Analytic Representations in the 3-dim Frobenius Problem

Leonid G. Fel

Department of Civil and Environmental Engineering, Technion, Haifa 3200, Israel
e-mail: lfel@techunix.technion.ac.il

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Abstract

We consider the Diophantine problem of Frobenius for semigroup $S(d^3)$ where $d^3$ denotes the tuple $(d_1, d_2, d_3)$, $\gcd(d_1, d_2, d_3) = 1$. Based on the Hadamard product of analytic functions [17] we have found the analytic representation for the diagonal elements $a_{kk}(d^3)$ of the Johnson’s matrix of minimal relations [12] in terms of $d_1, d_2, d_3$. Bearing in mind the results of the recent paper [10] this gives the analytic representation for the Frobenius number $F(d^3)$, genus $G(d^3)$ and the Hilbert series $H(d^3; z)$ for the semigroups $S(d^3)$. This representation does complement the Curtis’ theorem [13] on the non–algebraic representation of the Frobenius number $F(d^3)$. We also give a procedure to calculate the diagonal and off–diagonal elements of the Johnson’s matrix.

Key words: Semigroups, Frobenius problem, Hilbert series of a graded ring, Hadamard product of analytic functions.

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1 Introduction

Let $S(d_1, \ldots, d_m) \subset \mathbb{N}$ be the numerical semigroup generated by a minimal set \footnote{The set $\{d_1, \ldots, d_m\}$ is called minimal if there are no nonnegative integers $b_{i,j}$ for which the following linear dependence holds $d_i = \sum_{j \neq i} b_{i,j}d_j$, $b_{i,j} \in \{0, 1, \ldots\}$ for any $i \leq m$. It is classically known [1] that $d_1 \geq m$.} of integers $\{d_1, \ldots, d_m\}$ such that

$$m \leq d_1 < \ldots < d_m, \quad \gcd(d_1, \ldots, d_m) = 1, \quad m \geq 2.$$  \hspace{1cm} (1)

For short we denote the tuple $(d_1, \ldots, d_m)$ by $d^m$. The least positive integer $(d_1)$ belonging to $S(d^m)$ is called the multiplicity. The number $(m)$ of the minimal generators of $S(d_1, \ldots, d_m)$ is called the embedding dimension. The conductor $c(d^m)$ of $S(d^m)$ is defined by $c(d^m) := \min \{s \in S(d^m) \mid s + \mathbb{N} \cup \{0\} \subset S(d^m)\}$. The genus $G(d^m)$ of $S(d^m)$ is defined as the cardinality (#) of its complement $\Delta$ in $\mathbb{N}$, i.e. $\Delta(d^m) = \mathbb{N} \setminus S(d^m)$ and

$$G(d^m) := \# \Delta(d^m).$$  \hspace{1cm} (2)

Introduce the generating function $\Phi(d^m; z)$ for the set $\Delta(d^m)$ of unrepresentable integers

$$\Phi(d^m; z) = \sum_{s \in \Delta(d^m)} z^s.$$  \hspace{1cm} (3)

The semigroup ring $k[ X_1, \ldots, X_m ]$ over a field $k$ of characteristic 0 associated with $S(d^m)$ is a polynomial ring graded by $\deg X_i = d_i, 1 \leq i \leq m$ and generated by all monomials $z^{d_i}$. The Hilbert series $H(d^m; z)$ of a graded ring $k[ z^{d_1}, \ldots, z^{d_m} ]$ is defined by [2]

$$H(d^m; z) = \sum_{s \in S(d^m)} z^s = \frac{Q(d^m; z)}{\prod_{j=1}^m (1 - z^{d_j})}, \quad z < 1,$$  \hspace{1cm} (4)

where $Q(d^m; z)$ is a polynomial in $z$. Thus, we have

$$H(d^m; z) + \Phi(d^m; z) = \frac{1}{1 - z}, \quad z < 1.$$  \hspace{1cm} (5)

The number $F(d^m) := -1 + c(d^m)$ is referred to as Frobenius number. As follows from (3)

$$F(d^m) = \deg \Phi(d^m; z), \quad G(d^m) = \sum_{s \in \Delta(d^m)} 1^s = \Phi(d^m; 1).$$  \hspace{1cm} (6)

The determination of $F(d^m), G(d^m)$ and $H(d^m; z)$ is called the m–dimensional (mD) Frobenius problem. The last restriction $(m \geq 2)$ in (1) is essential since one can show that in the one dimensional case the problem is trivial

$$H(d; z) = \frac{1}{1 - z^d}, \quad F(d) = G(d) = \infty, \quad d \geq 2.$$  \hspace{1cm} (7)

The first non–trivial case $(m=2)$ was studied already by J. Sylvestre [3]

$$H(d^2; z) = \frac{1 - z^{d_1d_2}}{(1 - z^{d_1})(1 - z^{d_2})}, \quad F(d^2) = d_1d_2 - d_1 - d_2, \quad G(d^2) = \frac{1}{2} (d_1 - 1)(d_2 - 1).$$  \hspace{1cm} (8)

The next non–trivial case $(m=3)$ was extensively studied in the contents of commutative algebra [4]–[8] and algebraic geometry of monomial curves [9] where a partial progress was achieved (without
calculating the Hilbert series). A new diagrammatic procedure of construction of the set $\Delta(d^3)$ was developed recently in [10]. It has paved the way to calculate $Q(d^3; z)$ and, in accordance with (5) and (6), led to the complete solution of the 3D Frobenius problem. Based on Brauer’s lemma [11] on the matrix representation of the set $\Delta(d^2)$ and Johnson’s theorem [12] on the minimal relations, the author [10] was able to find the Frobenius number $F(d^3)$, genus $G(d^3)$ and the Hilbert series $H(d^3; z)$ of a graded ring for the non–symmetric and symmetric semigroups $S(d^3)$:

$$F(d^3) = \frac{1}{2} \left( \langle a, d \rangle + J(d^3) \right) - \sum_{k=1}^{3} d_k,$$

$$G(d^3) = \frac{1}{2} \left( 1 + \langle a, d \rangle - \sum_{k=1}^{3} d_k - \prod_{k=1}^{3} a_{kk} \right),$$

(9)

$$Q_n(d^3; z) = 1 - \sum_{k=1}^{3} z^{a_{kk}d_k} + z^{1/2}[\langle a, d \rangle - J(d^3)] + z^{1/2}[\langle a, d \rangle + J(d^3)],$$

