Simplifying 4d $\mathcal{N} = 3$ Harmonic Superspace

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ABSTRACT

We quantize super Yang-Mills action in $\mathcal{N} = 3$ harmonic superspace using “Fermi-Feynman” gauge and also develop the background field formalism. This leads to simpler propagators and Feynman rules, useful in performing explicit calculations. The superspace rules are used to show that divergences do not appear at 1-loop and beyond. We also compute a finite contribution to the effective action from a 4-point diagram at 1-loop, which matches the expected covariant result.
1 Introduction

\( \mathcal{N} = 3 \) harmonic superspace in four-dimensions was developed by GIKOS around three and a half decades ago \([1,2]\) and it provided the first successful off-shell formulation of 4d \( \mathcal{N} = 3 \) super Yang-Mills (SYM) theory. This theory was quantized in “Landau” gauge few years later by Delduc & McCabe \([3]\), however, the propagators obtained did not lend themselves to easier calculations. As is well-known, the field content of \( \mathcal{N} = 3 \) vector multiplet is the same as that of \( \mathcal{N} = 4 \) and Zupnik explicitly showed this hidden supersymmetry of the \( \mathcal{N} = 3 \) SYM in \([4]\). The \( \mathcal{N} = 3 \) superspace also manifests the full superconformal symmetry and using such symmetry arguments, low-energy effective action for \( \mathcal{N} = 3 \) and \( \mathcal{N} = 4 \) were considered by Zupnik and collaborators in \([5–8]\). A ‘twistorial’ perspective on the \( \mathcal{N} = 3 \) SYM action was presented in \([9]\) a few years ago but no concrete progress has yet happened “to bring the quantization scheme into a form suitable for computations” \([2]\).

We present some progress here in that direction of simplifying computations. We chose “Fermi-Feynman” gauge to drastically reduce the number (9 → 1) and simplify the form (\{chiral, antichiral, linear\}-analytic → just analytic) of propagators, compared to \([3]\). This simplifies the proof of the nonrenormalization theorem as one might expect. Moreover, we also introduce the background field formalism in the \( \mathcal{N} = 3 \) harmonic superspace to simplify computations further.

In Section 2, we review the basic ingredients of the \( \mathcal{N} = 3 \) harmonic superspace and the \( \mathcal{N} = 3 \) SYM action. In Section 3 we introduce the “Fermi-Feynman” gauge to gauge-fix this SYM action and derive the propagators. As an application, we prove the nonrenormalization theorem. In Section 4 we introduce the background field gauge to simplify the diagrammatic computations and present a sample calculation. Finally, we conclude with some discussion in Section 5.
2 Review

Our notation will closely follow [3] and we review it here for orientation purposes. The full 4d \( \mathcal{N} = 3 \) superspace has the usual set of ordinary bosonic \((x^{\alpha \dot{\alpha}})\) and fermionic \((\theta^{a}, \bar{\theta}^{\dot{a}})\) coordinates with \(i = 1, 2, 3\). The harmonic superspace augments these by six internal bosonic coordinates of the R-symmetry coset \(SU(3)/U(1) \times U(1)\), denoted collectively as \(u\). Using these internal coordinates, an ‘analytic’ subspace with 8 out of the 12 \(\theta\)’s of the full superspace is identified, which allows one to construct an off-shell action for the \(\mathcal{N} = 3\) SYM and proceed with its quantization. We discuss the internal coordinates in some detail first, then the fermionic ones, and finally the superspace action of \(\mathcal{N} = 3\) SYM in this section. The quantization is dealt with in subsequent sections.

2.1 Internal Coordinates

A \(SU(3)/U(1) \times U(1)\) coset element can be parameterized in matrix form as follows

\[
U = \left( u_{i}^{1}, u_{i}^{2}, u_{i}^{3} \right) \equiv \left( u_{i}^{(1,1)}, u_{i}^{(-1,1)}, u_{i}^{(0,-2)} \right) ; \quad \bar{U}^{\dagger} = \left( \bar{u}_{i}^{1}, \bar{u}_{i}^{2}, \bar{u}_{i}^{3} \right) \equiv \left( \bar{u}_{i}^{(-1,-1)}, \bar{u}_{i}^{(1,-1)}, \bar{u}_{i}^{(0,2)} \right)
\]  

(2.1)

Constraints: \(\bar{U}^{\dagger}U = UU^{\dagger} = 1\); \(\det U = 1\)

\[
\Rightarrow \bar{u}_{a} \cdot u^{b} = \bar{u}_{a}^{i}u_{i}^{b} = \delta_{a}^{b}, \quad u_{a}^{i}\bar{u}_{a}^{j} = \delta_{i}^{j}; \quad \epsilon^{ijk}u_{a}^{i}u_{j}^{a}u_{k}^{a} = 1.
\]  

(2.2)

The notation \((q_{1}, q_{2})\) denotes the charges corresponding to the two Cartan \(U(1)\) generators \(Q_{1}, Q_{2}\) of \(SU(3)\). Given the constraints in (2.2), we have eight independent coordinates in \(U\)-matrix as expected of \(SU(3)\) but also requiring that the two \(U(1)\) charges are fixed \(i.e., Q_{i}u = q_{i}u\) implements the \(U(1)^{2}\) quotient and effectively, we have six independent coordinates.

