Comment on ”the quantum dynamics for general time-dependent three coupled oscillators”

Zerimeche Rahma \(^{a,b,1}\), Mana Naima\(^{a,2}\) and Maamache Mustapha\(^{a,3\ast}\)

\(^a\)Laboratoire de Physique Quantique et Systèmes Dynamiques, Faculté des Sciences, Ferhat Abbas Sétif 1, Sétif 19000, Algeria.

\(^b\)Physics Department, University of Jijel, BP 98, Ouled Aissa, 18000 Jijel, Algeria.

Abstract

In a recent paper, Hassoul et al.\(^{[1]}\), the authors proposed an analysis of the quantum dynamics for general time-dependent three coupled oscillators through an approach based on their decoupling using the unitary transformation method. Thus, to diagonalize the transformed Hamiltonian, they introduce a new unitary operator corresponding to a three-dimensional rotation parameterized by Euler angles. Through this procedure, Hassoul et al.\(^{[1]}\) claim that the coupled oscillatory subsystems are completely decoupled. This last approach is partly wrong. In this brief note, we show that their method is indeed not correct and we try to explain what truly lies behind their mistakes. We also propose a brief discussion on an alternative method that might achieve satisfactory results.

1 Introduction

The study of the dynamic behavior of oscillating systems is a central issue in applied sciences and mathematics. The harmonic oscillator model has been studied so far for a wide range of mechanical systems. However, if the oscillator interacts with the environment or with other oscillators, the associated systems cannot be isolated.

Coupled oscillators are connected in such a way that energy can be transferred between them, and their motion is usually complex. Therefore, one can find a coordinate frame in which each oscillator oscillates with a well-defined frequency. Knowing that the coupled oscillators are very complex from the dynamic point of view and in particular when their parameters depend on time and/or when the number of coupled oscillatory subsystems is greater than two.

\(^\ast\)zerimecherahma@gmail.com,\(^{2}\)na3ima.nn@hotmail.fr,\(^{3}\)maamache@univ-setif.dz
Despite the key role of coupled oscillatory models in general mechanical descriptions, their studies have been mainly performed for cases where oscillator parameters, such as masses and frequencies, are independent of time. Given the need to develop the dynamics of non-conservative oscillatory physical systems, Hassoul et al [1] consider a system of three coupled oscillators where the parameters of the Hamiltonian are arbitrary time-dependent functions, their main idea is that the Hamiltonian can be written in a diagonal form and that the exact solutions of the Schrödinger equation can be obtained in a simple way. In this brief note, we show that the method used in Ref. [1] is not adapted to the system studied because the rotation matrix as a function of Euler angles cannot diagonalize the Hamiltonian (13) Ref. [1]. We end with a brief discussion of an alternative method that gives satisfactory results for diagonalizing the Hamiltonian.

2 A short review: Rotation matrix and diagonalization of Hamiltonian

We begin this section by showing, with a direct computation, that the paper [1] contains an essential (and, in fact, trivial) mistake, which makes their results incorrect. The considered system is described by the time-dependent Hamiltonian (Eq. (13)) of Ref. [1]:

\[ H(t) = \frac{1}{2} \sum_{i=1}^{3} \left[ P_i^2 + \omega_i^2(t) X_i^2 \right] + K_{12} X_1 X_2 + K_{13} X_1 X_3 + K_{23} X_2 X_3 \]

\[ = \frac{1}{2} \sum_{i,j=1}^{3} P_i \delta_{ij} P^j + \frac{1}{2} \sum_{i,j=1}^{3} X_i \Gamma_{ij} X^j, \]  

(1)

where

\[ \Gamma(t) = \begin{pmatrix} \omega_1^2(t) & K_{12} & K_{13} \\ K_{12} & \omega_2^2(t) & K_{23} \\ K_{13} & K_{23} & \omega_3^2(t) \end{pmatrix}, \]  

(2)

and

\[ \omega_i^2(t) = \frac{1}{4} \left( \frac{\dot{m}_i^2(t)}{m_i^2(t)} - \frac{2\dot{\tilde{m}}_i(t)}{m_i(t)} \right) + \frac{c_i(t)}{m_i(t)}, \]  

(3)

\[ K_{12} = \frac{c_{12}(t)}{2 \sqrt{m_1(t)m_2(t)}}, \quad K_{13} = \frac{c_{13}(t)}{2 \sqrt{m_1(t)m_3(t)}}, \quad \text{and} \quad K_{23} = \frac{c_{23}(t)}{2 \sqrt{m_2(t)m_3(t)}}, \]  