(10)

$$J^2(d^3) = \langle a, d \rangle^2 - 4 \sum_{i>j} a_{kk}a_{jj}d_kd_j + 4d_1d_2d_3,$$

$$\langle a, d \rangle = \sum_{k=1}^{3} a_{kk}d_k,$$

(11)

$$Q_s(d^3; z) = \left( 1 - z^{a_{22}d_2} \right) \left( 1 - z^{a_{33}d_3} \right),$$

if $a_{11}d_1 = a_{22}d_2,$

(12)

where subscripts ”$n$” in (10) and ”$s$” in (12) stand for non–symmetric and symmetric semigroups [10], respectively. The matrix $((a_{ij}))$ was introduced by Johnson [12]. Its elements uniquely define the minimal relations for given $d_1, d_2, d_3$

$$a_{11}d_1 = a_{12}d_2 + a_{13}d_3,$$

$$a_{22}d_2 = a_{21}d_1 + a_{23}d_3,$$

$$a_{33}d_3 = a_{31}d_1 + a_{32}d_2,$$

(13)

where

$$a_{11} = \min \{v_{11} \mid v_{11} \geq 2, v_{11}d_1 = v_{12}d_2 + v_{13}d_3, v_{12}, v_{13} \in \mathbb{N} \cup \{0\} \},$$

$$a_{22} = \min \{v_{22} \mid v_{22} \geq 2, v_{22}d_2 = v_{21}d_1 + v_{23}d_3, v_{21}, v_{23} \in \mathbb{N} \cup \{0\} \},$$

$$a_{33} = \min \{v_{33} \mid v_{33} \geq 2, v_{33}d_3 = v_{31}d_1 + v_{32}d_2, v_{31}, v_{32} \in \mathbb{N} \cup \{0\} \},$$

and

$$\gcd(a_{11}, a_{12}, a_{13}) = 1, \quad \gcd(a_{21}, a_{22}, a_{23}) = 1, \quad \gcd(a_{31}, a_{32}, a_{33}) = 1.$$  

Notice that in formulas (9) – (12) only the diagonal elements of the Johnson’s matrix (14) appear. Also notice that formulas (9) – (12) contain algebraic functions of $a_{ij}$ and $d_j, 1 \leq j \leq 3$. However the $a_{ij}$ cannot be algebraic functions of $d_j$ because of the following theorem of Curtis [13]:

There is no finite set of polynomials with integer coefficients

$$\{f_j(x_1, \ldots, x_m)\}, \quad j = 1, \ldots, n$$

such that for each choice of $d^m, m \geq 3$, there is $j$ such that $f_j(d_1, \ldots, d_m) = F(d^m)$.

On the other hand, it would be pretty interesting to build such representations for $F(d^3), G(d^3)$ and $H(d^3; z)$ in the frameworks of analytic number theory avoiding any auxiliary invariants like the minimal relations (13) and (14).

Our main result is the analytic representation for the diagonal elements $a_{kk}(d^3)$ of the Johnson’s matrix (14) in terms of $d_1$, $d_2$ and $d_3$:

$$a_{kk}(d^3) = \lim_{d_k \to 0} \frac{\ln \Psi_k(d^3; z)}{\ln z}, \quad \Psi_k(d^3; z) = \frac{1}{2\pi} \int_{0}^{2\pi} \frac{(1 - e^{id_kt})}{(1 - e^{id_kt})(1 - z^{d_k}e^{-id_kt})} dt - 1,$$

(15)

where $(k, j, l) = (1, 2, 3), (2, 3, 1), (3, 1, 2)$.

By (9) – (12) this gives the Frobenius number $F(d^3)$, genus $G(d^3)$ and the Hilbert series $H(d^3; z)$, and due to (5) it leads to the generating function $\Phi(d^3; z)$ for both symmetric and non–symmetric semigroups $S(d^3)$.  

3
2 Matrix representation of the set \( \Delta(d^2) \) and the map \( \tau \)

First, recall the main statements about matrix representation of the set \( \Delta(d^2) \).

**Lemma 1** ([11]) Let \( d_1 \) and \( d_2 \) be relatively prime positive integers. Then every positive integer \( s \) not divisible by \( d_1 \) or by \( d_2 \) is representable either in the form \( s = xd_1 + yd_2 \), \( x > 0, y > 0 \) or in the form \( s = d_1d_2 - pd_1 - qd_2 \), \( p, q \in \mathbb{N} \).

**Definition 1** Let \( 2 < d_1 < d_2 \) with \( \gcd(d_1, d_2) = 1 \). Define function \( \sigma(p, q) \) as follows

\[
\sigma(p, q) := d_1d_2 - pd_1 - qd_2, \quad p, q \in \mathbb{N}.
\]

(16)

**Lemma 2** ([10]) Let \( t \) be an integer and \( d_2 > d_1 \). Then \( t \in \Delta(d_1, d_2) \) iff \( t \) is uniquely representable by

\[
t = \sigma(p, q), \quad \text{where}
\]

\[
1 \leq p \leq \left\lfloor \frac{d_2 - d_2}{d_1} \right\rfloor, \quad 1 \leq q \leq d_1 - 1, \quad \text{and} \quad d_1 - 1 \leq \left\lfloor \frac{d_2 - d_2}{d_1} \right\rfloor,
\]

where \( \lfloor c \rfloor \) denotes an integer part of the number \( c \).

**Lemma 3** ([10]) Let \( 2 < d_1 < d_2 \) and \( \gcd(d_1, d_2) = 1 \). Every integer \( d_3 \in \Delta(d^2) \) gives rise to the minimal generating set \( \{d_1, d_2, d_3\} \) for the semigroup \( S(\mathbb{S}^3) \) such that \( F(\mathbb{S}^3) < F(d^2) \).

The representation (17) of all integers \( \sigma(p, q) \in \Delta(d_1, d_2) \) is called the matrix representation of the set \( \Delta(d_1, d_2) \) and is denoted by \( M \{\Delta(d_1, d_2)\} \) (see Figure 1), so

\[
\sigma \{M \{\Delta(d_1, d_2)\}\} = \Delta(d_1, d_2),
\]

(19)

i.e. \( \sigma \) maps the entries of \( M \{\Delta(d_1, d_2)\} \) onto \( \Delta(d_1, d_2) \) in a bijective manner (Lemma 2).

\( \sigma(p, q) \) is the integer which occurs in the row \( p \) and the column \( q \) of \( M \{\Delta(d_1, d_2)\} \), e.g. \( \sigma(1, 1) = d_1d_2 - d_1 - d_2 \). Note that the latter coincides with the Frobenius number \( F(d_1, d_2) \).

![Figure 1: Typical matrix representation \( M \{\Delta(d_1, d_2)\} \) of the set \( \Delta(d_1, d_2) \). The lowest cells in every column of \( M \{\Delta(d_1, d_2)\} \) (gray color) is occupied exclusively by the integers \( 1, \ldots, d_1 - 1 \) not in a necessarily consecutive order [10].](image)

Throughout the paper we will make use of another entity, a map \( \tau \), which was introduced in [10] in order to constitute the relationship between the set \( \Delta(d^3) \) and the generating function \( \Phi(d^3; z) \).

