Dressing chain for the acoustic spectral problem

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Abstract

The iterations are studied of the Darboux transformation for the generalized Schrödinger operator. The applications to the Dym and Camassa-Holm equations are considered.

1 Introduction

In this paper we consider the generalized Schrödinger problem

$$\varphi_{yy} = (q(y) - \lambda r^4(y))\varphi$$

and associated Dym \cite{1} and Camassa-Holm equations \cite{2,3}. Although this problem can be brought into the standard form $r = 1$ via the Liouville transformation, the reformulating of the known results is not a quite trivial job, due to the inevitable change of independent variable. This results, in particular, in the fact that all known exact solutions cannot be given explicitly and are represented in parametric form. The scattering theory was developed, among the others, in the papers \cite{4,5,6}. The papers \cite{7,8,9,10,11,12} and others were devoted to the construction of algebraic-geometric, multisoliton and peakon solutions. The Darboux-Bäcklund transformations were considered in the papers \cite{13,14,15}. However, from our point of view, the techniques of these transformations remains so far inadequate. Meanwhile, the Darboux transformations for the Schrödinger operator have proven themselves as an effective tool for constructing of exactly solvable potentials and explicit solutions of the KdV equation, see eg. \cite{16,17,18,19} and many other works. It is not always possible to copy the results obtained here, because of the actual differences in the settings of the spectral problems, but the main ideas should be reproduced. This is what we try to do in this article.

The presence of two potentials allows to combine the Darboux and Liouville transformations. This possibility yields interesting consequences and deserves detailed study. The key observation of the presented paper is that one of such combinations is almost as simple as the Darboux transformation itself but corresponds to the gauge $q = 0$ rather than $r = 1$. 
2 Darboux transformations

The goal of this Section is to bring into the consideration the transforms acting on the set of the generalized Schrödinger equations

\[
\varphi_{yy} = (q(y) - \lambda r^4(y))\varphi.
\]

(1)

Let us start from the classical transformation which eliminates the factor \(r\). We reserve the own notation for this form of equation, because of its importance.

**Statement 1 (Liouville transformation).** Equation (1) takes the form

\[
\psi_{xx} = (u(x) - \lambda)\psi
\]

(2)

under the change

\[
dx = r^2 dy, \quad \psi = r\varphi, \quad u = q/r^4 + r_{xx}/r.
\]

(3)

Another classical transformation is defined by any particular solution \(\psi^{(a)}\) of equation (2) at \(\lambda = \alpha\). It is not difficult to give it in the general gauge (1) as well.

**Statement 2 (Darboux transformation).** Equation (2) is form invariant under the transformation

\[
\hat{\psi} = \psi_x - f\psi, \quad f := \psi_y^{(a)}/\psi^{(a)}, \quad f_x + f^2 = u - \alpha, \quad \dot{u} = u - 2f_x.
\]

(4)

Equation (1) is form invariant under the transformation

\[
\hat{\varphi} = \frac{1}{r^2}(\varphi_y - f\varphi), \quad f := \varphi_y^{(a)}/\varphi^{(a)}, \quad f_y + f^2 = q - \alpha r^4,
\]

\[
\hat{r} = r, \quad \hat{q} = q - 2f_y + \frac{4r_y f}{r} + \frac{6r^2 r_y}{r^2} - \frac{2r_{yy}}{r}.
\]

(5)

Now let us consider the other canonical form of the equation (1),

\[
\varphi_{yy} = -\lambda r^4(y)\varphi.
\]

(5)

The so called acoustic spectral problem corresponds to the potential \(r\) bounded away from 0 and tending to 1 rapidly enough at \(|y| \to \infty\), [5]. However, we will not care of the analytic properties at first and we will refer to [5] just as to acoustic equation. Obviously, it is obtained from (2) by inverse Liouville transformation, if one choose \(q = 0\) and take a wave function at \(\lambda = 0\) as \(r\):

\[
dx = r^2 dy, \quad \psi = r\varphi, \quad u = r_{xx}/r.
\]

(6)

It is easy to show that the arbitrariness in the choice of \(r\) results in the linear-fractional changes on the set of equations (5):

\[
\bar{y} = \frac{c_1 y + c_2}{c_3 y + c_4}, \quad \bar{\varphi} = \frac{1}{c_3 y + c_4} \varphi, \quad \bar{r} = \frac{c_3 y + c_4}{\Delta^{1/2}} r, \quad \Delta = c_1 c_4 - c_2 c_3.
\]

(7)

The recomputation of the Darboux transform brings to unexpectedly simple formulae.
Statement 3. Equation (5) is form invariant under the transformation

\[ \hat{\phi} = \phi_y/p - \phi, \quad p := \phi_y^{(a)}/\phi^{(a)}, \quad p_y + p^2 = -\alpha r^4, \]
\[ \hat{r} = p/r, \quad \hat{r}^2 \hat{y} = r^2dy. \]

Proof. Application of \( D_{\hat{y}} = p^2 r^{-4} D_y \) and elimination of \( p_y \) in virtue of Riccati equation yields first \( \hat{\phi}_y = \alpha \phi_y - \lambda p \phi \), and next \( \hat{\phi}_{\hat{y}} = -\lambda \hat{r} r^{-4} \hat{\phi} \).

