On virtual singular braid groups

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Abstract

Let \( n \geq 2 \). In this paper we study some properties of the virtual singular braid group \( VSG_n \). We define some numerical invariants for virtual singular braids arising from exponents of words in \( VSG_n \) and describe the kernel of these homomorphisms. We determine all possible homomorphisms, up to conjugation, from the virtual singular braid group \( VSG_n \) to the symmetric group \( S_n \), and for the particular case \( n = 2 \) we give a presentation (and a description) for the kernel in each case. For all possible homomorphisms we obtain a decomposition of \( VSG_n \) as a semi-direct product of the kernel of the homomorphism and the symmetric group. Finally, we show that there are some forbidden relations in \( VSG_n \) and then we introduce some quotients of it (e.g. the welded singular braid group, the unrestricted virtual singular braid group, among other related groups) and then we state for them similar results obtained for \( VSG_n \).

1 Introduction

There are several generalizations of the Artin braid group \( B_n \), both from a geometric and algebraic point of view, and its study constitute an active line of research. Recently, in [9], Capraru, De la Pena and McGahan introduced virtual singular braids as a generalization of classical singular braids defined by Birman [8] and Baez [2] for the study of Vassiliev invariants, and virtual braids defined by Kauffman [17] and Vershinin [20]. In [9] the authors proved an Alexander and Markov Theorem for virtual singular braids and gave two presentations for the monoid of virtual singular braids, denoted by \( VSB_n \). In a more recent paper [10] Capraru and Yeung showed that the monoid \( VSB_n \) embeds in the group \( VSG_n \) called the virtual singular braid group on \( n \) strands. They also gave a presentation of the virtual singular pure braid group \( VSPG_n \) and showed that \( VSG_n \) is a semi-direct product of \( VSPG_n \) and the symmetric group \( S_n \).

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The interest for these objects is growing and some progress on its study has been made in recent years. For instance, Caprau and Zepeda [11] constructed a representation of $VSB_n$, and using the Reidemeister-Schreier algorithm they found a presentation for the virtual singular pure braid monoid. Also, Cisneros de la Cruz and Gandolfi [12] studied algebraic, combinatorial and topological properties of virtual singular braids.

In this paper we want to study some properties of the virtual singular braid group $VSG_n$ as well as of some of its subgroups and quotients. In Theorem 5 we show some properties of $VSG_n$, for instance that it is not residually nilpotent for $n \geq 3$ and that its commutator subgroup is perfect for $n \geq 5$, as happens with the classical, virtual and singular case (see [5] and [14]). We define some numerical invariants for virtual singular braids arising from exponents of words in $VSG_n$ and then describe in Theorem 8 the kernel of these homomorphisms, being one of these kernels the normal closure of the virtual braid group in $VSG_n$. Then we determine all possible homomorphisms, up to conjugation, from the virtual singular braid group $VSG_n$ to the symmetric group $S_n$, see Proposition 10 and Theorem 13. This study was made in [7] for virtual braid groups. In particular, in Proposition 11, we give descriptions of $VSG_n$ as a semi-direct product of the kernel of these homomorphisms and the symmetric group. In Theorem 14 and Corollary 16 we study the particular case $n = 2$, giving a presentation (and a description) for the kernel of $VSG_2 \to S_2$ in each case.

We show in Theorem 19 that there are some forbidden relations in $VSG_n$. Then we introduce some quotients of the virtual singular braid group, similar to the ones studied in the virtual case (see [4], [6], [13], [17], [18]), for instance we introduce the welded singular braid group, the unrestricted virtual singular braid group, among other related groups. Finally, we state for the quotients of $VSG_n$ considered in this paper similar results obtained for $VSG_n$ in this work.

This paper is organized as follows. In Section 2 we recall the definitions and presentations of the main groups that we use in this paper and study properties of the virtual singular braid group and also of some subgroups. In Section 3 we show that are some forbidden relations in $VSG_n$, then define some quotients of it and also describe some of their properties, similar to the ones of $VSG_n$.

Finally, we would like to mention a convention that we use in several places of this paper. Let $G$ be a group and let $N$ be a normal subgroup of $G$. By an abuse of notation, in this paper we use the same symbols for elements in $G$ and its equivalence classes in the quotient $G/N$.

## 2 The virtual singular braid group

We start this section giving the definitions and presentations of the main groups considered in this paper. Then we show properties of the virtual singular braid group, exhibit some invariants and also we describe all possible homomorphisms, up to conjugation, from the virtual singular braid group $VSG_n$ to the symmetric group $S_n$. The case $n = 2$ will be treated in details. For all possible homomorphisms we obtain a decomposition of $VSG_n$ as a semi-direct product of the kernel of the homomorphism and the symmetric group.
2.1 Definitions

Let \( n \geq 2 \) a positive integer. For the first definition we shall consider the classical presentation of the Artin braid group \( B_n \) (see [1]) and the presentation of the virtual braid group \( VB_n \) that was formulated in [20, p.798] and restated in [5].

**Definition 1.** The *braid group on \( n \) strands*, denoted by \( B_n \), is the abstract group generated by \( \sigma_i \), for \( i = 1, 2, \ldots, n - 1 \), with the following relations:

\[
(AR1) \quad \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad i = 1, 2, \ldots, n - 2;
\]

\[
(AR2) \quad \sigma_i \sigma_j = \sigma_j \sigma_i, \quad |i - j| \geq 2.
\]

The *virtual braid group on \( n \) strands*, denoted by \( VB_n \), is the abstract group generated by \( \sigma_i \) (classical generators) and \( v_i \) (virtual generators), for \( i = 1, 2, \ldots, n - 1 \), with relations (AR1), (AR2) and:

\[
(PR1) \quad v_i v_{i+1} v_i = v_{i+1} v_i v_{i+1}, \quad i = 1, 2, \ldots, n - 2;
\]

\[
(PR2) \quad v_i v_j = v_j v_i, \quad |i - j| \geq 2;
\]

\[
(PR3) \quad v_i^2 = 1, \quad i = 1, 2, \ldots, n - 1;
\]

\[
(MR1) \quad \sigma_i v_j = v_j \sigma_i, \quad |i - j| \geq 2;
\]

\[
(MR2) \quad v_i v_{i+1} \sigma_i = \sigma_{i+1} v_i v_{i+1}, \quad i = 1, 2, \ldots, n - 2.
\]

The generator \( \sigma_i \) corresponds to the diagram represented on the left of Figure 1, the generator \( \sigma_i^{-1} \) is obtained from \( \sigma_i \) by making a crossing change. The geometric generator \( v_i \) is illustrated in Figure 2.

![Figure 1: Classical crossings, for \( i = 1, \ldots, n - 1 \).](image)

The singular braid monoid with \( n \) strands, denoted by \( SB_n \), was introduced independently by Baez [2] and Birman [8]. It was shown by Fenn, Keyman and Rourke [15] that the singular braid monoid embeds in a group called the singular braid group and denoted by \( SG_n \). The singular braid monoid has been shown to be a useful object to consider, for example, in the context of Vassiliev invariants, see [3] and [21]. The elements of this group have a geometric interpretation as singular braids with two types of singular crossings that cancel: \( \tau_i \) and \( \tau_i^{-1} \). As in [10, Section 2] we will adopt the representation of the new type of singularity as an open blob, see Figure 2.

**Definition 2 ([15, Section 2]).** The *singular braid group on \( n \) strands*, denoted by \( SG_n \), is the abstract group generated by \( \sigma_i \) (classical generators) and \( \tau_i \) (singular generators), for \( i = 1, 2, \ldots, n - 1 \), with relations (AR1), (AR2) from Definition 1 and:
Virtual singular braids are similar to classical braids. In addition to having classical crossings, they also have virtual and singular crossings. The multiplication of two virtual singular braids $\alpha$ and $\beta$ on $n$ strands is given using vertical concatenation. The braid $\alpha\beta$ is formed by placing $\alpha$ on top of $\beta$ and gluing the bottom endpoints of $\alpha$ with the top endpoints of $\beta$. Under this binary operation, the set of isotopy classes of virtual singular braids on $n$ strands forms a monoid. Following [10, Definition 3] we will call an element in $VSG_n$ an extended virtual singular braid or simply a virtual singular braid for short. Geometrically, besides a classical Reidemeister move, now we also have an extended Reidemeister move, see Figure 3.

