CYCLOTOMIC BIRMAN–WENZL–MURAKAMI ALGEBRAS, II: ADMISSIBILITY RELATIONS AND FREENESS

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Abstract. The cyclotomic Birman-Wenzl-Murakami algebras are quotients of the affine BMW algebras in which the affine generator satisfies a polynomial relation. We study admissibility conditions on the ground ring for these algebras, and show that the algebras defined over an admissible integral ground ring $S$ are free $S$-modules and isomorphic to cyclotomic Kauffman tangle algebras. We also determine the representation theory in the generic semisimple case, obtain a recursive formula for the weights of the Markov trace, and give a sufficient condition for semisimplity.

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1. Introduction

This paper and the companion paper [10] continue the study of affine and cyclotomic Birman–Wenzl– Murakami (BMW) algebras, which we began in [9].

1.1. Background. The origin of the BMW algebras was in knot theory. Kauffman defined [14] an invariant of regular isotopy for links in $S^3$, determined by certain skein relations. Birman and Wenzl [6] and independently Murakami [22] then defined a family of quotients of the braid group algebras, and showed that

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Kauffman’s invariant could be recovered from a trace on these algebras. These (BMW) algebras were defined by generators and relations, but were implicitly modeled on certain algebras of tangles whose definition was subsequently made explicit by Morton and Traczyk [20], as follows: Let $S$ be a commutative unital ring with invertible elements $\rho$, $q$, and $\delta_0$ satisfying $\rho^{-1} - \rho = (q^{-1} - q)(\delta_0 - 1)$. The Kauffman tangle algebra $KT_{n,S}$ is the $S$–algebra of framed $(n,n)$–tangles in the disc cross the interval, modulo Kauffman skein relations:

1. Crossing relation:
   $$\begin{array}{c}
   \begin{array}{c}
   \bigcirc
   \end{array}
   \end{array}
   =
   \begin{array}{c}
   \begin{array}{c}
   \bigcirc
   \end{array}
   \end{array}
   =
   \begin{array}{c}
   \begin{array}{c}
   \bigcirc
   \end{array}
   \end{array}
   (q^{-1} - q) \left( \begin{array}{c}
   \begin{array}{c}
   \bigcirc
   \end{array}
   \end{array} - \begin{array}{c}
   \begin{array}{c}
   \bigcirc
   \end{array}
   \end{array} \right).$$

2. Untwisting relation:
   $$\begin{array}{c}
   \begin{array}{c}
   \bigcirc
   \end{array}
   = \rho
   \end{array}$$
   and
   $$\begin{array}{c}
   \begin{array}{c}
   \bigcirc
   \end{array}
   = \rho^{-1}.$$  

3. Free loop relation: $T \cup \bigcirc = \delta_0 T$.

Morton and Traczyk [20] showed that the $n$–strand algebra $KT_{n,S}$ is free of rank $(2n - 1)!!$ as a module over $S$, and Morton and Wassermann [21] proved that the BMW algebras and the Kauffman tangle algebras are isomorphic.

It is natural to “affinize” the BMW algebras to obtain BMW analogues of the affine Hecke algebras of type $A$, see [1]. The affine Hecke algebra can be realized geometrically as the algebra of braids in the annulus cross the interval, modulo Hecke skein relations; this suggests defining the affine Kauffman tangle algebra $\hat{KT}_{n,S}$ as the algebra of framed $(n,n)$–tangles in the annulus cross the interval, modulo Kauffman skein relations. Turaev [28] showed that the resulting algebra of $(0,0)$–tangles is a (commutative) polynomial algebra in infinitely many variables, so it makes sense to absorb this polynomial algebra into the ground ring. (The ground ring gains infinitely many parameters $\delta_j$ ($j \geq 1$) corresponding to the generators of the polynomial algebra.) On the other hand, Häring–Oldenburg [12] defined an affine version of the BMW algebras by generators and relations. In [9], we showed that Häring–Oldenburg’s affine BMW algebras are isomorphic to the affine Kauffman tangle algebras, and we showed that these algebras are free modules over their ground ring, with a basis reminiscent of a well–known basis of affine Hecke algebras.

The affine BMW algebras arise naturally in several different contexts: knot theory in the solid torus [28, 16, 17], representations of Artin braid group of type $B$ by $R$ matrices of quantum groups [24], and representations of ordinary BMW algebras. See [10], Section 1.1 for more detail.
1.2. Cyclotomic algebras. In this paper and the companion paper [10] we consider cyclotomic BMW algebras, which are the BMW analogues of cyclotomic Hecke algebras [1]. The affine BMW algebras have a distinguished generator $x_1$, which, in the geometric (Kauffman tangle) picture is represented by a braid with one strand wrapping around the hole in the annulus cross interval. The cyclotomic BMW algebra $W_{n,S,r}$ is defined to be the quotient of the affine BMW algebra $\hat{W}_{n,S}$ in which the generator $x_1$ satisfies a monic polynomial equation

$$x_1^r + \sum_{k=0}^{r-1} a_k x_1^k = 0.$$  

with coefficients in $S$.\footnote{Actually, we will assume that the polynomial splits in $S$.} The cyclotomic BMW algebras were also introduced by Häring-Oldenburg in [12].

In the geometric (Kauffman tangle) picture, it is more natural to convert this relation into a local skein relation:

$$T_r + \sum_{k=0}^{r-1} a_k T_k = 0,$$

whenever $T_0, T_1, \ldots, T_r$ are affine tangle diagrams that are identical in the exterior of some disc $E$ and $T_k \cap E$ consists of one strand wrapping $k$ times around the hole in the annulus cross interval; i.e. $T_k \cap E$ “equals” $x_1^k$. The cyclotomic Kauffman tangle algebra $KT_{n,S,r}$ is defined to be the quotient of the affine Kauffman tangle algebra $\hat{K}T_{n,S}$ by the cyclotomic skein relation. See [10], Section 1.2, for an alternative description of the cyclotomic Kauffman tangle algebras in terms of the affine Kauffman tangle category.

A priori, the ideal in $\hat{K}T_{n,S} \cong \hat{W}_{n,S}$ generated by the cyclotomic skein relation (1.2) is larger than the ideal generated by the polynomial relation (1.1), so we have a surjective, but not evidently injective homomorphism

$$\varphi : W_{n,S,r} \to KT_{n,S,r}.$$  

1.3. Admissibility. The cyclotomic BMW algebras and Kauffman tangle algebras can be defined over an arbitrary commutative unital ring $S$ with parameters $\rho$, $q$, $\delta_j$ ($j \geq 0$), and $u_1, \ldots, u_r$ (the eigenvalues of $y_1$), assuming that $\rho$, $q$, $\delta_0$ and $u_1, \ldots, u_r$ are invertible, and $\rho^{-1} - \rho = (q^{-1} - q)(\delta_0 - 1)$. However, unless the parameters satisfy additional relations, the identity element 1 of the cyclotomic Kauffman tangle algebras will be a torsion element over $S$; if $S$ is a field (and
the additional relations do not hold) then $1 = 0$, so $KT_{n,S,r}$ becomes trivial.

The additional conditions are called “weak admissibility;” see Section 2.5. Weak admissibility is thus a minimal condition for the non–triviality of the cyclotomic algebras.

In order to obtain substantial results about the cyclotomic BMW and Kauffman tangle algebras, it seems necessary to impose a condition on the ground ring $S$ that is stronger than weak admissibility. An appropriate condition was introduced by Wilcox and Yu in [31]. Their condition has a simple formulation in terms of the representation theory of the 2–strand algebra $W_{2,S,r}$, and also translates into explicit relations on the parameters. The condition of Wilcox and Yu, called “admissibility,” is the subject of Section 3 of this paper.

1.4. Results. The main results of this paper and [10] is that if the ground ring $S$ is an integral domain and admissible in the sense of Wilcox and Yu, then $W_{n,S,r} \cong KT_{n,S,r}$, and, moreover, these algebras are free $S$–modules of rank $r^n(2n - 1)!!$. The proof of these results has a topological component, which is given in [10], and an algebraic component, which is given in this paper.

The topological arguments in [10] have two essential consequences: first they show that the cyclotomic BMW algebra $W_{n,S,r}$ has a spanning set $A'_r$ of cardinality $r^n(2n - 1)!!$. Second, they show that the $(0,0)$–tangle algebra $KT_{0,S,r}$ is free of rank 1, assuming that $S$ is weakly admissible. This implies the existence of the “Markov trace” on the cyclotomic BMW and Kauffman tangle algebras (and, hence, Kauffman type invariants for links in the solid torus).

To achieve the main results, it remains to show that the spanning set $\varphi(A'_r)$ of the cyclotomic Kauffman tangle algebra $KT_{n,S,r}$ is linearly independent, when the ground ring $S$ is an admissible integral domain.

To do this we first show that there exists a universal admissible integral domain $\overline{R}_+$ with the property that every admissible integral domain is a quotient of $\overline{R}_+$. Moreover, field of fractions $F$ of $\overline{R}_+$ is the field of rational functions $\mathbb{Q}(q,u_1,\ldots,u_r)$.

We then analyze the representation theory of the cyclotomic BMW algebras defined over $F$, by adapting the inductive method of Wenzl from [6, 29, 30]. This analysis shows that $KT_{n,F,r} \cong W_{n,F,r}$, and that these algebras are split simisimple of dimension $r^n(2n - 1)!!$. The irreducible representations of $W_{n,F,r}$ are indexed by $r$-tuples of Young diagrams of total size $\leq n$ and congruent to $n$ mod 2.
A relatively simple argument then shows that for any admissible integral domain $S$, we have $KT_{n,S,r} \cong W_{n,S,r}$, and these algebras are free of rank $r^n(2n - 1)!!$ over $S$.

As a byproduct of our argument, we obtain a sufficient condition for semisimplicity of the cyclotomic BMW algebras defined over an admissible field.

The elements

$$y_j = g_{j-1} \cdots g_1 y_1 g_1 \cdots g_{j-1},$$

are important for the representation theory of the cyclotomic BMW algebras. They are BMW analogues of Jucys-Murphy elements; see, for example, [18, 19]. Indeed, in our treatment of the representation theory over the generic field $F$, the action of the Jucys-Murphy elements plays an essential role.

1.5. Related work, and acknowledgments. Wilcox and Yu have been studying the same material independently and have obtained similar (and, in some instances, slightly stronger) results [31, 33, 32]. We are indebted to Wilcox and Yu for pointing out an error in a previous preprint version of our work, which required us to substantially rework the algebraic (linear independence) component of the arguments. In fact, we had to adopt a completely different strategy for proving linear independence. Meanwhile, Wilcox and Yu [33, 32] were able to make our original strategy work, using a refined analysis of their admissibility condition.

We would also like to mention here recent work of Ariki, Mathas and Rui on the “cyclotomic Nazarov–Wenzl algebras” [3], which are cyclotomic quotients of the degenerate affine BMW algebras introduced by Nazarov [23]. The idea of deriving an admissibility condition on the ground ring from an assumption on the representation theory of the 2–strand algebra originates in [3], and is essential to our work and the work of Wilcox and Yu.

We rely heavily on ideas from Beliakova and Blanchet [5] (and indirectly on Nazarov [23]) for our computation of the weights of the Markov trace and of the action of conjugates of the affine generator $y_1$ on the basis of up–down tableaux.

It is shown in [7] and [33] that cyclotomic BMW algebras defined over integral admissible ground rings are cellular in the sense of Graham and Lehrer [11]. The proof in [7] depends on the isomorphism of cyclotomic BMW algebras and cyclotomic Kaufmann tangle algebras established here.
Finally, a recent preprint of Rui and Xu [26] concerns the representation theory of cyclotomic BMW algebras over a $u$–admissible field.

2. Preliminaries

2.1. The affine BMW algebra. We recall the definition of the affine Birman–Wenzl–Murakami (BMW) algebra. Let $S$ be a commutative unital ring with elements $\rho$, $q$, and $\delta_j$ ($j \geq 0$), such that $\rho$, $q$, and $\delta_0$ are invertible, and the relation $\rho^{-1} - \rho = (q^{-1} - q)(\delta_0 - 1)$ holds.

Definition 2.1. The affine Birman–Wenzl–Murakami algebra $\hat{W}_{n,S}$ is the $S$ algebra with generators $y_1^{\pm 1}$, $g_i^{\pm 1}$ and $e_i$ ($1 \leq i \leq n - 1$) and relations:

1. (Inverses) $g_i g_i^{-1} = g_i^{-1} g_i = 1$ and $y_1 y_1^{-1} = y_1^{-1} y_1 = 1$.
2. (Idempotent relation) $e_i^2 = \delta_0 e_i$.
3. (Affine braid relations)
   a. $g_ig_{i+1}g_i = g_{i+1}g_ig_i+1$ and $g_ig_j = g_jg_i$ if $|i - j| \geq 2$.
   b. $y_1g_1y_1g_1 = g_1y_1g_1y_1$ and $y_1g_j = g_jy_1$ if $j \geq 2$.
4. (Commutation relations)
   a. $g_ie_j = e_jg_i$ and $e_ie_j = e_je_i$ if $|i - j| \geq 2$.
   b. $y_1e_j = e_jy_1$ if $j \geq 2$.
5. (Affine tangle relations)
   a. $e_ie_{i\pm 1}e_i = e_i$,
   b. $g_ig_{i\pm 1}e_i = e_{i\pm 1}e_i$ and $e_ig_{i\pm 1}g_i = e_ie_{i\pm 1}$.
   c. For $j \geq 1$, $e_1y_je_1 = \delta_j e_1$.
6. (Kauffman skein relation) $g_i - g_i^{-1} = (q - q^{-1})(1 - e_i)$.
7. (Untwisting relations) $g_ie_i = e_ig_i = \rho^{-1}e_i$ and $e_ig_{i\pm 1}e_i = \rho e_i$.
8. (Unwrapping relation) $e_1y_1g_1y_1 = \rho e_1 = y_1g_1y_1e_1$.

