An unconstrained binary quadratic programming for the maximum independent set problem

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Abstract. For a given graph \( G = (V, E) \) the maximum independent set problem is to find the largest subset of pairwise nonadjacent vertices. We propose a new model which is a reformulation of the maximum independent set problem as an unconstrained quadratic binary programming, and we resolve it afterward by means of a genetic algorithm. The efficiency of the approach is confirmed by results of numerical experiments on DIMACS benchmarks.

Keywords: maximum independent set, unconstrained binary quadratic programming, genetic algorithm.

1 Introduction

The maximum independent set problem (MIS) is one of the central combinatorial optimization problems and shown to be NP-hard [1]. This problem is relevant for many practical applications in computer science and operation research, and engineering [2], such as register allocation in a compiler, assigning channels to radio station, scheduling exam, graph coloring, and the reader collision problem [3]. Some studies were made on the basis of the greedy algorithms and tabu search [4, 5], or using the intersection graph of an axis-parallel rectangles [6]. In [7] the authors proposed a method based on an improvement of the maximum independent set algorithm given by F. Glover in [8]. Other studies based on a method that utilizes the polynomially solvable critical independent set problem [9]. The unconstrained binary quadratic programming problem is to maximize (or minimize) the function:

\[
f(x) = x^T Q x,
\]

where \( Q = (q_{ij}) \) is an \( n \times n \) matrix of constants and \( x \) is an \( n \)-vector of binary variables. This formulation has an ability to model a wide range of different problems on many areas as traffic management [10], machine scheduling [11]. UBQP gives evidence to its relevance and effectiveness in the face of known problems by their complexity such
as the set-partitioning problem [12], the set packing problem [13], the vertex coloring problem [14], and the linear ordering problem [15]. Given its NP-hard nature [1], various approaches have been proposed for solving this model using exact methods [16, 17] and metaheuristic methods as memetic algorithms [18, 19], scatter search [20], simulated annealing [21], adaptive memory approaches based on tabu search [22–24], and recently combination of GRASP and tabu search [25].

This paper presents an efficient method for solving the maximum independent set problem (MIS) via its modeling as an unconstrained binary quadratic programming (UBQP).

This paper is organized as follows: we define the maximum independent set problem in Section 2. Section 3 presents the transformation of linear programming to unconstrained binary quadratic programming associated to the maximum independent set problem. In Section 4 the ingredients of our algorithm are described, including an adapted genetic algorithm, and Section 5 draws some conclusions.

2 Problem definition

The maximum independent set problem (MIS) may be written in a form of a linear problem with binary variables as follows:

\[
\begin{align*}
\text{(LP}_{\text{MIS}}) \quad & \max x_0 = \sum_{i=1}^{n} x_i, \\
& x_i + x_j \leq 1, \quad \{i, j\} \in E, \\
& x \text{ binary,}
\end{align*}
\]

(1)

where \(x_0^2 = x_i\) and \(x_i \in (0, 1)\).

The problem \((\text{LP}_{\text{MIS}})\) is constituted of a linear objective function with one type of constraints \(x_i + x_j \leq 1\) and the number of these inequality constraints is equal to \(\text{card}(E)\). Our purpose is to reformulate the problem \((\text{LP}_{\text{MIS}})\) in a binary quadratic problem without constraints in the form:

\[
\begin{align*}
\text{(UBQP}_{\text{MIS}}) \quad & \max x_0 = x^T Q x, \\
& x \text{ binary,}
\end{align*}
\]

where \(Q\) is a square symmetric matrix of dimension \(\text{card}(V)\). For this we apply a transformation on the constraints set.

