THE LOCALISED BOUNDED $L^2$-CURVATURE THEOREM

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ABSTRACT. In this paper, we prove a localised version of the bounded $L^2$-curvature theorem of Klainerman-Rodnianski-Szeftel [8]. More precisely, we consider initial data for the Einstein vacuum equations posed on a compact spacelike hypersurface $\Sigma$ with boundary, and show that the time of existence of a classical solution depends only on an $L^2$-bound on the Ricci curvature, an $L^4$-bound on the second fundamental form of $\partial\Sigma \subset \Sigma$, an $H^1$-bound on the second fundamental form, and a lower bound on the volume radius at scale 1 of $\Sigma$. Our localisation is achieved by first proving a localised bounded $L^2$-curvature theorem for small data posed on $B(0,1)$, and then using the scaling of the Einstein equations and a low regularity covering argument on $\Sigma$ to reduce from large data on $\Sigma$ to small data on $B(0,1)$. The proof uses the author’s previous work [6], [7], and the bounded $L^2$-curvature theorem [8] as black boxes.

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1. INTRODUCTION

1.1. The Cauchy problem of general relativity. A Lorentzian 4-manifold $(\mathcal{M}, g)$ solves the Einstein vacuum equations if

$$\text{Ric} = 0,$$

(1.1)
where \(\text{Ric}\) denotes the Ricci tensor of the Lorentzian metric \(g\). The Einstein vacuum equations are invariant under diffeomorphisms, and therefore one considers equivalence classes of solutions. Expressed in general coordinates, (1.1) is a non-linear geometric coupled system of partial differential equations of order 2 for \(g\). In a suitable gauge, namely the so-called \textit{wave coordinates}\(^1\), it can be shown that (1.1) is hyperbolic and hence corresponds to an evolution problem.

Initial data for the Einstein vacuum equations is specified by a triple \((\Sigma, g, k)\) where \((\Sigma, g)\) is a Riemannian 3-manifold and \(k\) a symmetric 2-tensor on \(\Sigma\) satisfying the \textit{constraint equations},

\[
R_{\text{scal}} = |k|^2_g - (\text{tr}_g k)^2, \\
\text{div}_g k = d(\text{tr}_g k).
\]

(1.2)

Here \(R_{\text{scal}}\) and \(d\) denote the scalar curvature of \(g\) and the exterior derivative on \(\Sigma\), respectively, and

\[
|k|^2_g := g^{ij}g^{lm}k_{il}k_{jm}, \\
\text{tr}_g k := g^{ij}k_{ij}, \\
(\text{div}_g k)_l := g^{ij}\nabla_i k_{jl},
\]

where \(\nabla\) denotes the covariant derivative on \((\Sigma, g)\) and we tacitly use, as in the rest of this paper, the Einstein summation convention.

In the seminal [2] it is shown that the above initial value formulation is well-posed. In the future development \((\mathcal{M}, g)\) of given initial data \((\Sigma, g, k)\), the 3-manifold \(\Sigma \subset \mathcal{M}\) is a spacelike Cauchy hypersurface with induced metric \(g\) and second fundamental form \(k\). See for example [17] or [10] for details.

In the rest of this paper, we assume that the initial hypersurface \(\Sigma\) is \textit{maximal}, that is,

\[\text{tr}_g k = 0 \quad \text{on} \quad \Sigma.\]

This assumption is sufficiently general for our purposes, see [1]. In particular, on a maximal hypersurface \(\Sigma\), the constraint equations (1.2) reduce to the \textit{maximal constraint equations},

\[
R_{\text{scal}} = |k|^2_g, \\
\text{div}_g k = 0, \\
\text{tr}_g k = 0.
\]

\(^1\)On a Lorentzian 4-manifold \((\mathcal{M}, g)\), wave coordinates \((x^0, x^1, x^2, x^3)\) satisfy by definition

\[
\Box_g x^\alpha = 0 \quad \text{for} \quad \alpha = 0, 1, 2, 3.
\]

The Einstein equations reduce in wave coordinates to

\[
\Box_g (g_{\alpha \beta}) = \mathcal{N}_{\alpha \beta}(g, \partial_\mu g), \quad \text{for} \quad \alpha, \beta = 0, 1, 2, 3,
\]

where \(\mathcal{N}_{\alpha \beta}(g, \partial g)\) is a non-linearity that is linear in \(g\) and quadratic in \(\partial_\mu g\), \(\mu = 0, 1, 2, 3\).
1.2. **Weak cosmic censorship and the bounded $L^2$-curvature theorem.** One of the main open questions of mathematical relativity is the so-called weak cosmic censorship conjecture formulated by Penrose, see [9].

**Conjecture 1.1 (Weak cosmic censorship conjecture).** For a generic solution to the Einstein equations, all singularities forming in the context of gravitational collapse are covered by black holes.

In the ground-breaking [5], Christodoulou proves Conjecture 1.1 for the Einstein-scalar field equations under the assumption of spherical symmetry. In Christodoulou’s proof, a low regularity control of the Einstein equations is essential for analysing the dynamical formation of black holes. More precisely, in [4] Christodoulou proves a well-posedness result for initial data which is bounded only in a scale-invariant BV-norm, and subsequently uses this framework to establish the formation of trapped surfaces in [5].

The result in [5] strongly suggests that a crucial step to prove the weak cosmic censorship in the absence of symmetry is to control solutions to the Einstein vacuum equations in very low regularity\(^2\). The current state-of-the-art with respect to low regularity control of solutions to the Einstein vacuum equations is the bounded $L^2$-curvature theorem by Klainerman-Rodnianski-Szeftel, see Theorem 2.2 in [8]. We refer to the introduction of [8] for a historical account of the developments leading to this result.

**Theorem 1.2 (The bounded $L^2$-curvature theorem, [8]).** Let $(\mathcal{M}, g)$ be an asymptotically flat solution to the Einstein vacuum equations together with a maximal foliation by spacelike hypersurfaces $\Sigma_t$ defined as level sets of a time function $t$. Assume that the initial slice $(\Sigma_0, g, k)$ is such that $\Sigma_0 \simeq \mathbb{R}^3$ and

\[
\|\text{Ric}\|_{L^2(\Sigma_0)} < \infty, \|\nabla k\|_{L^2(\Sigma_0)} < \infty \text{ and } r_{\text{vol}}(\Sigma_0, 1) > 0,
\]

where $r_{\text{vol}}(\Sigma_0, 1)$ is the volume radius\(^3\) of $(\Sigma_0, g)$ at scale 1, and $\text{Ric}$ denotes the Ricci tensor of $g$. Then

1. **$L^2$-regularity.** There exists a time $T = T(\|\text{Ric}\|_{L^2(\Sigma_0)}, \|\nabla k\|_{L^2(\Sigma_0)}, r_{\text{vol}}(\Sigma_0, 1)) > 0$, and a constant $C = C(\|\text{Ric}\|_{L^2(\Sigma_0)}, \|\nabla k\|_{L^2(\Sigma_0)}, r_{\text{vol}}(\Sigma_0, 1)) > 0$,

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\(^2\)Note that bounded variation norms are not suitable outside of spherical symmetry. In the absence of spherical symmetry, regularity should be measured with respect to $L^2$-based spaces, see [11].

