JSJ-DECOMPOSITIONS OF FINITELY PRESENTED GROUPS AND COMPLEXES OF GROUPS

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Dedicated to Professor David Epstein for his sixtieth birthday

Abstract. A JSJ-splitting of a group $G$ over a certain class of subgroups is a graph of groups decomposition of $G$ which describes all possible decompositions of $G$ as an amalgamated product or an HNN extension over subgroups lying in the given class. Such decompositions originated in 3-manifold topology. In this paper we generalize the JSJ-splitting constructions of Sela, Rips-Sela and Dunwoody-Sageev and we construct a JSJ-splitting for any finitely presented group with respect to the class of all slender subgroups along which the group splits. Our approach relies on Haefliger’s theory of group actions on CAT(0) spaces.

1. Introduction

The type of graph of groups decompositions that we will consider in this paper has its origin in 3-dimensional topology. Waldhausen in \cite{W} defined the characteristic submanifold of a 3-manifold $M$ and used it in order to understand exotic homotopy equivalences of 3-manifolds (i.e., homotopy equivalences that are not homotopic to homeomorphisms). Here is a (weak) version of the characteristic submanifold theory used by Waldhausen that is of interest to us: Let $M$ be a closed, irreducible, orientable 3-manifold. Then there is a finite collection of embedded 2-sided incompressible tori such that each piece obtained by cutting $M$ along this collection of tori is either a Seifert fibered space or atoroidal and acylindrical. Furthermore every embedded incompressible torus of $M$ is either homotopic to one of the cutting tori or can be isotoped into a Seifert fibered space piece. We note that embedded incompressible tori of $M$ correspond to splittings of the fundamental group of $M$ over abelian subgroups of rank 2. So from the algebraic point of view we have a ‘description’ of all splittings of $\pi_1(M)$ over abelian groups of...
rank 2. Waldhausen in \cite{W} did not give a proof of this theorem; it was proven later independently by Jaco-Shalen \cite{JS} and Johannson \cite{J} (this explains the term JSJ-decomposition).

We recall that by Grushko’s theorem every finitely generated group $G$ can be decomposed as a free product of finitely many indecomposable factors. Now if $G$ has no $\mathbb{Z}$ factors any other free decomposition of $G$ is simply a product of a rearrangement of conjugates of these indecomposable factors. One can see JSJ-decomposition as a generalization of this description for splittings of groups over certain classes of subgroups.

We recall that a group is termed small if it has no free subgroups of rank 2. Our paper deals with splittings over slender groups which are a subclass of small groups. We recall that a finitely generated group $G$ is slender if every subgroup of $G$ when it acts on a tree either leaves an infinite line invariant or it fixes a point. It turns out that a group is slender if and only if all its subgroups are finitely generated (see \cite{DS}). For example finitely generated nilpotent groups are slender.

To put our results on JSJ-decompositions in perspective we note that Dunwoody has shown that if $G$ is a finitely presented group then if $\Gamma$ is a graph of groups decomposition of $G$ with corresponding $G$-tree $T_\Gamma$ then there is a $G$-tree $T'$ and a $G$-equivariant map $\alpha : T' \to T_\Gamma$ such that $T'/G$ has at most $\delta(G)$ essential vertices (see \cite{BF}, lemma 1). We recall that a vertex in a graph of groups is not essential if it is adjacent to exactly two edges and both edges and the vertex are labelled by the same group. In other words one can obtain all graph of groups decompositions of $G$ by ‘folding’ from some graph of group decompositions which have less than $\delta(G)$ vertices.

We remark that in general there is no bound on the number of vertices of the graph of groups decompositions that one obtains after folding. However in the special case of decompositions with small edge groups Bestvina and Feighn (\cite{BF}. See Thm 5.3 in this paper) have strengthened this result showing that every reduced decomposition $\Gamma$ of a finitely presented group $G$ with small edge groups has at most $\gamma(G)$ vertices. Essentially they showed that in the case of small splittings the number of ‘foldings’ that keep the edge groups small is bounded. The JSJ decomposition that we present here complements the previous results as it gives a description of a set of decompositions with slender edge groups from which we can obtain any other decomposition by ‘foldings’. Roughly this set is obtained as follows: we start with the JSJ decomposition and then we refine it by picking for each enclosing group some splittings that correspond to disjoint simple closed curves on the underlying surface. Of course there are infinitely many such possible
refinements but they are completely described by the ‘surfaces’ that correspond to the enclosing groups.

Sela in [S] was the first to introduce the notion of a JSJ-decomposition for a generic class of groups, namely for hyperbolic groups. Sela’s JSJ-decomposition of hyperbolic groups describes all splittings of a hyperbolic group over infinite cyclic subgroups and was used to study the group of automorphisms of a hyperbolic group. Sela’s result was subsequently generalized by Rips and Sela ([RS]) to all finitely presented groups. Dunwoody and Sageev ([DS]) generalized this result further and produced a JSJ-decomposition which describes all splittings of a finitely presented group over slender groups under the assumption that the group does not split over groups ‘smaller’ than the ones considered. Bowditch in [B] gives a different way of constructing the JSJ-decomposition of a hyperbolic group using the boundary of the group. In particular this shows that the JSJ-decomposition is invariant under quasi-isometries.

In this paper we produce for every finitely presented group $G$ a JSJ-decomposition of $G$ that describes all splittings of $G$ over all its slender subgroups.

Our approach to JSJ-decompositions differs from that of [S], [RS] and of [DS] in that we use neither $\mathbb{R}$-trees nor presentation complexes. We use instead Haefliger’s theory of complexes of groups and actions on products of trees. To see how this can be useful in studying splittings of groups consider the following simple example: Let $G$ be the free abelian group on two generators $a, b$. Then $G$ splits as an HNN extension over infinitely many of its cyclic subgroups. Consider now two HNN decompositions of $G$, namely the HNN decomposition of $G$ over $\langle a \rangle$ and over $\langle b \rangle$. The trees corresponding to these decompositions are infinite linear trees. Consider now the diagonal action of $G$ on the product of these two trees. The quotient is a torus. Every splitting of $G$ is now represented in this quotient by a simple closed curve. We see therefore how we can arrive at a description of infinitely many splittings by considering an action on a product of trees corresponding to two splittings.

Before stating our results we give a brief description of our terminology: Let $T_A, T_B$ be Bass-Serre trees for one edge splittings of a group $G$ over subgroups $A, B$. We say that the splitting over $A$ is elliptic with respect to the splitting over $B$ if $A$ fixes a vertex of $T_B$. If the splitting over $A$ is not elliptic with respect to the splitting over $B$ we say that it is hyperbolic. We say that the pair of two splittings is hyperbolic-hyperbolic if they are hyperbolic with respect to each other. We define similarly elliptic-elliptic etc (see def [2.1]). If a splitting over a slender
group $A$ is not hyperbolic-elliptic with respect to any other splitting over a slender group then we say it is minimal. Finally we use the term enclosing group (def. 4.5) for what Rips-Sela call quadratically hanging group and Dunwoody-Sageev call hanging $K$-by-orbifold group.

This paper is organized as follows. In section 2 we prove some preliminary results and recall basic definitions from [RS]. In section 3 we introduce the notion of ‘minimality’ of splittings and prove several technical lemmas about minimal splittings that are used in the sequel. In section 4 we apply Haefliger’s theory to produce ‘enclosing groups’ for pairs of hyperbolic-hyperbolic minimal splittings. Proposition 4.7 is the main step in our construction of JSJ decompositions. It says that we can always find a graph decomposition that contains both splittings of a given pair of splittings. We note that, although in our main theorem we consider only finitely presented groups, proposition 4.7 is valid for groups that are only finitely generated. Moreover proposition 4.7 holds also for pairs of hyperbolic-hyperbolic splittings over small groups.

In section 5 using the same machinery as in section 4 we show that there is a graph of groups that ‘contains’ all splittings from a family of hyperbolic-hyperbolic minimal splittings (proposition 5.4). Using this we describe a refinement process that produces the JSJ-decomposition of a finitely presented group over all its slender subgroups. Because of the accessibility results of Bestvina-Feighn ([BF]. See Thm 5.3), there is an upper bound on the complexity of graph decompositions that appear in the refinement process, therefore this process must terminate. The terminal graph decomposition must “contain” all minimal splittings.

The graph decomposition has special vertex groups (maybe none) which are called maximal enclosing groups with adjacent edge groups to be peripheral (see Def 4.5). Each of them is an extension of the orbifold fundamental group of some compact 2-orbifold with boundary (maybe empty) by a slender group, $F$. Examples are surface groups ($F$ is trivial) and the fundamental group of a Seifert space ($F \simeq \mathbb{Z}$), which is a 3-manifold. We produce a graph decomposition using minimal splittings (see Def 3.1) of $G$. See Def 2.1 for the definition of the type of a pair of splittings, namely, hyperbolic-hyperbolic, elliptic-elliptic.

**Theorem 5.13.** Let $G$ be a finitely presented group. Then there exists a graph decomposition, $\Gamma$, of $G$ such that

(1) all edge groups are slender.
(2) Each edge of $\Gamma$ gives a minimal splitting of $G$ along a slender group. This splitting is elliptic-elliptic with respect to any minimal splitting of $G$ along a slender subgroup.

(3) Each maximal enclosing group of $G$ is a conjugate of some vertex group of $\Gamma$, which we call a (maximal) enclosing vertex group. The edge group of an edge adjacent to the vertex of a maximal enclosing vertex group is a peripheral subgroup of the enclosing group.

(4) Let $G = A \ast_C B$ or $A * C$ be a minimal splitting along a slender group $C$, and $T_C$ its Bass-Serre tree.
   
   (a) If it is elliptic-elliptic with respect to all minimal splittings of $G$ along slender groups, then all vertex groups of $\Gamma$ are elliptic on $T_C$.
   
   (b) If it is hyperbolic-hyperbolic with respect to some minimal splitting of $G$ along a slender group, then there is an enclosing vertex group, $S$, of $\Gamma$ which contains a conjugate of $C$, which is unique among enclosing vertex groups of $\Gamma$. $S$ is also the only one among enclosing vertex groups which is hyperbolic on $T_C$. There exist a base 2-orbifold, $\Sigma$, for $S$ and an essential simple closed curve or a segment on $\Sigma$ whose fundamental group (in the sense of complex of groups) is a conjugate of $C$.
   
   All vertex groups except for $S$ of $\Gamma$ are elliptic on $T_C$.
   
   In particular, there is a graph decomposition, $S$, of $S$ whose edge groups are in conjugates of $C$, which we can substitute for $S$ in $\Gamma$ such that all vertex groups of the resulting refinement of $\Gamma$ are elliptic on $T_C$.

Although we produce $\Gamma$, called a JSJ-decomposition, using only minimal splittings, it turns out that it is also good for non-minimal splittings.

**Theorem 5.15.** Let $G$ be a finitely presented group, and $\Gamma$ a graph decomposition we obtain in Theorem 5.13. Let $G = A \ast_C B, A * C$ be a splitting along a slender group $C$, and $T_C$ its Bass-Serre tree.

   (1) If the group $C$ is elliptic with respect to any minimal splitting of $G$ along a slender group, then all vertex groups of $\Gamma$ are elliptic on $T_C$.

   (2) Suppose the group $C$ is hyperbolic with respect to some minimal splitting of $G$ along a slender group. Then
   
   (a) all non-enclosing vertex groups of $\Gamma$ are elliptic on $T_C$.
(b) For each enclosing vertex group, \( V \), of \( \Gamma \), there is a graph decomposition of \( V \), \( V \), whose edge groups are in conjugates of \( C \), which we can substitute for \( V \) in \( \Gamma \) such that if we substitute for all enclosing vertex groups of \( \Gamma \) then all vertex groups of the resulting refinement of \( \Gamma \) are elliptic on \( T_C \).

The first version of this paper is written in 1998. Since then a very important application of JSJ-decompositions is found by Z.Sela on Tarski’s conjecture on the equivalence of the elementary theory of \( F_2, F_3 \) (see [S1] and the following papers of Sela on this). He uses JSJ-decompositions along abelian subgroups. We note also that the question of ‘uniqueness’ of JSJ-splittings has been treated in [F]. We would like to thank M.Bestvina, M.Feighn, V.Guirardel, B.Leeb, M.Sageev, Z.Sela and G.A.Swarup for discussions related to this work. We would like to thank A. Haefliger for his interest in this work and many suggestions that improved the exposition. Finally we would like to thank the referee for detailed suggestions which we found very helpful.

2. Pairs of splittings

In this section we recall and generalize notation from [RS].

**Definition 2.1** (types of a pair). Let \( A_1 \star_{C_1} B_1 \) (or \( A_1 \star_{C_1} C_1 \)), \( A_2 \star_{C_2} B_2 \) (or \( A_2 \star_{C_2} C_2 \)) be two splittings of a finitely generated group \( G \) with corresponding Bass-Serre trees \( T_1, T_2 \). We say that the first splitting is hyperbolic with respect to the second if there is \( c_1 \in C_1 \) acting as a hyperbolic element on \( T_2 \). We say that the first splitting is elliptic with respect to the second if \( C_1 \) fixes a point of \( T_2 \). We say that this pair of splittings is hyperbolic-hyperbolic if each splitting is hyperbolic with respect to the other. Similarly we define what it means for a pair of splittings to be elliptic-elliptic, elliptic-hyperbolic and hyperbolic-elliptic.

It is often useful to keep in mind the ‘geometric’ meaning of this definition: Consider for example a closed surface. Splittings of its fundamental group over \( \mathbb{Z} \) correspond to simple closed curves on the surface. Two splittings are hyperbolic-hyperbolic if their corresponding curves intersect and elliptic-elliptic otherwise. Consider now a punctured surface and two splittings of its fundamental group: one corresponding to a simple closed curve (a splitting over \( \mathbb{Z} \)) and a free splitting corresponding to an arc having its endpoints on the puncture such that the two curves intersect at one point. This pair of splittings is hyperbolic-elliptic.
**Proposition 2.2.** Let $A_1 *_{C_1} B_1$ (or $A_1 *_{C_1}$), $A_2 *_{C_2} B_2$ (or $A_2 *_{C_2}$) be two splittings of a group $G$ with corresponding Bass-Serre trees $T_1$, $T_2$. Suppose that there is no splitting of $G$ of the form $A *_{C} B$ or $A *_{C}$ with $C$ an infinite index subgroup of $C_1$ or of $C_2$. Then this pair of splittings is either hyperbolic-hyperbolic or elliptic-elliptic.

**Proof.** We treat first the amalgamated product case. Let $T_1$, $T_2$ be the Bass-Serre trees of the two splittings $A_1 *_{C_1} B_1$, $A_2 *_{C_2} B_2$. Suppose that $C_1$ does not fix any vertex of $T_2$ and that $C_2$ does fix a vertex of $T_1$. Without loss of generality we can assume that $C_2$ fixes the vertex stabilized by $A_1$. Consider the actions of $A_2$, $B_2$ on $T_1$. Suppose that both $A_2$, $B_2$ fix a vertex. If they fix different vertices then $C_2$ fixes an edge, so it is a finite index subgroup of a conjugate of $C_1$. But then $C_1$ can not be hyperbolic with respect to $A_2 *_{C_2} B_2$. On the other hand it is not possible that they fix the same vertex since $A_2$, $B_2$ generate $G$. So at least one of them, say $A_2$, does not fix a vertex. But then the action of $A_2$ on $T_1$ induces a splitting of $A_2$ over a group $C$ which is an infinite index subgroup of $C_1$. Since $C_2$ is contained in a vertex group of this splitting we obtain a splitting of $G$ over $C$ which is a contradiction.

We consider now the case one of the splittings is an HNN-extension: say we have the splittings $A_1 *_{C_1} B_1$, $A_2 *_{C_2}$ with Bass-Serre trees $T_1$, $T_2$. Assume $C_1$ is hyperbolic on $T_2$ and $C_2$ elliptic on $T_1$. Again it is not possible that $A_2$ fix a vertex of $T_1$. Indeed $C_2 = A_2 \cap tA_2 t^{-1}$ and if $A_2$ fixes a vertex $C_2$ is contained in a conjugate of $C_1$ which is impossible (note that $t$ can not fix the same vertex as $A_2$). We can therefore obtain a splitting of $G$ over an infinite index subgroup of $C_1$ which is a contradiction. If $C_1$ is elliptic on $T_2$ and $C_2$ hyperbolic on $T_1$ we argue as in the first case. The case where both splitting are HNN extension is treated similarly.

**Remark 2.3.** In the proof of the Proposition 2.2 one shows in fact that if $A_2 *_{C_2} B_2$ is elliptic with respect to $A_1 *_{C_1} B_1$ then either $A_1 *_{C_1} B_1$ is elliptic too, or there is a splitting of $G$ over a subgroup of infinite index of $C_1$.

### 3. Minimal splittings

**Definition 3.1** (Minimal splittings). We call a splitting $A *_{C} B$ (or $A *_{C}$) of a group $G$ minimal if it is not hyperbolic-elliptic with respect to any other splitting of $G$ over a slender subgroup.

**Remark 3.2.** Remark 2.3 implies that if $G$ splits over $C$ but does not split over an infinite index subgroup of $C$ then the splitting of $G$ over $C$
is minimal. There are examples of minimal and non-minimal splittings over a common subgroup. For example let $H$ be a group which does not split and let $G = \mathbb{Z}^2 \ast H$. If $a, b$ are generators of $\mathbb{Z}^2$ the splitting of $\mathbb{Z}^2$ over $\langle a \rangle$ induces a minimal splitting of $G$ over $\langle a \rangle$. On the other hand the splitting of $G$ given by $G = \mathbb{Z}^2 \ast \langle a \rangle (H \ast \langle a \rangle)$ is not minimal. Indeed it is hyperbolic-elliptic with respect to the splitting of $G$ over $\langle b \rangle$ which is induced from the splitting of $\mathbb{Z}^2$ over $\langle b \rangle$.

We collect results on minimal splittings we need. We first show the following:

**Lemma 3.3.** Suppose that a group $G$ splits over the slender groups $C_1, C_2$ and $K \subset C_2$. Assume moreover that the splittings over $C_1, C_2$ are hyperbolic-hyperbolic, the splitting over $C_1$ is minimal and that $G$ admits an action on a tree $T$ such that $C_2$ acts hyperbolically and $K$ fixes a vertex. Then the splittings over $C_1$ and $K$ are not hyperbolic-hyperbolic.

**Proof.** We will prove this by contradiction. Let $T_1, T_2, T_3$ be, respectively, the Bass-Serre trees of the splittings over $C_1, C_2, K$. Without loss of generality we can assume the axes of $C_2, K$ when acting on $T_1$ contain an edge stabilized by $C_1$. Let $t \in K \subset C_2$ be an element acting hyperbolically on $T_1$. Similarly let $u \in C_1$ be an element acting hyperbolically on $T_2, T_3$ and $y \in C_2$ be an element acting hyperbolically on $T$. We distinguish 2 cases:

**Case 1:** $y$ acts elliptically on $T_1$. Then either $y$ fixes the axis of translation of $C_2$ or it acts on it by a reflection (in the dihedral action case). In both cases $y^2 \in C_1$. Since $y^2 \not\in K$ we have that $y^2$ acts hyperbolically on $T_3$. Therefore there are $m, n \in \mathbb{Z}$ such that $y^m u^n$ fixes an edge of $T_3$ and $y^m u^n \in C_1 \cap \langle zKz^{-1} \rangle$, hence $y^m u^n \in C_1 \cap xC_2 x^{-1}$. This is clearly impossible since $y$ is elliptic when acting on $T_2$ while $u$ is hyperbolic.

**Case 2:** $y$ acts hyperbolically on $T_1$. Without loss of generality we assume that $t$ fixes the axis of translation of $y$ on $T$. Indeed if this is not so we can replace $t$ by $t^2$. Since both $t$ and $y$ act hyperbolically on $T_1$ there are $m, n \in \mathbb{Z}$ such that $t^m y^n \in C_1$. On the other hand $t^m y^n$ acts hyperbolically on $T$ since $t$ fixes the axis of translation of $y$. So $t^m y^n$ does not lie in a conjugate of $K$. For the same reason $(t^m y^n)^2$ does not lie in a conjugate of $K$. We consider now the action of $C_1$ on $T_3$. If $t^m y^n$ is elliptic then $(t^m y^n)^2$ fix the axis of translation of $C_1$ and therefore lies in a conjugate of $K$, which is impossible as we noted above. Therefore both $t^m y^n$ and $u$ acts hyperbolically on $T_3$. We infer that there are $p, q \in \mathbb{Z}$ such that $(t^m y^n)^p u^q$ lies in a conjugate of $K$. Therefore this element fixes the translation axis of $C_1$ when acting on $T_2$. This is however impossible since $t^m y^n \in C_1 \cap C_2$ so it fixes the
axis while \( u \) acts hyperbolically on \( T_2 \). This finishes the proof of the lemma. \( \square \)

Using lemma 3.3, we show the following.

