Scaling Asymptotics for Szegő Kernels on Grauert Tubes

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Abstract
Let $M_\tau$ be the Grauert tube of radius $\tau$ of a closed, real analytic Riemannian manifold $M$. Associated to the Grauert tube boundary is the orthogonal projection $\Pi_\tau : L^2(\partial M_\tau) \to H^2(\partial M_\tau)$, called the Szegő projector. Let $D_\sqrt{\rho}$ denote the Hamilton vector field of the Grauert tube function $\sqrt{\rho}$ acting as a differential operator. We prove scaling asymptotics for the spectral localization kernel of the Toeplitz operator $\Pi_\tau D_\sqrt{\rho} \Pi_\tau$. We also prove scaling asymptotics for the smoothed spectral projection kernel $P_{\chi,\lambda}(z, w) = \sum_{\lambda_j \leq \lambda} \chi(\lambda_j) e^{-2\tau \lambda_j} \varphi_{\lambda_j}^C(z) \overline{\varphi_{\lambda_j}^C(w)}$, where $\varphi_{\lambda_j}^C$ are CR holomorphic functions on the Grauert tube boundary $\partial M_\tau$, which are obtained by analytically continuing Laplace eigenfunctions on $M$.

Keywords Bergman kernel · Szegő kernel · Grauert tube · Scaling asymptotics

Mathematics Subject Classification 32A25

1 Statement of Main Results

Let $(M, g)$ be a closed, real analytic Riemannian manifold. Its Grauert tube $M_\tau$ is a Kähler manifold with boundary that is diffeomorphic to the co-ball bundle consisting of co-vectors of length at most $\tau$. The Szegő projector $\Pi_\tau$ associated to the boundary of a Grauert tube, that is, the orthogonal projection

$$\Pi_\tau : L^2(\partial M_\tau) \to H^2(\partial M_\tau)$$

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onto the space of CR holomorphic functions on the Grauert tube boundary that are square integrable. Fix a positive, even Schwartz function $\chi$ whose Fourier transform is compactly supported with $\hat{\chi}(0) = 1$. We study the smoothed out spectral localizations

$$\Pi_{\chi, \lambda} := \Pi_{\tau} \chi (\Pi_{\tau} D_{\sqrt{\rho}} \Pi_{\tau} - \lambda) = \int_{\mathbb{R}} \hat{\chi}(t) e^{-it\lambda} \Pi_{\tau} e^{it\lambda} D_{\sqrt{\rho}} \Pi_{\tau} \, dt$$

(1)

of the Toeplitz operator

$$\Pi_{\tau} D_{\sqrt{\rho}} \Pi_{\tau} : H^2(\partial M_{\tau}) \to H^2(\partial M_{\tau}),$$

(2)

where $D_{\sqrt{\rho}} = \frac{1}{i} \sqrt{\rho}$ is a constant multiple of the Hamilton vector field of the Grauert tube function $\sqrt{\rho}$ acting as a differential operator; see Sect. 2.1 for background on Grauert tubes. As explained in Sect. 1.1, the operator (1) is an analog of the Fourier components (12) of the Szegő kernel in the line bundle setting.

The scaling asymptotics is expressed in Heisenberg coordinates on $\partial M_{\tau}$ in the sense of [13, Sect. 14, 18]. We give the precise definition of the coordinates here; additional geometric background is recalled in Sect. 3.

**Definition 1.1** By Heisenberg coordinates (relative to the orthonormal frame $Z_1, \ldots, Z_{m-1}$) at $T_p^{1,0} \partial M_{\tau}$, we mean holomorphic coordinates $z_0, \ldots, z_{m-1}$ in a neighborhood of $p \in M_{\tau}$ with the following properties.

(i) If we set $\theta = \text{Re} z_0$ and $z_j = x_j + iy_j$, then $x_1, \ldots, x_{m-1}, y_1, \ldots, y_{m-1}, \theta$ form a coordinate system in a neighborhood of $p$ in $\partial M_{\tau}$.

(ii) Moreover, we have

$$Z_j = \frac{\partial}{\partial z_j} + i \frac{\partial}{\partial t} + \sum_{k=1}^{m-1} O^1 \frac{\partial}{\partial z_k} + O^2 \frac{\partial}{\partial \theta},$$

$$T = \frac{\partial}{\partial \theta} + \sum_{k=1}^{m-1} O^1 \frac{\partial}{\partial z_k} + \sum_{k=1}^{m-1} O^1 \frac{\partial}{\partial \bar{z}_k} + O^1 \frac{\partial}{\partial \theta}.$$  

Here, the Heisenberg-type order $O^k$ is defined inductively by

$$f = O^1 \text{ if } f(\eta) = O \left( \sum_{k=1}^{m-1} (|x_k(\eta)| + |y_k(\eta)|) + |\theta(\eta)|^{\frac{1}{2}} \right) \text{ as } \eta \to p,$$

$$f = O^k \text{ if } f = O(O^1 \cdot O^{k-1}).$$

(3)

**Theorem 1.2** (Scaling asymptotics for $\Pi_{\chi, \lambda}$) Let $\Pi_{\chi, \lambda}$ be the spectral projection (1) associated to the Grauert tube boundary $\partial M_{\tau}$ of a closed, real analytic Riemannian manifold $M$ of dimension $m$. Fix $p \in \partial M_{\tau}$. Then, in Heisenberg coordinates centered at $p = 0$, the distribution kernel has the following scaling asymptotics:

$$\Pi_{\chi, \lambda} \left( \frac{\theta}{\lambda}, \frac{u}{\sqrt{\lambda}}, \frac{\varphi}{\lambda}, \frac{v}{\sqrt{\lambda}} \right)$$
\[
\frac{C_{m,M} \left( \frac{\lambda}{\tau} \right)}{\tau} m^{-1} e^{\frac{i}{\tau} \left( \frac{1}{2} \theta^2 - \frac{|w^2|}{2} - \frac{|v|^2}{2} + v \cdot \overline{w} \right)}
\times \left( 1 + \sum_{j=1}^{N} \lambda^{-j} P_j(p, u, \theta, \varphi) + \lambda^\frac{N+1}{2} R_N (p, \theta, u, \varphi, v, \lambda) \right),
\]

where \( \| R_N (p, \theta, u, \varphi, v, \lambda) \|_{C_j \left( \{ |\theta| + |\varphi| + |u| + |v| \leq \rho \} \right)} \leq C_{K, j, \rho} \) for \( \rho > 0 \), \( j = 1, 2, 3, \ldots \) and \( P_j \) is a polynomial in \( u, v, \theta, \varphi \).

To give some context to our scaling asymptotics, recall the reduced Heisenberg group \( H_{m-1}^{\text{red}} = H_{m-1} / \{(0, 2\pi k) : k \in \mathbb{Z}\} = S^1 \times \mathbb{C}^{m-1} \) of degree \( m-1 \), which is obtained as a discrete quotient of the Heisenberg group \( H_{m-1} = \mathbb{R} \times \mathbb{C}^{m-1} \). The group law on \( H_{m-1}^{\text{red}} \) is given by

\[
(e^{i t}, \zeta) \cdot (e^{i s}, \eta) = (e^{i (t + s + \text{Im}(\zeta \cdot \overline{\eta}) + \frac{t}{2} |\zeta|^2 - \frac{1}{2} |\zeta|^2 - \frac{1}{2} |\eta|^2) \cdot (\zeta + \eta).)
\]

The level one Szegő projector \( \Pi_1^H : L^2(H_{m-1}^{\text{red}}) \to H_1^2 \) is the orthogonal projection onto the space \( H_1^2 \) of CR holomorphic functions square integrable with respect to \( e^{-|z|^2} \). Its Schwartz kernel has an exact formula:

\[
\Pi_1^H (\theta, z, \varphi, w) = \frac{1}{\pi^{m-1}} e^{i (\theta - \varphi)} e^{-\frac{1}{2} |z|^2 - \frac{1}{2} |w|^2}.
\]

We refer to [2] for more details. It is known (cf (16)) that, in correctly chosen coordinates, the Szegő kernel in the line bundle case is to leading order some multiple of (4); see for example [2, Theorem 3.1] or [31, Theorem 2.1]. Theorem 1.2 shows that the same phenomenon holds in the Grauert tube setting:

\[
\Pi_{\chi, \lambda} \left( \frac{\theta, u}{\lambda}, \frac{\varphi, v}{\sqrt{\lambda}} \right) = \Pi_1^H \left( \frac{\theta}{2\tau}, \frac{u}{2\tau}, \frac{\varphi}{\sqrt{\tau}}, \frac{v}{\sqrt{\tau}} \right)
\times \left( 1 + \sum_{j=1}^{N} \lambda^{-j} P_j(p, u, \theta, \varphi) + \lambda^\frac{N+1}{2} R_N (p, \theta, u, \varphi, v, \lambda) \right)
\]

Another natural operator to study is the spectral projection kernel defined using analytic continuation of Laplace eigenfunctions. Let

\[
\Delta = -\frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x_j} \left( \sqrt{|g|} g^{ij} \frac{\partial}{\partial x_k} \right)
\]

be the (positive) metric Laplacian. It is well known that \( \Delta \) has a discrete spectrum:

\[
0 = \lambda_0^2 < \lambda_1^2 \leq \cdots \lambda_j^2 \leq \cdots \to \infty
\]
with associated $L^2$-normalized eigenfunctions
\[ \Delta \varphi_{\lambda,j} = \lambda_j^2 \varphi_{\lambda,j}, \quad \| \varphi_{\lambda,j} \|_{L^2(M)} = 1. \] (5)

Our next result concerns the complex analog of the partial spectral projection kernel
\[ E_\lambda(x, y) = \sum_{\lambda,j \leq \lambda} \varphi_{\lambda,j}(x) \overline{\varphi_{\lambda,j}(y)}. \] (6)

The Poisson wave operator is used to analytically continue eigenfunctions. One starts with the half-wave operator $U(t) = e^{it\sqrt{\Delta}} : L^2(M) \to L^2(M)$ whose Schwartz kernel can be expressed as a sum of eigenfunctions as well as an oscillatory integral (for instance the Lax–Hörmander parametrix):
\[ U(t, x, y) = \sum_{j=0}^{\infty} e^{it\lambda_j} \varphi_{\lambda,j}(x) \overline{\varphi_{\lambda,j}(y)} = \int_{T^* M} A(t, x, y, \xi) e^{it|\xi|_y} e^{i(\xi, \exp^{-1}(x))} d\xi. \] (7)

Analytically continuing the oscillatory integral representation in the time variable $t \mapsto t+i\tau \in \mathbb{C}$ and the spatial variable $x \to z \in \partial M_\tau$ yields the Poisson wave kernel (see Theorem 2.4):
\[ U(i\tau) : L^2(M) \to \mathcal{O}(\partial M_\tau). \] (8)

Here, $\mathcal{O}(\partial M_\tau) = \ker(\overline{\partial}_b)$ denotes the space of CR holomorphic functions on the Grauert tube boundary. Comparing (7) with (8), we find
\[ U(i\tau, z, y) = \sum_{j=0}^{\infty} e^{-2\tau\lambda_j} \varphi^C_{\lambda,j}(z) \overline{\varphi^C_{\lambda,j}(y)}. \]

Hence, the formula
\[ \varphi^C_{\lambda,j} = e^{\tau\lambda_j} U(i\tau) \varphi_{\lambda,j} \in \mathcal{O}(\partial M_\tau) \]
defines the analytic extension of Laplace eigenfunctions (5).

