The fragmentation of protostellar discs: the Hill criterion for spiral arms

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ABSTRACT

We present a new framework to explain the link between cooling and fragmentation in gravitationally unstable protostellar discs. This framework consists of a simple model for the formation of spiral arms, as well as a criterion, based on the Hill radius, to determine if a spiral arm will fragment. This detailed model of fragmentation is based on the results of numerical simulations of gravitationally unstable protostellar discs, including those found in the literature, as well as our new suite of three-dimensional radiation hydrodynamics simulations of an irradiated, optically thick protostellar disc surrounding an A star. Our set of simulations probes the transition to fragmentation through a scaling of the physical opacity. This model allows us to directly calculate the critical cooling time of Gammie, with results that are consistent with those found from numerical experiment. We demonstrate how this model can be used to predict fragmentation in irradiated protostellar discs. These numerical simulations, as well as the model that they motivate, provide strong support for the hypothesis that gravitational instability is responsible for creating systems with giant planets on wide orbits.

Key words: hydrodynamics – radiative transfer – methods: numerical – protoplanetary discs – planetary systems.

1 INTRODUCTION

The fragmentation of protostellar discs through gravitational instability (GI) is a possible mechanism for the formation of gas-giant planets and brown dwarfs. For a disc to be prone to fragmentation, there are generally thought to be two criteria that must be satisfied. The first criterion is that a disc must be gravitationally unstable. This can be characterized by the Toomre Q parameter (Toomre 1964), which is the result of a linear stability analysis for a differentially rotating thin disc:

\[ Q = \frac{c_s \kappa_e}{\pi G \Sigma} \]

In the above, \( c_s \) is the sound speed of the gas, \( \kappa_e \) is the epicyclic frequency, \( \kappa_e = \Omega \) for Keplerian rotation, \( G \) is the gravitational constant, and \( \Sigma \) is the surface density. For low values of \( Q \sim 1 \), such as would be found in a massive, cold disc, a disc will be gravitationally unstable.

The second criterion for fragmentation is that a disc, in addition to being gravitationally unstable, must also cool quickly. Gammie (2001), using two-dimensional (2D) shearing-box simulations, examined the stability of a local patch of a protostellar disc with a simplified cooling prescription:

\[ t_{\text{cool}} \Omega = \beta, \]

where \( t_{\text{cool}} \) is the cooling time and \( \beta \) is a constant. The cooling time here is defined through the evolution of the specific internal gas energy, \( u \):

\[ \frac{D u}{D t} = -\frac{u}{t_{\text{cool}}}, \]

where \( D/Dt = (\partial/\partial t) + v \cdot \nabla \) is the comoving derivative.

By performing simulations with different values for \( \beta \), the author found the cooling criterion for fragmentation to be

\[ t_{\text{cool}} \Omega \leq \beta_{\text{crit}} = 3; \]

this particular value of \( \beta_{\text{crit}} \) is valid only for the specific 2D power-law equation of state used by the author. Subsequent work, using global 3D smoothed particle hydrodynamics (SPH) simulations (Rice, Lodato & Armitage 2005), as well as global 3D grid-based simulations (Mejía et al. 2005), has also found fragmentation for fixed cooling times faster than a critical rate.

Gammie (2001), as well as Rice et al. (2005), outlined a physical argument for the existence of a critical cooling time. If GI can be well characterized by an \( \alpha \)-viscosity model (Shakura & Sunyaev 1973; see Lodato & Rice 2004, 2005 for the applicability of this), then a steady state can exist if the viscous heating by GI is balanced by the prescribed cooling. If this balance is achieved, then the required viscosity of the disc is determined by

\[ \alpha = \frac{1}{9/4 \gamma_{2D} (\gamma_{2D} - 1) \beta}. \]

Here, \( \gamma_{2D} \) is the 2D adiabatic index. As discussed by Gammie (2001), this can be related to the 3D adiabatic index, \( \gamma \), for the
non-self-gravitating case, where \( \gamma_{2D} = (3\gamma - 1)(\gamma + 1) \), as well as for the strongly self-gravitating case, where \( \gamma_{2D} = 3 - 2\gamma / \gamma \). If GI has a maximum \( \alpha \) that it can attain, then it also has a maximum heating rate (Rice et al. 2005). If the prescribed \( \beta \) cools the disc faster than this maximum heating rate, then no balance between heating and cooling can be achieved, and the disc fragments.

The \( \beta \)-prescription of cooling, however, is a simplified model; more generally, \( \beta \) will evolve with \( \alpha \) in a given disc. In a realistic disc, heating and cooling are linked to the physical state of the disc. In this work, we consider fragmentation in this more realistic case, whereas the \( \beta \)-prescription of cooling does not allow Gammie (2001) to have done so.

Furthermore, the applicability of a single value of \( \beta_{\text{crit}} \) (or \( \alpha_{\text{max}} \)) is in some doubt. Rice et al. (2005) found that the value of \( \beta_{\text{crit}} \) depends on the state of the gas (the value of \( \gamma \)), but that this dependence is consistent with a unique value of \( \alpha_{\text{max}} \sim 0.06 \). However, other simulation results in the literature have found a non-unique value for the critical cooling time, even for a fixed state. Clarke, Harper-Clark & Lodato (2007), using a time-dependent \( \beta(t) \), found that the thermal history of the gas is important in determining \( \beta_{\text{crit}} \), and can result in its reduction by a factor of 2. Meru & Bate (2011) found \( \beta_{\text{crit}} \) to be a function of the distance from the central star, the local surface density and the stellar mass.

Complicating the applicability of a unique \( \beta_{\text{crit}} \) are questions of numerical convergence. Meru & Bate (2010) recently found that their 3D SPH simulations had not converged numerically; they found larger values of \( \beta_{\text{crit}} \) with increasing resolution. A similar increase in the critical cooling time with enhanced resolution was found by Paardekooper, Baruteau & Meru (2011) using 2D grid-based simulations. However, these authors were able to achieve numerical convergence if they slowly decreased the cooling rate with time, which allowed the entire disc to be in a state of gravitoturbulence while gradually approaching the fragmentation boundary. Lodato & Clarke (2011) have proposed that the non-convergence seen by Meru & Bate (2010) may be the result of heating via artificial viscosity, or smoothing of density enhancements in simulations with insufficient resolution, while Rice, Forgan & Armitage (2012) have proposed that the non-convergence is the result of using a particle’s specific internal energy in the cooling rate, rather than using a smoothed version of this quantity.

It is also worth noting that when considering the stability of protostellar discs using the cooling criterion (e.g. Rafikov 2005), it is important to consider a disc as it would be in the non-linear state of GI, rather than as it would be in the axisymmetric state. The radiative cooling time is a function of temperature, opacity and surface density, all of which are different in spiral arms, where fragmentation takes place, as compared to an axisymmetric disc. For example, Johnson & Gammie (2003) found that cooling times in the non-linear phase of GI could differ substantially from the cooling time of the initial axisymmetric initial condition.

In this work, we present a set of 3D radiation hydrodynamic simulations of a massive, optically thick, protostellar disc, unstable near 100 au, around an A star. Rather than using a \( \beta \)-prescription for the cooling, these simulations include realistic heating and cooling of the disc, including cooling from the disc photosphere and irradiation from the central star. We do, however, vary the cooling rate in this set of simulations by scaling the dust opacity table by a constant factor. By reducing the opacity (which reduces the cooling time for an optically thick disc) over this set, we observe a transition from discs that are stable against fragmentation to discs that do fragment; this is consistent with the cooling criterion work of Gammie (2001).

