Semi-parallelism of normal Jacobi operator
for Hopf hypersurfaces in complex two-plane Grassmannians

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Abstract. It is proved the non-existence of Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, whose normal Jacobi operator is semi-parallel, if the principal curvature of the Reeb vector field is non-vanishing and the component of the Reeb vector field in the maximal quaternionic subbundle $\mathcal{D}$ or its orthogonal complement $\mathcal{D}^\perp$ is invariant by the shape operator.

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1 Introduction

A complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ is the set of all 2-dimensional linear subspaces in $\mathbb{C}^{m+2}$. It is a symmetric space and is equipped with both a Kaehler structure $J$ and a quaternionic Kaehler structure $J'$ with a canonical local basis $\{J_1, J_2, J_3\}$, which does not contain $J$.

Let $M$ be a real hypersurface in $G_2(\mathbb{C}^{m+2})$, $N$ a unit normal vector field of $M$ and $A$ the shape operator of $M$ with respect to $N$. The Reeb vector field of $M$ is the structure vector field given by $\xi = -JN$. Apart from the Reeb vector field, there are three more vector fields given by $\xi_\nu = -J_\nu N$, $\nu = 1, 2, 3$. Consequently, we have two distributions on $M$ given by $[\xi] = \text{Span}\{\xi\}$ and $\mathcal{D}^\perp = \text{Span}\{\xi_1, \xi_2, \xi_3\}$. We denote by $\mathcal{D}$ the orthogonal complement of the distribution $\mathcal{D}^\perp$ such that $T_p M = \mathcal{D}_p \oplus \mathcal{D}^\perp_p$, for each point $p \in M$.

An important geometric condition for real hypersurfaces is the invariantness of the distributions $[\xi]$ and $\mathcal{D}^\perp$ under the action of the shape operator. Under this condition, using a result due to Alekseevskii [1], Berndt and Suh classified the real hypersurfaces in the following:

**Theorem 1.1** (Theorem 1, [4]) Let $M$ be a connected real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then both the distributions $[\xi]$ and $\mathcal{D}^\perp$ are invariant under the shape operator of $M$ if and only if either

- $M$ is of type (A), that is $M$ is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$, or
- $M$ is of type (B), that is $m$ is even, say $m = 2n$, and $M$ is an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{2n+2})$.

A real hypersurface $M$ in $G_2(\mathbb{C}^{m+2})$ is said to be a Hopf hypersurface if the Reeb vector field $\xi$ is principal, that is $A\xi = \alpha\xi$, where $\alpha = g(A\xi, \xi)$ is the corresponding principal curvature to $\xi$. In such a case the integral curves of the Reeb vector field $\xi$ are geodesics.
are Hopf hypersurfaces. Of course, all of hypersurfaces in $G_2(\mathbb{C}^{m+2})$ mentioned in Theorem 1.1 are Hopf hypersurfaces.

In [2], Berndt introduced the notion of normal Jacobi operator
\[
\overline{R}_N(X) = \overline{R}(X, N)N \in \text{End}(T_xM), \quad x \in M,
\]
for a real hypersurface $M$ in quaternionic projective spaces $\mathbb{H}P^m$ and in quaternionic hyperbolic spaces $\mathbb{H}H^m$, where $\overline{R}$ is the curvature tensor of the ambient space. He also proved the equivalence of the commutation of $\overline{R}_N$ with the shape operator $A$ with the fact that the distributions $\mathcal{D}$ and $\mathcal{D}^\perp$ are invariant under the shape operator $A$.

The classification of real hypersurfaces in $G_2(\mathbb{C}^{m+2})$, whose normal Jacobi operator $\overline{R}_N$ satisfies certain geometric conditions, is one of great importance in the area of Differential Geometry. In [15], Perez et. al. proved that $\mathcal{D}^\perp$-invariant real hypersurfaces in $G_2(\mathbb{C}^{m+2})$, whose normal Jacobi operator commutes with both the structure tensor $\varphi$ and the shape operator $A$ are locally congruent to one of type (A). Recently in [11], Jeong, Suh and the second author considered Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ which satisfy the following two commuting conditions
\[
\varphi A\overline{R}_N X = \overline{R}_N \varphi AX, \quad X \in T M \quad \text{and} \quad A\varphi_1 X = \varphi_1 AX, \quad X \in \mathcal{D}^\perp;
\]
and proved that such real hypersurfaces are locally congruent to one of type (A). The first condition is equivalent to $(\mathcal{L}_\xi \overline{R}_N)X = (\nabla_\xi \overline{R}_N)X$.

