A product construction for hyperbolic metric spaces

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Abstract

Given two pointed Gromov hyperbolic metric spaces \((X_i, d_i, z_i), i = 1, 2,\) and \(\Delta \in \mathbb{R}_+^+\), we present a construction method, which yields another Gromov hyperbolic metric space \(Y_\Delta = Y_\Delta((X_1, d_1, z_1), (X_2, d_2, z_2))\). Moreover, it is shown that once \((X_i, d_i)\) is roughly geodesic, \(i = 1, 2,\) then there exists a \(\Delta' \geq 0\) such that \(Y_\Delta\) also is roughly geodesic for all \(\Delta \geq \Delta'\).

1 Introduction

A metric space \((X, d)\) is called \(\delta\)-hyperbolic, \(\delta \geq 0,\) if for all \(x, y, z, w \in X\) it holds

\[
d(x, y) + d(z, w) \leq \max\{d(x, z) + d(y, w), d(x, w) + d(y, z)\} + 2\delta
\]

(1)

and said to be Gromov hyperbolic, if it is \(\delta\)-hyperbolic for some \(\delta \geq 0.\)

Let \((X_i, d_i)\) be Gromov hyperbolic metric spaces and fix \(z_i \in X_i, i = 1, 2.\) For \(\Delta \geq 0\) consider the set \(Y_\Delta = Y_\Delta(X_1, d_1, z_1, X_2, d_2, z_2)\) defined via

\[
Y_\Delta := \{(x_1, x_2) \in X_1 \times X_2 \mid |d_1(x_1, z_1) - d_2(x_2, z_2)| \leq \Delta\} \subset X := X_1 \times X_2.
\]

On \(Y_\Delta\) we consider the metric \(d_m|_{Y_\Delta \times Y_\Delta}\) which is the restriction of the \(l_\infty\)-product metric \(d_m : X \times X \rightarrow \mathbb{R}_0^+\)

\[
d_m((x_1, x_2), (x_1', x_2')) := \max\{d_1(x_1, x_1'), d_2(x_2, x_2')\},
\]

for all \(x_1, x_1' \in X_1, x_2, x_2' \in X_2,\) to \(Y_\Delta \times Y_\Delta \subset X \times X.\)

Our paper is based on the following elementary observation which we refer to as

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Theorem 1 Let \((X_i, d_i)\) be Gromov hyperbolic metric spaces and \(z_i \in X_i, i = 1, 2\). Then \((Y_\Delta, d_m|_{Y_\Delta \times Y_\Delta})\), as introduced above, is Gromov hyperbolic.

With Theorem 1 at hand we further prove the

Theorem 2 Let \((X_i, d_i)\) be roughly geodesic, Gromov hyperbolic metric spaces, \(z_i \in X_i, i = 1, 2\), then there exists \(\tilde{\Delta} \geq 0\) such that \((Y_\Delta, d_m|_{Y_\Delta \times Y_\Delta})\) is roughly geodesic for all \(\Delta \geq \tilde{\Delta}\) (and hyperbolic due to Theorem 1). Moreover, its boundary at infinity is naturally homeomorphic to \(\partial_\infty(X_1, d_1) \times \partial_\infty(X_2, d_2)\).

For precise definitions of the boundary at infinity and rough geodesics see Section 3.

Remark 1 (i) Both theorems can be formulated for a finite number of factors.

(ii) For metric spaces \((X_i, d_i)\) with nonempty boundaries at infinity, the Theorems 1 and 2 have analogues in the limit case, that the fixed points \(z_i \in X_i\) converge at infinity. Those will precisely be stated in Section 5.

(iii) An analogue of Theorem 2 in the setting of geodesic metric spaces has been studied in [FS2].

Outline of the paper: In Section 2 we prove Theorem 1. In Section 3 we recall some basic definitions and facts on Gromov hyperbolic metric spaces, which will be used in Section 4 when proving Theorem 2 and Section 5 when we state the above mentioned “limit case analogues” of the Theorems 1 and 2.

