NOTES ON THE TIGHTNESS OF $G_\delta$-MODIFICATIONS

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Abstract. We construct a countably tight normal $T_1$ space $X$ with $t(X_\delta) > 2^\omega$. This is an answer to the question posed by Dow-Juhász-Soukup-Szentmiklóssy-Weiss [5]. We also show that if the continuum is not so large, then the tightness of $G_\delta$-modifications of countably tight spaces can be arbitrary large up to the least $\omega_1$-strongly compact cardinal.

1. Introduction

For a topological space $X$, let $X_\delta$ be the $G_\delta$-midification of $X$, that is, $X_\delta$ is the space $X$ equipped with topology generated by all $G_\delta$-subsets of $X$.

Bella and Spadaro [1] studied the connection between the values of various cardinal functions taken on $X$ and $X_\delta$, respectively. In the paper they posed the following question: Is $t(X_\delta) \leq 2^{t(X)}$ true for every (compact) $T_2$ space $X$? Recall that $t(X)$, the tightness number of $X$, is the least infinite cardinal $\kappa$ such that for every $A \subseteq X$ and $p \in \overline{A}$, there is $B \in [A]^{\leq \kappa}$ with $p \in \overline{B}$. If $t(X) = \omega$, $X$ is said to be countably tight.

For this question, Dow-Juhász-Soukup-Szentmiklóssy-Weiss [5] answered as follows:

Fact 1.1 ([5]). (1) If $X$ is a regular $T_1$ Lindelöf space, then $t(X_\delta) \leq 2^{t(X)}$.

(2) Under $V = L$, for every cardinal $\kappa$ there is a Fréchet-Urysohn space $X$ with $t(X_\delta) \geq \kappa$.

The clause (1) of Fact 1.1 is a theorem of ZFC. However (2) is a consistency result, and they asked the following natural question:

Question 1.2. Is there a ZFC example of a countably tight Hausdorff (or regular, or Tychonoff) space $X$ for which $t(X_\delta) > 2^{t(X)}$?

In this paper we give a positive answer to their question.

Theorem 1.3. There is a countably tight normal $T_1$ space $X$ such that $t(X_\delta) > 2^\omega$.

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We will also observe some connection between \( \omega_1 \)-strongly compact cardinal and the tightness of \( G_\delta \)-modifications. Usuba [11] studied the Lindelöf number of \( G_\delta \)-modifications of compact spaces, and proved the following equality:

the least \( \omega_1 \)-strongly compact = sup\( \{ L(X_\delta) \mid X \text{ is compact } T_2 \} \).

Under some assumption, we will prove similar results for the tightness of \( G_\delta \)-modifications.

**Theorem 1.4.**  
(1) Suppose \( \kappa \) is the least \( \omega_1 \)-strongly compact cardinal. Then for every countably tight space \( X \) we have \( t(X_\delta) \leq \kappa \).

(2) Suppose there is no weakly Mahlo cardinal < \( 2^{\omega} \) (e.g., CH holds).

(a) If there is no \( \omega_1 \)-strongly compact cardinal, then for every cardinal \( \nu \), there is a countably tight normal \( T_1 \) space \( X \) such that \( t(X_\delta) \geq \nu \).

(b) If \( \kappa \) is the least \( \omega_1 \)-strongly compact cardinal, then for every cardinal \( \nu < \kappa \), there is a countably tight normal \( T_1 \) space \( X \) such that \( t(X_\delta) \geq \nu \).

Thus, assuming that \( 2^{\omega} \) is not so large, we have the following equality:

the least \( \omega_1 \)-strongly compact = sup\( \{ t(X_\delta) \mid X \text{ is countably tight normal } T_1 \} \).

Here we present some notations, definitions, and facts.

For a topological space \( X \) and \( A \subseteq X \), let \( A_\delta \) be the closure of \( A \) in \( X_\delta \).

For a filter \( F \) over the set \( S \) and a cardinal \( \kappa \), let us say that \( F \) is \( \kappa \)-complete if for every family \( A \subseteq F \) of size < \( \kappa \), we have \( \bigcap A \in F \). A filter \( F \) is \( \kappa \)-incomplete if \( F \) is not \( \kappa \)-complete.

The concept of \( \omega_1 \)-strongly compact cardinal is introduced by Bagaria and Magidor.

**Definition 1.5** (Bagaria-Magidor [3, 4]). An uncountable cardinal \( \kappa \) is \( \omega_1 \)-strongly compact if for every set \( S \) and every \( \kappa \)-complete filter \( F \) over \( S \), the filter \( F \) can be extended to an \( \omega_1 \)-complete ultrafilter over \( S \).

