Extra symmetry in the field equations in $5D$ with spatial spherical symmetry

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Abstract

We point out that the field equations in $5D$, with spatial spherical symmetry, possess an extra symmetry that leaves them invariant. This symmetry corresponds to certain simultaneous interchange of coordinates and metric coefficients. As a consequence a single solution in $5D$ can generate very different scenarios in $4D$, ranging from static configurations to cosmological situations. A new perspective emanates from our work. Namely, that different astrophysical and cosmological scenarios in $4D$ might correspond to the same physics in $5D$. We present explicit examples that illustrate this point of view.

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1 Introduction

Most of the recent advances in theoretical physics deal with models of our universe in more than four dimensions. Theories of the Kaluza-Klein type in many dimensions are used in different branches of physics. Superstrings (10D) and supergravity (11D) are well known examples. In gravitation and cosmology braneworld models [1]-[6] in 5D as well as space-time-matter (STM) theory [7] have become quite popular.

All these theories face the same challenge, namely the prediction of observable effects from the extra dimensions [8]. The success of this mission depends on the “correct” identification of the physical or observable space-time metric from the multidimensional one. This is not a trivial task in higher-dimensional cosmologies, like braneworld and STM, where the extra dimension is non-compact. In this scenario all the coordinates are alike, in the sense that the metric tensor is allowed to depend explicitly on the extra coordinate, usually called fifth coordinate.

In this regard, the crucial question is: given an arbitrary 5-dimensional metric, that depends on all five coordinates, how do we decide which one is the “extra” coordinate?. The answer to this question seems to be far from obvious. Even in the simple case of spherical symmetry, in ordinary three space, there are various possible options leading to different scenarios in 4D.

In the present paper we point out that the field equations in 5D, with spatial spherical symmetry, possess an extra symmetry that leaves them invariant. This extra symmetry corresponds to certain simultaneous interchange of coordinates and metric coefficients. As a consequence, for every solution of the field equations in 5D, there are different options for the identification of the extra dimension.

Our main conclusion is that, by virtue of the abovementioned extra symmetry, a single solution in 5D can generate very different scenarios in 4D, ranging from static configurations to cosmological situations. Said another way, the additional symmetry allows the unification in 5D of diverse, apparently different, physical scenarios in 4D.

The paper is organized as follows. In section 2 we give a brief summary of the general splitting formalism in 5D which leads to an effective, or induced, energy-momentum in 4D. In section 3 we discuss the stated additional symmetry of the field equations for the case of spatial spherical symmetry.

In section 4, in the scenario where the metric is independent on the fifth coordinate, we show that the extra coordinate can be either spacelike or timelike, without affecting the interpretation in 4D. At this point it is worth to mention that physical conditions, imposed on the 4D effective matter, do not preclude the existence of a large timelike extra dimension1. Then, we generate “new” solutions by the appropriate interchange of coordinates and metric coefficients. We illustrate the discussion with a specific solution of the 5D field equations that produces the following scenarios in 4D: (i) static configurations, (ii) evolution of inhomogeneities, and (iii) spatially-homogeneous cosmological models of the Kantowski-Sachs type.

In section 5 we consider the case of 5D metrics with explicit dependence on the extra dimension and which allow both signatures. We show that the various options for the identification of the extra dimension correspond to different looking metrics in 5D with distinct interpretation in 4D.

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1For a critical review of some objections commonly raised against the timelike signature of the extra coordinate see Ref. [9]
2 Field equations

In order to make the paper self-consistent and set the notation, in this section we give a brief review of the formalism in 5D. The line element is given by

\[ dS^2 = \gamma_{AB} dx^A dx^B, \quad (1) \]

where \( A, B = (0 - 4) \) and \( \gamma_{AB} \) is allowed to depend on all five coordinates. The dynamics in 5D is assumed to be governed by the Einstein equations

\[ (5) G_{AB} = R_{AB} - \frac{1}{2} \gamma_{AB} R = k_{(5)}^2 T_{AB}, \quad (2) \]

where \( k_{(5)}^2 \) is a constant introduced for dimensional considerations and \( (5) T_{AB} \) is the five-dimensional energy-momentum tensor.