Furthermore, the six harmonic covariant derivatives acting on these coordinates are

\[
D_{a}^{b} = u_{i}^{a} \frac{\partial}{\partial u_{i}^{b}} - \bar{u}_{i}^{a} \frac{\partial}{\partial \bar{u}_{i}^{a}} \quad (a \neq b).
\]  

(2.3)

These derivatives satisfy the \(SU(3)\) Lie algebra given by

\[
[D_{b}^{a}, D_{d}^{c}] = \delta_{b}^{c}D_{d}^{a} - \delta_{d}^{c}D_{b}^{a}.
\]  

(2.4)

We note here that the two Cartan generators are given in terms of \(D_{a}^{a}\) \((\text{no sum over } a)\) as follows

\[
Q_{1} = D_{1}^{1} - D_{2}^{2}, \quad \Rightarrow \quad D_{1}^{1} - D_{3}^{3} = \frac{1}{2} (Q_{1} + Q_{2})
\]

\[
Q_{2} = D_{1}^{1} + D_{2}^{2} - 2D_{3}^{3}, \quad \Rightarrow \quad D_{2}^{2} - D_{3}^{3} = \frac{1}{2} (Q_{2} - Q_{1}).
\]  

(2.5)

Their commutators with the harmonic derivatives are \([Q_{i}, D_{a}^{b}] = q_{i}D_{a}^{b}\) with the charges given by

| \(U(1)^{2}\) | \(D_{1}^{1}\) | \(D_{2}^{2}\) | \(D_{1}^{3}\) | \(D_{3}^{3}\) | \(D_{1}^{2}\) | \(D_{2}^{3}\) |
|---|---|---|---|---|---|---|
| \(q_{1}\) | 1 | 1 | 2 | -1 | -1 | -2 |
| \(q_{2}\) | 3 | -3 | 0 | -3 | 3 | 0 |
In what follows, we will mostly be dealing with functions defined at 2 different points in this coset space, labelled as \( u \) and \( v \). We denote their products by the notation \( U_{ab}^a = u^a \cdot \bar{v}_b \) and \( \bar{U}_{ab}^b = \bar{u}_a \cdot v^b \) such that the covariant derivatives in this basis simply read

\[
D_b^a = U^a_c \frac{\partial}{\partial U_b^c} - \bar{U}_b^c \frac{\partial}{\partial \bar{U}_a^c} \quad (a \neq b).
\]  

(2.6)

Finally, the integration over this coset space is defined such that only a \( SU(3) \) singlet integrand gives a non-vanishing result, \( i.e., \)

\[
\int du \, 1 = 1; \quad \int du \, D_b^a f(u) = 0.
\]  

(2.7)

The latter integral allows one to integrate by parts in the \( u \)-space.

### 2.2 Fermionic Coordinates

We make a coordinate transformation of the usual \( \theta \)'s with \( SU(3) \) indices to \( \theta \)'s having definite \( U(1) \) charges as follows

\[
\theta^a = \bar{u}_i^a \theta_i^a, \quad \bar{\theta}^a = u^a_i \bar{\theta}_i^a.
\]

(2.8)

The index \( a \) identifies the \( U(1) \) charges straightforwardly via (2.1). Then, the corresponding spinorial covariant derivatives satisfy the following commutators

\[
\{ D_a^a, D_b^b \} = 0, \quad \{ D_a^a, D_b^\beta \} = 0, \quad \{ D_a^a, D_b^\beta \} = i \delta_\beta^\alpha \partial_{\alpha\beta}.
\]

(2.9)

\[
[D_a^a, D_b^\beta] = \delta_\beta^\alpha D_a^\alpha, \quad [D_a^a, \bar{D}_b^\beta] = -\delta_\beta^\alpha \bar{D}_b^\alpha.
\]

(2.10)

Explicitly, the \( U(1)^2 \) charges of the spinorial derivatives are

\[
\begin{array}{c|cccccc}
U(1)^2 & D_1^a & D_2^a & D_3^a & \bar{D}_1^\alpha & \bar{D}_2^\alpha & \bar{D}_3^\alpha \\
\hline
q_1 & 1 & -1 & 0 & -1 & 1 & 0 \\
q_2 & 1 & 1 & -2 & -1 & -1 & 2
\end{array}
\]

The harmonic superspace is an analytic subspace of the full superspace, where the coordinates \( \theta_1^a \) and \( \bar{\theta}^{2\dot{a}} \) do not appear explicitly in a given harmonic superfield \( \Phi^{(q_1, q_2)}(x, \theta, u) \), \( i.e., \)

\[
D_a^a \Phi^{(q_1, q_2)} = \bar{D}_2^{2\dot{a}} \Phi^{(q_1, q_2)} = 0.
\]

(2.11)

Note that these analytic constraints are preserved by the three harmonic derivatives: \( D_1^1, D_3^1, D_2^3 \).

Finally, we can define an analytic measure \( \int du \, d\zeta \) on harmonic superspace via the full superspace as follows

\[
\int d^4 x \, d^4 \theta \, du \equiv \int du \, d\zeta_{11}^{22} (D_1^1)^2 (D_2^2)^2 \Rightarrow \int d\zeta_{11}^{22} = \int d^4 x A (D^2)^2 (D^3)^2 (\bar{D}_1)^2 (\bar{D}_3)^2.
\]

(2.12)

We frequently use the notation \( [D_1^{(1)}]_{22}^{11} = (D_1^1)^2 (D_2^2)^2 \) to denote the four \( \theta \)'s that are not part of the harmonic superspace. The \( [D_1^{(1)}]_{22}^{11} \) has \( U(1)^2 \) charge \( (4, 0) \), negative of that for the measure \( d\zeta_{11}^{22} \).
2.3 SYM Action

