Unitary Representations of Noncompact Quantum Groups at Roots of Unity

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Abstract

Noncompact forms of the Drinfeld–Jimbo quantum groups $U_q^{\text{fin}}(g)$ with $H_i^s = H_i$, $X_i^\pm = s_i X_i^\mp$ for $s_i = \pm 1$ are studied at roots of unity. This covers $g = \mathfrak{so}(n,2p)$, $\mathfrak{su}(n,p)$, $\mathfrak{so}^*(2l)$, $\mathfrak{sp}(n,p)$, $\mathfrak{sp}(l,\mathbb{R})$, and exceptional cases. Finite–dimensional unitary representations are found for all these forms, for even roots of unity. Their classical symmetry induced by the Frobenius–map is determined, and the meaning of the extra quasi–classical generators appearing at even roots of unity is clarified. The unitary highest weight modules of the classical case are recovered in the limit $q \to 1$. 

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1 Introduction

Quantum groups allow to generalize the concept of symmetry, which has proved to be of great importance in physics. Up to this date, most of the work on quantum groups has been done for the compact case. However noncompact groups are important as well, for example the Lorentz group, or the Anti–de Sitter group $SO(2, n)$ which has attracted much attention recently in the context of string theory [1].

We consider the Drinfeld–Jimbo quantized universal enveloping algebra $U^r_{\text{res}}(g)$ [2, 3, 4] corresponding to finite–dimensional semisimple Lie algebras. In the $q$–deformed case, there are several possibilities to define real, in particular noncompact forms of these algebras. If $q$ is real, the representation theory is largely parallel to the classical case, but more complicated; for some results in this case see [3, 4]. In the present paper, we consider instead the case where $q$ is a root of unity, which provides additional structure that does not exist in the classical case. This turns out to be much simpler, rather than more difficult than the undeformed case. We study unitary representations of (a slight extension of) the so–called “finite” quantum group $U^{fin}_q(g) \subset U^r_{\text{res}}(g)$ at roots of unity, with real structure of the form $H_i^* = H_i$ and $X_i^{\pm*} = s_i X_i^{\pm}$, where $s_i = \pm 1$. This covers $so(n, 2p), su(n, p), so^*(2l), sp(n, p), sp(l, \mathbb{R})$, as well as various forms for the exceptional groups. Even though this real form corresponds to a non–standard Hopf algebra $*$–structure, it is appropriate for our purpose, and leads to a large class of unitary representations.

Generalizing the method of [7], we find unitary representations for all these noncompact forms, provided $q$ is an even roots of unity. It is shown that all of them can be related to unitary representations of the compact form in a simple way. As opposed to the classical case, they are finite–dimensional, which means that the problem is a purely algebraic one. In many cases, they can be viewed as regularizations of classical, infinite–dimensional representations. In particular, we show how almost all classical unitary highest weight modules (with the possible exception of a certain “small”, discrete set of highest weights) can be obtained as the limit $q \to 1$ of unitary representations of $U^{fin}_q$. In the example of the Anti–de Sitter group $SO(2, 3)$, this was already studied for special cases in [8, 9], and more generally in [11]. Not all the representations found however have a classical limit in an obvious way; to understand this better is an interesting open problem. Moreover, it turns out that the unitary representations of $U^{fin}_q \subset U^r_{\text{res}}(g)$ are very different from the ones studied in [10], where a different specialization of $U_q(sl(2, \mathbb{R}))$ to roots of unity is considered, leading to an infinite–dimensional algebra.

This paper is organized as follows. After reviewing the definitions and basic concepts in Section 2, the unitary representations of the compact case are studied in Section 3, and the particular features appearing at roots of unity are discussed. In Section 4, the remarkable classical symmetry $U(\tilde{g})$ arising from $U^r_{\text{res}}(g)$ at roots of unity due to the Frobenius map
is discussed, including the case of even roots of unity which turns out to be most important. The extra generators arising at even roots of unity which extend the classical universal enveloping algebra find a natural interpretation here.

In Section 5, the noncompact forms are defined, and unitary representations are found for all of them in a rather simple way. It turns out that only a subgroup of the classical $U(\tilde{g})$ preserves the noncompact form, which is determined in Section 6. Finally in Section 7, the connection with the classical case is made, and it is shown how the classical unitary highest–weight representations are recovered in the limit $q \to 1$.

In the appendix, an explicit, self–contained approach to the classical symmetry arising from the Frobenius map is given including the case of even roots of unity, which was treated only implicitly in [12].

2 Definitions and basic properties

We first collect the basic definitions, in order to fix the notation. Let $A_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)}$ be the Cartan matrix of a classical simple Lie algebra $g$ of rank $r$, where $(,)$ is the inner product in root space and $\{\alpha_i, \ i = 1, ..., r\}$ are the simple roots. The positive roots will be denoted by $Q^+$, and $\rho = \frac{1}{2} \sum_{\alpha \in Q^+} \alpha$ is the Weyl vector.

For $q \in \mathbb{C}$, the quantized universal enveloping algebra $U_q(g)$ is the Hopf algebra with generators $\{Y_\pm^i, K_i, K_i^{-1}; \ i = 1, ..., r\}$ and relations [4, 2, 3]

\[
[K_i, K_j] = 0,
\]

\[
K_iY_j^\pm = q^{\pm A_{ji}}Y_j^\pm K_i,
\]

\[
[Y_i^+, Y_j^-] = \delta_{ij}\frac{K_i - K_i^{-1}}{q^{d_i} - q^{-d_i}},
\]

\[
\sum_{k=0}^{1-A_{ji}} \left[ 1 - \frac{A_{ji}}{k} \right]_{q_i} (Y_i^\pm)^k Y_j^\pm (Y_i^\pm)^{-1-A_{ji} - k} = 0, \ i \neq j,
\]

where the $d_i = (\alpha_i, \alpha_i)/2$ are relatively prime, $q_i = q^{d_i}$, $[n]_{q_i} = \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}$, and

\[
\left[ \begin{array}{c} n \\ m \end{array} \right]_{q_i} = \frac{[n]_{q_i}!}{[m]_{q_i}![n-m]_{q_i}!}.
\]

We assume that $q^{d_i} \neq q^{-d_i}$. The comultiplication is defined by

\[
\Delta(K_i) = K_i \otimes K_i,
\]
\[
\Delta(Y_i^+) = 1 \otimes Y_i^+ + Y_i^+ \otimes K_i,
\Delta(Y_i^-) = K_i^{-1} \otimes Y_i^- + Y_i^- \otimes 1.
\]

(2.6)

Antipode and counit exist as well, but will not be needed. The Borel subalgebras \(U^\pm_q(\mathfrak{g})\) are defined in the obvious way.

In this paper, \(q\) will always be a complex number, rather than a formal variable. Moreover, since we are mainly interested in representations, it is more intuitive to use the generators \(\{X^\pm_i, H_i\}\) defined by

\[
K_i = q^{d_i H_i}, \quad Y_i^+ = X_i^+ q^{d_i H_i / 2}, \quad Y_i^- = q^{-d_i H_i / 2} X_i^-,
\]

(2.7)

so that the relations take the more familiar form

\[
\begin{align*}
[H_i, H_j] &= 0, \\
[H_i, X^\pm_j] &= \pm A_{ij} X^\pm_j, \\
[X^+_i, X^-_j] &= \delta_{i,j} q^{d_i H_i} - q^{-d_i H_i} = \delta_{i,j} [H_i]_{q_i}.
\end{align*}
\]

(2.8)

(2.9)

(2.10)

The comultiplication is now

\[
\begin{align*}
\Delta(H_i) &= H_i \otimes 1 + 1 \otimes H_i, \\
\Delta(X^\pm_i) &= X^\pm_i \otimes q^{d_i H_i / 2} + q^{-d_i H_i / 2} \otimes X^\pm_i.
\end{align*}
\]

(2.11)

The classical case is recovered for \(q = 1\). Generators \(X^\pm_\alpha\) corresponding to the other positive roots \(\alpha\) can be defined using the braid group action [13]; we will quote some properties as they are needed. A Poincare–Birkhoff–Witt (P.B.W.) basis is then given as classically in terms of ordered monomials of the raising and lowering operators corresponding to all positive respectively negative roots.

