ON POISSON TRANSFORMS FOR SPINORS

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Abstract. Let \((\tau, V_\tau)\) be a spinor representation of \(\text{Spin}(n)\) and let \((\sigma, V_\sigma)\) be a spinor representation of \(\text{Spin}(n-1)\) that occurs in the restriction \(\tau|_{\text{Spin}(n-1)}\).

We consider the real hyperbolic space \(H^n(\mathbb{R})\) as the rank one homogeneous space \(\text{Spin}_0(1,n)/\text{Spin}(n)\) and the spinor bundle \(\Sigma H^n(\mathbb{R})\) over \(H^n(\mathbb{R})\) as the homogeneous bundle \(\text{Spin}_0(1,n) \times_{\text{Spin}(n)} V_\tau\).

Our aim is to characterize eignespinsors of the algebra of invariant differential operators acting on \(\Sigma H^n(\mathbb{R})\) which can be written as the Poisson transform of \(L^p\)-sections of the bundle \(\text{Spin}(n) \times_{\text{Spin}(n-1)} V_\sigma\) over the boundary \(S^{n-1} \simeq \text{Spin}(n)/\text{Spin}(n-1)\) of \(H^n(\mathbb{R})\).

1. Introduction

In this article we go on with the investigation of Poisson transforms on vector bundles over the real hyperbolic space \(H^n(\mathbb{R})\) which we started in [2] for the case of the bundle of differential forms. Since the Clifford algebra of \(\mathbb{R}^n\) can be realized as the exterior algebra \(\bigwedge \mathbb{R}^n\) (via the quantization map), it becomes natural to extend the results of [2] to the case of the spinor bundle over \(H^n(\mathbb{R})\). To be more explicit, we shall realize \(H^n(\mathbb{R}) \simeq G/K = \text{Spin}_0(1,n)/\text{Spin}(n)\) as the unit ball in \(\mathbb{R}^n\) and let \(S^{n-1} = \partial H^n(\mathbb{R}) \simeq K/M = \text{Spin}(n)/\text{Spin}(n-1)\) be its boundary. Let \(\tau_\eta\) be the complex spin representation of \(\text{Spin}(n)\), which is irreducible when \(n\) is odd and splits into two irreducible components \(\tau_\eta = \tau_\eta^+ \oplus \tau_\eta^-\) (half-spin representations) when \(n\) is even. It is known by classical arguments that, if \(n\) is even, then each restriction \(\tau_\eta^+|_{\text{Spin}(n-1)}\) is a unitary irreducible representation of \(\text{Spin}(n-1)\), and therefore coincides with the spin representation of \(\text{Spin}(n-1)\) which we will denote by \(\sigma_{n-1}\). Note that \(\sigma_{n-1} = \tau_{n-1}\) and we will use in the sequel the letter \(\tau\) (resp. \(\sigma\)) for the spin or half-spin representations of \(K = \text{Spin}(n)\) (resp. \(M = \text{Spin}(n-1)\)). On the other hand, if \(n\) is odd, then \(\tau_\eta|_{\text{Spin}(n-1)}\) splits into two irreducible representations of \(\text{Spin}(n-1)\). More explicitly, \(\tau_\eta|_{\text{Spin}(n-1)} = \sigma_{n-1}^+ \oplus \sigma_{n-1}^-\), where \(\sigma_{n-1}^\pm\) are the half-spin representations of \(\text{Spin}(n-1)\).

To make the notations in this introduction less cumbersome, we let \(\tau\) (resp. \(\sigma\)) for the spin or half-spin representations of \(K = \text{Spin}(n)\) (resp. \(M = \text{Spin}(n-1)\)). On the other hand, if \(n\) is odd, then \(\tau_\eta|_{\text{Spin}(n-1)}\) splits into two irreducible representations of \(\text{Spin}(n-1)\). More explicitly, \(\tau_\eta|_{\text{Spin}(n-1)} = \sigma_{n-1}^+ \oplus \sigma_{n-1}^\pm\), where \(\sigma_{n-1}^\pm\) are the half-spin representations of \(\text{Spin}(n-1)\).

To make the notations in this introduction less cumbersome, we let \(\tau = \tau_\eta\) or \(\tau_\eta^\pm\) whether \(n\) is odd or even and we put \(\hat{M}(\tau) = \{\sigma_{n-1}^+, \sigma_{n-1}^\pm\}\) or \(\{\sigma_{n-1}\}\) accordingly.

For \(\lambda \in \mathbb{C}\) and \(\sigma \in \hat{M}(\tau)\), we consider the Poisson transform

\[
P_{\sigma, \lambda} : C^\omega(K/M, \sigma) \to C^\infty(G/K, \tau)
\]
given by

\[ \mathcal{P}_{\sigma, \lambda}^\tau f(g) = \mathcal{P} \int_K e^{-i(\lambda+\rho)H(g^{-1}k)\tau(\kappa(g^{-1}k))} \iota^\tau_\sigma(f(k))dk \]

where \( \iota^\tau_\sigma \) is the embedding of \( \sigma \) in \( \tau \) and \( \mathcal{P} \) is the constant given by (4.2). Above \( C^{-\omega}(K/M, \sigma) \) is the space of hyperfunction sections of the bundle \( K \times_M V_\sigma \) viewed as the space of \( V_\sigma \)-valued covariant hyperfunctions on \( K \) and \( C^\infty(G/K, \tau) \) is the space of smooth sections of the bundle \( G \times_K V_\tau \) viewed as the space of \( V_\tau \)-valued smooth covariant functions on \( G \).

It is proved by Gaillard [5] that the commutative algebra \( D(G, \tau) \) of \( G \)-invariant differential operators acting on \( C^\infty(G/K, \tau) \) is generated explicitly as

\[ \begin{cases} D(G, \tau_\pm^n) \simeq \mathbb{C}[\mathcal{D}^2] & \text{if } n \text{ is even} \\ D(G, \tau_n) \simeq \mathbb{C}[\mathcal{D}] & \text{if } n \text{ is odd} \end{cases} \]

where \( \mathcal{D} \) is the Dirac operator. Moreover, it is well known (see e.g. Camporesi and Pedon [4], Olbrich [10]) that the Poisson integrals \( \mathcal{P}_{\sigma, \lambda}^\tau f \), for \( f \in C^{-\omega}(K/M, \sigma) \), are eigenfunctions of \( D(G, \tau) \).

The main result of this paper is to characterize the image of the spaces \( L^p(K/M, \sigma) \), for \( 1 < p < \infty \), by the Poisson transform \( \mathcal{P}_{\sigma, \lambda}^\tau \). To do so, we introduce the spaces \( \mathcal{E}_{\sigma, \lambda}^p(G/K, \tau) \) consisting of eigensections of \( D(G, \tau) \) satisfying a Hardy-type growth condition, see (4.5) and (4.6) for the precise definition.

**Theorem** (Main result). Let \( 1 < p < \infty \) and let \( \lambda \) in \( \mathbb{C} \) such that \( \Re(i\lambda) > 0 \).

1. If \( n \) is even, \( \mathcal{P}_{\sigma_n-1, \lambda}^\tau \) is a topological isomorphism of the space \( L^p(K/M, \sigma_{n-1}) \) onto the space \( \mathcal{E}_{\sigma_{n-1}, \lambda}^p(G/K, \tau_n) \).

2. If \( n \) is odd, \( \mathcal{P}_{\sigma_n-1, \lambda}^\tau \) is a topological isomorphism of the space \( L^p(K/M, \sigma_{n-1}^\pm) \) onto the space \( \mathcal{E}_{\sigma_{n-1}, \lambda}^p(G/K, \tau_n) \).

The paper is organized as follow. In section 2 we recall some useful facts on Clifford modules and spinor representations. In section 3 we define the principal series representations of the Lie group \( \text{Spin}_0(1, n) \) and in section 4 we set up the Poisson transform. In the remaining sections we prove the main result where we will follow the same steps as in [2]. More precisely, in section 5 we prove a Fatou-type theorem, which will allows us, in particular, to compute the vector-valued Harish-Chandra \( c \)-function associated with \( \tau \) in terms of Gamma functions. The Fatou-type theorem is essentially used in section 6 to establish the main result for \( p = 2 \). In the same section an inversion formula for Poisson transforms is also proved. Section 7 is devoted to the proof of the main result for every \( 1 < p < \infty \).

The proof is based on a reduction argument to the case \( p = 2 \) followed by the use of the inversion formula.

