MODULAR CLASS OF A LIE ALGEBROID WITH A NAMBU STRUCTURE

APURBA DAS, SHILPA GONDHALI, AND GOUTAM MUKHERJEE

Abstract. In this paper, we introduce the notion of modular class of a Lie algebroid $A$ equipped with a Nambu structure satisfying some suitable hypothesis. We also introduce cohomology and homology theories for such Lie algebroids and prove that these theories are connected by a duality isomorphism when the modular class is null.

1. Introduction

For a Poisson manifold $P$, it is well known that the cotangent bundle $T^*P$ has a natural Lie algebroid structure (see [14] for details). It is well known that for a Poisson manifold $P$ there is a distinguished characteristic class, called the modular class of $P$ which is defined as a first Lie algebroid cohomology class of the Lie algebroid $T^*P$. Nambu-Poisson manifolds are generalization of Poisson manifolds and just like Poisson manifolds, the notion of modular class of a Nambu-Poisson manifold exists. In [7], the authors proved that for an oriented Nambu-Poisson manifold $M$ of order $n$, $n \geq 3$, the vector bundle $\Lambda^{n-1}(T^*M)$ is a Leibniz algebroid and as a result the space $\Omega^{n-1}(M)$ of smooth sections of the bundle $\Lambda^{n-1}(T^*M)$ is a Leibniz algebra (a non anti-symmetric version of Lie algebra). The modular class of $M$ is then defined as an element of the first Leibniz algebroid cohomology of $\Lambda^{n-1}(T^*M)$ which is by definition, the first cohomology class of the Leibniz algebra $\Omega^{n-1}(M)$ with coefficients in $C^\infty(M)$. In [6], the authors introduced homology and cohomology theories for Nambu-Poisson manifolds and when the modular class vanishes, the authors proved a duality theorem connecting Nambu cohomology and canonical Nambu homology modules.

The notion of a Nambu structure on a Lie algebroid was introduced in [16] as a generalization of the notion of a Nambu-Poisson manifold. It turns out that for a Lie algebroid $A$ over a smooth manifold $M$ equipped with a Nambu structure of order $n$, the vector bundle $\Lambda^{n-1}A^*$ is a Leibniz algebroid. The aim of this article is to generalize the methods developed in [6], [7], in the context of Lie algebroid equipped with a Nambu structure. To do this, we introduce modular class of a Lie algebroid $A$ equipped with a Nambu structure of order $n$ as an element of the first Leibniz algebroid cohomology of the associated Leibniz algebroid $\Lambda^{n-1}A^*$, under the assumption that the space of smooth sections of $A^*$ is locally spanned by elements of the form $d_A f$, $f \in C^\infty(M)$, where $d_A$ denotes the coboundary operator in the Lie algebroid cohomology complex of $A$ with trivial coefficients. Moreover, we introduce Nambu cohomology and Nambu homology theories, generalizing the corresponding results for Nambu-Poisson manifolds, in the context of Lie algebroids equipped with regular Nambu...

\begin{flushleft}
Date: January 31, 2014.
2010 Mathematics Subject Classification. 53Cxx, 53C15, 53D17, 81S10.
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structures and prove that if the modular class is null then these theories are connected by
duality isomorphisms. It may be mentioned that in [5], the authors introduced a notion of
modular class of an arbitrary Lie algebroid which arise from a representation on a specific
line bundle, extending the notion of the modular class of Poisson manifolds.

The paper is organized as follows. In §2, we review some known facts and fix our notation.
In §3, we define the notion of modular class of a Lie algebroid which is orientable as a vector
bundle and equipped with a Nambu structure. In §4, we first introduce Nambu cohomology
for a Lie algebroid $A$ equipped with a regular Nambu structure and give an equivalent
formulation of this cohomology which is a variant of the notion of foliated cohomology
[12,13]. Next, under the assumption that $A$ is oriented as a vector bundle, we introduce the
notion of canonical Nambu homology theory of $A$. Finally, in this section we connect these
theories by proving a duality isomorphism theorem under the assumption that the modular
class of $A$ is null. In §5, the last section of the paper, we show that we can do away with the
orientability assumption on the vector bundle $A$ and introduce modular class and recover
the results of §4 by using the notion of density on $A$.

2. Preliminaries

In this section we briefly recall some known facts and fix notations (see [10] for details).
All vector bundles we consider are smooth vector bundles over smooth paracompact mani-
folds. The notion of Lie algebroids was introduced by J. Pradines [11] as generalization of
tangent bundles of smooth manifolds and Lie algebras.

2.1. Definition. Let $M$ be a smooth manifold. A Lie algebroid $(A, [, ], A, a)$ over $M$ is a
vector bundle $A$ over $M$ together with a vector bundle map $a: A \to TM$, called anchor
of $A$ and a bilinear map $[,]_A: \Gamma A \times \Gamma A \to \Gamma A$ on the space $\Gamma A$ of smooth sections of $A$,
which makes $\Gamma A$ a Lie algebra such that for all $X, Y \in \Gamma A$ and $f \in C^\infty(M)$ following holds:

$$[X,fY]_A = f[X,Y]_A + a(X)(f)Y.$$ 

Recall that the $C^\infty(M)$-linear map $\Gamma A \to \chi(M)$ induced by $a$, where $\chi(M)$ denote the
space of vector fields on $M$, is a Lie algebra homomorphism, that is,

$$a([X,Y]_A) = a(X)a(Y) - a(Y)a(X) = [a(X), a(Y)].$$

A representation $\rho: A \to D(E)$ of a Lie algebroid $A$ over $M$ on a vector bundle $E$ over $M$ is a Lie algebroid morphism, where $D(E)$ is the Lie algebroid of derivations on $E$. In
particular, the representation of $A$ on the trivial bundle $M \times \mathbb{R} \to M$ given by

$$a^0 : (X, f) \mapsto a(X)(f)$$

for all $f \in C^\infty(M) = \Gamma(M \times \mathbb{R})$ and $X \in \Gamma A$ is called the trivial representation.

We now briefly recall the definition of cohomology of a Lie algebroid with coefficients in
a given representation. Given a representation $\rho$ on a vector bundle $E$ over $M$ of a Lie
algebroid $A$ over $M$, we have a sequence of $C^\infty(M)$- modules $\Gamma(\Lambda^k A^* \otimes E)$, $k \geq 0$, and
$\mathbb{R}$-linear maps

$$d_A : \Gamma(\Lambda^k A^* \otimes E) \to \Gamma(\Lambda^{k+1} A^* \otimes E)$$
which forms a cochain complex. The homology of this complex defines the Lie algebroid cohomology of $A$ with coefficients in $\rho$. Note that an element of $\Gamma(\Lambda^k A^* \otimes E)$ may be viewed as a $C^\infty(M)$-multilinear and alternating map

$$\phi: \underbrace{\Gamma A \times \cdots \times \Gamma A}_{k \text{ times}} \to \Gamma E$$

and then $d_A \phi$ is given by the formula

$$d_A \phi(X_1, \cdots, X_{k+1}) = \sum_{r=1}^{n+1} (-1)^{r+1} \rho(X_r)(\phi(X_1, \cdots, X_{r-1}, \overline{X_r}, X_{r+1}, \cdots, X_{k+1}))$$

$$+ \sum_{1 \leq r < s \leq n+1} (-1)^{r+s} \phi([X_r, X_s]_A, X_1, \cdots, \overline{X_r}, \overline{X_s}, \cdots, X_{k+1}).$$

for all $X_1, \cdots, X_{k+1} \in \Gamma A$. The $k^{th}$-cohomology of $A$ with coefficients in $\rho$ is denoted by $\mathcal{H}^k(A, \rho)$. In particular, when $E = M \times \mathbb{R}$ and $\rho = \alpha^0$, we denote the above cochain complex simply by $\{\Gamma(\Lambda^k A^*), d_A\}_{k \geq 0}$ and the corresponding cohomology is denoted by $\mathcal{H}^*(A)$.

Recall that we have two useful operators defined as follows \cite{10}.

2.2. Definition. Let $M$ be a smooth manifold. Let $(A, [,]_A, a)$ be a Lie algebroid over $M$.
(1) Let $X \in \Gamma A$. The Lie derivative $\mathcal{L}_X: \Gamma(\Lambda^n A^*) \to \Gamma(\Lambda^n A^*)$ is defined by

$$\mathcal{L}_X(\phi)(X_1, \cdots, X_n) = a(X)(\phi(X_1, \cdots, X_n)) - \sum_{r=1}^n \phi(X_1, \cdots, [X, X_r]_A, \cdots, X_n)$$

where $\phi \in \Gamma(\Lambda^n A^*), X_r \in \Gamma A$ for $r = 1, \cdots, n$.

(2) Let $X \in \Gamma A$. The interior multiplication , also known as contraction

$$\iota_X: \Gamma(\Lambda^{n+1} A^*) \to \Gamma(\Lambda^n A^*)$$

is defined by

$$\iota_X(\phi)(X_1, \cdots, X_n) = \phi(X, X_1, \cdots, X_n)$$

where $n \in \mathbb{N}, \phi \in \Gamma(\Lambda^{n+1} A^*)$ and $X_1, \cdots, X_n \in \Gamma A$.

The operators $\mathcal{L}_X, \iota_X$ and $d_A$ satisfy a set of formulas similar to those which hold in the calculus of vector-valued forms on manifolds (see \cite{10} for details).

Let `$\wedge$' be the standard wedge product of $k$-multisections. For $\phi \in \Gamma(\Lambda^k A^*)$ and a $k$-multisection $\xi \in \Gamma(\Lambda^k A)$, let $\langle \phi, \xi \rangle$ denote the standard pairing. Then the contraction operator extends to give a well defined map $\iota_\xi: \Gamma(\Lambda^k A^*) \to \Gamma(\Lambda^{k-1} A^*)$ for $\xi \in \Gamma(\Lambda^k A)$ verifying

$$\langle \iota_\xi(\phi), \eta \rangle = (-1)^{k-\frac{1}{2}} \langle \phi, \xi \wedge \eta \rangle$$

for $\xi \in \Gamma(\Lambda^k A), \eta \in \Gamma(\Lambda^j A), \phi \in \Gamma(\Lambda^{k+j} A^*)$.

Next, given $\phi \in \Gamma A^*$, define $\iota_\phi: \Gamma(\Lambda^k A) \to \Gamma(\Lambda^{k-1} A)$ for $\xi \in \Gamma(\Lambda^k A)$ and $\psi \in \Gamma(\Lambda^{k-1} A^*)$ by

$$\langle \psi, \iota_\phi(\xi) \rangle = \langle \phi \wedge \psi, \xi \rangle.$$  

This again extends to a well-defined map $\iota_\phi: \Gamma(\Lambda^k A) \to \Gamma(\Lambda^{k-1} A)$ for $\phi \in \Gamma(\Lambda^k A^*)$. 

For $\phi \in \Gamma(\Lambda^i A^*)$, $\psi \in \Gamma(\Lambda^j A^*)$, $\xi \in \Gamma(\Lambda^k A^*)$, we have

$$\langle \psi, \iota_\phi(\xi) \rangle = (-1)^{i} \langle \phi \wedge \psi, \xi \rangle.$$ 

In particular, when $\xi \in \Gamma(\Lambda^k A)$ is of the form $\xi = X_1 \wedge \cdots \wedge X_k$ then $\iota_\xi$ is simply the compositions of the operators $\iota_{X_i}$, $1 \leq i \leq k$. Similar is the case for $\iota_\phi$ for $\phi \in \Gamma(\Lambda^i A^*)$.

