QUASILINEAR ELLIPTIC EQUATIONS WITH
SUB-NATURAL GROWTH TERMS AND
NONLINEAR POTENTIAL THEORY

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Dedicated to Vladimir Maz’ya with affection and admiration

Abstract. We discuss recent advances in the theory of quasi-linear equations of the type $-\Delta_p u = \sigma u^q$ in $\mathbb{R}^n$, in the case $0 < q < p - 1$, where $\sigma$ is a nonnegative measurable function, or measure, for the $p$-Laplacian $\Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u)$, as well as more general quasilinear, fractional Laplacian, and Hessian operators.

Within this context, we obtain some new results, in particular, necessary and sufficient conditions for the existence of solutions $u \in \text{BMO}(\mathbb{R}^n)$, $u \in L^r_{\text{loc}}(\mathbb{R}^n)$, etc., and prove an enhanced version of Wolff’s inequality for intrinsic nonlinear potentials associated with such problems.

Contents

1. Introduction 1
2. Nonlinear potentials 9
3. Main lemmas 11
4. Proofs of the main theorems and corollaries 19
References 26

1. Introduction

We present recent advances, along with some new results, in the existence, regularity, and nonlinear potential theory associated with

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the quasilinear elliptic equation

\begin{equation}
\begin{aligned}
-\Delta_p u &= \sigma u^q, \quad u \geq 0 \quad \text{in} \quad \mathbb{R}^n, \\
\liminf_{x \to \infty} u(x) &= 0,
\end{aligned}
\end{equation}

where $\sigma \geq 0$ is a locally integrable function, or Radon measure (locally finite) in $\mathbb{R}^n$, in the sub-natural growth case $0 < q < p - 1$. Such equations, together with the inhomogeneous problem

\begin{equation}
\begin{aligned}
-\Delta_p u &= \sigma u^q + \mu, \quad u \geq 0 \quad \text{in} \quad \mathbb{R}^n, \\
\liminf_{x \to \infty} u(x) &= c,
\end{aligned}
\end{equation}

where $\sigma, \mu$ are nonnegative Radon measures, and $c \geq 0$ is a constant, have been treated in [CV1]–[CV3], [SV1]–[SV3], [V3]. When $p = 2$, these are sublinear elliptic equations (see [BK], [QV], [V2], and the literature cited there).

The case $q \geq p - 1$, which comprises Schrödinger type equations with natural growth terms when $q = p - 1$, and superlinear type equations when $q > p - 1$, is quite different (see, for example, [AP], [JMV], [JV], [PV1], [PV2]).

In this paper, we will be using weak solutions (possibly unbounded). More precisely, all solutions are understood to be $p$-superharmonic (or equivalently, locally renormalized) solutions (see [KKT]). We will assume that $u \in L^q_{loc}(\mathbb{R}^n, d\sigma)$, so that the right-hand side of (1.1) is a Radon measure.

Among the new results obtained in this paper are necessary and sufficient conditions on $\sigma$ for the existence of a nontrivial solution $u \in L^r_{loc}(\mathbb{R}^n)$ to (1.1) for $\frac{n(p-1)}{n-p} \leq r < \infty$. Notice that for $0 < r < \frac{n(p-1)}{n-p}$, every $p$-superharmonic function $u \in L^r_{loc}(\mathbb{R}^n)$ ([HKM], [MZ]).

We will also characterize solutions $u \in \text{BMO}(\mathbb{R}^n)$, as well as solutions in the more restricted class

\begin{equation}
\int_K |\nabla u|^p dx \leq C \text{cap}_p(K), \quad \text{for all compact sets} \ K \subset \mathbb{R}^n.
\end{equation}

Here $\text{cap}_p(K)$ is the $p$-capacity defined by

\begin{equation}
\text{cap}_p(K) = \inf \left\{ \int_K |\nabla h|^p dx : h \in C_0^\infty(\mathbb{R}^n), \ h \geq 1 \text{ on } K \right\}.
\end{equation}

We observe that in general, for the existence of a nontrivial solution $u$ to (1.1), $\sigma$ must be absolutely continuous with respect to $p$-capacity, that is, $\sigma(K) = 0$ whenever $\text{cap}_p(K) = 0$. More precisely, if $u$ is a nontrivial solution to (1.1), then, for all compact sets $K \subset \mathbb{R}^n$, we have
\[ \sigma(K) \leq C \left[ \text{cap}_p(K) \right]^{\frac{1}{p-1}} \left( \int_K u^q \, d\sigma \right)^{\frac{p-1-q}{p-1}}. \]

The existence of solutions \( u \in L^\infty(\mathbb{R}^n) \) to (1.1) was characterized by Brezis and Kamin [BK] in the case \( p = 2 \). They also proved uniqueness of bounded solutions. However, a complete characterization of solutions \( u \in L^r(\mathbb{R}^n) \) with \( r < \infty \) turned out to be more complicated (see [V3] and the discussion below). Some sharp sufficient conditions for \( u \in L^r(\mathbb{R}^n) \) were established recently in [SV3]. See also [CV1], [SV1], [SV2], where finite energy solutions and their generalizations are treated.

Our main tools include certain nonlinear potentials associated with (1.1). Let \( \mathcal{M}^+(\mathbb{R}^n) \) denote the class of all (locally finite) Radon measures on \( \mathbb{R}^n \). Given a measure \( \sigma \in \mathcal{M}^+(\mathbb{R}^n) \), \( 1 < r < \infty \) and \( 0 < \alpha < \frac{n}{r} \), the Havin-Maz’ya-Wolff potential, introduced in [HM], is defined by

\[ W_{\alpha,r} \sigma(x) = \int_0^\infty \left[ \frac{\sigma(B(x, \rho))}{\rho^{n-\alpha r}} \right]^{\frac{1}{r-1}} \frac{d\rho}{\rho}, \quad x \in \mathbb{R}^n, \]

where \( B(x, \rho) \) is a ball centered at \( x \in \mathbb{R}^n \) of radius \( \rho > 0 \).

The nonlinear potential \( W_{\alpha,r} \sigma \), often called Wolff potential, appears in harmonic analysis, approximation theory and Sobolev spaces, in particular spectral synthesis problems, as well as quasilinear and fully nonlinear PDE (see [AH], [HW], [KM], [Lab], [Maz], [MZ], [PV1]).

In the linear case \( r = 2 \), we have \( W_{\alpha,2} \sigma = I_{2\alpha} \sigma \) (up to a constant multiple), where the Riesz potential of order \( \beta \in (0, n) \) is defined by

\[ I_{\beta} \sigma(x) = \int_{\mathbb{R}^n} \frac{d\sigma(y)}{|x-y|^{n-\beta}}, \quad x \in \mathbb{R}^n. \]

A related nonlinear potential is defined, for \( 1 < r < \infty \), \( 0 < \alpha < \frac{n}{r} \), by

\[ V_{\alpha,r} \sigma(x) = I_{\alpha}[I_{\alpha} \sigma]^{r-1}(x), \quad x \in \mathbb{R}^n. \]

This is the Havin-Maz’ya potential, which serves as the core notion of the nonlinear potential theory developed in [HM]. It is easy to see that, for all \( x \in \mathbb{R}^n \),

\[ V_{\alpha,r} \sigma(x) \geq c(\alpha, r, n) W_{\alpha,r} \sigma(x). \]

The converse pointwise inequality holds only for \( 2 - \frac{\alpha}{n} < r < \infty \) (see [HM], [Maz]).
Nonlinear potentials $W_{1,p}\sigma$ ($1 < p < \infty$) are intimately related to the equation

$$
\begin{cases}
-\Delta_p u = \sigma, & u \geq 0 \text{ in } \mathbb{R}^n, \\
\liminf_{x \to \infty} u(x) = 0,
\end{cases}
$$

where $\sigma \in \mathcal{M}^+ (\mathbb{R}^n)$.

The following important result is due to T. Kilpeläinen and J. Malý [KiMa]: Suppose $u \geq 0$ is a $p$-superharmonic solution to (1.9). Then

$$
(1.10) \quad K^{-1}W_{1,p}\sigma(x) \leq u(x) \leq KW_{1,p}\sigma(x),
$$

where $K = K(n, p)$ is a positive constant.

It is known that a nontrivial solution $u$ to (1.9) exists if and only if

$$
(1.11) \quad \int_1^\infty \left( \frac{\sigma(B(0, \rho))}{\rho^{n-p}} \right)^{\frac{p}{p-1}} \frac{1}{\rho} \, d\rho < \infty.
$$

This is equivalent to $W_{1,p}\sigma(x) < \infty$ for some $x \in \mathbb{R}^n$, or equivalently quasi-everywhere (q.e.) on $\mathbb{R}^n$. In particular, (1.11) may hold only in the case $1 < p < n$, unless $\sigma = 0$.

The following bilateral pointwise estimates of nontrivial (minimal) solutions $u$ to (1.1) in the case $0 < q < p - 1$ are fundamental to our approach [CV2]:

$$
(1.12) \quad c^{-1}[(W_{1,p}\sigma)^{\frac{p-1}{p-1-q}} + K_{1,p,q}\sigma] \leq u \leq c[(W_{1,p}\sigma)^{\frac{p-1}{p-1-q}} + K_{1,p,q}\sigma],
$$

where $c > 0$ is a constant which depends only on $p$, $q$, and $n$.