2. THE SPINOR REPRESENTATIONS

Consider the Euclidean vector space \( \mathbb{R}^n \) equipped with the standard inner product

\[ \langle x, y \rangle = \sum_{i=1}^{n} x_i y_i. \quad (2.1) \]
Choose an orthonormal basis \( \{e_1, \ldots, e_n\} \) on \( \mathbb{R}^n \). The Clifford algebra \( \mathcal{C}(n) \) is the algebra over \( \mathbb{R} \) generated by the vector space \( \mathbb{R}^n \) and the relations

\[
xy + yx = -2\langle x, y \rangle \quad \text{for } x, y \in \mathbb{R}^n,
\]

where \(-2\langle x, y \rangle\) is identified with \(-2\langle x, y \rangle 1 \) and \(1 \) being the algebra identity element of \( \mathcal{C}(n) \). In particular, we have

\[
e_i^2 = -1 \quad \text{and} \quad e_i e_j = -e_j e_i \quad \text{for} \quad i \neq j.
\]

As a real vector space, \( \mathcal{C}(n) \) has a basis consisting of

\[
e_0 = 1, \quad e_I = e_{i_1} e_{i_2} \cdots e_{i_k}
\]

with \( I = \{i_1, \ldots, i_k\} \subset \{1, \ldots, n\} \) and \( i_1 < i_2 < \cdots < i_k \).

We denote by \( a', a^* \) and \( \bar{a} \) the conjugations of any \( a \in \mathcal{C}(n) \). By linearity, it is sufficient to define the effect on the basis elements \( e_I = e_{i_1} e_{i_2} \cdots e_{i_k} \). The conjugations are

\[
e_I \mapsto e_I' = (-1)^k e_I \quad \text{(main involution)}
\]

\[
e_I \mapsto e_I^* = (-1)^{\frac{k(k-1)}{2}} e_I \quad \text{(reversion)}
\]

\[
e_I \mapsto \bar{e}_I = (-1)^{\frac{k(k+1)}{2}} e_I \quad \text{(Clifford conjugation)}.
\]

The later conjugation will be denoted by \( \alpha(a) := \bar{a} \). Observe that \( a \mapsto a' \) is an involution \( \langle (ab)' = a'b' \rangle \) while the conjugations \( a \mapsto a^* \) and \( a \mapsto \bar{a} = a^{*t} \) are anti-involutions \( \langle (ab)^* = b^* a^*, \bar{a}b = b\bar{a} \rangle \). In particular

\[
(e_{j_1} \cdots e_{j_k})(e_{j_1} \cdots e_{j_k})' = \begin{cases} 
1 & \text{if } k \text{ is even} \\
-1 & \text{if } k \text{ is odd}
\end{cases}
\]

Under the main involution, the Clifford algebra splits into a direct sum of even and odd elements:

\[
\mathcal{C}(n) = \mathcal{C}(n)^+ \oplus \mathcal{C}(n)^-.
\]

For \( a \in \mathcal{C}(n) \) define the norm \( |a| = \alpha(a)a = \bar{a}a \). In particular, if \( a \in \mathbb{R}^n \), then \( \bar{a} = -a \) and from (2.2) we get \( |a| = \langle a, a \rangle \).

The spin groupe \( \text{Spin}(\mathbb{R}^n) \) is the group of elements in \( \mathcal{C}(n) \) of the form

\[
g = x_1 \cdots x_{2k}, \quad x_i \in \mathbb{R}^n, \quad |x_i| = 1 \quad \text{for} \quad i = 1, \ldots, 2k.
\]

In particular, \( \text{Spin}(\mathbb{R}^n) \) is a Lie group which is a twofold covering of \( \text{SO}(n) \). We will often write \( \text{Spin}(n) \) in place of \( \text{Spin}(\mathbb{R}^n) \).

Denote by \( \mathbb{C} \ell(n) \) the complex Clifford algebra of \( \mathbb{C}^n \), which can be identified with \( \mathcal{C}(n) \otimes \mathbb{C} \).

A Clifford module \( (\tau, S_n) \) is a complex vector space \( S_n \) together with an action \( \tau \) of \( \mathbb{C} \ell(n) \) on \( S_n \). As \( \text{Spin}(n) \subset \mathbb{C} \ell(n) \subset \mathcal{C}(n) \), the representation \( \tau \) of \( \mathbb{C} \ell(n) \) restricts to a representation of \( \text{Spin}(n) \).

The Clifford modules \( S \) have different realizations. In the sequel we will use the Lagrangian space realization, where we shall distinguish the difference between an even dimension and an odd dimension. For more details, we refer the reader to [3,9].
2.1. The even-dimensional case. Assume that \( n = 2m \). In terms of the orthonormal basis \( e_1, \ldots, e_{2m} \) of \( \mathbb{R}^{2m} \), define for each \( j = 1, \ldots, m \),

\[
  f_j = \frac{1}{\sqrt{2}} (e_{2j-1} + i e_{2j}),
\]

\[
  \alpha(f_j) = \tilde{f}_j = -\frac{1}{\sqrt{2}} (e_{2j-1} - i e_{2j}).
\]

Set

\[
  W_m = \text{span}(f_1, \ldots, f_m) \quad \text{and} \quad \overline{W}_m = \text{span}(\tilde{f}_1, \ldots, \tilde{f}_m).
\]

Then \( W_m \) is a maximal isotropic subspaces of \( \mathbb{C}^n \),

\[
  \mathbb{C}^n = W_m \oplus \overline{W}_m,
\]

and the inner product (2.1) extends to \( \mathbb{C}\ell(n) \) and induces a non-degenerate duality between \( W_m \) and \( \overline{W}_m \).

In these circumstances, the Dirac spinor space \( S \) is defined as the exterior algebra \( \wedge W_m \), that is \( S = \wedge W_m \). If we want to emphasize the dimension we write \( S_n \) or \( S_{2m} \).

There is a unique \( \mathbb{C}\ell(n) \)-module structure on \( S_{2m} = \wedge W_m \) such that

\[
  \tau(u + \bar{v})x = \sqrt{2} \varepsilon(u)x - \sqrt{2} \iota(\bar{v})x,
\]

for \( u \in W_m \), \( v \in \overline{W}_m \) and \( x \in S_{2m} \). Above, \( \varepsilon(u) \) and \( \iota(\bar{v}) \) denote, respectively, the exterior and the interior products on \( \mathbb{C}\ell(2m) \). The action \( \tau \) gives rise to a representation of the Clifford algebra \( \mathbb{C}\ell(2m) \), which we will denote by \( \tau_{2m} \), and called the spinor representation. The representation \( (\tau_{2m}, S_{2m}) \) is of dimension \( 2^m \) and, up to equivalence, it is the unique irreducible representation of \( \mathbb{C}\ell(2m) \).

Let \( S^+_{2m} = \wedge^+ W_m \) be the even part and \( S^-_{2m} = \wedge^- W_m \) be the odd part of \( \wedge W_m \), which are stable under the action of \( \mathbb{C}\ell^+(n) \). As \( \text{Spin}(2m) \subset \mathbb{C}\ell^+(2m) \), the representation \( \tau_{2m} \) splits as a direct sum of two non-equivalent irreducible representations \( (\tau^+_{2m}, S^+_{2m}) \) and \( (\tau^-_{2m}, S^-_{2m}) \) of \( \text{Spin}(2m) \), called the half-spinor representations. Both representations are of dimension \( 2^{m-1} \).

2.2. The odd-dimensional case. Assume that \( n = 2m - 1 \). Then we have,

\[
  \mathbb{C}^n = W_{m-1} \oplus \overline{W}_{m-1} \oplus \mathbb{C}e_{2m-1}
\]

where \( W_{m-1} = \text{span}(f_1, \ldots, f_{m-1}) \). The spinor space \( S_{2m-1} = \wedge W_{m-1} \) has two module structures over \( \mathbb{C}\ell(2m - 1) \). Indeed, \( W_{m-1} \oplus \overline{W}_{m-1} \) operates by the same formula as in the even case above while the element \( e_{2m-1} \) acts on \( x \in S_{2m-1} \) either by multiplication by \( i(-1)^{\deg x} \) or by \( -i(-1)^{\deg x} \). As representations of \( \text{Spin}(2m - 1) \), the above mentioned modules are irreducible, equivalent, and thus leading to a unique spinor representation \( \tau_{2m-1} \) acting on the space \( S_{2m-1} = \wedge W_{m-1} \) by

\[
  \tau_{2m-1}(u + \bar{v} + \lambda e_{2m-1})x = \sqrt{2} \varepsilon(u)x - \sqrt{2} \iota(\bar{v})x + i(-1)^{\deg x} \lambda x
\]

for \( u \in W_{m-1} \), \( \bar{v} \in \overline{W}_{m-1} \), \( \lambda \in \mathbb{C} \) and \( x \in S_{2m-1} \).
3. Principal series representation of $\text{Spin}_0(1,n)$

Let $H^n(\mathbb{R})$ be the $n$-dimensional real hyperbolic space, $n \geq 2$. We identify $H^n(\mathbb{R})$, via the Poincaré model, with the unit ball of $\mathbb{R}^n$ and its topological boundary $\partial H^n(\mathbb{R})$ with the unit sphere $S^{n-1}$ of $\mathbb{R}^n$. We shall realize $H^n(\mathbb{R})$ as the rank one symmetric space $G = \text{Spin}_0(1,n)/\text{Spin}(n)$ and $S^{n-1}$ with the compact symmetric space $\text{Spin}(n)/\text{Spin}(n-1)$.

