A CERTAIN ESTIMATE OF VOLATILITY THROUGH RETURN FOR STOCHASTIC VOLATILITY MODELS

M. A. MARTYNOV, O. S. ROZANOVA

Аннотация. Мы исследуем зависимость волатильности от цен акций в стохастической модели волатильности на примере модели Гестона. Для более точной формулировки, мы рассмотрели условные ожидания волатильности (квадрат волатильности) при фиксированном значении изменения цен акций как функция изменения цен акций и времени. Величина этой функции зависит от начального распределения волатильности. В частности, мы показали, что график условного ожидания волатильности является выпуклым вниз около среднего значения цены акций. Для нормального распределения этот эффект силен, но он ослабевает и становится незначительным при медленном убывании распределения при неограниченном значении.

1. Introduction

Статистическая волатильность (СВ) модели являются популярной в последние десятилетия из-за необходимости качественного анализа финансовых данных. Наиболее популярными являются модели Гестона [11], Stein-Stein [20], Schöble-Zhu [18], Hull-White [12] и Scott [19]. Мы рекомендуем для обзоров [13], [14], [7]. Основная причина использования СВ моделей в том, чтобы найти реалистичный альтернативный подход к оценке опционов, учитывая изменение волатильности, которая считалась постоянной в модели Блэка-Шоулза.

Несмотря на это, СВ модели могут быть использованы для исследования других характеристик финансовых рынков. Например, в [4] исследовалась зависимость волатильности от изменений цен акций. В то же время, изменения в цене акций уже известны, поэтому волатильность (квадрат волатильности цен акций) не является прямым показателем, поэтому она служит случайной переменной. В [4] была найдена совместная плотность вероятности изменений цен акций и волатильности, затем интегрирование по волатильности было выполнено и получена функция распределения вероятности изменений цен акций, зависящая от волатильности. Последняя PDF может быть непосредственно сравнена с данными Dow-Jones за 20-летний период с 1982 по 2001 год, и было найдено отличное согласие. Конец PDF уменьшается медленнее, чем предсказывает лог-нормальное распределение (так называемый "fat-tails" эффект).

Технически наша работа связана с [4]. Однако, мы исследуем зависимость волатильности от изменений цен акций, то есть, мы оцениваем случайную переменную через переменную, которую можно легко получить из финансовых данных. Результат зависит от начального распределения изменений и волатильности. Это естественно, что распределения меняют свою форму со временем. В частности, мы показали, что для нормального распределения изменений цен акции и волатильности, ожидание волатильности демонстрирует выпуклость вниз около среднего значения изменений цен акций.
2. General formulas for the conditional expectation and variance

Let us consider the stochastic differential equation system:

\[\begin{align*}
  dF_t &= Adt + \sigma dW_1, \quad dV_t = Bdt + \lambda dW_2, \\
  F_0 &= f, \quad V_0 = v, \quad t \geq 0, \quad f, v \in \mathbb{R},
\end{align*}\]

where \(W(t) = (W_1(t), W_2(t))\) is a two-dimensional standard Wiener process, \(A = A(t, F_t, V_t), \quad B = B(t, F_t, V_t), \quad \sigma = \sigma(t, F_t, V_t), \quad \lambda = \lambda(t, F_t, V_t)\) are prescribed functions.

The joint probability density \(P(t, f, v)\) of random values \(F_t\) and \(V_t\) obeys the Fokker–Plank equation (e.g., [17]):

\[
\frac{\partial P}{\partial t} = -\frac{\partial}{\partial f} (AP) - \frac{\partial}{\partial v} (BP) + \frac{1}{2} \frac{\partial^2}{\partial f^2} (\sigma^2 P) + \frac{1}{2} \frac{\partial^2}{\partial v^2} (\lambda^2 P)
\]

with initial condition

\[P(0, f, v) = P_0(f, v),\]

determined by initial distributions of \(F_t\) and \(V_t\).

