Type checking data structures more complex than trees

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Abstract: Graphs are a generalized concept that encompasses more complex data structures than trees, such as difference lists, doubly-linked lists, skip lists, and leaf-linked trees. Normally, these structures are handled with destructive assignments to heaps, which is opposed to a purely functional programming style and makes verification difficult. We propose a new purely functional language, $\lambda_{GT}$, that handles graphs as immutable, first-class data structures with a pattern matching mechanism based on Graph Transformation and developed a new type system, $F_{GT}$, for the language. Our approach is in contrast with the analysis of pointer manipulation programs using separation logic, shape analysis, etc. in that (i) we do not consider destructive operations but pattern matchings over graphs provided by the new higher-level language that abstract pointers and heaps away and that (ii) we pursue what properties can be established automatically using a rather simple typing framework.

Keywords: Functional programming, graph grammar, type system, program verification, heap analysis

Fig. 1 Examples of complex graph structures

1. Introduction

In this study, we propose a new functional language that handles graphs as a first-class data structure. Graphs are a generalized concept that encompasses more complex data structures than trees, such as difference lists, doubly-linked lists, skip lists, and leaf-linked trees. However, graph structures cannot be handled succinctly in purely functional languages. Although such structures can be handled with references, this style implies imperative programming with destructive assignments, which makes it hard to read and write programs and also makes verification more difficult. In addition, classic type systems can only verify the types of the referenced data and cannot verify the shape of the data structure. Therefore, we aim to incorporate Graph Transformation [16] to a functional language and to develop a new type system for that. Our approach is in contrast with the analysis of pointer manipulation programs using separation logic [15], shape analysis, etc. in that (i) we consider graph structures formed by higher-level languages that abstract pointers and heaps away and guarantee low-level invariants such as the absence of dangling pointers and that (ii) we pursue what properties can be established automatically using a rather simple typing framework.

1.1 HyperLMNtal: Hypergraph rewriting language

Graph Transformation Systems (GTSs) are computational models and programming (or modeling) languages based on graphs and their rewritings [3], [16]. Of various GTSs, HyperLMNtal [21] is a rewriting language that supports hypergraphs. With hypergraphs, we can express structures more complex than trees, e.g., difference lists, doubly-linked lists, skip lists, and leaf-linked trees.

HyperLMNtal allows us to handle these data structures declaratively with rewrite rules that are activated by pattern matching. Furthermore, GTS has cultivated a unique style of type checking frameworks such as Structured Gamma [5]. However, GTSs are in general based on destructive rewriting and do not support higher-order functions. In contrast, functional languages basically work with immutable data structures and support higher-order functions, making them highly modular. This motivates us to study how we can incorporate the data structure of HyperLMNtal into the $\lambda$-calculus.

1.2 The $\lambda_{GT}$ language

We propose a new functional language, $\lambda_{GT}$, that features graphs as a first-class data structure. The $\lambda_{GT}$ language allows us to handle complex data structures declaratively with a static type system. Intuitively, the core language is a call-by-value $\lambda$-calculus that employs hypergraphs as values and supports pattern matching for them.

In order to formalize hypergraphs in a syntax-directed manner, we employ the techniques developed in a hypergraph rewrit-
We propose the formal syntax and semantics of $\lambda_{GT}$. While various different formalisms have been proposed to handle the shapes of graphs, including bisimulation (to handle “equivalence” of cyclic structures) and morphism (in a category-theoretic approach), we believe that our approach enables type checking relatively easily. We also propose a new type-checking algorithm that automatically performs this verification using structural induction.

1.3 Contributions
The main contributions of this paper are twofold.

1. We propose the formal syntax and semantics of $\lambda_{GT}$, a pure functional language that handles data structures beyond algebraic data types.

2. We propose a typing framework for the $\lambda_{GT}$ language and develop a new algorithm that can successfully handle the manipulations of graphs, which could not be handled in a previous study, Structured Gamma.

1.4 Structure of the Paper
The rest of this paper is organized as follows. Section 2 introduces HyperLMNtal, a calculus model based on hypergraph transformation. Section 3 gives the syntax and the operational semantics of the proposing language $\lambda_{GT}$. Section 4 introduces the new type system, $F_{GT}$ proposed for $\lambda_{GT}$. Section 5 extends the system $F_{GT}$ to cover powerful operations based on graph transformation. Section 6 discusses the algorithm for the extended $F_{GT}$. Section 7 describes related work.

1.5 Syntactic conventions
Throughout the paper, we use the following syntactic conventions.

For some syntactic entity $E$, $[E]$ stands for a sequence $E_1, \ldots, E_n$ for some $n \geq 0$. When we wish to mention the indices explicitly, $E_1, \ldots, E_n$ will also be denoted as $E_i$. The length of the sequence $E$ is denoted as $|E|$. For a set $S$, the form $S[s]$ stands for the set $S$ such that $s \in S$ (or equivalently, $S = S \cup \{s\}$).

For some syntactic entities $E$, $p$ and $q$, a substitution $E[q/p]$ stands for $E$ with all the (free) occurrences of $p$ replaced by $q$. An explicit definition will be given if the substitution should be capture-avoiding.

In order to focus on novel and/or non-obvious aspects of the language, constructs and properties that can be defined/derived in the same manner as those of standard functional languages will be described rather briefly.

2. HyperLMNtal
HyperLMNtal is extended from LMNtal [20]. LMNtal is a computational model and a programming language based on hierarchical graph rewriting. Flat LMNtal is a subset of LMNtal which does not allow a hierarchy of graphs. Links in graphs that LMNtal handles are restricted to have at most two endpoints. On the other hand, HyperLMNtal [21] allows hyperlinks, apart from normal links, which can interconnect an arbitrary number of endpoints. Flat HyperLMNtal is a subset of HyperLMNtal that disallow normal links and hierarchies of hypergraphs: the data structure of Flat HyperLMNtal is formed only by hyperlinks and nodes.

In the previous study, we have given syntax-directed semantics for Flat HyperLMNtal [17], [18]. As far as we have surveyed, Flat HyperLMNtal is the only computational model that has syntax-directed semantics which handles hypergraph matching and rewriting. Since the $\lambda$-calculus and many other computational models derived from the $\lambda$-calculus are defined as Structural Operational Semantics (SOS) [13], it would be smoother to incorporate Flat HyperLMNtal than other graph transformation formalisms based on algebraic approaches [16].

The following subsections are based on Flat HyperLMNtal, except that hypergraphs and rewrite rules are separated from each other for the sake of formulation. Hereinafter we simply refer to this language as HyperLMNtal, hyperlinks as links, and hypergraphs as graphs.

2.1 Syntax of graphs and rewrite rules
HyperLMNtal is composed of two syntactic categories.

- $X$ denotes a Link Name.
- $p$ denotes an Atom Name.

The only preserved atom name is $\star$, where an atom $X \rightsquigarrow Y$, called a fusion, fuses the link $X$ and the link $Y$ into a single link.

The syntax of HyperLMNtal is given in Fig. 2. We abbreviate $\nu X_1, \ldots, \nu X_n, G$ to $\nu X_1 \ldots X_n, G$, which can be denoted as $\nu X.G$. The pair of the name $p$ and the arity $n = |X|$ of an atom $p(X)$ is referred to as the function $^1$ of the atom and is written as $p/n$.

The set of free link names in hypergraph $G$ is denoted as $\text{fn}(G)$, which is defined inductively in Fig. 3.

**Definition 2.1** (Abbreviation). We introduce the following abbreviation schemes:

| Graph                  | $G ::= 0 \quad \text{Null}$ |
|------------------------|-------------------------------|
|                        | $p(X) \quad \text{Atom}$     |
|                        | $(G, G) \quad \text{Molecule}$|
|                        | $\nu X.G \quad \text{Hyperlink creation}$ |

**Rewrite Rule**

$r ::= G \rightarrow G \quad \text{Rule}$

$\text{Fig. 2} \quad \text{Syntax of HyperLMNtal}$

$\text{fn}(0) = 0$

$\text{fn}(p(X)) = [X]$

$\text{fn}(G_1, G_2) = \text{fn}(G_1) \cup \text{fn}(G_2)$

$\text{fn}(\nu X.G) = \text{fn}(G) \setminus [X]$

**Fig. 3** The set of free link names

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$^1$ Synonym of function symbol and function object; not to be confused with functors in category theory.
Definition 2.3 (Link Substitution) defines structural congruence.

Graphs related by structural congruence are essentially the same. This subsection describes the syntax and the semantics of HyperLMNtal graphs.

$$\nu X(G(Y/X)) \equiv \nu Y(X)$$

where $X \in fn(G)$ and $Y \notin fn(G)$.

2. Structural Concurrency

The semantics of Flat HyperLMNtal comes with two major ingredients, structural congruence $\equiv$ and reduction relation $\leadsto$ on graphs. Structural congruence defines what graphs (represented in the syntax of Fig. 2) are essentially the same. This subsection defines structural congruence.

Definition 2.3 (Link Substitution). $G(Y_1, \ldots, Y_n/X_1, \ldots, X_m)$ is a link substitution that replaces all free occurrences of $X_i$ with $Y_i$ as defined in Fig. 4. Here, the $X_1, \ldots, X_m$ should be mutually distinct. Note that, if a free occurrence of $X_i$ occurs at a location where $Y_i$ would not be free, $\alpha$-conversion may be required.

Definition 2.4 (Structural Concurrency). We define the relation $\equiv$ on graphs as the minimal equivalence relation satisfying the rules shown in Fig. 5. Two graphs related by $\equiv$ are essentially the same and are convertible to each other in zero steps. (E1), (E2) and (E3) are the characterization of molecules as multisets. (E4) and (E5) are structural rules that make $\equiv$ a congruence. (E6) and (E7) are concerned with fusions. (E7) says that a closed fusion is equivalent to 0. (E6) is an absorption law of $\leadsto$. (E8) says that an atom can emit a fusion as well. (E8), (E9) and (E10) are concerned with hyperlink creations.

We give two important theorems showing that the symmetry of $\equiv$ and $\alpha$-conversion can be derived from the rules of Fig. 5.

Theorem 2.1 (Symmetry of $\equiv$).

$X \leadsto Y \equiv Y \leadsto X$

Thus, (E6) can be used also when we have a local link on the right-hand side of $\leadsto$.

