Cosmological Singularities, Billiards and Lorentzian Kac-Moody Algebras

Thibault Damour

Institut des Hautes Études Scientifiques, 35 route de Chartres, 91440 Bures-sur-Yvette, France

Abstract

The structure of the general, inhomogeneous solution of (bosonic) Einstein-matter systems in the vicinity of a cosmological singularity is considered. We review the proof (based on ideas of Belinskii-Khalatnikov-Lifshitz and technically simplified by the use of the Arnowitt-Deser-Misner Hamiltonian formalism) that the asymptotic behaviour, as one approaches the singularity, of the general solution is describable, at each (generic) spatial point, as a billiard motion in an auxiliary Lorentzian space. For certain Einstein-matter systems, notably for pure Einstein gravity in any spacetime dimension \( D \) and for the particular Einstein-matter systems arising in String theory, the billiard tables describing asymptotic cosmological behaviour are found to be identical to the Weyl chambers of some Lorentzian Kac-Moody algebras. In the case of the bosonic sector of supergravity in 11-dimensional spacetime the underlying Lorentzian algebra is that of the hyperbolic Kac-Moody group \( E_{10} \), and there exists some evidence of a correspondence between the general solution of the Einstein-threeform system and a null geodesic in the infinite dimensional coset space \( E_{10}/K(E_{10}) \), where \( K(E_{10}) \) is the maximal compact subgroup of \( E_{10} \).

It is a pleasure to dedicate this review to Stanley Deser, a dear friend and a great physicist to whom I owe a lot.

1 Introduction and overview

A remarkable connection between the asymptotic behavior of certain Einstein-matter systems near a cosmological singularity and billiard motions in the Weyl chambers of some corresponding Lorentzian Kac–Moody algebras was
uncovered in a series of works [1, 2, 3, 4, 5, 6, 7]. This simultaneous appearance of billiards (with chaotic properties in important physical cases) and of an underlying symmetry structure (infinite-dimensional Lie algebra) is an interesting fact, which deserves to be studied in depth. Before explaining the techniques (notably the Arnowitt-Deser-Misner Hamiltonian formalism [8]) that have been used to uncover this fact, we will start by reviewing previous related works, and by stating the main results of this billiard/symmetry connection.

The simplest example of this connection concerns the pure Einstein system in $D = 3+1$-dimensional space-time. The Einstein equations ($R_{\mu\nu}(g_{\alpha\beta}) = 0$) are non-linear PDE’s for the metric components. Near a cosmological spacelike singularity, here chosen as $t = 0$, the spatial gradients are expected to become negligible compared to time derivatives ($\frac{\partial}{\partial x} \ll \frac{\partial}{\partial t}$); this then suggests the decoupling of spatial points and allows for an approximate treatment in which one replaces the above partial differential equations by a 3-dimensional family of ordinary differential equations. Within this simplified context, Belinskii, Khalatnikov and Lifshitz (BKL) gave a description [9, 10, 11] of the asymptotic behavior of the general solution of Einstein’s equations, close to the singularity, and showed that it can be described as a chaotic [12, 13] sequence of generalized Kasner solutions. The Kasner metric is of the type

$$g_{\alpha\beta}(t)dx^\alpha dx^\beta = -N^2 dt^2 + A_1 t^{2p_1} dx_1^2 + A_2 t^{2p_2} dx_2^2 + A_3 t^{2p_3} dx_3^2$$

(1.1)

where the constants $p_i$ obey

$$\overrightarrow{p}^2 = p_1^2 + p_2^2 + p_3^2 - (p_1 + p_2 + p_3)^2 = 0.$$  

(1.2)

An exact Kasner solution, with a given set of $A_i$’s and $p_i$’s, can be represented by a null line in a 3-dimensional auxiliary Lorentz space with coordinates $p_1, p_2, p_3$ equipped with the metric given by the quadratic form $\overrightarrow{p}^2$ above. The auxiliary Lorentz space can be radially projected on the unit hyperboloid or further on the Poincaré disk (i.e. on the hyperbolic plane $H_2$): the projection of a null line is a geodesic on the hyperbolic plane.

BKL showed that, because of non-linearities in Einstein’s equations, the generic solution behaves as a succession of Kasner epochs, i.e., to a broken null line in the auxiliary Lorentz space, or a broken geodesic on the Poincaré disk. This broken geodesic motion is a “billiard motion” (seen either in Lorentzian space or in hyperbolic space).

\[^1\text{In the } N = 1 \text{ gauge, they also obey } p_1 + p_2 + p_3 = 1.\]
The billiard picture naturally follows from the Hamiltonian approach to cosmological behavior and was first obtained in the homogeneous (Bianchi IX) four-dimensional case [14, 15] and then extended to higher space-time dimensions with $p$-forms and dilatons [3, 6, 7, 18, 19, 20, 21]. Recent work [7] has improved the derivation of the billiard picture by using the Iwasawa decomposition of the spatial metric. Combining this decomposition with the Arnowitt-Deser-Misner Hamiltonian formalism highlights the mechanism by which all variables except the scale factors and the dilatons get asymptotically frozen. The non-frozen variables (logarithms of scale factors and dilatons) then undergo a billiard motion. This billiard motion can be seen either in Lorentzian space or, after radial projection, on hyperbolic space (see below for details).

A remarkable connection was also established [1, 2, 3, 4, 5, 6, 7] between certain specific Einstein-matter systems and Lorentzian Kac-Moody (KM) algebras [22]. In the leading asymptotic approximation, this connection is simply that the Lorentzian billiard table within which the motion is confined can be identified with the Weyl chamber of some corresponding Lorentzian KM algebra. This can happen only when many conditions are met: in particular, (i) the billiard table must be a Coxeter polyhedron (the dihedral angles between adjacent walls must be integer submultiples of $\pi$) and ii)
the billiard must be a simplex. Surprisingly, this occurs in many physically interesting Einstein-matter systems. For instance, pure Einstein gravity in $D$ dimensional space-time corresponds to the Lorentzian KM algebra $AE_{D-1}$ [4] which is the overextension of the finite Lie algebra $A_{D-3}$: for $D = 4$, the algebra is $AE_3$ the Cartan matrix of which is given by

$$A = \begin{pmatrix}
2 & -1 & 0 \\
-1 & 2 & -2 \\
0 & -2 & 2
\end{pmatrix}$$

Chaotic billiard tables have finite volume in hyperbolic space, while non-chaotic ones have infinite volume; as a consequence, chaotic billiards are associated with hyperbolic KM algebras; this happens to be the case for pure gravity when $D \leq 10$.