We have used results from Cossins, Lodato & Clarke (2009), and from this set of simulations to develop a simple, yet detailed, physical model for the fragmentation of a gravitationally unstable protostellar disc. In this model, spiral arms develop in an unstable disc on a characteristic scale related to the disc scale height (Cossins et al. 2009). The heating rate per unit mass of the disc from GI is proportional to the square of the amplitude of the surface density variations in the disc (Cossins et al. 2009); as spiral arms become more condensed, the heating rate per unit mass is increased. The cooling rate per unit mass of the disc from photospheric cooling is inversely proportional to the square of the surface density; as spiral arms become more condensed, the cooling rate per unit mass in the arms decreases. There is therefore a natural scale for the thickness of a spiral arm in a gravitationally unstable disc. This scale is set by a balance between heating from spiral waves and radiative cooling. It is worth noting that for faster cooling rates (shorter cooling times), this thickness will be decreased.

A second scale of interest in this model is the Hill radius, which, for an object of mass \( M \), is

\[
H_{\text{Hill}} = \left( \frac{GM}{3\Omega^2} \right)^{1/3}.
\]

The importance of the Hill radius can be understood within the context of planet formation in a disc of planetesimals around a star. If a protoplanet embryo has formed in this disc, then it is of interest to determine the radius over which it may further accrete planetesimals. The Hill radius sets this embryo’s sphere of influence: material within the Hill radius is bound to the embryo and will be accreted. In essence, material within the Hill radius of an object is dominated by that object’s gravity, which is equivalent to the role of the local shear in the Toomre criterion. For the purpose of this discussion, we define the Hill thickness as twice the Hill radius.

In this framework, we can extend the cooling criterion of Gammie (2001) with the Hill criterion for spiral arms. In a gravitationally unstable disc, the natural thickness of the spiral arms is set by a balance between heating and cooling. Fragmentation occurs in this disc if there is a section of an arm whose natural thickness is smaller than that section’s Hill thickness. Essentially, if a section of a spiral arm lies within its own Hill thickness, then shear will be unable to prevent the collapse of the arm, and fragmentation can take place.

In a gaseous disc, pressure can prevent fragmentation from taking place. By considering the Hill radius, we have not addressed the role that pressure plays in determining fragmentation and how it may modify the critical thickness of spiral arms necessary for fragmentation to take place. The correct determination of this scale requires the solution to a stability analysis of a spiral arm in a differentially rotating system. Since we do not have such a solution, we have chosen to consider the Hill thickness. The analysis of our simulations, however, does indicate that the Hill thickness is the correct scale to examine. The Hill criterion for fragmentation is consequently an empirical criterion.

Our model is consistent with the cooling criterion: as the cooling time decreases, spiral arms become thinner and more overdense, becoming more likely to reside within their own Hill thickness, and consequently more likely to fragment. With the Hill criterion, however, we have developed a more detailed, and more complete, physical picture of fragmentation. This picture can be applied to the general case of a disc with physical heating and cooling, or the more specific case of a disc with \( \beta \)-prescription cooling. In fact, it
offers a means to calculate what the critical cooling time is for a given region of a given disc.

There have been previous attempts to characterize the fragmentation of spiral arms using a physical model. Durisen, Hartquist & Pickett (2008) examined the stability of a spiral arm, considered as an isothermal sheet, and found that fragmentation was expected for low values of $Q$; in addition, they argued that strong compression in a spiral shock acts to suppress fragmentation. This implies that fragmentation should occur near corotation.

Boley et al. (2010) proposed that fragmentation takes place when the Toomre length (the most unstable radial wavelength) in a spiral arm lies inside the region around corotation found by Durisen et al. (2008). The physical models of Durisen et al. (2008) and Boley et al. (2010) are strictly only applicable to isothermal discs; however, Boley et al. (2010) did find agreement between the fragment masses predicted in the model and those found in a simulation with radiative cooling. In contrast, our model of fragmentation presented in this work is derived for discs with cooling.

The structure of the paper is as follows. In Section 2, we overview our numerical methods as well as our set of simulations of gravitationally unstable, irradiated protostellar discs. In Section 3, we give a detailed picture of our model of protostellar disc fragmentation and the Hill criterion. In addition, we demonstrate the model’s consistency with the simulations of Section 2. In Section 4, we show that the Hill criterion is quantitatively consistent with the cooling criterion and discuss the predictive qualities of the model. Finally, in Section 5 we give our conclusions.

2 NUMERICAL SIMULATIONS OF GRAVITATIONALLY UNSTABLE IRRADIATED DISCS

2.1 Numerical methods

Our simulations were performed with the TreeSPH code GASOLINE (Wadsley, Stadel & Quinn 2004), with the addition of radiative transfer in the flux-limited diffusion approximation (FLD; Rogers & Wadsley 2011). As described by the authors, FLD is able to model the transfer of energy only in regions in which SPH particles reside. Because of limited resolution, any SPH representation of a protostellar disc naturally has two edges, representing the upper and lower atmospheres. Radiative cooling from the disc atmospheres is modelled by means of a photosphere boundary condition: the SPH particles on the ‘edge’ of the disc (the edge particles) are found, robust surface areas (the area of the photosphere for which an edge particle is responsible) are calculated using a 2D SPH estimate and a plane-parallel cooling term is added to the radiative energy equation for the edge particles. The radiative hydrodynamics has been tested on a number of standard problems, including the relaxation test of Boley et al. (2007b), which is particularly suited to protostellar disc simulations.

The conditions in the outer regions of discs (roughly 100 au and beyond) are expected to be favourable to gravitational fragmentation, since the cooling criterion is likely satisfied there (Rafikov 2007). As pointed out by Kratter & Murray-Clay (2011), the heating of the outer regions is expected to be dominated by the irradiation of the disc’s surface by the central star, rather than by viscous heating. Since our simulations focus on fragmentation at these large radii, it is fundamentally important to account for this heating via irradiation.

The photosphere boundary condition of Rogers & Wadsley (2011) offers a straightforward means by which this can be done. In addition to the cooling term in the specific radiation energy equation for each edge particle, we have added a heating term of

$$\frac{D\xi}{Dr}_{irrad} = \frac{A_i}{m_i} \sigma (T_{irrad})^4,$$

where $\xi$ is the specific radiation energy, $A_i$ is the surface area of the edge particle, $m_i$ is the particle mass, $\sigma$ is the Stefan–Boltzmann constant, and $T_{irrad}$ is the temperature of the stellar irradiation.

Kratter, Murray-Clay & Youdin (2010b) used the passive flared disc model of Chiang & Goldreich (1997), along with a stellar evolution model, to determine the equilibrium temperature distribution for a disc surrounding a 1.35-M$_\odot$ star, which they found to be

$$T = 40 \text{ K} \left(\frac{R}{70 \text{ au}}\right)^{-3/7}.$$

Since we are not able to treat the superheated dust layer of optical depth $\tau < 1$ in our simulations, it is appropriate to use this equilibrium temperature distribution as $T_{irrad}$ in the irradiation heating term, equation (7). In addition, we implement a floor of $T_{irrad} = 20$ K to take into account the background radiation field.

There have been previous multidimensional simulations of protostellar discs that have made efforts to account for the effects of stellar and/or envelope irradiation. These include 3D simulations (Cai et al. 2008; Stamatellos & Whitworth 2008; Boley 2009), axisymmetric simulations (Zhu et al. 2009) and thin-disc simulations (Vorobyov & Basu 2010).