Moreover, for $X \in \Gamma A$, $\theta \in \Gamma A^*$, $\phi, \psi \in \Gamma(\Lambda^i A^*)$ and $\xi, \eta \in \Gamma(\Lambda^j A)$ with $\phi$ and $\xi$ of degree $i$, the following hold:

$$\iota_X(\phi \wedge \psi) = \iota(\phi) \wedge \psi + (-1)^i \phi \wedge \iota_X(\psi),$$

$$\iota_\theta(\xi \wedge \eta) = \iota(\xi) \wedge \eta + (-1)^j \xi \wedge \iota_\theta(\eta).$$

The notion of Leibniz algebras was introduced by J-L Loday \[8\], as a non anti-symmetric analogue of Lie algebras. Recall that a left Leibniz bracket on a real vector space $L$ is a $\mathbb{R}$-bilinear operation $\ll, \gg_L : L \times L \rightarrow L$ such that the following identity, known as Leibniz identity

$$\ll a, \ll b, c \gg_L = \ll \ll a, b \gg_L, c \gg_L + \ll b, \ll a, c \gg_L \gg_L$$

hold for all $a, b, c \in L$. The vector space $L$ equipped with a left Leibniz bracket is called a left Leibniz algebra. Through out the paper, by a Leibniz algebra, we shall always mean a left Leibniz algebra. Note that the bracket $\ll, \gg_L$ becomes a Lie bracket, if in addition, it is anti-symmetric. Thus every Lie algebra is, in particular, a Leibniz algebra.

Recall \[9\] that a representation of a Leibniz algebra $L$ is a $\mathbb{R}$-module $E$ equipped with a $\mathbb{R}$-bilinear map $L \times E \rightarrow E$, $(a, e) \mapsto ae$, such that $a_1(a_2 e) = \ll a_1, a_2 \gg_L e + a_2(a_1 e)$, for all $a_1, a_2 \in L$ and $e \in E$.

Leibniz algebroids are generalizations of Leibniz algebras.

2.3. Definition. Let $M$ be a smooth manifold. A Leibniz algebroid over $M$ is a vector bundle $A$ over a smooth manifold $M$ together with a vector bundle map $a : A \rightarrow TM$ over $M$, called anchor of $A$ and a $\mathbb{R}$-bilinear map $\ll, \gg_A : \Gamma A \times \Gamma A \rightarrow \Gamma A$ such that for all $X, Y, Z \in \Gamma A$ and $f \in C^\infty(M)$ following hold.

1. $\ll X, f Y \gg_A = f \ll X, Y \gg_A + a(X)(f)Y$;
2. $\ll X, \ll Y, Z \gg_A \gg_A = \ll \ll X, Y \gg_A, Z \gg_A + \ll Y, \ll X, Z \gg_A \gg_A$.

It is a consequence (cf. Proposition 3.1, \[1\]) of the Definition 2.3 that

$$a(\ll X, Y \gg_A) = [a(X), a(Y)]_{TM}, \text{ for all } X, Y \in \Gamma A.$$ 

We will often use the notation $(A, \ll, \gg_A, a)$ to denote a Leibniz algebroid over a smooth manifold $M$. 
2.4. Example.

1. Any Lie algebroid is a Leibniz algebroid.
2. Every Leibniz algebra may be considered as a Leibniz algebroid over a one point space.
3. Let $M$ be a smooth manifold and set $A = TM \oplus T^*M$. Then $A$ is a vector bundle over $M$. Define $\langle \cdot, \cdot \rangle_A : \Gamma A \times \Gamma A \to \Gamma A$ as
\[
\langle X + \xi, Y + \eta \rangle_A = [X, Y]_{TM} + \mathcal{L}_X \eta - \iota_Y (d\xi)
\]
where $X, Y \in \Gamma(TM) = \mathfrak{X}(M)$ and $\xi, \eta \in \Gamma(T^*M) = \Omega^1(M)$. Then it is easy to check that $A$ is a Leibniz algebroid with the projection onto the first factor as the anchor.

The notion of morphism between two Leibniz algebroids over the same base manifold is similar to the definition of morphism between Lie algebroids. Just like Lie algebroids, the notion of representation of a Leibniz algebroid may be defined as follows.

2.5. Definition. A representation of a Leibniz algebroid $(A, \langle \cdot, \cdot \rangle_A, a)$ over $M$ on a vector bundle $E$ over $M$ is a Leibniz algebroid morphism

\[
\rho : A \to D(E),
\]
where $D(E)$ is the Lie algebroid of derivations on $E$.

In other words, a representation of $A$ consists of a $\mathbb{R}$-bilinear map

\[
\Gamma A \times \Gamma E \to \Gamma E; \ (X, s) \mapsto \rho_X s
\]
such that for any $X, Y \in \Gamma A$, $f \in C^\infty(M)$ and $s \in \Gamma E$ the following hold:

1. $\rho f X s = f \rho_X s$,
2. $\rho_X (fs) = f \rho_X s + (a(X)f)s$,
3. $\rho_X (\rho_Y s) - \rho_Y (\rho_X s) = \rho_{\langle X, Y \rangle_A} s$.

Note that when considered over a point, the above notion is precisely the notion of the representation of a Leibniz algebra.

2.6. Example. Let $A$ be a Leibniz algebroid on $M$. The representation $a^0$ of $A$ on the trivial line bundle

\[
M \times \mathbb{R} \to M
\]
given by $a^0(X)(f) = a(X)(f)$ for all $f \in C^\infty(M)$ and $X \in \Gamma A$ is called the trivial representation.

2.7. Definition. Let $(A, \langle \cdot, \cdot \rangle_A, a)$ be a Leibniz algebroid over $M$. Let $\rho$ be a representation of $A$ on a vector bundle $E$ over $M$. Define a sequence of complex as follows. For $n \in \mathbb{N}$,

\[
C^n(A, \rho) := \{ \phi : \Gamma A \times \cdots \times \Gamma A \to \Gamma E | \phi \text{ is } \mathbb{R}\text{-multilinear} \}.
\]
Define coboundary operator \(d_{A,\rho}: C^n(A,\rho) \to C^{n+1}(A,\rho)\) by the formula

\[
d_{A,\rho}\phi(X_1, \cdots, X_{n+1}) = \sum_{r=1}^{n+1} (-1)^{r+1} \rho_X r(X_1, \cdots, X_{r-1}, \hat{X}_r, X_r, X_{r+1}, \cdots, X_{n+1}) + \sum_{1 \leq r < s \leq n+1} (-1)^r \phi(X_1, \cdots, \hat{X}_r, \cdots, \ll X_r, X_s \gg A, \cdots, X_{n+1}).
\]

for all \(\phi \in C^n(A;\rho)\) and \(X_1, \cdots, X_{n+1} \in \Gamma A\). One can check that \(d_{A,\rho} \circ d_{A,\rho} = 0\).

Let

\[
Z^n(A,\rho) := \ker(d_{A,\rho}: C^n(A,\rho) \to C^{n+1}(A,\rho))
\]

\[
B^n(A,\rho) := \text{im}(d_{A,\rho}: C^{n-1}(A,\rho) \to C^n(A,\rho))
\]

where \(B^0(A,\rho) = \{0\}\) and \(n \in \mathbb{N}\). Then the \(n\)th cohomology of \(A\) with coefficients in the representation \(\rho\) is defined by the quotient

\[
\mathcal{H}L^n(A,\rho) = Z^n(A,\rho)/B^n(A,\rho).
\]

**Notation:** In the case, when the representation \(\rho\) is the trivial representation, we denote the cohomology operator by \(d_A\) and the cohomology modules simply by \(\mathcal{H}L^\bullet(A)\).

2.8. *Remark.*

(1) In the case, when \(M\) is a point, then \(A\) is a Leibniz algebra and \(\rho\) reduces to a representation of \(A\) as a Leibniz algebra and in this case, the definition of cohomology reduces to the Leibniz algebra cohomology with coefficients in a representation.

(2) When \(\rho\) is the trivial representation then the cohomology as defined above is precisely the Leibniz algebroid cohomology as introduced in [7].

3. LIE ALGEBROID WITH A NAMBU STRUCTURE AND THE MODULAR CLASS

Let \(M\) be a smooth manifold of dimension \(m\). Recall that a *Nambu-Poisson bracket* of order \(n, n \leq m\), on \(M\) is an \(n\)-multilinear mapping

\[
\{, \cdots, \}: C^\infty(M) \times \cdots \times C^\infty(M) \to C^\infty(M)
\]

satisfying the following:

1. Alternating:

\[
\{f_1, \cdots, f_n\} = (-1)^{\varepsilon(\sigma)} \{f_{\sigma(1)}, \cdots, f_{\sigma(n)}\},
\]

for all \(f_1, \cdots, f_n \in C^\infty(M)\) and \(\sigma \in \Sigma_n\), where \(\Sigma_n\) is the symmetric group of \(n\) elements and \(\varepsilon(\sigma)\) is the parity of the permutation \(\sigma\),

2. Leibniz rule:

\[
\{f_1 g_1, f_2, \cdots, f_n\} = f_1 \{g_1, f_2, \cdots, f_n\} + g_1 \{f_1, f_2, \cdots, f_n\}.
\]
(3) Fundamental identity:

\[ \{f_1, \ldots, f_{n-1}, g_1, \ldots, g_n\} = \sum_{i=1}^{n} \{g_1, \ldots, \{f_1, \ldots, f_{n-1}, g_i\}, \ldots, g_n\} \]

for all \( f_1, \ldots, f_{n-1}, g_1, \ldots, g_n \in C^\infty(M) \).

Given a Nambu-Poisson bracket, one can define an \( n \)-vector field \( \Lambda \in \Gamma(\Lambda^n TM) \) as

\[ \Lambda(df_1, \ldots, df_n) = \{f_1, \ldots, f_n\} \]

for \( f_1, \ldots, f_n \in C^\infty(M) \). The pair \( (M, \Lambda) \) is called a Nambu-Poisson manifold of order \( n \).

Nambu structure on a Lie algebroid ([15]) is a generalization of Nambu-Poisson manifold. In this section, we introduce the notion of the modular class of a Lie algebroid equipped with a Nambu structure under some suitable assumptions.

Recall that for a Lie algebroid \( (A, [\cdot, \cdot]_A, a) \) over a smooth manifold \( M \), the algebra \( \Gamma(\Lambda^\bullet A) \) endowed with the standard wedge product \( \wedge \) and the generalized Schouten bracket extending the bracket on \( \Gamma A \) (also denoted by \( [\cdot, \cdot]_A \)) is a Gerstenhaber algebra.

3.1. Definition. Let \( M \) be a smooth manifold. Let \( (A, [\cdot, \cdot]_A, a) \) be a Lie algebroid over \( M \). Let \( n \in \mathbb{N}, \ n \leq m = \text{rank} \ A \) and \( \Pi \in \Gamma(\Lambda^n A) \) a smooth section of the vector bundle \( \Lambda^n A \).

We say that \( \Pi \) is a Nambu structure of order \( n \) if

\[ [\Pi \alpha, \Pi]_A \beta = -\Pi(\iota_\Pi d_A \alpha), \quad \text{for any} \ \alpha, \beta \in \Gamma(\Lambda^{n-1} A^\ast). \]

where \( d_A \) denote coboundary operator for the Lie algebroid cohomology of \( A \) with trivial coefficients and \( \Pi \alpha := \iota_\alpha \Pi \) for all \( \alpha \in \Gamma(\Lambda^\bullet A^\ast) \).

3.2. Remark. The notion Nambu structure on Lie algebroids is a generalization of Nambu-Poisson manifolds in the sense that given a Lie algebroid \( (A, [\cdot, \cdot]_A, a) \) over a smooth manifold \( M \), the algebra \( \Gamma(\Lambda^\bullet A) \) endowed with the standard wedge product \( \wedge \) and the generalized Schouten-Nijenhuis bracket \( [\cdot, \cdot]_{SN} \) on multi-vector fields [2] and \( d \) the de Rham differential operator and hence \( \Lambda \) is a Nambu-structure on \( TM \).

3.3. Definition. (i) Let \( (A, [\cdot, \cdot]_A, a) \) be a Lie algebroid over a smooth manifold \( M \). A smooth section \( \Pi \in \Gamma(\Lambda^j A) \) is called locally decomposable if for any \( x \in M \), either \( \Pi(x) = 0 \) or in a neighborhood of \( x \), \( \Pi \) can be expressed as \( \Pi = X_1 \wedge \cdots \wedge X_j \) where \( X_1, \ldots, X_j \) are local sections of \( A \) defined on that neighbourhood.