In [10, Definition 3] was defined an abstract group called the virtual singular braid group such that the virtual singular braid monoid $VSB_n$ (defined in [9]) embeds in it. We shall use the presentation as stated in [10, Pages 5-6]. We note that the relations of the groups $B_n, VB_n$ and $SG_n$ given in their respective presentations (see Definition 1 and 2) appears in in the following presentation.

**Definition 3.** The virtual singular braid group, denoted by $VSG_n$, is the abstract group generated by $\sigma_i$ (classical generators), $\tau_i$ (singular generators) and $v_i$ (virtual generators), where $i = 1, 2, \ldots, n - 1$, subject to the following relations:

(2PR) Two point relations: $v_i^2 = 1$ and $\sigma_i \tau_i = \tau_i \sigma_i$, for $i = 1, 2, \ldots, n - 1$ for all $i = 1, 2, \ldots, n - 1$. 

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Figure 2: Singular and virtual crossings, for $i = 1, \ldots, n - 1$.

(SR1) $\tau_i \tau_j = \tau_j \tau_i$ for $|i - j| \geq 2$.

(MR1) $\tau_i \sigma_j = \sigma_j \tau_i$ for $|i - j| \geq 2$.

(MR2) $\tau_i \sigma_i = \sigma_i \tau_i$ for $i = 1, 2, \ldots, n - 1$.

(MR3) $\sigma_i \sigma_{i+1} \tau_i = \tau_{i+1} \sigma_i \sigma_{i+1}$, for $i = 1, 2, \ldots, n - 2$.

(MR4) $\sigma_i \sigma_{i+1} \tau_{i+1} = \tau_i \sigma_{i+1} \sigma_i$, for $i = 1, 2, \ldots, n - 2$.

Figure 3: Classical and extended Reidemeister move, for $i = 1, \ldots, n - 1$. 

$$v_i^2 = 1 \quad \sigma_i \tau_i = \tau_i \sigma_i$$
(3PR) Three point relations, for all $|i - j| = 1$:

(3PR1) $\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j$,
(3PR2) $v_i v_j v_i = v_j v_i v_j$,
(3PR3) $v_i \sigma_j v_i = v_j \sigma_i v_j$,
(3PR4) $v_i \tau_j v_i = v_j \tau_i v_j$,
(3PR5) $\sigma_i \sigma_j \tau_i = \tau_j \sigma_i \sigma_j$.

(CR) Commuting relations: $g_i h_j = h_j g_i$ for $|i - j| \geq 2$, where $g_i, h_i \in \{\sigma_i, \tau_i, v_i\}$.

Remark 4. Let $n \geq 3$. From [12, Proposition 3.1] and [10, Theorem 4] we conclude that the groups $B_n, VB_n$ and $SG_n$ are contained in the virtual singular braid group $VSG_n$.

We may define the virtual singular braid group through braid-like diagrams as for the classical case of the Artin braid group [1], the virtual braid group [16], [18] and the singular braid group [8], [15]. Recall that a topological interpretation of braids is as isotopy classes of strands embedded in the unital cube of the euclidian space $\mathbb{R}^3$, such that they move monotonically with respect to the $z$-axis, joining $n$ marked points on opposite faces, and that an isotopy is a continuous path of boundary-fixing diffeomorphisms of the unital cube, starting at the identity and preserving monotony of the strands. Furthermore, braids can be represented also through diagrams in the unital square of the plane $\mathbb{R}^2$, such that they move monotonically with respect to the $y$-axis, joining $n$ marked points on opposite faces, and identified up to isotopy and the Reidemeister moves. A similar topological interpretation are given for virtual braid groups, singular braid groups and virtual singular braid groups. We illustrate here the case of the virtual singular braid group (see also [10] and [12]). These elements have three types of crossings, classical ($\sigma_i$), singular ($\tau_i$) and virtual ($v_i$), which correspond to the generators illustrated in Figures 1 and 2. The diagrams of virtual singular braids are identified up to isotopy and the braided version of Reidemeister-type moves described in Figures 3, 4, 5 and 6.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{diagram.png}
\caption{Two point relations, for $i = 1, \ldots, n - 1$.}
\end{figure}

2.2 Some properties of the virtual singular braid group

Given a group $G$ we define the lower central series of $G$ recursively as $\Gamma_1(G) = G$ and for $i \geq 2 \Gamma_i(G) = [\Gamma_{i-1}(G), G]$. Let $\mathcal{P}$ be any group-theoretic property. We say that a group is residually $\mathcal{P}$ if for any (non-trivial) element $g \in G$, there exists a group $H$ with the property $\mathcal{P}$ and a surjective homomorphism $\varphi: G \rightarrow H$ such that $\varphi(g) \neq 1$. It is well known that a group $G$ is residually nilpotent if and only if $\bigcap_{i \geq 1} \Gamma_i(G) = \{1\}$. Also
\[ \sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j \quad \text{and} \quad \sigma_i \sigma_j \tau_i = \tau_j \sigma_i \sigma_j \]

Figure 5: Three point relations, for \(|i - j| = 1\).

\[ \sigma_i \sigma_j v_i = v_i \sigma_j v_j \quad \text{and} \quad \sigma_i \sigma_j v_i = v_i \sigma_j v_j \]

\[ \tau_i \sigma_j v_i = v_i \tau_j v_i \quad \text{and} \quad \tau_i \sigma_j v_i = v_i \tau_j v_i \]

Figure 6: Commuting relations, \(g_i h_j = h_j g_i\) for \(|i - j| \geq 2\) and \(g_i, h_i \in \{\sigma_i, \tau_i, v_i\}\), where in the blank spaces of the figure we have any of the crossings.

recall that we said that a group \(G\) is perfect if \(G\) is equal to its commutator subgroup, i.e. \(G = \Gamma_2(G)\).

In the following result we obtain properties of the virtual singular braid group, in some steps of the proof we use the presentation given in Definition 3.

**Theorem 5.** Let \(VSG_n\) be the virtual singular braid group on \(n\) strands. The following properties holds:

1. The group \(VSG_2\) is isomorphic to \((\mathbb{Z} \oplus \mathbb{Z}) \ast \mathbb{Z}_2\).
2. The abelianization of \(VSG_n\) is isomorphic to \(\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_2\) for \(n \geq 2\).
3. If \(n \geq 3\) the group \(VSG_n\) is not residually nilpotent.
4. If \(n \geq 4\) then \(\Gamma_2(VSG_n) = \Gamma_3(VSG_n)\).
5. If \(n \geq 5\) the group \(\Gamma_2(VSG_n)\) is perfect.

**Proof.**

1. It follows immediately from the presentation of the group \(VSG_2\).
2. It is straightforward the proof of this item using the presentation of the group \(VSG_n\).
3. Since the virtual braid group \( VB_n \) is not residually nilpotent (see [5, Proposition 7(d)]) and it is contained in the virtual singular braid group (see Remark 4) then \( VSG_n \) is not residually nilpotent.

4. Let \( n \geq 4 \). Consider the following short exact sequence

\[
1 \longrightarrow \frac{\Gamma_2(VSG_n)}{\Gamma_3(VSG_n)} \longrightarrow \frac{VSG_n}{\Gamma_3(VSG_n)} \longrightarrow \frac{\pi VSG_n}{\Gamma_2(VSG_n)} \longrightarrow 1.
\]

Recall from the second item of this result that \( \frac{\Gamma_2(VSG_n)}{\Gamma_3(VSG_n)} \) is isomorphic to \( \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_2 \) with generators \( \sigma_1, \tau_1 \) and \( v_1 \), respectively. Avoiding a lot of notation we shall write an element and its respective class in a quotient group using the same symbol. The map \( \pi \) is defined, for all \( i = 1, \ldots, n - 1 \), as: \( \pi(\sigma_i) = \sigma_1, \pi(\tau_i) = \tau_1 \) and \( \pi(v_i) = v_1 \).

