A NATURAL CONNECTION ON A BASIC CLASS OF RIEMANNIAN PRODUCT MANIFOLDS

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ABSTRACT. A Riemannian manifold $M$ with an integrable almost product structure $P$ is called a Riemannian product manifold. Our investigations are on the manifolds $(M, P, g)$ of the largest class of Riemannian product manifolds, which is closed with respect to the group of conformal transformations of the metric $g$. This class is an analogue of the class of locally conformal Kähler manifolds in almost Hermitian geometry. In the present paper we study a natural connection $D$ on $(M, P, g)$ (i.e. $DP = Dg = 0$). We find necessary and sufficient conditions the curvature tensor of $D$ to have properties similar to the Kähler tensor in Hermitian geometry. We pay attention to the case when $D$ has a parallel torsion. We establish that the Weyl tensors for the connection $D$ and the Levi-Civita connection coincide as well as the invariance of the curvature tensor of $D$ with respect to the usual conformal transformation. We consider the case when $D$ is a flat connection. We construct an example of the considered manifold by a Lie group where $D$ is a flat connection with non-parallel torsion.

Key words: Riemannian manifold, almost product structure, integrable structure, linear connection, parallel torsion, Lie algebra, Lie group.

2010 Mathematics Subject Classification: 53C05, 53C15, 53C25, 53B05, 22E60.

INTRODUCTION

The systematic development of the theory of Riemannian almost product manifolds, i.e. Riemannian manifolds with almost product structure, was started by K. Yano in [14]. The geometry of a Riemannian almost product manifold $(M, P, g)$ is a geometry of the Riemannian metric $g$ and the almost product structure $P$. There are important in this geometry the linear connections with respect to which the parallel transport determines an isomorphism of the tangent spaces of the manifold $M$ with structures $g$ and $P$. Such connections are called natural in [10]. They are an analogue of the Hermitian connections in almost Hermitian geometry.

If the almost product structure $P$ is integrable, i.e. the torsion tensor of $P$ (the Nijenhuis tensor) is zero, then the manifold $(M, P, g)$ is called a Riemannian product manifold. In the present work we study a natural connection $D$ on a manifold $(M, P, g)$ belonging to the largest class of Riemannian product manifolds, which is closed with respect to the group of the conformal transformations of the metric $g$. This is the class $W_1$ of the classification in [13].

The present paper is organized as follows. In Section 1 we give some necessary facts about the Riemannian almost product manifolds and the natural connections on them. In Section 2 we consider a natural connection $D$ from the 2-parametric family of all natural connections on $(M, P, g) \in W_1$ obtained in [3]. We find a relation between the curvature tensors of the Levi-Civita connection $\nabla$ and the
considered connection $D$. As a corollary we obtain a relation between the Ricci tensors and between the scalar curvatures for $\nabla$ and $D$. In Section 3 we find necessary and sufficient conditions for a connection $D$ whose curvature tensor is a Riemannian $P$-tensor. The notion of a Riemannian $P$-tensor is an analogue of the notion of a Kähler tensor in Hermitian geometry. In Section 4 we consider the case of a connection $D$ with parallel torsion. In Section 5 we establish that the Weyl tensors for $\nabla$ and $D$ coincide. In Section 6 we prove that the curvature tensor of $D$ is invariant with respect to the usual conformal transformation of the metric $g$. In Section 7 we consider the case of flat connection $D$. In Section 8 we construct an example of the considered manifold by a Lie group, where $D$ is a flat connection with non-parallel torsion.

1. Preliminaries

Let $(M, P, g)$ be a Riemannian almost product manifold, i.e. a differentiable manifold $M$ with a tensor field $P$ of type $(1,1)$ and a Riemannian metric $g$ such that

\[ P^2 x = x, \quad g(Px, Py) = g(x, y) \]

for arbitrary $x, y$ of the algebra $\mathfrak{X}(M)$ of the smooth vector fields on $M$. Obviously $g(Px, y) = g(x, Py)$.

Further $x, y, z, w$ will stand for arbitrary elements of $\mathfrak{X}(M)$ or vectors in the tangent space $T_p M$ at $p \in M$.

In this work we consider Riemannian almost product manifolds with $\text{tr} P = 0$. In this case $(M, P, g)$ is an even-dimensional manifold. Let $\dim M = 2n$. Then the associated metric $\tilde{g}$ of $g$, determined by $\tilde{g}(x, y) = g(x, Py)$, is an indefinite metric of signature $(n, n)$. We suppose that $\dim M \geq 4$.

