QUANTITATIVE ESTIMATES ON THE ENHANCED BINDING FOR THE PAULI-FIERZ OPERATOR

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Abstract. For a quantum particle interacting with a short-range potential, we estimate from below the shift of its binding threshold, which is due to the particle interaction with a quantized radiation field.

1. Introduction

Recently, the question of enhanced binding in nonrelativistic QED has been extensively studied in several publications [10, 9, 6, 3, 2, 4]. Dressing a charged particle with photons increases the ability of a potential to confine it. For the Pauli-Fierz operator which describes a nonrelativistic particle interacting with a radiation field, this effect was proved for small values of the fine structure constant $\alpha$, first under the simplifying assumption that the spin of the particle is absent [9], and later generalized to the case of a particle with spin [6, 3]. In [2], it was shown that the effect of the enhanced binding is asymptotically small in $\alpha$ in the sense that the binding threshold for the Pauli-Fierz operator tends to the binding threshold for the corresponding Schrödinger operator as $\alpha$ tends to zero. Some quantitative estimates on this effect were obtained in [4] where it was proved that the difference between the binding threshold for the Schrödinger operator and the corresponding Pauli-Fierz operator with spin zero is at least of the order $\alpha$. In the work at hand, using a different method, we prove similar results for the more general case of a particle with spin zero or one half. Notice that studying the enhanced binding effect in the case of a particle with spin requires recovering one more term of the energy’s expansion in powers of $\alpha$ than in the spinless case.

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The method of the proof is a further development of a method used in \cite{9} and \cite{6}. We prove that the Pauli-Fierz operator has a ground state even for some value of the potential coupling constant that is smaller than the binding threshold for the corresponding Schrödinger operator. To do so, we construct a trial function for which the quadratic form of the Pauli-Fierz operator with this coupling constant takes a value strictly less than the self-energy. Then we apply \cite[Theorem 2.1]{8} which tells us that this implies the existence of a ground state. The trial function we use is similar to the one in \cite{6} with some modifications necessary to obtain quantitative estimates in the case with spin. It is constructed using the ground state of the self-energy operator with total momentum zero.

As in all previous papers \cite{9,6,3,2,4}, our method is asymptotic in $\alpha$. Therefore, the problem of establishing the enhanced binding effect and estimating its strength for the physical value of $\alpha \approx 1/137$ still remains open.

2. Definitions and main result

The Pauli-Fierz Hamiltonian $H$ for a charged particle with or without spin in an external electrostatic potential and coupled to the quantized electromagnetic radiation field is defined by

\begin{equation}
H = (-i\nabla_x \otimes I_f + \sqrt{\alpha} A(x))^2 + g \sqrt{\alpha} \sigma \cdot B(x) + \lambda W(x) \otimes I_f + I_{el} \otimes H_f - c_{n.o.} \alpha.
\end{equation}

The operator $H$ acts on the Hilbert space $\mathcal{H} := \mathcal{F}^{\text{el}} \otimes \mathfrak{F}$. The Hilbert space $\mathcal{F}^{\text{el}}$ of the nonrelativistic particle is $L^2(\mathbb{R}^3) \otimes \mathbb{C}^2$ in the case $g = 1$ and $L^2(\mathbb{R}^3)$ in the case $g = 0$. Here $\mathbb{R}^3$ is the configuration space of a single particle, while $\mathbb{C}^2$ accommodates its spin in the case $g = 1$.

We will describe the quantized electromagnetic field by use of the Coulomb gauge condition. Accordingly, the one-photon Hilbert space is given by $L^2(\mathbb{R}^3) \otimes \mathbb{C}^2$, where $\mathbb{R}^3$ denotes either the photon momentum or configuration space, and $\mathbb{C}^2$ accounts for the two independent transversal polarizations of the photon. The photon Fock space is then defined by

$$\mathfrak{F} = \bigoplus_{n=0}^{\infty} \mathfrak{F}^{(n)}_s,$$

where the n-photons space $\mathfrak{F}^{(n)}_s = \bigotimes_n (L^2(\mathbb{R}^3) \otimes \mathbb{C}^2)$ is the symmetric tensor product of $n$ copies of $L^2(\mathbb{R}^3) \otimes \mathbb{C}^2$. 
We use units such that $\hbar = c = 1$, and where the mass of the particle equals $m = 1/2$. The particle charge is then given by $e = \sqrt{\alpha}$. As usual, we will consider $\alpha$ as a small parameter.

The operator that couples a particle to the quantized vector potential is given by

$$A(x) = \sum_{\lambda=1,2} \int_{\mathbb{R}^3} \frac{\zeta(|k|)}{2\pi |k|^{1/2}} \varepsilon_\lambda(k) \left[ e^{ikx} \otimes a_\lambda(k) + e^{-ikx} \otimes a_\lambda^*(k) \right] dk =: D(x) + D^*(x),$$

where $\text{div} A = 0$ by the Coulomb gauge condition. The operators $a_\lambda, a_\lambda^*$ satisfy the usual commutation relations

$$[a_\nu(k), a_\lambda^*(k')] = \delta(k - k')\delta_{\lambda,\nu}, \quad [a_\nu(k), a_\lambda(k')] = 0.$$

The vectors $\varepsilon_\lambda(k) \in \mathbb{R}^3$ are the two orthonormal polarization vectors perpendicular to $k$,

$$\varepsilon_1(k) = \frac{(k_2, -k_1, 0)}{\sqrt{k_1^2 + k_2^2}}$$

and

$$\varepsilon_2(k) = \frac{k}{|k|} \wedge \varepsilon_1(k).$$

The function $\zeta(|k|)$ describes the ultraviolet cutoff on the wavenumbers $k$. We assume $\zeta$ to be of class $C^1$ and to have compact support.

