ON A POWER SERIES INVOLVING CLASSICAL ORTHOGONAL POLYNOMIALS

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We investigate a class of power series occurring in some problems in quantum optics. Their coefficients are either Gegenbauer or Laguerre polynomials multiplied by binomial coefficients. Although their sums have been known for a long time, we employ here a different method to recover them as higher-order derivatives of the generating function of the given orthogonal polynomials. The key point in our proof consists in exploiting a specific functional equation satisfied by the generating function in conjunction with Cauchy’s integral formula for the derivatives of a holomorphic function. Special or limiting cases of Gegenbauer polynomials include the Legendre and Chebyshev polynomials. The series of Hermite polynomials is treated in a straightforward way, as well as an asymptotic case of either the Gegenbauer or the Laguerre series. Further, we have succeeded in evaluating the sum of a similar power series which is a higher-order derivative of Mehler’s generating function. As a prerequisite, we have used a convenient factorization of the latter that enabled us to employ a particular Laguerre expansion. Mehler’s summation formula is then applied in quantum mechanics in order to retrieve the propagator of a linear harmonic oscillator.

1. INTRODUCTION

The problem of evaluating the $l$th-order correlation function $\langle (\hat{a}^\dagger)^l \hat{a}^l \rangle$ for a one-mode squeezed thermal state of the quantum electromagnetic field by employing the photon-number distribution led one of us [1] to consider the following power series involving Legendre polynomials:

$$G_l^{(1)}(t,w) = \sum_{n=0}^{\infty} \binom{n}{l} t^n P_n(w), \quad (w := \cos \theta, \quad |t| < 1, \quad l = 0,1,2,3,\ldots). \quad (1.1)$$

In eq. (1.1), $\binom{n}{l}$ are binomial coefficients. For subsequent convenience, we introduce the square root of a complex variable as follows:

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The sum of any series of ascending powers (1.1) is the product of three factors: the generating function of the Legendre polynomials, the \( l \)th power of a specific variable, and the Legendre polynomial of degree \( l \) of another specific variable:

\[
G^{(l)}(t,w) = \frac{1}{q} \left( \frac{t}{q} \right)^l P_l \left( \frac{w-t}{q} \right), \quad (l = 0,1,2,3,\ldots) .
\]  

(1.3)

Note that the series expansion (1.1) reduces for \( l = 0 \) to the power series of the generating function of the Legendre polynomials.

A similar result has been found later for the Laguerre polynomials [2]. We obtained a formula which is useful when the general solution of the quantum optical master equation describing dissipation is applied to a single-mode displaced squeezed thermal state [3].

When we performed the above-mentioned summations we were not aware of the fact that the corresponding formulae and the similar ones for the Gegenbauer and Hermite polynomials have been known for a long time [4]. This happened because such formulae are not mentioned in any book devoted to special functions except for the cited textbook. Rainville derived them by making combined use of an appropriate double power series and of the generating function of the orthogonal polynomials under discussion. Owing to their importance for some physical problems, we find it useful to give in this work a comprehensive overview based on an alternative approach. We prove that a property of the type (1.3) is shared by two one-parameter families of classical orthogonal polynomials, namely, the Gegenbauer and Laguerre polynomials. The main ingredients of our method are Cauchy’s integral formula for the derivatives of a holomorphic function and specific functional equations satisfied by the generating functions of these well-known polynomials.

The outline of the article is as follows. Section 2 deals with the Gegenbauer series, while in Sec. 3 the obtained formula is applied to the particular cases of the Legendre and Chebyshev polynomials. In Sec. 4 we review the Laguerre series. The sum of the series of Hermite polynomials is recovered in Sec. 5 in a straightforward manner, as well as an asymptotic limit of both the Gegenbauer and Laguerre series. In Sec. 6 we compute a similar power series of an arbitrary derivative of Mehler’s generating function, finding a new formula. By way of illustration, Mehler’s summation formula is employed in Sec. 7 to write in closed form the familiar propagator of a linear harmonic oscillator. Concluding remarks are presented in Sec. 8.