We define tempered spectral projections $P_\lambda$ by the eigenfunction partial sums
\[ P_\lambda : \mathcal{O}(\partial M_\tau) \to \mathcal{O}(\partial M_\tau), \quad P_\lambda(z, w) = \sum_{j: \lambda,j \leq \lambda} e^{-2\tau\lambda_j} \varphi^C_{\lambda,j}(z) \overline{\varphi^C_{\lambda,j}(w)}. \]

The prefactor $e^{-2\tau\lambda_j}$ is introduced because of the exponential growth estimate ([35, Corollary 3]) on complexified eigenfunctions
\[ \lambda_j^{m-1} e^{\tau\lambda_j} \lesssim \sup_{\xi \in M_\tau} |\varphi^C_{\lambda,j}(z)| \lesssim \lambda_j^{m-1} e^{\tau\lambda_j}. \]
As before, fix a positive, even Schwartz function $\chi$ whose Fourier transform is compactly supported with $\hat{\chi}(0) = 1$. The techniques for proving Theorem 1.2 are easily adapted to prove scaling asymptotics for the smoothed sum

$$P_{\chi, \lambda}(z, w) = \chi \ast d_{\lambda} P_{\lambda}(z, w) = \sum_{j: \lambda_j \leq \lambda} \chi(\lambda - \lambda_j)e^{-2\tau \lambda_j \phi_{\lambda_j}}(z)\overline{\phi_{\lambda_j}(w)}.$$

(9)

**Theorem 1.3** (Scaling asymptotics for $P_{\chi, \lambda}^\tau$) Let $P_{\chi, \lambda}^\tau$ be the spectral projection (9) associated to the Grauert tube boundary $\partial M_\tau$ of a closed, real analytic Riemannian manifold $M$ of dimension $m$. Fix $p \in \partial M_\tau$. Then, in Heisenberg coordinates centered at $p = 0$, the distribution kernel has the following scaling asymptotics:

$$P_{\chi, \lambda}^\tau\left(\frac{\theta}{\lambda}, \frac{u}{\sqrt{\lambda}}, \frac{\phi}{\lambda}, \frac{v}{\sqrt{\lambda}}\right) = \frac{C_{m, M} \lambda^{m-1}}{\tau^m} \prod_1^H \left(\frac{\theta}{2\tau}, \frac{u}{\sqrt{\tau}}, \frac{\phi}{2\tau}, \frac{v}{\sqrt{\tau}}\right) \times \left(1 + \sum_{j=1}^N \lambda^{-\frac{1}{2}} P_j(p, u, v, \theta, \phi) + \lambda^{-\frac{N+1}{2}} R_N(p, \theta, u, \phi, v, \lambda)\right)$$

where $\|R_N(p, \theta, u, \phi, v, \lambda)\|_{C_j(|\theta|+|\phi|+|u|+|v|\leq \rho)} \leq C_{K, j, \rho}$ for $\rho > 0$, $j = 1, 2, 3, \ldots$ and $P_j$ is a polynomial in $u, v, \theta, \phi$.

**Remark 1.4** The leading order asymptotics of Theorems 1.2 and 1.3 differ by a factor of $\lambda^{-\frac{m-1}{2}}$ because the analytic extensions $\phi_{\lambda_j}^C$ are not $L^2$-normalized on $\partial M_\tau$.

Theorem 1.3 is the complex analog of [3, Proposition 10], where it was shown that the rescaled real spectral projection kernel $E_\lambda(x + \frac{u}{\sqrt{\lambda}}, y + \frac{v}{\sqrt{\lambda}})$ from (6) exhibits Bessel-type scaling asymptotics as $\lambda \to \infty$. The Heisenberg-scaling of the present paper is performed with parameter $\sqrt{\lambda}$ rather than $\lambda$. The complex geometry of the Grauert tube wins over the real Riemannian geometry since as $\tau \to 0$ our scaling asymptotics do not resemble Bessel asymptotics. It should also be mentioned that, unlike the authors of [3], we do not study the sharp (i.e., without the smoothing $\chi$) spectral projection in this paper.

**1.1 Comparison to the Line Bundle Setting**

Let $(L, h) \to (M, \omega)$ be a positive Hermitian line bundle over a closed Kähler manifold. The **Bergman projections** are orthogonal projections

$$\Pi_{hk} : L^2(M, L^k) \to H^0(M, L^k)$$

(10)

from the space of $L^2$ sections of the $k$th tensor power of the line bundle onto the space of square integrable holomorphic sections.
The Bergman projections are related to the Szegő projection

$$\Pi_h: L^2(\partial D) \to H^2(\partial D),$$

(11)

where $\partial D = \{ \ell \in L^*: \|\ell\|_{L^*} = 1 \}$ is the unit co-circle bundle. Indeed, let $r_\theta$ denote the circle action on $\partial D$, then the map

$$H^0(M, L^k) \to \{ f \in H^2(\partial D) : f(r_\theta x) = e^{ik\theta} f(x) \}

s_k(z) \mapsto f(x) = (\ell^\otimes k, s_k(z)) \text{ where } x = (\ell, z) \in \pi^{-1}(z) \times M$$

defines a unitary equivalence between the space of holomorphic sections of $L^k$ and the subspace of equivariant functions in $H^2(\partial D)$. Since the circle action on $\partial D$ commutes with $\partial / \partial \theta$, we conclude that the kernel of (10) are Fourier coefficients of the kernel of (11), that is,

$$\Pi_{h^k}(x, y) = \frac{1}{2\pi} \int_0^{2\pi} e^{-ik\theta} \Pi_h(r_\theta x, y) d\theta = \left( \frac{1}{2\pi} \int_0^{2\pi} e^{-ik\theta} r_\theta^* \Pi_h d\theta \right)(x, y).$$

(12)

Note that the Fourier decomposition of $\Pi_h$ coincides with the spectral decomposition of $D_\theta = \frac{\partial}{\partial \theta} / \partial b$ on $\partial D$.

In the Grauert tube setting, $\partial M_\tau$ plays the role of the strongly pseudoconvex CR hypersurface $\partial D \subseteq L^*$ and the Hamilton vector field $\Xi \sqrt{\rho}$ of the Grauert tube function on $\partial M_\tau$ plays the role of the Reeb vector field $\partial / \partial \theta$ on $\partial D$. With this analogy in mind, the direct analog of (12) in the Grauert tube setting is

$$\int_{\mathbb{R}} \hat{\chi}(t)e^{-i\lambda t} \Pi_\tau(G^\tau_t(z), w) dt = \left( \int_{\mathbb{R}} \hat{\chi}(t)e^{-i\lambda t} (G^\tau_t)^* \Pi_\tau dt \right)(z, w),$$

(13)

in which

- integration against (the Fourier transform of) a suitably chosen Schwartz function $\hat{\chi}$ replaces integration on the circle in the line bundle setting;
- pullback by the Hamilton flow (20) on the Grauert tube boundary replaces that of the circle action in the line bundle setting;
- the orthogonal projection (22) onto the Grauert tube boundary replaces the Szegő projection in the line bundle setting.

It might appear that (13) is the correct object to study, but this is not quite right. Unlike the circle action on $\partial D$ generated by $\frac{\partial}{\partial \theta}$, the geodesic flow on a Grauert tube is never holomorphic, nor does its orbits form a fiber bundle over a quotient space. In particular, $G^\tau_t$ need not commute with $\Pi_\tau$, so (13) fails to be CR holomorphic in the $z$ variable. This issue of holomorphy can be fixed: It is shown in [32, Proposition 5.3] (see Proposition 2.7 for the statement) that there exists a polyhomogeneous pseudodifferential operator $\hat{\sigma}$ on $\partial M_\tau$ so that

$$\Pi_\tau e^{i\pi t} D_\sqrt{\rho} \Pi_\tau = \Pi_\tau \hat{\sigma} (G^\tau_t)^* \Pi_\tau \text{ modulo smoothing Toeplitz operators.}$$
Evidently, the kernel of (14) is CR holomorphic in both variables. Thus, we are led to the CR holomorphic analog of (12) that is

\[ \int_{\mathbb{R}} \check{\chi}(t) e^{-i\lambda t} \Pi_\tau \tilde{\sigma} (G_t^\tau)^* \Pi_\tau \, dt = \int_{\mathbb{R}} \check{\chi}(t) e^{-i\lambda t} \Pi_\tau e^{i\Pi_\tau D_{\sqrt{\rho}} \Pi_\tau} \, dt \]

to leading order.

(15)

**Remark 1.5** We note that (15) is essentially the spectral decomposition of the elliptic Toeplitz operator \( D_{\sqrt{\rho}} \Pi_\tau \) introduced in (2). An interesting problem is to compare the spectrum of the Toeplitz operator with that of \( \sqrt{\Delta} \), as

\[ U(i\tau) \sqrt{\Delta} U(i\tau)^* = \Pi_\tau D_{\sqrt{\rho}} \Pi_\tau \]

modulo zeroth-order Toeplitz operator

where \( U(i\tau) \) is the Poisson wave operator discussed in Sect. 2.4. We include further discussions in this direction in a subsequent paper [4, Sect. 5].

### 1.2 Related Results in the Line Bundle Setting

Scaling asymptotics in the line bundle setting have been proved in varying degrees of generality; see [2, 17, 25–27, 31]. The simplest setup is a closed Kähler manifold \( M \) polarized by an ample line bundle \( L \). Endow \( L \) with a Hermitian metric \( h \) so that the curvature two-form \( \Omega \) is positive, and let \( \omega = \frac{1}{2} \Omega \) be the Kähler form on \( M \).

