THE FRÖLICHER-NIJENHUIS BRACKET AND THE GEOMETRY OF $G_2$-AND Spin(7)-MANIFOLDS

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Abstract. We extend the characterization of the integrability of an almost complex structure $J$ on differentiable manifolds via the vanishing of the Frölicher-Nijenhuis bracket $[J, J]^{FN}$ to an analogous characterization of torsion-free $G_2$-structures and torsion-free Spin(7)-structures. We also explain the Fernández-Gray classification of $G_2$-structures and the Fernández classification of Spin(7)-structures in terms of the Frölicher-Nijenhuis bracket.

1. Introduction

A $G_2$-structure on a 7-dimensional manifold $M^7$ is a 3-form $\varphi \in \Omega^3(M^7)$ which at each point $p \in M^7$ is contained in a certain open subset of $\Lambda^3 T^*_p M^7$; similarly, a Spin(7)-structure on an 8-dimensional manifold $M^8$ is given by a 4-form $\Phi \in \Omega^4(M^8)$ which at each point is contained in a certain subset of $\Lambda^4 T^*_p M^8$. Such structures induce both an orientation and a Riemannian metric on the underlying manifold, denoted by $g_\varphi$ and $g_\Phi$, respectively, and $\Phi$ is self-dual w.r.t. this metric.

Manifolds with $G_2$-structures have first been investigated by Fernández and Gray [FG1982], and Spin(7)-structures by Fernández [Fernandez1986] who showed that the covariant derivatives $\nabla \varphi \in \Omega^1(M^7, \Lambda^3 T^* M^7)$ and $\nabla \Phi \in \Omega^1(M^8, \Lambda^4 T^* M^8)$, respectively, decompose into four irreducible components in case of $G_2$-structures and into two irreducible components in the case of Spin(7)-structures. Thus, the conditions of the vanishing of some of these components yield $2^4 = 16$ classes of $G_2$-structure and $2^2 = 4$ classes of Spin(7)-structures, respectively, and the underlying geometries were discussed in [FG1982] and [Fernandez1986]; see also Section 5 below.

A $G_2$-structure (Spin(7)-structure, respectively) is called torsion-free, if $\varphi$ ($\Phi$, respectively) is parallel. As it turns out, the parallelity of $\varphi$ and $\Phi$, respectively, is equivalent to $\varphi$ and $\Phi$ being harmonic forms.

Alternatively, $G_2$- and Spin(7)-structures may be characterized via certain (2-fold or 3-fold) cross products on the tangent bundle. These are given as
the sections
\[ Cr_\varphi := \delta_g \varphi \in \Omega^2(M^7, TM^7), \quad \chi_\varphi := -\delta_g \ast \varphi \in \Omega^3(M^7, TM^7) \]
in case of $G_2$-structures, and as
\[ P_\Phi := -\delta_g \Phi \in \Omega^3(M^8, TM^8), \]
where $\delta_g : \Omega^{k+1}(M) \to \Omega^k(M, TM)$ is the contraction of a differential form with the Riemannian metric $g$ and have natural interpretations via octonion multiplication. The triple cross product $\chi \in \Omega^3(M^7, TM^7)$ on a manifold with a $G_2$-structure has been introduced by Harvey-Lawson [HL1982] and was used in many papers on deformation of associative submanifolds, see e.g. [McLean1998], [Kawai2014], [LV2016]. The 3-fold cross product $P$ on $\mathbb{R}^8$ has been first explicitly constructed by Brown and Gray [BG1967]. They also proved that (up to the $G_2$-action) there are exactly two non-equivalent 3-fold cross products on $\mathbb{R}^8 = \mathbb{O}$. In [HL1982] Harvey and Lawson intensively used the 3-fold cross product on $\mathbb{R}^8$ which is related to the Cayley 4-form and hence is invariant under the action of Spin(7). Fernandez showed the uniqueness of a Spin(7)-invariant 4-form on $\mathbb{R}^8$ (up to a multiplicative constant) and used the associated 3-fold cross product to classify Spin(7)-structures on 8-manifolds [Fernandez1986].

In this article, we view these cross products as elements of the Frölicher-Nijenhuis Lie algebra $\Omega^*(M, TM)$. Namely, it was shown by Frölicher-Nijenhuis in [FN1956] that $\Omega^*(M, TM)$ can be given the structure of a graded Lie algebra using the Frölicher-Nijenhuis bracket $[\cdot, \cdot]_{FN}$ in a natural way. Thus, given a manifold with a $G_2$-structure $(M^7, \varphi)$, we may consider the Frölicher-Nijenhuis brackets
\[ [Cr_\varphi, \chi_\varphi]_{FN} \in \Omega^5(M^7, TM^7), \quad [\chi_\varphi, \chi_\varphi]_{FN} \in \Omega^6(M^7, TM^7), \]
(observe that $[Cr, Cr]_{FN} = 0$ due to graded skew-symmetry), and analogously, for a manifold with a Spin(7)-structure $(M^8, \Phi)$ we may consider
\[ [P_\Phi, P_\Phi]_{FN} \in \Omega^6(M^8, TM^8). \]
These brackets may be regarded as a natural generalization of the Nijenhuis tensor of an almost complex structure $J$. Indeed, regarding such a structure as an element $J \in \Omega^1(M, TM)$, it turns out that $[J, J]_{FN} \in \Omega^2(M, TM)$ coincides – up to a constant multiple – with the Nijenhuis tensor of $J$, whence $J$ is integrable if and only if $[J, J]_{FN} = 0$ [FN1956b].

Our main result is that the Frölicher-Nijenhuis bracket also characterizes the torsion-freeness of $G_2$- and Spin(7)-structures, respectively. Namely, we show the following.

**Theorem 1.1.** Let $(M^7, \varphi)$ be a manifold with a $G_2$-structure and the associated Riemannian metric $g = g_\varphi$, and let $\nabla$ be the Levi-Civita connection of $g$. Then for every $p \in M^7$ the following are equivalent.

1. The $G_2$-structure is torsion-free at $p$, i.e., $(\nabla \varphi)_p = 0$.
2. $[Cr_\varphi, \chi_\varphi]_{FN} = 0 \in \Lambda^5 T_p^* M^7 \otimes T_p M^7$. 

(3) \([\chi_\varphi, \chi_\varphi]^{FN}_p = 0 \in \Lambda^6 T^*_p M^7 \otimes T_p M^7\).

In fact, we show in Theorem 3.7 that \((\nabla \varphi)_p\) is characterized by either \([\chi_\varphi, \chi_\varphi]^{FN}_p\), or by the projection of \([Cr_\varphi, \chi_\varphi]^{FN}_p\) onto a subspace isomorphic to \(T_p M^7 \otimes T_p M^7\).

**Theorem 1.2.** Let \((M^8, \Phi)\) be a manifold with a Spin(7)-structure, and let \(\nabla\) be the Levi-Civita connection of the associated Riemannian metric \(g = g_\Phi\). Then for every \(p \in M^8\) the following are equivalent.

1. The Spin(7)-structure is torsion-free at \(p\), i.e., \((\nabla \Phi)_p = 0\).
2. \([P_\Phi, P_\Phi]^{FN}_p = 0 \in \Lambda^6 T^*_p M^8 \otimes T_p M^8\).

Namely, we show in Theorem 4.6 that \((\nabla \Phi)_p\) is characterized by the projection of \([P_\Phi, P_\Phi]^{FN}_p\) onto a subspace isomorphic to \(W^7_p \otimes T_p M^8\) for some rank-7 bundle \(\Lambda^6 T^* M^8 \supset W^7 \to M^8\).

These explicit descriptions also allow us to give a complete characterization of the 16 cases of \(G_2\)-structures in terms of \([Cr_\varphi, \chi_\varphi]^{FN}_p\) and of the 4 classes of Spin(7)-structures in terms of \([P_\Phi, P_\Phi]^{FN}_p\); cf. Section 4.

Our paper is organized as follows. In Section 2 we recall the Frölicher-Nijenhuis bracket on \(\Omega^*(M, TM)\). Then we turn to the case of \(G_2\)-structures in Section 3 characterizing the torsion endomorphism and showing the results that lead us to Theorem 1.1. In Section 4 we repeat this discussion for the case of Spin(7)-structures which leads to Theorem 1.2. Finally, the characterization of the 16 classes of \(G_2\)-structures and the 4 classes of Spin(7)-structures in terms of the Frölicher-Nijenhuis bracket is given in Section 5. The appendix then contains the proofs of some identities on representations of \(G_2\) and Spin(7) which are used throughout the paper.

2. Preliminaries

2.1. The Frölicher-Nijenhuis bracket. Let \(M\) be a manifold and \((\Omega^*(M), \wedge) = (\bigoplus_{k \geq 0} \Omega^k(M), \wedge)\) be the graded algebra of differential forms. We shall use superscripts to indicate the degree of a form, i.e., \(\alpha^k\) denotes an element of \(\Omega^k(M)\).

Evidently, contraction \(\iota_X : \Omega^k(M) \to \Omega^{k-1}(M)\) with a vector field \(X \in \mathfrak{X}(M)\) is a derivation of degree \(-1\). More generally, for \(K \in \Omega^k(M, TM)\) we define \(\iota_K \alpha^l\) as the contraction of \(K\) with \(\alpha^l \in \Omega^l(M)\) pointwise by

\[\iota_{K \otimes X} \alpha^l := \kappa^k \wedge (\iota_X \alpha^l) \in \Omega^{k+l-1}(M),\]

where \(\kappa^k \in \Omega^k(M)\) and \(X \in \mathfrak{X}(M)\) is a vector field, and this is a derivation of \(\Omega^*(M)\) of degree \(l-1\). Thus, the Nijenhuis-Lie derivative along \(K \in \Omega^k(M, TM)\) defined as

\[\mathcal{L}_K(\alpha^l) := [\iota_K, d](\alpha^l) = \iota_K(d\alpha^l) + (-1)^k d(\iota_K \alpha^l) \in \Omega^{k+l}(M)\]

is a derivation of \(\Omega^*(M)\) of degree \(k\).
Observe that for $k = 0$ in which case $K \in \Omega^0(M, TM)$ is a vector field, both $i_K$ and $\mathcal{L}_K$ coincide with the standard notion of contraction with and Lie derivative along a vector field.