Let $\mathbb{R}^{1,n}$ be the real vector space of dimension $n+1$ equipped with the symmetric bilinear form

$$Q(x, y) = x_0y_0 - x_1y_1 - \cdots - x_ny_n,$$

and fix an orthonormal basis $\{e_0, e_1, \ldots, e_n\}$ of $\mathbb{R}^{n+1}$. Denote by $\text{Cl}(\mathbb{R}^{1,n})$ the corresponding Clifford algebra generated by $\mathbb{R}^{1,n}$ and subject to the relation

$$xy + yx = 2Q(x, y).$$

As in the previous section, we may define on $\text{Cl}(\mathbb{R}^{1,n})$ the main involution, the reversion, and the Clifford conjugaison. Define the spinorial norm $\mathcal{N}$ by

$$\mathcal{N}(a) = \alpha(a)a, \quad a \in \text{Cl}(\mathbb{R}^{1,n})$$

and let $\text{Spin}_0(1,n)$ be the group defined by

$$\text{Spin}_0(1,n) = \{g = x_1 \cdots x_{2k} \mid x_j \in \mathbb{R}^{1,n}, \ Q(x_j) = \pm 1, \# \{j, \ Q(x_j) = -1\} \text{ is even} \}. $$

Note that an element $a \in \text{Cl}(\mathbb{R}^{1,n})$ is in $\text{Spin}_0(1,n)$ if and only if $a$ is invertible and $\mathcal{N}(a) = 1$. Further, the group $G = \text{Spin}_0(1,n)$ is a connected Lie group which turns out to be a twofold covering of $SO_0(\mathbb{R}^{1,n}) = SO_0(1,n)$ (see [3]).

The Lie algebra $\mathfrak{g} = \sigma(1,n)$ of $G$ can be realized as the subspace $\text{Cl}^2(\mathbb{R}^{1,n}) \cong \bigwedge^2 \mathbb{R}^{n+1}$ of bi-vectors in $\text{Cl}(\mathbb{R}^{1,n})$ spanned by $\{e_ie_j, 0 \leq i < j \leq n\}$. Let

$$\mathfrak{t} = \bigoplus_{1 \leq i < j \leq n} \mathbb{R}e_ie_j, \quad \mathfrak{p} = \bigoplus_{j=1}^n \mathbb{R}e_0e_j. \quad (3.1)$$

Moreover, let $\mathfrak{a}$ be the Cartan subspace of $\mathfrak{p}$ given by

$$\mathfrak{a} = \mathbb{R}H,$$

where $H = e_0e_n$.

The analytic Lie subgroups of $G$ associated to the Lie sub-algebras $\mathfrak{t}$, $\mathfrak{a}$, $\mathfrak{m}$ and $\mathfrak{n}$ will be denoted, respectively, by $K$, $A$, $M$ and $N$. We pin down that $K \cong \text{Spin}(n)$ and $M \cong \text{Spin}(n-1)$. It is known that the homogenous space $G/K$ can be seen as the real hyperbolic space $H^n(\mathbb{R})$, which we may realize as the open unit ball in $\mathbb{R}^n$. Further, the homogeneous space $K/M$ can be seen as the boundary $\partial H^n(\mathbb{R})$ and realized as the unit sphere $S^{n-1}$ in $\mathbb{R}^n$.

Henceforth, we will use the Greek letter $\tau$ to denote the spin representations of $K$ and the Greek letter $\sigma$ for those of $M$.

Let $(\tau_n, V_{\tau_n})$ be a unitary complex spin representation of $K = \text{Spin}(n)$, where $V_{\tau_n} = \mathbb{S}_n$ is the space of spinors associated with the complexification of $\mathfrak{p} \cong T_0(G/K)$. The spinor bundle $\Sigma H^n(\mathbb{R}) \cong G \times_K \mathbb{S}_n$
over $H^n(\mathbb{R})$ associated with $\tau_n$.

The space $C^\infty(\Sigma H^n(\mathbb{R}))$ of smooth sections of this bundle will be identified with the space $C^\infty(G/K, \tau_n)$ of smooth $\mathbb{S}_n$-valued functions on $G$ such that

$$f(gk) = \tau_n(k^{-1})f(g)$$

for any $g \in G$ and $k \in K$.

Recall the following branching law of $(K, M) = (\text{Spin}(n), \text{Spin}(n - 1))$.

**Lemma 3.1** (See [6]). Let $\hat{M}$ be the set of unitary equivalence classes of irreducible representations of $M$. Using the same notation as in the previous section, we have:

1. If $n$ is even, then $\tau_n^\pm |_M = \sigma_{n-1}^+ \in \hat{M}$.
2. If $n$ is odd, then $\tau_n^+ |_M = \sigma_{n-1}^+ \oplus \sigma_{n-1}^-$ with $\sigma_{n-1}^+ \in \hat{M}$.

The decomposing factors occur with multiplicity one.

To be more explicit, the above lemma says:

1. If $n = 2m$, the two half-spin representations $(\tau_{2m}^\pm, \mathbb{S}_{2m}^\pm)$ of $\text{Spin}(2m)$ are non-equivalent irreducible representations. Fix the following identifications

$$\iota_{\tau_{2m}^\pm} : (\sigma_{2m-1}^\pm, \mathbb{S}_{2m-1}^\pm) \hookrightarrow (\tau_{2m}^\pm, \mathbb{S}_{2m}^\pm)$$

(3.2)

to be the identity map.

2. If $n = 2m + 1$, the spin representation $(\tau_{2m+1}^\pm, \mathbb{S}_{2m+1}^\pm)$ is irreducible. Its restriction to $\text{Spin}(2m)$ splits as $\tau_{2m+1}^\pm |_{\text{Spin}(2m)} = \sigma_{2m}^+ \oplus \sigma_{2m}^-$, where $(\sigma_{2m}^\pm, \mathbb{S}_{2m}^\pm)$ are the two half-spin representations of $\text{Spin}(2m)$. That is $\mathbb{S}_{2m+1} = \mathbb{S}_{2m} = \mathbb{S}_{2m}^+ \oplus \mathbb{S}_{2m}^-$. For any $x = x_1 \wedge \cdots \wedge x_k \in \mathbb{S}_{2m+1}$ we let

$$\gamma(x_1 \wedge \cdots \wedge x_k) = (-1)^k x_1 \wedge \cdots \wedge x_k.$$

Then the projection of $(\tau_{2m+1}^\pm, \mathbb{S}_{2m+1}^\pm)$ onto $(\sigma_{2m}^\pm, \mathbb{S}_{2m}^\pm)$ is given by

$$\pi_{\tau_{2m}^\pm} : \mathbb{S}_{2m+1} \rightarrow \mathbb{S}_{2m}^\pm$$

$$x \mapsto \frac{1}{2}(x \pm \gamma(x)),$$

while the embedding

$$l_{\tau_{2m}^\pm} : (\sigma_{2m}^\pm, \mathbb{S}_{2m}^\pm) \hookrightarrow (\tau_{2m+1}^\pm, \mathbb{S}_{2m+1})$$

(3.3)

is the identity map. One can easily check that the adjoint of $\pi_{\tau_{2m+1}^\pm}$ with respect to the extended inner product $\langle \cdot, \cdot \rangle$ coincides with $l_{\tau_{2m}^\pm}$.

**Convention 3.1.** Hereafter we shall use the following convention:

- Let $\tau = \tau_n$ or $\tau_n^\pm$. Denote by $\hat{M}(\tau)$ the set of representations $\sigma \in \hat{M}$ that occur in the restriction of $\tau$ to $M$ with multiplicity one.
- When $n$ is even, the notation $(\tau, \text{V}_\tau)$ refers to the two half-spin representations $(\tau_n^\pm, \mathbb{S}_n^\pm)$, and the set $\hat{M}(\tau)$ reduces to the single $\{\sigma_{n-1}^\pm\}$.
- When $n$ is odd, the notation $(\tau, \text{V}_\tau)$ refers to the unique spin representation $(\tau_n, \mathbb{S}_n)$, and the set $\hat{M}(\tau)$ becomes $\{\sigma_{n-1}^+, \sigma_{n-1}^-\}$. 

Let $\tau = \tau_n$ or $\tau_n^\pm$. For $\sigma \in \hat{M}(\tau)$ and $\lambda \in \mathfrak{a}_n^* \simeq \mathbb{C}$, let $\sigma_\lambda$ be the representation of the parabolic subgroup $P = MAN$ given by

$$\sigma_\lambda(ma_t n) := (\sigma \otimes e^{it\lambda} \otimes 1)(ma_t n) = e^{(n-1)t\lambda} \sigma(m),$$

where $\rho = (n-1)/2$. The principal series representation $\pi_{\sigma,\lambda}$ of $G$ is the associated induced representation from $P$ to $G$,

$$\pi_{\sigma,\lambda} = \text{Ind}_P^G \sigma_\lambda.$$

Let $S_{\sigma,\lambda} = G \times_P V_{\sigma} = G \times_P S_{n-1}$ be the associated homogeneous bundle over $G/P$. Its space of hyperfunction sections will be identified with the space $C^\omega(G/P, \sigma_\lambda)$ of $V_\sigma$-valued hyperfunctions $f$ on $G$ such that

$$f(gma_t n) = e^{(i\lambda-\rho)t} \sigma(m^{-1})f(g),$$

for any $g \in G$, $m \in M$, $n \in N$ and $a_t \in A$.