We note that for each \( i \in \{1, \ldots, n - 1\} \) there exists \( k_i \in \frac{\Gamma_2(VSG_n)}{\Gamma_3(VSG_n)} \) such that \( \sigma_i = k_i \sigma_1 \), with \( k_1 = 1 \). Hence, considering the relation (3PR1) in the presentation of \( VSG_n \) we get \( k_i \sigma_1 k_{i+1} \sigma_1 k_1 = k_{i+1} \sigma_1 k_i \sigma_1 k_1 \). Since the elements \( k_i \) are central in \( \frac{\Gamma_2(VSG_n)}{\Gamma_3(VSG_n)} \) then we conclude that \( k_i = k_{i+1} \) for all \( i \in \{1, \ldots, n - 2\} \). It follows that \( \sigma_1 = \sigma_2 = \cdots = \sigma_{n-1} \), because \( k_1 = 1 \). Applying the same argument, using relation (3PR2), we prove that \( v_1 = v_2 = \cdots = v_{n-1} \).

In a similar way, for each \( i \in \{1, \ldots, n - 1\} \) there exists \( l_i \in \frac{\Gamma_2(VSG_n)}{\Gamma_3(VSG_n)} \) such that \( \tau_i = l_i \tau_1 \), with \( l_1 = 1 \). Therefore, from relation (3PR5) in the presentation of \( VSG_n \) we obtain \( \sigma_i^2 \tau_1 = l_{i+1} \tau_1 \sigma_i^2 \), using the fact that \( \sigma_1 = \sigma_2 = \cdots = \sigma_{n-1} \) in \( \frac{\Gamma_2(VSG_n)}{\Gamma_3(VSG_n)} \). As before, the elements \( l_i \) are central in \( \frac{\Gamma_2(VSG_n)}{\Gamma_3(VSG_n)} \) and so \( l_i = l_{i+1} = 1 \) for all \( i \in \{1, \ldots, n - 2\} \). Hence \( \tau_1 = \tau_2 = \cdots = \tau_{n-1} \).

Finally, from relations (CR) in the presentation of \( VSG_n \) we have \( gh = hg \) for \( g, h \in \{\sigma_1, \tau_1, v_1\} \) in \( \frac{VSG_n}{\Gamma_3(VSG_n)} \). Then, for \( n \geq 4 \), \( \pi \) is an isomorphism and \( \Gamma_2(VSG_n) = \Gamma_3(VSG_n) \).

5. From [14, Theorem 1.1] the singular braid group \( \Gamma_2(SG_n) \) is perfect for \( n \geq 5 \). It is well known that \( \Gamma_2(S_n) = A_n \) is perfect for \( n \geq 5 \). We note that the virtual singular braid group is generated by the singular braid group and the symmetric group, \( VSG_n = \langle SG_n, S_n \rangle \).

It is clear that \( \Gamma_2(VSG_n), \Gamma_2(VSG_n) \subseteq \Gamma_2(VSG_n) \), so we just need to verify that \( \Gamma_2(VSG_n) \subseteq \Gamma_2(VSG_n), \Gamma_2(VSG_n) \).

Let \( [u, w] \in \Gamma_2(VSG_n) \). From the well-known commutator identities

\[
[ab, c] = [a, c]b [a, b], \quad [a, bc] = [a, c][a, b]c, \quad [a, b] = [b, a]^{-1}, \quad [a^{-1}, b] = [b, a]a^{-1}
\] (2.1)

and from the equality \( VSG_n = \langle SG_n, S_n \rangle \) we can rewrite the element \( [u, w] \in \Gamma_2(VSG_n) \) as a product of commutators

\[
[x, y]^\alpha, \quad \text{where} \quad x, y \in \{\sigma_i, \tau_i, v_i \ | \ 1 \leq i < j \leq n - 1\} \quad \text{and} \quad \alpha \in VSG_n.
\]

The elements \( [\sigma_i, \sigma_j], [\tau_i, \tau_j], [\sigma_i, \tau_j], [v_i, v_j] \) belongs to \( \Gamma_2(VSG_n), \Gamma_2(VSG_n) \), for \( 1 \leq i, j \leq n - 1 \), because the groups \( SG_n \) and \( S_n \) are perfect for \( n \geq 5 \). So, it remains to show that \( [x_i, v_j] \) belongs to \( \Gamma_2(VSG_n), \Gamma_2(VSG_n) \), where \( x_i = \sigma_i, \tau_i \), and \( 1 \leq i, j \leq n - 1 \). If \( |i - j| \geq 2 \) then \( [x_i, v_j] = 1 \) where \( x_i = \sigma_i, \tau_i \).
Note that in the abelianization of the singular braid group \( SG_n / \Gamma_2(SG_n) = \mathbb{Z} \times \mathbb{Z} \) the equalities \( \sigma_i = \sigma_k \) and \( \tau_i = \tau_k \) holds, for all \( 1 \leq i, k \leq n - 1 \). Also, \( v_i = v_l \) in \( S_n / \Gamma_2(S_n) \), for all \( 1 \leq i, l \leq n - 1 \). Now, suppose that \( |i - j| \leq 1 \) and recall that \( n \geq 5 \). So, there are \( c_{i,k} \in \Gamma_2(SG_n) \) and \( d_{j,l} \in \Gamma_2(S_n) \) for \( k, l \in \{1, \ldots, n-1\} \) with \( |k - l| \geq 2 \) such that \( x_i = c_{i,k} x_k \) and \( v_j = d_{j,l} v_l \), respecting the correspondence \( x_k = \sigma_k \) if \( x_i = \sigma_i \) and \( x_k = \tau_k \) if \( x_i = \tau_i \). Therefore, using commutator identities (see equation (2.1)) we obtain

\[
[x_i, v_j] = [c_{i,k} x_k, d_{j,l} v_l] = [c_{i,k}, v_l]^{x_k} [c_{i,k}, d_{j,l}]^{x_k v_l} [x_k, d_{j,l}]^{v_l}.
\]

Since \( c_{i,k} \in \Gamma_2(SG_n) \) and \( d_{j,l} \in \Gamma_2(S_n) \) then \( [c_{i,k}, d_{j,l}]^{x_k v_l} \in \Gamma_2(VSG_n) \). Also, we note that

\[
[c_{i,k}, v_l]^{x_k} = (c_{i,k}^{-1} c_{i,k})^{x_k} \in \text{ and } [x_k, d_{j,l}]^{v_l} = ((d_{j,l})^{-1})^{x_k} d_{j,l})^{v_l}
\]

belongs to \( [\Gamma_2(VSG_n), \Gamma_2(VSG_n)] \) because \( c_{i,k} \in [\Gamma_2(SG_n), \Gamma_2(SG_n)] \) and \( d_{j,l} \in [\Gamma_2(S_n), \Gamma_2(S_n)] \), since the singular braid group \( SG_n \) and the symmetric groups are perfect groups.

With this we concluded that, for \( n \geq 5 \), the group \( \Gamma_2(VSG_n) \) is perfect.

Remark 6. The last result, and its proof, was motivated from [5, Propositions 7 and 8] and adapted for the case of the virtual singular braid group.

### 2.3 Invariants of virtual singular braids

Let \( n \geq 2 \). Let \( \text{Abmap}: VSG_n \rightarrow \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \) be the abelianization map where the first copy of \( \mathbb{Z} \) in the group \( \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \) is generated by \( \sigma_1 \), the second copy of \( \mathbb{Z} \) is generated by \( \tau_1 \) and the group \( \mathbb{Z} \times \mathbb{Z} \) is generated by \( v_1 \). Consider the following homomorphisms \( \psi: \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z} \) defined by \( \psi(x, y, z) = x + y \), \( \pi_1: \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z} \) defined by \( \pi_1(x, y, z) = x \) and \( \pi_2: \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z} \) defined by \( \pi_2(x, y, z) = y \).