Here $K_{1,p,q}$ is the so-called intrinsic nonlinear potential associated with (1.1), which was introduced in [CV2]. It is defined in terms of the localized weighted norm inequalities,

$$
(1.13) \quad \left( \int_B |\varphi|^q \, d\sigma \right)^{\frac{1}{q}} \leq \varkappa(B) \|\Delta_p \varphi\|_{L^1(B(x, \rho))}^{\frac{1}{p-1}},
$$

for all test functions $\varphi$ such that $-\Delta_p \varphi \geq 0$, $\liminf_{x \to \infty} \varphi(x) = 0$. Here $\varkappa(B)$ denotes the least constant in (1.13) associated with the measure $\sigma_B = \sigma|_B$ restricted to a ball $B = B(x, \rho)$. Then the intrinsic nonlinear potential $K_{1,p,q}$ is defined by

$$
(1.14) \quad K_{1,p,q}\sigma(x) = \int_0^\infty \left( \frac{\varkappa(B(0, \rho))^{\frac{q(p-1)}{p-1-q}}}{\rho^{n-p}} \right)^{\frac{1}{p-1-q}} \frac{1}{\rho} \, d\rho, \quad x \in \mathbb{R}^n.
$$

As was shown in [CV2], $K_{1,p,q}\sigma \neq +\infty$ if and only if

$$
(1.15) \quad \int_1^\infty \left( \frac{\varkappa(B(0, \rho))^{\frac{q(p-1)}{p-1-q}}}{\rho^{n-p}} \right)^{\frac{1}{p-1-q}} \frac{1}{\rho} \, d\rho < \infty.
$$
Consequently, a nontrivial $p$-superharmonic solution $u$ to (1.1) exists if and only if both $K_{1,p,q} \sigma \not\equiv +\infty$ and $W_{1,p} \sigma \not\equiv +\infty$, that is,

\[
\int_1^\infty \left[ \left( \frac{\sigma(B(0,\rho))}{\rho^{n-p}} \right)^{\frac{1}{p-1}} + \left( \frac{\mathcal{I}(B(0,\rho))}{\rho^{n-p}} \right)^{\frac{1}{p-1}} \right] d\rho < \infty.
\]

Wolff’s inequality [HW], which holds for all $1 < r < \infty$, $0 < \alpha < \frac{n}{r}$, states

\[
E_{\alpha,r}[\sigma] = \int_{\mathbb{R}^n} (I_{\alpha} \sigma)^{r'} dx \leq C(\alpha, r, n) \int_{\mathbb{R}^n} W_{\alpha,r} \sigma d\sigma,
\]

where $r' = \frac{r}{r-1}$, and $E_{\alpha,r}[\sigma]$ is the $(\alpha,r)$-energy. The converse inequality holds as well, since by Fubini’s theorem and (1.8),

\[
\int_{\mathbb{R}^n} (I_{\alpha} \sigma)^{r'} dx = \int_{\mathbb{R}^n} V_{\alpha,r} \sigma d\sigma \geq c(\alpha, r, n) \int_{\mathbb{R}^n} W_{\alpha,r} \sigma d\sigma.
\]

Thus, Wolff’s inequality shows that, for all $1 < r < \infty$, $0 < \alpha < \frac{n}{r}$,

\[
E_{\alpha,r}[\sigma] \approx \int_{\mathbb{R}^n} W_{\alpha,r} \sigma d\sigma = \int_{\mathbb{R}^n} \int_0^\infty \left[ \frac{\sigma(B(x,\rho))}{\rho^{n-r\alpha}} \right]^{\frac{1}{r'-1}} \frac{d\rho}{\rho} d\sigma(x),
\]

where the constants of equivalence depend only on $\alpha, r,$ and $n$.

Several proofs of (1.17) are known, starting with the original proof due to Th. Wolff [HW] (see also [AH], [HJ], [V1]). In particular, it can be deduced from an inequality of Muckenhoupt and Wheeden for fractional integrals and maximal functions [MW] in weighted $L^r$ spaces (with $A_\infty$ weights). A two-weight version and applications can be found in in [COV3], [HV1], [HV2].

It follows from (1.12) that a necessary and sufficient condition for the existence of a solution $u \in L^r(\mathbb{R}^n)$ to (1.1) is given by:

\[
W_{1,p} \sigma \in L^{r(\frac{p-1}{p-1-q})}(\mathbb{R}^n) \quad \text{and} \quad K_{1,p,q} \sigma \in L^r(\mathbb{R}^n).
\]

Actually, the first condition in (1.19) is a consequence of the second one. Moreover, the second condition in (1.19) can be simplified using an analogue of Wolff’s inequality for potentials $K_{1,p,q}^d \sigma$ [V3, Theorem 1.1]:

\[
\|K_{1,p,q} \sigma\|_{L^r(\mathbb{R}^n)} \approx \int_{\mathbb{R}^n} \int_0^\infty \left( \frac{\mathcal{I}(B(x,\rho))}{\rho^{n-p}} \right)^{\frac{r}{p-1-q}} \frac{d\rho}{\rho} dx.
\]

In this paper, we obtain the following enhanced form of (1.20).
Theorem 1.1. Let \( 1 < p < n \), \( 0 < q < p - 1 \), \( \frac{n(p-1)}{n-p} < r < \infty \), and \( \sigma \in \mathcal{M}^+(\mathbb{R}^n) \). Then

\[
\|K_{1,p,q}\sigma\|_{L^r(\mathbb{R}^n)} \approx \int_{\mathbb{R}^n} \sup_{\rho>0} \left( \frac{[\kappa(B(x,\rho))]^{\frac{n(p-1)}{p-1-q}}}{\rho^{n-p}} \right)^{\frac{r}{p-1}}\,dx,
\]

where the constants of equivalence depend only on \( p, q, r \), and \( n \).

If \( n \leq p < \infty \), or \( 1 < p < n \) and \( 0 < r \leq \frac{n(p-1)}{n-p} \), then \( K_{1,p,q}\sigma \in L^r(\mathbb{R}^n) \) only if \( \sigma = 0 \).

As a corollary of Theorem 1.1, together with the results of [V3], we deduce that (1.1) has a nontrivial solution \( u \in L^r(\mathbb{R}^n) \) if and only if

\[
\int_{\mathbb{R}^n} \sup_{\rho>0} \left( \frac{[\kappa(B(x,\rho))]^{\frac{n(p-1)}{p-1-q}}}{\rho^{n-p}} \right)^{\frac{r}{p-1}}\,dx < \infty.
\]

A necessary (but generally not sufficient) condition for the existence of a nontrivial solution \( u \in L^r(\mathbb{R}^n) \) to (1.1) follows from (1.21),

\[
\int_{\mathbb{R}^n} \sup_{\rho>0} \left( \frac{\sigma(B(x,\rho))}{\rho^{n-p}} \right)^{\frac{r}{p-1-q}}\,dx < \infty.
\]

In fact, (1.23) is equivalent to the condition \( W_{1,p}\sigma \in L^{\frac{r(p-1)}{p-1-q}}(\mathbb{R}^n) \) by the Muckenhoupt and Wheeden inequality [MW] and its extensions (see [HJ], [JPW], [V3]).

Using Theorem 1.1, we deduce the following existence results for equation (1.1).

Theorem 1.2. Let \( 1 < p < n \), \( 0 < q < p - 1 \), and \( \sigma \in \mathcal{M}^+(\mathbb{R}^n) \) with \( \sigma \neq 0 \). Suppose that \( \frac{n(p-1)}{n-p} \leq r < \infty \). Then there exists a nontrivial solution \( u \in L^r_{\text{loc}}(\mathbb{R}^n) \) to (1.1) if and only if condition (1.16) holds, and additionally

\[
\int_{B(0,R)} \sup_{0<\rho<R} \left( \frac{[\kappa(B(x,\rho))]^{\frac{n(p-1)}{p-1-q}}}{\rho^{n-p}} \right)^{\frac{r}{p-1}}\,dx < \infty,
\]

for all \( R > 0 \).

If \( 0 < r < \frac{n(p-1)}{n-p} \), then there exists a nontrivial solution \( u \in L^r_{\text{loc}}(\mathbb{R}^n) \) to (1.1) whenever condition (1.16) holds.

The following corollary is deduced from Theorem 1.2 under the additional assumption that there exists a constant \( C = C(\sigma, p, n) \) so that

\[
\sigma(K) \leq C \text{cap}_p(K), \quad \text{for all compact sets } K \subset \mathbb{R}^n.
\]
In the case $0 < q < p - 1$, condition (1.25) ensures that solutions $u$ to (1.1) satisfy the Brezis–Kamin type pointwise estimates ([BK], [CV3]):

$$
c^{-1}(W_{1,p})^{\frac{p-1}{p-1-q}} \leq u \leq c(W_{1,p}^{-1})^{\frac{p-1}{p-1-q}} + W_{1,p},$$

where $c = c(p,q,n)$ is a positive constant.

We remark that condition (1.25) is also essential in the natural growth case $q = p - 1$ (see, for instance, [JMV]).

Corollary 1.3. Let $1 < p < n$ and $0 < q < p - 1$. If $\sigma \in M^+(\mathbb{R}^n)$ satisfies condition (1.25), then there exists a nontrivial solution $u \in L^r_{\text{loc}}(\mathbb{R}^n)$ to (1.1), for any $0 < r < \infty$, if and only if $W_{1,p} \neq \infty$, that is, when (1.11) holds.

Condition (1.25) in Corollary 1.3 can be relaxed in a substantial way so that estimates (1.26) still hold (see [CV3]).

Theorem 1.4. Let $1 < p < n$, $0 < q < p - 1$, and $\sigma \in M^+(\mathbb{R}^n)$ with $\sigma \neq 0$. If there exists a nontrivial solution $u \in \text{BMO}(\mathbb{R}^n)$ to (1.1), then there exists a constant $C = C(p,q,n)$ such that the following three conditions hold:

(i) For all $x \in \mathbb{R}^n$ and $R > 0$,

$$
\left[\mathcal{K}(B(x,R))\right]^{q/(p-1)} \leq C R^{n-p}.
$$

(ii) For all $x \in \mathbb{R}^n$ and $R > 0$,

$$
\sigma(B(x,R)) \left( \int_R^\infty \left( \frac{\mathcal{K}(B(x,\rho))}{\rho^{\alpha-p}} \right)^{q/(p-1)} \frac{1}{\rho} \, d\rho \right)^{1/q} \leq C R^{n-p}.
$$

(iii) For all $x \in \mathbb{R}^n$ and $R > 0$,

$$
\sigma(B(x,R)) \left( \int_R^\infty \left( \frac{\sigma(B(x,\rho))}{\rho^{\alpha-p}} \right)^{q/(p-1)} \frac{1}{\rho} \, d\rho \right)^{q/(p-1)} \leq C R^{n-p}.
$$

Conversely, if conditions (i), (ii), and (iii) hold, then there exists a nontrivial solution $u \in \text{BMO}(\mathbb{R}^n)$ to (1.1), provided $2 - \frac{1}{n} < p < n$. When $p \geq n$, there exists only a trivial solution to (1.1).

