Scars of Invariant Manifolds in Interacting Chaotic Few–Body Systems

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We present a novel extension of the concept of scars for the wave functions of classically chaotic few–body systems of identical particles with rotation and permutation symmetry. Generically there exist manifolds in classical phase space which are invariant under the action of a common subgroup of these two symmetries. Such manifolds are associated with highly symmetric configurations. If sufficiently stable, the quantum motion on such manifolds displays a notable enhancement of the revival in the autocorrelation function which is not directly associated with individual periodic orbits. Rather, it indicates some degree of localization around an invariant manifold which has collective characteristics that should be experimentally observable.

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During the last decade, spectral fluctuation properties of classically fully chaotic systems ("quantum chaos" for short) have been largely understood, and interest has shifted to the non–generic properties of wave functions. In systems with few degrees of freedom, these functions may display scars [1–5], i.e. possess increased intensities along short and not very unstable periodic orbits of the corresponding classical system.

In this Letter, we extend the study of scars to systems with identical particles. Such systems possess symmetry properties which are generically absent in systems with two or three degrees of freedom. Particle identity leads to permutational symmetry. In addition, a self–bound system possesses rotational symmetry. We show that the combination of these two symmetries leads to a novel mechanism for the formation of scars.

In a previous paper, two of the authors studied the influence of both symmetries on the periodic–orbit structure of chaotic systems [6]. In the present Letter, we proceed differently. The combination of rotational and permutational symmetry allows us to construct invariant manifolds in classical phase space. We show that such manifolds may lead to scarred wave functions of the corresponding quantum system. The scars are not related to individual periodic orbits but are similar to the ones found by Prosen in billiards [7]. We believe that our results are important for finite many–body systems such as atoms, molecules or atomic nuclei where this generalized scarring has obvious implications for collective motion. Our results derive from the numerical study of a specific Hamiltonian. For reasons given below, we believe our results to be generic, however.

The paper is organized as follows. We define a rotationally invariant few–body system of four interacting identical particles in two dimensions. We prove the existence of an invariant manifold and analyze its classical stability. Finally, we investigate the corresponding quantum system. This is done by semiclassical propagation of wave packets.

Our Hamiltonian has the form

$$ H = \sum_{i=1}^{4} \left( \frac{1}{2m_i} \mathbf{p}_i^2 + 16\alpha |\mathbf{r}_i|^4 \right) - \alpha \sum_{1 \leq i < j \leq 4} |\mathbf{r}_i - \mathbf{r}_j|^4, \quad (1) $$

where $\mathbf{p}_i = (p_{ix}, p_{iy})$ and $\mathbf{r}_i = (x_i, y_i)$ with $i = 1, \ldots, 4$ are the two–dimensional momentum and position vectors of the $i^{th}$ particle, respectively. We use units where $m = \alpha = 1, \hbar = 0.01$; then coordinates and momenta are given in units of $\hbar^{1/3} \alpha^{1/6} m^{-1/6}$ and $\hbar^{2/3} \alpha^{1/6} m^{1/6}$, respectively. For the Hamiltonian, this leads to the scaling relation $H(\gamma \mathbf{p}, \gamma \mathbf{r}) = \gamma H(\mathbf{p}, \mathbf{r})$. This shows that the structure of classical phase space is independent of energy. Moreover, energy and total angular momentum are the only integrals of motion, and the system is non–integrable.

Construction of the invariant manifold. The Hamiltonian \[\text{[8–13]}\] is invariant under the symmetric group $S_4$ and the orthogonal group $O(2)$. Thus, we may apply the ideas of ref. \[\text{[6]}\]. By imposing suitable symmetry requirements onto the initial conditions, we can restrict classical motion to a low–dimensional invariant manifold. Many such manifolds exist and it suffices to consider one of them. We choose the initial conditions in such a way that positions and momenta exhibit the symmetry of a rectangle. Such a configuration is shown in Fig.\[\text{[4]}\]. Positions and momenta of the particles are indicated by points and arrows, respectively. Obviously, particle 3 is the image of particle 1 under inversion, whereas particle 2 and particle 4 are the mirror images of particle 1 under a reflection at appropriately chosen axes, respectively. The manifold is spanned by the two two–dimensional vectors $\mathbf{p} = (p_{x}, p_{y})$ and $\mathbf{r} = (x, y)$ giving the momentum and position of particle 1. The associated Hamiltonian $\tilde{H}(\mathbf{p}, \mathbf{r}) = \frac{1}{2} \mathbf{p}^2 + 16\alpha |\mathbf{r}|^4$ has been studied extensively in the literature, both in the classical and the quantum case \[\text{[8–13]}\]. The classical system is essentially chaotic. Only one stable periodic orbit is known which is surrounded by a very small island of stability \[\text{[14]}\]. The periodic orbits can be enumerated by means of an (incomplete) symbolic code consisting of three letters \[\text{[13]}\]. Application of rotations and/or permutations to the configuration shown in Fig.\[\text{[4]}\] yields further equivalent invariant manifolds.
Stability of motion close to the invariant manifold. For particle positions far from the origin, we may expand the potential of the Hamiltonian around the “channel” configuration defined by the position \( x = r \gg E^{3/4}, \ y = 0 \) of particle 1. This yields a quadratic form with non-negative eigenvalues proportional to \( r \). Thus, far out in the channel, the particles perform almost harmonic oscillations with high frequencies in the directions transverse to the channel while moving slowly in the direction of the channel. This confines the motion to a small vicinity of the invariant manifold. Moreover, this motion is regular. Close to the central region, the motion is chaotic. Hence, we expect our few-body system to display intermittency.