For virtual singular braids we define the classical-singular exponent sum map \( \exp^{CS} = \psi \circ \text{Abmap}: VSG_n \rightarrow \mathbb{Z} \), the classical exponent sum map \( \exp^C = \pi_1 \circ \text{Abmap}: VSG_n \rightarrow \mathbb{Z} \) and the singular exponent sum map \( \exp^S = \pi_2 \circ \text{Abmap}: VSG_n \rightarrow \mathbb{Z} \).

Proposition 7. The classical-singular exponent sum, classical exponent sum and singular exponent sum maps are invariants of virtual singular braid groups.

Proof. The proof follows verifying each map in every relation of the presentation given for the group \( VSG_n \) in Definition 3.

We note that the classical exponent sum \( \exp^C \) was also defined for elements in the virtual singular braid monoid \( VSB_n \) by Cisneros and Gandolfi [12, Definition 3.4] under the name of degree of the virtual singular braid element.

Let \( \langle S \rangle^G \) denote the normal closure in \( G \) of a subset \( S \) of a group \( G \). In the following result we shall use the presentation of \( VSG_n \) given in Definition 3.

Theorem 8. The following sequences are short exact:

1. \( 1 \rightarrow \langle B_n \rangle^{VSG_n} \rightarrow VSG_n \rightarrow \mathbb{Z} \times S_n \rightarrow 1 \).
2. \(1 \rightarrow \langle \{\tau_i | 1 \leq i \leq n - 1\} \rangle_{VSG_n} \rightarrow VSG_n \rightarrow VB_n \rightarrow 1.\)

3. \(1 \rightarrow \langle \{\tau_i, v_i | 1 \leq i \leq n - 1\} \rangle_{VSG_n} \rightarrow VSG_n \rightarrow \mathbb{Z} \rightarrow 1.\) The surjective homomorphism coincides with the classical exponent sum homomorphism and \(\text{Ker}\left(\exp^C\right) = \langle \{\tau_i, v_i | 1 \leq i \leq n - 1\} \rangle_{VSG_n}.\)

4. \(1 \rightarrow \langle VB_n \rangle_{VSG_n} \rightarrow VSG_n \rightarrow \mathbb{Z} \rightarrow 1.\) In this case the projection coincides with the singular exponent sum homomorphism and \(\text{Ker}\left(\exp^S\right) = \langle VB_n \rangle_{VSG_n}.\)

5. \(1 \rightarrow \langle \{\sigma_i \tau_i^{-1}, v_i | 1 \leq i \leq n - 1\} \rangle_{VSG_n} \rightarrow VSG_n \rightarrow \mathbb{Z} \rightarrow 1.\) The surjective homomorphism coincides with the classical-singular exponent sum homomorphism and \(\text{Ker}\left(\exp^{CS}\right) = \langle \{\sigma_i \tau_i^{-1}, v_i | 1 \leq i \leq n - 1\} \rangle_{VSG_n}.\)

Proof. We shall analyze the group \(VSG_n/\langle B_n \rangle_{VSG_n}\) taking the presentation of \(VSG_n\) given in Definition 3 and considering the additional relations \(\sigma_i = 1\) for \(i = 1, \ldots, n - 1\) coming from the presentation of \(B_n\) given in Definition 1. From relation (3PR5) of \(VSG_n\) follows the equality \(\tau_i = \tau_{i+1}\) in \(VSG_n/\langle B_n \rangle_{VSG_n}\), for \(i = 1, \ldots, n - 1\). From the commuting relations (CR) follows that in \(VSG_n/\langle B_n \rangle_{VSG_n}\) the element \(\tau_1\) commutes with every generator of \(S_n\). Hence, \(VSG_n/\langle B_n \rangle_{VSG_n} = \langle \tau_1 \rangle \times S_n = \mathbb{Z} \times S_n\) and the result of the first item follows.

The proof of the second item is immediate from the presentation of \(VSG_n\) when taking the additional relations \(\tau_i = 1\), for \(i = 1, \ldots, n - 1\), and comparing with the presentation of the virtual braid group \(VB_n\) given in Definition 1.

The third, fourth and fifth items have similar proofs. We only verify the fourth item, the proof of the others are analogous. We analyze the group \(VSG_n/\langle VB_n \rangle_{VSG_n}\) taking the presentation of \(VSG_n\) given in Definition 3 and considering the additional relations \(\sigma_i = 1\) and \(v_i = 1\), for \(i = 1, \ldots, n - 1\). From relation (3PR4) of \(VSG_n\) follows the equality \(\tau_i = \tau_{i+1}\) in \(VSG_n/\langle VB_n \rangle_{VSG_n}\), for \(i = 1, \ldots, n - 1\). So, \(VSG_n/\langle VB_n \rangle_{VSG_n} = \langle \tau_1 \rangle = \mathbb{Z}\) and the first part of the fourth item follows. The second part of this item is obvious from the definition of the singular exponent sum homomorphism.

2.4 Homomorphisms from \(VSG_n\) to the symmetric group \(S_m\)

Consider the presentation of the virtual singular braid group \(VSG_n\) given in Definition 3.

Definition 9. Let \(n \geq 2\) and let \(\varepsilon_k \in \{0, 1\}\), for \(k = 1, 2, 3\). Let \(\varphi_{\varepsilon_1, \varepsilon_2, \varepsilon_3}: VSG_n \rightarrow S_n\) be the map defined, for all \(i = 1, \ldots, n - 1\), by

\[\varphi_{\varepsilon_1, \varepsilon_2, \varepsilon_3}(\sigma_i) = (i \quad i + 1)^{\varepsilon_1}, \quad \varphi_{\varepsilon_1, \varepsilon_2, \varepsilon_3}(\tau_i) = (i \quad i + 1)^{\varepsilon_2}\]

and \(\varphi_{\varepsilon_1, \varepsilon_2, \varepsilon_3}(v_i) = (i \quad i + 1)^{\varepsilon_3}\).

For \(n = 2\) the map \(\varphi_{\varepsilon_1, \varepsilon_2, \varepsilon_3}: VSG_2 \rightarrow S_2\) is a homomorphism for all eight triples, but this is not the case in general as the next result shows.

Proposition 10. Let \(n \geq 3\). The map \(\varphi_{\varepsilon_1, \varepsilon_2, \varepsilon_3}: VSG_n \rightarrow S_n\) is a homomorphism if and only if \((\varepsilon_1, \varepsilon_2, \varepsilon_3)\) is equal to one of the following triples:

- \((0, 0, 0)\), and in this case \(\varphi_{0, 0, 0}\) is the trivial homomorphism.
- \((1, 1, 1)\), and in this case \(\text{Ker}(\varphi_{1,1,1})\) is equal to the virtual singular pure braid group \(VSPG_n\).
(1, 0, 1).
(0, 0, 1).

Proof. Let \( n \geq 3 \) and let \( \varepsilon_k \in \{0, 1\} \), for \( k = 1, 2, 3 \). Let \( \varphi_{\varepsilon_1, \varepsilon_2, \varepsilon_3} : VSG_n \to S_n \) the map given in Definition \ref{def:virtual_sing}. In order to prove this result first we show the triples such that \( \varphi_{\varepsilon_1, \varepsilon_2, \varepsilon_3} \) is not a homomorphism.

- \( (\varepsilon_1, \varepsilon_2, \varepsilon_3) = (1, 1, 0) \): The map \( \varphi_{\varepsilon_1, \varepsilon_2, \varepsilon_3} \) is not a homomorphism since the relation \( v_1 \tau_2 v_1 = v_2 \tau_1 v_2 \) of \( VSG_n \) is not preserved under the map, once we get in \( S_n \) \( \varphi_{\varepsilon_1, \varepsilon_2, \varepsilon_3}(v_1 \tau_2 v_1) = (2 \ 3) \) that is different from \( \varphi_{\varepsilon_1, \varepsilon_2, \varepsilon_3}(v_2 \tau_1 v_2) = (1 \ 2) \).
- \( (\varepsilon_1, \varepsilon_2, \varepsilon_3) = (1, 0, 0) \): In this case one relation that is not preserved is \( v_1 \sigma_2 v_1 = v_2 \sigma_1 v_2 \).
- \( (\varepsilon_1, \varepsilon_2, \varepsilon_3) = (0, 1, 1) \): One relation that is not preserved is \( \sigma_1 \sigma_2 \tau_1 = \tau_2 \sigma_1 \sigma_2 \).
- \( (\varepsilon_1, \varepsilon_2, \varepsilon_3) = (0, 1, 0) \): For this item we may take the same relation as in the last item \( \sigma_1 \sigma_2 \tau_1 = \tau_2 \sigma_1 \sigma_2 \) and verify easily that \( \varphi_{\varepsilon_1, \varepsilon_2, \varepsilon_3} \) is not a homomorphism.