If \(P(t, f, v)\) is known, one can find \(E(F_t|F_i = f)\), which is the conditional expectation of value \(F_t\) at a fixed \(F_i\) at the moment \(t\). This value can be found by the following formula (see, [3]):

\[
E(F_t|F_i = f) = \lim_{L \to +\infty} \frac{\int_{(-L, L)} vP(t, f, v)dv}{\int_{(-L, L)} P(t, f, v)dv}.
\]

Let us also define the variance of \(V_t\) at a fixed \(F_t\) as

\[
\text{Var}(V_t|F_i = f) = \lim_{L \to +\infty} \frac{\int_{(-L, L)} v^2P(t, f, v)dv}{\int_{(-L, L)} P(t, f, v)dv} - E^2(V_t|F_i = f).
\]

In this both formulae the improper integrals from numerator and denominator are assumed to converge. The assumption imposes a restriction on the coefficients \(A, B, \sigma, \lambda\).

Note that if we choose \(P_0(f, v) = \delta(v - v_0(f))g(f)\), where \(v_0(f)\) and \(g(f)\) are arbitrary smooth functions, then \(E(F_t|F_i = f)|_{t = 0} = v_0(f)\).

For some classes of systems [1] the conditional expectation \(V(t, f)\) was found in [17, 1, 2] within an absolutely different context.

Let us remark that sometimes it is easier to find the Fourier transform of \(P(t, f, v)\) function over \(f, v\) variables, than the function itself. We will get formula allowing to express \(E(F_t|F_i = f)\) in terms of Fourier transform of \(P(t, f, v)\) and will apply it for finding an average variance of the stock price, which depends on known return rate.

**Proposition 1.** Let \(\hat{P}(t, \mu, \xi)\) be the Fourier transform of function \(P(t, f, v)\) over \((f, v)\) variables, which is the solution of problem [2, 3], and both integrals from [1] converge. Assume that \(\hat{P}(t, \mu, 0)\) and \(\partial_\mu \hat{P}(t, \mu, 0)\) are decreasing over \(\mu\) at infinity faster than any power. Then \(E(F_t|F_i = f)\) and \(\text{Var}(V_t|F_i = f)\) determined by [1] and [4] can be found as

\[
E(F_t|F_i = f) = \frac{i\mathcal{F}_\mu^{-1} [\partial_\mu \hat{P}(t, \mu, 0)](t, f)}{\mathcal{F}_\mu^{-1} [P(t, \mu, 0)](t, f)}, \quad t \geq 0, \quad f \in \mathbb{R},
\]

where \(\mathcal{F}_\mu^{-1}\) is the inverse Fourier transform and \(\hat{P}(t, \mu, \xi)\) is the Fourier transform of \(P(t, f, v)\).
A certain estimate of volatility through return

\[ \text{Var} (V_t | F_t = f) = \frac{(F^{-1}_\mu \partial_\xi \hat{P}(t, \mu, 0))^2 - F^{-1}_\mu \partial_\xi^2 \hat{P}(t, \mu, 0) F^{-1}_\mu \partial_\mu \hat{P}(t, \mu, 0)}{(F^{-1}_\mu \hat{P}(t, \mu, 0))^2} (t, f), \]

where \( F^{-1}_\mu \) and \( F^{-1}_\xi \) mean the inverse Fourier transforms over \( \mu \) and \( \xi \), respectively.

The proof is a simple exercise in the Fourier analysis.

3. Example: the Heston model

Of course, there is no explicit formula for the joint probability density function \( P(t, f, v) \) for arbitrary system (1). We will consider a particular, but important case of the Heston model [11]:

\[ df_t = \left( \alpha - \frac{v_t}{2} \right) dt + \sqrt{v_t} dW_1, \]
\[ dv_t = -\gamma(v_t - \theta) dt + \sqrt{k} \sqrt{v_t} dW_2. \]

Here \( \alpha, \gamma, k, \theta \) are arbitrary positive constants.

Equation (9) describes the process that in financial literature is called Cox-Ingersoll-Ross (CIR) process, and in mathematical statistics — the Feller process \[7, 6\]. In \[6\] it is shown that this equation has a nonnegative solution for \( t \in [0, +\infty) \) when \( 2\gamma \theta > k^2 \).

The first equation describes a return \( f_t \) on the stock price, in assumption that the stock price itself obeys a geometric Brownian motion with stochastic volatility. The second equation describes the square of volatility \( \sigma^2_t = v_t \).