Another connection between physically interesting Einstein-matter systems and KM algebras concerns the low-energy bosonic effective actions arising in string and $M$ theories. Bosonic string theory in any space-time dimension $D$ is related to the Lorentzian KM algebra $DE_D$ [3, 5]. The latter algebra is the canonical Lorentzian extension of the finite-dimensional algebra $D_{D-2}$. The various superstring theories (in the critical dimension $D = 10$) and $M$-theory have been found [3] to be related either to $E_{10}$ (when there are two supersymmetries in $D = 10$, i.e. for type IIA, type IIB and $M$-theory) or to $BE_{10}$ (when there is only one supersymmetry in $D = 10$, i.e. for type I and II heterotic theories), see the table below. A construction of the Einstein-matter systems related to the canonical Lorentzian extensions of all finite-dimensional Lie algebras $A_n$, $B_n$, $C_n$, $D_n$, $G_2$, $F_4$, $E_6$, $E_7$ and $E_8$ (in the above “billiard” sense) is presented in Ref. [5]. See also Ref. [23] for the identification of all hyperbolic KM algebras whose Weyl chambers are Einstein billiards.

The correspondence between the specific Einstein–three-form system (including a Chern–Simons term) describing the bosonic sector of 11-dimensional supergravity (also known as the “low-energy limit of $M$-theory”) and the hyperbolic KM group $E_{10}$ was studied in more detail in [6]. Reference [6] introduces a formal expansion of the field equations in terms of positive roots, i.e. combinations $\alpha = \Sigma_i n^i \alpha_i$ of simple roots of $E_{10}$, $\alpha_i$, $i = 1, \ldots, 10$, where the $n^i$’s are integers $\geq 0$. It is then useful to order this expansion according to the height of the positive root $\alpha = \Sigma_i n^i \alpha_i$, defined as $ht(\alpha) = \Sigma_i n^i$. The correspondence discussed above between the leading asymptotic evolution near a cosmological singularity (described by a billiard) and Weyl chambers of KM algebras involves only the terms in the field equation whose height is $ht(\alpha) \leq 1$. By contrast, the authors of Ref. [6] managed to show,
by explicit calculation, that there exists a way to define, at each spatial point \( x \), a correspondence between the field variables \( g_{\mu\nu}(t, x) \), \( A_{\mu\nu\lambda}(t, x) \) (and their gradients), and a (finite) subset of the parameters defining an element of the (infinite-dimensional) coset space \( E_{10}/K(E_{10}) \) where \( K(E_{10}) \) denotes the maximal compact subgroup of \( E_{10} \), such that the (PDE) field equations of supergravity get mapped onto the (ODE) equations describing a null geodesic in \( E_{10}/K(E_{10}) \) up to terms of height 30. This tantalizing result suggests that the infinite-dimensional hyperbolic Kac–Moody group \( E_{10} \) may be a “hidden symmetry” of supergravity in the sense of mapping solutions onto solutions (the idea that \( E_{10} \) might be a symmetry of supergravity was first suggested by Julia long ago [24, 25]). Note that the conjecture here is that the continuous group \( E_{10}(\mathbb{R}) \) be a hidden symmetry group of classical supergravity. At the quantum level, i.e. for M theory, one expects only a discrete version of \( E_{10} \), say \( E_{10}(\mathbb{Z}) \), to be a quantum symmetry. See [26] for recent work on extending the identification of [6] between roots of \( E_{10} \) and symmetries of supergravity/M-theory beyond height 30, and for references about previous suggestions of a possible role for \( E_{10} \). For earlier appearances of the Weyl groups of the \( E \) series in the context of \( U \)-duality see [27, 28, 29]. A series of recent papers [30, 31, 32, 33, 34] has also explored the possible role of \( E_{11} \) (a nonhyperbolic extension of \( E_{10} \)) as a hidden symmetry of M theory.

It is also tempting to assume that the KM groups underlying the other (special) Einstein-matter systems discussed above might be hidden (solution-generating) symmetries. For instance, in the case of pure Einstein gravity in \( D = 4 \) space-time, the conjecture is that \( AE_3 \) be such a symmetry of Einstein gravity. This case, and the correspondence between the field variables and the coset ones is further discussed in [7].

Rigorous mathematical proofs [17, 35, 16] are however only available for ‘non chaotic’ billiards.

In the remainder of this paper, we will outline various arguments explaining the above results; a more complete derivation can be found in [7].
2 General Models

The general systems considered here are of the following form

\[
S[g_{MN}, \phi, A^{\langle p \rangle}] = \int d^D x \sqrt{-g} \left[ R(g) - \partial_M \phi \partial^M \phi - \frac{1}{2} \sum_p \frac{1}{(p+1)!} \epsilon_{\lambda p} \phi F^{\langle p \rangle}_{M_1 \cdots M_{p+1}} F^{\langle p \rangle}_{M_1 \cdots M_{p+1}} \right] + \ldots \quad (2.4)
\]

Units are chosen such that \(16\pi G_N = 1\), \(G_N\) is Newton’s constant and the space-time dimension \(D \equiv d + 1\) is left unspecified. Besides the standard Einstein–Hilbert term the above Lagrangian contains a dilaton \(\phi\) field and a number of \(p\)-form fields \(A^{\langle p \rangle}_{M_1 \cdots M_p}\) (for \(p \geq 0\)). The \(p\)-form field strengths \(F^{\langle p \rangle} = dA^{\langle p \rangle}\) are normalized as

\[
F^{\langle p \rangle}_{M_1 \cdots M_{p+1}} = (p+1) \partial[M_1 A^{\langle p \rangle}_{M_2 \cdots M_{p+1}}] \equiv \partial[M_1 A^{\langle p \rangle}_{M_2 \cdots M_{p+1}}] \pm p \text{ permutations}.
\] (2.5)

As a convenient common formulation we adopt the Einstein conformal frame and normalize the kinetic term of the dilaton \(\phi\) with weight one with respect to the Ricci scalar. The Einstein metric \(g_{MN}\) has Lorentz signature \((- + \cdots +)\) and is used to lower or raise the indices; its determinant is denoted by \(g\). The dots in the action (2.4) above indicate possible modifications of the field strength by additional Yang–Mills or Chapline–Manton-type couplings [36, 37]. The real parameter \(\lambda_p\) measures the strength of the coupling of \(A^{\langle p \rangle}\) to the dilaton. When \(p = 0\), we assume that \(\lambda_0 \neq 0\) so that there is only one dilaton.

3 Dynamics in the vicinity of a spacelike singularity

The main technical points that will be reviewed here are the following

- near the singularity, \(t \to 0\), due to the decoupling of space points, the Einstein’s (PDE) equations become ODE’s with respect to time.

- The study of these ODE’s near \(t \to 0\), shows that the \(d\) diagonal spatial metric components ”\(g_{ii}\)” and the dilaton \(\phi\) move on a billiard in an auxiliary \(d + 1\) dimensional Lorentz space.

\(^2\)The generalization to any number of dilatons is straightforward.
• All the other field variables \((g_{ij}; i \neq j, A_{i_1...i_p}, \pi^{i_1...i_p})\) freeze as \(t \to 0\).

• In many interesting cases, the billiard tables can be identified with the fundamental Weyl chamber of an hyperbolic KM algebra.

