MARTINGALE PROPERTY OF GENERALIZED STOCHASTIC EXPONENTIALS

ALEKSANDAR MIJATOVIĆ, NIKA NOVAK, AND MIKHAIL URUSOV

Abstract. For a real Borel measurable function $b$, which satisfies certain integrability conditions, it is possible to define a stochastic integral of the process $b(Y)$ with respect to a Brownian motion $W$, where $Y$ is a diffusion driven by $W$. It is well know that the stochastic exponential of this stochastic integral is a local martingale. In this paper we consider the case of an arbitrary Borel measurable function $b$ where it may not be possible to define the stochastic integral of $b(Y)$ directly. However the notion of the stochastic exponential can be generalized. We define a non-negative process $Z$, called generalized stochastic exponential, which is not necessarily a local martingale. Our main result gives deterministic necessary and sufficient conditions for $Z$ to be a local, true or uniformly integrable martingale.

1. Introduction

A stochastic exponential of $X$ is a process $\mathcal{E}(X)$ defined by

$$\mathcal{E}(X)_t = \exp \left\{ X_t - X_0 - \frac{1}{2} \langle X \rangle_t \right\}$$

for some continuous local martingale $X$, where $\langle X \rangle$ denotes a quadratic variation of $X$. It is well known that the process $\mathcal{E}(X)$ is also a continuous local martingale. The characterisation of the martingale property of $\mathcal{E}(X)$ has been studied extensively in the literature because this question appears naturally in many situations.

In the case of one dimensional processes, necessary and sufficient conditions for the process $\mathcal{E}(X)$ to be a martingale were recently studied by Engelbert and Senf in [3], Blei and Engelbert in [1] and Mijatović and Urusov in [7]. In [3] $X$ is a general continuous local martingale and the characterisation is given in terms of the Dambis-Dubins-Schwartz time-change that turns $X$ into a Brownian motion. In [1] $X$ is a strong Markov continuous local martingale and the condition is deterministic, expressed in terms of the speed measure of $X$.

In [7] the local martingale $X$ is of the form $X_t = \int_0^t b(Y_u) \, dW_u$ for some measurable function $b$ and a one-dimensional diffusion $Y$ with drift $\mu$ and volatility $\sigma$ driven by a Brownian motion $W$. In order to define the stochastic integral $X$, an assumption that the function $\frac{b^2}{2\sigma^2}$ is locally integrable on the entire state space of the process $Y$ is required. Under this restriction the characterization of the martingale property of $\mathcal{E}(X)$ is studied in [7], where the necessary and sufficient conditions are deterministic and are expressed in terms of functions $\mu, \sigma$ and $b$ only.

In the present paper we consider an arbitrary Borel measurable function $b$. In this case the stochastic integral $X$ can only be defined on some subset of the probability space.

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However, it is possible to define a non-negative possibly discontinuous process $Z$, known as a generalized stochastic exponential, on the entire probability space. It is a consequence of the definition that, if the function $b$ satisfies the required local integrability condition, the process $Z$ coincides with $\mathcal{E}(X)$. We show that the process $Z$ is not necessarily a local martingale. In fact $Z$ is a local martingale if and only if it is continuous. We find a deterministic necessary and sufficient condition for $Z$ to be a local martingale, which is expressed in terms of local integrability of the quotient $\frac{b^2}{\sigma^2}$ multiplied by a linear function.

We also characterize the processes $Z$ that are true martingales and/or uniformly integrable martingales. All the necessary and sufficient conditions are deterministic and are given in terms of functions $\mu, \sigma$ and $b$.

The paper is structured as follows. In Section 2 we define the notion of generalized stochastic exponential and study its basic properties. The main results are stated in Section 3, where we give a necessary and sufficient condition for the process $Z$ defined by (3) and (6) to be a local martingale, a true martingale or a uniformly integrable martingale. Finally, in Section 4 we prove Theorem 3.4 that is central in obtaining the deterministic characterisation of the martingale property of the process $Z$. Appendix A contains an auxiliary fact that is used in Section 2.

2. Definition of Generalized Stochastic Exponential

Let $J = (l, r)$ be our state space, where $-\infty \leq l < r \leq \infty$. Let us define a $J$-valued diffusion $Y$ on a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, \infty)}, \mathbb{P})$ driven by a stochastic differential equation

$$dY_t = \mu(Y_t) \, dt + \sigma(Y_t) \, dW_t, \quad Y_0 = x_0 \in J,$$

where $W$ is a $(\mathcal{F}_t)$-Brownian motion and $\mu$ and $\sigma$ real, Borel measurable functions defined on $J$ that satisfy the Engelbert-Schmidt conditions

1. $\sigma(x) \neq 0 \quad \forall x \in J,$
2. $\frac{1}{\sigma^2}, \frac{\mu}{\sigma^2} \in L^1_{\text{loc}}(J).$

With $L^1_{\text{loc}}(J)$ we denote the class of locally integrable functions, i.e. real Borel measurable functions defined on $J$ that are integrable on every compact subset of $J$. Engelbert-Schmidt conditions guarantee existence of a weak solution that might exit the interval $J$ and is unique in law (see [5, Chapter 5]). Denote by $\zeta$ the exit time of $Y$. In addition, we assume that the boundary points are absorbing, i.e. the solution $Y$ stays at the boundary point at which it exits on the set $\{\zeta < \infty\}$. Let us note that we assume that $(\mathcal{F}_t)$ is generated neither by $Y$ nor by $W$.

