Schrödinger symmetry in cosmology and black hole mechanics

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Abstract

We show that the (symmetry-reduced) gravitational sectors describing i) the flat FRLW cosmology filled with a massless scalar field and ii) the Schwarzschild black hole mechanics both admit the two dimensional centrally extended Schrödinger group as a dynamical symmetry, a symmetry shared by the compressible Navier-Stokes equation. To this end, we use the Eisenhart-Duval (ED) lift method to identify the Schrödinger observables. The Casimirs of the observables algebra coincide respectively with the conserved kinetic energy of the scalar field, for cosmology, and with the mass of the black hole, for the Schwarzschild mechanics. Moreover, in both models, the central extension is found to encode the ratio between the IR and UV scales of the gravitational systems. We pursue this analysis by comparing the ED method to the superspace approach and demonstrate their complementarity. We use the superspace approach to show that these two models possess also an infinite dimensional symmetry whose conserved charges organize in two copies of a Witt algebra. Finally, we consider the anisotropic Bianchi I dynamics and show that it admits a dynamical symmetry under the SO(4, 2) conformal group, within which one can identify another Schrödinger algebra of observables. These new symmetries provide a new way to algebraically characterize these homogeneous gravitational sectors, to guide their quantization or emergence from quantum gravity models, and they suggest new dictionaries with non-linear Schrödinger systems and fluid mechanics.
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1 Introduction

Uncovering symmetries of general relativity and gravitational systems beyond the gauge symmetry under space-time diffeomorphisms is a crucial endeavour to understand the theory, both at the classical and quantum levels. It has thus a clear intrinsic interest. Beyond that, this exercise opens also the road to build fruitful dictionaries with other physical systems sharing the same symmetries, at least in some given regime. Among them, the interplay between (relativistic) fluid dynamics and gravitational physics plays a key role. Since the early works on the membrane paradigm \cite{1}, the relation between relativistic fluids and gravity has been extensively studied leading eventually to the so called fluid/gravity correspondence, formulated first in the AdS framework and more recently in the asymptotically flat context \cite{2–7}. It is an interesting question whether a similar correspondence can be uncovered between the dynamics of the gravitational field and that of non-relativistic fluids, and whether it exists in a non-holographic context, i.e. where the fluid system is not understood as defined in the boundary of the gravitational one\footnote{See \cite{8,9} for a recent discussion on ultra-relativistic and non-relativistic fluids.}.

In this work, we show that the non-relativistic conformal symmetry of the compressible fluid mechanics \cite{11}, known as the Schrödinger group, is also realized as a field space dynamical symmetry for suitable gravitational (symmetry-reduced) models describing either cosmology or black holes, thus revealing a new corner between the structure shared by fluids and gravity.

Gravitational mini-superspace models are obtained by imposing homogeneity conditions to the gravitational field. Such restriction selects the zero modes of the geometry. The dynamics of the corresponding gravitational field then reduces to a simple mechanical system. While such symmetry reduction can appear drastic, it nevertheless provides relevant models for both cosmology and astrophysics. Within this context, it turns out that one can identify new symmetries of the gravitational zero mode’s dynamics which fully encode the evolution of the homogeneous geometry \cite{12–21}. Interestingly, while such symmetries are bona fide Noetherian symmetries of the homogeneously reduced gravitational field, they are not realized in terms of residual space-time diffeomorphisms and the presence of a boundary does not play any (direct) role in their derivation. The appearance of such non-standard dynamical symmetries in gravity thus begs for further investigations in order to fully exploit them. The goal of this work is to i) provide a unified and systematic derivation of these new symmetries and ii) reveal a larger set of such dynamical symmetries, demonstrating in that way a new interplay between non-relativistic fluid and gravitational mini-superspaces. To set the stage, let us first review the body of results obtained on these symmetries to contrast with the present results.

A systematic investigation of these symmetries was initiated in \cite{14–18} for isotropic cosmological models and extended to black hole models in \cite{19–21}. More concretely, following the early work by Pioline and Waldron \cite{12}, it was shown in \cite{14–18} that the isotropic FRW cosmology filled with a massless scalar field enjoys an SL(2, \text{R}) symmetry which fully encodes the cosmological dynamics in
a way very similar to the conformal particle. This finding was further extended to the whole family of isotropic cosmological models in [16, 17], including the effect of a cosmological constant and an arbitrary homogeneous spatial curvature as well as a detailed discussion on the role of the gauge fixing. By considering the anisotropic Kantowski-Sachs model, which describes the Schwarzschild black hole interior, it was further shown that the SL(2, R) symmetry gets enlarged to a ISO(2, 1) symmetry group [19, 20]. More recently, this conclusion was extended to the Schwarzschild-(A)dS black hole mechanics in [21], showing that the associated charge algebra characterizes both the exterior and interior of the black hole geometry, as well as the region beyond the cosmological horizon for the Schwarzschild-dS case. An interesting feature of these symmetries is that they are not realized as standard spacetime-diffeomorphisms. For instance, the SL(2, R) symmetry amounts at a Möbius transformation of the time (or radial) coordinate while the components of the metric transform as primary fields with different conformal weights, leading to an anisotropic Weyl rescaling of the metric. Physically, these symmetries mapped solutions of the homogeneous Einstein’s equations onto another gauge-inequivalent one, as expected from a non-trivial (non-gauge) symmetry. From this perspective, this body of results raises several questions. For instance, is there a systematic approach to identify these dynamical symmetries of homogeneous gravity? If yes, does it provide a geometrical origin for them? And finally, can we use this approach to identify larger symmetries and in particular find the maximal symmetry group for each model (more ambitiously, in a long-term perspective, for GR as a whole)? Adopting a broader vision on these issues, the identification of physical symmetries at full diffeomorphism invariant level would help us characterizing the physical solution space of GR and offer a solid guide for extracting physics from quantum gravity. In particular, they would help characterizing or identifying diffeo-invariant observables, and relations between different ways of writing GR (and QG) in relational language, in correspondence with different physical reference frames.

Interestingly, at least in symmetry-reduced contexts, these questions can be addressed in a fairly elegant manner thanks to the geometrical techniques developed for mechanical systems. The general idea amounts at geometrizing the dynamics of the system under consideration in term of geodesic motion on an auxiliary background. Identifying the conformal isometries of this background provides a geometrization of the symmetries of the system. A first approach consists in considering the space of dynamical fields, i.e. the superspace, whose metric is constructed from the kinetic matrix of the system [13]. This is the approach developed in [22–26] and extended more recently in [27] to systematically explore the dynamical symmetries of homogeneous gravity models. This is a very natural approach as it can be motivated also by the fact that the general relativistic dynamics of the gravitational field can be recast in the form of geodesic motion (of a point particle) on superspace [28]. A second approach makes uses of an extended notion of superspace, the so-called Eisenhart-Duval lift [29]. See [30] for an introduction and [31] for a detailed presentation. This elegant method has been applied to a large variety of systems, from time-dependent systems, celestial mechanics, inflation and more exotic systems [32–41]. An interesting fact about this for-
malism is that the lift metric is equipped with a covariantly constant null Killing vector. It follows that the lift metric can be viewed as a pp-wave, leading to a correspondence between the conformal isometries of pp-waves and dynamical symmetries of mechanical systems [42, 43]. When applied to gravitational mini-superspace models, which already describe homogeneous gravitational fields, this lift can thus be understood as an elegant second geometrization of gravity. Moreover, this peculiar structure provides a natural host for the geometrical realization of non-relativistic conformal symmetry [44, 45]. In this work, we shall provide a first application of the Eisenhart-Duval lift to cosmological and black hole dynamics. Comparing the two approaches, one finds that the two formalisms reveal in general different symmetries of the underlying system. In this work, we shall illustrate these methods on explicit examples and demonstrate that the two approaches are indeed complementary tools to explore the dynamical symmetries of mechanical system, and, in the present case, of homogeneous gravity.

Let us now present the main results obtained. Our first step is to apply the ED lift method to two relevant examples of gravitational mini-superspaces with two degrees of freedom: the flat FLRW cosmological model filled with a massless scalar field and the Schwarzschild black hole mechanics. In both case, we find that the gravitational system admits the two dimensional Schrödinger group Sch(2) as a symmetry group. To our knowledge, this is the first appearance of this non-relativistic conformal symmetry in relativistic symmetry-reduced homogeneous gravity. We present in details the algebra of observables, the finite symmetry transformations and discuss the role of the Casimirs. A detailed discussion on the Schrödinger group, its geometrical realization in terms of Bargmann space, its realization in fluid dynamics and its relevance for non-relativistic holography can be found in [11, 44, 60]. The identification of this group as a dynamical symmetry of suitable homogeneous sectors of gravity provides the first main result of the present work.

A remarkable feature of this new symmetry is that the central extension of the Schrödinger algebra is given by the ratio between the IR scale, introduced as a regulator in the symmetry reduction process, and the UV scale given by the Planck length. If the latter is interpreted as corresponding to some sort of fundamental discrete spacetime structure, this ratio can thus be interpreted as the effective number of fundamental Planckian cells. If the same discretization corresponds, as in some quantum gravity approaches, to “atomic constituents” of quantum spacetime [46], the ratio would also acquire the interpretation as the average number of such constituents, whose dynamics is collectively described by the sector of GR under consideration, in an emergent gravity scenario [47] fully grounded in a fundamental quantum theory [48]. While this remains a suggestion at this stage, it nevertheless matches the situation in real fluids. In the standard approach to non-relativistic systems with a Schrödinger symmetry, the central charge plays indeed the role of the conserved number of particles or the mass of the system. Hence, the identification of the new Schrödinger algebra of observables naturally introduced a new notion of “number of particles” for cosmology and black hole mechanics through the central charge which fits remarkably well with the standard interpretation of these systems as thermodynamical (or hydrodynamical) objects.
As a second step, we show that the superspace approach reveals an infinite-dimensional symmetry for these two models. We construct the infinite set of conserved charges and show that they organize into two copies of a Witt algebra. These charges are weak Dirac observables which originate from the 2-dimensional nature of the super-metric for these models. This additional symmetry allows us to illustrate the complementarity of the superspace and the ED lift approaches in revealing the symmetry content of our gravitational systems.

Finally, we consider the case of the simplest anisotropic cosmological model: the Bianchi I model. When applying the ED lift method and focussing on the conformal Killing vectors of the field space, the anisotropies break the Schrödinger symmetry identified in the isotropic case. Nevertheless, the $\text{SL}(2, \mathbb{R})$ subgroup of transformation is preserved and encodes the isotropic motion. At this level, the ED lift reveals a $\text{SL}(2, \mathbb{R}) \times \text{ISO}(3)$ symmetry group. The persistence of the $\text{SL}(2, \mathbb{R})$ symmetry parallels the initial results discussed first in [83] and more recently in [27] in investigating the whole Bianchi class. On top of this result, the superspace approach unveils a new larger symmetry under the $\text{SO}(4, 2)$ conformal group. We derive the associated fifteen conserved charges and discuss the (Casimir) conditions between them. We further show that a sub-set of this charge algebra can be identified to the 2d centrally extended Schrödinger algebra. However, the symmetry transformations which they generate are very different from the ones associated to the Schrödinger charges identified for the isotropic FLRW cosmology and the Schwarzschild black hole as they are generated solely by strong Dirac observables.

All together, these results reveal new algebraic structures which organize the phase space of both cosmological and black hole mini-superspace models. As emphasized earlier, they also reveal new connections between homogeneous gravity and non-relativistic fluid systems. Indeed, the Schrödinger symmetry realized in the flat FLRW model and in the Schwarzschild black hole mechanics is precisely the dynamical symmetry of the compressible Navier-Stokes equation [11], and is also realized in the free Schrödinger equation and in some of its non-linear extensions [49–51]. See also [44, 52, 53]. From that point of view, our findings suggest that the dynamics of these gravitational mini-superspace models could be recast, based on their shared symmetries, in terms of Schrödinger systems, or using the Madelung fluid representation, in terms of non-relativistic fluid dynamics hosted on an auxiliary background, giving rise to a new form of duality. This idea resonates with some investigations on the non-relativistic regime of the AdS/CFT and fluid/gravity correspondences [57–60], the construction of the aging/gravity duality [61–63], and parallels some on-going efforts devoted to construct a mapping between cosmology and non-linear Schrödinger equations [64–66]. It might also reveals useful in the construction of analogue models of quantum (symmetry reduced) gravity. We shall come back on this point in the discussion.

This works is organized as follows. In Section 2, we briefly review the key points entering in the construction of gravitational mini-superspace models. Then, we provide a review of the geometrization of the model first in term of the superspace, and then in term of the ED lift. In Section 3, we present the Schrödinger symmetry of the flat FLRW cosmology filled with a
massless scalar field. We derive the Schrödinger observables, the symmetry transformations and algebraically solve the dynamics. In Section 3.4.2, we derive the \( \mathfrak{ Witt} \) charges for this cosmological model. Section 3.4 is devoted to an explicit derivation of the conformal Killing vectors (CKVs) of the ED lift’s superspace and their relation to the conserved charges. The next Section 4 presents the results for the Schwarzschild black hole mechanics, while Section 5 is devoted to the Bianchi I cosmological model. Finally, Section 6 summarizes our results and discuss the open perspectives. We have relegated most of the technical detailed to the appendixes. In Appendix A, we show that the CKVs of the ED lift of isotropic FLRW cosmology form indeed an \( \mathfrak{ so}(4,2) \) Lie algebra. Appendix B presents a short derivation of the \( \mathfrak{ Witt} \) charges for any mechanical system with two degrees of freedom. Finally, we provide the explicit expressions of the CKVs of the lift and the superspace for the last two models in appendix C and D.

2 Mini-superspace symmetries from field space geometry

Gravitational mini-superspaces are reduced models of general relativity restricted to classes of space-time metrics defined in terms of a finite number of degrees of freedom, based on the assumption of homogeneity. They describe specific sectors of the full theory, relevant to cosmology and astrophysics. See [54] for a pedagogical presentation. They are simple mechanical systems, and we review in this section how their dynamics can be given a further geometric translation, not in terms of spacetime geometry, but in terms of motion on the field space (a.k.a. superspace). In the case of a free mechanical model, the classical trajectories are identified as null geodesics of the field space and the symmetries of the system are given by the conformal Killing vectors of the (super-)metric of the field space. In the general case, the non-vanishing potential of the mini-superspace will drive the system away from null geodesics. Nevertheless, one can use the Eisenhart-Duval lift [30, 31], embedding the field space into a lifted space with 1+1 extra dimensions, to identify once again classical trajectories as null geodesics and symmetries as conformal Killing vectors of the lifted super-metric.

2.1 Gravitational mini-superspaces and super-metric

We consider mini-superspaces of general relativity defined by restricting the space-time metrics to a specific ansatz with a finite number of components allowed to vary in terms of a chosen coordinate while the rest of the metric is held fixed and considered as background data. For instance, a typical choice in cosmology is an ansatz of the type:

\[
\text{d}s^2 = \gamma_{\mu
u} \text{d}x^\mu \text{d}x^\nu = -N^2(t) \text{d}t^2 + \gamma_{ij}(t, x^i) \text{d}x^i \text{d}x^j ,
\]

(2.1)

with no cross terms in \( \text{d}t \text{d}x^i \), where the time component is the lapse function \( N \) and the spatial metric \( \gamma_{ij} \) is defined in terms of fields \( \chi^a(t, x^i) \) allowed to vary with the time coordinate \( t \) and with a fixed dependance with respect to the space coordinates \( x^i \). More generally, allowing for both
gravitational and matter degrees of freedom, and denoting $\chi^a(t)$, with $a$ running from 1 to an integer $d$, the degrees of freedom of the mini-superspace ansatz, the Einstein-Hilbert action for general relativity plus matter typically reduces to a mechanical system of the form:

$$S[N, \chi^a, \dot{\chi}^a; t] = c\ell_P \int dt \left( \frac{1}{2N} g^{ab}(\chi) \dot{\chi}^a \dot{\chi}^b - NV(\chi) \right),$$

(2.2)

where the dynamical fields depend on the coordinate $t$. The action is for a point particle on minisuperspace, and is expressed in terms of an affine parameter on the particle trajectories on minisuperspace. These trajectories can be of any signature, with respect to the supermetric, thus the affine parameter has no universal spacetime interpretation. In some cases, in particular when one is considering Friedmann dynamics, it can be identified with the “time coordinate” in which the spacetime metric is expressed, but this does not have to be always the case. We nevertheless use, unless specified otherwise, the convention that $t$ is a time coordinate and that we have integrated over the spatial coordinates $x^i$.

We write $\ell_P$ for the Planck length. The pre-factor $c\ell_P$ comes from the standard $1/\ell_P^2 \propto 1/\hbar G$ factor in front of the Einstein-Hilbert action, combined with the 3d volume $V_0$ coming from the integration over the space coordinates (with appropriate cut-off). Since the Planck length $\ell_P$ defines the UV scale and the fiducial volume $V_0$ defines the IR scale, the dimensionless factor $c$ plays the role of the ratio between UV and IR scales and turns out to be the central charge of the Schrödinger symmetry algebra for cosmology and black holes, as shown in sections 3 and 4.

