The increasing impact of algorithmic decisions on people's lives compels us to scrutinize their fairness and, in particular, the disparate impacts that ostensibly-color-blind algorithms can have on different groups. Examples include credit decisioning, hiring, advertising, criminal justice, personalized medicine, and targeted policymaking, where in some cases legislative or regulatory frameworks for fairness exist and define specific protected classes. In this paper we study a fundamental challenge to assessing disparate impacts in practice: protected class membership is often not observed in the data. This is particularly a problem in lending and healthcare. We consider the use of an auxiliary dataset, such as the US census, that includes class labels but not decisions or outcomes. We show that a variety of common disparity measures are generally unidentifiable aside for some unrealistic cases, providing a new perspective on the documented biases of popular proxy-based methods. We provide exact characterizations of the sharpest-possible partial identification set of disparities either under no assumptions or when we incorporate mild smoothness constraints. We further provide optimization-based algorithms for computing and visualizing these sets, which enables reliable and robust assessments – an important tool when disparity assessment can have far-reaching policy implications. We demonstrate this in two case studies with real data: mortgage lending and personalized medicine dosing.

Key words: Disparate Impact and Algorithmic Bias; Partial Identification; Proxy Variables; Fractional Optimization; Bayesian Improved Surname Geocoding
Table 1: Protected classes defined under US fair lending laws.

| Law                                      | FHA | ECOA |
|------------------------------------------|-----|------|
| age                                      |     | X    |
| color                                    | X   | X    |
| disability                               | X   |      |
| exercised rights under CCPA              | X   |      |
| familial status (household composition)  | X   |      |
| gender identity                          |     |      |
| marital status (single or married)       | X   |      |
| national origin                          | X   | X    |
| race                                     | X   | X    |
| recipient of public assistance           | X   |      |
| religion                                 | X   | X    |
| sex                                      | X   | X    |

Skeem (2016), while recent studies have revealed systematic race-based disparities in error rates (Angwin et al. 2016, Chouldechova 2017). In healthcare, algorithms that allocate resources like care management have been shown to exhibit racial biases (Obermeyer and Mullainathan 2019) and personalized medicine algorithms can offer disparate benefits to different groups (Goodman et al. 2018, Rajkomar et al. 2018). In lending, prescriptive algorithms optimize credit decisions using predicted default risks and their induced disparities are regulated by law (Comptroller of the Currency 2010), leading to legal cases against discriminatory lending (Consumer Financial Protection Bureau 2013).

For regulated decisions, there are two major legal theories of discrimination:

- **Disparate treatment** (Zimmer 1996): informally, intentionally treating an individual differently on the basis of membership in a protected class; and

- **Disparate impact** (Rutherglen 1987): informally, adversely affecting members of one protected class more than another even if by an ostensibly neutral policy.

Thus, prescriptive algorithms that do not take race, gender, or other sensitive attributes as an input may satisfy equal treatment but may still induce disparate impact (Kleinberg et al. 2017). Indeed, many of the disparities found above take the form of unintended disparate impact of ostensibly class-blind prescriptive algorithms. In some contexts, such as hiring, any disparate impact is prohibited, whereas in other contexts, such as lending, disparate impact is a basis for heightened scrutiny and sometimes sanction while some disparities may be justifiable as “business necessary;” see Section 4. Table 1 summarizes the protected classes codified by two US fair lending laws: the Fair Housing Act (FHA) and Equal Credit Opportunity Act (ECOA).

Assessing the disparate impacts of a prescriptive algorithm involves evaluating the difference in the distributions of decision outcomes received by different groups defined in terms of values of the
protected class of interest and potentially some ground truth. For example, the demographic disparity metric might measure the difference between the fraction of black loan applicants approved and white loan applicants approved. The opportunity (or, true-positive-rate) disparity metric might measure this difference after restricting to non-defaulting applicants in order to account for baseline differences in business-relevant variables like income (Hardt et al. 2016). (We define precisely these disparity metrics in Section 2.1 and discuss them in Section 4.) Note that what size of disparity counts as unacceptable depends on the context. While US employment law often uses the “four-fifths rule” (Equal Employment Opportunity Commission 1978), no such rules of thumb exist in fair lending (Comptroller of the Currency 2010, Consumer Financial Protection Bureau 2018). Therefore, any statistical measures of disparity must be considered in the appropriate legal, ethical, and regulatory context. In any case, they must first be measured.

In this paper, we study a fundamental challenge to assessing the disparity induced by prescriptive algorithms in practice:

*protected class membership is often not observed in the data.*

There may be many reasons for this missingness in practice, both legal, operational, and behavioral. In the US financial service industry, lenders are not permitted to collect race and ethnicity information on applicants for non-mortgage products such as credit cards, auto loans, and student loans. This considerably hinders auditing fair lending for non-mortgage loans, both by internal compliance officers and by regulators (Zhang 2016). Similarly, health plans and health care delivery entities lack race and ethnicity data on most of their enrollees and patients, as a consequence of high data-collection costs and people’s reluctance to reveal their race information for fear of potential discrimination (Weissman and Hasnain-Wynia 2011). This data collection challenge makes monitoring of racial and ethnic differences in care impractical and impedes the progress of healthcare equity reforms (Gaffney and McCormick 2017).

To address this challenge, some methods heuristically use observed proxies to predict and impute unobserved protected class labels. The most (in)famous example is the Bayesian Improved Surname Geocoding (BISG) method. BISG estimates conditional race membership probabilities given surname and geolocation (e.g., census tract or ZIP code) using data from the US census, and then imputes the race labels based on the estimated probabilities. Since its invention (Elliott et al. 2008, 2009), the BISG method has been widely used in assessing racial disparities in health care (Brown et al. 2016, Fremont et al. 2016, 2005, Haas et al. 2019, Weissman and Hasnain-Wynia 2011). In 2009, the Institute of Medicine also suggested it as an interim strategy until routine collection of relevant data is feasible (Nerenz et al. 2009). Later on, this methodology was adopted by the

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1 The US Home Mortgage Disclosure Act (HMDA) authorizes lenders to collect such information for mortgage applicants and co-applicants.
Consumer Financial Protection Bureau (CFPB) (Consumer Financial Protection Bureau 2014), a regulator in the US financial industry. In March 2013, CFPB’s analysis based on the BISG method supported a $98-million settlement against Ally Bank for harming minority borrowers in the auto loan market (Consumer Financial Protection Bureau 2013).

However, the validity of using proxies for the unobserved protected class for disparity assessment is controversial, and relevant research is still limited. Although advanced proxy methods like BISG are shown to outperform previous proxy methods that use only surname or only geolocation (Consumer Financial Protection Bureau 2014, Elliott et al. 2008, 2009), some researchers recently employed mortgage datasets to reveal biased disparity assessment resulting from using the race proxies in place of true race labels (Baines and Courchane 2014, Zhang 2016). Chen et al. (2019) further attempted to unveil the underlying mechanism for the disparity assessment bias, and they attributed the bias to the joint dependence among lending outcome, geolocation, and race. However, a systematic understanding of the precise limitations of using proxy methods in disparity assessment in general, and possible remedies to the potential biases, are still lacking. Filling in this gap is an important and urgent need, especially given the wide use of proxy methods and the high impact of disparity assessment in the settings where they are used, which motivates our current work.

In this paper, we study the basic statistical identification limits for assessing disparities when protected class labels are unobserved and provide new optimization-based algorithms for obtaining sharp partial-identification bounds on said disparities, which can enable robust and reliable auditing of the disparate impact of prescriptive algorithms. We first formulate the problem from the perspective of combining two datasets:

- A main dataset with the decision outcomes, (potentially) true outcomes, and proxy variables, but where the protected class labels are missing; and
- An auxiliary dataset with the proxy variable and protected class label, but without the outcomes.

Based on this formulation, we prove that disparity measures are generally unidentifiable from the observed data. In particular, we give necessary and sufficient conditions for identifiability, and we argue that they are too stringent to hold in practice. This implies that, even with state-of-the-art proxy methods, the observed datasets generally do not contain enough information to measure the disparity metrics of interest, no matter how large the sample sizes. Instead, we fully and exactly characterize the partial identification sets for some common disparity measures, i.e., the sets of all simultaneously valid values of the disparity measures that are compatible with the observed data (and optionally some mild assumptions). These sets provide the sharpest-possible pinpointing of true disparities given the data. Using convex optimization techniques, we provide algorithmic procedures to compute and visualize these partial identification sets (or, their convex hulls). In other
words, our approach gives set estimates for the disparity measures, as opposed to the spurious point estimates given by previous proxy methods for disparity assessment. Given the unidentifiability of the disparity measures, the point estimates of disparity measures provided by previous approaches are generally biased, and the bias may be highly dependent on ad-hoc modelling specifications and thus have unpredictable behavior (Chen et al. 2019). In contrast, our approach fully acknowledges the intrinsic uncertainty in learning disparity without direct measurements of the protected class, and fully exploits all information available in the observed data (and assumptions).

1.1. Contribution
We highlight our primary contributions below:

Problem formulation. We formulate disparity assessment with proxies for unobserved protected class as a data combination problem. This formulation facilitates a principled analysis of the identifiability of the disparity measures.

Identification Conditions. We characterize the necessary and sufficient conditions for point-identifiability using data combination of various disparity measures, which are nonlinear functionals. We argue that these conditions are generally too strong to assume in practice.

Characterizing the Partial Identification Set. We exactly characterize the sharp partial identification sets of some common disparity measures under data combination, that is, the smallest set containing all possible values that disparity measures may simultaneously take while still agreeing with the observed data. We further show how to extend this to incorporate additional mild smoothness assumptions that help reduce uncertainty.

Computing the Partial Identification Set. We propose procedures to compute these partial identification sets (or, their convex hulls) based on linear and linear-fractional optimization. These procedures enable us to visualize the disparity sets and to assess the fairness of prescriptive algorithms in practice.

Robust Auditing. These tools facilitate robust and reliable fairness auditing. Since the sets we describe are sharp in that they are the tightest possible characterization of disparity given the data, their size generally captures the amount of uncertainty that remains in evaluating disparity when the protected class is unobserved and only proxies are available. When the observed data is very informative about the disparity measures, the set tends to be small and may still lead to meaningful conclusions regarding the sign and magnitudes of disparity, despite unidentifiability.

2 This uncertainty is about the lack of identification of the disparity measures from the observed data because some essential information is unobserved (the protected class). It is very different from the uncertainty resulting from finite-sample variability involved in confidence interval or statistical hypothesis testing. When the sample size grows to infinity, the finite-sample uncertainty shrinks to zero, but the identification uncertainty remains. In this paper, we exclusively focus on the more problematic identification uncertainty and, for simplicity, ignore the eventually-vanishing finite-sample uncertainty.
In contrast, when the observed data is insufficient, the set tends to be large and gives a valuable warning about the risk of drawing conclusions from the fundamentally limited observed data.

**Empirical Analysis.** We apply our approach in two real case studies: evaluating the racial disparities (1) in mortgage lending decisions and (2) in personalized Warfarin dosing. We demonstrate how adding extra assumptions may decrease the size of partial identification sets of disparity measures, and illustrate how stronger proxies – either for race or for outcomes – can lead to small partial identification sets and informative conclusions on disparities.

### 2. Problem Setup

We mainly consider four types of relevant variables:

- **True outcome**, $Y \in \{0, 1\}$, is a target variable that justifies an optimal decision. In the lending example (Section 3), we denote $Y = 1$ for loan applicants who would not default on loan payment if the loan application were approved. $Y$ is not known to decision makers at the time of decision making.

- **Decision outcome**, $\hat{Y} \in \{0, 1\}$, is the prescription by either human decisions makers or machine learning algorithms, often based on imperfect predictions of $Y$. For example, $\hat{Y} = 1$ represents approval of a loan application, which is often based on some prediction of default risk. We call $\hat{Y} = 1$ the positive decision, even if it is not favorable in terms of utility (e.g., high medicine dosage).

- **Protected attribute**, $A \in \mathcal{A}$, is a categorical variable (e.g., race or gender). For clear exposition, our convention is to write $A = a$ for a group understood to be generally advantaged and $A = b$ for a disadvantaged group. Take race as an example: the advantaged group usually refers to the majority class (White), and the disadvantaged group refers to any of the minority classes (Black, Hispanic, API, etc.).

- **Proxy variables**, $Z \in \mathcal{Z}$, are a set of additional observed covariates. In proxy methods, these are used to predict $A$. In the BISG example (Section 3), $Z$ stands for surname and geolocation (census tract, zip code, county, etc.). The proxy variables can be categorical, continuous, or mixed.

In this paper, we mainly present the binary outcomes (true outcome and decision outcome), but our results can be straightforwardly extended to multi-leveled outcomes.

We formulate the problem of using proxy methods from a data combination perspective. Specifically, we assume we have two datasets: the main dataset with observations of $(\hat{Y}, Y, Z)$, and the
auxiliary dataset with observations of \((A,Z)\). Because we focus on identification uncertainty rather than finite-sample uncertainty we characterize this by our knowledge of two probability distributions: \(P(\hat{Y},Y,Z)\) and \(P(A,Z)\). Given a sample, these can then be estimated in some way and plugged in, as we will actually do in Section 8.

We cannot simply join these two datasets directly for many possible reasons. For example, no unique identifier for individuals (e.g., social security number) exists in both datasets (if one did, it would fall under the setting of a proxy with perfect prediction; see Section 5.2). Alternatively, we might not even have individual-level observations but only summary frequency statistics. This is for example the case for BISG proxy (see Section 3). Because we have only these two separate, unconnected datasets, we do not know the combined joint distribution \(P(A,\hat{Y},Y,Z)\).

2.1. Disparity measures

In this paper, we focus on assessing the disparity in the decision \(\hat{Y}\) with respect to the protected attribute \(A\). There exist a myriad of disparity measures in the literature (Section 4), and the appropriate disparity measure may have to vary depending on the context. Thus it is impossible to present all disparity measures. In this paper, we focus on so-called observational group disparity measures, which are widely used in the fair machine learning literature (Berk et al. 2018, Corbett-Davies and Goel 2018). These disparity measures are often formalized as some measure of classification error, and, if we were given observations of true class labels, they could be computed from a \(2 \times 2 \times |A|\) within-class confusion matrix of the decision and true outcome.

Specifically, we consider the following disparity measures:

- **Demographic Disparity (DD):**
  \[
  \delta_{DD}(a,b) = P(\hat{Y} = 1 \mid A = a) - P(\hat{Y} = 1 \mid A = b).
  \]

- **True Positive Rate Disparity (TPRD; aka opportunity disparity):**
  \[
  \delta_{TPRD}(a,b) = P(\hat{Y} = 1 \mid A = a, Y = 1) - P(\hat{Y} = 1 \mid A = b, Y = 1).
  \]

- **True Negative Rate Disparity (TNRD):**
  \[
  \delta_{TNRD}(a,b) = P(\hat{Y} = 0 \mid A = a, Y = 0) - P(\hat{Y} = 0 \mid A = b, Y = 0).
  \]

- **Positive Predictive Value Disparity (PPVD):**
  \[
  \delta_{PPVD}(a,b) = P(Y = 1 \mid A = a, \hat{Y} = 1) - P(Y = 1 \mid A = b, \hat{Y} = 1).
  \]

- **Negative Predictive Value Disparity (NPVD):**
  \[
  \delta_{NPVD}(a,b) = P(Y = 0 \mid A = a, \hat{Y} = 0) - P(Y = 0 \mid A = b, \hat{Y} = 0).
  \]
We interpret these disparity measures using the running example of making lending decisions. DD measures the disparity in within-class average loan approval rate. TPRD (respectively, TNRD) measure the disparity in the proportions of people who correctly get approved (respectively, rejected) in loan applications between two classes, given their true npn-default or default outcome. Compared to DD, TPRD and TNRD only measure the disparity unmediated by existing base disparities in true outcome \( Y \), and for this reason is considered as a more appropriate measure for unfairness than demographic disparity (Hardt et al. 2016). In particular, TPRD can be interpreted as disparity in opportunities offered to deserving or qualified individuals. PPVD (respectively, NPVD) measure the disparity in the proportions of approved applicants who pay back their loan (respectively, rejected applicants who default) between two classes. Such disparities can be interpreted as “disparate benefit of the doubt” in an individual having the positive label.