By restriction to $K$, $C^\omega(G/P, \sigma_\lambda)$ is isomorphic (as a $K$-module) to the space $C^\omega(K/M, \sigma)$ of $V_\sigma$-valued hyperfunctions $f$ on $K$ satisfying

$$f(km) = \sigma(m^{-1})f(k),$$

for any $k \in K$ and $m \in M$. The space $C^\omega(K/M, \sigma)$ can also be seen as the space of hyperfunction sections of the homogeneous bundle $K \times_M V_\sigma$ over $K/M$ corresponding to $\sigma$.

Let $G = KAN$ be the Iwasawa decomposition for $G$, so each $g \in G$ can be written as

$$g = \kappa(g)e^{H(g)}n(g).$$

Then the compact model of the principal series representation $\pi_{\sigma,\lambda}$ is given by

$$\pi_{\sigma,\lambda}(g)f(k) = e^{(i\lambda-\rho)H(g^{-1}k)}f(\kappa(g^{-1}k)),$$

with $g \in G$ and $k \in K$.

4. Poisson transforms

We continue with the same notations as above. Let $\iota_\sigma^\tau : V_\sigma \rightarrow V_\tau$ be the natural embedding of $V_\sigma$ into $V_\tau$ (see (3.2) for the even case and (3.3) for the odd case). The Poisson transform on $C^\omega(K/M, \sigma)$ is the map

$$\mathcal{P}_{\sigma,\lambda}^\tau : C^\omega(K/M, \sigma) \longrightarrow C^\infty(G/K, \tau)$$

given by

$$\mathcal{P}_{\sigma,\lambda}^\tau f(g) = \sqrt{\frac{\dim \tau}{\dim \sigma}} \kappa \int_K e^{-(i\lambda+\rho)H(g^{-1}k)}(\kappa(g^{-1}k))\iota_\sigma^\tau(f(k))dk,$$

where

$$\kappa = \sqrt{\frac{\dim \tau}{\dim \sigma}} = \begin{cases} 1 & \text{if } n \text{ is even} \\ \sqrt{2} & \text{if } n \text{ is odd}. \end{cases}$$

Let $D(G, \tau)$ be the left algebra of $G$-invariant differential operators acting on $C^\infty(G/K, \tau)$. In [5], the author proved that if $n$ is even, then $D(G, \tau_n^\pm) \simeq \mathbb{C}[\mathcal{D}^2]$, and if $n$ is odd, then $D(G, \tau_n) \simeq \mathbb{C}[\mathcal{D}]$. Here $\mathcal{D}$ is the Dirac operator.

The action of the algebra $D(G, \tau)$ on Poisson integrals is described as follows.
Proposition 4.1 (see [4]). For $\lambda \in \mathbb{C}$, we have:

1. If $n$ is even and $f \in C^{-\omega}(K/M, \sigma_{n-1})$, then $\mathcal{P}_{\sigma_{n-1}, \lambda}^n f$ is an eigenfunction of the algebra $D(G, \tau_n^\pm)$. More precisely, we have

$$\mathcal{D}^2 \mathcal{P}_{\sigma_{n-1}, \lambda}^n f = \lambda^2 \mathcal{P}_{\sigma_{n-1}, \lambda}^n f.$$

2. If $n$ is odd and $f \in C^{-\omega}(K/M, \sigma_n^\pm)$, then $\mathcal{P}_{\sigma_n, \lambda}^n$ is an eigenfunction of algebra $D(G, \tau_n)$. More precisely, we have

$$\mathcal{D} \mathcal{P}_{\sigma_n, \lambda}^n f = \mp \lambda \mathcal{P}_{\sigma_n, \lambda}^n f.$$

Let us introduce the following eigenspaces:

$$\mathcal{E}_\lambda(G/K, \tau_n^\pm) = \left\{ F \in C^\infty(G/K, \tau_n^\pm) \mid \mathcal{D}^2 F = \lambda^2 F \right\} \quad \text{whenever } n \text{ is even,} \quad (4.3)$$

$$\mathcal{E}_{\pm, \lambda}(G/K, \tau_n) = \left\{ F \in C^\infty(G/K, \tau_n) \mid \mathcal{D} F = \pm \lambda F \right\} \quad \text{whenever } n \text{ is odd.} \quad (4.4)$$

For $1 < p < \infty$ and $\sigma \in \hat{M}(\tau)$, let $L^p(K/M, \sigma)$ be the space of $V_\sigma$-valued functions $f$ on $K$ satisfying (3.4) and such that

$$\|f\|_{L^p(K/M, \sigma)} = \left( \int_K \|f(k)\|_\sigma^p \, dk \right)^{1/p} < \infty.$$  

(Here $\| \cdot \|_\sigma = \| \cdot \|_{V_\sigma}$.) The main goal of the paper is to characterize the image of $L^p(K/M, \sigma)$ by the Poisson transform $\mathcal{P}_{\sigma, \lambda}^\tau$. To state the main result, let us introduce the following Hardy type spaces:

$$\mathcal{E}_\lambda^p(G/K, \tau_n^\pm) = \left\{ F \in \mathcal{E}_\lambda(G/K, \tau_n^\pm) \mid \|F\|_{\mathcal{E}_\lambda^p} < \infty \right\} \quad \text{whenever } n \text{ is even,} \quad (4.5)$$

where

$$\|F\|_{\mathcal{E}_\lambda^p} = \sup_{t > 0} e^{(\sigma - \Re(i\lambda))t} \left( \int_K \|F(k\alpha_t)\|_{\tau_n^\pm}^p \, dk \right)^{\frac{1}{p}}.$$

and

$$\mathcal{E}_{\pm, \lambda}^p(G/K, \tau_n) = \left\{ F \in \mathcal{E}_{\pm, \lambda}(G/K, \tau_n) \mid \|F\|_{\mathcal{E}_{\pm, \lambda}^p} < \infty \right\} \quad \text{whenever } n \text{ is odd,} \quad (4.6)$$

where

$$\|F\|_{\mathcal{E}_{\pm, \lambda}^p} = \sup_{t > 0} e^{(\sigma - \Re(i\lambda))t} \left( \int_K \|F(k\alpha_t)\|_{\tau_n^\pm}^p \, dk \right)^{\frac{1}{p}}.$$

Our main result is

**Theorem 4.2.** Let $1 < p < \infty$ and let $\lambda \in \mathbb{C}$ such that $\Re(i\lambda) > 0$.

1. If $n$ is even, then $\mathcal{P}_{\sigma_{n-1}, \lambda}^n$ is a topological isomorphism of the space $L^p(K/M, \sigma_{n-1})$ onto the space $\mathcal{E}_{\lambda}^p(G/K, \tau_n^\pm)$. Furthermore, there exists a positive constant $\gamma_\lambda$ such that, for every $f \in L^p(K/M, \sigma)$ we have

$$\|c(\lambda, \tau_n^\pm)\|_{L^p(K/M, \sigma)} \leq \|\mathcal{P}_{\sigma, \lambda}^\tau f\|_{\mathcal{E}_{\lambda}^p} \leq \kappa \gamma_\lambda \|f\|_{L^p(K/M, \sigma)}.$$
(2) If \( n \) is odd, then \( P_{\sigma_n}^{\lambda_n} \) is a topological isomorphism of the space \( L^p(K/M, \sigma_n^{+}) \) onto the space \( \mathcal{E}_{\pm, \lambda_n}(G/K, \tau_n) \). Furthermore, there exists a positive constant \( \gamma_\lambda \) such that, for every \( f \in L^p(K/M, \sigma) \) we have
\[
\sqrt{2} |c^\pm(\lambda, \tau_n)||f||_{L^p(K/M, \sigma)} \leq ||P_{\sigma, \lambda}^\tau f||_{\mathcal{E}_{\pm, \lambda}} \lesssim \kappa \gamma_\lambda ||f||_{L^p(K/M, \sigma)}
\]
The rest of the paper is devoted to the proof of the above statement.

5. Intermediate Results

In the light of the Convention 3.1, let \( \tau = \tau_n \) or \( \tau_n^\pm \) and let \( \sigma \in \widehat{M}(\tau) \).

**Proposition 5.1.** For any \( \lambda \in \mathbb{C} \) with \( \Re(i\lambda) > 0 \), there exists a positive constant \( \gamma_\lambda \) such that, for every \( f \in L^p(K/M, \sigma) \) we have
\[
\left( \int_K ||P_{\sigma, \lambda}^\tau f(ka_t)||^p_{\tau} dk \right)^{1/p} \leq \kappa \gamma_\lambda \text{e}^{(\Re(i\lambda) - \rho)t} ||f||_{L^p(K/M, \sigma)}
\]
where \( \kappa \) is given by (4.2).

