DESERT OF DELIGNE GROUPOIDS

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Abstract. To any non-negatively graded dg Lie algebra \( \mathfrak{g} \) over a field \( k \) of characteristic zero we assign a functor \( \Sigma_\mathfrak{g} : \text{art}/k \to \text{Kan} \) from the category of commutative local artinian \( k \)-algebras with the residue field \( k \) to the category of Kan simplicial sets. There is a natural homotopy equivalence between \( \Sigma_\mathfrak{g} \) and the Deligne groupoid corresponding to \( \mathfrak{g} \).

The main result of the paper claims that the functor \( \Sigma \) commutes up to homotopy with the "total space" functors which assign a dg Lie algebra to a cosimplicial dg Lie algebra and a simplicial set to a cosimplicial simplicial set. This proves a conjecture of Schechtman [S1, S2, HS3] which implies that if a deformation problem is described "locally" by a sheaf of dg Lie algebras \( \mathfrak{g} \) on a topological space \( X \) then the global deformation problem is described by the homotopy Lie algebra \( \mathbf{R} \Gamma(X, \mathfrak{g}) \).

1. Introduction

1.1. Let \( \mathfrak{g} \) be a dg Lie algebra over a field \( k \) of characteristic zero concentrated in non-negative degrees. The algebra \( \mathfrak{g} \) defines a functor

\[ \mathcal{C}_\mathfrak{g} : \text{art}/k \to \text{Grp} \]

from the category of local artinian \( k \)-algebras with the residue field \( k \) to the category of groupoids — see [2,2] or [GM1], sect. 2.

A common belief is that any "reasonable" formal deformation problem can be described by the functor \( \mathcal{C}_\mathfrak{g} \) where \( \mathfrak{g} \) is an appropriate "Lie algebra of infinitesimal automorphisms". This would imply, for instance, that if \( H^0(\mathfrak{g}) = 0 \) (i.e. if the automorphism group of the deformed object is discrete) then the completion of the local ring of a moduli space at a given point is isomorphic to the completion of the 0-th cohomology group of \( \mathfrak{g} \).

If we are dealing with the deformations of algebraic structures (associative, commutative or Lie algebras or so), the Lie algebra \( \mathfrak{g} \) is just the standard complex calculating the cohomology of the appropriate type.

In this paper we prove the following claim conjectured by V. Schechtman in [S1, S2, HS3]. It allows one to construct a dg Lie algebra which governs various formal deformations in the non-affine case.

Let \( X \) be a topological space and let \( \mathfrak{g} \) be a sheaf of dg Lie \( k \)-algebras. Let \( \mathcal{U} = \{ U_i \} \) be an locally finite open covering of \( X \), \( \mathcal{D}_i = \mathcal{C}_\mathcal{U}(U_i, \mathfrak{g}) \) and let \( \mathcal{D} \) be the groupoid of "descent data" for the collection \( \mathcal{D}_i \) (see [2.2, 12]). Then \( \mathcal{D} \) is naturally equivalent to the groupoid \( \mathcal{C}_L \) where \( L \) is a dg Lie algebra representing the Cech complex \( \check{C}(\mathcal{U}, \mathfrak{g}) \).
This result implies, in particular, Corollary 5.2 which claims that Theorem 8.3 of [HS2] remains valid without the assumption of formal smoothness.

1.2. Structure of the Sections. In Section 2 we define the contents \( \Sigma(g) \) of a nilpotent dg Lie algebra \( g \). It is a Kan simplicial set homotopically equivalent to the Deligne groupoid \( C(g) \). In Section 3 we recall the definition of the total space functor in different categories. We also present Proposition 3.4.8 giving a sufficient condition for a map of bisimplicial dg algebras to be an acyclic fibration in the sense of 3.4. Now the claim 1.1 can be interpreted as the commutativity of \( \Sigma \) with the functors \( \text{Tot} \). The main Theorem 4.1 is proven in Section 4. In the last Section 5 we deduce from Theorem 4.1 an application to formal deformation theory.

The idea that the generalization of the result of [HS2] to the non-smooth case should follow from a descent property for Deligne groupoids belongs to V. Schechtman. I am very grateful to him for helpful discussions on the subject. Proposition 2.2.3 claiming that the content \( \Sigma(g) \) and the Deligne groupoid \( C(g) \) are homotopy equivalent is consonant to the Main Homotopy Theorem of Schlessinger–Stasheff, cf. [SS]. I am greatly indebted to J. Stasheff who read the first draft of the manuscript and made some important remarks.

1.3. Notations. Throughout this paper \( k \) is a fixed field of characteristic zero.

\( \text{art}/k \) denotes the category of commutative local artinian \( k \)-algebras having the residue field \( k \).

\( \text{dglie}(k) \) (resp., \( \text{cdga}(k) \)) is the category of non-negatively graded dg Lie (resp., commutative) algebras over \( k \).

\( \Delta \) is the category of ordered sets \( [n] = \{0, \ldots, n\}, \ n \geq 0 \) and monotone maps; \( \Delta^0\text{Ens} \) is the category of simplicial sets; \( \Delta^n \in \Delta^0\text{Ens} \) are the standard \( n \)-simplices.