We will focus our attention just on DD, TPRD, and TNRD. Indeed, by swapping the roles of \( Y \) and \( \hat{Y} \) in TPRD and TNRD, all our results can straightforwardly be extended to PPVD and NPVD, respectively. Similarly, disparities based on false negative rate and false positive rate simply differ with TPRD and TNRD by a minus sign, i.e., are given by swapping \( a \) and \( b \). Finally, our results can be extended to composite measures that combine any of the above disparities with some weights.

The main challenge in this paper is to estimate the disparity measures when the protected class \( A \) cannot be observed simultaneously with \( \hat{Y} \) and \( Y \). Instead, we need to rely on proxies \( Z \) for \( A \) from the auxiliary data. Note in particular that we focus on auditing these measures of disparate impact, not necessarily on adjusting algorithms to achieve parity with respect to these. Some robust forms of parity may potentially be achieved using our new tools but it may not necessarily be desirable (see Corbett-Davies and Goel 2018 and discussion in Section 4).

2.2. Notation

Generally we use \( \mathbb{P}(\cdot) \) to represent probability measures that are clear from the context. We often use \( \alpha, z, \hat{y}, y \) as generic values of the random variables \( A, Z, \hat{Y}, Y \), respectively. We also use \( a \) and \( b \) as additional generic values for \( A \), where \( a \) is generally understood to be a majority or advantaged class label.

We further define

\[
\mu(\alpha) := \mathbb{P}(\hat{Y} = 1 \mid A = \alpha), \quad \mu_{\hat{y}, y}(\alpha) := \mathbb{P}(\hat{Y} = 1 \mid A = \alpha, Y = y),
\]

So that \( \delta_{\text{DD}}(a, b) = \mu(a) - \mu(b) \), \( \delta_{\text{TPRD}}(a, b) = \mu_{11}(a) - \mu_{11}(b) \), and \( \delta_{\text{TNRD}}(a, b) = \mu_{00}(a) - \mu_{00}(b) \).

\(^3\) Strictly speaking, demographic disparity is not based on classification “error” but it can be also computed from the within-class confusion matrices.
3. Example: Disparity Assessment in Lending with Geolocation Proxies for Unobserved Race

In this section we present an example of the problem in the case of assessing disparate impacts in lending. We discuss previous proxy approaches based on BISG and study a mortgage lending dataset. We apply the tools we will develop in the paper to describe the set of possible values of disparity that agree with the observed data. We will revisit this case study to provide more details in Section 8.1. In summary, finding that this set is quite large, because the proxies are rather weak, explains the large and spurious biases observed previously in this dataset (Baines and Courchane 2014, Chen et al. 2019, Zhang 2016). This is in contrast to the personalized medicine case study we explore in Section 8.2 where our robust assessment using strong but imperfect proxies suggests conclusions about disparity can be drawn.

The BISG proxy method used for disparity assessment in lending (Consumer Financial Protection Bureau 2014) estimates the conditional probability of race labels, \( P(A = \alpha | Z_s, Z_g) \), given an individual’s surname \( Z_s \) and residence geolocation \( Z_g \), either as the census tract or ZIP code. Specifically, BISG uses a naïve Bayes classifier (Friedman et al. 2001, §6.6.3): it assumes surname and geolocation are independent given race and uses Bayes’s law to combine two separate estimates of the conditional probability of races labels given surname and given geolocation. \( P(A = \alpha | Z_s) \) is typically estimated from a census surname list that includes the fraction of different races for surnames occurring at least 100 times (Comenetz 2016). And, \( P(A = \alpha | Z_g) \) is typically estimated from census Summary File I (US Census Bureau 2010). Even if the naïve Bayes assumption holds and probabilities are perfectly estimated, this only gives half the picture.

We also have access to a main dataset including loan applicants’ information. The main dataset typically includes lending decision outcome (approval or not), actual outcome (e.g., default or not within 2 years), proxy variables (surname and geolocation), and other related variables, but it does not include race information. Figure 1 visualizes the two available datasets. Using the proxies, we can use the BISG model to compute conditional race label probabilities.

Consider assessing the demographic disparity – the simplest measure (see Sections 2.1 and 4 for others) and one that has often been considered for this problem (Chen et al. 2019, Zhang 2016). Here, it measures the discrepancy in marginal approval rates between groups. Here we will consider White, Black, and API (Asian and Pacific Islander). There are many ways to the above to compute demographic disparity (and it is not known what specific method was used by Consumer Financial Protection Bureau 2013). One way is to impute the most likely race, and possibly to additionally discard any data point where the highest conditional probability is below a specified certainty.

---

4 See Baines and Courchane (2014) for more implicit assumptions in constructing the BISG proxy probabilities.
threshold. Another way is to duplicate every data point $|\mathcal{A}|$ times, each with a different label $a \in \mathcal{A}$, and weight each by its corresponding conditional probability. Using a publicly available dataset of mortgage applications, the Home Mortgage Disclosure Act (HMDA) dataset, which includes self-reported race/ethnicity labels and geolocation, and using geolocation as a proxy, various authors have demonstrated that all of the above methods lead to biased estimates (Baines and Courchane 2014, Chen et al. 2019, Zhang 2016).

Indeed, as we will show in Section 5, without additional assumptions, demographic disparity is unidentifiable from this data, even with infinite samples. Furthermore, we will develop tools to study exactly how unidentifiable the disparity is by computing the set of all disparities that agree with the data. To demonstrate this, we apply this method to the same dataset above. We consider proxies consisting of either geolocation (county), income, or both and plot the resulting uncertainty sets in Fig. 2 along with the true value of disparity (only known in this case because we are using a mortgage lending dataset). In the case of income alone, we consider both imposing and not imposing smoothness constraints (see Section 6.2).

First, we can see that all partial identification sets contain the true value of disparity (the only set that makes an assumption not implied by the data is income proxy with smoothness constraint). Second, we can see that the sets are all quite large. This captures the intrinsic uncertainty in learning the demographic disparity from this very limited data. Because our sets are sharp (see Section 5), there is no possible way to further pin down identification without making additional (untestable) assumptions that are not implied by the observed data. When seen in this light, the spurious biases previously reported appear inevitable and serves as a warning sign for drawing any conclusions. In contrast, in Section 8.2 we will find smaller sets that support reliable, uncertainty-robust conclusions. As the disparity assessment may often involve high-impact policy implications, we believe that the uncertainty quantification our tools provide is invaluable in the challenging setting of unobserved protected class.

4. Related Literature

Proxy Methods for Unobserved Race Membership. Fremont et al. (2005) provide a comprehensive review on methods that use only geolocation or surname to impute unobserved race information and comment on their relative strengths for different groups in a US context. As surname and geolocation proxies complement each other, hybrid approaches like BISG were proposed to combine both (Elliott et al. 2008, 2009) and extended to further include first name (Voicu 2018). In terms of the accuracy of race imputation, BISG has been shown to outperform surname-only and geolocation-only analysis in many datasets, including medicare administration data (Dembosky et al. 2019), mortgage data (Consumer Financial Protection Bureau 2014), and voter registration records (Imai and Khanna 2016).
Figure 2 The set of possible simultaneous values for demographic disparity in loan approval rates in the HMDA dataset as determined by different proxies. Positive values correspond to disparity in favor of White. The true demographic disparity is shown as a red star.

However, these evaluations focus on classification accuracy, which is never perfect, and do not consider impact on downstream disparity assessment, mostly because this is usually unknowable. In contrast, Baines and Courchane (2014), Zhang (2016) assessed disparity on a mortgage dataset, and found that using imputed race tends to overestimates the true disparity. Chen et al. (2019) provided a full analysis of this bias and provide sufficient conditions to determine its direction. The analysis and additional empirics show that imputed-race estimators are extremely sensitive to tuning parameters like imputation threshold. As we show in Section 5, disparity is generally unidentifiable from proxies when protected class is unobserved, Consequently, all previous point estimators are generally biased unless very strong assumptions are satisfied.

Algorithmic Fairness. Over the last several years, a large body of literature have proposed more than twenty mathematical definitions of fairness to facilitate risk assessment for algorithmic decision making (Barocas et al. 2018, Cowgill and Tucker 2019, Narayanan 2018, Verma and Rubin 2018). The appropriate definition clearly depends on the context. There is also no clear agreement on when adjusting for parity is a justified a priori constraint. Selecting fairness criteria to enforce is further complicated by the fact that some are incompatible (Chouldechova 2017, Feller et al. 2016, Kleinberg et al. 2017) and many are closely correlated (Friedler et al. 2019). In this paper, we consider auditing two measures of fairness that have received considerable attention in the fair machine learning community: demographic (dis)parity and classification (dis)parity.

Demographic disparity compares the average decision outcome across different protected groups. This is closely related to the “four-fifths rule” in fair hiring, which states that “a selection rate of any protected group that is less than 80% of the highest rate for other groups is an evidence of disparate impact” (Commission et al. 1978, Feldman et al. 2015). Demographic parity and its variants are the focus of numerous early papers in fair machine learning (e.g., Calders et al. 2009).
Louizos et al. 2015, Zafar et al. 2015, Zemel et al. 2013, Zliobaite 2015). However, Hardt et al. (2016) argue that demographic parity is at odds with the utility goal of decision making. Take lending as an example: if default rates differ across groups, demographic parity would rule out the ideal decision according to true default outcome, which can hardly be considered discriminatory and moreover is based on business-relevant differences.

Classification disparity, including TPRD, TNRD, PPVD, and NPVD, compares some measures of classification accuracy (or error) across different protected groups (Corbett-Davies and Goel 2018). In contrast to DD, both TPRD and TNRD measure disparities conditional on the true underlying outcome and thus alleviate the drawbacks of demographic disparity. Classification disparity measures are widely used to characterize disparate impact (Chouldechova 2017) as in the scrutiny of the COMPAS recidivism risk score (Angwin et al. 2016, Feller et al. 2016).

We emphasize that we focus on auditing, not adjusting, disparity measures. Whether observed disparities warrant adjustments depends on the legal, ethical, and regulatory context. For example, as fairness criteria, both demographic and classification parity have been criticized for their inframarginality, i.e., they average over individual risk far from the decision boundary (Corbett-Davies and Goel 2018). However, inframarginality may be unavoidable when outcomes are binary. There may be no true individual “risk,” only the stratified frequencies of binary outcomes (default or recidivation) over strata defined by predictive features, which are in turn chosen by the decision maker. Regardless, disparate impact metrics measure the actual average impact on different groups. Adjudicating whether or not disparities are justifiable (e.g., based on business-relevant factors) still depends on the ability to assess them in the first place.

Comparison to fairness notions studied in the OR literature. Different notions of fairness have been studied in assessing performance guarantees for operational decision-making, with solution concepts such as proportional fairness, lexicographic, or min-max fairness (Adler 2012, Bertsimas et al. 2011, Luss 1999, Ogryczak et al. 2014). Proportional fairness is a solution concept from the fair bargaining literature; while lexicographic or min-max fairness corresponds to a notion of fairness related to equity (Rawls 2001, Young 1995). In contrast to these notions of fairness from the literature on fair division or inequity-aversion in social welfare, we focus on algorithmic fairness definitions that have developed and formalized notions of disparate impact discrimination, often for assessing decisions based on predictive models.

Partial Identification and Data Combination. There is an extensive literature on partial identification of unidentifiable parameters (e.g., Beresteau et al. 2011, Manski 2003). There are many reasons parameters may be unidentifiable, including confounding (e.g., Kallus et al. 2019), missingness (e.g., Manski 2005), and multiple equilibria (e.g., Ciliberto and Tamer 2009). One prominent example is data combination, also termed the “ecological inference problem,” where joint
distributions must be reconstructed from observation of marginal distributions (Freedman 1999, Jiang et al. 2018, Schuessler 1999, Wakefield 2004). One key tool for studying this problem is the Fréchet-Hoeffding inequalities, which give sharp bounds on joint cumulative distributions and super-additive expectations given marginals (Cambanis et al. 1976, Fan et al. 2014, Ridder and Moffitt 2007). Such tools are also used in risk analysis in finance, where the distribution of returns of a portfolio can be analyzed based only on marginal return distributions (Rüschendorf 2013). In contrast to much of the above work, we focus on assessing nonlinear functionals of partially identified distributions, namely, true positive and negative rates, as well as on leveraging conditional information to integrate marginal information across proxy-value levels with possible smoothness constraints.

5. The Identifiability of Disparity Under Data Combination

In this section we study the fundamental limits of our two separate datasets to identify – i.e., pinpoint – the disparity measures of interest.

We first introduce the concept of identification (Lewbel 2018). We call a parameter of interest (either finite-dimensional or infinite-dimensional) identifiable if it can be uniquely determined by unlimited amount of observed data. In other words in the case of iid data, if it is a function of the generating distribution of the data, since this distribution is the most we can hope to learn from samples from it. Conversely, it is unidentifiable if multiple different values of of this parameter all simultaneously agree with the observed data, i.e., it is not a function of the data generating distribution. The set of all these values is called the partial identification set for the parameter. Any value within the partial identification set is equally valid, as the observed data cannot distinguish one from the other. Identification is equivalent to the partial identification set being a singleton.

In our setting, the data is fully described by the two joint distributions \( \mathbb{P}(\hat{Y}, Y, Z) \) and \( \mathbb{P}(A, Z) \). In particular, as samples grow infinitely, we can learn these distributions exactly, but we cannot hope to learn more than that. Therefore, we can only hope to learn parameters that are functions of these distributions.

Note that the disparity measures of interest are functions of the full joint distribution \( \mathbb{P}(A, \hat{Y}, Y, Z) \) and so would immediately be identifiable if we observed the full data \((A, \hat{Y}, Y, Z)\) simultaneously. However, when the protected class \(A\) is not observed directly, the identifiability of the disparity measures is not guaranteed. In particular, we only have partial information about this joint via the marginals learned from each dataset.

Analyzing the identifiability of disparity measures is very important. Unidentifiability of the disparity measures means that it is impossible to pin down the exact values of the disparity measures, even if we have infinite amount samples in the main and auxiliary datasets. Consequently,
in the absence of some additional knowledge that ensure identification, any point estimate is in some sense spurious. It will in general be biased and may be very sensitive to ad-hoc modeling specifications (D’Amour 2019). In this case, generally one must be very cautious about drawing conclusions based on point estimates of disparity measures.

In Sections 5.1 and 5.2, we first give two sets of sufficient conditions for identifiability – one about the unknown joint and one about the marginals. We argue these conditions are too stringent in practice. In Section 5.3, we show that the latter condition is minimal in that in any instance where it is not satisfied, disparity measures are necessarily unidentifiable. In other words, the condition is both necessary and sufficient for identification, barring any additional (untestable) assumptions about unobservables. In Section 6, we will characterize the partial identification set of the disparity measures.

5.1. Identification Under Conditional Independence

Proposition 1. (i) If \( \hat{Y} \perp \perp A \mid Z \), then \( \delta_{DD}(a,b) \) is identifiable from \( \mathbb{P}(\hat{Y}, Y, Z) \), \( \mathbb{P}(A, Z) \).

(ii) If \( Y, \hat{Y} \perp \perp A \mid Z \), then \( \delta_{TPRD}(a,b) \) and \( \delta_{TNRD}(a,b) \) are identifiable from \( \mathbb{P}(\hat{Y}, Y, Z) \), \( \mathbb{P}(A, Z) \).

Proof. When \( \hat{Y} \perp \perp A \mid Z \) and \( Y, \hat{Y} \perp \perp A \mid Z \), respectively, the joint probability distributions factor:

\[
\mathbb{P}(\hat{Y} = \hat{y}, Y = y, A = \alpha, Z = z) = \mathbb{P}(\hat{Y} = \hat{y}, Y = y \mid Z = z) \mathbb{P}(A = \alpha \mid Z = z) \mathbb{P}(Z = z),
\]

\[
\mathbb{P}(\hat{Y} = \hat{y}, A = \alpha, Z = z) = \mathbb{P}(\hat{Y} = \hat{y} \mid Z = z) \mathbb{P}(A = \alpha \mid Z = z) \mathbb{P}(Z = z).
\]

Since \( \delta_{DD}(a,b) \) is a function of the former joint and \( \delta_{TPRD}(a,b), \delta_{TNRD}(a,b) \) of the latter, the conclusion follows by the identifiability of the above marginal conditional probabilities.