In [FN1956] [FN1956b], it was shown that $\Omega^r(M, TM)$ can be given a unique Lie algebra structure, called the Frölicher-Nijenhuis bracket and denoted by $[\cdot, \cdot]^{FN}$, such that $\mathcal{L}$ defines an action of $\Omega^r(M, TM)$ on $\Omega^r(M)$, that is,

$$\mathcal{L}_{[K_1, K_2]}^{FN} = [\mathcal{L}_{K_1}, \mathcal{L}_{K_2}] =: \mathcal{L}_{K_1} \circ \mathcal{L}_{K_2} - (-1)^{|K_1||K_2|} \mathcal{L}_{K_2} \circ \mathcal{L}_{K_1}. \quad (2.2)$$

It is given by the following formula for $\alpha^k \in \Omega^k(M)$, $\beta^l \in \Omega^l(M)$, $X_1, X_2 \in \mathfrak{x}(M)$ [KMS1993 Theorem 8.7 (6), p. 70]:

$$[\alpha^k \otimes X_1, \beta^l \otimes X_2]^{FN} = \alpha^k \wedge \beta^l \otimes [X_1, X_2]$$

$$+ \alpha^k \wedge \mathcal{L}_{X_1} \beta^l \otimes X_2 - \mathcal{L}_{X_2} \alpha^k \wedge \beta^l \otimes X_1$$

$$+ (-1)^{|k|} (d\alpha^k \wedge (\iota_{X_1} \beta^l) \otimes X_2 + (\iota_{X_2} \alpha^k) \wedge d\beta^l \otimes X_1). \quad (2.3)$$

In particular, for a vector field $X \in \mathfrak{x}(M)$ and $K \in \Omega^r(M, TM)$ we have [KMS1993 Theorem 8.16 (5), p. 75]

$$\mathcal{L}_X(K) = [X, K]^{FN},$$

that is, the Frölicher-Nijenhuis bracket with a vector field coincides with the Lie derivative of the tensor field $K \in \Omega^r(M, TM)$ which means that $\exp(tX) : \Omega^r(M, TM) \to \Omega^r(M, TM)$ is the action induced by (local) diffeomorphisms of $M$.

**Example 2.1.** Let $A \in \Omega^1(M, TM)$ be an endomorphism field on $M$. Then [KMS1993 Remark 8.17, p. 75]

$$[A, A]^{FN} = 2[A, A]_N,$$

where $[A, A]_N$ is the Nijenhuis tensor of $A$. W.r.t. a local frame $(e_i)$ with dual frame $(e^i)$ we can write $A = e^i \otimes A e_i$, whence

$$\mathcal{L}_A \alpha^k \overset{2.1}{=} e^i \wedge (\iota_{A e_i} d\alpha^k) - d(e^i \wedge (\iota_{A e_i} \alpha^k))$$

$$= A \cdot d\alpha^k - d(A \cdot \alpha^k), \quad (2.4)$$

where we denote by $\cdot$ the pointwise action of $A_p \in \text{End}(T_pM)$ on $\Lambda^k T^*_p M$. Observe that by (2.2) we have $\mathcal{L}_{[A, A]}^{FN} = 2(\mathcal{L}_A)^2$, so that the derivation $\mathcal{L}_A : \Omega^k(M) \to \Omega^{k+1}(M)$ is a differential iff $[A, A]_N = 0$.

For instance, if $A = Id$ then $I \cdot \alpha^k = e^i \wedge (\iota_{e_i} \alpha^k) = k\alpha^k$, so that

$$\mathcal{L}_I \alpha^k \overset{2.4}{=} I \cdot d\alpha^k - d(I \cdot \alpha^k) = (k+1)d\alpha^k - d(\alpha^k) = d\alpha^k.$$

To see another example, let $A = J$ be an almost complex structure. Then $[J, J]^{FN} = 2[J, J]_N = 0$ iff $J$ is integrable, and in this case one calculates from (2.4) that $\mathcal{L}_J = -d^c = i(\partial - \bar{\partial})$ is the negative of the complex differential, where $d = \partial + \bar{\partial}$ is the decomposition into the holomorphic and
anti-holomorphic part of $d$. In particular, $H^*_j(\Omega^*(M)) \cong H^*_{dR}(M)$ coincides with the deRham cohomology.

We end this section by providing a formula for the Frölicher-Nijenhuis bracket for those types of forms which we shall be concerned with. Recall from the introduction that on a Riemannian manifold $(M, g)$ we define the map

$$\delta = \delta_g : \Lambda^k V^* \to \Lambda^k V \otimes V, \quad \delta_g(\alpha^k) := (\iota_{e_i} \alpha^k) \otimes (e^i)^\#,$$

(2.5) taking the sum over some basis $(e_i)$ of $T_p M$ with dual basis $(e^i)$ of $T^*_p M$. This implies that to each $\Psi \in \Omega^{k+1}(M)$ we may associate a section $\delta_g(\Psi) \in \Omega^k(M, TM)$.

**Proposition 2.2.** Let $(M, g)$ be an $n$-dimensional Riemannian manifold of dimension $n$ and let $\Psi_l \in \Omega^{k+1}(M)$, $l = 1, 2$. Moreover, let

$$K_l := \delta_g(\Psi_l) \in \Omega^k(M, TM)$$

with the map $\delta_g$ from (2.5).

Then the Frölicher-Nijenhuis bracket at $p \in M$ is given as

$$[K_1, K_2]^{FN}_{p} = \left((\iota_{e_i} \Psi_1) \wedge (\iota_{e_j} \nabla_{e_i} \Psi_2) - (-1)^{k_1}(\iota_{e_j} \iota_{e_i} \Psi_1) \wedge e^k \wedge \iota_{e_i} \nabla_{e_k} \Psi_2ight)$$

$$- \left((-1)^{k_1} \iota_{e_j} \nabla_{e_i} \Psi_1) \wedge (\iota_{e_i} \Psi_2) - (-1)^{k_1} e^k \wedge \iota_{e_i} \nabla_{e_k} \Psi_1 \wedge (\iota_{e_j} \iota_{e_i} \Psi_2)\right) \otimes (e^j)^\#,$$

where $(e_i)$ is an arbitrary basis of $T_p M$ with dual basis $(e^i)$ of $T^*_p M$. In particular, if $K_1 = K_2 =: K$ and $k_1 = k_2$ is odd, then

$$[K, K]^{FN}_p = 2\left((\iota_{e_i} \Psi) \wedge (\iota_{e_j} \nabla_{e_i} \Psi) + (\iota_{e_j} \iota_{e_i} \Psi) \wedge e^k \wedge \iota_{e_i} \nabla_{e_k} \Psi\right) \otimes (e^j)^\#.$$

**Remark 2.3.** If $K_1 = K_2 = K$ and $k_1 = k_2$ is even, then $[K, K]^{FN}_p = 0$ due to the graded skew symmetry of the bracket. Furthermore, observe that $(e^j)^\# = e_j$ in case $(e_i)$ is an orthonormal basis.

**Proof.** Evidently, if this formula holds for some basis $(e_j)$ with dual basis $(e^j)$, then it holds for any basis. Therefore, it suffices to show the assertion for an orthonormal basis $(e_j)$ in which case $(e^j)^\# = e_j$.

Choose geodesic normal coordinates $(x^i)$ around $p \in M$ in such a way that $(\partial_i)_p := (\partial/\partial x^i)_p$ is an orthonormal basis of $T_p M$. The dual basis of $\partial_i$ is $dx^i$, whence $(dx^i)^\# = g^{ij}\partial_j$. Thus,

$$K_l = (\iota_{\partial_i} \Psi_l) \otimes (dx^i)^\# = g^{ij}(\iota_{\partial_i} \Psi_l) \otimes \partial_j.$$
Thus, by (2.3)

\[ [K_1, K_2]^{FN} = [g^{ij}(t\partial_i\Psi_1) \otimes \partial_j, g^{rs}(t\partial_r\Psi_2) \otimes \partial_s]^{FN} \]

\[ = (g^{ij}(t\partial_i\Psi_1) \wedge \mathcal{L}_{\partial_j}(g^{rs}(t\partial_r\Psi_2)) \otimes \partial_s \]

\[ - \mathcal{L}_{\partial_j}(g^{ij}(t\partial_i\Psi_1)) \wedge g^{rs}(t\partial_r\Psi_2) \otimes \partial_s \]

\[ + (-1)^{k_1}d(g^{ij}(t\partial_i\Psi_1)) \wedge \nabla_{\partial_j}(g^{rs}(t\partial_r\Psi_2)) \otimes \partial_s \]

\[ + (-1)^{k_1}(t\partial_j(g^{ij}(t\partial_i\Psi_1)) \wedge d(g^{rs}(t\partial_r\Psi_2)) \otimes \partial_s. \]

Since at \( p \), \( g_{ij} = g^{ij} = \delta_{ij}, \partial_p g_{ij} = 0, \mathcal{L}_{\partial_j}\Psi = \nabla_{\partial_j}\Psi, \nabla_{\partial_j}\partial_j = 0, \) and \( \partial_j = (e^j)^\# \), the asserted formula follows. \( \square \)

3. Cross products and \( G_2 \)-structures

3.1. \( G_2 \)-structures and associated cross products. In this section we collect some basic facts on \( G_2 \)-structures, see e.g. [Humphreys1978], [Bryant1987], [FG1982], [HL1982] for references.

Let \( M \) be an oriented 7-manifold. A \( G_2 \)-structure on \( M \) is a 3-form \( \varphi \in \Omega^3(M) \) such that at each \( p \in M \) there is a positively oriented basis \( (e_i) \) of \( T_pM \) with dual basis \( (e^i) \) such that

\[ \varphi_p = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356}, \]

where \( e^{i_1 \cdots i_k} \) is short for \( e^{i_1} \wedge \cdots \wedge e^{i_k} \). We call such a basis a \( G_2 \)-frame. The stabilizer of \( \varphi_p \) is isomorphic to the exceptional group \( G_2 \), and there is a unique \( G_2 \)-invariant Riemannian metric \( g_\varphi \) on \( M \) such that each \( G_2 \)-frame is orthonormal. In particular, the Hodge-dual of \( \varphi \) w.r.t. \( g_\varphi \) is given by

\[ *_{g_\varphi} \varphi = e^{4567} + e^{2367} + e^{2345} + e^{1357} - e^{1346} - e^{1256} - e^{1247}. \]

The set of \( G_2 \)-frames yields a principal \( G_2 \)-bundle

\[ G_2(M) = G_2(M, \varphi) \to M, \]

whence for each \( G_2 \)-module \( V \) we denote by

\[ (3.3) \quad V(M) := G_2(M) \times_{G_2} V \to M \]

the associated vector bundle over \( M \). For instance,

\[ V_7(M) \cong TM \cong T^*M. \]

**Definition 3.1.** [HL1982] Let \( (M, \varphi) \) be an oriented manifold with a \( G_2 \)-structure \( \varphi \) and the induced Riemannian metric \( g = g_\varphi \). Then the \( TM \)-valued forms \( Cr_\varphi \in \Omega^2(M, TM) \) and \( \chi_\varphi \in \Omega^3(M, TM) \) are defined by

\[ Cr_\varphi := \delta_{g_\varphi}(\varphi) \quad \text{and} \quad \chi_\varphi := -\delta_g(*\varphi), \]

and are called the 2-fold and 3-fold cross product on \( M \), respectively. That is, for \( x, y, z, w \in TM \) we have

\[ g_\varphi(Cr_\varphi(x, y), z) = \varphi(x, y, z), \quad g_\varphi(\chi_\varphi(x, y, z), w) = *_\varphi(x, y, z, w). \]
We shall usually suppress the indices \( \varphi \) for \( g, Cr \) and \( \chi \) if it is clear from the context which \( G_2 \)-structure \( \varphi \) is used.