Choose coordinates \( z = (z_1, \ldots, z_m) \) in a neighborhood \( U \) centered at \( p \in M \) so that the Kähler potential is locally of the form \( \phi(z) = |z|^2 + O(|z|^3) \). We identify the Bergman kernel with the Fourier components of the Szegő kernel using (11) and write

\[ \Pi_{hk}(x, y) = \Pi_{hk}(z, \theta, w, \varphi), \]

where \( x = (z, \theta) \) are linear coordinates on \( T_z M \times S^1 \). Then, the Bergman kernel scaling asymptotics is of the form

\[ k^{-m} \Pi_{hk} \left( \frac{z}{\sqrt{k}}, \frac{\theta}{\sqrt{k}}, \frac{w}{\sqrt{k}}, \frac{\varphi}{k} \right) = \frac{1}{\pi^m} e^{i(\theta-\varphi) + i3(z \cdot \bar{w}) - \frac{1}{4}|z-w|^2} \]

\[ \times \left[ 1 + \sum_{n=1}^{N} k^{-\frac{n}{2}} b_k(z, w) + k^{-\frac{N+1}{2}} R_N(z, w) \right] \]

(16)

with

\[ \| R_N \|_{C^j(|z| + |w| \leq \delta)} \leq C(N, j, \delta). \]

The remainder estimate implies Gaussian decay of the Bergman kernel in a neighborhood of the diagonal:

\[ \left| \Pi_{hk} (z, \theta, w, \varphi) \right| \leq \frac{k^m}{\pi^m} e^{-\frac{kD(z, w)}{2}} \left( 1 + O \left( \frac{1}{k} \right) \right) \]

whenever \( d(z, w) \lesssim \sqrt{\log k} \).
Here, $D(z, w)$ is the Calabi diastasis (29), which is bounded above and below by the square $d^2(z, w)$ of the Riemannian distance function. A complete asymptotic expansion far away from the diagonal is available thanks to the recent papers [10, 11, 19, 30] on analytic Bergman kernel. For instance, [18, Theorem 1.1] shows (under analyticity assumption of the Kähler potential) that

$$\left| \Pi_{h_k} (z, \theta, w, \varphi) \right| = \frac{k^m}{\pi^m} e^{-\frac{kD(z, w)}{2}} \left( 1 + O \left( \frac{1}{k} \right) \right) \text{ whenever } d(z, w) \lesssim \frac{1}{k}.$$ 

In more general settings (including when the Kähler potential is no longer assumed to be analytic), global Agmon-type estimates

$$\left| \Pi_{h_k} (z, \theta, w, \varphi) \right| \leq C e^{-c\sqrt{k}d(z, w)} \text{ for all } z, w \in M$$

have been established in [1, 5, 6, 12, 24, 28, 37].

In the presence of a real Hamiltonian $f \in C^\infty (M)$, there is a Hamiltonian flow $\Phi^M_\tau$ and its lift $\Phi^L_\tau$ to $L$ in addition to the circle action. In [29], the phase space scaling asymptotics is computed near a fixed point $m$ and time $\tau$ in the graph of $\Phi^M_\tau$ for the dynamical Toeplitz operators

$$U_\tau = R_\tau \circ \Pi \circ (\Phi^L_\tau \circ \Pi),$$

where $R_\tau$ is a 0th order Toeplitz operator. These asymptotics have an additional dependence on the derivative of $\Phi^M_{-\tau}$ at $m$. The nature of the computation in our proof is similar to the one in [29].

### 2 Geometry of and Analysis on Grauert Tubes

We assume throughout that $(M, g)$ is a closed, real analytic Riemannian manifold of dimension $m$. Grauert tubes were first studied by [15, 16, 22, 23] in relation to the complex Monge–Ampère equation and complexified geodesics. The Poisson wave operator and the tempered spectral projection (33), as well as other related operators, have been studied in [32, 34, 35].

#### 2.1 Grauert Tube and the Cotangent Bundle

Bruhat–Whitney proved that a real analytic manifold $M$ admits a complexification $M_\mathbb{C}$ such that $M \subseteq M_\mathbb{C}$ is a totally real submanifold, that is, $T_p M \cap JT_p M = \{0\}$ for all $p \in M$.

In a neighborhood of $M$ in $M_\mathbb{C}$, there exists a unique strictly plurisubharmonic function $\rho: U \subseteq M_\mathbb{C} \to [0, \infty)$ whose square root is the solution of the Monge–Ampère equation

$$\det \left( \frac{\partial^2 \sqrt{\rho}}{\partial z_j \partial \bar{z}_k} \right) = 0 \text{ on } U \setminus M,$$
with the initial condition that the metric induced by the Kähler form \( i \partial \bar{\partial} \rho \) restricts to the Riemannian metric \( g \) on \( M \). In fact, \( \sqrt{\rho} \) is given by

\[
\sqrt{\rho} : U \subseteq M_C \to \mathbb{R}, \quad \sqrt{\rho}(z) = \frac{1}{2i} \sqrt{r_C^2(z, \bar{z})},
\]

where \( r_C^2(z, \bar{w}) \) is the analytic extension of the square of the Riemannian distance function \( r : M \times M \to \mathbb{R} \) to a neighborhood of the diagonal in \( M_C \times M_C \). We call (17) the Grauert tube function. For each \( 0 < \tau < \tau_{\text{max}} \), the sublevel set

\[
M_\tau = \{ z \in M_C : \sqrt{\rho}(z) < \tau \}
\]

is called the Grauert tube of radius \( \tau \).

The complexified exponential map can be used to identify (18) with the co-ball bundle of radius \( \tau \):

\[
B^*_\tau M = \{ (x, \xi) : |\xi| < \tau \}.
\]

To explain this, we introduce notation for geodesic flows.

- The (geometer’s) geodesic flow \( g^t : T^*M \to T^*M \) is the Hamilton flow of the metric norm squared function \( H(x, \xi) = |\xi|^2_\rho \).

- The homogeneous geodesic flow \( G^t : T^*M - 0 \to T^*M - 0 \) is the Hamilton flow of the metric norm function \( \sqrt{H}(x, \xi) = |\xi|_x \). Note that \( G^t(x, \varepsilon \xi) = \varepsilon G^t(x, \xi) \).

Let \( \pi : T^*M \to M \) be the natural projection. The exponential map \( \exp : T^*M \to M \) is defined by \( \exp_x \xi = \pi g^1(x, \xi) \). Analyticity of the metric allows us to complexify \( t \mapsto t + i \tau \) in the time variable. For \( 0 < \tau < \tau_{\text{max}} \), the imaginary time geodesic flow \( E \) is a diffeomorphism ([36, Lemma 1.1])

\[
E : B^*_\tau M \to M_\tau, \quad E(x, \xi) = \exp^C_x i\xi.
\]

With \( \sqrt{H}(x, \xi) = |\xi| \) and \( \sqrt{\rho} \) the Grauert tube function, we have

\[
H = \rho \circ E \quad \text{and} \quad \sqrt{H} = \sqrt{\rho} \circ E.
\]

It follows that \( E^{-1} \) conjugates the geodesic flow \( G^t \) (resp. \( g^t \)) on the cotangent bundle to the Hamilton flow of Hamiltonian vector field \( \Xi_{\sqrt{\rho}} \) (resp. \( \Xi_\rho \)) on the Grauert tube. In particular, the transfer

\[
G^t : \partial M_\tau \to \partial M_\tau, \quad G^t = E \circ G^t \circ E^{-1}|_{\partial M_\tau}
\]

of the geodesic flow from the energy surface \( \partial B^*_\tau M \) to the Grauert tube boundary \( \partial M_\tau \) coincides with the restriction of the Hamilton flow of \( \Xi_{\sqrt{\rho}} \) to \( \partial M_\tau \).

Finally, if we write \( \alpha_{T^*M} = \xi \, dx \) and \( \omega_{T^*M} = d\xi \wedge dx \) for the canonical 1-form and the symplectic form on the cotangent bundle, then we also have

\[
(E^{-1})^* \alpha_{T^*M} = \text{Im} \, \partial \rho = d^c \sqrt{\rho} \quad \text{and} \quad (E^{-1})^* \omega_{T^*M} = -i \partial \bar{\partial} \rho.
\]
Note that the right-hand sides of the two equalities are the canonical 1-form and the Kähler form on the Grauert tube $M_\tau$.

**Remark 2.1** It is useful to think of the Grauert tube as the co-ball bundle endowed with an *adapted complex structure* $J = J_g$ induced from the Riemannian metric on $M$. This complex structure is characterized as follows. Let $\gamma: \mathbb{R} \to M$ be a geodesic (i.e., a $g^\tau$ orbit) and let $\gamma_C: \{ t + i \tau \in \mathbb{C} : \tau < \tau_{\max} \} \to M_C$ be its analytic continuation to a strip. Then, $J_g$ is the complex structure on $M_\tau = B^*_\tau M$ so that $\gamma_C$ is a holomorphic map; see [22].

2.2 Contact and CR Structure on the Boundary of a Grauert Tube

The boundary of a Grauert tube, being a level set of the strictly plurisubharmonic function $\rho$, is strongly pseudoconvex and real analytic. We endow $\partial M_\tau$ with the volume form $d\mu_\tau$ obtained by pulling back the standard Liouville form $\alpha \wedge \omega^{m-1}_{T^*M}$ on $\partial B^*_\tau M_\tau$ under the symplectic diffeomorphism (19):

$$d\mu_\tau = (E^{-1})^* (\alpha \wedge \omega^{m-1}_{T^*M}) \bigg|_{\partial M_\tau}.$$  

(21)

Let $J = J_g$ denote the adapted complex structure on a Grauert tube (see Remark 2.1). The boundary $\partial M_\tau$, being a real hypersurface of the complex manifold $M_\tau$, carries a CR structure. The tangent space admits the decomposition

$$T \partial M_\tau = H \oplus \mathbb{R} T,$$

where $H = \ker \alpha = JT \partial M_\tau \cap T \partial M_\tau$ is a real, $J$-invariant hyperplane bundle and $T = \mathbb{Z}_{\sqrt{\rho}}$, called the *Reeb vector field*, is the Hamilton vector field of the Grauert tube function. Equivalently, $JT = \nabla \sqrt{\rho}$.

Complexifying the tangent space yields

$$T^C \partial M_\tau = H^{(1,0)} \partial M_\tau \oplus H^{(0,1)} \partial M_\tau \oplus \mathbb{C} T$$

where $H^{(1,0)}$ and $H^{(0,1)}$ are the $J$-holomorphic and $J$-anti-holomorphic subspaces.