2. GEGENBAUER POLYNOMIALS

The generating function of the Gegenbauer polynomials of order \( \lambda \),

\[
q := \left( 1 - 2\nu t + t^2 \right)^{\frac{1}{2}}, \quad (t \to q = 1).
\]  

(1.2)
On a power series involving classical orthogonal polynomials

\[ G_0^{(r)} (t, w) := \left(1 - 2wt + t^2\right)^r, \quad \left(\lambda \neq 0, \quad \lambda > -\frac{1}{2}, \quad w := \cos \theta\right), \quad (2.1) \]

is holomorphic in the variable \( t \) for \(|t| < 1\) and has the Taylor expansion \([5]\]

\[ G_0^{(r)} (t, w) = \sum_{n=0}^{\infty} t^n C_n^r (w), \quad \left(w := \cos \theta, \quad |t| < 1\right). \quad (2.2) \]

Accordingly, Cauchy’s integral formula for the derivatives of a holomorphic function (CIFD) \([6,7]\) gives the following integral representation of the Gegenbauer polynomial \( C_l^\lambda (w) \) of degree \( l \) and order \( \lambda \): \[ C_l^\lambda (w) = \frac{1}{2\pi i} \int_{\gamma} dt \frac{1}{t^{l+1}} G_0^{(l+1)} (t, w). \quad (2.3) \]

The contour of integration in eq. (2.3) is a simple loop encircling the origin \( t = 0 \) counterclockwise and situated entirely within the open disk \(|t| < 1\). Note that the sum of the power series of interest,

\[ G_l^{(\lambda)} (t, w) := \sum_{n=l}^{\infty} \binom{n}{l} t^n C_n^\lambda (w), \quad \left(w := \cos \theta, \quad |t| < 1, \quad l = 0,1,2,3,\ldots\right), \quad (2.4) \]

involves the \( \ell \)-th-order derivative of the generating function:

\[ G_l^{(\lambda)} (t, w) = \frac{t^\ell}{l!} \left(\frac{\partial}{\partial t}\right)^l G_0^{(l+1)} (t, w). \quad (2.5) \]

We employ CIFD again and find, after a translation of the variable of integration,

\[ G_l^{(\lambda)} (t, w) = \frac{t^\ell}{2\pi i} \int_{\gamma} ds \frac{1}{s^{l+1}} G_0^{(l+1)} (s + t, w). \quad (2.6) \]

We now take advantage of the following functional equation for the generating function (2.1):

\[ G_0^{(\lambda)} (s + t, w) = G_0^{(\lambda)} (t, w) G_0^{(\lambda)} (\xi, \eta), \quad (2.7) \]

where we have denoted

\[ \xi := \frac{s}{q}, \quad \eta := \frac{w - t}{q}. \quad (2.8) \]

Insertion of eq. (2.7) into eq. (2.6), followed by the change of the integration variable to \( \xi \) defined in eq. (2.8), yields
\[ G_t^{(\lambda)}(t, w) = G_0^{(\lambda)}(t, w) \left( \frac{t}{q} \right)^{\lambda} \frac{1}{2\pi i} \int d\xi \frac{1}{\xi^{1+\lambda}} G_0^{(\lambda)}(\xi, \eta). \] (2.9)

For real \( t \), the obvious inequality \( \eta^2 \leq 1 \) allows us to use eqs. (2.1) and (2.3) and find the formula

\[ G_t^{(\lambda)}(t, w) = q^{-2\lambda} \left( \frac{t}{q} \right)^{\lambda} C^\lambda_l \left( \frac{w-t}{q} \right), \] (2.10)

\[ \left( \lambda \neq 0, \quad \lambda > -\frac{1}{2}, \quad w = \cos \theta, \quad |t| < 1, \quad l = 0, 1, 2, 3, \ldots \right). \]

The validity of eq. (2.10) can be extended by analytic continuation to any complex value \( t \) inside the unit circle. An equivalent form of eq. (2.10),

\[ \sum_{m=0}^{\infty} \binom{l+m}{l} t^n C^\lambda_{l+m} (w) = q^{-2\lambda-n} C^\lambda_l \left( \frac{w-t}{q} \right), \] (2.11)

coincides with the formula written by Rainville [8].