Remark 2.2. The presentation differs slightly from the one we used in [9]. There we used the generator $x_1 = \rho^{-1}y_1$, and parameters $\vartheta_j = \rho^{-j}\delta_j$ (so that $e_1x_1e_1 = \vartheta_j e_1$). We also used the parameter $z$ in place of $(q^{-1} - q)$.

In $\hat{W}_{n,S}$, define for $1 \leq j \leq n$,

$$y_j = g_{j-1} \cdots g_1 y_1 g_1 \cdots g_{j-1}$$

and

$$y'_j = g_{j-1} \cdots g_1 y_1 g_1^{-1} \cdots g_{j-1}^{-1}.$$

We record some elementary results from [9]:
Lemma 2.3. The following relations hold in $\hat{W}_{n,S}$. For $1 \leq k, j \leq n$,

1. For $j \not\in \{k, k - 1\}$, $g_j y_k = y_k g_j$.
2. For $j \not\in \{k, k - 1\}$, $e_j y_k = y_k e_j$.
3. $y_j y_k = y_k y_j$.
4. For $k < n$, $g_k y_k = y_{k+1} g_k^{-1}$ and $g_k^{-1} y_{k+1} = y_k g_k$.
5. For $k < n$,
   \[
   g_k y_{k+1} = y_k g_k + (q - q^{-1}) y_{k+1} - (q - q^{-1}) \rho e_k y_{k+1},
   \]
   \[
   g_k^{-1} y_k = y_{k+1} g_k^{-1} - (q - q^{-1}) y_k + (q - q^{-1}) e_k y_k.
   \]
6. For $k < n$, $e_k y_k = e_k y_{k+1}^{-1}$ and $e_k y_k^{-1} = e_k y_{k+1}$.
7. $g_k y_k g_k y_k = y_k g_k y_k g_k$.
8. $e_k e_k y_k y_k = \rho e_k$.
9. For $s \geq 1$, $e_k y_k^s g_k y_k = e_k y_k^{s-1} g_k^{-1}$, and $e_k y_k^{-s} g_k^{-1} y_k^{-1} = e_k y_k^{-s+1} g_k$.

Remark 2.4. There is a unique antiautomorphism of the affine BMW algebra leaving each of the generators $y_1, e_i, g_i$ invariant; the antiautomorphism simply reverses the order of a word in the generators. This follows from the symmetry of the relations. Using this, one obtains versions of points (6), (8) and (9) of the Lemma with $e_k$ written on the right, for example: $y_k g_k y_k^s e_k = g_k^{-1} y_k^{s-1} e_k$.

Lemma 2.5. For $j \geq 1$, there exist elements $\delta_{-j} \in \mathbb{Z}[\rho^\pm 1, q^\pm 1, \delta_0, \ldots, \delta_j]$ such that $e_1 y_1^{-j} e_1 = \delta_{-j} e_1$. Moreover, the elements $\delta_{-j}$ are determined by the recursion relation:

\[
\delta_{-1} = \rho^{-2} \delta_1
\]

\[
\delta_{-j} = \rho^{-2} \delta_j + (q^{-1} - q) \rho^{-1} \sum_{k=1}^{j-1} (\delta_k \delta_{k-j} - \delta_{2k-j}) \quad (j \geq 2).
\]

Proof. Follows from [9], Corollary 3.13, and [10], Lemma 2.6. □

2.2. The affine Kauffman tangle algebra. Let $S$ be a commutative unital ring with elements $\rho, q$, and $\delta_j$ ($j \geq 0$), such that $\rho, q$, and $\delta_0$ are invertible, and the relation $\rho^{-1} - \rho = (q^{-1} - q)(\delta_0 - 1)$ holds. The affine Kauffman tangle algebra $\hat{KT}_{n,S}$ is the $S$-algebra of framed $(n, n)$ tangles in $A \times I$, where $A$ is the annulus and $I$ the interval, modulo the Kauffman skein relations. The reader is referred to [9] for a detailed definition.

The affine Kauffman tangle algebra $\hat{KT}_{n,S}$ can be described in terms of affine tangle diagrams, which are quasi-planar diagrams representing framed tangles.
Here is a picture of a typical affine tangle diagram. The heavy vertical line represents the hole in $A \times I$; we refer to it as the flagpole.

Multiplication is by stacking of such diagrams; our convention is that the product $ab$ is given by stacking $b$ over $a$. The affine Kauffman tangle algebra is generated by the following affine tangle diagrams:

$$
X_1 = \begin{array}{c}
\includegraphics[scale=0.5]{affine_diagram1}
\end{array} \\
G_i = \begin{array}{c}
\includegraphics[scale=0.5]{affine_diagram2}
\end{array} \\
E_i = \begin{array}{c}
\includegraphics[scale=0.5]{affine_diagram3}
\end{array}
$$

**Theorem 2.6 ([9]).** The affine BMW algebra $\hat{W}_{n,S}$ is isomorphic to the affine Kauffman tangle algebra $\hat{KT}_{n,S}$ by a map determined by the assignments $y_1 \mapsto \rho X_1$, $e_i \mapsto E_i$, and $g_i \mapsto G_i$ for $1 \leq i \leq n - 1$.

2.3. The cyclotomic BMW and Kauffman tangle algebras. We will consider algebras defined over ground rings with parameters satisfying certain conditions. In order to avoid repeating these conditions, we establish the following convention:

**Definition 2.7.** A ground ring $S$ is a commutative unital ring with parameters $\rho$, $q$, $\delta_j$ $(j \geq 0)$, and $u_1, \ldots, u_r$, with $\rho$, $q$, $\delta_0$, and $u_1, \ldots, u_r$ invertible, and with $\rho^{-1} - \rho = (q^{-1} - q)(\delta_0 - 1)$.

**Definition 2.8.** Let $S$ be a ground ring with parameters $\rho$, $q$, $\delta_j$ $(j \geq 0)$, and $u_1, \ldots, u_r$. The cyclotomic BMW algebra $W_{n,S,r}(u_1, \ldots, u_r)$ is the quotient of $\hat{W}_{n,S}$ by the relation

$$(y_1 - u_1)(y_1 - u_2) \cdots (y_1 - u_r) = 0.
$$

**Remark 2.9.** The assignment $e_i \mapsto e_i$, $g_i \mapsto g_i$, $y_1 \mapsto y_1$ defines a homomorphism $\iota$ from $W_{n,S,r}$ to $W_{n+1,S,r}$, since the relations are preserved. It is not evident that $\iota$ is injective. However, when $S$ is an admissible integral domain, we will show that $W_{n,S,r}$ is isomorphic to a cyclotomic version of the Kauffman tangle algebra (defined below), and in this case, it is true that $\iota$ is injective.
Remark 2.10. Let $S$ be a ring with parameters $\rho, q$, etc., as above, and let $S'$ be another ring with parameters $\rho', q'$, etc. Suppose there is a ring homomorphism $\psi : S \to S'$ mapping $\rho \mapsto \rho'$, $q \mapsto q'$, etc. Then $W_{n,S',r}$ can be regarded as an algebra over $S$, with $sx = \psi(s)x$ for $s \in W_{n,S',r}$ and $x \in W_{n,S',r}$, and there is an $S$–algebra homomorphism $\tilde{\psi} : W_{n,S,r} \to W_{n,S',r}$ taking generators to generators.

We claim that $W_{n,S,r} \otimes_S S' \cong W_{n,S',r}$ as $S'$–algebras. In fact, we have the $S'$–algebra homomorphism $\tilde{\psi} \otimes \text{id} : W_{n,S,r} \otimes_S S' \to W_{n,S',r} \otimes_S S'$ mapping $g_i \mapsto g_i \otimes 1$, etc. The maps $\tilde{\psi} \otimes \text{id}$ and $\theta$ are inverses.

Of course, this remark applies in general to algebras defined by generators and relations.\footnote{We could have slightly simplified some arguments in [9] using this remark.}

Now we consider how to define a cyclotomic version of the Kauffman tangle algebra. Rewrite the relation (2.2) in the form

$$\sum_{k=0}^{r} (-1)^{r-k} \varepsilon_{r-k}(u_1, \ldots, u_r) y_1^k = 0,$$

where $\varepsilon_j$ is the $j$–th elementary symmetric function. The corresponding relation in the affine Kauffman tangle algebra is

$$\sum_{k=0}^{r} (-1)^{r-k} \varepsilon_{r-k}(u_1, \ldots, u_r) \rho^k X_1^k = 0,$$

Now we want to impose this as a local skein relation.

Definition 2.11. Let $S$ be a ground ring with parameters $\rho, q, \delta_j$ $(j \geq 0)$, and $u_1, \ldots, u_r$. The \textit{cyclotomic Kauffman tangle algebra} $KT_{n,S}(u_1, \ldots, u_r)$ is the quotient of the affine Kauffman tangle algebra $\hat{KT}_{n,S}$ by the cyclotomic skein relation:

$$\sum_{k=0}^{r} (-1)^{r-k} \varepsilon_{r-k}(u_1, \ldots, u_r) \rho^k X_1^k = 0,$$

The sum is over affine tangle diagrams which differ only in the interior of the indicated disc and are identical outside of the disc; the interior of the disc contains an interval on the flagpole and a piece of an affine tangle diagram isotopic to $X_1^k$.\footnote{We could have slightly simplified some arguments in [9] using this remark.}
We continue to write $E_i$, $G_i$, $X_1$ for the image of these elements of the affine Kauffman tangle algebra in the cyclotomic Kauffman tangle algebra. We write

$$Y_1 = \rho X_1.$$ 

The ideal in the affine Kauffman tangle algebra by which we are taking the quotient contains the ideal generated by $(Y_1 - u_1)(Y_1 - u_2) \cdots (Y_1 - u_r)$, but could in principle be larger. It follows that there is a homomorphism

$$\varphi : W_{n,S,r}(u_1, \ldots, u_r) \to KT_{n,S,r}(u_1, \ldots, u_r)$$

determined by $y_1 \mapsto Y_1$, $e_i \mapsto E_i$, $g_i \mapsto G_i$. Moreover, the following diagram (in which the vertical arrows are the quotient maps) commutes.

![Diagram](image)

The homomorphism $\varphi : W_{n,S,r}(u_1, \ldots, u_r) \to KT_{n,S,r}(u_1, \ldots, u_r)$ is surjective because the diagram commutes and $\varphi : \hat{W}_{n,S} \to \hat{KT}_{n,S}$ is an isomorphism. 

**Remark 2.12.** Let $S$ be a ring with parameters $\rho$, $q$, etc., as above, and let $S'$ be another ring with parameters $\rho'$, $q'$, etc. Suppose there is a ring homomorphism $\psi : S \to S'$ mapping $\rho \mapsto \rho'$, $q \mapsto q'$, etc. Any $S'$–algebra can be regarded as an $S$–algebra using $\psi$. We have a map of monoid rings $\psi : S \hat{U}(n,n) \to S' \hat{U}(n,n)$, and this map respects regular isotopy, the Kauffman skein relations, and the cyclotomic relations, so induces an $S$–algebra homomorphism from $\psi : KT_{n,S,r} \to KT_{n,S',r}$. As in Remark 2.10, we have $KT_{n,S,r} \otimes_S S' \cong KT_{n,S',r}$ as $S'$–algebras.

2.4. **Morphisms of ground rings.** We consider what are the appropriate morphisms between ground rings for cyclotomic BMW or Kauffman tangle algebras. The obvious notion would be that of a ring homomorphism taking parameters to parameters; that is, if $S$ is a ring with parameters $\rho$, $q$, etc., and $S'$ a weakly admissible ring with parameters $\rho'$, $q'$, etc., then a morphism $\varphi : S \to S'$ would be required to map $\rho \mapsto \rho'$, $q \mapsto q'$, etc.

However, it is better to require less, for the following reason: The parameter $q$ enters into the cyclotomic BMW or Kauffman tangle relations only in the
expression \( q^{-1} - q \), and the transformation \( q \mapsto -q^{-1} \) leaves this expression invariant. Moreover, the transformation \( g_i \mapsto -g_i \), \( \rho \mapsto -\rho \), \( q \mapsto -q \) (with all other generators and parameters unchanged) leaves the cyclotomic BMW relations unchanged. Likewise, for the cyclotomic Kauffman tangle algebras, the map taking an affine tangle diagram \( T \) to \((-1)^{\kappa(T)} T\), where \( \kappa(T) \) is the number of crossings of ordinary strands of \( T \), determines an isomorphism \( KT_{n,S,r}(\rho,q,\ldots) \cong KT_{n,S,r}(-\rho,-q,\ldots) \).

