Superposition of FLRW Universes

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We show that a nonlinear superposition of two distinct homogeneous Friedman-Lemaitre-Robertson-Walker (FLRW) metrics produces an inhomogeneous universe as a solution to the cosmological Einstein equations with a perfect fluid source. In this superposed model, (1) the early and the late stages of the universe are described by two different FLRW metrics; (2) a change of the topology of the 3-space during the evolution of the universe is possible; (3) there is a scale factor duality symmetry proposed first in string cosmology.

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Recently an exact solution of Einstein-perfect fluid system in an isotropic spherically symmetric spacetime which represents an inhomogeneous cosmological model that allows various interpretations was given in [1]. See [2, 3] for the classification and interpretations of the previously found inhomogeneous cosmological solutions. One of the interpretations is the generalization of the McVittie solution representing a black hole immersed in a FLRW universe [3, 5, 6]. In this work we propose a different interpretation of our solution. The inhomogeneous cosmological model of [1] is a nonlinear superposition of generically two distinct FLRW universes with different spatial curvatures. Nonlinear superposition in general relativity has been studied in the context of Bäcklund transformations [7, 11] for spacetimes that possess two Killing vector fields. Here we give a novel example of a nonlinear superposition in general relativity. The FLRW phases show up in the early and late stages of the universe. Since each FLRW metric has a different spatial curvature, then in the early and late eras of the universe the spatial curvatures are different in general and a topology change is possible. The superposed model has a scale factor duality symmetry akin to the one observed earlier in string cosmology [12].

Let us first recapitulate the basic essentials of the inhomogeneous model before we give several examples and discuss the features of our solution in the context of Big Bang and cyclic cosmological models [13, 14].

We consider the spherically symmetric metric in isotropic coordinates

$$ds^2 = -a^2 dt^2 + c^4 (dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2),$$  

where $a$ and $c$ are generic differentiable functions of time $t$ and radial coordinate $r$. In [1], it was proven that the Einstein-perfect fluid field equations for $\dot{c}(t, r) \neq 0$ reduce to

$$2r cc'' - 6rc'^2 - 2cc' - h = 0,$$

$$a(t, r) = 2q \frac{\dot{r}}{c},$$

where $h$ is an arbitrary function of $r$ and $q = q(t)$ is an arbitrary function of $t$; the fluid velocity is given as $u_\mu = a \delta_\mu^0$. Here a dot and a prime denote differentiation with respect to $t$ and $r$, respectively. The energy density $\rho(t, r)$ and the pressure $p(t, r)$, respectively read as

$$8\pi \rho(t, r) = \frac{3}{q^2} \frac{2}{rc^6} (6rc'^2 + 6cc' + \dot{h}) - \Lambda,$$

$$8\pi p(t, r) = -\frac{3}{q^2} \frac{1}{rq^3 c^6} (2q^3 (-rc'^2 \dot{c} + rcc' \dot{c}' + c' \dot{c}^2 + c^2 \dot{c}') + rcc' + r^2 \dot{q}) + \Lambda,$$

where $\Lambda$ is the cosmological constant. In [1], it was shown that

$$c(t, r) = \frac{\sqrt{R(t)}}{c_0 + c_1 r^2} + \frac{\gamma}{\sqrt{R(t)}} \frac{1}{c_2 + c_3 r^2},$$

is a solution to the ordinary nonlinear differential equation [2] with $c_0, c_1, c_2, c_3$ and $\gamma$ as arbitrary constants; and $R(t)$ is an at least twice differentiable function. For this solution, $h(r)$ is given by

$$h(r) = \frac{6 \gamma (c_0 c_3 - c_1 c_2) r^3}{(c_0 + c_1 r^2)^2 (c_2 + c_3 r^2)^2},$$

and furthermore choosing $q(t) = R(t)/\dot{R}(t)$, the lapse function reads as

$$a(t, r) = \frac{1 - \gamma}{R(t)} \frac{\sqrt{R(t) + c_0 + c_1 r^2}}{\sqrt{c_2 + c_3 r^2}} + \frac{\gamma}{R(t)} \frac{1}{\sqrt{c_2 + c_3 r^2}},$$

