INTRODUCTION TO: NON-ASSOCIATIVE FINITE INVERTIBLE LOOPS

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Abstract. Non-associative finite invertible loops (NAFIL) are loops whose every element has a unique two-sided inverse. Not much is known about the class of NAFIL loops which includes the familiar IP (Inverse Property), Moufang, and Bol loops. Our studies have shown that they are involved in such diverse fields as combinatorics, finite geometry, quasigroups and related systems, Cayley algebras, as well as in theoretical physics. This paper presents an introduction to the class of NAFIL loops as the starting point for the development of the theory of these interesting structures.

1. Introduction

Many associative algebraic structures like groups have been studied extensively by numerous mathematicians [2]. It appears, however, that not much is known in the current literature about non-associative group-like structures other than special classes of quasigroups and loops because the theory of these structures [13]. "... is a fairly young discipline which takes its roots from geometry, algebra and combinatorics."

In 1981, we became interested in studying a class of non-associative finite loops in which every element has a unique inverse [3]. Several structures of this type have been intensively studied like IP (Inverse Property), Moufang, and Bol loops. However, there are many interesting loops in this class other than these. A literature search has shown that there is no standard term for this class of loops. Because of this, we have decided to introduce the acronym NAFIL which stands for: non-associative finite invertible loop.

This paper presents an introduction to the class of NAFIL loops in an attempt to start the development of the theory of these structures. It is based on the results of various studies started in 1981 that led the way to a two-year research project supported by the National Research Council of the Philippines (NRCP)\(^1\). This project [6] which was conducted at the SciTech R&D Center of the Polytechnic University of the Philippines from 1996 to 1998 is still being actively pursued at present.

To our knowledge, NAFIL loops have not been systematically studied as a specific class of finite loops. Moreover, very little is known about their applications in other fields of mathematics and in theoretical physics. For instance, our studies have shown that these loops are involved in the theories of non-associative real algebras

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and rings. Thus, among other things, we have shown [4] that the basis vectors of all non-associative $2^r$-dimensional Cayley algebras form NAFIL loops of order $2^{r+1}$. These loops define real algebras that constitute an infinite family which includes all known alternative real division algebras as members. One of these algebras (the Cayley numbers or octonions) has also been applied by physicists to the quantum theory of elementary particles in connection with quark models and in string theory [12].

In this paper, we shall introduce the class of NAFILs by defining the concept of the non-associative finite invertible loop (NAFIL) and presenting some of its fundamental properties. As a sequel to this introductory paper, we shall deal with the structure of NAFILs, develop tools and methods (algorithms) for their construction and analysis, identify fundamental problems to be solved, and present some of the important results we have so far obtained from our researches in an attempt to lay the foundations of the Theory of Non-Associative Finite Invertible Loops.

2. The Finite Invertible Loop

To fully appreciate what the invertible loop is, it is helpful to recall the idea of an abstract mathematical system. Such a system essentially consists of: a non-empty set $S$ of distinct elements, at least one binary operation $*$, an equivalence relation $=$, a set of axioms, as well as a set of definitions and theorems and is usually denoted by $(S,*)$. The heart of the system is the set of axioms from which all the theorems are derived. Most algebraic systems like groups, rings, and fields satisfy some or all of the following axioms or postulates:

**Axiom 1.** For all $a, b \in S$, $a * b \in S$. (Closure axiom)

**Axiom 2.** There exists a unique element $e \in S$, called the identity, such that $e * a = a = a * e$ for all $a \in S$. (Identity axiom)

**Axiom 3.** Given an identity element $e \in S$, for every $a \in S$ there exists a unique element $a^{-1} \in S$, called its inverse, such that $a * a^{-1} = a^{-1} * a = e$. (Inverse axiom)

**Axiom 4.** For every $a, b \in S$ there exists unique $x, y \in S$ such that $a * x = b$ and $y * a = b$. (Unique Solution axiom)

**Axiom 5.** For every $a, b \in S$, $a * b = b * a$. (Commutative axiom)

**Axiom 6.** For all $a, b, c \in S$, $(a * b) * c = a * (b * c)$. (Associative axiom)

The simplest algebraic system is the groupoid; it is only required to satisfy A1. This is a trivial system and not much can be said about it. A groupoid that also satisfies A4 is called a quasigroup; and a quasigroup that satisfies A2 is called a loop [9, 13].

One of the most important systems is the group which can be defined as any system satisfying axioms A1, A2, A3, A4, and A6. Hence, the group is a loop that also satisfies A3 and A6. Alternatively, we can say that the group is a quasigroup that satisfies A2, A3, and A6. It must be noted that the axioms A1, A2, A3, A4, A5, and A6 are independent. (Of course A3 is meaningful only if A2 is assumed.) Any algebraic system that satisfies A5 (Commutative axiom) is called abelian (or commutative).

A loop that also satisfies A3 is called an invertible loop. Such a loop can be either associative or non-associative.
Remark 1. The class of loops forms a variety\(^2\) [9, 10]. defined by the set of operations \(\Omega = \{e, -1, \star\}\) (where \(e, -1\), and \(\star\) are nullary, unary, and binary operations, respectively), a set of identities \(I = \{e \star x = x, x \star e = x\}\) (where \(e\) is a unique identity element), and the equations: \(ax = b, ya = b\).

Definition 1. An invertible loop \((L, \star)\) is a non-empty set \(L\) with a binary operation \(\star\) satisfying axioms A1, A2, A3, and A4. If it also satisfies A5, it is called abelian and if \(L\) has a finite number of elements, it is called finite.