There are many interesting results concerning the non-existence of real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ under certain geometric conditions on the normal Jacobi operator. In [7], Jeong and Suh examined cases of real hypersurfaces in $G_2(\mathbb{C}^{m+2})$, when the normal Jacobi operator is Lie $\xi$-parallel, that is $\mathcal{L}_\xi \overline{R}_N = 0$. More precisely, they proved the non-existence of real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with $\mathcal{L}_\xi \overline{R}_N = 0$ and one of the conditions $\xi \in \mathcal{D}^\perp$ and $\xi \in \mathcal{D}$. They also proved the non-existence of Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with $\mathcal{L}_\xi \overline{R}_N = 0$ and commuting shape operator on the distribution $\mathcal{D}^\perp$.

In [9], it was proved that a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ does not exist if the normal Jacobi operator is Lie parallel and the integral curves of $\mathcal{D}$- and $\mathcal{D}^\perp$- components of the Reeb vector field are totally geodesic. In [13], Machado et. al. proved the non-existence of Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ whose normal Jacobi operator is of Codazzi type (that is, $(\nabla_X \overline{R}_N)Y = (\nabla_Y \overline{R}_N)X$ for any $X, Y \in TM$) and $\mathcal{D}$- or $\mathcal{D}^\perp$-component of $\xi$ is invariant by the shape operator. In [8], Jeong et. al. proved the non-existence of Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with parallel normal Jacobi operator, that is $\nabla_X \overline{R}_N = 0$. In [10], the non-existence of Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ whose normal Jacobi operator is $(\{\xi\} \cup \mathcal{D}^\perp)$-parallel, which is a weaker condition then the previous one, was proved.

A tensor field $P$ of type $(1, s)$ on a Riemannian manifold is said to be semi-parallel if $R \cdot P = 0$, where $R$ is the curvature tensor of the manifold and acts as a derivation on $P$ [6]. In the geometry of real hypersurfaces in complex space form the following results concerning the semi-parallelism conditions have been proved. In [16], Perez and Santos proved that there exist no real hypersurfaces in complex projective space $CP^n$, $n \geq 3$, with semi-parallel structure Jacobi operator (that is $R \cdot R_\xi = 0$, where $R_\xi = R(\cdot, \xi)\xi$ and $\xi$ is the structure vector field). Later, Cho and Kimura [6] generalized this work and proved that there do not
exist real hypersurfaces in complex space forms equipped with semi-parallel structure Jacobi operator. Finally, Niebergall and Ryan in [14] studied real hypersurfaces in complex space forms equipped with the semi-parallel shape operator \( A \).

Motivated by these studies the following question is raised naturally:

**Problem 1.2** Do there exist real hypersurfaces in \( G_2(\mathbb{C}^{m+2}) \), \( m \geq 3 \), whose normal Jacobi operator, structure Jacobi operator or shape operator is semi-parallel?

In the present paper we give the answer partially and prove the following:

**Theorem 1.3** There does not exist any connected Hopf hypersurface \( M \) in \( G_2(\mathbb{C}^{m+2}) \), \( m \geq 3 \), equipped with semi-parallel normal Jacobi operator, if \( \alpha \neq 0 \) and \( \mathcal{D} \)- or \( \mathcal{D}^\perp \)-component of the Reeb vector field \( \xi \) is invariant by the shape operator \( A \).

The paper is organized as follows. In section 2, we give a brief description of complex two plane Grassmanians. In section 3 basic relations for real hypersurfaces in \( G_2(\mathbb{C}^{m+2}) \) are presented. Section 4 contains some key results for further use. Finally, in section 5, we give the proof of Theorem 1.3.