We do not review here the procedure for finding prepotentials of the $\mathcal{N} = 3$ SYM in harmonic superspace but just state the results. The $\mathcal{N} = 3$ prepotentials are the gauge connections of the analyticity-preserving harmonic derivatives, \textit{i.e.}, we have 3 connections defined by $\nabla = D + iA$. The gauge transformations read as usual: $\delta A = -\nabla \lambda$. The field strengths are introduced via the ‘flat’ commutation relations as follows

$$\left[\nabla^1_2, \nabla^3_4\right] = F^{11}_{23}, \quad \left[\nabla^3_2, \nabla^1_4\right] = F^{31}_{22}, \quad \left[\nabla^1_3, \nabla^3_2\right] = \nabla^1_2 + F^1_2. \quad (2.13)$$

The equations of motion are, of course, all $\text{gauge transformations read as usual: } \delta A = -\nabla \lambda$. The field strengths are introduced via the ‘flat’ Chern-Simons-like action

$$S = \text{tr} \int du \, d\xi_{11} \left( A^1_2 F^1_2 + A^3_2 F^{11}_{23} + A^1_2 F^{31}_{22} + iA^1_2 [A^1_3, A^3_2]\right). \quad (2.14)$$

Also, notice that one of the 3 prepotentials is related algebraically to the other two on-shell

$$D^1_3 A^2_2 - D^2_3 A^1_3 + i[A^1_3, A^3_2] = A^1_2, \quad (2.15)$$

from where we start the quantization procedure in the next section.

3 Quantizing SYM in “Fermi-Feynman” Gauge

The $\mathcal{N} = 3$ SYM action (2.14) after substituting the algebraic equation defining $A^1_2$ (2.15) depends only on two harmonic connections and reads

$$S = \text{tr} \int du \, d\xi_{11} \left\{ (D^1_3 A^3_2)^2 + (D^3_2 A^3_1)^2 + 2A^1_3 (D^2_3 D^3_2 A^3_2) - 2A^1_3 D^1_3 A^3_2 \right. \left. + 2i[A^1_3, A^3_2] \left( D^1_3 A^3_2 - D^2_3 A^3_1 \right) - [A^1_3, A^3_2]^2 \right\}. \quad (3.1)$$

We choose the following gauge-fixing function

$$S_{gf} = -\text{tr} \int du \, d\xi_{11} \left( D^1_3 A^3_2 + D^3_2 A^3_1 \right)^2 = -\text{tr} \int du \, d\xi_{11} \left\{ (D^1_3 A^3_2)^2 + (D^3_2 A^3_1)^2 - 2A^1_3 (D^2_3 D^3_2 A^3_2) \right\}, \quad (3.2)$$

such that the gauge-fixed action for SYM in “Fermi-Feynman” gauge becomes

$$S + S_{gf} = \text{tr} \int du \, d\xi_{11} \left\{ 2A^1_3 \left(D^1_3 D^3_2 + D^3_2 D^3_1 - 2D^1_2 \right) A^3_2 + 2i[A^1_3, A^3_2] \left( D^1_3 A^3_2 - D^3_2 A^3_1 \right) - [A^1_3, A^3_2]^2 \right\}, \quad (3.3)$$

where we used $[D^1_3, D^3_2] = D^1_2$ once. The ghost action follows from the BRST formalism straightforwardly by using $\delta A = -D\lambda - i[A, \lambda]$:

$$S_{gh} = -\text{tr} \int du \, d\xi_{11} \left\{ b^1_2 \left( D^1_3 D^3_2 + D^3_2 D^3_1 \right) c + i(D^1_2 b^1_2 [A^1_3, c] + D^3_2 b^3_2 [A^1_3, c]) \right\}. \quad (3.4)$$
Having just introduced new fields, let us recap the $U(1)^2$ charges of all the fields here (which are straightforwardly deduced from the covariant derivatives):

| $U(1)^2$ | $A^1_3$ | $A^3_2$ | $A^1_2$ | $\lambda$ | $c$ | $b^1_2$ |
|----------|--------|--------|--------|--------|-----|--------|
| $q_1$    | 1      | 1      | 2      | 0      | 0   | 2      |
| $q_2$    | 3      | -3     | 0      | 0      | 0   | 0      |

### 3.1 Propagators

From the gauge-fixed SYM and ghost actions given above, we can derive the equations to solve for the Green’s functions for vector and ghost superfields:

\[
(K^2_0)_{00}^0(1, 2) = |\delta A^1_{22} (1, 2) \\
(K^2_0)_{10}^{10}(1, 2) = |\delta A^{11}_{220}(1, 2) \\
(K^2_1)_{+1}^{13}(1, 2) = |\delta A^{11}_{32}(1, 2) \\
(K^2_1)_{-1}^{13}(1, 2) = |\delta A^{11}_{23}(1, 2)
\]

(3.5)

where, $(K^2_0)_{a} = (1 + |a|) \{D^3_3, D^3_2\} + 2aD^1_2$ and the analytic delta functions explicitly read

\[
[\delta A^1_{22}(1, 2) = \delta(x_{12})[D^1_{v\theta}]_{22}\delta^{12}(\theta_{12})] \delta^{12}(u, v) = \delta(x_{12})[D^1_{v\theta}]_{22}\delta^{12}(\theta_{12}) (U^1_1 \bar{U}_2^2)^2 \delta^6(u, v) \\
[\delta A^{11}_{220}(1, 2) = \delta(x_{12})[D^1_{v\theta}]_{22}\delta^{12}(\theta_{12}) \delta^{12}(u, v) = \delta(x_{12})[D^1_{v\theta}]_{22}\delta^{12}(\theta_{12}) (U^1_1 \bar{U}_2^2)^2 \delta^6(u, v) \\
[\delta A_{23}(1, 2) = \delta(x_{12})[D^1_{v\theta}]_{22}\delta^{12}(\theta_{12}) \delta^{12}(u, v) = \delta(x_{12})[D^1_{v\theta}]_{22}\delta^{12}(\theta_{12}) (U^1_1 \bar{U}_2^2)^2 \delta^6(u, v).
\]

