Explicit Families Of Elliptic Curves With The Same Mod 6 Representations

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Abstract
Let $E$ be an elliptic curve over $\mathbb{Q}$. In this paper we study two certain modular curves which parameterize families of elliptic curves which are directly (resp. reverse) 6-congruent to $E$ together with the explicit parametrizations. The equations for the direct case has been worked out but the parametrization is only given for very few cases. In this paper we use a new method to obtain the equations for both direct and reverse cases together with full (and simpler) parametrizations.

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1 Introduction and Notations

1.1 Introduction and General Background

If $n$ is a positive integer and $E$ is an elliptic curve over $\mathbb{Q}$ with algebraic closure $\overline{\mathbb{Q}}$, let $E[n]$ be the $n$-torsion subgroup of $E$, which is the kernel of multiplication by $n$ on $E(\overline{\mathbb{Q}})$. Throughout the essay, each elliptic curve will be assumed to be defined over $\mathbb{Q}$.

Elliptic curves $E_1$ and $E_2$ are said to be $n$-congruent if their $E_1[n]$ and $E_2[n]$ are isomorphic as Galois modules. They are directly $n$-congruent if the isomorphism $\phi : E_1[n] \cong E_2[n]$ respects the Weil pairing and reverse $n$-congruent if $e_n(\phi P, \phi Q) = e_n(P, Q)^{-1}$ for all $P, Q \in E_1[n]$ where $e_n$ is the Weil pairing. The elliptic curves directly (resp. reverse) $n$-congruent to a given elliptic curve $E$ are parameterized by the modular curve $Y_E(n) = X_E(n) \{ \text{cusps} \}$ (resp. $Y_{E^-}(n) = X_{E^-}(n) \{ \text{cusps} \}$).

The existence of these modular curves and how to obtain these modular curves were explained in [S]. Let $W_E(n)$ (resp. $W_{E^-}(n)$) be the elliptic surface over $X_E(n)$ (resp. $X_{E^-}(n)$) whose fibers above points of $X_E(n)$ (resp. $X_{E^-}(n)$) are elliptic curves corresponding to these points and we aim to understand the structures of $X_{E^\pm}(n)$ and $W_{E^\pm}(n)$.

It was shown in [S] that these modular curves are twists of the classical modular curves $X(n)$ and hence have the same genus. For the case $n = 2, 3, 4$ or $5$, the modular curve $X(n)$ has genus 0 and there are infinite number of rational points on $X_E(n)$ for $n \leq 5$ and the explicit parametrizations were computed by Rubin and Silverberg in [RS1],[RS2] and [S]. The reverse case $X_{E^-}(n)$ was computed by Fisher using invariant theory [F1], [F2].

If $n = 6$, the curve $X(6)$ has genus one and so $X_E(6)$ and $X_{E^-}(6)$ are both elliptic curves over $\overline{\mathbb{Q}}$. A model for $X_E(6)$ is given independently by Papadopoulos in [P] and by Rubin and Silverberg in [RS3]. Some examples of the parametrization were computed by Roberts in [R]. In this paper we describe the complete parametrizations of $X_E(6)$ as well as those for $X_{E^-}(6)$.

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1.2 Main Theorem

We now list the main theorem in this paper
Theorem 1.1. Let $E : y^2 = x^3 + ax + b$ be an elliptic curve defined over $\mathbb{Q}$. Then a model for $X_E(6)$ is given by the intersection of 9-quadrics in $\mathbb{P}^5$ with coordinates $(x_1 : \ldots : x_6)$

\[

t = 9x_1 x_5 + 24ax_1 x_6 - 6x_2^2 + 24ax_2 x_3 - 6x_2 x_4 + 24ax_3 x_4
+ 72bx_4 x_6 + 2ax_5^2 + 8a^2 x_5 x_6 + Dx_6^2,
\]

\[
s_2 = -6x_1 x_3 + x_2 x_5 + 2ax_2 x_6 - 36bx_3^2 + 2ax_3 x_5 + 16a^2 x_3 x_6 + x_4 x_5
+ 2ax_4 x_6 + 12abx_6^2,
\]

\[
s_3 = 12ax_1 x_3 + 18bx_1 x_6 + 18bx_3 x_4 - 2ax_4 x_5 - 4a^2 x_4 x_6 + 3bx_5^2,
\]

\[
s_4 = -12ax_2 x_3 - 18bx_2 x_6 - 18bx_3 x_5 - 3x_4^2 - ax_5^2 + 4a^2 x_5 x_6,
\]

\[
s_5 = 3x_2^2 - 48a^2 x_3^2 - 144abx_3 x_6 - 36bx_4 x_6 + ax_5^2 - 8a^2 x_5 x_6 + 16a^3 x_6^2,
\]

\[
s_6 = -3x_1 x_4 + 18bx_2 x_3 - ax_2 x_5 - 4a^2 x_2 x_6 - 4a^2 x_3 x_5 - 6abx_5 x_6,
\]

\[
s_7 = -108bx_1 x_3 + 6ax_2^2 - 24a^2 x_2 x_3 + 18bx_2 x_5 - 36abx_4 x_6 - 2a^2 x_5^2
\]

\[- 8a^3 x_5 x_6 - aDx_6^2,
\]

\[
s_8 = 3x_1 x_2 - 72abx_3^2 - 216b^2 x_3 x_6 + ax_4 x_5 + 8a^2 x_4 x_6 - 12abx_5 x_6 + 24a^2 bx_6^2,
\]

\[
s_9 = 36x_1^2 + 12ax_2^2 + 12ax^2_3 + 4a^2 x_5^2 + Dx_5 x_6.
\]

The forgetful morphism $X_E(6) \to X_E(3)$ is given by

\[(x_1 : \ldots : x_6) \mapsto (x_3/3 : x_6)\]

By using the above theorem and studying the function field of the modular curve we obtain the following corollary which gives a much simpler model for $X_E(6)$

Corollary 1.2. The curve $X_E(6)$ is birational to the following curve defined by two equations in $\mathbb{A}^3$ with coordinates $(x, y, z)$:

\[
f = z^3 - (36ax^2 + 12a^2)z + 216bx^3 - 144a^2 x^2 - 216abx
\]

\[- (16a^3 + 216b^2) + y(64abx + 96b^2)27/\Delta_E,
\]

\[
g = y^2 - D(ax^4 + 6bx^3 - 2a^2 x^2 - 2abx + (-a^3/3 - 3b^2)).
\]

with forgetful morphism

$X_E(6) \to X_E(3) : (x, y, z) \mapsto x/3.$
2 Preliminary Material

In this section we will briefly give some material and preliminary results. Everything in this section has appeared elsewhere and can be found in the references. We mainly focus on the level two, level three and level six structures of the classical modular curves which we need for our method.

2.1 Modular Curves

Let \( n \geq 2 \) be an integer. Let \( Y(n) \) denote the classical modular curve which parametrizes isomorphism classes of pairs \((E, \phi)\) where \( E \) is an elliptic curve and

\[
\phi : \mathbb{Z}/n\mathbb{Z} \times \mu_n \to E[n]
\]

is an isomorphism such that

\[
e_n(\phi(a_1, \zeta_1), \phi(a_2, \zeta_2)) = \zeta_2^{a_1} / \zeta_1^{a_2}.
\]

Lemma 2.1. Equivalently, \( Y(n) \) parametrizes triples \((E, P, C)\) where \( E \) is an elliptic curve, \( P \) is a point of exact order \( n \) and \( C \) is a cyclic subgroup of order \( n \) which does not contain \( P \).

Proof. Given a triple \((E, P, C)\) define \( \phi(a, \zeta^b) = aP + bQ \) where \( Q \in C \) such that \( e_n(P, Q) = \zeta \). Conversely, given \((E, \phi)\) define \( P \) to be \( \phi(1, 1) = P \), \( \phi(0, \zeta) = Q \) and \( C = \langle Q \rangle \). \( \square \)

Let \( X(n) \) be the compactification of \( Y(n) \). Then by definition we can identify \( X(1) \) with \( \mathbb{P}^1 \) by the map which sends an elliptic curve \( E \) to its \( j \)-invariant.

Lemma 2.2. The forgetful morphism \( X(n) \to X(1) \) induced by \((E, \phi) \mapsto E \) (or \((E, P, C) \mapsto E\)) has degree \(|\text{PSL}_2(\mathbb{Z}/n\mathbb{Z})|\) where by our convention the special projective linear group \( \text{PSL}_2(\mathbb{Z}/n\mathbb{Z}) = \text{SL}_2(\mathbb{Z}/n\mathbb{Z}) / \{ \pm I \} \).

Proof. For each \( \alpha \in \text{SL}_2(\mathbb{Z}/n\mathbb{Z}) \), \( \alpha \) acts on \( Y(n) \) by \( \alpha \circ (E, \phi) = (E, \alpha \circ \phi) \) and \(-I\) acts trivially because it is induced by \([-1]\) which is an automorphism of \( E \). So \( \text{PSL}_2(\mathbb{Z}/n\mathbb{Z}) \) acts on \( Y(n) \) and the action extends to \( X(n) \). Finally, the quotient map corresponds to \((E, \phi) \to E\) which is just taking the \( j \)-invariant and so the degree of the forgetful morphism is the same as \(|\text{PSL}_2(\mathbb{Z}/n\mathbb{Z})|\). \( \square \)

Let \( W_n \) be a compactification of the elliptic surface associated to the universal elliptic curve over \( Y(n) \). Then \( W_n \) is an elliptic surface over \( X(n) \) with a projection map

\[
\pi_n : W_n \to X(n)
\]
and a zero-section

\[ \iota : X(n) \to W_n \]

and both maps are defined over \( \mathbb{Q} \). For each \( u \in Y(n) \), let \( A_u \) be the fiber of \( W_n \) above \( u \). Then there exist a point \( P_u \in A_u[n] \) and a cyclic subgroup \( C_u \) of \( A_u[n] \) of order \( n \) such that \( (A_u, P_u, C_u) \) is a representative of the isomorphism class corresponding to \( u \).

We now recall the main results for \( n = 2, 3 \) and 6.

