The Unruh effect for the rate of emission and absorption of neutral massive Majorana spinor particles – plausible constituents of the Dark Matter – in a Rindler space-time is thoroughly investigated. The corresponding Bogolyubov coefficients are explicitly calculated and the consistency with Fermi-Dirac statistics and the Pauli principle is actually verified.

Keywords: Unruh Effect; Rindler Spacetime.

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1. Introduction

There is little doubt that the very existence, the nature as well as the eventual detection of non-baryonic Dark Matter is the most intriguing issue of modern Cosmology and Particle Physics. There is no surprise that, in spite of the fact that its particle structure is insofar unknown – certainly rather different from any other kind of ordinary Visible Matter – the nearly exhaustive list of the existing references attempts to frame Dark Matter within the nearest extensions of the Standard Model of (visible) Particle Physics or its super-symmetric extensions. In our opinion, this might be a somewhat conventional wisdom and widespread prejudice which, strictly speaking, does not lie neither upon any solid ground nor any actual phenomenological evidence. Nonetheless, the non-baryonic Dark Matter is usually assumed to be a weakly interacting massive particle (WIMP) yet undiscovered. Moreover, the
Dark Matter is also assumed to be stable on the scale of the cosmological structures formation. More or less well motivated Particle Physics candidates have been proposed all of which arise from specific models beyond the Standard Model of Particle Physics. Certainly, if the Dark Matter really exists, the massive, neutral, spin $\frac{1}{2}$ Majorana particles are pretty available candidates for its constituents. Moreover, it is now quite evident that galaxies do (uniformly) accelerate one respect to each other. Thus it appears to be quite plausible that the eventual detection of the Dark Matter constituents will be shaped like a (very weakly) accelerated phenomenon. Accelerated observers are expected to experience the so-called Unruh effect. This entails that, for instance, a cosmological thermal bath of WIMP particles is expected to be produced by the cosmic acceleration, with a characteristic temperature called the Unruh temperature $T = \frac{\hbar}{2\pi k_B}$, where $k_B$ is the Boltzmann’s constant and $H$ the (time dependent) Hubble’s parameter, which is tightly related to the cosmic acceleration $a_{\text{cosmic}} = cH \approx 2.1 \times 10^{-9} \text{ m s}^{-2}$. In other words, it means that for the uniformly accelerated observers co-moving with the galaxies, the environment is populated by some thermal distribution of particles created by the underlying "inertial vacuum" where the relativistic quantum laws of Physics are presumably holding true. If this is the picture, the corresponding WIMP production is determined by the so-called Bogolyubov coefficients for the inertial to accelerated observers transformation law.

The Bogolyubov coefficients for the production of spinless particles out of the inertial vacuum have been known for a long time and have been recently rigorously reobtained. It is the main purpose of this paper to calculate the Bogolyubov coefficients for the emission out of the inertial vacuum of a thermal distribution of neutral spin $\frac{1}{2}$ massive particles, i.e. the most plausible WIMP candidates for the Dark Matter structure. As a matter of fact, to the best of our knowledge the Fulling modes expansion and the ensuing Bogolyubov coefficients have never been explicitly calculated for the important case of a Majorana spinor field – see e.g. the recent up-to-date review paper. It is just the aim of this paper to fill this gap.

Another purpose of the present paper is to present a first-principles and direct calculation of the effect: in the light of recent controversies regarding the very existence of the Unruh effect, raised by et seq., we aim for a detailed and exhaustive derivation.

In Section 2 we carefully study the Majorana field in a Rindler spacetime. In particular we obtain the explicit solutions to the field equation in the accelerated frame (also known as Fulling modes), we then perform explicit checks of orthonormality and completeness. With such solutions at hand, and upon checking self-adjointness of the Majorana Hamiltonian in the accelerated frame, we proceed with canonical quantization, establishing the Fock space for the Rindler observer. Section 3 contains the explicit calculation of the Bogolyubov coefficients for the massive Majorana field in a Rindler space-time. As a crucial and nontrivial check, we show that the Bogolyubov coefficients indeed agree with the Pauli exclusion principle, hence providing a consistency check for the canonical anticommutation
relations adopted in the Rindler space-time. A short discussion is contained in the concluding Section.

2. Majorana Equation in Rindler Coordinates

This section is devoted to the detailed study of the generally covariant Majorana equation in a Rindler space-time and of its solutions. Our conventions on tetrads and the spin-connection are summarized in Appendix A.

2.1. A short review of Rindler geometry

Consider the four dimensional Minkowski space with the line element

\[ ds^2 = \eta_{\alpha\beta} dX^\alpha dX^\beta = g_{\mu\nu}(x) dx^\mu dx^\nu \]  

where the constant metric tensor \( \eta_{\alpha\beta} = \text{diag}(+, -, -, -) \) is relative to an inertial coordinate system in the Minkowski space labelled by the so called anholonomic indices denoted with the greek letters from the first part of the alphabet, while the Einstein convention on the sum over repeated indices is understood. We employ natural units \( \hbar = c = 1 \) unless explicitly stated. If we set

\[ X^\alpha = (c\tau, X, Y, Z) \quad x^\mu = (ct, x, y, z) \]  

Then we shall denote the following space-like region of the Minkowski space, viz.,

\[ \mathcal{W}_R = \{ X^\mu \in \mathbb{R}^4 | X \geq 0, c^2\tau^2 \leq X^2 \} \]  

as the right Rindler wedge. Here we introduce the so called Rindler curvilinear coordinate system, which describes an accelerated observer: namely,

\[ c\tau = x \sinh(a t/c) \quad X = x \cosh(a t/c) \quad (a > 0 \lor x \geq 0) \]  
\[ Y = y \quad Z = z \]  

where \( [a] \) is the constant acceleration. The above coordinate transformations can be readily inverted, viz.,

\[ t = \frac{c}{a} \text{Arth} \frac{c\tau}{X} \quad x = \sqrt{X^2 - c^2 \tau^2} \geq 0 \quad y = Y \quad z = Z \]  

in such a manner that we can also write

\[ ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu = x^2(a/c)^2 dt^2 - dx^2 - dY^2 - dZ^2 \]  

whence we obtain

\[ g_{\mu\nu}(x) = \begin{pmatrix} a^2 x^2 / c^4 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \]  

so that

\[ g = \det g_{\mu\nu}(x) = [\det g^{\mu\nu}(x)]^{-1} = -a^2 x^2 / c^4 \]
Moreover we find, for \( \xi = ax/c^2 \) and \( \eta = at/c \),

\[
\frac{\partial X^\alpha}{\partial x^\nu} = \begin{pmatrix}
\xi \cosh \eta & \sinh \eta & 0 & 0 \\
\xi \sinh \eta & \cosh \eta & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} \equiv \zeta_\alpha^\nu(\xi, \eta)
\] (10)

which endorses

\[
\det \left( \frac{\partial X^\alpha}{\partial x^\nu} \right) = \sqrt{-g} = \xi = ax/c^2 > 0
\]

In order to set notation, we denote the inverse of (10) by

\[
\frac{\partial x^\nu}{\partial X^\alpha} = \zeta^\nu_\alpha[X(\xi, \eta)].
\] (11)

Notice that, by changing both signs in the definitions (4), we shall cover the other space-like region of the Minkowski space, i.e. the left Rindler wedge

\[
\mathcal{W}_L = \{ X^\alpha \in \mathbb{R}^4 | X \leq 0, c^2 \tau^2 \leq X^2 \}.
\] (12)

2.2. The generally covariant Majorana equation

The Majorana equation, in its generally covariant form, is

\[
V_\mu^\alpha(x) \gamma_\mu \{ \partial_\mu - i \Gamma_\mu(x) \} \psi_M(x) + im \psi_M^c(x) = 0 \quad \psi_M = \psi_M^c
\] (13)

The particular case of the uniformly accelerated noninertial frame referred to the Rindler’s curvilinear coordinates system can be handled as follows. As in the previous section, let \( X \) denote the coordinates in the frame of an inertial observer, and \( x \) those of of the accelerated observer, then we define

\[
\Lambda(t) = \begin{pmatrix}
\cosh \eta & \sinh \eta & 0 & 0 \\
\sinh \eta & \cosh \eta & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} \quad \Lambda(0) = 1
\] (14)

\[
\Psi_M[X(x)] = D[\Lambda(t)] \psi_M(x)
\] (15)

\[
D[\Lambda(t)] = \exp \left\{ -i \sigma_M^{01} \omega_1(t) \right\} = \exp \left\{ \frac{1}{2} \alpha_M^1 \eta \right\}
\]

\[
= \cosh \frac{1}{2} \eta + \alpha_M^1 \sinh \frac{1}{2} \eta = D^\dagger[\Lambda(t)]
\] (16)

together with

\[
\zeta_\mu^\alpha(\eta, \xi) = \Lambda_\mu^\alpha(t) V_\mu^\beta(x)
\] (17)

\[
V_0^\mu(x) = \xi^{-1} \delta_0^\mu \quad V_j^\mu = \delta_j^\mu
\] (18)

where the one-parameter rank-four square matrix \( D[\Lambda(t)] \) represents a local boost along the \( OX \)-direction. Then it is clear that the only contribution to (13) comes from the term

\[
\frac{\partial}{\partial t} \{ D[\Lambda(t)] \psi_M(x) \} = D[\Lambda(t)] \left\{ \frac{\partial}{\partial t} - i \Gamma_0 \right\} \Psi_M(x)
\]
Neutral Massive Spin $\frac{1}{2}$ Particles Emission in a Rindler Space