The constant $c_{\text{n.o.}}$ is

$$c_{\text{n.o.}} = [D, D^*] = \frac{2}{\pi} \int_0^\infty r|\zeta(r)|^2 dr,$$

and subtraction of the constant $c_{\text{n.o.}}\alpha$ amounts to normal ordering of the operator $A^2$.

The operator that couples a particle to the magnetic field $B = \text{curl} A$ is given by

$$B(x) = \sum_{\lambda=1,2} \int_{\mathbb{R}^3} \frac{\zeta(|k|)}{2\pi |k|^{1/2}} k \times i\varepsilon_\lambda(k) \left[ e^{ikx} \otimes a_\lambda(k) - e^{-ikx} \otimes a_\lambda^*(k) \right] dk =: K(x) + K^*(x).$$

In Equation (1), $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ is the 3-component vector of Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The photon field energy operator $H_f$ is given by

$$H_f = \sum_{\lambda=1,2} \int_{\mathbb{R}^3} |k| a_\lambda^*(k) a_\lambda(k) dk.$$
The multiplicative potential \( W \) is assumed to be short range and in \( L^4_{\text{loc}}(\mathbb{R}^3) \), and \( \lambda \) is a positive coupling constant. If the negative part of \( W \) is nontrivial, then there exists a critical value \( \lambda_0 \) such that the Schrödinger operator \(-\Delta + \lambda W\) has discrete spectrum for all \( \lambda > \lambda_0 \), but does not have any discrete spectrum for \( 0 \leq \lambda < \lambda_0 \). Analogously, the Pauli-Fierz operator also has a critical coupling constant \( \lambda_1 \), which depends on the fine structure constant \( \alpha \). It is known [2] that \( \lambda_1 \) converges to \( \lambda_0 \) from below as \( \alpha \) goes to zero.

Before stating our main result, let us introduce some notations. For \( v \) a measurable function in \( \mathbb{R}^3 \), we define

\[
\langle 3 \rangle \quad d_v = \frac{1}{2\pi} \left( \int \frac{|v(x)||v(y)|}{|x-y|^2} dx dy \right)^{\frac{1}{2}},
\]

if \( v \) is not spherically symmetric and

\[
\langle 4 \rangle \quad d_v = \min \{ \frac{1}{2\pi} \left( \int \frac{|v(x)||v(y)|}{|x-y|^2} dx dy \right)^{\frac{1}{2}}, \int_0^\infty t|v(t)| dt \}
\]

if \( v \) is spherically symmetric.

Our main result is thus

**Theorem 2.1.** Assume that \( W(x) \) satisfies the following conditions: \( W \in L^4_{\text{loc}}(\mathbb{R}^3) \) and there exists \( a > 0, c > 0 \) and \( \delta > 0 \) such that for all \( |x| > a, \ |W(x)| \leq c(1 + |x|)^{-2-\delta} \). Then

\[
\lambda_1 \leq \lambda_0 (1 - \alpha \eta^2 + O(\alpha^{\frac{5}{4}}))
\]

with

\[
\eta^2 = \frac{1}{6\pi^2} \int_{\mathbb{R}^3} \frac{\zeta(|k|)}{|k|(k^2 + |k| + C_W)} dk,
\]

and

\[
C_W = \lambda_0^2 (1 + \lambda_0 d_{W^+}) d_{W^2} \quad \text{and} \quad W^+ = (|W| + W)/2.
\]

3. **Proof of the main Theorem**

In this section, we will prove the main theorem in the case of particle with spin \( g = 1 \). The proof for \( g = 0 \) can easily be deduced with several simplifications.

We start with establishing some useful preliminary estimates.

3.1. **Properties of the self-energy operator \( T(0) \) with zero total momentum.** This section addresses the main properties of the self-energy operator \( T(0) \). Let us consider the case of a free particle coupled
to the quantized electromagnetic field. The self-energy operator $T$ is given by

$$T = (-i \nabla_x \otimes I_f + \sqrt{\alpha} A(x))^2 + g \sqrt{\alpha} \sigma \cdot B(x) + I_{el} \otimes H_f - c_{n.o.} \alpha. $$

We note that this system is translationally invariant, that is, $T$ commutes with the operator of total momentum $P_{tot} = p_{el} \otimes I_f + I_{el} \otimes P_f$.

Let $\mathcal{H}_f \cong \mathbb{C}^2 \otimes \mathcal{F}$ denotes the fibre Hilbert space corresponding to conserved total momentum $P$. For any fixed value $P$ of the total momentum, the restriction of $T$ to the fibre space $\mathcal{H}_f$ is given by (see e.g. [5])

$$T(P) = (P - P_f + \sqrt{\alpha} A(0))^2 + g \sqrt{\alpha} \sigma \cdot B(0) + H_f - c_{n.o.} \alpha. $$

We denote $\Sigma_0 := \inf \sigma(T(0))$.

For the reader convenience, we first collect in the following theorem different known facts regarding the ground state of the operator $T(0)$, which will be used in the proof of the main theorem.

From now on, we will denote by $\Pi_n$ the projection onto the subspace of $\mathbb{C}^2 \otimes \mathcal{F}$ corresponding to vectors which have all components zero except the $n$-photon components. We also define $\Pi_0^2 = 1 - \sum_{i=1}^{n-1} \Pi_n$.

For vectors in $\mathbb{C}^2 \otimes \mathcal{F}$, the norm $\lVert \cdot \rVert$ will refer to the standard norm in $\mathbb{C}^2 \otimes \mathcal{F}$.