## 2 Riemannian Geometry of \( G_2(\mathbb{C}^{m+2}) \)

The complex two-plane Grassmannian \( G_2(\mathbb{C}^{m+2}) \) is the Grassmann manifold of all complex 2-dimensional linear subspaces in \( \mathbb{C}^{m+2} \). The special unitary group \( G = SU(m + 2) \) acts transitively on \( G_2(\mathbb{C}^{m+2}) \) with stabilizer isomorphic to \( K = S(U(2) \times U(m)) \subset G \). Thus \( G_2(\mathbb{C}^{m+2}) \) can be identified with the homogeneous space \( G/K \), which can be equipped with the unique analytic structure for which the natural action of \( G \) on \( G_2(\mathbb{C}^{m+2}) \) becomes analytic. Denote by \( \mathfrak{g} \) and \( \mathfrak{l} \) the Lie algebra of \( G \) and \( K \), respectively. Let \( \mathfrak{m} \) be the orthogonal complement of \( \mathfrak{l} \) in \( \mathfrak{g} \) with respect to the Cartan-Killing form \( B \) of \( \mathfrak{g} \). Then \( \mathfrak{g} = \mathfrak{l} \oplus \mathfrak{m} \) is an \( Ad(K) \)-invariant reductive decomposition of \( \mathfrak{g} \). We put \( o = eK \) and identify \( T_oG_2(\mathbb{C}^{m+2}) \) with \( \mathfrak{m} \) in the usual manner. Since \( B \) is negative definite on \( \mathfrak{g} \), therefore the restriction \( -B|_{\mathfrak{m} \times \mathfrak{m}} \) yields a positive definite inner product on \( \mathfrak{m} \). By \( Ad(K) \)-invariance of \( B \) this inner product can be extended to a \( G \)-invariant Riemannian metric \( g \) on \( G_2(\mathbb{C}^{m+2}) \). In this manner \( G_2(\mathbb{C}^{m+2}) \) becomes a Riemannian homogeneous symmetric space. For computational reasons we normalize the Riemannian metric \( g \) such that the maximal sectional curvature of \( (G_2(\mathbb{C}^{m+2}), g) \) becomes 8.

When \( m = 1 \), \( G_2(\mathbb{C}^3) \) is isometric to the 2-dimensional complex projective space \( \mathbb{C}P^2 \) with constant holomorphic sectional curvature 8. When \( m = 2 \), the isomorphism \( Spin(6) \cong SU(4) \) provides an isometry between \( G_2(\mathbb{C}^4) \) and the real Grassmann manifold \( G_2^+(\mathbb{R}^6) \) of oriented 2-dimensional linear subspaces of \( \mathbb{R}^6 \). Therefore, we usually assume that \( m \geq 3 \).

The Lie algebra \( \mathfrak{l} \) has the direct sum decomposition \( \mathfrak{l} = \mathfrak{su}(m) \oplus \mathfrak{su}(m) \oplus \mathfrak{R} \), where \( \mathfrak{R} \) is the center of \( \mathfrak{l} \). Regarding \( \mathfrak{l} \) as the holonomy algebra of \( G_2(\mathbb{C}^{m+2}) \), the center \( \mathfrak{R} \) induces a Kaehler structure \( J \) and the \( \mathfrak{su}(2) \)-part induces a quaternionic Kaehler structure \( \mathcal{J} \) on \( G_2(\mathbb{C}^{m+2}) \). If \( J_\nu \) is any almost Hermitian structure in \( \mathcal{J} \), then \( JJ_\nu = J_\nu J \), and \( JJ_\nu \) is a symmetric endomorphism with \( (JJ_\nu)^2 = I \) and \( \text{tr}(JJ_\nu) = 0 \).
A canonical local basis \( \{ J_1, J_2, J_3 \} \) of \( \mathfrak{J} \) consists of three local almost Hermitian structures \( J_\nu \) in \( \mathfrak{J} \) such that \( J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1} J_\nu \), where the index is taken modulo 3. Since \( \mathfrak{J} \) is parallel with respect to the Riemannian connection \( \nabla \) of \( (G_2(\mathbb{C}^{m+2}), g) \), there exist for any canonical local basis \( J_1, J_2, J_3 \) of \( \mathfrak{J} \) three local 1-forms \( q_1, q_2, q_3 \), such that
\[
\nabla_X J_\nu = q_{\nu+2}(X) J_{\nu+1} - q_{\nu+1}(X) J_{\nu+2} \tag{2.1}
\]
for all vector fields \( X \) on \( G_2(\mathbb{C}^{m+2}) \).

The Riemann curvature tensor \( \overline{R} \) of \( G_2(\mathbb{C}^{m+2}) \) is locally given by [3]
\[
\overline{R}(X, Y)Z = g(Y, Z)X - g(X, Z)Y + g(JY, Z)JX - 2g(JX, Y)JZ + \sum_{\nu=1}^{3} \{ g(J_\nu Y, Z) J_\nu X - g(J_\nu X, Z) J_\nu Y - 2g(J_\nu X, Y) J_\nu Z \} + \sum_{\nu=1}^{3} \{ g(J_\nu JY, Z) J_\nu JX - g(J_\nu JX, Z) J_\nu JY \} \tag{2.2}
\]
for all vector fields \( X, Y, Z \) on \( G_2(\mathbb{C}^{m+2}) \), where \( \{ J_1, J_2, J_3 \} \) is any canonical local basis of \( \mathfrak{J} \). This expression involves the Riemannian curvature tensor of \( S^{4m}, \mathbb{C}P^{2m} \) and \( \mathbb{H}P^{m} \).