(ii) A point \( x \in M \) is called a singular point of \( \Pi \) if \( \Pi(x) = 0 \) and is called a regular point of \( \Pi \) if \( \Pi(x) \neq 0 \).

3.4. Definition. Let \( (A, [\cdot, \cdot]_A, a) \) be a Lie algebroid over \( M \) with a Nambu structure \( \Pi \) of order \( n \). We say \( \Pi \) is a regular Nambu structure if \( \Pi(x) \neq 0 \) for all \( x \in M \) i.e. each point of \( M \) is a regular point of \( \Pi \).
3.5. **Remark.** Let \((A, [\cdot, \cdot]_A, a)\) be a Lie algebroid over a smooth manifold \(M\). Let \(\Pi \in \Gamma(\Lambda^j A)\), where \(j \in \mathbb{N}, j < \text{rank} \ A\). Let \(x \in M\) be a regular point of \(\Pi\). Let \(\Pi\) be locally decomposable at \(x\). Then we may choose a neighbourhood \(U\) of \(x\) and linearly independent local sections \(X_1, \ldots, X_j\) of \(A\) defined on \(U\) such that \(\Pi|_U\) is of the form \(\Pi|_U = X_1 \wedge \cdots \wedge X_j\).

We have the following result \([15]\).

3.6. **Proposition.** Let \((A, [\cdot, \cdot]_A, a)\) be a Lie algebroid over \(M\) equipped with a Nambu structure \(\Pi\) of order \(n\) with \(n \geq 3\). Then for any \(f \in C^\infty(M)\), \(\Pi \left(d_A f\right)\) is locally decomposable. In particular, if \(\Gamma A^*\) is locally spanned by elements of the form \(d_A f\) where \(f \in C^\infty(M)\) then \(\Pi\) is locally decomposable.

Let \((A, [\cdot, \cdot]_A, a)\) be a Lie algebroid over \(M\) with a Nambu structure \(\Pi\) of order \(n\). Then \(\Pi\) induces a morphism of vector bundles \(\Pi_k: \Lambda^k A^* \to \Lambda^{n-k} A\) for \(k \leq n\) defined by

\[
\Pi_k(\alpha) = i_a \Pi(x) \quad \text{for all } \alpha \in \Lambda^k A^*_x
\]

Hence we have a \(C^\infty(M)\)-linear map \(\Pi_k: \Gamma(\Lambda^k A^*) \to \Gamma(\Lambda^{n-k} A)\) given by

\[
\Pi_k(\alpha) := \Pi \alpha = i_a \Pi \quad \text{for all } \alpha \in \Gamma(\Lambda^k A^*)
\]

3.7. **Remark.** Note that the induced map \(\Pi_k: \Gamma(\Lambda^k A^*) \to \Gamma(\Lambda^{n-k} A)\) is a \(C^\infty(M)\)-linear module isomorphism where \([\alpha] \in \Gamma(\Lambda^k A^*)/\ker \Pi_k\).

3.8. **Lemma.** Let \((A, [\cdot, \cdot]_A, a)\) be a Lie algebroid over \(M\) equipped with a regular Nambu structure \(\Pi\) of order \(n\). Assume further that \(\Gamma A^*\) is locally spanned by elements of the form \(d_A f\) where \(f \in C^\infty(M)\). Then

\[
\dim_{\mathbb{R}} \Gamma_{n-1}(\Lambda^{n-1} A^*_x) = n, \quad x \in M.
\]

Let \(D = \bigcup_{x \in M} D_x\) of \(A\) where \(D_x = \Gamma_{n-1}(\Lambda^{n-1} A^*_x)\). Thus \(D\) is a subbundle of \(A\).

**Proof.** Let \(x \in M\). Then by Lemma 3.5, we can find a trivializing neighbourhood \(U\) of \(x\) and linearly independent local sections \(X_1, \ldots, X_n\) on \(U\) such that

\[
\Pi|_U = X_1 \wedge \cdots \wedge X_n
\]

Now extend \(X_1, \ldots, X_n\) to linearly independent sections \(X_1, \ldots, X_n, X_{n+1}, \ldots, X_m\) on \(U\), \(m\) being the rank of the vector bundle \(A\), so that at each point \(y \in U\), \(X_1(y), \ldots, X_m(y)\) is a basis of \(A_y\).

Let \(\alpha \in \Lambda^{n-1} A^*_x\). Then

\[
\alpha = \sum_{1 \leq i_1 < \cdots < i_{n-1} \leq m} \alpha_{i_1 \cdots i_{n-1}} X_{i_1}^*(x) \wedge \cdots \wedge X_{i_{n-1}}^*(x) \quad \text{for some } \alpha_{i_1 \cdots i_{n-1}} \in \mathbb{R}
\]

After rearranging terms and renaming, we have

\[
\alpha = \sum_{i=1}^n \alpha_i X_i^*(x) \wedge \cdots \wedge \hat{X_i^*}(x) \wedge \cdots \wedge X_n^*(x) + \alpha' \quad \text{where } \Pi_{n-1}(\alpha') = 0.
\]
Applying $\Pi_{n-1}$ on $\alpha$ we get,

$$\Pi_{n-1}(\alpha) = \sum_{i=1}^{n} (-1)^{n-i} \alpha_i X_i(x)$$

Consider $\beta_i \in \Lambda^{n-1} A_x^*$ where $\beta_i = (-1)^{n-i} X_1^*(x) \wedge \cdots \wedge X_i^*(x) \wedge \cdots \wedge X_n^*(x)$.

Then $\Pi_{n-1}(\beta_i) = X_i(x)$. Hence $\dim \Pi_{n-1}(\Lambda^{n-1} A_x^*) \geq n$.

Clearly, $\dim \Pi_{n-1}(\Lambda^{n-1} A_x^*) \leq n$.

Hence the lemma. \qed

3.9. Remark. Let $(A, [\cdot, \cdot], a)$ be a Lie algebroid over a smooth manifold $M$ equipped with a regular Nambu structure $\Pi$ of order $n \leq m = \text{rank } A$. Assume that $\Gamma(A^*)$ is locally spanned by elements of the form $d Af$, $f \in C^\infty(M)$.

(1) For all $k$, $k \leq n$, $\ker \Pi_k$ is a vector subbundle of $\Lambda^k A^* \to M$ of rank

$$\left( \begin{array}{c} m \\ k \end{array} \right) - \left( \begin{array}{c} n \\ k \end{array} \right).$$

Similarly, $\Pi_k(\Lambda^k A^*)$ is a subbundle of $\Lambda^{n-k} A \to M$ of rank $\left( \begin{array}{c} n \\ k \end{array} \right)$.

(2) Since $M$ is paracompact, we may assume that the vector bundle $A$ is Euclidean. Let $D^0(x)$ be the annihilator of the subspace $D(x)$ of $A_x$. Thus $D^0(x)$ consists of all elements $u^* \in A_x^*$ such that $u^*(v) = 0$ for all $v \in D(x)$. Note that $D^0(x) = \ker (\Pi_1 A_x^*)$ for all $x \in M$. An $(n-k)$-multisection $P$ is orthogonal to $D$ if $\iota_u P(x) = 0$, for all $u^* \in D^0(x)$, $x \in M$. Then the space of smooth sections of the vector bundles $\Pi_k(\Lambda^k A^*) \to M$ are precisely the set of all $(n-k)$-multisections of $A$ that are orthogonal to $D$.

We have the following proposition \cite{15}.

3.10. Proposition. Let $(A, [\cdot, \cdot], a)$ be a Lie algebroid over $M$ with a Nambu structure $\Pi$ of order $n$. In that case, the triplet $(A = \Lambda^{n-1} A^*, \ll, \gg, a \circ \Pi_{n-1})$ determines a Leibniz algebroid, where $\ll \cdot, \cdot \gg$ is defined by

$$\ll \alpha, \beta, \gg = \mathcal{L}_{\Pi \alpha} \beta - \iota_\beta d_A \alpha, \text{ for any } \alpha, \beta \in \Gamma A.$$

3.11. Lemma. Let $(A, [\cdot, \cdot], a)$ be a Lie algebroid over $M$ with a Nambu structure $\Pi$ of order $n$. Let $\eta_1, \eta_2 \in \Gamma A^{n-1}(A)$. Then

$$[\Pi \eta_1, \Pi \eta_2]_A = [\Pi \eta_1, \Pi] A \eta_2 + \Pi (\mathcal{L}_{\Pi \eta_1} \eta_2)$$

Proof. It is enough to verify the equality for $\Pi = X_1 \wedge X_2 \wedge \cdots \wedge X_n$, $\eta_1 = Y_1 \wedge \cdots \wedge Y_{n-1}$ and $\eta_2 = Z_1 \wedge \cdots \wedge Z_{n-1}$.

Also, by induction, it is enough to verify this for $n = 2$, that is, for $\Pi = X_1 \wedge X_2$ where $X_1, X_2 \in \Gamma A$ and $\eta_1, \eta_2 \in \Gamma A^*$.

Consider

$$[\Pi \eta_1, \Pi \eta_2]_A = [\Pi \eta_1, \iota_{\eta_2} (X_1 \wedge X_2)]_A$$
We know that for \( \theta \in \Gamma A^* \) and \( \xi_1, \xi_2 \in \Gamma (\Lambda^* A) \),
\[
\iota_\theta (\xi_1 \wedge \xi_2) = \iota_\theta (\xi_1) \wedge \xi_2 + (-1)^{\deg(\xi_1)} \xi_1 \wedge \iota_\theta (\xi_2)
\]
Hence we have
\[
[\Pi \eta_1, \Pi \eta_2]_A = [\Pi \eta_1, \iota_{\eta_2} (X_1) \wedge X_2 - X_1 \wedge \iota_{\eta_2} (X_2)]_A
= [\Pi \eta_1, \iota_{\eta_2} (X_1) \wedge X_2]_A - [\Pi \eta_1, X_1 \wedge \iota_{\eta_2} (X_2)]_A
\]
By the definition of Gerstenhaber bracket \([\cdot, \cdot]_A\) on \( \Gamma (\Lambda^* A) \), we have
\[
[\Pi \eta_1, \Pi \eta_2]_A = [\Pi \eta_1, \iota_{\eta_2} (X_1)]_A \wedge X_2 + \iota_{\eta_2} (X_1)[\Pi \eta_1, X_2]_A - [\Pi \eta_1, X_1]_A \wedge \iota_{\eta_2} (X_2) - X_1 \wedge [\Pi \eta_1, \iota_{\eta_2} (X_2)]_A
= a(\Pi \eta_1) (\iota_{\eta_2} (X_1)) \wedge X_2 + \iota_{\eta_2} (X_1)[\Pi \eta_1, X_2]_A - [\Pi \eta_1, X_1]_A \wedge \iota_{\eta_2} (X_2) - X_1 \wedge a(\Pi \eta_1) (\iota_{\eta_2} (X_2)).
\]
Notice that \([\Pi \eta_1, \iota_{\eta_2} (X_1)]_A = a(\Pi \eta_1) (\iota_{\eta_2} (X_1)).\)
Similarly, we get
\[
[\Pi \eta_1, \Pi \eta_2]_A = \iota_{\eta_2} (X_1)[\Pi \eta_1, X_2]_A - X_1 \wedge \iota_{\eta_2} ([\Pi \eta_1, X_2]_A).
\]
Now consider
\[
\Pi (\mathcal{L}_{\Pi \eta_1} \eta_2) = \iota_{\mathcal{L}_{\Pi \eta_1} \eta_2} P = \iota_{\mathcal{L}_{\Pi \eta_1} \eta_2} (X_1 \wedge X_2)
= (\iota_{\mathcal{L}_{\Pi \eta_1} \eta_2} X_1) \wedge X_2 - X_1 \wedge (\iota_{\mathcal{L}_{\Pi \eta_1} \eta_2} X_2)
\]
By the definition of Lie derivative and contraction, we get
\[
\iota_{\mathcal{L}_{\Pi \eta_1} \eta_2} X_1 = (\mathcal{L}_{\Pi \eta_1} \eta_2)(X_1)
= a(\Pi \eta_1) (\eta_2 (X_1)) - \eta_2 ([\Pi \eta_1, X_1]_A)
= a(\Pi \eta_1) (\eta_2 (X_1)) - \eta_2 ([\Pi \eta_1, X_1]_A)
\]
By similar arguments we get,
\[
\Pi (\mathcal{L}_{\Pi \eta_1} \eta_2) = a(\Pi \eta_1) (\eta_2 (X_1)) \wedge X_2 - \iota_{\eta_2} ([\Pi \eta_1, X_1]_A) \wedge X_2
- X_1 \wedge a(\Pi \eta_1) (\eta_2 (X_2)) + X_1 \wedge \iota_{\eta_2} ([\Pi \eta_1, X_2]_A).
\]
By combining the above facts we get the result. \(\square\)

3.12. Corollary. Let \( (A, [\cdot, \cdot]_A, \alpha) \) be a Lie algebroid over \( M \) with a Nambu structure \( \Pi \) of order \( n \). Then the vector bundle map \( \Pi_{n-1}: \Lambda^{n-1} A^* \to A \) given by
\[
\alpha \mapsto \Pi \alpha, \ \alpha \in \Gamma \Lambda^{n-1} A^*
\]
as defined above is a morphism of Leibniz algebroid.

