NEW SELF-SIMILAR EULER FLOWS: GRADIENT CATASTROPHE WITHOUT SHOCK FORMATION

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Abstract. We consider self-similar solutions to the full compressible Euler system for an ideal gas in two and three space dimensions. The system admits a 2-parameter family of similarity solutions depending on parameters $\lambda$ and $\kappa$. Requiring locally finite amounts of mass, momentum, and energy imply certain constraints on $\lambda$ and $\kappa$. Further constraints are imposed for particular types of flows. E.g., Guderley’s pioneering construction of an unbounded converging shock wave invading a quiescent fluid, requires $\kappa = 0$ and $\lambda > 1$.

In this work we analyze the regime $0 < \lambda < 1$, which does not appear to have been addressed previously. Our findings include: (i) non-existence of Guderley shock solutions; (ii) existence of bounded and continuous incoming similarity flows in 3-d provided $\kappa$ takes the value $\hat{\kappa} = \frac{2(1-\lambda)}{\gamma-1}$, $\lambda$ is sufficiently small, and $\gamma$ is sufficiently large; (iii) continuation of the latter flows beyond collapse as globally defined and continuous similarity solutions.

A key feature of these solutions is that they, in contrast to Guderley solutions, remain bounded at time of collapse, while the density, velocity, and sound speed all suffer gradient blowup. It is noteworthy that, notwithstanding infinite gradients at collapse, no shock wave appears. The analysis is based on a combination of analytical and numerical calculations.

Key words. Compressible fluid flow, multi-d Euler system, radial symmetry, similarity solutions, singularity formation

AMS subject classifications. 35L45, 35L67, 76N10, 35Q31

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1. INTRODUCTION

The non-isentropic (full) compressible Euler system expresses conservation of mass, momentum, and energy in fluid flow in the absence of second order effects:

\[ \rho_t + \text{div}_x (\rho \mathbf{u}) = 0 \]
\[ (\rho \mathbf{u})_t + \text{div}_x [\rho \mathbf{u} \otimes \mathbf{u}] + \text{grad}_x p = 0 \]
\[ (\rho E)_t + \text{div}_x [(\rho E + p)\mathbf{u}] = 0. \]

The independent variables are time \( t \) and position \( x \in \mathbb{R}^n \), and the primary dependent variables are density \( \rho \), fluid velocity \( \mathbf{u} \), and internal energy \( e \); the total energy density is \( E = e + \frac{1}{2} |\mathbf{u}|^2 \). We restrict attention to ideal gases with pressure \( p \) given by

\[ p(\rho, e) = (\gamma - 1) \rho e. \]

Throughout, the adiabatic constant \( \gamma \) is assumed to satisfy \( \gamma > 1 \). The local speed of sound \( c \) is

\[ c = \sqrt{\frac{\gamma p}{\rho}} = \sqrt{\gamma(\gamma - 1)e}. \]

We consider radial flows in space dimension \( n = 2 \) or \( n = 3 \), i.e., the flow variables depend on position only through the distance \( r = |x| \) to the origin, and the velocity field is purely radial, viz. \( \mathbf{u} = r^{\frac{n-1}{2}} \). With these assumptions, and within smooth regions of the flow, (1.1)-(1.3) read

\[ \rho_t + u \rho_r + \rho (u_r + \frac{m}{r} u) = 0 \]
\[ u_t + uu_r + \frac{1}{r} (\rho c^2)_r = 0 \]
\[ c_t + uc_r + \frac{n-1}{2} c (u_r + \frac{m}{r} u) = 0, \]

where \( \rho = \rho(t, r), \quad u = u(t, r), \quad c = c(t, r), \) and \( m = n - 1 \).

1.1. Self similar Euler flows. We next specialize further by imposing self similarity \[9,11,25,26\]. For this we follow \[9,17\] and introduce the similarity variables

\[ x = \frac{t}{r^\lambda}, \quad \rho(t, r) = r^\kappa R(x), \quad u(t, r) = -\frac{r^{1-\lambda} V(x)}{\lambda} \frac{1}{x}, \quad c(t, r) = -\frac{r^{1-\lambda} C(x)}{\lambda} \frac{1}{x}. \]

At this stage the similarity parameters \( \lambda \) and \( \kappa \) are free.

Substitution of (1.9) into (1.6)-(1.7) yield three coupled ODEs for \( R(x), V(x), C(x) \). The density variable \( R \) can be eliminated to give two coupled ODEs for only \( V \) and \( C \), viz.

\[ \frac{dV}{dx} = \frac{1}{\lambda x} \frac{G(V, C)}{D(V, C)} \]
\[ \frac{dC}{dx} = \frac{1}{\lambda x} \frac{F(V, C)}{D(V, C)}, \]

which in turn yield a single, autonomous ODE

\[ \frac{dC}{dV} = \frac{F(V, C)}{G(V, C)}. \]
relating \( V \) and \( C \) along self-similar solutions. The functions \( D, G, F \) are given by
\[
D(V, C) = (1 + V)^2 - C^2, \\
G(V, C) = nC^2(V - V_s) - V(1 + V)(\lambda + V), \\
F(V, C) = C \left\{ C^2(1 + \frac{\alpha}{1 + V}) - k_1(1 + V)^2 + k_2(1 + V) - k_3 \right\},
\]
where
\[
V_s = \frac{\kappa - 2(\lambda - 1)}{n\gamma}, \\
\alpha = \frac{1}{2\gamma} [\kappa(\gamma - 1) + 2(\lambda - 1)],
\]
and
\[
k_1 = 1 + \frac{(n-1)(\gamma - 1)}{2}, \\
k_2 = \frac{(n-1)(\gamma - 1) + (\gamma - 3)(\lambda - 1)}{2}, \\
k_3 = \frac{(\gamma - 1)(\lambda - 1)}{2}.
\]

Evidently, the construction of radial self-similar Euler flows requires an analysis of the phase portrait for (1.12) in the \((V, C)\)-plane. However, since the ODEs (1.10)-(1.11) are singular along the two critical lines defined by \( D(V, C) = 0 \), only certain trajectories of (1.12) yield physically meaningful flows. Specifically, any trajectory crossing a critical line can do so only at points where all three of \( F, G, \) and \( D \) vanish.

Having identified an admissible trajectory \( \Gamma \) of (1.12) connecting some of its equilibria, it may be used to generate a solution to the system (1.10)-(1.11). Finally, it must be checked that the signs of \( \lambda, \alpha, k \) and \( k_1, k_2, k_3 \) are of this type.

For later reference we note that the choice
\[
\kappa = \kappa := \frac{2(1 - \lambda)}{\gamma - 1},
\]
makes \( \alpha \) and hence \( q \) vanish. The entropy integral (1.19) then reduces to
\[
\left( \frac{C(x)}{\rho(x)} \right)^2 R(x)^{1-\gamma} \equiv \text{constant} > 0.
\]

In terms of temperature \( \theta \propto \rho^2 \) and density \( \rho \), this amounts to \( \theta \rho^{1-\gamma} \) being constant, i.e., the specific entropy takes a constant value throughout any region of continuity. Thus, continuous similarity flows with \( \kappa = \kappa \) provide isentropic solutions to the Euler system. The solutions we build in Section 3 are of this type.
1.2. **Outline and main results.** The present work addresses a particular type of converging-diverging flows with $\lambda \in (0, 1)$, in which an incoming radially symmetric wave collapses on the center of motion and reflects an outgoing wave. Without loss of generality, the time of collapse is chosen as $t = 0$, a choice which is built into the definition of the similarity variable $x$ in (1.9).

Before describing our findings we briefly review some earlier results. The pioneering study [11] of Guderley provided examples of unbounded converging-diverging shock waves in an ideal gas. An incoming spherical shock wave approaches the origin by invading a quiescent fluid (homogeneous and at rest), while gaining strength. At collapse it has infinite speed and the velocity, sound speed, and pressure in its immediate wake are unbounded. The subsequent flow accommodates the infinite amplitudes at the center of motion by generating an expanding shock wave, which then slows down and weakens as it interacts with the still-incoming flow ahead of it.

In what follows, solutions in which a converging shock invades a quiescent fluid, collapses at the origin, and then generates an expanding shock wave, will be referred to as Guderley solutions. Their construction depends on resolving a nonlinear eigenvalue problem for the similarity parameter $\lambda$ (see Section 3). It turns out that, in a Guderley solution, the similarity parameter $\kappa$ must necessarily be zero and that the temperature in the quiescent part of the fluid vanishes identically. The allowed $\lambda$ values depend on both the geometry ($n$) and the gas ($\gamma$), and are dictated by the requirement that a certain ODE-trajectory pass through a particular equilibrium of (1.12). Their determination must be done numerically, a task that has been carried out to considerable accuracy in a number of works (for $n = 2$ or 3 and various $\gamma > 1$); see [1, 3, 11–13, 17, 24] and references therein.

**Remark 1.1.** In the applied literature on self-similar Euler flows the emphasis has been on Guderley solutions due to their relevance to inertial fusion research, [1, 10, 22, 23]. The closely related construction of unbounded self-similar cavity flows has been analyzed in [3, 14, 17].

In all works on self-similar solutions to the Euler system that we are aware of, attention is restricted to similarity parameters $\lambda > 1$ or, as a limiting case, $\lambda = 1$. Our first objective in this work is to consider the possibility of Guderley shock solutions when $\lambda \in (0, 1)$. Since a shock in a similarity flow propagates along a path with $x = \frac{r}{\lambda^t} \equiv constant$, $\lambda \in (0, 1)$ would yield a “glancing” shock wave that weakens and slows down, reaching the center of motion with vanishing speed. However, as described in Section 3, it does not appear possible to generate a Guderley solution when $\lambda \in (0, 1)$: the relevant ODE-trajectories simply do not reach the required equilibrium.

We then turn to the possibility of constructing continuous self-similar radial Euler flows. For $\lambda > 1$ such solutions have recently been constructed, up to time of collapse, in the works [15, 16, 20]. These solutions suffer amplitude blowup at the $t = 0$ and are propagated to positive times in [15, 16] by having a shock emerge from the center of motion, similar to what occurs in Guderley solutions.

**Remark 1.2.** The work [20] addresses the subtle issue of constructing smooth ($C^\infty$) self-similar isentropic flows (up to collapse). The recent work [2] provides numerical evidence that these solutions are linearly unstable with respect to 1-d radial perturbations.

We note that the continuous solutions considered in [15, 16, 20] demonstrate in particular that amplitude blowup does not require a central region of vanishing pressure, as is the case in Guderley solutions and cavity flows.

The main contribution of the present work is the construction and analysis of globally continuous radial self-similar flows for the full Euler system with similarity parameter $\lambda \in (0, 1)$. This parameter range yields very different behavior compared to those of Guderley solutions, or those in [15, 16, 20]: instead of suffering amplitude blowup, the primary flow variables $\rho$, $u$, $c$ remain bounded near the center of motion, and instead suffer gradient catastrophes at time of collapse $t = 0$. However, notwithstanding the infinite gradients, the solutions propagate as continuous flows.

\footnote{We note that $\lambda = 1$ provides the setting for the study of multi-d Riemann problems, [27].}
to positive times. We find it noteworthy that this can occur even in cases where all fluid particles move toward the origin at time \( t = 0 \).

The issue of shock formation and propagation in multi-d Euler flows has recently been analyzed in great detail, providing fundamental new results in the field, see [4–8, 18, 19] and references therein. In this connection, the solutions we obtain here simply point out that singularity formation (i.e., some of the primary flow variables suffer a gradient catastrophe), does not necessarily give rise to a shock wave; for further detail see Remark 1.5.

**Remark 1.3.** We have not addressed the stability of the solutions we obtain. However, we note that their pressure fields do not suffer gradient blowup. In fact, at time of collapse the pressure vanishes super-linearly as \( r \downarrow 0 \) (see Section 4.3), which might provide a stabilizing effect.

The construction of globally continuous self-similar flows with \( \lambda \in (0, 1) \) follows the standard strategy of building solutions from trajectories of the ODE (1.12) connecting some of its equilibria. However, the requirements of continuity and \( 0 < \lambda < 1 \) impose additional constraints. First, as in [16], we show that continuity of the flow (specifically, boundedness of \( \rho \) and \( c \) at the center of motion prior to collapse) requires the similarity parameter \( \kappa \) to take the “isentropic” value \( \kappa = \tilde{\kappa} \) in (1.21). As noted above, this choice renders the flow globally isentropic. We verify that it also guarantees the absence of a gradient catastrophe prior to \( t = 0 \) (Section 4.3).