### 2.2 Level Two Structure

Every elliptic curve \( E \) over \( \mathbb{Q} \) with the trivial \( G_\mathbb{Q} \)-action on \( E[2] \) is a quadratic twist of

\[ A_u : y^2 = x(x-1)(x-u), \quad u \neq 0, 1. \]

Putting \( A_u \) in Weierstrass form, we have

\[ A_u : y^2 = x^3 + a_4(u)x + a_6(u) \]

where

\[ a_4(u) = -\frac{1}{3}(u^2 - u + 1), \quad a_6(u) = -\frac{1}{27}(2u^3 - 3u^2 - 3u + 2). \]

### 2.3 Level Three Structure

It is well-known that the explicit parametrization of \( X(3) \) is the Hesse cubic:

\[ B_u : x^3 + y^3 + z^3 = 3uXYZ, \quad u \in Y(3). \]

The Weierstrass form of \( B_u \) is given by

\[ B'_u : y^2 = x^3 + b_4(u)x + b_6(u) \]

where

\[ b_4(u) = -27u(u^3 + 8), \quad b_6(u) = 54(u^6 - 20u^3 - 8). \]

A brief argument to see \( X(3) \) has parametrization \( B_u \) by parameter \( u \) can be found in [RS1]. There is an alternative approach that proves the above statement directly, which can be found in [IP] section 1 and 2.
2.4 Level Six Structure

We give a model for $X(6)$. Since $E[6] = E[2] \oplus E[3]$, each point $(E, \phi)$ on $Y(6)$ can be identified with $(E, \phi_2, \phi_3)$ where

$$
\phi_2 : \mathbb{Z}/2\mathbb{Z} \times \mu_2 \to E[2], \quad \phi_2(1, 1) = \phi(3, 1), \phi_2(0, -1) = \phi(0, -1)
$$

and

$$
\phi_3 : \mathbb{Z}/3\mathbb{Z} \times \mu_3 \to E[3], \quad \phi_3(1, 1) = \phi(2, 1), \phi_3(0, \zeta_3) = \phi_6(0, \zeta_3).
$$

**Theorem 2.3.** For any triple $(E, \phi_2, \phi_3) \in Y(6)$ as above, there exists a unique pair $(v, \tau) \in \mathbb{Q}^2$ with $2v^2\tau^2 = v + \tau, v(v^3 - 1)(8v^3 + 1) \neq 0$ such that $(E, \phi_2, \phi_3)$ is isomorphic to $(B_u, \phi_u, \phi_{\nu, \tau})$ where $3u = 2v + v^{-2} = 2\tau + \tau^{-2}$, $(B_u, \phi_u)$ as in level three structure with $\phi_u : \mathbb{Z}/3\mathbb{Z} \times \mu_3 \to B_u[3]$ and

$$
\phi_{\nu, \tau} : \mathbb{Z}/2\mathbb{Z} \times \mu_2 \to B_u[2], \phi_{\nu, \tau}(1, 1) = (v, v, 1), \phi_{\nu, \tau}(0, -1) = (\tau, \tau, 1).
$$

Further, the isomorphism from $(E, \phi_2, \phi_3)$ to $(B_u, \phi_u, \phi_{\nu, \tau})$ is unique.

**Proof.** This follows by starting with the parametrization of the level three structure and an explicit computation of the 2-torsion points on the model.

The above theorem identifies the curve $Y(6)$ with

$$2v^2\tau^2 = v + \tau, \quad v(8v^3 + 1)(v^3 - 1) \neq 0.
$$

On a simple change of variable $X = 2v, Y = 4v^2\tau - 1$, we have

$$Y^2 = X^3 + 1, \quad X^3 \neq \{0, -1, 8\}
$$

and so by taking compactification we can take a model for $X(6) : y^2 = x^3 + 1$.

2.5 Modular Elliptic Curves

Let $E$ be an elliptic curve. The general theoretic approach (but in many cases not practical) to get $X_E(n)$ can be found in [S], where it is shown that $X_E(n)$ is a twist of $X(n)$ and hence they have the same structure as abstract curves and in particular they have the same genus. To briefly illustrate the idea of the proof, we state the following result.
Theorem 2.4. Let $E$ be an elliptic curve and $V = E[N]$, viewed as $\mathbb{G}_\mathbb{Q}$-module. Define a bilinear pairing $\langle , \rangle$ on $\mathbb{Z}/n\mathbb{Z} \times \mu_n$ by
\[
\langle (a_1, \zeta_1), (a_2, \zeta_2) \rangle = \zeta_1^{a_2}/\zeta_2^{a_1}.
\]
Now fix an isomorphism $\phi : \mathbb{Z}/n\mathbb{Z} \times \mu_n \rightarrow V$ such that the above pairing is compatible with the Weil pairing under $\phi$. Then the cocycle $\tau \mapsto \phi^{-1} \tau(\phi)$ take values in $\text{Aut}(\mathbb{Z}/n\mathbb{Z} \times \mu_n, \langle , \rangle)$ which is the set of automorphisms of $\mathbb{Z}/n\mathbb{Z} \times \mu_n$ which preserve the bilinear pairing.

By Lemma 2.1 the identification induces an inclusion
\[
\text{Aut}(\mathbb{Z}/n\mathbb{Z} \times \mu_n, \langle , \rangle) \hookrightarrow \text{Aut}(W_n)
\]
and also there is a natural map $\text{Aut}(W_n) \rightarrow \text{Aut}(X(n))$. Thus, the cocycle above induces cocycles $c$ and $c_0$, taking values in $\text{Aut}(W_n)$ and $\text{Aut}(X(n))$ respectively. Then by general theory of twist, we obtain curves $W$ and $X$, and induced isomorphisms $\psi$ and $\psi_0$ defined over $\overline{\mathbb{Q}}$ together with a projection map $\pi : W \rightarrow X$ defined over $\mathbb{Q}$ such that the following diagram commutes
\[
\begin{array}{ccc}
W & \xrightarrow{\psi} & W_n \\
\downarrow{\pi} & & \downarrow{\pi_n} \\
X & \xrightarrow{\psi_0} & X(n)
\end{array}
\]
Then the curve $X$ is a model for $X_E(n)$ over $\mathbb{Q}$.

Proof. See [S], page 449-450. \qed

The formula for $X_E(2)$ and $X_E(3)$ can be found in [RS1] and [RS2]. However in this paper we also require the results of $X_E^{-1}(3)$ and so we use the following results for $X_E(3)$ and $X_E^{-1}(3)$ instead from [F1],[F2], which is also simpler for the computational purpose. Note that $X_E^{-1}(2)$ is the same as $X_E(2)$ because the reciprocal of $-1$ is again $-1$.

Theorem 2.5. Let $E/\mathbb{Q}$ be an elliptic curve with Weierstrass form $E : y^2 = x^3 + ax + b$. Then the family of elliptic curves $F_{u,v}$ which are 2-congruent to $E$ is given by
\[
F_{u,v} : y^2 = x^3 + 3(3av^2+9bu\nu-a^2u^2)x+27bv^3-18a^2uv^2-27abu^2v-(2a^3+27b^2)u^3,
\]
where $u, v \in \mathbb{Q}_2$ with discriminant $\Delta(F_{u,v}) = 3^6(v^3 + au^2v + bv^3)^2 \Delta(E)$. 

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Proof. See [RS2] Theorem 1.

Note that varying the ratio \((u : v)\) would replace the curve \(F_{u,v}\) by a quadratic twist but as taking quadratic twist does not change the mod 2 representation thus \(X_E(2)\) is viewed as \(\mathbb{P}^1\) with coordinate \((u : v)\).

**Theorem 2.6.** Let \(E\) be an elliptic curve with Weierstrass form \(E : y^2 = x^3 + ax + b\). Define \(c_4 = -a/27, c_6 = -b/54\) and

\[
\begin{align*}
c_4(\lambda, \mu) &= c_4 \lambda^4 + 4c_6 \lambda^3 \mu + 6c_4^2 \lambda^2 \mu^2 + 4c_4c_6 \lambda \mu^3 - (3c_4^3 - 4c_6)\mu^4, \\
c_6(\lambda, \mu) &= c_6 \lambda^6 + 6c_4^2 \lambda^5 \mu + 15c_4c_6 \lambda^4 \mu^2 + 20c_4^2 \lambda^3 \mu^3 + 15c_4^2c_6 \lambda^2 \mu^4 \\
&\quad + 6(3c_4^4 - 2c_4c_6^2)\lambda^2 \mu^5 + (9c_4^3c_6 - 8c_4^3)\mu^6, \\
c_4^*(\lambda, \mu) &= -4(\lambda^4 - 6c_4^2 \lambda^2 \mu^2 - 8c_4c_6 \lambda \mu^3 - 3c_4^2 \mu^2)/(c_4^3 - c_6^2), \\
c_6^*(\lambda, \mu) &= -8c_6(\lambda, \mu)/(c_4^3 - c_6^2)^2.
\end{align*}
\]

Then the family of elliptic curves \(E_{\lambda,\mu}\) which are directly 3-congruent to \(E\) is given by

\[E_{\lambda,\mu} : y^2 = x^3 - 27c_4(\lambda, \mu)x - 54c_6(\lambda, \mu).\]

The family of elliptic curves \(E'_{\lambda,\mu}\) which are reverse 3-congruent to \(E\) is given by

\[E'_{\lambda,\mu} : y^2 = x^3 - 27c_4^*(\lambda, \mu)x - 54c_6^*(\lambda, \mu).\]

Proof. See [F2] Theorem 1.1.

**Remark 2.7.** In theory, the reverse congruent case can be done in a similar way for the case \(n = 3\). Theorem 2.4 gives, in theory (but not practical), a general method to obtain \(X_E(n)\) and if we switch the sign of the bilinear pairing, which will give another bilinear pairing on \(\mathbb{Z}/n\mathbb{Z} \times \mu_n\) then we would obtain the curve \(X_{E'}(n)\). However this method does not work well in practise as constructing explicit (and simple) model of the twist corresponding to a given cocycle is complicated.
3 Explicit Families of Directly 6-Congruent Elliptic Curves

In this section we give a new approach to find a model for $X_E(6)$ which was previously computed by Papadopoulos [P] and Rubin and Silverberg [RS3]. Our method gives simpler parametrization and we will use the same method to compute a model of $X_E^{-1}(6)$ in the next section.

3.1 The Action of Special Projective Linear Group

We have seen the action of $\text{PSL}_2(\mathbb{Z}/n\mathbb{Z})$ on $X(n)$ in the previous section and the quotient of the action induces the forgetful morphism $X(n) \to X(1)$. On the other hand, the curve $X_E(n)$ is a certain twist of $X(n)$ and hence the action of $\text{PSL}_2(\mathbb{Z}/n\mathbb{Z})$ on $X(n)$ induces a twisted action on $X_E(n)$. Explicitly, we have

**Lemma 3.1.** Fix an isomorphism $\psi_n : X_E(n) \to X(n)$. Let $\alpha \in \text{PSL}_2(\mathbb{Z}/n\mathbb{Z})$ then the action of $\alpha$ on $X_E(n)$ is given by

$$\alpha P = \psi_n^{-1}(\alpha(\psi_n(P))), P \in X_E(n).$$

Further, the forgetful morphism $X_E(n) \to X(1)$ is the quotient map by the action defined above.

**Proof.** This is an immediate consequence of Theorem 2.4. \(\square\)

The action in the reverse case is defined in a similar way as above. We now focus on the case $n = 2$ and $3$. The following is a standard result about the structure of $\text{PSL}_2(\mathbb{Z}/2\mathbb{Z})$ and $\text{PSL}_2(\mathbb{Z}/3\mathbb{Z})$.

**Lemma 3.2.** We have group isomorphisms

$$\text{PSL}_2(\mathbb{Z}/2\mathbb{Z}) \cong S_3, \text{PSL}_2(\mathbb{Z}/3\mathbb{Z}) \cong A_4$$

where $S_3$ is the permutation group of 3 elements and $A_4$ is the alternating group of 4 elements. Hence there are exact sequences

$$0 \to C_3 \to \text{PSL}_2(\mathbb{Z}/2\mathbb{Z}) \to C_2 \to 0,$$

$$0 \to V_4 \to \text{PSL}_2(\mathbb{Z}/3\mathbb{Z}) \to C_3 \to 0$$

where $V_4$ is the unique $C_2 \times C_2$ subgroup of $A_4$. 