2.2 Initial conditions and input parameters

The initial axisymmetric model of $5 \times 10^5$ SPH particles was created in a manner similar to that of Shen et al. (2010) – see Fig. 1 for the disc properties. The surface density profile has the form $\Sigma \propto r^{-1}$ in the region of 20–70 au (there is a smooth increase of $\Sigma$ from 10 to 20 au). There is a smooth functional form of $\Sigma \propto r^{-1} \exp[-4 \log(R)[0.5 \log(R) - \log(R_m)]/\log(R/R_m)]$, with $R_m = 70$ au and $R_o = 160$ au, from 70 to 160 au, after which there is a steep drop off of $\Sigma \propto r^{-1.5}$. There is roughly 0.61 M$_\odot$ within 200 au.

![Figure 1](https://academic.oup.com/mnras/article-abstract/423/2/1896/975843)

**Figure 1.** The physical quantities of the initial disc profile. The midplane number density (in units of $10^{13}$ cm$^{-3}$) is given by the solid, black line; the midplane temperature (in units of 10 K) is given by the red, dashed line; the optical depth (in units of 10) is given by the purple long-dashed line; and $Q$ is given by the blue, dot-dashed line. The horizontal dotted line is a reference for $Q = 1$. 

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This particular surface density distribution is motivated by the initial temperature profile, which is given by the equilibrium temperature in equation (8). The combination of temperature and surface density results in a broad region of the disc having an initial \( Q \) of roughly unity.

Using the initial surface density and temperature profiles, the vertical structure satisfying hydrostatic equilibrium was calculated iteratively, taking into account both the gravity from the central star and the self-gravity of the disc. We have taken care to design this initial condition to ensure that it is ‘quiet’, even though it is gravitationally unstable; that is, to ensure that there is a smooth evolution to spiral structure, without any strong radial redistribution of mass on short time-scales (roughly the orbital period) due to transients from the initial set-up. In this way, the azimuthally averaged properties of the initial condition are reflective of the evolved disc.

The central star is modelled as a 1.35-\( M_\odot \) sink particle with a radius of 10 au. We use a mean molecular weight of 2.3 and realistic Rosseland mean opacities (D'Alessio, Calvet & Hartmann 1997). Although the code is capable of using a consistent treatment of the internal energy of molecular hydrogen that takes into account translational, rotational and vibrational modes (Boley et al. 2007a), to simplify the analysis of our simulations, we use a fixed adiabatic index of \( \gamma = 7/5 \). In the outer regions of protostellar discs, of particular interest here, temperatures are typically below 100 K. Therefore, a choice of \( \gamma = 5/3 \) would be more physical; however, for the purposes of this study, which aims to understand the process of fragmentation in detail, the distinction is not important. The scale height is resolved by at least three smoothing lengths outside of 25 au, and the Jeans length is resolved until shortly after fragmentation takes place. We define one ORP, one outer rotation period, to be the Keplerian period at our fiducial radius of 100 au, roughly 863 years.

2.3 Simulations

We present a set of five simulations, each of which uses the initial conditions described above, the only difference being the opacity used. Simulation (A, B, C, D and E) has an opacity table that is scaled by a constant value of (1/10, 1/3, 1, 3 and 10). Thus, simulation C has an estimated physical opacity for solar metallicity, while simulations A and B have reduced opacities, and simulations D and E have increased opacities. Physical changes in opacity could be the result of grain growth (Birnstiel, Dullemond & Brauer 2010), grain evolution via the passage of spiral arms (Podolak, Mayer & Quinn 2011), or formation in an environment with a non-solar metallicity. Our goal, however, is not to reproduce different physical environments, but rather to explore the necessary conditions for gravitational fragmentation. In this context, a simple scaling of the opacity table is both sufficient and desirable.

All five simulations evolved in a similar fashion over the first 2.5 ORPs. High mode-number spiral structure developed slowly from SPH Poisson noise in each of the discs over this time until settling down to a transitioning state of two or three spiral arms. The transition from axisymmetric initial condition to spiral structure is observed to be smooth, with no strong transients.

The final states of the simulations are shown in Fig. 2. Simulations C, D and E have been evolved for roughly 8.5 ORPs without fragmentation having taken place (although strong spiral overdensities may persist), while simulation A has fragmented with two objects forming, and simulation B has fragmented with one object forming. This set of simulations, therefore, demonstrates a transition...
from non-fragmentation to fragmentation, as a function of the opacity scaling.

For a patch of an optically thick disc, the cooling time is approximately

$$t_{\text{cool}} = \frac{1}{4 \gamma - 1} \frac{c_s^2 \kappa}{\Sigma^2}, \quad (9)$$

where $\kappa$ is the opacity and $T$ is the midplane temperature (Rafikov 2007). As can be observed in Fig. 1, our initial condition is optically thick, with $\tau > 1$, out to at least 150 au, even when the opacity is scaled by 1/10, as it is in simulation A. This extends over the entire gravitationally unstable region (where $Q \lesssim 1.7$; Durisen et al. 2007). Hence, the cooling time is directly proportional to the opacity, and our set of simulations offers a means of exploring the fragmentation boundary as a function of cooling time in a manner similar to the simulations of Gammie (2001). The difference is that our simulations use realistic radiative cooling (even though the opacities may be scaled), rather than $\beta$-prescription cooling.

The cooling criterion suggests that reducing the cooling time (by reducing the opacity) will eventually lead to fragmentation. Thus, that simulations A and B fragment is consistent with this picture. However, it is possible to use these simulations to better understand why exactly fragmentation takes place.

Fig. 3 shows the five simulations at roughly the same time, shortly before fragmentation took place in simulations A and B. The difference in the structure of the five discs at this time offers evidence of a detailed description for why simulations A and B fragment, while simulations C–E do not. As can be observed, as the cooling time decreases (as the opacity decreases), spiral arms in a disc become thinner and more overdense and this makes fragmentation more likely to occur.

3 FRAGMENTATION MODEL AND THE HILL CRITERION

We present a model of spiral arm fragmentation in gravitationally unstable discs, based on the observation from the simulations of the previous section that reduced cooling times lead to thinner arms, which are more likely to fragment. This model can be broken into two components: the first is a model for the (roughly) steady-state spiral structure in an unstable disc; while the second is a criterion for the fragmentation of these spirals. Many of the details of our model are empirical in nature: we have used results from the simulations of Cossins et al. (2009), as well as our own set of simulations, in determining some of the important parameters.

3.1 Spiral structure

We begin with a model for the spiral structure that results in a gravitationally unstable disc. We consider a patch of an initially axisymmetric disc that will develop spiral structure, such as our initial condition for the simulations of the preceding section. This patch is located at some distance, $R$, away from the central star and is of a radial extent $l_0$, with a characteristic surface density $\Sigma_0$. GI acts on the scale of $l_0$ to collapse mass radially, resulting in the formation of a spiral arm of thickness $l_1$ and characteristic surface density $\Sigma_1$. This process is demonstrated in Fig. 4.

Figure 3. The simulated discs before fragmentation: surface density plots of simulations E, D, C, B and A are shown from top to bottom. As the opacity scaling is reduced, the spiral arms become thinner and more overdense. The discs are shown at a time of 2.5 ORPs.
What is the appropriate scale for \( l_0 \)? Cossins et al. (2009) performed a number of simulations of \( Q \sim 1 \) discs and found that, from a radial Fourier transform of these discs, the dominant radial wavenumber was typically

\[
k \approx \frac{1}{H} = \frac{\pi G \Sigma}{c_s^2}.
\]

Therefore, we expect the scale of spiral arm formation to be \( l_0 = 2\pi H \). We have tested that this is consistent with our own simulations. Fig. 5 shows the results of a radial Fourier transform of simulation B at a time of 2.5 ORPs.