\( \text{Kan} \subseteq \Delta^0\text{Ens} \) is the full subcategory of Kan simplicial sets.

\( \text{Ab} \) is the category of abelian groups. \( C(A) \) (sometimes \( C(R) \)) is the category of complexes over an abelian category \( A \) (over the category of \( R \)-modules). \( C_{\geq 0} \) denotes the full subcategory of non-negatively graded complexes.

2. Contents of a nilpotent Lie dg algebra

2.1. For any \( n \geq 0 \) denote by \( \Omega_n \) the \( k \)-algebra of polynomial differential forms on the standard \( n \)-simplex \( \Delta^n \) — see [BG].

One has

\[ \Omega_n = k[t_0, \ldots, t_n, dt_0, \ldots, dt_n]/(\sum t_i - 1, \sum dt_i). \]

The algebras \( \Omega_n \) form a simplicial commutative dg algebra: a map \( u : [p] \to [q] \) induces the map \( \Omega(u) : \Omega_q \to \Omega_p \) defined by the formula \( \Omega(u)(t_i) = \sum_{u(j)=i} t_j \).

If \( g \) is a dg Lie \( k \)-algebra and \( A \) is a commutative dg \( k \)-algebra then the tensor product \( A \otimes g \) is also a dg Lie \( k \)-algebra. Thus, any dg Lie algebra \( g \) gives rise to a simplicial dg Lie algebra

\[ g_\bullet = \{ \Omega_n \otimes g \}_{n \geq 0}. \]
For any dg Lie algebra \( g \) denote by \( \text{MC}(g) \) the set of elements \( x \in g^1 \) satisfying the Maurer-Cartan equation:

\[
dx + \frac{1}{2}[x,x] = 0.
\]

2.1.1. **Definition.** Let \( g \in \text{dglie}(k) \) be nilpotent. Its contents \( \Sigma(g) \in \Delta^0\text{Ens} \) is defined as 

\[
\Sigma(g) = \text{MC}(g^*).
\]

2.1.2. Recall (see [BG]) that the collection of commutative dg algebras \( \Omega_n \) defines a contravariant functor

\[
\Omega : \Delta^0\text{Ens} \to \text{cdga}(k)
\]
so that \( \Omega(\Delta^n) = \Omega_n \) and \( \Omega \) carries direct limits in \( \Delta^0\text{Ens} \) to inverse limits.

**Lemma.** Let \( S \in \Delta^0\text{Ens} \). There is a natural map

\[
\text{MC}(\Omega(S) \otimes g) \to \text{Hom}(S, \Sigma(g))
\]
which is bijective provided \( S \) is finite (i.e., has a finite number of non-degenerate simplices).

**Proof.** This is because tensoring by \( g \) commutes with finite limits — compare to [BG], 5.2.

2.1.3. **Definition.** A map \( f : g \to h \) of nilpotent algebras in \( \text{dglie}(k) \) will be called an acyclic fibration if it is surjective and induces a quasi-isomorphism of the corresponding lower central series.

2.1.4. **Lemma.** Let \( f : g \to h \) be an acyclic fibration of nilpotent dg Lie algebras in \( \text{dglie}(k) \). Then the induced map \( \text{MC}(f) : \text{MC}(g) \to \text{MC}(h) \) is surjective.

**Proof.** Induction by the nilpotence degree of \( g \) — similarly to [GM1], Th. 2.4.

2.1.5. **Lemma.** Let \( f : A \to B \) be a surjective map in \( \text{cdga}(k) \) and \( g : g \to h \) be a surjective map in \( \text{dglie}(k) \). Then the map

\[
A \otimes g \to (A \otimes h) \times_{(B \otimes h)} (B \otimes g)
\]
is an acyclic fibration provided either (a) \( f \) is quasi-isomorphism or (b) \( g \) is acyclic fibration.

**Proof.** Since for any commutative dg algebra \( A \) (with 1) the functor \( A \otimes \_ \) transforms the lower central series of \( g \) into the lower central series of \( A \otimes g \), it suffices to check that the above map is a surjective quasi-isomorphism. This is fairly standard.
2.1.6. Proposition. Let $f$ be as above. Then $\Sigma(f) : \Sigma(g) \to \Sigma(h)$ is an acyclic fibration of simplicial sets.

Proof. Lemmas 2.1.2 and 2.1.4 reduce the question to the following. Let $K \to L$ be an injective map of simplicial sets. Then one has to show that the induced map of nilpotent Lie algebras

$$\Omega(L) \otimes g \to \Omega(L) \otimes h \times_{\Omega(K) \otimes h} \Omega(K) \otimes g$$

is an acyclic fibration. This follows from 2.1.5(b). \hfill \Box

2.2. Deligne groupoid. Recall (cf. [GM1]) that for a nilpotent dg Lie algebra $g \in \text{dglie}(k)$ defines the Deligne groupoid $C(g)$ as follows.

