Global solutions of the random vortex filament equation

Z Brzeźniak\textsuperscript{1}, M Gubinelli\textsuperscript{2} and M Neklyudov\textsuperscript{1}

\textsuperscript{1} Department of Mathematics, University of York, Heslington, UK
\textsuperscript{2} Ceremade and CNRS UMR 7534, Université Paris-Dauphine, France

Received 2 October 2012, in final form 5 July 2013
Published 5 August 2013
Online at stacks.iop.org/Non/26/2499

Abstract
We prove the existence of a global solution for the filament equation with the initial condition given by a geometric rough path in the sense of Lyons (1998 \textit{Rev. Mat. Iberoamericana} \textbf{14} 215–310). Our work gives a positive answer to a question left open in recent publications: Berselli and Gubinelli (2007 \textit{Commun. Math. Phys.} \textbf{269} 693–713) showed the existence of a global solution for a smooth initial condition while Bessaih \textit{et al} (2005 \textit{Ann. Probab.} \textbf{33} 1825–55) proved the existence of a local solution for a general initial condition given by a rough path.

Mathematics Subject Classification: 76B47, 35Q35, 37K05

1. The filament equation

In this paper we prove the existence of a global solution for the following rough filament equation

\[
\begin{aligned}
\frac{d\gamma(t)(\xi)}{dt} &= u^{\nu(t)}(\gamma(t)(\xi)), \quad t \in [0, \infty), \xi \in [0, 1], \\
\gamma(0) &= \gamma_0,
\end{aligned}
\]  

(1.1)

where the initial condition \(\gamma_0 : [0, 1] \rightarrow \mathbb{R}^3\) is a geometric \(\nu\)-rough path (for some \(\nu \in (\frac{1}{3}, 1)\)), see assumption 2.6. Here \(\gamma : [0, \infty) \rightarrow D_{\gamma_0} \subset \mathcal{C}\) is some trajectory in the subset \(D_{\gamma_0}\) of the space \(\mathcal{C} \subset \mathcal{C}(\{0, 1\}; \mathbb{R}^3)\) of continuous closed curves in \(\mathbb{R}^3\) parametrized by the interval \([0, 1]\). Finally \(u^\nu, \gamma \in D_{\gamma_0}\) is a vector field given by

\[
u^\nu(x) = \int_0^1 \nabla \phi(x - \gamma(\xi)) \times d\gamma(\xi),
\]  

(1.2)

where \(\phi : \mathbb{R}^3 \rightarrow \mathbb{R}\) is a smooth function which satisfies certain assumptions (see hypothesis 3.2). The exact meaning of the line integral above and the definition of the set \(D_{\gamma_0}\) will be given below. Equation (1.1) has its origin in the theory of the three-dimensional...
Euler equations. It is well known that for the two dimensional Euler equations the vorticity \( \omega = \text{curl } u \) is transported along the fluid flow. The situation changes drastically in the three-dimensional case. The additional ‘stretching’ term in the equation defining the vorticity leads to the possibility of its blow up in finite time. Furthermore, a result of Beale \textit{et al} [2] suggests that a possible singularity of the Euler equations appears when the vorticity field of the fluid blows up. Consequently, understanding the behaviour of the vorticity of an ideal incompressible fluid is an important problem in fluid dynamics.

The properties of the motion of the vorticity have been studied for the last 150 years, beginning with the works of Helmholtz [15] and Kelvin [16]. It has been suggested by Kelvin to use the so-called Biot–Savart law

\[
u(x) = \int_{\mathbb{R}^3} \frac{x - y}{|x - y|^3} \times \omega(y) \, dy, \quad x \in \mathbb{R}^3
\]

where \( \times \) denotes the vector product in \( \mathbb{R}^3 \), combined with an assumption that the vorticity is supported by a smooth closed curve evolving in time \( \gamma : \mathbb{R} \times [0, 1] \to \mathbb{R}^3 \) via a formula

\[
\omega(x, t) = \Gamma \int_0^1 \delta(x - \gamma(t, \xi)) \frac{\partial \gamma(t, \xi)}{\partial \xi} \, d\xi, \quad x \in \mathbb{R}^3, \ t \geq 0
\]

and definition of the flow

\[
\begin{align*}
\frac{dX_t(x)}{dt} &= u(X_t(x), t), \quad t \geq 0, \\
X_0(x) &= x.
\end{align*}
\]

to formally deduce the \textit{filament equation}

\[
\frac{\partial \gamma}{\partial t}(t, \xi) = -\frac{\Gamma}{4\pi} \int_0^1 \gamma(t, \xi) - \gamma(t, \eta) \left( \frac{|\gamma(t, \xi) - \gamma(t, \eta)|^2}{|\gamma(t, \xi) - \gamma(t, \eta)|^3} \times \frac{\partial \gamma(t, \eta)}{\partial \eta} \right) d\eta.
\]

The assumption that the vorticity is supported by some curve is coherent with the numerical simulations of 3D turbulent fluids which show that regions of large vorticity have the form of a ‘filament’; see for instance [3, 21].

Equation (1.4) has singularity when \( \xi \) and \( \eta \) are close to each other and the initial curve \( \gamma \) is smooth. As a consequence, the energy of the solution of this equation given by the formula

\[
E(t) = \frac{\Gamma^2}{8\pi} \int_0^1 \int_0^1 \frac{1}{|\gamma(t, \xi) - \gamma(t, \eta)|} \left( \frac{\partial \gamma(t, \eta)}{\partial \eta} \right) \cdot \left( \frac{\partial \gamma(t, \xi)}{\partial \xi} \right) d\xi \, d\eta, \quad t \geq 0
\]

is infinite for any smooth curve \( \gamma(t, \cdot) \). In order to avoid the singularity Rosenhead [20] introduced the regularized model

\[
\frac{\partial \gamma}{\partial t}(t, \xi) = -\frac{\Gamma}{4\pi} \int_0^1 \gamma(t, \xi) - \gamma(t, \eta) \left( \frac{|\gamma(t, \xi) - \gamma(t, \eta)|^2 + \mu^2}{|\gamma(t, \xi) - \gamma(t, \eta)|^3} \times \frac{\partial \gamma(t, \eta)}{\partial \eta} \right) d\eta
\]

which is widely used in aeronautics to study the dynamics of trailing vortices at the tips of airplane wings.

The problem (1.1) has been studied by Berselli and Bessaih [4] and then by Berselli and Gubinelli [5]. It contains the equation (1.5) as a particular case where, for some \( \mu > 0 \),

\[
\phi(x) = \frac{\Gamma}{(|x|^2 + \mu^2)^{\frac{3}{2}}}, \quad x \in \mathbb{R}^3.
\]

Equation (1.1) is in fact a nonlinear PDE for a function \( \gamma : [0, \infty) \times [0, 1] \to \mathbb{R}^3 \). A natural setting for the well-posedness of the corresponding Cauchy problem is obtained by requiring the vector field \( u \) to be well defined and Lipschitz with respect to the space variable. To this effect the approach followed in [4] is to set up the equation as an evolution problem in
the Sobolev space $H^1$ of closed absolutely continuous curves in $\mathbb{R}^3$ with a square integrable first derivative (with respect to the parameter). This approach implies that the vector field (once lifted to $H^1$) does not allow strong enough estimates to deduce the global existence. This issue is ultimately due to the fact that in the three-dimensional incompressible flows, the vortices stretch and undergo a complex dynamics and that a priori this could lead to an explosion of the $H^1$ norm. Such a difficulty should be compared to the more stable behaviour of 2D vortex points which, under incompressible flows, are simply transported along the flow lines. Exploiting the conservation of the kinetic energy of the flow and a control of the velocity field generated by the vortex line via the associated kinetic energy Berselli and Gubinelli [5] showed the existence of a global solution to equation (1.1) with initial conditions in $H^1$.

Bessaih et al [6], partially motivated by the random filament models suggested by Gallavotti [11] and Chorin [8], considered the above evolution problem where the initial data is a random closed curve. For definiteness they considered the initial data sampled from the law of a three-dimensional Brownian loop (a Brownian motion starting at $0 \in \mathbb{R}^3$ and conditioned to return to $0 \in \mathbb{R}^3$ at time 1). In this case it is no longer possible to set up the problem in the Sobolev space $H^1$ since Brownian trajectories almost certainly do not belong to this space. A more serious problem is the meaning to be given to the generalized Biot–Savart relation (1.2) since the line integral along a Brownian trajectory does not allow a straightforward definition. Moreover, stochastic integration (à la Itô or Stratonovich) does not provide a good framework to study this problem since it does not possess a natural filtration on the parameter space stable under the time evolution. For these reasons [6] identified the spaces of controlled rough paths as a natural framework to have a well-posed problem.