In [11] A.M. Naveira gives a classification of Riemannian almost product manifolds with respect to the tensor $F$ of type $(0,3)$, defined by

\[ F(x, y, z) = g((\nabla_x P)y, z), \]

where $\nabla$ is the Levi-Civita connection of $g$. The tensor $F$ has the following properties:

\[ F(x, y, z) = F(x, z, y) = -F(x, Py, Pz), \]
\[ F(x, y, Pz) = -F(x, Py, z). \]

Using the Naveira classification, in [13] M. Staikova and K. Gribachev give a classification of Riemannian almost product manifolds $(M, P, g)$ with $\text{tr} P = 0$. The basic classes of the classification in [13] are $W_1$, $W_2$ and $W_3$. Their intersection is the class $W_0$ of the Riemannian $P$-manifolds, determined by the condition $F = 0$ or equivalently $\nabla P = 0$ [12]. This class is the analogue of the class of Kähler manifolds in the geometry of almost Hermitian manifolds.

A Riemannian almost product manifold $(M, P, g)$ is a Riemannian product manifold if it has a local product structure. This means that the almost product structure $P$ is integrable, i.e. the Nijenhuis tensor $N$ determined by

\[ N(x, y) = [Px, Py] + [x, y] - P[Px, y] - P[x, Py] \]

is zero. The Riemannian product manifolds form the class $W_1 \oplus W_2$ from the classification in [13]. This class is an analogue of Hermitian manifolds in almost Hermitian geometry.
The class $W_1$ from the classification in [13] consist of the manifolds which are locally conformal equivalent to Riemannian $P$-manifolds. This class plays a similar role of the role of the class of the conformal Kähler manifolds in almost Hermitian geometry [2]. The characteristic condition for the class $W_1$ is the following

$$F(x, y, z) = \frac{1}{2} \left\{ g(x, y)\theta(z) + g(x, z)\theta(y) - g(x, P_y)\theta(P_z) - g(x, P_z)\theta(P_y) \right\},$$

(2)

where $\theta$ is the associated Lee 1-form for $F$ determined by

$$\theta(x) = g^{ij}F(e_i, e_j, x).$$

Here and further $g^{ij}$ will stand for the components of the inverse matrix of $g$ with respect to a basis $\{e_i\}$ of $T_p M$ at $p \in M$.

The curvature tensor $R$ of $\nabla$ is determined by $R(x, y)z = \nabla_x \nabla_y z - \nabla_y \nabla_x z - \nabla_{[x, y]} z$ and the corresponding tensor of type $(0, 4)$ is defined as follows $R(x, y, z, w) = L(x, y, z, w)$. We denote the Ricci tensor and the scalar curvature for $\nabla$ by $\rho$ and $\tau$, respectively, i.e. $\rho(y, z) = g^{ij}R(e_i, y, z, e_j)$ and $\tau = g^{ij}\rho(e_i, e_j)$.

In [9], a tensor $L$ of type $(0, 4)$ with properties

$$L(x, y, z, w) = -L(y, x, z, w) = -L(x, y, w, z),$$

(3)

$$\delta_{x,y,z} L(x, y, z, w) = 0,$$

(4)

$$L(x, y, P_z, P_w) = L(x, y, z, w),$$

(5)

is called a Riemannian $P$-tensor. This notion is an analogue of the notion of a Kähler tensor in Hermitian geometry.

The linear connections in our investigations have a torsion. Let $\nabla'$ be a linear connection with a tensor $Q$ of the transformation $\nabla \to \nabla'$ and a torsion $T$, i.e.

$$\nabla'_x y = \nabla_x y + Q(x, y), \quad T(x, y) = \nabla'_x y - \nabla'_y x - [x, y].$$

The corresponding $(0,3)$-tensors are defined by

$$Q(x, y, z) = g(Q(x, y), z), \quad T(x, y, z) = g(T(x, y), z).$$

(6)

The symmetry of the Levi-Civita connection implies

$$T(x, y) = Q(x, y) - Q(y, x),$$

(7)

$$T(x, y) = -T(y, x).$$

(8)

**Definition 1.1** ([10]). A linear connection $\nabla'$ on a Riemannian almost product manifold $(M, P, g)$ is called a natural connection if $\nabla' P = \nabla' g = 0$.