In particular, under the conditional independence condition we have that

\[
\mu(\alpha) = \frac{\mathbb{E}[\mathbb{I}[\hat{Y} = 1] \mathbb{P}(A = \alpha \mid Z)]}{\mathbb{E} \left[ \mathbb{P}(A = \alpha \mid Z) \right]}, \quad \mu_{\hat{y}y}(\alpha) = \frac{\mathbb{E}[\mathbb{I}[\hat{Y} = \hat{y}, Y = y] \mathbb{P}(A = \alpha \mid Z)]}{\mathbb{E} \left[ \mathbb{I}[Y = y] \mathbb{P}(A = \alpha \mid Z) \right]},
\]

which, given two datasets, can be consistently estimated by replacing expectations by empirical sums over the main datasets and replacing proxy conditional probabilities \( \mathbb{P}(A = \alpha \mid Z) \) by any consistent estimate based on the auxiliary dataset. For example, doing this for \( \mu(\alpha) \) corresponds exactly to the weighted estimator of Chen et al. (2019).

The conditional independence condition holds if \( Z \) includes all variables that can mediate the dependence between the protected class \( A \), and the decision outcome \( \hat{Y} \) and the true outcome \( Y \). However, this condition is indefensible in real applications. In practice, often the number of proxy variables \( Z \) existing in both the main dataset and the auxiliary dataset is too small to account for the joint dependence completely. In the BISG example (Section 3), \( Z \) only includes surname and geolocation, which are unlikely to capture all dependence between race \( A \), loan approval \( \hat{Y} \) and
default behavior \( Y \). For example, both \( \hat{Y} \) and \( A \) may be highly correlated with FICO score or other socio-economic status factors, even after conditioning on surname and geolocation. In this case, the weighted estimator for DD usually produces biased estimates for the disparity measures. Sufficient conditions to identify the direction of this bias were studied by Chen et al. (2019). Furthermore, the structure of dependence among \( \hat{Y}, Y, A \) may vary across different levels of geolocation and surname, which renders the overall bias in the weighted estimators for TPRD and TNRD largely unpredictable a priori.

### 5.2. Identification Under Perfect Prediction

**Proposition 2.** (i) If for almost all \( z \in Z \), we have either \( \mathbb{P}(\hat{Y} = \hat{y} \mid Z = z) \in \{0, 1\} \) for \( \hat{y} \in \{0, 1\} \) or \( \mathbb{P}(A = \alpha \mid Z = z) \in \{0, 1\} \) for \( \alpha \in A \), then \( \delta_{\text{DD}}(a, b) \) is identifiable from \( \mathbb{P}(\hat{Y}, Y, Z), \mathbb{P}(A, Z) \).

(ii) If for almost all \( z \in Z \), we have either \( \mathbb{P}(\hat{Y} = \hat{y}, Y = y \mid Z = z) \in \{0, 1\} \) for \( \hat{y}, y \in \{0, 1\} \) or \( \mathbb{P}(A = \alpha \mid Z = z) \in \{0, 1\} \) for \( \alpha \in A \), then \( \delta_{\text{TPRD}}(a, b) \) and \( \delta_{\text{TNRD}}(a, b) \) are identifiable from \( \mathbb{P}(\hat{Y}, Y, Z), \mathbb{P}(A, Z) \).

**Proof.** Consider the statement (ii), about \( \delta_{\text{TPRD}}(a, b) \) and \( \delta_{\text{TNRD}}(a, b) \). By the Law of Total Probability, for any \( \hat{y}, y \in \{0, 1\} \) and \( \alpha \in A \),

\[
\mathbb{P}(\hat{Y} = \hat{y}, Y = y \mid Z = z) = \mathbb{P}(\hat{Y} = \hat{y}, Y = y, A = \alpha \mid Z = z) + \mathbb{P}(\hat{Y} = \hat{y}, Y = y, A \neq \alpha \mid Z = z) \tag{1}
\]

\[
\mathbb{P}(A = \alpha \mid Z = z) = \mathbb{P}(A = \alpha, \hat{Y} = \hat{y}, Y = y \mid Z = z) + \mathbb{P}(A = \alpha, \hat{Y} \neq \hat{y} \text{ or } Y \neq y \mid Z = z) \tag{2}
\]

If \( \mathbb{P}(\hat{Y} = \hat{y}, Y = y \mid Z = z) = 0 \), then by Eq. (1), \( 0 \leq \mathbb{P}(\hat{Y} = \hat{y}, Y = y, A = \alpha \mid Z = z) \leq \mathbb{P}(\hat{Y} = \hat{y}, Y = y \mid Z = z) \), which implies that \( \mathbb{P}(\hat{Y} = \hat{y}, Y = y, A = \alpha \mid Z = z) = 0 = \mathbb{P}(A = \alpha \mid Z = z) \mathbb{P}(\hat{Y} = \hat{y}, Y = y \mid Z = z) \). In contrast, if \( \mathbb{P}(\hat{Y} = \hat{y}, Y = y \mid Z = z) = 1 \), then \( 0 \leq \mathbb{P}(A = \alpha, Y \neq y \text{ or } \hat{Y} \neq \hat{y} \mid Z = z) \leq \mathbb{P}(Y \neq y \text{ or } \hat{Y} \neq \hat{y} \mid Z = z) = 0 \), thus by Eq. (2), \( \mathbb{P}(\hat{Y} = \hat{y}, Y = y, A = \alpha \mid z) = \mathbb{P}(A = \alpha \mid Z = z) \cdot 1 = \mathbb{P}(A = \alpha \mid Z = z) \mathbb{P}(\hat{Y} = \hat{y}, Y = y \mid Z = z) \). Analogous conclusions hold if \( \mathbb{P}(A = \alpha \mid Z = z) \in \{0, 1\} \).

Since this holds almost surely we have \( Y, \hat{Y} \perp \! \! \! \perp A \mid Z \), so the result follows by Proposition 1.

A similar argument holds for statement (i), about \( \delta_{\text{DD}}(a, b) \). \( \square \)

The condition in Proposition 2 require that the proxy variables \( Z \) can *perfectly* predict either the protected attribute \( A \) or the decision outcome \( \hat{Y} \) and true outcome \( Y \). This is strictly stronger than the conditional independence condition of Proposition 1. Unlike the latter, the perfect prediction condition is *checkable* as it only involves each dataset separately.

However, the perfect prediction condition almost never holds in practice: neither race nor true outcome can be *perfectly* predicted by any observable predictive features, let alone the few features that can be simultaneously found in the two separate datasets. The decision outcome may be predictable if it is deterministic given features (e.g., given by a machine learning algorithm), but it would require that the same features be observed in the auxiliary dataset. It is unrealistic (and
illegal) that surname and geolocation determine loan approval. Therefore, point estimators may lead to misleading conclusions. In contrast, the partial identification sets we develop in Section 5 will capture this uncertainty and, if it so happens that proxies are indeed perfect, this set will capture this and exactly recover the unique disparity measures.

Note that the comparative predictiveness of different proxies for protected class labels, such as race, has been thoroughly studied (Consumer Financial Protection Bureau 2014, Dembosky et al. 2019, Elliott et al. 2008, 2009, Imai and Khanna 2016) as it was understood this impacts the quality of corresponding proxy assessments of disparity. The above result highlights that the proxies’ predictiveness of outcomes is also crucial, or even sufficient. We discuss the impact of better predictiveness for either outcome or class when neither is perfect in Section 6.3.

First, in the next subsection, we show that the above condition is minimal in that, in any instance where it is violated, disparity is necessarily unidentifiable. That is, it is both necessary and sufficient.

5.3. General Unidentifiability

Since the disparity measures are functions of the full joint distribution $P(A, \hat{Y}, Y, Z)$, to prove the unidentifiability of the disparity measures we show that there generally exist multiple valid full joint distributions that give rise to different disparities but at the same time agree with the marginal joint distributions $P(\hat{Y}, Y, Z)$ and $P(A, Z)$ identifiable from the main dataset and the auxiliary dataset, respectively. To formalize the validity of full joint distributions, we introduce the coupling of two marginal distributions (Villani 2008). Because outcomes and protected classes are discrete, we focus on couplings of discrete distributions.

**Definition 1.** Given two discrete probability spaces $(S, \sigma)$ and $(T, \tau)$ (i.e., $\sigma(s) \geq 0, \tau(t) \geq 0, \sum_{s \in S} \sigma(s) = 1, \sum_{t \in T} \tau(t) = 1$), a distribution $\pi$ over $S \times T$ is a coupling of $(\sigma, \tau)$ if the marginal distributions of $\pi$ coincide with $\sigma$, $\tau$. The set of all possible couplings is denoted $\Pi(\sigma, \tau)$. Specifically:

$$\Pi(\sigma, \tau) = \left\{ \pi \in \mathbb{R}^{S \times T} : \sum_{t \in T} \pi(s, t) = \sigma(s), \sum_{s \in S} \pi(s, t) = \tau(t), \pi(s, t) \geq 0, \forall s \in S, t \in T \right\}.$$ (3)

With the knowledge of marginal distributions, the bounds of the couplings can be rephrased using the Frechet-Hoeffding inequality (Cambanis et al. 1976, Fan et al. 2014, Ridder and Moffitt 2007). This characterization will prove convenient when characterizing the size of partial identification sets in Section 6.3 and computing the partial identification sets in Section 7.

**Proposition 3 (Frechet-Hoeffding).** The coupling set is equivalently given by

$$\Pi(\sigma, \tau) = \left\{ \pi \in \mathbb{R}^{S \times T} : \sum_{t \in T} \pi(s, t) = \sigma(s), \sum_{s \in S} \pi(s, t) = \tau(t), \right. \left. \max\{\sigma(s) + \tau(t) - 1, 0\} \leq \pi(s, t) \leq \min\{\sigma(s), \tau(t)\}, \forall s \in S, t \in T \right\}.$$ (3)

5 Proxies can still be continuous, which we will leverage when we impose smoothness constraints.
Figure 3 Unidentifiability of joint distributions given marginal distributions. The gray region denotes unknown joint probabilities. Row and column sums are known. Even with binary protected class and outcome, this leaves one degree of freedom in the unknowns, unless one of the marginals is degenerate.

The coupling set includes all valid joint distributions that agree with marginal distributions. Whether the coupling set contains more than one valid joint distribution is crucial for the identifiability of parameters of the joint distribution. For example, demographic disparity is determined by the joint distribution \( P(A, \hat{Y}, Z) = P(A, \hat{Y} | Z) P(Z) \), therefore the identifiability of the joint \( P(A, \hat{Y} | Z) \), which is a coupling of \( (P(A | Z), P(\hat{Y} | Z)) \), is a sufficient condition for the identifiability of demographic disparity.

An illustration of this is given in Fig. 3. With binary protected class and outcomes, marginal information provides only three independent constraints on four unknowns. If one of the marginals is equal to 1, the nonnegativity of probabilities forces the unknowns to a single point. Indeed this is what drives Proposition 2. This also straightforwardly extends to the coupling set \( \Pi(P(A | Z), P(Y | Z)) \).

It remains to be shown that when the conditions of Proposition 2 are violated, not only can we always necessarily have multiple different couplings, but also that having these for only certain values of \( Z \) is sufficient to render the disparities, which are differences of nonlinear functions of the couplings, unidentifiable. This shows, that barring assumptions about unobservables such as those in Proposition 1, the conditions of Proposition 2 on the known marginals are both necessary and sufficient for point-identification of disparity measures.

**Proposition 4.** Let \( \mathcal{A} = \{a, b\} \). Let any marginal distributions \( P(\hat{Y}, Y, Z), P(A, Z) \) be given. As long as there exists a set of \( z \)'s with positive probability such that the assumptions of Proposition 2(i) are not satisfied, then \( \delta_{DD}(a, b) \) is unidentifiable. That is, there exist two different joint distributions \( P(A, \hat{Y}, Y, Z) \) that agree with these marginals but give rise to different values of \( \delta_{DD}(a, b) \).

Similarly, as long as there exists a set of \( z \)'s with positive probability such that the assumptions of Proposition 2(ii) are not satisfied, then both \( \delta_{TPRD}(a, b) \) nor \( \delta_{TNRD}(a, b) \) are unidentifiable.

Note that by exchanging \( \hat{Y} \) and \( Y \) the same conclusions hold for PPVD, NPVD.

\( ^6 \) Note \( P(Z) \) can be easily identified from either the main dataset or the auxiliary dataset.
To prove Proposition 4 we show that given any distributions that violate the assumptions of Proposition 2, we can still construct feasible couplings that lead to different disparities. In particular, since discrepancies are differences of nonlinear functions of the coupling, we need to show that the variation in probabilities can be chosen so not to cancel for any given set of marginals. Since the feasible choices depend on the given marginals, which can be arbitrary as long as they violate Proposition 2, our proof proceeds by considering six exhaustive cases and proving the conclusion in each. See Appendix A.1 for the proof.

6. Partial Identification of Disparity Under Data Combination

In the last section we showed that DD, TPRD, and TNRD (and symmetrically also PPVD and NPVD) are generally not identifiable from the two separate datasets. Next, we will characterize exactly how identifiable or unidentifiable they are by describing the set of all disparity values that agree with the data, and possibly any additional assumptions.

6.1. Sharp Partial Identification Sets Without Additional Assumptions

We first consider the case where we impose no additional assumptions other than what is given directly from the data in terms of marginal distributions. Toward that end, for any functions $w_\alpha(\hat{y}, z)$ and $\tilde{w}_\alpha(\hat{y}, y, z)$, define, respectively,

$$
\mu(\alpha; w) := \frac{\mathbb{E}[w_\alpha(\hat{Y}, Z)\hat{Y}]}{\mathbb{E}[w_\alpha(\hat{Y}, Z)]},
$$

(4)

$$
\mu_{\tilde{y}y}(\alpha; \tilde{w}) := \frac{\mathbb{E}[\tilde{w}_\alpha(\hat{Y}, Y, Z)\mathbb{I}(Y = y)\mathbb{I}(\hat{Y} = \hat{y})]}{\mathbb{E}[\tilde{w}_\alpha(\hat{Y}, Y, Z)\mathbb{I}(Y = y)]}.
$$

(5)

Furthermore define

$$
w_\alpha^*(\hat{y}, z) := \mathbb{P}(A = \alpha | \hat{Y} = \hat{y}, Z = z), \quad \tilde{w}_\alpha^*(\hat{y}, y, z) := \mathbb{P}(A = \alpha | \hat{Y} = \hat{y}, Y = y, Z = z)
$$

such that $\mu(\alpha) = \mu(\alpha; w^*)$ and $\mu_{\tilde{y}y}(\alpha) = \mu_{\tilde{y}y}(\alpha; \tilde{w}^*)$, recalling the definitions from Section 2.2.

We make two important observations. First, for any fixed functions $w, \tilde{w}$, both $\mu(\alpha; w)$ and $\mu_{\tilde{y}y}(\alpha; \tilde{w})$ are identifiable from just the marginal $\mathbb{P}(\hat{Y}, Y, Z)$ since every term is just an expectation over that distribution. Second, the special functions $w^*, \tilde{w}^*$ depend upon the unknown full joint distribution $\mathbb{P}(A, \hat{Y}, Y, Z)$. Indeed, $\mu(\alpha)$ and $\mu_{\tilde{y}y}(\alpha)$ are not identifiable from the data, or else disparities would be too.

We use this reformulation to characterize the range of possible disparities. We will fix one class, $a \in A$, to measure all disparities against. Therefore, define $A_0 = A \setminus \{a\}$ and, given sets $\mathcal{W}, \tilde{\mathcal{W}}$,

$$
\Delta_{DD}(\mathcal{W}) := \{(\mu(a; w) - \mu(b; w))_{b \in A_0} : w \in \mathcal{W}\},
$$

$$
\Delta_{TPRD}(\tilde{\mathcal{W}}) := \{(\mu_{11}(a; \tilde{w}) - \mu_{11}(b; \tilde{w}))_{b \in A_0} : \tilde{w} \in \tilde{\mathcal{W}}\},
$$

$$
\Delta_{TNRD}(\tilde{\mathcal{W}}) := \{(\mu_{00}(a; \tilde{w}) - \mu_{00}(b; \tilde{w}))_{b \in A_0} : \tilde{w} \in \tilde{\mathcal{W}}\}.
$$
Any disparity between \( b, c \in A_0 \) can be given by contrasting the disparities for \( a, b \) and \( a, c \). Similarly, PPVD and NPVD are given by swapping \( \hat{Y} \) and \( Y \). Moreover, we can extend the above to sets combining multiple disparities at the same time.