(4)

and the parameters \( m_i(t) \) (\( i = 1, 2, 3 \)) and \( c_i(t) \) (\( i = 1, 2, 3 \)) are arbitrary functions of time.
The eigenvalues of the $3 \times 3$ matrix $\Gamma(t)$ are

$$\begin{align*}
\Omega_1^2 &= \frac{1}{3} \left[ (\omega_1^2 + \omega_2^2 + \omega_3^2) + 2\Omega \cos \left( \frac{\Phi}{3} \right) \right], \\
\Omega_2^2 &= \frac{1}{3} \left[ (\omega_1^2 + \omega_2^2 + \omega_3^2) + 2\Omega \cos \left( \frac{\Phi + 2\pi}{3} \right) \right], \\
\Omega_3^2 &= \frac{1}{3} \left[ (\omega_1^2 + \omega_2^2 + \omega_3^2) + 2\Omega \cos \left( \frac{\Phi - 2\pi}{3} \right) \right],
\end{align*}$$

(5)

$$\Phi = \arccos \left( \frac{\Delta}{2\sqrt{\Omega^2}} \right),$$

(6)

the expressions of $\Omega$ and $\Delta$ are

$$\Omega = \frac{1}{2} \left[ (\omega_1^2 - \omega_2^2)^2 + (\omega_1^2 - \omega_3^2)^2 + (\omega_2^2 - \omega_3^2)^2 \right] + 3 \left( K_{23}^2 + K_{13}^2 + K_{12}^2 \right),$$

(7)

$$\Delta = 18 (\omega_1^2 \omega_2^2 \omega_3^2 + 3K_{12}K_{13}K_{23}) + 2 (\omega_1^3 + \omega_2^3 + \omega_3^3) + 9 (\omega_1^2 + \omega_2^2 + \omega_3^2) (K_{23}^2 + K_{13}^2 + K_{12}^2)$$

$$- 3 (\omega_1^2 + \omega_2^2) (\omega_1^2 + \omega_3^2) (\omega_2^2 + \omega_3^2) - 27 (\omega_1^2 K_{23}^2 + \omega_2^2 K_{13}^2 + \omega_3^2 K_{12}^2).$$

(8)

Knowing, from classical mechanics, that an arbitrary rotation of a rigid body can be expressed in terms of three consecutive rotations, called the Euler rotations. In order to diagonalize the matrix Hamiltonian $H(t)$ Hassoul et al perform a unitary transformation that corresponds to a three-dimensional rotation parameterized by three Euler angles $(\phi, \theta, \varphi)$ introduced as

$$\mathbb{R} = \mathbb{R}_{X_1}(\phi)\mathbb{R}_{X_2}(\theta)\mathbb{R}_{X_3}(\varphi),$$

$$= \begin{pmatrix}
1 & 0 & 0 \\
0 & \cos \phi & -\sin \phi \\
0 & \sin \phi & \cos \phi
\end{pmatrix}
\begin{pmatrix}
\cos \theta & 0 & \sin \theta \\
0 & 1 & 0 \\
-\sin \theta & 0 & \cos \theta
\end{pmatrix}
\begin{pmatrix}
\cos \varphi & -\sin \varphi & 0 \\
\sin \varphi & \cos \varphi & 0 \\
0 & 0 & 1
\end{pmatrix},
$$

(9)

seeing the rotation matrix $\mathbb{R}$, we can determine how vectors transform under rotations; in a three-dimensional space, vectors $X$ and $P$ are rotated as $q = \mathbb{R}^{-1}X$ and $p = \mathbb{R}^{-1}P$. It is therefore more convenient, in quantum mechanics, to parametrize, as in classical mechanics in terms of the three Euler angles $(\phi, \theta, \varphi)$, the rotation operator $\Lambda(t)$ that corresponds to the rotation matrix $\mathbb{R}_{X_1}(\phi)\mathbb{R}_{X_2}(\theta)\mathbb{R}_{X_3}(\varphi)$ in the form

$$\Lambda(t) = \exp[i\phi(t)J_1] \exp[i\theta(t)J_2] \exp[i\varphi(t)J_3].$$

(10)

The components of rotation generators obey the commutation relations

$$[J_i, J_j] = i\hbar \varepsilon_{ijk} J_k, \quad [J_i, X_j] = i\hbar \varepsilon_{ijk} X_k, \quad [J_i, P_j] = i\hbar \varepsilon_{ijk} P_k,$$

(11)
$\varepsilon_{ijk}$ is the Levi-Civita symbol.

