On optimal matching of Gaussian samples III

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Abstract

This article is a continuation of the papers [8, 9] in which the optimal matching problem, and the related rates of convergence of empirical measures for Gaussian samples are addressed. A further step in both the dimensional and Kantorovich parameters is achieved here, proving that, given $X_1, \ldots, X_n$ independent random variables with common distribution the standard Gaussian measure $\mu$ on $\mathbb{R}^d$, $d \geq 3$, and $\mu_n = \frac{1}{n} \sum_{i=1}^{n} \delta_{X_i}$ the associated empirical measure,

$$\mathbb{E}(W^p_p(\mu_n; \mu)) \approx \frac{1}{n^{p/d}}$$

for any $1 \leq p < d$, where $W_p$ is the $p$-th Kantorovich metric. The proof relies on the pde and mass transportation approach developed by L. Ambrosio, F. Stra and D. Trevisan in a compact setting.

1 Introduction and main results

Given $p \geq 1$, the Kantorovich distance (cf. [16] e.g.) between two probability measures $\nu$ and $\mu$ on the Borel sets of $\mathbb{R}^d$ with a finite $p$-th moment is defined by

$$W_p(\nu, \mu) = \inf \left( \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^p d\pi(x, y) \right)^{1/p}$$

(1.1)

where the infimum is taken over all couplings $\pi$ on $\mathbb{R}^d \times \mathbb{R}^d$ with respective marginals $\nu$ and $\mu$. Here $|x - y|$ denotes the Euclidean distance between $x$ and $y$ in $\mathbb{R}^d$.

Denote by $X_1, \ldots, X_n, n \geq 1$, independent random variables in $\mathbb{R}^d$ with common distribution $\mu$ and let

$$\mu_n = \frac{1}{n} \sum_{i=1}^{n} \delta_{X_i}$$
be the empirical measure on the sample \((X_1, \ldots, X_n)\). The question investigated here is the order of decay in \(n\) of the expectations

\[
\mathbb{E}(W_p^p(\mu_n, \mu))
\]

when the random variables \(X_1, \ldots, X_n\) are independent with the same standard Gaussian distribution \(d\mu(x) = e^{-|x|^2 / 2} \frac{dx}{(2\pi)^d/2}\) on \(\mathbb{R}^d\). The optimal matching problem would consist in the study of \(\mathbb{E}(W_p^p(\mu_n, \nu_n))\) for two independent samples \(X_1, \ldots, X_n\) and \(Y_1, \ldots, Y_n\) with \(\mu_n = \frac{1}{n} \sum_{i=1}^n \delta_{X_i}\) and \(\nu_n = \frac{1}{n} \sum_{i=1}^n \delta_{Y_i}\), which is easily compared to (1.2).

As a continuation of [8], we refer to this article for more background and motivation in the study of the optimal matching problem and rates of convergence in Kantorovich metrics of empirical measures of Gaussian samples. To introduce the results of this work, we nevertheless recall the picture for the uniform distribution on \([0, 1]^d\), as well as the known results so far in the Gaussian (and more general) setting.

Throughout the paper \(A \lesssim B\) between two real positive numbers \(A\) and \(B\) means that \(A \leq CB\) where \(C > 0\) is either numerical or depends on \(p, d\), but not on anything else. In the same way, a sentence like “\(A\) is bounded from above by \(B\)” has the same meaning. The equivalence sign \(A \approx B\) indicates that \(A \lesssim B\) and \(B \lesssim A\). In particular, these extended inequalities will then hold uniformly over \(n \geq 1\). Actually, \(n\) may always be assumed to be larger than some fixed integer \(n_0\), large enough for obvious inequalities to hold true.

If \(\mu\) is uniform on \([0, 1]^d\), it holds true that (cf. e.g. [14]), for every \(1 \leq p < \infty\),

\[
\mathbb{E}(W_p^p(\mu_n, \mu)) \approx \begin{cases} 
\frac{1}{n^{p/2}} & \text{if } d = 1, \\
\frac{(\log n)^{p/2}}{n} & \text{if } d = 2, \\
\frac{1}{n^{p/d}} & \text{if } d \geq 3.
\end{cases}
\]

The particular, and critical, case \(d = 2\) is the famous Ajtai-Komlós-Tusnády theorem [1].

Before addressing the Gaussian model, it is worthwhile mentioning that, by a simple contraction argument (cf. [8]), the expected cost \(\mathbb{E}(W_p^p(\mu_n, \mu))\) when \(\mu\) is uniform is bounded from above by the corresponding quantity when \(\mu\) is the standard Gaussian measure. Hence, the rates in the uniform case provide lower bounds for the Gaussian model. This comparison is used implicitly in the following descriptions.

Let now \(\mu\) be the standard Gaussian measure on the Borel sets of \(\mathbb{R}^d\). It has been shown in [5] that in dimension \(d = 1\),

\[
\mathbb{E}(W_p^p(\mu_n, \mu)) \approx \begin{cases} 
\frac{1}{n^{p/2}} & \text{if } 1 \leq p < 2, \\
\frac{\log \log n}{n} & \text{if } p = 2, \\
\frac{1}{n(\log n)^{p/2}} & \text{if } p > 2.
\end{cases}
\]

With respect to (1.3), it therefore appears that, already in dimension one, the rates for \(p \geq 2\) are rather sensitive to the underlying distribution. The proof of (1.4) in [5] relies on monotone rearrangement transport and explicit one-dimensional distributional inequalities.
In dimension $d = 2$, 
\[
\mathbb{E}(W_p^p(\mu_n, \mu)) \approx \begin{cases} 
\left(\frac{\log n}{n}\right)^{p/2} & \text{if } 1 \leq p < 2, \\
\left(\frac{\log n}{n}\right)^2 & \text{if } p = 2.
\end{cases}
\] (1.5)

Again, a specific new feature appears as $p = 2$. As we will see, the case $p > 2$ is essentially open. The proof of the case $1 \leq p < 2$, and the upper bound for $p = 2$, given in [8] is based on the pde and mass transportation approach developed next, while the lower bound for $p = 2$ in [15] relies on the generic chaining ideas of [14] together with a scaling argument. An alternate proof of this lower bound using the pde-transportation method has been provided soon after in [9]. When $p = 1$, the upper bound $\mathbb{E}(W_1(\mu_n, \mu)) \lesssim \sqrt{\frac{\log n}{n}}$ has been shown in [17] to hold for distributions $\mu$ with a mild moment assumption.

In higher dimension $d \geq 3$, a general bound using dyadic decompositions in the spirit of the Ajtai-Komlós-Tusnády theorem, and actually holding for distributions $\mu$ with enough moments, has been obtained in [6, 7] expressing that 
\[
\mathbb{E}(W_p^p(\mu_n, \mu)) \lesssim \frac{1}{n^{p/d}}
\] (1.6)
whenever $1 \leq p < \frac{d}{2}$. For Gaussian samples, it is extended up to $p < 2$ in dimension 3 in [8]. It is also known from the works [6, 7] that 
\[
\mathbb{E}(W_p^p(\mu_n, \mu)) \lesssim \frac{1}{\sqrt{n}}
\] (1.7)
when $p > \frac{d}{2}$, with an extra $\log n$ at the numerator when $p = \frac{d}{2}$. The used methodology however will never produce anything better than this $\frac{1}{\sqrt{n}}$ rate, and the upper bound (1.7) is actually far from the potential lower bound deduced from (1.3), and already not satisfactory when $d = 1$.

The purpose of the work is to propose some progress in the understanding of the rates in this Gaussian setting when $d \geq 3$ with the following statements.

**Theorem 1.** Let $X_1, \ldots, X_n$ be independent with common law the standard Gaussian distribution $\mu$ on $\mathbb{R}^d$, $d \geq 3$, and set $\mu_n = \frac{1}{n} \sum_{i=1}^{n} \delta_{X_i}$, $n \geq 1$. Then, for $1 \leq p < d$,
\[
\mathbb{E}(W_p^p(\mu_n, \mu)) \approx \frac{1}{n^{p/d}}.
\]

In this range $1 \leq p < d$, $d \geq 3$, the rates for the Gaussian are therefore the same as the ones for the compact uniform model. The result extends (1.6) from $p < \frac{d}{2}$ to $p < d$. This might look as only a small step, but it overcomes the $\frac{1}{\sqrt{n}}$ rate and, as the proof will amply demonstrate, the amount of work to reach this conclusion is rather significant. Due to the results in [6, 7, 8] when $1 \leq p < 2$ mentioned above, only the values $2 \leq p < d$ have to be considered.

As identified by (1.5) when $p = d = 2$, the case $p = d$ might be of special interest. We have been able to reach the following conclusion, not definitive however.
Theorem 2. Let $X_1, \ldots, X_n$ be independent with common law the standard Gaussian distribution $\mu$ on $\mathbb{R}^d$, $d \geq 2$, and set $\mu_n = \frac{1}{n} \sum_{i=1}^n \delta_{X_i}$, $n \geq 1$. Then,

$$\mathbb{E}(W_d^d(\mu_n, \mu)) \lesssim \frac{(\log n)^\kappa}{n}$$

where

$$\kappa = \begin{cases} 
\frac{d^2 + 6d}{8} & \text{if } d = 2, 3, \\
\frac{d^2}{2} - \frac{3d}{4} & \text{if } d \geq 4.
\end{cases}$$

The case $d = 2$ recovers the claim in (1.5) (although the proof developed here to study every $d \geq 2$ is more involved than the ones in [8] and [15]). The lower bound provided by the uniform model is $\frac{1}{n}$ when $d \geq 3$. A possible conjecture might be that

$$\mathbb{E}(W_d^d(\mu_n, \mu)) \approx \frac{(\log n)^\frac{d}{2}}{n}$$

for $d \geq 3$. This is what is suggested as a lower bound in [15].

At this point, we do not have any reasonable conjecture for $p > d$ ($\geq 2$).

Our proof of the preceding main results relies on the pde and transportation approach developed by L. Ambrosio, F. Stra and D. Trevisan [2] towards the exact limit for the two-dimensional uniform model in $W_2$, already exploited in the Gaussian case in [8]. This approach relies on a heat kernel regularization argument together with transportation bounds in terms of dual Sobolev norms. With respect to the compact case, the Gaussian model involves the so-called Mehler kernel as underlying dynamics, which is unbounded. One of the features of the work [8] was the introduction of a localization step in order to take into account the infinite support of the Gaussian measure and the associated Gaussian tails. A further main step is achieved here by a randomization of the regularization time, together with estimates on the Mehler kernel.

The note [15] by M. Talagrand gave a proof of the exceptional case $p = d = 2$ in (1.5) by means of a scaling argument (for more general distributions than the Gaussian). It is certainly possible that the tools developed therein could lead to the main results presented here, as well as answer some of the left open rates, in particular in the case $p = d$. However, this note is rather difficult to grasp and we could not extract further conclusions at this stage. On the other hand, the pde and transportation method may be used to recover the main result of [15] for $p = d = 2$ as was shown in [9] soon after [15]. For completeness and convenience, this proof is briefly recalled here as Section 7.

Turning to the content of the paper, Section 2 collects several formulas and bounds on the Mehler kernel as substitutes of uniform bounds in the compact case. The subsequent paragraph briefly recalls, in this Gaussian context, the transportation inequalities needed for the proofs already emphasized in [2], [8]. The proof of the main Theorem 1 is divided in the two next sections, addressing respectively the case $p = 2$ and $p > 2$. It could have been possible to present the proof directly for $p \geq 2$, but the case $p = 2$ is easier to handle and provides a good
warm-up for the general case. In Section 6, we discuss how the parameters may be adjusted to reach Theorem 2. As announced, the last Section 7 presents the pde and transportation proof of the lower bound in (1.5) first achieved by M. Talagrand [15] using combinatorial and scaling properties.

2 Properties of the Mehler kernel

This short section collects basic and classical properties of the Mehler kernel and the associated Ornstein-Uhlenbeck semigroup which will be of use throughout this work. The reference [4] (and the bibliography therein) covers most of the claims emphasized below. With respect to the compact case, the main issue here is that the Mehler kernel is spatially unbounded, and so various pointwise and integral controls have to be identified.

Let $d\mu(x) = \frac{dx}{(2\pi)^{d/2}}$ be the standard Gaussian measure on the Borel sets of $\mathbb{R}^d$. The Mehler kernel is given, for $t > 0$, $x, y \in \mathbb{R}^d$, by

$$p_t(x, y) = p_t(y, x) = \frac{1}{(1 - e^{-2t})^{d/2}} \exp \left( - \frac{e^{-2t}}{2(1 - e^{-2t})} \left[ |x|^2 + |y|^2 - 2e^t x \cdot y \right] \right). \quad (2.1)$$

Here $|x|$ denotes the Euclidean length of $x \in \mathbb{R}^d$. It holds true that $\int_{\mathbb{R}^d} p_t(x, y) d\mu(y) = 1$ for all $t > 0$ and $x \in \mathbb{R}^d$. The Mehler kernel satisfies besides the basic semigroup property with respect to $\mu$,

$$\int_{\mathbb{R}^d} p_s(x, z) p_t(z, y) d\mu(z) = p_{s+t}(x, y) \quad (2.2)$$

for all $s, t > 0$ and $x, y \in \mathbb{R}^d$.

To ease the notation, set below $a = e^{-t} \in (0, 1)$ in some of the statements.