We remark that, for $2 - \frac{1}{n} < p < n$, we actually deduce (see Lemma 3.1 below) that, under assumptions (i)-(iii) of Theorem 1.4, solutions
$u$ to (1.1) satisfy
\[
\int_{B(x,R)} |\nabla u|^s dy \leq C R^{n-s},
\]
for all $0 < s < p$. The restriction $p > 2 - \frac{1}{n}$ for this estimate can be extended to the range $\frac{3n-2}{2n-1} < p \leq 2 - \frac{1}{n}$, using recent gradient estimates obtained in [NP].

The next corollary characterizes the existence of BMO solutions, for all $1 < p < n$, in terms of potentials $W_{1,p}\sigma$ under assumption (1.25).

**Corollary 1.5.** Let $1 < p < n$ and $0 < q < p - 1$. Suppose $\sigma \in M^+(\mathbb{R}^n)$ satisfies condition (1.25). Then there exists a nontrivial solution $u \in \text{BMO}(\mathbb{R}^n)$ to (1.1) if and only if, for all $x \in \mathbb{R}^n$ and $R > 0$,
\[
\int_{B(x,R)} (W_{1,p}\sigma)^{\frac{q(p-1)}{p-1-q}} d\sigma \leq C R^{n-p},
\]
or, equivalently, condition (1.29) holds.

In a similar way, using arbitrary compact sets $K$ in place of balls $B(x,R)$ in (1.31), we characterize solutions $u$ to (1.1) in the smaller class (1.3), which by Poincaré’s inequality is contained in BMO(\mathbb{R}^n).

**Theorem 1.6.** Let $1 < p < n$, $0 < q < p - 1$, and $\sigma \in M^+(\mathbb{R}^n)$ with $\sigma \not\equiv 0$. Then there exists a nontrivial solution $u$ to (1.1) which satisfies condition (1.3) if and only if, for all compact sets $K$ in $\mathbb{R}^n$,
\[
\int_{K} (W_{1,p}\sigma)^{\frac{q(p-1)}{p-1-q}} d\sigma \leq C \text{cap}_p(K).
\]

We remark that condition (1.32) is stronger than (1.25).

Our methods are applicable to intrinsic nonlinear potentials of fractional order related to nonlinear integral equations of the type
\[
u = W_{\alpha,p}(u^\sigma d\sigma) \quad \text{in} \quad \mathbb{R}^n.
\]
Here, a solution $u \geq 0$ is understood in the sense that $u \in L^q_{\text{loc}}(\mathbb{R}^n,\sigma)$ satisfies (1.33) $d\sigma$-a.e., or equivalently q.e. with respect to the $(\alpha,p)$-capacity (see [AH]). In the special case $p = 2$, this integral equation, namely $u = I_{2\alpha}(u^\sigma d\sigma)$, is equivalent to the corresponding problem for the fractional Laplacian (1.35) considered below.

Bilateral pointwise estimates of solutions to (1.33), similar to (1.12), in terms of fractional nonlinear potentials $W_{\alpha,p}\sigma$ and intrinsic potentials $K_{\alpha,p,q}$ defined in Sec. 2 below, are obtained in [CV2].

The following theorem is an analogue of Theorem 1.1.
Theorem 1.7. Let $1 < p < \infty$, $0 < q < p - 1$, $0 < \alpha < \frac{n}{p}$, and $\sigma \in \mathcal{M}^+(\mathbb{R}^n)$. Suppose that $\frac{n(p-1)}{n-\alpha p} < r < \infty$. Then there exists a positive solution $u \in L^r(\mathbb{R}^n)$ to (1.33) if and only if $K_{\alpha,p,q} \sigma \in L^r(\mathbb{R}^n)$. Moreover,

\begin{equation}
\|K_{\alpha,p,q} \sigma \|^r_{L^r(\mathbb{R}^n)} \approx \int_{\mathbb{R}^n} \sup_{\rho > 0} \left( \frac{\kappa(B(x,\rho))}{\rho^{n-\alpha p}} \right)^{\frac{r}{p-1}} \rho^{-r} \, dx,
\end{equation}

where the constants of equivalence depend only on $\alpha, p, q, r, n$.

If $0 < r \leq \frac{n(p-1)}{n-\alpha p}$, then there is only a trivial supersolution $u \in L^r(\mathbb{R}^n)$ to (1.33).

In (1.34), we employ the localized embedding constants $\kappa(B)$ with $B = B(x, \rho)$ associated with certain weighted norm inequalities for potentials $W_{\alpha,p}$. They are used to define the intrinsic potentials $K_{\alpha,p,q} \sigma$, along with their dyadic analogues $K_{d_{\alpha,p,q}} \sigma$ (see Sec. 2).

There are also analogues of Theorems 1.2, 1.4, and Corollaries 1.3, 1.5 for equation (1.33). In particular, in the special case $p = 2$, similar results hold for the fractional Laplace problem

\begin{equation}
\begin{cases}
(-\Delta)^\alpha u = \sigma u^q, & u \geq 0 \text{ in } \mathbb{R}^n, \\
\liminf_{x \to \infty} u(x) = 0,
\end{cases}
\end{equation}

where $0 < q < 1$ and $0 < \alpha < \frac{n}{2}$.

Other direct applications of Theorem 1.7 and related results for equation (1.33) in the case $\alpha = \frac{2k}{k+1}$, $p = k + 1$ and $q < k$ involve $k$-Hessian equations ($k = 1, 2, \ldots, n$), based upon the nonlinear potential theory developed in [Lab], [TW], similar to the case $q \geq k$ considered in [JV], [PV2].

This paper is organized as follows. In Sec. 2, we give definitions of nonlinear potentials $K_{\alpha,p,q}$ and discuss some of their properties. Certain lemmas on the existence of solutions $u$ to (1.9) in $\text{BMO}(\mathbb{R}^n)$ and in the class (1.3), along with a dyadic version of Theorem 1.1, are proved in Sec. 3. They are used in Sec. 4, where we prove Theorems 1.1, 1.2 and 1.4, and their analogues for equation (1.33).

2. Nonlinear potentials

Let $1 < p < \infty$, $0 < \alpha < \frac{n}{p}$, and $0 < q < p - 1$. Let $\sigma \in \mathcal{M}^+(\mathbb{R}^n)$. We denote by $\kappa$ the least constant in the weighted norm inequality

\begin{equation}
||W_{\alpha,p} \nu||_{L^q(\mathbb{R}^n, d\sigma)} \leq \kappa \nu(\mathbb{R}^n)^{\frac{1}{p-1}}, \quad \forall \nu \in \mathcal{M}^+(\mathbb{R}^n).
\end{equation}
We will also need a localized version of (2.1) for $\sigma_E = \sigma|_E$, where $E$ is a Borel subset of $\mathbb{R}^n$, and $\kappa(E)$ is the least constant in

\[(2.2) \quad \|W_{\alpha,p}\|_{L^q(d\sigma_E)} \leq \kappa(E) \nu(\mathbb{R}^n)^{\frac{1}{p-1}}, \quad \forall \nu \in \mathcal{M}^+(\mathbb{R}^n).\]

In applications, it will be enough to use $\kappa(E)$ where $E \subseteq Q$ is a dyadic cube, or $E = B$ is a ball in $\mathbb{R}^n$.

It is easy to see using estimates (1.10) that embedding constants $\kappa(B)$ in the case $\alpha = 1$ are equivalent to the constants $\kappa(B)$ in (1.13).

We define the intrinsic potential of Wolff type in terms of $\kappa(B(x, \rho))$, the least constant in (2.2) with $E = B(x, \rho)$:

\[(2.3) \quad K_{\alpha,p,q}\sigma(x) = \int_0^\infty \left[ \frac{\kappa(B(x, \rho))^{\frac{q(p-1)}{p-1-q}}}{\rho^{n-\alpha p}} \right]^\frac{1}{p-1} \frac{d\rho}{\rho}, \quad x \in \mathbb{R}^n.\]

It is easy to see that $K_{\alpha,p,q}\sigma(x) \not\equiv \infty$ if and only if

\[(2.4) \quad \int_a^\infty \left[ \frac{\kappa(B(0, \rho))^{\frac{q(p-1)}{p-1-q}}}{\rho^{n-\alpha p}} \right]^\frac{1}{p-1} \frac{d\rho}{\rho} < \infty, \quad \text{for any (all)} \quad a > 0.\]

This is similar to the condition $W_{\alpha,p}\sigma(x) \not\equiv \infty$, which is equivalent to (see, for instance, [CV2, Corollary 3.2])

\[(2.5) \quad \int_a^\infty \left[ \frac{\sigma(B(0, \rho))}{\rho^{n-\alpha p}} \right]^\frac{1}{p-1} \frac{d\rho}{\rho} < \infty.\]

In the case of potentials $W_{\alpha,p}$, sometimes a dyadic version $W_{\alpha,p}^d$ of nonlinear potentials is more convenient (see [HW]). In the same way, we find useful the dyadic version $K_{\alpha,p}^d\sigma$ of the intrinsic potential $K_{\alpha,p,q}$ defined by (see [V3])

\[(2.6) \quad K_{\alpha,p,q}^d\sigma(x) = \sum_{Q \in \mathcal{Q}} \left[ \frac{\kappa(Q)^{\frac{q(p-1)}{p-1-q}}}{|Q|^{1-\frac{\alpha p}{p}}} \right]^\frac{1}{p-1} \chi_Q(x), \quad x \in \mathbb{R}^n,\]

where the sum is taken over all dyadic cubes (cells) $\mathcal{Q}$. It is easy to see that, similarly to (2.4), $K_{\alpha,p,q}^d\sigma \not\equiv \infty$ if and only if, for all $P \in \mathcal{Q}$,

\[(2.7) \quad \sum_{R \supseteq P} \left[ \frac{\kappa(R)^{\frac{q(p-1)}{p-1-q}}}{|R|^{1-\frac{\alpha p}{p}}} \right]^\frac{1}{p-1} < \infty,\]

where $R \in \mathcal{Q}$. 
3. Main lemmas

We start with some lemmas on regularity of solutions to equation (1.9) based on certain pointwise and integral gradient estimates (see [AP], [DM], [KM]). The sufficiency part of the following lemma for the existence of BMO solutions to (1.9) (in bounded domains) is known (see [Mi1, Theorem 1.11], [Mi2, Theorem 4.3]).