**Proposition 3.4** (dual-minimality). Let \( A_1 \ast_{C_1} B_1 \) (or \( A_1 \ast_{C_1} \)) be a minimal splitting of \( G \) over a slender group \( C_1 \). Suppose that \( A_1 \ast_{C_1} B_1 \) (or \( A_1 \ast_{C_1} \)) is hyperbolic-hyperbolic with respect to another splitting of \( G \), \( A_2 \ast_{C_2} B_2 \) (or \( A_2 \ast_{C_2} \)), where \( C_2 \) is slender. Then \( A_2 \ast_{C_2} B_2 \) (or \( A_2 \ast_{C_2} \)) is also minimal.

**Proof.** We denote the Bass-Serre trees for the splittings over \( C_1, C_2 \) by \( T_1, T_2 \) respectively.

Suppose that the splitting over \( C_2 \) is not minimal; then it is hyperbolic-elliptic with respect to another splitting over a slender subgroup \( C_3 \).

We distinguish 2 cases:

1st case: The splitting over \( C_3 \) is an amalgamated product, say \( A_3 \ast_{C_3} B_3 \).

We let \( A_3, B_3 \) act on \( T_2 \) and we get graph of groups decompositions for \( A_3, B_3 \), say \( \Gamma_1, \Gamma_2 \). Since \( C_3 \) is elliptic when acting on \( T_2 \) we can refine \( A_3 \ast_{C_3} B_3 \) by replacing \( A_3, B_3 \) by \( \Gamma_1, \Gamma_2 \). We collapse then the edge labelled by \( C_3 \) and we obtain a new graph of groups decomposition that we call \( \Gamma \). We note that all vertex groups of \( \Gamma \) fix vertices of \( T_2 \).

This implies that \( C_1 \) is not contained in a conjugate of a vertex group of \( \Gamma \). Therefore we can collapse all edges of \( \Gamma \) except one and obtain a splitting over a subgroup \( K \) of \( C_2 \) such that \( C_1 \) is hyperbolic with respect to this splitting. Since \( C_1 \) is minimal the pair of splittings over \( C_1, K \) is hyperbolic-hyperbolic. This however contradicts lemma 3.3 since \( K \) fixes a vertex of the Bass-Serre tree of \( A_3 \ast_{C_3} B_3 \) while \( C_2 \) acts hyperbolically on this tree.

2nd case: The splitting over \( C_3 \) is an HNN-extension, say \( A_3 \ast_{C_3} \).

The argument is similar in this case but a bit more delicate. We let \( A_3 \) act on \( T_2 \) and we obtain a graph decomposition for \( A_3 \), say \( \Gamma_1 \). Since \( C_3 \) fixes a vertex of \( T_2 \) we can refine \( A_3 \ast_{C_3} \) by replacing \( A_3 \) by \( \Gamma_1 \). Let \( e \) be the edge of \( A_3 \ast_{C_3} \). If \( e \) stays a loop in \( \Gamma_1 \) we argue as in the amalgamated product case. After we collapse \( e \) in \( \Gamma_1 \) we obtain a graph of groups such that \( C_1 \) is not contained in the conjugate of any vertex group. We arrive then at a contradiction as before.

Assume now that \( e \) connects to two different vertices in \( \Gamma_1 \). Let \( V, U \) be the vertex groups of these vertices. Clearly \( C_2 \) is not contained in a conjugate of either \( V \) or \( U \). We remark that the vertex we get after collapsing \( e \) in \( \Gamma_1 \) is labelled by \( \langle V_1, V_2 \rangle \). By Bass-Serre theory \( C_2 \) is not contained in a conjugate of \( \langle V_1, V_2 \rangle \) either. Let’s call \( \Gamma \) the graph of groups obtained after collapsing \( e \). Let \( T_\Gamma \) be its Bass-Serre...
tree. Clearly $C_2$ does not fix any vertex of $T$. It follows that we can collapse all edges of $\Gamma$ except one and obtain an elliptic-hyperbolic splitting with respect to $C_2$. What we have gained is that this splitting is over a subgroup of $C_2$, say $K$. Of course if this splitting is an amalgam we are done by case 1 so we assume it is an HNN-extension $A \ast_K$. Let’s call $e_1$ the edge of this HNN-extension and let $T_K$ be its Bass-Serre tree. If $C_1$ acts hyperbolically on $T$ then we are done as before by lemma 3.3 since $C_2$ is hyperbolic on $T$ and $K$ elliptic. Otherwise we let $A$ act on $T_2$ and we refine $A \ast_K$ as before. Let’s call $\Gamma'$ the graph of groups obtained. If $e_1$ stays a loop in $\Gamma'$ we are done as before. Otherwise by collapsing $e_1$ we obtain a graph of groups such that no vertex group contains a conjugate of $C_1$. We can collapse this graph further to a one edge splitting over, say $K_1 < C_2$ such that its vertex groups do not contain a conjugate of $C_1$. Since the splitting over $C_1$ is minimal this new splitting is hyperbolic-hyperbolic with respect to the splitting over $C_1$. Moreover $K_1$ fixes a vertex of $T$ while $C_2$ acts hyperbolically on $T$. This contradicts lemma 3.3.

We prove an accessibility result for minimal splittings.

**Proposition 3.5** (accessibility of minimal splittings). Let $G$ be a finitely generated group. There is no infinite sequence of splittings of $G$ of the form $A_n \ast C_n B_n$ or of the form $A_n \ast C_n$ where $C_{n+1}$ is a subgroup of $C_n$, $C_1$ is a (finitely generated) slender group, such that $C_n$ acts hyperbolically on some $G$-tree $T_n$ while $C_{n+1}$ fixes a vertex of $T_n$.

**Proof.** We define a sequence of homomorphisms $f_n$ from $C_1$ to $\mathbb{Z}/2\mathbb{Z}$ as follows: Consider the graph of groups corresponding to the action of $C_1$ on $T_n$. The fundamental group of this graph of groups is $C_1$. If the underlying graph of this graph of groups is a circle, map this group to $\mathbb{Z}/2\mathbb{Z}$ by mapping all vertex groups to 0 and the single loop of the graph to 1. If $C_1$ acts by a dihedral type action on $T_n$, map $C_1$ to $\mathbb{Z}/2\mathbb{Z} \ast \mathbb{Z}/2\mathbb{Z}$ in the obvious way and then map $\mathbb{Z}/2\mathbb{Z} \ast \mathbb{Z}/2\mathbb{Z}$ to $\mathbb{Z}/2\mathbb{Z}$ so that $C_{n+1}$ is mapped to 0. This is possible since $C_{n+1}$ is elliptic. By construction $f_{n+1}(C_{n+1}) = \mathbb{Z}/2\mathbb{Z}$. It follows that the map $\Phi_n : C_1 \to (\mathbb{Z}/2\mathbb{Z})^n$ is onto for every $n$. This contradicts the fact that $C_1$ is finitely generated.

Each edge, $e$, of a graph decomposition, $\Gamma$, of a group $G$ gives (rise to) a splitting of $G$ along the edge group of $e$, $E$, by collapsing all edges of $\Gamma$ but $e$. We do this often in the paper. To state the main result of this section, we give one definition.

**Definition 3.6** (Refinement). Let $\Gamma$ be a graph of groups decomposition of $G$. We say that $\Gamma'$ is a refinement of $\Gamma$ if each vertex group of $\Gamma'$ is contained in a conjugate of a vertex group of $\Gamma$. We say that
Γ′ is a proper refinement of Γ if Γ′ is a refinement of Γ and Γ is not a refinement of Γ′.

Let Γ be a graph of groups and let V be a vertex group of Γ, which admits a graph of groups decomposition ∆ such that a conjugate of each edge group adjacent to V in Γ is contained in a vertex group of ∆. Then one can obtain a refinement of Γ by replacing V by ∆. In the refinement, each edge, e, in Γ adjacent to the vertex for V is connected to a vertex of ∆ whose vertex group contains a conjugate of the edge group of e. The monomorphism from the edge group of e to the vertex group of ∆ is equal to the corresponding monomorphism in Γ modified by conjugation.

This is a special type of refinement used often in this paper. We say that we substitute ∆ for V in Γ.

**Proposition 3.7** (Modification to minimal splittings). Let G be a finitely presented group. Suppose that Γ is a graph of groups decomposition of G with slender edge groups. Then there is a graph of groups decomposition of G, Γ′, which is a refinement of Γ such that all edges of Γ′ give rise to minimal splittings of G. All edge groups of Γ′ are subgroups of edge groups of Γ.

**Proof.** We give two proofs of this proposition. We think that in the first one the idea is more transparent, which uses actions on product of trees. Since we use terminologies and ideas from the part we construct "enclosing groups" in Proposition 4.7, one should read the first proof after reading that part. The second proof uses only classical Bass-Serre theory and might be more palatable to readers not accustomed to Haefliger’s theory.

1st proof. We define a process to produce a sequence of refinements of Γ which we can continue as long as an edge of a graph decomposition in the sequence gives a non-minimal splitting, and then show that it must terminate in a finite step.

If every edge of Γ gives a minimal splitting, nothing to do, so suppose there is an edge, e, of Γ with the edge group E which gives a splitting which is is hyperbolic-elliptic with respect to, say, G = P * R Q or P* R. We consider the action of G on the product of trees TΓ, TR where TΓ is the Bass-Serre tree of Γ and TR the tree of the splitting over R. We consider the diagonal action of G on TΓ × TR, then produce a G-invariant subcomplex of TΓ × TR, Y, such that Y/G is compact as we will do in the proof of Proposition 4.7. Y/G has a structure of a complex of groups whose fundamental group is G. In the construction of Y/G, we give priority (see Remark 4.9) to the decomposition Γ over the splitting of G along R, so that we can recover Γ from Y/G by
collapsing the complex of groups obtained. To fix ideas we think of $T_\Gamma$ as horizontal (see the paragraph after Lemma 4.1). Since the slender group $E$ acts hyperbolically on $T_R$, there is a line, $l_E$, in $T_R$ which is invariant by $E$. Then, $l_E/E$ is either a segment, when the action of $E$ is dihedral, or else, a circle. Put $c_E = l_E/E$, which is the core for $E$. Then, $l_E/E$ is either a segment, when the action of $E$ is dihedral, or else, a circle. Put $c_E = l_E/E$, which is the core for $E$. There is a map from $c_E \times [0, 1]$ to $Y/G$, and let’s call the image, $b_E$, the band for $E$. $b_E$ is a union of finite squares. For other edges, $e_i$, of $\Gamma$ than $e$ with edge groups $E_i$, we have similar objects, $b_{E_i}$, which can be a segment, when the action of $E_i$ on $T_R$ is elliptic. As in Proposition 4.7, the squares in $Y/G$ is exactly the union of the squares contained in $b_E$ and other $b_{E_i}$’s. Note that there is at least one square in $Y/G$, which is contained in $b_E$.

On the other hand since $R$ is elliptic on $T_\Gamma$, $T_\Gamma/R$ is a tree, so that the intersection of $T_\Gamma/R$ (note that this is naturally embedded in $T_\Gamma \times T_R$) and $Y/G$ is a forest. Therefore, we can remove squares from $Y/G$ without changing the fundamental group, which is $G$. We then collapse all vertical edges which are left. In this way, we obtain a graph decomposition of $G$, which we denote $\Gamma_1$. By construction, all vertex groups of $\Gamma_1$ are elliptic on $T_\Gamma$, so that $\Gamma_1$ is a refinement of $\Gamma$. Also edge groups of $\Gamma_1$ are subgroups of edge groups of $\Gamma$. There is no edge group of $\Gamma_1$ which is hyperbolic-elliptic with respect to the splitting over $R$. If all edge of $\Gamma_1$ gives a minimal splitting of $G$, then $\Gamma_1$ is a desired one. If not, we apply the same process to $\Gamma_1$, and obtain a refinement, $\Gamma_2$. But this process must terminate by Prop 3.5 and Theorem 5.3, which gives a desired one.

2nd proof. If all edges of $\Gamma$ correspond to minimal splittings then there is nothing to prove. Assume therefore that an edge $e$ of $\Gamma$ corresponds to a splitting which is not minimal. We will construct a finite sequence of refinements of $\Gamma$ such that the last term of the sequence is $\Gamma'$. Let’s say that $e$ is labelled by the slender group $E$. Let $A \ast_e B$ (or $A \ast_e E$) be the decomposition of $G$ obtained by collapsing all edges of $\Gamma$ except $e$. Since this splitting is not minimal it is hyperbolic-elliptic with respect to another splitting of $G$ over a slender group, say $P \ast_R Q$ (or $P \ast_R$).

Let $T_E, T_R$ be the Bass-Serre tree of the splittings of $G$ over $E, R$ and let $T_\Gamma$ be the Bass-Serre tree corresponding to $\Gamma$. We distinguish two cases:

Case 1: $R$ is contained in a conjugate of $E$.
In this case we let $P, Q$ act on $T_\Gamma$. We obtain graph of groups decompositions of $P, Q$ and we refine $P \ast_R Q$ by substituting $P, Q$ by these graphs of groups decompositions. In this way we obtain a graph of groups decomposition $\Gamma_1$. If some edges of $\Gamma_1$ (which are not loops) are labelled by the same group as an adjacent vertex we collapse them.
For simplicity we still call $\Gamma_1$ the graph of groups obtained after this collapsing. We remark that all edge groups of $\Gamma_1$ fix a vertex of the tree of the splitting $P \ast_R Q$. We argue in the same way if the splitting over $R$ is an HNN-extension.

Case 2: $R$ is not contained in any conjugate of $E$.
We let $P, Q$ act on $T_E$ and we obtain graph of groups decompositions of these groups. We refine $P \ast_R Q$ as before by substituting $P, Q$ by the graph of groups obtained. We note that by our hypothesis in case 2 $P, Q$ do not fix both vertices of $T_E$. Let’s call $\Delta$ the graph of groups obtained in this way. Since $C$ fixes a vertex of $T_E$ we can assume without loss of generality that $C \subset P$. We collapse the edge of $\Delta$ labelled by $R$ and we obtain a graph of groups $\Delta_1$. We note now that if $E$ acts hyperbolically on the Bass-Serre tree of $\Delta_1$ we are in the case 1 (i.e., we have a pair of hyperbolic-elliptic splittings where the second splitting is obtained by appropriately collapsing all edges of $\Delta_1$ except one). So we can refine $\Gamma$ and obtain a decomposition $\Gamma_1$ as in case 1.

We suppose now that this is not the case. Let’s denote by $P'$ the group of the vertex obtained after collapsing the edge labelled by $R$. We let $P'$ act on $T_\Gamma$ and we obtain a graph of groups decomposition of $P'$, say $\Delta_2$. We note that the vertex obtained after this collapsing is now labelled by a subgroup of $P$, say $P'$. We let $P'$ act on $T_E$ and we obtain a graph of groups decomposition of $P'$, say $\Delta_2$. If every edge group of $\Delta_1$ acts elliptically on $T_R$ then we let all other vertices of $\Delta_1$ act on $T_\Gamma$ and we substitute all these vertices in $\Delta_1$ by the graphs obtained. We also substitute $P'$ by $\Delta_2$. We call the graph of groups obtained in this way $\Gamma_1$.

Finally we explain what we do if some edge of $\Delta_2$ acts hyperbolically on $T_R$. We note that $P'$ splits over $R$, indeed $P'$ corresponds to a one-edge subgraph of $\Delta$. Abusing notation we call still $T_R$ the tree of the splitting of $P'$ over $R$. We now repeat with $P'$ the procedure applied to $G$. We note that we are necessarily in case 2 as $R$ can not be contained in a conjugate of an edge group of $\Delta_2$. As before we either obtain a refinement of $\Delta_2$ such that all edge groups of $\Delta_2$ act elliptically on $T_R$ or we obtain a non-trivial decomposition of $P'$, say $\Delta'$, such that an edge of $\Delta'$ is labelled by $R$ and the following holds: If we collapse the edge of $\Delta'$ labelled by $R$ we obtain a vertex $P''$ which has the same property as $E'$. Namely if $\Delta_3$ is the decomposition of $P''$ obtained by acting on $T_\Gamma$ then some edge group of $\Delta_3$ acts hyperbolically on $T_R$. If we denote by $\Delta'_1$ the decomposition of $P'$ obtained after the collapsing we remark that we can substitute $P'$ by $\Delta'_1$ in $\Delta_1$ and obtain a decomposition of $G$ with more edges than $\Delta_1$. Now we repeat the same procedure to $P''$. 

By Theorem 5.3 this process terminates and produces a refinement of $\Gamma$ which we call $\Gamma_1$.

By the argument above we obtain in both cases a graph of groups decomposition of $G$ $\Gamma_1$ which has the following properties:

1) $\Gamma_1$ is a refinement of $\Gamma$ and
2) There is an action of $G$ on a tree $T$ such that some edge group of $\Gamma$ act on $T$ hyperbolically while all edge groups of $\Gamma_1$ act on $T$ elliptically.

Now we repeat the same procedure to $\Gamma_1$ and we obtain a graph of groups $\Gamma_2$ etc. One sees that this procedure will terminate using Prop 3.5 and Theorem 5.3. The last step of this procedure produces a decomposition $\Gamma'$ as required by this proposition.

One finds an argument similar to the 1st proof, using product of trees and retraction in the paper [DF].

4. Enclosing groups for a pair of hyperbolic-hyperbolic splittings

4.1. Product of trees and core. Producing a graph of groups which 'contains' a given pair of hyperbolic-hyperbolic splittings along slender groups is the main step in the construction of a JSJ-decomposition of a group. This step explains also what type of groups should appear as vertex groups in a JSJ-decomposition of a group. We produce such a graph of groups in proposition 4.7.

We recall here the definitions of a complex of groups and the fundamental group of such a complex. They were first given in the case of simplicial complexes in [H] and then generalized to polyhedral complexes in [BH]. Here we will give the definition only in the case of 2-dimensional complexes. We recommend [BH] ch.III.C for a more extensive treatment.

Let $X$ be a polyhedral complex of dimension less or equal to 2.

We associate to $X$ an oriented graph as follows: The vertex set $V(X)$ is the set of $n$-cells of $X$ (where $n = 0, 1, 2$). The set of oriented edges $E(X)$ is the set $E(X) = \{(\tau, \sigma)\}$ where $\sigma$ is an $n$-cell of $X$ and $\tau$ is a face of $\sigma$. If $e \in E(X)$, $e = (\tau, \sigma)$, we define the original vertex of $e$, $i(e)$ to be $\tau$ and the terminal vertex of $e$, $t(e)$ to be $\sigma$.

If $a, b \in E(X)$ are such that $i(a) = t(b)$ we define the composition $ab$ of $a, b$ to be the edge $ab = (i(b), t(a))$. If $t(b) = i(a)$ for edges $a, b$ we say that these edges are composable. We remark that the set $E(X)$ is in fact the set of edges of the barycentric subdivision of $X$. Geometrically one represents the edge $e = (\tau, \sigma)$ by an edge joining the barycenter of $\tau$ to the barycenter of $\sigma$. Also $V(X)$ can be identified with the set of vertices of the barycentric subdivision of $X$, to a cell $\sigma$ there
corresponds a vertex of the barycentric subdivision, the barycenter of \( \sigma \).

A complex of groups \( G(X) = (X, G_\sigma, \psi_\sigma, g_{a,b}) \) with underlying complex \( X \) is given by the following data:
1. For each \( n \)-cell of \( X \), \( \sigma \) we are given a group \( G_\sigma \).
2. If \( a \) is an edge in \( E(X) \) with \( i(a) = \sigma, t(a) = \tau \) we are given an injective homomorphism \( \psi_a : G_\sigma \to G_\tau \).
3. If \( a, b \) are composable edges we are given an element \( g_{a,b} \in E_{t(a)} \) such that

\[
g_{a,b}\psi_{ab}g_{a,b}^{-1} = \psi_a\psi_b
\]

We remark that when \( \dim(X) = 1 \), \( G(X) \) is simply a graph of groups. In fact in Haefliger’s setup loops are not allowed so to represent a graph of groups with underlying graph \( \Gamma \) one eliminates loops by passing to the barycentric subdivision of \( \Gamma \). In this case there are no composable edges so condition 3 is void.

We define the fundamental group of a complex of groups \( \pi_1(G(X), \sigma_0) \) as follows:

Let \( E^\pm(X) \) be the set of symbols \( a^+, a^- \) where \( a \in E(X) \). Let \( T \) be a maximal tree of the graph \((V(X), E(X))\) defined above. \( \pi_1(G(X), \sigma_0) \) is the group with generating set:

\[
\coprod G_\sigma, \, \sigma \in V(X), \coprod E^\pm(X)
\]

and set of relations:

- relations of \( G_\sigma \), \( (a^+)^{-1} = a^-, (a^-)^{-1} = a^+ \), \( \forall a \in E(X) \)
- \( a^+b^+ = g_{a,b}(ab)^+, \forall a, b \in E(X), \psi_a(g) = a^+ga^- \), \( \forall g \in G_{i(a)}, a^+ = 1, \forall a \in T \)

It is shown in [BH] that this group does not depend up to isomorphism on the choice of maximal tree \( T \) and its elements can be represented by ‘homotopy classes’ of loops in a similar way as for graphs of groups.