When $x = y$,

$$p_t(x, x) = \frac{1}{(1 - a^2)^{d/2}} e^{\frac{a^2}{1 - a^2} |x|^2} \leq \frac{1}{(1 - a^2)^{d/2}} e^{\frac{|x|^2}{2}}. \quad (2.3)$$

It is easily seen besides that for all $x, y \in \mathbb{R}^d$,

$$p_t(x, y) \leq \frac{1}{(1 - a^2)^{d/2}} e^{\frac{|x|^2}{2}}. \quad (2.4)$$

Combining (2.4), (2.2) and (2.3) also shows that for every $q \geq 2$ and all $x \in \mathbb{R}^d$,

$$\int_{\mathbb{R}^d} p_t(x, y)^q d\mu(y) = \int_{\mathbb{R}^d} p_t(x, y)^{q-2} p_t(x, y)^2 d\mu(y) \leq \frac{1}{(1 - a^2)^{(q-1)d/2}} e^{(q-1)|x|^2/2}. \quad (2.5)$$

The Mehler kernel generates the Ornstein-Uhlenbeck semigroup

$$P_t f(x) = \int_{\mathbb{R}^d} f(y) p_t(x, y) d\mu(y) = \int_{\mathbb{R}^d} f(e^{-t} x + \sqrt{1 - e^{-2t}} y) d\mu(y) \quad (2.6)$$
for all $t > 0$, $x \in \mathbb{R}^d$, and any suitable measurable function $f : \mathbb{R}^d \to \mathbb{R}$, with the natural extension $P_0 = \text{Id}$. The family $(P_t)_{t \geq 0}$ defines a Markov semigroup, symmetric in $L^2(\mu)$, with infinitesimal generator $\mathbb{L} = \Delta - x \cdot \nabla$ for which the integration by parts formula

$$\int_{\mathbb{R}^d} f(-\mathbb{L}g)d\mu = \int_{\mathbb{R}^d} \nabla f \cdot \nabla g \, d\mu \quad (2.7)$$

holds true for every smooth functions $f, g : \mathbb{R}^d \to \mathbb{R}$. The spectrum of the operator $\mathbb{L}$ is $\mathbb{N}$.

The semigroup $(P_t)_{t \geq 0}$ is a contraction in all $L^p(\mu)$-spaces with norms $\| \cdot \|_p$, $1 \leq p \leq \infty$. The spectral gap induces an exponential decay for mean zero functions $f$ in $L^2(\mu)$,

$$\|P_t f\|_2 \leq e^{-t} \|f\|_2, \quad t \geq 0. \quad (2.8)$$

The hypercontractivity property on the other hand expresses that whenever $1 < p < q < \infty$ and $e^{2t} \geq \frac{q-1}{p-1}$,

$$\|P_t f\|_q \leq \|f\|_p. \quad (2.9)$$

It will be convenient later on to combine the preceding smoothing properties in the following form: for any $f : \mathbb{R}^d \to \mathbb{R}$ in $L^p(\mu)$, $p \geq 2$, with mean zero,

$$\|P_t f\|_p \leq C e^{-t/2} \|f\|_p, \quad t \geq 0, \quad (2.10)$$

where $C > 0$ only depends on $p$. The decay $e^{-t/2}$ is far from optimal but good enough for our purpose. For the proof, let $t_0 > 0$ be such that $e^{t_0} = p - 1 > 1$, so that by (2.9), for every $t \geq t_0$,

$$\|P_t f\|_p = \|P_{t/2}(P_{t/2} f)\|_p \leq \|P_{t/2} f\|_2.$$  

By the exponential decay in $L^2(\mu)$-norm (2.8), $\|P_{t/2} f\|_2 \leq e^{-t/2} \|f\|_2$, $\leq e^{-t/2} \|f\|_p$ and hence

$$\|P_t f\|_p \leq e^{-t/2} \|f\|_p.$$  

If $t \leq t_0$, $\|P_t f\|_p \leq \|f\|_p \leq C e^{t_0/2}$. The claim (2.10) is established.

The following pseudo-Poincaré inequality is another useful tool: for any $1 \leq p < \infty$, there exists $C = C(p, d) > 0$ such that for every smooth function $f : \mathbb{R}^d \to \mathbb{R}$ and every $0 < t \leq 1$,

$$\int_{\mathbb{R}^d} |P_t f - f|^p d\mu \leq C t^{p/2} \int_{\mathbb{R}^d} |\nabla f|^p d\mu. \quad (2.11)$$  

In case $p = 2$, this inequality may be easily deduced spectrally. For a proof in the general case, let $g : \mathbb{R}^d \to \mathbb{R}$ be smooth with $\|g\|_q \leq 1$ such that

$$\|P_t f - f\|_p = \int_{\mathbb{R}^d} g(P_t f - f) d\mu$$

where $\frac{1}{p} + \frac{1}{q} = 1$. By symmetry of the semigroup $(P_t)_{t \geq 0}$ and integration by parts

$$\int_{\mathbb{R}^d} g(P_t f - f) d\mu = \int_0^t \int_{\mathbb{R}^d} g \mathbb{L} P_s f d\mu ds = -\int_0^t \int_{\mathbb{R}^d} \nabla P_s g \cdot \nabla f d\mu ds.$$
The integral representation \(2.6\) and one more integration by parts indicate that, at any point \(x \in \mathbb{R}^d\),
\[
\nabla P_s g(x) = e^{-s} \int_{\mathbb{R}^d} \nabla g(e^{-s} x + \sqrt{1-e^{-2s}} y) d\mu(y) \\
= \frac{e^{-s}}{\sqrt{1-e^{-2s}}} \int_{\mathbb{R}^d} y g(e^{-s} x + \sqrt{1-e^{-2s}} y) d\mu(y).
\]
Hence, by Hölder’s inequality,
\[
\int_{\mathbb{R}^d} \nabla P_s \cdot \nabla f d\mu = \frac{e^{-s}}{\sqrt{1-e^{-2s}}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} y \cdot \nabla f(x) g(e^{-s} x + \sqrt{1-e^{-2s}} y) d\mu(x) d\mu(y) \\
\leq \frac{e^{-s}}{\sqrt{1-e^{-2s}}} \left( \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left| y \cdot \nabla f(x) \right|^p d\mu(x) d\mu(y) \right)^{1/p}
\]
where it is used that
\[
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g(e^{-s} x + \sqrt{1-e^{-2s}} y) |y|^q d\mu(x) d\mu(y) = \int_{\mathbb{R}^d} |g|^q d\mu \leq 1.
\]
Finally, after partial integration in \(d\mu(y)\),
\[
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |y \cdot \nabla f(x)|^p d\mu(x) d\mu(y) = C \int_{\mathbb{R}^d} |\nabla f|^p d\mu
\]
where \(C > 0\) only depends on \(p\). The claim \(2.11\) then easily follows.

We will make use of an important and delicate property, the Riesz transform bounds. In this Gaussian setting, there were established by P. A. Meyer [10] (see also [3]) and express that, for every \(1 < p < \infty\) and every \(f : \mathbb{R}^d \to \mathbb{R}\) in the suitable domain,
\[
\int_{\mathbb{R}^d} |\nabla f|^p d\mu \approx \int_{\mathbb{R}^d} |(-L)^{-1/2} f|^p d\mu \tag{2.12}
\]
\((\approx\) only depending on \(p\)) where \((-L)^{-1/2}\) is defined spectrally on mean zero functions \(f\), for example by the classical formula
\[
(-L)^{-1/2} = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{1}{\sqrt{s}} P_s ds. \tag{2.13}
\]

Finally, some technical tools related to energy estimates will be requested. The reverse Poincaré inequality for the Gaussian measure ([4, Theorem 4.7.2]) expresses that for every Borel set \(A\) in \(\mathbb{R}^d\) and every \(s > 0\),
\[
|\nabla P_s 1_A|^2 \leq \frac{1}{e^{2s} - 1} \left[ P_s (1_A^2) - (P_s 1_A)^2 \right] \leq \frac{1}{e^{2s} - 1}.
\]
Combining with \(2.11\) for \(f = P_s 1_A\), for every \(s > 0\), \(t \in (0, 1)\) and \(p \geq 1\),
\[
\int_{\mathbb{R}^d} |P_t (P_s 1_A) - P_s 1_A|^p d\mu \leq \frac{C t^{p/2}}{(e^{2s} - 1)^{(p-1)/2}} \int_{\mathbb{R}^d} |\nabla P_s 1_A| d\mu \leq \frac{C t^{p/2} e^{-s}}{(e^{2s} - 1)^{(p-1)/2}} \mu(\partial A) \tag{2.14}
\]
where, in the last step, it is used that \(\nabla P_s = e^{-s} P_s \nabla\) and \(\limsup_{\varepsilon \to 0} \int_{\mathbb{R}^d} |\nabla P_\varepsilon 1_A| d\mu \leq \mu(\partial A)\) for a measurable subset \(A\) of \(\mathbb{R}^d\) with smooth boundary \(\partial A\).
3 Mass transportation bounds

This short paragraph presents the key functional analytic tool to bound Kantorovich distances in this context. It was already put forward in [8] (see also [11, 13]).

**Proposition 3.** Let $p \geq 1$. For any $d\nu = fd\mu$ with $f - 1$ in $H^{-1,p}(\mu)$, we have

$$W_p(\nu, \mu) \leq p \|f - 1\|_{H^{-1,p}(\mu)}$$

where the $H^{-1,p}(\mu)$ negative Sobolev norm is defined by

$$\|g\|_{H^{-1,p}(\mu)} = \left(\int_{\mathbb{R}^d} |\nabla((-L)^{-1}g)|^p d\mu\right)^{1/p}$$

for any $g : \mathbb{R}^d \to \mathbb{R}$ with mean zero such that the right-hand side makes sense.

It may be emphasized that in the particular case $p = 2$,

$$\|g\|_{H^{-1,2}(\mu)}^2 = 2 \int_0^\infty \int_{\mathbb{R}^d} (P_s g)^2 d\mu ds. \hspace{1cm} (3.1)$$

Indeed, since $(-L)^{-1} = \int_0^\infty P_s ds$, by integration by parts,

$$\int_{\mathbb{R}^d} |\nabla((-L)^{-1}g)|^2 d\mu = \int_{\mathbb{R}^d} g(-L)^{-1} g d\mu = \int_0^\infty \int_{\mathbb{R}^d} g P_s g d\mu ds$$

from which the claim follows from the symmetry of $P_s$.

When $p \neq 2$, it will be necessary to rely on the Riesz transform bound (2.12), for $1 < p < \infty$, after which Proposition 3 takes the form

$$W_p^p(\nu, \mu) \lesssim \int_{\mathbb{R}^d} |(-L)^{-1/2}(f - 1)|^p d\mu. \hspace{1cm} (3.2)$$

4 The case $2 = p < d$

We therefore address here the proof of Theorem 1 for $2 = p < d$. Although the proof of this case is actually contained in the more general section $2 \leq p < d$, it is easier to access due to the semigroup representation (3.1) of the negative Sobolev norm.

The first step in the investigation is the localization argument introduced in [8]. For $R > 0$, let $d\mu^R = \frac{1}{\mu(B_R)} 1_{B_R} d\mu$ where $B_R$ is the Euclidean ball centered at 0 with radius $R$. Define independent random variables $X_i^R$, $i = 1, \ldots, n$, with common distribution $\mu^R$ by

$$X_i^R = \begin{cases} X_i & \text{if } X_i \in B_R, \\ Z_i & \text{if } X_i \notin B_R, \end{cases}$$

$\ldots$
where $Z_1, \ldots, Z_n$ are independent with distribution $\mu^R$, independent of the $X_i$'s. Setting $\mu_n^R = \frac{1}{n} \sum_{i=1}^n \delta_{X_i^R}$, by definition of the coupling,

$$W_2^2(\mu_n^R, \mu_n^R) \leq \frac{1}{n} \sum_{i=1}^n |X_i - X_i^R|^2 \leq \frac{4}{n} \sum_{i=1}^n |X_i|^2 \mathbf{1}_{\{|X_i| \geq R\}}.$$

Therefore

$$\mathbb{E}(W_2^2(\mu_n^R, \mu_n^R)) \leq 4 \int_{\{|x| > R\}} |x|^2 d\mu.$$

Since

$$\int_{\{|x| > R\}} |x|^2 d\mu = C_d \int_0^\infty r^{d+1} e^{-r^2/2} dr$$

is of the order of $R^d e^{-R^2/2}$ as $R \to \infty$, choose $R = \sqrt{2c \log n}$ for some $c \in (\frac{2}{d}, 1)$ so that

$$\mathbb{E}(W_2^2(\mu_n^R, \mu_n^R)) \lesssim \frac{1}{n^{2/d}}. \quad (4.1)$$

As a result, the investigation is concentrated on the study of $\mathbb{E}(W_2^2(\mu_n^R, \mu))$. Note furthermore that $\mu(B_R) \geq \frac{1}{2}$ for $n$ large enough so that this normalization factor may essentially be neglected throughout the investigation. For the further developments, it will be convenient to refer to a random variable $X$ with distribution $\mu$ and to $X^R$ with distribution $\mu^R$.

A new step with respect to the former investigations is the introduction of a randomized regularization time by means of the decomposition of $B_R$ as the union of $m$ annuli

$$D_k = \{x \in \mathbb{R}^d; r_{k-1} \leq |x| < r_k\}, \quad k = 1, \ldots, m,$$

where $0 = r_0 < r_1 < \cdots < r_m = R$ with $r_k = \sqrt{k}$, $k = 1, \ldots, m$. In particular $m = R^2 = 2c \log n$. Define then a map $T : B_R \to (0, 1)$ as $T(x) = t_k$ if $x \in D_k$ where the $0 < t_1 < \cdots < t_m < 1$ will be specified later.