The Fokker–Planck equation (2) for the joint density function \( P(t, f, v) \) of return \( f_t \) and variance \( v_t \) takes here the following form:

\[ \frac{\partial P(t, f, v)}{\partial t} = \gamma P(t, f, v) + \left( \gamma(v - \theta) + k^2 \right) \frac{\partial P(t, f, v)}{\partial v} + \left( \frac{v}{2} - \alpha \right) \frac{\partial P(t, f, v)}{\partial f} + \]
\[ \frac{k^2 v}{2} \frac{\partial^2 P(t, f, v)}{\partial v^2} + \frac{v}{2} \frac{\partial^2 P(t, f, v)}{\partial f^2}. \]

Now we can choose different initial distributions for return and variance. Note that it is natural to assume that initially the variance does not depend on return.

Below we denote \( E (v_t | f_t = f) \) as \( V(t, f) \) for short.

The function \( \hat{P}(t, \mu, \xi) \), the Fourier transform of \( P \) over \( (f, v) \), satisfies the equation

\[ \frac{\partial \hat{P}(t, \mu, \xi)}{\partial t} + \frac{1}{2} \left( \mu + i \mu^2 + 2\gamma \xi + ik^2 \xi^2 \right) \frac{\partial \hat{P}(t, \mu, \xi)}{\partial \xi} + i (\gamma \theta + \xi \mu \alpha) \hat{P}(t, \mu, \xi) = 0. \]

The first-order PDE (11) can be integrated, the solution has the following form:

\[ \hat{P}(t, \mu, \xi) = e^{ \left( \frac{-\mu \alpha + k^2 + \gamma \mu}{2} \right) t} \left( \frac{k^2 (i (2 \gamma \xi + \mu) - \mu^2 - k^2 \xi^2)}{q} \right)^{-\frac{q}{k^2}} \]
\[ * \frac{F \left( \mu, -t + \frac{2 \arctan \left( \frac{-k^2 + i \gamma}{\sqrt{q}} \right)}{\sqrt{q}} \right)}{\sqrt{q}}, \]
where \( q = -ik^2\mu + k^2\mu^2 + \gamma^2 \), \( F \) is an arbitrary differentiable function of two variables.

### 3.1. The uniform initial distribution of returns

We begin with the simplest and almost trivial case. Let us assume that initially the rate of return is distributed uniformly in the interval \((-L, L)\), \((L = \text{const} > 0)\), and volatility is equal to some constant \( a \geq 0 \). Then the initial joint density distribution of \( f_t \) and \( v_t \) is

\[
P(0, f, v) = \frac{1}{2L}\delta(v - a).
\]

To simplify further calculations we will exclude randomness for \( t = 0 \), i.e. we will assume \( a = 0 \).

The respective initial condition for the Fourier transform is

\[
\hat{P}(0, \mu, \xi) = \frac{\pi}{L}\delta(\mu).
\]

The solution of problem (11), (15) takes the form

\[
\hat{P}(t, \mu, \xi) = \frac{\pi}{L}\delta(\mu) \left( \frac{4\gamma^2 e^{2\gamma t}}{(2\gamma e^{\gamma t} + ik^2\xi (e^{\gamma t} - 1))^2} \right)^{\frac{\alpha}{4\pi}}
\]

It is easy to calculate that

\[
\hat{P}(t, \mu, 0) = \frac{\pi}{L}\delta(\mu), \quad \partial_\xi \hat{P}(t, \mu, 0) = \frac{\pi}{L}\delta(\mu)i\theta e^{-\gamma t}.
\]

Finally from (6) and (17) we get

\[
E(v_t|f_t = f) = \theta \left( 1 - e^{-\gamma t} \right),
\]

\[
\text{Var}(v_t|f_t = f) = \frac{\theta k^2}{2\gamma} \left( 1 - e^{-\gamma t} \right)^2.
\]

It is evident that here there is no dependence on \( f \) and the result is the same as we could obtain from calculation of mathematical expectation and variance of \( v_t \) from equation (6).

### 3.2. The Gaussian initial distribution of returns

Let us assume that initially rate of return is distributed according to the Gaussian law. Then we have the following initial condition:

\[
P(0, f, v) = \frac{m}{\sqrt{\pi}} e^{-m^2f^2}\delta(v), \quad m > 0.
\]

When \( a = 0 \), the Fourier transform of initial data over \((f, v)\) is \( \hat{P}(0, \mu, \xi) = e^{-\frac{\mu^2}{4m^2}} \).