**Remark 3.2.** If we use the \( G_2 \)-structure to identify each \( T_p M \cong Im \mathbb{O} \) with the imaginary octonians, then \( Cr \) and \( \chi \) can be interpreted w.r.t. the octonian product \( \cdot : \mathbb{O} \times \mathbb{O} \to \mathbb{O} \) as

\[
Cr(x, y) := (x \cdot y)_{Im \mathbb{O}} \quad \text{and} \quad \chi(x, y, z) := ((x \cdot y) \cdot z - x \cdot (y \cdot z))_{Im \mathbb{O}}.
\]

We summarize important known facts about the decomposition of tensor products of \( G_2 \)-modules into irreducible summands which are well known, see e.g. [Kar2005, Section 2]. We denote by \( V_k \) the \( k \)-dimensional irreducible \( G_2 \)-module if there is a unique such module. For instance, \( V_7 \) is the irreducible 7-dimensional \( G_2 \)-module from above, and \( V_7^* \cong V_7 \). For its exterior powers, we obtain the decompositions

\[
\Lambda^0 V_7 \cong V_1, \quad \Lambda^1 V_7 \cong V_7, \quad \Lambda^2 V_7 \cong V_7 \oplus V_14,
\]

where \( \Lambda^k V_7 \cong \Lambda^{7-k} V_7 \) due to the Hodge isomorphism. We denote by \( \Lambda^k V_7 \subset \Lambda^k V_7 \) the subspace isomorphic to \( V_7 \) in the above notation. Evidently, \( \Lambda^1 V_7 \) and \( \Lambda^2 V_7 \) are spanned by \( \varphi \) and \(*\varphi\), respectively. For the decompositions of \( \Lambda^2 V_7 \) and \( \Lambda^3 V_7 \) the following descriptions are well known.

\[
\Lambda^2 V_7 = \{v \varphi \mid v \in V_7\},
\]

\[
\Lambda^2 V_7 = \{\alpha^2 \in \Lambda^2 V_7 \mid *\varphi \wedge \alpha^2 = 0\},
\]

\[
\Lambda^2 V_7 = \mathbb{R} \varphi,
\]

\[
\Lambda^3 V_7 = \{v \varphi \mid v \in V_7\},
\]

\[
\Lambda^3 V_7 = \mathbb{R} \ast \varphi,
\]

\[
\Lambda^3 V_7 = \varphi \wedge V_7 = \{\varphi \wedge v \mid v \in V_7\},
\]

\[
\Lambda^3 V_7 = \ast \varphi \wedge V_7 = \{\ast \varphi \wedge v \mid v \in V_7\},
\]

\[
\Lambda^5 V_7 = \{\alpha^5 \in \Lambda^5 V_7 \mid \alpha^5 \wedge (t_v \varphi) = 0 \text{ for all } v \in V_7\}.
\]

We also point out that all representations of \( G_2 \) are of real type, meaning that for any real irreducible representation \( V \) of \( G_2 \) the complexified space \( V^C := V \otimes \mathbb{C} \) is (complex) irreducible; equivalently, a real irreducible representation of \( G_2 \) does not admit a \( G_2 \)-invariant complex structure.

These decompositions are used in the appendix to obtain many formulas which will be used in the sequel.

### 3.2. The torsion of manifolds with a \( G_2 \)-structure

Let \((M, \varphi)\) be a manifold with a \( G_2 \)-structure with the corresponding Riemannian metric \( g = g_\varphi \), and let \( \nabla \) be the Levi-Civita connection of \( g \). In general, \( \varphi \) and \(*\varphi\) will not be parallel w.r.t. \( \nabla \), and the failure of their parallelity can be
described in the following way which is essentially a reformulation of the intrinsic torsion of a $G_2$-structure discussed in [Bryant2005] and [Kar2005].

**Proposition 3.3.** Let $(M, \varphi)$ be a manifold with a $G_2$-structure with associated Riemannian metric $g = g_\varphi$ and Levi-Civita connection $\nabla$. Then there is a section $T \in \Omega^1(M, TM) = \Gamma(\text{End}(TM))$ such that for all $v \in TM$ we have

\begin{equation}
\nabla_v \varphi = \iota_{T(v)} \ast \varphi \quad \text{and} \quad \nabla_v \ast \varphi = -(T(v))^\flat \wedge \varphi.
\end{equation}

Thus, the section $T \in \Omega^1(M, TM)$ measures how $\varphi$ fails to be parallel, and this has been described in Fernández and Gray [FG1982] by slightly different means. In fact, it contains the same information as the intrinsic torsion of the $G_2$-structure in the sense of [Bryant2005], whence we use the following terminology.

**Definition 3.4.** Let $(M, \varphi)$ be a manifold with a $G_2$-structure. The section $T \in \Omega^1(M, TM)$ for which (3.6) holds is called the torsion endomorphism of the $G_2$-structure.

For an orthonormal frame $(e_i)$ of $T_pM$ we define the coefficients of $T$ by

\begin{equation}
t_{ij} := \langle T(e_i), e_j \rangle, \quad \text{so that} \quad T(e_i) = t_{ij} e_j.
\end{equation}

Furthermore, we define the form

\begin{equation}
\tau := t_{ij} e^{ij} = \frac{1}{2}(t_{ij} - t_{ji}) e^{ij} = e_i \wedge T(e_i)^\flat \in \Lambda^2 V_7^*.
\end{equation}

For the exterior derivatives of $\varphi$ and $\ast \varphi$, we have

\begin{equation}
d \varphi_p = T_p^\top (e_i)^\flat \wedge (\iota_{e_i} \ast \varphi_p) \quad \text{and} \quad d \ast \varphi_p = -\tau_p \wedge \varphi_p,
\end{equation}

where we sum over an orthonormal basis $(e_i)$ of $T_pM$ in the first equation and where $T_p^\top$ denotes the transpose matrix of $T_p$. In particular, it is now a straightforward calculation to show that $(M, \varphi)$ is torsion free at $p \in M$ (i.e., $T_p = 0$) iff $d \varphi_p = 0$ and $d \ast \varphi_p = 0$ (cf. [FG1982]).

### 3.3. The Frölicher-Nijenhuis brackets on a manifold with a $G_2$-structure

In this section, we shall compute part of their Frölicher-Nijenhuis brackets of the sections $Cr = \delta_\varphi \in \Omega^2(M, TM)$ and $\chi = -\delta_\varphi \ast \varphi \in \Omega^3(M, TM)$. from Definition 3.1 on a manifold $M$ with a $G_2$-structure $\varphi$.

The Frölicher-Nijenhuis bracket $[Cr, Cr]^{FN}$ vanishes identically due to the graded skew-symmetry of the bracket. On the other hand, the Frölicher-Nijenhuis brackets $[Cr, \chi]^{FN}$ and $[\chi, \chi]^{FN}$ are elements of $\Omega^5(M, TM)$ and $\Omega^6(M, TM)$, respectively.

Due to the decomposition $\Lambda^5 V_7 = \Lambda^5_7 V_7 \oplus \Lambda^5_{14} V_7$ as a $G_2$-module, we may decompose

\begin{equation}
\Omega^5(M, TM) = \Gamma(M, \Lambda^5_7 T^* M \otimes TM) \oplus \Gamma(M, \Lambda^5_{14} T^* M \otimes TM),
\end{equation}

where $\Lambda^5_7 V_7$ and $\Lambda^5_{14} V_7$ are the $(3,2)$ and $(1,4)$-forms, respectively.
and we denote the projections onto the two summands by \( \pi_7 \) and \( \pi_{14} \), respectively. We now wish to show Theorem 1.1 from the introduction. In order to work towards the proof, we first calculate \( \pi_7([CR, \chi]_{p}^{FN}) \).

**Proposition 3.5.** Let \((M, g, \varphi)\) be a manifold with a \(G_2\)-structure and let \(T \in \Omega^1(M, TM)\) be its torsion endomorphism. Then for each \(p \in M\),

\[
\pi_7([CR, \chi]_{p}^{FN}) = 2 \ast \varphi \wedge \left( T_p^\top - 2T_p - tr(T_p) \right) e_i \otimes e_i,
\]

summing over an orthonormal basis \((e_i)\) of \(T_p M\), where \(T_p^\top\) denotes the transpose of \(T\). In particular, \(\pi_7([CR, \chi]_{p}^{FN}) = 0\) if and only if \(T_p = 0\) and only if \([CR, \chi]_{p}^{FN} = 0\).

**Proof.** We fix \(p \in M\) and use normal coordinates around \(p\). Then in order to calculate \([CR, \chi]_{p}^{FN}\) we apply Proposition 2.2 to \(\Psi_1 = \varphi\) and \(\Psi_2 = \ast \varphi\) and obtain

\[
- [CR, \chi]_{p}^{FN} = \left[ (t_{e_i} \varphi) \otimes e_i, (t_{e_j} \ast \varphi) \otimes e_j \right]_{p}^{FN} =: \beta_j \otimes e_j,
\]

where

\[
\beta_j = (t_{e_k} \varphi) \wedge (t_{e_j} \nabla_{e_k} \ast \varphi) - (t_{e_k} \ast \varphi) \wedge (t_{e_j} \nabla_{e_k} \varphi) - (t_{e_k} \ast \varphi) \wedge (t_{e_j} \nabla_{e_k} \varphi) - (t_{e_j} \ast \varphi) \wedge (t_{e_k} \nabla_{e_j} \varphi).
\]

Decomposing \(\Lambda^5 T_p^\ast M\) according to \(\ref{3.3} \), we write \(\beta_j = \ast \varphi \wedge v_j^5 + \beta_j^{14}\) with \(v_j \in T_p M\) and \(\beta_j^{14} \in \Lambda^5 T_p^\ast M\), so that \(\pi_7([CR, \chi]_{p}^{FN}) = \ast \varphi \wedge v_j^5 \otimes e_j\). Let

\[
\ast = (t_{e_i} \varphi) \wedge \beta_j.
\]

Then as \((t_{e_i} \varphi) \wedge \beta_j^{14} = 0\) by \(\ref{3.3}\) and \((t_{e_i} \varphi) \wedge \ast = 3(e_i, v_j)\) vol by \(\ref{6.1}\), it follows that

\[
\pi_7(-[CR, \chi]_{p}^{FN}) = \frac{1}{3} b_{ij} (\ast \varphi \wedge e_i) \otimes e_j.
\]

In order to determine the coefficients \(b_{ij}\), we decompose \(\beta_j\) into the four summands from \(\ref{3.11}\). Then from the first summand we get

\[
(t_{e_i} \varphi) \wedge (t_{e_k} \varphi) \wedge (t_{e_j} \nabla_{e_k} \ast \varphi) = -(t_{e_i} \varphi) \wedge (t_{e_k} \varphi) \wedge (t_{e_j} (T(e_k)^a \wedge \varphi))
= -(t_{e_i} \varphi) \wedge (t_{e_k} \varphi) \wedge (t_{kj} \varphi - T(e_k)^a \wedge (t_{e_j} \varphi))
\]

\[
= -6 t_{kj} \delta_{ik} \text{vol} + 2(t_{ik} \delta_{kj} + t_{kk} \delta_{ij}) \text{vol}
\]

\[
= 2(t_{ji} - 2t_{ij} + tr(T) \delta_{ij}) \text{vol}.
\]