We would like to define a process $X$ as a stochastic integral of a process $b(Y)$ with respect to Brownian motion $W$, where $b : J \to \mathbb{R}$ is an arbitrary Borel measurable function. Before further discussion, we should establish if the stochastic integral can be defined.

Define a set

$$A = \{x \in J; \frac{b^2}{\sigma^2} \notin L^1_{\text{loc}}(x)\},$$

where $L^1_{\text{loc}}(x)$ denote a set of real, Borel measurable functions $f$ such that $\int_{x-\varepsilon}^{x+\varepsilon} f(y) \, dy < \infty$ for some $\varepsilon > 0$. Then $A$ is closed and its complement is a union of open intervals. Let
\[ \tau_A = \inf\{t \geq 0; Y_t \in A\} \] and \( \zeta^A = \zeta \wedge \tau_A \). Then
\[ \int_0^t b^2(Y_u)\,du < \infty \quad \mathbb{P}\text{-a.s. on } \{t < \zeta^A\}. \]

This follows from Proposition [A.1] and the fact that a continuous process \( Y \) on \( \{t < \zeta^A\} \) reaches only values in an open interval that is a component of the complement of \( A \), where \( \frac{b^2}{2} \) is locally integrable.

Let us define \( A_n = \{x \in J; \rho(x, A \cup \{t, r\}) \leq \frac{1}{n}\} \), where \( \rho(x, y) = |\arctan x - \arctan y|, x, y \in J \), and set \( \zeta_n^A = \inf\{t \geq 0; Y_t \in A_n\} \). Since \( \zeta_n^A < \zeta^A \) on the set \( \{\zeta^A < \infty\} \), we have \( \int_0^{\zeta^A} b^2(Y_u)\,du < \infty \) \( \mathbb{P}\text{-a.s.} \). Thus, we can define the stochastic integral \( \int_0^{\zeta^A} b(Y_u)\,dW_u \) for every \( n \). Since the integrals \( \int_0^{\zeta^A} b(Y_u)\,dW_u \) and \( \int_0^{\zeta_{n+1}^A} b(Y_u)\,dW_u \) coincide on \( \{t < \zeta^A\} \) and \( \zeta_n^A \uparrow \zeta^A \), we can define \( \int_0^{\zeta^A} b(Y_u)\,dW_u \) as a limit of the integrals \( \int_0^{\zeta^A} b(Y_u)\,dW_u \).

In the case where \( A \) is not empty or \( Y \) exits the interval \( J \), the stochastic exponential cannot be defined. However, we can define a generalized stochastic exponential \( Z \) in the following way for every \( t \in [0, \infty) \)

\[
Z_t = \begin{cases} 
\exp\left\{ \int_0^t b(Y_u)\,dW_u - \frac{1}{2} \int_0^t b^2(Y_u)\,du \right\}, & t < \zeta^A \\
\exp\left\{ \int_0^\zeta b(Y_u)\,dW_u - \frac{1}{2} \int_0^\zeta b^2(Y_u)\,du \right\}, & t \geq \zeta^A = \zeta, \int_0^\zeta b^2(Y_u)\,du < \infty \\
0, & t \geq \zeta^A = \tau_A \text{ or } \zeta^A = \zeta, \int_0^\zeta b^2(Y_u)\,du = \infty
\end{cases}
\]

The different behaviour of \( Z \) on \( \{t \geq \zeta^A = \zeta\} \) and \( \{t \geq \zeta^A = \tau_A\} \) follows from the fact, that after the exit time \( \zeta \) the process \( Y \) is stopped, while this does not happen after \( \tau_A \). From the definition of the set \( A \), the integral \( \int_0^t b^2(Y_u)\,du \) is infinite for every \( t > \tau_A \). Therefore, we set \( Z = 0 \) on the set \( \{t \geq \zeta^A = \tau_A\} \).

Let us define the processes

\[
\tilde{Z}_t = \exp\left\{ \int_0^{t \wedge \zeta^A} b(Y_u)\,dW_u - \frac{1}{2} \int_0^{t \wedge \zeta^A} b^2(Y_u)\,du \right\},
\]

where we set \( \tilde{Z}_t = 0 \) for \( t \geq \zeta^A \) on \( \{\zeta^A < \infty, \int_0^{\zeta^A} b^2(Y_u)\,du = \infty\} \), and

\[
S_t = \exp\left\{ \int_0^{\tau_A} b(Y_u)\,dW_u - \frac{1}{2} \int_0^{\tau_A} b^2(Y_u)\,du \right\} \mathbb{1}_{\{t \geq \zeta^A = \tau_A, \int_0^{\tau_A} b^2(Y_u)\,du < \infty\}}.
\]

Then we can write

\[
Z = \tilde{Z} - S.
\]

Now \( Z \) is not necessarily a continuous process. Furthermore, \( \tilde{Z} \) is positive local martingale and therefore a supermartingale. The process \( S \) has increasing paths. Hence,

\[
\mathbb{E}[Z_t|\mathcal{F}_s] \leq \tilde{Z}_s - \mathbb{E}[S_t|\mathcal{F}_s] \leq \tilde{Z}_s - S_s = Z_s.
\]

It follows that \( Z \) is a supermartingale and we can define

\[
Z_{\infty} = \lim_{t \to \infty} Z_t.
\]

Remark. Note that we should not use (3) for \( t = \infty \) because in (3) \( Z_{\infty} \) is not well defined on \( \{\zeta^A = \infty\} \).
We may assume that \( x_0 \not\in A \). Otherwise, \( Z \equiv 0 \) and hence it is a martingale.

