From Kardar-Parisi-Zhang scaling to explosive desynchronization in arrays of limit-cycle oscillators

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We study the synchronization physics of 1D and 2D oscillator lattices subject to noise and predict a dynamical transition that leads to a sudden drastic increase of phase diffusion. Our analysis is based on the widely applicable Kuramoto-Sakaguchi model, with local couplings between oscillators. For smooth phase fields, the time evolution can initially be described by a surface growth model, the Kardar-Parisi-Zhang (KPZ) theory. We delineate the regime in which one can indeed observe the universal KPZ scaling in 1D lattices. For larger couplings, both in 1D and 2D, we observe a stochastic dynamical instability that is linked to an apparent finite-time singularity in a related KPZ lattice model. This has direct consequences for the frequency stability of coupled oscillator lattices, and it precludes the observation of non-Gaussian KPZ-scaling in 2D lattices.

Networks and lattices of coupled limit-cycle oscillators do not only represent a paradigmatic system in nonlinear dynamics, but are also highly relevant for potential applications. This significance derives from the fact that the coupling can serve to counteract the effects of the noise that is unavoidable in real physical systems. Synchronization between oscillators can drastically suppress the diffusion of the oscillation phases, improving the overall frequency stability. Experimental implementations of coupled oscillators include laser arrays [1] and coupled electromagnetic circuits, e.g. [2, 3], as well as the modern recent example of coupled electromechanical and optomechanical oscillators [4–8]. In this work, we will be dealing with the experimentally most relevant case of locally coupled 1D and 2D lattices. Naive arguments indicate that the diffusion rate of the collective phase in a coupled lattice of $N$ synchronized oscillators is suppressed as $1/N$, which leads to the improvement of frequency stability mentioned above. However, it is far from guaranteed that this ideal limit is reached in practice [9, 10]. The nonequilibrium nonlinear stochastic dynamics of the underlying lattice field theory is sufficiently complex that a more detailed analysis is called for. In this context, it has been conjectured earlier that there is a fruitful connection [11] between the synchronization dynamics of a noisy oscillator lattice and the Kardar-Parisi-Zhang (KPZ) theory of stochastic surface growth [12, 13].

We have been able to confirm that this is indeed true in a limited regime, particularly for 1D lattices. However, the most important prediction of our analysis consists in the observation that a certain dynamical instability can take the lattice system out of this regime in the course of the time evolution. As we will show, this instability is related to an apparent finite-time singularity in the evolution of the related KPZ lattice model. It has a significant impact on the phase dynamics, increasing the phase spread by several orders of magnitude. As such, this phenomenon represents an important general feature of the dynamics of coupled oscillator lattices.

Before we turn to a definition of the model, it is helpful to briefly outline the wider context of this study. We will be dealing with phase-only models, which can often be used to describe systems of coupled limit-cycle oscillators effectively (see also Fig. 1), whenever the amplitude degree of freedom is irrelevant. The most prominent examples are the Kuramoto model and extensions thereof [14–17]. These deterministic models are studied intensely for their synchronization properties [11], mainly for globally coupled systems with disorder, as well as for pattern formation (for locally coupled systems) [18]. In the latter case, interesting effects show up even in the absence of disorder [19].

Adding noise to these models can significantly influence the synchronization properties [17]; see [20] for an example in globally coupled systems. In locally cou-
pled systems, the delicate interplay between the nonlinear coupling, the noise, and the spatial patterns can lead to even more complex dynamics. In contrast to the related XY model [21], which is used to describe systems in thermodynamic equilibrium, driven nonlinear oscillator lattices are usually far from equilibrium. This is reflected in an additional, “non-variational” coupling term [15, 16]. As long as the phase field is smooth, one can employ a continuum description of the oscillator lattice [11]. This is important to make the connection to the theory of surface growth. The continuum description also links our research to recent developments in the study of (non-equilibrium) driven dissipative condensates [22, 23], where the connection to the physics of surface growth was highlighted in an additional, “non-variational” coupling term in thermodynamic equilibrium, driven nonlinear oscillators. In real physical systems, there will be noise acting on the physical system, with recent examples including electromagnetic [27] and optomechanical [28, 29] oscillators. In real physical systems, there will be noise acting on the oscillators. This can be modeled by a Langevin noise term in the right hand side of the effective phase equation.