How many spiral arms are likely to form in our disc? Numerical studies (Lodato & Rice 2004, 2005) have shown that as the disc-to-star mass ratio increases, \( Q \sim 1 \) discs show fewer spiral arms. Our simulations are of quite massive discs, with a disc-to-star mass ratio of \( M_d/M_\star = 0.45 \). This high disc mass is necessary for the disc to have \( Q \sim 1 \) near 100 au for our realistic irradiation temperature, and results in a typical arm number of \( m = 2 \) or 3.

The number of arms in a disc is likely the result of swing amplification, with significant amplification of a mode, \( m \), requiring the swing amplification parameter,

\[
X_m = \frac{\Omega^2 R}{2\pi G \Sigma m}
\]

to satisfy \( 1 < X_m < 3 \) (Binney & Tremaine 2008). From the above, we can see that \( X_m \) roughly scales with the disc-to-star mass ratio:

\[
X_m \propto M_\star/M_d.
\]

Thus, for low disc-to-star mass ratios, only high-order modes will satisfy \( 1 < X_m < 3 \), while for high disc-to-star mass ratios, only low-order modes will.

What is the steady-state thickness, \( l_i \), of the newly formed spiral arm? We posit that this scale is the result of a balance between heating of the disc through the spiral waves and radiative cooling. Assuming that the spiral density wave deposits a fixed fraction, \( \epsilon \), of its energy into the disc per dynamical time, Cossins et al. (2009) showed that the heating rate per unit mass from spiral arms can be written as

\[
Q^+ = \epsilon \frac{c_s^2 M \dot{M} \Omega}{2} \left( \frac{\delta \Sigma}{\Sigma} \right)^2,
\]

where

\[
\dot{M} = \frac{m \Omega_\star}{k c_s} \quad \text{and} \quad \dot{M} = \frac{m}{k c_c}(\Omega_\star - \Omega).
\]

are the radial phase Mach number and the Doppler-shifted radial phase Mach number, \( \Omega_\star \) is the pattern speed, and \( (\delta \Sigma/\Sigma) \) is the spiral overdensity, the fractional increase in surface density in an arm compared to the average surface density at that radius.

As outlined by the authors, the pattern speed, and hence the Mach numbers, can be calculated from the dispersion relation for a finite-thickness disc:

\[
m^2 (\Omega_\star - \Omega)^2 = c_s^2 k^2 + \Omega^2 - \frac{2\pi G M |k|}{1 + |k|H},
\]

if the radial and azimuthal wavenumbers are known. From their simulations, the authors found a relatively constant value of \( \epsilon \approx 0.2 \) (see their fig. 15). For \( Q \sim 1 \) discs, which have dominant radial wavenumbers of \( k \approx 1/H \), solving the above dispersion relation for the ratio of the pattern speed to the rotation rate yields \( \Omega_\star/\Omega = 1 + 1/m \). We therefore expect the disc to be near corotation throughout the gravitationally unstable region of \( Q \sim 1 \), consistent with the results of Cossins et al. (2009).

In an irradiated disc, there is additional heating from the stellar irradiation, so that

\[
Q^+ = \epsilon \frac{c_s^2 M \dot{M} \Omega}{2} \left( \frac{\delta \Sigma}{\Sigma} \right)^2 + \frac{2\pi T_\text{rad}^4}{\Sigma}.
\]

In our simple analytic model, the spiral overdensity can be calculated by assuming that some fraction, \( f \), of the total mass per length within \( l_0 \), with a characteristic surface density \( \Sigma_0 \), is compressed into the spiral arm of thickness \( l_1 \), and characteristic surface density \( \Sigma_1 \):

\[
\left( \frac{\delta \Sigma}{\Sigma} \right) = \frac{\Sigma_1 - \Sigma_0}{\Sigma_0} = \left( \frac{l_0}{l_1} - 1 \right).
\]

In the above, and in the analytic calculations of Section 4, we have made use of a simplified top-hat surface density profile for a spiral arm, since we are unable to predict the true surface density profiles that will result from GI. The surface density \( \Sigma_1 \) can be thought of as the integrated average of the true surface density within the arm’s thickness, \( l_1 \); \( \Sigma_1 = \int_{l_1} \Sigma(R) dR / l_1 \).
The heating from equation (16) is balanced by radiative cooling, for which the cooling rate per unit mass, using equation (9), is

$$Q^- = u/\dot{\epsilon}_{\text{cool}} = \frac{4aT^4}{\gamma \kappa \Sigma^2},$$

(18)

where we have used $u = c_s^2/[\gamma(\gamma - 1)]$. Setting the above cooling rate equal to the heating rate of equation (16), and using the other information in this section, as well as the initial axisymmetric properties of our disc, leaves us with an equation with only one unknown: the thickness of the spiral arm, $l_1$, in our patch of interest (assuming that we know the proper midplane temperature for radiative cooling, $T$, in the spiral arm; this will be elaborated upon in Section 4).

### 3.2 Determining fragmentation

Once we know the steady-state thickness of the spiral arm, $l_1$, we can determine the mass of the section of the arm within that thickness and use equation (6) to calculate the Hill radius for this section of the arm. If the section of the arm has a thickness satisfying $l_1/(2H_{\text{Hill}}) < 1$, then the section lies within its own Hill thickness. In the absence of pressure forces, this means that the section is bound, as the tidal force from the central star (manifested as rotational shear) is less than the self-gravity of the section. Once the section is bound, fragmentation occurs. Conversely, if $l_1/(2H_{\text{Hill}}) > 1$, then the section of the arm is not bound and fragmentation does not occur.

In our simple analytic model, with a top-hat surface density profile as in equation (17), the section of the arm has a mass of $M = \Sigma l_1^2$ and equation (6) gives a Hill radius of

$$H_{\text{Hill}} = \left[\frac{G\Sigma l_1^2}{3\Omega^2}\right]^{1/3}.$$  

(19)

The Hill thickness tells one about the ability of shear to prevent the fragmentation of the arm and is therefore expected to be an important scale. However, in comparing the radial thickness of the arm to the Hill thickness we are ignoring the role of pressure, despite strong radial pressure gradients present across the arm. It is therefore reasonable to expect that the critical thickness for arm fragmentation may be modified from the Hill thickness. There are, however, no strong pressure gradients along the arm (the azimuthal direction); thus, fragmentation occurring in this direction should be determined by the Hill criterion.

Determining the correct scale for fragmentation requires a detailed calculation of the stability of a spiral arm accounting for differential rotation. In the absence of such a calculation, we posit that the correct scale to consider is indeed the Hill thickness. As described below, the results from our simulations are consistent with this. The Hill criterion for fragmentation, demonstrated in Fig. 6, is thus empirically based.

### 3.3 Consistency with simulations

An analysis of the spiral arms formed in the simulations of Section 2 shows that their thicknesses and stability are consistent with the Hill criterion for fragmentation. In Fig. 7, we show two examples of this analysis to illustrate this consistency. Our analysis focuses on the surface density of a radial slice of the disc (with a typical angular width of 5°). Over this slice, a spiral arm is evident as a large overdensity. We find that arms are often asymmetric; consequently, we determine a thickness for an arm by fitting each side of the arm (with respect to the radius of highest $\Sigma$, $R_{\text{peak}}$), with a Gaussian of the form

$$\Sigma_{\text{arm}} = \Sigma_{\text{base}} + \Sigma_{\text{peak}} e^{-\frac{(x-x_{\text{peak}})^2}{2\sigma^2}},$$

(20)

where $\Sigma_{\text{base}}$ is the value of the surface density adjacent to the arm and $\Sigma_{\text{peak}}$ is the maximum of the arm’s surface density. The thickness of the arm is taken to be $l_1 = a(b_{\text{left}} + b_{\text{right}})$, with $a = 2$, and the mass of the section of the arm is determined using a numerical evaluation of

$$M_{\text{arm}} = \int_{R_{\text{peak}}-2b_{\text{left}}}^{R_{\text{peak}}+2b_{\text{right}}} \Theta(R) R \Sigma(R) dR,$$

(21)

where $\Theta(R) = l_1/R$ is the angular extent of the section of the arm. In contrast to our analytic model presented in the preceding section, we do not use a simplified top-hat model here, but integrate the fit to the measured arm profile over the thickness of the arm, since we know what the arm profile is in this case. This allows us to verify the Hill criterion’s ability to determine fragmentation. With this verification, we can then be confident in applying the Hill criterion to our analytic model, as we do in Section 4.

