VIBRATION CHARACTERISTICS OF NONUNIFORM BLADES MADE OF FUNCTIONALLY GRADED MATERIAL

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ABSTRACT

The purpose of this study is to examine the vibration characteristics of a rotating blade whose material distribution varies in the spanwise direction. Formulations for functionally graded materials and beam structural models are carried out in detail and the results are displayed in several figures and tables which is a significant source of information for the authors working in this area. Different parameters such as angular speed, radius of the hub, material properties, power law index parameter, boundary conditions and slenderness ratio are considered in the formulation. Finite Element Method where the element matrices are obtained from potential and kinetic energy expressions is applied as the solution procedure. Results of the study are validated with open literature in several tables and figures.

Keywords: Functionally Graded, Helicopter Blade, Finite Element Analysis, Structural Vibration

1. INTRODUCTION

Helicopters are subjected to vibration for various reasons during the flight. Determination of the frequency values and the normal modes is required to perform the vibration analysis in the rotor blade design process correctly. Many numerical and approximate calculation methods are used in the vibration and natural frequency calculations and the Finite Element Analysis (FEA) is among the most efficient methods.

Air and space vehicles, wind turbines, helicopter blades, turbine rotors, defence and civil industries, ship and automotive sectors are among the engineering areas where composite materials have been mostly preferred due to their advantages, i.e. light weight and high strength/stiffness-to-weight ratios. However, composite materials have some limitations. For instance, stress concentration near interlayer surfaces is high because of the sudden changes in mechanical properties and this limitation may cause severe material failures. Moreover, the adhesive layer may get cracked when the temperature is low and it may creep at high temperature. Functionally graded materials, FGMs, are considered to be the new generation composite materials. The variation character of their material properties is continuous through the structure so stress concentrations do not occur. Survivability in high temperatures by maintaining structural integrity is among the outstanding properties of FGMs. Although many different material combinations have been studied for FGMs, the most widely used one is the ceramic-metal combination where the ceramic reduces heat transfer to protect metal from corrosion and oxidation, whereas metal provides strength, higher fracture toughness, etc.

Structural components used in engineering are mostly beams and beam structures. Different material types, i.e. homogeneous, composite, functionally graded, etc. are used in these structural components to...
meet different engineering design requirements. Both in conventional structural applications and in advanced structural applications, including electric-thermal-structural systems, FGMs are commonly used as harvesters, sensors and actuators. Therefore, many researchers have been studying these materials for different application areas. Due to the increasing application trend of FGMs, their vibration properties have been examined by applying different beam theories.

The concept of FGMs was originated from a team of material scientists working on thermal barrier materials [1] and nowadays, production areas and application fields are increasing day by day with the development of additive manufacturing technology and powder metallurgy of the material properties. Sankar [2] studied FGM beams with simply-simply supported end conditions under the effect of transverse loading where the beam elasticity modulus changes through the beam thickness. Aydogdu and Taskin [3] studied FGM beams with simply supported end conditions where Modulus of Elasticity changes with respect to a power and an exponential rule in the transverse direction. Chakraborty et al. [4] developed a new beam element to study the thermoelastic behavior of functionally graded beam structures. Goupee and Senthil [5] optimized the natural frequencies of FGM beams by changing the material distribution via a genetic algorithm methodology. Xiang and Yang [6] analysed a thermally prestressed nonuniform FGM beam in the free and forced vibration cases. Piovan and Sampoia [7] employed formulas considering shear-deformation and nonlinear relationship between strain and displacement to study the dynamic behavior of rotating FGM beams. Both the free vibration and the harmonically forced vibration of FGM Euler-Bernoulli beams are studied by Simsek and Kocaturk [8]. Free vibration of curved beams made of FGM in the out of plane direction is analysed by Malekzadeh et al. [9] where temperature dependent material properties are considered. Huang and Li [10] studied free vibration of nonuniform axially functionally graded beams with variable flexural rigidity and mass density. Free vibration and stability analyses of Timoshenko beams with nonuniform cross-sections was studied by Shahba et al [11] by employing an energy based finite element solution. Additionally, several review papers have been published in recent years about the modelling, buckling, stability and vibration characteristics of FGM structures [12, 13].

In this study, FG blades whose material distribution changes in the spanwise direction are modeled and vibration analyses are performed. In these studies, beam models with fixed-free and fixed-fixed boundary conditions and different material distribution properties are investigated. For developing the mathematical models and for the solution, finite element method (FEM) is used. The blade formulations are derived for both Euler-Bernoulli and Timoshenko beam theories to inspect the effect of different parameters on the vibration characteristics. For each beam theory, both the stiffness matrix and the mass matrix are derived from the energy expressions. In the solution part, effects of different parameters such as hub rotating speed, material properties, power law index parameter, different boundary conditions, rotary inertia and shear deformation are investigated. Results of the study are validated with open literature in several tables and figures.

2. MATERIAL and METHOD

2.1. Functionally Graded Nonuniform Blade Model

In this study, vibration analysis is carried out for an Axially Functionally Graded nonuniform blade model which is shown in Figure 1.
Here a blade having a constant rotational velocity, $\Omega$, is fixed to a rigid hub of radius $R$ at point $O$. The origin of the right-handed Cartesian coordinate system is located at the root of the blade and the $x$-axis is directed along the blade while the rotational axis and the $z$-axis are parallel.