3 Real hypersurfaces in \( G_2(\mathbb{C}^{m+2}) \)

Let \( M \) be a real hypersurface in \( G_2(\mathbb{C}^{m+2}) \), that is a hypersurface of \( G_2(\mathbb{C}^{m+2}) \) with real codimension one. The induced Riemannian metric on \( M \) is denoted by \( g \) and \( \nabla \) denotes the induced Riemannian connection of \( (M, g) \). Let \( N \) be a local unit normal field of \( M \) and \( A \) the shape operator of \( M \) with respect to \( N \).

Now let us put
\[
JX = \varphi X + \eta(X)N, \quad J_\nu X = \varphi_\nu X + \eta_\nu(X)N \tag{3.1}
\]
for any tangent vector \( X \) of a real hypersurface \( M \) in \( G_2(\mathbb{C}^{m+2}) \).

The Kaehler structure \( J \) of \( G_2(\mathbb{C}^{m+2}) \) induces a local almost contact metric structure \( (\varphi, \xi, \eta, g) \) on \( M \) in the following way
\[
\varphi^2 X = -X + \eta(X)\xi, \quad \eta(X) = 1, \quad \varphi \xi = 0, \quad \eta(X) = g(x, \xi).
\]

If \( M \) is orientable then \( \xi \) is globally defined and is the induced Reeb vector field on \( M \). Furthermore, let \( \{ J_1, J_2, J_3 \} \) be a canonical local basis of \( \mathfrak{J} \). Then each \( J_\nu \) induces an almost contact metric structure \( (\varphi_\nu, \xi_\nu, \eta_\nu, g) \) on \( M \). Locally, the orthogonal complement of the real span of \( \xi \) in \( TM \) is denoted by \( \mathfrak{J} \) and the orthogonal complement of the real span of \( \xi_1, \xi_2, \xi_3 \) in \( TM \) is denoted by \( \mathfrak{D} \).
In view of (2.2), the Gauss equation is given by

\[ R(X, Y)Z = g(Y, Z)X - g(X, Z)Y + g(\varphi Y, Z)\varphi X - g(\varphi X, Z)\varphi Y - 2g(\varphi X, Y)\varphi Z + \sum_{\nu=1}^{3} \left\{ g(\varphi_{\nu} Y, Z)\varphi_{\nu} X - g(\varphi_{\nu} X, Z)\varphi_{\nu} Y - 2g(\varphi_{\nu} X, Y)\varphi_{\nu} Z \right\} + \sum_{\nu=1}^{3} \left\{ g(\varphi_{\nu} Y, Z)\varphi_{\nu} \varphi X - g(\varphi_{\nu} \varphi X, Z)\varphi_{\nu} \varphi Y \right\} - \sum_{\nu=1}^{3} \left\{ \eta(Y)\eta_{\nu}(Z)\varphi_{\nu} \varphi X - \eta(X)\eta_{\nu}(Z)\varphi_{\nu} \varphi Y \right\} - \sum_{\nu=1}^{3} \left\{ \eta(X)g(\varphi_{\nu} \varphi Y, Z) - \eta(Y)g(\varphi_{\nu} \varphi X, Z) \right\} \xi_{\nu} + g(AY, Z)AX - g(AX, Z)AY \]  

(3.2)

where \( R \) denotes the curvature tensor of the real hypersurface \( M \) in \( G_{2}(\mathbb{C}^{m+2}) \).

It is straightforward to verify the following identities

\[ \varphi_{\nu} \xi_{\nu+1} = \xi_{\nu+2}, \quad \varphi_{\nu+1} \xi_{\nu} = -\xi_{\nu+2}, \]
\[ \varphi_{\nu} \xi_{\nu} = \varphi_{\nu} \xi, \quad \eta_{\nu}(\varphi X) = \eta(\varphi_{\nu} X), \]
\[ \varphi_{\nu} \varphi_{\nu+1} X = \varphi_{\nu+2} X + \eta_{\nu+1}(X)\xi_{\nu}, \]
\[ \varphi_{\nu+1} \varphi_{\nu} X = -\varphi_{\nu+2} X + \eta_{\nu}(X)\xi_{\nu+1}. \]

(3.3)

In view of (3.1), (2.1) and (3.3), it is known that

\[ (\nabla_X \varphi) Y = \eta(Y) AX - g(AX, Y) \xi, \quad \nabla_X \xi = \varphi AX, \]
\[ \nabla_X \xi_{\nu} = q_{\nu+2}(X)\xi_{\nu+1} - q_{\nu+1}(X)\xi_{\nu+2} + \varphi_{\nu} AX, \]
\[ (\nabla_X \varphi_{\nu}) Y = -q_{\nu+1}(X)\varphi_{\nu+2} Y + q_{\nu+2}(X)\varphi_{\nu+1} Y + \eta_{\nu}(Y) AX - g(AX, Y) \xi_{\nu}. \]

