Abstract

M-theory on K3 × K3 with non-supersymmetry-breaking G-flux is dual to M-theory on a Calabi–Yau threefold times a 2-torus without flux. This allows for a thorough analysis of the effects of flux without relying on supergravity approximations. We discuss several dual pairs showing that the usual rules of G-flux compactifications work well in detail. We discuss how a transition can convert M2-branes into G-flux. We see how new effects can arise at short distances allowing fluxes to obstruct more moduli than one expects from the supergravity analysis.
1 Introduction

Flux compactifications along the lines of [1–3] have received a good deal of attention recently. One aspect about flux compactification that might make one a little uneasy is that the analysis tends to depend on supergravity arguments. Effects due to finite sizes which require the analysis of flux compactifications may need to be treated more carefully.

In this paper we will undertake a rather random perusal of some aspects of fluxes in the context of M-theory on $K3 \times K3$. This seems to be the most accessible nontrivial case of fluxes. As we will see, there are vast possibilities even in this case. In addition, we will show how there are inevitably some fluxes which cannot be analyzed at large radius. These fluxes can fix the volumes of the $K3$ surfaces.

The nice thing about M-theory on $K3 \times K3$ with M2-branes or $G$-flux that does not break any supersymmetry, is that it is dual to a flux-free and brane-free compactification of M-theory on a Calabi–Yau threefold, $X$, times a 2-torus. This allows us to check some aspects of $G$-flux in a more rigorous context.

Our main plan is to follow extremal “conifold”-like transitions in the Calabi–Yau threefold $X$ and see what happens in the dual $K3 \times K3$ picture. Some of the basic ideas of this analysis are not new (see particularly [4], for example) but we try to give a more complete picture of the interplay between the geometry of $X$ and the fluxes on $K3 \times K3$. One of the most useful tools will be the “stable degeneration” picture of F-theory [5, 6]. The yields, for example, an explicit demonstration of how the moduli of the $K3$ surfaces can be obstructed when $G$-flux is turned on. Mirror symmetry also plays an interesting rôle since it corresponds to exchanging the two $K3$ surfaces.

The $G$-flux will change as one passes through an extremal transition. Since $G$ is in integral cohomology, the only way this can happen is if the $K3$ surfaces become singular. This is indeed the case as we shall see.

A full analysis of the complete web of Calabi–Yau threefolds is still out of reach since it involves, as yet, poorly-understood non-perturbative effects. Thus, even in this simplest case, the subject of flux compactification is highly nontrivial.

In section 2 we will give several examples of how extremal transitions are mapped into $G$-flux transitions. A particularly interesting case concerns changing M2-branes into smooth $G$-flux where the initial and final $K3$ surfaces are smooth.

Section 3 is more speculative in nature and concerns the parts of the moduli space where nonperturbative effects become important. We argue that new types of moduli obstructions can appear because of fluxes and that the possibilities of M-theory compactifications on $K3 \times K3$ must exceed the number of types of Calabi–Yau threefolds.

2 Calabi–Yau to $K3\times K3$ Dualities

Let us recall some well-known general facts about M-theory compactifications to three-dimensions with $N = 4$ supersymmetry. Table 1 shows the various possibilities for how many supersymmetries arise from the holonomy of a given manifold. The case of $N = 4$ is
noteworthy since it is the highest value for $N$ which arises in two different ways.

Suppose we are given an M-theory compactification on $S_1 \times S_2$, where $S_1$ and $S_2$ are K3 surfaces. M-theory on $S_1$ is known [7] to be dual to the heterotic string on $T^3$. The heterotic string on $T^2 \times S_2$ is frequently dual to the type IIA string on a Calabi–Yau threefold $X$, and, since the type IIA string in ten-dimensions is dual to M-theory on a circle, we complete the chain of dualities to arrive at M-theory compactified on $X \times T^2$.\(^1\)

Thus we may explicitly map between dual pairs of $S_1 \times S_2$ and $X \times T^2$. The number of families of Calabi–Yau threefolds up to birational equivalence is known to be at least in the thousands. What’s more, most (although not all) of these Calabi–Yau’s have moduli spaces which are connected into a big web. Thus, the M-theory compactifications can be followed through transitions into thousands of possibilities. The same must therefore be true on the K3 × K3 side of the duality. But there is only one class of K3 × K3! This mismatch is solved by allowing for $G$-fluxes and M2-branes\(^2\) on the K3 × K3 side. It turns out that there are many, many ways of turning on G-flux for M-theory on K3 × K3 while preserving the $N = 4$ supersymmetry [9].

Before turning to many examples of these possibilities, let us review a bit more about the moduli space of these M-theory compactifications. The $R$-symmetry of $N = 4$ in three dimensions is SO(4) ∼ $Sp(1) \times Sp(1)$. We therefore expect the moduli space to be (locally) of the form $\mathcal{M}_1 \times \mathcal{M}_2$, where $\mathcal{M}_1$ and $\mathcal{M}_2$ are quaternionic Kähler manifolds.

This structure arises in an obvious way for M-theory compactified on $S_1 \times S_2$ — we associate $\mathcal{M}_1$ to the moduli space of $S_1$ and $\mathcal{M}_2$ to the moduli space of $\mathcal{M}_2$. In the case of M-theory on $X \times T^2$, we argue as follows. The type IIA string on a Calabi–Yau threefold has a moduli space of the form $\mathcal{M}_V \times \mathcal{M}_H$, where $\mathcal{M}_V$ is the special Kähler moduli space of vector multiplets and $\mathcal{M}_H$ is the quaternionic Kähler moduli space of hypermultiplets. We refer to [10] and references therein for more details. Upon compactification of this four-dimensional theory on a circle to three dimensions, $\mathcal{M}_V$ becomes “quaternionified” and $\mathcal{M}_H$.

\(^1\)Since there heterotic string is actually compactified on $T^3$, rather than $T^2$, there are more possibilities for choices of Wilson lines as explained in [8]. We ignore this fact.

\(^2\)To simplify discussion, we will often refer to “a choice of M2-branes and G-fluxes” simply as a “choice of G-fluxes”.

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| $N$ | Holonomy       | Manifold                                       |
|-----|----------------|-----------------------------------------------|
| 1   | Spin(7)       | Spin(7)-manifold                              |
| 2   | $G_2$         | $G_2$-manifold × $S^1$                        |
| 2   | SU(4)         | Calabi–Yau fourfold                           |
| 3   | Sp(2)         | Hyperkähler                                    |
| 4   | $Sp(1) \times Sp(1)$ | K3 × K3                          |
| 4   | SU(3)         | Calabi–Yau × $T^2$                            |
| 8   | Sp(1)         | K3 × $T^4$                                    |
| 16  | 1             | $T^8$                                         |

Table 1: Three-dimensional compactifications of M-theory.
is unchanged. Exchanging these factors amounts to mirror symmetry on $X$. Thus, the mirror symmetry of a Calabi–Yau threefold can be understood, via the above duality as an exchange of two K3 surfaces.

### 2.1 No $G$-flux

We first review the case of M-theory on $S_1 \times S_2$, where both $S_1$ and $S_2$ are smooth and we have $G = 0$. The standard tadpole cancellation rule states that \([1, 9, 11]\)

$$n_{M2} + \frac{1}{2}G^2 = \frac{X}{24} = 24,$$

where $n_{M2}$ is the number of M2-branes, i.e., points, on $S_1 \times S_2$. Thus, in the initial case of interest, $n_{M2} = 24$.

We would like to find a Calabi–Yau threefold $X_0$, such that this compactification of M-theory is equivalent to M-theory on $X_0 \times T^2$. This may be done using the F-theory picture of \([12, 13]\).

First, M-theory on $S_1$ is dual to the heterotic string (either $E_8 \times E_8$ or Spin(32)/$\mathbb{Z}_2$) on $T^3$ \([7]\). If $S_1$ is smooth, then the resulting gauge group in seven dimensions is $U(1)^{22}$ and we have a moduli space

$$\mathcal{M}_{1,7\text{-dim}} = O(\Gamma_{3,19}) \backslash O(3,19)/(O(3) \times O(19)) \times \mathbb{R}_+,$$

where $H^2(S^1) \cong \Gamma_{3,19}$, the even self-dual lattice of signature (3,19). Since we will obtain several moduli spaces of the above form, let us use the shorthand notation $\text{Gr}(\Lambda)$ for the Grassmannian of maximal space-like planes in the space spanned by the lattice $\Lambda$, divided by the automorphisms of $\Lambda$. That is, we denote the above moduli space by $\text{Gr}(\Gamma_{3,19}) \times \mathbb{R}_+$.