Rough path theory has been introduced by Lyons in the seminal paper [17] (see also [10, 18, 19]) as a way to overcome certain difficulties of stochastic integration theories and have a robust analytical framework to solve stochastic differential equations and similar problems involving the integration of non-regular vector-fields. It turns out that rough path theory and in particular the notion of controlled paths introduced in [12] allows one to give a natural interpretation to the Biot–Savart relation (1.2) and obtain a well-posed problem. Using this approach Bessaih et al [6] obtained the existence of a local solution to equation (1.1) where the initial data is a closed curve of Hölder class with exponent $\nu \in (1/3, 1]$ (suitably lifted to rough path space).

The aim of the present paper is to extend the energy method of Berselli and Gubinelli to the rough path setting and obtain a global existence result for the equation (1.1) where the initial data is a geometric rough path, thus completing the analysis of [6].

Recently there has been some good progress in the study of evolution equations in the space of controlled rough paths. In particular Hairer [13] showed how to use controlled path theory to obtain the well-posedness of a multidimensional Burgers type equation driven by additive space–time white noise, and later [14] used similar ideas to tackle the longstanding open problem of the Kardar–Parisi–Zhang equation, for which he described the notion of a solution for which existence and uniqueness can be proven. The key technical tool for obtaining these results has been the observation that the more singular nonlinear term involved in the fixed-point argument has the same structure of the Biot–Savart relation (1.2) and thus can be similarly handled via controlled path techniques.

A major open problem which remains largely unexplored is the rigorous study of the unregularized filament equation (1.4), whose singularity seems to defy any reasonable analytic approach. On this respect the very suggestive heuristic computations of Gallavotti [11] seems to hint at the fact that rapid oscillations of the curve $\gamma$ could provide a natural regularization. Another very interesting problem is the study of the rough filament equation as a Hamiltonian system on the space of parametrization invariant paths on the lines of [1, 7].
2. Definition and properties of rough path integrals

In this section we will present the controlled path framework and we will define the rough path integral and state some of its properties. We will mainly follow the papers \[6, 12\].

In what follows by \(V, W\) we will usually denote Banach spaces, by \(L(V, W)\) the Banach space of bounded linear maps from \(V\) to \(W\) and by \(C(X, Y)\) the space of the continuous function from \(X\) to \(Y\). Moreover we let \(\mathbb{T} = [0, 1]\) and \(C_n(V) = \{f \in C(\mathbb{T}^n, V) : f(t_1, t_2, \ldots, t_n) = 0\text{ if } t_i = t_{i+1}\text{ for some } i = 1, \ldots, n-1\}\). We will understand that \(C(V) = C_1(V)\) and we will consider the Hölder norm

\[
|f|_\mu = \sup_{a \in \mathbb{T}} |f(a)| + \sup_{0 \leq a < b \leq 1} \frac{|f(a) - f(b)|_V}{|a - b|^{\mu}}, \quad f \in C_1(V);
\]
on \(C_1(V)\) and the following semi-norms on the spaces \(C_2(V)\) and \(C_3(V)\)

\[
|f|_\mu = \sup_{0 \leq a < b \leq 1} \frac{|f(a, b)|_V}{|a - b|^{\mu}}, \quad f \in C_2(V), \quad |g|_\mu = \sup_{0 \leq a < b < c \leq 1} \frac{|g(a, b, c)|_V}{|a - c|^{\mu}}, \quad g \in C_3(V).
\]

The associated normed spaces are defined by

\[
C^\mu(V) = C_1^\mu(V) = \{f \in C_1(V) : |f|_\mu < \infty\},
\]

\[
C_2^\mu(V) = \{f \in C_2(V) : |f|_\mu < \infty\}, \quad C_3^\mu(V) = \{g \in C_3(V) : |g|_\mu < \infty\}.
\]

For \(n = 1, 2\) we define the operators \(\delta - \delta_n : C_n(V) \to C_{n+1}(V)\) as

\[
\delta_1 f(a, b) = f(b) - f(a), \quad \delta_2 g(a, b, c) = g(a, c) - g(a, b) - g(b, c)
\]

for all \(a, b, c \in \mathbb{T}\). They satisfy the following fundamental property \(\delta \delta f = 0\), for all \(f \in C_1(V)\). Let \(ZC_2^\mu(V) = C_2^\mu(V) \cap \ker \delta\) and \(BC_2^\mu(V) = C_2^\mu(V) \cap \im \delta\). If \(g \in ZC_2^\mu(V)\) then \(g \in BC_2^\mu(V)\), i.e. there exists \(f \in C_1(V)\) such that \(\delta f = g\). The following result has been proved in \[12\]:

**Proposition 2.1 (Sewing lemma).** For any \(\mu > 1\) there exists an unique linear map (sewing map) \(\Lambda : ZC_2^\mu(V) \to C_2^\mu(V)\) such that

\[
\delta \Lambda = \text{id}_{ZC_2^\mu(V)}.
\]

This map is continuous and we have

\[
\|\Lambda h\|_\mu \leq \frac{1}{2\mu - 2} \|h\|_\mu, \quad h \in ZC_2^\mu(V).
\]

Now, we define a class of paths for which the rough path integral will be defined.

**Definition 2.2.** Assume that \(v \in (0, 1)\) and fix \(X \in C^v(V)\). We say that a path \(Y \in C(V)\) is weakly controlled by \(X\) if there exists a function \(Y' \in C^0(L(V, V))\) such that the function

\[
R(\xi, \eta) = Y'(\xi) - Y'(\eta)(X(\xi) - X(\eta)), \quad \xi, \eta \in \mathbb{T},
\]

belongs to \(C_2^\mu(V)\). We denote \(D_X\) as the set of all the pairs \((Y, Y')\) satisfying (2.1). This is a vector space which can be endowed with the semi-norm \(\|\cdot\|_{D_X}\) given by the following formula

\[
\|(Y, Y')\|_{D_X} = \|Y'\|_{C^v} + \|R\|_{C_2^\mu}.
\]

Furthermore, let us define a norm \(\|(\cdot, \cdot)\|_{D_X}^2\) in \(D_X\) by

\[
\|(Y, Y')\|_{D_X}^2 = \|Y\|_{D_X}^2 + \sup_{\xi \in \mathbb{T}} |Y'(\xi)|^2.
\]

With these definitions \((D_X, \|(\cdot, \cdot)\|_{D_X}^2)\) is a Banach space. While not necessary for the integration theory it will be convenient for our intended application to assume that all our controlled paths take values in \(V = \mathbb{R}^2\) and are closed, that is \(Y(0) = Y(1)\).
Furthermore, there exists a constant \( \varphi \) specifying \( Y, \) when it is clear from the context and write \( \| Y \|_{\mathcal{D}_X} \) instead of \( \| (Y, Y') \|_{\mathcal{D}_X} \).

We will repeatedly use the basic estimate
\[
\| Y \|_{C^1} \leq \| Y \|_{\mathcal{D}_X}(1 + \| X \|_{C^0}).
\] (2.4)

Controlled paths are stable by mapping via sufficiently regular functions.

**Lemma 2.3.** Let \( \phi \in C^2(\mathbb{R}^3, \mathbb{R}^3) \) and \( (Y, Y') \in \mathcal{D}_X \). Then
\[
(W, W') := (\phi(Y), \phi'(Y)Y') \in \mathcal{D}_X
\] (2.5)

and the corresponding remainder has the following representation
\[
R^W(\xi, \eta) = \phi'(Y(\xi))R^Y(\xi, \eta) + Y(\eta) - Y(\xi)
\]
\[
\times \int_0^1 [\nabla \phi(Y(\xi)) + r(\eta) - Y(\xi)) - \nabla \phi(Y(\xi))] \, dr, \quad \xi, \eta \in \mathbb{T}.
\] (2.6)

Furthermore, there exists a constant \( K \geq 1 \) such that
\[
\| \phi(Y) \|_{\mathcal{D}_X} \leq K\| \nabla \phi \|_{C^1}\| Y \|_{\mathcal{D}_X}(1 + \| Y \|_{\mathcal{D}_X})(1 + \| X \|_v)^2.
\] (2.7)

Moreover, if \( (\tilde{Y}, \tilde{Y}') \in \mathcal{D}_X \) and \( (W, \tilde{W}') := (\phi(\tilde{Y}), \phi'(\tilde{Y})\tilde{Y}') \) then
\[
\| W' - \tilde{W}' \| + \| R^W - R^\tilde{W} \|_{2v} + \| W - \tilde{W} \|_v \leq C(\| X - \tilde{X} \|_v + \| Y - \tilde{Y} \|_v + \| R^Y - R^\tilde{Y} \|_{2v} + \| Y - \tilde{Y} \|_v)
\] (2.8)

with
\[
C = K\| \phi \|_{C^1}(1 + \| X \|_{C^0} + \| \tilde{X} \|_{C^0})^3(1 + \| Y \|_{\mathcal{D}_X} + \| \tilde{Y} \|_{\mathcal{D}_X})^2
\] (2.9)