For this map $T$, consider then the (random) probability density

$$f(y) = f_{n,T}^{R} (y) = \frac{1}{n} \sum_{i=1}^n p_{T(X_i^R)}(X_i^R, y), \quad y \in \mathbb{R}^d,$$

and set $d\mu_{n,T}^R = f_{n,T}^{R} d\mu$. By convexity of the Kantorovich metric $W_2^2$ [16, Theorem 4.8] and the representation formula for the Ornstein-Uhlenbeck semigroup [22,6,], conditionally on the $X_i^R$'s,

$$W_2^2(\mu_n^R, \mu_{n,T}^R) \leq \frac{1}{n} \sum_{i=1}^n W_2^2(\delta_{X_i^R}, p_{T(X_i^R)}(X_i^R, \cdot)) d\mu$$

$$= \frac{1}{n} \sum_{i=1}^n \int_{\mathbb{R}^d} |X_i^R - y|^2 p_{T(X_i^R)}(X_i^R, y) d\mu(y)$$

$$= \frac{1}{n} \sum_{i=1}^n \left[ (1 - e^{-T(X_i^R)}) |X_i^R|^2 + d(1 - e^{-2T(X_i^R)}) \right]$$

$$\lesssim \frac{1}{n} \sum_{i=1}^n \left[ T(X_i^R)^2 |X_i^R|^2 + T(X_i^R) \right]$$

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where we used in the last step that $T(X_i^R) \in (0, 1)$. Averaging over the $X_i^R$'s, and dropping $X^R$ in $T(X^R)$ for simplicity,

\[
\mathbb{E}(W_2^2(\mu_{n}^{R}, \mu_{n}^{R,T})) \lesssim \mathbb{E}(T^2|X^R|^2) + \mathbb{E}(T)
\]

\[
\lesssim (t_m R^2 + 1) \mathbb{E}(T) = (t_m R^2 + 1) \sum_{k=1}^{m} t_k \mu(D_k). \tag{4.2}
\]

From the localization \[4.1\] and regularization \[4.2\] arguments, and the triangle inequality for $W_2$, we have therefore obtained at this stage that

\[
\mathbb{E}(W_2^2(\mu_{n}, \mu)) \lesssim \mathbb{E}(W_2^2(\mu_{n}^{R,T}, \mu)) + \frac{1}{n^{2/d}} + (t_m R^2 + 1) \sum_{k=1}^{m} t_k \mu(D_k). \tag{4.3}
\]

From here, we thus concentrate on $\mathbb{E}(W_2^2(\mu_{n}^{R,T}, \mu))$ for which we make use of Proposition \[3\] and \[4.1\] with

\[
g = g(y) = f_{n,T}^R(y) - 1 = \frac{1}{n} \sum_{i=1}^{n} [p_{T(X_i^R)}(X_i^R, y) - 1], \quad y \in \mathbb{R}^d,
\]

to get that

\[
\mathbb{E}(W_2^2(\mu_{n}^{R,T}, \mu)) \leq 4 \mathbb{E}(\|f_{n,T}^R - 1\|_{H^{-1/2}(\mu)}) = 8 \int_{0}^{\infty} \int_{\mathbb{R}^d} \mathbb{E}((P_s g)^2) d\mu ds.
\]

In order to develop probabilistic arguments, it is convenient to center the elements $p_{T(X_i^R)}(X_i^R, y)$ in the definition of $g$. Write therefore, for every $y \in \mathbb{R}^d$,

\[
g(y) = \frac{1}{n} \sum_{i=1}^{n} \left[ p_{T(X_i^R)}(X_i^R, y) - \mathbb{E}(p_{T(X_i^R)}(X_i^R, y)) \right] + \mathbb{E}(p_{T}(X^R, y)) - 1 = \bar{g}(y) + \phi(y)
\]

where $\phi(y) = \mathbb{E}(p_{T}(X^R, y)) - 1$. Recall that here $T = T(X^R)$. For every $s > 0$, $\mathbb{E}((P_s g)^2) \leq 2 \mathbb{E}((P_s \bar{g})^2) + 2(P_s \phi)^2$ so that

\[
\mathbb{E}(W_2^2(\mu_{n}^{R,T}, \mu)) \leq 16 \int_{0}^{\infty} \int_{\mathbb{R}^d} \mathbb{E}((P_s \bar{g})^2) d\mu ds + 16 \int_{0}^{\infty} \int_{\mathbb{R}^d} (P_s \phi)^2 d\mu ds.
\]

Now, since

\[
P_s \bar{g}(y) = \frac{1}{n} \sum_{i=1}^{n} \left[ p_{s+T(X_i^R)}(X_i^R, y) - \mathbb{E}(p_{s+T(X_i^R)}(X_i^R, y)) \right]
\]

and since the $X_i^R$'s, $i = 1, \ldots, n$, are independent and identically distributed, for each $y \in \mathbb{R}^d$,

\[
\mathbb{E}((P_s \bar{g})^2) = \frac{1}{n} \mathbb{E} \left( \left[ p_{s+T}(X^R, y) - \mathbb{E}(p_{s+T}(X^R, y)) \right]^2 \right).
\]

But $\mathbb{E}(p_{s+T}(X^R, y)) = P_s \phi(y) + 1$, and as we aim to control $\int_{0}^{\infty} \int_{\mathbb{R}^d} (P_s \phi)^2 d\mu ds$, we may as well replace back $\mathbb{E}(p_{T+s}(X^R, y))$ by 1 in the latter, to obtain that

\[
\mathbb{E}(W_2^2(\mu_{n}^{R,T}, \mu)) \lesssim \frac{1}{n} \int_{0}^{\infty} \int_{\mathbb{R}^d} \mathbb{E} \left( \left[ p_{s+T}(X^R, y) - 1 \right]^2 \right) d\mu(y) ds + \int_{0}^{\infty} \int_{\mathbb{R}^d} (P_s \phi)^2 d\mu ds. \tag{4.4}
\]
The most important term on the right-hand side of (4.4) is the first one on which we concentrate next. Divide the integral in $s$ according as $s \in (0, 1)$ or $s \in (1, \infty)$ so to get

$$\frac{1}{n} \int_0^\infty \int_{\mathbb{R}^d} \mathbb{E}( [p_{s+T}(X^R, y) - 1]^2 ) \, d\mu(y) \, ds$$

$$= \frac{1}{n} \int_0^1 \int_{\mathbb{R}^d} \mathbb{E}( [p_{s+T}(X^R, y) - 1]^2 ) \, d\mu(y) \, ds$$

$$+ \frac{1}{n} \int_1^\infty \int_{\mathbb{R}^d} \mathbb{E}( [p_{s+T}(X^R, y) - 1]^2 ) \, d\mu(y) \, ds. \tag{4.5}$$

Recall in addition that by definition of $T$,

$$\mathbb{E}( [p_{s+T}(X^R, y) - 1]^2 ) = \frac{1}{\mu(B_R)} \sum_{k=1}^m \int_{D_k} [p_{s+t_k}(x, y) - 1]^2 \, d\mu(x).$$

The first piece on the right-hand side of (4.5) may then be simply upper-bounded as

$$\frac{1}{n} \int_0^1 \int_{\mathbb{R}^d} \mathbb{E}( [p_{s+T}(X^R, y) - 1]^2 ) \, d\mu(y) \, ds \lesssim \frac{1}{n} \sum_{k=1}^m \int_{D_k} \int_0^1 \int_{\mathbb{R}^d} p_{s+t_k}(x, y)^2 \, d\mu(y) \, ds \, d\mu(x).$$

By (2.2), (2.3) and integration in $s$, for every $x \in \mathbb{R}^d$ and $k = 1, \ldots, m$,

$$\int_0^1 \int_{\mathbb{R}^d} p_{s+t_k}(x, y)^2 \, d\mu(y) \, ds = \int_0^1 p_{2(s+t_k)}(x, x) \, ds \lesssim \frac{1}{t_k^{(d/2)-1}} e^{\|x\|^2/2}.$$

Hence, since $\lambda(D_k) \approx k^{(d/2)-1}$,

$$\frac{1}{n} \int_0^1 \int_{\mathbb{R}^d} \mathbb{E}( [p_{s+T}(X^R, y) - 1]^2 ) \, d\mu(y) \, ds \lesssim \frac{1}{n} \sum_{k=1}^m \int_{D_k} \frac{1}{t_k^{(d/2)-1}} e^{\|x\|^2/2} \, d\mu(x)$$

$$\lesssim \frac{1}{n} \sum_{k=1}^m \lambda(D_k)$$

$$\lesssim \frac{1}{n} \sum_{k=1}^m \left( \frac{k}{t_k} \right)^{(d/2)-1} \tag{4.6}.$$

On the other hand, towards the second piece on the right-hand side of (4.5), by the exponential decay (2.8) in $L^2(\mu)$,

$$\frac{1}{n} \int_1^\infty \int_{\mathbb{R}^d} \mathbb{E}( [p_{s+T}(X^R, y) - 1]^2 ) \, d\mu(y) \, ds$$

$$= \frac{1}{n} \frac{1}{\mu(B_R)} \sum_{k=1}^m \int_{D_k} \int_0^\infty \int_{\mathbb{R}^d} [p_{s+t_k}(x, y) - 1]^2 \, d\mu(y) \, ds \, d\mu(x)$$

$$\lesssim \frac{1}{n} \sum_{k=1}^m \int_{D_k} \int_0^\infty e^{-2s} \int_{\mathbb{R}^d} [p_1(x, y) - 1]^2 \, d\mu(y) \, ds \, d\mu(x)$$

$$\lesssim \frac{1}{n} \sum_{k=1}^m \lambda(D_k) = \frac{1}{n} \lambda(B_R) \tag{4.7}.$$
where it is used again that $\int_{\mathbb{R}^d} p_1(x, y)^2 d\mu(y) = p_2(x, x) \lesssim e^{\|x\|^2/2}$.

Summarizing (4.6) and (4.7) in (4.5), it follows that

$$\frac{1}{n} \int_0^\infty \int_{\mathbb{R}^d} \mathbb{E}\left(\left[ p_{s+T}(X^R, y) - 1 \right]^2 \right) d\mu(y) ds \lesssim \frac{1}{n} \sum_{k=1}^m \left( \frac{k}{t_k} \right)^{(d/2) - 1} + \frac{1}{n^{2/d}}$$

(recall that $d > 2$). Collecting this estimate in (4.4) and (4.3), it holds true that

$$\mathbb{E}(W^2_2(\mu_n, \mu)) \lesssim \frac{1}{n} \sum_{k=1}^m \left( \frac{k}{t_k} \right)^{(d/2) - 1} + (t_m R^2 + 1) \sum_{k=1}^m t_k \mu(D_k)$$

$$+ \frac{1}{n^{2/d}} + \int_0^\infty \int_{\mathbb{R}^d} (P_s\phi)^2 d\mu ds. \quad (4.8)$$

The choice of the $t_k$’s is now determined by optimization between the first two terms on the right-hand side of (4.5). Using that $\mu(D_k) \lesssim k^{(d/2) - 1} e^{-k/2}$, set $t_k = n^{-2/d} c^{k/d}$, $k = 1, \ldots, m$. In particular,

$$t_k \leq t_m = \frac{1}{n^{2(1-c)/d}}$$

for every $k = 1, \ldots, m = R^2 = 2c \log n$, $c \in (2/d, 1)$. For these values, and since $d > 2$, it follows that

$$\frac{1}{n} \sum_{k=1}^m \left( \frac{k}{t_k} \right)^{(d/2) - 1} + (t_m R^2 + 1) \sum_{k=1}^m t_k \mu(D_k) \lesssim \frac{1}{n^{2/d}}$$

Therefore (4.8) yields

$$\mathbb{E}(W^2_2(\mu_n, \mu)) \lesssim \frac{1}{n^{2/d}} + \int_0^\infty \int_{\mathbb{R}^d} (P_s\phi)^2 d\mu ds. \quad (4.9)$$

We are left with the study of the centering term $\int_0^\infty \int_{\mathbb{R}^d} (P_s\phi)^2 d\mu ds$. To this task, recall that

$$P_s\phi = \mathbb{E}\left( p_{s+T}(X^R, \cdot) - 1 \right)$$

$$= \frac{1}{\mu(B_R)} \sum_{k=1}^m \left[ p_{s+t_k} 1_{D_k} - \mu(D_k) \right]$$

$$= \frac{1}{\mu(B_R)} \sum_{k=1}^m \left[ p_{s+t_k} 1_{D_k} - p_s 1_{D_k} \right] + \frac{1}{\mu(B_R)} \sum_{k=1}^m \left[ p_s 1_{D_k} - \mu(D_k) \right]$$

$$= \frac{1}{\mu(B_R)} \sum_{k=1}^m \left[ p_{s+t_k} 1_{D_k} - p_s 1_{D_k} \right] + \frac{1}{\mu(B_R)} \left[ p_s 1_{B_R} - \mu(B_R) \right].$$

Hence, for every $s > 0$,

$$\int_{\mathbb{R}^d} (P_s\phi)^2 d\mu \lesssim \int_{\mathbb{R}^d} \left[ \sum_{k=1}^m \left[ p_{s+t_k} 1_{D_k} - p_s 1_{D_k} \right] \right]^2 d\mu + \int_{\mathbb{R}^d} \left[ p_s 1_{B_R} - \mu(B_R) \right]^2 d\mu$$

and we treat separately the two expressions on the right-hand side. The second one is easy since by the exponential decay (2.8) in $L^2(\mu)$,

$$\int_{\mathbb{R}^d} \left[ p_s 1_{B_R} - \mu(B_R) \right]^2 d\mu \leq e^{-2s} \left[ 1 - \mu(B_R) \right] \lesssim \frac{e^{-2s}}{n^{2/d}}$$

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by the choice of $R$. For the first one, by the triangle inequality and (2.14),
\[
\int_{\mathbb{R}^d} \left[ \sum_{k=1}^{m} [P_{s+t_k} \mathbb{1}_{D_k} - P_s \mathbb{1}_{D_k}] \right]^2 d\mu \lesssim \left( \sum_{k=1}^{m} \left[ \frac{t_k e^{-s}}{\sqrt{e^{2s} - 1}} \mu(\partial D_k) \right]^{1/2} \right)^2 \lesssim \frac{e^{-s}}{\sqrt{e^{2s} - 1}} \left( \sum_{k=1}^{m} \sqrt{t_k} \sqrt{\mu(\partial D_k)} \right)^2.
\]
Combining the preceding,
\[
\int_0^\infty \int_{\mathbb{R}^d} (P_s \phi)^2 d\mu ds \lesssim \left( \sum_{k=1}^{m} \sqrt{t_k} \sqrt{\mu(\partial D_k)} \right)^2 + \frac{1}{n^{2/d}}.
\]
Since $\mu(\partial D_k) \lesssim k^{(d-1)/2}e^{-k/2}$, the choice of $t_k = n^{-2/d}e^{k/d}$, $k = 1, \ldots, m$, easily shows that this centering contribution is at most $\frac{1}{n^{2/d}}$. Inserting this claim into (4.9) concludes the proof of the theorem for $2 = p < d$.

5 The case $2 \leq p < d$

The pattern of the proof will be similar to the one of Section 4 but with significant increase of the technicalities since (3.1) is now more available and the arguments go through the more involved Riesz transform bound (3.2). The scheme of proof is then similar to the one developed in [8] in the compact case (1.3) but, again, unboundedness of the Mehler kernel requires several delicate estimates.