We next show that appropriately defining the sets \( \mathcal{W}, \hat{\mathcal{W}} \) gives the sharp partial identification sets. Let (where “LTP” refers to the Law of Total Probability)

\[
\mathcal{W}_{\text{LTP}} = \left\{ w : \sum_{y \in \{0,1\}} w_{\alpha}(\hat{y}, y, z) \mathbb{P}(\hat{Y} = \hat{y} \mid Y = y, Z = z) = \mathbb{P}(A = \alpha \mid Z = z), \quad \sum_{\alpha \in A} w_{\alpha}(\hat{y}, z) = 1, w_{\alpha}(\hat{y}, z) \geq 0, \text{ for any } \alpha, z, \hat{y} \right\}
\]

\[
\hat{\mathcal{W}}_{\text{LTP}} = \left\{ \hat{w} : \sum_{y \in \{0,1\}} \hat{w}_{\alpha}(\hat{y}, y, z) \mathbb{P}(\hat{Y} = \hat{y}, Y = y \mid Z = z) = \mathbb{P}(A = \alpha \mid Z = z), \quad \sum_{\alpha \in A} \hat{w}_{\alpha}(\hat{y}, y, z) = 1, \hat{w}_{\alpha}(\hat{y}, y, z) \geq 0, \text{ for any } \alpha, z, \hat{y}, y \right\}
\]

and note that these sets only depend on the known marginals \( \mathbb{P}(A \mid Z) \), \( \mathbb{P}(\hat{Y}, Y, Z) \).

**Proposition 5.** Given marginals \( \mathbb{P}(A \mid Z) \), \( \mathbb{P}(\hat{Y}, Y, Z) \), the sets \( \Delta_{\text{DD}}(\mathcal{W}_{\text{LTP}}) \), \( \Delta_{\text{TPRD}}(\hat{\mathcal{W}}_{\text{LTP}}) \), \( \Delta_{\text{TNRD}}(\hat{\mathcal{W}}_{\text{LTP}}) \) are sharp for the true disparities. That is, for each set, every element corresponds to the true disparity given by some full joint \( \mathbb{P}(A, \hat{Y}, Y, Z) \) agreeing with the given marginals and, conversely, every such full joint gives rise to one of the elements.

The proof is given in Appendix A.2. Proposition 5 shows that the given sets are the tightest-possible characterization of the possible simultaneous values of the disparities of interest when we are given only the main and auxiliary datasets and no other additional assumptions that are not already implied by the data itself. In particular, we necessarily have that if the conditions of Proposition 5 hold then these sets are singletons.

### 6.2. Sharp Partial Identification Sets With Smoothness Assumptions

For DD, the formulation given above is constructed by using \( \mathbb{P}(Z) \) to average over different couplings \( \mathbb{P}(A, Y \mid Z = z) \), each in \( \Pi(\mathbb{P}(A \mid Z = z), \mathbb{P}(Y \mid Z = z)) \). Lacking any additional assumptions, each coupling can in principle be chosen independently of the others; i.e., the sets we get in this way are sharp. However, one might expect that, for two similar values \( z, z' \), the two true joints \( \mathbb{P}(A, Y \mid Z = z), \mathbb{P}(A, Y \mid Z = z') \) are also similar (some limited amount of similarity is already implied by the Law of Total Probability when the given marginals are smooth). Truly, there is no way to know from the separate datasets only (again, the sets above are sharp), but such an assumption may be defensible and can help narrow the possible values disparities may take.

We therefore further consider sharp partial identification sets of disparities when we impose the following additional assumptions:

\[
\mathbb{P}(A = \alpha \mid Y = y, Z = z) - \mathbb{P}(A \mid Y = y, Z = z') \leq d(z, z') \quad \forall \alpha, y, z, z' \quad (6)
\]

\[
\mathbb{P}(A = \alpha \mid \hat{Y} = \hat{y}, Y = y, Z = z) - \mathbb{P}(A, Y \mid \hat{Y} = \hat{y}, Y = y, Z = z') \leq d(z, z') \quad \forall \alpha, \hat{y}, y, z, z' \quad (7)
\]
where \( d(z, z') \) is some given metric. In particular, we encode the implicit Lipschitz constant within the metric \( d \) itself.

Let

\[
W_{\text{Lip}} := \{ w : w_\alpha(\hat{y}, z) - w_\alpha(\hat{y}, z') \leq d(z, z') \forall z, z', \hat{y} \},
\]

\[
\tilde{W}_{\text{Lip}} := \{ \tilde{w} : \tilde{w}_\alpha(\hat{y}, y, z) - \tilde{w}_\alpha(\hat{y}, y, z') \leq d(z, z') \forall z, z', \hat{y}, y \}.
\]

**Proposition 6.** Given marginals \( P(A \mid Z), P(\hat{Y}, Y, Z) \), \( \Delta_{\text{DD}}(W_{\text{LTP}} \cap W_{\text{Lip}}) \) is sharp for DD assuming Eq. (6) and \( \Delta_{\text{TPRD}}(\tilde{W}_{\text{LTP}} \cap \tilde{W}_{\text{Lip}}) \), \( \Delta_{\text{TNRD}}(\tilde{W}_{\text{LTP}} \cap \tilde{W}_{\text{Lip}}) \) are sharp for TPRD, TNRD, respectively, assuming Eq. (7). That is, for each set, every element corresponds to the true disparity given by some full joint \( P(A, \hat{Y}, Y, Z) \) agreeing with the given marginals and satisfying Eq. (6) or Eq. (7) (respectively) and, conversely, every such full joint gives rise to one of the elements.

The proof is given in Appendix A.2.

6.3. The Size of the Partial Identification Sets

The partial identification sets given in Propositions 5 and 6 are sharp, i.e., they are the smallest sets containing all possible values of the disparity measures that are compatible with the data and possibly any assumptions. A natural question is when are these smallest possible sets also actually small. We next discuss different scenarios where the sets can be small or large and the implications.

**Informative proxies.** If the proxies are very predictive, then the observed data may be informative enough to sufficiently pin down the the disparity measures. At the extreme, if proxies are perfectly predictive, Proposition 2 showed the sets will become singletons. Indeed, considering the Fréchet-Hoeffding bounds, Eq. (3), we see that if either marginal is zero or one, the bounds collapse. If proxies are not perfect but are very predictive, either of protected class, of class, or of both, then lower and upper bounds are not equal but they are still close. Consequently, the partial identification sets will be small. This is the case we observe in Section 8.2 when using genetic proxies.

**Assumptions on joint.** If the data is not very informative by itself, we can still combine it with assumptions on the unknown joint to narrow down the options. In Section 6.2, we proposed to use smoothness assumptions. These assumptions are both rather mild and possibly defensible even if not verifiable. We observed in Section 3 that imposing this assumption narrowed down the set based on income proxies slightly. On the other hand, imposing the much stronger assumptions in Proposition 1 would necessarily collapse the partial identification set to a singleton. One could potentially impose some relaxed version of this conditional independence assumption, requiring that conditionals be close rather than equal. This, for example, can easily be incorporated into many
of the optimization formulations we present in the next section. However, we do not believe this is advisable in practice and therefore do not present it. The conditional independence assumption is highly unrealistic and so it does not make sense to conduct a sensitivity analysis on its relaxation: we do not expect it to hold even slightly. Therefore, although such strong assumptions are indeed very informative, they may also lead to misleading conclusions if the assumptions are wrong.

Uninformative proxies and weak or no assumptions. If the observed datasets and statistical law alone are not sufficiently informative, and we are not willing to impose overly stringent assumptions, we generally end up with partial identification sets with nontrivial size. In this case, the size of partial identification sets exactly captures the uncertainty in learning disparity measures based on the observed data and imposed assumptions. Large sets are not meaningless: they serve as an important warning about drawing any conclusions from highly flawed data.

7. Computing the Partial Identification Sets
In the previous section we described the \textit{sharp}, i.e., tightest-possible, sets of disparity values that can simultaneously realize given all that we know from the data and any assumptions. However, it is not immediately clear what to do with these definitions. In this section we show how to actually compute these sets. Specifically, we consider computing their \textit{support functions}.

Given a set $\Theta \subseteq \mathbb{R}^d$ its support function is given by $h_{\Theta}(\rho) = \sup_{\theta \in \Theta} \rho^T \theta$. Not only does the support function provide the maximal and minimal contrasts in a set, it also exactly characterizes its convex hull \cite{Rockafellar2015}. That is, $\text{Conv} (\Theta) := \{\lambda \theta + (1-\lambda)\theta' : \theta, \theta' \in \Theta, \lambda \in [0,1]\} = \{\theta : \rho^T \theta \leq h_{\Theta}(\rho) \ \forall \rho\}$. So computing $h_{\Theta}(\rho)$ allows us to compute $\text{Conv} (\Theta)$, and ranging over a $\rho$ grid allows us to visualize the set as the polyhedron given by the corresponding hyperplanes, which gives a safe outer approximation. In the next sections we therefore consider how to compute the support functions for the sets we developed in the above sections. Then, we can both compute maximal/minimal disparities between any two class labels and visualize the whole set of simultaneously realizable disparities between any two class labels.

7.1. Demographic Disparity, Binary Protected Class Without Smoothness
When considering a binary protected class, $A = \{a, b\}$, the convex hull of partial identification sets are simply intervals. For the case of demographic disparity, the endpoints take a particularly simple form.

**Proposition 7.** Let

$$w_{\alpha}^L(\hat{y}, z) = \max \left\{0, \frac{1}{\mathbb{P}(\hat{Y} = \hat{y} \mid Z = z)} \right\}, \quad w_{\alpha}^U(\hat{y}, z) = \min \left\{1, \frac{\mathbb{P}(A = \alpha \mid Z = z)}{\mathbb{P}(\hat{Y} = \hat{y} \mid Z = z)} \right\},$$

Then

$$\text{Conv}(\Delta_{DD}(W_{LTP})) = [\mu(a; w^L) - \mu(b; w^L), \mu(a; w^U) - \mu(b; w^U)].$$
Notice that \( \mathbb{P}(\hat{Y} = \hat{y} | Z = z)w^L_\alpha(\hat{y}, z), \mathbb{P}(\hat{Y} = \hat{y} | Z = z)w^U_\alpha(\hat{y}, z) \) exactly correspond to the endpoints of the Fréchet-Hoeffding inequalities in Eq. (3). The key observations driving the proof are: (a) that \( w \in \mathcal{W}_{\text{LTP}} \) must imply that for the denominator in (4), \( \mathbb{E}[w_\alpha(\hat{Y}, Z)] = \mathbb{P}(A = \alpha) \), which does not depend on \( w \), and (b) that when \( A = \{a, b\} \) we must have \( w^L_\alpha(\hat{y}, z) + w^U_\alpha(\hat{y}, z) = 1 \) by complementarity of the labels. See Appendix [A.3] for the full proof.

7.2. Classification Disparity, Binary Protected Class Without Smoothness

Next we consider the same setting for the classification disparities TPRD and TNRD. Again, the convex hull is an interval and we can express its endpoints in closed form, but they are slightly more intricate.

**Proposition 8.** Let

\[
\mu'_{\alpha y}(\alpha; \tilde{w}, \tilde{w}') := \frac{\mathbb{E}[\tilde{w}_\alpha(\hat{Y}, Y, Z)I(Y = y)I(\hat{Y} = \hat{y})]}{\mathbb{E}[\tilde{w}_\alpha(\hat{Y}, Y, Z)I(Y = y)I(\hat{Y} = \hat{y}) + \mathbb{E}[\tilde{w}'_\alpha(\hat{Y}, Y, Z)I(Y = y)I(\hat{Y} \neq \hat{y})]},
\]

\[
\bar{w}^L_\alpha(\hat{y}, y, z) = \max \left\{ 0, 1 + \frac{\mathbb{P}(A = \alpha | Z = z) - 1}{\mathbb{P}(Y = \hat{y}, Y = y | Z = z)} \right\},
\]

\[
\bar{w}^U_\alpha(\hat{y}, y, z) = \min \left\{ 1, \frac{\mathbb{P}(A = \alpha | Z = z)}{\mathbb{P}(Y = \hat{y}, Y = y | Z = z)} \right\}.
\]

Then

\[
\text{Conv}(\Delta_{\text{TPRD}}(\mathcal{W}_{\text{LTP}})) = [\mu'_{11}(a; \bar{w}^L, \bar{w}^U) - \mu'_{11}(b; \bar{w}^U, \bar{w}^L), \mu'_{11}(a; \bar{w}^U, \bar{w}^L) - \mu'_{11}(b; \bar{w}^L, \bar{w}^U)],
\]

\[
\text{Conv}(\Delta_{\text{TNRD}}(\mathcal{W}_{\text{LTP}})) = [\mu'_{00}(a; \bar{w}^L, \bar{w}^U) - \mu'_{00}(b; \bar{w}^U, \bar{w}^L), \mu'_{00}(a; \bar{w}^U, \bar{w}^L) - \mu'_{00}(b; \bar{w}^L, \bar{w}^U)].
\]

To sketch the proof of Proposition 8, consider the case of TPRD. Notice, again, that \( \mathbb{P}(\hat{Y} = \hat{y}, Y = y | Z = z)\bar{w}^L_\alpha(\hat{y}, y, z), \mathbb{P}(\hat{Y} = \hat{y}, Y = y | Z = z)\bar{w}^U_\alpha(\hat{y}, y, z) \) exactly correspond to the endpoints of the Fréchet-Hoeffding inequalities in Eq. (3). Therefore, the box given by these coordinate-wise bounds, \( \bar{w}^L_\alpha(\hat{y}, y, z), \bar{w}^U_\alpha(\hat{y}, y, z) \), contains \( \mathcal{W}_{\text{LTP}} \). If we consider, say, maximizing \( \mu_{11}(a; \bar{w}) - \mu_{11}(b; \bar{w}) \) over this box, we arrive at the bounds above. Moreover note that \( \mu_{11}(a; \bar{w}) - \mu_{11}(b; \bar{w}) \) only depends on the restriction of \( \bar{w} \) to \( y = 1 \), i.e., \( \bar{w}_\alpha(\hat{1}, 1, z) : \alpha \in \mathcal{A}, \hat{y} \in \{0, 1\}, z \in \mathcal{Z} \). So, to prove Proposition 8, we consider four exhaustive cases and show that in each case, even though it may not be possible to achieve all coordinate bounds simultaneously, we can still find a feasible \( \bar{w} \in \mathcal{W}_{\text{LTP}} \) that achieves the necessary coordinate bounds for the restriction to \( y = 1 \). See Appendix [A.4] for detailed proof.

7.3. Demographic Disparity, General Case

When we either have more than two class labels or we wish to impose smoothness constraints, computing identification sets is no longer closed form. We first consider the far simpler case of demographic disparity.
Proposition 9. Suppose $\mathcal{W}_{\text{LTP}} \subseteq \mathcal{W}$. Then

$$h_{\Delta_{DD}(\mathcal{W})}(\rho) = \max_{w \in \mathcal{W}} \sum_{b \in A_0} \rho_b \left( \frac{\mathbb{E}[w_a(\hat{Y}, Z)|\hat{Y}]}{\mathbb{P}(A = a)} - \frac{\mathbb{E}[w_b(\hat{Y}, Z)|\hat{Y}]}{\mathbb{P}(A = b)} \right).$$

When either $\mathcal{W} = \mathcal{W}_{\text{LTP}}$ or $\mathcal{W} = \mathcal{W}_{\text{LTP}} \cap \mathcal{W}_{\text{Lip}}$, the above gives an infinite linear program since the constraints are linear in $w$. When either $Z$ is discrete or when expectations are estimated by empirical sample averages, this becomes a regular linear program, which is easily solvable with off-the-shelf software (we use Gurobi).

Proposition 9 follows directly by our sharp characterization of the partial identification set in Section 6, using the characterization Eq. (4) of within-group mean outcome, and noting that $w \in \mathcal{W}$ implies $w \in \mathcal{W}_{\text{LTP}}$ which in turn implies that $\mathbb{E}[w_\alpha(\hat{Y}, Z)|\hat{Y}] = \mathbb{P}(A = \alpha)$, which is identifiable from the auxiliary dataset.

7.4. Classification Disparity, General Case

We next consider the case of classification disparities in the general case. For a concise and clear exposition, we focus on the case of TPRD. The case of TNRD can be symmetrically handled.