The general form of $G$’s which satisfy the Green’s function equations then looks like

\[
\langle c(1)b^1_2(2) \rangle \equiv G^0_{00}^0(1, 2) = \frac{1}{2}D^4_{v\theta}\delta^{12}(\theta_{12}) \delta^{12}(u, v) \\
\langle b^1_2(1)c(2) \rangle \equiv G^0_{02}^1(1, 2) = \frac{1}{2}D^4_{v\theta}\delta^{12}(\theta_{12}) \delta^{12}(u, v) \\
\langle A^1_3(1)A^2_3(2) \rangle \equiv G^1_{33}^1(1, 2) = \frac{1}{2}D^4_{v\theta}\delta^{12}(\theta_{12}) \\
\langle A^3_2(1)A^3_2(2) \rangle \equiv G^3_{33}^3(1, 2) = \frac{1}{2}D^4_{v\theta}\delta^{12}(\theta_{12}) \\
\langle A^3_2(1)A^3_2(2) \rangle = \frac{1}{2}D^4_{v\theta}\delta^{12}(\theta_{12})
\]

(3.7)

such that

\[
(K^1_2)_{a} F^{1}_{\alpha}(u, v) = \frac{1}{2}(D^1_2)^2 \delta_{\alpha}(u, v).
\]

(3.11)

The $\delta_{\alpha}(u, v)$ functions are the same $\delta$-functions appearing in the corresponding $[\delta A^1_{\alpha}]^\alpha(1, 2)$ defined in (3.6). The equation (3.11) is motivated by the identity\(^1\)

\[
(D^1_2)^2 D^4_{\alpha\theta} D^4_{v\theta} \delta^6(u, v) = 2 \Box D^4_{v\theta} \delta^6(u, v),
\]

(3.12)

which can be used to prove that (3.7)-(3.10) indeed satisfy (3.5).

\(^1\text{We will suppress the SU(3) ‘indices’ on } [D^4_{\alpha\theta}] \text{ from now on.}\)
In order to make the above equations simpler and more tractable, we choose the following ‘independent’ coordinates from the $U$-matrix:

$$U_1, U_3, U_2, U_3, \bar{U}_2, \bar{U}_2, \bar{U}_3, \bar{U}_3,$$

and the ‘zero charge’ $\delta$-function in these coordinates reads

$$\delta^6(u, v) = \pi U_1 \bar{U}_2 \delta(U_3) \delta(U_3) \delta(U_3) \delta(U_3) \delta(U_3) \delta(U_3).$$  \hfill (3.13)

The rest of the $U$-coordinates can be written in terms of the chosen ones as follows

\begin{align*}
U_1 &= -\frac{U_1 \bar{U}_2 + U_1^3 U_2^3}{U_2^2}, \\
U_2 &= -\frac{U_1^2 U_1 \bar{U}_2 - U_1^3 U_2^3 - \bar{U}_2 U_3^2}{U_1^2}, \\
U_3 &= \frac{U_1^3 U_3^2 + \bar{U}_2}{U_1^3}, \\
\bar{U}_1 &= -\frac{U_2 \bar{U}_2 + U_3 \bar{U}_3}{U_1}, \\
\bar{U}_2 &= -\frac{U_2^2 \bar{U}_2 + U_3^2 \bar{U}_3 + U_1 U_1^3}{U_2^2}, \\
\bar{U}_3 &= \frac{\bar{U}_2 \bar{U}_3 + U_1}{U_2}.
\end{align*}

The differential operators get modified as well leading to the following expressions

\begin{align*}
D_1^1 &= U_1^1 \frac{\partial}{\partial U_1^1}, \\
D_1^3 &= U_1^1 \frac{\partial}{\partial U_1^3}, \\
D_2^2 &= U_2^2 \frac{\partial}{\partial U_2^2}, \\
D_3^2 &= U_3^2 \frac{\partial}{\partial U_3^2}, \\
D_1^1 &= U_1^1 \frac{\partial}{\partial U_1^1} + U_3^3 \frac{\partial}{\partial U_3^3} - \bar{U}_2 \frac{\partial}{\partial U_2^2} - \bar{U}_3 \frac{\partial}{\partial U_3^3}, \\
D_1^3 &= U_1^3 \frac{\partial}{\partial U_1^3} + U_3^3 \frac{\partial}{\partial U_3^3} - \bar{U}_2 \frac{\partial}{\partial U_2^2} - \bar{U}_3 \frac{\partial}{\partial U_3^3}, \\
D_2^3 &= U_2^3 \frac{\partial}{\partial U_2^3} - U_3^2 \frac{\partial}{\partial U_3^2} - \bar{U}_2 \frac{\partial}{\partial U_2^2} - \bar{U}_3 \frac{\partial}{\partial U_3^3}, \\
D_3^2 &= U_2^2 \frac{\partial}{\partial U_2^2} - U_3^2 \frac{\partial}{\partial U_3^2} - \bar{U}_2 \frac{\partial}{\partial U_2^2} - \bar{U}_3 \frac{\partial}{\partial U_3^3}.
\end{align*}