Proof. Observe that
\[
[\Pi \alpha, \Pi \beta]_A = [\Pi \alpha, \Pi]_A \beta + \Pi (\iota_{\Pi \beta} d A \alpha) + \Pi (\mathcal{L}_{\Pi \alpha} \beta - \iota_{\Pi \beta} d A \alpha)
= \Pi (\mathcal{L}_{\Pi \alpha} \beta - \iota_{\Pi \beta} d A \alpha) = \Pi(\ll \alpha, \beta \gg).
\]
\(\square\)
The following is a consequence of the above facts.

3.13. Corollary. The subbundle $\Pi_{n-1}(\Lambda^{n-1}A^*)$ is a Lie subalgebroid of $A$.

Proof. Let $\alpha,\beta \in \Gamma(\Pi_{n-1}(\Lambda^{n-1}A^*))$.

Note that

$$[[\Pi_{n-1}\alpha,\Pi_{n-1}\beta]]_A = ([\Pi\alpha,\Pi]_A + \Pi(\mathcal{L}_{\Pi\alpha}\beta)
= [\Pi\alpha,\Pi]_A\beta + \Pi(\iota_{\Pi\beta}d_A\alpha + \Pi(\mathcal{L}_{\Pi\alpha}\beta - \iota_{\Pi\beta}d_A\alpha)
= \Pi(\mathcal{L}_{\Pi\alpha}\beta - \iota_{\Pi\beta}d_A\alpha)
= \Pi \iota_{\alpha\beta}.$$ 

Hence the result. □

Let $(A,[\cdot,\cdot]_A,a)$ be a Lie algebroid over $M$ with a Nambu structure $\Pi$ of order $n$. Recall that then $M$ is a Nambu-Poisson manifold of order $n$ with the Nambu-Poisson structure on $M$ being the push-forward $\Lambda = a \circ \Pi$ of $\Pi$ via the anchor $a$. In other words,

$$\Lambda(df_1,\ldots,df_n) = \{f_1,\ldots,f_n\}_\Pi = \Pi(d_Af_1,\ldots,d_Af_n), \ f_1, f_2, \ldots, f_n \in C^\infty(M).$$

Moreover, $\Lambda$ is locally decomposable.

We have the following result.

3.14. Lemma. Assume that $\Gamma A^*$ is locally spanned by elements of the form $d_Af$ where $f \in C^\infty(M)$. Then

$$\mathcal{L}_{\alpha \Pi(\alpha)}\Lambda = (-1)^n(\iota_{d_A\alpha}\Pi)\Lambda \text{ for all } \alpha \in \Gamma\Lambda^{n-1}A^*.$$ 

Proof. It is enough to check the equality for regular points. Let $x \in M$ be a regular point of $\Lambda$. There exists a coordinate chart $(U,x_1,\ldots,x_n,x_{n+1},\ldots,x_m)$ around $x$ such that $\Lambda/U$ can be expressed as

$$\Lambda|_U = \frac{\partial}{\partial x_1} \wedge \cdots \wedge \frac{\partial}{\partial x_n}.$$ 

We may assume that on $\alpha/U = f d_Af_1 \wedge d_Af_2 \wedge \cdots \wedge d_Af_{n-1}$, where $f, f_i \in C^\infty(M)$. Consider the $(n-1)$-form $\alpha_0 = f d_f_1 \wedge \cdots \wedge d_f_{n-1}$ on $U$. Note that

$$\iota_{\alpha_0}\Lambda = f \iota_{(d_f_1 \wedge d_f_2 \wedge \cdots \wedge d_f_{n-1})}\Lambda = f \Lambda(d_f_1 \wedge d_f_2 \wedge \cdots \wedge d_f_{n-1}).$$

Observe that $\Lambda(df_1 \wedge df_2 \wedge \cdots \wedge df_{n-1})(f) = \Lambda(df_1 \wedge df_2 \wedge \cdots \wedge df_{n-1} \wedge df) = \Pi(df_1 \wedge df_2 \wedge \cdots \wedge df_{n-1} \wedge d_Af)$ for any $f \in C^\infty(M)$. Therefore,

$$\iota_{\alpha_0}\Lambda = f a \circ \Pi(df_1 \wedge df_2 \wedge \cdots \wedge d_Af_{n-1}) = a \circ \Pi(df_1 \wedge df_2 \wedge \cdots \wedge d_Af_{n-1}) = a \circ \Pi(\alpha).$$

On the other hand,

$$\iota_{d_A\alpha_0}\Lambda = \Lambda(df \wedge df_1 \wedge \cdots \wedge df_{n-1}) = \Pi(df \wedge df_1 \wedge \cdots \wedge d_Af_{n-1}) = \iota_{d_Af \wedge d_Af_1 \wedge \cdots \wedge d_Af_{n-1}}\Pi = \iota_{d_A\alpha}\Pi.$$ 

The proof is now complete by [7, Equation (3.3)]. □
3.15. Remark. Given a multisection $\Pi \in \Gamma(\Lambda^q A)$ of a Lie algebroid $A$ over $M$, we have a $\mathbb{R}$-multilinear alternating map

$$\Pi : \underbrace{C^\infty(M) \times \cdots \times C^\infty(M)}_{q \text{ times}} \longrightarrow C^\infty(M)$$

defined by

$$\Pi(f_1, \ldots, f_q) := \Pi(d_Af_1, \ldots, d_Af_q)$$

which satisfies the derivation property in each argument. The converse is true when $\Gamma A^*$ is locally spanned by elements of the form $d_Af$, $f \in C^\infty(M)$. Because, in that case, we may define a multisection $\Pi \in \Gamma(\Lambda^q A)$ by

$$\langle \Pi, d_Af_1 \wedge \cdots \wedge d_Af_q \rangle := \Pi(f_1, \ldots, f_q).$$

As in the proof of Lemma 1.2.2 [L], we need only to check that the value of $\Pi(f_1, \ldots, f_q)$ at a point $x \in M$ depends only on the value of $d_Af_1, \ldots, d_Af_q$ at $x$. Note that for any $f \in C^\infty(M)$, $d_Af$ is given by $d_Af(X) = a(X)(f)$, $X \in \Gamma A$. Thus, if $d_Af$ is zero at a point $x \in M$, then on a neighbourhood of $x$, we may express $f$ in the form $f = c + \sum_i g_i h_i$ where $c$ is a constant and $g_i$, $h_i$ are smooth functions which vanish at $x$. The rest of the proof is similar to the proof of [L Lemma 1.2.2].

Let $(A, [\cdot, \cdot]_A, a)$ be a Lie algebroid over a smooth manifold $M$ equipped with a Nambu structure $\Pi$ of order $n$, $n \leq m = \text{rank } A$. Assume that the vector bundle $A$ is orientable. Let $\nu \in \Gamma(\Lambda^m A^*)$ be the nowhere vanishing form representing the orientation. We will call it the orientation form. Further assume that $\Gamma A^*$ is locally spanned by elements of the form $d_Af$ where $f \in C^\infty(M)$ and $d_A$ denote coboundary operator for the Lie algebroid cohomology of $A$ with trivial coefficients.

Consider the mapping

$$\mathcal{M}^\nu : \underbrace{C^\infty(M) \times \cdots \times C^\infty(M)}_{n-1 \text{ times}} \longrightarrow C^\infty(M)$$

given by

$$\mathcal{L}_\Pi(d_Af_1 \wedge \cdots \wedge d_Af_{n-1})^\nu = \mathcal{M}^\nu(f_1, \ldots, f_{n-1})^\nu$$

where $f_1, \ldots, f_{n-1} \in C^\infty(M)$ for all $i = 1, \ldots, n-1$. Then $\mathcal{M}^\nu$ defines an $(n-1)$ $\mathbb{R}$-multilinear alternating map and satisfies Leibniz rule in each argument with respect to usual product of functions. Then by the above remark, $\mathcal{M}^\nu$ arises from an $(n-1)$-multisection $\tilde{\mathcal{M}}^\nu_\Pi \in \Gamma(\Lambda^{n-1} A)$, so that

$$\mathcal{M}^\nu(f_1, \ldots, f_{n-1}) = \langle \tilde{\mathcal{M}}^\nu_\Pi, d_Af_1 \wedge \cdots \wedge d_Af_{n-1} \rangle.$$

3.16. Lemma. For all $\alpha \in \Gamma(\Lambda^{n-1} A)$,

$$\mathcal{L}_\Pi \alpha^\nu = (t_\alpha \tilde{\mathcal{M}}^\nu_\Pi + (-1)^{n-1} t_{d_A}\alpha \Pi)^\nu$$