• For SUGRA_{11}, the KM algebra is \(E_{10}\). Moreover, the PDE’s are equivalent to the equations of a null geodesic on the coset space \(E_{10}/K(E_{10})\), up to height 30.

3.1 Arnowitt-Deser-Misner Hamiltonian formalism

To focus on the features relevant to the billiard picture, we assume here that there are no Chern–Simons and no Chapline–Manton terms and that the curvatures \(F^{(p)}\) are abelian, \(F^{(p)} = dA^{(p)}\). That such additional terms do not alter the analysis has been proven in [7]. In any pseudo-Gaussian gauge and in the temporal gauge \((g_{0i} = 0 \text{ and } A_{0i_2...i_p} = 0, \forall p)\), the Arnowitt-Deser-Misner Hamiltonian action [8] reads

\[
S \left[ g_{ij}, \pi^{ij}, \phi, \pi_{ij}, A^{(p)}_{j_1...j_p}, \pi^{(p)}_{j_1...j_p} \right] = 
\int dx^0 \int d^d x \left( \pi^{ij} \dot{g}_{ij} + \pi_{ij} \dot{\phi} + \frac{1}{p!} \sum_p \pi^{(p)}_{j_1...j_p} \dot{A}^{(p)}_{j_1...j_p} - H \right) \tag{3.6}
\]

where the Hamiltonian density \(H\) is

\[
H = \tilde{\mathcal{N}} \mathcal{H}, \tag{3.7}
\]

\[
\mathcal{H} = \mathcal{K} + \mathcal{M}, \tag{3.8}
\]

\[
\mathcal{K} = \pi^{ij} \pi_{ij} - \frac{1}{d-1} \pi^i \pi_j + \frac{1}{4} \pi_{ij}^2 + \sum_p e^{-\lambda_p \phi} \frac{1}{2p!} \pi^{(p)}_{j_1...j_p} \pi_{(p) j_1...j_p}, \tag{3.9}
\]

\[
\mathcal{M} = -g R + gg^{ij} \partial_i \phi \partial_j \phi + \sum_p \frac{e^{\lambda_p \phi}}{2 (p+1)!} g F^{(p)}_{j_1...j_{p+1}} F^{(p)}_{j_1...j_p} + \mathcal{M} \tag{3.10}
\]

and \(R\) is the spatial curvature scalar. \(\tilde{\mathcal{N}} = N/\sqrt{g^{(d)}}\) is the rescaled lapse. The dynamical equations of motion are obtained by varying the above action with respect to the spatial metric components, the dilaton, the spatial \(p\)-form components and their conjugate momenta. In addition, there are constraints on the dynamical variables,
\[ H \approx 0 \quad \text{("Hamiltonian constraint"),} \quad (3.11) \]

\[ H_i \approx 0 \quad \text{("momentum constraint"),} \quad (3.12) \]

\[ \varphi_j^{j_1 \cdots j_{p-1}} \approx 0 \quad \text{("Gauss law" for each \( p \)-form),} \quad (3.13) \]

with

\[ H_i = -2\pi^j_{ij} + \pi_\phi \partial_i \phi + \sum_p \frac{1}{p!} \pi_j^{j_1 \cdots j_p} F_{ij_1 \cdots j_p}^{(p)}, \quad (3.14) \]

\[ \varphi_j^{j_1 \cdots j_{p-1}}(p) = \pi_j^{j_1 \cdots j_{p-1}j_p} |_{j_p}, \quad (3.15) \]

where the subscript \(| j \) stands for a spatially covariant derivative.

### 3.2 Iwasawa decomposition of the spatial metric

We systematically use the Iwasawa decomposition of the spatial metric \( g_{ij} \) and write

\[ g_{ij} = \sum_{a=1}^{d} e^{-2\beta^a} N^a_i N^a_j \quad (3.16) \]

where \( N \) is an upper triangular matrix with 1’s on the diagonal. We will also need the Iwasawa coframe \( \{\theta^a\} \),

\[ \theta^a = N^a_i dx^i, \quad (3.17) \]

as well as the vectorial frame \( \{e_a\} \) dual to the coframe \( \{\theta^a\} \),

\[ e_a = N^a_i \frac{\partial}{\partial x^i}, \quad (3.18) \]

where the matrix \( N^a_i \) is the inverse of \( N^a_i \), i.e., \( N^a_i N^b_j = \delta^a_b \). It is again an upper triangular matrix with 1’s on the diagonal. Let us now examine how the Hamiltonian action gets transformed when one performs, at each spatial point, the Iwasawa decomposition (3.16) of the spatial metric. The kinetic terms of the metric and of the dilaton in the Lagrangian (2.4) are given by the quadratic form

\[ G_{\mu \nu} d\beta^\mu d\beta^\nu = \sum_{a=1}^{d} (d\beta^a)^2 - \left( \sum_{a=1}^{d} d\beta^a \right)^2 + d\phi^2, \quad \beta^\mu = (\beta^a, \phi). \quad (3.19) \]
The change of variables \((g_{ij} \to \beta^a, N^a)\) corresponds to a point transformation and can be extended to the momenta as a canonical transformation in the standard way via

\[
\pi^i \dot{g}_{ij} = \sum_a \pi_a \dot{\beta}^a + \sum_a P^i \dot{N}^a \,.
\] (3.20)

Note that the momenta

\[
P^i_a = \frac{\partial L}{\partial \dot{N}^a} = \sum_b e^{2(\beta^b - \beta^a)} N^a_j N^i_b N^b_i
\] (3.21)

conjugate to the nonconstant off-diagonal Iwasawa components \(N^a\) are only defined for \(a < i\); hence the second sum in (3.20) receives only contributions from \(a < i\).

### 3.3 Splitting of the Hamiltonian

We next split the Hamiltonian density \(\mathcal{H}\) (3.7) in two parts: \(\mathcal{H}_0\), which is the kinetic term for the local scale factors \(\beta^a = (\beta^a, \phi)\), and \(\mathcal{V}\), a “potential density” of weight 2, which contains everything else. Our analysis below will show why it makes sense to group the kinetic terms of both the off-diagonal metric components and the \(p\)-forms with the usual potential terms, i.e. the term \(\mathcal{M}\) in (3.8). Thus, we write

\[
\mathcal{H} = \mathcal{H}_0 + \mathcal{V}
\] (3.22)

with the kinetic term of the \(\beta\) variables

\[
\mathcal{H}_0 = \frac{1}{4} G^{\mu \nu} \pi_\mu \pi_\nu,
\] (3.23)

where \(G^{\mu \nu}\) denotes the inverse of the metric \(G_{\mu \nu}\) of Eq. (3.19). In other words, the right hand side of Eq. (3.23) is defined by

\[
G^{\mu \nu} \pi_\mu \pi_\nu \equiv \sum_{a=1}^d \pi_a^2 - \frac{1}{d-1} \left( \sum_{a=1}^d \pi_a \right)^2 + \pi_\phi^2,
\] (3.24)

where \(\pi_\mu \equiv (\pi_a, \pi_\phi)\) are the momenta conjugate to \(\beta^a\) and \(\phi\), respectively, i.e.

\[
\pi_\mu = 2 \tilde{N}^{-1} G_{\mu \nu} \dot{\beta}^\nu = 2 G_{\mu \nu} \frac{d\beta^\nu}{d\tau}.
\] (3.25)
The total (weight 2) potential density,

\[ \mathcal{V} = \mathcal{V}_S + \mathcal{V}_G + \sum_p \mathcal{V}_p + \mathcal{V}_\phi \],

(3.26)
is naturally split into a “centrifugal” part \( \mathcal{V}_S \) linked to the kinetic energy of the off-diagonal components (the index \( S \) referring to “symmetry”), a “gravitational” part \( \mathcal{V}_G \), a term from the \( p \)-forms, \( \sum_p \mathcal{V}_p \), which is a sum of an “electric” and a “magnetic” contribution and also a contribution to the potential coming from the spatial gradients of the dilaton \( \mathcal{V}_\phi \).