In order to check that the presented transformation is equivalent to (4), apply the change (6) to \( \phi \) and \( \hat{\phi} \). Taking into account the relation \( f = \psi_x^{(a)}/\psi^{(a)} = r_x/r + p/r^2 \) yields

\[ \hat{\psi} = \hat{r} \hat{\phi} = \frac{p}{r} \left( \frac{1}{p} \phi_y - \phi \right) = r \phi_x - \frac{p}{r} \phi = r \left( \frac{\psi}{r} \right)_x - \frac{p \psi}{r^2} = \psi_x - f \psi. \]

The equation (5) is considered on a finite interval as well [4]. It is possible to convert this case to the spectral problem on the whole axis

\[ \chi_{zz} = (1 - \lambda R^4(z)) \chi \]

by means of the following Liouville transformation [5]:

\[ y = \tanh z, \quad \varphi = \chi \sech z, \quad r = R \cosh z. \]

Here we assume that \( r \in C^\infty([-1, 1]), \ r > 0, \ r(-1) = r(1). \) The importance of this gauge is caused by the relation to Camassa-Holm equation which is discussed in Section 5. However, the Darboux transformation looks rather awkward in these variables, and it is more convenient to keep using the formulae (5), recalculating the answer by the indicated substitution.

In conclusion, we mention two more simple auto-transformations of the acoustic equation.

Statement 4. The form of equation (5) does not change under the transformations

\[ \bar{\phi} = \phi_y, \quad \bar{r} = 1/r, \quad d\bar{y} = r^4 dy, \]
\[ \tilde{\varphi} = \varphi/\varphi^{(a)}, \quad \tilde{r} = r \varphi^{(a)}, \quad (\varphi^{(a)})^2 d\tilde{y} = dy, \quad \tilde{\lambda} = \lambda - \alpha. \]

Proof. One has for the first transformation

\[ \bar{\phi}_y = r^{-4} \phi_{yy} = -\lambda \phi, \quad \bar{\phi}_{\bar{y}} = -\lambda r^{-4} \varphi_y = -\lambda r \bar{\phi}. \]

This transformation is equivalent to the Darboux transformation (4) generated by the function \( \psi^{(0)} = r \). Indeed,

\[ \tilde{\psi} = \tilde{r} \tilde{\varphi} = \varphi_y/r = r \varphi_x = r(\psi/r)_x = \psi_x - r \varphi/\varphi. \]

Notice that the formulae (8) correspond to the choice \( \psi^{(0)} = (y + c)r = r \int r^{-2} dx + cr. \)
For the second transformation, one has
\[ \varphi_y = \varphi_y \varphi^{(a)} - \varphi \varphi_y^{(a)}, \]
\[ \varphi_{yy} = (\varphi_{yy} \varphi^{(a)} - \varphi \varphi_{yy}^{(a)}) (\varphi^{(a)})^2 = (\alpha - \lambda) r^4 \varphi^{(a)} = (\alpha - \lambda) r^4 \varphi. \]

The change (6) gives
\[ \tilde{\psi} = \tilde{\varphi} = \psi, \quad \tilde{u} = \frac{\tilde{r}_{xx}}{r} = \frac{\varphi}{r} \left( \frac{2r_x \varphi^{(a)} + \varphi^{(a)}}{r^x_{xx}} \right) = u - \alpha, \]
that is, this transformation is equivalent to the shift
\[ \tilde{\psi} = \psi, \quad \tilde{u} = u - \alpha, \quad \tilde{\lambda} = \lambda - \alpha \]
in the Schrödinger equation.

3 Dressing chains

When studying the iterations of the Darboux transformation (8) it is convenient to introduce an additional scaling by setting \( \tilde{r}_n = -\gamma_n r_{n+1}, \quad \tilde{y}_n = \gamma_n^{-2} y_{n+1}. \) Introduce the parameter \( x \) accordingly to equation \( dx = r^2_n dy_n \) and eliminate \( p_n \) from relations \( p_n = -\gamma_n r_{n+1} r_n, \quad p_n, y_n + p_n^2 = -\alpha_n r^4_n, \) then the sequence of equations \( (r_{n+1} r_n)_{x} = \gamma_n r_{n+1}^{2} + \alpha_n / \gamma_n r_{n}^{2} \) appears, which looks more symmetric under the choice
\[ -\gamma_n^2 = \alpha_n. \]
Accepting this we lose the Darboux transformation corresponding to \( \alpha = 0. \) However, it is clear that this transformation actually differs from the other ones and it should be considered separately. This will be done in Section 5. So, we come to equations
\[ (r_{n+1} r_n)_{x} = \gamma_n (r_{n+1}^{2} - r_n^{2}), \quad y_n, x = r_n^{-2}. \]
(12)

