On the irreducibility of the commuting variety of a symmetric pair associated to a parabolic subalgebra with abelian unipotent radical

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Abstract

In this paper, we study the commuting variety of symmetric pairs associated to parabolic subalgebras with abelian unipotent radical in a simple complex Lie algebra. By using the “cascade” construction of Kostant, we construct a Cartan subspace which in turn provides, in certain cases, useful information on the centralizers of non p-regular semisimple elements. In the case of the rank 2 symmetric pair (so_{p+2}, so_p × so_2), p ≥ 2, this allows us to apply induction, in view of previous results of the authors [11], and reduce the problem of the irreducibility of the commuting variety to the consideration of evenness of p-distinguished elements. Finally, via the correspondence of Kostant-Sekiguchi, we check that in this case, p-distinguished elements are indeed even, and consequently, the commuting variety is irreducible.

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1 Introduction and notations

Let \( \mathfrak{g} \) be a complex simple Lie algebra and \( \theta \) an involutive automorphism of \( \mathfrak{g} \). Let \( \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \) be the decomposition of \( \mathfrak{g} \) into eigenspaces with respect to \( \theta \), where \( \mathfrak{k} = \{ X \in \mathfrak{g} \mid \theta(X) = X \} \), \( \mathfrak{p} = \{ X \in \mathfrak{g} \mid \theta(X) = -X \} \). In this case, we say that \( (\mathfrak{g}, \mathfrak{k}) \) is a symmetric pair.
Let $G$ be the adjoint group of $\mathfrak{g}$ and $K$ the connected algebraic subgroup of $G$ whose Lie algebra is $\mathfrak{k}$.

Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$ consisting of semisimple elements. Any such subspace is called a Cartan subspace of $\mathfrak{p}$. All the Cartan subspaces are $K$-conjugate. Its dimension is called the rank of the symmetric pair $(\mathfrak{g}, \mathfrak{k})$.

We define the commuting variety of $(\mathfrak{g}, \mathfrak{k})$ as the following set:

$$C(\mathfrak{p}) = \{(x, y) \in \mathfrak{p} \times \mathfrak{p} \mid [x, y] = 0\}.$$

We may also consider the commuting variety $C(\mathfrak{g})$ of $\mathfrak{g}$, defined in the same way. Richardson proved in [10] that, if $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, then $C(\mathfrak{g}) = G.(\mathfrak{h} \times \mathfrak{h})$. In particular, the commuting variety $C(\mathfrak{g})$ is an irreducible algebraic variety.

On the other hand, the commuting variety of any semisimple symmetric pair is not irreducible in general. Panyushev showed in [7] that in the case of the symmetric pair $(\mathfrak{sl}_n, \mathfrak{gl}_{n-1})$, $n > 2$, associated to the involutive automorphism, defined via conjugation by the diagonal matrix $\text{diag}(-1, \ldots, -1, 1)$. The corresponding commuting variety has three irreducible components of dimension, respectively, $2n - 1$, $2n - 2$, $2n - 2$.

Nevertheless, in some cases, the irreducibility problem has been solved.

- As an obvious consequence of the classical case proved by Richardson, the symmetric pair $(\mathfrak{g} \times \mathfrak{g}, \Delta(\mathfrak{g}))$, associated to the automorphism $(X, Y) \mapsto (Y, X)$, has an irreducible commuting variety.

- If the rank of the symmetric pair $(\mathfrak{g}, \mathfrak{k})$ is equal to the semisimple rank of $\mathfrak{g}$ (called the maximal rank case), then Panyushev proved in [7] that the corresponding commuting variety is irreducible.

- The rank 1 case has been considered independently by the authors [11] and Panyushev [S]. In this case, it has been proved that $(\mathfrak{so}_{m+1}, \mathfrak{so}_m)$ is the only symmetric pair whose commuting variety is irreducible.

- In [8], Panyushev proves the irreducibility of the commuting variety for the symmetric pairs $(\mathfrak{sl}_{2n}, \mathfrak{sp}_{2n})$ and $(\mathfrak{E}_6, \mathfrak{F}_4)$.

For a symmetric pair of rank strictly larger than one, we observe that due to the rank 1 case, the inductive arguments used by Richardson in the classical case [10] do not apply. However, if $\mathfrak{a}$ is a Cartan subspace, then it is
well-known that $C_0 = \overline{K.(a \times a)}$ is the unique irreducible component of $C(p)$ of maximal dimension, which is equal to $\dim p + \dim a$. The main problem is therefore to determine if there exist components other than the maximal one.

In \cite{8}, it has been conjectured that $C(p)$ is irreducible if the rank of the symmetric pair is greater than or equal to 2.

In this paper, we consider symmetric pairs $(g, \mathfrak{k})$ associated to parabolic subalgebras with abelian unipotent radical. In this case, by using the “cascade” construction of Kostant, we obtain a particular Cartan subspace of $p$ which, in certain cases, turns out to be a useful tool for the description of symmetric subpairs of $(g, \mathfrak{k})$, associated to centralizers of semisimple elements of $p$.

The information on the centralizers of semisimple elements of $p$ allows us to apply the induction step of Richardson in the case of the rank 2 symmetric pair $(\mathfrak{so}_{p+2}, \mathfrak{so}_p \times \mathfrak{so}_2)$, $p \geq 2$. As a result, we obtain that all the irreducible components of $C(p)$ other than $C_0$ are necessarily related to $K$-orbits of $p$-distinguished elements in $p$.

By using the Kostant-Sekiguchi correspondence, we are able to establish that in the case of symmetric pair $(\mathfrak{so}_{p+2}, \mathfrak{so}_p \times \mathfrak{so}_2)$, $p \geq 2$, every $p$-distinguished elements is even. We prove finally that this condition is sufficient for the irreducibility of the commuting variety of $(\mathfrak{so}_{p+2}, \mathfrak{so}_p \times \mathfrak{so}_2)$, $p \geq 2$.

We shall conserve the notations above in the sequel. The reader may refer to \cite{12} for basic definitions and properties of symmetric pairs. The paper is organized as follows. In Sections 2 and 3, we study symmetric pairs associated to parabolic subalgebras with abelian unipotent radical and the description of subpairs associated to centralizers of semisimple elements. We recall some well-known results relating sheets and the commuting variety in Section 4. In particular, we prove that an even nilpotent element of $p$ belongs to a sheet containing non-zero semisimple elements. Section 5 is dedicated to the rank 2 symmetric pair $(\mathfrak{so}_{p+2}, \mathfrak{so}_p \times \mathfrak{so}_2)$, $p \geq 2$. Finally, Section 6 describes certain computations used in Section 3.
2 Symmetric pair associated to a parabolic subalgebra with abelian unipotent radical

Let us fix a Cartan subalgebra \( h \) of \( g \) and a Borel subalgebra \( b \) containing \( h \). Denote by \( R \supset R^+ \supset \Pi \) the corresponding set of roots, positive roots and simple roots. Let us also fix root vectors \( X_\alpha, \alpha \in R \), and for \( \alpha \in R \), we set \( g_\alpha = \mathbb{C}X_\alpha \).

For any subset \( S \subseteq \Pi \), we denote by \( R_S = R \cap ZS \). Then \( R_S \) is a root system in the vector subspace that it spans in \( h^* \). Moreover \( S \) is a base of simple roots for \( R_S \).

