Subgraphs with large minimum $\ell$-degree in hypergraphs where almost all $\ell$-degrees are large

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Abstract

Let $G$ be an $r$-uniform hypergraph on $n$ vertices such that all but at most $\varepsilon(n^\ell)$ $\ell$-subsets of vertices have degree at least $p(n^{-\ell})$. We show that $G$ contains a large subgraph with high minimum $\ell$-degree.

Keywords: $r$-uniform hypergraphs, $\ell$-degree, extremal hypergraph theory

1 Introduction

Given $r \in \mathbb{N}$ and a set $A$, we write $A^{(r)}$ for the collection of all $r$-subsets of $A$ and $[n]$ for the set $\{1, 2, \ldots, n\}$. An $r$-graph, or $r$-uniform hypergraph, is a pair $G = (V, E)$, where $V = V(G)$ is a set of vertices and $E = E(G) \subseteq V^{(r)}$ is a collection of $r$-subsets, which constitute the edges of $G$. We say $G$ is nonempty if it contains at least one edge and set $v(G) = |V(G)|$ and $e(G) = |E(G)|$. A subgraph of $G$ is an $r$-graph $H$ with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. The subgraph of $G$ induced by a set $X \subseteq V(G)$ is $G[X] = (X, E(G) \cap X^{(r)})$.

Let $\mathcal{F}$ be a family of nonempty $r$-graphs. If $G$ does not contain a copy of a member of $\mathcal{F}$ as a subgraph, we say that $G$ is $\mathcal{F}$-free. The Turán number $\text{ex}(n, \mathcal{F})$ of a family $\mathcal{F}$ is the maximum number of edges in an $\mathcal{F}$-free $r$-graph on $n$ vertices, and its Turán density is the limit $\pi(\mathcal{F}) = \lim_{n \to \infty} \text{ex}(n, \mathcal{F})/\binom{n}{r}$ (this is easily shown to exist). Let $K_t^{(r)} = ([t], [t]^{(r)})$ denote the complete $r$-graph on $t$ vertices. Determining $\pi(K_t^{(r)})$ for any $t > r \geq 3$ is a major problem in extremal combinatorics. Turán [19] famously conjectured in 1941 that $\pi(K_4^{(3)}) = 5/9$, and despite much research effort this remains open [8]. In this paper we shall be interested in some variants of Turán density.

The neighbourhood $N(S)$ of an $\ell$-subset $S \in V(G)^{(\ell)}$ is the collection of $(r - \ell)$-subsets $T \in V(G)^{(r-\ell)}$ such that $S \cup T$ is an edge of $G$. The degree of $S$ is the number $\deg(S)$ of edges of $G$ containing $S$, that is, $\deg(S) = |N(S)|$. The minimum $\ell$-degree of $G$, $\delta_\ell(G)$, is defined to be the

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minimum of $\deg(S)$ over all $\ell$-subsets $S \in V(G)^{(\ell)}$. The Turán $\ell$-degree threshold $\ex_{\ell}(n, \mathcal{F})$ of a family $\mathcal{F}$ of $r$-graphs is the maximum of $\delta_{\ell}(G)$ over all $\mathcal{F}$-free $r$-graphs $G$ on $n$ vertices. It can be shown [11, 9] that the limit $\pi_{\ell}(\mathcal{F}) = \lim_{n \to \infty} \ex_{\ell}(n, \mathcal{F})/(\binom{n}{r-\ell})$ exists; this quantity is known as the Turán $\ell$-degree density of $\mathcal{F}$. A simple averaging argument shows that

$$0 \leq \pi_{\ell-1}(\mathcal{F}) \leq \cdots \leq \pi_{1}(\mathcal{F}) = \pi(\mathcal{F}) \leq 1,$$

and it is known that $\pi_{\ell}(\mathcal{F}) \neq \pi(\mathcal{F})$ in general (for $\ell \notin \{0, 1\}$). In the special case where $(r, \ell) = (r, r - 1)$, $\pi_{r-1}(\mathcal{F})$ is known as the codegree density of $\mathcal{F}$.