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\[ \Gamma_0 = \frac{i}{2} i a \gamma_M^0 \gamma_M^1 = \frac{i}{2} i a \sigma_3 = \frac{i}{2} i a \begin{pmatrix} -\sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix} \]

Thus, in this specific case we can write

\[ \begin{aligned} 
\Lambda^\beta_0(t) V^0_\beta(x) D^{-1}[\Lambda(t)] \gamma_M^0_\alpha \{ i \partial_t D[\Lambda(t)] \} \\
= \Lambda^\beta_0(t) V^0_\beta(x) D^{-1}[\Lambda(t)] \gamma_M^0 D[\Lambda(t)] \Gamma_0 \\
= \Lambda^\beta_0(t) V^0_\beta(x) \gamma_M^0 \Gamma_0 = V^0_\beta(x) \gamma_M^0 \Gamma_0 \\
= \frac{c^2}{a x} \gamma_M^0 \Gamma_0 
\end{aligned} \]

in such a manner that we eventually obtain

\[ \{ V^0_\alpha(x) \gamma_M^0 (\partial_0 - i \Gamma_0) + \gamma_M^0 \partial_k + im \} \psi_M(x) = 0 \quad \psi_M = \psi_M^c \]

It is worthwhile to rederive this result in a manifestly generally covariant formalism. Turning to a general curved space and to some generic curvilinear coordinate system $x^\mu = (ct, x, y, z)$, then we come to the generally covariant Majorana bispinor wave equation

\[ V^\mu_\alpha(x) \gamma_M^0 \nabla_\mu \psi_M(x) + i m \psi_M^c(x) = 0 \quad \psi_M = \psi_M^c \]  

(19)

where

\[ \nabla_\mu \equiv \partial_\mu - i \Gamma_\mu = \partial_\mu - \frac{i}{2} i \sigma^{\alpha \beta}_{\alpha \beta} \omega_{\alpha \beta ; \mu} \]  

(20)

\[ \sigma^{\alpha \beta}_{\alpha \beta} \equiv \frac{1}{2} i [ \gamma_M^0, \gamma_M^0 ] \]  

(21)

\[ \omega_{\alpha \beta ; \mu} \equiv V^\mu_\alpha(x) V^0_\beta(x) \]  

(22)

\[ V_{\beta \nu ; \mu}(x) \equiv D_\mu V_{\beta \nu}(x) = \partial_\mu V_{\beta \nu}(x) - \Gamma_\mu^\lambda \beta \nu \lambda(x) \]  

(23)

Again, if we turn to the purely imaginary Majorana representation for the gamma matrices $\gamma_M^0 + \gamma_M^0 = 0$, then the above covariant bispinor equation (19) admits real self-conjugated Majorana bispinor solutions

\[ V^\mu_\alpha(x) \gamma_M^0 \nabla_\mu \psi_M(x) + i m \psi_M^c(x) = 0 \quad \psi_M = \psi_M^c = \psi_M^s \]  

(24)

It turns out that, taking into account the simplest choice (17) for the vierbeine or tetrad fields, the matrix-valued spin connection components are given by

\[ \begin{aligned} 
\Gamma_0(x) &= \frac{1}{2} \gamma_M^0 \gamma_M^1 [ V^\nu_0(x) V_{1\nu ; 0}(x) - V^\nu_1(x) V_{0\nu ; 0}(x) ] \\
\Gamma_1(x) &= \frac{1}{2} \gamma_M^0 \gamma_M^1 [ V^\nu_0(x) V_{1\nu ; 1}(x) - V^\nu_1(x) V_{0\nu ; 1}(x) ] \\
\Gamma_2(x) &= \Gamma_3(x) = 0 
\end{aligned} \]  

(25)

(26)

(27)

Now we have

\[ \begin{aligned} 
V^\nu_0 V_{1\nu ; 0}(x) &= - V^\nu_0 \Gamma^8_0 \nu V_{1\nu ; 0}(x) = a \\
V^\nu_1(x) V_{0\nu ; 0}(x) &= - V^\nu_1 \Gamma^8_0 V_{0\nu ; 0}(x) = - a 
\end{aligned} \]  

(28)

(29)
that yields once again as before

$$\Gamma_0 = \frac{1}{2} i a \gamma^0_M \gamma^1_M = \frac{1}{2} i a \alpha^1_M = \frac{1}{2} i a \begin{pmatrix} -\sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}$$

Finally one gets

$$\Gamma_1(x) \equiv 0 \hspace{1cm} (31)$$

in such a manner that we can eventually write

$$\nabla_0 = \partial_t + \frac{1}{2} a \alpha^1_M \nabla_j = \partial_j \hspace{1cm} (j = x, y, z) \hspace{1cm} (32)$$

Hence, for a uniformly accelerated noninertial observer employing a Rindler’s curvilinear coordinate system, the covariant Majorana massive operator definitely reads

$$iD /M - m \equiv iV^\mu_{\alpha} \gamma^\mu_M \nabla_\mu - m = \gamma^0_M i \partial_0 - \gamma^1_M \hat{p}_x - \gamma^2_M \hat{p}_y - \gamma^3_M \hat{p}_z - m$$

in which we have set

$$\hat{p}_x = \hat{x} - i \hbar \left( \frac{\partial}{\partial x} + \frac{1}{2x} \right) \hspace{1cm} \hat{p}_j = -i \hbar \partial_j \hspace{1cm} (j = x, y, z) \hspace{1cm} (33)$$

2.3. **Self-adjointness of the self-conjugated Hamiltonian**

On the one hand, from the above explicit expression of the covariant Dirac operator in the Majorana representation of the gamma matrices, it keeps manifestly true that even the covariant Majorana spinor equation \[24\] always admits real solutions, i.e. the self-conjugated spinors, as expected from the general covariance and the equivalence principles. On the other hand, the Rindler time evolution of a real Majorana bispinor is governed by the 1-particle Majorana Hamiltonian

$$i \hbar \partial_t \psi_M(t, x) = H_M \psi_M(t, x)$$

which can be readily extracted from the covariant equation \[24\] and reads

$$H_M = \sqrt{-\hbar} \left\{ \alpha^1_M \left( \hat{p}_x - i \frac{1}{2x} \right) + \alpha^2_M \hat{p}_y + \alpha^3_M \hat{p}_z + m \beta_M \right\} = H_M^\dagger \hspace{1cm} (34)$$

where, as customary,

$$\alpha^k_M = \gamma^0_M \gamma^k_M \hspace{1cm} \hat{p}_k = -i \hbar \partial_k \hspace{1cm} \beta_M = \gamma^0_M \hspace{1cm} (k = 1, 2, 3) \hspace{1cm} (35)$$

Just like in the Dirac case, the 1-particle Majorana Hamiltonian is self-adjoint in the right Rindler wedge \(\mathbb{M}_R\) iff

$$\lim_{\lambda \to 1^+} \lim_{x \to 0^+} \int d^2x_\perp x^\lambda \psi_1^\dagger(t, x, x_\perp) \alpha^1_M \psi_1(t, x, x_\perp) = 0 \hspace{1cm} \forall \psi_1, \psi_2 \in \mathcal{D}_M (36)$$

the order of the two limits being not interchangeable.
As a matter of fact the Rindler time evolution of a complex Dirac bispinor is governed by the 1-particle Dirac Hamiltonian
\[ i\hbar \partial_t \psi(t, x) = H_D \psi(t, x) \]
which can be readily extracted from the covariant equation (24) and reads
\[ H_D = \sqrt{-g} \left\{ \alpha^1 \left( \hat{p}_x - \frac{i}{2x} \right) + \alpha^2 \hat{p}_y + \alpha^3 \hat{p}_z + m \beta \right\} \]  \hspace{1cm} (37)
where, as customary,
\[ \alpha^k = \gamma^0 \gamma^k \quad \hat{p}_k = -i\hbar \partial_k \quad \beta = \gamma^0 \quad (k = 1, 2, 3) \]  \hspace{1cm} (38)
Let us investigate a little bit closer the hermiticity property of the above Dirac Hamiltonian operator \( H_D \).

