Leptonic and semileptonic decays of heavy mesons

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The leptonic decay of a pseudoscalar meson with total momentum \( P \) is described by

\[
\langle 0|\bar{q}_f \gamma_\mu q_i |\Phi_M(P)\rangle := f_M P_\mu = N_c \int \frac{d^4k}{(2\pi)^4} \text{tr}_D [\gamma_\mu S_f(k)\Gamma_M(k; P)S_b(k - P)] ,
\]

which defines the leptonic decay constant, \( f_M \). (With this normalisation, \( f_\pi \simeq 131 \text{ MeV} \).

In \( (1) \), \( S_f \) is the dressed quark propagator and \( \Gamma_M(k; P) \) is the Bethe-Salpeter amplitude for the bound state; \( M \) labels the meson whose flavour content is made explicit by the quark flavour labels, \( f_i \). For example, for the \( B^- \)-meson: \( f_1 = u \) and \( f_2 = b \).

The calculation of \( f_M \) requires a knowledge of \( S_f \) and \( \Gamma_M \). The dressed-quark propagator has the general form \( S_f(p) = 1/[i\gamma \cdot pA_f(p^2) + B_f(p^2)] \): a bare-quark is described by \( A(p^2) \equiv 1 \) and \( B(p^2) = m \), where \( m \) is the current-quark mass. As described in Ref. [3], it is a characteristic of QCD elucidated in Dyson-Schwinger equation studies that for light-quarks; i.e., \( u \)-, \( d \)- and \( s \)-quarks, \( A_f(p^2) \) and particularly \( B_f(p^2) \) have a strong momentum-dependence for \( p^2 < 1 \text{ GeV}^2 \). This momentum-dependence is nonperturbative in origin.

For the \( b \)-quark, however, the momentum-dependence of \( A_b(p^2) \) and \( B_b(p^2) \) for all spacelike-\( p^2 \) is weak and mainly perturbative in origin. This suggests that, in phenomenological applications, it is a good approximation to write the dressed-\( b \)-quark propagator as

\[
S_b(p) = \frac{1}{i\gamma \cdot p + \hat{M}_b} ,
\]

where \( \hat{M}_b \) is approximately the Euclidean constituent-quark mass.\textsuperscript{[4]} As observed in Ref. [3], this is the origin of “heavy-quark symmetry” in the Dyson-Schwinger equation [DSE] approach. For the \( c \)-quark, \( A_c(p^2) \) and \( B_c(p^2) \) have a stronger momentum-dependence. Hence representing \( S_c \) analogously to \( (2) \) is only, at best, a first, exploratory step in the study of heavy meson properties. To proceed we write the heavy-meson total-momentum as \( P := (\hat{M}_{fQ} + E) v_H \), where \( E = M_H - \hat{M}_{fQ} \) and \( v^2 = -1 \). It follows that the heavy-quark propagator in \( (1) \) becomes

\[
S_{f_2} = \delta(k - P) = \left( \frac{1}{2} \frac{1 + i\gamma \cdot v}{k \cdot v + E} + O \left( \frac{|k|}{\hat{M}_{fQ}}, \frac{E}{\hat{M}_{fQ}} \right) \right) .
\]

The Bethe-Salpeter amplitude, \( \Gamma_M(k; P) \), in \( (1) \) is a function of the light-quark’s momentum, \( k \). It can be obtained as the solution of a Bethe-Salpeter equation.\textsuperscript{[5]} These studies have not yet been completed hence herein we employ the Ansatz

\[
\Gamma_{B,D}(k; P) = \gamma_5 \left( 1 - \frac{1}{2} i\gamma \cdot v \right) \frac{\varphi(k^2)}{\mathcal{N}_{B,D}} ,
\]

whose Dirac structure is motivated by Ref. [4]. Here \( \mathcal{N}_H \) is the canonical normalisation constant for the Bethe-Salpeter amplitude. In this study we interpret an insensitivity of our results to details of the form of \( \varphi(k^2) \) as indicating that they are robust.
At this point the calculation of the leptonic decay constants and \( \phi \) includes specification of the light-quark propagators and the function \( \phi \) and \( \sigma \). One quantity characterising the function \( \phi \) in impulse approximation the hadronic matrix element for the \( B^0 \to D^- \ell \nu \) decay is:

\[
\langle D^- (k) | \bar{b} \gamma_{\mu} c | B^0 (P) \rangle := f_+(t)(K + P)_\mu - f_-(t)(K - P)_\mu
\]

where \( t = -(P - K)^2 \) and \( \bar{\Gamma}_{B^0, D^-} (k; P)^T = C^\dagger \Gamma_{B^0, D^-} (k; P) C \), with \( C = \gamma_2 \gamma_4 \). In \((8)\) we have used the fact that in the heavy-quark limit the vector piece of the dressed-quark-W-boson vertex is \( V_{\mu}^{bc} = \gamma_{\mu} \). Substituting \((3)\) and \((4)\) into \((8)\) we obtain

\[
f_\pm (t) = \frac{1}{M_B} \frac{M_D \pm M_B}{2 \sqrt{M_D M_B}} \xi (w),
\]

with \( W = 1 + 2 \tau (1 - \tau) (w - 1) \) and \( z_W = u - 2 \sqrt{u/W} \). In \((9)\), \( w = \frac{M_B^2 + M_D^2 - t}{2M_B M_D} = v_B \cdot v_D \) and the physically accessible region is \( 1.0 < w < 1.6 \). The canonical normalisation of the Bethe-Salpeter amplitude, \((9)\), ensures that \( \xi (w = 1) = 1 \).

One quantity characterising the function \( \xi (w) \) is its slope at \( w = 1 \), the point of minimal heavy meson recoil: \( \rho^2 := -\xi'(w) \big|_{w=1} \). It follows from \((10)\) that \( \rho^2 \geq 1/3 \) for any \( \varphi (z) \) and \( \sigma_{\mu/S} (z) \) non-negative, non-increasing, convex-up functions of their argument, which includes \( \varphi = \text{constant} \) and a free-particle propagator.

At this point the calculation of the leptonic decay constants and \( \xi (w) \) wants only the specification of the light-quark propagators and the function \( \varphi (k^2) \). The light-quark propagators have been fixed in Ref. [2]:

\[
\sigma_{S}^f (x) = 2 \bar{m}_{f} F (2 (x + \bar{m}_{f}^2)) + F (b_{1} x) F (b_{2} x) \left( b_{1}^2 + b_{2}^2 F (\epsilon x) \right), \tag{11}
\]

\[
\sigma_{V}^f (x) = \frac{2 (x + \bar{m}_{f}^2) - 1 + e^{-2 (x + \bar{m}_{f}^2)}}{2(x + \bar{m}_{f}^2)^2}, \tag{12}
\]
Fig. 1. Experiment: points, Ref. [7]; dashed line, (13); short-dashed line, the linear fit \[ \xi(w) = 1 - \rho^2(w - 1), \] \[ \rho^2 = 0.91 \pm 0.15 \pm 0.06. \] Our calculations using \( \varphi_{A-C} \) from (16) are represented by the solid line with the dot-dash line being the result we obtain assuming a point-like heavy-meson, \( \varphi_D \). Importantly, there is significant curvature in each case, which is a manifestation of the role played by the light-quarks. The light solid line is described in Fig. 2.

where \( F(y) \equiv (1 - e^{-y})/y, x = p^2/(2D) \) and: \( \sigma^f_V(x) = 2D \sigma_V^f(p^2); \sigma^f_S(x) = \sqrt{2D} \sigma_S^f(p^2); m_f = m_f/\sqrt{2D}, \) with \( D \) a mass scale. This form is motivated by extensive studies of the DSE for the dressed-quark propagator and combines the effects of confinement and dynamical chiral symmetry breaking with free-particle behaviour at large spacelike-\( p^2 \).

The parameters \( m_f, b_{0,3}^f \) in (11), (12) were determined in a \( \chi^2 \)-fit to a range of light-hadron observables, which is described in Ref. [4] and leads to the values in (13)

\[
\begin{align*}
u & : \quad 0.00897 \quad 0.131 \quad 2.90 \quad 0.603 \quad 0.185 \\
s & : \quad 0.224 \quad 0.105 \quad 2.90 \quad 0.740 \quad 0.185
\end{align*}
\]

The values of \( b_{1,3}^f \) are underlined to indicate that the constraints \( b_{1,3}^f = b_{1,3}^t \) were imposed in the fitting. The scale parameter \( D = 0.160 \text{ GeV}^2 \).