From the second summand we obtain
Thus, adding (3.14) through (3.17), we get from (3.11) that
\[
-(t_{e_i} \varphi) \wedge (t_{e_j} t_{e_k} \varphi) \wedge e^l \wedge (t_{e_k} \nabla_{e_l} \varphi)
\]
\[
= (t_{e_i} \varphi) \wedge (t_{e_j} t_{e_k} \varphi) \wedge e^l \wedge (t_{e_k} (T(e_l))^\varphi \wedge \varphi)
\]
\[
= (t_{e_i} \varphi) \wedge (t_{e_j} t_{e_k} \varphi) \wedge e^l \wedge (t_{l k} \varphi - T(e_l))^\varphi \wedge (t_{e_k} \varphi)
\]
\[
= t_{l k} \left(2(\delta_{k l} \delta_{j i} - \delta_{j i} \delta_{k l}) \text{vol} - 2e^{j k l i} \wedge \varphi \right)
\]
\[
- (t_{e_i} \varphi) \wedge (t_{e_j} t_{e_k} \varphi) \wedge e^l \wedge T(e_l))^\varphi \wedge (t_{e_k} \varphi)
\]
\[
= 2(t_{r l} \delta_{j i} - t_{j i}) \text{vol} - 2e^{j i} \wedge \tau \wedge \varphi
\]
\[
- \frac{1}{2}(t_{e_i} \varphi) \wedge t_{e_j} ((t_{e_k} \varphi) \wedge (t_{e_k} \varphi)) \wedge \tau
\]
\[
= 2(t_{r l} \delta_{j i} - t_{j i}) \text{vol} - 2e^{j i} \wedge \tau \wedge \varphi
\]
\[
- 3(t_{e_i} \varphi) \wedge (t_{e_j} \varphi) \wedge \tau
\]
\[
= 2(t_{r l} \delta_{j i} - t_{j i}) \text{vol} + e^{j i} \wedge \tau \wedge \varphi
\]
\[
+ 2t_{j i} (\delta_{j k} \delta_{k l} - \delta_{j l} \delta_{k l}) \text{vol}
\]
\[
= 2(t_{r l} \delta_{j i} - t_{j i}) \text{vol} + e^{j i} \wedge \tau \wedge \varphi.
\]

From the third term in (3.11) we get
\[
-(t_{e_i} \varphi) \wedge (t_{e_j} \nabla_{e_k} \varphi) \wedge (t_{e_k} \varphi)
\]
\[
= - (t_{e_i} \varphi) \wedge (t_{e_j} T(e_k))^\varphi \wedge (t_{e_k} \varphi)
\]
\[
= -2(t_{k k} \delta_{j i} - \delta_{j k} t_{k l}) \text{vol} - e^j \wedge T(e_k)^\varphi \wedge e^{k i} \wedge \varphi
\]
\[
= -2(t_{r l} \delta_{j i} - t_{j i}) \text{vol} - e^{j i} \wedge \tau \wedge \varphi.
\]

Finally, from the last term in (3.11) we get
\[
-(t_{e_i} \varphi) \wedge e^l \wedge (t_{e_k} \nabla_{e_l} \varphi) \wedge (t_{e_j} t_{e_k} \varphi)
\]
\[
= -(t_{e_i} \varphi) \wedge e^l \wedge (t_{e_k} T(e_l))^\varphi \wedge (t_{e_j} t_{e_k} \varphi)
\]
\[
= 2(t_{e_i} \varphi) \wedge e^l \wedge (T(e_l)\varphi) \wedge (t_{e_j} \varphi)
\]
\[
= 4(t_{l l} \delta_{j i} + \delta_{l i} t_{j l} + t_{l l} \delta_{l i}) \text{vol}
\]
\[
= 4(t_{j i} + t_{i j} + t r(T) \delta_{i j}) \text{vol}.
\]

Thus, adding (3.14) through (3.17), we get from (3.11) that
Proof. if and only if \( \Omega^2 \) summing over an orthonormal basis

\[
(3.18) \quad [\chi, \chi]_{p}^{FN} = -4 * (T_p + T_p^\top)(e_i) \otimes e_i + 6e^i \wedge \tau_p \wedge \varphi \otimes e_i,
\]

summing over an orthonormal basis \((e_i)\) of \( T_p M \). In particular, \([\chi, \chi]_{p}^{FN} = 0\) if and only if \( T_p = 0 \).

\[\text{Proposition 3.6.}\] Let \((M, g, \varphi)\) be a manifold with a \(G_2\)-structure and let \( T \in \Omega^1(M, TM)\) be its torsion endomorphism with the associated form \( \tau \in \Omega^2(M) \) from \((3.8)\). Then for each \( p \in M \),

\[
(3.18) \quad [\chi, \chi]_{p}^{FN} = -4 * (T_p + T_p^\top)(e_i) \otimes e_i + 6e^i \wedge \tau_p \wedge \varphi \otimes e_i,
\]

Next, let us consider the bracket \([\chi, \chi]_{p}^{FN}\).

Proof. According to Proposition 2.2 we have

\[
[\chi, \chi]_{p}^{FN} = \gamma_j \otimes e_j,
\]

where

\[
(3.19) \quad \gamma_j = 2((\tau_{ek} \ast \varphi) \wedge (e_j \nabla_{ek} \ast \varphi) + (\tau_{ek} \ast \varphi) \wedge e^i \wedge (\tau_{ek} \nabla_{ei} \ast \varphi)).
\]

Now let \( c_{ij} := *e^i \wedge \gamma_j \). Then

\[
(3.20) \quad [\chi, \chi]_{p}^{FN} = c_{ij} \ast e^i \otimes e_j,
\]

In order to evaluate the coefficients \( c_{ij} \), we consider the two summands in \((3.19)\) separately, and obtain from the first one

\[
e^i \wedge (\tau_{ek} \ast \varphi) \wedge (e_j \nabla_{ek} \ast \varphi) \overset{(3.6)}{=} -e^i \wedge (\tau_{ek} \ast \varphi) \wedge (e_j (T(\epsilon_k)^\flat \wedge \varphi)) = -t_{kj} e^i \wedge (\tau_{ek} \ast \varphi) \wedge \varphi + e^i \wedge (\tau_{ek} \ast \varphi) \wedge T(\epsilon_k)^\flat \wedge (\epsilon_j \varphi) \overset{(6.2), (6.8)}{=} -4t_{kj} \delta_{ik} \text{vol} + 2(\delta_{ik} t_{kj} - t_{kk} \delta_{ij}) \text{vol} + T(\epsilon_k)^\flat \wedge e^{ij} \wedge \varphi \overset{(3.21)}{=} -2(t_{ij} + tr(T) \delta_{ij}) \text{vol} + e^{ij} \wedge \tau \wedge \varphi.
\]

From the second summand in \((3.19)\) we calculate
\begin{align*}
e^i \wedge (t_{e_j} t_{e_k} \ast \varphi) & \wedge e^t \wedge (t_{e_k} \nabla_{e_t} \ast \varphi) \\
\quad & \quad \quad \text{(3.6)}
& = -e^i \wedge (t_{e_j} t_{e_k} \ast \varphi) \wedge (t_{e_k} (T(e_t))^b \wedge \varphi)) \\
& = -t_{lk} e^i \wedge (t_{e_j} t_{e_k} \ast \varphi) \wedge \varphi + e^t \wedge (t_{e_j} t_{e_k} \ast \varphi) \wedge \tau \wedge (t_{e_k} \varphi) \\
\quad & \quad \quad \text{(6.9)} \quad \quad \text{(6.14)}
& = -t_{lk} (2 \delta_{ij} \delta_{lk} - \delta_{ij} \delta_{lk}) \text{vol} - e^{ijk} \wedge \varphi) + 3 e^{ij} \wedge \tau \wedge \varphi
\quad & \quad \quad \text{(3.22)}
& = -2(t_{ij} - tr(T) \delta_{ij}) \text{vol} + 2 e^{ij} \wedge \tau \wedge \varphi.
\end{align*}

Thus, adding (3.21) and (3.22), equation (3.19) yields

\[ c_{ij} \text{vol} = e^i \wedge \gamma_j = 2 \left( -2(t_{ij} + tr(T) \delta_{ij}) \text{vol} + e^{ij} \wedge \tau \wedge \varphi \\
\quad - 2(t_{ji} - tr(T) \delta_{ij}) \text{vol} + 2 e^{ij} \wedge \tau \wedge \varphi \right) \\
\quad = 2(-2(t_{ij} + t_{ji}) + 3(*e^i, e^j \wedge \tau \wedge \varphi)) \text{vol}, \]

and from this and (3.20), the formula (3.18) follows.

In order to show the last statement, observe that \([\chi, \chi]_p^{FN} = 0\) iff \(c_{ij} = 0\) for all \(i, j\). Since then \(c_{ij} + c_{ji} = -8(t_{ij} + t_{ji})\), it follows that \(t_{ij} + t_{ji} = 0\) and hence \(0 = c_{ij} = 6 \ast (e^{ij} \wedge \tau \wedge \varphi)\) for all \(i, j\) which implies that \(\tau = t_{kl} e^{kl} = 0\) and hence, \(t_{kl} = t_{lk}\). All of this together implies that \(t_{ij} = 0\) for all \(i, j\), and hence, \(T_p = 0\) as asserted. \qed

We are now ready to show the following result which immediately implies Theorem 1.1 from the introduction.

**Theorem 3.7.** Let \((M^7, \varphi)\) be a manifold with a \(G_2\)-structure with associated metric \(g = g_{\varphi}\), let \(\nabla\) be the Levi-Civita connection of \(g\), and let \(T \in \Omega^1(M^7, TM^7)\) be its torsion endomorphism defined in Definition 3.4. Then for every \(p \in M^7\) the following are equivalent.

1. \(T_p = 0 \in T^*_p M^7 \otimes T_p M^7\).
2. The \(G_2\)-structure is torsion-free at \(p\), i.e., \((\nabla \varphi)_p = 0\).
3. \(\pi_7([\mathcal{C} \mathcal{R}, \chi]_p^{FN}) = 0 \in \Lambda^5 T^*_p M^7 \otimes T_p M^7\).
4. \([\mathcal{C} \mathcal{R}, \chi]_p^{FN} = 0 \in \Lambda^5 T^*_p M^7 \otimes T_p M^7\).
5. \([\chi, \chi]_p^{FN} = 0 \in \Lambda^6 T^*_p M^7 \otimes T_p M^7\).

**Proof.** The equivalence of the first two statements is well known, see e.g. [FC1982]. Proposition 3.5 shows the equivalence of the first and the third, whereas Proposition 3.6 shows the equivalence of the first and the last statement. That \((\nabla \varphi)_p = 0\) implies \([\mathcal{C} \mathcal{R}, \chi]_p^{FN} = 0\) is immediate from the formula of the bracket in Proposition 2.2 and obviously, \([\mathcal{C} \mathcal{R}, \chi]_p^{FN} = 0\) implies \(\pi_7([\mathcal{C} \mathcal{R}, \chi]_p^{FN}) = 0\). \qed
4. Cross products and Spin(7)-structures

4.1. Spin(7)-structures and associated cross products. The exposition in this section mainly follows the references [Bryant1987], [Fernandez1986], [HL1982].