The blade is modeled as a beam structure with variable cross sectional dimensions and material properties along the $x$-axis. The beam model has two different material properties, i.e. ceramic and metal, in different compositions from the fixed to the free end. Additionally, the beam tapers linearly from a height of $h_0$ at the root to $h$ at the free end in the $xz$ plane and from a breadth $b_0$ to $b$ in the $xy$ plane.

Beam material properties vary continuously in the longitudinal direction, i.e. $x$-axis, via a simple power law. The rule of mixture states that $T(x)$, i.e. the effective material property such as the Elasticity modulus and material density. The other properties can be expressed as given by Equations (1a)-(1e) where $\alpha$ is the power law index parameter that is a positive number and that defines the material variation characteristic along the $x$-axis.

$$T(x)=(T_R-T_L)\left(\frac{x}{L}\right)^\alpha+T_L, \quad \alpha \geq 0 \quad (1a)$$

$$E(x) = (E_R - E_L)\left(\frac{x}{L}\right)^\alpha + E_L \quad (1b)$$

$$G(x) = (G_R - G_L)\left(\frac{x}{L}\right)^\alpha + G_L \quad (1c)$$

$$\nu(x) = (\nu_R - \nu_L)\left(\frac{x}{L}\right)^\alpha + \nu_L \quad (1d)$$

$$\rho(x) = (\rho_R - \rho_L)\left(\frac{x}{L}\right)^\alpha + \rho_L \quad (1e)$$
where \( R \) and \( L \) are the material properties, i.e., elasticity modulus \( E \), shear modulus \( G \), Poisson’s ratio, \( \nu \) and material density, \( \rho \) at the right hand side and left hand side of the beam, respectively as given in Figure 2.

![Figure 2. Material variation in an axially functionally graded beam model](image)

The expressions for the geometrical dimensions and the cross-sectional preoperties of a beam that is double tapered are

\[
b(x) = b_0 \left( 1 - c_b \frac{x}{L} \right)^m \tag{2a}
\]

\[
h(x) = h_0 \left( 1 - c_h \frac{x}{L} \right)^n \tag{2b}
\]

\[
A(x) = A_0 \left( 1 - c_h \frac{x}{L} \right)^n \left( 1 - c_b \frac{x}{L} \right)^m \text{ where } A_0 = b_0 h_0 \tag{2c}
\]

\[
l_y(x) = l_{yo} \left( 1 - c_h \frac{x}{L} \right)^{3n} \left( 1 - c_b \frac{x}{L} \right)^{m} \text{ where } l_{yo} = \frac{1}{12} b_0 h_0^3 \tag{2d}
\]

Equation (3a) expresses the breadth taper ratio, \( c_b \) while Equation (3b) expresses the height taper ratio, \( c_h \). The beam formulation is achieved in a way to let the beam get different taper ratio values in different planes so \( c_b \) and \( c_h \) do not have to be the same.

\[
c_h = 1 - \frac{h_0}{h} \tag{3a}
\]

\[
c_b = 1 - \frac{b_0}{b} \tag{3b}
\]

The exponents \( n \) and \( m \) get values depending on the taper type of the beam. In this study, a beam structure that tapers in two planes in a linear manner is considered so \( n = 1 \) and \( m = 1 \) for this study.

### 2.2. Energy Expressions

In this section, energy expressions are given both for rotating, nonuniform AFG Euler-Bernoulli and Timoshenko beam models. Details of the derivation can be found in Ref. [14-16] in great detail by using several explanatory figures and tables.

The potential energy expressions are given for Euler-Bernoulli and Timoshenko beam models in Equation (4a) and Equation (4b), respectively.
\[ U_{Euler} = \frac{1}{2} \int_{0}^{L_e} [E(x)I_y(x)(\theta')^2 + F_{CF}(x)(w')^2] dx + C_1 \]  

\[ U_{Timoshenko} = \frac{1}{2} \int_{0}^{L_e} [E(x)I_y(x)(\theta')^2 + kA(x)G(x)(w' - \theta)^2 + F_{CF}(x)(w')^2] dx + C_2 \]  

where the centrifugal force is

\[ F_{CF}(x) = \int_{x}^{L} \rho A \Omega^2 (R + x) dx \]  

In Equation (4b), the first term is the strain energy due to transverse displacement while the second term is the strain energy due to shear which is a result of the Timoshenko beam formulation.

The kinetic energy expressions are given for Euler-Bernoulli and Timoshenko beam models in Equation (6a) and Equation (6b), respectively.

\[ T_{Euler} = \frac{1}{2} \int_{0}^{L_e} \left( \rho A \dot{w}^2 + \rho I_y (\dot{w}')^2 + \rho I_y \Omega^2 (w')^2 \right) dx + D_1 \]  

\[ T_{Timoshenko} = \frac{1}{2} \int_{0}^{L_e} \left( \rho A \dot{w}^2 + \rho I_y \dot{\theta}^2 + \rho I_y \Omega^2 \theta^2 \right) dx + D_2 \]  

In the Equations (4a)-(6b), \( C_1, C_2, D_1 \) and \( D_2 \) are the integration constants.

2.3. Finite Element Modeling

Finite element representation of the functionally graded and nonuniform rotating beam model is shown in Figure 3.

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**Figure 3.** FE model of a FG nonuniform rotating beam

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where $L_i$ is the offset of each element from the rotational axis and $L_e$ is the element length. Depending on the analysis, $L_e$ may get different values for each element but in this study, the beam model is divided into elements of the same length. Here $XYZ$ and $x'y'z'$ are the global and local coordinates, respectively.