If \(q\) is allowed to be a root of unity, we will instead consider the “restricted specialization” \(U^\text{res}_q(\mathfrak{g})\) [13] with generators \(X^\pm_i^{(k)} = \frac{(X^\pm_i)^k}{[k]_{q_i}}\) for \(k \in \mathbb{N}\) as well as \(H_i\). For generic \(q\), i.e. \(q\) not a root of unity, this is the same as before. However if \(q\) is a root of unity,\(^1\)

\[
q = e^{2\pi i n / m}
\]

(2.12)

with \(m\) and \(n\) relatively prime, then \([k]_{q}\) becomes 0 for certain \(k\). Denote with \(M\) the smallest positive integer such that \(q^{2M} = 1\), i.e. \([M]_{q} = 0\). Thus \(M = m\) if \(m\) is odd, and \(M = m/2\) if \(m\) is even. In the first case \(q^M = 1\), and we will say that \(q\) is an “odd” root of unity. In the second case \(q^M = -1\), and \(q\) will be called “even”. More generally for \(q_i = e^{2\pi i d_i n / m}\), let \(M_i\) be the the smallest integer such that

\[
[M_i]_{q_i} = 0.
\]

(2.13)

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\(^1\) If \(q\) is a root of unity, then \(q^M = 1\) and \(q^M = -1\) as well. In the first case \(q^M = 1\), and we will say that \(q\) is an “odd” root of unity. In the second case \(q^M = -1\), and \(q\) will be called “even”. More generally for \(q_i = e^{2\pi i d_i n / m}\), let \(M_i\) be the the smallest integer such that

\[
[M_i]_{q_i} = 0.
\]
Then $M_i$ divides $M$; similarly, we define $M_\alpha$ and $d_\alpha$ for the other roots. $U_q^{\text{res}}(\mathfrak{g})$ contains the additional generators $X_i^{\pm(M_i)}$, which have a well–defined coproduct, and thus are defined on tensor products of representations. Verma modules can also be defined in the usual way, for integral highest weights \[14\]. We will only consider these types of representations of $U_q^{\text{res}}(\mathfrak{g})$. In particular, $(X_i^{\pm})^{M_i} = 0$ in $U_q^{\text{res}}(\mathfrak{g})$. Therefore $U_q^{\text{res}}(\mathfrak{g})$ contains a remarkable sub–Hopf algebra $U_q^{\text{res}}(\mathfrak{g})$.

The right–hand side is indeed an element of $U_q^{\text{res}}(\mathfrak{g})$, as can be seen from the commutation relations (A.14).

Finally, we quote the following useful relation:

$$[X_i^+, (X_i^-)^k] = (X_i^-)^{k-1}[k]_q[H_i - k + 1]_q.$$ (2.14)

3 Representations of $U_q^{\text{fin}}$ and weight space

The Cartan generators can be evaluated on weights $\lambda$, such that

$$< H_i, \lambda >= \frac{\langle \alpha_i, \lambda \rangle}{d_i} = (\alpha_i^\vee, \lambda),$$

where as usual $\alpha^\vee = \frac{2\alpha}{(\alpha, \alpha)}$ is the coroot of $\alpha$. The fundamental weights $\Lambda_i$ satisfy $(\Lambda_i, \alpha_j^\vee) = \delta_{i,j}$, therefore

$$< H_i, \Lambda_j >= \delta_{ij},$$

and span the lattice of integral weights. The Weyl group $W$ is defined as usual, and $D = \{ \sum_i r_i \Lambda_i; r_i \in \mathbb{R}_{\geq 0} \}$ is the dominant Weyl chamber.

It is well–known \[15\] that for generic $q$, the representation theory is essentially the same as in the classical case. In particular, the finite–dimensional representations (=modules) of $U_q^{\text{res}}(\mathfrak{g})$ are direct sums of irreducible representations $L^{\text{res}}(\lambda)$, which are highest–weight representations with dominant integral highest weight $\lambda$. Their character

$$\chi(L^{\text{res}}(\lambda)) = e^\lambda \sum_{\eta > 0} \dim L^{\text{res}}(\lambda)_{\eta} e^{-\eta} =: \chi(\lambda)$$

is given by Weyls formula. Here $L^{\text{res}}(\lambda)_{\eta}$ is the weight space of $L^{\text{res}}(\lambda)$ with weight $\lambda - \eta$. Irreducible highest weight representations of $U_q^{\text{fin}}$ are denoted by $L^{\text{fin}}(\lambda)$.
3.1 Singlets, special points, and the dual algebra $\tilde{g}$

One important feature at roots of unity is the existence of nontrivial one–dimensional representations $L^{fin}(\lambda_z)$ of $U^{fin}_q$, with weights

$$\lambda_z = \sum_i z_i M_i \Lambda_i$$

for $z_i \in \mathbb{Z}$; this follows from (2.14). There also exist similar representations with $z_i \notin \mathbb{Z}$ which will be considered in Section 5, but for now we concentrate on the case of integral weights. These weights $\lambda_z$ will be called special points. They span a lattice which is the weight lattice of a dual Lie algebra $\tilde{g}$, rescaled by $M$. In particular, it contains the root lattice of $\tilde{g}$, which is generated by the $M_i \alpha_i$ or equivalently $M \alpha_i$. Indeed, consider a second metric on weight space defined by

$$(\alpha_i, \alpha_j)_d := (M_i \alpha_i, M_j \alpha_j),$$

with associated matrix

$$\tilde{A}_{ij} := 2 \frac{(\alpha_i, \alpha_j)_d}{(\alpha_j, \alpha_j)_d} = \frac{M_i}{M_j} A_{ij}. \quad (3.6)$$

In particular,

$$M_i A_{ij} = \tilde{A}_{ij} M_j. \quad (3.7)$$

$\tilde{A}_{ij}$ is always a Cartan Matrix: it is clearly nondegenerate, and $\tilde{A}_{ii} = 2$. To see that $A_{ij} \in -N_0$ for $i \neq j$, observe that by the definition of $M$, $M_j d_j$ is the smallest integer which is divisible by both $M$ and $d_j$. Similarly $A_{ji} d_i M_i$ is divisible by $M$ because $A_{ji}$ is an integer, and also by $d_j$, since $A_{ji} d_i = A_{ij} d_j$. Therefore $\frac{A_{ij} d_i M_i}{M_j d_j}$ is an integer, equal to $\frac{M_i}{M_j} A_{ij} = \tilde{A}_{ij}$.

We shall determine $\tilde{A}_{ij}$ explicitly. In the simply laced case, all $M_i$ are equal, therefore $(\ , \ )_d$ is proportional to the Killing metric, and $\tilde{A}_{ij} = A_{ij}$. Thus the lattice of special points is nothing but the weight lattice rescaled by $M$, and $\tilde{g} = g$.

For $B_n$, $C_n$ and $F_4$, there are roots with 2 different lengths $d_s = 1$ and $d_l = 2$. Again, if $M$ is not divisible by 2, i.e. if $q$ is odd, then clearly $M_i = M$ is odd for all $i$, and $\tilde{A}_{ij} = A_{ij}$. On the other hand if $q$ is even, then $M_i = M/2 =: M_i$ if $\alpha_i$ is long, and $M_i = M =: M_s$ if $\alpha_i$ is short. Thus $\frac{M_i}{M_j} = \frac{d_i}{d_j}$ and $\tilde{A}_{ij} = A_{ji}$, which means that $M_i \alpha_i$ are the short roots and $M_s \alpha_s$ the long roots in the lattice of special points. Therefore the dual algebra of $B_n$ is $C_n$ and vice versa, while $F_4$ remains $F_4$ except that the roots change their role.

For $G_2$, the roots have lengths $d_s = 1$ and $d_l = 3$. If $M$ is not divisible by 3, then $M_i = M$ for $i = 1, 2$, and again $\tilde{A}_{ij} = A_{ij}$. On the other hand if $M$ is divisible by 3, let $\alpha_1 := \alpha_1$ be the long simple root, and $\alpha_s := \alpha_2$ be the short one. Then $M_l = M_1 = M/3$, $M_s = M_2 = M$,
and \( \tilde{A}_{ij} = A_{ji} \). Thus the dual lattice is again of type \( G_2 \), but now \( M_s\alpha_s \) is the long root, and \( M_l\alpha_l \) the short one.

To summarize, \( \tilde{g} = g \), except for \( \tilde{B}_n = C_n \) and \( \tilde{C}_n = B_n \) if \( q \) is even. For all cases, the Weyl group of \( \tilde{g} \) is the same as that of \( g \). In Section 4, we will see that in some sense, \( U^\text{res}_q(g) \) contains indeed a classical algebra associated with the lattice of special points.

The hyperplanes

\[
H_\alpha^z := \{ \lambda; (\lambda, \alpha^\vee) = M_\alpha z \}, \quad (3.8)
\]

where \( \alpha \) is any root and \( z \in \mathbb{Z} \), divide weight space into simplices called alcoves. The alcove of dominant weights with the origin on its boundary is called the fundamental alcove. The reflections on these hyperplanes generate the affine Weyl group, which plays an important role in the representation theory at roots of unity. Notice that every special point is in some \( H_\alpha^z \) for every root \( \alpha \). To see this, we have to show that \( (M_i\Lambda_i, \alpha^\vee) \in M_\alpha \mathbb{Z} \) for every root \( \alpha \). Since the Weyl group preserves the lattice generated by \( M_i\Lambda_i \), this follows from the fact that \( (\sum z_i M_i\Lambda_i, \alpha^\vee_j) \in M_j \mathbb{Z} \), for a suitable \( \alpha_j \). In fact, the special points are the intersection points of a maximal number of hyperplanes.