\(^3\)The volume radius of $(\Sigma_0, g)$ at scale 1 is defined as

\[
r_{\text{vol}}(\Sigma_0, 1) := \inf_{p \in \Sigma_0} \inf_{0 < r < 1} \frac{\text{vol}_g(B_g(p, r))}{\frac{4\pi}{3} r^3},
\]

where $B_g(p, r)$ denotes the geodesic ball of radius $r$ centered at the point $p$. 
such that the following control holds on $0 \leq t \leq T$.

$$\|R\|_{L^\infty_t L^2(\Sigma_t)} \leq C, \quad \|\nabla k_t\|_{L^\infty_t L^2(\Sigma_t)} \leq C, \quad \inf_{0 \leq t \leq T} r_{\text{vol}}(\Sigma_t, 1) \geq \frac{1}{C},$$

where $R$ denotes the Riemann curvature tensor of $(\mathcal{M}, g)$, and $g_t$ and $k_t$ are the induced metric and the second fundamental form of $\Sigma_t$, respectively.

(2) **Higher regularity.** In case of higher regularity of the initial data, we have for integers $m \geq 1$, within the same time interval as in part (1), the higher derivative estimate

$$\sum_{|\alpha| \leq m} \|D^{(\alpha)} R\|_{L^\infty_t L^2(\Sigma_t)} \leq C_m \sum_{|i| \leq m} \left( \|\nabla^{(i)} \text{Ric}\|_{L^2(\Sigma_0)} + \|\nabla^{(i)} \nabla k\|_{L^2(\Sigma_0)} \right),$$

where the constant $C_m > 0$ depends only on the previous $C$ and $m$.

**Remark 1.3.** In the above theorem, as in the rest of this paper, the statement should be understood as a continuation result for smooth solutions. That is, a solution to the Einstein vacuum equation developed from smooth initial data can smoothly be continued as long as condition (1.3) holds. For details, see the introduction in [8].

The proof of Theorem 1.2 is based on bilinear estimates, see [8], as well as Strichartz estimates, see [16], in a low regularity spacetime where the Riemann curvature tensor is only assumed to be in $L^2$. The proof of these estimates relies crucially on a plane wave representation formula for the wave equation on low regularity spacetimes constructed in [12]-[16]. This plane wave representation formula is built as a Fourier integral operator which necessitates the assumption $\Sigma_0 \simeq \mathbb{R}^3$.

Given that the Einstein equations are hyperbolic and have finite speed of propagation, the assumption $\Sigma_0 \simeq \mathbb{R}^3$ in Theorem 1.2 seems unnatural. Furthermore, gravitational collapse is studied in local domains of dependence, that is, given a *compact* initial data set with boundary, one considers the development inside the future domain of dependence of the initial data set, see for example [5] and [3]. For these reasons, it appears important to localise Theorem 1.2, that is, to relax the condition $\Sigma_0 \simeq \mathbb{R}^3$, which is the main goal of this paper.

### 1.3. The localised bounded $L^2$ curvature theorem.

The following is the main result of this paper.

**Theorem 1.4** (The localised bounded $L^2$-curvature theorem). Let $(\Sigma, g, k)$ be a maximal initial data set such that $(\Sigma, g)$ is a compact complete\(^4\) smooth Riemannian manifold with boundary and assume that

$$\|\text{Ric}\|_{L^2(\Sigma)} < \infty, \quad \|k\|_{L^4(\Sigma)} < \infty, \quad \|\nabla k\|_{L^2(\Sigma)} < \infty, \quad \|\Theta\|_{L^4(\partial \Sigma)} < \infty, \quad r_{\text{vol}}(\Sigma, 1) > 0,$$

where $\Theta$ denotes the second fundamental form of $\partial \Sigma \subset \Sigma$. Then,

\(^4\)A smooth Riemannian manifold with boundary is called complete if it is complete as a metric space.
Theorem 1.4 in three steps.

1.4. Overview of the proof of Theorem 1.4. In this section, we sketch the proof of Theorem 1.4 in three steps.

1. **L^2-regularity.** There exists a radius
   \[ r = r(\|\text{Ric}\|_{L^2(\Sigma)}, \|k\|_{L^4(\Sigma)}, \|\nabla k\|_{L^2(\Sigma)}, \|\Theta\|_{L^4(\partial \Sigma)}, r_{vol}(\Sigma, 1) > 0, \]
   a time
   \[ T = T(\|\text{Ric}\|_{L^2(\Sigma)}, \|k\|_{L^4(\Sigma)}, \|\nabla k\|_{L^2(\Sigma)}, \|\Theta\|_{L^4(\partial \Sigma)}, r_{vol}(\Sigma, 1) > 0, \]
   and a constant
   \[ C = C(\|\text{Ric}\|_{L^2(\Sigma)}, \|k\|_{L^4(\Sigma)}, \|\nabla k\|_{L^2(\Sigma)}, \|\Theta\|_{L^4(\partial \Sigma)}, r_{vol}(\Sigma, 1) > 0, \]
   such that for every point \( p \in \Sigma \), the future domain of dependence \( \mathcal{D} \) of the geodesic ball \( B_g(p, r) \) admits a time function \( t \) whose level sets \( \Sigma_t \) are spacelike maximal hypersurfaces and foliate \( \mathcal{D} \) with \( \Sigma_0 = B_g(p, r) \subset \Sigma \), and the following control holds on \( 0 \leq t \leq T \),
   \[ \|\text{R}\|_{L^p_{T}L^2(\Sigma_t)} \leq C, \|k_i\|_{L^p_{T}L^4(\Sigma_t)} \leq C, \|\nabla k_i\|_{L^p_{T}L^2(\Sigma_t)} \leq C, \inf_{0 \leq t \leq T} r_{vol}(\Sigma_t, r) \geq \frac{1}{C}. \]

2. **Higher regularity.** In case of higher regularity, we have for \( m \geq 1 \), within the same time interval as in part (1), the higher derivative estimate
   \[ \sum_{|\alpha| \leq m} \|D^{(\alpha)}R\|_{L^p_{T}L^2(\Sigma_t)} \leq C_m \sum_{|i| \leq m} \left( \|\nabla^{(i)}\text{Ric}\|_{L^2(\Sigma)} + \|\nabla^{(i)}\nabla k\|_{L^2(\Sigma)} + 1 \right), \]
   where the constant \( C_m > 0 \) depends only on the previous \( C \) and \( m \).

1.4. **Construction of a cover of \( \Sigma \) by coordinate systems \( B(0, r_1) \) and \( B(0, r_2) \).**

In Section 4, we cover \( \Sigma \) by coordinate systems \( B(0, r_1) \) and \( B(0, r_2) \), where the radii \( r_1, r_2 > 0 \) are chosen small and depend only on low regularity bounds assumed in Theorem 1.4.

The cover is constructed such that its Lebesgue number\(^5\) is bounded from below and depends only on low regularity bounds assumed in Theorem 1.4.

The construction uses the existence of boundary harmonic coordinates on manifolds with boundary\(^7\) as black boxes. We remark that the coordinate systems \( B(0, r_1) \) cover an open neighbourhood of \( \partial \Sigma \) in \( \Sigma \), and the \( B(0, r_2) \) cover the rest of \( \Sigma \).

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\(^5\)Given a cover \( (C_i)_{i \in I} \) of \( \Sigma \), its Lebesgue number \( \ell \) is defined as the largest number such that for each point \( p \in \Sigma \), the geodesic ball \( B_g(p, \ell) \) is completely contained in \( C_i \) for some \( i \in I \).
(3) **Scaling to small data.** In Section 5, we use the scaling invariance of the Einstein equations to rescale for $\lambda > 0$

$$(B(0, r), g, k) \rightarrow (B(0, \lambda^{-1}r), g_\lambda, k_\lambda).$$

We show that for $\lambda > 0$ sufficiently small, depending only on low regularity bounds assumed in Theorem 1.4, the rescaled initial data is small in $H^2 \times H^1$, see Lemma 5.1.

The proof of Theorem 1.4 is then concluded in Section 6 by combining the above three steps.

1.5. **Overview of the paper.** In Section 2, we introduce notations and preliminaries. In Sections 3-5, we prove Steps (1)-(3) as outlined above. In Section 6, we conclude the proof of Theorem 1.4.