3. LEGENDRE AND CHEBYSHEV POLYNOMIALS

Recall that the Legendre polynomials are the Gegenbauer polynomials of order \( \lambda = \frac{1}{2} \):

\[ P_l^\lambda (w) = C^\frac{1}{2}_l (w). \] (3.1)

Therefore, by setting \( \lambda = \frac{1}{2} \) in eqs. (2.4) and (2.10), we recover eqs. (1.1) and (1.3), respectively. For \( \lambda = \frac{1}{2} \), eq. (2.11) reduces to the equivalent formula chosen by Rainville [9]:

\[ \sum_{m=0}^{\infty} \binom{l+m}{l} t^n P_{l+m} (w) = \left( \frac{1}{q} \right)^{1+l} P_l \left( \frac{w-t}{q} \right). \] (3.2)

The Chebyshev polynomials of the first kind,

\[ T_l (w) := \cos(l\theta), \quad (w := \cos \theta, \quad l = 0, 1, 2, 3, \ldots), \] (3.3)

are the limiting case \( \lambda = 0 \) of the Gegenbauer polynomials, as follows [10]:
\[ T_0(w) = C_0^0(w), \] (3.4)
\[ T_l(w) = \frac{l}{2} C_l^0(w), \quad (l = 1, 2, 3, \ldots), \] (3.5)

with [11]
\[ C_0^0(w) := 1, \] (3.6)
\[ C_l^0(w) := \lim_{\lambda \to 0} \left[ \frac{1}{\lambda} C_l^1(w) \right], \quad (l = 1, 2, 3, \ldots). \] (3.7)

Hence the generating function of the polynomials (3.7) is the derivative
\[ G_0^{(0)}(t, w) := \frac{\partial G_0^{(1)}(t, w)}{\partial \lambda} \bigg|_{\lambda=0} , \] (3.8)
while the functions studied here are given by the limit
\[ G_l^{(0)}(w) := \lim_{\lambda \to 0} \left[ \frac{1}{\lambda} G_l^{(1)}(t, w) \right], \quad (l = 1, 2, 3, \ldots). \] (3.9)

The generating function (3.8), whose explicit form is the principal determination of the logarithm
\[ G_0^{(0)}(w) = -\ln \left(1 - 2wt + t^2\right), \] (3.10)
is the sum of the power series [12]
\[ G_0^{(0)}(t, w) = \sum_{n=1}^{\infty} t^n \frac{2}{n} T_n(w), \quad (w := \cos \theta, \quad |w| < 1). \] (3.11)

In turn, eq. (2.4) leads us to write a function (3.9) as the sum of the series
\[ G_l^{(0)}(t, w) = \sum_{n=l}^{\infty} \binom{n}{l} t^n \frac{2}{n} T_n(w), \quad (w := \cos \theta, \quad |w| < 1, \quad l = 1, 2, 3, \ldots), \] (3.12)
while eq. (2.10) gives its explicit expression:
\[ G_l^{(0)}(t, w) = \left(\frac{t}{q}\right)^l \frac{2}{l} T_l\left(\frac{w-t}{q}\right), \quad (l = 1, 2, 3, \ldots). \] (3.13)

In particular, for \(l=1[13],\)
\[
\sum_{n=0}^{\infty} t^n T_n(w) = \frac{1-wt}{1-2wt+t^2}.
\] (3.14)

Equations (3.12) and (3.13) also read:
\[
\sum_{n=0}^{\infty} \binom{l+m}{l} t^m \frac{1}{l+m} T_{l+m}(w) = \left(\frac{1}{q}\right)^l \frac{1}{l} T_l\left(\frac{w-t}{q}\right), \quad (l=1,2,3,\ldots). \] (3.15)

The Chebyshev polynomials of the second kind,
\[
U_l(w) := \frac{\sin\left[(l+1)\theta\right]}{\sin\theta}, \quad (w := \cos\theta, \quad l=0,1,2,3,\ldots), \] (3.16)
are the Gegenbauer polynomials of order \(\lambda = 1\) [14]:
\[
U_l(w) = C_l(w). \] (3.17)

Accordingly, the sum of the series
\[
G_l^{(1)}(t,w) = \sum_{n=0}^{\infty} \binom{n}{l} t^n U_n(w), \quad (w := \cos\theta, \quad |t| < 1, \quad l=0,1,2,3,\ldots), \] (3.18)
is the special case \(\lambda = 1\) of eq. (2.10):
\[
G_l^{(1)}(t,w) = q^{-2} \left(\frac{1}{q}\right)^l U_l\left(\frac{w-t}{q}\right), \quad (l=0,1,2,3,\ldots). \] (3.19)