In addition, to guarantee the existence of suitable trajectories when \( \lambda \in (0, 1) \), further restrictions must be imposed. These are dictated by the requirement that a certain critical point (\( P_8 \) in what follows) be a proper node with a suitable primary direction. It turns out that this requires the space dimension to be 3, and that \( \lambda \) belongs to the restricted range \((0, \frac{1}{9})\). Finally, the adiabatic constant needs to be sufficiently large, viz. \( \gamma > \gamma_3(\lambda) \), where the latter is an increasing function satisfying

\[
\lim_{\lambda \downarrow 0} \gamma_3(\lambda) = \gamma_* \approx 8.72, \quad \lim_{\lambda \uparrow \frac{1}{9}} \gamma_3(\lambda) = +\infty;
\]

see Figure 3. With these assumptions met, we verify numerically the existence of suitable ODE trajectories. Our main findings are as follows:

**Main Results.** Consider radial self-similar solutions of the form (1.9) to the full multi-d Euler system (1.1)-(1.3) in space dimension 2 or 3, with similarity variables \( \lambda \) and \( \kappa \). Then:

1. No Guderley solutions (converging shock invading a quiescent state) appear possible when \( \lambda \in (0, 1) \).
2. The existence of continuous self-similar solutions requires that \( \kappa \) takes the “isentropic” value \( \kappa = \tilde{\kappa} \) in (1.21); in turn, this choice renders the flow globally isentropic and without singularities (gradient catastrophes) prior to collapse.
3. With \( \kappa = \tilde{\kappa} \) there is a 1-parameter family of continuous self-similar solutions (1.9) which describe a converging wave collapsing at the origin at time \( t = 0 \). Our construction of this type of solution requires \( n = 3 \), \( \lambda \in (0, \frac{1}{9}) \), and sufficiently large values of \( \gamma \), viz. \( \gamma > \gamma_3(\lambda) \).
4. The solutions described in (3), while locally bounded, are such that \( \rho, u, c \) all suffer gradient catastrophes at the origin at time of collapse. The pressure is \( C^1 \)-smooth and vanishes super-linearly as the center of motion is approached at time \( t = 0 \).
5. Notwithstanding infinite gradients in \( \rho, u, \) and \( c \) at collapse, we provide examples of solutions that extend as continuous similarity solutions to positive times. Numerical evidence suggests that no outgoing shock is generated whenever \( \lambda \) and \( \gamma \) are as described in (3).

Two remarks are in order.

**Remark 1.4.** The solutions described in parts (3)-(5) have locally bounded mass, momentum, and energy. On the other hand, it is readily verified that they are unbounded as \( r \to \infty \) at any fixed...
the Main Results (i.e., $n = 3$)
in $\lambda > 1$ and $\gamma > 1$. However, it appears reasonable that the same local behavior near the center of motion can be obtained in solutions with bounded mass, momentum, and energy. This could be achieved by fixing a time $t_0 < 0$ and modifying the self-similar solution outside of a sufficiently large ball $B_{R_0}(0)$. Specifically, $R_0$ should be larger than the radial position $r_c(t_0)$, where $r_c(t)$ denotes the critical 1-characteristic (sonic line) passing through the origin at $t = 0$. This would ensure that the modified part of the solution remain causally independent of the flow near $r = 0$, provided the modified solution remains continuous up to time $t = 0$. It is reasonable that this scenario can be achieved (e.g., by having the modification at time $t_0$ generate a suitable expanding rarefaction wave), but we stress that we do not have a rigorous proof of this.

Remark 1.5. Concerning the absence of shocks, it is of interest to consider the behavior of 1-characteristics near the center of motion in the continuous solutions described above. For this, fix a time $t_0 < 0$ and let $r(t, \xi)$ denote the 1-characteristic that passes through location $r = \xi$ at time $t_0$, i.e.,

$$
\partial_t r(t, \xi) = (u - c)|_{(t, r(t, \xi))}, \quad r(t_0, \xi) = \xi.
$$

The critical 1-characteristic (sonic line) which arrives at the origin at time of collapse, propagates through the origin at $t = 0$.

Since $\lambda < 1$, the solutions have infinite total mass, momentum, and energy. However, it appears reasonable that the same local behavior near the center of motion can be obtained in solutions with bounded mass, momentum, and energy. This could be achieved by fixing a time $t_0 < 0$ and modifying the self-similar solution outside of a sufficiently large ball $B_{R_0}(0)$. Specifically, $R_0$ should be larger than the radial position $r_c(t_0)$, where $r_c(t)$ denotes the critical 1-characteristic (sonic curve) passing through the origin at $t = 0$. This would ensure that the modified part of the solution remain causally independent of the flow near $r = 0$, provided the modified solution remains continuous up to time $t = 0$. It is reasonable that this scenario can be achieved (e.g., by having the modification at time $t_0$ generate a suitable expanding rarefaction wave), but we stress that we do not have a rigorous proof of this.

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use of the singular points at infinity \((P_{\pm\infty})\) and at the origin \((P_1)\). For the resulting flows we then analyze the restrictions placed on \(\lambda\) and \(\kappa\) by integrability and continuity constraints. These are dealt with in Section 3 where it is found that the latter constraint fixes \(\kappa = \mathring{\kappa}\). We also verify that no gradient catastrophe occurs prior to collapse in the resulting flows.

The construction of the relevant trajectories is detailed in Section 5. For this we want that one of the critical points, \(P_8\), is a proper node, guaranteeing that an infinite number of trajectories are drawn to it. This requires a detailed analysis of various quantities defined in terms of the partial derivatives of \(F\) and \(G\) at \(P_8\). We then show how the requirement that the saddle point \(P_{+\infty}\) be joined to the node at \(P_8\) via a trajectory \(\Gamma_1\) of (1.10)-(1.11) imposes the additional constraints \(n = 3, \lambda \in (0, \frac{2}{3})\), and \(\gamma > \gamma_3(\lambda)\) (Sections 5.3-5.4). We next describe how to select suitable trajectories \(\Gamma_2\) joining \(P_8\) to its reflection \(P_9\) about the \(V\)-axis (Section 5.5). Such trajectories must pass through the proper node \(P_1\) at the origin; there is typically an infinite number of such solutions. Finally, we add the reflection \(\Gamma_3\) of \(\Gamma_1\) about the \(V\)-axis to define the complete solution trajectory \(\Gamma := \Gamma_1 \cup \Gamma_2 \cup \Gamma_3\) of (1.10)-(1.11). The corresponding flow variables defined via (1.9) and (1.22) then provide global, 3-dimensional, self-similar, and continuous Euler flows. Section 5.6 summarizes the construction and the required numeric tests, which are done in Section 5.7.

To illustrate the construction in Section 5, we provide figures displaying the various trajectories for the particular case \(n = 3, \lambda = 0.02, \gamma = 12\). We finally illustrate graphically the absence of a shock wave in the flow after collapse in this case (Section 5.8).

2. Critical points

Throughout this section \(n = 2\) or \(3\), and \(\gamma > 1\). The goal is to identify the critical points of (1.12) and to determine how their presence depends on the parameters \(\lambda, \kappa, \gamma,\) and \(n\); see Section 2.2.3. Further requirements are imposed in Section 4.

Remark 2.1. The presence of some of the critical points (viz. \(P_0 - P_9\) in the notation introduced below) places certain constraints on the parameters \(\lambda, \kappa, \gamma,\) and \(n\); see Section 2.2. Further requirements are imposed in Section 4.

We introduce the critical lines

\[
\mathcal{L}_\pm := \{(V, C) \mid C = \pm(1 + V)\},
\]

and note the relation

\[
F(V, \pm(1 + V)) \equiv \pm \left(\frac{\gamma - 1}{2}\right) G(V, \pm(1 + V)). \tag{2.1}
\]

The critical points of (1.12) are the points of intersection between the zero-level sets

\[
\mathcal{F} := \{(V, C) : F(V, C) = 0\} \quad \text{and} \quad \mathcal{G} := \{(V, C) : G(V, C) = 0\}
\]

of the functions \(F\) and \(G\) defined in (1.13) and (1.14), respectively. Note that \(V = V_*\) (see (1.16)) is a vertical asymptote for \(\mathcal{G}\). The following symmetries will be important in assembling trajectories of (1.10)-(1.11),

\[
G(V, -C) = G(V, C), \quad F(V, -C) = -F(V, C). \tag{2.2}
\]

It turns out that there are up to nine points of intersection between \(\mathcal{F}\) and \(\mathcal{G}\), and we follow [17] in numbering these \(P_i = (V_i, C_i), i = 1, \ldots, 9\). In addition there are two critical points at infinity,

\[
P_{\pm\infty} := (V_*, \pm\infty), \tag{2.3}
\]

both of which are used in the construction of continuous Euler flows in Section 5.
2.1. Critical points \( P_1 - P_3 \). We begin by observing that there are always three critical points located along the \( V \)-axis:

\[
P_1 := (0,0), \quad P_2 := (-1,0), \quad \text{and} \quad P_3 := (-\lambda,0).
\]

Of these only \( P_1 \) is relevant for our purposes. The linearization of \( (1.12) \) at \( P_1 \) is \( \frac{dC}{dV} = \frac{C}{V} \) (for all values of \( n, \gamma, \kappa, \) and \( \lambda \), showing that \( P_1 \) is a star point (proper node). Thus, for any straight line \( \ell \) from the origin, there is a unique trajectory \( (V,C(V)) \) of \( (1.12) \) which approaches the origin tangent to \( \ell \).

Assume now that a solution \( (V(x),C(x)) \) of \( (1.10)-(1.11) \) approaches \( P_1 \) with slope \( k \). The corresponding trajectory \( (V,C(V)) \) of \( (1.12) \) then satisfies \( C(V) \approx kV \) for \( V \approx 0 \), and an inspection of \( (1.10)-(1.11) \) yields

\[
\frac{dV}{dx} \approx \frac{V}{x} \quad \text{and} \quad \frac{dC}{dx} \approx \frac{C}{x}
\]
as \( P_1 \) is approached. It follows from this that any solution of \( (1.10)-(1.11) \) reaching \( P_1 \) must do so for \( x = 0 \), and also that the limits

\[
\nu := \lim_{x \to 0} \frac{V(x)}{x} \quad \text{and} \quad \omega := \lim_{x \to 0} \frac{C(x)}{x}
\]
exist as finite numbers. (2.4)

This last property is a minimal requirement for \( (1.9) \) to yield a meaningful flow at time \( t = 0 \). (In the limiting case that \( P_1 \) is reached with infinite slope, \( \nu \) vanishes.)

Observe also that due to the requirement that \( c(t,r) \geq 0 \), we get from \( (1.9) \) that a solution \( (V(x),C(x)) \) of \( (1.10)-(1.11) \) must necessarily pass from the upper half-plane \( \{ C > 0 \} \) to the lower half-plane \( \{ C < 0 \} \) as \( x \) increases from negative to positive values.

2.2. Critical points \( P_4-P_9 \). The critical points \( P_4-P_9 \) are obtained by solving \( G(V,C) = 0 \) for \( C^2 \) in terms of \( V \), and substituting the result into the equation \( F(V,C) = 0 \); this yields a cubic polynomial in \( V \) (see below). According to the symmetries in \( (2.2) \), the critical points \( P_4-P_9 \) come in pairs located symmetrically about the \( V \)-axis. The ones located above (below) the \( V \)-axis are \( P_4 \) \( (P_5) \), \( P_6 \) \( (P_7) \), and \( P_8 \) \( (P_9) \). It turns out that among these, \( P_4 \) and \( P_5 \) are present for all values of \( n, \kappa, \lambda, \) and \( \gamma \), while \( P_6-P_9 \) may or may not be present.

Restricting attention to \( P_4, P_6, \) and \( P_8 \), we proceed to determine when and where these occur. From \( G(V,C) = 0 \) we have

\[
C^2 = \frac{V(1+V)(\lambda+V)}{n(V-V_s)}.
\]

Substituting (2.5) into \( F(V,C) = 0 \), and recalling that we now seek critical points off the \( V \)-axis, give the following cubic equation for \( W := 1+V \):

\[
[nk_1 - 1]W^3 - [nk_2 - \beta k_1 + \alpha + (\lambda - 2)]W^2
+ [nk_3 - \beta k_2 - (\lambda - 2)\alpha + (\lambda - 1)]W + [k_3 + (\lambda - 1)\alpha] = 0,
\]

where \( \alpha \) and the \( k_i \) are given in \( (1.17) \) and \( (1.18) \), and \( \beta = -n(1+V_s) \). This cubic always has one real root, denoted \( W_4 \), and two possibly complex roots \( W_6 \) and \( W_8 \). The root \( W_4 = 1+V_4 \) is given by

\[
V_4 = -\frac{\lambda}{1+\frac{2}{\gamma-1}},
\]

(cf. Eqn. (3.3) in [17]). We note that \( V_4 \) is independent of \( \kappa \); however, the corresponding \( C \)-value \( C_4 > 0 \), given by (2.5), does depend on \( \kappa \) through \( V_s \). The two remaining roots \( V_6 \equiv V_- \) and \( V_8 \equiv V_+ \) are given by

\[
V_{\pm} = \frac{1}{2m\gamma} \left[ (\gamma - 2)\mu + \kappa + m\gamma \pm \sqrt{(\gamma - 2)^2\mu^2 - 2\mu(\gamma + 2) - \kappa(\gamma - 2)} \right],
\]

where we have set

\[
m := n - 1 \quad \text{and} \quad \mu := \lambda - 1.
\]
Therefore, whenever $P_0$ and $P_8$ are present, they are necessarily located on one of the critical lines $L_{\pm}$. By symmetry, the same applies to $P_7$ and $P_9$.