The general diffeomorphisms invariant inner product between any arbitrary pair of square integrable complex solutions of the covariant Dirac equation is defined by
\[ (\psi_2, \psi_1) \equiv \oint_{\Sigma} V_{\alpha \mu}(x) \bar{\psi}_2(x) \gamma^\alpha \psi_1(x) \\mathrm{d}\Sigma^\mu \]  \hspace{1cm} (39)
where \( \bar{\psi}(x) \equiv \psi^\dagger(x) \gamma^0 = \psi^\dagger(x) \beta \) whereas \( \Sigma \) is a three dimensional future oriented space-like hyper-surface. Again, for the initial time three dimensional hyper-surface in the right Rindler wedge \( \mathcal{M}_R \) we get
\[ \mathrm{d}\Sigma^0 = -\frac{\theta(x) \mathrm{dx}}{\sqrt{(-g)}} \quad \mathrm{d}\Sigma^i = 0 \quad (i = 1, 2, 3) \]  \hspace{1cm} (40)
and since we have \( V_{00} = \sqrt{-g} \) we eventually obtain
\[ (\psi_2, \psi_1) = -\int_0^{\infty} \mathrm{dx} \int \mathrm{d}^2 x_\perp \psi_2^\dagger(t, x, x_\perp) \psi_1(t, x, x_\perp) \]  \hspace{1cm} (41)
Thus we find
\[ (\psi_2, H_D \psi_1) = -\int_0^{\infty} \mathrm{dx} \int \mathrm{d}^2 x_\perp \psi_2^\dagger(t, x, x_\perp) H_D \psi_1(t, x, x_\perp) \]
\[ = -a \int \mathrm{d}^2 x_\perp \int_0^{\infty} \mathrm{dx} \psi_2^\dagger(t, x, x_\perp) \]
\[ \times \left\{ \alpha^1 \left( \hat{p}_x - \frac{i\hbar}{2x} \right) + \alpha^2 \hat{p}_y + \alpha^3 \hat{p}_z + m \beta \right\} \psi_1(t, x, x_\perp) \]
Let us focus our attention on the expression
\[ i\hbar a \int_0^{\infty} x \psi_2^\dagger(t, x, x_\perp) \alpha^1 \left( \partial_x + \frac{1}{2x} \right) \psi_1(t, x, x_\perp) \mathrm{dx} \]
\[ = i\hbar a \left[ x \psi_2^\dagger(t, x, x_\perp) \alpha^1 \psi_1(t, x, x_\perp) \right]_0^{\infty} \]
\[ - i\hbar a \int_0^{\infty} x \left[ \left( \partial_x + \frac{1}{2x} \right) \psi_2(t, x, x_\perp) \right] \alpha^1 \psi_1(t, x, x_\perp) \mathrm{dx} \]  \hspace{1cm} (42)
in such a manner that we can write
\[
(\psi_2, H_D \psi_1) = \int d^2x_\perp \left[ x^\lambda \psi_2^\dagger(t, x, x_\perp) \alpha^1 \psi_1(t, x, x_\perp) \right]_0^\infty
\]
and
\[
\int d^2x_\perp \left[ x^\lambda \psi_2^\dagger(t, x, x_\perp) \alpha^1 \psi_1(t, x, x_\perp) \right]_0^\infty + \int d^2x_\perp \left[ \alpha^1 \left( \hat{p}_x - \frac{i\hbar}{2x} \right) \psi_2(t, x, x_\perp) \right]_0^\infty.
\]
This last equality means that we can identify
\[
H_D^\dagger = \sqrt{-\hat{g}} \left\{ \alpha^1 \left( \hat{p}_x - \frac{i\hbar}{2x} \right) + \alpha^2 \hat{p}_y + \alpha^3 \hat{p}_z + m \beta \right\} \psi_2, \psi_1 = H_D
\]
which means that the symmetric Dirac Hamiltonian operator is actually self-adjoint on the right Rindler wedge provided the domain \(D\) of the Dirac complex bispinors \(\psi_1, \psi_2\) is such that
\[
\lim_{\lambda \to 1^+} \int d^2x_\perp \left[ x^\lambda \psi_2^\dagger(t, x, x_\perp) \alpha^1 \psi_1(t, x, x_\perp) \right]_0^\infty = 0 \quad \forall \psi_1, \psi_2 \in D
\]

### 2.4. Solutions of the generally covariant Majorana wave equation

Consider the second order differential operator acting on the Majorana spinors, which can be cast into the form
\[
- (i \mathcal{D}_M - m) (i \mathcal{D}_M + m)
= \left( V_\alpha^\mu(x) \gamma_\mu^\alpha \nabla_\mu + im \right) \left( V_\beta^\lambda(x) \gamma_\lambda^\beta \nabla_\lambda - im \right)
= \frac{1}{a^2 x^2} \partial_t^2 + \hat{p}_x^2 + m^2 + \frac{\alpha_1^1}{a x^2} \partial_t
= \frac{1}{a^2 x^2} (\partial_t^2 + \alpha_1^1 a \partial_t) - \left( \partial_x + \frac{1}{2x} \right)^2 - \partial_\perp^2 + m^2
\]
It is convenient to obtain the spinor solutions of the Majorana equation from the solutions of the second order differential equation
\[
- (i \mathcal{D}_M - m) (i \mathcal{D}_M + m) f(t, x, y, z) \Upsilon = 0
\]
where \(f(t, x, y, z)\) is an invariant scalar function, whereas \(\Upsilon\) is a constant eigenspinor of the matrix
\[
\alpha_1^1 = \gamma_M^0 \gamma_1^1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}
\]
There are two degenerate real eigenvalues \(\lambda = \pm 1\) of the Hermitean matrix \(\alpha_1^1\) and for each eigenvalue one pair of degenerate constant eigenbispinors. Moreover, consider the \(OX-\)component of the spin in the particle comoving instantaneous rest frame, that is nothing but the helicity Hermitean matrix along the direction of the acceleration
\[
\frac{1}{2} \Sigma^1_M = \frac{i}{4} [\gamma^2_M, \gamma^3_M] \quad \Sigma^1_M = \begin{pmatrix} 0 & i\sigma_3 \\ -i\sigma_3 & 0 \end{pmatrix}
\]
which obviously satisfies \( \alpha_M^1, \Sigma_M^1 \) = 0, in such a manner that we can set

\[
\Upsilon^\pm = \frac{1}{2} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \quad \Upsilon^\pm = \frac{1}{2} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}
\]

\[
\Upsilon^\pm = \frac{1}{2} \begin{pmatrix} i & 1 \\ -1 & -i \end{pmatrix} \quad \Upsilon^\pm = \frac{1}{2} \begin{pmatrix} -i & 1 \\ 1 & -i \end{pmatrix}
\]

The above four bispinors do realize a complete and orthonormal set and fulfill by construction

\[
\alpha_M^1 \Upsilon^\pm = \pm \Upsilon^\pm \quad \alpha_M^1 \Upsilon^\pm = \pm \Upsilon^\pm
\]

\[
\Sigma_M^1 \Upsilon^\pm = \Upsilon^\pm \quad \Sigma_M^1 \Upsilon^\pm = -\Upsilon^\pm
\]

Notice that if we set

\[
\chi^+ = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \chi^- = \begin{pmatrix} i \\ i \end{pmatrix}
\]

then we can write

\[
\Upsilon^\pm = \frac{1}{2} \begin{pmatrix} \chi^\pm & -\sigma_2 \chi^\pm \end{pmatrix} = \Upsilon^L \quad \Upsilon^\pm = \frac{1}{2} \begin{pmatrix} \sigma_2 \chi^\pm & -\chi^\pm \end{pmatrix} = \Upsilon^R
\]

which means that the constant self-conjugated Majorana bispinors \( \Upsilon \) of positive helicity do correspond to the left handed Weyl spinors, while the negative helicity self-conjugated Majorana bispinors to the right handed Weyl spinors. Thus we can eventually identify up arrow with left handed \( \uparrow = L \) and down arrow with right handed \( \downarrow = R \). Moreover we can check by direct inspection that

\[
(\Upsilon^L)^* = \Upsilon^R
\]

There are in general four linearly independent complex solutions of the covariant bispinor equation (19) which can always be written in the form

\[
\psi^r(t, x, y, z) = (i \partial_M + m) f^r(t, x, y, z) \Upsilon^r \quad (r = \uparrow, \downarrow = L, R)
\]

but since we have

\[
(i \partial_M + m)^* = i \partial_M + m
\]

thanks to the purely imaginary form of the \( \gamma^- \)-matrices in the Majorana representation, there exists only two types of real massive bispinor solutions, viz.

\[
\chi^r(t, x) = (i \partial_M + m) \left[ f^r(t, x) \Upsilon^r + \text{c.c.} \right] \quad (r = L, R)
\]
where $f_{\pm}(t, x, y, z)$ are arbitrary complex solutions of the second order differential equations
\[
\left\{-\frac{1}{a^2 x^2} \left( \partial_t^2 \pm a \partial_t \right) + \left( \partial_x + \frac{1}{2x} \right)^2 + \partial_\perp^2 - m^2 \right\} f_{\pm}(t, x, y, z) = 0 \quad (59)
\]
In fact, after taking the partial Fourier transform
\[
f_{\pm}(t, x, x_\perp) \equiv \int_{-\infty}^{\infty} dp_0 \int dp \tilde{f}_{\pm}(p_0, p; x) \exp\{-ip_0 t + ip \cdot x_\perp\} \quad (60)
\]
and setting
\[
\nu = \frac{p}{ia} \quad \kappa = \sqrt{p^2 + (mc/\hbar)^2} \quad (61)
\]
we recover the modified Bessel equations
\[
\left[ \frac{d^2}{dx^2} + \frac{1}{x} \frac{d}{dx} - \frac{1}{2x^2} \left( \frac{\bar{p}}{a} + \frac{1}{2} \right)^2 - \kappa \right] \tilde{f}_{\pm}(\bar{p}, p; x) = 0 \quad (63)
\]
the most general solutions of which are the modified Bessel functions
\[
\tilde{f}_{\pm}(\bar{p}, p; x) = C_{\pm} K_{i\nu \mp \frac{1}{2}}(\kappa x) + C'_\pm I_{i\nu \mp \frac{1}{2}}(\kappa x) \quad (x > 0) \quad (64)
\]
The solution $I_{i\nu \mp \frac{1}{2}}(\kappa x)$ must be discarded for $x > 0$ so that we are left with the sets
\[
\tilde{f}_{\pm}(p_0, p; x) = \tilde{f}_{\pm}(\nu, p) K_{i\nu \mp \frac{1}{2}}(\kappa x) \quad (x > 0) \quad (65)
\]
and if we take into account that the solutions keep the very same under the substitutions
\[
\tilde{f}_{\pm}(\nu, p) \mapsto \tilde{f}_{\mp}(\nu, -p) \mapsto \tilde{f}_{\mp}(\nu, p)
\]
we come to the two sets of the real bispinor normal modes on the right Rindler wedge $\mathcal{W}_R$ which are solutions of the second order covariant equation (47), i.e.,
\[
\phi_\pm^R(t, x, y, z) = \mathcal{T}_{\pm}^L K_{i\nu \mp \frac{1}{2}}(\kappa x) \exp\{-ia\nu t/c + ip \cdot x_\perp\}
\]
\[
+ \mathcal{T}_{\mp}^R K_{i\nu \mp \frac{1}{2}}(\kappa x) \exp\{ia\nu t/c - ip \cdot x_\perp\} \quad (x > 0)
\]
where the suffix p stands for the triple $(E, p)$ or $(\nu, p)$ with $\nu = cE/ha$. The above sets of real bispinor normal modes are eigenstates of the matrix $\alpha^R_{\pm}$ because
\[
\alpha^R_{\pm} \phi_\pm^R(t, x, y, z) = \pm \phi_\pm^R(t, x, y, z)
\]
2.5. Majorana bispinor normal modes