Set \( R^+_S = R_S \cap R^+ \), and

\[
q_S = h \oplus \bigoplus_{\alpha \in R_S \cup R^+} g_\alpha
\]

the standard parabolic subalgebra associated to \( S \). The unipotent radical \( u_S \) of \( q_S \) is therefore \( \bigoplus_{\alpha \in R^+_S} g_\alpha \) where \( R^+_S = R^+ \setminus R_S \).

Let \( S \subseteq \Pi \) be such that the unipotent radical \( u_S \) of \( q_S \) is abelian. Then it is easy to check that \( q_S \) is maximal. Moreover, if we set \( u^-_S = \bigoplus_{\alpha \in R^+_S} g_{-\alpha} \),

the decomposition

\[
g = u_S \oplus (h \oplus \bigoplus_{\alpha \in R_S} g_\alpha) \oplus u^-_S
\]

induces a \( \mathbb{Z} \)-grading on \( g \). This in turn defines a symmetric pair with

\[
e_S = h \oplus \bigoplus_{\alpha \in R_S} g_\alpha, \quad p_S = u_S \oplus u^-_S.
\]

These symmetric pairs correspond to inner automorphisms of \( g \) with non semisimple fixed point sets. A list of all such parabolic subalgebras and the corresponding symmetric pairs is given in Table 1 (see also Table 7 of the Reference chapter of [6]).

We shall construct a nice Cartan subspace for these symmetric pairs by using the following properties of the root system, sometimes referred to as the “cascade” construction of Kostant. We shall recall this construction and certain useful properties. For more details, the reader may refer to [4], [5] or [13]

Let \( T \subset \Pi \). If \( R_T \) is irreducible, then \( T \) is connected, and we shall denote by \( \varepsilon_T \) the highest root of \( R_T \).

We define a set \( \mathcal{K}(T) \) by induction on the cardinality of \( T \) as follows :
\( \mathcal{K}(\emptyset) = \emptyset. \)

- If \( T_1, \ldots, T_r \) are the connected components of \( T \), then:

\[
\mathcal{K}(T) = \mathcal{K}(T_1) \cup \cdots \cup \mathcal{K}(T_r).
\]

- If \( T \) is connected, then

\[
\mathcal{K}(T) = \{ T \} \cup \mathcal{K} (\{ \alpha \in T \mid \alpha \text{ and } \varepsilon_T \text{ are orthogonal} \}).
\]

Note that any \( K \in \mathcal{K}(T) \) is connected, and the roots \( \varepsilon_K, K \in \mathcal{K}(T) \), are pairwise strongly orthogonal (two distinct positive roots \( \alpha, \beta \) are strongly orthogonal if \( \alpha \pm \beta \) are not roots).

For \( K \in \mathcal{K}(T) \), denote by \( \Gamma^K \) the set of positive roots \( \alpha \in R^+_K \) which are not orthogonal to \( \varepsilon_K \).

We have the following properties of the sets \( \Gamma^K \).

- \( R^+_T = \bigcup_{K \in \mathcal{K}(T)} \Gamma^K \) (disjoint union).

- Let \( K \in \mathcal{K}(T) \). If \( \alpha \in \Gamma^K \setminus \{ \varepsilon_K \} \), then there exists a unique \( \beta \in \Gamma^K \) such that \( \alpha + \beta = \varepsilon_K \).

- Let \( K, K' \in \mathcal{K}(T) \), \( \alpha \in \Gamma^K \) and \( \beta \in \Gamma^{K'} \) be such that \( \alpha + \beta \in R \), then either \( K \subset K' \) and \( \alpha + \beta \in \Gamma^{K'} \) or \( K' \subset K \) and \( \alpha + \beta \in \Gamma^K \).

Let \( S \subset \Pi \) be such that \( u_S \) is abelian. We set

\[
\mathcal{E} = \{ K \in \mathcal{K}(\Pi) \mid \varepsilon_K \in R^1_S \},
\]

and for \( K \in \mathcal{E} \), \( X_K = X_{\varepsilon_K} + X_{-\varepsilon_K} \). Note that the elements \( X_K \) are semisimple. Let \((\mathfrak{g}, t_S)\) be the corresponding symmetric pair as defined above.

**Proposition 2.1**

1. We have \( R^1_S \subset \bigcup_{K \in \mathcal{E}} \Gamma^K \).

2. Let \( K \in \mathcal{E} \) and \( \alpha \in R^1_S \cap \Gamma^K \). Then \( \varepsilon_K - \alpha \notin R^1_S \).

3. The subspace \( \mathfrak{a} \) spanned by the \( X_K, K \in \mathcal{E}, \) is a Cartan subspace of \( \mathfrak{p}_S \), and the rank of \((\mathfrak{g}, t_S)\) is \(|\mathcal{E}|.\)
Proof. The first two parts are easy consequences of the properties of $\Gamma^K$ stated above and of the fact that $u_S$ is an abelian ideal of $q_S$.

Since the $\varepsilon_K$, $K \in \mathcal{E}$, are pairwise strongly orthogonal, the subspace $a$ of $p_S$ consists of pairwise commuting semisimple elements, and its dimension is $\mathcal{E}$. Using Table 1, we check that $a$ has the right dimension for a Cartan subspace.

We list below all the standard parabolic subalgebras $q_S$ with abelian unipotent radical, the unique simple root in $R^l_S$, the corresponding symmetric pair, and the elements of the set $\mathcal{E}$. We follow the numbering of simple roots in $\Pi$.

| ($A_n, \alpha_i$) $n \geq 1$ | (sl$_{n+1}$, sl$_{n+1-i} \times$ sl$_i \times \mathbb{C}$) | $\{\alpha_j, \ldots, \alpha_{n+1-j}\}$, $1 \leq j \leq \min(i, n + 1 - i)$ |
| ($B_n, \alpha_1$) $n \geq 2$ | (so$_{2n+1}$, so$_{2n-1} \times$ so$_2$) | $\{\alpha_1\}, \Pi$ |
| ($C_n, \alpha_n$) $n \geq 2$ | (sp$_{2n}$, gl$_n$) | $K(\Pi)$ |
| ($D_n, \alpha_1$) $n \geq 4$ | (so$_{2n}$, so$_{2n-2} \times$ so$_2$) | $\{\alpha_1\}, \Pi$ |
| ($D_n, \alpha_i$), $n \geq 4$ | (so$_{2n}$, gl$_n$) | $\{\alpha_m, \ldots, \alpha_n\}, m \leq n - 2$ odd and $\{\alpha_i\}$ if $n \in 2\mathbb{Z}$ |
| ($E_6, \alpha_1$), ($E_6, \alpha_6$) | ($E_6, D_5 \times \mathbb{C}$) | $\Pi \setminus \{\alpha_2\}, \Pi$ |
| ($E_7, \alpha_7$) | ($E_7, E_6 \times \mathbb{C}$) | $\{\alpha_7\}, \Pi \setminus \{\alpha_1\}, \Pi$ |

Table 1

3 Symmetric pair associated to the centralizer of a semisimple element

In this section, ($\mathfrak{g}, \mathfrak{t}_S$) is a symmetric pair associated to a standard parabolic subalgebra $q_S$ with abelian unipotent radical. Let $a$ be the Cartan subspace in $p_S$ as defined in part 3 of Proposition 2.1.
Let $X \in \mathfrak{a}$. Then $\mathfrak{g}^X$ is a Levi factor of a parabolic subalgebra of $\mathfrak{g}$. Denote by $I = [\mathfrak{g}^X, \mathfrak{g}^X]$ the semisimple part of $\mathfrak{g}^X$, and set $I_+ = I \cap \mathfrak{k}^X$, $I_- = I \cap \mathfrak{p}^X$ and $\mathfrak{r}_+ = [I_-, I_-]$. Then the decompositions

$$\mathfrak{g}^X = \mathfrak{k}^X \oplus \mathfrak{p}^X, \quad I = I_+ \oplus I_- \quad \text{and} \quad \mathfrak{r} = \mathfrak{r}_+ \oplus I_-$$

define symmetric subpairs of $(\mathfrak{g}, \mathfrak{k}_S)$, and the ranks of the pairs $(I, I_+)$ and $(\mathfrak{r}, \mathfrak{r}_+)$ are strictly inferior to that of $(\mathfrak{g}, \mathfrak{k}_S)$.