**Theorem 3.1.** [7][8][9][11] For $\alpha$ sufficiently small we have:

- $\Sigma_0$ is an eigenvalue bordering to continuous spectrum of $T(0)$ and $\Sigma_0 = \inf \sigma(T)$.
- For any $\Omega_0 \in \ker(T(0) - \Sigma_0)$, its projection $\Pi_0 \Omega_0$ onto the zero-photon sector of $\mathbb{C}^2 \otimes \mathcal{F}$ fulfills $\lVert \Pi_0 \Omega_0 \rVert \neq 0$. If $\Omega_0$ is normalized by $\lVert \Pi_0 \Omega_0 \rVert = 1$, then the following inequalities are satisfied: $\lVert \Omega_0 \rVert = 1 + \mathcal{O}(\alpha^{1/2})$, $\lVert D(0) \Omega_0 \rVert = \mathcal{O}(\alpha^{1/2})$, and $\lVert H_f^{1/2} \Omega_0 \rVert = \mathcal{O}(\alpha^{1/2})$.
- For the photon number operator $N_f := \sum_{\lambda=1,2} \int a^*_\lambda(k) a_\lambda(k) dk$, we have $\lVert N_f^{1/2} \Omega_0 \rVert = \mathcal{O}(\alpha^{1/2})$.

**Corollary 3.2.** For any vector $\Omega_0 \in \ker(T(0) - \Sigma_0)$ normalized by $\lVert \Pi_0 \Omega_0 \rVert = 1$, we have $\lVert \Omega_0 \rVert = 1 + \mathcal{O}(\alpha)$, $\lVert D^*(0) \Pi_0^2 \Omega_0 \rVert = \mathcal{O}(\alpha^{1/2})$ and $\lVert \sigma \cdot K^*(0) \Pi_0^2 \Omega_0 \rVert = \mathcal{O}(\alpha^{1/2})$.

In the following, we consider two 4-vectors in $\mathbb{C}^2 \otimes (L^2(\mathbb{R}^3) \otimes \mathbb{C}^2)$, of the form $(\xi(\uparrow, k, \lambda_1), \xi(\uparrow, k, \lambda_2), \xi(\downarrow, k, \lambda_1), \xi(\downarrow, k, \lambda_2))$, where $\uparrow$ and $\downarrow$
refer to the spin up and spin down of the particle, and \( \lambda_1, \lambda_2 \) refer to the two polarizations of the transverse photons.

\[
\Gamma_{a,b} := \left( \begin{array}{c}
\Gamma(\uparrow, k, \lambda_1) \\
\Gamma(\uparrow, k, \lambda_2) \\
\Gamma(\downarrow, k, \lambda_1) \\
\Gamma(\downarrow, k, \lambda_2)
\end{array} \right) := \left( \begin{array}{c}
\frac{\zeta(k)}{|k|^2} \left(-a\sqrt{k_1^2 + k_2^2} + b\frac{(k_1 - ik_2)k_3}{\sqrt{k_1^2 + k_2^2}}\right) \\
b\zeta(k)\frac{-k_2 - ik_1}{\sqrt{k_1^2 + k_2^2}}|k|^2 \\
\frac{\zeta(k)}{|k|^2} \left(b\sqrt{k_1^2 + k_2^2} + a\frac{(k_1 + ik_2)k_3}{\sqrt{k_1^2 + k_2^2}}\right) \\
a\zeta(k)\frac{-k_2 + ik_1}{\sqrt{k_1^2 + k_2^2}}|k|^2
\end{array} \right).
\]

Let
\[
\varphi_{a,b} = \sqrt{\alpha} \frac{i}{2\pi |k|(1 + |k|)} \Gamma_{a,b}.
\]

**Proposition 3.3** (Approximate ground state of the Pauli-Fierz operator). For \( a \) and \( b \) in \( \mathbb{C} \) such that \( |a|^2 + |b|^2 = 1 \), we consider the family of real-valued functionals \( L_{a,b} \) defined on \( \mathbb{C}^2 \otimes L^2(\mathbb{R}^3) \otimes \mathbb{C}^2 \) by

\[
L_{a,b}(\xi) = \langle (k^2 + |k|)\xi, \xi \rangle + 2\sqrt{\alpha}\Re\langle \xi, \Pi_1 \sigma \cdot K^*(0) \left( \begin{array}{c} a \\
b \end{array} \right) 0,0,\cdots \rangle,
\]

where as before \( B(0) = K(0) + K^*(0) \). Then we have

i) The vector \( \varphi_{a,b} \) defined by (6) is the unique minimizer of \( L_{a,b} \).

ii) \( |\Sigma_0 - \inf L_{a,b}(\xi)| = \mathcal{O}(\alpha^{3/2}) \).

iii) Let \( \Omega_0 \in \ker(T(0) - \Sigma_0) \) be normalized by \( \|\Pi_0 \Omega_0\| = 1 \). Let us denote by \( (a,b) := \Pi_0 \Omega_0 \). We define the scalar product \( \langle \cdot, \cdot \rangle_1 \) on the one-photon sector \( \Pi_1(\mathbb{C}^2 \otimes \mathbb{F}) = \mathbb{C}^2 \otimes L^2(\mathbb{R}^3) \otimes \mathbb{C}^2 \) by \( \langle f, g \rangle_1 = \langle (k^2 + |k|)f, g \rangle_{\Pi_1(\mathbb{C}^2 \otimes \mathbb{F})} \). Then for \( \gamma \in \mathbb{R} \) and \( R \in \mathbb{C}^2 \otimes L^2(\mathbb{R}^3) \otimes \mathbb{C}^2 \) such that

\[
\Pi_1 \Omega_0 = \gamma \varphi_{a,b} + R
\]

and \( \langle \varphi_{a,b}, R \rangle_1 = 0 \), we have

\[
\langle R, R \rangle_1 = \mathcal{O}(\alpha^{3/2}) \quad \text{and} \quad |\gamma - 1| = \mathcal{O}(\alpha^{3/4})
\]