A path of the process \( Z \) defined by (3) and (6) is equal to a path of a stochastic exponential if \( \zeta^A = \infty \). Otherwise, if \( \zeta^A < \infty \), it has one of the following forms:

(i) \( \tau_A < \zeta \) and \( \int_0^{\tau_A} b^2(Y_t)dt < \infty \) (see Figure 1);

(ii) \( \zeta^A < \infty \) and \( \int_0^{\zeta} b^2(Y_t)dt = \infty \) (see Figure 2);

(iii) \( \zeta < \tau_A \) and \( \int_0^{\zeta} b^2(Y_t)dt < \infty \) (see Figure 3).
Figure 3. If $\zeta < \tau_A$, the process $Z$ is stopped after the exit time. Since $\int_0^\zeta b^2(Y_u)\,du$ is finite, $Z_t$ is equal to a positive constant for $t \geq \zeta$.

3. Main Results

The case $A = \emptyset$ was studied by Mijatović and Urusov in [7]. We generalize their result for the case where $A \neq \emptyset$.

3.1. The Case $A = \emptyset$. In this case we have

$$b^2 \sigma^2 \in L^1_{\text{loc}}(J).$$

The generalized stochastic exponential $Z$ defined by (3) and (6) can now be written as

$$Z_t = \exp \left\{ \int_0^{t \wedge \zeta} b(Y_u)\,dW_u - \frac{1}{2} \int_0^{t \wedge \zeta} b^2(Y_u)\,du \right\},$$

where we set $Z_t = 0$ for $t \geq \zeta$ on $\{ \zeta < \infty, \int_0^{\zeta} b^2(Y_u)\,du = \infty \}$. Note that in this case $Z$ is a local martingale.

Let us now define an auxiliary $J$-valued diffusion $\widetilde{Y}$ governed by the SDE

$$d\widetilde{Y}_t = (\mu + b\sigma) \left( \widetilde{Y}_t \right) dt + \sigma \left( \widetilde{Y}_t \right) d\widetilde{W}_t, \quad \widetilde{Y}_0 = x_0,$$

on some probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t \in [0,\infty)}, \tilde{\mathbb{P}})$. The coefficients $\mu + b\sigma$ and $\sigma$ satisfy Engelbert-Schmidt conditions since $\frac{b}{\sigma} \in L^1_{\text{loc}}(J)$ (this follows from (7)). Hence the SDE has a unique in law, possibly explosive, weak solution. As with diffusion $Y$, we denote by $\tilde{\zeta}$ the exit time of $\widetilde{Y}$ and assume that the boundary points are absorbing.
For an arbitrary $c \in J$ we define the scale functions $s, \tilde{s}$ and their derivatives $\rho, \tilde{\rho}$:

$$\rho(x) = \exp \left\{ - \int_c^x \frac{2\mu(y)}{\sigma^2(y)} \, dy \right\}, \quad x \in J,$$

$$\tilde{\rho}(x) = \rho(x) \exp \left\{ - \int_c^x \frac{2\sigma(b(y))}{\sigma(y)} \, dy \right\}, \quad x \in J,$$

$$s(x) = \int_c^x \rho(y) \, dy, \quad x \in J,$$

$$\tilde{s}(x) = \int_c^x \tilde{\rho}(y) \, dy, \quad x \in J.$$  \hspace{1cm} (8)

Denote by $L^1_{\text{loc}}(r^-)$ the set of all Borel measurable functions $f : J \to \mathbb{R}$ such that $\int_{r^-}^r |f(x)| \, dx$ is finite for some $\varepsilon > 0$. Similarly, we denote by $L^1_{\text{loc}}(l^+)$ the set of all Borel measurable functions such that $\int_l^{l+\varepsilon} |f(x)| \, dx$ is finite for some $\varepsilon > 0$.

We say that the endpoint $r$ is good if

$$s(r) < \infty \quad \text{and} \quad \frac{(s(r) - s)b^2}{\rho \sigma^2} \in L^1_{\text{loc}}(r^-).$$

It is equivalent to show that

$$\tilde{s}(r) < \infty \quad \text{and} \quad \frac{(\tilde{s}(r) - \tilde{s})b^2}{\tilde{\rho} \sigma^2} \in L^1_{\text{loc}}(r^-).$$

The endpoint $l$ is good if

$$s(l) > -\infty \quad \text{and} \quad \frac{(s - s(l))b^2}{\rho \sigma^2} \in L^1_{\text{loc}}(l^+),$$

or equivalently

$$\tilde{s}(l) > -\infty \quad \text{and} \quad \frac{(\tilde{s} - \tilde{s}(l))b^2}{\tilde{\rho} \sigma^2} \in L^1_{\text{loc}}(l^+).$$

If an endpoint is not good, we say it is bad. The good and bad endpoints were introduced in [7], where one can also find the proof of equivalences above.