The kinetic terms comes from the extrinsic curvature (of the constant $t$ hypersurfaces) contribution to the Einstein-Hilbert action plus the kinetic terms of the matter Lagrangians. It is expressed in terms of the $d$-dimensional field space metric $g_{ab}$, or super-metric in short. The potential $V(\chi^a)$ reflects both the intrinsic curvature (of the constant $t$ hypersurfaces) and the self-interaction of the matter fields. The mini-superspace action is defined by the super-metric and the potential.

The equations of motion of the mini-superspace descend from the stationarity of the action with respect to lapse variations $\delta N$ and to field variations $\delta \chi^a$. These should amount to imposing the Einstein equations on the chosen space-time metric ansatz plus the field equations for matter evolving on that metric. For an arbitrarily chosen metric ansatz, this is clearly a non-trivial statement. Nevertheless, if the space-time metric ansatz is defined by symmetry reduction, i.e. requiring the considered metric to be invariant under a symmetry of the full theory, for instance by imposing a certain number of Killing vectors (thus descending from the diffeomorphism invariance of the Einstein-Hilbert action), then it is automatic that the solutions of the equations of motion of the reduced action are also solutions of the equations of motion of the full theory. Here the equation of motion $\delta S/\delta N = 0$ gives the Hamiltonian constraint of the ADM formalism, giving the time evolution of the spatial metric, while the equations of motion $\delta S/\delta \chi^a = 0$ should lead to the momentum constraints, giving the projection of the Einstein equations on the spatial slices. Although we will not attempt a full classification of gravitational mini-superspaces here, we will make sure case by case that the classical solutions of the mini-superspaces studied satisfy indeed
the full Einstein equations.

In particular, these models describe straightforward parametrized mechanical systems, invariant under time reparametrizations

\[
\begin{align*}
  t & \mapsto \tilde{t} = f(t), \\
  N(t) & \mapsto \tilde{N}(\tilde{t}) = \frac{1}{f'(t)}N(t), \\
  \chi^a(t) & \mapsto \tilde{\chi}^a(\tilde{t}) = \chi^a(t).
\end{align*}
\]

(2.3)

This means that the Hamiltonian of the system vanishes. Indeed, the Legendre transform defines the canonical momenta \(\pi_a\) and Hamiltonian \(H\):

\[
\begin{align*}
  \pi_a &= \frac{1}{N}g_{ab}\dot{x}^b, \\
  H &= \pi_a \chi^a - L = Nh \quad \text{with} \quad h = \frac{1}{2}g^{ab}\pi_a \pi_b + V,
\end{align*}
\]

(2.4)

where \(g^{ab}\) is the inverse super-metric and the \(\chi\) dependence of the Hamiltonian comes from both the super-metric and the potential. The equation of motion with respect to lapse variations simply imposes the Hamiltonian constraint \(h = 0\). This zero-energy condition is a crucial property of those systems and their symmetries.

We focus in this paper on three specific mini-superspace models:

- **flat FLRW cosmology of a scalar field**: This model contains two dynamical fields \(\chi^a = \{a, \varphi\}\). The former plays the role of the scale factor in the metric

\[
\begin{align*}
  ds^2 &= -N^2(t)dt^2 + a^2(t)\delta_{ab}dx^a dx^b
\end{align*}
\]

(2.5)

while the second dynamical field corresponds to an homogeneous scalar field coupled to this metric.

- **Schwarzschild mechanics**: The geometry contains two dynamical fields denoted \(\chi^a = \{A, B\}\) and the line element reads

\[
\begin{align*}
  ds^2 &= \epsilon \left(-N^2(t)dt^2 + \frac{dy^2}{A^2(t)}\right) + L_s^2B^2(t) \left( d\theta^2 + \sin^2 \theta d\phi^2 \right)
\end{align*}
\]

(2.6)

This geometry describes both the interior and the exterior of the black hole. When \(\epsilon = +1\), the coordinate \(t\) is time-like and labels space-like hypersurface foliating the black hole interior, while for \(\epsilon = -1\), it is space-like and it labels the time-like foliation of the exterior region, thus playing the role of the radius.

- **Bianchi I cosmology**: Finally, the third model corresponds to the simplest anisotropic cosmology. It contains four dynamical fields \(\chi^a = \{\alpha, \beta, \gamma, \varphi\}\). The first three enters in the metric as

\[
\begin{align*}
  ds^2 &= -N^2(t)dt^2 + \alpha^2(t)dx^2 + \beta^2(t)dy^2 + \gamma(t)dz^2
\end{align*}
\]

(2.7)

while the last one plays the role an homogeneous scalar matter field.
On top of being physically relevant gravitational models, in cosmology and astrophysics, these systems have rather generic mathematical properties. Both flat FLRW cosmology and Bianchi I cosmologies are free systems, with a vanishing potential and a conformally flat super-metric. Schwarzschild mechanics is then the typical example of a two-dimensional field space, thus with a conformally flat super-metric, but with a non-vanishing potential.

For free systems with vanishing potential, the mini-superspace action reads:

\[ S[N, \chi^a, \dot{\chi}^a; t] = c\ell_P \int dt \left[ \frac{1}{2N} g_{ab}(\chi) \dot{\chi}^a \dot{\chi}^b \right] . \] (2.8)

This is simply a geodesic Lagrangian: the equations of motion impose that the space-time metric components \( \chi^a \) follow a null geodesic in field space provided with the super-metric \( g_{ab} \). The fact that we must consider null geodesics is imposed by the equation of motion with respect to lapse variations. Symmetries of the theory will then map the set of null geodesics in field space onto itself.

For general mini-superspaces with non-vanishing potential, one can absorb the potential into a field-dependent redefinition of the lapse by defining a proper time coordinate:

\[ d\eta \equiv NV(\chi) dt . \] (2.9)

This choice of lapse leads to the gauge-fixed action:

\[ S_{gf}[\chi_a, \dot{\chi}_a; \eta] = c\ell_P \int d\eta \left[ \frac{1}{2} G_{ab}(\chi) d\eta \chi^a d\eta \chi^b - 1 \right] , \quad \text{with} \quad G_{ab} = V(\chi) g_{ab} . \] (2.10)

We have absorbed the potential into a conformal rescaling of the super-metric. The constant shift \(-1\) in the Lagrangian should usually be dropped out since it would not contribute to the (bulk) equations of motion. Here, it plays the non-trivial role of shifting the physical value of the Hamiltonian. Indeed, putting aside that constant shift, we recognize once again a geodesic Lagrangian: classical solutions \( \chi^a(\eta) \) now follows geodesics of the conformally-rescaled super-metric \( G_{ab} \). Nevertheless, even though we gauge-fixed the lapse function, one must not forget the original equation of motion with respect to lapse variations. It is indeed “hidden” in the definition of the \( \eta \) coordinate and it implies the constraint:

\[ \frac{1}{2} G_{ab}(\chi) d\eta \chi^a d\eta \chi^b - 1 = 0 , \] (2.11)

i.e. it restricts to massive geodesics with fixed mass. In that case, we anticipate the introduction of correction terms proportional to the “super-mass” that will have to be added to the conserved charges for null geodesics.

Another avenue to treat such general mini-superspaces with non-vanishing potential is to use the Eisenhart-Duval lift. Indeed, as we shall see in the following, we can embed such systems with potential in a \((d + 2)\)-dimensional extended field space in order to map them once again on null geodesic Lagrangians. This is will be reviewed in section 2.3.
2.2 Symmetries from field space Killing vectors

Let us first study (gravitational) mechanical models with vanishing potential. As we saw above, these free systems are geodesic Lagrangians, whose classical trajectories are null geodesics of the super-metric $g_{ab}$. Since conformal isometries of the super-metric map the set of null geodesics onto itself, and thus define automorphisms of the set of solutions to the equations of motion, conformal Killing vectors (CKVs) of the super-metric naturally define symmetries of the mini-superspace.

More precisely, the action of a free mini-superspace read in Lagrangian form:

$$S[N, \chi^a, \dot{\chi}^a; t] = c\ell_P \int \frac{1}{2N} g_{ab}(\chi) \dot{\chi}^a \dot{\chi}^b, \quad (2.12)$$

and in Hamiltonian form, after Legendre transform:

$$S[N, \chi^a, \pi_a; t] = c\ell_P \int \frac{1}{2} \pi_a \dot{\chi}^a - Nh[\chi, \pi], \quad \text{with} \quad h[\chi, \pi] = \frac{1}{2} g^{ab} \pi_a \pi_b. \quad (2.13)$$

Considering a conformal Killing vector $\xi^a \partial_a$ on the field space, with the $\xi^a$’s functions of the $\chi^a$’s, satisfying

$$\mathcal{L}_\xi g_{ab} = \varphi g_{ab} \quad (2.14)$$

with the conformal rescaling factor $\varphi(\chi)$, it is a standard result that the scalar product between the velocity vector and the Killing vector is conserved along null geodesics: it is thus a constant of motion. Here, introducing $O_\xi = \xi^a \pi_a$, a straightforward calculation allows to check that

$$\{O_\xi, H\} = -\frac{1}{2} \pi^a \pi^b \mathcal{L}_\xi g_{ab} = -\varphi H. \quad (2.15)$$

This shows that $O_\xi$ is a weak Dirac observable of the system, it is conserved along classical trajectories and, as a conserved charge, it generates a symmetry of the mini-superspace by Noether’s theorem. Listing all the CKVs of the field space metric therefore provides a simple way to identify conserved charges and symmetries of the mini-superspace model.

However, this procedure only reveals a subset of the symmetries of the system. First, the observables constructed above, as $O_\xi$ with a (conformal) Killing vector $\xi$, are always linear in the momenta $\pi_a$. The method clearly misses conserved charges of higher order in the momenta. This could probably be remedied by looking at (conformal) Killing tensors of the super-metric. Second, this approach cannot produce time-dependent conserved charges, or in other words evolving constants of motion, which satisfy:

$$\frac{dO}{d\tau} = \partial_\tau O + \{O, H\} = 0, \quad (2.16)$$

because the CKVs $\xi$ depend on the field space variables but not directly on time (and space coordinates). To obtain such observables, one has either to adopt the approach followed in [27], or to develop an extension of the field space allowing to treat the affine parameter $t$ on the same footing of the fields $\chi^a$. This is actually realized by the Eisenhart-Duval lift which we now review below, in the next section.
2.3 Eisenhart-Duval lift: back to null geodesics

The Eisenhart-Duval lift is a general method to geometrize any mechanical system, by mapping them onto null geodesic Lagrangians \[29\]. The interested reader can refer to \[30, 31\] for a detailed presentation of the formalism. In the following, we shall review the main points relevant for our investigation of gravitational mini-superspaces. Let us start with the Lagrangian ansatz:

\[
S[\chi^a, \dot{\chi}^a; \tau] = c \ell P \int d\tau \left( \frac{1}{2} g_{ab}(\chi) \ddot{\chi}^a \ddot{\chi}^b - V(\chi) \right),
\]

where we have gauge-fixed the lapse to \(N = 1\) by introducing the proper time \(d\tau = N dt\). Despite the gauge-fixing, we will keep the Hamiltonian constraint in mind, i.e. that the value of the Hamiltonian on physical trajectories vanishes, \(h = 0\).

The idea of the Eisenhart-Duval lift is to introduce another variable \(u\) and add a factor \(\dot{u}^2\) in front of the potential, thereby yielding a geodesic Lagrangian. More precisely, we introduce a pair of new coordinates \((u, w)\) with a lifted (super-)metric:

\[
ds^2_{(ED)} = 2 du dw - 2 V(\chi) du^2 + g_{ab} d\chi^a d\chi^b,
\]

leading to a lifted action defined as an integration over a parameter \(\lambda\):

\[
S_{(ED)}[\chi(\lambda), u(\lambda), w(\lambda); \lambda] \equiv \int d\lambda \frac{1}{N} \left( \dot{u} \dot{w} - V(\chi) \dot{u}^2 + \frac{1}{2} g_{ab}(\chi) \dot{\chi}^a \dot{\chi}^b \right),
\]

where the dot denotes here the derivative with respect to \(\lambda\). This looks very much like the original Lagrangian and, in fact, take the same form if \(\dot{u} = 1\). This is actually the role of the extra coordinate \(w\), whose equation of motion imposes that \(\dot{u}\) is indeed constant on-shell.

Let us have a closer look at the lifted metric and derive the equations of motion of the lifted Lagrangian. By construction, the lifted metric is very similar to the pp-wave ansatz, with \(\partial_w\) and \(\partial_u\) both null Killing directions. The classical trajectories are geodesics of the lifted metric written in terms of the affine parameter \(\lambda\). The lifted lapse \(N\) imposes that we focus on null geodesics and ensures the invariance of the lifted action under \(\lambda\)-reparametrization.

We compute the canonical momenta,

\[
p_u = \frac{\delta L}{\delta \dot{u}} = \frac{(\ddot{w} - 2 V(\chi) \dot{u})}{N}, \quad p_w = \frac{\delta L}{\delta \dot{w}} = \frac{\dot{u}}{N}, \quad p_a = \frac{\delta L}{\delta \dot{\chi}^a} = \frac{1}{N} g_{ab}(\chi) \dot{\chi}^b,
\]

and get the Hamiltonian governing the geodesic motion on the lift,

\[
\mathcal{H} = \mathcal{L} = N \left[ p_u p_w + \frac{1}{2} g_{ab}(\chi) p_a p_b + V(\chi) p_w^2 \right].
\]

Stationarity with respect to variation of the lifted lapse \(N\) imposes the condition that the vanishing Hamiltonian, \(\mathcal{H} = 0\). This restricts the dynamics to null geodesics on the lifted field space. The
remaining equations of motion read

\[ \frac{d\lambda}{p_u} = \{p_u, \mathcal{H}\} = 0, \quad \frac{d\lambda}{u} = \{u, \mathcal{H}\} = Np_w, \tag{2.22} \]
\[ \frac{d\lambda}{p_w} = \{p_u, \mathcal{H}\} = 0, \quad \frac{d\lambda}{w} = \{w, \mathcal{H}\} = N(p_u + 2V), \tag{2.23} \]
\[ \frac{d\lambda}{p} = \{\chi^a, \mathcal{H}\} = Ng^{ab}p_b, \quad \frac{d\lambda}{a} = \{p_a, \mathcal{H}\} = -N \left[ \frac{1}{2} p_b p_c \partial_a g^{bc} + p_w^2 \partial_a V \right]. \tag{2.24} \]

Let us integrate these equations of motion for the null coordinates \((u, w)\) and their momenta to show the lifted system is equivalent to the initial system \((2.17)\). First, the coordinate \(w\) is a cyclic variable, i.e. its momentum \(p_w\) is a constant of motion. For the sake of simplicity, we can set

\[ p_w = 1. \]

Then the lifted Hamiltonian reads:

\[ \mathcal{H} = p_w = 1 \]
\[ \mathcal{H} = N(p_u + h), \quad h = \frac{1}{2} g^{ab}(\chi)p_a p_b + V(\chi), \tag{2.25} \]

where \(h\) is the Hamiltonian of the original action. Thus, imposing that the lifted Hamiltonian vanishes \(\mathcal{H} = 0\) amounts to identifying the \(u\)-momentum to the Hamiltonian of the initial system (up to a sign switch):

\[ p_u = \frac{\mathcal{H}}{\mathcal{H}} = -h. \tag{2.26} \]

This means that the variable \(u\) can be identified to the original time coordinate \(\tau\). In particular, we can deparametrize the equations of motion of the lifted system in terms of \(u\) and recover the equations of motion of the original system:

\[ \frac{d\chi^a}{du} = \frac{d\lambda}{\lambda u} \bigg|_{p_w=1} = g^{ab}p_b, \quad \frac{dp_a}{du} = \frac{d\lambda}{\lambda u} \bigg|_{p_w=1} = -\left[ \frac{1}{2} p_b p_c \partial_a g^{bc} + \partial_a V \right], \tag{2.27} \]

where we recognize the Hamilton equations of motion of the original system \((2.17)\) for the geodesics of the metric \(g\). This shows that the geodesic equation for the massless test particle on the lift indeed reproduces the equations of the initial mechanical system.