To analyze the stability of the central region, we confine ourselves to computing the full phase-space monodromy matrices of certain periodic orbits within the manifold. All orbits up to code length three (12 in total) are taken into account. Together they explore a significant part of the central region. For each periodic orbit there are five pairs of stability exponents that correspond to the directions perpendicular to the invariant manifold. A sixth pair of stability exponents corresponds to the motion within the invariant manifold. The remaining two pairs of stability exponents vanish because of the continuous symmetries of the Hamiltonian. Our computations show that the motion in the directions perpendicular to the manifold is unstable. However, the corresponding stability exponents are significantly smaller than the stability exponents which correspond to the chaotic motion within the invariant manifold, see Table II. This combination of large stability exponents within the manifold and small stability exponents perpendicular to the manifold leads us to conjecture that the manifold may scar wave functions of the corresponding quantum system.

Clearly this is a particular property of the invariant manifold we have chosen. If we consider the collinear configuration with mirror symmetry about the center of the system, we obtain a different invariant manifold with very large transverse Lyapunov exponents. On this invariant manifold we do not expect scarring.

Quantum computation. To prove our conjecture we consider the time evolution of a Gaussian wave packet

\[
\Psi(r, t) = C \exp \left[ -\frac{1}{2} (r - r_0)^T A (r - r_0) + i \frac{E}{\hbar} p^T (r - r_0) \right]
\]

where \( < p > = p_0 \) and \( < r > = r_0 \) define a point on the invariant manifold. We have used the shorthand notation \( r = (r_1, r_2, r_3, r_4) \) and \( p = (p_1, p_2, p_3, p_4) \) for configuration and momentum space vectors, respectively. The autocorrelation function \( C(t) = < \Psi(t = 0) | \Psi(t) > \) is computed in semiclassical approximation. Within the manifold we used Heller’s cellular dynamics which takes into account the nonlinearity of the classical motion. In the transverse direction the time-propagation was done using linearized dynamics only. This approximation neglects any recurrences from the transverse directions and implies a permanent flux of probability out of the manifold and its vicinity. On the time scales considered here, the linearization is justified since the classical return probability to the manifold of transversely escaping trajectories is negligible. It is also important to note that the loss of probability inside the manifold is not severe since the transverse stability exponents are not too large.

We launch wave packets along periodic or aperiodic orbits lying within the invariant manifold and consider their revival as measured by the autocorrelation function. To achieve shorter recurrence times the initial packet was symmetrized with respect to the reflection symmetry of the system within the invariant manifold. This effects a partial projection onto the symmetric and eliminates the antisymmetric representation of \( \mathcal{S}_4 \); antisymmetrizing with respect to the reflections we could have eliminated the symmetric rather than the antisymmetric representation.

We propagate such wave packets for both, the manifold with small and with large transversal exponents. For not too unstable periodic orbits we expect a fairly strong revival after one period, known as the linear revival \([13]\). As an example, we show in Fig. 2 the real part of the autocorrelation function, calculated for the more stable invariant manifold and starting on the periodic orbit 2. We indeed find strong linear revival. However, at larger times we find randomly scattered strong revivals, the revival corresponding to the second period not being dominant. This implies that a significant fraction of the original amplitude remains within the invariant manifold, and that this fact is not related to the periodic orbit we started on. Revivals calculated for packets started on aperiodic orbits show similar features except for the obvious absence of the linear revival. The collinear and more unstable of the two manifolds discussed above, on the contrary, shows practically no revival at all. Even the linear revival is negligible, see Fig. 3, despite the fact that the packet was launched along a short periodic orbit.

