A NOTE ON THE NUMBER OF $k$-ROOTS IN $S_n$

YUVAL ROICHMAN

Abstract. The number of $k$-roots of an arbitrary permutation is expressed as a weighted sum of $k$-roots of the identity permutation.

1. A Combinatorial Identity

1.1. $\mu$-unimodal permutations. Let $\mu = (\mu_1, \ldots, \mu_t)$ be a partition of $n$ with $t$ nonzero parts. Denote

$$\mu(0) := 0$$

and

$$\mu(i) := \sum_{j=1}^{i} \mu_j \quad (1 \leq i \leq t)$$

and

$$(1) \quad S(\mu) := (\mu(1), \ldots, \mu(t)).$$

A permutation $\pi \in S_n$ is $\mu$-unimodal if for every $0 \leq i < t$ there exist $0 \leq \hat{i} \leq \mu(i) + 1$ such that

$$\pi(\mu(i) + 1) > \pi(\mu(i) + 2) > \cdots > \pi(\mu(i) + 1) < \pi(\mu(i) + 1 + 1) < \cdots < \pi(\mu(i + 1)).$$

Denote the set of $\mu$-unimodal permutations in $S_n$ by $U_\mu$.

For example, let $\mu = (\mu_1, \mu_2, \mu_3) = (4, 3, 1)$ then $S(\mu) = (\mu(1), \mu(2), \mu(3)) = (4, 7, 8)$. The permutations 53687142 and 35687412 are $\mu$-unimodal but 53681742 and 53867142 are not.

1.2. $k$-roots in $S_n$. For $n \geq 1$ and $k \geq 0$ denote

$$I_n^k := \{\pi \in S_n : \pi^k = 1\}$$

the set of $k$-roots of the identity permutation in $S_n$.

Theorem 1.1. For every $n \geq 1$, $k \geq 0$, partition $\mu \vdash n$ and $\pi \in S_n$ of cycle type $\mu$ the following holds:

$$(2) \quad \# \{\sigma \in S_n : \sigma^k = \pi\} = \sum_{\sigma \in I_n^k \cap U_\mu} (-1)^{|\Des(\sigma)|} |S(\mu)|.$$  

It follows that the set of $k$-roots of the identity permutation is a fine set in sense of [3]. The case $k = 2$ follows from [2, Prop.1.2.3]. Note that the proof there does not apply to a general $k$.

Date: May 23, '13.
2. Proof of Theorem 1.1

2.1. Induced representations. For every $n \geq 1$ and $k \geq 0$ let $\theta^{k,n} : S_n \to\mathbb{N} \cup \{0\}$ be the enumerator of $k$-roots of a permutation $\pi$ in $S_n$

$$\theta^{(k,n)}(\pi) := \#\{\sigma \in S_n : \sigma^k = \pi\}.$$ 

Clearly, $\theta^{(k,n)}$ is a class function. By a classical result of Frobenius and Schur $\theta^{(2,n)}$ is not virtual. It was conjectured by Kerber and proved by Scharf [8] that for every $k \geq 0$, $\theta^{(k,n)}$ is a non virtual character.

Let $Z_{\lambda}$ be the centralizer of a permutation of cycle type $\lambda$ in $S_n$. $Z_{\lambda}$ is isomorphic to the direct product $\times_{i=1}^n C_{k_i} \wr S_{k_i}$, where $k_i$ is the multiplicity of the part $i$ in $\lambda$. Denote by $\rho_i$ the one dimensional representation of $C_{k_i} \wr S_{k_i}$ indexed by the $i$-tuple of partitions $(\emptyset, (k_i), \emptyset, \ldots, \emptyset)$. Let

$$\rho^\lambda := \bigotimes_{i=1}^n \rho_i$$

a one-dimensional representation of $Z_{\lambda}$ and

$$\psi^\lambda = \rho^\lambda \uparrow_{Z_{\lambda}}^{S_n}.$$ 

the corresponding induced $S_n$-representation.

Denote by $\phi^{k,n}$ the representation whose character is $\theta^{(k,n)}$. The following theorem implies that $\phi^{k,n}$ is not virtual.

**Theorem 2.1.** [8] For every $n \geq 1$ and $k \geq 0$

$$\phi^{k,n} = \bigoplus_{\lambda \vdash n \text{ all parts divide } k} \psi^\lambda.$$ 

See also [9, Ex. 7.69(c)]. Note that letting $k = 2$ gives Inglis-Richardson-Saxl’s well known construction of a Gelfand model for $S_n$ [5].

2.2. Descents over conjugacy classes. Let $C_{\lambda}$ be the conjugacy class of cycle type $\lambda$ in $S_n$ and SYT($\nu$) be the set of all standard Young tableaux of shape $\nu$. Denote the multiplicity of the Specht module $S^\nu$ in $\psi^\lambda$ by $m(\nu, \lambda)$. The following is a reformulation of [4, Thm. 2.1].

**Theorem 2.2.** For every $\lambda \vdash n$

$$\sum_{\pi \in C_{\lambda}} x^{\Des(\pi)} = \sum_{\nu \vdash n} m(\nu, \lambda) \sum_{T \in \text{SYT}(\nu)} x^{\Des(T)}.$$ 

**Proof.** Denote by $L_{\lambda}$ the image of $\psi^\lambda$ under the Frobenius characteristic map. For an explicit description of this symmetric function see e.g. [9, Ex. 7.89]. For $J \subseteq [n-1]$ let $z_J$ be the skew Schur function which corresponds to the zigzag skew shape with down steps on positions which belong to $J$. By [4, Thm. 2.1], the coefficient of $x^J$ in the left hand side of Equation (4), which is the number of permutations of cycle type $\lambda$ and descent set $J$, is equal to $\langle L_{\lambda}, z_J \rangle$.