We proceed to determine when the critical points $P_0$ and $P_8$ are present. This amounts to deciding whether the radicand in (2.11) is non-negative. To do so we consider two situations: either $\kappa$ is a free parameter, or $\kappa = \hat{\kappa}$ is given in terms of $\lambda$ and $\gamma$ by (1.21). We start with the general case where $\kappa$ is independent of $\lambda$, $\gamma$, and $n$.

2.2.1. General case: $\kappa$ free. Consider the radicand in (2.7) as a polynomial in $\mu = \lambda - 1$; to organize the analysis we consider four sub-cases:

(i) For $\gamma = 2$ the radicand in (2.7) is linear in $\mu$, with the single root corresponding to

$$\lambda = \lambda_{\max} := 1 + \frac{(2m+\kappa)^2}{16m}. \quad (2.10)$$

In this case, $V_{\pm}$ are real if and only if $\lambda \leq \lambda_{\max}$. The limiting case $\lambda = \lambda_{\max}$ yields $V_+ = V_- = \frac{\kappa}{4m} - \frac{1}{2}$.

Next, a direct calculation shows that for $\gamma \neq 2$ the radicand in (2.7) has the roots

$$\lambda = 1 + \frac{(m\gamma + \kappa)^2}{\left(\gamma \sqrt{\mu} \pm \sqrt{2\gamma m - \kappa(\gamma - 2)}\right)^2}. \quad (2.11)$$

Here the $\pm$ signs are unrelated to those in (2.7). The expressions in (2.11) generalize the expressions recorded by Lazarus who treated the cases $\kappa = 0$ and $\kappa = \hat{\kappa}$ (Section 3 in [17]).

(ii) When $\gamma \neq 2$ and the radicand $2\gamma m - \kappa(\gamma - 2)$ in (2.11) is strictly negative, then the radicand in (2.7) has no real $\mu$-root. Therefore, since the coefficient of $\mu^2$ in (2.11) is positive, the radicand in (2.7) is then strictly positive. Consequently, $V_{\pm}$ are necessarily real and distinct numbers in this case.

If $\gamma \neq 2$ and the radicand in (2.11) satisfies $2\gamma m - \kappa(\gamma - 2) \geq 0$, there are two further sub-cases depending on whether the minus-sign in (2.11) gives a vanishing denominator:

(iii) When $\gamma \neq 2$ and $\kappa = -\gamma m$ (in particular, the radicand in (2.11) is strictly positive, but the minus-sign gives a $\frac{\mu}{4}$ expression), substitution of the latter $\kappa$-value directly into (2.7) gives

$$V_{\pm} = \frac{1}{4m\gamma} \left(\gamma - 2\right) \mu - 2m\gamma \pm \sqrt{(\gamma - 2)^2 \mu^2 - 4\gamma^2 m^2 \mu}. \quad (2.12)$$

In this case $V_{\pm}$ are real numbers if and only if $\mu \leq 0$ or $\mu \geq \frac{4m\gamma^2}{(\gamma - 2)^2}$, i.e., if and only if

$$\lambda \leq \lambda_{\min} := 1 + \frac{4m\gamma^2}{(\gamma - 2)^2}. \quad (2.13)$$

We have that $V_+ = V_-$ if and only if $\lambda$ takes one of the values $\lambda_{\min}$ or $\lambda_{\max}$.

(iv) Finally, consider the case when $\gamma \neq 2$, $\kappa \neq -\gamma m$, and the radicand $2\gamma m - \kappa(\gamma - 2)$ in (2.11) is non-negative. We set

$$\lambda_{\max} := 1 + \frac{(m\gamma + \kappa)^2}{\left(\gamma \sqrt{\mu} + \sqrt{2\gamma m - \kappa(\gamma - 2)}\right)^2}, \quad (2.14)$$

and

$$\lambda_{\min} := 1 + \frac{(m\gamma + \kappa)^2}{\left(\gamma \sqrt{\mu} - \sqrt{2\gamma m - \kappa(\gamma - 2)}\right)^2}, \quad (2.15)$$

so that $V_{\pm}$ are real if and only if, either

$$\lambda \leq \lambda_{\max} \quad \text{or} \quad \lambda \geq \lambda_{\min}. \quad \text{9}$$
Figure 1. The zero-level curves of $F(V, C)$ (solid, including the $V$-axis) and $G(V, C)$ (dashed), together with the critical lines $L_{\pm} = \{C = \pm(1 + V)\}$ and the vertical asymptote $V = V_*$ (dotted). The parameters are $n = 3$, $\gamma = \frac{5}{3}$, $\lambda = \frac{2}{3}$, and $\kappa = \hat{\kappa} = 1$. All of the singular points $P_1$-$P_9$ are present in this case (solid dots).

Again, $V_+ = V_-$ if and only if $\lambda$ takes one of the values $\lambda_{\text{min}}$ or $\lambda_{\text{max}}$.

2.2.2. The case $\kappa = 0$. For later use we consider separately the case when $\kappa = 0$. $V_{\pm}$ are then real provided

$$\lambda \leq \lambda_{\text{max}} = 1 + \frac{m\gamma}{(\sqrt{\gamma} + \sqrt{2})} \quad \text{or} \quad \lambda \geq \lambda_{\text{min}} = 1 + \frac{m\gamma}{(\sqrt{\gamma} - \sqrt{2})}.$$ 

For the special value $\gamma = 2$, we have $V_{\pm}$ real whenever

$$\lambda \leq \lambda_{\text{max}} = 1 + \frac{m}{4}.$$ 

Note that when $\kappa = 0$, we necessarily have $\lambda_{\text{max}} > 1$.

2.2.3. Isentropic case: $\kappa = \hat{\kappa}$. In this case $\kappa$ is fixed according to (1.21) for given $\lambda$ and $\gamma$. In terms of $\mu = \lambda - 1$ we have

$$\hat{\kappa} = -\frac{2\mu}{\gamma - 1},$$

and substitution of this $\kappa$-value into (2.7) gives

$$V_{\pm} = \frac{1}{2}(a \pm \sqrt{Q}), \quad \text{(2.16)}$$

where

$$a = \frac{(\gamma - 3)}{m(\gamma - 1)}\mu - 1 \quad \text{and} \quad Q = \left(\frac{(\gamma - 3)}{m(\gamma - 1)}\right)^2\mu^2 - 2\frac{(\gamma + 1)}{m(\gamma - 1)}\mu + 1. \quad \text{(2.17)}$$

To have $V_{\pm}$ real requires $Q \geq 0$. Regarding $Q$ as a polynomial in $\mu$ there are two cases:

(a) When $\gamma = 3$, $Q$ is linear in $\mu$ and $Q \geq 0$ if and only if $\mu \leq \frac{m}{2}$. In terms of $\lambda$ this means that $V_{\pm}$ are real if and only if $\lambda \leq \lambda_{\text{max}} := 1 + \frac{m}{2}$. Also, $V_- = V_+$ if and only if $\lambda = \lambda_{\text{max}}$.

(b) For $\gamma \neq 3$, $Q$ is a quadratic in $\mu$ with a positive leading coefficient. A direct calculation shows that $Q \geq 0$ if and only if, either $\lambda \leq \lambda_{\text{max}}$ or $\lambda \geq \lambda_{\text{min}}$, where

$$\lambda_{\text{max}} = 1 + \frac{m(\gamma - 1)}{\sqrt{8(\gamma - 1)}} \quad \text{(2.18)}$$
\[ \lambda_{\text{min}} = 1 + \frac{n(\gamma-1)}{(\gamma+1)-\sqrt{8(\gamma-1)}} \]  

Finally, \( V_+ = V_- \) if and only if \( \lambda \) takes one of the values \( \lambda_{\text{min}} \) or \( \lambda_{\text{max}} \).

We note that, in either case (a) or case (b), \( \lambda_{\text{max}} > 1 \) holds due to our assumptions \( n \geq 2 \) and \( \gamma > 1 \). In particular, when \( \kappa = \hat{\kappa} \), \( P_6 \) and \( P_8 \) are present whenever \( \lambda < 1 \). In particular, when \( \kappa = \hat{\kappa} \), \( P_6 \) and \( P_8 \) are present whenever \( \lambda < 1 \).

Figure 1 displays a representative case with \( \kappa = \hat{\kappa} \) and all critical points present.

2.3. Critical points \( P_{\pm\infty} \). The critical points at infinity are \( P_{\pm\infty} = (V_*, \pm\infty) \). To analyze these we consider \( P_{+\infty} \) (sufficient according to (2.2)) and change to the variables \( W := V - V_* \) and \( Z := C^{-2} \). Linearizing the resulting equation for \( \frac{dZ}{dW} \) about \( (W, Z) = (0, 0) \) yields

\[ \frac{dZ}{dW} = -\frac{AZ}{nW - BZ}, \]  

where

\[ A = 2(1 + \alpha \frac{\alpha}{1 + V_*}), \quad B = V_*(1 + V_*)(\lambda + V_*). \]  

For later reference we note that \( P_{+\infty} \) is a saddle point if and only if \( A > 0 \). The latter condition is satisfied when \( \kappa = \hat{\kappa} \), since \( \alpha \) then vanishes (see (1.17) and (1.21)).

3. Absence of Guderley solutions when \( 0 < \lambda < 1 \)

Recall from Section 1.2 that a Guderley solution refers to a radial self-similar solution of the form (1.9) to the Euler system (1.6)-(1.8), defined (at least) for all negative times, and in which a converging shock wave approaches the origin by propagating into a quiescent fluid near the center of motion (i.e., the fluid is at rest and at constant pressure and density there).

It is further assumed that the parameters \( \kappa \) and \( \lambda \) are the same inside and outside of the converging shock, and that the shock follows a path with \( \alpha = \frac{x_s}{x} \equiv x_s \), where \( x_s \) is a negative constant. As pointed out by Lazarus [17], the constant density inside the converging shock implies that the parameter \( \kappa \) must be zero for a Guderley solution. In this work we assume \( \lambda \neq 1 \), and it follows from (1.9) that the sound speed \( c \) must vanish within the quiescent region \( x < x_s \). For the case of an ideal gas, this means that the temperature vanishes identically there. (We stress that the density within the quiescent region does not vanish in a Guderley solution; the collapse of a spherical vacuum region is a different problem which also admits similarity solutions, [3, 14, 17].)

Therefore, for a Guderley solution, we have \( (V(x), C(x)) \equiv (0, 0) \) for \( -\infty < x < x_s \). To the best of our knowledge, starting with [11], all works on Guderley solutions assume that \( \lambda > 1 \). In order that the shock accelerates and collapses with infinite speed one must have \( \lambda > 1 \).

Among the many works on Guderley solutions (and collapsing cavities) we have found only a few that address the choice of range for \( \lambda \). Among these, [14][17] simply choose to disregard cases where the shock collapses with vanishing speed, while [3] (p. 16) claims that \( \lambda < 1 \) “is incompatible with any finite pressure before the wave.” However, we do not see any a priori reason to exclude cases with \( 0 < \lambda < 1 \). If the Euler system admitted converging similarity shocks for this range, they would provide examples of “glancing” shocks that weaken, slow down, and reach the origin with zero speed.

However, based on numerical tests, we conjecture that the Euler system (for an ideal gas in 2 or 3 space dimensions) does not admit Guderley solutions with this type of glancing similarity shock. In the rest of this section we briefly describe the analysis leading to this conclusion. Thus, for the remainder of this section, the assumptions

\[ 0 < \lambda < 1 \quad \text{and} \quad \kappa = 0 \]
are in force. First, the Rankine-Hugoniot relations for a discontinuity propagating in a similarity solution along a curve \( x \equiv x_0 \) in the \((r,t)\)-plane are

\[
1 + V_+ = \frac{2-1}{\gamma+1} (1 + V_-) + \frac{2C_0^2}{\gamma+1} \quad (3.1)
\]

\[
C_0^2 = C_+^2 + \frac{2(1+V_-)(1+V_+)}{\gamma+1} - 2 = (1 + V_-)^2 - (1 + V_+)^2 \quad (3.2)
\]

\[
R_+(1 + V_+) = R_-(1 + V_-), \quad (3.3)
\]

where the subscripts \( - \) and \( + \) refer to states immediately prior to and after passing through the shock, respectively. Note that, in a Guderley solution the converging shock invades a quiescent state where \((V_-, C_-) = (0, 0)\), and it follows from (3.1) and (3.2) that

\[
P_+ = (V_+, C_+) = ( - \frac{2}{\gamma+1}, \sqrt{2\gamma(-1)}), \quad (3.4)
\]

Note that \( P_+ \) depends only on \( \gamma \) and is located on the graph of the function \( C_+(V) := \sqrt{(1 + V)(2 + V)} \) for \(-1 < V < 0\).

Next, the entropy conditions for a converging similarity 1-shock defined for negatives times take the form

\[
C_- < 1 + V_- \quad \text{and} \quad C_+ > 1 + V_+. \quad (3.5)
\]

(It may be shown from the entropy conditions that any converging self-similar shock with a quiescent inner state is necessarily a 1-shock when \( \lambda > 0 \).) As is evident from (3.4), \( P_+ \) is located above the critical line \( L_+ \), so that the 1-shock in a Guderley solution is entropy admissible.