Now we need to check that for the other four triples the map \( \varphi_{\varepsilon_1, \varepsilon_2, \varepsilon_3} \) is a homomorphism. It is clear for \( (\varepsilon_1, \varepsilon_2, \varepsilon_3) \) equal to \( (0, 0, 0) \) since we get the trivial map, and also for \( (\varepsilon_1, \varepsilon_2, \varepsilon_3) \) equal to \( (1, 1, 1) \) since the map \( \varphi_{1,1,1} : VSG_n \to S_n \) is exactly the homomorphism \( \pi \) defined in \cite[Page 6]{R} and its kernel is the so-called virtual singular pure braid group, see \cite[Definition 5]{R}. Finally, that \( \varphi_{\varepsilon_1, \varepsilon_2, \varepsilon_3} \) is a homomorphism for \( (\varepsilon_1, \varepsilon_2, \varepsilon_3) \) equal to \( (1, 0, 1) \) or \( (0, 0, 1) \) follows from a simple computation using the canonical presentations of the groups involved.

From Proposition \ref{prop:sk_n} the maps \( \varphi_{1,0,1} \) and \( \varphi_{0,0,1} \) are also homomorphisms for \( n \geq 3 \). We shall denote the kernel of \( \varphi_{1,0,1} \) by \( VST_n \) and the kernel of \( \varphi_{0,0,1} \) by \( VSK_n \).

Proposition 11. Let \( n \geq 2 \). There are decompositions of the virtual singular braid group as semi-direct products:

- \( VSG_n = VSPG_n \rtimes S_n \),
- \( VSG_n = VST_n \rtimes S_n \), and
- \( VSG_n = VSK_n \rtimes S_n \).

Proof. Let \( n \geq 2 \). Let \( (\varepsilon_1, \varepsilon_2, \varepsilon_3) \) one of the following triples \( (1, 1, 1) \), \( (1, 0, 1) \) or \( (0, 0, 1) \). The following short exact sequence

\[
1 \to \text{Ker} (\varphi_{\varepsilon_1, \varepsilon_2, \varepsilon_3}) \to VSG_n \overset{\varphi_{\varepsilon_1, \varepsilon_2, \varepsilon_3}}\to S_n \to 1
\]

admits a natural section \( \iota : S_n \to VSG_n \) defined by \( \iota((i \ i + 1)) = v_i \) for every \( i = 1, \ldots, n - 1 \), obtaining the desired result.

Remark 12. The decomposition \( VSG_n = VSPG_n \rtimes S_n \) was already proved in \cite[Corollary 13]{R}.
In order to state the next theorem first we recall the following definitions. Let $G, H$ be two groups. For every $h \in H$ we have the group homomorphism $c_h : H \to H$, defined by $c_h(x) = hxh^{-1}$. We say that two homomorphisms $\psi_1, \psi_2 : G \to H$ are conjugate if there exists an element $h \in H$ such that $\psi_2 = c_h \circ \psi_1$, which means that $\psi_2(g) = h\psi_1(g)h^{-1}$, for every $g \in G$. We say that a group homomorphism $\psi : G \to H$ is abelian if its image $\psi(G)$ is an abelian subgroup of $H$. In the following result we determine all possible homomorphisms, up to conjugation, from the virtual singular braid group $VSG_n$ to the symmetric group $S_m$. It is similar to [7, Theorem 2.1] for virtual braid groups.

**Theorem 13.** Let $n, m$ such that $n \geq 5$, $m \geq 2$ and $n \geq m$. Let $\psi : VSG_n \to S_m$ be a homomorphism. Then, up to conjugation, one of the following possibilities holds

1. $\psi$ is abelian,
2. $n = m$ and $\psi \in \{\varphi_{1,1,1}, \varphi_{1,0,1}, \varphi_{0,0,1}\}$;
3. $n = m = 6$ and $\psi \in \{\nu_6 \circ \varphi_{1,1,1}, \nu_6 \circ \varphi_{1,0,1}, \nu_6 \circ \varphi_{0,0,1}\}$.

**Proof.** This proof, in the case of the virtual singular braid group, is easily adapted from the one given for virtual braid groups in [7, Theorem 2.1].

### 2.5 Kernel of $\varphi_{\varepsilon_1, \varepsilon_2, \varepsilon_3} : VSG_2 \to S_2$

Let $\varepsilon_k \in \{0, 1\}$, for $k = 1, 2, 3$. Let $\varphi_{\varepsilon_1, \varepsilon_2, \varepsilon_3} : VSG_n \to S_n$ be the map given in Definition 9. From [10, Definition 5] the kernel of $\varphi_{1,1,1}$ is called the virtual singular pure braid group, denoted by $VSPG_n$. We know from Proposition 10 that from $n \geq 3$ the maps $\varphi_{1,0,1}$ and $\varphi_{0,0,1}$ are also homomorphisms and its kernels are denoted by $VST_n$ and $VSK_n$, respectively. In the next result we give presentations for the kernel of $\varphi_{\varepsilon_1, \varepsilon_2, \varepsilon_3} : VSG_2 \to S_2$.

We shall use the following notations for some elements in the group $VSG_2$:

$$a_{1,2} = \sigma_1 v_1, \quad b_{1,2} = \tau_1 v_1, \quad c_{1,2} = v_1 \sigma_1, \quad d_{1,2} = v_1 \tau_1, \quad e_{1,2} = \sigma_1 \tau_1, \quad f_{1,2} = \sigma_1^{-1} \tau_1$$

and

$$A_{1,2} = \sigma_1^2, \quad a = \sigma_1 v_1 \sigma_1^{-1}, \quad b = \tau_1 v_1 \tau_1^{-1}, \quad c = v_1 \sigma_1 v_1^{-1}, \quad d = v_1 \tau_1 v_1^{-1}.$$

**Theorem 14.** Let $\varepsilon_k \in \{0, 1\}$, for $k = 1, 2, 3$. The kernel of $\varphi_{\varepsilon_1, \varepsilon_2, \varepsilon_3} : VSG_2 \to S_2$ admits the following presentation, depending on the triples $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$:

\[ (1,1,1) : \text{Ker}(\varphi_{1,1,1}) = VSPG_2 \text{ has generators } a_{1,2}, b_{1,2} \text{ and } c_{1,2} \text{ subject to the relation } a_{1,2} c_{1,2} b_{1,2} = b_{1,2} c_{1,2} a_{1,2}. \]

\[ (1,1,0) : \text{Ker}(\varphi_{1,1,0}) \text{ has generators } A_{1,2}, e_{1,2}, v_1 \text{ and a subject to the relations } v_1^2 = 1 \text{ and } a^2 = 1. \]

\[ (1,0,1) : \text{Ker}(\varphi_{1,0,1}) = VST_2 \text{ has generators } a_{1,2}, c_{1,2} \text{ and } \tau_1 \text{ subject to the relation } a_{1,2} c_{1,2} \tau_1 = \tau_1 a_{1,2} c_{1,2}. \]

\[ (1,0,0) : \text{Ker}(\varphi_{1,0,0}) \text{ has generators } A_{1,2}, \tau_1, v_1 \text{ and a subject to the relations } v_1^2 = 1, a^2 = 1 \text{ and } A_{1,2} \tau_1 = \tau_1 A_{1,2}. \]

\[ (0,1,1) : \text{Ker}(\varphi_{0,1,1}) \text{ has generators } b_{1,2}, d_{1,2} \text{ and } \sigma_1 \text{ subject to the relation } b_{1,2} d_{1,2} \sigma_1 = \sigma_1 b_{1,2} d_{1,2}. \]
(0,1,0): \(\text{Ker}(\varphi_{(0,1,0)})\) has generators \(\sigma_1, \tau_1^2, v_1\) and \(b\) subject to the relations \(v_1^2 = 1, b^2 = 1\) and \(\sigma_1 \tau_1^2 = \tau_1^2 \sigma_1\).