It is clear from Definition 1 that an invertible loop is a loop in which every element has a unique (or two-sided) inverse. Hence the class of all invertible loops belongs to the variety of loops and is defined by the set of operations \(\Omega = \{e, -1, \star\}\), the equations \(ax = b, ya = b\), and the identities:

\[
e \star x = x, \quad x \star e = x, \quad x \star x^{-1} = e, \quad x^{-1} \star x = e
\]

where \(e\) is the identity element and \(x^{-1}\) is the unique inverse of \(x\). Thus groups are also invertible loops. The term invertible loop therefore applies to both associative and non-associative types. Moreover, it can be shown that the class of invertible loops also forms a variety of loops.

3. The Non-Associative Finite Invertible Loop (NAFIL)

Having defined what the invertible loop is, let us now focus our attention on the class of non-associative finite invertible loops whose abstract theory we are attempting to develop.

Definition 2. A finite invertible loop that is non-associative is called a NAFIL (non-associative finite invertible loop) while one that is associative is called a group.

To emphasize the fact that the acronym NAFIL refers to a loop, we shall allow ourselves some grammatical freedom by often calling it a NAFIL loop. Although NAFIL loops are invertible loops, the class of NAFILs does not form a variety.

The class of NAFIL loops includes, among others, the following:

- Loops with inverse properties: IP (Inverse Property), LIP/RIP (Left/Right Inverse Property), CIP (Crossed Inverse Property), etc.
- Moufang loops and Bol loops
- Plain loops (anti-associative)

Remark 2. We note that a distinction is sometimes made between the terms non-associative and not-associative. A system is “non-associative” if A6 (associative axiom) is not assumed to hold while it is “not-associative” if A6 is required to be not satisfied by the system as a whole. The NAFIL, as defined above, is therefore not-associative in this sense. However, unless otherwise indicated, the term non-associative will henceforth be taken to mean not-associative when referring to NAFIL loops.

\(^2\)A variety is a class of algebraic structures of the same signature satisfying a given set of identities which is closed under the taking of homomorphic images (H), subalgebras (S), and direct products (P). Thus, the class of invertible loops also forms a variety. The class of NAFIL loops, however, does not form a variety because it is not closed under H and P; a NAFIL can have homomorphic images and subloops that are groups (which are not NAFILs).
In this paper, our main concern will be on finite invertible loops that are not-
associative (NAFIL loops). This does not imply, however, that A6 does not hold
at all in such a loop; it may be satisfied in a limited way within the system. Since
all finite algebraic systems we will consider satisfy A1 (Closure axiom), they are
essentially groupoids. Henceforth, we shall therefore use groupoid as a generic
term for finite quasigroups, loops, groups, and related structures. For such finite
systems, it is both convenient and useful to define them in constructive
terms.

A finite algebraic system or groupoid like the invertible loop \((\mathcal{L}, \star)\) is completely
defined by its multiplication or Cayley table. Such a table is a listing of the \(n^2\)
possible binary products, \(\ell_i \star \ell_j\), of its \(n\) elements and it can be represented by an
\(n \times n\) matrix \(S(\mathcal{L}) = (\ell_{ij})\), called its structure matrix with entries \(\ell_{ij} = \ell_i \star \ell_j\) from
its set \(\mathcal{L}\) of elements. All abstract properties of the groupoid \((\mathcal{L}, \star)\) are embodied
in this matrix.

**Definition 3.** Let \((\mathcal{L}, \star)\) be any finite groupoid of order \(n\), where \(\mathcal{L} = \{\ell_x \mid x = 1, \ldots, n\}\) and \(\star\) is a closed binary operation on \(\mathcal{L}\). The \(n \times n\) matrix \(S(\mathcal{L}) = (\ell_{ij})\),
where \(\ell_{ij} = \ell_i \star \ell_j \in \mathcal{L}\) for all \(i, j = 1, \ldots, n\), is the structure matrix (or Cayley
table) of \((\mathcal{L}, \star)\).

\[
\begin{bmatrix}
\ell_{11} & \ell_{12} & \cdots & \ell_{1j} & \cdots & \ell_{1n} \\
\ell_{21} & \ell_{22} & \cdots & \ell_{2j} & \cdots & \ell_{2n} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
\ell_{i1} & \ell_{i2} & \cdots & \ell_{ij} & \cdots & \ell_{in} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\ell_{n1} & \ell_{n2} & \cdots & \ell_{nj} & \cdots & \ell_{nn}
\end{bmatrix}
\]

(A) Structure Matrix

\[
\begin{array}{ccccccc}
\ell_1 & \ell_2 & \cdots & \ell_j & \cdots & \ell_n \\
\ell_1 & \ell_{11} & \ell_{12} & \cdots & \ell_{1j} & \cdots & \ell_{1n} \\
\ell_2 & \ell_{21} & \ell_{22} & \cdots & \ell_{2j} & \cdots & \ell_{2n} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
\ell_i & \ell_{i1} & \ell_{i2} & \cdots & \ell_{ij} & \cdots & \ell_{in} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\ell_n & \ell_{n1} & \ell_{n2} & \cdots & \ell_{nj} & \cdots & \ell_{nn}
\end{array}
\]

(B) Cayley Table

Figure 1. General form of the structure matrix (or simply \(S-
matrix\)) \(S(\mathcal{L}) = (\ell_{ij})\) of the groupoid \((\mathcal{L}, \star)\) of order \(n\). The entries
\(\ell_{ij} = \ell_i \star \ell_j\) are elements of \(\mathcal{L}\). This matrix, when provided with row
and column headings, is also commonly called a Cayley table.