Taking this into account, we arrive at the following notion:

**Definition 2.13.** Let \( S \) be a ground ring with parameters \( \rho, q, \delta_j \ (j \geq 0) \), and \( u_1, \ldots, u_r \). Let \( S' \) be another ground ring with parameters \( \rho', q' \), etc.

A unital ring homomorphism \( \varphi : S \to S' \) is a morphism of ground rings if it maps

\[
\begin{align*}
\rho &\mapsto \rho', \text{ and} \\
q &\mapsto q' \text{ or } q \mapsto -q'^{-1},
\end{align*}
\]

or

\[
\begin{align*}
\rho &\mapsto -\rho', \text{ and} \\
q &\mapsto -q' \text{ or } q \mapsto q'^{-1},
\end{align*}
\]

and strictly preserves all other parameters.

Suppose there is a morphism of ground rings \( \psi : S \to S' \). Then \( \psi \) extends to a homomorphism from \( W_{n,S,r} \) to \( W_{n,S',r} \), and likewise to a homomorphism from \( KT_{n,S,r} \) to \( KT_{n,S',r} \). Moreover, \( W_{n,S,r} \otimes_S S' \cong W_{n,S',r} \) and \( KT_{n,S,r} \otimes_S S' \cong KT_{n,S',r} \) as \( S' \)-algebras. This follows from the observations above and Remarks 2.10 and 2.12.

2.5. **Weak admissibility.** As noted in [10], Section 4.1, unless the parameters of the ground ring \( S \) satisfy the weak admissibility condition defined below, the identity element \( 1 \) of the cyclotomic Kauffman tangle algebras will be a torsion element over \( S \); if \( S \) is a field (and the additional relations do not hold) then \( 1 = 0 \), so \( KT_{n,S,r} = \{0\} \). Similarly, in the absence of weak admissibility, a computation done in the cyclotomic BMW algebra shows that \( e_1 \in W_{2,S,r} \) is a torsion element over \( S \).

**Definition 2.14.** Let \( S \) be a ground ring with parameters \( \rho, q, \delta_j, j \geq 0 \), and \( u_1, \ldots, u_r \). We say that the parameters are weakly admissible (or that the ring \( S \)
is weakly admissible) if the following relation holds:

\[
\sum_{k=0}^{r} (-1)^{r-k} \varepsilon_{r-k}(u_1, \ldots, u_r) \delta_{k+a} = 0,
\]

for \( a \in \mathbb{Z} \), where for \( j \geq 1 \), \( \delta_{-j} \) is defined by the recursive relations of Lemma 2.5.

2.6. *Inclusion, conditional expectation, and trace.* Let \( S \) be a ground ring with parameters as above and write \( KT_{n,S,r} \) for \( KT_{n,S,r}(u_1, \ldots, u_r) \). The affine Kauffman tangle algebras have inclusion maps \( \iota : \hat{KT}_{n-1} \to \hat{KT}_n \), defined on the level of affine tangle diagrams by adding an additional strand on the right without adding any crossings:

\[
\iota : \quad \begin{array}{c}
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\end{array}
\quad \mapsto \quad \begin{array}{c}
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\end{array}.
\]

Since these maps respect the cyclotomic relation (2.4), they induce homomorphisms

\[
\iota : KT_{n-1,S,r} \to KT_{n,S,r}.
\]

Recall also that the affine Kauffman tangle algebras have a conditional expectation \( \varepsilon_n : \hat{KT}_n \to \hat{KT}_{n-1} \) defined by

\[
\varepsilon_n(T) = \delta_0^{-1} c_n(T),
\]

where \( c_n \) is the map of affine \((n,n)\)-tangle diagrams to affine \((n-1,n-1)\)-tangle diagrams that “closes” the rightmost strand, without adding any crossings:

\[
c_n : \quad \begin{array}{c}
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\end{array}
\quad \mapsto \quad \begin{array}{c}
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\text{ } \\
\end{array}.
\]

These maps respect the cyclotomic relation (2.4), so induce conditional expectations

\[
\varepsilon_n : KT_{n,S,r} \to KT_{n-1,S,r}.
\]

Since \( \varepsilon_n \circ \iota \) is the identity on \( KT_{n-1,S,r} \), it follows that \( \iota : KT_{n-1,S,r} \to KT_{n,S,r} \) is injective.

It is shown in the companion paper [10], Proposition 4.3 that if the ground ring \( S \) is weakly admissible, then the cyclotomic Kauffman tangle algebra \( KT_{0,S,r} \) is
a free $S$–module of rank $1$. Define

$$\varepsilon = \varepsilon_1 \circ \cdots \circ \varepsilon_n : KT_{n,S,r} \to KT_{0,S,r} \cong S$$

It follows from [9], Proposition 2.14, that $\varepsilon$ is a trace. We also define $\varepsilon : W_{n,S,r} \to S$ by $\varepsilon = \varepsilon \circ \varphi$, where $\varphi : W_{n,S,r} \to KT_{n,S,r}$ is the canonical homomorphism. Then $\varepsilon$ is a trace on $W_{n,S,R}$ with the Markov property: for $b \in W_{n-1,S,r}$,

(a) $\varepsilon(bg_{n-1}^{\pm 1}) = (\rho^{\pm 1}/\delta_0)\varepsilon(b)$,

(b) $\varepsilon(be_{n-1}) = (1/\delta_0)\varepsilon(b)$,

(c) $\varepsilon(b(y_n')^r) = \delta_r\varepsilon(b)$, and

for $r \in \mathbb{Z}$, where $y_n' = (g_{n-1} \cdots g_1)g_1(g_1^{-1} \cdots g_{n-1}^{-1})$. See [9], Corollary 6.16.

**Lemma 2.15.** Let $S$ be a weakly admissible ring. For $x \in KT_{n,S,r}$,

1. $E_n x E_n = \delta_0 \varepsilon_n(x)$.
2. $\varepsilon(E_n x E_n) = \varepsilon(x)$.
3. $\varepsilon(E_n x) = \delta_0^{-1}\varepsilon(x)$.

**Proof.** Straightforward. 

**Lemma 2.16.** Let $S$ be a weakly admissible ring. Then for all $n \geq 1$, $E_n$ and $G_n$ are non-zero elements in $KT_{n,S,r}$.

**Proof.** $\varepsilon(E_n) = \delta_0^{-1}$ and $\varepsilon(G_n) = \rho^{-1}\delta_0^{-2}$. 

### 2.7. Inclusions of split semisimple algebras

A general source for the material in this section is [8]. The Jones basic construction (discussed below) is from [13].

A finite dimensional split semisimple algebra over a field $F$ is one which is isomorphic to a finite direct sum of full matrix algebras over $F$. Suppose $A$ and $B$ are finite dimensional split semisimple algebras over $F$. Let $A_i$, $i \in I$, be the minimal ideals of $A$ and $B_j$, $j \in J$, the minimal ideals of $B$. Let $\psi : A \to B$ be a unital homomorphism of algebras.

We associated a $J \times I$ multiplicity matrix $\Lambda_\psi$ to $\psi$, as follows. Let $W_j$ be a simple $B_j$–module. Then $W_j$ becomes an $A$–module via $\psi$, and $\Lambda_\psi(j,i)$ is the multiplicity of a simple $A_i$–module in the decomposition of $W_j$ as an $A$–module. In particular, if $A \subseteq B$ (and both algebras have the same identity element) the multiplicity matrix for the inclusion is called the inclusion matrix for $A \subseteq B$.

An equivalent characterization of the multiplicity matrix $\Lambda_\psi$ of $\psi$ is the following. Let $q_i$ be a minimal idempotent in $A_i$ and let $z_j$ be the identity of $B_j$ (a
minimal central idempotent in $B$). Then $\psi(q_i)z_j$ is the sum of $\Lambda_{\psi}(j, i)$ minimal idempotents in $B_j$.

Sometimes it is convenient to encode an inclusion matrix (or multiplicity matrix) by a bipartite graph, called the **branching diagram**; the branching diagram has vertices labeled by $I$ arranged on one horizontal line, vertices labelled by $J$ arranged along a second (higher) horizontal line, and $\Lambda_{\psi}(j, i)$ edges connecting $j \in J$ to $i \in I$.

If $A^{(1)} \subseteq A^{(2)} \subseteq A^{(3)} \cdots$ is a sequence of inclusions of finite dimensional split semisimple algebras over $F$, then the branching diagram for the sequence is obtained by stacking the branching diagrams for each inclusion, with the upper vertices of the diagram for $A^{(i)} \subseteq A^{(i+1)}$ being identified with the lower vertices of the diagram for $A^{(i+1)} \subseteq A^{(i+2)}$.

Let $A \subseteq B$ be a pair of finite dimensional split semisimple algebras over $F$. The **Jones basic construction** for $A \subseteq B$ is the pair $B \subseteq \text{End}(B_A)$. That is, $B$ is regarded as acting on itself by left multiplication, and is thus included in the endomorphisms of $B$ as a right $A$–module. Then $\text{End}(B_A)$ is again a split semisimple algebra whose minimal ideals are in one–to–one correspondence with those of $A$, and the inclusion matrix for the pair $B \subseteq \text{End}(B_A)$ is the transpose of that for $A \subseteq B$.

Suppose now that $B$ has a faithful trace $\varepsilon$ with faithful restriction to $A$. Then there is a unique trace preserving conditional expectation (i.e. $A$–$A$ bimodule map) $\varepsilon_A : B \rightarrow A$ determined by $\varepsilon(ba) = \varepsilon(\varepsilon_A(b)a)$ for all $a \in A$ and $b \in B$. Note that $\varepsilon_A$ is an idempotent in $\text{End}(B_A)$. The Jones basic construction has a particular realization in terms of this data. In fact, $\text{End}(B_A) = \langle B, \varepsilon_A \rangle = B\varepsilon_A B$, where $\langle B, \varepsilon_A \rangle$ denotes the subalgebra of $\text{End}(B_A)$ generated by $B$ and $\varepsilon_A$. Moreover, if $q_i$ is a minimal idempotent in $A_i$, then $q_i\varepsilon_A = \varepsilon_A q_i$ is a minimal idempotent in the corresponding minimal ideal of $\text{End}(B_A)$.

**Theorem 2.17** (Wenzl [29]). Suppose $A \subseteq B \subseteq C$ are $F$–algebras with $A$ and $B$ finite dimensional split semisimple. Suppose that $B$ has a trace $\varepsilon \in B^*$ that is non-degenerate on $B$ and has non-degenerate restriction to $A$. Let $\varepsilon_A$ denote the corresponding trace preserving conditional expectation $\varepsilon_A : B \rightarrow A$.

Suppose that $e \in C$ is an idempotent such that $exe = \varepsilon_A(x)e$ for all $x \in B$ and $a \mapsto ae$ is an injective homomorphism from $A$ to $C$. Let $\langle B, e \rangle$ be the subalgebra
of $C$ generated by $B$ and $e$. Then $BeB$ is an ideal in $(B, e)$, and
\[ BeB \cong B \varepsilon_A B = \langle B, \varepsilon_A \rangle = \text{End}(B_A). \]

**Remark 2.18.** In the situation of the theorem, it follows that $BeB$ is split semisimple, with minimal ideals in one–to–one correspondence with those of $A$. If $q_i$ is a minimal idempotent in a minimal ideal $A_i$ of $A$, then $q_i e$ is a minimal idempotent in the corresponding minimal ideal of $BeB$. $BeB$ has an identity element $z$, necessarily a central idempotent in $(B, e)$. The multiplicity matrix for the homomorphism $b \mapsto zb$ from $B$ to $BeB$ is the transpose of the inclusion matrix for $A \subseteq B$.

We now discuss *path idempotents* for a sequence $A^{(0)} \subseteq A^{(1)} \subseteq A^{(2)} \cdots$ of split semisimple algebras, under the simplifying assumption (sufficient for our purposes) that $A^{(0)} = F$ and each inclusion $A^{(i)} \subseteq A^{(i+1)}$ is *multiplicity–free*, that is, the branching diagram has at most one edge connecting any pair of vertices. Let $\Gamma_k$ be an index set for the minimal ideals of $A^{(k)}$; denote the minimal ideals of $A^{(k)}$ by $A^{(k)}_\lambda$ and the minimal central idempotents by $z^{(k)}_\lambda$ ($\lambda \in \Gamma_k$). Denote the unique element of $\Gamma_0$ by $\emptyset$. For $\mu \in \Gamma_j$ and $\lambda \in \Gamma_{j+1}$, write $\mu \subseteq \lambda$ if $\mu$ and $\lambda$ are connected by an edge of the branching diagram. A *path* of length $k$ in the branching diagram is a sequence $(\emptyset, \lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(k)})$ with $\lambda^{(j)} \subseteq \lambda^{(j+1)}$ for all $j$. Let $\mathcal{T}(k)$ be the set of all paths of length $k$, and $\mathcal{T}(k, \lambda)$ the set of paths of length $k$ with final vertex $\lambda$. If $T = (\emptyset, \lambda^{(1)}, \ldots, \lambda^{(k)})$ is a path of length $k$, let $T'$ denote its truncation of length $k - 1$, $T' = (\emptyset, \lambda^{(1)}, \ldots, \lambda^{(k-1)})$. Moreover, if $\lambda \in \Gamma_{k+1}$ and $\lambda^{(k)} \subseteq \lambda$, let $T + \lambda$ denote the extension of $T$ of length $k + 1$ with final vertex $\lambda$. We claim that for each $k$, there is a canonical family of idempotents \{\(p_T : T \in \mathcal{T}(k)\)\}, in $A^{(k)}$, with the properties:

1. The elements $p_T$ are mutually orthogonal minimal idempotents in $A^{(k)}$, and the sum of the $p_T$ with $T \in \mathcal{T}(k, \lambda)$ is the minimal central idempotent $z^{(k)}_\lambda$.
2. $p_T p_{T'} = p_T p_{T'} = p_T$.