We shall determine, in certain cases, the symmetric pair $(\mathfrak{r}, \mathfrak{r}_+)$ for any non-zero non $\mathfrak{p}_S$-regular element $X \in \mathfrak{a}$, i.e. $\mathfrak{p}_S^X$ contains a non-zero nilpotent element.

To simplify notations, $B_0$ will correspond to the Lie algebra $\{0\}$, $B_1 = A_1$, $D_2 = A_1 \times A_1$ and $D_3 = A_3$.

We shall need the classification of simple symmetric pairs of rank less than or equal to 2.

**List 3.1 Rank 1 :**

- $(\mathfrak{sl}_{n+1}, \mathfrak{sl}_n \times \mathbb{C})$,
- $(\mathfrak{so}_{n+1}, \mathfrak{so}_n)$,
- $(\mathfrak{sp}_{2n}, \mathfrak{sp}_{2n-1} \times \mathfrak{sp}_2)$,
- $(F_4, B_4)$.

**List 3.2 Rank 2 :**

- $(\mathfrak{sl}_{n+2}, \mathfrak{sl}_n \times \mathfrak{sl}_2 \times \mathbb{C})$,
- $(\mathfrak{sl}_3, \mathfrak{so}_3)$,
- $(\mathfrak{sl}_6, \mathfrak{sp}_6)$,
- $(\mathfrak{so}_{n+2}, \mathfrak{so}_n \times \mathfrak{so}_2)$,
- $(\mathfrak{sp}_{2n+4}, \mathfrak{sp}_{2n} \times \mathfrak{sp}_4)$,
- $(\mathfrak{sp}_4, \mathfrak{sl}_2 \times \mathbb{C})$,
- $(\mathfrak{so}_{10}, \mathfrak{sl}_5 \times \mathbb{C})$,
- $(E_6, F_4)$,
- $(E_6, D_5 \times \mathbb{C})$,
- $(G_2, A_1 \times A_1)$.

**Lemma 3.3** Suppose that we are in the case $(B_n, \alpha_1)$ of Table 1. Then there exists $m \in \mathbb{N}$ such that $(\mathfrak{r}, \mathfrak{r}_+) = (\mathfrak{so}_{m+1}, \mathfrak{so}_m)$.

**Proof.** From the definition of $\mathfrak{a}$, we verify easily that $\mathfrak{a}$ commutes with the Lie subalgebra $\mathfrak{s}$ generated by the root vectors $X_{\pm \alpha}$, $\alpha \in \Pi \setminus \{\alpha_1, \alpha_2\}$. So $\mathfrak{t}_S^X$ contains $\mathfrak{s}$. Note that $\mathfrak{s}$ is simple of type $B_{n-2}$.

If $\mathfrak{t}$ is a Cartan subalgebra of the $\mathfrak{s}$, then $\mathfrak{a} \oplus \mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}^X$. It follows that the root system of the semisimple part $\mathfrak{l}$ of $\mathfrak{g}^X$ contains as a subsystem the root system of $\mathfrak{s}$. In particular, the semisimple rank of $\mathfrak{l}$ is equal to $n - 2$ or $n - 1$.

Since $X$ is not $\mathfrak{p}$-regular, $\mathfrak{p}^X$ contains a nilpotent element, and so $\mathfrak{l}$ contains strictly $\mathfrak{s}$. It follows that the root system $R(\mathfrak{l})$ of $\mathfrak{l}$ is of one of the following types : $A_{n-1}$, $B_{n-1}$ or $A_1 \times B_{n-2}$.
Suppose that $R(l)$ is not of type $A_{n-1}$. Then $l_+$ contains a simple Lie subalgebra of type $B_{n-2}$. Consequently, using the classification of rank 1 symmetric pairs (see List 3.1), we deduce that $r$ has the required form.

Suppose now that $R(l)$ is of type $A_{n-1}$. Then from the fact that $R(l)$ contains a subsystem of type $B_{n-2}$, we deduce that $n = 3$. Explicit computations as described in the Section 6 show that this case does not occur.

**Lemma 3.4** Suppose that we are in the case $(D_n, \alpha_1)$ of Table 1. Then there exists $m \in \mathbb{N}$ such that $(r, r_+) = (\mathfrak{so}_{m+1}, \mathfrak{so}_m)$.

**Proof.** The proof is more or less the same as in the case $(B_n, \alpha_1)$. However, note that $\mathfrak{s}$ is simple except when $n = 4$, in which case it is semisimple. The root system $R(l)$ is of one of the following types: $A_{n-1}$, $D_{n-1}$ or $A_1 \times D_{n-2}$. Here, if $R(l)$ is of type $A_{n-1}$, then $n = 4$ or $n = 5$. In the former, we use the classification of rank 1 symmetric pairs to obtain a contradiction, while in the latter, a direct computation is required (see Section 6).

**Remark 3.5** Note that we may extend Lemma 3.4 to the symmetric pairs $(\mathfrak{so}_4, \mathfrak{so}_2 \times \mathfrak{so}_2)$ and $(\mathfrak{so}_6, \mathfrak{so}_4 \times \mathfrak{so}_2)$.

In the first case, we have $(\mathfrak{so}_4, \mathfrak{so}_2 \times \mathfrak{so}_2) = (\mathfrak{so}_3, \mathfrak{so}_2) \times (\mathfrak{so}_3, \mathfrak{so}_2)$, while in the second case, we have $(\mathfrak{so}_6, \mathfrak{so}_4 \times \mathfrak{so}_2) = (\mathfrak{sl}_4, \mathfrak{sl}_2 \times \mathfrak{sl}_2 \times \mathbb{C})$. In both cases, we may obtain the same conclusion by direct computations (see Section 6).

Summarizing, since Cartan subspaces are $K$-conjugate, we have therefore obtained the following result:

**Proposition 3.6** Let $(\mathfrak{g}, \mathfrak{k})$ be the symmetric pair $(\mathfrak{so}_{p+2}, \mathfrak{so}_p \times \mathfrak{so}_2)$, $p \geq 2$. For any non-zero non $\mathfrak{p}$-regular semisimple element $X$ in $\mathfrak{p}$, the subpair $(r, r_+)$ is of the form $(\mathfrak{so}_{m+1}, \mathfrak{so}_m)$ for some $m \in \mathbb{N}$.

**Lemma 3.7** Suppose that we are in the case $(E_7, \alpha_7)$ of Table 1. Then the symmetric pair $(r, r_+)$ is the product of symmetric pairs of the form $(\mathfrak{so}_{m+1}, \mathfrak{so}_m)$ or $(\mathfrak{so}_{m+2}, \mathfrak{so}_m \times \mathfrak{so}_2)$ for some integer $m \in \mathbb{N}$. 