When \( X = \tilde{X} \) we have
\[
\| \phi(Y) - \phi(\tilde{Y}) \|_{\mathcal{D}_X} \leq K\| \nabla \phi \|_{C^1}\| Y \|_{\mathcal{D}_X}(1 + \| Y \|_{\mathcal{D}_X} + \| \tilde{Y} \|_{\mathcal{D}_X})^2\| Y - \tilde{Y} \|_{\mathcal{D}_X}.
\] (2.10)

**Proof.** See [12], proposition 4 for all statements of the lemma, except (2.6) (which is actually also proven, though not stated explicitly). Let us show (2.6). Denote \( y(r) = Y(\xi) + r(Y(\eta) - Y(\xi)), r \in [0, 1] \). Then
\[
\phi(y(1)) - \phi(y(0)) = \int_0^1 \phi'(y(r))y'(r) \, dr
\]
\[
= \sum_k \frac{\partial \phi}{\partial x_k}(Y^k(\eta) - Y^k(\xi)) \int_0^1 \frac{\partial}{\partial x_k}(y(r)) \, dr = \sum_k \frac{\partial \phi}{\partial x_k}(Y(\xi)) \frac{\partial}{\partial x_k}(Y^k(\eta) - Y^k(\xi))
\]
\[
+ \sum_k \frac{\partial \phi}{\partial x_k}(Y^k(\eta) - Y^k(\xi)) \int_0^1 \left[ \frac{\partial}{\partial x_k}(y(r)) - \frac{\partial}{\partial x_k}(Y(\xi)) \right] \, dr
\]
\[
= \sum_{k,l} \frac{\partial \phi}{\partial x_k}(Y(\xi))(Y^l(\eta) - X^l(\xi)) \int_0^1 \left[ \frac{\partial}{\partial x_k}(y(r)) - \frac{\partial}{\partial x_k}(Y(\xi)) \right] \, dr
\]
\[
+ \sum_k \frac{\partial \phi}{\partial x_k}(Y^k(\eta) - Y^k(\xi)) \int_0^1 \left[ \frac{\partial}{\partial x_k}(y(r)) - \frac{\partial}{\partial x_k}(Y(\xi)) \right] \, dr.
\] (2.11)

and the result follows. \( \square \)
In order to define the integral for paths controlled by $X$ we need a further ingredient: the existence of basic integrals of $X$ with respect to $X$ itself. These objects are encoded into what is referred to as a rough path as follows.

**Definition 2.4.** Assume that $\nu > 1/3$. We say that couple $X = (X, X^2)$ where $X \in C^6(\mathbb{T}, \mathbb{R}^3)$ and $X^2 \in C^2(\mathbb{L}(\mathbb{R}^3, \mathbb{R}^3))$ is a $\nu$-rough path if the following condition is satisfied:

$$X^2(\xi, \rho) - X^2(\xi, \eta) - X^2(\eta, \rho) = (X(\xi) - X(\eta)) \otimes (X(\eta) - X(\rho)), \quad \xi, \eta, \rho \in \mathbb{T}.$$  

(2.12)

**Remark 2.5.** If $\nu > 1/2$ the solution $X^2$ to (2.12) is unique. Indeed, assume that there exists another $X^2_1$ which satisfies definition 2.4. Put $G(\xi) = X^2(\xi, 0) - X^2_1(\xi, 0)$. Then by condition (2.12)

$$X^2(\xi, \rho) - X^2_1(\xi, \rho) = G(\xi) - G(\rho),$$

and, since $X^2 \in C^2(\nu)$, $G$ is a Hölder function of order greater than 1. Hence, $G = 0$ and thus $X^2 = X^2_1$.

Indeed, in this case $X$ is the Hölder function with exponent greater than 1 i.e. $X$ is equal to a constant function $X(0)$ and the only possible choice for the second component is $X^2 = 0$.

If $\nu > 1$ and $X$ is a $\nu$-rough path, then $X$ is given by the pair of constants $(X(0), 0)$. Indeed, in this case $X$ is the Hölder function with exponent greater than 1 i.e. $X$ is equal to a constant function $X(0)$ and the only possible choice for the second component is $X^2 = 0$.

If $\nu \in (1/2, 1]$ then $X^2$ is uniquely determined by $X$. Indeed, if we let

$$X^{2, ij}(\xi, \eta) = \int_{\xi}^{\eta} (X^i_t - X^i_{\xi}) \, dX^j_t, \quad \xi, \eta \in \mathbb{T}, i, j = 1, 2, 3,$$  

(2.13)

where the integral is understood in the sense of Young (see [22]) it is possible to show that $X^2 \in C^2(\nu)$ and that this definition satisfies conditions of definition 2.4. Since $\nu > 1/2$ it is the only possible choice.

**Definition 2.6.** We say that our $\nu$-rough path $(X, X^2)$ is a geometric $\nu$-rough path if there exists a sequence $(X_n, X^2_n)$ such that $X_n \in C(\mathbb{T}, \mathbb{R}^3)$ is piecewise smooth, and

$$\lim_{n \to \infty} \left[ \|X_n - X\|_{\nu} + \|X^2_n - X^2\|_{2\nu} \right] = 0,$$

(2.14)

where

$$X^{2, ij}_n(\xi, \eta) = \int_{\xi}^{\eta} (X^i_n(\rho) - X^i_{\xi}(\eta)) \, dX^j_n(\rho), \quad \xi, \eta \in \mathbb{T}, i, j = 1, 2, 3.$$

**Example 2.7.** Let $\{B_t\}_{t \in [0,1]}$ be the standard three-dimensional Brownian bridge such that $B_0 = B_1 = x_0$ and let $\mathbb{B}^{3, ij}, i, j = 1, 2, 3,$ be the area processes defined by

$$\mathbb{B}^{3, ij}(\xi, \eta) = \int_{\xi}^{\eta} (B^i_{\rho} - B^i_{\xi}) \, dB^j_{\rho},$$

where the integral can be understood either in the Stratonovich or in the Itô sense. Then, the couple $(B, \mathbb{B})$ is a $\nu$-rough path for all $\nu \in (1/3, 1/2)$ (see [6, p 1849]). Moreover, if the integral is understood in the Stratonovich sense, then $(B, \mathbb{B})$ is also a geometric $\nu$-rough path. Indeed, it follows from theorem 3.1 in [9] that one can approximate $X$ with piecewise linear dyadic $(X_n)_{n \geq 1}$ in the sense of definition 2.6 where the limit is understood in the almost sure sense.

From now on we suppose that the geometric $\nu$-rough path $X = (X, X^2)$ with $\nu > 1/3$ and the corresponding Banach space $D_X$ are fixed. For a finite partition $\pi = \{\xi_0 = \xi < \xi_1 < \cdots < \xi_n = \eta\}$ to be of the interval $[\xi, \eta]$, let $d(\pi) = \sup \{\xi_{i+1} - \xi_i\}$ denote the mesh size of the partition $\pi$. 
Lemma 2.8. If \( Y, Z \in \mathcal{D}_X \) then the limit
\[
\lim_{d(\pi) \to 0} \sum_{i=0}^{n-1} \left[ Y(\xi_i)(Z(\xi_{i+1}) - Z(\xi_i)) + Y'(\xi_i)Z'(\xi_i)\mathbb{X}^2(\xi_{i+1}, \xi_i) \right] =: \int_\xi^n Y \, dZ \tag{2.15}
\]
evaluates and defines the integral on the rhs.

**Proof.** This statement is a direct consequence of the existence and properties of the sewing map and of the definition of the controlled path space; see [12, theorem 1]. For the sake of presentation we sketch a proof. Let us denote
\[
A(\xi', \eta') = Y(\xi')(Z(\eta') - Z(\xi')) + Y'(\xi')Z'(\xi')(\eta' - \xi'), \quad \eta', \xi' \in T.
\]
Let us note that, with the assumptions of the lemma, \( \delta A \in C_3^{3\nu} \). Hence, provided that \( 3\nu > 1 \), we can apply the sewing map and we have the decomposition \( A = \delta I + \Lambda \delta A \) where \( I \in C_1 \). But this implies that
\[
\lim_{d(\pi) \to 0} \sum_{i=0}^{n-1} A(\xi_i, \xi_{i+1}) = \lim_{d(\pi) \to 0} \sum_{i=0}^{n-1} (I(\xi_{i+1}) - I(\xi_i)) + \lim_{d(\pi) \to 0} \sum_{i=0}^{n-1} (\Lambda \delta A(\xi_i, \xi_{i+1})).
\]
The first term on the rhs telescopes and since \( |\Lambda \delta A(\xi_i, \xi_{i+1})| \lesssim |\xi_{i+1} - \xi_i|^{3\nu} \) the second term on the rhs converges to zero proving the existence of the limit and moreover
\[
\lim_{d(\pi) \to 0} \sum_{i=0}^{n-1} A(\xi_i, \xi_{i+1}) = I(\eta) - I(\xi). \tag*{□}
\]

The explicit representation in terms of the sewing map can be used to prove the various estimates for the rough integral below.