In order to obtain the bispinor solutions [50] it is convenient to start with

\[
(i\partial_t + m) \exp\{-iEt + ip \cdot x_\perp\} = (\mathcal{P}_M + m) \exp\{-i(Et - p \cdot x_\perp)\}
\]

where

\[
\mathcal{P}_M + m = \gamma_1^M iD_x + \frac{E}{\hbar x} \gamma_0^M - \gamma_2^M p_y - \gamma_3^M p_z + m = \begin{pmatrix} D_x + m & -i(p_z - \nu/x) & 0 & ip_y \\ -i(p_z + \nu/x) & D_x + m & -ip_y & 0 \\ 0 & -ip_y & D_x + m & -i(p_z + \nu/x) \\ ip_y & 0 & -i(p_z - \nu/x) & D_x + m \end{pmatrix}
\]

Next we find

\[
\begin{align*}
(\mathcal{P}_M + m) \Upsilon^L_\pm &= m \Upsilon^L_\pm + (p_y + ip_z) \Upsilon^R_\pm - \Upsilon^L_\pm i\delta_+ \\
(\mathcal{P}_M + m) \Upsilon^R_\pm &= m \Upsilon^R_\pm - (p_y - ip_z) \Upsilon^L_\pm + \Upsilon^R_\pm i\delta_-
\end{align*}
\]

in which we have set

\[
\delta_\pm \equiv i\hat{p}_x \pm \frac{i\nu}{x} = \frac{d}{dx} + \frac{1}{x} \left( \frac{1}{x} \pm i\nu \right) = \delta^+_\pm
\]

Now, from the recursion formulæ

\[
\delta_\pm K_{i\nu \pm \frac{1}{2}}(\kappa x) + \kappa K_{i\nu + \frac{1}{2}}(\kappa x) = 0
\]

we get

\[
\delta_\pm \hat{f}_\pm(p_0, p; x) = \hat{f}_\pm(\nu, p) \delta_\pm K_{i\nu \mp \frac{1}{2}}(\kappa x) = -\kappa \hat{f}_\pm(\nu, p) K_{i\nu \pm \frac{1}{2}}(\kappa x)
\]

and after the suitable introduction of the transverse momentum dependent spin states

\[
\begin{align*}
u_\pm(p) &= m \Upsilon^L_\pm + (p_y + ip_z) \Upsilon^R_\pm \\
u_\pm(p) &= m \Upsilon^R_\pm - (p_y - ip_z) \Upsilon^L_\pm
\end{align*}
\]

we can eventually write the two sets of bispinor solutions: namely,

\[
\begin{align*}
(\mathcal{P}_M + m) \Upsilon^L_+ \hat{f}_+(p_0, p; x) &= u_+(p) K_{i\nu \mp \frac{1}{2}}(\kappa x) + i\kappa \Upsilon^L_- K_{i\nu \pm \frac{1}{2}}(\kappa x) \\
(\mathcal{P}_M + m) \Upsilon^R_+ \hat{f}_+(p_0, p; x) &= u_+(p) K_{i\nu - \frac{1}{2}}(\kappa x) - i\kappa \Upsilon^R_- K_{i\nu + \frac{1}{2}}(\kappa x)
\end{align*}
\]

together with

\[
\begin{align*}
(\mathcal{P}_M + m) \Upsilon^L_- \hat{f}_-(p_0, p; x) &= u_-(p) K_{i\nu \mp \frac{1}{2}}(\kappa x) - i\kappa \Upsilon^L_+ K_{i\nu \pm \frac{1}{2}}(\kappa x) \\
(\mathcal{P}_M + m) \Upsilon^R_- \hat{f}_-(p_0, p; x) &= u_-(p) K_{i\nu - \frac{1}{2}}(\kappa x) + i\kappa \Upsilon^R_+ K_{i\nu + \frac{1}{2}}(\kappa x)
\end{align*}
\]

Thus \( \forall p_0 \in \mathbb{R} \) we find four bispinor solutions describing the spin \( \frac{1}{2} \) quantum states of definite transverse momentum, which propagate in the Rindler wedges. Of course they can be always set into 1:1 correspondence with the left and right massive Weyl spinor with positive and negative helicities which are experienced by an inertial
observer, as it does. It can be readily checked by direct inspection that the spin states do satisfy the following orthonormality relations, viz.,

\[ \overline{u}_\pm(p) \gamma^0_M u_\pm(p) = \kappa^2 \]

(77)

\[ \overline{u}_\pm(p) \gamma^0_M \gamma^l_+ = \bar{v}_\pm(p) \gamma^0_M \gamma^l_+ = 0 \]

(78)

\[ \overline{u}_\pm(p) \gamma^0_M v_\pm(p) = \bar{v}_\pm(p) \gamma^0_M u_\pm(p) = 0 \]

(79)

(80)

Thus we can build up the two sets of complex plane wave solutions of the massive spinor covariant equation (19) in the right Rindler wedge \( \mathcal{M}_R \) which read

\[
\psi^i_p(t, x) = \sqrt{N/x} \exp\{-i\nu t (a/c) + i p \cdot x_\perp \}
\]

\[
\times \begin{cases} u_+(p) K_{i\nu - \frac{1}{2}(kx)} + ik \gamma^l_+ K_{i\nu + \frac{1}{2}(kx)} & \text{for } i = 1 \\ v_+(p) K_{i\nu + \frac{1}{2}(kx)} - ik \gamma^l_+ K_{i\nu - \frac{1}{2}(kx)} & \text{for } i = 2 \end{cases}
\]

(81)

\[
\phi^i_p(t, x) = \sqrt{N/x} \exp\{-i\nu t (a/c) + i p \cdot x_\perp \}
\]

\[
\times \begin{cases} u_-(p) K_{i\nu + \frac{1}{2}(kx)} - ik \gamma^l_- K_{i\nu - \frac{1}{2}(kx)} & \text{for } i = 1 \\ v_-(p) K_{i\nu - \frac{1}{2}(kx)} + ik \gamma^l_- K_{i\nu + \frac{1}{2}(kx)} & \text{for } i = 2 \end{cases}
\]

(82)

\( N \) being a normalization constant to be determined here below. As a matter of fact, the invariant inner product between any two complex solutions of the covariant Dirac equation is defined by

\[ (\psi_2, \psi_1) \equiv \oint_{\Sigma} V_{a\mu}(x) \bar{\psi}_2(x) \gamma^\alpha \psi_1(x) d\Sigma^\mu \]

(83)

where \( \Sigma \) is a three dimensional future oriented space-like hyper-surface. Again, for the initial time three dimensional hyper-surface in the right Rindler wedge \( \mathcal{M}_R \) we get

\[ d\Sigma^0 = \sqrt{-g} \theta(x) dx d^2x_\perp \quad d\Sigma^i = 0 \quad (i = 1, 2, 3) \]

and consequently

\[ (\psi_2, \psi_1) = \int_0^\infty dx \int d^2x_\perp \psi_2^i(t, x, x_\perp) \psi_1(t, x, x_\perp) \]

(84)

For \( p = (\mu, p) \) and \( q = (\nu, q) \) we find for example

\[ (\phi_q^i, \phi_p^j) = N (2\pi\kappa)^2 \delta_{ij} \delta(p - q) e^{-it(\mu - \nu)a/c} \]

\[ \times \lim_{\lambda \to 0^-} \int_0^\infty dx x^{-\lambda} \left\{ K^*_{i\nu + \frac{1}{2}(kx)} K_{i\mu - \frac{1}{2}(kx)} + \mu \leftrightarrow \nu \right\} \]