If $\nabla'$ is a linear connection with a tensor $Q$ of the transformation $\nabla \to \nabla'$ on a Riemannian almost product manifold, then it is a natural connection if and only if the following conditions are valid [10]:

$$F(x, y, z) = Q(x, y, P_z) - Q(x, P_y, z),$$

(9)

$$Q(x, y, z) = -Q(x, z, y).$$

(10)

Let $R'$ be the curvature tensor of a natural connection $\nabla'$ on Riemannian almost product manifold $(M, P, g)$. Then, according to the definitional equalities for $R$
\[ R(x, y, z, w) = R'(x, y, z, w) - Q(T(x, y), z, w) \]
\[ - (\nabla'_z Q)(y, z, w) + (\nabla_y Q)(x, z, w) \]
\[ + g(Q(x, z), Q(y, w)) - g(Q(y, z), Q(x, w)) \]
\[ (12) \]

2. A natural connection \( D \) on \((M, P, g) \in W_1\)

In the classification of Riemannian almost product manifolds given in [13], the class \( W_1 \) is the only class, where the basic tensor \( F \) is expressed in terms of the tensor \( g \otimes \theta \), namely (2).

Let \( T \) be the torsion of an arbitrary natural connection on a Riemannian product manifold \((M, P, g) \in W_1\). In [3], it is obtained the following expression of \( T \) by \( g \otimes \theta \):

\[ T(x, y, z) = \frac{1}{2n} \{ g(y, z)\theta(Px) - g(x, z)\theta(Py) \} \]
\[ + \lambda \{ g(y, z)\theta(x) - g(x, z)\theta(y) + g(y, Pz)\theta(Px) - g(x, Pz)\theta(Py) \} \]
\[ + \mu \{ g(y, Pz)\theta(x) - g(x, Pz)\theta(y) + g(y, z)\theta(Px) - g(x, z)\theta(Py) \} , \]

where \( \lambda, \mu \in \mathbb{R} \). Let us note that for \( \lambda = 0 \) and \( \mu = -\frac{1}{4n} \) we have the torsion of the canonical connection investigated in [13]. The canonical connection on an arbitrary Riemannian almost product manifold is introduced in [10] as an analogue of the Hermitian connection in Hermitian geometry ([7], [8], [14]).

The goal of the present work is the investigation of the natural connection \( D \) whose torsion \( T \) is determined by (13) for \( \lambda = \mu = 0 \), i.e.

\[ T(x, y, z) = \frac{1}{2n} \{ g(y, z)\theta(Px) - g(x, z)\theta(Py) \} . \]

**Proposition 2.1.** The connection \( D \) is determined by the following equality

\[ g(D_x y, z) = g(\nabla_x y, z) + Q(x, y, z), \]

where

\[ Q(x, y, z) = \frac{1}{2n} \{ g(x, Pz)\theta(Py) - g(x, Pz)\theta(Py) \} . \]

**Proof.** Since \( D \) is a natural connection, according to [4], there is valid the following equality for the tensor \( Q \) of the transformation \( \nabla \rightarrow D \):

\[ Q(x, y, z) = \frac{1}{2n} \{ T(x, y, z) - T(y, z, x) + T(z, x, y) \} . \]

Applying (14) to (17), we have (15). \( \square \)

For the corresponding tensor \( Q \) of type (1,2), according to (16), we obtain

\[ Q(x, y) = \frac{1}{2n} \{ g(x, y)P\Omega - \theta(Py)x \} , \]

where \( \Omega \) is determined by \( g(\Omega, x) = \theta(x) \). Then (15) implies

\[ D_x y = \nabla_x y + \frac{1}{2n} \{ g(x, y)P\Omega - \theta(Py)x \} , \]

and \( R' \), bearing in mind \( \nabla'g = 0 \) and equalities (7), (8), (9), (10) and (11), we have the following relation for \( R \) and \( R' \):

\[ R(x, y, z, w) = R'(x, y, z, w) - Q(T(x, y), z, w) \]
\[ - (\nabla'_z Q)(y, z, w) + (\nabla_y Q)(x, z, w) \]
\[ + g(Q(x, z), Q(y, w)) - g(Q(y, z), Q(x, w)) . \]
which yields the following relation between the covariant derivatives of $\theta$ with respect $D$ and $\nabla$:

$$
(D_x \theta) y = (\nabla_x \theta) y - \frac{1}{2n} \left\{ g(x, y)\theta(P\Omega) - \theta(Py)\theta(x) \right\}.
$$

We have the following immediately consequence of (18):

$$
T(x, y) = \frac{1}{2n} \left\{ \theta(Px)y - \theta(Py)\theta(x) \right\}.
$$

The equality (14) implies the properties

$$
S_{x, y, z} T(x, y, z) = 0 \\
S_{x, y, z} T(x, y, z) = 0
$$

where $S_{x, y, z}$ is the cyclic sum by $x, y, z$.