An equivalent formula is given by eq. (2.11) written for \(\lambda = 1\):
\[
\sum_{m=0}^{\infty} \binom{l+m}{l} t^m U_{l+m}(w) = \left(\frac{1}{q}\right)^{2+l} U_l\left(\frac{w-t}{q}\right). \] (3.20)

### 4. LAGUERRE POLYNOMIALS

For the sake of completeness, we prove the similar property of the Laguerre polynomials, repeating step by step the argument from Section 2. The generating function of the Laguerre polynomials of order \(\alpha\),
\[
S_0^{(\alpha)}(t,u) := (1-t)^{-\alpha} \exp\left(-\frac{ut}{1-t}\right), \quad (\alpha > -1), \] (4.1)
is holomorphic in the variable $t$ inside the circle $|t|=1$ and has the Taylor expansion

\[ S_0^{(\alpha)}(t,u) = \sum_{n=0}^{\infty} t^n L_n^{\alpha}(u), \quad (|t|<1). \]  

(4.2)

Making use of CIFD, we write the Laguerre polynomial $L_n^{\alpha}(u)$ of degree $l$ and order $\alpha$ as an integral representation[16]:  

\[ L_n^{\alpha}(u) = \frac{1}{2\pi i} \int_{(0)} ds \frac{1}{t^{\alpha+1}} S_0^{(\alpha)}(t,u). \]  

(4.3)

Just as in eq. (2.3), the contour of integration encircles the origin $t = 0$ once, counterclockwise, inside the unit circle. Now the power series under investigation,

\[ S_0^{(\alpha)}(t,u) = \sum_{n=0}^{\infty} \frac{n!}{l^n} t^n L_n^{\alpha}(u), \quad (|t|<1, \quad l = 0,1,2,3,\ldots), \]  

(4.4)

has the sum

\[ S_0^{(\alpha)}(t,u) = \left( \frac{t}{\partial t} \right)^l S_0^{(\alpha)}(t,u). \]  

(4.5)

Applying once again CIFD, we get with a translation of the variable of integration:

\[ S_0^{(\alpha)}(s+t,u) = \frac{t^l}{2\pi i} \int_{(0)} ds \frac{1}{s^{\alpha+1}} S_0^{(\alpha)}(s+t,u). \]  

(4.6)

The generating function (4.1) satisfies the functional equation

\[ S_0^{(\alpha)}(s+t,u) = S_0^{(\alpha)}(t,u) S_0^{(\alpha)}(s,1-t). \]  

(4.7)

Combining it with the change of variable

\[ \xi := \frac{s}{1-t} \]  

(4.8)

in the integral, eq. (4.6) becomes

\[ S_0^{(\alpha)}(t,u) = S_0^{(\alpha)}(t,u) \left( \frac{t}{1-t} \right)^l \frac{1}{2\pi i} \int_{(0)} ds \frac{1}{s^{\alpha+1}} S_0^{(\alpha)}(\xi, u/(1-t)). \]  

(4.9)

On account of eqs. (4.1) and (4.3), eq. (4.9) eventually yields the sum of the series (4.4):
\[ S_j^{(\alpha)}(t,u) = (1-t)^{-\alpha-1} \exp\left(-\frac{ut}{1-t}\right) \left(\frac{t}{1-t}\right)^j \left(\frac{u}{1-t}\right), \]  
(\alpha > -1, \ |t| < 1, \ i = 0,1,2,3,\ldots). \tag{4.10} \]

Note that eqs. (4.4) and (4.10) are equivalent to the formula preferred by Rainville [17]:

\[ \sum_{m=0}^{\infty} \binom{l+m}{l} t^m L_{l+m}^{(\alpha)}(u) = (1-t)^{-\alpha-1} \exp\left(-\frac{ut}{1-t}\right) \left(\frac{1}{1-t}\right)^j \left(\frac{u}{1-t}\right). \]  
(4.11) \]