### 3.2 Unitary representations of the compact form

To define unitary representations, one first has to specify the real form of the algebra, or group in the classical case.

A real form or \( * \)–structure is an antilinear involution (=anti–algebra map) on \( U^\text{res}_q(g) \). In the classical case, the \( * \) is acting on the complexified Lie algebra, and the real Lie algebra is by definition its eigenspace with eigenvalue \(-1\). The interpretation of a real form at \( q \neq 1 \) is given by its classical limit.

In this section, we only consider the compact form. It is defined by \( * = \theta \) where \( \theta(X_i^{\mp}) = X_i^{\mp}, \theta(H_i) = H_i \) is the Cartan–Weyl involution, thus

\[
(X_i^{\pm})^* = X_i^{\mp}, \quad H_i^* = H_i, \quad (3.9)
\]

extended as an antilinear anti–algebra map. This is consistent for \( q \) real and \( |q| = 1 \).

A representation of \( U^\text{res}_q(g) \) on a Hilbert space \( V \) is said to be unitary if the star is implemented as the adjoint on the Hilbert space, i.e. \( (v, x\cdot w) = (x^*\cdot v, w) \) for any \( x \in U^\text{res}_q(g) \) and \( v, w \in V \). In particular, \( (\ , \ ) \) is positive definite. In the classical case, this means precisely that the adjoint (=star) of a group element is its inverse. Since all unitary representations are completely reducible, we only need to consider irreducible ones. Unitary and unitarizable will be used synonymously. On unitary highest weight modules with (3.9) or (5.1), the inner product can be calculated recursively, descending from the highest weight state. In particular, it is unique up to normalization.
Finite-dimensional unitary representations of noncompact forms with the correct classical limit are possible only at roots of unity. Therefore we will concentrate on that case from now on, in particular $q^* = q^{-1}$. Even though (3.3) is then a “nonstandard” Hopf algebra $\ast$–structure, it is appropriate for our purpose.

All finite-dimensional representations of $U^{\text{res}}_q(g)$ have integral weights, even at roots of unity. While this is not true for $U^{\text{fin}}_q$ any more, we nevertheless start with studying the unitary representations of $U^{\text{fin}}_q$ with integral weights. The following well-known fact [16] is useful:

**Theorem 3.1** Assume that $\lambda$ is a dominant integral weight with $(\lambda + \rho, \alpha^\vee) \leq M_\alpha$ for all positive roots $\alpha$. Then the highest weight representation $L^{\text{res}}(\lambda)$ has the same character $\chi$ as in the classical case, given by Weyl’s character formula.

In other words, $\lambda + \rho$ is in the fundamental alcove. This follows from the strong linkage principle, which was first shown in [17]; for a more elementary approach, see [1]. Moreover, $L^{\text{fin}}(\lambda) = L^{\text{res}}(\lambda)$ for these weights $\lambda$, since the $X_i^{\pm(M_i)}$ act trivially. If the above bound is not satisfied, then the Verma module with highest weight $\lambda$ contains additional highest weight submodules besides the classical ones.

Now we can show the following:

**Theorem 3.2** Let $\lambda$ be a dominant integral weight, and $q = e^{2\pi i n/m}$. Then $L^{\text{fin}}(\lambda)$ is a unitary representation of the compact form (3.9) of $U^{\text{fin}}_q$ if the character of $L^{\text{fin}}(\lambda)$ is given by Weyl’s formula for all $q' = e^{2\pi i \varphi'}$ with $0 \leq \varphi' < n/m$. In particular, this holds if $(\lambda + \rho, \alpha^\vee) \leq \left\lceil \frac{m}{2n\alpha_i} \right\rceil + 1$ for all positive roots $\alpha$, where $[c]$ denotes the largest integer $\leq c$ for $c \in \mathbb{R}$.

**Proof** Consider $L^{\text{fin}}(\lambda)$ for all $q' \in B := \{e^{2\pi i \varphi}; 0 \leq \varphi < n/m\}$. If the character of $L^{\text{fin}}(\lambda)$ is the same for all $q \in B$, one can identify the $L^{\text{fin}}(\lambda)$ as vector spaces. Their inner product matrix is smooth (in fact analytic) in $q'$, and positive definite at $q' = 1$ since we consider the compact case. This implies that all eigenvalues are positive on $B$: assume to the contrary that the matrix were not positive definite for some $q' \in B$. Then it would have a zero eigenvalue for some $q_0 \in B$, which implies that its null space is a submodule of $L^{\text{fin}}(\lambda)$. But this is impossible, since the $L^{\text{fin}}(\lambda)$ are irreducible by definition. For $q' = e^{2\pi i n/m}$, some eigenvalues may vanish; but then $L^{\text{fin}}(\lambda)$ is the quotient of $\lim_{q' \to q} L^{\text{fin}}(\lambda)$ modulo its null space, which again has a positive definite inner product.

\[2\text{since } (S(x))^* = S(x^*) \text{ here where } S \text{ is the antipode, rather than } (S(x))^* = S^{-1}(x^*) \text{ for } x \in U^{\text{res}}_q(g)\]

\[3\text{or even better, view them as trivial vector bundle over } B, \text{ with local trivializations given in terms of the P.B.W. basis}\]
In particular, assume that \((\lambda + \rho, \alpha^\vee) \leq \left\lceil \frac{m_{2\alpha}}{2m} \right\rceil + 1\). Let \(M'_\alpha\) be the smallest integer greater than \(\frac{m_{2\alpha}}{2m}\), which is \(\left\lceil \frac{m_{2\alpha}}{2m} \right\rceil + 1\). Then \(M'_\alpha\) is associated to \(q' := e^{\frac{2m}{m_{2\alpha}} M'_\alpha} \in B\) as defined in Section 2. Therefore by Theorem 3.1, \(L^{\text{fin}}(\lambda)\) at \(q'\) has the same character as for \(q = 1\), since \((\lambda + \rho, \alpha^\vee) \leq \left\lceil \frac{m_{2\alpha}}{2m} \right\rceil + 1 = M'_\alpha\). For all other roots of unity \(q'' \in B\), the character is again the same since the associated \(M''_\alpha\) is larger than \(M'_\alpha\). Thus the above argument applies. \(\Box\)

For some highest weights \(\lambda\) on the boundary of the domain specified in Theorem 3.2, the character of the unitary representation \(L^{\text{fin}}(\lambda)\) is smaller than the classical one. The reason is that the generic representations develop null–submodules; this can be interpreted in the context of gauge theories, see [7].

One may ask if all the unitary representations have been found in Theorem 3.2. As will be discussed in Section 7, it is possible that there exist certain unitary representations with integral and nonintegral weights which do not even satisfy the first condition in Theorem 3.2, as suggested by the classical noncompact case. This would have to be studied by different methods. Other unitary representations with integral and nonintegral weights will be obtained in Theorem 3.1, which however do not have a classical limit.

4 Frobenius map and the quasi–classical symmetry \(\tilde{g}\)

The modules \(L^{\text{res}}(\lambda) = L^{\text{fin}}(\lambda)\) in Theorem 3.2 are irreducible representations of \(U^f_\text{fin}\). For larger \(\lambda\), \(L^{\text{res}}(\lambda)\) decomposes into a direct sum of irreducible modules of \(U^f_\text{fin}\), which will be described now. This involves the special points introduced in Section 3.1.

The basic observation is the following. Consider a highest–weight module \(U^{-\text{res}}(g) \cdot v_\lambda\) with highest weight \(\lambda = \sum_i z_i M_i \lambda_i\) and \(z_i \in \mathbb{Z}\). From (2.14), it follows that all \(X_i^- \cdot v_\lambda\) are highest weight vectors. Therefore \(X_i^- \cdot v_\lambda = 0\) in \(L^{\text{res}}(\lambda_\alpha)\), because it is irreducible by definition. Using the P.B.W. basis, one can see that any element of \(U^{-\text{res}}(g)\) can be written as a sum of terms of the form \((X_i^{-\lambda_\alpha})\) \(\otimes \cdots \otimes (X_j^{-\lambda_\alpha})\) \(U^{-\text{fin}}\). It follows that all weights of \(L^{\text{res}}(\lambda_\alpha)\) have the form \(\lambda' = \lambda - \sum_i n_i i M_i \lambda_i\) with \(n_i \in \mathbb{N}\). In other words, \(L^{\text{res}}(\lambda_\alpha)\) is a direct sum of one–dimensional representations \(L^{\text{fin}}(\lambda_\alpha)\) of \(U^{\text{fin}}\), since \(\lambda_\alpha\) is a special point. However the “large” generators \(X_i^{\pm M_i}\) do act nontrivially, as we will see.

