Covariantization and canonical quantization in the light-cone gauge

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ABSTRACT

In the light-cone gauge choice for Abelian and non-Abelian gauge fields, the vector boson propagator carries in it an additional “spurious” or “unphysical” pole intrinsic to the choice requiring a careful mathematical treatment. Research in this field over the years has shown us that mathematical consistency only is not enough to guarantee physically meaningful results. Whatever the prescription invoked to handle such an object, it has to preserve causality in the process. On the other hand the covariantization technique is a well suited one to tackle gauge dependent poles in the Feynman integrals, dispensing the use of ad hoc prescriptions. In this work we show that the covariantization technique in the light-cone gauge is a direct consequence of the canonical quantization of the theory.

IFT-01-2002

January 2002

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1 Introduction.

The historical development of light-cone gauge started well back in the 1940’s with the pioneering work of P.A.M.Dirac [1]. Since its début into quantum field theory it has known days of both glory and oblivion for varied reasons. On the one hand it seemed a solid grounded and more convenient approach to studying quantum fields, e.g., the only setting where a proof of the finiteness of the $N = 4$ supersymmetric Yang-Mills theory could be carried out successfully was in the light-cone gauge (a facet of its glory) [2]. But on the other hand, manifest Lorentz covariance is lost and non-local terms sneak into the renormalization program (the other side of the coin that charges us with a price to pay).

Having this very brief historical overview of the light-cone gauge as our hindsight, we can say that some of the advantages of using it in quantum field theory has to do with the possibility of decoupling the ghost fields in the non-Abelian theories, since it is an axial type gauge, as shown by J. Frenkel [3], a property that can simplify Ward-Takahashi identities [4] and problems involving operator mixing or diagram summation [5]. The annoying thing is that actual calculations of Feynman diagrams involved
in any physical process of interest give rise to “spurious” poles and these need to be treated carefully. By carefully we mean that any device or prescription called upon to handle the singularity needs not only to be mathematically sound but must guarantee no violation of basic underlying physical principles such as causality \[^{3}\]. Besides, even with causal prescriptions such as the Mandelstam-Leibbrandt one to handle it, there emerges a lot of nonlocal terms in the calculation proper that could pose challenges to the renormalization program \[^{7}\].

Of course, the machinery developed to handle covariant poles in propagators is on the road for a longer period of time, and from the start the procedure there has been such that the methodology sought would preserve unitarity, causality and so on, i.e., the analytical properties being rightly understood and applied for that case, ensuring the possibility of Wick rotation. Therefore if we could somehow build a bridge between the light-cone and the covariant pole, that could bring some fruition to our understanding of the problem inherent to the light-cone. This is exactly what has been proposed in \[^{8}\] and the bridging of the gap between the two kinds of singularities was named “covariantization”.

Covariantization then is a procedure to treat the so called “spurious” poles
that appear in the light-cone gauge choice transforming them into covariant-type poles, by the use of a “dispersion” relation. This enables us to take advantage of well known properties belonging to covariant poles in the propagators, eliminating the need to construct an “ad hoc” prescription to handle specifically the light-cone pole. As far as we know there is only one other prescriptionless method to handle light-cone integrals: the negative dimensional integration method (NDIM) which is an altogether different approach developed later by the turn of the last century [9].

The above mentioned covariantization method was applied from an intuition that the similarity of covariant-type pole generated by the dispersion relation would work correctly for the light-cone pole thus covariantized. In order to be formal and consistent, it also has to be possible to deduce it from the canonical quantization of the theory, and this is the aim and purpose of the present work. Since ghosts in a non-Abelian theory decouple in the light-cone gauge and without loss of generality, it is sufficient for us to consider an Abelian theory for this purpose.

The outline for our paper is as follows: in the next short section we review and consider the main ideas behind covariantization method whereas in the following section we consider the canonical quantization of a model Abelian
quantum gauge field constrained in the light-cone choice. The last section is
devoted to the final comments and conclusions.