Equalities (22) and (17) yield

$$
Q(x, y, z) = T(z, y, x).
$$

According to (21), it is follows $\theta(PT(x, y)) = 0$ and then, bearing in mind (14), we obtain

$$
T(T(x, y), z) = \frac{1}{4n^2} \left\{ \theta(Py)\theta(Pz)x - \theta(Px)\theta(Pz)y \right\}.
$$

Therefore we have the property

$$
S_{x, y, z} T(T(x, y), z) = 0.
$$

**Remark 2.1.** According to property (23), the case $T(x, y) = -[x, y]$, i.e. $D_x y = D_y x$, is possible ([5], [6]). Then the manifold has a structure of a Lie group.

According to (12), (16), (18) and (21), we obtain the following identity for the curvature tensors $R$ and $R'$ of the connections $\nabla$ and $D$:

$$
R(x, y, z, w) = R'(x, y, z, w) - \frac{\theta(\Omega)}{4n^2} \pi_1(x, y, z, w)
$$

$$
- \frac{1}{2n} \left\{ g(y, z) (D_x \theta) Pw - g(x, z) (D_y \theta) Pw \\
+ g(x, w) (D_y \theta) Pz - g(y, w) (D_x \theta) Pz \right\},
$$

where

$$
\pi_1(x, y, z, w) = g(y, z)g(x, w) - g(x, z)g(y, w).
$$

Since $S_{x, y, z} R(x, y, z, w) = 0$ and $S_{x, y, z} \pi_1(x, y, z, w) = 0$, then equality (24) implies

$$
S_{x, y, z} R'(x, y, z, w) = \frac{1}{2n} \left\{ (D_y \theta) Pz - (D_z \theta) Py \right\}.
$$

In [13], it is introduced the following curvature-like tensor $\psi_1$ on a Riemannian product manifold

$$
\psi_1(S)(x, y, z, w) = g(y, z)S(x, w) - g(x, z)S(y, w) \\
+ S(y, z)g(x, w) - S(x, z)g(y, w),
$$

where

$$
S(x, y) = (D_x \theta) Py + \frac{\theta(\Omega)}{2n} g(x, y).
$$
Using (24), (25), (27) and (28), we obtain the following

**Theorem 2.2.** The curvature tensors $R$ and $R'$ of $\nabla$ and $D$ are related via the formula

$$R(x, y, z, w) = R'(x, y, z, w) - \frac{1}{2n} \psi_1(S)(x, y, z, w).$$

□

**Corollary 2.3.** The Ricci tensors $\rho$ and $\rho'$ as well as the scalar curvatures $\tau$ and $\tau'$ for $\nabla$ and $D$, respectively, are related as follows

$$\rho(y, z) = \rho'(y, z) - \frac{1}{2n} \{g(y, z) \text{tr} S + 2(n - 1)S(y, z)\},$$

$$\tau = \tau' - \frac{2n - 1}{n} \text{tr} S.$$

□

3. **Connection $D$ with Riemannian $P$-tensor of curvature**

Conditions (4) are satisfied for the curvature tensor $R'$ of the connection $D$ because of (24). Moreover, condition (6) is also valid for $R'$, because of $DP = 0$. Therefore, $R'$ is a Riemannian $P$-tensor if and only if $R'$ satisfies condition (5).

**Theorem 3.1.** The connection $D$ has a Riemannian $P$-tensor of curvature if and only if the following condition is valid

$$(D_y \theta) Pz = (D_z \theta) Py.$$

**Proof.** Let the curvature tensor $R'$ of the connection $D$ be a Riemannian $P$-tensor. Then (6) is valid for $R'$ and because of (26) we have

$$S_{x, y, z} g(x, w) \{(D_y \theta) Pz - (D_z \theta) Py\} = 0.$$

Since we have supposed that $\dim M \geq 4$, then (33) implies (32).

Vice versa, if (32) is satisfied, then (26) yields condition (5) for $R'$ and therefore $R'$ is a Riemannian $P$-tensor.