Determining the thickness of a spiral arm is not a trivial task. Our use of Gaussian fitting functions is an attempt to be objective and reproducible and is motivated by the reasonably accurate fits that we achieve. Our choice of $a = 2$ as the coefficient in determining $l_1$ is motivated by having the majority (95 per cent) of the overdensity of the arm (the mass in the arm that is at $\Sigma > \Sigma_{\text{base}}$) contained within the arm. To contain all of the mass in a Gaussian fit requires integrating out to an infinite distance, which is clearly problematic in defining the finite thickness of a spiral arm. We have found that using values of $a > 2$ in fitting the arms often results in including material that is clearly outside of the arm; that is, the calculated arm thickness is obviously too large when the radial surface density profiles are examined by eye.

Fig. 7 (top) shows this analysis, at roughly 2.5 ORPs, for an arm in simulation B, shortly before fragmentation. Consistent with the Hill criterion, the arm thickness is less than the Hill thickness and therefore fragmentation is expected to occur in this arm; indeed, this arm fragmented a short time after the time-step used for this analysis. In contrast, Fig. 7 (bottom) shows this analysis, at roughly 2.3 ORPs, for an arm in simulation D that never fragmented.

Figure 6. The Hill criterion for spiral arm fragmentation: if a section of the spiral arm lies within its own Hill thickness, then that section of the arm is free to collapse and fragmentation takes place. If a section of the spiral arm lies outside of its own Hill thickness, then shear stabilizes the arm and fragmentation does not take place.
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3.4 Detailed evolution of spiral arms

The simplicity of the model described above is desirable, as it gives a straightforward, physical picture of the formation of spiral arms in a gravitationally unstable disc, as well as the physical criterion that determines whether or not those spiral arms fragment. However, unstable discs do exhibit a great deal of complexity; here, we discuss this complexity and comment on its implications for our model.

We have described an individual spiral arm’s stability as being the result of whether or not it is contained within its Hill radius. In addition, we have described the steady-state thickness of a single spiral arm as being the result of a balance between heating and cooling. However, the simulations show that the spiral structure in the disc evolves with time: the number of arms in the disc is not constant, nor is the overdensity of each arm.

An analysis of spiral arms in our simulations shows that individual spiral arms typically exist for a number (roughly three) of sound-crossing times; that is, they last approximately a dynamical time (orbital period). The sound-crossing times of arms are shorter than the radiative cooling time (2–30 times shorter, depending on the opacity scaling), suggesting that the arms can be treated as evolving in a quasi-static manner. This is important in applying the Hill criterion to determine stability.

Material in the arm is shock-heated, and radiatively cools. From the observed density contrasts between arms and inter-arm regions, we estimate the expected temperature increase in an adiabatic shock, and calculate the radiative cooling time to reach the observed to fall within their own Hill thickness shortly before fragmentation. Indeed, all arms that smoothly evolve to lie within their own Hill thickness fragment.

Consistent with the Hill criterion, the arm thickness is greater than the Hill thickness and therefore fragmentation is not expected. The positions of the radial slices for this analysis are shown in Fig. 8.

We use these two examples to illustrate the consistency of the fragmentation criterion with the simulations performed. More generally, we have found that lower opacity discs have arms that are consistently smaller with respect to their Hill thickness in comparison with higher opacity discs. Despite a large range in the opacity scalings used in the simulations (a factor of 100), we find that the arm thickness in the discs is generally similar to the Hill thickness. This can be observed for the two arms examined in Fig. 7. This small range for the ratio of the arm thickness to the Hill thickness observed in the simulations may seem surprising; however, it can be understood through our analytic calculations for radiatively cooled discs, presented in Section 4.2. The ratio of the arm thickness to the Hill thickness is weakly dependent on the opacity.

In each of the cases where fragmentation takes place (this occurs in the reduced opacity discs), the arms that fragment are observed...
temperature contrasts. We find that this cooling time is within the lifetime of the arms (it is equal to or up to five times shorter than the dynamical time, depending on the opacity scaling). This suggests that the arms have time to come to a balance between heating and cooling. In addition, the ability of the arm to readjust hydrodynamically on the short sound-crossing time also suggests that heating and balance can come into balance, since the arm can quickly adjust to an imbalance.

Evolution of the disc as a whole, such as heating during a phase of pronounced spiral activity such as a ‘burst’ (Mejía et al. 2005), does not prevent the application of our model. As long as the evolution of the disc takes place on a time-scale that is longer than the sound-crossing time of a spiral arm, the balance between heating and cooling in a spiral arm is a reasonable approximation. Since a burst phase heats the disc on roughly the orbital time, the model is applicable. A balance between heating and cooling will be invalid; however, in cases where there are strong hydrodynamic interactions between spiral arms (or between a spiral and a fragment, see below), since this will produce heating on a time-scale of roughly the sound-crossing time.

The Hill criterion describes the fragmentation of a spiral arm, but it does not necessarily determine whether or not this fragmentation leads to the formation of a long-lived object; this also depends on the cooling of the fragment and the complex environment of the disc in which the initial fragmentation takes place. Simulation C, for example, demonstrates an instance of ‘failed fragmentation’, as shown in Fig. 9. One of the spiral arms appears to have fragmented; however, the fragment is only short-lived: it shortly thereafter collided with the next spiral arm, without surviving.

The Hill criterion describes the formation of a fragment based on the inability of shear to stabilize a section of a spiral arm. Further collapse occurs on the radiative cooling time-scale of the fragment. If this time-scale is long, then the fragment may still be quite diffuse, and easily disrupted by collisions with subsequent spiral arms. Indeed, simulation B, which generally has shorter cooling times than simulation C because of its opacity scaling, shows a fragment which formed, but then subsequently collided twice with spiral arms; in contrast to simulation C, this object survived, as observed in Fig. 2. Fragmentation, therefore, can be well characterized by our model; however, whether or not fragments survive also depends on the complex non-linear interactions between collapsing fragments and the spiral structure in the disc.

As described, the fragment of simulation C was disrupted through a collision with the subsequent spiral arm. This resulted in a strong compression of the spiral arm; in fact, the compression was strong enough that the arm was observed to lie within its own Hill thickness. Nevertheless, the arm did not fragment. This does not conflict with the Hill criterion because in this instance, the arm was not in a near steady state. Since the time-scale for the collision was much shorter than the sound-crossing time of the arm, the arm could not adjust. As a result, the increased overdensity of the arm led to an increased heating rate, see equation (13), and a reduced cooling rate, see equation (18). As a consequence of the imbalance between heating and cooling, pressure forces caused the arm thickness to expand on roughly the sound-crossing time, with the result that no fragmentation took place.