We will use the following terminology:

$\tilde{Y}$ exits at $r$ means $\tilde{\mathbb{P}}(\tilde{\zeta} < \infty, \lim_{t \uparrow \tilde{\zeta}} \tilde{Y}_t = r) > 0$;

$\tilde{Y}$ exits at $l$ means $\tilde{\mathbb{P}}(\tilde{\zeta} < \infty, \lim_{t \downarrow \tilde{\zeta}} \tilde{Y}_t = l) > 0$.

Define

$$\tilde{v}(x) = \int_c^x \frac{\tilde{s}(x) - \tilde{s}(y)}{\tilde{\rho}(y) \sigma^2(y)} \, dy, \quad x \in J,$$  \hspace{1cm} (9)

and

$$\tilde{v}(r) = \lim_{x \uparrow r} \tilde{v}(x), \quad \tilde{v}(l) = \lim_{x \downarrow l} \tilde{v}(x).$$  \hspace{1cm} (10)

Feller’s test for explosions (see [5, Chapter 5, Theorem 5.29]) tells us that:
(i) \( \tilde{Y} \) exits at the boundary point \( r \) if and only if
\[ \bar{v}(r) < \infty. \]
It is equivalent to check (see [2, Chapter 4.1])
\[ \tilde{s}(r) < \infty \quad \text{and} \quad \frac{\tilde{s}(r) - \bar{s}}{\rho \sigma^2} \in L^1_{\text{loc}}(r-); \]
(ii) \( \tilde{Y} \) exits at the boundary point \( l \) if and only if
\[ \bar{v}(l) > -\infty, \]
which is equivalent to
\[ \tilde{s}(l) > -\infty \quad \text{and} \quad \frac{\bar{s} - \tilde{s}(l)}{\rho \sigma^2} \in L^1_{\text{loc}}(l+). \]

Remark. The endpoint \( r \) (resp. \( l \)) is bad whenever one of the processes \( Y \) and \( \tilde{Y} \) exits at \( r \) (resp. \( l \)) and the other does not.

**Theorem 3.1.** Let the functions \( \mu, \sigma \) and \( b \) satisfy conditions (1), (2) and (7). Then the process \( Z \) is a martingale if and only if \( \tilde{Y} \) does not exit at the bad endpoints.

**Theorem 3.2.** Let the functions \( \mu, \sigma \) and \( b \) satisfy conditions (1), (2) and (7). Then \( Z \) is a uniformly integrable martingale if and only if one of the conditions (a) – (d) below is satisfied:

(a) \( b = 0 \) a.e. on \( J \) with respect to the Lebesgue measure;
(b) \( r \) is good and \( \tilde{s}(l) = -\infty; \)
(c) \( l \) is good and \( \tilde{s}(r) = \infty; \)
(d) \( l \) and \( r \) are good.

3.2. The Case \( A \neq \emptyset \). The following example shows that even when \( A \) is not empty we can get a martingale or a uniformly integrable martingale defined by \( [3] \) and \( [6] \).

**Example 3.3.** (i) Let us consider the case \( J = \mathbb{R}, \mu = 0, \sigma = 1 \) and \( b(x) = \frac{1}{x} \). Then \( A = \{0\} \) and \( Y_t = W_t, W_0 = x_0 > 0 \). Using Itô’s formula and the fact that Brownian motion does not exit at infinity, we get for \( t < \tau_0 \)
\[
Z_t = \exp \left\{ \int_0^t \frac{1}{W_u} \, dW_u - \frac{1}{2} \int_0^t \frac{1}{W_u^2} \, du \right\} \\
= \frac{1}{x_0} W_t
\]
and \( Z_t = 0 \) for \( t \geq \tau_0 \). Hence, \( Z_t = \frac{1}{x_0} W_{t \wedge \tau_0} \) that is a martingale.

(ii) Using the same functions \( \mu, \sigma \) and \( b \) as above on a state space \( J = (-\infty, x_0 + 1) \) we get
\[
Z_t = \frac{1}{x_0} W_{t \wedge \tau_{0,x_0+1}},
\]
which is a uniformly integrable martingale.

Define maps \( \alpha \) and \( \beta \) on \( J \setminus A \) so that
\[
(11) \quad \alpha(x), \beta(x) \in A \cup \{l, r\} \quad \text{and} \quad x \in (\alpha(x), \beta(x)) \subset J \setminus A.
\]
So, $\alpha(x)$ is the point in $A$ that is closest to $x$ from the left side and $\beta(x)$ is the closest point in $A$ from the right side. Then $\frac{b^2}{\sigma^2} \in L_1^{\text{loc}}(\alpha(x), \beta(x))$. Therefore, on $(\alpha(x), \beta(x))$ functions $\mu, \sigma$ and $b$ satisfy the same conditions as in previous subsection.

We can define an auxiliary diffusion $\tilde{Y}$ with values in $(\alpha(x_0), \beta(x_0))$ driven by the SDE

$$d\tilde{Y}_t = (\mu + b\sigma)(\tilde{Y}_t)\,dt + \sigma(\tilde{Y}_t)\,d\tilde{W}_t, \quad \tilde{Y}_0 = x_0,$$

on some probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t \in [0, \infty)}, \tilde{\mathbb{P})}$. There exists a unique weak solution of this equation since coefficients satisfy the Engelbert-Schmidt conditions.