Using all these expressions, we can write (3.11) explicitly in the following form

\begin{align*}
(1 + |a|) \left[ U_3^3 \frac{\partial^2}{\partial U_2^2 \partial U_3^3} - \bar{U}_2^2 \frac{\partial^2}{\partial U_3^2 \partial U_3^3} + \left( a + \frac{1}{2} \right) \frac{\partial}{\partial U_2^2} \right] F_{d^\dagger} = \Delta^\dagger_d \delta(U_3^3) \delta(U_2^3) \delta(U_3^3), \quad (3.19)
\end{align*}
where

\[ \Delta_{\|} = \frac{\pi}{2} (U_1^1)^{n_1} (U_2^2)^{n_2} \delta(U_1^3) \delta(U_2^3) \delta''(U_1^1) \quad \text{with} \quad (n_1, n_2) = \begin{cases} 
(1, 2) & \text{for } \langle c(1)b_1^2(2) \rangle \\
(0, 3) & \text{for } \langle b_2^1(1)c(2) \rangle \\
(1, 1) & \text{for } \langle A_3^2(1)A_2^3(2) \rangle \\
(-2, 4) & \text{for } \langle A_2^3(1)A_3^2(2) \rangle .
\end{cases} \]

Relabelling \( U_3^1 = x, U_1^3 = \hat{x}, U_2^1 = y, \bar{U}_2^1 = \bar{y}, \bar{U}_3^3 = z, \bar{U}_2^3 = \hat{z}, \bar{U}_2^2 = A, U_1^1 = B \) and \( a + \frac{1}{2} = b \), we get a simple looking partial differential equation

\[
(x \partial_x \partial_y - A \partial_x \partial_z + b \partial_y) F_{\|}^+ = \frac{\pi}{2(1 + |a|)} B^{n_1} A^{n_2} \delta(x) \delta(y) \delta''(y) \delta(z) \delta(\hat{z}). \tag{3.20}
\]

We solve it by choosing an Ansatz of the form

\[
F_{\|}^+ = C_a A^p B^q \left( \frac{A y + x z}{B} \right)^t \left( \frac{B \bar{y} + \hat{x} \hat{z}}{A} \right)^{-1}, \tag{3.21}
\]

where \( \epsilon \) is an infinitesimal parameter and the exponents \( p, q, r, s, t \) along with the normalization factor \( C_a \) are to be determined. Plugging (3.21) into LHS of (3.20), we find that the values \( t = -b, r = b - 1, s = 0 \) simplify the expression to a single term as follows:

\[
\text{LHS of (3.20)} \bigg|_{t = -b, r = b - 1, s = 0} = \frac{3C_ab A^{p+2} B^{q+b} y^{b+2} (A y + x z)^{-b} \epsilon^2}{(A y + x z)(B \bar{y} + \hat{x} \hat{z})(y \bar{y} + \epsilon^2)^4} \cdot
\]

\[
= \frac{C_a b A^{p+2} B^{q+b} (A + \frac{xz}{y})^{-b}}{2(A y + x z)(B \bar{y} + \hat{x} \hat{z})} - \pi \delta(y) \delta''(y) \\
= \frac{\pi}{2} C_a b A^{p+2} B^{q+b} (A + \frac{xz}{y})^{-b} \delta(x) \delta(z) \delta(\hat{x}) \delta(\hat{z}) \delta(y) \delta''(y) \\
= \frac{\pi}{2} C_a b A^{p+2} B^{q+b} \delta(x) \delta(y) \delta''(y) \delta(z) \delta(\hat{z}), \tag{3.22}
\]

where we used that \( \delta \)'s for \( y, \bar{y} \) are produced in the limit of \( \epsilon \to 0 \) and \( y \to 0 \) via the following identity

\[
\frac{(m + 1)!(-y)^m \epsilon^2}{(y \bar{y} + \epsilon^2)^{m+2}} \to \pi \delta(y) \delta^{(m)}(\bar{y}) \tag{3.23}
\]

and \( \delta(z) = \frac{1}{z} \), etc. for rest of the four non-conjugate complex variables. Now, comparing (3.22) to RHS of (3.20), we can deduce that \( p = n_2 + b - 2, q = n_1 - b \) and \( C_a = \frac{1}{b(1 + |a|)} \). Thus, the final form of \( F_{\|}^+ \) that solves (3.20) reads

\[
F_{\|}^+ = \frac{1}{b(1 + |a|)} A^{n_2+b-2} B^{n_1-b} \left( \frac{A y + x z}{B} \right)^{-b} \left( \frac{B \bar{y} + \hat{x} \hat{z}}{A} \right)^{-1}. \tag{3.24}
\]

Finally, the complete propagators in terms of \( U \)-variables read as follows:

\[
G_{\|}^+ (1, 2) = \frac{(\bar{U}_2^2)^{n_2+b-2}(U_1^1)^{n_1-b}}{b(1 + |a|)} \frac{U_2^1)^{b-1}(-\bar{U}_1^1)^{-b}}{(\bar{U}_2^1 + \frac{x^2}{U_2^1})^3 (-U_1^2)^{b-1}} \frac{1}{D_{u\theta} D_{v\theta} \delta^{12}(\theta_{12}) \delta(x_{12})}. \tag{3.25}
\]
3.2 Feynman Rules

The Feynman rules are now derived as usual. The vector and ghost propagators are given in (3.25) but we reproduce them here individually with explicit harmonic factors in momentum space (replace $\frac{1}{k^2} \delta(x_{12}) \rightarrow \frac{1}{x_{12}}$):

$$\langle c(1)b_1^i(2) \rangle = \frac{2(U_2)^{\frac{1}{2}}}{(U_1)^{\frac{1}{2}}} \frac{(U_1)^{\frac{1}{2}} (U_2)^{-\frac{1}{2}}}{(U_1 + c^2 v^2)^{\frac{3}{2}}} \frac{-U_1^2}{U_{12}^2} \frac{1}{k^2} D_{u\bar{a}} D_{v\bar{a}} \delta^{12}(\theta_{12});$$