**Remark 2.9.** In the case of \( \nu > 1/2 \) the line integral defined in lemma 2.8 is reduced to the Young definition of the line integral \( \int Y \, dZ \). Indeed, it is enough to notice that the second term in formula (2.15) is of the order \( O(|\xi_{i+1} - \xi_i|^{2\nu}) \), \( 2\nu > 1 \). In this case the line integral does not depend upon \( Y', Z' \).

Integrals of controlled paths have a very nice continuity property with respect to variations of the controlled path and also with respect to variations of the rough path on which the controlled path space is modelled.

In the following \( \mathbb{X} = (X, \mathbb{X}^2) \) and \( \mathbb{X} = (\hat{X}, \mathbb{X}^2) \) are two \( \nu \)-rough paths and \( Y \in \mathcal{D}_X \) and \( \hat{Y} \in \mathcal{D}_\hat{X} \) two path respectively controlled by \( X \) and \( \hat{X} \). In this case we introduce the following distance
\[
D(Y, \hat{Y}) = \|X - \hat{X}\|_{c^\nu} + \|\mathbb{X}^2 - \mathbb{X}^2\|_{c^\nu} + \|Y' - \hat{Y}'\|_{c^\nu} + \|R^Y - R^\hat{Y}\|_{c_{Y^c}} + \|Y - \hat{Y}\|_{c^\nu}.
\]

**Lemma 2.10.** Assume \( Y, W \in \mathcal{D}_X \), \( \hat{Y}, \hat{W} \in \mathcal{D}_\hat{X} \). Define maps \( Q, \hat{Q} : T^2 \to \mathbb{R} \) by the following identities
\[
Q(\eta, \xi) := \int_\xi^n Y \, dW - Y(\xi)(W(\eta) - W(\xi)) - Y'(\xi)W'(\xi)\mathbb{X}^2(\eta, \xi), \quad \eta, \xi \in T,
\]
\[
\hat{Q}(\eta, \xi) := \int_\xi^n \hat{Y} \, d\hat{W} - \hat{Y}(\xi)(\hat{W}(\eta) - \hat{W}(\xi)) - \hat{Y}'(\xi)\hat{W}'(\xi)\mathbb{X}^2(\eta, \xi), \quad \eta, \xi \in \hat{T},
\]
and a similar expression for \( \hat{Q} \) with \( Y, Y', W, W', X^2 \) replaced by \( \hat{Y}, \hat{Y}', \hat{W}, \hat{W}', \hat{X}^2 \). Then \( Q, \hat{Q} \in C_3^{3\nu} \) and there exists a constant \( C = C(\nu) > 0 \) such that for all \( Y, W \in \mathcal{D}_X \)
\[
\|Q\|_{C_3^{3\nu}} \leq C(1 + \|X\|_{c^\nu} + \|\mathbb{X}^2\|_{c^\nu})\|Y\|_{\mathcal{D}_X} \|W\|_{\mathcal{D}_X}.
\]

**Proof.**
Furthermore,
\[ \| Q - \hat{Q} \|_{c^\nu} \leq C(1 + \| X \|_{c^\nu} + \| X^2 \|_{c^\nu}) \]
\[ (\| Y \|_{\Delta x} + \| \tilde{Y} \|_{\Delta x} )\varepsilon_Y + (\| W \|_{\Delta x} + \| \tilde{W} \|_{\Delta x} )\varepsilon_X. \]
(2.19)

where

\[ \varepsilon_Y = \| Y' - \tilde{Y}' \|_{c^\nu} + \| RY - \tilde{R}\tilde{Y} \|_{c^\nu} + \| Y - \tilde{Y} \|_{c^\nu}, \]
\[ \varepsilon_W = \| W' - \tilde{W}' \|_{c^\nu} + \| RW - \tilde{R}\tilde{W} \|_{c^\nu} + \| W - \tilde{W} \|_{c^\nu}, \]
\[ \varepsilon_X = (\| Y \|_{\Delta x} + \| \tilde{Y} \|_{\Delta x} )(\| W \|_{\Delta x} + \| \tilde{W} \|_{\Delta x} )(\| X - \tilde{X} \|_{c^\nu} + \| X^2 - \tilde{X}^2 \|_{c^\nu}). \]

**Proof.** See [12, theorem 1]. For formula (2.19) see [12, p 104] formula (27). □

By lemmata 2.8 and 2.3 for any \( A \in C^2(\mathbb{R}^3, L(\mathbb{R}^3, \mathbb{R}^3)) \), \( Y \in \mathcal{D}_X \) we can define a map \( V^Y : \mathbb{R}^3 \to \mathbb{R} \) by means of the rough path integral as follows

\[ V^Y(x) := \int_0^1 A(x - Y(\xi)) \, dY(\xi), \quad x \in \mathbb{R}^3. \]
(2.20)

For maps \( F : \mathbb{R}^3 \to W \) and \( n \geq 0 \) we denote by \( \| F \|_{c^\nu} \) the usual norms

\[ \| F \|_{c^\nu} = \sup_{0 \leq |a| \leq n} \sup_{x \in \mathbb{R}^3} |\nabla^a F(x)|. \]

We have the following bounds on the regularity of \( V^Y \).

**Lemma 2.11.** There exists \( C_1 = C_1(v) \), \( C_2 = C_2(\mathbb{X}) \) such that for any integer \( n \geq 0 \) and all \( Y \in \mathcal{D}_X, \tilde{Y} \in \tilde{\mathcal{D}}_X \),

\[ \| \nabla^n V^Y \|_{L^\infty} \leq 4C_1C_2^2\| \nabla^{n+1} A \|_{c^\nu} \| Y \|_{\Delta x}^2 (1 + \| Y \|_{\Delta x}) \]
(2.21)

and

\[ \| \nabla^n V^Y - \nabla^n \tilde{V} \tilde{Y} \|_{L^\infty} \leq C(v)\| A \|_{c^\nu} C_X^1 (1 + \| Y \|_{\Delta x} + \| \tilde{Y} \|_{\Delta x})^3 D(Y, \tilde{Y}) \]
(2.22)

where

\[ C_X = 1 + \| X \|_{c^\nu} + \| \tilde{X} \|_{c^\nu} + \| X^2 \|_{c^\nu} + \| \tilde{X}^2 \|_{c^\nu}. \]

In the case of \( X = \tilde{X} \), inequality (2.22) can be rewritten as

\[ \| \nabla^n V^Y - \nabla^n \tilde{V} \tilde{Y} \|_{L^\infty} \leq 16C_1C_2^2\| \nabla^{n+1} A \|_{c^\nu} \| Y \|_{\Delta x}^2 (1 + \| Y \|_{\Delta x})^2 Y - \tilde{Y} \|_{\Delta x}^2. \]
(2.23)

**Proof.** By the results [6, lemma 7] it is known that \( V^Y \in C^n \) if \( A \in C^{n+2} \) and also that equations (2.21) and (2.23) hold. Now we will show (2.22). It is enough to consider the case of \( n = 0 \). By equation (2.18) we have

\[ V^Y(x) - \tilde{V}^Y(x) = A(x - Y(0))(Y(1) - Y(0)) - A(x - \tilde{Y}(0))(\tilde{Y}(1) - \tilde{Y}(0)) \]
\[ + (A(x - Y))'Y(0)X^2(0, 1) - (A(x - \tilde{Y}))'(0)\tilde{Y}(0)\tilde{X}^2(0, 1) \]
\[ + Q^{\nu}(0, 1) - \tilde{Q}^{\nu}(0, 1), \]

where \( Q^{\nu} \) and \( \tilde{Q}^{\nu} \) (given by the equation (2.18)) satisfy inequality (2.19). Recall that \( Y(1) = Y(0) \) and \( \tilde{Y}(1) = \tilde{Y}(0) \), hence we have