Now that we are back to a system of null geodesics, we know that conformal Killing vectors will give symmetries of the system. More precisely, writing \(G_{AB}\) for the lifted metric,

\[ G_{uu} = -2V, \quad G_{uw} = 1, \quad G_{ab} = g_{ab}, \tag{2.28} \]

we consider a conformal Killing vector,

\[ X = X^A \partial_A = X^u \partial_u + X^w \partial_w + \xi^a \partial_a, \tag{2.29} \]

thus satisfying the Lie derivative equation \(\mathcal{L}_X G = \varphi G\) or explicitly, if we distinguish the null coordinates \((u, w)\) from the original coordinates \(\chi^a\):

\[ \varphi = \partial_u X^u + \partial_u X^w - 2V \partial_u X^u, \]
\[ \varphi V = 2V \partial_u X^u - \partial_u X^w + \xi^a \partial_a V, \]
\[ 0 = \partial_u X^w - 2V \partial_u X^u + g_{ab} \partial_u \xi^b \]
\[ 0 = \partial_u X^u + g_{ab} \partial_u \xi^b, \]
\[ \varphi g_{ab} = \mathcal{L}_\xi g_{ab}, \tag{2.30} \]
Let us underline that, although the last equation, giving the projection of the conformal Lie derivative onto the original space, strictly reads as before \( L_\xi g = \varphi g \), now the vector components \( \xi^a \) not only depend on the original coordinates \( \chi^a \) but can also depend on \( u \) and \( w \). We know that the scalar product \( X^A p_A \) is a constant of motion along all null geodesics. Translating this lifted Poisson bracket \( \{ X^A p_A, \mathcal{H} \}_{ED} = -\varphi \mathcal{H} \hat{=} 0 \) back to the original system, using the mapping derived above from the equations of motion in \((u, w)\),

\[
p_w = 1, \quad p_u = -h, \quad u = \tau,
\]

we introduce the corresponding observable:

\[
O_X \equiv X^w - hX^u + \xi^a p_a \bigg|_{u=\tau}.
\]

Upon further assuming that the vector does not depend on the coordinate \( w \), \( \partial_w X = 0 \), a straightforward calculation allows to check that this is indeed a constant of motion for the original system:

\[
\partial_w X = 0 \quad \mid \quad \mathcal{L}_X g = \varphi g \quad \Rightarrow \quad d_\tau O_X = \partial_\tau O_X + \{ O_X, h \} = 0.
\]

A key remark is that this method produces, as desired, evolving constants of motion, i.e. conserved charges which explicitly depend on time, which was not possible before. This follows directly the \( u \)-dependence of the Killing vector field \( X \). This time dependence of the conserved charges seems to be a natural feature of systems with non-trivial potential.

Let us point out that the Eisenhart-Duval lift method also works on free systems, when the potential \( V \) vanishes. The lifted metric is the straightforward embedding of the original \( d \)-dimensional metric into a \( (d+2) \)-dimensional metric by adding a mere \( du dw \) contribution for the new null coordinates. Despite being straightforward, the charges derived from the Eisenhart-Duval lift CKVs are non-trivial and explicitly time-dependent and thus come on top of the ones descending from the CKVs of the original (super-)metric.

We would like to conclude this discussion with the remark that it is tempting to interpret free systems, being themselves null geodesic systems, as Eisenhart-Duval lifts. Actually realizing them as Eisenhart-Duval lifts would mean to be able to recast the \( d \)-dimensional (super-)metric \( g_{ab} \) in the lifted ansatz \((2.18)\), that is:

\[
g_{ab} d\chi^a d\chi^b = 2dU dW - 2\Upsilon(U, \zeta) dU^2 + g_{\alpha\beta}^{\text{red}} d\zeta^\alpha d\zeta^\beta,
\]

similarly to the pp-wave ansatz, in terms of a pair of null coordinates \((U, W)\), a \((d-2)\)-dimensional reduced metric \( g^{\text{red}} \) and an effective potential \( \Upsilon \). As we will see later, it turns out that the 2d mini-superspace for FLRW cosmology can be recast exactly in those terms. We leave the general case for future investigation, but nevertheless point out that such a reformulation of relativistic free systems as Eisenhart-Duval lifts begs the intriguing question of the possibility of a general \( d \to (d-2) \) deparametrization of the relativistic models.
2.4 On conformal Killing vectors

The strategy that we shall adopt in the next sections to find the dynamical symmetries of a given mini-superspace is to write down its super-metric and its Eisenhart-Duval lift and identify all their CKVs. This raises two basic questions: (i) What is the maximal number of (linearly independent) CKVs (resp. KVs) for a given curved manifold of dimension $d$? (ii) Under which condition a given curved manifold of dimension $d$ is maximally symmetric, i.e. have the maximal number of CKVs?

These questions have been thoroughly studied in the general relativity literature and we quickly overview known results. The interested reader can refer to [67] for detailed proofs. The maximal number of Killing vectors (KVs) for a $d$-dimensional geometry is given by

$$n_{KV} = \frac{d(d + 1)}{2},$$

while the maximal number of CKVs that a $d$-dimensional curved manifold can admit is given by

$$n_{CKV} = \frac{(d + 1)(d + 2)}{2}, \quad \text{for} \quad d \geq 3.$$

The $d = 2$-dimensional case is special since such a geometry can admit at most $n_{KV} = 3$ Killing vectors while it can admit an infinity of linearly independent conformal Killing vectors.

Finally, a $d$-dimensional curved background is maximally symmetric, i.e. has the maximal number of CKVs, if and only if its Weyl and Cotton tensors vanish identically. Notice that the Cotton tensor is defined only for $d \geq 3$ while the Weyl tensor always vanishes in $d = 2$ and $d = 3$ dimensions. Therefore, in $d = 3$ dimensions, we only have to check for the Cotton tensor to vanish or not in order to determine whether the geometry is maximally symmetric.

Let us stress that these properties further underline the differences between the superspace and the Eisenhart-Duval lift. Indeed, for a system with $d$ degrees of freedom, the former is $d$-dimensional while the latter is a higher $d + 2$-dimensional manifold. It follows that their curvature properties and thus the maximal number of CKVs they can have can be radically different. By construction, this reflects on the underlying symmetries of the mechanical system which stands as their shadow. Perhaps the most striking example is for a system with two degrees of freedom, such as the flat FLRW cosmology filled with a massless scalar field or the Schwarzschild black hole mechanics. In those cases, the associated superspace is a 2-dimensional background which is thus conformally flat. Therefore, it possesses an infinite set of linearly independent CKVs. On the contrary, the Eisenhart-Duval lift of these systems is a 4-dimensional background which possesses at most fifteen linearly independent CKVs. Hence, some symmetries of the superspace can not have their counterpart in the ED lift and vice-versa. Thus identifying the underlying symmetries of a given mechanical system via the CKVs of its superspace or its Eisenhart-Duval lift should be considered as two distinct approaches which can reveal different sets of symmetries.

In the following sections, we study in details three gravitational mini-superspaces: (i) the FLRW cosmological mini-superspace for general relativity coupled to a homogeneous isotropic scalar field;
(ii) the Schwarzschild mechanics for gauge-fixed spherically symmetric space-times formulated as Kantowski-Sachs metrics; (iii) the Bianchi I mini-superspace for homogeneous anisotropic cosmology.

3 Flat FLRW cosmology

In this section, we present the dynamical symmetries of the simplest cosmological model consisting in the flat FLRW geometry filled with a massless scalar field. This model was initially studied in [14–17].

3.1 Action and phase space

Consider the line element given by

$$ds^2 = -N^2(t)dt^2 + a^2(t)\delta_{ij}dx^i dx^j$$

where $N$ is the lapse and $a$ the scale factor. The reduced action of the Einstein-Scalar system is given by

$$S[g, \phi] = \int_M d^4x \sqrt{|g|} \left[ \frac{\mathcal{R}}{16\pi G} - \frac{1}{2} g^\mu\nu \phi_\mu \phi_\nu \right]$$

$$= V_0 \int dt \left[ \frac{a^3}{2N} \dot{\phi}^2 - \frac{3}{8\pi G} \frac{a \dot{\phi}^2}{N} + \frac{3}{8\pi G} \frac{d}{dt} \left( \frac{a^2 \dot{\phi}}{N} \right) \right]$$

where $V_0$ is the fiducial volume of the cell on which we restrict the spatial integration. Introducing the volume $v = a^3$, the action becomes

$$S[N, v, \phi; t] = c \ell_P \int dt \left[ \ell_P^2 \frac{\dot{v}^2}{N} - \frac{\dot{v}^2}{Nv} + \frac{d}{dt} \left( \frac{3\dot{v}}{N} \right) \right]$$

where $\ell_P = \sqrt{12\pi G}$ is the Planck length and we have introduced the dimensionless constant

$$c = \frac{V_0}{\ell_P^3}$$

which encodes the ratio between the IR and UV cut-offs of our symmetry-reduced system. At this stage, we can omit the total derivative term in the action since it will not modify the Friedmann dynamics nor its symmetries.

In order to discuss the dynamical symmetries of this cosmological system, let us slightly simplify the FLRW action (3.4) by introducing the field redefinition

$$z = \sqrt{v}$$

such that the FLRW action becomes

$$S[N, z, \phi; t] = c \ell_P \int dt \frac{1}{2N} \left( \ell_P^2 z^2 \dot{\phi}^2 - 4z^2 \right)$$
It corresponds to a free system of the form (2.12) with $\chi^a = \{\phi, z\}$ where the super-metric is given by $g_{\phi\phi} = \ell_p^2 z^2$ and $g_{zz} = -4$. This field redefinition removes the denominator in the gravitational kinetic term which will be useful when computing the finite symmetry transformations. With this new variable, the momenta and the hamiltonian take the following form

$$
\begin{align*}
    \dot{p} &= -4c\ell_P N^{-1}\dot{z}, \\
    \dot{\pi} &= c\ell_P^3 N^{-1}z^2\dot{\phi}, \\
    H[N] &= Nh = \frac{N}{2c\ell_P^3} \left[ \frac{\pi^2}{z^2} - \ell_P^2 p^2 \right].
\end{align*}
$$

and the symplectic structure reads

$$
\{z, p\} = \{\phi, \pi\} = 1
$$

The equations of motion governing the cosmological dynamics read

$$
\begin{align*}
    \dot{z} &= -\frac{p}{4c\ell_P}, \\
    \dot{\pi} &= \frac{\pi^2}{c\ell_P^3 z^3}, \\
    \dot{\phi} &= \frac{\pi}{c\ell_P^3 z^2}, \\
    \dot{\pi} &= 0
\end{align*}
$$

supplemented with the constraint $h = 0$ enforced by the non-dynamical lapse field. Having reviewed the phase space of the system, we can now present its hidden symmetries and discuss the associated algebra of observables. As we shall see, this simple cosmological system can be algebraically characterized by different sets of observables.

### 3.2 Schrödinger observables

In this section, we show that the gauge-fixed FLRW system admits a set of non-independent observables which form a two-dimensional centrally extended Schrödinger algebra. We discuss the role of the Casimirs and algebraically solve the cosmological dynamics from the knowledge of the charges. The derivation of these Schrödinger observables using the Eisenhart-Duval lift is discussed in the next section.

#### 3.2.1 Charge algebra

Consider the first three conserved charges defined by

$$
\begin{align*}
    Q_+ &= \ell_P h, \\
    Q_0 &= \frac{1}{2}zp + \tau h, \\
    Q_- &= cz^2 - \frac{1}{2\ell_P} (\tau z p + \tau^2 h)
\end{align*}
$$

They correspond to the charges already discussed at length in [14–17] which form an $\mathfrak{sl}(2, \mathbb{R})$ algebra. Consider now the five additional charges

$$
\begin{align*}
P_\pm &= e^{\mp\ell_P \phi/2} \left[ \frac{p}{2} \pm \frac{\pi}{\ell_P z} \right], \\
B_\pm &= e^{\mp\ell_P \phi/2} \left[ 2cz - \frac{\tau}{\ell_P} \left( \frac{p}{2} \pm \frac{\pi}{\ell_P z} \right) \right], \\
J &= \frac{\pi}{\ell_P}.
\end{align*}
$$

We have $[Q] = [P] = [B] = [J] = 1$. It is straightforward to check that these charges are indeed conserved. In particular, while the charges $P$ and $J$ are strong Dirac observables, the charges $Q$
and $B$ depend explicitly on time and play the role of evolving constants of motion which generate dynamical symmetries of the system, i.e. they satisfy (2.16). Altogether, they form the following 9-dimensional Lie algebra

\[
\{Q_+, Q_-\} = Q_0, \quad \{Q_0, Q_\pm\} = \pm Q_\pm, \quad \{Q_- P_\pm\} = \frac{1}{2} B_\pm, \quad \{Q_0, P_\pm\} = \frac{1}{2} P_\pm, \quad \{J, P_\pm\} = \frac{1}{2} P_\pm, \\
\{Q_+, B_\pm\} = P_\pm, \quad \{Q_0, B_\pm\} = -\frac{1}{2} B_\pm, \quad \{J, B_\pm\} = \frac{1}{2} B_\pm, \quad \{J, P_\pm\} = \frac{1}{2} P_\pm, \quad \{J, B_\pm\} = \frac{1}{2} P_\pm,
\]

while

\[
\{B_\mp, P_\pm\} = n
\]

where the central extension is given by $n = 2c$. We recognize a two dimensional centrally extended Schrödinger algebra

\[
\mathfrak{sh}(2) = (\mathfrak{sl}(2, \mathbb{R}) \times \mathfrak{so}(2)) \ltimes (\mathbb{R}^2 \times \mathbb{R}^2)
\]

The generators $P_\pm$ and $B_\pm$ are the two $\mathbb{R}^2$ subalgebras. They generate respectively the translations and galilean boosts. The generator $J$ is the $\mathfrak{so}(2)$ sector and generates the rotation in both the $P_\pm$ plane and the $B_\pm$ plane. This might be a surprising feature since $J$ is simply the dimensionless scalar field momentum, but it reflects the peculiar exponential factor $e^{\mp \ell_P \phi/2}$ appearing in the $P_\pm$’s and the $B_\pm$’s. We underline that this is not a rotation at all in space or space-time but a rotation in field space. Finally, the observables $Q_0$ and $Q_\pm$ form the $\mathfrak{sl}(2, \mathbb{R})$ sector and generate Möbius transformations.

There are three independent Casimirs associated to this 2d Schrödinger algebra. The first one is given by

\[
C_1 = (P_+ B_- - P_- B_+) - nJ = 2c \frac{\pi}{\ell_P}
\]

which corresponds to the conserved kinetic energy of the scalar matter field. The second Casimir is cubic in the generators and its expression can be found in [68]. It can be shown that it is proportional to $\pi^2$. Finally, the third Casimir corresponds to the central extension $n$.

At this stage, let us discuss the interpretation of this central extension in our gravitational mini-superspace model. For standard massive non-relativistic systems which enjoy the Schrödinger symmetry, this quantity encodes the mass of the system or, in the case of quantum many-body systems, e.g. quantum fluids, its (conserved) number of particles. In the present gravitational context, the central charge is given explicitly by the dimensionless constant

\[
n = \frac{2V_0}{\ell_P^3}
\]

introduced in (3.5) when enforcing the symmetry reduction. It corresponds to the ratio between the IR and the UV cut-offs of the system, which can be thought as the effective number of fundamental
Planckian cells contained in the region of volume $V_0$. The number of Planckian cells may be seen as a feature of an underlying discrete structure (as in lattice quantum gravity approaches or loop quantum gravity) or, if they are in turn associated to fundamental ‘atomic’ constituents (as in group field theories), each contributing at least a Planckian contribution to the total volume of the universe, as measuring directly the average population of the quantum many-body states whose dynamics collectively described by our gravitational model. Thus, this central extension allows one to associate a notion of “number of particles” to this gravitational mini-superspace which is intimately related to the symmetry-reduction procedure we have used. Indeed, the scale $V_0$ fixes the spatial extension of the region of interest, all the information on the boundary of that region turn out to be encapsulated in this single parameter. This interpretation parallels the standard interpretation of such gravitational mini-superspace as a many-body system emerging from an underlying microscopic theory, and provides a new symmetry-based argument for a hydrodynamic-like description of classical and quantum cosmology [47, 48]. As we shall see, the same conclusion holds also for black hole mechanics.

Now, let us present the explicit symmetry transformations induced by the Schrödinger observables.

### 3.2.2 Symmetry transformations

Consider the action (3.7) for the flat FLRW cosmology, $S = \frac{c\ell_p}{2} \int N^{-1} (\ell_p^2 z^2 \dot{\phi}^2 - 4 \dot{z}^2) dt$. Here, we give the finite symmetry transformations generated by the conserved charges identified above, and check explicitly the invariance of action.

The translations and Galilean boosts generated respectively by the conserved charges $P_\pm$ and $B_\pm$ can be compactly written as

\[
\begin{align*}
\tau & \to \tilde{\tau} = \tau \\
\dot{\tau} & = \dot{\tau} \\
z & \to \tilde{z}(\tilde{\tau}) = z(\tau) + \frac{\xi(\tau)}{2} e^{\epsilon \ell_p \phi/2} \\
\dot{z} & = \dot{z}(\tau) + \frac{\epsilon \ell_p \xi(\tau)}{2} e^{\epsilon \ell_p \phi/2} \\
\phi & \to \tilde{\phi}(\tilde{\tau}) = \phi(\tau) - \frac{\epsilon \ell_p z}{\epsilon \ell_p z} \xi(\tau) e^{\epsilon \ell_p \phi/2}
\end{align*}
\]

where $\xi(\tau)$ is an arbitrary function and $\epsilon = \pm 1$. Considering the infinitesimal version of these transformations, the action transforms as

\[
\delta_\xi S = \frac{c\ell_p}{2} \int d\tau \left\{ \tilde{\xi} z e^{\epsilon \ell_p \phi/2} - \frac{d}{d\tau} \left( \tilde{\xi} z e^{\epsilon \ell_p \phi/2} \right) + O(\xi^2) \right\}
\]

which shows that the above transformations are indeed Noether symmetries provided $\ddot{\xi} = 0$. This implies that

\[
\xi(\tau) = \lambda \tau + \eta
\]
The remaining symmetries generated by the charges $Q_\pm$ and $Q_0$ correspond to the following conformal transformations

\[ \tau \to \tilde{\tau} = f(\tau) \quad (3.23) \]
\[ z \to \tilde{z}(\tilde{\tau}) = f^{1/2}z(\tau) \quad (3.24) \]
\[ \phi \to \tilde{\phi}(\tilde{\tau}) = \phi(\tau) \quad (3.25) \]

At the infinitesimal level, i.e. for $f(\tau) \simeq \tau + \chi(\tau)$, the action transforms as

\[ \delta_\chi S = c\ell_P \int d\tau \left\{ \chi\dot{z}^2 + \frac{d}{d\tau} \left[ \frac{1}{2} \left( 4\dot{z}^2 - \ell_P^2 z^2 \phi^2 \right) - \chi z^2 \right] + O(\chi^2) \right\} \quad (3.26) \]

This variation is a Noether symmetry provided $\chi = 0$ which imposes

\[ \chi(\tau) = \alpha_+ + \alpha_0 \tau + \alpha_- \tau^2 \quad (3.27) \]

where $(\alpha_\pm, \alpha_0)$ are real constants parametrizing this symmetry. We recover the symmetries presented initially in [14] for the flat FLRW cosmological model and extended to the (A)dS cosmology and the closed and open universes in [16, 17].