There has been much research on Turán $\ell$-degree threshold for $r$-graphs when $(r, \ell) = (3, 2)$. In the late 1990s, Nagle [12] and Nagle and Czygrinow [2] conjectured that $\pi_{2}(K_{4}^{(3)}) = 1/4$ and $\pi_{2}(K_{4}^{(2)}) = 1/2$, respectively. Here $K_{4}^{(3)}$ denotes the 3-graph obtained by removing one edge from $K_{4}$. Falgas-Ravry, Pikhurko, Vaughan and Volec [6, 7] recently proved $\pi_{2}(K_{4}^{(3)}) = 1/4$, settling the conjecture of Nagle, and showed all near-extremal constructions are close (in edit distance) to a set of quasirandom tournament constructions of Erdős and Hajnal [3]. The lower bound $\pi_{2}(K_{4}^{(3)}) \geq 1/2$ also comes from a quasirandom construction, which is due to Rödl [17]. For $t > r \geq 3$, the codegree density $\pi_{r-1}(K_{t}^{(r)})$ has been studied by Falgas-Ravry [4], Lo and Markström [9] and Sidorenko [18]. Recently, Lo and Zhao [10] showed that $1 - \pi_{r-1}(K_{t}^{(r)}) = \Theta(\ln t/t^{r-1})$ for $r \geq 3$.

One variant of $\ell$-degree Turán density is to study $r$-graphs in which almost all $\ell$-subsets have large degree. To be precise, given $\varepsilon > 0$, let $\delta_{\ell}^{(\varepsilon)}(G)$ be the largest integer $d$ such that all but at most $\varepsilon^{(v(G))}$ of the $\ell$-subsets $S \in V(G)^{(\ell)}$ satisfy $\deg(S) \geq d$. Note that $r$-graphs with large $\delta_{\ell}(G)$ but with small $\delta_{\ell}(G)$ arise naturally. For instance, the reduced graphs $R$ obtained from $r$-graphs with large minimum $\ell$-degree after an application of hypergraph regularity lemma have large $\delta_{\ell}(R)$.

**Definition 1** ($(r, \ell)$-sequence). Let $1 \leq \ell < r$. We say that a sequence $G = (G_n)_{n \in \mathbb{N}}$ of $r$-graphs is an $(r, \ell)$-sequence if

(i) $v(G_n) \to \infty$ as $n \to \infty$ and

(ii) there is a constant $p \in [0, 1]$ and a sequence of nonnegative reals $\varepsilon_n \to 0$ as $n \to \infty$ such that $\delta_{\ell}^{(\varepsilon_n)}(G_n) \geq p\binom{v(G_n)}{r-\ell}$ for each $n$.

We refer to the supremum of all $p \geq 0$ for which (ii) is satisfied as the density of the sequence $G$ and denote it by $\rho(G)$.

We can define the analogue of Turán density for $(r, \ell)$-sequences.

**Definition 2.** Let $1 \leq \ell < r$. Let $\mathcal{F}$ be a family of nonempty $r$-graphs. Define

$$\pi_{\ell}^{\mathcal{F}}(G) := \sup\{\rho(G) : G \text{ is an } (r, \ell)\text{-sequence of } \mathcal{F}\text{-free } r\text{-graphs}\}.$$

Our main result show that every large $r$-graph $G$ contains a ‘somewhat large’ subgraph $H$ with minimum $\ell$-degree satisfying $\delta_{\ell}(H)/(\binom{v(H)}{r-\ell}) \approx \delta_{\ell}(G)/(\binom{v(G)}{r-\ell})$. Here ‘somewhat large’ means $v(H) = \Omega(\varepsilon^{1/\ell})$.

**Theorem 3.** Let $1 \leq \ell < r$. For any fixed $\delta > 0$, there exists $m_0 > 0$ such that any $r$-graph $G$ on $n \geq m \geq m_0$ vertices with $\delta_{\ell}(G) \geq p\binom{n}{r-\ell}$ for some $\varepsilon \leq m^{-\ell}/2$ contains an induced subgraph $H$ on $m$ vertices with

$$\delta_{\ell}(H) \geq (p - \delta)\binom{m}{r-\ell}.$$
This immediate implies the $\pi_i^*(\mathcal{F}) = \pi_{\ell}(\mathcal{F})$ for all families $\mathcal{F}$ of $r$-graphs.