Theorem 2.2 ($\alpha$-conversion of hyperlinks). Bound link names are $\alpha$-convertible in HyperLMNtal, i.e.,

$$\nu X.G \equiv \nu Y.G$$

where $X \in fn(G)$ and $Y \notin fn(G)$.

It is a subject for future work to elucidate the relationship between the structural congruence rules and graph isomorphism, including the completeness and the soundness of the structural congruence rules. However, these properties are irrelevant to the validity of the semantics of HyperLMNtal and $\lambda_{GT}$, and the verification upon them. The graphs handled in HyperLMNtal and $\lambda_{GT}$ are the graphs of HyperLMNtal defined inductively from the beginning, not the graphs in ordinary algebraic graph transformation formalisms [3]. Therefore, there is no need for the structural congruence rules to correspond to graph isomorphism, and relating them would require another new formulation of HyperLMNtal graphs in the style of standard graph theory, which is beyond the scope of the present work. We have run examples to confirm that the equivalence using structural congruence is practical on HyperLMNtal and $\lambda_{GT}$.

2.3 Reduction Relation

We give the reduction relation of Flat HyperLMNtal that defined the small-step semantics of the language. Note, however, that $\lambda_{GT}$ described in the next section has its own operational semantics without incorporating the reduction relation described here. We nevertheless introduce the reduction relation of Flat HyperLMNtal here because it serves as the basis of the graph types of $\lambda_{GT}$ described in Section 4.

Definition 2.5 (Reduction relation). For a set $P$ of rewrite rules, the reduction relation $\leadsto_P$ on graphs is defined as the minimal relation satisfying the rules in Fig. 6.

3. Syntax and semantics of $\lambda_{GT}$

This section describes the syntax and the semantics of $\lambda_{GT}$.
which is a small, call-by-value functional language that employs hypergraphs as values and supports pattern matching for them. The main design issue is how to represent and manipulate hypergraphs in the setting of a functional language and how to let hypergraphs and abstractions co-exist in a unified framework.

### 3.1 Syntax of $\Lambda_{GT}$

The $\Lambda_{GT}$ language is composed of the following syntactic categories.

- $X$ denotes a Link Name.
- $C$ denotes a Constructor Name.
- $x$ denotes a Graph Context Name.

The syntax of the language is given in Fig. 7. $T$ is a template of a graph. It extends graphs in HyperLMNtal defined in Fig. 2 with graph contexts. A graph context $x(X)$, where $X$ is a sequence of different links, is a wildcard in pattern matching corresponding to a variable in functional languages. It matches any graph with free links $X$. Free links of a graph could be thought of as named parameters (or ‘access points’) of the graph. $C(X)$ is a constructor atom. Intuitively, it is a node of a data structure with links $X$. We allow $\lambda$-abstractions as the names of atoms in graph templates $T$ (and its subclass $G$ to be defined shortly). The $\lambda$-abstraction atoms have the form $(\lambda x[X].e(Y))$. Intuitively, the atom takes a graph with free links $X$, binds it to the graph context $x[X]$ and returns the value (defined in Fig. 8) obtained by evaluating the expression $e$ with the bound graph context. Notice that the $\lambda$-abstraction $(\lambda x[X].e)$ is just the name of an atom: $\lambda$-abstraction atoms can be incorporated into data structures just like atoms with constructor names. This is how $\Lambda_{GT}$ supports first-class functions in a graph setting. The free link(s) $Y$ of the atom can be used to connect the atom to other structures such as lists to form a graph structure containing a first-class function. The links $X$ and the links appearing in the graphs of the body expression $e$ are not the free links of the atom.

**(case** $e_1$ **of** $T$ **→** $e_2$ **| otherwise** **→** $e_3$) **evaluates** $e_1$, **checks** whether this matches the graph template $T$, and reduces to $e_2$ or $e_3$. The details are described in Sections 3.2.1–3.2.3. The case expression covers just two cases in pattern matching, but we can nest the expression to handle more cases. ($e_1$ $e_2$) is an application.

Note that some graph rewriting languages including Interaction Nets [10] and HyperLMNtal have encodings of the $\lambda$-calculus [9], [24] in which both abstractions and applications are encoded using explicit graph nodes (hyper) links. In contrast, $\Lambda_{GT}$ features abstractions and applications at the language level so as to retain the standard framework of functional languages.

$G$ stands for a value of the language $\Lambda_{GT}$, which is $T$ not containing graph contexts. Henceforth, we may call both $G$ and $T$ a graph when the distinction is not important.

**Definition 3.1** (Syntactic condition on expressions). A $\lambda$-abstraction atom is not allowed to appear in the pattern $T$ of the case expression **case** $e_1$ **of** $T$ **→** $e_2$ **| otherwise** **→** $e_3$.

**Definition 3.2** (Abbreviation rules for graph contexts). We introduce the following abbreviation schemes to graph contexts as well as we have done to atoms.

(1) The parentheses of nullary graph contexts can be abbreviated. For example, $x()$ can be abbreviated as $x$.

(2) Term Notation: $\nu X.(v(\ldots,x\ldots),x(\ldots,X))$ can be abbreviated as $v(\ldots,x\ldots)$. The same can be done for embedding atoms (or graph contexts) in the argument of a graph context (or atoms), respectively.

**Definition 3.3** (Free functors of an expression). We define free functors of an expression $e$, $ff(e)$, in Fig. 9. Free functors are not to be confused with free link names.

### 3.2 Operational semantics of $\Lambda_{GT}$

First, we define the congruence rules ($\equiv$) and the link substitutions, $T(Y\times X)$ and $G(Y\times X)$, for $T$ and $G$ in the same manner as we have defined in Section 2. Although there is no graph context in Flat HyperLMNtal, the link substitution for $x[X]$ in $T$ can be defined in the same way as the one for atoms in HyperLMNtal.
\[
\frac{\text{ff}(\text{case } e_1 \text{ of } T \rightarrow e_2 | \text{otherwise } \rightarrow e_3) = \text{ff}(e_1) \cup (\text{ff}(e_2) \setminus \text{ff}(T)) \cup \text{ff}(e_3)}{\text{ff}((e_1, e_2)) = \text{ff}(e_1) \cup \text{ff}(e_2)}
\]

\[\text{ff}(\lambda x[X]) = \{x/\overline{X}\}\]

\[\text{ff}(\nu X) = \emptyset\]

\[\text{ff}((\lambda x[X].e)(\overline{Y})) = \text{ff}(e) \setminus \{x/\overline{X}\}\]

\[\text{ff}((T_1, T_2)) = \text{ff}(T_1) \cup \text{ff}(T_2)\]

\[\text{ff}(\nu X.T) = \text{ff}(T)\]

\[\text{ff}((\lambda x[X].e)(\overline{Z})) = \left\{ \begin{array}{ll}
\text{if } x/\overline{X} = y/\overline{Y} \text{ then } T(\overline{X}/\overline{Y}) & \\
\text{else } x/\overline{X} \neq e/\overline{E} \text{ then } & \\
\text{else } \lambda z[Z].e[z/\overline{Z}]/x[\overline{X}]/(T/y[\overline{Y}])/Z & \\
\end{array} \right. \]

\[\text{ff} fixes an expression}\]

\[\text{Fig. 9}\]

\[\begin{align*}
(T_1, T_2) & = (T_1\theta, T_2\theta) \\
(vX.T) & = vX.T\theta \\
(x[X])[T/y[\overline{Y}]] & = \text{if } x/\overline{X} = y/\overline{Y} \text{ then } T(\overline{X}/\overline{Y}) \\
&\text{else } x/\overline{X} \\
(C(X)) & = C(\overline{X}) \\
(\lambda x[X].e)[T/y[\overline{Y}]] & = \left\{ \begin{array}{ll}
\text{if } x/\overline{X} = y/\overline{Y} \text{ then } \lambda x[X].e(\overline{Z}) & \\
\text{else } x/\overline{X} \neq e/\overline{E} \text{ then } & \\
\text{else } \lambda z[Z].e[z/\overline{Z}]/x[\overline{X}]/(T/y[\overline{Y}])/Z & \\
\end{array} \right. \\
\text{where } z/\overline{X} \neq e/\overline{E}.
\end{align*}\]

\[\text{(case } e_1 \text{ of } T \rightarrow e_2 | \text{otherwise } \rightarrow e_3)\theta = \text{case } e_1\theta \text{ of } T \rightarrow e_2\theta | \text{otherwise } \rightarrow e_3\theta\]

\[\text{Fig. 10}\]

\[\begin{align*}
(T_1, T_2) & = (T_1\theta, T_2\theta) \\
\text{Graph Substitution}\]

We define graph substitution, which replaces a graph context whose functor occurs free by a given subgraph. The substitution avoids clashes with any bound functors by implicit α-conversion (capture-avoiding substitution). Graph substitution is not to be confused with hyperlink substitution. Intuitively, hyperlink substitution just reconnects hyperlinks. On the other hand, graph substitution performs deep copying at the semantics level (though it could or should be implemented with sharing whenever possible).

We define capture-avoiding substitution \(\theta\) of a graph context \(x[\overline{X}]\) with a template \(T\) in \(e\), written \(e[T/x[\overline{X}]]\), as in \text{Fig. 10}. The definition is standard except that it handles the substitution of the free links of graph contexts in the third rule.

\[\text{3.2.2 Matching}\]

We say that \(T\) matches a graph \(G\) if there exists graph substitutions \(\theta\) such that \(G \equiv T\theta\). The graphs in the range of substitutions should not contain free occurence of graph contexts: i.e., the substitution should be ground. Since the matching of \(\lambda GT\) does not involve abstractions (by Def. 3.1), in which case \(G\) of \(\lambda GT\) is essentially the same as \(G\) of HyperLMNtal, we employ the \(\equiv\) defined in \text{Fig. 5}.

\[\text{Note that the matching of } \lambda GT \text{ is not subgraph matching (as is standard in graph rewriting systems) but the matching with the entire graph } G \text{ (as is standard in pattern matching of functional languages). For this reason, the free link names appearing in a template } T \text{ must exactly match the free links in the graph } G \text{ to be matched. This is to be contrasted with free links of HyperLMNtal rules that are effectively } \alpha \text{-convertible since the rules can match subgraphs by supplementing fusion atoms ([18], Section 4.4).}\

\[\text{The matching can be done non-deterministic. We are planning to put constraints over the graph templates in case expressions to ensure deterministic matching but it is a future task.}\

\[\text{3.2.3 Reduction}\]

We choose the call-by-value evaluation strategy. The reason we did not choose call-by-need (or call-by-name) is to avoid infinite graphs to use infinite-descent in the verification later in Section 6.