(85)
This means that one has to first consider the integral\[\int_0^\infty dx \, x^{-\lambda} K_{i\mu + \frac{i}{2}(\kappa x)} K_{i\nu - \frac{i}{2}(\kappa x)} = \frac{2^{-2-\lambda} \kappa^{\lambda-1}}{\Gamma(1-\lambda)} \times \Gamma\left(\frac{1-\lambda + i\mu + i\nu}{2}\right) \times \Gamma\left(\frac{2-\lambda + i\mu - i\nu}{2}\right) \times \Gamma\left(\frac{1-\lambda - i\mu - i\nu}{2}\right) \left( \Re \lambda < 0 \right) \]
and realize that the right hand side of the above equality is analytic for $\lambda \in \mathbb{R}$. Thus the most convenient and simple way one can understand the value of the integral appearing in the right hand side of (85) is in terms of analytic regularization. Namely, one can take the limit for $\lambda \to 0^-$ of the analytic function in the right hand side of (86) that yields
\[
\int_0^\infty dx \, K_{i\mu + \frac{i}{2}(\kappa x)} K_{i\nu - \frac{i}{2}(\kappa x)} = i \pi^2 / 4 \kappa \cosh \frac{1}{2}(\mu + \nu) \sinh \frac{1}{2}(\mu - \nu)
\]
which is purely imaginary or, equivalently, manifestly antisymmetric under the exchange $\mu \leftrightarrow \nu$. Hence, adding the complex conjugate, we immediately obtain the orthogonality relation, viz.,
\[
\int_0^\infty dx \, K_{i\mu + \frac{i}{2}(\kappa x)} K_{i\nu - \frac{i}{2}(\kappa x)} + \text{c.c.} = 0 \quad (\mu \neq \nu) \quad (87)
\]
Moreover, if we set $\mu - \nu = \xi + i\varepsilon$ then we can write for $\varepsilon \to 0^+$
\[
\int_0^\infty dx \, K_{i\mu + \frac{i}{2}(\kappa x)} K_{i\nu - \frac{i}{2}(\kappa x)} \sim \frac{i}{2\kappa \cosh \pi \mu} \frac{1}{\xi + i\varepsilon} = \left\{ \text{CPV} \left( \frac{1}{\xi} \right) - i\pi\delta(\xi) \right\} \frac{i}{2\kappa \cosh \pi \mu} \frac{\pi^2}{2\kappa} \delta(\mu - \nu) \sech \pi \mu \quad (88)
\]
in such a manner that we can eventually write
\[
\langle \phi^j_{\theta^i}, \phi^i_p \rangle = N (2\pi \kappa)^2 \delta_{ij} \delta(p - p') \delta(\mu - \nu) \frac{\pi^2}{\kappa \cosh \pi \mu} = \frac{4\pi^4 N \kappa \hbar a}{c \cosh(\pi c E / \hbar a)} \delta(p - p') \delta(E - E') \delta_{ij} \quad (89)
\]
where $i,j = 1,2$, whereas use has been made of the orthonormality relations of the spin states. Hence we eventually come to the two complete and orthonormal sets of normal modes for the massive spin $\frac{1}{2}$ field in the right Rindler wedge $\mathcal{M}_R$

\[
\phi^i_p(t, x, \perp) = c \sqrt{\frac{c^2}{\kappa a} \cosh \pi \mu} \exp\left\{-i\mu t (a/c) + i p \cdot x_\perp \right\} \varphi^i_p(x) = \sqrt{\frac{c^2}{\kappa a} \cosh \frac{\pi c E}{\hbar a}} \exp\left\{-\frac{i}{\hbar} E t + i p \cdot x_\perp \right\} \varphi^i_p(x) \quad (90)
\]
\[
\varphi^i_p(x) = \frac{\theta(x)}{2\pi^2} \times \begin{cases} u_{-}(p) K_{i\mu + \frac{i}{2}(\kappa x)} - i \kappa \Upsilon^L_{-} K_{i\mu - \frac{i}{2}(\kappa x)} & \text{for } i = 1 \\ v_{-}(p) K_{i\nu + \frac{i}{2}(\kappa x)} + i \kappa \Upsilon^R_{-} K_{i\nu - \frac{i}{2}(\kappa x)} & \text{for } i = 2 \end{cases} \quad (91)
\]
In the very same way we get the other complete orthonormal set of bispinors in the right Rindler wedge \( \mathcal{W}_R \):

\[
\psi_\alpha^i(t, x, x_\perp) = \sqrt{\frac{c^2}{\kappa a}} \cosh \pi \mu \exp\left\{ -i \mu t (a/c) + i p \cdot x_\perp \right\} \chi_\alpha^i(x)
\]

\[
\chi_\beta^i(x) \equiv \frac{\theta(x)}{2\pi^2} \times \begin{cases} u_+(p) K_{iv - \frac{1}{2} (\kappa x)} + i \kappa \Upsilon^L K_{iv + \frac{1}{2} (\kappa x)} & \text{for } i = 1 \\ v_+(p) K_{iv - \frac{1}{2} (\kappa x)} - i \kappa \Upsilon^R K_{iv + \frac{1}{2} (\kappa x)} & \text{for } i = 2 \end{cases}
\]

\[
p \equiv (\mu, p) = \left( \frac{E c}{\hbar a}, p \right) \quad q \equiv (\nu, q) = \left( \frac{E' c}{\hbar a}, q \right)
\]

\[
(\psi_\alpha^i, \phi_\beta^j) = \hbar c \delta(p - q) \delta(E - E') \delta_{ij}
\]

The above normalized bispinors belonging to both complete and orthonormal sets of solutions of the covariant massive Majorana spinor wave equation are dimensionless and turn out to be, as already noticed, equivalent and independent because

\[
(\psi_\alpha^i, \phi_\beta^j) = 0 \quad (\forall \mu, \nu \in \mathbb{R}, p, p' \in \mathbb{R}^2, i, j = 1, 2)
\]

It is also important to gather that the normal modes (90) and (93) of the accelerated massive spin \( \frac{1}{2} \) field do exhibit two helicity states, i.e. \( i = 1, 2 \).

2.6. Canonical quantization in the Rindler wedge

To proceed further on, it is convenient to simplify a little bit the notations by introducing a multi-index which collectively labels all the quantum numbers of the covariant massive Majorana spinor solutions. To this purpose, we shall use the indices \( i, \kappa, \ell, \omega, q, \varsigma, \vartheta, \ldots \) to label the quartets of quantum numbers

\[
\Omega = \{ (E, p, r) \mid E \in \mathbb{R}, (p_y, p_z) \in \mathbb{R}^2, i = 1, 2 \}
\]

together with

\[
\sum_{i \in \Omega} \equiv \int_{-\infty}^{\infty} \frac{dE}{\hbar c} \int d^2 p \sum_{j = 1, 2} = \frac{a}{c^2} \int_{-\infty}^{\infty} d\nu \int d^2 p \sum_{j = 1, 2}
\]

\[
\delta_{\omega \vartheta} = \hbar c \delta(E - E') \delta(p - p') \delta_{ij} = \frac{c^2}{a} \delta(\nu - \nu') \delta(p - p') \delta_{ij} = \delta^{\omega \vartheta}
\]
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in such a manner that we can write the normal modes expansion of the quantized massive Majorana spinor hermitean fields in the right Rindler wedge $\mathcal{W}_R$ in the form

$$\psi(x) = \sum_{\iota \in \Omega} \left[ a_\iota \psi_\iota(x) + a_\iota^\dagger \psi^*_\iota(x) \right] = \psi^\dagger(x) \quad [\psi] = eV^2$$

(98)

with

$$\psi_\iota(x) \equiv \psi_\iota^P(t,x,x_\perp) \quad \psi^*_\iota(x) \equiv \overline{\psi}_\iota^P(t,x,x_\perp) \beta_M$$

where the canonical anticommutation relations hold true, viz.,

$$\{a_\kappa , a_\ell^\dagger \} = \delta^\kappa_\ell \quad [a_\kappa] = cm^2$$

(99)

all other anticommutators being null. Notice that the canonical anticommutation relations drive towards the closure or completeness relation for the normal modes that reads

$$\{\psi(x), \overline{\psi}(y)\} = \sum_{\iota \in \Omega} \sum_{\kappa \in \Omega} \{a_\iota \psi_\iota(x) + a_\iota^\dagger \psi^*_\iota(x), a_\kappa \overline{\psi}_\kappa(y) + a_\kappa^\dagger \overline{\psi}^*_\kappa(y)\}$$

$$= \sum_{\iota \in \Omega} [\psi_\iota(x) \overline{\psi}_\iota(y) + \psi^*_\iota(x) \overline{\psi}^*_\iota(y)] = S(x,y)$$

(100)

In a similar way we find

$$\phi(x) = \sum_{\iota \in \Omega} \left[ b_\iota \phi_\iota(x) + b_\iota^\dagger \phi^*_\iota(x) \right] = \phi^\dagger(x) \quad [\phi] = eV^2$$

(101)

with

$$\phi_\iota(x) \equiv \phi_\iota^P(t,x,x_\perp) \quad \phi^*_\iota(x) \equiv \overline{\phi}_\iota^P(t,x,x_\perp) \beta_M$$

where the canonical anticommutation relations hold true, viz.,

$$\{b_\kappa , b_\ell^\dagger \} = \delta^\kappa_\ell \quad [a_\kappa] = cm^2$$

(102)

all other anticommutators being equal to zero.