\[ \| V^Y - \tilde{V}^Y \|_{L^\infty} \leq \sup_x |(A(x - Y))'Y(0)X^2(0, 1) - (A(x - \tilde{Y}))'(0)\tilde{Y}(0)\tilde{X}^2(0, 1)| \]
\[ + \sup_x |Q^{\nu}(0, 1) - \tilde{Q}^{\nu}(0, 1)|. \]
(2.24)
For the first term on the rhs we have
\[
|\langle A(x - Y)^2(0)X^2(0, 1) - (A(x - Y))'(0)\tilde{Y}'(0)\tilde{X}'(0, 1)\rangle|
\leq |\nabla A(x - Y(0))Y'(0) - Y'(0)\tilde{Y}'(0)\tilde{X}'(0, 1) - 2\tilde{X}'(0, 1)|
\]
\[
+ |\nabla A(x - Y(0))\tilde{Y}'(0)\tilde{X}'(0, 1) - \tilde{X}'(0, 1)|
\leq \|X^2\|_C^2|\nabla A(x - Y(0))Y'(0) - Y'(0)\tilde{Y}'(0)\tilde{X}'(0, 1)|
\]
\[
+ A\|Y\|^2_{L^\infty}\|X^2 - \tilde{X}^2\|_C^2
\leq \|X^2\|_C^2|\nabla A(x - Y(0))Y'(0) - Y'(0)\tilde{Y}'(0)\tilde{X}'(0)|
\]
\[
+ \|\tilde{Y}'\|_{L^\infty}\|Y' - \tilde{Y}'\|_{L^\infty} + A\|\nabla Y\|^2 \|X^2 - \tilde{X}^2\|_C^2. \tag{2.25}
\]
By (2.19) we can estimate the second term as follows
\[
|Q - \tilde{Q}|_{C^2} \leq C\left[\|A(x - Y)\|_D^2 + \|A(x - \tilde{Y})\|_D^2\right]\epsilon_Y
\]
\[
+ \|Y\|_D + \|\tilde{Y}\|_D \epsilon_A + \epsilon_X,
\]
where
\[
\epsilon_Y = \|Y - \tilde{Y}\|_{C^2} + \|Y' - \tilde{Y}'\|_{C^2},
\]
\[
\epsilon_A = \|A(x - Y) - A(x - \tilde{Y})\|_{C^2} + \|A(x - Y)' - A(x - \tilde{Y})\|_{C^2} + \|R^A(x - Y) - R^A(x - \tilde{Y})\|_{C^2},
\]
\[
\epsilon_X = \|A(x - Y)\|_{D^2} + \|A(x - \tilde{Y})\|_{D^2} \times \|Y\|_{D^2} + \|\tilde{Y}\|_{D^2} \times \|X - \tilde{X}\|_{C^2} + |X^2 - \tilde{X}^2|_{C^2}.
\]
By formula (2.8) we can estimate \(\epsilon_A\) as follows
\[
\epsilon_A \leq K\|A\|_{C^2}(1 + \|X\|_{C^2} + \|\tilde{X}\|_{C^2})^3(1 + \|Y\|_{D^2} + \|\tilde{Y}\|_{D^2})^2
\]
\[
	imes \|X - \tilde{X}\|_{C^2} + \|Y - \tilde{Y}\|_{C^2} + \|Y' - \tilde{Y}'\|_{C^2} + \|R^Y - \tilde{R}^Y\|_{C^2}. \tag{2.26}
\]
By inequality (2.7) we infer that
\[
\|A(x - Y)\|_{D^2} \leq K\|A\|_{C^2}\|Y\|_{D^2}(1 + \|Y\|_{D^2})(1 + \|X\|_{C^2})^2, \tag{2.27}
\]
and similarly,
\[
\|A(x - \tilde{Y})\|_{D^2} \leq K\|A\|_{C^2}\|\tilde{Y}\|_{D^2}(1 + \|\tilde{Y}\|_{D^2})(1 + \|\tilde{X}\|_{C^2})^2. \tag{2.28}
\]
Therefore, combining inequalities (2.26) with (2.26), (2.27) and (2.28) we get
\[
\|Q - \tilde{Q}\|_{C^2} \leq C\|\tilde{Y}\|_{C^2}(1 + \|X\|_{C^2} + |X|_{C^2})^4(1 + \|Y\|_{D^2} + \|\tilde{Y}\|_{D^2})^3
\]
\[
\times \|X - \tilde{X}\|_{C^2} + \|X^2 - \tilde{X}^2\|_{C^2} + \|Y' - \tilde{Y}'\|_{C^2} + \|R^Y - \tilde{R}^Y\|_{C^2}. \tag{2.29}
\]
Hence, the result follows from (2.25) and (2.29).

3. The evolution problem for a rough filament

Let \(D_{X,T} = C([0, T], D_X)\), where the space \(D_X\) has been defined in definition 2.2, be a vector space with the usual supremum norm
\[
\|F\|_{D_{X,T}} = \sup_{t \in [0, T]} \|F(t)\|_{D_X}. \tag{3.1}
\]
Obviously \(D_{X,T}\) is a Banach space. For an element \(\gamma \in D_{X,T}\), the time-dependent vector field
\[
(t, x) \mapsto u^{\gamma(t)}(x) = \int_0^1 \nabla \phi(x - \gamma(t, \xi)) \times d\gamma(t, \xi)
\]
is well defined by the rough integral and it is meaningful to consider the Cauchy problem (1.1) in \(D_{X,T}\). The following local existence and uniqueness result been proved in [6, see theorem 3, p1842].
Theorem 3.1. Assume \( \phi \in C^6(\mathbb{R}^3, \mathbb{R}) \), \( v \in (1/3, 1) \), \( X = (X, X^2) \) is a \( v \)-rough path, \( \gamma_0 \in D_X \). Then there exists a time \( T_0 = T_0(v, \|\phi\|_{C^6}, X) > 0 \) such that the problem (1.1) has a unique solution in the space \( D_{\gamma_0, T_0} \subset D_{X, T_0} \). Moreover, if \( X, X^2 \) are two \( v \)-rough paths, \( \gamma_0 \in D_X \), \( \tilde{\gamma}_0 \in D_{X^2} \) and \( \gamma, \tilde{\gamma} \) two solutions of (1.1) respectively in \( D_{X, T_0} \) and \( D_{X^2, T_0} \) then there exists a constant \( C = C(X, X^2, \phi, v, \|\gamma_0\|_{D_X}, \|\tilde{\gamma}_0\|_{D_{X^2}}) \) such that

\[
\sup_{0 \leq t \leq T_0, \tilde{\gamma}(t) \in \gamma(t)} D(\gamma(t), \tilde{\gamma}(t)) \leq C D(\gamma(0), \tilde{\gamma}(0)).
\]

This result can be easily proven by applying the Banach fixed point theorem in the space \( D_{X, T} \) with a sufficiently small \( T \). The requirements on \( \phi \) can be easily understood as follows. In order for \( u^v \) to be well defined the function \( \nabla \phi \) has to be twice differentiable. Moreover in order to apply the fixed point theorem the function \( (t, \xi) \mapsto u^v(t, \xi) \) has to belong to \( D_{X, T} \) and this can be enforced by requiring that the map \( x \mapsto u^v(t, x) \) has to be \( C^2 \) in space, uniformly in time. This requires \( \nabla \phi \) to be at least four times differentiable. Furthermore we need a strict contraction of the fixed-point map which holds at the price of an additional derivative of \( \nabla \phi \) ending up with the requirement that \( \phi \) has to be six times differentiable. It is not obvious how to improve on this requirement for general rough initial data and \( v \in (1/3, 1/2) \).

The arguments used in theorem 3.1 do not allow a global solution. The aim of the present paper is to enforce further structure conditions on the kernel \( \phi \) in order to be able to use the energy method introduced in [5].

To prove the global existence result for the problem (1.1) we need the following additional hypothesis.

Hypothesis 3.2. Assume that the function \( \phi \) appearing in equation (1.2) has a Fourier transform \( \hat{\phi} : \mathbb{R}^3 \to \mathbb{R} \) which is real, even and non-negative and satisfying the integrability condition

\[
\int_{\mathbb{R}^3} (1 + |k|^2)^2 \hat{\phi}(k) \, dk < \infty.
\]

Example 3.3. The function \( \phi_\mu, \mu > 0 \) defined by

\[
\phi_\mu(v) = \frac{1}{(|v|^2 + \mu^2)^{3/2}}
\]

is smooth and satisfies hypothesis 3.2; see p 6 of [5]. This function is also of \( C^\infty \)-class so it also satisfies the assumptions of theorem 3.1 and as a result is an explicit example which satisfies all the assumptions needed for the following global existence result.

Theorem 3.4. Assume that \( \gamma_0 \) is a geometric \( v \)-rough path, \( \phi \in C^6(\mathbb{R}^3, \mathbb{R}) \) satisfies hypothesis 3.2 and \( v \in (1/3, 1) \). Then for every \( T > 0 \), the problem (1.1) has a unique solution in \( D_{\gamma_0, T} \).

We will need the following definition.