The boundary Cauchy–Riemann operators are defined by

$$\partial_b f = df \big|_{H^{(1,0)}} \quad \text{and} \quad \overline{\partial}_b f = df \big|_{H^{(0,1)}}.$$

2.3 The Szegő Projector and the Boutet de Monvel–Sjöstrand Parametrix

The Szegő projector is a complex Fourier integral operator (FIO) with a positive complex canonical relation. We recall the Boutet de Monvel–Sjöstrand parametrix construction for the Szegő kernel in the context of Grauert tubes, but the same construction holds with $\partial M_\tau$ replaced by any bounded, strongly pseudoconvex domain.
Definition 2.2 The Szegő projector $\Pi_\tau$ is the orthogonal projection

$$\Pi_\tau : L^2(\partial M_\tau, d\mu_\tau) \to H^2(\partial M_\tau, d\mu_\tau)$$

onto the Hardy space consisting of boundary values of holomorphic functions in $M_\tau$ that are square integrable with respect to the volume form (21). Its distributional kernel $\Pi_\tau(z, w)$ is defined by the relation

$$\Pi_\tau f(z) = \int_{\partial M_\tau} \Pi_\tau(z, w) f(w) d\mu_\tau(w) \quad \text{for all } f \in L^2(\partial M_\tau).$$

(22)

The Szegő projector is a Fourier integral operator with a positive complex canonical relation. The real points of this canonical relation are the graph of the identity map on the symplectic cone $\Sigma_\tau$ spanned by the contact form $\alpha := d\sqrt{\rho}|_{T^*\partial M_\tau}$, i.e.,

$$\Sigma_\tau = \{(\zeta, r\alpha \zeta) : r \in \mathbb{R}_+\} \subseteq T^*(\partial M_\tau).$$

(23)

Using the imaginary time geodesic flow $E$ in (19), we can construct the symplectic equivalence $\iota_\tau$ between the cotangent bundle and the symplectic cone:

$$\iota_\tau : T^*M - 0 \to \Sigma_\tau, \quad \iota_\tau(x, \xi) = \left(E\left(x, \tau \xi \frac{\xi}{|\xi|}\right), \frac{|\xi|\alpha}{E(x, \tau \frac{\xi}{|\xi|})}\right).$$

(24)

We now briefly describe the symbol of the Szegő projector. Details can be found in [7, Theorem 11.2] or [33, Sect. 3]. Let $\Sigma_\tau^+ \otimes \mathbb{C}$ be the complexified normal bundle of (23). The symbol $\sigma(\Pi_\tau)$ of $\Pi_\tau$ is a rank one projection onto a ground state $e_{\Lambda_\tau}$, which is annihilated by a Lagrangian system of Cauchy–Riemann equations corresponding to a Lagrangian subspace $\Lambda_\tau \subseteq \Sigma_\tau^+ \otimes \mathbb{C}$. The time evolution $\Pi_\tau \mapsto G_{\tau}^{-t} \Pi_\tau G_{\tau}^t$ under the Hamilton flow (20) yields another rank one projection onto some time-dependent ground state $e_{\Lambda_\tau^t}$, where $\Lambda_\tau^t$ is the pushforward of $\Lambda_\tau$ under the flow. The quantity

$$\sigma_{t, \tau, 0} = \langle e_{\Lambda_\tau^t}, e_{\Lambda_\tau}\rangle^{-1}$$

appears in Propositions 2.5 and 2.7. It was shown in [33] that $\langle e_{\Lambda_\tau^t}, e_{\Lambda_\tau}\rangle$ is the $L^2$ inner product of two Gaussians, hence nowhere vanishing.

Finally, we discuss an oscillatory integral representation of the Szegő kernel. Recall from Sect. 2.1 that $\rho$ is the real analytic Kähler potential on the Grauert tube. We introduce the defining function for the boundary $\partial M_\tau$ of a Grauert tube:

$$\varphi_\tau : M_{\tau_{\text{max}}} \to [0, \infty), \quad \varphi_\tau(z) := \rho(z) - \tau^2$$

(26)

Let $\varphi_\tau(z, \overline{w})$ be the analytic extension of $\varphi_\tau(z, \overline{z})$ to $M_\tau \times M_\tau$ obtained by polarization. Put (c.f. the formula (17) for the Grauert tube function)

$$\psi_\tau(z, w) = \frac{1}{i} \varphi_\tau(z, \overline{w}) = \frac{1}{i} \left(-\frac{1}{4} \rho_\tau^2(z, \overline{w}) - \tau^2\right).$$

(27)
By construction, \( \psi \) is homolomorphic in \( z \), anti-holomorphic in \( w \), and satisfies \( \psi(z, w) = -\overline{\psi}(z, w) \).

**Theorem 2.3** (The Boutet de Monvel–Sjöstrand parametrix, [9, Theorem 1.5]) With \( \psi \) as in (27), there exists a classical symbol

\[
s \in S^{m-1}(\partial M_\tau \times \partial M_\tau \times \mathbb{R}^+) \quad \text{with} \quad s(z, w, \sigma) \sim \sum_{k=0}^\infty \sigma^{m-1-k} s_k(z, w)
\]

so that the Szegő kernel (22) has the oscillatory integral representation

\[
\Pi_\tau(z, w) = \int_0^\infty e^{i\sigma \psi_\tau(z, w)} s(z, w, \sigma) d\sigma \quad \text{modulo a smoothing kernel.} \quad (28)
\]

We record a key estimate for \( \varphi_\tau \) (or equivalently for the phase function \( \psi \)). Define the Calabi diastatis function by

\[
D(z, w) = \varphi_\tau(z, \overline{z}) + \varphi_\tau(w, \overline{w}) - \varphi_\tau(z, w) - \varphi_\tau(w, z). \quad (29)
\]

On \( \partial M_\tau \), the diastatis simplifies to

\[
D(z, w) = -\varphi_\tau(z, \overline{w}) - \varphi_\tau(w, \overline{z}) = -2 \text{Re} \\varphi_\tau(z, \overline{w}) \quad \text{for} \quad z, w \in \partial M_\tau.
\]

In the closure of the Grauert tube, [9, Corollary 1.3] gives the lower bound

\[
D(z, w) \geq C \left( d(z, \partial M_\tau) + d(w, \partial M_\tau) + d(z, w)^2 \right) \quad \text{for} \quad z, w \in \overline{M_\tau}. \quad (30)
\]

**2.4 Poisson Wave Operator and Tempered Spectral Projection**

Recall the eigenequation (5) for the Laplacian on \( M \). The eigenfunction expansion of the Schwartz kernel of the half-wave operator \( U(t) := e^{it\sqrt{\Delta}} \) is given by

\[
U(t, x, y) = \sum_{j=0}^\infty e^{it\lambda_j} \varphi_{\lambda_j}(x) \overline{\varphi_{\lambda_j}(y)}. \quad (31)
\]

It is well known (see for instance [16, Theorem]) that for \( 0 \leq \tau \leq \tau_{\text{max}} \), the Schwartz kernel \( U(t, x, y) \) admits an analytic extension \( U(t + i\tau, x, y) \) in the time variable \( t \mapsto t + i\tau \in \mathbb{C} \). Note that the corresponding operator \( U(i\tau) = e^{-\tau\sqrt{\Delta}} \) is the Poisson operator. Moreover, for \( y \) and \( \tau \) fixed, the Poisson kernel can be further analytically extended in the spacial variable \( x \mapsto z \in M_\tau \). The resulting operator is a Fourier integral operator of complex type. More precisely, denote by \( \mathcal{O}(\partial M_\tau) \) the space of CR holomorphic functions on the Grauert tube boundary.

**Theorem 2.4** ([8, 14, 16, 35]) For \( 0 < \tau < \tau_{\text{max}} \), the Poisson operator

\[
U(i\tau) = e^{-\tau\sqrt{\Delta}} : L^2(M) \to \mathcal{O}(\partial M_\tau)
\]
whose Schwartz kernel $U(i\tau, z, y)$ is obtained from analytically continuing the half-wave kernel $U(t, x, y)$ of (31) is a Fourier integral operator of order $-(m-1)/4$ with complex phase associated to the canonical relation

$$\{(y, \eta, \iota_\tau(y, \eta)) \subseteq T^*M \times \Sigma_\tau,$$

where $\iota_\tau$ and $\Sigma_\tau$ are defined in (24) and (23).

To introduce the complexified spectral projection kernels, we need to further continue $U(i\tau, z, y)$ anti-holomorphically in the $y$ variable. Consider the operator

$$U_C(t + 2i\tau) = U(i\tau)U(t)U(i\tau)^*: O(\partial M_\tau) \to O(\partial M_\tau)$$

with Schwartz kernel

$$U_C(t + 2i\tau, z, w) = \sum_{j=0}^{\infty} e^{i(t+2i\tau)j} \phi_{C_{\lambda_j}}^{\lambda_j}(z) \bar{\phi}_{C_{\lambda_j}}^{\lambda_j}(w).$$

**Proposition 2.5** [32, Proposition 7.1] Let $(G_t^i)^*$ denote the pullback by the Hamilton flow (20) of $\Xi_{\sqrt{\rho}}$ on $\partial M_\tau$. There exists a classical polyhomogeneous pseudodifferential operator $\hat{\sigma}_{t_\tau}(w, D_{\sqrt{\rho}})$ on $\partial M_\tau$ so that

$$U_C(t + 2i\tau) = \Pi_\tau \hat{\sigma}_{t_\tau}(G_t^i)^* \Pi_\tau \mod \text{a smoothing Toeplitz operator.}$$

The symbol $\sigma_{t_\tau}$ of $\hat{\sigma}_{t_\tau}$ admits a complete asymptotic expansion

$$\sigma_{t_\tau}(w, r) \sim \sum_{j=0}^{\infty} \sigma_{t_\tau, j}(w) r^{-\frac{m-1}{2} - j}, \quad (32)$$

in which $\sigma_{t_\tau, 0} = \langle e_{\lambda^*_t}, e_{\lambda_t} \rangle^{-1}$ is the reciprocal of the overlap of two Gaussians, as in (25).

Finally, we define the tempered spectral projection kernel by

$$P_\lambda(z, w) = \sum_{j: \lambda_j \leq \lambda} e^{-2\tau\lambda_j} \phi_{\lambda_j}^{C}(z) \bar{\phi}_{\lambda_j}^{C}(w). \quad (33)$$

Fix $\varepsilon > 0$ and let $\chi$ be a positive even Schwartz function such that $\hat{\chi}(0) = 1$ and supp $\hat{\chi} \subseteq [-\varepsilon, \varepsilon]$. Then,

$$P_{\chi, \lambda}(z, w) = \chi * d_\lambda P_\lambda(z, w) = \int_{\mathbb{R}} \hat{\chi}(t)e^{-i\lambda t} U_C(t + 2i\tau, z, w) dt \quad (34)$$
Remark 2.6 It was shown in \cite{35} that the complexified spectral function
\[ \sum_{\lambda_j \leq \lambda} |\varphi_{\lambda_j}^{C}(\zeta)|^2 \quad (\zeta \in M_{\tau}) \]
grows exponentially at a rate of \( e^{2\lambda \sqrt{\rho}(\zeta)} \). We introduce the exponentially damping prefactor to obtain polynomial growth in the tempered version (33).

2.5 The Toeplitz Operator \( \Pi_{\tau} D_{\sqrt{\rho}} \Pi_{\tau} \)

Let \( D_{\sqrt{\rho}} = \frac{1}{i} \mathcal{E}_{\sqrt{\rho}} \) denote the Hamilton vector field of the Grauert tube function acting as a differential operator. The symbol of \( D_{\sqrt{\rho}} \) is nowhere vanishing on \( \Sigma_{\tau} - 0 \), where \( \Sigma_{\tau} \) is the symplectic cone (23). Thus, \( \Pi_{\tau} D_{\sqrt{\rho}} \Pi_{\tau} \) is an elliptic Toeplitz operator with discrete spectrum.