It will be useful for us to define barycentric subdivisions of complexes of groups \( G(X) \). This will be an operation that leaves the fundamental group of the complex of groups unchanged but substitutes the underlying complex \( X \) with its barycentric subdivision \( X' \). We explain this first in the case of graphs of groups. If \( G(X) \) is a graph of groups then we have a group \( G_v \) for each vertex \( v \) of the barycentric subdivision of \( X \). If \( v \) is the barycenter of an edge \( e \), \( G_v \) is by definition in Haefliger’s notation the group associated to the edge \( e \), \( G_e \). Now if \( v \) is a vertex of the second barycentric subdivision then \( v \) lies in some edge \( e \) of \( X \) so we define \( G_v = G_e \). The oriented edges \( E(X) \) are of two types:
1) an edge $e$ from a barycenter $v$ of the second barycentric subdivision to a barycenter $w$ of the first barycentric subdivision. If this case the map $\psi_e : G_v \to G_w$ is the identity.

2) an edge $e$ from a barycenter $v$ of the second barycentric subdivision to a vertex $w$ of $X$. In this case $v$ is the barycenter of an edge $a$ of the first barycentric subdivision and $G_v$ is isomorphic to $G_{i(a)}$. We define then $\psi_e : G_v \to G_w$ to be $\psi_e$.

Let’s call $G(X')$ the graph of groups obtained by this operation. It is clear that the fundamental group of $G(X')$ is isomorphic to the fundamental group of $G(X)$.

Let now $X$ be a 2-dimensional complex and $G(X)$ a complex of groups with underlying complex $X$. Let $X'$ be the barycentric subdivision of $X$. We associate to $X'$ a graph $((V(X'), E(X')))$ as we did for $X$. Now we explain what are the groups and maps associated to $((V(X'), E(X')))$.

In order to describe the groups associated to $V(X')$ it is convenient to recall the geometric representation of $V(X), E(X)$.

The vertices of $V(X)$ correspond geometrically to barycenters of $n$-cells of $X$, i.e. they are just the vertices of the barycentric subdivision of $X$. Similarly the edges $E(X)$ are the edges of the barycentric subdivision and the orientation of an edge is from the barycenter of a face of $X$ to a vertex of $X$.

$V(X')$ analogously can be identified with the set of vertices of the second barycentric subdivision of $X$ and the edges $E(X')$ with the edge set of the second barycentric subdivision of $X$. All 2-cells of the barycentric subdivision of $X$ are 2-simplices.

If $v$ is a vertex of $V(X')$ which is the barycenter of the 2-simplex $\sigma$ then there is a single 2-cell $\tau$ of $X$ containing $\sigma$. If $w$ is the barycenter of $\tau$ we define $G_v = G_w$. If $v$ is a vertex of $V(X')$ which is the barycenter of an edge $a$ we define $G_v = G_{i(a)}$.

We explain now what are the homomorphisms corresponding to $E(X')$. If $a$ is an edge of $E(X')$ then there are two cases:

1) $i(a)$ is the barycenter of a 2-simplex $\sigma$ of $X'$. Then if $t(a)$ is the barycenter of a 2-cell $\tau$ of $X$ by definition $G_{i(a)} = G_{t(a)}$ and we define $\psi_a$ to be the identity. Otherwise if $w$ is the barycenter of the 2-cell of $X$ containing $\sigma$ we have that $G_{i(a)} = G_w$ and there is an edge $e$ in $E(X)$ from $w$ to $t(a)$. We define then $\psi_a$ to be $\psi_e$.

2) $i(a)$ is the barycenter of an edge $e$ of $X'$. If $t(a) = i(e)$ we define $\psi_a$ to be the identity. Otherwise $t(a) = t(e)$ and we define $\psi_a$ to be $\psi_e$.

It remains to define the ‘twisting elements’ for pairs of composable edges of $G(X')$. We remark that if $a', b'$ are composable edges of $X'$ then either $\psi_{a'} = id$ and $\psi_{a'b'} = \psi_{a'}$ in which case we define $g_{a', b'} = e$.
or there are composable edges $a, b$ of $G(X)$ and $\psi_a' = \psi_a, \psi_b' = \psi_b$. In this case we define $g_{a'b'} = g_{a,b}$.

One can see using presentations that the fundamental group of $G(X')$ is isomorphic to the fundamental group of $G(X)$. We explain this in detail now. It might be helpful for the reader to draw the barycentric subdivision of a 2-simplex while following our argument.

Let $T$ be the maximal tree that we pick for the presentation of the fundamental group of $G(X)$. We will choose a maximal tree for $G(X')$ that contains $T$. We focus now on the generators and relators added by a subdivision of a 2-simplex of $X$, $\sigma$. $\sigma$ in $X$ has three edges which correspond to generators. After subdivision we obtain 4 new vertex groups and 11 edges (6 of which are subdivisions of old edges). 4 of the edges correspond, by definition, to the identity homomorphism from old vertex groups to the four new vertex groups. To obtain $T'$ We add these 4 edges to $T$ (some of course might already be contained in $T$). The relations $\psi_a(g) = a^+ ga^-$ for these 4 edges together with $a^+ = 1$, for $a^+ \in T'$ show that the new vertex groups do not add any new generators. Let’s call the 4 edges we added to $T$, $a_1, a_2, a_3, a_4$. We remark that 2 more edges, say $b_1, b_2$ correspond by definition to the identity map and the relations $a^+ b^+ = g_{a,b}(ab)^+$ show that these two new generators are also trivial (the corresponding $g_{a,b}$'s here are trivial as the maps that we compose are identity maps). We define now a homomorphism from the fundamental group of $G(X)$ to the fundamental group of $G(X')$ (the presentations given with respect to $T, T'$ respectively). We focus again on the generators of the 2-simplex $\sigma$. Vertex groups $G_\tau$ are mapped by the identity map to themselves. Each edge of $\sigma$ is subdivided in two edges, one of which we added to $T'$.

We map each edge to the edge of the subdivision that we did not add to $T'$. By the definition of $G(X')$ all relators are satisfied so we have a homomorphism. It remains to see that it is onto. As we remarked before 6 of the new edges are trivial in the group. The other ones can be obtained by successive compositions of the edges contained in the image (together with edges that are trivial). The relations $a^+ b^+ = g_{a,b}(ab)^+$ for composable edges show that all generators corresponding to edges are contained in the image. It is clear that the homomorphism we defined is also 1-1. So it is an isomorphism.

We return now to our treatment of pairs of splittings.

Let $A_1 \ast C_1 B_1$ (or $A_1 \ast C_1$), $A_2 \ast C_2 B_2$ (or $A_2 \ast C_2$) be a pair of hyperbolic-hyperbolic splittings of a group $G$ with corresponding Bass-Serre trees $T_1$, $T_2$. We consider the diagonal action of $G$ on $Y = T_1 \times T_2$ given by

$$g(t_1, t_2) = (gt_1, gt_2)$$
where \( t_1, t_2 \) are vertices of, respectively, \( T_1 \) and \( T_2 \) and \( g \in G \). We consider the quotient complex of groups in the sense of Haefliger. If \( X \) is the quotient complex \( Y/G \) we denote the quotient complex of groups by \( G(X) \).

We give now a detailed description of \( G(X) \). We assume for notational simplicity that the two splittings are \( A_1 \ast C_1 B_1 \) and \( A_2 \ast C_2 B_2 \) (i.e., they are both amalgamated products). One has similar descriptions in the other two cases. In the following, if, for example, the splitting along \( C_1 \) is an HNN-extension, namely, \( G = A_1 \ast C_1 \), then one should just disregard \( B_1, B_1 \), etc. When there are essential differences in the HNN-case we will explain the changes.

Let \( A_1 = T_2/A_1, B_1 = T_2/B_1, A_2 = T_1/A_2, B_2 = T_1/B_2, \Gamma_1 = T_2/C_1, \Gamma_2 = T_1/C_1 \) be the quotient graphs of the actions of \( A_1, B_1, C_1 \) on \( T_2 \) and of \( A_2, B_2, C_2 \) on \( T_1 \). Let \( A_1(A_1), B_1(B_1), C_1(\Gamma_1), A_2(A_2), B_2(B_2), C_2(\Gamma_2) \) be the corresponding Bass-Serre graphs of groups. We note now that if \( e \) is an edge of \( X \) which lifts to an edge of \( T_1 \) in \( Y \) then the subgraph of the barycentric subdivision of \( X \) perpendicular to the midpoint of this edge is isomorphic to \( \Gamma_1 \), and if we consider it as a graph of groups using the groups assigned to the vertices and edges by \( G(X) \) then we get a graph of groups isomorphic to \( C_1(\Gamma_1) \). We identify therefore this one-dimensional subcomplex of \( G(X) \) with \( C_1(\Gamma_1) \) and in a similar way we define a subcomplex of \( G(X) \) isomorphic to \( C_2(\Gamma_2) \) and we call it \( C_2(\Gamma_2) \).

We have the following:

**Lemma 4.1** (Van-Kampen theorem). Let \( \Gamma \) be a connected 1-subcomplex of the barycentric subdivision of \( X \) separating locally \( X \) in two pieces. We consider \( \Gamma \) as a graph of groups where the groups of the 0 and 1-cells of this graph are induced by \( G(X) \). Let \( C \) be the image of the fundamental group of this graph into the fundamental group of \( G(X) \). Then the fundamental group of \( G(X) \) splits over \( C \).

**Proof.** It follows easily from the presentation of the fundamental group of \( G(X) \) given in [H] or in [BH]. A detailed explanation is given in [BH], ch. III, 3.11 (5), p. 552. \( \square \)

Since \( Y = T_1 \times T_2 \) is a product we sometimes use terms "perpendicular" and "parallel" for certain one-dimensional subsets in \( Y \). Formally speaking, let \( p_1, p_2 \) be the natural projections of \( Y \) to \( T_1, T_2 \). Let \( e \) be an edge of \( T_1 \) and \( v \) a vertex of \( T_2 \). For a point \( x \in (e \times v) \subset Y \), we say that the set \( p_1^{-1}(p_1(x)) \) is perpendicular to \( (e \times v) \) at \( x \).

For convenience we say that the \( T_2 \)-direction is 'vertical' and the \( T_1 \) direction is 'horizontal'.
More formally a set of the form \( p_2^{-1}(p_2(x)) \) \((x \in Y)\) is 'horizontal'. We also say that two vertical sets are "parallel". In the same way, all "horizontal sets", which are of the form \( p_2^{-1}(p_2(x)) \), are parallel to each other. Also, those terms make sense for the quotient \( Y/G \) since the action of \( G \) is diagonal, so we may use those terms for the quotient as well.

**Definition 4.2** (Core subgraph). Let \( A \) be a finitely generated group acting on a tree \( T \). Let \( A = T/A \) be the quotient graph and let \( A(\mathcal{A}) \) be the corresponding graph of groups. Let \( T' \) be a minimal invariant subtree for the action of \( A \) on \( T \). We define the core of \( A(\mathcal{A}) \) to be the subgraph of groups of \( A(\mathcal{A}) \) corresponding to \( T'/A \).

Note that the core of \( A(\mathcal{A}) \) is a finite graph. If \( A \) does not fix a point of \( T \) this subgraph is unique. Otherwise it is equal to a single point whose stabilizer group is \( A \). In what follows we will assume that \( C_1, C_2 \) are slender groups. Therefore the core of \( C_1(\Gamma_1) \) (resp. \( C_2(\Gamma_2) \)) is a circle unless \( C_1 \) (resp. \( C_2 \)) acts on \( T_2 \) (resp. \( T_1 \)) by a dihedral action in which case the core is a segment (which might contain more than one edge).

We give now an informal description of the quotient complex of groups \( G(X) \). This description is not used in the sequel but we hope it will help the reader gain some intuition for \( G(X) \). We consider the graphs of groups \( A_1(\mathcal{A}_1), B_1(\mathcal{B}_1), C_1(\Gamma_1) \). (Disregard \( B_1(\mathcal{B}_1) \) if the splitting along \( C_1 \) is an HNN-extension. In what follows, this kind of trivial modification should be made). There are graph morphisms from \( \Gamma_1 \) to \( A_1, B_1 \) coming from the inclusion of \( C_1 \) into \( A_1, B_1 \). (If HNN, both morphisms are to \( A_1 \)). We consider the complex \([0, 1] \times \Gamma_1 \). We glue \( 0 \times \Gamma_1 \) to \( A_1 \) using the morphism from \( \Gamma_1 \) to \( A_1 \) and \( 1 \times \Gamma_1 \) to \( B_1 \) using the morphism from \( \Gamma_1 \) to \( B_1 \). The complex we get this way is equal to \( X \). The vertex groups are the vertex groups of \( A_1(\mathcal{A}_1), B_1(\mathcal{B}_1) \). There are two kinds of edges: the (vertical) edges of \( A_1(\mathcal{A}_1), B_1(\mathcal{B}_1) \) and the (horizontal) edges of the form \([0, 1] \times v \) where \( v \) is a vertex of \( \Gamma_1 \). The groups of the edges of the first type are given by \( A_1(\mathcal{A}_1), B_1(\mathcal{B}_1) \). The group of an edge \([0, 1] \times v \) is the group of \( v \) in \( C_1(\Gamma_1) \).

Finally the group of a 2-cell \([0, 1] \times e \) is just the group of \( e \) where \( e \) is an edge of \( \Gamma_1 \).

**Proposition 4.3** (Core subcomplex). There is a finite subcomplex \( Z \) of \( X \) such that the fundamental group of \( G(Z) \) is equal to the fundamental group of \( G(X) \).
Proof. We will show that there is a subcomplex $\tilde{Z}$, of $T_1 \times T_2$ which is invariant under the action of $G$ and such that the quotient complex of groups corresponding to the action of $G$ on $\tilde{Z}$ is finite. We denote this quotient complex by $G(Z)$. Clearly the fundamental group of $G(Z)$ is equal to the fundamental group of $G(X)$.

We describe now how one can find such a complex $\tilde{Z}$. Let $e = [a, b]$ be an edge of $T_1$ stabilized by $C_1$ and let $p_1 : T_1 \times T_2 \to T_1$ the natural projection. $p_1^{-1}(e)$ is equal to $T_2 \times [a, b]$ and $C_1$ acts on $T_2$ leaving invariant a line $l$, because $C_1$ is slender. $A_1$ acts on $p_1^{-1}(a)$ and $B_1$ acts on $p_1^{-1}(b)$. Let $S_1, S_2$ be, respectively, minimal invariant subtrees of $p_1^{-1}(a), p_1^{-1}(b)$ for these actions. We note that $l \subset S_1, S_2$ since $C_1$ is contained in both $A_1, B_1$. We can take then $\tilde{Z} = G(S_1 \cup \{l \times [0, 1]\} \cup S_2)$.

The construction is similar if the splitting over $C_1$ is an HNN-extension ($G = A_1 \ast C_1$); we simply take $\tilde{Z} = G(S_1 \cup \{l \times [0, 1]\})$ is this case.

Note that $\{l \times (0, 1)\}/C_1$ embeds in $Z$. If the action of $C_1$ on the line $l$ is not dihedral, then it is an (open) annulus and if the action is dihedral then it is a rectangle. In $Z$, some identification may happen at $\{(l \times \{0\}) \cup (l \times \{1\})\}/C_1$, so that, for example, $\{l \times [0, 1]\}/C_1$ can be a closed surface in $Z$.

It is easy to see that $Z = \tilde{Z}/G$ is a finite complex. Vertices of $Z$ are in 1-1 correspondence with the union of vertices of $S_1/A_1 \cup S_2/B_1$ and the latter set is finite. Edges of $Z$ correspond to edges of $S_1/A_1 \cup S_2/B_1$ and vertices of $l/C_1$ while 2-cells are in 1-1 correspondence with edges of $l/C_1$.

One can define $Z$ also using our previous description of $G(X)$: We take $\tilde{Z}$ to be the union of the cores of $A_1(A_1), B_1(B_1)$ and $\{\text{core of } C_1(\Gamma_1) \times [0, 1]\}$. We have then that the fundamental group of $G(Z)$ is $A_1 \ast C_1 B_1$.

To see this, consider the (vertical) graph perpendicular to the midpoint of an edge of the form $v \times [0, 1], (v \in \Gamma_1)$. The fundamental group of the graph is $C_1$. This graph separates $Z$ in two pieces. The fundamental groups of these pieces are $A_1, B_1$. So from Lemma (Van-Kampen theorem) we conclude that the fundamental group of $G(Z)$ is $A_1 \ast C_1 B_1$.

4.2. Enclosing groups.

Definition 4.4 (a set of hyperbolic-hyperbolic splittings). A set, $I$, of splittings of $G$ over slender subgroups is called a set of hyperbolic-hyperbolic splittings if for any two splittings in $I$ there is a sequence of splittings in $I$ of the form $A_i \ast C_i B_i$ or $A_i \ast C_i$, $i = 1, \ldots, n$, such that the first and the last splitting of the sequence are the given splittings and any two successive splittings of the sequence are hyperbolic-hyperbolic.
We remark that a pair of splittings is hyperbolic-hyperbolic (def. 2.1) if and only if the set containing the 2 splittings is a set of hyperbolic-hyperbolic splittings. This follows from prop. 3.4.

**Definition 4.5** (Enclosing graph decomposition). Let $I$ be a set of hyperbolic-hyperbolic minimal splittings of a group $G$ along slender groups. An enclosing group of $I$, denoted by $S(I)$, is a subgroup in $G$, which is a vertex group of some graph decomposition of $G$ with the following properties:

1. There is a graph of groups decomposition, $\Gamma$, of $G$ with a vertex, $v$, such that $S(I)$ is the vertex group of $v$, all edges are adjacent to $v$ and their stabilizers are slender and peripheral subgroups of $S(I)$ (see below for the definition of peripheral subgroups). $S(I)$ contains conjugates of all the edge groups of the splittings of $G$ contained in $I$. Each edge of $\Gamma$ gives a splitting which is elliptic with respect to all splittings in $I$. $\Gamma$ is called an enclosing graph decomposition.

2. (rigidity) Suppose $\Gamma'$ is a graph decomposition of $G$ such that any edge group is slender and gives a splitting which is elliptic-elliptic to any of the splittings in $I$. Then $S(I)$ is a subgroup of a conjugate of a vertex group of $\Gamma'$.

3. $S(I)$ is an extension of the (orbifold) fundamental group of a compact 2-orbifold, $\Sigma$, by a group $F$ which is a normal subgroup of some edge group of a splitting contained in $I$. We say that $\Sigma$ is a base orbifold of $S(I)$, and $F$ is the fiber group. A subgroup of a group in $S(I)$ which is the induced extension of the (orbifold) fundamental group of $\partial \Sigma$ by $F$ is called a peripheral (or boundary) subgroup. We also consider subgroups of $F$ and of induced extensions of the (finite cyclic) groups of the singular points of $\Sigma$, to be peripheral subgroups as well.

4. Each edge of $\Gamma$ gives a minimal splitting of $G$.

**Remark 4.6.**

1. Peripheral subgroups are always proper subgroups of infinite index of an enclosing group.

2. An enclosing groups is not slender except when its base 2-orbifold has a fundamental group isomorphic to $\mathbb{Z} \times \mathbb{Z}$ or $(\mathbb{Z}_2 \ast \mathbb{Z}_2) \times \mathbb{Z}$, i.e., the orbifold is a torus or an annulus whose two boundary circles are of cone points of index 2). Those two cases are the only tricky ones that an enclosing group may have more than one “seifert structure”, i.e., the structure of the extension might be not unique. For example, $\mathbb{Z}^3$ has more than one structures of an extension of $\mathbb{Z}^2$ by $\mathbb{Z}$. 

4.3. Producing enclosing group. We will show that an enclosing graph decomposition with an enclosing vertex group $S(I)$ exists (in fact we construct it) for $I$ given. We start with the simplest case where there are only two splittings in $I$. As a first step, in the following proposition using products of trees, we produce a graph decomposition $\Gamma$ with a vertex group which has properties (1),(2),(3) in Def 4.5. Later we will show that one can also ensure that $\Gamma$ satisfies (4) as well.

One may wonder what happens if we use $A_2, B_2, C_2$ instead of $A_1, B_1, C_1$ to construct $Z$ in Prop 4.3. In fact, if both splittings are minimal, then we get the same finite complex. This is the idea behind the next proposition.

**Proposition 4.7** (Enclosing groups for a pair of splittings). Let $A_1 \star C_1 B_1$ (or $A_1 \star C_1$) and $A_2 \star C_2 B_2$ (or $A_2 \star C_2$) be a pair of hyperbolic-hyperbolic splittings of a finitely generated group $G$ over two slender groups $C_1, C_2$. Suppose that both splittings are minimal. Then there is a graph decomposition of $G$ with a vertex group which has the properties 1, 2, 3 in Def 4.5 for $C_1, C_2$.

**Remark 4.8.** If $G$ does not split over a subgroup of infinite index in $C_1, C_2$ then the two splittings along $C_1, C_2$ are minimal (see remark 2.3). This hypothesis is used in [RS] and [DS] instead of minimality.