□

Bearing in mind (32) and (20), we have immediately the following

**Corollary 3.2.** The connection $D$ has a Riemannian $P$-tensor of curvature if and only if the following condition is valid

$$(\nabla_x \theta) Py = (\nabla_y \theta) Px,$$

i.e. if and only if the 1-form $\theta \circ P$ is closed.

□

4. **Connection $D$ with parallel torsion**

Equality (21) implies

$$(D_z T)(y, z) = \frac{1}{2n} \{(D_z \theta) Py, z - (D_z \theta) Pz, y\},$$

which yields immediately the following

**Proposition 4.1.** The connection $D$ has parallel torsion if and only if the associated Lee 1-form $\theta$ is also parallel with respect to $D$.

□
Corollary 4.2. The connection $D$ has parallel torsion if and only if the following condition is satisfied
\[
(\nabla_x \theta) y = \frac{1}{2n} \{ g(x, y) \theta(P\Omega) - \theta(Py) \theta(x) \}.
\]

Let $D$ have a parallel torsion. Then, according to (24) we obtain
\[
R(x, y, z, w) = R'(x, y, z, w) - \frac{\theta(\Omega)}{4n^2\pi_1(x, y, z, w)},
\]
which implies condition (3) for $R'$. Therefore, it is valid the following

Proposition 4.3. If the connection $D$ has parallel torsion then the curvature tensor of $D$ is a Riemannian $\mathcal{P}$-tensor which satisfies condition (34).

5. The Weyl tensor of the transformation $\nabla \to D$

Let $W$ and $W'$ be the Weyl tensors for the connections $\nabla$ and $D$, respectively, i.e.
\[
W = R - \frac{1}{2(n-1)} \left\{ \psi_1(\rho) - \tau \frac{\pi_1}{2n-1} \right\},
\]
\[
W' = R' - \frac{1}{2(n-1)} \left\{ \psi_1(\rho') - \tau' \frac{\pi_1}{2n-1} \right\}.
\]

Theorem 5.1. The Weyl tensor is invariant with respect to the transformation $\nabla \to D$.

Proof. We determine $\text{tr} S$ from (31) and replace it in (30). So we obtain
\[
S = \left\{ \rho' - \frac{\tau'}{2(2n-1)} g \right\} - \left\{ \rho - \frac{\tau}{2(2n-1)} g \right\}.
\]

After that we determine $S$ from the latter equality and replace it in (29). In the result we take into account (27) and the equality $\psi_1(g) = 2\pi_1$ and obtain $W = W'$ by appropriate regrouping.

6. Conformal transformation of the curvature tensor of $D$

Now we will establish the way of transforming of the curvature tensor $R'$ of the connection $D$ using the usual conformal transformation of the metric $g$. This transformation is determined via the formula
\[
\bar{g} = e^{2u} g,
\]
where $u$ is a smooth function on the considered manifold.

It is known that the Levi-Civita connection is transformed by (36) as follows
\[
\bar{\nabla}xy = \nabla xy + du(x)y + du(y)x - g(x, y)L,
\]
where $L = \text{grad} u$.

In [13], it is proved that $(M, P, g) \in \mathcal{W}_1$ implies $(M, P, \bar{g}) \in \mathcal{W}_1$, such as the associated Lee 1-forms $\theta$ and $\bar{\theta}$ as well as their corresponding vectors $\Omega$ and $\bar{\Omega}$ are related via the formulas
\[
\bar{\theta}(x) = \theta(x) + 2udu(Px),
\]
\[
\bar{\Omega} = e^{-2u}(\Omega + 2nL).
\]
Analogously to (19) it is valid the following
\[ \bar{D}x y = \bar{\nabla}x y + \frac{1}{2n} \{ \bar{g}(x, y) P\bar{\Omega} - \bar{\theta}(P)y x \}, \]
The latter equality and equalities (36), (37), (38) and (39) yield (40)
\[ \bar{D}x y = Dx y + du(x)y. \]
Using the definitional equality \( \bar{R}'(x, y)z = D_x D_y z - D_y D_z x - D_{[x,y]} z \) for the curvature tensor \( \bar{R}' \) of \( \bar{D} \) and (40), we obtain \( \bar{R}' = R' \), i.e. it is valid the following

**Theorem 6.1.** The curvature tensor of the connection \( D \) is invariant with respect to the usual conformal transformation (36) of the metric \( g \). □