Solution of the problem (11), (20) takes the form:

\[
\hat{P}(t, \mu, \xi) = \frac{\sqrt{\pi}}{m} \left( \frac{\mu(\mu - i) + \frac{\gamma^2}{k^2}}{\mu^2 + k^2\gamma^2 - i(2\gamma\xi + \mu)} \right)^{\frac{\alpha}{4\pi}} \exp \left( -\frac{\mu^2}{4m^2} - (\alpha\mu + \frac{\gamma^2\theta}{k^2})t \right) *
\]

\[
\left( -\cosh \left( \frac{t}{2} \sqrt{k^2\mu(\mu - i) + \gamma^2} - i \arctan \left( \frac{-k^2\xi + i\gamma}{\sqrt{k^2\mu(\mu - i) + \gamma^2}} \right) \right) \right)^{\frac{2\gamma^2}{k^2}}.
\]
We see that \( \hat{P}(t, \mu, \xi) \) exponentially decreases over \( \mu \). That is why we can use formula (6) and obtain (after cumbersome transformations) the following integral expression:

\[
V(t, f) = 2\gamma\theta \frac{\int \Phi(t, \mu, f) d\mu}{\int \Psi(t, \mu, f) d\mu},
\]

where

\[
\Psi(t, \mu, f) = e^{-\mu^2 + \mu(4f - 4\alpha - 1)} \left( \frac{\lambda}{(\lambda \cosh (\frac{\lambda}{4}) + 2\gamma \sinh (\frac{\lambda}{4}))} \right)^{2\gamma\theta} \frac{\sinh (\frac{\lambda}{4})}{(\lambda \cosh (\frac{\lambda}{4}) + 2\gamma \sinh (\frac{\lambda}{4}))},
\]

\[
\Phi(t, \mu, f) = \Psi(t, \mu, f) \frac{\sinh (\frac{\lambda}{4})}{(\lambda \cosh (\frac{\lambda}{4}) + 2\gamma \sinh (\frac{\lambda}{4}))},
\]

\[
\lambda^2 = k^2(4\mu^2 + 1) + 4\gamma^2.
\]

Let us remark that if \( a \neq 0 \), we can also get a similar formula, but it will be more cumbersome.

The limit case as \( m \to \infty \) for (20) is

\[
P(0, f, v) = \delta(f)\delta(v), \quad m > 0.
\]

For this case the formula (22) modifies as follows: the exponential factor in the expression for \( \Psi \) takes the form \( e^{i\mu(f - \alpha)} \).

3.3. “Fat-tails” initial distribution of returns. Integral formula, analogous to (22) can be obtained for initial distributions intermediate between uniform and Gaussian ones. For example, as initial distribution we can take

\[
P(0, f, v) = K(1 + m^2 f^2)^q \delta(v), \quad m > 0, \quad q < 0,
\]

with an appropriate constant \( K \). Exact formula for the Fourier transform \( \hat{P}(t, \mu, \xi) \) can be found for \( q = -\frac{1}{2}, -n, n \in \mathbb{N} \). For all these cases \( \hat{P}(t, \mu, \xi) \) decays as \( |\mu| \to \infty \) sufficiently fast and Proposition 11 can be applied for calculation of \( V(t, f) \).

For example, for \( q = -1 \) the difference with (21) is only in the multiplier \( e^{-\mu^2/m^2} \): it should be changed to

\[
\left( e^{-\mu/m} - e^{\mu/m} \right) H(\mu) + e^{\mu/m},
\]

with the Heaviside function \( H \).

3.4. Convexity downward of the volatility curve and asymptotic behavior for small time. It turns out that if in the Heston model the average volatility is considered as a function of the rate on return, we will observe a deflection of the plot. The effect appears in numerical calculation of both integrals in (22) with the use of standard algorithms. The numerical calculation of the integrals over an infinite interval is based on the QUADPACK routine QAGI [16], where the entire infinite integration range is first transformed to the segment \([0, 1]\). For example, Fig. 1 presents the graph of function \( V(t, f) \) at three consequent moments of time for the following values of parameters: \( \gamma = 1, k = 1, \theta = 1, \alpha = 1, m = 1 \).