The Lie algebra $g^0$ acts on $MC(g)$ by vector fields:

$$\rho(y)(x) = dy + [x, y] \quad \text{for } y \in g^0, x \in g^1.$$ 

This defines the action of the nilpotent group $G = \exp(g^0)$ on the set $MC(g)$. Then the groupoid $C = C(g)$ is defined by the formulas

$$\text{Ob} C = MC(g)$$

$$\text{Hom}_C(x, x') = \{ g \in G | x' = g(x) \}.$$ 

2.2.1. Lemma. Let $g \in \text{dglie}(k)$ be nilpotent. The natural map $g \to g_n = \Omega_n \otimes g$ induces an equivalence of groupoids $C(g) \to C(g_n)$.

Proof. It suffices to check the claim when $n = 1$. In this case an element $z = x + dt \cdot y \in g_1 = g[t, dt]$ with $x \in g^1[t], y \in g^0[t]$ satisfies MC iff $x(0) \in MC(g)$ and $x$ satisfies the differential equation

$$\dot{x} = dy + [x, y].$$ 

This means precisely that $x = g(x(0))$ where $g \in G_1 = \exp(g_1)$ is given by the differential equation

$$\dot{g} = g(y)$$

with the initial condition $g(0) = 1$. \hfill \Box

2.2.2. Explicit description of $\Sigma(g)$. The notations are as in 2.2.

For any $n \geq 0$ let $G_n = \exp(g^0_n)$ be the group of polynomial maps from the standard $n$-simplex $\Delta_n$ to the group $G$. The collection $G_\bullet = \{G_n\}$ forms a simplicial group. Right multiplication defines on $G_\bullet$ a right $G$-action.

Proposition. There is a natural bijection

$$G_\bullet \times^G MC(g) \to \Sigma(g)$$

of simplicial sets. Here, as usual, $X \times^G Y$ is the quotient of the cartesian product $X \times Y$ by the relation

$$(xg, y) \sim (x, gy) \quad \text{for } x \in X, y \in Y, g \in G.$$ 

Proof. This immediately follows from Lemma 2.2.1. \hfill \Box
Let $NC(g) \in \Delta^0\text{Ens}$ be the nerve of $C(g)$.

Define the map $\tau : \Sigma(g) \to NC(g)$ by the formula
$$\tau(g, x) = (g_0(x), g_1g_0^{-1}, \ldots, g_ng_{n-1})$$
where $g \in G_n, x \in MC(g), g_i = v_i(g)$ with $v_i$ being the $i$-th vertex.

Proposition 2.2.2 implies the following

2.2.3. **Proposition.** $\Sigma(g)$ is a Kan simplicial set. The map $\tau$ is an acyclic fibration identifying $C(g)$ with the Poincaré groupoid of $\Sigma(g)$.

More generally, if $f : g \to h$ is a surjective map of nilpotent dg Lie algebras, then the induced map
$$\Sigma(g) \to C(g) \times_{C(h)} \Sigma(h)$$
is an acyclic fibration.

**Proof.** The question reduces to the following. Given a pair of polynomial maps $\alpha : \partial \Delta^n \to g^0, \beta : \Delta^n \to h^0$ satisfying $f\alpha = \beta|_{\partial \Delta^n}$ find a map $\gamma : \Delta^n \to g^0$ such that $\alpha = \gamma|_{\partial \Delta^n}, \beta = f\gamma$. This is always possible since the canonical map $\Omega_n \to \Omega(\partial \Delta^n)$ is surjective. \Halmos

2.2.4. **Remark.** If one does not require $g$ to be non-negatively graded, its contents is still a Kan simplicial set. In this case it can probably be considered as a generalization of the notion of Deligne groupoid.

3. "Total space" functor and $\mathcal{M}$-simplicial sets

3.1. **Generalities.**

3.1.1. **The category $\mathcal{M}$.** Here and below $\mathcal{M}$ denotes the following category of morphisms of $\Delta$:

The objects of $\mathcal{M}$ are morphisms $[p] \to [q]$ in $\Delta$. A morphism from $[p] \to [q]$ to $[p'] \to [q']$ is given by a commutative diagram
$${\begin{array}{ccc}
[p] & \longrightarrow & [q] \\
\alpha \uparrow & & \beta \downarrow \\
[p'] & \longrightarrow & [q']
\end{array}}$$

The morphism in $\mathcal{M}$ corresponding to $\alpha = \text{id}, \beta = \sigma^i$, is denoted by $\sigma^i$; the one corresponding to $\alpha = \text{id}, \beta = \partial^i$, is denoted by $\partial^i$. "Dually", the morphism corresponding to $\alpha = \partial^i, \beta = \text{id}$, is denoted by $d_i$ and the one with $\alpha = \sigma^i, \beta = \text{id}$, is denoted by $s_i$.

3.1.2. **Total space.** Let $\mathcal{C}$ be a simplicial category having inverse limits and functorial function objects $\text{hom}(S, X) \in \mathcal{C}$ for $X \in \mathcal{C}, S \in \Delta^0\text{Ens}$ — see [BK], IX.4.5 and the examples below. The total space $\text{Tot}(X)$ of a cosimplicial object $X \in \Delta\mathcal{C}$ is defined by the formula
$$\text{Tot}(X) = \lim_{\phi \in \mathcal{M}} \text{hom}(\Delta^p, X^q)$$
where $\phi : [p] \to [q]$. 
3.2. **Examples.** We will use three instances of the described construction.

