CHSH is not supported with supermartingale statistics.

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Abstract

It is demonstrated that the supermartingale statistics approach of Gill to the CHSH contrast contains a physical (and statistical) unrealistic assumption.

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1 Introduction

The CHSH statistic arises in discussions of locality and causality with additional parameters in quantum mechanics. The argument starts with Bell’s correlation expression [1]

\[ E(a, b) = \int_{\lambda \in \Lambda} \rho_\lambda A_\lambda(a) B_\lambda(b) d\lambda \]  

(1)

Here \( \rho_\lambda \) is the probability density function of the hidden variables \( \lambda \in \Lambda \). The \( A_\lambda(a) \) and \( B_\lambda(b) \) are measurement functions that project in \{\(-1, 1\)\}. The \( a \) and \( b \) refer to the setting parameters. In CHSH studies \( a \) and \( b \) refer to labels in \{1, 2\} that, in turn, refer to unit parameter vectors. From (1) it can be derived that a contrast \( S = E(1, 1) - E(1, 2) - E(2, 1) - E(2, 2) \) exist such that \( |S| \leq 2 \). This is called the CHSH contrast.

In [2] and [3] it is argued that violation with CHSH with local hidden variables is impossible because the CHSH statistics follows a supermartingale structure. This comprises Gill’s formulation of Bell’s fifth position: “It will be impossible to simulate a CHSH violation using strict Einstein locality with a computer model”. Local hidden variables are additional parameters that in a local manner explain the correlation between distant particles.
In the present paper the supermartingale argument against local hidden causal-
ity will be found to be in conflict with observation.

2 Preliminary definitions counting and response probability

In the first place we follow Gill in his definition of a finite elementary statistical counting measure.

2.1 Counting measure

\[ \Delta_n(a, b) = 1\{(X_n, Y_n) = (x_n, y_n) : x_n = y_n\}1\{(A_n, B_n) = (a, b)\} \]  

(2)

The \((a, b)\) refer to the physical unit parameter vectors \(|a| = 1\) and \(|b| = 1\). Later the notation will be replaced with labels \((a, b)\). We see that \(\Delta_n(a, b) \in \{0, 1\}\). The pair \((x_n, y_n) \in \{-1, 1\}\) refer to the outcome of the measurements. \(X_n\) at Alice’s measuring instrument and \(Y_n\) at Bob’s. \(1\{(X_n, Y_n) = (x_n, y_n) : x_n = y_n\}\) is unity when \(x_n = y_n\) and zero otherwise. The uppercase letters indicate random variables. The lowercase letters refer to the value of random variables. The index \(n\) is the trial number. If there are e.g. \(N\) trials then e.g. the number of equal measurements at setting pair \((a, b)\) is \(N = (a, b) = \sum_{n=1}^{N} \Delta_n(a, b)\) \(\)  

(3)

From the definition of the \(1\{(A_n, B_n) = (a, b)\}\) we can then derive that \(N(a, b) = \sum_{n=1}^{N} 1\{(A_n, B_n) = (a, b)\}\). Hence, there are \(N^-(a, b) = N(a, b) - N^+(a, b)\) unequal, i.e. \(x_n \neq y_n\) measurements when 'no measurements are lost'. The product moment \(\hat{\rho}(a, b) = \hat{\rho}(a, b)\) is defined by

\[ \hat{\rho}(a, b) = \frac{N^-(a, b) - N^+(a, b)}{N(a, b)} = \frac{2N^-(a, b)}{N(a, b)} - 1 \]  

(4)

Here we will use 'labels' that refer to parameter vectors.

2.2 CHSH transform

Let us defined a CHSH weight based on \(2\) like \(C_{(1,1)}(\hat{\rho}(a, b))\) such that

\[ C_{(1,1)}(\hat{\rho}(a, b)) = \hat{\rho}(1, 1) - \hat{\rho}(1, 2) - \hat{\rho}(2, 1) - \hat{\rho}(2, 2) \]  

(5)

With this notation let us subsequently introduce \(\Delta_n = C_{(1,1)}(\Delta_n(a, b))\). We have \(\Delta_n \in \{-1, 0, 1\}\). For completeness, the outcome zero occurs when \((a, b)\)
has been selected at the \( n \)-th trial but \( x_n \neq y_n \). In a proper experiment the probability of this event will be low. Let us subsequently define the random variable \( Z_N \) as a sum of the \( \Delta_n \) for \( n = 1, 2, \ldots \)

\[
Z_N = \sum_{n=1}^{N} \Delta_n \tag{6}
\]

When \( N \) is the total number of trials and is large enough then we can safely assume that \( N(a, b) \approx N/4 \) then the following relation between \( Z_n \) and \( S \) can be obtained: \( Z_N \approx (S - 2)N/8 \). The aim of Gill’s argument is to show that \( Z_n \) finally decreases in \( n = 1, 2, \ldots, N \) and will likely be \( \leq 0 \) in the end.

### 2.3 Expectation value & probability

Looking at the expected behavior of \( Z_n \sim < 0 \) for increasing \( n \), Gill introduces the supermartingale assumption.

\[
E(Z_{n+1} \mid Z_1 = z_1, Z_2 = z_2, \ldots, Z_n = z_n) \leq z_n \tag{7}
\]

and tries to show with a Markov equation that the probability of \( \max(Z_n) > 0 \) decreases with increasing \( n \).