Applying the Noether theorem, we recover the conserved Noether charges from the total derivative terms of the action variations computed above. This leads back to the Schrödinger charges derived in the previous section. An interesting question which goes beyond the scope of this work is to fully characterize the role of the gauge-fixing in the realization of this Schrödinger symmetry. This point was discussed in detailed in [16] for the restricted SL(2, $R$) symmetry.

### 3.2.3 Integration of the dynamics

Now, let us show that the knowledge of this charge algebra allows one to algebraically solve the FLRW dynamics. One can distinguish between the information encoded on one hand in the strong Dirac observables, such as $J$ and $P_\pm$, and on the other hand in the evolving constants of motion, i.e. the charges which depend explicitly on the time coordinate.

The first charges allows one to solve for the deparametrized dynamics. For instance, the translational charges $P_\pm = e^{\mp \ell_P/2} \left[ \frac{P}{2} \pm \frac{p}{\ell_P} \right]$ allow to directly solve the deparametrized trajectories $v(\phi)$ which read

\[ v(\phi) \equiv V_\pm^2 e^{\pm \ell_P \phi}, \quad \text{with} \quad V_\pm = \frac{Q_0 \pm J}{P_\pm} \quad (3.28) \]

These conserved charges are to be understood as relational observables, in the sense that they encode (time independent) relations between the dynamical variables. These relations tell how variables varies in terms of the others. If one has a complete set of such observables, then one can deparametrize the system and tell express the evolution of all dynamical variables in terms of any single variable of the system. Identifying those relational observables are thus exactly equivalent to solving for the timeless deparametrized trajectories of the system.
On the contrary, the evolving constants of motion allow one to algebraically determine the physical trajectories w.r.t. the time coordinate \( \tau \). Remember that on-shell, the hamiltonian of the system has to vanish which reflects the relativistic nature of the system. We have therefore \( Q_+ = 0 \). Combining the expressions of \( Q_- \) and \( Q_0 \) allows one to solve for the physical trajectory \( v(\tau) \) which reads

\[
v(\tau) := z^2(\tau) = \frac{2}{N} \left[ Q_0 \frac{\tau}{\ell_P} + Q_- \right],
\]

and we recover the linear growth of the volume in presence of a scalar field driving the cosmic expansion. Notice that the sign of \( Q_0 \) determines the expanding/contracting behavior of the universe. This linear evolution of the spatial volume clearly leads to a finite time crash, either in the past if \( Q_0 \) is positive or in the future if \( Q_0 \) is negative. This singularity is a standard feature of flat FLRW cosmology.

In order to solve the profile of the scalar field, notice that

\[
Q_+ = -\frac{1}{2c} P_+ P_- = 0,
\]

so that physical trajectories correspond to either \( P_- \simeq 0 \) and \( P_+ \neq 0 \), or \( P_+ \simeq 0 \) and \( P_- \neq 0 \). Choosing the first case, we can use the profile of \( z(\tau) \) and the charge \( P_+ \) to find

\[
\phi(\tau) \equiv -\frac{1}{\ell_P} \log \left[ \frac{2}{N} \left( Q_0 \frac{\tau}{\ell_P} + Q_- \right) \right] + \frac{2}{\ell_P} \log \left[ \frac{Q_0 + J}{P_+} \right],
\]

Thus, as expected, the conserved charges provide a complete set of observables which fully encode the FLRW dynamics. As a result, one can algebraically solve the cosmological dynamics solely based on the knowledge of the Schrödinger cosmological observables.

Before explaining how to derive these charges, let us first present another set of interesting observables associated to this system.

### 3.3 Witt observables

Consider the charges generating the translations, i.e \( P_\pm \). Since these charges do not depend on time, they stand as strong Dirac observables of the system. Interestingly, one can construct from them an infinite tower of weak Dirac observables as follows. Consider for example the phase space function

\[
W^\pm_F = \frac{1}{2} \left( \frac{\pi}{\ell_P} \pm \frac{z}{2} \right) F_\pm(\ell_P \phi \pm 2 \log z)
\]

where \( F_\pm \) are arbitrary functions. It is straightforward to show that they represent weak Dirac observables of the system:

\[
\{W^\pm_F, h\} = \left( F'_\pm \pm \frac{F_\pm}{2} \right) h \equiv 0
\]
This shows that the FLRW cosmological system is equipped with an infinite set of such observables. They satisfy the following Witt algebras

\[
\{W^\pm_F, W^\pm_G\} = W^\pm_{[F,G]}, \quad \text{where} \quad [F,G] = F'G - G'F, 
\]

\[
\{W^+_F, W^-_G\} = 0.
\]

(3.34)

(3.35)

Here the parameter functions \( F(\tau) \) and \( G(\tau) \) are to be understood as vectors fields in \( \tau \), then the commutator \([F,G]\) is simply the Lie derivative.

As we shall see, such infinite tower of weak observables is a universal property of any mechanical system with two degrees of freedom. This can be understood from the infinite number of conformal Killing vectors of their two dimensional super-metric. From that perspective, this structure will be broken as soon as we add any new degrees of freedom to the system, and it is therefore not clear whether this symmetry can serve to characterize the physics of this gravitational system.

Finally, let us point that demanding strong Dirac observables from this construction imposes that \( F^\pm_\pm = \mp F/2 \). In terms of the canonical variables \((\phi, z, \pi, p)\), the resulting charges read

\[
W^\pm = \frac{1}{2} \left( \frac{\pi}{\ell p z} \pm \frac{p}{2} \right) e^{\mp \ell p \phi/2} \quad (3.36)
\]

which reproduce the generators of translations \( P_\pm \) introduced earlier in (3.12). Then the weak observables for arbitrary parameter functions \( F \) and \( G \) are interpreted as the equivalent of super-translations, if we compare our construction to the BMS symmetry structure of the dynamics of asymptotically flat space-times.

Having presented this last family of observables, we now present how to derive the different charges introduced in this section from a geometrical point of view, using either the conformal isometries of the Eisenhart-Duval lift or the ones of the superspace as reviewed in Section 2.

### 3.4 Deriving the symmetry generators

In this section, we show that the Schrödinger observables can be derived from conformal Killing vector fields of the Eisenhart-duval lift of the FLRW system. We also show that the Witt observables descend from the infinite set of CKVs of the superspace. This exercise provides a concrete way to contrast between the ED lift and the superspace formalism, showing indeed that these two structures are complementary when investigating the physical symmetries of a given mechanical system.

#### 3.4.1 Observables from the Eisenhart-Duval lift

By construction, the ED lift of the flat FLRW cosmology has the following line element

\[
ds^2 = g_{AB}dX^AdX^B = 2du dw + c\ell p (\ell^2 p^2 z^2 d\phi^2 - 4dz^2) \quad (3.37)
\]
Let us denote \( N^A \partial_A = \partial_w \) the covariant constant null vector. It is straightforward to show that the Weyl and Cotton tensor associated to this four dimensional geometry vanish. Therefore, it is conformally flat and thus maximally symmetric. It follows that it possesses the maximal number of CKVs, i.e. fifteen, which form the \( \mathfrak{so}(4,2) \) Lie algebra. This set of CKVs can be organized with respect to their action on the null Killing field \( N^A \partial_A = \partial_w \).

Consider firstly the set of CKVs which commute with \( \partial_w \). There are eight such CKVs. The first three are given by

\[
Q^A_+ \partial_A = -c \partial_u \quad (3.38)
\]
\[
Q^A_0 \partial_A = \frac{1}{2} z \partial_z + u \partial_u \quad (3.39)
\]
\[
Q^A_- \partial_A = c \ell_P z^2 \partial_w + \frac{1}{2} (uz \partial_z + u^2 \partial_u) \quad (3.40)
\]

while the five remaining vectors fields are given by

\[
P^A_\pm \partial_A = e^{\pm \ell_P \phi/2} \left[ \frac{\ell_P}{2} \partial_z \pm \frac{1}{z} \partial_{\phi} \right], \quad (3.41)
\]
\[
B^A_\pm \partial_A = e^{\pm \ell_P \phi/2} \left[ 2 c \ell_P^2 z \partial_w + u \left( \frac{\ell_P}{2} \partial_z \pm \frac{1}{z} \partial_{\phi} \right) \right], \quad (3.42)
\]
\[
J^A \partial_A = \partial_{\phi} \quad (3.43)
\]

Together with \( N^A \partial_A \), these eight vectors fields form the 2-dimensional Schrödinger algebra. Moreover, applying the projections rules reviewed in section 2.3 reproduces the Schrödinger observables presented in section 3.2.1. These charges are at most quadratic in the momenta. Let us present one example of such projection. Consider the CKV \( Q^A_- \partial_A \). The associated conserved charge for the null geodesic motion on the lift is given by

\[
Q_- = Q^A_- p_A = c \ell_P z^2 p_w + \frac{1}{2} (uz p_z + u^2 p_u) \quad (3.44)
\]

Imposing the projection rules

\[
p_w = c, \quad p_u = -\frac{h}{c}, \quad u = -c \tau \quad (3.45)
\]

one obtains

\[
Q_- = c \ell_P \left[ cz^2 - \frac{1}{2 \ell_P} (\tau z p_z + \tau^2 h) \right] \quad (3.46)
\]

which reproduces the third charge in (3.11) up to an irrelevant global factor \( c \ell_P \). Proceeding the same way, one can derive the other Schrödinger observables in a straightforward manner.

The remaining CKVs which complete the \( \mathfrak{so}(4,2) \) algebra do not commute with the constant null vector. Since the metric of the lift is invariant under the switch \( u \leftrightarrow w \), we have four dual
charges which read
\[ \tilde{Q}_0^A \partial_A = \frac{1}{2} z \partial_z + w \partial_w \]  
(3.47)
\[ \tilde{Q}_-^A \partial_A = c \ell_p z^2 \partial_u + \frac{1}{2} (w z \partial_z + w^2 \partial_u) \]  
(3.48)
\[ \tilde{B}_-^A \partial_A = e^{\pm \ell_p \phi / 2} \left[ 2 c \ell_p^2 z \partial_u + w \left( \frac{\ell_p}{2} \partial_z \pm \frac{1}{z} \partial_\phi \right) \right], \]  
(3.49)

It is interesting to notice that upon imposing the projection rules, the charges $\tilde{Q}_-$ and $\tilde{B}_-$ turn out to be cubic in the momenta, while $\tilde{Q}_0 = Q_0$ and does not provide a new information on the system. Finally, the two last CKVs are given by
\[ Y_\pm^A = e^{\pm \ell_p \phi / 2} \left[ z (u \partial_u + w \partial_w) + \frac{1}{2 c \ell_p^2} (2 c \ell_p z^2 + uw) \left( \frac{\ell_p}{2} \partial_z \pm \frac{1}{z} \partial_\phi \right) \right] \]  
(3.50)
which are also associated to charges cubic in the momenta. Those charges are clearly conserved since they are polynomial combinations of the already derived constants of motion with extra Hamiltonian factors in $\hbar$.

It is straightforward to check that these fifteen vectors fields indeed form a $\mathfrak{so}(4,2)$ algebra, see appendix A for the details. We will not look into the finite symmetry transformations that they generate. Indeed, this $\mathfrak{so}(4,2)$ algebra of observables plays a similar role as the $\mathfrak{so}(4,2)$ conformal symmetries of the quantum 2-body problems, as for the spectral analysis of the Hydrogen atom [69]. Thus the finite transformations do not seem especially relevant at the classical level, although it could be interesting to see how they act on classical trajectories, but they will definitely be relevant to the quantization and spectrum of the quantum theory.

### 3.4.2 Observables from the superspace

Now, let us analyze the differences between the ED lift symmetries with the direct superspace symmetries. The superspace associated to the flat FLRW cosmology filled with a massless scalar field is given by the super-metric
\[ ds^2 = g_{ab} dx^a dx^b = z^2 \left( \ell_p^2 d\phi^2 - \frac{4 dz^2}{z^2} \right) \]  
(3.51)
This is a two-dimensional manifold which is therefore conformally flat. As any 2d space, it admits an infinite set of CKVs which provides an infinite set of observables for this cosmological system. This property explains the existence of the observables presented earlier in (3.32).

To see this, it is convenient to consider the conformally rescaled super-metric
\[ ds^2 = dx_+ dx_- = \ell_p^2 d\phi^2 - \frac{4 dz^2}{z^2}, \quad \text{with} \quad x_\pm = \ell_p \phi \pm 2 \log z . \]  
(3.52)
A straightforward computation shown in appendix B shows that this geometry admits the following set of CKVs
\[ X_{F_+} = F_+ (x_+) \partial_+, \quad X_{F_-} = F_- (x_-) \partial_- , \]  
(3.53)
with conformal factor $\varphi_\pm = \partial_\pm F_\pm$. The function $F_\pm$ are free and we have therefore an infinite set of such CKVs. These form a closed Witt algebra:

$$[X_{F_+}, X_{F_-}] = 0, \quad [X_{F_\pm}, X_{G_\pm}] = 2X_{[F,G]_\pm}, \quad \text{with} \quad [F,G]_\pm = F_\pm \partial_\pm G_\pm - G_\pm \partial_\pm F_\pm. \quad (3.54)$$

Then, these conformal isometries translate into an infinite set of Witt observables for the FLRW cosmology. To derive them explicitly, one can write down the momenta $p_\pm$

$$p_\pm = \frac{1}{2} \left( \frac{\pi}{\ell_P} \pm \frac{zp}{2} \right)$$

which satisfy the canonical brackets:

$$\{x_\pm, p_\pm\} = 1, \quad \{x_\pm, p_\mp\} = 0. \quad (3.56)$$

The conserved charges associated with the general CKVs (3.53) thus read

$$W^\pm_F = p_\pm F_\pm = \frac{1}{2} \left( \frac{\pi}{\ell_P} \pm \frac{zp}{2} \right) F_\pm (\ell_P \phi \pm 2 \log z) \quad (3.57)$$

which reproduce the expression (3.32) in terms of the cosmological canonical variables $(\phi, z, \pi, p)$. Then, it is straightforward to show they are indeed weak Dirac observables whatever form of the function $F_\pm$ is chosen, i.e. $\{W^\pm_F, h\} = 0$, and that they form the Witt algebra (3.34). The superspace formalism thus provides a geometrical interpretation for the origin of these Witt cosmological observables. To conclude, notice that since the existence of an infinite tower of CKVs is a consequence of the 2d nature of the superspace, any free mechanical system with two degrees of freedom with a vanishing Hamiltonian will inherit an infinite set of such observables.

4 Schwarzschild black hole mechanics

In this section, we consider the second example of interest: the Schwarzschild black hole mechanics. This mini-superspace possesses again two degrees of freedom. We show that just as the flat FLRW cosmology filled with a massless scalar field, it enjoys again a set of Schrödinger observables which fully encodes the Schwarzschild geometry, both in the exterior and interior regions of the black hole. Finally, we show that it also admit an infinite set of Witt observables with some subtle difference with the FLRW case. Previous investigations on the dynamical symmetries of the Schwarzschild mechanics were presented in [19–21].

4.1 Action and phase space

Consider the spherically symmetric geometry with line element

$$ds^2 = \epsilon \left(-N^2(x)dx^2 + A^2(x)dy^2\right) + \ell^2 \delta(x)d\Omega^2 \quad (4.1)$$

25
where \( N(x) \) is the lapse, \( \ell_s \) is a fiducial length scale and \( \epsilon = \pm 1 \) is a parameter which will allow us to treat both the interior and the exterior black hole regions at once. To map the line element onto the Schwarzschild metric, the coordinate \( x \) is the radial coordinate, while the \( y \) coordinate is the time coordinate \( t \). The sector \( \epsilon = -1 \) describes the usual exterior of the Schwarzschild metric, for which \( x \) is space-like and \( y \) is time-like and for which we use a time-like foliation of the bulk. The sector \( \epsilon = +1 \) describes the interior of the Schwarzschild black hole, for which \( x \) becomes time-like and \( y \) space-like and where we use a spacelike foliation of the geometry. See [21] for detail.