As in the preceding section for $p = 2$, we truncate in the same way on a ball $B_R$ with $R = \sqrt{2c \log n}$ for some $c \in (\frac{d}{2}, 1)$ for which we get similarly that
\[
\mathbb{E}(W_p^p(\mu_n, \mu_n^R)) \lesssim \frac{1}{n^{p/d}}. \quad (5.1)
\]
We decompose again $B_R$ as the union of $m$ annuli $D_k = \{x \in \mathbb{R}^d; r_{k-1} \leq |x| < r_k\}$, $k = 1, \ldots, m$, where $0 = r_0 < r_1 < \cdots < r_m = R$, and consider also the map $T : B_R \to (0, 1)$ defined by $T(x) = t_k$ if $x \in D_k$. We will use the same choices $r_k = \sqrt{k}$ and $t_k = n^{-2/d}e^{k/d}$, $k = 1, \ldots, m$. In particular, for the further purposes, it is important to notice again that
\[
t_k \leq t_m \leq \frac{1}{n^{2(1-c)/d}}, \quad k = 1, \ldots, m, \quad (5.2)
\]
small enough therefore for a number of subsequent small issues (recall that $n \geq n_0$ is assumed large enough throughout the investigation).

For this map $T : B_R \to (0, 1)$, set similarly
\[
f(y) = f^{R,T}_n(y) = \frac{1}{n} \sum_{i=1}^{n} p_{T(X_i^R)}(X_i^R, y), \quad y \in \mathbb{R}^d,
\]
and \(d \mu_{n}^{R,T} = f_{n}^{R,T} d \mu\). By convexity [16, Theorem 4.8] and the representation formula for the Ornstein-Uhlenbeck semigroup (2.6), conditionally on the \(X_i^R\)'s,

\[
W_p^p(\mu_n^R, \mu_n^{R,T}) \leq \frac{1}{n} \sum_{i=1}^{n} W_p^p(\delta_{X_i^R}, p_{T(X_i^R)}(X_i^R, \cdot)) d \mu
\]

\[
= \frac{1}{n} \sum_{i=1}^{n} \int_{\mathbb{R}^d} |X_i^R - y|^p p_{T(X_i^R)}(X_i^R, y) d \mu(y)
\]

\[
\lesssim \frac{1}{n} \sum_{i=1}^{n} \left[(1 - e^{-T(X_i^R)})^p |X_i^R|^p + (1 - e^{-2T(X_i^R)})^{p/2}\right].
\]

Noticing that \(T(X_i^R) \in (0, 1)\), it follows after taking expectation that

\[
\mathbb{E}(W_p^p(\mu_n^R, \mu_n^{R,T})) \lesssim \mathbb{E}(T^p |X|^p) + \mathbb{E}(T^{p/2}) \lesssim (t_m^{p/2} R^p + 1) \mathbb{E}(T^{p/2})
\]

(5.3)

where we recall that \(T = T(X^R)\).

From (5.1), (5.3) and the triangle inequality, and the fact (5.2) that \(t_m \leq \frac{1}{n^{d(1-c)/d} d} \), we have therefore obtained at this point that

\[
\mathbb{E}(W_p^p(\mu_n, \mu)) \lesssim \mathbb{E}(W_p^p(\mu_n^{R,T}, \mu)) + \frac{1}{n^{p/d}} + \sum_{k=1}^{m} t_k^{p/2} \mu(D_k)
\]

\[
\lesssim \mathbb{E}(W_p^p(\mu_n^{R,T}, \mu)) + \frac{1}{n^{p/d}}
\]

(5.4)

where we used that \(t_k = n^{-2/d} e^{k/d} \) and \(\mu(D_k) \lesssim k^{(d/2)-1} e^{-k/2}, k = 1, \ldots, m\).

As in the previous section, we thus concentrate on the study of \(\mathbb{E}(W_p^p(\mu_n^{R,T}, \mu))\) that we control from Proposition [3] together with the Riesz transform bound (3.2). To this task, set

\[
g = g(y) = f_n^{R,T}(y) - 1 = \frac{1}{n} \sum_{i=1}^{n} \left[p_{T(X_i^R)}(X_i^R, y) - 1\right], \quad y \in \mathbb{R}^d.
\]

Therefore

\[
\mathbb{E}(W_p^p(\mu_n^{R,T}, \mu)) \leq p^p \mathbb{E}\left(\left\|f_n^{R,T} - 1\right\|_{P^{p,\mu}}^p\right) \lesssim \mathbb{E}\left(\int_{\mathbb{R}^d} |(-L)^{-1/2} g|^p d \mu\right).
\]

Center then the terms \(p_{T(X_i^R)}(X_i^R, y)\) in the definition of \(g\) with respect to randomness in the \(X_i^R\)'s. To this task, write for every \(y\),

\[
g(y) = \frac{1}{n} \sum_{i=1}^{n} \left[p_{T(X_i^R)}(X_i^R, y) - \mathbb{E}(p_{T(X_i^R)}(X_i^R, y))\right] + \mathbb{E}(p_{T(X_i^R, y))} - 1 = \bar{g}(y) + \phi(y)
\]

with \(\phi(y) = \mathbb{E}(p_{T(X_i^R, y))} - 1\) so that

\[
\mathbb{E}(W_p^p(\mu_n^{R,T}, \mu)) \lesssim \mathbb{E}\left(\int_{\mathbb{R}^d} |(-L)^{-1/2} \bar{g}|^p d \mu\right) + \int_{\mathbb{R}^d} |(-L)^{-1/2} \phi|^p d \mu.
\]

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Rosenthal’s inequality [12] for independent centered random variables \( V_1, \ldots, V_n \) with a \( p \)-th moment, \( p \geq 2 \), expresses that

\[
\mathbb{E}\left( \left| \sum_{i=1}^{n} V_i \right|^p \right) \leq C_p \sum_{i=1}^{n} \mathbb{E}(|V_i|^p) + C_p \left( \sum_{i=1}^{n} \mathbb{E}(V_i^2)^{p/2} \right) \quad (5.5)
\]

where \( C_p > 0 \) only depends on \( p \). For each fixed \( y \in \mathbb{R}^d \), apply this inequality to the independent identically distributed and centered random variables

\[
(-L_y)^{-1/2}[p_T(X^R, y) - \mathbb{E}(p_T(X^R, y))], \quad i = 1, \ldots, n,
\]
to get that

\[
\mathbb{E}\left( \left| (-L_y)^{-1/2} \tilde{g}(y) \right|^p \right) \leq \frac{1}{n^{p-1}} \mathbb{E}\left( \left| (-L_y)^{-1/2} \left[ p_T(X^R, y) - \mathbb{E}(p_T(X^R, y)) \right] \right|^p \right) + \frac{1}{n^{p/2}} \mathbb{E}\left( \left| (-L_y)^{-1/2} \left[ p_T(X^R, y) - \mathbb{E}(p_T(X^R, y)) \right] \right|^2 \right)^{p/2}. \quad (5.6)
\]

Since \( \mathbb{E}(p_T(X^R, y) = \phi(y) + 1 \), and we eventually aim to control \( \int_{\mathbb{R}^d} \left| (-L)^{-1/2} \phi \right|^p d\mu \), we may replace back \( \mathbb{E}(p_T(X^R, y)) \) by 1 in the preceding. That is, we have at that point

\[
\mathbb{E}(W_p^R(\mu, \mu)) \leq \frac{1}{n^{p-1}} \int_{\mathbb{R}^d} \mathbb{E}\left( \left| (-L_y)^{-1/2} [p_T(X^R, y) - 1] \right|^p \right) d\mu(y) + \frac{1}{n^{p/2}} \int_{\mathbb{R}^d} \mathbb{E}\left( \left| (-L_y)^{-1/2} [p_T(X^R, y) - 1] \right|^2 \right)^{p/2} d\mu(y) + \int_{\mathbb{R}^d} \left| (-L)^{-1/2} \phi \right|^p d\mu. \quad (5.6)
\]

In this expression, the random variables \( (-L_y)^{-1/2} [p_T(X^R, y) - 1] \) will be studied with the help of the spectral representation \( (2.13) \), that is

\[
(-L_y)^{-1/2} [p_T(X^R, y) - 1] = \frac{1}{\sqrt{\pi}} \int_0^{\infty} \frac{1}{\sqrt{s}} [p_{s+T}(X^R, y) - 1] ds
\]

and similarly for \( (-L)^{-1/2} \phi \).

According to \( (5.4) \) and \( (5.6) \), the proof of the theorem will therefore be achieved once it may be established that

\[
\frac{1}{n^{p-1}} \int_{\mathbb{R}^d} \mathbb{E}\left( \left| \int_0^{\infty} \frac{1}{\sqrt{s}} \left[ p_{s+T}(X^R, y) - 1 \right] ds \right|^p \right) d\mu(y) \lesssim \frac{1}{n^{p/d}}, \quad (5.7)
\]

\[
\frac{1}{n^{p/2}} \int_{\mathbb{R}^d} \mathbb{E}\left( \left[ \int_0^{\infty} \frac{1}{\sqrt{s}} \left[ p_{s+T}(X^R, y) - 1 \right] ds \right]^2 \right)^{p/2} d\mu(y) \lesssim \frac{1}{n^{p/d}}, \quad (5.8)
\]

and

\[
\int_{\mathbb{R}^d} \left| \int_0^{\infty} \frac{1}{\sqrt{s}} P_s \phi ds \right|^p d\mu \lesssim \frac{1}{n^{p/d}}. \quad (5.9)
\]

The centering term \( (5.9) \) will be examined at the end of the proof. We concentrate on the first two terms, starting with the investigation of \( (5.8) \) which is the most delicate one.
Study of (5.8). Fix $y \in \mathbb{R}^d$ to begin with; by definition of the map $T$,

$$
\mathbb{E} \left( \left[ \int_0^\infty \frac{1}{\sqrt{s}} [p_{s+T}(X^R, y) - 1] \, ds \right]^2 \right)
= \frac{1}{\mu(B_R)} \sum_{k=1}^m \int_{D_k} \left[ \int_0^\infty \frac{1}{\sqrt{s}} [p_{s+t_k}(x, y) - 1] \, ds \right]^2 \, d\mu(x).
$$

Given $s_k \in (0, 1)$, $k = 1, \ldots, m$, to be specified, we decompose the integral in $s$ on $(0, s_k)$ and $(s_k, \infty)$ and study separately, by the triangle inequality, the resulting two pieces in (5.8), showing that

$$
\frac{1}{n^{p/2}} \int_{\mathbb{R}^d} \left( \sum_{k=1}^m \int_{D_k} \left[ \int_0^{s_k} \frac{1}{\sqrt{s}} [p_{s+t_k}(x, y) - 1] \, ds \right]^2 \, d\mu(x) \right)^{p/2} \, d\mu(y) \lesssim \frac{1}{n^{p/d}} \quad (5.10)
$$

and

$$
\frac{1}{n^{p/2}} \int_{\mathbb{R}^d} \left( \sum_{k=1}^m \int_{D_k} \left[ \int_{s_k}^\infty \frac{1}{\sqrt{s}} [p_{s+t_k}(x, y) - 1] \, ds \right]^2 \, d\mu(x) \right)^{p/2} \, d\mu(y) \lesssim \frac{1}{n^{p/d}}. \quad (5.11)
$$

Concerning (5.10), for each $k = 1, \ldots, m$ and $y \in \mathbb{R}^d$, by Fubini’s theorem,

$$
\int_{D_k} \left[ \int_0^{s_k} \frac{1}{\sqrt{s}} [p_{s+t_k}(x, y) - 1] \, ds \right]^2 \, d\mu(x)
= \int_{D_k} \int_0^{s_k} \int_0^{s_k} \frac{1}{\sqrt{ss'}} [p_{s+t_k}(x, y) - 1] \, ds \, ds' \, d\mu(x)
\lesssim \int_0^{s_k} \int_0^{s_k} \frac{1}{\sqrt{ss'}} \int_{D_k} p_{s+t_k}(x, y) \, ds \, ds' \, d\mu(x) \, d\mu(y) + \mu(D_k).
$$

Summing over $k$, it is clear that the contribution $\mu(D_k)$ will be irrelevant for the final bound and it is therefore ignored below. Now, a standard calculation on the explicit expression of the Mehler kernel $p_t(x, y)$ yields

$$
\int_{D_k} p_{s+t_k}(x, y) \, d\mu(x) = p_{s+s'+2t_k}(y, y) \mu(\tilde{D}_k)
$$

where \( \tilde{D}_k = -\frac{\beta}{\alpha} y + \alpha D_k \),

$$
\alpha^2 = 1 + \frac{a^2}{1 - a^2} + \frac{b^2}{1 - b^2}, \quad \beta = \frac{a}{1 - a^2} + \frac{b}{1 - b^2},
$$

with $a = e^{-s-t_k}$, $b = e^{-s'-t_k}$. For the further purposes, note that

$$
\frac{\alpha^2}{\beta} - 1 = \frac{(1 - a)(1 - b)}{a + b}. \quad (5.13)
$$

We examine separately the expression $p_{s+s'+2t_k}(y, y) \mu(\tilde{D}_k)$ in (5.12) according as $y \in \frac{a^2}{\beta} D_k$ or not via the bound

$$
p_{s+s'+2t_k}(y, y) \mu(\tilde{D}_k) \leq p_{s+s'+2t_k}(y, y) \left[ 1_{\frac{a^2}{\beta} D_k}(y) + 1_{(\frac{a^2}{\beta} D_k)^c}(y) \mu(\tilde{D}_k) \right]. \quad (5.14)
$$
As such, the study of (5.10) is divided into two parts, and the task is to show that

\[ \frac{1}{n^{p/2}} \int_{\mathbb{R}^d} \left[ \sum_{k=1}^{m} \int_0^{s_k} \int_0^{s_k} \frac{1}{\sqrt{s_s'}} p_{s+s'+2t_k}(y,y) \mathbb{1}_{\alpha^2 D_k}(y) dsds' \right]^{p/2} d\mu(y) \lesssim \frac{1}{n^{p/d}} \]  

and

\[ \frac{1}{n^{p/2}} \int_{\mathbb{R}^d} \left[ \sum_{k=1}^{m} \int_0^{s_k} \int_0^{s_k} \frac{1}{\sqrt{s_s'}} p_{s+s'+2t_k}(y,y) \mathbb{1}_{(\alpha^2 D_k)c}(y) \mu(\tilde{D}_k) dsds' \right]^{p/2} d\mu(y) \lesssim \frac{1}{n^{p/d}}. \]  