Now we further compactify this seven-dimensional heterotic theory on $S_2$. This compactification requires a choice of a bundle $E \to S_2$. Since we chose $S_1$ generically, there are no non-abelian groups to be used in the construction of $E$. Furthermore, as we will see in section 2.3, we may not use nontrivial line bundles for $E$ either. So $E$ must be a completely trivial bundle. That said, in order to satisfy anomaly cancellation, this bundle should have $c_2 = 24$. This apparent contradiction may be evaded by using point-like instantons \([14, 15]\) (perhaps more properly thought of as ideal sheaves \([16]\)). We therefore require that the heterotic compactification on $S_2$ has 24 point-like instantons.

The moduli space of $N = (4,4)$ conformal field theories on $S_2$ is given by $\text{Gr}(\Gamma_{4,20})$ \([17–19]\). Here $\Gamma_{4,20}$ is given by the total cohomology $H^0 \oplus H^2 \oplus H^4$ of $S_2$. However, the heterotic string on a K3 surface is not an $N = (4,4)$ theory. Instanton effects, from the heterotic 5-brane wrapping $K3 \times T^2$, will deform the metric of this moduli space. Let us, for the time being, ignore these instanton effects. This would make the moduli space

$$\mathcal{M}_2 = \text{Gr}(\Gamma_{4,20}) \times \text{Sym}^{24}(S_2),$$

where the $\text{Sym}^{24}(S_2)$ factor arises from the location of the 24 identical instantons on the K3 surface $S_2$. The symbol “$\times$” is used to denote a warped product — the shape of $S_2$ depends on the moduli in the first factor.
The instantons will, of course, warp the moduli space (3). We only expect this form of the moduli space to be accurate in a neighbourhood where the volume of $S_2$ is large.

After compactifying on $S_2$, we now have a theory with gauge group $U(1)^{22}$. In three dimensions, a vector field may be dualized into a periodic scalar field. Thus, we acquire 22 more moduli. This enhances the moduli space (2) to

$$\text{Gr}(\Gamma_{3,19}) \times U(1)^{22} \times \mathbb{R}_+.$$  \hspace{1cm} (4)

Up to some discrete identifications, this is exactly the decomposition of $\text{Gr}(\Gamma_{4,20})$ used in studying the moduli space of strings on K3 surfaces (see [19] for example). Usually the $U(1)^{22}$ factor represents the $B$-field degree of freedom. Here the same rôle is played by the scalars dual to the $U(1)^{22}$ gauge group.

There are still more moduli in the three-dimensional field theory due to the 24 point-like instantons. If these instantons are of the $E_8$ variety, they produce massless tensor fields in six dimensions [15]. This supermultiplet contains one real scalar, plus upon compactification on $T^3$, we obtain $b_1(T^3) = 3$ more scalars. That is, there are 4 real moduli per point-like instanton. Similarly, such an instanton of the $\text{Spin}(32)/\mathbb{Z}_2$ persuasion produces a $\text{Sp}(1)$ gauge symmetry in six dimensions [14]. Wilson lines for this on $T^3$ produce 3 moduli, and dualizing the resulting $U(1)$ in three dimensions produces a fourth. Thus, again, each point-like instanton yields 4 real moduli.

Finally, it is clear that exchanging $S_1$ and $S_2$ in the compactification of M-theory on $S_1 \times S_2$ should be a symmetry. This must make the moduli space of the compactification associated to $S_1$, i.e., $\mathcal{M}_1$, isomorphic to $\mathcal{M}_2$ given in (3). The full moduli space must therefore be of the form

$$\mathcal{M}_{G=0} = (\mathcal{M}_1 \times \mathcal{M}_2)/\mathbb{Z}_2$$
$$= (\text{Gr}(\Gamma_{4,20}) \times \text{Gr}(\Gamma_{4,20}) \times \text{Sym}^{24}(S_1 \times S_2))/\mathbb{Z}_2.$$  \hspace{1cm} (5)

Note that we see clearly the moduli space $\text{Sym}^{24}(S_1 \times S_2)$ of the 24 M2-branes. Having said that, let us emphasize again that this moduli space will be deformed by instantons. In the language of M-theory, these are M5-branes wrapped around $S_1$ times a 2-sphere in $S_2$, or $S_2$ times a 2-sphere in $S_1$.

Now we turn to the question of finding a Calabi–Yau threefold $X_{G=0}$ such that the above compactification is dual to M-theory on $X_{G=0} \times T^2$. The answer to this question has been known for some time [12]. In order to be fairly explicit, we will use the language of hypersurfaces in toric varieties following [20] to describe our Calabi–Yau threefolds. Here, a Calabi–Yau is described in terms of a reflexive lattice polytope in some lattice $\mathbf{N}$. This polytope describes a toric 4-fold. The Calabi–Yau is then realized as a smooth representative of the anticanonical divisor of this toric variety. It is by no means true that all Calabi–Yau manifolds can be realized as a hypersurface in a toric 4-fold but, fortunately, all the manifolds we require in this paper are of this type.

It was established in [5, 6, 12, 13, 21] that the $E_8 \times E_8$ heterotic string on a smooth $K3 \times T^2$ with a generic $E_8 \times E_8$ bundle with $c_2 = (12 + n, 12 - n)$ is dual to a type IIA string
compactified on a Calabi–Yau threefold $X$ specified by a generic elliptic fibration, with a section, over the Hirzebruch surface $\mathbb{F}_n$.

An elliptic fibration with a section over $\mathbb{F}_2$ is provided by the resolution of the hypersurface

$$x_0^2 + x_1^3 + x_2^{12} + x_3^{24} + x_4^{24} = 0,$$

in the weighted projective space $\mathbb{P}^4_{128211}$ as studied in [22]. This corresponds to the lattice polytope with vertices

$$(1,0,0,0) \quad (0,1,0,0) \quad (0,0,1,0) \quad (0,0,1) \quad (-12,-8,-2,-1).$$

Note that (6) represents the special “Fermat form” of the hypersurface. This may be deformed to include many more monomials. In lattice language, the terms written in (6) represent the vertices of the Newton polytope of all possible monomials which appear in the defining equation for the hypersurface. The Newton polytope is the polar polytope of that given by (7).

F-theory compactified on the Calabi–Yau given by (6) yields no gauge symmetry in six dimensions. The bundle structure group has broken the entire $E_8 \times E_8$. We want the opposite extreme where the bundle is given by point-like instantons and the $E_8 \times E_8$ is unbroken. This may be achieved [12] by deforming the above Calabi–Yau such that it acquires two curves of $E_8$ singularities. The following form of the hypersurface achieves this:

$$x_0^2 + x_1^3 + x_2^7x_3^{10} + x_2^7x_4^{10} + x_2^{14} + x_2^{14} = 0.$$  

This singular Calabi–Yau threefold may be blown-up (corresponding to giving vacuum expectation values to the tensor moduli). This extremal transition can be described in terms of lattice polytopes using the ideas of [23, 24]. The monomials in (8) represent the vertices of a new smaller Newton polytope. The polar of this Newton polytope provides the new reflexive polytope for the new manifold. This latter polytope has vertices

$$(1,0,0,0) \quad (0,1,0,0) \quad (0,0,1,0) \quad (-12,-8,-2,-1) \quad (15,10,6,0) \quad (-21,-14,-6,0).$$

Let us denote this new Calabi–Yau manifold $X_{G=0}$. Since it corresponds to 24 point-like instantons, it must be true that the above compactification of M-theory on $S_1 \times S_2$ is dual to the type IIA string compactified on $X_{G=0} \times S^1$, i.e., M-theory compactified on $X_{G=0} \times T^2$.

Lest the reader doubt the construction, let us check that the Hodge numbers agree. M-theory compactified on $X \times T^2$ has a moduli space generically of the form $\mathcal{M}_1 \times \mathcal{M}_2$, where $\mathcal{M}_1$ and $\mathcal{M}_2$ are quaternionic Kähler manifolds of quaternionic dimension $h^{1,1}(X) + 1$ and $h^{2,1}(X) + 1$ respectively. Thus, in order to match (5) we require $h^{1,1}(X_{G=0}) = h^{2,1}(X_{G=0}) = 43$. It is a simple matter [20, 25] to check that this is so.