- “centrifugal” potential

\[ \mathcal{V}_S = \frac{1}{2} \sum_{a<b} e^{-2(\beta^b - \beta^a)} (P^j_{b} \nabla^a j)^2, \]

(3.27)

- “gravitational” (or “curvature”) potential

\[ \mathcal{V}_G = -gR = \frac{1}{4} \sum_{a \neq b \neq c} e^{-2\alpha_{abc}(\beta)} (C^a_{bc})^2 - \sum_a e^{-2\mu_a(\beta)} F_a, \]

(3.28)

where

\[ \alpha_{abc}(\beta) = \sum_{c} \beta^c + \beta^a - \beta^b - \beta^c, \quad a \neq b, b \neq c, c \neq a \]

(3.29)

and

\[ d\theta^a = -\frac{1}{2} C^a_{bc} \theta^b \wedge \theta^c \]

(3.30)

while \( F_a \) is a polynomial of degree two in the first derivatives \( \partial \beta \) and of degree one in the second derivatives \( \partial^2 \beta \).

- \( p \)-form potential

\[ \mathcal{V}_p = \mathcal{V}^{el}_{(p)} + \mathcal{V}^{magn}_{(p)}, \]

(3.31)

which is a sum of an “electric” \( \mathcal{V}^{el}_{(p)} \) and a “magnetic” \( \mathcal{V}^{magn}_{(p)} \) contribution. The “electric” contribution can be written as

\[ \mathcal{V}^{el}_{(p)} = \frac{e^{-\kappa_p \phi}}{2p!} \pi^j_{(p)} \pi_{(p)} j_1 \cdots j_p \]

\[ \mathcal{V}^{el}_{(p)} = \frac{1}{2p!} \sum_{a_1,a_2,\ldots,a_p} e^{-2\epsilon_{a_1 \cdots a_p}(\beta)} (\mathcal{E}^{a_1 \cdots a_p})^2, \]

(3.32)
where $E_{a_1 \cdots a_p} \equiv N^{a_1 j_1}N^{a_2 j_2} \cdots N^{a_p j_p} \pi^{j_1 \cdots j_p}$, and $e_{a_1 \cdots a_p}(\beta)$ are the “electric wall” forms, 

$$e_{a_1 \cdots a_p}(\beta) = \beta^{a_1} + \cdots + \beta^{a_p} + \frac{\lambda_p}{2} \phi.$$  

(3.33)

And the “magnetic” contribution reads,

$$\gamma^{magn}_{(p)} = \frac{e^{\lambda_p \phi}}{2 (p+1)!} g F_{j_1 \cdots j_{p+1}}^{(p)} F^{(p)}_{j_1 \cdots j_{p+1}}$$

$$= \frac{1}{2 (p+1)!} \sum_{a_1, a_2, \cdots, a_{p+1}} e^{-2m_{a_1 \cdots a_{p+1}}(\beta)} (F_{a_1 \cdots a_{p+1}})^2$$  

(3.34)

where $F_{a_1 \cdots a_{p+1}} = N^{j_1 a_1} \cdots N^{j_{p+1} a_{p+1}} F^{(j_1 \cdots j_{p+1})}$ and the $m_{a_1 \cdots a_{p+1}}(\beta)$ are the magnetic linear forms

$$m_{a_1 \cdots a_{p+1}}(\beta) = \sum_{b \in \{a_1, a_2, \cdots, a_{p+1}\}} \beta^b - \frac{\lambda_p}{2} \phi,$$  

(3.35)

- dilaton potential

$$V_{\phi} = gg^{ij} \partial_i \phi \partial_j \phi$$

$$= \sum_a e^{-\mu_a(\beta)} (N_a^i \partial_i \phi)^2,$$  

(3.36)

(3.37)

$$\mu_a(\beta) = \sum_e \beta^e - \beta^a$$  

(3.38)

### 3.4 Appearance of sharp walls in the BKL limit

In the decomposition of the hamiltonian as $H = H_0 + V$, $H_0$ is the kinetic term for the $\beta^\mu$’s while all other variables now only appear through the potential $V$ which is schematically of the form

$$V(\beta^\mu, \partial_x \beta^\mu, P, Q) = \sum_A c_A (\partial_x \beta^\mu, P, Q) \exp \left(-2w_A(\beta)\right).$$  

(3.39)
where \((P, Q) = (N^a_i, P^i_a, \mathcal{E}_a^{1\cdots a_p}, F_{a_1\cdots a_{p+1}})\). Here \(w_A(\beta) = w_{A\mu}^{\beta^\mu}\) are the linear wall forms already introduced above:

- **symmetry walls** : 
  \[ w_{ab}^S \equiv \beta^b - \beta^a; \quad a < b \]

- **gravitational walls** : 
  \[ \alpha_{abc}(\beta) \equiv \sum_e \beta^e + \beta^a - \beta^b - \beta^c, \quad a \neq b, b \neq c, c \neq a \]

- **electric walls** : 
  \[ e_{a_1\cdots a_p}(\beta) \equiv \beta^{a_1} + \cdots + \beta^{a_p} + \frac{1}{2} \lambda_p \phi, \]

- **magnetic walls** : 
  \[ m_{a_1\cdots a_{p+1}}(\beta) \equiv \sum_e \beta^e - \beta^{a_1} - \cdots - \beta^{a_{p+1}} - \frac{1}{2} \lambda_p \phi. \]

In order to take the limit \(t \to 0\) which corresponds to \(\beta^\mu\) tending to future time-like infinity, we decompose \(\beta^\mu\) into hyperbolic polar coordinates \((\rho, \gamma^\mu)\), i.e.

\[ \beta^\mu = \rho \gamma^\mu \]  

(3.40)

where \(\gamma^\mu\) are coordinates on the future sheet of the unit hyperboloid which are constrained by

\[ G_{\mu\nu} \gamma^\mu \gamma^\nu \equiv \gamma^\mu \gamma_\mu = -1 \]  