Now, let us write the action governing its dynamics. The Einstein-Hilbert action reduces to

\[
S_\epsilon[N, A, B; x] = \int dx \frac{R}{2\ell_p^2} = \frac{\ell_0 \ell_s^2}{\ell_p^2} \int dx \left\{ \frac{\epsilon NA}{\ell_s^2} - \frac{A(B')^2 + 2BB'A'}{N} + \frac{d}{dx} \left( \frac{B^2A' + 2ABB'}{N} \right) \right\}
\]

where we have introduced the length \( \ell_0 \) which sets the size of the system in the \( y \)-direction. At this stage, we introduce a new time coordinate \( Ndx = N\,d\eta/A \) such that the metric reads

\[
ds^2 = \epsilon \left( -\frac{N^2(\eta)d\eta^2}{A^2(\eta)} + A^2(\eta)d\eta^2 \right) + \ell_s^2 B^2(\eta)d\Omega^2
\]

This field redefinition allows one to simplify the potential term in the action, such that once we gauge-fixed the \( \eta \)-reparametrization invariance, i.e. when working with the coordinate \( d\tau = N\,d\eta \), the potential term reduces to a simple constant in the mechanical action. Proceeding that way, the symmetry-reduced action becomes

\[
S_\epsilon[N, A, B; \tau] = c\ell_p \int d\tau \left[ \frac{\epsilon}{\ell_s^2} - \left( A^2 \hat{B}^2 + 2ABB' \right) \right]
\]

In this mechanical action, we have omitted the total derivative term and a dot refers to a derivative w.r.t. the new coordinate \( \tau \). Notice also that we have introduced the dimensionless parameter

\[
c = \frac{\ell_0 \ell_s^2}{\ell_p^2}.
\]

Let us point out that we have two independent IR regulators \( \ell_0 \neq \ell_s \), defining integration cut-off respectively in the \( y \) and \( x \) directions. \( \ell_s \) is the fiducial radius of the 2-sphere, while \( \ell_0 \) is the interval for the \( y \)-coordinate.

We can now solve the equations of motion w.r.t. the coordinate \( d\tau = N\,d\eta \). A straightforward computation gives the following profiles for the \( A \)-field and \( B \)-field:

\[
A^2(\tau) = -\frac{\epsilon}{C^2 \ell_s^2} \frac{\tau - \tau_1}{\tau - \tau_0}, \quad B(\tau) = C(\tau - \tau_0)
\]

where \( C \) and \( (\tau_0, \tau_1) \) are constants of integration. Upon performing (i) a translation \( \tau \to \tau + \tau_0 \) and (ii) a rescaling of the coordinates \( \tau \to C\tau/\ell_s \) and \( y \to Cy/\ell_s \), the line element reads

\[
ds^2 = \left( 1 - \frac{\ell_M}{\tau} \right) d\tau^2 + \left( 1 - \frac{\ell_M}{\tau} \right)^{-1} dy^2 + \tau^2 d\Omega^2
\]
where $\ell_M = \tau_1 - \tau_0$. This corresponds to the Schwarzschild black hole geometry with Komar mass $\ell_M$, showing that the simple mechanical action (4.5) reproduces indeed the Schwarzschild mechanics.

Now, in order to identify the different relevant families of observables of the black hole mechanics, it will be useful to rewrite the action (4.5) in a more appropriate form. Consider the field redefinition

$$V_1 := B^2, \quad V_2 := \frac{A^2 B^2}{2}$$

(4.9)

In term of these new fields, the (gauge-fixed) Schwarzschild mechanics (4.5) can be recast as

$$S_s[V_1, V_2; \tau] = \epsilon \epsilon \ell P \int d\tau \left[ \frac{\epsilon}{\ell_s^2} + \frac{V_2 \dot{V}_1^2 - 2V_1 \dot{V}_1 \dot{V}_2}{2V_1^2} \right]$$

(4.10)

The momenta are given by

$$P_1 = \frac{\epsilon \epsilon \ell P}{V_1^2} \left( V_2 \dot{V}_1 - V_1 \dot{V}_2 \right), \quad P_2 = -\epsilon \epsilon \ell P \frac{\dot{V}_1}{V_1}$$

(4.11)

and we have

$$\dot{V}_1 = -\frac{1}{\epsilon \epsilon \ell P} V_1 P_2, \quad \dot{V}_2 = -\frac{1}{\epsilon \epsilon \ell P} (V_1 P_1 + V_2 P_2)$$

(4.12)

The Hamiltonian reads

$$h = -\frac{1}{\epsilon \epsilon \ell P} \left[ V_1 P_1 P_2 + \frac{1}{2} V_2 P_2^2 \right] - \frac{\epsilon \ell P}{\ell_s^2}$$

(4.13)

Because the potential term of the Hamiltonian is a mere constant, it is convenient to work with the shifted Hamiltonian $\tilde{h} = h + \epsilon \ell P / \ell_s^2$ such that on-shell, one has $\tilde{h} \approx \epsilon \ell P / \ell_s^2$. The equations of motion are given by

$$\dot{V}_1 = \{V_1, H\} = -\frac{1}{\epsilon \epsilon \ell P} V_1 P_2, \quad \dot{P}_1 = \{V_1, H\} = \frac{1}{\epsilon \epsilon \ell P} P_1 P_2$$

(4.14)

$$\dot{V}_2 = \{V_2, H\} = -\frac{1}{\epsilon \epsilon \ell P} (V_1 P_1 + V_2 P_2), \quad \dot{P}_2 = \{P_2, H\} = \frac{P_2^2}{2\epsilon \epsilon \ell P}$$

(4.15)

Before introducing the Schrödinger observables of the system, it is useful to first introduce the following phase space function

$$C := \epsilon \ell P \{V_2, H\} = -\epsilon (V_1 P_1 + V_2 P_2)$$

(4.16)

which is a weak Dirac observable of the system satisfying

$$\{C, h\} = -\epsilon h$$

(4.17)

Having presented the mechanical system which reproduces the Schwarzschild mechanics, we now show that it is equipped with a set of Schrödinger observables but also with an infinite tower of Witt observables in a way similar to the FLRW cosmological model.
4.2 Schrödinger observables

In this section, we present the Schrödinger observables of the Schwarzschild mechanics and the associated Casimirs. Then, we show that the Schwarzschild solution can be recovered solely from the knowledge of the conserved charges.

4.2.1 Charge algebra

Consider the first three observables

\[ Q_+ = \tilde{h} \] (4.18)
\[ Q_0 = \tau \tilde{h} + \epsilon C \] (4.19)
\[ Q_- = \tau^2 \tilde{h} + 2\epsilon \tau C - 2\epsilon \ell p V_2 \] (4.20)

where \( \tilde{h} \) is the shifted hamiltonian. They correspond to the conserved charges already discussed in [19–21]. Next, we obtain the four additional conserved charges given by

\[ P_+ = 2\sqrt{V_1}P_1 + \frac{V_2}{\sqrt{V_1}}P_2 , \quad B_+^\epsilon = \frac{2\epsilon \ell p V_2}{\sqrt{V_1}} + \tau \left( 2\sqrt{V_1}P_1 + \frac{V_2}{\sqrt{V_1}}P_2 \right) \] (4.21)
\[ P_- = 2\sqrt{V_1}P_2 , \quad B_-^\epsilon = 4\epsilon \ell p \sqrt{V_1} + 2\tau \sqrt{V_1}P_2 , \] (4.22)

to which we can add

\[ J = V_1P_1 . \] (4.23)

Notice that \( [Q] = [B] = L \) and \( [J] = [P] = 1 \). A straightforward computation shows that these quantities are indeed conserved charges. In particular, the charges \( Q_0, Q_- \) and \( B_\pm \) are evolving constants of motion and depend explicitly on the \( \tau \)-coordinate. Therefore, they do not commute with the hamiltonian and change the energy. On the other hand, the other charges are strong Dirac observables of the system. The first three charges form an \( \mathfrak{sl}(2, \mathbb{R}) \) algebra:

\[ \{ Q_+, Q_- \} = 2Q_0^\epsilon \], \[ \{ Q_0, Q_+ \} = -Q_+ \], \[ \{ Q_0, Q_- \} = +Q_- \] (4.24)

while the other brackets read

\[ \{ Q_-^\epsilon, P_\pm \} = -B_\pm^\epsilon \], \[ \{ Q_0, P_\pm \} = -\frac{1}{2}P_\pm \], \[ \{ J, P_\pm \} = \pm \frac{1}{2}P_\pm \] (4.25)
\[ \{ Q_\pm, B_\pm^\epsilon \} = P_\pm \], \[ \{ Q_0^\epsilon, B_\pm \} = \frac{1}{2}B_\pm^\epsilon \], \[ \{ J, B_\pm^\epsilon \} = \pm \frac{1}{2}B_\pm^\epsilon \].

Finally, the boosts and translations form the 2d Heisenberg algebra:

\[ \{ B_\pm^\epsilon, P_\pm \} = n , \] (4.26)

where \( n \) is the central extension. We recognize again the two-dimensional centrally extended Schrödinger algebra \( \mathfrak{sh}(2) \) which shows that the Schwarzschild mechanics can indeed be equipped with a set of Schrödinger observables.
Now, let us discuss the interpretation of the Casimirs. For such \( \mathfrak{sh}(2) \) algebra, there are three independent Casimirs \([68]\). The first one corresponds to the central extension \( n \). In the present context, we find that \( n \) is given by

\[
\ell_0 \ell_2 \ell_3 P
\]

(4.27)

Therefore, as anticipated, the interpretation proposed for the flat FLRW cosmological system extends to the Schwarzschild mechanics, too. Indeed, this generator coincides again with the parameter \( c \) introduced in (4.6), such that it can be understood as encoding the ratio between the IR and UV scales of the symmetry-reduced system. The second Casimir quadratic in the generators reads

\[
C_1 = (P_+ B_- - P_- B_+) - n J = 4 c V_1 P_1 ,
\]

(4.28)

and one can show that \( V_1 P_1 \) is proportional to the Schwarzschild mass, i.e \( V_1 P_1 \propto \ell_M \), where \( \ell_M \) has been introduced in (4.8). Finally, the third Casimir of the algebra is cubic in the momenta and one can show that it is proportional to the squared black hole mass. Thus, the invariant Casimir operators labelling the state of the black hole correspond to its mass as expected but also encode the information on the effective size of the system through the central extension \( n \).

The Schrödinger observables presented in this section can be derived from the conformal isometries of the Ehrenharts-Duval lift of the Schwarzschild action, as explained in the first section. Since a first example has been treated in detail in the previous section, we do not reproduce the computation of the CKVs here but we refer the reader to appendix C.1 where the expressions of the CKVs are given explicitly. An interesting outcome of this investigation is that the ED lift of the black hole is again a four dimensional conformally flat manifold, such that its CKVs form again an \( \text{ISO}(2,1) \) Lie algebra. The Schrödinger charges presented above correspond to the charges at most quadratic in the momenta.

At this stage, let us point that an additional symmetry have been found in earlier works \([19, 20]\) which, together with the \( \text{SL}(2, \mathbb{R}) \) charges, form an \( \text{ISO}(2,1) \) symmetry group. This symmetry group turns out to be also a symmetry of the Schwarzschild-(A)dS mechanics \([21]\). Interestingly, the translational charges which build up the Poincaré group are quadratic in the momenta \( P_2 \). For that reason, they cannot be derived by focusing only on the CKVs of the ED lift. Instead, one has to investigate the Killing tensors of the lift to identify its geometrical origin, as discussed in \([31]\). A geometrical derivation of these charges and the extension to the Schwarzschild-(A)dS case will be presented elsewhere.

### 4.2.2 Symmetry transformations

We now show that the Schrödinger charges presented above generate well defined Noether symmetries of the gauge-fixed Schwarzschild mechanical action (4.5). The symmetries induced by the \( \mathfrak{sl}(2, \mathbb{R}) \) charges \( Q_\pm \) and \( Q_0 \) have been derived previously in \([19–21]\) and we shall not reproduce this
computation here. The symmetry transformations are given explicitly by

\begin{align}
\tau &\rightarrow \tilde{\tau} = f(\tau) \\
V_1 &\rightarrow \tilde{V}_1(\tilde{\tau}) = \tilde{f}(\tau)V_1(\tau) \\
V_2 &\rightarrow \tilde{V}_2(\tilde{\tau}) = \tilde{f}(\tau)V_2(\tau)
\end{align}

where the function \( f(\tau) \) is given by

\[ f(\tau) = \frac{a\tau + b}{c\tau + d}, \quad \text{with} \quad ad - bc \neq 0 \] (4.32)

The interested reader is referred to [19, 21] for an explicit expression of the transformed action.

Now, consider the symmetry transformations generated by \( P_\pm \) and \( B_\pm \) which represent the newly introduced charges w.r.t. the previous works [19, 21]. They can again be compactly written as

\begin{align}
\tau &\rightarrow \tilde{\tau} = \tau \\
V_1 &\rightarrow \tilde{V}_1(\tilde{\tau}) = V_1(\tau) + 2\chi(\tau)\sqrt{V_1} \\
V_2 &\rightarrow \tilde{V}_2(\tilde{\tau}) = V_2(\tau) + \xi(\tau)\sqrt{V_1(\tau)} + \chi(\tau)\frac{V_2(\tau)}{\sqrt{V_1(\tau)}}
\end{align}

where \((\xi, \chi)\) are two functions of the \( \tau \)-coordinate. The infinitesimal transformation of the action is given by

\[ \delta_{\xi,\chi} S = 2\epsilon c\ell P \int d\tau \left\{ \dot{\xi} \sqrt{V_1} + \dot{\chi} \frac{V_2}{\sqrt{V_1}} - \frac{d}{d\tau} \left( \dot{\xi} \sqrt{V_1} - \dot{\chi} \frac{V_2}{\sqrt{V_1}} - \frac{\epsilon}{2\ell_s^2} (f - \tau) \right) + \mathcal{O}(\chi^2, \xi^2) \right\} \] (4.36)

such that we have a symmetry provided \( \ddot{\xi} = \ddot{\chi} = 0 \). We obtain thus

\[ \chi(\tau) = \alpha_+ + \alpha_-\tau, \quad \xi(\tau) = \beta_+ + \beta_-\tau \] (4.37)

which provides a four-parameter Noether symmetry of the action. The \( P \)'s and \( B \)'s thus act as translations and boosts in field space. This concludes the presentation of the symmetry transformations.

### 4.2.3 Integrating the dynamics

We now want to show that the Schrödinger charges fully encode the Schwarzschild geometry, i.e. that their knowledge is enough to reconstruct the geometry. It implies that the constants of integration \((C, \tau_1, \tau_0)\) introduced in (4.7) can be expressed in terms of the conserved charges. To see this, remember that the gauge-fixed metric is given by

\[ ds^2 = e \left( -\frac{d\tau^2}{A^2(\tau)} + A^2(\tau)dy^2 \right) + \ell_s^2 B^2(\tau)d\Omega^2 \] (4.38)
where the \( A \) and \( B \)-fields are given in terms of the dynamical fields \( V_1 \) and \( V_2 \) by (4.9). In order to algebraically determine their profiles, one can use the translations and boosts (4.21) and (4.22). One finds

\[
\sqrt{V_1} = \frac{1}{4eclp} \left( B_+ - \tau P_+ \right), \quad \frac{V_2}{\sqrt{V_1}} = \frac{1}{2eclp} \left( B'_+ - \tau P'_+ \right)
\]

These solutions allows one to reconstruct the profiles of the \( A \) and \( B \)-fields such that

\[
A'^2(\tau) = \frac{B'_+ - \tau P'_+}{B'_- - \tau P'_-}, \quad 4eclp B(\tau) = B'_- - \tau P'_- \quad (4.40)
\]

With this profile at hand, one can rewrite the constants of integration in (4.7) in terms of the conserved charges:

\[
\frac{\ell^2}{clp} = \frac{1}{Q^+_+}, \quad C = -\frac{P_-}{4eclp}, \quad \tau_1 = \frac{B'_+}{P'_+}, \quad \tau_0 = \frac{B'_-}{P'_-} \quad (4.41)
\]

This provides the dictionary between the observables of the Schrödinger algebra and the constants of integration directly entering the expressions of the classical trajectories. This underlines the role of constants of integration as constants of motion, and vice-versa the interpretation of conserved charges as initial conditions for the trajectories. This concludes the reconstruction of the Schwarzschild geometry from the Schrödinger observables.