As discussed in Sect. 1.1, the Grauert tube analog of Fourier coefficients of the Szegö kernel is given by the spectral localization operator
\[ \Pi_{\chi, \lambda} = \Pi_{\tau} \chi (\Pi_{\tau} D_{\sqrt{\rho}} \Pi_{\tau} - \lambda) = \int_{\mathbb{R}} \hat{\chi}(t) e^{-it\lambda} \Pi_{\tau} e^{it\Pi_{\tau} D_{\sqrt{\rho}} \Pi_{\tau}} dt \]

**Proposition 2.7** [32, Proposition 5.3] Let \((G_{\tau}^t)^{*}\) denote the pullback by the Hamilton flow (20) of \( \mathcal{E}_{\sqrt{\rho}} \) on \( \partial M_{\tau} \). There exists a classical polyhomogeneous pseudodifferential operator \( \hat{\sigma}_{t, \tau}(w, D_{\sqrt{\rho}}) \) on \( \partial M_{\tau} \) so that
\[ \Pi_{\tau} e^{it\Pi_{\tau} D_{\sqrt{\rho}} \Pi_{\tau}} = \Pi_{\tau} \hat{\sigma}_{t, \tau} (G_{\tau}^t)^{*} \Pi_{\tau} \text{ modulo a smoothing Toeplitz operator.} \]

The symbol \( \sigma_{t, \tau} \) of \( \hat{\sigma}_{t, \tau} \) admits a complete asymptotic expansion
\[ \sigma_{t, \tau}(w, r) \sim \sum_{j=0}^{\infty} \sigma_{t, \tau, j}(w) r^{-j}, \quad (35) \]
in which \( \sigma_{t, \tau, 0} = \langle e_{\Lambda_{\tau}^t}, e_{\Lambda_{\tau}^t} \rangle^{-1} \) is the reciprocal of the overlap of two Gaussians, as in (25).

**Remark 2.8** The difference between Propositions 2.7 and 2.5 is that the symbol (32) is of order zero and (35) is of order \(- (m - 1)/2\). This is because the partial sums (33) are not divided through by \( \| e^{-\tau \lambda \varphi_{\lambda}} \|_{L^2(\partial M_{\tau})} \). For an in-depth discussion of complexified eigenfunctions and dynamical Toeplitz operators, see [21, 32, 35].

3 Heisenberg Coordinates on the Boundary of a Grauert Tube

This section reviews the construction and properties of Heisenberg coordinates following [13, Sect. 18]. Roughly speaking, in these coordinates, the strongly pseudoconvex
boundary $\partial M_\tau \subseteq M_C$ is, to a first approximation, the Heisenberg group viewed as a hypersurface in complex Euclidean space. We briefly recall the CR structure on the Heisenberg group.

The Heisenberg group $H^{m-1}$ of degree $m - 1$ is the Lie group $\mathbb{R} \times \mathbb{C}^{m-1}$ with group law

$$(\theta, z) \cdot (\varphi, w) = (\theta + \varphi + 2 \text{Im}(z \cdot w), z + w).$$

(36)

**Remark 3.1** It is more common to use $t$ for the $\mathbb{R}$-coordinate but, since $t$ is already reserved for the time parameter of the geodesic or Hamiltonian flow, we use $\theta$ instead.

For $1 \leq j \leq m - 1$, the vector fields

$$Z_j = \frac{\partial}{\partial z_j} + i \bar{z}_j \frac{\partial}{\partial \varphi}, \quad \bar{Z}_j = \frac{\partial}{\partial \bar{z}_j} + i z_j \frac{\partial}{\partial \varphi}, \quad T = \frac{\partial}{\partial \varphi}$$

satisfy the commutation relations

$$[Z_j, \bar{Z}_k] = -2i \delta_{jk} T \quad \text{and} \quad [Z_j, Z_k] = [\bar{Z}_j, \bar{Z}_k] = [Z_j, T] = [\bar{Z}_j, T] = 0.$$

Furthermore, the 1-form

$$\alpha = d\varphi - i \sum_{j=1}^{m-1} (\bar{z}_j \, dz_j - z_j \, d\bar{z}_j)$$

annihilates $Z_j$ and $\bar{Z}_j$ for all $1 \leq j \leq m - 1$. Hence, the complexified tangent space of the Heisenberg group admits the decomposition

$$T^C H^{m-1} = H^{(1,0)} \oplus H^{(0,1)} \oplus \mathbb{C} T,$$

in which $H^{(1,0)}$ and $H^{(0,1)}$ are spanned by the holomorphic vector fields $Z_j$ and the anti-holomorphic vector fields $\bar{Z}_j$, respectively. Direct verification shows that the subspace $H^{(1,0)}$ defines a CR structure on $H^{m-1}$.

The Levi form $\langle , \rangle_L$ is the Hermitian form on $H^{(1,0)}$ defined by

$$(Z, W)_L = -i \langle d\alpha, Z \wedge \bar{W} \rangle \quad \text{for all } Z, W \in H^{(1,0)}.$$

Direct computation shows that the Levi form with respect to the basis $Z_j$ is the identity. Thus, $H^{m-1}$ is strongly pseudoconvex.

Set

$$D_m = \left\{ \zeta = (\xi_0, \ldots, \xi_{m-1}) \in \mathbb{C}^m : \sum_{j=1}^{m-1} |\xi_j|^2 < \text{Im} \, \xi_0 \right\}.$$

$$\partial D_m = \left\{ \zeta = (\xi_0, \ldots, \xi_{m-1}) \in \mathbb{C}^m : \sum_{j=1}^{m-1} |\xi_j|^2 = \text{Im} \, \xi_0 \right\}.$$

(37)
The Heisenberg group $\mathbb{H}^{m-1}$ acts on $\mathbb{C}^m$ by holomorphic affine transformations; given $(\theta, z) = (\theta, z_1, \ldots, z_{m-1}) \in \mathbb{H}^{m-1}$ and $\zeta = (\zeta_0, \ldots, \zeta_{m-1}) \in \mathbb{C}^m$, we have $(\theta, z) \cdot \zeta = (\zeta_0', \ldots, \zeta_{m-1}')$, where

$$
\begin{align*}
\zeta_0' &= \zeta_0 + \theta + i|z|^2 + 2i \sum_{j=1}^{m-1} \zeta_j \bar{z}_j, \\
\zeta_j' &= \zeta_j + z_j \quad \text{for } 1 \leq j \leq m-1.
\end{align*}
$$

(38)

Note that the group action preserves $D_m$ (which is biholomorphic to the unit ball in $\mathbb{C}^m$) and $\partial D_m$. Indeed, $\mathbb{H}^{m-1}$ acts simply and transitively on $\partial D_m$. Thus, comparing (36) and (38), we find that the Heisenberg group may be identified to $\partial D_m$ via the correspondence

$$
\mathbb{H}^{m-1} \ni (\theta, z) \mapsto (\theta + i|z|^2, z_1, \ldots, z_{m-1}) \in \partial D_m.
$$

On any nondegenerate CR manifold (in particular the boundary of a Grauert tube), there exists a Heisenberg-like coordinate system $(\theta, z_1, \ldots, z_{m-1})$ so that $Z_j = (\partial/\partial z_j) + i\bar{z}_j(\partial/\partial \theta)$ and $T = (\partial/\partial \theta)$ up to lower order terms. The precise statement, Definition 1.1, is already given in the introduction, so we do not reproduce it here. An explicit construction of such “normal coordinates” was presented in [13]. We call the same coordinates “Heisenberg coordinates” in this paper to emphasize the osculating Heisenberg structure (see Remark 3.4).

**Remark 3.2** If $f$ is smooth, then

$$
\begin{align*}
f &= O^1 \text{ if and only if } f(\eta) = O \left( \sum_{k=1}^{m-1} (|x_k(\eta)| + |y_k(\eta)| + |\theta(\eta)|) \right), \\
f &= O^2 \text{ if and only if } f(\eta) = O \left( \sum_{k=1}^{m-1} (|x_k(\eta)|^2 + |y_k(\eta)|^2 + |\theta(\eta)|) \right).
\end{align*}
$$

**3.1 Taylor Expansions in Heisenberg Coordinates**

We fix notation. Let $p \in \partial M_\tau$ and let $z, w$ be two points in $M_{\tau_{\max}}$ that lie in a Heisenberg coordinate patch (see Definition 1.1) containing $p$. We write

$$
\begin{align*}
z &= (z_0, \ldots, z_{m-1}), \quad \theta = \text{Re } z_0, \quad u = (z_1, \ldots, z_{m-1}), \\
w &= (w_0, \ldots, w_{m-1}), \quad \varphi = \text{Re } w_0, \quad v = (w_1, \ldots, w_{m-1}).
\end{align*}
$$

(39)

Note that if $z \in \partial M_\tau$, then $z = (\theta, u)$.

We record the Taylor expansions of the boundary defining function (26) and the Hamilton flow (20) on the boundary in these coordinates. Detailed computations in the setting of a real hypersurface in a complex manifold whose induced CR structure is strongly pseudoconvex (as is the case of $\partial M_\tau \subseteq M_\tau$) are found in [13, Sect. 18].
Lemma 3.3 Let $\varphi_\tau$ be the defining function (26) of the Grauert tube boundary. Denote by $\varphi_\tau(z, \bar{w})$ be its analytic extension obtained by polarization. Then, in Heisenberg coordinates near $p$ in $M_\tau$, we have

$$\varphi_\tau(z, \bar{w}) = \frac{i}{2} (z_0 - \bar{w}_0) + \sum_{j=1}^{m-1} z_j \bar{w}_j + R(z, \bar{w}).$$

The remainder $R$ takes the form

$$R(z, \bar{w}) = R_{\varphi_\tau}(z_0, u, \bar{w}_0, \bar{v}) = R_2(z_0, \bar{w}_0) + R_2(z_0, u, \bar{w}_0, \bar{v}) + R_3(u, \bar{v}),$$

where

- $R_2(z_0, \bar{w}_0)$ contains only of mixed terms of the form $z_0^\alpha \bar{w}_0^\beta$ with $|\alpha| + |\beta| \geq 2$.
- $R_2(z_0, u, \bar{w}_0, \bar{v})$ contains only of mixed terms of the form $z_0^\alpha \bar{v}^\beta$ or $\bar{w}_0^\alpha u^\beta$ with $|\alpha| + |\beta| \geq 2$.
- $R_3(u, \bar{v})$ contains only of mixed terms of the form $u^\alpha \bar{v}^\beta$ with $|\alpha| + |\beta| \geq 3$.

Proof Direct computation (see [13, (18.4)]) yields

$$\varphi_\tau(z) = \varphi_\tau(z_0, u) = -\text{Im } z_0 + |u|^2 + \mathcal{O}(|z_0| |u| + |z_0|^2 + |u|^3),$$

so

$$d\varphi_\tau \big|_p = -\text{Im } dz_0 \big|_p \quad \text{and} \quad \frac{\partial^2 \varphi_\tau}{\partial z_j \partial \bar{z}_k} \big|_p = \delta_{jk} \quad (1 \leq j, k \leq m - 1).$$

Combined with the near-diagonal Taylor expansion

$$\varphi_\tau(p + h, p + k) = \sum_{\alpha, \beta} \frac{\partial^{\alpha + \beta} \varphi_\tau}{\partial z^\alpha \partial \bar{z}^\beta} (p) \frac{h^\alpha \bar{k}^\beta}{\alpha! \beta!},$$

of $\varphi_\tau$, we obtain the desired result. \qed

Remark 3.4 Note that the formula (40) for the defining function implies $\partial M_\tau$ is highly tangent at $p$ to the hypersurface $\text{Im } z_0 = \sum_{k=1}^{m-1} |z_k|^2$, which is the geometric model (37) for the Heisenberg group.