2 Covariantizaton.

Here is a brief review of the “covariantization” technique which was proposed
by A.T.Suzuki [8]. The idea is quite simple. In light-cone coordinates, the
square of a four-momentum is:

\[ q^2 = 2q^+ q^- - \hat{q}^2. \]  \( \text{(1)} \)

Therefore, as long as \( q^- \neq 0 \) (and this is a key point, as we shall shortly see),
we can write \( q^+ \) as

\[ q^+ = \frac{q^2 + \hat{q}^2}{2q^-} \]  \( \text{(2)} \)

We note that this dispersion relation \textit{almost} guarantees that real gauge
fields for which \( q^2 = 0 \) (real photons or real gluons for example) are trans-
verse; the residual gauge freedom, that is left to be dealt with so that fields
be manifestly transverse comes from the presence of the \( q^- \) in the denomi-
nator of the expression above. This implies that in the light-cone gauge the
characteristic pole becomes

\[ \frac{1}{q^+} = \frac{2q^-}{q^2 + \hat{q}^2}, \quad q^- \neq 0. \] (3)

Note that the mapping between the light-cone pole in \( q^+ \) and the covariant-type pole \( q^2 + \hat{q}^2 \) is restricted by the condition \( q^- \neq 0 \). This means that, in the usual coordinate notation \( q^- = q^0 - q^3 \neq 0 \), or \( q^0 \neq q^3 \).

If we remind ourselves of the pure covariant case, this is equivalent to the condition \( q^0 \neq 0 \), the zero-mode extracted out from the range of meaningful frequencies allowed for the quanta: only negative energy quanta \( q^0 < 0 \) propagates into the past and only positive energy quanta \( q^0 > 0 \) propagates into the future. This is the causal connection. In the light-cone case, this translates into two distinct physically allowed regions, namely \( q^0 < q^3 \) and \( q^0 > q^3 \) respectively.

The important thing here is that the condition \( q^- \neq 0 \) warranties the causal structure of the covariantization technique since it eliminates the troublesome \( q^- = 0 \) modes. Elimination of these modes restores the physically acceptable results as can manifestly be seen in the causal prescription [10] for the light-cone gauge.
3 Canonical quantization.

As explained earlier, it is sufficient for us to consider here the Abelian gauge fields. Our procedure is similar to that used by A. Burnel in relating the canonical formalism to the Mandelstam-Leibbrandt prescription for the light-cone gauge.

Let us start with:

\[ \mathcal{L} = -\frac{1}{4}(F_{\mu\nu})^2 + h (m \cdot \partial)(n \cdot A) + \frac{1}{2}\alpha h^2, \]  

(4)

where \( h \) is an auxiliary field, \( \alpha \) is a parameter, \( n_\mu \) and \( m_\mu \) are the light-cone vectors that define the gauge (note that we have two null vectors that define the gauge, even though the usual gauge condition mentions only \( n \cdot A = 0 \)), and the field strength tensor

\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \]

Using the Euler-Lagrange conditions, we obtain the equations of motion:

\[ \partial^\mu F_{\mu\nu} - h (m \cdot \partial) n_\nu = 0, \]

\[ (m \cdot \partial) (n \cdot A) = \alpha h. \]

Following the canonical formalism, the equal-time commutation relations are:

\[ [A_\mu(x), \pi^\nu(y)]_{x_0=y_0} = i\delta^\nu_\mu \prod_{i=1}^{3} \delta(x^i - y^i), \]  

(5)
\[
\begin{align*}
[A_\mu(x), A_\nu(y)]_{x_0=y_0} &= 0, \\
[\pi^\mu(x), \pi^\nu(y)]_{x_0=y_0} &= 0,
\end{align*}
\]

(6) (7)

where the canonical momenta are

\[
\begin{align*}
\pi^0 &= \hbar n_0 m_0, \\
\pi^k &= F^{k0} + \hbar m_0 n^k.
\end{align*}
\]

(8) (9)

Using the equal-time relations (6) and the equations of motions, it is possible to get the commutation relations for any time. This was done by Burnel in 1989 [11]. That we are interested in is on momentum space operators, so we will leave space-time in the usual way, i.e. using the Fourier transforms

\[
A_\mu(x) = \frac{1}{(2\pi)^{2/3}} \int d^4k \theta(k_0) [a_\mu(k) e^{-ik\cdot x} + b_\mu(k) e^{ik\cdot x}],
\]

(10)

with the following commutation relations for \(a_\mu\) and \(b_\mu\):

\[
\theta(k_0)[a_\mu(k), b_\nu(k')] = \delta^4(k - k') \left[ -g_{\mu\nu}\delta(k^2) \\
- \alpha k_\mu k_\nu \delta(k \cdot n k \cdot m) \\
+ n^2(k \cdot m)^2 k_\mu k_\nu \left( \frac{k \cdot n k \cdot m}{k^2} \right) \\
+ (n_\mu k_\nu + n_\nu k_\mu) k \cdot m \left( \frac{\delta(k \cdot n k \cdot m)}{k^2} + \frac{\delta(k^2)}{k \cdot n k \cdot m} \right) \right]
\]

(11)
with also \([a_\mu(k), a_\nu(k')] = 0\) e \([b_\mu(k), b_\nu(k')] = 0\).