In order to eliminate the coupled terms $X_iX_j$, Hassoul et al [1] evaluate $R^{-1}(t)\Gamma(t)R(t)$ and affirm that it is possible to verify the relation $R(t)\{\text{diag} [\Omega_1^2, \Omega_2^2, \Omega_3^2]\}R^{-1}(t)$ in terms of the new diagonal matrix

$$\Gamma(t) = R(t) \begin{pmatrix} \Omega_1^2 & 0 & 0 \\ 0 & \Omega_2^2 & 0 \\ 0 & 0 & \Omega_3^2 \end{pmatrix} R^{-1}(t). \quad (12)$$

This statement is incorrect and to be convinced, it suffices to evaluate $R^{-1}(t)\Gamma(t)R(t)$. So let’s embark on the calculation of

$$R^{-1}(t)\Gamma(t)R(t) = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix}, \quad (13)$$

where the coefficients $M_{ij}$ are given in the appendix. To get the diagonal matrix $\text{diag} [\Omega_1^2, \Omega_2^2, \Omega_3^2]$, we put $M_{ii} = \Omega_i^2$ and $M_{ij} = 0$ for $(i \neq j)$ and in this case we deduce that $K_{12} = K_{13} = K_{23} = 0$ which implies that $\bar{\omega}_1^2 = \bar{\omega}_2^2 = \bar{\omega}_3^2$ and $\Omega_1^2 = \Omega_2^2 = \Omega_3^2 = \bar{\omega}_1^2 (t)$ which does not coincide with Eq. (36) of Ref. [1], therefore the quantum system described by Hamiltonian [1] has not been solved as claimed by Hassoul et al [1]. In order to remedy this situation, we will adopt the approach of diagonalization described in [2] but before that we shall emphasize that the rotated-coordinate column vector $X_R$ are

$$\Lambda^{-1}(t)X_1\Lambda(t) = \cos \theta \cos \varphi X_1 + \cos \theta \sin \varphi X_2 - \sin \theta X_3, \quad (14)$$
$$\Lambda^{-1}(t)X_2\Lambda(t) = (\sin \varphi \sin \theta \cos \varphi - \sin \varphi \cos \phi) X_1 + (\sin \varphi \sin \theta \sin \varphi + \cos \varphi \cos \phi) X_2 + \sin \phi \cos \theta X_3, \quad (15)$$
$$\Lambda^{-1}(t)X_3\Lambda(t) = (\sin \theta \cos \phi \cos \varphi + \sin \phi \sin \varphi) X_1 + (\sin \theta \cos \phi \sin \varphi - \sin \phi \cos \varphi) X_2 + \cos \phi \cos \theta X_3, \quad (16)$$

which we can write as

$$\Lambda^{-1}(t)X_k\Lambda(t) = \sum_{k=1}^{3} \bar{R}_{ik} X_k, \quad (17)$$

where the matrix $\bar{R}$ has the form

$$\bar{R} = \begin{pmatrix} \cos \theta \cos \varphi & \cos \theta \sin \varphi & -\sin \theta \\ \sin \varphi \sin \theta \cos \varphi - \sin \varphi \cos \phi & \sin \varphi \sin \theta \sin \varphi + \cos \varphi \cos \phi & \sin \phi \cos \theta \\ \sin \theta \cos \phi \cos \varphi + \sin \phi \sin \varphi & \sin \theta \cos \phi \sin \varphi - \sin \phi \cos \varphi & \cos \phi \cos \theta \end{pmatrix}, \quad (18)$$

we note that this matrix is different from the matrix $R$ given in [1] with equation (22), and consequently formulae of the rotated-coordinates (Eqs.(23,33 and 34)) are incorrect.