Start with (5.15) and fix now \( s \in \mathbb{R}^d \) so that \( 0 < s, s' \leq s_k \),

\[ \alpha^2 \beta D_k \subset E_k = \{ x \in \mathbb{R}^d ; r_{k-1} \leq |x| < (1+\frac{s}{k})r_k \}. \]

Hence, together with (2.3) and a simple integration in \( s, s' \), for every \( y \in \mathbb{R}^d \),

\[ \int_0^{s_k} \int_0^{s_k} \frac{1}{\sqrt{s_s'}} p_{s+s'+2t_k}(y,y) \mathbb{1}_{\alpha^2 D_k}(y) dsds' \lesssim \mathbb{1}_{E_k}(y) \frac{1}{t_k^{(d/2)-1}} e^{2|y|^2/2}. \]

It follows that the left-hand side of (5.15) is bounded from above by

\[ \frac{1}{n^{p/2}} \int_{\mathbb{R}^d} \left[ \sum_{k=1}^{m} \mathbb{1}_{E_k}(y) \frac{1}{t_k^{(d/2)-1}} \right]^{p/2} e^{p|y|^2/4} d\mu(y). \]

Observe that for every \( k = 1, \ldots, m, E_k \subset \bigcup_{\ell=k}^{k+s_0} D_\ell \) (with the obvious extension of \( D_k \) when \( k \geq m \)). As a consequence, the latter is bounded from above by

\[ \frac{1}{n^{p/2}} \sum_{k=1}^{m+s_0} \frac{1}{t_k^{(d/2)-1}} \int_{D_k} e^{p|y|^2/4} d\mu(y) \lesssim \frac{1}{n^{p/2}} \sum_{k=1}^{m+s_0} \frac{1}{t_k^{(d/2)-1}} k^{\frac{d}{2} - 1} e^{2(\frac{d}{2} - 1) \frac{|y|^2}{2}}. \]

Now \( t_k = n^{-2/d} e^{k/d}, k = 1, \ldots, m \), so that since \( p < d \) the latter is of the order \( \frac{1}{n^{p/2}} \), proving (5.15).

We turn next to (5.16). Fix \( k = 1, \ldots, m \). When \( y \notin \frac{\alpha^2}{\beta} D_k \), then (draw a picture), it is clear that

\[ \inf \{ |z| \in \mathbb{R}^d ; z \in \tilde{D}_k \} = \left| \frac{\beta}{\alpha} |y| - \alpha \tilde{r}_k \right| \]

where \( \tilde{r}_k = r_k \) or \( r_{k-1} \). Thus

\[ \mu(\tilde{D}_k) \lesssim \left( 1 + \left| \frac{\beta}{\alpha} |y| - \alpha \tilde{r}_k \right|^{d-2} \right) \exp \left( - \frac{1}{2} \left( \frac{\beta}{\alpha} |y| - \alpha \tilde{r}_k \right)^2 \right). \]

To get rid of the prefactors in front of the exponential, let \( \sigma \in (0, 1) \) (which will depend on \( p \) and \( d \) only) so that

\[ \mu(\tilde{D}_k) \lesssim \exp \left( - \frac{1 - \sigma}{2} \left( \frac{\beta}{\alpha} |y| - \alpha \tilde{r}_k \right)^2 \right). \]
Hence, together with (2.3),
\[
p_{s+s'+2t_k}(y, y)1_{(\frac{a^2}{D_k}D_k)^c}(y)\mu(\tilde{D}_k) \lesssim \frac{1}{(1 - a^2b^2)^{d/2}} \exp\left(-\frac{1}{2} - \frac{\sigma}{\alpha} |y - \alpha \tilde{r}_k|^2 + \frac{ab}{1 + ab} |y|^2\right).
\]

Now, since
\[
\frac{\beta^2}{\alpha^2} - \frac{2ab}{1 + ab} = \alpha^2 - 1,
\]
it is easily seen that
\[
-\frac{1}{2} - \frac{\sigma}{\alpha} |y|^2 - \alpha \tilde{r}_k |y| + \frac{ab}{1 + ab} |y|^2 \leq (1 - \sigma) A + \frac{\sigma}{2} |y|^2
\]
where
\[
A = -\frac{1}{2} + (\alpha^2 + 1) |y|^2 + \beta \tilde{r}_k |y| - \frac{1}{2} \alpha^2 \tilde{r}_k^2.
\]
We would like to choose numerical non-negative constants \( K \) and \( L \) such that
\[
A \leq A' = \frac{K}{2} |y|^2 + \frac{L}{2} \tilde{r}_k^2
\]
for all \( y \in \mathbb{R}^d \) and \( k = 1, \ldots, m \). This may be achieved provided that \( K + L \geq 1 \) and \( L \leq 1 \). Indeed, the quadratic form
\[
B = A' - A = \frac{1}{2} (\alpha^2 - 1 + K) |y|^2 - \beta \tilde{r}_k |y| + \frac{1}{2} (\alpha^2 + L) \tilde{r}_k^2
\]
is positive semi-definite if \( Q = \beta^2 - (\alpha^2 - 1 + K)(\alpha^2 + L) \leq 0 \). But when \( K + L \geq 1 \) and \( L \leq 1 \),
\[
Q = -\frac{1 - a^2b^2}{(1 - a^2)(1 - b^2)} \left( K + L - \frac{2ab}{1 + ab} \right) + L(1 - K)
\]
\[
\leq -\frac{1 - a^2b^2}{(1 - a^2)(1 - b^2)} \left( 1 - \frac{2ab}{1 + ab} \right) + L^2
\]
\[
\leq -\frac{(1 - ab)^2}{(1 - a^2)(1 - b^2)} + L^2 \leq -1 + L^2 \leq 0.
\]

As a consequence of this analysis, we have obtained that, provided \( K + L \geq 1 \) and \( L \leq 1 \),
\[
p_{s+s'+2t_k}(y, y)1_{(\frac{a^2}{D_k}D_k)^c}(y)\mu(\tilde{D}_k) \lesssim \frac{1}{(1 - a^2b^2)^{d/2}} e^{A''} \lesssim \frac{1}{(s + s' + 2t_k)^{d/2}} e^{A''}
\]
uniformly over \( s, s' \leq s_k \) and \( y \in \mathbb{R}^d \), where
\[
A'' = (1 - \sigma) A' + \frac{\sigma}{2} |y|^2 = \frac{(1 - \sigma) K + \sigma}{2} |y|^2 + \frac{(1 - \sigma) L}{2} \tilde{r}_k^2.
\]

It may now be integrated in \( 0 < s, s' \leq s_k \) for every \( k \) to get that the left-hand side of (5.16) is bounded from above by
\[
\frac{1}{n^{p/2}} \int_{\mathbb{R}^d} \left[ \sum_{k=1}^m \frac{1}{t_k^{(d-2)/2}} e^{A''} \right]^{p/2} d\mu(y).
\]

(5.17)
The first step is a Minkowski integral inequality to exchange the order of integration. To this we write here
\[ S \sum_{k=1}^{m} \frac{1}{t_k^{(d/2)-1}} e^{A''} d\mu(y) \]
\[ = \frac{1}{n^{p/2}} \int_{\mathbb{R}^d} \left[ \sum_{k=1}^{m} \frac{1}{t_k^{(d/2)-1}} e^{(1-\sigma)L_k^{2}} \right]^{p/2} e^{\|\cdot\|^2/4} |y|^2 d\mu(y) \]
\[ \lesssim \frac{1}{n^{p/2}} \left( 1 - \frac{p}{2} \right) \left( 1 - \sigma \right) \frac{d}{2} \frac{d}{2} \left( 1 - \sigma \right)^{d/2} \left[ \sum_{k=1}^{m} \frac{1}{t_k^{(d/2)-1}} e^{(1-\sigma)L_k^{2}} \right]^{p/2} \]
provided that \( \frac{p}{2} \left( 1 - \sigma \right) K + \sigma \) < 1. Since \( t_k = n^{-2/d} e^{k/d}, k = 1, \ldots, m, \) and \( r_k = \sqrt{k} \) or \( \sqrt{k-1}, \) it is necessary in addition that \( (1-\sigma)L < 1 - \frac{2}{d} \) in order that the preceding bound yields the correct rate \( \frac{1}{n^{p/2}}. \) Provided that \( \sigma \in (0, \frac{2}{d}) \), the two preceding conditions are indeed compatible with \( K + L \geq 1 \) and \( L \leq 1. \) As an interesting example, we can always set \( K = 0, L = 1 \) and \( \sigma \in (\frac{2}{d}, \frac{2}{d}), \) simplifying therefore the exposition. The preceding construction will however be needed later in the case \( p = d \) (Section 6). As a conclusion, (5.16) is established.

As announced, the two controls (5.15) and (5.16) yield together the expected bound (5.10).

We turn to the analysis of the second part (5.11) concerned with the values of \( s \geq s_k. \) The first step is a Minkowski integral inequality to exchange the order of integration. To this purpose, it is convenient to rewrite the left-hand side of (5.11) as
\[ \frac{1}{n^{p/2}} \int_{\mathbb{R}^d} \left( \mathbb{E} \left( \left( \int_{S} \frac{1}{\sqrt{s}} \left[ (p+sT(X^R, y) - 1) ds \right]^{2} \right) \right) \right)^{p/2} d\mu(y) \]
where \( S : B_R \to (0,1) \) is defined by \( S(x) = s_k \) if \( x \in D_k, k = 1, \ldots, m. \) As for \( T = T(X^R), \) we write here \( S = S(X^R) \) to ease the notation. As announced, by Minkowski’s inequality since \( p \geq 2, \) the latter is less than or equal to
\[ \frac{1}{n^{p/2}} \mathbb{E} \left( \left( \int_{\mathbb{R}^d} \left( \int_{S} \frac{1}{\sqrt{s}} \left[ (p+sT(X^R, y) - 1) ds \right]^{2} \right)^{p/2} d\mu(y) \right)^{p/2} \right). \]
Now, and conditionally on the randomness of \( X^R, \) also by the triangle inequality,
\[ \int_{\mathbb{R}^d} \left( \int_{S} \frac{1}{\sqrt{s}} \left[ (p+sT(X^R, y) - 1) ds \right]^{2} \right)^{p/2} d\mu(y) \leq \left( \int_{S} \frac{1}{\sqrt{s}} \left\| (p+sT(X^R, \cdot) - 1) \right\|_{p} ds \right)^{p} \]
where the norm \( \| \cdot \|_{p} \) is in \( L^p(d\mu(y)). \) By (2.10), for every \( s \geq S, \)
\[ \| (p+sT(X^R, \cdot) - 1 \|_{p} \lesssim e^{-s/2} \| pS(X^R, \cdot) - 1 \|_{p} \]
so that
\[ \left( \int_{S} \frac{1}{\sqrt{s}} \left\| (p+sT(X^R, \cdot) - 1 \right\|_{p} ds \right)^{p} \lesssim \| pS(X^R, \cdot) - 1 \|_{p} \lesssim \frac{1}{S^{(p-1)d/2}} e^{(p-1)|X^R|^2/2} \]
(5.19)
where (2.5) is used in the last step. Therefore, with the choice of \( s_k = \frac{1}{\sqrt{k}} \), (5.18) is bounded from above by

\[
\frac{1}{n^{p/2}} \mathbb{E} \left( \frac{1}{S(1-\frac{1}{p})} e^{(1-\frac{1}{p})|x|^2} \right)^{p/2} \lesssim \frac{1}{n^{p/2}} \left( \sum_{k=1}^m k^{(1-\frac{1}{p})d} \int_{D_k} e^{(1-\frac{1}{p})|x|^2} d\mu(x) \right)^{p/2} \lesssim \frac{1}{n^{p/2}} \left( \sum_{k=1}^m k^{(1-\frac{1}{p})d-1} e^{(\frac{1}{2}-\frac{1}{p})k} \right)^{p/2}.
\]

Recalling that \( m = R^2 = 2c \log n \) with \( c < 1 \) yields an expression of at most the order \( \frac{1}{n^{p/d}} \) so that (5.11) is established.

As a consequence of (5.10) and (5.11), the bound (5.8) is established.

**Study of (5.7).** We address here (5.7) following the steps developed for (5.8) but in a simplified way. Decomposing the integral in \( s \), simply here on \((0, 1)\) and \((1, \infty)\) as in the case \( p = 2 \), (5.7) will hold as soon as

\[
\frac{1}{n^{p-1}} \mathbb{E} \left( \int_{\mathbb{R}^d} \left| \int_0^1 \frac{1}{\sqrt{s}} \left[ p_{s+T}(X^R, y) - 1 \right] ds \right|^p d\mu(y) \right) \lesssim \frac{1}{n^{p/d}}
\]

and

\[
\frac{1}{n^{p-1}} \mathbb{E} \left( \int_{\mathbb{R}^d} \left| \int_1^\infty \frac{1}{\sqrt{s}} \left[ p_{s+T}(X^R, y) - 1 \right] ds \right|^p d\mu(y) \right) \lesssim \frac{1}{n^{p/d}}.
\]

The main simplification here with respect to (5.8) is that Fubini’s theorem applies between the expectation \( \mathbb{E} \) and integration in \( d\mu(y) \).