When a beam that rotates about a fixed axis is studied, new terms are added to the element stiffness matrices resulting from the centrifugal force. Considering the finite element model, given in Figure 3, the centrifugal force, i.e. Equation (5), can be expressed as follows where $N_e$ is the number of elements used in the FE formulation.

$$ F_{CF}(x) = \frac{\rho A \Omega^2}{2} \left[ R(L_i - x') + \frac{1}{2} (L_i - x') (L_i + x') \right] $$

(7a)

$$ L_i = (i - 1) \frac{L}{N_e}, \quad i = 1, 2, \ldots, N_e $$

(7b)

**Euler Bernoulli beam finite element modeling**

In Figure 4, an Euler Bernoulli beam finite element model which has four degrees of freedom is shown. Due to the Euler beam theory, shear effects are not considered and the degree of freedom is two, i.e. transverse displacement, $w$ and rotation $\theta$ at each node. The rotation angle is defined as the slope at each node, so $\theta = w'$.

![Figure 4: FE model of an Euler Bernoulli beam](image)

Polynomials are defined to express the displacement field of the Euler-Bernoulli beam [17]

$$ w = a_0 + a_1 x + a_2 x^2 + a_3 x^3 $$

(8a)

$$ \theta = w' = a_1 + 2a_2 x + 3a_3 x^2 $$

(8b)

Considering the displacement field polynomials given by Equation (8a) and Equation (8b), the nodal displacements are defined at the 1st node and at the 2nd node, respectively as follows
Here, ( )₁ are the displacement values of the 1st node while ( )₂ are the displacements on the 2nd node.

The displacement field vector, \( \{ q \} \) and the nodal displacement vector, \( \{ q_e \} \) are related to each other by the matrix of shape functions, \( [N] \).

\[
\{ q \} = [N] \{ q_e \}
\]  \hspace{1cm} (10)

where

\[
\{ q \} = \{ w \theta \}^T \hspace{1cm} (11a)
\]

\[
\{ q_e \} = \{ w_1 \theta_1 w_2 \theta_2 \}^T \hspace{1cm} (11b)
\]

\[
[N] = \begin{bmatrix} N_w & N_\theta \end{bmatrix}^T \hspace{1cm} (11c)
\]

Here the expressions of the shape functions are

\[
[N_w] = \begin{bmatrix} 1 - \frac{3x^2}{L^2} - \frac{2x^3}{L^3} & x - \frac{2x^2}{L} + \frac{x^3}{L^2} & \frac{3x^2}{L^2} - \frac{2x^3}{L^3} & \frac{x^2}{L} + \frac{x^3}{L^2} \end{bmatrix} \hspace{1cm} (12a)
\]

\[
[N_\theta] = \begin{bmatrix} \frac{6x}{L^2} + \frac{6x^2}{L^3} & 1 - \frac{4x}{L} + \frac{3x^2}{L^2} & \frac{6x}{L^2} & \frac{6x^2}{L^3} - \frac{2x}{L} + \frac{3x^2}{L^2} \end{bmatrix} \hspace{1cm} (12b)
\]

Here, \( [N_w] \) and \( [N_\theta] \) are the normal modes associated with the transverse motion \( w \) and the rotation angle, \( \theta \), respectively and \( [\cdot]^T \) is the transpose of a matrix.

Considering the effect of the centrifugal force and substituting the shape functions, i.e. Equation (12a) and Equation (12b), into the energy expressions the element stiffness and mass matrices, i.e. \( [K^e] \) and \( [M^e] \), are obtained as follows

\[
[K^e] = \frac{1}{2} \int_{0}^{l_e} \left( E(x)I_y(x) \begin{bmatrix} \frac{dN_\theta}{dx}^T \\ \frac{dN_\theta}{dx} \end{bmatrix} + F_{MK}(x) \begin{bmatrix} \frac{dN_w}{dx}^T \\ \frac{dN_w}{dx} \end{bmatrix} \right) dx \hspace{1cm} (13a)
\]
\[ [M^e] = \frac{1}{2} \int_0^{L_e} (\rho(x)A(x)[N_w]^T[N_w] + \rho(x)I_y(x)[N_\theta]^T[N_\theta]) \, dx \]  

(13b)

Timoshenko beam finite element modeling

In Figure 5, a Timoshenko beam finite element model which has six degrees of freedom is shown. Due to the Timoshenko beam theory, shear effects are considered and the degree of freedom is three, i.e. transverse displacement, \( w \), rotation angle \( \theta \) and shear angle, \( \phi \) at each node.

Figure 5: FE model of a Timoshenko beam

Polynomials are defined to express the displacement field of the Timoshenko beam [17]

\[ w = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \]  

(14a)

\[ \varphi = a_4 + a_5 x \]  

(14b)

\[ \theta = w' - \varphi = a_1 - a_4 + (2a_2 - a_5)x + 3a_3 x^2 \]  

(14c)

Considering the displacement field polynomials given by Equations (14a)-(14c), the nodal displacements are defined at the 1\textsuperscript{st} node and at the 2\textsuperscript{nd} node, respectively as follows.

\[
\begin{bmatrix}
  w_1 \\
  \theta_1 \\
  \varphi_1 \\
  w_2 \\
  \theta_2 \\
  \varphi_2
\end{bmatrix} =
\begin{bmatrix}
  1 & 0 & 0 & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 & -1 & 0 \\
  0 & 0 & 0 & 0 & 1 & 0 \\
  1 & L_e & L_e^2 & L_e^3 & 0 & 0 \\
  0 & 1 & 2L_e & 3L_e^2 & -1 & -L_e \\
  0 & 0 & 0 & 0 & 1 & L_e
\end{bmatrix}
\begin{bmatrix}
  a_0 \\
  a_1 \\
  a_2 \\
  a_3 \\
  a_4 \\
  a_5
\end{bmatrix}
\]