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2 The Proof of Theorem 1

Proof of Theorem 1 First of all note that if for a metric space \((X, d)\) there exist \(z \in X\) and \(\delta \geq 0\) such that for all \(x, y, w \in X\) it holds

\[
d(x, y) + d(w, z) \leq \max\{d(x, w) + d(y, z), d(w, z) + d(x, y)\} + 2\delta
\]

then \((X, d)\) is \(2\delta\)-hyperbolic (see e.g. [G]).

Let now \((X_i, d_i)\) be \(\delta_i\)-hyperbolic metric spaces and set \(\delta := \max\{\delta_1, \delta_2\}\). In order to show that \((Y_\Delta, d_m)\) is hyperbolic, we show that inequality (2) holds for \(d = d_m, z = (z_1, z_2) \in Y_\Delta, \delta := \delta + \frac{\Delta}{2}\) and \(x, y, z \in Y_\Delta\) arbitrary:

Without loss of generality we assume \(d_1(x_1, y_1) \geq d_2(x_2, y_2)\). Now, due to the definition of \(Y_\Delta\) we find

\[
d_m(w, z) = \max\{d_1(w_1, z_1), d_2(w_2, z_2)\} \leq d_1(w_1, z_1) + \Delta.
\]
Thus with the $\delta$-hyperbolicity of the first factor we get
\[
d_{m}(x, y) + d_{m}(w, z) \\
\leq d_{1}(x, y_{1}) + d_{1}(w_{1}, z_{1}) + \Delta \\
\leq \max\{d_{m}(x, w) + d_{m}(y, z), d_{m}(x, z) + d_{m}(y, z)\} + 2\delta + \Delta.
\]

\[\square\]

3 Some basic definitions

3.1 Hyperbolicity and rough geodesics

Let $(X_{1}, d_{1}), (X_{2}, d_{2})$ be metric spaces. A map $f : X_{1} \rightarrow X_{2}$ is called a quasi-isometric embedding, if there exist $\lambda \geq 1, k \geq 0$ such that for all $x, x' \in X_{1}$ it holds
\[
\frac{1}{\lambda} d_{1}(x, x') - k \leq d_{2}(f(x), f(x')) \leq \lambda d_{1}(x, x') + k.
\]
If $k = 0$, $f$ is called bilipschitz, while, for $\lambda = 1$, $f$ is said to be a $k$-rough-isometric embedding. For $k = 0$ and $\lambda = 1$ the embedding is called isometric.

Let $f : X_{1} \rightarrow X_{2}$ be a rough isometric embedding. Then, if $(X_{2}, d_{2})$ is hyperbolic, so is $(X_{1}, d_{1})$. In case $d_{1}$ and $d_{2}$ are length metrics, then the same holds when replacing the rough-isometric embedding through a quasi-isometric embedding (This non-trivial but by now standard result may be found in, for instance, [BriH] or [BuBuI]).

A $(k$-rough) geodesic $\gamma$ in $(X, d)$ connecting $x \in X$ to $x' \in X$ is a $(k$-rough) isometric embedding $\gamma : [\alpha, \omega] \rightarrow X$ such that $\gamma(\alpha) = x$ and $\gamma(\omega) = x'$. The metric space $(X, d)$ is called $(k$-rough) geodesic if for all $x, x' \in X$ there exists a $(k$-rough) geodesic connecting $x$ to $x'$. If there exists a $k \geq 0$ such that $(X, d)$ is $k$-rough geodesic, then $(X, d)$ is said to be rough geodesic.

According to [BoS] a metric space $(X, d)$ is called $k$-almost geodesic, if for every $x, y \in X$ and every $t \in [0, d(x, y)]$, there exists $w \in X$ such that $|d(x, w) - t| \leq k$ and $|d(y, w) - (d(x, y) - t)| \leq k$. $(X, d)$ is said to be almost geodesic, if there exists $k \geq 0$ such that it is $k$-almost geodesic. Note that a hyperbolic metric space $(X, d)$ is almost geodesic if and only if it is rough geodesic (compare Proposition 5.2 in [BoS]).