Starting with (5.20), recall that by definition of the map \( T \), for every \( y \in \mathbb{R}^d \),

\[
\mathbb{E} \left( \left| \int_0^1 \frac{1}{\sqrt{s}} \left[ p_{s+T}(X^R, y) - 1 \right] ds \right|^p \right) = \frac{1}{\mu(B_R)} \sum_{k=1}^m \int_{D_k} \left| \int_0^1 \frac{1}{\sqrt{s}} \left[ p_{s+t_k}(x, y) - 1 \right] ds \right|^p d\mu(x).
\]

Then, by the triangle inequality, for every \( k = 1, \ldots, m \),

\[
\int_{D_k} \left| \int_0^1 \frac{1}{\sqrt{s}} \left[ p_{s+t_k}(x, y) - 1 \right] ds \right|^p d\mu(x) \lesssim \int_{D_k} \left[ \int_0^1 \frac{1}{\sqrt{s}} p_{s+t_k}(x, y) ds \right]^p d\mu(x) + \mu(D_k)
\]

and it is clear again that we may ignore the contribution \( \mu(D_k) \) in what follows. By (2.4), for all \( x, y \in \mathbb{R}^d \),

\[
p_{s+t_k}(x, y) \leq \frac{1}{(1 - e^{-2(s+t_k)})^{d/2}} e^{|x|^2/2} \lesssim \frac{1}{(s + t_k)^{d/2}} e^{|x|^2/2}
\]

in the range \( s \leq 1 \). Hence, after integration in \( s \), for every \( x, y \in \mathbb{R}^d \),

\[
\int_0^1 \frac{1}{\sqrt{s}} p_{s+t_k}(x, y) ds \lesssim \frac{1}{t_k^{(d-1)/2}} e^{|x|^2/2}.
\]
Therefore, for every $k$, 

$$
\int_{D_k} \int_{\mathbb{R}^d} \left[ \int_0^1 \frac{1}{\sqrt{s}} \rho_{s+t_k}(x,y) ds \right]^p \, d\mu(y) \, d\mu(x)
$$

\[
\lesssim \frac{1}{t_k^{(d-1)(p-2)/2}} \int_{D_k} \int_{\mathbb{R}^d} \left[ \int_0^1 \frac{1}{\sqrt{s}} \rho_{s+t_k}(x,y) ds \right]^2 \, d\mu(y) \, e^{(p-2)|x|^2/2} \, d\mu(x).
\]

Now \( \int_{\mathbb{R}^d} \left[ \int_0^1 \frac{1}{\sqrt{s}} \rho_{s+t_k}(x,y) ds \right]^2 \, d\mu(y) = \int_0^1 \int_0^1 \frac{1}{\sqrt{s} \sqrt{s'}} \rho_{s+s'+2t_k}(x,x) ds ds' \) is, by (2.3), of the order of at most \( \frac{1}{t_k^{d/2}} \) e\( |x|^2/2 \). It follows that

\[
\frac{1}{n^{p-1}} \sum_{k=1}^m \int_{D_k} \int_{\mathbb{R}^d} \left[ \int_0^1 \frac{1}{\sqrt{s}} \rho_{s+t_k}(x,y) ds \right]^p \, d\mu(y) \, d\mu(x)
\]

\[
\lesssim \frac{1}{n^{p-1}} \sum_{k=1}^m \frac{k^{(d/2)-1}}{t_k^{(d-p-1)/2}} \int_{D_k} e^{(p-1)|x|^2/2} \, d\mu(x)
\]

\[
\lesssim \frac{1}{n^{p-1}} \sum_{k=1}^m \frac{k^{(d/2)-1}}{t_k^{(d-p-1)/2}} e^{(p-2)k/2}
\]

\[
\lesssim \frac{1}{n^{p/d}}
\]

by the choice of \( t_k = n^{-2/d} e^{k/d} \), \( k = 1, \ldots, m \), together with the fact that \( p < d \). Hence (5.20) holds true.

Concerning (5.21), the arguments developed for (5.18) may essentially be repeated. In particular, making use of (5.19), the left-hand side of (5.21) may be seen to be bounded from above by

\[
\frac{1}{n^{p-1}} \mathbb{E}(e^{(p-1)|X^R|^2/2}) \lesssim \frac{1}{n^{p-1}} \sum_{k=1}^m \int_{D_k} e^{(p-1)|x|^2/2} \, d\mu(x) \lesssim \frac{1}{n^{p-1}} \sum_{k=1}^m \frac{k^{(d/2)-1}}{t_k^{(d-p-1)/2}} e^{(p-2)k/2}.
\]

Again since \( m = R^2 = 2c \log n \) with \( c < 1 \), this contribution is at most the order \( \frac{1}{n^{p/d}} \) proving (5.21).

As announced, as a consequence of (5.20) and (5.21), the bound (5.7) is established.

**Study of (5.9).** In the final part of the proof, we thus take care of the centering term

\[
\int_{\mathbb{R}^d} \int_0^{\infty} \frac{1}{\sqrt{s}} P_s \phi \, ds \, d\mu
\]

of (5.9). Recall that by definition

\[
\phi = \mathbb{E}(p_T(X^R, \cdot)) - 1 = \frac{1}{\mu(B_R)} \sum_{k=1}^m [P_{t_k}(1_{D_k}) - \mu(D_k)],
\]

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and write then
\[
P_s \phi = \frac{1}{\mu(B_R)} \sum_{k=1}^{m} \left[ P_{s+t_k}(1_{D_k}) - P_s(1_{D_k}) \right] + \frac{1}{\mu(B_R)} \left[ P_s(1_{B_R}) - \mu(B_R) \right] \\
= \frac{1}{\mu(B_R)} (\phi_{s,1} + \phi_{s,2}).
\]

By means of Hölder’s inequality (in ds), for any \(0 < \kappa < \frac{p}{2} - 1\),
\[
\left| \int_0^\infty \frac{1}{\sqrt{s}} P_s \phi \, ds \right|^p \lesssim \int_0^\infty s^{\frac{p}{2} - 1 - \kappa} e^{\kappa s} |P_s \phi|^p \, ds \\
\lesssim \int_0^\infty s^{\frac{p}{2} - 1 - \kappa} e^{\kappa s} |\phi_{s,1}|^p \, ds + \int_0^\infty s^{\frac{p}{2} - 1 - \kappa} e^{\kappa s} |\phi_{s,2}|^p \, ds. \tag{5.22}
\]

We examine successively the contributions of \(\phi_{s,1}\) and \(\phi_{s,2}\) in the preceding. By the triangle inequality,
\[
\int_{\mathbb{R}^d} |\phi_{s,1}|^p \, d\mu = \int_{\mathbb{R}^d} \left| \sum_{k=1}^{m} \left[ P_{s+t_k}(1_{D_k}) - P_s(1_{D_k}) \right] \right|^p \, d\mu \\
\leq \left( \sum_{k=1}^{m} \| P_{s+t_k}(1_{D_k}) - P_s(1_{D_k}) \|_p \right)^p.
\]
Using (2.14), for any \(s > 0\) and since \(t_k < 1\),
\[
\| P_{s+t_k}(1_{D_k}) - P_s(1_{D_k}) \|_p \lesssim \frac{e^{-s/p}}{(e^{2s} - 1)^{(p-1)/2p}} \sqrt{t_k} \mu(\partial D_k)^{1/p}.
\]
For \(\kappa > 0\) small enough, it follows that
\[
\int_0^\infty s^{\frac{p}{2} - 1 - \kappa} e^{\kappa s} \int_{\mathbb{R}^d} |\phi_{s,1}|^p \, d\mu \, ds \lesssim \left( \sum_{k=1}^{m} \sqrt{t_k} \mu(\partial D_k)^{1/p} \right)^p \lesssim \frac{1}{n^{p/d}} \tag{5.23}
\]
since \(t_k = n^{-2/d} e^{k/d}\) and \(\mu(\partial D_k) \lesssim k^{(d-1)/2} e^{-k/2}\), \(k = 1, \ldots, m\).

On the other hand, it is easily seen as in the case \(p = 2\) that, again for \(\kappa > 0\) small enough,
\[
\int_0^\infty s^{\frac{p}{2} - 1 - \kappa} e^{\kappa s} \int_{\mathbb{R}^d} |\phi_{s,2}|^p \, d\mu \, ds \lesssim \int_0^\infty s^{\frac{p}{2} - 1 - \kappa} e^{\kappa s} \int_{\mathbb{R}^d} |\phi_{s,2}|^2 \, d\mu \, ds \\
\lesssim \int_0^\infty s^{\frac{p}{2} - 1 - \kappa} e^{\kappa s - 2s} \mu(B_R^c) \, ds \lesssim \frac{1}{n^{p/d}} \tag{5.24}
\]
by the choice of \(R = \sqrt{2c \log n}\).

Together with (5.22), it follows from (5.23) and (5.24) that the bound (5.9) is established. Altogether, the proof of Theorem \(\text{[1]}\) is complete.
6 The case \( p = d \)

This section addresses the proof of Theorem 2 for, thus, \( p = d \geq 2 \). The proof for \( p = d = 2 \) was actually provided in \([8]\), as a simpler version of what is developed here. (In the next section, we present the pde-transportation argument for the lower bound in this case.) The proof here for \( p = d \) carefully adjusts several parameters in the various steps of the one developed in Section 5.

The first step is truncation on a ball \( B_R \) this time with \( R = \sqrt{2 \log n} \) for which it holds similarly that

\[
\mathbb{E}(W^d_d(\mu_n, \mu_R^R)) \lesssim \frac{(\log n)^{\frac{d}{2}}}{n}. \tag{6.1}
\]

The ball \( B_R \) is decomposed again as the union of \( m \) annuli \( D_k = \{ x \in \mathbb{R}^d; r_{k-1} \leq |x| < r_k \} \), \( k = 1, \ldots, m \), where \( 0 = r_0 < r_1 < \cdots < r_m = R \). Consider as well the map \( T : B_R \to (0,1) \) defined by \( T(x) = t_k \) if \( x \in D_k \). We will use the same choices \( r_k = \sqrt{k} \), but modify the values of \( t_k \) as

\[
t_k = \frac{e^{k/d}}{n^{2/d} \sqrt{k}}, \quad k = 1, \ldots, m.
\]

Setting\[ f(y) = f_n^{R,T}(y) = \frac{1}{n} \sum_{i=1}^{n} p_{T(X^R_i)}(X^R_i, y), \quad y \in \mathbb{R}^d, \]
and \( d\mu_{n,R,T} = f_n^{R,T} d\mu \), the preceding choice of the \( t_k \)'s now yields in (5.3) that

\[
\mathbb{E}(W^d_d(\mu_n^R, \mu_{R,T}^R)) \lesssim \frac{(\log n)^{\max(\frac{d}{2}-1, \frac{d}{4})}}{n}. \tag{6.2}
\]

Next, to estimate \( \mathbb{E}((W^d_d(\mu_{n,R,T}^R, \mu^R)) \) as in Section 5 we need to control the terms

\[
\frac{1}{n^{d-1}} \int_{\mathbb{R}^d} \mathbb{E} \left( \left| \int_0^\infty \frac{1}{\sqrt{s}} \left[ p_{s+T}(X^R, y) - 1 \right] ds \right|^d \right) d\mu(y), \tag{6.3}
\]

\[
\frac{1}{n^{d/2}} \int_{\mathbb{R}^d} \mathbb{E} \left( \left[ \int_0^\infty \frac{1}{\sqrt{s}} \left[ p_{s+T}(X^R, y) - 1 \right] ds \right]^2 \right)^{d/2} d\mu(y) \tag{6.4}
\]

and

\[
\int_{\mathbb{R}^d} \left| \int_0^\infty \frac{1}{\sqrt{s}} P_s \phi ds \right|^d d\mu
\]

where \( \phi(y) = \mathbb{E} \left( p_T(X^R, y) \right) - 1, \quad y \in \mathbb{R}^d. \)

**Study of (6.4).** Given \( s_k = \frac{1}{\sqrt{k}}, \quad k = 1, \ldots, m \), it is sufficient to estimate separately

\[
\frac{1}{n^{d/2}} \int_{\mathbb{R}^d} \left( \sum_{k=1}^{m} \int_{D_k} \left[ \int_0^{s_k} \frac{1}{\sqrt{s}} \left[ p_{s+t_k}(x, y) - 1 \right] ds \right]^2 d\mu(x) \right)^{d/2} d\mu(y) \tag{6.6}
\]
and

$$\frac{1}{n^{d/2}} \int_{\mathbb{R}^d} \left( \sum_{k=1}^{m} \int_{D_k} \left[ \int_{s_k}^{\infty} \frac{1}{\sqrt{s}} \left[ p_{s+\epsilon} (x, y) - 1 \right] ds \right] \right)^{d/2} d\mu(x) \ d\mu(y). \quad (6.7)$$

Concerning (6.6), with the notation of the previous section, we need to investigate

$$\frac{1}{n^{d/2}} \int_{\mathbb{R}^d} \left[ \sum_{k=1}^{m} \int_{0}^{s_k} \int_{0}^{s_k} \frac{1}{\sqrt{s} \sqrt{t}} p_{s+t} (x, y) \mathbf{1}_{s_t} D_k (y) ds dt \right]^{d/2} d\mu(y) \quad (6.8)$$

and

$$\frac{1}{n^{d/2}} \int_{\mathbb{R}^d} \left[ \sum_{k=1}^{m} \int_{0}^{s_k} \int_{0}^{s_k} \frac{1}{\sqrt{s} \sqrt{t}} p_{s+t} (x, y) \mathbf{1}_{(s^2 + \sigma^2) D_k} (y) \mu (\widetilde{D}_k) ds dt \right]^{d/2} d\mu(y). \quad (6.9)$$

Arguing as for (5.15), (6.8) is upper bounded by

$$\frac{1}{n^{d/2}} \sum_{k=1}^{m+80} \frac{1}{(k^{d/2} - 1)^2} e^{\frac{(d-2)k}{2} \frac{n}{\sqrt{n}}} \lesssim \frac{(\log n)^{d^2/8 + \delta d}}{n}. \quad (6.10)$$

Turning to (6.9), fix \( k = 1, \ldots, m \). The proof proceeds as for (5.16) in Section 5 but now with a choice of \( K = K_k \) and \( L = L_k \) this time depending on \( k \), to reach that (6.9) is upper bounded by

$$\frac{1}{n} \left[ \sum_{k=1}^{m} k^{d/2 - \frac{3}{2}} e^{\left[ \frac{(1-\sigma) L_k - 1 + \frac{3}{2} \sqrt{k} \sigma \right] \frac{n}{\sqrt{n}}} \frac{1 - \frac{d}{2} (1 - \sigma) K_k + \sigma)}{1 - \frac{d}{2} (1 - \sigma) K_k + \sigma} \right]^{d/2} \lesssim \frac{1}{n} \left( \sum_{k=1}^{m} k^{d/2 - \frac{3}{2}} \right)^{d/2} \lesssim \frac{(\log n)^{d^2/8 + \delta d}}{n}. \quad (6.10)$$

As a consequence of the previous analysis, (6.9) is thus controlled by \( \frac{1}{n} \left( \log n \right)^{d^2/8 + \delta d} \). Concerning (6.7), it is handled as in Section 5 for the study of (5.11) and upper bounded by

$$\frac{1}{n^{d/2}} \left( \sum_{k=1}^{m} k^{d/2 - \frac{3}{2}} e^{\left[ \frac{(1-\sigma) L_k - 1 + \frac{3}{2} \sqrt{k} \sigma \right] \frac{n}{\sqrt{n}}} \frac{1 - \frac{d}{2} (1 - \sigma) K_k + \sigma)}{1 - \frac{d}{2} (1 - \sigma) K_k + \sigma} \right)^{d/2} \lesssim \frac{1}{n} \left( \log n \right)^{d^2/8 + \delta d/4}. \quad (6.10)$$

These two bounds lead to the dichotomy \( d = 2, 3 \) and \( d \geq 4 \) in the statement of Theorem 2.