Let $M$ be an oriented 8-manifold. A Spin(7)-structure on $M$ is a 4-form $\Phi \in \Omega^4(M)$ such that at each $p \in M$ there is a positively oriented basis $(e_\mu)_{\mu=0}^7$ of $T_pM$ with dual basis $(e^\mu)_{\mu=0}^7$ such that $\Phi_p \in \Lambda^4 T_pM$ is of the form

$$\Phi_p := e^{0123} + e^{0145} + e^{0246} - e^{0257} - e^{0347} - e^{0356} + e^{4567} + e^{2367} + e^{2345} + e^{1357} - e^{1346} - e^{1256} - e^{1247}.$$  

Throughout this section, we shall use Greek indices $\mu, \nu, \ldots$ to run over $0, \ldots, 7$, whereas Latin indices $i, j, \ldots$ range over $1, \ldots, 7$.

A basis $(e_\mu)$ of $T_pM$ whose dual basis $(e^\mu)$ satisfies (4.1) is called a Spin(7)-frame. Observe that if we define for a Spin(7)-frame $(e_\mu)$ the forms $\varphi_p$ and $\ast_7 \varphi_p$ on $V_p := \text{span}(e_i)_{i=1}^7 \subset T_pM$ as in (3.1) and (3.2), then

$$\Phi_p = e^0 \wedge \varphi_p + \ast_7 \varphi_p.$$  

The stabilizer of $\Phi_p$ is the group Spin(7) acting on $T_pM$ via the spinor representation, and there is a unique Spin(7)-invariant Riemannian metric $g_\Phi$ on $M$ such that each Spin(7)-frame is orthonormal. In particular, $\Phi$ is self-dual w.r.t. $g_\Phi$. The set of all Spin(7)-frames forms a principal Spin(7)-bundle

$$\text{Spin}(7)_M = \text{Spin}(7)_{(M,\Phi)} \longrightarrow M,$$

and again, for each Spin(7)-module $W$ we obtain the associated vector bundle

$$W(M) := \text{Spin}(7)_M \times_{\text{Spin}(7)} W \longrightarrow M.$$  

For instance, if we denote the $k$-dimensional irreducible Spin(7)-module by $W_k$ (in case the dimension uniquely specifies this module), then

$$W_8(M) \cong TM \cong T^*M.$$  

It is well known that the action of Spin(7) on $W_8$ is transitive on the unit sphere $S^7 \subset W_8$, and the stabilizer of an element is isomorphic to $G_2 \subset \text{Spin}(7)$. In analogy of the products $C\tau$ and $\chi$ on manifolds with a $G_2$-structure in Definition 3.1, we define on a Spin(7)-manifold $M$ a triple product as follows.

**Definition 4.1.** Let $(M, \Phi)$ be manifold with a Spin(7)-structure, and let $g = g_\Phi$ be the induced Riemannian metric. Then the $TM$-valued form $P = P_\Phi \in \Omega^3(M, TM)$ is defined by

$$P_\Phi := -\delta_{g_\Phi}(\Phi),$$

where $\delta_{g_\Phi}$ denotes the $g_\Phi$-cyclic contraction.
and is called the 3-fold cross product on $M$. That is, for $x, y, z, w \in TM$ we have

\[ g(P(x, y, z), w) = \Phi(x, y, z, w). \]

We shall usually suppress the indices $\Phi$ for $g$ and $P$ if it is clear from the context which Spin(7)-structure $\Phi$ is used.

4.2. Spin(7)-representations. In this section, we shall discuss the decomposition of symmetric and anti-symmetric powers of $W_8$ as Spin(7)-modules. For its exterior powers, we obtain the decompositions

\begin{align}
\Lambda^0 W_8 &\cong \Lambda^8 W_8 &\cong W_1, & \Lambda^2 W_8 &\cong \Lambda^6 W_8 &\cong W_7 \oplus W_{21}, \\
\Lambda^1 W_8 &\cong \Lambda^7 W_8 &\cong W_8, & \Lambda^3 W_8 &\cong \Lambda^5 W_8 &\cong W_8 \oplus W_{18}, \\
\Lambda^4 W_8 &\cong W_1 \oplus W_7 \oplus W_{27} \oplus W_{35}
\end{align}

where $\Lambda^k W_8 \cong \Lambda^{8-k} W_8$ via the Hodge-$\ast$. Again, we denote by $\Lambda^k W_8 \subset \Lambda^k W_8$ the subspace isomorphic to $W_1$ in the above notation.

Moreover, there are also irreducible decompositions of the symmetric powers of $W_7$ and $W_8$ as

\begin{align}
\odot^2 W_7 &\cong W_1 \oplus W_{27}, & \odot^2 W_8 &\cong W_1 \oplus W_{35}
\end{align}

into the induced Spin(7)-invariant metric and the trace free symmetric tensors; see [Humphreys1978].

**Lemma 4.2.** Let $e^0 \in W_8$ be a unit vector, let $V_7 := e^0_\perp$ on which Spin(7) acts as the double cover of $SO(7)$, so that $V_7 \cong W_7$ as a Spin(7)-module. Then the following maps are Spin(7)-equivariant embeddings.

\begin{align}
\lambda^2(v) &:= e^0 \wedge v^b + (v^b \varphi) \\
\lambda^4(v) &:= e^0 \wedge (v^b \ast_7 \varphi) - v^b \wedge \varphi \\
\lambda^6(v) &:= \Phi \wedge \lambda^2(v) = 3 \ast \lambda^2(v)
\end{align}

Here $\ast$ and $\ast_7$ denote the Hodge-$\ast$ in $W_8$ and $V_7$, respectively.

**Proof.** The decompositions in (4.5) imply that there are Spin(7)-equivariant maps $\lambda^k : W_7 \to \Lambda^k W_8$, and these are unique up to rescaling.

The equivariance of $\lambda^2$ follows from [SW2017] p.68, and thus, $\Phi \wedge \lambda^2(v) \in \Lambda^6 W_8$, whence $\Phi \wedge \lambda^2(v) = 3 \ast \lambda^2(v)$ follows from [SW2017] Theorem 9.8. This shows the statement on $\lambda^6$.

By [SW2017] Theorem 9.8, $\Lambda^4 W_8$ is the infinitesimal orbit of $\Phi$ under the action of $so(W_8) \cong \Lambda^2 W_8$. That is,

$$
\Lambda^4 W_8 = \{(u^b \wedge v^b) \cdot \Phi \mid u, v \in W_8\} = \{u^b \wedge (v^b \Phi) - v^b \wedge (u^b \Phi) \mid u, v \in W_8\}.
$$

Setting $u := e_0$ and picking $v \in e^0_\perp \cong W_7$ for a Spin(7)-frame $(e_\mu)$, it follows that the image of $\lambda^4$ equals $\Lambda^4 W_8$, and since $\lambda^4$ is evidently $G_2$-equivariant, it must coincide with the Spin(7)-equivariant map $W_7 \to \Lambda^4 W_8$. \qed
From this lemma, we obtain the following descriptions of the decompositions, which essentially recapitulates [SW2017, Theorem 9.8].

\[ \Lambda^k_{ij} W_8 = \{ \lambda^k (v) \mid v \in V_7 \} \quad \text{for } k = 2, 4, 6, \]
\[ \Lambda^2_{ij} W_8 = \{ \alpha^2 \in \Lambda^2 W_8 \mid \alpha^2 \wedge \Phi \wedge \lambda^2 (v) = 0 \text{ for all } v \in V_7 \}, \]
\[ \Lambda^6_{ij} W_8 = \{ \alpha^6 \in \Lambda^6 W_8 \mid \alpha^6 \wedge \lambda^2 (v) = 0 \text{ for all } v \in V_7 \}, \]
\[ \Lambda^3_{ij} W_8 = \{ i_a \Phi \mid a \in W_8 \}, \]
\[ \Lambda^5_{ij} W_8 = \{ \alpha^5 \wedge \Phi \mid a \in W_8 \}, \]
\[ \Lambda^3_{ij8} W_8 = \{ \alpha^3 \in \Lambda^3 W_8 \mid \Phi \wedge \alpha^3 = 0 \}, \]
\[ \Lambda^5_{ij8} W_8 = \{ \alpha^5 \in \Lambda^5 W_8 \mid \Phi \wedge \alpha^5 = 0 \}, \]
\[ \Lambda^3_{ij1} W_8 = \mathbb{R} \Phi, \]
\[ \Lambda^4_{ij8} W_8 = \text{span}\{ \lambda^2 (v) \wedge \lambda^2 (w) \mid v, w \in V_7, \langle v, w \rangle = 0 \} \]
\[ \Lambda^4_{ij35} W_8 = \{ \alpha^4 \mid \alpha^4 = -\alpha^4 \}. \]

We also recall the decomposition of the tensor product

\[ \text{Lin}(W_8, W_7) := W_8^* \otimes W_7 = W_8 \otimes W_{48}. \]

Here, the summand isomorphic to \( W_8 \) is given as

\[ \{(i_a \lambda^2 (e_i)) \otimes e_i \mid a \in W_8 \}, \]

where the sum is taken over an orthonormal basis \( (e_i) \) of \( V_7 \cong W_7 \). Finally, we define the Spin(7)-invariant tensor \( \sigma \in (W_8 \otimes W_7 \otimes W_8 \otimes W_7)^* \) by

\[ \sigma(a, u, b, v) := \frac{1}{2} \ast (a^b \wedge b^v \wedge \lambda^4 (u) \wedge \lambda^2 (v)). \]

Contraction with the inner products on \( W_7 \) and \( W_8 \) induces a Spin(7)-equivariant map

\[ \phi_{\sigma} : \text{Lin}(W_8, W_7) \longrightarrow \text{Lin}(W_8, W_7) \]
\[ \phi_{\sigma} (A)(a) := \sigma(a, A(e_\mu), e_\mu, e_i) e_i. \]

We calculate

\[ \lambda^4(u) \wedge \lambda^2(v) = (e^0 \wedge (i_u \ast_7 \varphi) - u^b \wedge \varphi) \wedge (e^0 \wedge v^b + i_v \varphi) \]
\[ = e^0 \wedge (i_u \ast_7 \varphi) \wedge (i_v \varphi) + e^0 \wedge u^b \wedge v^b \wedge \varphi - u^b \wedge \varphi \wedge (i_v \varphi) \]
\[ \overset{(6.10)}{=} \overset{(6.11)}{=} 2u^b \wedge v^b \wedge (e^0 \wedge \varphi - \ast_7 \varphi) - 2e^0 \wedge (\ast_7 (u^b \wedge v^b)). \]

**Lemma 4.3.** The map \( \phi_{\sigma} \) has eigenvalues \(-1\) and \(6\) with multiplicity \(48\) and \(8\), respectively.