To better understand the physical meaning of the above solutions of the massive Majorana spinor wave equation in the Rindler accelerated coordinates system, it is utmost convenient to select the special class of noninertial reference frames in which $p_y = p_z = 0$. For any such an observer, that will be named transverse momentum rest frame instantaneous observer, we have for example

$$\psi^P(x) = \sqrt{\frac{c^2}{\kappa a}} \cosh \frac{\pi cE}{\hbar a} \exp \left\{ - \frac{i}{\hbar} Et \right\} \chi^P_\lambda(x)$$

(103)

$$\chi^P_\lambda(x) = \frac{m}{2\pi^2} \theta(x) \left\{ \begin{array}{l} Y^L_\lambda K_{\mu -} + \frac{1}{2} (\kappa x) + i Y^L_{\mu +} K_{\lambda +} (\kappa x) \quad \text{for } \lambda = L \\ Y^R_\lambda K_{\mu -} + \frac{1}{2} (\kappa x) - i Y^R_{\mu +} K_{\lambda +} (\kappa x) \quad \text{for } \lambda = R \end{array} \right\}$$

(104)
\[ cH_M = ax \left( \alpha_M^\dagger \hat{P}_x + mc \beta_M \right) \]
\[ \hat{P}_x = \begin{cases} -i\delta_+ - E/ax \\ -i\delta_- + E/ax \end{cases} \]
\[ \beta_M \Upsilon^L_\pm = -\Upsilon^L_\mp \quad \beta_M \Upsilon^R_\pm = \Upsilon^R_\mp \]
and consequently
\[ H_M \psi^\dagger_E(t,x) = E \psi^\dagger_E(t,x) \quad (i = L, R) \]
\[ \Sigma^i_M \psi^\dagger_E(t,x) = \begin{cases} +\psi^\dagger_E(t,x) & \text{for } i = L \\ -\psi^\dagger_E(t,x) & \text{for } i = R \end{cases} \]

It means that, for example, the 1-particle states
\[ a_{\ell}^\dagger |0\rangle = |\ell^+\rangle = |E p_j\rangle \quad (\forall \ell = E, p, j) \]
will actually describe massive, neutral, spin \( \frac{1}{2} \) quanta, called pseudo-particles, with indefinite energy, i.e., \(-\infty < E < \infty\), transverse wave numbers \( p_y, p_z \in \mathbb{R} \) and definite helicities \( j = 1, 2 \), i.e. spin projection along and versus the direction of the accelerated observer, while
\[ b_{\varrho}^\dagger |0\rangle = |\varrho^-\rangle = |E p_\iota\rangle \quad (\forall \varrho = E, p, \iota) \]
do describe massive neutral spin \( \frac{1}{2} \) particles with opposite helicities.

3. The Bogolyubov Coefficients For Majorana Spinors

Now we are ready to generalize the method developed for the spinless and chargeless quantum field to the neutral Majorana bispinor quantum field. To this purpose, let us recall the normal modes expansion of a quantized Majorana Hermitean bispinor in an inertial reference frame of the Minkowski space, i.e.,
\[ \Psi_M(X) = \sum_{P, r} \left[ A_{P, r} \Psi_{P}^R(X) + A_{P, r}^\dagger \Psi_{P}^L(X) \right] = \Psi_M^\dagger(X) \]
\[ \bar{\Psi}_M(X) = \sum_{\bar{P}, r} \left[ A_{\bar{P}, r} \bar{\Psi}_{\bar{P}}^R(X) + A_{\bar{P}, r}^\dagger \bar{\Psi}_{\bar{P}}^L(X) \right] = \bar{\Psi}_M^\dagger(X) \]

Now we have to express the above hermitean quantum field in Rindler’s curvilinear coordinates according to (15), in which, however, we shall use the slightly modified and more convenient notation
\[ \Psi_K^\dagger[X(x)] = \left[ (2\pi)^3 2\omega_K \right]^{-1/2} u_K(t) \exp \left\{ \text{i} k_x x + i K_{\perp} \cdot x_{\perp} \right\} \]
\[ \bar{\Psi}_K^\dagger[X(x)] = \left[ (2\pi)^3 2\omega_K \right]^{-1/2} \bar{u}_K(t) \exp \left\{ -\text{i} k_x x - i K_{\perp} \cdot x_{\perp} \right\} \]
where
\[ \sum_{K_{\perp}, r} = \int d^2 K_{\perp} \int_{-\infty}^{\infty} dK \sum_{r = \uparrow, \downarrow} \]
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\[ k_t = K \cosh \eta - \omega_K \sinh \eta \]

the spin states being now time dependent and denoted by

\[ u_K(t) \equiv D[\Lambda(t)] u_r(K) \quad [K^\mu = (\omega_K, K), \ r = \uparrow, \downarrow] \]

where \( u_r(K) (r = \uparrow, \downarrow) \) are the usual spin states, defined as

\[ u_r(K) \equiv \begin{pmatrix} 1 & -i \\ 0 & 0 \end{pmatrix}, \quad u_r^*(K) = \begin{pmatrix} 0 & 0 \\ 1 & i \end{pmatrix} \]

which are moreover the eigenstates of the positive energy projector

\[ E_M^+(K) u_r(K) = u_r(K) \quad (r = \uparrow, \downarrow) \]

where

\[ E_M^\pm(K) = \begin{pmatrix} m \pm \gamma_M^\mu K_\mu / 2m \end{pmatrix} \]

are the eigen-bispinors of the rest frame Majorana Hamiltonian, both with eigenvalue \( m \).

The canonical anticommutation relations hold true

\[ \{ A_{P, r}, A_{K, s} \} = \{ A_{P, r}^\dagger, A_{K, s}^\dagger \} = 0 \]

\[ \{ A_{P, r}, A_{K, s}^\dagger \} = \delta_{rs} \delta(P - K) \]

Notice that from the general invariant and time-independent normalization in the space-like region of the right Rindler’s wedge

\[ (\Psi_P^*, \Psi_K^\dagger) = \int d^3X \ \overline{\Psi}_P(\tau, X) \beta_M \Psi_K^\dagger(\tau, X) \]

\[ = \int_0^\infty dx \int d^2x_\perp \overline{\Psi}_P[X(x)] \beta_M \Psi_K^\dagger[X(x)] \]

\[ = \delta^{rs} \delta(P - K) \]

we can readily obtain the normal modes expansions of the observables involving the Majorana massive spinor field. The Majorana field equation, which is satisfied by the real self-conjugated bispinor field for any inertial observer, reads

\[ \gamma_M^\mu \partial_\mu \Psi_M(X) + im \Psi_M^* \Psi_M = 0 \]

\[ \Psi_M = \Psi_M^* = \Psi_M^\dagger \]
To proceed further on, consider the basic integrals

\[ I_{i\nu \pm \frac{1}{2}}(\kappa, P) = \int_0^\infty dx \, e^{-iPx} K_{i\nu \pm \frac{1}{2}}(\kappa x) = \frac{\pi \cos(\pi \nu \pm \pi/2)}{\sqrt{(-\kappa^2 - P^2)}} \]

\[ \times \left[ \kappa^{-i\nu \pm \frac{1}{2}} \left( \sqrt{-\kappa^2 - P^2} + iP \right)^{i\nu \pm \frac{1}{2}} - \kappa^{i\nu \pm \frac{1}{2}} \left( \sqrt{-\kappa^2 - P^2} + iP \right)^{-i\nu \pm \frac{1}{2}} \right] \]

\[ = \frac{\mp \pi i}{\omega_p \cosh(\pi \nu)} \left[ e^{-\frac{\pi i}{2} \nu \pm \frac{\pi}{4} \theta} \left( \frac{P + \omega_p}{\kappa} \right)^{i\nu \pm \frac{1}{2}} - e^{\frac{\pi i}{2} \nu \mp \frac{\pi}{4} \theta} \left( \frac{P + \omega_p}{\kappa} \right)^{-i\nu \mp \frac{1}{2}} \right] \]

so that if we set

\[ \omega_p = \kappa \cosh \theta \quad P = \kappa \sinh \theta \quad (122) \]

\[ \kappa^2 = \omega_p^2 - P^2 \quad \theta = \arctan(P/\omega_p) \quad (123) \]

we can recast the basic integral in the form\[ ]

\[ I_{i\nu \pm \frac{1}{2}}(\kappa, P) = \frac{\pi}{2i\omega_p \cosh(\pi \nu)} \left[ (ie^\theta)^{i\nu \pm 1/2} - (ie^\theta)^{-i\nu \mp 1/2} \right] \]

\[ = \frac{\pi}{\kappa \cosh \theta \cosh(\pi \nu)} \sin \left[ \theta \nu + \frac{\pi i \nu}{2} \pm \frac{i \theta}{2} \pm \frac{\pi}{4} \right] \]

(124)

The accelerated noninertial observer does instead employ e.g. the other complete and orthonormal set (93) of bispinors in the right Rindler wedge \[ ]

\[ \psi_p^i(x) = \sqrt{\frac{1}{\kappa a}} \cosh \pi \mu \exp\{-ia \mu t + i p \cdot x_\perp\} \chi_p^i(x) \]

\[ \chi_p^i(x) \equiv \frac{1}{\sqrt{2\pi^2}} \times \begin{cases} u_+(p) K_{i\mu - \frac{1}{2}}(\kappa x) + i \kappa \Upsilon_+ K_{i\mu + \frac{1}{2}}(\kappa x) & \text{for } i = 1 \\ v_+(p) K_{i\mu - \frac{1}{2}}(\kappa x) - i \kappa \Upsilon_+ K_{i\mu + \frac{1}{2}}(\kappa x) & \text{for } i = 2 \end{cases} \quad (x \geq 0) \]

\[ p \equiv (\mu, p) = (\frac{E}{a}, p) \quad q \equiv (\nu, q) = (\frac{E'}{a}, q) \]

\[ (\psi_q^i, \psi_p^i) = \frac{1}{a} \delta(p - q) \delta(\mu - \nu) \delta^{ij} \]