Now

$$\langle L_{\lambda}, z_J \rangle = \langle L_{\lambda}, \sum_{\nu \vdash n} (s_\nu, z_J) s_\nu \rangle = \sum_{\nu \vdash n} \langle L_{\lambda}, s_\nu \rangle (s_\nu, z_J) = \sum_{\nu \vdash n} m(\nu, \lambda) (s_\nu, z_J).$$
A NOTE ON THE NUMBER OF k-ROOTS IN $S_n$

Since $\langle s_\nu, z_J \rangle$ is equal to number of SYT of shape $\nu$ and descent set $J$, see e.g. [1, Thm. 4.1], this is equal to the coefficient of $x^J$ in the right hand side of Equation (3).

□

Corollary 2.3. For every partition $\mu \vdash n$ the value of $\psi^\lambda$ at a permutation of cycle type $\mu$ is

\begin{equation}
\psi^\lambda_\mu = \sum_{\sigma \in C_\lambda \cap U_\mu} (-1)^{|\text{Des}(\sigma) \setminus S(\mu)|}.
\end{equation}

Proof. Let $\chi^\nu_\mu$ be the character value of the Specht module $S^\nu$ on a conjugacy class of cycle type $\mu$. By [7, Theorem 4] [6],

$$
\chi^\nu_\mu = \sum_{T \in \text{SYT}(\nu) \cap \text{SYT}_\mu} (-1)^{|\text{Des}(T) \setminus S(\mu)|},
$$

where $\text{SYT}(\nu)$ is the set of all SYT of shape $\nu$ and $\text{SYT}_\mu$ is the set of $\mu$-unimodal SYT of size $n$.

Combining this with Theorem 2.2 gives

$$
\psi^\lambda_\mu = \sum_{\nu \vdash n} m(\nu, \lambda) \chi^\nu_\mu = \sum_{\nu \vdash n} m(\nu, \lambda) \sum_{T \in \text{SYT}(\nu) \cap \text{SYT}_\mu} (-1)^{|\text{Des}(T) \setminus S(\mu)|} = \sum_{\sigma \in C_\lambda \cap U_\mu} (-1)^{|\text{Des}(\sigma) \setminus S(\mu)|}.
$$

□

2.3. Conclusion. By Theorem 2.1 together with Corollary 2.3 for every $\pi \in S_n$ of cycle type $\mu$

$$
\# \{ \sigma \in S_n : \sigma^k = \pi \} = \theta^{(k,n)}(\pi) = \sum_{\lambda \vdash n} \psi^\lambda_\mu = \sum_{\lambda \vdash n, \text{ all parts divide } k} \sum_{\sigma \in C_\lambda \cap U_\mu} (-1)^{|\text{Des}(\sigma) \setminus S(\mu)|} = \sum_{\sigma \in I^{kn}_k \cap U_\mu} (-1)^{|\text{Des}(\sigma) \setminus S(\mu)|},
$$

completing the proof of Theorem 1.1.

□

3. Remarks and questions

It is desired to prove Theorem 1.1 via generalizations of the explicit combinatorial construction of Gelfand models described in [2].

Question 3.1. Find a “simple” $S_n$-linear action on a basis of $\phi^{k,n}$ indexed by $I^k$, which will imply the character formula given in the right hand side of Equation (2).

Another desired approach to prove Theorem 1.1 is purely combinatorial.

Question 3.2. Define an involution on the set of $k$-roots of the identity permutation, which changes the parity of $\text{Des}(\cdot) \setminus S(\mu)$ on non-fixed points, and cardinality of fixed point set is equal to LHS of Equation (3).

Question 3.3. Prove Theorem 2.2 by constructing a map from $C_\lambda$ to SYT of size $n$, under which for every $\nu \vdash n$ and $T \in \text{SYT}(\nu)$ the cardinality of the preimage of $T$ is exactly $m(\nu, \lambda)$.

Note that for $\lambda = (2^k, 1^{n-2k})$, $0 \leq k \leq n/2$, the RSK map satisfies this property.

Acknowledgements: Thanks to Ron Adin for useful discussions.
References

[1] R. M. Adin, F. Brenti and Y. Roichman, Descent representations and multivariate statistics, Trans. Amer. Math. Soc. 357 (2005), 3051-3082.
[2] R. M. Adin, A. Postnikov and Y. Roichman, Combinatorial Gelfand models, J. Algebra 320 (2008), no. 3, 1311-1325.
[3] R. M. Adin, Y. Roichman, Characters and descents, arXiv:1301.1675.
[4] I. M. Gessel and C. Reutenauer, Counting permutations with given cycle structure and descent set, J. Combin. Theory Ser. A 64 (1993), 189-215.
[5] N. F. J. Inglis, R. W. Richardson and J. Saxl, An explicit model for the complex representations of $S_n$, Arch. Math. (Basel) 54 (1990), 258-259.
[6] A. Ram, An elementary proof of Roichman’s rule for irreducible characters of Iwahori-Hecke algebras of type A, in: Mathematical essays in honor of Gian-Carlo Rota, Progr. Math., 161, Birkhäuser, Boston, 1998, 335–342.
[7] Y. Roichman, A recursive rule for Kazhdan-Lusztig characters, Adv. in Math. 129 (1997), 24-45.
[8] T. Scharf, Die Wurzelanzahlfunktion in symmetrischen Gruppen, (German) [The root number function in symmetric groups] J. Algebra 139 (1991), 446-457.
[9] R. P. Stanley, Enumerative combinatorics. Vol. 2. Cambridge Studies in Advanced Mathematics, 62. Cambridge University Press, Cambridge, 1999.

Department of Mathematics, Bar-Ilan University, 52900 Ramat-Gan, Israel
E-mail address: yuvalr@math.biu.ac.il