The change \( r_n = e^{\beta_n} \) brings the first one to the form
\[ g_{n+1, x} + g_{n, x} = 2\gamma_n \sinh(g_{n+1} - g_n) \]
(13)
which defines the \( x \)-part of the Bäcklund transformation for the pot-mKdV and sinh-Gordon equations (see eg. [17])
\[ g_t = g_{xxx} - 2g^3, \quad g_{x} = \sinh 2g. \]

We will call the differential-difference equations of such type the dressing chains. This terminology was suggested in the papers [20, 18, 19] in connection with the equations
\[ f_{n+1, x} + f_{n, x} = f_n^2 - f_{n+1}^2 + \alpha_n - \alpha_{n+1} \]
(14)

describing the iterations of Darboux transformation (4). The transformations (4) and (8) were shown in the previous Section to be conjugated by Liouville transformation (6).
This suggests that some relation exists between the chains \((12)\) and \((14)\). In order to find it we will use the well-known commutativity property of Darboux transformations, or the nonlinear superposition principle (see eg. \([17]\)). This property is most conveniently expressed in terms of the pre-potential introduced by the formulae

\[ 2v_{n,x} = u_n, \quad v_n - v_{n+1} = f_n \]

which bring to the following form of the dressing chain \((14)\):

\[ v_{n+1,x} + v_{n,x} = (v_{n+1} - v_n)^2 + \alpha_n. \quad (15) \]

**Statement 5.** Let \(v, v^{(\alpha)}\) and \(v^{(\beta)}\) be related by Darboux transformations

\[ v_x^{(\alpha)} + v_x = (v^{(\alpha)} - v)^2 + \alpha, \quad v_x^{(\beta)} + v_x = (v^{(\beta)} - v)^2 + \beta. \]

Then the quantity \(v^{(\alpha,\beta)} = v - \frac{\alpha - \beta}{v^{(\alpha)} - v^{(\beta)}}\) is related to \(v^{(\alpha)}\) and \(v^{(\beta)}\) by Darboux transformations

\[ v_x^{(\alpha,\beta)} + v_x = (v^{(\alpha,\beta)} - v)^2 + \alpha, \quad v_x^{(\alpha,\beta)} + v_x^{(\alpha)} = (v^{(\alpha,\beta)} - v^{(\alpha)})^2 + \beta. \]

Now, consider two copies of the dressing chain \((15)\) with respect to the variables \(v_n\) and \(\bar{v}_n\) related by Darboux transform with zero parameter:

\[ v_{n,x} + \bar{v}_{n,x} = (v_n - \bar{v}_n)^2. \quad (16) \]

The consistency of these equations with both copies of the chain is provided by the **Statement 5** and moreover the following relations are fulfilled

\[ (v_{n+1} - \bar{v}_n)(v_n - \bar{v}_{n+1}) = \alpha_n. \quad (17) \]

The differences \(f_n, \bar{f}_n\) are identified with the oriented horizontal edges of the lattice shown on the **fig. 1**. It is not difficult to prove that the differences \(F_n = v_n - \bar{v}_n\) corresponding to the vertical edges satisfy the chain

\[ F_{n+1,x} + F_{n,x} = (F_{n+1} - F_n)\sqrt{(F_{n+1} + F_n)^2 - 4\alpha_n}. \quad (18) \]
Indeed,

\[(F_{n+1} + F_n)_x = (v_{n+1} + v_n - \bar{v}_{n+1} - \bar{v}_n)_x = (v_{n+1} - v_n)^2 - (\bar{v}_{n+1} - \bar{v}_n)^2 \]
\[= (v_{n+1} - v_n - \bar{v}_{n+1} + \bar{v}_n)(v_{n+1} - v_n + \bar{v}_{n+1} - \bar{v}_n) \]
\[= (F_{n+1} - F_n)\sqrt{(v_{n+1} + v_n - \bar{v}_{n+1} - \bar{v}_n)^2 - 4(v_{n+1} - \bar{v}_n)(v_n - \bar{v}_{n+1})} \]
\[\overset{(17)}{=} (F_{n+1} - F_n)\sqrt{(F_{n+1} + F_n)^2 - 4\alpha_n}. \]

The chains (18) and (12) are related by the substitution $F_n = r_{n,x}/r_n$. Remind, that the operator $D_x - r_x/r$ defines the special Darboux transformation at $\alpha = 0$ (see the proof of the Statement 4).

Coming back to the chain (14), notice that the relation (17) is equivalent to the quadratic equation with respect to $f_n$. Its solution provides the substitutions into the chain (14) and its copy for the variables $\bar{f}_n$.