Substituting [15] in [1] and [15], one can find that for the limit $R(t) \rightarrow 0$ and keeping the leading orders, $p(t, r)$ and $\rho(t, r)$ become homogeneous and reduce to

$$8\pi \rho(t) \rightarrow 3 \frac{\dot{R}^2}{R^2} + \frac{12c_0 c_3 R^2}{\gamma^4} - \Lambda,$$

$$8\pi p(t) \rightarrow \frac{2}{R} \frac{\dot{R}}{\gamma^2} - \frac{12c_0 c_3 R^2}{\gamma^4} + \Lambda.$$
Similarly, \( \rho(t, r) \) and \( \rho(t, r) \) become also homogeneous in the limit \( R(t) \to \infty \) and read as

\[
8\pi \rho(t) \to \frac{R^2}{R^2 + \frac{12 c_0 c_1}{R^2}} - \Lambda, \tag{11}
\]

\[
8\pi p(t) \to -2 \frac{\dot{R}}{R} - \frac{R^2}{R^2 - \frac{4 c_0 c_1}{R^2}} + \Lambda. \tag{12}
\]

Observe that the metric, i.e \[6, 7\], is invariant under the scale factor duality symmetry \( R \to R^2/R \), \( c_0 \leftrightarrow c_2 \) and \( c_1 \leftrightarrow c_3 \). This duality symmetry can also be observed in the field equations: under this symmetry, the pair of equations \(9\) and \(10\) go to \(11\) and \(12\) and vice versa.

**Interpretations of the solution:**

**A: A generalized McVittie metric**

Without losing any generality, one can choose the arbitrary constants \( c_0, c_1, c_2, c_3 \) and \( \gamma \) in such a way that the function \( c(t, r) \) becomes

\[
c(t, r) = \frac{\sqrt{R(t)}}{\sqrt{\mu + r^2}} + \frac{M}{2 \sqrt{R(t)}} \frac{1}{\sqrt{1 + kr^2}}, \tag{13}
\]

with three constants \( M, k, \) and \( \mu \) that represent the mass, spatial curvature of the background FLRW universe and the reduction parameter, respectively and \( R(t) \) is the same function as in \(6\) up to a multiplicative constant. Letting \( \mu = 0 \) the solution reduces to the uncharged Vaidya–Shah solution while letting \( \mu = 0 \) and \( k = 0 \), it gives the McVittie solution \(4, 5\). The McVittie solution can be interpreted as a black hole in the spatially flat FLRW universe \(4, 5\). Then, one novel interpretation of the solution \(13\) is that it is a generalization of the McVittie solution representing a black hole in a FLRW universe with a non-zero spatial curvature.

**B: Superposition of two FLRW universes**

One can arrange the constants \( c_0, c_1, c_2 \) and \( c_3 \) in \(6\) so that \( c(t, r) \) takes the form

\[
c(t, r) = \frac{\sqrt{R(t)}}{\sqrt{1 + k_1 r^2}} + \frac{\gamma}{\sqrt{R(t)}} \frac{1}{\sqrt{1 + k_2 r^2}}, \tag{14}
\]

where \( k_1, k_2 \) and \( \gamma \) are arbitrary constants and and \( R(t) \) is the same function as in \(6\) up to a multiplicative constant.

To show that two FLRW metrics are nonlinearly superposed, let us first recast the standard FLRW metric

\[
ds^2 = -dt^2 + R^2(t) \left[ \frac{dr^2}{1 - k_1 v^2} + v^2 d\Omega^2 \right], \tag{15}
\]

in the isotropic coordinates. For this purpose, let us do the following coordinate transformation

\[
v = \frac{r}{1 + k r^2}, \tag{16}
\]

which brings the metric \(13\) to the form

\[
ds^2 = -dt^2 + \frac{R^2(t)}{(1 + kr^2)^2} \left( dr^2 + r^2 d\Omega^2 \right), \tag{17}
\]

where \( k = k_2/4 \). Here, \( R(t) \) is the scale factor and \( k_2 \) is the curvature constant of 3-space which, after making the coordinate unitless, can take the values \( \pm 1 \) and 0. To discuss that our new solution represents the superposition of two FLRW universes, we consider the following two different FLRW universes in the isotropic coordinates \((t, r, \theta, \phi)\).