Being a matrix, \(S(\mathcal{L})\) can be formally subjected under certain conditions to
some matrix operations like addition, transpose, etc. The labeling of the elements
of \(\mathcal{L}\) is arbitrary; it is determined simply by considerations of convenience and may
be changed without affecting the structure of the system. However, if \(\mathcal{L}\) has an
identity element \(\ell_1\), it is convenient to arrange its entries such that \(\ell_{1k} = \ell_{k1} = \ell_k\)
for all \(k = 1, \ldots, n\). We shall call this arrangement the standard or normal form
of \(S(\mathcal{L})\) and often simplify the notation for \(S(\mathcal{L})\) entries by writing \(i \equiv \ell_i\) for all
\(i = 1, \ldots, n\).

### 3.1. Existence of Non-Associative Finite Invertible Loops (NAFIL loops)

Now that we have defined what NAFIL loops are, our first main task is to establish
their existence. It can be shown [1] that any invertible loop of order \(n \leq 4\) is a
group and that if \(n = 5\), there exists a unique non-associative invertible loop [3,
Hence, this loop is the smallest NAFIL. If $n \geq 5$, there are two special families of NAFIL loops whose members cover all ODD and EVEN orders from $n = 5$ to infinity [3, 6]. The $S$-matrices (Cayley tables) of the first members of each of these families are shown in Figure 2. Thus, we have

**Theorem 1.** There exists at least one NAFIL (non-associative finite invertible loop) of every finite order $n \geq 5$.

This is the Fundamental Theorem of the Theory of Non-Associative Finite Invertible Loops. From this theorem we find that the smallest NAFIL is of order $n = 5$. Moreover, it can be shown that there is exactly one (up to isomorphism) such loop of this order whose Cayley table is that shown in Figure 2 for $n = 5$.

**Remark 3.** From 1996 to 2001, we undertook a study [8, 15] to determine all nonisomorphic NAFIL loops of small order $n = 5, 6, 7$. This study showed that there is exactly one NAFIL loop of order $n = 5$, 33 of order $n = 6$, and 2,333 of order $n = 7$.

3.2. **Subsystem Composition of NAFIL Loops.** Most algebraic systems contain smaller systems (called subsystems) in their structures [13]. Thus algebras may contain subalgebras and groups can have subgroups. Similarly, certain loops (like NAFILs) may also have subloops.

By definition, a non-empty subset $H$ of a set $G$ is a subgroupoid (subloop, subgroup) of a groupoid $(G, \circ)$ iff $(H, \circ)$ is a groupoid (loop, group) [9, 13]. This means that $(H, \circ)$ satisfies all axioms satisfied by $(G, \circ)$. In the case of invertible loops like the NAFILs, we have shown that $m \leq \frac{n}{2}$, where $m$ is the order of $H$ and $n$ is the order of $G$. Not every groupoid, however, can have subsystems.

In general algebraic systems can be classified into two types: composite (with at least one non-trivial subsystem) and non-composite or plain [14] (without any non-trivial subsystem).

4. **Basic Concepts and Terminology**

The NAFIL is a natural generalization of the group because it satisfies all group axioms except A6 (associative axiom). Hence, all theorems and basic algebraic concepts of group theory that do not depend on A6 (e.g. homomorphisms, isomorphisms, cosets, subsystems, quotients, etc.) apply to NAFIL loops. Those that depend on A6 like the fundamental theorem of Lagrange on the order of a subgroup are not satisfied in general by NAFIL loops. Similarly, it is not always possible to
uniquely define in the traditional sense such concepts as the *power* and *order* of a NAFIL element.

Because of the applications of NAFIL loops in non-associative algebras and general loop theory, we shall also adapt several basic ideas and terms from these fields like those involved in alternative and power-associative algebras, Moufang, Bol, and IP loops, etc. Moreover, we shall also introduce new concepts and ideas in addition to well-established basic algebraic concepts.

In what follows, unless otherwise indicated, we shall mainly consider NAFIL loops. And, for convenience of notation, we shall sometimes represent \((L, \ast)\) simply by \(L\) and the product \(x \ast y\) by \(xy\) (juxtaposition) if no confusion arises.

### 4.1. Products and Powers of NAFIL Elements

Because NAFIL loops are by definition non-associative, expressions involving products of more than two elements are meaningful only if the grouping of the factors is defined. Such groupings (or parenthesizings) must be clearly indicated in all expressions. To simplify matters, it becomes necessary to define special products of NAFIL elements.

**Definition 4.** Let \((L, \ast)\) be a NAFIL and let \(\ell_1, \ell_2, \ell_3, \ldots, \ell_m \in L\). The *left* and *right products* of these \(m\) elements are defined by the expressions:

Left: \(L(\ell_1\ell_2\ell_3\ldots\ell_m) \equiv \ell_m(\cdots(\ell_3(\ell_2\ell_1))\cdots)\)

Right: \(R(\ell_1\ell_2\ell_3\ldots\ell_m) \equiv (\cdots((\ell_1\ell_2)\ell_3)\cdots)\ell_m\)

As noted earlier, the idea of the power of a NAFIL element is not always meaningful. However, certain special forms of repeated products of an element by itself can be defined that are useful in many contexts. If we set \(\ell \equiv \ell_x\) for all \(x = 1, \ldots, m\) in Definition 4, then we have:

**Definition 5.** The expression \(L(\ell^m) \equiv \ell(\cdots(\ell(\ell\ell))\cdots)\), where \(m\) is any positive integer, is the \(m\)-th *left power* of \(\ell \in L\) and \(R(\ell^m) \equiv (\cdots((\ell\ell)\ell)\cdots)\ell\) is the \(m\)-th *right power* of \(\ell \in L\). If \(R(\ell^m) = L(\ell^m)\), we shall simply write \(\ell^m\) to represent either \(R(\ell^m)\) or \(L(\ell^m)\).