Note that if \( \kappa \) is \( \omega_1 \)-strongly compact, then every cardinal greater than \( \kappa \) is \( \omega_1 \)-strongly compact.

**Definition 1.6.**  
(1) For an uncountable cardinal \( \kappa \) and a set \( A \), let \( \mathcal{P}_\kappa A = \{ x \subseteq A \mid |x| < \kappa \} \).

(2) A filter \( F \) over \( \mathcal{P}_\kappa A \) is fine if for every \( a \in A \), we have \( \{ x \in \mathcal{P}_\kappa A \mid a \in x \} \in F \).

**Fact 1.7** (Bagaria-Magidor [3, 4]).  
(1) An uncountable cardinal \( \kappa \) is \( \omega_1 \)-strongly compact if and only if for every cardinal \( \lambda \geq \kappa \), there exists an \( \omega_1 \)-complete fine ultrafilter over \( \mathcal{P}_\kappa \lambda \).

(2) If \( \kappa \) is the least \( \omega_1 \)-strongly compact, then \( \kappa \) is a limit cardinal and there exists a measurable cardinal \( \leq \kappa \).
(3) It is possible that the least $\omega_1$-strongly compact is a singular cardinal.

Now we give the proof of (1) in Theorem 1.4. The proof is essentially the same to in Dow-Juhász-Soukup-Szentmiklóssy-Weiss [5], but we give it for the completeness.

**Proposition 1.8.** Let $\kappa$ be the least $\omega_1$-strongly compact. Then for every countably tight topological space $X$, $A \subseteq X$, and $p \in \overline{A}$, there is $B \subseteq A$ with $|B| < \kappa$ and $p \in \overline{B}$. Hence $t(X_\delta) \leq \kappa$.

**Proof.** We may assume that $A = \lambda$ for some cardinal $\lambda \geq \kappa$. By Fact 1.7, there is an $\omega_1$-complete fine ultrafilter $U$ over $\mathcal{P}_\kappa \lambda$.

Suppose to the contrary that $p \notin \overline{B}$ for every $B \subseteq \lambda$ with $|B| < \kappa$. For $B \subseteq \lambda$ with $|B| < \kappa$, there are open neighborhoods $O_B^n (n < \omega)$ of $p$ with $B \cap \bigcap_{n<\omega} O_B^n = \emptyset$. Since $U$ is $\omega_1$-complete, for each $\alpha < \lambda$, there is $n < \omega$ with $
 \{ B \in \mathcal{P}_\kappa \lambda \mid \alpha \in B \setminus \bigcap_{i<n} O_B^i \} \in U$. For $n < \omega$, let $A_n$ be the set of all $\alpha < \lambda$ with $\{ B \in \mathcal{P}_\kappa \lambda \mid \alpha \in B \setminus \bigcap_{i<n} O_B^i \} \in U$. We have $\lambda = \bigcup_{n<\omega} A_n$. On the other hand, we have $p \notin \bigcup_{n<\omega} A_n$; If $p \in \bigcup_{n<\omega} A_n$, there is a countable $C \subseteq A_n$ with $p \in \overline{C}$. Since $U$ is $\omega_1$-complete, we can find $B \in \mathcal{P}_\kappa \lambda$ with $\alpha \in B \setminus \bigcap_{i<n} O_B^i$ for every $\alpha \in C$, this means that $p \in \bigcap_{i<n} O_B^i$, but $C \cap \bigcap_{i<n} O_B^i = \emptyset$, this is impossible. Thus $p \notin \bigcup_{n<\omega} A_n$, and this immediately implies that $p \notin \overline{A}$. □

2. Construction of the spaces

For the sake of constructing our spaces, we use the function spaces. Let us recall some definitions and basic facts. For a Tychonoff space $X$, let $C(X)$ be the set of all continuous functions from $X$ into the real line $\mathbb{R}$. $C_p(X)$ is the space $C(X)$ endowed with the point-wise convergence, that is, the topology of $C_p(X)$ is generated by the family $\{ V(x_0, \ldots, x_n, O_0, \ldots, O_n) \mid x_0, \ldots, x_n \in X, O_0, \ldots, O_n \subseteq \mathbb{R} \}$ where $V(x_0, \ldots, x_n, O_0, \ldots, O_n)$ is the set of all $f \in C(X)$ with $f(x_i) \in O_i$ for every $i \leq n$.