Fig. 4 shows the Fourier transform of the autocorrelation function whose real part was displayed in Fig. 2. Our time sequences are not long enough to resolve the spectrum, and it seems difficult to obtain sequences of sufficient length with a reasonable numerical effort. Nevertheless the Fourier transform in Fig. 4 is very informative. It shows a peak in energy (frequency) corresponding to the energy of the original wave packet. We have chosen the initial conditions \((p_0, r_0)\) such that the energy of the initial wavepacket is centered around \( E = 1 \). An integration over classical phase space shows that this energy is chosen around the 10^{10th} level. This implies that we are deep within the semiclassical region. The expected peak in energy has a superposed equidistant structure with spacing \( \Delta E = 0.034 \) corresponding to the period of the
orbit on which we launched the wave packet. In addition there is a significant superposition of other frequencies resulting in a finer, non-equidistant structure.

This indicates that other trajectories in the manifold contribute. The structure under the peak is typical for the more stable invariant manifold and practically absent for the collinear one (not shown).

Our argument suggests that the enhanced revival seen on sufficiently stable invariant manifolds should not depend on the irreducible representation of \( S_4 \) we choose, unless the invariant manifold itself limits the possible representations. We could not check this point since our calculation did not project onto a specific irreducible representation. Nevertheless, the localization effect implicit in the large revival of the autocorrelation function is not of classical nature. Indeed, an ensemble of trajectories launched near the invariant manifold did not return near it on the time scale we consider. We note that this observation also justifies the linear approximation for cellular dynamics in directions transversal to the invariant manifold.

In summary, we have shown that wave packets may have unusually long life times on certain invariant manifolds characterized by small classical transverse instabilities. This is a quantum and not a classical phenomenon and constitutes a novel and very exciting extension of the concept of a scar to which we attribute considerable significance. Indeed, invariant manifolds of the type considered above occur generically in many–body systems of physical interest like nuclei, atoms, molecules, or small metallic clusters. The importance of these manifolds for quantum properties will hinge on their stability properties: Stability will determine the degree to which scars actually exist in such systems. Stability is a system–specific property, of course, and generic statements are at least very difficult. We recall, however, that in Helium the collinear manifold \( [4] \) (which is linearly stable in the perpendicular direction) supports doubly excited states. This shows why we expect our results to be generic for interacting few–body systems.

| Code | \( T \) | \( u_\parallel \) | \( \sum_{i=1}^{n} u_\perp^{(i)} \) | \( n \) |
|------|-------|-------|-----------------|------|
| 1    | 3.313 | 5.74  | 2               | 0    |
| 2    | 2.622 | 4.86  | 3.29            | 2    |
| 01   | 2.958 | 4.46  | 1.22            | 2    |
| 02   | 4.036 | 6.99  | 4.18            | 3    |
| 12   | 7.933 | 11.67 | —               | 0    |
| 001  | 8.015 | 9.64  | —               | 0    |
| 002  | 5.074 | 8.07  | 6.84            | 4    |
| 011  | 4.541 | 7.30  | 0.35            | 2    |
| 012  | 5.753 | 10.00 | 1.63            | 1    |
| 022  | 6.623 | 11.37 | 7.13            | 5    |
| 112  | 2.860 | 4.79  | 0.68            | 1    |
| 122  | 13.253| 23.03 | 21.41           | 5    |

TABLE I. Periods and stability exponents of periodic orbits up to code length three lying inside the invariant manifold. \( T \) is the period, \( u_\parallel \) the stability exponent inside the manifold, \( \sum_{i=1}^{n} u_\perp^{(i)} \) the sum of all \( n \) (in total) real positive stability exponents perpendicular to the manifold. The energy of the orbits is \( E = 4 \).

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FIG. 1. Collective configuration
FIG. 2. Autocorrelation function $C(t)$ of a symmetrized wave packet launched on a periodic orbit with period $T$ inside the weakly unstable manifold. In addition to the linear revival around $t = \frac{T}{2}$, a strong nonlinear revival is seen for larger times.

FIG. 3. Autocorrelation function $C(t)$ of a symmetrized wave packet launched on a short periodic orbit with period $T$ inside the very unstable collinear manifold. Practically no revival is seen.

FIG. 4. Fourier transform $S(E)$ of the autocorrelation function $C(t)$ shown in Fig. 2. The equidistant structure corresponding to half the period of the periodic orbit is accompanied by a fine structure resulting from the nonlinear revival.