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We have seen that how $\text{PSL}_2(\mathbb{Z}/n\mathbb{Z})$ acts on $X(n)$ and $X_E(n)$ and the quotient map induces the forgetful morphism to $X(1)$. Further, for $m|n$, viewing $\text{PSL}_2(\mathbb{Z}/m\mathbb{Z})$ as a subgroup of $\text{PSL}_2(\mathbb{Z}/n\mathbb{Z})$ and $\mathbb{Z}/m\mathbb{Z} \times \mu_n$, we conclude that the quotient by the action of $\text{PSL}_2(\mathbb{Z}/m\mathbb{Z})$ induces the forgetful morphisms

$$X(n) \to X(n/m), X_E(n) \to X_E(n/m).$$

Thus, we have forgetful morphisms

$$\psi_2 : X_E(6) \to X_E(2), \psi_3 : X_E(6) \to X_E(3)$$

induced by the action of $\text{PSL}_2(\mathbb{Z}/3\mathbb{Z})$ and $\text{PSL}_2(\mathbb{Z}/2\mathbb{Z})$ respectively. Therefore, $X_E(6)$ can be obtained by taking the fiber product of $X_E(3)$ and $X_E(2)$. Explicitly, to have the correct explicit description of $X_E(6)$, with respect to the forgetful morphisms $\psi_2$ and $\psi_3$, we have the following commutative diagram:

$$
\begin{array}{ccc}
X_E(6) & \xrightarrow{\psi_2} & X_E(2) \\
\downarrow{\psi_3} & & \downarrow{f_3} \\
X_E(3) & \xrightarrow{f_2} & X(1)
\end{array}
$$

where $f_3, f_2$ are the forgetful morphisms from $X_E(3)$ (resp. $X_E(2)$) to $X(1)$.

### 3.2 A Commutative Diagram

Throughout this subsection, we will work over the algebraic closure $\overline{\mathbb{Q}}$. Consider the forgetful morphism $\psi_3 : X_E(6) \to X_E(3)$, which is the quotient by the action of $\text{PSL}_2(\mathbb{Z}/2\mathbb{Z})$. By Lemma 3.2, there is a normal subgroup $H$ of order 3 inside $\text{PSL}_2(\mathbb{Z}/2\mathbb{Z})$. Thus, the morphism $\psi_3$ factors through $\chi_3$ and $\rho_3$ where $\chi_3$ is the morphism obtained by quotient out the action of $H$.

We define the image of the quotient by:

$$X_E(6) \xrightarrow{\chi_3} X \xrightarrow{\rho_3} X_E(3)$$

Similarly, as $\text{PSL}_2(\mathbb{Z}/3\mathbb{Z})$ contains a normal subgroup of order 4 which is isomorphic to $H' = C_2 \times C_2$, the morphism $\psi_2$ factors in the following way:

$$X_E(6) \xrightarrow{\chi_2} Y \xrightarrow{\rho_2} X_E(2)$$

where $\chi_2$ is the morphism obtained by quotient out the action of $H'$.

Further, we can take the quotient of $Y$ by the action of $H'$ (with quotient map $\alpha$) and the quotient of $X$ by the action of $H$ (with quotient map $\beta$).
This is the same as the quotient of $X_E(6)$ by the action of $HH'$ and so they have the same image and we define the image by the following commutative diagram

$$
\begin{array}{ccc}
X_E(6) & \xrightarrow{\chi^2} & Y \\
\downarrow{\chi^3} & & \downarrow{\alpha} \\
X & \xrightarrow{\beta} & Z
\end{array}
$$

and thus $X_E(6)$ can be understood as the fiber product of $X$ and $Y$ with the above commutative diagram.

We now establish some properties of the diagram.

**Proposition 3.3.** The curves $X, Y, Z$ all have genus 1.

**Proof.** Working over algebraically closed field, for any surjective morphism $\phi : C_1 \to C_2$ between two smooth projective curves, we have, by Riemann-Hurwitz formula,

$$2 - 2g(C_1) = \deg(\phi)(2 - 2g(C_2)) - \sum_{P \in C_1} (e_P - 1)$$

where $g(C_i)$ is the genus of $C_i$ and $e_P$ is the ramification index at $P$. Further as we work over $\bar{\mathbb{Q}}$, the extension of the function fields is Galois and hence we apply tower law on ramification indices.

To compute the genus of $X$, we take $C_1$ to be $X_E(6)$ and so $g(C_1) = 1$ and $\deg(\phi) = \deg(\chi^3) = 3$. Suppose $P \in X_E(6)$ is a ramified. Then it will be a ramified under the map

$$\psi_3 : X_E(6) \to X_E(3)$$

and if we fix an isomorphism $\psi_6 : X_E(6) \to X(6)$ then the image $\psi_6(P)$ will be a ramified under the forgetful map $X(6) \to X(3)$ and hence is ramified under the forgetful morphism $X(6) \to X(1)$ and it is well-known that this is ramified at the $P \in X(n)$ where $P$ is sent to 0, 1728 or $\infty$. Let $e_0, e_1, e_\infty$ be the ramification index of points above 0, 1728 and $\infty$ respectively, then $e_0 = 3, e_1 = 2, e_\infty = n$. It is immediate by tower law that the points above 0 and 1728 are not ramified under the morphism

$$X(6) \to X(3)$$

and the points above $\infty$ have ramification degree 2. This shows that ramification degree of any points $P$ under $X_E(6) \to X_E(3)$ is at most 2 and hence
the same holds for the morphism $\chi_3 : X_E(6) \to X$. But the map $\chi_3$ has degree 3 and as 2 is prime to 3 we conclude that $\chi_3$ is unramified.

Thus we have

$$0 = \deg(\chi_3)(2 - 2g(X)) - \sum_{P \in X_E(6)} (e_P - 1) = 3(2 - 2g(X))$$

and this shows $g(X) = 1$.

To compute the genus of $Y$, we repeat the above procedure and now we have $\deg(\phi) = \deg(\chi_2) = 4$. Then again by tower law, the points above 0, 1728 are unramified under the map $X(6) \to X(2)$ and the ramification degree of the points above $\infty$ of $X(6) \to X(2)$ is 3. This implies that the ramification degree of any points $P \in X_E(6)$ of $X_E(2)$ is a factor of 3 and hence the same holds for $\chi_2 : X_E(6) \to Y$ by tower law. But $\chi_2$ has degree 4 which is prime to 3.

Thus we conclude that $\chi_2$ is unramified and so

$$0 = \deg(\chi_2)(2 - 2g(Y)) - \sum_{P \in X_E(6)} (e_P - 1) = 4(2 - 2g(Y))$$

and this shows $g(Y) = 1$.

Finally, to compute the genus of $Z$, we use the commutative diagram

\[
\begin{array}{ccc}
X_E(6) & \xrightarrow{\chi_2} & Y \\
\downarrow \chi_3 & & \downarrow \alpha \\
X & \xrightarrow{\beta} & Z \\
\downarrow \beta & & \downarrow \gamma \\
X_E(3) & \xrightarrow{\gamma_5} & X_E(3)/H' & \xrightarrow{\gamma_6} & X(1)
\end{array}
\]

Since we have seen that $X_E(6) \to X$ and $X_E(6) \to Y$ are unramified, so the map $X_E(6) \to Z$ is unramified, because the degrees of the maps $X \to Z$ and $Y \to Z$ are coprime and by Riemann-Hurwitz we conclude that $g(Z) = 1$. □

### 3.3 Galois Theory and Function Fields

Our next step is computing explicit equations for $X, Y$ and $Z$. We establish the following diagram:

\[
\begin{array}{ccc}
X_E(6) & \xrightarrow{x_2} & Y & \xrightarrow{\rho_2} & X_E(2) \\
\downarrow \chi_3 & & \downarrow \alpha & & \downarrow \gamma_1 \\
X & \xrightarrow{\beta} & Z & \xrightarrow{\gamma_2} & X_E(2)/H \\
\downarrow \rho_3 & & \downarrow \gamma_3 & & \downarrow \gamma_4 \\
X_E(3) & \xrightarrow{\gamma_5} & X_E(3)/H' & \xrightarrow{\gamma_6} & X(1)
\end{array}
\]
where $\chi_2, \chi_3, \rho_2, \rho_3, \alpha, \beta$ are the morphisms defined above and $\gamma_1$ corresponds to the quotient map $X_E(2) \to X_E(2)/H$ and $\gamma_4 \circ \gamma_1$ is the forgetful morphism from $X_E(2) \to X(1)$, and $\gamma_5$ corresponds to the quotient map $X_E(3) \to X_E(3)/H'$ and $\gamma_6 \circ \gamma_5$ is the forgetful morphism $X_E(3) \to X(1)$.

We modify the above diagram, in terms of function fields of the curve $s$. Since $X_E(2)$ and $X_E(3)$ are both isomorphic to $\mathbb{P}^1$ over $\mathbb{Q}$ (for explicit parametrization one refers back to Theorem 2.5 and 2.6), so we identify the function field of $X_E(2)$ with $\mathbb{Q}(u)$ and the function field of $X_E(3)$ with $\mathbb{Q}(\lambda)$, in the sense that in Theorem 2.5 we take the affine piece of $\mathbb{P}^1$ with coordinate $(u : v)$ by setting $v = 1$ and in Theorem 2.6 we take the affine piece of $\mathbb{P}^1$ with coordinate $(\lambda : \mu)$ by setting $\mu = 1$.

Fix an elliptic curve $E$. Suppose $F$ is an elliptic curve which is directly $3$-congruent to $E$. If $F$ is also $2$-congruent to $E$, then by Theorem 2.5, the ratio $\Delta_F/\Delta_E$ is a rational square. Thus, we consider the family of curves which are directly $3$-congruent to $E$, whose discriminant differs from $\Delta_E$ by a rational square. Then the curve parameterizing families of these curves is given by a quotient of $X_E(6)$, with function fields $\mathbb{Q}(\lambda, y)$ such that

$$\Delta_E y^2 = \Delta_{E, \mu}$$

where $E_{\lambda, \mu}$ is the elliptic curve with parameters $\lambda, \mu$ as in Theorem 2.6. Further, by considering the degree of the extension of function fields and some explicit computations, we obtain the following proposition

**Proposition 3.4.** Let $E$ have Weierstrass equation $y^2 = x^3 + ax + b$. Then a model of $X$ in weighted projective space is given by

$$C_X : y^2 = \lambda^4 + 2a\lambda^2\mu^2 + 4b\lambda\mu^3 - \frac{1}{3}a^2\mu^4$$

with $\lambda, \mu, y$ having weights $1, 1, 2$ respectively. The morphism from $X$ to $X_E(3)$ by taking the model $C_X$ is

$$C_X \to X_E(3), \ (\lambda : \mu : y) \mapsto (\lambda/3 : \mu).$$

An affine version of the map, is

$$C_X \to X_E(3), \ (\lambda, y) \mapsto \lambda/3 \text{ if } \mu = 1 \quad \text{and} \quad (1 : 0 : \pm 1) \mapsto (1 : 0).$$

**Proof.** We will verify this curve is indeed the same as $X$ in the last step. We firstly compute a model for the curve $C$ which parameterizes elliptic curves which are directly $3$-congruent to $E$ whose discriminant differs from $\Delta_E$ by a rational square.
Let \( E_{y,\lambda,\mu} \) be a curve parametrized by \( X \), which is directly 3-congruent to \( E \), whose discriminant differs \( \Delta_E \) by a rational square \( y^2 \), and thus, we have

\[
C : \Delta_E y^2 = \Delta_{E_{y,\lambda,\mu}} = \Delta_E \left( \lambda^4 + \frac{2a}{9} \lambda^2 \mu^2 + \frac{4}{27} b \lambda \mu^3 - \frac{1}{243} a^2 \mu^4 \right)^3.
\]

Cancelling \( \Delta_E \) on both sides yields the first equality. Writing above equation in the form

\[
81 y^2 \left( \lambda^4 + \frac{2a}{9} \lambda^2 \mu^2 + \frac{4}{27} b \lambda \mu^3 - \frac{1}{243} a^2 \mu^4 \right)^2 = (3\lambda)^4 + 2a(3\lambda)^2 \mu^2 + 4b(3\lambda) \mu^3 - \frac{1}{3} a^2 \mu^4
\]

we see the curve is isomorphic to \( C_X \).