**Fact 2.1** (Arhangel'skii-Pytkheev [2, 9]). Let $X$ be a Tychonoff space, and $\nu$ a cardinal. Then $L(X^\nu) \leq \nu$ for every $n < \omega$ if and only if $t(C_p(X)) \leq \nu$. In particular, each finite product of $X$ is Lindelöf if and only if $C_p(X)$ is countably tight.

**Proposition 2.2.** Let $\kappa$ be an uncountable cardinal and $\lambda \geq \kappa$ a cardinal. Suppose there is no $\omega_1$-complete fine ultrafilter over $\mathcal{P}_\kappa \lambda$. In addition we suppose that, for every countable family $\{ U_n \mid n < \omega \}$ of fine ultrafilters over $\mathcal{P}_\kappa \lambda$, there is a countable partition $\mathcal{A}$ of $\mathcal{P}_\kappa \lambda$ such that $A \notin U_n$ for every $A \in \mathcal{A}$ and $n < \omega$. Then there is a countably tight Tychonoff space $X$ with $t(X_\delta) \geq \kappa$.

**Proof.** Identifying $\mathcal{P}_\kappa \lambda$ as a discrete space, let $\text{Fine}(\mathcal{P}_\kappa \lambda)$ be the closed subspace of the Stone-Čech compactification $\beta(\mathcal{P}_\kappa \lambda)$ consisting of all fine ultrafilters over
$\mathcal{P}_\kappa \lambda$. Let $X = C_\mu(\Fine(\mathcal{P}_\kappa \lambda))$. Since $\Fine(\mathcal{P}_\kappa \lambda)$ is compact Hausdorff, each finite product of $\Fine(\mathcal{P}_\kappa \lambda)$ is compact. Hence $X$ is countably tight by Fact 2.1. We shall show that $t(X_\delta) \geq \kappa$.

Let $\{A^\alpha \mid \alpha < \nu\}$ be an enumeration of all countable partitions of $\mathcal{P}_\kappa \lambda$. For $\alpha < \nu$, let $S^{A^\alpha} = \{U \subseteq \Fine(\mathcal{P}_\kappa \lambda) \mid A \notin U \text{ for every } A \in A^\alpha\}$. $S^{A^\alpha}$ is a closed $G_\delta$-subset of $\Fine(\mathcal{P}_\kappa \lambda)$. Since there is no $\omega_1$-complete fine ultrafilter over $\mathcal{P}_\kappa \lambda$, the family $\{S^{A^\alpha} \mid \alpha < \nu\}$ is a cover of $\Fine(\mathcal{P}_\kappa \lambda)$. Furthermore, by our assumption, for every $U_n \subseteq \Fine(\mathcal{P}_\kappa \lambda) \ (n < \omega)$ there is $\alpha < \nu$ such that $U_n \subseteq S^{A^\alpha}$ for every $n < \omega$.

We use the following fact:

**Fact 2.3** (Usuba [11]). If $\{S^{A^\alpha} \mid \alpha \in E\}$ is a cover of $\Fine(\mathcal{P}_\kappa \lambda)$ for some $E \subseteq \nu$, then $|E| \geq \kappa$.

**Sketch of the proof.** Take $E \subseteq \nu$ with size $< \kappa$. Take a sufficiently large regular $\theta$, and take $M \prec H_\theta$ containing all relevant objects such that $|M| < \kappa$ and $E \subseteq M$. For each $\alpha \in E$, there is a unique $A_\alpha \in A^\alpha$ with $M \cap \kappa \subseteq A_\alpha$. By the elementarity, we have that for every $\alpha_0, \ldots, \alpha_n \in E$ and $\beta_0, \ldots, \beta_m \in \lambda$, the family $\{x \in \mathcal{P}_\kappa \lambda \mid x \in A_\alpha \text{ for every } i < n \text{ and } \beta_i \in x \text{ for every } i < m\}$ is non-empty. Thus we can take a fine ultrafilter $U$ over $\mathcal{P}_\kappa \lambda$ such that $U \notin S^{A^\alpha}$ for every $\alpha \in E$. Hence $\{S^{A^\alpha} \mid \alpha \in E\}$ is not a cover. \(\square\)

For $\alpha < \nu$, since $S^{A^\alpha}$ is a closed $G_\delta$-subset of the compact Hausdorff space $\Fine(\mathcal{P}_\kappa \lambda)$, there is a continuous map $f_\alpha : \Fine(\mathcal{P}_\kappa \lambda) \to [0, 1]$ such that $S^{A^\alpha} = f_\alpha^{-1}[0]$. Let $D = \{f_\alpha \mid \alpha < \nu\}$. Let $g : \Fine(\mathcal{P}_\kappa \lambda) \to \{0\}$ be the constant function. We shall show that $g$ and $D$ witness $t(X_\delta) \geq \kappa$.