Lemma 3.1. Let $0 < q < p - 1$, and $2 - \frac{1}{n} < p < n$. Suppose $\sigma \in M^+(\mathbb{R}^n)$ satisfies the condition

$$\sigma(B(x, R)) \leq C R^{n-p}, \quad \text{for all } x \in \mathbb{R}^n, \; R > 0,$$

and (1.11) holds. Then there exists a nontrivial solution $u \in \text{BMO}(\mathbb{R}^n)$ to (1.9). Moreover, any solution to (1.9) satisfies (1.30) for $0 < s < p$.

Conversely, for all $1 < p < n$, if there exists a solution $u \in \text{BMO}(\mathbb{R}^n)$ to (1.9), then both conditions (1.11) and (3.1) hold.

Proof. We first prove the sufficiency of condition (3.1) for the existence of a solution $u \in \text{BMO}(\mathbb{R}^n)$, provided (1.11) holds, that is, $W_{1,p} \sigma \not\equiv \infty$. The latter condition ensures (see [PV2]) that there exists a solution $u$ to (1.9), which satisfies pointwise bounds (1.10). Next, we invoke the known pointwise gradient estimates for solutions $u$ to (1.9) in the case $2 - \frac{1}{n} < p < n$, when $u \in W^{1,s}_{\text{loc}}(\mathbb{R}^n)$ for $1 \leq s < \frac{n(p-1)}{n-1}$ (see [DM], [KM]):

$$|\nabla u| \leq C(I_1 \sigma)^{\frac{1}{p-1}}.
$$

By Poincare’s inequality and (3.2), for $B = B(x, R)$ and $s \geq 1$, we have

$$\left( \frac{1}{|B|} \int_B |u - \bar{u}_B|^s dy \right)^{\frac{1}{s}} \leq C(n, s) R \left( \frac{1}{|B|} \int_B |\nabla u|^s dy \right)^{\frac{1}{s}} \leq C(n, s) R \left( \frac{1}{|B|} \int_B (I_1 \sigma)^{\frac{s}{p-1}} dy \right)^{\frac{1}{s}}.
$$

We next prove that, for $1 < s < \frac{n(p-1)}{n-1}$,

$$\int_B (I_1 \sigma)^{\frac{s}{p-1}} dy \leq C |B|^{1-\frac{s}{p}},
$$

where $C$ does not depend on $B = B(x, R)$. Clearly,

$$I_1 \sigma(y) = (n-1) \int_0^R \frac{\sigma(B(y, \rho))}{\rho^{n-1}} d\rho + (n-1) \int_R^\infty \frac{\sigma(B(y, \rho))}{\rho^{n-1}} d\rho.
$$

Hence, we can write

$$\int_B (I_1 \sigma)^{\frac{s}{p-1}} dy = (n-1)^{\frac{s}{p-1}} (I + II),$$
where
\[
I = \int_B \left( \int_0^R \frac{\sigma(B(y, \rho)) \, d\rho}{\rho^{n-1}} \right)^{\frac{p}{p-1}} dy,
\]
\[
II = \int_B \left( \int_R^\infty \frac{\sigma(B(y, \rho)) \, d\rho}{\rho^{n-1}} \right)^{\frac{p}{p-1}} dy.
\]
By (3.1),
\[
\int_R^\infty \frac{\sigma(B(y, \rho)) \, d\rho}{\rho^{n-1}} \leq C \int_R^\infty \rho^{n-p} \, d\rho = \frac{C}{p-1} R^{1-p}.
\]
Hence, for term II we have
\[
II \leq C R^{n-s}.
\]
We next prove a similar estimate for term I with \( s \geq 1 \). Since \( 2 - \frac{1}{n} < p < n \), we can assume without loss of generality that
\[
\max(1, p-1) \leq s < \frac{(p-1)n}{n-1}.
\]
Notice that, for \( y \in B(x, R) \) and \( 0 < \rho < R \), we have \( B(y, \rho) \subset B(x, 2R) \). Using the integral Minkowski inequality with \( \frac{s}{p-1} \geq 1 \) and taking into account (3.5), we estimate
\[
I = \int_{B(x,R)} \left( \int_0^R \int_{B(x,2R)} \chi_{B(y,\rho)}(z) \, d\sigma(z) \, \frac{d\rho}{\rho^n} \right)^{\frac{p}{p-1}} dy
\]
\[
\leq \left[ \int_{B(x,2R)} \left( \int_0^R \left( \int_{B(z,\rho)} \, dy \right)^{\frac{p}{p-1}} \frac{d\rho}{\rho^n} \, d\sigma(z) \right)^{\frac{p}{p-1}} \right]
\]
\[
= |B(0,1)| \left[ \int_{B(x,2R)} \int_0^R \rho^{\frac{n(p-1)}{s-1} - n} \, d\rho \, d\sigma(z) \right]^{\frac{s}{s-1}}
\]
\[
= C(p, s, n) \sigma(B(x,2R)) R^{n-\frac{(n-1)s}{p-1}}.
\]
Consequently, by (3.1),
\[
I \leq C |B|^{1-\frac{s}{p}}.
\]
Combining the preceding estimates for terms I and II, we obtain (3.4), for any ball \( B = B(x, R) \).

In fact, estimate (3.4), and consequently (1.30), holds for all \( 0 < s < p \). Indeed, by Jensen’s inequality, we may assume without loss of generality that \( p-\epsilon \leq s < p \). Then by pointwise Hedberg’s inequalities (see [AH, Sec. 3.1]), there exists a constant \( c = c(p, n, \epsilon) \) such that, for all \( \epsilon \in (0, p) \),
\[
I_1 \sigma \leq c \left( M_p \mu \right)^{\frac{1}{p-\epsilon}} (I_1 \sigma)^{\frac{p-1}{p-\epsilon}},
\]
where \( \sigma \in \mathcal{M}^+(\mathbb{R}^n) \), and \( M_p\sigma \) is the fractional maximal function of order \( p \), which is uniformly bounded by (3.1). Consequently, by the preceding estimate and Jensen’s inequality, for \( p - \epsilon \leq s < p \) we have

\[
\frac{1}{|B|} \int_B (I_1 \sigma)^{s-1} dy \leq \frac{c^{s-1}}{|B|} \int_B (I_\sigma)^{s-1} dy \leq c^{s-1} \left( \frac{1}{|B|} \int_B (I_\sigma) dy \right)^{s-1}. 
\]

Clearly,

\[
\frac{1}{|B|} \int_B (I_\sigma) dy \leq c(\epsilon, n) \left[ \sigma(2B) R^{n-\epsilon} + \int^\infty_R \sigma(B(x, \rho)) \rho^{n-\epsilon+1} d\rho \right]. 
\]

Invoking (3.1), we deduce that the right-hand side is bounded by \( C R^{n-\epsilon} \), which yields (3.4) for all \( 0 < s < p \).

Hence, by (3.3) with \( s \geq 1 \), we have

\[
\left( \frac{1}{|B|} \int_B |u - \bar{u}_B|^s dx \right)^{\frac{1}{s}} \leq C,
\]

where \( C \) does not depend on \( B \). Thus, \( u \in \text{BMO}(\mathbb{R}^n) \).

Let us now prove the necessity of (3.1) for all \( 1 < p < n \). Notice that if a solution \( u \) to (1.9) exists, then \( W_{1,p} \sigma \neq \infty \) by (1.10). Suppose \( u \in \text{BMO}(\mathbb{R}^n) \) is a solution to (1.9). Without loss of generality we may assume that \( u \in W^{1,p}_{\text{loc}}(\mathbb{R}^n) \). Otherwise, we replace \( u \) with \( u_k = \min(u, k) \), for \( k > 0 \). Since \( u \geq 0 \) is \( p \)-superharmonic, it follows that the same is true for \( u_k \), and \( u_k \in W^{1,p}_{\text{loc}}(\mathbb{R}^n) \) [HKM]. Moreover, we clearly have \( u_k \in \text{BMO}(\mathbb{R}^n) \), and

\[
\|u_k\|_{\text{BMO}(\mathbb{R}^n)} \leq \|u\|_{\text{BMO}(\mathbb{R}^n)}.
\]

The corresponding \( p \)-measures \( \sigma_k \) of the supersolutions \( u_k \) converge weakly to \( \sigma \) as \( k \to +\infty \). Consequently, it suffices to prove (3.1) with \( \sigma_k \) and \( u_k \) in place of \( \sigma \) and \( u \), respectively.

Let \( B = B(x, R) \), and let \( \eta \in C^\infty_0(\mathbb{R}^n) \) be a smooth cut-off function supported in \( 2B \) such that \( 0 \leq \eta \leq 1 \), \( \eta = 1 \) on \( B \), with \( \|
abla \eta \| \leq \frac{C}{R} \).