The construction is simple: Let $\emptyset$ denote the path of length 0. Necessarily $p_\emptyset = 1$.

If the idempotents $p_T$ have been constructed for paths of length $j$, $T \in \mathcal{T}(j, \mu)$, and $\lambda \in \Gamma_{j+1}$ satisfies $\mu \subseteq \lambda$, then put $p_{T+\lambda} = p_T z^{(j+1)}_\lambda$, which is a minimal idempotent in $A^{(j+1)}$ because of the assumption of multiplicity–free inclusions.

It follows that if $T = (\emptyset, \lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(k)})$ is a path of length $k$, then for all $j \leq k$, $A^{(j)} p_T = A^{(j)}_{\lambda(j)} p_T$ affords the simple $A^{(j)}_{\lambda(j)}$–module.
Lemma 2.19. Suppose $A^{(0)} \subseteq A^{(1)} \subseteq A^{(2)} \subseteq \cdots \subseteq A^{(n)}$ is a sequence of split semisimple algebras, with $A^{(0)} = F$ and with each inclusion $A^{(i)} \subseteq A^{(i+1)}$ multiplicity-free. Suppose we have a trace $\varepsilon$ on $A_n$, not necessarily faithful, but with faithful restriction to $A_{n-1}$. Suppose also that we have a trace-preserving conditional expectation $\varepsilon_n : A_n \to A_{n-1}$. Let $T$ be a path of length $n$ on the branching diagram for the sequence of inclusions. Then

$$\varepsilon_n(p_T) = \left(\varepsilon(p_T)/\varepsilon(p_T')\right)p_{T'}.$$  

Proof. We have $\varepsilon_n(p_T) = \varepsilon_n(p_T' p_T p_{T'}) = p_{T'} \varepsilon_n(p_T)p_{T'}$. Since $p_{T'}$ is a minimal idempotent in $A_{n-1}$, we have $\varepsilon_n(p_T) = \kappa p_{T'}$ for some $\kappa \in F$. Taking traces, we have $\varepsilon(p_T) = \kappa \varepsilon(p_T')$, so $\kappa = \varepsilon(p_T)/\varepsilon(p_T')$. □

3. Representations of the 2–strand algebra and admissibility conditions

To obtain any substantial results about the cyclotomic BMW and Kauffman tangle algebras, it appears to be necessary to impose a stronger condition on the ground ring than weak admissibility. Appropriate conditions can be found by considering the representation theory of the 2–strand algebra, in particular representations in which the generator $e_1$ acts non–trivially.

This idea was used by Ariki, Mathas, and Rui in a related context [3]; the analogue of their condition, translated to our context, is useful, but too restrictive. The “correct” notion of admissibility for cyclotomic BMW algebras was discovered by Wilcox and Yu [31]. The left module $W_{2,S,r} e_1$ is spanned over the ground ring by $\{e_1, y_1 e_1, \ldots, y_1^{r-1} e_1\}$, as is easily checked. Generically, one would expect this spanning set to be linearly independent over $S$, and the requirement of linear independence is the module–theoretic formulation of the admissibility condition of Wilcox and Yu.

3.1. Admissibility. Consider a ground ring $S$ with parameters $\rho$, $q$, $\delta_j$ ($j \geq 0$) and $u_1, \ldots, u_r$. Let $a_j$ denote the signed elementary symmetric function in $u_1, \ldots, u_r$, $a_j = (-1)^{r-j} \varepsilon_{r-j}(u_1, \ldots, u_r)$. Let $W_2$ denote the cyclotomic BMW algebra $W_2 = W_{2,S,r}(u_1, \ldots, u_r)$. Write $e$ for $e_1$ and $g$ for $g_1$.

Lemma 3.1. The left ideal $W_2 e$ in $W_2$ is equal to the $S$–span of $\{e, y_1 e, \ldots, y_1^{r-1} e\}$. 
Theorem 3.2 (Wilcox-Yu, [31]). Let $S$ be a ground ring with parameters $\rho$, $q$, $\delta_j$ ($j \geq 0$) and $u_1, \ldots, u_r$. Assume that $(q - q^{-1})$ is non-zero and not a zero-divisor in $S$. The following conditions are equivalent:

1. $S$ is weakly admissible, and $\{e, y_1 e, \ldots, y_1^{r-1} e\} \subseteq W_2$ is linearly independent over $S$.

2. The parameters satisfy the following relations:

\[
\begin{align*}
\rho(a_\ell - a_{r-\ell}/a_0) + (q - q^{-1}) \left[ \sum_{j=1}^{r-\ell} a_j + \delta_j - \sum_{j=\max(\ell+1, [r/2])}^{\lfloor (\ell+r)/2 \rfloor} a_{2j-\ell} + \sum_{j=\lceil \ell/2 \rceil}^{\min(\ell, [r/2]-1)} a_{2j-\ell} \right] &= 0, \\
&\text{for } 1 \leq \ell \leq r - 1,
\end{align*}
\]

3. $S$ is weakly admissible, and $W_2$ admits a module $M$ with an $S$–basis $\{v_0, y_1 v_0, \ldots, y_1^{r-1} v_0\}$ such that $ev_0 = \delta_0 v_0$.

Definition 3.3 (Wilcox and Yu, [31]). Let $S$ be a ground ring with parameters $\rho$, $q$, $\delta_j$ ($j \geq 0$) and $u_1, \ldots, u_r$. One says that $S$ is admissible (or that the parameters are admissible) if $(q - q^{-1})$ is non-zero and not a zero-divisor in $S$ and if the equivalent conditions of Theorem 3.2 hold.

In the following statement, we omit the assumption that $q - q^{-1}$ is not a zero-divisor, but we impose the assumption that $S$ is weakly admissible. The statement can be proved by a minor modification of the proof of Wilcox and Yu.

Corollary 3.4. Let $S$ be a weakly admissible ring with parameters $\rho$, $q$, $\delta_j$ ($j \geq 0$), and $u_1, \ldots, u_r$. The following conditions are equivalent.

1. $\{e, y_1 e, \ldots, y_1^{r-1} e\} \subseteq W_2$ is linearly independent over $S$.

2. The parameters satisfy the relations (3.1) and (3.2) of Wilcox and Yu.

3. $W_2$ admits a cyclic module $M$ with an $S$–basis $\{v_0, y_1 v_0, \ldots, y_1^{r-1} v_0\}$ such that $ev_0 = \delta_0 v_0$. 
**Remark 3.5.**

(1) Let $S$ be a ground ring with admissible parameters $\rho$, $q$, $\delta_j$ ($j \geq 0$), and $u_1, \ldots, u_r$. Then

$$\rho, -q^{-1}, \delta_j \ (j \geq 0), \text{ and } u_1, \ldots, u_r$$

and

$$-\rho, -q, \delta_j \ (j \geq 0), \text{ and } u_1, \ldots, u_r$$

are also sets of admissible parameters. This follows from condition (2) of Theorem 3.2.

(2) If $S$ is a ground ring with admissible parameters and $\varphi : S \rightarrow S'$ is a morphism of ground rings in the sense of Definition 2.13, such that $\varphi(q - q^{-1})$ is non-zero and not a zero divisor, then $S'$ is also admissible.

**Remark 3.6.** Assume that $S$ is a ground ring with admissible parameters. We observe that the admissibility relation (3.1) can be solved for

$$(q - q^{-1})\delta_1, \ldots, (q - q^{-1})\delta_{r-1}$$

in terms of $\rho$, $a_0^\pm, a_1, \ldots, a_{r-1}$, and $q^\pm$. Rewrite the relations (3.1) in the form

$$\sum_{j=1}^{r-\ell} a_{j+\ell} \delta_j = p_\ell(q^\pm, \rho, a_0^\pm, a_1, \ldots, a_{r-1}) \quad (1 \leq \ell \leq r - 1),$$

where the $p_\ell$ are specific polynomials. This is a triangular system of linear equations in the variables $(q - q^{-1})\delta_j$ with a unitriangular matrix of coefficients

$$
\begin{bmatrix}
1 & & & \\
0 & 1 & & \\
0 & 0 & 1 & \\
\vdots & \ddots & \ddots & \\
a_r & a_{r-1} & \cdots & 1
\end{bmatrix},
$$

so it has a solution of the form

$$(q - q^{-1})\delta_j = Q_j(q^\pm, \rho, a_0^\pm, a_1, \ldots, a_{r-1}), \quad \text{for } 1 \leq j \leq r - 1,$$

where the $Q_j$ are again some specific polynomials. Because $q - q^{-1}$ is assumed not to be a zero-divisor in $S$, we can append an inverse of $q - q^{-1}$ to $S$. In $S[(q - q^{-1})^{-1}]$, we have

$$\delta_j = (q - q^{-1})^{-1} Q_j(q^\pm, \rho, a_0^\pm, a_1, \ldots, a_{r-1}), \quad \text{for } 1 \leq j \leq r - 1,$$
Remark 3.7. Equation (3.2) is equivalent to
\[
\begin{cases}
(a_0 - \rho)(a_0 + \rho) = 0 & \text{if } r \text{ is odd, and} \\
(\rho - q^{-1}a_0)(\rho + qa_0) = 0 & \text{if } r \text{ is even.}
\end{cases}
\]
If \( S \) is an integral domain, we have \( \rho = \pm a_0 \) if \( r \) is odd, and \( \rho = -qa_0 \) or \( \rho = q^{-1}a_0 \) if \( r \) is even. However, we do not need to entertain both solutions. Considering Remark 3.5, we can assume
\[
\rho = -a_0 = \prod_{j=1}^{r} u_j,
\]
if \( r \) is odd, and
\[
\rho = q^{-1}a_0 = q^{-1} \prod_{j=1}^{r} u_j,
\]
if \( r \) is even.

We have the following consequence of this discussion:

Corollary 3.8. Let \( S \) be an integral ground ring with admissible parameters \( \rho \), \( q \), etc. Then \( \rho \pm 1 \) and \( \delta_j \) for \( j \geq 0 \) are elements of the subring of \( S[[q - q^{-1}]^{-1}] \) generated by \( q^{\pm 1} \), \( q^{-1} - q \), and \( u_1^{\pm 1}, \ldots, u_r^{\pm 1} \).

3.2. \( u \)-admissibility. We maintain the notation established at the beginning of Section 3.1. In this section, we assume that the ring \( S \) is an integral domain, that \( q - q^{-1} \neq 0 \), that the parameters \( u_i \) are distinct, and that \( u_iu_j \neq 1 \) for all \( i, j \).

Lemma 3.9. Let \( F \) be field and let \( u_1, \ldots, u_r \) be distinct non-zero elements of \( F \) with \( 1 \neq u_iu_j \) for all \( i, j \). Let \( \rho \) and \( q \) be non-zero elements of \( F \) with \( q^2 \neq 1 \). Then the unique solution to the system of linear equations
\[
(3.5) \quad \sum_j \frac{1}{1-u_iu_j} \gamma_j = \frac{1}{1-u_i^2} + \frac{1}{\rho(q-1-q)} \quad (1 \leq i \leq r)
\]
is
\[
(3.6) \quad \gamma_j = \prod_{\ell \neq j} \frac{(u_\ell u_j - 1)}{u_j - u_\ell} \left( \frac{1 - u_j^2}{\rho(q^{-1} - q)} \right) \prod_{\ell \neq j} u_\ell + \begin{cases} 
1 & \text{if } r \text{ is odd} \\
-u_j & \text{if } r \text{ is even}
\end{cases}
\]
We defer the proof of the lemma to Remark 3.16 in Section 3.3.

Theorem 3.10. Let \( S \) be an integral ground ring with parameters \( \rho \), \( q \), \( \delta_j \) \( (j \geq 0) \) and \( u_1, \ldots, u_r \). Assume that \( (q - q^{-1}) \neq 0 \), that the elements \( u_i \) are distinct, and that \( u_iu_j \neq 1 \) for all \( i, j \). Define \( \gamma_j \) in the field of fractions of \( S \) by (3.6), for \( 1 \leq j \leq r \).
The following conditions are equivalent:

1. $S$ is weakly admissible, and $\{e, y_1e, \ldots, y_1^{r-1}e\} \subseteq W_{2,S}$ is linearly independent over $S$.

2. For all $a \geq 0$, we have $\delta_a = \sum_{j=1}^r \gamma_j u_j^a$.

3. $S$ is weakly admissible, and $W_{2,S}$ admits a module $M$ with an $S$-basis $\{v_0, y_1v_0, \ldots, y_1^{r-1}v_0\}$ such that $ev_0 = \delta_0v_0$.