For geodesic metric spaces there are a number of equivalent characterizations of hyperbolicity using the geometry of geodesic triangles (see e.g. [BriH] and [FS2]). All of those characterizations have analogues in the rough geodesic setting. Here we state the corresponding results, we are going to make use of in the following:
Definition 1 A $k$-roughly geodesic metric space $(X, d)$ is said to be $(\delta, k)$-hyperbolic if each side of any $k$-roughly geodesic triangle in $(X, d)$ is contained in the $\delta$-neighborhood of the union of the other two sides.

Just along the lines of the proof of the corresponding statement for geodesic spaces (see e.g. [BriH]) one proves the

Proposition 1 Let $(X, d)$ be a $k$-roughly geodesic space. Then the following are equivalent:

1. $(X, d)$ is Gromov-hyperbolic.
2. there exists a $\delta \in \mathbb{R}_0^+$ such that $(X, d)$ is $(\delta, k)$-hyperbolic.

Let $X$ be a metric space and $x, y, z \in X$. Then there exist unique $a, b, c \in \mathbb{R}_0^+$ such that

$$d(x, y) = a + b, \quad d(x, z) = a + c \quad \text{and} \quad d(y, z) = b + c.$$  

In fact those numbers are given through

$$a = (y \cdot z)_x, \quad b = (x \cdot z)_y \quad \text{and} \quad c = (x \cdot y)_z,$$  

where, for instance,

$$(y \cdot z)_x = \frac{1}{2} \left[ d(y, x) + d(z, x) - d(y, z) \right].$$

In the case that $X$ is $k$-roughly geodesic we may consider a $k$-roughly geodesic triangle $\overline{xy} \cup \overline{xz} \cup \overline{yz} \subset X$, where for example $\overline{xy}$ denotes a $k$-roughly geodesic segment connecting $x$ to $y$. Given such a triangle we write $\tilde{x} := \gamma_{yz}(b)$, $\tilde{y} := \gamma_{xz}(a)$ and $\tilde{z} := \gamma_{xy}(a)$. Note that for geodesic triangles it holds $\gamma_{xz}(a) = \gamma_{yz}^{-1}(c)$. In the case that $(X, d)$ is only $k$-roughly geodesic we still have e.g. $d(\gamma_{xz}(a), \gamma_{yz}^{-1}(c)) \leq 2k$.

Similar to Lemma 1 i) in [FS2] one proves the

Lemma 1 If $(X, d)$ is $(\delta, k)$-hyperbolic, then

$$d(z, \tilde{z}) \leq c + 2\delta + 4k, \quad d(\gamma_{xy}(t), \gamma_{xz}(t)) \leq 4\delta + 15k \quad \forall t \in [0, a]$$

and the points $\tilde{x}$, $\tilde{y}$ and $\tilde{z}$ have pairwise distance $\leq 4\delta + 15k$.

3.2 The boundary at infinity and Busemann functions

Given a hyperbolic space $(X, d)$ there are various ways to attach a boundary at infinity $\partial_\infty X$ to $X$. In this paper we define $\partial_\infty X$ in the following way:

We choose a basepoint $z \in X$ and say that a sequence $\{x^i\}_{i \in \mathbb{N}}$ of points in $X$ converges to infinity, if

$$\liminf_{i, j \to \infty} (x^i \cdot x^j)_z = \infty.$$
Two sequences \( \{x^i\}_{i \in \mathbb{N}} \) and \( \{y^i\}_{i \in \mathbb{N}} \) converging to infinity are equivalent, \( \{x^k\}_{k \in \mathbb{N}} \sim \{y^k\}_{k \in \mathbb{N}} \) if
\[
\liminf_{i,j \to \infty} (x^i \cdot y^j)_z = \infty.
\]
One shows that \( \sim \) is an equivalence relation and defines \( \partial X \) as the set of equivalence classes. We write \([\{x^i\}] \in \partial X\) for the corresponding class.