(3.26)

$$\langle b_2^i(1)c(2) \rangle = \frac{2(U_2)^{\frac{1}{2}}}{(U_1)^{\frac{1}{2}}} \frac{(U_1)^{\frac{1}{2}} (U_2)^{-\frac{1}{2}}}{(U_1 + c^2 v^2)^{\frac{3}{2}}} \frac{-U_1^2}{U_{12}^2} \frac{1}{k^2} D_{u\bar{a}} D_{v\bar{a}} \delta^{12}(\theta_{12});$$

(3.27)

$$\langle A_3^i(1)A_2^3(2) \rangle = \frac{(U_2)^{\frac{1}{2}}}{3(U_1)^{\frac{1}{2}}} \frac{(U_1)^{\frac{1}{2}} (U_2)^{-\frac{1}{2}}}{(U_1 + c^2 v^2)^{\frac{3}{2}}} \frac{-U_1^2}{U_{12}^2} \frac{1}{k^2} D_{u\bar{a}} D_{v\bar{a}} \delta^{12}(\theta_{12});$$

(3.28)

$$\langle A_2^i(1)A_3^1(2) \rangle = -\frac{(U_2)^{\frac{1}{2}}}{(U_1)^{\frac{1}{2}}} \frac{(U_1)^{\frac{1}{2}} (U_2)^{-\frac{1}{2}}}{(U_1 + c^2 v^2)^{\frac{3}{2}}} \frac{-U_1^2}{U_{12}^2} \frac{1}{k^2} D_{u\bar{a}} D_{v\bar{a}} \delta^{12}(\theta_{12}),$$

(3.29)

where we have kept the $\epsilon$-prescription explicit. The vertices can be read from (3.3) and (3.4):

$$\langle (A_3)^a (A_2)^b (A_2^3)^c \rangle / \langle (A_2)^a (A_3)^b (A_2^3)^c \rangle \rightarrow 2 \int d\theta \theta^2 \epsilon^{abc} \left[ D_3^1 / D_2^3 \right];$$

$$\langle (A_3)^a (A_2)^b (A_2^3)^c \rangle / \langle (A_2)^a (A_2)^b (A_3^d)^c \rangle \rightarrow i \int d\theta \theta^2 \epsilon^{abc} \epsilon^{cde} \left[ D_3^1 / D_2^3 \right];$$

(3.30)

$$\langle (A_3)^a (A_2)^b (A_2^3)^c \rangle / \langle (A_2)^a (A_2)^b (A_2^3)^c \rangle \rightarrow - \int d\theta \theta^2 \epsilon^{abc} \left[ D_3^1 / D_2^3 \right],$$

where $\int d\theta \equiv (D^2)^2 (D^3)^2 (D_1)^2 (D_3)^2$, $\epsilon^{abc}$ are structure constants of the gauge group, and the harmonic derivatives act on the leg corresponding to the group index ‘c’.

Evaluating loop graphs. Let us first focus on 1-loop graphs. We can generate $\int d\theta \theta^2$ from the analytic measure at vertices by taking off one factor of $D_3^1$ from the propagators. After this we are left with $(v_3 + v_4) \int d\theta \theta^2$ integrals from 3- and 4-point vertices, $p \ D_3^1 \delta^{12}(\theta)$’s from propagators. As usual, we need to saturate all but one $\int d\theta \theta^2$, which means we need to kill one of the $\delta^{12}(\theta)$. This is achieved by using three $D_3^1$’s and the identity

$$D_{u\bar{a}} D_{v\bar{a}} D_{u\bar{a}} = \left[ (W_1^2)^2 (W_2^2)^2 (V_2^2)^2 (V_1^2)^2 + (W_1^2)^2 (W_2^2)^2 (V_1^2)^2 (V_2^2)^2 + (W_1^2)^2 (W_2^2)^2 (V_1^2)^2 (V_2^2)^2 + (W_1^2)^2 (W_2^2)^2 (V_1^2)^2 (V_2^2)^2 \right]$$

$$- \frac{1}{2} \Box \left[ (W_1^2)^2 (W_2^2)^2 (V_1^2)^2 (V_2^2)^2 (D^2)^2 (D_1^2)^2 + (W_1^2)^2 (W_2^2)^2 (V_1^2)^2 (V_2^2)^2 (D^2)^2 (D_1^2)^2 \right]$$

$$+ (W_1^2)^2 (W_2^2)^2 (V_1^2)^2 (V_2^2)^2 (D^2)^2 (D_1^2)^2 + (W_1^2)^2 (W_2^2)^2 (V_1^2)^2 (V_2^2)^2 (D_1^2)^2 (D_1^2)^2 \right] \right] \ D^4_{u\bar{a}} \right],$$

(3.31)

where $D^4_\theta \equiv (D^2)^2 (D^3)^2 (D_1)^2 (D_3)^2$, $V_{a\bar{b}} \equiv v^a \bar{u}_b$, $W_{a\bar{b}} \equiv \bar{w}_a w^b$, etc. The first term which contains $D^4_\theta$ (recall that $D^4_{\bar{a}} \equiv (D_1)^2 (D_2)^2$) can be used to kill one $\delta^{12}(\theta)$. This means a 2-point function trivially vanishes as it does not have enough $D^4_\theta$’s and a 3-point function cannot have any divergent piece due to the presence of three $\Box$’s in the denominator, i.e., $\int \frac{d^4 k}{(2\pi)^4}$ is finite. In fact, no higher-point function can have any divergent piece at 1-loop because the numerator can generate at most $(p - 3) \Box$’s (together with a $D^4_\theta$) compared to $p \Box$’s in the denominator and so the difference is always $n \geq 3$, etc.
\[ \int \frac{d^4k}{(2\pi)^n} \text{ is finite.} \]

This power-counting readily generalizes to multi-loop graphs because each 1-loop subgraph needs to follow this procedure of \( D \)-algebra and hence the whole graph is rendered finite. This proves the nonrenormalization theorem at all loops for \( \mathcal{N} = 3 \) SYM, or equivalently, \( \mathcal{N} = 4 \) SYM.

