Spherical P-spin glass at $P \to \infty$ and the information storing by continuous spins

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Abstract

The $P \to \infty$ limit was considered in the spherical P-spin glass. It is possible to store information in the vacuum configuration of ferromagnetic phase. Maximal allowed level of noise was calculated in ferromagnetic phase.

Derrida’s model [1,2] has been applied for optimal coding [3-6]. One chooses couplings $J_{i_1 \cdots i_p}^0$ for the N spin hamiltonian $H(\sigma)$ to have single and given vacuum configuration $\{\xi_i\}$:

$$J_{i_1 \cdots i_p}^0 = \xi_{i_1} \cdots \xi_{i_p} \quad (1)$$

$$H = - \sum_{1 \leq i_1 < i_2 \cdots < i_p \leq N} \sum_{k=1}^{\alpha N} C_{i_1 \cdots i_p}^k J_{i_1 \cdots i_p}^0 \sigma_{i_1} \cdots \sigma_{i_p} \quad (2)$$

where $C_{i_1 \cdots i_p}^k$ is a connectivity matrix. It has only one nonzero element (equal to 1) at any $k$ for some choice of indices $(i_1 \cdots i_p)$. Spins $\sigma_i, \xi_i$ taking values $\pm 1$. Our hamiltonian $H(\sigma)$ has a minimum at the configuration $\{\sigma_i\} = \{\xi_i\}$.

Our original message (with length N) $A$ has been proven in [4], it stays a true vacuum, even when one makes our couplings noisy (our original couplings $J_{i_1 \cdots i_p}^0$ with the probability $\frac{1+m}{2}$ stay correct and with probability $\frac{1-m}{2}$ change their sign), if

$$\alpha [\ln 2 + \frac{1+m}{2} \ln \frac{1+m}{2} + \frac{1-m}{2} \ln \frac{1-m}{2}] \geq \ln 2 \quad (3)$$
This inequality, derived as a condition for the existence of ferromagnetic phase of the model (1),(2) in the limit $P \to \infty$ coincides with Shannon inequality from information theory. Only Derrida’s hamiltonian saturates Shannon limit. For the other choice of hamiltonian $H(\sigma)$ one needs in more couplings (than $\alpha N$ from (3)) to support ferromagnetic phase with the full magnetization $< \sigma_i \xi_i > \sim 1$.

Our original message $\{\xi_i\}$ (with length N) was transformed into encoded message (with length $\alpha N$) $\{J^0_{i_1,..i_P}\}$. So we performed coding. One can extract(decoding) our original message $\{\xi_i\}$ (decoding) by vacuum search of hamiltonian (even with noisy couplings). When with probability close to 1 vacuum of this hamiltonian is our old one, our coding scheme is succesfull. On the language of statistical physics we need in a full magnetization (at low temperatures) at our given configuration. To suppress finite size corrections we need in large $P$.

Is it possible to construct similar hamiltonian for the continuous spins?

Let us consider $N$ spins $\sigma_i$ under constraint of

$$\sum_{i=1}^{N} (\sigma_i)^2 = N$$

and the same for the $\xi_i$. Again we consider hamiltonian (1),(2). If our connectivity matrix has been chosen symmetric by $i_\alpha$ and one enlareges sum by indices $i_\alpha$ in (2) till $1 \leq i_\alpha \leq N$, then $H(\sigma)$ goes to

$$H(\sigma) = -\left(\sum_{i} \sigma_i \xi_i\right)P\frac{\alpha N}{NP}$$

This function has a minimum at $\{\sigma_i\} = \{\xi_i\}$. In the limit $N \gg P \gg 1$ it will be the minimum of the hamiltonian (2) with the constraint sum for indices in (2) (instead of $1 \leq i_\alpha \leq N$).

To search something similar to inequality (3) one needs to find phase structure of spherical P-spins glass [7-10]. The static limit of model was solved in [8], we need in a little modification of their calculation to consider our case $P \to \infty$ and ferromagnetic phase.

We see, that our hamiltonian has a correct minimum, even in the class of continuous spins with spherical constraint for the homogeneous choice of connectivity matrix $C$. The
situation is different from the case of discrete spins, where the condition of homogeneity is not necessary.

The situation will be the same for NN models (again one could look for minimum in the class of continuous spins with spherical constraint).

To solve the case of nontrival connectivity matrix we need in a solution of diluted spherical P-spin glass. As a first step let us consider the simple case of fully connected model (matrix $C$ disappears).

Let us consider hamiltonian

$$H = - \sum_{1 \leq i_1 < i_2 < \ldots < i_p \leq N} (J_0 N/C_N^p + J_{i_1 \ldots i_p} \sqrt{N/C_N^p}) \sigma_{i_1} \ldots \sigma_{i_p}$$  \hspace{1cm} (6)$$

where $J_0$ -ferromagnetic coupling and quenched couplings $J_{i_1 \ldots i_p}$ (noise) have a 0 mean and variance $<(j_{i_1 \ldots i_p})^2> = J^2/2$. The signal/noise ratio $J_0/J$ is similar to $m$ in (4).