Clearly exchanging $S_1$ and $S_2$ swaps the two factors of the moduli space and thus corresponds to mirror symmetry. It must therefore be that $X_{G=0}$ is self-mirror. Since the polytope (9) is not self-polar, this fact is not manifest from the toric description.
2.2 A singular K3×K3 with G-flux

A Calabi–Yau threefold can undergo extremal transitions (e.g. conifold transforms) changing the topology but without destroying any finiteness of a string compactification [26]. Indeed the resulting connected “web” of components of the moduli space seems to contain a very large number of the possible Calabi–Yau threefolds. This must mean that the M-theory compactified on $S_1 \times S_2$ needs to undergo similar transformations.

The simplest transformation corresponds to giving the point-like instantons of the previous section nonzero size to yield a smooth bundle for the heterotic string. This picture is again fairly well-known but we will review the ideas once more to fix notions for later sections.

In order that the heterotic string on $S_2$ has a nonabelian structure group, we must have some nonabelian gauge group before compactification. Thus, M-theory on $S_1$ gives some nonabelian factors to the gauge group which, in turn, implies that $S_1$ acquires at least one ADE-like singularity.

Let $H$ denote a subgroup of $\text{SL}(2, \mathbb{Z})$ and let $\mathcal{H}$ denote the corresponding Lie group. If $S_1$ acquires a singularity of the form $\mathbb{C}^2/H$, then the heterotic bundle $E$ on $S_2$ may have structure group $\mathcal{H}$. Since $c_2(E) > 0$, some of the point-like instantons must have been eaten up by this bundle. The point-like instantons that remain uneaten will still correspond to M2-branes in the M-theory picture of the compactification. These M2-branes are not associated with the transition and therefore are still free to wander about $S_1 \times S_2$. In order for this description to be consistent, exactly $c_2(E)$ of the corresponding points on $S_1$ must have gone into the singularity $\mathbb{C}^2/H$ on $S_1$. We show an example of such a transition in figure 1 for an SU(2)-bundle with $c_2 = 4$.

The interpretation of this transition in terms of $G$-flux is straight-forward. The supergravity analysis of supersymmetry-preserving $G$-flux on a four-fold dictates that $G$ must be
of type (2, 2) and primitive [1]. Following [27], one achieves this by putting

\[ G = \sum_{\alpha} \omega_{1}^{(\alpha)} \wedge \omega_{2}^{(\alpha)}, \]

(10)

where \( \omega_{j}^{(\alpha)} \) is a primitive (1,1)-form on \( S_{i} \).

The classical moduli space of Ricci-flat metrics on a K3 surface \( S \) is given by \( \text{Gr}(\Gamma_{3,19}) \) — the Grassmannian of space-like 3-planes \( \Sigma \) in \( H^{2}(S, \mathbb{R}) \). The 3-plane \( \Sigma \) is spanned by the real and imaginary parts of the holomorphic 2-form on \( S \) together with the Kähler form \( J \) (see [19] for more details). The statement that a 2-form \( \omega \) is primitive and of type (1,1) is therefore equivalent to the statement that \( \omega \) is perpendicular to \( \Sigma \) in \( H^{2}(S, \mathbb{R}) \).

A K3 surface is hyperkähler. Thus, for a fixed choice of Ricci-flat metric, there is a whole \( S^{2} \) of compatible complex structures. The statement that \( \omega \) is a primitive (1,1)-form is unaffected by this choice of complex structure. This means that the hyperkähler structure is unaffected by the presence of flux. Since the supersymmetry generators can be constructed from the complex structures, this means that the \( G \)-flux given in (10) breaks none of the extended supersymmetry — we still have \( N = 4 \) in three dimensions. Conversely, the only choice of \( G \)-flux which preserves \( N = 4 \) supersymmetry must be of the form (10). An example of a flux which breaks supersymmetry to \( N = 2 \) was given in [27].

Since we are looking for transitions given by extremal transitions of Calabi–Yau manifolds with no flux involved, we are not breaking any supersymmetry and so we need the \( G \)-flux to be of the form (10) on \( S_{1} \times S_{2} \).

Since \( G \) is integral, the 2-forms \( \omega_{j}^{(\alpha)} \) can be chosen to be integral. If \( \omega \) is a primitive integral (1,1)-form on a K3 surface \( S \), then \( \omega^{2} \) must be a negative even integer. The minimal flux is therefore given by \( \omega^{2} = -2 \). In this case let \( L \) be a line bundle with \( c_{1}(L) = \omega \). The Riemann-Roch theorem yields

\[ \chi(L) = h^{0}(L) - h^{1}(L) + h^{2}(L) = h^{0}(L) - h^{1}(L) + h^{0}(L^{-1}) = \int_{S} \text{ch}(L) \text{td}(S) = \frac{1}{2} \omega^{2} + 2 = 1. \]

Thus, either \( L \) or \( L^{-1} \) has nontrivial sections. The zero-set of such sections will be an algebraic curve in \( S \) Poincaré dual to \( \pm \omega \). But the fact that \( \omega \) is primitive means that this algebraic curve has area \( J.\omega = 0 \). This implies that \( S \) must be singular.

The M-theory interpretation of the case in hand is therefore that we have a \( G \)-flux given, in part, by a primitive (1,1)-form \( \omega_{1} \) on \( S_{1} \) of length-squared \(-2\). The rest of the \( G \)-flux is given by the curvature of the bundle \( E \) on \( S_{2} \). This part of the \( G \)-flux is Lie algebra valued and has therefore become nonabelian in some sense. This is not unreasonable since the connection on the nontrivial bundle \( E \) arises from the M-theory 3-form potential compactified on a vanishing 2-sphere in \( S_{1} \).
The statement that $\omega_2$ is primitive of type $(1,1)$ should therefore be naturally stated in terms of $c_1(E)$ being primitive. Note that since $c_1(E) = 0$ for a semi-simple structure group, this condition is trivial. Therefore, the $G$-flux restricts the moduli of $S_1$ forcing it to be singular, whereas $S_2$ is still allowed to be any K3 surface. Indeed, the nonabelian $G$-flux corresponding to the curvature of the bundle on $S_2$ gives more moduli associated to $S_2$ as one would expect from an extremal transition.

Clearly in this case of $G$-flux, the $\frac{1}{2}G^2$ term in the tadpole condition (1) is replaced by $c_2(E)$. It is also apparent that we may mix the roles of $S_1$ and $S_2$. That is, both $S_1$ and $S_2$ may be forced to be singular by $G$-flux and both surfaces may be endowed with nontrivial gauge bundles with semi-simple structure groups.

Finally let us note in this section that this picture of M-theory on $S_1 \times S_2$ gives a nice explanation of an observation in [28]. As we have already argued, mirror symmetry of the Calabi–Yau threefold $X$ corresponds to exchanging the K3 surfaces $S_1$ and $S_2$. As seen in figure 1, on $S_1$ a number, $n$, of point-like instantons have merged into a quotient singularity $\mathbb{C}^2/H$, whereas on $S_2$ we have obtained a smooth bundle with structure group $\mathcal{H}$ and with $c_2(n)$. Thus mirror symmetry exchanges $n$-point like instantons embedded in a quotient singularity $\mathbb{C}^2/H$, with a smooth bundle with structure group $\mathcal{H}$ and with $c_2(n)$. Also, a point-like instanton at a smooth point is exchanged with another point-like instanton at a smooth point. This is exactly the phenomenon experimentally observed in [28].

2.3 Smooth K3 surfaces and $U(1)$-bundles

The picture of the last section gave a fairly nice interpretation of $G$-flux obstructing moduli, in that it forced $S_1$ to be singular. It would be more satisfying, though, to give an example of moduli obstruction where the K3 surfaces were smooth.

To do this, let us consider a specific Calabi–Yau threefold and prove in detail an interpretation of F-theory compactified on this threefold conjectured in [29].

The type IIA string compactified on any Calabi–Yau threefold $X$ is, in some sense, always dual to a heterotic string compactified on $K3 \times T^2$. Having said that, the heterotic string cannot be weakly-coupled at any point in the moduli space unless $X$ is a K3 fibration [30,31]. Furthermore, the $T^2$ can only be decompactified to produce a six-dimensional, F-theory, compactification if each K3 fibre is itself an elliptic fibration with a section. Thus we will restrict ourselves to the case where $X$ is an elliptic fibration with a section.