3 Transformation

We introduce the constraints in the objective function in the following way. The objective is transformed under shape \(x^T D x\) by (1), then the constraints \(x_i + x_j \leq 1\) are introduced by the product \(x_i x_j\). The problem \((\text{LP}_{\text{MIS}})\) is thus replaced by the quadratic problem without constraint:

\[
\begin{align*}
\text{(UBQP}_{\text{MIS}}) \quad & \max x_0 = \sum_{i=1}^{n} x_i - \sum_{\{i, j\} \in E} x_i x_j, \\
& x \text{ binary.}
\end{align*}
\]
Example. We consider the undirected graph in Fig. 1, on this graph we obtain three independent sets: $C_1 = (2, 4, 5, 7, 9), C_2 = (1, 6, 8)$ and $C_3 = (3).

![Fig. 1. Independent sets of a graph.](image)

We reformulate it into an unconstrained binary quadratic programming problem. The example satisfies the following linear programming:

$$\begin{align*}
\text{(LP}_{\text{MIS}}) \quad & \max x_0 = \sum_{i=1}^{9} x_i, \\
& x_3 + x_6 \leq 1, \\
& x_1 + x_2 \leq 1, \\
& x_1 + x_9 \leq 1, \\
& x_2 + x_3 \leq 1, \\
& x_3 + x_4 \leq 1, \\
& x_3 + x_5 \leq 1, \\
& x_3 + x_7 \leq 1, \\
& x_3 + x_8 \leq 1, \\
& x_5 + x_6 \leq 1, \\
& x_8 + x_9 \leq 1, \\
& x \text{ binary.}
\end{align*}$$

We apply the transformation and obtain:

$$Q = \begin{pmatrix}
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
-1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 1 & -1 & -1 & -1 & -1 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & -1 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1
\end{pmatrix}.$$  

The independent set $C_1 = (2, 4, 5, 7, 9)$, i.e., $x = (010110101)$ gives an optimal solution for $UQP_{\text{MIS}}$ with $x_0 = 5$, if we add or remove one or more vertex to $C_1$ the value of $UQP_{\text{MIS}}$ will be inferior, for example if we add to $C_1$ the vertex 3, i.e., $x = (011110101)$ we find $x_0 = -2$.

4 Solving UBQP_{MIS}

After having transformed our MIS problem in an unconstrained binary quadratic form, we try to solve it using an adapted genetic algorithm well by choosing aptly the operators which will be ideal to the MIS problem.
4.1 Genetic algorithm

Our resolution approach of UBQP\textsubscript{MIS} problem is based on genetic algorithm (GA). The GA is a research process based on the laws of natural selection and genetics. Generally, a GA consists of three simple operations. Their functioning is extremely simple, we start with an initial population, we evaluate the performance of each individual, and we create a new population of potential solutions using evolutionary operators: selection, crossover and mutation. GA cycle is repeated until a desired stopping criterion is reached.

The individuals of initial population are generated randomly with equal probability such as the genes are assigned a value of 0 or 1. To build a diversified initial population, each individual added to the population must be different to all the existing solutions of the population. Using the best known lower bound based on degrees of vertices \(d_i\) given by Caro and Tuza [26]:

\[
\sum_{i \in V} \frac{1}{d_i + 1} \leq \text{card}(\text{MIS}(G))
\]

During the initialization of the population we fix the size of the independent set to \(p = \left\lceil \sum_{i \in V} 1/(d_i + 1) \right\rceil\) that is, the number of 1 in each individual of the population.

We consider the objective function \(x^TQx\) as evaluation function (fitness) of each individual in the population.

For the selection, first, we randomly choose two individuals, then we apply the tournament selection operator in order to keep the best individual. The comparison between two individuals is carried out according to their fitness.

Crossover is a recombination operator that combines parts of two individual parents to produce offspring that contain some genetic information from both parents. A probability parameter \(p_c\), is set to determine the crossover rate. We opt for a special crossing, we choose two random integers \(r, s\) in each parents \(P_1\) and \(P_2\) such as \(r, s \in [1, n]\) who represent points inter-genes. We opt for a crossing which aims to permute two blocks of genes of each parents pair. The two selected blocks have the same number of genes which are worth one and the same size. By this crossing we always keep the number of 1 on each new individual obtained.