### 4 Quantizing SYM in Background Field Gauge

The computation of finite terms for loop graphs is still cumbersome with the Feynman rules discussed in the previous section because manifestly covariant expressions are not obtained for individual graphs. For that purpose, we develop the background field formalism in this section.

Let us gauge-covariantize all the differential operators \((D \rightarrow \nabla = D + iA)\). Then we do a background splitting of the connections in a straightforward manner: \( A \rightarrow A_{bg} + a_q \), where the subscripts will be suppressed in favour of self-explanatory fonts. Next, we choose different representations for these connections: ‘real’ rep for background \( A \)'s meaning the three harmonic connections vanish \((A_1^a = A_2^a = A_3^a = 0)\) and ‘analytic’ rep for quantum \( a \)'s meaning the four fermionic connections vanish \((a_1^a = \bar{a}_2 \alpha = 0 \Rightarrow D_{\vartheta} \rightarrow \nabla_{\vartheta} \equiv D_{\vartheta})\).\(^2\) Let us write down the consequences of these choices on connections and field strengths from various commutators:

\[
\begin{align*}
\{\nabla_a^a, \nabla_{b\beta}\} &= i\delta^a_\alpha \nabla_\alpha \beta \quad \text{("unchanged")}
\{\nabla_a^b, \nabla_{b\beta}\} &= \epsilon_{\alpha\beta} W_{ab} \\
\{\nabla_{a\alpha}, \nabla_{b\beta}\} &= \epsilon_{\alpha\beta} W_{ab} \\
[\nabla_b^a, \nabla_c^a] &= \delta^a_\beta \nabla_\alpha \\
[\nabla_b^a, \nabla_{c\alpha}] &= -\delta^a_\alpha \nabla_{b\alpha} \\
[\nabla_b^a, \bar{\nabla}_d^c] &= \delta^a_d \nabla_b^c - \delta^a_d \nabla_b^c. 
\end{align*}
\]

(4.1)

Of course, \( W \)'s are anti-symmetric in the two indices and they satisfy a few Bianchi identities along with some analytic + harmonic constraints \([5, 8]\). The most relevant identity for us is

\[
D_\alpha^a W_{bc} = \frac{1}{2}(\delta^a_\beta D_\alpha^k W_{kc} - \delta^a_\epsilon D_\alpha^k W_{kb}).
\]

(4.2)

Another thing to note is that the spinorial background-covariant derivatives \( D_\alpha^1 \) & \( D_\alpha^{2\alpha} \) still possess the structure of \((2.9)\) so that ‘background analytic’ superfields can be defined: \( D_\alpha^1 \Phi = D_\alpha^{2\alpha} \Phi = 0 \). Moreover, the fourth & fifth equations of \((4.1)\) tell us that the harmonic connections we are most interested in are now background analytic. We will also need the following identity defining a generalized d’Alembertian

\[
D_\vartheta^4 (\nabla^2_\vartheta) D_\vartheta^4 = 2 \big[ (\nabla_\vartheta - 2\bar{W}^{12} W_{21}) - \nabla_1 \cdot \nabla_\vartheta \bar{W}^{1\alpha} + 2\nabla^2 \cdot D_1^4 W_{21} + \nabla_1^2 (D_1^4)^2 W_{21} \big] D_\vartheta^4. \quad (4.3)
\]

\(^2\)Such a choice was used in the case of \( \mathcal{N} = 2 \) projective superspace to construct the background field formalism \([10]\). It ensures that the effective action is independent of background fields with dimension 0 (like the harmonic connections), which is required for the nonrenormalization theorems to hold \([11]\).
4.1 Feynman Rules

Since the harmonic connections appearing in the action as well as the ghosts are purely quantum superfields but background analytic, only $\Box$ changes to $\hat{\Box}$ in the propagators derived in the previous section whereas the vertices in the Lagrangian remain the same. However, we can expand the $\hat{\Box}$ to get vertices with explicit field strengths, which leads to covariant results in the loop calculations directly. With this structure, we can write down the background Feynman rules as follows:

\[
\text{Propagators: } \begin{cases} 
\mathcal{F}_W^{++}(u,v) \frac{1}{k^2} D_{u\theta}^1 D_{v\phi}^1 \delta^{12}(\theta_{12}) & \text{ignore background vertices} \\
\frac{1}{k^2} \delta^6(\theta_{12}) \delta^6(u,v) & \text{consider all such vertices but one} 
\end{cases}
\]

All but one background vertices: $\int du d^8\theta (\hat{\Box} - \Box)$

One background vertex: $\int du dv d^8\theta (K_2^1)_{a\bar{a}} \delta^6(u,v)$

All quantum vertices: Same as (3.30).

Note that the background d’Alembertian is expanded as $\Box \equiv D_{\alpha\bar{\alpha}} D_{a\bar{a}} = \Box_0 + \cdots$, where $\Box_0$ corresponds to $-k^2$ in the momentum space.

1-loop graph computation. Let us focus on the computation of a 4-point function here. It is finite and from symmetry arguments of [8] (Section 5.5), it is known to look like $(\bar{W}^{13} W_{23})^2$ in $\mathcal{N} = 3$ harmonic superspace.