In order to define the small-step reduction relation, we extend the syntax with evaluation contexts defined as follows:

\[E ::= [] | \text{case } E \text{ of } T \rightarrow e | \text{otherwise } \rightarrow e | (E \ e) | (G \ E) \ |

As usual, \(E[e]\) stands for \(E\) whose hole is filled with \(e\).

We define the reduction relation in \text{Fig. 11}.

\[\text{Definition 3.4 (Abbreviation rules for } \lambda \text{-abstraction atom). We introduce a shorthand notation similar to the } \lambda \text{-calculus.}\

\[\begin{align*}
(1) & \text{ Application is left-associative.} \\
(2) & (\lambda x[\overline{X}].(\lambda y[\overline{Y}].e)(\overline{Z})) \text{ can be abbreviated as } (\lambda x[\overline{X}].y[\overline{Y}].e)(\overline{Z}). \\
(3) & (((\lambda x[\overline{X}].e_1)(\overline{Y}).e_2) \text{ can be abbreviated as } \text{let } x[\overline{X}] = e_1 \text{ in } e_2. \\
& \text{The } \overline{Y} \text{ will disappear immediately after evaluating the expression, doing nothing, in } \beta \text{-reduction. Thus, we omit the links in the abbreviation.}\

\text{For example, we can describe a program to append two singleton difference lists as follows (detailed description of difference lists will be given in Section 4.2):}\

\[\text{let append}[Z] = (\lambda x[Y,X].y[Y,X].x[y[Y],X]) (Z)\]

\[\text{in append}[Z] \text{ Cons}(1, Y, X) \text{ Cons}(2, Y, X)\]

We show the whole process of reduction of this program in
4. Type System

In this section, we propose a type system, \( F_{GT} \), for the \( \lambda_{GT} \) language. We define the type of graphs using graph grammar. This can be regarded as an extension of regular tree grammar, on which algebraic data types are based.

4.1 Syntax and rules for \( F_{GT} \)

Let \( \alpha \) be a syntactic category denoting the identifier of a type name. The syntax of types is given in Fig. 14. It can be observed that the definition of a type employs both inductive definition (standard in programming languages) and production rules (standard in formal grammar). The reason for doing so is that, unlike ADTs, types of graphs cannot be defined inductively in general. Thus we employed generative grammar as a well-established formalism for defining graphs. Integrating it into \( F_{GT} \) is the research question of the present work.

We extend the \( \lambda \)-expression \( \lambda x[\overline{x}] \cdot e \) with type annotation \( \tau(x[\overline{x}]) \) as \( \lambda x[\overline{x}] : \tau(x[\overline{x}]) \cdot e \).

**Definition 4.1** (Abbreviation rule for an arrow atom). We introduce a shorthand notation similar to an arrow in the typed \( \lambda \)-calculus, that is,

\[
(\tau_1(x[\overline{x}]) \rightarrow \tau_2(\overline{Y}) \rightarrow \tau_3(\overline{Z})(\overline{W}))(\overline{W})
\]

can be abbreviated as

\[
(\tau_1(x[\overline{x}]) \rightarrow \tau_2(\overline{Y}) \rightarrow \tau_3(\overline{Z}))(\overline{W}).
\]

**Definition 4.2** (Syntactic constraints). A production rule \( \alpha(x[\overline{x}]) \rightarrow \tau \) should satisfy \( fn(\tau) = \{x[\overline{x}]\} \).

Let \( \Gamma \) be a typing context which is a set of the form \( x[\overline{x}] : \tau(x[\overline{x}]) \), where the \( x \)'s are mutually distinct and \( t \) should be a type variable or an arrow. The typing relation \( (\Gamma, \mathcal{P}) \vdash e : \tau(x[\overline{x}]) \) denotes that \( e \) has the type \( \tau(x[\overline{x}]) \) under the type environment \( \Gamma \) and a set \( \mathcal{P} \) of production rules, whose typing rules are defined as follows.

**Definition 4.3** (Rules for \( F_{GT} \)). Typing rules for \( F_{GT} \) is given in Fig. 15.

Ty-App, Ty-Arrow, and Ty-Var are essentially the same as...
Ty-App

\[(\Gamma, P) + e_1 : \tau_1(\overrightarrow{X}) \rightarrow \tau_2(\overrightarrow{Y}))\]

\[(\Gamma, P) + e_2 : \tau_1(\overrightarrow{X})\]

\[(\Gamma, P) + e_1 + e_2 : \tau_1(\overrightarrow{X})\]

Ty-Arrow

\[((\Gamma, x[\overrightarrow{X}] : \tau_1(\overrightarrow{X})), P) + e : \tau_2(\overrightarrow{Y})\]

\[(\Gamma, P) + (\lambda x[\overrightarrow{X}] : \tau_1(\overrightarrow{X}), e)(\overrightarrow{Z}) : (\tau_1(\overrightarrow{X}) \rightarrow \tau_2(\overrightarrow{Y}))(\overrightarrow{Z})\]

Ty-Var

\[\Gamma[\overrightarrow{X}] : \tau(\overrightarrow{X}), P + x[\overrightarrow{X}] : \tau(\overrightarrow{X})\]

Ty-Cong

\[\Gamma, T : \tau(\overrightarrow{X}) \quad T \equiv T'\]

\[\Gamma, P + T : \tau(\overrightarrow{X})\]

where \(Z \notin \text{fn}(T)\)

Ty-Alpha

\[\Gamma, P + T : \tau(\overrightarrow{X})\]

\[\Gamma, P + T(Z/Y) : \tau(\overrightarrow{X})(Z/Y)\]

Ty-Prod

\[\Gamma, P + T_1 : \tau_1(\overrightarrow{X}) \ldots \ldots \Gamma, P + T_n : \tau_n(\overrightarrow{X}_n)\]

\[\Gamma, P(\alpha(\overrightarrow{X}) \rightarrow \overrightarrow{T}) + T_1/T_1(\overrightarrow{X_1}), \ldots, T_n/T_n(\overrightarrow{X_n}) : \alpha(\overrightarrow{X})\]

where \(\tau_i(\overrightarrow{X})\) are all the type atoms appearing in \(T\)

Ty-Case

\[\Gamma, P + e_1 : \tau_1(\overrightarrow{X}) \quad ((\Gamma, \Gamma') + P) + e_2 : \tau_2(\overrightarrow{Y}) \quad (\Gamma, P) + e_3 : \tau_3(\overrightarrow{Y})\]

\[\Gamma, P + (\text{case } e_1 \text{ of } T \rightarrow e_2 \mid \text{otherwise } e_3) : \tau_3(\overrightarrow{Y})\]

Fig. 15 Typing rules for \(F_{GT}\)

Algebraic data types (ADTs) can be easily expressed in the same way as in this example: our language and the type system is a natural extension of functional languages and their type systems.

Example 4.2 (Type of a difference list). The \(\lambda_{GT}\) language can handle some data structures that algebraic data types cannot handle. A difference list can be understood as a list with an additional link to the last element. This is a popular data structure since the early days of logic programming in which the links are represented as logical variables. It allows us to append two lists in constant time. In functional programming, a difference list can be implemented using a higher-order function that receives a subsequent list and returns the entire list, but we wish to represent such data structures in the first-order setting.

The production rules for a difference list can be defined as follows.

\[\text{nodes}(Y, X) \rightarrow X \bowtie Y\]

\[\text{nodes}(Y, X) \rightarrow \text{Cons}(\text{nat}, \text{nodes}(Y), X)\]

Example 4.3 (Typing a difference list with functions). Since \(\lambda_{GT}\) and its type system \(F_{GT}\) treat functions as first-class citizens, it is even possible to have a difference list with functions as its elements. Figure 16 shows that the graph \(G = \text{Cons}(\text{succ}, Y, X)\) has type \(\text{nodes}(Y, X)\) under type environment \(\Gamma = \text{succ}(Z_1) : (\text{nat}(X) \rightarrow \text{nat}(X))(Z_1)\) and production rules \(P = \{P_1, P_2\}\) where

\[\text{nodes}(Y, X) \rightarrow X \bowtie Y\]

\[\text{nodes}(Y, X) \rightarrow \text{Cons}(\text{nat}(X) \rightarrow \text{nat}(X), \text{nodes}(Y), X) \ldots \ldots P_1\]

Example 4.4 (Type of a doubly-linked difference list). A doubly-linked difference list is a list with four free links, two different links for each end. Although the (hyper)links of \(\lambda_{GT}\) and HyperLMNtal are undirected, we are interested in using them to model directed hyperlinks (roughly corresponding to pointers in imperative languages) that are to be ‘followed’ in one direction. As with difference lists, the addition of elements to the tail of the list can be done in constant time, as desired in representing dequeues. Of course, doubly-linked lists that are not difference lists can also be handled in an obvious way.

\[\text{nodes}(F', B, B', F) \rightarrow F \bowtie B, B' \bowtie F'\]

\[\text{nodes}(F', B, B', F) \rightarrow vX.\text{Cons}(\text{nat}, F', \text{nodes}(X, B, B'), X, F)\]

Example 4.5 (Type of difference skip lists). By extending the type definition of difference lists, the type of unbounded-level skip lists can be defined. This implies that we can also define a type for skip lists with a nil node at the end and/or whose level is fixed.

\[\text{nodes}(Y, X) \rightarrow X \bowtie Y\]

\[\text{nodes}(Y, X) \rightarrow \text{Cons}(\text{nat}, \text{forks}(Y), X)\]

\[\text{forks}(Y, X) \rightarrow \text{Next}(\text{nodes}(Y), X)\]

\[\text{forks}(Y, X) \rightarrow vZ.\text{Fork}(Z, \text{forks}(Z), X, \text{nodes}(Y, Z))\]

We also show the visualized version of production rules in Fig. 17 and an example difference skip list in Fig. 18.