Actually, the map \( \tau \) serves for such relationships in any dimension \( m \geq 1 \) as well.
Definition 2 The function $\tau$ maps each power series (polynomials including)
\[
\sum_{s \in \mathbb{N}} c_s [\Delta] z^s \in \mathbb{N}[z]
\]
with $c_s [\Delta] \in \{0, 1\}$ onto the set $\Delta$ of degrees $\{s \in \Delta \subset \mathbb{N} \mid c_s [\Delta] \neq 0\}$.
The coefficient $c_s [\Delta]$ is a characteristic function for the set $\Delta \subset \mathbb{N}$
\[
c_s [\Delta] = \begin{cases} 1, & \text{if } s \in \Delta, \\ 0, & \text{if } s \notin \Delta, \end{cases}
\] (20)
which satisfies the following properties:

Let two sets, $\Delta_1$ and $\Delta_2$, be given such that $\Delta_1, \Delta_2 \subset \mathbb{N}$. Then for set intersections and set unions, we have [14]
\[
c_s [\Delta_1 \cap \Delta_2] = c_s [\Delta_1] c_s [\Delta_2], \quad c_s [\Delta_1 \cup \Delta_2] = c_s [\Delta_1] + c_s [\Delta_2] - c_s [\Delta_1 \cap \Delta_2].
\] (21)

It also follows from Definition 2 and from representations (3) and (4) that
\[
\tau [\Phi (d^m; z)] = \Delta (d^m), \quad \tau [H (d^m; z)] = S (d^m), \quad \tau [0] = \emptyset,
\]
\[
\tau^{-1} [\Delta (d^m)] = \Phi (d^m; z), \quad \tau^{-1} [S (d^m)] = H (d^m; z), \quad \tau^{-1} [0] = 0,
\] (22) (23)
where $\emptyset$ denotes an empty set. Observe that since all coefficients of the polynomial $\Phi (d^m; z)$ are $1$ or $0$, we can uniquely reconstruct $\Phi (d^m; z)$ from $\Delta (d^m)$ and vice versa. In this sense $\tau$ is an isomorphic map. The map $\tau$ is also linear in the following sense:

Lemma 4 Let $d^m$ be given and a set $\Delta (d^m)$ be related to its generating function $\Phi (d^m; z)$ by the isomorphic map $\tau$ defined in (22). Let $\Delta_1$ and $\Delta_2$, be subsets $\Delta (d^m)$. Then the following holds
\[
\tau^{-1} [\Delta_1 \cup \Delta_2] = \tau^{-1} [\Delta_1] + \tau^{-1} [\Delta_2] - \tau^{-1} [\Delta_1 \cap \Delta_2].
\] (24)

Proof By (3), (20), (21) and Definition 2 we have
\[
\tau^{-1} [\Delta_1 \cup \Delta_2] = \sum_{s \in \mathbb{N}} c_s [\Delta_1 \cup \Delta_2] z^s = \sum_{s \in \mathbb{N}} (c_s [\Delta_1] + c_s [\Delta_2] - c_s [\Delta_1 \cap \Delta_2]) z^s = \tau^{-1} [\Delta_1] + \tau^{-1} [\Delta_2] - \tau^{-1} [\Delta_1 \cap \Delta_2],
\]
that proves the Lemma. $\square$

The following statement is a consequence of Lemma 4 and generalizes the result obtained in [10] (formula (64))

Lemma 5 Let $d^m$ be given and a set $\Delta (d^m)$ be related to its generating function $\Phi (d^m; z)$ by the map $\tau$ according to (22) and (23). Let $n$ sets $\Delta_i$, $i = 1, \ldots, n$ be given such that
\[
\Delta_i \subset \Delta (d^m), \quad \Delta_i \cap \Delta_j = \emptyset, \quad \text{for } i \neq j = 1, \ldots, n.
\] (25)

Then the following holds
\[
\tau^{-1} \left[ \bigcup_{i=1}^n \Delta_i \right] = \sum_{i=1}^n \tau^{-1} [\Delta_i].
\] (26)

Proof By induction on $n$ applying Lemma 4 to the left hand side of (26) consecutively and making use of (25) we obtain
\[
\tau^{-1} \left[ \bigcup_{i=1}^n \Delta_i \right] = \tau^{-1} \left[ \bigcup_{i=1}^{n-1} \Delta_i \right] + \tau^{-1} [\Delta_n] - \tau^{-1} \left[ \bigcap_{i=1}^{n-1} \Delta_i \cap \Delta_n \right] = \tau^{-1} \left[ \bigcup_{i=1}^{n-1} \Delta_i \right] + \tau^{-1} [\Delta_n] - \tau^{-1} \left[ \bigcup_{i=1}^{n-1} \left( \Delta_i \cap \Delta_n \right) \right] = \tau^{-1} \left[ \bigcup_{i=1}^{n-1} \Delta_i \right] + \tau^{-1} [\Delta_n] - \tau^{-1} \left[ \bigcup_{i=1}^{n-1} \Delta_i \right] = \tau^{-1} \left[ \bigcup_{i=1}^{n-1} \Delta_i \right] + \tau^{-1} [\Delta_n] = \cdots = \sum_{i=1}^n \tau^{-1} [\Delta_i]
\]
that proves the Lemma. $\square$
2.1 The intersection set $\Delta(d_1, d_2) \cap S(d_3)$ and its generating function

In this Section we construct the intersection set $\Delta(d_1, d_2) \cap S(d_3)$. This set is essential to determine an explicit non–algebraic expression for $a_{33}(d_1, d_2, d_3)$ (see next Section).

First, define a 1D numerical semigroup $S(d_3)$ of integers $\sigma = 0 \mod (d_3)$

$$S(d_3) := \{ \sigma \mid \sigma = jd_3, \ j \in \mathbb{N} \cup \{0\}, \ d_3 \geq 2 \}, \quad (27)$$

which is generated by the Hilbert series $H(d_3; z)$ given by (7).

Consider the intersection set $\Delta(d_1, d_2) \cap S(d_3)$ which consists exclusively of the integers $\sigma = jd_3$, the index $j$ runs with jumps from 1 to $N_3$ such that $N_3d_3 \in \Delta(d_1, d_2)$ still holds. Notice that $N_3$ satisfies

$$N_3 \leq \frac{d_1d_2 - d_1 - d_2}{d_3} < d_1 - 1,$$

that follows from $N_3d_3 \leq F(d_1, d_2)$ and $d_2(d_1 - 1) - d_1 < d_3(d_1 - 1)$.