Finally, by construction the morphism from \( C \) to \( X_E(3) \) is given by \( E_{y,\lambda,\mu} \mapsto E_{\lambda,\mu} \) and the morphism from \( C_X \to C \) is given by

\[
(\lambda : \mu : y) \mapsto \left( \frac{\lambda}{3} : \mu : \frac{y}{9} \left( \lambda^4 + \frac{2a}{9} \lambda^2 \mu^2 + \frac{4}{27} b \lambda \mu^3 - \frac{1}{243} a^2 \mu^4 \right) \right)
\]

and therefore, we have the correct morphism \( C_X \to X_E(3) \) as described in the proposition.

To see \( C_X \) is the same curve as \( X \) (which means isomorphic to \( X \) over \( \mathbb{Q} \)), we consider the degree of the function fields. Let \( F(C_X), F(X) \) be the function fields of \( C_X \) and \( X \) respectively. Then they are both subfields of \( F(X_E(6)) \) which is the function field of \( X_E(6) \). The extension \( F(X_E(6))/F(X_E(3)) = F(X_E(6))/\mathbb{Q}(\lambda) \) has degree 6 and the Galois group is \( \text{PSL}_2(\mathbb{Z}/2\mathbb{Z}) \) with a unique subgroup of order 3. Further, both extensions \( F(C_X)/F(X_E(3)) \) and \( F(X)/F(X_E(3)) \) have degree 2 and so

\[
[F(X_E(6)) : F(X)] = [F(X_E(6)) : F(C_X)] = 3.
\]

Thus, \( F(X) = F(C_X) \) by Galois theory and by the identification of smooth projective curves with their function fields, we conclude that \( X \) is isomorphic to \( C_X \) over \( \mathbb{Q} \).

The method of finding a model of \( Y \) is similar. We start with \( X_E(2) \) and consider an elliptic curve \( F \) with the same mod 2 representation as \( E \). If \( F \) is also directly 3-congruent to \( E \), then it corresponds to \( E_{\lambda,\mu} \) for some \( \lambda, \mu \in X_E(3) \). Thus, by a direct computation, we have

\[
\Delta_F = \Delta_{E_{\lambda,\mu}} = \Delta_E \left( \lambda^4 + \frac{2a}{9} \lambda^2 \mu^2 + \frac{4}{27} b \lambda \mu^3 - \frac{1}{243} a^2 \mu^4 \right)^3,
\]

and hence \( \Delta_F \) differs from \( \Delta_E \) by a rational cube.
Note that, for \((u : v) \in \mathbb{P}^1(\mathbb{Q})\), the explicit equations
\[ F_{u,v} : y^2 = x^3 + 3(3av^2 + 9bw - a^2u^2)x + 27bv^3 - 18a^2uv^2 - 27abu^2v - (2a^3 + 27b^2)u^3 \]
parameterizes family of elliptic curves, up to a quadratic twist (instead of \(\mathbb{Q}\)-isomorphism), which have the same mod 2 representation of \(E\). Thus we now consider a modular curve which parameterizes families of elliptic curves with the same mod 2 representation as \(E\), up to a quadratic twist, whose discriminant differs from \(\Delta_E\) by a rational cube. From Theorem 2.5, the discriminant of \(F_{u,v}\) is given by
\[ \Delta(F_{u,v}) = 3^6(v^3 + au^2v + bu^3)^2\Delta_E. \]
Thus if \(\Delta(F_{u,v})\) differs from \(\Delta_E\) by a cube say \(y^3\), then we obtain the following proposition

**Proposition 3.5.** Let \(E\) be an elliptic curve with Weierstrass model \(E : y^2 = x^3 + ax + b\). Then a model of \(Y \subset \mathbb{P}^2\) is
\[ C_Y : y^3 = v^3 + au^2v + bu^3 \]
and the morphism \(Y \to X_E(2)\) is \((u : v : y) \mapsto (u : v)\).

**Proof.** Following the argument and the description of the modular curve above we have \(y^3\Delta_E = \Delta_{F_{u,v}}\) and so
\[ \left( \frac{9(v^3 + au^2v + bu^3)}{y} \right)^3 = v^3 + au^2v + bu^3. \]
This is clearly birational to \(C_Y\) by
\[ (u : v : y) \mapsto (u : v : 9(v^3 + au^2v + bu^3)/y). \]
Finally this curve is birational to \(Y\) over \(\mathbb{Q}\) by considering the function field and a similar argument as in Proposition 3.4.

To find a model of \(Z\), we need to recall the diagram
\[
\begin{array}{cccccc}
X_E(6) & \xrightarrow{x^2} & Y & \xrightarrow{\rho_2} & X_E(2) \\
\downarrow x^3 & & \downarrow \alpha & & \downarrow \gamma_1 \\
X & \xrightarrow{\beta} & Z & \xrightarrow{\gamma_2} & X_E(2)/H \\
\downarrow \rho_3 & & \downarrow \gamma_3 & & \downarrow \gamma_4 \\
X_E(3) & \xrightarrow{\gamma_5} & X_E(3)/H' & \xrightarrow{\gamma_6} & X(1) \\
\end{array}
\]
We will show that $Z$ is the Jacobian of $X$ by the following two lemmas.

**Lemma 3.6.** Recall the map $\beta$ is the quotient map by $H' = C_2 \times C_2$ as a subgroup in $\text{PSL}_2(\mathbb{Z}/3\mathbb{Z})$. Working over $\overline{\mathbb{Q}}$ and identifying $X$ with its Jacobian $J_X$, we conclude the action of $H'$ on $J_X$ is translation by 2-torsion points of $J_X$. Further, the quotient map $J_X \to Z$ by the action of $H'$ is multiplication-by-2 map in the sense that we can identify $J_X \cong Z$ over $\overline{\mathbb{Q}}$.

By exactly the same argument, we conclude the map $X_{E(6)} \to Y$ is multiplication-by-2 map when we define $Y \cong X_{E(6)}$ over $\overline{\mathbb{Q}}$.

**Proof.** Identify $X$ with $J_X$ over $\overline{\mathbb{Q}}$ (as $X$ has genus one). For each element $f \in H'$, the action on $J_X$ induces an automorphism $f : J_X \to J_X$.

We have shown the map $X \to Z$ is unramified and so does $J_X \to Z$. As it is induced by the quotient of the action of $H'$ on $J_X$ thus we conclude that $\forall f \in H'$, $f$ does not fix any point of $J_X$, otherwise the quotient has non-trivial ramification index at the fixed point. Now consider the map $f - 1$ where 1 is the identity map on $J_X$ and so $(f - 1)(P) \neq O$ for all $P \in J_X$.

Any morphism over $\overline{\mathbb{Q}}$ between smooth projective curves is either surjective or constant. Hence we conclude $f - 1$ is constant and thus $f$ is translation on $J_X$. Since $f$ has order 2 so the action is translation by 2-torsion points.

Now we consider the quotient map $J_X \to Z$ or in fact $J_X \to J_Z$ where we identify $Z$ with its Jacobian $J_Z$ over $\overline{\mathbb{Q}}$. As the action of $H'$ on $J_X$ is translation by 2-torsion points, we conclude that the kernel of $J_X \to J_Z$ is $J_X[2]$. By Proposition 4.12 of [AEC], there is a unique elliptic curve (unique up to $\overline{\mathbb{Q}}$-isomorphism) $E'$ and a separable isogeny

$$\phi : J_X \to E'$$

such that $\ker \phi = J_X[2]$ and in our notation, $E'$ is $\mathbb{Q}$-isomorphic to $Z$.

The morphism

$$J_X \to J_X, \quad P \mapsto 2P$$

is separable and has kernel $J_X[2]$ and thus we conclude that $Z$ is isomorphic to $J_X$ over $\overline{\mathbb{Q}}$ and the map $\beta$ is multiplication-by-2 map over $\overline{\mathbb{Q}}$. $\square$

**Lemma 3.7.** The curve $Z$ has a $\mathbb{Q}$-rational point and thus it is an elliptic curve and in particular, $Z$ is the Jacobian of $X$. A model of $Z \subset \mathbb{P}^2$ is given by

$$y^2z = x^3 - 27\Delta_Ez^3.$$
**Proof.** Take the model $C_X$ for $X$ which is given by

$$y^2 = \lambda^4 + 2\lambda^2 \mu^2 + 4b \lambda \mu^3 - \frac{1}{3} a^2 \mu^4 = p(\lambda, \mu)$$

and let $t_i, i = 1, 2, 3, 4$ be the roots of the polynomial $p(\lambda, 1)$. The 2-torsion points of $J_X$ corresponds to $(t_i, 0), i = 1, 2, 3, 4$ as if we move one of $(t_i, 0)$ to infinity then the model $C_X$ becomes the standard Weierstrass model. Alternatively one can check this by referring to section 5.1 of [AKM3P]. Since the action of $H'$ on $X$ permutes $(t_i, 0), i = 1, 2, 3, 4$ and hence the quotient map $\beta$ sends $(t_i, 0), i = 1, 2, 3, 4$ to the same image, say $T = \beta((t_i, 0))$.

Let $\sigma \in G_{\bar{Q}}$. Since $\beta$ is defined over $\bar{Q}$, we have

$$\sigma(T) = \sigma(\beta(t_i, 0)) = \beta(\sigma(t_i, 0)) = \beta((t_j, 0)) = T, \quad i, j = 1, 2, 3, 4.$$ 

Thus, $T$ is fixed by $\sigma$ and since $\sigma$ is arbitrary we conclude that $T$ is fixed by $G_{\bar{Q}}$ and hence a $\bar{Q}$-rational point. Thus $Z$ is an elliptic curve with identity $T$ and by previous lemma, over $\bar{Q}$ $Z$ is isomorphic to $J_X$ and thus we may identify $Z$ with the Jacobian of $X$. The model in the statement of $Z$ is an immediate consequence of direct computation, by taking the Jacobian of $C_X$ where the standard formula can be found in 3.1 of [AKM3P].

**Remark 3.8.** Lemma 3.6 shows that $Y$ is the Jacobian of $X_{E}(6)$. Recall that $X_{E}(6)$ always admits a rational point which corresponds to $E$ itself. Therefore $X_{E}(6)$ is isomorphic to $Y$ over $\bar{Q}$. Lemma 3.7 is obvious in the sense that the curve $X$ admits a rational point which corresponds to the curve $E$ itself and the image of that point on $Z$ is again rational. But for the purpose of the reverse case where the analogue of the curve $X$ does not always admit a $\bar{Q}$-rational point, we will need Lemma 3.7 (see section 4 for detail).

An immediate consequence of the above remark is

**Theorem 3.9.** A model of $X_{E}(6) \subset \mathbb{P}^2$ is given by

$$y^2 z = x^3 + \Delta_E z^3.$$ 

**Proof.** Taking the Jacobian of the model $C_Y$ of $Y$ and using the formula in 3.2 of [AKM3P], we obtained

$$y^2 z = x^3 + \Delta_E z^3$$ 

and by the above remark $X_{E}(6)$ is isomorphic to $Y$ over $\bar{Q}$. Hence a model of $X_{E}(6)$ is given by

$$y^2 z = x^3 + \Delta_E z^3.$$ 

We have now computed a model for $X_{E}(6)$ but we are more interested in the explicit parametrization of this modular curve. We will illustrate this in the next subsection where we give an explicit formula.
3.4 Explicit Equations

The main idea is to compute the morphism $X_E(6) \to X_E(3)$ explicitly. We begin with the following lemma.