Consider \(L^{\text{res}}(\lambda_\alpha) \otimes L^{\text{res}}(\lambda_0)\) for \(\lambda_\alpha\) as above and integral \(\lambda_0\) with \(0 \leq (\lambda_0, \alpha^\vee) < M_i\). Now the generators \(Y_i^{\pm} (2.1)\) are useful. Using the coproduct, one finds

\[ Y_i^{\pm} \cdot (v \otimes w) = v \otimes (Y_i^{\pm} \cdot w), \quad (4.1) \]

and

\[ Y_i^{\pm M_i} \cdot (v \otimes w) = (Y_i^{\pm M_i} \cdot v) \otimes (K_i^{M_i} \cdot w). \quad (4.2) \]
for \(v \in L^{\text{res}}(\lambda_z)\) and \(w \in L^{\text{res}}(\lambda_0)\), because \(Y_i^{\pm(M_i)} \cdot w = 0\) by the bound on \(\lambda_0\). For the same reason, \(L^{\text{res}}(\lambda_0)\) is an irreducible representation of \(U_{q}^{\text{fin}}\). Together with (4.1) and (4.2), it follows that \(L^{\text{res}}(\lambda_z) \otimes L^{\text{res}}(\lambda_0)\) is an irreducible highest weight module of \(U_{q}^{\text{res}}(\mathfrak{g})\), and we have verified [14].

**Theorem 4.1** Let \(\lambda_z\) and \(\lambda_0\) be integral weights as above with \(0 \leq (\lambda_0, \alpha_i^\vee) < M_i\), and \(\lambda = \lambda_0 + \lambda_z\). Then

\[
L^{\text{res}}(\lambda) = L^{\text{res}}(\lambda_0) \otimes L^{\text{res}}(\lambda_z).
\] (4.3)

In particular, \(L^{\text{res}}(\lambda)\) decomposes into a direct sum of irreducible representations \(L^{\text{fin}}(\lambda - \sum_i n_i M_i \alpha_i)\) of \(U_{q}^{\text{fin}}\). Moreover, (4.1) and (4.2) show that \(Y_i^{\pm}\) commutes with \(Y_i^{\pm(M_i)} K_i^{M_i}\) on \(L^{\text{res}}(\lambda)\).

We will now see that the latter generators acting on \(L^{\text{res}}(\lambda_z)\) provide a representation of the classical universal enveloping algebra \(U(\tilde{\mathfrak{g}})\) corresponding to the Cartan matrix \(\tilde{A}_{ij}\). This is the essence of a remarkable result of Lusztig [11, 12]. For odd roots of unity, it states that there is a surjective algebra homomorphism

\[
U_{q}^{\text{res}}(\tilde{\mathfrak{g}}) \rightarrow U(\mathfrak{g}),
\]

\[
Y_i^{\pm} \rightarrow 0, \\
K_i \rightarrow 1, \\
Y_i^{\pm(M_i)} \rightarrow X_i^{\pm}.
\] (4.4)

This is the so–called Frobenius map (recall that \(\tilde{\mathfrak{g}} = \mathfrak{g}\) for odd roots of unity). It is generalized to even roots of unity in [12]; unfortunately the results given there are not very explicit. Since this case is of central importance to us, we will give an elementary, self–contained approach, and show explicitly how the action of \(U(\tilde{\mathfrak{g}})\) on \(L^{\text{res}}(\lambda_z)\) is given in terms of the \(X_j^{\pm(M_j)}\). The complications arise because at even roots of unity, \(K_i\) cannot be set to 1, while \(K_i^{2}\) must be, since \([Y_i^+, Y_i^-] = \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}\). Indeed,

\[
< K_i, \sum_j z_j \Lambda_j M_j > = q_i^{z_i M_i} = \pm 1.
\] (4.5)

These extra, “quasiclassical” generators \(K_i\) in some cases anticommute with \(X_j^{\pm(M_j)}\), and will extend the algebra \(U(\tilde{\mathfrak{g}})\). They will play an important role in the noncompact case.
Let \( a_i \in \{0, 1\} \) such that \( a_i + a_j = 1 \) if \( \tilde{A}_{ij} \neq 0 \) and \( i \neq j \); this is always possible. Define \( \tilde{K}_i = K_i^{M_i} \), and

\[
\begin{align*}
\tilde{X}_i^+ &= X_i^{+(M_i)} \tilde{K}_i^{a_i}, \\
\tilde{X}_i^- &= X_i^{-(M_i)} \tilde{K}_i^{1-a_i} q_i^{M_i^2}, \\
\tilde{H}_i &= [\tilde{X}_i^+, \tilde{X}_i^-]
\end{align*}
\]

(4.6)

Then we can show the following:

**Theorem 4.2** For all special points \( \lambda_z \), \( L^{res}(\lambda_z) \) is an irreducible highest–weight representation of the classical \( \tilde{U}(\tilde{g}) \), with generators \( \tilde{X}_i^\pm \) and \( \tilde{H}_i \). If \( v_{z'} \in L^{res}(\lambda_z) \) has weight \( \sum_j z_j' M_j \Lambda_j \), then \( \tilde{H}_i \cdot v_{z'} = z_i' v_{z'} \). Moreover,

\[
\tilde{X}_i^\pm \tilde{K}_j = s_{ij} \tilde{K}_j \tilde{X}_i^\pm
\]

(4.7)

where \( s_{ij} = q_i^{M_i M_j A_{ji}} = q_i^{M_i M_j (\alpha_i, \alpha_j)} = \pm 1 \). For dominant integral \( \lambda \), \( L^{res}(\lambda) \) is a direct sum of such irreducible representations, by (4.2).

This is proved in the appendix. Root vectors \( \tilde{X}_i^\pm \in U^{res}_q(\tilde{g}) \) for the remaining roots \( \tilde{\alpha} \in \tilde{g} \) are then obtained as classically; see in particular (A.13). From Section 3.1, the classical algebras are \( \tilde{B}_n = C_n \) and \( \tilde{C}_n = B_n \) if \( M \) is even, and \( \tilde{g} = g \) otherwise. This shows explicitly the refinements of (4.4) which arise for even roots of unity, in the most important case of finite–dimensional representations. Notice that for odd roots of unity, \( K_i \) evaluates to 1 on the special points \( \lambda_z \), and Theorem 4.2 essentially reduces to (1.4). The general, abstract result is given in [12].

To summarize the results of this section, any \( L^{res}(\lambda) \) for dominant integral \( \lambda \) is a direct sum of irreducible representations of \( U^{fin}_q \), which are related by an action of the classical \( U(\tilde{g}) \), extended by parity generators \( K_i \) for even roots of unity. In particular, this holds for unitary representations.

### 5 Noncompact forms and unitary representations

We first recall some concepts in the classical case, see e.g. [18, 19]. Consider a not necessarily compact semisimple Lie group \( G \) with real Lie algebra \( g \). Let \( -\sigma \) be the conjugation on the complexification \( g_C \) with respect to \( g \) extended as an involution, by which we mean an anti-linear anti–algebra map whose square is the identity; one could equally well consider algebra maps. On the other hand, the compact form \( g_K \) of \( g_C \) is the eigenspace with eigenvalue \(-1\).
of the Cartan–Weyl involution $\theta$. By a theorem of Cartan (see [18], Theorem 7.1), one can assume that $\sigma = \phi \circ \theta$, where $\phi$ is a linear automorphism of $\mathfrak{g}_K$ with $\phi^2 = 1$. Let $\mathfrak{k}$ be the eigenspace of $\phi$ with eigenvalue $+1$, and $\mathfrak{p}$ the eigenspace with eigenvalue $-1$. Then $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{p}$ is the Cartan decomposition of $\mathfrak{g}$, and $\mathfrak{k}$ is a maximal compact subalgebra. A root $\alpha$ is called compact if the corresponding root vector is in $\mathfrak{k}$. The star structure is then defined as $* = \sigma$.

Now there are two cases, depending on if $\phi$ is an inner automorphism or an outer automorphism [19]. In this work, we only consider the first type, which covers $\text{so}(n, 2p), \text{su}(n, p), \text{so}^*(2l), \text{sp}(n, p), \text{sp}(l, \mathbb{R})$, and various forms for the exceptional groups. We will find quantum versions and unitary representations for all them, even though not all of the representations will have a classical limit. The second type includes $\text{sl}(l+1, \mathbb{R}), \text{su}^*(l+1), \text{so}^*(2l-2p-1, 2p+1)$, and exceptional cases.