In any such case, one can very systematically [5, 6, 12, 13, 21] establish precisely to which $E_8 \times E_8$ heterotic string compactification this F-theory compactification is dual.\(^3\)

The type IIA string compactified on $X$ will generically produce $h^{1,1}(X)$ vector multiplets in four dimensions. These have two sources from a six-dimensional theory compactified on $T^2$. They may originate either from six-dimensional vector or tensor multiplets. Which is the case was explained in [12,13]. The result is as follows. Consider an F-theory compactification on $X$, where $X$ is an elliptic fibration $\pi : X \to B$. The associated spectral sequence for cohomology implies that $H^2(X, \mathbb{Z})$ has three sources which have the effect:

\(^3\)The techniques are not quite so well-developed for the Spin(32)/$\mathbb{Z}_2$ case at present.
1. $H^2(B, \mathbb{Z})$ from the base which yield tensor multiplets (except for two),

2. $H^0(B, R^2 \pi_* \mathbb{Z})$ from the fibres which yield vector multiplets (except for one), and

3. $H^1(B, R^1 \pi_* \mathbb{Z})$ which also yield vector multiplets.

The $H^0(B, R^2 \pi_* \mathbb{Z})$ contribution, beyond one coming from the generic fibre, comes from singular fibres with more than one component. These produce nonabelian gauge symmetries in six dimensions. The $H^1(B, R^1 \pi_* \mathbb{Z})$ contribution does not produce nonabelian symmetries and therefore is associated to $U(1)$ factors in the six-dimensional gauge group.

A key notion in the study of elliptic fibrations is the Mordell-Weil group $\Phi$. This is the group of sections (viewing an elliptic curve as the group $U(1) \times U(1)$). Being a finitely generated abelian group, $\Phi$ will have some finite torsion part and a free part of finite rank, $\text{rank}(\Phi)$. For an elliptic fibration, the part of $H^1(B, R^1 \pi_* \mathbb{Z})$ that is of type $(1, 1)$ will provide the free part of $\Phi$ (see [32] for example). Thus, the number of $U(1)$ factors of the six-dimensional gauge group in F-theory is given by the rank of the Mordell–Weil group as stated in [13].

One can also show [33] that the torsion part of $\Phi$ is associated to the gauge group being not simply-connected. Thus $\pi_1$ of the six-dimensional gauge group is isomorphic to $\Phi$.

Armed with the necessary facts, let us consider the Calabi–Yau given by the hypersurface

$$x_0^2 + x_1^4 + x_2^8 + x_3^{16} + x_4^{16},$$

in $\mathbb{P}^4_{\{8,4,2,1,1\}}$. If we fix the values of $[x_2, x_3, x_4]$ we are left with the elliptic curve of degree 4 in $\mathbb{P}^2_{\{2,1,1\}}$. Thus, this threefold contains a “net” (i.e., a two-dimensional family) of elliptic curves. This does not mean that our threefold is elliptic however.

The space (12) contains two singularities of the form $\mathbb{C}^3/\mathbb{Z}_4$ at the two points $[x_0, x_1, x_2, x_3, x_4] = [\pm i, 1, 0, 0, 0]$. Every elliptic curve in our net passes through these two points. Blowing up these two points (together with a curve connecting them) will resolve the hypersurface. It will also make each elliptic curve in the net disjoint and thus yield an elliptic fibration. Let us denote this elliptic threefold $X_1$.

The two exceptional divisors of this resolution each provide a section of the elliptic fibration. Thus $\Phi$ contains at least two elements. If we use a generic equation for the hypersurface (rather than the Fermat form (12)) then all the fibres have only one component and thus $H^0(B, R^2 \pi_* \mathbb{Z})$ is rank one. The base of the fibration is $\mathbb{F}_2$ and thus $H^2(B, \mathbb{Z})$ is rank two. It is easy to show that $h^{1,1}(X_1) = 4$ and thus $H^1(B, R^1 \pi_* \mathbb{Z})$ is of rank one. That is, the group of sections is infinite. We can view one of the exceptional divisors as the “zero section” $\sigma_0$. The other exceptional divisor $\sigma_1$ generates $\Phi \cong \mathbb{Z}$.

For special choices of complex structure, such as the one corresponding to the Fermat form (12), $\sigma_1$ actually corresponds to a torsion section and the Mordell–Weil group is only $\mathbb{Z}_2$. One also acquires fibres with more than one component. The standard F-theory interpretation of this would then have at least an $SO(3)$ gauge group. To avoid this subtlety for now, we will assume that $X_1$ has a generic complex structure but we will visit this issue again at the end of section 2.4.
Thus, F-theory compactified on $X_1$ yields a gauge group $U(1)$ in six-dimensions as conjectured in [29]. Indeed, every one of the conjectured models in [29] can be analyzed similarly. For example, if the elliptic curves are cubic equations in $\mathbb{P}^2$ one will have to blow up three base points of the net of curves producing a Mordell-Weil group isomorphic to $\mathbb{Z}^2 \oplus \mathbb{Z}$ and thus a gauge group of $U(1)^2$.

We can now follow the arguments of [29, 36, 37] to see how this $U(1)$ gauge group can arise in the language of the $E_8 \times E_8$ heterotic string. The effective action of a heterotic compactification contains the term $H^2$, where

$$H = dB + \omega_L - \omega_Y. \quad (13)$$

A generic $U(1)$-bundle with a generic $c_1$ will produce a vacuum expectation value for the bundle curvature $\langle F \rangle$. This, in turn, will produce a term proportional to $AdB$ in the effective action. By analogy with the Higgs’ mechanism, a coordinate change in these fields will have the effect of making the $U(1)$ photon massive while removing a zero mode of the $B$-field. That is, the $U(1)$ does not actually appear as a gauge symmetry. In order to circumvent this effect we take a $U(1)$-bundle with nonzero $c_1$ and embed it identically into both $E_8$ factors in the heterotic string. The diagonal combination of these $U(1)$’s will gain a mass by the above mechanism, but the anti-diagonal combination will not, thus leaving a single $U(1)$ gauge group as a low-energy symmetry.

This idea is confirmed if we use the stable degeneration method of [5, 6, 21] to find precisely the heterotic dual of F-theory on $X_1$. One takes a limit in the complex structure of $X_1$ which is dual to taking the large $T^2$ limit of the heterotic string on $K3 \times T^2$. $X_1$ then degenerates into a pair of elliptic threefolds $X^{(1)}_1$ and $X^{(2)}_1$ which intersect along a K3 surface $S_H$. The latter K3 surface is identified as the K3 surface on which the dual heterotic string is compactified.

In order to find the bundle on $S_H$ for the heterotic string, one notices that $X^{(1)}_1$ and $X^{(2)}_1$ are fibrations over $\mathbb{P}^1$ with fibre given by a rational elliptic surface. The generators of the Mordell–Weil group of these rational elliptic surfaces intersect $S_H$ along curves $C^{(1)}$ and $C^{(2)}$. These curves provide “cameral” curves that encode the data of $E_8$-bundles. We refer to [5, 21, 38] for more information.

The key point is that the Mordell–Weil group of $X_1$ has a non-trivial generator. This generator is preserved by the stable degeneration and continues to have its effect on the cameral curve. To be precise, $S_H$ is itself an elliptic K3 surface with a section $\sigma_1$ generating a free component of its own Mordell–Weil group. The rôle of our global section in the Mordell–Weil groups of $X^{(1)}_1$ and $X^{(2)}_1$ will mean that both $C^{(1)}$ and $C^{(2)}$ contain a component given by $\sigma_1$.

---

4 This fact is disputed in [34] where it is asserted that $U(1)$ gauge groups arise from the monodromy group of the elliptic fibration lying in a proper subgroup of $\text{SL}(2, \mathbb{Z})$. We believe that the authors of [34] were misled by the fact that, as noted above, for special values of complex structure, a free section may indeed become a torsion section. Torsion sections are associated with the restriction of the monodromy group but free sections are not (see [35] for examples).