Our 4D spacetime is orthogonal to the extra dimension. The unit \((n_A n^A = \epsilon)\) vector, along the fifth dimension is given by

\[ n^A = \frac{\delta^A_4}{\sqrt{\epsilon \gamma_{44}}}, \quad n_A = \frac{\gamma^{A4}}{\sqrt{\epsilon \gamma_{44}}}, \quad (3) \]

where \( \epsilon = -1 \) or \( \epsilon = +1 \) depending on whether the extra dimension is spacelike or timelike, respectively. The metric induced in 4D is given by \( g_{AB} = \gamma_{AB} - n_A n_B \). Since \( \gamma_{44} = 0 \), denoting \( \gamma_{4A} = \epsilon \Phi^2 \) and \( \gamma_{4\mu} = \epsilon \Phi^2 A_{\mu} \), the line element \((1)\) can be written as

\[ dS^2 = g_{\mu\nu}(x^\rho, y) dx^\mu dx^\nu + \epsilon \Phi^2 (A_{\mu} dx^\mu + dx^4)^2. \quad (4) \]

In absence of off-diagonal terms \((\gamma_{4\mu} = 0)\), the vector \( n^A \) defined in \((3)\) becomes orthogonal to hypersurfaces \( y = const. \), which provides a clean separation between the spacetime section and the extra coordinate, viz.,

\[ dS^2 = g_{\mu\nu}(x^\rho, y) dx^\mu dx^\nu + \epsilon \Phi^2 (x^\rho, y) dy^2, \quad (5) \]

here and in what follows the extra coordinate will be denoted as \( y \equiv x^4 \). In this case the dimensional reduction of the five-dimensional equations is particularly simple \((10)\). In fact, using the Gauss-Codacci-Mainardi relations, the 15 equations \((2)\) can be split up into three parts.

The first part consists of 10 equations which are interpreted as the effective field equations in 4D. They are

\[ (4) G_{\alpha\beta} = \frac{2}{3} k_{(5)}^2 \left[ (5) T_{\alpha\beta} + (5) T_{44} - \frac{1}{4} (5) T g_{\alpha\beta} \right] - \epsilon \left( K_{\alpha\lambda} K^\lambda_{\beta} - K^\lambda_{\alpha} K_{\lambda\beta} \right) + \frac{\epsilon}{2} g_{\alpha\beta} \left( K_{4\rho} K^{\rho\nu} - (K^\lambda)^2 \right) - \epsilon E_{\alpha\beta}, \quad (6) \]

where \( K_{\mu\nu} \) is the extrinsic curvature

\[ K_{\alpha\beta} = \frac{1}{2} L_n g_{\alpha\beta} = \frac{1}{2 \Phi} \frac{\partial g_{\alpha\beta}}{\partial y}, \quad K_{44} = 0, \quad (7) \]

\[ \text{We note that this vector is not orthogonal to the hypersurfaces } y = const., \text{ except in the case where } \gamma_{4\mu} = 0. \]
and $E_{\mu\nu}$ is the projection of the bulk Weyl tensor $(^5)C_{ABCD}$ orthogonal to $n^A$, i.e., “parallel” to spacetime, viz.,

$$E_{\alpha\beta} = (^5)C_{\alpha\lambda\beta\rho} n^\lambda n^\rho$$

$$= -\frac{1}{\Phi} \frac{\partial K_{\alpha\beta}}{\partial y} + K_{\alpha\rho} K_{\beta}^\rho - \epsilon \frac{\Phi_{\alpha\beta}}{\Phi} - \epsilon \frac{k_2}{3} \left[ (^5)T_{\alpha\beta} + (^5)T^4_4 - \frac{1}{2} (^5)T g_{\alpha\beta} \right].$$

The second part is an inhomogeneous wave equation for $\Phi$, which follows from the fact that $E_{\mu\nu}$ is traceless. Indeed, the requirement $E_{\mu\mu} = 0$ is equivalent to $(^5)G_{44} = k_2 (^5)T_{44}$ from (2) and gives

$$\Phi_{\mu\nu} = -\epsilon \frac{\partial K}{\partial y} - \Phi (\epsilon K_{\lambda\rho} K^{\lambda\rho} + (^5)R_4).$$

Finally, the remaining four equations are

$$D_\mu \left( K_\alpha - \delta_\alpha K^\lambda \right) = k_2 (^5)T^{4\alpha} \Phi$$

In the above expressions, the covariant derivatives are calculated with respect to $g_{\alpha\beta}$, i.e., $Dg_{\alpha\beta} = 0$.