We first construct a graph decomposition of $G$ and show that it is the desired one for Prop 4.7 later. Let $T_i$ be the Bass-Serre tree of the splitting over $C_i$. Consider the diagonal action of $G$ on $Y = T_1 \times T_2$. Let $G(X)$ be the quotient complex in the sense of Haefliger. Consider the finite subcomplex of $G(X)$, $G(Z)$, constructed in proposition 4.3. Let $e$ be an edge of $A_1(\Gamma_1)$ lying in the core of $C_1(\Gamma_1) \times \{0,1\}$. In other words $e \in A_1 \cap \{\Gamma_1 \times \{0\}\}$. Consider the (horizontal) graph in $G(Z)$ perpendicular to $e$ at its midpoint. In other words this is the maximal connected graph passing through the midpoint of $e$ whose lift to $T_1 \times T_2$ is parallel to $T_1$. The fundamental group of the graph of groups corresponding to this graph is a subgroup of a conjugate of $C_2$. Since both the assumptions and the conclusions of Prop 4.7 do not change if we take conjugates in $G$ of the splittings along $C_1, C_2$, without loss of generality (by substituting the splitting along $C_2$ by a conjugate in $G$) we can assume that this graph is a subgraph of $C_2(\Gamma_2)$.

Claim. Consider the squares intersecting the core of $C_2(\Gamma_2)$. Then, this set of squares contains all squares of $G(Z)$. To argue by contradiction, we distinguish two cases regarding the set $Z \cap (T_1/C_2 \times \{1/2\})$. We naturally identify the core of $C_2(\Gamma_2)$ with a subgraph in $T_1/C_2 \times \{1/2\}$. 

(i) If $Z$ does not contain the core of $C_2(\Gamma_2)$ then, by Lemma 4.1 (Van-Kampen), $G$ splits over an infinite index subgroup of $C_2$. Moreover, $A_1 \ast_{C_1} B_1$ (or $A_1 \ast_{C_1}$) is hyperbolic-elliptic with respect to this new splitting contradicting our hypothesis. We explain this in more detail. The type of the splitting over $C_1$, i.e., either an amalgamation or HNN extension, does not make difference in this discussion. On the other hand, we may need to make a minor change according to the type of the splitting along $C_2$, which we pay attention to. What requires more attention, because the topology of the base 2-orbifold of $S(I)$ becomes different, is the type of the action of $C_2$ on $T_1$, i.e., dihedral or not, although the type of the action of $C_1$ on $T_2$ is not important once the subcomplex $Z$ is constructed.

Let’s first assume that the splitting along $C_2$ is not dihedral. Let $l_2 \subset T_1$ be the invariant line of the action by $C_2$. The core of $C_2(\Gamma_2)$ is $l_2/C_2$, which is a circle by the assumption we made. The circle $l_2/C_2$ is a retract of a graph $T_1/C_2$, so that $T_1/C_2 - l_2/C_2$ is a forest, i.e., each connected component is a tree. Therefore, if $Z$ does not contain the core of $T_1/C_2 \times \{1/2\}$, then $Z \cap (T_1/C_2 \times \{1/2\})$ is a forest. Let $U_1, \ldots, U_n$ be the connected components of the forest. Then, if we cut $Z$ along each $U_i$, and apply Lemma 4.1 we get a splitting of $G$ along the fundamental group (in the sense of graph of groups) of $U_i$, $K_i$, which is a subgroup of infinite index in $C_2$.

By construction, $K_i$ is contained in $C_1$. Therefore this splitting along $K_i$ is elliptic with respect to $A_1 \ast_{C_1} B_1$ (or $A_1 \ast_{C_1}$). Moreover, $C_1$ is hyperbolic to at least one of the splittings along $K_i$’s. This is because if not, then $C_1$ is contained in a conjugate of $A_2$ or $B_2$ (or $A_2$ in the case that the splitting along $C_2$ is an HNN-extension), which is impossible, since $C_1$ is hyperbolic with respect to $A_2 \ast_{C_2} B_2$ (or $A_2 \ast_{C_2}$). The last claim does not require the theory of complex of groups, but just Bass-Serre theory; since each $U_i$ is a tree, $U_i$ contains a vertex, $u_i$, whose vertex group is $K_i$. Let $e_i = u_i \times [0, 1] \subset Z$. If we delete all (open) squares and edges parallel to $e_i$ (except for $e_i$) in $U_i \times [0, 1]$ from $Z$, the fundamental group (in the sense of Haefliger) does not change, and also the edge $e_i$ gives the splitting of $G$ along $K_i$. If we do the same thing for all $U_i$’s, the subcomplex of $Z$ we obtain is indeed a graph, $\Lambda$, where the edges $e_i$’s are parallel to each other, and no other edges are parallel to them. Note that all of those other edges are the ones which were in the graph $A_2 \cup B_2 \subset Z$ (or just $A_2$ in the case of HNN-extension). Therefore, if $C_1$ is elliptic with respect to all the splittings along $K_i$, it means that $C_1$ is conjugated to the fundamental group (in the sense of Bass-Serre) of a connected component of $\Lambda - \cup_i U_i$, which is a subgraph of either $A_2$ or $B_2$ (or just $A_2$ in the case of HNN-extension). This
means that $C_1$ is a conjugate of a subgroup of either $A_2$ or $B_2$ (or $A_2$ in the case of HNN-extension). This is what we want.

We are still left with the case that $C_2$ is dihedral on $T_1$. The argument only requires a notational change. $T_1/C_2 \times \{1/2\}$ is a forest and we look at each connected component, and appropriately delete all squares and some edges from $Z$ without changing the fundamental group, which is $G$, and get a graph of groups at the end as before. We omit details.

We remark that our argument does not change if the action of $C_1$ on $T_2$ is dihedral or not. So we treated all possibilities in terms of the type of the splittings along $C_1, C_2$ and also the type of the actions of $C_1, C_2$.

(ii) If on the other hand $Z \cap (T_1/C_2 \times \{1/2\})$ is bigger than the core of $C_2(\Gamma_2)$ we can delete from $G(Z)$ the 2-cells (i.e., squares) containing edges of this graph which do not belong to the core of $C_2(\Gamma_2)$ without altering the fundamental group. To explain the reason, let’s first suppose that the action of $C_2$ on $T_1$ is not dihedral. Then the core is topologically a circle, $c_2$. The connected component, $U$, of the finite graph $Z \cap (T_1/C_2 \times \{1/2\})$ which contains the core is topologically the circle with some trees attached. The fundamental group (in the sense of graph of groups) of not only the circle $c_2$, but also the graph $U$ is $C_2$. Therefore, one can remove those trees from $U$ without changing the fundamental group, which is $C_2$.

In $Z$, $U \times (0, 1)$ embeds, and one can remove the part $(U - c_2) \times (0, 1)$ from $Z$ without changing its fundamental group. One can see this using the presentation of the fundamental group of $G(Z)$. For the reader’s convenience we give also an argument using the action of $G$ on $\tilde{Z}$. Let $p_2: \tilde{Z} \to T_2$ be the natural projection from $\tilde{Z}$ to $T_2$. If $e$ is an edge of $T_2$, $p_2^{-1}(e)$ is a connected set of the form $L_e \times e$. Let $\text{Stab}(e)$ be the stabilizer of $e$ in $T_2$ (which is a conjugate of $C_2$) and $l_e$ the line invariant under $\text{Stab}(e)$ on $T_1$. Then by the discussion above $L_e$ contains $l_e$ and is connected. We will show that $L_e = l_e$. Indeed if not we consider the subcomplex of $\tilde{Z}$ obtained by the union of $l_e \times e$ over all edges $e \in T_2$ with $p_2^{-1}(v)$ over all vertices $v \in T_2$. Let’s call this complex $\tilde{Z}_1$. It is clear that $\tilde{Z}_1$ is connected, simply connected and invariant under the action of $G$. The quotient complex of groups $G(Z_1)$ is a subcomplex of $G$. If for some $e$, $L_e \neq l_e$ $G(Z_1)$ is properly contained in $G(Z)$. By the preceding discussion it follows that the splittings over $C_2, C_1$ are hyperbolic-elliptic, a contradiction.

In the case when the action of $C_2$ on $T_1$ is dihedral, then the core is a segment, and $Z \cap (T_1/C_2 \times \{1/2\})$ is a graph which is the segment with some finite trees attached. In this case one can delete the squares
which contain those trees from $Z$ without changing the fundamental group of $Z$, which is $G$.

But then the complex obtained, after deleting those unnecessary squares, does not contain the core of $C_1(\Gamma_1)$, because there are no other squares in $Z$ than the ones which contains the core of $C_1(\Gamma_1)$, which implies that $G$ splits over an infinite index subgroup of $C_1$, and this new splitting is elliptic-hyperbolic with respect to $A_2 \ast C_2$, $B_2$ (or $A_2 \ast C_2$), which is a contradiction. The last part follows from the same consideration as the last part of the case (i), so we omit the details. We showed the claim.

From this claim it follows that if we apply the same construction of $Z$ in Prop 4.3 using $A_2, B_2$ instead of $A_1, B_1$, the resulting complex contains the same set of squares.

We describe the topology of $Z$. If none of $C_1, C_2$ acts as a dihedral group the above implies that every edge in $Z$ which is a side of a 2-cell lies on exactly two 2-cells. Therefore the link of every vertex of $Z$ is a union of disjoint circles and points. It then follows that the union of 2-cells in $Z$ is topologically a closed surface with, possibly, some (vertex) points identified. $Z$ is this 2-dimensional object with some graphs attached at vertices; if one deletes from $Z$ those graphs including attaching vertices and identified vertices, one obtains a compact surface with punctures.

If at least one of $C_1, C_2$ acts as a dihedral group then $Z$ is topologically a compact surface with boundary with, possibly, some points identified and some graph attached. Therefore the links of vertex points on this surface are disjoint unions of circles, segments and points. The boundary components come from the dihedral action(s), and there are at most 4 connected components. To see this, suppose that only $C_1$ is dihedral on $T_2$, and let $l_1$ be its invariant line. Then the rectangle $l_1/C_1 \times (0, 1)$ embeds in the surface. Let $u_1, u_2$ be the boundary points of the segment $l_1/C_1$. Then the edges $u_1 \times [0, 1], u_2 \times [0, 1]$ are exactly the boundary of the surface. Note that $u_1 \times (0, 1)$ embeds, but possibly, $u_1 \times [0, 1]$ may become a circle in $Z$. $u_1 \times [0, 1]$ and $u_2 \times [0, 1]$ may become one circle in $Z$ as well. Therefore, the surface has at most two boundary components in this case. If $C_2$ is dihedral on $T_1$ as well, then there are two more edges which are on the boundary, so that there are at most 4 boundary components.

**Remark 4.9** (Priority among splittings). Let $Z$ be the complex we constructed in the proof of Prop. 4.3. Let $l_1$ be the invariant line in $T_2$ by $C_1$ and $c_1 = l_1/C_1$. We may call $c_1$ the core of $C_1$. If the action of $C_1$ is dihedral, then $c_1$ is a segment, or else, a circle. The core $c_1$ embeds
in $Z$, and if we cut $Z$ along $c_1$ we get (not a conjugate, but exactly) the splitting $A_1 *_{C_1} B_1$ (or $A_1 *_{C_1}$). Similarly, let $l_2$ be the invariant line in $T_1$ by $C_2$, and $c_2 = l_2/C_2$. As before, this core $c_2$ is either a segment or a circle, and embeds in $Z$. Cutting $Z$ along $c_2$, we get a splitting of $G$ along a conjugate of $C_2$. But, unlike the splitting along $C_1$, this splitting may be different from the original splitting along $C_2$. This point becomes important later, that we can keep at least one splitting unchanged (along $C_1$ in this case) in $Z$, because we gave priority to the splitting along $C_1$ over $C_2$ when we constructed $Z$.

However, it is true that if $G$ does not split along a subgroup in $C_1$ of infinite index, then the new splitting along $C_2$ obtained by cutting $Z$ along $c_2$ is the same (up to conjugation) as the original one. It is because that under this assumption, $Z$ does not have any graphs attached, and it is just a squared complex.

Although the new splitting along $C_2$ may be different from the original one, it is hyperbolic-hyperbolic with respect to the splitting along $C_1$. It follows from Lemma 3.4 that the new splitting along $C_2$ is minimal.

We now explain how to obtain the desired graph of groups decomposition of $G$. First, let the group $S = S(C_1, C_2)$ be the subgroup of the fundamental group of $G(Z)$ corresponding to the subcomplex of $G(Z)$ which is the union of the cores of $C_1(\Gamma_1)$ and $C_2(\Gamma_2)$, namely, $S$ is the image in $G$ of the fundamental group (in the sense of a graph of groups) of this union. Here we use Haefliger’s notation; the cores of $\Gamma_1, \Gamma_2$ are contained in the barycentric subdivision of $Z$ which is used in the definition of its fundamental group.

Using lemma 4.1 (Van-Kampen theorem) we show that $G(Z)$ is the fundamental group of a graph of groups, which we call $\Gamma$. The vertices of this graph are as follows: there is a vertex for each connected component of $Z$ minus the cores of $C_1(\Gamma_1)$ and $C_2(\Gamma_2)$. The vertex group is the fundamental group of the component in Haefliger’s sense. We remark that each such component contains exactly one vertex group of the ‘surface’ piece of $Z$ with, possibly, a graph attached at the vertex. The fundamental group of the component is then the fundamental group of the graph of groups of the attached graph (and is equal to the group of the vertex if there is no graph attached).

There is also a vertex with group $S(C_1, C_2)$. There is an edge for each component of the intersection between the union of the cores of $C_1(\Gamma_1)$, $C_2(\Gamma_2)$ and each vertex component.

Note that such an intersection is topologically a circle or a segment (this happens only when at least one of the actions of $C_i$ is dihedral).
As this intersection is a subgraph of the union of the cores of $C_1(\Gamma_1)$, $C_2(\Gamma_2)$ there is a group associated to it, namely the image of the fundamental group of this subgraph in $G(Z)$. Note that the graph of groups that we described here is a graph of groups in a generalized sense, i.e., the edge groups do not necessarily inject into the vertex groups. Note that every vertex group except $S$ injects in $G$.

To understand the group $S$ better, let $U$ be the union of the cores of $C_1(\Gamma_1), C_2(\Gamma_2)$, which is a graph in $Z$. If we consider a small closed neighborhood, $\bar{U}$, of $U$ in $Z$, it is a compact surface with boundary in general. We may consider the graph, $P$, corresponding to an edge, $e$, of $\Gamma$ as a subset in the boundary of $\bar{U}$, which is either a circle or a segment. Let $F < G$ be the stabilizer of a square in $Z$. Then, the fundamental group, in the sense of Haefliger, of $P$ is an extension of $\mathbb{Z}$ (when $P$ is a circle) or $\mathbb{Z}_2 * \mathbb{Z}_2$ (when $P$ is a segment) by $F$. Since the group $F$ is a subgroup of $G$, the image of the fundamental group of $P$ in $G$ is an extension of (1) $\mathbb{Z}$, (1') $\mathbb{Z}_n$, (1'') the trivial group; (2) $\mathbb{Z}_2 * \mathbb{Z}_2$, (2') $\mathbb{Z}_2$, or (2'') a finite dihedral group of order $2n$ by $F$. As a consequence, the image of the fundamental group of $U$ (as well as $\bar{U}$) in $G$, which is $S$ by our definition, is an extension of the orbifold fundamental group of a 2-orbifold, $\Sigma$, by $F$ such that $\Sigma$ is obtained from the compact surface $\bar{U}$ by adding to each $P$ (1) nothing, (1') a disk with a cone point of index $n$ at the center, (1'') a disk; (2) nothing, (2') a half disk such that the diameter consists of cone points of index 2 (in other words, we just collapse the segment $P$ to a point), or (2'') a half disk such that the diameter consists of cone points of index 2 except for the center whose index is $2n$. The orbifold fundamental group of this 2-orbifold is $S$. Note that $\Sigma$ is no more embedded in $Z$, but the surface $\bar{U}$ is a subsurface of $\Sigma$.

**Remark 4.10.** By our construction of $Z, U, \Sigma$, there is a simple closed curve or a segment (the core) on $\Sigma$ which corresponds to each of $C_1, C_2$. If we cut $\Sigma$ along it, we obtain a splitting of $S$ along $C_1$ or $C_2$, respectively, which also gives a splitting of $G$, as we do by cutting $Z$. Although one of them may be different from the original one, both of them are minimal (use Prop 3.4).

### 4.4. Proof of Prop 4.7

**Proof.** We will show that the graph decomposition $\Gamma$ with $S(C_1, C_2)$ we constructed satisfies the properties 1,2,3 in Def 4.5. In fact $S(C_1, C_2)$ is an enclosing group for $C_1, C_2$ although we may need to modify $\Gamma$ so that the property (4) holds as well. We will discuss this point later.
(3) is clear. By construction, \( S \) is an extension of the orbifold fundamental group of the compact 2-orbifold \( \Sigma \) by a group \( F \), which is the edge stabilizer subgroup in \( C_1 \) when it acts on the tree \( T_2 \), hence a normal subgroup of \( C_1 \). \( F \) is slender since it is a subgroup of a slender group \( C_1 \).

(1). Let \( v \) be the vertex of \( \Gamma \) whose vertex group is \( S \). By our construction, all edges are adjacent to \( v \). The edge group, \( E \), of an edge is slender since there is the following exact sequence: \( 1 \to F \to E \to Z \to 1 \) such that the group \( Z \) is either the trivial group, the fundamental group of one of the singular points of \( \Sigma \), so that isomorphic to \( \mathbb{Z}_n \), or a subgroup of the (orbifold) fundamental group of \( \partial \Sigma \), so that isomorphic to \( \mathbb{Z}_2 \ast \mathbb{Z}_2 \). In any case, \( E \) is slender and a peripheral subgroup of \( S \). Clearly \( S \) contains conjugates of \( C_1, C_2 \), because \( \Sigma \) contains the graph \( U \), which is the union of the cores for \( C_1, C_2 \). By construction of \( \Gamma \), all vertex groups except for \( S \) are elliptic on both \( T_1 \) and \( T_2 \), so that all edge groups of \( \Gamma \) are elliptic on \( T_1, T_2 \) since they are subgroups of vertex groups. We showed (1).

To prove that enclosing groups are ‘rigid’, namely the property (2) in Def 4.5, we recall some results from Bass-Serre theory.

**Proposition 4.11** (Cor 2 in §6.5 of [Se]). Suppose \( G \) acts on a tree. Assume \( G \) is generated by \( s_1, \ldots, s_l \) and all \( s_i \) and \( s_i s_j (i \neq j) \) are elliptic on the tree. Then \( G \) is elliptic.

Let \( c \) be a simple closed curve on \( Z \) which avoids vertices of \( Z \). Using repeated barycentric subdivisions of \( G(Z) \) we see that \( c \) is homotopic in \( Z - Z^{(0)} \) to a curve lying in the 1-skeleton of the iterated barycentric subdivision. Let’s assume then that \( c \) is a curve lying in the 1-skeleton of an iterated barycentric subdivision. Then cutting \( Z \) along \( c \), we get a splitting of \( G \) along the fundamental group (in the sense of graph of groups) of \( c \), (4.11). If the fundamental group of \( c \) is not contained to a vertex group, then the splitting induced by \( c \) is hyperbolic-hyperbolic with respect to either the splitting along \( C_1 \) or \( C_2 \), so that in particular, it is non trivial and a minimal splitting. We call such simple closed curves \( c \) essential.

In the case \( Z \) has a boundary (i.e., there exists an edge which is contained in only one square), let \( c \) be an embedded segment whose boundary points are in the boundary of \( Z \). Cutting \( Z \) along \( c \), one also obtains a splitting of \( G \) along the fundamental group of \( c \). If the fundamental group of \( c \) is not contained to the fundamental group of \( \partial Z \) then this splitting is hyperbolic-hyperbolic with respect to at least one of the splittings along \( C_1 \) and \( C_2 \), so it is non trivial and minimal. We call such segments \( c \) essential. We remark that essential simple
closed curves and essential segments correspond to subgroups of the fundamental group of \( \Sigma \).

If \( \partial \Sigma \) contains segments of singular points of index two (reflection points) we denote this set by \((\partial \Sigma; 2)\).

**Corollary 4.12.** If all splittings over the slender groups which are represented by essential simple closed curves on \( \Sigma \) and essential embedded segments are elliptic on \( \Gamma \), then \( S = S(C_1, C_2) \) is elliptic on \( \Gamma \).

**Proof.** We explain how to choose a finite set of generators of \( S \) so that we can apply Prop 4.11. First choose a finite set of generators \( f_i \) of \( F \) (\( F \) is slender, so that finitely generated).

Let's first assume that \((\partial \Sigma; 2) = \emptyset \). Then, one can choose a set of non-boundary simple closed curves \( c_1, \ldots, c_l \) on \( \Sigma \) so that all \( c_ic_j(i \neq j) \) are also represented by simple closed curves and that the elements corresponding to \( c_i \) generate the fundamental group of \( \Sigma \). Each \( c_i \) or \( c_ic_j(i \neq j) \) represents a slender subgroup in \( G \) with the fiber group \( F \), which gives a splitting of \( S \). By assumption all of those splittings are elliptic on \( \Gamma \). Therefore we apply Prop 4.11 to \( S \) with the generating set of \( \{f_i, c_j\} \) and conclude that \( S \) is elliptic on \( \Gamma \).

In the case \((\partial \Sigma; 2) \neq \emptyset \), we need extra elements represented by embedded segments \((s, \partial s) \subset (\Sigma, (\partial \Sigma; 2)) \). Put an order to the connected components of \((\Sigma, (\partial \Sigma; 2)) \), and take a finite set of embedded segments, \( s_i \), such that any adjacent (in the order) pair of components of \((\partial \Sigma; 2) \) is joined by a segment. Then the set \( \{f_i, c_j, s_k\} \) generates a subgroup \( S' \subset S \) of finite index. By Prop 4.11 \( S' \) is elliptic on \( \Gamma \), so that so is \( S \).

