Upper Covers of 2-Chains and of 2-Antichains in Sets of Indecomposable Subsets

Bernd S. W. Schröder
School of Mathematics and Natural Sciences
The University of Southern Mississippi
118 College Avenue, #5043
Hattiesburg, MS 39406

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Abstract

We prove that there are arbitrarily large indecomposable ordered sets $T$ with a 2-chain $C \subset T$ such that the smallest indecomposable proper superset $U$ of $C$ in $T$ is $T$ itself. Subsequently, we characterize all such indecomposable ordered sets $T$ and 2-chains $C$. We also prove the same type of result for 2-antichains.

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1 Introduction

In [3], Schmerl and Trotter proved the following (in a more general context).

Theorem 1.1 (See Theorem 2.2 in [3].) Let $T$ be a finite indecomposable ordered set and let $P \subset T$ be an indecomposable ordered subset of $T$ with $4 \leq |P| \leq |T| + 2$. Then there is an indecomposable ordered set $U$ such that $P \subset U \leq T$ and $|U| = |P| + 2$.

2-chains and 2-antichains satisfy the definition of indecomposability and, for them, Theorem 1.1 fails. Hence the requirement that $|P| \geq 4$ is needed. This note characterizes all the ways in which Theorem 1.1 fails for 2-chains and for 2-antichains.
Figure 1: The first four elements \((P, a, b)\) in the class \(\mathcal{X}\) from Definition 2.1. We have \((N, a, b) \in \mathcal{X}\) by part 1 of Definition 2.1 and, for \(H \in \{X, Y, Z\}\), \((H, a, b) \in \mathcal{X}\) is obtained by applying part 2 or 3 of Definition 2.1 to the set to the left of \(H\). The labeling of \(X\) is used in the proof of Lemma 2.2.

2 Upper Covers of Nonadjacent 2-Chains

We first consider certain 2-chains \(C_2 = \{a < b\}\) such that \(a\) is not a lower cover of \(b\). Note that, for the family \(\mathcal{X}\) below, the ordered sets \(P\) such that there are \(a, b\) with \((P, a, b) \in \mathcal{X}\) include the ordered sets that are obtained as finite convex indecomposable subsets with at least 4 elements of the infinite ordered sets in [1] as well as the ordered sets obtained from the 3-irreducible ordered sets \(G_n, J_n\) and \(H_n\) (see [4], p. 65) by deleting the elements \(c\) and \(d\).

Definition 2.1 We define the family \(\mathcal{X}\) of triples \((P, a, b)\) of a finite ordered set \(P\), a minimal element \(a \in P\) and a maximal element \(b \in P\) by saying that \((P, a, b) \in \mathcal{X}\) iff one of the following hold.

1. \(P\) is an \(N\) and \(a\) and \(b\) are placed as in Figure 1.

2. There is a \((\tilde{P}, \tilde{a}, \tilde{b}) \in \mathcal{X}\) such that \(P\) is obtained from \(\tilde{P}\) by attaching \(a\) as a new minimal element below \(\tilde{P} \setminus \{\tilde{a}\}\) and \(b = \tilde{b}\).

3. There is a \((\tilde{P}, \tilde{a}, \tilde{b}) \in \mathcal{X}\) such that \(P\) is obtained from \(\tilde{P}\) by attaching \(b\) as a new maximal element above \(\tilde{P} \setminus \{\tilde{b}\}\) and \(a = \tilde{a}\).

We could immediately show that, if \((P, a, b) \in \mathcal{X}\), then the only indecomposable subset of \(P\) that contains \(a\) and \(b\) is \(P\) itself. However, a direct argument is more technical than needed. Hence we delay this discussion until after the proof of Proposition 3.2. We start by proving that certain 2-chains \(\{a, b\}\), in which the elements are not covers of each other, will be contained
in sets $P \subseteq T$ such that $(P, a, b) \in \mathcal{X}$. The hypothesis is a bit technical, but the overall situation for 2-chains will be resolved in the proof of Proposition 3.2.