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Proof. Proceeding as in the proof of Lemma 3.3, we check easily that \( k_X \) contains the Lie subalgebra \( \mathfrak{s} \) generated by the root vectors \( X_{\pm \alpha}, \alpha \in \Pi \setminus \{ \alpha_1, \alpha_6, \alpha_7 \} \). Note that \( \mathfrak{s} \) is simple of type \( D_4 \).

Hence \( R(l) \) contains a subsystem of type \( D_4 \). We deduce therefore that \( R(l) \) is of one of the following types: \( D_5 \), \( D_4 \times C \), \( D_6 \) or \( D_5 \times A_1 \). We may conclude by using the classification of symmetric pairs of rank less than or equal to 2 (see Lists 3.1 and 3.2).

Remark 3.8 Note that in all the other rank 2 cases, there exists \( X \in \mathfrak{a} \) non-p-regular such that \( p^X \) contains two non-proportional commuting nilpotent elements. Hence by the result of [11, Proposition 3], the conclusion of the previous lemmas fails. See Section 6 for more details.

4 Sheets and commuting varieties

Let \((g, \mathfrak{t})\) be a symmetric pair. Recall that the connected algebraic group \( K \) acts on \( \mathfrak{p} \). For \( n \in \mathbb{N} \), we set:

\[
\mathfrak{p}^{(n)} = \{ X \in \mathfrak{p} ; \dim K.X = n \}.
\]

The set \( \mathfrak{p}^{(n)} \) is locally closed, and an irreducible component of \( \mathfrak{p}^{(n)} \) shall be called a \( K \)-sheet of \( \mathfrak{p} \). Clearly, \( K \)-sheets are \( K \)-invariant, and by [12], each \( K \)-sheet contains a nilpotent element.

Let \( \pi_1 : C(\mathfrak{p}) \to \mathfrak{p} \) be the projection \((X,Y) \mapsto X\). Recall the following result concerning the commuting variety of \( \mathfrak{p} \).

Theorem 4.1 There exist \( K \)-sheets \( S_1, \ldots, S_r \) of \( \mathfrak{p} \) such that \( \pi_1^{-1}(S_i), \ i = 1, \ldots, r \), are the irreducible components of \( C(\mathfrak{p}) \).

The proof of Theorem 4.1 is a simple consequence of the following result. For the sake of completeness, we have included a proof.

Lemma 4.2 Let \( V \) be a vector space, \( E \subset V \times V \) a locally closed subvariety and for \( i = 1, 2 \), \( \pi_i : E \to V \) be the projection \((x_1, x_2) \mapsto x_i \). Suppose that:

1. \( \pi_1(E) \) is locally closed.

2. There exists \( r \in \mathbb{N} \) such that for all \( x \in \pi_1(E) \), \( \pi_2(\pi_1^{-1}(x)) \) is a vector subspace of dimension \( r \).

If \( \pi_1(E) \) is irreducible, then so is \( E \).
Proof. Let $G$ be the Grassmann variety of $r$-dimensional subspaces of $V$, $x \in \pi_1(E)$ and $W = \pi_2(\pi_1^{-1}(x)) \in G$. Fix a complementary subspace $U$ of $W$ in $V$ and set:

$$F = \{T \in G ; T \cap U = \{0\}\} = \{T \in G ; T + U = V\}.$$ 

Clearly, $F$ is an open subset of $G$ containing $W$. For $\tau \in \text{Hom}(W, U)$ the set of linear maps from $W$ to $U$, we define:

$$T(\tau) = \{w + \tau(w) ; w \in W\}.$$ 

Then we check easily that $T(\tau) \in F$, and we have a map $\text{Hom}(W, U) \to F$, $\tau \mapsto T(\tau)$. We claim that this map is an isomorphism.

Since $w_1 + \tau_1(w_1) = w_2 + \tau_2(w_2)$ is equivalent to $w_1 - w_2 = \tau_2(w_2) - \tau_1(w_1)$, we deduce that the above map is injective.

Now if $T \in F$, then for $w \in W$, we define $\tau(w)$ to be the unique element in $U$ such that $w + \tau(w) \in T$. We then verify easily that $T(\tau) = T$. So we have proved our claim.

The map

$$\Phi : \pi_1(E) \to G, y \mapsto \pi_2(\pi_1^{-1}(y))$$

is a morphism of algebraic varieties. So $F = \Phi^{-1}(F)$ is an open subset of $\pi_1(E)$ containing $x$. The above claim says that we have a well-defined map:

$$\Psi : F \times W \to E, (y, w) \mapsto (y, w + \tau(w))$$

where $T(\tau) = \Phi(y)$. It is then a straightforward verification that $\Psi$ is an isomorphism of the algebraic varieties $F \times W$ and $\pi_1^{-1}(F)$.

It follows that the map $\pi_1 : E \to \pi_1(E)$ is an open map whose fibers are irreducible. Hence by a classical result on topology [3, T.5], if $\pi_1(E)$ is irreducible, then $E$ is irreducible.

Since the set of $p$-generic elements and the set $p_{\text{reg}}$ of $p$-regular elements are open subsets of $p$, we have the following corollary:

**Corollary 4.3** Let $a$ be a Cartan subspace in $p$. The set

$$C_0 = \overline{K.(a \times a)} = \pi_1^{-1}(p_{\text{reg}}) = \pi_2^{-1}(p_{\text{reg}})$$

is the unique irreducible component of $C(p)$ of maximal dimension.
We shall finish this section with a result on even nilpotent elements. Let $X \in \mathfrak{p}$ be a nilpotent element, and $(H, Y) \in \mathfrak{k} \times \mathfrak{p}$ be such that $(X, H, Y)$ is a normal $\text{sl}_2$-triple (called a normal $S$-triple in [12]). Recall that $X$ is even if the eigenvalues of $\text{ad}_p H$ are even. In fact, this is equivalent to the condition that the eigenvalues of $\text{ad}_p H$ are even.

**Proposition 4.4** Let $X \in \mathfrak{p}$ be an even nilpotent element, then $X$ belongs to a $K$-sheet containing semisimple elements.

**Proof.** Let $(X, H, Y)$ be a normal $\text{sl}_2$-triple and $\mathfrak{s} = \mathbb{C}X + \mathbb{C}H + \mathbb{C}Y$. Then $\mathfrak{g}$ decomposes into a direct sum of $\mathfrak{s}$-modules, say $V_i$, $i = 1, \ldots, r$. Since $X$ is even, $\dim V_i$ is odd for $i = 1, \ldots, r$.

For $\lambda \in \mathbb{C}$, we set $X_\lambda = X + \lambda Y \in \mathfrak{p}$. If $\lambda \neq 0$, then $X_\lambda$ is semisimple because $X_\lambda$ is $G$-conjugate to a multiple of $H$. We claim that $\dim \mathfrak{p}^{X_\lambda} = \dim \mathfrak{p}^X$ for all $\lambda \in \mathbb{C}$.

First of all, observe that $\mathfrak{p}^{X_\lambda} = \bigoplus_{i=1}^r (V_i \cap \mathfrak{p})^{X_\lambda}$ because $V_i = (V_i \cap \mathfrak{k}) \oplus (V_i \cap \mathfrak{p})$. Moreover $\dim (V_i \cap \mathfrak{p})^{X_\lambda} \leq 1$.

Now if $(V_i \cap \mathfrak{p})^{X_\lambda} \neq \{0\}$, then a simple weight argument shows that $(V_i \cap \mathfrak{p})^X \neq \{0\}$.