**Proof.** By the definition (4.1) of the Poisson integrals, we have
\[
\|P_{\sigma, \lambda}^\tau f(ka_t)\|_{V_\tau} \leq \kappa \int_K \text{e}^{-(\Re(i\lambda) + \rho)t} \|T_{\sigma}(f(h))\|_{V_\tau} dh,
\]
and let
\[
eq \kappa \int_K \text{e}^{-(\Re(i\lambda) + \rho)t} \|f(h)\|_{V_\sigma} dh = \kappa \text{e}_{\lambda, t} * \|f(\cdot)\|_{V_\sigma}(k),
\]
where \( \text{e}_{\lambda, t}(k) = \text{e}^{-(\Re(i\lambda) + \rho)t} \text{e}^{(a_t^{-1}k)^{-1}} \) and \( * \) is the convolution product over \( K \). To conclude, we use Young’s inequality and the fact that for \( \Re(i\lambda) > 0 \) we have
\[
\|e_{\lambda, t}\|_{L^1(K/M, \sigma)} = \int_K \text{e}^{-(\Re(i\lambda) + \rho)t} \|f(\cdot)\|_{V_\sigma}(k) dh,\]
where \( \phi_{(\alpha, \beta)}(\cdot) \) is the Jacobi function (5.2) and \( c_{\alpha, \beta}(\lambda) \) is the constant (5.4). For the last identity, we refer to (5.3). \( \square \)

Let \( \overline{N} = \theta(N) \), where \( \theta \) is the Cartan involution of \( G \) corresponding to (3.1). Define the generalized Harish-Chandra \( c \)-function by
\[
c(\lambda, \tau) = \int_{\overline{N}} \text{e}^{-(i\lambda + \rho)t(\kappa(\overline{\pi}))} d\overline{\pi} \in \text{End}_M(V_\tau),
\]
where \( d\overline{\pi} \) is the Haar measure on \( \overline{N} \) with the normalization
\[
\int_{\overline{N}} \text{e}^{-2\rho(H(\overline{\pi}))} d\overline{\pi} = 1.
\]
It is well known that the integral (5.1) converges for \( \lambda \in \mathbb{C} \) such that \( \Re(i\lambda) > 0 \) and it has a meromorphic continuation to \( \mathbb{C} \).
Proposition 5.2 (Fatou Lemma). Let $\lambda \in \mathbb{C}$ such that $\Re(i\lambda) > 0$. Then

$$\lim_{t \to \infty} e^{(p-\Re(i\lambda))t} I^{\tau}_{\sigma, \lambda} f(ka_t) = \mathcal{F}(\lambda, \tau)\nu_{\sigma} f(k)$$

(i) uniformly for $f \in C^\infty(K/M, \sigma)$;
(ii) in the $L^p(K, V_\sigma)$-sense for $f \in L^p(K/M, \sigma)$ with $1 < p < \infty$.

Proof. The proof follows the same arguments as in [2, Theorem 4.3].

Since the restriction $c(\lambda, \tau)|_{V_\sigma}$ commutes with the representation $\sigma$, then by Schur’s lemma we have

$$c(\lambda, \tau_n) = c(\lambda, \tau_n)\text{Id}_{V_{\sigma_n-1}}$$

whenever $n$ even,

$$c(\lambda, \tau_n) = c^+(\lambda, \tau_n)\text{Id}_{V_{\sigma_n+1}} + c^-(\lambda, \tau_n)\text{Id}_{V_{\sigma_n-1}}$$

whenever $n$ odd

for some complex coefficients $c(\lambda, \tau_n^\pm)$ and $c^\pm(\lambda, \tau_n)$. To compute explicitly these scalar components, we will study the asymptotic behaviour of the so-called $\tau$-spherical functions.

A continuous function $F: G \to \text{End}(V_\tau)$ is called elementary $\tau$-spherical if $F$ satisfies

(a) $F(k_1gk_2) = \tau(k_2)^{-1}F(g)\tau(k_1^{-1})$,

(b) $F$ is a joint-eigenfunction for $D(G, \tau)$ with $F(e) = \text{Id}$.

In view of the Convention 3.1, for $\sigma \in \hat{M}(\tau)$ and $\lambda \in \mathbb{C}$, we consider the function $\Phi^\tau_{\sigma}(\lambda, \cdot): G \to \text{End}(V_\tau)$ defined by

$$\Phi^\tau_{\sigma}(\lambda, g)v = \kappa^2 \int_K e^{-i\lambda(\rho_0)}H(g^{-1}k)\tau(\rho_0(g^{-1}k))\nu_{\sigma}(\tau(k^{-1})v))dk, \forall v \in V_\tau$$

is a $\tau$-spherical function. Using the Cartan decomposition $G = KAK$, it is clear that $\Phi^\tau_{\sigma}(\lambda, \cdot)$ is completely determined by its restriction to $A$. Since $A$ and $M$ commute, $\Phi^\tau_{\sigma}(\lambda, a_t) \in \text{End}_M(V_\sigma)$ for all $a_t \in A$. Hence, by Schur’s lemma $\Phi^\tau_{\sigma}(\lambda, a_t)$ is scalar on each $M$-type $V_\sigma$ of $V_\tau$. That is

$$\Phi^\tau_{\sigma_n-1}(\lambda, a_t) = \varphi^\pm(\lambda, t)\text{Id}_{V_{\sigma_n-1}}$$

whenever $n$ is even,

$$\Phi^\tau_{\sigma_n+1}(\lambda, a_t) = \varphi^\pm(\lambda, t)\text{Id}_{V_{\sigma_n+1}} + \varphi^\pm(\lambda, t)\text{Id}_{V_{\sigma_n-1}}$$

whenever $n$ is odd.

In [4], the scalar components $\varphi^\pm$, $\varphi^\mp$ and $\varphi^\pm$ are given in terms of the Jacobi function

$$\phi^\alpha_\lambda(\alpha, \beta)(t) = 2F_1\left(\frac{i\lambda + \alpha + \beta + 1}{2}, \frac{-i\lambda + \alpha + \beta + 1}{2}; \alpha + 1; -\sinh^2 t\right)$$

where $\alpha, \beta, \lambda \in \mathbb{C}$ with $\alpha \neq -1, -2, \ldots$, (see, e.g. [8]).
**Theorem 5.3** (see [4, Theorem 5.4]). We have:

1. When \( n \) is even,
   \[
   \varphi^\pm(\lambda, t) = \left( \cosh \frac{t}{2} \right) \phi_{2\lambda}^{(n/2-1,n/2)} \left( \frac{t}{2} \right).
   \]

2. When \( n \) is odd,
   \[
   \varphi^+_n(\lambda, t) = \left( \cosh \frac{t}{2} \right) \phi_{2\lambda}^{(n/2-1,n/2)} \left( \frac{t}{2} \right) \pm i \frac{2\lambda}{n} \left( \sinh \frac{t}{2} \right) \phi_{2\lambda}^{(n/2,n/2-1)} \left( \frac{t}{2} \right),
   \]
   \[
   \varphi^-_n(\lambda, t) = \left( \cosh \frac{t}{2} \right) \phi_{2\lambda}^{(n/2-1,n/2)} \left( \frac{t}{2} \right) \mp i \frac{2\lambda}{n} \left( \sinh \frac{t}{2} \right) \phi_{2\lambda}^{(n/2,n/2-1)} \left( \frac{t}{2} \right).
   \]

Next, we will compute the scalar components of the Harish-Chandra \( c \)-function \( c(\lambda, \tau) \).