**Lemma 2.2** Let $T$ be a finite indecomposable ordered set with $|T| > 2$ and let $C_2 = \{a < b\} \subset T$ be a chain with 2 elements such that $T \setminus \{t \in T : a < t < b\}$ is series-decomposable. Then $C_2$ is contained in a subset $H$ of $T$ such that $(H, a, b) \in \mathcal{X}$ and $H$ is not isomorphic to $N$.

**Proof.** Let $R := \{t \in T : a < t < b\}$ and suppose, for a contradiction, that $T$ is a finite indecomposable ordered set with $|T| > 2$ such that the result does not hold and such that $R$ is as small as possible.

Consider the ordered set $S := T \setminus R$. By assumption, $S$ is series-decomposable. Hence, there are nonempty subsets $L, U \subseteq S$ such that $S = L \oplus U$. If $b \in L$, then $T = (L \cup R) \oplus U$, which is not possible. If $a \in U$, then $T = L \oplus (R \cup U)$, which is not possible. Thus $a \in L$ and $b \in U$. Because $S$ contains no elements strictly between $a$ and $b$, $a$ is maximal in $L$ and $b$ is minimal in $U$. Moreover, because $T$ is indecomposable, we conclude that $R \neq \emptyset$.

Let $R_a := \{r \in R : r \ngeq L\}$ and let $R_b := \{r \in R : r \not\leq U\}$. Note that both sets are nonempty, because otherwise $T$ would be series-decomposable into $L \oplus (R \cup U)$ or into $(L \cup R) \oplus U$, respectively. Pick $r_a \in R_a$ and $r_b \in R_b$ such that, if $R_a \cap R_b \neq \emptyset$, then $r_a = r_b = : r$. Then there is an $a'' \in L$ such that $r_a \not\geq a''$. Let $a' \in L$ be a maximal element of $L$ such that $a' \geq a''$. Then $a'$ is not comparable to $r_a$, because $a' > r_a$ implies $a' > r_a > a$ (contradicting maximality of $a$ in $L$) and $a' < r_a$ implies $r_a > a' \geq a''$ (contradicting the choice of $a''$). Similarly, there is a $b' \in U$ that is not comparable to $r_b$ and minimal in $U$.

If $R_a \cap R_b \neq \emptyset$, then $\{a, a', r, b, b'\}$ is isomorphic to the ordered set $X$ in Figure 1 and $(X, a, b) \in \mathcal{X}$, a contradiction to the choice of $T$. For the remainder, we can assume that $R_a \cap R_b = \emptyset$. That is, every element of $R$ is below $U$ or above $L$. Therefore, because $T = L \cup R \cup U$, every element of $T$ is below $U$ or above $L$.

Now let $\tilde{a} \in L$ be maximal in $L$ and let $\tilde{b} \in U$ be minimal in $U$, chosen so that $\tilde{R} := \{t \in T : \tilde{a} < t < \tilde{b}\}$ is as small as possible. Note that $\tilde{R}$ intersects neither $L$ nor $U$, which means that $\tilde{R} \subseteq R$. Because $R'_a := \{t \in T : a' < t < b'\} \subseteq R \setminus \{r_a, r_b\}$, we have that $|\tilde{R}| \leq |R'| < |R|$, which means $\tilde{R} \subseteq R$ and hence we have $\tilde{a} \neq a$ or $\tilde{b} \neq b$. 

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Because every element of $T$ is below $U$ or above $L$, every element of $T$ is comparable to $\tilde{a}$ or to $\tilde{b}$. Because $\tilde{R} \subseteq R$, we have $a < \tilde{R} \setminus \{\tilde{a}\}$ and $b > \tilde{R} \setminus \{\tilde{b}\}$.