The propagator is defined by

\[
\tilde{G}_{\mu\nu}(x) = \langle 0 | T[A_\mu(x)A_\nu(0)] | 0 \rangle.
\]  

(12)

Using the equations (10) and (11), we obtain

\[
(\Box g^{\mu\nu} - \partial^\mu \partial^\nu - n^\mu n^\nu \frac{(m \cdot \partial)^2}{\alpha})\tilde{G}_{\mu\nu} = i\delta^\mu_\rho \delta^4(x),
\]

and inverting this last equation we arrive at

\[
G_{\mu\nu}(k) = -\frac{i}{k^2} \left[ g_{\mu\nu} - \frac{n_\mu k_\nu + n_\nu k_\mu}{k \cdot n \cdot k \cdot m} k \cdot m - \frac{\alpha k^2 - n^2 (k \cdot m)^2}{k \cdot n \cdot k \cdot m} k_{\mu\nu} \right].
\]  

(13)

That for the \(\alpha \rightarrow 0\) limit in the light-cone gauge \((n^2 = 0)\) goes exactly to the covariantization proposal.

\[
G_{\mu\nu}(k) = -\frac{i}{k^2} \left[ g_{\mu\nu} - \frac{n_\mu k_\nu + n_\nu k_\mu}{k \cdot n \cdot k \cdot m} k \cdot m \right].
\]  

(14)

Note that this preserves the causality of the propagator since the \(k^- = k \cdot m\) mode does not vanish nor cannot be canceled against the \(k^+ k^- = k \cdot n \cdot k \cdot m\) bilinear term in the denominator. This fact is also observed in the definition of the light-like planar gauge as was done in [14].

4 Conclusions.

In this work we have demonstrated that the canonical quantization of the theory in the light-cone gauge leads to the covariantization proposal for the
gauge dependent pole. The question that we now ask is related to the importance of the poles with the bilinear terms \((k \cdot n k \cdot m)^{-1}\) instead of the usual ones \((k \cdot n)^{-1}\). The raising of such a question becomes natural since that bilinear structure is the one that preserves causality of the theory. This is also related to the form of the propagator in the case of the light-like planar gauge \([14]\), where the important fact was to observe the discrete symmetry between the light-cone vectors \(n_\mu\) and \(m_\mu\).

However, here, our starting term for the gauge fixing Lagrangian, \(L_{\text{fix}}\), does not display this kind of discrete symmetry, since we took a general class III linear gauge. So, what seems the logical step now is to seek a gauge fixing Lagrangian term of the form

\[
L_{\text{fix}} = L_{\text{fix}}(n, m, A),
\]

that is, a term with such discrete symmetry built into it. The easiest way to construct or incorporate this symmetry is to take the factors of the form \((n \cdot A)(m \cdot A)\) alongside some non-propagating auxiliary fields. Then, some tentative forms would be terms such as these:

- First proposal:

\[
L_{\text{fix}} = h [(n \cdot A)(m \cdot A)]^{\frac{1}{2}} - \frac{1}{2} h^2,
\]
• Second proposal:
\[ \mathcal{L}_{\text{fix}} = h (n \cdot A) - \frac{1}{2} \frac{(n \cdot A)^2}{(n \cdot A)(m \cdot A)} h^2, \]

• Third proposal:
\[ \mathcal{L}_{\text{fix}} = h (m \cdot A) - \frac{1}{2} \frac{(m \cdot A)^2}{(n \cdot A)(m \cdot A)} h^2. \]

where for simplicity we have taken the gauge parameter \( a = 1 \). The first option has the built-in symmetry \( n \leftrightarrow m \), but it is practically prohibited by the very presence of a square root in it. The second and third options does generate the desired gauge fixing term \( \mathcal{L}_{\text{fix}} \), but they lack the manifest symmetry \( n \leftrightarrow m \). So, is it the best we can arrive at? Hopefully not, since upon examining the three proposals above, one notes that they have one feature in common: they make use of just one non-propagating auxiliary field \( h \). If we insert a second auxiliary field, we can construct a Lagrangian density of the form
\[ \mathcal{L} = -\frac{1}{4} (F_{\mu\nu})^2 + h_1 (n \cdot A) + h_2 (m \cdot A) - h_1 h_2 \]  
which, of course, is more complicated, but nonetheless bears the desired features we want. Just in passing we mention that the Lagrangian density in (15) yields a propagator of the exactly same form as found in [14]. Our
interest now is the canonical quantization of the Lagrangian density [13] whose result is to appear shortly elsewhere.

Acknowledgments: A.T.S. acknowledges partial funding from CNPq and R.B. wishes to thank FAPESP for financial support.

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