### 2.1 Effective energy-momentum tensor in 4D

Following the usual procedure, we define the energy-momentum tensor in four-dimensions (on the hypersurface $y = \text{const}$) through (6), the effective Einstein field equations in 4D, namely

$$8\pi G T^{(\text{eff})}_{\mu\nu} \equiv -\epsilon \left( K_{\mu\lambda} K^\lambda_{\nu} - K_{\lambda} K^\lambda_{\nu} \right) + \frac{\epsilon}{2} g_{\mu\nu} \left( K_{\lambda\rho} K^{\lambda\rho} - (K^\lambda)^2 \right) - \epsilon E_{\mu\nu},$$

which, in terms of the metric, is given by (in what follows $f \equiv \partial f / \partial y$),

$$8\pi G T^{(\text{eff})}_{\mu\nu} = -\frac{\epsilon}{2} \left[ \frac{\Phi_{\alpha\beta}}{\Phi} - g_{\alpha\beta} + g^{\lambda\mu} g^{\mu\nu} g_{\alpha\lambda} g_{\beta\mu} - \frac{1}{2} g^{\mu\nu} g_{\mu\nu} g_{\alpha\beta} + \frac{1}{4} g_{\alpha\beta} \left( g_{\mu\nu} + (g^{\mu\nu} g^{\nu\lambda})^2 \right) \right],$$

$$+ \Phi_{\alpha\beta}. \tag{12}$$

The energy-momentum tensor in 5D is usually taken as

$$(^5)T_{AB} = \Lambda_{(5)} \gamma_{AB}, \tag{13}$$

where $\Lambda_{(5)}$ is the cosmological constant in 5D. Therefore, the first term on the r.h.s. of (10) yields

$$T^{(\text{eff})}_{\mu\nu} = \frac{1}{2} k_2 (^5)\Lambda_{(5)} g_{\mu\nu} \equiv \Lambda g_{\mu\nu}, \tag{14}$$

which defines an effective cosmological constant in 4D as $\Lambda = k_2 (^5)\Lambda_{(5)}/2$. 

4
As a consequence of the contracted Bianchi identities in 4D, \( C^{\mu}_{\nu\rho\sigma} = 0 \), and \( \Box \), the effective energy-momentum tensor satisfies the standard general relativity conservation equations, viz.,

\[
D_{\mu} T^\text{(eff)}_{\mu\nu} = 0.
\] (15)

If in the bulk there were scalar and/or other fields, then this would be no longer true, in general.

In addition, the field equations in 5D reduce to

\[
R_{AB} = -\frac{2}{3} k_5^2 \Lambda_5 \gamma_{AB}.
\] (16)

3 Spherically symmetric spacetime

In this section we will discuss the extra symmetry of the field equations in 5D, in the case of spatial spherical symmetry. Thus, in what follows we will consider the metric

\[
dS^2 = e^\nu dt^2 - e^\lambda dr^2 - R^2(d\theta^2 + \sin^2 \theta d\phi^2) + \epsilon e^\mu dy^2,
\] (17)

where the metric coefficients \( \nu, \lambda, R \) and \( \mu \) are allowed to be functions of the “extra” coordinate \( y \).

For the sake of clarity we provide here the non-zero components of the Ricci tensor. In the usual notation \( \dot{f} \equiv \partial f / \partial t \), \( f' \equiv \partial f / \partial r \) and \( \dot{f} \equiv \partial f / \partial y \), they are

\[
R_{00} = \left( \frac{\dot{\nu} \dot{\lambda}}{4} + \frac{\dot{\nu} \dot{\mu}}{4} + \frac{\dot{\nu} \dot{R}}{R} - \frac{\dot{\lambda}}{2} - \frac{\dot{\mu}}{2} - 2 \frac{\ddot{R}}{R} - \frac{\dot{\lambda}^2}{4} - \frac{\dot{\mu}^2}{4} \right) + e^{\nu-\lambda} \left( \frac{\nu''}{2} + \frac{\nu'^2}{4} - \nu' \lambda' \right) + e^{\nu-\mu} \left( \frac{\nu''}{2} + \frac{\nu'^2}{4} - \nu' \mu' + \frac{\ddot{\nu} R}{R} \right),
\] (18)

\[
R_{11} = \left( \frac{\chi \mu'}{4} + \frac{\chi \nu'}{4} + \frac{\chi' R'}{R} - \frac{\mu''}{2} - 2 \frac{\ddot{R}}{R} - 2 \frac{\mu^2}{4} - \frac{\nu'^2}{4} \right) + e^{\lambda-\mu} \left( \frac{\lambda' \mu'}{4} + \frac{\lambda' \nu'}{4} + \frac{\lambda' R'}{R} \right),
\] (19)