(3.41)

and \(\rho\) is the time-like variable defined by

\[ \rho^2 \equiv -G_{\mu\nu} \beta^\mu \beta^\nu \equiv -\beta_\mu \beta^\mu > 0, \]  

(3.42)

which behaves like \(\rho \sim -\ln t \to +\infty\) at the BKL limit. In terms of these variables, the potential term looks like

\[ \sum_A c_A (\partial_x \beta^\mu, P, Q) \rho^2 \exp \left( -2 \rho w_A(\gamma) \right). \]  

(3.43)

The essential point now is that, since \(\rho \to +\infty\), each term \(\rho^2 \exp \left( -2 \rho w_A(\gamma) \right)\) becomes a sharp wall potential, i.e. a function of \(w_A(\gamma)\) which is zero when \(w_A(\gamma) > 0\), and \(+\infty\) when \(w_A(\gamma) < 0\). To formalize this behavior we define the sharp wall \(\Theta\)-function\(^3\) as

\[ \Theta(x) := \begin{cases} 
  0 & \text{if } x < 0, \\
  +\infty & \text{if } x > 0.
\end{cases} \]  

(3.44)

\(^3\)One should more properly write \(\Theta_\infty(x)\), but since this is the only step function encountered here, we use the simpler notation \(\Theta(x)\).
A basic formal property of this Θ-function is its invariance under multiplication by a positive quantity. Because all the relevant prefactors $c_A(\partial x \beta^\mu, P, Q)$ are generically positive near each leading wall, we can formally write

$$\lim_{\rho \to \infty} \left[ c_A(\partial x \beta^\mu, Q, P)\rho^2 \exp \left( -\rho w_A(\gamma) \right) \right] = c_A(Q, P)\Theta \left( -2w_A(\gamma) \right) \equiv \Theta \left( -2w_A(\gamma) \right)$$

valid in spite of the increasing of the spatial gradients [7]. Therefore, the limiting dynamics is equivalent to a free motion in the $\beta$-space interrupted by reflections against hyperplanes in this $\beta$-space given by $w_A(\beta) = 0$ which correspond to a potential described by infinitely high step functions

$$\mathcal{V}(\beta, P, Q) = \sum_A \Theta \left( -2w_A(\gamma) \right)$$

The other dynamical variables (all variables but the $\beta^\mu$’s) completely disappear from this limiting Hamiltonian and therefore they all get frozen as $t \to 0$.

### 4 Cosmological singularities and Kac–Moody algebras

Two kinds of motion are possible according to the volume of the billiard table on which it takes place, i.e. the volume (after projection on hyperbolic space) of the region where $\mathcal{V} = 0$ for $t \to 0$, also characterized by the conditions,

$$w_A(\beta) > 0 \quad \forall A.$$  

Depending on the fields present in the Lagrangian, on their dilaton-couplings and on the spacetime dimension, the (projected) billiard volume is either finite or infinite. The finite volume case corresponds to never-ending, chaotic oscillations for the $\beta$’s while in the infinite volume case, after a finite number of reflections off the walls, they tend to an asymptotically monotonic Kasner-like behavior, see figure 3.

In figure 3 the upper panels are drawn in the Lorentzian space spanned by $(\beta^\mu) = (\beta^a, \phi)$. The billiard tables are represented as “wedges” in $(d+1)$-dimensional (or $d$-dimensional, if there are no dilatons) $\beta$-space, bounded by
hyperplanar walls \( w_A(\beta) = 0 \) on which the billiard ball undergoes specular reflections. The upper left panel is a (critical) “chaotic” billiard table (contained within the \( \beta \)-space future light cone), while the upper right one is a (subcritical) “nonchaotic” one (extending beyond the light cone). The lower panels represent the corresponding billiard tables (and billiard motions) after projection onto hyperbolic space \( H_d \) (\( H_{d-1} \) if there are no dilatons). The latter projection is defined in the text by central projection onto \( \gamma \)-space (i.e. the unit hyperboloid \( G_{\mu\nu} \gamma^\mu \gamma^\nu = -1 \), see the upper panels), and is represented in the lower panels by its image in the Poincaré ball (disk).

In fact, not all the walls are relevant for determining the billiard table. Some of the walls stay behind the others and are not met by the billiard ball. Only a subset of the walls \( w_A(\beta) \), called dominant walls and here denoted \( \{w_i(\beta)\} \) are needed to delimit the hyperbolic domain. Once the dominant walls are found, one can compute the following matrix

\[
A_{ij} \equiv 2 \frac{w_i.w_j}{w_i.w_i}
\]

(4.48)

where \( w_i.w_j = G^{\mu\nu}w_i{\mu}w_j{\nu} \). By definition, the diagonal elements are all equal to 2. Moreover, in many interesting cases, the off-diagonal elements
happen to be non positive integers. These are precisely the characteristics of a generalized Cartan matrix, namely that of an infinite KM algebra (see appendix). As recalled in the introduction, for pure gravity in $D$ space-time dimensions, there are $D - 1$ dominant walls and the matrix $A_{ij}$ is exactly the generalized Cartan matrix of the hyperbolic KM algebra $AE_{D-1} \equiv A_{D-3}^{+} \equiv A_{D-3}^{-}$ which is hyperbolic for $D \leq 10$. More generally, bosonic string theory in $D$ space-time dimensions is related to the Lorentzian KM algebra $DE_{D}$ [3, 5] which is the canonical Lorentzian extension of the finite-dimensional Lie algebra $D_{D-2}$. The various superstring theories, in the critical dimension $D = 10$, and $M$-theory have been found [3] to be related either to $E_{10}$ (when there are two supersymmetries, i.e. for type IIA, type IIB and $M$-theory) or to $BE_{10}$ (when there is only one supersymmetry, i.e. for type I and II heterotic theories), see the table.

The hyperbolic KM algebras are those relevant for chaotic billiards since their fundamental Weyl chamber has a finite volume.

| Theory                                   | Corresponding Hyperbolic KM algebra |
|------------------------------------------|-------------------------------------|
| Pure gravity in $D \leq 10$              | ![Diagram](image)                   |
| M-theory, IIA and IIB Strings            | ![Diagram](image)                   |
| type I and heterotic Strings             | ![Diagram](image)                   |
| closed bosonic string in $D = 10$        | ![Diagram](image)                   |

This table displays the Coxeter–Dynkin diagrams which encode the geometry of the billiard tables describing the asymptotic cosmological behavior of General Relativity and of three blocks of string theories: $B_{2} = \{M$-theory, type IIA and type IIB superstring theories$, \}$, $B_{1} = \{type \ I \ and \ the \ two \ heterotic \ superstring \ theories$, and $B_{0} = \{closed \ bosonic \ string \ theory \ in \ D = 10$. Each node of the diagrams represents a dominant wall of the cosmological billiard. Each Coxeter diagram of a billiard table corresponds to the Dynkin diagram of a (hyperbolic) KM algebra: $E_{10}$, $BE_{10}$ and $DE_{10}$.