Lemma 3.5 Let $G_t^\tau$ be the Hamilton flow (20) of the Grauert tube function on $\partial M_\tau$. Then, in Heisenberg coordinates near $p$ in $\partial M_\tau$, we have

$$G_t^\tau(z) = G_t^\tau(\theta, u) = \left( \theta + 2\pi t + t \cdot O^1 + O(t^2), u + t \cdot O^1 + O(t^2) \right),$$

where $O^1$ denotes the Heisenberg-type order (3).
Proof} By \cite[Theorem 18.5]{13}, the Reeb vector field is of the form

\[
\frac{1}{2\tau} H = \frac{\partial}{\partial \theta} + \sum_{j=1}^{m-1} O^1 \frac{\partial}{\partial z_j} + \sum_{j=1}^{m-1} O^1 \frac{\partial}{\partial \overline{z_j}} + O^1 \frac{\partial}{\partial \overline{\theta}}.
\]

The Lemma then follows by Taylor expanding $G_t^\tau$ near $t = 0$. \hfill \Box

4 Proof of Main Results

We briefly outline the proof of Theorem 1.2 before delving into the computations. As indicated in a subsequent Section, the same techniques work almost verbatim for proving Theorem 1.3.

Since we are interested in scaling asymptotics, variables are rescaled according to their Heisenberg-type order (cf (3)). Namely, given $(\theta, u) \in \partial M^{\tau}$ in Heisenberg coordinates near $p \in \partial M^{\tau}$ (recall Definition 1.1 and (39)), we consider

\[
(\theta, u) \mapsto \left( \frac{\theta}{\lambda}, \frac{u}{\sqrt{\lambda}} \right) \in \partial M^{\tau}.
\]

We rewrite the (rescaled) spectral localization kernel of (1) in terms of the kernel of a “dynamical Toeplitz operator” using Proposition 2.7. Then, substituting the Boutet de Monvel–Sjöstrand parametrix (28) for the Szeg"o projectors, we are left with an oscillatory integral. Taylor expanding the phase function using Lemma 3.3, Lemma 3.5, and then applying the method of stationary phase complete the proof.

4.1 Proof of Theorem 1.2

From (1) and Proposition 2.7, we have

\[
\Pi_{\chi,\lambda} = \int_{\mathbb{R}} \hat{\chi}(t) \Pi_\tau \hat{\sigma}_{t,\tau}(G_t^{\tau})^* \Pi_\tau dt \quad \text{modulo a smoothing Toeplitz operator}.
\]

Fix a point $p \in \partial M^{\tau}$ and a Heisenberg coordinate chart centered at $p = 0$. Fix two points $(\theta, u), (\varphi, v) \in \partial M^{\tau}$ in Heisenberg coordinates. Then, substituting the parametrix (28) for each instance of $\Pi_\tau$ above and composing the resulting kernels, we arrive at the oscillatory integral representation

\[
\Pi_{\chi,\lambda} \left( \frac{\theta}{\lambda}, \frac{u}{\sqrt{\lambda}}, \frac{\varphi}{\lambda}, \frac{v}{\sqrt{\lambda}} \right) \sim \int_{\mathbb{R} \times \partial M^{\tau} \times \mathbb{R}^+ \times \mathbb{R}^+} e^{i\lambda \Psi} A d\sigma_1 d\sigma_2 d\mu^{\tau}(w) dt, \quad (41)
\]

in which the phase $\Psi$ and the amplitude $A$ are given by

\[
\Psi = -t + \frac{1}{\lambda} \sigma_2 \psi_\tau \left( \left( \frac{\theta}{\lambda}, \frac{u}{\sqrt{\lambda}} \right), w \right) + \frac{1}{\lambda} \sigma_1 \psi_\tau \left( G_t^{\tau}(w), \left( \frac{\varphi}{\lambda}, \frac{v}{\sqrt{\lambda}} \right) \right).
\]
\[ A = \hat{\chi}(t)s\left(\frac{\theta}{\lambda}, \frac{u}{\sqrt{\lambda}}, w, \sigma_2\right)s\left(G^\prime_t(w), \frac{\varphi}{\lambda}, \frac{v}{\sqrt{\lambda}}, \sigma_1\right)\sigma_{t, \tau}(w, \sigma_1). \] (42)

**Remark 4.1** Recall \( \sigma_{t, \tau} \) is the unitarization symbol of order zero from Proposition 2.7, \( s = s(z, w, \sigma) \) is the symbol of order \( m - 1 \) in the Boutet de Monvel–Sjöstrand parametrix (28), and \( \sigma = \sigma_j \in \mathbb{R}^+ \) are parameters of integration in the parametrix. We do not yet put the spatial variable \( x \in \partial M_\tau \) of integration in Heisenberg coordinates.

Make the change-of-variables \( \sigma_j \mapsto \lambda \sigma_j \). Homogeneity of the symbols implies

\[ \Pi_{\chi, \lambda}\left(\frac{\theta}{\lambda}, \frac{u}{\sqrt{\lambda}}, \frac{\varphi}{\lambda}, \frac{v}{\sqrt{\lambda}}\right) \sim \lambda^{2m} \int_{\mathbb{R} \times \partial M_\tau \times \mathbb{R}^+ \times \mathbb{R}^+} e^{i\lambda \tilde{\Psi}} \tilde{A} d\sigma_1 d\sigma_2 d\mu\tau(w) dt, \] (43)

in which the phase \( \tilde{\Psi} \) and the amplitude \( \tilde{A} \) are given by

\[ \tilde{\Psi} = -t + \sigma_2 \psi_t\left(\left(\frac{\theta}{\lambda}, \frac{u}{\sqrt{\lambda}}\right), w\right) + \sigma_1 \psi_t\left(G^\prime_t(w), \left(\frac{\varphi}{\lambda}, \frac{v}{\sqrt{\lambda}}\right)\right), \]

\[ \tilde{A} = \lambda^{-2m+2}. \] (44)

**Remark 4.2** It follows from Remark 4.1 that \( \tilde{A} \) is a symbol of order 0.

We begin by localizing in \((w, t) \in \partial M_\tau \times \mathbb{R}\). Fix \( C > 0 \) and \( 0 < \delta \ll 1 \). Set

\[ V_\lambda = \left\{(w, t) : \max\left\{d\left(w, \left(\frac{\theta}{\lambda}, \frac{u}{\sqrt{\lambda}}\right)\right), d\left(G^\prime_t(w), \left(\frac{\varphi}{\lambda}, \frac{v}{\sqrt{\lambda}}\right)\right)\right\} < \frac{2C}{3} \lambda^{\delta - \frac{1}{2}} \right\}, \]

\[ W_\lambda = \left\{(w, t) : \max\left\{d\left(w, \left(\frac{\theta}{\lambda}, \frac{u}{\sqrt{\lambda}}\right)\right), d\left(G^\prime_t(w), \left(\frac{\varphi}{\lambda}, \frac{v}{\sqrt{\lambda}}\right)\right)\right\} > \frac{C}{2} \lambda^{\delta - \frac{1}{2}} \right\}. \] (45)

Let \( \{\varrho_\lambda, 1 - \varrho_\lambda\} \) be a partition of unity subordinate to the cover \( \{V_\lambda, W_\lambda\} \) and decompose the integral (43) into

\[ \Pi_{\chi, \lambda}\left(\frac{\theta}{\lambda}, \frac{u}{\sqrt{\lambda}}, \frac{\varphi}{\lambda}, \frac{v}{\sqrt{\lambda}}\right) \sim I_1 + I_2, \]

\[ I_1 = \lambda^{2m} \int e^{i\lambda \tilde{\Psi}} \varrho_\lambda(t, w) \tilde{A} d\sigma_1 d\sigma_2 d\mu\tau(w) dt, \]

\[ I_2 = \lambda^{2m} \int e^{i\lambda \tilde{\Psi}} (1 - \varrho_\lambda(t, w)) \tilde{A} d\sigma_1 d\sigma_2 d\mu\tau(w) dt. \]

**Lemma 4.3** We have \( I_2 = O(\lambda^{-\infty}). \)

**Proof** By definition of \( W_\lambda \), on the support of \( 1 - \varrho_\lambda \) either

\[ |d\sigma_2 \tilde{\Psi}| = \left|\psi_t\left(\left(\frac{\theta}{\lambda}, \frac{u}{\sqrt{\lambda}}\right), w\right)\right| \geq 2D\left(\left(\frac{\theta}{\lambda}, \frac{u}{\sqrt{\lambda}}\right), w\right) \geq C' \lambda^{2\delta - 1}. \]
or
\[ |d_{\sigma_1} \tilde{\Psi}| = \left| \psi_\tau \left( \left( G_\tau^i(w), \frac{\varphi}{\lambda}, \frac{v}{\sqrt{\lambda}} \right) \right) \right| \geq 2D \left( \left( G_\tau^i(w), \frac{\varphi}{\lambda}, \frac{v}{\sqrt{\lambda}} \right) \right) \geq C' \lambda^{2\delta - 1} \]

where \( D \) is the Calabi diastasis (29) and the inequalities follow from (30). Repeated integration by parts in \( \sigma_1 \) or \( \sigma_2 \) as appropriate completes the proof. \( \square \)

So far, we have shown

\[ \Pi_{\chi, \lambda} \left( \frac{\theta}{\lambda}, \frac{u}{\sqrt{\lambda}}, \frac{\varphi}{\lambda}, \frac{v}{\sqrt{\lambda}} \right) \sim \lambda^{2m} \int e^{i \lambda \tilde{\psi}_{\psi_\lambda} \tilde{A}} d\sigma_1 d\sigma_2 d\mu_\tau(w) dt, \quad (46) \]

with \( \tilde{\psi}, \tilde{A} \) as in (44) and \( \psi_\lambda \) a smooth cutoff to a neighborhood of \( V_\lambda \) as in (45). The next step is to Taylor expand the phase function and absorb the remainder into the amplitude. Since the expansion (40) is stated in terms of ambient coordinates on the Grauert tube (as opposed to coordinates on the boundary), we need to introduce some notation. Recall in Heisenberg coordinates, we have \( z = (z_0, \ldots, z_{m-1}) \) in terms of coordinates in the ambient Grauert tube with the property that if in addition \( z \in \partial M_\tau \), then \( z = (\Re z_0, z_1, \ldots, z_{m-1}) \) in terms of coordinates on the Grauert tube boundary. Here, we have a point \( (\frac{\theta}{\lambda}, \frac{u}{\sqrt{\lambda}}) \) in coordinates on \( \partial M_\tau \). In terms of the ambient coordinates on \( M_{\tau_{\max}} \), we denote the same point by \( (\Theta_1(\lambda), \frac{u}{\sqrt{\lambda}}) \), where \( \Re \Theta_1(\lambda) = \frac{\theta}{\lambda} \) and \( \Im \Theta_1(\lambda) \) are determined by \( \varphi_{\tau}(\Theta_1(\lambda), \frac{u}{\sqrt{\lambda}}) = 0 \).