5 The reference [4] also used spectral curves to analyze the $U(1)$ bundles but in a different way to the one presented here.
A rational curve component to the cameral curve signifies a U(1) structure group of a factor of the gauge bundle. Thus we explicitly see a U(1)-gauge bundle embedded identically into both $E_8$ factors. We may do even better than this. The gauge bundle can be constructed as a Fourier–Mukai transform of the cameral cover data (at least in the case of holomorphic bundles with structure group $U(N)$ which is good enough for our purposes here). Thus produces the Chern class data for the bundle. We refer to [16] for the relevant information. The only fact we require here is that the Fourier–Mukai transform only acts on a 4-dimensional sublattice of $H^*(K3)$ generated by $H^0$, $H^4$, the 2-form dual to the elliptic fibre of the K3 surface, and the 2-form dual to the zero section. Directions in $H^2$ orthogonal to this lattice are unchanged by the transformation. This allows us to compute $c_1$ of the heterotic bundle. The result is (at least up to unimportant fibre contributions) $c_1 = \sigma_1 - \sigma_0$ — the difference between the two sections.

For the case of the Calabi–Yau threefold $X_1$, the bundle structure group is generically\(^6\) $((E_7 \times U(1))/\mathbb{Z}_2)^{\otimes 2}$ leaving an unbroken gauge group of $U(1)$. Returning to the language of M-theory on $S_1 \times S_2$, we saw in section 2.2 that bundles with a nonabelian structure group corresponded to singular K3 surfaces. We would like to make the K3 surfaces $S_1 \times S_2$ smooth. In analogy with section 2.1, we would therefore like to reduce the structure group of the heterotic bundle as much as possible and produce point-like instantons.

If we reduce the $E_7$ parts of the structure group completely we would expect to obtain an $(E_7 \times E_7 \times U(1)/\mathbb{Z}_2)$ six-dimensional gauge group from F-theory. This may be achieved by deforming the complex structure of $X_1$ to acquire two curves of type III* fibres by choosing a hypersurface with equation

$$x_0^2 + x_4^4 + x_2^5 x_3^6 + x_2^5 x_4^6 + x_2^3 x_3^{10} + x_2^3 x_4^{10}. \quad (14)$$

As in section 2.1, this produces an extremal transition to a new manifold we call $X_{U(1)}$, which is a hypersurface in a toric variety associated to a polytope with vertices:

\[
\begin{align*}
    (1, 0, 0, 0) & \quad (0, 1, 0, 0) & \quad (0, 0, 0, 1) \\
    (-8, -4, -2, -1) & \quad (6, 3, 4, 0) & \quad (-10, -5, -4, 0).
\end{align*}
\]

It is an easy matter to show that $h^{1,1}(X_{U(1)}) = h^{2,1}(X_{U(1)}) = 34$ and thus, for M-theory compactified on $X_{U(1)} \times T^2$, $\dim \mathcal{M}_1 = \dim \mathcal{M}_2 = 35$. Examination of the discriminant locus of the elliptic fibration (14) shows that there are 16 point-like instantons of the $E_8$ kind.

We now claim that M-theory compactified on $X_{U(1)} \times T^2$ is dual to a particular flux-compactification of M-theory compactified on $S_1 \times S_2$. The $G$-flux is given by $\omega_1 \wedge \omega_2$, where $\omega_j = \alpha_j - \beta_j$, and $\alpha_j$ and $\beta_j$ are dual to two disjoint $(-2)$-curves (i.e., $S^2$’s) in $S_j$. Thus, $G = 16$, consistent with the rest of the tadpole cancellation being given by 16 M2-branes.

This model first appeared in [27]. The 2-forms $\alpha_1$ and $\beta_1$ are associated with the embedding of the U(1) bundles in each $E_8$. The 2-forms $\alpha_2$ and $\beta_2$ are directly identified with the sections $\sigma_0$ and $\sigma_1$. We now explain the correspondence in detail.

Let us first make a general statement about the M-theory interpretation of moduli obstructions due to $G$-flux. The statement that $\omega_1$ or $\omega_2$ is (1,1) and primitive can never give

\(^6\)See section 2.4 for an explanation of the $\mathbb{Z}_2$ quotient.
Obstruction in Moduli space  |  G-flux language  |  Heterotic Language  
---|---|---
$\Sigma_1 \perp \omega_1$  | $\omega_1$ is (1,1) and primitive  | There is a $U(1) \times U(1)$-bundle $L$ on $S_2$  
$\Pi_1 \perp \omega_1$  | new effect  | In addition, one $U(1)$ photon acquires a mass  
$\Sigma_2 \perp \omega_2$  | $\omega_2$ is (1,1) and primitive  | $c_1(L)$ is primitive.  
$\Pi_2 \perp \omega_2$  | new effect  | In addition, one component of $B$-field is eaten  

Table 2: Obstructing the moduli space in two languages.

Obstructions in the moduli space. This is because such a statement removes real metric moduli in multiples of 3 (2 from the (1,1) statement and 1 for the primitive statement). The moduli space is quaternionic Kähler and so obstructions should be removed in multiples of 4. Thus, there should always be some extra effect that removes moduli beyond the primitive-(1,1) condition. We find this to be the case in this example.

Recall that earlier our moduli spaces contained factors like $Gr(\Lambda_{4,20})$ corresponding a Grassmannian of space-like 4-planes. When explicitly constructing these Grassmannians in the current context, the space like 4-plane, which we denote $\Pi$, is naturally described as being spanned by a space-like 3-plane $\Sigma$ (associated to the metric on the K3 surface) and fourth direction. We use this language below.

Let us first interpret $\omega_1 = \alpha_1 - \beta_1$. $\alpha_1$ and $\beta_1$ represent two-spheres in $S_1$ which map to directions in the $E_8 \times E_8$ lattice for the heterotic string. $\alpha_1$ is in one $E_8$ and $\beta_1$ is in the other. By the rules of G-flux compactifications, $\omega_1 = \alpha_1 - \beta_1$ is type (1,1) and $J.\omega = 0$. This puts the space-like 3-plane $\Sigma$ perpendicular to $\omega_1$. The fact that $\omega^2 = -4$ means that $S_1$ need not be singular in contrast to the example in section 2.2, since Riemann–Roch no longer implies that $\omega$ corresponds to the class of an algebraic curve. Since one of the $U(1)$ gauge fields becomes massive via the Higgs’ mechanism, we lose one of the $U(1)$’s in (4). This means, in the language of the Grassmannian $Gr(\Gamma_{4,20})$, the space-like 4-plane $\Pi$ is perpendicular to $\omega_1$. Thus $\omega_1$ obstructs one quaternionic deformation of $S_1$.

The form $\omega_2$ will similarly obstruct $S_2$. In fact, we may see this very explicitly. When we blew-up the hypersurface (12) in $\mathbb{P}^4_{4,2,1,1}$, there was a fixed curve of $\mathbb{C}^2/\mathbb{Z}_2$ singularities along $x_3 = x_4 = 0$. This gets resolved along with the two $\mathbb{C}^3/\mathbb{Z}_4$ singularities discussed above. When we reach the stable degeneration, this resolved curve hits $S_H$ twice. That is, $S_H$ contains two $\mathbb{P}^1$’s arising from this resolution. This forces these two homologically independent curves in $S_H$ to have equal area. We identify these two curves as dual to $\alpha_2$ and $\beta_2$ and it immediately follows that $J.(\alpha_2 - \beta_2) = 0$ as desired. Obviously $\omega_2 = \alpha_2 - \beta_2$ is of type (1,1) since it is dual to algebraic cycles.

In addition, the Higgs’ mechanism ate up one of the $B$-field zero modes associated with
the U(1)-bundles $c_1$ — namely $\omega_2$. Thus $M_2$ is also associated with a Grassmannian of 4-planes $\Pi$ perpendicular to $\omega_2$. The remaining 16 point-like instantons are free to move about as M2-branes on $S_1 \times S_2$ in a similar way to section 2.1. The dimensions of $M_1$ and $M_2$ should therefore be $16 + 19$ in agreement with the Hodge numbers above.

Since this story might be a little hard to follow, we review the key ideas in table 2. The entries labeled “new effect” refer to the extra obstructions beyond the primitive-(1,1) condition required to make the obstructions a multiple of 4.