First we check that $g \in \overline{D^\delta}$. Take an open neighborhood $O$ of $g$ in $X_\delta$. By the definition of the topology of $X_\delta$, there are $U_n \in \Fine(\mathcal{P}_\kappa \lambda) \ (n < \omega)$ such that $\{h \in X \mid h(U_n) = 0 \text{ for } n < \omega\} \subseteq O$. By the assumption, there is $\alpha < \nu$ such that $U_n \subseteq S^{A^\alpha}$ for every $n < \omega$. Then $f_\alpha(U_n) = 0$ for $n < \omega$, hence $f_\alpha \in D \cap O$.

Finally we show that if a set $E \subseteq \nu$ has cardinality $< \kappa$, then $g \notin \overline{\{f_\alpha \mid \alpha \in E\}}$ which means that $t(X_\delta) \geq \kappa$. Suppose to the contrary that $g \in \overline{\{f_\alpha \mid \alpha \in E\}}$. For each $U \subseteq \Fine(\mathcal{P}_\kappa \lambda)$, the set $\{h \in C_\mu(X) \mid h(U) = 0\}$ is an open neighborhood of $g$ in $X_\delta$. So we can pick $\alpha \in E$ with $f_\alpha(U) = 0$. Since $S^{A^\alpha} = f_\alpha^{-1}[0]$, we have $U \subseteq S^{A^\alpha}$. This shows that $\{S^{A^\alpha} \mid \alpha \in E\}$ is a cover of $\Fine(\mathcal{P}_\kappa \lambda)$, contradicting to Fact 2.3. \(\square\)

3. On the assumptions of Proposition 2.2

Now let us discuss when the assumptions of Proposition 2.2 hold.

For a filter $F$ over the set $S$, let $F^+ = \{X \in \mathcal{P}(S) \mid S \setminus X \notin F\}$. $F^+$ is the complement of the dual ideal of $F$. An element of $F^+$ is called an $F$-positive set. For $X \in F^+$, let $F \upharmon X = \{Y \subseteq S \mid Y \cup (S \setminus X) \in F\}$. $F \upharmon X$ is the filter over $S$ generated by $F \cup \{X\}$.
Lemma 3.1. Let $S$ be an uncountable set, and $\{U_n \mid n < \omega\}$ a family of ultrafilters over $S$. Let $F = \bigcap_{n<\omega} U_n$. Then the following are equivalent:

1. There is a countable partition $A$ of $S$ such that $A \notin U_n$ for every $A \in A$ and $n < \omega$.
2. For every $X \in F^+$, the filter $F \upharpoonright X$ is $\omega_1$-incomplete.

Proof. (1) $\Rightarrow$ (2) is clear. For (2) $\Rightarrow$ (1), we define $C_\alpha \subseteq S$ ($\alpha < \omega_1$) as follows:

First, let $C_0 = S \in F$. Suppose $C_\gamma$ is defined for every $\gamma < \alpha$ so that:

1. $\langle C_\gamma \mid \gamma < \alpha \rangle$ is a $\subseteq$-decreasing sequence of $F$-positive sets.
2. $C_\gamma = \bigcap_{\delta < \gamma} C_\delta$ if $\gamma$ is limit.
3. If $\gamma + 1 < \alpha$, then there are $C_{\gamma,i} \in F \upharpoonright C_\gamma$ for $i < \omega$ such that $C_\gamma = \bigcap_{i < \omega} C_{\gamma,1} \supseteq \cdots$ and $C_{\gamma+1} = \bigcap_{i < \omega} C_{\gamma+1,i}$.

Suppose $\alpha = \beta + 1$. Since $C_\beta \in F^+$ and $F \upharpoonright C_\beta$ is $\omega_1$-incomplete, we can find $C_{\beta,i} \subseteq C_\beta$ for $i < \omega$ such that $C_\beta = C_{\beta,0} \supseteq C_{\beta,1} \supseteq \cdots$, $C_{\beta,i} \in F \upharpoonright C_\beta$ for every $i < \omega$, and $\bigcap_{i < \omega} C_{\beta,i} \notin F \upharpoonright C_\beta$. If $\bigcap_{i < \omega} C_{\beta,i} \notin F^+$, then we finish this construction.