3.2.1. Let $C = \Delta^0\text{Ens}$. Then the above definition coincides with the standard one given in *loc. cit.*, XI.3.

Let $G \in \Delta\text{Grp}$ be a (strict) cosimplicial groupoid. We will consider $\text{Grp}$ as a full subcategory of $\Delta^0\text{Ens}$, so a simplicial set $\text{Tot}(G)$ is defined.

**Lemma.** $T = \text{Tot}(G)$ is a groupoid. The objects of $T$ are collections $\{a, \theta\in \text{Ob} G_0, \theta : \partial^1(a) \simeq \partial^0(a)\}$ with $\theta$ satisfying the cocycle condition:

$$\sigma^0(\theta) = \text{id}(a); \quad \partial^1\theta = \partial^0\theta \circ \partial^2\theta.$$

A morphism in $T$ from $\{a, \theta\}$ to $\{b, \theta'\}$ is a morphism $a \to b$ compatible with $\theta, \theta'$.

Thus, $\text{Tot}(G)$ is “the groupoid of descent data” for $G$.

3.2.2. Let $C = C(k)$ be the category of complexes over $k$. For $S \in \Delta^0\text{Ens}$ and $X \in C(k)$ the complex $\text{hom}(S,X)$ is defined to be $\text{Hom}(C_\bullet(S), X)$ where $C_\bullet$ is the complex of normalized chains of $S$ with coefficients in $k$. The above definition of the functor $\text{Tot}$ coincides with the standard one.

3.2.3. Let $C = \text{dglie}(k)$. For $S \in \Delta^0\text{Ens}$ and $g \in \text{dglie}(k)$ define $\text{hom}(S, g) = \Omega(S) \otimes g$. Then the functor $\text{Tot} : \Delta\text{dglie}(k) \to \text{dglie}(k)$ coincides with the Thom-Sullivan functor described in [HS2], 5.2.4. The De Rham theorem (see, e.g., *loc. cit.* , 5.2.8) shows that the functor $\text{Tot}$ commutes up to homotopy with the forgetful functor $\# : \text{dglie}(k) \to C(k)$.

3.3. **$M$-simplicial sets.** In the sequel functors $X : M \to \Delta^0\text{Ens}$ will be called $M$-simplicial sets. We now wish to find a sufficient condition for a map $f : X \to Y$ of $M$-simplicial sets to induce an acyclic fibration $\varprojlim f$ of the inverse limits. This will be an important technical tool to prove the main theorem [4.3].

3.3.1. **Matching spaces.** Fix $n \in \mathbb{N}$. Let $\partial^i : [n-1] \to [n], i = 0, \ldots, n$, be the standard face maps and let $\sigma^i : [n] \to [n-1], i = 0, \ldots, n-1$, be the standard degeneracies.

The $n$-th matching space of a $M$-simplicial set $X$ is a simplicial subset $\mu_n(X)$ of the product

$$\prod_{i=0}^{n} X(\partial^i) \times \prod_{i=0}^{n-1} X(\sigma^i)$$

consisting of the collections $(x_i \in X(\partial^i), y_i \in X(\sigma^i))$ satisfying the following three conditions:

1. **(d):** $d_i x_j = d_{j-1} x_i$ for $i < j$.
2. **(σ):** $\sigma^j y_i = \sigma^i y_{j+1}$ for $i \leq j$.
3. **(dσ):** $\sigma^j x_i = d_i y_j$ for all $i, j$.

One has a canonical map $X(\text{id}_n) \to \mu_n(X)$ which sends an element $x \in X(\text{id}_n)$ to the collection $(d_0 x, \ldots, d_n x, \sigma^0 x, \ldots, \sigma^{n-1} x)$.
3.3.2. **Definition.** A map \( f : X \to Y \) of \( \mathcal{M} \)-simplicial sets is called an *acyclic fibration* if for any \( n \in \mathbb{N} \) the commutative square

\[
\begin{array}{ccc}
X(\text{id}_n) & \longrightarrow & Y(\text{id}_n) \\
\downarrow & & \downarrow \\
\mu_n(X) & \longrightarrow & \mu_n(Y)
\end{array}
\]

defines an acyclic fibration

\[
X(\text{id}_n) \to Y(\text{id}_n) \times_{\mu_n(Y)} \mu_n(X).
\]

3.3.3. **Lemma.** Let \( f : X \to Y \) be an acyclic fibration of \( \mathcal{M} \)-simplicial sets. Then the induced map of the corresponding inverse limits, \( \lim_{\leftarrow} f \), is an acyclic fibration.