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2. **Notations, definitions and prerequisites**

In this section, we introduce notations, definitions and preliminary results that are used in this paper.

In this work, lowercase Latin indices run through $i, j = 1, 2, 3$. Greek indices run through $\mu, \nu = 0, 1, 2, 3$. We write $A \lesssim B$ if there exists a universal constant $C > 0$ such that $A \leq CB$.

Let the closed upper half-space of $\mathbb{R}^3$ be denoted by

$$\mathbb{H}^+ := \{ x \in \mathbb{R}^3 \mid x^3 \geq 0 \}.$$

For a point $x \in \mathbb{H}^+$ and a real number $r > 0$, let

$$B(x, r) := \{ y \in \mathbb{R}^3 \mid |x - y| < r \}, \quad B^+(x, r) := B(x, r) \cap \mathbb{H}^+.$$

**Definition 2.1** (Function spaces). Let $m \geq 1$ be an integer. Let $\Omega \subset \mathbb{R}^3$ be an open subset, and let $f$ be a scalar function on $\Omega$.

1. Let the norm

$$\| f \|_{H^m(\Omega)}^p := \sum_{|\alpha| \leq m} \| \partial^\alpha f \|_{L^2(\Omega)}^2,$$

and define the function space $H^m(\Omega)$ by

$$H^m(\Omega) := \{ f \in L^2(\Omega) : \| f \|_{H^m(\Omega)} < \infty \}.$$
(2) Let the norm
\[ \| f \|_{C^m(\Omega)} := \max_{|\beta| \leq m} \sup_{x \in \Omega} |\partial^\beta f|. \]
and define \( C^m(\Omega) \) to be the function space of \( m \)-times differentiable functions on \( \Omega \) equipped with norm \( \| \cdot \|_{C^m(\Omega)} \).

Here \( \alpha = (\alpha_1, \alpha_2, \alpha_3) \), \( \beta = (\beta_1, \beta_2, \beta_3) \in \mathbb{N}^3 \), and \( \partial^\alpha := \partial_1^{\alpha_1} \partial_2^{\alpha_2} \partial_3^{\alpha_3} \), \( |\alpha| := \sum_{i=1}^{3} |\alpha_i| \).

**Definition 2.2 (Tensor spaces).** Let \( \Omega \subset \mathbb{R}^3 \) be an open subset and let \( T \) be a tensor on \( \Omega \). For integers \( m \geq 1 \) and reals \( 1 < p < \infty \), define
\[ H^m(\Omega), L^p(\Omega) \text{ and } C^m(\Omega) \]
equipped with the natural norm, that is, for example, for an \((l, m)\)-tensor \( T \) on \( \Omega \),
\[ \|T\|_{H^m(\Omega)} := \sum_{i_1, \ldots, i_l=1}^{3} \sum_{j_1, \ldots, j_m=1}^{3} \|T_{i_1 \ldots i_l j_1 \ldots j_m}\|_{H^m(\Omega)}, \]
where \( T_{i_1 \ldots i_l j_1 \ldots j_m} \) denote the coordinate components.

**Definition 2.3 (Volume radius at scale \( r \)).** Let \((M, g)\) be a Riemannian 3-manifold with boundary. For a real \( r > 0 \) and a point \( p \in M \), the volume radius at scale \( r \) at \( p \) is defined as
\[ r_{vol}(r, p) := \inf_{r' < r} \frac{\text{vol}_g(B_g(p, r'))}{\frac{4\pi}{3}(r')^3}. \]
The volume radius of \((M, g)\) at scale \( r \) is defined as
\[ r_{vol}(M, r) := \inf_{p \in M} r_{vol}(r, p). \]

**Definition 2.4 (Lebesgue number).** Let \((M, g)\) be a Riemannian manifold. Given a covering \( (C_i)_{i=1}^{N} \) of \( M \), the Lebesgue number \( \ell \) is defined to be the largest real number such that for each point \( p \in M \), there is an \( i \in \{1, \ldots, N\} \) such that \( B_g(p, \ell) \subset C_i \).

3. **The localised bounded \( L^2 \)-curvature theorem for small data on \( B(0, 1) \)**

In this section, we prove the following result.

**Proposition 3.1 (Localised bounded \( L^2 \) curvature theorem for small data on \( B(0, 1) \)).** Let \((\bar{g}, \bar{k})\) be maximal initial data for the Einstein vacuum equations on \( B(0, 1) \subset \mathbb{R}^3 \) and assume that for some \( \varepsilon > 0 \),
\[ \|\bar{g} - \varepsilon\|_{H^2(B(0,1))} + \|\bar{k}\|_{H^1(B(0,1))} < \varepsilon. \]
Let \((\mathcal{D}, \mathbf{g})\) be the solution to the Einstein vacuum equations in the future domain of dependence \(\mathcal{D}\) of \(B(0, 1)\), and let \(t\) be a time function in \(\mathcal{D}\) such that its level sets \(\Sigma_t\) are spacelike maximal hypersurfaces and foliate \(\mathcal{D}\) with \(\Sigma_0 = B(0, 1)\). Then, the following holds.

1. **\(L^2\)-regularity.** There is an \(\varepsilon_0 > 0\) small such that if \(\varepsilon < \varepsilon_0\), then the following control holds on \(0 \leq t \leq 1/2\),
\[
\|\text{Ric}\|_{L^2_t L^2(\Sigma_t)} \lesssim \varepsilon, \quad \|k_i\|_{L^2_t L^4(\Sigma_t)} \lesssim \varepsilon, \quad \|\nabla k_i\|_{L^2_t L^2(\Sigma_t)} \lesssim \varepsilon, \quad \inf_{0 \leq t \leq 1/2} r_{\text{vol}}(\Sigma_t, 1) \geq 1/8.
\]

2. **Higher regularity.** Let \(m \geq 1\) be an integer. In case of higher regularity, we have the following higher regularity estimate on \(0 \leq t \leq 1/2\),
\[
\sum_{|\alpha| \leq m} \|D^{(\alpha)} \mathbf{R}\|_{L^\infty_t L^2(\Sigma_t)} \leq C_m \left( \|\bar{g} - e\|_{\mathcal{H}^{m+2}(B(0,1))} + \|\bar{k}\|_{\mathcal{H}^{m+1}(B(0,1))} \right),
\]
where the constant \(C_m > 0\) depends on \(m\).

The proof of Proposition 3.1 is based on the following two literature results.

**Theorem 3.2** (An extension procedure for the constraint equations, [6]). Let \((\bar{g}, \bar{k})\) be maximal initial data for the Einstein vacuum equations on \(B(0, 1) \subset \mathbb{R}^3\) and assume that for some \(\varepsilon > 0\) it holds that
\[
\|\bar{g}_{ij} - e_{ij}\|_{\mathcal{H}^{2}(B(0,1))} + \|\bar{k}_{ij}\|_{\mathcal{H}^{1}(B(0,1))} < \varepsilon.
\]
Then, the following holds.

1. **\(L^2\)-regularity.** There is a universal \(\varepsilon_0 > 0\) such that if \(\varepsilon < \varepsilon_0\), then there exists asymptotically flat maximal initial data \((g, k)\) on \(\mathbb{R}^3\) with \((g, k)|_{B(0,1)} = (\bar{g}, \bar{k})\) and
\[
\|g - e\|_{\mathcal{H}^{-1/2}_{-1/2}(\mathbb{R}^3)} + \|k\|_{\mathcal{H}^{-1/2}_{-3/2}(\mathbb{R}^3)} \lesssim \|\bar{g} - e\|_{\mathcal{H}^{2}(B(0,1))} + \|\bar{k}\|_{\mathcal{H}^{1}(B(0,1))}.
\]

2. **Higher regularity.** In case of higher regularity, we have for integers \(m \geq 1\) the following higher regularity estimates,
\[
\|g - e\|_{\mathcal{H}^{m+2}_{-1/2}(\mathbb{R}^3)} + \|k\|_{\mathcal{H}^{m+1}_{-3/2}(\mathbb{R}^3)} \leq C_m \left( \|\bar{g} - e\|_{\mathcal{H}^{m+2}(B(0,1))} + \|\bar{k}\|_{\mathcal{H}^{m+1}(B(0,1))} \right),
\]
where the constant \(C_m > 0\) depends on \(m\).