Let $\tilde{L} := \{t \in T \setminus \tilde{R} : t < \tilde{b}\}$ and let $\tilde{U} := \{t \in T \setminus \tilde{R} : t > \tilde{a}\}$. Then $\tilde{a}$ is maximal in $\tilde{L}$, $\tilde{b}$ is minimal in $\tilde{U}$, and $\tilde{L} \cap \tilde{U} = \emptyset$. Moreover, because every element of $T$ is comparable to $\tilde{a}$ or $\tilde{b}$, we have $\tilde{L} \cup \tilde{U} = T \setminus \tilde{R}$.

Consider the case that $\tilde{a}, \tilde{b}$ can be chosen so that $T \setminus \tilde{R}$ is co-connected. If $\tilde{L} < \tilde{U}$, then we would have $T \setminus \tilde{R} = \tilde{L} \oplus \tilde{U}$, which cannot be. Hence $\tilde{L} \not= \tilde{U}$, which means that there is a maximal $\tilde{u} \in \tilde{L}$ that is not comparable to a minimal $\tilde{u} \in \tilde{U}$. If $\tilde{u} \in L$, then $\tilde{u} \not= L$, but $\tilde{u} \in \tilde{U}$ also gives that $\tilde{a} < \tilde{u}$, so that, because $\tilde{u} \not\in \tilde{R}$, $\tilde{a}$ is not smaller than $\tilde{b}$, implying that $\tilde{a} \not\in \tilde{U}$, which contradicts the fact that $\tilde{a}$ must be above $L$ or below $U$. Similarly we exclude $\tilde{u} \in U$. Thus, $\tilde{u}, \tilde{u} \in R$, and, in particular, $\tilde{u} \not= \tilde{a}$ and $\tilde{u} \not= \tilde{b}$. However, then $\tilde{N} := \{\tilde{\ell} < \tilde{b} > \tilde{a} < \tilde{u}\}$ is such that $(\tilde{N}, \tilde{a}, \tilde{b}) \in \mathcal{X}$, and subsequently $(\tilde{N} \cup \{a, b\}, a, b) \in \mathcal{X}$, contradicting the choice of $T$.

Therefore, $\tilde{a}, \tilde{b}$ can only be chosen so that $T \setminus \tilde{R}$ is series-decomposable. Hence, by choice of $T$ and because $|\tilde{R}| < |R|$, there is an ordered set $\tilde{P} \subseteq T$ such that $(\tilde{P}, \tilde{a}, \tilde{b}) \in \mathcal{X}$. With $P := \tilde{P} \cup \{a, b\}$, we have $(P, a, b) \in \mathcal{X}$, independent of whether $|P| = |\tilde{P}| + 1$ or $|P| = |\tilde{P}| + 2$, a final contradiction to the choice of $T$.

### 3 Upper Covers of Adjacent 2-Chains

Now we prove that, if a chain in which both elements cover each other is contained in an indecomposable ordered set, then the ordered set must contain one of the ordered sets in Figure 2 or its dual.

**Lemma 3.1** Let $T$ be a finite indecomposable ordered set with $|T| > 2$ and let $C_2 = \{a < b\} \subset T$ be a chain with 2 elements such that $a$ is a lower cover of $b$. Then $C_2$ is contained in a subset of $T$ that is isomorphic to one of the ordered sets in Figure 2 or its dual.

**Proof.** First note that, if every strict upper bound of $a$ is an upper bound of $b$, and if every strict lower bound of $b$ is a lower bound of $a$, then $C_2$ is an order-autonomous subset of $T$, which is not possible. Thus, without loss of generality, there is a $p \in T$ such that $p > a$ and $p \not\geq b$. (The dual case is not addressed in Figure 2, but, obviously, runs along similar lines.) Because $b$ is
Figure 2: The forbidden sets in Lemma 3.1, with one possible placement of $b$ and $x$ indicated. The only other possible placement of $b$ and $x$ is obtained by $b$ taking the place of $x$ and vice versa and keeping all other points fixed. The labelings in the sets correspond to cases in the proof of Lemma 3.1.

an upper cover of $a$, $b$ is not comparable to any element $x$ with $a < x \leq p$. In particular, $a$ has an upper cover $x \neq b$. Let

$$C := \{x \in T : x \text{ is an upper cover of } a\}$$

and note that $|C| \geq 2$.