The precise links between a chaotic billiard and its corresponding Kac–Moody algebra can be summarized as follows

- the scale factors $\beta^{\mu}$ parametrize a Cartan element $h = \sum_{\mu=1}^{r} \beta^{\mu} h_{\mu}$,
• the dominant walls \( w_i(\beta), (i = 1, \ldots, r) \) correspond to the simple roots \( \alpha_i \) of the KM algebra,

• the group of reflections in the cosmological billiard is the Weyl group of the KM algebra, and

• the billiard table can be identified with the Weyl chamber of the KM algebra.

5 \( E_{10} \) and a “small tension” limit of SUGRA_{11}

The main feature of the gravitational billiards that can be associated with the KM algebras is that there exists a group theoretical interpretation of the billiard motion: the asymptotic BKL dynamics is equivalent (in a sense to be made precise below), at each spatial point, to the asymptotic dynamics of a one-dimensional nonlinear \( \sigma \)-model based on a certain infinite-dimensional coset space \( G/K \), where the KM group \( G \) and its maximal compact subgroup \( K \) depend on the specific model. As we have seen, the walls that determine the billiards are the dominant walls. For the KM billiards, they correspond to the simple roots of the KM algebra. As we discuss below, some of the subdominant walls also have an algebraic interpretation in terms of higher-height positive roots. This enables one to go beyond the BKL limit and to see the beginnings of a possible identification of the dynamics of the scale factors and of all the remaining variables with that of a nonlinear \( \sigma \)-model defined on the cosets of the KM group divided by its maximal compact subgroup \([6, 7]\).

For concreteness, we will only consider one specific example here: the relation between the cosmological evolution of \( D = 11 \) supergravity and a null geodesic on \( E_{10}/K(E_{10}) \) \([6]\) where \( KE_{10} \) is the maximally compact subgroup of \( E_{10} \). The \( \sigma \)-model is formulated in terms of a one-parameter dependent group element \( \mathcal{V} = \mathcal{V}(t) \in E_{10} \) and its Lie algebra value derivative

\[
v(t) := \frac{d\mathcal{V}}{dt} \mathcal{V}^{-1}(t) \in e_{10}.
\] (5.49)

The action is

\[
S_{1}^{E_{10}} = \int \frac{dt}{n(t)} < v_{\text{sym}}(t) | v_{\text{sym}}(t) >
\] (5.50)

with a lapse function \( n(t) \) whose variation gives rise to the Hamiltonian constraint ensuring that the trajectory is a null geodesic. The symmetric
projection
\[ v_{\text{sym}} := \frac{1}{2} (v + v^T) \]  
(5.51)
is introduced in order to define an evolution on the coset space. Here \(<,>\) is the standard invariant bilinear form on \(E_{10}\); \(v^T\) is the “transpose” of \(v\) defined with the Chevalley involution\(^4\) as \(v^T = -\omega(v)\). This action is invariant under \(E_{10}\),

\[ \mathcal{V}(t) \rightarrow k(t) \mathcal{V}(t) g \quad \text{where} \quad k \in KE_{10} \quad g \in E_{10} \]  
(5.52)
Making use of the explicit Iwasawa parametrization of the generic \(E_{10}\) group element \(\mathcal{V} = KAN\) together with the gauge choice \(K = 1\) (Borel gauge), one can write

\[ \mathcal{V}(t) = \exp X_h(t) \cdot \exp X_A(t) \]

with \(X_h(t) = h^a_b K_b^a\) and

\[ X_A(t) = \frac{1}{3!} A_{abc} E^{abc} + \frac{1}{6} A_{a1...a6} E^{a1...a6} + \frac{1}{9!} A_{a0|a1...a8} E^{a0|a1...a8} + \ldots. \]

Using the \(E_{10}\) commutation relations in \(GL(10)\) form together with the bilinear form for \(E_{10}\), one obtains up to height 30\(^5\),

\[ n\mathcal{L} = \frac{1}{4} (g^{ac} g^{bd} - g^{ab} g^{cd}) \dot{g}_{ab} \dot{g}_{cd} + \frac{1}{2} DA_{a1a2a3} DA^{a1a2a3} \]
\[ + \frac{1}{2} DA_{a1...a6} DA^{a1...a6} + \frac{1}{2} DA_{a0|a1...a8} DA^{a0|a1...a8}, \]  
(5.53)
where \(g^{ab} = e^a_c e^b_c\) with

\[ e^a_b = (\exp h)^a_b, \]

and all “contravariant indices” have been raised by \(g^{ab}\). The “covariant” time derivatives are defined by (with \(\partial A \equiv \dot{A}\))

\[ DA_{a1a2a3} := \partial A_{a1a2a3}, \]
\[ DA_{a1...a6} := \partial A_{a1...a6} + 10 A_{[a1a2a3} \partial A_{a4a5a6]} , \]
\[ DA_{a0|a1...a9} := \partial A_{a1|a2...a9} + 42 A_{(a1a2a3} \partial A_{a4...a9)} - 42 \partial A_{(a1a2a3} A_{a4...a9)} + 280 A_{(a1a2a3} A_{4a5a6} \partial A_{7a8a9)}. \]  
(5.54)

\(^4\)The Chevalley involution is defined by \(\omega(h_i) = -h_i; \quad \omega(e_i) = -f_i; \quad \omega(f_i) = -e_i\).

\(^5\)We keep only the generators \(E^{abc}, E^{a1...a6}\) and \(E^{a0|a1...a8}\) corresponding to the \(E_{10}\) roots \(\alpha = \sum n_i \alpha_i\) with height \(\sum_i n_i \leq 29\) (\(\alpha_i\) are simple roots and \(n_i\) integers)
Here antisymmetrization \([\ldots]\), and projection on the \(\ell = 3\) representation \(\langle \ldots \rangle\), are normalized with strength one (e.g. \([[\ldots]] = [\ldots]]\)). Modulo field redefinitions, all numerical coefficients in (5.53) and in (5.54) are uniquely fixed by the structure of \(E_{10}\).

In order to compare the above coset model results with those of the bosonic part of \(D = 11\) supergravity, we recall the action

\[
S_{11}^{\text{sugra}} = \int d^{11}x \left[ \sqrt{-G} R(G) - \frac{\sqrt{-G}}{48} F_{\alpha\beta\gamma\delta} F^{\alpha\beta\gamma\delta} 
+ \frac{1}{(12)^4} \varepsilon^{\alpha_1 \ldots \alpha_{11}} F_{\alpha_1 \ldots \alpha_4} F_{\alpha_5 \ldots \alpha_8} A_{\alpha_9 \alpha_{10} \alpha_{11}} \right].
\]