Here \(\mathcal{H}^{m+2}_{-1/2}(\mathbb{R}^3)\) and \(\mathcal{H}^{m+1}_{-3/2}(\mathbb{R}^3)\) denote Sobolev spaces of tensors equipped with weights \(-1/2\) and \(-3/2\), respectively, corresponding to the asymptotic flatness of the initial data, see [6] for details.

**Theorem 3.3** (The bounded \(L^2\)-curvature theorem for small data, [8]). Let \((\mathcal{M}, \mathbf{g})\) an asymptotically flat solution to the Einstein vacuum equations together with a maximal foliation by space-like hypersurfaces \(\Sigma_t\) defined as level hypersurfaces of a time function \(t\). Assume that the initial slice \((\Sigma_0, g, k)\) is such that \(\Sigma_0 \simeq \mathbb{R}^3\) and
\[
\|\text{Ric}\|_{L^2(\Sigma_0)} \leq \varepsilon, \quad \|\nabla k\|_{L^2(\Sigma_0)} \leq \varepsilon \quad \text{and} \quad r_{\text{vol}}(\Sigma_0, 1) \geq \frac{1}{2}.
\]
Then,

(1) **\(L^2\)-regularity.** There exists a small universal constant \(\varepsilon_0 > 0\) such that if \(0 < \varepsilon < \varepsilon_0\), then the following control holds on \(0 \leq t \leq 1\),

\[
\| R \|_{L^\infty_t L^2(\Sigma_t)} \lesssim \varepsilon, \quad \| k \|_{L^\infty_t L^2(\Sigma_t)} \lesssim \varepsilon, \quad \| \nabla k \|_{L^\infty_t L^2(\Sigma_t)} \lesssim \varepsilon, \quad \inf_{0 \leq t \leq 1} r_{\text{vol}}(\Sigma_t, 1) \geq \frac{1}{4}.
\]

(2) **Higher regularity.** In case of higher regularity, we have for integers \(m \geq 1\) the following higher regularity estimates on \(0 \leq t \leq 1\),

\[
\sum_{|\alpha| \leq m} \| D^{(\alpha)} R \|_{L^\infty_t L^2(\Sigma_t)} \lesssim \sum_{|i| \leq m} \left( \| \nabla^{(i)} \text{Ric} \|_{L^2(\Sigma_0)} + \| \nabla^{(i)} \nabla k \|_{L^2(\Sigma_0)} \right).
\]

For the rest of this section, we prove Proposition 3.1. For \(\varepsilon > 0\) sufficiently small, the initial data \((\bar{g}, \bar{k})\) on \(B(0, 1)\) can be extended by Theorem 3.2 to an asymptotically flat, maximal initial data set \((g, k)\) on \(\mathbb{R}^3\) such that \((g, k)|_{B_1} = (\bar{g}, \bar{k})\) and

\[
\| g - e \|_{H^2_{-1/2}(\mathbb{R}^3)} + \| k \|_{H^1_{-3/2}(\mathbb{R}^3)} \lesssim \varepsilon.
\]

In particular, for \(\varepsilon > 0\) sufficiently small, the extension \((g, k)\) on \(\mathbb{R}^3\) satisfies the assertions of Theorem 3.3.

Let therefore \((\mathcal{M}, g)\) denote the future development of \((\mathbb{R}^3, g, k)\) and \(t\) be the time function in \(\mathcal{M}\) such that its level sets \(\Sigma_t\) are spacelike maximal hypersurfaces with \(\Sigma_0 = \mathbb{R}^3\). By Theorem 3.3, we have on \(0 \leq t \leq 1\),

\[
\| \text{Ric}_t \|_{L^\infty_t L^2(\Sigma_t)} \lesssim \varepsilon, \quad \| k_t \|_{L^\infty_t L^2(\Sigma_t)} \lesssim \varepsilon, \quad \| \nabla k_t \|_{L^\infty_t L^2(\Sigma_t)} \lesssim \varepsilon, \quad \inf_{0 \leq t \leq 1} r_{\text{vol}}(\Sigma_t, 1) \geq 1/4,
\]

where \(\text{Ric}_t\) denotes the Ricci curvature of the induced metric \(g_t\) and \(k_t\) the second fundamental form of \(\Sigma_t\).

By restricting to the domain of dependence \(\mathcal{D}\) of \(B(0, 1)\), it follows that for \(0 \leq t \leq 1/2\),

\[
\| \text{Ric}_t \|_{L^\infty_t L^2(\Sigma_t \cap \mathcal{D})} \lesssim \varepsilon, \quad \| k_t \|_{L^\infty_t L^2(\Sigma_t \cap \mathcal{D})} \lesssim \varepsilon, \quad \| \nabla k_t \|_{L^\infty_t L^2(\Sigma_t \cap \mathcal{D})} \lesssim \varepsilon, \quad \inf_{0 \leq t \leq 1/2} r_{\text{vol}}(\Sigma_t \cap \mathcal{D}, 1) \geq 1/8.
\]

We remark that the control of the volume radius follows as in the proof of Theorem 3.3 by a control of \(g_{\mu\nu}\) in \(C^0\); for details we refer the reader to the estimates in Section 4 of [14].
It remains to prove the higher regularity estimate of Proposition 3.1. By the higher regularity estimates of Theorems 3.2 and 3.3, we have, for integers $m \geq 1$, on $0 \leq t \leq 1/2$,

$$
\sum_{|\alpha| \leq m} \| D^{(\alpha)} R \|_{L^\infty L^2(\Sigma, t)} \lesssim \sum_{|i| \leq m} \left( \| \nabla^{(i)} \text{Ric} \|_{L^2(\mathbb{R}^3)} + \| \nabla^{(i)} k \|_{L^2(\mathbb{R}^3)} \right)
$$

$$
\lesssim \| g - e \|_{\mathcal{H}^m+2(\mathbb{R}^3)} + \| k \|_{\mathcal{H}^{m+1}(\mathbb{R}^3)}
$$

$$
\leq C_m \left( \| \bar{g} - e \|_{\mathcal{H}^m+2(B(0,1))} + \| k \|_{\mathcal{H}^{m+1}(B(0,1))} \right),
$$

where the constant $C_m > 0$ depends on $m$. Restriction to the future domain of dependence $\mathcal{D}$ of $B(0,1)$ then proves the higher regularity estimates of Proposition 3.1. This finishes the proof of Proposition 3.1.

4. Construction of the cover of $\Sigma$ by coordinate systems

In this section, we cover of $\Sigma$ by coordinate systems $\overline{B(0, r_1)}$ and $B(0, r_2)$, where the radii $r_1, r_2 > 0$ are small, depending only on the low regularity geometric bounds assumed in Theorem 1.4. In particular, the radii $r_1$ and $r_2$ are sufficiently small such that subsequent rescaling to the unit ball leads to small data, see Sections 5 and 6.

At first, we construct coordinate systems $\overline{B(0, r_1)}$ near the boundary $\partial \Sigma$ of $\Sigma$. Then, we cover the rest of $\Sigma$ by coordinate systems $B(0, r_2)$. Finally, we prove that the Lebesgue constant $\ell$ of the constructed cover of $\Sigma$ is bounded from below.