The integers $jd_3$ are distributed inside the matrix representation $M\{\Delta(d_1, d_2)\}$ as shown in Figure 2. Let us prove an important Lemma.

![Figure 2: Integers $jd_3 \in \Delta(d_1, d_2) \cap S(d_3)$ (black boxes) are distributed inside the matrix representation $M\{\Delta(d_1, d_2)\}$.](image)

**Lemma 6** Let $d^3$ be given, $d^3 = (d_1, d_2, d_3)$, and assume that the minimal relations are defined by (13) and (14). Then

$$jd_3 \in \Delta(d_1, d_2) \cap S(d_3), \ j = 1, \ldots, a_{33} - 1, \quad (28)$$

and

$$a_{33}d_3 \notin \Delta(d_1, d_2) \cap S(d_3). \quad (29)$$

Notice that $j$ runs in (28) without jumps.

**Proof** The proof follows from two obvious relations. First, by (27) we have

$$jd_3 \in S(d_3), \ j = 1, \ldots, a_{33}, \ldots, \quad (30)$$

and, next, by the 3rd line of (14) we have

$$jd_3 \in \Delta(d_1, d_2), \ j = 1, \ldots, a_{33} - 1, \text{ and } a_{33}d_3 \notin \Delta(d_1, d_2), \quad (31)$$
that leads to (28) and (29). □

Keeping in mind Lemma 6 we write the generating function of the set $\Delta (d_1,d_2) \cap S (d_3)$.

According to (23) we get

$$\tau^{-1} [\Delta (d_1,d_2) \cap S (d_3)] = \sum_{j=1}^{a_{33}-1} z^{jd_3} + \sum_{j>a_{33}} z^{jd_3} ,$$

(32)

where index $j$ runs with jumps in the second summation of (32). Define an auxiliary function $\Psi_3 (d^3; z)$ by

$$\Psi_3 (d^3; z) = \frac{z^{d_3}}{1-z^{d_3}} - \tau^{-1} [\Delta (d_1,d_2) \cap S (d_3)] .$$

(33)

It plays a key role in calculating the diagonal element $a_{33}$.

**Theorem 1** Let $d^3$ be given, $d^3 = (d_1,d_2,d_3)$, and assume that the minimal relations are defined by (13) and (14). Then the diagonal element $a_{33} (d^3)$ is given by

$$a_{33} (d^3) = \frac{1}{d_3} \lim_{z \to 0} \frac{\ln \Psi_3 (d^3; z)}{\ln z} .$$

(34)

**Proof** A series expansion for $\Psi_3 (d^3; z)$ reads

$$\Psi_3 (d^3; z) = z^{a_{33}d_3} + \sum_{j<N_3} z^{jd_3} + \sum_{j=N_3+1} \infty z^{jd_3} ,$$

(35)

where index $j$ runs with jumps in the second summation of (35). The power series (35) converges for $z < 1$

$$\Psi_3 (d^3; z) = z^{a_{33}d_3} + \sum_{j>a_{33}} z^{jd_3} + \frac{z^{(N_3+1)d_3}}{1-z^{d_3}} .$$

The first term in (35) turns to be the leading term if $z \to 0$, i.e. $\Psi_3 (d^3; z) \overset{z \to 0}{\to} z^{a_{33}d_3}$. Hence, formula (34) follows. □

Formulas (33) and (34) have nothing specific concerned with arrangements of the $d_i$ in the tuple $(d_1,d_2,d_3)$. Therefore the similar formulas can be obtained for the diagonal elements $a_{11} (d^3)$ and $a_{22} (d^3)$

$$a_{kk} (d^3) = \frac{1}{d_k} \lim_{z \to 0} \frac{\ln \Psi_k (d^3; z)}{\ln z} , \quad k = 1,2 ,$$

(36)

where

$$\Psi_1 (d^3; z) = \frac{z^{d_1}}{1-z^{d_1}} - \tau^{-1} [\Delta (d_2,d_3) \cap S (d_1)] , \quad \Psi_2 (d^3; z) = \frac{z^{d_2}}{1-z^{d_2}} - \tau^{-1} [\Delta (d_3,d_1) \cap S (d_2)] .$$

3 **Analytic representation of generating functions**

In the previous Section we have found the diagonal elements $a_{jj} (d^3) , j = 1,2,3$, through the auxiliary functions $\Psi_j (d^3; z) , j = 1,2,3$. Thus, the non–algebraic expressions for $a_{jj} (d^3)$ will be given in the closed form if the analytic representation of corresponding generating functions

$$\tau^{-1} [\Delta (d_j,d_l) \cap S (d_k)] , \quad (k,j,l) = (1,2,3), (2,3,1), (3,1,2) ,$$

will be found. The present Section is completely devoted to this question. In order to resolve it we make use of the Hadamard product of analytic functions.
3.1 Hadamard product of analytic functions

Let \( u(z) = \sum_{n=0}^{\infty} a_n z^n \) and \( v(z) = \sum_{n=0}^{\infty} b_n z^n \) be analytic functions in the unit disk \( |z| < 1 \). Define the \( \otimes \)-product of \( u(z) \) and \( v(z) \) by

\[
u(z) \otimes v(z) = \frac{1}{2\pi} \int_{0}^{2\pi} u(ze^{it}) v(e^{-it}) \, dt = \sum_{n=0}^{\infty} a_n b_n z^n , \quad |z| < 1 . \tag{37}\]

This product was introduced by J. Hadamard [15] to discuss the singularities of the analytic function \( f(z) \) defined by the series \( \sum_{n=0}^{\infty} a_n b_n z^n \) in terms of those of the functions \( u \) and \( v \) ("multiplication of singularities" theorem [16]).

Another representation, which is similar to (37), can be found for the analytic function \( f(z^2) = \sum_{n=0}^{\infty} a_n b_n z^{2n} \) through the integral convolution "\( \circ \)" of the functions \( u \) and \( v \),

\[
u(z) \circ v(z) = \frac{1}{2\pi} \int_{0}^{2\pi} u(ze^{it}) v(ze^{-it}) \, dt = \sum_{n=0}^{\infty} a_n b_n z^{2n} , \quad |z| < 1 . \tag{38}\]

This product was also used by J. Hadamard [15] and is strongly related to the \( \otimes \)-product

\[
u(\overline{z}^2) \otimes v(\overline{z}^2) = u(z) \circ v(z) , \tag{39}\]

e.g. for the analytic function \( u(z) \) we have

\[
u(z) \otimes (1-z)^{-1} = u(z) , \quad \text{but} \quad u(z) \circ (1-z)^{-1} = u(\overline{z}^2) . \tag{40}\]

Both \( \otimes \)- and \( \circ \)-products have been variously referred to as the Hadamard product, quasi inner product or the termwise product (see survey [17]). The \( \otimes \)- and \( \circ \)-products appear naturally in functional analysis, e.g. the Bieberbach conjecture for univalent analytic functions \([18]\) and the Polya-Schoenberg conjecture for analytic convex mappings \([19]\).

