Applications of an Approximate Kerr-like Metric with Quadrupole

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Abstract

In this contribution, we calculate the light deflection, perihelium shift, time delay and gravitational redshift using an approximate metric that contains the Kerr metric and an approximation of the Erez-Rosen spacetime. The results were obtained directly using Mathematica. The results agree with the ones presented in the literature, but they are extended until second order terms of mass, angular momentum and mass quadrupole.

1 Introduction

In 1916, Karl Schwarzschild discovered a solution to the Einstein field equations in vacuum, suitable for describing the spacetime in the empty space surrounding a spherical, static object [16]. Ever since then, this metric has been used to describe a wide range of phenomena, including light deflection close to a massive star, planetary precession of the perihelion, time delay and gravitational redshifts for weak fields. G. Erez and N. Rosen introduced the effects of mass quadrupole as exact solution in 1959 [2, 5]. This derivation had some errors which were corrected by Doroshkevich et al. [3], Winicour et al. [17] and Young and Coulter [19]. The exact solution for a rotating black hole could only be solved as late as 1963 by Roy P. Kerr [8]. There are exact solutions containing the Erez-Rosen and Kerr features, such spacetimes are cumbersome. A new approximate metric representing the spacetime of a rotating deformed body is obtained by perturbing the Kerr metric to include up to the second order of the quadrupole moment. These kinds of approximations are valid because the quadrupole moment is small generally for a variety of astrophysical objects.

In the literature, calculations that include the mass quadrupole are only done using (parameterized) post Newtonian metrics. To introduce the mass quadrupole, the gravitational potential is expressed as a multipolar expansion [9, 10, 11, 12, 13, 14, 15]. In our calculation we perform no such expansion of the gravitational potential. The quadrupole parameter is introduced from the metric.

Now, it is possible to do such calculation in a straightforward manner using software like Mathematica [18]. In this contribution, we present the results of light deflection, perihelium shift, time delay and gravitational redshift using this software. The results were compared with the ones obtained from the Reduce software.

This paper is organized as follows. The classical tests of general relativity are described in section 2. The parametric post-Newtonian formalism is introduced in section 3. The approximate metric with three parameters (M, J = ma, q) is described in section 4. The metric potentials are expanded in a Taylor series up to second order of J, M and q. The resulting metric is transformed into a Hartle-Thorne form. In section 5 we calculate the angle of the deflection of light in traveling in the equatorial plane of our metric. In section 6 we present the necessary calculations to obtain the angle of Precession of the perihelion of the orbit of a planet in the presence of a space-time described by our metric. In section 7 we calculate the time delay of light traveling between two points and in section 8 we obtain the expression for the gravitational redshift in two different positions in our space-time. The Mathematica notebook is available upon request. Our concluding remarks are presented in the last section.

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2 The classical tests

In the solar system, most of the Newtonian mechanics predictions are in good agreement with observations. However, there are a few situations where general relativity (GR) is positioned as a more precise theory. Traditionally, they are Mercury’s perihelion precession, the light deflection by the Sun, the gravitational redshift of light and the time delay of light.

The first classical test was first noted by Le Verrier in 1859, and can be explained by the special relativistic mass of the planet, as well as the mass associated with the energy density of the Sun’s gravitational field. In this phenomenon, classical contributions such as the planetary perturbations influence yet it remains a discrepancy of 42.7″ per century. The contributions from GR reports a value of 42.95″ per century. During the 1960’s and 1970’s there was a considerable controversy on the importance of the contribution of the solar oblateness mass quadrupole \( J_2 \) on the perihelion precession. This discussion has relaxed as the value of the solar quadrupole has been inferred to be small, on the order of \( J_2 = (2.25 \pm 0.09) \times 10^{-7} \) [20]. Using this value, it has been estimated that the contribution to the precession from the solar oblateness is of 0.0286 ± 0.0011″ per century. Yet, its importance can not be specified until a reliable value of the quadrupole is known. The second test, the light deflection due to the massive body of the Sun, was famously first observed during the Eddington’s expedition in 1919, but it was not observed with precision until the 70’s using radio wave interferometry. By this time, it was reported that the mean gravitational deflection was 1.007 ± 0.009 times the value predicted by GR [20]. The deflection caused by the solar oblateness can be treated as a small correction. Typically, it could modify the path of ray of light in 0.2μarcseconds. Other physical property that influences light deflection is the Sun’s angular momentum, as it has been calculated that the Sun’s amount of \( L \approx 2 \times 10^{45} \text{g cm}^2/\text{s} \) can be responsible for a deflection of 0.7μarcseconds [4].

