Star-Triangle Relation for a Three Dimensional Model.

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Abstract. The solvable $sl(n)$-chiral Potts model can be interpreted as a three-dimensional lattice model with local interactions. To within a minor modification of the boundary conditions it is an Ising type model on the body centered cubic lattice with two- and three-spin interactions. The corresponding local Boltzmann weights obey a number of simple relations, including a restricted star-triangle relation, which is a modified version of the well-known star-triangle relation appearing in two-dimensional models. We show that these relations lead to remarkable symmetry properties of the Boltzmann weight function of an elementary cube of the lattice, related to spatial symmetry group of the cubic lattice. These symmetry properties allow one to prove the commutativity of the row-to-row transfer matrices, bypassing the tetrahedron relation. The partition function per site for the infinite lattice is calculated exactly.

Keywords: Three-dimensional solvable models, Zamolodchikov model, Generalized chiral Potts model, Symmetry relations, Symmetry group of the cube, Commuting transfer matrices, Star-triangle relation, Yang-Baxter equation, Star-star relation.

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Introduction and Summary

There is a large numbers of solvable models in statistical mechanics where the bulk free energy and possibly other quantities such as order parameters and the correlation length can be calculated exactly. Mostly these models are two-dimensional, and only a few three-dimensional examples are known. The first such model was the two-dimensional Ising model, solved by Onsager in 1944 [1]. Then, it took some two decades to realize that two important ingredients of the Ising model — the star-triangle relation (STR) and the resulting commutativity of the transfer matrices — can be used to solve the other two-dimensional models [2,3].

There is now quite rich theory of solvable two-dimensional lattice models. There are several different (but related) methods, mostly based on the commutativity of the transfer matrices, and thereby on the Yang-Baxter relation which have been developed (for a review see [4,5]).

Can we find three-dimensional models with commuting transfer matrices? It is known that the tetrahedron relation [6] replaces the Yang-Baxter relation as a commutativity condition [7,8] for the three-dimensional cubic lattice. This relation contains thousands of distinct algebraic equations, and, obviously, it is very difficult to solve them. The only non-trivial solution obtained by direct analysis of these equations known so far is that of Zamolodchikov [6,9]. In fact, even in the simplest two state spin case one has to solve $2^{14}$ equations, instead of $2^6$ in two dimensions. The symmetry properties could slightly reduce this number, but the jump in complexity is still enormous. Therefore one would like to find an alternative approach to the commutativity which would be based on more simple algebraic relations.

In the present paper we discuss one such scheme. We consider the solvable interaction-round-a-cube model [10] on the cubic lattice with $N$-valued spins ($N \geq 2$) at each site. To within a minor modification of the boundary condition it is equivalent to the $sl(n)$-chiral Potts model [11] (this model has an interesting history which can be traced in [12-17]). On the other hand it can be regarded as a multistate generalization of the Zamolodchikov model [6], reducing to it when $N = 2$. The Boltzmann weight function of eight corner spins around a cube have a very special form, such that by introducing an auxiliary center spin for each cube the model can be viewed as an Ising-type one on the body centered cubic lattice with two- and three-spin interactions only. The corresponding Boltzmann weights are of course not arbitrary, and obey a number of simple relations. The most complicated among them is a “restricted star-triangle relation” (RSTR) which relates two- and three-spin weights. Remarkably enough this relation is a simplified version of the two-dimensional star-triangle relation (for the two-state spin case it can be obtained simply by specialization of rapidities in the STR of the 2d Ising model).

Starting from these relations we derive symmetry properties of a Boltzmann weight function of the elementary cube of the lattice under transformations from the symmetry group of the cube. As in the Zamolodchikov model this Boltzmann weight function depends on three parameters which can conveniently be chosen to be the dihedral angles $\theta_1, \theta_2, \theta_3$ between three “rapidity planes” passing through the cube. The symmetry properties are entirely consistent with such geometric interpretation of these angles.
The implication of the symmetry properties is twofold. First, they contain the three-dimensional star-star relation \[[10,18]\] which results in the commutativity of the transfer matrices. The other symmetry relations imply certain symmetry properties of the partition function per site for the lattice infinite in the all three directions. These two features — commutativity and symmetry — allow us to find the partition function per site extending the $N = 2$ Zamolodchikov case calculations of \([18]\) to the general $N$ case. The result is quite surprising:

$$\log \kappa_N(\theta_1, \theta_2, \theta_3) = \frac{2(N-1)}{N} \log \kappa_2(\theta_1, \theta_2, \theta_3)$$

where $\kappa_N$ is the partition function per site for the $N$-state model. In addition we show that there exists a three-dimensional free boson (Gaussian) model whose partition function per site is given by

$$\log \kappa_B(\theta_1, \theta_2, \theta_3) = -2 \log \kappa_2(\theta_1, \theta_2, \theta_3)$$

This means that $\kappa_N$ can be expressed as a (rational) power of some free boson determinant.

It would be very interesting to understand the reason for this fact.

The sequence of our working is illustrated on Fig.1. We hope this will assist the reader.

1. Formulation of the model

1.1 The interaction-round-a-cube model

Consider a simple cubic lattice $L$ of $M$ sites with the periodical boundary conditions in each direction. At each site of $L$ place a spin variable $s$ taking $N \geq 2$ distinct values $s = 0, \ldots, N - 1$, and allow all possible interactions of the spins within each elementary cube. The partition function reads

$$Z = \sum_{\text{spins}} \prod_{\text{cubes}} V_1(s_0|s_1, s_2, s_3|s_1, s_2, s_3|s_0)$$

(1.1)

where $s_0, \ldots, s_0$ are the eight spins of the cube arranged as in Fig. 2, and $V(s_0|s_1, s_2, s_3|s_1, s_2, s_3|s_0)$ is the Boltzmann weight of the spin configuration $s_0, \ldots, s_0$. Note that the spins $s_0, s_1, s_2, s_3, \bar{s}_1, \bar{s}_2, \bar{s}_3, \bar{s}_0$ correspond respectively to $a, b, c, d, e, f, g, h$ in Fig. 2 of reference \([10]\). The product is over all elementary cubes in $L$.

Taking the lattice to have $m$ horizontal layers and letting $\phi_i$ denote all spins in layer $i$, one can rewrite (1.1) as

$$Z = \sum_{\phi_1} \sum_{\phi_2} \cdots \sum_{\phi_m} T_{\phi_1, \phi_2, \phi_3, \ldots, \phi_m} = \text{Tr } T_m$$

(1.2)

where $T$ is a layer-to-layer transfer matrix whose elements are the products of all the $V$ functions of cubes between two adjacent layers. Clearly, $T$ depends on the Boltzmann weight function $V$, so we can write it as $T(V)$.