**Lemma 3.10.** The map $\chi_3 : X_E(6) \to X$ is 3-isogeny over $\overline{\mathbb{Q}}$.

**Proof.** Over $\overline{\mathbb{Q}}$ any morphism between genus one curves is an isogeny because we simply match up the identity points under the morphism. Further, the map $\chi_3$ has degree 3 and so it is a 3-isogeny.

We now need to give an explicit equation of the map $\chi_3$. Firstly, recall from Proposition 3.4, that the map $\rho_3 : C_X \to X_E(3)$ is

$$(\lambda, y) \mapsto \lambda/3, \quad \text{if } \mu = 1, \quad \text{and} \quad (1 : 0 : \pm 1) \mapsto (1 : 0).$$

We want to take $C_X$ as the model of $X$ and gives a morphism $X_E(6) \to C_X$ such that the composition is the forgetful morphism $X_E(6) \to X_E(3)$. Recall from Theorem 3.9 our convention for the model

$$y^2z = x^3 + \Delta_E z^3$$

of $X_E(6)$ is we identify the point at infinity $(0 : 1 : 0)$ as the isomorphism class containing $E$. Since $\chi_3$ is geometrically a 3-isogeny, so the image of $(0 : 1 : 0)$ under $\chi_3$ should also corresponds to $E$, under the parametrization of $X$. Finally, the image under $\rho_3$ should send the point corresponding to $E$ on $X$ to the point corresponding to $E$ on $X_E(3)$.

From Theorem 2.6, the point corresponding to $E$ on $X_E(3)$ is the point at infinity $(\lambda : \mu) = (1 : 0)$. Thus, the point corresponding to $E$ on $X$ should be one of the preimages of $(1 : 0)$, namely $(\lambda : \mu : y) = (1 : 0 : \pm 1)$. We want to match up the points on $X_E(6)$ and $X$ which correspond to $E$ itself.

Our strategy to compute the explicit equation of the forgetful morphism is the following: we firstly compute a map $f$ which is a 3-isogeny from $X_E(6)$ to $J_X$. Then compute an isomorphism $g$ which sends $J_X$ to $X$ in a way such that the identity point $O$ on $J_X$ is sent to one of $(1 : 0 : \pm 1)$. Finally, take the quotient map $X \to X_E(3)$ by using Proposition 3.4. Thus we need to decide which one of $(1 : 0 : \pm 1)$ on $X$ is $O \in J_X$ sent to. In this case, it does not matter which one we pick because if $E$ corresponds to $(1 : 0 : 1)$ on $X$, then we compose $\chi_3$ with the automorphism $[-1]$ on $X_E(6)$, we would have $E$ corresponding to $(1 : 0 : -1)$ on $X$. We take $(1 : 0 : 1)$ for convention.

Another issue we need to consider is Lemma 3.10 is proved in terms of function fields and thus it is possible that $\chi_3$ differs from the usual 3-isogeny by an automorphism $\chi'$ of $X$. However, as in this case we fixes the
convention that the point of infinity should always corresponds to $E$ itself and so the possible automorphism $\chi'$ will be those which fix the the identity point of the Weierstrass form of $X$. Thus there are six possible automorphisms, generated by $[\zeta_6]$. But as $X_E(6)$ itself is a curve of $j$-invariant 0, composing $\chi_3$ with an automorphism $[\zeta_6]$ is the same as applying an automorphism $[\zeta_6]^3$ on $X_E(6)$, followed by $\chi_3$. In other words, the following diagram commutes

$$
\begin{array}{ccc}
X_E(6) & \xrightarrow{[\zeta_6]} & X_E(6) \\
\downarrow \chi_3 & & \downarrow \chi_3 \\
X & \xrightarrow{[\zeta_6]} & X
\end{array}
$$

It does not matter if our resulting morphisms differ from the required one by an automorphism on $X_E(6)$ because it will just change the coordinate of $X_E(6)$.

Thus, following the above argument and computational detail, we are ready to state the main theorem of the section

**Theorem 3.11.** Let $E$ be an elliptic curve with Weierstrass equation $E : y^2 = x^3 + ax + b$. The point $O$ on $X_E(6)$ corresponds to $E$, and we consider the affine model $y^2 = x^3 + \Delta_E$ of $X_E(6)$ where $\Delta_E = -16(4a^3 + 27b^2)$. For $(x, y) \in X_E(6)$, let

$$
E_{x,y} : y^2 = x^3 + AX + B
$$

where we define

$$
v = \frac{-\frac{1}{6}x^3y - 18bx^3 + \frac{4\Delta_E}{3}y}{x^4 + 12ax^3 + 4\Delta_Ex}, \quad \lambda = v/3, \quad c_4 = -a/27, \quad c_6 = -b/54,$$

$$
c_4 = c_4\lambda^4 + 4c_6\lambda^3 + 6c_4^2\lambda^2 + 4c_4c_6\lambda - (3c_4^3 - 4c_6^2),$$

$$
c_6 = c_6\lambda^6 + 6c_4^2\lambda^5 + 15c_4c_6\lambda^4 + 20c_4^2\lambda^3 + 15c_4c_6\lambda^2 + 6(3c_4^3 - 2c_4c_6^2)\lambda + (9c_4^3c_6 - 8c_6^3),$$

$$
A = -27c_4, \quad B = -54c_6.
$$

Then the families of elliptic curves which are directly 6-congruent to $E$ are given by $E_{x,y}$.

**Proof.** The proof basically follows from the argument above. We give some computational detail. The map

$$
(x, y) \mapsto \left(\frac{x^3 + 4\Delta_E}{x^2}, \frac{x^3y - 8\Delta_Ey}{x^3}\right)
$$
is a 3-isogeny and the map

\[ J_X \rightarrow X, \quad (x, y) \mapsto \left( \frac{-\frac{b}{6} - 18b}{x + 12a}, \frac{1}{18}x^3 + ax^2 - \frac{1}{36}y^2 - 6by - 48a^3 - 324b^2}{(x + 12a)^2} \right) \]

where \( J_X \) has equation \( y^2 = x^3 - 27\Delta_E \), is an affine version of the map from the Weierstrass form of \( X \) to \( X \) which sends \( O \) to \( (1 : 0 : 1) \), using the standard formula for taking a binary quartic to a Weierstrass form. Taking composition we have the map \( \chi_3 \). Then since we are only interested in the first coordinate, and so taking the image of the first coordinate we have a map \( (x, y) \rightarrow v \) where \( v \) is defined in the statement. Finally, taking \( v/3 \) we have the forgetful morphism

\[ X_E(6) \rightarrow X_E(3), \quad (x, y) \mapsto v/3 = \lambda. \]

Then use the parametrization of \( X_E(3) \) from Theorem 2.6 and take the relevant polynomials \( c_4, c_6 \) with \( \mu = 1 \). Finally, substitute the value of \( \lambda \) into \( c_4 \) and \( c_6 \).

**Remark 3.12.**

1. In fact, the explicit actions of \( H, H' \) on \( X_E(6) \) with the affine model were computed in [P] which agrees with our conclusion.

2. One can similarly use the forgetful morphism \( X_E(6) \rightarrow X_E(2) \) to give explicit parametrization. However, in that case, one needs to be careful about the fact mod 2 representation is unchanged by taking quadratic twist, and therefore one might need to pick a suitable quadratic twist in the final step of the calculation if the morphism \( X_E(6) \rightarrow X_E(2) \) is used.

3. The observation that \( X, Y \) have rational points is immediate by investigating the models \( C_X, C_Y \). However, for the purpose of the next section where we deduce corresponding results for the reverse congruent case, we gave a more detailed argument and did not insist \( X, Y \) being elliptic curves in the first place.

4. Compared to the results in [R], the formula we got is a bit simpler and works for all \( j \)-invariant.
4 Explicit Families of Reverse 6-Congruent Elliptic Curves

We now use a similar approach to compute a model for $X_E^{-1}(6)$. In fact if we only want a model of the Jacobian of $X_E^{-1}(6)$ then we can modify the method used in [P]. The idea is to identify each point of $X(6)$ by a pair $(E', \phi)$ where $E'$ is an elliptic curve and $\phi : E'[6] \cong \mathbb{Z}/6\mathbb{Z} \times \mu_6$ such that $\phi$ takes the Weil pairing to $\langle \cdot, \cdot \rangle^{-1}$ where $\langle \cdot, \cdot \rangle$ is the bilinear pairing defined as

$$\langle (a_1, \zeta_1), (a_2, \zeta_2) \rangle = \zeta_1^{a_2}/\zeta_2^{a_1}.$$ 

But we will see later that $X_E^{-1}(6)$ in general does not necessarily have a $\mathbb{Q}$-rational point so we will use the method described in the previous section to derive a model of $X_E^{-1}(6)$. In theory one could achieve a model of $X_E^{-1}(6)$ as a fiber product of $X_E(2)$ and $X_E^{-1}(3)$ which is complicated and we do insist to find as simple model as possible to study the structure of the curve.

4.1 Setup

We establish a commutative diagram as in the previous section as follows:

$$
\begin{array}{cccccc}
X_E^{-}(6) & \xrightarrow{\chi_2} & Y & \xrightarrow{\rho_2} & X_E^{-}(2) \\
\downarrow \chi_3 & & \downarrow \alpha & & \downarrow \gamma_1 \\
X^{-} & \xrightarrow{\beta} & Z^{-} & \xrightarrow{\gamma_2} & X_E^{-}(2)/H \\
\downarrow \rho_3 & & \downarrow \gamma_3 & & \downarrow \gamma_4 \\
X_E^{-}(3) & \xrightarrow{\gamma_5} & X_E^{-}(3)/H' & \xrightarrow{\gamma_6} & X_0(1) \\
\end{array}
$$

Note that $X_E^{-}(2)$ is the same as $X_E(2)$ and $H, H'$ are the same as in the previous section and we define the quotient maps in a similar way as in the previous section. We now compute models for $X^-, Y^-$ and $Z^-.$

**Proposition 4.1.** Let $E$ have Weierstrass equation $y^2 = x^3 + ax + b$. Then a model of $X^-$ in weighted projective space is given by

$$C_{X^-} : y^2 = \Delta_E(a\lambda^4 + 6b\lambda^3\mu - 2a^2\lambda^2\mu^2 - 2ab\lambda\mu^3 + (-a^3/3 - 3b^2)\mu^4).$$

The morphism from $X^-$ to $X_E(3)$ by taking the model $C_{X^-}$ is

$$C_{X^-} \rightarrow X_E^{-}(3), \quad (\lambda : \mu : y) \mapsto (\lambda/3 : \mu).$$
An affine version of the map is

\[ C_X^- \rightarrow X_E^-(3), \quad (\lambda, y) \mapsto \lambda/3, \text{ if } \mu = 1 \text{ and } (1 : 0 : \pm \sqrt{a \Delta_E}) \mapsto (1 : 0). \]

**Proof.** The proof is similar to that of Proposition 3.4. Note that \( X^- \) parameterizes families of elliptic curves which are reverse 3-congruent to \( E \) whose discriminant differs from \( \Delta_E \) by a rational square, and the argument is exactly the same as in the proof of Proposition 3.4. Thus, by using the explicit parametrization of \( X_E^-(3) \) as in Theorem 2.6, a model for \( X^- \) in weighted projective space \((\lambda : \mu : y)\) is

\[ C : y^2 \Delta = \frac{2^{36}3^{36}a^2}{\Delta_E^4}h(\lambda, \mu)^3, \]

where

\[ h(\lambda, \mu) = a\lambda^4 + 2b\lambda^3\mu - \frac{2}{9}a^2\lambda^2\mu^2 - \frac{2}{27}ab\lambda\mu^3 + \left(-\frac{1}{243}a^3 - \frac{1}{27}b^2\right)\mu^4, \]