We start our analysis by briefly motivating the model that we will use. Consider a large number of self-sustained nonlinear oscillators moving on their respective limit cycles. When they are coupled, they influence each other’s amplitude and phase dynamics. Provided that the amplitudes do not deviate much from the limit cycle, effective models can be derived, which describe the phase dynamics only [14–17], see also Fig. 1a. The transition from the microscopic model to the phase model depends on the physical system, with recent examples including electromechanical [27] and optomechanical [28, 29] oscillators. In real physical systems, there will be noise acting on the oscillators. This can be modeled by a Langevin noise term in the right hand side of the effective phase equation.

We will be interested in understanding the competition between noise and coupling in the context of synchronization dynamics. To bring out fully this competition, we focus on the ideal case of a non-disordered lattice, with uniform natural frequencies. In that case, the derivation of an effective model leads to the noisy Kuramoto-Sakaguchi model [16, 17] for the oscillator phases \( \varphi_j(t) \):

\[
\dot{\varphi}_j = S \sum_{\langle k,j \rangle} \sin(\varphi_k - \varphi_j) + C \sum_{\langle k,j \rangle} \cos(\varphi_k - \varphi_j) + \xi_j, \tag{1}
\]

where \( \xi_j(t) \) is a Gaussian white noise term with correlator \( \langle \xi_j(t)\xi_k(0) \rangle = 2D \varphi \delta(t) \delta_{jk} \), and \( S \) and \( C \) are the coupling parameters. The sums run over nearest neighbors. We will often call this model the “phase model”, for brevity. In this article, we focus on the time evolution of the phase field from a homogeneous initial state. Hence, we set \( \varphi_j(0) = 0 \) on all sites. As an illustrative example, we show the (smoothed) snapshots of the phase field from a simulation of Eq. (1) in Fig. 1b. In this figure, the color (and the mesh geometry) encode the phase value at lattice site \( j = (j_x,j_y) \) for three different points in time.

At this point it should be noted that a lattice of coupled self-sustained oscillators actually gives rise to an additional term in the effective phase equation. This term couples the phases of three adjacent sites. This was derived in [28, 29], and the resulting dynamics of that extended model (without noise) has been explored by us in a previous work [30]. However, the additional term is irrelevant for long-wavelength dynamics and moreover can be tuned experimentally. Therefore, in the present article we focus entirely on the important limiting case of the Kuramoto-Sakaguchi model, Eq. (1).

How does the interplay of noise and coupling affect the frequency stability of the oscillators? This is a central question for synchronization and metrology. It can be discussed in terms of the average frequencies, defined as \( \Omega_j(t) = t^{-1} \int_0^t \! dt' \varphi_j(t') = \varphi_j(t)/t \). Here the \( \varphi_j(t) \) are the phases accumulated during the full time evolution (see also [17, 31]). As the definition of \( \Omega_j \) shows, they have an important physical meaning in the present setting, essentially indicating the number of cycles that have elapsed. This is in contrast to other physical scenarios that also involve phase models like the one shown here. For example, in studies of superfluids, the phase is defined only up to multiples of \( 2\pi \). Then, the total number of phase windings during the time evolution does not have any direct physical significance. This distinction is important when trying to make the connections we are going to point out below.