For \( v \in \partial X \) and \( r > 0 \) one defines
\[
U(v, r) := \left\{ w \in \partial X \mid \exists \{x^k\}, \{y^k\} \text{ s.t. } [\{x^k\}] = v, [\{y^k\}] = w, \liminf_{k,l \to \infty} (x^k \cdot y^l)_z > r \right\}.
\]
On \( \partial X \) we consider the topology generated by \( U(v, r) \), \( v \in \partial X \), \( r > 0 \).

Let now \( (X, d) \) be \( k \)-roughly geodesic, then there exists a \( k' = k'(k, \delta) \) with \( \delta \) as in equation (1) such that for every \( x \in X \) there exists a \( k' \)-rough geodesic \( \gamma_{zu} : [0, \infty) \to X \) with \( \gamma_{zu}(0) = x \) and \([\{\gamma_{zu}(i)\}] = u \) (see [BoS]). Such rays are said to connect \( x \) to \( u \).

We now fix such a \( k' \)-roughly geodesic ray \( \gamma_{zu} \) connecting \( z \in X \) to \( u \in \partial \infty X \) and define the Busemann function \( B_{\gamma_{zu}} : X \to \mathbb{R} \) associated to the ray \( \gamma_{zu} \) via
\[
B_{\gamma_{zu}}(x) := \liminf_{t \to \infty} \left[ d(x, \gamma_{zu}(t)) - t \right].
\]
Note that the limit inferior always exists, while the limit itself necessarily only exists once \( \gamma_{zu} \) is a geodesic.

4 The proof of Theorem 2

In this section we provide the

**Proof of Theorem 2** Let \( (X_i, d_i) \) be \( k_i \)-roughly geodesic, \( \delta_i \)-hyperbolic, \( i = 1, 2 \), and set \( k := \max\{k_1, k_2\} \) as well as \( \delta := \max\{\delta_1, \delta_2\} \). We show that for \( \Delta \geq 4k \) the space \( (Y_\Delta, d_m) \) is \( K(\Delta, k, \delta) \)-almost geodesic and therefore roughly geodesic due to Theorem 1 and Proposition 5.2 in [BoS].

Thus we have to show that for \( \Delta \geq 4k \) there exists \( K \geq 0 \) such that for all \( x, y \in Y_\Delta, t \in [0, d_m(x, y)] \) there exists \( w \in Y_\Delta \) such that
\[
t - K \leq d_m(x, w) \leq t + K \quad \text{and} \quad (*)
\]
\[
d_m(x, y) - t - K \leq d_m(y, w) \leq d_m(x, y) - t - K.
\]

(1) W.l.o.g. we assume \( d_m(x, y) = d_1(x_1, y_1) \geq d_2(x_2, y_2) \).
(2) W.l.o.g. we assume \( \Delta < t < d_m(x, y) - \Delta \). This can be done, since for \( \Delta < t \) we may set \( w := x \) while for \( t > d_m(x, y) - \Delta \) we may set \( w := y \) and the inequalities above trivially hold once \( K \geq 2\Delta \).
(3) Now set

\[a_i := \frac{1}{2} \left( d_i(x_i, y_i) + d_i(x_i, z_i) - d_i(y_i, z_i) \right),\]

\[b_i := \frac{1}{2} \left( d_i(y_i, x_i) + d_i(y_i, z_i) - d_i(x_i, z_i) \right),\]

\[c_i := \frac{1}{2} \left( d_i(z_i, x_i) + d_i(z_i, y_i) - d_i(x_i, y_i) \right),\]

\[i = 1, 2. \] W.l.o.g. we assume \( \Delta < t \leq a_1 \) and \( \gamma_{z_i x_i} : [0, d_i(z_i, x_i)] \to X_i \) be 2\( k \)-rough geodesics connecting \( x_i \) to \( z_i \), \( i = 1, 2 \) and set

\[ w := \left( \gamma_{z_1 x_1}(d_1(x_1, z_1) - t), \gamma_{z_2 x_2}(d_1(x_1, z_1) - t) \right). \]

Note that since \( \Delta \geq 4k \) it follows that \( w \in Y_\Delta \).