(0,0,1): \(\text{Ker}(\varphi_{(0,0,1)}) = VSK_2\) has generators \(\sigma_1, \tau_1, c\) and \(d\) subject to the relations \(\sigma_1 \tau_1 = \tau_1 \sigma_1\) and \(cd = dc\).

(0,0,0): The same presentation as \(VSG_2\).

\textit{Proof.} The proof follows applying the Reidemeister-Schreier process (see \cite[Section 2.3]{Ker}).

In this case we considered the Schreier set of coset representatives for \(\text{Ker}(\varphi_{(1,1,1)})\). We shall write the details of the process for the presentation of the group \(\text{Ker}(\varphi_{(1,1,1)}) = VSPG_2\) and for the other cases we just mention few steps used in the algorithm.

The first general step is given considering a set \(\Lambda\) of coset representatives for the kernel of \(\varphi_{\varepsilon_1,\varepsilon_2,\varepsilon_3}\) in \(VSG_2\), so following the Reidemeister-Schreier process, the group \(\text{Ker}(\varphi_{\varepsilon_1,\varepsilon_2,\varepsilon_3})\) is generated by the set:

\[
\{S_{\lambda,a} = (\lambda a)(\overline{\lambda a})^{-1} \mid \lambda \in \Lambda, a \in \{\sigma_1, \tau_1, v_1\}\}.
\]

(1,1,1): Consider the Schreier set of coset representatives for \(VSPG_2\) in \(VSG_2\):

\[
\Lambda = \{1, v_1\}.
\]

Hence, we obtain the following generators for \(VSPG_2\):

- \(a_{1,2}: S_{1,\sigma_1} = \sigma_1 v_1^{-1}\)
- \(d_{1,2}: S_{\sigma_1,\tau_1} = v_1 \tau_1\)
- \(b_{1,2}: S_{1,\tau_1} = \tau_1 v_1^{-1}\)
- \(c_{1,2}: S_{v_1,\sigma_1} = v_1 \sigma_1\)
- \(v_1^2: S_{v_1, v_1} = v_1^2\)

Find the set of relations. From relation \(r_1 = v_1^2\) of \(VSG_2\) we have that \(S_{v_1, v_1} = 1\). Now consider the relation \(r_2 = \sigma_1 \tau_1 \sigma_1^{-1} \tau_1^{-1}\) of \(VSG_2\). We obtain:

\[
r_{2,1} = \sigma_1 \tau_1 \sigma_1^{-1} \tau_1^{-1} = \sigma_1 (v_1^{-1} v_1) \tau_1 \sigma_1^{-1} (v_1^{-1} v_1) \tau_1^{-1} = a_{1,2} d_{1,2} c_{1,2}^{-1} b_{1,2}^{-1}
\]

and

\[
r_{2,2} = v_1 \sigma_1 \tau_1 \sigma_1^{-1} \tau_1^{-1} v_1^{-1} = v_1 \sigma_1 \tau_1 (v_1^{-1} v_1) \sigma_1^{-1} \tau_1^{-1} v_1^{-1} = c_{1,2} b_{1,2} a_{1,2}^{-1} d_{1,2}^{-1}.
\]

From this we get

\[
c_{1,2} b_{1,2} a_{1,2}^{-1} = d_{1,2} = a_{1,2}^{-1} b_{1,2} c_{1,2}.
\]

So, \(\text{Ker}(\varphi_{(1,1,1)}) = VSPG_2\) has generators \(a_{1,2}, b_{1,2}\) and \(c_{1,2}\) subject to the relation \(a_{1,2} c_{1,2} b_{1,2} = b_{1,2} c_{1,2} a_{1,2}\).

(1,1,0): In this case we considered the Schreier set of coset representatives for \(\text{Ker}(\varphi_{(1,1,0)})\) in \(VSG_2\):

\[
\Lambda = \{1, \sigma_1\}.
\]

So, we get the following generators for \(\text{Ker}(\varphi_{(1,1,0)})\), after a simplification using the relations obtained in the next step of the process:
\[ A_{1,2} : S_{\sigma_1, \sigma_1} = \sigma_1^2 \quad v_1 : S_{1,v_1} = v_1 \]
\[ e_{1,2} : S_{\sigma_1, \tau_1} = \sigma_1 \tau_1 \]

In this case, and the others below, we find the defining relations from a routine computation.

(1,0,1): We considered the Schreier set of coset representatives for \( Ker(\varphi_{(1,0,1)}) \) in \( VSG_2 \):
\[ \Lambda = \{1, v_1\}. \]

So, we get the following generators for \( Ker(\varphi_{(1,0,1)}) \), after a simplification using the relations obtained in the next step of the process:
\[ a_{1,2} : S_{1,\sigma_1} = \sigma_1 v_1^{-1} \quad \tau_1 : S_{1,\tau_1} = \tau_1 \]
\[ c_{1,2} : S_{v_1,\sigma_1} = v_1 \sigma_1 \]

(1,0,0): In this case we considered the Schreier set of coset representatives for \( Ker(\varphi_{(1,0,0)}) \) in \( VSG_2 \):
\[ \Lambda = \{1, \sigma_1\}. \]

So, we get the following generators for \( Ker(\varphi_{(1,0,0)}) \), after a simplification using the relations obtained in the next step of the process:
\[ A_{1,2} : S_{\sigma_1, \sigma_1} = \sigma_1^2 \quad v_1 : S_{1,v_1} = v_1 \]
\[ \tau_1 : S_{1,\tau_1} = \tau_1 \]
\[ a : S_{\sigma_1, v_1} = \sigma_1 v_1 \sigma_1^{-1} \]

(0,1,1): We considered the Schreier set of coset representatives for \( Ker(\varphi_{(0,1,1)}) \) in \( VSG_2 \):
\[ \Lambda = \{1, v_1\}. \]

Generators for \( Ker(\varphi_{(0,1,1)}) \), after a simplification using the relations obtained in the next step of the process:
\[ b_{1,2} : S_{1,\tau_1} = \tau_1 v_1^{-1} \quad \sigma_1 : S_{1,\sigma_1} = \sigma_1 \]
\[ d_{1,2} : S_{v_1,\tau_1} = v_1 \tau_1 \]

(0,1,0): We considered the Schreier set of coset representatives for \( Ker(\varphi_{(0,1,0)}) \) in \( VSG_2 \):
\[ \Lambda = \{1, \tau_1\} \]

and we get the following generators for \( Ker(\varphi_{(0,1,0)}) \), after a simplification using the relations obtained in the next step of the process:
\[ \sigma_1 : S_{1,\sigma_1} = \sigma_1 \quad v_1 : S_{1,v_1} = v_1 \]
\[ \tau_1^2 : S_{\tau_1, \tau_1} = \tau_1^2 \]
\[ b : S_{\tau_1, v_1} = \tau_1 v_1 \tau_1^{-1} \]

(0,0,1): In this case we considered the Schreier set of coset representatives for \( Ker(\varphi_{(0,0,1)}) \) in \( VSG_2 \):
\[ \Lambda = \{1, v_1\}. \]

and after a simplification using the relations obtained in the next step of the process, we get the following generators for \( Ker(\varphi_{(0,0,1)}) \):
\[
\sigma_1: S_{1, \sigma_1} = \sigma_1 \\
\tau_1: S_{1, \tau_1} = \tau_1 \\
c: S_{v_1, \sigma_1} = v_1 \sigma_1 v_1^{-1} \\
d: S_{v_1, \tau_1} = v_1 \tau_1 v_1^{-1}
\]

(0,0,0): This case is obvious.

**Remark 15.** We note that the presentation given here for \(VSPG_2\) is slightly different from the one given in \([10, Theorem 14]\).