The above definition can be stated in the equivalent form: Let \(\ell^{\lambda(m)} \equiv L(\ell^m)\) and \(\ell^{\rho(m)} \equiv R(\ell^m)\), where \(\lambda\) means *left power* and \(\rho\) means *right power*.

It is often convenient to define \(R(\ell^m)\) recursively as follows: Let \(R(\ell^1) = \ell\), \(R(\ell^2) = \ell\ell\), \(R(\ell^3) = \ell(\ell\ell)\ell\), and \(R(\ell^m) = R(\ell^{m-1})\ell\) whenever \(m > 1\). Then we can write \(R(\ell^m) = R(\ell^{m-1})\ell\) in the form: \(\ell^{\rho(m)} = \ell^{\rho(m-1)}\ell\). Similarly, we can also write \(L(\ell^m) = \ell L(\ell^{m-1})\) as \(\ell^{\lambda(m)} = \ell^{\lambda(m-1)}\ell\). If \(R(\ell^m) = L(\ell^m)\) as in the case of abelian loops, then we simply drop the \(R\), \(L\), \(\rho\) and \(\lambda\) and write: \(\ell^m = \ell^{m-1}\ell = \ell^{m-1}\).

### 4.2. Generators and Order

In finite group theory where the power of a group element \(\ell \in L\) is uniquely defined, if \(n\) is the smallest positive integer such that \(\ell^n = 1\), where 1 is the identity, then the elements of the set \(L = \{\ell^x \mid x = 1, \ldots, n\}\) are distinct and \(L\) is said to be generated by \(\ell\). If \((L, \ast)\) is a group, any element \(\ell \in L\) whose powers generate \(L\) is therefore called a *generator* of the group. Moreover, if \(m \leq n\) is the smallest positive integer for which \(\ell^m = 1\), then \(m\) is called the *order* of the element \(\ell \in L\). In this case, the set \(\{\ell^x \mid x = 1, \ldots, m\}\) of powers of \(\ell\) always forms a subgroup of order \(m\). This is not true in general for loops like
the NAFILs where the set of powers of an element does not necessarily form a subsystem. Nevertheless, there are many interesting cases where this condition is satisfied.

The idea of the order of an element and a generator of a set can also be extended to NAFIL loops in a modified form as follows:

**Definition 6.** Let \((L, \star)\) be a NAFIL loop of order \(n\) whose identity is 1. (a) The order of an element \(\ell \in L\) is the order \(m \leq n\) of the smallest subsystem generated by \(\ell\). If \(m = n\), then we say that \(\ell\) is a generator of \((L, \star)\). (b) The left (right) power-order of \(\ell\) is the least positive integer \(m \leq n\) such that \(\ell^{\lambda(m)} = 1\) \((\ell^{\rho(m)} = 1)\). If \(m = n\), then we say that \(\ell\) is a left (right) power-generator of \(L\).

Definition 7(a) is the standard definition of order which is based on the fact that in a loop every element generates a subsystem. This definition applies to all loops. However, unlike groups, the elements of the generated subsystem are not restricted to the powers of an element. Because of this we introduce in Definition 7(b) the idea of the power-order (or simply \(p\)-order) of an element and that of a \(p\)-generator (or simply \(p\)-generator) of a set which are analogous to the ideas of order and generator in group theory. If \(m\) is the least positive integer such that \(\ell^{\lambda(m)} = 1\), then by Definition 7(b) the order \(m\) of the set \(\{\ell^{\lambda(x)} \mid x = 1, \ldots, m\}\) of left powers of \(\ell\) is called the Left \(p\)-order of \(\ell\) to distinguish it from the standard order given in Definition 7(a): the former is simply the order of the set of left powers of the element, whereas the latter is the order of the smallest subsystem generated by the element. The Right \(p\)-order of \(\ell\) is similarly defined. This distinction is important because for non-abelian loops the Left and Right \(p\)-orders of an element are not necessarily equal. Moreover, the set of powers of an element does not necessarily form a subsystem. If the set of Left (Right) powers of \(\ell\) forms a subsystem, then its Left (Right) \(p\)-order is equal to its standard order. This condition is completely satisfied by loops that are power-associative. For such loops, there is no fundamental distinction between order and \(p\)-order as well as between generator and \(p\)-generator.

A NAFIL loop \((L, \star)\) is called monogenic [13] if it can be generated by a single element. If a monogenic loop can be generated by the powers of a given element \(\ell \in L\), then we can distinguish two kinds of generators: left \(p\)-generator (\(LpG\)) and right \(p\)-generator (\(RpG\)) according as \(L\) is generated by the left or right powers of the element \(\ell\). The distinction between left and right \(p\)-generator (as well as between left and right \(p\)-order) is useful when dealing with loops with left- or right-handed properties.

It must be pointed out that the set \(L\) may not have any generator at all other than the set itself. In many cases, it may not also have a single element that can generate it. However, it could happen that it may be generated by several elements that constitute a set of generators. Also, the set \(L\) may have more than one set of generators.

Following conventional notation, if \(\ell \in L\), then the set of all elements generated by \(\ell\) shall be denoted by \(\langle \ell \rangle\). If a set is generated by two or more elements \(q_1, q_2, \ldots, q_m\), we shall denote this by \(\langle q_1, q_2, \ldots, q_m \rangle\). However, the set \(\langle \ell \rangle\) of elements generated by the element \(\ell\) is understood to mean not only the powers of \(\ell\) but may also include elements that are not powers of \(\ell\). In this case, some elements of \(\langle \ell \rangle\) may be products of the powers of \(\ell\), their inverses, etc.
Remark 4. The terms order of an element and generator of a set as given in Definition 7(a) do not indicate how they are determined in specific cases. For instance, given a loop \((\mathcal{L}, \star)\) of order \(n\), where \(\mathcal{L} = \{\ell_i \mid i = 1, 2, \ldots, n\}\), the set of elements generated by an element \(\ell_i\) (denoted by \(\langle \ell_i \rangle\)) consists of all powers (left and right) of \(\ell_i\), the products of these powers, the products of \(\ell_i\) and its powers, etc. Computationally, this is very difficult to determine for each element of \(\mathcal{L}\). The p-order concept given in Definition 7(b), on the other hand, is easier to determine and its use is often more appropriate in most cases than the standard order.