**Proof.** Let \( \mathcal{M}_{\leq n} \) be the full subcategory of \( \mathcal{M} \) consisting of morphisms \( \alpha : [p] \to [q] \) with \( p \leq n, q \leq n \). Put \( X(n) = \lim_{\leftarrow} X_{|\mathcal{M}_{\leq n}} \). Then it is easy to see that \( X(n) = X(\text{id}_n) \times_{\mu_n(X)} X(n-1) \). This immediately proves the lemma. \( \square \)

3.4. **Bisimplicial algebras.** Now we will present a source of various \( \mathcal{M} \)-simplicial sets in this paper.

3.4.1. Fix a cosimplicial nilpotent dg Lie algebra \( g \). Any bisimplicial commutative dg algebra \( A \in (\Delta^0)^2\text{cdga} \) defines a \( \mathcal{M} \)-simplicial set \( \Sigma(A, g) \) as follows:

For \( a : [p] \to [q], n \in \mathbb{N} \) one has

\[
\Sigma(A, g)(a)_n = \text{MC}(A_{np} \otimes g^q).
\]

Define also a simplicial set \( \sigma(A, g) \) to be the inverse limit of \( \Sigma(A, g) \) as a functor from \( \mathcal{M} \) to \( \Delta^0\text{Ens} \). We provide now a sufficient condition for a map \( f : A \to B \) of bisimplicial commutative dg algebras to induce an acyclic fibration \( \Sigma(f, g) \) for any cosimplicial dg Lie algebra. According to Lemma 3.3.3 this implies that \( \sigma(f, g) \) is also an acyclic fibration.

3.4.2. Any bisimplicial abelian group \( A \) gives rise to a functor

\[
A : \Delta^0\text{Ens} \times \Delta^0\text{Ens} \longrightarrow \text{Ab}
\]

which is uniquely described by the following properties:

\begin{itemize}
  \item \( A(\Delta^m, \Delta^n) = A_{mn} \)
  \item \( A \) carries direct limits over each one of the arguments to inverse limit.
\end{itemize}

We will identify bisimplicial abelian groups with the functors they define.

3.4.3. **Definition.** The matching space \( M_{mn}(A) \) of a bisimplicial abelian group \( A \) is defined to be

\[
M_{mn}(A) = A(\partial \Delta^m, \Delta^n) \times_{A(\partial \Delta^m, \partial \Delta^n)} A(\Delta^m, \partial \Delta^n)
\]

where \( \partial \Delta^n \) is the boundary of the \( n \)-simplex. One has a canonical map \( A_{mn} \to M_{mn}(A) \).
3.4.4. **Definition.** A map \( f : A \to B \) in \((\Delta^0)^2C(\mathbb{Z})\) is called an *acyclic fibration* if for any \( m, n \) the canonical map
\[
A_{mn} \to B_{mn} \times_{M_{mn}(B)} M_{mn}(A)
\]
is a surjective quasi-isomorphism.

3.4.5. **Proposition.** Let \( f : A \to B \) in \((\Delta^0)^2\text{cdga}(k)\) be an acyclic fibration. Let \( \mathfrak{g} \) be a cosimplicial nilpotent dg Lie algebra. Then the induced map \( \Sigma(f, \mathfrak{g}) \) is an acyclic fibration of \( \mathcal{M} \)-simplicial sets.

**Proof.** This is a direct calculation using 2.1.4, 2.1.5(a). Here we use that the natural map from \( \mathfrak{g}^{n+1} \) to the \( n \)-th matching space \( M^n(\mathfrak{g}) \) (see [BK], X.5) is surjective. \( \square \)

Now we wish to formulate a sufficient condition for \( f : A \to B \) to be an acyclic fibration of bisimplicial dg algebras. The following trivial lemma will be useful.

3.4.6. **Lemma.** Let in a commutative square
\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
g \downarrow & & \downarrow h \\
C & \to & D
\end{array}
\]
of abelian groups the map \( g \) and the map \( \text{Ker}(g) \to \text{Ker}(h) \) be surjective. Then the induced map \( A \to B \times_D C \) is also surjective.

We start with a simplicial case. Recall that a simplicial abelian group \( A \in \Delta^0\text{Ab} \) defines a functor \( A : \Delta^0\text{Ens} \to \text{Ab} \) by the formula
\[
A(S) = \text{Hom}(S, A).
\]

3.4.7. **Lemma.** Let \( f : A \to B \) be a map in \( \Delta^0C_{\geq 0}(\mathbb{Z}) = C_{\geq 0}(\Delta^0\text{Ab}) \). Suppose that
(a) \( f_n : A_n \to B_n \) is a quasi-isomorphism in \( C(\mathbb{Z}) \)
(b) for any \( S \in \Delta^0\text{Ens} \) the map \( f(S) : A(S) \to B(S) \) is surjective
(c) for any \( d \in \mathbb{Z} \) the \( d \)-components \( A^d \) and \( B^d \) are contractible simplicial abelian groups.
Then for any injective map \( \alpha : S \to T \) the induced map
\[
A(T) \to A(S) \times_{B(S)} B(T)
\]
is a surjective quasi-isomorphism.