Up to equivalence, the inner automorphisms of a simple Lie algebra of rank $r$ are given by $2^r$ “chief” inner automorphisms of the form $\phi(H_i) = H_i$, $\phi(X_i^\pm) = s_iX_i^\mp$, for $s_i = \pm 1$ ([19], Ch. 14). They define the real forms
\begin{align*}
H_i^* &= H_i, \\
(X_i^\pm)^* &= s_iX_i^\mp, \quad \text{for } s_i = \pm 1.
\end{align*}
They are not necessarily inequivalent; the compact case corresponds to all $s_i = 1$. It should however be noted that real forms which are equivalent classically are not necessarily equivalent in the $q$–deformed case. For example, the real form $(X^\pm)^* = -X^\pm$, $H^* = -H$ for $|q| = 1$ of the ”non–restricted” $U_q(\text{sl}(2, \mathbb{R}))$ considered in [10] is classically equivalent to the form $(X^\pm)^* = -X^\mp, H^* = H$, which is a special case of (5.1). Nevertheless, the first form has no unitary representations at roots of unity if imposed on $U_q^{\text{fin}}(\text{sl}(2))$, while the second does.

We consider $U_q^{\text{fin}}$, which becomes a $*$–algebra for any of the forms (5.1) for $q$ a root of unity. Now we allow non–integral weights as well (it should be noted that the weights must be integral if working with $U_q^{\text{res}}(\mathfrak{g})$). Then there exist one–dimensional representations $L^{\text{fin}}(\lambda_r)$ of $U_q^{\text{fin}}$ with weight $\lambda_r = \sum_i r_iM_i\Lambda_i$ generalizing (3.4), where $r_i \in \mathbb{Q}$ such that $[r_iM_i]_{qi} = 0$, or equivalently $q^{2r_iM_id_i} = 1$ for all $i$. This follows immediately from (2.14). Explicitly,
\begin{equation}
\lambda_r = \sum_i \frac{m}{2nd_i}p_i\Lambda_i \tag{5.2}
\end{equation}
with $p_i \in \mathbb{Z}$.

Let $L^{\text{fin}}(\lambda)$ be a unitary representation of the compact form (such as in Theorem 3.2) with inner product $(\ , \ )$, and consider $L^{\text{fin}}(\lambda) \otimes L^{\text{fin}}(\lambda_r)$. This is again an irreducible
representation of $U_q^{\text{fin}}$, and we can define an inner product on it by
\[(v \otimes \rho_r, w \otimes \rho_r) := (v, w)\] (5.3)
where $\rho_r \in L^{\text{fin}}(\lambda_r)$. It is positive definite by definition. Let us calculate the adjoint of $X_i^\pm$ on this Hilbert space:
\[(v \otimes \rho_r, X_i^\pm \cdot (w \otimes \rho_r)) = (v \otimes \rho_r, X_i^\pm \cdot w \otimes q_i^{-r_iM_i/2}\rho_r) = q_i^{-r_iM_i/2}(v, X_i^\pm \cdot w).\] (5.4)
On the other hand,
\[(X_i^\mp \cdot (v \otimes \rho_r), w \otimes \rho_r) = (X_i^\mp \cdot v \otimes q_i^{-r_iM_i/2}\rho_r, w \otimes \rho_r) = (q_i^{-r_iM_i/2}v, X_i^\pm \cdot w)\] (5.5)
by unitarity of $L^{\text{fin}}(\lambda)$. By definition, the inner product is antilinear in the first argument.

Now there are 2 cases: first, if $q_i^{r_iM_i} = 1$, then $q_i^{-r_iM_i/2} = \pm 1$, and the adjoint of $X_i^\pm$ becomes $(X_i^\pm)^* = X_i^\mp$. Second, if $q_i^{r_iM_i} = -1$, then $q_i^{-r_iM_i/2} = \pm i$, and the adjoint of $X_i^\pm$ is $(X_i^\pm)^* = -X_i^\mp$. Therefore we have proved

**Theorem 5.1** Let $L^{\text{fin}}(\lambda)$ be a unitary representation of the compact form of $U_q^{\text{fin}}$, and $L^{\text{fin}}(\lambda_r)$ a one–dimensional representation of $U_q^{\text{fin}}$ with weight $\lambda_r = \sum r_iM_i\Lambda_i$ as in (5.2). Then $L^{\text{fin}}(\lambda+\lambda_r) = L^{\text{fin}}(\lambda) \otimes L^{\text{fin}}(\lambda_r)$ with inner product (5.3) is a unitary representation of the real form (5.1) of $U_q^{\text{fin}}$, where $s_i = q_i^{M_i} = \langle K_i, \lambda_r \rangle = \pm 1$. All unitary representations of that real form can be obtained in this way.

The last statement follows since the noncompact representations can similarly be “shifted” back to the compact form.

This explains the role of the extra, “quasiclassical” generators $K_i$ at even roots of unity: they determine the real form of a representation. While the symmetric form of the coproduct (2.11) was useful in the proof, it is irrelevant for the result.

For the remainder of this section we concentrate on the case of integral weights, i.e. $\lambda_r = \lambda_z$ as in (3.4), and determine which of the classical noncompact forms actually occur in this way.

In the simply laced case, $M_i = M$, and $q_i^{M_i} = -1$ precisely if $q$ is an even root of unity. Thus for odd roots of unity, $s_i = 1$ for all $i$, whereas for even roots of unity, $s_i = (-1)^{z_i}$, so that there are unitary representations for all the noncompact forms considered.

In the non–simply laced case, consider first $B_n, C_n$ and $F_4$. If $q$ is odd, i.e. $q_i^{M_*} = 1$ with odd $M_* = M$, then $M_i = M$, and $s_i = 1$ for all $i$. Therefore only the compact form occurs.
For even \( q \), one has to distinguish whether \( M = m/2 \) is even or odd. If \( M \) is odd, then \( M_l = M_s = M \), therefore \( q_{M_l}^{M_l} = 1 \) and \( q_{M_s}^{M_s} = -1 \). This means that only those noncompact forms with \( s_i = 1 \) for \( \alpha_i \) a long root and \( s_i = (-1)^{z_i} \) for \( \alpha_i \) a short root occur. If \( M \) is even, then \( M_l = M_s/2 \), and \( q_{M_l}^{M_l} = -1 \) for all \( i \). Therefore \( s_i = (-1)^{z_i} \) for all \( i \), and again all noncompact forms considered are realized (to recover the results in [7], notice that the conventions there are such that \( d_s = \frac{1}{2} \)).

Finally consider \( G_2 \). If \( q \) is odd, then \( M_l = M/3 \) if \( M \) is a multiple of 3, and \( M_l = M \) otherwise. In either case, \( q_{M_l}^{M_l} = 1 \) for all \( i \), and \( s_i = 1 \) for all \( i \). If \( q \) is even, then \( q_{M_l}^{M_l} = -1 \) for all \( i \), thus \( s_i = (-1)^{z_i} \) for all \( i \), and again all noncompact forms considered are realized.

The classical limit of these unitary representations will be discussed in Section 7. Notice that Theorem 5.1 also yields additional unitary representations of the compact form with generally non–integral weights, for \( q_{M_i}^{M_i} = 1 \). We will see in Proposition 7.1 however that the distance of their weights from the origin becomes infinite as \( q \) approaches 1. In that sense, they are non–classical.

6 Reality–preserving algebra on \( L^{res}(\lambda) \)

Consider a dominant integral weight \( \lambda_0 \) such that \( L^{fin}(\lambda_0) \) is a unitary representation of the compact form with \( 0 \leq (\lambda_0, \alpha_i^\vee) < M_i \) for all \( i \), and a special point \( \lambda_z \). By Theorem 4.1, \( L^{res}(\lambda_0 + \lambda_z) = \oplus_{\lambda_{z'}} (L^{fin}(\lambda_0) \otimes L^{fin}(\lambda_{z'})) \) is a direct sum of irreducible representations of \( U_q^{fin} \), where the \( L^{fin}(\lambda_{z'}) \) are one–dimensional components of \( L^{res}(\lambda_z) \). These sectors are unitary representations of various real forms of \( U_q^{fin} \), according to Theorem 5.1. Moreover by Theorem 4.2, the “large” generators \( \tilde{X}_i^\pm \) connect the various sectors with different \( \lambda_{z'} \). It is natural to ask which subalgebra of the classical \( U(\tilde{\mathfrak{g}}) \) connects only those sectors with the same real form. This will be called reality–preserving algebra. Of course, the \( (\tilde{X}_i^\pm)^2 \) always preserve the real form, but they do not form a closed algebra.

\( X_\alpha^\pm \) as defined below Theorem 1.2 preserves the real form if and only if

\[
[X_\alpha^\pm, K_i] = 0 \quad \text{for all } i, \tag{6.1}
\]

or

\[
q^{M_i(\alpha, \beta)} = 1 \quad \text{for all } i \tag{6.2}
\]

This is equivalent to \( q^{M_i(\alpha, \beta)} = 1 \) for all roots \( \beta \). Using the Weyl group, we can assume that \( \alpha = \alpha_j \) is a simple root, since all other \( \alpha \) satisfying (6.2) are then obtained as the image under the Weyl group of the simple ones.