Consider the case that there are $x, y \in C$ that do not have the same strict lower bounds. Because we can, if needed, replace one of $x, y$ with $b$ and rename elements, we can assume that $y = b$. If $b$ has a lower bound $\ell < b$ that is not a lower bound of $x$, then $\{\ell < b > a < x\}$ is an ordered set $N$, see Figure 2. If $x$ has a lower bound $\ell < x$ that is not a lower bound of $b$, then $\{\ell < x > a < b\}$ is an ordered set isomorphic to $N$, but, compared to Figure 2, the positions of $b$ and $x$ are interchanged. Thus, from here on, we can assume that any two $x, y \in C$ have the same strict lower bounds.

The argument above shows two characteristics of this proof. First, we will continue to add hypotheses on the set $T$, typically indicated by “from here on, we can assume.” These hypotheses typically are regarding the comparability of sets. In Figure 2, elements of sets that will be introduced in the future will be denoted with the corresponding lowercase letter and, possibly,
a superscript. Second, although $b$ is an element of $C_2$ and $x$ will be used to denote a generic upper cover of $a$ that is not equal to $b$, the roles of $x$ and $b$ in the sets depicted in Figure 2 will be interchangeable, similar to the two versions of $N$ above. This interchangeability will be indicated as needed.

Because any two $x, y \in C$ have the same strict lower bounds (and because any element of $C$ is above $a$), any $\ell \in T$ that is not comparable to either of $a$ or $b$ is not comparable to any element of $C$.

Suppose, for a contradiction, that, for every upper bound $w \in A := \uparrow a \setminus \{a\}$, every lower bound $\ell < w$ is comparable to $a$ or $b$. Let $p \in T \setminus A$ be such that $p$ is comparable to an element of $A$. By definition of $A$, there is a $w \in A$ such that $p < w$. By assumption, $p$ is comparable to $a$ or $b$. By definition of $A$, we have $p \leq a$ or $p < b$. In either case, because any two elements of $C$ have the same strict lower bounds, we infer that $p < C$, and hence $p < A$. We conclude that $A$ is nontrivially order-autonomous in $T$, a contradiction. Thus there is an upper bound $w \in A = \uparrow a \setminus \{a\}$ that has a lower bound $\ell < w$ that is not comparable to either of $a$ or $b$. Let

$$W := \{w \in T : w > a \land (\exists \ell < w) \ell \not\sim a \land \ell \not\sim b\} \neq \emptyset.$$ 

Because any two $x, y \in C$ have the same strict lower bounds, we have that $W \cap C = \emptyset$ and that any $\ell$ as in the definition of $W$ is not comparable to any element of $C$. Also note that any element that is above an element of $W$ must be an element of $W$.

If there is a $w \in W$ such that $w \not\sim C$, let $\ell < w$ be not comparable to either of $a$ and $b$ and choose $x \in C$ such that $w$ is comparable to one of $b$ and $x$, but not the other. Because $\ell$ is not comparable to any element of $C$, we have that $\{a, b\}$ is contained in an ordered set isomorphic to $\tilde{N}$, see Figure 2 for the case $w > b$ (as indicated earlier, the case $w > x$ is obtained by switching $b$ and $x$). Thus, from here on, we can assume that every $w \in W$ is comparable to all elements of $C$, that is, $W > C$.