\[
C_1(i) = \begin{cases} P_1(i) & \text{if } i \notin [r_1, s_1], \\ P_2(j) & \text{if } i \in [r_1, s_1] \text{ and } j \in [r_2, s_2], \end{cases}
\]

\[
C_2(i) = \begin{cases} P_2(i) & \text{if } i \notin [r_2, s_2], \\ P_1(j) & \text{if } i \in [r_2, s_2] \text{ and } j \in [r_1, s_1]. \end{cases}
\]

Table 1 presents a comparison between the results obtained using a 1-point crossover, a 2-point crossover and the proposed crossover for 25 tests. The tests were performed for a crossover rate equal to \(p_c = 0.8\) and a mutation rate equal to \(p_m = 0.2\). The average values \((X_{\text{avg}})\) solutions obtained with the proposed crossover are clearly superior compared to the two types of crossover.

The mutation is used with the aim to further explore the search space and reaching solutions that the crossing cannot touch them. We choose randomly two distinct genes that we permute them after, usually with small probability \(p_m\).
Table 1. Comparison of solution quality for different type of crossover operations.

| Graph   | One point crossover | Two point crossover | Proposed crossover |
|---------|---------------------|---------------------|--------------------|
| anna    | 77.84               | 76.53               | 79.04              |
| david   | 33.71               | 33.60               | 34.18              |
| huck    | 86.73               | 32.19               | 94.07              |
| fpsol2.i.1 | 301.28          | 303.59               | 306.47             |
| initlx3.i.1 | 563.06         | 561.25               | 564.80             |
| zero1n.i.3 | 118.72            | 120.55               | 122.36             |
| mycel4   | 764.27             | 47                  | 47                 |
| mycel5   | 20.61              | 18.27               | 23                 |
| mycel6   | 11                 | 11                  | 11                 |
| mycel7   | 93.65              | 93.12               | 94.07              |
| mug100-25 | 32.19             | 31.45               | 31.86              |
| mulsol.i.5 | 86.04             | 84.27               | 86.73              |

Remark. We fix \( p = \lfloor \sum_{i \in V} 1/(d_i + 1) \rfloor \), then we apply the transformation to obtain UBQPMIS problem, after we run the genetic algorithm, and each time we find a maximum independent set we increment the number \( p \) by one in order to find an independent set larger.

Algorithm 1. UBQPMIS algorithm.

1. Give the problem UBQPMIS.
2. Put \( p = \lfloor \sum_{i \in V} 1/(d_i + 1) \rfloor \).
3. While \( T \leq T_{\text{max}} \).
4. Evaluate each individual by calculating its fitness \( x_0 = x^T Q x \).
5. Make a selection by tournament.
6. Apply the crossing.
7. Apply the mutation with a very low probability.
8. Evaluate the individuals.
9. If we find a feasible solution \( x_0 \) then \( p \leftarrow p + 1 \).
10. End while.

Fig. 2. Used crossover operator.
4.2 Obtained results

In this section, we present the experimental results after running our algorithm on instances available in the literature (see the web page http://mat.gsia.cmu.edu/COLOR03/ for a complete description of the instances) and compare them with those given by [7,9]. The platform of our experiments is a personal computer windows 7, AMD Athlon(tm) X2 dual-core QL-65 (2cpus) 2 GHz with 4 GB RAM. The results obtained by our algorithm are reported in the Table 2. For each instance, we indicate the number of vertices $n$, the number of edges $d_0$, the values $X_S$, $X_{DBG}$ and $X_{UBQP}$ respectively denote the cardinal of the maximal independent set given by S. Butenko et al. [9], DBG algorithm [7], and UBQP algorithm, the symbol ‘–’ means that the information is not available. The parameters of the GA for the UBQP algorithm are: population size $= 100$, crossover rate $p_c = 0.8$, mutation rate $p_m = 0.2$, and the maximum iterations number $T_{max} = 200$.