First consider the simply laced case. Then for any \( j \), there is an \( i \) such that \( (\alpha_i, \alpha_j) = -1 \), therefore \( q^{M_j(\alpha_i, \alpha_j)} = 1 \) only if \( q \) is odd. But then all sectors are all compact. Therefore the reality–preserving algebra is \( \mathfrak{g} \) for odd \( q \), and trivial otherwise.
Next consider the non–simply laced case. If $q$ is odd, then all forms are compact, and the reality–preserving algebra is clearly $\mathfrak{g}$.

Thus assume $q$ is even. For $G_2$, $q^{M_j(\alpha_i, \alpha_j)} = q^{-3M_j} = -1$ if $i \neq j$, and the reality–preserving algebra is trivial.

For $B_n, C_n$ and $F_4$, observe first that if $A_{ij} \neq 0$ and $d_j \geq d_i$, then $(\alpha_i, \alpha_j) = -d_j$. Therefore $q^{M_j(\alpha_i, \alpha_j)} = q^{M_{\max\{d_i, d_j\}}}$.

One has to distinguish $M = m/2$ even and odd. Assume $M$ is even, so that $M_i = M_j/2 = M/2$. Then the only way that $q^{M_{\max\{d_i, d_j\}}} = 1$ for all $i \neq j$ with $A_{ij} \neq 0$ is $M_j = M$ and $\max\{d_i, d_j\} = 2$, i.e. $j$ is short and is connected only to long nodes in the Dynkin diagram. The only case where this happens is $B_n$, which has one short simple root. By the Weyl group, it follows that the reality–preserving algebra is generated by all $X_{\alpha_s}$ where $\alpha_s$ are the short roots of $B_n$. Since $q$ is even, the dual algebra $\mathfrak{g}$ of $B_n$ is $C_n$, i.e. these $X_{\alpha_s}$ correspond precisely to the long roots of $\mathfrak{g}$. Now $C_n$ has precisely $n$ long roots which are all orthogonal, and the corresponding root vectors commute. Therefore for even $M$, the reality–preserving algebra for $B_n$ is $(su(2))^n$, generated by the $X_{\alpha_s}$ which commute with each other (on $L^{res}(\lambda)$). For $C_n$ and $F_4$, it is trivial except for $C_2 \cong B_2$.

If $M$ is odd, then $M_i = M_j = M$, thus $q^{M_{\max\{d_i, d_j\}}} = 1$, and $q^{M_{\max\{d_i, d_j\}}} = -1$. Therefore if $q^{M_{\max\{d_i, d_j\}}} = 1$ for all $i \neq j$ with $A_{ij} \neq 0$, then either $j$ must be long, or $j$ is connected only to long nodes in the Dynkin diagram. For $B_n$ this holds for all $j$, for $C_n$ this holds for the one long simple root, and for $F_4$ this holds for the 2 long simple roots. Therefore the reality–preserving algebra for $B_n$ is again $B_n$, with generators $X_{\alpha_s}$ for all $\alpha$. For $C_n$, it is $(su(2))^n$ with generators $X_{\alpha_s}$ which commute with each other, where $\alpha_l$ are the long roots of $C_n$. For $F_4$, it is the algebra generated by all long roots, which is $D_4$.

7 Non–integral weights and the classical limit

In this section, we want to determine which of the unitary representations of $U_q^{fin}$ in Theorem 5.1 have a well–defined classical limit. The idea is to consider them as highest–weight modules in a suitable way with fixed highest weight, and let $q$ approach 1.

For dominant integral $\lambda_0$ and $\lambda_r = \sum p_j \frac{m}{2n-d}, \Lambda_j$ as in (5.2), consider

$$L^{fin}(\lambda) = L^{fin}(\lambda_0) \otimes L^{fin}(\lambda_r) \quad \text{with}$$

$$\left(\lambda_0 + \rho, \alpha^\vee\right) \leq \left[\frac{m}{2n-d}\right] + 1 \quad \text{for all} \quad \alpha \in Q^+, \quad (7.1)$$

where $\lambda = \lambda_0 + \lambda_r$. According to Theorems 3.2 and 5.1, this is a unitary representation of a certain noncompact form determined by $\lambda_r$. We want to understand the location of the weights of $L^{fin}(\lambda)$ in weight space, and in particular if they are close enough to the origin.
so that they can have a classical limit as a highest weight module. For \( q \neq 1 \) of course, they can always be viewed as highest weight modules.

The bound (7.1) for \( L_{\text{fin}}(\lambda) \) being unitary can be stated more geometrically as follows. Divide weight space into alcoves separated by the hyperplanes

\[
h_{\alpha}^z := \{ \mu; (\mu, \alpha) = m2nz \}
\]

(7.2)

for all roots \( \alpha \) and \( z \in \mathbb{Z} \), similar as in Section 3.1. Then \( \lambda_0 \) is in the fundamental alcove by (7.1), using the fact that \( (\rho, \alpha^\vee) \geq 1 \) for all positive roots \( \alpha \); the latter can be seen using \( \rho = \sum \Lambda_i \).

By the Weyl group, all weights of \( L_{\text{fin}}(\lambda_0) \) are therefore contained in the union of those alcoves which have the origin as corner, more precisely within a certain distance from its walls as determined by (7.1). Since the set of hyperplanes (7.2) is invariant under translations by \( \lambda_r \), the weights of \( L_{\text{fin}}(\lambda) \) are contained in the union of those alcoves which have \( \lambda_r \) as corner. In particular, they are contained in a half–space with the origin on its boundary. Since the distance between parallel hyperplanes goes to infinity as \( q \to 1 \), \( L_{\text{fin}}(\lambda) \) can have a classical limit only if \( \lambda_r \) and the origin belong to the same alcove. This puts a restriction on the possible real forms as determined by \( \lambda_r \).

To make this more precise, recall the definition of compact roots in Section 5, and the definition of the Coxeter labels \( a_i \) which are the coefficients of the highest root \( \theta = \sum a_j \alpha_j \), and satisfy \( a_i \geq 1 \) for all \( i \).

**Proposition 7.1** Consider \( \lambda_r = \sum p_j \frac{m}{2nd_j} \Lambda_j \) with \( p_j \in \mathbb{Z} \), as in (7.2). It belongs to the closure of an alcove as defined above which also contains the origin as a corner, if and only if there is a set of simple roots (denoted again by \( \alpha_i \)) such that \( \alpha_{i_0} \) has Coxeter label \( a_{i_0} = 1 \) and \( p_j = \pm \delta_{j,i_0} \), hence

\[
\lambda_r = \pm \frac{m}{2nd_{i_0}} \Lambda_{i_0}.
\]

(7.3)

In that case, the remaining \( r-1 \) simple roots are compact with respect to the real form defined in Theorem (5.1).

**Proof** By using an element \( \sigma \) of the Weyl group if necessary, we can assume that \( \lambda_r \) is an anti–dominant weight. Then \( \lambda_r = -\sum p_j \frac{m}{2nd_j} \Lambda_j \) with \( p_j \in \mathbb{N} \), and \( (\lambda_r, \theta) = -\sum p_j a_j \frac{m}{2nd_j} \).

This implies that \( (\lambda_r, \theta) \leq -\frac{m}{2n} (h_\theta^{-1}, \theta) \) where \( h_\theta^{-1} \) is a hyperplane as defined in (7.2), and equality holds precisely if \( \lambda_r = -\frac{m}{2nd_{i_0}} \Lambda_{i_0} \) and \( a_{i_0} = 1 \) for some \( i_0 \). The desired set of simple roots is obtained by applying \( \sigma \) to the original simple roots. Then all (new) simple roots \( \alpha_j \) for \( j \neq i_0 \) are compact according the definition in Theorem (5.1). \( \square \)
We will always use this set of simple roots from now on. The corresponding generators in $U_{q}^{res}(g)$ could be obtained via the braid group action \[13\], but this is not needed since we will only make statements about the characters below. The corresponding real form is

\[(X_{i_{0}}^{\pm})^* = -X_{i_{0}}^{\mp}, \quad \text{and} \quad (X_{j}^{\pm})^* = X_{j}^{\mp} \quad \text{for} \quad j \neq i_{0}.\] (7.4)

In the classical limit, the center of $k$ is then one–dimensional and generated by an element of the Cartan subalgebra dual to $\Lambda_{i_{0}}$, which is orthogonal to the compact roots. Explicitly, this leads to the following cases:

- $i_{0} = 1, 2, \ldots, l$ for $A_l$, corresponding to $su(l + 1 - p, p)$ for all $p$
- $i_{0} = 1$ for $B_l$, corresponding to $so(2l - 1, 2)$
- $i_{0} = 1$ for $D_l$, corresponding to $so(2l - 2, 2)$
- $i_{0} = l$ or equivalently $i_{0} = l - 1$ for $D_l$, corresponding to $so^{*}(2l)$
- $i_{0} = l$ for $C_l$, corresponding to $sp(l, \mathbb{R})$
- $i_{0} = 1$ or equivalently $i_{0} = 5$ for $E_6$, and $i = 6$ for $E_7$.

see for example \[19\], table 14.1. Not surprisingly, these are precisely the cases where highest weight modules exist in the classical limit, see \[20, 21\] and references therein. We will restrict ourselves to (7.4) from now on, and show how to recover the classical unitary highest weight representations from the $L^{\text{fin}}(\lambda)$. To do that, we choose a minus sign in (7.3), since then it is possible to consider highest–weight modules $L^{\text{fin}}(\lambda)$ with fixed highest weight $\lambda$ independent of $q$, in particular as $q \to 1$. The plus sign would correspond to lowest–weight modules in the classical limit.