### 4.3 Witt observables

To complete the discussion on the observables of this model, let us show that the Schwarzschild mechanics admits an infinite tower of Witt charges. This is expected from the fact that the system possesses only two degrees of freedom. Consider the phase space functions given by

\[
W^+_F = F(\sqrt{V_1}) \left[ P_+ - \frac{\alpha}{P_-} \right], \quad (4.42)
\]

\[
W^-_G = G \left( \frac{V_2}{\sqrt{V_1}} \right) \left[ P_- - \frac{\beta P_-}{Q^+_+} \right], \quad (4.43)
\]

where

\[
\alpha = 2eclp \frac{\ell^2}{l^2 s}, \quad \beta = -\frac{clp}{l^2 s} \quad (4.44)
\]

and the functions \( F \) and \( G \) are arbitrary. \( P_\pm \) are the two translation charges of the Schrödinger algebra introduced in (4.21) and (4.22) which are related to the hamiltonian via \( P_+ P_- = -4eclp Q^+_+ \). It is straightforward to show that

\[
\{W^+_F, h\} = F' \left( \tilde{h} + \frac{clp}{l^2 s} \right) \equiv 0, \quad \{W^-_G, h\} = 2G' \left( \tilde{h} + \frac{clp}{l^2 s} \right) \equiv 0 \quad (4.45)
\]

where \( \tilde{h} \) is the shifted hamiltonian, i.e. \( \tilde{h} \equiv -clp/l^2 s \). Each set of charges form a Witt algebra, i.e.

\[
\{W^+_F, W^-_G\} = W^\pm_{[F,G]} \quad (4.46)
\]
but they do not commute with each other. Indeed, one can show that

\[
\{W_F^+, W_G^\pm\} = \alpha \beta \frac{4 \varepsilon c \ell_P}{P_+ P_-} \left( \frac{G^F F}{P_-} + \frac{G F^I}{P_+} \right) + \alpha \frac{G^F F}{2 P_-} - 4 \varepsilon c \ell_P \beta \frac{G F^I}{P_+} \neq 0 .
\]  

(4.47)

We observe that the anomalies originate from the corrective terms, labelled by \((\alpha, \beta)\), that we have introduced in order to obtain well defined conserved charges. From that perspective, these \(W_{\text{ Witt}}\) charges are different from the ones derived for the flat FLRW case. This can be understood from the potential contribution of the symmetry-reduced action of each model. In the FLRW case, the action does not contain any potential term, such that the cosmological dynamics can be mapped to null geodesics in superspace. It follows that any CKVs of the cosmological super-metric is a well defined charge of the dynamics. On the contrary, the symmetry reduced action \((4.5)\) for the black hole mechanics contains a constant potential. At the level of the superspace, this term plays the role of a non-vanishing mass and the black hole dynamics is mapped onto space-like or time-like geodesics on the super-metric. As explained in the first section, for such model, the CKVs of the super-metric do not provide anymore conserved charges of the system. Instead, one needs to add corrective terms proportional to the “mass” of the system to recover well defined weak observables from the CKVs. This procedure is presented in detailed in appendix C.2 for the black hole case. A consequence of adding these corrective terms is that, in general, the two copies of \(W_{\text{ Witt}}\) charges one can derive do not commute anymore. See [70] for a related discussion.

5 Bianchi I cosmology

Finally, as the third and last example, let us consider the simplest anisotropic cosmological model given by the Bianchi I universe filled with a massless scalar field. Treating this model allows us to illustrate the role played by the dimensionality and the curvature properties of both the superspace and the ED lift in the realization of the dynamical symmetries. As we shall see, the Schrödinger symmetry found for the two previous models is no longer realized. However, we show that another Schrödinger charge algebra can be identified which is embedded in a larger \(so(4,2)\) set of observables. This provides a new algebraic characterization of the Bianchi I model. We refer the reader to [71,72] for pedagogical introductions on the Bianchi cosmologies. Symmetries of this family of spacetimes have been investigated in [73–76] and more recently in [27], while recent discussions on the quantum Bianchi I model can be found in [77,78].

5.1 Action and phase space

Consider the anisotropic line element

\[
\text{d}s^2 = -N^2 \text{d}t^2 + e^{3\sqrt{3} \lambda/4} \left[ e^{\beta_2-\beta_1/\sqrt{3}} \text{d}x^2 + e^{-\beta_2-\beta_1/\sqrt{3}} \text{d}y^2 + e^{2\beta_1/\sqrt{3}} \text{d}z^2 \right]
\]  

(5.1)
where $N$ is the lapse and $(\lambda, \beta_1, \beta_2)$ are the Misner variables\(^2\) [79]. All the fields depend only on the time coordinate and the coefficients have been chosen in order to simplify the resulting symmetry-reduced action. Introducing the variable

$$z = e^{3\sqrt{3}\lambda/16}$$

which encodes the square root of the volume and corresponds to the (3.6). The dynamics of this geometry is governed by the symmetry-reduced Einstein-Scalar action which reads

$$S[N, z, \beta_1, \beta_2, \phi; t] = \frac{c\ell_P}{2} \int \frac{dt}{N} \left[ \left( \ell_P^2 \dot{\phi}^2 + \dot{\beta}_1^2 + \dot{\beta}_2^2 \right) z^2 - \dot{z}^2 \right]$$

The reduction to the isotropic case is simply obtained by demanding that $\partial_t \beta_1 = \partial_t \beta_2 = 0$. These two additional modes thus account for the anisotropies and behave in a way similar to two additional scalar fields. For now on, we consider the gauge fixed system and work with the time $d\tau = N dt$. The momenta are given by

$$p_z = \frac{\delta L}{\delta \dot{z}} = -c\ell_P \dot{z}, \quad p_i = \frac{\delta L}{\delta \dot{\beta}_i} = c\ell_P z^2 \dot{\beta}_i, \quad \pi = \frac{\delta L}{\delta \dot{\phi}} = c\ell_P^3 z^2 \dot{\phi}$$

and the Hamiltonian reads

$$H = \frac{1}{2c\ell_P} \left[ \ell_P^2 \pi^2 + p_1^2 + p_2^2 \right] - p_z^2$$

which reduces to the FLRW hamiltonian for $p_1 = p_2 = 0$. We can now present the symmetries of this system. In the next section, we present the charges and their algebra. The explicit derivation of these observables is presented in appendix D.

### 5.2 SO(4,2) observables

Using the form of the super-metric (D.1), one can show that it is a conformally flat four dimensional manifold. It follows that the four dimensional conformal group $SO(4,2)$ stands as a physical symmetry of the Bianchi I universe filled with a massless scalar field. The standard techniques discussed above allow to derive the $so(4,2)$ observables of this system. This provides us with strong Dirac observables. These are (time-independent) constants of motion, giving algebraic relations between the dynamical variables. They fully characterize how the variables evolve with respect to each other and give the deparametrized trajectories, i.e. the evolution of all the degrees of freedom.

\[^2\]Writing the metric in the standard variables $(\alpha_{\pm}, \gamma)$ as

$$ds^2 = -N^2(t)dt^2 + \alpha_{\pm}^2(t)dx^2 + \alpha_+^2(t)dy^2 + \gamma^2(t)dz^2$$

one has to following definitions for the scale factors:

$$\log \alpha_{\pm} = \frac{1}{2\sqrt{3}} \left( \frac{3}{4} \lambda + \pm \sqrt{3} \beta_2 - \beta_1 \right), \quad \log \gamma = \frac{1}{2\sqrt{3}} \left( \frac{3}{4} \lambda + 2\beta_1 \right)$$
freedom in terms of an arbitrary singled-out variable. In this sense, they play the role of relational observables.

The translation charges read
\[
T_1 = p_1, \quad T_2 = p_2, \quad T_\varphi = \pi, \quad T_0 = z p_z
\] (5.8)
while the rotations and boosts generators are given by
\[
\begin{align*}
M_{12} &= \beta_1 p_2 - \beta_2 p_1, \\
M_{1\varphi} &= \beta_1 \pi - \varphi p_1, \\
M_{2\varphi} &= \beta_2 \pi - \varphi p_2,
\end{align*}
\] (5.9)
Finally, the four generators of special conformal transformations read
\[
\begin{align*}
K_0 &= -2 \log z \left[ \left( \log z + \frac{F}{2 \log z} \right) z p_z + \beta_1 p_1 + \beta_2 p_2 + \varphi \pi \right] \\
K_1 &= 2\beta_1 \left[ \log z z p_z + \left( \beta_1 - \frac{F}{2 \pi} \right) p_1 + \beta_2 p_2 + \varphi \pi \right] \\
K_2 &= 2\beta_2 \left[ \log z z p_z + \beta_1 p_1 + \left( \beta_2 - \frac{F}{2 \pi} \right) p_2 + \varphi \pi \right] \\
K_\varphi &= 2\varphi \left[ \log z z p_z + \beta_1 p_1 + \beta_2 p_2 + \left( \varphi - \frac{F}{2 \pi} \right) \pi \right]
\end{align*}
\] (5.10)
where the function \( F \) is given by
\[
F(z, \beta_1, \beta_2, \varphi) := \beta_1^2 + \beta_2^2 + \varphi^2 - \log^2 z
\] (5.11)
Finally, the dilatation generator is given by
\[
D = \log z z p_z + \beta_1 p_1 + \beta_2 p_2 + \varphi \pi
\] (5.12)
One can check that these phase space functions indeed form a \( \mathfrak{so}(4, 2) \) algebra
\[
\begin{align*}
\{M_{\alpha\beta}, M_{\gamma\delta}\} &= \eta_{\alpha\delta} M_{\beta\gamma} + \eta_{\beta\gamma} M_{\alpha\delta} + \eta_{\alpha\gamma} M_{\beta\delta} + \eta_{\beta\delta} M_{\alpha\gamma}; \\
\{T_\alpha, M_{\beta\gamma}\} &= \eta_{\beta\gamma} T_\alpha - \eta_{\alpha\gamma} T_\beta; \quad \{T_\alpha, D\} = T_\alpha; \\
\{K_\alpha, M_{\beta\gamma}\} &= \eta_{\beta\alpha} K_\gamma - \eta_{\gamma\alpha} K_\beta; \quad \{D, K_\alpha\} = K_\alpha; \\
\{T_\alpha, K_\beta\} &= 2 \left( \eta_{\alpha\beta} D - M_{\alpha\beta} \right)
\end{align*}
\] (5.13)
This \( \mathfrak{so}(4, 2) \) charge algebra provides a new way to organize the phase space of the Bianchi I model. Let us point that if one takes the isotropic limit, the only charges which survive are given by \( T_\varphi \) and \( T_0 \) while all the other charges are no longer conserved when \( p_1 = p_2 = 0 \). Thus, the presence of the anisotropic modes drastically modify the set of observables of the system. In particular, turning
off the anisotropies will kill some symmetries while allowing for the emergence of the Schrödinger symmetries of the isotropic model.

Finally, let us point that the fifteen charges are not independent and satisfy the following relations

\[
T_0 M_{12} + T_1 M_{20} + T_2 M_{01} = 0 \quad (5.14)
\]
\[
T_\varphi M_{12} + T_1 M_{2\varphi} + T_2 M_{\varphi 1} = 0 \quad (5.15)
\]
\[
M_{01} M_{2\varphi} + M_{02} M_{\varphi 1} + M_{0\varphi} M_{12} = 0 \quad (5.16)
\]
\[
T_1 K_2 - T_2 K_1 + 2 M_{12} D = 0 \quad (5.17)
\]
\[
T_0 K_\varphi - T_\varphi K_0 + 2 M_{0\varphi} D = 0 \quad (5.18)
\]

Together with the three Casimirs of the \( \mathfrak{so}(4,2) \) algebra, this gives 8 conditions that reduce the \( \mathfrak{so}(4,2) \) space to the original phase space with 8 variables constrained by the vanishing Hamiltonian. Now, we show that among these charges, one can identify a new Schrödinger charge algebra.

### 5.3 Schrödinger observables

By combining the \( \mathfrak{so}(4,2) \) generators, one can introduce the following \( \mathfrak{sl}(2,\mathbb{R}) \) charges

\[
q_- = \frac{1}{2} (T_\varphi + T_0) , \quad q_0 = \frac{1}{2} (D + M_{\varphi,0}) , \quad q_+ = \frac{1}{2} (K_\varphi - K_0) ,
\]

(5.19)

One can complete this algebra by the translation, boosts and rotations generators given by

\[
\begin{align*}
P_+ &= T_1 , & B_+ &= M_{10} + M_{\varphi 1} , & J &= -M_{12} , \\
P_- &= T_2 , & B_- &= M_{20} + M_{\varphi 2} , & n &= \frac{1}{2} (T_0 - T_\varphi) ,
\end{align*}
\]

(5.20)

These nine charges form a two dimensional centrally extended Schrödinger algebra

\[
\{q_+, q_-\} = 2q_0 , \quad \{q_0, q_\pm\} = \pm q_\pm , \quad (5.21)
\]

\[
\{q_+, P_\pm\} = B_\pm , \quad \{q_0, P_\pm\} = -\frac{1}{2} P_\pm , \quad \{J, P_\pm\} = \pm P_\pm , \\
\{q_-, B_\pm\} = P_\pm , \quad \{q_0, B_\pm\} = +\frac{1}{2} B_\pm , \quad \{J, B_\pm\} = \pm B_\pm
\]

with the central extension given by

\[
\{P_\pm, B_\pm\} = 2n
\]

(5.22)

This new Schrödinger charge algebra is remarkably different from the ones identified for the two previous models. Indeed, it is constructed from strong Dirac observables and it does not include any evolving constants of motion, such that the symmetry transformations do not involve the time coordinate, contrary to the Schrödinger transformations (3.19 - 3.20) realized in the isotropic
case. Moreover, the Casimirs labelling the classical state of this system have a different interpretation. The central extension \( n \) corresponds to the center-of-mass of the isotropic motion, while the quadratic Casimir vanishes, i.e.

\[
C_1 = P_+ B_- - P_- B_+ + 2n J = 0.
\] (5.23)

This Schrödinger algebra provides thus a different algebraic structure to organize the phase space of this anisotropic model.

At this stage, a natural question is which of the Schrödinger charges found in the isotropic FLRW model are realized in this anisotropic extension? This is achieved below.

### 5.4 Further dynamical symmetries

As it turns out, the Bianchi I model also possesses a set of evolving constants of motion given by

\[
Q_- = H, \quad Q_0 = \frac{z}{2} p_z + \tau H, \quad Q_+ = \frac{c \ell p_z^2}{2} + \tau z p_z + \tau^2 H
\] (5.24)

They reproduce exactly the time-dependent charges of the isotropic FLRW model and govern the isotropic motion. These charges are associated to the CKVs of the Eisenhart-Duval lift of the Bianchi I model, see appendix D.2. Then, among the KVs of the lift, one recovers the six charges \( T_1, T_2, T_\varphi, M_{1,2}, M_{1,\varphi} \) and \( M_{2,\varphi} \) presented above. One thus obtains the following \( sl(2, \mathbb{R}) \times \text{iso}(3) \) algebra

\[
\{ M_{\alpha\beta}, M_{\gamma\delta} \} = \eta_{\alpha\delta} M_{\beta\gamma} + \eta_{\beta\gamma} M_{\alpha\delta} + \eta_{\alpha\gamma} M_{\delta\beta} + \eta_{\beta\delta} M_{\gamma\alpha};
\]

\[
\{ T_\alpha, M_{\beta\gamma} \} = \eta_{\beta\alpha} T_\gamma - \eta_{\gamma\alpha} T_\beta;
\]

\[
\{ Q_0, Q_\pm \} = \pm Q_\pm, \quad \{ Q_+, Q_- \} = Q_0
\] (5.25)

where the indices run as \( \alpha \in \{1, 2, \varphi\} \).

The remaining charges entering in the \( so(4, 2) \) algebra do not commute with \( Q_\pm \) and \( Q_0 \). This is expected from the fact that they do not descend from the same set of CKVs. Finally, let us stress that as shown in appendix D.2, the main difference between the anisotropic and isotropic case is that the lift of the former is a six dimensional manifold which is no conformally flat. It possesses only twelve CKVs among which nine of them generate the \( sl(2, \mathbb{R}) \times \text{iso}(3) \) symmetry. Let us point that the presence of the \( sl(2, \mathbb{R}) \) algebra of evolving constants of motion fits with the results discussed initially in [83] and more recently in [27].

### 5.5 Integration of the dynamics

To complete the discussion on this model, let us show how the dynamics of the four degrees of freedom, i.e. the trajectories \( (z(\tau), \varphi(\tau)) \) and \( (\beta_1(\tau), \beta_2(\tau)) \), can be integrated using the conserved charges. Consider firstly the isotropic modes \( z(\tau) \) and \( \varphi(\tau) \). Just as for the FLRW case, combining
the charges $Q_+$ and $Q_0$, we recover the linear growth of the volume (3.29) w.r.t. the time coordinate $\tau$, i.e.

$$z^2(\tau) = \frac{2}{c\ell_P} [Q_+ - 2\tau Q_0]$$

(5.26)

Then, using the charges $M_{\varphi 0}$ and $T_\varphi$, we obtain the deparametrized relation $\varphi(z)$ from which we can deduce the physical trajectory $\varphi(\tau)$ which reads

$$\varphi(\tau) = \frac{1}{T_0} \left[ M_{\varphi 0} - \frac{T_\varphi}{2} \log z^2(\tau) \right],$$

(5.27)

The deparametrized dynamics of the anisotropic modes is obtained using the charges $M_{i,0}$, $T_0$ and $T_i$, from which we deduce again the physical trajectories $\beta_i(t)$. They read

$$\beta_i(\tau) = \frac{1}{T_0} \left[ M_{i,0} - \frac{T_i}{2} \log z^2(\tau) \right]$$

(5.28)

and one recovers that the trajectories of the anisotropic modes are qualitatively the very same as the one of the scalar matter field. One readily checks that

$$T_0 M_{12} + T_1 M_{20} + T_2 M_{01} = 0,$$

(5.29)

where the conserved charges $T_i$ are the initial conditions for the velocities of the anisotropic modes $\beta_i$ while $M_{i,0}$ are their (rescaled) initial values. This relation which corresponds to (5.14), similar to a Jacobi identity, thus gives the initial conditions for the shear in terms of the initial conditions for the anisotropies. It reflects the intricate relations existing between the observables. This concludes the algebraic integration of the Bianchi I dynamics from the charges entering in the $\mathfrak{sl}(2,\mathbb{R}) \times \mathfrak{iso}(3)$ algebra, i.e. it shows how we fully integrate the model using the identified conserved charges.