Proof. We will prove the equation for $\alpha$ of the form

$$f d_Af_1 \wedge \cdots \wedge d_Af_{n-1},$$

where $f, f_1, \ldots, f_{n-1} \in C^\infty(M)$. 
Consider
\[
\mathcal{L}_{\Pi\alpha
}^\nu = \mathcal{L}_{\Pi(f_{dA f_1 \wedge \cdots \wedge dA f_{n-1}})\nu} \\
= f \mathcal{L}_{\Pi(f_{dA f_1 \wedge \cdots \wedge dA f_{n-1})}\nu} + dA f \wedge \iota_{\Pi(f_{dA f_1 \wedge \cdots \wedge dA f_{n-1}})}\nu \\
= f \mathcal{M}^\nu(f_1, \ldots, f_{n-1})\nu + dA f \wedge \iota_{\Pi(f_{dA f_1 \wedge \cdots \wedge dA f_{n-1}})}\nu.
\]
So, we have
\[
(3.2) \quad \mathcal{L}_{\Pi\alpha
}^\nu = f \mathcal{M}^\nu(f_1, \ldots, f_{n-1})\nu + dA f \wedge \iota_{\Pi(f_{dA f_1 \wedge \cdots \wedge dA f_{n-1}})}\nu.
\]
Now,
\[
f \mathcal{M}^\nu(f_1, \ldots, f_{n-1}) = f(\widetilde{\mathcal{M}}_{\Pi
}^\nu, dA f_1 \wedge \cdots \wedge dA f_{n-1}) \\
= f_{dA f_1 \wedge \cdots \wedge dA f_{n-1}} \widetilde{\mathcal{M}}_{\Pi
}^\nu \\
= \iota_{dA f_1 \wedge \cdots \wedge dA f_{n-1}} \widetilde{\mathcal{M}}_{\Pi
}^\nu
\]
that is,
\[
(3.3) \quad f \mathcal{M}^\nu(f_1, \ldots, f_{n-1}) = \iota_{dA} \widetilde{\mathcal{M}}_{\Pi
}^\nu.
\]
Note that \(dA f \wedge \nu = 0\) and hence \(\iota_{\Pi(f_{dA f_1 \wedge \cdots \wedge dA f_{n-1}})}(dA f \wedge \nu) = 0\). Since
\[
\iota_X(\omega \wedge \tau) = \iota_X(\omega) \wedge \tau + (-1)^k \omega \wedge \iota_X(\tau), \quad \omega \in \Gamma(\Lambda^k A^*),
\]
we obtain
\[
dA f \wedge \iota_{\Pi(f_{dA f_1 \wedge \cdots \wedge dA f_{n-1}})}\nu = (\iota_{\Pi(f_{dA f_1 \wedge \cdots \wedge dA f_{n-1}})}dA f) \wedge \nu \\
= (\iota_{\Pi(f_{dA f_1 \wedge \cdots \wedge dA f_{n-1}})}dA f)\nu \\
= (\Pi(f_{dA f_1 \wedge \cdots \wedge dA f_{n-1}}), dA f)\nu \\
= (\iota(f_{dA f_1 \wedge \cdots \wedge dA f_{n-1}}) \Pi, dA f)\nu \\
= (dA f_1 \wedge \cdots \wedge dA f_{n-1} \wedge dA f)\nu \\
= (-1)^{n-1} \Pi(dA f \wedge dA f_1 \wedge \cdots \wedge dA f_{n-1})\nu \\
= (-1)^{n-1} \Pi(dA(f \wedge dA f_1 \wedge \cdots \wedge dA f_{n-1}))\nu \\
= (-1)^{n-1} \iota_{dA}(f \wedge dA f_1 \wedge \cdots \wedge dA f_{n-1})\nu.
\]
Hence, we get
\[
(3.4) \quad dA f \wedge \iota_{\Pi(f_{dA f_1 \wedge \cdots \wedge dA f_{n-1}})}\nu = (-1)^{n-1} \iota_{dA} \Pi\nu.
\]
Thus (3.1) follows from (3.2) – (3.4).

We need the following lemma.

3.17. Lemma. For all \(\alpha, \beta \in \Gamma(\Lambda^{n-1} A)\),
\[
\iota_{dA} \in \alpha, \beta \cdot \Pi = ((\alpha \circ \Pi)(\alpha))\iota_{dA} \beta \Pi - ((\alpha \circ \Pi)(\beta))\iota_{dA} \alpha \Pi.
\]
Proof. Let \( \Lambda \) be the Nambu-Poisson structure on \( M \) induced by \( \Pi \). Then, by Lemma 3.14 we have

\[
(t_{d_A \ll \alpha, \beta \gg} \Pi) \Lambda = (-1)^n \mathcal{L}_{(\ll \alpha, \beta \gg)}(\ll \alpha, \beta \gg) \Lambda
\]

\[
= (-1)^n \mathcal{L}_{(\ll \alpha \Pi)(\ll \alpha, \beta \gg) \Pi} \Lambda
\]

\[
= (-1)^n \mathcal{L}_{(\ll \alpha \Pi)(\ll \alpha, \beta \gg) \Pi} \Lambda - (-1)^n \mathcal{L}_{(\ll \alpha \Pi)(\ll \alpha, \beta \gg) \Pi} \Lambda
\]

\[
= (-1)^n \mathcal{L}_{(\ll \alpha \Pi)(\ll \alpha, \beta \gg) \Pi} ((-1)^n(t_{d_A \beta} \Pi) \Lambda) - (-1)^n \mathcal{L}_{(\ll \alpha \Pi)(\ll \alpha, \beta \gg) \Pi} ((-1)^n(t_{d_A \alpha} \Pi) \Lambda)
\]

\[
= \mathcal{L}_{(\ll \alpha \Pi)(\ll \alpha, \beta \gg) \Pi} ((t_{d_A \beta} \Pi) \Lambda) - \mathcal{L}_{(\ll \alpha \Pi)(\ll \alpha, \beta \gg) \Pi} ((t_{d_A \alpha} \Pi) \Lambda)
\]

\[
= (t_{d_A \beta} \Pi) ((t_{d_A \beta} \Pi) \Lambda) + ((a \circ \Pi)(\alpha))(t_{d_A \beta} \Pi) \Lambda
\]

\[
- (t_{d_A \alpha} \Pi) ((t_{d_A \beta} \Pi) \Lambda) - ((a \circ \Pi)(\beta))(t_{d_A \alpha} \Pi) \Lambda
\]

\[
= ((a \circ \Pi)(\alpha))(t_{d_A \beta} \Pi) \Lambda - ((a \circ \Pi)(\beta))(t_{d_A \alpha} \Pi) \Lambda.
\]

So, we conclude that

\[
t_{d_A \ll \alpha, \beta \gg} \Pi = ((a \circ \Pi)(\alpha))(t_{d_A \beta} \Pi) - ((a \circ \Pi)(\beta))(t_{d_A \alpha} \Pi).
\]

\[
(\ll \alpha, \beta \gg) \Pi
\]

3.18. Theorem. (1) The mapping \( \mathcal{M}^\nu_{\Pi} : \Gamma(\Lambda^{n-1} A) \longrightarrow C^\infty(M) \) given by \( \alpha \mapsto t_{\alpha} \widetilde{\mathcal{M}}^\nu_{\Pi} \) defines a 1-cocycle in the cohomology complex associated to the Leibniz algebroid

\[
(\mathcal{A} = \Lambda^{n-1} A^*, \ll \gg, a \circ \Pi_{n-1}).
\]

(2) The cohomology class \( \mathcal{M}_{\Pi} = [\mathcal{M}^\nu_{\Pi}] \in \mathcal{H}L^1(\mathcal{A}) \) does not depend on the chosen orientation form.

Proof. (1) We need to prove that for all \( \alpha, \beta \in \Gamma(\mathcal{A}) \), \( d_A \mathcal{M}^\nu_{\Pi}(\alpha, \beta) = 0 \).

Recall that for all \( \alpha, \beta \in \Gamma(\mathcal{A}) \),

\[
(d_A \mathcal{M}^\nu_{\Pi})(\alpha, \beta) = (a \circ \Pi)(\alpha)\mathcal{M}^\nu_{\Pi}(\beta) - (a \circ \Pi)(\beta)\mathcal{M}^\nu_{\Pi}(\alpha) - \mathcal{M}^\nu_{\Pi} \ll \alpha, \beta \gg
\]

\[
= (a \circ \Pi)(\alpha)\mathcal{M}^\nu_{\Pi} - (a \circ \Pi)(\beta)\mathcal{M}^\nu_{\Pi} - (a \circ \Pi)(\beta)\mathcal{M}^\nu_{\Pi} + (a \circ \Pi)(\alpha)\mathcal{M}^\nu_{\Pi}.
\]

So it is enough to prove that

\[
(t_{\ll \alpha, \beta \gg} \Pi) \mathcal{M}^\nu_{\Pi} = (a \circ \Pi)(\alpha)\mathcal{M}^\nu_{\Pi} - (a \circ \Pi)(\beta)\mathcal{M}^\nu_{\Pi}.
\]
By Lemma 3.16, Corollary 3.12 and Cartan formula, we get
\[
(i_{\ll a, b \gg}, \tilde{M}^\nu_{\Pi}) \nu = \mathcal{L}_{\Pi} < a, b \gg, \nu + (-1)^n (i_{d_a \ll a, b \gg} \Pi) \nu \\
= \mathcal{L}_{[\Pi a, \Pi b] \Pi} ^\nu \nu + (-1)^n (i_{d_a \ll a, b \gg} \Pi) \nu \\
= \mathcal{L}_{\Pi a} \mathcal{L}_{\Pi b} ^\nu \nu - \mathcal{L}_{\Pi b} \mathcal{L}_{\Pi a} ^\nu \nu + (-1)^n (i_{d_a \ll a, b \gg} \Pi) \nu \\
= \mathcal{L}_{\Pi a} ((i_b \tilde{M}^\nu_{\Pi}) + (-1)^n i_{d_a \Pi} \Pi) \nu \\
- \mathcal{L}_{\Pi b} ((i_a \tilde{M}^\nu_{\Pi}) + (-1)^n i_{d_a \Pi} \Pi) \nu \\
+ (-1)^n (i_{d_a \ll a, b \gg} \Pi) \nu.
\]

The last equality follows from Equation (3.1).

Simplifying further using Equation (3.5), we have
\[
(i_{\ll a, b \gg}, \tilde{M}^\nu_{\Pi}) \nu = (i_b \tilde{M}^\nu_{\Pi} + (-1)^n i_{d_a \Pi} \Pi) \mathcal{L}_{\Pi a} \nu + ((a \circ \Pi)/(a)) ((\Pi_{b, \Pi} + (-1)^n i_{d_a \Pi} \Pi) \nu \\
- (i_a \tilde{M}^\nu_{\Pi} + (-1)^n i_{d_a \Pi} \Pi) \mathcal{L}_{\Pi b} \nu - ((a \circ \Pi)/(\beta)) (i_a \tilde{M}^\nu_{\Pi} + (-1)^n i_{d_a \Pi} \Pi) \nu \\
+ (-1)^n (((a \circ \Pi)/(\alpha)) i_{d a} \Pi) \nu - (((a \circ \Pi)/(\beta)) i_{d a} \Pi) \nu.
\]

After rearranging and canceling and using Equation (3.1), we have
\[
(i_{\ll a, b \gg}, \tilde{M}^\nu_{\Pi}) \nu = (i_b \tilde{M}^\nu_{\Pi} + (-1)^n i_{d_a \Pi} \Pi) (i_a \tilde{M}^\nu_{\Pi} + (-1)^n i_{d_a \Pi} \Pi) \nu - ((a \circ \Pi/(\beta)) i_{d a} \Pi) \nu.
\]

Therefore,
\[
(i_{\ll a, b \gg}, \tilde{M}^\nu_{\Pi}) \nu = (((a \circ \Pi)/(\alpha)) i_{d b} \Pi) \nu - (((a \circ \Pi)/(\beta)) i_{d a} \Pi) \nu.
\]

This proves the first part.

(2) Let \(\nu'\) be another orientation form representing the given orientation of \(A\). Then there exists \(f \in C^\infty(M)\), \(f > 0\) everywhere such that \(\nu' = f \nu\). Let \(\mathcal{M}^\nu_{\Pi} = [\mathcal{M}^\nu_{\Pi}]\). We need to prove that \(\mathcal{M}^\nu_{\Pi} = \mathcal{M}_{\Pi}\).

We have
\[
\mathcal{L}_{\Pi a} \nu' = (i_a \tilde{M}^\nu_{\Pi} + (-1)^n i_{d_a \Pi} \Pi) \nu' = f (i_a \tilde{M}^\nu_{\Pi} + (-1)^n i_{d_a \Pi} \Pi) \nu.
\]

On the other hand,
\[
\mathcal{L}_{\Pi a} \nu' = \mathcal{L}_{\Pi a} f \nu = f \mathcal{L}_{\Pi a} \nu + ((a \circ \Pi)/(\alpha) f) \nu = f (i_a \tilde{M}^\nu_{\Pi} + (-1)^n i_{d_a \Pi} \Pi) \nu + ((a \circ \Pi)/(\alpha) f) \nu
\]

Hence, we have
\[
f (i_a \tilde{M}^\nu_{\Pi}) \nu = f (i_a \tilde{M}^\nu_{\Pi}) \nu + ((a \circ \Pi)/(\alpha) f) \nu.
\]

Thus we obtain,
\[
i_a \tilde{M}^\nu_{\Pi} = i_a \tilde{M}^\nu_{\Pi} + \frac{1}{f} ((a \circ \Pi)/(\alpha) f).
\]
Notice that \( \frac{1}{\pi}((a \circ \Pi)(\alpha)f) = ((a \circ \Pi)(\alpha))(ln(f)) = d_A(ln(f))(\alpha). \) Hence we have,

\[
\mathcal{M}^{\prime}_\Pi = \mathcal{M}_\Pi + d_A(ln(u)).
\]

This implies \( \mathcal{M}^{\prime}_\Pi = \mathcal{M}_\Pi. \)

3.19. **Definition.** Let \( (A, [\cdot, \cdot]_A, a) \) be a Lie algebroid over a smooth manifold \( M \) equipped with a Nambu structure \( \Pi \) of order \( n \geq 3 \). Let \( A \) be orientable as a vector bundle and \( \Gamma(A^*) \) be locally spanned by elements of the form \( d_A f, f \in C^\infty(M) \). Then the cohomology class \( \mathcal{M}_\Pi = [\mathcal{M}^{\prime}_\Pi] \in H^1(A) \) as defined above is called the *the modular class* of \( A \).