To make the connection with the literature on the classical case \[20\], consider the character $\chi(L(\lambda + z\Lambda_{i_{0}}))e^{-z\Lambda_{i_{0}}}$ for $z \in \mathbb{R}$, where $L(\lambda + z\Lambda_{i_{0}})$ is the classical irreducible highest weight module with highest weight $\lambda + z\Lambda_{i_{0}}$. It is independent of $z$ for sufficiently negative $z$, which can be seen from the strong linkage principle (see e.g. \[22\]): by writing $\lambda = c_{0}\Lambda_{i_{0}} + \sum_{j \neq i_{0}} n_{j}\Lambda_{j}$ and noticing that the compact roots are orthogonal to $\Lambda_{i_{0}}$, it follows that for sufficiently negative $z$, all weights strongly linked to $\lambda + z\Lambda_{i_{0}}$ are in the orbit of the compact Weyl group acting on $\lambda + z\Lambda_{i_{0}}$. The first reduction point $z_{0}$ is the maximal value of $z$ where this is no longer the case. Clearly $L(\lambda + z\Lambda_{i_{0}})$ can only be unitary with respect to (7.4) if $\lambda$ is a dominant integral weight with respect to $\mathfrak{k}$, i.e. $n_{j} \in \mathbb{N}$ in the above notation. Provided this is the case, $L(\lambda + z\Lambda_{i_{0}})$ is unitary \[20\] if and only if $z \leq z_{0}$, or $z$ is in a certain finite set of $z > z_{0}$.

In the $q$–deformed case, we can show the following:
Proposition 7.2 Let λ be a rational weight which is dominant integral with respect to ℱ. If the first reduction point of \( L(\lambda + z\Lambda_{i_0}) \) is at \( z \geq 0 \) for \( q = 1 \), then there exists a series of roots of unity \( q_k \rightarrow 1 \) such that \( L^{\text{fin}}(\lambda) \) is unitary with respect to (7.4) for all \( q = q_k \). In particular, this holds if \( (\lambda + \rho,\alpha^\vee) \leq [(\lambda,\alpha^\vee) + (\lambda,\alpha^\vee)] + 1 \) for all positive noncompact roots \( \alpha \).

Of course, this generalizes to irrational \( \lambda \) which can be approximated by rational weights as above.

**Proof** Assume that \( \lambda \) is as required. Then there are \( m,n \in \mathbb{N} \) such that \( (\lambda - \lambda_r,\alpha^\vee_{i_0}) \in \mathbb{N} \) where \( \lambda_r = -\frac{m}{2n\alpha_{i_0}} \Lambda_{i_0} \), thus \( \lambda - \lambda_r \) is dominant integral. For \( k \in \mathbb{N} \), let \( m_k := m + 2nk\alpha_{i_0} \), \( \lambda_{r,k} := -\frac{m}{2n\alpha_{i_0}} \Lambda_{i_0} \), and \( q_k := e^{2\pi i/n_k} \). We claim that for sufficiently large \( k \), the character of \( L^{\text{fin}}(\lambda - \lambda_{r,k}) \) is given by Weyl’s formula for all \( q' \) between 1 and \( q_k \). Then the first part of the proposition follows from Theorems 3.2 and 5.1.

Let \( q' = e^{2\pi i/n'}/m' \) with \( \frac{n'}{m'} < \frac{n}{m} \), with associated \( M'_\alpha \) as in Section 2. By the strong linkage principle \([3,7]\), the character of \( L^{\text{fin}}(\lambda - \lambda_{r,k}) \) can differ from \( \chi(\lambda - \lambda_{r,k}) \) only by the sum of classical characters \( \chi(\mu_n) \) \([3,3]\) with dominant \( \mu_n \), which are “strongly linked” to \( \lambda - \lambda_{r,k} \) by a series of reflections by hyperplanes \( H_{\alpha'}^{\text{fin}} \) defined in \([3,8]\) using \( M'_\alpha \), but shifted by \(-\rho\). They again divide weight space into (shifted) alcoves, with corresponding special points for \( q' \), also shifted by \(-\rho\).

Now \( \alpha_{i_0} = 1 \) implies that \(-\rho \) and \(-\lambda_{r,k} - \rho \) are in the same shifted alcove for any \( q' \) between 1 and \( q_k \), because \( (-\lambda_{r,k},\alpha^\vee) \leq \frac{m}{2n\alpha_{i_0}} \leq M'_\alpha \) for all positive \( \alpha \). Moreover, the union of the alcoves which have \(-\rho \) as a corner is a convex set of weights, and invariant under the Weyl group action with center \(-\rho\). Therefore all weights in that set which are strongly linked to \(-\lambda_{r,k} - \rho \) are obtained by the action of the classical Weyl group with center \(-\rho\); in particular, \(-\rho \) is not. Thus if \( k \) is large enough, all dominant \( \mu_n \) strongly linked to \( \lambda - \lambda_{r,k} \) can be obtained by reflections of \( \lambda - \lambda_{r,k} \) by those \( H_{\alpha'}^{\text{fin}} \) which contain the special point \( M'_{i_0} \Lambda_{i_0} - \rho \). However using the assumption, the character of \( L^{\text{fin}}(\lambda - \lambda_{r,k}) \) is not affected by these \( \mu_n \); indeed, by a shift by \(-\frac{n'}{2n'd_{i_0}} \Lambda_{i_0} \) as in Section 5, \( L^{\text{fin}}(\lambda - \lambda_{r,k}) \) can be related to \( L^{\text{fin}}(\lambda + z\Lambda_{i_0}) \) for \( z = \frac{m}{2n\alpha_{i_0}} - \frac{m'}{2n'd_{i_0}}, \leq 0 \). The special point \( M'_{i_0} \Lambda_{i_0} - \rho \) is then moved to \( (M'_{i_0} - \frac{m'}{2n'd_{i_0}}) \Lambda_{i_0} - \rho \), and is only relevant for large \( k \) if \( M'_{i_0} = \frac{m'}{2n'd_{i_0}} \), when it becomes \(-\rho\). However by the assumption on the first reduction point, the character of the classical \( L(\lambda + z\Lambda_{i_0}) \) is not affected by the hyperplanes through \(-\rho\) for \( z < 0 \). Using the fact that \( U_q(\mathfrak{g}) \) is the same as \( U(\mathfrak{g}) \) as algebra over \( \mathbb{C}[q - 1] \) \([28]\), the character of \( L^{\text{fin}}(\lambda + z\Lambda_{i_0}) \) is not affected by the hyperplanes through \(-\rho\) either. Combining all this, it follows that the character of \( L^{\text{fin}}(\lambda - \lambda_{r,k}) \) is given by Weyl’s formula for all \( q' \) between 1 and \( q_k \).

In particular, this holds if \( (\lambda - \lambda_{r,k} + \rho,\alpha^\vee) \leq [(\lambda,\alpha^\vee) + (\lambda,\alpha^\vee)] + 1 \) for all positive noncompact roots \( \alpha \), by Theorem 3.2. This is certainly satisfied for compact \( \alpha \) if \( k \) is sufficiently large.
Using \( \frac{m}{2d_{t_0}} = -(\lambda, \alpha_{t_0}') + (\lambda - \lambda_{r,k}, \alpha_{t_0}') \in -(\lambda, \alpha_{t_0}') + \mathbb{N} \) and the fact that \( d_{t_0} = 2 \) for the non–simply laced cases, this bound follows from the given condition. \( \square \)

Therefore we recover the classical results on unitary highest weight representations, except for the small, finite set of \( z > z_0 \), which we cannot address here. It is quite possible that there exist unitary representations of \( U_{q}^{fin} \) corresponding to these remaining cases; this would have to be studied by other methods. By Theorem 5.1, they would correspond to additional unitary representations of the compact form, as was pointed out in Section 3.

In the example of the Anti–de Sitter group, the highest–weight property corresponds to positivity of the energy \( \tilde{h} \), which is an important physical requirement.