Important insights can be obtained from studying the evolving spread of the average frequencies. This turns out to be directly related to the spread of the phase field, \( w_{\varphi}(t) \),

\[
w^2_{\varphi}(t) = \left\langle \frac{1}{N} \sum_{j=1}^N (\varphi_j(t) - \bar{\varphi}(t))^2 \right\rangle
= t^2 \left\langle \frac{1}{N} \sum_{j=1}^N (\Omega_j(t) - \bar{\Omega}(t))^2 \right\rangle, \tag{2}
\]

where \( \bar{\varphi}(t) = N^{-1} \sum_{j=1}^N \varphi_j(t) \) is the mean (spatially averaged) phase and \( \bar{\Omega}(t) = \bar{\varphi}/t \) is the mean average frequency of a lattice with \( N \) sites. The angular brackets denote an ensemble average over different realizations of the noise.

For the simple case of uncoupled identical oscillators subject to noise, the phase spread grows diffusively, \( w_{\varphi}(t) = \sqrt{2D_{\varphi}t} \). Hence, the spread of time-averaged frequencies decreases as \( t^{-1/2} \). This reflects the fact that the averaged frequencies are identical in the long-time limit because there is no disorder. In this sense, the oscillators are always synchronized. However, if coupling is included, we will find different exponents in the time-dependency of the phase field spread. For example, a smaller exponent means that the tendency towards synchronization is stronger. Hence, we will see that the coupling between the oscillators can either enhance or hinder the synchronization process, depending on the parameter
regime. We expect that this translates to systems with small disorder in the natural frequencies.

Much of our discussion of the initial stages of evolution will hinge on the approximations that become possible when the phases on neighboring sites are close. Then the phase model, Eq. (1), is well approximated by a second-order expansion in the phase differences [11]. This expansion can be recast in dimensionless form using a single parameter $g_{1d,2d} = 4D_x C^2 / S^3$. In a one-dimensional array, for example, the resulting model reads

$$
\frac{\partial h_j}{\partial \tau} = (h_{j+1} + h_{j-1} - 2h_j) + \frac{1}{4}[(h_{j+1} - h_j)^2 + (h_{j-1} - h_j)^2] + \gamma_{1d} \eta_j,
$$

where we have rescaled both the time, $\tau = S \tau$, and the phase field, $h_j = -(2C/S)(\varphi_j - 2Ct)$. The noise correlator is $\langle \eta_j(\tau) \eta_k(0) \rangle = 2\delta_{jk} \delta(\tau)$. The generalization to two dimensions is straightforward.

Eq. (3) can be readily identified as a lattice version of the Kardar-Parisi-Zhang (KPZ) model [12, 13, 32], a universal model for surface growth and other phenomena. This nonlinear stochastic continuum field theory describes the evolution of a height field $h(\vec{r}, t)$,

$$\dot{h} = \nu \Delta h + \frac{\lambda}{2}(\nabla h)^2 + \eta,$$

with white noise $\eta(\vec{r}, t)$, where $\langle \eta(\vec{r}_1,t)\eta(\vec{r}_2,0) \rangle = 2D \delta(\vec{r}_2 - \vec{r}_1) \delta(t)$. The diffusive term tries to smooth the surface, while both the noise and the nonlinear gradient term tend to induce a roughening.

The relation of the KPZ model to coupled oscillator lattices has been pointed out before [11]. However, up to now it has remained unclear how far this formal connection is really able to predict universal features of the synchronization dynamics. In the present article, we will indeed observe transient behavior where universal KPZ dynamics is applicable, but we will also find that this is invariably followed by phenomena that lead into completely different dynamical regimes. All the numerical results discussed in this article will refer either to the full phase model, Eq. (1), or to its approximate version, the “lattice KPZ model” Eq. (3). From the comparison of these models, we will be able to extract valuable predictions for the synchronization dynamics.

It is straightforward to make the connection between Eq. (3) and a one-dimensional lattice version of the KPZ model more precise. Starting from Eq. (4), and given a lattice constant $a$, we have to rescale time, $\tau = (\nu/a^2)t$, and height, $h_j(\tau) = (\lambda/\nu)h(x, \tau)$, and choose a particular discretization of the derivatives. Note that in the continuum model in one dimension, it would even be possible to get rid of all parameters by rescaling time, height and space. In contrast, for the lattice model, we are left with the one dimensionless parameter $g_{1d} = aD x^2 / \nu^3$ [13, 33].