**Statement 6.** The general solutions of the chains (14), (18) and (12) are related by the substitutions

\[F_n = \frac{r_{n,x}}{r_n}, \quad s_n := \sqrt{(F_{n+1} + F_n)^2 - 4\alpha_n} = \gamma_n\left(\frac{r_{n+1}}{r_n} + \frac{r_n}{r_{n+1}}\right), \quad \alpha_n = -\gamma_n^2, \]
\[2f_n = F_n - F_{n+1} - s_n, \quad 2\bar{f}_n = F_{n+1} - F_n - s_n, \]
\[f_n = \frac{r_{n,x} - \gamma_n r_{n+1}}{r_n}, \quad \bar{f}_n = -\frac{r_{n,x} - \gamma_n r_n}{r_n}. \quad (19)\]

Till now we have not paid attention to the variables $y_n$. It turns out that the dressing chain can be rewritten in terms of these variables as well. One obtains, after dividing (12) by $r_{n+1}^2 r_n^2$ and integrating (the integration constants are unessential here and are eliminated by the shift $y_n \rightarrow y_n + c_n$), the relation

\[\frac{1}{\gamma_n r_{n+1} r_n} = y_{n+1} - y_n \quad (20)\]

which means that all $y_n$ can be recovered by the single quadrature. These equations can be rewritten also in the form

\[y_{n+1,x} y_{n,x} = \gamma_n^2 (y_{n+1} - y_n)^2. \quad (21)\]

Finally, notice that the change $\bar{r}_n = 1/r_n$ arising from the transformation (11) is equivalent to the reversion of the vertical arrows on the fig. 1 and it leads to the chains

\[(\bar{r}_{n+1} \bar{r}_n)_x = \gamma_n(\bar{r}_{n+1}^2 - \bar{r}_n^2), \quad \bar{y}_{n+1,x} \bar{y}_{n,x} = \gamma_n^2 (\bar{y}_{n+1} - \bar{y}_n)^2, \quad \bar{y}_{n,x} = \bar{r}_n^{-2}. \]

The variables $\bar{y}_n$ satisfy the recurrent relation analogous to (20):

\[\bar{y}_{n+1} - \bar{y}_n = r_{n+1} r_n / \gamma_n. \quad (22)\]
4 Dym equation

The dressing chains introduced in the previous Section belong to the rather important class of the Bäcklund transformations of the general form

\[ u_{n+1,x} = b(u_{n,x}, u_n, u_{n+1}, \alpha_n) \]  

(23)

which correspond to the KdV type equations

\[ u_t = A(u_{xxx}, u_{xx}, u_x, u, x, \alpha). \]  

(24)

For example, the chains (14) and (18) define the \( x \)-parts of the Bäcklund transformations for the modified KdV equations

\[ f_t = f_{xxx} - 6(f^2 + \alpha) f_x, \quad F_t = F_{xxx} - 6F^2F_x, \]

while the chains (21) and (15) correspond to the Schwarz-KdV and pot-KdV equations

\[ y_t = y_{xxx} - \frac{3y_{xx}^2}{2y_x}, \quad v_t = v_{xxx} - 6v_x^2. \]

As we have already mentioned, the chain (13) corresponds to the pot-mKdV equation or, in the \( r \) variables,

\[ r_t = r_{xxx} - \frac{3r_{xx}r_x}{r}. \]  

(25)

Although the list can be easily expanded, it does not fall outside the subclass of equations of the form

\[ u_t = u_{xxx} + a(u_{xx}, u_x, u, x, \alpha). \]  

(26)

Indeed, the compatibility of the equations (23), (24) means that the identity holds

\[ D_x(A[n + 1]) = b_{u_{n,x}} D_x(A[n]) + b_{u_n} A[n] + b_{u_{n+1}} A[n + 1] \]

where the derivatives of \( u_{n+1} \) are eliminated in virtue of (23). Equating the coefficients at \( u_{n,xxx} \) gives the relation \( A_{u_{xxx}}[n + 1] = A_{u_{xxx}}[n] \). The quantity \( A_{u_{xxx}}^{-1/3} \) is called the separant of equation (24). If it depends on \( x \) only then the equation can be brought to the form (26). Otherwise the chain (23) does not lead out of the finite-parametric family of solutions of the ordinary differential equation of the form \( A_{u_{xxx}} = A_{u_{xxx}}[0] = c(x) \). It is clear that such type of Bäcklund transformation must be considered as defective.

Nevertheless, a chain of the form (23) may define a Bäcklund transformation for an equation with variable separant if it can be extended by some equation for the independent variable, which we will denote as \( y_n \) from now on. The variable \( x \) now will play the role of an auxiliary parameter. The chain (12) gives the example of such an extension, corresponding to the Dym equation for the variable \( w = r^{-2} \):

\[ w_t = w^3 w_{yyy}. \]  

(27)
Indeed, this equation is related to (25) by the following composition of the hodograph transform and two differential substitutions like introducing of the potential:

\[
\begin{align*}
    x_y &= \frac{1}{w}, & x_t &= \frac{1}{2}w_y^2 - ww_{yy}, & y_x &= \frac{1}{r^2}, & y_t &= -\frac{2r_{xx}}{r^3} \\
    \Downarrow & & \Downarrow & & \Downarrow
    \quad x_t &= \frac{x_{yyy}}{x_y^3} - \frac{3x_{yy}^2}{2x_y} \quad \Leftrightarrow \quad y_t = y_{xxx} - \frac{3y_{xx}^2}{2y_x}
\end{align*}
\]

(such type of changes can be written a little bit more compactly as a reciprocal transformation). The relation \( w = r^{-2} \) is just a consequence of the identity \( y_x x_y = 1 \).