**Study of (6.3)**. This term is simpler than (6.4). Following the analysis of (5.7) in Section 5, the leading term is

$$\frac{1}{n^{d-1}} \sum_{k=1}^{m} k^{(d/2) - 1} t_k^{(d-2d)/2} e^{(d-2)k/2} \lesssim \frac{(\log n)^{d^2}}{n}. \quad (6.10)$$
by the choice of \( t_k = \frac{e^{k/d}}{n^{2/d} \sqrt{k}} \), \( k = 1, \ldots, m \).

**Study of (6.5).** Using the method in the previous section, it is bounded from above by
\[
\left( \sum_{k=1}^{m} \sqrt{t_k} \mu(\partial D_k)^{1/d} \right)^d \lesssim \frac{(\log n)^{\frac{5d-4}{4}}}{n}.
\]
since \( t_k = \frac{e^{k/d}}{n^{2/d} \sqrt{k}} \) and \( \mu(\partial D_k) \lesssim k^{(d-1)/2} e^{-k/2} \), \( k = 1, \ldots, m \).

To conclude, it may be checked that all the logarithmic exponents are less than or equal to
\[ \kappa = \max \left( \frac{d^2 + 6d}{8}, \frac{d^2}{2} - \frac{3d}{4} \right), \]
thereby completing the proof of Theorem 2.

### 7 Lower bound \( p = d = 2 \)

The purpose of this paragraph is to provide an alternate proof based on the pde-transportation method of [2] of the lower bound
\[
\mathbb{E}(W_2^2(\mu_n, \mu)) \gtrsim \frac{(\log n)^2}{n} \tag{7.11}
\]
of (1.5) in dimension 2, which has been established in [15] by different means. This proof already appeared in [9], and is included here for completeness and convenience.

The first step is a two-sided bound on the Kantorovich metric \( W_2 \) in terms of Sobolev norms. It is developed in [9] in weighted Riemannian manifolds under the curvature condition \( CD(K, \infty) \) for some \( K \in \mathbb{R} \), but for simplicity is restricted here to the Gaussian model (for which \( K = 1 \)). Let thus \( \mu \) be the standard Gaussian measure on the Borel sets of \( \mathbb{R}^d \), and \( L \) be the Ornstein-Uhlenbeck operator as presented in Section 2. Proposition 5 is taken from [2]. Proposition 4 (a slight extension of Proposition 3) is not used below but included for comparison.

**Proposition 4.** Let \( dv = f d\mu \) and \( f = 1 + g \), and let \( 0 < c \leq 1 \). If \( g \geq -c \), then
\[
W_2^2(\nu, \mu) \leq \frac{4}{c^2} \left[ 1 - \sqrt{1 - c} \right]^2 \int_{\mathbb{R}^d} g(-L)^{-1} g d\mu \tag{7.12}
\]
(where \( g \) is assumed to belong to the suitable domain so that the left-hand side makes sense).

**Proposition 5.** Let \( dv = f d\mu \) and \( f = 1 + g \). Then, whenever \( g \) and \( h : \mathbb{R}^d \to \mathbb{R} \) belong to the suitable domain and \( h \) is such that \( \int_{\mathbb{R}^d} h d\mu = 0 \) and \( h \leq c \) uniformly for some \( c > 0 \),
\[
W_2^2(\nu, \mu) \geq 2 \int_{\mathbb{R}^d} g(-L)^{-1} h d\mu - \frac{e^{c-1}}{c} \int_{\mathbb{R}^d} h(-L)^{-1} h d\mu. \tag{7.13}
\]
In particular, if \( g \leq c \),
\[
W_2^2(\nu, \mu) \geq \left( 2 - \frac{e^{c-1}}{c} \right) \int_{\mathbb{R}^d} g(-L)^{-1} g d\mu. \tag{7.14}
\]
Recall that by integration by parts
\[
\int_{\mathbb{R}^d} g(-L)^{-1}g \, d\mu = \int_{\mathbb{R}^d} |\nabla((L)^{-1}g)|^2 \, d\mu
\]
which is the Sobolev norm of Proposition 3. Note also that as \(c \to 0\),
\[
\frac{4}{c^2} \left[1 - \sqrt{1 - c}\right]^2 \sim 1 + \frac{c}{2} \quad \text{and} \quad 2 - \frac{e^c - 1}{c} \sim 1 - \frac{c}{2}
\]
so that the bounds (7.12) and (7.14) are sharp in this regime.

**Proof of Proposition 4.** It is shown in [8] that for every (smooth) increasing \(\theta : [0, 1] \to [0, 1]\) with \(\theta(0) = 0\), \(\theta(1) = 1\),
\[
W_2^2(\nu, \mu) \leq \int_{\mathbb{R}^d} \left|\nabla((L)^{-1}g)\right|^2 \int_0^1 \frac{\theta'(s)^2}{1 + \theta(s)g} \, ds \, d\mu.
\]
Using that \(g \geq -c\),
\[
W_2^2(\nu, \mu) \leq \int_0^1 \frac{\theta'(s)^2}{1 - \theta(s)c} \, ds \int_{\mathbb{R}^d} \left|\nabla((L)^{-1}g)\right|^2 \, d\mu.
\]
The claim (7.12) follows from the (optimal) choice
\[
\theta(s) = \frac{1 - \sqrt{1 - c}}{c} \left(2s - \left[1 - \sqrt{1 - c}\right]s^2\right), \quad s \in [0, 1].
\]
When \(c = 1\), the conclusion amounts to Proposition 3.

**Proof of Proposition 5.** As announced, we follow [2]. By the Kantorovich dual description of the Kantorovich metric \(W_2\) (cf. [16]), for any bounded continuous \(\varphi : \mathbb{R}^d \to \mathbb{R}\),
\[
\frac{1}{2} W_2^2(\nu, \mu) \geq \int_{\mathbb{R}^d} \varphi \, f \, d\mu - \int_{\mathbb{R}^d} \widehat{Q}_1 \varphi \, d\mu
\]
\[= \int_{\mathbb{R}^d} \varphi \, g \, d\mu - \left(\int_{\mathbb{R}^d} \widehat{Q}_1 \varphi \, d\mu - \int_{\mathbb{R}^d} \varphi \, d\mu\right)
\]
where \(\widehat{Q}_1\) is the supremum convolution
\[
\widehat{Q}_1 \varphi(x) = \sup_{y \in \mathbb{R}^d} \left[\varphi(y) - \frac{1}{2} |x - y|^2\right].
\]
Choose then \(\varphi = (-L)^{-1}h\). Now
\[
\int_{\mathbb{R}^d} \widehat{Q}_1 \varphi \, d\mu - \int_{\mathbb{R}^d} \varphi \, d\mu = \frac{1}{2} \int_0^1 \int_{\mathbb{R}^d} |\nabla \widehat{Q}_s \varphi|^2 \, d\mu \, ds.
\]
It is shown in [2] that since \(-L\varphi = h \leq c\) uniformly, under a \(CD(0, \infty)\) curvature condition,
\[
\int_{\mathbb{R}^d} |\nabla \widehat{Q}_s \varphi|^2 \, d\mu \leq e^{cs} \int_{\mathbb{R}^d} |\nabla \varphi|^2 \, d\mu, \quad 0 \leq s \leq 1.
\]
Therefore
\[ \int_{\mathbb{R}^d} \hat{Q}_1 \varphi \, d\mu - \int_{\mathbb{R}^d} \varphi \, d\mu \leq \frac{e^c - 1}{c} \int_{\mathbb{R}^d} |\nabla \varphi|^2 \, d\mu. \]

Since
\[ \int_{\mathbb{R}^d} |\nabla \varphi|^2 \, d\mu = \int_{\mathbb{R}^d} \varphi (-L \varphi) \, d\mu = \int_{\mathbb{R}^2} h (-L)^{-1} h \, d\mu, \]
the assertion (7.13) follows. \qed

On the basis of Proposition 5 we address the proof of (7.11). The first part of the discussion develops in \( \mathbb{R}^d, \, d \geq 1 \).

The first step is the Kantorovitch contraction property under a \( CD(0, \infty) \) curvature condition (cf. [10, 14]), which holds in Gaussian space for the Mehler kernel \( p_t(x, y) \),

\[ W_2^2(\mu_n, \mu) \geq W_2^2(\mu'_n, \mu) \]

(7.15)

where we recall that \( d\mu'_n = f \, d\mu, \, f(y) = 1 + g(y), \, g = g(y) = \frac{1}{n} \sum_{i=1}^n [p_t(X_i, y) - 1], \, t > 0. \)

Next we use the truncation argument on a ball \( B_R \) with radius \( R > 0 \) to be specified later on, and recall the random variables \( X_i^R, \, i = 1, \ldots, n, \) with common distribution \( d\mu^R = \frac{1}{\nu(B_R)} 1_{B_R} \, d\mu. \)

Let
\[ \tilde{g} = \tilde{g}(y) = \frac{1}{n} \sum_{i=1}^n [p_t(X_i^R, y) - \mathbb{E}(p_t(X_i^R, y))], \]

and, for \( c > 0, \)
\[ \tilde{g}_c = (\tilde{g} \wedge c) \vee (-c) - \int_{\mathbb{R}^d} [(\tilde{g} \wedge c) \vee (-c)] \, d\mu \]

(so that \( |\tilde{g}_c| \leq 2c \) and \( \int_{\mathbb{R}^d} \tilde{g}_c \, d\mu = 0). \)

In (7.13) of Proposition 5 choose \( h = \tilde{g}_c. \) It holds true that
\[ \int_{\mathbb{R}^d} \tilde{g}_c (-L)^{-1} \tilde{g}_c \, d\mu = \int_{\mathbb{R}^d} \tilde{g} (-L)^{-1} \tilde{g} \, d\mu + \int_{\mathbb{R}^d} (\tilde{g} - \tilde{g}_c) (-L)^{-1} (\tilde{g} - \tilde{g}_c) \, d\mu \]
\[ - 2 \int_{\mathbb{R}^d} (\tilde{g} - \tilde{g}_c) (-L)^{-1} \tilde{g} \, d\mu. \]

Therefore, after some algebra, and with \( c \leq \frac{1}{2} \) for example,
\[ W_2^2(\mu'_n, \mu) \geq 2 \int_{\mathbb{R}^d} \tilde{g} (-L)^{-1} g \, d\mu - \frac{e^{2c} - 1}{2c} \int_{\mathbb{R}^d} \tilde{g} (-L)^{-1} \tilde{g} \, d\mu \]
\[ - 2 \int_{\mathbb{R}^d} (\tilde{g} - \tilde{g}_c) (-L)^{-1} g \, d\mu \]
\[ - 2 \int_{\mathbb{R}^d} (\tilde{g} - \tilde{g}_c) (-L)^{-1} (\tilde{g} - \tilde{g}_c) \, d\mu - 4 \int_{\mathbb{R}^d} (\tilde{g} - \tilde{g}_c) (-L)^{-1} \tilde{g} \, d\mu \]

(7.16)

The three last terms on the right-hand side of (7.16) are error terms which may are handled by the exponential decay (2.13) in \( L^2(\mu) \). Indeed, since \( (-L)^{-1} = \int_0^\infty P_s \, ds, \)
\[ \int_{\mathbb{R}^d} (\tilde{g} - \tilde{g}_c) (-L)^{-1} (\tilde{g} - \tilde{g}_c) \, d\mu = 2 \int_0^\infty \| P_s (\tilde{g} - \tilde{g}_c) \|_2^2 \, ds \leq \| \tilde{g} - \tilde{g}_c \|_2^2. \]

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In the same way,
\[ \int_{\mathbb{R}^d} (\tilde{g} - \tilde{g}_c)(-L)^{-1}g \, d\mu \leq \|g\|_2 \|\tilde{g} - \tilde{g}_c\|_2 \]
and
\[ \left| \int_{\mathbb{R}^d} (\tilde{g} - \tilde{g}_c)(-L)^{-1}g \, d\mu \right| \leq \|\tilde{g}\|_2 \|\tilde{g} - \tilde{g}_c\|_2. \]
Putting things together, and since
\[ |\tilde{g} - \tilde{g}_c| \leq |\tilde{g}| \mathbf{1}_{\{|\tilde{g}| \geq c\}} + \int_{\mathbb{R}^d} |\tilde{g}| \mathbf{1}_{\{|\tilde{g}| \geq c\}} \, d\mu, \]
it is deduced from (7.15) and (7.16) that for every \( 0 < c \leq \frac{1}{2} \),
\[ W^2_2(\mu^n, \mu) \geq 2 \int_{\mathbb{R}^d} \tilde{g}(-L)^{-1}g \, d\mu - \frac{e^{2c}}{2c} \int_{\mathbb{R}^d} \tilde{g}(-L)^{-1}g \, d\mu \]
\[ - 8 \int_{\{|\tilde{g}| \geq c\}} |\tilde{g}|^2 d\mu - 8 (\|g\|_2 + \|\tilde{g}\|_2) \left( \int_{\{|\tilde{g}| \geq c\}} |\tilde{g}|^2 d\mu \right)^{1/2}. \] (7.17)