Finally note that everything is symmetric between $S_1$ and $S_2$ in this construction. Thus we expect $X_{U(1)}$ to be self-mirror. To summarize, we have shown in detail that M-theory on $X_{U(1)} \times T^2$ is dual to M-theory on $S_1 \times S_2$ where $G = \omega_1 \wedge \omega_2$ as above and the moduli space is

$$ M_{U(1)} = (M_1 \times M_2)/\mathbb{Z}_2 $$

$$ = (\text{Gr}(\Gamma_{4,19}) \times \text{Gr}(\Gamma_{4,19}) \ltimes \text{Sym}^{16}(S_1 \times S_2)) / \mathbb{Z}_2. \quad (16) $$

We may extend this analysis to more than one U(1) group. For example, using another example from [29], M-theory on a 2-torus times the Calabi–Yau given by the polytope with vertices

$$(1, 0, 0, 0) \quad (0, 1, 0, 0) \quad (0, 0, 1, 0) \quad (-4, -4, -2, -1) \quad (2, 2, 3, 0) \quad (-4, -4, -3, 0), \quad (17)$$

is dual to M-theory on $S_1 \times S_2$ with

$$ G = (\alpha_1 - \gamma_1) \wedge (\alpha_2 - \gamma_2) - (\beta_1 - \delta_1) \wedge (\beta_2 - \delta_2), \quad (18) $$

where $\alpha_j, \beta_j, \gamma_j, \delta_j$ are dual to $(-2)$-curves on $S_j$ with $\alpha_j, \beta_j = \gamma_j, \delta_j = 1$ and with all other intersections zero; and 12 M2-branes. In this case, the six-dimensional F-theory model would have a gauge group $U(1) \times U(1)$. The moduli space (before instantons corrections) is

$$ (\text{Gr}(\Gamma_{4,18}) \times \text{Gr}(\Gamma_{4,18}) \ltimes \text{Sym}^{12}(S_1 \times S_2)) / \mathbb{Z}_2, \quad (19) $$

where the 4-plane $\Pi_j$ is perpendicular to $\alpha_j - \gamma_j$ and $\beta_j - \delta_j$.

Before closing this section we should point out a subtlety when the heterotic bundle structure group contains both abelian and non-abelian factors. Consider the Calabi–Yau $X_1$ given by (12). As stated above, the six-dimensional gauge group for F-theory on $X_1$ has a gauge group given only by $U(1)$. Denoting the structure group of the heterotic bundle by $H$, this means that $H$ must centralize $U(1) \times U(1)$ in $E_8 \times E_8$ (since one $U(1)$ was eaten by the Higgs’ mechanism). One might carelessly assert that $U(1) \times E_7$ is a subgroup of $E_8$ but this is not true. A more careful analysis shows that the correct subgroup is $((U(1) \times E_7)/\mathbb{Z}_2$, where the $\mathbb{Z}_2$ is generated by $-1$ in $U(1)$ and acts as the centre of $E_7$. An obvious choice for $H$ would therefore be $((U(1) \times E_7)/\mathbb{Z}_2)^2$.

In the case of $X_{U(1)}$ we allowed the heterotic bundle to degenerate so that its structure group was only $U(1)^2$. The first Chern class of these line bundles then contributed towards
the G-flux the equivalent of 8 M2-branes. For $X_1$ we would over-count the contribution to the tadpole if we added 8 to $c_2$ of the $E_7 \times E_7$ precisely because of this $\mathbb{Z}_2$ quotient. Instead, the effective $c_1$ of the abelian part of the bundle is halved and thus $c_2^1$ only contributes 2 to the tadpole. The result is, as observed in [29], that one requires an $E_7 \times E_7$ bundle with $c_2 = 22$ to produce an anomaly-free theory. Only then do we get the dimension of the moduli space to agree with the Hodge numbers of $X_1$.

2.4 A brane to flux transition

In section 2.1 we had M-theory on $K3 \times K3$ with no flux and 24 M2-branes; and in section 2.3 we had a theory with fluxes and only 16 M2-branes. We may follow extremal transitions through the dual picture of M-theory on a Calabi–Yau threefold times a 2-torus to see how M2-branes may turn into fluxes. Some of the analysis we do also appeared in [4] and a similar transition appeared in [34].

Consider deforming the hypersurface of (6) to the singular hypersurface $X^\sharp$ given by

$$x_0^2 + x_1^3 + x_1 x_2^5 x_3^6 + x_1 x_2^5 x_4^6 + x_1 x_2^3 x_3^{10} + x_1 x_2^3 x_4^{10}.$$  \hspace{1cm} (20)

The F-theory interpretation of this is as follows. The base $B$ of this fibration is $\mathbb{F}_2$ (before some of the blow-ups). We have lines of type III* fibres in $B$ along $x_2 = 0$ and along $x_2 = \infty$. We also have a curve of III fibres along a curve which intersects the two III* lines 6 times and 10 times respectively. The Mordell–Weil group is $\mathbb{Z}_2$. Thus, the six-dimensional gauge group is $(E_7 \times E_7 \times \text{SU}(2))/\mathbb{Z}_2$. There are 16 point-like instantons coming from the fibre collisions.

If we follow the stable degeneration, the heterotic string K3 surface, $S_H$ intersects the line of III fibres a total of 8 times and thus has $\mathbb{C}^2/\mathbb{Z}_2$ quotient singularities. Following [33] we interpret this as a heterotic string compactification on $S_H$ in the following way. Let $S_0$ be a smooth K3 surface with a $\mathbb{Z}_2$ symmetry that preserves the holomorphic 2-form. $S_H$ is then constructed as the orbifold $S_0/\mathbb{Z}_2$. Put a bundle on $S_H$ that is trivial except for monodromy around the 8 orbifold points. This monodromy acts as the $\mathbb{Z}_2$ subgroup of $E_8$ that centralizes $(E_7 \times \text{SU}(2))/\mathbb{Z}_2$ in each $E_8$. In addition, the curve of III fibres tells us that the SU(2) group is diagonally embedded in both $E_8$’s. The only way to express this in terms of centralizing a group action is to include an exchange of the two SU(2) subgroups of $E_8 \times E_8$ in the monodromy of the bundle to obtain the correct unbroken gauge group.\footnote{This is a little odd since the apparent structure group of the bundle is now not in $(E_8 \times E_8) \times \mathbb{Z}_2$. This point deserves to be better-understood.}

As in previous sections, we may resolve the hypersurface $X^\sharp$ given by (20) to form a smooth Calabi–Yau threefold we denote $X'$. The smooth threefold $X'$ is then given by a hypersurface in a toric variety associated to a polytope with vertices

$$(1,0,0,0) \quad (1,2,0,0) \quad (0,0,0,1) \quad (−12,−8,−2,−1) \quad (9,6,4,0) \quad (−15,−10,−4,0).$$  \hspace{1cm} (21)
We now wish to prove that $X'$ is isomorphic to $X_{U(1)}$ of section 2.3. To do this, note that

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
-8 & -4 & -2 & -1 \\
-10 & -5 & -4 & 0
\end{pmatrix}
= \begin{pmatrix}
1 & 0 & 0 & 0 \\
-12 & -8 & -2 & -1 \\
0 & 0 & 0 & 1 \\
16 & -12 & -2 & -1
\end{pmatrix}
\begin{pmatrix}
-2 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

(22)

The matrix on the left hand side of this equation consists of the lattice vectors (15) used to make $X_{U(1)}$. The first factor on the right hand side are the lattice vectors (21) used to make $X'$. The second factor on the right is an element of $\text{GL}(4, \mathbb{Z})$ and so just a simple change of basis of the lattice $N$.

The last factor on the right of (22) is manifestly not in $\text{GL}(4, \mathbb{Z})$. Thus, the two toric varieties in which $X_{U(1)}$ and $X'$ are embedded are not manifestly isomorphic. They differ by a $\mathbb{Z}_2$ refinement of the lattice $N$. Such a refinement corresponds to orbifolding by a $\mathbb{Z}_2$ subgroup of the $(\mathbb{C}^*)^4$ torus action. In the case above, it is easy to show that this amounts to the statement that $X'$ is a resolution of the $\mathbb{Z}_2$ orbifold of $X_{U(1)}$ given by the generator

\[g : [x_0, x_1, x_2, x_3, x_4] \rightarrow [-x_0, -x_1, x_2, x_3, x_4].\]

(23)

Curiously this orbifolding turns out to have no effect. The action given by (23) clearly acts purely on the fibre of the elliptic fibration leaving the base $\mathbb{F}_2$ untouched. On the smooth elliptic fibres, the action of $g$ is free and thus we retain the structure of an elliptic fibration. The only bad fibres in this model are of type III or III*. These bad fibres do have fixed points but the process of quotienting the fibre and then blowing up leaves these fibres where they started. For example, we show what happens to the III* fibre in figure 2. The $\mathbb{Z}_2$-action is roughly the left-right symmetry in this figure. The dots on the right-hand side denote fixed points to be resolved.