We now show that enclosing groups are ‘rigid’.

**Lemma 4.13** (Rigidity). Let \( A_1 \ast C_1 B_1 \) (or \( A_1 \ast C_1 \)) and \( A_2 \ast C_2 B_2 \) (or \( A_2 \ast C_2 \)) be as in proposition 4.11 and let \( S = S(C_1, C_2) \) be the group constructed above. Suppose \( \Gamma' \) is a graph decomposition of \( G \) such that any edge group is slender and elliptic to both of the splittings over \( C_1, C_2 \). Then \( S \) is a subgroup of a conjugate of a vertex group of \( \Gamma' \).

**Proof.** \( \Gamma \) denotes the graph of groups decomposition we constructed with \( S \) as a vertex group. As we pointed out in Remark 4.13, although the splitting of \( G \) over \( C_2 \) which we obtain by cutting \( \Sigma \) along the core curve for \( C_2 \) may be different from the original one, this splitting is minimal, because it is hyperbolic-hyperbolic to the (original) splitting over \( C_1 \), (see Prop. 3.3). Let \( T, T' \) be the Bass-Serre trees of \( \Gamma, \Gamma' \). Our goal is to show that \( S \) is elliptic on \( T' \). Let \( c \subset \Sigma \) be an essential simple closed curve or \((s, \partial s) \subset (\Sigma, (\partial \Sigma; 2)) \) an embedded essential segment, and \( C < S \) the
group represented by it. If we show that \( C \) is elliptic on \( T' \), then Prop 2.9 implies that \( S \) is elliptic on \( T' \). The splitting of \( G \) over \( C \) by cutting \( \Sigma \) along \( c \) or \( s \) is minimal by Prop. 3.4 since it is hyperbolic-hyperbolic to one of the minimal splittings over \( C_1, C_2 \).

Let \( e \) be an edge of \( \Gamma' \) with edge group, \( E \). Since the subgroup \( E \) is elliptic with respect to the splittings over \( C_1, C_2 \), it fixes a vertex of \( T_1 \times T_2 \). Therefore it is contained in a conjugate of a vertex group of \( G(Z) \), which is not \( S \). It follows that the group \( E \) is elliptic with respect to the splitting of \( G \) over \( C \). Since the splitting along \( C \) is minimal, by Prop 3.4 it is elliptic-elliptic with respect to the splitting of \( G \) over \( E \) obtained from \( \Gamma' \) by collapsing all edges but \( e \). Since \( e \) was arbitrary, the subgroup \( C \) is elliptic on \( T' \). □ (lemma 4.13).

Lemma 4.13 implies (2) in Def 4.5. We have verified the items (1), (2), (3) in Def 4.5 for \( S(C_1, C_2) \) which finishes the proof of Prop 4.7. □ (Prop 4.7).

Remark 4.14 (Maximal peripheral subgroup). Note that the subset of \( \partial \Sigma \) which is produced by the cutting of \( Z \) is exactly \( \partial \Sigma - \) interior of(\( \partial \Sigma; 2 \)). Let \( c \) be a connected component of this subset, and \( E \) the corresponding (peripheral) subgroup of \( S \). Let’s call such peripheral subgroup of \( S \) maximal. \( E \) is an edge group of \( \Gamma \) by our construction, so that any maximal peripheral subgroup of \( S \) is an edge group of \( \Gamma \). For example when the fiber group \( F \) is trivial, \( \Sigma \) is a 2-manifold with boundary. Then the infinite cyclic subgroup in \( S = \pi_1(\Sigma) \) corresponding to each boundary component of \( \Sigma \) is an edge group of \( \Gamma \). In this sense, \( \Sigma \) does not have any free boundary points.

4.5. Producing enclosing graph decomposition. We now discuss the property (4) of Def 4.5. In general the edges of \( \Gamma \) we obtained in Prop 4.7 may give non-minimal splittings. See the example.

However, by applying Prop 5.7 to \( \Gamma \), there is a refinement (see Def 3.6) of \( \Gamma \) such that all edges give minimal splittings of \( G \). We then verify that the refinement satisfies all the properties of Def 4.5, most importantly, \( S \) remains a vertex group, and is the enclosing group of the decomposition we get.

Example. This example is suggested by V. Guirardel to us. We thank him. Let \( G = Z^3 * A \) such that \( A \) is a non-trivial group. Fix free abelian generators \( a_1, a_2, a_3 \) of \( Z^3 \). Write \( Z^3 \) as an HNN-extension \( Z^2 *_{Z^2} \) such that \( Z^2 = \langle a_2, a_3 \rangle \) and the stable letter is \( a_1 \). This extends to an HNN-extension \( G = (Z^2 * A) *_{Z^2} \) over \( Z^2 = \langle a_2, a_3 \rangle \). Let \( T_1 \) be the Bass-Serre tree of this splitting. We abuse the notation and call the splitting \( T_1 \) as well. Similarly, we obtain HNN-extensions \( T_2 \) and \( T_3: G = (Z^2 * A) *_{Z^2} \)
over \( \mathbb{Z}^2 = \langle a_1, a_3 \rangle \), and \( \langle a_1, a_2 \rangle \), with Bass-Serre trees \( T_2, T_3 \). For \( i \neq j \), the pair of splittings \( T_i, T_j \) is hyperbolic-hyperbolic. Each splitting \( T_i \) is minimal, because if \( G = (\mathbb{Z}^2 * A) * \mathbb{Z}^2 \) was not minimal, then the corresponding HNN-extension \( \mathbb{Z}^3 = \mathbb{Z}^2 * \mathbb{Z}^2 \) would give (for example, by taking product of trees) a splitting of \( \mathbb{Z}^3 \) over \( \mathbb{Z} \) or the trivial group, which is impossible. We obtain a graph decomposition for the pair \( T_1, T_2 \) by taking product of trees: 

\[
G = \mathbb{Z}^3 * \mathbb{Z} * (\mathbb{Z} * A)
\]

such that this is an amalgamation over \( \langle a_3 \rangle \) with two vertex groups \( \mathbb{Z}^3 \) and \( \langle a_3 \rangle * A \). Let’s call this decomposition, and its Bass-Serre tree \( T \). The vertex group \( \mathbb{Z}^3 \) is the enclosing group such that the base is a torus with the fundamental group \( \langle a_1, a_2 \rangle \) and the fiber group is \( \langle a_3 \rangle \). This splitting along \( \langle a_3 \rangle \) is not minimal, because it is hyperbolic-elliptic to \( T_3 \). We now demonstrate how to handle this problem using Prop 3.7. Following the first proof of Prop 3.7, we refine \( T \) such that all edge gives a minimal splitting. Take product of trees of \( T, T_3 \). The core is topologically an annulus, which contains one square, and three edges, where two of them are vertical, and they are loops. Since \( A \) fixes a vertex of \( T \times T_3 \), \( A \) is a vertex group of the core. We remove the (open) square, and also one vertical loop, whose edge group is trivial, appropriately without changing the fundamental group. We obtain a graph decomposition with two edges, \( G = A * \mathbb{Z}^2 * \mathbb{Z}^2 \) such that both \( \mathbb{Z}^2 \) are \( \langle a_1, a_2 \rangle \). Next, collapse the other vertical loop, whose edge group is \( \langle a_1, a_2 \rangle \), in the graph decomposition. We are left with the horizontal edge, whose edge group is trivial, and obtain \( G = \mathbb{Z}^3 * A \), which is a refinement of \( T \). This is an enclosing decomposition for \( T_1, T_2 \) with the enclosing vertex group \( \mathbb{Z}^3 \).

**Lemma 4.15** (Refinement of \( \Gamma \)). There is a refinement, \( \Gamma' \), of \( \Gamma \) such that

1. each edge of \( \Gamma' \) gives a minimal splitting of \( G \),
2. each edge group of \( \Gamma' \) is a subgroup of some edge group of \( \Gamma \),
3. \( S = S(C_1, C_2) \) remains a vertex group of \( \Gamma' \),
4. each edge group is a peripheral subgroup of \( S \).

*Proof.* Apply Prop 3.7 to \( \Gamma \) and obtain a refinement \( \Gamma' \) such that each edge of \( \Gamma' \) gives a minimal splitting of \( G \). Each edge group, \( E \), of \( \Gamma' \) is a subgroup of some edge group of \( \Gamma \). Therefore, by Prop 4.7, \( E \) is elliptic to both splittings of \( G \) along \( C_1, C_2 \). Since both of the splitting along \( C_1, C_2 \) are minimal, each of them is elliptic-elliptic with respect to the splitting over \( E \). Therefore by Lemma 4.13, \( S = S(C_1, C_2) \) is a subgroup of a conjugate of some vertex group, \( V \), of \( \Gamma' \). But since \( \Gamma' \) is a refinement of \( \Gamma \) and \( S \) is a vertex group of \( \Gamma \), \( S \) is a conjugate of \( V \).
$E$ is a peripheral subgroup of $S$ since it is a subgroup of a peripheral subgroup.

We collapse all edges in $\Gamma'$ which are not adjacent to the vertex whose vertex group is $S$, and still call it $\Gamma'$. Then by Prop 4.7 and Lemma 4.15, $\Gamma'$ is an enclosing graph decomposition with an enclosing vertex group $S(C_1, C_2)$ for the splittings along $C_1, C_2$. We have shown the following.

**Proposition 4.16** (Enclosing decomposition for a pair of splittings). Let $A_1 \ast_{C_1} B_1$ (or $A_1 \ast_{C_1}$) and $A_2 \ast_{C_2} B_2$ (or $A_2 \ast_{C_2}$) be a pair of hyperbolic-hyperbolic splittings of a finitely generated group $G$ over two slender groups $C_1, C_2$. Suppose that both splittings are minimal. Then an enclosing graph decomposition of $G$ exists for those two splittings.

**5. JSJ-decomposition**

5.1. **Dealing with a third splitting.** Let $G$ be a finitely presented group. We want to show that an enclosing graph decomposition exists for a set, $I$, of hyperbolic-hyperbolic minimal splittings of $G$ along slender groups. We already know this when $I$ contains only two elements by Prop 4.16. We now discuss the case when $I$ has three elements.

**Proposition 5.1** (Enclosing group). Let $I$ be a set of hyperbolic-hyperbolic splittings (Def 4.4) of a finitely generated group $G$. Suppose all of them are minimal splittings. Suppose that $I$ consists of three splittings. Then an enclosing graph decomposition of $G$ exists for $I$.

**Proof.** Suppose that the three splittings in $I$ are along $C_1, C_2, C_3$. We may assume that the pair of the splittings along $C_1, C_2$, and also the pair for $C_2, C_3$ are hyperbolic-hyperbolic. Apply Prop 4.7 to the first pair, and obtain an enclosing graph decomposition, $\Gamma$, with the vertex group $S = S(C_1, C_2)$. We remark that $S(C_1, C_2)$ depends on the two splittings, not only the two subgroups. Note that by cutting the 2-orbifold $\Sigma$ for $S$ along a simple closed curve or a segment corresponding to each of $C_1, C_2$, we obtain a minimal splitting of $G$ along $C_1$, and $C_2$, respectively. Although this splitting along $C_2$ is possibly different from the original one, it is still a minimal splitting so it is hyperbolic-hyperbolic with respect to the splitting along $C_3$.

Let's assume first that the group $C_3$ is elliptic with respect to $\Gamma$. Then $C_3$ is a subgroup of a conjugate of $S$. This is because if $C_3$ was a subgroup of a conjugate of a vertex group of $\Gamma$ which is not $S$, then the group $C_3$ is elliptic with respect to both of the (new) splittings of $G$ along $C_1, C_2$ which we obtain by cutting $\Sigma$. It then follows that the splitting along $C_3$ would be elliptic-elliptic with respect to both of the
original splittings of $G$ along $C_1, C_2$, which is a contradiction. Let $\Gamma'$ be a refinement of $\Gamma$ which we obtain by Prop 4.16 which is an enclosing graph decomposition for the splittings along $C_1, C_2$. We claim that $\Gamma'$ with an enclosing vertex group $S$ is an enclosing decomposition for the three splittings. First, the properties 2,3,4 are clear. To verify (the non-trivial part of) the property 1, let $e$ be an edge of $\Gamma'$ with edge group, $E$. We want to show that the group $E$ is elliptic with respect to the splitting in $I$ along, $C_3$. We know that $C_3 < S$ by our assumption. Since the group $S$ is elliptic with respect to the splitting of $G$ along $E$ which the edge $e$ gives, so is $C_3$. Since both of the splittings along $C_3$ and $E$ are minimal, it follows that the group $E$ is elliptic with respect to the splitting along $C_3$. This proves property 1.

We treat now the case that $C_3$ is hyperbolic with respect to $\Gamma$. This is the essential case. Let $T_\Gamma, T_3$ be, respectively, the Bass-Serre trees of $\Gamma$ and the splitting over $C_3$. Since the splitting along $C_3$ is minimal, there is at least one edge, $e$, of $\Gamma$ such that the splitting of $G$ which the edge $e$ gives, along the edge group, $E$, is hyperbolic-hyperbolic with respect to the splitting along $C_3$. The group $E$ acts hyperbolically on $T_3$. $F$ denotes the fiber group of $S(C_1, C_2)$. We have the following lemma:

**Lemma 5.2 (Elliptic fiber).** Letting $E$ act on $T_3$, we obtain a presentation

$$E = \langle t, F | tFt^{-1} = \alpha(F) \rangle,$$

where $\alpha \in \text{Aut}(F)$ or

$$E = L *_F M,$$

where $[L : F] = [M : F] = 2$.

**Proof.** We first show that $F$ is elliptic on $T_3$. To argue by contradiction, assume that there is $a \in F$ acting hyperbolically on $T_3$. Then the pairs $C_1, C_3$ and $C_2, C_3$ are hyperbolic-hyperbolic. Let $F_1$ be the fiber of the enclosing group $S(C_1, C_3)$ corresponding to the pair $C_1, C_3$ and let $F_2$ be the fiber of $S(C_2, C_3)$. We claim that there is $w_1 \in F_1$ which does not lie in any conjugate of $F$. Indeed $F_1$ is contained in a conjugate of $C_1$. $C_1$ acts on $T_2$ hyperbolically preserving an axis which is stabilized by $F$. If $F_1$ contains an element, $w_1$, that acts hyperbolically on $T_2$ then $w_1$ does not lie in any conjugate of $F$. Otherwise $F_1$ fixes an axis and it is contained in a conjugate of $F$. In this case consider the actions of $C_1$ on $T_2$ and $T_3$. By passing, if necessary (in the dihedral action case), to a subgroup of index 2 we can assume that $C_1$ is generated by $\langle t, F \rangle$ where $t$ acts hyperbolically on $T_2$. Similarly $C_1$ is generated by some $x$ acting hyperbolically on $T_3$ and a conjugate of $F_1$ which is contained
in $F$. Since $C_1$ acts hyperbolically on $T_2$, $x$ acts hyperbolically on $T_2$ and $x = tf$ where $f \in F$. Since $a$ acts hyperbolically on $T_3$, we have $a = x^k f'$ with $f' \in F$. Then $t^{-k} a$ acts elliptically on $T_3$, therefore it lies in $F$. But this is a contradiction since $t \not \in F$.

Now if $b \in C_3$ either $b \in F_2$ or $b^k w_1^i \in F_2$. This is because if $b \not \in F_2$, $w_1, b$ act both as hyperbolic elements on $T_2$ and they fix the same axis (since $w_1, b \in C_3$ and $C_3$ is slender). But then $b^k \in S(C_1, C_2)$. Since the translation length of any hyperbolic element of $C_3$, for its action on $T_2$, is a multiple of a fixed number we can pick the same $k$ for all $b \in C_3$. So one has $C_3^k \subset S(C_1, C_2)$. Therefore if we consider the graph of groups corresponding to $S(C_1, C_2)$ and its Bass-Serre tree then $C_3$ fixes a vertex of this tree. Therefore either it fixes the vertex stabilized by $S(C_1, C_2)$ or $C_3^k$ is contained in the edge stabilizer of an edge adjacent to the vertex stabilized by $S(C_1, C_2)$. But in the first case we have that $C_3 \subset S(C_1, C_2)$ and in the second it is impossible that $C_3$ is hyperbolic-hyperbolic with respect to, say, $C_1$. We conclude that there is no $a \in F$ acting hyperbolically on $T_3$. Therefore $F$ is elliptic on $T_3$.

On the other hand the splitting over $C_3$ is hyperbolic-hyperbolic with respect one of the splittings used to construct $\Gamma$. Let’s say that it is hyperbolic-hyperbolic with respect to the splitting over $C_1$. Since $F \subset C_1$ and $F$ fixes an axis of $T_3$ a conjugate of $F$ is contained in $C_3$. Therefore since the splitting over $C_3$ is hyperbolic-hyperbolic with respect to the splitting over $E$, $E$ contains a conjugate of $F$. Moreover this conjugate of $F$ is an infinite index subgroup of $E$. This clearly implies that

$E = \langle t, F | tFt^{-1} = \alpha(F) \rangle$

where $\alpha$ is an automorphism of $F$ or that

$E = L \ast_F M,$

where $[L : F] = [M : F] = 2$. \hfill \Box (Lemma 5.2).

Let $\{e_i\}$ be the collection of the edges of $\Gamma$ whose edge groups, $E_i$, are hyperbolic on $T_3$, and $\{d_j\}, \{D_j\}$ the collections of the rest of the edges and their edge groups. Let $T_{E_i}$ be the Bass-Serre tree of the splitting of $G$ we obtain by collapsing all edges of $\Gamma$ but $e_i$. The group $C_3$ is hyperbolic on $T_{E_i}$ by the way we took $e_i$, and the splitting along $C_3$ is minimal. Consider the diagonal action of $G$ on $T_3 \times T_\Gamma$. In the same way as in the proof of Proposition 1.13 we can show that there is a subcomplex $\tilde{Z}$ of $T_3 \times T_\Gamma$ which is invariant by $G$ such that $\tilde{Z}/G$ is finite. We explain this in detail: Let $S$ be the enclosing group of $\Gamma$ and let $T_S$ be the minimal invariant subtree of $T_3$ for the action of $S$. For each $E_i \subset S$ let $l_i$ be the invariant line for the action of $E_i$ on $T_3$. 

Finally for each $D_j$ we pick a vertex $v_j$ on $T_3$ fixed by $D_j$. Let $\tilde{e}_i$ be a lifting of $e_i$ to $T_1 \times T_2$ with an endpoint on $l_i$ and $\tilde{d}_j$ a lifting of $d_j$ on $T_1 \times T_2$ with an endpoint on $v_j$. Let

$$Z_1 = T_3 \cup (l_i \times \tilde{e}_i) \cup (\tilde{d}_j)$$

where the union is over all the $e_i's, d_j's$. We take then $Z$ to be the complex obtained by the translates $GZ_1$.

As before, we give a description of $Z$ using gluings of graphs. Let $\{k_i\}$ be the vertices of $\Gamma$ other than the one for $S$, and $\{K_i\}$ their vertex groups. Let $T_3 / S = \mathcal{S}, T_3 / E_i = \mathcal{E}_i, T_3 / D_i = \mathcal{D}_i, T_3 / K_i = \mathcal{K}_i$ be the quotient graph of groups. Since the action of $E_i$ on $T_3$ is hyperbolic and $E_i$ is slender, there is an invariant line $l_i$ in $T_3$ by $E_i$ and the core of $\mathcal{E}_i$ is $c_i = l_i / E_i$, which is topologically a segment if the action of $E_i$ on $T_3$ is dihedral, or else a circle. A core of $\mathcal{D}_i$ is a vertex since the action is elliptic. Let’s denote a core of a graph of groups, $A$, as $\text{co}(A)$.

A core complex, $Z$, of the diagonal action of $G$ on $T_1 \times T_3$ is given as follows:

$$Z = \text{co}(\mathcal{S}) \bigcup \bigcup_i \text{co}(K_i) \bigcup \bigcup_i (\text{co}(\mathcal{E}_i) \times [0, 1]) \bigcup \bigcup_i (\text{co}(\mathcal{D}_i) \times [0, 1])$$

Note that each $\text{co}(\mathcal{E}_i) \times [0, 1]$ and $\text{co}(\mathcal{D}_i) \times [0, 1]$ is attached to $\text{co}(\mathcal{S}) \bigcup \bigcup_i \text{co}(K_i)$ by the graph morphism induced by the homomorphism of each of $E_i, D_j$ to $S$ and to $K_k$ given in $\Gamma$.

$Z$ is a finite complex, and the fundamental group in the sense of complexes of groups (let’s call such fundamental group $H$-fundamental group in this proof) is $G$. Let $C_3 = T_1 / C_3$. Since $C_3$ is slender, and the action of $C_3$ on $T_1$ is hyperbolic, there is an invariant line $l$ in $T_1$. Since the splittings of $G$ along $E_i, C_3$ are minimal, we can conclude, as in Prop. 4.7, that $\bigcup_i (\text{co}(\mathcal{E}_i) \times [0, 1]) = \text{co}(C_3) \times [0, 1]$ in $(T_1 \times T_3) / G$. Although $l / C_3$ is embedded in $Z$, which locally separates $Z$, the splitting of $G$ along $C_3$ which we obtain by cutting $Z$ along $l / C_3$ might be different from the original splitting along $C_3$.