There is a variety of algebraic and other analytic properties associated with \( \circ \)- and \( \otimes \)-products \([20]\). The commutative and distributive laws can be easily verified

\[
u(z) \circ v(z) = v(z) \circ u(z) , \quad u(z) \circ [v(z) + w(z)] = u(z) \circ v(z) + u(z) \circ w(z) , \quad u(z) \otimes v(z) = v(z) \otimes u(z) , \quad u(z) \otimes [v(z) + w(z)] = u(z) \otimes v(z) + u(z) \otimes w(z) . \tag{41}\]

Notice that (38) manifests the non–associativity of the \( \circ \)-product

\[
[u(z) \circ v(z)] \circ w(z) \neq u(z) \circ [v(z) \circ w(z)] ,
\]
e.g. \([z \circ z] \circ z^2 = z^4\), but \(z \circ [z \circ z^2] = 0\). However the \( \otimes \)-product is associative,

\[
u(z) \otimes v(z) \otimes w(z) = u(z) \otimes [v(z) \otimes w(z)] , \tag{42}\]
e.g. \([z \otimes z] \otimes z^2 = z \otimes [z \otimes z^2] = 0\). The last property is very important to generalize the Hadamard product. We give an appropriate extension of \( \otimes \)-product referred to as the Hadamard multiple product \([21]\).

Let \( u_k(z) = \sum_{n=0}^{\infty} a_n^{(k)} z^n \), \( k = 1, \ldots, N \), be analytic functions in the unit disk \( |z| < 1 \). Define the Hadamard multiple product \( \bigotimes \) of \( u_k(z) \) by

\[
\bigotimes_{k=1}^{N} u_k(z) = u_1(z) \otimes u_2(z) \otimes \ldots \otimes u_N(z) = \sum_{n=0}^{\infty} a_n^{(1)} \cdot a_n^{(2)} \cdot \ldots \cdot a_n^{(N)} \cdot z^n . \tag{43}\]
The $\otimes$–product is well defined due to the associativity of the $\otimes$–product (42). Its integral representation has the form
\[
\sum_{k=1}^{N} u_k(z) = \frac{1}{(2\pi)^{N-1}} \int_{0}^{2\pi} \cdots \int_{0}^{2\pi} u_1 \left( ze^{i\sum_{k=2}^{N} t_k} \right) \prod_{k=2}^{N} u_k(e^{-it_k}) \, dt_k.
\] (44)

We give some examples of the $\otimes$–product operating with rational functions [16]
\[
\sum_{k=1}^{N} \frac{1}{a_k - z} = \frac{1}{\prod_{k=1}^{N} a_k - z}, \quad \sum_{k=1}^{N} \frac{1}{1 - z^{n_k}} = \frac{1}{1 - z^L}, \quad n_k \in \mathbb{N}, \quad L = \text{lcm}(n_1, \ldots, n_N)
\] (45)

and also of the $\circ$–product operating with transcendental functions [20], [21]
\[
\cosh \frac{z}{2} \circ \cosh \frac{z}{2} + \sinh \frac{z}{2} \circ \sinh \frac{z}{2} = I_0(z), \quad \cosh \frac{z}{2} \circ \cosh \frac{z}{2} - \sinh \frac{z}{2} \circ \sinh \frac{z}{2} = J_0(z),
\] (46)
\[
e^{z/2} \circ e^{z/2} = I_0(z), \quad e^{-z} \circ (1 + z)^n = L_n(z^2), \quad \frac{1}{(1 - z)^{\alpha}} \circ \frac{1}{1 - z^{\beta}} = 2F_1(\alpha; \beta; 1; z^2)
\] (47)

where $\alpha, \beta > 0$. In (46) and (47) the functions $I_0(z)$, $J_0(z)$, $L_n(z)$ and $2F_1(\alpha; \beta; 1; z^2)$ denote the modified and non–modified Bessel functions of the 1st kind, the Laguerre polynomial and the hypergeometric function, respectively.

Because of its connection with the integral convolution (38), the Hadamard product leads to elegant evaluations of complicated trigonometric integrals and provide analytic derivations of combinatorial identities [20], [21]. Finally, the Hadamard product was used in [22] to construct solutions for a variety of Cauchy–type problems.

It turns out that the $\otimes$–product is a sensitive tool to deal with an intersection of discrete numerical sets. This will be the subject for discussion in the next Section 3.2. In order to be technically equipped we derive here two important formulas which will be useful further.

**Lemma 7** Let $u(z) = \sum_{k=0}^{\infty} a_k z^k$ be an analytic function in the unit disk $|z| < |c|$. Then
\[
u(z) \otimes \frac{1}{c - z} = \frac{1}{c} u \left( \frac{z}{c} \right), \quad c \neq 0.
\] (48)

**Proof** Calculating the $\otimes$–product we get
\[
u(z) \otimes \frac{1}{c - z} = \frac{1}{c} u(z) \otimes \frac{1}{1 - z/c} = \frac{1}{c} \sum_{k=0}^{\infty} a_k z^k \otimes \sum_{k=0}^{\infty} \left( \frac{z}{c} \right)^k = \frac{1}{c} \sum_{k=0}^{\infty} a_k \frac{z^k}{c^k} = \frac{1}{c} u \left( \frac{z}{c} \right),
\]
that proves formula (48). □

Before going to the next formula (Lemma 8) we prove an elementary identity
\[
\frac{n}{z^n - 1} = \sum_{l=1}^{n} \frac{w^l}{z - w^l}, \quad w_n = \exp \left( \frac{2\pi i}{n} \right).
\] (49)

Rewrite the right hand side of (49) in the form
\[
\sum_{l=1}^{n} \frac{w^l}{z - w^l} = \frac{1}{z^n - 1} \sum_{k=1}^{n} (-1)^{k-1} k \Pi_k z^{n-k} = \frac{\Pi_1 z^{n-1} - 2\Pi_2 z^{n-2} + \ldots + n(-1)^{n-1}\Pi_n}{z^n - 1},
\] (50)

where $\Pi_k$ denote the basic symmetric polynomials
\[
\Pi_k = \sum_{l_1 > l_2 > \ldots > l_k = 1} w_{l_1}^{l_1} w_{l_2}^{l_2} \ldots w_{l_k}^{l_k}, \text{ i.e. } \Pi_1 = \sum_{l=1}^{n} w^l, \quad \Pi_2 = \sum_{l_1 > l_2 = 1} w_{l_1}^{l_1} w_{l_2}^{l_2}, \ldots, \quad \Pi_n = \prod_{l=1}^{n} w^l.
\] (51)
Recall Vieta’s formula for the sum $S_k$ of the products of distinct roots $z_i$ of the polynomial equation of degree $n$

$$a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0 = 0,$$

which reads [23]

$$S_k = (-1)^k \frac{a_n - k}{a_n}, \quad \text{where} \quad S_k = \sum_{l_1 > l_2 > \ldots > l_k = 1}^{n} z_{l_1} z_{l_2} \ldots z_{l_k}. \quad (52)$$