Example 4.6 (Type of a leaf-linked tree). A leaf-linked tree is a graph with three free links (say X, L, R) which is a tree whose root is represented by X and whose leaves form a difference list represented by L and R.
ty-Alpha(, ty-Prod(, (a) X X X X 2 1 1 1 nodes Y.

Fig. 16 Type checking a difference list

\[
\begin{align*}
X &\xrightarrow{\text{nodes}} Y \\
\text{forks} &\xrightarrow{} X \\
\text{forks} &\xrightarrow{} Y
\end{align*}
\]

Fig. 17 The production rules for the type of difference skip list

\[
\begin{align*}
\text{lltree} (L, R, X) &\rightarrow L \equiv X , \text{Leaf}(\text{nat}, R, X) \\
\text{lltree} (L, R, X) &\rightarrow \nu Y \cdot \text{Node}(\text{lltree} (L, Y), \text{lltree} (Y, R), X)
\end{align*}
\]

Example 4.7 (Type of a threaded tree). A threaded tree is somewhat similar to a leaf-linked tree but each non-terminal node has access to the rightmost leaf the left subtree and the leftmost leaf of the right subtree.

\[
\begin{align*}
\text{three} (L, R, X) &\rightarrow L \equiv X , \text{Leaf}(\text{nat}, R, X) \\
\text{three} (L, R, X) &\rightarrow \text{Node}(\text{three} (L, X), \text{three} (X, R), X)
\end{align*}
\]

4.3 Properties of \( F_{\text{GT}} \)

This section discusses some properties of \( \lambda_{\text{GT}} \) and \( F_{\text{GT}} \). As mentioned in Section 3, we keep the language small to focus on the handling of graph structures, more specifically the handling of graphs by pattern matching with graph contexts. In particular, it has no explicit mechanism (such as \texttt{let rec} or \texttt{fix}) to deal with recursive functions. This is because those features can be achieved essentially in the same way as other functional languages do.

4.3.1 Soundness of \( F_{\text{GT}} \)

Lemma 4.1 (Progress). If \((\emptyset, P) \vdash e : \tau (X)\), then \(e\) is a value or \(\exists e'. e \rightarrow_{\text{val}} e'\).

Proof. By induction on the derivation of \((\emptyset, P) \vdash e : \tau (X)\). Notice that the only new extension from other functional languages in expressions (Fig. 7) is Case, and the Case expression is never stuck because if matching fails, it just branches to \texttt{otherwise} and evaluation proceeds.

Lemma 4.2 (Substitution). If

\[
((\Gamma, P) \vdash e_1 : \tau_1 (Y_1))
\]

and

\[
((\Gamma, x [X_1] : \tau_1 (Y_1), P) \vdash e_2 : \tau_2 (Y_2))
\]

then

\[
((\Gamma, P) \vdash e_1 [e_1/x [X_1]] : \tau_2 (Y_2)).
\]

Proof. By induction on the derivation of \(((\Gamma, x [X_1] : \tau_1 (Y_1), P) \vdash e_2 : \tau_2 (Y_2))\).

Lemma 4.3 (Preservation). If \((\Gamma, P) \vdash e : \tau (X)\) and \(e \rightarrow_{\text{val}} e'\), then \((\Gamma, P) \vdash e' : \tau (X)\).

Proof. Proved using the Lemma 4.2.

Theorem 4.1 (Soundness). If \((\emptyset, P) \vdash e : \tau (X)\) and \(e \rightarrow_{\text{val}} e'\) then \(e'\) is a value or \(\exists e'' : e' \rightarrow_{\text{val}} e''\).

Proof. Follows from Lemma 4.1 and Lemma 4.3.

4.3.2 Relation with graph reduction

Structured Gamma is a first-order graph rewriting system developed to represent and reason about the shapes of pointer data structures. The framework of Structured Gamma was then adapted to LMNtal (whose graph structures are dual to those of Structured Gamma, roughly speaking) to design and implement LMNtal ShapeType. Despite several syntactic variations (such as the duality of nodes/links and the presence/absence of hyperlinks), Structured Gamma and LMNtal ShapeType can essentially handle graphs of \( \lambda_{\text{GT}} \) without \( \lambda \)-abstraction atoms. The typing relation à la Structured Gamma and LMNtal ShapeType is defined as follows.

Definition 4.4 (Typing relation in Structured Gamma/LMNtal ShapeType). \( P \vdash \tau : \alpha (X) \) if \( \alpha (X) \rightarrow_{\text{p}} \tau \) and \( \tau \) does not contain type variables or arrow atoms.

We have shown that the typing relation in our type system \( F_{\text{GT}} \) subsumes the one in Structured Gamma in the following sense.

Theorem 4.2 \((F_{\text{GT}} \text{ and HyperLMNtal reduction}).

\[
(\Gamma, P) \vdash T : \tau (X) \\
\Leftrightarrow \tau (X) \leadsto_{\text{p}} T [\tau (Y_1)/\tau (X)].
\]

where

- \( \Gamma = x [X_1] : \tau_1 (Y_1) \).
- \( (\lambda \ldots)(\overline{W}_1) \) are all the \( \lambda \)-abstraction atoms in \( T \), and
(Γ, P) ⊢ (λ \ldots)\lambda(W) : \tau(Z)

\textbf{Proof.} For ⊢, we can prove by induction on the last applied \textit{F}_{CT} rules. For \equiv, We prove by induction on the length of the reduction \equiv\tau. □

Note that if no graph contexts or \lambda-expressions appear in \textit{T}, by Theorem 4.2, the typing relation in \textit{F}_{CT} is equivalent to the one in Structured Gamma. In other words, our type system is an extension of Structured Gamma to allow graph contexts and \lambda-abstraction atoms. This allows us to take advantage of research results on Structured Gamma, its derivative LMNtal ShapeType, and parsing of graphs using graph grammar.

\textbf{Example 4.8 (Theorem 4.2 on the difference list example)}

Here, we see that Theorem 4.2 holds on Example 4.3. Recall that (\text{succ}[Z_1] : (\text{nat}(X) \rightarrow \text{nat}(X))(Z_1), P) + \text{Cons(succ, Y, X)} : \text{nodes}(Y, X) holds in \textit{F}_{CT}, which can also be shown using HyperLMNtal reduction as follows.

\begin{align*}
\text{nodes}(Y, X) & \equiv \nu Z_1 Z_2. (\text{Cons}(Z_1, Z_2, X), (\text{nat}(X) \rightarrow \text{nat}(X))(Z_1), \text{nodes}(Y, Z_2)) \\
& = \nu Z_1 Z_2. (\text{Cons}(Z_1, Z_2, X), (\text{nat}(X) \rightarrow \text{nat}(X))(Z_1), Z_2 \Rightarrow Y) \\
& = \text{Cons}(\text{succ, Y, X}[(\text{nat}(X) \rightarrow \text{nat}(X))(Z_1)/\text{succ}[Z_1]])
\end{align*}

\subsection{4.4 Type checking case expressions}

\textit{A}_{CT} allows pattern matching of graphs. In pattern matching, graph contexts can be used as wildcards. Since a graph context can match any graph as long as the sets of free links are the same, we cannot naively ensure that the type of the graph matches the intended type of the context. Therefore, we allow the typing annotation of graph contexts.

To allow type annotation in pattern matching, we extend the syntax of the graph template \textit{T}. A type annotation \textit{T} : \tau(\overline{X}) ensures that the type of the graph matched with \textit{T} is of type \tau(\overline{X}).

To evaluate pattern matching with annotations, we extend the matching mechanism. Match(Γ, T, \overline{θ}) denotes that (i) the graph context \textit{T} can match the graph \textit{G} with graph substitutions \overline{θ} and that (ii) each subgraph of \textit{G} matched by a subcontext of \textit{T} satisfies the type constraint attached to the subcontext. Match(Γ, T, \overline{θ}) is defined inductively as in Fig. 19. It is a straightforward inductive argument to see that Fig. 19 extends the matching defined in Section 3.2.2 with the rule Mt-Ty for type checking.

The type annotations that do not match the type definitions of production rules could be reported as bugs, which could be analyzed easily and statically. Also, for simplicity, henceforth we will assume that all the graph contexts are type-annotated and make it a future task to support unannotated graph contexts.

The matching can be non-deterministic; that is, given a graph and a pattern, there may in general be more than one way in which graph contexts in the pattern are bound to subgraphs. However, the non-determinacy of the matching does not affect the soundness of the type system since the system proves that every execution path is type-safe.

The program that pops the last element of a difference list we have introduced in Section 3.2.3 can be handled with type-annotations as follows.

\begin{align*}
\text{fn}(G) & = \overline{X} \\
\text{Mt-Var} \\
\text{Match}(\text{G, } x[\overline{X}], [\text{G}/x[\overline{X}])] \\
\text{Mt-Triv} \\
\text{Match}(\text{G, G}, []) \\
\text{Mt-Mol} \\
\text{Match}(G_1, T_1, \overline{θ}_1) & \quad \text{Match}(G_2, T_2, \overline{θ}_2) \\
\text{Mt-Cong} \\
\text{Match}(G_1, T_1, \overline{θ}_1) & \quad G_1 \equiv G_2 \\
\text{Mt-Ty} \\
\text{Match}(G_2, T, \overline{θ}) & \quad \text{G} : \tau(\overline{X})
\end{align*}

Fig. 19 Matching with a template and graph substitutions

\begin{align*}
(\Gamma, P) & \vdash \\
(\lambda x[Y, X]) : \text{nodes}(Y, X), \\
\text{case } x[Y, X] \text{ of} \\
\nu Z_1 Z_2. (g[Z_1, X] : \text{nodes}(Z_1, X), & \quad \text{Cons}(Z_2, Y, Z_1), \quad \text{(nodes}(Y, X) \rightarrow \text{nodes}(Y, X)) \\
\nu Z_1 Z_2. \tau(Z_2) & \quad : \tau(Z_2)) \\
\nu Z_1 Z_2. \tau(Z) & \quad \rightarrow y[Y, X] \\
\text{otherwise} & \rightarrow x[Y, X]) \rightarrow (Z)
\end{align*}

This can be typed using Ty-Case where the \textit{Γ}’ stands for the annotated typing relations \text{g}[Z_1, X] : \text{nodes}(Z_1, X), \tau[Z_2] : \text{nat}(Z_2).