No two $x, y \in C$ can have the same strict upper bounds, because, otherwise, $\{x, y\}$ would be order-autonomous in $T$, which cannot be. Hence, for any two $x, y \in C$, there is a $u \in T$ such that $u$ is a strict upper bound of one, but not the other. Let

$$U := \{u \in T : (\exists x \in C)[u > b \land u \not\sim x] \lor [u > x \land u \not\sim b]\} \neq \emptyset.$$ 

Note that, because $W > C$, we have that $U \cap W = \emptyset$. If there is a $u \in U$ that is not comparable to an element $w \in W$, let $x \in C$ be such that $u$
is above one of $b$ and $x$, but not the other. Then $\{a, b\}$ is contained in an ordered set isomorphic to $B$, see Figure 2 for the case $u > b$ (again, the case $u \not> b$ is obtained by switching $b$ and $x$). Thus, from here on, we can assume that $U < W$. Next, we claim that

$$U = \{u \in T : (\exists x, y \in C) u > x \land u \not> y\}.$$ 

The containment “$\subseteq$” follows from the definition. For the containment “$\supseteq$,”

Let $t \in T$ and $x, y \in C$ be such that $t > x$ and $t \not> y$. In case $t > b$, we conclude that $t \in U$ because of the presence of $y$, and, in case $t \not> b$, we conclude that $t \in U$ because of the presence of $x$. This proves the equality.

Let

$$V_1 := \{v_1 \in T \setminus W : v_1 > C, v_1 \not> U\}.$$ 

Note that no element of $V_1$ can be greater than or equal to any element of $V_2$.

Suppose, for a contradiction, that $V_1 = \emptyset$. Let $H := C \cup U$ and let $p \in T \setminus H$ be comparable to an $h \in H$. If $p < h$, then, by definition of $H$ and because $U \cap W = \emptyset$, $c$ is below an upper cover of $a$ and hence $p < C$ and then $p < H$. If $p > h$, then $p$ is above an element of $C$. Because $p \not\in U$, we obtain $p > C$, and then, because $V_1 = \emptyset$ and $W > U$, we have $p > U$, which means that $p > H$. We conclude that $H$ is nontrivially order-autonomous in $T$, a contradiction. Thus $V_1 \neq \emptyset$. Note that, because $W > U$, no element of $V_1$ is above any element of $W$.

If $V_1 \not< W$, then there are a $v_1 \in V_1$ and a $w \in W$ that are not comparable. Because $v_1 \not> U$, there is a $u \in U$ that is not comparable to $v_1$. For this $u \in U$, there is an $x \in C$ such that $u$ is above one of $b$ and $x$, but not the other. This means that $\{a, b\}$ is contained in an ordered set isomorphic to $\hat{B}$, see Figure 2 for the case that $u > b$ (the case $u \not> b$ is obtained by switching $b$ and $x$). Thus, from here on, we can assume that $V_1 < W$.

Let

$$V_2 := \{v_2 \in T \setminus W : v_2 > U\}.$$ 

Note that no element of $V_2$ is above any element of $W$ and that no element of $V_1$ is above any element of $V_2$. Moreover, because $V_2 \cap U = \emptyset$, we have $V_2 > C$.

Suppose, for a contradiction, that $V_1 < V_2$, and consider the set $H := C \cup U \cup V_1$. Let $p \in T \setminus H$. If there is an $h \in H$ such that $p > h$, then $p$ is above an element of $C$, so, because $p \not\in U$, we have $p > C$, because $p \not\in V_1$ and $W > U$, we have $p > U$, and hence $p \in V_2 \cup W > H$. If there is an
\( h \in H \) such that \( p < h \), then (because \( h \not\in W \)) \( p \) is below an upper cover of \( a \). Hence \( p < C \), which implies \( p < H \) and \( H \) is nontrivially order-autonomous, a contradiction. Thus, \( V_1 \not< V_2 \).

Let

\[
V_2^< := \{ v_2 \in V_2 : v_2 < W \} \\
V_2^\not< := V_2 \setminus V_2^<
\]

Note that no element of \( V_2^\not< \) is below any element of \( V_2^< \).