This behavior of the volatility plot can be studied by analytical methods as well. Indeed, let us fix rate of return \( f \). Then from (22) by expansion of integrand
functions into formal series as $t \to 0$ up to the forth component and by further term-wise integration (series converge at least for small $f$ and $m$) we will get that 

\[
V(t, f) = \gamma \theta - \frac{1}{2} \gamma^2 \theta t^2 + \frac{1}{6} \gamma^3 \theta (\gamma^2 + 2f^2m^4k^2 - fm^2k^2 + m^2k^2) t^3
\]

\[-\frac{1}{6} \gamma \theta (8 \gamma k^2 f^2 m^4 - 4 (\gamma + 4 \alpha m^2) k^2 m^2 f - 4 (\gamma + \alpha) m^4 k^2 + \gamma^3) t^4 + O(t^5).
\]

Let us justify a possibility to expand $V(t, f)$ into the Taylor series. We should prove that both integrals in the numerator and denominator of (22) can be differentiated with respect to $t$. Indeed, let $t \in \Omega_t = (-\tau, \tau)$, $0 < \tau < \infty$. It can be readily shown that both integrands in (22), $\Phi$ and $\Psi$, are continuous with respect to $\mu$ and $t$ on $\mathbb{R} \times \Omega_t$, the derivatives of any order $\partial_t^n \Phi$, $\partial_t^n \Psi$, $n = 0, 1, \ldots$ are also continuous on $\mathbb{R} \times \Omega_t$. Moreover, $|\partial_t^n \Phi|$, $|\partial_t^n \Psi|$ can be estimated from above by $c_1 \cdot e^{-c_2 \mu^2}$, with positive constants $c_1$ and $c_2$. Therefore $\int_{\mathbb{R}} \partial_t^n \Phi d\mu$ and $\int_{\mathbb{R}} \partial_t^n \Psi d\mu$ converge uniformly on $\Omega_t$. Thus, according to the classical theorem of calculus the numerator and denominator in (22) can be differentiated on $\Omega_t$ under the integral sign. Since for

\[
\partial_t^n \int_{\mathbb{R}} Q(t, \mu, f) d\mu|_{t=0} = \int_{\mathbb{R}} \partial_t^n Q(t, \mu, f)|_{t=0} d\mu,
\]

$Q = \Psi$ or $\Phi$, the Taylor coefficients in the expansion of $\int_{\mathbb{R}} Q(t, \mu, f) d\mu$ can be obtained by integration of the respective coefficient of $Q(t, \mu, f)$ of the Taylor series in $t$ with respect to $\mu$. The latter integrals can be explicitly calculated. This gives expansion (24).

Hence for $t \to 0$ we find that

\[
V(t, f) \sim \frac{1}{6} \gamma \theta t^3 (1 - \gamma t) m^4 k^2 f^2 - \left(\frac{2}{3} \alpha m^2 t + \frac{1}{6} (1 - \gamma t) t^3 \gamma \theta m^2 k^2 f + \frac{1}{6} \gamma^3 \theta t^5\right), \quad t > 0.
\]

is a quadratic trinomial over $f$ with a minimum in point $f = \frac{4 m^2 \alpha t - \gamma t + \theta}{4 m^2 (1 - \gamma t)}$, for $t > 0$.

The effect holds for initial “fat-tails” power initial distributions as well. Nevertheless, this effect weakens as the decay of the distribution at infinity becomes slower.

Fig.2 presents the function $V(t, f)$ for three consequent moments of time for the initial distribution of return $p(f)$ given by formula

\[
p(f) = \frac{1}{\pi} \frac{1}{1 + f^2}.
\]