\[
R_{22} = 1 + R^2 e^{-\nu} \left[ \frac{\ddot{R}}{R^2} + \frac{\ddot{R}}{R} - \frac{\ddot{R}}{2R} (\dot{\nu} - \dot{\lambda} - \dot{\mu}) \right] - R^2 e^{-\lambda} \left[ \frac{R'^2}{R^2} + \frac{R''}{R} + \frac{R'}{2R} (\nu' - \lambda' + \mu') \right] + eR^2 e^{-\mu} \left[ \frac{\dot{R}^2}{R^2} + \frac{\ddot{R}'}{R} + \frac{R'}{2R} (\nu' + \dot{\lambda} + \dot{\mu}) \right],
\] (20)

\[
R_{33} = \sin^2 \theta R_{22},
\] (21)
\[ R_{44} = \left( \frac{\mu \lambda}{4} + \frac{\mu \nu}{4} + \frac{\mu R}{R} - \frac{\lambda^2}{2} - \frac{\nu^2}{2} - 2 \frac{\dot{R}^2}{R} - \frac{\lambda}{4} - \frac{\nu}{4} \right) + \epsilon \tau \left( \frac{\mu''}{2} + \frac{\mu'^2}{4} + \frac{\mu' \nu'}{4} + \frac{\mu' R'}{R} \right) + \epsilon \tau \left( \frac{\mu''}{2} + \frac{\mu'^2}{4} - \frac{\mu' \lambda'}{4} - \frac{\mu' R'}{R} \right), \tag{22} \]

and

\[
\begin{align*}
R_{01} &= \frac{\nu' \dot{\mu}}{4} + \frac{\dot{\lambda} \mu'}{4} + \frac{\dot{\lambda} R'}{R} + \frac{\nu' \dot{R}}{R} - \frac{\mu'}{2} - \frac{\mu' \nu'}{4} - 2 \frac{\dot{R}'}{R}, \\
R_{41} &= \frac{\mu' \nu'}{4} + \frac{\dot{\lambda} \nu'}{4} + \frac{\dot{\lambda} R'}{R} + \frac{\mu' \dot{R}}{R} - \frac{\nu'}{2} - \frac{\nu' \nu'}{4} - 2 \frac{\dot{R}'}{R}, \\
R_{04} &= \frac{\nu \dot{\lambda}}{4} + \frac{\mu' \dot{\lambda}}{4} + \frac{\mu' \dot{R}}{R} + \frac{\nu' \dot{R}}{R} - \frac{\lambda}{2} - \frac{\lambda \lambda}{4} - 2 \frac{\dot{R}}{R}. \tag{23} \end{align*}
\]

Let us notice some properties of the above equations:

1. If we assume that the metric coefficients are independent on \( y \), then the extra dimension can be either spacelike or timelike, without affecting the effective matter distribution in 4D.

   Indeed, a simple examination of (18)-(23) with \( \tau = \lambda = \mu = R = 0 \), indicates that \( \epsilon \), the signature of the extra dimension, enters nowhere, except in \( R_{44} \), which becomes \( R_{44} = \epsilon \times (\text{some function of} t \text{ and} r) \). But \( \gamma_{44} = e \Phi^2 \). Then, from (12) and (16) it follows that both the field equations and the effective 4D matter are invariant with respect to the change of sign of \( \epsilon \).

2. If \( y \) is spacelike (\( \epsilon = -1 \)), then we find that the field equations in 5D are invariant under the transformation

\[
r \leftrightarrow y, \quad \lambda \leftrightarrow \mu, \quad \frac{\partial}{\partial r} \leftrightarrow \frac{\partial}{\partial y}. \tag{24} \]

Indeed, \( R_{00}, R_{22} \) and \( R_{14} \) remain invariant and \( R_{01} \leftrightarrow R_{04}, R_{11} \leftrightarrow R_{44} \). Consequently, if (17) with \( \epsilon = -1 \) is a solution of the field equation, then

\[
dS^2 = e^\nu dt^2 - e^\mu dr^2 - R^2 (d\theta^2 + \sin^2 \theta d\phi^2) - e^\lambda dy^2, \tag{25} \]

is also a solution. We note from (12) that the effective matter is \textit{not} invariant under such transformation, which means that (17) and (25) lead to different scenarios in 4D.