**Proof.** Observe that the Spin(7)-invariant inner products on \( W_7 \) and \( W_8 \) induce an inner product on \( \text{Lin}(W_8, W_7) = W_8^* \otimes W_7 \) for which \( (e^\mu \otimes e_i) \) is
an orthonormal basis whenever \((e_\mu)\) is an orthonormal basis of \(W_8\) so that \(V_7 = e_0^\perp\) is spanned by \((e_i)\). This induced inner product satisfies

\[
\langle \phi_\sigma(e^\mu \otimes e_i), e^\nu \otimes e_j \rangle_{\text{Lin}(W_8, W_7)} = \langle \phi_\sigma(e^\mu \otimes e_i)(e_\nu), e_j \rangle_{W_7} = \sigma(e_\nu, e_i, e_\mu, e_j),
\]

and since \(\sigma(e_\mu, e_i, e_\mu, e_j) = 0\), it follows that the matrix representation of \(\phi_\sigma\) w.r.t. the basis \((e^\mu \otimes e_i)\) has 0’s on the diagonal, whence \(\text{tr}(\phi_\sigma) = 0\). Furthermore, \(\phi_\sigma\) is self-adjoint since \(\sigma(a, u, b, v) = \sigma(b, v, a, u)\) by (4.10) and (4.12), whence has real eigenvalues.

Decomposing \(\text{Lin}(W_8, W_7) \cong W_8 \oplus W_{48}\), (4.9) implies that the elements in the summand congruent to \(W_8\) are given by the maps

\[
A_a : W_8 \to W_7, \quad A_a(b) := \langle (i_a \lambda^2(e_i))^\# , b \rangle e_i
\]

for a fixed \(a \in W_8\). In order to calculate \(\phi_\sigma(A_a)\), observe that \(\text{Spin}(7)\) acts transitively on the unit sphere, whence we may assume w.l.o.g. that \(a = e_0\), so that

\[
A_{e_0}(b) = \langle e_i, b \rangle e_i = pr_{e_0^\perp}(b),
\]

where \(pr_{e_0^\perp} : W_8 \to e_0^\perp = V_7\) is the orthogonal projection. Thus,

\[
\phi_\sigma(A_{e_0})(b) = \sigma(b, A_{e_0}(e_\mu), e_\mu, e_i) e_i = \sigma(b, e_j, e_j, e_i) e_i
\]

\[
\overset{(4.12)}{=} \ast \left( b^j \wedge e^j \wedge \left( e^i \wedge (e^0 \wedge \varphi - *_7 \varphi) - e^0 \wedge (*_7 e^j) \right) \right) e_i
\]

\[
= \ast \left( e^0 \wedge b^j \wedge e^j \wedge (*_7 e^j) \right) e_i
\]

\[
= (1 - \delta_{ij}) \ast (e^0 \wedge b^j \wedge *_7 e^j) e_i
\]

\[
= 6\langle b, e_i \rangle e_i = 6A_{e_0}(b),
\]

so that \(\phi_\sigma(A_a) = 6A_a\) for all \(A_a\). By Schur’s lemma and since \(\phi_\sigma\) is self-adjoint, \(\phi_\sigma|_{W_{48}} = cI d_{W_{48}}\) for some \(c \in \mathbb{R}\), whence

\[
0 = \text{tr}(\phi_\sigma) = 6 \dim W_8 + c \dim W_{48},
\]

and from this, \(c = -1\) and the lemma follows.

For a manifold with a \(\text{Spin}(7)\)-structure \((M, \Phi)\) and induced metric \(g = g_\Phi\), the covariant derivative \(g_\Phi \nabla_v \Phi\) w.r.t. the Levi-Civita connection is contained in the infinitesimal orbit of \(\mathfrak{so}(T_p M, g_p)\) [Bryant1987] and hence in \(\Lambda^2 T^* M\). That is, there is a section \(T \in \Omega^1(M, W_7(M)) = \Gamma(\text{Lin}(TM, W_7(M)))\) such that

\[
\nabla_v \Phi = \lambda^4(T(v)) = e^0 \wedge (\iota_{T(v)} * \varphi) - (T(v))^\# \wedge \varphi
\]

with the map \(\lambda^4 : W_7 \to \Lambda^2 T_p M\) from (4.6). In analogy to Definition 3.4, we use the following terminology.

**Definition 4.4.** Let \((M, \Phi)\) be a manifold with a \(\text{Spin}(7)\)-structure. The section \(T \in \Omega^1(M, W_7(M))\) for which (4.13) holds is called the **torsion endomorphism** of the \(\text{Spin}(7)\)-structure.
4.3. **The Frölicher-Nijenhuis brackets on a manifold with a Spin(7)-structure.** Recall the section $P = -\delta_y \Phi \in \Omega^3(M, TM)$ on a manifold with a Spin(7)-structure $(M, \Phi)$ from Definition 4.1. We wish to relate its Frölicher-Nijenhuis bracket to its torsion. In order to do this, recall that $[P, P]_{FN} \in \Omega^6(M, TM)$.

Due to the decomposition $\Lambda^6 W_8 = \Lambda_3^6 W_8 \oplus \Lambda_{21}^6 W_8$ as a $G_2$-module, we may decompose

$$\Omega^6(M, TM) = \Gamma(M, \Lambda_3^6 T^* M \otimes TM) \oplus \Gamma(M, \Lambda_{21}^6 T^* M \otimes TM),$$

and we denote the projections onto the two summands by $\pi_7$ and $\pi_{21}$, respectively.

**Proposition 4.5.** Let $(M, \Phi)$ be a manifold with a Spin(7)-structure with the torsion endomorphism $T \in \Omega^1(M, W_7(M))$ from (4.13), and let $P = -\delta_y \Phi \in \Omega^3(M, TM)$ be as before. Then for $p \in M$,

$$\pi_7([P, P]_{FN}) = -\frac{2}{3} \Phi \wedge \lambda^2 \left(4T_p + \phi_\rho(T_p)\right)(e_\mu) \otimes e_\mu. \tag{4.14}$$

In particular, $\pi_7([P, P]_{FN}) = 0$ iff $T_p = 0$.

**Proof.** By Proposition 2.2, $[P, P]_{FN} = \gamma_\mu \otimes e_\mu$, where $\gamma_\mu \in \Lambda^6 T^*_p M$ is given by

$$\gamma_\mu = 2((ie_\nu \Phi) \wedge \iota_{e_\nu}(\nabla_{e_\nu} \Phi) + (ie_\nu \iota_{e_\nu} \Phi) \wedge e^0 \wedge (ie_\nu \nabla_{e_\nu} \Phi)). \tag{4.15}$$

If we decompose $\gamma_\mu = \Phi \wedge \lambda^2(v_\mu) + \gamma_\mu^{21}$ with $\gamma_\mu^{21} \in \Lambda_{21}^6 T^*_p M$, then for any $v \in V_7 = e_0^\perp$ we have $\gamma_\mu^{21} \wedge \lambda^2(v) = 0$ by (4.7) and hence,

$$\gamma_\mu \wedge \lambda^2(v) = \Phi \wedge \lambda^2(v_\mu) \wedge \lambda^2(v) = (e^0 \wedge \varphi + \star \varphi) \wedge (e^0 \wedge v^0_\mu + \iota_{v_\mu} \varphi) \wedge (e^0 \wedge v^0 + \iota_v \varphi)$$

$$= e^0 \wedge \varphi \wedge (\iota_{v_\mu} \varphi) \wedge (\iota_v \varphi) + \star \varphi \wedge e^0 \wedge v^0_\mu \wedge (\iota_v \varphi)$$

$$+ \star \varphi \wedge (\iota_{v_\mu} \varphi) \wedge e^0 \wedge v^0 \tag{6.3}, \tag{6.1}$$

$$= 6\langle v_\mu, v \rangle \text{vol} + 3\langle v_\mu, v \rangle \text{vol} + 3\langle v_\mu, v \rangle \text{vol} = 12\langle v_\mu, v \rangle \text{vol}.$$

Thus,

$$\pi_7([P, P]_{FN}) = \frac{1}{12} \star (\gamma_\mu \wedge \lambda^2(e_i)) \Phi \wedge \lambda^2(e_i) \otimes e_\mu. \tag{4.16}$$

For arbitrary $v \in V_7 = e_0^\perp$ we compute
\[ \gamma_{\mu} \wedge \lambda^2(v) = 2((\iota_{\epsilon_{\mu}} \Phi) \wedge \iota_{\epsilon_{\mu}}(\nabla_{\epsilon_{\mu}} \Phi)) \wedge \lambda^2(v) \]
\[ + (\iota_{\epsilon_{\mu}} \iota_{\epsilon_{\mu}} \Phi) \wedge e^\rho \wedge (\iota_{\epsilon_{\mu}} \nabla_{\epsilon_{\mu}} \Phi) \wedge \lambda^2(v)) \]
\[ = 2((\iota_{\epsilon_{\mu}} \Phi) \wedge \iota_{\epsilon_{\mu}} \lambda^4(T_p(e_{\mu})) \wedge \lambda^2(v) \]
\[ + (\iota_{\epsilon_{\mu}} \iota_{\epsilon_{\mu}} \Phi) \wedge e^\rho \wedge (\iota_{\epsilon_{\mu}} \lambda^4(T_p(e_{\rho}))) \wedge \lambda^2(v)) \]
\[ = 2(-4\delta_{\mu\nu}(T(e_{\nu}), v)\text{vol} + e^{\nu\mu} \wedge \lambda^4(T_p(e_{\nu})) \wedge \lambda^2(v)) \]
\[ + (-12\delta_{\mu\rho}(T_p(e_{\rho}), v)\text{vol} + e^{\rho\mu} \wedge \lambda^4(T_p(e_{\rho})) \wedge \lambda^2(v)) \]
\[ = 2(-16(T_p(e_{\mu}), v)\text{vol} - 2e^{\mu\nu} \wedge \lambda^4(T_p(e_{\nu})) \wedge \lambda^2(v)) \]
\[ = 2\left(-16(T_p(e_{\mu}), v) - 4\sigma(e_{\mu}, T_p(e_{\nu}), e_{\nu}, v)\right)\text{vol} \]
\[ \geq -8\langle(4T_p + \phi_{\sigma}(T_p))(e_{\mu}), v\rangle\text{vol}, \]
and this together with (4.16) implies (4.14) and completes the proof. \[ \square \]

With this, we are now ready to prove the following which immediately implies Theorem 1.2 from the introduction.

**Theorem 4.6.** Let \((M^8, \Phi)\) be a manifold with a \(\text{Spin}(7)\)-structure \((M^8, \Phi)\), let \(\nabla\) be the Levi-Civita connection of \(g = g_{\Phi}\), and let \(T \in \Omega^1(M^8, W_7(M^8))\) be its torsion endomorphism defined in Definition 4.4. Then for every \(p \in M^8\) the following are equivalent.

1. \(T_p = 0 \in T^*_p M^8 \otimes W^7(M)_p.\)
2. The \(\text{Spin}(7)\)-structure is torsion-free at \(p\), i.e., \((\nabla \Phi)_p = 0.\)
3. \(\pi_7([P, P]^F_N) = 0 \in \Lambda^6 T^*_p M^8 \otimes T_p M^8.\)
4. \([P, P]^F_N = 0 \in \Lambda^6 T^*_p M^8 \otimes T_p M^8.\)

**Proof.** The equivalence of the first two statements was shown in [Fernandez1986]. Also, \(T_p = 0\) implies \((\nabla \Phi)_p = 0\), whence by Proposition 2.2 \([P, P]^F_N = 0\), and this trivially implies \(\pi_7([P, P]^F_N) = 0.\)

By (4.14), \(\pi_7([P, P]^F_N) = 0\) iff \(4T_p + \phi_{\sigma}(T_p) = 0\), and since \(\phi_{\sigma}\) does not have \(-4\) as an eigenvalue by Lemma 4.3 this implies that \(T_p = 0.\) \[ \square \]

5. The 16 classes of \(G_2\)- and 4 classes of \(\text{Spin}(7)\)-structures

In this section, we shall interpret the classification of \(G_2\)-structures and of \(\text{Spin}(7)\)-structures ([FG1982] and [Fernandez1986]) in terms of the Frölicher-Nijenhuis bracket.