Since the $w_n^j$-roots are associated with the polynomial equation $z^{n-1} = 0$ then the Vieta’s formula (52) gives for $\Pi_k$-polynomials

$$\Pi_k = \begin{cases} 0, & \text{if } 1 \leq k < n, \\ (-1)^{n+1}, & \text{if } k = n. \end{cases} \quad (53)$$

Substituting (53) into (50) we arrive at (49).

**Lemma 8** Let $u(z) = \sum_{k=0}^{\infty} a_k z^k$ be an analytic function in the unit disk $|z| < 1$. Then

$$u(z) \otimes \frac{1}{1 - z^n} = \frac{1}{n} \sum_{k=0}^{n-1} u \left( z w_n^k \right). \quad (54)$$

**Proof** Making use of the identity (49) we obtain by Lemma 7

$$u(z) \otimes \frac{1}{1 - z^n} = \frac{1}{n} \sum_{k=1}^{n} \left( u(z) \otimes \frac{w_n^k}{w_n^n - z} \right) = \frac{1}{n} \sum_{k=1}^{n} \frac{w_n^k}{w_n^n} u \left( \frac{z}{w_n^n} \right) = \frac{1}{n} \sum_{l=1}^{n} u \left( zw_n^{n-l} \right) = \frac{1}{n} \sum_{k=0}^{n-1} u \left( zw_n^k \right)$$

that proves formula (54). $\square$

Henceforth we use the $\otimes$-product and refer to as the Hadamard product.

### 3.2 Representation of $\tau^{-1}[\Delta (d_1, d_2) \cap S (d_3)]$

In this Section we derive the analytic representation of the generating function for the intersection of two sets, $\Delta (d_1, d_2)$ and $S (d_3)$. Start with the following proposition which utilizes the Hadamard product for the intersecting sets $\Delta_1$ and $\Delta_2$.

**Lemma 9** Let $\Delta_1, \Delta_2$ be subsets of $\mathbb{N}$, and let their corresponding generating functions $\tau^{-1}[\Delta_1]$ and $\tau^{-1}[\Delta_2]$ be

$$\tau^{-1}[\Delta_i] = \sum_{s \in \mathbb{N}} c_s [\Delta_i] z^s, \quad i = 1, 2, \quad (55)$$

where $c_s [\Delta_i]$ stands for corresponding characteristic function of the set $\Delta_i$ and satisfies (20). Then the set $\Delta_1 \cap \Delta_2$ is generated by

$$\tau^{-1}[\Delta_1 \cap \Delta_2] = \tau^{-1}[\Delta_1] \otimes \tau^{-1}[\Delta_2]. \quad (56)$$

**Proof** By Definition 2 of generating function $\tau^{-1}[\Delta]$ and the 1st relation in (21) we have

$$\tau^{-1}[\Delta_1 \cap \Delta_2] = \sum_{s \in \mathbb{N}} c_s [\Delta_1 \cap \Delta_2] z^s = \sum_{s \in \mathbb{N}} c_s [\Delta_1] c_s [\Delta_2] z^s. \quad (57)$$
On the other hand, by definition (37) of the Hadamard product we have
\[
\tau^{-1} [\Delta_1] \otimes \tau^{-1} [\Delta_2] = \sum_{s \in \mathbb{N}} c_s [\Delta_1] z^s \otimes \sum_{s \in \mathbb{N}} c_s [\Delta_2] z^s = \sum_{s \in \mathbb{N}} c_s [\Delta_1] c_s [\Delta_1] z^s. \tag{58}
\]
A comparison of (57) and (58) proves the Lemma. □

Lemma 9 has a simple generalization
\[
\tau^{-1} \left[ \bigcap_{j=1}^N \Delta_j \right] = \bigotimes_{j=1}^N \tau^{-1} [\Delta_j]. \tag{59}
\]
Returning to \(\tau^{-1} [\Delta (d_1, d_2) \cap S (d_3)]\) we can verify that both generating functions \(\tau^{-1} [\Delta (d_1, d_2)]\) and \(\tau^{-1} [S (d_3)]\) are representable in the form (55)
\[
\tau^{-1} [\Delta (d_1, d_2)] = \Phi \{\{d_1, d_2\}; z\} = \sum_{s \in \Delta (d_1, d_2)} z^s, \quad \tau^{-1} [S (d_3)] = \frac{1}{1 - z^{d_3}} = \sum_{s \in S (d_3)} z^s.
\]
Therefore we obtain
\[
\tau^{-1} [\Delta (d_1, d_2) \cap S (d_3)] = \tau^{-1} [\Delta (d_1, d_2)] \otimes \tau^{-1} [S (d_3)] = \Phi \{\{d_1, d_2\}; z\} \otimes \frac{1}{1 - z^{d_1}}. \tag{60}
\]
Formula (60) can be slightly simplified by utilizing the relationship (5) between the generating function \(\Phi\) and the Hilbert series \(H\) and of distributive law (41) for the Hadamard product
\[
\tau^{-1} [\Delta (d_1, d_2) \cap S (d_3)] = \frac{1}{1 - z} \otimes \frac{1}{1 - z^{d_3}} - H_{12}(z) \otimes \frac{1}{1 - z^{d_1}}, \tag{61}
\]
where \(H_{12}(z)\) denotes for short the Hilbert series \(H \{\{d_1, d_2\}; z\}\). On the last step we make use of the 1st equality in (40) and get the generating function for the intersection of two sets, \(\Delta (d_1, d_2)\) and \(S (d_3)\),
\[
\tau^{-1} [\Delta (d_1, d_2) \cap S (d_3)] = \frac{1}{1 - z^{d_3}} - H_{12}(z) \otimes \frac{1}{1 - z^{d_1}}, \tag{62}
\]
and also the function \(\Psi_3 (d^3; z)\) introduced in (33)
\[
\Psi_3 (d^3; z) = \frac{z^{d_3}}{1 - z^{d_3}} - \tau^{-1} [\Delta (d_1, d_2) \cap S (d_3)] = H_{12}(z) \otimes \frac{1}{1 - z^{d_1}} - 1. \tag{63}
\]
Applying now Lemma 8 to formula (63) we get
\[
\Psi_3 (d^3; z) = \frac{1}{d_3} \sum_{k=0}^{d_3 - 1} H_{12} \left( zw_{d_3}^k \right) - 1, \quad w_{d_3} = \exp \left( \frac{2\pi i}{d_3} \right). \tag{64}
\]
Simplifying the expression (8) for \(H_{12}(z)\)
\[
H_{12}(z) = \sum_{p=0}^{d_2 - 1} \frac{z^{pd_1}}{1 - z^{d_2}} = \sum_{p=0}^{d_2 - 1} \sum_{q=0}^{\infty} z^{p(d_1 + qd_2)}, \tag{65}
\]
and substituting it into (64) we obtain
\[
\Psi_3 (d^3; z) = \frac{1}{d_3} \sum_{p=0}^{d_2 - 1} \sum_{q=0}^{\infty} z^{p(d_1 + qd_2)} \sum_{k=0}^{d_3 - 1} w_{d_3}^k - 1. \tag{66}
\]
The inner sum in (66) does vanish for \( p, q \) such that \( pd_1 + qd_2 \) is not divisible by \( d_3 \). Indeed,