To construct a Guderley solution (defined for all \( t < 0 \)) it is then necessary to find a solution \((V(x), C(x))\) of (1.10)-(1.11) which starts out from \( P_+ \) with \( x = x_0 < 0 \), and reaches the critical point \( P_1 \) at the origin with \( x = 0 \). In particular, it must cross the critical line \( L_+ \). Since the denominator \( D(V, C) \) in (1.10)-(1.11) vanishes there, the only possibility is that the trajectory crosses at one of the critical points \( P_0 \) or \( P_8 \). For fixed \( n = 2 \) or \( 3 \) and \( \gamma > 1 \) it turns out that only certain values of \( \lambda \) makes this happen.

This non-linear eigenvalue problem for \( \lambda \) was first addressed by Guderley [11], and later by several authors, see [24]. As far as we know, the most comprehensive treatment (always for \( \lambda \geq 1 \)) is due to Lazarus [17], who also carried out detailed numerical calculations. One conclusion of these works is that, for \( n = 2 \) or \( 3 \) and \( \gamma > 1 \), there is always at least one \( \lambda \)-value \( \lambda > 1 \) for which the trajectory starting at \( P_+ \) passes through either \( P_0 \) or \( P_8 \), and then proceeds to reach \( P_1 = (0, 0) \). (In fact, depending on \( \gamma \), there can be whole intervals of allowed \( \lambda \)-values; also, once \( P_0 \) or \( P_8 \) has been reached from \( P_+ \), there may be infinitely many trajectories connecting to the origin; see [17].)

With this background we now turn to the possibility of generating a Guderley solution when \( 0 < \lambda < 1 \). As \( \kappa = 0 \) it follows from Section 2.2.2 that \( V_0 = V_- \) and \( V_8 = V_+ \) are real, so that the critical points \( P_0, P_8 \) are necessarily present. A direct calculation using (2.6) and (2.7) shows that

\[
V_6 < -1 < -\lambda < V_4 < 0 < V_8 < V_8
\]

in this case. (We omit the details; similar computations are detailed in the proof Lemma 5.1 below.) It follows that \( P_0 \) is located to the left of the vertical line \( V = -1 \), and therefore belongs to \( L_+ \), while \( P_8 \) is located to the right of the vertical asymptote \( V = V_* \) of \( G \), and lies on \( L_+ \). Also, in the case under consideration, \( 0 < C_4 < 1 + V_4 \), so that \( P_4 \) is located strictly below \( L_+ \).

Let \( \Gamma_+ \) denote the sought-for trajectory (i.e., staring at \( P_+ \) and ending at \( P_1 = (0, 0) \)). Since \( x < 0 \) along \( \Gamma_+ \), and since its starting point \( P_+ \) lies above \( L_+ \), it follows from (1.10)-(1.11) that the trajectory moves in the direction of the vector field \((-G(V, C), -F(V, C))\) as \( x \) increases from \( x_s < 0 \) toward 0. It may be verified that \( G(V, C) < 0 \) whenever \((V, C)\) belongs to the region

\footnote{See [24]. There is an apparent third possibility in the exceptional case that \( P_1 \) happens to lie on \( L_+ \). However, this is not a separate case: it can be shown that if \( P_1 \in L_+ \), then \( P_4 \) necessarily coincides with either \( P_0 \) or \( P_8 \).}

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\( \mathcal{R} := \{ -1 < V < 0 \text{ and } C > 1 + V \} \). Since \( \Gamma_+ \) starts at \( P_+ \in \mathcal{R} \), it starts out moving to the right, and it follows that the only possibility for \( \Gamma_+ \) to reach \( P_1 \) is by crossing \( \mathcal{L}_+ \) at \( P_8 \).

An inspection of the ODE system (1.10)-(1.11) shows that, depending on the value of \( \gamma > 1 \), this could potentially happen in one of two ways:

(A) either \( P_+ \) is located near \( P_2 = (-1, 0) \) and above \( \mathcal{F} = \{(V, C) : F(V, C) = 0\} \) (this happens for \( \gamma \)-values sufficiently close to 1), and \( \Gamma_+ \) would start out by moving up in a North-East direction, then cross \( \mathcal{F} \) horizontally, before moving down in a South-East direction toward \( P_8 \); or,

(B) \( P_+ \) is located below \( \mathcal{F} \), and \( \Gamma_+ \) would move monotonically in a South-East direction toward \( P_8 \). (This could only occur for \( \gamma \) large, so that \( P_+ \) is located above \( P_8 \).

However, numerical tests with various choices for \( \lambda \in (0, 1) \) and \( \gamma > 1 \) indicate that neither of these scenarios actually occurs. In all cases we have considered the trajectory \( \Gamma_+ \) hits \( \mathcal{L}_+ \) well to the left of \( P_8 \). To have \( \Gamma_+ \) reach \( P_8 \) it appears advantageous to choose \( \gamma \gg 1 \), so that \( P_+ \approx (0, \sqrt{2}) \) is as close as possible to \( P_8 \). However, even with extreme values for \( \gamma \) (of order \( 10^6 \), say), we have not been able to find cases where \( \Gamma_+ \) even crosses into the right half-plane (where \( P_8 \) is located) before hitting \( \mathcal{L}_+ \).

We therefore abandon the search for Guderley solutions when \( \lambda \in (0, 1) \), and instead turn to the construction of shock-free solutions for this parameter regime. To do so we first need to consider constraints imposed on the similarity parameters \( \lambda, \kappa \).

4. Restrictions on \( \lambda \) and \( \kappa \)

In this section the similarity parameters \( \lambda \) and \( \kappa \) are at the outset free, while \( \gamma > 1 \) is fixed and \( n = 2, 3 \). The goal is to obtain restrictions on \( \lambda \) and \( \kappa \) from physically relevant constraints as described below. Some of the arguments in this section are similar to those in [16], for completeness we include the details.

4.1. Restrictions from integral bounds. Referring to the discussion in Section 2.1 we restrict attention to solutions \((V(x), C(x))\) of (1.10)-(1.11), which pass through the origin with (2.4) satisfied. It follows from (1.9) that the flow variables at time of collapse are given by

\[
\rho(0, r) = R(0)r^\kappa, \quad u(0, r) = -\frac{\nu}{\lambda} r^{1-\lambda}, \quad c(0, r) = -\frac{\mu}{\lambda} r^{1-\lambda}.
\]

As a minimal, physical requirement we insist that the resulting flow has locally finite mass, momentum, and total energy, i.e., for each \( \bar{r} > 0 \), we have

\[
\int_0^{\bar{r}} \rho(t, r)r^m dr, \quad \int_0^{\bar{r}} \rho(t, r)|u(t, r)|r^m dr, \quad \int_0^{\bar{r}} \rho(t, r) \left( e(t, r) + \frac{1}{2}|u(t, r)|^2 \right) r^m dr < \infty.
\]

Using (4.1) it is straightforward to verify that, at time \( t = 0 \), these integral bounds imply

(I) \( \kappa + n > 0 \)

(II) \( \lambda < 1 + \kappa + n \)

(III) \( \lambda < 1 + \frac{2\kappa}{n} \),

respectively. Note that (II) is a consequence of (I) and (III). For later reference we record the following consequence: According to (III) and the standing assumption \( \gamma > 1 \), we have

\[
0 < \frac{n+\kappa-2(\lambda-1)}{n\gamma} < \frac{n\gamma+\kappa-2(\lambda-1)}{n\gamma} = 1 + V_0.
\]

(4.2)
4.2. Restrictions from pointwise bounds in a continuous flow. The restrictions (I)-(III) above are now in force; in particular, (1.2) holds. We then consider any solution \((V(x), C(x))\) of the similarity ODEs (1.10)-(1.11) which is defined for all \(x < 0\), and with the property that it defines a continuous Euler flow for all \(t < 0\). As far as we are aware, the only way for this to occur is by having the solution \((V(x), C(x))\) approach the critical point \(P_{+\infty}\) in the upper half-plane:
\[
(V(x), C(x)) \rightarrow P_{+\infty} = (V_\ast, +\infty) \quad \text{as } x \downarrow -\infty.
\]
(4.3)
The latter property will hold, by construction, for the continuous solutions we analyze in Section 5 and (4.3) is assumed for the remainder of the present section.

Remark 4.1. Strictly speaking, there may be another type of continuous similarity flows with \(0 < \lambda < 1\) violating (4.3), viz. flows describing a spherical cavity (vacuum region) being filled by an inflowing gas. In this work we restrict attention to flows without open vacuum regions (but see Remark 4.2).

By imposing continuity of the flow for \(t < 0\), we require that the primary flow variables \(\rho, u, c\) are locally bounded at any fixed time strictly prior to collapse. In particular, \(\rho(t, r), u(t, r),\) and \(c(t, r)\) should remain bounded as \(r \downarrow 0\) whenever \(\tilde{t} < 0\). We proceed to analyze the implications of these requirements. For \(\tilde{t} < 0\) fixed we have
\[
u(\tilde{t}, r) = -\frac{r^{1-\lambda} V(x)}{\lambda} = -\frac{1}{\lambda} V(x) r \propto V(x) r.
\]
From (1.3) it follows that \(u(\tilde{t}, r) \sim r\) as \(r \downarrow 0\). This shows that the speed of the fluid particles, at any time \(\tilde{t} < 0\), approach zero at a linear rate as the center of motion is approached. Thus, no additional constraint is imposed on the similarity parameters \(\lambda\) and \(\kappa\) by requiring bounded (indeed, vanishing) fluid speed at the center of motion.

Next, to analyze \(c(\tilde{t}, r)\) as \(r \downarrow 0\), we need the leading order behavior of \(C(x)\) as \(x \downarrow -\infty\). Applying (4.3) in (1.11) gives
\[
\frac{1}{C} \frac{dC}{dx} \sim \frac{1}{\lambda}(1 + \frac{\alpha}{1 + V_\ast}) \frac{1}{x} \quad \text{as } x \downarrow -\infty,
\]
so that
\[
C(x) \sim |x|^\sigma \quad \text{as } x \downarrow -\infty, \quad \text{where} \quad \sigma = \frac{1}{\lambda}(1 + \frac{\alpha}{1 + V_\ast}).
\]
(4.4)
As \(\tilde{t}\) is fixed, we have \(x \propto -r^{-\lambda}\) and (1.9) gives
\[
c(\tilde{t}, r) \sim r^{1-\sigma \lambda} \quad \text{as } r \downarrow 0.
\]
(4.5)
Boundedness of \(c(\tilde{t}, r)\) as \(r \downarrow 0\) therefore imposes the constraint \(1 - \sigma \lambda \geq 0\). According to (4.4) and (4.2), this amounts to \(\alpha \leq 0\), or, according to (1.17),
\[
2(\lambda - 1) + \kappa(\gamma - 1) \leq 0.
\]
(4.6)
Next, to obtain the behavior of \(\rho(\tilde{t}, r)\) as \(r \downarrow 0\), we use the exact integral (1.19) together with \(V(x) \sim V_\ast\), \(C(x) \sim |x|^\sigma\), and \(x \propto r^{-\lambda}\), to get that
\[
\rho(\tilde{t}, r) \sim r^{\kappa + \frac{2(\sigma - 1)}{1 - \gamma + q}} \quad \text{as } r \downarrow 0,
\]
(4.7)
where \(q\) is given by (1.20). Boundedness of \(\rho(\tilde{t}, r)\) as \(r \downarrow 0\) therefore requires
\[
\kappa + \frac{2(\sigma - 1)}{1 - \gamma + q} \geq 0.
\]
(4.8)
We claim that (4.8), together with requirement (I) in Section 4.1 (1.2), and (4.6), imply that \(\kappa\) must take the “isentropic” value \(\hat{\kappa}\) given in (1.21). To see this, note that (1.20), (I), and (4.6) (i.e., \(\alpha \leq 0\)) give \(q \leq 0\). Therefore, the denominator in (4.8) satisfies \(1 - \gamma + q < 0\), and (4.8) holds if and only if
\[
\kappa(1 - \gamma + q) + 2\lambda(\sigma - 1) \leq 0.
\]
Recalling that boundedness of $c(t, r)$ near $r = 0$ requires (4.6), i.e., $\alpha \leq 0$. If $\alpha < 0$ (4.9) simplifies to
\[
\frac{\alpha}{1 + \nu} \leq \frac{n}{\kappa - n} (\lambda - 1 + \frac{\kappa}{2}(\gamma - 1)) \equiv \frac{n^\gamma \alpha}{\kappa - n}.
\] (4.9)

Recall that boundedness of $c(t, r)$ near $r = 0$ requires (4.6), i.e., $\alpha \leq 0$. If $\alpha < 0$ (4.9) simplifies to
\[
\frac{\alpha}{1 + \nu} \geq \frac{n^\gamma}{\kappa - n},
\]
which, according to (4.2), (I) and (1.16), reduces to
\[
n(\gamma - 1) \leq 2(\lambda - 1).
\] (4.10)

However, $\alpha < 0$ also gives $2(\lambda - 1) < -\kappa(\gamma - 1)$, so that (1.10) yields $n(\gamma - 1) < -\kappa(\gamma - 1)$, or $n + \kappa < 0$. This contradicts the integrability condition (I), and we conclude that $\alpha$ must vanish, i.e., we must have $\kappa = \bar{\kappa}$. We observe that, with $\kappa = \bar{\kappa}$, (4.7) and (4.5) indeed provide bounded values for both $\rho(t, r)$ and $c(t, r)$ as $r \downarrow 0$. As detailed above (after (1.21)), it follows that the resulting flow in this case is globally isentropic. We sum up our findings in the following proposition:

**Proposition 4.1.** Let $n = 2$ or $3$ and fix $\gamma > 1$ and $\lambda > 0$. Consider any solution $(V(x), C(x))$ of the similarity ODEs (1.10)-(1.11), defined for $x < 0$ and satisfying (4.3) and (2.4) (with $\mu$ and $\nu$ finite and nonzero). Finally, let $R \geq 0$ be given by (1.19), and define the flow variables $\rho, u, c$ according to (1.9).