Proposition 10. Suppose $\tilde{\mathcal{W}} = \tilde{\mathcal{W}}_{\text{LTP}} \cap \tilde{\mathcal{W}}'$ where $\tilde{\mathcal{W}}' = \{w : w_\alpha \in \tilde{\mathcal{W}}'_\alpha \forall \alpha \in A\}$. Then

$$h_{\Delta_{\text{TPRD}}(\tilde{\mathcal{W}})}(\rho) = \max_{\tilde{u}, t} \sum_{b \in A_0} \rho_b \left( \mathbb{E}[\tilde{u}_a(\hat{Y}, Y, Z)|\hat{Y}] - \mathbb{E}[\tilde{u}_b(\hat{Y}, Y, Z)|\hat{Y}] \right)$$

s.t.

$$\mathbb{E}[\tilde{u}_\alpha(\hat{Y}, Y, Z)|Y] = 1 \quad \forall \alpha \in A, \quad (9)$$

$$\sum_{\alpha \in A} \tilde{u}_\alpha(\hat{y}, y, z) t_\alpha = 1 \quad \forall \hat{y} \in \{0, 1\}, y \in \{0, 1\}, z \in Z, \quad (10)$$

$$\sum_{\hat{y}, y \in \{0, 1\}} \tilde{u}_\alpha(\hat{y}, y, z) \mathbb{P}(Y = \hat{y}, Y = y | Z = z) = \mathbb{P}(A = \alpha | Z = z) t_\alpha \quad \forall \alpha \in A, z \in Z, \quad (11)$$

$$\tilde{u}_\alpha(\hat{y}, y, z) \geq 0 \quad \forall \alpha \in A, \hat{y} \in \{0, 1\}, y \in \{0, 1\}, z \in Z, \quad (12)$$

$$u_\alpha / t_\alpha \in \tilde{\mathcal{W}}'_\alpha \quad \forall \alpha \in A. \quad (13)$$

Equation (8) is generally a non-convex infinite program. When either $Z$ is discrete or when expectations are estimated by empirical sample averages, this becomes a finite program but still non-convex. Note that the objective is linear in $\tilde{u}$ and that Eq. (13) is a perspective constraint, which is convex if $\tilde{\mathcal{W}}'_\alpha$ is convex and linear if $\tilde{\mathcal{W}}'_\alpha$ is a polyhedron, as in the case of $\tilde{\mathcal{W}}'_\alpha = \tilde{\mathcal{W}}_{\text{Lip}}$. Thus, the only non-convex constraint is Eq. (10). To handle this non-convexity, we note that the constraint becomes linear once we fix $(t_\alpha)_{\alpha \in A}$.

To solve Eq. (8) we therefore propose to perform a grid search over values $(t_\alpha)_{\alpha \in A}$ and solve the convex (usually linear) program that remains for each, until the maximal value is found. Note that
Eqs. (9) and (10) together imply that \( \sum_{\alpha \in \mathcal{A}} 1/t_{\alpha} = \mathbb{P}(Y = 1) \). Therefore, it suffices to search over a \((|\mathcal{A}| - 1)\)-dimensional grid.

Proposition 10 follows by our sharp characterization of the partial identification set in Section 6, using Eq. (5), and applying several Charnes-Cooper-type transformations (Charnes and Cooper 1962). See Appendix A.5 for detail.

8. Case Studies

In the subsequent sections we consider applying our results and methods in two different case studies: mortgage credit decisioning and personalized Warfarin dosing. In both cases, we estimate all probabilities and plug them into our methods above, focusing on the identification uncertainty rather than sampling uncertainty, especially given the datasets are fairly large relative to the dimension of estimated probability parameters.

8.1. Mortgage Credit Decisioning

We first demonstrate the partial identification set of demographic disparity with the public HMDA (Home Mortgage Disclosure Act) data set\(^7\), which has been used in the literature to evaluate proxy methods for race (Baines and Courchane 2014, Chen et al. 2019, Zhang 2016). This dataset includes mortgage loan application records in the U.S., which include self-reported race/ethnicity, loan origination outcome, geolocation (state, county, and census tract), annual income, loan amount, among other variables. We use a sample containing 14903 loan application records for White, Black, and API applicants with annual income no more than $100K during 2011-2012. We denote \( \hat{Y} = 1 \) if a loan application was approved or originated, and \( \hat{Y} = 0 \) if it was denied. The dataset does not contain default outcomes. We therefore restrict our attention to demographic disparity and consider a ternary protected class: \( \mathcal{A} = \{\text{White}, \text{Black}, \text{API}\} \).

We consider three different set of proxy variables for the unobserved race: only geolocation (county), only annual income, and both geolocation and annual income. The dataset is anonymized and does not include surname information, so we could not evaluate the popular BISG method. Instead we used income as an additional proxy. The distribution of race/ethnicity by both these proxies can likely be estimated from public records. For example, U.S. census Summary File I (US Census Bureau 2010) contains race distributions for different geolocation levels, and the Annual Population Survey (United States Census Bureau 2018) contains race distributions for different income brackets. To illustrate the effect of the smoothness constraint in Section 6.2, we estimate the conditional probabilities of race and decision outcome directly on the sample of interest where

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\(^7\) The dataset can be downloaded from [https://www.consumerfinance.gov/data-research/hmda/explore](https://www.consumerfinance.gov/data-research/hmda/explore).

\(^8\) We limit the annual income to better demonstrate the income proxy, which is easier to train on a restricted range.
the income variable is continuous. When only geolocation is used as the proxy variable, we use the within-county race proportions and average loan acceptance rate to estimate the conditional probabilities of race and loan acceptance respectively. When only income is used as the proxy variable, we fit a logistic regression to estimate the conditional probability of loan acceptance, and a multinomial logistic regression to estimate the conditional probabilities of races. When both income and geolocation are used as the proxy variables, we fit the logistic regression and multinomial logistic regression with respect to income within each county.

Recall that the size of the partial identification set depends on the informativeness of the proxies about both protected class and outcomes (Section 6.3). In Fig. 4, we display the histograms of the conditional probability predictions for each race and, separately, for the positive outcome. We also report the (negative) entropy ($\sum_{\alpha \in A} P(A = \alpha \mid Z)log P(A = \alpha \mid Z)$), which summarizes how predictive the proxies are. We find that, in terms of outcome, all proxies are equally uninformative (the unconditional entropy is $-0.500$). In terms of protected class, we find geolocation more informative than income and that combining them adds very little.

Figure 2 shows the resulting partial identification sets of the demographic disparities when different proxy variables are used. For the income-only proxy, we show the partial identification sets both without the smoothness constraint and with the smoothness constraint, where the Lipschitz constant is set as the minimal feasible one. We also show the true demographic disparity computed based on the self-reported race and considering the dataset as a whole as opposed to combining two separate datasets. We can observe that all partial identification sets are fairly large, and all of them correctly contain the ground truth demographic disparity. Using income as proxy without smoothness constraint seems quite weak in terms of identifying the demographic disparity. Income-only proxy without smoothness results in the largest partial identification set, and using income
8.2. Personalized Warfarin Dosing

In this section, we demonstrate an example of informative proxy variables in a PharmGKB dataset of 5700 patients treated with warfarin, the most commonly used oral anticoagulant agent worldwide (Consortium 2009). Finding appropriate warfarin dosage is very challenging and important, since it can vary drastically among patients and incorrect dose can possibly lead to serious adverse outcomes. This challenge attracts considerable interest in designing personalized warfarin dosage algorithms, including linear regression (Consortium 2009), LASSO (Bastani and Bayati 2015), and decision trees (Kallus 2017). However, it was shown that the personalized dosage algorithms may show disparate performance for different ethnic groups (see, e.g., Supplementary Appendix 9 in Consortium 2009). Here we apply our method to audit the potential racial disparity of a personalized warfarin dosage algorithm.

9 The dataset can be downloaded from https://www.pharmgkb.org/downloads
The data for each patient includes demographics (sex, ethnicity, age, weight, height, and smoker), reason for treatment (e.g., atrial fibrillation), current medications, co-morbidities (e.g., diabetes), genetic factors (presence of genotype variants of CYP2C9 and VKORC1). All of these variables are categorical, and we treat missing value of each variable as a separate value. Moreover, this dataset contains the true patient-specific optimal warfarin doses determined by physicians’ adjustment over a few weeks. We focus on the subsample of 4891 White, Black, and Asian patients whose optimal warfarin doses are not missing. We dichotomize the optimal doses into high dosage (more than 35mg/week, denoted $Y = 1$), and low dosage (less than 35mg/week, denoted $Y = 0$). To develop a personalized dosage algorithm, we follow Consortium (2009) and fit a linear regression to predict the optimal dosage based on all other variables, and recommend high dosage if the predicted optimal dosage is more than 35mg/week ($\hat{Y} = 1$) and recommend low dosage ($\hat{Y} = 0$) otherwise. Our goal is to evaluate the partial identification sets for true positive rates of this personalized dosage algorithm.

We consider three sets of discrete proxy variables: only genetic factors, only current medications, and both genetic factors and current medications. The conditional probabilities of race, optimal dosage indicator $Y$, and recommended dosage indicator $\hat{Y}$, given these proxy variables can be easily estimated by corresponding sample averages within each level of the proxy variables. Among the proxy variables, the genetic factors are particularly strong candidates since they are found to be highly predictive for the optimal warfarin dosage (Consortium 2009). At the same time, genotype variants of CYP2C9 and VKORC1 are known to also be highly correlated with race. For example, Consortium (2009) even recommended imputing missing values of the genotypes based on race labels. In Fig. 5, we display the histograms of the conditional probability predictions for both race and outcomes, for each proxy. For outcomes, we show all four combinations of true outcome and decision outcome. For race, we separate the probabilities by label. We note that current medications and genetic factors together form a highly informative proxy, both for race and for outcomes.
In Fig. 6 we use the support function formulation of Proposition 10 to compute the convex hull of the partially identified set for the simultaneous values for TPRD measures. The true vector of simultaneous TPRD values, derived using the whole single dataset, is plotted as a red star and is contained in each partial identification set, as guaranteed by the theory. Positive disparities indicate higher TPR for the White group.

We first observe that using genetic factor proxies, whether in combination with current medication or not, provides clear evidence that the TPRD between White and Asian is positive in favor of White. At the same time, in every case, both the direction and magnitude of the disparity between White and Black is unclear, suggesting that given only proxy data no conclusion should be drawn. Using genetic factors in combination with current medication adds a very clear sense of the significant magnitude of the TPRD between White and Asian, not just its direction. It limits the magnitude of the White-Black disparity but gives no indication of direction. This is consistent with the different quality of the proxies: while the genetic proxy is stronger than the medicine proxy, combining the proxies adds additional information that tightens the partially identified set. Therefore studying the partially identified plots allows a practitioner to assess the value of additional information and, in some cases, the direction of disparities.

9. Conclusion
Assessing the fairness of algorithmic decisions is a fundamentally difficult task: it is now well-understood that even when algorithms do not take sensitive information as an input they can still be biased in various worrisome ways, but what counts as “unfair” can be very context-dependent. But any such adjudication and scrutiny must start from understanding how different groups are disparately impacted by such decisions. For example, disparate impact has been codified in US law and regulation as evidentiary basis for closer review and even sanction. We here studied a further complication: membership in protected groups is usually not even seen in the data, requiring the use of auxiliary data where such labels are present. This limitation hinders both fair lending and healthcare reforms and it is important to address it.

We formulated this problem from the perspective of data combination and studied the fundamental limits of identification. This provided a new perspective on the commonplace usage of proxy models and a way to assess what can and cannot be learned from the data. The tools we developed allow one to compute exactly the sharpest possible bounds on disparity that could possibly be learned from the data. We believe this is an invaluable tool given that disparate impact assessments can have far-reaching policy implications.

Beyond the specific tools we presented here, we also hope our work will inspire other researchers to consider fundamental statistical ambiguities in the definitions of fairness, beyond just the ambiguities between the different definitions. Given the sensitivity of such matters, truly understanding
the limits of what cannot actually be measured, and what on the other hand can be said with certainty, is critical for any reliable assessment of the fairness of any decision-making algorithm.

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Online Appendix:
Assessing Algorithmic Fairness with Unobserved Protected Class Using Data Combination

Appendix A: Omitted Proofs

A.1. Proof of Proposition 4

Proof. For binary protected group, for \( y, y' \in \{0, 1\} \)

\[
\mathbb{P}(A = b, \hat{Y} = y' | Z = z) = \mathbb{P}(\hat{Y} = y | Z = z) - \mathbb{P}(A = a, \hat{Y} = y' | Z = z)
\]  

(14)

\[
\mathbb{P}(A = a, \hat{Y} = y, Y = y | Z = z) = \mathbb{P}(\hat{Y} = y, Y = y | Z = z) - \mathbb{P}(A = a, \hat{Y} = y, Y = y | Z = z)
\]  

(15)

Demographic disparity. We first illustrate the unidentifiability of demographic disparity.

\[
\delta_{DD}(a, b) = \left( \frac{\int \mathbb{P}(A = a, \hat{Y} = 1 | Z = z) dP(z)}{\mathbb{P}(A = a)} - \frac{\int \mathbb{P}(A = b, \hat{Y} = 1 | Z = z) dP(z)}{\mathbb{P}(A = b)} \right)
\]

(16)

This formulation means that \( \delta_{DD}(a, b) \) is a bijective map of \( \int \mathbb{P}(A = a, \hat{Y} = 1 | Z = z) dP(z) \). We will construct two valid distributions \( \hat{P}_1 \) and \( \hat{P}_2 \) such that \( \int \hat{P}_1(A = a, \hat{Y} = 1 | Z = z) dP(z) \neq \int \hat{P}_2(A = a, \hat{Y} = 1 | Z = z) dP(z) \). As a result, \( \delta_{DD}(a, b) \) induced by these two distributions are different.

Under the given assumptions, it is easy to show that for \( z \in \mathcal{Z}_0 \), the Frechet Hoeffding bounds for \( \mathbb{P}(A = a, \hat{Y} = \hat{y} | Z = z) \) satisfy that

\[
\max \left\{ \mathbb{P}(A = a | Z = z) + \mathbb{P}(\hat{Y} = 1 | Z = z) - 1, 0 \right\} < \min \left\{ \mathbb{P}(A = a | Z = z), \mathbb{P}(\hat{Y} = 1 | Z = z) \right\}
\]

(17)

We now prove that a valid distribution \( \hat{P}_{A, \hat{Y} | Z} \in \Pi(\mathbb{P}(A | Z), \mathbb{P}(\hat{Y} | Z)) \) can take any value within the Frechet-Hoeffding bounds \(16\). The Law of Total Probability in the coupling set \(3\) further requires \( \hat{P} \) to satisfy that

\[
\mathbb{P}(A = a | Z = z) = \hat{P}(A = a, \hat{Y} = 1 | Z = z) + \hat{P}(A = a, \hat{Y} = 0 | Z = z)
\]

(17)

\[
\mathbb{P}(A = b | Z = z) = \hat{P}(A = b, \hat{Y} = 1 | Z = z) + \hat{P}(A = b, \hat{Y} = 0 | Z = z)
\]

(18)

\[
\mathbb{P}(A = b, \hat{Y} = \hat{y} | Z = z) = \hat{P}(\hat{Y} = \hat{y} | Z = z) - \mathbb{P}(A = a, \hat{Y} = \hat{y} | Z = z)
\]

(19)

For any value of \( \hat{P}(A = a, \hat{Y} = 1 | Z = z) \) within the Frechet-Hoeffding bounds \(16\), we can always set \( \hat{P}(A = a, \hat{Y} = 0 | Z = z) = \mathbb{P}(A = a | Z = z) - \hat{P}(A = a, \hat{Y} = 1 | Z = z) \) and \( \hat{P}(A = b, \hat{Y} = 0 | Z = z) = \mathbb{P}(\hat{Y} = 0 | Z = z) - \hat{P}(A = b, \hat{Y} = 0 | Z = z) \), so that \(17\)(18) are satisfied, and \( \hat{P}(A = b, \hat{Y} = \hat{y} | Z = z) = \hat{P}(\hat{Y} = \hat{y} | Z = z) - \hat{P}(A = a, \hat{Y} = \hat{y} | Z = z) \) for \( \hat{y} \in \{0, 1\} \) so that \(19\) is satisfied. Obviously these choices are within \([0, 1]\),
and the resulting \( \hat{P} \) the Law of Total Probability. Therefore, the distribution \( \hat{P} \) defined in this way is a valid distribution according to Definition 1. In particular, \( \hat{P}(A = a, \hat{Y} = 1 \mid Z = z) \) can attain any value within the Frechet Hoeffding bounds in \( [0, 1] \) for \( z \in \mathcal{Z}_0 \), as long as other probabilities are chosen appropriately to meet the Law of Total Probability.