The Bogolyubov coefficients for the Hermitian self-conjugated Majorana quantized bispinor field are thereby defined as follows: namely,

\[ \alpha(P, r; p, i) = \left( \Psi_p^i, \psi_p^i \right) = 2 \pi \delta(P_\perp - p) \]

\[ \times \sqrt{\frac{\cosh \pi \mu}{4\pi \kappa a \omega_p}} \int_0^\infty \tilde{u}_p(0) \beta_M \chi_p^i(x) \, e^{-iPx} \, dx \]

\[ = \delta(P_\perp - p) \sqrt{\frac{e^3}{2\pi \kappa a \omega_p}} \]

\[ \times \left\{ R_p^i P I_{i\mu - \frac{1}{2}}(\kappa, P) + R_p^{i*} P I_{i\mu + \frac{1}{2}}(\kappa, P) \right\} \]

(125)
where we have introduced the spin correlation matrices

\[ \mathbf{R}_{\mathbf{p}} = \frac{1}{\pi} \left( \begin{array}{c} \tilde{u}_{\mathbf{p}}^\dagger \beta_{\mathbf{p}} \mu_{\mathbf{p}}(\mathbf{p}) \tilde{u}_{\mathbf{p}}^\dagger \beta_{\mathbf{p}} \nu_{\mathbf{p}}(\mathbf{p}) \end{array} \right) \sqrt{\cosh \pi \mu} \] (126)

\[ \mathbf{R}_{\mathbf{p}}^\dagger = \frac{1}{\pi} \left( \begin{array}{c} \tilde{u}_{\mathbf{p}}^\dagger \beta_{\mathbf{p}} \mu_{\mathbf{p}}(\mathbf{p}) \tilde{u}_{\mathbf{p}}^\dagger \beta_{\mathbf{p}} \nu_{\mathbf{p}}(\mathbf{p}) \end{array} \right) \sqrt{\frac{1}{2} \kappa \cosh \pi \mu} \] (127)

Notice that we have suitably factorized the classical volume factor

\[ \delta(\mathbf{P}_\perp - \mathbf{p}) \sqrt{\frac{c^3}{2 \pi \hbar \omega_{\mathbf{p}}}} \]

just like in the spinless case. Then the Bogolyubov coefficients read

\[ \alpha(\mathbf{P}, \uparrow; \mathbf{p}, 1) = \left( \begin{array}{c} \Psi_{\mathbf{p}}^\dagger \psi_1^\mathbf{p} \end{array} \right) \] (128)

\[ \alpha(\mathbf{P}, \downarrow; \mathbf{p}, 2) = \left( \begin{array}{c} \Psi_{\mathbf{p}}^\dagger \psi_2^\mathbf{p} \end{array} \right) \] (129)

\[ \alpha(\mathbf{P}, \downarrow; \mathbf{p}, 1) = \left( \begin{array}{c} \Psi_{\mathbf{p}}^\dagger \psi_1^\mathbf{p} \end{array} \right) \] (130)

\[ \alpha(\mathbf{P}, \downarrow; \mathbf{p}, 2) = \left( \begin{array}{c} \Psi_{\mathbf{p}}^\dagger \psi_2^\mathbf{p} \end{array} \right) \] (131)
Thus we get for example coefficients

\[ \sum_{i=1,2} \sum_{r=\uparrow,\downarrow} \left| \mathbb{R}^{ir}_{\mathbf{P}} I_{ir-\frac{1}{2}}(\kappa, \mathbf{P}) + \mathbb{R}^{ir}_{\mathbf{P}} I_{ir+\frac{1}{2}}(\kappa, \mathbf{P}) \right|^2 \]

\[ = \frac{k^2}{\pi^2} \cosh(\pi \nu) \left\{ e^{-\theta} |I_{ir+\frac{1}{2}}(\kappa, \mathbf{P})|^2 + e^{\theta} |I_{ir-\frac{1}{2}}(\kappa, \mathbf{P})|^2 + i \left[ I_{ir+\frac{1}{2}}^*(\kappa, \mathbf{P}) I_{ir-\frac{1}{2}}(\kappa, \mathbf{P}) - I_{ir-\frac{1}{2}}^*(\kappa, \mathbf{P}) I_{ir+\frac{1}{2}}(\kappa, \mathbf{P}) \right] \right\} \]

\[ = \frac{2}{1 + e^{2\pi \nu}} \equiv \bar{N}_\nu \]

where the hyperbolic parameter \( \theta \) – call it rapidity – is defined by eqs. (122) and (123).

In order to determine the complementary Bogolyubov coefficients \( \beta(\mathbf{P}, r; \mathbf{p}, i) \), we need to employ Minkowskian normal modes of negative frequencies that read

\[ \beta(\mathbf{P}, r; \mathbf{p}, i) \equiv \left( \Psi^*_{\mathbf{P}} \right| \psi^i_{\mathbf{P}} \right) = 2\pi \delta(\mathbf{P}_\perp + \mathbf{p}) \times \sqrt{\frac{\cosh \pi \mu}{4\pi \kappa \omega_{\mathbf{P}}}} \int_0^\infty [\bar{u}^*_\mathbf{P}(0)]^* \beta^{M\perp}_\mathbf{P} x e^{ip_x} \, dx \]

\[ = \delta(\mathbf{P}_\perp + \mathbf{p}) \sqrt{\frac{c^3}{2\pi \kappa \omega_{\mathbf{P}}}} \times \left\{ \mathbb{S}^{ir}_{\mathbf{P}} I_{ir+\frac{1}{2}}^*(\kappa, \mathbf{P}) + \mathbb{S}^{ir}_{\mathbf{P}} I_{ir-\frac{1}{2}}^*(\kappa, \mathbf{P}) \right\} \]

(133)

where we have introduced the spin correlation matrices

\[ \mathbb{S}^{ir}_{\mathbf{P}} = \frac{1}{\pi} \begin{pmatrix} 
\bar{u}^*_\mathbf{P} \beta^M u^+_{\mathbf{P}} & \bar{u}^*_\mathbf{P} \beta^M v^+_{\mathbf{P}} \\
\bar{u}^*_\mathbf{P} \beta^M v^+_{\mathbf{P}} & \bar{u}^*_\mathbf{P} \beta^M u^+_{\mathbf{P}} 
\end{pmatrix} \sqrt{\frac{\cosh \pi \mu}{2\kappa}} \]

(134)

\[ \mathbb{S}^{ir}_{\mathbf{P}} = \frac{1}{\pi} \begin{pmatrix} 
\bar{u}^*_\mathbf{P} \beta^M \Upsilon^L_{\mathbf{P}} & \bar{u}^*_\mathbf{P} \beta^M \Upsilon^R_{\mathbf{P}} \\
-\bar{u}^*_\mathbf{P} \beta^M \Upsilon^R_{\mathbf{P}} & \bar{u}^*_\mathbf{P} \beta^M \Upsilon^L_{\mathbf{P}} 
\end{pmatrix} \sqrt{\frac{\kappa \cosh \pi \mu}{2}} \]

(135)

Thus we get for example

\[ \beta(\mathbf{P}, \uparrow; \mathbf{p}, 1) = \left( \Psi^*_{\mathbf{P}} \right| \psi^1_{\mathbf{P}} \right) \]

\[ = \frac{1}{\pi} (1 + i) \delta(\mathbf{P}_\perp + \mathbf{p}) \sqrt{\frac{\cosh \pi \nu}{(2\pi)^3 \kappa \omega_{\mathbf{P}} (m + \omega_{\mathbf{P}}^2)}} \times \left\{ \left[ m^2 + (m - p_y) \omega_{\mathbf{P}} + p_y^2 + p_z (m - p_y - ip_x) + p_x (p_z - i\omega_{\mathbf{P}}) \right] \right\} \]

\[ \times \left[ \frac{m^2 + (m - p_y) \omega_{\mathbf{P}} + p_y^2 + p_z (m - p_y - ip_x) + p_x (p_z - i\omega_{\mathbf{P}})}{m^2 + (m - p_y) \omega_{\mathbf{P}} + p_y^2 + p_z (m - p_y - ip_x) + p_x (p_z - i\omega_{\mathbf{P}})} \right] \]

\[ \times I_{ir+\frac{1}{2}}^*(\kappa, \mathbf{P}) + in (m + \omega_{\mathbf{P}} - p_y - ip_x) I_{ir-\frac{1}{2}}^*(\kappa, \mathbf{P}) \]

and quite closer formulas for the three remaining components. Once again, just like in the case of the \( \alpha \)-type Bogolyubov coefficients, it is straightforward although
tedious to check the following quite remarkable sum rule for the dimensionless coefficients

\[
\sum_{i=1,2} \sum_{r=\uparrow,\downarrow} \left| \mathbb{S}_{\mathbf{pP}}^{ir} I_{iv+\frac{1}{2}}(\kappa, P) + \mathbb{S}_{\mathbf{pP}}^{ir} I_{iv-\frac{1}{2}}(\kappa, P) \right|^2 \\
= \frac{\kappa^2}{\pi^2} \cosh(\pi \nu) \left\{ e^{-\theta} |I_{iv-\frac{1}{2}}(\kappa, P)|^2 + e^{\theta} |I_{iv+\frac{1}{2}}(\kappa, P)|^2 \\
+ i \left( I_{iv-\frac{1}{2}}^{*}(\kappa, P) I_{iv+\frac{1}{2}}(\kappa, P) - I_{iv-\frac{1}{2}}(\kappa, P) I_{iv+\frac{1}{2}}^{*}(\kappa, P) \right) \right\} \\
= \frac{2}{1 + e^{-2\pi \nu}} \equiv N_\nu
\]
which drives to the expected Pauli principle and spin sum relations

\[
N_\nu + \bar{N}_\nu = 2S + 1 \quad \forall \nu = \frac{Ec}{\hbar a} \quad (S = \frac{1}{2})
\]
and provides the crucial test for the correctness of our long and delicate derivation.