**Remark 1.** An extension of the chain may be not unique. Indeed, equation (25) admits a more general potential:

\[
\begin{align*}
    y_x &= ar^{-2} + br^2, & y_t &= -2ar_{xx}r^{-3} + 2b(rr_{xx} - 2r_x^2).
\end{align*}
\]

It can be used in order to preserve the real structure when \( r \sim e^{ig} \), that is in the case of sine-Gordon equation and mKdV equation with the plus sign before the nonlinear term.

In particular, the chain

\[
g_{n+1,x} + g_{n,x} = 2\gamma_n \sin(g_{n+1} - g_n)
\]

can be extended by setting \( y_{n,x} = c + \sin 2g_n \), where the constant \( c \geq 1 \) is added in order to provide the one-to-one correspondence \( x \leftrightarrow y \) on the whole axis. This leads to the following sequence of the transformations:

\[
\begin{align*}
    w_t &= w^3 w_{yy} - \frac{3}{2}w^2 D_y \left( \frac{w_y^2(1 - c^2 + cw)}{(w - c)^2 - 1} \right) \quad & g_t &= g_{xxx} + 2g_x^3 \\
    \Updownarrow & & \Updownarrow & & \Updownarrow \quad x_y &= \frac{1}{w}, & x_t &= \frac{1}{2}w_y^2 - ww_{yy} + \frac{3w_y^2(1 - c^2 + cw)}{2((w - c)^2 - 1)} \quad & y_x &= c + \sin 2g, & y_t &= 2g_{xx} \cos 2g + 2g_x^2 \sin 2g \\
    \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \quad x_t &= \frac{1}{x_y^3} \left( x_{yyy} - \frac{3x_{yy}^2(1 - 3cx_y + 2(c^2 - 1)x_y^2)}{2x_y((1 - cx_y)^2 - x_y^2)} \right) \quad \Leftrightarrow \quad y_t &= y_{xxx} - \frac{3(y_x - c)y_{xx}^2}{2((y_x - c)^2 - 1)}
\end{align*}
\]

By construction, the equation for \( w \) must support the breather type solutions. However, it is not clear, if it possesses some applications, so we will not discuss it further on.

The construction of the extended dressing chain can be based on the zero curvature representation

\[
\Phi_y = M\Phi, \quad \Phi_t = N\Phi \quad \Rightarrow \quad M_t = N_y + [N, M].
\]

Since, in the case of the variable separant, the Bäcklund transformation involves the independent variable, hence it is natural to replace \( \partial_y \) by differentiation with respect to some parameter \( x \) independent on \( n \), assuming

\[
\partial_x = \rho_n \partial_{y_n}, \quad \partial_T = \partial_{t_n} + \sigma_n \partial_{y_n}, \quad \rho_n = y_{n,x}, \quad \sigma_n = y_{n,T}.
\]
Then the compatibility condition of the auxiliary linear problems
\[ \Phi_{n,x} = \rho_n M_n \Phi_n, \quad \Phi_{n+1} = L_n \Phi_n \]
defines the extended dressing chain
\[ L_{n,x} = \rho_{n+1} M_{n+1} L_n - \rho_n L_n M_n, \quad y_{n,x} = \rho_n. \] (28)
If the matrix \( M \) is given then this equation allows to find constructively both the factor \( \rho_n \) and the matrix \( L_n \). Analogously, the \( t \)-part of the Bäcklund transformation is derived from the consistency with the linear problem
\[ \Phi_{n,T} = (N_n + \sigma_n M_n) \Phi_n, \quad y_{n,T} = \sigma_n. \]
For example, the Dym equation defines the isospectral deformation of the acoustic problem (5):
\[ \varphi_{yy} = -\lambda w^{-2} \varphi, \quad \varphi_t = 2\lambda w_y \varphi - 4\lambda w \varphi_y. \]
In the matrix form,
\[ \Phi = \begin{pmatrix} \varphi \\ \varphi_y \end{pmatrix}, \quad M = \begin{pmatrix} 0 & 1 \\ -\lambda w^{-2} & 0 \end{pmatrix}, \quad N = 2\lambda \begin{pmatrix} w_y & -2w \\ 2\lambda w^{-1} + w_{yy} & -w_y \end{pmatrix}. \]
Extension of the Darboux transformation (8) on \( \hat{\varphi}_y \) brings to the matrix
\[ L_n = \begin{pmatrix} \gamma_n & (r_{n+1}r_n)^{-1} \\ -\lambda r_{n+1}r_n & \gamma_n \end{pmatrix}, \]
and the substitution into (28) yields the equations of the chain along with the constraint
\[ \rho_n = w_n = r_n^{-2}. \]
Conversely, it is easy to check that if we assume \( \deg_\lambda L_n = \deg_\lambda \det L_n = 1 \) then both this constraint and the matrix \( L_n \) itself are found uniquely from (28).