4.3. Association Properties. Weaker forms of A6, called weak associative laws, are known to play important roles in loop theory and many of these also apply to NAFIL loops. These are special identities (or identical relations) which have the general form of the associative relation, \((ab)c = a(bc)\), but which hold true only under certain conditions. These weak associative laws shall also be called association properties.

In general, a weak associative law \([11]\) is defined as a universally quantified equation of the form \(\alpha = \beta\), where for some variables, \(V_1, V_2, \ldots, V_n\) (not necessarily distinct), \(\alpha\) and \(\beta\) are both products of the form \(V_1V_2\ldots V_n\) (with some distribution of parentheses). The number \(n\) of variables is also called the size of the equation. Such a law is called non-trivial iff \(\alpha\) and \(\beta\) do not have the same distribution of parentheses. For instance, the relation \((x(yz))x = (xy)(zx)\), called a Moufang identity, is an equation of size \(n = 4\) that is non-trivial. Here, only three of the four variables are distinct. If the equation \(\alpha = \beta\) is such that any variable in it appears exactly once on each side, then it is called a balanced identity. The simplest example of this is the associative relation: \((xy)z = x(yz)\).

4.3.1. Inverse Properties. Many NAFIL loops are known to satisfy certain important identities \([9]\) involving weak forms of A6, called inverse properties, such as those given in

Definition 7. Let \((\mathcal{L}, \star)\) be a NAFIL and let \(q, \ell, \ell^{-1} \in \mathcal{L}\). If \(\ell^{-1}(\ell q) = q\) and \((q\ell)\ell^{-1} = q\), then \((\mathcal{L}, \star)\) is said to have the Left Inverse Property (LIP) and Right Inverse Property (RIP), respectively. If \((\mathcal{L}, \star)\) satisfies both LIP and RIP properties, then \(\ell^{-1}(\ell q) = (q\ell)\ell^{-1} = q\) and it is said to have the Inverse Property (IP).

The equations that define the above inverse properties are balanced identities of size three. Thus, the LIP identity has the form \(\ell^{-1}(\ell q) = (\ell^{-1}\ell)q = q\) since \((\ell^{-1}\ell) = 1\), where 1 is the identity element. Similarly, the RIP identity has the form \((q\ell)\ell^{-1} = q(\ell\ell^{-1}) = q\) since \((\ell\ell^{-1}) = 1\). This implies that in an IP loop, \(\ell^{-1}\ell = \ell\ell^{-1} = 1\) so that every element has a unique two-sided inverse. Therefore, all IP loops are NAFILs but the converse is not true: there are NAFIL loops that are not IP loops.

In most loops, A3 (Inverse axiom) is not assumed to hold. These loops are not invertible but their elements satisfy weak forms of A3. Thus, if \((\mathcal{L}, \star)\) is a loop whose identity element is \(\ell_1 \equiv 1\), then there exists elements \(\ell^{-\lambda}\) and \(\ell^{-\rho}\), called the left inverse and right inverse of \(\ell \in \mathcal{L}\), respectively, such that \(\ell^{-\lambda} \star \ell = 1\) and \(\ell \star \ell^{-\rho} = 1\). If \((\mathcal{L}, \star)\) is an invertible loop, however, then \(\ell^{-\lambda} = \ell^{-\rho} \equiv \ell^{-1}\) such that \(\ell^{-1}\ell = \ell\ell^{-1} = 1\) because in this case every \(\ell \in \mathcal{L}\) has a unique (or two-sided)
**inverse** \( ℓ^{-1} \in ℒ \). For loops that are not invertible, the inverse properties given in Definition 7 will still hold provided that the weak forms of A3 are considered.

It is easy to show that in an IP NAFIL, the linear equations \( ax = b \) and \( ya = b \) have the unique solutions \( x = a^{-1}b \) and \( y = ba^{-1} \), respectively. On the other hand, in a LIP NAFIL only the equation \( ax = b \) has the unique solution \( x = a^{-1}b \) while in a RIP NAFIL only the equation \( ya = b \) has the unique solution \( y = ba^{-1} \).

### 4.3.2. Other Weak Associative Properties

**Definition 8.** Let \((ℒ,⋆)\) be any NAFIL and let \(ℓ_i, ℓ_k \in ℒ \). If \((ℒ,⋆)\) satisfies the identities

\[
(ℓ_i(ℓ_iℓ_k)) = ℓ_i(ℓ_iℓ_k) \quad \text{[left alternative property (LAP)]}
\]

\[
(ℓ_iℓ_kℓ_i) = ℓ_iℓ_kℓ_i \quad \text{[right alternative property (RAP)]}
\]

for all elements \(ℓ_i, ℓ_k \in ℒ \), then \((ℒ,⋆)\) is called **alternative** and is said to have the **Alternative Property (AP)**. If \((ℒ,⋆)\) satisfies the identity

\[
ℓ_i(ℓ_kℓ_i) = (ℓ_kℓ_iℓ_i) \quad \text{[flexible law (FL)]}
\]

then it is called **flexible**.