4. Discussion And Conclusions

In this paper we have explicitly derived the Bogolyubov coefficients which express the probability amplitude for particle creation out of the inertial vacuum for a neutral, massive, spin $\frac{1}{2}$ field in a Rindler coordinate system, i.e., as experienced by some uniformly accelerated observer. Our method for the evaluation of the probability amplitudes is nothing but a straightforward though highly non-trivial application of the very definition. In particular, it relies on the explicit knowledge of the Fulling-type modes for the Majorana field. The very evaluation of the Bogolyubov coefficients, in the case of spinless particles, has often been carried out in the literature by indirect methods involving, sometimes, somewhat ambiguous mathematical trickeries. This, in turn, sparked some controversy regarding the validity of such derivations, raised by [10] et seq. Hence, it is worthwhile to stress that our method is exempt from the criticism raised in those references, because of the direct and straightforward techniques adopted, therefore directly addressing this issue.

The Bogolyubov coefficients can be eventually expressed in closed parametric form in terms of the rapidity $\theta = \text{Arsh}(P/\kappa)$, where $P$ and $\kappa$ are the longitudinal and transverse momenta of the particle in an inertial reference frame – transverse and longitudinal are understood with respect the the direction of the acceleration.

Among the possible applications of our results, we briefly mention the simple situation of a de Sitter universe with constant Hubble parameter. We consider an observer located at fixed distance from (but inside) the Hubble sphere of another, geodesic, observer. The first observer is then non-geodesic, and is known as a Ko-dama observer [13]. Then our considerations can be applied locally around the location of the non-geodesic observer: the relation between tangent spaces at that point, from the point of view of the two different frames, is precisely the same as Minkowski-Rindler. The relative acceleration would then be given by the acceleration of the
Hubble horizon, namely
\[ a_{\text{cosmic}} = c H \approx 2.1 \times 10^{-9} \text{ m s}^{-2} \]

Then the spin \( \frac{1}{2} \) WIMP candidates would be arranged in a Fermi-Dirac distribution at the thermal equilibrium with an absolute temperature

\[ T_{\text{cosmic}} = \frac{\hbar H}{2 \pi k_B} \approx 8.0 \times 10^{-30} \text{ eK} \]

the Unruh cosmic temperature, in agreement with.\(^\text{13}\)

On the one hand, taking into account that \( k_B T_{\text{cosmic}} \approx 1 \times 10^{-33} \text{ eV} \), it appears that the energy density induced by the Unruh effect is very much suppressed. Instead, in order to obtain an appreciable contribution to the energy density, one would need to consider an acceleration much larger than \( a_{\text{cosmic}} \). It must be noted that, in the case of two relatively accelerating galaxies, these would be both geodesic observers. Nevertheless, it can be shown\(^\text{14}\) that the effect still exists in that context, the temperature being the same as the one obtained above (for a mutual distance of the order of the Hubble radius).

On the other hand, a quite close phenomenon occurs, \textit{mutatis mutandis}, in the vicinity of a black-hole horizon\(^\text{15}\), so that the present calculation might be also useful in the Quantum Gravity framework.

Last but not least, our method of calculation can be further applied to the evaluation of the Bogolyubov coefficients for the emission of charged spin \( \frac{1}{2} \) pairs and vector particles in a uniformly accelerated reference frame, which are still unknown.

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Appendix A. Tetrads and the spin-connection for the Majorana field
This section provides a concise summary of our conventions for the tetrads and the spin connection of the Majorana field. First we define

\[
X^\alpha (x^\mu) = \begin{cases} 
  c \tau (ct, x, y, z) \\
  X (ct, x, y, z) \\
  Y (ct, x, y, z) \\
  Z (ct, x, y, z)
\end{cases}
\]

\[
\frac{\partial X^\alpha}{\partial x^\mu} \equiv \zeta^\alpha_\mu (x) = \Lambda^\alpha_\beta (x) V^\beta_\mu (x)
\]

\[
ds^2 = \eta_{\alpha\beta} dX^\alpha dX^\beta = \eta_{\alpha\beta} V^\alpha_\mu (x) V^\beta_\nu (x) dx^\mu dx^\nu \equiv g_{\mu\nu} (x) dx^\mu dx^\nu
\]

where the local Lorentz transformation \( \Lambda (x) \) is a rank-four square matrix belonging to the irreducible vector representation \( D_{\frac{3}{2}} \) of the Lorentz group, its matrix
elements $\Lambda_\alpha^\beta(x)$ being dependent upon the curvilinear coordinates. Next we set

$$\Psi_M[X(x)] = D[\Lambda(x)]\psi_M(x)$$

$$D[\Lambda(x)] = \exp \left\{ -\frac{1}{2} i \sigma_M^{\alpha\beta} \omega_{\alpha\beta}(x) \right\}$$

$$\sigma_M^{\alpha\beta} \equiv \frac{1}{4} i [\gamma_M^\alpha, \gamma_M^\beta]$$

$$D[\Lambda(x)] = \exp \left\{ -\frac{1}{2} i \sigma_M^{\alpha\beta} \omega_{\alpha\beta}(x) \right\}$$

$$\frac{\partial}{\partial X^\alpha} = \left( \frac{\partial x^\mu}{\partial X^\alpha} \right) \frac{\partial}{\partial x^\mu} = \Lambda_{\beta}^\alpha(x)V^{\mu}_\beta[X(x)] \frac{\partial}{\partial x^\mu}$$

$$V^{\mu}_\alpha[X(x)]V_{\alpha}^{\beta}(x) = \delta^\beta_{\alpha}$$

where $D[\Lambda(x)]$ denotes the four dimensional spinor local representation $D[\Lambda(x)] = D_{\pm}[\Lambda(x)] \oplus D_{\pm}[\Lambda(x)]$ of the Lorentz group in the purely imaginary Majorana representation for the gamma matrices

$$\gamma_M^0 = \begin{pmatrix} -\sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix} \quad \gamma_M^1 = \begin{pmatrix} -i\sigma_3 & 0 \\ 0 & -i\sigma_3 \end{pmatrix}$$

$$\gamma_M^2 = \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} \quad \gamma_M^3 = \begin{pmatrix} i\sigma_1 & 0 \\ 0 & i\sigma_1 \end{pmatrix}.$$ 

$D[\Lambda(x)]$ satisfies

$$D^{-1}[\Lambda(x)] = D[\Lambda^{-1}(x)] = \exp \left\{ \frac{1}{2} i \sigma_M^{\alpha\beta} \omega_{\alpha\beta}(x) \right\}$$

Then we get

$$\frac{i\partial}{\partial X^\alpha} \Psi_M(X) = \gamma_M^\alpha \Lambda_\alpha^\beta(x)V^{\mu}_\beta[X(x)] \frac{i\partial}{\partial X^\alpha} \{D[\Lambda(x)]\psi_M(x)\}$$

$$= \Lambda_\alpha^\beta(x)V^{\mu}_\beta[X(x)] \gamma_M^\alpha \{i\partial_\mu D[\Lambda(x)]\} \psi_M(x)$$

$$+ \Lambda_\alpha^\beta(x)V^{\mu}_\beta[X(x)] \gamma_M^\alpha D[\Lambda(x)] \{i\partial_\mu\psi_M(x)\}.$$ 

Thus, if we multiply from the left by the inverse matrix $D^{-1}[\Lambda(x)]$ we find

$$m\psi_M(x) = \Lambda_\alpha^\beta(x)V^{\mu}_\beta[X(x)] D^{-1}[\Lambda(x)] \gamma_M^\alpha \{i\partial_\mu D[\Lambda(x)]\} \psi_M(x)$$

$$+ \Lambda_\alpha^\beta(x)V^{\mu}_\beta[X(x)] D^{-1}[\Lambda(x)] \gamma_M^\alpha D[\Lambda(x)] \{i\partial_\mu \psi_M(x)\}.$$ 

and from the well known transformation properties of the gamma matrices

$$D^{-1}[\Lambda(x)] \gamma_M^\alpha D[\Lambda(x)] = \Lambda_\alpha^\beta(x) \gamma_M^\beta \quad \Lambda_\alpha^\beta(x) \Lambda^\eta_\beta(x) = \delta^\alpha_\eta$$

together with the definition

$$\Lambda_\alpha^\beta(x)V^{\mu}_\beta[X(x)] D^{-1}[\Lambda(x)] \gamma_M^\alpha \{i\partial_\mu D[\Lambda(x)]\} = V^{\mu}_\alpha(x) \gamma_M^\alpha \Gamma_\mu(x) \quad (A.1)$$

we eventually come to the Majorana bispinor field equation for a noninertial frame referred to a curvilinear coordinate system: namely,

$$V^{\mu}_\alpha(x) \gamma_M^\alpha \{\partial_\mu - i\Gamma_\mu(x)\} \psi_M(x) + im\psi_M(x) = 0 \quad \psi_M = \psi_M^c \quad (A.2)$$

$$\psi_M^c \equiv \psi_M(x)$$
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