The values of parameters are $\gamma = 10$, $k = 1$, $\theta = 0.1$, $\alpha = 10$. It seems that the curves are strait lines, but the analysis of numerical values shows that the deflection still persists near the mean value of return. Acting as in the case of the Gaussian initial distribution one can find the Taylor expansion of $V(t, f)$ as $t \to 0$,

\[
V(t, f) = \gamma \theta t - \frac{1}{2} \gamma^2 \theta t^2 - \gamma \theta R_4(f, \gamma, k) R_6(f) - \gamma^2 \theta R_6(f, \gamma, k, \alpha) R_4(f) t^4 + O(t^5),
\]

where we denote by $R_k$ a polynomial of order $k$ with respect to $f$. We do not write down these polynomial, let us only note that $\frac{R_4(f, \gamma, k)}{R_6(f)} \sim \frac{8}{127} (k^2 + \gamma^2 - \gamma^2)$ and

\[
\frac{R_6(f, \gamma, k, \alpha)}{R_6(f)} \sim \frac{3}{16} \left(2 \gamma^2 - k^2\right)\quad \text{as} \quad |f| \to \infty.
\]

It is very interesting to study the asymptotic behaviour of $V(t, f)$ as $|f| \to \infty$ and $t \to \infty$. We do not dwell here on this quite delicate question at all and reserve it for future research. Some hints can be found in [4], [9], [10].
3.5. **Modifications of the Heston model.** Let us analyze the situation when the coefficient $\gamma$ from equation (9) depends on time. For some interesting cases of this dependence one can find the Fourier transform of $P(t, f, v)$ and formula for $V(t, f)$. For example, if we set $\gamma = \frac{1}{T-t}$, then we get a Brownian bridge-like equation (see, [15] describing square of volatility behavior with start at $v_0 = a > 0$ and end at $v_T = b \geq 0$. Here the solution will be represented in terms of integrals of Bessel functions and the solution is cumbersome.

It may seem that the described approach, which helps to find the conditional expectation of volatility under fixed returns in the Heston model, can be successfully applied in other variations of this model. This is true when initial rate on return has a uniform distribution. However, this situation is trivial, because the answer does not contain $f$ and is equal to the expectation of return obtained from the second equation of model. In the case of non-uniform initial distribution of return (for instance, Gaussian) formula (6) may be non-applicable, even when explicit expression for $\hat{P}(t, \mu, \xi)$ can be found. The cause is that $\hat{P}(t, \mu, \xi)$ increases as $|\mu| \to \infty$. For example, if we replace equation (9) with

\begin{equation}
 dv_t = -\gamma(v_t - \theta)dt + kdW_2, \quad \gamma, \theta, k > 0,
\end{equation}

under initial data (20), $a = 0$, we will get

\begin{equation}
 \hat{P}(t, \mu, \xi) = \frac{\sqrt{\pi}}{m} \exp \left( \frac{k^2 t}{8\gamma^2} \mu^4 - i \frac{k^2 t}{4\gamma^2} \mu^3 - \left( \frac{\theta t}{2} + \frac{k^2 t}{8\gamma^2} + \frac{1}{4m^2} \right) \mu^2 + i \left( \frac{\theta}{2} - \alpha \right) t \mu \right),
\end{equation}

whence it follows that the coefficient of $\mu^4$ in exponent power is positive when $t$ is positive. This means that integrals from (6) are divergent.

4. **Possible application**

Basing on our results one can introduce a rule for estimation of the company’s rating based on stock prices. The natural presumption is that company’s rating increases when return on assets increases and volatility decreases. Hence for estimation of the company’s rating one can use (very rough) index $R(t, f) = f/V(t, f)$, where $V(t, f)$ is calculated by formula (22). Figs. 3 and 4 shows the plot function $R(t, f)$ for three consequent time points for Gaussian and power distributions, respectively. Parameters as in Figs. 1 and 2. We can see that in the Gaussian case the index does not rise monotonically with return.
5. Conclusion and further work

In this article we obtain an estimate of volatility given rate on return data in the frame of the Heston model. This problem has been solved by calculation of the average volatility under a fixed rate on return and under the supplementary condition on initial distribution of return and volatility. Namely, different cases of initial distribution of returns have been studied: uniform, Gaussian and “fattails” distributions, intermediate between them. We revealed that the graph of the averaged volatility is convex downwards near the mean value of the stock price return for the Gaussian initial distribution and for certain distributions decreasing at infinity slower than the Gaussian one (for which we succeed to find the Fourier transform of the joint probability density of return and variance explicitly). For the Gaussian distribution this effect is strong, but it weakens and becomes negligible as the decay of distribution at infinity slows down.