**Proposition 5.4.** We have:

1. When \( n \) is even,
   \[
   c(\lambda, \tau^+_n) = c(\lambda, \tau^-_n) = 2^{n-i2\lambda} \frac{\Gamma(n/2)\Gamma(i2\lambda)}{\Gamma(i\lambda+n/2)\Gamma(i\lambda)}.
   \]

2. When \( n \) is odd,
   \[
   c^+(\lambda, \tau_n) = c^-(\lambda, \tau_n) = 2^{n-1-i2\lambda} \frac{\Gamma(n/2)\Gamma(i2\lambda)}{\Gamma(i\lambda+n/2)\Gamma(i\lambda)}.
   \]

**Proof.** Let us consider the case when \( n \) is odd. It is well known that for \( \Re(i\lambda) > 0 \), the Jacobi function satisfies

\[
\phi^{(a,b)}_\lambda(t) = e^{(\lambda-a-b+1)t} (c_{a,b}(\lambda) + o(1)) \text{ as } t \to \infty
\]

where

\[
c_{a,b}(\lambda) = \frac{2^{-i\lambda+a+\beta+1} \Gamma(\alpha + 1) \Gamma(i\lambda)}{\Gamma \left( \frac{i\lambda+a+\beta+1}{2} \right) \Gamma \left( \frac{i\lambda+\alpha-\beta+1}{2} \right)}.
\]

In the light of (5.4), one may check the identity

\[
c_{\frac{n}{2}, \frac{n}{2} - 1}(2\lambda) = \frac{n}{2i\lambda} c_{\frac{n}{2} - 1, \frac{n}{2}}(2\lambda).
\]

Let us first consider the case of \( \Phi_{\tau_n}^{\sigma_{n-1}} \). Since

\[
\Phi_{\tau_n}^{\sigma_{n-1}}(\lambda, a_t) = \varphi^+_n(\lambda, t) \text{Id}_{\sigma_{n-1}} + \varphi^-_n(\lambda, t) \text{Id}_{\sigma_{n-1}},
\]

then by Theorem 5.3 together with (5.3), (5.4) and (5.5), we deduce that

\[
\lim_{t \to \infty} e^{(\rho - i\lambda)t} \varphi^+_n(\lambda, t) = c_{\frac{n}{2} - 1, \frac{n}{2}}(2\lambda) = 2^{n-1-i2\lambda} \frac{\Gamma(n/2)\Gamma(i2\lambda)}{\Gamma(i\lambda+n/2)\Gamma(i\lambda)}
\]

and

\[
\lim_{t \to \infty} e^{(\rho - i\lambda)t} \varphi^-_n(\lambda, t) = 0.
\]

On the other hand, \( \Phi_{\tau_n}^{\sigma_{n-1}}(\lambda, ka_t) \) can be written in terms of the Poisson transform as \( \Phi_{\sigma_{n-1}}^{\tau_n}(\lambda, ka_t) = \mathcal{P}_{\sigma_{n-1}}^{\tau_n} \left( \pi_{\sigma_{n-1}}^{\tau_n} \left[ \tau_n(k^{-1})v \right] \right)(a_t) \). Thus, by Proposition 5.2 we
get
\[
\lim_{t \to \infty} e^{(\sigma - i\lambda)t} \Phi_{\sigma_n}^\tau (\lambda, a_t) = \gamma^2 c^+(\lambda, \tau_n) \text{Id}_{\sigma_{n-1}^+} = 2c^+(\lambda, \tau_n) \text{Id}_{\sigma_{n-1}^+}.
\]
Comparing this with (5.6) we obtain
\[
c^+(\lambda, \tau_n) = 2^{n-1-i2\lambda} \frac{\Gamma(n/2)\Gamma(i2\lambda)}{\Gamma(i\lambda + n/2)\Gamma(i\lambda)}.
\]
Now if we consider the case of \(\Phi_{\sigma_n}^\tau\), then by the same arguments as in the proof of \(c^+(\lambda, \tau_n)\) we get
\[
c^-(\lambda, \tau_n) = 2^{n-1-i2\lambda} \frac{\Gamma(n/2)\Gamma(i2\lambda)}{\Gamma(i\lambda + n/2)\Gamma(i\lambda)}.
\]
The case when \(n\) is even is handled in the same way. \(\square\)

**Proposition 5.5.** (1) If \(n\) is even, then there exists a positive constant \(\gamma_\lambda\) such that, for every \(f \in L^p(K/M, \sigma)\) we have
\[
|c(\lambda, \tau_n^\pm)||f||_{L^p(K/M, \sigma)} \leq \|P^\tau_{\sigma, \lambda} f\|_{L^p_{\pm, \lambda}} \leq \gamma_\lambda||f||_{L^p(K/M, \sigma)}
\]
(2) If \(n\) is odd, then there exists a positive constant \(\gamma_\lambda\) such that, for every \(f \in L^p(K/M, \sigma)\) we have
\[
\sqrt{2}c^\pm(\lambda, \tau_n)||f||_{L^p(K/M, \sigma)} \leq \|P^\tau_{\sigma, \lambda} f\|_{L^p_{\pm, \lambda}} \leq \gamma_\lambda||f||_{L^p(K/M, \sigma)}
\]

**Proof.** The proof is similar to [2, Proposition 4.4]. \(\square\)
By abuse of notation, we will denote the scalar components \(c(\lambda, \tau_n^\pm)\) and \(c^\pm(\lambda, \tau_n)\) by \(c(\lambda, \tau)\) where the meaning is clear from the context.

6. **Proof of the main theorem for** \(p = 2\)

6.1. **Auxiliary results.** Recall the Convention 3.1 and let \((\sigma, V_\sigma) \in \widehat{M}(\tau)\) of dimension \(d_\sigma\). Let \(\tilde{K}(\sigma) \subset \tilde{K}\) be the subset of unitary equivalence classes of irreducible representations containing \(\sigma\) upon restriction to \(\tilde{K}\). Consider an element \((\delta, V_\delta)\) in \(\tilde{K}(\sigma)\). From [1] or [7] it follows that \(\sigma\) occurs in \(\delta_{1M}\) with multiplicity one, and therefore \(\dim \text{Hom}_{M}(V_\delta, V_\sigma) = 1\). Choose the orthogonal projection \(P_\delta : V_\delta \to V_\sigma\) to be a generator of \(\text{Hom}_{M}(V_\delta, V_\sigma)\).

Let \(\{v_j : j = 1, \ldots, d_\delta = \dim V_\delta\}\) be an orthonormal basis for \(V_\delta\). Then the set of functions \(\{\phi^\delta_j : 1 \leq j \leq d_\delta, \ \delta \in \tilde{K}(\sigma)\}\) defined by
\[
k \mapsto \phi^\delta_j(k) = P_\delta(\delta(k^{-1})v_j)
\]
is an orthogonal basis of the space \(L^2(K/M; \sigma)\), see, e.g., [11]. Hence, the Fourier series expansion of each \(f\) in \(L^2(K/M; \sigma)\) is given by
\[
f(k) = \sum_{\delta \in \tilde{K}(\sigma)} \sum_{j=1}^{d_\delta} a^\delta_j \phi^\delta_j(k),
\]
with
\[ \|f\|_{L^2(K/M; \sigma)}^2 = \sum_{\delta \in K(\sigma)} \frac{d_\sigma}{d_\delta} \sum_{j=1}^{d_\delta} |a_j^\delta|^2. \]

Define the following Eisenstein integrals \( \Phi_{\lambda, \delta} \) by
\[ \Phi_{\lambda, \delta}(g)(v) = \langle f, T(6.4) \rangle_{K} e^{-(i\lambda + \rho)(g^{-1}k)} \tau(k(g^{-1}k)) \tau_{\sigma} P_\delta(\delta(k^{-1})v)dk, \] for \( g \in G \) and \( v \in V_\delta \). One may check that \( \Phi_{\lambda, \delta}(k_1 g k_2) = \tau(k_2^{-1}) \Phi_{\lambda, \delta}(g) \delta(k_1^{-1}) \) for every \( g \in G \) and \( k_1, k_2 \in K \).

**Lemma 6.1.** We have
\[ \sup_{t>0} e^{(\rho - R(\lambda))t} \|\Phi_{\lambda, \delta}(a_I)\|_{HS} \leq \kappa \gamma \|P_\delta\|_{HS} = \kappa \gamma \sqrt{d_\sigma}, \] and
\[ \lim_{t \to \infty} e^{2(\rho - R(\lambda))t} \|\Phi_{\lambda, \delta}(a_I)\|_{HS}^2 = \kappa^2 |c(\lambda, \tau)|^2 d_\sigma. \]

**Proof.** The proof follows the same lines as in [2, Lemma 5.3, Lemma 5.4], so we omit it. \( \square \)

A functional on \( G \times G/P V_\sigma \) is a linear form \( T \) on \( C^\infty(G/P; \sigma) \). For a such functional \( T \), we define \( \overline{P}_{\sigma, \lambda}(T) \) by
\[ \langle v, \overline{P}_{\sigma, \lambda} T(g) \rangle_{V_\sigma} = \kappa(T, \pi_\sigma^L g \Phi_\lambda v), \; \forall v \in V_\tau \] (6.4)
where \( L_g \) is the left regular action, and \( \Phi_\lambda : G \to \text{End}(V_\tau) \) is given by
\[ \Phi_\lambda(g) = e^{(i\lambda - \rho)(g^{-1}k)} \tau^{-1}(\kappa(g)). \] (6.5)
Notice that \( \Phi_\lambda(g^{-1}k)^* = P_{\sigma, \lambda}^r(g, k) \), where \( P_{\sigma, \lambda}^r : G \times K \to \text{End}(V_\tau) \) is the Poisson kernel given by
\[ P_{\sigma, \lambda}^r(g, k) = e^{-(i\lambda + \rho)(g^{-1}k)} \tau(k(g^{-1}k)). \] (6.6)
If \( T = T_f \) is a functional given by a smooth function \( f \in C^\infty(G/P; \sigma) \), then by using (6.4) together with the facts that \( \Phi_\lambda(g^{-1}k)^* \) coincides with the Poisson kernel and that the adjoint of \( \pi_\sigma^L \) is the embedding \( \iota_\sigma^L \), one may check that
\[ \overline{P}_{\sigma, \lambda}(T_f) = P_{\sigma, \lambda}^r(f). \] (6.7)

**6.2. Proof of the main Theorem for** \( p = 2 \). The necessary condition follows from Theorem 4.1 and Proposition 5.1.