The third test, the gravitational redshift, measures the wavelength shift between two identical clocks placed at rest at different positions in a gravitational field. This was first tested by Pound, Rebka and Snider in the 1960s, as they measured the gamma radiation emitted by \(^{57}\text{Fe} \), as they ascended or descended the Jefferson Physical Laboratory tower [20]. The fourth test, the gravitational time delay, was first observed by Shapiro in 1964 when he discovered that a ray of light propagating in the gravitational field of a massive body will take more time traveling a given distance, than if the field were absent [20]. Gravitational time delay can be observed by measuring the round trip of a radio signal emitted from Earth and reflected from another body, such as another planet or a satellite. To properly measure the effect, it is necessary to do a differential measurement in the variations in the round trip as the target object moves through the sun’s gravitational field. This task is particularly difficult as it involves taking into account the variations in the round trip as a result of the orbital motion of the target relative to Earth [26].

3 The parametrized post-Newtonian formalism

The parametrized post-Newtonian (PPN) formalism is a device that allows the comparison between different theories of gravitation and experiments. It is motivated by the advent of alternative theories of gravitation other than GR during the second half of the twentieth century. It has provided a common framework to quantify deviations from GR which are small in the post-Newtonian order.

As the various theories of gravitation involve mathematical objects such as coordinates, mass variables and and metric tensors, PPN formalism is provided with a set of ten parameters which describe the physical effects of these theories. The so called Eddington-Robertson-Schiff parameters \( \gamma \) and \( \beta \) are the only non-zero parameters, hence they are significant in the study of classical tests. \( \beta \) measures whether gravitational fields do interact with each other, while \( \gamma \) quantifies the space-curvature produced by unit rest mass, and both their values is one in GR [20].

In this context, it is very important to mention Gaia, the ESA space astrometry mission launched in late 2013. Through its detectors, it will perform Eddington-like experiments through the comparison between the pattern of the starfield observed with and without Jupiter. For this purpose, it is vital to have a formula relevant for the monopole and quadrupole light deflection for an oblate planet. These results will provide a new independent determination of \( \gamma \) and evidence of the bending effect of the mass quadrupole of a planet [21], [22]. It is currently accepted that \( |1 - \gamma| \) is less than \( 2 \times 10^{-5} \).

It is also relevant to highlight the use of radiometric range measurements to the MESSENGER spacecraft in orbit around Mercury to estimate the precession of Mercury’s perihelion. Knowing a suitable relationship between this
classical test and the quadrupole allows to decouple $\beta$ and the solar quadrupole $J_2$ to yield $(\beta - 1) = (-2.7 \pm 3.9) \times 10^{-5}$ [24]. It has been conjectured that there is another additional contribution to the perihelion advance from the relativistic cross terms in the post-Newtonian equations of motion between Mercury’s interaction with the Sun and with the other planets, as well from the interaction between Mercury’s motion and the gravitomagnetic field of the moving planets. These effects are planned to be detected by the BepiColombo mission, launched in late 2018 [25].

4 The Metric

The metric, we will employ to do the calculations was generated in a perturbative form using the Kerr spacetime as seed metric. This approximate rotating spacetime with quadrupole moment written in standard form is as follows [6,7]:

$$ds^2 = -\frac{\Delta}{\rho^2} [e^{-\psi} dt - ae^{\psi} \sin^2 \theta d\phi]^2 + \frac{\sin^2 \theta}{\rho^2} [(r^2 + a^2)e^{\psi} d\phi - ae^{\psi} dt]^2 + \rho^2 e^{2\chi} \left( \frac{dr^2}{\Delta} + d\theta^2 \right),$$

where

$$\Delta = r^2 - 2Mr + a^2,$$
$$\rho^2 = r^2 + a^2 \cos^2 \theta,$$
$$\psi = \frac{q}{r^3} P_2 + \frac{3Mq}{r^4} P_2,$$
$$\chi = \frac{qP_2}{r^3} + \frac{1}{3} \frac{Mq}{r^4} (5P_2^2 + 5P_2 - 1) + \frac{1}{9} \frac{q^2}{r^5} (25P_2^3 - 21P_2^2 - 6P_2 + 2).$$