5. HERMITE POLYNOMIALS

The generating function of the Hermite polynomials is the exponential

\[ W_0(z,x) := \exp\left(2xz - z^2\right). \]  
(5.1) \]

Being an entire function, the Taylor series [18]

\[ W_0(z,x) = \sum_{n=0}^{\infty} \frac{z^n}{n!} H_n(x) \]  
(5.2) \]

converges in the whole complex \( z \)-plane. The more general series we are interested in,

\[ W_l(z,x) := \sum_{n=0}^{\infty} \binom{n}{l} \frac{z^n}{n!} H_n(x), \quad (l = 0,1,2,3,\ldots), \]  
(5.3) \]

has the sum

\[ W_l(z,x) = \frac{z^l}{l!} \left(\frac{\partial}{\partial z}\right)^l W_0(z,x). \]  
(5.4) \]

In order to evaluate it, we apply the method of Secs. 2 and 4, taking advantage of the functional equation satisfied by the generating function (5.1):

\[ W_0(v+z,x) = W_0(z,x) W_0(v,x-z). \]  
(5.5) \]

The result is

\[ W_l(z,x) = \exp\left(2xz - z^2\right) \frac{t^l}{l!} H_l(x-z), \quad (l = 0,1,2,3,\ldots), \]  
(5.6) \]

and may be written in the equivalent form chosen by Rainville [19]:
\[ \sum_{m=0}^{\infty} \frac{z^m}{m!} H_{j+m} (x) = \exp \left( 2xz - z^2 \right) H_j(x - z). \] (5.7)

However, eq. (5.6) can be established as an asymptotic case of the similar formula either for the Gegenbauer polynomials, eq. (2.10), or for the Laguerre polynomials, eq. (4.10). In the first case, our starting point is the limit [20]

\[ W_0 (z, x) = \lim_{\lambda \to \infty} \left[ G^{(k)}_0 \left( \frac{1}{\lambda \frac{3}{2} z}, \lambda \frac{1}{2} x \right) \right]. \] (5.8)

Hence

\[ \frac{1}{l!} H_l (x) = \lim_{\lambda \to \infty} \left[ \frac{1}{\lambda \frac{3}{2} z} C^\lambda_l \left( \lambda \frac{1}{2} x \right) \right]. \] (5.9)

Making use of eqs. (5.4) and (2.5), we get

\[ W_i (z, x) = \lim_{\lambda \to \infty} \left[ G^{(k)}_i \left( \frac{1}{\lambda \frac{3}{2} z}, \lambda \frac{1}{2} x \right) \right]. \] (5.10)

Substitution of eqs. (2.10), (5.8), and (5.9) into eq. (5.10) yields again the result (5.6).

In the second case, we start from the identity [21]

\[ W_0 (z, x) = \lim_{\alpha \to \infty} \left\{ S^{(\alpha)}_0 \left( \frac{2}{\alpha} \frac{1}{\alpha} z, \alpha \left[ 1 - \left( \frac{2}{\alpha} \frac{1}{\alpha} \right) x \right] \right) \right\}, \] (5.11)

which gives the limit

\[ \frac{1}{l!} H_l (x) = \lim_{\alpha \to \infty} \left\{ \frac{2}{\alpha} \frac{1}{\alpha} l^\alpha \left[ \alpha \left[ 1 - \left( \frac{2}{\alpha} \frac{1}{\alpha} \right) x \right] \right] \right\}. \] (5.12)

On the other hand, taking note of eqs. (5.4) and (4.5), we find:

\[ W_i (z, x) = \lim_{\alpha \to \infty} \left\{ S^{(\alpha)}_i \left( \frac{2}{\alpha} \frac{1}{\alpha} z, \alpha \left[ 1 - \left( \frac{2}{\alpha} \frac{1}{\alpha} \right) x \right] \right) \right\}. \] (5.13)
By inserting eq. (4.10) as well as the limits (5.11) and (5.12) into eq. (5.13), we recover once more eq. (5.6).