**Remark 2.** The hypothesis exists that all integrable equations of the form (24) can be brought via the differential substitutions and contact or point transformations to the form with constant separant. If this is true then the Bäcklund transformations for equations (24) may be obtained from the Bäcklund transformations for equations (26) by means of suitable extension, like in the above examples. The integrable equations of the form (26) are very well studied and it is known that any of them can be reduced (via the transformations not involving \( x \)) either to KdV or Krichever-Novikov or linear equation. Correspondingly, all dressing chains can be reduced finally to few basic ones. On the contemporary state of the problem of classification of the equations (24) see [21].
5 Camassa-Holm equation

The Camassa-Holm equation \[2, 3\]

\[4h_t - h_{zzt} + 2\varepsilon h_z = hh_{zzz} + 2h_z h_{zz} - 12hh_z\]  \hspace{1cm} (29)

appears as the compatibility condition for the linear problems

\[\chi_{zz} = (1 - \lambda R^4(z))\chi, \quad \chi_t = \frac{h_z}{2}\chi + \left(\frac{1}{2\lambda} - h\right)\chi_z, \quad R^4 := h_{zz} - 4h - \varepsilon.\]

The dressing chain for this equation, with respect to the variables \(z, R\), may be obtained simply by applying the Liouville transformation (10) to the chain (21):

\[z_{n+1,x}z_{n,x} = \gamma_n^2 \sinh^2(z_{n+1} - z_n), \quad z_{n,x} = R_n^{-2}.\]  \hspace{1cm} (30)

The equation (29) is equivalent to the conservation law \((R^2)_t + (R^2 h)_z = 0\) which allows to apply the reciprocal transformation

\[dx = R^2 dz - R^2 h dt \quad \Rightarrow \quad z_x = R^{-2}, \quad z_t = h.\]

Eliminating \(R\) and \(h\) from the equality \(R^4 = (z_x^{-1} D_x)^2(h) - 4h - \varepsilon\) yields the equation

\[z_{xxt}z_x - z_{xt}z_{xx} = (4z_t + \varepsilon)z_x^3 + z_x\]  \hspace{1cm} (31)

which is one of the equivalent forms of the so called associated Camassa-Holm equation [13, 14, 15]. The first equation of the extended chain (30) defines the \(x\)-part of the Bäcklund transformation for the equation (31).

Along with this transformation, equation (31) admits one more Bäcklund transformation which does not contain the parameter and may be considered as a limiting case. The interesting fact is that the corresponding \(t\)-part can be written as a Volterra type lattice. In order to avoid confusion we will denote the iterations of this Bäcklund transformation by superscript.

**Statement 7.** The following two lattices commute:

\[z_x^m z_x^{m+1} = e^{2z^m - 2z^{m+1}}, \quad -8z_x^m = 2\varepsilon + e^{2z^{m+1}} - 2z^{m} + e^{2z^m} - 2z^{m-1}.\]  \hspace{1cm} (32)

The variables \(z^m\) solve equation (31) in virtue of these lattices.

Several equivalent equations can be written down. Coming back to the Schrödinger gauge, assume \(\psi = R \chi\) then

\[\psi_{xx} = (u - \lambda)\psi, \quad 2\lambda \psi_t = R^2 \psi_x - RR_x \psi, \quad u = \frac{R_{xx}}{R} + \frac{1}{R^4}, \quad u_t = -2RR_x.\]

The variable \(v\) is introduced accordingly to the relations \(u = 2v_x, \quad R^2 = -2v_t\) and elimination of \(R\) yields the equation

\[2v_tv_{xxt} - v_{xt}^2 - 8v_x v_t^2 + 1 = 0\]
which is represented by the consistent pair of the lattices

\[ v^m_x + v^{m+1}_x = (v^m - v^{m+1})^2, \quad v^m_t = (v^{m+1} - v^{m-1})^{-1}. \]  

(33)

The differences \( F^m = v^m - v^{m+1} \) correspond to the pair

\[ F^m_x + F^{m+1}_x = (F^m)^2 - (F^{m+1})^2, \quad F^m_t = (F^{m+1} + F^m)^{-1} - (F^m + F^{m-1})^{-1}. \]

Its consistency (and few other equivalent forms) was stated in the papers 20 22, although the associated equation

\[ FF_{xx} - F_xF_t - 4F^3F + 2F_x = 0 \]

was not explicitly presented.

Obviously, the first of the chains (33) governs the iterations of the Darboux transformations with zero parameter 16. Therefore, this Bäcklund transformation is interpreted as the reproduction of the Fig. 1 in vertical direction. It is easy to check that the totally discrete equation implied by (17)

\[(v^m_{n+1} - v^{m+1}_n)(v^m_n - v^{m+1}_{n+1}) = \alpha_n\]

is consistent with the dynamics on \( t \).