Moreover, from the definition of \( w \) it is clear that we have

\[ d_2(w_2, x_2) \leq d_1(w_1, x_1) + \Delta + 4k. \] (3)

\( (X_i, d_i) \) is hyperbolic, \( i = 1, 2. \) Thus, due to Proposition [3] there exists \( \delta_i \) such that \( (X_i, d_i) \) is \( (\delta_i, 2k) \)-hyperbolic. Setting \( \delta := \max\{\delta_1, \delta_2\} \) and \( \delta' := 4\delta + 30k \) yields, due to Lemma [4]

\[ t - 2k - \delta' \leq d_1(w_1, x_1) \leq t + 2k + \delta' \] and \[(**)
\[ d_1(x_1, y_1) - t - 2k - \delta' \leq d_1(w_1, y_1) \leq d_1(x_1, y_1) - t + 2k - \delta'.\]

Now we consider the following two cases:

(i) \( d_2(x_2, z_2) - [d_1(x_1, z_1) - t] \leq a_2 \): In this case Lemma [4] yields

\[ t - 2k - \delta' \leq d_2(x_2, y_2) \leq t + 2k + \delta' \] and

\[ d_2(x_2, y_2) - t - 2k - \delta' \leq d_2(w_2, y_2) \leq d_2(x_2, y_2) - t + 2k - \delta'.\]

(ii) \( d_2(x_2, z_2) - [d_1(x_1, z_1) - t] > a_2 \): Of course we have \( d_1(x_1, z_1) - t \geq d_1(x_1, z_1) - a_1 = c_1 \), hence

\[ d_2(y_2, z_2) - [d_1(x_1, z_1) - t] \leq d_2(y_2, z_2) - c_1 \leq d_1(y_1, z_1) + \Delta - c_1 = b_1 + \Delta. \]

Thus, due to \( d_2(x_2, z_2) - [d_1(x_1, z_1) - t] > a_2 \) and Lemma [4] we conclude

\[ d_2(w_2, y_2) \leq b_1 + \Delta + \delta'. \]

From (i) and (ii) and the inequalities (**) as well as [3] it follows that the inequalities (*) hold for a \( K \geq 0 \) sufficiently large.

For the part of the proof concerning the boundary at infinity, we refer the reader to [FS2], where the geodesic case is treated. \( \Box \)
5 The limit case and final remarks

In this section we state the Theorems 3 and 4 corresponding to the Theorems 1 and 2 when fixing points in the boundary at infinity of the factors rather than points in the interior.

Let therefore \((X_1, d_1)\) and \((X_2, d_2)\) be roughly geodesic hyperbolic metric spaces with non-empty boundaries at infinity. Fix \(u_i \in \partial_\infty X_i\) as well as Busemann functions \(B_i : X_i \to \mathbb{R}\), associated to roughly geodesic rays \(\gamma_i\) converging to \(u_i\), \(i = 1, 2\). This time we consider the sets

\[
Y_\Delta := \{(x_1, x_2) \in X_1 \times X_2 \mid |B_1(x_1) - B_2(x_2)| \leq \Delta\}, \quad \Delta \geq 0.
\]

With this notation the following theorems hold:

**Theorem 3** Let \((X_i, d_i)\) be Gromov hyperbolic metric spaces with nonempty boundaries at infinity such that two Busemann functions \(B_i : X_i \to \mathbb{R}\) associated to rough geodesic rays are defined. Then, for all \(\Delta \geq 0\), \((Y_\Delta, d_m|_{Y_\Delta \times Y_\Delta})\) also is hyperbolic.

**Theorem 4** Let \((X_i, d_i)\) be roughly geodesic, Gromov hyperbolic metric spaces and \(B_i : X_i \to \mathbb{R}\) Busemann functions on \(X_i\), \(i = 1, 2\). Then there exists \(\tilde{\Delta} \geq 0\) such that \((Y_\Delta, d_m|_{Y_\Delta \times Y_\Delta})\) also is roughly geodesic and hyperbolic for all \(\Delta \geq \tilde{\Delta}\). Moreover, \(\partial_\infty(Y_\Delta, d_m|_{Y_\Delta \times Y_\Delta})\) is naturally homeomorphic to the smashed product \(\partial_\infty(X_1, d_1) \vee \partial_\infty(X_2, d_2)\).