The construction is based on the following existence result from [7].

**Theorem 4.1** (Existence of regular coordinate systems). Let $(M, g)$ be a smooth Riemannian 3-manifold with boundary such that

$$
\| \text{Ric} \|_{L^2(M)} < \infty, \| \Theta \|_{L^1(\partial M)} < \infty, r_{\text{vol}}(M, 1) > 0.
$$

Then, the following holds.

1. **$L^2$-regularity.** There is $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$, there exists a radius $r = r(\| \text{Ric} \|_{L^2(M)}, \| \Theta \|_{L^1(\partial M)}, r_{\text{vol}}(M, 1), \varepsilon) > 0$

such that for every $p \in \Sigma$, there is a chart $\varphi : B^+(x, r) \to U \subset M$ with $\varphi(x) = p$

such that

$$
(1 - \varepsilon)e_{ij} \leq g_{ij} \leq (1 + \varepsilon)e_{ij}
$$

(4.1)

and

$$
r^{-1/2}\| \partial g \|_{L^2(B^+(x,r))} + r^{1/2}\| \partial^2 g \|_{L^2(B^+(x,r))} \leq \varepsilon.
$$
(2) Higher regularity. In case of higher regularity, we have for integers \( m \geq 1 \),
\[
\|g\|_{H^{m+2}(B^+(x,r))} \leq C_r \sum_{i=0}^{m} \|\nabla^{(m)} \text{Ric}\|_{L^2(M)} + C_{r,m} \varepsilon.
\]

Remark 4.2. The bound (4.1) allows to compare geodesic length on \( M \) with coordinate length in the chart \( \varphi \). In particular, it holds that for \( r' < r \) and \( \varepsilon > 0 \) small,
\[
B_g(p, (1 - \varepsilon)r') \subset \varphi(B^+(x, r')) \subset B_g(p, (1 + \varepsilon)r'),
\]
\[
\varphi(B^+(x, (1 - \varepsilon)r')) \subset B_g(p, r') \subset \varphi(B^+(x, (1 + \varepsilon)r')).
\]

Construction of coordinate systems \( B(0, r_1) \) near \( \partial \Sigma \). First, for a given point \( p \in \partial \Sigma \), we construct a coordinate system \( B(0, r_1) \) in \( \Sigma \) containing \( p \). Then we pick points \( (p_i)_{i=1}^N \subset \partial \Sigma \) such that the corresponding constructed coordinate systems \( B(0, r_1) \) cover an open neighbourhood of \( \partial \Sigma \) in \( \Sigma \).

Let thus \( p \in \partial \Sigma \), and let \( \varepsilon > 0 \) small to be determined. By Theorem 4.1, there is a radius
\[
r = r(\|\text{Ric}\|_{L^2(M)}, \|\Theta\|_{L^4(\partial M), r \text{vol}(M, 1, \varepsilon)} > 0
\]
and a chart \( \varphi : B^+(0, r) \to U \subset \Sigma \) with \( \varphi(0) = p \) and
\[
(1 - \varepsilon)e_{ij} \leq g_{ij} \leq (1 + \varepsilon)e_{ij},
\]
\[
r^{-1/2} \|\partial g\|_{L^2(B^+(x,r))} + r^{1/2} \|\partial^2 g\|_{L^2(B^+(x,r))} \leq \varepsilon, \tag{4.2}
\]
and in case of higher regularity,
\[
\|g\|_{H^{m+2}(B^+(x,r))} \leq C_r \sum_{i=0}^{m} \|\nabla^{(m)} \text{Ric}\|_{L^2(M)} + C_{r,m} \varepsilon. \tag{4.3}
\]

We define the radius \( r_1 > 0 \) by
\[
2r_1 = \min \left\{ r, \frac{\varepsilon^4}{\|k\|_{L^4(M)} + \|\nabla k\|_{L^2(M)}} \right\},
\]
and let the chart
\[
\varphi_p : B^+(0, 2r_1) \to U_p \subset \Sigma
\]
be defined as the restriction of \( \varphi \) to \( B^+(0, 2r_1) \).

The following technical lemma is used to put a coordinate system \( B(0, r_1) \) into \( B^+(0, 2r_1) \) such that in addition it covers an open neighbourhood of the origin in \( \{x^3 = 0\} \).

Lemma 4.3. There is \( \delta_0 > 0 \) such that for all reals \( 0 < \delta < \delta_0 \), there is a smooth diffeomorphism
\[
\Psi_\delta : B(0, r_1) \to P_\delta \subset B^+(0, 2r_1)
\]
such that
\[ B^+(0, r_1 \delta) \subset \subset P_\delta, \]
\[ B^+(0, r_1 \delta) \subset \subset P_\delta, \] (4.4)
and for every integer \( m \geq 0 \),
\[ \| D\Psi_\delta - I \|_{C^m(B(0,r_1))} + \| D(\Psi_\delta)^{-1} - I \|_{C^m(P_\delta)} \leq C_{r_1,m}\delta, \] (4.5)
where \( I \) denotes the identity matrix.

Proof. The diffeomorphism \( \Psi_\delta \) is constructed by smoothly deforming the ball \( B((0,0,r_1), r_1) \subset B^+(0,2r_1) \). Details are left to the reader, see the next figure.

\[ \text{Figure 1. The closed set } P_\delta \subset B^+(0,2r_1) \text{ with smooth boundary is depicted as the shaded region.} \]

Using \( \Psi_\delta \) of Lemma 4.3, we define the chart \( \varphi'_p \) as
\[ \varphi'_p = \varphi \circ \Psi_\delta : \overline{B(0,r_1)} \to V \subset \Sigma. \]
Letting \( \delta > 0 \) sufficiently small depending on \( \varepsilon > 0 \), it holds by (4.2), (4.3) and (4.5) that for \( g' = (\varphi'_p)^* \),
\[ (1 - 2\varepsilon)e_{ij} \leq g'_{ij} \leq (1 + 2\varepsilon)e_{ij} \text{ on } \overline{B(0,r_1)}, \]
\[ (r_1)^{1/2}\| \partial^2 g' \|_{L^2(B(0,r_1))} \lesssim \varepsilon, \]
and in case of higher regularity, for integers \( m \geq 1 \), for \( \varepsilon > 0 \) sufficiently small,
\[ \| g' \|_{H^{m+2}(B(0,r_1))} \leq C_{r_1,m} \left( \sum_{i=0}^{m} \| \nabla^{(m)} \text{Ric} \|_{L^2(M)} + 1 \right). \]

Lemma 4.4. The g-area of \( \varphi_p(P_\delta) \cap \partial \Sigma \) is bounded from below by
\[ \text{area}_g(\varphi_p(P_\delta) \cap \partial \Sigma) \geq (1 - \varepsilon) (r_1 \delta)^2. \]
Proof. By the fact that $B^+(0,r_1\delta) \subset P_\delta$, see (4.4), and (4.2), we have
\[
\text{area}_g(\varphi_p(P_\delta) \cap \partial\Sigma) \geq \text{area}_g(\varphi_p(B^+(0,r_1\delta))) \geq (1-\varepsilon)(r_1\delta)^2.
\]
□

We now turn to pick points $(p_i)_{i=1}^{N_1} \subset \Sigma$. Let the integer $N_1 \geq 1$ and $(p_i)_{i=1}^{N_1} \subset \partial\Sigma$ be such that
\[
\partial\Sigma \subset \bigcup_{i=1}^{N_1} \varphi_{p_i}(B^+(0,\frac{r_1\delta}{2})) = \bigcup_{i=1}^{N_1} \varphi'_{p_i}(B(0,r_1)).
\]
(4.6)

By Lemma 4.4, the smallest necessary integer $N_1 \geq 1$ for (4.6) depends only on area$_g(\partial\Sigma)$, $r_1$, $\delta$ and $\varepsilon$. Define the sets $V_i$ as
\[
V_i = \varphi'_{p_i}(B(0,r_1)).
\]
(4.7)

In the next lemma, we use the following definition.