**Proof.** Let $F$ denote the field of fractions of $S$, and let $(1_F)$ and $(3_F)$ indicate the analogue of conditions (1) and (3) over $F$, i.e.

$(1_F) \quad F$ is weakly admissible, and $\{e, y_1e, \ldots, y_1^{r-1}e\} \subseteq W_{2,F}$ is linearly independent over $F$.

$(3_F) \quad F$ is weakly admissible, and $W_{2,F}$ admits a module $M$ with an $F$-basis $\{v_0, y_1v_0, \ldots, y_1^{r-1}v_0\}$ such that $ev_0 = \delta_0v_0$.

According to Theorem 3.2, conditions (1) and (3) are equivalent, and moreover, they are equivalent to the second condition of Theorem 3.2. Since that condition is unchanged if we replace $S$ by $F$, it follows that (1) and (3) are equivalent to $(1_F)$ and $(3_F)$. Therefore, to prove the theorem, it suffices to show that

$$(1_F) \implies (2) \implies (3_F).$$

Assume condition $(1_F)$. Recall that the span of $\{e, y_1e, \ldots, y_1^{r-1}e\}$ is equal to the left ideal $W_{2,F}e$. Moreover, $ey_1^je = \delta_j e$, so $eW_{2,F}e = Fe$.

For each $j$ ($1 \leq j \leq r$) define $p_j$ by

$$p_j = \prod_{i \neq j} \frac{y_1 - u_i}{u_j - u_i} \in W_{1,F}.$$

Then we have $y_1 p_j = u_j p_j$, $p_j^2 = p_j$, and $\sum_j p_j = 1$.

Let $m_j = p_j e$. Then $m_j \neq 0$, by the linear independence of $\{e, y_1e, \ldots, y_1^{r-1}e\}$ over $F$, $y_1 m_j = u_j m_j$, and $\sum_j m_j = e$. The set $\{m_1, \ldots, m_r\}$ is linearly independent, since the $m_j$ are eigenvectors for $y_1$ with distinct eigenvalues. Because the dimension of $W_{2,F}e$ is $r$, $\{m_1, \ldots, m_r\}$ is a basis of $W_{2,F}e$.

Now we define $\kappa_j \in F$ by $em_j = \kappa_j e = \kappa_j \sum_i m_i$ and $c_{i,j}$ by $gm_j = \sum_i c_{i,j} m_i$. Note that $y_1 g(y_1^i e) = g^{-1} y_1^{-1} (y_1^i e)$, by Remark 2.4. Therefore,

$$y_1 gm_j = g^{-1} y_1^{-1} m_j = \left(g + (q - q^{-1})(e - 1)\right) y_1^{-1} m_j.$$

We have

$$y_1 gm_j = y_1 \sum_i c_{i,j} m_i = \sum_i u_i c_{i,j} m_i.$$
On the other hand,
\[(g+(q-q^{-1})(e-1))y_1^{-1}m_j = u_j^{-1} \sum_i c_{i,j} m_i + u_j^{-1}(q-q^{-1})(\kappa_j \sum_i m_i - m_j)\]

Equating the coefficient of \(m_i\) in the two expressions gives
\[(u_i - u_j^{-1})c_{i,j} = (q-q^{-1})u_j^{-1}(\kappa_j - \delta(i,j)),\]
where \(\delta(i,j)\) denotes the Kronecker delta. Solving, we get
\[(3.7) \quad c_{i,j} = (q^{-1} - q)\frac{\kappa_j - \delta(i,j)}{1 - u_iu_j}.\]

If we apply both sides of the equation \(ge = \rho^{-1}e\) to \(m_k\), we get
\[\kappa_k \sum_{i,j} c_{i,j} m_i = \rho^{-1} \kappa_k \sum_i m_i.\]

Choosing \(k\) so that \(\kappa_k \neq 0\), and equating the coefficients of \(m_i\) on both sides gives
\[\sum_j c_{i,j} = \rho^{-1} \quad \text{for all} \quad i.\]
Taking (3.7) into account, we have for all \(i\),
\[(3.8) \quad \sum_j \frac{1}{1 - u_iu_j} \kappa_j = \frac{1}{1 - u_i^2} + \frac{1}{\rho(q^{-1} - q)}.\]

It follows from (3.8) and Corollary 3.9 that \(\kappa_j\) is given by
\[(3.9) \quad \kappa_j = \gamma_j = \prod_{\ell \neq j} \frac{(u_\ell u_j - 1)}{u_j - u_\ell} \left( \frac{1 - u_j^2}{\rho(q^{-1} - q)} \prod_{\ell \neq j} u_\ell + \begin{cases} 1 & \text{if} \ r \ \text{is odd} \\ -u_j & \text{if} \ r \ \text{is even} \end{cases} \right).\]

Next, note that for \(a \geq 0\)
\[\delta_a e = ey_1^a e = ey_1^a \sum m_i = e \sum_i u_i^a m_i = (\sum_i \gamma_i u_i^a) e.\]

Therefore, for \(a \geq 0\), \(\delta_a = \sum_i \gamma_i u_i^a\). This completes the proof of the implication \((1_F) \implies (2)\).

We now assume condition (2) and prove (3_F). Let \(\hat{e}_1, \ldots, \hat{e}_r\) denote the standard basis of \(M = F^r\). Define linear maps \(E, G, Y\) on \(M\) by
\[
Y \hat{e}_j = u_j \hat{e}_j, \\
E \hat{e}_j = \gamma_j(\hat{e}_1 + \cdots + \hat{e}_r), \\
G \hat{e}_j = (q^{-1} - q) \sum_{i=1}^r \frac{\gamma_j - \delta(i,j)}{1 - u_iu_j} \hat{e}_i,
\]
where for \(1 \leq j \leq r\), \(\gamma_j\) is given by (3.6).
Applying the definitions of \(Y, G, E\), and the \(u\)-admissibility assumption, we have, for \(a \geq 0\),
\[
EY^a E = \delta_a E,
\]
and, in particular, \(E^2 = \delta_0 E\). It is straightforward to compute that
\[
YG Y = G + (q^{-1} - q)(1 - E), \quad \text{and} \quad GE = EG = \rho^{-1} E.
\]
We give some details for the computation of \(YG Y\); applying \(YG Y\) to \(\hat{e}_j\) gives
\[
YG Y \hat{e}_j = (q^{-1} - q) \sum_k \frac{u_j u_k (\gamma_j - \delta(j, k))}{1 - u_j u_k} \hat{e}_k.
\]
If we write \(u_j u_k = (u_j u_k - 1) + 1\) in the numerator and simplify, we get
\[
(q^{-1} - q)(-\gamma_j \sum_k \hat{e}_k + \hat{e}_j) + (q^{-1} - q) \sum_k \frac{(\gamma_j - \delta(j, k))}{1 - u_j u_k} \hat{e}_k
\]
\[
= (q^{-1} - q)(-E + 1) \hat{e}_j + G \hat{e}_j.
\]
Taking into account (3.12), we see that \(YG Y\) commutes with both \(E\) and \(G\),
\[
G Y G Y = Y G Y G, \quad \text{and} \quad E Y G Y = G Y E Y.
\]
Similarly, we compute
\[
Y G E \hat{e}_j = (q^{-1} - q) \sum_i \sum_k \frac{u_i u_k (\gamma_i - \delta(i, k))}{1 - u_i u_k} \hat{e}_k.
\]
Simplifying this as above, we get
\[
(q^{-1} - q) \gamma_j (-\sum_i \gamma_i + 1) \sum_k \hat{e}_k + (q^{-1} - q) \gamma_j \sum_i \sum_k \frac{(\gamma_i - \delta(i, k))}{1 - u_i u_k} \hat{e}_k
\]
Using \(\sum_i \gamma_i = \delta_0\) and (3.5), we can simplify this to
\[
((q^{-1} - q)(-\delta_0 + 1) + \rho^{-1}) \gamma_j \sum_k \hat{e}_k = \rho \gamma_j \sum_k \hat{e}_k = \rho E \hat{e}_j
\]
Thus, we have
\[
E Y G Y = Y G E Y = \rho E.
\]
Set \(m = \sum_i \hat{e}_i\). We claim that \(\{m, Y m, \ldots, Y^{r-1} m\}\) is a basis of \(M\). In fact, if 
\[
\sum_{j=0}^{r-1} \alpha_j Y^j m = 0,
\]
then \(\sum_{i=1}^{r} (\sum_{j=0}^{r-1} \alpha_j u_i^j) \hat{e}_i = 0\). Since the \(u_i\) are distinct, the Vandermonde matrix \((u_i^j)\) is invertible, so \(\alpha_j = 0\) for all \(j\). This shows that \(\{m, Y m, \ldots, Y^{r-1} m\}\) is linearly independent, and therefore a basis of \(M\).
We claim that $GYGY$ is the identity transformation. By the last paragraph, it is enough to show that $GYGY(Y^k m) = Y^k m$ for $0 \leq k \leq r - 1$. Note that $GYGE = E$, by (3.12) and (3.14), and also $Em = \delta_0 m$. According to (3.13), $Y$ commutes with $GYG$, so

$$GYGY(Y^k m) = Y^k YGYGm = (1/\delta_0)Y^k YGYGEm = (1/\delta_0)Y^k Em = Y^k m.$$ 

Using (3.12), we have

\[(3.15) \quad G^{-1} = YGY = G + (q^{-1} - q)(1 - E).\]

It follows from (3.11)-(3.15) that $E, G, Y$ satisfy the relations of the cyclotomic BMW algebra with parameters $\rho, q, \delta_j \ (j \geq 0)$, and $u_1, \ldots, u_r$. In particular, since we are working over a field, and $E \neq 0$, the parameters must be weakly admissible; see the statement just before Definition 2.14. \hfill \Box

**Definition 3.11.** Let $S$ be an integral ground ring with parameters $\rho, q, \delta_j \ (j \geq 0)$, and $u_1, \ldots, u_r$. Say that the parameters are $u$-admissible or that $S$ is $u$-admissible if:

1. $q - q^{-1} \neq 0$, $u_i \neq u_j$ for $i \neq j$ and $u_i u_j \neq 1$ for any $i, j$, and
2. the equivalent conditions of Theorem 3.10 hold.

**Corollary 3.12.** Let $S$ be an integral ground ring with parameters $\rho, q, \delta_j \ (j \geq 0)$, and $u_1, \ldots, u_r$. Then $S$ is $u$-admissible if, and only if, $S$ is admissible, the $u_i$ are distinct and $u_i u_j \neq 1$ for all $i, j$.

*Proof.* This follows from Theorems 3.2 and 3.10. \hfill \Box

**Remark 3.13.** Let $S$ be a ground ring with $u$-admissible parameters $\rho, q, \delta_j \ (j \geq 0)$, and $u_1, \ldots, u_r$. Then

$$\rho, -q^{-1}, \delta_j \ (j \geq 0), \text{ and } u_1, \ldots, u_r$$

and

$$-\rho, -q, \delta_j \ (j \geq 0), \text{ and } u_1, \ldots, u_r$$

are also sets of $u$-admissible parameters. Moreover, the quantities $\gamma_j$ computed from any of these sets of parameters according to (3.6) are the same.

**Remark 3.14.** Let $S$ be an integral ground ring with $u$-admissible parameters $\rho, q, \delta_j \ (j \geq 0)$, and $u_1, \ldots, u_r$. By Remarks 3.7 and 3.13, we can assume $\rho = \prod_j u_j$.
if \( r \) is odd and \( \rho = q^{-1} \prod_j u_j \) if \( r \) is even. Using this, we can rewrite the formula for \( \gamma_j \) (3.6) as

\[
\gamma_j = \frac{\rho(u_j - q^{-1})(u_j + q)}{u_j^2(q - q^{-1})} \prod_{\ell \neq j} \frac{u_j - u_{\ell}^{-1}}{u_j - u_{\ell}} \quad \text{if } r \text{ is odd,}
\]

and

\[
\gamma_j = \frac{\rho(u_j - q)(u_j + q)}{u_j^2(q - q^{-1})} \prod_{\ell \neq j} \frac{u_j - u_{\ell}^{-1}}{u_j - u_{\ell}} \quad \text{if } r \text{ is even.}
\]

Note that if \( u_j \not\in \{\pm q^{\pm 1}\} \), then \( \gamma_j \neq 0 \).

3.3. Some generating functions. We will investigate some generating functions that appear naturally in the study of cyclotomic BMW algebras. Let \( u_1, \ldots, u_r, q, \) and \( \rho \) be indeterminants over \( \mathbb{Z} \). We will need to consider rational functions \( \gamma_j \) in these variables that are solutions to the system of linear equations:

\[
\sum_j \frac{1}{1 - u_i u_j} \gamma_j = \frac{1}{1 - u_i^2} + \frac{1}{\rho(q^{-1} - q)} \quad (1 \leq i \leq r).
\]

Lemma 3.15. Let \( F \) be field and let \( u_1, \ldots, u_r \) be distinct non–zero elements of \( F \) with \( 1 \neq u_i u_j \) for all \( i, j \).

1. The matrix

\[
\left( \frac{1}{1 - u_i u_j} \right)_{1 \leq i, j \leq r}
\]

is invertible.