As in the previous subsection we can define good and bad endpoints. We say that the endpoint $\beta(x_0)$ is good if

$$s(\beta(x_0)) < \infty \quad \text{and} \quad \frac{(s(\beta(x_0)) - s)b^2}{\rho\sigma^2} \in L_1^{\text{loc}}(\beta(x_0)-).$$

It is equivalent to show and sometimes easier to check that

$$\tilde{s}(\beta(x_0)) < \infty \quad \text{and} \quad \frac{(\tilde{s}(\beta(x_0)) - \tilde{s})b^2}{\tilde{\rho}\tilde{\sigma}^2} \in L_1^{\text{loc}}(\beta(x_0)-).$$

The endpoint $\alpha(x_0)$ is good if

$$s(\alpha(x_0)) > -\infty \quad \text{and} \quad \frac{(s - s(\alpha(x_0)))b^2}{\rho\sigma^2} \in L_1^{\text{loc}}(\alpha(x_0)+),$$

or equivalently

$$\tilde{s}(\alpha(x_0)) > -\infty \quad \text{and} \quad \frac{\tilde{s} - \tilde{s}(\alpha(x_0)))b^2}{\tilde{\rho}\tilde{\sigma}^2} \in L_1^{\text{loc}}(\alpha(x_0)+).$$

If an endpoint is not good, we say it is bad.

**Remark.** Functions $\rho, \tilde{\rho}, s, \tilde{s}$ and $\tilde{v}$ are defined as in [8], [9] and (10). Now, we only need to take $c$ from the interval $(\alpha(x_0), \beta(x_0))$.

Define a set

$$B = \left\{ x \in A; \int_0^{\tau_x} b^2(Y_t)\,dt = \infty \; \mathbb{P}\text{-a.s.} \right\},$$

where $\tau_x = \inf\{t \geq 0; Y_t = x\}$. The following theorem characterizes the set $B$ in a deterministic way.

**Theorem 3.4.** Let $\alpha$ be the function defined by (11), $\alpha(x_0) > l$ and let us write shortly $\alpha = \alpha(x_0)$. Then:

(a) $$(x - \alpha)\frac{b^2}{\sigma^2}(x) \in L_1^{\text{loc}}(\alpha+) \iff \int_0^{\tau_x} b^2(Y_t)\,dt < \infty \; \mathbb{P}\text{-a.s. on } \{\tau_\alpha = \tau_A < \infty\};$$

(b) $$(x - \alpha)\frac{b^2}{\sigma^2}(x) \notin L_1^{\text{loc}}(\alpha+) \iff \int_0^{\tau_x} b^2(Y_t)\,dt = \infty \; \mathbb{P}\text{-a.s. on } \{\tau_\alpha = \tau_A < \infty\}.$$

Note that the assertions (a) and (b) in Theorem 3.4 are not the negation of each other. If the integral $\int_0^{\tau_x} b^2(Y_t)\,dt$ is not finite $\mathbb{P}$-a.s. on $\{\tau_\alpha = \tau_A < \infty\}$, then it is infinite on some subset of $\{\tau_\alpha = \tau_A < \infty\}$ with positive probability. Observe that $\mathbb{P}(\tau_\alpha = \tau_A < \infty) > 0$.

Clearly, Theorem 3.4 has its analogue for $\beta(x_0) < r$.

Now we can show when a generalized stochastic exponential is a local martingale and when it is a true martingale.
Theorem 3.5. (i) The generalized stochastic exponential $Z$ is a local martingale if and only if $\alpha(x_0), \beta(x_0) \in B \cup \{l, r\}$.

(ii) The generalized stochastic exponential $Z$ is a martingale if and only if $Z$ is a local martingale and at least one of the conditions (a)-(b) below is satisfied and at least one of the conditions (c)-(d) below is satisfied:

(a) $\tilde{Y}$ does not exit at $\beta(x_0)$, i.e. $\tilde{v}(\beta(x_0)) = \infty$ or equivalently,

$$
\tilde{s}(\beta(x_0)) = \infty \quad \text{or} \quad \left( \tilde{s}(\beta(x_0)) < \infty \quad \text{and} \quad \frac{\tilde{s}(\beta(x_0)) - \bar{s}}{\bar{\rho} \sigma^2} \notin L^1_{\text{loc}}(\beta(x_0)) \right);
$$

(b) $\beta(x_0)$ is good,

(c) $\tilde{Y}$ does not exit at $\alpha(x_0)$, i.e. $\tilde{v}(\alpha(x_0)) = -\infty$ or equivalently,

$$
\tilde{s}(\alpha(x_0)) = -\infty \quad \text{or} \quad \left( \tilde{s}(\alpha(x_0)) > -\infty \quad \text{and} \quad \frac{\bar{s} - \tilde{s}(\alpha(x_0))}{\bar{\rho} \sigma^2} \notin L^1_{\text{loc}}(\alpha(x_0)) \right);
$$

(d) $\alpha(x_0)$ is good.

Remark. Part (ii) of Theorem 3.5 says that $Z$ is a martingale if and only if the $(\alpha(x_0), \beta(x_0))$-valued process $\tilde{Y}$ can exit only at the good endpoints.