$^1$ Interestingly, this relation corresponds to the "central inversion" of the cubic lattice.
Obviously, we cannot solve the model (1.1) for an arbitrary Boltzmann weight function $V$. What we apparently can do is to solve a restricted class of models whose transfer matrices form commuting families, i.e., such that any two transfer matrices $T(V)$ and $T(V')$ belonging to the same family commute,

$$[T(V), T(V')] = 0 \quad .$$

(1.3)

In all known cases this commutativity condition can always be reformulated as a relation for the local Boltzmann weight functions $V$ and $V'$. In two dimensions the corresponding relation is well known: it is the Yang-Baxter equation for the local Boltzmann weights. This equation involves three different Boltzmann weight functions. A straightforward generalization [7,8] of this construction for a 3-dimensional lattice leads to the tetrahedron equation [6], involving four different Boltzmann weight functions. Even in the simplest two-valued spin case this equation involves thousands of distinct relations and, obviously, is very difficult to solve. The only known non-trivial solution obtained from the direct analysis of these relations is the Zamolodchikov’s one [6,9].

In this paper we present a somewhat different approach to the commutativity in three dimensions, which could, however, be applied to the Zamolodchikov model as well. We consider the interaction-round-a-cube model of ref [10] where the Boltzmann weight function of eight corner spins around a cube, $V$, has a very special form, such that by introducing an auxiliary center spin for each cube the model can be viewed as an Ising-type one on the body centered cubic lattice with two- and three-spin interactions only. The corresponding 2-spin and 3-spin Boltzmann weights obey a number of simple relations including “a restricted star-triangle relation” (RSTR) which allow us to prove the commutativity of the transfer matrices and calculate the partition function per site by use of the symmetry properties of the model.

Before ending this subsection note that there are simple transformations of $V$ that do not change (1.1). For instance, if we multiply $V(s_0, \ldots, s)$ by $F(s_0, s_2, s_1, s_3)/F(s_1, s_3, s_0, s_2)$ then each horizontal face of $\mathcal{L}$ acquires an $F$-factor from the cube below it, and a canceling $(1/F)$ factor from the cube above. The effect on the transfer matrix $T$ is to apply a diagonal similarity transformation. Provided $F$ is the same for $T(V)$ and $T(V')$ the commutation relation is unaffected. This is an example of a “face-factor” transformation. Similarly, one could apply “edge-factor” and “site-factor” transformations that leave (1.1) and (1.3) unchanged.

1.2 The Boltzmann weights

Let $a, b, c$ be integers and $v$ be a complex variable. Introduce the following notation

$$\omega = \exp(2\pi i / N) \quad , \quad \omega^{1/2} = \exp(i\pi / N) \quad ,$$

(1.4)

$$\gamma(a, b) = \omega^{ab} \quad , \quad \Phi(a) = (\omega^{1/2})^{a(N+a)} \quad ,$$

(1.5)

$$\Delta(v) = (1 - v^N)^{1/N} \quad .$$

(1.6)
Note that
\[ \gamma(a, b) = \gamma(b, a) = \gamma(a + N, b) \]
\[ \Phi(a + N) = \Phi(a) \]  
(1.7)

Define a function \( w(v, a) \) such that
\[ \frac{w(v, a)}{w(v, 0)} = (\Delta(v))^a \prod_{j=1}^{a} (1 - \omega^j v)^{-1} \]  
(1.8)

Obviously, the LHS of this equation is a multivalued function of \( v \), but it is a singlevalued function of a point \((v, \Delta(v))\) on the algebraic curve
\[ v^N + \Delta^N = 1 \]  
(1.9)

Below we shall interpret \( w \) in this latter way, suppressing, however, an explicit dependence on the phase of \( \Delta(v) \) in the arguments of \( w \).

Now, fix four complex parameters \( p, p', q, q' \), and define
\[ v_1 = q'/(\omega p') \quad v_2 = q'/p \quad v_3 = p/q \quad v_4 = p'/q \]  
(1.10)

Obviously
\[ \omega v_1 v_4 = v_2 v_3 \]  
(1.11)

The Boltzmann weight function of the model is given by the equations (2.9), (2.10) of ref.[10]. Taking into account equation (1.18) (given below) and omitting equivalence transformation factors (which do not change the partition function) one can write it in the form
\[ V(s_0|\overline{s}_1, \overline{s}_2, \overline{s}_3|s_1, s_2, s_3|\overline{s}_0|v_1, v_2, v_3, v_4) = \]
\[ = \rho \sum_{\sigma=0}^{N-1} \frac{w(v_2, s_1 - \overline{s}_2 + \sigma) w(v_3, s_3 - \overline{s}_0 - \sigma) \gamma(s_0, \sigma) \gamma(\overline{s}_0, \sigma)}{w(v_1, \overline{s}_3 - s_0 + \sigma) w(v_4, \overline{s}_1 - s_2 - \sigma) \gamma(s_2, \sigma) \gamma(\overline{s}_2, \sigma)} \]  
(1.12)

where \( \rho \) is a normalization factor depending on \( v_1, \ldots, v_4 \). Note that the Boltzmann weight function (1.12) describes a very special type of interaction of eight spins around the cube. A typical cube with its centre spin, \( \sigma \), is shown in Figure 2. There are three-spin interactions on the shaded triangles, described by \( w(v, a) \) or by \( 1/w(v, a) \) in (1.12). There are also two-spin interactions such as \( \gamma(s_0, \sigma) \), or \( 1/\gamma(s_2, \sigma) \) associated with the edges linking \( \sigma \) to \( s_0, \overline{s}_0, s_2, \overline{s}_2 \) (these edges are denoted by heavy lines in Figure 2.). In addition, the three-spin and two-spin interaction weights \( w \) and \( s \) obey a number of important relations which we now discuss. These relations, rather than the explicit form of \( w \) and \( \gamma \) given by (1.5) and (1.8), enable us to execute the program stated in the introduction and calculate the partition function per site in the thermodynamic limit.
1.3 Restricted star-triangle relation.

Consider the following automorphism of the curve (1.9)

\[ (v, \Delta(v)) \rightarrow (\tilde{v}, \Delta(\tilde{v})) \quad , \]

\[ \tilde{v} = \frac{1}{\omega v} \quad , \quad \Delta(\tilde{v}) = \frac{w^{-1/2}\Delta(v)}{v} . \] (1.14)

One can easily show that \( \gamma(a, b) \), \( \Phi(a) \) and \( w(v, a) \) satisfy the following properties

\[ \gamma(a, b + c) = \gamma(a, b) \gamma(a, c) \] (1.15)

\[ \sum_{b=0}^{N-1} \gamma(a, b) \gamma(-b, c) = N \delta_{ac} \] (1.16)

\[ \Phi(a + b) = \Phi(a) \Phi(b) \gamma(a, b) \] (1.17)

\[ \frac{w(v, a) w(\tilde{v}, -a)}{w(v, 0) w(\tilde{v}, 0)} = \Phi^{-1}(a) \] (1.18)

where \( \tilde{v} \) is related to \( v \) by equation (1.14). Moreover these functions satisfy two less trivial relations. For the moment let \( v_1, v_2, v_3, v_4 \) denote four arbitrary complex variables, then

\[ F_1(v_1, v_2|a, b) \equiv \sum_{\ell=0}^{N-1} \frac{w(v_2, a - \ell)}{w(v_1, -\ell)} \gamma(b, \ell) = \varphi_1(v_1, v_2) \frac{w(v_2', -b) w(v_2/(\omega v_1), a)}{w(v_1', a - b)} , \] (1.19)