**Proof.** For any \( S \in \Delta^0\text{Ens} \) the map \( f(S) : A(S) \to B(S) \) is quasi-isomorphism — this follows from [L], remark at the end of III.2, applied to [Ha], Thm. 1.5.1. This immediately implies that for any \( \alpha : S \to T \) the induced map
\[
A(T) \to A(S) \times_{B(S)} B(T)
\]
is a quasi-isomorphism. Let us prove that it is surjective if \( \alpha \) is injective.
Since $A^d$ are contractible (and Kan), the map $A(\alpha): A(T) \to A(S)$ is surjective. Thus, by Lemma 3.4.6 it suffices to check that the map

$$\text{Ker } A(\alpha) \to \text{Ker } B(\alpha)$$

is surjective.

Since the functors $A, B : \Delta^0\text{Ens} \to C(\mathbb{Z})$ carry colimits to limits, the map $\text{Ker } A(\alpha) \to \text{Ker } B(\alpha)$ is a direct summand of the map $f(T/S): A(T/S) \to B(T/S)$ which is surjective by (b).

Now we are able to prove the following criterion for a map of bisimplicial complexes of abelian groups to be an acyclic fibration.

3.4.8. Proposition. Let $f: A \to B$ be a map in $(\Delta^0)^2 C^{\geq 0}(\mathbb{Z})$ satisfying:

(a) for any $S, T \in \Delta^0\text{Ens}$ the map $f(S, T)$ is surjective
(b) for any $S \in \Delta^0\text{Ens}$, $p \in \mathbb{N}$ the map $f(S, \Delta^p): A(S, \Delta^p) \to B(S, \Delta^p)$ is a quasi-isomorphism in $C(\mathbb{Z})$.
(c) for any $S \in \Delta^0\text{Ens}$ and any $d$ the simplicial abelian groups $A^d(S, \_)$, $B^d(S, \_)$ are contractible.
(d) for any $p, d$ the simplicial abelian group $A^d(S', \Delta^p)$ is contractible.

Then $f$ is an acyclic fibration.

Proof. Apply Lemma 3.4.7 to the map $f(S, \_): A(S, \_) \to B(S, \_)$. We immediately get that for any $S \in \Delta^0\text{Ens}$ and any injective map $\alpha: T \to T'$ the induced map

$$A(S, T') \to A(S, T) \times_{B(S, T)} B(S, T')$$

is a surjective quasi-isomorphism. Put $T' = \Delta^p, T = \partial \Delta^p$. The map $A_{np} \to B_{np} \times_{M_{np}(B)} M_{np}(A)$ is then automatically quasi-isomorphism, an we have only to check it is surjective. According to (d) the map $A(S', T') \to A(S, T')$ is surjective for $T' = \Delta^p$. Define

$$X(T) = \text{Ker}(A(S', T) \to A(S, T)), \ Y(T) = \text{Ker}(B(S', T) \to B(S, T)).$$

The groups $X(T)$ and $Y(T)$ are direct summands of $A(S'/S, T)$ and of $B(S'/S, T)$ respectively. Hence the map $X \to Y$ satisfies the hypotheses of Lemma 3.4.7. Therefore the map

$$X(T') \to X(T) \times_{Y(T)} Y(T')$$

is surjective by Lemma 3.4.6. Proposition is proven.

4. The main theorem

Now we came to the main result of the paper.

Let $g \in \Delta\text{dglie}(k)$ be a cosimplicial nilpotent dg Lie $k$-algebra. Suppose that $g$ is finitely dimensional in the cosimplicial sense, i.e. that the normalization

$$N^n(g) = \{x \in g^n | \sigma^i(x) = 0 \text{ for all } i\}$$

vanishes for sufficiently big $n$. 

4.1. **Theorem.** There is a natural homotopy equivalence
\[ \Sigma(\text{Tot}(g)) \rightarrow \text{Tot}(\Sigma(g)) \]
in Kan.

Taking into account 2.2.3, we easily get

4.1.1. **Corollary.** Let \( g \in \Delta_{dglie}(k) \) be a nilpotent cosimplicial dg Lie \( k \)-algebra. Suppose that \( g \) is finitely dimensional in the cosimplicial sense.

Then there is a natural equivalence of groupoids
\[ C(\text{Tot}(g)) \rightarrow \text{Tot}(C(g)) \].

**Proof.** One has to check that the functor Tot carries the map \( \tau : \Sigma(g) \rightarrow C(g) \) to a homotopy equivalence. By [BK], X.5, it suffices to check \( \tau \) is a fibration in sense of loc. cit.

This follows from (the second claim of) Proposition 2.2.3 since for any \( n \) the natural map from \( g^{n+1} \) to the \( n \)-th matching space \( M^n(g) \) (in notations of loc. cit) is surjective.

4.2. **Proof of the theorem.** The set of \( n \)-simplices of the left-hand side is
\[ \text{MC}(\Omega_n \otimes \lim_{\leftarrow} \Omega_p \otimes g^q) \]
and for the right-hand side:
\[ \lim_{\leftarrow} \text{MC}(\Omega(\Delta^n \times \Delta^p) \otimes g^q) \].

Here the inverse limits are taken over the category \( M \) defined in 3.1.1 and Lemma 2.1.2 is used to get the second formula.