Table 2: Computational results.

| Graph                  | $n$ | $d_0$ | $X_S$ | $X_{DBG}$ | $X_{UBQP}$ | Time $UBQP$ (s) |
|------------------------|-----|-------|-------|-----------|-------------|-----------------|
| Anna                   | 138 | 493   | 80    | 80        | 80          | 0.87            |
| david                  | 87  | 406   | 36    | 36        | 36          | 0.49            |
| fpsol2.i.1             | 496 | 11654 | 307   | 307       | 307         | 12.39           |
| fpsol2.i.2             | 451 | 8691  | 261   | 261       | 261         | 10.77           |
| fpsol2.i.3             | 425 | 8688  | 238   | 238       | 238         | 9.18            |
| huck                   | 74  | 301   | 27    | 27        |             | 0.51            |
| inithx.i.1             | 864 | 18707 | 566   | 566       | 566         | 15.26           |
| inithx.i.2             | 645 | 13979 | 365   | 365       | 365         | 10.94           |
| inithx.i.3             | 621 | 13969 | 360   | 360       | 360         | 7.89            |
| jean                   | 80  | 254   | 38    | 38        | 38          | 0.60            |
| zeroin.i.1             | 211 | 4100  | 120   | 120       | 120         | 4.22            |
| zeroin.i.2             | 211 | 3541  | 127   | 127       | 127         | 4.05            |
| zeroin.i.3             | 206 | 3540  | 123   | 123       | 123         | 5.13            |
| 1-FullIns5             | 282 | 3247  | –     | 138       | 138         | 6.67            |
| 3-FullIns4             | 405 | 3524  | –     | 193       | 193         | 8.42            |
| 2-Inser4               | 149 | 541   | –     | 74        | 74          | 1.05            |
| 3-Inser3               | 56  | 110   | –     | 27        | 27          | 0.38            |
| 4-Inser3               | 79  | 156   | –     | 30        | 30          | 0.96            |
| games120               | 120 | 638   | –     | 22        | 22          | 1.20            |
| mulsol.i.1             | 197 | 3925  | –     | 100       | 100         | 5.26            |
| mulsol.i.5             | 186 | 3973  | –     | 88        | 88          | 4.81            |
| mug88-1                | 88  | 146   | –     | –         | 29          | 1.17            |
| mug88-25               | 88  | 146   | –     | –         | 29          | 0.95            |
| mug100-1               | 100 | 166   | –     | –         | 33          | 1.78            |
| mug100-25              | 100 | 166   | –     | –         | 33          | 2.34            |
| mycie13                | 11  | 20    | –     | –         | 5           | 0.21            |
| mycie4                | 23  | 71    | –     | –         | 11          | 0.33            |
| mycie5                | 47  | 236   | –     | –         | 23          | 0.57            |
| mycie6                | 95  | 755   | –     | –         | 47          | 0.84            |
| Graph       | n   | \(d_0\) | \(X_S\) | \(X_{DBG}\) | \(X_{UBQP}\) | Time_{UBQP} (s) |
|------------|-----|--------|--------|-------------|-------------|-----------------|
| myciel7    | 191 | 2360   | –      | –           | 95          | 3.96            |
| dsjc125.1  | 125 | 736    | –      | –           | 32          | 2.03            |
| dsjc250.1  | 250 | 3218   | –      | –           | 39          | 5.49            |
| le450_15a  | 450 | 8168   | –      | –           | 71          | 9.41            |
| le450_15d  | 450 | 16750  | –      | –           | 37          | 12.16           |
| le450_25a  | 450 | 8260   | –      | –           | 85          | 11.07           |
| le450_25d  | 450 | 17425  | –      | –           | 40          | 13.22           |

5 Conclusion

In this paper we have presented a method to reformulate the linear program of maximum independent set problem as an unconstrained binary quadratic problem, the proposed algorithm proves to be highly effective in solving a range of benchmark instances from the literature. Most of these instances have been easily resolved with a reasonable execution time. For the first 13 instances we obtained identical results than those in [9], our results are also identical to the first 21 instances given in [7], to show the effectiveness of our algorithm we tested 15 other instances.

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