\[ F_2(v_3, v_4|a, b) \equiv \sum_{\ell=0}^{N-1} \frac{w(v_3, -\ell) \gamma(b, \ell)}{w(v_4, a - \ell)} = \varphi_2(v_3, v_4) \frac{w(v_3', a - b)}{w(v_4', -b) w(v_4/v_3, a)} , \] (1.20)

where \( \varphi_1 \) and \( \varphi_2 \) are scalar functions and

\[ v_1' = \frac{v_2 \Delta(v_1)}{\omega v_1 \Delta(v_2)} , \quad \Delta(v_1') = \frac{\Delta(v_2/(\omega v_1))}{\Delta(v_2)} , \] (1.21a)

\[ v_2' = \frac{\Delta(v_1)}{\Delta(v_2)} , \quad \Delta(v_2') = \frac{v_2 \Delta(v_2/(\omega v_1))}{\Delta(v_2)} , \] (1.21b)

\[ v_3' = \frac{v_4 \Delta(v_3)}{v_3 \Delta(v_4)} , \quad \Delta(v_3') = \frac{\Delta(v_4/v_3)}{\Delta(v_4)} , \] (1.21c)

\[ v_4' = \frac{\Delta(v_3)}{\omega \Delta(v_4)} , \quad \Delta(v_4') = \frac{v_3 \Delta(v_4/v_3)}{\Delta(v_4)} . \] (1.21d)
Each of the relations (1.19), (1.20) is a corollary of another and the properties (1.15) - (1.18). The phases of $\Delta(v_4/v_3)$ and $\Delta(v_2/(\omega v_1))$ can be chosen arbitrarily since they cancel out the RHSs of (1.19) and (1.20).

The reader may have noticed that relation (1.19) (or (1.20)) is a particular case of the usual star-triangle relation. Indeed, the LHS of (1.19) is the sum of the product of the three functions (each depending on two spins) while the RHS is the product of three such function. Unlike the usual star-triangle relation there is some asymmetry in the LHS of (1.19): the function $\gamma$ does not depend on any continuous parameters. It is quite possible that (1.19) is a particular case of a more general relation and $\gamma$ is just a limiting value of a more complex function. In fact, this is exactly so for $N = 2$ when (1.19) and (1.20) can be obtained be a specialization of rapidities in the the star-triangle relation of the two-dimensional Ising model [1]. This is the reason why we call the (1.19), (1.20) as the “restricted star-triangle relations”.

2. The symmetry relations

2.1 The cube symmetry

Consider the cube $C$ shown in Fig. 2. The eight spins $s_0, \ldots, \bar{s}_0$ at the corners of $C$ can be grouped into four ordered pairs

$$d_j = (s_j, \bar{s}_j) \quad j = 0, \ldots, 3$$

(2.1)

corresponding to four diagonals $d_0, \ldots, d_3$, which we assume to be oriented and directed from $s_j$ to $\bar{s}_j$, $j = 0, \ldots, 3$. The spatial symmetry group of the cube $G(C)$, (consisting of all possible reflection and rotations which map the cube to itself) has a structure of the direct product

$$G(C) = C_2 \times S_4$$

(2.2)
of the cyclic group of order 2, $C_2$, generated by the central inversion, $P$ (which reverses the directions of all four diagonals $d_0, \ldots, d_3$) and the symmetric group, $S_4$, of order 24, consisting of the transformations which permute the diagonals preserving their directions.\(^2\)

Let

$$s_{0123} = (s_0|s_1, s_2, s_3|s_1, s_2, s_3|s_0)$$

(2.3)

denote the sequence of the spins $s_0, \ldots, \bar{s}_0$, corresponding to their basic arrangement at the corners of $C$ as shown in Fig.2.

Obviously, the transformations from $G(C)$ induce some permutations of the spins $s_0, \ldots, \bar{s}_0$. According to the above discussion of the structure (2.2) of $G(C)$ these transformations map the sequence (2.3) either to $s_{ijk\ell}$ or to $\bar{s}_{ijk\ell}$,

$$s_{ijk\ell} = (s_i|s_j, s_k, s_\ell|s_j, s_k, s_\ell|s)$$

(2.4)

$$\bar{s}_{ijk\ell} = (\bar{s}_i|s_j, s_k, s_\ell|\bar{s}_j, \bar{s}_k, \bar{s}_\ell|s)$$

(2.5)

\(^2\) For the even permutation the corresponding transformation is a pure rotation of the cube, while for the odd permutation it is a rotation followed by the central inversion, $P$. 
where \((i, j, k, \ell)\) is some permutation of \((0, 1, 2, 3)\).

Consider two elements of \(G(C)\), specifying their action on the spins. Let \((i, j, k, \ell)\) be any permutation of \((0, 1, 2, 3)\), and let \(R \in G(C)\),

\[
R s_{ijk\ell} = \bar{s}_{\ell kij}, \quad R \bar{s}_{ijk\ell} = s_{\ell kij}, \quad R^4 = 1
\]  

(2.6)
denote the \(90^\circ\) rotation around the axes passing through the centers of the top and bottom faces of \(C\), while \(T \in G(C)\),

\[
T s_{ijk\ell} = s_{ikj\ell}, \quad T \bar{s}_{ijk\ell} = \bar{s}_{ikj\ell}, \quad T^2 = 1,
\]

(2.7)
denotes the reflection with respect to the plane passing through the corners of \(C\) occupied by the spins \(s_0, s_3, \bar{s}_o, \bar{s}_3\) in Fig.2.

One could easily check that these two elements \(R\) and \(T\) generate the whole group \(G(C)\). In particular, the central inversion, \(P\),

\[
P s_{ijk\ell} = \bar{s}_{ijk\ell}, \quad P \bar{s}_{ijk\ell} = s_{ijk\ell}, \quad P^2 = 1,
\]

(2.8)
can be expressed as

\[
P = (RT)^3.
\]

(2.9)

2.2 The angle parameterization

Let \(\theta_1, \theta_2, \theta_3\) denote angles of a spherical triangle and \(a_1, a_2, a_3\) denote three sides of this triangle opposite to the angles \(\theta_1, \theta_2, \theta_3\). Define the related variables

\[
\alpha_0 = (\theta_1 + \theta_2 + \theta_3 - \pi)/2, \quad \alpha_i = \theta_i - \alpha_0,
\]

(2.10)
\[
\beta_0 = (2\pi - a_1 - a_2 - a_3)/2, \quad \beta_i = \pi - \beta_0 - a_i,
\]
for \(i = 1, 2, 3\). Choose \(\theta_1, \theta_2, \theta_3\), so that \(\theta_1, \theta_2, \theta_3, \alpha_0, \ldots, \alpha_3, \beta_0, \ldots, \beta_3\) are all real, between 0 and \(\pi\). Further, define (taking real positive values of roots)

\[
S_i = [\sin(\theta_i/2)]^{1/N}, \quad C_i = [\cos(\theta_i/2)]^{1/N},
\]

(2.11)
\[
T_i = [\tan(\theta_i/2)]^{1/N}, \quad z_i = \exp(ia_i/N),
\]
for \(i = 1, 2, 3\) and

\[
u_i = \exp(i\beta_i/N), \quad c_i = [\cos(\alpha_i/2)]^{1/2}, \quad i = 0, \ldots, 3
\]