6. A GENERALIZATION OF MEHLER’S SUMMATION FORMULA

We intend to evaluate the sum of the power series

$$M_l(z,x,y) := \sum_{n=0}^{\infty} \frac{n^l}{n!} \frac{z^n}{2^n} H_n(x)H_n(y), \quad (|z|<1, \ l = 0,1,2,3,\ldots). \quad (6.1)$$

Its special case $l = 0$ is Mehler’s power series expansion,

$$M_0(z,x,y) = \sum_{n=0}^{\infty} \frac{z^n}{n!} H_n(x)H_n(y), \quad (|z|<1), \quad (6.2)$$

of the generating function [22, 23, 24]

$$M_0(z,x,y) = \left(1-z^2\right)^{1/2} \exp\left\{\frac{1}{1-z^2}\left[2xyz - \left(x^2+y^2\right)z^2\right]\right\}. \quad (6.3)$$

The analytic function (6.3) factors in the following way [25]:

$$M_0(z,x,y) = S_0^{(-1/2)}(z,1/2(x-y)^2)S_0^{(-1/2)}(-z,1/2(x+y)^2), \quad (|z|<1), \quad (6.4)$$

where $S_0^{(-1/2)}(t,u)$ is the generating function (4.1) of the Laguerre polynomials of order $\alpha = -1/2$. On the other hand, the function (6.1) involves a $l$th-order partial derivative of Mehler’s generating function (6.3):

$$M_l(z,x,y) = \frac{z^l}{l!} \left(\frac{\partial}{\partial z}\right)^l M_0(z,x,y). \quad (6.5)$$

Insertion of the factorization (6.4) into eq. (6.5), followed by application of Leibniz’ rule and use of eq. (4.5) with $\alpha = -1/2$, leads us to a finite expansion:

$$M_l(z,x,y) = \sum_{m=0}^{l} S_l^{(-1/2)}\left(z,\frac{1}{2}(x-y)^2\right) S_m^{(-1/2)}\left(-z,\frac{1}{2}(x+y)^2\right), \quad (|z|<1, \ l = 0,1,2,3,\ldots). \quad (6.6)$$
By substituting the series sum (4.10) and employing again eq. (6.4), we establish
the following formula:

\[ M_l(z, x, y) = \left( \frac{z}{1 - z} \right)^l M_0(z, x, y) \]

\[ \times \sum_{m=0}^{l} \left( \frac{-1}{1 + z} \right)^m \frac{1}{L_m^2} \left( \frac{(x - y)^2}{2(1 - z)} \right) \frac{1}{L_m^2} \left( \frac{(x + y)^2}{2(1 + z)} \right), \]  

(6.7)

\[ (|z| < 1, \ l = 0, 1, 2, 3, \ldots). \]

The function (6.7) is a finite sum involving Laguerre polynomials of order 
\( \alpha = -\frac{1}{2} \). It can equally be expressed in terms of Hermite polynomials, via the
identity [26]

\[ H_{2m}(x) = (-1)^m (2^m m!) \left( \frac{x^2}{2^m} \right). \]

(6.8)

We get thus an alternative explicit formula:

\[ M_l(z, x, y) = \left( \frac{-1}{1 + z} \right)^l \left( \frac{z}{1 - z} \right)^l M_0(z, x, y) \]

\[ \times \sum_{m=0}^{l} \binom{l}{m} \left( \frac{-1}{1 + z} \right)^m H_{2(l - m)} \left( \frac{x - y}{2(1 - z)} \right) \frac{1}{L_m^2} \left( \frac{x + y}{2(1 + z)} \right), \]  

(6.9)

\[ (|z| < 1, \ l = 0, 1, 2, 3, \ldots). \]

7. THE PROPAGATOR OF A LINEAR HARMONIC OSCILLATOR

In nonrelativistic quantum mechanics it is instructive to evaluate
the propagator of a linear harmonic oscillator by employing Mehler’s summation
formula, eqs. (6.2)–(6.3). In the coordinate representation, the propagator is defined
as the probability amplitude for finding the particle at some point \( x_b \) at time \( t_b \),
when it has originally been at another point \( x_a \), at time \( t_a \):

\[ K(x_b, t_b; x_a, t_a) := \langle x_b, t_b | x_a, t_a \rangle. \]

(7.1)
For an oscillator of mass \( m \) and classical angular frequency \( \omega \) it is convenient to denote \( \alpha := \left( \frac{m\omega}{\hbar} \right)^{\frac{1}{2}} > 0 \). We recall the eigenvalues

\[
E_n = \left( n + \frac{1}{2} \right) \hbar \omega, \quad (n = 0, 1, 2, 3, \ldots),
\]  