**Corollary 4.** For any $1 \leq \ell < r$ and any family $\mathcal{F}$ of nonempty $r$-graphs, $\pi_i^*(\mathcal{F}) = \pi_{\ell}(\mathcal{F})$.

We note that the (tight) upper bounds for codegree densities $\pi_2(F)$ for 3-graphs $F$ obtained by flag algebraic methods in [5, 6, 7] actually relied on giving upper bounds for $\pi_1^*(F)$. Corollary 4 provides theoretical justification for why this strategy could give optimal bounds.

### 1.1 Quasirandomness in 3-graphs

One of the main motivations for this note comes from recent work of Reiher, Rödl and Schacht [13, 14, 15, 16] on extremal questions for quasirandom hypergraphs. These authors studied the following notion of quasirandomness for 3-graphs.

**Definition 5** ($(1,2)$-quasirandomness). A 3-graph $G$ is $(p, \varepsilon, (1, 2))$-quasirandom if for every set of vertices $X \subseteq V$ and every set of pairs of vertices $P \subseteq V^{(2)}$, the number $e_{1,2}(X, P)$ of pairs $(x, uv) \in X \times P$ such that $\{x\} \cup \{uv\} \in E(G)$ satisfies:

$$|e_{1,2}(X, P) - p| |X| \cdot |P| \leq \varepsilon v(G)^3.$$

We define a $(1,2)$-quasirandom sequence and the corresponding extremal density, denoted by $\pi_{(1,2)}(\mathcal{F})$, analogously to the way we defined $(r, \ell)$-sequences and $\pi_i^*(\mathcal{F})$ in Definitions 1 and 2. It is not difficult to see that $\pi_{(1,2)}(\mathcal{F}) \leq \pi(\mathcal{F})$ for all families $\mathcal{F}$ of 3-graphs. Moreover, a $(p, \varepsilon, (1, 2))$-quasirandom 3-graph $G$ satisfies $\delta_2^\mathcal{F}(G) \geq (p - 4\sqrt{\varepsilon})v(G)$. Hence, Theorem 3 and Corollary 6 imply the following.

**Corollary 6.** For any family of nonempty 3-graphs $\mathcal{F}$, $\pi_{(1,2)}(\mathcal{F}) \leq \pi_2(\mathcal{F})$.

Consider a $(p, \varepsilon, (1, 2))$-quasirandom 3-graph $G$ for some $p > 4 \sqrt{\varepsilon} > 0$. As noted above, $\delta_2^\mathcal{F}(G) \geq (p - 4\sqrt{\varepsilon})v(G)$. Thus provided $v(G)$ is sufficiently large, Theorem 3 tells us we can find a subgraph $H$ of $G$ on $m = \Omega(\varepsilon^{-1/4})$ vertices with strictly positive minimum codegree (at least $(p - 4\sqrt{\varepsilon})m$).

However, as we show below, we cannot guarantee the existence of any subgraph with strictly positive codegree on more than $2/\varepsilon + 1$ vertices: our lower bound on $m$ above in terms of an inverse power of the error parameter $\varepsilon$ is thus sharp up to the value of the exponent.

**Proposition 7.** For every $p \in (0, 1)$ and every $\varepsilon > 0$, there exists $n_0$ such that for all $n \geq n_0$ there exist $(p, 2\varepsilon, (1, 2))$-quasirandom 3-graphs in which every subgraph on $m \geq \lceil \varepsilon^{-1} \rceil + 1$ vertices has minimum codegree equal to zero.

**Proof.** Let $G = (V, E)$ be a $(p, \varepsilon, (1, 2))$-quasirandom 3-graph on $n$ vertices. Such a 3-graph can be obtained for example by taking a typical instance of an Erdős–Rényi random 3-graph with edge probability $p$. Consider a balanced partition of $V$ into $N = \lceil \varepsilon^{-1} \rceil$ sets $V_i = \bigcup_{x \in V_i} V_i$ with $\lceil n/N \rceil \leq |V_1| \leq |V_2| \leq \ldots \leq |V_N| \leq \lceil n/N \rceil$. Now let $G'$ be the 3-graph obtained from $G$ by deleting all triples that meet some $V_i$ in at least two vertices for some $i$: $1 \leq i \leq N$.