**FLRW\textsubscript{1}**: The metric for this case is given by

\[
ds^2 = -dt^2 + c_1(t, r) \left( dr^2 + r^2 d\Omega^2 \right), \tag{18}\]

where

\[
c_1(t, r) = \frac{1}{\sqrt{R(t)}} \frac{1}{\sqrt{1 + k_1 r^2}}. \tag{19}\]

This metric represents a FLRW universe with the scale factor \( R(t) \) and the spatial curvature \( k_1 = 4k_2 \). The homogeneous matter density and pressure profiles for this case read as

\[
8\pi \rho_1(t) = 3 \frac{\dot{R}^2}{R^2} + 3 \frac{k_1}{R^2} - \Lambda, \tag{20}\]

\[
8\pi p_1(t) = -\frac{\dot{R}}{R} - \frac{R^2}{R^2} - 4k_1 + \Lambda. \tag{21}\]

**FLRW\textsubscript{2}**: The metric for this case is given by

\[
ds^2 = -dt^2 + c_2(t, r) \left( dr^2 + r^2 d\Omega^2 \right), \tag{22}\]

where

\[
c_2(t, r) = \frac{1}{\sqrt{R(t)}} \frac{1}{\sqrt{1 + k_2 r^2}}. \tag{23}\]

Then it represents a FLRW metric with the scale factor \( R^{-1}(t) \) and the spatial curvature \( k_2 = 4k_2 \). Here, the matter density and pressure profiles are

\[
8\pi \rho_2(t) = 3 \frac{\dot{R}^2}{R^2} + 12k_2 R^2 - \Lambda, \tag{24}\]

\[
8\pi p_2(t) = 2 \frac{\dot{R}}{R} - \frac{5 R^2}{R^2} - 4k_2 R^2 + \Lambda. \tag{25}\]

Now, regarding the above two cases, we have the following theorem indicating the second novel interpretation of the general solution \(6\) which we mainly discuss below.

**Theorem:** The linear superposition of \( c_1(t, r) \) and \( c_2(t, r) \), i.e. \( c(t, r) = c_1(t, r) + \gamma c_2(t, r) \), with \( c_1(t, r) \) and \( c_2(t, r) \) given in \(14\) and \(15\) solves the Einstein field equations \(2\), \(4\), \(6\) and \(8\). Here, \( \gamma \) is an arbitrary
constant. Then, this represents a nonlinear superposition of two particular solutions [14] for which each part is generically a distinct FLRW universe with different spatial curvatures \( k_1 \) and \( k_2 \). If \( k_1 = k_2 = k \), this general solution reduces to a single FLRW solution with the spatial curvature \( k = 4k \).

The model has a Big Bang singularity if \( R \to 0 \) as \( t \to 0 \) without further assumption on the energy density and pressure, this is the only constraint on the function \( R \). Hence, each choice of \( R \) generates a different cosmological model. For instance, assuming the scale factor to be \( R(t) = R_0(e^{\lambda t} - 1)^n \) where \( \lambda \) and \( n \) are positive constants, then from (9) and (10), one finds \( \rho(t) \to \infty \) and \( p(t) \to -\infty \) as \( t \to 0 \), respectively. In the limit \( t \to \infty \), \( R(t) \to \infty \) and from (11) and (12), one finds \( \rho(t) \to 3n^2\lambda^2 - \Lambda \) and \( p(t) \to -3n^2\lambda^2 + \Lambda \), respectively, which corresponds to a (anti)de Sitter space. In this case, the universe starts from a Big Bang and evolves to a pure (anti)de Sitter phase at late times.

Three consequences of the above theorem are as follows: (i) the new exact solution unifies two different homogeneous FLRW solutions in a single superposed solution which generally is inhomogeneous, (ii) regarding (9)-(12) and (20), (21), (24) and (25), the universe is approximated FLRW as \( R(t) \to 0 \) and approximately FLRW\(_1\) as \( R(t) \to \infty \). Then the solution represents a phase transition from FLRW\(_2\) to FLRW\(_1\) with possibly a topology change and (iii) there exists a scale factor dual symmetry in the metric as \( R \to \gamma^2/R \) then \( k_1 \leftrightarrow k_2 \). Under this symmetry, FLRW\(_2 \) ↔ FLRW\(_1\).