4. A DUALITY THEOREM

Let \( (A, [\cdot, \cdot]_A, a) \) be a Lie algebroid over a smooth manifold \( M \) equipped with a regular Nambu structure \( \Pi \) of order \( n, 3 \leq n \leq m, m \) being the rank of the vector bundle \( A \). In this section, we first define Nambu cohomology modules of \( A \) and give an equivalent formulation of this cohomology which is a variant of foliated cohomology \cite{12, 13} in the present context.

Then we define oriented Nambu homology of \( A \), when \( A \) is oriented as a vector bundle and prove a duality theorem connecting oriented Nambu homology modules and Nambu cohomology modules. Throughout, in this section, we assume that \( \Gamma(A^*) \) is locally spanned by elements of the form \( d_A f, f \in C^\infty(M) \).

Recall from Lemma 3.8 that we have a subbundle \( D = \bigcup_{x \in M} D_x \) of \( A \) where \( D_x = \Pi_{n-1}(\Lambda^{n-1}A^*_x) \). Therefore, the vector bundle morphism \( \Pi_{n-1}: \Lambda^{n-1}A^* \to A \) (we also denote by \( \Pi_{n-1} \) the corresponding \( C^\infty(M) \)-linear map \( \Gamma(\Lambda^{n-1}A^*) \to \Gamma(A) \)) induces a vector bundle isomorphism

\[
\Pi_{n-1}: \frac{\Lambda^{n-1}A^*}{\ker \Pi_{n-1}} \to \Pi_{n-1}(\Lambda^{n-1}A^*).
\]

Note that \( \Gamma\left(\frac{\Lambda^{n-1}A^*}{\ker \Pi_{n-1}}\right) \) can be identified with \( \Gamma(\Lambda^{n-1}A^*) \) and similarly, the space of smooth sections of \( \Pi_{n-1}(\Lambda^{n-1}A^*) \) can be identified with \( \Pi_{n-1}(\Gamma(\Lambda^{n-1}A^*)) \). With the above identifications, the \( C^\infty(M) \)-linear map

\[
\Pi_{n-1}: \frac{\Gamma(\Lambda^{n-1}A^*)}{\ker \Pi_{n-1}} \to \Pi_{n-1}(\Gamma(\Lambda^{n-1}A^*))
\]

may be described simply by \( \Pi_{n-1}([a]) = \Pi_{n-1}(a) \) for \( [a] \in \frac{\Gamma(\Lambda^{n-1}A^*)}{\ker \Pi_{n-1}} \).

Thus, we have an associated Lie algebroid

\[
\left( \frac{\Lambda^{n-1}A^*}{\ker \Pi_{n-1}}, \langle \cdot, \cdot \rangle; a \circ \Pi_{n-1} \right)
\]

where the Lie bracket on the space of smooth sections and the anchor are given respectively as follows.

\[
\langle [\alpha], [\beta] \rangle := [\langle \alpha, \beta \rangle] = a \circ \Pi_{n-1}([\alpha]) : \Gamma(\Lambda^{n-1}A^*) \to \Gamma(\Lambda^{n-1}A^*)
\]

for all \( \alpha, \beta \in \frac{\Gamma(\Lambda^{n-1}A^*)}{\ker \Pi_{n-1}} \). Moreover, note that \( D \) is a Lie subalgebroid of \( A \) and

\[
\Pi_{n-1}: \frac{\Lambda^{n-1}A^*}{\ker \Pi_{n-1}} \to \Pi_{n-1}(\Lambda^{n-1}A^*)
\]
is a Lie algebroid isomorphism.

Consider the trivial representation of the Lie algebroid \((\Lambda^{n-1}A^*, \{, \}, a \circ \Pi_{n-1})\) given by
\[
\begin{align*}
\frac{\Gamma(\Lambda^{n-1}A^*)}{\ker \Pi_{n-1}} \times C^\infty(M) &\rightarrow C^\infty(M), \quad ([\alpha], f) \mapsto (a \circ \Pi_{n-1}(\alpha))f.
\end{align*}
\]

4.1. Definition. Nambu cohomology of the Lie algebroid \((A, [,], a)\) is by definition the Lie algebroid cohomology of the associated Lie algebroid
\[
(\frac{\Lambda^{n-1}A^*}{\ker \Pi_{n-1}}, \{, \}, a \circ \Pi_{n-1})
\]
with trivial representation and will be denoted by \(H^*_N(A)\).

Next we give an equivalent formulation Nambu cohomology. We define a cochain complex as follows.

Let \((\Gamma(\Lambda^kA^*), d_A)\) be the cochain complex defining the Lie algebroid cohomology of \(A\). Set
\[
\Omega^k(A, \Pi) = \{\alpha \in \Gamma(\Lambda^kA^*)|\alpha(X_1, \ldots, X_k) = 0, \quad \text{for all } X_i \in \Pi_{n-1}(\Gamma(\Lambda^{n-1}A))\}.
\]
Suppose \(\alpha \in \Omega^k(A, \Pi)\), then it is straightforward to verify using Corollary 3.12 that \(d_A \alpha \in \Omega^{k+1}(A, \Pi)\). Define
\[
\Omega^k_\Pi(A) := \Gamma(\Lambda^kA^*)/\Omega^k(A, \Pi).
\]

Then the coboundary operator \(d_A\) induces a square zero operator
\[
\tilde{d}_A : \Omega^k_\Pi(A) \rightarrow \Omega^{k+1}_\Pi(A)
\]
where \(\tilde{d}_A([\alpha]) = [d_A \alpha]\) for all \([\alpha] \in \Omega^k_\Pi(A)\). We shall denote the corresponding cohomology by \(H^*_\Pi(A)\).

Define a map \(\Gamma(\Lambda^kA^*) \rightarrow \Gamma(\Lambda^kD^*)\) given by
\[
i^k(\alpha)(X_1, \ldots, X_k) = \alpha(X_1, \ldots, X_k), \quad \text{for } \alpha \in \Gamma(\Lambda^kA^*), \ X_i \in \Gamma(D).
\]
Note that this map is surjective, because, we may assume that the vector bundle \(A\) is Euclidean and hence every sub bundle admits a complement. Therefore, for every \(f \in \Gamma(\Lambda^kD^*)\), we may find \(\tilde{f} \in \Gamma(\Lambda^kA^*)\) with the property \(i^k(\tilde{f}) = f\). It is also clear from the definition that the kernel of \(i^k\) is \(\Omega^k(A, \Pi)\). Therefore, for each \(k\), we have a \(C^\infty(M)\)-linear isomorphism
\[
\pi^k : \Omega^k_\Pi(A) \rightarrow \Gamma(\Lambda^kD^*).
\]
Moreover, we have \(\pi^k \circ \tilde{d}_A = d_D \circ \pi^k\), because, for any sections \(X_0, X_1, \ldots, X_k \in \Gamma(D)\)
and \(\alpha \in \Gamma(\Lambda^kA^*), \ d_D \circ \pi^k([\alpha])(X_0, \ldots, X_k)
\]
\[
= \sum_{i=0}^{k} (-1)^i a(X_i) \pi^k([\alpha])(X_0, \ldots, \widehat{X_i}, \ldots, X_k)
+ \sum_{i<j} (-1)^{i+j} \pi^k([\alpha])(X_i, X_j)_A, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_k)
\]
\[
= \sum_{i=0}^{k} (-1)^i a(X_i) \alpha(X_0, \ldots, \widehat{X_i}, \ldots, X_k)
+ \sum_{i<j} (-1)^{i+j} \alpha([X_i, X_j]_A, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_k)
\]
\[
= (d_A \alpha)(X_0, \ldots, X_k)
\]
Thus we have the following result.

4.2. Proposition. Let \((A, [,]_A, a)\) be a Lie algebroid with a regular Nambu structure \(\Pi\) of order \(n\) with \(n \geq 3\). Assume that \(\Gamma A^*\) is locally spanned by elements of the form \(d_A f\) where \(f \in \mathcal{C}^\infty(M)\). Then the \(\mathcal{C}^\infty(M)\)-linear homomorphisms

\[i^k(\alpha)(X_1, \ldots, X_k) = \alpha(X_1, \ldots, X_k), \quad \text{for} \ \alpha \in \Gamma(\Lambda^k A^*), \ X_i \in \Gamma(D)\]

induce an isomorphism of complexes \(\pi^*: (\Omega^\bullet \Pi(A), d_A) \rightarrow (\Gamma(\Lambda^\bullet D^*), d_D)\). Thus,

\[\mathcal{H}^*_{\Pi}(A) \cong \mathcal{H}^*_{\Pi}(A).\]

The following result follows from the definition of regularity of Nambu structure (see Proposition 4.2., [6] for details).

4.3. Proposition. Let \((A, [,]_A, a)\) be a Lie algebroid over a smooth manifold \(M\). Let \(\Pi\) be a regular Nambu structure on \(A\) of order \(n\). Then

\[\Omega^k(A, \Pi) = \ker \Pi_k, \quad \text{for all} \ k = 1, \ldots, n.\]

Next we define the notion of canonical Nambu homology of a Lie algebroid which is oriented as a vector bundle and equipped with a Nambu structure. Let \(A\) be an oriented Lie algebroid with rank \(A = m\). Let \(\nu\) be the chosen orientation form on \(A\). Then we have a vector bundle isomorphism \(b_\nu: \Lambda^k A \rightarrow \Lambda^{m-k} A^*\) given by

\[b_\nu(P) = \iota_P \nu \quad \text{for all} \ P \in \Gamma(\Lambda^k A).\]

Define an operator \(\delta_\nu\) as

\[\delta_\nu := b_\nu^{-1} \circ d_A \circ b_\nu: \Gamma(\Lambda^k A) \rightarrow \Gamma(\Lambda^{k-1} A)\]

Clearly, \(\delta_\nu^2 = 0\). The homology of the complex \((\Gamma(\Lambda^\bullet A), \delta_\nu)\) is denoted by \(\mathcal{H}^\nu_*(A)\).

Notice that the cohomology \(\mathcal{H}^\nu_*(A)\) is dual of the Lie algebroid cohomology of \(A\) with trivial coefficients. So, \(\mathcal{H}^\nu_*(A)\) does not depend on the choice of orientation form on \(A\). Note that for \(X \in \Gamma A\), we have \(\mathcal{L}_X \nu = (\delta_\nu(X)) \nu\). This is because,

\[\delta_\nu(X) = (b_\nu^{-1} \circ d_A \circ b_\nu)(X) = b_\nu^{-1} \circ d_A \circ \iota_X \nu,\]

and since we have \(\mathcal{L}_X = d_A \circ \iota_X + \iota_X \circ d_A\), it follows that \(\delta_\nu(X) = b_\nu^{-1} \circ \mathcal{L}_X \nu\). Therefore,

\[\mathcal{L}_X \nu = \nu (\delta_\nu(X)) = (\delta_\nu(X)) \nu = (\delta_\nu(X)) \nu.\]

In view of this, we may write

\[\delta_\nu(X) = \text{div}_\nu(X) \quad \text{for all} \ X \in \Gamma A.\]

The proofs of the following two results are analogous to the proofs of Lemma 5.1 and Proposition 5.2., [6] and hence we omit the details.
4.4. **Lemma.** Let $A$ be an oriented Lie algebroid of rank $m$ and $\nu$ be the chosen orientation form on $A$. Then for all $P \in \Gamma(\Lambda^k A)$ and $X \in \Gamma A$, we have
\[ L_X \flat_{\nu}(P) = \flat_{\nu}(L_X P) + \delta_{\nu}(X) \flat_{\nu}(P) \]

4.5. **Proposition.** Let $A$ be an oriented Lie algebroid of rank $m$ and $\nu$ be the chosen orientation form on $A$. Then
\[ \iota_{\alpha}(\delta_{\nu}(P)) = \delta_{\nu}(\iota_{\alpha}P) + (-1)^k \iota_{d\alpha} \alpha P \text{ for all } P \in \Gamma(\Lambda^k A), \alpha \in \Gamma(\Lambda^{k-1} A^*) \]

We define a subcomplex of $(\Gamma(\Lambda^* A), \delta_{\nu})$ as follows.