We will use a Caccioppoli type estimate for supersolutions \( u \geq 0 \) to (1.9) on \( 4B \) [MZ, Lemma 2.113], which is based on the weak Harnack inequality:

\[
\int_{2B} |\nabla u|^{p-1} \eta^{p-1} |\nabla \eta| dy \leq C R^{n-p} (\inf_B u)^{p-1}.
\]
In particular, by replacing $u$ in (3.6) with $u - \inf_{4B} u$, a nonnegative supersolution on $4B$, we deduce

\[
\int_{2B} |\nabla u|^{p-1} |\nabla \eta| \, dy \leq C R^{n-p} \frac{1}{|B|} \int_B \left[ u(y) - \inf_{4B} u \right]^{p-1} \, dy.
\]

Integrating by parts and using (3.7), we estimate

\[
\sigma(B) = \int_{2B} \eta \, d\sigma = p \int_{2B} \eta^{p-1} \nabla \eta \cdot \nabla u |\nabla u|^{p-2} \, dy \leq C R^{n-p} \frac{1}{|B|} \int_B \left[ u(y) - \inf_{4B} u \right]^{p-1} \, dy.
\]

On the other hand, if $v$ is a weak subsolution on $2B$ and $s > p-1$, we have by [MZ, Lemma 2.111],

\[
\sup_B v \leq C \left( \frac{1}{|2B|} \int_{2B} |v(y)|^s \, dy \right)^{\frac{1}{s}}.
\]

Letting $v = \bar{u}_{4B} - u$, we obviously have

\[
\sup_{4B} v = \bar{u}_{4B} - \inf_{4B} u.
\]

Hence, by (3.9) with $4B$ in place of $B$, and $s > p-1$,

\[
0 \leq \bar{u}_{4B} - \inf_{4B} u \leq C \left( \frac{1}{|8B|} \int_{8B} |u - \bar{u}_{4B}|^s \, dy \right)^{\frac{1}{s}}.
\]

Using the well-known estimates for BMO functions,

\[
|\bar{u}_{4B} - \bar{u}_{8B}| \leq C(n) \|u\|_{BMO(\mathbb{R}^n)},
\]

we see that, for any $s > 0$,

\[
\left( \frac{1}{|8B|} \int_{8B} |u(y) - \bar{u}_{4B}|^s \, dy \right)^{\frac{1}{s}} \leq C \|u\|_{BMO(\mathbb{R}^n)}.
\]

Combining the preceding estimates, we deduce

\[
0 \leq \bar{u}_{4B} - \inf_{4B} u \leq C \|u\|_{BMO(\mathbb{R}^n)},
\]
where $C$ depends on $p$, $s$, and $n$. Thus, using (3.8) together with (3.10), we estimate
\[
\sigma(B) \leq C R^{n-p} \frac{1}{|B|} \int_B \left[ u(y) - \inf_{4B} u \right]^{p-1} dy \\
\leq C R^{n-p} \left( \frac{1}{|B|} \int_B |u(y) - \bar{u}_{4B}|^{p-1} dy + \left[ \bar{u}_{4B} - \inf_{4B} u \right]^{p-1} \right) \\
\leq C R^{n-p} \left( \frac{1}{|4B|} \int_{4B} |u(y) - \bar{u}_{4B}|^{p-1} dy + \|u\|^{p-1}_{\text{BMO}(\mathbb{R}^n)} \right) \\
\leq C R^{n-p} \|u\|^{p-1}_{\text{BMO}(\mathbb{R}^n)}.
\]

\[ \square \]

**Remark 3.2.** An analogue of Lemma 3.1 in the case $p = 2$ is known for the fractional Laplacian $(-\Delta)^\alpha$ in place of the $p$-Laplacian. It can be deduced from the fact that if $u = I_{2\alpha}^\sigma$, where $\sigma \in M^+(\mathbb{R}^n)$ and $I_{2\alpha}^\sigma \not\equiv \infty$, then $u^\sharp \approx M_{2\alpha}^\sigma$, where $u^\sharp = M^\sharp(u)$ is the sharp maximal function of $u$, and $M_{2\alpha}$ is the fractional maximal function of order $2\alpha < n$; this estimate is due to D. Adams (see [AH]). It follows that $u \in \text{BMO}(\mathbb{R}^n)$ if and only if $M_{2\alpha}^\sigma \in L^\infty(\mathbb{R}^n)$, and $I_{2\alpha}^\sigma \not\equiv \infty$, for all $0 < \alpha < \frac{n}{2}$.

The next lemma concerns $\sigma$ satisfying the capacity condition (1.25), which is stronger than (3.1). As a result, solutions $u$ to (1.9) belong to the more narrow class (1.3). Notice that this lemma (see [AP] and the literature cited there) holds for all $1 < p < n$. In the case $2 - \frac{1}{n} < p < n$ it follows from the pointwise gradient estimates (3.2).

**Lemma 3.3.** Let $0 < q < p - 1$, and $1 < p < n$. Then (1.9) has a solution $u$ in the class (1.3) if and only if $\sigma \in M^+(\mathbb{R}^n)$ satisfies (1.11) and (1.25).

**Proof.** Suppose $u$ satisfies condition (1.3) and is a solution to (1.9), so that (1.10), and consequently (1.11), holds. Let $v \in C_0^\infty(\mathbb{R}^n)$, $v \geq 0$, and $v \geq 1$ on a compact set $K \subset \mathbb{R}^n$. Then, integrating by parts, we estimate
\[
\sigma(K) \leq \int_{\mathbb{R}^n} v^p d\sigma = p \int_{\mathbb{R}^n} v^{p-1} \nabla v \cdot \nabla u |\nabla u|^{p-2} dx \\
\leq \left( \int_{\mathbb{R}^n} v^p |\nabla u|^p dx \right)^\frac{1}{p} \|
abla v\|_{L^p(\mathbb{R}^n)} \left( \int_{\mathbb{R}^n} v^p |\nabla u|^p dx \right)^{\frac{1}{p}}.
\]

It follows from (1.3) (see [Maz, Sec. 2.4.1])
\[
\int_{\mathbb{R}^n} v^p |\nabla u|^p dx \leq C \int_{\mathbb{R}^n} |\nabla v|^p dx.
\]
Hence, $$\sigma(K) \leq C\|\nabla v\|_{L^p(\mathbb{R}^n)}^p.$$ Minimizing the right-hand side over all such \( v \) yields (1.25).

To prove the converse statement, notice that there exists a solution \( u \) to (1.9), in view of (1.11), which satisfies (1.10) (see, for example, [PV2]). Moreover, such a solution is known to be unique (see [KiMa], [KM]), since \( \sigma \) is absolutely continuous with respect to the \( p \)-capacity by (1.25).

Clearly, (1.25) yields (3.1), that is, \( \sigma(B(x, R)) \leq CR^{n-p} \) for all \( x \in \mathbb{R}^n \) and \( R > 0 \). In particular, \( I_1 \sigma \neq \infty \), for all \( 1 < p < n \). As was shown in [HMV, Lemma 2.5], for such \( \sigma \) there exists a solution \( v \) (not necessarily positive) to the Poisson equation \( -\Delta v = \sigma \) such that \( |\nabla v| \leq C I_1 \sigma \). Moreover, by [MV, Theorem 2.1] with \( l = 1 \) (see also [V1, Theorem 1.7]), condition (1.25) yields that there exists a positive constant \( c = c(p, n) \) such that, for all compact sets \( K \subset \mathbb{R}^n \),

\[
\int_K |\nabla v|^p \, dx \leq c \text{cap}_p(K),
\]

where \( C \) is the constant in (1.25). Setting \( F = -\nabla v \), so that \( \text{div} F = \sigma \), and consequently \( -\Delta_p u = \text{div} F \), we deduce using [AP, Lemma 2.7] that, in view of (3.11), the solution \( u \) satisfies (1.3).

We next prove an enhanced Wolff inequality for intrinsic nonlinear potentials \( K_{\alpha,p,q} \sigma \) in the dyadic case. The dyadic version \( K_{\alpha,p,q}^d \sigma \) is defined by (2.6). We will also need a localized version of \( K_{\alpha,p,q}^d \sigma \), for a cube \( P \in Q \):

\[
K_{\alpha,p,q}^d \sigma = \sum_{Q \subseteq P} \left[ \frac{\kappa(Q)^{q(p-1)/p}}{|Q|^{1-\alpha p/n}} \right]^{\frac{1}{p-1}} \chi_Q.
\]

By \( W_{\alpha,p}^d \sigma \) and \( I_{\alpha}^d \sigma \) we denote the corresponding localized dyadic versions of the potentials \( W_{\alpha,p} \sigma \) and \( I_{\alpha} \sigma \), respectively:

\[
W_{\alpha,p}^d \sigma = \sum_{Q \subseteq P} \left[ \frac{\sigma(Q)}{|Q|^{1-\alpha p/n}} \right]^{\frac{1}{p-1}} \chi_Q, \quad I_{\alpha}^d \sigma = \sum_{Q \subseteq P} \frac{\sigma(Q)}{|Q|^{1-\alpha p/n}} \chi_Q.
\]

**Lemma 3.4.** Let \( \sigma \in M^+(\mathbb{R}^n) \), and let \( 0 < q < p - 1 \), \( 0 < \alpha < \frac{n}{p} \), and \( r > \frac{n(p-1)}{n-\alpha p} \). Then

\[
(3.12) \quad \int_{\mathbb{R}^n} (K_{\alpha,p,q}^d \sigma)^r \, dx \approx \int_{\mathbb{R}^n} \sup_{P \in \mathbb{Q}, x \in P} \left( \frac{\kappa(P)^{q(p-1)/p}}{|P|^{1-\alpha p/n}} \right)^{\frac{r}{p-1}} \, dx,
\]

with constants of equivalence that do not depend on \( \sigma \).
Proof. The lower bound in (3.12) is obvious, since clearly

\[ K^d_{\alpha,p,q} \sigma \geq \sup_{Q \in \mathcal{Q} : x \in Q} \left[ \frac{\kappa(Q)^{q(p-1)}}{|Q|^{1-\frac{q(p-1)}{n}}} \right]^{\frac{1}{p-1}}. \]

Let us prove the upper bound. For \( r > 1 \), we have (see [COV2, Proposition 2.2]):

\[ \int_{\mathbb{R}^n} (K^d_{\alpha,p,q} \sigma)^r \, dx \approx \int_{\mathbb{R}^n} \sup_{P \in \mathcal{Q}, x \in P} \left( \frac{1}{|P|} \sum_{Q \subseteq P} \left[ \frac{\kappa(Q)^{q(p-1)}}{|Q|^{1-\frac{q(p-1)}{n}}} \right] \right)^{\frac{1}{r-1}} |P| \, dx. \]

We will need the following estimates (see [CV2, Lemma 4.2 and Corollary 4.3]): for every \( Q \subseteq P \), we have

\[ C(\alpha, p, q, n)[\kappa(Q)]^{-\frac{q}{p-1-q}} \leq \left[ \int_{Q} u_Q^q \, d\sigma \right]^{\frac{1}{p-1}} \leq \left[ \int_{P} u_P^q \, d\sigma \right]^{\frac{1}{p-1}} \leq [\kappa(P)]^{-\frac{q}{p-1-q}}. \]  

(3.13)

Here \( u_P \) is a solution to (1.1) with \( \sigma_P \) in place of \( \sigma \).