Let us expound on the difference of the early and late eras as (\( R \to 0 \) and \( R \to \infty \)) of the universe as different FLRW universes. We can normalize only one of the parameters \( k_1 \) or \( k_2 \) by scaling the coordinate \( r \). By such a scaling \( k_1 \), \( k_2 \) and \( r \) become unitless. If the beginning of the universe is FLRW\(_2\) with a normalized \( k_2 \) and ends as FLRW\(_1\) with spatial curvature \( k_1 \), we have two possibilities, either sign \( (k_1) = \text{sign} \ (k_2) \) (two universes having the same topology) and sign \( (k_1) \neq \text{sign} \ (k_2) \) indicating a change of topology. In these asymptotic stages, the universe is approximately homogeneous.

In the following, we elaborate on the properties of the solution for \( k_1 \neq k_2 \) with two specific cosmological scenarios.

I. Expanding universe scenario

For an ever expanding universe with \( R(t) \to 0 \) as \( t \to 0 \) and \( R(t) \to \infty \) as \( t \to \infty \), or a universe with the minimum size \( R_{\min} \) (\( R_{\min} \ll 1 \)) for \( t \to 0 \) and maximum size \( R_{\max} \) (\( R_{\max} \gg 1 \)) for \( t \to \infty \), we note the following points.

(i) In the early era \((t \to 0)\), \( c_2(t,r) \) dominates in \( c(t,r) \). Then, the universe is effectively FLRW\(_2\) and hence becomes homogeneous according to the matter density and pressure given by (23) and (25), respectively. See the left plot in Figure 1.

(ii) In the intermediate era \((0 < t < \infty)\), both \( c_1(t,r) \) and \( c_2(t,r) \) are effective and hence we have a mixture of FLRW\(_1\) and FLRW\(_2\) universes which indeed is an inhomogeneous universe according to the matter density and pressure given by (3) and (5), respectively. In this case, cosmological inhomogeneities emerge and contribute to the formation of structures in the universe. See Figure 1 where the plots A and C represent initial and final homogeneous FLRW universe while the plot B indicates a mixed inhomogeneous universe.

(iii) In the late times \((t \to \infty)\), \( c_1(t,r) \) dominates in \( c(t,r) \). Therefore, the universe tends effectively to FLRW\(_1\) and hence becomes homogeneous according to (20) and (21). See the left plot in Figure 1.

(iv) An important consequence of the phase transition between FLRW\(_2\) and FLRW\(_1\) is the topology change in the universe. Since FLRW\(_2\) dominating at early times and FLRW\(_1\) dominating at late times have different spatial curvatures in general, the universe may undergo a topology change from \( k_2 \) to \( k_1 \) during its evolution. For this topological transition, one realizes the following two points.

1. If \( k_1 = 0 \) and \( k_2 > 0 \) (or \( k_1 > 0 \) and \( k_2 = 0 \)), hence \( r \in (0, \infty) \). This case represents the topology change from a closed universe to a flat universe (or vice versa).

2. If \( k_1 < 0 \) and \( k_2 \geq 0 \) (or \( k_2 < 0 \) and \( k_1 \geq 0 \)), which represents a topology change from a spatially closed or flat universe to an open universe (or vice versa). For this case, there is a restriction on the coordinate patch as \(-\frac{1}{\sqrt{|k_1|}} < r < \frac{1}{\sqrt{|k_1|}}\).

Figure 1: The left plot represents two different initial and final FLRW universes. The plots A and C represent two different initial and final homogeneous universes while B denotes growing inhomogeneities in the intermediate mixed phase.
A particular example: Initially inflating and finally accelerating expanding universe scenario

Considering the scale factor \( R(t) = R_0 \left( e^{\sqrt{3}t} - 1 \right) \), one observes that in the early universe as \( R(t) \to 0 \), FLRW\(_2\) dominates in the superposition. Then, the evolution of the universe is governed by (24) and (25). According to (24) and (25), we have \( p_2(t) \to \infty \) and \( p_1(t) \to 0 \) as \( t \to 0 \), respectively. The first represents the initial singularity in the matter sector while the latter represents a self-driven inflation in the early times. The late times, as \( t \to \infty \), \( R(t) \to R_0 e^{\sqrt{3}t} \), FLRW\(_1\) is dominant in the superposition and the universe evolves according to (20) and (21) to an accelerating expanding a pure de Sitter phase, i.e. \( \rho_1(t) \to 0 \) and \( p_1(t) \to 0 \) as \( t \to \infty \). Then, one observes an interesting property in this exact solution to the Einstein field equations. Indeed, this solution provides a scenario including the early time inflation, inhomogeneous structure formation in the intermediate era, and the late time accelerating expansion in a unified model possessing a possible topology change. In the intermediate phase where both FLRW\(_1\) and FLRW\(_2\) contribute effectively to the evolution of the universe, the inhomogeneities emerge. This possibility cannot be achieved in the usual standard FLRW cosmology with only time dependent field equations.