Proof. (i) We can write $Z = \tilde{Z} - S$ as in (5). Since $\left( \int_0^{t \wedge \zeta} b(Y_u) \, dW_u \right)_t$ is a continuous local martingale, the process $\tilde{Z}$ is a continuous local martingale. Suppose that $Z$ is a local martingale. Then $S$ can be written as a sum of two local martingales and therefore, it is also a local martingale. It follows that $S$ is a supermartingale (since it is positive). Since $\zeta > 0$ and $S_0 = 0$, $S$ should be almost surely equal to 0. This happens if and only if $\alpha(x_0), \beta(x_0) \in B \cup \{l, r\}$.

(ii) To get at least a local martingale $S$ needs to be zero $\mathbb{P}$-a.s. Then $Z = \tilde{Z}$. Since the values of $Y$ on $[0, \zeta]$ do not exit the interval $(\alpha(x_0), \beta(x_0))$, the conditions of Theorem 3.1 are satisfied and the result follows.

Similarly, we can characterize uniformly integrable martingale. We can use characterization in Theorem 3.2 for the process $\tilde{Z}$ defined by (4). As above, for $\alpha(x_0), \beta(x_0) \in B \cup \{l, r\}$ the process $Z$ defined by (3) and (6) coincides with $\tilde{Z}$. Otherwise, $Z$ is not even a local martingale.

Theorem 3.6. The process $Z$ is a uniformly integrable martingale if and only if $Z$ is a local martingale and at least one of the conditions (a) – (d) below is satisfied:

(a) $b = 0$ a.e. on $(\alpha(x_0), \beta(x_0))$ with respect to the Lebesgue measure;

(b) $\alpha(x_0)$ is good and $\tilde{s}(\beta(x_0)) = \infty$;

(c) $\beta(x_0)$ is good and $\tilde{s}(\alpha(x_0)) = -\infty$;

(d) $\alpha(x_0)$ and $\beta(x_0)$ are good.

Remark. If $\alpha(x_0) \in B$, then $\alpha(x_0)$ is not a good endpoint. Indeed, if $s(\alpha(x_0)) > -\infty$, then we can write

$$
\frac{(s(x) - s(\alpha(x_0)))b^2(x)}{\rho(x)\sigma^2(x)} = \frac{(s(x) - s(\alpha(x_0)))(x - \alpha(x_0))}{(x - \alpha(x_0))\rho(x)}(x - \alpha(x_0)) \frac{b^2}{\sigma^2(x)}.
$$

The first fraction is bounded away from zero, since it is continuous for $x > \alpha(x_0)$ and has a limit equal to 1 as $x$ approaches $\alpha(x_0)$. Therefore, $\frac{(s-s(\alpha(x_0)))b^2}{\rho \sigma^2} \notin L^1_{\text{loc}}(\alpha(x_0)+)$.

Similarly, $\beta(x_0) \in B$ implies that $\beta(x_0)$ is not a good endpoint.
This remark simplifies the application of Theorems 3.5 and 3.6 in specific situations.

4. PROOF OF THEOREM 3.4

For the proof of Theorem 3.4 we first consider the case of Brownian motion. Let \( W \) be a Brownian motion with \( W_0 = x_0 \). Denote by \( L^y_t(W) \) a local time of \( W \) at time \( t \) and level \( y \). Let \(-\infty < \alpha < x_0\) and consider a Borel function \( b : (\alpha, \infty) \rightarrow \mathbb{R} \) that is locally integrable on the interval \((\alpha, \infty)\).

**Lemma 4.1.** If \((x - \alpha)b^2(x) \in L^1_{\text{loc}}(\alpha+),\) then \( \int_0^{\tau_n} b^2(W_t) \, dt < \infty \) \( \mathbb{P} \)-a.s.

**Proof.** Let \((\beta_n)\) be an increasing sequence such that \( x_0 < \beta_n < \infty \) and \( \beta_n \uparrow \infty \). By [9, Chapter VII, Corollary 3.8] we get

\[
\mathbb{E}_{x_0} \left[ \int_0^{\tau_n \wedge \tau_{\beta_n}} b^2(W_t) \, dt \right] = 2\frac{\beta_n - x_0}{\beta_n - \alpha} \int_0^{x_0} (y - \alpha)b^2(y) \, dy + 2\frac{x_0 - \alpha}{\beta_n - \alpha} \int_{x_0}^{\tau_n} (\beta_n - y)b^2(y) \, dy
\]

for every \( \beta_n \). Both integrals are finite since \( b^2 \in L^1_{\text{loc}}(\alpha, \infty) \) and \((x - \alpha)b^2(x) \in L^1_{\text{loc}}(\alpha+)\). Thus, we have \( \mathbb{E}_{x_0}[\int_0^{\tau_n \wedge \tau_{\beta_n}} b^2(W_t) \, dt] < \infty \) and therefore \( \int_0^{\tau_n \wedge \tau_{\beta_n}} b^2(W_t) \, dt < \infty \) almost surely for every \( n \).