We recall that in \([6, Section 5]\) was studied the flat virtual pure braid group \(FVP_n\). The particular case of \(n = 3\) was considered in \([6, Remark 5.2]\). In the next result we obtain an algebraic description of the kernel of \(\varphi_{\epsilon_1, \epsilon_2, \epsilon_3}: VSG_2 \to S_2\). In particular, the virtual singular pure braid group \(VSPG_2\) is isomorphic to \(FVP_3\).

**Corollary 16.** Let \(\epsilon_k \in \{0, 1\}\), for \(k = 1, 2, 3\). The kernel of \(\varphi_{\epsilon_1, \epsilon_2, \epsilon_3}: VSG_2 \to S_2\) has the following properties, depending on the triples \((\epsilon_1, \epsilon_2, \epsilon_3)\):

\[(1,1,1): \text{Ker}(\varphi_{(1,1,1)}) = VSPG_2\text{ is the HNN-extension of the free group of rank two generated by } b_{1,2}, c_{1,2}\text{ with stable element } a_{1,2}\text{ and with associated subgroups } A = \langle c_{1,2} b_{1,2} \rangle\text{ and } B = \langle b_{1,2} c_{1,2} \rangle, \text{ which are isomorphic to the infinite cyclic group. Furthermore, } VSPG_2 \text{ is isomorphic to the flat virtual pure braid group } FVP_3\text{ and as a consequence it is isomorphic to the free product } \mathbb{Z}^2 \ast \mathbb{Z}.

(1,1,0): The group \(\text{Ker}(\varphi_{(1,1,0)})\) is isomorphic to the free product \(F_2 \ast \mathbb{Z}_2 \ast \mathbb{Z}_2\), where \(F_2\) is a free group of rank 2.

(1,0,1): The kernel \(\text{Ker}(\varphi_{(1,0,1)}) = VST_2\text{ is isomorphic to the virtual singular pure braid group } VSPG_2\).

(1,0,0): The group \(\text{Ker}(\varphi_{(1,0,0)})\) is isomorphic to the free product \(\mathbb{Z}^2 \ast \mathbb{Z}_2 \ast \mathbb{Z}_2\).

(0,1,1): \(\text{Ker}(\varphi_{(0,1,1)})\) is isomorphic to the virtual singular pure braid group \(VSPG_2\).

(0,1,0): \(\text{Ker}(\varphi_{(0,1,0)})\) is isomorphic to the group \(\text{Ker}(\varphi_{(1,0,0)})\).

(0,0,1): The group \(\text{Ker}(\varphi_{(0,0,1)}) = VSK_2\text{ is a right-angled Artin Group (RAAG) and it is isomorphic to the free product } \mathbb{Z}^2 \ast \mathbb{Z}^2\).

(0,0,0): The virtual singular braid group with two strings \(VSG_2\) is isomorphic to \(\mathbb{Z}^2 \ast \mathbb{Z}_2\).

**Proof.** The proof of this result follows from a routine computation using the presentations given in Theorem 14.

**Question 17.** It was proved in Corollary 16 that \(VST_2\) is isomorphic to \(VSP_2\). It is natural to ask if in general is true that \(VST_n\) is isomorphic to \(VSP_n\).

**Corollary 18.** Let \(\epsilon_k \in \{0, 1\}\), for \(k = 1, 2, 3\). For any triple \((\epsilon_1, \epsilon_2, \epsilon_3)\) the kernel of \(\varphi_{\epsilon_1, \epsilon_2, \epsilon_3}: VSG_2 \to S_2\) has trivial center. In particular, it holds for \(VSG_2\).

**Proof.** The results follows immediately since free products of groups have trivial center.
3 Some quotients of virtual singular braid groups

Many quotients of virtual braid groups are very interesting, in particular the welded braid group $WB_n$, the unrestricted virtual braid group $UVB_n$, the flat virtual braid group $FVB_n$, the flat welded braid groups $FWB_n$, and the Gauss virtual braid group $GVB_n$. For more details about these groups and its properties see [6], [13],[16] and [17].

As doing for virtual braid groups it is very natural to define similar quotients for virtual singular braid groups. First, we show that there are some relations in the virtual singular braid group $VSG_n$ that are not allowed.

Theorem 19. For $n \geq 3$ and for every $i = 1, \ldots, n - 2$, the relations

1. $v_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i v_{i+1}$,
2. $v_i \sigma_i \sigma_{i+1} = \sigma_i \sigma_{i+1} v_i$,
3. $v_i \tau_i \tau_i = \tau_i \tau_i v_{i+1}$, and
4. $v_i \tau_i \tau_i = \tau_i \tau_i v_i$

are forbidden relations in the virtual singular braid group $VSG_n$.

Proof. From Theorem 8 the map $\eta: VSG_n \rightarrow VB_n$, defined by $\eta(\sigma_i) = \sigma_i, \eta(\tau_i) = e, \eta(v_i) = v_i$ for all $i = 1, \ldots, n - 1$ is a surjective homomorphism.

Let $n \geq 3$. Since, for every $i = 1, \ldots, n - 2$, $v_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i v_{i+1}$ and $v_i \sigma_i \sigma_{i+1} = \sigma_i \sigma_{i+1} v_i$ are forbidden relations in the virtual braid group $VB_n$ (see [17]) and $\eta: VSG_n \rightarrow VB_n$ is an epimorphism then follows that these are also forbidden relations in the group $VSG_n$.

Now we prove that the last two relations are also forbidden in $VSG_n$. From Theorem 8 the map $\psi: VSG_n \rightarrow \mathbb{Z} \times S_n$, defined by $\eta(\sigma_i) = e, \eta(\tau_i) = \tau, \eta(v_i) = v_i$ for all $i = 1, \ldots, n - 1$ is a surjective homomorphism. Hence, if the relations $v_i \tau_i \tau_i = \tau_i \tau_i v_{i+1}$ and $v_i \tau_i \tau_i = \tau_i \tau_i v_i$ holds in $VSG_n$ then it implies that, for all $i = 1, \ldots, n - 1$, in $\mathbb{Z} \times S_n$, the equality $v_i \tau^2 = \tau^2 v_i$ is valid, i.e. $v_i = v_i$ for all $i = 1, \ldots, n - 1$ in $S_n$, that is a contradiction. Then $v_i \tau_i \tau_i = \tau_i \tau_i v_{i+1}$ and $v_i \tau_i \tau_i = \tau_i \tau_i v_i$ are forbidden relations in $VSG_n$.

Motivated from Theorem 19 and some natural quotients of the virtual braid groups $VB_n$ (see [6], [13], [17]) we introduce the following definitions.

Definition 20. The classical welded singular braid group, denoted by $W^C S_n$, is the quotient of $VSG_n$ obtained by adding the relations $v_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i v_{i+1}$, for $i = 1, \ldots, n - 2$. The classical unrestricted virtual singular braid group, denoted by $U^C VSG_n$, is the quotient of $W^C S_n$ obtained by adding the relations $v_i \sigma_i \sigma_{i+1} = \sigma_i \sigma_{i+1} v_i$, for $i = 1, \ldots, n - 2$.

The welded singular braid group, denoted by $WS_n$, is the quotient of $VSG_n$ obtained by adding the relations $v_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i v_{i+1}$ and $v_i \tau_i \tau_i = \tau_i \tau_i v_{i+1}$, for $i = 1, \ldots, n - 2$. The unrestricted virtual singular braid group, denoted by $UVSG_n$, is the quotient of $WS_n$ obtained by adding the relations $v_i \sigma_i \sigma_{i+1} = \sigma_i \sigma_{i+1} v_i$ and $v_i \tau_i \tau_i = \tau_i \tau_i v_i$, for $i = 1, \ldots, n - 2$.

Remark 21. We note that $VSG_2 = W^C S_2 = WSG_2 = U^C VSG_2 = UVSG_2$. 