This resolved orbifold also has a section since the zero section of $X_{U(1)}$ is identified with a torsion section. We have therefore shown that $X_{U(1)}$ and $X'$ are both elliptic fibrations with a section with identical fibres in the same configuration over the same base. It follows [39] that $X_{U(1)}$ and $X'$ are birationally equivalent. In this case, since there are generically no special fibres appearing in codimension 2 over the base, it follows that $X_{U(1)}$ and $X'$ are isomorphic.

This completes the proof that we have obtained the required extremal transition. One begins with the smooth $S_1 \times S_2$ with 24 M2-branes dual to $X_{G=0}$. One then lets $S_1$ acquire two $\mathbb{C}^2/\mathbb{Z}_2$ singularities and we push 4 M2-branes into each of these singularities. This produces degrees of freedom over $S_2$ corresponding to an $(\text{SU}(2) \times \text{SU}(2)) \times \mathbb{Z}_2$-bundle. By making $S_2$ an orbifold $\text{K3}/\mathbb{Z}_2$ we make the monodromy of this bundle exchange the $\text{SU}(2)$
factors. This produces a model dual to $X_1$ before it is resolved. Now we may allow all the singularities to be resolved resulting in a smooth $S_1 \times S_2$ with a $G$-flux and only 16 M2-branes. We depict this transition in figure 3, where dots represent “free” point-like instantons, and circles represent $\mathbb{C}^2/\mathbb{Z}_2$ singularities (containing point-like instantons).

Some subtleties of the geometry of this kind of transition were discussed in [40]. The curve of type III fibres in $X^g$ is a curve of genus 7. This means that F-theory on $X^g$ will have 7 hypermultiplets in the adjoint representation of the corresponding enhanced SU(2) gauge symmetry. Part of the resolution to $X_{U(1)}$ corresponds to giving vacuum expectation values to these hypermultiplets which breaks the SU(2) to the observed U(1) gauge group.

These 7 hypermultiplets also account for the following. The precise form of $X_{U(1)}$ given by (14) has a curve of III fibres and therefore might be associated with an SU(2) gauge group. Moving to a generic complex structure this curve of fibres is replaced by more generic $I_1$’s. Thus, when the complex structure takes the form (14) there is a divisor of the form $\mathbb{P}^1 \times C$ for a curve $C$ of genus 7. A generic deformation turns this divisor into a collection of 12 disjoint rational curves as described in [40]. This is the geometric picture of the SU(2) gauge group being Higgs’ed to a U(1) subgroup.

3 Nonperturbative Effects

So far we have skated around the boundary of the moduli space that is accessible from F-theory techniques. Now we will try to make a couple of general statements about our compactifications regarding the deep interior of the moduli space.
3.1 Volume-obstructing $G$-fluxes

All of the $G$-fluxes discussed so far do not obstruct an overall metric rescaling of either K3 surface $S_1$ or $S_2$. The supergravity analysis of $G$-fluxes [1] necessarily happens at the large volume limit and thus is unlikely to observe any other kind of obstruction. In this section we will look for $G$-fluxes which do obstruct such dilatations.

In the previous section we restricted attention to Calabi–Yau threefolds which are elliptic and K3 fibrations. This is an artificial restriction and there most certainly exist extremal transitions in and out of this class.

A simple example can be borrowed from [41]. Consider the intersection $X_1$ of two hypersurfaces

\[
(x_1^4 + x_3^4 - x_4^4)y_0 + (x_0^4 + x_2^4 + x_4^4)y_1 = 0 \\
x_1y_1 + x_2y_2 = 0,
\]

in $\mathbb{P}^4 \times \mathbb{P}^1$. A projection onto the $\mathbb{P}^1$ factor manifestly gives $X_1$ the structure of a K3 fibration. This fibration has 16 $\mathbb{P}^1$'s corresponding to sections. We may contract these $\mathbb{P}^1$'s to produce 16 nodes in $X_1$ which may then be deformed to produce the quintic threefold [41].

The type IIA string compactified on $X_1$ is dual to some $E_8 \times E_8$ heterotic string compactified on $K3 \times T^2$. It is true that, since $X_1$ is not an elliptic fibration with a section, one is not free to vary the size of the heterotic string’s $T^2$. It will be the kind of heterotic string compactification considered in section 4.2 of [42]. To find precisely the heterotic dual one would find an extremal transition to an elliptic fibration with a section, use the F-theory rules we used above, and then follow the transition back. As the details will not concern us, we choose not to perform this exercise and just note that a solution exists.

The vector multiplet moduli space, $\mathcal{M}_V$, of the type IIA string compactified on $X_1$ is determined in the usual way from the prepotential $\mathcal{F}$. To leading order in $\alpha'$, this prepotential is cubic. If this were the case exactly, the moduli space would be locally of the form

\[
\frac{O(2, n - 1)}{O(2) \times O(n - 1)} \times \frac{SL(2, \mathbb{R})}{U(1)},
\]

where $n = h^{1,1}(X_1)$. We refer to [10] for more details. The second factor of (25) corresponds to $B + iJ$ integrated over a section of the K3 fibration. That is, it controls the size of the section. The first factor corresponds to $B + iJ$ integrated over the monodromy-invariant part of $H_2$ of the K3 fibres.

Upon compactification on a circle to three dimensions, $\mathcal{M}_V$ is elevated to the quaternionic Kähler manifold $\mathcal{M}_1$. This proceeds via the so-called “c-map” of [43]. Ignoring all nonperturbative effects, we may model this via a parabolic subgroup decomposition

\[
\mathcal{M}_1 = \frac{O(4, n + 1)}{O(4) \times O(n + 1)} = \frac{O(2, n - 1)}{O(2) \times O(n - 1)} \times \frac{O(2, 2)}{O(2) \times O(2)} \times \mathbb{R}^{2n+2} \\
= \frac{O(2, n - 1)}{O(2) \times O(n - 1)} \times \frac{SL(2, \mathbb{R})}{U(1)} \times \frac{SL(2, \mathbb{R})}{U(1)} \times \mathbb{R}^{2n+2}.
\]
The last factor corresponds to the Wilson lines (with the vector dualized to a scalar) of the \( n + 1 \) U(1)’s we had in four dimensions. The latter SL(2, \( \mathbb{R} \))/U(1) contains degrees of freedom associated to the radius of the circle and the B-field.

Now, the important thing to notice is that there are world-sheet instanton corrections to the prepotential as seen, for example, in [42]. The instantons generically arise from instantons wrapped around the sections of the K3-fibration. This is one way of seeing that the size of the section is dual to the coupling of the dual heterotic string.

Thus, if we follow \( X_1 \) through the conifold transition described above, we necessarily enter into the deep “strongly-coupled” part of the moduli space away from the limit where it looks like the symmetric space (26). The compactification on the quintic threefold is therefore “stuck” in this region as the degree of freedom corresponding to the size of the section is lost.

What is the “strongly-coupled” region of the moduli space in the language of M-theory on \( S_1 \times S_2 \)? The nonperturbative effects contributing to the moduli space metric can only arise as divisors with holomorphic Euler characteristic 2 on \( S_1 \times S_2 \) following the arguments of [44]. Such divisors correspond to K3 × \( \mathbb{P}^1 \) (or, obviously, \( \mathbb{P}^1 \times \text{K3} \)). The “coupling” associated to the moduli space \( \mathcal{M}_1 \) is therefore determined by \( \text{Vol}(S_1) \times \text{Vol}(S_2)^2 \).

We may derive the (mirror of the) latter result in another way. M-theory on \( S_1 \times S_2 \) is dual to the heterotic string on \( T^3 \times S_2 \). The moduli space of the heterotic string on \( S_2 \) will contain a factor which looks something like \( O(4,20)/\left( O(4) \times O(20) \right) \) when the K3 surface \( S_2 \) is large. Worldsheet instanton corrections wrapping \( \mathbb{P}^1 \)'s inside \( S_2 \) will warp this moduli space along the lines of [45]. The coupling is therefore measured by the volume of \( S_2 \), as measured by the heterotic string. The relationship between the metric of the heterotic string and the metric on M-theory was determined in [7]. The result is that the volume of \( S_2 \) as measured by the heterotic string is replaced by \( \text{Vol}(S_1)^2 \text{Vol}(S_2) \).