Since \( \{\tau_n < \tau_{\beta_n}\} \uparrow \{\tau_n < \infty\} \) almost surely as \( n \) tends to infinity and \( \mathbb{P}(\{\tau_n < \infty\}) = 1 \), we get

\[
\int_0^{\tau_n} b^2(W_t) \, dt < \infty \quad \mathbb{P}\text{-a.s.},
\]

which concludes the proof. \( \square \)

**Lemma 4.2.** If \( \int_0^{\tau_n} b^2(W_t) \, dt < \infty \) on a set \( U \) with \( \mathbb{P}(U) > 0 \), then \((x - \alpha)b^2(x) \in L^1_{\text{loc}}(\alpha+)\).

**Proof.** The idea of the proof comes from [4]. Using the occupation times formula we can write

\[
\int_0^{\tau_n} b^2(W_t) \, dt = \int_0^{\infty} b^2(y)L^y_{\tau_n}(W) \, dy \\
\geq \int_0^{x_0} b^2(y)L^y_{\tau_n}(W) \, dy.
\]

Let us define a process \( R_y = \frac{1}{y - \alpha}L^y_{\tau_n}(W) \). Then \( R \) is positive and we have

\[
\int_0^{\tau_n} b^2(W_t) \, dt \geq \int_0^{x_0} R_y(y - \alpha)b^2(y) \, dy.
\]

By [9 Chapter VI, Proposition 4.6], Laplace transform of \( R_y \) is

\[
\mathbb{E}[^{-\lambda R_y}] = \frac{1}{1 + 2\lambda} \quad \text{for every } y.
\]

Hence, every random variable \( R_y \) has exponential distribution with \( \mathbb{E}[R_y] = 2 \).

Denote by \( L \) an indicator function of a measurable set. We can write

\[
\mathbb{E}[LR_y] = \mathbb{E} \left[ L \int_0^{\infty} 1_{(R_y > u)} \, du \right] = \int_0^{\infty} \mathbb{E}[L1_{(R_y > u)}] \, du.
\]
By Jensen’s inequality we get a lower bound for the integrand
\[ \mathbb{E}[L \mathbbm{1}_{\{R_y > u\}}] = \mathbb{E}[(L - \mathbbm{1}_{\{R_y \leq u\}})^+] \]
\[ \geq (\mathbb{E}[L] - \mathbb{P}[R_y \leq u])^+ \]
\[ = (\mathbb{E}[L] + e^{-\frac{u}{2}} - 1)^+. \]

Hence,
\[ (13) \quad \mathbb{E}[LR_y] \geq \int_0^\infty (\mathbb{E}[L] + e^{-\frac{u}{2}} - 1)^+ du = C, \]

where \( C \) is a strictly positive constant if \( \mathbb{E}[L] \) is strictly positive.

Then we choose \( L \), so that \( \mathbb{E}[L \int_0^\tau b^2(W_t) \, dt] \) is finite. Using Fubini’s Theorem and inequalities (12) and (13), we get
\[ \mathbb{E} \left[ L \int_0^\tau b^2(W_t) \, dt \right] \geq \int_0^\infty \mathbb{E}[LR_y](y - \alpha)b^2(y) \, dy \]
\[ \geq C \int_\alpha^{x_0} (y - \alpha)b^2(y) \, dy. \]

Therefore, \((y - \alpha)b^2(y) \in L^1_{\text{loc}}(\alpha+)\) if we can find an indicator function \( L \) such that \( \mathbb{E}[L] \) is strictly positive and \( \mathbb{E}[L \int_0^\tau b^2(W_t) \, dt] \) is finite.

Since \( \int_0^\tau b^2(W_t) \, dt < \infty \) on a set with positive measure, such \( L \) exists. Indeed, denote by \( L_n \) an indicator function of the set \( U_n = \{ \int_0^\tau b^2(W_t) \, dt \leq n \} \). Then \( \mathbb{E}[L_n \int_0^\tau b^2(W_t) \, dt] < \infty \) for every integer \( n \). Since the sequence \( (U_n)_{n \in \mathbb{N}} \) is increasing, \( U \subseteq \bigcup_{n \in \mathbb{N}} U_n \) and \( \mathbb{P}(U) > 0 \), there exists an integer \( N \) such that \( \mathbb{P}(U_N) > 0 \) and therefore \( \mathbb{E}[L_N] > 0 \). \( \square \)

Now we return to the setting of Section 2.

**Proof of Theorem 3.4.** First, suppose that \( \mu = 0 \) and \( \sigma = 1 \). In this case our diffusion \( Y_t \) is equal to a (possibly stopped) Brownian motion \( W_t \) with \( W_0 = x_0 \). The equivalences in (a) and (b) follow from Lemmas 4.1 and 4.2.

Now suppose that \( \mu = 0 \) and \( \sigma \) is arbitrary. Since \( Y_t \) is a continuous local martingale, by Dambis–Dubins–Schwartz we have \( Y_t = B_{\langle Y \rangle_t} \) for a Brownian motion \( B \) with \( B_0 = Y_0 \). Using the substitution \( s = \langle Y \rangle_t \), we get
\[ \int_0^\tau b^2(Y_t) \, dt = \int_0^\tau \frac{b^2}{\sigma^2}(Y_t) \, d\langle Y \rangle_t \]
\[ = \int_0^{\langle Y \rangle_\tau} \frac{b^2}{\sigma^2}(B_s) \, ds. \]

Since \( B_{\langle Y \rangle_\tau} = Y_\tau = \alpha \) and \( \langle Y \rangle_\tau = \inf \{ s \geq 0; B_s = \alpha \} \), we can use the first part of the proof to show the assertions.