Without loss of generality, we can always construct two valid distributions \( \hat{P}_1 \) and \( \hat{P}_2 \) such that \( \hat{P}_1(A = a, \hat{Y} = 1 \mid Z = z) < \hat{P}_2(A = a, \hat{Y} = 1 \mid Z = z) \) for \( z \in \mathcal{Z}_0 \) and \( \hat{P}_1(A = a, \hat{Y} = 1 \mid Z = z) = \hat{P}_2(A = a, \hat{Y} = 1 \mid Z = z) \) for \( z \in \mathcal{Z}_0^c \). Obviously,

\[
\int \hat{P}_1(A = a, \hat{Y} = 1 \mid Z = z) dP(z) < \int \hat{P}_2(A = a, \hat{Y} = 1 \mid Z = z) dP(z),
\]

which implies \( \delta_{\text{DD}}(a, b) \) under the two distributions \( \hat{P}_1 \) and \( \hat{P}_2 \) are different.

**True positive rate disparity.** Now we prove the unidentifiability for true positive rate disparity, and the conclusion for true negative rate disparity can be proved analogously. We start with a formulation of \( \delta_{\text{TPRD}}(a, b) \):

\[
\delta_{\text{TPRD}}(a, b) = \frac{\int P(A = a, \hat{Y} = 1, Y = 1 \mid Z = z) dP(z)}{\int P(A = a, \hat{Y} = 1, Y = 1 \mid Z = z) dP(z)} - \frac{\int P(A = b, \hat{Y} = 1, Y = 1 \mid Z = z) dP(z)}{\int P(A = b, \hat{Y} = 1, Y = 1 \mid Z = z) dP(z)} - \frac{\int P(A = a, \hat{Y} = 1, Y = 1 \mid Z = z) dP(z)}{\int P(A = a, \hat{Y} = 1, Y = 1 \mid Z = z) dP(z)} - \frac{\int P(A = a, \hat{Y} = 1, Y = 1 \mid Z = z) dP(z)}{\int P(A = a, \hat{Y} = 1, Y = 1 \mid Z = z) dP(z)} - \frac{\int P(A = a, \hat{Y} = 1, Y = 1 \mid Z = z) dP(z)}{\int P(A = a, \hat{Y} = 1, Y = 1 \mid Z = z) dP(z)}.
\]

Here \( \delta_{\text{TPRD}}(a, b) \) depends on both \( \int P(A = a, \hat{Y} = 1, Y = 1 \mid Z = z) dP(z) \) and \( \int P(A = a, \hat{Y} = 1, Y = 1 \mid Z = z) dP(z) \). However, if we can fix \( \int P(A = a, \hat{Y} = 0, Y = 1 \mid Z = z) dP(z) + \int P(A = a, \hat{Y} = 1, Y = 1 \mid Z = z) dP(z) \) while varying \( \int P(A = a, \hat{Y} = 1, Y = 1 \mid Z = z) dP(z) \), then \( \delta_{\text{TPRD}}(a, b) \) is also a bijective map of \( \int P(A = a, \hat{Y} = 1, Y = 1 \mid Z = z) dP(z) \). We will construct two valid distributions \( \hat{P}_1 \) and \( \hat{P}_2 \) such that

\[
\int \hat{P}_1(A = a, \hat{Y} = 0, Y = 1 \mid Z = z) dP(z) + \int \hat{P}_1(A = a, \hat{Y} = 1, Y = 1 \mid Z = z) dP(z) = \int \hat{P}_2(A = a, \hat{Y} = 0, Y = 1 \mid Z = z) dP(z) + \int \hat{P}_2(A = a, \hat{Y} = 1, Y = 1 \mid Z = z) dP(z),
\]

and

\[
\int \hat{P}_1(A = a, \hat{Y} = 1, Y = 1 \mid Z = z) dP(z) \neq \int \hat{P}_2(A = a, \hat{Y} = 1, Y = 1 \mid Z = z) dP(z).
\]

As a result, TPRD \( \delta_{\text{TPRD}}(a, b) \) induced by the two distributions are different.

Under the given assumption in Proposition 4, the Frechet Hoeffding bounds for \( P(A = \alpha, \hat{Y} = \hat{y}, Y = y \mid Z = z) \) for \( \hat{y}, y \in \{0, 1\} \) and \( \alpha \in \mathcal{A} \) satisfy that

\[
\max \left\{ P(A = \alpha \mid Z = z) + P(\hat{Y} = \hat{y}, Y = y \mid Z = z) - 1, 0 \right\} < \min \left\{ P(A = \alpha \mid Z = z), P(\hat{Y} = \hat{y}, Y = y \mid Z = z) \right\}.
\]

We denote \( U(\alpha, \hat{y}, y, z) = \{ P(A = \alpha \mid Z = z), P(\hat{Y} = \hat{y}, Y = y \mid Z = z) \} \) and \( L(\alpha, \hat{y}, y, z) = \{ P(A = \alpha \mid Z = z) + P(\hat{Y} = \hat{y}, Y = y \mid Z = z) - 1, 0 \} \).
We now construct a valid distribution \( \hat{P}_{A, Y, Z} \in \Pi(\mathbb{P}(A | Z), \mathbb{P}(\hat{Y}, Y | Z)) \) such that \( \hat{P}(A = a, \hat{Y} = 0, Y = 1 | Z = z) + \hat{P}(A = a, \hat{Y} = 1, Y = 1 | Z = z) \) only depends on fixed marginal distributions \( \mathbb{P}(A, Z) \) and \( \mathbb{P}(\hat{Y}, Y, Z) \) and a priori fixed function \( \epsilon(z) \), but \( \hat{P}(A = a, \hat{Y} = 1, Y = 1 | Z = z) \) can vary arbitrarily. The law of total probability requires \( \hat{P} \) to satisfy

\[
\sum_{\hat{y}, y \in \{0, 1\}} \hat{P}(A = a, \hat{Y} = \hat{y}, Y = y | Z = z) = \mathbb{P}(A = a | Z = z), \tag{21}
\]

\[
\hat{P}(A = a, \hat{Y} = \hat{y}, Y = y | Z = z) + \hat{P}(A = b, \hat{Y} = \hat{y}, Y = y | Z = z) = \mathbb{P}(\hat{Y} = \hat{y}, Y = y | Z = z). \tag{22}
\]

We can set \( \hat{P}(A = b, \hat{Y} = \hat{y}, Y = y | Z = z) = \mathbb{P}(\hat{Y} = \hat{y}, Y = y | Z = z) - \hat{P}(A = a, \hat{Y} = \hat{y}, Y = y | Z = z) \) for \( \hat{y}, y \in \{0, 1\} \) so that (22) is satisfied. Moreover, given (22), (21) is satisfied for \( \alpha = b \) automatically if (21) is satisfied for \( \alpha = a \). Therefore, for any \( \hat{P}(A = a, \hat{Y} = 1, Y = 1 | Z = z) \) with fixed \( \hat{P}(A = a, \hat{Y} = 0, Y = 1 | Z = z) + \hat{P}(A = a, \hat{Y} = 1, Y = 1 | Z = z) \), we only need to choose \( \hat{P}(A = a, \hat{Y} = \hat{y}, Y = 0 | Z = z) \) for \( \hat{y} \in \{0, 1\} \) such that (21) is satisfied for \( \alpha = a \). Without loss of generality, we suppose \( \mathbb{P}(A = a | Z = z) > 1/2 \), otherwise \( \mathbb{P}(A = b | Z = z) > 1/2 \) and we can work with \( \hat{P}(A = b, \hat{Y} = \hat{y}, Y = y | Z = z) \) instead.

We now enumerate all possible cases.

- Case I:

\[
G_1 = \{ z \in Z : \mathbb{P}(\hat{Y} = 1, Y = 1 | Z = z) > \mathbb{P}(A = a | Z = z) \}.
\]

For \( z \in G_1 \), \( \mathbb{P}(\hat{Y} = 0, Y = 1 | Z = z) \leq 1 - \mathbb{P}(\hat{Y} = 1, Y = 1 | Z = z) \leq 1 - \mathbb{P}(A = a | Z = z) \leq \mathbb{P}(A = a | Z = z) \), and similarly \( \mathbb{P}(\hat{Y} = 0, Y = 0 | Z = z) \), \( \mathbb{P}(\hat{Y} = 1, Y = 0 | Z = z) \leq 1 - \mathbb{P}(A = a | Z = z) \leq \mathbb{P}(A = a | Z = z) \). In this case, the Frechet Hoeffding bounds are

\[
\mathbb{P}(\hat{Y} = 1, Y = 1 | Z = z) + \mathbb{P}(A = a | Z = z) - 1 \leq \hat{P}(A = a, \hat{Y} = 1, Y = 1 | Z = z) \leq \mathbb{P}(A = a | Z = z)
\]

\[
L(a, 0, 1, z) \leq \hat{P}(A = a, \hat{Y} = 0, Y = 1 | Z = z) \leq \hat{P}(\hat{Y} = 0, Y = 1 | Z = z)
\]

Any choice of the four conditional probabilities that are within the Frechet Hoeffding bounds and satisfy (21) for \( \alpha = a \) are valid. In particular, for \( t \in [0, 1] \), we set

\[
\hat{P}(A = a, \hat{Y} = 1, Y = 1 | Z = z) = \mathbb{P}(\hat{Y} = 1, Y = 1 | Z = z) + \mathbb{P}(A = a | Z = z) - 1 + t\epsilon(z),
\]

\[
\hat{P}(A = a, \hat{Y} = 0, Y = 1 | Z = z) = \mathbb{P}(\hat{Y} = 0, Y = 1 | Z = z) - t\epsilon(z),
\]

\[
\hat{P}(A = a, \hat{Y} = 0, Y = 0 | Z = z) = \mathbb{P}(\hat{Y} = 0, Y = 0 | Z = z),
\]

\[
\hat{P}(A = a, \hat{Y} = 0, Y = 0 | Z = z) = \mathbb{P}(\hat{Y} = 1, Y = 0 | Z = z),
\]

where \( \epsilon(z) \) is very small so that the choices of \( \hat{P}(A = a | \hat{Y} = 1, Y = 1 | Z = z) \) and \( \hat{P}(A = a | \hat{Y} = 0, Y = 1 | Z = z) \) are within their Frechet Hoeffding bounds. When \( t \) changes, obviously \( \hat{P}(A = a | \hat{Y} = 1, Y = 1 | Z = z) \) changes, yet \( \hat{P}(A = a | \hat{Y} = 1, Y = 1 | Z = z) + \hat{P}(A = a | \hat{Y} = 0, Y = 1 | Z = z) = \mathbb{P}(A = a | Z = z) + \mathbb{P}(Y = 1 | Z = z) - 1 \) is fixed. And it is easy to verify that (21) holds.

- Case II:

\[
G_2 = \left\{ z \in Z : \hat{P}(\hat{Y} = 1, Y = 1 | Z = z) \leq \mathbb{P}(A = a | Z = z), \right. \\
\left. \hat{P}(\hat{Y} = 0, Y = 1 | Z = z) > 1 - \mathbb{P}(A = a | Z = z) \right\}.
\]
For $z \in G_2$, the Frechet Hoeffding bounds are

$$L(a, 1, 1, z) \leq \hat{P}(A = a, \hat{Y} = 1, Y = 1 \mid Z = z) \leq \mathbb{P}(\hat{Y} = 1, Y = 1 \mid Z = z)$$

$$\mathbb{P}(\hat{Y} = 0, Y = 1 \mid Z = z) + \mathbb{P}(A = a \mid Z = z) - 1 \leq \hat{P}(A = a, \hat{Y} = 0, Y = 1 \mid Z = z) \leq U(a, 0, 1, z)$$

Any choice of the four conditional probabilities that are within the Frechet Hoeffding bounds and satisfy (21) for $\alpha = a$ are valid. In particular, for $t \in [0, 1]$, we can set

$$\hat{P}(A = a, \hat{Y} = 1, Y = 1 \mid Z = z) = \mathbb{P}(\hat{Y} = 1, Y = 1 \mid Z = z) - t \epsilon(z),$$

$$\hat{P}(A = a, Y = 0 \mid Z = z) = \mathbb{P}(a, Y = 0 \mid Z = z) + \mathbb{P}(Y = 1 \mid Z = z) - 1 + t \epsilon(z),$$

$$\hat{P}(A = a, \hat{Y} = 0, Y = 0 \mid Z = z) = \mathbb{P}(\hat{Y} = 0, Y = 0 \mid Z = z),$$

$$\hat{P}(A = a, \hat{Y} = 0, Y = 0 \mid Z = z) = \mathbb{P}(\hat{Y} = 1, Y = 0 \mid Z = z).$$

When $t$ changes, obviously $\hat{P}(A = a \mid \hat{Y} = 1, Y = 1 \mid Z = z)$ changes, yet $\hat{P}(A = a \mid \hat{Y} = 1, Y = 1 \mid Z = z) + \hat{P}(A = a \mid \hat{Y} = 0, Y = 1 \mid Z = z)$ is fixed.

- Case III:

$$G_3 = \left\{ z \in Z : 1 - \mathbb{P}(A = a \mid Z = z) \leq \hat{P}(\hat{Y} = 1, Y = 1 \mid Z = z) \leq \mathbb{P}(A = a \mid Z = z), \right.$$  

$$\hat{P}(\hat{Y} = 0, Y = 1 \mid Z = z) \leq 1 - \mathbb{P}(A = a \mid Z = z) \right\}$$

For $z \in G_3$, the Frechet Hoeffding bounds are

$$\mathbb{P}(\hat{Y} = 1, Y = 1 \mid Z = z) + \mathbb{P}(A = a \mid Z = z) - 1 \leq \hat{P}(A = a, \hat{Y} = 1, Y = 1 \mid Z = z) \leq \mathbb{P}(\hat{Y} = 1, Y = 1 \mid Z = z)$$

$$0 \leq \hat{P}(A = a, \hat{Y} = 0, Y = 1 \mid Z = z) \leq \mathbb{P}(\hat{Y} = 0, Y = 1 \mid Z = z)$$

We can handle this case by following Case I.

- Case IV:

$$G_4 = \left\{ z \in Z : \hat{P}(\hat{Y} = 1, Y = 1 \mid Z = z) \leq 1 - \mathbb{P}(A = a \mid Z = z), \right.$$  

$$\hat{P}(\hat{Y} = 0, Y = 1 \mid Z = z) \leq 1 - \mathbb{P}(A = a \mid Z = z), \right.$$  

$$\mathbb{P}(\hat{Y} = 1, Y = 0 \mid Z = z) > \mathbb{P}(A = a \mid Z = z) \text{ or } \mathbb{P}(\hat{Y} = 0, Y = 0 \mid Z = z) > \mathbb{P}(A = a \mid Z = z) \right\}$$

Without loss of generality, we consider $\mathbb{P}(A = a, \hat{Y} = 0, Y = 0 \mid Z = z) > \mathbb{P}(A = a \mid Z = z)$. For $z \in G_4$, the Frechet Hoeffding upper bounds are

$$0 \leq \hat{P}(A = a, \hat{Y} = 1, Y = 1 \mid Z = z) \leq \mathbb{P}(\hat{Y} = 1, Y = 1 \mid Z = z)$$

$$0 \leq \hat{P}(A = a, \hat{Y} = 0, Y = 1 \mid Z = z) \leq \mathbb{P}(\hat{Y} = 0, Y = 1 \mid Z = z)$$

$$L(a, 0, 0, z) \leq \mathbb{P}(A = a, \hat{Y} = 0, Y = 0 \mid Z = z) \leq \mathbb{P}(A = a \mid Z = z)$$

For $t \in [0, 1]$, we can set

$$\mathbb{P}(A = a, \hat{Y} = 0, Y = 0 \mid Z = z) = \mathbb{P}(A = a \mid Z = z) - \epsilon(z)$$

$$\mathbb{P}(A = a, \hat{Y} = 0, Y = 1 \mid Z = z) = 0$$

$$\mathbb{P}(A = a, \hat{Y} = 1, Y = 1 \mid Z = z) = t \epsilon(z)$$

$$\mathbb{P}(A = a, \hat{Y} = 0, Y = 1 \mid Z = z) = (1 - t) \epsilon(z)$$

Fairness Using Data Combination
where $\epsilon(z)$ is small so that $P(A = a, \hat{Y} = 0, Y = 0 \mid Z = z), P(A = a, \hat{Y} = 1, Y = 1 \mid Z = z), P(A = a, \hat{Y} = 0, Y = 1 \mid Z = z)$ are within their Frechet Hoeffding bounds. It is easy to verify that (21) for $a = a$ holds. When $t$ changes, $P(A = a, \hat{Y} = 1, Y = 1 \mid Z = z) + P(A = a, \hat{Y} = 0, Y = 1 \mid Z = z) = \epsilon(z)$ is fixed, yet $P(A = a, \hat{Y} = 1, Y = 1 \mid Z = z)$ varies.