If \(ℓ_i = ℓ_k = ℓ \) in the LAP and RAP identities, then \(L(ℓ^3) = ℓℓ^2 = ℓ^2ℓ = R(ℓ^3) \) and we find that \(L(ℓ^3) = R(ℓ^3) \). In general, we find that \(L(ℓ^a) = ℓℓ^{a-1} = ℓ^{a-1}ℓ = R(ℓ^a) \) so that \(L(ℓ^a) = R(ℓ^a) \). The smallest Moufang loop satisfies the requirements of Definition 8 for alternativity.

**Definition 9.** Let \((ℒ,⋆)\) be a NAFIL and let \(ℓ_i, ℓ_j, ℓ_k \in ℒ \). If

\[
ℓ_i[ℓ_j(ℓ_iℓ_k)] = [(ℓ_iℓ_j)ℓ_i]ℓ_k
\]

then \((ℒ,⋆)\) is called a **Moufang loop** and is said to have the **Moufang Property (MP)**. The expression

\[
(D9.1) \quad ℓ_i[ℓ_j(ℓ_iℓ_k)] = [(ℓ_iℓ_j)ℓ_i]ℓ_k
\]

is known as the **Moufang identity** and it is equivalent to each of the identities:

\[
(D9.2) \quad ℓ_i[ℓ_j(ℓ_kℓ_j)] = [(ℓ_iℓ_j)ℓ_k]ℓ_j
\]

\[
(D9.3) \quad (ℓ_iℓ_j)(ℓ_kℓ_i) = ℓ_i[(ℓ_jℓ_k)ℓ_i]
\]

It can be shown that a Moufang loop satisfies the alternative property (AP) and therefore also the power-associative property (PAP). For if we let \(ℓ_k = 1 \) (identity element) in (D9.1), (D9.2) and (D9.3), then we obtain the identities:

\[
ℓ_i(ℓ_iℓ_i) = ℓ_iℓ_i, \quad ℓ_i(ℓ_jℓ_j) = ℓ_jℓ_i, \quad \text{and} \quad ℓ_i(ℓ_jℓ_j) = ℓ_iℓ_j. \]

These three identities satisfy the requirements of Definition 8 for alternativity.

Another interesting structure is the Bol loop [13] which is closely related to the Moufang loop. In a sense, it is a generalization of the Moufang loop.
Definition 10. Let $(\mathcal{L}, \ast)$ be a NAFIL and let $\ell_i, \ell_j, \ell_k \in \mathcal{L}$. If
\[ [\ell_i, \ell_j] \ast \ell_k = \ell_i [\ell_j, \ell_k] \]
then $(\mathcal{L}, \ast)$ is called a right Bol loop (RBol). If it satisfies the identity
\[ [\ell_i, (\ell_j \ast \ell_k)] \ast \ell_k = \ell_i [(\ell_j \ast \ell_k), \ell_k] \]
then $(\mathcal{L}, \ast)$ is called a left Bol loop (LBol).

It is known that there is a duality between the right and left Bol loops. A right Bol loop is RIP and RAP and a left Bol loop is LIP and LAP. Given the Cayley table of a right Bol loop, its transpose [9] is the Cayley table of a left Bol loop. Thus the distinction between them is not fundamental. The smallest right (left) Bol loop is of order $n = 8$; and there are exactly 6 right (left) Bol loops of this order. Because of this duality, we shall simply call such loops as Bol loops.

There is an interesting variety of loops, called extra loops, satisfying the following equivalent identities
\[
(x(yz))y = (xy)(zy) \quad \text{and} \quad (yz)(yx) = y((zy)x)
\]
\[
((xy)z)x = x(yzx)
\]
Thus, if a loop satisfies any of these identities, it will also satisfy the others. These are weak associative laws of size 4 in 3 distinct variables. If we set $x = 1$ in the first two equations and $z = 1$ in the in the third, we find that $(yz)y = y(zy)$ and $(xy)x = x(yx)$, respectively. Hence, extra loops are also flexible. Note that the first two equations are the mirrors [11] of each other, that is, one can be obtained from the other by writing it backwards. A loop that satisfies any of the above identities is said to have the extra loop property (ELP).

Because the left and right powers of a NAFIL element are not always equal, the familiar law of exponents, $\ell^a \ast \ell^b = \ell^{a+b}$, is not always satisfied. However, there are certain NAFIL loops in which this law holds.

Definition 11. Let $(\mathcal{L}, \ast)$ be a NAFIL and let $\ell \in \mathcal{L}$. If $\ell^a \ast \ell^b = \ell^{a+b}$, where $a$ and $b$ are any two positive integers, then $(\mathcal{L}, \ast)$ is called power-associative and is said to have the Power Associative Property (PAP).

Although the PAP identity does not have the explicit form of a weak associative law, it is a consequence of the associative axiom A6. The defining identities are infinite in number; they are special cases of A6 involving one variable (e.g. $x^2x = xx^2, (x^2x)x = x^2x^2 = x(xx^2),...$) [9]. This indicates that in a power-associative NAFIL, A6 holds in a limited way.

4.4. Other Loop Properties. There are other loop properties defined by identities that do not have the form of the weak associative properties. Among these are the following [9]: Weak Inverse Property (WIP), Automorphic Inverse Property (AIP), Semiautomorphic Inverse Property (SAIP), Anti-automorphic Inverse Property (AAIP), and the Crossed Inverse Property (CIP). These are defined as follows:

Definition 12. A NAFIL $(\mathcal{L}, \ast)$ is said to have: (a) the Crossed Inverse Property (CIP) if it satisfies the identity
\[
(\ell q)\ell^{-1} = q
\]
for all \(\ell, q \in L\), (b) the **Weak Inverse Property (WIP)** if it satisfies the identity
\[
\ell(q\ell)^{-1} = q^{-1}
\]
for all \(\ell, q \in L\), and (c) the **Automorphic Inverse Property (AIP)** if it satisfies
the identity
\[
(\ell q)^{-1} = \ell^{-1}q^{-1}
\]
for all \(\ell, q \in L\).