\section{5. Extending the type system}

In this section, we deal with an example in which the type system in Section 4 fails to verify. The type system in Section 4 was actually for parsing when dealing with graphs; it just checks if the graph can be generated from the annotated type variable atom, i.e., the start symbol. Algebraic data types can be handled in this manner because they can only be generated according to the grammar that defines the type. However, in the case of graphs, more powerful operations are possible, for example the concatenation of difference lists. In this section, we propose an extended verification framework to deal with such cases.

\subsection{5.1 Motivation}

As a running example, we consider a typed version of the following program for appending two difference lists introduced in Section 3.2.3.

\begin{align*}
(\lambda x[Y, X]) : \text{nodes}(Y, X) \\
g[Y, X] : \text{nodes}(Y, X), \\
x[y[Y, X]] \\
)Z
\end{align*}
It seems natural that the following typing relation holds, where \( \text{append}[Z] \) is the \( \lambda \)-abstraction atom above.

\[
(F, P) \vdash \text{append}[Z] : (\text{nodes}(Y, X) \to \text{nodes}(Y, X) \to \text{nodes}(Y, X))(Z)
\]

However, this program cannot be verified by directly using the rules in the type system in Section 4.

**Theorem 5.1.** The append operation on difference lists fails to verify on the previously defined \( F_{\Gamma T} \).

**Proof.** We need to prove

\[
((x, y), y) \vdash \text{append}(x, y) : \text{nodes}(Y, X)
\]

to verify the present example. Theorem 4.2 states that, if we can successfully prove the typing relation using \( F_{\Gamma T} \), we should be able to prove \( \text{nodes}(Y, X) \to \text{nodes}(Y, X) \). However, applying the production rules of difference lists cannot increase the number of \( \text{nodes} \). Therefore, applying the production rules to the annotated type variable atom \( \text{nodes}(Y, X) \) will never yield \( \text{nodes}(\text{nodes}(Y, X)) \).

However, it is obvious that appending two difference lists returns a difference list, and this operation should be supported. We extend the previously defined \( F_{\Gamma T} \) to enable such verification.

### 5.2 Extension on \( F_{\Gamma T} \)

#### Definition 5.1 (Extension on \( F_{\Gamma T} \) (Unrefined)).

For a graph template \( T \), it is sufficient if the typing succeeds after replacing each graph context in \( T \) by all possible values of the types attached to the graph context, or more formally, as in Fig. 20.

The (apparently intuitive) rule in Definition 5.1 has \( \Rightarrow \) on the antecedent and the typing relation we are going to define (the parameter of the generating function) appears on the left-hand side of the \( \Rightarrow \). Unfortunately, then, we cannot ensure the monotonicity of the generating function and the existence of a least fixed point, which is the typing relation we want to define.

Now we consider how to fix this, which we have found is not trivial or straightforward. If we define a typing relation, say \( R_0 \), without Ty-Subst and define the left-hand side of the \( \Rightarrow \) of Ty-Subst with \( R_0 \), we can ensure the monotonicity of the generating function and the typing relation becomes well-defined.

First, we prepare two sets of typing rules, one with all the :’s in Fig. 15 rewritten as \( \omega \) and the other as \( \iota \). Then, we can define \( R_0 \) only with typing rules with \( \omega \), which is well-defined.

Next, we define the typing relation, say \( R_1 \), using typing rule with \( \iota \) and the rule in Fig. 21 (Ty-Subst with \( \omega \) and \( \iota \)). Since the left-hand side of \( \Rightarrow \) in Fig. 21 uses the already defined \( R_0 \), it can be interpreted as a monotonic function and the typing relation \( R_1 \) is well-defined.

However, we cannot ensure the soundness if we define Ty-Subst in such a way because the antecedent of the rule ensures the safety if \( G_i \) has a type \( \tau_i(\overrightarrow{x}) \) in \( R_0 \), but the antecedent does not ensure the safety when \( G_i \) has a type \( \tau_i(\overrightarrow{x}) \) only in \( R_1 \). Since \( R_1 \) can handle more programs than \( R_0 \), this may violate the soundness of the system.

For example, consider the case where \( G_i \) is a graph containing a function that concatenates difference lists and \( \tau_i(\overrightarrow{x}) \) contains an arrow type for a function that takes two difference lists (as curried arguments) and return a difference list. Since it is not verifiable in \( R_0 \) that a function that concatenates difference lists returns a difference list, we can make the left-hand side of \( \Rightarrow \) on Ty-Subst false. In such a case, the antecedent of Ty-Subst is satisfied no matter what the right-hand side of \( \Rightarrow \) is. Thus, we cannot ensure safety for the case where we bound a graph containing a function that concatenates difference lists. However, since the consequent of Ty-Subst uses \( \iota \), it allows the \( x_i(\overrightarrow{X}) \) to be bound to a graph that includes a function that concatenates difference lists.

Therefore, we need a more refined framework that allows the indices we have attached to : previously to be different for each type. Accordingly, we introduce the notion of \( \text{ranks} \) for types and Ty-Subst.

If the typing on the left-hand side of \( \Rightarrow \) does not use Ty-Subst, which we will define, it can be interpreted as a monotonic function and is well-defined. Therefore, we introduce ranks into Ty-Subst so that the left-hand side typing of \( \Rightarrow \) can only use Ty-Subst with a lower rank that has already been defined.

We first introduce ranks to the type. We denote the type with Rank \( n (n \geq 0) \) as \( \tau^r(\overrightarrow{X}) \). We extend the rules in Fig. 15 so that the types have ranks.

**Definition 5.2 (Rules for \( F_{\Gamma T} \) with ranks).** Typing rules for \( F_{\Gamma T} \) with ranks are given in Fig. 22.

Notice that we have added a new typing rule Ty-Sub, a rule for subtyping.

**Definition 5.3 (Extension on \( F_{\Gamma T} \) (Refined)).** The refined version of Definition 5.1 is shown in Fig. 23.

**Proposition 5.1.** The typing rules are well-defined even if we add Definition 5.3.

**Proof.** In Definition 5.3, the ranks of the type on the left-hand side of \( \Rightarrow \), \( n_r \), are always smaller than the rank of the type \( \tau^r(\overrightarrow{X}) \) on the consequent of the rule, \( n \). The typing rules in Fig. 22 are defined so that the ranks do not increase when we read the rules upwards. Thus, the typing relation used for the left-hand side of \( \Rightarrow \) in the antecedent of Ty-Subst (of rank \( n \)) can be established using Ty-Subst with smaller ranks \( m (m < n) \) only (which may actually be used when, for example, the \( G_i \)’s contain abstraction atoms). Suppose all typing relations involving smaller ranks are well-defined. Then, since the typing relation we are about to define does not appear in the left-hand side of \( \Rightarrow \), we can ensure the well-definedness of the typing relation involving ranks up to \( n \). Because the typing relation containing types with rank 0 only does not involve Ty-Subst and is therefore well-defined, by mathematical induction on rank, we can define a typing relation for all ranks.

The existence of the typing rule defined in Definition 5.3 does not violate the soundness since using the rule ensures that a program can be typed without such a rule for all the possible graphs bound to graph contexts.

Let us consider the typing of a function that concatenates difference lists. From now on, we omit the “\( (\emptyset, P) \uparrow \)” for brevity. Suppose we have already proven the following (we will prove...
this in Section 5.3).

\[
\forall G_1, G_2, \forall (\mathcal{G} \colon \text{nodes}^a(Y, X) \land G_2 : \text{nodes}^m(Y, X) \Rightarrow \nu Z(x[Z, X], y[Z], [G_1/x[Y, X]][G_2/x[Y, X]])[G_2/x[Y, X]](Y, X) 
\]

Then, we can type the function using Ty-Sub, Ty-Subst (rank \( \max(n, m) + 1 \)), and Ty-Arrow as shown in Fig. 21.

Since the type system is monomorphic, we cannot type the following program.

\[
\begin{align*}
\forall G_1, G_2, \forall (\mathcal{G} \colon \text{nodes}^a(Y, X) \land G_2 : \text{nodes}^m(Y, X) \\
\Rightarrow \nu Z(x[Z, X], y[Z], [G_1/x[Y, X]][G_2/x[Y, X]])[G_2/x[Y, X]](Y, X) 
\end{align*}
\]

(1)

We need to satisfy \( n = \max(n, m) + 1 \), which is unsatisfiable.

Such programs can be typed introducing polymorphism for ranks. However, this paper does not go into this and leaves it as future work.

5.3 Proving the antecedent of the rule

In order to apply Definition 5.3 to the present example, we need to prove that, for any graphs to which \( X \), and \( Y \), can be mapped, the substituted result must have the type \( \text{nodes}^a(Y, X) \), that is,

\[
\forall G_1, G_2, \forall (\mathcal{G} \colon \text{nodes}^a(Y, X) \land G_2 : \text{nodes}^m(Y, X) \\
\Rightarrow \nu Z(x[Z, X], y[Z], [G_1/x[Y, X]][G_2/x[Y, X]])[G_2/x[Y, X]](Y, X) 
\]

The above can be rewritten using Ty-Alpha as follows.

\[
\forall G_1, G_2, \forall (\mathcal{G} \colon \text{nodes}^a(Y, X) \land G_2 : \text{nodes}^m(Y, Z) \\
\Rightarrow \nu Z(x[Z, Y], y[Z], [G_1/x[Y, Z]][G_2/x[Y, Z]])[G_2/x[Y, Z]](Y, X) 
\]

We prove this by induction on the derivation of the antecedents. To do this, we need a lemma and a theorem.

**Lemma 5.1.** For \( G : a^s(\mathcal{X}) \), the rule Ty-Prod with \( a/\mathcal{X} \) on its LHS is used in the derivation. Furthermore, only Ty-Cong and Ty-Alpha are used after the last application of Ty-Prod.