Then the requirements (I)-(III) in Section 4.1, together with boundedness of $\rho(t, r)$ and $c(t, r)$ as $r \downarrow 0$ at fixed times $t < 0$, imply that the similarity parameter $\kappa$ in (1.9) must have the value $\bar{\kappa}$ given in (2.11). Finally, with $\kappa = \bar{\kappa}$ the resulting Euler flow (defined for $t < 0$) is necessarily isentropic.

From now on $\kappa = \bar{\kappa}$ is assumed. We note that the integrability conditions (I)-(III) in Section 4.1 then reduce to the single requirement (III), which now reads
\[
\lambda < \bar{\lambda}(\gamma, n) := 1 + \frac{2}{\gamma}(1 - \frac{1}{\gamma}).
\] (4.11)

This is trivially satisfied if $0 < \lambda < 1$, a fact we make use of in Section 5.

4.3. **Isentropic behavior near** $r = 0$. The arguments above show that with $\kappa = \bar{\kappa}$, any continuous solution of (1.10)-(1.11) which is defined for all $x < 0$ and satisfies (4.3) and (2.4), generates flow variables $u(t, r), c(t, r), \rho(t, r)$ that approach finite values as $r \downarrow 0$ at each fixed $t < 0$. We now verify that these finite values are approached with bounded gradients. In particular, no gradient catastrophe occurs in the flow prior to collapse at time $t = 0$.

First, consider $u_r(t, r)$; according to (1.9) and (1.10) we have
\[
u u_r(\bar{t}, r) = - \frac{1}{M} \left( V(x) + \frac{G(V(x), C(x))}{D(V(x), C(x))} \right), \quad \text{where} \quad x = \frac{\bar{t}}{r^\bar{\kappa}}.
\]

Recalling that $C(x) \uparrow +\infty$ and $V(x) \to V_*$ as $r \downarrow 0$, we get from (1.14) that
\[
\frac{G(V(x), C(x))}{D(V(x), C(x))} \to 0 \quad \text{as} \quad r \downarrow 0.
\]

It follows from this that
\[
u u_r(\bar{t}, r) \sim - \frac{V_*}{M} \quad \text{as} \quad r \downarrow 0.
\]

Similarly, using (1.9) and (1.11), we have
\[
u c_r(\bar{t}, r) = \frac{C(x)}{M} \left[ \frac{1-k_1(1+V(x))^2}{C(x)^{2k_2}} - (1+V(x))k_3 \right],
\]
and it follows from (1.3) that $c_r(\bar{t}, r) \sim 0$ as $r \downarrow 0$. In particular, to leading order, $c(\bar{t}, r)$ is constant as $r \downarrow 0$. Finally, since $\rho \propto c^{\frac{2}{7-k}}$ in isentropic flow, the same applies to the density field.

We conclude that, in the isentropic setting under consideration, at any fixed time $\bar{t} < 0$, all of $u_r(\bar{t}, r), c_r(\bar{t}, r), \rho_r(\bar{t}, r)$, and hence also $\rho_r(\bar{t}, r)$, remain bounded as $r \downarrow 0$. In particular, no gradient catastrophe occurs at $r = 0$ at strictly negative times.
On the other hand, at time of collapse \( t = 0 \), (2.4) and (1.9) give
\[
\rho(0, r) = r^\kappa R(0), \quad u(0, r) = -\frac{\omega}{\rho} r^{1-\lambda}, \quad c(0, r) = -\frac{\omega}{\rho} r^{1-\lambda}.
\]
In particular, provided \( \nu \) and \( \omega \) are finite and nonzero, and \( 0 < \lambda < 1 \), we see that both the velocity and sound speed suffer a gradient catastrophe at the origin at \( t = 0 \).

The same applies to the density field provided \( \kappa < 1 \), i.e., \( \lambda > \frac{3-\gamma}{2} \). Note that the latter inequality is satisfied whenever \( \lambda > 0 \) and \( \gamma > 3 \), as will be the case for the solutions we construct in Section 5. On the other hand, the pressure field at time of collapse is, by (1.5),
\[
p(0, r) = \frac{1}{\gamma} \rho(0, r)c^2(0, r) \propto r^{\kappa+2(1-\lambda)},
\]
which suffers a gradient catastrophe at \( r = 0 \) provided \( \kappa + 2(1-\lambda) < 1 \), or equivalently,
\[
\lambda > \frac{1}{2}(1 + \frac{1}{\gamma}). \tag{4.12}
\]
As we shall see, (4.12) will be violated for all solutions we construct below: their pressure fields are at least \( C^1 \)-smooth at time of collapse.

**Remark 4.2.** We note that, with \( \lambda \in (0, 1) \) and \( \kappa := \kappa \) the density field at time of collapse satisfies \( \rho(0, r) \propto r^\kappa \), which vanishes at the origin. The resulting Euler flow therefore has a one-point vacuum at the origin at time of collapse.

5. Construction of continuous flows with \( 0 < \lambda < 1 \) and \( \kappa = \kappa \)

We now turn to the construction of continuous, and in particular, locally bounded radial Euler flows with similarity variable \( \lambda \in (0, 1) \). As explained in Section 4.2, we restrict attention to solutions satisfying (1.3), and Proposition 4.1 then shows that we must choose \( \kappa = \kappa \) in order to meet the physical constraints (I)-(III) in Section 4.1. Thus, for the remainder of the paper it is assumed that
\[
0 < \lambda < 1 \quad \text{and} \quad \kappa = \kappa = \frac{2(1-\lambda)}{\gamma-1}. \tag{5.1}
\]
We observed at the end of Section 2.2.3 that \( V_0 = V_- \) and \( V_8 = V_+ \) are both real under assumptions (5.1), so that the critical points \( P_1-P_3 \) are all present.

5.1. **Outline of construction.** The continuous flows are built by identifying solution trajectories \( \Gamma_1-\Gamma_3 \) of (1.10)-(1.11) with the properties
- (\( \Pi_1 \)) \( \Gamma_1 \) connects \( P_+ \) to \( P_8 \);
- (\( \Pi_2 \)) \( \Gamma_2 \) connects \( P_8 \) to \( P_0 \) and passes through \( P_1 = (0, 0) \);
- (\( \Pi_3 \)) \( \Gamma_3 \) connects \( P_0 \) to \( P_- \).

The symmetries recorded in (2.2) effectively reduce our task to identifying only \( \Gamma_1 \) and \( \Gamma_2 \): \( \Gamma_3 \) will simply be the the reflection of \( \Gamma_1 \) about the \( V \)-axis, connecting \( P_0 \) to \( P_- \).

Recall from Section 2.3 that \( P_\pm \) are saddle points. In searching for a trajectory \( \Gamma_1 \) satisfying (\( \Pi_1 \)) it is therefore advantageous that \( P_8 \) be a nodal point into which a large family of trajectories are drawn. Indeed, a key part of the following analysis (Section 5.4) concerns the identification of a \( (\lambda, \gamma) \)-regime for which \( P_8 \) is a proper node.

Care must be taken that \( \Gamma_1 \) reaches \( P_8 \) without crossing the critical line \( \mathcal{L}_+ \). It turns out that the latter requirement fixes the spatial dimension to be \( n = 3 \) (see Section 5.4.2). Likewise, the trajectory \( \Gamma_2 \) must connect to the origin without crossing \( \mathcal{L}_+ \); this however, will not impose any further constraints on the parameters.

Having identified a suitable \( (\lambda, \gamma) \)-regime (for \( n = 3 \)) we find it necessary to verify numerically that there are cases in which \( \Gamma_1 \) connects \( P_+ \) to \( P_8 \) without first crossing \( \mathcal{L}_+ \). With \( \Gamma_1 \), and hence \( \Gamma_3 \), thus determined, it remains to determine a suitable trajectory \( \Gamma_2 \) satisfying (\( \Pi_2 \)). It is unproblematic to reach the origin \( P_1 \) from \( P_8 \): among the trajectories leaving the node at \( P_8 \), there are infinitely many that connect to the origin. However, as the trajectory is continued through the
origin, it should subsequently be drawn into $P_9$. It turns out that this last requirement determines a range of possible slopes with which $\Gamma_2$ can reach $P_1$; see Section 5.5. Again, we verify numerically the existence of trajectories $\Gamma_2$ meeting these constraints.

We note that it is necessary to make a final check on the selected trajectories $\Gamma_1-\Gamma_3$: they must provide admissible solutions trajectories for the original ODE system (1.10)-(1.11) (as opposed to the single ODE (1.12)). As explained in Section 2.1, any solution $(V(x), C(x))$ of physical relevance must necessarily pass through the origin $P_1$ with $x = 0$, and from the upper half-plane to the lower half-plane as $x$ increases. Also, they must move along $\Pi_1-\Pi_2-\Pi_3$ in the correct direction given by (1.10)-(1.11) as $x$ increases from $-\infty$ to $+\infty$.

Figure 2 provides a representative case of the vector field $\left(\frac{-1}{\lambda x} G(V,C), \frac{-1}{\lambda x} F(V,C)\right)$ corresponding to the ODE system (1.10)-(1.11), with $x < 0$ ($x > 0$) in the upper (lower) half-plane. Notice that the arrows provide the actual direction of flow for solutions $(V(x), C(x))$ as $x$ increases.

Remark 5.1. Figure 2 illustrates the impossibility of having a solution of the ODE system (1.10)-(1.11) cross the critical lines $L_\pm = \{C = \pm(1 + V)\}$ at a non-singular point: such trajectories of the autonomous ODE (1.12) fail to yield relevant solutions to (1.10)-(1.11) since the vector field corresponding to (1.10)-(1.11) points in opposite directions on either side of $L_\pm$.

Note that the parameters in Figure 2 are chosen for illustrative purposes; in particular, the behavior near $P_8$ is such that property $(\Pi_1)$ fails in this case. It will be shown below that to satisfy $(\Pi_1)$, we need to choose $n = 3$, $\lambda$ sufficiently small, and $\gamma$ sufficiently large. It turns out that with such parameter values it is necessary to zoom in at the critical points $P_8$, $P_1$, and $P_9$ in order to display the behavior there; see Figures 6 and 7.

5.2. Location of critical points. We start by determining the relative $V$-locations of the critical points under the assumptions in (5.1). For convenience we repeat the expressions for $V_\pm$ (see (2.16)
and \((2.17)\) and \(V_\ast\) in terms of \(\mu = \lambda - 1:\)

\[
V_\pm = \frac{1}{2}(a \pm \sqrt{Q}) \quad \text{and} \quad V_\ast = \frac{-2\mu}{n(\gamma - 1)},
\]

(5.2)

where

\[
a = \frac{(\gamma - 3)}{m(\gamma - 1)}\mu - 1 \quad \text{and} \quad Q = \left(\frac{(\gamma - 3)}{m(\gamma - 1)}\right)^2 \mu^2 - 2\frac{(\gamma + 1)}{m(\gamma - 1)}\mu + 1.
\]

(5.3)

Recalling \((2.6)\) and introducing the positive constant

\[
k := \frac{m^2}{2}[n(\gamma - 1) + 2],
\]

(5.4)

we have

\[
V_4 = -\frac{2\mu}{k}(1 + \mu).
\]

(5.5)

For later reference we note that

\[
V_8 - V_6 \equiv V_+ - V_- = \sqrt{Q},
\]

(5.6)

and

\[
V_8 - V_4 = \frac{1}{2} \left[\left(\frac{(\gamma - 3)}{m(\gamma - 1)} + \frac{2m}{k}\right)\mu + \left(\frac{2m}{k} - 1\right) + \sqrt{Q}\right].
\]

(5.7)

Lemma 5.1. Assuming \(n = 2 \text{ or } 3, \gamma > 1, \text{ and that (5.7) holds, we have}\)

\[
V_6 < V_2 = -1 < V_3 = -\lambda < V_4 < 0 < V_\ast < V_8.
\]

(5.8)

Proof. We consider each inequality in turn, from left to right:

- \(V_6 = V_- < -1: \) According to (5.2) this inequality amounts to

\[
a + 2 < \sqrt{Q}.
\]

(5.9)

It is immediate to verify that \(a + 2 > 0\) if and only if \((\gamma - 1)(\mu + m) > 2\mu, \) which holds since \(m \geq 1, \) \(-1 < \mu < 0, \) and \(\gamma > 1.\) It follows that (5.9) is equivalent to \((a + 2)^2 < Q; \) substituting from (5.3) shows that the latter inequality reduces to \(\gamma > 1, \) establishing the first inequality.