Definition 3.5. Let \( \phi \in C^6(\mathbb{R}^3, \mathbb{R}), \gamma \in D_X \). Let

\[
\psi^\gamma(x) = \int_0^1 \phi(x - \gamma(\eta)) \, d\gamma(\eta).
\]

and

\[
H^\phi_X(\gamma) = \frac{1}{2} \int_0^1 \psi^\gamma(\gamma(\xi)) \cdot d\gamma(\xi) = \frac{1}{2} \int_0^1 \int_0^1 \phi(\gamma(\xi) - \gamma(\eta))(d\gamma(\xi) \cdot d\gamma(\eta)).
\]

(3.2)

The quantity \( H^\phi_X(\gamma) \) is called the energy of path \( \gamma \). We will omit \( \phi \) below.
Remark 3.6. Definition (3.2) is well posed. Indeed, by lemma 2.11 $\psi'(y) \in C^2(\mathbb{R}^3, \mathbb{R}^3)$ and, therefore, it follows by lemma 2.3 that $\psi'(y) \in D_X$. Moreover, if $\nu > 1/2$ and $y \in C^1(T, \mathbb{R}^3)$, then by remark 2.9 the line integrals in the definition of the energy are understood in the sense of Young.

Lemma 3.7. Assume that $\phi \in C^4(\mathbb{R}^3, \mathbb{R})$. Then there exists a constant $C = C(\nu, \mathcal{H})$ such that for all $y \in D_X$

$$|H_X(y)| \leq C \|\phi\|_{C^4} \|y\|^2_{D_X} (1 + \|y\|_{D_X})^2. \quad (3.3)$$

Furthermore, for any $R > 0$ there exists $C = C(R)$ such that for any $y \in D_X$, $\tilde{y} \in D_\mathcal{H}$ satisfying

$$\|y\|_{D_X} \leq R, \|\tilde{y}\|_{D_\mathcal{H}} \leq R,$$

$$C_{X, \tilde{x}} = \|X\|_{C^\nu} + \|\tilde{X}\|_{C^\nu} + \|X^2\|_{C^\nu} + \|\tilde{X}^2\|_{C^\nu} < R$$

we have

$$|H_X(y) - H_X(\tilde{y})| \leq C(R)D(\gamma, \tilde{\gamma}). \quad (3.4)$$

In particular the map $H_X : D_X \to \mathbb{R}$ is Lipschitz on balls, i.e. for any $R > 0$ there exists $C = C(R)$ such that for any $y, \tilde{y} \in D_X$: $\|y\|_{D_X} \leq R, \|\tilde{y}\|_{D_X} \leq R$ we have

$$|H_X(y) - H_X(\tilde{y})| \leq C(R)\|y - \tilde{y}\|_{D_X}^3. \quad (3.5)$$

Proof. First we will show inequality (3.3). By representation (2.17) we have

$$H_X(y) = \frac{1}{2}[\psi'(y(0))(y(1) - y(0)) + [\nabla \psi'(y(0))y'(0)]y'(0)\mathcal{H}(1, 0) + Q(0, 1)$$

$$= I + II + III. \quad (3.6)$$

Since $\gamma(1) = \gamma(0)$ we infer that $I = 0$. Concerning the second term by lemma 2.11 we have the following estimate

$$|II| \leq \|X^2\|_{C^\nu} \|\nabla \psi'(y(0))\|_{L^\infty} \|y'(0)\|_{L^2} \leq C(\nu, \mathcal{X})\|\phi\|_{C^\nu} \|y\|^2_{D_X} (1 + \|y\|_{D_X}). \quad (3.7)$$

For the third term, from inequality (2.18) we infer that

$$|III| \leq \|Q\|_{C^\nu} \leq C(\nu, \mathcal{X})\|\psi'(y)\|_{D_X} \|y\|_{D_X}. \quad (3.8)$$

Then by lemmata 2.3 and 2.11 we have

$$\|\psi'(y)\|_{D_X} \leq C(\nu, \mathcal{X})\|\psi'(y)\|_{D_X} \|y\|_{D_X} (1 + \|y\|_{D_X}) \leq C(\nu, \mathcal{X})\|\phi\|_{C^\nu} \|y\|_{D_X} (1 + \|y\|_{D_X})^2. \quad (3.9)$$

Combining inequalities (3.7), (3.8) and (3.9) we get inequality (3.3). To prove inequality (3.4) we start by formula (2.18) to get

$$H_X(y) - H_X(\tilde{y}) = \frac{1}{2}\left[\nabla \psi'(y(0))y'(0) - \nabla \psi'(\tilde{y}(0))\tilde{y}'(0)\right]X(1, 0)$$

$$+ \nabla \psi'(\tilde{y}(0))\tilde{y}'(0)(X^2(1, 0) - \tilde{X}^2(1, 0)) + Q(0, 1) - \tilde{Q}(0, 1) \right]$$

$$= : I + II + III. \quad (3.10)$$

The first term in (3.10) can be represented as follows:

$$I = \nabla \psi'(y(0))y'(0) - \nabla \psi'(\tilde{y}(0))\tilde{y}'(0)X^2(1, 0) = A + B + C, \quad (3.11)$$

where
and the first term in (3.11) can be estimated as follows:
\[ |A| = |(\nabla \psi^\gamma(\nu(0)) - \nabla \bar{\psi}^\gamma(\nu(0)))\nu(0)X^\gamma(1, 0)| \]
\[ \leq \|\nabla^2_c\|_{C^2_\nu} \|\nu\|_{D_\nu}^2 (\|\nabla^2_c\|_{C^2_\nu} + |\nabla \psi^\gamma(\nu(0)) - \nabla \bar{\psi}^\gamma(\nu(0))|) \]
\[ \leq \|\nabla^2_c\|_{C^2_\nu} \|\nu\|_{D_\nu}^2 (\|\nu\|_{C^1} + \|\nu\|_{D_\nu} + \|\bar{\nu}\|_{D_\nu})^3 D(\gamma, \bar{\gamma}). \]

Here the second inequality follows from inequality (2.22) and the third one from inequality (2.21). For the second term in (3.11) we have by inequality (2.21)
\[ |B| \leq C\|\nabla\bar{\psi}^\gamma\|_{L^\infty} \|\nu\|_{D_\nu} \|\nabla\bar{\psi}^\gamma\|_{D_\nu} (1 + \|\bar{\nu}\|_{D_\nu}) D(\gamma, \bar{\gamma}) \]
\[ \leq CC_X (1 + \|\gamma\|_{D_\nu} + \|\bar{\gamma}\|_{D_\nu}^3) D(\gamma, \bar{\gamma}). \]

Similarly, we have for the third term
\[ |C| \leq C(v, X, \|\gamma\|_{D_\nu}, \|\bar{\gamma}\|_{D_\nu}) D(\gamma, \bar{\gamma}). \]

Going back to the term II in (3.10) we observe that it can be estimated as follows:
\[ |II| \leq \|\nabla \bar{\psi}^\gamma\|_{L^\infty} \|\gamma\|_{D_\nu} \|\bar{\psi}^\gamma\|_{D_\nu} D(\gamma, \bar{\gamma}) \]
\[ \leq C_X \|\phi\|_{C^1} (1 + \|\gamma\|_{D_\nu} + \|\bar{\gamma}\|_{D_\nu})^3 D(\gamma, \bar{\gamma}). \]

Thus it remains to estimate the third term of equality (3.10). By inequality (2.19) we have
\[ |Q(0, 1) - \tilde{Q}(0, 1)| \leq \|Q - \tilde{Q}\|_{C^\gamma}. \]

By the triangle inequality we have
\[ D(\psi^\gamma(\gamma), \psi^\gamma(\nu)) \leq D(\psi^\gamma(\gamma), \bar{\psi}^\gamma(\nu)) + D(\bar{\psi}^\gamma(\nu), \psi^\gamma(\nu)) \]
\[ = I + II. \]

The first term can be estimated by using inequality (2.8) as follows
\[ |I| \leq KC_X \|\bar{\psi}^\gamma\|_{C^1} (1 + \|\bar{\gamma}\|_{D_\nu} + \|\gamma\|_{D_\nu})^3 D(\gamma, \bar{\gamma}). \]

Combining inequalities (3.17) and (3.18) we obtain the necessary estimate for I. It now remains to find an estimate for the term II. By inequalities (2.7) and (2.23) we have
\[ II = D(\psi^\gamma(\gamma), \psi^\gamma(\nu)) \leq (1 + \|X\|_{C^1}) \|\psi^\gamma(\gamma) - \psi^\gamma(\nu)\|_{D_\nu} \]
\[ \leq K \|\nabla \psi^\gamma - \nabla \psi^\gamma\|_{C^1} \|\nu\|_{D_\nu} (1 + \|\nu\|_{D_\nu})^3 \]
\[ \leq K \|\phi\|_{C^1} C_X^2 (1 + \|\gamma\|_{D_\nu} + \|\bar{\gamma}\|_{D_\nu})^3 D(\gamma, \bar{\gamma}). \]

Hence the inequality (3.4) follows. Finally the bound (3.5) is a consequence of equation (3.4). □
Let us recall the definition (1.2) of the vector field $u^\gamma$ generated by a controlled path $\gamma$:

$$u^\gamma(x) = \int_0^1 \nabla \phi(x - \gamma(\xi)) \times d\gamma(\xi), \quad \gamma \in D_X. \quad (3.20)$$

Now we will show that if the energy functional of $\gamma$ is bounded then the associated velocity field is a smooth function. We have the following.