\[
\sum_{k=0}^{d_3-1} \frac{1}{w_d} k^{pd_1 + qd_2} = \frac{\exp [2\pi i (pd_1 + qd_2)] - 1}{\exp (2\pi i \frac{pd_1 + qd_2}{d_3}) - 1} = 0, \quad \text{if} \quad d_3 \nmid pd_1 + qd_2.
\]

Thus, it remains

\[
\Psi_3 (d^3; z) = \sum_{p=0}^{d_3-1} \sum_{q=0}^{\infty} z^{pd_1 + qd_2} - 1 = \sum_{p=0}^{d_3-1} \sum_{q=0}^{\infty} z^{pd_1 + qd_2} = \sum_{j=1}^{\infty} z^{b_j d_3}, \tag{67}
\]

where \( b_j \in \mathbb{N} \) is defined as the integer which satisfies the Diophantine equations in \( b_j \)

\[
pd_1 + qd_2 = b_j d_3, \quad p = 0, \ldots, d_2 - 1, \quad q = 0, \ldots
\]

at least with one solution. Recalling the definition (13) and (14) of the Johnson's minimal relations we conclude that

\[
b_1 = a_{33}, \tag{69}
\]

and the first term in series expansion (67) reads \( z^{a_{33}d_3} \). Hence, formula (34) follows. Notice that (67) can be obtained by straightforward calculation of the Hadamard product (63) according to Lemma 9 and representation (65)

\[
\Psi_3 (d^3; z) = \left( \sum_{p=0}^{d_3-1} \sum_{q=0}^{\infty} z^{pd_1 + qd_2} \right) \otimes \sum_{r=0}^{\infty} z^{rd_3} - 1 = \left( \sum_{p=0}^{d_3-1} \sum_{q=0}^{\infty} z^{pd_1 + qd_2} \right) \otimes \sum_{r=0}^{\infty} z^{rd_3} = \sum_{j=1}^{\infty} z^{b_j d_3},
\]

where \( b_j \) is defined in (68).

We finish this Section by giving an integral representation for (63) according to (37)

\[
\Psi_3 (d^3; z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - e^{id_1 d_2 t}}{1 - e^{id_1 t}} \frac{1 - e^{id_2 d_3 t}}{1 - e^{id_3 t}} dt - 1. \tag{70}
\]

The other two functions, \( \Psi_1 (d^3; z) \) and \( \Psi_2 (d^3; z) \), can be written by the cyclic permutation of the indices \( (1, 2, 3) \) in (70)

\[
\Psi_1 (d^3; z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - e^{id_2 d_3 t}}{1 - e^{id_3 t}} \frac{1 - e^{id_3 d_1 t}}{1 - e^{id_1 t}} dt - 1, \tag{71}
\]

\[
\Psi_2 (d^3; z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - e^{id_3 d_1 t}}{1 - e^{id_1 t}} \frac{1 - e^{id_1 d_2 t}}{1 - e^{id_2 t}} dt - 1. \tag{72}
\]

4 The explicit calculation of the entries of the Johnson's matrix

The complexity of expressions (34) and (36) in conjunction with (70) – (72) makes further evaluation of the diagonal elements \( a_{kk} (d^3) \) of the Johnson's matrix excessively difficult. Therefore, we develop in this Section another approach representing \( a_{kk} (d^3) \) as zeroes of some functions.

Combining (67) and (70) we have

\[
\frac{1}{2\pi} \int_0^{2\pi} \frac{1 - e^{id_1 d_2 t}}{1 - e^{id_1 t}} \frac{1 - e^{id_2 d_3 t}}{1 - e^{id_2 t}} \frac{1 - e^{id_3 d_1 t}}{1 - e^{id_3 t}} dt - 1 = \sum_{j=1}^{\infty} z^{b_j d_3}, \tag{73}
\]
where \( b_1, b_2, \ldots \) present all positive integers such that in accordance with (68) every integer \( b_3 d_3 \) is representable by \( d_1 \) and \( d_2 \).

Fix the index \( j = k \) and differentiate with respect to \( z^{d_3} \) both sides of the equality (73) \( b_k \) times, and take their values at \( z = 0 \)

\[
\frac{b_k!}{2\pi} \int_0^{2\pi} (1 - e^{i d_3 z t}) e^{-i b_k d_3 t} dt = b_k!
\]  

(74)

Define a new function \( \Xi_3 (d^3; b) \) by

\[
\Xi_3 (d^3; b) = 1 - \frac{1}{2\pi} \int_0^{2\pi} \frac{(1 - e^{i d_3 z t}) e^{-i b d_3 t}}{(1 - e^{i d_3 t}) (1 - e^{i d_2 t})} dt .
\]  

(75)

Thus \( b_k \) is its zero, \( \Xi_3 (d^3; b_k) = 0 \), and \( a_{33} (d^3) \) is its minimal integer zero according to (69)

\[
a_{33} (d^3) = \min \left\{ b_k \mid \Xi_3 (d^3; b_k) = 0, \ b_k \in \mathbb{N} \right\} .
\]  

(76)

Notice that \( 2 \leq a_{33} (d^3) \leq d_1 - 1 \) according to [10], that bound the range of \( b \) where the first zero \( b_1 \) does appear. In Figure 3 we present the typical plot of the function \( \Xi_3 (d^3; b) \) for the triple \( d_1 = 23 \), \( d_2 = 29 \) and \( d_3 = 44 \) which was considered numerically in [24]