4 CONSISTENCY WITH THE CRITICAL COOLING CRITERION AND PREDICTIVE ABILITY

The physical model developed in the preceding section can be used to examine whether or not fragmentation is likely to take place in a disc. In this section, we demonstrate that the predictions of this model are consistent with previous numerical results of discs evolved using the $\beta$-prescription of cooling, as well as the results of our suite of simulations discussed in Section 2. Specifically, we analyse the initial condition of our simulated disc described in Section 2 and adopt general values for the parameters. We use $I_0 = 2\pi H, \epsilon = 0.2$ (Cossins et al. 2009), $m = 2$ and $f = 1.5$ (characteristic of measurements from our set of simulations). We calculate the value of $M \dot{M}$, equation (14), as outlined in Cossins et al. (2009), by solving the dispersion equation, (15); we use equation (10) for $k$, which results in $M \dot{M} = Q^2 (m + 1)$. The values for these parameters will be correct to within $O(1)$, but will likely have variation depending on the physical properties of a disc.

Caution is warranted when considering the stability of a disc based on its initial condition. Johnson & Gammie (2003) examined the stability of discs using 2D shearing-box simulations that included radiative cooling. They found that the cooling times in the non-linear phase of the disc could differ by orders of magnitude from the cooling times in the initial condition. As a consequence, assessing the stability of the disc in its non-linear phase, via the cooling criterion, based on the initial cooling times can lead to the wrong conclusion.

In the following calculations, we use the initial condition of our disc simulations as a starting point in analytic calculations of stability. However, our model takes into account the increased surface density in the spiral arms that is characteristic of the non-linear phase. In addition, for irradiated discs, we take into account the increase in temperature and opacity in the spiral arms, based on results from our set of simulations.

4.1 Calculating the critical cooling time

The critical cooling time, $\beta_{\text{crit}}$, for a $Q \sim 1$ disc with $\gamma = 7/5$ has been found to be $\beta_{\text{crit}} = 12$ from numerical experiments (Rice et al. 2005; with the caveat that numerical convergence has not been clearly demonstrated). If we adopt a heating rate without irradiation (consistent with the aforementioned simulations), given by equation (13) and balance this heating with a $\beta$-prescription cooling
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4.2 Predictive ability of the model for irradiated discs

It is useful to check our model against the results of previous work using cooling in the form of a $\beta$-prescription. However, it is of particular interest to apply the model to the more realistic case of an irradiated disc with radiative cooling. Without considering GI, an irradiated disc has a natural equilibrium state in which the heating of a particular patch of the disc from stellar irradiation is balanced by the radiative cooling of that patch. Here, we consider deviations from this equilibrium state due to GI.

Specifically, there is an additional heating of the disc from the spiral arms, given by equation (13), which will result in an increase in the midplane temperature of $\delta T$. We consider spiral arms in which this excess heating is balanced by a perturbative radiative cooling. Following Kratter et al. (2010b), for excess heating transported to the photosphere via radiative diffusion, the perturbative radiative flux is

$$ F = \frac{16\sigma}{3\kappa} \frac{C}{\Sigma} \left( T^4 - T_{\text{irrad}}^4 \right), $$

which results in a cooling rate per unit mass of

$$ Q^- = \frac{F}{\Sigma} = \frac{16\sigma}{3\kappa\Sigma^2} \left( T^4 - T_{\text{irrad}}^4 \right), $$

where the midplane temperature is $T = T_{\text{irrad}} + \delta T$.

We calculate the natural arm thickness for our disc initial condition from Section 2 by balancing the perturbative heating and cooling in the arm. This results in the following quartic:

$$(C) l_1^4 - (B) l_1^2 + \left( \frac{2B l_0}{f} \right) l_1 - \left( B \left( \frac{l_0}{f} \right)^2 \right) = 0,$$

where

$$ B = \frac{\epsilon c_s^2 M \dot{M} \Omega}{2} $$

and

$$ C = \frac{16\sigma}{3\kappa} \left( \frac{T^4 - T_{\text{irrad}}^4}{\Sigma/l_0} \right). $$

Once $l_1$ is calculated, we can calculate the ratio

$$ \frac{l_1}{2 H_{\text{Hill}}} = \frac{l_1^{2/3}}{2} \left( \frac{3 f \Omega^2}{G \Sigma/l_0} \right)^{1/3} $$

to determine the stability of the arm.

Computing the arm thickness requires knowledge of the midplane temperature of the spiral arm during the non-linear phase. Since we are unable to self-consistently predict this temperature, this unfortunately limits the predictive ability of the model’s application to radiatively cooled discs. However, we can use the results of our simulations in Section 2 to test the consistency of the model with our simulation results. We have measured the maximum midplane temperature for each opacity case as a function of radius, averaged over the one ORP before fragmentation took place in simulations A and B. We use these measurements of $T$ in determining the arm thickness from equation (26). In addition to the temperature perturbation, we take into account the resulting changes in sound speed and opacity.

Johnson & Gammie (2003) found that cooling times in the non-linear phase can differ by orders of magnitude from the cooling times in the initial disc, largely because of sharp variations in the opacity with temperature. However, these sharp variations occur in discs that have high temperatures; in particular, temperatures near the opacity gap, $T \sim 1200$ K. In contrast, the outer regions of irradiated discs typically have much lower temperatures of $T < 100$ K. For these temperatures, there are no sharp variation in $\kappa(T)$ and $\epsilon$ is not a function of the density. For the typical increase of $\delta T \sim T_{\text{irrad}}/2$ observed in the region of fragmentation in our simulations, the resulting change in opacity is only about a factor of 1.25. In
the outer regions of irradiated discs, opacity changes during the non-linear phase do not have a large effect; however, we have taken them into account in our calculations.

The results of the arm thickness calculations, over the range of opacity scalings used in our simulations, are shown in Fig. 11. We expect discs to fragment for \( l_1/(2H_{\text{Hill}}) \leq 1 \), and the trend in Fig. 11 is consistent with this picture: as found in our simulations, the increased opacity discs are less likely to fragment [have larger \( l_1/(2H_{\text{Hill}}) \)] than the reduced opacity discs. However, even though the increased opacity discs are not expected to fragment, the arm thickness is expected to be within a factor of 2 of the Hill thickness in the region of \( Q \sim 1 \). The small range in \( l_1/(2H_{\text{Hill}}) \) over the different opacity scalings, as seen in Fig. 11, may seem counterintuitive given that the opacity variation is a factor of 100. However, we can understand this weak dependence on opacity as the result of the non-linear nature of the self-regulation of heating and cooling in a spiral arm.

As described in Section 3.1, the region of the initial disc that goes into the formation of the arm, \( l_0 \), is the result of the initial GI and does not depend on the opacity of the disc. The thickness of the spiral arm is then the result of the balance of heating and cooling in the arm. For radiative cooling in a spiral arm, the cooling rate per unit mass is \( Q^- \propto 1/\nu^2 \), while the heating rate per unit mass is \( Q^+ \propto \Sigma^2 \). A lower opacity results in a faster cooling rate, which requires a faster heating rate for balance to be achieved. This faster heating rate requires a higher surface density in the arm, but this acts to decrease the cooling rate. One can see then that the balance between heating and cooling, which determines \( l_1/(2H_{\text{Hill}}) \), has a non-linear dependence on the opacity. In fact, if we parametrize this dependence as a power law, \( l_1/(2H_{\text{Hill}}) \propto \nu^{-a \Sigma^2} \), in the region of instability, we find a weak opacity dependence with \( a \sim 0.1 \).