2. The unique solution to the linear system of equations

\[
\sum_j \frac{1}{1 - u_i u_j} \gamma_j^{(1)} = 1 \quad (1 \leq i \leq r)
\]

is

\[
\gamma_j^{(1)} = (1 - u_j^2) \prod_{\ell \neq j} u_{\ell} \prod_{\ell \neq j} \frac{u_{\ell} u_j - 1}{u_j - u_{\ell}}.
\]

3. The unique solution to the linear system of equations

\[
\sum_j \frac{1}{1 - u_i u_j} \gamma_j^{(2)} = \frac{1}{1 - u_i^2} \quad (1 \leq i \leq r)
\]

is

\[
\gamma_j^{(2)} = \prod_{\ell \neq j} \frac{u_{\ell} u_j - 1}{u_j - u_{\ell}} \quad \text{if } r \text{ is odd,}
\]
and
\[ \gamma_j^{(2)} = -u_j \prod_{\ell \neq j} \frac{u_\ell u_j - 1}{u_j - u_\ell} \] if \( r \) is even.

Proof. Recall Cauchy’s determinant identity [15], Section 2.1,
\[ \det \left( \frac{1}{1 - x_i y_j} \right) = \prod_{i<j} (x_i - x_j)(y_i - y_j) \prod_{i,j} (1 - x_i y_j), \]
which specializes to
\[ \det \left( \frac{1}{1 - u_i u_j} \right) = \prod_{i<j} (u_i - u_j)^2 \prod_{i,j} (1 - u_i u_j), \]
Since \( u_i \neq u_j \) for \( i \neq j \), this implies statement (1).

For part (2), we have to prove the identity, for all \( i \) \((1 \leq i \leq r)\):
\[ \sum_j \frac{1}{1 - u_i u_j} (1 - u_j^2) \prod_{\ell \neq j} \frac{u_\ell (u_\ell u_j - 1)}{u_j - u_\ell} = 1, \]
or
\[ (3.18) \sum_j \prod_{\ell \neq i} (u_\ell u_j - 1) \prod_{\ell \neq j} \frac{u_\ell}{u_j - u_\ell} = 1. \]

By the principal of permanence of identities (see [4], p. 456) it suffices to prove this when the \( u_i \) are distinct non–zero complex numbers satisfying \( u_i u_j \neq 1 \) for all \( i, j \). Put
\[ f_i(\zeta) = \frac{1}{\zeta} \prod_{\ell \neq i} (\zeta u_\ell - 1) \prod_{\ell=1}^r \frac{u_\ell}{\zeta - u_\ell}. \]
Check that the residue of \( f_i(\zeta) \) at \( \zeta = u_j \) is the summand on the left side of (3.18). The other possible poles of \( f_i(\zeta) \) are at \( \zeta = 0 \) and at \( \zeta = \infty \). The residue at \( \zeta = 0 \) is \(-1\) and the residue at \( \zeta = \infty \) is zero. The identity of (3.18) follows, since the sum of the residues of \( f_i(\zeta) \) at its finite poles and at \( \infty \) is zero.

For part (3), we have to prove the identity for all \( i \) \((1 \leq i \leq r)\):
\[ (3.19) \sum_j \frac{1 - u_i^2}{1 - u_i u_j} \prod_{\ell \neq j} \frac{u_\ell u_j - 1}{u_j - u_\ell} = 1 \quad \text{if } r \text{ is odd}, \]
and
\[ (3.20) \sum_j \frac{1 - u_i^2}{1 - u_i u_j} (-u_j) \prod_{\ell \neq j} \frac{u_\ell u_j - 1}{u_j - u_\ell} = 1 \quad \text{if } r \text{ is even}. \]
Again, we may assume that the \( u_i \) are distinct non-zero complex numbers. When \( r \) is odd, we put
\[
f_i(\zeta) = \frac{1 - u_i^2}{(\zeta^2 - 1)(1 - \zeta u_i)} \prod_{\ell=1}^{r} \frac{\zeta u_\ell - 1}{\zeta - u_\ell},
\]
and observe that the residue of \( f_i(\zeta) \) at \( \zeta = u_j \) is the summand on the left side of (3.19). The other possible poles of \( f_i(\zeta) \) are at \( \zeta = \pm 1 \) and \( \zeta = \infty \). The residue at \( \zeta = 1 \) is \((-1/2)(1 + u_i)\), the residue at \( \zeta = -1 \) is \((-1/2)(1 - u_i)\), and the residue at \( \zeta = \infty \) is zero. Since the sum of these residues is \(-1\), Equation (3.19) follows.

The proof of (3.20) is similar, and we omit it. \( \square \)

**Remark 3.16.** Lemma 3.9 from Section 3.2 is a corollary of Lemma 3.15. Namely, the unique solution to the system of equations
\[
\sum_j \frac{1}{1 - u_i u_j} \gamma_j = \frac{1}{1 - u_i^q} + \frac{1}{\rho(q^{-1} - q)} \quad (1 \leq i \leq r)
\]
is
\[
\gamma_j = \frac{1}{\rho(q^{-1} - q)} \gamma_j^{(1)} + \gamma_j^{(2)}.
\]

Fix a natural number \( r \). Let \( u_1, \ldots, u_r \) and \( t \) be indeterminants over \( \mathbb{Z} \), and define
\[
G(t) = G(u_1, \ldots, u_r; t) = \prod_{\ell=1}^{r} \frac{t - u_\ell}{tu_\ell - 1}.
\]

Let \( \mu_a = \mu_a(u_1, \ldots, u_r) \) denote the coefficients of the formal power series expansion of \( G(t) \),
\[
G(t) = \sum_{a=0}^{\infty} \mu_a t^a.
\]
Notice that each \( \mu_a \) is a symmetric polynomial in \( u_1, \ldots, u_r \) and that \( G(t^{-1}) = G(t)^{-1} \).

Now suppose that \( \rho \) and \( q \) are additional indeterminants, and for \( 1 \leq j \leq r \), define rational functions \( \gamma_j \) by (3.6). Moreover, for \( a \geq 0 \), define \( \delta_a = \sum_{j=1}^{r} \gamma_j u_j^a \).

Let \( Z_1(t) \) be the generating function for the \( \delta_a \),
\[
Z_1(t) = \sum_{a \geq 0} \delta_a t^{-a} = \sum_{a \geq 0} \sum_{j=1}^{r} \gamma_j u_j^a t^{-a} = \sum_{j=1}^{r} \gamma_j \sum_{a \geq 0} u_j^a t^{-a} = \sum_{j=1}^{r} \gamma_j \frac{t}{t - u_j}.
\]
In the following, we use the notation

\[ \delta(P) = \begin{cases} 
1 & \text{if } P \text{ is true} \\
0 & \text{if } P \text{ is false.} 
\end{cases} \]

**Lemma 3.17.** Let \( p \) denote \( \prod_{j=1}^{r} u_j \).

1. \( Z_1(t) = \frac{1}{\rho(q^{-1} - q)} + \frac{t^2}{t^2 - 1} + A(t) G(t^{-1}) \), where

\[
A(t) = \begin{cases} 
\rho^{-1} p/(q - q^{-1}) + t/(t^2 - 1) & \text{if } r \text{ is odd, and} \\
\rho^{-1} p/(q - q^{-1}) - t^2/(t^2 - 1) & \text{if } r \text{ is even.}
\end{cases}
\]

2. If \( r \) is odd, then for \( a \geq 0 \),

\[ \delta_a = \delta(a = 0) \rho^{-1} (q - q^{-1}) + \delta(a \text{ is even}) + \mu_a \rho^{-1} p/(q - q^{-1}) + \mu_{a-1} + \mu_{a-3} + \cdots. \]

3. If \( r \) is even, then for \( a \geq 0 \),

\[ \delta_a = \delta(a = 0) \rho^{-1} (q - q^{-1}) + \delta(a \text{ is even}) + \mu_a \rho^{-1} p/(q - q^{-1}) - \mu_a - \mu_{a-2} - \mu_{a-4} + \cdots. \]

4. \( \delta_0 = \frac{1 - p^2}{\rho(q^{-1} - q)} + 1 \begin{cases} 
0 & \text{if } r \text{ is odd} \\
p & \text{if } r \text{ is even.}
\end{cases} \)

5. For all \( a \geq 0 \), \( \delta_a \) is an element of the ring

\[ \mathbb{Z}[u_1, \ldots, u_r, \rho^{-1}, (q^{-1} - q)^{-1}], \]

and is symmetric in \( u_1, \ldots, u_r \).

**Proof.** To prove the identity of part (1), it suffices to suppose that the quantities \( u_1, \ldots, u_r, t, q, \rho \) are complex numbers, algebraically independent over \( \mathbb{Q} \). Set

\[ h(\zeta) = G(\zeta^{-1}) \left( \zeta^{-1} \frac{p}{\rho(q - q^{-1})} + \begin{cases} 
1/(\zeta^2 - 1) & \text{if } r \text{ is odd} \\
-\zeta/(\zeta^2 - 1) & \text{if } r \text{ is even.}
\end{cases} \right). \]

It is straightforward to check that the residue of \( h(\zeta)t/(t - \zeta) \) at \( \zeta = u_j \) is

\[ \gamma_j t/(t - u_j). \]

Thus, \( Z_1(t) \) is the sum of the residues of \( h(\zeta)t/(t - \zeta) \) at \( u_1, \ldots, u_r \). The other possible poles of \( h(\zeta)t/(t - \zeta) \) are at 0, ±1, ∞, and \( t \). Independent of the parity of \( r \), the residue of \( h(\zeta)t/(t - \zeta) \) at \( \zeta = 0 \) is \( 1/(\rho(q - q^{-1})) \), the residue at \( \zeta = 1 \) is \( -(1/2)(t/(t - 1)) \), the residue at \( \zeta = -1 \) is \( -(1/2)(t/(t + 1)) \), and the residue at \( \infty \) is zero. For \( r \) odd, the residue of \( h(\zeta)t/(t - \zeta) \) at \( \zeta = t \) is

\[ G(t^{-1}) \left( p/(\rho(q^{-1} - q)) - t/(t^2 - 1) \right). \]
For $r$ even, the residue of $h(\zeta)t/(t - \zeta)$ at $\zeta = t$ is

$$G(t^{-1}) \left( p/(\rho(q^{-1} - q)) + t^2/(t^2 - 1) \right).$$

$Z_1(t)$ is the negative of the sum of the residues of $h(\zeta)t/(t - \zeta)$ at 0, $\pm 1$, $\infty$, and $t$. This yields the formula of part (1).

For part (2), write

$$Z_1(t^{-1}) = \sum_{a \geq 0} \delta_at^a = \frac{1}{\rho(q^{-1} - q)} + \frac{1}{1 - t^2}$$

$$+ G(t) \begin{cases} 
\frac{p}{\rho(q^{-1} - q)} + \frac{t}{1 - t^2} & \text{if } r \text{ is odd} \\
\frac{1}{1 - t^2} - \frac{1}{1 - t^2} & \text{if } r \text{ is even}
\end{cases}.$$

One can now read off the coefficient of $t^a$ on the right hand side.

Part (3) follows from part (2), using that $\mu_0 = G(0) = p$, and part (4) follows from part (2) as well. \qed

Note that if $F$ is a field with $u$–admissible parameters $\rho$, $q$, $\delta_j$, and $u_1, \ldots, u_r$, and $Z_1(t) = \sum_{a \geq 0} \delta_at^{-a}$, then Lemma 3.17 applies, but we also can assume that $\rho = p$ if $r$ is odd, and $\rho = q^{-1}p$ if $r$ is even. So the formulas of the lemma simplify as follows:

$$Z_1(t) = \frac{1}{\rho(q^{-1} - q)} + \frac{t^2}{t^2 - 1} + A(t) G(t^{-1}),$$

where

$$A(t) = \begin{cases} 
1/(q - q^{-1}) + t/t^2 - 1 & \text{if } r \text{ is odd, and} \\
q/(q - q^{-1}) - t^2/t^2 - 1 & \text{if } r \text{ is even.}
\end{cases}$$

3.4. The universal admissible ring. We begin by constructing a universal $u$–admissible integral domain. Let $q$, $u_1$, $\ldots$, $u_r$ be algebraically independent indeterminants over $\mathbb{Z}$. Let $S_0$ be the Laurent polynomial ring in $q, u_1, \ldots, u_r$, with $(q^{-1} - q)^{-1}$ adjoined,

$$S_0 = \mathbb{Z}[q^{\pm 1}, u_1^{\pm 1}, \ldots, u_r^{\pm 1}][(q^{-1} - q)^{-1}].$$

Let $p = \prod_j u_j$. If $r$ is odd, put $\rho = p$, and if $r$ is even, put $\rho = q^{-1}p$. Define $\gamma_j$ for $1 \leq j \leq r$ (in the field of fractions of $S_0$) by (3.6), and set $\delta_a = \sum_j \gamma_j u_j^a$. According to Lemma 3.17, $\delta_a \in S_0$, and, moreover,

$$\delta_0 = 1 + \frac{1 - p^2}{\rho(q^{-1} - q)} - \begin{cases} 
0 & \text{if } r \text{ is odd} \\
p & \text{if } r \text{ is even.}
\end{cases}$$

Taking into account that $\rho = p$ or $\rho = q^{-1}p$, we have $\delta_0 \neq 0$ by the algebraic independence of $q$ and $u_1, \ldots, u_r$. Moreover, one can check that
\( \rho^{-1} - \rho = (q^{-1} - q)(\delta_0 - 1) \). Finally, define \( \overline{S} = S_0[\delta_0^{-1}] \). Then \( \overline{S} \) is \( u \)-admissible.