Now consider the case that there is a \( v_2^\not< \in V_2^\not< \) that is not an upper bound of \( V_1 \). Then there is a \( w \in W \) that is not above \( v_2^\not< \), there is a \( v_1 \in V_1 \) that is not below \( v_2^\not< \) (and hence not comparable to it) and there is a \( u \in U \) that is not comparable to \( v_1 \). Consequently, using the comparabilities between the various sets that are already established, we conclude that \( \{a, b\} \) is contained in an ordered set isomorphic to \( \bar{B} \), see Figure 2 for the case that \( u > b \) (the case \( u \ngtr b \) is obtained by switching \( b \) and \( x \)). Thus, from here on, we can assume that \( V_1 < V_2^\not< \). Because \( V_1 \not< V_2 \), this means that \( V_1 \not< V_2^< \).

Suppose, for a contradiction, that \( V_2^< \not< V_2^\not< \), and consider the set \( H := C \cup U \cup V_1 \cup V_2^< \). Let \( p \in T \setminus H \). If there is an \( h \in H \) such that \( p > h \), then \( p \) is above an element of \( C \), so, because \( p \not\in U \), we have \( p > C \), because \( p \not\in V_1 \) and \( W > U \), we have \( p > U \), so, because \( p \not\in V_2^< \), we have \( p \in V_2^\not< \cup W > H \). If there is an \( h \in H \) such that \( p < h \), then (because \( h \not\in W \)) \( p \) is below an upper cover of \( a \). Hence \( p < C \), which implies \( p < H \) and \( H \) is nontrivially order-autonomous, a contradiction. Thus, \( V_2^< \not< V_2^\not< \), which means that there are \( v_2^< \in V_2^< \) and \( v_2^\not< \in V_2^\not< \) that are not comparable. In particular, and this is all that will be used in the following, this means that neither set is empty.

Let \( v_2^< \in V_2^< \) be such that there is a \( v_1 \in V_1 \) that is not below \( v_2^< \) (and hence not comparable to it). Then there is a \( u \in U \) that is not comparable to \( v_1 \).

First, consider the case that there is a \( v_2^\not< \in V_2^\not< \) that is not comparable to \( v_2^< \). Then there is a \( w \in W \) that is not above \( v_2^\not< \). Consequently, using the comparabilities between the various sets that are already established, we conclude that \( \{a, b\} \) is contained in an ordered set \( B' \), see Figure 2 for the case that \( u > b \) (the case \( u \npreceq b \) is obtained by switching \( b \) and \( x \)).

Finally, if this is not the case, then there is a \( v_2^\not< \in V_2^\not< \) that is greater than \( v_2^< \). Then there is a \( w \in W \) that is not above \( v_2^\not< \). Consequently, using the comparabilities between the various sets that are already established, we conclude that \( \{a, b\} \) is contained in an ordered set \( B' \), see Figure 2 for the case that \( u > b \) (the case \( u \npreceq b \) is obtained by switching \( b \) and \( x \)).
We can now summarize the situation for 2-chains.

**Proposition 3.2** Let $T$ be a finite indecomposable ordered set with $|T| > 2$, let $C_2 = \{a < b\} \subset T$ be a chain with 2 elements and let $H$ be an indecomposable ordered subset of $T$ that properly contains $C_2$ such that there is no indecomposable ordered subset $U$ of $T$ with $C_2 \subsetneq U \subsetneq H$. Then $H$ is isomorphic to one of the ordered sets in Figure 2 or $H$ is isomorphic to one of their duals, or $(H, a, b) \in \mathcal{X}$.