Conversely, suppose that $(V_i \cap \mathfrak{p})^X \neq \{0\}$. Let $\dim V_i = 2n + 1$ and $v_{-n}, \ldots, v_n$ be a basis weight vectors of $V_i$ such that $H v_k = 2k v_k$, $k = -n, \ldots, n$. Then $(V_i \cap \mathfrak{p})^X = \mathbb{C} v_n$.

So $v_k \in \mathfrak{t}$ (resp. $v_k \in \mathfrak{p}$) when $n - k$ is odd (resp. even). In particular, $v_{-n} \in \mathfrak{p}$. It follows that for $k$ such that $n - k$ odd, $\lambda Y v_{k+1} = -a_k X v_{k-1}$ for some $a_k \in \mathbb{C}$. We may therefore renormalize the $v_k$’s so that $v = v_{-n} + v_{-n+2} + \cdots + v_{n-2} + v_n$ verifies $X_\lambda v = 0$.

We have therefore proved that $\dim \mathfrak{p}^{X_\lambda} = \dim \mathfrak{p}^X$ for all $\lambda$.

Now, consider the morphism $\Phi : K \times \mathbb{C} \to \mathfrak{p}$, $(k, \lambda) \mapsto k.X_\lambda$. The image of $\Phi$ is irreducible and contains semisimple elements, so it contains strictly $K.X$. Consequently, $K.X$ is contained strictly in a $K$-sheet with semisimple elements. 

**5 The rank 2 case**

In this section, $(\mathfrak{g}, \mathfrak{k})$ is a symmetric pair of rank 2.
Lemma 5.1 Let $X \in \mathfrak{p}$ be an element which is non-semisimple and non-nilpotent, and $X_s, X_n \in \mathfrak{p} \setminus \{0\}$ be its semisimple and nilpotent components. If $(X, Y) \in C(\mathfrak{p})$, then the semisimple component of $Y$ belongs to $\mathbb{C}X_s$.

Proof. Let $Y = Y_s + Y_n$ be the corresponding decomposition into semisimple and nilpotent components. Since $\mathfrak{g}^X = \mathfrak{g}^{X_s} \cap \mathfrak{g}^{X_n}$, it follows easily that $\mathfrak{p}^X = \mathfrak{p}^{X_s} \cap \mathfrak{p}^{X_n}$. Hence $X_s, X_n, Y_s, Y_n$ commute pairwise. If $X_s$ and $Y_s$ are not proportional, then they span a Cartan subspace because the rank is two. But this would imply that $X$ commutes with a Cartan subspace, thus $X$ belongs to a Cartan subspace, which is absurd because $X$ is not semisimple.

In the rest of this section, we consider the rank 2 symmetric pair $(\mathfrak{g}, \mathfrak{t}) = (\mathfrak{so}_{p+2}, \mathfrak{so}_p \times \mathfrak{so}_2), p \geq 2$. In particular, if $p \neq 2$, then the symmetric pair comes from a parabolic subalgebra with abelian unipotent radical.

Let $\mathfrak{a}$ be a Cartan subspace in $\mathfrak{p}$. Recall that $C_0 = K.(\mathfrak{a} \times \mathfrak{a})$.

Theorem 5.2 The commuting variety $C(\mathfrak{p})$ is irreducible.

Proof. We proceed as in the proof of Richardson in the classical case (see [11]) by using inductive arguments. Let $(X, Y) \in C(\mathfrak{p})$.

1. If $X$ is semisimple, then $X$ commutes with a $\mathfrak{p}$-regular semisimple element $Z$. The line $L_Z = \{(X, tY + (1 - t)Z), t \in \mathbb{C}\}$ is contained in $C(\mathfrak{p})$. Since $\{(tY + (1 - t)Z), t \in \mathbb{C}\}$ meets the set of $\mathfrak{p}$-regular semisimple elements which is open in $\mathfrak{p}$, we conclude that $L_Z$, and hence $(X, Y)$, are contained in $C_0$ (Corollary 4.3).

2. We may assume that neither $X$ nor $Y$ is semisimple.

Suppose that $X$ is not nilpotent. Writing $X = X_s + X_n$, $Y = Y_s + Y_n$, we deduce from the Lemma 5.1 that $Y_s \in \mathbb{C}X_s, X_n, Y_n \in \mathfrak{p}^{X_s}$ and $[X_n, Y_n] = 0$.

By Proposition 3.6 we deduce that $X_n, Y_n$ are commuting elements contained in a symmetric pair of the form $(\mathfrak{so}_{m+1}, \mathfrak{so}_m)$ for some $m \in \mathbb{N}$. Consequently, $Y_n \in \mathbb{C}X_n$ by [11] Proposition 3].

Thus, there exists $\lambda, \mu \in \mathbb{C}$ such that $Y = \lambda X_s + \mu X_n$. In particular, $X$ is $\mathfrak{p}$-regular, and therefore $(X, Y) \in C_0$ by Corollary 4.3.

3. So we may further assume that $X$ and $Y$ are both nilpotent. If $X$ commutes with a non-zero semisimple element $Z \in \mathfrak{p}$, then the same argument as in 1) works because the set of non-nilpotent elements is open.
4. Recall that an element of $\mathfrak{p}$ is said to be $\mathfrak{p}$-distinguished if its centralizer in $\mathfrak{p}$ does not contain any non-zero semisimple element. In particular, a $\mathfrak{p}$-distinguished element is nilpotent. So the number of $K$-orbits of $\mathfrak{p}$-distinguished elements is finite.

Denote by $\pi_1 : C(\mathfrak{p}) \to \mathfrak{p}$ the projection $(X_1, X_2) \mapsto X_1$, $\mathcal{O}$ the set of non $\mathfrak{p}$-distinguished elements in $\mathfrak{p}$, and $\Omega_1, \ldots, \Omega_r$ the set of $K$-orbits of $\mathfrak{p}$-distinguished elements in $\mathfrak{p}$. Thus $\mathfrak{p} = \mathcal{O} \cup \Omega_1 \cup \cdots \cup \Omega_r$, and $C(\mathfrak{p}) = \pi_1^{-1}(\mathcal{O}) \cup \pi_1^{-1}(\Omega_1) \cup \cdots \cup \pi_1^{-1}(\Omega_r)$.

From the previous paragraph, we obtain that $\pi_1^{-1}(\mathcal{O}) \subseteq \mathfrak{c}^0$. Consequently, $C(\mathfrak{p})$ is the union of $\mathfrak{c}^0$ with $\pi_1^{-1}(\Omega_1), \ldots, \pi_1^{-1}(\Omega_r)$. Now we check easily that for $X \in \mathfrak{p}$, $\pi_1^{-1}(K.X) = K.(X, \mathfrak{p}X)$ is an irreducible subset of $C(\mathfrak{p})$ of dimension $\dim \mathfrak{f} - \dim \mathfrak{f}^X + \dim \mathfrak{p}^X = \dim \mathfrak{p}$.

It follows that all irreducible components of $C(\mathfrak{p})$ other than $\mathfrak{c}^0$, if they exist, are of dimension $\dim \mathfrak{p}$.

Suppose that $C(\mathfrak{p})$ is not irreducible. By the previous discussion, there exists a $\mathfrak{p}$-distinguished element $X$ such that $\pi_1^{-1}(K.X)$ is an irreducible component of dimension $\dim \mathfrak{p}$.

If $X$ is even, then by Proposition 4.3, $X$ belongs to a $K$-sheet $\mathcal{S}$ containing non-zero semisimple elements. So $\dim \mathcal{S} > \dim K.X$. Now Lemma 4.2 says that $\dim \pi_1^{-1}(\mathcal{S}) > \dim \mathfrak{p}$. Contradiction.