The space-time indices \(\alpha, \beta, \ldots\) take the values \(0, 1, \ldots, 10\); \(\varepsilon^{01\ldots10} = +1\), and the four-form \(F\) is the exterior derivative of \(A\), \(F = dA\). Note the presence of the Chern–Simons term \(\mathcal{F} \wedge \mathcal{F} \wedge \mathcal{A}\) in the action (5.55). Introducing a zero-shift slicing \((N^1 = 0)\) of the eleven-dimensional space-time, and a *time-independent* spatial zehnbein \(\theta^a(x) \equiv E^a_i(x) dx^i\), the metric and four-form \(\mathcal{F} = d\mathcal{A}\) become

\[
\begin{align*}
\mathcal{F} &= \frac{1}{3!} F_{0abc} dx^0 \wedge \theta^a \wedge \theta^b \wedge \theta^c + \frac{1}{4!} F_{abcd} \theta^a \wedge \theta^b \wedge \theta^c \wedge \theta^d. \\

\end{align*}
\]

We choose the time coordinate \(x^0\) so that the lapse \(N = \sqrt{G}\), with \(G := \det G_{ab}\) (note that \(x^0\) is not the proper time\(^6\) \(T = \int N dx^0\); rather, \(x^0 \to \infty\) as \(T \to 0\)). In this frame the complete evolution equations of \(D = 11\) supergravity read

\[
\begin{align*}
\partial_0 (G^{a\beta} \partial_0 G_{cb}) &= \frac{1}{6} G F^{a\beta\gamma} F_{b\gamma\delta} - \frac{1}{72} G F^{a\beta\gamma} F_{\alpha\beta\gamma\delta} \delta^\alpha_0 - 2 G R^a_0 (\Gamma, C), \\
\partial_0 (GF^{0abc}) &= \frac{1}{144} \varepsilon^{abca_1 a_2 a_3 b_1 b_2 b_3 b_4} F_{0a_1 a_2 a_3} F_{b_1 b_2 b_3 b_4} \\
&\quad + \frac{7}{2} G F^{de[ab} C^{c]} d e - GC_d e F^{dabc} - \partial_d (G F^{deabc}), \\
\partial_0 F_{abcd} &= 6 F_{0e[ab} C^{c} d f] + 4 \partial_{[a} F_{0bc]d},
\end{align*}
\]

where \(a, b \in \{1, \ldots, 10\}\) and \(\alpha, \beta \in \{0, 1, \ldots, 10\}\), and \(R_{ab}(\Gamma, C)\) denotes the spatial Ricci tensor; the (frame) connection components are given by \(2 G_{ad} \Gamma_{bc} = C_{abc} + C_{bca} - C_{cab} + \partial_0 G_{ca} + \partial_a G_{ab} - \partial_b G_{bc} = \partial_a G_{bc} - \partial_b G_{ac}\), with \(C_{a}^{bc} \equiv G^{ad} C_{dabc}\) being the structure coefficients of the zehnbein \(d\theta^a = \frac{1}{2} C_{a}^{bc} \theta^b \wedge \theta^c\). (Note

\(^6\)In this section, the proper time is denoted by \(T\) while the variable \(t\) denotes the parameter of the one-dimensional \(\sigma\)-model introduced above.)
the change in sign convention here compared to above.) The frame derivative is \( \partial_\alpha \equiv E^i_\alpha(x) \partial_i \) (with \( E^a_i E^b_j = \delta^a_b \)). To determine the solution at any \textit{given} spatial point \( x \) requires knowledge of an infinite tower of spatial gradients; one should thus augment (5.57) by evolution equations for \( \partial_a G_{bc}, \partial_a F_{0bcd}, \partial_a F_{bcde}, \) etc., which in turn would involve higher and higher spatial gradients.

The main result of concern here is the following: there exists a \textit{map} between geometrical quantities constructed at a given spatial point \( x \) from the supergravity fields \( G_{\mu\nu}(x^0, x) \) and \( A_{\mu\nu\rho}(x^0, x) \) and the one-parameter-dependent quantities \( g_{ab}(t), A_{abc}(t), \ldots \) entering the coset Lagrangian (5.53), under which the supergravity equations of motion (5.57) become \textit{equivalent}, \textit{up to 30th order in height}, to the Euler-Lagrange equations of (5.53). In the gauge (5.56) this map is defined by

\[
t = x_0 \equiv \int \frac{dT}{\sqrt{G}} \quad \text{and} \quad g_{ab}(t) = G_{ab}(t, x),
\]
\[
\partial A_{a_1 a_2 a_3}(t) = F_{0a_1 a_2 a_3}(t, x),
\]
\[
\partial A^{a_1 \ldots a_6}(t) = -\frac{1}{12} e^{a_1 \ldots a_6 b_1 b_2 b_3 b_4} F_{b_1 b_2 b_3 b_4}(t, x),
\]
\[
\partial A^{b|a_1 \ldots a_8}(t) = \frac{3}{2} e^{a_1 \ldots a_8 b_1 b_2} \left( C^b_{b_1 b_2}(x) + \frac{2}{5} \delta^b_{[b_1} C^c_{b_2 b_3]}(x) \right).
\]

(5.58)

Let us also mention in passing (from [39]) that the \( E_{10} \) coset action is not compatible with the addition of an eleven-dimensional cosmological constant in the supergravity action (an addition which has been proven to be incompatible with supersymmetry in [40]).

6 Conclusions

We have reviewed the finding that the general solution of many physically relevant (bosonic) Einstein-matter systems, in the vicinity of a space-like singularity, exhibits a remarkable mixture of chaos and symmetry. Near the singularity, the behavior of the general solution is describable, at each (generic) spatial point, as a billiard motion in an auxiliary Lorentzian space or, after a suitable “radial” projection, as a billiard motion on hyperbolic space. This motion appears to be chaotic in many physically interesting cases including pure Einstein gravity in any space-time dimension \( D \leq 10 \) and the particular Einstein-matter systems arising in string theory. Also, for these cases, the billiard tables can be identified with the Weyl chambers of some Lorentzian Kac–Moody algebras. In the case of the bosonic sector
of supergravity in 11-dimensional space-time the underlying Lorentzian algebra is that of the hyperbolic Kac–Moody group $E_{10}$, and there exists some evidence of a correspondence between the general solution of the Einstein-three-form system and a null geodesic in the infinite-dimensional coset space $E_{10}/K(E_{10})$, where $K(E_{10})$ is the maximal compact subgroup of $E_{10}$.

**Acknowledgement**

It is a pleasure to thank Sophie de Buyl and Christiane Schomblond for their help in trimming the manuscript and in improving the figures.

## A Kac-Moody algebras

A KM algebra $G(A)$ can be constructed out of a generalized Cartan matrix $A$, (i.e. an $r \times r$ matrix such that $A_{ii} = 2, i = 1, \ldots, r$, ii) $-A_{ij} \in \mathbb{N}$ for $i \neq j$ and iii) $A_{ij} = 0$ implies $A_{ji} = 0$) according to the following rules for the Chevalley generators $\{h_i, e_i, f_i\}, i = 1, \ldots, r$:

- $[e_i, f_j] = \delta_{ij}h_i$
- $[h_i, e_j] = A_{ij}e_j$
- $[h_i, f_j] = -A_{ij}f_j$
- $[h_i, h_j] = 0$.