By construction, every set of $N + 1$ vertices in $G'$ must contain at least two vertices from the same $V_i$, and thus must induce a subgraph of $G'$ with minimum codegree zero. Note that $e(G) - e(G') \leq N n \left(\frac{n/N}{2}\right) \leq n^2/N \leq \varepsilon n^3$. Since $G$ is $(p, \varepsilon, (1, 2))$-quasirandom, it follows that $G'$ is $(p, 2\varepsilon, (1, 2))$-quasirandom. \(\square\)
2 Finding high minimum $\ell$-degree subgraphs in $r$-graphs with large $\delta^\varepsilon$

In this section we show how we can extract arbitrarily large subgraphs with high minimum $\ell$-degree from sufficiently large $r$-graphs with sufficiently small error $\varepsilon$. To do so, we will need Azuma’s inequality (see e.g. [1]).

**Lemma 8** (Azuma’s inequality). Let $\{X_i : i = 0, 1, \ldots\}$ be a martingale with $|X_i - X_{i-1}| \leq c_i$ for all $i$. Then for all positive integers $N$ and $\lambda > 0$,

$$\mathbb{P}(X_N \leq X_0 - \lambda) \leq \exp \left( -\frac{\lambda^2}{2 \sum_{i=1}^{\infty} c_i^2} \right).$$

**Proof of Theorem 3.** We may assume without loss of generality that $\delta > 0$ is small enough to ensure $\delta^{-1} \geq 26\ell(r - \ell)^2 \log(1/\delta)$ and $\ell \log(1/\delta) \geq \log 2$ as this only makes our task harder. Set $m_0 = \lceil 26\ell(r - \ell)^2 \delta^{-2} \log(1/\delta) \rceil$. Note that this implies that

$$2\ell \log m_0 \leq 4\ell \log (26\ell(r - \ell)^2 \delta^{-2} \log(1/\delta)) \leq 12\ell \log(1/\delta).$$

Fix $m \geq m_0$. Let $n \geq m \geq m_0$ and $\varepsilon = m^{-\ell}/2$.

Suppose $G = (V, E)$ is an $r$-graph on $n$ vertices with $\delta^\varepsilon(G) \geq p^{(n-\ell)}_{m-\ell}$. We claim that it contains an induced subgraph on $m$ vertices with minimum $\ell$-degree at least $(p - \delta)(m-\ell)$. For $p \leq \delta$, we have nothing to prove, so we may assume that $1 \geq p > \delta$.

Call an $\ell$-subset $S \in V^{(\ell)}$ poor if $\deg(S) < p^{(n-\ell)}_{m-\ell}$, and rich otherwise. Let $\mathcal{P}$ be the collection of all poor $\ell$-subsets. By our assumption on $\delta^\varepsilon(G)$, $|\mathcal{P}| \leq \varepsilon \binom{n}{\ell}$. As each poor $\ell$-subset is contained in $\binom{n-\ell}{m-\ell}$ $m$-subsets, it follows that there are at least

$$\binom{n}{m} - |\mathcal{P}| \binom{n-\ell}{m-\ell} > (1 - \varepsilon m^\ell) \binom{n}{m} = \frac{1}{2} \binom{n}{m}$$

$m$-subsets of vertices which do not contain any poor $\ell$-subsets.