So we may assume that $X$ is not even and the theorem follows from the following result.

**Proposition 5.3** All $\mathfrak{p}$-distinguished elements of the symmetric $(\mathfrak{so}_{p+2}, \mathfrak{so}_p \times \mathfrak{so}_2)$, $p \geq 2$, are even.

We shall first classify the nilpotent $K$-orbits in $\mathfrak{p}$.

- Each element $X$ of $\mathfrak{g}$ can be represented by a matrix of the following type:

  $$X = \begin{pmatrix} A_p & B \\ -tB & A_2 \end{pmatrix}, \quad A_p \in \mathfrak{so}_p, A_2 \in \mathfrak{so}_2, B \in M_{p,2}(\mathbb{C}).$$

The involutive automorphism $\theta$ is defined by:

$$\theta(X) = J_{p,2}XJ_{p,2}, \text{ where } J_{p,2} = \begin{pmatrix} I_p & 0 \\ 0 & -I_2 \end{pmatrix}.$$ 

It follows that:

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & B \\ -tB & 0 \end{pmatrix}, \quad B \in M_{p,2}(\mathbb{C}) \right\}.$$
The dimension of $\mathfrak{p}$ is therefore $2p$.

Set:

$$H_i = E_{i,n-i+1} - E_{n-i+1,i}, \quad 1 \leq i \leq 2.$$ 

The subspace $\mathfrak{a}$, spanned by $H_1, H_2$, is a Cartan subspace of $\mathfrak{p}$.

• Let $\mathcal{N}$ be the set of $G$-nilpotent orbits in $\mathfrak{g}$, $\text{YD}_{p+2}$ the set of all Young diagrams corresponding to the partitions of $p+2$ and satisfying the two following properties:

(P$_1$) Each row of even length occurs with even multiplicity.

(P$_2$) If all rows have even length, then Roman numerals are attached to the diagram, namely I or II.

Recall (see for example [2]) that $\mathcal{N}$ is in one-to-one correspondence with $\text{YD}_{p+2}$.

• Let $\mathcal{N}_p$ be the set of nilpotent $K$-orbits in $\mathfrak{p}$. To describe this set, we use the Kostant-Sekiguchi correspondence between $\mathcal{N}_p$ and the set $\mathcal{N}_0$ of nilpotent $G_0$-orbits in $\mathfrak{g}_0$, where $\mathfrak{g}_0$ is a real form of $\mathfrak{g}$ and $G_0$ the corresponding connected real Lie group. Let us recall briefly this correspondence.

We need to fix some notations:

- $\mathfrak{g}_0 = \text{so}(p, 2)$ is the set of matrices $X_0$ of the following type:

$$X_0 = \begin{pmatrix} A_p & B \\ \overset{t}{B} & A_2 \end{pmatrix}, \quad A_p \in \text{so}_p(\mathbb{R}), \quad A_2 \in \text{so}_2(\mathbb{R}), \quad B \in \text{M}_{p,2}(\mathbb{R}).$$

- A Cartan decomposition of $\mathfrak{g}_0$ is given by:

$$\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$$

where the Cartan involution $\theta_0$ is defined by: $\theta_0(X_0) = -^t X_0$.

$$\mathfrak{k}_0 = \left\{ \begin{pmatrix} A_p & 0 \\ 0 & A_2 \end{pmatrix} \right\}, \quad A_p \in \text{so}_p(\mathbb{R}), \quad A_2 \in \text{so}_2(\mathbb{R})$$

$$\mathfrak{p}_0 = \left\{ \begin{pmatrix} 0 & B \\ \overset{t}{B} & 0 \end{pmatrix} \right\}, \quad B \in \text{M}_{p,2}(\mathbb{R})$$

- $G_0$ is the adjoint group of $\mathfrak{g}_0$. 
The Lie algebra $\mathfrak{g}_0$ is embedded in $\mathfrak{g}$, via the map $\phi$ defined by:

$$\phi(X_0) = \tilde{J}_{p,2}X_0\tilde{J}_{p,2}^{-1}$$

where:

$$\tilde{J}_{p,2} = \begin{pmatrix} I_p & 0 \\ 0 & -iI_2 \end{pmatrix}$$

We check easily that $\phi \circ \theta_0 = \theta \circ \phi$. It follows that $\mathfrak{g}_0$ can be identified with a real form of $\mathfrak{g}$. Hence, we have:

$$(\mathfrak{k}_0)_C = \mathfrak{k}, \quad (\mathfrak{p}_0)_C = \mathfrak{p}.$$ 

- The next step is to describe $\mathcal{N}_0$, using the method described in [2, Chapter 9].

First of all, we define a signed Young diagram of signature $(p, 2)$ to be a Young diagram in which each box is labeled with a $+$ or a $-$ sign in such a way that:

- Signs alternate across rows.

- Two diagrams are equivalent if and only if one can be obtained from the other by interchanging rows of equal length.

- The number of boxes labeled $+$ is $p$, the number of boxes labeled $-$ is 2.

We define an orthogonal signed Young diagram of signature $(p, 2)$ to be a signed Young diagram satisfying $(P_1)$, $(P_2)$ and the three following properties:

$(P_3)$ Each row of even length has its leftmost box labeled $+$. 

$(P_4)$ If all rows have even length, then two Roman numerals, each I or II, are attached to the corresponding diagram (giving then four orbits).

$(P_5)$ If at least one row has odd length and if each row of odd length has an even number of boxes labeled $+$ or if each row of odd length has an even number of boxes labeled $-$, then one numeral I or II is attached to the corresponding diagram.
Let us denote by \( \text{DYO}_{p,2} \) the set of all orthogonal signed Young diagram.

By a classical result (see [2, 9.3.4]), we know that \( \mathcal{N}_0 \) is in one-to-one correspondence with \( \text{DYO}_{p,2} \).

**Remark 5.4** If \( \Omega_0 \in \mathcal{N}_0 \), then \( \Omega = (\Omega_0)_C \in \mathcal{N} \). The Young diagram corresponding to \( \Omega \) is obtained by removing the signs in the orthogonal signed Young diagram corresponding to \( \Omega_0 \).

**Remark 5.5** It follows from the definition of elements of \( \text{DYO}_{p,2} \) that if \( \Omega_0 \in \mathcal{N}_0 \) and \( \Omega_0 = (d_1, \ldots, d_s) \), then \( d_1 \leq 5 \).

- We may deduce now, from the previous statements, all the nilpotent orbits in \( \text{so}(p,2) \).
  
  If \( p \geq 4 \), we have :

\[
\begin{align*}
\begin{array}{c}
\begin{array}{c}
+ - & + - & + \\
+ & . & . \\
+ & . \\
+ & . \\
\end{array} & (5,1,\ldots,1), & \\
\begin{array}{c}
+ - \\
. \\
. \\
+ \\
\end{array} & (3,3,1,\ldots,1), & (\text{I or II if } p = 4)
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\begin{array}{c}
+ - \\
- \\
. \\
. \\
. \\
+ \\
\end{array} & (3,1,1,\ldots,1), & \\
\begin{array}{c}
. \\
. \\
. \\
+ \\
\end{array} & (3,1,1,\ldots,1), & (\text{I or II})
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\begin{array}{c}
+ - \\
. \\
. \\
. \\
\end{array} & (2,2,1,\ldots,1).
\end{array}
\end{align*}
\]
If $p = 3$, the orbits are given by the partitions $(5), (3, 1), (2, 2, 1)$ and the corresponding diagrams above.