**Lemma 4.4** We have

\[ \Im \Theta_1(\lambda) = \frac{1}{\lambda} \sum_{j=1}^{m-1} |u_j|^2 + O(\lambda^{-\frac{3}{2}}). \]

In particular, \( \Im \Theta_1(\lambda) = O(\lambda^{-1}) = \Im \Theta(\lambda) \).

**Proof** The implicit function theorem applied to \( \varphi_{\tau}(\Theta_1(\lambda), \frac{u}{\sqrt{\lambda}}) = 0 \) shows \( \Im \Theta = O(\frac{1}{\sqrt{\lambda}}) \). Substituting \( |z_0| = |\Theta_1(\lambda)| = O(\frac{1}{\sqrt{\lambda}}) \) in the remainder term of (40) shows \( \Im \Theta = O(\frac{1}{\lambda}) \). Since \( \Re \Theta = O(\frac{1}{\lambda}) \), we can in fact substitute \( |z_0| = |\Theta_1(\lambda)| = O(\frac{1}{\lambda}) \) in the remainder term of (40), proving the Lemma. \( \square \)

We write the spatial variable \( w \) of integration as

\[ w = (w_0, w') \quad \text{with} \quad w' = (w_1, \ldots, w_{m-1}) \quad \text{in coordinates on} \quad M_\tau, \]

\[ w = (\Re w_0, w') \quad \text{in coordinates on} \quad \partial M_\tau, \]

\[ w(t) = G_\tau^i(w). \]
We are now ready to Taylor expand $i \psi_\tau = \varphi_\tau$ that appears in the phase (44). By Lemma 3.3, Taylor expansions in the ambient coordinates take the form

$$i \psi_\tau \left( \left( \Theta(\lambda), \frac{u}{\sqrt{\lambda}} \right), w \right) = \frac{i}{2} (\Theta(\lambda) - \overline{w}_0) + \frac{1}{\sqrt{\lambda}} \sum_{j=1}^{m-1} u_j \overline{w}_j$$

$$+ R \left( \Theta(\lambda), \overline{w}_0, \frac{u}{\sqrt{\lambda}}, \overline{w}' \right), \quad (47)$$

$$i \psi_\tau \left( G^\lambda_\tau(w), \left( \Phi(\lambda), \frac{v}{\sqrt{\lambda}} \right) \right) = \frac{i}{2} (w_0(t) - \overline{\Phi}(\lambda)) + \frac{1}{\sqrt{\lambda}} \sum_{j=1}^{m-1} w_j(t) \overline{v}_j$$

$$+ R \left( w_0(t), \overline{\Phi}(\lambda), w'(t), \frac{v}{\sqrt{\lambda}} \right). \quad (48)$$

**Remark 4.5** The obvious notation $(\Phi(\lambda), \frac{v}{\sqrt{\lambda}})$ denotes the ambient coordinates of $(\varphi_{\lambda, \sqrt{\lambda}})$. The form of the remainder $R$ is explicitly stated in Lemma 3.3.

To apply stationary phase, the variables $t \in \mathbb{R}$ and $w \in \partial M_\tau$ need to be rescaled:

$$t \mapsto \frac{t}{\sqrt{\lambda}} \quad \text{and} \quad (\text{Re } w_0, \text{w}') \mapsto \left( \frac{\text{Re } w_0}{\sqrt{\lambda}}, \frac{w'}{\sqrt{\lambda}} \right). \quad (49)$$

We write $w_0(\lambda)$ for the image of $w_0$ under this rescaling, so that $\text{Re } w_0(\lambda) = \frac{\text{Re } w_0}{\sqrt{\lambda}}$ and $\text{Im } w(\lambda)$ are determined by $\varphi_\tau(w_0(\lambda), \frac{w'}{\sqrt{\lambda}}) = 0$.

**Remark 4.6** Since $w$ is only an integration variable, it is not necessary that rescaling of $w$ respects Heisenberg-type order. Indeed, the uniform $\sqrt{\lambda}$-rescaling (49) is needed for stationary phase.

Under the change-of-variables (49), the expansion (47) turns into

$$i \psi_\tau \left( \left( \Theta(\lambda), \frac{u}{\sqrt{\lambda}} \right), w(\lambda) \right) = \frac{i}{2} (\Theta(\lambda) - \overline{w}_0(\lambda)) + \frac{1}{\lambda} \sum_{j=1}^{m-1} u_j \overline{w}_j$$

$$+ R \left( \Theta(\lambda), \overline{w}_0(\lambda), \frac{u}{\sqrt{\lambda}}, \overline{w}' \right).$$

Multiplying through by $\lambda$ and using Lemma 3.3, Lemma 4.4 to compute the remainder, we find

$$i \lambda \psi_\tau \left( \left( \Theta(\lambda), \frac{u}{\sqrt{\lambda}} \right), w(\lambda) \right) = -\sqrt{\lambda} \frac{i}{2} \text{Re } w_0 + \tilde{R} \left( \frac{\theta}{\lambda}, \frac{\text{Re } w_0}{\sqrt{\lambda}}, \frac{u}{\sqrt{\lambda}}, \frac{w'}{\sqrt{\lambda}} \right), \quad (50)$$

where

$$\tilde{R} = \frac{i}{2} \theta - \frac{|u|^2}{2} - \frac{|w|^2}{2} + u \cdot \overline{w} + \lambda Q \left( \frac{\theta}{\lambda}, \frac{u}{\sqrt{\lambda}}, \frac{\text{Re } w_0}{\sqrt{\lambda}}, \frac{w'}{\sqrt{\lambda}} \right) \quad (51)$$
and $Q$ has the following asymptotic expansion

$$Q\left(\frac{\theta}{\sqrt{\lambda}}, \frac{u}{\sqrt{\lambda}}, \frac{\Re w_0}{\sqrt{\lambda}}, \frac{w'}{\sqrt{\lambda}}\right) \sim \sum_{j=1}^{\infty} \sum_{j+k+l=2} \sum_{j+k+l+m=2} P_{j,k,l,m} \left(\frac{\theta}{\sqrt{\lambda}}, \frac{u}{\sqrt{\lambda}}, \frac{\Re w_0}{\sqrt{\lambda}}, \frac{w'}{\sqrt{\lambda}}\right).$$

Here, $P_{j,k,l}$ is a polynomial in three variables that is of degree $j, k, l$ in the first, second, and third slot, respectively. (The polynomial $P_{j,k,l,m}$ is similarly defined.)

Similarly, using in addition Lemma 3.5, under the change-of-variables (49) the expansion (48) turns into

$$i \lambda \psi_t \left( G_{\tau}^{\sqrt{\lambda}} (w(\lambda)), \left( \Phi(\lambda), \frac{v}{\sqrt{\lambda}} \right) \right) = \sqrt{\lambda} \frac{i}{2} \left( \Re w_0 + 2 \tau t \right) + \tilde{S} \left( \frac{\varphi}{\lambda}, \frac{\Re w_0}{\sqrt{\lambda}}, \frac{t}{\sqrt{\lambda}}, \frac{v}{\sqrt{\lambda}}, \frac{w'}{\sqrt{\lambda}} \right),$$

(52)

where

$$\tilde{S} = -i \frac{\varphi}{2} - \frac{|v|^2}{2} - \frac{|w|^2}{2} + \bar{v} \cdot w + \lambda T \left( \frac{\varphi}{\sqrt{\lambda}}, \frac{t}{\sqrt{\lambda}}, \frac{\Re w_0}{\sqrt{\lambda}} \right).$$

(53)

and $T$ has the following asymptotic expansion

$$T \left( \frac{\varphi}{\sqrt{\lambda}}, \frac{v}{\sqrt{\lambda}}, \frac{t}{\sqrt{\lambda}}, \frac{\Re w_0}{\sqrt{\lambda}}, \frac{w'}{\sqrt{\lambda}} \right) \sim \sum_{i+j=0}^{i=j} \sum_{i+j+k+l=2} P_{i,j,k,l} \left( \frac{t}{\sqrt{\lambda}}, \frac{\Re w_0}{\sqrt{\lambda}}, \frac{v}{\sqrt{\lambda}}, \frac{w'}{\sqrt{\lambda}} \right) + \sum_{i+j+k+l+m=2} P_{i,j,k,l,m} \left( \frac{\varphi}{\sqrt{\lambda}}, \frac{v}{\sqrt{\lambda}}, \frac{t}{\sqrt{\lambda}}, \frac{\Re w_0}{\sqrt{\lambda}}, \frac{w'}{\sqrt{\lambda}} \right).$$

(54)

It follows from (52) and (50) that under the change-of-variables (49), the integral (46) becomes an oscillatory integral with parameter $\sqrt{\lambda}$:

$$\Pi_{\lambda, \alpha} \left( \frac{\theta}{\sqrt{\lambda}}, \frac{u}{\sqrt{\lambda}}, \frac{\varphi}{\sqrt{\lambda}}, \frac{v}{\sqrt{\lambda}} \right) \sim \lambda^m \int e^{i \sqrt{\lambda} \tilde{\Psi}} \tilde{A} d\sigma_1 d\sigma_2 d\mu_t (w) dt,$$

(55)

where

$$\tilde{\Psi} = -t - \frac{\sigma_2}{2} \Re w_0 + \frac{\sigma_1}{2} (\Re w_0 + 2 \tau t),$$

$$\tilde{A} = e^{\sigma_2 \tilde{R}_t + \sigma_1 \tilde{S}} Q_0 \tilde{A} \left( \frac{\theta}{\sqrt{\lambda}}, \frac{u}{\sqrt{\lambda}}, \frac{\varphi}{\sqrt{\lambda}}, \frac{v}{\sqrt{\lambda}}, \frac{\Re w_0}{\sqrt{\lambda}}, \frac{w'}{\sqrt{\lambda}}, \frac{t}{\sqrt{\lambda}}, \sigma_1, \sigma_2 \right) J(w(\lambda)),$$

with $J(\eta)$ the volume density in Heisenberg coordinates.
**Remark 4.7** In (49), the spatial rescaling introduces a factor of $\lambda^{-m+1/2}$, while the time rescaling introduces a factor of $\lambda^{-1/2}$. Together with the overall prefactor of $\lambda^{2m}$ in (46), we obtain the $\lambda^m$ prefactor in (54). The explicit descriptions of $\tilde{R}$ and $\tilde{S}$ imply that $e^{\sigma_2 \tilde{R} + \sigma_1 \tilde{S}}$ may be absorbed into the amplitude.

We make one final localization argument. Let $\{\eta, 1 - \eta \}$ be a partition of unity subordinate to the cover

$$\left\{ (\sigma_1, \sigma_2) : 0 < \sigma_1, \sigma_2 < \frac{2}{\tau} \right\} \quad \text{and} \quad \left\{ (\sigma_1, \sigma_2) : \sigma_1, \sigma_2 > \frac{3}{2\tau} \right\}.$$ 

Decompose (54) into two integrals:

$$\Pi_{\chi, \lambda} \left( \frac{\theta}{\lambda}, \frac{u}{\sqrt{\lambda}}, \frac{\varphi}{\sqrt{\lambda}}, \frac{v}{\sqrt{\lambda}} \right) \sim I'_1 + I'_2,$$

$$I'_1 = \lambda^m \int e^{i\sqrt{\lambda} \tilde{\Psi} (\sigma_1, \sigma_2) \tilde{A}} d\sigma_1 d\sigma_2 d\mu_\tau(w) dt,$$

$$I'_2 = \lambda^m \int e^{i\sqrt{\lambda} (1 - \eta(\sigma_1, \sigma_2)) \tilde{A}} d\sigma_1 d\sigma_2 d\mu_\tau(w) dt,$$

with $\tilde{A}$ and $\tilde{\Psi}$ as in (55).