There are two other known inverse properties that are related to AIP:

- **SAIP (semiautomorphic inverse property):** There are two forms
  
  RSAIP: \(((\ell q)\ell)^{-1} = (\ell^{-1}q^{-1})\ell^{-1}\) and
  
  LSAIP: \((\ell(q\ell))^{-1} = \ell^{-1}(q^{-1}\ell^{-1})\).

  A loop that satisfies both RSAIP and LSAIP is simply called SAIP.

- **AAIP (antiautomorphic inverse property):** \((\ell q)^{-1} = q^{-1}\ell^{-1}\).

The above definitions of CIP, WIP, and AIP hold for certain invertible loops.
An example of a NAFIL that is CIP, WIP, and AIP is the non-abelian loop \((L_5, \ast)\) of order 5 whose Cayley table is shown in Figure 1. This NAFIL is also a PAP and an FL loop. Any NAFIL in which every element is self-inverse (unipotent) is trivially power-associative. In such a loop, every element generates a subgroup of order 2 and is also called monassociative. Thus, all loops belonging to the EVEN family of NAFIL loops are monassociative.

Examples of loops that are LSAIP and RSAIP are as follows:

\[
\begin{array}{cccccccccc}
\circ & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 1 \\
2 & 3 & 4 & 1 & 6 & 7 & 8 & 9 & 5 & 6 \\
3 & 4 & 1 & 2 & 7 & 8 & 9 & 5 & 6 & 7 \\
4 & 1 & 2 & 3 & 8 & 9 & 5 & 6 & 7 & 4 \\
5 & 6 & 7 & 9 & 1 & 2 & 3 & 4 & 8 & 5 \\
6 & 7 & 8 & 5 & 9 & 1 & 2 & 3 & 4 & 6 \\
7 & 8 & 9 & 6 & 4 & 5 & 1 & 3 & 3 & 7 \\
8 & 9 & 5 & 7 & 3 & 4 & 6 & 1 & 2 & 8 \\
9 & 5 & 6 & 8 & 2 & 3 & 4 & 7 & 1 & 9
\end{array}
\quad
\begin{array}{cccccccccc}
\circ' & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 1 \\
2 & 3 & 4 & 1 & 6 & 7 & 8 & 9 & 5 & 6 \\
3 & 4 & 1 & 2 & 7 & 8 & 9 & 5 & 6 & 7 \\
4 & 1 & 2 & 3 & 9 & 5 & 6 & 7 & 8 & 4 \\
5 & 6 & 7 & 9 & 1 & 2 & 3 & 4 & 8 & 5 \\
6 & 7 & 8 & 5 & 9 & 1 & 2 & 3 & 4 & 6 \\
7 & 8 & 9 & 6 & 4 & 5 & 1 & 3 & 3 & 7 \\
8 & 9 & 5 & 7 & 3 & 4 & 6 & 1 & 2 & 8 \\
9 & 5 & 6 & 8 & 2 & 3 & 4 & 7 & 1 & 9
\end{array}
\]

**LSAIP NAFIL of order 9**

\[
\begin{array}{cccccccccc}
\circ & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 1 \\
2 & 3 & 4 & 1 & 6 & 7 & 8 & 9 & 5 & 6 \\
3 & 4 & 1 & 2 & 7 & 8 & 9 & 5 & 6 & 7 \\
4 & 1 & 2 & 3 & 9 & 5 & 6 & 7 & 8 & 4 \\
5 & 6 & 7 & 9 & 1 & 2 & 3 & 4 & 8 & 5 \\
6 & 7 & 8 & 5 & 9 & 1 & 2 & 3 & 4 & 6 \\
7 & 8 & 9 & 6 & 4 & 5 & 1 & 3 & 3 & 7 \\
8 & 9 & 5 & 7 & 3 & 4 & 6 & 1 & 2 & 8 \\
9 & 5 & 6 & 8 & 2 & 3 & 4 & 7 & 1 & 9
\end{array}
\quad
\begin{array}{cccccccccc}
\circ' & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 1 \\
2 & 3 & 4 & 1 & 6 & 7 & 8 & 9 & 5 & 6 \\
3 & 4 & 1 & 2 & 7 & 8 & 9 & 5 & 6 & 7 \\
4 & 1 & 2 & 3 & 9 & 5 & 6 & 7 & 8 & 4 \\
5 & 6 & 7 & 9 & 1 & 2 & 3 & 4 & 8 & 5 \\
6 & 7 & 8 & 5 & 9 & 1 & 2 & 3 & 4 & 6 \\
7 & 8 & 9 & 6 & 4 & 5 & 1 & 3 & 3 & 7 \\
8 & 9 & 5 & 7 & 3 & 4 & 6 & 1 & 2 & 8 \\
9 & 5 & 6 & 8 & 2 & 3 & 4 & 7 & 1 & 9
\end{array}
\]

**RSAIP NAFIL of order 9**

These loops are the transpose of each other. Both of them are PAP and have the same subsystems: \(\{1,2,3,4\}, \{1,3\}, \{1,5\}, \{1,6\}, \{1,7\}, \{1,8\}, \{1,9\}\).

Another type of loop is called **totally symmetric (TS)** (also called a Steiner loop) which satisfies the identities
\[
\ell \ell x = \ell y \ell \ell x \quad \text{and} \quad \ell x (\ell x \ell y) = \ell y
\]
If such a loop is a NAFIL, we shall call it a **TS NAFIL**. Here again, the equation \(\ell x (\ell x \ell y) = \ell y\) has the form \(\ell x (\ell x \ell y) = (\ell x \ell x) \ell y = \ell y\), where \(\ell x \ell x = 1\). It is clear from this that a TS NAFIL is an abelian IP loop such that every element is self-inverse.
4.4.1. Summary of Loop Properties. Other weak associative laws have also been found useful in the study of loops and quasigroups. However, many of these do not apply to NAFIL loops nor do they contribute much to the understanding of these structures.