If $\bigcap_{i < \omega} C_{\beta,i} \in F^+$, then let $C_\alpha = \bigcap_{i < \omega} C_{\beta,i}$.

If $\alpha$ is limit and $\bigcap_{\gamma < \alpha} C_\gamma \notin F^+$, then we finish the construction, otherwise let $C_\alpha = \bigcap_{\gamma < \alpha} C_\gamma$.

We claim that this construction have to be finished at some $\gamma < \omega_1$. If not, then let $D_\alpha = C_\alpha \setminus C_{\alpha+1}$ for $\alpha < \omega_1$. Since $C_{\alpha+1} \notin F \upharpoonright C_\alpha$, we have $D_\alpha \in F^+$. For $\alpha < \omega_1$, there is some $n_\alpha < \omega$ with $D_\alpha \in U_{n_\alpha}$. Hence there is some $n < \omega$ such that the set $\{\alpha < \omega_1 \mid n_\alpha = n\}$ is uncountable. Pick $\alpha < \beta < \omega_1$ with $n = n_\alpha = n_\beta$. We have $D_\alpha, D_\beta \in U_n$, hence $D_\alpha \cap D_\beta \notin \emptyset$. This is impossible.

Now suppose $\{C_\alpha \mid \alpha < \gamma\}$ is defined as above but $C_\gamma$ cannot be defined. If $\gamma$ is limit, we have $\bigcap_{\alpha < \gamma} C_\alpha \notin F^+$. By shrinking each $C_\alpha$, we may assume that $\bigcap_{\alpha < \gamma} C_\alpha = \emptyset$. By the construction, for $\alpha < \gamma$, there is a sequence $\langle C_{\alpha,i} \mid i < \omega \rangle$ as before. Let $A_{\alpha,i} = C_{\alpha,i} \setminus C_{\alpha,i+1}$. We check that $A_{\alpha,i} \notin F^+$; since $C_{\alpha,i}, C_{\alpha,i+1} \in F \upharpoonright C_\alpha$, we have that $A_{\alpha,i} = C_{\alpha,i} \setminus C_{\alpha,i+1} \notin (F \upharpoonright C_\alpha)^+$.

Furthermore, since $A_{\alpha,i} \subseteq C_{\alpha,i} \subseteq C_\alpha$, we have $A_{\alpha,i} \notin F^+$. Now the family $\{A_{\alpha,i} \mid \alpha < \gamma, i < \omega\}$ is a countable partition of $S$ such that $A_{\alpha,i} \notin F^+$, so $A_{\alpha,i} \notin U_n$ for every $n < \omega$. Thus (1) holds.

If $\beta$ is successor, say $\gamma = \beta + 1$, then there are $C_{\beta,i} \in (F \upharpoonright C_\beta)^*$ ($i < \omega$) such that $C_\beta \supseteq C_{\beta,0} \supseteq C_{\beta,1} \supseteq \cdots$ and $\bigcap_{i < \omega} C_{\beta,i} \notin F^+$. As in the limit case, we may assume that $\bigcap_{i < \omega} C_{\beta,i} = \emptyset$. For each $\alpha < \beta$, let $A_{\alpha,i} = C_{\alpha,i} \setminus C_{\alpha,i+1}$. Then $\{A_{\alpha,i} \mid i < \omega, \alpha \leq \beta\}$ is a required partition.

We will use the generic ultrapower argument. See Foreman [6] for the generic ultrapower. We present some basic definitions and facts.

Let $F$ be a filter over the set $S$. For a cardinal $\kappa$, we say that $F$ is $\kappa$-saturated if for every family $\{X_\alpha \mid \alpha < \kappa\}$ of $F$-positive sets, there are $\alpha < \beta < \kappa$ with $X_\alpha \cap X_\beta \in F^+$. For $F$-positive sets $X$ and $Y$, define $X \leq_F Y$ if $X \setminus Y \notin F^+$. Let
$\mathbb{P}_F$ be the poset $F^+$ with the order $\leq_F$. Note that for $X, Y \in \mathbb{P}_F$, $X$ is compatible with $Y$ in $\mathbb{P}_F$ if and only if $X \cap Y \in F^+$, and $\mathbb{P}_F$ has the $\kappa$-c.c. if and only if $F$ is $\kappa$-saturated.