### 6 Formulae with Wronskians

Let the wave functions \( r_1 = \psi_1^{(0)} \) and \( \psi_1^{(\alpha_n)} \) are known for the Schrödinger operator \(-D^2_x + u_1\), corresponding to the pairwise different values of the spectral parameter (in the case of the multiple parameters all formulae below remain valid, but one should to use the associated functions in addition to the wave ones). The Darboux transformation defined by the function \( \psi_1^{(\alpha_1)} \) leads to the potential \( u_2 \) with the wave functions \( \psi_2^{(\alpha_n)} = (D_x - f_1)(\psi_1^{(\alpha_n)}), n \neq 1 \), where \( f_1 = \psi_{1,x}/\psi_1^{(\alpha_1)} \). Further on, application of the transformations defined by the functions \( \psi_2^{(\alpha_2)}, \psi_3^{(\alpha_3)} \) etc. generates the sequence of the multiple Darboux transformations. Their result is expressed in terms of the Wronskians of the original functions \( \psi_1^{(\alpha_n)} \) accordingly to the Crum formulae 16

\[
\psi_1^{(\lambda)} = \Delta_n(\psi_1^{(\lambda)})/\Delta_n, \quad f_{n+1} = D_x \log(\Delta_{n+1}/\Delta_n), \\
u_{n+1} = u_1 - 2D_x^2 \log \Delta_n, \quad v_{n+1} = v_1 - D_x \log \Delta_n, \\
\Delta_0 = 1, \quad \Delta_n = \langle \psi_1^{(\alpha_1)}, \ldots, \psi_1^{(\alpha_n)} \rangle, \quad \Delta_n(g) = \langle \psi_1^{(\alpha_1)}, \ldots, \psi_1^{(\alpha_n)} , g \rangle, \\
\langle g_1, \ldots, g_n \rangle := \det(D_x^{k-1}(g_j))_{j,k=1}^n.
\]

The proof follows immediately from the definition of the Darboux transformation 11 if one takes into account the identity

\[
\langle g_1, \ldots, g_n \rangle = g_1 \langle A(g_2), \ldots, A(g_n) \rangle, \quad A = D_x - g_{1,x}/g_1.
\]
The Crum formulae can be easily extended on the other variables introduced in Section 3. Since \( r_1 \) is a wave function at \( \lambda = 0 \), hence
\[
\bar{v}_n = v_1 - D_x \log \Delta_{n-1}(r_1), \quad F_n = D_x \log(\Delta_{n-1}(r_1)/\Delta_{n-1}).
\]
Therefore \( r_n = c_n \Delta_{n-1}(r_1)/\Delta_{n-1} \), and the multiplier can be found from the formula for \( f_n \) rewritten in the form
\[
\langle \Delta_n, \Delta_{n-1}(r_1) \rangle = \gamma_n \frac{c_{n+1}}{c_n} \Delta_{n-1} \Delta_n(r_1).
\]

The Jacobi identity
\[
\langle (g_1, \ldots, g_n, g), (g_1, \ldots, g_n, h) \rangle = \langle g_1, \ldots, g_n \rangle g_1, \ldots, g_n, g, h,
\]
implies \( c_{n+1} = c_n/\gamma_n = (\gamma_n \cdots \gamma_1)^{-1} \). Finally, we come to the following parametric representation of the potentials and wave functions of the acoustic problem:
\[
\begin{align*}
\varphi_{n+1}^{(\lambda)} &= \frac{\Delta_n(r_1)}{\gamma_n \cdots \gamma_1 \Delta_n}, \quad \varphi_{n+1}^{(\lambda)} = \frac{\Delta_n(\psi_1^{(\lambda)})}{\Delta_n(r_1)}, \quad y_{n+1} = \sum_{k=1}^{n} \frac{1}{\gamma_k r_k + k} + \int r_1^2(\xi) d\xi, \\
\bar{\varphi}_{n+1} &= \frac{1}{r_{n+1}}, \quad \bar{y}_{n+1} = \sum_{k=1}^{n} \frac{r_k + k r_k}{\gamma_k} + \int r_1^2(\xi) d\xi.
\end{align*}
\]

**Example 1.** Let \( u_1 = c^2, \alpha_n = c^2 - \kappa_n^2 \) where \( 0 < \kappa_1 < \ldots < \kappa_{N-1} < \kappa_N = c \). The wave functions are assumed to be
\[
\psi_1^{(\alpha_n)} = \begin{cases} 
\cosh(\kappa_n x + \delta_n), & n = 2k - 1 \\
\sinh(\kappa_n x + \delta_n), & n = 2k
\end{cases}, \quad r_1 = \psi_1^{(\alpha_n)} = \psi_1^{(0)}.
\]

Then the determinants \( \Delta_n \) do not vanish on the real axis and asymptotically \( \Delta_n \sim c^{(\kappa_1 + \cdots + \kappa_n)|x|}, \quad x \to \infty \). Since \( \Delta_{N-1}(r_1) = \Delta_N \) in virtue of the definition of \( r_1 \), hence the potentials \( u_N = c^2 - 2D_x^2 \log \Delta_{N-1} \) and \( \bar{u}_N = c^2 - 2D_x^2 \log \Delta_N \) are the usual multisoliton potentials of the Schrödinger operator raised on \( c^2 \).