This spacetime has three parameters, namely mass $M$, spin, $J = Ma$ (a as the Kerr rotation parameter) and $q$, the mass quadrupole. It contains the Kerr and the Schwarzschild metrics. This metric is an approximation to the Erez-Rosen metric ($q^3 \sim 0$).

Expanding in Taylor series the metric components of (1) up to second order of $a, J, M$ and $q$ and transforming the metric into a Hartle-Thorne (HT) spacetime form, one gets
\[ g_{\alpha\beta} = - \left( 1 - 2U + \frac{2Q}{r^3}P_2 - \frac{2}{3} \frac{j^2}{r^4} (2P_2 + 1) \right) + \frac{2MQ}{r^4} P_2 + \frac{2Q^2}{r^5} P_2^2 \] 
\[ g_{t\phi} = -\frac{J}{r} \sin^2 \theta \]  \hspace{1cm} (3)
\[ g_{rr} = 1 + 2U + 4U^2 - 2 \frac{Q}{r^2} P_2 \]
\[ + \frac{2}{r^4} \left( 8P_2 - 1 \right) - 10 \frac{MQ}{r^4} P_2 \]
\[ + \frac{1}{12} \frac{Q^2}{r^6} (8P_2^2 - 16P_2 + 77) \]
\[ g_{\theta\theta} = r^2 \left( 1 - 2 \frac{Q}{r^3} P_2 + \frac{J^2}{r^4} P_2 - 5 \frac{MQ}{r^4} P_2 \right) \]
\[ + \frac{1}{36} \frac{Q^2}{r^6} (44P_2^2 + 8P_2 - 43) \]
\[ g_{\phi\phi} = r^2 \sin^2 \theta \left( 1 - \frac{2}{r^3} P_2 + \frac{J^2}{r^4} - 5 \frac{MQ}{r^4} P_2 \right) \]
\[ + \frac{1}{36} \frac{Q^2}{r^6} (44P_2^2 + 8P_2 - 43) \],

where \( U = M/r \) and \( P_2 = (3 \cos^2 \theta - 1)/2 \). The HT form is a more convenient way to calculate the quantities we are going to obtain, because it is in Schwarzschild spherical coordinates.

### 5 The Geodesic Equation

The space-time interval between two events is defined as,

\[ ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta. \] \hspace{1cm} (4)

We can equate the interval with a proper time \( d \tau \) and so write down the following equation,

\[ \mu = g_{\alpha\beta} \frac{dx^\alpha}{d \tau} \frac{dx^\beta}{d \tau}, \] \hspace{1cm} (5)

where \( \mu \) is a parameter to be defined. For massive particles moving across spacetime its trajectories are described by time-like intervals (\( ds^2 < 0 \)), so we set \( \mu = +1 \), while light trajectories are described by light-like intervals (\( ds^2 = 0 \)) and so we set \( \mu = 0 \). The former case is suitable for describing planetary motion, as its the case for planetary perihelium, while light deflection and time delay, which are light related, are described by the later. The geodesic equations help to calculate the path with the shortest proper time between two points,

\[ \frac{d}{d \tau} \left( g_{\alpha\beta} \frac{dx^\beta}{d \tau} \right) - \frac{1}{2} \partial_{\alpha} g_{\mu\nu} \frac{dx^\mu}{d \tau} \frac{dx^\nu}{d \tau} = 0. \] \hspace{1cm} (6)

The geodesic equation is related to conserved quantities, as in our case when we set \( \alpha = t \),

\[ \frac{d}{d \tau} \left( g_{tt} \frac{dx^t}{d \tau} + g_{t\phi} \frac{dx^\phi}{d \tau} \right) = 0. \] \hspace{1cm} (7)