At the end of this section, let us note that the Hassoul et al’s [1] paper is largely inspired from the incoherent results of [3].
3 Brief discussion: the right diagonalization of Hamiltonian

The existence of a basis of eigenvectors makes possible to diagonalize the Hamiltonian in equation (11), then it is necessary to seek the corresponding eigenvectors as

\[ v(t) = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad v_\pm(t) = A_\pm \begin{pmatrix} K_{12} - K_{23} \mp z \\ K_{13} - K_{12} \pm z \\ K_{23} - K_{13} \end{pmatrix}, \quad (19) \]

where

\[ A_\pm(t) = \frac{1}{\sqrt{2z[2z - (J_{13} + J_{23} - 2J_{12})]}} \]

\[ z = \sqrt{K_{12}^2 + K_{13}^2 + K_{23}^2 - (K_{12}K_{13} + K_{12}K_{23} + K_{13}K_{23})}. \]

The eigenvalues associated to the eigenvectors \( v(t) \) and \( v_\pm(t) \) are

\[ \lambda = \frac{1}{3} \left[ \omega_1^2(t) + \omega_2^2(t) + \omega_3^2(t) + 2(K_{12} + K_{13} + K_{23}) \right] \]

\[ \lambda_+ = \frac{1}{3} \left[ \omega_1^2(t) + \omega_2^2(t) + \omega_3^2(t) - (K_{12} + K_{13} + K_{23}) \right] + z \]

\[ \lambda_- = \frac{1}{3} \left[ \omega_1^2(t) + \omega_2^2(t) + \omega_3^2(t) - (K_{12} + K_{13} + K_{23}) \right] - z \quad (21) \]

and do not coincide with the eigenvalues \( \Omega_i^2 \) (expressions (41-43) in [1]).

Thus, \( \Gamma(t) \) can be diagonalized as \( \Gamma(t) = U^+(t)\Gamma_D(t)U(t) \), where

\[ U(t) = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ A_+ (K_{12} - K_{23} - z) & A_+ (K_{13} - K_{12} + z) & A_+ (K_{23} - K_{13}) \\ A_- (K_{12} - K_{23} + z) & A_- (K_{13} - K_{12} - z) & A_- (K_{23} - K_{13}) \end{pmatrix}. \]

(22)

Now, we introduce new coordinates

\[ \begin{pmatrix} \tilde{q}_1 \\ \tilde{q}_2 \\ \tilde{q}_3 \end{pmatrix} = U(t) \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}, \quad \begin{pmatrix} \tilde{p}_1 \\ \tilde{p}_2 \\ \tilde{p}_3 \end{pmatrix} = U(t) \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}, \quad (23) \]

in terms of the new coordinates the Hamiltonian \( (11) \) can be diagonalized in the form

\[ H_1(t) = \frac{1}{2} \sum_{i,j=1}^3 P_i U_i^+(t)\delta_{ij}U_{ji}(t)P_j + \frac{1}{2} \sum_{i,j=1}^3 X_i U_i^+(t)\Gamma_{ij}U_{ji}(t)U_{ji}(t)X_j \]

\[ = \frac{1}{2} \sum_{i=1}^3 \tilde{p}_i^2 + \frac{1}{2} \sum_{i,j=1}^3 \Omega_i^2 \tilde{q}_i^2 \quad (24) \]

where \( \tilde{p}_i \) are the conjugate momenta of \( \tilde{q}_i \).
Finally let us note that the solutions to the original problem can not be obtained because the author has ignored in the calculations the term which comes from the time derivative of the unitary operator transformation (2.8) in [2]. We believe that this apparently natural procedure is ill founded. To transform back to the original variables $X_i$, we first note that since $\mathcal{U}(t)$ performs the scale change (23), the states are related by $\langle \tilde{q} | = \langle X \mathcal{U}^{-1}(t) [4, 5, 6, 7]$.

References

[1] S. Hassoul, S. Menouar, H. Benseridi and J. Ryeol Choi, Quantum dynamics for general time-dependent three coupled oscillators based on an exact decoupling, Physica A 604 (2022) 127755.

[2] D. Park, Dynamics of Entanglement in Three Coupled Harmonic Oscillator System with Arbitrary Time-Dependent Frequency and Coupling Constants, Quantum Information Processing 18, 282 (2019).

[3] R. Habarrih, A. Jellal, A. Merdaci, Dynamics and redistribution of entanglement and coherence in three time-dependent coupled harmonic oscillators, Int. J. Geometric Methods Mod. Phys. 18, 2150120 (2021).

[4] L. S. Brown, Quantum Motion in a Paul Trap, Phys. Rev. Lett. 66, 527 (1991).

[5] J. Y. Ji, J. K. Kim, S. P. Kim, and K. S. Soh, Exact wave functions and nonadiabatic Berry phases of a time-dependent harmonic oscillator, Phys. Rev. A 52, 3352 (1995).