(15)

From Equation (10),

\[ \{ q \} = \{ w \ \theta \ \varphi \}^T \]  

(16a)

\[ \{ q_e \} = \{ w_1 \ \theta_1 \ \varphi_1 \ \ w_2 \ \theta_2 \ \varphi_2 \}^T \]  

(16b)

\[ [N] = [N_w \ N_\theta \ N_\varphi]^T \]  

(16c)
Here \( [N_w] \), \( [N_\theta] \) and \( [N_\phi] \) are the shape functions associated with the transverse displacement, \( w \), rotation angle due to transverse displacement, \( \theta \) and shear angle, \( \phi \), respectively.

\[
[N_w] = \begin{bmatrix}
1 - \frac{3x^2}{L_e^2} + \frac{2x^3}{L_e^3} & -\frac{2x^2}{L_e^2} + \frac{x^3}{L_e^3} & x - \frac{2x^2}{L_e^2} + \frac{x^3}{L_e^3} \\
\frac{3x^2}{L_e^2} & -\frac{2x^3}{L_e^3} & -\frac{x^2}{L_e^2} + \frac{x^3}{L_e^3}
\end{bmatrix}
\] (17a)

\[
[N_\theta] = \begin{bmatrix}
-\frac{6x}{L_e^2} + \frac{6x^2}{L_e^3} & 1 - \frac{4x}{L_e^2} + \frac{3x^2}{L_e^3} & -\frac{3x}{L_e^2} + \frac{3x^2}{L_e^3} \\
\frac{6x}{L_e^2} & -\frac{6x^2}{L_e^3} - \frac{2x}{L_e^2} + \frac{3x^2}{L_e^3} & -\frac{3x}{L_e^2} + \frac{3x^2}{L_e^3}
\end{bmatrix}
\] (17b)

\[
[N_\phi] = \begin{bmatrix}
0 & 0 & 1 - \frac{x}{L_e} & 0 & 0 & \frac{x}{L_e}
\end{bmatrix}
\] (17c)

Considering the effect of the centrifugal force and substituting the shape functions into the potential and kinetic energy expressions, the element stiffness matrix, \( [K^e] \), and element mass matrix, \( [M^e] \), are obtained as follows

\[
[K^e] = \frac{1}{2} \int_0^L \left( E(x) I_y(x) \left[ \frac{dN_\theta}{dx} \right]^T \left[ \frac{dN_\theta}{dx} \right] + kA(x) G(x) \left[ \frac{dN_w}{dx} - N_\theta \right]^T \left[ \frac{dN_w}{dx} - N_\theta \right] \right) + F_{CF}(x) \left[ \frac{dN_w}{dx} \right]^T \left[ \frac{dN_w}{dx} \right] \, dx
\] (18a)

\[
[M^e] = \frac{1}{2} \int_0^L \left( \rho(x) A(x) [N_w]^T [N_w] + \rho(x) I_y(x) [N_\theta]^T [N_\theta] \right) \, dx
\] (18b)

Reduced global matrices and modal analysis

Depending on the element number used in the FE modeling, all the element matrices are assembled by considering the finite element rules to get the global matrices. The BC’s are applied to the global matrices to get the reduced matrix system of equations

\[
[M] \{ \ddot{q} \} + [K] \{ q \} = \{ 0 \}
\] (19)

where \([M]\) and \([K]\) are the reduced global mass and reduced global stiffness matrices, respectively.

Modal analysis is applied to Equation (19) to calculate the natural frequencies, \( \omega \) as follows.

\[
\det([K] - \omega^2[M]) = 0
\] (20)
3. THE RESEARCH FINDINGS AND DISCUSSION

In this section, flapwise bending vibration analysis of both Euler-Bernoulli and Timoshenko beams that taper in both planes and that have material variation in the axial direction are carried out. Several parameters, i.e. taper ratios for nonuniformity, power law index parameter for material distribution, slenderness ratio, rotational speed, hub radius parameter, etc. are considered for the vibration analysis. The normalized parameters used in the tables and graphics are given by

\[ \bar{\omega} = \omega \sqrt{\frac{\rho A_0 L^4}{EI_{y_0}}} \]  
\[ r = \sqrt{\frac{I_{y_0}}{A_0 L^2}} \]  
\[ \sigma = \frac{R}{L} \]

where the properties given in the paranthesis \((...)_0\) are the ones defined at the root of the blade where \(x=0\). Here, \(\bar{\omega}\) is the normalized natural frequency, \(r\) is the slenderness ratio and \(\sigma\) is the normalized hub radius.

Several tables and figures which are expected to be a good source for the researchers who study in this field to analyse the initial models for helicopter blades are presented in this study. When the results are compared with the ones in open literature, it is noticed that there is a very good agreement between the results which proves the correctness and accuracy of the studies in this paper.

3.1. Homogeneous Nonuniform Beams

In this section, vibration characteristics are examined for both Euler Bernoulli and Timoshenko beams that have taper effects and homogeneous material properties.