Notice that the unitary representations of noncompact forms of \( U_{q}^{fin} \) in general have non–integer, but rational weights. Those with integral weights can have a classical limit only if \( \frac{m}{2d_{t_0}} \in \mathbb{Z} \), in particular \( q \) must be an even root of unity.

The question arises if and how the unitary representations of \( U_{q}^{fin} \) in those cases where there exists no classical unitary highest weight representation might be related to other classical series of unitary representations, and how the latter may be obtained from the quantum case at roots of unity. The answer may be related to the fact that there do exist other types of unitary representations of the non–restricted specialization for \( |q| = 1 \), such as \( U_{q}(sl(2, \mathbb{R})) \) \([10]\), as was mentioned in Section 5. This certainly deserves further investigation.

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\section{Appendix}

We prove Theorem 4.2 by verifying the classical relations of the Chevalley basis \( \tilde{X}_i^\pm \) and \( \tilde{H}_i \).

To calculate \( \tilde{H}_i \) on the weights \( \lambda_z = \sum_j z_j M_j \Lambda_j \), one can use the standard commutation relation

\[ [X_i^{+(M_i)}, X_i^{-(M_i)}] = \left[ \frac{H_i}{M_i} \right]_{q_i} + u^{fin}_{q_i} \]  

where the last term vanishes on \( L^{res}(\lambda_z) \). We also need the following identity \([12]\) which can be checked directly: If \( q^{2M} = 1 \), then

\[ \left[ aM + b \atop cM + d \right]_{q} = q^{M^2c(a+1)+M(ad-bc)} \left[ b \atop d \right]_{q} \left[ a \atop c \right]_{1}. \]  

(A.2)
Furthermore $[K_i, X_j^{\pm(M_j)}] = 0$, and therefore $[\tilde{X}_i^+, \tilde{X}_i^-] = [X_i^{+(M_i)}, X_i^{-(M_i)}] K_i q_i^M$. Using (A.1), (A.2) and (A.3), this evaluates on $v_{\lambda_{z'}}$ to

$$\tilde{H}_i \cdot v_{\lambda_{z'}} = \left[ z_i' M_i \
\frac{M_i}{q_i} \right] q_i^M, \quad v_{\lambda_{z'}} = z_i' v_{\lambda_{z'}}, \quad (A.3)$$

as claimed.

We next check that

$$[\tilde{X}_i^+, \tilde{X}_j^-] = 0 \quad (A.4)$$

for $i \neq j$. This is clear if $A_{ij} = 0$. Otherwise, one can write

$$X_i^{+(M_i)} K_j = s_{ij} K_j X_i^{+(M_i)}, \quad (A.5)$$

where $s_{ij} = q_i^{M_i M_j A_{ij}} = q_i M_j(\alpha_i, \alpha_j) = s_{ji} = \pm 1$. Then $s_{ij} s_{ij}^{-1} = q_i M_j(\alpha_i, \alpha_j)^2 \alpha_i = 1$, since $A_i = (1 - A_j)$ if $A_{ij} \neq 0$ and $i \neq j$. Therefore

$$X_i^{+} X_j^{-} = s_{ij} s_{ij}^{-1} X_j^{-} X_i^{+} = X_j^{-} X_i^{+}. \quad (A.6)$$

Next, to verify

$$[\tilde{H}_i, \tilde{X}_j^{\pm}] = \pm \tilde{A}_{ji} \tilde{X}_j^{\pm}, \quad (A.7)$$

replace again $\tilde{H}_i$ by $\left[ H_i \\frac{M_i}{q_i} \right] q_i K_i q_i^M$, and observe using (A.7) that

$$[H_i] q_i X_j^{\pm(M_j)} = X_j^{\pm(M_j)} [H_i \pm M_j A_{ji}] q_i = X_j^{\pm(M_j)} [H_i \pm M_i \tilde{A}_{ji}] q_i. \quad (A.8)$$

We first show

$$\left[ H_i \\frac{M_i}{q_i} \right] K_i q_i^M X_j^{+(M_j)} \tilde{K}_j = X_j^{+(M_j)} \tilde{K}_j \left[ H_i \\frac{M_i}{q_i} \right] K_i q_i^M + \tilde{A}_{ji}. \quad (A.9)$$

Using the above, this becomes

$$X_j^{+(M_j)} \tilde{K}_j \left[ H_i + \tilde{A}_{ji} M_i \frac{M_i}{q_i} \right] K_i s_{ji} q_i^M = X_j^{+(M_j)} \tilde{K}_j \left[ H_i \frac{M_i}{q_i} \right] K_i q_i^M + \tilde{A}_{ji}. \quad (A.10)$$

Restricting on a weight $\lambda_{z'}$, it remains to show

$$\left[ (z_i' + \tilde{A}_{ji}) M_i \right] q_i^{M_{z'}^2} s_{ji} q_i^M = \left[ z_i' M_i \frac{M_{z'}^2}{q_i} \right] q_i^{M_{z'}^2} q_i^M + \tilde{A}_{ji}. \quad (A.11)$$
Now $s_{ji} = q_i^{M^2_i} {\tilde A}_{ji}$, and the claim follows from (A.2). The calculation for $[\tilde{H}_i, \tilde{X}_j] = -\tilde{A}_{ji} \tilde{X}_j$ is completely analogous.

Finally, the Serre relations are

$$[\tilde{X}_i^+, \ldots, [\tilde{X}_i^+, \tilde{X}_j^+]]^{1-\tilde{A}_{ji}} = 0 \quad \text{(A.12)}$$

(1 - $\tilde{A}_{ji}$ brackets) on $L^{res}(\lambda_z)$, and similarly for the negative roots.

To prove this, consider a P.B.W. basis of $U_q^{+res}(\mathfrak{g})$, which is given by the expressions $X^{+ (t_N)}_{\alpha N} \ldots X^{+ (t_1)}_{\alpha 1}$ where $\{\alpha_1, \ldots, \alpha_N\}$ is an ordered basis of the positive roots, obtained e.g. by the braid group action [12]. Let $\tilde{Q} = \{\tilde{\alpha} = M_{\alpha} \alpha\}$ be the set of roots of the lattice of special points. For $k \in \mathbb{N}$ such that $\tilde{\beta} = k \tilde{\alpha}_i + \tilde{\alpha}_j \in \tilde{Q}$, define $\tilde{X}^+_\beta := X^{+(M_{\beta})} \tilde{K}_{i}^{k \alpha_i} \tilde{K}_j^{a_j}$ generalizing (4.6), and $\tilde{X}^+_\beta := 0$ if $\tilde{\beta} \notin \tilde{Q}$. We claim that

$$[\tilde{X}^+_i, \tilde{X}^+_j] = c\tilde{X}^+_{\tilde{\alpha}_i + \tilde{\beta}} \quad \text{(A.13)}$$

if acting on $L^{res}(\lambda_z)$, for some constant $c$. This clearly implies the Serre relations. The proof is by induction on $k$, using the well–known commutation relations [16]

$$X^{+(\alpha r \cdot \alpha s)}_{\alpha r \cdot \alpha s} X^+_{\alpha r} X^+_{\alpha s} = \sum c(t_{r+1}, \ldots, t_{s-1}) X^+_{\alpha_{r+1}} \ldots X^+_{\alpha_{s-1}} \quad \text{(A.14)}$$

for $r < s$, with some constant $c(t_{r+1}, \ldots, t_{s-1})$.

We want to order the expression $\tilde{X}^+_\beta \tilde{X}^+_i$ (or the reversed form) as in the P.B.W. basis, using (A.14). The leading term is

$$q^{M_i M_{\beta}(\alpha_i, \beta)} s^a_i s^{a_j}_{ji} \tilde{X}^+_i \tilde{X}^+_\beta \quad \text{(A.15)}$$

since $[\tilde{K}_{i}^{a_i}, X^{+(M_{\beta})}_i] = 0$. We claim that the only other term on the rhs of (A.14) which may not vanish on $L^{res}(\lambda_z)$ is proportional to $\tilde{X}^+_{\tilde{\alpha}_i + \tilde{\beta}}$. This is so because only products of “large” generators $X^{+(M_{\alpha})}_\alpha$ are nonzero on $L^{res}(\lambda_z)$, and in fact only one “large” generator can occur on the rhs of (A.13), because only a simple (formal) pole in $q$ can arise by the derivation property mentioned in Section 2. Moreover using $\tilde{\beta} = M_{\beta} \beta = k M_{i} \alpha_i + M_{j} \alpha_j$, it follows that $q^{M_i M_{\beta}(\alpha_i, \beta)} = q^{2k M^2_i a_i} q^{M_i M_j(\alpha_j, \alpha_i)} = q^{M_i M_j(\alpha_j, \alpha_i)} = s_{ij}$. Thus the overall coefficient in front of (A.13) is $s_{ij}^{1-a} s^{a_j}_{ji}$, which is 1 as above. This concludes the proof.

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