7. **The case of a flat connection \( D \)**

Let \( D \) be a flat connection, i.e. \( R' = 0 \). Then \( W' = 0 \) and because of Theorem 5.1 we have \( W = 0 \). Therefore it is valid the following

**Proposition 7.1.** If \( D \) is a flat connection then the manifold is conformally flat with respect to \( \nabla \). □

Further, we have the following

**Theorem 7.2.** Let the connection \( D \) be a flat connection with parallel torsion. Then the following propositions are valid:

(i) \( R = -\frac{1}{4n} \theta(\Omega) \pi_1, \quad \rho = -\frac{2n-1}{4n} \theta(\Omega) g, \quad \tau = -\frac{2n-1}{2n} \theta(\Omega); \)
(ii) The tensor \( R \) is parallel with respect to \( D \);
(iii) The manifold is a space form;
(iv) The scalar curvature \( \tau \) is negative.

**Proof.** Let the connection \( D \) be a flat connection with parallel torsion. Then, according to the Proposition 4.3, we obtain the first equality in (i), which implies the other two equalities of (i).

Bearing in mind (15), we have
\[ R(x, y)z = R'(x, y)z - (D_x Q)(y, z) + (D_y Q)(x, z) \]
\[ + Q(x, Q(y, z)) - Q(y, Q(x, z)). \]
Since \( DT = 0 \), using (17), we obtain \( DQ = 0 \). Then relation (11) implies \( DR = DR' \). Hence, because of \( DR' = 0 \), we obtain \( DR = 0 \), i.e. (ii) is valid.

The formulas in (i) imply directly (iii).

Since \( g \) is a Riemannian metric, then \( \theta(\Omega) = g(\Omega, \Omega) > 0 \). Therefore, because of the latter equality in (i), we have that \( \tau < 0 \), i.e. (iv) is valid. □

8. **Example**

8.1. **A Lie group \( G \) as a Riemannian product \( W_1 \)-manifold \( (G, P, g) \).** Let \( G \) be a 4-dimensional real connected Lie group and \( g \) be its Lie algebra with a basis \{\( X_i \)\}.

We introduce a structure \( P \) and left invariant metric \( g \) as follows
\[ PX_1 = X_3, \quad PX_2 = X_4, \quad PX_3 = X_1, \quad PX_4 = X_2, \]
\[ g(X_i, X_j) = \begin{cases} 1, & i = j; \\ 0, & i \neq j. \end{cases} \]
Thus, \((G, P, g)\) becomes a Riemannian almost product manifold with \(\text{tr} P = 0\).

We will consider the case when \(P\) is an Abelian almost product structure \([1]\), i.e.

\begin{equation}
[PX_i, PX_j] = -[X_i, X_j].
\end{equation}

Then the manifold \((G, P, g)\) has a zero Nijenhuis tensor and therefore \((G, P, g)\) belongs to the class \(W_1 \oplus W_2\).

**Theorem 8.1.** The manifold \((G, P, g)\) is a Riemannian product manifold belonging to the class \(W_1\) if and only if the Lie algebra \(g\) is determined by the conditions:

\begin{equation}
\begin{aligned}
[X_1, X_2] &= -[X_3, X_4] = \lambda_1 X_1 + \lambda_2 X_2 + \lambda_3 X_3 + \lambda_4 X_4, \\
[X_1, X_3] &= [X_2, X_4] = \lambda_4 X_1 - \lambda_3 X_2 + \lambda_2 X_3 - \lambda_1 X_4, \\
[X_2, X_3] &= [X_1, X_4] = 0, \\
\end{aligned}
\end{equation}

\(\lambda_i \in \mathbb{R}; \ i = 1, 2, 3, 4\).

**Proof.** Because of (43) we have

\begin{equation}
2g(\nabla X_i X_j X_k) = g([X_i, X_j], X_k) + g([X_k, X_i], X_j)
\end{equation}

which implies the following equality, according to (44) and (1):

\begin{equation}
\begin{aligned}
2F(X_i, X_j, X_k) &= g([X_i, PX_j] - P[X_i, X_j], X_k) \\
&+ g([X_k, PX_i] - P[X_k, X_i], X_j) \\
&+ 2g([X_k, PX_j], X_i).
\end{aligned}
\end{equation}

The comparing of condition (47) with the characteristic condition (2) for the class \(W_1\), taking into account (42), (43), (44) and the Jacobi identity for the commutators \([X_i, X_j]\), yields conditions (45). \(\square\)

The comparing of (47) and (2) give also the following formulas for the associated Lee 1-form \(\theta\):

\begin{equation}
\theta_1 = 4\lambda_1, \quad \theta_2 = -4\lambda_3, \quad \theta_3 = -4\lambda_2, \quad \theta_4 = 4\lambda_1.
\end{equation}

Further, \((G, P, g)\) will stand for the manifold determined by (45).