Definition 4.5. For every real $s > 0$, define the annulus $A_s$ by
\[
A_s := \{ p \in \Sigma : d_g(p,\partial\Sigma) < s \} \subset \Sigma,
\]
where $d_g$ denotes the geodesic distance.

Lemma 4.6. For $\varepsilon > 0$ sufficiently small, the constructed $(V_i)_{i=1}^{N_1}$ satisfy the following.
1. $\partial\Sigma \subset \bigcup_{i=1}^{N_1} V_i$.
2. For all $i \in \{1,\ldots,N_1\}$, $B_g(p_i,(1-\varepsilon)r_1\delta) \subset V_i$.
3. $A_{\frac{r_1\delta}{4}} \subset \bigcup_{i=1}^{N_1} V_i$.
4. The Lebesgue number$^6$ $\ell$ of the cover $(V_i)_{i=1}^{N_1}$ of $A_{\frac{r_1\delta}{4}}$ is bounded from below by
\[
\ell \geq \frac{r_1\delta}{16}.
\]

Proof. Proof of (1). By (4.4), $B^+(0,\frac{r_1\delta}{2}) \subset P_\delta$, so together with (4.6), we have
\[
\partial\Sigma \subset \bigcup_{i=1}^{N_1} \varphi_{p_i}(B^+(0,\frac{r_1\delta}{2})) \subset \bigcup_{i=1}^{N_1} \varphi_{p_i}(P_\delta) = \bigcup_{i=1}^{N_1} \varphi'_{p_i}(B(0,r_1)) = \bigcup_{i=1}^{N_1} V_i.
\]

Proof of (2). By Remark 4.2, (4.2), (4.4) and (4.7), we have
\[
B_g(p_i,(1-\varepsilon)r_1\delta) \subset \varphi_{p_i}(B^+(0,(1-\varepsilon^2)r_1\delta)) \subset \varphi_{p_i}(B^+(0,r_1\delta)) \subset V_i.
\]

Proof of (3). Let $p \in A_{\frac{r_1\delta}{4}}$. By definition of $A_{\frac{r_1\delta}{4}}$, there is a point $p' \in \partial\Sigma$ such that
\[
d_g(p,p') < \frac{r_1\delta}{4}.
\]

$^6$Given a covering $(C_i)_{i=1}^{N}$ of $M$, the Lebesgue number $\ell$ is defined to be the largest real number such that for each point $p \in M$, there is an $i \in \{1,\ldots,N\}$ such that $B_g(p,\ell) \subset C_i$. 
Further, by (4.6) and Remark 4.2, there is a $p_i \in \partial \Sigma$ such that

$$d_g(p', p_i) < (1 + \varepsilon) \frac{r_1 \delta}{2}.$$  

By the above two, using the triangle inequality,

$$d_g(p_i, p) \leq d_g(p_i, p') + d_g(p', p) < (1 + \varepsilon) \frac{3r_1 \delta}{4}.$$  

Consequently, using (2) of this lemma, we have for $\varepsilon > 0$ sufficiently small,

$$p \in B_g \left(p_i, (1 + \varepsilon) \frac{3r_1 \delta}{4}\right) \subset B_g (p_i, (1 - \varepsilon) r_1 \delta) \subset \varphi_{p_i}(B^+(0, r_1 \delta)) \subset V_i \subset \bigcup_{i=1}^{N_1} V_i.$$  

**Proof of (4).** Let $p \in A_{\frac{r_1 \delta}{4}}$, and let $\bar{p} \in B_g (p, \frac{r_1 \delta}{4})$. By definition of $A_{\frac{r_1 \delta}{4}}$, there is a point $p' \in \partial \Sigma$ such that

$$d_g(p, p') < \frac{r_1 \delta}{4}.$$  

Further, by (4.6) and Remark 4.2, there is a $p_i \in \partial \Sigma$ such that

$$d_g(p', p_i) < (1 + \varepsilon) \frac{r_1 \delta}{2}.$$  

Therefore, by using the triangle inequality,

$$d_g(p_i, \bar{p}) \leq d_g(p_i, p') + d_g(p', p) + d_g(p, \bar{p}) < (1 + \varepsilon) \frac{r_1 \delta}{2} + \frac{r_1 \delta}{4} + \frac{r_1 \delta}{16} < (1 + \varepsilon) \frac{15r_1 \delta}{16}.$$  

Consequently, using (2) of this lemma, for $\varepsilon > 0$ sufficiently small,

$$\bar{p} \in B_g \left(p_i, (1 + \varepsilon) \frac{15r_1 \delta}{16}\right) \subset B_g (p_i, (1 - \varepsilon) r_1 \delta) \subset V_i.$$  

This finishes the proof of Lemma 4.6. 

**Construction of coordinate balls** $B(0, r_2)$ **away from** $\partial \Sigma$. First, for a given $p \in \Sigma \setminus A_{\frac{r_1 \delta}{4}}$, we construct a coordinate system $B(0, r_2)$ in $\Sigma$. Then we pick points $(p_i)_{i=N_1+1}^{N_2} \subset \Sigma$ such that the corresponding constructed coordinate systems cover $\Sigma \setminus A_{\frac{r_1 \delta}{4}}$.

Let thus $p \in \Sigma \setminus A_{\frac{r_1 \delta}{4}}$, and let $\varepsilon > 0$ small to be determined. By Theorem 4.1, there is a radius

$$r = r(\|\text{Ric}\|_{L^2(\Sigma)} , \|\Theta\|_{L^4(\partial \Sigma)} , r_{vol}(\Sigma, 1), \varepsilon) > 0$$.
and a chart \( \varphi : B^+(0, r) \to U \subset \Sigma \) with \( \varphi(0) = p \) and
\[
(1 - \varepsilon)e_{ij} \leq g_{ij} \leq (1 + \varepsilon)e_{ij},
\]
\[
r^{-1/2}\|\partial g\|_{L^2(B^+(x, r))} + r^{1/2}\|\partial^2 g\|_{L^2(B^+(x, r))} \leq \varepsilon,
\]
and in case of higher regularity,
\[
\|g\|_{H^{m+2}(B^+(x, r))} \leq C_r \sum_{i=0}^{m} \|\nabla^{(m)} Ric\|_{L^2(M)} + C_{r,m}\varepsilon.
\]

We define the radius \( r_2 > 0 \) by
\[
r_2 = \min \left\{ r, \frac{r_1\delta}{8}, \frac{\varepsilon^4}{\|k\|_{L^2(M)} + \|\nabla k\|_{L^2(M)}^2} \right\}.
\]

For \( \varepsilon > 0 \) sufficiently small, by Remark 4.2 and the fact that \( \text{dist}_g(p, \partial \Sigma) > r_2 \), the chart \( \varphi_p : B(0, r_2) \to U \subset \Sigma \), defined as the restriction of \( \varphi \) to \( B(0, r_2) \), is well-defined. Moreover, by (4.8) and (4.9),
\[
(1 - \varepsilon)e_{ij} \leq g_{ij} \leq (1 + \varepsilon)e_{ij},
\]
\[
(r_2)^{1/2}\|\partial^2 g\|_{L^2(B(0, r_2))} \leq \varepsilon,
\]
and in case of higher regularity, for integers \( m \geq 1 \),
\[
\|g\|_{H^{m+2}(B(0, r_2))} \leq C_{r_2} \sum_{i=0}^{m} \|\nabla^{(m)} Ric\|_{L^2(M)} + C_{r_2,m}\varepsilon.
\]

We now turn to pick the points \((p_i^j)_{i=N_1+1}^{N_2}\). Let the integer \( N_2 \geq 0 \) and \((p_i)_{i=N_1+1}^{N_2}\) be such that
\[
\Sigma \setminus \mathcal{A}_{r_2\delta} \subset \bigcup_{i=1}^{N} \varphi_{p_i}(B\left(0, \frac{r_2}{2}\right)).
\]
The integer \( N_2 \geq 1 \) depends only on area\(_g\)(\(\partial \Sigma\)), \(\text{vol}_g\Sigma\), \(\varepsilon\) and the low regularity geometric bounds assumed in Theorem 1.4. Define the sets \( U_i \) as
\[
U_i := \varphi_{p_i}(B(0, r_2)).
\]

We have the next result. Its proof is similar to Lemma 4.6 and left to the reader.