We estimate using the lower bound in (3.13),

\[ \sum_{Q \subseteq P} \left[ \frac{\kappa(Q)^{q(p-1)}}{|Q|^{1-\frac{q(p-1)}{n}}} \right]^{\frac{1}{p-1}} |Q| \leq \sum_{Q \subseteq P} \left[ \frac{\int_{Q} u_Q^q \, d\sigma}{|Q|^{1-\frac{q(p-1)}{n}}} \right]^{\frac{1}{p-1}} |Q|. \]

Let \( r > \frac{n(p-1)}{n-\alpha p} \). If \( r \leq 1 \), then \( p < \frac{2n}{n+\alpha} < 2 \). This case will be considered below.

For \( p \geq 2 \), we have:

\[ \sum_{Q \subseteq P} \left[ \frac{\int_{Q} u_Q^q \, d\sigma}{|Q|^{1-\frac{q(p-1)}{n}}} \right]^{\frac{1}{p-1}} |Q| = \int_{P} W^{d,p}_{\alpha,p}(u_P^q \, d\sigma_P) \, dx \approx \int_{P} \left( I^{d,p}_{\alpha,p}(u_P^q \, d\sigma_P) \right)^{\frac{1}{p-1}} \, dx \]

\[ \leq \left( \frac{1}{|P|} \sum_{Q \subseteq P} \left[ \frac{\int_{Q} u_Q^q \, d\sigma}{|Q|^{1-\frac{q(p-1)}{n}}} \right] \right)^{\frac{1}{p-1}} |P| \]

\[ = \left( \frac{1}{|P|} \sum_{Q \subseteq P} \left[ \frac{\int_{Q} u_Q^q \, d\sigma}{|Q|^{1-\frac{q(p-1)}{n}}} \right] \right)^{\frac{1}{p-1}} |P|. \]
Thus, as in the case above, we have

\[
\sum_{Q \subseteq P} \left| \frac{Q_{\alpha p}}{n} \right| = (1 - 2^{-\alpha p})^{-1} |P|_{\alpha p} \frac{|Q|_{\alpha p}}{n}.
\]

Consequently, we have

\[
\sum_{Q \subseteq P} \frac{\int_0^1 u_p \, d\sigma(Q)}{|Q|^{1 - \frac{\alpha p}{n}}} |Q| = \sum_{Q \subseteq P} \frac{|Q|_{\alpha p}}{n} \int_0^1 u_p \, d\sigma(Q)
\]

\[
= (1 - 2^{-\alpha p})^{-1} |P|_{\alpha p} \sum_{Q \subseteq P} \frac{|Q|_{\alpha p}}{n} \int_0^1 u_p \, d\sigma(Q)
\]

\[
\leq C |P|_{\alpha p} \kappa(P) \frac{g(p-1)}{p-1 - q},
\]

where in the last line we used the upper estimate in (3.13).

Thus, in the case \( p \geq 2 \) and \( r > 1 \), we have

\[
\int_{\mathbb{R}^n} (K^{d}_{\alpha p, q} \sigma)^r \, dx \leq C \int_{\mathbb{R}^n} \sup_{P \in Q; x \in P} \left( \frac{\kappa(P) \frac{g(p-1)}{p-1 - q}}{|P|^{1 - \frac{\alpha p}{n}}} \right)^{\frac{r}{p-1}} \, dx.
\]

In the case \( 1 < p < 2 \) we have \( \frac{1}{p-1} > 1 \). Hence, clearly,

\[
\int_{\mathbb{R}^n} \left( \sum_Q \left[ \frac{\kappa(Q) \frac{g(p-1)}{p-1 - q}}{|Q|^{1 - \frac{\alpha p}{n}}} \right]^{\frac{1}{p-1}} \chi_Q \right)^r \, dx \leq \int_{\mathbb{R}^n} \left( \sum_Q \frac{\kappa(Q) \frac{g(p-1)}{p-1 - q}}{|Q|^{1 - \frac{\alpha p}{n}}} \chi_Q \right)^{\frac{r}{p-1}} \, dx.
\]

Since \( \frac{r}{p-1} > \frac{n-\alpha p}{n} > 1 \), we deduce using [COV2, Proposition 2.2] again,

\[
\int_{\mathbb{R}^n} \left( \sum_Q \frac{\kappa(Q) \frac{g(p-1)}{p-1 - q}}{|Q|^{1 - \frac{\alpha p}{n}}} \chi_Q \right)^{\frac{r}{p-1}} \, dx
\]

\[
\leq \int_{\mathbb{R}^n} \sup_{P \in Q; x \in P} \left( \frac{1}{|P|} \sum_{Q \subseteq P} \frac{\kappa(Q) \frac{g(p-1)}{p-1 - q}}{|Q|^{1 - \frac{\alpha p}{n}}} \right)^{\frac{r}{p-1}} \, dx.
\]

We estimate as above, using (3.14),

\[
\frac{1}{|P|} \sum_{Q \subseteq P} \frac{\kappa(Q) \frac{g(p-1)}{p-1 - q}}{|Q|^{1 - \frac{\alpha p}{n}}} |Q| \leq \frac{1}{|P|} \sum_{Q \subseteq P} \int_{\mathbb{R}^n} u_p \, d\sigma(Q)
\]

\[
= (1 - 2^{-\alpha p})^{-1} \int_{\mathbb{R}^n} u_p \, d\sigma(Q) \leq C \frac{\kappa(P) \frac{g(p-1)}{p-1 - q}}{|P|^{1 - \frac{\alpha p}{n}}}
\]

Thus, as in the case \( p \geq 2 \) and \( r > 1 \) above, we have

\[
\int_{\mathbb{R}^n} \left( \sum_Q \frac{\kappa(Q) \frac{g(p-1)}{p-1 - q}}{|Q|^{1 - \frac{\alpha p}{n}}} \chi_Q \right)^{\frac{r}{p-1}} \, dx \leq C \int_{\mathbb{R}^n} \sup_{P \in Q; x \in P} \left( \frac{\kappa(P) \frac{g(p-1)}{p-1 - q}}{|P|^{1 - \frac{\alpha p}{n}}} \right)^{\frac{r}{p-1}} \, dx.
\]

\( \square \)
There is a localized version of Lemma 3.4.

**Lemma 3.5.** Let \( \sigma \in M^+(\mathbb{R}^n) \), and let \( 0 < q < p - 1 \), \( 0 < \alpha < \frac{n}{p} \), and \( r > \frac{n(p-1)}{n-\alpha p} \). Let \( P \in Q \). Then

\[
(3.15) \quad \int_P (K_{\alpha,p,q}^{d,p})^r dx \approx \int_P \sup_{R \leq P} \left( \kappa(R) \frac{(p-1)}{p-1-q} \right)^{\frac{r}{p-1}} dx,
\]

with constants of equivalence that do not depend on \( \sigma \) and \( P \).

The proof of Lemma 3.5 is essentially the same as that of Lemma 3.4, and we omit it here.

4. **Proofs of the main theorems and corollaries**

In this section, we prove the main theorems and corollaries stated in the Introduction.

It is shown in [CV2] that \((1.1)\) has a positive (super) solution if and only if the same is true for \((1.33)\) in the case \( \alpha = 1 \). Moreover, the conditions in Theorems 1.1 and 1.7 are equivalent, since one can use embedding constants \( \kappa(B) \) in place of \( \kappa(B) \) if \( \alpha = 1 \) (see Sec. 2). Thus, it suffices to prove only Theorem 1.7.

**Proof of Theorem 1.7.** Notice that, for all \( R > 0 \) and \( x \in \mathbb{R}^n \), we obviously have

\[
\frac{[\kappa(B(x, R))]^{p-1-q}}{R^{n-\alpha p}} \leq 2^{\frac{n-\alpha p}{p-1}} (\log 2) \int_R^{2R} \left( \frac{[\kappa(B(x, \rho))]^{\frac{(p-1)}{p-1-q}}}{\rho^{n-\alpha p}} \right)^{\frac{1}{p-1}} d\rho.
\]

Hence,

\[
(4.1) \quad \sup_{\rho > 0} \frac{[\kappa(B(x, \rho))]^{p-1-q}}{\rho^{n-\alpha p}} \leq 2^{\frac{n-\alpha p}{p-1}} (\log 2) K_{\alpha,p,q}(x).
\]

Consequently,

\[
K_{\alpha,p,q} \in L^r(\mathbb{R}^n) \Rightarrow \sup_{\rho > 0} \frac{[\kappa(B(x, \rho))]^{p-1-q}}{\rho^{n-\alpha p}} \in L^r(\mathbb{R}^n).
\]

It remains to prove the converse statement.

Let \( u \in \text{L}^q_{\text{loc}}(\sigma) \) \((u \geq 0)\) be a solution to \((1.33)\). In [CV2], the following analogue of the bilateral pointwise estimates \((1.12)\) was obtained for nontrivial (minimal) solutions \( u \) to \((1.33)\) in the case \( 0 < q < p - 1 \):

\[
(4.2) \quad c^{-1}[(W_{\alpha,p})^{\frac{p-1}{p-1-q}} + K_{\alpha,p,q}] \leq u \leq c[(W_{\alpha,p})^{\frac{p-1}{p-1-q}} + K_{\alpha,p,q}],
\]
where $c > 0$ is a constant which depends only on $\alpha$, $p$, $q$, and $n$. Moreover a nontrivial (super) solution exists if and only if both $W_{\alpha,p,q} \neq \infty$ and $K_{\alpha,p,q} \neq \infty$.