3. If \( y \) is timelike (\( \epsilon = +1 \)), we find that the field equations are invariant with respect to the transformation

\[
t \leftrightarrow y, \quad \nu \leftrightarrow \mu, \quad \frac{\partial}{\partial t} \leftrightarrow \frac{\partial}{\partial y}. \tag{26} \]
In this case $R_{11}$, $R_{22}$ and $R_{04}$ are invariant, while $R_{00} \leftrightarrow R_{44}$ and $R_{01} \leftrightarrow R_{41}$. Therefore, from (17) with $\epsilon = 1$, it follows that

$$dS^2 = e^\mu dt^2 - e^\lambda dr^2 - R^2 (d\theta^2 + \sin^2 \theta d\phi^2) + e^\nu dy^2,$$

(27)

also satisfies the field equations. This metric and (17) with time like extra dimension yield different scenarios in 4D.

4 5D metrics with “no” dependence on the extra dimension

This is an important case because in 5D there are a number of solutions to the field equations obtained under the assumption that $\mathbf{v} = \mathbf{\lambda} = \mathbf{\mu} = \mathbf{R} = 0$.

Usually, the extra coordinate $y$ is assumed to be spacelike, nevertheless this is not a requirement of the field equations. Rather, we have seen that

$$dS^2_{(\pm)} = e^{\nu(t,r)} dt^2 - e^{\lambda(t,r)} dr^2 - R^2(t,r)(d\theta^2 + \sin^2 \theta d\phi^2) + \epsilon e^{\mu(t,r)} dy^2,$$

(28)

solves the field equations for both signatures ($\epsilon = \pm 1$), without affecting 4D. Therefore, there are two “other” solutions associated with (28), depending on whether we choose $\epsilon = -1$ or $\epsilon = +1$. These are

$$dS^2_{(-)} = e^{\nu(t,y)} dt^2 - e^{\mu(t,y)} dr^2 - R^2(t,y)(d\theta^2 + \sin^2 \theta d\phi^2) - e^{\lambda(t,y)} dy^2,$$

(29)

and

$$dS^2_{(+) =} e^{\mu(r,y)} dt^2 - e^{\lambda(r,y)} dr^2 - R^2(r,y)(d\theta^2 + \sin^2 \theta d\phi^2) + e^{\nu(r,y)} dy^2,$$

(30)

for $\epsilon = -1$ and $\epsilon = +1$, respectively. Thus, we have started from a metric that does not depend on the extra dimension and finished with two metrics that do depend on it. Clearly, the property of being or not dependent of the extra dimension depends on how we define it.

**Interpretation in 4D:** For the four-dimensional interpretation of five-dimensional metrics we identify our spacetime with a hypersurface orthogonal to the extra dimension, located at some value of $y$.

Thus, from (28), (29) and (30) we find three different scenarios in 4D, viz.,

$$ds^2 = e^{\nu(t,r)} dt^2 - e^{\lambda(t,r)} dr^2 - R^2(t,r)(d\theta^2 + \sin^2 \theta d\phi^2),$$

(31)

$$ds^2 = e^{\nu(t)} dt^2 - e^{\mu(t)} dr^2 - R^2(t)(d\theta^2 + \sin^2 \theta d\phi^2),$$

(32)

and

$$ds^2 = e^{\mu(r)} dt^2 - e^{\lambda(r)} dr^2 - R^2(r)(d\theta^2 + \sin^2 \theta d\phi^2).$$

(33)

The first scenario (31) represents a non-static and spatially non-uniform spherical distribution of matter. The second one (32) is a cosmological metric of the Kantowski-Sachs type. The third scenario (33) corresponds to some static spherical distribution of matter.
4.1 Time-depending solutions

In order to illustrate the above discussion let us consider the 5D metric

\[
dS^2(\pm) = B^2 \left[ \frac{3r^2}{a^2} dt^2 - t^2 dr^2 - \frac{t^2 r^2}{(3 - a^2)} (d\theta^2 + \sin^2 \theta d\phi^2) \right] \pm C^2 r^{2(\alpha+1)} y^{2(\alpha+3)/\alpha} dy^2, \tag{34}
\]

which is a solution of the 5D field equations \[16\] with \(\Lambda(5) = 0\). Here \(\alpha\) is a dimensionless parameter in the range \(0 < \alpha^2 < 3\), whereas \(B\) and \(C\) are arbitrary constants with the dimensions \(L^{-1}\) and \(L^{-2(\alpha+3)/\alpha}\), respectively.

The metric (34) works for both signatures. Let us discuss the different possible scenarios.