On the other hand, let \( F \) in \( E_1^2(G/K; \tau_n) \) (when \( n \) is even) or in \( E_2^2(G/K; \tau_n) \) (when \( n \) is odd), with the \( K \)-type expansion
\[ F(g) = \sum_{\delta \in K(\sigma)} F_\delta(g) \]
then, by [12, Corollary 10.8], there exists a $K$-finite vector $f_δ$ in $C^∞(G/P; σ_λ)$ such that $F_δ = P^χ_σ f_δ$. This implies

$$f_δ(k) = \sum_{j=1}^{d_δ} a_j^δ P_δ(\delta(k^{-1}) v_j).$$

It follows from [2, Proposition 5.1] that the functional $T$ on $C^∞(G/P; σ_{\overline{λ}})$ defined by

$$(T, φ) = \sum_{δ ∈ \hat{K}(σ)} \sum_{j=1}^{d_δ} \overline{a_j^δ} \int_K \langle φ(k), P_δ(\delta(k^{-1}) v_j) \rangle_{V_σ} dk,$$  (6.8)

satisfies $F = P_{σ,Λ}^χ T$. Therefore,

$$F(g) = \hat{κ} \sum_{δ ∈ \hat{K}(σ)} \sum_{j=1}^{d_δ} a_j^δ \int_K e^{-ι(λ + ρ) H(g^{-1} k) T ϕ} (\delta(k^{-1}) v_j) dk,$$

Define $Φ_λ,δ$ by

$$Φ_λ,δ(g)(v) = \hat{κ} \int_K e^{-ι(λ + ρ) H(g^{-1} k) T ϕ} (\delta(k^{-1}) v_j) dk,$$  (6.9)

for $g ∈ G$ and $v ∈ V_δ$. One may check that $Φ_λ,δ(k_1 k_2) = τ(k_2^{-1}) Φ_λ,δ(g) δ(k_1^{-1})$ for every $g ∈ G$ and $k_1, k_2 ∈ K$. Further, using the covariance property of $Φ_λ,δ$ and Schur’s orthogonality relations, we get

$$\int_K \|F(ka_t)\|_τ^2 dk = \sum_{δ ∈ \hat{K}(σ)} \frac{1}{d_δ} \sum_{j=1}^{d_δ} |a_j^δ|^2 \text{tr} (Φ_λ,δ( a_t )^* Φ_λ,δ( a_t )) ,$$

$$= \frac{1}{d_δ} \sum_{δ} \|Φ_λ,δ( a_t )\|_{HS}^2 \sum_{j} |a_j^δ|^2 ,$$

where $\| \cdot \|_{HS}$ is the Hilbert-Schmidt norm.

Let $Λ$ be a finite subset of $\hat{K}(σ)$, then

$$\sum_{δ ∈ Λ} \frac{1}{d_δ} \sum_{j} |a_j^δ|^2 \leq \frac{1}{d_δ} \sum_{j} |a_j^δ|^2 \leq \|F\|^2_{L^2_λ},$$

By (6.3) we have

$$κ^2 |c(λ, τ)|^2 \sum_{δ ∈ Λ} \frac{d_δ}{d_δ} \sum_{j} |a_j^δ|^2 \leq \|F\|^2_{L^2_λ}.$$  

Since the subset $Λ ⊂ \hat{K}(σ)$ is arbitrary, it follows that

$$κ^2 |c(λ, τ)|^2 \sum_{δ ∈ \hat{K}(σ)} \frac{d_δ}{d_δ} \sum_{j} |a_j^δ|^2 \leq \|F\|^2_{L^2_λ}.$$
This shows that the functional \( T(k) \sim \sum_{\delta \in \mathcal{K}(\sigma)} \sum_{j=1}^{d_\delta} a_{\delta}^j P_\delta(\delta(k^{-1})) v_j \) defines a function \( f \in L^2(K/M; \sigma) \) and by (6.7) we deduce that \( F = P_{\sigma,\lambda}^r f \) with
\[
\| f(c(\lambda, \tau)) \|_{L^2(K/M; \sigma)} \leq \| F \|_{C^2}.
\]
This finishes the proof of the main Theorem 4.2 for \( p = 2 \).

6.3. The inversion formula. We close this section by the following inversion formula which will be needed in the proof of the main Theorem 4.2 for arbitrary \( p \).

**Theorem 6.2** (Inversion formula). Let \( \tau = \tau_n \) or \( \tau_n^\pm \) and \( \sigma \in \hat{M}(\tau) \). Assume \( \lambda \in \mathbb{C} \) with \( \Re(i\lambda) > 0 \). Let \( F \) be an element in \( E_\lambda^2(G/K; \tau_n^\pm) \) (when \( n \) is even) or in \( E_{\pm,\lambda}^2(G/K; \tau_n) \) (when \( n \) is odd), and let \( f \in L^2(K/M; \sigma) \) such that \( F = P_{\sigma,\lambda}^r f \). Then the following inversion formula holds in \( L^2(K/M; \sigma) \)
\[
f(k) = \kappa^{-1} | c(\lambda, \tau) |^{-2} \lim_{t \to \infty} e^{2(\rho-R(i\lambda))t} \pi_{\sigma}^r \left( \int_K P_{\sigma,\lambda}^r (ha_t, k) F(ha_t) \, dh \right),
\]
where \( P_{\sigma,\lambda}^r (\cdot, \cdot)^* \) is adjoint of the Poisson kernel (6.6) and \( \pi_{\sigma}^r \) is the projection of \( V_\tau \) onto \( V_\sigma \).

**Proof.** For the convenience of the reader we will outline the proof which follows the same arguments as in [2, Theorem 5.5].

Let \( F \) be as in the statement. By the main theorem for \( p = 2 \), there exists a unique \( f \in L^2(K/M; \sigma) \) such that \( F = P_{\sigma,\lambda}^r f \). Using the notation introduced at the beginning of this section, we write

\[
f(k) = \sum_{\delta \in \mathcal{K}(\sigma)} \sum_{j=1}^{d_\delta} a_{\delta}^j P_\delta(\delta(k^{-1})) v_j.
\]

Then
\[
F(k a_t) = \sum_{\delta} \sum_{j} a_{\delta}^j \Phi_{\lambda,\delta}(a_t) \delta(k^{-1}) v_j,
\]
and therefore
\[
\int_K \| F(k a_t) \|_{L^2}^2 \, dk = \sum_{\delta} \sum_{j} \frac{|a_{\delta}^j|^2}{d_\delta} \| \Phi_{\lambda,\delta}(a_t) \|_{L^2}^2.
\]

Then, by lemma 6.1, we obtain
\[
\lim_{t \to \infty} e^{2(\rho-R(i\lambda))t} \int_K \| P_{\sigma,\lambda}^r f(k a_t) \|_{L^2}^2 \, dk = \kappa^2 | c(\lambda, \tau) |^2 \| f \|_{L^2(K/M; \sigma)}^2,
\]
which implies
\[
\lim_{t \to \infty} \langle g_t, \varphi \rangle_{L^2(K/M; \sigma)} = \langle f, \varphi \rangle_{L^2(K/M; \sigma)}
\]
for all \( \varphi \in L^2(K/M; \sigma) \), where \( g_t \) is the \( V_\sigma \)-valued function defined by
\[
g_t(k) = \kappa^{-1} | c(\lambda, \tau) |^{-2} e^{2(\rho-R(i\lambda))t} \pi_{\sigma}^r \int_K P_{\sigma,\lambda}^r (ha_t, k) F(ha_t) \, dh.
\]
On the other hand, following the same arguments as in [2, Theorem 5.5], the Fourier coefficients $c^\delta_j(g_t)$ of $g_t$ are given by

$$c^\delta_j(g_t) = \frac{d\delta}{d\sigma} \int_K \langle g_t(k), P_\delta(k^{-1})\nu_j \rangle_{V_\sigma} dk$$

$$= k^{-2} |c(\lambda, \tau)|^{-2} e^{2(\rho - \Re(i\lambda))t} \frac{d\delta}{d\sigma} \Phi_{\lambda, \delta}(a_t) \|_{HS}^2.$$ 

Hence,

$$\|g_t\|^2_{L^2(K/M, \sigma)} = \left( e^{2(\rho - \Re(i\lambda))t} k^{-2} |c(\lambda, \tau)|^{-2} \sum_\delta \frac{d\sigma}{d\delta} \sum_j \frac{1}{\|\Phi_{\lambda, \delta}(a_t)\|_{HS}^4} \right)^2 \sum_\delta \frac{d\sigma}{d\delta} \sum_j |a_j^\delta|^2 \|\Phi_{\lambda, \delta}(a_t)\|_{HS}^4.$$ 

Now, using (6.3) we conclude that

$$\lim_{t \to \infty} \|g_t\|^2_{L^2(K/M, \sigma)} = \sum_\delta \frac{d\sigma}{d\delta} \sum_j |a_j^\delta|^2 = \|f\|^2_{L^2(K/M, \sigma)}.$$ 

The interchange of $\lim_{t \to \infty}$ and $\sum_\delta$ is justified by the fact that

$$\sum_\delta \frac{d\sigma}{d\delta} \sum_j |e^{2(\rho - \Re(i\lambda))t} k^{-2} |c(\lambda, \tau)|^{-2} \sum_\delta \frac{d\sigma}{d\delta} \sum_j |a_j^\delta|^2 \|\Phi_{\lambda, \delta}(a_t)\|_{HS}^4,$$

is uniformly convergent, by (6.2). \qed

7. Proof of the main theorem for $1 < p < \infty$

As in the case $p = 2$, the necessary condition follows from Theorem 4.1 and Proposition 5.1. Let us turn our attention to the sufficiency condition.