Given an $\ell$-subset $S \in V^{(\ell)} \setminus \mathcal{P}$, we call an $m$-subset $T$ of $V$ bad for $S$ if $S \subseteq T$ and $|N(S) \cap T^{(r-\ell)}| \leq (p - \delta)(m-\ell)$. Let $\phi_S$ be the number of bad $m$-subsets for $S$. We claim that

$$\phi_S \leq \binom{n-\ell}{m-\ell} \exp \left( -\frac{\delta^2 m}{2(r-\ell)^2} \right).$$

Observe that

$$\phi_S = \left\{ T \in (V \setminus S)^{(m-\ell)} : |N(S) \cap T^{(r-\ell)}| \leq (p - \delta)(m-\ell) \right\}.$$

Let $X$ be the random variable $|N(S) \cap T^{(r-\ell)}|$, where $T$ is an $(m-\ell)$-subset of $V \setminus S$ picked uniformly at random. We consider the vertex exposure martingale on $T$. Let $Z_i$ be the $i$th exposed vertex in $T$. Define $X_i = \mathbb{E}(X | Z_1, \ldots, Z_i)$. Note that $\{X_i : i = 0, 1, \ldots, m-\ell\}$ is a martingale and $X_0 \geq p^{(m-\ell)}_{r-\ell}$. Moreover, $|X_i - X_{i-1}| \leq (m-\ell-1) < (m-\ell-1)$. Thus, by Lemma 8 applied with $\lambda = \delta^{(m)}_{r-\ell}$ and $c_i = \delta^{(m-1)}_{r-\ell-1}$, we have

$$\mathbb{P} \left( X_m \leq (p - \delta)(m-\ell) \right) \leq \mathbb{P}(X_m \leq X_0 - \lambda) \leq \exp \left( -\frac{\delta^2 (m-\ell)^2}{2m (r-\ell)} \right) = \frac{(-\delta^2 (m-\ell))}{2(r-\ell)}$$

$$\leq \exp \left( -\frac{\delta^2 m}{2(r-\ell)^2} \right).$$
Hence (3) holds.

An $m$-subset $T$ of $V$ is called bad if it is bad for some $S \in V^{(\ell)} \setminus P$. The number of bad $m$-subsets is at most

$$\sum_{S \in V^{(\ell)} \setminus P} \phi_S \leq \binom{n}{\ell} \binom{n-\ell}{m-\ell} \exp \left( -\frac{\delta^2 m}{2(r-\ell)^2} \right) = \binom{n}{m} \binom{m}{\ell} \exp \left( -\frac{\delta^2 m}{2(r-\ell)^2} \right) \leq \binom{n}{m} m_0^\ell \exp \left( -\frac{\delta^2 m_0}{2(r-\ell)^2} \right) \leq \binom{n}{m} \exp \left( -\ell \log(1/\delta) \right) \leq \frac{1}{2} \binom{n}{m},$$

where the last three inequalities hold by our choice of $m_0$, by inequality (1), and by our assumption on $\delta$, respectively. Together with (2), this shows there exists an $m$-subset inside which there is no poor $\ell$-subsets and in which every rich $\ell$-subset has degree at least $(p-\delta) \binom{m-\ell}{r-\ell}$. Such a set clearly gives us an induced subgraph of $G$ on $m$ vertices with minimum $\ell$-degree at least $(p-\delta) \binom{m-\ell}{r-\ell}$. \(\square\)

3 Concluding remarks

A 3-graph $G$ is $(p, \varepsilon, (1,1,1))$-quasirandom if for every triple of sets of vertices $X$, $Y$ and $Z \subseteq V$, the number $e_{1,1,1}(X,Y,Z)$ of triples $(x,y,z) \in X \times Y \times Z$ such that $xyz \in E(G)$ satisfies

$$|e_{1,1,1}(X,Y,Z) - p|X| \cdot |Y| \cdot |Z|| \leq \varepsilon v(G)^3.$$ Define $\pi_{(1,1,1)}-(qr)(F)$ analogously to $\pi_{(1,2)}-(qr)(F)$. Note that $\pi_{(1,2)}-(qr)(F) \leq \pi_{(1,1,1)}-(qr)(F) \leq \pi(F)$ for all 3-graph families $F$. An obvious open question is whether we have

$$\pi_{(1,1,1)}-(qr)(F) \leq \pi_2(F).$$

Even more: can one always extract subgraphs with large minimum codegree from $(1, 1, 1)$-quasirandom graphs? Even obtaining large subgraphs with non-zero minimum codegree remains an open problem for this weaker notion of quasirandomness.

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