**Lemma 4.7.** For \( \varepsilon > 0 \) sufficiently small, the constructed \((U_i)_{i=N_1+1}^{N_2}\) satisfy the following.

- \( \Sigma \setminus \mathcal{A}_{r_2\delta} \subset \bigcup_{i=N_1+1}^{N_2} U_i \).
- The Lebesgue number \( \ell \) of the cover \((U_i)_{i=N_1+1}^{N_2}\) of \( \Sigma \setminus \mathcal{A}_{r_2\delta} \) is bounded by
\[
\ell \geq \frac{r_2}{16}.
\]
To summarise the above, we have constructed a cover of $\Sigma$ by coordinate systems $B(0, r_1)$ and $B(0, r_2)$.

**Lemma 4.8.** The Lebesgue number of the constructed cover $(U_i)_{i=1}^{N_1}, (V_i)_{i=N_1+1}^{N_2}$ of $\Sigma$ is bounded by

$$\ell \geq \frac{r_2}{16}.$$  

**Proof.** For every point $p \in \Sigma$, either $p \in A_{16}$ or $p \in \Sigma \setminus A_{16}$. Therefore, by construction of $(U_i)_{i=1}^{N_1}, (V_i)_{i=1}^{N_2}$ and the definition of $r_2 \leq r_1$, see (4.10), there exists an $i \in \{1, \ldots, N_2\}$ such that either $B_g(p, \frac{r_2}{16}) \subset V_i$ or $B_g(p, \frac{r_2}{16}) \subset U_i$. $\square$

5. **The scaling to small data**

The Einstein equations (1.1) are invariant under the scaling

$$g(t, x) \rightarrow g_\lambda(t, x) := g(\lambda t, \lambda x), \quad (5.1)$$

where $\lambda > 0$ is a real number. As a consequence, the constraint equations are invariant under the scaling

$$(g, k)(x) \rightarrow (g_\lambda, k_\lambda)(x) := (g, \lambda^4 k)(\lambda x). \quad (5.2)$$

The main result of this section is the following.

**Lemma 5.1** (Scaling to small data). Let $r > 0$ and let $(B(0, r), g, k)$ be initial data such that for some $0 < \varepsilon < 1$,

$$(1 - \varepsilon)e_{ij} \leq g_{ij} \leq (1 + \varepsilon)e_{ij}$$

$$r^{1/2}\|\partial^2 g\|_{L^2(B(0, r))} \leq \varepsilon.$$

Then, for

$$\lambda = \min \left\{ r, \varepsilon^4 \left[ \|k\|^4_{L^4(B(0, r))} + \|\nabla k\|^2_{L^2(B(0, r))} \right] \right\},$$

the rescaled initial data set $(B(0, \lambda^{-1}r), g_\lambda, k_\lambda)$ satisfies

$$\|g_\lambda - e\|_{H^2(B(0, \lambda^{-1}r))} + \|k\|_{H^4(B(0, \lambda^{-1}r))} \lesssim \varepsilon.$$  

In case of higher regularity, we have for integers $m \geq 0$

$$\|g_\lambda - e\|_{H^{m+2}(B(0, \lambda^{-1}r))} \leq C_{\lambda, r} \sum_{i=0}^{m} \|\nabla^{(m)}Ric\|_{L^2(M)} + C_{\lambda, r, m} \varepsilon.$$
Proof. By the scaling (5.2), we have
\[ \|k(\lambda)\|_{L^2(B(0,\lambda^{-1}r))} \lesssim \|k(\lambda)\|_{L^4(B(0,r))} \lesssim \lambda^{1/4}\|k\|_{L^4(B(0,r))} \lesssim \varepsilon, \]
\[ \|\nabla k(\lambda)\|_{L^2(B(0,\lambda^{-1}r))} = \lambda^{1/2}\|\nabla k\|_{L^2(B(0,r))} \leq \varepsilon, \]  
(5.3)
\[ \|\partial^2 g(\lambda)\|_{L^2(B(0,\lambda^{-1}r))} = \lambda^{1/2}\|\partial^2 g\|_{L^2(B(0,r))} \leq \lambda^{1/2} \frac{\varepsilon}{r^{1/2}} \leq \varepsilon. \]

Moreover, we have on \(B(0,\lambda^{-1}r)\)
\[ (1 - \varepsilon)e_{ij} \leq (g(\lambda))_{ij} \leq (1 + \varepsilon)(g(\lambda))_{ij}. \]  
(5.4)
It follows from (5.3) and (5.4) by interpolation that
\[ \|g(\lambda) - e\|_{L^2(B(0,\lambda^{-1}r))} + \|\partial g(\lambda)\|_{L^2(B(0,\lambda^{-1}r))} \lesssim \varepsilon. \]

The higher regularity estimate is similar and left to the reader. This finishes the proof of Lemma 5.1. \(\square\)

6. The conclusion of the proof of Theorem 1.4

In this section we conclude the proof of Theorem 1.4 by combining the results of the previous sections. We first have the following proposition.

Proposition 6.1. Let \((\Sigma, g, k)\) be a maximal initial data set such that \((\Sigma, g)\) is a compact Riemannian manifold with boundary and
\[ \|\text{Ric}\|_{L^2(\Sigma)} < \infty, \|\Theta\|_{L^4(\partial\Sigma)} < \infty, \|k\|_{L^4(\Sigma)} < \infty, \|\nabla k\|_{L^2(\Sigma)} < \infty, r_{\text{vol}}(\Sigma, 1) > 0. \]

Let
\[ (\varphi_i : B(0,r_1) \to V_i \subset \Sigma)_{i=1}^{N_1}, (\varphi_i : B(0,r_2) \to U_i \subset \Sigma)_{i=N_1+1}^{N_2} \]
be the constructed cover of \(\Sigma\), see Section 4. Then, the future domains of dependence \(\mathcal{D}(V_i)\) and \(\mathcal{D}(U_i)\) are each foliated by spacelike maximal hypersurfaces \(\Sigma_t\) defined as level sets of a time function \(t\) with \(\Sigma_0 = U_i\) (or \(\Sigma_0 = V_i\)) such that

1. \(L^2\)-regularity. there is a constant
\[ C = C(\|\text{Ric}\|_{L^2(\Sigma)}, \|k\|_{L^4(\Sigma)}, \|\nabla k\|_{L^2(\Sigma)}, \|\Theta\|_{L^4(\partial\Sigma)}, r_{\text{vol}}(\Sigma, 1)) > 0 \]
and a time
\[ T = T(\|\text{Ric}\|_{L^2(\Sigma)}, \|k\|_{L^4(\Sigma)}, \|\nabla k\|_{L^2(\Sigma)}, \|\Theta\|_{L^4(\partial\Sigma)}, r_{\text{vol}}(\Sigma, 1)) > 0 \]
such that on \(0 \leq t \leq T\), we have
\[
\|\text{Ric}\|_{L^\infty_t L^2(\Sigma_t)} \leq C, \|k_t\|_{L^\infty_t L^4(\Sigma_t)} \leq C, \inf_{0 \leq t \leq T} r_{\text{vol}}(\Sigma_t, 1) \geq \frac{1}{C}.
\] (6.1)

(2) **Higher regularity.** In case of higher regularity, for integers \(m \geq 0\), on \(0 \leq t \leq T\),
\[
\sum_{|\alpha| \leq m} \|D^{(\alpha)}R\|_{L^\infty_t L^2(\Sigma_t)} \leq C_m \left( \sum_{|i| \leq m} \|\nabla^{(i)}\text{Ric}\|_{L^2(\Sigma)} + \|\nabla^{(i)}k\|_{L^2(\Sigma)} + 1 \right).
\] (6.2)

The above proposition implies the proof of Theorem 1.4 as follows.