- Case V:
  \[ G_5 = \left\{ z \in \mathbb{Z} : \hat{P}(\hat{Y} = 1, Y = 1 \mid Z = z) \leq 1 - P(A = a \mid Z = z), \right. \]
  \[ \hat{P}(\hat{Y} = 0, Y = 1 \mid Z = z) \leq 1 - P(A = a \mid Z = z), \]
  \[ P(\hat{Y} = 1, Y = 0 \mid Z = z) \leq 1 - P(A = a \mid Z = z), \]
  \[ P(\hat{Y} = 0, Y = 0 \mid Z = z) \leq P(A = a \mid Z = z) \right\} \]

For $z \in G_5$, the Frechet Hoeffding bounds are

\[ 0 \leq \hat{P}(A = a, \hat{Y} = 1, Y = 1 \mid Z = z) \leq P(\hat{Y} = 1, Y = 1 \mid Z = z) \]
\[ 0 \leq \hat{P}(A = a, \hat{Y} = 0, Y = 1 \mid Z = z) \leq P(\hat{Y} = 0, Y = 1 \mid Z = z) \]
\[ 0 \leq \hat{P}(A = a, \hat{Y} = 1, Y = 0 \mid Z = z) \leq P(\hat{Y} = 1, Y = 0 \mid Z = z) \]

We can set $\hat{P}(A = a, \hat{Y} = 0, Y = 0 \mid Z = z) = P(\hat{Y} = 0, Y = 0 \mid Z = z)$ and $\hat{P}(A = a, \hat{Y} = 1, Y = 0 \mid Z = z)$. We only need to satisfy (21) for $a = a$:

\[ \hat{P}(A = a, \hat{Y} = 1, Y = 1 \mid Z = z) + \hat{P}(A = a, \hat{Y} = 0, Y = 1 \mid Z = z) = P(A = a \mid Z = z) - P(\hat{Y} = 0, Y = 0 \mid Z = z) \]

Obviously this can be satisfied by infinitely many combinations of $\hat{P}(A = a, \hat{Y} = 1, Y = 1 \mid Z = z)$ and $\hat{P}(A = a, \hat{Y} = 0, Y = 1 \mid Z = z)$ within their Frechet Hoeffding bounds, as long as

\[ 0 \leq P(A = a \mid Z = z) - P(\hat{Y} = 0, Y = 0 \mid Z = z) \leq \hat{P}(\hat{Y} = 1, Y = 1 \mid Z = z) + \hat{P}(\hat{Y} = 0, Y = 1 \mid Z = z). \]

The first inequality is satisfied by $P(\hat{Y} = 0, Y = 0 \mid Z = z) \leq P(A = a \mid Z = z)$. The second inequality is satisfied because

\[ P(A = a \mid Z = z) - P(\hat{Y} = 0, Y = 0 \mid Z = z) = \hat{P}(\hat{Y} = 1, Y = 1 \mid Z = z) - \hat{P}(\hat{Y} = 0, Y = 1 \mid Z = z) \]
\[ = P(A = a \mid Z = z) + 1 - P(\hat{Y} = 1, Y = 0 \mid Z = z) \leq 0. \]

So in this case, we can also find different valid $\hat{P}(A = a, \hat{Y} = 1, Y = 1 \mid Z = z)$ while $\hat{P}(A = a, \hat{Y} = 1, Y = 1 \mid Z = z) + \hat{P}(A = a, \hat{Y} = 0, Y = 1 \mid Z = z) = P(A = a \mid Z = z) - P(\hat{Y} = 0, Y = 0 \mid Z = z)$ is fixed.

- Case VI:
  \[ G_6 = \left\{ z \in \mathbb{Z} : \hat{P}(\hat{Y} = 1, Y = 1 \mid Z = z) \leq 1 - P(A = a \mid Z = z), \right. \]
  \[ \hat{P}(\hat{Y} = 0, Y = 1 \mid Z = z) \leq 1 - P(A = a \mid Z = z), \]
  \[ 1 - P(A = a \mid Z = z) \leq P(\hat{Y} = 1, Y = 0 \mid Z = z) \leq P(A = a \mid Z = z), \]
  \[ P(\hat{Y} = 0, Y = 0 \mid Z = z) \leq P(A = a \mid Z = z) \right\} \]
For $z \in G_6$, the Frechet Hoeffding bounds are

$$0 \leq \hat{P}(A = a, \hat{Y} = 1, Y = 1 \mid Z = z) \leq P(\hat{Y} = 1, Y = 1 \mid Z = z)$$

$$0 \leq \hat{P}(A = a, \hat{Y} = 0, Y = 1 \mid Z = z) \leq P(\hat{Y} = 0, Y = 1 \mid Z = z)$$

$$P(A = a \mid Z = z) + P(\hat{Y} = 1, Y = 0 \mid Z = z) - 1 \leq \hat{P}(A = a, \hat{Y} = 1, Y = 0 \mid Z = z) \leq P(\hat{Y} = 1, Y = 0 \mid Z = z)$$

$$L(a, 0, 0, z) \leq \hat{P}(A = a, \hat{Y} = 0, Y = 0 \mid Z = z) \leq P(\hat{Y} = 0, Y = 0 \mid Z = z)$$

For $t \in [0, 1]$, we set

$$\hat{P}(A = a, \hat{Y} = 1, Y = 0 \mid Z = z) = \mathbb{P}(A = a \mid Z = z) + \mathbb{P}(\hat{Y} = 1, Y = 0 \mid Z = z) - 1 + \epsilon(z)$$

$$\hat{P}(A = a, \hat{Y} = 0, Y = 0 \mid Z = z) = \mathbb{P}(\hat{Y} = 0, Y = 0 \mid Z = z)$$

$$\hat{P}(A = a, \hat{Y} = 1, Y = 1 \mid Z = z) = \mathbb{P}(\hat{Y} = 1, Y = 1 \mid Z = z) - t\epsilon(z)$$

$$\hat{P}(A = a, \hat{Y} = 0, Y = 1 \mid Z = z) = \mathbb{P}(\hat{Y} = 0, Y = 1 \mid Z = z) - (1 - t)\epsilon(z)$$

which satisfies (21) for $\alpha = a$. Meanwhile, as $t$ varies, $\hat{P}(A = a, \hat{Y} = 1, Y = 1 \mid Z = z)$ also varies but $\hat{P}(A = a, \hat{Y} = 1, Y = 1 \mid Z = z)$ is always independent of $t$.

Since $\bigcup_{i=1}^6 G_i = \mathcal{Z}$ and $\mathbb{P}(Z \in \mathcal{Z}_0) > 0$, there must exist $1 \leq k \leq 6$ such that $\mathbb{P}(Z \in \mathcal{Z}_0 \cap G_k) > 0$. From the discussion above, we can always construct two valid distributions $\hat{P}_1$ and $\hat{P}_2$ to satisfy that

- for $z \in \mathcal{Z}_0 \cap G_k$, $\hat{P}_1(A = a, \hat{Y} = 1, Y = 1 \mid Z = z) < \hat{P}_2(A = a, \hat{Y} = 1, Y = 1 \mid Z = z)$ but $\hat{P}_1(A = a, \hat{Y} = 1, Y = 1 \mid Z = z) + \hat{P}_1(A = a, \hat{Y} = 0, Y = 1 \mid Z = z) = \hat{P}_2(A = a, \hat{Y} = 1, Y = 1 \mid Z = z) + \hat{P}_2(A = a, \hat{Y} = 0, Y = 1 \mid Z = z)$.
- for $z \in (\mathcal{Z}_0 \cap G_k)^C$, $\hat{P}_1(A = a, \hat{Y} = 1, Y = 1 \mid Z = z) = \hat{P}_2(A = a, \hat{Y} = 1, Y = 1 \mid Z = z)$ and $\hat{P}_1(A = a, \hat{Y} = 0, Y = 1 \mid Z = z) < \hat{P}_2(A = a, \hat{Y} = 0, Y = 1 \mid Z = z)$.

As a result,

$$\int_{z \in \mathcal{Z}} \hat{P}_1(A = a, \hat{Y} = 1, Y = 1 \mid Z = z) d\mathbb{P}(z) - \int_{z \in \mathcal{Z}} \hat{P}_2(A = a, \hat{Y} = 1, Y = 1 \mid Z = z) d\mathbb{P}(z)$$

$$= \int_{z \in \mathcal{Z}_0 \cap G_k} (\hat{P}_1(A = a, \hat{Y} = 1, Y = 1 \mid Z = z) - \hat{P}_2(A = a, \hat{Y} = 1, Y = 1 \mid Z = z)) d\mathbb{P}(z) < 0$$

and

$$\int_{z \in \mathcal{Z}} \hat{P}_1(A = a, \hat{Y} = 1, Y = 1 \mid Z = z) d\mathbb{P}(z) + \int_{z \in \mathcal{Z}} \hat{P}_2(A = a, \hat{Y} = 0, Y = 1 \mid Z = z) d\mathbb{P}(z)$$

$$= \int_{z \in \mathcal{Z}} \hat{P}_2(A = a, \hat{Y} = 1, Y = 1 \mid Z = z) d\mathbb{P}(z) + \int_{z \in \mathcal{Z}} \hat{P}_2(A = a, \hat{Y} = 0, Y = 1 \mid Z = z) d\mathbb{P}(z)$$

Therefore, the TPRD $\delta_{\text{TPRD}}(a, b)$ induced by $\hat{P}_1$ and $\hat{P}_2$ are different. \hfill \Box

### A.2. Proofs of Propositions 5 and 6

**Proof.** Consider the case of TPRD. Since the true TPR values, $(\mu_{11}(\alpha))_{i \in A} = (\mu_{11}(\alpha; \hat{w}^*))_{i \in A}$, depend only on the the known and fixed $\mathbb{P}(\hat{Y}, Y, Z)$ and the unknown and variable $\hat{w}^*_i(\hat{g}, y, z) = \mathbb{P}(A = a \mid \hat{Y} = \hat{g}, Y = y, Z = z)$, to prove the results it suffices to show that the set of distributions $\mathbb{P}(A \mid \hat{Y}, Y, Z = z)$ induced by couplings in $\Pi(\mathbb{P}(A \mid Z = z), \mathbb{P}(\hat{Y}, Y \mid Z = z))$ is exactly the set $\mathcal{W}_{\text{LTP}}$. We proceed to show this.
Note that \( P(A \mid \hat{Y}, Y, Z = z) = \frac{P(A, Y, Z = z)}{\sum_{\alpha \in A} P(A = \alpha, Y, Z = z)} \), which we can conceive of as a map with domain \( \Pi(P(A \mid Z = z), P(Y, Y \mid Z = z)) \). Since \( \hat{W}_{LTP} \) is exactly the image of this map, we have our conclusion.

Finally, the smoothness constraints in Eqs. (6) and (7) are by definition constraints on \( \hat{Y}^{\ast} \), which correspond to the constraints in \( \hat{W}_{LTP} \).

Analogous arguments hold for both DD and TNRD. □

A.3. Proof of Proposition 7

Proof. We are interested in computing

\[
\delta = \sup_{w \in \mathcal{W}_{LTP}} \left( \frac{E[w_a(\hat{y}, z)\hat{Y}]}{E[w_a(\hat{y}, z)]} - \frac{E[w_b(\hat{y}, z)\hat{Y}]}{E[w_b(\hat{y}, z)]} \right).
\]

Exchanging the roles of \( a, b \) gives the inf.

First, note that \( w \in \mathcal{W}_{LTP} \) implies \( E[w_a(\hat{y}, z)] = P(A = \alpha) \). Next, let

\[
w^{U/L}_a(\hat{y}, z) = \begin{cases} w^U_a(\hat{y}, z) & \text{if } \alpha = a, \\ w^L_a(\hat{y}, z) & \text{if } \alpha = b,
\end{cases}
\]

and note that it is feasible in \( w \in \mathcal{W}_{LTP} \), which can be verified by checking each constraint. In particular, notice that \( w^{U}_a(\hat{y}, z) + w^{L}_a(\hat{y}, z) = 1 \) because \( A = \{a, b\} \) so the two labels are complementary. Therefore, \( \delta \geq \mu(a; w^U) - \mu(b; w^L) \).

Next, note that \( \sup_{w \in \mathcal{W}_{LTP}} w_a(\hat{y}, z) = w^U_a(\hat{y}, z) \) and \( \inf_{w \in \mathcal{W}_{LTP}} w_a(\hat{y}, z) = w^L_a(\hat{y}, z) \), which can be verified by Proposition 3. Therefore,

\[
\delta \leq \sup_{w \in \mathcal{W}_{LTP}} \frac{E[w_a(\hat{y}, z)\hat{Y}]}{E[w_a(\hat{y}, z)]} - \inf_{w \in \mathcal{W}_{LTP}} \frac{E[w_b(\hat{y}, z)\hat{Y}]}{E[w_b(\hat{y}, z)]} = \mu(a; w^U) - \mu(b; w^L),
\]

which completes the proof. □

A.4. Proof of Proposition 8

Proof. In this proof, we use the notation \( p(\alpha \mid z) = P(A = \alpha \mid Z = z) \) for \( \alpha = a, b, \) and \( p(\hat{y}, y \mid z) = P(\hat{Y} = \hat{y}, Y = y \mid Z = z) \). We prove the result for TPR and the result for TNR can be proved analogously.

Recall the formulation in (5) gives TPRD the following representation

\[
\delta_{TPRD}(a, b; \hat{w}) = \mu_{11}(a; \hat{w}) - \mu_{11}(b; \hat{w})
\]

(23)

where

\[
\mu_{11}(a; \hat{w}) = \frac{E[\hat{w}_a(\hat{Y}, Y, Z)I(Y = y)I(\hat{Y} = \hat{y})]}{E[\hat{w}_a(\hat{Y}, Y, Z)I(Y = y)]} = \frac{E[\hat{w}_a(\hat{y} = 1, y = 1, Z)I(Y = 1)I(\hat{Y} = 1)]}{E[\hat{w}_a(\hat{y} = 1, y = 1, Z)I(Y = 1)I(\hat{Y} = 1)] + E[\hat{w}_a(\hat{y} = 0, y = 1, Z)I(Y = 1)I(\hat{Y} = 0)]}
\]

Our aim is to compute the partial identification bounds of TPRD:

\[
[\inf_{\hat{w} \in \mathcal{W}_{LTP}} \mu_{11}(a; \hat{w}), \sup_{\hat{w} \in \mathcal{W}_{LTP}} \mu_{11}(a; \hat{w})].
\]

(24)

and we want to prove that the given solutions in Proposition 8 exactly compute bounds in (24).

Step 1: we first show that the given solutions in Proposition 8 correspond to bounds of TPRD under a constraint \( \hat{W}_{FH} \) larger than the Law of Total Probability Constraint. So the bounds given by the solutions
in Proposition 8 are at least as wide as the true partial identification bounds in (24). The constraint \( \tilde{W}_{FH} \) requires Frechet Hoeffding bounds on all components of \( \tilde{w} \):

\[
\tilde{W}_{FH} = \{ \tilde{w} : \tilde{w}^L_a(\hat{y}, 1, z) \leq \tilde{w}_a(\hat{y}, 1, z) \leq \tilde{w}^U_a(\hat{y}, 1, z) \text{ for } \hat{y} \in \{0, 1\}, z \in Z \}.
\]

Obviously \( \delta_{\text{TPRD}}(a, b; \tilde{w}) \) is monotonically increasing in \( \tilde{w}_a(\hat{y}, y = 1, Z) \) and \( \tilde{w}_b(\hat{y}, y = 1, Z) \), while \( \delta_{\text{TPRD}}(a, b; \tilde{w}) \) does not depend on the components of \( \tilde{w} \) with \( y = 0 \) at all. Thus the solutions given in Proposition 8 are at least as wide as the actual partial identification bounds.