\(\epsilon = \pm 1\): The induced metric in 4D can be interpreted as a spherically symmetric dissipative distribution of matter, with heat flux, whose effective density and pressure are nonstatic, nonuniform, and satisfy the equation of state of radiation. This interpretation is not affected whatsoever by the signature of the extra dimension \[11\].

\(\epsilon = -1\): If we take \(y\) as a spacelike coordinate, and choose \(3B^2 = \alpha^2\), then (34) yields

\[
dS^2(-) = y^2 dt^2 - \tilde{C}^2 y^{2(\alpha+1)} t^{2(\alpha+3)/\alpha} dr^2 - \frac{\alpha^2 t^2 y^2}{3(3 - \alpha^2)} (d\theta^2 + \sin^2 \theta d\phi^2) - \frac{\alpha^2 t^2}{3} dy^2. \tag{35}
\]

In 4D this solution corresponds to a cosmological model of Kantowski-Sachs type. These cosmologies have extensively been studied by the present author; they can be relevant to describe “bubbles” of new phases, in phase transitions \[12\]-\[13\]. In particular, the 4D part of (35) is the only Kantowski-Sachs universe that has the property of self-similarity. It represents a homogeneous universe which is expanding with shear (rather than shear-free as in FRW cosmologies). The expansion \(\Theta\) and shear \(\sigma\) are given by \[13\]

\[
\Theta = \frac{3(\alpha + 1)}{\alpha t}, \quad \sigma = \frac{\sqrt{3}}{\alpha t}. \tag{36}
\]

We note that in (35) the coordinate \(y\) is dimensionless and \(\tilde{C} \sim L^{-(\alpha+3)/\alpha}\).

\(\epsilon = +1\): If we take \(y\) as a timelike coordinate, choose \(B^2 = (3 - \alpha^3)\) and denote \(C = r_0^{-(\alpha+1)}\), then \(y\) becomes dimensionless and (34) yields

\[
dS^2(+) = \left( \frac{r}{r_0} \right)^{2(\alpha+1)} y^{2(\alpha+3)/\alpha} dt^2 - (3 - \alpha^2) y^2 dr^2 - y^2 r^2 (d\theta^2 + \sin^2 \theta d\phi^2) + 3(3\alpha^2 - 1) r^2 dy^2. \tag{37}
\]

This solution was extensively studied by Billyard and Wesson \[14\]. In 4D, it represents a spherical static cloud of matter with density profiles similar to those of cluster of galaxies.

\[3\]We note that, for this choice of the constants, the Kantowski-Sachs line element (35) looks very similar to the standard FRW-flat cosmological model embedded in 5D. Namely, \(dS^2 = y^2 dt^2 - A^2 t^{2/\alpha} y^{2/(1-\alpha)} [dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)] - \alpha^2 (1 - \alpha)^{-2} t^2 dy^2 \[17\].
There are many other spherical solutions of the five-dimensional field equations that depend on \( t \) and \( r \), but not on \( y \) \cite{15}-\cite{16}. All of them were obtained under the assumption of a spacelike extra dimension. Therefore, using the symmetries \cite{24} and \cite{26}, we can use them to generate new solutions of the type discussed above.

### 4.2 Static solutions

In the case of spherical symmetry in ordinary 3D space, there is only one family of exact solutions of \( (16) \), with \( \Lambda(5) = 0 \), which are (i) static and (ii) independent of the fifth coordinate; if one relaxes any of these conditions, then there are many solutions in 5D vacuum, with spherical 3-space.

This solution has been rediscovered many times and is known in the literature under different names, viz., as Kramer’s solution \cite{18}:

\[
dS^2 = \left(1 - \frac{r_g}{r}\right)^{(A-B)} dt^2 - \left(1 - \frac{r_g}{r}\right)^{-(A+B)} dr^2 - r^2 \left(1 - \frac{r_g}{r}\right)^{(1-A-B)} d\Omega^2 - \left(1 - \frac{r_g}{r}\right)^{2B} dy^2, \quad (38)
\]

with \( A^2 + 3B^2 = 1 \) and \( r_g = \text{const} \), as the Davidson-Owen class of solutions \cite{19}:

\[
dS^2 = \left(\frac{ar - 1}{ar + 1}\right)^{2\sigma k} dt^2 - \frac{1}{a^4r^4} \left(\frac{ar + 1}{ar - 1}\right)^{2[\sigma(k-1)+1]} [dr^2 + r^2 d\Omega^2] - \left(\frac{ar + 1}{ar - 1}\right)^{2\sigma} dy^2, \quad (39)
\]

where \( \sigma^2(k^2 - k + 1) = 1 \) and \( a = \text{const} \), and, although in another context, as Gross and Perry solutions \cite{20}. These solutions were discovered under the assumption of a spacelike extra coordinate, but they also work for a timelike extra dimension.