As discussed in Section 3.3, this result is consistent with our simulations. Fig. 7 (bottom) shows the analysis of an arm from simulation D. This arm is stable, in that it does not lie within its Hill thickness; however, its thickness only exceeds the Hill thickness by a relatively small factor. All of the arms analysed in the simulations have thicknesses that are similar to their Hill thicknesses.

From our calculation, only the lowest opacity case, corresponding to simulation A, has \( l_1/(2H_{\text{Hill}}) < 1 \), and would be expected to fragment; in fact, both simulation A and the second lowest opacity case, simulation B, have shown fragmentation. This discrepancy is likely simply the result of the choices of parameters used in the calculation. We have chosen values for a number of parameters in our model \( (l_0, f, \epsilon, M, \dot{M}) \) that are expected to be correct to within \( \mathcal{O}(1) \); however, the exact values will likely have some variation. A change in these parameters shifts the curves of \( l_1/(2H_{\text{Hill}}) \) vertically, but has little effect on the small range of \( l_1/(2H_{\text{Hill}}) \) over opacity scalings. With an improved understanding of the growth of spiral structure, and the heating of spiral arms, the model’s predictive abilities will be improved.

Furthermore, our analytic model considers the stability of a disc based on the stability of a statistically averaged arm, as our choices of parameters, \( (l_0, f, \epsilon, M, \dot{M}) \), are characteristic of the average. For example, the value of \( \epsilon \) was determined by Cossins et al. (2009) by considering the overall self-regulation of their \( Q \sim 1 \) discs. It is unclear how much the value of \( \epsilon \) in a given arm is expected to vary from this statistical average. For a disc to fragment, however, only a single arm is required to be unstable, rather than the statistically averaged arm. It may then be important to understand the variance of the model parameters away from their characteristic values, in order to more accurately predict fragmentation.

As with the application of the model to the case of \( \beta \)-prescription cooling in Section 4.1, we find, from our numerical solution of equation (26), that fragmentation is more likely for smaller values of \( M, \dot{M} \) in the case of radiative cooling. This implies that fragmentation should take place near corotation.

4.3 Predicted masses of fragments in irradiated discs

The initial mass of fragments can be a useful constraint on the physical model of fragmentation (Boley et al. 2010; Forgan & Rice 2011). From the natural arm thickness, \( l_1 \), calculated using equation (26), the fragment mass can be estimated as \( M = \Sigma f^2 \). The model predictions are lower bounds for the final mass of fragments, since significant accretion of disc material can occur after fragmentation has taken place.

Fig. 12 demonstrates the initial fragment mass expected for the range of opacity scalings used in our simulations. We compare our model’s fragment mass with the Toomre mass, \( M_T = \pi \Sigma (l_0/2)^2 \), as well as with the fragmentation model of Boley et al. (2010). In addition, we compare the model predictions with the initial fragments observed in simulations A and B, for which masses have been determined using the \( \text{SKID} \) group finder (Stadel 2001). \( \text{SKID} \) determines groups of SPH particles based on the gradient of density, and then performs an iterative unbinding procedure, which we have modified to include thermal energy.

As can be observed in Fig. 12, our model predicts fragments with initial masses in the gas-giant regime, significantly less than the Toomre mass. The masses of the fragments from simulations A and B are consistent with the predicted masses to within a factor of 2. It is interesting to note that the fragmentation model of Boley et al. (2010), strictly valid only for isothermal simulations, predicts similar mass fragments in the region where fragmentation is observed.

5 DISCUSSION AND CONCLUSIONS

5.1 Implications for planet formation

Direct imaging observations have shown the existence of gas-giant planets at large distances from their host A star, including HR
The predicted initial fragment mass (black curves), in Jupiter masses, of an irradiated disc, as calculated for our disc initial condition. The fragment mass is calculated for the range of opacities used in the simulations. From the curve of greatest mass to the curve of smallest mass, the opacity scalings are 10, 3, 1, 1/3 and 1/10 the physical opacity (the solid black curve). The Toomre mass is given by the red, dashed curve, while the predicted mass of Boley et al. (2010) is given by the blue, triple-dot-dashed curve. The asterisks represent the initial fragments from simulation A, while the diamond represents the initial fragment from simulation B.

8799b, 7M_{Jup} at a distance of 68 au (Marois et al. 2008), and Fomalhautb, 3M_{Jup} at a distance of 119 au (Kalas et al. 2008). It is difficult to explain the existence of gas giants at such distances from their host star in the core-accumulation scenario, since the surface densities are typically too low to form the necessary rocky cores within the lifetime of the gas disc (Dodson-Robinson et al. 2009; Rafikov 2011). However, more investigation is warranted in order to determine if such planets can be explained in the core-accumulation scenario.

In comparison, fragmentation via GI has been shown to be a viable formation mechanism at large distances from the host star both from theoretical arguments (Rafikov 2007; Nero & Bjorkman 2009; Kratter et al. 2010b) and from numerical simulations with radiative transfer (Boley 2009; Vorobyov & Basu 2010; Boss 2011; Stamatellos et al. 2011).

The particular set of 3D radiation hydrodynamic simulations presented here was designed to investigate fragmentation at large radii (~100 au) around A stars. At these distances, heating from stellar irradiation is expected to be the dominant heating source; we have included irradiation using $T_{\text{irrad}}$ expected for a 1.35-M$_{\odot}$ A star (Kratter et al. 2010b).

The results of our simulations show that GI can produce gravitationally bound objects at large distances from the star, given opacities on the low side of the expected range. Such opacities could be the result of grain growth (Birnstiel et al. 2010), grain evolution via the passage of spiral arms (Podolak et al. 2011), or formation in an environment with a non-solar metallicity (HR 8799 is roughly 1/3 solar metallicity; Marois et al. 2008). Our simulations do not take these physical mechanisms into account, but rather use a simple scaling of the opacity table.

Although our simulations do not have the resolution to follow the evolution of bound objects as their central densities run away, it is interesting to consider the objects at the end state of our simulations, as shown in Fig. 2. At the end of simulation A (with an opacity scaled by 1/10), there are two brown dwarfs of masses 21 and 15 M_{Jup}, at respective distances of 62 and 95 au; while at the end of simulation B (with an opacity scaled by 1/3), there is one brown dwarf of mass 40 M_{Jup}, at a distance of 95 au. Neither the masses nor the distances of these objects represent their final state: all of the objects are acquiring mass and migrating inwards at the end of the simulation. The long-term evolution and migration of objects formed at large distances through GI is an active area of research. In addition to providing a mechanism to form the gas giants observed at large distances from their star, the formation of objects through GI coupled with inward migration and tidal disruption (Boley & Durisen 2010; Nayakshin 2010) represents an interesting channel for the formation of rocky cores at smaller distances.

We conclude that GI in discs can produce brown dwarfs at large distances from A stars. We have, of course, only shown fragmentation for a single surface density and temperature profile. It is of interest to investigate a greater region of the parameter space with numerical simulations in order to explore the possibility of low-mass companions such as those observed by Marois et al. (2008) and Kalas et al. (2008).

5.2 Physical model of fragmentation

We have presented a new framework to explain the link between cooling and fragmentation in protostellar discs. This framework consists of two components. The first is a simple model for the formation of spiral arms, in which the thickness of a spiral arm is set by a balance between heating (through gravitational instability and irradiation) and radiative cooling. The second is a criterion for fragmentation: spiral arms that have a natural thickness smaller than their Hill thickness fragment, resulting in objects that may survive to become gas-giant planets or brown dwarfs.