6 Discussion

We have presented new dynamical symmetries of cosmological and black hole mechanics. We have considered three different symmetry-reduced sectors of GR: i) the FLRW cosmology filled with a massless scalar field, ii) the Bianchi I model of cosmology which accounts for the effects of anisotropies and finally iii) the Schwarzschild black hole mechanics. Each models is obtained by a suitable symmetry reduction such that the gravitational field is homogeneous and described only by a finite number of degrees of freedom, leading in the end to a (gauge-invariant) mechanical system. In order to systematically investigate the associated dynamical symmetries, we have employed the geometrization approach which allows one to recast the (gauge fixed) dynamics of these gravitational systems as a geodesic on an auxiliary background, i.e. a “second geometrization” of (homogeneous) gravity [28]. We have considered two such geometrizations, based on the well-known superspace metric on the one hand, and on its extended version known as the Eisenhart-Duval lift on the other end. The key point of such geometrization approach is that, together with the dynamics, the
symmetries of the system are also lifted to the conformal isometries of these field-space backgrounds. This allows a systematic procedure to extend the previous investigations presented in [14–21] and more recently in [27].

Following this strategy, we have revealed a larger set of dynamical symmetries for each model. We have presented the algebra of observables generating this symmetry and we have discussed the interpretation of the Casimirs which label the classical state of the associated gravitational system. Finally, we have shown how the knowledge of these charges allows one to algebraically integrate the gravitational (homogeneous) dynamics. The results obtained in this work can be summarized as follows:

- The geometrization approach employed in this work provide a geometrical origin to the dynamical symmetries first uncovered in [14–17] and later in [19, 21]. Because their generators can be understood as CKVs on the field space (or its extended version given by the ED lift), they are realized as standard diffeomorphism in field space, Lie dragging the field space metric. However, their realization on a space-time solution appear less transparent and can be understood as a type of anisotropic conformal transformations since the different components of the spacetime metric do not share the same conformal weight.

- Solving the conformal Killing equation on superspace and on the ED lift for each models has revealed a larger symmetry group containing the previous results obtained so far [14–21]. We have found that the FRW cosmology and the Schwarzschild mechanics are invariant under the two dimensional centrally extended Schrödinger group, which appears as the dynamical symmetry group of the compressible Navier-Stokes equation [11]. The associated algebra of observable decomposes as $\mathfrak{sh}(2) = (\mathfrak{sl}(2, \mathbb{R}) \times \mathfrak{so}(2)) \rtimes (\mathbb{R}^2 \times \mathbb{R}^2)$. These transformations appear naturally as the chrono-projective vectors fields of the ED lift, i.e. the conformal isometries which preserve the covariantly constant null vector of the lift [80]. The $\mathfrak{sl}(2, \mathbb{R})$ conformal sector is the charge algebra identified earlier in [14, 17] as the (time-dependent) CVH algebra introduced in [81–83], thus providing an extended structure for this previous finding. We stress that the fact that such non-relativistic conformal symmetry is realized in such relativistic gravitational system is made possible because it is realized in field space and not in spacetime per se. We have further shown that, for both models, this 2d Schrödinger algebra is embedded in a larger $\mathfrak{so}(4, 2)$ charge algebra which can be identified again by mean of the Eisenhart-Duval lift. This $\mathfrak{so}(4, 2)$ algebra descends directly from the conformal flatness of the lift for each models. Indeed, the lifted field space being four dimensional, it possesses indeed the same conformal isometries as the 4d Minkowski geometry.

- An interesting outcome of this new symmetry and their associated charge algebra is that they provide a new way to structure the phase space form a group theoretical point of view especially relevant for quantization. In particular, the Casimirs of the charge algebra label the state of the system. In both cases, one of the Casimirs is given by the central charge of
the Heisenberg algebra, i.e. either (3.5) for the FRW cosmology or (4.6) for the Schwarzschild mechanics. This central charge descends from the symmetry reduction process, on the one hand, since it depends on the IR volume used as a regulator, but it also knows about the underlying quantum nature of the system, since it depends also on the UV (Planck) minimal volume. It amounts exactly to the ratio between these two quantities, and thus sets the scale for how classical or quantum is the system. Interestingly, this central charge shows up precisely like the conserved number of particle (or the mass) for quantum many-body systems governed by the same Schrödinger algebra, e.g. some quantum fluids. This is remarkable, as it fits nicely with the interpretation of gravitational minisuperspace as describing the hydrodynamic, collective approximation of a quantum many-body system providing the fundamental structure of spacetime. Moreover, it underlines the key role played by such cut-offs at the quantum level, when representing this symmetry, and from the renormalization viewpoint, when changing the observation scale of the system. We refer the interested reader to [89–91] for related discussions on this last point and to [48, 84–88] for a realization of this idea in some approaches to quantum gravity.

Then, the Schrödinger algebra contains two additional Casimirs, beside the central charge. In the FRW model, they encode the conserved kinetic energy of the scalar field (and its square), while the Casimirs for the Schwarzschild mechanics give the mass of the black hole, as expected.

- It is interesting to point that the conformal isometries of the superspace or the ED lift do not reveal the same symmetry content, thus making these two structures complementary. This is easily understood from that fact that the only common symmetries shared by the ED lift and the superspace are given by the Killing vectors of the superspace. From that point of view, it is instructive the compare the present derivation of the Schrödinger observables with the recent work [27]. Starting from the superspace CKVs, and integrating the weak Dirac observables in a way similar to [24], the authors identified the translational charges \( P_\pm \) which correspond to the Killing vectors of the superspace, and also further reconstruct the \( \mathfrak{sl}(2, \mathbb{R}) \) charges. However, the boosts \( B_\pm \) which correspond to Killing vectors of the ED lift are missed by this approach.

- Another interesting structure revealed by our work is the existence of an infinite dimensional charge algebra for both the FRW cosmological model and the Schwarzschild mechanics. We find that both are equipped with two copies of Witt charges which stand as weak Dirac observables. The existence of these charges is easily understood as descending from the fact that the corresponding superspaces are two dimensional and therefore possess an infinite set of conformal isometries. Because of this, we do not expect these charges to be especially relevant to characterize these mini-superspaces, since adding any additional degrees of freedom will break this infinite dimensional symmetry. Thus, whether this structure can reveal useful at
the quantum level remains to be checked. Nevertheless, it illustrates well the complementarity of the superspace and the ED lift approaches in revealing different symmetry structure of the underlying system.

- Finally, we have shown that the case of the Bianchi I cosmology reveals some key differences which can be traced back to the dimensionality and the curvature properties of its superspace and ED lift. Indeed, one finds that the Bianchi I lift is six dimensional and not conformally flat and thus not maximally symmetric. Physically, these differences are triggered by the presence of anisotropies in the universe. An inspection of the conformal isometries of its lift reveals that the \( sl(2, \mathbb{R}) \) charges associated to the CVH algebra are still present for this model in agreement with [27, 83], but the remaining Schrödinger charges found for the two other models are no longer present. The system possesses instead a charge algebra given by \( sl(2, \mathbb{R}) \times iso(3) \).

Quite remarkably, the Bianchi I superspace reveals a much larger symmetry. Indeed, this geometry is four dimensional and conformally flat, leading to a \( so(4, 2) \) charge algebra. We provide the different conditions which reduces the fifteen charges to the physical number of d.o.f. The emergence of the 4d conformal group as a dynamical symmetry of the Bianchi I model is surprising and reveals a new symmetry criterion to perform a group quantization of this cosmological model.

Finally, we have shown that among the \( so(4, 2) \) charges, one can identify a subset which form again a centrally extended Schrödinger algebra. However, we stress that such a structure is radically different from the one found for the isotropic FRW cosmology. In particular, both the \( so(4, 2) \) algebra and its Schrödinger subalgebra are build solely from strong Dirac observables for the Bianchi I case, while the Schrödinger algebra found for the isotropic FRW model is also build from evolving constants of motion, which generate very different symmetry transformations involving the time coordinate.

The results summarized above provide new powerful symmetry description of these gravitational mini-superspaces, relevant both at classical and quantum levels. Moreover, the knowledge of these new symmetries and their associated charges algebra suggest new directions to be further explored:

- A natural question is to which extent the symmetries found in this work generalize to more complex gravitational systems. It would be desirable to further extend the present investigations to more complex black hole and cosmological models including additional matter or gauge fields, while generalizing the ED lift approach to the whole class of Bianchi cosmologies. A further discussion on the role of the gauge fixing would be also needed to completely characterize these symmetries.

- A second question which calls for further exploration is how the systematical approach employed in this work to identify the dynamical symmetries can be applied to inhomogeneous
gravitational systems. At the moment, the previous works and the present results have focused only on homogenous gravity. One obstacle to generalize the present approach is to find a field theory formulation of the ED lift. Few attempts in that directions have been presented in [92,93] but further investigations are required to obtain well defined formalism. This would be especially relevant to generalize the present results to the case of rotating black hole which are described by a (symmetry reduced) field theory.

- Another way to approach this question is to consider perturbative inhomogeneities and investigate how the symmetry of the background found in our work can constrain their dynamics. Initial results in this direction have been discussed in [55] in the case of static inhomogeneities, with interesting application to the vanishing of Love numbers for 4d GR black holes, but a more systematic approach would be desirable, especially in the context of cosmological perturbations.

- In the case of the FRW cosmology and the Schwarzschild mechanics, the appearance of the dynamical Schrödinger symmetry suggests that one could construct a new type of dictionary between their gravitational mechanics on the one hand, and non-relativistic fluids, on the other hand. Interestingly, in the fluid picture, this symmetry is realized as a space-time symmetry transforming an inhomogeneous fluid configuration [11,49–53], while on the gravity side, it acts on the ED lift. This suggests that one can use this interplay to recast the quantum dynamics of these gravitational models as a fluid propagating on the lift (or on mini-superspace), opening a new type of dictionary for symmetry-reduced classical and quantum gravity.

From that perspective, it is also interesting to consider whether such dictionary can offer a new way to construct analogue gravity models beyond the kinematical level, i.e. able to reproduce in realistic fluid systems also (a sector, at least) of gravitational dynamics. This is an exciting direction for further investigations, that we plan to investigate in the future.

Finally, let us point that this suggestion is further corroborated by recent works on the extraction of an emergent cosmological dynamics from quantum gravity [48,84–88], where indeed a dynamics from cosmological observables is extracted from hydrodynamic equations on minisuperspace (that can also be seen as a non-linear extension of quantum cosmology) obtained as a coarse-grained approximation of the fundamental quantum gravity dynamics.

- Finally, using the new group theoretical formulation of these gravitational mini-superspace to address their quantization and narrow the standard quantization ambiguities provide yet another interesting direction to develop. In the case of the Schrödinger symmetry, the representation theory is not straightforward to use but useful results can be found in [94]. See also [95] for the representation theory of the infinite-dimensional extension of the Schrödinger symmetry.
A Conformal isometries of the FLRW lift

In this appendix, we explicitly show that the conformal isometries of the Eisenhart-Duval lift associated to the flat FLRW model form indeed an $so(4,2)$ Lie algebra. Consider the flat metric and the coordinates

$$
\eta_{\alpha\beta} = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}
$$

$$
x^\alpha = \begin{pmatrix}
-u + w \\
u + w \\
ze^\phi \\
ze^{-\phi}
\end{pmatrix}
$$

(A.1)

A given vector field can be decomposed as

$$
\partial_\alpha = \frac{\partial}{\partial x^\alpha} = \left(-\partial_u + \partial_w, \partial_u + \partial_w, e^{-\phi} \left[\partial_z + \frac{\partial \phi}{z}\right], e^{+\phi} \left[\partial_z - \frac{\partial \phi}{z}\right]\right)
$$

(A.2)

The generators of the $so(4,2)$ Lie algebra are given by

4 translations

$$
T_\alpha = \partial_\alpha
$$

(A.3)

6 rotations

$$
M_{\alpha\beta} = x_\alpha \partial_\beta - x_\beta \partial_\alpha
$$

(A.4)

1 dilation

$$
D = x^\alpha \partial_\alpha
$$

(A.5)

4 inversions

$$
K_\alpha = 2x_\alpha x^\beta \partial_\beta - x^\beta x_\beta \partial_\alpha
$$

(A.6)

which satisfy the standard commutation relations

$$
[M_{\alpha\beta} M_{\gamma\delta}] = \eta_{\alpha\delta} M_{\beta\gamma} + \eta_{\beta\gamma} M_{\alpha\delta} + \eta_{\alpha\gamma} M_{\delta\beta} + \eta_{\beta\delta} M_{\gamma\alpha}
$$

$$
[T_\alpha, T_\beta] = 0; \quad [T_\alpha, M_{\beta\gamma}] = \eta_{\beta\gamma} T_\alpha - \eta_{\alpha\gamma} T_\beta
$$

$$
[K_\alpha, K_\beta] = 0; \quad [K_\alpha, M_{\beta\gamma}] = \eta_{\beta\gamma} K_\alpha - \eta_{\alpha\gamma} K_\beta
$$

$$
[D, K_\alpha] = K_\alpha; \quad [D, M_{\alpha\beta}] = 0; \quad [T_\alpha, D] = T_\alpha
$$

$$
[T_\alpha, K_\beta] = 2(\eta_{\alpha\beta} D - M_{\alpha\beta})
$$

(A.7)

In term of the coordinates of the lift, one can write the rotational generators $M_{\alpha\beta}$ as follows

$$
M_{01} = 2(u \partial_u - w \partial_w)
$$

(A.8)

$$
M_{0\pm} = e^{\mp \phi} \left[(u - w) \left(\partial_z \pm \frac{\partial \phi}{z}\right) - z (-\partial_u + \partial_w)\right]
$$

(A.9)

$$
M_{1\pm} = e^{\mp \phi} \left[(u + w) \left(\partial_z \pm \frac{\partial \phi}{z}\right) - z (\partial_u + \partial_w)\right]
$$

(A.10)

$$
M_{++} = -2 \partial_\phi
$$

(A.11)
while the generators $K_\alpha$ are given by

\begin{align*}
K_0 &= 4 \left( u^2 + \frac{z^2}{2} \right) \partial_u - 4 \left( w^2 + \frac{z^2}{2} \right) \partial_w + 4z (u - w) \partial_z \\
K_1 &= 4 \left( u^2 - \frac{z^2}{2} \right) \partial_u + 4 \left( w^2 - \frac{z^2}{2} \right) \partial_w + 4z (u + w) \partial_z \\
K_\pm &= e^{\mp \phi} \left[ 4z (u \partial_u + w \partial_w) + (2z^2 - 4uw) \partial_z \mp \frac{(2z^2 + 4uw)}{z} \partial \phi \right]
\end{align*}

(A.12)  

(A.13)  

(A.14)

Finally, the dilatation takes the form

\[ D = 2u \partial_u + 2w \partial_w + 2z \partial_z \]  

(A.15)

At this point, we can reorganize the so(4,2) generators to extract the Schrödinger ones. These vector fields are given by

\[ Q_+ = \frac{1}{8} (K_1 + K_0) \quad Q_0 = -\frac{1}{4} (D + M_{01}) \quad Q_- = \frac{1}{2} (T_1 - T_0) \]

together with

\[ P_\pm = P_\pm \quad B_\pm = -\frac{1}{2} (M_{1\pm} + M_{0\pm}) \quad J = -\frac{1}{2} M_{+-} \]

Finally, the dual generators coming from the switch $u \leftrightarrow w$ in the lifted metric read

\[ \tilde{Q}_+ = \frac{1}{8} (K_1 - K_0) \quad \tilde{Q}_- = \frac{1}{2} (T_1 + T_0) \quad \tilde{B}_\pm = -\frac{1}{2} (M_{1\pm} - M_{0\pm}) \]

The so(4,2) algebra is completed by the three remaining generators

\[ \mathcal{O} = -\frac{1}{2} M_{01} \quad Y_\pm = \frac{1}{2} K_\pm \]

This concludes the presentation of the so(4,2) algebra of conformal isometries of the Eisenhart-Duval lift associated to the flat FLRW cosmology filled with a massless scalar field.