Summing up these formulas, we also find the following

\[ \nabla_X (\varphi_{\nu} \xi) = (\nabla_X \varphi_{\nu}) \xi + \varphi_{\nu} (\nabla_X \xi) = -q_{\nu+1}(X)\varphi_{\nu+2} \xi + q_{\nu+2}(X)\varphi_{\nu+1} \xi + \eta_{\nu}(\xi) AX - g(AX, \xi) \xi_{\nu} + \varphi_{\nu} \varphi AX. \]

Moreover, from \( JJ_{\nu} = J_{\nu} J, \nu = 1, 2, 3 \), it follows that

\[ \varphi_{\nu} \varphi X = \varphi_{\nu} X - \eta_{\nu}(X) \xi + \eta(X) \xi_{\nu}. \]

For more details we refer to [1], [3], [4] and [5].
4 Key Lemmas

We consider a connected, orientable, Hopf hypersurface \( M \) in \( G_2(C^{m+2}) \) with \( \alpha \neq 0 \) and semi-parallel normal Jacobi operator. The normal Jacobi operator \( \overline{R}_N \) for a real hypersurface \( M \) in \( G_2(C^{m+2}) \) is given by

\[
\overline{R}_N(X) = X + 3\eta(X)\xi + \sum_{\nu=1}^{3} \eta_{\nu}(X)\xi_{\nu}
\]

\[
- \sum_{\nu=1}^{3} \{ \eta_{\nu}(\xi) (\varphi_{\nu}\varphi X - \eta(X)\xi_{\nu}) - \eta_{\nu}(\varphi X)\varphi_{\nu}\xi \} \tag{4.1}
\]

for any vector field \( X \) tangent to \( M \). Furthermore, semi-parallelism condition of it, that is \( R(X,Y) \cdot \overline{R}_N = 0 \), implies

\[
R(X,Y)\overline{R}_NZ = \overline{R}_N(R(X,Y)Z) \tag{4.2}
\]

for all vector fields \( X, Y, Z \) tangent to \( M \).

**Lemma 4.1** Let \( M \) be a Hopf hypersurface in \( G_2(C^{m+2}) \) such that \( \mathcal{D} \)- or \( \mathcal{D}^\perp \)-component of \( \xi \) is invariant by the shape operator \( A \) and \( \alpha \neq 0 \). If the normal Jacobi operator is semi-parallel, then \( \xi \in \mathcal{D} \) or \( \xi \in \mathcal{D}^\perp \).

**Proof.** Suppose that \( \xi \) is written as

\[
\xi = \eta(U)U + \eta(\xi_1)\xi_1 + \eta(\xi_2)\xi_2 + \eta(\xi_3)\xi_3, \tag{4.3}
\]

where \( U \) is a unit vector in \( \mathcal{D} \) and \( \eta(U) \neq 0 \) and \( \eta(\xi_\kappa) \neq 0 \) for at least one \( \kappa \in \{1,2,3\} \). Then relation (4.3) implies that

\[
\varphi_\kappa \xi = \eta(U)\varphi_\kappa U + \eta(\xi_{\kappa+1})\xi_{\kappa+2} - \eta(\xi_{\kappa+2})\xi_{\kappa+1}. \tag{4.4}
\]

From (4.1), we get

\[
\overline{R}_N(\xi) = 4\xi + 4 \sum_{\nu=1}^{3} \eta(\xi_{\nu})(\xi_{\nu}), \tag{4.5}
\]

\[
\overline{R}_N(\xi_\kappa) = 4\xi_\kappa + 4\eta(\xi_\kappa)\xi + 2\eta(\xi_{\kappa+1})\varphi_{\kappa+2}\xi - 2\eta(\xi_{\kappa+2})\varphi_{\kappa+1}\xi, \tag{4.6}
\]

\[
\overline{R}_N(\varphi_\kappa \xi) = 2\eta(\xi_{\kappa+1})\xi_{\kappa+2} - 2\eta(\xi_{\kappa+2})\xi_{\kappa+1}. \tag{4.7}
\]

Since the normal Jacobi operator is semi-parallel, from (4.2) and (4.5), we get

\[
\overline{R}_N(R(\xi,\xi_\kappa)\xi) = 4R(\xi,\xi_\kappa)\xi + 4 \sum_{\nu=1}^{3} \eta(\xi_{\nu})R(\xi,\xi_\kappa)\xi_{\nu}. \tag{4.8}
\]