**Lemma 3.8.** Assume that $\phi \in \mathcal{C}^4(\mathbb{R}^3, \mathbb{R})$ and hypothesis 2.6 holds. Then the energy function $H_X : D_X \to \mathbb{R}$ is continuous. Furthermore, if $\gamma$ is a geometric rough path then

$$H_X(\gamma) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} |\phi(k)|^2 \left( \int_0^1 e^{i(k \cdot \gamma(\xi))} \, d\gamma(\xi) \right)^2 \, dk \geq 0. \quad (3.21)$$

Moreover, if in addition the integral $\int_{\mathbb{R}^3} |k|^{2(1+n)} \hat{\phi}(k) \, dk$ is finite and $\phi \in \mathcal{C}^{n+4}(\mathbb{R}^3, \mathbb{R})$, then for any $n \in \mathbb{N}_0$, we have the following bound:

$$\|\nabla^n u^\gamma\|_{L^\infty} \leq \frac{1}{(2\pi)^3} \left[ \int_{\mathbb{R}^3} |k|^{2(1+n)} \hat{\phi}(k) \, dk \right] H_X(\gamma), \quad \gamma \in D_X, \quad (3.22)$$

provided that

**Proof.** For a smooth curve $\gamma$ lemma 3.8 has been proved in [5, lemma 3]. In the general case, when $\gamma \in D_X$, it is enough to notice that both sides of equation (3.21) and of inequality (3.22) are locally Lipschitz and therefore, continuous w.r.t. distance $D(Y, \tilde{Y})$, $Y \in D_X$, $\tilde{Y} \in D_{\tilde{X}}$ defined in (2.16). Indeed, the continuity of $H_X$ and the continuity of $\|\nabla^n u^\gamma\|_{L^\infty}$ readily follows from lemma 2.11. \qed

Now we are going to show that energy is a local integral of motion for problem (1.1).

**Lemma 3.9.** Let $\gamma_0$ be a geometric $v$-rough path and $\gamma \in D_{\gamma_0, r_0}$ be a local solution of problem (1.1) (such a solution exists by theorem 3.1). Then

$$\frac{dH_{\gamma_0}(\gamma(s))}{ds} = 0, \quad s \in [0, T_0).$$

**Proof.** Since $\gamma(0) = \gamma_0 \in D_{\gamma_0}$ is a geometric rough path (we will denote its area component by $\Gamma_0$) there exists the sequence $\{\gamma_0^n\}_{n=1}^\infty$ of piecewise smooth closed curves in $\mathbb{R}^3$ such that if we denote by $(\gamma_0^n, \Gamma_0^n)$ their canonical lift to the space of $v$-rough paths we have

$$\|\gamma_0^n - \gamma_0\|_{C^2} + \|\Gamma_0^n - \Gamma_0\|_{C^2} \to 0, \quad n \to \infty.$$

Now observe that $\gamma_0^n \in D_{\gamma_0^n}, \gamma_0 \in D_{\gamma_0}$ since we can take $(\gamma_0^n)' = (\gamma_0)' = 1$ and $R^{\gamma_0^n} = R^{\gamma_0} = 0$. Hence we deduce that

$$D(\gamma_0^n, \gamma_0) \to 0, \quad n \to \infty.$$

Denote by $\gamma^n \in C([0, \infty), H^1(\mathbb{T}, \mathbb{R}^3))$ the global solution of problem (1.1) with the initial condition $\gamma_0^n$. The existence of such a solution has been proved in theorem 2 of [5]. Moreover, note that for smooth functions of $\gamma^n$ controlled by the rough path $(\gamma_0^n, \Gamma_0^n)$, the integral defined via rough paths coincides with the standard Lebesgue integral, and the solution of the Cauchy problem in $C([0, \infty), H^1(\mathbb{T}, \mathbb{R}^3))$ coincides with the rough solution whose local existence is stated in theorem 3.1. The locally Lipschitz dependence of the rough solution on the initial data stated in theorem 3.1 implies that

$$\lim_{n \to \infty} \sup_{t \in [0, T_0]} D(\gamma^n(t), \gamma(t)) \lesssim \lim_{n \to \infty} D(\gamma^n(0), \gamma(0)) = 0$$

where $D(\gamma^n(t), \gamma(t)) = D(\gamma^n, \gamma)$. \qed
for a strictly positive $T_0$ which depends only on the rough path norm of $\gamma_0$. Therefore, by the continuity of the energy functional $\mathcal{H}_{\gamma_0}$ we have
\[
\mathcal{H}_{\gamma_0}(\gamma(s)) = \lim_{n \to \infty} \mathcal{H}_{\gamma_0}^n(\gamma^n(s)), \quad s \in [0, T_0].
\] (3.23)

Furthermore, by lemma 2 of [5], we have
\[
\mathcal{H}_{\gamma_0}^n(\gamma^n(s)) = \mathcal{H}_{\gamma_0}(\gamma_0^n), \quad s \in [0, T_0].
\] (3.24)

As a result, combining inequalities (3.23) and (3.24) we get the statement of the lemma. □

Now we are ready to prove theorem 3.4.

**Proof of theorem 3.4.** According to theorem 3.1 there exists a unique local solution of problem (1.1). Then, we can find $T^* > 0$ and a unique maximal local solution $\gamma : [0, T^*) \to \mathcal{D}_{\gamma_0}$ which then satisfies
\[
\lim_{t \nearrow T^*} \|\gamma(t)\|_{\mathcal{D}_{\gamma_0}} = \infty.
\] (3.25)

We need to show that $T^* = \infty$. Therefore, it is enough to prove
\[
\sup_{t \in (0, T^*)} \|\gamma(t)\|_{\mathcal{D}_{\gamma_0}} < \infty.
\]

Indeed, by contradiction with (3.25), the result will follow. In the rest of the proof we show such an estimate. Notice that we will have
\[
\mathcal{H}_{\gamma_0}(\gamma(s)) = \mathcal{H}_{\gamma_0}(\gamma_0), \quad s \in [0, T^*),
\] (3.26)

and recall that
\[
\gamma(t) = \gamma_0 + \int_0^t u^{\gamma(s)}(\gamma(s)) \, ds.
\] (3.27)

Firstly we have
\[
\|\gamma(t)\|_{L^\infty} \leq \|\gamma_0\|_{L^\infty} + \int_0^t \|u^{\gamma(s)}\|_{L^\infty} \, ds \leq \|\gamma_0\|_{L^\infty} + C \int_0^t \mathcal{H}_{\gamma_0}^{1/2}(\gamma(s)) \, ds
\]
\[
\leq \|\gamma_0\|_{L^\infty} + CH_{\gamma_0}^{1/2}(\gamma_0)t, \quad t \in [0, T^*).
\] (3.28)

It follows from (3.27) that
\[
\gamma'(t) = \gamma_0' + \int_0^t \nabla u^{\gamma(s)}(\gamma(s))\gamma'(s) \, ds, \quad t \in [0, T^*).
\] (3.29)

Therefore, by lemmata 3.9 and 3.8 we have
\[
\|\gamma'(t)\|_{L^\infty} \leq \|\gamma_0'\|_{L^\infty} + \int_0^t \|\nabla u^{\gamma(s)}\|_{L^\infty} \|\gamma'(s)\|_{L^\infty} \, ds \leq \|\gamma_0'\|_{L^\infty} + \int_0^t C\mathcal{H}_{\gamma_0}^{1/2}(\gamma_0) \|\gamma'(s)\|_{L^\infty} \, ds
\]
\[
= \|\gamma_0'\|_{L^\infty} + \int_0^t C\mathcal{H}_{\gamma_0}^{1/2}(\gamma_0) \|\gamma'(s)\|_{L^\infty} \, ds, \quad t \in [0, T^*).
\] (3.30)

Then by the Gronwall lemma we infer our second estimate
\[
\|\gamma'(t)\|_{L^\infty} \leq \|\gamma_0'\|_{L^\infty} e^{C\mathcal{H}_{\gamma_0}^{1/2}(\gamma_0)t}, \quad t \in [0, T^*).
\] (3.31)