Let $(A, [\cdot, \cdot], a)$ be a Lie algebroid of rank $m$ over a smooth manifold $M$. Let $\Pi$ be a regular Nambu structure on $A$ of order $n$, $n \leq m$. Then consider the subset of $\Gamma(\Lambda^k A)$ given by
\[ V^k(A, \Pi) = \{ P \in \Gamma(\Lambda^k A) | \iota_{\alpha} P = 0 \text{ for all } \alpha \in \Gamma(A^*), \alpha \in \ker \Pi \} \]

Set $V^0(A, \Pi) = C^\infty(M)$. Note that by Remark 3.9, $V^k(A, \Pi)$ is just the space of all $k$-multisections of $A$ which are orthogonal to $\mathcal{D}$. As a consequence, we have

4.6. **Lemma.** Let $(A, [\cdot, \cdot], a)$ be a Lie algebroid over $M$ with regular Nambu structure $\Pi$ of order $n$. Then
\[ V^k(A, \Pi) = \Pi_{n-k}(\Gamma(\Lambda^{n-k} A^*)) \text{ for all } k = 1, \ldots, n. \]

Now proceeding as in the proof of Proposition 5.5, [6] and using Proposition (4.3) and Proposition (4.5), we obtain

4.7. **Proposition.** Let $(A, [\cdot, \cdot], a)$ be a Lie algebroid over a smooth manifold $M$ with a regular Nambu structure $\Pi$ of order $n$. Let $A$ be oriented as a vector bundle and $\nu$ be the chosen orientation form on $A$ representing the orientation. Then
\[ \delta_{\nu}(V^k(A, \Pi)) \subseteq V^{k-1}(A, \Pi) \text{ for all } k = 1, \ldots, n. \]

Therefore, $(V^*(A, \Pi), \delta_{\nu})$ is a subcomplex of $(\Gamma(\Lambda^* A), \delta_{\nu})$ and the canonical Nambu homology modules of $A$, denoted by $\mathcal{H}^{\text{canN}}_*(A)$ are by definition the homology modules of this subcomplex.

4.8. **Remark.** Note that the canonical Nambu homology of $A$ does not depend on the choice of an orientation form. To see this, let $\nu$ and $\nu'$ be two orientation form on $A$ representing the orientation. Then there exists a nowhere zero function $f \in C^\infty(M)$ such that $\nu' = f \nu$. Define an isomorphism of $C^\infty(M)$ modules
\[ \Phi^k : V^k(A, \Pi) \longrightarrow V^k(A, \Pi) \]
\[ P \mapsto \frac{1}{f} P \text{ for all } k = 0, \ldots, n. \]

Then using definition of $\delta_{\nu}$ and $\delta_{\nu'}$, it is straightforward to prove that
\[ \delta_{\nu'} \circ \Phi^k = \Phi^{k-1} \circ \delta_{\nu} \]

Therefore, the mapping $\Phi^k$ induces an isomorphism between associated canonical Nambu homologies.
Let $A$ be a Lie algebroid equipped with a regular Nambu structure $\Pi \in \Gamma(\Lambda^nA)$ of order $n$, $n \leq m = \text{rank } A$. Assume that $A$ is oriented with $\nu \in \Gamma(\Lambda^nA^\ast)$ being a chosen orientation form representing the orientation on $A$. Then we have the following results.

4.9. **Proposition.** The graded vector space $\Pi_\ast(\Gamma(\Lambda^k A^\ast)) = \oplus_{k=0}^n \Pi_k(\Gamma(\Lambda^k A^\ast))$ defines a subcomplex of the chain complex $(\Gamma(\Lambda^\ast A), \delta_\nu)$ if and only if $\tilde{\mathcal{M}}_\nu^k \subseteq \Pi_1(\Gamma(A^\ast))$.

**Proof.** Let $\alpha \in \Gamma(\Lambda^{n-k-1}A^\ast)$ and $\beta \in \Gamma(\Lambda^k A^\ast)$ be arbitrary. Then, by Lemma (4.3)

$$\iota_\alpha \delta_\nu(\Pi_k \beta) = \delta_\nu(\iota_\alpha \Pi_k(\beta)) + (-1)^{n-k} \iota d_A \iota(\Pi_k(\beta))$$

$$= \delta_\nu(\iota_\alpha \iota \beta \nu) + (-1)^{n-k} \iota d_A \iota(\iota_\beta \nu)$$

$$= \delta_\nu(\Pi_{n-1}(\beta \wedge \alpha)) + (-1)^{n-k} \Pi_n(\beta \wedge d_A \alpha)$$

$$= \iota_\alpha(\iota_\beta \tilde{\mathcal{M}}_\nu^k + (-1)^{n-1} \Pi_{k+1}(d_A \beta))$$

Using injectivity of contraction map, we have

$$\delta_\nu(\Pi_k \beta) = \iota_\beta \tilde{\mathcal{M}}_\nu^k + (-1)^{n-1} \Pi_{k+1}(d_A \beta).$$

Hence, we conclude that $\delta_\nu(\Pi_k(\Gamma(\Lambda^k A^\ast))) \subseteq \Pi_{k+1}(\Gamma(\Lambda^{k+1} A^\ast))$ for all $k = 1, \ldots, n$ if and only if $\tilde{\mathcal{M}}_\nu^k \subseteq \Pi_1(\Gamma(A^\ast))$. \qed

4.10. **Proposition.** If $\Pi_\ast(\Gamma(\Lambda^\ast A^\ast))$ is a subcomplex of $(\Gamma(\Lambda^\ast A), \delta_\nu)$, then the homology of this complex is independent of the choice of the orientation form.

**Proof.** Let $\nu'$ be another choice of orientation form. Then, there exists a nowhere vanishing real valued function $f \in C^\infty(M)$ so that $\nu' = f \nu$. Without loss of generality, we may assume that $f > 0$. Then,

$$\Psi^k: \Pi_k(\Omega^k(A)) \longrightarrow \Pi_k(\Omega^k(A))$$

defined by $P \mapsto \frac{1}{f} P$ is an isomorphism.

Since, $\delta_\nu' \circ \Psi^k = \Psi^{k-1} \circ \delta_\nu$, the result follows. \qed

4.11. **Proposition.** If the modular class of $A$ is null, then the graded vector space $\Pi_\ast(\Gamma(\Lambda^\ast A^\ast))$ is a subcomplex of $(\Gamma(\Lambda^\ast A), \delta_\nu)$ and

$$\bar{\mathcal{H}}_k^\text{can}(A) \cong \mathcal{H}_k^{n-k}(A)$$

for all $k = 0, \ldots, n$.

where $\bar{\mathcal{H}}_k^\text{can}(A)$ denotes the $k$-th homology of the subcomplex $\Pi_\ast(\Gamma(\Lambda^\ast A^\ast))$.

**Proof.** Suppose the modular class of $A$ is null, then there exists $f \in C^\infty(M)$ such that $d_A f = \mathcal{M}_\nu^k$. Therefore, for all $\alpha \in \Gamma(\Lambda^{n-k}A^\ast)$,

$$\iota_\alpha \tilde{\mathcal{M}}_\nu^k = \mathcal{M}_\nu^k(\alpha) = (d_A f) \alpha = ((a \circ \Pi_{n-1})(\alpha)) f = (a(\iota_\alpha \Pi)) f = (d_A f)(\iota_\alpha \Pi)$$

$$= \iota d_A f(\iota_\alpha \Pi) = \iota_\alpha((-1)^{n-1} d_A f \Pi) = \iota_\alpha((-1)^{n-1} \Pi_1(d_A f)).$$

Thus,

$$\tilde{\mathcal{M}}_\nu^k = \Pi_1((-1)^{n-1} d_A f).$$
Proposition (4.9) now implies that \( \Pi^\bullet(\Gamma(\Lambda^\bullet A^*)) \) is a subcomplex of \((\Gamma(\Lambda^\bullet A), \delta_\nu)\). Next using Proposition (4.3), we define isomorphisms of \( C^\infty(M) \)-modules

\[
h_k: \Omega^{n-k}_\Pi(A) \rightarrow \Pi^{n-k}(\Gamma(\Lambda^{n-k}A^*)), \quad h_k([\alpha]) = e^{-f} \Pi^{n-k}(\alpha)
\]
such that \( h_k \circ \tilde{d}_A = (-1)^{n-1} \delta_\nu \circ h_{k+1} \). Required isomorphism in cohomologies is induced by the isomorphisms \( h_k \). \( \square \)

Finally using the above results and Lemma (4.6) we obtain the following theorem.

**4.12. Theorem.** If the modular class of \( A \) is null then

\[
\mathcal{H}^k_N(A) \cong \mathcal{H}^k_\Pi(A) \cong \mathcal{H}^{canN}_{n-k}(A) \text{ for all } k = 0, \ldots, n.
\]

### 5. Density and the Modular Class

To define the modular class of a Lie algebroid \( A \) equipped with a regular Nambu structure and to prove the duality theorem connecting Nambu cohomology and canonical nambu homology as developed in the previous sections we assumed that \( A \) is orientable as a vector bundle. The aim of this last section is show that we can do away with the orientability assumption using the notion of density.

Recall that orientation bundle \( O(A) \) of a smooth vector bundle \( A \) over a smooth manifold \( M \) is defined as follows [3].

**5.1. Definition.** Let \( A \) be a smooth vector bundle over a smooth manifold \( M \). Let \( \{\{U_\alpha, \phi_\alpha\}\} \) be a smooth atlas and \( h_{\alpha\beta} \) be the corresponding transition functions for \( A \). The orientation bundle of \( A \), denoted by \( O(A) \), is the real line bundle over \( M \) with transition functions \( \text{sgn}(J(h_{\alpha\beta})) \), where \( \text{sgn}: \mathbb{R} \rightarrow \mathbb{R} \) is given by

\[
\text{sgn}(x) = \begin{cases} 
1 & \text{for } x \geq 0, \\
0 & \text{for } x = 0, \\
-1 & \text{for } x \leq 0
\end{cases}
\]

and \( J(h_{\alpha\beta}) \) is the Jacobian determinant of the matrix of partial derivatives of \( h_{\alpha\beta} \).

**5.2. Definition.** The density bundle, denoted by \( D(A) \), of \( A \) is defined to be \( \Lambda^{top} A^* \otimes O(A) \). A section of \( D(A) \) is called a density on \( A \).

The following is an alternative way to introduce density bundle.

**5.3. Definition.** Let \( V \) be an \( n \)-dimensional vector space. A density function of \( V \) is a function

\[
\mu: V \times \cdots \times V \rightarrow \mathbb{R}
\]
satisfying the following condition: If \( T: V \rightarrow V \) is any linear map then

\[
\mu(T(X_1), \ldots, T(X_n)) = |\det(T)| \mu(X_1, \ldots, X_n), \quad X_i \in V
\]

Let \( D(V) \) denote set of all densities on \( V \).
5.4. **Definition.** A *positive* density on $V$ is a density $\mu$ such that $\mu(X_1, \ldots, X_n) > 0$, for all linearly independent vectors $X_1, \ldots, X_n \in V$.