For the \(G_2\)-case, this classification is given by determining which components of the torsion endomorphism \(T\) vanish, where \(T\) is regarded as a section of the endomorphism bundle \(\text{End}(TM^7)\) which is \(G_2\)-equivariantly isomorphic to

\[ V_7(M^7) \otimes V_7(M^7) \cong V_1(M^7) \oplus V_7(M^7) \oplus V_{14}(M^7) \oplus V_{27}(M^7). \]
Since this decomposition has 4 summands, the classification consists of $2^4 = 16$ cases.

Observe that both $\Lambda_5^2 T^* M^7 \otimes T M^7$ and $\Lambda_6^6 T^* M^7 \otimes T M^7$ are $G_2$-equivariantly isomorphic to $V_7(M^7) \otimes V_7(M^7)$, where explicit isomorphisms are given by

$$K : \Lambda_5^2 T^* M^7 \otimes T M^7 \ni (\varphi \wedge \psi) \otimes w \mapsto \psi \otimes w \in T^* M^7 \otimes T M^7$$
$$L : \Lambda_6^6 T^* M^7 \otimes T M^7 \ni (\varphi) \otimes w \mapsto \psi \otimes w \in T^* M^7 \otimes T M^7.$$

If $(M^7, \varphi)$ is a manifold with a $G_2$-structure and the cross products $Cr$ and $\chi$, then we define the sections

$$K_{\pi_7([Cr,\chi]^{FN})}, \quad L_{[\chi,\chi]^{FN}} \in \Gamma(End(TM)).$$

Therefore, by Propositions 3.5, 3.6 there are $G_2$-equivariant vector bundle isomorphisms

$$\tau_1, \tau_2 : \text{End}(TM^7) \longrightarrow \text{End}(TM^7)$$

such that for the torsion endormorphism $T \in \Gamma(End(TM))$ we have

$$(5.2) \quad \tau_1(T) = K_{\pi_7([Cr,\chi]^{FN})} \quad \text{and} \quad \tau_2(T) = L_{[\chi,\chi]^{FN}},$$

where by a slight abuse of notation we denote the map $\tau_i : \Gamma(End(TM)) \rightarrow \Gamma(End(TM))$ applying $\tau_i$ pointwise by the same symbol.

For an element $A = a_{ij} e^i \otimes e_j \in \text{End}(V_7)$ let us denote its skew-symmetrization by

$$\sigma_A := a_{ij} e^{ij} \in \Lambda^2 V_7^*.$$  

With this notation, it follows from (3.10) and (3.18) that $\tau_1$ and $\tau_2$ take the form

$$\tau_1(T) = 2 \left( T - 2T^T - tr(T) id \right),$$
$$\tau_2(T) = -4(T + T^T) + 6 \ast (e^i \wedge \sigma_T \wedge \varphi) \otimes e_i,$$

summing over some basis $(e_i)$ with dual basis $(e^i)$.

The $G_2$-equivariance of $\tau_1$ and $\tau_2$ and (5.2) now implies that the $V_k(M)$-component of $T$ vanishes if and only if the $V_k(M)$-component of $K_{\pi_7([Cr,\chi]^{FN})}$ vanishes if and only if the $V_k(M)$-component of $L_{[\chi,\chi]^{FN}}$ vanishes. Since the cases in the Fernandez-Gray classification are determined by the vanishing of the components of $T$, we obtain the interpretation of these cases given in Table 1.

The interpretation of manifolds $(M^8, \Phi)$ with a Spin(7)-structure is analogous. Again, the torsion $T$ and the projection $\pi_7([P, P]^{FN})$ are sections of the Spin(7)-equivariantly isomorphic bundles $T^* M^8 \otimes W_7(M^8)$ and $\Lambda_6^6 T^* M^8 \otimes TM^8$, respectively, with an explicit identification given by

$$H : \Lambda_6^6 T^* M^8 \otimes W_7(M^8) \ni \Phi \wedge (\Lambda^2(a)) \otimes v \mapsto v^b \otimes a \in T^* M^8 \otimes W_7(M^8),$$
Here, by abuse of notation we regard $\phi_M$ and if (4.14) then by (4.14)

$W_{7}^{(k)}(M) \otimes T^{*}M^{8} \cong \text{Lin}(W_{8}(M^{8}), W_{7}(M^{8}))$.

By (4.8), $W_{7}(M^{8}) \otimes T^{*}M^{8}$ can be decomposed as $W_{8}(M^{8}) \oplus W_{48}(M^{8})$ whence by the Spin(7)-equivariance of $\tau_{3}$, the $W_{k}(M^{8})$-component of $T$ vanishes if and only if the $W_{8}(M^{8})$-component of $H_{\pi_{7}([P,P]F^{N})}$ does. Since the classification of Fernández into $2^{4} = 4$ different cases is given by the vanishing of the components of the torsion $T$, it follows that these cases can be also interpreted by the vanishing of the components of $H_{\pi_{7}([P,P]F^{N})}$, which leads to the interpretation of the classes of Spin(7)-manifolds given in Table 2 where $pr_{k} : W_{7}(M^{8}) \otimes T^{*}M^{8} \rightarrow W_{k}(M^{8})$ is the canonical projection.

| Classes | Relation on $A \in \{T, K_{\pi_{7}([Cr,\chi]F^{N})}, L_{[\chi,\lambda]F^{N}}\}$ |
|---------|------------------------------------------------------------------|
| $V_{1}(M) \oplus V_{2}(M) \oplus V_{4}(M) \oplus V_{27}(M)$ | no relation |
| $V_{7}(M) \oplus V_{4}(M) \oplus V_{27}(M)$ | $\text{tr}(A) = 0$ |
| $V_{1}(M) \oplus V_{7}(M) \oplus V_{27}(M)$ | $\sigma_{A} \in \Omega_{14}^{2}(M)$ |
| $V_{1}(M) \oplus V_{7}(M) \oplus V_{27}(M)$ | $A + A^{\dagger} - \frac{2}{7}\text{tr}(A)\text{id}_{TM} = 0$, $\sigma_{A} \in \Omega_{14}^{2}(M)$ |
| $V_{14}(M) \oplus V_{27}(M)$ | $\sigma_{A} \in \Omega_{14}^{2}(M)$, $\text{tr}(A) = 0$ |
| $V_{7}(M) \oplus V_{14}(M)$ | $A + A^{\dagger} = 0$ |
| $V_{1}(M) \oplus V_{27}(M)$ | $A - A^{\dagger} = 0$, $\sigma_{A} \in \Omega_{14}^{2}(M)$ |
| $V_{27}(M)$ | $A - A^{\dagger} = 0$, $\text{tr}(A) = 0$ |
| $V_{14}(M)$ | $A + A^{\dagger} = 0$, $\sigma_{A} \in \Omega_{14}^{2}(M)$ |
| $V_{7}(M)$ | $A + A^{\dagger} = 0$, $\sigma_{A} \in \Omega_{14}^{2}(M)$ |
| $V_{1}(M)$ | $A = \frac{2}{7}\text{tr}(A)\text{id}_{TM}$ |
| $\{0\}$ | $A = 0$ |

and if $(M^{8}, \Phi)$ is a manifold with a Spin(7)-structure and the 3-fold product $P$, then by (4.14)

$$H_{\pi_{7}([P,P]F^{N})} = \tau_{3}(T), \quad \text{where} \quad \tau_{3}(T) = -\frac{2}{3}(4T + \phi_{\sigma}(T)).$$

Here, by abuse of notation we regard $\phi_{\sigma}$ as the pointwise application of the map from (4.11) to sections of $W_{7}(M^{8}) \otimes T^{*}M^{8} \cong \text{Lin}(W_{8}(M^{8}), W_{7}(M^{8}))$.

6. Appendix

In this appendix, we shall collect some of the formulas which we needed in the calculations in this paper. Most of them are known and can be found in a similar form e.g. in [SW 2017, Lemma 4.37], but we shall collect them here for the reader’s convenience.
### Table 2.

| Classes | Relation on $A \in \{ T, H_{g^2((P,P)^F,N)} \} $ |
|---------|-------------------------------------------------|
| $W^8 \oplus W_{48}$ | no relation |
| $W_{48}$ | $pr_8(A) = 0$ |
| $W_{8}$ | $pr_{48}(A) = 0$ |
| $\{0\}$ | $A = 0$ |

**Lemma 6.1.** For all $u, v, w, r \in V_{7}$ and any orthonormal basis $(e_i)$ of $V_{7}$ the following identities hold.

1. $u^b \wedge (t_v \varphi) \wedge * \varphi = 3 \langle u, v \rangle \text{vol}$  
2. $u^b \wedge (t_v \star \varphi) \wedge \varphi = 4 \langle u, v \rangle \text{vol}$  
3. $(t_u \varphi) \wedge (t_v \varphi) \wedge \varphi = 6 \langle u, v \rangle \text{vol}$  
4. $u^b \wedge (t_v \varphi) \wedge (t_w \varphi) \wedge (t_r \varphi) = 2 \left( \langle v, w \rangle \langle u, r \rangle + \langle u, v \rangle \langle w, r \rangle + \langle u, w \rangle \langle v, r \rangle \right) \text{vol}$  
5. $(t_u t_v \star \varphi) \wedge w^b \wedge * \varphi = -2 u^b \wedge v^b \wedge w^b \wedge * \varphi$  
6. $(t_u t_v \varphi) \wedge w^b \wedge (t_r \varphi) \wedge \varphi = 2 \left( \langle v, w \rangle \langle u, r \rangle - \langle u, w \rangle \langle v, r \rangle \right) \text{vol}$  
7. $(t_u t_v \star \varphi) \wedge (t_w \star \varphi) \wedge (t_r \varphi) = 2 \left( \langle v, w \rangle \langle u, r \rangle - \langle u, w \rangle \langle v, r \rangle \right) \text{vol}$  
8. $u^b \wedge v^b \wedge (t_w \star \varphi) \wedge (t_r \varphi) = 2 \left( \langle v, w \rangle \langle u, r \rangle - \langle u, w \rangle \langle v, r \rangle \right) \text{vol}$  
9. $u^b \wedge v^b \wedge (t_w t_r \star \varphi) \wedge \varphi = 2 \left( \langle v, w \rangle \langle u, r \rangle - \langle u, w \rangle \langle v, r \rangle \right) \text{vol}$  
10. $(t_u \star \varphi) \wedge (t_v \varphi) = -2 * (u^b \wedge v^b) + u^b \wedge v^b \wedge \varphi$  
11. $\varphi \wedge (t_u \varphi) = 2 u^b \wedge * \varphi$  
12. $(t_{e_i} \varphi)^2 = 6 * \varphi$  
13. $(t_u t_{e_i} \star \varphi) \wedge (t_v t_{e_i} \star \varphi) = 2 \langle t_u \varphi, t_v \varphi \rangle$  
14. $(t_u t_{e_i} \star \varphi) \wedge (t_{e_i} \varphi) = 3 u^b \wedge \varphi$