**Proof.** Suppose we build a proof tree of \( G : a^s(\mathcal{X}) \) bottom-up. Since \( G \) is a value, we can only use Ty-Cong, Ty-Alpha, Ty-Subst, Ty-Subst, and Ty-Arrow until a \( \lambda \)-abstraction atom appears. Ty-Alpha, Ty-Subst, Ty-Subst, and Ty-Arrow only inherit the annotated type from the antecedent (although they may change the rank) so they alone cannot make the annotated type a type variable atom. If Ty-Prod does not appear but a \( \lambda \)-abstraction atom appears and Ty-Arrow is used, the annotated type becomes an arrow and not a type variable. Therefore, there must exist a Ty-Prod whose annotated type has the function \( a/\mathcal{X} \). \( \square \)

**Theorem 5.2.** For \( G : a^s(\mathcal{X}) \), if the production rule used by last Ty-Prod was \( a(\mathcal{X}) \rightarrow T \), there exists \( G_j \) such that \( G \equiv T\Gamma(G_j/\tau_j(\mathcal{X})) \).

- \( T \) or \( T(\mathcal{X}) \).
- \( \tau_j(\mathcal{X}) \) are all the type atoms appearing in \( T \).
- \( G_j \) is \( a^m(\mathcal{X}) \) and
- \( \max n_j = n \).

**Proof.** By induction on the derivation of \( G : a^s(\mathcal{X}) \) after the last application of Ty-Prod using Lemma 5.1. \( \square \)
For brevity, we denote the graph $G$ of the type $\alpha^n(X)$ as $G^n(X)$ and omit $VG$. Then eq. (2) can be rewritten as

$$vZ.(\text{nodes}^n(Z,X), \text{nodes}^m(Y,Z)) : \text{nodes}^{\max(n,m)}(Y,X).$$

The inference rule (or rule scheme, precisely speaking) that splits the cases by the last application of Ty-Prod to derive $\beta^n_{/j}$ in $G$ is expressed in the following form. Here, $G_i$ is the graph such that the last production rule used in the derivation of $\beta^n$ is $P_i$.

$$G_1 : \alpha^n(X) \quad \ldots \quad G_m : \alpha^n(X) \quad \text{Case } \beta^n_{/j} \quad G : \alpha^n(X)$$

The concatenation of difference lists can be verified as shown in Fig. 25, where the arrow $\iff$ refers to using the induction hypothesis.

6. Automatic verification on the extended type system

In Section 5, we typed the program by manually applying structural induction to the target program. In this section, we describe a method to do this automatically. From now on, we handle the cases where ranks are all zero and omit them. Extending the algorithm to handle general rank is future work.

We construct a proof tree like what we have shown in Fig. 25 bottom-up. Given $G : \alpha^n(X)$, we can use the following strategies to verify those programs.

**Ty-Prod**: If we get a constructor atom $C/n$ from $G$, we can check whether an annotated type name atom $\alpha(X)$ can be derived from the target graph using Prod with a production rule with $\alpha(X)$ on the LHS and $C/n$ on the RHS. However, in order to use a production rule, the subgraphs in $G$ must have the types necessary for the derivation. For this reason, the type checker is performed inductively on the subgraphs.

**Case $\beta^n_{/j}$**: If we get a type annotated graph $\beta^n$ from $G$, we decompose it using Theorem 5.2. Then check if $G$ with its subgraph $\beta^n$ thus decomposed has type $\alpha^n(X)$.

$\iff$: Induction hypotheses are used when applicable.

However, it is not that easy to do this automatically. Especially for more complex examples.

(1) We cannot easily separate a graph into subgraphs when using a production rule. It is difficult to automatically separate and guess the type of a subgraph, prove it as a subproblem, and proceed with the proof using it without any prior preparation.

(2) The possibility that links may be fused later makes it difficult to get the correspondence of link names in the target graph and the applying production rule.

Remember the production rules for leaf-linked trees in Example 4.6. Here, we want to type check the following graph.

$$vY.i\text{Node}(L, l\text{tree}(Y,R), X), \text{Leaf}((\text{nat}, L, Y)) : l\text{tree}(L, R, X)$$

In this example, we try to apply the second rule...
A strategy is also needed for the decomposition of annotated nodes into subgraphs by introducing the notion of a root link. Therefore, we restrict the production rules to facilitate disassembly into subgraphs by traversing the root links. Since a spanning tree can be found for any connected graph, we can arrange the ordering of links of individual atoms in such a way that the root link form the edges of a spanning tree. Thus the restriction on production rules will not essentially sacrifice the expressive power of the data structure for practical programs. We call a link the last link of each atom as its root link.

**Definition 6.1** (Root link). We call the last link of each atom as its root link.

In order to handle graphs inductively with production rules easier, we introduce the notion of root links.

A production rule should have the form $\alpha(\overline{X}, R) \rightarrow \tau$, where the $\tau$ should be one of the following.

1. one or more fusions.
2. has one constructor atom $C(\overline{Y}, R)$, zero or more type variable atoms $\alpha_i(\overline{Y}_i)$, zero or more fusions, and zero or more arrow atoms and satisfies all the following conditions.
   a. The root link $R$ of $C(\overline{Y}, R)$ occurs free in $\tau$. on production rules, even disconnected graphs (multisets) could be handled. However, here, we design the type system to efficiently support data structures of practical importance.

### 6.1 Constraints on production rules

The type system $F_{GT}$ defined so far has imposed no restriction on production rules, even disconnected graphs (multisets) could be handled. However, here, we design the type system to efficiently support data structures of practical importance.

In order to handle graphs inductively with production rules easier, we introduce the notion of root links.

**Definition 6.1** (Root link). We call the last link of each atom as its root link.

We give a restriction on production rules so that we can find a spanning tree of a graph by traversing the root links. Since a spanning tree can be found for any connected graph, we can arrange the ordering of links of individual atoms in such a way that the root links form the edges of a spanning tree. Thus the restriction on production rules will not essentially sacrifice the expressive power of the data structure for practical programs. We call a link $X$ the root link of a graph $G$ if every atom in the graph can be reached through their root links from $X$.

**Definition 6.2** (Constraints on production). A production rule should have the form $\alpha(\overline{X}, R) \rightarrow \tau$, where the $\tau$ should be one of the following.

1. one or more fusions.
2. has one constructor atom $C(\overline{Y}, R)$, zero or more type variable atoms $\alpha_i(\overline{Y}_i)$, zero or more fusions, and zero or more arrow atoms and satisfies all the following conditions.

   a. The root link $R$ of $C(\overline{Y}, R)$ occurs free in $\tau$. on production rules, even disconnected graphs (multisets) could be handled. However, here, we design the type system to efficiently support data structures of practical importance.

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We give a restriction on production rules so that we can find a spanning tree of a graph by traversing the root links. Since a spanning tree can be found for any connected graph, we can arrange the ordering of links of individual atoms in such a way that the root links form the edges of a spanning tree. Thus the restriction on production rules will not essentially sacrifice the expressive power of the data structure for practical programs. We call a link $X$ the root link of a graph $G$ if every atom in the graph can be reached through their root links from $X$.

**Definition 6.2** (Constraints on production). A production rule should have the form $\alpha(\overline{X}, R) \rightarrow \tau$, where the $\tau$ should be one of the following.

1. one or more fusions.
2. has one constructor atom $C(\overline{Y}, R)$, zero or more type variable atoms $\alpha_i(\overline{Y}_i)$, zero or more fusions, and zero or more arrow atoms and satisfies all the following conditions.

   a. The root link $R$ of $C(\overline{Y}, R)$ occurs free in $\tau$. on production rules, even disconnected graphs (multisets) could be handled. However, here, we design the type system to efficiently support data structures of practical importance.

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The root link $R_i$ of a type variable atom $\alpha_i(\ldots, R_i)$ should satisfy $R_i \in \{Y\}$ and all the $R_i$'s are mutually distinct.

All the examples we have introduced in Section 4.2 satisfy these constraints. Therefore, we claim that most of the practical examples are covered even with the restrictions.

### 6.2 Fusion elimination

Since fusion (\(\bowtie\)) is difficult to handle, we attempt to eliminate fusions (\(\bowtie\)) except when they are generated directly from the annotated type variable atom by merging production rules.

**Definition 6.3** (Fusion elimination). Let $P_{\bowtie}$ denote the set of production rules that include fusion. And let $P_{\bowtie}$ denote the set of production rules without fusion. For each production rule in $P_{\bowtie}$, we apply the production rules in $P_{\bowtie}$ to the type variable atom in the RHS of the rule. This is done in $n^2$ ways for $n$ type variable atoms to cover all combinations. We add the newly created rules, which includes the original one, to $P'$. We also add the rules that have no type variable atoms on RHS to $P'$. If there exist rules in $P'$ and $P_{\bowtie}$ which have the annotated type variable $\alpha(X)$ on the LHS, we add the rule whose LHS is replaced with $\alpha(X)$ to $P'$.

Finally, we replace the annotated type variable $\alpha(X)$ with $\alpha_m(X)$.

We have observed that it is not always possible to eliminate fusion in this way. However, all of our practical examples can be successfully transformed by this method. A more refined method of fusion elimination and a rigorous proof that the production rules obtained by this operation are equivalent to the original ones will be the subject of future work.

If fusion elimination succeeds, we can say that fusion will not appear “later” when the production rule is applied backwards (Ty-Prod). On the other hand, we cannot deny the possibility of occurrence of unabsorbable fusion when applying production rules to decompose graphs (Case). However, this did not happen in our examples.

Once we eliminate fusions, it will be easy to check the correspondence of links. Firstly, we $\alpha$-convert link names so that all the link names are distinct. Then, the correspondence of links in the target graph and the annotated type can be checked as follows. If they are free links, check if they have the same name. If the links are local links, we check the correspondence between the link in the target graph and the link in the annotated type based on mapping. If the correspondence has not yet been established, add a new correspondence. If the correspondence is already in place, we check that it is satisfied. Figure 26 shows the algorithm to check the correspondence of links.

### 6.3 The algorithm

It will be a little troublesome to implement the backward application of a production rule to handle the reverse execution of Ty-Prod. Thus, we will first apply the production rule to the annotated type and then remove the constructor atom both on the target graph and the annotated type. Note that this will result in allowing graphs in the annotation during the execution of this algorithm, which we refer to as an annotated graph.