If $p = 2$, we have only the partitions $(3, 1)$ and $(2, 2)$.

- Let $\Omega_0 \in \mathcal{N}_0$. We may choose a sl$_2$-triple $(H_0, X_0, Y_0)$ such that:

\[
X_0 \in \Omega_0, \quad \theta_0(H_0) = -H_0, \quad \theta_0(X_0) = -Y_0, \quad \theta_0(Y_0) = -X_0.
\]

Triples with these properties are called Cayley triples.

Set:

\[
H_S = i(X_0 - Y_0), \quad X_S = \frac{1}{2}(X_0 + Y_0 + iH_0), \quad Y_S = \frac{1}{2}(X_0 + Y_0 - iH_0)
\]

Then $(H_S, X_S, Y_S)$ is a normal sl$_2$-triple (that is, $H_S \in \mathfrak{k}$ and $X_S, Y_S \in \mathfrak{p}$), called the Cayley transform of $(H_0, X_0, Y_0)$.

**Theorem 5.6 (Kostant-Sekiguchi correspondence [2])** There is a bijection from $\mathcal{N}_0$ onto $\mathcal{N}_p$ which sends the orbit of the positive element of a Cayley triple to the orbit of the positive element of its Cayley transform.

Via this correspondence, we classify the nilpotent $K$-orbits in $\mathfrak{p}$ using the set $\text{DYO}_{p,2}$.

- Let $r = \left[\frac{p + 2}{2}\right]$ be the rank of $\mathfrak{g}$ and $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{t}$ a Cartan subalgebra of $\mathfrak{g}$, where $\mathfrak{t}$ is a Cartan subalgebra of the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. We may complete the basis $(H_1, H_2)$ of $\mathfrak{a}$ to a basis $(H_i), 1 \leq i \leq r$, of $\mathfrak{h}$, where $(H_i, 3 \leq i \leq r)$, is a basis of $\mathfrak{t}$. Let $(e_i)_{1 \leq i \leq r}$ be the corresponding dual basis in $\mathfrak{h}^*$, $R = R(\mathfrak{g}, \mathfrak{h})$ the corresponding root system and $\Pi$ a base of simple roots of $R$, defined as follows:

\[
p + 2 = 2r + 1, \quad R = \{\pm e_i \pm e_j, 1 \leq i \neq j \leq r\} \cup \{\pm e_i, 1 \leq i \leq r\}
\]

\[
\Pi = \{e_i - e_j, 1 \leq i < j \leq r\} \cup \{e_r\}
\]

\[
p + 2 = 2r, \quad R = \{\pm e_i \pm e_j, 1 \leq i \neq j \leq r\}
\]

\[
\Pi = \{e_i - e_j, 1 \leq i < j \leq r\} \cup \{e_{r-1} + e_r\}
\]

In the same way, we choose a Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}_0$, such that $\mathfrak{h} = (\mathfrak{h}_0)_C$. In particular, $\mathfrak{a}_0 = \mathbb{R}H_1 \oplus \mathbb{R}H_2$ is a Cartan subspace of $\mathfrak{p}_0$ such that $\mathfrak{a} = (\mathfrak{a}_0)_C$.

- Given a nilpotent orbit $\Omega$ in $\mathcal{N}$, there is a unique sl$_2$-triple $(h^+, e, f)$ such that $e \in \Omega$ and for all $\alpha_i \in \Pi$, $\alpha_i(h^+) \in \{0, 1, 2\}$.

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Recall that the sequence \( C(\Omega) = (\alpha_1(h^+), \ldots, \alpha_r(h^+)) \) is called the characteristic of \( \Omega \).

Given a Young diagram, we can compute now the characteristic of the corresponding nilpotent orbit \( \Omega \) by using the following method:

- First of all, we suppose that \( p + 2 = 2r + 1 \). Let \( \Omega = (d_1, \ldots, d_s) \) be a nilpotent \( G \)-orbit. We consider the sequence of integers \((d_i - 1, d_i - 3, \ldots, -d_i + 1)\), \(1 \leq i \leq s\), and we rearrange it so that a 0 comes first, followed by the other positive terms in non-increasing order, followed by the negative terms. Hence, the new sequence takes the form:
  \[
  (0, h_1, h_2, \ldots, h_r, -h_1, \ldots, -h_r) , \ h_1 \geq h_2 \geq \cdots \geq h_r.
  \]

Then, we have:
  \[
  C(\Omega) = (h_1 - h_2, h_2 - h_3, \ldots, h_{r-1} - h_r, h_r).
  \]

- Suppose now that \( p + 2 = 2r \).
  If \((d_1, \ldots, d_s)\) does not contain even terms, we use the same recipe as in the previous case and we obtain the sequence:
  \[
  (h_1, h_2, \ldots, h_r, -h_1, \ldots, -h_r) , \ h_1 \geq h_2 \geq \cdots \geq h_r.
  \]

Then, we have:
  \[
  C(\Omega) = (h_1 - h_2, \ldots, h_{r-1} - h_r, h_r, h_r + h_r).
  \]

If \((d_1, \ldots, d_s)\) contains only even terms, we have the same sequence:
  \[
  (h_1, h_2, \ldots, h_r, -h_1, \ldots, -h_r) , \ h_1 \geq h_2 \geq \cdots \geq h_r.
  \]

Then, we set: \( a = 0 \), if \( r \) is a multiple of 4, and \( a = 2 \), otherwise. In this situation, there are two orbits \( \Omega'' \) and \( \Omega''' \). We have:
  \[
  C(\Omega'') = (h_1 - h_2, \ldots, h_{r-2} - h_{r-1}, a, 2 - a),
  \]
  \[
  C(\Omega''') = (h_1 - h_2, \ldots, h_{r-2} - h_{r-1}, 2 - a, a).
  \]

Clearly, a nilpotent element is even if all the terms of its characteristic are even.
\textbf{Proof.} (of Proposition 5.3) Let $X$ be a nilpotent element of $p$, $\Omega_K = K.X$, $\Omega = G.X$ and $\Omega_0 = G_0.X_0$, the corresponding real nilpotent orbit, via the Kostant-Sekiguchi correspondence. By [2, Remark 9.5.2], we have $G.X_0 = \Omega$.

Let $(d_1, d_2, \ldots, d_s)$ be the Young diagram of $\Omega$. By Remark 5.4 and the description of nilpotent orbits in $so(p, 2)$, we only need to consider the two following cases:

- $(d_1, \ldots, d_s) \neq (2, 2, 1, \ldots, 1)$. Hence, $(d_1, \ldots, d_s)$ contains only odd terms. We deduce from the discussions above that $C(\Omega)$ contains only even terms, and therefore, $X$ is even.

- $(d_1, \ldots, d_s) = (2, 2, 1, \ldots, 1)$. The two distinct corresponding real orbits are those of minimal dimension. Defining, from $(e_i)$, $1 \leq i \leq 2$, a system of restricted roots for $g_0$, we know by a classical result (see [2, §4.3]) that these two orbits are respectively generated by $X_0 = X_{e_1 - e_2}$ and $-X_0$. Consider now the element $H_0 = H_{e_1 + e_2}$. Then, $H_0$ is an element of $p$ belonging to the centralizer of $X_0$. Applying the Cayley transform, we check easily that $H_0 \in p^X$. So $X$ is not $p$-distinguished.

We have therefore proved that $p$-distinguished elements are even. \hfill $\blacksquare$

\section{Explicit computations}

We shall describe in this section some of the explicit computations stated in the proofs of Lemmas 3.3, 3.4, and Remarks 3.5, 3.8.