Since \( \mathcal{D} \)- or \( \mathcal{D}^\perp \)-component of \( \xi \) is assumed to be invariant by the shape operator \( A \), we obtain

\[
AU = \alpha U \quad \text{and} \quad A\xi_\kappa = \alpha \xi_\kappa, \quad \kappa \in \{1,2,3\}. \tag{4.9}
\]
In view of (4.9), from relation (3.2) we get
\[ R(\xi, \xi_\kappa)\xi = \alpha^2 \eta(\xi_\kappa)\xi - \alpha^2 \xi_\kappa + 2\eta(\xi_{\kappa+1})\varphi_{\kappa+2}\xi - 2\eta(\xi_{\kappa+2})\varphi_{\kappa+1}\xi. \] (4.10)

Substituting (4.10) in (4.8), we lead to the following
\[ 4 \sum_{\nu=1}^{3} \eta(\xi_\nu)R(\xi, \xi_\kappa)\xi_\nu = \alpha^2 \eta(\xi_\kappa)R_N(\xi) - \alpha^2 R_N(\xi_\kappa) \\
+ 2\eta(\xi_{\kappa+1})R_N(\varphi_{\kappa+2}\xi) - 2\eta(\xi_{\kappa+2})R_N(\varphi_{\kappa+1}\xi) \] (4.11)
\[ - 4\alpha^2 \eta(\xi_\kappa)\xi + 4\alpha^2 \xi_\kappa \\
- 8\eta(\xi_{\kappa+1})\varphi_{\kappa+2}\xi + 8\eta(\xi_{\kappa+2})\varphi_{\kappa+1}\xi. \]

Taking the inner product of (4.11) with \( U \), in view of (4.6), (4.7) and (4.4) we obtain
\[ \sum_{\nu=1}^{3} \eta(\xi_\nu)g(R(\xi, \xi_\kappa)\xi_\nu, U) = -\alpha^2 \eta(\xi_\kappa)\eta(U). \] (4.12)

We calculate \( R(\xi, \xi_\kappa)\xi_\nu \) from relation (3.2) taking into account (4.9) and then we take the inner product with \( U \) and we lead to the following relation
\[ g(R(\xi, \xi_\kappa)\xi_\nu, U) = \alpha^2 \eta_\kappa(\xi_\nu)\eta(U). \] (4.13)

From (4.12) and (4.13) we get
\[ \alpha^2 \eta(\xi_\kappa)\eta(U) = 0, \quad \kappa \in \{1, 2, 3\}, \]
which is a contradiction. □

Now, we examine the case when the Reeb vector field \( \xi \) belongs to the distribution \( \mathcal{D}^\perp \). In fact, we have the following

**Lemma 4.2** Let \( M \) be a Hopf hypersurface in \( G_2(\mathbb{C}^{m+2}) \) and \( \alpha \neq 0 \), with semi-parallel normal Jacobi operator and \( \xi \in \mathcal{D}^\perp \) then \( g(A\mathcal{D}, \mathcal{D}^\perp) = 0 \).

**Proof.** Let \( W \in \mathcal{D} \) arbitrarily. In order to prove that \( g(A\mathcal{D}, \mathcal{D}^\perp) = 0 \), it suffices to prove that \( g(AW, \xi_\kappa) = 0, \kappa = 1, 2, 3 \). Since \( \xi \in \mathcal{D}^\perp \), we have that \( JN \in \mathfrak{j}N \). Let \( J_1 \) be an almost Hermitian structure of \( \mathfrak{j} \) such that \( JN = J_1N \). Then we obtain that \( \xi = \xi_1 \) and \( \eta(\xi_2) = \eta(\xi_3) = 0 \). Furthermore, \( \varphi_{\xi_2} = -\xi_3, \varphi_{\xi_3} = \xi_2 \) and \( \varphi(\mathcal{D}) \subset \mathcal{D} \).

Due to the fact that \( M \) is a Hopf hypersurface, we have that \( A\xi = \alpha \xi \) and so \( g(AW, \xi) = g(AW, \xi_1) = 0 \). Thus, it remains to prove that \( g(AW, \xi_\kappa) = 0, \quad \kappa = 2, 3 \).