- \(-1 < -\lambda: \) Immediate by (5.1).

- \(-\lambda < V_4: \) Substituting from (5.4) and (5.5), and recalling that \(\mu + 1 = \lambda > 0, \) this inequality reduces to \(n(\gamma - 1) > 0, \) which holds since \(\gamma > 1.\)

- \(V_4 < 0: \) Immediate from (5.5) since \(m, k (\text{see (5.4)}), \) and \(\mu + 1 = \lambda \) are all positive.

- \(0 < V_\ast: \) Immediate from (5.2) since \(\mu < 0 < \gamma - 1.\)

- \(V_\ast < V_8 = V_4: \) By substituting from (5.2) we obtain the equivalent inequality

\[
-\frac{4\mu}{n(\gamma - 1)} - a < \sqrt{Q}.
\]

(5.10)

Using the expression for \(a\) in (5.3) and rearranging, we get that the left-hand side of (5.10) is positive provided \(n(\gamma + 1) - 4 > \frac{mn(\gamma - 1)}{\mu}, \) which holds since the left-hand side in the latter inequality is positive (because \(n \geq 2\) and \(\gamma > 1), \) while the right-hand side is negative. It follows that (5.10) is equivalent to

\[
\left(-\frac{4\mu}{n(\gamma - 1)} + a\right)^2 < Q.
\]

(5.11)

Substituting from (5.3) for \(a\) and \(Q, \) and simplifying the result, we obtain that (5.11) is equivalent to \(n(\gamma - 1) > -\mu(n(\gamma - 1) - 1), \) which holds since \(\gamma > 1\) and \(-\mu = 1 - \lambda < 1.\)

\(\square\)
5.3. **The critical point** $P_{+\infty}$. This analysis was done in Section 2.3, and it was noted there that $P_{+\infty} = (V_*, +\infty)$ is necessarily a saddle point when $\kappa = \hat{\kappa}$. It follows that there is a unique trajectory $\Gamma_1$ of (1.12) which approaches $P_{+\infty}$. An inspection of $F(V, C)$ and $G(V, C)$ shows that the solutions of (1.12) have negative slopes within the the region

$$\Omega := \{(V, C) \mid V_* < V < V_8, 1 + V < C < \bar{C}(V)\}$$

(5.12)

where $C = \bar{C}(V)$ denotes the $V$-parametrization of the zero-level $G$ of $G$. Furthermore, their slopes are finite along $V = V_*$ and infinite along $C = \bar{C}(V)$. It follows that $\Gamma_1$ is located within $\Omega$ and can reach its boundary only along $C = 1 + V$ for some $V \in (V_*, V_8)$, or at $P_8$. In order to be useful for our purpose of building a globally defined fluid flow, we must have that $\Gamma_1$ passes through $P_8$.

5.4. **Behavior near** $P_8$; **construction of** $\Gamma_1$. In this subsection, unless indicated differently, all quantities are evaluated at $P_8 = (V_8, C_8)$, and the subscript ‘8’ is suppressed in most of the expressions. To determine the type of the critical point $P_8$ we shall need the signs of various quantities given in terms of the partial derivatives of $F$ and $G$ there. First, since $P_8 \in L_{+} \cap F \cap G$ we have

$$C = 1 + V$$

(5.13)

$$C^2 = k_1(1 + V)^2 - k_2(1 + V) + k_3$$

(5.14)

$$C^2 = \frac{V(1+V)(\lambda + V)}{n(V-V_*)}.$$  (5.15)

At $P_8$ we then have

$$F_C = 2C^2$$

(5.16)

$$F_V = C(2k_2 - 2k_1(1 + V))$$

(5.17)

$$G_C = 2nC(V - V_*) \equiv 2V(\lambda + V)$$

(5.18)

$$G_V = C(n - \lambda + nV_* - 2V).$$

(5.19)

Here, $F_C$ and $F_V$ are calculated from (1.15) (using that $\alpha = 0$ when $\kappa = \hat{\kappa}$), while $G_C$ and $G_V$ are calculated from (1.14), and using that

$$nC(V - V_*) = V(\lambda + V),$$

the latter being a consequence of (5.13) and (5.15). We note that (5.16), (5.18), and (5.19) give

$$F_C > 0 \quad \text{and} \quad G_C > 0.$$  (5.20)

Next, using the expressions above, we obtain that

$$G_V + G_C = C(2mV + m - \mu - nV_*),$$

(5.21)

and substitution of the expressions in (5.2) for $V = V_8 = V_+$ and $V_*$ then yields

$$G_V + G_C = mC \sqrt{Q} > 0.$$  (5.22)

Applying (2.1), we therefore obtain

$$F_V + F_C = -\frac{(\gamma - 1)}{2}(G_V + G_C) < 0,$$

(5.23)

so that

$$F_V < -F_C < 0.$$  (5.24)

Finally, we note that the expressions above give

$$F_C + G_V = C \left(n + 1 - \frac{\gamma + 1}{\gamma - 1} \mu\right) > 0.$$  (5.25)
We next recall some notation and results from Lazarus [17]. The Wronskian is defined by
\[ W := F_C G_V - F_V G_C, \]
and the discriminant by
\[ R^2 := (F_C - G_V)^2 + 4F_V G_C \equiv (F_C + G_V)^2 - 4W. \]  
(5.26)

In the following, whenever \( R^2 > 0 \), we set \( R := +\sqrt{R^2} > 0 \). Next, with
\[ L_{1,2} = \frac{1}{2G_C} (F_C - G_V \pm R) \]  
(5.27)

and
\[ E_{1,2} = \frac{1}{2G_C} (F_C + G_V \pm R), \]  
(5.28)

and signs chosen so that
\[ |E_1| < |E_2|, \]  
(5.29)

we have that integrals of (1.12) near \( P_8 \) approach one of the curves
\[ (c - L_1v)^{E_1} = \text{constant} \times (c - L_2v)^{E_2}, \]
where \( v = V - V_8 \) and \( c = C - C_8 \). Note that the signs \( \pm \) in (5.27) and in (5.28) agree; \( L_1 \) and \( L_2 \) are referred to as the primary and secondary slopes (or directions), respectively. Provided that \( R^2 > 0 \) (so that \( R \) is real and positive) and \( W > 0 \), \( P_8 \) is a proper node. In this case all solution curves approaching \( P_8 \) do so with slope equal to the primary slope \( L_1 \), except one which approaches \( P_8 \) with slope \( L_2 \).

An elegant argument by Lazarus [17] shows that \( W \equiv W_8 \) is given as
\[ W = 2kC_8^2(V_8 - V_4)(V_8 - V_6), \]  
(5.30)

where \( k > 0 \) is given in (5.1). It follows from Lemma 5.1 that
\[ W > 0. \]  
(5.31)

We now assume that \( R^2 > 0 \) (this requirement is addressed below in Section 5.4.1) and proceed to determine the signs to be used in (5.27) so that (5.29) is satisfied. According to (5.28), (5.29) holds if and only if
\[ |F_C + G_V \pm R| < |F_C + G_V \mp R|. \]  
(5.32)

From (5.25) we have that \( F_C + G_V > 0 \), and since \( W > 0 \) we have
\[ 0 < R = \sqrt{(F_C + G_V)^2 - 4W} < F_C + G_V. \]  
(5.33)

It follows that the minus-sign should be used on the left hand side of (5.32), and the plus-sign should be used on the right hand side of (5.32). That is, under the condition that \( R^2 > 0 \), together with the standing assumption (5.1), we have
\[ L_1 = \frac{1}{2G_C} (F_C - G_V - R) \quad \text{and} \quad L_2 = \frac{1}{2G_C} (F_C - G_V + R). \]  
(5.34)

With this we have that \( P_8 \) is a proper node and that all but one of the integrals of (1.12) approaching \( P_8 \) do so with slope \( L_1 \).

We next want to determine how the primary slope \( L_1 \) compares to those of the curves \( \mathcal{G} \) and \( \mathcal{L}_+ \) at \( P_8 \). Let \( \mathcal{C}(V) \) be the \( V \)-parametrization of the zero-level curve \( \mathcal{G} \) for \( G(V,C) \), so that \( \mathcal{C}'(V_8) = -\frac{G_C}{G_V} \). Together with (5.34), and the fact that \( G_C > 0 \) (by (5.20)2), this implies that the inequality \( L_1 > \mathcal{C}'(V_8) \) is equivalent to \( F_C + G_V > R \), which holds according to (5.33). Therefore, near \( P_8 \), the straight line
\[ \mathcal{L}_1 : \quad C = C_8 + L_1 (V - V_8) \]
is located below \( \mathcal{G} \) for \( V < V_8 \) and above \( \mathcal{G} \) for \( V > V_8 \).
Before proceeding we also note the following: with $\tilde{C}(V)$ denoting the $V$-parametrization of the zero-level curve $F$ for $F(V,C)$, we have $\tilde{C}'(V) = -\frac{F_V}{C}$, and it follows from (5.20) and (5.23) that $\tilde{C}'(V_8) > 1$. Similarly, using (5.20) and (5.22), we have $\tilde{C}'(V_8) < 1$.

Finally, since $\frac{dC}{dV} = \frac{2}{\sqrt{V}} < 0$ within the region $\Omega$ given in (5.12), a necessary condition for having $\Gamma_1$ approach $P_3$ is that $L_1 < 0$. As $G_C > 0$, (5.34) shows that this condition amounts to

$$F_C < R + G_V.$$  \hfill (5.35)

Our goal now is to identify a parameter regime $(\lambda, \gamma) \in (0,1) \times (1,\infty)$ for which both of the two requirements $R^2 > 0$ and $L_1 < 0$ are satisfied. As demonstrated below in Lemma 5.3, this requires $n = 3$, $0 < \lambda < \frac{1}{9}$, and $\gamma$ sufficiently large. For such parameter values we then verify numerically that there are cases in which

- there are integrals of (1.12) passing through the node $P_8$ along the primary direction $L_1$ and crossing the vertical line $V = V_2$; and
- there are other integrals of (1.12) passing through the node $P_8$ along the primary direction $L_1$ and crossing $G$ vertically.

It then follows by continuity that the unique integral $\Gamma_1$ of (1.12) which approaches the critical point $P_{+\infty} = (V_+,+\infty)$, also passes through $P_3$ along the primary direction $L_1$.

5.4.1. The requirement $R^2(\lambda, \gamma) > 0$. The discriminant $R^2$ is given in (5.26); fixing $n = 2$ or 3, it is a function $R^2(\lambda, \gamma)$. According to (5.25), (5.26), and (5.30), we have

$$R^2(\lambda, \gamma) > 0$$

if and only if

$$\left(n + 1 - \frac{n + 3}{\gamma - 1} \mu\right)^2 > 8k(V_8 - V_4)(V_8 - V_6).$$  \hfill (5.36)

For each choice of $n = 2$ or 3 this inequality defines a certain region in the $(\lambda, \gamma)$-plane, which, according to our standing assumption (5.1), is located within the half-strip $\{0 < \lambda < 1, \gamma > 1\}$. As the next lemma shows, the location of this region depends sensitively on $n$.