### 8.2. Some geometrical characteristics of the manifold \((G, P, g)\).

By virtue of (46) and (45) we get the non-zero components \(\nabla X_i X_j\) of the Levi-Civita connection \(\nabla\):

\begin{equation}
\begin{aligned}
\nabla_{X_1} X_1 &= \nabla_{X_2} X_4 = -\lambda_1 X_2 - \lambda_4 X_3, \\
\nabla_{X_2} X_2 &= \nabla_{X_3} X_3 = \lambda_2 X_1 + \lambda_3 X_4, \\
\nabla_{X_1} X_2 &= -\nabla_{X_3} X_4 = \lambda_1 X_1 + \lambda_3 X_3, \\
\nabla_{X_2} X_1 &= -\nabla_{X_4} X_3 = -\lambda_2 X_2 - \lambda_4 X_4, \\
\nabla_{X_1} X_3 &= \nabla_{X_2} X_4 = \lambda_4 X_1 - \lambda_3 X_2, \\
\nabla_{X_3} X_1 &= \nabla_{X_4} X_2 = -\lambda_2 X_3 + \lambda_1 X_4.
\end{aligned}
\end{equation}
Using (19) and (15) we obtain the non-zero components \( R_{ijk} = R(X_i, X_j, X_k, X_s) \) of the curvature tensor \( R \) for \( \nabla \):

\[
\begin{align*}
R_{1212} &= \lambda_1^2 + \lambda_2^2, & R_{1313} &= \lambda_2^2 + \lambda_3^2, \\
R_{1414} &= \lambda_1^2 + \lambda_4^2, & R_{2323} &= \lambda_2^2 + \lambda_3^2, \\
R_{2424} &= \lambda_1^2 + \lambda_3^2, & R_{3434} &= \lambda_2^2 + \lambda_4^2, \\
R_{1213} &= \lambda_1 \lambda_4, & R_{1214} &= \lambda_2 \lambda_4, \\
R_{1232} &= \lambda_1 \lambda_3, & R_{1242} &= \lambda_2 \lambda_3, \\
R_{1341} &= \lambda_1 \lambda_2, & R_{1442} &= \lambda_2 \lambda_4.
\end{align*}
\]

(50)

The rest of the non-zero components are obtained by the properties

\[
R_{ijk} = R_{kij}, \quad R_{ijk} = -R_{jik} = -R_{ijk}.
\]

Using (50) for the non-zero components \( \rho_{ij} = \rho(X_i, X_j) \) of the Ricci tensor \( \rho \) we compute:

\[
\begin{align*}
\rho_{11} &= -2 (\lambda_1^2 + \lambda_2^2 + \lambda_3^2), & \rho_{22} &= -2 (\lambda_2^2 + \lambda_3^2 + \lambda_4^2), \\
\rho_{33} &= -2 (\lambda_2^2 + \lambda_3^2 + \lambda_4^2), & \rho_{44} &= -2 (\lambda_2^2 + \lambda_3^2 + \lambda_4^2), \\
\rho_{12} &= 2 \lambda_3 \lambda_4, & \rho_{13} &= -2 \lambda_1 \lambda_3, & \rho_{14} &= -2 \lambda_2 \lambda_3, \\
\rho_{34} &= 2 \lambda_1 \lambda_2, & \rho_{23} &= -2 \lambda_1 \lambda_4, & \rho_{24} &= -2 \lambda_2 \lambda_4.
\end{align*}
\]

(51)

The rest of the non-zero components are obtained by the property \( \rho_{ij} = \rho_{ji} \).