A concise summary of various loop properties known to be satisfied by NAFILs is given in the table below. These properties are defined by universally quantified equations called identities or identical relations.

| Special Loop Property | Acronym | Defining Equation |
|-----------------------|---------|-------------------|
| Left Inverse Property | LIP     | $ℓ^{-1}(ℓq) = (ℓ^{-1}ℓ)q = q$ |
| Right Inverse Property| RIP     | $(qℓ)^{-1} = q(ℓ^{-1}) = q$ |
| Inverse Property      | IP      | LIP and RIP       |
| Left Alternative Property | LAP   | $ℓ_i(ℓ_iℓ_k) = (ℓ_iℓ_i)ℓ_k$ |
| Right Alternative Property | RAP   | $(ℓ_iℓ_k)ℓ_k = ℓ_i(ℓ_iℓ_k)$ |
| Alternative Property  | AP      | LAP and RAP       |
| Flexible Law          | FL      | $ℓ_i(ℓ_iℓ_i) = (ℓ_iℓ_k)ℓ_i$ |
| Moufang Property      | MP      | $ℓ_i(ℓ_iℓ_jℓ_k) = ([ℓ_iℓ_j]ℓ_iℓ_k$ |
| Left Bol              | LBol    | $ℓ_i(ℓ_jℓ_k)ℓ_k = ℓ_i[ℓ_j(ℓ_iℓ_k)]$ |
| Right Bol             | RBol    | $([ℓ_iℓ_j]ℓ_i)ℓ_j = ℓ_i([ℓ_iℓ_j]ℓ_k)$ |
| Extra Loop Property   | ELP     | $(ℓ_iℓ_jℓ_k)ℓ_j = (ℓ_iℓ_j)[ℓ_k(ℓ_iℓ_j)]$ |
| C Loop Property       | CP      | $ℓ_i[ℓ_j(ℓ_iℓ_k)]ℓ_k = ([ℓ_jℓ_ℓ_i]ℓ_i)ℓ_k$ |
| RIF Loop Property     | RIFP    | $ℓ_i[ℓ_j(ℓ_iℓ_k)]ℓ_k = ([ℓ_jℓ_iℓ_k]ℓ_i)[ℓ_k]$ |
| A sub m Loop Property | A_m P   | $ℓ_i[ℓ_j(ℓ_iℓ_k)]ℓ_k = ([ℓ_j(ℓ_iℓ_k)]ℓ_i)ℓ_k$ |
| Power Associative Property | PAP | $ℓ^a * ℓ^b = ℓ^{a+b}$ |
| Totally Symmetric     | TS      | $ℓ_i ℓ_j = ℓ_j ℓ_i$ and $ℓ_i(ℓ_i ℓ_j) = ℓ_j$ |
| Weak Inverse Property | WIP     | $ℓ(qℓ)^{-1} = q^{-1}$ |
| Automorphic Inverse Property | AIP | $(qℓ)^{-1} = ℓ^{-1}q^{-1}$ |
| Anti-Automorphic Inverse Property | AAIP | $(ℓq)^{-1} = q^{-1}ℓ^{-1}$ |
| Left Semi-Automorphic Inverse Property | LSAIP | $((ℓq)ℓ)^{-1} = ℓ^{-1}q^{-1}ℓ^{-1}$ |
| Right Semi-Automorphic Inverse Property | RSAIP | $(ℓ(ℓq))^{-1} = ℓ^{-1}(q^{-1}ℓ^{-1})$ |
| Semi-Automorphic Inverse Property | SAIP | LSAIP and RSAIP |
| Crossed Inverse Property | CIP    | $(ℓq)^{-1} = ℓ^{-1}q^{-1}$ |
| Left Conjugacy Closed  | LCC     | $ℓ_i(ℓ_iℓ_k) = [ℓ_i(ℓ_iℓ_k)](ℓ_iℓ_k)$ |
| Right Conjugacy Closed | RCC     | $(ℓ_iℓ_jℓ_i)ℓ_k = (ℓ_iℓ_k)[(ℓ_iℓ_j]ℓ_k)$ |
| Conjugacy Closed Loop Property | CCP | LCC and RCC |

Table 3. List of known special loop properties used to test NAFIL loops of orders $n = 5, 6, 7$.

5. Construction and Analysis of NAFIL Loops

So far, we have defined the NAFIL and presented some of its basic properties. Our next important question is: How do we construct such a loop? Finally, how do we analyze the constructed loop to determine its properties?

To answer these questions we must first specify the kind of loop we propose to construct. Usually, a groupoid (loop, group, quasigroup) is constructed by means of a set of independent generators and a set of relations. However, this procedure is often difficult to carry out. We mentioned earlier that a finite groupoid is completely
defined by its structure matrix or Cayley table. We know that the structure matrix of a quasigroup is a Latin square. Since a loop is a quasigroup with an identity element, then its Cayley table is a Latin Square in normal (or reduced) form. Thus, the construction of a loop (like the NAFIL or group) is equivalent to the construction of a Latin square. This method of construction is highly developed and it can be done manually or by means of computer programs [5].

Because there are many kinds of NAFILs, there is no general method of constructing them. A survey of the literature has shown that there are numerous ad hoc methods of construction [9]. We have, however, developed a simple and efficient method of construction and analysis of finite loops called the Structure Matrix Method. This method can be carried out manually as well as by a computer program called FINITAS.[7]

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