**Euler-Bernoulli beam results**

In this case, vibration characteristics of a rotating/nonrotating, tapered Euler-Bernoulli beam having clamped free boundary conditions is examined. The geometrical and material properties of the beam model are given in Table 1.

| Beam Height, \(h_0 = 0.01\) m | Material Density, \(\rho = 7850\) kg/m³ |
|--------------------------------|----------------------------------------|
| Beam Breadth, \(b_0 = 0.1\) m | Elasticity Modulus, \(E = 206.8\) GPa  |
| Beam Length, \(L = 2\) m    | Poisson’s Ratio, \(\nu = 0.3\)         |
| Hub radius, \(R=0\) m (Clamped beam) |                                         |

Variation of the normalized natural frequencies with respect to taper ratios, \(c_h\) and \(c_b\) is introduced in Table 2 and Table 3 for homogeneous Euler-Bernoulli beams model with fixed-free end conditions. The calculated results are compared with the ones in open literature and a very good agreement between the results is observed.
Table 2. Effects of taper ratios on the natural frequencies of a nonrotating Euler-Bernoulli beam

| $c_b$ | Present | Ref.[18] | Present | Ref.[18] | Present | Ref.[18] | Present | Ref.[18] |
|-------|---------|----------|---------|----------|---------|----------|---------|----------|
| $c_b=0$ | 3.5160 | 3.5160 | 3.6656 | 3.6667 | 3.9258 | 3.9343 | 4.2631 | 4.2925 |
| 0.3 | 22.0338 | 22.0345 | 19.8903 | 19.8806 | 17.5318 | 17.4879 | 15.8180 | 15.7427 |
| 0.6 | 61.6924 | 61.6972 | 53.3599 | 53.3222 | 44.2046 | 44.0248 | 37.2405 | 36.8846 |
| 0.8 | 120.8850 | 120.9020 | 103.3440 | 103.2670 | 83.9397 | 83.5541 | 68.9108 | 68.1164 |

| $c_b=0.5$ | 3.9135 | 3.9160 | 4.0632 | 4.0669 | 4.3250 | 4.3362 | 4.6665 | 4.6991 |
| 0.3 | 22.7801 | 22.7860 | 20.5609 | 20.5555 | 18.1217 | 18.0803 | 16.3487 | 16.2744 |
| 0.6 | 62.4253 | 62.4361 | 54.0474 | 54.0152 | 44.8340 | 44.6583 | 37.8171 | 37.4635 |
| 0.8 | 121.6240 | 121.6480 | 104.0450 | 103.9750 | 84.5900 | 84.2101 | 69.5110 | 68.7209 |

| $c_b=0.8$ | 4.5670 | 4.5853 | 4.7175 | 4.7372 | 4.9929 | 5.0090 | 5.3471 | 5.3761 |
| 0.3 | 23.9833 | 24.0211 | 21.6466 | 21.6699 | 19.1775 | 19.0649 | 17.2032 | 17.1657 |
| 0.6 | 63.6990 | 63.7515 | 55.2170 | 55.2224 | 45.8302 | 45.7384 | 38.6497 | 38.4392 |
| 0.8 | 122.9480 | 123.0250 | 105.2630 | 105.2410 | 85.5567 | 85.3438 | 70.2244 | 69.7438 |

In Table 2, it is noticed that increasing taper ratios have increasing effects on the natural frequencies and the effect of the breadth taper ratio, $c_b$, is more dominant.

Variation of the normalized natural frequencies with respect to the taper ratios and the rotational speed parameter is given in Table 3.

Table 3. Effects of taper ratios and rotational speed on the natural frequencies of a rotating Euler-Bernoulli beam

| $c_b$ | Present | Ref.[19] | Present | Ref.[19] | Present | Ref.[19] | Present | Ref.[19] |
|-------|---------|----------|---------|----------|---------|----------|---------|----------|
| $c_b=0$ | 4.527 | 4.4368 | 6.1826 | 5.8788 | 8.1936 | 7.6551 | 10.3337 | 9.5539 |
| 0.3 | 19.109 | 18.9366 | 21.1283 | 20.6851 | 24.1151 | 23.3093 | 27.7481 | 26.5437 |
| 0.6 | 48.3031 | 47.8717 | 50.4098 | 49.6456 | 53.7245 | 52.4632 | 58.0238 | 56.1595 |
| 0.8 | 91.9318 | 91.0625 | 94.1133 | 92.873 | 97.6306 | 95.809 | 102.332 | 99.7638 |
| $c_b=0.5$ | 5.3099 | 5.1564 | 7.054 | 6.4726 | 9.2283 | 8.1663 | 11.5754 | 10.0192 |
| 0.3 | 20.2771 | 20.0733 | 22.283 | 21.5749 | 25.2655 | 23.8684 | 28.9117 | 26.7454 |
| 0.6 | 49.5292 | 49.0900 | 51.6321 | 50.5938 | 54.9446 | 53.0018 | 59.2468 | 56.1941 |
| 0.8 | 93.1673 | 92.3243 | 95.345 | 93.8415 | 98.8577 | 96.3142 | 103.555 | 99.6673 |

Here Table 3 reveals that as the rotational speed $\Omega$, increases, the natural frequencies increase because the centrifugal force, i.e. Equation (5), which is proportional to the square of the rotational speed makes the beam stiffer.