According to the Proposition 3, \( \tilde{W}_{LTP} \subset \tilde{W}_{FH}, \) which implies

\[
\begin{align*}
\inf_{\tilde{w} \in \tilde{W}_{LTP}} \mu_{11}(\alpha; \tilde{w}) & \geq \inf_{\tilde{w} \in \tilde{W}_{FH}} \delta_{\text{TPRD}}(a, b; \tilde{w}), \\
\sup_{\tilde{w} \in \tilde{W}_{LTP}} \mu_{11}(\alpha; \tilde{w}) & \leq \sup_{\tilde{w} \in \tilde{W}_{FH}} \delta_{\text{TPRD}}(a, b; \tilde{w}).
\end{align*}
\]

These means that the bounds given in Proposition 8 are at least as wide as the actual partial identification bounds.

**Step 2:** we prove that the optimal \( \tilde{w} \) attaining \( \inf_{\tilde{w} \in \tilde{W}_{FH}} \delta_{\text{TPRD}}(a, b; \tilde{w}) \) and \( \sup_{\tilde{w} \in \tilde{W}_{FH}} \delta_{\text{TPRD}}(a, b; \tilde{w}) \) respectively are feasible in \( \tilde{W}_{LTP} \). This means that the solutions given in Proposition 8, namely (25) and (26), can be also achieved only under the Law of Total Probability Constraint \( \tilde{W}_{LTP} \). Thus the solutions given in Proposition 8 are exactly the same as the actual partial identification bounds.

We now prove the feasibility of the optimal \( \tilde{w} \) attaining \( \sup_{\tilde{w} \in \tilde{W}_{FH}} \delta_{\text{TPRD}}(a, b; \tilde{w}) \) with respect to \( \tilde{W}_{LTP} \). The feasibility for the optimal \( \tilde{w} \) attaining \( \inf_{\tilde{w} \in \tilde{W}_{FH}} \delta_{\text{TPRD}}(a, b; \tilde{w}) \) can be proved analogously.

The optimal \( \tilde{w} \) attaining \( \sup_{\tilde{w} \in \tilde{W}_{FH}} \delta_{\text{TPRD}}(a, b; \tilde{w}) \), according to (25) satisfies the following condition: for \( z \in Z \),

\[
\begin{align*}
\tilde{w}_a(1, 1, z) &= \tilde{w}_a^U(1, 1, z), & \tilde{w}_a(0, 1, z) &= \tilde{w}_a^L(0, 1, z), \\
\tilde{w}_b(1, 1, z) &= \tilde{w}_b^U(1, 1, z), & \tilde{w}_b(0, 1, z) &= \tilde{w}_b^L(0, 1, z),
\end{align*}
\]

where

\[
\begin{align*}
\tilde{w}_a^L(\hat{y}, y, z) &= \max \left\{ 0, 1 + \frac{\mathbb{P}(A = \alpha \mid Z = z) - 1}{\mathbb{P}(Y = \hat{y}, Y = y \mid Z = z)} \right\}, & \tilde{w}_a^U(\hat{y}, y, z) &= \min \left\{ 1, \frac{\mathbb{P}(A = \alpha \mid Z = z)}{\mathbb{P}(Y = \hat{y}, Y = y \mid Z = z)} \right\}.
\end{align*}
\]

Obviously for each \( z \in Z \), \( \tilde{w} \) contains 8 different components. (27) and (28) fix four components of the optimal \( \tilde{w} \) attaining \( \sup_{\tilde{w} \in \tilde{W}_{FH}} \delta_{\text{TPRD}}(a, b; \tilde{w}) \). We need to specify other four components for \( \tilde{w} \) such that the law of total probability constraint is satisfied: for \( \hat{y}, y \in \{0, 1\}, z \in Z \),

\[
\begin{align*}
\tilde{w}_a(\hat{y}, y, z) + \tilde{w}_a(\hat{y}, y, z) &= 1, & \sum_{\hat{y}, y \in \{0, 1\}} \tilde{w}_a(\hat{y}, y, z) \mathbb{P}(\hat{Y} = \hat{y}, Y = y \mid Z = z) &= \mathbb{P}(A = \alpha \mid Z = z) & \mathbb{P}(A = \alpha \mid Z = z) & \mathbb{P}(A = \alpha \mid Z = z)
\end{align*}
\]

\[0 \leq \tilde{w}_a(\hat{y}, y, z) \leq 1, \quad \sum_{\hat{y}, y \in \{0, 1\}} \tilde{w}_a(\hat{y}, y, z) \leq 1, \quad \sum_{\hat{y}, y \in \{0, 1\}} \tilde{w}_b(\hat{y}, y, z) \leq 1,
\]

where

\[
\begin{align*}
\tilde{w}_a^{\prime}(\hat{y}, y, z) &= \max \left\{ 0, \frac{\mathbb{P}(A = \alpha \mid Z = z) - 1}{\mathbb{P}(Y = \hat{y}, Y = y \mid Z = z)} \right\}, & \tilde{w}_a^{\prime}(\hat{y}, y, z) &= \min \left\{ 1, \frac{\mathbb{P}(A = \alpha \mid Z = z)}{\mathbb{P}(Y = \hat{y}, Y = y \mid Z = z)} \right\}.
\end{align*}
\]

These means that the bounds given in Proposition 8 are at least as wide as the actual partial identification bounds.
We first notice that that $\tilde{w}_a(\hat{y}, 1, z)$ for $\alpha = a, b$ and $\hat{y} \in \{0, 1\}$ given in (27) and (28) satisfy (29). We can then guarantee (29) by enforcing

$$\hat{w}_a(1, 0, z) = 1 - \tilde{w}_a(1, 0, z), \quad \tilde{w}_a(0, 0, z) = 1 - \tilde{w}_a(0, 0, z).$$

Furthermore, if we can specify $0 \leq \hat{w}_a(1, 0, z), \tilde{w}_a(0, 0, z) \leq 1$ such that (30) is satisfied for $\alpha = a$. Then (30) for $\alpha = b$ is automatically satisfied according to (29) for $\hat{y}, y \in \{0, 1\}$. Therefore, all we need to do is to find appropriate $0 \leq \tilde{w}_a(1, 0, z), \tilde{w}_a(0, 0, z) \leq 1$ to accommodate (30). In the following part, we enumerate all possible cases to show that we can always do so.

Without loss of generality, we first assume that $p(a \mid z) > \frac{1}{2}$. Otherwise, $p(b \mid z) > \frac{1}{2}$, and we can choose to work with $\tilde{w}_b(1, 0, z), \tilde{w}_b(0, 0, z)$ instead. Given $p(a \mid z) > \frac{1}{2} > 1 - p(a \mid z)$,

$$\begin{cases} (\tilde{w}_a(\hat{y}, y, z), \tilde{w}_b(\hat{y}, y, z)) = \begin{cases} (1 + \frac{P(A = a \mid Z = z) - 1}{P(Y = \hat{y}, Y = y \mid Z = z)}, \frac{P(A = a \mid Z = z)}{P(Y = \hat{y}, Y = y \mid Z = z)}), & p(\hat{y}, y \mid z) > p(\alpha \mid z) \\ (1 + \frac{P(A = a \mid Z = z)}{P(Y = \hat{y}, Y = y \mid Z = z)}, 1), & 1 - p(\alpha \mid z) < p(\hat{y}, y \mid z) \leq p(\alpha \mid z) \\ (0, 1), & p(\hat{y}, y \mid z) \leq 1 - p(\alpha \mid z) \end{cases} \end{cases}$$

We consider all possible cases:

- **Case I:** $p(\hat{y} = 1, y = 1 \mid z) > p(a \mid z)$. In this case, $p(\hat{y} = 0, y = 1 \mid z) \leq 1 - p(\hat{y} = 1, y = 1 \mid z) \leq 1 - p(a \mid z)$.

  Thus

  $$\tilde{w}_a(1, 1, z) = \frac{p(a \mid z)}{p(\hat{y} = 1, y = 1 \mid z)}, \quad \tilde{w}_a(0, 1, z) = 0,$$

  $$\tilde{w}_b(1, 1, z) = 1 + \frac{p(b \mid z) - 1}{p(\hat{y} = 1, y = 1 \mid z)}, \quad \tilde{w}_b(0, 1, z) = 1,$$

  Then we can set $\hat{w}_a(1, 0, z) = \tilde{w}_a(0, 0, z) = 0$, which obviously satisfy (30) for $\alpha = a$.

- **Case II:** $p(\hat{y} = 1, y = 1 \mid z) \leq p(a \mid z)$ and $p(\hat{y} = 0, y = 1 \mid z) > 1 - p(a \mid z)$. In this case,

  $$\hat{w}_a(1, 1, z) = 1, \quad \tilde{w}_a(0, 1, z) = 1 + \frac{p(a \mid z) - 1}{p(\hat{y} = 0, y = 1 \mid z)},$$

  $$\hat{w}_b(1, 1, z) = 0, \quad \tilde{w}_b(0, 1, z) = \frac{p(b \mid z)}{p(\hat{y} = 0, y = 1 \mid z)},$$

  Then we can set $\hat{w}_a(1, 0, z) = \tilde{w}_a(0, 0, z) = 1$ to satisfy (30) for $\alpha = a$:

  $$\sum_{\hat{y}, y \in \{0, 1\}} \tilde{w}_a(\hat{y}, y, z) \mathbb{P}(\hat{Y} = \hat{y}, Y = y \mid Z = z) = p(\hat{y} = 1, y = 1 \mid z) + p(\hat{y} = 1, y = 0 \mid z) + p(\hat{y} = 0, y = 0 \mid z) + p(\hat{y} = 0, y = 1 \mid z) + p(\hat{y} = 0, y = 0 \mid z) = p(a \mid z)$$

- **Case III:** $p(\hat{y} = 1, y = 1 \mid z) \leq p(a \mid z), P(\hat{y} = 0, y = 1 \mid z) \leq 1 - p(a \mid z) < p(a \mid z)$, and there exists $\hat{y} \in \{0, 1\}$ such that $p(\hat{y}, y = 0 \mid z) > p(a \mid z)$. Without loss of generality, we assume that $p(\hat{y} = 1, y = 0 \mid z) > p(a \mid z)$. In this case, $p(\hat{y} = 1, y = 1 \mid z) \leq 1 - p(\hat{y} = 1, y = 0 \mid z) < 1 - p(a \mid z) < p(a \mid z)$, thus

  $$\hat{w}_a(1, 1, z) = 1, \quad \tilde{w}_a(0, 1, z) = 0,$$

  $$\hat{w}_b(1, 1, z) = 0, \quad \tilde{w}_b(0, 1, z) = 1,$$

  We set

  $$\hat{w}_a(1, 0, z) = \frac{p(a \mid z) - p(\hat{y} = 1, y = 1 \mid z)}{p(\hat{y} = 1, y = 0 \mid z)}, \quad \tilde{w}_a(0, 0, z) = 0.$$
Since \( p(\hat{y} = 1, y = 1 \mid z) \leq p(a \mid z) \), we know that \( \hat{w}_a(1, 0, z) > 0 \). Plus,

\[
\hat{w}_a(1, 0, z) = \frac{p(a \mid z) - p(\hat{y} = 1, y = 1 \mid z)}{p(\hat{y} = 1, y = 0 \mid z)} < \frac{p(a \mid z)}{p(\hat{y} = 1, y = 0 \mid z)} < 1.
\]

Furthermore, (30) for \( \alpha = \alpha \) is satisfied:

\[
\sum_{\tilde{y}, y, z \in \{0, 1\}} \hat{w}_a(\tilde{y}, y, z) \mathbb{P}(\tilde{Y} = \tilde{y}, Y = y \mid Z = z) = p(\hat{y} = 1, y = 1 \mid z) + 0 + p(a \mid z) - p(\hat{y} = 1, y = 1 \mid z) = p(a \mid z)
\]

- Case IV: \( p(\hat{y} = 1, y = 1 \mid z) \leq p(a \mid z) \), \( \mathbb{P}(\hat{y} = 0, y = 1 \mid z) \leq 1 - p(a \mid z) < p(a \mid z) \) and \( p(\hat{y} = 1, y = 0 \mid z) \), \( p(\hat{y} = 0, y = 0 \mid z) \leq p(a \mid z) \). In this case, \( p(\hat{y} = 1, y = 1 \mid z) + p(\hat{y} = 1, y = 0 \mid z) + p(\hat{y} = 0, y = 0 \mid z) = 1 - \mathbb{P}(\hat{y} = 0, y = 1 \mid z) \geq p(a \mid z) \). As in Case III, we still have

\[
\hat{w}_a(1, 1, z) = 1, \quad \hat{w}_a(0, 1, z) = 0,
\]

\[
\hat{w}_b(1, 1, z) = 0, \quad \hat{w}_b(0, 1, z) = 1.
\]

In order to guarantee (30) for \( \alpha = \alpha \), we need \( \hat{w}_a(1, 0, z) \) and \( \hat{w}_a(0, 0, z) \) to satisfy

\[
\hat{w}_a(1, 0, z)p(\hat{y} = 1, y = 0 \mid z) + \hat{w}_a(0, 0, z)p(\hat{y} = 0, y = 0 \mid z) = p(a \mid z) - p(\hat{y} = 1, y = 1 \mid z).
\]

(33)

Since \( \hat{w}_a(1, 0, z) \) and \( \hat{w}_a(0, 0, z) \) can take any values within \([0, 1]\), we can find \( 0 \leq \hat{w}_a(1, 0, z), \hat{w}_a(0, 0, z) \leq 1 \) to satisfy (33), as long as

\[
0 \leq p(a \mid z) - p(\hat{y} = 1, y = 1 \mid z) \leq p(\hat{y} = 1, y = 0 \mid z) + p(\hat{y} = 0, y = 0 \mid z).
\]

This is satisfied automatically, since \( p(\hat{y} = 1, y = 1 \mid z) \leq p(a \mid z) \) and \( p(\hat{y} = 1, y = 1 \mid z) + p(\hat{y} = 1, y = 0 \mid z) + p(\hat{y} = 0, y = 0 \mid z) \geq p(a \mid z) \).

Since the cases are exhaustive, the conclusion follows. \( \square \)

### A.5. Proof of Proposition 10

**Proof.** We begin by observing that, using Eq. (5),

\[
h_{\Delta_{\text{TPRD}}}^{(\hat{w})}(\rho) = \sup_{\hat{w} \in \hat{W}} \sum_{b \in A_0} \rho_b \left( \mathbb{E} \left[ \hat{w}_a(\hat{Y}, Y, Z)Y\hat{Y} \right] - \mathbb{E} \left[ \hat{w}_b(\hat{Y}, Y, Z)Y\hat{Y} \right] \right).
\]

We next proceed to make a change of variables. Specifically, for each \( \alpha \in A \), we let

\[
t_\alpha = \frac{1}{\mathbb{E} \left[ \hat{w}_a(\hat{Y}, Y, Z) \right]},
\]

\[
\hat{w}_\alpha(\hat{y}, y, z) = t_\alpha \hat{w}_a(\hat{y}, y, z).
\]

Following Charnes and Cooper [1962], we find that

\[
h_{\Delta_{\text{TPRD}}}^{(\hat{w})}(\rho) = \max_{\hat{w} \in \hat{W}_{\text{LTP}}} \sum_{b \in A_0} \rho_b \left( \mathbb{E} \left[ \hat{w}_a(\hat{Y}, Y, Z)Y\hat{Y} \right] - \mathbb{E} \left[ \hat{w}_b(\hat{Y}, Y, Z)Y\hat{Y} \right] \right)
\]

s.t. \( \mathbb{E} \left[ \hat{w}_\alpha(\hat{Y}, Y, Z) \right] = 1 \quad \forall \alpha \in A \),

\[
\hat{w} \in \hat{W},
\]

\[
\hat{w}_\alpha(\hat{y}, y, z) = t_\alpha \hat{w}_a(\hat{y}, y, z).
\]

Projecting away the constraints \( \hat{w} \in \hat{W} = \hat{W}_{\text{LTP}} \cap \{ w : w_\alpha \in \hat{W}_{\alpha} \quad \forall \alpha \in A \} \), we arrive at the formulation in the result. \( \square \)