**Remark 3.4.** In the above proposition, and in the sequel, we use the same notation for \( \Pi_1 \Omega_0 \) as a vector in \( \mathbb{C}^2 \otimes \mathbb{F} \) which has all components zero except its one-photon component \( (\Pi_1 \Omega_0)^{(1)} \), as well as for the vector \( (\Pi_1 \Omega_0)^{(1)} \) in \( \mathbb{C}^2 \otimes L^2(\mathbb{R}^3) \otimes \mathbb{C}^2 \).

**Proof.** In this proof, for the sake of simplicity of notations, we will drop the argument 0 in the operators \( A(0), B(0), D(0), K(0) \) and their adjoint. We first prove i). Denoting

\[
g_{a,b} := \frac{1}{(k^2 + |k|)} \Pi_1 \sigma \cdot K^*(0) \left( \begin{array}{c} a \\
b \end{array} \right) 0,0,\cdots ,
\]
we have

\begin{equation}
L_{a,b}(\xi) = \langle \xi, \xi \rangle_1 + 2 \sqrt{aR} \Re \langle \xi, g_{a,b} \rangle_1 = \| \xi + \sqrt{a} g_{a,b} \|_1^2 - \| \sqrt{a} g_{a,b} \|_1^2,
\end{equation}

where \( \| \cdot \|_1 \) is the norm associated to the scalar product \( \langle \cdot, \cdot \rangle_1 \). Therefore, the minimizer of \( L_{a,b} \) is \(-\sqrt{a} g_{a,b} \). A straightforward computation shows that \(-\sqrt{a} g_{a,b} = \varphi_{a,b} \). This implies that

\begin{equation}
\inf L_{a,b} = L_{a,b}(\varphi_{a,b}) = -\| \varphi_{a,b} \|_1^2.
\end{equation}

We now prove ii). We have

\begin{equation}
\langle T(0)\Omega_0, \Omega_0 \rangle = \langle P_f^2 \Omega_0, \Omega_0 \rangle - \sqrt{a} 2R \Re \langle P_f \Omega_0, A \Omega_0 \rangle + \alpha \langle A^2 \Omega_0, \Omega_0 \rangle + \sqrt{a} \langle \sigma \cdot B \Omega_0, \Omega_0 \rangle + \langle H_f \Omega_0, \Omega_0 \rangle - c_{n,o} \alpha
\end{equation}

Let us estimate the terms in the above equality in order to identify those who are of order \( \alpha^{3/2} \) and higher.

\begin{equation}
\langle P_f^2 \Omega_0, \Omega_0 \rangle = \langle P_f^2 \Pi_0 \Omega_0, \Pi_0 \Omega_0 \rangle + \langle P_f^2 \Pi_1 \Omega_0, \Pi_1 \Omega_0 \rangle + \langle P_f^2 \Pi_2^\infty \Omega_0, \Pi_2^\infty \Omega_0 \rangle
\end{equation}

\begin{equation}
= \langle P_f^2 \Pi_0 \Omega_0, \Pi_0 \Omega_0 \rangle + \langle P_f^2 \Pi_1 \Omega_0, \Pi_1 \Omega_0 \rangle + \langle P_f^2 \Pi_2^\infty \Omega_0, \Pi_2^\infty \Omega_0 \rangle.
\end{equation}

Now, using the fact that \( n \)-photon sectors are invariant under \( P_f \), \( P_f \Pi_0 \Omega_0 = 0 \), and \( \langle P_f \Omega_0, A \Omega_0 \rangle = \langle A P_f \Pi_1^\infty \Omega_0, \Omega_0 \rangle = \langle A \Pi_1^\infty \Omega_0, P_f \Omega_0 \rangle = \langle A \Pi_1^\infty \Omega_0, P_f \Pi_1^\infty \Omega_0 \rangle \), we get

\begin{equation}
| \langle P_f \Omega_0, A \Omega_0 \rangle | = | \langle P_f \Pi_1^\infty \Omega_0, A \Pi_1^\infty \Omega_0 \rangle |
\leq | \langle P_f \Pi_1 \Omega_0, D \Pi_2 \Omega_0 \rangle + | \langle P_f \Pi_2^\infty \Omega_0, D^* \Pi_1^\infty \Omega_0 \rangle |
\leq | \langle P_f \Pi_1 \Omega_0, D \Pi_2 \Omega_0 \rangle | + | \langle P_f \Pi_2^\infty \Omega_0, D \Pi_3 \Omega_0 \rangle |
+ | \langle P_f \Pi_1 \Omega_0, D^* \Pi_1^\infty \Omega_0 \rangle |
\leq \| P_f \Pi_1 \Omega_0 \| \| D \Pi_2 \Omega_0 \| + \frac{1}{2} \| P_f \Pi_2^\infty \Omega_0 \|^2 + \frac{1}{2} \| D \Pi_3 \Omega_0 \|^2 + 2 \| D \Pi_2 \Omega_0 \|^2 + 2 \| D^* \Pi_1^\infty \Omega_0 \|^2.
\end{equation}