Consider the following subcomplex, $W$, of $Z$,

$$W = \text{co}(\mathcal{S}) \bigcup \bigcup_i \text{co}(\mathcal{E}_i) \times [0, 1] \bigcup \bigcup_i (\text{co}(\mathcal{D}_i) \times [0, 1])$$

Let $\{p_j\}$ be the set of vertices in $W$ which are not contained in $\text{co}(\mathcal{S})$. Let $m_j$ be the link of $p_j$ in $W$, which we denote by $Lk(p_j, W)$. Since each $\text{co}(\mathcal{D}_i)$ is a point, if $p_j$ is in $\bigcup_i (\text{co}(\mathcal{D}_i) \times [0, 1])$, then $m_j$ is a point, whose fundamental group (in the sense of graph of groups) is one of the $D_i's$ (the group corresponding to the edge which contains $p_j$). The point $m_j$ locally separates $W$, and also $Z$. If the vertex $p_j$ is in $\bigcup_i (\text{co}(\mathcal{E}_i) \times [0, 1])$, then the link $m_j$ is the finite union of circles and segments, such that each of them locally separates $W$, and also $Z$. 


If we cut $Z$ along the union of those links $\bigcup_j \text{Lk}(p_j, W)$, we obtain a graph decomposition of $G$ by Lemma 4.1 such that edge groups are the image in $G$ of the H-fundamental groups of connected components of $\bigcup_j \text{Lk}(p_j, W)$. Let $V$ be the connected component of $W - \bigcup_j \text{Lk}(p_j, W)$ which contains $\text{co}(S)$. The image in $G$ of the H-fundamental group of $V$ contains $S = S(C_1, C_2)$. Let’s denote it by $S'$. We claim that $S'$ is an extension of the fundamental group of some 2-orbifold, $\Sigma'$, by $F$, the fiber group of $S$ such that $\Sigma \subset \Sigma'$. To see it, let $U = \bigcup_i (\text{co}(E_i) \times [0,1])$, which is a squared surface possibly with some vertices identified. Note $U \subset W$. Let $\{q_i\}$ be the vertices in $U \cap \text{co}(S)$. Define $l_i = \text{Lk}(q_i, U)$ for each $i$. Note that each $p_i \in U$ and also $m_i \subset U$. If we cut $U$ along $\bigcup_i l_i$ and $\bigcup_i m_i$, we obtain a graph decomposition along slender groups, which are the image (in $G$) of the H-fundamental group of $l_i$’s and $m_i$’s. Let $U' \subset U$ be the connected component of $U - (\bigcup_i l_i \cup \bigcup_i m_i)$ which does not contain any of $p_i$, $q_i$, i.e., the vertices of $U$. We know that $U'$ is a surface with boundary. Also $U' \subset V$. Cutting $V$ along $\bigcup_i l_i$, where $U'$ is one of the connected component after the cutting, we obtain a graph decomposition of $S'$ along the slender groups corresponding to $l_i$’s. The vertex group, $S_0$, corresponding to $U'$ is an extension of the fundamental group of some 2-orbifold, $\Sigma_0$, by $F$ by our construction. $\Sigma_0$ is obtained from the 2-manifold $U'$ attaching a disks or a half disk with cone points appropriately each time if the fundamental group of $m_i$ does not inject to $G$ (cf. we did the same thing when we constructed $\Sigma$ for $S$ previously). A vertex group other than $S_0$ is not only a subgroup of $S$, but also it corresponds to a suborbifold in $\Sigma$, the 2-orbifold for $S$. To see it, consider a small neighborhood, $\text{co}(S)$, of $\text{co}(S)$ in $W$. To be concrete, for example, we take a barycentric subdivision of $W$ and collect all cells which intersect $\text{co}(S)$. The H-fundamental group of $\text{co}(S)$ is $S$. One can consider that $\text{co}(S)$ is a deformation retract of $\text{co}(S)$. We may assume that each $l_i$ is in $\text{co}(S)$. Cutting $\text{co}(S)$ along $\bigcup_i l_i$, we obtain a graph decomposition of $S$ along slender groups. This decomposition is realized by cutting $\Sigma$ along simple closed curves and segments. (Consider the quotients by $F$ of the H-fundamental groups of $\text{co}(S)$ and $l_i$’s and obtain a decomposition of the orbifold fundamental group of $\Sigma$ along slender groups, and reduce the argument to surface topology. Note that all maximal peripheral subgroups of $S$ are elliptic with respect to the graph decomposition, cf. Rem 4.14 so that $\Sigma$ does not have any free boundary points in the decomposition, and the H-fundamental group of $l_i$ injects in $G$). Let $S_i$ be the H-fundamental group of the connected component of $\text{co}(S) - \bigcup_i l_i$ which contains $q_i$. Note that this is identical to the connected component of $V - \bigcup_i l_i$.
which contains $q_i$. Let $\Sigma_i \subset \Sigma$ be the sub-orbifold such that $S_i$ is the $H$-fundamental group of $\Sigma_i$ (each $\Sigma_i$ is a connected component of $\Sigma$ after the cutting we obtained in the above). Then, $S_i$ is an extension of the orbifold fundamental group of $\Sigma_i$ by $F$. Since the graph decomposition of $S'$ we obtained by cutting $V$ along $\cup_i l_i$ has vertex groups $S_i$'s (corresponding to $q_i$'s) and $S_0$, with edge groups corresponding to the $H$-fundamental groups of $l_i$'s, and $S_0$ is also an extension of the orbifold fundamental group of $\Sigma_0$ by $F$, we conclude that $S'$ is an extension of the orbifold fundamental group of $\Sigma'$ by $F$. Since the graph decomposition of $S'$ obtained by cutting $V$ along $\cup_i l_i$ has vertex groups $S_i$'s (corresponding to $q_i$'s) and $S_0$, with edge groups corresponding to the $H$-fundamental groups of $l_i$'s. By construction, $\Sigma \subset \Sigma'$.

Let $\Gamma'$ be the graph decomposition of $G$ obtained by cutting $Z$ along $\cup_j Lk(p_j, W)$, with a vertex group $S'$. We first show that $\Gamma'$ satisfies the properties 1,2,3 of Def 4.5 for the three splittings (cf. Prop 4.7). Then we apply Prop 3.7 to $\Gamma'$ and obtain a refinement, $\Gamma''$, such that each edge of $\Gamma''$ gives a minimal splitting of $G$. We will show that $\Gamma''$ satisfies all the properties of Def 4.5 so that it is an enclosing graph decomposition for the three splittings, with an enclosing vertex group $S'$. The argument is similar to Prop 4.16.

The group $S'$ has the property 3 by the construction. Regarding the property 1 of $\Gamma'$, it is clear that $S'$ contains some conjugates of $C_1, C_2, C_3$.

To verify the property 2(rigidity) of $S'$, let $\Lambda$ be a graph decomposition of $G$ such that the splitting of $G$ which any edge of $\Lambda$ gives is elliptic-elliptic with respect to any of the three splittings. We argue in the same way as in the proof of Prop 4.7 to show that $S'$ is elliptic on the Bass-Serre tree of $\Lambda$, $T_\Lambda$. Let $c$ be either an essential simple closed curve on $\Sigma'$ or an essential embedded segment on $(\Sigma', (\partial \Sigma', 2))$. Let $C < S'$ be the fundamental group for $c$. Cutting $\Sigma'$ along $c$, we obtain a splitting of $G$ along $C$. This splitting is minimal by Prop 3.4. Let $T_C$ be its Bass-Serre tree. For our purpose, by Cor 4.12, it suffices for us to show that the group $C$ is elliptic on $T_\Lambda$ to conclude that so is $S'$. Let $e$ be an edge of $\Lambda$, which gives a splitting of $G$ along its edge group, $E$. To conclude $C$ is elliptic on $T_\Lambda$, we will show that $C$ is elliptic with respect to the splitting along $E$. Since the group $E$ is elliptic with respect to the (original) splittings of $G$ along $C_1, C_2$, it is elliptic on $T_{C_1} \times T_{C_2}$, i.e., $E$ fixes a vertex. Therefore $E$ is elliptic on $T_{\Gamma}$, which is the Bass-Serre tree of the enclosing graph decomposition, $\Gamma$, we constructed for the splittings along $C_1, C_2$. Moreover, we know that $E$ is not in a conjugate of $S$ (cf. the proof of Prop 4.7). Since the group $E$ is elliptic on $T_{C_3}$ as well by our assumption, it fixes a vertex when it acts on $T_{\Gamma} \times T_{C_3}$. It follows that $E$ is elliptic on $T_{\Gamma'}$, the Bass-Serre tree for $\Gamma'$, by the way we constructed it. Therefore $E$ is in a conjugate
of a vertex group of $\Gamma'$, which is not $S'$. This implies that the group $E$ is elliptic on $T_C$. Since the splitting along $C$ is minimal, we find that the pair of splittings along $C$ and $E$ is elliptic-elliptic. But, the edge $e$ was an arbitrary edge of $\Lambda$, so that the group $C$ is elliptic on $T_\Lambda$. We showed the property 2 for $S'$.

So far, we have shown that the graph decomposition $\Gamma'$ with a vertex group $S'$ satisfies the properties 2, 3 and a part of the property 1. As we obtain Prop 4.7 from Prop 4.16 for a pair of splittings, we apply Prop 3.7 to $\Gamma'$ and obtain a graph decomposition $\Gamma''$ such that each edge gives a minimal splitting. We now claim that $\Gamma''$ has $S'$ as an enclosing vertex group and satisfies all the properties to be an enclosing decomposition for the three splittings. The argument is same as when we show Prop 4.10 from Prop 4.7, so we omit some details. By construction, $\Gamma''$ has the property 4. $S'$ is a vertex group of $\Gamma''$ because the edge groups of $\Gamma''$ are in edge groups of $\Gamma'$ and the rigidity of $S'$. Therefore, $\Gamma''$ with $S'$ satisfies the properties 2, 3, and the property 1 except for the last item, which we did not verify for $\Gamma'$.

To verify the rest of the property 1 for $\Gamma''$, let $e$ be an edge of $\Gamma''$ with the edge group, $E$. Let $T_{C_i}$ be the Bass-Serre tree of the (original) splitting of $G$ along $C_i$, $i = 1, 2, 3$. We want to show that the edge group $E$ is elliptic on all $T_{C_i}$. The splitting of $G$ along $E$ which the edge $e$ gives is minimal. Let $T_E$ be the Bass-Serre tree of this splitting. By the property 2 (rigidity) of $S'$, $S'$ is elliptic on $T_E$, so that the subgroups $C_i$ are elliptic as well. It follows that the group $E$ is elliptic on all $T_{C_i}$ because the splitting along $E$ is minimal. This is what we want. We showed all the properties for $\Gamma''$ with $S'$, so that the proof of Prop 5.1 is complete. □ (Prop 5.1)

5.2. Maximal enclosing decompositions. Following the previous subsection, we produce an enclosing graph decomposition of a set, $I$, of hyperbolic-hyperbolic minimal splittings of $G$ along slender subgroups. We put an order to the elements in $I$ such that if $I_i$ denotes the set of the first $i$ elements, then each $I_i$ is a set of hyperbolic-hyperbolic splittings. Then we produce a sequence of graph decompositions, $\Gamma_i$, of $G$ such that $\Gamma_i$ is an enclosing graph decomposition for $I_i$ with enclosing vertex group $S_i$. We already explained how to construct $\Gamma_2$, then $\Gamma_3$ using it. In the same way as we produce $\Gamma_3$ from $\Gamma_2$ from the splitting along $C_3$, we produce $\Gamma_{i+1}$ from $\Gamma_i$. Note that $\Gamma_{i+1}$ is identical to $\Gamma_i$ if the edge group, $C_{i+1}$, of the $(i + 1)$-th splitting is contained in a conjugate of $S_i$.

Although $\Gamma_i$ is an infinite sequence in general, there exists a number $N$ such that $\Gamma_i$ is identical if $i \geq N$ by the following result. We recall
that a graph of groups, whose fundamental group is $G$, is reduced if its Bass-Serre tree does not contain any proper subtree which is $G$-invariant, and the vertex group of any vertex of the graph of valence 2 properly contains the edge groups of the associated edges.

**Theorem 5.3** (Bestvina-Feighn accessibility [BF]). Let $G$ be a finitely presented group. Then there exists a number $\gamma(G)$ such that if $\Gamma$ is a reduced graph of groups with fundamental group isomorphic to $G$, and small edge groups, then the number of vertices of $\Gamma$ is at most $\gamma(G)$.

Let $\Sigma_i$ be the 2-orbifold for the enclosing vertex group $S_i$. Then, $\Sigma_i \subset \Sigma_{i+1}$ as 2-orbifolds. Any system, $F$, of disjoint essential simple closed curves on $\Sigma_i$ and essential segments on $(\Sigma_i, (\partial \Sigma_i; 2))$ such that any two of them are not homotopic to each other gives a reduced graph decomposition of not only $S_i$ but also $G$. Because the number of the connected components of $\Sigma_i \backslash F$ are bounded by $\gamma(G)$ by Theorem 5.3 there exists $N$ such that $\Sigma_i$ is constant if $i \geq N$. This implies, by the way we constructed $\{\Gamma_i\}$, $\Gamma_i$ is also constant if $i \geq N$. $\Gamma_N$ is an enclosing graph decomposition for $I$. We have shown the following.

**Proposition 5.4.** Let $G$ be a finitely presented group. Let $I$ be a set of hyperbolic-hyperbolic minimal splittings of $G$ along slender subgroups. Then, an enclosing graph decomposition of $G$, $\Gamma_I$, exists for $I$.

If a set of hyperbolic-hyperbolic minimal splitting along slender groups, $I$, is maximal, we call $\Gamma_I$ maximal. A maximal enclosing graph decomposition has the following property.

**Lemma 5.5** (Maximal enclosing graph decomposition). Let $G$ be a finitely generated group. Let $\Gamma$ be a maximal enclosing decomposition of $G$. Suppose $A \ast_C B, A \ast_C$ is a minimal splitting of $G$ along a slender subgroup $C$. Then the splitting along $C$ is elliptic-elliptic with respect to the splitting of $G$ which each edge of $\Gamma$ gives. In particular, the group $C$ is elliptic on $T_\Gamma$, the Bass-Serre tree for $\Gamma$.

**Remark 5.6.** Although the existence of maximal enclosing group is guaranteed only for a finitely presented group, the lemma is true if a maximal enclosing group exists for a finitely generated group.

**Proof.** Suppose not. Then there exists an edge, $e$, of $\Gamma$ with edge group, $E$, such that the minimal splitting of $G$ the edge $e$ gives is hyperbolic-hyperbolic with respect to the splitting along $C$. Suppose $\Gamma$ is an enclosing decomposition for a maximal set $I$. Since $E < S$, $S$ is hyperbolic on $T_C$, the Bass-Serre tree for the splitting along $C$. Then by the rigidity (Def 4.5) of $S$, there is a splitting in $I$ which is hyperbolic-hyperbolic with respect to the splitting along $C$. Let $I'$ be the union...
of $I$ and the splitting along $C$, which is a set of hyperbolic-hyperbolic splittings. If we produce an enclosing decomposition for $I'$ using $\Gamma$ and the decomposition along $C$, we obtain a graph decomposition with a different enclosing vertex group (namely, the 2-orbifold is larger) from $\Gamma$, because the group $C$ is hyperbolic with respect to the splitting along $E$, which is impossible since $\Gamma$ is maximal. The last claim is clear from Bass-Serre theory. □

**Proposition 5.7** (Rigidity of maximal enclosing group). Let $G$ be a finitely generated group. Let $\Gamma, \Gamma'$ be maximal enclosing decompositions of $G$ with enclosing groups $S, S'$. Then the group $S$ is elliptic on $T_{\Gamma'}$, the Bass-Serre tree of $\Gamma'$, so that $S$ is a subgroup of a conjugate of a vertex group of $\Gamma'$. If $S'$ is a subgroup of a conjugate of $S$, then it is a conjugate of $S$.

**Proof.** Let $I, I'$ be maximal sets of hyperbolic-hyperbolic minimal splittings of $G$ for $\Gamma, \Gamma'$. We may assume $I \neq I'$. Since they are maximal, if they have a common splitting, then $I = I'$, so that $I \cap I' = \emptyset$. Also, a pair consisting of any splitting in $I$ and any splitting in $I'$ is elliptic-elliptic. Moreover, there is no (minimal) splitting of $G$ along a slender subgroup which is hyperbolic-hyperbolic with respect to some splittings in both of $I, I'$, because, then such splitting and $I \cup I'$ would violate the maximality of $I$.

Let $\Sigma$ be the 2-orbifold for the enclosing group $S$. Let $d$ be a simple closed curve on $\Sigma$ or a segment on $(\Sigma, (\partial \Sigma; 2))$ which is essential, then cutting $\Sigma$ along $d$, we obtain a splitting of $G$ along the group, $D$, which is the fundamental group of $d$. $D$ is slender, and the splitting of $G$ along $D$ is minimal since it is hyperbolic-hyperbolic with respect to one of the minimal splittings in $I$ (Prop 3.4). Therefore, the splitting along $D$ is elliptic-elliptic to all splittings in $I$.

By Cor 14, it suffices for us to show that the group $D$ is elliptic on $T_{\Gamma'}$, the Bass-Serre tree of $\Gamma'$ to show that $S$ is elliptic on it. Let $e$ be an edge of $\Gamma'$, with edge group $E$. Collapsing all edges of $\Gamma'$ except $e$, we obtain a splitting of $G$ along $E$, which is minimal. Let $T_E$ be the Bass-Serre tree of this splitting. Then, it is enough for us to show that the group $D$ is elliptic on $T_E$. Since the splitting along $E$ is minimal, it suffices to show that the group $E$ is elliptic on $T_D$, the Bass-Serre tree for the splitting along $D$. Since $E < S'$, it is enough if we show that $S'$ is elliptic on $T_D$. By the property 2 (rigidity) of $S'$, it suffices to show that the splitting along $D$ is elliptic-elliptic with respect to all splittings in $I'$, which we already know. We have shown that $S$ is elliptic on $T_{\Gamma}$. 


By the same argument, $S'$ is elliptic on $T$. Suppose $S$ is in a conjugate of $S'$, i.e., $S < gS'g^{-1}, g \in G$. Then $S'$ is also in a conjugate of $S$, since, otherwise, $S'$ is in a conjugate of a vertex group of $\Gamma$ which is not $S$. Then this vertex group contain a conjugate of $S$, which is impossible since all edge of $\Gamma$ which is adjacent to the vertex whose vertex group is a conjugate of $S$ has an edge group which is a proper subgroup of the conjugate of $S$. Suppose $S' < hSh^{-1}, h \in G$. Therefore, $S < ghS(gh)^{-1}$. This implies that $gh \in S$, and $S = ghS(gh)^{-1}$, so that $S = gS'g^{-1}$.

\[ \Box \]

5.3. **JSJ-decomposition for hyperbolic-hyperbolic minimal splittings.** Using maximal enclosing graph decompositions, we produce a graph decomposition of $G$, $\Lambda$, which "contains" all maximal enclosing groups. $\Lambda$ will deal with all minimal splittings of $G$ along slender subgroups which are hyperbolic-hyperbolic with respect to some (minimal) splittings along slender subgroups.

Consider all maximal enclosing decompositions, $\Gamma_i$, of $G$ enclosing groups, $S_i$. Let $T_i$ be the Bass-Serre tree of $\Gamma_i$. We construct a sequence of refinements $\{\Lambda_i\}$ such that $\Gamma_1 = \Lambda_1$. We then show that after a finite step, the graph decompositions stay the same. We denote the decomposition obtained after this step by $\Lambda$.

We put $\Lambda_1 = \Gamma_1$. We consider now $\Gamma_2$. By Prop 5.7, $S_2$ is elliptic on $T_1$. If $S_2$ is a subgroup of a conjugate of $S_1$, then we do nothing and put $\Lambda_2 = \Gamma_1$. If $S_2$ is conjugate into a vertex group, $A$, of $\Gamma_1$ which is not $S_1$, then we let $A$ act on $T_2$ and obtain a refinement, $\Gamma_2'$, of $\Gamma_1$. Namely, let $A$ be the graph decomposition of $A$ we get. We substitute $A$ to the vertex, $a$, for $A$ in $\Gamma_1$ (see Def 3.6 and the following remarks). We can do this since each edge group, $E$, of $\Gamma_1$ is elliptic on $T_2$ (because $E$ is a subgroup of $S_1$, which is elliptic on $T_2$), so that $E$ is a subgroup of a conjugate of a vertex group of $A$. Note that all edge groups of $\Gamma_2'$ are slender since they are subgroups of conjugates of edge groups of $\Gamma_1, \Gamma_2$. $\Gamma_2'$ has conjugates of $S_1, S_2$ as vertex groups. Also they are peripheral subgroups of either $S_1$ or $S_2$.