Note that the field of fractions of \( \overline{S} \) is \( \mathbb{Q}(q, u_1, \ldots, u_r) \).

Let \( S \) be another \( u \)-admissible integral domain with parameters \( \rho, q, \) etc. Using Remark 3.13, we can assume that \( \rho = \prod_j u_j \) if \( r \) is odd and \( \rho = q^{-1} \prod_j u_j \) if \( r \) is even. Because of the algebraic independence of \( q, u_1, \ldots, u_r \), there is a ring homomorphism \( \varphi : \overline{S} \to S[(q - q^{-1})^{-1}] \) mapping \( q \mapsto q \) and \( u_i \mapsto u_i \) for each \( i \). It follows that \( \varphi : \rho \mapsto \rho \). Since \( \overline{S} \) and \( S \) are \( u \)-admissible, we have \( \delta_a \mapsto \delta_a \) for \( a \geq 0 \) as well.

We have shown the following result:

**Lemma 3.18.** There is a “universal” integral domain \( \overline{S} \) with \( u \)-admissible parameters \( \rho, q, \delta_j, (j \geq 0), u_1, \ldots, u_r \), with the properties:

1. \( q, u_1, \ldots, u_r \) are algebraically independent over \( \mathbb{Z} \).
2. \( \overline{S} \) is generated as a unital ring by \( q^{\pm 1}, (q^{-1} - q)^{-1}, u_1^{\pm 1}, \ldots, u_r^{\pm 1} \), and \( \delta_0^{-1} \).
3. The field of fractions of \( \overline{S} \) is \( \mathbb{Q}(q, u_1, \ldots, u_r) \).
4. Whenever \( S \) is an integral domain with \( u \)-admissible parameters \( \rho, q, \) etc., there exists a morphism of ground rings (see Definition 2.13)

\[
\phi : \overline{S} \to S[(q - q^{-1})^{-1}] .
\]

Next we consider the construction of universal admissible rings. Let \( q, \rho, \delta_0, \ldots, \delta_{-1}, u_1, \ldots, u_r \) be indeterminants over \( \mathbb{Z} \). Let \( a_j \) be the signed elementary symmetric polynomials in the variables \( u_i \). For \( a \geq r \), define \( \delta_a \) by the recursion \( \delta_a = - \sum_{j=0}^{r-1} a_j \delta_{a-r+j} \).

Set

\[
A = \mathbb{Z}[q^{\pm 1}, \rho^{\pm 1}, \delta_0^{\pm 1}, \delta_1, \ldots, \delta_{r-1}, u_1^{\pm 1}, \ldots, u_r^{\pm 1}] .
\]

If \( r \) is odd, set \( f_+ = \rho + a_0 \) and \( f_- = \rho - a_0 \). If \( r \) is even, set \( f_+ = \rho - q^{-1}a_0 \) and \( f_- = \rho + qa_0 \). For \( \epsilon \in \{0, +, -\} \), let \( J_\epsilon \) be the ideal in \( A \) generated by \( f_\epsilon, \rho^{-1} - \rho - (q^{-1} - q)(\delta_0 - 1) \), and the elements

\[
\rho(a_\ell - a_{r-\ell}/a_0) + (q - q^{-1}) \left[ \sum_{j=0}^{r-\ell} a_{j+\ell} \delta_j - \sum_{j=(\ell+1)\ell/2)}^{\left[(\ell+1)r/2\right] - 1} a_{2j-\ell} + \sum_{j=[\ell/2]}^{\min(\ell, [r/2] - 1)} a_{2j-\ell} \right] ,
\]

for \( 1 \leq \ell \leq r - 1 \). Note that \( J_0 \subseteq J_+ \cap J_- \). Set \( R_\epsilon = A/J_\epsilon \). Since \( J_0 \subseteq J_\pm \), we have maps \( R_0 \to R_\pm \) defined by \( x + J_0 \mapsto x + J_\pm \). Moreover, we can define an automorphism \( \theta \) of \( A \) taking \( J_+ \) to \( J_- \). (For \( r \) odd, require \( \theta(\rho) = -\rho, \theta(q) = -q \),
\[ \theta(\delta_j) = \delta_j \text{ and } \theta(u_i) = u_i. \] For \( r \) even, require \( \theta(\rho) = \rho, \theta(q) = -q^{-1}, \theta(\delta_j) = \delta_j \) and \( \theta(u_i) = u_i. \) Then \( \theta \) induces an isomorphism \( \theta: R_+ \to R_- \).

The rings \( R_\varepsilon \) have the properties of an admissible ring, except that the image of \((q - q^{-1})\) in \( R_\varepsilon \) might be a zero-divisor. Consequently, we will introduce quotients of \( R_\varepsilon \) in which the image of \((q - q^{-1})\) is not a zero divisor.

If \( S \) is an admissible integral domain, then there is a morphism of ground rings from \( R_0 \) to \( S \), and this homomorphism must factor through \( R_+ \) or \( R_- \) according to Remark 3.7. In particular, there is a morphism of ground rings from \( R_0 \) to the universal \( u \)-admissible ring \( \overline{S} \) that factors through \( R_+ \). It follows from this that \((q - q^{-1})^n + J_0 \neq 0 \) in \( R_0 \) and \((q - q^{-1})^n + J_+ \neq 0 \) in \( R_+ \), for all natural numbers \( n \). Using the isomorphism \( \theta: R_+ \to R_- \), we have also that \((q - q^{-1})^n + J_- \neq 0 \) in \( R_- \) for all natural numbers \( n \).

For \( \varepsilon \in \{0, +, -\} \), let

\[ P_\varepsilon = \{ x + J_\varepsilon \in R_\varepsilon : \exists n \geq 1 \text{ such that } (x + J_\varepsilon)(q - q^{-1} + J_\varepsilon)^n = 0 \}. \]

Then \( P_\varepsilon \) is an ideal in \( R_\varepsilon \). Set \( \overline{R}_\varepsilon = R_\varepsilon / P_\varepsilon \). Let \( \overline{\rho}, \overline{q}, \) etc. denote the images of the parameters in \( \overline{R}_\varepsilon \). Since the quotient maps \( R_0 \to R_\pm \) take \( P_0 \) into \( P_\pm \), they induce maps \( \overline{R}_0 \to \overline{R}_\pm \). Moreover, the isomorphism \( \theta: R_+ \to R_- \) maps \( P_+ \) to \( P_- \), so induces an isomorphism \( \theta: \overline{R}_+ \to \overline{R}_- \).

If \( S \) is any admissible ring and \( \varphi: R_\varepsilon \to S \) is a morphism of ground rings from \( R_\varepsilon \) to \( S \), then \( \varphi(P_\varepsilon) = 0 \). Hence \( \varphi \) induces a morphism of ground rings from \( \overline{R}_\varepsilon \) to \( S \). In particular, we have a morphism of ground rings from \( R_0 \) to the universal \( u \)-admissible ring \( \overline{S} \), that factors through \( R_+ \), and this map induces a morphism of ground rings from \( \overline{R}_0 \) to \( \overline{S} \), factoring through \( \overline{R}_+ \). It follows from this that \( \overline{q} - \overline{q}^{-1} \neq 0 \) in \( \overline{R}_\varepsilon \) (\( \varepsilon = 0, + \)). Using the isomorphism \( \theta: \overline{R}_+ \to \overline{R}_- \), we see that \( \overline{q} - \overline{q}^{-1} \neq 0 \) in \( \overline{R}_- \) as well.

Finally we check that \( \overline{q} - \overline{q}^{-1} \) is not a zero-divisor in \( \overline{R}_\varepsilon \), for \( \varepsilon \in \{0, +, -\} \). Let \( \overline{x} \in \overline{R}_\varepsilon \) such that \( \overline{x}(\overline{q} - \overline{q}^{-1}) = 0 \). Let \( x \) be a preimage of \( \overline{x} \) in \( R_\varepsilon \). Then \( x(q - q^{-1} + J_\varepsilon) \in P_\varepsilon \). Hence there exists an \( n \geq 1 \) such that

\[ x((q - q^{-1}) + J_\varepsilon)((q - q^{-1}) + J_\varepsilon)^n = 0. \]

But this means that \( x \in P_\varepsilon \), so \( \overline{x} = 0 \).

We claim that \( \overline{R}_+ \) is an integral domain.
Since there is a morphism of ground rings from $\overline{R}_+$ to the universal $u$–admissible ring $\overline{S}$, it follows that the parameters $\bar{q}, \bar{u}_1, \ldots, \bar{u}_r$ in $\overline{R}_+$ are algebraically independent over $\mathbb{Z}$. Consider $\overline{R}_+[(\bar{q} - \bar{q}^{-1})^{-1}]$ and its subring

$$A_0 = \mathbb{Z}[\bar{q}^{\pm 1}, \bar{u}_1^{\pm 1}, \ldots, \bar{u}_r^{\pm 1}][(\bar{q} - \bar{q}^{-1})^{-1}]$$

$A_0$ is an integral domain, since $\bar{q}, \bar{u}_1, \ldots, \bar{u}_r$ are algebraically independent over $\mathbb{Z}$. According to Corollary 3.8, $A_0$ also contains $\bar{\rho}^{\pm 1}$ and $\bar{\delta}_0, \bar{\delta}_1, \ldots, \bar{\delta}_{r-1}$. Therefore,

$$A_0[\bar{\delta}_0^{-1}] = \overline{R}_+[(\bar{q} - \bar{q}^{-1})^{-1}]$$

Thus, $\overline{R}_+[(\bar{q} - \bar{q}^{-1})^{-1}]$ and its subring $\overline{R}_+$ are integral domains. According to Corollary 3.12, $\overline{R}_+$ is $u$–admissible. It is now clear that $\overline{R}_+[(\bar{q} - \bar{q}^{-1})^{-1}]$ is isomorphic to the universal $u$–admissible ring $\overline{S}$ by a isomorphism of ground rings.

We summarize this discussion with the following theorem:

**Theorem 3.19.** There exist a universal admissible ring $\overline{R}_0$ and a universal admissible integral domain $\overline{R}_+$ with the following properties:

1. There is morphism of ground rings from $\overline{R}_0$ to $\overline{R}_+$.
2. $\overline{R}_0$ (and hence also $\overline{R}_+$) are generated as unital rings by the parameters $\bar{q}^{\pm 1}$, $\bar{\rho}^{\pm 1}$, $\bar{\delta}_0^{\pm 1}$, $\bar{\delta}_1$, $\ldots$, $\bar{\delta}_{r-1}$, $\bar{u}_1^{\pm 1}$, $\ldots$, $\bar{u}_r^{\pm 1}$.
3. The parameters $\bar{q}, \bar{u}_1, \ldots, \bar{u}_r$ of $\overline{R}_+$ (hence also of $\overline{R}_0$) are algebraically independent over $\mathbb{Z}$.
4. Whenever $S$ is a ring with admissible parameters, there exists a morphism of ground rings from $\overline{R}_0$ to $S$. If $S$ is also an integral domain, then the morphism of ground rings can be chosen to factor through $\overline{R}_+$.
5. $\overline{R}_+[(\bar{q} - \bar{q}^{-1})^{-1}]$ is isomorphic to the universal $u$–admissible ring, by an isomorphism of ground rings.

### 3.5. More generating functions.

Let $F$ be a field with $u$–admissible parameters $\rho$, $q$, $\delta_j$, $u_1, \ldots, u_r$. Let $KT_{n,r}$ denote $KT_{n,F,r}(u_1, \ldots, u_r)$. Recall that $\varepsilon_n$ denotes the conditional expectation from $KT_{n,r}$ to $KT_{n-1,r}$, and $\varepsilon$ denotes the canonical trace on $KT_{n,r}$. For $n \geq 1$, write $Y_n$ for $\varphi(y_n)$. Note that $\varepsilon_n(Y_n^a) \in Z(KT_{n-1,r})$ for $a \geq 0$, since $Y_n^a$ commutes with $KT_{n-1,r}$. Define $\omega_n^{(a)}$ by $\omega_n^{(a)} = \delta_0 \varepsilon_n(Y_n^a)$, so

$$E_n Y_n^a E_n = \omega_n^{(a)} E_n.$$

Define

$$Z_n(t) = \sum_{a \geq 0} \omega_n^{(a)} t^{-a},$$

(3.25)
a formal power series with coefficients in the center of $KT_{n-1,r}$. This is consistent with the previous definition of $Z_1$ in 3.23. Moreover, define

$$(3.26) \quad Q_n(t) = Z_n(t) - \rho^{-1}/(q^{-1} - q) - t^2/(t^2 - 1).$$

In particular

$$(3.27) \quad Q_1(t) = G(t^{-1})A(t),$$

according to (3.24).

We have the following remarkable recursion for $Q_n(t)$, which is due to Beliakova and Blanchet [5], Lemma 7.4.