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Definition 22. The classical flat virtual singular braid group, denoted by $F^C VSG_n$, is the quotient of $VSG_n$ obtained by adding the relations $\sigma_i^2 = 1$, for $i = 1, \ldots, n-1$. The classical flat welded singular braid group (resp. flat welded singular braid group), denoted by $F^C WSG_n$ (resp. $FWSG_n$), is the quotient of $W^C SG_n$ (resp. $WSG_n$) obtained by adding the relations $\sigma_i^2 = 1$, for $i = 1, \ldots, n-1$. The classical Gauss virtual singular braid group, denoted by $G^C VSG_n$, is the quotient of $F^C VSG_n$ obtained by adding the relations $\sigma_i v_i = v_i \sigma_i$, for $i = 1, \ldots, n-1$.

We summarize the last definitions in the diagram given in Figure 7 where the arrows

![Diagram](image-url)

**Figure 7: Quotients of virtual singular braid groups**

are quotient homomorphisms obtained by adding relations according to the enumeration

1. $v_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i v_{i+1}$, for $i = 1, \ldots, n-2$;
2. $v_{i+1} \sigma_i \sigma_{i+1} = \sigma_i \sigma_{i+1} v_i$, for $i = 1, \ldots, n-2$;
3. $v_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i v_{i+1}$, for $i = 1, \ldots, n-2$;
4. $v_{i+1} \tau_i \tau_{i+1} = \tau_i \tau_{i+1} v_i$, for $i = 1, \ldots, n-2$;
5. $\sigma_i^2 = 1$, for $i = 1, \ldots, n-1$;
6. $\sigma_i v_i = v_i \sigma_i$, for $i = 1, \ldots, n-1$.

Now, we state some results related to these quotients, whose proofs are similar to the ones given for the virtual singular braid group in each case. The proof of the following result is similar to the one given for Theorem 8, so we omit it here.

**Theorem 23.** Let $B_n$ (resp. $P_n$) denote the Artin braids group (resp. the pure Artin braid group). The following sequences are short exact:

1. $1 \rightarrow \langle B_n \rangle^X \rightarrow X \rightarrow \mathbb{Z} \times \mathbb{Z}_2 \rightarrow 1$, where $X$ is one of the groups: $W^C SG_n$, $WSG_n$, $UC VSG_n$ or $UVSG_n$.

2. $1 \rightarrow \langle P_n \rangle^{VSG_n} \rightarrow VSG_n \rightarrow F^C VSG_n \rightarrow 1$,
   $1 \rightarrow \langle P_n \rangle^{W^C SG_n} \rightarrow W^C SG_n \rightarrow FW^C SG_n \rightarrow 1$ and
   $1 \rightarrow \langle P_n \rangle^{WSG_n} \rightarrow WSG_n \rightarrow FWSG_n \rightarrow 1$. 

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3. \( 1 \rightarrow \langle \{ \tau_i \mid 1 \leq i \leq n-1 \} \rangle_{W^C SG_n} \rightarrow W^C SG_n \rightarrow WB_n \rightarrow 1 \) and 
\( 1 \rightarrow \langle \{ \tau_i \mid 1 \leq i \leq n-1 \} \rangle_{U^C VSG_n} \rightarrow U^C VSG_n \rightarrow U VB_n \rightarrow 1. \)

Also, we would like to analyze homomorphisms from the groups given in Figure 7 into the symmetric group.

**Definition 24.** Let \( n \geq 2 \) and let \( \varepsilon_k \in \{0,1\} \), for \( k = 1, 2, 3 \). Let \( \varphi^n_{X_n} : X_n \rightarrow S_n \) be the map defined, for all \( i = 1, \ldots, n-1 \), by

\[
\varphi^n_{X_n, \varepsilon_1, \varepsilon_2, \varepsilon_3} (\tau_i) = (i \ i + 1)^{\varepsilon_1}, \quad \varphi^n_{X_n, \varepsilon_1, \varepsilon_2, \varepsilon_3} (\sigma_i) = (i \ i + 1)^{\varepsilon_2} \text{ and } \varphi^n_{X_n, \varepsilon_1, \varepsilon_2, \varepsilon_3} (v_i) = (i \ i + 1)^{\varepsilon_3}.
\]

where \( X_n \) is one of the nine groups given in Figure 7.

It is clear that \( \varphi^n_{0,0,0} : X_n \rightarrow S_n \) is a homomorphism (in fact it is the trivial one). The next result has a proof similar to the one given for Proposition 10.

**Proposition 25.** Let \( n \geq 3 \). Let \( X_n \) be one of the nine groups given in Figure 7.

1. The map \( \varphi^n_{1,1,1} : X_n \rightarrow S_n \) is a surjective homomorphism for every \( X_n \). Furthermore, the kernel of \( \varphi^n_{1,1,1} \) is the pure respective group in each case and \( X_n \) admits a semi-direct product decomposition \( X_n = \text{Ker}(\varphi^n_{1,1,1}) \rtimes S_n \).

2. The map \( \varphi^n_{1,0,1} : X_n \rightarrow S_n \) is a surjective homomorphism if and only if \( X_n \) is different from \( WSG_n, FWSG_n \) and \( UVSG_n \). Furthermore, there is a semi-direct product decomposition \( X_n = \text{Ker}(\varphi^n_{1,0,1}) \rtimes S_n \).

3. The map \( \varphi^n_{0,0,1} : X_n \rightarrow S_n \) is a surjective homomorphism if and only if \( X_n \) is one of the groups \( VSG_n, FC VSG_n \) or \( GC VSG_n \). Furthermore, there is a semi-direct product decomposition \( X_n = \text{Ker}(\varphi^n_{0,0,1}) \rtimes S_n \).

4. For the other triples \( (\varepsilon_1, \varepsilon_2, \varepsilon_3) \) not considered in the items above, with \( \varepsilon_k \in \{0,1\} \) for \( k = 1, 2, 3 \), the map \( \varphi^n_{X_n} : X_n \rightarrow S_n \) is not a homomorphism.

We end this section with a formulation of Theorem 13 for the quotients of \( VSG_n \) considered in this paper, its proof is also similar to the one given for [7, Theorem 2.1].

**Theorem 26.** Let \( n, m \) such that \( n \geq 5, m \geq 2 \) and \( n \geq m \).

1. Let \( X_n \) be one of the groups \( VSG_n, FC VSG_n \) or \( GC VSG_n \). Let \( \psi^n_{X_n} : X_n \rightarrow S_m \) be a homomorphism. Then, up to conjugation, one of the following possibilities holds

   (a) \( \psi^n_{X_n} \) is abelian,
   (b) \( n = m \) and \( \psi^n_{X_n} \in \{ \varphi^n_{1,1,1}, \varphi^n_{1,0,1}, \varphi^n_{0,0,1} \} \),
   (c) \( n = m = 6 \) and \( \psi^n_{X_n} \in \{ \nu_6 \circ \varphi^n_{1,1,1}, \nu_6 \circ \varphi^n_{1,0,1}, \nu_6 \circ \varphi^n_{0,0,1} \} \).

2. Let \( X_n \) be one of the groups \( W^C SG_n, FC WSG_n \) or \( UC VSG_n \). Let \( \psi^n_{X_n} : X_n \rightarrow S_m \) be a homomorphism. Then, up to conjugation, one of the following possibilities holds

   (a) \( \psi^n_{X_n} \) is abelian,
   (b) \( n = m \) and \( \psi^n_{X_n} \in \{ \varphi^n_{1,1,1}, \varphi^n_{1,0,1} \} \),
(c) \( n = m = 6 \) and \( \psi^X_n \in \{ \nu_6 \circ \varphi_{1,1,1}^X, \nu_6 \circ \varphi_{1,0,1}^X \} \).

3. Let \( X_n \) be one of the groups \( \text{WSG}_n \), \( \text{FWSG}_n \) or \( \text{UVSG}_n \). Let \( \psi^X_n : X_n \rightarrow S_m \) be a homomorphism. Then, up to conjugation, one of the following possibilities holds

(a) \( \psi^X_n \) is abelian,

(b) \( n = m \) and \( \psi^X_n = \varphi_{1,1,1}^X \),

(c) \( n = m = 6 \) and \( \psi^X_n = \nu_6 \circ \varphi_{1,1,1}^X \).

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