We can set the conserved quantity with energy \( E \).
When we set $\alpha = \phi$ we obtain a conserved quantity related to angular momentum along the $z$-axis, $L_z$,

$$g_{\phi\phi} \frac{dx^\phi}{d\tau} + g_{\phi\phi} \frac{d\phi}{d\tau} = L_z. \quad (9)$$

These relations can be reversed to obtain:

$$\frac{dt}{d\tau} = -\frac{1}{\rho^2} [-Eg_{\phi\phi} - g_{\phi\phi} L_z], \quad (10)$$

$$\frac{d\phi}{d\tau} = -\frac{1}{\rho^2} [g_{\phi\phi} L_z + Eg_{\phi\phi}], \quad (11)$$

where $\rho^2 = g_{\phi\phi}^2 - g_{\phi\phi} g_{\phi\phi}$. Equations (10) and (11) can be combined to,

$$\frac{d\phi}{dt} = \frac{d\phi}{d\tau} \frac{d\tau}{dt} = \frac{g_{\phi\phi} L_z + Eg_{\phi\phi}}{-Eg_{\phi\phi} - g_{\phi\phi} L_z}. \quad (12)$$

## 6 Light deflection

We set $\mu = 0$ in (5) and rearranging provides an equation for $dr/dt$. We can use the substitution $u = 1/r$ to obtain up to order $O(M^2, Q^2, J^2)$:

$$\frac{d^2 u}{d\phi^2} = -2J^2 \frac{E^3}{L_z^3} + \left(12J^2 \frac{E^4}{L_z^4} - 8JM \frac{E^3}{L_z^3} - 1\right) u$$

$$+ \left(3Q \frac{E^2}{L_z^2} + 3M\right) u^2$$

$$+ \left(-24JQ \frac{E^3}{L_z^3} + 34J^2 \frac{E^2}{L_z^2} + 10MQ \frac{E^2}{L_z^2}\right) u^3$$

$$+ \left(- \frac{81J^2}{2} + \frac{3MQ}{2} - \frac{93}{4} Q^2 \frac{E^2}{L_z^2}\right) u^5 + 33Q^2 u^7. \quad (13)$$

This equation can only be solved by perturbation theory. For this purpose, we propose a solution of the form
\[ u = u_0 \cos \phi + c_0 u_m \]
\[ + \ J \left( u_0^2 u_{J1} + u_3^2 u_{J2} u_m + u_0 u_{J3} u_m^2 + u_{J4} u_m^3 \right) \]
\[ + \ M \left( u_0^2 u_{M1} + u_0 u_m u_{M2} + u_m^2 u_{M3} \right) \]
\[ + \ Q \left( u_0^2 u_{Q1} + u_3^2 u_m u_{Q2} + u_0 u_m^2 u_{Q3} + u_0 u_m^3 u_{Q4} + u_m^4 u_{Q5} \right) \]
\[ + \ J^2 \left( u_0^2 u_{JJ1} + u_3^2 u_{JJ2} u_m + u_{JJ3} u_m^3 \right) \]
\[ + \ u_0^2 u_{JJ4} u_m^5 + u_0 u_{JJ5} u_m^5 + u_{JJ6} u_m^5 \]
\[ + \ J M \left( u_0^5 u_{MJ1} + u_0 u_m u_{MJ2} + u_m^3 u_{MJ3} \right) \]
\[ + \ u_0 u_m^3 u_{MJ4} + u_m^4 u_{MJ5} \]
\[ + \ M^2 \left( u_0^5 u_{MM1} + u_0^3 u_m u_{MM2} + u_0 u_m^2 u_{MM3} + u_m^3 u_{MM4} \right) \]
\[ + \ M Q \left( u_0^5 u_{MQ1} + u_0^3 u_m u_{MQ2} + u_m^3 u_{MQ3} + u_m^4 u_{MQ4} \right) \]
\[ + \ u_0 u_m^4 u_{MQ5} + u_m^5 u_{MQ6} \]
\[ + \ Q^2 \left( u_0^5 u_{QQ1} + u_0^3 u_m u_{QQ2} + u_0 u_m^2 u_{QQ3} + u_0 u_m^3 u_{QQ4} + u_0^5 u_m^4 u_{QQ5} + u_0^3 u_m^5 u_{QQ6} + u_0 u_m^6 u_{QQ7} + u_m^7 u_{QQ8} \right) \]
\[ + \ Q J \left( u_0^5 u_{QJ1} + u_0^3 u_m u_{QJ2} + u_0 u_m^2 u_{QJ3} + u_0 u_m^3 u_{QJ4} + u_0^5 u_m^4 u_{QJ5} + u_0 u_m^5 u_{QJ6} + u_m^6 u_{QJ7} \right). \]