Each edge, $e$, of $\Gamma_2'$ gives a minimal splitting of $G$ along its edge group, $E$, which is a subgroup of a conjugate of the edge group, $E'$, of an edge, $e'$, of $\Gamma_i, (i = 1 \text{ or } 2)$. We show this by contradiction: suppose that the edge $e$ gives a non-minimal splitting, which is hyperbolic-elliptic with respect to a splitting $G = P *_D Q, (or P *_D)$ such that $D$ is slender. Let $T_D$ be its Bass-Serre tree. Since the group $E$ is hyperbolic on $T_D$, so is $E'$. Because the splitting of $G$ along $E'$, the one $e'$ gives, is minimal, it is hyperbolic-hyperbolic with respect to $G = P *_D Q, (or P *_D)$. By Prop 3.4, the splitting along $D$ is minimal. On the other hand, by
Lemma 5.5, the group $D$ is elliptic on $T_1$, the Bass-Serre tree of $\Gamma_i$ since it is maximal. It follows that the group $D$ is elliptic on $T_{E'}$, the Bass-Serre tree of the splitting along $E'$, a contradiction.

We collapse all edges of this decomposition which are not adjacent to the vertices with vertex groups $S_1, S_2$. If the resulting graph decomposition is not reduced (cf. Theorem 5.3) at some vertex, then we collapse one of the associated two edges, appropriately, to make it reduced. Note that it is reduced at a vertex whose vertex group is an enclosing group since all edge groups are proper subgroups at the vertex of an enclosing group. We denote the resulting reduced graph decomposition by $\Lambda_2$. We remark that $\Lambda_2$ is a refinement of $\Lambda_1$.

By our construction, all edge groups of $\Lambda_2$ are conjugates of edge groups of $\Gamma_1, \Gamma_2$, and each edge of $\Lambda_2$ is connected to the vertex of an enclosing group. This is not obvious, we recall that we only know that edge groups of $\Gamma_1$ are elliptic on $T_1$. When we substitute $A$ for $A$, some of these edge groups are connected to (a conjugate of) $S_2$. We have to show that these edge groups are peripheral in $gS_2g^{-1}$. To see it, let $E$ be an edge group, and suppose that $E < S_2$, where we assume that $g = 1$ for notational simplicity. (In general, just take conjugates by $g$ appropriately in the following argument). We will show $E$ is peripheral in $S_2$. Note that this is the only essential case since $E$ can be only peripheral in $S_1$ because we don’t do anything around the vertex for $S_1$ when we construct $\Lambda_2$. Also we may assume $E < S_1$. Let $\Sigma_2$ be the 2-orbifold for $S_2$, and $d$ an essential simple closed curve/segment on it with the group $D$ represented by $d$. Cutting $\Sigma_2$ along $d$, we obtain a splitting of $G$ along $D$, with Bass-Serre tree $T_D$. It suffices to show that the group $E$ is elliptic with respect to $T_D$ to conclude that $E$ is peripheral in $S_2$. Suppose not. Then, the splittings of $G$ along $E$ and $D$ are hyperbolic-hyperbolic since both of them are minimal. Then $S_1$ is hyperbolic on $T_D$ since $E < S_1$. Let $\Sigma_1$ be the 2-orbifold for $S_1$. It follows from Cor 4.12 that there exists an essential simple closed curve or a segment, $d'$, on $\Sigma_1$ such that cutting $\Sigma_1$ along $d'$ gives a splitting of $G$ along the group for $d'$, $D'$, such that the splittings along $D$ and $D'$ are hyperbolic-hyperbolic. Then, the set $I_1 \cup I_2$ with the two splittings along $D, D'$ is a set of hyperbolic-hyperbolic, which is impossible because the enclosing group for this set must be strictly bigger than $S_1$, and $S_2$ as well, which is impossible since they are maximal. We have show that all edge groups of $\Lambda_2$ are peripheral subgroups of enclosing vertex groups.

We continue similarly and obtain a sequence of reduced graph decompositions of $G$; $\Lambda_1, \Lambda_2, \Lambda_3, \cdots$. Namely, we first show that the enclosing group $S_3$ is elliptic with respect to $\Lambda_2$, using the maximality of
S_i and rigidity. If S_3 is a subgroup of a conjugate of S_1 or S_2, then Λ_3 is Λ_2. Otherwise, there exists a vertex group, A_2, of Λ_2 which is different from S_1, S_2 and contains a conjugate of S_3. We let A_2 act on the Bass-Serre tree of Γ_3 and obtain a graph decomposition, A_2, of A_2. We then substitute A_2 to the vertex for A_2 in Λ_2, which is Γ'_3. We show that all edges of Γ'_3 give minimal splittings of G along slender subgroups. We then collapse all edges which are not adjacent to the vertices with the vertex groups conjugates of S_1, S_2, S_3, and also collapse edges appropriately at non-reduced vertices, to obtain a reduced graph decomposition, Λ_3. The edge groups of Λ_3 which are adjacent to some conjugates of S_i are the conjugates of peripheral subgroups of S_i. In this way, we obtain Λ_{n+1} from Λ_n using Γ_{n+1}. This is a sequence of refinements.

We claim that there exists a number N such that if n ≥ N then Λ_{n+1} = Λ_n. Indeed, if not, then the number of vertices in Λ_n whose vertex groups are enclosing groups S_i tends to infinity as n goes to infinity. This is impossible since the number of the vertices of Λ_n is at most γ(G) by Theorem 5.3. Note that slender groups are small. Let's denote Λ_N by Λ, and state some of the properties we have shown as follows.

**Proposition 5.8 (JSJ-decomposition for hyp-hyp minimal splittings along slender groups).** Let G be a finitely presented group. Then there exists a reduced graph decomposition, Λ, with the following properties:

1. All edge groups are slender.

2. Each edge of Λ gives a minimal splitting of G along a slender group. This splitting is elliptic-elliptic with respect to any minimal splitting of G along a slender subgroup.

3. Each maximal enclosing group of G is a conjugate of some vertex group of Λ, which we call a (maximal) enclosing vertex group. The edge group of any edge adjacent to the vertex of a maximal enclosing vertex group is a peripheral subgroup of the enclosing group.

4. Each edge of Λ is adjacent to some vertex group whose vertex group is a maximal enclosing group.

5. Let G = A ∗_C B or A∗_C be a minimal splitting of G along a slender subgroup C, and T_C its Bass-Serre tree.
   
   (a) If it is hyperbolic-hyperbolic with respect to a minimal splitting of G along a slender subgroup, then
   
   (i) a conjugate of C is a subgroup of a unique enclosing vertex group, S, of Λ. S is also the only one among enclosing vertex groups which is hyperbolic.
on $T_C$. There exists a base 2-orbifold, $\Sigma$, for $S$ and an essential simple closed curve or a segment on $\Sigma$ whose fundamental group (in the sense of complex of groups) is a conjugate of $C$.

(ii) Moreover, if $G$ does not split along a group which is a subgroup of $C$ of infinite index, then all non-enclosing vertex groups of $\Lambda$ are elliptic on $T_C$.

(b) If it is elliptic-elliptic with respect to any minimal splitting of $G$ along a slender subgroup, then all vertex groups of $\Lambda$ which are maximal enclosing groups are elliptic on $T_C$.

**Proof.** By the previous discussion we know that properties 1,3,4, and a part of property 2 hold. Let’s show the rest of the property 2. To argue by contradiction, suppose that the edge, $e$, of $\Lambda$ gives a minimal splitting along the edge group, $E$, which is hyperbolic-hyperbolic with respect to a minimal splitting of $G$ along a slender subgroup, $C$. But then $C$ would be contained in an enclosing vertex group of some graph decomposition $\Gamma_i$ and it would not be a peripheral group in $\Gamma_i$, a contradiction.

We show now (5-a). There is a maximal enclosing group which contains a conjugate of $C$, such that $C$ is the fundamental group of an essential simple closed curve or a segment of the 2-orbifold for the enclosing group. This is because we start with the splitting along $C$ to construct the enclosing group. By the construction of $\Lambda$, this enclosing group is a conjugate of some vertex group, $S$, of $\Lambda$. To argue by contradiction, suppose there is another enclosing vertex group which contains a conjugate of $C$. Then, by Bass-Serre theory, there must be an edge associated to each of those two vertices whose edge group contains a conjugate of $C$. But the edge and its edge group has the property 2, which contradicts the assumption on the splitting along $C$ that it is hyperbolic-hyperbolic. One can show that all enclosing vertex groups of $\Lambda$ except $S$ are hyperbolic on $T_C$ using Cor 4.12, and we omit details since similar arguments appeared repeatedly.

To show the last claim, suppose that there is a non-enclosing vertex group, $V$, of $\Lambda$ which is hyperbolic on $T_C$. Letting $V$ act on $T_C$, we obtain a graph decomposition of $V$, which we can substitute for $V$ in $\Lambda$. All edge groups of this graph decomposition are conjugates of subgroups of $C$, which have to be of finite index by our additional assumption. Since a conjugate of $C$ is contained in $S$, which is different from $V$, by Bass-Serre theory, there must be an edge in $\Lambda$ adjacent to the vertex for $S$ whose edge group, $E$, is a conjugate of a subgroup of $C$ of finite index. But the edge and its edge group $E$, so that the group
C as well, satisfies the property 2, which contradicts the assumption on C that it is hyperbolic-hyperbolic.

We show (5-b). Let S be a maximal enclosing vertex group of \( \Lambda \). Let \( \Gamma \) be a maximal enclosing decomposition which has S as the maximal enclosing group. Let \( \Sigma \) be the 2-orbifold for S, and \( c \) an essential simple closed curve or an essential segment on \( \Sigma \). Cutting \( \Sigma \) along \( d \), we obtain a splitting of \( G \) along the slender group, \( D \), which corresponds to \( d \). This splitting is minimal by Prop 3.4. By the assumption on the splitting along \( C \), the group \( D \) is elliptic on \( T_C \). It follows from Cor 4.12 that \( S \) is elliptic on \( T_C \). □

5.4. Elliptic-Elliptic splittings. Let \( G = A_n \ast_{C_n} B_n \) (or \( A_n \ast C_n \)) be all minimal splittings of \( G \) along slender groups, \( C_n \), which are elliptic-elliptic with respect to any minimal splitting of \( G \) along slender subgroups. To deal with them as well, we refine \( \Lambda \) which we obtained in Prop 5.8. As we constructed a sequence of refinements \( \{ \Lambda_n \} \) to obtain \( \Lambda \), we construct a sequence, \( \{ \Delta_n \} \), of refinements of \( \Lambda \) using the sequence of splittings along \( C_n \). Then we show that after a finite step the sequence stabilizes in some sense, again by Theorem 5.3, and obtain the desired graph decomposition of \( G, \Delta \).

We explain how to refine \( \Lambda \) in the first step. Let \( G = A \ast_C B \) or \( A \ast C \) be a minimal splitting along a slender group \( C \) which is elliptic-elliptic with respect to any minimal splitting of \( G \) along a slender group. By Prop 5.8 all enclosing vertex groups and all edge groups of \( \Lambda \) are elliptic on \( T_C \), the Bass-Serre tree of the splitting along \( C \). Let \( U \) be a vertex group of \( \Lambda \) which is not an enclosing vertex group. Letting \( U \) act on \( T_C \), we obtain a graph decomposition of \( U, \mathcal{U} \), which may be a trivial decomposition. Substituting \( \mathcal{U} \) for the vertex for \( U \) in \( \Lambda \), which one can do since all edge groups of \( \Lambda \) are elliptic on \( T_C \), we obtain a refinement of \( \Lambda \). We do this to all non-enclosing vertex groups of \( \Lambda \). Then we apply Prop 3.7 to this graph decomposition, and obtain a further refinement of \( \Lambda \), which we denote \( \Delta_1 \), such that each edge of \( \Delta_1 \) gives a minimal splitting of \( G \) along a slender group. By construction, all vertex groups of \( \Delta_1 \) are elliptic on \( T_C \). Although when we apply Prop 3.7, a vertex group may become smaller, all enclosing vertex groups of \( \Lambda \) stay as vertex groups in \( \Delta_1 \). We see this by an argument similar to the one in the proof of Prop 5.8. We omit details, but just remark that all edge groups of \( \Delta_1 \) are edge groups of \( \Lambda \) or subgroups of conjugates of \( C \).

Note that \( \Delta_1 \) might not be reduced. This may cause a problem when we want to apply Theorem 5.3 later. To handle this problem, if there is a vertex of \( \Delta_1 \) of valence two such that one of the two edge group is same as the vertex group, we collapse that edge. Note that
the other edge group is properly contained in the vertex group in our case. We do this to all such vertices of $\Delta_1$ at one time, and obtain a reduced decomposition, which we keep denoting $\Delta_1$. In general, we can obtain a reduced decomposition from a non-reduced decomposition in this way. We call the inverse of this operation an *elementary unfolding*. By definition, a composition of elementary unfoldings is an elementary unfolding. If we obtain a graph decomposition, $\Gamma'$, by an elementary folding from $\Gamma$, we may say $\Gamma'$ is an elementary unfolding of $\Gamma$.

**Example 5.9** (Elementary unfolding). Let $\Gamma$ be a graph decomposition of $G$ which is $G = A \ast_p B \ast Q C$ and suppose $B$ has a graph decomposition $\mathcal{B}$ which is $P \ast_{p'} B' \ast_{Q'} Q$. Then one can substitute $\mathcal{B}$ to the vertex of $B$ in $\Gamma$ and obtains a new graph decomposition $\Gamma'$, which is an elementary unfolding. One can substitute $\mathcal{B}$ to $\Gamma$ because each edge group adjacent to the vertex for $B$ in $\Gamma$ ($P, Q$ in this case) is a subgroup of some vertex group of $\mathcal{B}$. If we refine $\Gamma$ using $\mathcal{B}$, we obtain $\Gamma'$: $G = A \ast_p P \ast_{p'} B' \ast_{Q'} Q \ast_Q C$. Although $\Gamma'$ has two more vertices than $\Gamma$, $\Gamma'$ is not reduced. And if we collapse edges of $\Gamma'$ to obtain a reduced decomposition we get $G = A \ast_p B' \ast_{Q'} C$, which has the same number of vertices as $\Gamma$.

As this example shows Theorem 5.3 can not control a sequence of reduced graph decomposition which is obtained by elementary unfoldings. But we have another accessibility result to control this, which we prove later.

As we said, we now produce a sequence of refinements $\Delta_n$ using the splittings of $G$ along $C_n$. We may assume that the first splitting is the splitting along $C$, with which we already constructed $\Delta_1$. We now refine $\Delta_1$ using the splitting along $C_2$. Same as $\Lambda$, all edge groups and all enclosing vertex groups of $\Delta_1$ are elliptic on $T_2$, the Bass-Serre tree of the splitting along $C_2$. As before, we let a non-enclosing vertex group of $\Delta_1$, $U$, act on $T_2$ and obtain a graph decomposition of $U$, which we substitute for the vertex labelled by $U$ in $\Delta_1$. We do this for all non-enclosing vertex groups of $\Delta_1$, then we apply Prop 3.7. If the resulting graph decomposition is an elementary unfolding of $\Delta_1$, then we put $\Delta_2 = \Delta_1$. Otherwise, if the graph decomposition is not reduced, then we collapse one edge at a vertex where it is not reduced, and obtain a reduced graph decomposition of $G$, which we denote $\Delta_2$. $\Delta_2$ has following properties:

1. $\Delta_2$ is a refinement of $\Delta_1$. $\Delta_2$ is identical to $\Delta_1$, or has more vertices.
(2) Each edge of $\Delta_2$ gives a minimal splitting of $G$ along a slender group. The edge group is a subgroup of a conjugate of either an edge group of $\Delta_1$ or $C_2$.

(3) Each maximal enclosing group is a conjugate of some vertex group of $\Delta_2$.

(4) After, if necessary, performing an elementary unfolding to $\Delta_2$, each vertex group is elliptic on $T_2$, and also $T_1$, the Bass-Serre tree of the splitting along $C_1$.

We repeat the same process; refine $\Delta_n$ using the splitting along $C_n$ to $\Delta_{n+1}$. Because of the property 1 in the above list, by Theorem 5.3, there exists a number, $N$, such that if $n \geq N$, then $\Delta_{n+1}$ is equal to or an elementary unfolding of $\Delta_n$. Let’s denote $\Delta_N$ by $\Delta$. We state some properties of $\Delta$.

**Proposition 5.10** (JSJ decomposition for minimal splittings with elementary unfoldings). Let $G$ be a finitely presented group. Let $G = \langle A_n \ast C_n \mid B_n \rangle$ (or $\langle A_n \ast C_n \rangle$) be all minimal splittings of $G$ along slender groups, $C_n$, which are elliptic-elliptic with respect to any minimal splittings of $G$ along slender subgroups. Let $T_n$ be their Bass-Serre trees. Then there exists a graph decomposition, $\Delta$, of $G$ such that

1, 2, 3. Same as the properties 1, 2, 3 of Prop 5.8.

4. For each $n$, there exists an elementary unfolding of $\Delta$ such that each vertex group is elliptic on $T_n$.

5. Let $G = A \ast C B$ or $G = A \ast C$ be a minimal splitting along a slender group $C$ which is hyperbolic-hyperbolic with respect to some minimal splitting along a slender group, and $T_C$ its Bass-Serre tree. Then,

(a) same as 5(a) of Prop 5.8.

(b) There exists an elementary unfolding of $\Delta$, at non-enclosing vertex groups, such that all vertex groups in the elementary unfolding except for $S$ are elliptic on $T_C$, the Bass-Serre tree of the splitting along $C$.

**Remark 5.11.** In fact we do not need an elementary unfolding in the properties 4 and 5 in the proposition, if we construct a more refined $\Delta$. We show this in Theorem 5.13.

**Proof.** We already know 1, 2, 3, 4 from the way we constructed $\Delta$. Also the property 5(a) is immediate from Prop 5.8. To show 5(b), let $U$ be a non-enclosing vertex group of $\Delta$ which is not elliptic on $T_C$. If such vertex does not exist, we are done. As usual, letting $U$ act on $T_C$, we obtain a graph decomposition, $\mathcal{U}$, of $U$, then we substitute this for $U$ in $\Delta$ to obtain a refinement, $\Delta'$, of $\Delta$ whose edges give minimal splittings,
after we apply Prop 3.7 if necessary. But this resulting decomposition has to be an elementary unfolding of $\Delta = \Delta_N$, because otherwise we must have refined $\Delta_N$ further when we constructed $\Delta$. □

5.5. Elementary unfolding and accessibility. As we said in the remark after Prop 5.10, we do not need elementary unfoldings. But as we saw in Example 5.9, Theorem 5.3 cannot control a sequence of elementary unfoldings because they are not reduced. We prove another accessibility result. This result was suggested to us by Bestvina. The argument is similar to the one used by Swarup in a proof of Dunwoody’s accessibility result (Sw).

Proposition 5.12 (Intersection accessibility). Let $G$ be a finitely presented group. Suppose $\Gamma_i$ is a sequence of graph decompositions of $G$ such that all edge groups are slender. Suppose for any $i$, $\Gamma_{i+1}$ is obtained from $\Gamma_i$ by an elementary unfolding. Then there is a graph decomposition $\Gamma$ of $G$ with all edge groups slender such that for any $i$, $\Gamma$ is a refinement of $\Gamma_i$.

Proof. We define a partial order on the set of graph of groups decompositions of $G$. We say that $\Gamma < \Lambda$ if all vertex groups of $\Gamma$ act elliptically on the Bass-Serre tree corresponding to $\Lambda$, in other words, $\Gamma$ is a refinement of $\Lambda$. We have $\Gamma_{i+1} < \Gamma_i$ for all $i$.

We can apply Dunwoody’s tracks technique to obtain a graph of groups decomposition $\Gamma$ such that $\Gamma < \Gamma_i$ for all $i$. We describe briefly how this is done: Let $K$ be a presentation complex for $G$. Without loss of generality we assume that $K$ corresponds to a triangular presentation. Let $T_i$ be the Bass-Serre tree of $\Gamma_i$. As we noted earlier there are maps $\phi_i : T_{i+1} \to T_i$ obtained by collapsing some edges. We choose a sequence of points $(x_i)$ such that $x_i$ is a midpoint of an edge and $\phi(x_{i+1}) = x_i$.

We will define maps $\alpha_i : K \to T_i$. Each oriented edge of $K$ corresponds to a generator of $G$. Given an edge $e$ corresponding to an element $g \in G$ we map it by a linear map to the geodesic joining $x_i$ to $gx_i$. We extend linearly this map to the 2-skeleton of $K$. A track is a preimage of a vertex of $T_i$ under this map. We note that the tracks we obtain from $T_i$ are a subset of the tracks obtained from $T_{i+1}$ (or to be more formal each track obtained from $T_i$ is ‘parallel’ to a track obtained from $T_{i+1}$). We remark that for each $i$ the tracks obtained from $\alpha_i$ give rise to a decomposition $\Gamma'_i$ of $G$. $\Gamma_i$ is obtained from $\Gamma'_i$ by subdivisions and foldings.