Using Theorem 3.1 and Corollary 3.2 and the fact that \( \| P_f \Pi_1 \Omega_0 \| \leq c(\Lambda) \| \Pi_1 \Omega_0 \| = O(\alpha^{1/2}) \), where \( c(\Lambda) \) depends only on the ultraviolet cutoff, yields

\begin{equation}
| \langle P_f \Omega_0, A \Omega_0 \rangle | \leq \frac{1}{2} \| P_f \Pi_2^\infty \Omega_0 \|^2 + O(\alpha^2).
\end{equation}
We also have
\[
\langle A^2\Omega_0, \Omega_0 \rangle = \langle (D + D^*)^2\Omega_0, \Omega_0 \rangle = 2\Re \langle DD\Omega_0, \Omega_0 \rangle + 2\|D\Omega_0\|^2 + \|[D, D^*]\|\|\Omega_0\|^2 = O(\alpha^{1/2}) + O(\alpha) + \|[D, D^*]\|\|\Omega_0\|^2,
\]
where we used from Theorem 3.1 that \(\|\Omega_0\| = 1 + O(\alpha)\) and \(\|D\Omega_0\| = O(\alpha^{1/2})\). Since the commutator \([D, D^*]\) equals \(c_{n.o.}\) we arrive at
\[
\langle A^2\Omega_0, \Omega_0 \rangle = c_{n.o.} + O(\alpha^{1/2}).
\]
Finally we have, writing \(B = K + K^*\)
\[
\langle \sigma \cdot B\Omega_0, \Omega_0 \rangle = \langle \sigma \cdot K\Pi_1\Omega_0, \Pi_0\Omega_0 \rangle + \langle \sigma \cdot K\Pi_2\Omega_0, \Pi_1\Omega_0 \rangle + \langle \sigma \cdot K^*\Pi_0\Omega_0, \Pi_1\Omega_0 \rangle + \langle \sigma \cdot K^*\Pi_2\Omega_0, \Pi_1\Omega_0 \rangle = 2\Re \langle \sigma \cdot K\Pi_1\Omega_0, \Pi_0\Omega_0 \rangle + 2\Re \langle \sigma \cdot K^*\Pi_2\Omega_0, \Pi_1\Omega_0 \rangle.
\]
using Theorem 3.1 and Corollary 3.2 we obtain
\[
\langle \sigma \cdot B\Omega_0, \Omega_0 \rangle = 2\Re \langle \sigma \cdot K\Pi_1\Omega_0, \Pi_0\Omega_0 \rangle + O(\alpha).
\]
Collecting (10)-(15) and using \(H_f\Pi_2^2\Omega_0, \Pi_2^2\Omega_0 \geq 0\) we obtain
\[
\langle T(0)\Omega_0, \Omega_0 \rangle \geq \langle P_f^2\Pi_1\Omega_0, \Pi_1\Omega_0 \rangle + \langle H_f\Pi_1\Omega_0, \Pi_1\Omega_0 \rangle + 2\sqrt{\alpha}\Re \langle \sigma \cdot K\Pi_1\Omega_0, \Pi_0\Omega_0 \rangle + O(\alpha^2).
\]
Since on the one-photon sector the operator \(P_f^2\) reduces to multiplication by \(k^2\), and the operator \(H_f\) reduced to multiplication by \(|k|\), we obtain
\[
\Sigma_0 = \frac{\langle T(0)\Omega_0, \Omega_0 \rangle}{\|\Omega_0\|^2} \geq L_{a,b}(\Pi_1\Omega_0) + O(\alpha^{3/2}) \geq \inf L_{a,b} + O(\alpha^{3/2}).
\]
On the other hand, using i), and for \(\psi_{a,b} = (\begin{pmatrix} a \\ b \end{pmatrix}, \varphi_{a,b}, 0, 0, \cdots)\) we have
\[
\inf L_{a,b} = L_{a,b}(\varphi_{a,b}) = \langle T(0)\psi_{a,b}, \psi_{a,b} \rangle \geq \Sigma_0\|\psi_{a,b}\|^2 = \Sigma_0(1 + O(\alpha)) \geq \Sigma_0 + O(\alpha^2)
\]
Inequalities (17) and (18) conclude the proof of ii).
Eventually, we prove iii). Due to the Inequalities (17) and (18), we have \(\inf L_{a,b} + O(\alpha^{3/2}) = L_{a,b}(\Pi_1\Omega_0)\). Using (8), the fact that \(-\sqrt{\alpha}g_{a,b} = \varphi_{a,b}\), and (19), we thus get
\[
\inf L_{a,b} + O(\alpha^{3/2}) = L_{a,b}(\Pi_1\Omega_0) = \|\gamma \varphi_{a,b} + R - \varphi_{a,b}\|_1^2 - \|\varphi_{a,b}\|_1^2 = (\gamma - 1)^2\|\varphi_{a,b}\|_1^2 + \|R\|_1^2 + \inf L_{a,b},
\]
3.2. **Proof of Theorem 2.1** As it was mentioned in the introduction, we prove the theorem by constructing a trial function $\Psi$ for which the quadratic form of $H$ takes a value strictly smaller than $\Sigma_0\|\Psi\|^2$.

Let us start by proving an auxiliary result. For $\gamma \in (0, 1)$, we define $f_\gamma \in L^2(\mathbb{R}^3)$ to be a normalized real valued eigenfunction, with associated eigenvalue $e_\gamma$, of the Schrödinger operator

$$h_\gamma := -(1 - \gamma)\Delta + \lambda_0 W(x).$$

Here $\lambda_0$ is the critical coupling constant defined in Section 2.