(7.2)

as well as the corresponding eigenfunctions of its energy operator \( \hat{H} \),

\[
u_n(x) = \left( \frac{\alpha}{\pi^{1/4} n! 2^n} \right)^{1/2} \exp \left( -\frac{1}{2} \alpha^2 x^2 \right) H_n(\alpha x), \quad (n = 0, 1, 2, 3, \ldots).
\]  

(7.3)

Making use of the time-evolution operator in the Schrödinger picture,

\[
\hat{U}(t_b, t_a) = \exp \left[ -\frac{i}{\hbar}(t_b - t_a) \hat{H} \right],
\]  

(7.4)

the transition probability amplitude (7.1) reads:

\[
\langle x_b, t_b | x_a, t_a \rangle = \langle x_b | \hat{U}(t_b, t_a) | x_a \rangle.
\]  

(7.5)

Insertion of eq. (7.4) into eq. (7.5) yields, via the spectral decomposition of the energy \( \hat{H} \), the standard eigenfunction expansion of the propagator:

\[
K(x_b, t_b; x_a, t_a) = \sum_{n=0}^{\infty} \exp \left[ -i \frac{E_n}{\hbar}(t_b - t_a) \right] \nu_n(x_b) \nu_n^*(x_a).
\]  

(7.6)

Substitution of the energy levels (7.2) and eigenfunctions (7.3) into eq. (7.6) enables us to exploit Mehler’s expansion (6.2) to get a compact formula:

\[
K(x_b, t_b; x_a, t_a) = \frac{\alpha}{\pi^{1/4} \hbar} \exp \left[ -i \frac{\alpha^2}{2}(t_b - t_a) - \frac{1}{2} \alpha^2 (x_b^2 + x_a^2) \right]
\]  

\times M_0 \left( -i \alpha(x_b - x_a) \right).
\]  

(7.7)

Equations (6.3) and (7.7) then give the exact closed-form expression of the propagator of the one-dimensional harmonic oscillator [28]:

\[
K(x_b, t_b; x_a, t_a) = \left( \frac{-i \hbar}{2 \pi \sin[\omega(t_b - t_a)]} \right)^{1/2} \ exp \left[ -i \omega(t_b - t_a) - \frac{1}{2} \left( x_b^2 + x_a^2 \right) \cos \left( \omega(t_b - t_a) \right) - 2 x_b x_a \right].
\]  

(7.8)
Since Mehler’s summation of the eigenfunction series (7.6) does not always fulfill the convergence requirement from eq. (6.6), the propagator (7.8) is a distribution involving δ-type singularities when the time difference $t_b - t_a$ is a multiple of the classical half-period of oscillation $\pi/\omega$. For instance, eq. (7.5), and eq. (7.6) as well, displays the equal-time values

$$K(x_b, t; x_a, t) = \delta(x_b - x_a).$$  \hspace{1cm} (7.9)

We stress that the transition probability amplitudes of the type (7.1) are at heart of Feynman’s path-integral formulation of quantum mechanics. In particular, for the linear harmonic oscillator the propagator (7.1) has been evaluated exactly in this framework [29, 30]. However, the explicit formula (7.8) can also be obtained in an alternative simpler way [28]. Had one started from eq. (7.8), one could then find, conversely, the energy levels (7.2) and eigenfunctions (7.3). Indeed, insertion of Mehler’s power series (6.2) into eq. (7.7) allows one to compare the resulting formula with the general eigenfunction expansion (7.6). One thus readily gets the well-known solution of this energy eigenvalue problem.

8. CONCLUSIONS

The sum of a series of the type (1.1) involves the $l$th-order derivative of the generating function of the orthogonal polynomials. We have evaluated it in a straightforward way for both the Gegenbauer and Laguerre polynomials taking advantage of the analytic structure of their generating functions. The explicit sum is found to be a product of three factors: the generating function, the $l$th power of a specific variable, and the corresponding orthogonal polynomial of degree $l$ in another specific variable. However, the limiting case of the Chebyshev polynomials of the first kind stands out since their generating function is missing in the explicit formula. Note that the above-mentioned result is by no means general: it holds neither for arbitrary Jacobi polynomials, nor for some nonclassical orthogonal polynomials [27]. This is due to the analytic peculiarities of the generating functions in question. Along the same lines, we have succeeded in evaluating the higher-order derivatives of Mehler’s generating function as finite sums involving products of pairs of Hermite polynomials of even degrees, eq. (6.9). The importance of Mehler’s formula in quantum mechanics is finally emphasized.