Distinct forms of the solution are useful in different contexts, which include the generation of new solutions, the description of extended spherical objects in 4D called solitons, and the study of geodesics in 5D.

#### 4.2.1 New solutions

In the above solutions the extra coordinate is assumed to be spacelike. However, according to our discussion, a “new” solution arises from the symmetries of the field equations. Using \cite{26} in \cite{38}, we obtain

\[
dS^2 = \left(1 - \frac{r_g}{r}\right)^{2B} dt^2 - \left(1 - \frac{r_g}{r}\right)^{-(A+B)} dr^2 - r^2 \left(1 - \frac{r_g}{r}\right)^{(1-A-B)} d\Omega^2 + \left(1 - \frac{r_g}{r}\right)^{(A-B)} dy^2, \quad (40)
\]

and in the and Davidson-Owen form

\[
dS^2 = \left(\frac{ar + 1}{ar - 1}\right)^{2\sigma} dt^2 - \frac{1}{a^4r^4} \left(\frac{ar + 1}{ar - 1}\right)^{2[\sigma(k-1)+1]} [dr^2 + r^2 d\Omega^2] + \left(\frac{ar - 1}{ar + 1}\right)^{2\sigma k} dy^2, \quad (41)
\]

The four-dimensional interpretation of these metrics (at some \( y = \text{const} \)) differs from the one for \cite{38} and \cite{39}.
First, in (38) and (40) the Schwarzschild solution is recovered in curvature coordinates for distinct values of \( A \) and \( B \). Namely, \( A = 1, B = 0 \) for (38) and \( A = B = 1/2 \) for (40). The original Davidson-Owen solution yields the Schwarzschild solution in isotropic coordinates in the limit \( \sigma \to 0, k \to \infty \) and \( \sigma k \to 1 \), while in the new version (41) this occurs for \( \sigma = -1 \) and \( k = 0 \).

Second, the effective energy-momentum tensor in 4D calculated from (40) is distinct from the one calculated from (38), similarly for (41) and (39), respectively. Indeed, from (12) it follows that

\[
8\pi G T_{\mu\nu}^{(\text{eff})} = \frac{\Phi^{\alpha;\beta}}{\Phi},
\]

where \( \epsilon \Phi^2 \), is the metric coefficient in front of \( dy^2 \), which is different in (38), (39) and (40),(41), respectively.

The soliton solutions (38)-(39) play a central role in the discussion of many important observational problems, which include the classical tests of relativity, as well as the geodesic precession of a gyroscope and possible departures from the equivalence principle [7]. Therefore, a detailed study of metrics (40) and (41) is important. It should allow us to understand the influence of an extra timelike coordinate on ordinary four-dimensional physics. Such investigation is out of the scope of the present paper.

5 5D metrics with explicit dependence on the extra dimension and \( \epsilon = \pm 1 \)

The five-dimensional metrics (35) and (37) show explicit dependence on the extra dimension, but the signature is restricted to be either spacelike or timelike, respectively. On the other hand, there are a number of solutions of the 5D field equations that depend on the extra dimension and allow both signatures.

For solutions of this kind the extra symmetry produces a large “family” of different scenarios in 4D. As an illustration let us consider the metric

\[
dS^2(\pm) = \frac{A^2\alpha^2y^2}{(\alpha y^2 + \beta)}dt^2 - \frac{(\alpha y^2 + \beta)}{(ar^2 + b)^2}[dr^2 + r^2d\Omega^2] + \epsilon dy^2,
\]

which is an exact solution of the field equations in 5D, provided\(^4\) \( \alpha \epsilon = -4ab \). Here \( A \) is an arbitrary constant with dimensions of \( L \), \( \alpha \) and \( a \) are constants with dimensions \( L^{-2} \), while \( \beta \) and \( b \) are dimensionless constants. The properties of this solution were discussed in Ref. [21]. In 4D it represents a matter distribution which satisfies the equation of state\(^5\) \( \rho = -3p \), where \( 8\pi G \rho = 12ab(\alpha y^2 + \beta)^{-1} \).