**Lemma 3.5.** Then for $\lambda \leq \lambda_0$, we have

$$\sum_i \langle (-\Delta + \lambda W) \frac{\partial f_\gamma}{\partial x_i}, \frac{\partial f_\gamma}{\partial x_i} \rangle \leq C_W \|\nabla f_\gamma\|^2 + o_\gamma(1)\|\nabla f_\gamma\|^2,$$

with $C_W := \lambda_0^2(1 + \lambda_0 d_{W_+})d_{W_2}$, where $W_+ = (W + |W|)/2$ and $d_{W_2}$ and $d_{W_+}$ are defined by (3)-(4).

**Proof.** For a potential $V$ such that $V \in L^2_{\text{loc}}$ and short range we have

$$|\langle V\psi, \psi \rangle| \leq d_V \| \nabla \psi \|^2.$$

Moreover, we know that $f_\gamma$ is an eigenfunction of $-(1 - \gamma)\Delta + \lambda_0 W$ and that the associated eigenvalue $e_\gamma$ tends to zero as $\gamma$ tends to zero, since $\lambda_0$ is the critical coupling constant. Therefore we obtain the following sequence of inequalities

$$\sum_i \langle (-\Delta + \lambda W) \frac{\partial f_\gamma}{\partial x_i}, \frac{\partial f_\gamma}{\partial x_i} \rangle \leq \sum_i \langle (-\Delta + \lambda W_+) \frac{\partial f_\gamma}{\partial x_i}, \frac{\partial f_\gamma}{\partial x_i} \rangle$$

$$\leq \sum_i \langle (-\Delta + \lambda_0 W_+) \frac{\partial f_\gamma}{\partial x_i}, \frac{\partial f_\gamma}{\partial x_i} \rangle \leq \sum_i \langle -\Delta \frac{\partial f_\gamma}{\partial x_i}, \frac{\partial f_\gamma}{\partial x_i} \rangle (1 + d_{\lambda_0 W_+})$$

$$= (1 + d_{\lambda_0 W_+}) \langle -\Delta f_\gamma, -\Delta f_\gamma \rangle = (1 + d_{\lambda_0 W_+}) \langle \frac{e_\gamma - \lambda_0 W}{1 - \gamma} f_\gamma, -\Delta f_\gamma \rangle$$

$$= \frac{e_\gamma (1 + d_{\lambda_0 W_+})}{1 - \gamma} \langle f_\gamma, -\Delta f_\gamma \rangle = \frac{\lambda_0 (1 + d_{\lambda_0 W_+})}{1 - \gamma} \langle W f_\gamma, -\Delta f_\gamma \rangle$$

We estimate the last term in the right hand side by

$$-\langle W f_\gamma, -\Delta f_\gamma \rangle = \langle W f_\gamma, \frac{\lambda_0 W - e_\gamma}{1 - \gamma} f_\gamma \rangle$$

$$\leq -\frac{e_\gamma d_{W_+}}{1 - \gamma} \| \nabla f_\gamma \|^2 + \frac{\lambda_0}{1 - \gamma} d_{W_2} \| \nabla f_\gamma \|^2$$
The Inequalities (20) and (21) imply for \( \lambda \leq \lambda_0 \)
\[
\sum_i \langle (-\Delta + \lambda W) \frac{\partial f}{\partial x_i}, \frac{\partial f}{\partial x_i} \rangle \leq (\lambda_0^2 (1 + \lambda_0 d_W) d_W^2 + o(\gamma)) \| \nabla f \|^2
\]

In the rest of this section, we will mainly work in the space representation for both particle and photons. Following [6], let us introduce, for given \( x \in \mathbb{R}^3 \), the shift operator on the photon space variables \( \tau_x : \mathbb{C}^2 \otimes \mathfrak{f} \to \mathbb{C}^2 \otimes \mathfrak{f} \). For \( \phi = (\phi_0, \phi_1, \ldots, \phi_n, \ldots) \in \mathbb{C}^2 \otimes \mathcal{F} \), we have, writing by abuse of notation \( \tau_x \phi = (\tau_x \phi_0, \tau_x \phi_1, \ldots) \),
\[
\tau_x \phi_n(s; y_1, \ldots, y_n; \lambda_1, \ldots, \lambda_n) = \phi_n(s; y_1 - x, \ldots, y_n - x; \lambda_1, \ldots, \lambda_n),
\]
where \( s \) is the spin of the particle and takes value in \( \{\uparrow, \downarrow\} \).

We denote by \( \Omega^x_0 \) the ground state \( \Omega_0 \) written in space representation and shifted by \( x \), i.e.
\[
\Omega^x_0 := \tau_x \mathcal{F}^{-1} \Omega_0,
\]
where \( \mathcal{F} \) stands for the Fourier transform.

Recall that \( D^*(0) \) is an operator valued vector with 3 components which we denote by \( D^*(0)_i (i = 1, 2, 3) \). Then we consider the functions
\[
\theta_i = (0, \theta_i^{(1)}, 0, \ldots) \in \mathbb{C}^2 \otimes \mathfrak{f}
\]
with
\[
\theta_i^{(1)} = (k^2 + |k| + C_W)^{-1} \Pi_1 D^*(0)_i \left( \left( \begin{array}{c} a \\ b \end{array} \right), 0, \ldots \right)
\]
and
\[
\theta^x_i = \tau_x \mathcal{F}^{-1} \theta_i.
\]
We first state some properties of \( \theta_i \).