Putting \( C \) in the form

\[ \left(\frac{y\Delta_E^3}{2^{18}3^{16}ah(\lambda, \mu)}\right)^2 = \Delta_E((3\lambda)^4 + 6b(3\lambda)^3\mu - 2a^2(3\lambda^2)\mu^2 - 2ab(3\lambda)\mu^3 + \left(-a^3/3 - 3b^2\right)\mu^4), \]

we see that \( C \) is birational to \( C_X^- \) by

\[ (\lambda : \mu : y) \mapsto \left(3\lambda : \mu : \frac{y\Delta_E^3}{2^{18}3^{16}ah(\lambda, \mu)}\right). \]

The morphism from \( C \) to \( X_E^-(3) \) is \((\lambda : \mu : y) \mapsto (\lambda : \mu)\) so we have the correct morphism from \( C_X^- \) to \( X_E^-(3) \) as described in the statement. \( \square \)

To compute a model of \( Y \), observe that if elliptic curves are reverse 3-congruent, then the product of their discriminants is a rational cube. In fact, this is immediate from the computation of \( C \) in Proposition 4.1. Thus, by considering the degree of extension of the function fields (which is exactly the same argument we used in the previous section), we conclude that \( Y^- \) parameterizes families of elliptic curves (up to \( \overline{\mathbb{Q}} \)-isomorphism as we need to take care of the quadratic twists) which are 2-congruent to \( E \) whose discriminant multiplied by \( \Delta_E \) is a cube. Thus, we obtain the following:
Proposition 4.2. Let $E$ be an elliptic curve with Weierstrass model $E : y^2 = x^3 + ax + b$. Then a model of $Y^- \subset \mathbb{P}^2$ is birational to

$$C_{Y^-} : y^3 = \Delta_E(v^3 + au^2v + bu^3)$$

and the morphism $Y^- \to X_E(2)$ is $(u : v : y) \mapsto (u : v)$.

Proof. Using the explicit equations for $X_E(2)$ as in Theorem 2.5 and the argument above, we have

$$y^3 = \Delta_E \Delta_{F_{u,v}}$$

which gives

$$\left(\frac{9\Delta_E(v^3 + au^2v + bu^3)}{y}\right)^3 = (v^3 + au^2v + bu^3)^2 \Delta_E,$$

which is birational to $C_{Y^-}$ by

$$(u : v : y) \mapsto (u : v : 9\Delta_E(v^3 + au^2v + bu^3)/y).$$

We recall what we have shown in the previous section. The map $\beta : X \to Z$ is the quotient map by the action of $H$ in $\text{PSL}_2(\mathbb{Z}/2\mathbb{Z})$ and we proved this is geometrically multiplication-by-2 map by investigating the action explicitly. We used the fact that the map $\beta$ is unramified and deduce that the action is translation by 2-torsion points and hence determine the kernel of $\beta$. We have not used any properties of the direct congruence. Thus, we have the same situation here, that geometrically (which means over $\bar{\mathbb{Q}}$) the map $\beta^-$ is multiplication-by-2 map and similarly, the map $\alpha^-$ is 3-isogeny.

This shows that $Z^-$ has the same Jacobian as $X^-$, which is 3-isogenous to the Jacobian of $Y^-$. Similarly, $\chi^-_2$ is multiplication-by-2 map and $\chi^-_3$ is multiplication-by-3 map and the Jacobian of $X^-_E(6)$ is the same as the Jacobian of $Y^-$. Therefore,

Corollary 4.3. The Jacobian of $X^-_E(6)$ is isomorphic (over $\mathbb{Q}$) to

$$y^2z = x^3 + \frac{1}{\Delta_E}z^3.$$ 

Proof. This is an immediate calculation by using $C_{Y^-}$ as the model of $Y^-$, using reference in 3.2 of [AKM\textsuperscript{P}].

Recall from Lemma 3.7 that $Z$ has a rational point. The proof only uses the correspondence of points on $X$ with the 2-torsion points on the Jacobian. We have exactly the same situation here and so by the same argument we conclude that $Z^-$ has a rational point. Thus, $Z^-$ is an elliptic curve and
Proposition 4.4. A model of $Z^-$ is given by

$$y^2z = x^3 - \frac{27}{\Delta_E} z^3.$$  

Proof. This follows immediately from the computation of the Jacobian of $X^-$.  

4.2 Explicit Model

We now illustrate how to obtain an explicit model for $X^-_E(6)$. In this previous section, we compute a model of $X_E(6)$ by computing its Jacobian and since $X_E(6)$ always has a rational point which corresponds to $E$ itself, we can then give a model of $X_E(6)$. However, this is not the case in the reverse case. For example, if $X^-_E(6)$ has a rational point, then under the quotient map, the curves $X^-, Y^-$ should also have rational points. We now give examples where $X^-$ and $Y^-$ have no rational point.

Example 4.5. Let $a = 1, b = 0$ then the curve $C_{X^-}$ is isomorphic (over $\mathbb{Q}$) to

$$3y^2 = -3x^4 + 6x^2 + 1.$$ 

This is not locally soluble at 3, and hence no rational solution.

Now consider $a = 0$ then the model of $C_{Y^-}$ is

$$y^3 = -16(27b^2)(v^3 + bu^3).$$

This is isomorphic to

$$y^3 = -2\frac{v^3}{b} - 2u^3.$$ 

There are a lot of possible values of $b$ such that this has no solution. For example, it is well-known (first proved by Legendre in 1808) that

$$x^3 + 4y^3 + z^3 = 0$$

only has trivial solution with $y = 0$ and hence no solution in $\mathbb{P}^2(\mathbb{Q})$. Thus if $b = 1$ and it is clear that the curve $2x^3 + 2z^3 + y^3 = 0$ is isomorphic to $x^3 + z^3 + 4y^3 = 0$, so this gives an example that $C_{Y^-}$ has no $\mathbb{Q}$-rational point.

To compute a model of $X^-_E(6)$, consider that $X^-$ is a 2-covering of $Z^-$ and $Y^-$ is a 3-isogeny covering of $Z^-$ and $X^-_E(6)$ is the fiber product of $X^-$ and $Y^-$ over $Z^-$ with respect to the quotient maps. We construct curves $X^-_1, Z^-_1$ such that $X^-_1$ is a 2-covering of $Z^-_1$ and $Y^-$ is a 3-covering of $Z^-_1$. then we apply the algorithms in [F3], which gives a way to compute a model.
for the 6-covering of an elliptic curve as the fiber product of a 2-covering and a 3-covering.

Our first step is to compute models for $X_1^−$ and $Z_1^−$. To apply the algorithms in [F3] we need $X_E(6)$ to be a 6-covering of $Z_1^−$ which means $Z_1^−$ is the Jacobian of $X_E(6)$, that is

$$y^2z = x^3 + \frac{1}{\Delta_E}z^3.$$ 

Then we need to find a curve $X_1^−$ such that the following commutes

$$\begin{array}{ccc} X_E(6) & \longrightarrow & Y^- \\
\downarrow & & \downarrow \\
X_1^- & \longrightarrow & Z_1^- 
\end{array}$$

Since $Z_1^−$ is already a 3-isogeny covering of $Z_1^−$ we can compute $X_1^−$ by making the following diagram commutes

$$\begin{array}{ccc} X_E(6) & \longrightarrow & Y^- \\
\downarrow & & \downarrow \\
X^- & \longrightarrow & Z^- \\
\downarrow & & \downarrow \\
X_1^- & \longrightarrow & Z_1^- 
\end{array}$$

where the map $Z^- \rightarrow Z_1^−$ is geometrically the dual isogeny of $Y^- \rightarrow Z_1^−$.

**Lemma 4.6.** A model of $X_1^−$ is given by the equation

$$-3y^2 = \Delta_E(a\lambda^4 + 6b\lambda^3\mu - 2a^2\lambda^2\mu^2 - 2ab\lambda\mu^3 + (-a^3/3 - 3b^2)\mu^4).$$

Note that this only differs from the equation of $C_{X^-}$ by a factor of $-3$ on the left hand side.

**Proof.** Take the equation above and we show how to obtain the corresponding maps in the commutative diagram above. Note that the factor $-3$ comes from the fact that $Z_1^−$ is a quadratic twist of $Z^- \rightarrow -3$ and one checks that the Jacobian of $X_1^−$ is $Z_1^−$ by using formulae in 3.1 of [AKM^3P]. The map $\hat{\chi}_3 : X^- \rightarrow X_1^−$ is geometrically a 3-isogeny. Thus, it can be constructed by $\hat{\chi}_3 = I_2^{-1}fI_1$, where $f : Z^- \rightarrow Z_1^−$ is the usual isogeny between elliptic curves and the isomorphisms $I_1, I_2$, defined over $\mathbb{Q}$ are the ones which satisfy the following commutative diagrams:
where \( \pi_2, \pi'_2 \) are the 2-covering maps defined over \( \mathbb{Q} \). Then the resulting \( \hat{\chi}_3^- \) is geometrically a 3-isogeny and also satisfies the commutative diagram

\[
\begin{array}{ccc}
X^- & \xrightarrow{\pi_2} & Z^- \\
\downarrow I_1 & & \mid \\
Z^- & \xrightarrow{[2]} & Z^- \\
\end{array}
\]

\[
\begin{array}{ccc}
X_1^- & \xrightarrow{\pi'_2} & Z_1^- \\
\downarrow I_2 & & \mid \\
Z_1^- & \xrightarrow{[2]} & Z_1^- \\
\end{array}
\]

because

\[
\pi'_2 \hat{\chi}_3^- = [2](I_2 I_2^-)(f I_1) = ([2]f)I_1 = f([2]I_1) = f \pi_2
\]

as multiplication-by-2 map commutes with 3-isogeny.

Note that \( I_1 \) is not unique, for if we replace \( I_1 \) by \( T_i I_1 \) where \( T_i \) is translation by 2-torsion point then the resulting map will still satisfy the commutative diagram and similarly \( I_2 \) is not unique. Finally, we want the map \( \hat{\chi}_3^- \) to be defined over \( \mathbb{Q} \) and this is equivalent to

\[
I_2 \sigma(I_2^-) f \sigma(I_1^-) I_1^- = f.
\]

Explicitly, the map \( \sigma(I_1^-) I_1^- \) is a translation by some 2-torsion point of \( Z^- \) and \( I_2 \sigma(I_2^-) \) is a translation by some 2-torsion point of \( Z_1^- \). As \( f \) is a 3-isogeny, which induces an isomorphism between \( Z^- [2] \) and \( Z_1^- [2] \), we conclude that this holds if we pick the suitable choices for \( I_1 \) and \( I_2 \) in the sense we match up the coclyces (which means we match up the 2-torsion points induced by the coclyces).

One can explicitly compute the map \( \hat{\chi}_3^- \) by the method above but the resulting morphism turned out to be very complicated so we do not include the computational detail here.

Now we work through the algorithms in [F3]. Following the notation in [F3], we firstly embed our 2-covering in \( \mathbb{P}^5 \) with coordinates \( x_1, x_2, x_3, x_4, x_5, x_6 \) in the way

\[
\begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
x_6
\end{pmatrix}
\]
such that the action of translation by 2-torsion points is multiplication by
certain 2-by-2 matrices on the left and the action of translation by 3-torsion
points is multiplication by certain 3-by-3 matrices on the right. The algo-
rum to do this is described in Proposition 5.1 [F3].