**Proof.** Let $S = H \setminus \{t \in H : a < t < b\}$. If $S$ is series-decomposable, the statement follows from Lemma 2.2. If $S$ is co-connected, then $C_2$ is not contained in a nontrivial order-autonomous subset of $S$ (otherwise, $H$ would be decomposable). We conclude that $C_2$ is contained in an indecomposable subset $H'$ of $S$, and hence of $H$, that is isomorphic to the index set of the canonical decomposition of $S$. In particular, this means that $H'$ is not a 2-chain and, because $a$ is a lower cover of $b$ in $S$, that $a$ is a lower cover of $b$ in $H'$. Therefore, by Lemma 3.1, $C_2$ is properly contained in an indecomposable subset of $H' \subseteq H$ that is isomorphic to one of the ordered sets in Figure 2 or to one of their duals. The statement now follows from the fact that this subset cannot be properly contained in $H$. \hfill \blacksquare

It can be checked that none of the isomorphism types of the ordered subsets $H$ in Proposition 3.2 can be omitted: One can prove that, for any two chains $C = \{a < b\}$ and $C' = \{a' < b'\}$ and for ordered sets $H$ (for $C$) and $H'$ (for $C'$) as in Proposition 3.2, there is no embedding of $H$ into $H'$ that maps $a$ to $a'$ and $b$ to $b'$. Such an argument is tedious, but can essentially be done “by inspection.” Because of none of the isomorphism types of the ordered subsets $H$ in Proposition 3.2 can be omitted, all bounds in Corollary 3.3 below are sharp.

**Corollary 3.3** Let $S$ be a finite ordered set and consider the set $\mathcal{I}$ of all indecomposable subsets of $S$, ordered by containment. Let $C \subset S$ be a 2-chain and let $U$ be an upper cover of $C$ in $\mathcal{I}$. If $C = \{a < b\}$ and $K$ is the longest chain from $a$ to $b$, then $|U| \leq \max\{2|K|, 9\}$.

**Proof.** The statement follows from Proposition 3.2. \hfill \blacksquare
Figure 3: Visualization of indecomposable V-covers, see Definition 4.1.

4 Upper Covers of 2-Antichains in Sets of Indecomposable Ordered Subsets

Definition 4.1 An ordered set $C$ is called an indecomposable V-cover iff $C$ has exactly two minimal elements, $\ell$ and $d$, the set $\uparrow \ell \setminus \{\ell\}$ is composed of exactly two connected components, which are a singleton $\{a\}$ and a fence $F$ from an element $b$ to an element $h$ such that $h$ is maximal in $F$ (with the equality $b = h$ being allowed); and we have $\uparrow d \setminus \{d\} = \{h\}$. The two types of indecomposable V-covers are depicted in Figure 3.

The name “indecomposable V-cover” comes from the following facts. Clearly $a$, $b$ and $\ell$ form a fence that looks like a “V.” Moreover, it is easy to check that every indecomposable V-cover is indeed indecomposable. Finally, we can easily see that, if $C$ is an indecomposable V-cover, then the only indecomposable subset $I \subseteq C$ that properly contains $\{a, b\}$ is $C$ itself: Because such a subset $I$ must be connected, it must contain $\ell$. Now, if any element of $F \setminus \{b\}$ were not contained in $I$, then, with $B$ being the connected component of $F \cap I$ that contains $b$, we would have that $\{a\} \cup B$ is nontrivially order-autonomous in $I$, which cannot be. Hence $\{a, \ell\} \cup F \subseteq I$, and, because $\{a, \ell\} \cup F$ is series-decomposable, we must have $d \in I$ and hence $C = I$.

Proposition 4.2 Let $T$ be a finite indecomposable ordered set with $|T| > 2$ and let $A_2 = \{a, b\} \subset T$ be an antichain with 2 elements. If $d(a, b) > 2$, then any smallest indecomposable ordered subset $I$ of $T$ that contains $A_2$ is a fence from $a$ to $b$ with $d(a, b) + 1$ elements. If $d(a, b) = 2$, then any smallest indecomposable ordered subset $I$ of $T$ that contains $A_2$ is either a fence with at least 4 elements, or it is isomorphic or dually isomorphic to an indecomposable V-cover in which $a$ and $b$ are as in Definition 4.1 (see Figure 3) or in which the roles of $a$ and $b$ are interchanged.
The case \( d(a, b) > 2 \) is already discussed at the start of this section. We are left to consider the case \( d(a, b) = 2 \).