Timoshenko beam results

Table 4 gives the material and geometrical properties of the homogeneous nonuniform Timoshenko beam model. In Table 5, variation of the Timoshenko beam natural frequencies with respect to the taper ratio parameters are tabulated. The case, given in Table 5, demonstrates a beam that has the same taper in both planes; i.e. $c_b=c_b$, and that has fixed-free boundary conditions.
Table 4. Geometrical and material properties of homogeneous Timoshenko beam

| Beam Height, \( h = 0.037 \text{ m} \) | Material Density, \( \rho = 7860 \text{ kg/ m}^3 \) |
|----------------------------------------|----------------------------------|
|Beam Length, \( L = 0.24 \text{ m} \) | Elasticity Modulus, \( E = 210 \text{ GPa} \) |
| Slenderness Ratio, \( r=0.01 \)      | Poisson’s Ratio, \( \nu = 0.3 \) |
|                                       | Shear Correction Factor, \( k = 5/6 \) |

Table 5. Effects of taper ratios on the natural frequencies of a Timoshenko beam (fixed-free)

| \( c_h \) | Experimental Study Ref.[20] | Mathematical Modelling Ref.[20] | Present |
|-----------|-----------------------------|--------------------------------|---------|
| 0.1       | 3.4821                      | 3.4956                         | 3.4976  |
|           | 18.941                      | 19.1962                        | 19.2561 |
|           | 46.8812                     | 47.5057                        | 47.8338 |
|           | 80.8891                     | 82.0774                        | 82.9644 |
| 0.3       | 3.5962                      | 3.6076                         | 3.6068  |
|           | 17.9951                     | 18.2044                        | 18.2602 |
|           | 43.861                      | 44.3941                        | 44.6912 |
|           | 76.1284                     | 77.1983                        | 78.0033 |
| 0.5       | 3.7462                      | 3.7665                         | 3.7606  |
|           | 16.9821                     | 17.0617                        | 17.1278 |
|           | 39.9951                     | 40.7118                        | 41.0235 |
|           | 69.7561                     | 71.1402                        | 71.9314 |

Variation of the first four natural frequencies of homogeneous Euler-Bernoulli and Timoshenko beams with respect to the height taper ratio, \( c_h \) and the breadth taper ratio, \( c_b \) are displayed in Figure 6 and Figure 7, respectively. Here, \( R=0, r=0.08, \nu =0.3, k=0.85 \). The dashed lines show the variation of the Euler-Bernoulli beam natural frequencies while the solid lines show the variation of the Timoshenko beam natural frequencies.

Figure 6. Effect of the height taper ratio, \( c_h \), on the natural frequencies \( (c_b=0.2) \)
Figure 7. Effect of the breadth taper ratio, \( c_b \), on the natural frequencies (\( c_h = 0.2 \))

Here Figure 6 and Figure 7 reveal that \( c_b \) has less effect on the variation of the natural frequencies while \( c_h \) is more dominant. Especially, this difference is more obvious on higher modes. Moreover, Euler-Bernoulli beam frequencies are higher than the Timoshenko beam frequencies due to the decreasing effect of the inverse of the slenderness ratio.

3.2. Axially Functionally Graded Beams

In this section, vibration characteristics are examined for both Euler Bernoulli and Timoshenko beams that have axially functionally graded (AFG) material properties. The beam model used for the analysis is shown in Figure 8 where the beam material is pure \( \text{ZrO}_2 \) at the fixed end and it is pure \( \text{Al} \) at the free end.

![Figure 8. Rotating, axially functionally graded, cantilevered beam](image)

In Table 6, the material properties of Aluminum and Zirconia are displayed.

| Material Property       | Zirconia (\( \text{ZrO}_2 \)) | Aluminum (\( \text{Al} \)) |
|-------------------------|-------------------------------|-----------------------------|
| Elasticity Modulus, \( E \) | 200 GPa                       | 70 GPa                      |
| Material Density, \( \rho \) | \( 5700 \text{ kg/m}^3 \)      | \( 2702 \text{ kg/m}^3 \)    |
| Poisson’s Ratio, \( \nu \)  | 0.3                           | 0.3                         |
**Euler-Bernoulli beam results**

Vibration properties of an AFG Euler-Bernoulli beam is analyzed for hinged-hinged boundary conditions. In Table 7, effect of the taper ratio on the natural frequencies of homogeneous and AFG Timoshenko beam models are tabulated. The results are compared with the ones given by Ref. [21].

| cb | Homogeneous Euler Beam | AFG Euler Beam |
|----|------------------------|----------------|
|    | ch                     | ch             |
|    | 0          | 0.1  | 0.5  | 0.9  | 0    | 0.1  | 0.5  | 0.9  |
| 0  | 9.865      | 9.361| 7.118| 3.928| 9.569| 9.5994*| 9.044| 6.711| 3.425|
|    | 39.401     | 37.409| 28.912| 18.216| 38.265| 36.329| 28.090| 17.833|
|    | 88.435     | 83.980| 64.805| 40.210| 85.985| 81.616| 62.882| 39.028|
|    | 156.682    | 148.833| 114.831| 70.561| 152.736| 144.948| 111.408| 68.141|
| 0.1| 9.861      | 9.354| 7.095| 3.892| 9.550| 9.022| 6.678| 3.386|
|    | 39.392     | 37.405| 28.925| 18.252| 38.268| 36.336| 28.115| 17.882|
|    | 88.414     | 83.966| 64.820| 40.257| 85.986| 81.624| 62.915| 39.089|
|    | 156.644    | 148.804| 114.839| 70.609| 152.726| 144.946| 111.437| 68.203|
| 0.5| 9.808      | 9.281| 6.946| 6.9566* | 3.676| 9.413| 8.870| 6.470* | 6.5127* | 3.155|
|    | 39.389     | 37.423| 29.032| 18.494| 38.320| 36.410| 28.285| 18.206|
|    | 88.388     | 83.974| 64.969| 40.582| 86.075| 81.743| 63.162| 39.491|
|    | 156.563    | 148.766| 114.976| 70.959| 152.798| 145.054| 111.695| 68.631|
| 0.9| 9.525      | 8.967| 6.502| 30.974| 8.953| 8.387| 5.904| 2.559|
|    | 39.397     | 37.469| 29.259| 19.115| 38.350| 36.492| 28.614| 19.060|
|    | 88.539     | 84.184| 65.449| 41.553| 86.399| 82.132| 63.845| 40.691|
|    | 156.810    | 149.081| 115.610| 72.137| 153.337| 145.661| 112.605| 70.037|