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REFERENCES

1. Paulina Marian, Phys. Rev. A 45, 2044-2051 (1992), Appendix B.
2. Paulina Marian and Tudor A. Marian, Phys. Rev. A 47, 4487-4495 (1993), Appendix B.
3. Ref. [2], Appendix A and Section IV.
4. Earl D. Rainville, *Special Functions* (Macmillan, New York, 1960).
5. A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Higher Transcendental Functions* (McGraw-Hill, New York, 1953), Vol. 2, p. 177, eq. (29).
6. Lars V. Ahlfors, *Complex Analysis*, Third Edition, (McGraw-Hill, New York, 1979), pp. 114–123.
7. Einar Hille, *Analytic Function Theory*, Vol. 1, Second Edition (Chelsea, New York, 1982), pp. 175–182.
8. Ref. [4], p. 280, § 144, eq. (23).
9. Ref. [4], p. 169, § 95, eq. (7).
10. Ref. [5], Vol. 2, p. 184, eq. (5).
11. Ref. [5], Vol. 2, p. 174, eq. (6), where the limit should be read as $\lim_{\lambda \to 0}$.
12. Ref. [5], Vol. 2, p. 186, eq. (30): there is a mistake that can be corrected by omitting unity in the l.h.s. of this equation.
13. See Ref. [5], Vol. 2, p. 186, eq. (29): our eq. (3.14) is equivalent to this equation.
14. Ref. [5], Vol. 2, p. 184, eq. (6).
15. Ref. [5], Vol. 2, p. 189, eq. (17).
16. There is a mistake in eq. (B3) of Ref. [2]. The correct formula reads

$$L_{\nu}^{(s)}(u) = \frac{1}{2\pi t} \int_{-\infty}^{\infty} \frac{1}{e^{t^2} - 1} T_{\nu}^{(s)}(t,u)$$

and coincides with eq. (4.3) of the present work.

17. Ref. [4], p. 211, §119, eq. (9).
18. Ref. [5], Vol. 2, p. 194, eq. (19).
19. Ref. [4], p. 197, § 111, eq. (1).
20. George E. Andrews, Richard Askey, and Ranjan Roy, *Special Functions* (Cambridge University Press, Cambridge, UK, 2000), p. 306.
21. Ref. [20], p. 339.
22. Ref. [5], Vol. 2, p. 194, eq. (22).
23. Herbert Buchholz, *The Confluent Hypergeometric Function* (Springer, Berlin, 1969), p.147, eq. (12b). The original reference is: Ferdinand Gustav Mehler, *Über die Entwicklung einer Funktion von beliebig vielen Variablen nach Laplaceschen Funktionen höherer Ordnung*, J. reine angew. Math. 66, 161-176 (1866).
24. Ref. [20], pp. 280–282.
25. Ref. [23], p. 147, eq. (12b).
26. Ref. [5], Vol. 2, p. 193, eq. (2).
27. Ref. [20], pp. 348–350. See the generating functions of some nonclassical orthogonal polynomials listed there.
28. Eugen Merzbacher, *Quantum Mechanics*, Third Edition, (John Wiley, New York, 1998), p. 352, eq. (15.59).
29. R.P. Feynman and A.R. Hibbs, *Quantum Mechanics and Path Integrals*, (McGraw-Hill, New York, 1965), p. 198, eq. (8-1). Our eq. (7.8) coincides with their explicit expression of the propagator. In order to derive it, the authors of the book point out several intermediate steps to be achieved: p. 28, eq. (2-9); p. 63, eq. (3-59); p. 73, eq. (3-93).
30. Hagen Kleinert, *Path Integrals in Quantum Mechanics, Statistics, and Polymer Physics*, Second Edition, (World Scientific, Singapore, 1995), p. 93, eq. (2.148).