**Lemma 3.6.**

i) For \( i \neq j \) we have
\[
\langle \theta_i, \theta_j \rangle = 0 \quad \text{and} \quad \langle \theta_i, \theta_j \rangle_1 = 0.
\]

ii) For \( i = 1, 2, 3 \) holds
\[
\| \theta_i \sqrt{k^2 + |k| + C_W} \|^2 = \frac{1}{6\pi^2} \int_{\mathbb{R}^3} \frac{\zeta(|k|)}{|k|(k^2 + |k| + C_W)} dk,
\]

iii) For \( i = 1, 2, 3, \langle k_i \varphi_{a,b}, \Pi_1 \theta_i \rangle = 0 \).
To prove this Lemma we remind that $\theta_i$ has only a non zero component $\Pi_1 \theta_i$ in the one photon sector, and

$$
\Pi_1 \theta_i = \begin{pmatrix}
\epsilon_1(k) \zeta(k) \\
\epsilon_2(k) \zeta(k) \\
\epsilon_1(k) \zeta(k) \\
\epsilon_2(k) \zeta(k)
\end{pmatrix},
$$

where the two polarization vectors $\epsilon_1(k)$ and $\epsilon_2(k)$ are defined in (2). The properties stated in the Lemma follow straightforwardly from computations of the corresponding integrals.

We consider the trial function $\Psi \in L^2(\mathbb{R}^3) \otimes \mathbb{C}^2 \otimes \mathfrak{F}$:

$$
\Psi := \Psi_1 + \Psi_2 := f_\gamma(x) \Omega_0^\gamma + i\sqrt{\alpha} \sum_{i=1}^3 \theta_i \frac{\partial f_\gamma(x)}{\partial x_i}.
$$

Now we compute the expectation value of $H$ in the state $\Psi$. We have

$$
\langle H \Psi, \Psi \rangle = \langle H \Psi_1, \Psi_1 \rangle + \langle H \Psi_2, \Psi_2 \rangle + 2\Re \langle H \Psi_1, \Psi_2 \rangle.
$$

As usual (3), due to the orthogonality $\langle f, \partial f/\partial x_i \rangle = 0$, we have

$$
\langle H \Psi_1, \Psi_1 \rangle = \Sigma_0 \|\Psi_1\|^2 + \langle (-\Delta + \lambda W(x))f_\gamma, f_\gamma \rangle \|\Omega_0\|^2
$$

(26)

Since $\Psi_2$ has only a non zero component in the one photon sector, in the quadratic form $\langle H \Psi_2, \Psi_2 \rangle$, all the terms involving $A(0)$ or $B(0)$ vanish. Moreover, using Lemma 3.6 and the orthogonalties $\langle \partial f_\gamma, \partial f_\gamma \rangle = 0$ and $\langle \partial f_\gamma, \partial^2 f_\gamma \rangle = 0$, for $i \neq j$, we arrive at

$$
\langle H \Psi_2, \Psi_2 \rangle = \alpha \sum_l \|\theta_l^\gamma\|^2 \langle (-\Delta + \lambda W) \frac{\partial f_\gamma}{\partial x_l}, \frac{\partial f_\gamma}{\partial x_l} \rangle + \mathcal{O}(\alpha^2) \|\nabla f_\gamma\|^2
$$

(27)

$$
+ \alpha \sum_l \|\frac{\partial f_\gamma}{\partial x_l}\|^2 \langle (|k| + k^2) \Pi_1 \theta_l^\gamma, \Pi_1 \theta_l^\gamma \rangle.
$$

To compute the last term $\langle H \Psi_1, \Psi_2 \rangle$, we first note that

$$
\langle (-\Delta + \lambda W) f_\gamma, \frac{\partial f_\gamma}{\partial x_i} \rangle = \langle -(1 - \gamma)\Delta + \lambda W \rangle f_\gamma, \frac{\partial f_\gamma}{\partial x_i} \rangle - \gamma \langle \Delta f_\gamma, \frac{\partial f_\gamma}{\partial x_i} \rangle
$$

$$
= e_\gamma \frac{\partial f_\gamma}{\partial x_i} + \gamma \sum_j \frac{\partial^2 f_\gamma}{\partial x_j^2} \frac{\partial f_\gamma}{\partial x_i} = 0.
$$

$$
\langle H \Psi_1, \Psi_2 \rangle = 0.
$$
The last equality holds since \( f_\gamma \) is a real function vanishing at infinity. Moreover, all other terms in the quadratic form \( \langle H \Psi_1, \Psi_2 \rangle \) which contain \( \langle f_\gamma, \frac{\partial f_\gamma}{\partial x_i} \rangle \) vanish also. So we arrive at

\[
2 \text{Re} \langle H \Psi_1, \Psi_2 \rangle = -2 \text{Re} \langle P \cdot (P_f - \sqrt{\alpha} A(0)) \Psi_1, \Psi_2 \rangle
\]

\[
= 2 \sqrt{\alpha} \sum_i \left| \frac{\partial f_\gamma}{\partial x_i} \right|^2 \text{Re} \langle (P_f - \sqrt{\alpha} A(0)) \Omega_0, \theta_i \rangle
\]

The term with \( P_f \) on the right hand side is estimated as follows

\[
\text{Re} \langle (P_f)_i \Omega_0, \theta_i \rangle_{C^2 \otimes F} = \text{Re} \langle \Pi_1 (P_f)_i (\gamma \varphi_{a,b} + R), \Pi_1 \theta_i \rangle_{C^2 \otimes L^2(\mathbb{R}^3) \otimes C^2} = \text{Re} \langle k_i (\gamma \varphi_{a,b} + R), \Pi_1 \theta_i \rangle_{C^2 \otimes L^2(\mathbb{R}^3) \otimes C^2} \leq ||R||_1 ||k||_2 \Pi_1 \theta_i || + \gamma \text{Re} \langle k_i \varphi_{a,b}, \Pi_1 \theta_i \rangle_{C^2 \otimes L^2(\mathbb{R}^3) \otimes C^2}
\]