* Ref.[21]

Effect of the power law index parameter, $\alpha$ on the Modulus of Elasticity, $E$, of the beam is demonstrated in Figure 9. Here it is noticed that the percentage of Zirconia gets higher with the increasing value of the power law index parameter, $\alpha$. 

112
Figure 9. Effect of the power law index parameter, $\alpha$ on the Modulus of Elasticity, $E$

Timoshenko beam results

Vibration analysis of AFG Timoshenko beam is carried out for the hinged-hinged boundary condition. As an addition to the material properties given in Table 5, $r = 0.01 \, m$ is the slenderness ratio, $L= 5 \, m$ is the beam length and $k=5/6$ is the shear correction factor.

In Table 8, effect of the power law index parameter, $\alpha$ on the normalized frequencies is tabulated. Here, it is noticed that the natural frequencies increase with the power law index parameter.

Table 8. Natural frequencies of AFG Uniform Timoshenko beam (fixed-free)

| Power Law Index Parameter, $\alpha$ | Dimensionless Natural Frequencies |  |  |
|-----------------------------------|----------------------------------|---|---|
|                                   | Clamped Free                     | Clamped Clamped       |  |
|                                   | Ref.[22]                         | Present               | Ref.[22] | Present |
| 0.3                               | 3.500                            | 3.522                 | 12.870    | 13.065   |
|                                   | 14.250                           | 14.333                | 26.780    | 27.052   |
|                                   | 30.059                           | 30.059                | 43.300    | 43.283   |
|                                   | 45.070                           | 45.070                | 59.000    | 57.876   |
| 0.9                               | 3.900                            | 3.882                 | 12.730    | 13.107   |
|                                   | 15.000                           | 15.114                | 26.700    | 27.460   |
|                                   | 30.900                           | 31.228                | 43.490    | 44.298   |
|                                   | 46.000                           | 46.807                | 59.500    | 59.599   |
| 1.5                               | 3.940                            | 3.959                 | 12.650    | 12.968   |
|                                   | 15.150                           | 15.399                | 26.650    | 27.451   |
|                                   | 31.580                           | 31.827                | 43.580    | 44.653   |
|                                   | 47.700                           | 47.826                | 59.700    | 60.290   |
| 2.1                               | 3.920                            | 3.946                 | 12.600    | 12.852   |
|                                   | 15.250                           | 15.525                | 26.630    | 27.390   |
|                                   | 31.700                           | 32.215                | 43.620    | 44.821   |
|                                   | 48.200                           | 48.522                | 59.740    | 60.640   |
In Table 9, variation of the normalized natural frequencies of an AFG Timoshenko beam (α=2) with respect to the taper ratios; i.e. $c_h$ and $c_b$ is given.

**Table 9.** Natural frequencies of AFG Nonuniform Timoshenko beam (fixed-free)

| $c_b$ | Present 0 | Present 0.1 | Present 0.3 | Present 0.5 | Ref.[22] Present 0 | Ref.[22] Present 0.3 | Ref.[22] Present 0.5 |
|-------|-----------|-------------|-------------|-------------|-------------------|-------------------|-------------------|
| 0     | 3.896     | 4.008       | -           | 4.282       | -                 | 4.654             | -                 |
|       | 15.309    | 15.466      | -           | 15.837      | -                 | 16.329            | -                 |
|       | 31.727    | 31.838      | -           | 32.111      | -                 | 32.495            | -                 |
|       | 47.789    | 47.896      | -           | 48.172      | -                 | 48.587            | -                 |
| 0.1   | 3.948     | 4.060       | 4.049       | 4.334       | -                 | 4.708             | -                 |
|       | 15.197    | 15.349      | 15.313      | 33.000      | -                 | 16.191            | -                 |
|       | 31.323    | 31.433      | 31.380      | 31.703      | -                 | 32.081            | -                 |
|       | 47.732    | 47.833      | 47.823      | 48.088      | -                 | 48.468            | -                 |
| 0.3   | 4.073     | 4.186       | -           | 4.461       | 4.457             | 4.837             | -                 |
|       | 14.887    | 15.030      | -           | 15.373      | 15.358            | 15.832            | -                 |
|       | 30.328    | 30.437      | -           | 30.704      | 30.680            | 31.075            | -                 |
|       | 46.946    | 47.037      | -           | 47.266      | 47.291            | 47.600            | -                 |
| 0.5   | 4.244     | 4.357       | -           | 4.634       | -                 | 5.011             | 5.018             |
|       | 14.442    | 14.577      | -           | 14.901      | -                 | 15.340            | 15.349            |
|       | 29.007    | 29.117      | -           | 29.382      | -                 | 29.751            | 29.757            |
|       | 45.351    | 45.440      | -           | 45.659      | -                 | 45.973            | 46.029            |

Effects of the rotational speed parameter, $\bar{\Omega}$ and the hub radius parameter, $\sigma$ on the dimensionless frequencies of a AFG Timoshenko beam that rotates with a constant angular speed is analyzed in Figure 10 and Figure 11 for $\alpha=1$.