This method brings up a number of equations of the form:

\[ \frac{d^2 u_{04}}{d\phi^2} = -u_{04} + 4 \frac{E^3}{L^2} \cos \phi, \]  \hspace{1cm} (15)

or,

\[ \frac{d^2 u_{11}}{d\phi^2} = -u_{11} + 3 \cos^2 \phi, \]  \hspace{1cm} (16)

and so on. For this part, we stuck to the general solutions to the differential equation as in \([1]\).

\[ \frac{d^2 y}{dx^2} + y = \cos(nx) \]

to be

\[ y = -\frac{1}{n^2 - 1} \cos nx \]

for \(n \neq 1\) and

\[ y = \frac{\phi}{2} \sin \phi \]

for \(n = 1\). The approximate solution is:
\[ u = u_0 \cos \phi - 2Ju_m^3 + \frac{1}{2}Mu_0^2(3 - \cos 2\phi) \]
\[ + \frac{1}{2}Qu_0^2u_m^2(3 - \cos 2\phi) \]
\[ - \frac{81}{32}J^2u_0^5\left(5\phi \sin \phi + \frac{5}{2} \cos 3\phi + \frac{1}{24} \cos 5\phi\right) \]
\[ + J^2u_0^3u_m^2\left(\frac{51}{4} \phi \sin \phi + \frac{17}{16} \cos 3\phi\right) \]
\[ + 6J^2u_0^4u_m \phi \sin \phi \]
\[ + 2JM\phi_0^4 \phi \sin \phi \]
\[ + JQu_0^3u_m^3\left(\frac{3}{4} \cos 3\phi - 9\phi \sin \phi\right) \]
\[ - 6JQu_0^5u_m^4 \phi \sin \phi \]
\[ + M^2u_0^3\left(\frac{15}{4} \phi \sin \phi - \frac{3}{16} \cos 3\phi\right) \]
\[ + MQu_0^5\left(\frac{15}{2} \phi \sin \phi + \frac{15}{16} \cos 3\phi + \frac{1}{16} \cos 5\phi\right) \]
\[ + MQu_0^3u_m^2\left(\frac{45}{4} \phi \sin \phi - \frac{1}{16} \cos 3\phi\right) \]
\[ + \frac{33}{64}Q^2u_0^7\left(\frac{35}{2} \phi \sin \phi + \frac{21}{8} \cos 3\phi \right) \]
\[ + \frac{7}{24} \cos 5\phi + \frac{1}{48} \cos 7\phi \]
\[ - \frac{93}{64}Q^2u_0^4u_m^2\left(5\phi \sin \phi + \frac{5}{8} \cos 3\phi + \frac{1}{24} \cos 5\phi\right) \]
\[ + Q^2u_0^3u_m^4\left(\frac{15}{4} \phi \sin \phi - \frac{3}{16} \cos 3\phi\right) \] (17)

The closest approach \( u_m \) occurs when \( \phi = 0 \), so:

\[ u_m = u_0 - 2Ju_m^3 + Mu_0^2 + Qu_0^2u_m^2 \]
\[ - J^2u_0^3\left(\frac{27}{16}u_0^2 + \frac{17}{16}u_m^2\right) \]
\[ + \frac{3}{4}JQu_0^3u_m^3 - \frac{3M^2u_0^3}{16} \]
\[ + MQu_0^3\left(u_0^2 - \frac{1}{16}u_m^4\right) \]
\[ + Q^2u_0^3\left(\frac{1551u_0^4}{1024} - \frac{31}{32}u_0^4u_m^2 - \frac{3}{16}u_m^4\right) \] (18)