\(^4\)Under the transformation \( \bar{r} = Br, \bar{y} = Cy; \bar{\alpha} = \alpha C^2 \) and \( \bar{a} = aB^2 \), therefore \( \epsilon \bar{\alpha}(B/C)^2 = -4ab \).

\(^5\)It is interesting to note that this equation of state appears in very different contexts; in an alternative derivation of properties of matter from 5D geometry by Davidson and Owen [19]; in discussions of cosmic strings by Gott and Rees [22] and Kolb [23]; in certain sources (called “limiting configurations”) for the Reissner-Nordström field by the present author [24] and as the only equation of state consistent with the existence of quantum zero-point fields by Wesson [25]-[26].
We have several choices for the extra dimension here. If $\epsilon = -1$, from (24) we get
\[
dS^2(-) = \frac{A^2 \alpha^2 r^2}{(\alpha^2 + \beta)} dt^2 - dr^2 - \frac{y^2 (\alpha r^2 + \beta)}{(ay^2 + b)^2} d\Omega^2 - \frac{(\alpha^2 + \beta)}{(ay^2 + b)^2} dy^2.
\] (44)

If $\epsilon = 1$, from (26) we get
\[
dS^2(+) = dt^2 - \frac{(\alpha^2 + \beta)}{(ar^2 + b)^2} [dr^2 + r^2 d\Omega^2] + \frac{A^2 \alpha^2 t^2}{(\alpha^2 + \beta)} d\Omega^2 + \frac{A^2 \alpha^2 t^2}{(\alpha^2 + \beta)} dy^2.
\] (45)

This metric depends on $t$ and $r$ only. Therefore, it should solve the equations for both signatures, i.e.,
\[
dS^2(-) = dt^2 - \frac{(\alpha^2 + \beta)}{(ar^2 + b)^2} [dr^2 + r^2 d\Omega^2] - \frac{A^2 \alpha^2 t^2}{(\alpha^2 + \beta)} dy^2.
\] (46)
is also a solution. From here and (24) we get the Kantowski-Sachs metric
\[
dS^2(-) = dt^2 - \frac{A^2 \alpha^2 t^2}{(\alpha^2 + \beta)} dr^2 - \frac{y^2 (\alpha t^2 + \beta)}{(ay^2 + b)^2} d\Omega^2 - \frac{(\alpha^2 + \beta)}{(ay^2 + b)^2} dy^2.
\] (47)

Despite the similarity between the above line elements, each represents a different scenario in $4D$, which is obtained at some $y = \text{const.}$. But all of them represent the same five-dimensional solution (43).

### 6 Summary and concluding remarks

The field equations in $5D$, with spatial spherical symmetry, have an additional symmetry that leave the equations invariant. This symmetry reflects the fact that there are many ways of producing, or embedding, a $4D$ spacetime in a given five-dimensional manifold, while satisfying the field equations.

From a practical viewpoint this symmetry allows us to generate diverse physical scenarios in $4D$ from a given solution in $5D$. From a theoretical point of view it unifies in five-dimensions a number of, otherwise detached, dynamical models in $4D$.

Thus, our work provides a new perspective and new challenges. The new perspective is that different astrophysical and cosmological scenarios in $4D$ might correspond to the same higher dimensional system, with different choices of signature and extra coordinate.

The new challenge is to understand in more detail the relationship between apparently different $4D$ scenarios, belonging to the same five-dimensional family, as well as the nature and effects of a timelike extra dimension.

We would like to finish this paper with the following comments.

1. We have never seen in the literature the metric (40) nor (41). To our best knowledge they have never been used in the discussion of the classical and other tests in $5D$. Certainly, the study of these metrics in this context would give a better understanding of the subject.
2. It can be shown that the solution \( (43) \) also satisfies the 5D field equations (with \( \Lambda(5) = 0 \)) with the constants \( a, b \) replaced by functions of \( y \). Therefore, applying the transformations \( (24) \) and \( (26) \) to it, a large family of other 5D solutions will emerge that generate a variety of physical models in 4D. We hope to investigate these models in future work.

3. In both, compactified and non-compactified theories, the extra dimensions are usually assumed to be spacelike. However, there is no \textit{a priori} reason why extra dimensions cannot be timelike. As a matter of fact, the consideration of extra timelike dimensions in physics has a long and distinguished history \cite{27}, \cite{28}, \cite{29}, \cite{30} and currently it is a subject of considerable interest. For a more detailed discussion and references see for example \cite{9}.

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