(2.12)

Now parameterize \(p, p', q, q'\) in (1.10) as follows\(^3\)

\[
p = z_3^{-1} T_1, \quad p' = \omega^{-1/2} z_3^{-1} T_1^{-1}, \quad q = T_2^{-1}, \quad q' = \omega^{-1/2} T_2,
\]

\(^3\) This parameterization has been obtained by a formal generalization of eqs.(4.19),(4.24) of ref.[10] for arbitrary values of \(N\).
then
\[ v_1 = \omega^{-1} z_3 T_1 T_2, \quad v_2 = \omega^{-1/2} z_3 T_2 / T_1, \]
\[ v_3 = z_3^{-1} T_1 T_2, \quad v_4 = \omega^{-1/2} z_3^{-1} T_2 / T_1. \]

(2.13)

Now choose the phases of \( \Delta(v_1), \ldots, \Delta(v_4) \) such that
\[ \Delta(v_1) = S_3 / (C_1 C_2 u_3), \quad \Delta(v_2) = C_3 v_1 / (S_1 C_2), \]
\[ \Delta(v_3) = S_3 u_3 / (C_1 C_2), \quad \Delta(v_4) = C_3 / (S_1 C_2 u_1), \]
\[ \Delta(v_4 / v_3) = \Delta(v_2 / (\omega u_1)) = S_1^{-2}. \]

(2.14)

(2.15)

With these definitions the Boltzmann weight function (1.12) can be regarded as a function of the three independent variables \( \theta_1, \theta_2, \theta_3 \) or, equivalently, of the four dependent variables \( \alpha_0, \ldots, \alpha_3 \) constrained by the relation
\[ \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 = \pi. \]

(2.16)

so we can write the LHS of (21) as \( V(s_0, \ldots, s_0 \| \alpha_0, \ldots, \alpha_3) \).

Note, that the angles \( \theta_1, \theta_2, \theta_3 \) can be viewed as the dihedral angles between three “rapidity planes” passing through the cube exactly as it is in the Zamolodchikov model.

2.3 The normalization of the weights

Now we like to fix the normalization factors \( w(v, 0) \) in (1.8) and \( \rho \) in (1.12).

First let us choose \( w(v, 0) \) such that
\[ \prod_{a=0}^{N-1} w(v, a) = 1. \]

(2.17)

With this normalization define
\[ D_+(v) = (\det_N \| w(v, a - b) \|)^{1/N}, \]
\[ D_-(v) = (\det_N \| 1 / w(v, a - b) \|)^{1/N}. \]

(2.18)

Also, set
\[ S_+ = (\det_N \| \gamma(a, b) \|)^{1/N}, \]
\[ S_- = (\det_N \| 1 / \gamma(a, b) \|)^{1/N}. \]

(2.19)

\(^4\) For \( N = 2 \) the variables \( v_1, \ldots, v_4 \) here differ from those given by (7.19) of [18] merely by negating \( v_3 \) and \( v_4 \).
Regarding \( a, b \) in (1.19), (1.20) as matrix indices running the values 0, \ldots, \( N-1 \), and taking the determinants of the both sides of these equations one gets by using (2.17)–(2.19)

\[
\varphi_1(v_1, v_2) = D_+ (v_2) S_- / D_- (v'_1) \quad ,
\]

\[
\varphi_2(v_3, v_4) = D_- (v_4) S_+ / D_+ (v'_3) \quad ,
\]

where \( v'_1, v'_3 \) are given by (1.21). Explicit calculations with the equations (1.8) give

\[
D_{\pm} (v) = c_{\pm} (v/\Delta(v))^{(N-1)/2}
\]

\[
S_+ S_- = N
\]

where \( c_{\pm} \) are (inessential) constants. Note, in particular, that when \( v_1, \ldots, v_4 \) are parameterized by (2.13)-(2.15) we have

\[
\varphi_1(v_1, v_2) \varphi_2(v_3, v_4) = N \left( \frac{\sin \theta_2}{\sin \theta_3} \right)^{(N-1)/N}
\]

Further, set the normalization factor in (1.12) as

\[
\rho = (2\xi)^{2(N-1)/N} / N ,
\]

\[
2\xi = \left( \frac{1}{2} \sin \theta_3 \right)^{1/2} / (c_0 c_1 c_2 c_3)
\]

where \( c_0, \ldots, c_3 \) are given by (2.12).

We want to calculate the free energy, or equivalently the partition function per site

\[
\kappa = Z^{1/M} .
\]

Note that when \( \alpha_0 = \alpha_2 = 0 \) we have from (2.10) - (2.13)

\[
v_1 = v_2 = v_3 = v_4 = 0 \quad , \quad 2\xi = 1 ,
\]

while from (1.8)

\[
w(0, a) = w(0, 0) \quad , \quad \forall a .
\]

Substituting (2.26), (2.27) into (1.12) and using (1.15), (1.16) one gets

\[
V = \delta_{s_0 - \bar{s}_2, s_2 - \bar{s}_0} .
\]

Ignoring irrelevant boundary contributions, it follows that for \( M \) large

\[
\kappa = 1 \quad , \quad \text{when} \quad \alpha_0 = \alpha_2 = 0 .
\]
2.4 The symmetry properties of the Boltzmann weights

The definition (1.12) of the Boltzmann weight function $V$ is obviously rather asymmetric with respect to the orientation of the elementary cube (see Fig.3). We shall see, however, that despite this visual asymmetry, the Boltzmann weight function (1.12) has a remarkable hidden symmetry. Below we shall show that up to the equivalence transformation factors, which do not affect the partition function, the Boltzmann weight function remains unchanged upon the permutations the corner spins $s_0, \ldots, s_3$ induced by symmetry transformations of the cube complemented by corresponding transformation of the variables $v_1, v_2, v_3, v_4$ (or, conveniently, the variables $\alpha_0, \ldots, \alpha_3$). Obviously, it is enough to prove this just for two transformations (2.6) and (2.7) since they generate the whole symmetry group of the cube.

From (1.3), (1.18), (2.17) it follows that

$$w(v,0)w(\tilde{v},0) = e^{i\pi(N^2-1)/6}$$

(2.30)

is a constant independent of $v$. Let us of interchange $\alpha_1$ with $\alpha_2$ leaving $\alpha_0, \alpha_3$ intact. From (2.13) this is equivalent to the replacement of $v_1, v_2, v_3, v_4$ by $v_1, \tilde{v}_4, v_3, \tilde{v}_2$ respectively. Using (2.30) and noting that (2.23) remains unchanged one can easily trace the effect of this interchange on the weight function (1.12)

$$V(s_{0123}||\alpha_0, \alpha_1, \alpha_2, \alpha_3) = \frac{\Phi(s_2 - \bar{s}_1)}{\Phi(s_1 - \bar{s}_2)} V(s_{0213}||\alpha_0, \alpha_2, \alpha_1, \alpha_3)$$

(2.31)

where we have used the short notations (2.4) for the spin configuration of the corner spins. This gives the transformation law of $V$ under the reflection $T$, (2.7).