Let $F$ be in the Hardy type space and write $F(g) = \sum_i F_i(g)u_i$ where $\{u_i\}_i$ is an orthonormal basis of $V_\tau$. Consider an approximation of the identity $(\chi_m)_m$ in $C^\infty(K)$, and define the sequence $(F_{i,m})_m$ by $F_{i,m}(g) = \int_K \chi_m(h)F_i(h^{-1}g)dh$. As $(\chi_m)_m$ is an approximation of the identity, it follows that $(F_{i,m})_m$ converges pointwise to $F_i$. On the other hand, define $F_m : G \to V_\tau$ by $F_m(g) = \sum_i F_{i,m}(g)u_i$. That is,

$$F_m(g) = \int_K \chi_m(h)F(h^{-1}g)dh,$$

and we have $\|F_m(g) - F(g)\|^2_\tau \to 0$ as $m \to \infty$. Further, since the operators $\mathcal{D}$ and $\mathcal{D}^2$ are $K$-invariant, then $F_m$ belongs to either $\mathcal{E}_\chi$ or $\mathcal{E}_{\pm, \chi}$ according to whether $n$ is even or odd (see (4.3) and (4.4)). Further,

$$F_m(ka_t) = \int_K \chi_m(h)F(h^{-1}ka_t)dh,$$

where $F^t : K \to V_\tau$ is defined for any $t > 0$ by $F^t(k) = F(ka_t)$. Since

$$\|(\chi_m \ast F^t)(k)\|_\tau \leq \int_K |\chi_m(h)||F^t(h^{-1}k)||_\tau dh = |\chi_m(\cdot)| \ast \|F^t(\cdot)||_\tau(k),$$
then
\[ \|F^t_m\|_{L^p(K;V_\tau)} \leq \|\chi_m\| \AST \|F^t(\cdot)\|_{L^p(K)} . \]
Applying Young’s inequalities to the right-hand side of the above inequality we get
\[ \|F^t_m\|_{L^p(K;V_\tau)} \leq \|F^t\|_{L^p(K;V_\tau)}, \tag{7.1} \]
and
\[ \|F^t_m\|_{L^2(K;V_\tau)} \leq \|\chi_m\|_{L^2(K)} \|\|F^t(\cdot)\|_{V_\tau}\|_{L^1(K)}, \]
\[ \leq \|\chi_m\|_{L^2(K)} \|F^t\|_{L^p(K;V_\tau)}, \tag{7.2} \]
since \( p > 1 \). The last inequality says
\[ \sup_{t > 0} e^{(\rho - \Re(i\lambda))t} \left( \int_K \|F_m(ka_t)\|^2 dk \right)^{1/2} \leq \|\chi_m\|_{L^2(K)} \|F\|_{C_{\sigma,\lambda}^p} < \infty. \]
Hence, for every \( m \), \( F_m \in C_{\sigma,\lambda}^2(G/K;\tau) \) and by the previous section (the case \( p = 2 \)), there exists \( f_m \in L^2(K;M;\sigma) \) such that \( F_m = \mathcal{P}_{\sigma,\lambda}^r f_m \). To prove that in fact \( f_m \in L^p(K;M;\sigma) \) we follow a similar approach as in [2]. According to the inversion Theorem 6.2 we have, for any \( \varphi \in C(K/M;\sigma) \),
\[ \int_K (f_m(k), \varphi(k))_\sigma dk = \lim_{t \to \infty} \int_K (g^t_m(k), \varphi(k))_\sigma dk, \]
where
\[ g^t_m(k) := q^{-2}|c(\lambda, \tau)|^{-2}e^{2(\rho - \Re(i\lambda))t} \int_K (\mathcal{P}_{\sigma,\lambda}^r (ha_t, k)^* F_m(ha_t) dh. \]
Further,
\[ \left| \int_K (g^t_m(k), \varphi(k))_\sigma dk \right| \]
\[ = \left| q^{-3}|c(\lambda, \tau)|^{-2}e^{2(\rho - \Re(i\lambda))t} \int_K (F_m(ha_t), (\mathcal{P}_{\sigma,\lambda}^r \varphi)(ha_t))_\tau dh \right| \]
\[ \leq \left| q^{-3}|c(\lambda, \tau)|^{-2}e^{2(\rho - \Re(i\lambda))t} \int_K \|F_m(ha_t)\|_\tau \|\mathcal{P}_{\sigma,\lambda}^r \varphi(ha_t)\|_\tau dh. \]
By Hölder’s inequality (with \( \frac{1}{p} + \frac{1}{q} = 1 \)), we deduce
\[ \left| \int_K (g^t_m(k), \varphi(k))_\sigma dk \right| \]
\[ \leq \left| q^{-3}|c(\lambda, \tau)|^{-2}e^{2(\rho - \Re(i\lambda))t} \|F^t_m\|_{L^p(K;V_\tau)} \|\mathcal{P}_{\sigma,\lambda}^\tau \|_{L^q(K;V_\tau)}, \]
where \( (\mathcal{P}_{\sigma,\lambda}^\tau \varphi)^t(k) = (\mathcal{P}_{\sigma,\lambda}^r \varphi)(ka_t) \). Using (7.1) and Proposition 5.1 we get
\[ \left| \int_K (f_m(k), \varphi(k))_\sigma dk \right| \]
\[ \leq \left| q^{-2} \gamma_\lambda |c(\lambda, \tau)|^{-2} \|F_m\|_{C_{\sigma,\lambda}^p} \|\varphi\|_{L^q(K/M;\sigma)}, \]
\[ \leq \left| q^{-2} \gamma_\lambda |c(\lambda, \tau)|^{-2} \|F\|_{C_{\sigma,\lambda}^p} \|\varphi\|_{L^q(K/M;\sigma)}. \]
Taking the supremum over all \( \varphi \in C(K/M;\sigma) \) with \( \|\varphi\|_{L^q(K/M;\sigma)} = 1 \) gives
\[ \|f_m\|_{L^p(K/M;\sigma)} \leq q^{-2} \gamma_\lambda |c(\lambda, \tau)|^{-2} \|F\|_{C_{\sigma,\lambda}^p}, \]
which implies that $f_m$ is indeed in $L^p(K/M; \sigma)$.

For every integer $m$, define the linear form $T_m$ on $L^q(K/M; \sigma)$ by

$$T_m(\varphi) = \int_K \langle f_m(k), \varphi(k) \rangle_{\sigma} dk.$$  

The operator $T_m$ is continuous and satisfies

$$|T_m(\varphi)| \leq k^{-2} \gamma_\lambda |c(\lambda, \tau)|^{-2} \|F\|_{E_{p,\sigma,\lambda}} \|\varphi\|_{L^q(K/M; \sigma)}.$$  

This shows that the sequence $(T_m)_m$ is uniformly bounded in $L^q(K/M; \sigma)$, with

$$\sup_m \|T_m\|_{\operatorname{op}} \leq k^{-2} \gamma_\lambda |c(\lambda, \tau)|^{-2} \|F\|_{E_{p,\sigma,\lambda}}.$$  

Therefore, Riesz’s representation theorem implies the existence of a unique $f \in L^p(K/M; \sigma)$ such that

$$T(\varphi) = \int_K \langle \varphi(k), f(k) \rangle_{\sigma} dk.$$  

For an arbitrary given $v \in V_\tau$, taking $\varphi(k) = \varphi_g(k) = P^\tau_{\sigma,\lambda}(g, k)^* v$ we obtain

$$T(\varphi_g) = \langle v, P^\tau_{\sigma,\lambda} f(g) \rangle_\tau. \quad (7.3)$$

On the other hand, we have

$$T_m(\varphi_g) = \langle v, P^\tau_{\sigma,\lambda} f_m(g) \rangle_\tau = \langle v, F_m(g) \rangle_\tau.$$  

After taking the limit $j \to \infty$, the above identity together with (7.3) imply $F(g) = P^\tau_{\sigma,\lambda} f(g)$ for every $g \in G$. This finishes the proof of the main Theorem 4.2 for every $1 < p < \infty$.

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