Let \( I \) be a smallest indecomposable ordered subset of \( T \) that contains \( A_2 \). If \( I \) does not contain a common upper or lower bound of \( a \) and \( b \), then an argument similar to the argument at the start of this section shows that \( I \) is a fence. (Surprisingly, this case can occur: Consider \( N = \{a < f_2 > f_3 < b\} \) with an additional element \( \ell < a, b \) added.) This leaves the case that \( I \) contains a common upper or lower bound of \( a \) and \( b \). Without loss of generality, assume that \( I \) contains a common lower bound of \( a \) and \( b \). (The other case is handled with the dual argument.)

Let \( L := \{x \in I: x < a, b\} \neq \emptyset \), let \( U := \{x \in I: x > a, b\} \supseteq \{a, b\} \) be the set of all elements between \( L \) and \( U \) in \( I \). Let \( A \) be the connected component of \( H \) that contains \( a \) and let \( B \) be the connected component of \( H \) that contains \( b \). Note that \( A \) could be equal to \( B \). If \( |A| > 1 \), then, because \( B \) cannot be order-autonomous in \( I \), there must be an element in \( B \) that has a strict upper or lower bound in \( I \) that is not in \( L \cup B \cup U \). Similarly, if \( |B| > 1 \), there is an element of \( A \) that has a strict upper or lower bound in \( I \) that is not in \( L \cup A \cup U \). Finally, in case \( |A| = |B| = 1 \), because \( \{a, b\} \) cannot be order-autonomous in \( I \), \( a \) must have a strict upper or lower bound in \( I \) that is not in \( L \cup A \cup U \), or \( b \) must have a strict upper or lower bound in \( I \) that is not in \( L \cup B \cup U \). Note that all these upper and lower bounds are not in \( L \cup H \cup U \). Moreover, note that the existence of a strict upper bound \( s \) for an element of \( A \) (or \( B \)) such that \( s \notin L \cup H \cup U \) implies \( U \neq \emptyset \), because otherwise \( s \) would be in \( A \) (or \( B \)). Because we can switch the roles of \( a \) and \( b \), and because we can work with the dual ordered set if needed, without loss of generality, we can assume that \( B \) contains an element that has a strict lower bound in \( I \) that is not in \( L \cup B \cup U \).

Let \( s \) be the shortest distance in \( B \) from the element \( b \) to an element of \( B \) that has a strict lower bound in \( I \) that is not in \( L \cup B \cup U \). In case \( a \in B \), because the roles of \( a \) and \( b \) can be switched, we can assume that the distance in \( B \) from the element \( a \) to any element of \( B \) that has a strict lower bound in \( I \) that is not in \( L \cup B \cup U \) is at least \( s \). Let \( F \subseteq B \) be a fence of length \( s \) in \( B \) that goes from \( b \) to an element \( h \in B \) that has a lower bound \( d \in I \setminus (L \cup B \cup U) \). Because \( a \) and \( b \) do not have common upper or lower bounds in \( B \), \( F \) has length \( s \) and \( F \) is contained in \( B \), we conclude that \( a \) is not comparable to any element of \( F \cup \{d\} \) and that \( d \) is not comparable to any element of \( F \setminus \{h\} \). Because \( d \notin L \), no element of \( L \) is above \( d \).
Because \( d < h \), we have \( d < U \). Because \( d < h \in B, d \notin B \subseteq H \) and \( B \) is a connected component of \( H \), the point \( d \) cannot be in \( H \). Because \( d < U \), and \( d \notin H \), the point \( d \) cannot be above all elements of \( L \). Therefore, there is an \( \ell \in L \) that is incomparable to \( d \). This means that \( C := \{a, b, d, \ell\} \cup F \subseteq I \) is an indecomposable \( V \)-cover that contains \( a \) and \( b \). Because \( I \) is smallest possible, we obtain that \( I = C \). \[ \blacksquare \]

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