We will need one more auxiliary estimate. We have
\[
\|\gamma(t)\|_{C^0} \leq \|\gamma_0\|_{C^0} + \int_0^t \|u^{\gamma(s)}(\gamma(s))\|_{C^0} \, ds \leq \|\gamma_0\|_{C^0} + \int_0^t \|\nabla u^{\gamma(s)}\|_{L^\infty} \|\gamma(s)\|_{C^0} \, ds
\]
\[
\leq \|\gamma_0\|_{C^0} + \int_0^t C\mathcal{H}_{\gamma_0}^{1/2}(\gamma_0) \|\gamma(s)\|_{C^0} \, ds
\]
\[
= \|\gamma_0\|_{C^0} + \int_0^t C\mathcal{H}_{\gamma_0}^{1/2}(\gamma_0) \|\gamma(s)\|_{C^0} \, ds, \quad t \in [0, T^*).
\] (3.32)
Thus, by the Gronwall lemma we get
\[ \|y(t)\|_{C^2} \leq \|y_0\|_{C^2} e^{CH_{10}^{1/2}(\gamma_0)H}, \quad t \in [0, T^*). \] (3.33)
Now we can estimate the \( C^* \) norm of \( y' \). We have
\[
\|y'(t)\|_{C^*} \leq \|y_0\|_{C^*} + \int_0^t \|\nabla u^{y(s)}(y(s))y'(s)\|_{C^*} \, ds \\
\leq \|y_0\|_{C^*} + \int_0^t (\|\nabla u^{y(s)}\|_{L^\infty} \|y'(s)\|_{C^*} + \|y'(s)\|_{L^\infty} \|\nabla u^{y(s)}(y(s))\|_{C^*}) \, ds \\
\leq \|y_0\|_{C^*} + \int_0^t (\|\nabla u^{y(s)}\|_{L^\infty} \|y'(s)\|_{C^*} + \|y'(s)\|_{L^\infty} \|\nabla^2 u^{y(s)}\|_{L^\infty} \|y(s)\|_{C^*}) \, ds \\
\leq \|y_0\|_{C^*} + \int_0^t (CH_{10}^{1/2}(\gamma_0)(\|y'(s)\|_{C^*} + \|y'_0\|_{L^\infty} \|y_0\|_{C^*} e^{CH_{10}^{1/2}(\gamma_0)H}) \, ds, \\ t \in [0, T^*),
\] (3.34)
where the last inequality follows from lemmata 3.9 and 3.8. Then by the Gronwall Lemma we get the third estimate
\[ \|y'(t)\|_{C^*} \leq (\|y'_0\|_{C^*} + \|y_0\|_{L^\infty} \|y_0\|_{C^*}) e^{CH_{10}^{1/2}(\gamma_0)H}, \quad t \in [0, T^*). \] (3.35)
It remains to find an estimate for \( R^{y(s)}_{y(s)} \). We have
\[
R^{y(s)}_{y(s)} = R^{y_0} + \int_0^s R^{u^{y_0}(y(s))} \, ds, \quad t \in [0, T^*).
\] (3.36)
By identity (2.6) we have for \( s \in [0, T^*) 
\[
R^{u^{y_0}(y(s))}(\xi, \eta) = \nabla u^{y_0}(\gamma(s, \xi)) R^{y(s)}(\xi, \eta) + \sum_k (\gamma^k(s, \eta) - \gamma^k(s, \xi)) \\
\times \int_0^1 \left[ \frac{du^{y_0}(s)}{dx_k} (\gamma(s, \xi) + r(\gamma(s, \eta) - \gamma(s, \xi))) - \frac{\partial u^{y_0}(s)}{\partial x_k}(\gamma(s, \xi)) \right] \, ds.
\] (3.37)
Therefore,
\[ \|R^{u^{y_0}(y(s))}\|_{C^2} \leq \|\nabla u^{y_0}\|_{L^\infty} \|R^{y(s)}\|_{C^2} + \frac{1}{2} \|y(s)\|_{C^2} \|\nabla^2 u^{y(s)}\|_{L^\infty}, \quad s \in [0, T^*). \] (3.38)
Thus, by inequalities (3.38) and (3.33) we have for \( t \in [0, T^*) 
\[
\|R^{y(s)}\|_{C^2} \leq \|R^{y_0}\|_{C^2} + \int_0^s (\|\nabla u^{y_0}\|_{L^\infty} \|R^{y(s)}\|_{C^2} + \frac{1}{2} \|y(s)\|_{C^2} \|\nabla^2 u^{y(s)}\|_{L^\infty}) \, ds \\
\leq \|R^{y_0}\|_{C^2} + \int_0^s (\|\nabla u^{y_0}\|_{L^\infty} \|R^{y(s)}\|_{C^2} + \|y_0\|_{C^2} e^{CH_{10}^{1/2}(\gamma_0)H} \|\nabla^2 u^{y(s)}\|_{L^\infty}) \, ds \\
\leq \|R^{y_0}\|_{C^2} + C \|y_0\|_{C^2} e^{CH_{10}^{1/2}(\gamma_0)H} e^{CH_{10}^{1/2}(\gamma_0)H} \\
+ \int_0^s CH_{10}^{1/2}(\gamma_0) \|R^{y(s)}\|_{C^2} \, ds,
\] (3.39)
where in the last inequality we used lemmata 3.9 and 3.8. Hence, by the Gronwall lemma we get
\[ \|y(t)\|_{C^2} \leq (\|R^{y_0}\|_{C^2} + C(\|y_0\|_{C^2} + CH_{10}^{1/2}(\gamma_0)) e^{CH_{10}^{1/2}(\gamma_0)H}) e^{CH_{10}^{1/2}(\gamma_0)H}, \quad t \in [0, T^*), \] (3.40)
and combining estimates (3.28), (3.31), (3.35) and (3.40) we prove following \textit{a priori} estimate
\[ \|y(t)\|_{D_{y_0}} \leq K(1 + CH_{10}^{1/2}(\gamma_0))(1 + \|y_0\|_{D_{y_0}}) \|y_0\|_{D_{y_0}} e^{CH_{10}^{1/2}(\gamma_0)H}, \quad t \in [0, T^*), \] (3.41)
and the result follows. \[ \square \]
Acknowledgments

This research has been supported by the ANR Project ECRU (ANR-09-BLAN-0114-01/2). The authors would like to thank the anonymous referees for their careful reading and the detailed comments which helped us to improve the presentation.

References

[1] Arnold V I and Khesin B A 1998 Topological Methods in Hydrodynamics (Berlin: Springer)
[2] Beale J T, Kato T and Majda A 1984 Remarks on the breakdown of smooth solutions for the 3-D Euler equations Commun. Math. Phys. 94 61–6
[3] Bell J B and Marcus D L 1992 Vorticity intensification and transition to turbulence in the three-dimensional Euler equations Commun. Math. Phys. 147 371–94
[4] Bessaih H and Gubinelli M 2007 On the global evolution of vortex filaments, blobs, and small loops in 3D ideal flows Commun. Math. Phys. 269 693–713
[5] Bessaih H, Gubinelli M and Russo F 2005 The evolution of a random vortex filament Ann. Probab. 33 1825–55
[6] Brylinski J L 1993 Loop Spaces, Characteristic Classes and Geometric Quantization (Progress in Mathematics vol 107) (Boston, MA: Birkhäuser)
[7] Chernoff A J 1994 Vorticity and Turbulence (Applied Mathematical Sciences vol 103) (New York: Springer)
[8] Friz P K 2005 Continuity of the Itô-map for Hölder rough paths with applications to the support theorem in Hölder norm Probability and Partial Differential Equations in Modern Applied Mathematics (IMA Volumes in Mathematics and its Applications vol 140) (New York: Springer) pp 117–35
[9] Friz P K and Victoir N B 2010 Theory and applications Multidimensional Stochastic Processes as Rough Paths (Cambridge Studies in Advanced Mathematics vol 120) (Cambridge: Cambridge University Press)
[10] Gallavotti G 2002 Foundations of Fluid Dynamics (Texts and Monographs in Physics) (Berlin: Springer)
[11] Gubinelli M 2004 Controlling rough paths J. Funct. Anal. 216 86–140
[12] Hairer M 2013 Solving the KPZ equation Ann. Math. 178 559–64
[13] Helmholtz H 1858 Über Integrale der hydrodynamischen Gleichungen, welche den Wirbelbewegungen entsprechen J. Reine Angew. Math. (Crelles Journal) 1858 25–55
[14] Thomson W (Lord Kelvin) 1868 On vortex motion Trans. R. Soc. Edn 25 217–60
[15] Lyons T J 1998 Differential equations driven by rough signals Rev. Mat. Iberoamericana 14 215–310
[16] Lyons T J, Caruana M and Lévy T 2007 Differential Equations Driven by Rough Paths (Lecture Notes in Mathematics vol 1908) (Berlin: Springer)
[17] Meneguzzi M 1991 The spatial structure and statistical properties of homogeneous turbulence J. Fluid Mech. 225 1–20
[18] Vincent A and Meneguzzi M 1991 The spatial structure and statistical properties of homogeneous turbulence Acta Math. 67 251–82