Then with the above notations the density bundle $D(A)$ of $A$ may be described as

$$D(A) := \prod_{x \in M} D(A_x).$$

5.5. **Definition.** A smooth section of $\mu \in \Gamma(D(A))$ is called a *positive density* on $A$ if for all $x \in M$, $\mu_x$ is positive.

Using a partition unity argument one can show that there always exists a smooth positive density on $A$. More precisely, we have the following lemma.

5.6. **Lemma.** Let $A$ be a smooth vector bundle over a smooth manifold $M$. Then there exists a positive density on $A$.

**Proof.** Note that the set of positive elements of $D(A)$ is an open subset whose intersection with each fibre is convex. Then the usual partition unity argument allows us to piece together local positive densities to obtain a global one. □

Next we need to introduce *twisted Lie algebroid cochain complex*. The construction is similar to the construction of twisted de Rham complex of vector-valued differential forms on smooth manifolds and hence we mention it briefly (see [3] for details).

Let $A$ be a Lie algebroid and $E$ be a flat vector bundle on a smooth manifold $M$. Recall that a vector bundle $E$ is said to be flat if it admits an atlas $\{(U_\alpha, \phi_\alpha)\}$ relative to which the transition functions are locally constant. We may assume that the atlas $\{(U_\alpha, \phi_\alpha)\}$ on $A$ is induced from an atlas on $M$. Let $\alpha \in \Gamma(\Lambda^k A^* \otimes E)$ for all $k \in \mathbb{N}$. Then locally, on an open set $U, \alpha = \sum_i e_i \otimes f_i,$ where $e_i \in \Gamma(U, \Lambda^k A^*)$ and $f_i \in \Gamma(U, E)$ and tensor product is over $C^\infty(U)$ and for any vector bundle $E$, $\Gamma(U, E)$ denotes the space of smooth sections of $E$ on $U$. Define $d_A^1 : \Gamma(\Lambda^k A^* \otimes E) \rightarrow \Gamma(\Lambda^{k+1} A^* \otimes E)$ locally as

$$d_A^1 \left( \sum_i e_i \otimes f_i \right) = \sum_i d_A(e_i) \otimes f_i,$$

where $e_i \in \Gamma(U, \Lambda^k A^*)$ and $f_i \in \Gamma(U, E)$ and tensor product is over $C^\infty(U)$ for some open subset $U$ of $M$ and then extend $d_A^1$ globally using Leibniz rule and linearity to give a well defined map. Then $(\Gamma(\Lambda^* A^* \otimes E), d_A^1)$ gives the twisted Lie algebroid cochain complex of $A$. Next just like Definition [22], we have operators defined as follows.

5.7. **Definition.** Let $X \in \Gamma A$. Define $\mathcal{L}_X^1 : \Gamma(\Lambda^k A^* \otimes E) \rightarrow \Gamma(\Lambda^k A^* \otimes E)$ locally as

$$\mathcal{L}_X^1 \left( \sum_i e_i \otimes f_i \right) = \sum_i \mathcal{L}_X(e_i) \otimes f_i,$$

where $e_i \in \Gamma(U, \Lambda^k A^*)$ and $f_i \in \Gamma(U, E)$ and tensor product is over $C^\infty(U)$ for some open subset $U$ of $M$.

Similarly, for $X \in \Gamma(\Lambda^i A)$, we may define

$$\iota_X^1 : \Gamma(\Lambda^k A^* \otimes E) \rightarrow \Gamma(\Lambda^{k-i} A^* \otimes E).$$
5.8. Remark. Notice that if $E$ is the trivial line bundle, $d_A^j, L_X^j$ and $\iota_X$ are precisely $d_A, L_X$ and $\iota_X$, respectively. Also, note that the line bundle $O(A)$ is flat.

Let $(A, [\cdot, \cdot]_A, a)$ be a Lie algebroid over a smooth manifold $M$. Let $\omega_x \in \Lambda A_x$ and $\tau_x \otimes o_x \in \Lambda A_x \otimes O(A)_x, x \in M$. Define $\omega_x \wedge^j (\tau_x \otimes o_x) := (\omega_x \wedge \tau_x) \otimes o_x$. By extending it linearly, we have a well defined map

$$\wedge^j : \Gamma(\Lambda^A) \times \Gamma(\Lambda^j A^* \otimes O(A)) \longrightarrow \Gamma(\Lambda^{i+j} A^* \otimes O(A)).$$

The following lemma is easy to prove.

5.9. Lemma. Let $(A, [\cdot, \cdot]_A, a)$ be a Lie algebroid over a smooth manifold $M$. Let $X \in \Gamma(\Lambda^* A)$. Then for $\omega \in \Gamma(\Lambda^* A^*)$ and $\tau \otimes o \in \Gamma(\Lambda^* A^* \otimes O(A))$, we have

$$\iota_X (\omega \wedge^j (\tau \otimes o)) = \iota_X \omega \wedge^j (\tau \otimes o) + (-1)^{\deg(\omega)} \omega \wedge^j \iota_X (\tau \otimes o).$$

5.10. Definition. Let $A$ be a Lie algebroid over a smooth manifold $M$. Let $\mu$ be a fixed positive density on $A$. Then, for any $X \in \Gamma A, \text{div}_X X \in C^\infty(M)$ is defined by the equation

$$L_X^j \mu = (\text{div}_X X) \mu.$$

Let $(A, [\cdot, \cdot]_A, a)$ be a Lie algebroid over a smooth manifold $M$. Assume that $\Gamma(A^*)$ is locally spanned by elements of the form $d_A f, f \in C^\infty(M)$. Let $\Pi$ be a Nambu structure on $A$ order $n$. Fix a positive density $\mu$ on $A$. Consider the mapping

$$\mathcal{M}^\mu : C^\infty(M) \times \cdots \times C^\infty(M) \longrightarrow C^\infty(M)$$

given by

$$L_{\Pi}(d_A f_1 \wedge \cdots \wedge d_A f_{n-1}) \mu = \mathcal{M}^\mu(f_1, \ldots, f_{n-1}) \mu$$

where $f_1, \ldots, f_{n-1} \in C^\infty(M)$ for all $i = 1, \ldots, n - 1$. Then as before $\mathcal{M}^\mu$ defines an $(n - 1)$-multisection $\tilde{\mathcal{M}}^\mu_\Pi \in \Gamma(\Lambda^{n-1} A)$, so that

$$\mathcal{M}^\mu(f_1, \ldots, f_{n-1}) = (\tilde{\mathcal{M}}^\mu_\Pi, d_A f_1 \wedge \cdots \wedge d_A f_{n-1}).$$

Then as in Section 3, we have the following result.

5.11. Theorem. (1) The mapping $\mathcal{M}^\mu_\Pi : \Gamma(\Lambda^{n-1} A) \longrightarrow C^\infty(M)$ given by $\alpha \mapsto \iota_\alpha \tilde{\mathcal{M}}^\mu_\Pi$ defines a 1-cocycle in the cohomology complex associated to the Leibniz algebroid

$$(A = \Lambda^{n-1} A^*, \ll, \gg, a \circ \Pi_{n-1}).$$

(2) The cohomology class $\mathcal{M}^\mu_\Pi = [\mathcal{M}^\mu_\Pi] \in H^1 L(A)$ is independent of the choice of $\mu$.

5.12. Definition. The cohomology class $[\mathcal{M}^\mu_\Pi] \in H^1 L(A)$ as defined above is called the the modular class of $A$.

In order to prove a version of duality theorem in the present context, we need to have a version of canonical Nambu homology. To this end, we prove the following result.

5.13. Proposition. Let $A$ be a Lie algebroid with a regular Nambu structure $\Pi$ of order $n \leq m = \text{rank} A$ over a smooth manifold $M$. Let $\mu$ be a positive density on $A$. Let
\( b_\mu : \Gamma(\Lambda^k A) \to \Gamma(\Lambda^{m-k} A^* \otimes O(A)) \) be defined as \( b_\mu(P) = \iota^k_\mu \) for all \( P \in \Gamma(\Lambda^k A) \). Then \( b_\mu \) is an isomorphism.

**Proof.** For dimension reasons, it is enough to prove that \( b_\mu \) is injective. Let \( P \in \Gamma(\Lambda^k A) \) be non-zero. Then \( P(x) \neq 0 \) for some \( x \in M \). We know \( \mu(x) = X^*_1(x) \wedge \cdots \wedge X^*_m(x) \otimes \mu(x) \) where \( X^*_i(x) \in A^*_x \) and \( \mu(x) \in O(A)_x \) is non-zero. Write

\[
P(x) = \sum_{1 \leq i_1 < \cdots < i_k \leq m} \alpha_{i_1 \cdots i_k} X_{i_1}(x) \wedge \cdots \wedge X_{i_k}(x).
\]

Without loss of generality, we may assume that \( \alpha_{i_1 \cdots i_k} \neq 0 \) for all tuple \((i_1, \ldots, i_k)\) appearing in the expression above. Consider

\[
A_{i_1 \cdots i_k}(x) := X_1(x) \wedge \cdots \wedge X_{i_1}(x) \wedge \cdots \wedge X_{i_k}(x) \wedge \cdots \wedge X_m(x) \in \Lambda^{m-k} A_x.
\]

Then \( \iota^k_{P(x)} \mu(x)(A_{i_1 \cdots i_k}(x)) = (-1)^{a_{i_1 \cdots i_k}} \mu(x) \) for some \( a \in \mathbb{Z} \). Hence we conclude that \( b_\mu(P) \neq 0 \). In other words, \( b_\mu \) is injective. \( \square \)

Notice that \( b_\mu \) is \( C^\infty(M) \)-linear and hence yields a vector bundle isomorphism

\[
b_\mu : \Lambda^k A \to \Lambda^{m-k} A^* \otimes O(A).
\]

Now we use the above isomorphisms and proceed as in Section 4, to define an operator

\[
\delta_\mu := b^{-1}_\mu \circ d^*_A \circ b_\mu : \Gamma(\Lambda^k A) \to \Gamma(\Lambda^{k-1} A)
\]

to deduce a homology complex \((\Gamma^* A, \delta_\mu)\) from the twisted Lie algebroid cochain complex \((\Gamma(\Lambda^* A^* \otimes O(A)), d^*_A)\). Define

\[
\mathcal{V}^k_\Omega(A, \Pi) = \{ P \in \Gamma(\Lambda^k A) | \ i^k_\alpha P = 0 \ \text{for all} \ \alpha \in \Gamma(A^*), \alpha \in \ker \Pi_1 \}, \ k \geq 1
\]

Set \( \mathcal{V}^0_\Omega(A, \Pi) = C^\infty(M) \). Then one can verify that

\[
\delta_\mu(\mathcal{V}^k_\Omega(A, \Pi)) \subseteq \mathcal{V}^{k-1}_\Omega(A, \Pi) \quad \text{for all} \quad k,
\]

yielding a subcomplex \((\mathcal{V}^*_\Omega(A, \Pi), \delta_\mu)\) of \((\Gamma^* A, \delta_\mu)\). The canonical Nambu homology modules in the present context are defined by the above subcomplex.

We may now recover all the results in Section 4 and their proofs can be written verbatim with the tools as developed above and the duality theorem in the present context may be deduced in the following form.

**5.14. Theorem.** Let \((A, [\cdot], A, a)\) be a Lie algebroid over a smooth manifold \(M\) with a regular Nambu structure \(\Pi\) of order \(n\), \(3 \leq n \leq m = \text{rank } A\). Assume that \(\Gamma A^*\) is generated by elements of the form \(d_A f\) where \(f \in C^\infty(M)\). If the modular class of \(A\) is null then

\[
\mathcal{H}^k_N(A) \cong \mathcal{H}^k_{\Pi\Omega}(A) \cong \mathcal{H}^k_{\Pi_\Omega N}(A) \quad \text{for all} \quad k = 0, \ldots, n.
\]

**Acknowledgement:** Second author would like to thank Mahuya Datta for clearing some initial doubts while learning the subject.
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