This coupling constant will become important in the following.

We had derived our lattice model, Eq. (3), as an approximation to the phase model, Eq. (1), with its trigonometric coupling terms that are periodic in the phase variables. Hence, for the evaluation of the equation of motion, the configuration space of each phase variable may be restricted to the compact interval $[-\pi, \pi)$. In view of the foregoing discussion, one may then see the phase (Kuramoto-Sakaguchi) model as a “compact KPZ model”. This designation has indeed been proposed in a recent article [34] (see also [35]).

The rescaling of time and phase introduced above, for the approximate lattice model of Eq. (3), can also be employed in the full phase model, Eq. (1). Crucially, this leads to one more dimensionless parameter, $S/C$. For example, the sine term will be converted to $(2C/S) \sum \sin [(S/2C)(h_k - h_l)]$. This establishes that for given differences $h_k - h_l$ the approximation, Eq. (3), becomes better for smaller $S/C$. For this reason, we will focus on small values $S/C \ll 1$, where substantial findings can be expected from the connection of the phase model to KPZ dynamics.

First insights can be gained by direct numerical simulations of the phase model. For one-dimensional arrays, the outcome of a single simulation is displayed in Fig. 2a.
The typical time evolution of the phase spread $w_\varphi(t)$ is shown in Fig. 2b. We can distinguish two parameter regimes from the long time evolution. In one regime, we see that after initial transients, the phase spread evolves according to $w_\varphi(t) \sim t^{1/3}$ (see magenta curves). Hence, the synchronization is enhanced as compared to the case of uncoupled oscillators (where $w_\varphi(t) \sim t^{1/2}$ as discussed above).

The power-law growth of the phase field spread with exponent $1/3$ can be identified as universal KPZ behavior, as we will explain in the following. Luckily, in the context of KPZ dynamics, the best-studied quantity is the mean surface width $w$, which directly relates to the phase spread $w_\varphi$ introduced above:

$$w^2(L,t) = \left( \frac{1}{L^d} \int d^d r \left( h(\vec{r},t) - \bar{h}(t) \right)^2 \right),$$  \hspace{1cm} (5)

with the average surface height $\bar{h}(t) = L^{-d} \int d^d r h(\vec{r},t)$ in a system of linear size $L$. The surface width has been found to obey a scaling law $w^2(L,t) \sim L^{2\alpha} F(t/L^z)$ [36]. In particular, for $t \ll L^z$ (in appropriately rescaled units), we have $w^2(L,t) \sim t^{2\beta}$ with $\beta = \zeta/2$.

In one dimension, the scaling exponent can be calculated analytically and is $\beta = 1/3$ [12]. This means that the surface will become rougher with time, but less rapidly than for independent diffusive growth at individual sites. It is this exponent that is also observed in the evolution of the phase model, Fig. 2. Hence, we conclude that 1D arrays of limit-cycle oscillators, as described by the noisy Kuramoto-Sakaguchi phase model, indeed show KPZ scaling in certain parameter regimes.

Far more surprising is the other dynamical regime (red and green curves). In that regime, one observes diffusive growth, $w_\varphi(t) \sim t^{1/2}$ for long times, which may seem unremarkable except for clearly deviating from any KPZ predictions. However, at this point, it is worthwhile to emphasize that we are displaying curves averaged over many simulations. If instead we look at single simulation trajectories, we see an explosive growth of $w_\varphi(t)$ at some random intermediate time (gray lines). At these random times, the phase field suddenly grows its variance by several orders of magnitude. This corresponds to an explosive desynchronization of the oscillators.