[6] M. Maamache, K. Bencheikh, and H. Hachemi, Comment on “Harmonic oscillator with time-dependent mass and frequency and a perturbative potential”, Phys. Rev. A 59, 3124 (1999)

[7] M. Maamache, Comment on “Exact wave function of a harmonic plus an inverse harmonic potential with time-dependent mass and frequency” Phys. Rev. A 61, 026102 (2000).

Appendix: Derivation of the coefficients $M_{ij}$

\begin{align}
M_{11} &= \cos^2\theta \cos^2\varphi \omega_1^2 + (\cos \phi \sin \varphi + \sin \theta \sin \phi \cos \varphi)^2 \omega_2^2 + (\sin \varphi \sin \phi - \cos \varphi \cos \phi \sin \theta)^2 \omega_3^2 \\
&\quad + 2 \cos \theta \cos \varphi \left[ (\cos \phi \sin \varphi + \sin \theta \sin \phi \cos \varphi) K_{12} + (\sin \varphi \sin \phi - \cos \varphi \cos \phi \sin \theta) K_{13} \right] \\
&\quad + 2 \left[ \cos \phi \sin \phi \left( \sin^2 \varphi - \cos^2 \varphi \sin^2 \theta \right) - \sin \theta \sin \phi \cos \varphi \cos (2\phi) \right] K_{23} \\
\end{align}

(25)
\[ M_{12} = -\cos^2 \theta \cos \varphi \sin \varphi \omega_1^2 + \left[ \sin \varphi \cos \varphi \left( \cos^2 \phi - \sin^2 \theta \sin^2 \phi \right) + \cos \phi \sin \phi \sin \theta \cos (2\varphi) \right] \omega_2^2 \]
\[ + \left[ \cos \varphi \sin \varphi \left( \sin^2 \phi - \sin^2 \theta \cos^2 \phi \right) - \cos \phi \sin \phi \sin \theta \cos (2\varphi) \right] \omega_3^2 \]
\[ + \cos \theta \left[ \cos \phi \cos (2\varphi) - 2 \cos \varphi \sin \phi \sin \theta \sin \varphi \theta \right] K_{12} \]
\[ + \cos \theta \left[ \sin \phi \cos (2\varphi) + 2 \cos \varphi \cos \phi \sin \theta \sin \varphi \right] K_{13} \]
\[ + \left[ 2 \left( \sin^2 \theta + 1 \right) \cos \varphi \cos \phi \sin \varphi \sin \phi - \sin \theta \cos (2\phi) \cos (2\varphi) \right] K_{23} \quad (26) \]

\[ M_{13} = \sin \theta \cos \theta \cos \varphi \omega_1^2 - \sin \phi \cos \theta \left( \cos \phi \sin \varphi + \sin \theta \sin \phi \cos \varphi \right) \omega_2^2 + \cos \phi \cos \theta \left( \sin \varphi \sin \phi - \cos \varphi \cos \phi \sin \theta \right) \omega_3^2 \]
\[ + \left[ \cos \phi \sin \phi \sin \theta \cos (2\varphi) + \sin \varphi \cos \varphi \left( \sin^2 \phi - \cos^2 \phi \sin^2 \theta \right) \right] \omega_3^2 \]
\[ + \cos \theta \left[ \cos \phi \cos (2\varphi) - 2 \cos \varphi \sin \phi \sin \theta \sin \varphi \theta \right] K_{12} \]
\[ + \cos \theta \left[ \sin \phi \cos (2\varphi) + 2 \cos \varphi \sin \phi \sin \theta \sin \phi \right] K_{13} \]
\[ + \left[ 2 \sin \theta \sin \phi \cos \phi \cos \phi \sin \phi + \sin \varphi \cos \phi \cos (2\phi) \right] K_{23} \quad (27) \]

\[ M_{21} = -\cos^2 \theta \sin \varphi \omega_1^2 + \left[ \cos \phi \sin \phi \sin \theta \cos (2\varphi) + \sin \varphi \cos \varphi \left( \cos^2 \phi - \sin^2 \phi \sin^2 \theta \right) \right] \omega_2^2 \]
\[ + \left[ - \cos \phi \sin \phi \sin \theta \cos (2\varphi) + \sin \varphi \cos \varphi \left( \sin^2 \phi - \cos^2 \phi \sin^2 \theta \right) \right] \omega_3^2 \]
\[ + \cos \theta \left[ \cos \phi \cos (2\varphi) - 2 \cos \varphi \sin \phi \sin \theta \sin \varphi \theta \right] K_{12} \]
\[ + \cos \theta \left[ \sin \phi \cos (2\varphi) + 2 \cos \varphi \sin \phi \sin \theta \sin \phi \right] K_{13} \]
\[ + \left[ \sin \theta \left( \sin^2 \varphi \cos (2\phi) - \cos^2 \varphi \cos (2\phi) \right) \right] + 2 \left( 1 + \sin^2 \theta \right) \cos \varphi \sin \varphi \cos \phi \sin \phi \right] K_{23} \quad (28) \]