Figure 27 shows the outline of the algorithm. The function check($G, \alpha(X), R$) checks that $(\emptyset, P) \vdash G : \alpha(X), R$ where $G$ possibly includes $\beta(\overline{Y})$; type annotated graph $G_{\overline{Y}}$ where $G_{\overline{Y}} : \beta(\overline{Y})$. The algorithm runs recursively with helper function (line 6) on the atoms/type annotated graph with a root link $R$ of the target graph $G$ and the annotated graph $T$.

Line 12 checks that the graph $G$ has type $T$ trivially. For example, $G$ maybe the type annotated graph whose annotated type was $T$ or a $\lambda$-abstraction atom, whose typing relation can be checked as the same as the other functional language (except that we may need to apply this algorithm recursively for the graphs in its body expression).

From line 13, we split the cases by the atom with the root link of the target graph and the annotated graph. If both atoms have constructor names with the same function, then we remove the atoms and run the algorithm recursively to all the subgraphs traversable from their arguments.

If the atom in the annotated graph is a type variable atom $\alpha(X)$, then we first try to use induction hypotheses $H$ (line 23 and line 28). Notice that we can use congruence rules (Ty-Cong) and $\alpha$-conversion of free links (Ty-Alpha) to absorb the syntactic difference between $(G : T)$ and hypothesis in $H$.

If we cannot prove it by the hypothesis, then we should proceed with the construction of the proof tree with Ty-Prod or Case. If the root of the target graph is a constructor atom $C_G(X)$ (line 22), then we apply the production rules whose LHS is $\alpha/\overline{Y}$ and check...
7. Related work

Since graphs and its operations are more complex than trees, there are diverse formalisms for graphs and graph types.

7.1 Typing frameworks for graphs

Structured Gamma [5] is a typing framework for graphs, in which types are defined by production rules in context-free graph grammar. Shape Types [4] are similar but the following restrictions are imposed on type definitions to ensure completeness of type checking: (i) the state space of type checking must be confluent, and (ii) graphs supplemented during the type checking must consist only of a finite number of symbols. With context-free graph grammar, we can express a broad and expressive class of types. However, type checking becomes harder and hence it does not cover some practical operations. For example, the concatenation of difference lists and the pop operation from the tail of them cannot be checked by either Shape Types or Structured Gamma.

In this research, we restrict the target grammar so that we can verify practical operations by structural induction.

With Graph Types [8], we can define types of algebraic data structures accompanied by extra edges, where the destination of an extra edge is specified by a routing expression. A routing expression is a regular expression over small-step traverse operations, which describes the relative position of the destination of an extra edge, and the actual destination can be automatically computed based on it. In addition, Graph Types provide a decidable monadic second-order logic on the types as a way of formal verification and automatic program generation. For example, a constant-time concatenation of doubly-linked lists as modification of pointers can be deduced by the logic.

Our type system $F_{GT}$ and Graph Types share the ideas that typed graphs consist of a canonical spanning tree and auxiliary edges, and types are defined by production rules. On the other hand, auxiliary edges and their modification are computed based on the types in our method. In addition, pattern matching based on the types can be described in our language $\mathcal{A}_{GT}$.

7.2 Functional language with graphs

FUnCAL [11] is a functional language that supports graphs as a first-class data structure. This language is based on an existing graph rewriting language, UnCAL. In UnCAL (and FUnCAL), graphs may include back edges and their equality is defined based on bisimulation. FUnCAL comes with its type system but does not support pattern matching for user-defined data types, which classic functional languages support for ADTs.

Functional programming with structured graphs [12] can express recursive graphs using recursive functions, i.e., let rec statements. Since they employ ADTs as the basic structure, they can enjoy type-based analysis based on the traditional type system. On the other hand, we can do further detailed type analysis by our language and type system.

Initial algebra semantics for cyclic sharing tree structures [6]
discusses how to express graphs by \( \lambda \)-expressions. However, there is a large gap between \( \lambda \)-expressions and pointer structures. On the other hand, we defined a graph based on nodes and hyper-edges, which has a clear correspondence to a pointer structure. This style is rather suitable for future implementation. In addition, they do not support user-defined graph types or verification based on them.

### 7.3 Separation Logic

Our approach is in contrast with the analysis of pointer manipulation programs using Separation Logic [15], shape analysis [22], etc.

Firstly, the target languages differ in many ways. Separation Logic and shape analysis normally handle low-level imperative programs using heaps and pointers. In contrast, we dispense with destructive operations and adopt pattern matching over graphs provided by the new higher-level language \( \lambda_{GT} \), which abstracts address, pointers and heaps away, and features hyperlinks and operations on them including fusion and hiding.

Secondly, we pursue a lightweight, automatic type system for functional languages rather than Hoare-style general verification for imperative languages. Separation Logic allows us to use pure formulae that represent various non-spatial properties. The only thing that seems to correspond to pure formulae in our type system is fusion (which can be regarded as \( x = y \) in Separation Logic). This design choice reflects the fact that our goal is not a formal system for software verification but a programming language and its type system.

The problem discussed in Section 5, verification of an inductively defined structure with structural induction, is close to the entailment problem of inductive predicates with symbolic heaps in Separation Logic, sometimes referred to as SLRD (Separation Logic with Recursive Definitions). Cyclist [2] performs automatic verification of the problem. However, the algorithm requires dynamic checking of the soundness condition. On the other hand, we have restricted graph grammar and proved the soundness statically as a (meta-)theorem. Antonopoulos et al. [1] show that the entailment problem of general SLRD is undecidable. Therefore, decision procedures for them impose some restrictions on SLRD. Josif et al. [7] propose a sub-class of SLRD, \( \text{SLRD}_{\text{free}} \), which handles graphs with bounded treewidth. The restrictions imposed on the recursive definitions are similar to the restrictions we have introduced in Section 6.1. However, they do not allow empty graphs and cannot handle a difference list without elements. Tatsuta et al. [19] has imposed further restriction to \( \text{SLRD}_{\text{free}} \) which corresponds to the notion of root link in ours. A precise comparison of the algorithms in [19] and our technique will be the subject of future work.

### 8. Conclusions and further work

In this study, we proposed a new functional language \( \lambda_{GT} \) that handles graphs as a first-class data structure with declarative operations based on graph transformation.

First, we formalized the formal syntax and semantics of \( \lambda_{GT} \) in a syntax-directed manner, incorporating HyperLMNtal into a call-by-value \( \lambda \)-calculus.

Second, we developed a new type system \( F_{GT} \) that emplois HyperLMNtal rules as production rules to deal with data structures more complex than trees.

Third, we extended the type system to support more powerful verification such as concatenation of difference lists. Then we developed an algorithm to automatically verify programs with the extended type system using structural induction.

Finally, we address future work that is not mentioned in previous sections.

#### 8.1 Extend the type system to handle untyped graph contexts

In this paper, we introduced dynamic type checking (Section 4.4) and excluded untyped graph contexts. However, verification with untyped graph contexts is necessary not just to reduce the programmer’s extra effort since there exist programs that cannot be succinctly handled without untyped graph contexts. For example, matching the leftmost leaf in a leaf-linked tree is possible in \( \lambda_{GT} \) using a template consisting of the leftmost leaf and an untyped graph context for the rest of the tree. However, we cannot denote the type of the untyped graph context using the type of the leaf-linked tree because it is not a tree.

#### 8.2 Full implementation of the language and the type system

We have implemented the type checker to verify operations over graphs. However, implementation of the language with the full type system including arrows is a future work. We believe that it is straightforward to implement the type system. However, implementation of the efficient runtime has many things to be considered including deeper static analysis of programs (such as the guarantee of immutability using ownership checking) to allow destructive operations on graphs without forcing imperative programming on users.

#### 8.3 Extension on the type system: polymorphism and type inference

The proposed type system \( F_{GT} \) is monomorphic. We can only define difference lists with a specific element type, though introducing generic data types as in other functional languages could be done in the same way.

However, for more complex data structures, introducing polymorphism may be not that straightforward since we have introduced more powerful operations than the other languages such as concatenation of difference lists. In \( \lambda_{GT} \), concatenation of difference lists can be done without explicitly handling constructor atoms, which may be typeable as a generic function. However, since operations on data structures may not result in data structures of the same type, we may need to verify programs with the type information of the inputs, which seems to be a little incompatible with polymorphism.

The same thing can be said for type inference. Since we allow powerful operations over data structures without explicitly denoting constructor names, it may be more difficult than in other functional languages and may require some non-obvious ingenious techniques.
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Appendix

In this appendix, we give proofs for the propositions and theorems that appeared in this paper.

A.1 Proof of properties of HyperLMNtal

Lemma A.1.1 (Elimination of $v$ which bounds no link name),

$vXG ≡ G \text{ where } X \not \in \text{fn}(G)$

Proof.

Lemma A.1.2 (Elimination of a futile link substitution)

$G(X/X) ≡ G$

This is not as obvious as it may seem. The reason is that it cannot be naively ruled out that a $\alpha$-conversion may be performed during the hyperlink assignment, resulting in a congruent but syntactically different graphs.

Proof. We prove by induction on graphs. It is trivial for $0, p(X_1, \ldots, X_m), (G, Q), (G \rightarrow Q)$. Case $vYG$:

$$(vYG(X/X)) \overset{\text{def}}{=} \begin{cases} vYG & \text{if } Y = X \\ vYG(X/X) & \text{if } X \neq X \\ vYG & \text{induction hypothesis} \end{cases}$$

Since $Y \neq X \land Y = X$ can never happen, there is no possibility of $\alpha$-conversion of links (which could have resulted in loss of syntactic equality) to avoid variable capture.

Proof of Theorem 2.2. We are using Lemma A.1.1 and Lemma A.1.2. We consider the case where the free hyperlink to be substituted appears and the case where it does not. The latter case seems obvious, but it is not because of the possibility of $\alpha$-conversion due to hyperlink substitution. We prove the former first, and then transform the latter into a form that allows us to use the former.