**Remark 2** The smashed product \(\vee\) is a standard construction for pointed topological spaces (see e.g. [M]). Let \((U_1, u_1), (U_2, u_2)\) be two pointed spaces then the smashed product \(U_1 \vee U_2\) is defined as \(U_1 \times U_2 / (U_1 \times \{u_2\} \cup \{u_1\} \times U_2)\), where \(U_1 \times U_2\) is the usual product and

\[
U_1 \vee U_2 = \left(\{u_1\} \times U_2\right) \cup \left(U_1 \times \{u_2\}\right) \subset U_1 \times U_2
\]

is the wedge product canonically embedded in \(U_1 \times U_2\). Thus \(U_1 \vee U_2\) is obtained from \(U_1 \times U_2\) by collapsing \(U_1 \vee U_2\) to a point. For example \(S^m \vee S^n = S^{m+n}\).

The proofs of these theorems go just along the lines of the proofs of the corresponding Theorems 1 and 2 when fixing points in the interior rather than the boundary. For the part of Theorem 2 concerning the boundary at infinity we refer the reader to [FS2], where the analogue in the geodesic setting is proved.

We finally point out that when starting off with two proper geodesic metric spaces one has to consider the length metric \(d\) induced by \(d_m\) on \(Y_0\), in order to obtain a proper geodesic space again. In this case, we might as well endow \(Y_0\) with the length metric induced by the Euclidean product metric \(d_e\) instead of the maximum metric \(d_m\). Since both are length spaces which are bilipschitz related, one of them is Gromov hyperbolic if and only if the other one is.
In fact, when starting off with two Riemannian manifolds and fixing points at infinity, the construction using the Euclidean product metric has the advantage that it once again yields a Riemannian manifold (compare e.g. [FS1]). However, we emphasize that for neither of the Theorems 1, 2, 3 and 4 we might replace the maximum metric through the Euclidean metric. This is, for instance, seen in the

**Example 1** Consider two copies of the real hyperbolic space $\mathbb{H}^2$. Fix points $u_i \in \partial_\infty \mathbb{H}^2$, Busemann functions $B_i$ associated to geodesic rays $\gamma_i$ converging to $u_i$, $i = 1, 2$, and consider sequences of points $\{x^n = (x^n_1, x^n_2)\}$, $\{y^n = (y^n_1, y^n_2)\}$, $\{z^n = (z^n_1, z^n_2)\}$ and $\{w^n = (w^n_1, w^n_2)\}$ such that $x^n_1 = y^n_2$, $y^n_1 = y^n_2$, $z^n_1 = z^n_2$, $B_i(z^n_i) = B_i(y^n_i)$, $d_i(x^n_i, y^n_i) = d_i(x^n_i, z^n_i) = \frac{1}{2}d_i(y^n_i, z^n_i)$, $w^n_1 = y^n_1$ and $w^n_2 = z^n_2$ for all $n \in \mathbb{N}$, $i = 1, 2$, as well as $d_i(y^n_i, z^n_i) \xrightarrow{n\to \infty} \infty$, $i = 1, 2$.

We claim that $(Y_0(\mathbb{H}^2, u_i, B_i), d_e)$ is not hyperbolic. Suppose the contrary, then there exists a $\delta \geq 0$ such that for all $n \in \mathbb{N}$

$$d_e(y^n, z^n) + d_e(x^n, w^n) \leq \max\{d_e(x^n, y^n) + d(z^n, w^n), d_e(y^n, w^n) + d(x^n, z^n)\} + 2\delta \iff d_e(y^n, z^n) \leq \max\{d_e(z^n, w^n), d(y^n, w^n)\} + 2\delta$$

$$\iff \sqrt{2}d_1(y^n_1, z^n_1) \leq d_1(y^n_1, z^n_1) + 2\delta,$$

which contradicts our choices of sequences.

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