To understand this important dynamical feature better, we now briefly turn away from the full phase model and study the evolution of the lattice KPZ model, Eq. (3). This serves as an approximate description at small phase differences, so we can expect to learn something about the onset of the growth, but not about the long-time regime which involves large phase differences. As an example, we show the result of a simulation of Eq. (3) in Fig. 3a, where we plot the field $h_j(\tau)$ for several points in time. Clearly, even this simpler model already displays some kind of instability, which now leads to an apparent (numerical) finite-time singularity. It is worthwhile to note that such divergences had been identified before in numerical attempts to solve the KPZ dynamics by discretizing it on a lattice [37–39] (see also [40, 41]). In those simulations, this behavior was considered to be a numerical artifact depending on the details of the discretization. In contrast, in view of our phase model, the onset of the instabilities is a physical phenomenon which merits closer inspection.

The points in time, for which the snapshots are shown in Fig. 3a, approach the time of the singularity logarithmically. In addition to the normal roughening process, which we expect from the continuum theory, we see the rapid growth of single peaks. Those can send out shocks of large height differences, which then propagate through the system, as can be seen in the center of Fig. 3a. The collision of such shocks can produce larger peaks. We commonly observe that eventually very large shocks grow during propagation, which leads to the singularity in the numerical evolution (marked with a red star in the figure). In the inset, we show how the maximum phase difference between nearest neighbors, $\delta h_{\text{NN}}^{\text{max}}$, increases drastically just before the divergence. We also indicate the points in time for which we plotted the height field. The details of the instability development depend on the lattice size and the coupling parameter.

The occurrence of an instability is a random event. In Fig. 3b, we plot the probability of observing an instability during the evolution up to a time $\tau$, as a function of the coupling $g_{1d}$. In principle, instabilities can occur at all coupling strengths, but we find that for the lattice size employed here (1000 sites) they become much less likely (happen much later) for $g_{1d} < 40$. To extrapolate
to larger lattices, we may adopt the assumption that the stochastic seeds for these instabilities are planted independently in different parts of the system. In that case, the probability to encounter a divergence within a small time interval will just scale linearly in system size, and the present results for \( N = 1000 \) are therefore sufficient to predict the behavior at any \( N \).

As mentioned above, the instabilities in lattice KPZ models are considered unphysical in the surface growth context, because they do not show up in the continuum model, at least in one dimension [38]. On the contrary, our phase model, describing synchronization in discrete oscillator lattices, is a genuine lattice model from the start. Hence, the onset of instabilities has to be taken seriously. In the full phase model, Eq. (1), the incipient divergences are eventually cured by the periodicity of the coupling functions. Instead of resulting in a finite-time singularity, they will lead the system away from KPZ-like behavior and make it enter a new dynamical regime.

To find out for which parameters this happens, we have determined numerically the probability of encountering large growth of nearest-neighbor phase differences. We find that we can distinguish between a “stable” regime, where no large phase differences (> \( \pi \)) show up in most simulations, and an “unstable” regime, where large differences occur with a high probability. For small \( S/C (\leq 0.001) \), we indeed get quantitative agreement with the results discussed above for the lattice KPZ model, Fig. 3b.

In a single simulation in the unstable regime of the phase model, we typically observe a time evolution such as the one depicted in Fig. 4. Initially, the phase field develops as in the corresponding KPZ lattice model. Then, a KPZ-like instability induces large phase differences. As mentioned above, this does not lead to a divergence. Instead, we find that huge triangular structures develop rapidly. Afterwards, these structures get diffused on a much longer time scale. The time evolution is reflected in the phase spread, as shown previously in Fig. 2b (gray lines): The development of triangular structures leads to an explosive growth, whereas the subsequent diffusion leads to the asymptotic scaling \( w_\phi \sim t^{1/2} \).