Further, using (1.15), (1.16) rewrite (1.12) in the form

$$\frac{s(s_0 - \bar{s}_3, s_0 - \bar{s}_2) \rho}{s(\bar{s}_0 - s_3, \bar{s}_0 - s_2) N} \sum_{\mu = 0}^{N-1} \left\{ \frac{s(s_3 + \bar{s}_3, \mu)}{s(s_0 + \bar{s}_0, \mu)} \times \right.$$

$$F_1(v_1, v_2|s_0 + s_1 - \bar{s}_2 = \bar{s}_3, s_2 - s_0 + \mu) F_2(v_3, v_4|s_0 + \bar{s}_1 - s_2 - s_3, s_2 - \bar{s}_0 - \mu) \left\}$$

(2.32)

Applying now (1.19), (1.20) and taking into account (2.14), (2.15), (2.20)-(2.24) one obtains

$$V(s_{0123}||\alpha_0, \alpha_1, \alpha_2, \alpha_3) =$$

$$\frac{s(s_0 - \bar{s}_3, s_0 - \bar{s}_2) w(v_4/v_3, s_0 + s_1 - \bar{s}_2 - \bar{s}_3)}{s(\bar{s}_0 - s_3, \bar{s}_0 - s_2) w(v_4/v_3, \bar{s}_0 + \bar{s}_1 - s_2 - s_3)} V(\bar{s}_{3201}||\alpha_3, \alpha_2, \alpha_0, \alpha_1)$$

(2.33)

which gives the required transformation of $V$ under the rotation, $R$, (2.6). Combining now eqs.(2.32) and (2.33) and using (1.18), (2.30) for the functions $w$ in (2.33) one gets the relation corresponding to the $RT$-transformation

$$V(s_{0123}||\alpha_0, \alpha_1, \alpha_2, \alpha_3) =$$

$$\frac{\Phi(s_0) \Phi(s_3) s(\bar{s}_2, \bar{s}_0 - s_3) w(v_2v_3, s_1 + s_3 - \bar{s}_0 - \bar{s}_2)}{\Phi(\bar{s}_0) \Phi(\bar{s}_3) s(s_2, s_0 - \bar{s}_3) w(v_2v_3, \bar{s}_1 + \bar{s}_3 - s_0 - s_2)} V(\bar{s}_{3102}||\alpha_3, \alpha_1, \alpha_0, \alpha_2)$$

(2.34)
where the argument of $w$'s can be written as
\[ v_2v_3 = q'/q = \omega^{-1/2}T_2^2. \]  
(2.35)

Remembering eq.(2.9) and iterating (2.34) three times one obtains
\[ \frac{V(s_{123}||\alpha_0, \alpha_1, \alpha_2, \alpha_3)}{V(s_{123}||\alpha_0, \alpha_1, \alpha_2, \alpha_3)} = \Phi^2(s_0) \frac{s(s_2, s_0 - s_3) s(s_0, s_3 - s_2) s(s_3, s_2 - s_0)}{\Phi^2(s_0) s(s_2, s_0 - s_3) s(s_0, s_3 - s_2) s(s_3, s_2 - s_0)} \]
\[ w(v_2v_3, s_1 + s_3 - s_0 - s_2) w(v_5, s_1 + s_2 - s_0 - s_3) w(v_4/v_3, s_0 + s_1 - s_2 - s_3) \]
\[ w(v_2v_3, s_1 + s_3 - s_0 - s_2) w(v_5, s_1 + s_2 - s_0 - s_3) w(v_4/v_3, s_0 + s_1 - s_2 - s_3) \]
where
\[ v_5 = \frac{v_4\Delta(v_1)\Delta(v_3)}{v_3\Delta(v_2)\Delta(v_4)} = \omega^{-1/2}T_3^2, \quad \Delta(v_5) = C_3^2. \]  
(2.36)

One can check that (2.36) is exactly the three-dimensional star-star relation conjectured previously (eq.(6.1) of ref.[10]).

As we remarked before the angles $\theta_1, \theta_2, \theta_3$ can be interpreted as the dihedral angles between the three rapidity planes rigidly connected with the cube. Then, from geometric considerations these angles should be very simply transformed by the cube symmetry group $G(C)$. Namely, the related variables $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ given by (2.10) should just permute for any transformation from $G(C)$. This is entirely consistent with (2.31), (2.33).

3. Partition function

3.2 Factorization and commutativity.

Consider two successive layers of $L$ with $l$ spins $\phi$ on the lower layer, $\phi'$ on the upper. In the center of each intervening cube we have the central spin $\sigma$ as in Fig. 3. Let $\phi''$ denote the set of all these $\sigma$-spins between $\phi$ and $\phi''$. Then because the top spins $s_0, s_1, s_2, s_3$ in Fig. 3 interact only with one another and with $\sigma$, and similarity for the bottom spins, we can write the elements of the transfer matrix $T$ as
\[ T_{\phi,\phi'} = \rho^l \sum_{\phi''} X_{\phi,\phi''} Y_{\phi'',\phi'}, \]  
(3.1)
where
\[ X_{\phi,\phi''} = \prod_{\text{cubes}} \frac{w(v_3, s_3 - s_0 - \sigma)\gamma(s_0, \sigma)}{w(v_4, s_1 - s_2 - \sigma)\gamma(s_2, \sigma)}, \]
\[ Y_{\phi'',\phi} = \prod_{\text{cubes}} \frac{w(v_2, s_1 - s_2 + \sigma)\gamma(s_0, \sigma)}{w(v_1, s_3 - s_0 + \sigma)\gamma(s_2, \sigma)}. \]  
(3.2)

The products are over all the $l$ cubes between the two layers; for each cube $s_0, s_1, s_2, s_3, s_1, s_2, s_3, s_0$ are the eight spins shown in Fig 3.
Obviously we can regard $X_{\phi,\phi'}$ as the element of a matrix $X$. From (3.2) this matrix depends on $v_3$ and $v_4$, so it can be written as $X(v_3, v_4)$. With similar conventions for $Y$, (3.1) implies

$$T = \rho^l X(v_3, v_4)Y(v_1, v_2).$$

(3.3)

Thus $\rho^{-l}T$ factors into a product of two matrices, one dependent on $\theta_1, \theta_2, \theta_3$ only via $v_3$ and $v_4$, the other via $v_1$ and $v_2$.

In [10] it was shown that transfer matrices $T$ form two-parameter commuting families. More precisely, when parameterizing $p, p', q, q'$ through the angles (2.13), the eqs. (1.5), (2.11), (2.17), (2.22) of ref. [10] imply that two transfer matrices (with different values of $\theta_1, \theta_2, \theta_3$) commute provided they have the same value of $\theta_1$

$$[T(\theta_1, \theta_2, \theta_3), T(\theta_1, \theta'_2, \theta'_3)] = 0, \quad \forall \theta_2, \theta_3, \theta'_2, \theta'_3$$

(3.4)

The proof was based on the Yang-Baxter equation for the $sl(n)$-chiral Potts model [11, 17, 19]. Alternatively we can establish the commutativity property (3.4) directly from the 3-dimensional star-star relation (2.36) not referring to those results. This will be done below in this section.