### B Witt algebras of CKVs from 2d superspace

Consider a mechanical system with two degrees of freedom. One can always find a suitable field redefinition such that the action reads

\[ S[\chi^a, \tau] = \int \! d\tau \ g_{ab}(\chi) \dot{\chi}^a \chi^b = \int \! d\tau \ \dot{\chi}_+ \dot{\chi}_- \]  

(B.1)

In terms of these dynamical fields, the superspace metric becomes

\[ ds^2 = 2d\chi_+ d\chi_- \]  

(B.2)
The conformal killing equation leads to the three conditions

\[ \mathcal{L}_\xi g_{ab} = \varphi g_{ab}, \quad \Rightarrow \quad \partial_+ \xi^- = 0, \]
\[ \partial_- \xi^+ = 0, \]
\[ \partial_- \xi^- + \partial_+ \xi^+ = \varphi, \]  

(B.3)

Therefore, one obtains two families of CKVs given by

\[ X^a F \partial_a = F(\chi_+) \partial_+, \quad Z^a G \partial_a = G(\chi_-) \partial_- \]  

(B.4)

which form two copies of a Witt algebra:

\[ [X_F, X_G] = 2X_{[F,G]}, \quad [Z_F, Z_G] = 2Z_{[F,G]}, \quad [X_F, Z_G] = 0, \]  

(B.5)

where \([F, G] = [F, G] = FG' - GF'.\) Introducing the conjugated momenta \(p_\pm\) of the dynamical fields \(\chi_\pm\), one obtains an infinite tower of conserved charges given by

\[ X_F = F(\chi_+) p_+, \quad Z_G = G(\chi_-) p_- \]  

(B.6)

The hamiltonian of the system being given by \(h = p_+ p_-\) and if it descends from a relativistic action, i.e. if it vanishes on-shell \(h \equiv 0\), one can check that these sets of Witt charges are indeed weak Dirac observables of the system

\[ \{X_F, h\} = F' h \equiv 0, \quad \{Z_G, h\} = G' h \equiv 0 \]  

(B.7)

This is typically the case of the flat FLRW cosmological model filled with a massless scalar field.

C Geometrizing the Schwarzschild symmetry generators

In this appendix, we provide the explicit expressions of the CKVs of the Eisenhart-Duval lift associated to the black hole as well as the CKVs of the Schwarzschild superspace.

C.1 CKVs of the lift

The gauge-fixed action for the Schwarzschild mechanics is given by (4.10). The potential term being a mere constant, one can omit it and consider the modified mechanical action

\[ S_\epsilon[V_1, V_2; \tau] = \epsilon c\ell_P \int d\tau \left[ \frac{V_2 \dot{V}_1^2 - 2V_1 \dot{V}_1 \dot{V}_2}{2V_1^2} \right] \]  

(C.1)

Then, the ED lift is a four dimensional manifold with line element

\[ ds^2 = g_{AB} dX^A dX^B = 2dudw + \epsilon c\ell_P \left( \frac{V_2}{V_1^2} dV_1^2 - \frac{2dV_1 dV_2}{V_1} \right) \]  

(C.2)
Just as for the FLRW cosmological system, this manifold turns out to be conformally flat, i.e. its Weyl and Cotton tensors vanish, and it is thus maximally symmetric, i.e. it possesses fifteen CKVs. They can be organized w.r.t. the commutation properties with the constant null vector field $N^A \partial_A = \partial_w$.

The CKVs which commute with $\partial_w$ are given by

$$Q_0^A \partial_A = V_2 \partial_{V_2} + u \partial_u,$$

$$Q_-^A \partial_A = 2V_2 \partial_w + \frac{2u}{\epsilon c} (V_1 \partial_{V_1} + V_2 \partial_{V_2}) + \frac{1}{\epsilon c} u^2 \partial_u,$$  \hspace{1cm} (C.4)

and act on the metric of the lift as

$$\mathcal{L}_{Q_0^A} g_{\mu \nu} = g_{\mu \nu}, \quad \mathcal{L}_{Q_-^A} g_{\mu \nu} = \frac{4u}{c} g_{\mu \nu}.$$  \hspace{1cm} (C.6)

We also identify six Killing vectors which are given by

$$P_+^A \partial_A = 2\sqrt{V_1} \partial_{V_1} + \frac{V_2}{\sqrt{V_1}} \partial_{V_2}, \quad B_+^A \partial_A = \frac{2V_2}{\sqrt{V_1}} \partial_w + \frac{u}{\epsilon c} \left(2\sqrt{V_1} \partial_{V_1} + \frac{V_2}{\sqrt{V_1}} \partial_{V_2}\right),$$

$$P_-^A \partial_A = \sqrt{V_1} \partial_{V_2}, \quad B_-^A \partial_A = 2\sqrt{V_1} \partial_w + \frac{u}{\epsilon c} \sqrt{V_1} \partial_{V_2},$$  \hspace{1cm} (C.7)

Together with the null vector $N^A \partial_A = \partial_w$, they form the Schrödinger algebra from which one can derive the Schrödinger observables presented in Section 4.2.1.

Three other CKVs are easily identified as

$$\tilde{Q}_-^A \partial_A = \partial_u, \quad J^0 \partial_\alpha = V_1 \partial_{V_1},$$  \hspace{1cm} (C.11)

The dual of the CKV $Q_0^A \partial_A$ is recovered thanks to the generator of dilatation in the $(u, v)$-plane.

Notice that if we had kept the constant term in the action, the ED lift would have been given by

$$ds^2 = \frac{2\epsilon}{\ell^2} du^2 + 2ду dw + \epsilon \epsilon F \left(\frac{V_2 dV_2}{V_1} - \frac{2dV_1 dV_2}{V_1}\right).$$  \hspace{1cm} (C.3)

One can see that the potential term being constant, $\partial_u$ is still a Killing vector. Moreover, one can show that this additional term does not spoil the conformally flatness of the lift such that this alternative version admits again fifteen CKVs. While the form of these CKVs is slightly different starting from this lift, the conserved charges one builds from the CKVs are the same.
Finally, the last two CKVs are
\[ Y^A_{\pm} \partial_A = \frac{2V_2}{\sqrt{V_1}} (u \partial_u + w \partial_w) + 2uw \sqrt{V_1} \partial V_1 + (uw + 2V_2) \frac{V_2}{V_1} \partial V_2 \]  
(C.12)
\[ Y^A_{\pm} \partial_A = 2 \sqrt{V_1} (u \partial_u + w \partial_w) + 4V_1 \sqrt{V_1} \partial V_1 + (uw + 2V_2) \sqrt{V_1} \partial V_2 \]  
(C.13)

Altogether, these fifteen vectors fields from the so(4,2) Lie algebra associated to the Schwarzschild black hole mechanics.

### C.2 CKVs of the superspace

Let us now compare with the conformal isometries of the superspace. The super-metric of the Schwarzschild mechanics is given by
\[ ds^2 = g_{ab} dx^a dx^b = \frac{V_2}{V_1^2} dV_1^2 - \frac{2dV_1 dV_2}{V_1} \]  
(C.14)

Notice that the Schwarzschild dynamics does not correspond to a null geodesic on this geometry because the action contains a constant potential term. Therefore, depending on the sign of this contribution, the dynamics is either mapped to time-like or space-like geodesics of the super-metric (C.14). Thus, as explained in the first section, the CKVs of this geometry do not provide directly conserved charges for the system, and one needs to add corrective terms proportional to the mass of the system, i.e. proportional to the constant potential in the present case. With this subtlety in mind, we now present the KVs and the CKVs of this super-metric.

First, one can check that this superspace has constant curvature such that it admits three Killing vectors. They are given by
\[ P^a \partial_a = 2\sqrt{V_1} \partial V_1 + \frac{V_2}{\sqrt{V_1}} \partial V_2 , \quad P^a \partial_a = \sqrt{V_1} \partial V_2 , \quad J^a \partial_a = V_1 \partial V_1 \]  
(C.15)

which correspond to the KVs of the ED lift. Being Killing vectors, they are directly associated to strong Dirac observables of the system and no corrective terms need to be added.

Now, let us derive the general expression of the CKVs and the associated charges. Just as the flat FLRW cosmological model discussed in Section 3.4.2, the super-metric is two dimensional and therefore possesses an infinite tower of CKVs. To derive the associated charges, let us introduce the new coordinates
\[ x_+ = \sqrt{V_1} , \quad x_- = - \frac{V_2}{\sqrt{V_1}} \]  
(C.16)

such that the (rescaled) super-metric reads
\[ ds^2 = dx_+ dx_- \]  
(C.17)

The general expressions for the CKVs are given by
\[ X_F^a \partial_a = F(x_+) \partial_+ , \quad X_G^a \partial_a = G(x_-) \partial_- \]  
(C.18)
where $F$ and $G$ are arbitrary functions. In order to compute the associated charges, we first write the momenta $p_{\pm}$ which are given by

$$
p_+ = 2\sqrt{V_1}p_1 + \frac{V_2p_2}{\sqrt{V_1}}, \quad p_- = -\sqrt{V_1}p_2 \tag{C.19}
$$

They are nothing else but the translational charges entering in the Schrödinger algebra. Then, one obtains the following set of observables

$$X^+_F = p_+F(\sqrt{V_1}), \quad X^-_G = p_-G\left(\frac{V_2}{\sqrt{V_1}}\right) \tag{C.20}
$$

As expected, these phase space functions are not conserved on shell. A straightforward computation shows that

$$\{X^+_F, h\} = F'\tilde{h} \doteq -\frac{c\ell_P F'}{\ell_s^2}, \quad \{X^-_G, h\} = 2G'\tilde{h} \doteq -\frac{2c\ell_P G'}{\ell_s^2} \tag{C.21}
$$

where $\tilde{h}$ is the shifted hamiltonian whose value can be interpreted as the effective mass of the test particle moving on the superspace. To obtain well defined weak Dirac observables, we slightly correct the charges as follows. Consider

$$W^+_F = X_F - \frac{\alpha}{2\sqrt{V_1}p_2}F(\sqrt{V_1}), \quad \text{with} \quad \alpha = 2\epsilon c\ell_P^2 \frac{P}{\ell_s^2} \tag{C.22}
$$

$$W^-_G = X_G \left[2 - \frac{\beta}{\tilde{h}}\right] G\left(\frac{V_2}{\sqrt{V_1}}\right), \quad \text{with} \quad \beta = -\frac{c\ell_P}{\ell_s^2} \tag{C.23}
$$

With these corrections, the modified charges now satisfy

$$\{W^+_F, h\} = F'\left(\tilde{h} + \frac{c\ell_P}{\ell_s^2}\right) \doteq 0, \quad \{W^-_G, h\} = 2G'\left(\tilde{h} + \frac{c\ell_P}{\ell_s^2}\right) \doteq 0 \tag{C.24}
$$

and one recovers the charges presented in Section 4.3. Let us further point that the corrective terms spoil the commutation between the two set of charges as shown by (4.47). This concludes the discussion on the CKVs of the black hole superspace.

### D Geometrizing the Bianchi I symmetry generators

In this last appendix, we present the explicit expressions of the CKVs of the Bianchi I superspace and the ones derived from its ED lift. To simplify the notation, we use the new rescaled variable $d\varphi = \ell_P^2 d\phi$.

#### D.1 CKVs of the superspace

The superspace metric is given by

$$ds^2 = \frac{c\ell_P}{2} \left[ (d\varphi^2 + d\beta_1^2 + d\beta_2^2) z^2 - dz^2 \right] \tag{D.1}$$
This four dimensional geometry has vanishing Weyl and Cotton tensors and is thus conformally flat. It admits therefore the maximal number of CKVs, fifteen, among which ten are KVs. The first obvious KVs are given by

\[ T^\alpha_1 \partial_\alpha = \partial_1, \quad T^\alpha_2 \partial_\alpha = \partial_2, \quad T^\phi \partial_\alpha = \partial_\phi \quad \text{(D.2)} \]

We have further

\[ T^\alpha_0 \partial_\alpha = z \partial_z, \quad D^\alpha \partial_\alpha = \log z \partial_z + \beta_1 \partial_1 + \beta_2 \partial_2 + \phi \partial_\phi \quad \text{(D.3)} \]

and

\[
\begin{align*}
M^\alpha_1 \partial_\alpha &= \beta_1 \partial_2 - \beta_2 \partial_1, & M^\alpha_0 \partial_\alpha &= \beta_1 z \partial_z + \log z \partial_1, \\
M^\phi \partial_\alpha &= \beta_1 \partial_\phi - \phi \partial_1, & M^\alpha_2 \partial_\alpha &= \beta_2 z \partial_z + \log z \partial_2, \\
M^\alpha_0 \partial_\alpha &= \beta_2 z \partial_z + \log z \partial_\phi & M^\alpha_2 \partial_\alpha &= \phi \partial_z + \log z \partial_\phi
\end{align*}
\]

The remaining CKVs are given by

\[
\begin{align*}
K^\alpha_0 \partial_\alpha &= -2 \log z \left[ \left( \log z + \frac{F}{2 \log z} \right) z \partial_z + \beta_1 \partial_1 + \beta_2 \partial_2 + \phi \partial_\phi \right], \\
K^\alpha_1 \partial_\alpha &= 2 \beta_1 \left[ \log z \partial_z + \left( \frac{\beta_1}{\log z} \right) \partial_1 + \beta_2 \partial_2 + \phi \partial_\phi \right], \\
K^\alpha_2 \partial_\alpha &= 2 \beta_2 \left[ \log z \partial_z + \beta_1 \partial_1 + \left( \frac{\beta_2}{\log z} \right) \partial_2 + \phi \partial_\phi \right], \\
K^\phi \partial_\alpha &= 2 \phi \left[ \log z \partial_z + \beta_1 \partial_1 + \beta_2 \partial_2 + \left( \frac{\phi}{\log z} \right) \partial_\phi \right]
\end{align*}
\]

where

\[ F(z, \beta_1, \beta_2, \phi) := \beta_1^2 + \beta_2^2 + \phi^2 - \log^2 z \quad \text{(D.6)} \]

These fifteen vectors represent the maximal set of conformal isometries of the superspace. Being conformally flat, these vector fields form the standard \( so(4,2) \) Lie algebra which stands as the conformal structure of 4d Minkowski spacetime. Indeed, taken the usual Minkowski metric \( \eta_{\alpha \beta} \) mostly positive and \( x^\alpha \) to be

\[
\eta_{\alpha \beta} = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad x^\alpha = \begin{pmatrix}
\log z \\
\beta_1 \\
\beta_2 \\
\phi
\end{pmatrix} \quad \text{(D.7)}
\]

so that

\[
x_\alpha = \begin{pmatrix}
-\log z, \beta_1, \beta_2, \phi
\end{pmatrix}, \quad \partial_\alpha \frac{\partial}{\partial x^\alpha} = \left( \begin{array}{c}
z \partial_z, \quad \partial_1, \quad \partial_2, \quad \partial_\phi
\end{array} \right) \quad \text{(D.8)}
\]
it is possible to write the fifteen CKVs in the more compact and common way

4 translations \[ T_\alpha = \partial_\alpha \] (D.9)

6 rotations \[ M_{\alpha \beta} = x_\alpha \partial_\beta - x_\beta \partial_\alpha \] (D.10)

1 dilation \[ D = x^\alpha \partial_\alpha \] (D.11)

4 inversions \[ K_\alpha = 2x_\alpha x^\beta \partial_\beta - x^\beta x_\beta \partial_\alpha \] (D.12)

These indeed satisfy the \( \mathfrak{so}(4,2) \) algebra

\[
\begin{align*}
[M_{\alpha \beta}, M_{\gamma \delta}] &= \eta_{\alpha \delta} M_{\beta \gamma} + \eta_{\beta \gamma} M_{\alpha \delta} + \eta_{\alpha \gamma} M_{\delta \beta} + \eta_{\beta \delta} M_{\gamma \alpha} ; \\
[T_\alpha, T_\beta] &= 0 ; \quad [T_\alpha, M_{\beta \gamma}] = \eta_{\beta \alpha} T_\gamma - \eta_{\gamma \alpha} T_\beta ; \\
[K_\alpha, K_\beta] &= 0 ; \quad [K_\alpha, M_{\beta \gamma}] = \eta_{\beta \alpha} K_\gamma - \eta_{\gamma \alpha} K_\beta ; \\
[D, K_\alpha] &= K_\alpha ; \quad [D, M_{\alpha \beta}] = 0 ; \quad [T_\alpha, D] = T_\alpha ; \\
[T_\alpha, K_\beta] &= 2 (\eta_{\alpha \beta} D - M_{\alpha \beta})
\end{align*}
\] (D.13)

These CKVs automatically provide a \( \mathfrak{so}(4,2) \) algebra of Dirac observables for the Bianchi I model.

Let us now compare with the conformal isometries of the ED lift.

### D.2 CKVs of the lift

The Eisenhart-Duval lift of the Bianchi I system is given by

\[
ds^2 = 2dudw + c\ell P \left[(d\varphi^2 + d\beta_1^2 + d\beta_2^2) z^2 - dz^2\right]
\] (D.14)

which is six dimensional. Contrary to the superspace, the ED lift is not conformally flat because its Cotton tensor is not vanishing. This crucial difference implies that the lift is not maximally symmetric. We find only twelve CKVs for this geometry. The first three are given by

\[
Q_- = \partial_u \] (D.15)

\[
Q_0 = \frac{z}{2} \partial_z + u \partial_u \] (D.16)

\[
Q_+ = \frac{c\ell P z^2}{2} \partial_w + uz \partial_z + u^2 \partial_u
\] (D.17)

and are identical to the ones found for the FLRW model. We have also the generators dual under the \( u \leftrightarrow w \) change for \( Q_+ \) and \( Q_- ; \) \( Q_0 \) changes thanks to the generator

\[
O = w \partial_w - u \partial_u
\] (D.18)

while the six other CKVs coincide with the generators \( T_1, T_2, T_\varphi, M_{1,2}, M_{1,\varphi} \) and \( M_{2,\varphi} \) given in (D.2) and (D.4) and that fully commute with the \( Q \)’s since they are independent from \( u, w, z \). The fact that the CKVs of the superspace are not all CKVs of the ED lift can be understood by the fact that while the superspace is conformally flat, the lift is not.
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