The peculiar time evolution after the onset of the instability can be explained by considering the deterministic phase model. For the parameter value \( S/C \) employed here, this model is (at least for some time) turbulent for initial states with large phase differences. In the simulations of the full model, the stochastic dynamics induces an instability initially, which brings the phase field from a KPZ-like state to a turbulent state locally. After this, the dynamics can be understood deterministically. Because of the large phase differences in the turbulent region, this part of the lattice will have a very different phase velocity from the KPZ-like region (on average). At the same time, the turbulent region, which is the shaded region in the plots of Fig. 4, grows in space. These two processes lead to a triangular phase field shape covering the whole lattice. Additionally, the turbulent dynamics produces very large phase differences, including wrap-arounds by \( 2\pi \). This induces a diffusive growth of the phase field width \( w_\phi \) with a large diffusion coefficient. This can be seen in the red curve of Fig. 2b. The behavior of this curve after the rapid increase can be fitted well with \( w_\phi(t) = \sqrt{A + Bt} \) (blue dotted line). We checked that the diffusion coefficient \( B \) from this fit can also be found in simulations of the deterministic model with random initial conditions. The numerical value of \( B \) is much larger than the noise strength \( D_\phi \).

Hence, we conclude that in the unstable regime of the one-dimensional phase model, the onset of KPZ-like instabilities induces an explosive desynchronization of the oscillators. This is followed by diffusive growth of \( w_\phi(t) \). Note that there remains the large phase field spread resulting from the desynchronization, and the large long-time diffusion coefficient \( B \), which stems from the deterministic turbulent dynamics. All of this is relevant for small values of the parameter \( S/C \).

The physics of surface growth depends crucially on the dimensionality. Correspondingly, we ask how the synchronization dynamics in oscillator lattices changes when we proceed to 2D lattices, which can be implemented in experiments and which are expected to be favorable towards synchronization.

By using the same rescaling as above, the lattice KPZ model can be written in dimensionless units with a single parameter \( g_{2d} = D \lambda^2 / \nu^3 \). Interestingly, an appropriately rescaled form of the continuum KPZ model in 2D also contains this single parameter. That is in contrast to the
1D case, where the rescaled continuum model did not depend on any parameter. As a consequence, there are now different time regimes in the growth of the surface width [42]. In particular, KPZ power-law scaling \( w \sim t^\beta \) sets in beyond a time scale \( t^* \) that becomes exponentially large at small couplings, \( t^* \sim \exp(16\pi/g_{2d}) \). This has to be taken into account in numerical attempts to observe the scaling regime, as in [43]. In finite systems, the surface width saturates eventually, for times \( (\lambda^2/\nu)t \gg (\lambda L/\nu)^2 \).

The lattice version of the 2D KPZ model, as obtained by extending Eq. (3) to two dimensions, also develops instabilities. Like in 1D, we study the probability of encountering such instabilities, see Fig. 5c. We find qualitatively the same behavior as in 1D: The likelihood of an instability during a time \( \tau \) increases rapidly with larger \( g_{2d} \).

There is, however, a crucial difference with respect to the 1D situation: we find that the instabilities occur much earlier than the (exponentially late) onset of KPZ power-law scaling. This is illustrated in the inset of Fig. 5c, where the hatched region is the KPZ scaling regime expected from the continuum theory for infinite systems. In addition, at smaller couplings, the surface width would saturate long before the projected onset of KPZ scaling for any reasonable lattice sizes. As an example, the dotted line in the inset of Fig. 5c shows the saturation time for a lattice of size \( N = 10^5 \). Overall, we predict that in 2D the power-law KPZ scaling regime will be irrelevant for the synchronization dynamics of oscillator lattices.

These predictions are borne out in simulations of the full phase model, Eq. (1), in 2D (Fig. 5a and b). Like in one dimension, we focus on small parameter values of \( S/C \). As long as the phase differences remain small, which is the case for small \( g_{2d} = 4D_\phi C^2/S^3 \), the behavior is analogous to the lattice KPZ model, see Fig. 5a. As explained above, the exponentially large times of the KPZ power-law regime cannot be reached before instabilities set in. Instead, the evolution shows the behavior of the linearized KPZ equation, the so-called Edwards-Wilkinson model [44]. This produces a slow logarithmic growth of the surface width [42, 44]. In this linear model, we can also straightforwardly take into account the effects of the lattice discretization and the finite size of the lattice. The resulting analytical prediction is shown as the dashed line in Fig. 5a, with a good initial fit and some deviations only at later times (see also the appendix).