Calculating $Z^n$ by means of replica trick, as in [8], we derive

$$Z^n = \int_{-\infty}^{i\infty} \prod_{\alpha < \beta} N dQ_{\alpha \beta} d\lambda_{\alpha \beta} \prod_{\alpha} \int_{-\infty}^{i\infty} \frac{\sqrt{N}}{2\pi} d\lambda_{\alpha \alpha} \int_{-\infty}^{i\infty} \frac{dt_{\alpha} dm_{\alpha}}{2\pi} \exp(NG)$$  \hspace{1cm} (7)$$

where

$$G = J_0 B \sum_{\alpha} (m_{\alpha})^P + \frac{B^2 J^2}{4} \sum_{\alpha, \beta} (q_{\alpha \beta})^P - \frac{1}{2} \sum_{\alpha, \beta} q_{\alpha \beta} \lambda_{\alpha \beta} \sum_{\alpha} t_{\alpha} m_{\alpha} + \frac{1}{2} \ln 2\pi + \ln \int_{-\infty}^{i\infty} \prod_{\alpha} dx_{\alpha} \exp\left\{ \frac{1}{2} \sum_{\alpha, \beta} \lambda_{\alpha \beta} x_{\alpha} x_{\beta} + \sum_{\alpha} t_{\alpha} x_{\alpha} \right\}$$  \hspace{1cm} (8)$$

In this expression $m_{\alpha} = <x_{\alpha}>, q_{\alpha \beta} = <x_{\alpha} x_{\beta}>, t_{\alpha}$ and $\lambda_{\alpha \beta}$ are conjugate for them. Equation (4) gives $q_{\alpha \alpha} = 1$.

After integrating by $x_{\alpha}, t_{\alpha}$, we have

$$G = \frac{1}{2} \ln 2\pi + J_0 B (m)^P + \frac{B^2 J^2}{4} \sum_{\alpha, \beta} (q_{\alpha \beta})^P + \frac{1}{2} \sum_{\alpha, \beta} \lambda_{\alpha \beta} (m_{\alpha} m_{\beta} - q_{\alpha \beta}) - \frac{1}{2} \ln \det\{-\lambda^{-1}\}_{\alpha \beta}$$  \hspace{1cm} (9)$$

The saddle point condition for the $\lambda_{\alpha \beta}$ gives $q_{\alpha \beta} = m_{\alpha} m_{\beta} - \{\lambda^{-1}\}_{\alpha \beta}$

At the high temperetures the system lives in the paramagnetic phase, where $m_{\alpha} =$
0, $q_{\alpha \neq \beta} = 0$. It is easy to derive, as in [8]

$$G = \frac{1}{2} (1 + \ln 2\pi) + \frac{B^2 J^2}{4}$$  \hspace{1cm} (10)

In the SG phase we need in one-level breaking of replica symmetry, to block of indices of order $m_1$. We have $q_{\alpha \alpha} = 1, q_{\alpha \beta} = q$ for $\alpha \beta$ from one block of order $m_1$, and for other indice sets $q_{\alpha \beta} = 0$.

Then (9) goes to

$$G = \frac{1}{2} (1 + \ln 2\pi) + \frac{B^2 J^2}{4} [1 + (m_1 - 1)q^P] + \frac{m_1 - 1}{2m_1} \ln [1 + (m_1 - 1)q]$$  \hspace{1cm} (11)

Taking derivatives by $m_1$ and $q$ and dividing equatios we derive system

$$\begin{cases}
  \frac{q^2}{P} = \frac{(1-q)(1-q+mq)}{m^2} \ln (1 - q + mq)(1 - q) - \frac{q(1-q)}{m} \\
  \frac{PB^2 J^2}{2} q^P = \frac{1}{2(1-q)(1-q+mq)}
\end{cases}$$  \hspace{1cm} (12)

Let us consider limit $P \to \infty$. It is a reasonable anzats

$$q = 1 - \epsilon, \epsilon \to 0$$  \hspace{1cm} (13)

Then (16) goes to

$$\begin{cases}
  \frac{1}{P} = \frac{m}{m} \ln \frac{m}{\epsilon} \\
  \frac{PB^2 J^2}{2} = \frac{1}{\epsilon m}
\end{cases}$$  \hspace{1cm} (14)

Its solution is

$$\epsilon = \sqrt{\frac{2 \ln P}{PBJ}}$$  \hspace{1cm} (15)

$$m = \frac{\sqrt{2 \ln P}}{BJ}$$  \hspace{1cm} (16)

For the free energy we have

$$G = \frac{BJ \sqrt{2 \ln P}}{2} - \frac{1}{2} \ln \frac{PBJ}{\sqrt{2 \ln P}} + \frac{1}{2} (1 + \ln 2\pi)$$  \hspace{1cm} (17)

We find $B_c$ for the phase transition (paramagnetic-SG) by comparing (24) and (13):

$$B_c = \sqrt{2 \ln P} / J$$  \hspace{1cm} (18)
For the ferromagnetic phase we have in the bulk approximation (just enough for our purposes) \( \ln \frac{Z}{N} = J_0 B \). More accurate consideration gives

\[
\ln \frac{Z}{N} = J_0 B - \frac{\ln P}{2}
\]

(19)

Comparison with two other phases gives, that at high \( B \) ferromagnetic phase appears, when

\[
\begin{cases} 
J_0 > \sqrt{\frac{\ln P}{2} J} \\
B > B_c \equiv \sqrt{2 \ln P / J}
\end{cases}
\]

(20)

It is the main result of this work. It resembles the result for the discrete spins with Potts like interaction and color equal to \( P \). It will be very interesting to solve the case of diluted couplings, as well as to consider similar models for the pattern recognition like Hopiefield or Gardner. The distance in the space of pattern with our spins will be like euclidean and as a consequence, closely connected with the reality, than artifical discretization by means of \( Q \) color spins.

The multicritical point has coordinates

\[
\begin{cases} 
B_c = \sqrt{2 \ln P / J} \\
J_0 = \sqrt{\frac{\ln P}{2} J}
\end{cases}
\]

(21)

It will be interesting to search analogy with the Nishimori line. Direct enlargement for the methods of [11] is impossible, but the idea of resonance between Gibbs partition and inhomogeneous partition of couplings is too beatifull to throw it.

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