**Figure 10.** Fundamental frequency variation via hub radius and rotational speed.
As it is seen in Figure 10 and Figure 11, the increasing rotational speed increases the natural frequencies and as the hub radius parameter gets larger values, this increasing effect gets more dominant because the centrifugal force becomes more effective on the natural frequencies.

4. CONCLUSION

In this study, finite element formulation of axially functionally graded Euler Bernoulli and Timoshenko beams with different boundary conditions that undergo transverse displacement is derived. The beam models are tapered in one or two planes with different taper ratios. The calculated results are introduced in several figures and tables and compared with the ones in open literature.

Considering the calculated results, the following conclusions are reached:

- The hub rotating speed increases the natural frequencies. This effect becomes more dominant as the hub radius parameter gets larger values.
- Increasing taper ratios have increasing effects on the natural frequencies of Euler Bernoulli and Timoshenko beam models. The increasing effect of the breadth taper ratio, $c_b$, is more dominant.
- The power law index parameter, i.e. $\alpha$, increases the natural frequencies and this increasing effect becomes more dominant on higher frequencies.

CONFLICT OF INTEREST

The authors stated that there are no conflicts of interest regarding the publication of this article.

REFERENCES

[1] Loy CT. Lam KY and Reddy JN. Vibration of functionally graded cylindrical shells, Int. J. Mech. Sci. 1999, 41: 309-324.

[2] Sankar BV. An elasticity solution for functionally graded beams, Compos. Sc. Technol. 2001, 61: 689–696.
[3] Aydogdu M, Taskin V. Free vibration analysis of functionally graded beams with simply supported edges, Mater. Des. 2007, 28:1651–1656.

[4] Chakraborty A, Gopalakrishnan S, Reddy JN. A new beam finite element for the analysis of functionally graded materials, J. Mech.Sci. 2003, 45.

[5] Goupee AJ and Senthil SV, Optimization of natural frequencies of bidirectional functionally graded beams, Struct Multidiscip O 2006; 32:473–484.

[6] Xiang HJ and Yang J, Free and forced vibration of a laminated FGM Timoshenko beam of variable thickness under heat conduction, Compos Part B 2008; 39:292–303.

[7] Piovan MT and Sampoia R, A study on the dynamics of rotating beams with functionally graded properties, J Sound Vib 2009; 327:134–143.

[8] Simsek M and Kocaturk T, Free and forced vibration of a functionally graded beam subjected to a concentrated moving harmonic load, Compos Struct 2009; 90: 465–473.

[9] Malekzadeh P, Golbahar MR and Atashi MM, Out-of-plane free vibration of functionally graded circular curved beams in thermal environment, Compos Struct 2010; 92: 541–552.

[10] Huang Y and Li XF, A new approach for free vibration of axially functionally graded beams with non-uniform cross-section, J Sound Vib 2010; 329:2291–2303.

[11] Shahba A, Attarnejad R, Marvi MT and Hajilar S, Free vibration and stability analysis of axially functionally graded tapered Timoshenko beams with classical and non-classical boundary conditions, Compos Part B 2011; 42(4):801-808.

[12] Zahedinejad P, Zhang C, Zhang C and Shuaia J, A comprehensive review on vibration analysis of functionally graded beams, Int J Struct Stab Dy 2020; 20 (4), 2030002.

[13] Zhang N, Khan T, Guo H, Shi S, Zhong W and Zhang W, Functionally graded materials: An overview of stability, buckling, and free vibration analysis, Adv Mater Sci Eng 2019; 1354150.

[14] Özdemir O, Vibration analysis of rotating Timoshenko beams with different material distribution properties, Selçuk University, Int J Sci 2019; 7(2): 272-286.

[15] Kılıç B, Eksene Fonksiyonel Derecelendirilmiş Rotor Pallerinin Titreşim Analizi, Msc.Thesis, Department of Aeronautical Engineering, Istanbul Technical University, 2019.

[16] Şahin S, İki Eksende Daralan Helikopter Pallerinin Sonlu Elemanlar Metodu ile Titreşim Analizi, Msc.Thesis, Department of Aeronautical Engineering, Istanbul Technical University, 2019.

[17] Hartmann F and Katz C, Structural Analysis with Finite Elements, Springer, 2004.

[18] Downs B, Transverse vibrations of cantilever beams having unequal breadth and depth tapers, J Appl Mech 1977; 44(4): 737-742.

[19] Banerjee JR and Williams FW, Exact Bernoulli–Euler dynamic stiffness matrix for a range of tapered beams, Int J Numer Meth Eng 1985; 21: 2289–2302.
[20] Talebi S and Ariaei A, Vibration analysis of a rotating Timoshenko beam with internal and external flexible connections, Arch Appl Mech 2015; 85(5): 555-572.

[21] Soltani M and Asgarian B, New hybrid approach for free vibration and stability analyses of axially functionally graded Euler-Bernoulli beams with variable cross-section resting on uniform Winkler-Pasternak foundation, Lat Am J Solids Stru 2019; 16(3), e173.

[22] Shahba A, Attarnejad R, Marvi MT and Hajilar S, Free vibration and stability analysis of axially functionally graded tapered Timoshenko beams with classical and non-classical boundary conditions, Compos Part B-Eng 2011; 42(4): 801-808.