In the present argument of our paper we will be in need of conditional probabilities in a discrete probability space. Let us concentrate on the random variables \( \Delta_n \). A conditional density, equal to conditional probability in the discrete case, for the \( \Delta_n \) is

\[
f_{\Delta_{n+1} \mid \Delta_1, \ldots, \Delta_n}(\delta_{n+1} \mid \delta_1, \ldots, \delta_n) = \frac{f_{\Delta_1, \ldots, \Delta_{n+1}}(\delta_1, \ldots, \delta_n, \delta_{n+1})}{\sum_{\delta_{n+1} \in \{-1,0,1\}} f_{\Delta_1, \ldots, \Delta_{n+1}}(\delta_1, \ldots, \delta_n, \delta_{n+1})} \tag{8}
\]

In addition, we also will be in need to derive a density for \( \{\delta_1, \ldots, \delta_n\} \) from a density of \( \{\delta_1, \ldots, \delta_n, \delta_{n+1}\} \) with \[4\]

\[
f_{\Delta_1, \ldots, \Delta_n}(\delta_1, \ldots, \delta_n) = \sum_{\delta_{n+1} \in \{-1,0,1\}} f_{\Delta_1, \ldots, \Delta_{n+1}}(\delta_1, \ldots, \delta_n, \delta_{n+1}) \tag{9}
\]

We assume that the probabilities can be small but in all cases non-zero. This appears a reasonable assumption for this type of research.

### 3 Results and Discussion

These are the main results of the paper. From equation (6) the supermartingale condition in (7) can be transformed in terms of \( \Delta_n \), with \( n = 1, 2, \ldots, N \). In
terms of $\Delta$ the expectation can be rewritten like
\[
E(\Delta_{n+1} \mid \Delta_1 = \delta_1, \Delta_2 = \delta_2, \ldots, \Delta_n = \delta_n) = \sum_{\delta_{n+1} \in \{-1,0,1\}} \delta_{n+1} f_{\Delta_{n+1} \mid \Delta_1,\ldots,\Delta_n}(\delta_{n+1} \mid \delta_1, \ldots, \delta_n)
\] (10)

The following theorem can be easily proved. It shows the transformation of a $Z$ supermartingale into a $\Delta$ supermartingale.

**Theorem 3.1** The $Z$ form of the CHSH supermartingale can be transformed into a $\Delta$ form as.

\[
E(\Delta_{n+1} \mid \Delta_1 = \delta_1, \Delta_2 = \delta_2, \ldots, \Delta_n = \delta_n) \leq 0
\] (11)

This follows quite easily from equation (6) and equation (7) because the expectation is a linear function of a sum of random variables. Note: $z_{n+1} = \sum_{m=1}^{n} \delta_m$ and the value of $\Delta_m$ in a conditioned expectation where $\Delta_m = \delta_m$ is of course $\delta_m$. Hence $z_{n+1}$ on the right hand and the left hand side drop off when $Z_{n+1} = z_n + \Delta_{n+1}$. A consequence of this theorem is that

\[
f_{\Delta_{n+1} \mid \Delta_1,\ldots,\Delta_n}(-1\mid \delta_1, \ldots, \delta_n) \geq f_{\Delta_{n+1} \mid \Delta_1,\ldots,\Delta_n}(\delta_{n+1} \mid \delta_1, \ldots, \delta_n)
\] (12)

for $\delta_{n+1} \in \{-1,0,1\}$. From equation (8) it follows that with the unconditioned density we must have

\[
f_{\Delta_1,\ldots,\Delta_{n+1}}(\delta_1, \ldots, \delta_n, -1) \geq f_{\Delta_1,\ldots,\Delta_{n+1}}(\delta_1, \ldots, \delta_n, \delta_{n+1})
\] (13)

In order to see a steady decline in $\{Z_n\}_{n=1}^N$ i.e. in $\{\Delta_N\}_{n=1}^N$, the need is there for a strict supermartingale. Moreover, because in proper experimentation the $\delta_n = 0$ has a low probability we can derive, from (13), for a strict supermartingale that

\[
f_{\Delta_1,\ldots,\Delta_{n+1}}(\delta_1, \ldots, \delta_n, -1) > f_{\Delta_1,\ldots,\Delta_{n+1}}(\delta_1, \ldots, \delta_n, 1) > f_{\Delta_1,\ldots,\Delta_{n+1}}(\delta_1, \ldots, \delta_n, 0)
\] (14)

From (9) we may also derive that

\[
f_{\Delta_1,\ldots,\Delta_{n+1}}(\delta_1, \ldots, \delta_n, -1) \geq \sum_{\delta_{n+2} \in \{-1,0,1\}} f_{\Delta_1,\ldots,\Delta_{n+2}}(\delta_1, \ldots, \delta_n, \delta_{n+1}, \delta_{n+2})
\] (15)

Hence, $f_{\Delta_1,\ldots,\Delta_{n+1}}(\delta_1, \ldots, \delta_n, -1) > f_{\Delta_1,\ldots,\Delta_{n+2}}(\delta_1, \ldots, \delta_n, \delta_{n+1}, -1)$ when all probabilities are assumed unequal to zero.
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4 Conclusion

The conclusion from the previous analysis is that in order to have a strict supermartingale, the probability of the $\delta_i = -1$ response declines with the increase of the trial number. This is so because: $f_{\Delta_1, \ldots, \Delta_{n+1}}(\delta_1, \ldots, \delta_n, -1) > f_{\Delta_1, \ldots, \Delta_{n+2}}(\delta_1, \ldots, \delta_{n}, \delta_{n+1}, -1)$. The decline of probability to see a $\delta_i = -1$ is physically unrealistic.

The strict martingale is (every now and then) necessary in order to warrant a steady decline of $\Delta_n$ and hence of $Z_n$. Hence, Gill’s supermartingale argument and the subsequent application of Hoeffding’s inequality is based upon an unrealistic assumption. The assumption is unrealistic in a physical and probabilistical sense. Hence, there is no plausible case for a fifth position in the dispute on locality and causality in quantum mechanics. This finding supports previous results of the author [5], [6] and [7].

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