**Lemma 5.2.** For $\lambda \in (0,1) \text{ fixed}$, we have

$$\lim_{\gamma \uparrow \infty} R^2(\lambda, \gamma) = 0 \quad \iff \quad \begin{cases} \lambda \in (0, \frac{1}{9}) \quad \text{ when } n = 3 \\ \lambda \in (\frac{8}{9}, 1) \quad \text{ when } n = 2. \end{cases}$$

**Proof.** Sending $\gamma \uparrow \infty$ in the expressions in (5.3), we get

$$a \rightarrow a_{\infty} := \frac{2}{m} - 1 < 0, \quad Q \rightarrow Q_{\infty} := a_{\infty}^2.$$  

It follows from (5.2) that $V_8 = V_+ \rightarrow \frac{1}{2}(a_{\infty} + |a_{\infty}|) = 0$, and $V_6 = V_- \rightarrow \frac{1}{2}(a_{\infty} - |a_{\infty}|) = a_{\infty}$. Also, from (5.4) and (2.6) we have that $V_4 \rightarrow 0$, while $kV_4 = -m(1 + \mu)$. As $\gamma \uparrow \infty$ the requirement $R^2 > 0$ in (5.36) therefore reduces to the condition

$$\left(n + 1 - \mu\right)^2 > 8|m|(\lim_{\gamma \uparrow \infty} k(V_8 - V_4)) = 8(1 - \frac{\mu}{m})\left[m(1 + \mu) + \lim_{\gamma \uparrow \infty} kV_8\right],$$  \hfill (5.37)

where the last limit is of the form “$\infty \cdot 0$.” To analyze it we determine more precisely the distance between $Q$ and $Q_{\infty}$, and between $a$ and $a_{\infty}$, as $\gamma \uparrow \infty$. Rewriting the expression (5.3) for $Q$, we find that

$$Q = (1 - \frac{\mu}{m})^2 - \frac{4\mu}{m(\gamma - 1)} \left[1 + \frac{\mu}{m} \left(\frac{\gamma - 2}{\gamma - 1}\right)\right] \sim (1 - \frac{\mu}{m})^2 - \frac{4\mu(m + \mu)}{m(\gamma - 1)}\quad \text{as } \gamma \uparrow \infty.$$  

To leading order in $\gamma$ we therefore have

$$\sqrt{Q} \sim (1 - \frac{\mu}{m}) - \frac{2\mu(m + \mu)}{m(\gamma - 1)} \quad \text{as } \gamma \uparrow \infty.$$  

Combining this with

$$a = (\frac{\mu}{m} - 1) - \frac{2\mu}{m} \frac{1}{(\gamma - 1)},$$  

we finally conclude...
Figure 3. The graph of the function $\gamma_3(\lambda)$ which is defined for $0 < \lambda < \frac{1}{9}$; the discriminant $R^2(\lambda, \gamma)$ in (5.26) is positive when $\gamma > \gamma_3(\lambda)$; $\gamma_* = \gamma_3(0) \approx 8.72$.

gives
$$V_8 = \frac{1}{2}(a + \sqrt{\mathcal{C}}) \sim -\frac{2\mu}{(m-\mu)(\gamma-1)}, \quad \text{as } \gamma \uparrow \infty.$$ Recalling the expression (5.4) for $k$ we obtain
$$\lim_{\gamma \uparrow \infty} kV_8 = \frac{mn\mu}{m-\mu}.$$ Using this in (5.37) we conclude that, as $\gamma \uparrow \infty$, the requirement $R^2 > 0$ reduces to the condition
$$(n+1-\mu)^2 > 8[(m-\mu)(1+\mu) - n\mu]. \quad (5.38)$$ Finally, with $n = 3$ ($m = 2$), (5.38) becomes $8\mu + 9\mu^2 > 0$, which reduces to $\lambda = 1 + \mu < \frac{1}{9}$; with $n = 2$ ($m = 1$), (5.38) becomes $(1+\mu)(1+9\mu) > 0$, which reduces to $\lambda > \frac{8}{9}$. $\square$

Numerical plots of the curve in the $(\lambda, \gamma)$-plane defined by $R^2(\lambda, \gamma) = 0$ reveal that it is the graph of an:

• increasing function $\lambda \mapsto \gamma_3(\lambda)$ defined for $\lambda \in (0, \frac{1}{9})$ and with a vertical asymptote at $\lambda = \frac{1}{9}$ when $n = 3$;
• decreasing function $\lambda \mapsto \gamma_2(\lambda)$ defined for $\lambda \in (\frac{8}{9}, 1)$ and with a vertical asymptote at $\lambda = \frac{8}{9}$ when $n = 2$.

Figure 3 shows the situation for $n = 3$; the minimum value of $\gamma_3(\lambda)$ is $\gamma_* := \gamma_3(0) \approx 8.72$.

5.4.2. The requirement $L_1 < 0$. For $n$ fixed we now consider $R^2$ and $L_1$ as functions of $\lambda$ and $\gamma$. Recall that $L_1(\lambda, \gamma) < 0$ is a necessary condition for having the trajectory $\Gamma_1$ connect $P_+\infty$ to $P_8$, and that we want $P_8$ to be a node, i.e., we need $R^2(\lambda, \gamma) > 0$. The following lemma shows that only the case $n = 3$ is favorable in this regard.

**Lemma 5.3.** For $\lambda \in (0, 1)$ and $\gamma > 1$ we have

(1) for $n = 3$: if $R^2(\lambda, \gamma) > 0$, then $L_1(\lambda, \gamma) < 0$;
(2) for $n = 2$: if $R^2(\lambda, \gamma) > 0$, then $L_1(\lambda, \gamma) > 0$.

**Proof.** As detailed above, the requirement $L_1 < 0$ amounts to the inequality in (5.35). First, if $n = 3$ and $R^2 > 0$ (so that $R > 0$), then (5.35) follows once we verify that $F_C < G_V$. A direct calculation shows that the latter inequality (for $n = 3$) reduces to
$$4(\gamma - 3)(\gamma - 1) > \mu(3\gamma - 5)(\gamma + 1),$$
which is trivially satisfied whenever $\gamma > 3$ since $\mu = \lambda - 1 < 0$. According to the analysis above, $R^2 > 0$ implies $\gamma > \gamma_4 > 3$ when $n = 3$, establishing part (1) of the lemma.

Next consider the case $n = 2$. According to (5.24) and (5.20), the inequality $L_1 > 0$ amounts to $F_C - G_V > R$. For $n = 2$ it follows from (5.16), (5.19), and (5.13), that $F_C > G_V$ if and only if

$$2V_8 + \frac{3}{4} > V_4,$$

which is satisfied since $\lambda > 0$ and $V_8 > V_4$ (by Lemma 5.1). If $R^2 > 0$, so that $R > 0$, it follows that $F_C - G_V > R$ holds if and only if $(F_C - G_V)^2 > R^2$. Substituting from (5.26) for $R^2$ shows that the latter inequality reduces to $F_V G_C < 0$, which is satisfied according to (5.21) and (5.24). We conclude that, for $n = 2$, $L_1 > 0$ whenever $R^2 > 0$.

With this we have identified the relevant parameter regime in which to search for continuous similarity flows when $\lambda \in (0, 1)$: we need to choose $n = 3$, and $(\lambda, \gamma)$ so that $R^2(\lambda, \gamma) > 0$, i.e., $\lambda \in (0, \frac{1}{4})$ and $\gamma > \gamma_3(\lambda)$. This guarantees that $P_8$ is a node with a negative primary slope $L_1$. It remains to provide examples in which the trajectory $\Gamma_1$ from $P_{+\infty}$ is drawn into $P_8$. As noted above, a sufficient condition for this behavior is the existence of trajectories that enter the region $\Omega$ (see (5.12)) along its left edge at $V = V_8$, and from there continue on to reach $P_8$. The numerical verification of this condition is addressed in Section 5.7.

Before moving on to the behavior near $P_1$ we note the following consequence of the proof of Lemma 5.3: when $n = 3$ and $R^2(\lambda, \gamma) > 0$, then $L_1(\lambda, \gamma) < L_2(\lambda, \gamma) < 0$. Indeed, $L_1(\lambda, \gamma) < L_2(\lambda, \gamma)$ holds by definition, while the inequality $L_2(\lambda, \gamma) < 0$ amounts to $R < G_V - F_C$. The proof of Lemma 5.3 showed that when $n = 3$ and $R^2(\lambda, \gamma) > 0$, then $G_V - F_C > 0$. Therefore, $R < G_V - F_C$ holds provided $R^2 < (G_V - F_C)^2$, which, according to (5.20), amounts to $F_V G_C < 0$. The latter inequality is satisfied according to (5.20) and (5.24).

5.5. Behavior near $P_1$; construction of $\Gamma_2$. While there is (at most) a single trajectory $\Gamma_1$ joining $P_{+\infty}$ to $P_8$, there will be a continuum of trajectories joining $P_8$ to $P_9$ via the proper node $P_1$ at the origin. To identify these it is convenient to also classify the critical point $P_4$. Lazarus [17] shows that the the Wronskian there is given as

$$W_4 = 2kC_3^2(V_4 - V_6)(V_4 - V_8),$$

(5.39)

where $k$ is a positive constant (cf. (5.30)). It follows from Lemma 5.1 that $W_4 < 0$ so that $P_4$, and hence also $P_5$, are saddle points. An inspection of the $(V, C)$-plane reveals the presence of three relevant separatrices (see Figure 6):

- $\Theta$ joining $P_8$ to $P_4$;
- $\Phi$ joining $P_4$ to $P_1$; and
- $\Psi$ joining $P_1$ to $P_5$.

Let $\zeta > 0$ denote the slope of $\Psi$ at $P_1$; by symmetry, the slope of $\Phi$ at $P_1$ is then $-\zeta$.

The trajectories $\Gamma_2$ of interest to us (i.e., the ones leading to continuous Euler flows) are those that reach the origin $P_1$ from $P_8$, and then moves on to $P_9$ in the lower half-plane.

As is clear from Figure 6 in order for $\Gamma_2$ to reach the origin, it must be located to the right of the separatrix $\Theta$. Also, it follows from the analysis at the end of Section 5.4.2 that all trajectories leaving $P_8$ do so with a negative slope: either $L_1$ or $L_2$, where $L_1 < L_2 < 0$. Let $\Gamma_s$ denote the unique one among these which leaves with slope $L_2$. Now, all trajectories leaving $P_8$ (with $V(x)$ increasing) proceed to cross vertically that part of the zero-level $\mathcal{G} = \{G = 0\}$ which is located in the first quadrant. Among these, some reach the origin with negative slopes after having vertically crossed also that part of $\mathcal{G}$ located in the second quadrant within the half-strip

If we use the trajectory $\Gamma$, the resulting Euler flow will suffer a weak discontinuity (i.e., a discontinuity in the first derivatives of the flow variables) across a curve $r(t) = (\frac{1}{x_8})^{t}$ in the $(r, t)$-plane, where $x_8 < 0$ is such that $(V(x_8), C(x_8)) = P_8$. This curve is a 1-characteristic for the radial Euler system (1.6)-(1.8).
{(V, C) : V_4 < V < 0, C > 0}. In all cases we have considered, it is clear from numerical tests that there are other trajectories from \( P_8 \) which reach \( P_1 \) with positive slopes; see Remark 5.2 below. In the following discussion it is assumed that this is the case. Evidently, the smallest positive slope with which \( P_1 \) can be reached from \( P_8 \) is that with which \( \Gamma_s \) approaches \( P_1 \); we denote the latter slope by \( \epsilon > 0 \).

We next observe that the trajectory \( \Gamma_1 \) is not allowed to change its slope as it passes through \( P_1 \).\(^4\)

Recalling the symmetry \((2.2)\) of the phase portrait, and setting \( \delta := \max(\zeta, \epsilon) > 0 \), we have that any of the infinitely many trajectories from \( P_8 \) which arrives at \( P_1 \) with a slope \( s \in (-\infty, -\delta) \cup (\delta, +\infty] \), continues into the lower half-plane and reaches \( P_9 \). Indeed, the part of its trajectory in the lower half-plane will simply be the reflection about the \( V \)-axis of one of the other trajectories from \( P_8 \) to \( P_1 \), viz. the one arriving at \( P_1 \) with slope \(-s\). (In the limiting case that \( \Gamma_2 \) reaches \( P_1 \) vertically, the lower part of \( \Gamma_2 \) is simply the reflection of its upper part about the \( V \)-axis.) Any one of these trajectories may serve as \( \Gamma_2 \) in our construction of continuous Euler flows.

**Remark 5.2.** If \( \Gamma_s \) reached \( P_1 \) with slope \( \epsilon < 0 \), then none of the trajectories reaching the origin \( P_1 \) from \( P_8 \) would reach \( P_9 \). Again, we have not observed this scenario in any of our numerical tests. E.g., in the case displayed in Figures 5 and 6 (with \( n = 3, \lambda = 0.02, \gamma = 12 \), \( \epsilon \) is positive but so small that \( \Gamma_s \) is indistinguishable from the \( V \)-axis near the origin.

5.6. **Summary.** We briefly summarize our findings so far in this section. First, by imposing the conditions \( W > 0 \) and \( R^2 > 0 \) at the critical point \( P_8 \), we guarantee that \( P_8 \) is a node. The former requirement is automatically met once \((5.1)\) holds, while the second requirement puts an \( n \)-dependent constraint on \( \lambda \) and \( \gamma \). The further condition \( L_1(\lambda, \gamma) < 0 \), which is necessary in order that the trajectory \( \Gamma_1 \) connects to \( P_8 \) without first crossing \( L_+ \), implies that the space dimension \( n \) must be 3. With \( n = 3 \), the requirement \( R^2 > 0 \) is met, provided \( (\lambda, \gamma) \) lies above a certain graph \( \gamma_3(\lambda) \) defined for \( 0 < \lambda < \frac{1}{4} \). If this is the case, then \( L_1(\lambda, \gamma) < 0 \) is automatically satisfied.

Next, for all cases we have investigated numerically (see Section 5.7), there is an infinite number of trajectories \( \Gamma_2 \) joining \( P_8 \) to \( P_9 \) and passing through the origin \( P_1 \). Care must be taken that the trajectory from \( P_8 \) arrives at \( P_1 \) with a slope \( s \) for which another trajectory arrives at \( P_1 \) from \( P_8 \) with the slope \(-s\). When this holds, \( \Gamma_2 \) consists of the former trajectory together with the reflection of the latter about the \( V \)-axis.

Finally, the trajectory \( \Gamma_3 \) joining \( P_9 \) to \( P_{-\infty} \) is the reflection of \( \Gamma_1 \) about the \( V \)-axis.