Taking into account that the functor MC commutes with the inverse limits, a canonical map
\[ \Sigma \circ \text{Tot}(g) \rightarrow \text{Tot} \circ \Sigma(g) \]
is defined by the composition
\[ \Omega_n \otimes \lim_{\leftarrow} \Omega_p \otimes g^q \rightarrow \lim_{\leftarrow} \Omega_n \otimes \Omega_p \otimes g^q \rightarrow \lim_{\leftarrow} \Omega(\Delta^n \times \Delta^p) \otimes g^q \]
the latter arrow being induced by the canonical projections of \( \Delta^n \times \Delta^p \) to \( \Delta^n \) and to \( \Delta^p \).

We wish to prove that the map described induces a homotopy equivalence.

Since \( g \) is finitely dimensional, the first map in the composition is bijective — see [HS2], Thm. 6.11. In order to prove that the second map is a homotopy equivalence, let us fix \( n \) and \( p \) and present the map
\[ \alpha : \Omega_n \otimes \Omega_p \rightarrow \Omega(\Delta^n \times \Delta^p) \]
as the composition
\[ \Omega_n \otimes \Omega_p \xrightarrow{\beta} \Omega(\Delta^n \times \Delta^{p+1}) \otimes \Omega_p \xrightarrow{\pi} \Omega(\Delta^n \times \Delta^p) \]
where \( \beta \) is induced by the projection \( \Delta^n \times \Delta^{p+1} \rightarrow \Delta^n \) and \( \pi \) by the pair of maps \( \text{id} \times \partial^{p+1} : \Delta^n \times \Delta^p \rightarrow \Delta^n \times \Delta^{p+1} \), \( pr_2 : \Delta^n \times \Delta^p \rightarrow \Delta^p \).

We will check below that the maps \( \beta \) and \( \pi \) induce homotopy equivalences for different reasons: \( \beta \) induces a strong deformation retract and \( \pi \) induces an acyclic (Kan) fibration. This will prove the theorem.
4.2.1. Notations. Define $\Delta^+$ to be the cosimplicial simplicial set with $(\Delta^+)^n = \Delta^{n+1}$ whose cofaces and codegeneracies are the standard maps between the standard simplices (they all preserve the final vertex).

Put $A_{np} = \Omega_n \otimes \Omega_p$, $B_{np} = \Omega(\Delta^n \times \Delta^p)$, $C_{np} = \Omega(\Delta^n \times \Delta^{p+1}) \otimes \Omega_p$. $A$ and $B$ are bisimplicial commutative dg algebras by an obvious reason. Bisimplicial structure on $C$ is defined by the cosimplicial structure on $\Delta^+$.

Our aim is to prove that the maps $\beta : A \to C$ and $\pi : C \to B$ induce homotopy equivalences $\sigma(\beta, g)$ and $\sigma(\pi, g)$.

We will check immediately that $\sigma(\beta, g)$ is a strong deformation retract. Afterwards, using the criterion 3.4.8 we will get that $\sigma(\pi, g)$ is an acyclic fibration.

4.2.2. Checking $\beta$. For $S \in \Delta^0\text{Ens}$ and nilpotent $g \in \text{dglie}(k)$ denote by $\Sigma^S(g)$ (or just $\Sigma^S$ when $g$ is one and the same) the simplicial set whose set of $n$-simplicies is $\text{MC}(\Omega(\Delta^n \times S) \otimes g)$.

Any map $f : K \times S \to T$ in $\Delta^0\text{Ens}$ induces a map

$\Sigma^f : K \times \Sigma^T \to \Sigma^S$

as follows. Let $k \in K_n$, $x \in \Sigma^n_T = \text{MC}(\Omega(\Delta^n \times T) \otimes g)$. Denote by $F_k$ the composition

$\Delta^n \times S \xrightarrow{\text{diag} \times 1} \Delta^n \times \Delta^n \times S \xrightarrow{1 \times k \times 1} \Delta^n \times K \times S \xrightarrow{1 \times f} \Delta^n \times T$.

Then $\Sigma^f(k, x)$ is defined to be $(\Omega(F_k) \otimes \text{id}_g)(x)$.

Define the map $\Phi_p : \Delta^1 \times \Delta^{p+1} \to \Delta^{p+1}$ as the one given on the level of posets by the formula

$\Phi_p(i, j) = \begin{cases} j & \text{if } i = 0 \\ p + 1 & \text{if } i = 1 \end{cases}$

The maps $\Phi_p$ induce the maps $\Sigma^{\Phi_p}(\Omega_p \otimes g^q)$ which define for any $a : [p] \to [q]$ in $\mathcal{M}$ the simplicial set $\Sigma(A, g)(a) = \text{MC}(A_p \otimes g^q)$ as a strong deformation retract of $\Sigma(C, g)(a)$. The retraction $\Sigma^{\Phi_p}(\Omega_p \otimes g^q)$ are functorial in $a \in \mathcal{M}$, therefore the inverse image map $\sigma(\beta, g)$ is also strong deformation retract.

4.2.3. Checking $\pi$. Now we will prove that the map $\pi : C \to B$ of bisimplicial dg algebras defined above induces an acyclic fibration $\sigma(\pi, g)$.