We consider the following four forms for \( \varphi(k^2) \):

\[
\begin{align*}
\varphi_A(k^2) &= \exp \left(-\frac{k^2}{\Lambda^2} \right) \\
\varphi_B(k^2) &= \frac{\Lambda^2}{k^2 + \Lambda^2} \left( \frac{\Lambda^2}{k^2 + \Lambda^2} \right)^2 \theta \left(1 - \frac{k^2}{\Lambda^2} \right)
\end{align*}
\]

and with this we have 2 parameters in our study: the “binding energy”, \( E := M_H - \hat{M}_{fQ} \), and the width, \( \Lambda \), of the heavy meson Bethe-Salpeter amplitude.

Now we ask the question: “Is the heavy-quark limit of the DSE framework capable of describing heavy meson observables?” To answer this we perform a \( \chi^2 \)-fit of \( (E, \Lambda) \) to the following parametrisation of the experimental data on \( \xi(w) \):

\[
\xi(w) = \frac{2}{w+1} \exp \left( \frac{1 - 2\rho^2}{w+1} \right), \quad \rho^2 = 1.53 \pm 0.36 \pm 0.14,
\]

to \( f_D = 0.216 \pm 0.015 \text{ GeV} \) and \( f_B = 0.206 \pm 0.030 \text{ GeV} \), which, in the absence of experimental data, is our weighted average of lattice-QCD results. Using \( M_D = 1.87 \text{ GeV} \), \( M_{D_s} = 1.97 \text{ GeV} \) and \( M_B = 5.27 \text{ GeV} \), we obtain the results presented in Fig. 1 and (16), energies in GeV and \( \rho^2 \) dimensionless.
|   |   |   |   |   |
|---|---|---|---|---|
|   | $E$ | $A$ |  $f_D$ | $f_{Ds}$ | $f_B$ | $\rho^2$ |
| $A$ | 0.640 | 1.03 | 0.227 | 0.245 | 0.135 | 1.55 |
| $B$ | 0.567 | 0.843 | 0.227 | 0.239 | 0.135 | 1.56 |
| $C$ | 0.612 | 1.32 | 0.227 | 0.242 | 0.135 | 1.55 |
| $D$ | 0.643 | 1.02 | 0.272 | 0.296 | 0.162 | 1.21 |

Clearly, a good description is possible. The fitted values of $E$ are consistent with contemporary estimates of this binding energy in Bethe-Salpeter equation studies and the values of $\Lambda$ indicates that the heavy meson occupies a spacetime volume of only 4-20% that of the pion. We observe that $f_{Ds}/f_D = f_{Bs}/f_B \approx 1.07$. Comparing this with the value expected in the heavy-quark limit: $\sqrt{M_D/M_{Ds}} = 0.97$ and $\sqrt{M_B/M_{Bs}} = 0.99$, illustrates the influence that light-quarks have on real heavy-meson observables.

![Fig. 2. Calculated form of $f_+(q^2)$ for the decay $B \rightarrow \pi e \nu$ using $\varphi_A$, with $E = 0.47$ and $\Lambda = 1.1$ GeV, and $\Gamma_\pi$ from Ref. [1]: $f_+(0) = 0.48$. This gives a branching ratio of $2.3 \times 10^{-4}$ to be compared with the experimental value of $(1.8 \pm 0.4 \pm 0.3 \pm 0.2) \times 10^{-4}$. With this $\varphi_A$: $f_D = 0.224$, $f_{Ds} = 0.241$, $f_B = 0.133$ and $f_{Bs} = 0.146$ GeV. $\xi(w)$ is plotted as the thin solid line in Fig. 1, for which $\rho^2 = 1.0$. Requiring a simultaneous fit reduces $\rho^2$ and increases $\xi(w = 1.6)$. The data points are the results of the lattice simulations in Ref. [10].](image)

We are currently applying the formalism described herein to the simultaneous calculation of leptonic and heavy-to-heavy and heavy-to-light semileptonic decays. Our framework allows the calculation of each form factor at all $q^2$. The light-quark degrees of freedom are particularly important in heavy-to-light semileptonic decays, which probe the structure of the final-state light-meson Bethe-Salpeter amplitude and are inaccessible in heavy-quark effective theory. A uniformly good description of all these decays requires a refitting of the two parameters $E$ and $\Lambda$. We illustrate what is possible in Fig. 2.

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