Let us note that our formulas can be obtained in a different way, using the well-known expression for the joint characteristic function of the log-return and the variance in the Heston model [11] (in the correlated case). This expression was obtained exploiting the linearity of the coefficients in the respective PDE, in other word, the fact that the Heston model is affine [3]. Nevertheless, this way is not convenient for our purpose, since it requires an additional integration.

Formulas for the conditional variance at fixed return $V(t, f)$ are obtained in the present work in the integral form, we compute the integrals numerically using standard algorithms and study asymptotics of the formulas for small time. The questions on analysis of the formulas for larger $t$ and $f$ and on the asymptotics of $V(t, f)$ as $|f| \to \infty$ and $t \to \infty$ are open. Moreover, the dependence of the averaged variance on the properties of the initial distribution of returns has to be studied in general case, not only for separate examples, as it was done here.

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Список литературы

[1] S. Albeverio and O. Rozanova, The Non-Viscous Burgers Equation Associated with Random Positions in Coordinate Space: a Threshold for Blow up Behavior, Mathematical Models and Methods in Applied Sciences, 19(2009), pp 1–19.

[2] S. Albeverio and O. Rozanova, Suppression of Unbounded Gradients in a SDE Associated with the Burgers Equation, Proc. Amer. Math. Soc. 138 (2010), pp. 241–251.

[3] A. J. Chorin and O. H. Hald, Stochastic Tools in Mathematics and Science, Springer, New York, 2006.

[4] A.A. Dragulescu and V.M. Yakovenko, Probability Distribution of Returns in the Heston Model with Stochastic Volatility, Quantitative Finance, 2(2002), pp.443–453.

[5] D. Duffie, D. Filipović and W. Schachermayer, Affine processes and applications in finance, The Annals of Applied Probability, 13(2003), pp.984–1053.

[6] W. Feller, Two Singular Diffusion Problems, Annals of Mathematics, 54(1951), pp.173–182.

[7] J.P. Fouque, G. Papanicolaou, and K.R. Sircar, Derivatives in Financial Markets with Stochastic Volatility, Cambridge University Press, Cambridge, 2000.

[8] J. Gatheral, The Volatility Surface, Wiley and Sons, Inc., Hoboken, New Jersey 2006.

[9] A. Gulisashvili and E. M. Stein Asymptotic Behavior of the Stock Price Distribution Density and Implied Volatility in Stochastic Volatility Models, Mathematical Finance, 30 (2010), pp.447–477.

[10] A. Gulisashvili and E. M. Stein, Asymptotic Behavior of the Distribution of the Stock Price in Models with Stochastic Volatility: the Hull-White Model, C. R. Acad Sci. Paris, Ser.I, 343(2006), pp.519–523.

[11] S.L. Heston, A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options, The Review of Financial Studies, 6 (1993), pp. 327–343.

[12] J. Hull and A. White, The Pricing of Options on Asset with Stochastic Volatilities, J. Finance, 42(1987), pp.281–300.

[13] S. Miccichè, G. Bonanno, F. Lillo and R. N. Mantegna Volatility in Financial Markets: Stochastic Models and Empirical Results, Physica A, 314(2002), pp.756-761.

[14] S. Mitra, A Review of Volatility and Option Pricing, Available at http://arxiv.org/pdf/0904.1392

[15] B. Øksendal, Stochastic Differential Equations: An Introduction with Applications, 5th ed., Springer, Heidelberg, 2002.

[16] R. Piessens, E. de Doncker-Kapenga, C. Überhuber and D. Kahaner, QUADPACK, A Subroutine Package for Automatic Integration Springer–Verlag, Berlin, 1983.

[17] H. Risken, The Fokker-Planck Equation. Methods of solution and applications, 2ed, Springer, New York, 1989.

[18] R. Schöbel, and J. Zhu, Stochastic Volatility with an Ornstein-Uhlenbeck Process: An Extension, Europ. Finance Rev., 4(1999), pp.23–46.

[19] L. Scott, Option Pricing when the Variance Changes Randomly: Theory, Estimation and Applications, J. Finan. Quant. Anal., 22(1987), pp.419–438.

[20] E.M. Stein and J.C. Stein, Stock Price Distributions with Stochastic Volatility: An Analytic Approach, Rev. Finan. Stud., 4(1991), pp.727–752.