If $G$ is a $(V, \mathbb{P}_F)$-generic filter, then $G$ is a $V$-ultrafilter over $S$, that is the following hold:

- $S \in G$, $\emptyset \notin G$.
- $X \cap Y \in G$ for every $X, Y \in G$.
- For $X, Y \in V$, if $X \in G$ and $X \subseteq Y \subseteq S$ then $Y \in G$.
- For every $X \subseteq S$ with $X \in V$, either $X \in G$ or $S \setminus X \in G$.

Hence we can take the generic ultrapower of $V$ by $G$. For a $(V, \mathbb{P}_F)$-generic filter $G$, let $\text{Ult}(V, G)$ be the generic ultrapower of $V$ by $G$, and $j : V \to \text{Ult}(V, G)$ be the elementary embedding induced by $G$. If $\text{Ult}(V, G)$ is well-founded, we identify $\text{Ult}(V, G)$ with its transitive collapse. We say that $F$ is precipitous if for every $(V, \mathbb{P}_F)$-generic $G$, $\text{Ult}(V, G)$ is well-founded.

**Fact 3.2.** Let $\kappa$ be an uncountable cardinal, and $F$ a $\kappa$-complete filter over $S$. If $F$ is $\kappa^+$-saturated, then $F$ is precipitous.

**Proposition 3.3.** Let $\kappa$ be an uncountable cardinal, and $\lambda > 2^\kappa$. Let $\{U_n \mid n < \omega\}$ be a family of $\omega_1$-incomplete fine ultrafilters over $\mathcal{P}_\kappa \lambda$. If the filter $F = \bigcap_{n<\omega} U_n$ is $\omega_1$-complete, then there is a weakly Mahlo cardinal $< 2^\omega$, and $\kappa > (2^\omega)^+$.

**Proof.** First note that for $X \subseteq \mathcal{P}_\kappa \lambda$, $X \in \mathbb{P}_F$ if and only if $X \in U_n$ for some $n < \omega$. For each $X \in \mathbb{P}_F$, let $I_X = \{n < \omega \mid X \subseteq U_n\} \neq \emptyset$.

We shall prove a series of claims.

**Claim 3.4.** For $X, Y \in \mathbb{P}_F$, $X \leq_F Y \iff I_X \subseteq I_Y$, and $X$ is compatible with $Y \iff I_X \cap I_Y \neq \emptyset$.

**Proof.** If $I_X \nsubseteq I_Y$, pick $n \in I_X \setminus I_Y$. We know $X \subseteq U_n$ but $Y \nsubseteq U_n$, hence $X \setminus Y \subseteq U_n$, and $X \not\leq_F Y$. For the converse, suppose $X \not\leq_F Y$. Then $X \setminus Y \in F^+$, and there is $n < \omega$ with $X \setminus Y \subseteq U_n$. We have $X \subseteq U_n$ but $Y \nsubseteq U_n$, so $n \in I_X \setminus I_Y$ and $I_X \nsubseteq I_Y$.

If $X$ is compatible with $Y$, then there is $Z \leq_F X, Y$. We may assume $Z \subseteq X \cap Y$, then $I_Z \subseteq I_X \cap I_Y$, so $I_X \cap I_Y \neq \emptyset$. For the converse, if $I_X \cap I_Y \neq \emptyset$, take $n \in I_X \cap I_Y$. Then $X, Y \subseteq U_n$, so $X \cap Y \subseteq U_n$. Hence $X \cap Y \in F^+$, and we have $X \cap Y \leq_F X, Y$. □

**Claim 3.5.** $\mathbb{P}_F$ has the c.c.c., and has a dense subset of size $\leq 2^\omega$.

**Proof.** Take an uncountable family $\{X_\alpha \mid \alpha < \omega_1\} \subseteq \mathbb{P}_F$. Since $I_{X_\alpha} \subseteq \omega$, there must be $\alpha < \beta < \omega_1$ with $I_{X_\alpha} \cap I_{X_\beta} \neq \emptyset$. Hence $X_\alpha$ is compatible with $X_\beta$ by Claim 3.4.
For $X, Y \in \mathbb{P}_F$, define $X \approx Y$ if $X \leq_F Y$ and $Y \leq_F X$. By Claim 3.4, $X \approx Y$ if and only if $I_X = I_Y$. Hence there are at most $2^\omega$ many equivalence classes, and we can take a dense subset in $\mathbb{P}_F$ of size $\leq 2^\omega$.

From now on, we identify $\mathbb{P}_F$ with its dense subset of size $\leq 2^\omega$.