The generators must also obey the Serre’s relations, namely

$$
(ad e_i)^{1-A_{ij}}e_j = 0 \\
(ad f_i)^{1-A_{ij}}f_j = 0
$$

and the Jacobi identity. $G(A)$ admits a triangular decomposition

$$
G(A) = n_- \oplus h \oplus n_+ \quad (A.59)
$$

where $n_-$ is generated by the multicommutators of the form $[f_{i_1}, [f_{i_2}, \ldots]]$, $n_+$ by the multicommutators of the form $[e_{i_1}, [e_{i_2}, \ldots]]$ and $h$ is the Cartan subalgebra.

The algebras $G(A)$ build on a symmetrizable Cartan matrix $A$ have been classified according to properties of their eigenvalues

- if $A$ is positive definite, $G(A)$ is a finite dimensional Lie algebra;
- if $A$ admits one null eigenvalue and the others are all strictly positive, $G(A)$ is an Affine KM algebra;
• if $A$ admits one negative eigenvalue and all the others are strictly positive, $\mathcal{G}(A)$ is a Lorentzian KM algebra.

A KM algebra such that the deletion of one node from its Dynkin diagram gives a sum of finite or affine algebras is called an hyperbolic KM algebra. These algebras are all known; in particular, there exists no hyperbolic algebra with rank higher than 10.

References

[1] T. Damour and M. Henneaux, Phys. Rev. Lett. 85, 920 (2000) [hep-th/0003139]; see also a short version in Gen. Rel. Grav. 32, 2339 (2000).

[2] T. Damour and M. Henneaux, Phys. Lett. B 488, 108 (2000) [hep-th/0006171].

[3] T. Damour and M. Henneaux, Phys. Rev. Lett. 86, 4749 (2001) [hep-th/0012172].

[4] T. Damour, M. Henneaux, B. Julia and H. Nicolai, Phys. Lett. B 509, 323 (2001), [hep-th/0103094].

[5] T. Damour, S. de Buyl, M. Henneaux and C. Schomblond, JHEP 0208, 030 (2002) [hep-th/0206125].

[6] T. Damour, M. Henneaux and H. Nicolai, Phys. Rev. Lett. 89, 221601 (2002), [hep-th/0207267].

[7] T. Damour, M. Henneaux and H. Nicolai, Class. Quant. Grav. 20, R145 (2003), [hep-th/0212256].

[8] R. Arnowitt, S. Deser and C. W. Misner, “The Dynamics Of General Relativity,” [arXiv:gr-qc/0405109].

[9] V.A. Belinskii, I.M. Khalatnikov and E.M. Lifshitz, Adv. Phys. 19, 525 (1970).

[10] V.A. Belinskii, I.M. Khalatnikov and E.M. Lifshitz, Sov. Phys. JETP 35, 838 (1972).

[11] V.A. Belinskii, I.M. Khalatnikov and E.M. Lifshitz, Adv. Phys. 31, 639 (1982).
[12] E. M. Lifshitz, I. M. Lifshitz and I. M. Khalatnikov, Sov. Phys. JETP 32, 173 (1971).

[13] D. F. Chernoff and J. D. Barrow, Phys. Rev. Lett. 50, 134 (1983).

[14] D. M. Chitre, Ph. D. Thesis, University of Maryland, 1972.

[15] C.W. Misner, in: D. Hobill et al. (Eds), Deterministic chaos in general relativity, (Plenum, 1994), p. 317 [gr-qc/9405068].

[16] T. Damour, M. Henneaux, A. D. Rendall and M. Weaver, Annales Henri Poincaré 3, 1049 (2002), [gr-qc/0202069].

[17] L. Andersson and A.D. Rendall, Commun. Math. Phys. 218, 479 (2001) [gr-qc/0001047].

[18] A. A. Kirillov, Sov. Phys. JETP 76, 355 (1993).

[19] A. A. Kirillov and V. N. Melnikov, Phys. Rev. D 52, 723 (1995) [gr-qc/9408004].

[20] V. D. Ivashchuk, A. A. Kirillov and V. N. Melnikov, JETP Lett. 60, 235 (1994) [Pisma Zh. Eksp. Teor. Fiz. 60, 225 (1994)].

[21] V.D. Ivashchuk and V.N. Melnikov, Class. Quantum Grav. 12, 809 (1995).

[22] V.G. Kac, Infinite Dimensional Lie Algebras, Third Edition, (Cambridge University Press, 1990).

[23] S. de Buyl and C. Schomblond, Hyperbolic Kac Moody algebras and Einstein billiards, [hep-th/0403285].

[24] B. Julia, Report LPTENS 80/16, Invited Paper Presented at the Nuffield Gravity Workshop, Cambridge, England, June 22 – July 12, 1980.

[25] B. Julia, in Lectures in Applied Mathematics, AMS-SIAM, vol. 21 (1985), p. 355.

[26] J. Brown, O. J. Ganor and C. Helfgott, M-theory and E_{10}: Billiards, Branes, and Imaginary Roots, [hep-th/0401053].
[27] H. Lu, C. N. Pope and K. S. Stelle, Nucl. Phys. B 476, 89 (1996) [hep-th/9602140].

[28] N. A. Obers, B. Pioline and E. Rabinovici, Nucl. Phys. B 525, 163 (1998) [hep-th/9712084].

[29] T. Banks, W. Fischler and L. Motl, JHEP 9901, 019 (1999) [hep-th/9811194].

[30] P. C. West, Class. Quant. Grav. 18, 4443 (2001) [hep-th/0104081].

[31] I. Schnakenburg and P. C. West, Phys. Lett. B 517, 421 (2001) [hep-th/0107181].

[32] I. Schnakenburg and P. C. West, Phys. Lett. B 540, 137 (2002) [hep-th/0204207].

[33] F. Englert, L. Houart, A. Taormina and P. West, JHEP 0309, 020 (2003) [hep-th/0304206].

[34] F. Englert and L. Houart, JHEP 0405, 059 (2004) [hep-th/0405082].

[35] A. D. Rendall and M. Weaver, Class. Quant. Grav. 18, 2959 (2001). [gr-qc/0103102].

[36] E. Bergshoeff, M. de Roo, B. de Wit and P. van Nieuwenhuizen, Nucl. Phys. B 195, 97 (1982).

[37] G. F. Chapline and N. S. Manton, Phys. Lett. B 120, 105 (1983).

[38] E. Cremmer, B. Julia and J. Scherk, Phys. Lett. B 76, 409 (1978).

[39] T. Damour and H. Nicolai, “Eleven dimensional supergravity and the $E_{10}/K(E_{10})$ $\sigma$-model at low $A_9$ levels”, invited contribution to the XXV International Colloquium on Group Theoretical Methods in Physics, 2-6 August 2004, Cocoyoc, Mexico; to appear in the proceedings. [arXiv:hep-th/0410245.]

[40] K. Bautier, S. Deser, M. Henneaux and D. Seminara, Phys. Lett. B 406, 49 (1997) [arXiv:hep-th/9704131].

23