Since $G$ is finitely presented, so that $K$ is compact, there is a $\lambda(G)$ such that there are at most $\lambda(G)$ non-parallel tracks we conclude that
there is an $n$ such that each track obtained from $T_k$ ($k > n$) is parallel to a track obtained from $T_n$. We can then take as $\Gamma_\infty$ the graph of groups decomposition corresponding to the tracks obtained from $T_n$. It follows that $\Gamma_i > \Gamma_\infty$. Put $\Gamma = \Gamma_\infty$. □

5.6. JSJ-decomposition along slender groups. We state one of our main theorems.

**Theorem 5.13** (JSJ-decomposition for minimal splittings along slender groups). Let $G$ be a finitely presented group. Then there exists a graph decomposition, $\Gamma$, of $G$ such that

1. same as the properties 1, 2, 3 of Prop 5.8.
2. Let $G = A \ast_C B$ or $A \ast C$ be a minimal splitting along a slender group $C$, and $T_C$ its Bass-Serre tree.
   - (a) If it is elliptic-elliptic with respect to all minimal splittings of $G$ along slender groups, then all vertex groups of $\Gamma$ are elliptic on $T_C$.
   - (b) If it is hyperbolic-hyperbolic with respect to some minimal splitting of $G$ along a slender group, then there is an enclosing vertex group, $S$, of $\Gamma$ which contains a conjugate of $C$ and the property 5(a) of Prop 5.8 holds for $S$. All vertex groups except for $S$ of $\Gamma$ are elliptic on $T_C$.
   - In particular, there is a graph decomposition, $S$, of $S$ whose edge groups are in conjugates of $C$, which we can substitute for $S$ in $\Gamma$ such that all vertex groups of the resulting refinement of $\Gamma$ are elliptic on $T_C$.

**Proof.** Let $\Delta$ be the graph decomposition of $G$ which we have constructed for Prop 5.10. We will obtain $\Gamma$ as a refinement of $\Delta$ at non-enclosing vertex groups. Let $G = A_n \ast_{C_n} B_n$ (or $A_n \ast C_n$) be all minimal splittings of $G$ along slender groups, $C_n$, which are elliptic-elliptic with respect to any minimal splitting of $G$ along slender subgroups, and $T_n$ their Bass-Serre trees. We have defined a process to refine a graph decomposition using this collection to obtain a sequence $\{\Delta_n\}$ for Prop 5.10. We apply nearly the same process to $\Delta$ again using the splittings along $C_n$, and produce a sequence $\{\Gamma_n\}$. The only difference is that we do not make a graph decomposition reduced in each step. Let $T_n$ be the Bass-Serre tree of the splitting along $C_n$. To start with, put $\Gamma_0 = \Delta$. Letting all vertex groups act on $T_1$, we obtain graph decompositions, then substitute them for the corresponding vertex groups in $\Gamma_0$, which is $\Gamma_1$. $\Gamma_1$ is an elementary unfolding of $\Gamma_0$, because otherwise, we must have refined $\Delta$ farther in the proof of Prop 5.10. Note that $\Gamma_1$ is not reduced, but we do not collapse any edges. We repeat
the same process; we let all vertex groups of $\Gamma_1$ act on $T_2$, substitute those graph decompositions for the corresponding vertex groups in $\Gamma_1$. The resulting non-reduced graph decomposition is $\Gamma_2$, and so on. In this way, we obtain a sequence of graph decompositions $\Gamma_n$ such that $\Gamma_{n+1}$ is an elementary unfolding of $\Gamma_n$. We remark that in each step enclosing vertex groups stay unchanged since they are elliptic on all $T_n$. Note that each $\Gamma_n$ satisfies the properties 1, 2, 3 and 5(a)i of Prop 5.8.

Suppose that there exists $N$ such that for any $n \geq N$, $\Gamma_n = \Gamma_{n+1}$. Then $\Gamma_N$ satisfies the properties 4 and 5(b) of Prop 5.10 as well without elementary unfoldings, so that the property 4 of the theorem follows. Putting $\Gamma = \Gamma_N$, we obtain a desired $\Gamma$.

If such $N$ does not exist, then we apply Prop 5.12 to our sequence and obtain a graph decomposition which is smaller (or equal to), for the order defined in Prop 5.12, than all $\Gamma_n$. Let’s take a minimal element, $\Gamma$, with respect to our order. Such a decomposition exists by Zorn’s lemma. $\Gamma$ is the decomposition that we look for, because if we apply the process to refine $\Gamma$ using the sequence of decompositions along $C_n$ as before, nothing happens, because $\Gamma$ is minimal in our order. It follows that $\Gamma$ satisfies all the properties. □

We call a graph decomposition of $G$ we obtain in Theorem 5.13 a JSJ decomposition of $G$ for splittings along slender groups. We will prove that $\Gamma$ has the properties stated in this theorem not only for minimal splittings of $G$ along slender groups, but also non-minimal splittings as well in Theorem 5.15.

Corollary 5.14 (Uniqueness of JSJ decomposition). Let $G$ be a finitely presented group. Suppose a graph decomposition, $\Gamma$, of $G$ satisfies the properties 2 and 4(a) of Theorem 5.13.

1. Suppose $\Gamma'$ is a graph decomposition of $G$ which satisfies the properties 2 and 4(a) of Theorem 5.13. Then all vertex groups of $\Gamma'$ are elliptic on the Bass-Serre tree for $\Gamma$.
2. $\Gamma$ satisfies the property 3 of Theorem 5.13.
3. $\Gamma$ satisfies the property 4(b) if $G$ does not split along an infinite index subgroup of $C$.

Proof. 1. Let $T$ be the Bass-Serre tree for $\Gamma$. Let $V$ be a vertex group of $\Gamma'$. Let $e$ be an edge of $\Gamma$ with edge group $E$, and $T_e$ the Bass-Serre tree of the splitting of $G$ along $E$ which the edge $e$ gives. To show $V$ is elliptic on $T$, it suffices to show that it is elliptic on $T_e$ for all $e$. This splitting along the slender group $E$ is minimal, and elliptic-elliptic with respect to any minimal splitting of $G$ along a slender group by the property 2 of $\Gamma$. Therefore, by the property 4(a) of $\Gamma'$, all vertex
groups of $\Gamma'$ are elliptic on $T_e$. In particular $V$ is elliptic on $T_e$, so it is elliptic on $T$.

2. Let $T$ be the Bass-Serre tree of $\Gamma$. Let $S$ be the maximal enclosing vertex group in a maximal enclosing decomposition, $\Lambda$, of $G$. We first show that $S$ is elliptic on $T$. To show it, as usual, we use Cor 4.12. Let $\Sigma$ be the 2-orbifold for $S$, and $s$ an essential simple closed curve or a segment on $\Sigma$. Cutting $\Sigma$ along $s$, we obtain a splitting of not only $S$ but also $G$ along the slender group, $C$, represented by $s$. This splitting is minimal by Prop 3.3. By Cor 4.12 it suffices to show that the group $C$ is elliptic on $T$. By the property 1 of $\Gamma$, the pair of the splittings of $G$ along $E$ (from the previous paragraph) and $C$ is elliptic-elliptic. Therefore, the group $C$ is elliptic on $T_E$, so that it is elliptic on $T$ as well since the edge $e$ was arbitrary.

We already know that $S$ is in a conjugate of a vertex group, $V$, of $\Gamma$. We want to show that indeed $S$ is a conjugate of $V$. Let $T_\Lambda$ be the Bass-Serre tree of the maximal enclosing decomposition $\Lambda$ with $S$. It suffices to show that $V$ is elliptic on $T_\Lambda$ to conclude that $S$ is a conjugate of $V$, because $S$ is the only vertex group of $\Lambda$ which can contain a conjugate of $V$. This is because all edge groups adjacent to $S$ are peripheral subgroups, so they are proper subgroups of $S$. Let $d$ be an edge of $\Lambda$ with the edge group $D$. The splitting of $G$ along $D$ which the edge $d$ gives is minimal since $\Lambda$ is an enclosing decomposition, and elliptic-elliptic with respect to any minimal splitting along a slender group since $\Lambda$ is maximal. By the property 3 (a) of $\Gamma$, all vertex groups of $\Gamma$ are elliptic on $T_D$, the Bass-Serre tree of the splitting along $D$. Since the edge $d$ was arbitrary, all vertex groups of $\Gamma$ are elliptic on $T_\Lambda$, in particular, so is $V$.

3. Let $I$ be a maximal set of hyperbolic-hyperbolic minimal splittings of $G$ along slender groups which contains the splitting along $C$. Let $\Lambda$ be a maximal enclosing decomposition of $G$ for $I$ with enclosing vertex group $S$. Let $\Sigma$ be the 2-orbifold for $S$. We can assume that there is an essential simple closed curve or a segment, $s$, on $\Sigma$ such that by cutting $\Sigma$ along $s$ we obtain a splitting of $G$ along $C$. This is because when we construct $\Lambda$ for $I$ using a sequence of graph decomposition, we can start with the splitting along $C$. Although we do not know in general if this splitting is the same as the one we are given, it is the case under our extra assumption. By the property 3 of $\Gamma$, a conjugate of $S$ is a vertex group of $\Gamma$, which therefore contains a conjugate of $C$. No other vertex group of $\Gamma$ contains a conjugate of $C$ because if it did, then an edge group of $\Gamma$ has to contain a conjugate of $C$, which is a contradiction since the splitting along $C$ is hyperbolic-hyperbolic while $\Gamma$ has property 2.
Let \( w \) be the vertex of \( \Gamma \) with the vertex group, \( W \), which is a conjugate of \( S \). Let \( v \) be a vertex of \( \Gamma \) with vertex group, \( V \), such that \( v \neq w \). We want to show that \( V \) is elliptic on \( T_C \), the Bass-Serre tree of the splitting \( G = A \ast_C B \), or \( A \ast_C \), which we know is obtained by cutting \( \Sigma \) along \( s \). We first claim that \( V \) is elliptic on \( T_\Lambda \). This is because each edge of \( \Lambda \) gives a minimal splitting which is elliptic-elliptic since \( \Lambda \) is maximal, so that \( V \) is elliptic on \( T_\Lambda \) by the property 4(a). (Use it to each edge decomposition of \( \Lambda \)). Therefore \( V \) is in a conjugate of some vertex group, \( U \), of \( \Lambda \). If \( U \) is not \( S \), we are done, because then \( U \) is elliptic on \( T_C \). We have used that the original splitting along \( C \) is identical to the one we obtain by cutting along \( s \). Suppose \( U = S \), then \( V \) is in a conjugate of \( W \). By Bass-Serre theory, this means that there is an edge in \( \Gamma \) adjacent to \( v \) whose edge group is \( V \). It then follows from the property 2 for \( \Gamma \) that the edge group \( V \) is elliptic on \( T_\Lambda \). The proof is complete. □

As we said, \( \Gamma \) indeed can deal with non-minimal splittings of \( G \) along slender groups as well.

**Theorem 5.15** (JSJ decomposition for splittings along slender groups). Let \( G \) be a finitely presented group, and let \( \Gamma \) be the graph decomposition we obtain in Theorem 5.13. Let \( G = A \ast_C B \), \( A \ast_C \), be a splitting along a slender group \( C \), and \( T_C \) its Bass-Serre tree.

1. If the group \( C \) is elliptic with respect to any minimal splitting of \( G \) along a slender group, then all vertex groups of \( \Gamma \) are elliptic on \( T_C \).
2. Suppose the group \( C \) is hyperbolic with respect to some minimal splitting of \( G \) along a slender group. Then
   (a) all non-enclosing vertex groups of \( \Gamma \) are elliptic on \( T_C \).
   (b) For each enclosing vertex group, \( V \), of \( \Gamma \), there is a graph decomposition of \( V \), \( \mathcal{V} \), whose edge groups are in conjugates of \( C \), which we can substitute for \( V \) in \( \Gamma \) such that if we substitute for all enclosing vertex groups of \( \Gamma \) then all vertex groups of the resulting refinement of \( \Gamma \) are elliptic on \( T_C \).

**Proof.** 1. If the splitting along \( C \) is minimal, then nothing to prove (Theorem 5.13). Suppose not. Apply Prop 3.7 to the splitting and obtain a refinement, \( \Lambda \), such that each edge, \( e \), of \( \Lambda \) gives a minimal splitting of \( G \) along a slender group, \( E \), which is a subgroup of a conjugate of \( C \). Let \( G = P \ast_E Q \) (or \( P \ast_E \)) be the splitting along \( E \) which the edge \( e \) gives. Let \( T_E \) be its Bass-Serre tree. Let \( T_\Lambda \) be the Bass-Serre tree of \( \Lambda \). We want to prove that each vertex group, \( V \), of \( \Gamma \) is elliptic on \( T_\Lambda \), which implies that \( V \) is elliptic on \( T_C \), since \( \Lambda \) is a refinement of the splitting along \( C \). By Bass-Serre theory, it suffices to prove that
V is elliptic on $T_E$. By our assumption, the group $C$ is elliptic with respect to any minimal splitting of $G$ along a slender group, so that so is $E$ since it is a subgroup of a conjugate of $C$. Therefore, the minimal splitting $G = P \ast_E Q$ (or $P \ast_E$) is elliptic-elliptic with respect to any minimal splitting of $G$ along a slender group, so that by the property 4(a), Theorem 5.13 of $\Gamma$, all vertex groups of $\Gamma$, in particular $V$, are elliptic on $T_E$.

2. If the given splitting along $C$ is minimal, then nothing to prove because we have 4(b), Theorem 5.13. Suppose it is not minimal, and apply Prop 5.7 to obtain a refinement, $\Lambda$ such that each edge gives a minimal splitting of $G$. All edge groups of $\Lambda$ are in conjugates of $C$. Let $T_\Lambda$ be its Bass-Serre tree. Then all non-enclosing vertex groups of $\Gamma$ are elliptic on $T_\Lambda$. The argument is similar to the case 1 above and the proof for 3(b), Theorem 5.13. We omit details. Let $V$ be an enclosing vertex group of $\Gamma$. Letting $V$ act on $T_\Lambda$, we obtain a graph decomposition, $\mathcal{V}$, of $V$ such that all edge groups are in conjugates of $C$. Because each edge of $\Lambda$ gives a minimal splitting of $G$ along a slender group, by the property 1, Theorem 5.13 for $\Gamma$, all edge groups of $\Gamma$ are elliptic on $T_\Lambda$. Therefore we can substitute $\mathcal{V}$ for $V$ in $\Gamma$. If we substitute for all enclosing vertex groups in this way, we obtain the desired refinement of $\Gamma$. □

One can interpret our theorems using the language of foldings. What we show is that if $\Gamma$ is the JSJ-decomposition of a finitely presented group and $A \ast_C B$ (or $A \ast_C$) is a splitting of $G$ over a slender group $C$ with Bass-Serre tree $T_C$ then we can obtain a graph decomposition $\Gamma'$ from $\Gamma$ such that all vertex groups of $\Gamma'$ act elliptically on $T_C$. Let’s call $T$ the Bass-Serre tree of $\Gamma'$. Since all vertex groups of $\Gamma'$ fix vertices of $T_C$ we can define a $G$-equivariant simplicial map $f$ from a subdivision of $T$ to $T_C$. To see this pick a tree $S \subset T$ such that the projection from $S$ to $\Gamma$ is bijective on vertices. If $v \in S^0$ pick a vertex $u \in T_C$ such that $Stab(v) \subset Stab(u)$. Define then $f(v) = u$. We extend this to the edges of $S$ by sending the edge joining two vertices to the geodesic joining their images in $T_C$. This can be made simplicial by subdividing the edge. Finally extend this map equivariantly on $T$. It follows that the splitting over $C$ can be obtained from $\Gamma$ by first passing to $\Gamma'$ and then performing a finite sequence of subdivisions and foldings. In this sense $\Gamma$ ‘encodes’ all slender splittings of $G$.

6. Final remarks

For producing the JSJ-decomposition we did not put any restriction on $G$; in particular we did not assume that $G$ does not split over groups
‘smaller’ than the class considered. One of the difficulties in this is that there is no natural ‘order’ on the set of slender groups. Otherwise one could work inductively starting from the ‘smallest’ ones. This is the reason we introduced the notion of minimal splittings.

We remark that the situation is simpler if one restricts one’s attention to polycyclic groups as one can ‘order’ them.

It is a natural question whether there is a JSJ-decomposition over small groups. Our results (in particular proposition 4.7) might prove useful in this direction. The main difficulty for generalizing it to an arbitrary number of small splittings to produce a JSJ-decomposition over small groups is that the edge groups of the decomposition in prop. 4.7 are not small in general. So one can not apply induction in the case of small splittings.

We note however that if a JSJ-decomposition over small groups exist its edge groups are not small. We illustrate this by the following example.

**Example 6.1.** Let’s denote by $A$ the group given by $A = \langle a, t, s | t a t^{-1} = a^2, s a s^{-1} = a^2 \rangle$ and let $H$ be an unsplittable group containing $F_2$, e.g. $SL_3(\mathbb{Z})$.

Let’s consider a complex of groups $G(X)$ with underlying complex a sphere obtained by gluing two squares along their boundary. We label all 4 vertices (0-cells) by $A \times H$. We label 2-cells and 1-cells by an infinite cyclic group $\langle c \rangle$. In one of the squares all maps from the group of a 2-cell to a group of a 1-cell are isomorphisms while all maps to the group of 0-cell send $c$ to $a$.

We describe now the maps in the second square $\tau$. Let $e_1, e_2$ be two adjacent edges (1-cells) of the square. Let $a_{12}$ be the common vertex of $e_1, e_2$ and let $a_1$ be the other vertex of $e_1$ and $a_2$ the other vertex of $e_2$. Let finally $b$ be the fourth vertex of the square.

The monomorphisms $\psi_1 : G_{\tau} \to G_{e_1}, \psi_2 : G_{\tau} \to G_{e_2}$ are given by $c \to c^2$. All other maps are defined as in the first square. To satisfy condition 3 of the definition of a complex of groups (see subsec. 4.1) we define the ‘twisting’ element $g_{e,f}$ for two composable edges $e, f$ as follows:

For each vertex of the square there are two pairs of composable edges from the barycenter of $\tau$ to the vertex. For the vertex $b$ we put $g_{e,f} = 1$ for the first pair and $g_{e,f} = t^{-1}s$ for the second pair. We remark that $t^{-1}s$ commutes with $a$ so condition 3 is satisfied. For the vertex $a_1$ for the pair of composable edges that corresponds to an isomorphism from $G_{\tau}$ to $G_{a_1}$ we put $g_{e,f} = 1$ and for the other pair we put $g_{e,f} = t$. Similarly for the vertex $a_2$ for the pair of composable edges
that corresponds to an isomorphism from $G_\tau$ to $G_{a_2}$, we put $g_{e,f} = 1$ and for the other pair we put $g_{e,f} = t$. Finally for the vertex $a_{12}$ for one pair we put $g_{e,f} = t$ and for the other $g_{e,f} = s$.

It is now a straightforward computation to see that this complex is developable. In fact to show this here it is enough to show that links of vertices do not contain simple closed curves of length 2. Let $v$ be a vertex of $X$. The link of $v$, $Lk(v)$ has as set of vertices the pairs $(g\psi_a(G_i(e)), e)$ where $g \in G_v$ and $e \in E(X)$ with $t(e) = v$. The set of edges of the barycentric subdivision of $Lk(v)$ is the set of triples $(g\psi_e(G_i(f)), e, f)$ where $e, f$ are composable edges in $E(X)$ with $t(e) = v$. The initial and terminal vertices of an edge are given by:

$$i(g\psi_e(G_i(f)), e, f) = (g\psi_e(G_i(f)), ef)$$

$$i(g\psi_e(G_i(f)), e, f) = (gg_{e,f}^{-1}\psi_e(G_i(e)), e)$$

Our choice of twisting elements now insures that there are no curves of length 2 in the link. To see this notice that if e.g. one assigns $g_{e,f} = 1$ to all pairs of composable edges in $E(X)$ with $t(e) = b$ then condition 3 of the definition of the complex of groups is satisfied but now the link has a simple closed curve of length 2. Our choice of non-trivial twisting element insures that there are no length 2 curves in the link.

Let’s denote by $G$ the fundamental group of $G(X)$. We remark that the two simple closed curves perpendicular at the midpoints of $e_1, e_2$ give rise to two small splittings of $G$ over $BS(1, 2) = \langle x, y | xyx^{-1} = y^2 \rangle$. Note also that this pair of splittings is hyperbolic-hyperbolic.

We claim that this complex gives the JSJ-decomposition of $G$ over small groups. Let $\tilde{X}$ be a complex on which $G$ acts with quotient complex of groups $G(X)$. Let $T$ be the Bass-Serre tree of a splitting of $G$ over a small group $C$. Then all vertex stabilizers of $\tilde{X}$ fix vertices of $T$. This implies that there is a $G$-equivariant map $f : \tilde{X} \rightarrow T$. The preimage of a midpoint of an edge of $T$ is a graph (a tree) in $\tilde{X}$ projecting to an essential simple closed curve on $X$ corresponding to a splitting of $G$ over a conjugate of $C$. In other words this complex gives us a JSJ-decomposition for $G$. Now one of the edge groups of this decomposition has the presentation $\langle s, a | sa^2s^{-1} = a^2 \rangle$ (it is the edge corresponding to the vertex lying in both $e_1, e_2$), which clearly is not a small group. We remark that 2 edge groups are labelled by $BS(1, 2)$ and one by $\mathbb{Z} \times \mathbb{Z}$, so they are small.

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