**Proof of Theorem 1.4.** Let \((\Sigma, g, k)\) be maximal initial data such that \((\Sigma, g)\) is a compact Riemannian manifold with boundary such that
\[
\|\text{Ric}\|_{L^2(\Sigma)} < \infty, \|\Theta\|_{L^4(\partial\Sigma)} < \infty, \|k\|_{L^4(\Sigma)} < \infty, \|\nabla k\|_{L^4(\Sigma)} < \infty, r_{\text{vol}}(\Sigma, 1) > 0.
\]

Let \(V_i^{N_i}\) and \(U_i^{N_i+1}\) be the cover of \(\Sigma\) constructed in Section 4.

On the one hand, by Proposition 6.1, it follows that the future domains of dependence of \(V_i\) and \(U_i\) are controlled with quantitative bounds (6.1) and (6.2).

On the other hand, by the construction of the cover, see Lemma 4.8, there is
\[
\bar{r} = \bar{r}(\|\text{Ric}\|_{L^2(\Sigma)}, \|k\|_{L^4(\Sigma)}, \|\nabla k\|_{L^2(\Sigma)}, \|\Theta\|_{L^4(\partial\Sigma)}, r_{\text{vol}}(\Sigma, 1)) > 0
\]
such that for every \(p \in \Sigma\), there is a \(V_i\) or \(U_i\) such that
\[
B_g(p, \bar{r}) \subset V_i \text{ or } B_g(p, \bar{r}) \subset U_i.
\]

In particular, it holds that the future domain of dependence
\[
\mathcal{D}(B_g(p, r)) \subset \mathcal{D}(U_i) \text{ or } \mathcal{D}(B_g(p, r)) \subset \mathcal{D}(V_i),
\]
and thus is foliated by spacelike maximal hypersurfaces \(\Sigma_t\) defined as level sets of a time function \(t\) with bounds (6.1) and (6.2). We remark that the control of the volume radius follows as in the proof of Theorem 3.3 by a control of \(g_{\mu\nu}\) in \(C^0\). We refer the reader to the estimates in Section 4 of [14]. This finishes the proof of Theorem 1.4. \(\square\)

It remains to prove Proposition 6.1.

**Proof of Proposition 6.1.** Consider the future domain of dependence \(\mathcal{D}(U_i)\) of \(U_i \subset \Sigma\). By using the scaling (5.2) with \(\lambda = r_2\), it follows by Lemma 5.1 and our choice of \(r_2 > 0\) in (4.10), that the rescaled initial data \((B(0, 1), g_{(r_2)}; k_{(r_2)})\) satisfies
\[
\|g_{(r_2)} - e\|_{H^2(B(0, 1))} + \|k_{(r_2)}\|_{H^1(B(0, r))} \lesssim \varepsilon,
\]
and in case of higher regularity, for integers \(m \geq 1\),
\[
\|g_{(r_2)} - e\|_{H^{m+2}(B(0, 1))} + \|k_{(r_2)}\|_{H^{m+1}(B(0, 1))} \leq C_{r_2, m} \left( \sum_{i=1}^{m} \|\nabla^{(i)}\text{Ric}\|_{L^2(\Sigma)} + \|\nabla^{(i)}k\|_{L^2(\Sigma)} + 1 \right).
\]
Therefore for \( \varepsilon > 0 \) sufficiently small, by Proposition 3.1, the future domain of dependence of \( B(0,1) \) is locally foliated by spacelike maximal hypersurfaces \( \Sigma_t \) defined as level sets of a time function \( t \) such that on \( 0 \leq t \leq 1/2 \),

\[
\|\text{Ric}(r_2)\|_{L_t^\infty L^2(\Sigma_t)} \lesssim \varepsilon, \|k(r_2)\|_{L_t^\infty L^4(\Sigma_t)} \lesssim \varepsilon, \|\nabla k(r_2)\|_{L_t^\infty L^2(\Sigma_t)} \lesssim \varepsilon, \inf_{0 \leq t \leq 1/2} r_{vol}(\Sigma_t, 1) \geq 1/8,
\]

and in case of higher regularity, for integers \( m \geq 1 \), on \( 0 \leq t \leq 1/2 \),

\[
\sum_{|\alpha| \leq m} \|D^{(\alpha)}R(r_2)\|_{L_t^\infty L^2(\Sigma_t)} \lesssim C_m \left( \|g(r_2) - e\|_{H^{m+2}(B(0,1))} + \|k(r_2)\|_{H^{m+1}(B(0,1))} \right)
\]

\[
\leq C_{t_2, m} \left( \sum_{i=1}^m \|\nabla^{(i)}\text{Ric}\|_{L^2(\Sigma)} + \|\nabla^{(i)}\nabla k\|_{L^2(\Sigma)} + 1 \right).
\]

Using the spacetime scaling (5.1) with \( \lambda = (r_2)^{-1} \), it follows that the future domain of dependence of \( B(0, r_2) \) is controlled up to time

\[
T = \frac{r_2}{2} = \frac{r_2}{2} \left( \|\text{Ric}\|_{L^2(\Sigma)}, \|k\|_{L^4(\Sigma)}, \|\nabla k\|_{L^2(\Sigma)}, \|\Theta\|_{L^1(\partial \Sigma)}, r_{vol}(\Sigma, 1) \right) > 0
\]

with bounds on the interval \( 0 \leq t \leq T \),

\[
\|\text{Ric}\|_{L_t^\infty L^2(\Sigma_t)} \leq C, \|k_t\|_{L_t^\infty L^4(\Sigma_t)} \leq C, \|\nabla k_t\|_{L_t^\infty L^2(\Sigma_t)} \leq C, \inf_{0 \leq t \leq T} r_{vol}(\Sigma_t, 1) \geq \frac{1}{C},
\]

where

\[
C = C \left( \|\text{Ric}\|_{L^2(\Sigma)}, \|k\|_{L^4(\Sigma)}, \|\nabla k\|_{L^2(\Sigma)}, \|\Theta\|_{L^1(\partial \Sigma)}, r_{vol}(\Sigma, 1) \right) > 0. \tag{6.3}
\]

Moreover, in case of higher regularity, for integers \( m \geq 0 \), on \( 0 \leq t \leq T \),

\[
\sum_{|\alpha| \leq m} \|D^{(\alpha)}R\|_{L_t^\infty L^2(\Sigma_t)} \leq C_m \left( \sum_{|i| \leq m} \|\nabla^{(i)}\text{Ric}\|_{L^2(\Sigma)} + \|\nabla^{(i)}\nabla k\|_{L^2(\Sigma)} + 1 \right),
\]

where the constant \( C_m > 0 \) depends on \( m \) and the previous \( C > 0 \) in (6.3).

This finishes the control of the future domain of dependence \( D(U_t) \). The control of \( D(V_t) \) is similar and left to the reader. This finishes the proof of Proposition 6.1. \( \square \)

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