The potentials of the acoustic problem, as functions on \( x \), have the asymptotic behavior \( r_N \sim \cosh c x, \bar{r}_N \sim \text{sech } c x \). They are shown on the fig. 2a,d together with the corresponding arguments \( y_N, \bar{y}_N \). Both functions \( y_N, \bar{y}_N \) are monotonically increasing, but \( y_N \) is bounded while \( \bar{y}_N \) grows at infinity as \( \sinh 2c x \). Consider these two cases separately.

1) Choose the value of the antiderivative in \( (52) \) equal to \( c^{-1} \tanh(cx + \delta_N) \) if \( N = 2k - 1 \) and \( c^{-1} \coth(cx + \delta_N) \) if \( N = 2k \). Then it is easy to prove that the scaled variable \( y = cy_N \) changes in the limits from \(-1\) till \( 1 \) and the graph \( w(y) = cr_N^{-2} \) looks like a finite cap (fig. 2b). The phase dependence on \( t \) of the form
\[
\delta_n = \kappa_n(4\kappa_n^2 - 6c^2)t + \delta_n, \quad \tilde{\delta}_n = \text{const}
\]
brings to the solution of the Dym equation (27) on the interval \([-1, 1]\) with zero boundary conditions.
Next, application of the Liouville transformation \[10\] yields the function \(R(z)\) in the parametric form (fig. 2c)

\[ R = r_N(x)\sqrt{1 - y^2(x)}, \quad z = \frac{1}{2}\log\frac{1 + y(x)}{1 - y(x)}. \]

Notice that one of the solitons is “absorbed” by the transformation. One can prove that the phase dependence on \(t\) corresponding to Camassa-Holm equation is of the form

\[ \delta_n = \frac{\kappa_n t}{2c(c^2 - \kappa_n^2)} + \tilde{\delta}_n, \quad n = 1, \ldots, N - 1, \quad \delta_N = -\frac{(1 + \varepsilon c^2)t}{4c^2} + \tilde{\delta}_N. \]

2) The function \(w(y) = r_N^2\), \(y = \bar{y}_N\) has asymptotically linear behavior (fig. 2e). It defines the solution of Dym equation if the phases \(\delta_n\) depend on \(t\) accordingly to (36) and the value of the antiderivative is

\[ \bar{y}_1 = \frac{1}{4c}\sinh 2\delta_N + (-1)^{N-1}\left(\frac{X}{2} + 3c^2t\right) + \text{const}. \]

Example 2. Let \(u_1 = 0\) and \(\alpha_n = -\gamma_n^2\), where \(\gamma_N = 0 < \gamma_1 < \cdots < \gamma_{N-1}\),

\[ \psi^{(\alpha_n)}_1 = \begin{cases} \cosh(\gamma_n x + 4\gamma_n^3t + \delta_n), & n = 2k - 1, \\ \sinh(\gamma_n x + 4\gamma_n^3t + \delta_n), & n = 2k, \end{cases} \quad r_1 = \psi^{(0)}_1 = 1. \]

Then the function \(r_N(x) = \Delta_{N-1}(1)/\Delta_{N-1}\) has \(N - 1\) zeroes. Take \(\bar{y}_1 = x\), then the formulae \(w(y) = r_N^2\), \(y = \bar{y}_N\) define a solution of Dym equation with \(N - 1\) singularities (fig. 2f).
7 Concluding remarks

The main conclusion of the paper can be formulated as follows: in the case of an equation with variable separant it is convenient to represent the chain of Bäcklund transformations in the parametric form as a two-component chain for dependent and independent variables. We have illustrated this by the example of Dym equation where the dressing chain is an extension of the dressing chain for mKdV equation. It is generated by Darboux transformations for the Schrödinger operator in the acoustic gauge, in contrast to the dressing chain for the KdV equation which corresponds to the standard gauge. Both versions merge on the totally discrete level. Camassa-Holm equation finds its place in this picture as well.

Many problems remain beyond the scope of our paper, for example the ultrashort pulse equation [23] related to the sine-Gordon equation. Possibly, the Remark 1 suggests the way of construction of the Bäcklund transformations in this case as well, but the Zakharov-Shabat spectral problem is more likely to be convenient to start from. The two-component analogs of Dym and Camassa-Holm equations which have drawn the attention only recently (see eg. [24]) are also associated to this spectral problem. Comparing to the Schrödinger operator, this case admits more diversity in the Darboux transformations which generate, in particular, the lattices of the Toda and relativistic Toda lattice type.

The recent Degasperis-Procesi equation [25, 26] is of some interest as well. This equation is very close to Camassa-Holm one, but it is associated with the Kaup-Kupershmidt spectral problem of the third order. This makes the realization of the general scheme more difficult since the Darboux transformation for this problem is rather complicated, see eg. [27].

The construction of the extended chains of Darboux and Laplace transformations for $2+1$-dimensional equations with variable separant seems to be straightforward. As a typical example one can take the generalized Dym equation studied in the papers [28, 29].

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