By (51) we obtain the following

**Proposition 8.2.** The manifold \((G, P, g)\) has a negative scalar curvature for \( \nabla \):

\[
\tau = -6 (\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2).
\]

\[
\square
\]

For the Riemannian sectional curvatures of the \( P \)-invariant basis 2-planes \((X_1, X_3)\) and \((X_2, X_4)\), i.e. for the invariant sectional curvatures of the basis 2-planes, we get

\[
k_{13} = - (\lambda_2^2 + \lambda_3^2), \quad k_{24} = - (\lambda_1^2 + \lambda_3^2).
\]

(53)

The sectional curvatures of the rest of the basis 2-planes, i.e. the anti-invariant sectional curvatures of the basis 2-planes, are:

\[
k_{12} = - (\lambda_1^2 + \lambda_2^2), \quad k_{14} = - (\lambda_1^2 + \lambda_2^2), \\
k_{23} = - (\lambda_2^2 + \lambda_3^2), \quad k_{34} = - (\lambda_2^2 + \lambda_3^2).
\]

(54)

Conditions (53) and (54) imply the following

**Theorem 8.3.** The manifold \((G, P, g)\) has:

(i) a constant invariant sectional curvature if and only if

\[
\lambda_1^2 - \lambda_2^2 + \lambda_3^2 - \lambda_4^2 = 0;
\]

(ii) a constant anti-invariant sectional curvature if and only if

\[
\lambda_1^2 = \lambda_3^2, \quad \lambda_2^2 = \lambda_4^2;
\]

(iii) a constant sectional curvature if and only if

\[
\lambda_1^2 = \lambda_2^2 = \lambda_3^2 = \lambda_4^2.
\]

In this case \( R = \frac{\pi}{12} \).

\[
\square
\]
By virtue of (35), using (50), (51), (52), (25) and (27) for $S = \rho$, we obtain the following

**Proposition 8.4.** The manifold $(G, P, g)$ has a zero Weyl tensor, i.e. $(G, P, g)$ is conformally flat for $\nabla$. □

8.3. The connection $D$ on the manifold $(G, P, g)$. Further in our considerations we exclude the trivial case $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$, i.e. the case when $(G, P, g)$ is a Riemannian $P$-manifold.

By virtue of (19), (19) and (13) for the components of the connection $D$ on $(G, P, g)$ we obtain:

$$
D_{X_1} X_1 = -\lambda_3 X_4, \quad D_{X_1} X_2 = \lambda_3 X_3,
$$

$$
D_{X_1} X_3 = -\lambda_3 X_2, \quad D_{X_1} X_4 = \lambda_3 X_1,
$$

$$
D_{X_2} X_1 = -\lambda_4 X_4, \quad D_{X_2} X_2 = \lambda_4 X_3,
$$

$$
D_{X_2} X_3 = -\lambda_4 X_2, \quad D_{X_2} X_4 = \lambda_4 X_1,
$$

$$
D_{X_3} X_2 = -\lambda_1 X_3, \quad D_{X_3} X_1 = \lambda_1 X_4,
$$

$$
D_{X_3} X_4 = -\lambda_1 X_1, \quad D_{X_3} X_3 = \lambda_1 X_2,
$$

$$
D_{X_4} X_2 = -\lambda_2 X_3, \quad D_{X_4} X_1 = \lambda_2 X_4,
$$

$$
D_{X_4} X_4 = -\lambda_2 X_1, \quad D_{X_4} X_3 = \lambda_2 X_2.
$$

(55)

Using (55) and (15), we obtain that all components of the curvature tensor $R'$ of $D$ are zeros, i.e. the following proposition holds:

**Proposition 8.5.** The manifold $(G, P, g)$ has a flat connection $D$. □

By virtue of (21) and (48) we get the non-zero components $T_{ij} = T(X_i, X_j)$ of the torsion $T$ for the connection $D$:

$$
T_{12} = -\lambda_1 X_1 - \lambda_2 X_2, \quad T_{13} = -\lambda_4 X_1 - \lambda_2 X_3,
$$

$$
T_{14} = \lambda_3 X_1 - \lambda_2 X_4, \quad T_{23} = -\lambda_4 X_2 + \lambda_1 X_3,
$$

$$
T_{24} = \lambda_3 X_2 + \lambda_1 X_4, \quad T_{34} = \lambda_3 X_3 + \lambda_4 X_4.
$$

The rest of the non-zero components are obtained by the property $T_{ij} = -T_{ji}$.

According to Proposition 4.1 the connection $D$ has a parallel torsion if and only if $(DX, \theta) X_j = 0$. The latter equality is equivalent to $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$, because of (48) and (55). This case is excluded from our investigations. Therefore, the following proposition holds:

**Proposition 8.6.** The connection $D$ on the manifold $(G, P, g)$ is not parallel. □

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