If $\theta_1$ is known then $v_2$ can be determined from $v_1$ and $v_4$ from $v_3$ by corollaries of (2.14).

$$v_2 = \omega^{1/2} \left(\cot \frac{\theta_1}{2}\right)^{2/N} v_1,$$

$$v_4 = \omega^{-1/2} \left(\cot \frac{\theta_1}{2}\right)^{2/N} v_3.$$  (3.5)

The text step is to show that the factorization property (3.3) remains true when the matrices $T, X, Y$ are appropriately diagonalized. To ensure this, it is necessary to slightly modify the model.

There are two sorts of vertical faces in $\mathcal{L}$: those whose perpendiculars run in front-to-back direction (such as $s_0s_2s_3\bar{s}_1$ and $\bar{s}_3s_1\bar{s}_0s_2$ in Fig. 3), and those whose perpendiculars run right-to-left. Call the former type “FB”, the latter “RL”. At the center of each FB face place a spin $\mu$, with values $0, \ldots, N - 1$. Let the spins on the front and back faces in Fig.3 be $\mu$ and $\mu'$, respectively. Choose them so that

$$\sigma = \mu - \mu' \pmod{N}.$$  (3.6)

Do this for all cubes in $\mathcal{L}$. If $\sigma'$ is the spin behind $\sigma$, and $\sigma''$ is the spin behind that, etc.; then on using the cyclic boundary conditions we observe that

$$\sigma + \sigma' + \sigma'' + \cdots = (\mu - \mu') + (\mu' - \mu'') + (\mu'' - \mu'''') + \cdots = 0 \pmod{N}.$$  (3.7)

(Each $\mu$-spin occurs twice with the opposite signs. If $\mathcal{L}$ has $n$ layers perpendicular to the front-to-back direction then there are $n$ $\sigma$-spins on the LHS of (3.6)). Thus we can use (3.6) only if the sum of each horizontal front-to-back line of $\sigma$-spins is constrained to be zero. This is merely a change of boundary conditions, and in the limit of $n$ large it should
have no effect on the partition function per site $\kappa$. We shall refer to the model subject to these constraints as the “modified model”.

For the modified model (3.1) and (3.2) formally remains the same, but $\phi''$ is now the set of all $\mu$-spins and $\sigma$ in (3.2) is now given by (3.6). Note also that in this case the matrices $X, Y, T$ are unchanged under overall shifts of all $\mu$-spins, or all $s$-spins, on any front-to-back line. Therefore $X, Y$ and $T$ have non-zero entries only in the diagonal block with respect to the subspace invariant under all such shifts.

For the moment, let us ignore the constraints (3.5), and work with the original model. The elements of the transfer matrix $XY$ in (3.1)-(3.3) are obtained by taking the product over all cubes in a layer of the function $V(s_{0123} | \alpha_0, \ldots, \alpha_3)$ given by (1.12), (2.3). Let us replace the function $V$ by $V = V(s_{0123} | \alpha_0, \ldots, \alpha_3)$ with $s_{0123}$ given by (2.5) and denote the corresponding transfer matrix as $\hat{T}$. Then with the same arguments which led to (3.3) one can show that

$$\hat{T} = \rho \hat{X}(v_1, v_2) \hat{Y}(v_3, v_4),$$

(3.8)

where $\hat{X}, \hat{Y}$ are similar but not identical to $X, Y$ in (3.3). From (2.36) the functions $V$ and $V$ differ by the equivalence transformation factors only. If we take the product of these factors over all cubes in a layer then the result can be put in the form $L(\phi)/L(\phi')$, where $\phi$ is the set of spins on the lower layer, $\phi'$ on the upper.

We have therefore shown that

$$[\hat{X}(v_1, v_2)\hat{Y}(v_3, v_4)]_{\phi, \phi'} = L(\phi)[X(v_3, v_4)Y(v_1, v_2)]_{\phi, \phi'}/L(\phi')$$

(3.9)

or in matrix notation

$$\hat{X}(v_1, v_2)\hat{Y}(v_3, v_4) = LX(v_3, v_4)Y(v_1, v_2)L^{-1}$$

(3.10)

where $L$ is the diagonal matrix with elements $L(\phi)\delta(\phi, \phi')$. It depends on $\theta_1$, but not on any other parameters.

Now introduce the constraints (3.6). This is equivalent to introducing a factor

$$\prod \left\{ \frac{1}{N} \sum_{k=0}^{N-1} \omega^{k(\sigma+\sigma'+\sigma''+\ldots)} \right\}$$

(3.11)

into summand in (3.1), where $\sigma, \sigma', \sigma'', \ldots$ are the $n$ center spins an a horizontal front-to-back line, and the outer product is over $l/n$ such lines in the horizontal layer of spins $\phi''$.

Let $V_k$ be a function that differs from $V$ only in that an extra single spin factor $\omega^{k}\sigma$ is put into the summand in (1.12). Inserting (3.11) into (3.1) we obtain

$$T = \prod \left\{ \frac{1}{N} \sum_{k=0}^{N-1} \Pi V_k \right\}$$

(3.12)

where the inner product is over $n$ cubes in a horizontal front-to-back line and again the outer product is over all $l/n$ such lines in a layer. From (1.2)

$$V_k(s_0 | s_1, s_2, s_3 | s_1, s_2, s_3 | s_0) = V(s_0 + k | s_1, s_2, s_3 + k | s_1, s_2, s_3 | s_0)$$

(3.13)
Shifting $s_0$ and $\overline{s}_3$ in (2.36) we obtain

$$V_k = \frac{\gamma(k, s_0 - \overline{s}_3) w(v_5, s_1 + s_2 - s_0 - k) w(v_5, s_1 + s_2 - \overline{s}_0 - \overline{s}_3) \gamma(k, s_0 - \overline{s}_3) w(v_5, s_1 + s_2 - \overline{s}_0 - s_0 - s_3)}{\gamma(k, \overline{s}_0 - s_3) w(v_5, s_1 + s_2 - \overline{s}_0 - s_3 - k) w(v_5, s_1 + s_2 - \overline{s}_0 - s_3)} V_k$$  (3.14)

where $V_k = V_k(\overline{s}_{0123})$. If we take the product of each side of (3.14) over the $n$ cubes in a horizontal front-to-back line, the $w$ and $s$ factors cancel. It follows that we still have relations (3.10) in the modified model.

Now consider the case when $v_2 = v_3 = 1$. From (3.5) it follows then

$$v_1 = \frac{1}{\omega v_4} = \omega^{-1/2}(\tan \frac{\theta_1}{2})^2/N$$  (3.15)

while (1.12) gives

$$V(s_0, \ldots, s_0) = \frac{\lambda_1 \delta(s_2 - s_1, s_3 - s_0)}{w(v_1, \overline{s}_3 - s_0 - s_1 + s_2)w(v_4, s_1 - s_2 - s_3 - \overline{s}_0)}$$  (3.16)

where $\lambda_1$ is a scalar factor. (From (2.17) $\lambda_1$ is infinite, but this is just a feature of the normalization (2.17), and can readily be removed, leaving the following arguments intact). The function $V_k$ contains extra $\omega^{k(s_0 - \overline{s}_3)}$ factor but this cancel out of the product in (3.12), so in this case

$$T_{\phi,\phi'} = \prod_{\text{cubes}} V(s_0, \ldots, \overline{s}_0)$$  (3.17)

where $V$ is given by (3.16) and the product is over all cubes in a layer. Substituting (3.16) into (3.17) and taking into account (1.18), (2.30) one obtains

$$T_{\phi,\phi'} = \lambda_2 \prod_{\text{cubes}} \delta(\overline{s}_2 - s_1, s_3 - \overline{s}_0)$$  (3.18)

where $\lambda_2$ is a constant.