The natural claim, therefore, is that when we pass through a conifold transition, as above, that kills the K3 fibration structure, we force \( \text{Vol}(S_1)^2 \text{Vol}(S_2)^2 \) to be fixed. That is, we lose the moduli which allow us to make \( S_1 \) and \( S_2 \) simultaneously large. This necessarily takes us away from the supergravity analysis of G-flux.

Given the O(\( \Gamma_{4,20} \)) T-duality of K3 surfaces (at least for the spin connection embedded in the gauge group) the obvious thing to conjecture is that we have a G-flux similar to that considered in section 2.2, where \( \omega_1 \) has picked up some 0-form or 4-form component. Forcing the 4-plane \( \Pi \) to be perpendicular to \( \omega_1 \) would now force the volume of \( S_1 \) (corrected to \( \text{Vol}(S_1) \text{Vol}(S_2)^2 \) as above) to be fixed. This is rather like the way that type IIA strings compactified on a K3 surface may pick up nonabelian gauge groups associated with the finite size of the K3 surface, rather than singularities in the K3 surface [19].

Adding a 0-form or a 4-form to \( \omega_1 \) clearly violates the primitivity condition on \( G \). This should not concern us greatly however since we expect the supergravity analysis to fail.

It is difficult to be more quantitative at this stage because we are necessarily dealing with regions of the quaternionic Kähler moduli spaces where the nonperturbative effects are strong.

Finally, suppose we can find a Calabi–Yau \( Z \) in the web of possibilities such that neither
Z nor the mirror of Z is a K3 fibration. This would mean that both Vol($S_1$) Vol($S_2$)$^2$ and Vol($S_1$)$^2$ Vol($S_2$) are fixed. That is, each K3 surface has fixed size.

Given the statistics of K3 fibrations [46] it is surely likely that such a Calabi–Yau threefold $Z$ exists. A proof along the lines of [46] might be a little difficult however. Just because a Calabi–Yau threefold contains a K3 surface does not mean that this K3 surface need be compatible with the torus actions considered in [46]. Thus, one may have to go beyond toric methods to prove the non-existence of the K3 fibration.

### 3.2 Even more transitions

So far we have just looked at the geometry of the Calabi–Yau threefold $X$ for M-theory compactified on $X \times T^2$. Might it be that the 2-torus can also play a rôle to yield more possibilities?

Consider first the five-dimensional theory obtained from M-theory on a Calabi–Yau threefold $X$. Let $X$ be a K3 fibration without reducible fibres. The moduli space of vector multiplets will generically be of some form

$$M_{5,V} = \frac{O(1, n-2)}{O(n-2)} \times \mathbb{R}_+,$$

(27)

where $n = h^{1,1}(X)$. This moduli space may be viewed as the classical Kähler cone of the K3 fibre (with normalized volume) times a factor of $\mathbb{R}_+$ for the size of the base. Generically the Picard number of the K3 fibre will be given by the dimension of the monodromy-invariant part of $H^2$ of the fibres. Note that the real dimension of $\mathcal{M}_{5,V}$ is $h^{1,1}(X) - 1$ since one of the deformations of Kähler form (the overall volume of $X$) defects to the hypermultiplet moduli space.

M-theory on $X$ may acquire an enhanced nonabelian gauge symmetry in the usual way if the K3 fibres acquire an ADE singularity. Let us denote by $\Gamma_{1,n-2}$ the Picard lattice of the K3 fibre. It should then be clear that enhanced gauge symmetry corresponds to viewing $\mathcal{M}_{5,V}$ as $\text{Gr}(\Gamma_{1,n-2})$ and letting the space-like 1-plane be perpendicular to at least one vector of length squared $-2$ in $\Gamma_{1,n-2}$. Many extremal transitions of $X$ proceed through such enhanced gauge symmetry points in the moduli space.

Now consider M-theory on $X \times S^1$ or, in other words, the type IIA string on $X$. Now the vector multiplet moduli space becomes (ignoring instanton effects)

$$M_{6,V} = \frac{O(2, n-1)}{O(2) \times O(n-1)} \times \frac{\text{SL}(2, \mathbb{R})}{U(1)}.$$  

(28)

The first factor can be viewed as $\text{Gr}(\Gamma_{2,n-1})$ where $\Gamma_{2,n-1}$ is the Picard lattice of the K3 fibre plus $U$, where $U$ is generated by $H^0$ and $H^4$ of the fibre. Now we may have enhanced gauge symmetries whenever the corresponding space-like 2-plane is perpendicular to a vector in $\Gamma_{2,n-1}$ of length squared $-2$. This means, that in addition to enhanced gauge groups occurring whenever the K3 fibre acquires a singularity, one may also get an enhancement if
the K3 surface acquires a special overall volume. This is exactly how the SU(2) gauge group arises in the example of section 4.2 of [42].

Now go to M-theory on $X \times T^2$. Now the vector multiplet moduli space becomes (ignoring instanton effects)

$$M_{6, V} = \frac{O(4, n + 1)}{O(4) \times O(n + 1)}. \quad (29)$$

We will identify this as $\text{Gr}(\Gamma_{4,n+1})$. Again we expect to see an enhanced gauge symmetry if the space-like 4-plane is perpendicular to a vector of length squared $-2$ in $\Gamma_{4,n+1}$. The lattice $\text{Gr}(\Gamma_{4,n+1})$ should be viewed as one $U$-duality extended version of $H^{\text{even}}(K3, \mathbb{Z})$ as in [47,48]. These new gauge symmetries, not visible in four or five dimensional compactifications, cannot be associated with the Calabi–Yau threefold $X$ alone.

For M-theory on $K3 \times K3$, the identification of $\text{Gr}(\Gamma_{4,n+1})$ is manifest — we associate it with one of the K3 surfaces. Even though M-theory has no $B$-field, we always get a factor looking something like this associated to each K3 surface as we saw earlier in section 2.

This is all evidence for the following claim. If the K3 surface $S_1$ is such that the type IIA string compactified on $S_1$ yields an enhance gauge symmetry group, then an M-theory compactification on $S_1 \times S_2$ will see the same effect. That is, we obtain a gauge symmetry from the $S_1$ part but this may be broken by some bundle over $S_2$.

One may also argue for this proposal as follows. It was shown in [6] that the $E_8 \times E_8$ heterotic string compactified on a K3 surface with singularity $\mathbb{C}^2/H$ will always produce a nonperturbative gauge group enhancement by a group containing the associated $\mathcal{H}$ so long as enough point-like instantons are embedded in the singularity.\(^8\) We simply want to extend this naturally into the moduli space so that if a K3 surface is of a special volume that leads to enhanced gauge symmetries, then the heterotic string can also produce the enhanced symmetries so long as we deal with point-like instantons accordingly.

There is only one significant objection that can be made to the structure conjectured above. That is, we have not taken any instanton effects into account and the moduli spaces will not be of the Grassmannian form we used. In other words, the T-duality group of the K3 surface will be broken along the lines of [49]. Therefore it might be most unfair to treat the $H^0$ and $H^4$ directions in the cohomology the same as the $H^2$ directions.

We wish only to assert that there will be extremal transitions associated to these new enhanced gauge symmetries. While it is true that nonperturbative effects can break gauge symmetries as in [50], the extremal transitions (i.e., Higgs–Coulomb transitions) are not removed by these corrections.

We conjecture, therefore, that there are extremal transitions associated to M-theory on $X \times T^2$, that are not associated to extremal transitions on the Calabi–Yau threefold $X$ itself. The extra structure arising from the $T^2$ part of the compactification allow for these new transitions.

\(^8\)From our discussion in section 2.2 and the mirror symmetry analysis of [28] this number of point-like instantons is equal to $c_2$ for a principle $\mathcal{H}$-bundle. Thus puts a weak bound on this number of at least $\dim(\mathcal{H})/h^\vee$ where $h^\vee$ is the dual Coxeter number of $\mathcal{H}$.
This means that there are even more possibilities for choices of flux and M2-branes for M-theory on K3 \times K3 than there are Calabi–Yau threefolds!

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