From (4.1), we obtain
\[ \overline{R}_N(\xi) = 8\xi, \quad \overline{R}_N(W) = W - \varphi_1 \varphi W. \] (4.14)

Using (4.14) in (4.2) we get
\[ 8R(W, \xi)\xi = \overline{R}_N(R(W, \xi)\xi). \] (4.15)
In view of $A\xi = \alpha \xi$, from (3.2), it follows that

$$R(W, \xi)\xi = W + \alpha AW - \varphi_1 \varphi W.$$  

(4.16)

Substituting (4.16) in (4.15) and taking into consideration (4.14) we lead to the following

$$8W + 8\alpha AW - 8\varphi_1 \varphi W = R_N(W) + \alpha R_N(AW) - R_N(\varphi_1 \varphi W).$$  

(4.17)

From (4.1) we also get

$$R_N(AW) = AW + 2\eta_2(AW)\xi_2 + 2\eta_3(AW)\xi_3 - \varphi_1 \varphi AW,$$

$$R_N(\varphi_1 \varphi W) = \varphi_1 \varphi W - \varphi_1 \varphi (\varphi_1 \varphi W).$$

Substitution of the previous two relations in (4.17) gives

$$7W + 7\alpha AW - 6\varphi_1 \varphi W = 2\alpha \eta_2(AW)\xi_2 + 2\alpha \eta_3(AW)\xi_3 + \varphi_1 \varphi (\varphi_1 \varphi W) - \alpha \varphi_1 \varphi AW.$$  

Taking the inner product of the last relation with $\xi_\kappa$, $\kappa = 2, 3,$ and because of $\alpha \neq 0$ implies

$$\eta_\kappa(AW) = 0, \quad \kappa = 2, 3,$$

and this completes the proof. $\blacksquare$

Finally, in the case when the Reeb vector field $\xi$ belongs to the distribution $\frak{D}$, we refer to the following

**Proposition 4.3** (Proposition 3.1, [12]) Let $M$ be a connected orientable Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$. If the Reeb vector $\xi$ belongs to the distribution $\frak{D}$, then the distribution $\frak{D}$ is invariant under the shape operator $A$ of $M$, that is $g(A\frak{D}, \frak{D}^\perp) = 0$.

## 5 Proof of Theorem 1.3

In the previous section, because of Lemma 4.2, Proposition 4.3 and Theorem 1.1, we lead to the conclusion that real hypersurfaces in $G_2(\mathbb{C}^{m+2})$, under some additional assumptions, whose normal Jacobi operator is semi-parallel are locally congruent to real hypersurfaces of type (A) or (B). Now, we check if the normal Jacobi operator of such real hypersurfaces satisfies the semi-parallelism condition.

First, we recall the following proposition due to Berndt and Suh ([4]).

**Proposition 5.1** (Proposition 3, [4]) Let $M$ be a connected real hypersurface of $G_2(\mathbb{C}^{m+2})$. Suppose that $A\frak{D} \subset \frak{D}$, $A\xi = \alpha \xi$ and $\xi$ is tangent to $\frak{D}^\perp$. Let $J_1 \in \frak{J}$ be the almost Hermitian structure such that $JN = J_1 N$. Then $M$ has three (if $r = \frac{\pi}{2\sqrt{8}}$) or four (otherwise) distinct constant principal curvatures

$$\alpha = \sqrt{8} \cot(\sqrt{8}r), \quad \beta = \sqrt{2} \cot(\sqrt{2}r), \quad \lambda = -\sqrt{2} \tan(\sqrt{2}r), \quad \mu = 0,$$

with some $r \in (0, \frac{\pi}{2\sqrt{8}})$. The corresponding multiplicities are

$$m(\alpha) = 1, \quad m(\beta) = 2, \quad m(\lambda) = 2m - 2 = m(\mu),$$

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and the corresponding eigenspaces are
\[ T_\alpha = \mathbb{R} \xi = \mathbb{R} \xi_1 = \mathbb{R} J N = \text{Span} \{ \xi \} = \text{Span} \{ \xi_1 \}, \]
\[ T_\beta = \mathbb{C}^+ \xi = \mathbb{C}^+ N = \mathbb{R} \xi_2 \oplus \mathbb{R} \xi_3 = \text{Span} \{ \xi_2, \xi_3 \}, \]
\[ T_\gamma = \{ X/X \perp \mathbb{H} \xi, JX = J_1 X \}, \]
\[ T_\mu = \{ X/X \perp \mathbb{H} \xi, JX = -J_1 X \}, \]
where \( \mathbb{R} \xi, \mathbb{C} \xi \) and \( \mathbb{H} \xi \) respectively denotes real, complex, quaternionic span of the structure vector field \( \xi \) and \( \mathbb{C}^+ \xi \) denotes the orthogonal complement of the \( \mathbb{C} \xi \) in \( \mathbb{H} \xi \).