In simulations of the phase model with a larger parameter \( g_{2d} \), we see initially the same behavior, but followed by a rapid increase of the phase field spread with time (see Fig. 5b, red curve). This can be explained by the explosive growth in single simulations (gray lines), similar to the behavior in one dimension. For different parameters, where the instabilities occur earlier, we see that the phase spread approaches a diffusive square-root growth for long times (not shown here).

Figure 5. Dynamics in two-dimensional models. (a) Phase model, slow logarithmic growth of the phase spread for \( g_{2d} = 1 \) (red curve). The data is from an average over 300 simulations with parameters \( S/C = 0.001 \), \( D_\phi/S = 2.5 \times 10^{-7} \), \( N = 256^2 \), \( S\Delta t = 0.1 \). The linear theory would lead to a slightly different behavior, as shown by the dashed black lines in (a) and (b). (b) Same quantity for a slightly larger coupling, \( g_{2d} = 1.5 \) (red curve). Due to explosive instabilities (single trajectories shown as gray lines), there is a rapid increase to much larger values than in the linear theory. [Parameters: \( D_\phi/S = 3.75 \times 10^{-7} \), \( N = 64^2 \), otherwise like in (a)]. (c) Lattice KPZ: Probability of instability in the lattice KPZ model, the 2D version of Eq. (3). The inset shows that the power-law 2D KPZ scaling (hatched region) would be expected at much later times than the instabilities (note the logarithmic scaling of the time axis). This makes the scaling unobservable also in the phase model, where the instabilities induce a different dynamical regime.

Overall, we see that there is a parameter regime where lattice KPZ-like instabilities are not relevant in 2D arrays. Then, the phase field spreads very slowly (logarithmically) with time. According to Eq. (2), this means that the oscillators tend to synchronize quickly. However, if instabilities show up, which is the case for larger \( g_{2d} = 4D_\phi C^2/S^3 \), we find the same explosive desynchronization as in 1D.

In conclusion, we have studied the phase dynamics of one- and two-dimensional arrays of identical limit-cycle oscillators, described by the noisy Kuramoto-Sakaguchi model with local coupling. We have shown that, depending on parameters, the coupling can either enhance or hinder the synchronization when starting from homogeneous initial conditions. In 1D, for sufficiently small noise and at short times, one can observe roughening of the phase field as in the Kardar-Parisi-Zhang model of surface growth, with the corresponding universal power-law scaling. At larger noise, or for larger times, explosive desynchronization sets in, triggering a transition into a different dynamical regime. We have traced back this behavior to an apparent finite-time singularity of the approximate (KPZ-like) lattice model. This is especially relevant for two dimensions, where it will occur before the long-term KPZ scaling can be observed, although the initial slow logarithmic growth still makes 2D arrays more favorable for synchronization.

With these results, we have also made more precise the connection between phase-only models of limit-cycle os-
oscillators and the KPZ model, which was only established formally before [11]. In particular, we have shown that the lattice nature of the phase model, Eq. (1), is important, especially for large values of the coupling parameter $g_{1d,2d}$. The reason is that for small phase differences, we are led to a particular lattice KPZ model, Eq. (3), which, however, contains instabilities. These will destroy any resemblance between the phase dynamics and surface growth physics.

Our predictions will be relevant for all studies of synchronization in locally coupled oscillator lattices, when the phase-only description is applicable. This can be the case in optomechanical arrays (e.g. in extensions of the work presented in [5]). They may also become important for the study of driven-dissipative condensates, described by the stochastic complex Ginzburg-Landau equation or Gross-Pitaevskii-type equations, where a connection to the KPZ model has been explored recently [22, 24, 25] for the continuum case. Once these studies are extended to lattice implementations of such models (e.g. in optical lattices), one may encounter the physics predicted here.