II. Cyclic universe scenarios

There are various cyclic universe scenarios, as exemplified in [13, 15]. The old cyclic scenario is based on the possibility that the scale factor \( R(t) \) of the universe oscillates at regularly spaced intervals of time between maximum and minimum values [13]. Considering such a scale factor, the superposed universe may undergo a topology change during each of its expansion and contraction phases between those minimum and maximum values of the scale factor \( R(t) \), see Figure 2. As the scale factor reaches its minimum and maximum values, the universe becomes approximately homogeneous FLRW\(_2\) and FLRW\(_1\), respectively, and it is in the mixed phase between these extremum points where inhomogeneities emerge.

Another possible type of cyclic universe scenario was proposed recently by Ijjas and Steinhardt [14]. In this model, instead of the scale factor \( R(t) \), the Hubble parameter \( H(t) = \dot{R}(t)/R(t) \) oscillates periodically during the evolution of the universe. In the context of this model, the scale factor \( R(t) \) increases substantially during each cosmological era and then undergoes an ultra-slow contraction phase at the end of each cycle. Then, the next cycle of the universe begins with a non-singular bounce. Figure 3 represents a typical plot of this type of cyclic universe scenario. In the first cycle, from \( t_0 \) to \( t_2 \), the universe expands from FLRW\(_2\) at \( t_0 \) with the spatial curvature \( k_2 \) to FLRW\(_1\) at the local maximum of \( R(t) \) at \( t_1 \) with the spatial curvature of \( k_1 \). Then, it contracts between \( t_1 \) to \( t_2 \) and recover its previous curvature \( k_2 \) of the FLRW\(_2\) state provided that \( R_2 \ll R_1 \). In both the expansion and contraction phases, the universe becomes inhomogeneous in between the local maximum and minimum values of the scale factor. One also notes that by flattening the contraction phases, i.e. in the second cycle where \( R_3 \sim R_4 \), the universe keeps its topology in the contraction phase as in the past local maximum point but still evolves inhomogeneously toward the non-singular bouncing point.

As a final remark, let us note that although in Penrose’s conformal cyclic cosmological picture [15] there is no contraction phase, since in each aeon the universe is ever expanding \( (R(t_f) \gg R(t_i)) \) where \( i \) and \( f \) denote the initial and final states of an arbitrary aeon) the universe undergoes a change of spatial curvature from \( k_2 \) to \( k_1 \) during the evolution of each aeon. This requires a sudden change of spatial curvature and possibly topology change at the transition point from the past aeon to the present aeon from \( k_1 \) to \( k_2 \). This sudden transition point corresponds to the Big Bang of the present aeon.

Figure 2: The evolution of a cyclic universe undergoing a topology change during the oscillation between subsequent global maximum and minimum values of its scale factor \( R(t) \).

Figure 3: The evolution of a cyclic universe undergoing a topology change in transition through the local maximum and minimum values of a typical scale factor \( R(t) = e^{\lambda J_0^2(t)} + 4 \) where \( J_0 \) is the zeroth order Bessel function and we have set the dimensionless parameter \( \lambda = 0.009 \).
However, since the metric of the past aeon at its null infinity and the metric of the present aeon at its Big Bang surface are conformally related, occurrence of such a sudden topology change is forbidden in the conformal cyclic cosmology picture.

A summary of what we propose in this work is as follows: (1) we give a new nonlinear superposition in general relativity. This is a superposition of two different homogenous FLR W universes yielding an inhomogeneous cosmological model. (2) The metric is invariant under a scale factor duality. (3) If the scale factor $R$ is zero in the beginning of the universe and goes to infinity as $t \rightarrow \infty$ then the universe starts approximately as a FLRW universe and ends as a different FLRW universe. (4) During such phase transition the spatial curvature of 3-space changes in both magnitude and sign. If the sign changes then the topology of the 3-space also changes but if the sign remains intact, then the spatial curvature of the 3-space either increases or decreases.

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