This model of fragmentation is based on results from Cossins et al. (2009) as well as our suite of 3D radiation hydrodynamics simulations of gravitational instability in an irradiated, optically thick protostellar disc surrounding an A star. By reducing the opacity scaling, and consequently the cooling time, over the set of simulations, we have produced a suite that demonstrates the transition from non-fragmentation to fragmentation. From an analysis of these simulations, we have found that the critical scale for determining fragmentation is roughly the Hill thickness: those spiral arms that are found to fragment lie within twice their Hill radius, while those spiral arms that do not fragment extend beyond their Hill thickness. In the future, it would be of interest to have a robust calculation of the critical scale for fragmentation from a stability analysis of a spiral arm in a differentially rotating system.

In comparison to the critical cooling time picture, our model of fragmentation is a more detailed, and more general, physical picture of fragmentation that is applicable to discs with realistic heating and cooling, in addition to discs with $\beta$-prescription cooling. Indeed, by coupling the Hill criterion to our simple model of spiral arm formation, heating and cooling using a $\beta$-prescription, we have been able, for the first time, to calculate $\beta_{\text{crit}}$. We find that there is not a single value for $\beta_{\text{crit}}$, but that it depends on the local properties of the disc; in addition, our calculation is consistent with the value determined by numerical experiment.

We have also demonstrated how this model can be used to predict fragmentation, and fragment masses, in irradiated discs with radiative cooling. Applying the model to the initial condition of our simulated disc, for the various opacity scalings used, yields predictions that are consistent with the results of our simulations. An improvement in the predictive abilities of the model depends on a better understanding of several parameters that describe the formation and heating of the spiral arms.

Fragmentation, as determined by the Hill criterion, is weakly dependent on opacity, in agreement with Boss (2002) and Cai et al.
(2006). Despite the factor of 100 in opacity scaling across the suite of simulations, the thickness of spiral arms is observed to be near the Hill thickness, as predicted by our analytic calculations.

In addition, the survival of a fragment can depend on the outcome of the initial fragment’s collision with the remaining spiral arms. In one of our simulations, a fragment was destroyed through this process. The destruction of fragments through interactions with weak shocks has also recently been shown to be important in studies of GI employing the $\beta$-cooling prescription (Paardekooper 2012). The outcome of these interactions may be particularly sensitive to numerical methods.

This model has been developed in the context of protostellar discs; however, it may also be of use in the context of star formation in a disc near the Galactic Centre (Levin & Beloborodov 2003), as well as star cluster formation in optically thick starburst galaxies such as Arp 220.

In this work, we have considered isolated discs; that is, the effects of accretion from the surrounding envelope were ignored. However, accretion is expected to play an important role in gravitationally unstable discs (Boley 2009; Kratter et al. 2010a), since it is accretion that will push the mass of the disc towards being sufficient for instability to set in, keep it unstable despite mass transport and contribute to heating. In future work, we intend to investigate the effects of accretion on the stability of protostellar discs, in the context of our model of fragmentation.

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REFERENCES

Binney J., Tremaine S., 2008, Galactic Dynamics, 2nd ed. Princeton Univ. Press, Princeton, NJ
Birnstiel T., Dullemond C. P., Brauer F., 2010, A&A, 513, A79
Boley A. C., 2009, ApJ, 695, L53
Boley A. C., Durisen R. H., 2010, ApJ, 724, 618
Boley A. C., Hartquist T. W., Durisen R. H., Michael S., 2007a, ApJ, 656, L89
Boley A. C., Durisen R. H., Nordlund Å., Lord J., 2007b, ApJ, 665, 1254
Boley A. C., Hayfield T., Mayer L., Durisen R. H., 2010, Icarus, 207, 509
Boss A. P., 2002, ApJ, 567, L149
Boss A. P., 2011, ApJ, 731, 74
Cai K., Durisen R. H., Michael S., Boley A. C., Mejía A. C., Pickett M. K., D’Alessio P., 2006, ApJ, 636, L149
Cai K., Durisen R. H., Boley A. C., Pickett M. K., Mejía A. C., 2008, ApJ, 673, 1138
Chiang E. I., Goldreich P., 1997, ApJ, 490, 368
Clarke C. J., Harper-Clark E., Lodato G., 2007, MNRAS, 381, 1543
Cossins P., Lodato G., Clarke C. J., 2009, MNRAS, 393, 1157
D’Alessio P., Calvet N., Hartmann L., 1997, ApJ, 474, 397
Dodson-Robinson S. E., Veras D., Ford E. B., Beichman C. A., 2009, ApJ, 707, 79
Durisen R. H., Boss A. P., Mayer L., Nelson A. F., Quinn T., Rice W. K. M., 2007, in Reipurth B., Jewitt D., Keil K., eds, Protostars and Planets V. Univ. Arizona Press, Tucson, p. 607
Durisen R. H., Hartquist T. W., Pickett M. K., 2008, Ap&SS, 317, 3
Forgan D., Rice K., 2011, MNRAS, 417, 1928
Gammie C. F., 2001, ApJ, 553, 174
Johnson B. M., Gammie C. F., 2003, ApJ, 597, 131
Kalas P. et al., 2008, Sci, 322, 1345
Kratter K. M., Murray-Clay R. A., 2011, ApJ, 740, 1
Kratter K. M., Matzner C. D., Krumholz M. R., Klein R. I., 2010a, ApJ, 708, 1585
Kratter K. M., Murray-Clay R. A., Youdin A. N., 2010b, ApJ, 710, 1375
Levin Y., Beloborodov A. M., 2003, ApJ, 590, L33
Lodato G., Clarke C. J., 2011, MNRAS, 413, 2735
Lodato G., Rice W. K. M., 2004, MNRAS, 351, 630
Lodato G., Rice W. K. M., 2005, MNRAS, 358, 1489
Marois C., Macintosh B., Barman T., Zuckerman B., Song I., Patience J., Lafrenière D., Doyon R., 2008, Sci, 322, 1348
Mejía A. C., Durisen R. H., Pickett M. K., 2005, ApJ, 619, 1098
Meru F., Bate M. R., 2010, MNRAS, 406, 2279
Meru F., Bate M. R., 2011, MNRAS, 410, 559
Nayakshin S., 2010, MNRAS, 408, L36
Nero D., Bjorkman J. E., 2009, ApJ, 702, L163
Paardekooper S.-J., 2012, MNRAS, 421, 3286
Paardekooper S.-J., Baruteau C., Meru F., 2011, MNRAS, 416, L65
Podolak M., Mayer L., Quinn T., 2011, ApJ, 734, 56
Price D. J., 2007, PASA, 24, 159
Rafikov R. K., 2005, ApJ, 621, L69
Rafikov R. K., 2007, ApJ, 662, 642
Rafikov R. K., 2011, ApJ, 727, 86
Rice W. K. M., Lodato G., Armitage P. J., 2005, MNRAS, 364, L56
Rice W. K. M., Forgan D. H., Armitage P. J., 2012, MNRAS, 420, 1640
Rogers P. D., Wadsley J., 2011, MNRAS, 414, 913
Shakura N. I., Sunyaev R. A., 1973, A&A, 24, 337
Shen S., Wadsley J., Hayfield T., Ellens N., 2010, MNRAS, 401, 727
Stadel J. G., 2001, PhD thesis, Univ. Washington
Stamatellos D., Whitworth A. P., 2008, A&A, 480, 879
Stamatellos D., Maury A., Whitworth A., Andréû, 2011, MNRAS, 413, 1787
Toomre A., 1964, ApJ, 139, 1217
Vorobyov E. I., Basu S., 2010, ApJ, 719, 1896
Wadsley J. W., Stadel J., Quinn T., 2004, New Astron., 9, 137
Zhu Z., Hartmann L., Gammie C., McKinney J. C., 2009, ApJ, 701, 620

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