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**APPENDIX: METHODS**

The numerical time integration of the coupled Langevin equations on the lattice was performed with the algorithm presented in [45]. In the following, we provide further details on the parameters employed for the simulations whose results are shown in the figures.

For the simulations of the full phase model in one dimension in Fig. 2, we employed the following parameters. Fig. 2a: $S/C = 0.001$, $D_\varphi/S = 2 \times 10^{-6}$, $S\Delta t = 0.01$, $N = 5 \times 10^5$ (resulting in $g_{1d} = 8$). We only show a part of the phase field. Fig. 2b: Parameters for the upper magenta curve: $S/C = 0.001$, $D_\varphi/S = 2 \times 10^{-6}$, $S\Delta t = 0.01$, $N = 10^4$. Lower magenta curve: $S/C = 0.001$, $D_\varphi/S = 2.5 \times 10^{-7}$, $S\Delta t = 0.1$, $N = 10^4$. For both magenta curves, the average was taken over 300 simulations. For the red curve: $S/C = 0.001$, $D_\varphi/S = 1.25 \times 10^{-5}$, $S\Delta t = 0.001$, $N = 10^3$. For the green curve: $S/C = 0.1$, $D_\varphi/S = 0.0625$, $S\Delta t = 0.001$, $N = 10^3$. The average was taken over 120 simulations.

We now turn to the simulations of the KPZ model. In general, direct numerical simulations of this model where the scaling properties are extracted are always performed for stable evolution. Hence, they are done in the small-coupling regime, also for slightly different lattice realizations with quantitatively different stability properties, see [33]. There, it is also found that the parameter $g_{1d}$ has an influence on the transient dynamics in one dimension (see also [46]) which explains the transients that we observed in the phase model, in Fig. 2b (magenta curves).

In Fig. 3b, we plot the probability of encountering instabilities in the 1D KPZ lattice model as given by Eq. (3), for a wide range of the coupling parameter $g_{1d}$. The data is extracted from 300 simulations for each value of $g_{1d} = 1, 2, ..., 50$, running up to time $\tau = 100$, with a time step $\Delta \tau = 10^{-4}$. The probability of instability is just the ratio of unstable simulations. We checked that the results for this quantity do not change at $g_{1d} = 50$ if we go to a smaller time step of $\Delta \tau = 10^{-5}$. A simulation was considered unstable when the nearest-neighbor height difference at one lattice site exceeded a large value, which was chosen to be $10^5$. We used a lattice size of $N = 1000$. The probability of an instability generally increases for larger lattices. An exception are very small lattices, where boundary effects can become important.

Fig. 5c shows the results for the probability to find an unstable simulation in the 2D KPZ lattice model. The data for the plot is from 300 simulations for each value of $g_{2d} = 0.1, 0.2, ..., 4$, on a lattice of size $N = 64^2$ with time step $\Delta \tau = 0.01$. A simulation was considered unstable when one of the nearest neighbor height difference at one lattice site exceeded a large value, which was chosen to be $10^8$. As in 1D, the probability of instability depends on the lattice size.

Regarding the results for the two-dimensional phase model, shown in Fig. 5a, we commented in the main text on the analytical predictions from a finite-size lattice version of the linear Edwards-Wilkinson model (dashed curve in the figure). It can be seen that there are deviations between this curve and the simulation of the phase model (red curve) at later times. Further investigation shows that the two-dimensional lattice version of the KPZ model (in analogy to Eq. (3)) shows the same deviations. We checked that another lattice version of KPZ (as in [33]) does indeed agree with the result from the linear equation. The reason for the discrepancy in different lattice models might be more subtle influences of the nonlinearity, as also reported in [47].

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