Now, we know that $F$ is $\omega_1$-complete and $\omega_1$-saturated. Hence $F$ is precipitous by Fact 3.2. Take a $(V, \mathbb{P}_F)$-generic $G$, and let $j : V \rightarrow \text{Ult}(V, G)$ be the generic elementary embedding induced by $G$. Since $F$ is precipitous, we can identify $\text{Ult}(V, G)$ with its transitive collapse $M$. For a map $f : \mathcal{P}_\kappa \lambda \rightarrow V$ with $f \in V$, let $[f]$ be the equivalence class of $f$ by $G$. If $id : \mathcal{P}_\kappa \lambda \rightarrow \mathcal{P}_\kappa \lambda$ is the identity map, then we have $j[id] \subseteq j[\mathcal{P}_\kappa \lambda]$ because $F$ is a fine filter. Moreover $\lambda \leq ||id||^M < j(\kappa)$.

Let $\mu$ be the critical point of $j$. $\mu$ is regular uncountable in $V$. Since $\mathbb{P}_F$ has the c.c.c., $\mu$ remains regular in $V[G]$, and so does in $M$.

Claim 3.6. $\mu$ is weakly Mahlo in $V$.

Proof. Let $C \subseteq \mu$ be a club in $\mu$ with $C \in V$. Then $\mu \in j(C)$, hence it holds that the statement “$j(C)$ contains a regular cardinal” in $M$. By the elementarity of $j$, $C$ contains a regular cardinal in $V$.

Note that $\mu$ is in fact weakly $\mu$-Mahlo.

Claim 3.7. For every cardinal $\nu$, we have $(2^\nu)^V = (2^\nu)^{V[G]}$.

Proof. Since $\mathbb{P}_F$ has a dense subset of size $(2^\omega)^F$ and has the c.c.c., there are at most $(2^\omega)^\omega$-many antichains in $\mathbb{P}_F$, and at most $(2^\omega)^\nu = (2^\nu)^V$ many canonical names for subsets of $\nu$. Hence we have $(2^\nu)^V = (2^\nu)^{V[G]}$.

Thus for each cardinal $\nu$, we can let $2^\nu$ denote $(2^\nu)^V$ and $(2^\nu)^{V[G]}$. In addition, since $\mathbb{P}_F$ has the c.c.c., we have $(\nu^+)^V = (\nu^+)^{V[G]}$ for every cardinal $\nu$, and we can let $\nu^+$ denote $(\nu^+)^V$ and $(\nu^+)^{V[G]}$. Note that $(\nu^+)^M \leq \nu^+$ for every cardinal $\nu$.

Claim 3.8. $j(2^\omega) < (2^\omega)^+ = j((2^\omega)^+) < \kappa$.

Proof. If $j(2^\omega) \geq (2^\omega)^+$, then $M$ has at least $(2^\omega)^+$ many subsets of $\omega$, so we have $(2^\omega)^{V[G]} \geq (2^\omega)^+$, this contradicts to the previous claim, and we have $j(2^\omega) < (2^\omega)^+$. Thus $(2^\omega)^+ \leq j((2^\omega)^+) = j(j(2^\omega)^+) = j(2^\omega)^+ \leq (2^\omega)^+$, so we have $j((2^\omega)^+) = (2^\omega)^+$.

Finally, since $(2^\omega)^+ \leq (2^\omega)^+ \leq \lambda$, we have $j((2^\omega)^+) = (2^\omega)^+ \leq \lambda < j(\kappa)$. Then $(2^\omega)^+ < \kappa$ by the elementarity of $j$.

We completes the proof by showing the following:

Claim 3.9. $\mu < 2^\omega$.

Proof. Note that $j(\mu) > \mu^+$ since $\mathcal{P}(\mu)^V \subseteq \mathcal{P}(\mu)^M$ and $j(\mu)$ is a limit cardinal in $M$.  

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Since \( j(2^\omega) < (2^\omega)^+ \), we have \( \mu \neq 2^\omega \). Next we show \( 2^\omega > \mu \). If not, then \( 2^\omega < \mu \). Take \( X \in G \) such that \( X \models \text{“the critical point of } j \text{ is } \mu \” \). Then the filter \( F \upharpoonright X \) is in fact \( \mu \)-complete. We use the following well-known fact by Tarski:

**Fact 3.10** (Tarski, e.g. see Kanamori [8]). Let \( S \) be an uncountable set, and \( F \) a filter over \( S \). If \( F \) is \((2^\omega)^+\)-complete and \( \omega_1\)-saturated, then there is \( Y \in F^+ \) such that \( F \upharpoonright Y \) is a unltrafilter.