Fix an elliptic curve $E$ with Weierstrass equation $y^2 = x^3 + ax + b$ and
the 2-covering we have here is the one we got from Lemma 4.6, namely

$$-3y^2 = \Delta_E(a\lambda^4 + 6b\lambda^3\mu - 2a^2\lambda^2\mu^2 - 2ab\lambda\mu^3 + (-a^3/3 - 3b^2)\mu^4).$$

In the notation of Proposition 5.1 [F3], we have

$$U(\lambda, \mu) = -3\Delta_E(a\lambda^4 + 6b\lambda^3\mu - 2a^2\lambda^2\mu^2 - 2ab\lambda\mu^3 + (-a^3/3 - 3b^2)\mu^4).$$

The invariant $H$ is given in Definition 4.1 [F3], which is

$$H = \frac{1}{3} \det \left( \frac{\partial^2 U}{\partial \lambda \partial \mu} \right).$$

Thus we can compute the image $A_{2,3}$ as in Proposition 5.1 [F3]. Explicitly,
we have

$$A_{2,3} = \begin{pmatrix} -\frac{9\partial H}{\partial \mu} & -\frac{3\partial U}{\partial \mu} & \lambda y \\ \frac{\partial^2 U}{\partial \lambda \partial \mu} & \frac{\partial^2 U}{\partial \mu^2} & \mu y \end{pmatrix}.$$

It is well-known that a genus one curve in $\mathbb{P}^5$ embedded by a complete
linear system of degree 6 is given by the intersection of 9 quadrics, and we
are free to replace each of them by some invertible linear transformation
of them. To work out the quadrics, take a polynomial ring of 6 variables and
list all 21 monomials of degree 2 and call this set $T$. Then evaluate each
monomial in $T$ at entries $x_i$ above where $x_i$ will be given in terms of $y, \lambda$ and
$\mu$. Further, one can reduce these modulo $I$, where $I$ is the ideal given by
$\langle y^2 - U(\lambda, \mu) \rangle$. Now take the list of all the monomials in $y, \lambda, \mu$ which appear
after the reduction, and call this set $S$. Finally we work out the kernel of the
matrix, whose $i, j$-th entry is the coefficient of the $j$-th monomial in $S$ in the
$i$-th monomial of degree 2 in $T$.

One can apply invertible linear transformation to tidy up the resulting
equations for the quadrics. We will give our result and readers can check
they span the same 9-dimensional vector space by the method above. The
resulting 9-quadrics are given by:

\[
q_1 = -x_1x_4 + 8aD^3x_2x_5 - 3D^5x_3x_6 + 12bD^3x_5^2,
\]
\[
q_2 = x_1x_5 + 72bDx_2^2 + x_2x_4 - 32a^2Dx_2x_5 + 24aD^3x_3x_6
- 24abDx_5^2 + 36bD^3x_6^2,
\]
\[
q_3 = -72bDx_1x_5 - 72bDx_2x_4 - 576a^2D^4x_3x_6^2 - 1728abD^4x_3x_6 - x_4^2
+ 32a^2Dx_4x_5 - 8aD^3x_5^2 - 1296b^2D^4x_6^2,
\]
\[
q_4 = -24Dax_2x_3 - 36bDx_2x_6 - 36bDx_3x_5 + x_4x_6 + 8a^2Dx_5x_6,
\]
\[
q_5 = -3x_1^2 + 9D^5x_3^2 - ax_4^2 - D^2x_4x_5 + 3aD^5x_6^2,
\]
\[
q_6 = 24aDx_2^2 + 72bDx_2x_5 + 18D^3x_3^2 - x_4x_5 - 8a^2Dx_5^2 + 6aD^3x_6^2,
\]
\[
q_7 = x_1x_6 + 72bDx_2x_3 - 16a^2Dx_2x_6 + x_3x_4 - 16a^2Dx_3x_5 - 24abDx_5x_6,
\]
\[
q_8 = -3x_1x_2 + 36D^3ax_3^2 + 108bD^2x_3x_6 - ax_4x_5 + D^2x_5^2/2 - 12a^2D^3x_6^2,
\]
\[
q_9 = -3x_1x_3 - ax_4x_6 + D^2x_5x_6/2,
\]

where we use the notation \( D = \Delta_E = -16(4a^3 + 27b^2) \).

Then we follow the procedure [F3] section 6, and the curve \( X_E(6) \) is given by intersection of 9-quadrics in variables \( X_1, \ldots, X_6 \) where we have the relation

\[
\begin{pmatrix}
X_1 & X_2 & X_3 \\
X_4 & X_5 & X_6
\end{pmatrix} = 
\begin{pmatrix}
x_1 & x_2 & x_3 \\
x_4 & x_5 & x_6
\end{pmatrix} g
\]

and \( g \) is a change of coordinate matrix to put \( C_{Y^+} \) in Weierstrass form. The formula of \( g \) we use here can be found on page 24 in [F3]. The explicit algorithm to compute the flex matrix is given in Algorithm 6.2 and 6.3. Following the procedure, we obtain the inverse of the flex matrix (as in fact this is more useful for our calculation)

\[
g^{-1} = \begin{pmatrix}
-648D^2a\alpha - 972bD^2 & 0 & 1 \\
972bD^2\alpha - 216a^2D^2 & 0 & \alpha \\
0 & -18D\alpha^2 - 6aD & 0
\end{pmatrix},
\]

where \( \alpha \) is a root of \( x^3 + ax + b = 0 \).

Finally, as is described on page 24 in [F3], the new quadrics have coefficients in an extension of \( \mathbb{Q} \) but the space they span has a basis with coefficients in \( \mathbb{Q} \). So what we do is compute \( x_i \) in terms of \( X_i \) and then substitute these into the quadrics \( q_i \). Thus the new quadrics have coefficients in \( \mathbb{Q}(\alpha) \). But as we know they have a basis over \( \mathbb{Q} \) we collect the coefficients of \( \alpha^2, \alpha \) and the constant term instead, and the vanishing of them will give 3-quadrics. Thus, we obtain 27-quadrics with coefficients in \( \mathbb{Q} \) and we
covering is recovered by taking the 2-by-2 minors of the matrix 

\[ X \]

where our curve 

\[ \text{We compute the explicit morphism from } X \]

This gives a model for 

\[ X \]

equations which are reverse 6-congruent to the given one.

Then by using this morphism we can write down the family of explicit equation for the morphism we write down corresponds to the forgetful morphism. Then it can be checked by direct computation that the 27-quadrics obtained after twisted by the flex matrix span a 9-dimensional vector space, and a basis can be given by the following 9-quadrics (again after some linear transformation on the basis):

\[
\begin{align*}
s_1 &= -6x_1x_3 + 24ax_1x_6 - 6x_2^2 + 24ax_3x_3 - 6x_2x_4 + 24ax_3x_4 \\
&+ 72bx_4x_6 + 2ax_5^2 + 8a^2x_5x_6 + Dx_6^2, \\
s_2 &= -6x_1x_3 + x_2x_5 + 2ax_2x_6 - 36bx_3x_6 + 2ax_3x_5 + 16a^2x_3x_6 + x_4x_5 \\
&+ 2ax_4x_6 + 12abx_5^2, \\
s_3 &= 12ax_1x_3 + 18bx_1x_6 + 18bx_3x_4 - 2ax_4x_5 - 4a^2x_4x_6 + 3bx_5^2, \\
s_4 &= -12ax_2x_3 - 18bx_2x_6 - 18bx_3x_5 - 3x_4^2 - ax_5^2 + 4a^2x_5x_6, \\
s_5 &= 3x_2^2 - 48a^2x_3^2 - 144abx_3x_6 - 36bx_4x_6 + ax_5^2 - 8a^2x_3x_6 + 16a^3x_6^2, \\
s_6 &= -3x_1x_4 + 18bx_2x_3 - ax_2x_5 - 4a^2x_2x_6 - 4a^2x_3x_5 - 6abx_5x_6, \\
s_7 &= -108bx_1x_3 + 6ax_2^2 - 24a^2x_2x_3 + 18bx_2x_5 - 36abx_4x_6 - 2a^2x_5^2 \\
&- 8a^3x_5x_6 - aDx_6^2, \\
s_8 &= 3x_1x_2 - 72abx_3^2 - 216b^2x_3x_6 + ax_4x_5 + 8a^2x_4x_6 - 12abx_5x_6 + 24a^2bx_6^2, \\
s_9 &= 36x_1^2 + 12ax_2^2 + 12ax_4^2 + 4a^2x_5^2 + Dx_5x_6.
\end{align*}
\]

This gives a model for \( X^-(6) \) which proves the first part of Theorem 1.1.

### 4.3 Explicit Equations And Examples

We compute the explicit morphism from \( X^-(6) \) to \( X^-(3) \) in the sense the explicit equation for the morphism we write down corresponds to the forgetful morphism. Then by using this morphism we can write down the family of elliptic curves which are reverse 6-congruent to the given one.

In [F3], by Theorem 2.5 (or page 24), the map from the 6-covering to the 3-covering is recovered by taking the 2-by-2 minors of the matrix

\[
\begin{pmatrix}
x_1 & x_2 & x_3 \\
x_4 & x_5 & x_6
\end{pmatrix},
\]

where our curve \( X^-(6) \) is the intersection of 9 quadrics in \( \mathbb{P}^5 \) with coordinates \( x_1, \ldots, x_6 \). Thus, this gives a map from \( X^-(6) \) to \( Y^- \) by our construction:

\[
\chi^-(x_1 : \cdots : x_6) = (x_2x_6 - x_3x_5 : x_3x_4 - x_1x_6 : x_1x_5 - x_2x_4).
\]
We need to be careful about the matching of the coordinate here as we need to make the correct matching for the coordinate to make the morphism correct. By a direct computation, we obtain
\[(x_1 : \cdots : x_6) \mapsto (u : v : y), u = x_2 x_6 - x_3 x_5, v = x_3 x_4 - x_1 x_6, y = x_1 x_5 - x_2 x_4.\]

Our method to work out the forgetful morphism \(X_E^{-}(6) \to X_E^{-}(3)\) is based on the following observation. For any point \(P = (x_1 : \cdots : x_6) \in X_E^{-}(6)\), take the image \(\chi^{-2}(P)\) defined above and then recall the description of the map \(\rho^{-2}\) as in Proposition 4.2 we conclude that
\[X_E^{-}(6) \to X_E^{-}(2), \quad (x_1 : \cdots : x_6) \mapsto (u : v) = (x_2 x_6 - x_3 x_5 : x_3 x_4 - x_1 x_6)\]
is the forgetful morphism from \(X_E^{-}(6)\) to \(X_E^{-}(2)\) and let \(E'\) be a representative of the isomorphism class in \(X_E^{-}(2)\) which corresponds to
\[(u : v) = (x_2 x_6 - x_3 x_5 : x_3 x_4 - x_1 x_6).\]

Recall that each point in \(X_E^{-}(2)\) parameterizes family of elliptic curves (isomorphic over \(\overline{\mathbb{Q}}\)) which are congruent to the given one because taking quadratic twist does not change the mod 2 representation but it certainly does change the mod 3 representation. Thus, the image of \(P \in X_E^{-}(6)\) in \(X_E^{-}(3)\) must correspond to an elliptic curve which is a quadratic twist of \(E'\) and hence they have the same \(j\)-invariant. So we compute the \(j\)-invariant \(j(\lambda, \mu)\) of \(E'_{\lambda, \mu}\), the family of curves which are reverse 3-congruent to \(E\), as in Theorem 2.6, which is a rational function, homogenous in \(\lambda, \mu\) of degree 12. Finally, we solve the equation \(j(\lambda, \mu) = j(E')\) where \(j(E')\) will be given in terms of \(x_1, \ldots, x_6\).