**Lemma 4.8** We have $I'_2 = O(\lambda^{-\infty})$.

**Proof** Notice that

$$\left| \nabla \Re w_0, t \tilde{\Psi} \right|^2 \geq \left( \frac{\sigma_1}{2} - \frac{\sigma_2}{2} \right)^2 + (\tau \sigma_1 - 1)^2 \geq \frac{1}{4}$$

on the support of $1 - \eta$. Thus, the Lemma follows from repeated integration by parts in $(\Re w_0, t)$. $\square$

We have finally reduced the spectral localization kernel to the oscillatory integral

$$\Pi_{\chi, \lambda} \left( \frac{\theta}{\lambda}, \frac{u}{\sqrt{\lambda}}, \frac{\varphi}{\sqrt{\lambda}}, \frac{v}{\sqrt{\lambda}} \right) \sim \lambda^m \int_{\mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{C}^{m-1}} e^{i\sqrt{\lambda} \tilde{\Psi} B dw' d(\Re w_0)} d\sigma_1 d\sigma_2 dt,$$

with phase and amplitude

$$\tilde{\Psi} = -t - \frac{\sigma_2}{2} \Re w_0 + \frac{\sigma_1}{2}(\Re w_0 + 2\tau t),$$

$$\tilde{A} = e^{\sigma_2 \tilde{R} + \sigma_1 \tilde{S}} \eta \hat{\chi} \tilde{A} J.$$

**Remark 4.9** The cutoff functions $\varrho_\lambda, \eta, \hat{\chi}$ localize $B$ to a compact region. The $\lambda^{m-1}$ factor in $B$ comes from comparing the amplitude $\tilde{A}$ in the integral (54) (which already has an overall factor of $\lambda^m$ from the change-of-variables $\sigma_j \mapsto \lambda \sigma_j$ and (49)) with the amplitude $A$ in (41).
We are now in the position to integrate with respect to $\text{Re } w_0, \sigma_1, \sigma_2, t$ using the method of stationary phase, thereby reducing (56) to a Gaussian integral over $\mathbb{C}^{m-1}$. We note the following derivatives:

$$\partial_{\sigma_2} \tilde{\Psi} = -\frac{1}{2} \text{Re } w_0, \quad \partial_{\sigma_1} \tilde{\Psi} = \frac{1}{2} (\text{Re } w_0 + 2 \tau t),$$

$$\partial_t \tilde{\Psi} = -1 + \tau \sigma_1, \quad \partial_{\text{Re } w_0} \tilde{\Psi} = -\frac{\sigma_2}{2} + \frac{\sigma_1}{2}.$$ 

The critical set of the phase is the point $C = \{ \text{Re } w_0 = 0, t = 0, \sigma_1 = \sigma_2 = \frac{1}{\tau} \}$. The Hessian matrix and its inverse at the critical point are

$$\tilde{\Psi}_C'' = \begin{pmatrix} t & \sigma_1 & \sigma_2 & \text{Re } w_0 \\ \sigma_1 & 0 & \frac{1}{2} & 0 \\ \sigma_2 & 0 & 0 & -\frac{1}{2} \\ \text{Re } w_0 & 0 & -\frac{1}{2} & 0 \end{pmatrix}, \quad (\tilde{\Psi}_C'')^{-1} = \begin{pmatrix} 0 & \frac{1}{\tau} & \frac{1}{\tau} & 0 \\ \frac{1}{\tau} & 0 & 0 & 0 \\ \frac{1}{\tau} & 0 & 0 & -2 \\ 0 & 0 & -2 & 0 \end{pmatrix}.$$ 

Set

$$L_{\tilde{\Psi}} = (\tilde{\Psi}_C'')^{-1} D, D = \frac{2}{\tau} \partial_{\sigma_1} \partial_t + \frac{2}{\tau} \partial_{\sigma_2} \partial_t - 4 \partial_{\text{Re } w_0} \partial_{\text{Re } w_0}$$

By the method of stationary phase ([20, Theorem 7.75]), we have

$$\int_{\mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}} e^{i \sqrt{\lambda} \tilde{\Psi}} \tilde{A} \, dw' \, d\sigma_1 d\sigma_2 dt$$

$$= \gamma \lambda^\alpha \sum_{j=0}^{N-1} \lambda^{-\frac{j}{2}} \sum_{\nu - \mu = j, 2\nu \geq 3\mu} \frac{1}{i^{\nu / 2}} L^\nu_{\tilde{\Psi}} \left[ e^{\mu (\sigma_2 \tilde{R} + \sigma_1 \tilde{S})} \eta_0 \tilde{\chi} \tilde{A} \right] C + \hat{R}_N. \quad (57)$$

The leading coefficient $\gamma$ is given by

$$\gamma = e^{i \sqrt{\lambda} \tilde{\Psi}|_C} \left( \det \left( \sqrt{\lambda} \tilde{\Psi}_C'' / 2\pi i \right) \right)^{-\frac{1}{2}} = \frac{8\pi^2}{\lambda \tau}.$$ 

The stationary phase remainder estimate implies

$$\int_{\mathbb{C}^{m-1}} |\hat{R}_N| \, dw' \leq \int_{\mathbb{C}^{m-1}} \lambda^{-\frac{N}{2}} \sup_{|\sigma| \leq 2N} |D^\alpha (\eta \rho_0 \tilde{\chi} \tilde{A} J)| \, dw' \leq C_N \lambda^{-\frac{N}{2}}$$

because $\tilde{A}$ is a symbol of order zero by Remark 4.2. (Here, the supremum and the derivative $D^\alpha$ are taken over $t, \sigma_1, \sigma_2, \text{Re } w_0$ and the integral is with respect to the remaining variable $w'$.)

It remains to integrate the asymptotic expansion (57) in $w'$. The left-hand side is (56); the right-hand side can be integrated term-by-term thanks to the estimate above.
After substituting expressions (51) and (53) and performing a Taylor expansion near the \( p = (0, 0) \in \mathbb{R} \times \mathbb{C}^{m-1} \), we find an asymptotic expansion for the kernel in increasing half-powers of \( \lambda \):

\[
\Pi_{\chi, \lambda} \left( \frac{\theta}{\lambda}, \frac{u}{\sqrt{\lambda}}, \frac{\varphi}{\lambda}, \frac{v}{\sqrt{\lambda}} \right) \sim \frac{8\pi^2 \lambda^{m-1}}{\tau} \int_{\mathbb{C}^{m-1}} e^{\frac{1}{\tau} \left( \frac{i}{2}(\theta - \varphi) - \frac{|u|^2}{2\tau} - \frac{|v|^2}{2\tau} - |w'|^2 + u \cdot w' + w' \cdot \overline{v} \right)}
\times \sum_{j \geq 0} C_j \lambda^{-\frac{j}{2}} P_j (\theta, u, \varphi, v, w') \, dw'
\]

\[
= \frac{8\pi^2 \lambda^{m-1}}{\tau} e^{\frac{iu(\theta - \varphi)}{2\tau} - \frac{|u|^2}{2\tau} + \frac{v}{\sqrt{\lambda}}} \int_{\mathbb{C}^{m-1}} e^{\frac{1}{\tau} \left( -|z|^2 + z \cdot (\overline{v} - \pi) \right)}
\times \sum_{j \geq 0} C_j \lambda^{-\frac{j}{2}} P_j (\theta, u, \varphi, v, w') \, dz.
\]

The equality follows from the change-of-variables \( w = z + u \). The principal term in the series is given by

\[
P_0 = \tau^{-2(m-1)} s_0(p, p) \sigma_0, \tau(p) J(p).
\]

Since

\[
\int_{\mathbb{C}^{m-1}} e^{\frac{1}{\tau} \left( -|z|^2 + z \cdot (\overline{v} - \pi) \right)} \, dz = (\tau \pi)^{m-1},
\]

we conclude

\[
\Pi_{\chi, \lambda} \left( \frac{\theta}{\lambda}, \frac{u}{\sqrt{\lambda}}, \frac{\varphi}{\lambda}, \frac{v}{\sqrt{\lambda}} \right) \sim C_m \frac{\lambda^{m-1}}{\tau^{m-2}} e^{\frac{iu(\theta - \varphi)}{2\tau} - \frac{|u|^2}{2\tau} + \frac{v}{\sqrt{\lambda}} - \frac{N}{2}}
\times \left( 1 + \sum_{j=1}^{N} \lambda^{-\frac{j}{2}} P_j (u, v, \theta, \varphi) \right)
\times \lambda^{m-1} \frac{1}{\tau^{N/2}} R(\theta, u, \varphi, v, \lambda).
\]

4.2 Proof of Theorem 1.3

We look at the near-diagonal scaling asymptotics for the tempered spectral projection kernel (34) under Heisenberg-type scaling. Similar to the proof of Theorem 1.2, we write out the kernel using Proposition 2.5 and (28):

\[
P_{\chi, \lambda} \left( \frac{\theta}{\lambda}, \frac{u}{\sqrt{\lambda}}, \frac{\varphi}{\lambda}, \frac{v}{\sqrt{\lambda}} \right) \sim \int_{\mathbb{R} \times \partial M \times \mathbb{R}^+ \times \mathbb{R}^+} e^{i\lambda \Psi} B \, d\sigma_1 d\sigma_2 d\mu_\tau (w) dt.
\]
in which the phase $\Psi$ and the amplitude $B$ are given by

$$
\Psi = -t + \frac{1}{\lambda} \sigma_2 \psi_\tau \left( \left( \frac{\theta}{\sqrt{\lambda}}, \frac{u}{\sqrt{\lambda}} \right), w \right) + \frac{1}{\lambda} \sigma_1 \psi_\tau \left( G^\lambda_\tau (w), \left( \frac{\varphi}{\lambda}, \frac{v}{\sqrt{\lambda}} \right) \right),
$$

$$
B = \hat{\chi}(t)s \left( \frac{\theta}{\lambda}, \frac{u}{\sqrt{\lambda}}, w, \sigma_2 \right)s \left( G^\lambda_\tau (w), \frac{\varphi}{\lambda}, \frac{v}{\sqrt{\lambda}}, \sigma_1 \right) \sigma_{t, \tau}(w, \sigma_1).
$$

As we point out in Remark 2.8, despite identical notation, the unitarization symbol $\sigma_{t, \tau}$ in the expression for $B$ is of order $-(m-1)/2$, whereas $\sigma_{t, \tau}$ in the expression for $A$ in (42) is of order zero. Hence, the oscillatory integral expression (58) is exactly $\lambda^{-(m-1)/2}$ times the expression (41). The rest of the computations proceed in the same manner.

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