![Figure 1: 1-loop 4-point graph with external background field strengths.](image)

We can obtain such a contribution by evaluating a bubble graph and expanding the $(\hat{\Box} - \Box_0)$-vertex factors to get the relevant $W$’s using (4.3) and (4.2). The above Feynman rules give the following expression for the graph with vector loop shown in Figure 1 (after doing $\theta$ and $u$ integrals at two of the vertices):

\[
\Gamma_{4}^{(A)} \sim \hat{A}_4 \int du \int dv \int d^8\theta_{1,2} (\bar{\nabla}_1 \cdot \bar{\nabla}_3 \bar{W}^{13}(p_1)) (\bar{\nabla}_1 \cdot \bar{\nabla}_3 \bar{W}^{13}(p_2)) ((U_1^2 U_1^1) \bar{\nabla}_2 \cdot \nabla^3 W_{23}(p_3)) \delta^8(\theta_{12}) \]
\[
\times (K_2^1)_{-1} \mathcal{F}_W^{22|23}(u,v) D_{u\theta}^1 D_{v\phi}^1 \delta^{12}(\theta_{12}) \delta^6(u,v),
\]

where $\hat{A}_4 \sim \int d^4k \frac{1}{(k_1^2)(k_2^2)(k_3^2)(k_4^2)}$ is the scalar box integral with the subscripts on loop-momentum $k$ denoting the external momenta ($p_i$) dependence. Note that we had to partially integrate $(K_2^1)_{-1}$ so that it changed to $(K_2^1)_{-1}$ acting on $\mathcal{F}_W^{++}$, which gives using (3.19): $(K_2^1)_{-1} \mathcal{F}_W^{22|23}(u,v) = \frac{1}{2} (D_{12}^2)^2 \delta^6(u,v) - 4 \frac{\partial}{\partial k_1^2} \mathcal{F}_W^{22|23}$. The first term leads to a harmonic singularity with two $\delta^6(u,v)$-functions but this singularity will cancel with the analogous contribution $\Gamma_{4}^{(bc)}$ from the ghost loop in Figure 1. So we
focus only on the second term that has no analogue from the ghost loop graph (as $a = 0$) and thus gives the complete four-point function. To get rid of the $δ^{12}(θ_{12})$-function, eight spinorial derivatives should be gathered off of $W$‘s in addition to $D_φ^4$ as follows\(^3\)

\[
\Gamma_4 \sim \hat{A}_4 \int du \, dv \int d^8θ_{1,2} \bar{W}^{13}(p_1)\bar{W}^{13}(p_2)W_{23}(p_3)δ^8(θ_{12}) Δ^2_{13|11}(u, v) \\
\times (\nabla_1)^2(\nabla_3)^2((U^2_1U^2_2)\nabla^2 \cdot \nabla^3)(\bar{U}^2_1\bar{U}^2_1U^2_1U^2_2\nabla^2 \cdot \nabla^3W_{23}(p_4)) \, D^4_{\nu\rho}(p_4)δ^{12}(θ_{12})δ^6(u, v) \\
\sim \hat{A}_4 \int du \int d^8θ \bar{W}^{13}(p_1)\bar{W}^{13}(p_2)W_{23}(p_3)W_{23}(p_4) \left[\left(\frac{U^1_1}{(U^2_1)^4(U^2_2)^2}\right)^2\right]_{u \to v},
\]

where we used $\nabla^8_8 D^4_φ δ^{12}(θ_{12}) = 1$ in the second step. In the last step, the apparent harmonic singularity cancels as we take the limit $u \to v$, leading to the expected result for the 1-loop effective action

\[
\Gamma_4 \sim \int du \, dΩ^{22}_{11} \hat{A}_4 (\bar{W}^{13}W_{23})^2. \tag{4.6}
\]

5 Discussion

We have introduced a new gauge fixing action to quantize $N = 3$ SYM in $N = 3$ harmonic superspace. This leads to simpler (and fewer) propagators for vector and ghost superfields in a “Fermi-Feynman” gauge. These are sufficient to prove the nonrenormalization theorem for $N = 3, 4$ SYM at all loops. However, computation of loop graphs beyond the divergent terms can be simplified more with the background field formalism. With the background Feynman rules in hand, we have computed the 1-loop four-point contribution to the effective action, which gets a finite contribution purely from the vector loop diagram.

The way we derived the harmonic propagators here is reminiscent of how $N = 2$ projective superspace [12, 13] is derived from $N = 2$ harmonic superspace [2, 14] in [15]: by choosing a special parameterization of the harmonic R-symmetry coordinates on $SU(2)/U(1) \simeq S^2$ to obtain a single (complex) coordinate on $CP^1$ that forms the internal coordinate for the projective case.\(^4\) Though, we did not take this route to its full conclusion, it should be possible to derive a $N = 3$ projective superspace in such a way that simplifies the $\int du$-integrals to something more tractable. For example, viewing the R-symmetry coset $SU(3)/U(1)^2 \simeq [SU(3)/(SU(2) \times U(1))] \times [SU(2)/U(1)] \simeq CP^2 \times CP^1$, one can expect reducing the 6 ($x, \hat{x}, y, \hat{y}, z, \hat{z}$) R-symmetry coordinates of $N = 3$ harmonic superspace to only 3 ($x, y, z$) for a possible $N = 3$ projective superspace. This should then be followed by a projection of the harmonic gauge condition and equations of motion for the gauge and ghost fields to the projective superspace, which we leave for future work.

---

\(^3\)The requirement of collecting eight spinorial derivatives to get $\nabla^8_8$ (analogous to extracting $D^8_φ$ from (3.31) while evaluating graphs in “Fermi-Feynman” gauge) is sufficient to make $Γ_2 = Γ_3 = 0$ identically at 1-loop.

\(^4\)The relation between these two hyperspaces has been explored from different points of view in [16, 17].
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