Case $X \in \text{fn}(G)$:

$vX \cdot Y(X) \overset{\text{def}}{=} (X, X \to Y, G)$

$$(E5, E3) \quad vX \cdot Y(X) \overset{\text{def}}{=} (X, X \to Y, G)$$

$$(E5, E10) \quad vX \cdot Y(X) \overset{\text{def}}{=} (X, X \to Y, G)$$

$$(E5, E10) \quad X \not \in \text{fn}(G)$$

$$(E5, E6) \quad vX \cdot Y(X) \overset{\text{def}}{=} (X, X \to Y, G)$$

$$(E5, E10) \quad X \not \in \text{fn}(G)$$

$$(E6, \text{Lemma A.1.2}) \quad vX \cdot Y(X) \overset{\text{def}}{=} (X, X \to Y, G)$$

$$(E6, \text{Lemma A.1.2}) \quad G(X/X) = G$$

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$$(E6, \text{Lemma A.1.2}) \quad G(X/X) = G$$

$$(E6, \text{Lemma A.1.2}) \quad G(X/X) = G$$
Thus, \( vX.G \equiv vY.G(Y/X) \)

Case \( X \notin fn(G) \):

In this case, we use the previous proof by first adding a
free hyperlink \( X \) using (E7).

\[ \begin{align*}
  & vX.G \\
  \equiv & \text{Lemma A.1.1 } G \\
  \equiv & \text{E1 } (\emptyset, G) \\
  \equiv & \text{E4, E7 } (vX.vX.X \Rightarrow X, G) \\
  \equiv & \text{E4, Lemma A.1.1 } (vX.X \Rightarrow X, G) \\
  \vdash & X \notin fn(vX.X \Rightarrow X) \\
  \equiv & \text{E10 } vX.X \Rightarrow X, G \\
  \vdash & X \notin fn(G) \\
  \text{The formal proof } vY.(X \Rightarrow X, G)(Y/X) \\
  \vdash & X \notin fn(G) \\
  \equiv & vY.(Y \Rightarrow Y, G)(Y/X) \\
  \equiv & \text{Lemma A.1.1 } vY.G(Y/X) \\
  \equiv & \text{E1 } G(Y/X) \\
  \equiv & \text{Lemma A.1.1 } vY.G(Y/X)
\end{align*} \]

\( \square \)

Proof of Theorem 2.1.

\[ \begin{align*}
  & vZ.(Z \Rightarrow X, Z \Rightarrow Y) \\
  \equiv & \text{E5 } vZ.(X \Rightarrow Y) \\
  \vdash & (Z \Rightarrow Y)(X/Z) = X \Rightarrow Y \\
  \equiv & \text{Lemma A.1.1 } X \Rightarrow Y \\
  \text{and } \\
  & vZ.(Z \Rightarrow X, Z \Rightarrow Y) \\
  \equiv & \text{E2, E5 } vZ.(Z \Rightarrow Y, Z \Rightarrow X) \\
  \equiv & \text{E5 } vZ.(X \Rightarrow X) \\
  \vdash & (Z \Rightarrow X)(Y/Z) = Y \Rightarrow X \\
  \equiv & \text{Lemma A.1.1 } Y \Rightarrow X \\
  \text{Therefore, } X \Rightarrow Y \equiv Y \Rightarrow X.
\end{align*} \]

\( \square \)

A.2 Proof of properties of \( F_{GT} \)

Theorem 4.1 (Soundness of \( F_{GT} \)) can be derived in the same way as in the ordinary type systems for functional languages, so we omit the precise proof. Theorem 4.2 and Theorem 5.2 have a proof specific to \( F_{GT} \), which is supplemented in this appendix.

A.2.1 Theorem 4.2 (\( F_{GT} \) and HyperLMNTarl reduction)

Lemma A.2.1.

If \( G_1 \rightsquigarrow_p G_2 \) then \( G_1(Y/X) \rightsquigarrow_p G_2(Y/X) \)

Proof.

By (R1), (R2), (R3) and \( G_1 \rightsquigarrow_p G_2 \), we can show
\( vX.(X \Rightarrow Y, G_1) \rightsquigarrow_p vX.(X \Rightarrow Y, G_2) \). Thus \( G_1(Y/X) \rightsquigarrow_p G_2(Y/X) \) by (R3). Then we can obtain \( G_1(Y/X) \rightsquigarrow_p G_2(Y/X) \) by induction on the length of the reduction \( \rightsquigarrow_p \).

\( \square \)

Proof of Theorem 4.2.

We denote \( \tau_i(Y) = [x_i][\tau_i(Y)] \) as \( \theta_i \) and
\( \tau_i(Z) = [x_i][\tau_i(Z)] \) as \( \theta_i \).

We firstly prove \( \Rightarrow \). We split the cases by the last applied \( F_{GT} \) rules.

Case Ty-Ctx:

\[ T = \sigma[\tau_i(Y)], \text{ and } T[\sigma[\tau_i(Z)]] \in \Gamma. \text{ Thus } T[\tau_i(Y)] \text{, and } T[\tau_i(Z)] \text{, we can show } T \rightsquigarrow_p T \text{.} \]

Case Ty-Arrow:

\[ T = (\lambda \cdots \alpha)(\tau(X)) \text{ where } (\Gamma, P) \vdash T^p \text{.} \text{ By induction hypothesis, } T \rightsquigarrow_p T^p \text{.} \]

Case Ty-Cong:

Suppose the antecedent of Ty-Cong was \( (\Gamma, P) \vdash T' \text{.} \text{ By induction hypothesis, } T' \rightsquigarrow_p T' \text{.} \text{ Since } T \equiv T' \text{, we can show } T \rightsquigarrow_p T \text{ using (R3).} \]

Case Ty-Alpha:

Suppose the antecedent of Ty-Alpha was \( (\Gamma, P) \vdash T' \text{.} \text{ By induction hypothesis, } T' \rightsquigarrow_p T' \text{.} \text{ Here, we can show } T \equiv T' \text{ using (R3).} \text{ Therefore, by Lemma A.2.1, } T \equiv T' \text{.} \]

Case Ty-Prod:

Suppose the antecedents of Ty-Prod was
\[ (\Gamma, P) \vdash T_j : \tau_i(Y), \text{ and } T \equiv \tau_i(Y). \text{ By induction hypothesis, } T \rightsquigarrow_p T \text{.} \text{ Therefore, using (R1), (R2), and (R3), we can show } T' \rightsquigarrow_p T' \text{.} \text{ Here, we can have } T \equiv T' \text{ using (R3).} \text{ Thus, we can inductively define as } T_0 = T \text{ and } T_{i+1} = T_i[T_i(T_i \theta_i \theta_i)(\tau_i(Y))] \text{, in which } T_n = T \text{.} \]

Then, we prove \( \Leftarrow \). By induction on the length of the reduction \( \rightsquigarrow_p \).

We denote \( \tau_i(Y) = [x_i][\tau_i(Y)] \) as \( \theta_i \) and \( (\lambda \cdots \alpha)(\tau_i(Y)) \) as \( \theta_i \).

Then, the proposition can be rewritten as
\[ \tau_i(Y) \rightsquigarrow_p T \Rightarrow (\Gamma, P) \vdash T \theta_i \theta_i^{-1} : \tau_i(Y) \text{.} \]

Case \( \tau_i(Y) \equiv T \text{ (The length of } \rightsquigarrow_p \text{ is zero)} \):

Follows by Ty-Ctx or Ty-Arrow depending on whether \( \tau_i(Y) \text{ is replaced with the graph context in } \theta_i \text{ or the } \lambda \text{-abstraction atom in } \theta_i^{-1} \).

Case \( \tau_i(Y) \rightsquigarrow_p T' \equiv T \text{ (The length of } \rightsquigarrow_p \text{ is } n > 0) \):

Suppose the production rule applied to reduce from \( T' \) to \( T \text{ was } \alpha \Rightarrow T'' \text{.} \text{ Using (R1), (R2), and (R3), we can obtain (new) } \tau_i(Y) \rightsquigarrow_p T' \rightsquigarrow_p T \text{ which satisfies } T \equiv T'[T''[\alpha[Y]]]. \text{ By induction hypothesis, we can obtain the derivation tree of} \]
\[ (\Gamma, P) \vdash T''[\alpha[Y]] : \tau(X) \text{.} \quad \text{(A.1)} \]

Since \( T'' \) contains \( \alpha[Y] \), there exists a derivation of
\[ (\Gamma, P) \vdash \theta_i^{-1} \theta_i^{-1} : \alpha(X) \text{.} \quad \text{(A.2)} \]
in the tree. Since
holds immediately by Ty-Prod, we can replace the derivation tree of (A.2) with that of (A.3) in that of (A.1), which will result in the derivation tree of the desired typing relation.

□

A.2.2 Theorem 5.2 (decomposing graph with the last applied production rule)
We omit $(\emptyset, P) \vdash$ for brevity.

**Proof of Theorem 5.2.** We prove by induction on the derivation of $G : \alpha(\vec{Y})$ after the last application of Ty-Prod.

By Lemma 5.1, there exists the last Ty-Prod and only Ty-Cong and Ty-Alpha are used later on the derivation of $G : \alpha(\vec{Y})$.

Case Ty-Prod:
Trivial from the definition of Ty-Prod.

Case Ty-Cong:
The theorem holds on $G : \alpha(\vec{Y})$ by induction hypothesis. Therefore, it holds on $G' \equiv G$.

Case Ty-Alpha:
By induction hypothesis, we can assume for $G : \alpha(\vec{Y})$, there exists $\overline{G_j}$ such that $G \equiv T'[\overline{G_j/\theta_j(X_j)}]$ where

- $T' = T(\overline{Y_j/X_j})$.
- $\theta_j(X_j)$ are all the type atoms appearing in $T'$, and
- $G_j : \tau_j(\vec{X}_j)$.

For $G(Z|Y) : \alpha(\vec{Y})(Z|Y)$, we can obtain

- $T'' = T(\overline{Z_j/X_j})$, where $Z_j = Y_j(Z|Y)$.

The type atom $\tau_j(\vec{Z}_j)$ appearing in $T''$, corresponding to the atom $\tau_j(\vec{X}_j)$ in $T'$, may have substituted its links. Thus, we need to denote it as $\tau_j(\vec{X}_j)\theta_j$ where $\theta_j$ is a hyperlink substitution which satisfies $\tau_j(\vec{X}_j)\theta_j = \tau_j(\vec{Z}_j)$. Since $G_j : \tau_j(\vec{X}_j)$ holds by the induction hypothesis, we can show that $G_j\theta_j : \tau_j(\vec{X}_j)\theta_j$ holds using Ty-Alpha. Therefore, we can obtain $G_j\theta_j$ that satisfies the conditions.

□

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