It follows that $u \in L^r(\mathbb{R}^n)$ ($r > 0$) exists if and only the following analogue of (1.19) holds:

$$K_{\alpha,p,q} \sigma \in L^r(\mathbb{R}^n), \quad W_{\alpha,p,q} \sigma \in L^{r(p-1)/q}(\mathbb{R}^n).$$

The second condition here actually follows from the first one, both in (1.19) (in the case $\alpha = 1$), and in (4.3), that is,

$$K_{\alpha,p,q} \sigma \in L^r(\mathbb{R}^n) \implies W_{\alpha,p,q} \sigma \in L^{r(p-1)/q}(\mathbb{R}^n).$$

Indeed, suppose that $K_{\alpha,p,q} \sigma \in L^r(\mathbb{R}^n)$. Using the following trivial estimate for balls $B = B(x, \rho)$,

$$\sigma(B) \left| B \right|^{-\frac{n-\alpha p}{p-1}} \leq C [\kappa(B)]^q,$$

we see that

$$K_{\alpha,p,q} \sigma(x) \geq C \int_0^\infty \left[ \frac{\sigma(B(x, \rho))}{\rho^{n-\alpha p}} \right]^{\frac{1}{p-1}-q} d\rho.$$

Hence,

$$\int_0^\infty \left[ \frac{\sigma(B(x, \rho))}{\rho^{n-\alpha p}} \right]^{\frac{1}{p-1}-q} d\rho \in L^r(\mathbb{R}^n).$$

Estimates in [HJ], [JPW] yield that the preceding condition is equivalent to $W_{\alpha,p,q} \sigma \in L^{(r(p-1)/q)}(\mathbb{R}^n)$. This proves (4.4).

In the same way, one can prove that there exists a (super) solution $u \in L^r(\mathbb{R}^n)$ to the dyadic version of (1.33), that is,

$$u = W_{\alpha,p,q}^d(u^q d\sigma) \quad \text{in } \mathbb{R}^n,$$

if and only if $K_{\alpha,p,q}^d \sigma \in L^r(\mathbb{R}^n)$.

It is known [HW] that, for $\omega \in M^+(\mathbb{R}^n)$, the conditions $W_{\alpha,p,q}^d \omega \in L^r(\mathbb{R}^n)$ and $W_{\alpha,p,q}^d \omega \in L^r(\mathbb{R}^n)$ are equivalent. From this it is easy to deduce, as in [HW], that the conditions $K_{\alpha,p,q}^d \sigma \in L^r(\mathbb{R}^n)$ and $K_{\alpha,p,q} \sigma \in L^r(\mathbb{R}^n)$ are equivalent. Thus, to prove Theorem 1.7 it is enough to prove its dyadic version, that is, to show that

$$\sup_{\rho > 0} \frac{[\kappa(B(x, \rho))]^{\frac{1}{p-1}-q}}{\rho^{n-\alpha p}} \in L^r(\mathbb{R}^n) \implies K_{\alpha,p,q}^d \sigma \in L^r(\mathbb{R}^n).$$
By Lemma 3.4, $K_{α,p,q}^d σ ∈ L^r(\mathbb{R}^n)$ is equivalent to the right-hand side of (3.12), which is clearly dominated by its continuous version, that is,

$$\int_{\mathbb{R}^n} (K_{α,p,q}^d σ)^r dx \leq C \int_{\mathbb{R}^n} \sup_{ρ > 0} \left( \frac{κ(B(x, ρ))}{ρ^{n-αp}} \right)^{\frac{r}{p-1}} dx. \tag{4.8}$$

This completes the proof of Theorem 1.7, and consequently Theorem 1.1.

**Proof of Theorem 1.2.** In the case $0 < r < \frac{n(p-1)}{n-αp}$, it is known ([HKM], [MZ]) that every $p$-superharmonic function $u ∈ L^r_{loc}(\mathbb{R}^n)$, and so necessary and sufficient conditions for the existence of such a solution are given by (1.16).

It is enough to consider solutions to (1.33) in the special case $α = 1$, although we present the proof for all $0 < α < n$. Let $r ≥ \frac{n(p-1)}{n-αp}$. (It is easy to see that for $0 < r < \frac{n(p-1)}{n-αp}$, every solution $u ∈ L^r_{loc}(\mathbb{R}^n)$.) Notice that a solution $u ∈ L^r_{loc}(\mathbb{R}^n)$ to (1.33) exists if and only if the following analogue of (1.19) holds:

$$K_{α,p,q} σ ∈ L^r_{loc}(\mathbb{R}^n), \quad W_{α,p} σ ∈ L^{\frac{r(p-1)}{p-1-q}_{loc}}(\mathbb{R}^n). \tag{4.9}$$

Again, as in the proof of (4.4), the second condition here actually follows from the first one, that is,

$$K_{α,p,q} σ ∈ L^r_{loc}(\mathbb{R}^n) \implies W_{α,p} σ ∈ L^{\frac{r(p-1)}{p-1-q}}_{loc}(\mathbb{R}^n), \tag{4.10}$$

provided (2.5) and (2.4) hold, which are both necessary for the existence of any solution. To prove (4.10), let $B = B(0, R)$, and suppose $K_{α,p,q} σ ∈ L^r(B)$ for every $R > 0$. Let us show that $W_{α,p} σ ∈ L^{\frac{r(p-1)}{p-1-q}}(\mathbb{R}^n)$.

Notice that for all $x ∈ B$, we have

$$W_{α,p} σ_{2B}(x) ≤ \int_{R}^{∞} \left( \frac{σ(B(0, 2ρ))}{ρ^{n-αp}} \right)^{\frac{1}{p-1}} dρ,$$

$$= 2^{\frac{αp-n}{p-1}} \int_{2R}^{∞} \left( \frac{σ(B(0, t))}{t^{n-αp}} \right)^{\frac{1}{p-1}} dt.$$

Hence, by (2.5) we have $W_{α,p} σ_{2B} ∈ L^∞(B)$. It remains to show that $W_{α,p} σ_{2B} ∈ L^{\frac{r(p-1)}{p-1-q}}(B)$. In fact, we will prove that $W_{α,p} σ_{2B} ∈ L^{\frac{r(p-1)}{p-1-q}}(\mathbb{R}^n)$, which by Wolff’s inequality ([HJ], [JPW]) is equivalent to

$$\int_{\mathbb{R}^n} \left( \int_{0}^{∞} \left( \frac{σ(B(x, ρ) ∩ 2B)}{ρ^{n-αp}} \right)^{\frac{r}{p-1}} dρ \right)^{p-1} dx < \infty.$$
By (4.6), we see that $K_{\alpha,p,q}^\sigma 2B \in L^r(3B)$ yields
\[
\int_{3B} \left( \int_0^\infty \left( \frac{\sigma(B(x,\rho) \cap 2B)}{\rho^{n-\alpha p}} \right)^{\frac{1}{p-1-q}} \frac{d\rho}{\rho} \right)^r dx < \infty.
\]
Hence, it remains to prove that
\[
I = \int_{(3B)^c} \left( \int_0^\infty \left( \frac{\sigma(B(x,\rho) \cap 2B)}{\rho^{n-\alpha p}} \right)^{\frac{1}{p-1-q}} \frac{d\rho}{\rho} \right)^r dx < \infty.
\]
Notice that in this integral $3R \leq |x| < \rho + 2R$, and consequently $\rho > \frac{|x|}{3}$.

For $r \geq \frac{n(p-1)}{n-\alpha p}$ and $0 < q < p - 1$ we have $\frac{(n-\alpha p)r}{p-1-q} > n$, so that
\[
I \leq \int_{(3B)^c} \left( \int_{\frac{|x|}{3}}^\infty \left( \frac{\sigma(2B)}{\rho^{n-\alpha p}} \right)^{\frac{1}{p-1-q}} \frac{d\rho}{\rho} \right)^r \frac{dx}{|x|^{\frac{(n-\alpha p)r}{p-1-q}}} < \infty.
\]
This proves (4.10).

It remains to show that $K_{\alpha,p,q}^\sigma \in L^r_{\text{loc}}(\mathbb{R}^n)$. As above, it is enough to establish a dyadic version, $K_{\alpha,p,q}^d \sigma \in L^r_{\text{loc}}(\mathbb{R}^n)$. In other words, for any dyadic cube $P$, we need to show that
\[
\int_P (K_{\alpha,p,q}^d \sigma)^r dx < \infty.
\]
This condition naturally breaks into two parts: the first one is a localized condition
\[
I = \int_P (K_{\alpha,p,q}^d \sigma_P)^r dx < \infty,
\]
whereas the second one is
\[
II = |P| \left[ \sum_{R \subset P} \left( \frac{\kappa(R)^{\frac{q(p-1)}{2}}}{|R|^{\frac{2q}{p-1-q}}} \right)^{\frac{1}{p-1-q}} \right]^r < \infty.
\]
By Lemma 3.5, condition (1.24) ensures that $I < \infty$, whereas $II < \infty$ by (2.7). The converse statement is obvious, since all the conditions (2.5), (2.7), and (1.24) are clearly necessary for the existence of a solution $u \in L^r_{\text{loc}}(\mathbb{R}^n)$ in view of (4.1) and (4.10). This completes the proof of Theorem 1.2.