It only remains to prove the general case when both \( \mu \) and \( \sigma \) are arbitrary. Let \( Z_t = s(Y_t) \), where \( s \) is the scale function of \( Y \). Then \( Z \) satisfies SDE
\[ dZ_t = \tilde{\sigma}(Z_t) \, dW_t, \]
where \( \tilde{\sigma}(x) = s'(q(x))\sigma(q(x)) \) and \( q \) is the inverse of \( s \).
Define \( \tilde{b} = b \circ q \). Since \( s \) is increasing and \( Z_{\tau_\alpha} = s(Y_{\tau_\alpha}) = s(\alpha) \), we can also show that \( \tau_\alpha(Y) = \tau_{s(\alpha)}(Z) \). Then we have

\[
\int_0^{\tau_\alpha(Y)} b^2(Y_t) \, dt = \int_0^{\tau_{s(\alpha)}(Z)} \tilde{b}^2(Z_t) \, dt.
\]

Besides,

\[
\int_{s(\alpha)}^{s(\alpha + \epsilon)} \frac{\tilde{b}^2(x)}{\sigma^2(x)} (x - s(\alpha)) \, dx = \int_{\alpha}^{\alpha + \epsilon} \frac{b^2(y)}{\sigma^2(y)} (s(y) - s(\alpha)) \frac{s'(y)}{y - \alpha} \, dy.
\]

Fraction \( \frac{(s(y) - s(\alpha))s'(y)}{y - \alpha} \) is continuous for \( y > \alpha \) and has a positive limit in \( \alpha \). Hence it is bounded and bounded away from zero. It follows that \( (x - \alpha) \frac{b^2}{\sigma^2} (x) \in L^1_{loc}(\alpha+) \) if and only if \( (x - s(\alpha)) \frac{\tilde{b}^2}{\sigma^2} (x) \in L^1_{loc}(s(\alpha)+) \). Then the result follows from the second part of the proof. \( \square \)

**Appendix A.**

Let \( Y \) be a \( J \)-valued diffusion with a drift \( \mu \) and volatility \( \sigma \) that satisfy Engelbert-Schmidt conditions. Let \( b : J \to \mathbb{R} \) be a Borel-measurable function and let \( (c, d) \subseteq J \).

**Proposition A.1.** A condition

\[
\frac{b^2}{\sigma^2} \in L^1_{loc}(c, d)
\]

is equivalent to

\[
\int_{t}^{\tau_{c,d}} b^2(Y_u) \, du < \infty \text{ \( \mathbb{P} \)-a.s. on \( \{t < \tau_{c,d}\} \).
\]

**Proof.** Using the occupation times formula we get

\[
\int_{t}^{\tau_{c,d}} b^2(Y_u) \, du = \int_{0}^{\tau_{c,d}} \frac{b^2(Y_u)}{\sigma^2(Y_u)} \, d\langle Y \rangle_u
\]

\[= \int_{c}^{d} \frac{b^2(y)}{\sigma^2(y)} L^y_t(Y) \, dy.\]

Suppose first that \( \frac{b^2}{\sigma^2} \in L^1_{loc}(c, d) \). Since the function \( y \mapsto L^y_t(Y) \) is cádlág (see [9, Chapter VI, Theorem 1.7]), it is bounded for every \( t < \tau_{c,d} \) and has a compact support in \( (c, d) \). The implication follows directly.

Suppose now that \( \frac{b^2}{\sigma^2} \notin L^1_{loc}(c, d) \). Then there exists such \( \alpha \in (c, d) \) that we have

\[
\int_{\alpha - \epsilon}^{\alpha + \epsilon} \frac{b^2(y)}{\sigma^2(y)} \, dy = \infty \text{ for all } \epsilon > 0.
\]

It is well known that \( \mathbb{P}(\tau_\alpha < \infty) > 0 \), where \( \tau_\alpha = \{t \geq 0; Y_t = \alpha\} \). By [2, Theorem 2.7], we have

\[
L^\alpha_t(Y) > 0 \text{ and } \lim_{y \uparrow \alpha} L^y_t(Y) > 0 \text{ \( \mathbb{P} \)-a.s.}
\]

for any \( t \geq 0 \) on the set \( \{t > \tau_\alpha\} \). Then there exists \( \epsilon > 0 \), such that the function \( y \mapsto L^y_t(Y) \) is bounded away from zero \( \mathbb{P} \)-a.s. on \( \{t > \tau_\alpha\} \) on the interval \( (\alpha - \epsilon, \alpha + \epsilon) \).

It follows that \( \int_{t}^{\tau_{\alpha}} b^2(Y_u) \, du = \infty \) on \( \{t > \tau_\alpha\} \), which proves the assumption. \( \square \)
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Department of Mathematics, Imperial College London
E-mail address: a.mijatovic@imperial.ac.uk

Faculty of Mathematics and Physics, University of Ljubljana and Department of Mathematics, Imperial College London
E-mail address: nika.novak@fmf.uni-lj.si

Institute of Mathematical Finance, Ulm University, Germany
E-mail address: mikhail.urusov@uni-ulm.de