Using Proposition 3.3 yields the following bound for the first term in the right hand side of (30)

\[
\text{Re} \langle (P_f)_i \Omega_0, \theta_i \rangle_{C^2 \otimes F} = O(\alpha^{\frac{3}{4}})
\]

According to Lemma 3.6 iii), the second term in the right hand side of (30) equals zero. Therefore, collecting (29) - (31), we arrive at

\[
2 \text{Re} \langle H \Psi_1, \Psi_2 \rangle = -2\alpha \sum_i \left| \frac{\partial f_\gamma}{\partial x_i} \right|^2 \text{Re} \langle A(0) \Omega_0, \theta_i \rangle + O(\alpha^{\frac{3}{2}}) \|\nabla f_\gamma\|^2
\]

Now we have, using Theorem 3.1 and the fact that \( D(0) \) restricted to the 2-photon sector is a bounded operator

\[
\text{Re} \langle A(0) \Omega_0, \theta_i \rangle = \text{Re} \langle D(0) \Omega_0, \theta_i \rangle + \text{Re} \langle D^*(0) \Pi_0 \Omega_0, \theta_i \rangle = O(\alpha^{\frac{3}{2}}) + \text{Re} \langle D^*(0) \Pi_0 \Omega_0, \theta_i \rangle.
\]

Due to the definition (24) of \( \theta_i \), the second term on the right hand side of (33) is \( ||\theta_i \sqrt{k^2} + |k| + C_W||^2 \). Therefore, collecting the Equalities (27),

\[
\text{Re} \langle A(0) \Omega_0, \theta_i \rangle = O(\alpha^{\frac{3}{4}})
\]
(28), (32) and (33) we obtain

\[
\langle H \Psi, \Psi \rangle = \Sigma_0 \| \Psi_1 \|^2 + \| \Omega_0 \|^2 \langle (-\Delta + \lambda W) f_\gamma, f_\gamma \rangle \\
+ \alpha \sum_l \| \theta_l \|^2 \langle (-\Delta + \lambda W) \frac{\partial f_\gamma}{\partial x_l}, \frac{\partial f_\gamma}{\partial x_l} \rangle \\
- 2\alpha \sum_l \| \frac{\partial f_\gamma}{\partial x_l} \|^2 \| \theta_l \| \sqrt{k^2 + |\gamma| + C_W^2} + \mathcal{O}(\alpha^2) \| \nabla f_\gamma \|^2 \\
+ \alpha \sum_l \| \frac{\partial f_\gamma}{\partial x_l} \|^2 \| \theta_l \|_l^2.
\]

From Lemma 3.6, we know that \( \| \theta_l \sqrt{k^2 + |\gamma| + C_W^2} \| \) is independent of \( l \). We denote this constant by \( \eta^2 \). With Lemma 3.5 we thus arrive at

\[
\langle H \Psi, \Psi \rangle \leq \Sigma_0 \| \Psi \|^2 - \Sigma_0 \| \Psi_2 \|^2 + \| \Omega_0 \|^2 \left( \| \nabla f_\gamma \|^2 + \langle \lambda W f_\gamma, f_\gamma \rangle \right) \\
- \alpha \sum_l \| \frac{\partial f_\gamma}{\partial x_l} \|^2 \eta^2 + \alpha o_\gamma(1) \| \nabla f_\gamma \|^2 + \mathcal{O}(\alpha^2) \| \nabla f_\gamma \|^2.
\]

Note that \( \Sigma_0 \| \psi_2 \|^2 = \mathcal{O}(\alpha^2) \| \nabla f_\gamma \|^2 \). We thus obtain

\[
\langle H \Psi, \Psi \rangle - \Sigma_0 \| \Psi \|^2 \leq \| \Omega_0 \|^2 \left( (1 - \alpha \eta^2 + \alpha o_\gamma(1) + \mathcal{O}(\alpha^2)) \| \nabla f_\gamma \|^2 + \langle \lambda W f_\gamma, f_\gamma \rangle \right)
\]

Using from Corollary 3.2 that \( \| \Omega_0 \|^2 = 1 + \mathcal{O}(\alpha) \), we obtain

\[
\langle H \Psi, \Psi \rangle - \Sigma_0 \| \Psi \|^2 \leq \| \Omega_0 \|^2 \left( (1 - \alpha \eta^2 + \alpha o_\gamma(1) + \mathcal{O}(\alpha^2)) \| \nabla f_\gamma \|^2 + \langle \lambda W f_\gamma, f_\gamma \rangle \right).
\]

Therefore

\[
\langle H \Psi, \Psi \rangle - \Sigma_0 \| \Psi \|^2 \leq \| \Omega_0 \|^2 \left( (1 - \alpha \eta^2 + \alpha o_\gamma(1) + \mathcal{O}(\alpha^2)) \right) \times \left( \| \nabla f_\gamma \|^2 + (1 + \alpha \eta^2 + \alpha o_\gamma(1) + \mathcal{O}(\alpha^2))^{-1} \langle \lambda W f_\gamma, f_\gamma \rangle \right).
\]

If \( \lambda > \lambda_0(1 - \alpha \eta^2 + \mathcal{O}(\alpha^2)) \), choosing \( \gamma \) (depending on \( \alpha \)) small enough, we arrive at \( \langle H \Psi, \Psi \rangle - \Sigma_0 \| \Psi \|^2 < 0 \). Due to Griesemer et al. [8], this implies the existence of a ground state for \( H \).

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