Next, integrate over the samples \( X_1, \ldots, X_n \) and \( X_1^R, \ldots, X_n^R \) the first two terms on the right-hand side of (7.17). Recalling the definitions of \( g \) and \( \tilde{g} \), by independence and identical distribution,
\[ \mathbb{E} \left( \int_{\mathbb{R}^d} \tilde{g}(-L)^{-1}g \, d\mu \right) = \frac{1}{n} \int_t^\infty \int_{\mathbb{R}^d} \mathbb{E} \left[ \left[ p_t(X_1^R, y) - \mathbb{E}(p_t(X_1^R, y)) \right] p_s(X_1, y) \right] d\mu(y) ds. \]
By definition of \( X_1^R \),
\[ \mathbb{E} \left[ \left[ p_t(X_1^R, y) - \mathbb{E}(p_t(X_1^R, y)) \right] p_s(X_1, y) \right] = \mathbb{E} \left( \mathbf{1}_{\{X_1 \in B_R\}} [p_t(X_1, y) - \mathbb{E}(p_t(X_1^R, y))] p_s(X_1, y) \right) \]
\[ + \mathbb{E} \left( \mathbf{1}_{\{X_1 \notin B_R\}} [p_t(Z_1, y) - \mathbb{E}(p_t(X_1^R, y))] p_s(X_1, y) \right) \]
\[ = \mathbb{E} \left( \mathbf{1}_{\{X_1 \in B_R\}} [p_t(X_1, y) - \mathbb{E}(p_t(X_1^R, y))] p_s(X_1, y) \right) \]
since \( Z_1 \) is independent of \( X_1 \) and with the same law as \( X_1^R \). Hence, after integration in \( d\mu(y) \) and the semigroup property,
\[ \mathbb{E} \left( \int_{\mathbb{R}^d} \tilde{g}(-L)^{-1}g \, d\mu \right) \]
\[ = \frac{\mu(B_R)}{n} \int_t^\infty \left[ \int_{\mathbb{R}^d} p_{t+s}(x, x) d\mu_R(x) - \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} p_{t+s}(x, x') d\mu_R(x) d\mu_R(x') \right] ds \]
\[ = \frac{\mu(B_R)}{n} \int_0^\infty \left[ \int_{\mathbb{R}^d} p_s(x, x) d\mu_R(x) - \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} p_s(x, x') d\mu_R(x) d\mu_R(x') \right] ds \]
\[ = 28 \]
In the same way,

\[
\mathbb{E}\left( \int_{\mathbb{R}^d} \tilde{g}(-L)^{-1}g \, d\mu \right) \\
= \frac{1}{n} \int_t^\infty \int_{\mathbb{R}^d} \mathbb{E}\left( [p_t(X_1^R, y) - \mathbb{E}(p_t(X_1^R, y))] p_s(X_1^R, y) \right) \, d\mu(y) \, ds \\
= \frac{1}{n} \int_t^\infty \left[ \int_{\mathbb{R}^d} p_s(x, x) \, d\mu^R(x) - \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} p_s(x, x') \, d\mu^R(x) \, d\mu^R(x') \right] \, ds.
\]

As a consequence, if \( c > 0 \) is small enough and \( \mu(B_R) \) close to 1,

\[
\mathbb{E}\left( 2 \int_{\mathbb{R}^d} \tilde{g}(-L)^{-1}g \, d\mu - \frac{e^{2c} - 1}{2c} \int_{\mathbb{R}^d} \tilde{g}(-L)^{-1}g \, d\mu \right) \\
\geq \frac{1}{2n} \left( \int_{2t}^\infty \left[ \int_{\mathbb{R}^d} p_s(x, x) \, d\mu^R(x) - \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} p_s(x, x') \, d\mu^R(x) \, d\mu^R(x') \right] \, ds \right).
\]

Also, by the spectral gap inequality \([2,13]\),

\[
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} p_s(x, x') \, d\mu^R(x) \, d\mu^R(x') = \frac{1}{\mu(B_R)^2} \int_{\mathbb{R}^d} 1_{B_R} P_s 1_{B_R} \, d\mu \\
= 1 + \frac{1}{\mu(B_R)^2} \int_{\mathbb{R}^d} 1_{B_R} P_s (1_{B_R} - \mu(B_R)) \, d\mu \\
\leq 1 + \frac{1 - \mu(B_R)}{\mu(B_R)} \left( e^{-s} \right) \\
\leq 1 + 2 e^{-s}
\]

provided that \( \mu(B_R) \geq \frac{1}{2} \). As a conclusion at this stage,

\[
\mathbb{E}(W_2^2(\mu^n, \mu)) \geq \frac{1}{2n} \int_{2t}^\infty \int_{\mathbb{R}^d} [p_s(x, x) - 1] \, d\mu^R(x) \, ds - \frac{1}{n} \\
- 8 \mathbb{E} \left( \int_{\{|\tilde{g}| \geq c\}} |\tilde{g}|^2 \, d\mu \right) - 8 \mathbb{E} \left( \left( \|g\|_2 + \|\tilde{g}\|_2 \right) \left( \int_{\{|\tilde{g}| \geq c\}} |\tilde{g}|^2 \, d\mu \right)^{1/2} \right). \tag{7.18}
\]

The final part of the proof will be to take care of the correction terms on the right-hand side of the preceding \(7.18\). To this task, we develop some (crude) bounds on the Mehler kernel \( p_t(x, y) \) of Section 2. Consider for each \( y \in \mathbb{R}^d, t > 0 \) and \( q \geq 1 \),

\[
\int_{B_R} p_t(x, y)^q \, d\mu(x).
\]

After translation and a change of variable,

\[
\int_{B_R} p_t(x, y)^q \, d\mu(x) = \frac{1}{(1 - a^2)^{d/2}} e^{\tau(q-1)a |y|^2} \int_{B(-\tau y, R)} e^{-\frac{qa}{(1+q-1)a^2}} \frac{|x|^2}{(2\pi)^{d/2}} \, dx
\]

where \( a = e^{-t} \) and \( \tau = \frac{qa}{1+(q-1)a^2} \).

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Note that \( \tau(q - 1)a \leq q \), and that \( \tau \geq \frac{1}{2} \) at least provided that \( a \) is close to one which we may assume. Then, if \( |y| \geq 4R \) and \( x \in B(-\tau y, R) \), we have \( |x| \geq \frac{|y|}{4} \). Hence, whenever \( |y| \geq 4R \),

\[
\int_{B_R} p_t(x, y)^q d\mu(x) \leq \frac{1}{(1 - a^2)(q-1)d/2} e^{-\left(\frac{1}{32(1-a^2)} - \frac{q}{2}\right)|y|^2}.
\]

Otherwise, that is when \( |y| \leq 4R \),

\[
\int_{B_R} p_t(x, y)^q d\mu(x) \leq \frac{1}{(1 - a^2)(q-1)d/2} e^{\frac{q}{2}|y|^2}.
\]

Recall

\[
\tilde{g} = \tilde{g}(y) = \frac{1}{n} \sum_{i=1}^{n} [p_t(X_i^R, y) - \mathbb{E}(p_t(X_i^R, y))].
\]

By Rosenthal’s inequality \([5.4]\), for any \( q \geq 2 \) there exists \( C_q > 0 \) only depending on \( q \) such that

\[
\mathbb{E}(\tilde{g}(y)^q) \leq C_q \left( \frac{1}{n^{q-1}} \mathbb{E}(p_t(X_1^R, y)^q) + \frac{1}{n^{q/2}} \left[ \mathbb{E}(p_t(X_1^R, y)^2) \right]^{q/2} \right)
\]

\[
\leq 2^q C_q \left( \frac{1}{n^{q-1}} \int_{B_R} p_t(x, y)^q d\mu(x) + \frac{1}{n^{q/2}} \left[ \int_{B_R} p_t(x, y)^2 d\mu(x) \right]^{q/2} \right)
\]

where it is assumed that \( \mu(B_R) \geq \frac{1}{2} \).

In the following \( q \geq 2 \) is fixed. Then \( t > 0 \) may be chosen small enough (in terms of \( q \) but independently of \( n \)) such that \( \tau \geq \frac{1}{2} \) and \( \frac{1}{32(1-a^2)} - \frac{q}{2} \geq 0 \) (for example). By the previous step,

\[
\int_{\mathbb{R}^d} \left( \int_{B_R} p_t(x, y)^q d\mu(x) \right) d\mu(y) \leq \frac{1}{(1 - a^2)(q-1)d/2} (1 + e^{8qR^2}).
\]

In the same way,

\[
\int_{\mathbb{R}^d} \left( \int_{B_R} p_t(x, y)^2 d\mu(x) \right)^{q/2} d\mu(y) \leq \frac{1}{(1 - a^2)d/4} (1 + e^{4qR^2}).
\]

For simplicity (in order not to carry the two preceding expressions with \( q - 1 \) and \( \frac{q}{2} \)), assume in the following that \( (1 - a^2)d/2n \geq 1 \). Therefore, using that \( q - 1 \geq \frac{q}{2} \),

\[
\int_{\mathbb{R}^2} \mathbb{E}(\tilde{g}(y)^q) d\mu(y) \leq \frac{2^q C_q}{(1 - a^2)d/2n}^{q/2} (1 + e^{8qR^2}). \tag{7.19}
\]

We use the preceding bounds to control the error term

\[
\text{Er} = \mathbb{E} \left( \int_{\{\|\tilde{g}\| \geq c\}} \|\tilde{g}\|^2 d\mu \right) + \mathbb{E} \left( \left( \|\tilde{g}\|_2 + \|\tilde{g}\|_\infty \right) \left( \int_{\{\|\tilde{g}\| \geq c\}} \|\tilde{g}\|^2 d\mu \right)^{1/2} \right) \tag{7.20}
\]

of (7.18). By repeated use of the Young and Hölder inequalities, the latter is bounded from above for any \( \delta > 0 \) and \( \alpha > 1 \) by

\[
\delta \left[ \mathbb{E}(\|g\|^2) + \mathbb{E}(\|\tilde{g}\|^2) \right] + \frac{1 + 2\delta}{2c^2(q-1)\delta} \int_{\mathbb{R}^d} \mathbb{E}(\|\tilde{g}(y)\|^{2\alpha}) d\mu(y).
\]
Since \( p_t(x, x) = \frac{1}{1-\sigma^2} e^{\frac{1}{2} \sigma^2 |x|^2} \), again with \( \mu(B_R) \geq \frac{1}{2} \),

\[
\mathbb{E}(\|\tilde{g}\|_2^2) = \frac{1}{n} \int_{\mathbb{R}^d} \left[ \mathbb{E}(p_t(X_1^R, y))^2 - \mathbb{E}(p_t(X_1^R, y))^2 \right] d\mu(y)
\leq \frac{1}{n} \int_{\mathbb{R}^d} p_t(x, x) d\mu^R(x)
\leq \frac{1}{n \mu(B_R)(1-a)^d}
\leq \frac{1}{n(1-a)^d}.
\]

Similarly

\[
\mathbb{E}(\|g\|_2^2) \leq \frac{1}{n} \int_{\mathbb{R}^d} p_t(x, x) d\mu(x) \leq \frac{1}{n(1-a)^d}.
\]

On the other hand, (7.19) with \( q = 2\alpha \) yields

\[
\int_{\mathbb{R}^d} \mathbb{E}(|\tilde{g}(y)|^{2\alpha}) d\mu(y) \leq \frac{4^\alpha C_{2\alpha}}{(1-a^2)^{d/2}n^{\alpha}} (1 + e^{16\alpha R^2}).
\]

Hence, for any \( 0 < \delta \leq 1 \) and \( \alpha > 1 \), the error term (7.20) satisfies

\[
\text{Er} \leq \frac{3\delta}{n(1-a)^d} + \frac{4^{\alpha+1} C_{2\alpha}}{c^2(\alpha-1) \delta (1-a^2)^{d/2} n^{\alpha}} (1 + e^{16\alpha R^2}).
\]

Therefore, from (7.18),

\[
\mathbb{E}(W_2^n(\mu^n, \mu)) \geq \frac{1}{2n} \int_{2t}^{\infty} \int_{\mathbb{R}^d} [p_s(x, x) - 1] d\mu^R(x) ds - \frac{1}{n} \int_{2t}^{\infty} \int_{\mathbb{R}^d} (\tilde{g}(y))^{2\alpha} d\mu(y) ds
\geq \frac{24\delta}{n(1-a)^d} - \frac{4^{\alpha+3} C_{2\alpha}}{c^2(\alpha-1) \delta (1-a^2)^{d/2} n^{\alpha}} (1 + e^{16\alpha R^2}).
\]

(7.21)

In this last step, we fix the various parameters involved in the previous analysis. Basically, \( t \approx \frac{1}{n^t} \) and \( R \approx \varepsilon \sqrt{\log n} \) for some small \( \varepsilon > 0 \), and \( \alpha > 1 \) is chosen large enough. Take for example \( t = \frac{1}{n^{\varepsilon/2}} \) and \( R^2 = \frac{1}{64} \log n \). Then, for \( n \) large enough, the necessary conditions on \( a = e^{-t} \) or \( \mu(B_R) \) are fulfilled. After some details, the choice of \( \delta = \frac{1}{n} \) and \( \alpha = 8 \) in (7.21) yields that

\[
\mathbb{E}(W_2^n(\mu^n, \mu)) \geq \frac{1}{2n} \int_{2t}^{\infty} \int_{\mathbb{R}^d} [p_s(x, x) - 1] d\mu^R(x) ds - O\left(\frac{1}{n}\right).
\]

From the analysis of the upper bound in [8], it is known that as \( t << \frac{1}{R^2} \), for some \( \rho > 0 \),

\[
\int_{2t}^{\infty} \int_{\mathbb{R}^2} [p_s(x, x) - 1] d\mu^R(x) ds \geq \rho R^2 \log\left(\frac{1}{t}\right)
\]

when \( d = 2 \), and also

\[
\int_{2t}^{\infty} \int_{\mathbb{R}} [p_s(x, x) - 1] d\mu^R(x) ds \geq \rho \log(R^2)
\]

when \( d = 1 \). Therefore, for the preceding choices of \( t \) and \( R \), this establishes the claim (7.11), as well as the \( \frac{\log \log a}{n} \) lower bound in (1.4). The announced proof is complete.
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