In this case we have \( \xi = \xi_1 \). From (4.1) we obtain
\[ \mathcal{R}_N(\xi) = 8 \xi \quad \text{and} \quad \mathcal{R}_N(\xi_2) = 2 \xi_2. \] (5.1)
Since the normal Jacobi operator is semi-parallel, from (4.2) and the second relation of (5.1) we obtain:
\[ 2R(\xi_2, \xi)\xi_2 = \mathcal{R}_N(R(\xi_2, \xi)\xi_2), \] (5.2)
Relation (3.2) for \( X = \xi_2, Y = \xi \) and \( Z = \xi_2 \) taking into account the fact that \( A\xi = \alpha \xi \) and \( A\xi_2 = \beta \xi_2 \) implies
\[ R(\xi_2, \xi)\xi_2 = -(2 + \alpha \beta)\xi. \] (5.3)
Substitution of relation (5.3) in (5.2) leads to
\[ (2 + \alpha \beta)\xi = 0. \]
The last relation taking into account that \( \alpha = \sqrt{8} \cot(\sqrt{8} r) \) and \( \beta = \sqrt{2} \cot(\sqrt{2} r) \) implies
\[ \cot^2(\sqrt{2} r) = 0, \]
which is a contradiction. So real hypersurfaces of type (A) do not have semi-parallel normal Jacobi operator.

Next we check that whether real hypersurfaces of type (B) are equipped with semi-parallel normal Jacobi operator. We recall the following proposition due to Berndt and Suh ([4]).

**Proposition 5.2** (Proposition 2, [4]) Let \( M \) be a connected real hypersurface of \( G_2(\mathbb{C}^{m+2}) \). Suppose that \( AD \subset \mathcal{D} \), \( A\xi = \alpha \xi \) and \( \xi \) is tangent to \( \mathcal{D} \). Then the quaternionic dimension \( m \) of \( G_2(\mathbb{C}^{m+2}) \) is even, say \( m = 2n \), and \( M \) has five distinct constant principal curvatures
\[ \alpha = -2 \tan(2r), \quad \beta = 2 \cot(2r), \quad \gamma = 0, \quad \lambda = \cot(r), \quad \mu = -\tan(r), \]
with some \( r \in (0, \pi/4) \). The corresponding multiplicities are
\[ m(\alpha) = 1, \quad m(\beta) = 3 = m(\gamma), \quad m(\lambda) = 4n - 4 = m(\mu), \]
and the corresponding eigenspaces are
\[ T_\alpha = \mathbb{R} \xi = \text{Span} \{ \xi \}, \]
\[ T_\beta = \mathfrak{J} J \xi = \text{Span} \{ \xi_1, \xi_2, \xi_3 \}, \]
\[ T_\gamma = \mathfrak{J} \xi = \text{Span} \{ \varphi_1 \xi, \varphi_2 \xi, \varphi_3 \xi \}, \]
\[ T_\lambda, T_\mu, \]
where
\[ T_\lambda \oplus T_\mu = (\mathbb{H} \mathbb{C} \xi)^+, \quad \mathfrak{J} T_\lambda = T_\lambda, \quad \mathfrak{J} T_\mu = T_\mu, \quad J T_\lambda = T_\mu. \]
From (4.1) we obtain
\[ \mathcal{R}_N(W) = W, \quad \mathcal{R}_N(\xi) = 4\xi \quad \text{and} \quad \mathcal{R}_N(\xi_\nu) = 4\xi_\nu, \quad \nu = 1, 2, 3, \] (5.4)
where \( W \in T_\lambda \). Due to the semi-parallelism of the normal Jacobi operator, from (4.2) and the first relation of (5.4) we get:
\[ R(W, \xi)W = \mathcal{R}_N(R(W, \xi)W), \] (5.5)
The Gauss equation (3.2) for \( X = W, \ Y = \xi \) and \( Z = W \), because of \( A\xi = \alpha \xi \) and \( AW = \lambda W \) implies
\[ R(W, \xi)W = -(1 + \alpha \lambda) \xi + \sum_{\nu=1}^{3} g(\varphi_\nu W, W)\xi_\nu. \] (5.6)
Substituting (5.6) in (5.5) and taking into account relation (5.4), we lead to the following
\[ [1 + \alpha \lambda] \xi - \sum_{\nu=1}^{3} g(\varphi_\nu W, W)\xi_\nu = 0. \]
The inner product of the last relation with \( \xi \) and substitution of \( \alpha = -2\tan(2r) \) and \( \lambda = \cot(r) \) yield
\[ 1 - 2\tan(2r)\cot(r) = 0, \]
from which we obtain
\[ 3 + \tan^2(r) = 0, \]
which is a contradiction. So real hypersurfaces of type (B) do not admit semi-parallel normal Jacobi operator and this completes the proof. \( \blacksquare \)

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