**Proof.** For the proof of these identities, observe that the left hand side of each equation is a $G_2$-invariant element of some tensor power of $V_7$, and therefore it has to be a linear combination of the summands on the right hand side; the coefficients of this linear combination then can be determined by using the explicit formulas for $\varphi$ and $* \varphi$ in (3.1) and (3.2).
To pick one explicit example which is not among the identities shown in SW2017, observe that the left hand side of (6.17) is an element of \((V_7 \otimes \odot^3 V_7)^{G_2}\). Since \(\odot^3 V_7 \cong V_7 \oplus V_7^7\), we have dim\((V_7 \otimes \odot^3 V_7)^{G_2}\) = 1, and there is one \(G_2\)-invariant element of \(V_7 \otimes \odot^3 V_7\) given by deriving the square of the scalar product which lies in \(\odot^3 V_7\). Thus,

\[
(6.15) \quad u^b \wedge (i_v \varphi) \wedge (i_w \varphi) \wedge (i_r \varphi) = c \left( \langle v, w \rangle \langle u, r \rangle + \langle u, v \rangle \langle w, r \rangle \right) \text{vol}
\]

for some constant \(c \in \mathbb{R}\). Now setting \(u = v = w = r = e_1\) and using (6.11) implies that \(c = 2\), showing (6.1). The remaining identities are shown in a similar fashion. \( \square \)

The following two decompositions of \(G_2\)- and Spin(7)-representations is also well known, cf. SW2017 Theorem 8.5, 9.8], Kar2005 (4.7), (4.8)].

**Lemma 6.2.** Decompose \(\Lambda^2 V_7^* = \Lambda_7^2 V_7^* \oplus \Lambda_{14}^2 V_7^*\) according to (3.5). Then

\[
\Lambda_7^2 V_7^* = \left\{ \alpha^2 \in \Lambda^2 V_7^* \mid \star(\alpha^2 \wedge \varphi) = 2\alpha^2 \right\}, \quad \text{and}
\]

\[
\Lambda_{14}^2 V_7^* = \left\{ \alpha^2 \in \Lambda^2 V_7^* \mid \star(\alpha^2 \wedge \varphi) = -\alpha^2 \right\}.
\]

In particular,

\[
(6.16) \quad \Lambda_7^2 V_7^* = \left\{ \alpha^2 + \star(\alpha^2 \wedge \varphi) \mid \alpha^2 \in \Lambda^2 V_7^* \right\}, \quad \text{and}
\]

\[
\Lambda_{14}^2 V_7^* = \left\{ 2\alpha^2 - \star(\alpha^2 \wedge \varphi) \mid \alpha^2 \in \Lambda^2 V_7^* \right\}.
\]

**Lemma 6.3.** Decompose \(\Lambda^2 W_8^* = \Lambda_7^2 W_8^* \oplus \Lambda_{21}^2 W_8^*\) according to (4.4). Then

\[
\Lambda_7^2 W_8^* = \left\{ \alpha^2 \in \Lambda^2 W_8^* \mid \Phi \wedge \alpha^2 = 3 \ast \alpha^2 \right\}, \quad \text{and}
\]

\[
\Lambda_{21}^2 W_8^* = \left\{ \alpha^2 \in \Lambda^2 W_8^* \mid \Phi \wedge \alpha^2 = -\ast \alpha^2 \right\}.
\]

We shall also need the following result.

**Lemma 6.4.** For all \(u, v \in W_7\) and \(a, b \in W_8\) the following formulas hold:

\[
(t_a \Phi) \wedge (t_b \lambda^4(u)) \wedge \lambda^2(v) =
\]

\[
(6.17) \quad -4\langle a, b \rangle W_8 \langle u, v \rangle W_7 \text{vol} + a^b \wedge b^\flat \wedge \lambda^4(u) \wedge \lambda^2(v),
\]

\[
(6.18) \quad -12\langle a, b \rangle W_8 \langle u, v \rangle W_7 \text{vol} + a^b \wedge b^\flat \wedge \lambda^4(u) \wedge \lambda^2(v)
\]

where in (6.18) the sum is taken over an orthonormal basis \((e_\mu)\) of \(W_8\).

**Proof.** By (4.4) and (4.5), the decomposition of \(W_8 \otimes W_8\) and \(W_7 \otimes W_7\) into Spin(7)-irreducible summands yields

\[
W_8 \otimes W_8 = \odot^2 W_8 \oplus \Lambda^2 W_8 \cong (W_1 \oplus W_{35}) \oplus (W_7 \oplus W_{21}),
\]

\[
W_7 \otimes W_7 = \odot^2 W_7 \oplus \Lambda^2 W_7 \cong (W_1 \oplus W_{27}) \oplus W_{21},
\]

so that there are two inequivalent summands in common and hence, the space of Spin(7)-invariant tensors in \(W_8 \otimes W_8 \otimes W_7 \otimes W_7\) is 2-dimensional.
Since the left hand sides of (6.17) and (6.18) describe such tensors, it follows that there must be constants $c_1, \ldots, c_4 \in \mathbb{R}$ such that

$$
(t_a \Phi) \wedge (u_b \lambda^4(u)) \wedge \lambda^2(v) =
$$

(6.19) 
$$
c_1 \langle a, b \rangle W_8 \langle u, v \rangle W_7 \text{vol} + c_2 a^b \wedge b^b \wedge \lambda^4(u) \wedge \lambda^2(v),
$$

(6.20) 
$$
c_3 \langle a, b \rangle W_8 \langle u, v \rangle W_7 \text{vol} + c_4 a^b \wedge b^b \wedge \lambda^4(u) \wedge \lambda^2(v).
$$

In order to determine these constants, we first let $a = b := e_0$, so that

$$
(t_{e_0} \Phi) \wedge (t_{e_0} \lambda^4(u)) \wedge \lambda^2(v) = \varphi \wedge (t_u * \tau_7 \varphi) \wedge (e^0 \wedge v^b + (t_v \varphi))
$$

(6.2) 
$$
= - e^0 \wedge v^b \wedge (t_u * \tau_7 \varphi) \wedge \varphi
$$

and from this, $c_1 = -4$ follows. Whence if we let $a := e_0$ and $u, v \in W_7 = e^\perp_0$ then

$$
(t_{e_0} \Phi) \wedge (u_b \lambda^4(u)) \wedge \lambda^2(v) = \varphi \wedge (-e^0 \wedge (u_b t_u * \tau_7 \varphi) - u_b (u^b \wedge \varphi))
$$

(6.6) 
$$
\wedge (e^0 \wedge v^b + (t_v \varphi))
$$

(6.10) 
$$
= - \varphi \wedge e^0 \wedge (u_b t_u * \tau_7 \varphi) \wedge (t_v \varphi)
$$

(6.10) 
$$
\wedge \varphi \wedge u^b \wedge (u_b \varphi) \wedge e^0 \wedge v^b
$$

(6.11) 
$$
= 2 e^0 \wedge (u_b t_u * \tau_7 \varphi) \wedge v^b \wedge \tau_7 \varphi
$$

(6.5) 
$$
+ 2 b^b \wedge \tau_7 \varphi \wedge u^b \wedge e^0 \wedge v^b
$$

(6.4) 
$$
= - 2 e^0 \wedge b^b \wedge u^b \wedge v^b \wedge \tau_7 \varphi
$$

(6.4) 
$$
= e^0 \wedge b^b \wedge \lambda^4(u) \wedge \lambda^2(v),
$$

so that $c_2 = 1$ follows. Now substituting $a = b := e_0$ and $u = v := e_1$ into (6.20) and using the index $i$ to run from 1 to 7 yields

$$
e^0 \wedge (-i_{e_1} \varphi) \wedge (i_{e_1} (-e^1 \wedge \varphi)) \wedge (i_{e_1} \varphi) = e^0 \wedge (i_{e_1} \varphi) \wedge (\delta_1 i \varphi - e^1 \wedge (i_{e_1} \varphi)) \wedge (i_{e_1} \varphi)
$$

(6.13) 
$$
= e^0 \wedge (i_{e_1} \varphi) \wedge (i_{e_1} \varphi) \wedge \varphi
$$

(6.13) 
$$
- e^0 \wedge e^1 \wedge (i_{e_1} \varphi) \wedge (i_{e_1} \varphi) \wedge i_{e_1} \varphi
$$

(6.14) 
$$
6 \text{vol} - 6 e^0 \wedge e^1 \wedge i_{e_1} \varphi \wedge * \varphi
$$

(6.14) 
$$
= -12 \text{vol},
$$

where $\delta_1 i \varphi$ is the Frölicher-Nijenhuis bracket of $G_2$-and Spin(7)-manifolds.
so that \( c_3 = -12 \) follows. Finally, for \( a := e_0, u := e_1 \) and \( b, v \in W_7 = e_0^\perp \), \( (6.20) \) reads
\[
e^0 \wedge (t_b e_\mu \Phi) \wedge (t_{e_\mu} \lambda^4(e_1)) \wedge \lambda^2(v)
= e^0 \wedge (t_b \varphi) \wedge (t_{e_1} \ast_7 \varphi) \wedge (t_v \varphi)
+ e^0 \wedge (t_b e_{e_1} \ast_7 \varphi) \wedge (-t_{e_1} (e^1 \wedge \varphi)) \wedge (t_v \varphi)
= 0 + e^0 \wedge (t_b e_{e_1} \ast_7 \varphi) \wedge (-\delta_1 \varphi + e^1 \wedge (t_{e_1} \varphi)) \wedge (t_v \varphi)
= -e^0 \wedge (t_b e_{e_1} \ast_7 \varphi) \wedge \varphi \wedge (t_v \varphi)
+ e^{01} \wedge (t_b e_{e_1} \ast_7 \varphi) \wedge (t_{e_1} \varphi) \wedge (t_v \varphi)
\]
\( \equiv (6.11) = 6.14 \) \( -2e^0 \wedge (t_b e_{e_1} \ast_7 \varphi) \wedge b^v \wedge \ast_7 \varphi + 3e^{01} \wedge b^v \wedge \varphi \wedge (t_v \varphi) \)
\( \equiv (6.11) = 6.5 \) \( 4e^0 \wedge b^v \wedge e^1 \wedge v^b \wedge \ast_7 \varphi + 6e^{01} \wedge b^v \wedge v^b \wedge \ast_7 \varphi \)
\( \equiv (6.12) \) \( e^0 \wedge b^v \wedge \lambda^4(e_1) \wedge \lambda^2(v) \),
so that \( c_4 = 1 \) follows. At \( (\ast) \) we have used that the map
\[
(u, v, w) \mapsto \#(t_u \varphi) \wedge (t_u \ast_7 \varphi) \wedge (t_v \varphi)
\]
is a \( G_2 \)-invariant element of \( W_7 \otimes \ast^2 W_7 \), and since by \( (6.5) \) \( \otimes^2 W_7 \cong W_1 \oplus W_{27} \)
has no irreducible component isomorphic to \( W_7 \), there is no such element other than 0. \( \square \)

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