The deflection angle \( \Delta \phi = 2\delta \) can be found using the condition \( u(\pi/2 + \delta) = 0 \), that is:
\[
\Delta \phi = 4Mu_m - 4Ju_m^2 + 4Qu_m^3 \\
+ \left(8 + \frac{195}{32}\pi\right)J^2u_m^4 + 2(2 + \pi)JM^2u_m^3 \\
+ (4 - 15\pi)Ju_m^5 - \left(4 - \frac{15}{4}\pi\right)M^2u_m^2 \\
- \left(8 - \frac{75}{4}\pi\right)MQ^2u_m^4 \\
+ \left(\frac{705}{128}\pi - 4\right)Q^2u_m^6.
\] (19)

7 Precession of the perihelion

First, we use the geodesic equation (6) to find the conserved quantities, and the equations (10) and (11). Using these new identities, it is possible to calculate \(\frac{dr}{d\tau}\) setting \(\mu = 1\) in (5) and imposing a planar orbit \((\theta = \pi/2)\). After this, the well known variable change \(u = 1/r\) is used, so it is possible to find \(u = u(\phi)\) by means of:

\[
\frac{du}{d\phi} = \frac{du}{d\tau} \frac{d\tau}{d\phi}.
\] (20)

After taking the second derivative with respecto to \(\phi\), we found up to order \(O(M^2, Q^2, J^2)\), the result is:
\[
\frac{d^2u}{d\phi^2} = \frac{2J E}{L_z^2} - \frac{2J E^3}{L_z^2} + \frac{M}{L_z^2} + u \left( \frac{12J^2 E^4}{L_z^2} - 8JM \frac{E^3}{L_z^2} - 12J^2 E^2 \frac{L_z}{L_c^2} - 2M^2 \frac{E^2}{L_z^2} - 1 \right) + u^2 \left( 3M + 3Q \frac{E^2}{L_z^2} - 3 \frac{Q^2}{L_c^2} \right) + u^3 \left( -24JM \frac{E^3}{L_z^2} + 34J^2 \frac{E^2}{L_z^2} + 10MQ \frac{E^2}{L_z^2} - 16JM \frac{E}{L_z^2} - 34J^2 \frac{L_z}{L_c^2} - 4M^2 \right) + u^5 \left( \frac{93}{4} Q \frac{E^2}{L_z^2} - \frac{81}{2} \frac{J^2}{L_z^2} + \frac{111}{4} \frac{Q^2}{L_z^2} + \frac{3}{2} MQ \right) + 33Q^2 u^7
\]

(21)

As a first approximation, an orbit with constant radius \( r_c = 1/u_c = L_c/E \) meets the condition that \( \frac{d^2u}{d\phi^2} = 0 \), so we can consider a perturbation \( u = u_c + u_c w(\phi) \), where \( w \) is the wobble function we want to find. As such, \( w \) satisfies the harmonic equation:

\[
\frac{d^2w}{d\phi^2} + w = \left( \frac{6M}{r_c} + \frac{3Q}{r_c} \left( \frac{1}{r_c^2} - \frac{1}{L_z^2} \right) - \frac{3J^2}{r_c^2} \left( \frac{38}{L_z^2} + \frac{59}{2r_c^2} \right) - \frac{8JM}{r_c^3} + 24 \frac{JQ}{r_c^3} \left( \frac{2}{L_z^2} - \frac{3}{r_c^2} \right) - 2M^2 \left( \frac{1}{L_z^2} + \frac{1}{r_c^2} \right) + \frac{75}{2} \frac{MQ}{r_c^3} + \frac{3}{4} \frac{Q^2}{r_c^3} \left( \frac{185}{L_z^2} + \frac{153}{r_c^2} \right) \right) w
\]

(22)