- Case $(B_3, \alpha_1)$. We have $S = \{\alpha_2, \alpha_3\}$, and

\[ R_1^S = \{\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + 2\alpha_3, \alpha_1 + 2\alpha_2 + 2\alpha_3\}. \]

A simple computation shows that $X_{\{\alpha_1\}}$ and $X_\Pi$ are not $p_S$-regular. For $\lambda \in \mathbb{C}^*$, set $X_\lambda = X_{\{\alpha_1\}} + \lambda X_\Pi$. Let $Y \in p_\lambda^S$.

Writing $Y = Y_1 + Y_{-1}$ with $Y_1 \in u_S$ and $Y_{-1} \in u_{-S}$, we conclude that $Y = A + Z$ where $A \in a$ and $Z \in p_0^S = \bigoplus_{\alpha \in R_1^S} g_\alpha$ with $R_1^0 = (R_1^S \cup -R_1^S) \setminus \{\alpha_1, -\alpha_1, \varepsilon_\Pi, -\varepsilon_\Pi\}$.

So we may assume that $Y \in p_0^S$. It follows that

\[ 0 = [X_\lambda, Y] = [X_{\alpha_1}, Y_{-1}] + \lambda[X_{-\varepsilon_\Pi}, Y_1] + [X_{-\alpha_1}, Y_{1}] + \lambda[X_{\varepsilon_\Pi}, Y_{-1}]. \]
So
\[ [X_{\alpha_1}, Y_{-1}] + \lambda [X_{-\varepsilon\Pi}, Y_1] = [X_{-\alpha_1}, Y_1] + \lambda [X_{\varepsilon\Pi}, Y_{-1}] = 0. \]

Hence if \( Y = \sum_{\alpha \in R^0_1} c_{\alpha} X_{\alpha} \), then \([X_\lambda, Y] = 0\) is equivalent to
\[
c_{-\beta}[X_{\alpha_1}, X_{-\beta}] + \lambda c_{\gamma}[X_{-\varepsilon\Pi}, X_{\gamma}] = 0,
c_{\gamma}[X_{-\alpha_1}, X_{\gamma}] + \lambda c_{-\beta}[X_{\varepsilon\Pi}, X_{-\beta}] = 0.
\]

whenever we have \( \beta, \gamma \in R^1_3 \cap R^0_1 \) verifying \( \beta + \gamma = \alpha_1 + \varepsilon_{\Pi} \). The list of such pairs is as follows:
\[
(\beta_1, \gamma_1) = (\alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + 2\alpha_3),
(\beta_2, \beta_2) = (\alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3).
\]

Using the Chevalley basis from GAP4, we are reduced to the condition:
\[
c_{-\beta_1} + \lambda c_{\gamma_1} = 0 = c_{\gamma_1} + \lambda c_{-\beta_1},
c_{-\beta_2} - \lambda c_{\beta_2} = 0 = c_{\beta_2} - \lambda c_{-\beta_2}.
\]

These conditions can only be satisfied if \( Y = 0 \) or \( \lambda = \pm 1 \).

We have therefore checked that if \( X \in a \setminus \{0\} \) is not \( p_{S}-\)regular, then it belongs to one of the following lines: \( C_{X_{\{\alpha_1\}}} \), \( C_{X_{\Pi}} \), \( C_{X_1} \), \( C_{X_{-1}} \).

Using GAP4, we obtain that if \( X \) belongs to the first two lines, then \( \dim g^X = 7 \), thus its semisimple part is necessarily of type \( A_1 \times A_1 \). If \( X \) belongs to the other two lines, then \( \dim g^X = 11 \), and its semisimple part must be of type \( B_2 \).

- **Case \((D_5, \alpha_1)\).** The elements of \( R^1_3 \) are:
  \[
  \alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_5, \\
  \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5, \alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5, \\
  \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5.
  \]

Using the same argument above, the pairs are
\[
(\alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5),
(\alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5),
(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_5).
\]

Again, we may check using GAP4 that if \( X \in a \setminus \{0\} \) is not \( p_{S}-\)regular, then it belongs to one of the following lines: \( C_{X_{\{\alpha_1\}}} \), \( C_{X_{\Pi}} \), \( C_{X_1} \), \( C_{X_{-1}} \).
Using GAP4, we obtain that if $X$ belongs to the first two lines, then $\dim \mathfrak{g}^X = 19$, thus its semisimple part is necessarily of type $A_1 \times A_3$. If $X$ belongs to the other two lines, then $\dim \mathfrak{g}^X = 29$, and its semisimple part must be of type $D_4$.

- Computations are similar for cases $(\text{so}_6, \text{so}_4 \times \text{so}_2)$ and $(\text{so}_4, \text{so}_2 \times \text{so}_2)$.

- Case $(A_n, \alpha_2)$ or $(A_n, \alpha_{n-1})$ with $n \geq 4$. Note that the case $n = 3$ is just the symmetric pair $(\text{so}_6, \text{so}_4 \times \text{so}_2)$. By symmetry, we only need to consider the case $\alpha_2$.

Here $\mathcal{E}$ consists of $\Pi$ and $K = \{\alpha_2, \alpha_3, \ldots, \alpha_{n-1}\}$. Consider the element $X_K$. Its centralizer contains $X_{\xi_{\Pi}}$ and $X_{\alpha_1+\alpha_2}$ because $n \geq 4$. Since these two root vectors are not proportional, the subpair $(\mathfrak{r}, \mathfrak{r}_+)$ associated to $X_K$ can not be of type $(\text{so}_{m+1}, \text{so}_m)$ (see [1, Proposition 3]).

Via GAP4 computations, we see that the semisimple part of $\mathfrak{g}^{X_K}$ is of type $A_2$.

- Case $(D_5, \alpha_4)$ or $(D_5, \alpha_5)$. By symmetry, we may take $\alpha_5$. Recall that the two elements of $\mathcal{E}$ are $K = \{\alpha_3, \alpha_4, \alpha_5\}$ and $\Pi$. Consider the element $X_K$. Its centralizer contains $X_{\xi_{\Pi}}$ and $X_{\alpha_1+\alpha_2+\alpha_3+\alpha_5}$ which are not proportional. Consequently, the subpair $(\mathfrak{r}, \mathfrak{r}_+)$ associated to $X_K$ can not be of type $(\text{so}_{m+1}, \text{so}_m)$.

- Case $(E_6, \alpha_1)$ or $(E_6, \alpha_5)$. Again we may take $\alpha_1$. The set $\mathcal{R}_1^+$ has 16 elements. The two elements of $\mathcal{E}$ are $K = \Pi \setminus \{\alpha_2\}$ and $\Pi$. Again, consider the element $X_K$. Its centralizer contains $X_{\xi_{\Pi}}$ and $X_{\alpha_1+\alpha_2+\alpha_3+\alpha_4}$. So the subpair $(\mathfrak{r}, \mathfrak{r}_+)$ associated to $X_K$ can not be of type $(\text{so}_{m+1}, \text{so}_m)$.

GAP4 computations show that the semisimple part of $\mathfrak{g}^{X_K}$ is of type $A_5$.

7 A final remark

In view of Lemma 3.7, we tried to apply the same method for the symmetric pair $(E_7, E_6 \times \mathbb{C})$. Unfortunately, using the classification of $\mathfrak{p}$-distinguished orbits in $\mathfrak{p}$ as described in [2], we found a non even $\mathfrak{p}$-distinguished element. Namely, the orbit corresponding to label 3 of Table 13 of [2].

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