Since \( 2^\omega < \mu \), by Tarski’s theorem there is \( Y \in (F \upharpoonright X)^+ \) such that \( F \upharpoonright Y \) is a unltrafilter. Hence \( F \upharpoonright Y = U_n \) for some \( n \), and \( U_n \) is \( \mu \)-complete. This is a contradiction. \( \square \)

**Lemma 3.11.** Let \( X \) be a countably tight \( T_1 \) space. Then there is a countably tight normal \( T_1 \) space \( Y \) with \( t(Y_\delta) = t(X_\delta) \).

**Proof.** Fix a point \( p^* \in X \) such that there is \( A \subseteq X \) with \( p^* \in \overline{A}^\delta \), but no \( B \subseteq A \) with \( |B| < t(X_\delta) \) and \( p \in \overline{B}^\delta \). Let \( Y \) be the space \( X \) equipped with the following topology:

1. Every \( q \in X \setminus \{p^*\} \) is isolated in \( Y \).
2. A local base for \( p^* \) in \( Y \) is the same to in \( X \).

It is easy to check that \( Y \) is a countably tight normal \( T_1 \) space with \( t(Y_\delta) = t(X_\delta) \). \( \square \)

Now we have the theorems.

**Corollary 3.12.** There is a countably tight normal \( T_1 \) space \( X \) such that \( t(X_\delta) > 2^\omega \).

**Proof.** Let \( \kappa = (2^\omega)^+ \). \( \kappa \) is not \( \omega_1 \)-strongly compact, and there is \( \lambda \geq \kappa \) such that \( \mathcal{P}_{\kappa \lambda} \) has no \( \omega_1 \)-complete fine ultrafilter. \( \lambda \) can be arbitrary large, so we may assume \( \lambda > 2^\kappa \). By Proposition 2.2 and Lemmas 3.1, 3.11, it is easy to show that for every countable family \( \{U_n \mid n < \omega\} \) of fine ultrafilters over \( \mathcal{P}_{\kappa \lambda} \) and \( X \in F^+ = (\bigcap_{n<\omega} U_n)^+ \), the filter \( F \upharpoonright X \) is \( \omega_1 \)-incomplete. Let \( I = \{n < \omega \mid X \in U_n\} \). Then it is easy to check that \( F \upharpoonright X = \bigcap_{n \in I} U_n \). Because \( \kappa = (2^\omega)^+ \), we know that \( \bigcap_{n \in I} U_n \) is \( \omega_1 \)-incomplete by Proposition 3.3. \( \square \)

**Corollary 3.13.** Suppose there is no weakly Mahlo cardinal \( < 2^\omega \).

1. If there is no \( \omega_1 \)-strongly compact cardinal, then for every cardinal \( \nu \), there is a countably tight normal \( T_1 \) space \( X \) such that \( t(X_\delta) \geq \nu \).
2. If \( \kappa \) is the least \( \omega_1 \)-strongly compact cardinal, then for every cardinal \( \nu < \kappa \), there is a countably tight normal \( T_1 \) space \( X \) such that \( t(X_\delta) \geq \nu \).
Proof. If $\kappa$ is not $\omega_1$-strongly compact, there is a large $\lambda > \kappa$ such that $\mathcal{P}_\kappa \lambda$ cannot carry an $\omega_1$-complete fine ultrafilter. By the assumption and Proposition 3.3, there is no countable family of fine ultrafilters $\{U_n \mid n < \omega\}$ over $\mathcal{P}_\kappa \lambda$ with $\bigcap_{n < \omega} U_n$ $\omega_1$-complete. Again, by Proposition 2.2 and Lemmas 3.1, 3.11 we can take a countably tight normal $T_1$ space $X$ with $t(X_\delta) \geq \kappa$. □

Question 3.14. Is the equality that “the least $\omega_1$-strongly compact $= \sup\{t(X_\delta) \mid X$ is countably tight normal (regular) $T_1\}$” provable from ZFC without any assumptions?

Question 3.15. In the theorems, can we replace “countably tight” by “Fréchet-Urysohn”? For instance, is there a ZFC-example of a Fréchet-Urysohn space $X$ with $t(X_\delta) > 2^\omega$?

Note that our space $C_p(\text{Fine}(\mathcal{P}_\kappa \lambda))$ is not Fréchet-Urysohn; It is known that for a compact Hausdorff space $Y$, $C_p(Y)$ is Fréchet-Urysohn if and only if $Y$ is scattered (Pytkeev [10], Gerlits [7]). However the space $\text{Fine}(\mathcal{P}_\kappa \lambda)$ is not scattered.

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