Which provides an angular frequency \( \omega \) value for which \( w = A \cos(\omega \phi + \phi_0) \). The orbit perihelion \( \Delta \phi \) occurs when \( w(\phi) \) is a minimum, i.e. when the argument of the cosine function is \( \pi + 2\pi n \). \( \Delta \phi \) can be found using the condition \( \omega \Delta \phi = 2\pi \), this implies:

\[
\Delta \phi = 6\pi \frac{M}{r_c} - 3\pi \frac{Q}{r_c} \left( \frac{1}{L_z^2} - \frac{2}{r_c^2} \right) - 3\pi \frac{J^2}{r_c^2} \left( \frac{38}{L_z^2} + \frac{59}{2r_c^2} \right) - \frac{8\pi JM}{r_c^3} + 24 \pi \frac{JQ}{r_c^3} \left( \frac{2}{L_z^2} - \frac{3}{r_c^2} \right) - \pi M^2 \left( \frac{2}{L_z^2} - \frac{17}{r_c^2} \right) - 3\pi \frac{MQ}{r_c^3} \left( \frac{9}{L_z^2} - \frac{61}{2r_c^2} \right) + \frac{3}{4} \pi \frac{Q^2}{r_c^3} \left( \frac{9}{L_z^2} + \frac{149}{L_z^2 r_c^2} + \frac{189}{r_c^2} \right)
\]

(23)
8 Time delay

The curvature induced in the spacetime surrounding a massive body increases the travel time of light rays relative to what would be the case in flat space. Let $b$ be the maximum approach distance of a ray of light traveling near a massive body. If the beam traveled in a straight line, then $r \cos \phi = b$. This means $d\phi = \frac{bdr}{\sqrt{r^2 - b^2}}$. By using $d\theta = 0$, it is possible then to extract $dt$ from $0 = g_{\mu\nu}dx^\mu dx^\nu$, so we obtain:

$$
\begin{align*}
\frac{dt}{dr} &= \frac{dr}{\sqrt{r^2 - b^2}} \left[ 2M + r - 2Jb \frac{r}{r^2} - \frac{Mb^2}{r^2} - \frac{Q}{r^2} \
&\quad - 5J^2 \frac{r^3}{r^3} + 27J^2 b^2 \frac{r^3}{r^3} - 4 \frac{JMb}{r^3} - 2 \frac{JQb}{r^3} \
&\quad + 4 \frac{M^2}{r} - 2 \frac{M^2 b^2}{r^3} - \frac{1}{2} \frac{M^2 b^4}{r^5} - \frac{5MQ}{r^3} - \frac{5MQb^2}{4r^3} \right] \\
&\quad + \left[ \frac{31}{8} \frac{Q^2}{r^5} - \frac{33}{8} \frac{Q^2 b^2}{r^7} \right].
\end{align*}
$$

(24)