The product of Kronecker delta-functions is zero unless $\phi = \phi'$, or if $\phi, \phi'$ differ only by overall shifts of spins on some horizontal front-to-back lines. Restricting attention to the subspace invariant under these shifts we then obtain from (3.3), (3.18)

$$X(1, v_4)Y(v_1, 1) = \lambda I$$  (3.19)

where $\lambda$ is another (nonzero) constant, $I$ is the unit matrix and $v_1$ and $v_4$ are related by (3.15). For the modified model the matrices in (3.18) are square and hence non-singular and invertible.

5 Applying similar arguments to the product of the weight function $V$ along one front-to-back line of the cube, rather than the whole horizontal layer, one obtains precisely the two dimensional star-star relation of the $s(l(n))$-chiral Potts model (eq. (3.19) of [11]).

6 Curiously enough, it is this apparently innocuous statement that fails for the original model and is the reason for introducing the constraints (3.6).
Regard $\theta_1$ as fixed, $v_1$ and $v_3$ as independent variables, and $v_2$ and $v_4$ as given by (3.5). Then we can suppress $\theta_1$, $v_2$, $v_4$ dependence and write (3.10) as

$$X(v_1)\hat{Y}(v_3) = LX(v_3)Y(v_1)L^{-1}$$

(3.20)

Then (3.19) gives

$$X(1)Y(x) = \hat{X}(x)\hat{Y}(1) = \lambda I$$

(3.21)

where $x = \omega^{-1/2}(\tan \theta_1/2)^{2/N}$ is the value of $v_1$ when $v_2 = 1$.

Using (3.20), (3.21) we can now perform the following transformations

$$X(v_3)Y(x)X(1)Y(v_1) = \lambda X(v_3)Y(v_1) = \lambda L^{-1}\hat{X}(v_1)\hat{Y}(v_3)L = L^{-1}\hat{X}(v_1)\hat{Y}(1)\hat{X}(x)L\hat{Y}(v_3)L = X(1)Y(v_1)X(v_3)Y(x)$$

(3.22)

Hence the matrix products $X(1)Y(v_1)$ and $X(v_3)Y(x)$ commute for all values of $v_1$ and $v_3$. Assuming that these commuting matrix products can be simultaneously diagonalized it follows that there must exists diagonal matrices $A(v_3)$ and $B(v_1)$ and a non-singular matrix $P$ (independent of the variables $v_1$ and $v_3$) such that

$$X(v_3)Y(x) = PA(v_3)P^{-1}, \quad \lambda^{-1}X(1)Y(v_1) = PB(v_1)P^{-1}$$

(3.23)

Setting $Q = Y(x)P$ and remembering that all matrices depend implicitly on $\theta_1$, we finally obtain

$$X(v_3, v_4) = P(\theta_1)A(v_3, v_4)Q^{-1}(\theta_1)$$

$$Y(v_1, v_2) = Q(\theta_1)B(v_1, v_2)P^{-1}(\theta_1)$$

(3.24)[B]

Here all the matrices depend on $\theta_1, \theta_2, \theta_3$ only via the arguments explicitly shown.

Note that (3.23) imply the commutativity relation (3.4) for the modified model. (Using then simple arguments like those at the end of Sect. 2 of ref. [10] we can easily extend (3.4) to the original model as well).

Thus, the relations (1.15)-(1.20) imply the star-star relation (2.36) which in its turn imply (3.20) and the relation (3.24).

From (1.2), (3.3) and (3.24) it follows that

$$Z = \rho M \text{Tr} (A(v_3, v_4)B(v_1, v_2))^m$$

$$= \rho M \sum_j (A_{jj}(v_3, v_4)B_{jj}(v_1, v_2))^m$$

where the $j$-summation is over all the diagonal elements $A_{jj}$ and $B_{jj}$ of $A$ and $B$. Assuming the largest term in the summation is unique and writing $a^l$ for $A_{00}$ and $b^l$ for $B_{00}$ (choosing $j = 0$ for the largest term), it follows that from (3.25), (2.25)

$$\kappa = \rho a(v_3, v_4)b(v_1, v_2),$$

(3.25)

where $\rho$ is given by (2.23). This is the factorization property which will be used in the next Section.
3.2 Partition function per site.

If $\mathcal{L}$ is infinite in all directions, we can evaluate the partition function per site solely from the symmetry and the factorization properties. In fact, the definition (1.1) is invariant under the symmetry transformation of the lattice generated by the symmetry group of the cube $\mathcal{G}(\mathcal{C})$. On the other hand from (2.31) and (2.33) the effect of any such transformation on the weight function $V$ is to multiply it by some equivalence transformation factors and permute $\alpha_0, \alpha_1, \alpha_2, \alpha_3$. Since two such transformations in (2.31), (2.33) generate all possible permutations of $\alpha_0, \alpha_1, \alpha_2, \alpha_3$, the partition function obey the following symmetry relation

$$\kappa(\alpha_0, \alpha_1, \alpha_2, \alpha_3) = \kappa(\alpha_i, \alpha_j, \alpha_k, \alpha_l)$$

(3.26)

where $(i, j, k, l)$ is any permutation of $(0, 1, 2, 3)$.

Thus we have two functional equations for $\kappa$, the factorization property (3.25), (2.23) and the symmetry property (3.26). We shall see that these two functional equations define $\kappa$ up to multiplicative constant which is determined then from the normalization (2.29).

The partition function per site $\kappa$ obviously depends on the number of the spin states $N$. We can exhibit this writing it as $\kappa_N$. Let us concentrate on the $N$-dependence in the functional equations. The symmetry relation (3.26) does not involve $N$ at all. However, there are two possible sources of the $N$-dependence in the factorization equation (3.25). First $\rho$ in (3.25) depends on $N$ as given by (2.23). Second, the formulae (2.13) expressing the variables $v_1, \ldots, v_4$ through the angles $\theta_1, \theta_2, \theta_3$ have the $N$-dependence coming from (2.11). This, however, can be readily removed since the $N$-th powers of the variables $v_1, \ldots, v_4$ are the same for any $N$ (the $N$-th powers of the RHS’s of (2.13) do not have any $N$-dependence). So redefining the functions $a$ and $b$ we we can rewrite (2.25) in the form

$$\kappa = \rho a(x_3, x_4)b(x_1, x_2)$$

(3.27)

where $x_i = v_i^N$, $i = 1, \ldots, 4$,

$$x_1 = e^{ia_3} \tan(\theta_1/2) \tan(\theta_2/2), \quad x_2 = -e^{ia_3} \cot(\theta_1/2) \tan(\theta_2/2)$$
$$x_3 = e^{-ia_3} \tan(\theta_1/2) \tan(\theta_2/2), \quad x_4 = -e^{-ia_3} \cot(\theta_1/2) \tan(\theta_2/2)$$