![Figure 3: Typical plot of the function \( \Xi_3 \) and the distribution of its zeroes \( b_k \). The diagonal element \( a_{33} \) of the Johnson’s matrix reads \( a_{33} (23, 29, 44) = 5 \).](image)

The other two diagonal elements, \( a_{11} (d^3) \) and \( a_{22} (d^3) \), can be obtained by the cyclic permutation of the indices (1, 2, 3) in (75) and (76)

\[
a_{11} (d^3) = \min \left\{ b_k \mid \Xi_1 (d^3; b_k) = 0, \ b_k \in \mathbb{N} \right\} , \ a_{22} (d^3) = \min \left\{ b_k \mid \Xi_2 (d^3; b_k) = 0, \ b_k \in \mathbb{N} \right\}
\]  

(77)

where

\[
\Xi_1 (d^3; b) = 1 - \frac{1}{2\pi} \int_0^{2\pi} \frac{(1 - e^{i d_2 d_3 t}) e^{-i b d_3 t}}{(1 - e^{i d_3 t}) (1 - e^{i d_2 t})} dt , \ \Xi_2 (d^3; b) = 1 - \frac{1}{2\pi} \int_0^{2\pi} \frac{(1 - e^{i d_3 d_1 t}) e^{-i b d_2 t}}{(1 - e^{i d_3 t}) (1 - e^{i d_1 t})} dt .
\]

and \( 2 \leq a_{11} (d^3) \leq d_2 - 1 \), \( 2 \leq a_{22} (d^3) \leq d_1 - 1 \).

Performing the calculation for the triple \( d_1 = 23 \), \( d_2 = 29 \) and \( d_3 = 44 \) we can find in accordance with (77)

\[
a_{11} (23, 29, 44) = a_{22} (23, 29, 44) = 7 .
\]

Together with \( a_{33} (23, 29, 44) = 5 \) these are in full agreement with the Johnson’s matrix of minimal relations which was found in [10], Example 2.
4.1 The off–digonal elements of the Johnson's matrix

The uniqueness of the Johnson's matrix \(((a_{ij}))\) of minimal relations makes us possible to determine also its off–digonal elements for non–symmetric semigroup $S(d^3)$. As was shown by Johnson [12], six off–digonal elements of the matrix \(((a_{ij}))\) are related by six identities

$$a_{21} + a_{31} = a_{11}, \quad a_{12} + a_{32} = a_{22}, \quad a_{13} + a_{23} = a_{33},$$

$$a_{23}a_{32} = a_{22}a_{33} - d_1, \quad a_{13}a_{31} = a_{11}a_{33} - d_2, \quad a_{12}a_{21} = a_{11}a_{22} - d_3. \quad (78)$$

These identities give rise to six quadratic equations

$$d_3a_{23}^2 - (\langle a, d \rangle - 2a_{11}d_1)a_{23} + (a_{22}a_{33} - d_1)d_2 = d_3a_{13}^2 - (\langle a, d \rangle - 2a_{22}d_2)a_{13} + (a_{11}a_{33} - d_2)d_1 = 0,$$

$$d_2a_{32}^2 - (\langle a, d \rangle - 2a_{11}d_1)a_{32} + (a_{33}a_{22} - d_1)d_3 = d_2a_{12}^2 - (\langle a, d \rangle - 2a_{33}d_3)a_{12} + (a_{11}a_{22} - d_3)d_1 = 0,$$

$$d_1a_{31}^2 - (\langle a, d \rangle - 2a_{22}d_2)a_{31} + (a_{33}a_{11} - d_2)d_3 = d_1a_{21}^2 - (\langle a, d \rangle - 2a_{33}d_3)a_{21} + (a_{22}a_{11} - d_3)d_2 = 0,$$

where $\langle a, d \rangle$ is already defined in (11). Notice that all these equations have common discriminant

$$\langle a, d \rangle^2 - 4(a_{11}a_{22}d_1d_2 + a_{22}a_{33}d_2d_3 + a_{33}a_{11}d_3d_1) + 4d_1d_2d_3,$$

which can be recognized as $J^2(d^3)$ defined in (11). As was shown in [10] its square root, $J(d^3)$, is a positive integer,

$$J(d^3) = |a_{12}a_{23}a_{31} - a_{13}a_{32}a_{21}| \geq 1. \quad (79)$$

Therefore the solutions of all six quadratic equations are always rational numbers.

$$a_{23}^+ = \frac{1}{2d_3}(\langle a, d \rangle \pm J(d^3) - 2a_{11}d_1), \quad a_{32}^+ = \frac{1}{2d_2}(\langle a, d \rangle \pm J(d^3) - 2a_{11}d_1),$$

$$a_{31}^+ = \frac{1}{2d_1}(\langle a, d \rangle \pm J(d^3) - 2a_{22}d_2), \quad a_{13}^+ = \frac{1}{2d_3}(\langle a, d \rangle \pm J(d^3) - 2a_{22}d_2),$$

$$a_{12}^+ = \frac{1}{2d_2}(\langle a, d \rangle \pm J(d^3) - 2a_{33}d_3), \quad a_{21}^+ = \frac{1}{2d_1}(\langle a, d \rangle \pm J(d^3) - 2a_{33}d_3). \quad (80)$$

Show that one of two rational roots associated with every quadratic equation is always positive integer while another is necessarily not integer. Making use of the identity [10], formula (135),

$$\frac{1}{2}(\langle a, d \rangle \pm J(d^3)) = a_{11}a_{22}a_{33} + \frac{1}{2}(a_{12}a_{23}a_{31} + a_{13}a_{32}a_{21} \pm |a_{12}a_{23}a_{31} - a_{13}a_{32}a_{21}|),$$

and the identities (78), we have

$$a_{23}^+ = a_{23}, \quad a_{32}^+ = a_{32}, \quad a_{31}^+ = a_{31}, \quad a_{13}^+ = a_{13}, \quad a_{12}^+ = a_{12}, \quad a_{21}^+ = a_{21}. \quad (81)$$

$$a_{23}^- = a_{32}d_2/d_3, \quad a_{32}^- = a_{23}d_3/d_2, \quad a_{31}^- = a_{13}d_3/d_1, \quad a_{13}^- = a_{31}d_1/d_3, \quad a_{12}^- = a_{21}d_1/d_2, \quad a_{21}^- = a_{12}d_2/d_1. \quad (82)$$

There is only one way to satisfy the uniqueness of the Johnson’s matrix \(((a_{ij}))\) of minimal relations comprised exclusively of integer entries $a_{ij}$ if we require that $a_{ij}^- = a_{ij}^+ = a_{ij}$. However, as one can see from (80), this leads to $J(d^3) = 0$ that contradicts (79).

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