Performing an integration to go from a planet at position $r_e$, to another planet at $r_p$, to find the time delay:
\[ \Delta t = d_e + d_p \\
= 2J \left( \frac{b}{r_e d_e} + \frac{b}{r_p d_p} - \frac{r_e}{b d_e} - \frac{r_p}{b d_p} \right) \\
+ 2M \log \left( \frac{(r_e + d_e)(r_p + d_p)}{b^2} \right) \\
+ M \left( \frac{b^2}{r_e d_e} + \frac{b^2}{r_p d_p} - \frac{r_e}{d_e} - \frac{r_p}{d_p} \right) \\
+ Q \left( \frac{d_e}{b^2 r_e} + \frac{d_p}{b^2 r_p} \right) \\
- \frac{27}{16} J^2 \left( \frac{b^2}{r_e^2 d_e} - \frac{b^2}{r_p^2 d_p} \right) \\
+ \frac{1}{32} J^2 \left( \pi - \frac{\theta_e}{b^3} + \frac{\theta_p}{b^3} + \frac{1}{b^2 d_e} + \frac{1}{b^2 d_p} \right) \\
+ 2JM \left( \frac{b}{r_e^2 d_e} + \frac{b}{r_p^2 d_p} - \frac{1}{b d_e} - \frac{1}{b d_p} \right) \\
+ 2JM \left( \frac{\theta_e}{b^2} + \frac{\theta_p}{b^2} - \frac{\pi}{b^2} \right) \\
+ \frac{1}{2} JQ \left( \frac{b}{r_e^2 d_e} + \frac{b}{r_p^2 d_p} \right) \\
+ \frac{3}{4} JQ \left( \frac{1}{b^3 d_e} + \frac{1}{b^3 d_p} + \frac{\pi}{b^4} \right) \\
+ \frac{3}{4} JQ \left( \frac{1}{b^4 d_e} + \frac{1}{b^4 d_p} \right) \\
+ \frac{1}{8} M^2 \left( \frac{b^4}{r_e^2 d_e} + \frac{b^4}{r_p^2 d_p} + \frac{9}{16} M^2 \left( \frac{b^2}{r_e^2 d_e} + \frac{b^2}{r_p^2 d_p} \right) \right) \\
+ \frac{37}{16} M^2 \left( \frac{\pi}{b} - \frac{\theta_e}{b^3} - \frac{\theta_p}{b^3} - \frac{1}{b^2 d_e} + \frac{1}{b^2 d_p} \right) \\
+ \frac{5}{16} MQ \left( \frac{b^2}{r_e^2 d_e} + \frac{b^2}{r_p^2 d_p} \right) \\
+ \frac{65}{32} MQ \left( \frac{\pi}{b^3} - \frac{\theta_e}{b^3} - \frac{\theta_p}{b^3} + \frac{1}{b^2 d_e} + \frac{1}{b^2 d_p} \right) \\
- \frac{75}{32} MQ \left( \frac{1}{r_e^2 d_e} + \frac{1}{r_p^2 d_p} \right) \\
+ \frac{11}{16} Q^2 \left( \frac{b^2}{r_e^2 d_e} + \frac{b^2}{r_p^2 d_p} \right) \\
+ \frac{21}{128} Q^2 \left( \frac{\pi}{b^3} - \frac{\theta_e}{b^3} - \frac{\theta_p}{b^3} + \frac{1}{b^4 d_e} + \frac{1}{b^4 d_p} \right) \\
- \frac{7}{128} Q^2 \left( \frac{1}{b^2 r_e^2 d_e} + \frac{1}{b^2 r_p^2 d_p} \right) \\
- \frac{51}{64} Q^2 \left( \frac{1}{r_e^4 d_e} + \frac{1}{r_p^4 d_p} \right) \right) \"]
Figure 4: Gravitational redshift.

where \( d_e = \sqrt{r_e^2 - b^2} \), \( d_p = \sqrt{r_p^2 - b^2} \), \( \theta_e = \sin^{-1}(b/r_e) \), and \( \theta_p = \sin^{-1}(b/r_p) \).

9 Gravitational redshift

It is possible to calculate a redshift factor by comparing the proper time for observers located at two different values of \( r \).

\[
\frac{\lambda_r}{\lambda_{r_e}} = \sqrt{\frac{g_{tt}(r)}{g_{tt}(r_e)}} \approx 1 + M \left( \frac{1}{r_e} - \frac{1}{r_r} \right) + \frac{Q}{2} \left( \frac{1}{r_e^3} - \frac{1}{r_r^3} \right) + \frac{3}{2} M^2 \left( \frac{1}{r_e^2} - \frac{2}{r_e r_r} - \frac{1}{r_r^2} \right) + MQ \left( \frac{2}{r_e^3} - \frac{1}{2 r_e^3 r_r} - \frac{1}{2 r_e r_r^3} - \frac{1}{r_r^4} \right) + Q^2 \left( \frac{1}{8 r_e^6} - \frac{1}{4 r_e^3 r_r^3} + \frac{1}{8 r_r^6} \right)
\]

(26)

10 Conclusions

We reviewed the calculations of the classical experiments in GR with an approximative metric and taking in account all second order terms of mass, angular momentum and mass quadrupole. If we neglect these terms our results agree with the ones in the literature.

In PPN theory these results were obtained, but in this theory the quadrupole moment is introduced in the expansion of the mass potential. Here, this effect is introduced by the metric in a straightforward way.

As future work, it would be interesting to include the spin octupole and the mass hexadecapole, because now, these relativistic multipoles are currently considered in neutron stars calculations.

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