(3.28)

are expressed merely through the angles $\theta_1, \theta_2, \theta_3$ and the only $N$-dependence in the RHS of (3.27) comes from the factor $\rho$. Writing $\kappa_N$ in the form

$$\kappa_N = \lambda_N(\psi)^{2(N-1)/N}$$

(3.29)

where $\lambda_N$ is a numerical constant and substituting it into (3.27) we get the factorization equation for $\psi$

$$\psi = (2\xi)\overline{\pi}(x_3, x_4)\overline{b}(x_1, x_2)$$

(3.30)

where $\xi$ is given by (2.24). The symmetry relation for $\psi$ remains, clearly, the same as (3.26). Comparing (3.29) with eq.(7.33) of ref.[18] we see that $\psi$ satisfies the same factorization and symmetry relation as those for the partition function per site of the Zamolodchikov
model, $\kappa_2$ (which is the $N = 2$ case of the considered model). It was shown in [18] that these relations determine $\kappa_2$ to within a numerical constant, so we can conclude that

$$\psi = \lambda \kappa_2$$  \hspace{1cm} (3.31)

If we define the function $G(\beta)$ by

$$\log G(\beta) = \frac{1}{2\pi} \int_0^\beta \left[ x \cot x + \frac{\pi}{2} \tan \frac{x}{2} - \log(2 \sin x) \right] dx$$  \hspace{1cm} (3.32)

then the result of [18] can be written as

$$\kappa_2 = \frac{1}{2} G(\beta_0) G(\beta_1) G(\beta_2) G(\beta_3)$$  \hspace{1cm} (3.33)

where $\beta_0, \beta_1, \beta_2, \beta_3$ are given by (2.11). Note that (3.33) satisfies the normalization (2.29). Taking this into account and substituting (3.31) into (3.29) and then into (2.29) we obtain

$$\log \kappa_N(\theta_1, \theta_2, \theta_3) = \frac{2(N - 1)}{N} \log \kappa_2(\theta_1, \theta_2, \theta_3)$$  \hspace{1cm} (3.34)

which seems to be very simple and interesting result.

4. Free Boson Model

It is interesting to learn whether the method described above is unique to the considered model or it can be applied to the other models as well. In the other words how many solutions are there for the relations (1.15)–(1.20)? At the moment we do not know an answer to this question but can give one more example. It is the free boson (Gaussian) model on the three dimensional lattice.

Let now the spins at the sites of the lattice take continuous real values $-\infty < s < \infty$ and the summation over each spin $s$ is replaced by the integration $\int_{-\infty}^{\infty} \ldots ds$. With these modifications the definitions (1.1), (1.12) remain valid, but the weights $w$ and $\gamma$ in (1.12) are now given by

$$w(v, a) = \exp \left( -ia^2 \frac{v}{2(1 - v)} \right), \quad \gamma(a, b) = \exp(-iab), \quad \Phi(a) = \exp(-ia^2/2).$$  \hspace{1cm} (4.1)

Obviously,

$$w(v, 0) = 1.$$  \hspace{1cm} (4.2)

If we define $\tilde{v}$ and $\Delta(v)$ as

$$\tilde{v} = 1/v, \quad \Delta(v) = 1 - v,$$  \hspace{1cm} (4.3)

for any $v$ then the relations (1.15), (1.17), (1.18) remain valid while (1.16) is replaced with

$$\int_{-\infty}^{\infty} \gamma(a, b) \gamma(-b, c) db = 2\pi \delta(a - c).$$  \hspace{1cm} (4.4)
Further, if we also replace the spin summation by the integration in (1.19)–(1.21) then these formulae become valid as well (provided one formally sets $N = 1$ in (1.4), (1.6), i.e. $\Delta(v)$ is now given by (4.3) and $\omega = 1$). The scalar functions $\phi_1$ and $\phi_2$ are now given by

$$
\phi_1(v_1, v_2) = \left[ \frac{v_2 - v_1}{2\pi i(1 - v_1)(1 - v_2)} \right]^{-1/2}, \tag{4.5}
$$

$$
\phi_2(v_3, v_4) = \left[ \frac{v_3 - v_4}{2\pi i(1 - v_3)(1 - v_4)} \right]^{-1/2}.
$$

Let us parameterize $v_1, \ldots, v_4$ through the angles $\theta_1, \theta_2, \theta_3$ setting $v_i = x_i$, $i = 1, \ldots, 4$, where $x$’s are given by (3.28). Then, in particular, we have (cf. (2.22))

$$
\phi_1(v_1, v_2)\phi_2(v_3, v_4) = 2\pi \left( \frac{\sin \theta_2}{\sin \theta_3} \right)^{-1}. \tag{4.6}
$$

Choose the normalization factor $\rho$ in (1.12) as (cf. (2.23))

$$
\rho = (2\xi)^{-2}/(2\pi), \tag{4.7}
$$

where $\xi$ is given by (2.24). Finally the RHS of (2.28) is replaced by $\delta(s_0 - s_2 - \bar{s}_2 + \bar{s}_0)$ and we still have (2.26).

We have now all the required relations to reproduce the results (3.26), (3.27) for the free boson model and calculate its partition function per site $\kappa_B$. In fact all the calculations are identical to those we did before except that the exponents in (4.7) and (2.23) are different and the factors $N$ in (1.16) and (2.22) are replaced by $2\pi$ in (4.4), (4.6). (This factors are mutually cancel and therefore have no effect on the final result). Taking into account these minor differences we obtain

$$
\log \kappa_B = -2\log \kappa_2 \tag{4.8}
$$

A nice feature of the Gaussian model is that its partition function can be found by the elementary “Gaussian integration” directly from the definition (1.1). Together with (3.33) this means that the partition function per site of the $N$-state model $\kappa_N$ can be expressed as a rational power of a free boson determinant. We see no obvious reasons of this fact and it would be quite interesting to learn why it is so. We postpone the evaluation and the analysis of this determinant to the separate publication.

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Figure Captions

Fig.1. This figure explains the approach used in this paper.

Fig 2. Arrangements of the spins $s_0, \ldots, \bar{s}_0$ on the corner sites of an elementary cube of the simple cubic lattice $\mathcal{L}$; $d_0, d_1, d_2, d_3$ denote four oriented diagonals of the cube.

Fig.3. A typical elementary cube of $\mathcal{L}$, with corner spins $s_0, \ldots, \bar{s}_0$ and the center spin $\sigma$. 

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Special 2- and 3-spin interaction on the BCC lattice.

Properties of the Boltzmann weights: Restricted Star-Triangle Relation (RSTR).

Spatial Symmetry, 3-d Star-Star Relation.

Factorization of the Transfer Matrix.

Commutativity of the Transfer Matrices.

Partition Function per Site.

Bazhanov VV, Baxter RJ. Star-Triangle relation ...

Figure 1
Bazhanov VV, Baxter RJ. Star-Triangle relation ...

Figure 2
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