Proof of Corollary 1.3. We invoke estimates (1.26), which were proved in [CV3] under the assumption (1.25). Since $\frac{p-1}{p-1-q} > 1$, by Hölder’s
inequality it is enough to ensure that \( (W_{1,p}^r)^{p^{-1}} \in L^{r}_{\text{loc}}(\mathbb{R}^n) \). As above, it suffices to show that, for any dyadic cube \( P \),
\[
\int_P (W_{1,p}^r)^{r/(p-1)} \, dx = I + II < \infty,
\]
where
\[
I = \int_P (W_{1,p}^r)^{r/(p-1)} \, dx, \quad II = |P| \left[ \sum_{R \supseteq P} \left( \frac{\sigma(R)}{|R|^{1-\frac{p}{n}}} \right)^{\frac{r}{p-1-q}} \right]^{r/(p-1)}.
\]
The second term is finite by the necessary condition \((1.11)\), which ensures that \( W_{1,p} \sigma \not\equiv \infty \).

To show that the localized term \( I < \infty \), notice that by a localized version of Wolff’s inequality (see, for instance, \([V3]\)),
\[
I \approx \sum_{Q \subseteq P} \left( \frac{\sigma(Q)}{|Q|^{1-\frac{p}{n}}} \right)^{\frac{r}{p-1-q}} |Q|.
\]
On the other hand, \((1.25)\) yields the estimate (\([CV3, \text{Lemma 2.1 and Remark 2.2}]\))
\[
\int_P (W_{1,p}^r)^s \, d\sigma \leq C \sigma(P) < \infty,
\]
for any \( s > 0 \). In particular, for \( s \geq 1 \) we obviously have
\[
\sum_{Q \subseteq P} \left( \frac{\sigma(Q)}{|Q|^{1-\frac{p}{n}}} \right)^{\frac{r}{p-1-q}} \sigma(Q) \leq C \sigma(P) < \infty.
\]
Setting \( s = \frac{r-(p-1-q)(p-1)}{p-1-q} \), where without loss of generality we may assume \( s \geq 1 \) (for \( r \) large enough), we deduce
\[
\sum_{Q \subseteq P} \left( \frac{\sigma(Q)}{|Q|^{1-\frac{p}{n}}} \right)^{\frac{r}{p-1-q}} |Q| \leq |P|^{\frac{r}{p-1-q}} \sum_{Q \subseteq P} \left( \frac{\sigma(Q)}{|Q|^{1-\frac{p}{n}}} \right)^{\frac{r}{p-1-q}} \sigma(Q) < \infty.
\]
Hence, \( II < \infty \) as well, so that \( (W_{1,p}^r)^{n-1} \in L^{r}_{\text{loc}}(\mathbb{R}^n) \). It follows by \([CV3, \text{Theorem 1.2}]\) that there exists a nontrivial solution \( u \in L^{r}_{\text{loc}}(\mathbb{R}^n) \).

**Proof of Theorem 1.4.** By Lemma 3.1 with \( d\omega = u^q d\sigma \) in place of \( \sigma \), a solution \( u \in \text{BMO}(\mathbb{R}^n) \) to \((1.1)\) exists if (in the case \( 2 - \frac{1}{n} < p < n \)) and only if, for every ball \( B(x,R) \subset \mathbb{R}^n \),
\[
\omega(B(x,R)) = \int_{B(x,R)} u^q \, d\sigma \leq C R^{n-p}.
\]
Moreover, if \( p > 2 - \frac{1}{n} \), then such a solution actually satisfies \((1.30)\) for all \( 0 < s < \frac{n(p-1)}{n-1} \).
By estimates (1.12), it follows that (4.11) holds if and only if
\[ \int_{B(x,R)} [(W_{1,p}\sigma)^{\frac{q(p-1)}{p-1-q}} + (K_{1,p}\sigma)^q] d\sigma \leq C R^{n-p}. \]

Moreover, by [CV2, Lemma 4.2], for every ball \( B = B(x, R) \), we have
\[ \kappa(B)^{\frac{q(p-1)}{p-1-q}} \leq C(p, q, n) \int_B u^q d\sigma. \]

This proves the necessity of condition (1.27). To prove the necessity of condition (1.28), notice that, for all \( y \in B(x, R) \) and \( \rho > 2R \), we have \( B(y, \rho) \supseteq B(x, R) \). Letting \( t = \frac{\rho}{2} \), we estimate, for \( y \in B(x, R) \),
\[ K_{1,p,q}(y) \geq \int_{2R}^{\infty} \left( \frac{[\kappa(B(x, \rho))^{\frac{q(p-1)}{p-1-q}}]}{\rho^{n-p}} \right) \rho^{-1} d\rho \]
\[ = 2 \frac{p-n}{p-1} \int_R^{\infty} \left( \frac{[\kappa(B(x, t))^{\frac{q(p-1)}{p-1-q}}]}{t^{n-p}} \right) \frac{1}{t} dt. \]

Thus, (1.28) follows from (4.12). The necessity of (1.29) is deduced in a similar way.

To prove the sufficiency of conditions (1.27), (1.28) and (1.29), we first verify the estimate of the localized term in (4.12), with \( \sigma_{2B} \) in place of \( \sigma \) (here \( 2B = B(x, 2R) \)), that is,
\[ \int_B [(W_{1,p}\sigma_{2B})^{\frac{q(p-1)}{p-1-q}} + (K_{1,p,q}\sigma_{2B})^q] d\sigma \leq C R^{n-p}. \]

We invoke the estimate [CV2, Corollary 4.3],
\[ \int_{2B} u_{2B}^q d\sigma \leq [\kappa(2B)]^{\frac{q(p-1)}{p-1-q}}. \]

Here \( u_{2B} \) denotes a nontrivial solution to (1.1) with \( \sigma_{2B} \) in place of \( \sigma \). Combining (4.15) with the lower pointwise estimate (1.12) for \( u_{2B} \) in place of \( u \), namely,
\[ c(p, q, n) [(W_{1,p}\sigma_{2B})^{\frac{q(p-1)}{p-1-q}} + K_{1,p,q}\sigma_{2B}] \leq u_{2B}, \]

together with (1.27), yields (4.14).

To obtain similar estimates for \( \sigma(2B)^e \) (the portion of \( \sigma \) supported outside \( 2B \)) in place of \( \sigma \) in (4.12), notice that, for all \( y \in B(x, R) \), we have \( [B(x, 2R)]^c \cap B(y, \rho) = \emptyset \) if \( 0 < \rho < R \), and \( B(y, \rho) \subset B(x, 2\rho) \) if
\[ \rho > R. \] Hence, for \( y \in B = B(x, R) \),
\[
W_{1,p} \sigma (2B) (y) \leq \int_{R}^{\infty} \left( \frac{\sigma (B(x, 2\rho))}{\rho^{n-p}} \right)^{\frac{1}{p-1}} \frac{d\rho}{\rho},
\]
\[
K_{1,p} \sigma (2B) (y) \leq \int_{R}^{\infty} \left( \frac{[\kappa (B(x, 2\rho))]^{q(p-1)}}{\rho^{n-p}} \right)^{\frac{1}{p-1}} \frac{d\rho}{\rho}.
\]
Letting \( t = 2\rho \) in these integrals, we estimate
\[
\int_{B} (W_{1,p} \sigma (2B) (y))^{q(p-1)} \frac{1}{t^{n-p}} \frac{dt}{t} \leq C \sigma (B) \int_{2R}^{\infty} \left( \frac{\sigma (B(x, t))}{t^{n-p}} \right)^{\frac{1}{p-1}} \frac{dt}{t}.
\]
Using conditions (1.28) and (1.29), we deduce
\[
\int_{B} (W_{1,p} \sigma (2B) (y))^{q(p-1)} \frac{1}{t^{n-p}} \frac{dt}{t} \leq C R^{n-p}.
\]
This completes the proof of (4.12), and consequently, Theorem 1.4.

**Proof of Corollary 1.5.** As in the proof of Theorem 1.4, it follows from Lemma 3.1 that a nontrivial solution \( u \in \text{BMO}(\mathbb{R}^n) \) to (1.1) exists if and only if, for every ball \( B(x, R) \subset \mathbb{R}^n \), condition (4.11) holds.

Moreover, the upper estimate in (1.26), which holds for a minimal solution \( u \) under the assumption (1.25), yields that a sufficient condition for \( u \in \text{BMO}(\mathbb{R}^n) \) is given by
\[
(4.16) \quad \int_{B(x,R)} (W_{1,p} \sigma )^{q(p-1)} \frac{1}{t^{n-p}} \frac{dt}{t} \leq C R^{n-p}.
\]
Since by (1.25) we have \( \sigma (B(x, R)) \leq C R^{n-p} \) for any ball \( B(x, R) \), it follows by Hölder’s inequality that we can drop the second term in (4.16). In other words, the condition
\[
(4.17) \quad \int_{B} (W_{1,p} \sigma )^{q(p-1)} d\sigma \leq C R^{n-p},
\]
for all balls \( B = B(x, R) \), is sufficient. It is also necessary, since it follows from (4.11) and the lower estimate (1.26).

It remains to show that (4.17) is equivalent to (1.29). Clearly, for all \( y \in B(x, R) \), we have
\[
W_{1,p} \sigma (y) \geq C \int_{2R}^{\infty} \left( \frac{\sigma (B(x, t))}{t^{n-p}} \right)^{\frac{1}{p-1}} \frac{dt}{t}.
\]
Hence, $(4.17) \Rightarrow (1.29)$. To prove the converse, it suffices to estimate only the localized part of $(4.17)$, namely,

$$
(4.18) \quad \int_B (W_{1,p} \sigma_{2B}) \frac{q(p-1)}{p-1-q} d\sigma \leq C R^{n-p},
$$

since the term corresponding to $(\sigma_{2B})^c$ is estimated as above using $(1.29)$. Invoking again [CV3, Lemma 2.1 and Remark 2.2] with $s = \frac{q(p-1)}{p-1-q}$ we see that (1.25) yields

$$
\int_B (W_{1,p} \sigma_{2B})^{q(p-1)} d\sigma \leq C \sigma(2B) \leq C R^{n-p}.
$$

This shows that (4.18) holds, that is, $(1.29) \Rightarrow (4.17)$. \hfill \qed

The proof of Theorem 1.6, based on Lemma 3.3, is similar to the above arguments, and is omitted here.

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