To finish the argument for the existence of continuous, radial similarity Euler flows, as described in the Main Results, it remains to verify the following two points:

(i) with \( n = 3 \), there are choices of \( \lambda \in (0, \frac{1}{4}) \) and \( \gamma > \gamma_3(\lambda) \) such that the trajectory \( \Gamma_1 \) from \( P_{-\infty} \) reaches \( P_8 \), and

(ii) for these choices of the parameters there are trajectories \( \Gamma_2 \) from \( P_8 \) reaching the origin and with the following property: \( \Gamma_2 \) reaches the origin with slope \( s \), where \(|s| > \delta \) (\( \delta \) defined as above), and there is another trajectory from \( P_8 \) reaching the origin with slope \(-s\).

Below we describe the numerical verification of these points.

5.7. **Numerical verification of (i) and (ii).** The numerical verification is carried out with Maple. As explained in Sections 5.3 and 5.4 to verify (i) it suffices to show that there is at least one solution crossing into the region \( \Omega \) (see \((5.12)\)) along \( V = V_\ast \) and reaching \( P_8 \) without first crossing the critical line \( L_+ \). To numerically check this, it is convenient to switch to the variables \( W := V - V_\ast \) and \( Z := C^{-2} \) which were used in Section 2.3 to analyze the critical point \( P_{+\infty} \). The latter point is then located at the origin of the \((W, Z)\)-plane. The analysis in Section 2.3 shows

\(^4This points out a difference between \( P_1 \) and the critical points \( P_0-P_9 \). As noted above, a change of slope as \((V(x), C(x)) \) passes through \( P_8 \), say, results in a weak discontinuity in the corresponding Euler flow. In contrast, a change in slope at \( P_1 \) would generate, via \((1.9) \) and \((2.3) \), an un-physical jump discontinuity across \( t = 0 \).
Figure 4. The dash-dot curve is an approximation of the trajectory $\Gamma_1$ when plotted in the $(W,Z)$-plane, where $W = V - V_*$ and $Z = C^{-2}$. The dashed curve is the zero-level $G = \{G = 0\}$ (crossed vertically by trajectories), and the dotted curve is the critical line $L_+$. The two solid curves are solutions: the upper one passes through the point $(W, Z) = (0, 0.3)$ while the lower one starts out near $G$. Finally, the grey curve indicates the primary slope with which the solutions reach $P_8$. The parameter values are $n = 3$, $\lambda = 0.02$, and $\gamma = 12$. Evidently, $\Gamma_1$ reaches $P_8$ without first crossing $L_+$; this is confirmed by further numerical plots near $P_8$.

that this is a saddle point and that $\Gamma_1$ leaves $P_+ = (0, 0)$ with slope $\frac{dZ}{dW} = \frac{n+2}{B} = \frac{5}{B}$, where $B$ is given in (2.21). A sufficient condition for (i) to hold is that there are trajectories through points on the positive $Z$-axis which reach $P_8$ without first crossing $L_+$.

Figure 4 displays the situation in the $(W,Z)$-plane for $n = 3$, $\lambda = 0.02$, and $\gamma = 12$, and provides clear numerical evidence that this is indeed the case. (The approximation of $\Gamma_1$, the dash-dot curve in Figure 4 is obtained by starting very close to the origin along the straight line $Z = \frac{5}{B}W$.)

For numerical verification of (ii) we return to the $(V,C)$-plane and compute various trajectories joining the node at $P_8$ to the proper node (star point) $P_1$ at the origin. Figures 5 and 6 display the situation for the same parameter values as in Figure 4. Figure 5 shows the trajectory $\Gamma_2$ leaving $P_8$ along the secondary direction and reaching the origin with a very small slope $\epsilon > 0$. It also shows a complete $\Gamma_2$-trajectory, joining $P_8$ to $P_9$ via the origin, which appears to be symmetric about the $V$-axis. However, it is slightly un-symmetric and obscures the presence of the $P_4P_5$-separatrix $\Theta$; see Figure 6 for the detailed situation near the origin.

5.8. The flow at collapse and absence of shocks. Figures 5-6 display a continuous trajectory $\Gamma_2$ joining the critical points $P_8$ and $P_9$ via the critical point $P_1$ at the origin. When this is joined with the trajectories $\Gamma_1$ and $\Gamma_3$ (the latter being the reflection of $\Gamma_1$ about the $V$-axis), we obtain a global, continuous solution $(V(x), C(x))$ of (1.10)-(1.11) joining $P_{+\infty}$ to $P_{-\infty}$ as $x$ varies from $-\infty$ to $+\infty$. Finally, from this we obtain, according to (1.9), a globally defined, continuous, radial similarity Euler flow.

We now observe that whenever $\Gamma_2$ is a trajectory joining $P_8$ and $P_9$ via the origin, the same is true for its reflection $\Gamma_2'$ about the $V$-axis. Assuming $\Gamma_2$ reaches the origin with negative slope (as in Figures 5-6), the trajectory $\Gamma_2'$ will have positive slope at the origin. We therefore obtain two, physically distinct, Euler flows from these trajectories. In particular, since both $\Gamma_2$ and $\Gamma_2'$ reach the origin as $x \uparrow 0$, but with opposite signs of $V(x)$, the corresponding Euler flows display different
Figure 5. The dashed curves are the zero-levels \{F = 0\} and \{G = 0\}, and the dash-dot curve the \(\Gamma_s\)-trajectory which leaves \(P_8\) along the secondary direction. In addition there are two more solution trajectories: one is the separatrix \(\Theta\) joining \(P_4\) to \(P_8\), and the other is a complete \(\Gamma_2\)-trajectory (solid curve) joining \(P_8\) and \(P_9\) via the origin \(P_1\). However, at this resolution the latter two trajectories are indistinguishable, and the \(\Gamma_2\) trajectory appears symmetric about the \(V\)-axis. (Figure 6 displays a zoom-in near the origin which shows that this is not actually so.) The parameter values are as in Figure 4: \(n = 3, \lambda = 0.02,\) and \(\gamma = 12\).

Figure 6. Zoom-in near the origin of Figure 5. The dashed curves are the zero-levels \{F = 0\} and \{G = 0\} the dotted curves are the separatrices \(\Theta, \Phi,\) and \(\Psi\), and the solid curve is the \(\Gamma_2\)-trajectory joining \(P_8\) and \(P_9\) via the origin \(P_1\). Note that the \(\Gamma_s\)-trajectory is indistinguishable from the \(V\)-axis near the origin; thus \(\epsilon \gtrsim 0\) and \(\delta = \zeta\) in this case.
Figure 7. Zoom-in near the critical point $P_9$ with the same parameter values as in Figures 4-6: $n = 3$, $\lambda = 0.02$, and $\gamma = 12$. The solid straight line is the critical line $L_- = \{C = -1 - V\}$, which appears almost horizontal. The solid trajectory consists of parts of $\Gamma_2$ and $\Gamma_3$, located above and below $L_-$, respectively; these meet at $P_9$ along the primary direction (grey line). The dashed vertical line is the asymptote $V = V_*$ approached by $\Gamma_3$. Finally, the dotted curve is the Hugoniot locus $H$ consisting of states to which points along $\Gamma_2$ can jump. Note that $H$ is located below $L_-$ (as dictated by the entropy condition), but does not intersect $\Gamma_3$ before reaching $P_9$. As a consequence, no shock wave occurs in the corresponding Euler flow subsequent to collapse.

Behaviors at time of collapse: in the one built from $\Gamma_2$ the fluid is moving toward the origin at time of collapse, while in the one built from $\Gamma'_2$ it moves outward.

We find it noteworthy that the former flow remains continuous beyond collapse. The flow variables $\rho(0,r)$, $u(0,r)$, $c(0,r)$ all suffer gradient blowup at $r = 0$ (see Section 4.3), and in addition the fluid flow is directed inward. It would be reasonable to expect that such data would generate an expanding shock wave for $t > 0$.

However, we can observe numerically that this is not what occurs for the flow corresponding to $\Gamma_2$. In Figure 7 we have plotted the trajectory $\Gamma_2$ from Figure 5 together with the trajectory $\Gamma_3$, near $P_9$ (solid curve). We have also included the “Hugoniot-locus” $H$ of $\Gamma_2$ (dotted curve). This is the curve of points $(V_+,C_+)$ obtained from the Rankine-Hugoniot relations (3.1)-(3.2) as $(V_-,C_-)$ moves down from the origin along $\Gamma_2$. It may be deduced from the entropy condition that, for the solutions under consideration, a shock generated at collapse and propagating outward must necessarily be a 3-shock which connects the outer state $(V_-,C_-)$, located above $L_-$, to the inner state $(V_+,C_+)$ located below $L_-$. The presence of an expanding shock would then manifest itself by $H$ intersecting the trajectory $\Gamma_3$ at a point strictly below the critical line $L_-$. However, Figure 7 shows that $H$ (dotted curve) reaches $P_9$ without first intersecting $\Gamma_3$: no shock is formed.

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References

[1] S. Atzeni and J. Meyer-ter-Vehn, *The Physics of Inertial Fusion*, International Series of Monographs on Physics, vol. 125, Oxford University Press, Oxford, 2004.

[2] Anxo Biasi, *Self-similar solutions to the compressible Euler equations and their instabilities*, Commun. Nonlinear Sci. Numer. Simul. 103 (2021), Paper No. 106014, 28, DOI 10.1016/j.cnsns.2021.106014. MR4312224

[3] K. V. Bruslinškij and Ja. M. Každian, *Self-similar solutions of certain problems in gas dynamics*, Uspehi Mat. Nauk 18 (1963), no. 2 (110), 3–23 (Russian). MR0172577

[4] Tristan Buckmaster and Sameer Iyer, *Formation of unstable shocks for 2D isentropic compressible Euler*, Comm. Math. Phys. 389 (2022), no. 1, 197–271, DOI 10.1007/s00220-021-04271-z. MR4365141

[5] Tristan Buckmaster, Steve Shkoller, and Vlad Vicol, *Shock formation and vorticity creation for 3d Euler*, arXiv:2006.14789 (2020).

[6] Tristan Buckmaster, Theodore D. Drivas, Steve Shkoller, and Vlad Vicol, *Simultaneous development of shocks and cusps for 2D Euler with azimuthal symmetry from smooth data*, arXiv:2106.02143 (2021).

[7] Demetrios Christodoulou, *The formation of shocks in 3-dimensional fluids*, EMS Monographs in Mathematics, European Mathematical Society (EMS), Zürich, 2007. MR2284927

[8] R. Courant and K. O. Friedrichs, *Supersonic flow and shock waves*, Springer-Verlag, New York, 1976. Reprinting of the 1948 original; Applied Mathematical Sciences, Vol. 21. MR0421279 (54 #9284)

[9] J. Duderstadt and G. Moses, *Inertial Confinement Fusion*, Wiley, 1982.

[10] G. Guderley, *Starke kugelige und zylindrische Verdichtungsstöße in der Nähe des Kugelmittelpunktes bzw. der Zylinderachse*, Luftfahrtforschung 19 (1942), 302–311 (German). MR0008522

[11] Peter Hafner, *Strong convergent shock waves near the center of convergence: a power series solution*, SIAM J. Appl. Math. 48 (1988), no. 6, 1244–1261, DOI 10.1137/0148076. MR968828

[12] T. Hirschler and W. Gretler, *On the eigenvalue problem of imploding shock waves*, Z. Angew. Math. Phys. 52 (2001), no. 1, 151–166, DOI 10.1007/PL00001537. MR1818639

[13] Roger B. Lazarus, *Self-similar solutions for converging shocks and collapsing cavities*, SIAM J. Numer. Anal. 18 (1981), no. 2, 316–371.

[14] Jonathan Luk and Jared Speck, *Shock formation in solutions to the 2D compressible Euler equations in the presence of non-zero vorticity*, Invent. Math. 214 (2018), no. 1, 1–169, DOI 10.1007/s00222-018-0799-8. MR3858399

[15] Frank Merle, Pierre Raphaël, Igor Rodnianski, and Jeremie Szeftel, *On smooth self similar solutions to the compressible Euler equations*, arXiv:1912.10998 (2019).

[16] Scott D. Ramsey, James R. Kamm, and John H. Bolstad, *The Guderley problem revisited*, Int. J. Comput. Fluid Dyn. 26 (2012), no. 2, 79–99, DOI 10.1080/10618562.2011.647768. MR2892836

[17] L. I. Sedov, *Similarity and dimensional methods in mechanics*, “Mir”, Moscow, 1982. Translated from the Russian by V. I. Kisin. MR067290

[18] K. P. Stanyukovich, *Unsteady motion of continuous media*, Translation edited by Maurice Holt; literal translation by J. George Adasheko, Pergamon Press, New York-London-Oxford-Paris, 1960. MR0114423

[19] Yuxi Zheng, *Systems of conservation laws*, Progress in Nonlinear Differential Equations and their Applications, 38, Birkhäuser Boston Inc., Boston, MA, 2001. Two-dimensional Riemann problems. MR1839813 (2002c:35155)
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