\[ M_{22} = \cos^2 \theta \sin^2 \varphi \omega_1^2 + \left[ \cos \phi \cos \varphi - \sin \phi \sin \varphi \sin \theta \right] \omega_2^2 + \left[ \cos \varphi \sin \phi + \sin \theta \cos \phi \sin^2 \varphi \right] \omega_3^2 \]
\[ - 2 \cos \theta \sin \varphi \left( \cos \phi \cos \varphi - \sin \phi \sin \varphi \sin \theta \right) K_{12} - 2 \cos \theta \sin \varphi \left( \cos \varphi \sin \phi + \sin \theta \cos \phi \sin \varphi \right) K_{13} \]
\[ + 2 \left[ \sin \phi \cos \phi \left( \cos^2 \varphi - \sin^2 \varphi \sin^2 \theta \right) \right] + \sin \theta \cos \varphi \sin \varphi \cos (2\phi) \right] K_{23} \quad (29) \]

\[ M_{23} = - \sin \theta \cos \theta \sin \varphi \omega_1^2 + \left[ \cos \phi \sin \varphi \sin \theta - \cos \phi \cos \varphi \right] \omega_2^2 + \left[ \cos \varphi \sin \phi + \sin \theta \cos \phi \sin \varphi \right] \omega_3^2 \]
\[ + \left[ \cos \phi \cos \varphi \sin \phi + \sin \phi \sin \varphi \cos (2\theta) \right] K_{12} + \left[ \cos \varphi \sin \phi \sin \theta - \sin \phi \cos \varphi \cos (2\theta) \right] K_{13} + \cos \theta \left[ \cos \varphi \cos (2\phi) - 2 \sin \phi \sin \theta \cos \phi \sin \varphi \right] K_{23} \quad (30) \]

\[ M_{31} = \cos \theta \sin \theta \cos \varphi \omega_1^2 - \cos \theta \sin \phi \left( \sin \phi \sin \theta \cos \varphi + \cos \phi \sin \varphi \right) \omega_2^2 \]
\[ + \cos \theta \cos \phi \left( \sin \varphi \sin \phi - \cos \phi \sin \theta \cos \varphi \right) \omega_3^2 \]
\[ + \left[ \sin \phi \cos \phi \cos (2\theta) + 2 \sin \phi \sin \phi \sin \varphi \right] K_{12} + \left[ \cos \varphi \cos \varphi \cos (2\theta) + \sin \phi \sin \phi \sin \varphi \right] K_{13} \]
\[ + \cos \theta \left[ 2 \cos \phi \sin \phi \sin \varphi \cos \varphi \sin \varphi \sin \theta \right] K_{23} \quad (31) \]

\[ M_{32} = - \cos \theta \sin \theta \sin \varphi \omega_1^2 - \cos \theta \sin \phi \left( \cos \phi \cos \varphi - \sin \phi \sin \varphi \sin \theta \right) \omega_2^2 \]
\[ + \cos \phi \cos \theta \left( \sin \phi \cos \varphi + \cos \phi \cos \phi \sin \theta \right) \omega_3^2 \]
\[ + \left[ \sin \phi \cos \phi \cos (2\theta) + \sin \phi \sin \phi \cos \varphi \right] K_{12} + \left[ \cos \phi \cos \phi \cos (2\theta) + \sin \phi \sin \phi \sin \varphi \right] K_{13} \]
\[ + \cos \theta \left[ \cos \phi \cos (2\phi) - 2 \cos \phi \sin \phi \sin \varphi \sin \theta \right] K_{23} \quad (32) \]

\[ M_{33} = \sin^2 \theta \omega_1^2 + \cos^2 \theta \sin^2 \phi \omega_2^2 + \cos^2 \phi \cos^2 \theta \omega_3^2 \]
\[ - 2 \cos \theta \left[ \sin \phi \sin \phi K_{12} - \sin \theta \cos \phi K_{13} + \cos \theta \phi \sin \phi K_{23} \right] \quad (33) \]