Let us check the hypotheses of 3.4.8.

First of all, let us check that $\pi(S, T)$ is surjective. For this consider $D_{np} = \Omega(\Delta^n, \Delta^{p+1})$ and the map $\rho_{np} : D_{np} \to B_{np}$ which is the composition of $\pi$ with the natural embedding $D_{np} \to C_{np} = D_{np} \otimes \Omega_p$. Surely, it suffices to prove that $\rho(S, T)$ is surjective. For this we note that $D(S, T) = \Omega(S \times \hat{T})$ where $\hat{T}$ is the double dual of $T$. Then, since the map $S \times T \to S \times \hat{T}$ is injective, the map

$\Omega(S \times \hat{T}) \to \Omega(S \times T)$

is surjective.
Next, one has
\[ C(S, \Delta^p) = \Omega(S \times \Delta^{p+1}) \otimes \Omega_p, \]
\[ B(S, \Delta^p) = \Omega(S \times \Delta^p) \]
so the condition 3.4.8 (b) is fulfilled.

The simplicial abelian groups \( C^d(S, \Delta) \) are contractible by the Künneth formula. The abelian groups \( B^d(S, \Delta) = \Omega^d(S \times \Delta) \) are contractible since any injective map \( T \to T' \) induces a surjection \( \Omega(S \times T') \to \Omega(S \times T) \). The same reason proves the condition (d).

Therefore, the map \( \pi : C \to B \) of bisimplicial commutative dg algebras is an acyclic fibration by Proposition 3.4.8 and then by Proposition 3.4.3 the map \( \sigma(\pi, g) \) is an acyclic fibration in \( \Delta^0\text{-Ens} \).

The Theorem is proven.

5. APPLICATION TO DEFORMATION THEORY

In this Section we describe how to deduce from Corollary 4.1.1 the description of universal formal deformations for some typical deformation problems.

Consider, for example, three deformation problems which have been studied in [HS2].

Let \( X \) be a smooth separated scheme \( X \) over a field \( k \) of characteristic 0, \( G \) an algebraic group over \( k \) and \( p : P \to X \) a \( G \)-torsor over \( X \). Consider the following deformation problems.

**Problem 1.** Flat deformations of \( X \).

**Problem 2.** Flat deformations of the pair \((X, P)\).

**Problem 3.** Deformations of \( P \) (\( X \) being fixed).

To each problem one can assign a sheaf of \( k \)-Lie algebras \( g_i \) on \( X \) (these are the sheaves \( \mathcal{A}_i, i = 1, 2, 3 \) from loc. cit., Section 8).

According to Grothendieck, to each problem corresponds a (2-)functor of infinitesimal deformations
\[ F_i : \text{art}/k \to \text{Grp} \]
from the category of local artinian \( k \)-algebras with the residue field \( k \) to the (2-)category of groupoids.

In each case, \( g_i \) is ”a sheaf of infinitesimal automorphisms” corresponding to \( F_i \) (in the sense of [SGA1], Exp.III, 5, especially Cor. 5.2 for Problem 1; for the other problems the meaning is analogous).

If \( X \) is affine, the functor \( F_i \) is equivalent to the Deligne groupoid \( C_L, L = \Gamma(X, g_i) \) defined as the functor
\[ C_L : \text{art}/k \to \text{Grp} \]
which is given by the formula
\[ C_L(A) = C(\mathfrak{m} \otimes L) \]
where \( \mathfrak{m} \) is the maximal ideal of \( A \).
The deformation functor $F_i$ defines a stack $\mathcal{F}_i$ in the Zariski topology of $X$. The Deligne functor defines a fibered category $\mathcal{C}_{g_i}$ which assigns a groupoid $\mathcal{C}_{\Gamma(U,g_i)}(A)$ to each Zariski open set $U$ and to each $A \in \text{art}/k$.

We have a canonical map of fibered categories $\mathcal{C}_{g_i} \to \mathcal{F}_i$ so that $\mathcal{F}_i$ is equivalent to the stack associated to the fibered category $\mathcal{C}_{g_i}$.

Using Corollary 4.1.1 we immediately get

5.1. **Corollary.** The deformation groupoid $F_i$ is naturally equivalent to the Deligne groupoid associated with the dg Lie algebra $R^\Gamma_{\text{Lie}}(X, g_i)$.

In particular, the following generalization of [HS2], Thm. 8.3 takes place.

5.2. **Corollary.** Suppose that $H^0(X, g_i) = 0$; let $\mathcal{G} = \text{Spf}(R)$ be the base of the universal formal deformation for Problem $i$. Then we have a canonical isomorphism

$$R^* = H^0_{\text{Lie}}(R^\Gamma_{\text{Lie}}(X, g_i))$$

where $R^*$ denotes the space of continuous $k$-linear maps $R \to k$ ($k$ considered in the discrete topology) and $H^0_{\text{Lie}}$ denotes the 0-th Lie homology.

Recall that this result has been proven in loc. cit. for $\mathcal{G}$ formally smooth.

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