**Theorem 4.7.** The forgetful morphism \(X_E^{-}(6) \to X_E^{-}(3)\) is given by
\[(x_1 : \cdots : x_6) \mapsto (x_3/3 : x_6).\]

**Proof.** We implement the above algorithm and obtain a degree 12 homogenous polynomial in \(\lambda, \mu\) and let \(L = \mathbb{Q}(a, b)\) where \(a, b\) are meant to be the coefficients of a general elliptic curve \(y^2 = x^3 + ax + b\).

Now if \((x_1 : \cdots : x_6)\) is a \(L\)-rational point then the image must also be \(L\)-rational. Thus, we look for a \(L\)-rational point at which the degree 12 polynomial vanishes. A direct computation shows that the polynomial is equal to zero at the point \((\lambda : \mu) = (x_3/3 : x_6)\) (recall the point \((x_1 : \cdots : x_6)\) satisfies the equation of \(X_E^{-}(6)\)).

Finally, to see there is only one \(L\)-rational point, we can implement the algorithm for some explicit values of \(a, b\) and factorize the polynomial directly. For example if we set \(a = 1, b = 1\) then there is only one linear factor and so is the case for general \(a, b\). \(\square\)
Remark 4.8. In principal one could also try to factorize the polynomial directly to get the linear factor for general $a, b$ by defining a function field $K$ over $\mathbb{Q}$ with two variables $a, b$ and define the equation for $X_f(6)$ over $K$ and then factorize the polynomial over the function field of $X_f(6)$. However, this is computationally much more complicated.

Remark 4.9. There is also a theoretical reason why one should believe for general $a, b$ there is only one linear factor defined over our base field. Each point in $X_f(3)$ corresponds to a pair of $(E', \phi)$ such that $E'$ is reverse 3-congruent to $E$ and $\phi: E[3] \cong E'[3]$ is the explicit isomorphism between the 3-torsion subgroups. Also, the 12 roots of the polynomial are permuted by the action of $\text{PSL}_2(\mathbb{Z}/3\mathbb{Z})$ and hence they correspond to the same $\overline{\mathbb{Q}}$-isomorphism class with distinct isomorphism between the 3-torsion subgroups.

If there were two distinct linear factors, then it implies for any $E$ with $X_f(6)$ having a rational point, there exists another elliptic curve $E'$ with two distinct Galois equivariant isomorphisms between $E[3]$ and $E'[3]$ such that the sign of Weil pairing is switched, which is certainly one would not expect.

Finally, by Theorem 4.7, the family of curves which are reverse 6-congruent to $E$ is given by $E_{\lambda, \mu}^*$ as in Theorem 2.6 with $(\lambda: \mu) = (x_3/3 : x_6)$ for any rational point $(x_1 : \cdots : x_6) \in X_f(6)$. We now give some examples.

Example 4.10. Let $t \in \mathbb{Q}$ be any rational number and let $a = \frac{-8}{27}t^2, b = \frac{64}{729}t^3$. Then there is a rational point on $X_f(6)$ given by

$$x_1 := \frac{-2^4t^4}{2187}, x_2 = \frac{2^6t^3}{243}, x_3 = \frac{2}{9}t, x_4 := \frac{-2^6t^3}{243}, x_5 = \frac{2^5}{27}t^2, x_6 = 1.$$ 

Then the image of this point in $X_f(3)$ is $\frac{2}{27}t$. Hence we conclude that the curve $E : y^2 = x^3 + ax + b$ with $a, b$ given above is reverse 6-congruent to

$$F : y^2 = x^3 - 27A(\frac{2}{27}t)x - 54B(\frac{2}{27}t),$$

where

$$A(t) = c_1^*(t, 1), B(t) = c_0^*(t, 1)$$

as in Theorem 2.6. For example, take $t = \frac{2}{7}$ we have $E : y^2 = x^3 - 6x + 8$ and then $A(\frac{2}{7}) = -2187/2$ and $B(\frac{2}{7}) = -19683/2$. Taking simplified model, we get a model for the point corresponding to $t = \frac{2}{7}$, which is $y^2 = x^3 - 216x + 1728$ and these two are reverse 6-congruent. Also, one computes that the rank of $X_f(6)$ is 1 and as $X_f(6)$ has at least one $\mathbb{Q}$-rational point so $X_f(6)$ is isomorphic to its Jacobian $y^2 = x^3 + 1/\Delta_E$ and the rank of the curve is one so this is an example where $X_f(6)$ and $X_f(6)$ both have positive rank.
If \( E : y^2 = x^3 - 6x + 8 \) and \( F : y^2 = x^3 - 216x + 1728 \) are reverse 6-congruent then they must have the same traces of Frobenius mod 6. The following table lists the traces of Frobenius of the curves \( E \) and \( F \) over several primes, where ToF stands for Traces of Frobenius.

| Primes | ToF(E) | ToF(F) |
|--------|--------|--------|
| 2      | 0      | 0      |
| 3      | 0      | 0      |
| 5      | 3      | 3      |
| 7      | 3      | -3     |
| 11     | -3     | 3      |
| 13     | -6     | 6      |
| 17     | -6     | 6      |
| 19     | 2      | 2      |
| 23     | -6     | -6     |
| 29     | 6      | 6      |

**Example 4.11.** The curves considered in the previous examples are quadratic twists to each other. In this example, we give a list of infinitely many values of \( j \)-invariant such that for any curve with \( j \)-invariant in the list, \( X_E^{-}(6) \) has at least one \( \mathbb{Q} \)-rational point.

As for each elliptic curve with \( j \)-invariant \( j \) with \( j \neq 0, 1728 \) there exists a unique value \( t \) such that the elliptic curve \( E_t : y^2 = x^3 + tx + t \) has \( j \)-invariant \( j \). Consider for \([u : v] \in \mathbb{P}^1(\mathbb{Q})\),

\[
t = -\frac{27}{8} \cdot \frac{(24u - v)^3(24u + v)^3}{(576u^2 - 24uv + v^2)^2(576u^2 - 24uv - \frac{1}{2}v^2)}.
\]
Then by a direct computation one checks the following is a point on $X^-_E(6)$:

$$
\begin{align*}
    x_1 &= t^3 \left( \frac{2}{3} u^7 - \frac{7}{48} u^6 v + \frac{5}{384} u^5 v^2 - \frac{13}{20736} u^4 v^3 + \frac{11}{663552} u^3 v^4 \\
    &\quad - \frac{1}{5308416} u^2 v^5 - \frac{1}{573308928} u v^6 \right), \\
    x_2 &= -t^2 \left( \frac{3}{4} u^7 - \frac{1}{32} u^6 v + \frac{1}{192} u^5 v^2 - \frac{1}{9216} u^4 v^3 - \frac{5}{442368} u^3 v^4 \\
    &\quad + \frac{1}{10616832} u^2 v^5 + \frac{1}{127401984} u v^6 \right), \\
    x_3 &= t \left( \frac{3}{16} u^7 + \frac{1}{128} u^6 v - \frac{1}{1024} u^5 v^2 - \frac{1}{24576} u^4 v^3 + \frac{1}{589824} u^3 v^4 \\
    &\quad + \frac{1}{14155776} u^2 v^5 - \frac{1}{1019215872} u v^6 - \frac{1}{24461180928} u^7 \right), \\
    x_4 &= t^2 \left( \frac{3}{4} u^7 - \frac{1}{32} u^6 v + \frac{1}{192} u^5 v^2 + \frac{1}{9216} u^4 v^3 + \frac{5}{442368} u^3 v^4 \\
    &\quad - \frac{1}{10616832} u^2 v^5 - \frac{1}{127401984} u v^6 \right), \\
    x_5 &= t^2 \left( u^7 + \frac{7}{48} u^6 v + \frac{7}{1152} u^5 v^2 - \frac{7}{663552} u^4 v^3 + \frac{7}{15925248} u^3 v^4 \\
    &\quad - \frac{1}{191102976} u^2 v^5 \right), \\
    x_6 &= -t \left( \frac{1}{4} u^7 + \frac{1}{64} u^6 v + \frac{1}{4608} u^5 v^2 - \frac{5}{110592} u^4 v^3 + \frac{1}{663552} u^3 v^4 \\
    &\quad + \frac{1}{63700992} u^2 v^5 - \frac{1}{509607936} u v^6 + \frac{1}{36691771392} v^7 \right).
\end{align*}
$$

In particular, the above example shows that

**Corollary 4.12.** There are infinitely many pairs of elliptic curves which are reverse 6-congruent.

### 4.4 A Simpler Birational Model

In this section we derive a much simpler birational model of $X^-_E(6)$ based on what we obtained.

**Proposition 4.13.** The forgetful morphism from $X^-_E(6) \to X^-$ is given by

$$(x_1, x_2, x_3, x_4, x_5) \mapsto (x_3, \frac{-x_1 x_5 + x_2 x_4}{2}),$$

where to ease the notation we take an affine piece $x_6 = 1$ of $X^-_E(6)$ and $\mu = 1$ for $X^-$ with the model $C_X$. 

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Proof. We have obtained the map $X^-_E(6) \to X^-_E(3)$ and recall from Proposition that the map $X^- \to X^-_E(3)$ is given by $(\mu, y) \mapsto \mu/3$ where we again take the affine pieces. Thus we conclude that the map $X^-_E(6) \to X^-$ must be of the form

$$(x_1, x_2, x_3, x_4, x_5) \mapsto (x_3, y)$$

where $y$ satisfies the equation

$$y^2 = \Delta_E(ax^4 + 6bx^3 - 2a^2x^2 - 2abx + (-a^3/3 - 3b^2)).$$

We calculate explicitly $y$ by factorizing the polynomial

$$t^2 - \Delta_E(ax^4 + 6bx^3 - 2a^2x^2 - 2abx + (-a^3/3 - 3b^2))$$

over the function field of $X^-_E(6)$.

Finally, it does not matter which square root we pick because it differs from the other by applying the $[-1]$ map on $X^-_E(6)$.

We can now simplify our defining equations for $X^-_E(6)$.

**Corollary 4.14.** The curve $X^-_E(6)$ is birational to the following curve defined by two equations in $\mathbb{A}^3$ with coordinates $(x, y, z)$:

$$f = z^3 - (36ax^2 + 12a^2)z + 216bx^3 - 144a^2x^2 - 216abx - (16a^3 + 216b^2) + y(64abx + 96b^2)27/\Delta_E,$$

$$g = y^2 - D(ax^4 + 6bx^3 - 2a^2x^2 - 2abx + (-a^3/3 - 3b^2)).$$

with the forgetful morphisms

$$X^-_E(6) \to X^- : (x, y, z) \mapsto (x, y), \quad X^- \to X^-_E(3) : (x, y) \mapsto x/3.$$

**Proof.** The equation for $g$ is essentially the same as the defining equation for $X^-$. An explicit computations shows that by taking $z = x_5, x = x_3, y = (-x_1x_4 + x_2x_3)/2$, we obtain a relation defined by $f$ above. Also $x, y$ satisfy the equation defined by $g$, using Proposition 4.12. Since $X^-_E(6)$ is a curve and hence the function field has transcendental degree 1 and thus the field generated by $x, z, y$ is the same as the function field of $X^-_E(6)$. Therefore, we conclude that the curve defined by equations $f$ and $g$ is birational to $X^-_E(6)$. Finally, by using Theorem 4.7 and Proposition 4.12 we obtain the desired forgetful morphisms.
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