**Proposition 3.20.**

$$Q_{n+1}(t) = Q_n(t) \left( \frac{(t - Y_n)^2(t - q^{-2}Y_n^{-1})(t - q^2Y_n^{-1})}{(t - Y_n^{-1})^2(t - q^{-2}Y_n)(t - q^2Y_n)} \right).$$

**Proof.** Beliakova and Blanchet prove this for the ordinary BMW (or Kauffman tangle) algebras, that is, for $r = 1$. However, their proof remains valid in the general case.

4. The generic structure of the cyclotomic BMW algebra

In this section we obtain the “generic structure” of the cyclotomic BMW algebras over a field, following the method of Wenzl from [6], [29], and [30]. The argument also relies on ideas from Beliakova and Blanchet [5].

Let $F$ be the field of fractions of the universal admissible integral domain $\mathcal{R}_+$. We will show that for all $n$ the cyclotomic BMW algebra $W_{n,F,r}$ over $F$ is isomorphic to the cyclotomic Kauffman tangle algebra $\hat{KT}_{n,F,r}$ over $F$, and is split semisimple of dimension $r^n(2n - 1)!$. The irreducible representations of $W_{n,F,r}$ are labelled by $r$–tuples of Young diagrams of total size $n, n-2, n-4, \ldots$, and the dimension of the irreducible representation labelled by such an $r$–tuple $\mu$ is the number of up–down tableaux of length $n$ and shape $\mu$.

4.1. Semisimplicity and structure of $W_{n,F,r}$. Let $S$ be a ground ring. For each $n$, let $I_n$ be the two sided ideal in $W_{n,S,r}$ generated by $e_{n-1}$. Because of the relations $e_j e_{j \pm 1} e_j = e_j$, the ideal $I_n$ is generated by any $e_i \ (1 \leq i \leq n - 1)$ or by all of them.

**Lemma 4.1.** For all $n \geq 1$, $I_{n+1} = W_{n,S,r} e_n W_{n,S,r}$.
Proof. This follows from the corresponding result for the affine BMW algebras, [9], Proposition 3.20. □

Let us recall the definition of the affine and cyclotomic Hecke algebras, see [1].

**Definition 4.2.** Let $S$ be a commutative unital ring with an invertible element $q$. The affine Hecke algebra $\hat{H}_{n,S}(q^2)$ over $S$ is the $S$–algebra with generators $t_1, g_1, \ldots, g_{n-1}$, with relations:

1. The generators $g_i$ are invertible, satisfy the braid relations, and $g_i^{-1} = (q - q^{-1})$.
2. The generator $t_1$ is invertible, $t_1 g_1 t_1 g_1 = g_1 t_1 g_1 t_1$ and $t_1$ commutes with $g_j$ for $j \geq 2$.

Let $u_1, \ldots, u_r$ be additional elements in $S$. The cyclotomic Hecke algebra $H_{n,S,r}(q^2; u_1, \ldots, u_r)$ is the quotient of the affine Hecke algebra $\hat{H}_{n,S}(q^2)$ by the polynomial relation $(t_1 - u_1) \cdots (t_1 - u_r) = 0$.

**Remark 4.3.** Since the generator $t_1$ can be rescaled by an arbitrary invertible element of $S$, only the ratios of the parameters $u_i$ have invariant significance in the definition of the cyclotomic Hecke algebra.

**Lemma 4.4.** Let $S$ be a ground ring with parameters $\rho$, $q$, $\delta_j$ ($j \geq 0$), and $u_1, \ldots, u_r$. For all $n \geq 1$, the quotient of the cyclotomic BMW algebra $W_{n,S,r}(u_1, \ldots, u_r)$ by the ideal $I_n$ is isomorphic to the cyclotomic Hecke algebra $H_{n,S,r}(q^2; u_1, \ldots, u_r)$.

Proof. Evident. □

**Lemma 4.5.** Let $F$ be a field with $u$–admissible parameters $\rho$, $q$, $\delta_j$ ($j \geq 0$), and $u_1, \ldots, u_r$. Assume that $u_j \notin \{\pm q^{\pm 1}\}$ for all $j$.

1. $W_{1,F,r} \cong KT_{1,F,r} \cong F^r$.
2. Let $P_j = \prod_{\ell \neq j} \frac{Y_1 - u_\ell}{u_j - u_\ell} \in KT_{1,F,r}$. Then $EP_j E = \gamma_j E$, and $\varepsilon(P_j) = \gamma_j / \delta_0$, where $\gamma_j$ is defined by (3.6).

Proof. Let $\varphi : W_{2,F,r} \to KT_{2,F,r}$ be the surjective algebra homomorphism determined by $\varphi(y_1) = Y_1$, $\varphi(e) = E$, $\varphi(g) = G$. Define $p_j = \prod_{\ell \neq j} \frac{y_1 - u_\ell}{u_j - u_\ell} \in W_{1,F,r}$. It is shown in the proof of Theorem 3.10 that $ep_j e = \gamma_j e$. Hence, $EP_j E = \varphi(ep_j e) = \gamma_j E$. 


By Remark 3.14, \( \gamma_j \neq 0 \) for all \( j \), and by Lemma 2.16, \( E \neq 0 \). Therefore, \( P_j \neq 0 \). The elements \( P_j \) are mutually orthogonal non-zero idempotents, so they are linearly independent. Thus the restriction of \( \varphi \) to \( W_{1,F,r} \) is an isomorphism. Moreover, \( KT_{1,F,r} \cong F P_1 \oplus \cdots \oplus F P_r \cong F^r \), as algebras. \( \square \)

Let \( \lambda = (\lambda^{(1)}, \ldots, \lambda^{(r)}) \) be an \( r \)-tuple of Young diagrams. The total size of \( \lambda \) is \( |\lambda| = \sum_i |\lambda^{(i)}| \). If \( \mu \) and \( \lambda \) are \( r \)-tuples of Young diagrams of total size \( f-1 \) and \( f \) respectively, we write \( \mu \subset \lambda \) if \( \mu \) is obtained from \( \lambda \) by removing one box from one component of \( \lambda \).

**Theorem 4.6** ([1]). Let \( F \) be a field. The cyclotomic Hecke algebra \( H_{n,F,r}(q; u_1, \ldots, u_r) \) is split semisimple for all \( n \) as long as \( q \) is not a proper root of unity and \( u_i/u_j \) is not a power of \( q \) for all \( i \neq j \). In this case, the simple components of \( H_{n,F,r}(q; u_1, \ldots, u_r) \) are labeled by \( r \)-tuples of Young diagrams of total size \( n \), and a simple \( H_{n,F,r} \) module \( V_\lambda \) decomposes as a \( H_{n-1,F,r} \) module as the direct sum of all \( V_\mu \) with \( \mu \subset \lambda \).

In the following, let \( \Gamma_n \) denote the set of \( r \)-tuples of Young diagrams whose total size is no more than \( n \) and congruent to \( n \) mod 2. If \( \lambda \in \Gamma_n \) and \( \mu \in \Gamma_{n-1} \), write \( \mu \leftrightarrow \lambda \) if \( \mu \) is obtained from \( \lambda \) by either adding one box to, or removing one box from, one component of \( \lambda \). For \( \lambda \in \Gamma_n \), an *up–down tableau of length \( n \) and shape \( \lambda \) is a sequence \((\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(n)})\), where

1. for all \( i \), \( \lambda^{(i)} \in \Gamma_i \),
2. \( \lambda^{(0)} = (\emptyset, \ldots, \emptyset) \),
3. \( \lambda^{(n)} = \lambda \), and
4. for each \( i \), \( \lambda^{(i)} \leftrightarrow \lambda^{(i+1)} \).

Let \( T(n) \) denote the set of up–down tableaux of length \( n \), and \( T(n, \lambda) \) the set of up–down tableaux of length \( n \) and shape \( \lambda \).

**Definition 4.7.** Let \( \lambda \) be an \( r \)-tuple of Young diagrams. Label the nodes (or cells, or boxes) of \( \lambda \) by triples \((j, x, y)\), where \( j \) is the component \((1 \leq j \leq r)\), \( x \) is the row coordinate, and \( y \) is the column coordinate of the node. Define the *multiplicative content* of a node \( \alpha = (j, x, y) \) by \( c(\alpha) = u_j q^{2(y-x)} \).

An *addable node* of \( \lambda \) is a node which can be added to one component to give a new \( r \)-tuple of Young diagrams. A *removable node* is a node which can be removed with the result still being an \( r \)-tuple of Young diagrams.
For an addable node of \( \lambda \), set \( b(\alpha, \lambda) = \tilde{c}(\alpha) \), and for removable node, set \( b(\alpha, \lambda) = \tilde{c}(\alpha)^{-1} \).

If \( T = (\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(n)}) \) is an up–down tableau of length \( n \) and \( j \leq n \), let \( b(j, T) = b(\alpha_j, \lambda^{(j-1)}) \), where \( \alpha_j \) is the node that is added or removed in passing from \( \lambda^{(j-1)} \) to \( \lambda^{(j)} \).

The total number of addable and removable nodes of a single Young diagram is always odd, so the total number of addable and removable nodes of an \( r \)-tuple of Young diagrams has the same parity as \( r \).

In the following we work over the field of fractions \( F \) of the universal admissible integral domain \( \overline{R}_+ \). Since \( \overline{R}_+ \) imbeds in the universal \( u \)-admissible ring, according to Theorem 3.19, we will now denote the parameters of \( \overline{R}_+ \) by \( \rho \), \( q \), \( \delta_j \), and \( u_i \). Recall that \( q, u_1, \ldots, u_r \) are algebraically independent over \( \mathbb{Z} \).

Theorem 4.8. Let \( F \) denote the field of fractions of the universal admissible integral domain \( \overline{R}_+ \). Write \( W_n \) for \( W_{n,F,r}(u_1, \ldots, u_r) \), \( KT_n \) for \( KT_{n,F,r}(u_1, \ldots, u_r) \), and \( H_n \) for \( H_{n,F,r}(q^2; u_1, \ldots, u_r) \). The following statements hold for all \( n \geq 0 \).

1. \( \varphi : W_n \to KT_n \) is an isomorphism.
2. The Markov trace \( \varepsilon \) on \( KT_n \) is non-degenerate.
3. \( W_n \) is split semisimple, and \( W_n \cong I_n \oplus H_n \).
4. The minimal ideals of \( W_n \) are labeled by \( \Gamma_n \); more precisely, the minimal ideals of \( I_n \) are labeled by \( \{ \lambda \in \Gamma_n : |\lambda| < n \} \) and the minimal ideals in \( W_n/I_n \) are labeled by \( \{ \lambda \in \Gamma_n : |\lambda| = n \} \).
5. The branching diagram for \( W_{n-1} \subseteq W_n \) has a single edge connecting \( \mu \in \Gamma_{n-1} \) and \( \lambda \in \Gamma_n \) if \( \mu \leftrightarrow \lambda \), and no edges otherwise.
6. The set of paths on the branching diagram for the sequence \( W_0 \subseteq W_1 \subseteq \cdots \subseteq W_n \) is the set of up–down tableaux of length \( n \).
7. Let \( p_T \) be the path idempotent in \( W_n \) corresponding to an up–down tableau \( T \) of length \( n \). Then \( p_T \) is a common eigenvector for \( y_1, y_2, \ldots, y_n \),

\[
y_k p_T = p_T y_k = b(k, T) p_T.
\]

Proof. The assertions of the theorem hold trivially for \( n = 0 \), and the assertions for \( n = 1 \) are implied by Lemma 4.5. (We label the one minimal ideal of \( W_{0,F} \cong F \) by the \( r \)-tuple of empty diagrams, \( \emptyset = (\emptyset, \ldots, \emptyset) \). We label the minimal ideal \( Fp_j \) of \( W_1 \) by the the \( r \)-tuple \( \square_j \), with the Young diagram \( \square \) in the \( j \)-th position, and the empty diagram \( \emptyset \) in all other positions. Assertion (2) holds for \( n = 1 \) since
\[ \varepsilon(p_j) = \gamma_j / \delta_0 \] by Lemma 4.5 and \( \gamma_j \neq 0 \) by Remark 3.14. Assertion (7) holds for \( n = 1 \) because \( y_1 p_j = u_j p_j \), and \( u_j \) is the multiplicative content of the node added to \((\emptyset, \ldots, \emptyset)\) to get \( \Box_j \).

We proceed by induction on \( n \). Fix \( n \geq 1 \), and suppose that the assertions hold for \( W_f \) for \( f \leq n \).

We have a conditional expectation \( \varepsilon_n : W_n \rightarrow W_{n-1} \) that preserves the non-degenerate trace on \( W_n \) and \( W_{n-1} \); that is, \( \varepsilon \circ \varepsilon_n = \varepsilon \). (The conditional expectation results from the isomorphism \( W_f \cong KT_f \) for \( f \leq n \) and the conditional expectation on the cyclotomic Kauffman tangle algebras; see Section 2.6.) Moreover, the conditional expectation is implemented by the idempotent \( e = (1/\delta_0)e_n \in W_{n+1} \); that is \( ebe = \varepsilon_n(b)e \) for \( b \in W_n \). Therefore, by Wenzl’s Theorem 2.17 and the induction hypothesis, \( W_n e_n W_n \subseteq W_{n+1} \) is split semisimple, with minimal ideals labeled by \( \Gamma_{n-1} \). By Lemma 4.1, \( W_n e_n W_n = I_{n+1} \).

Since \( I_{n+1} = W_n e_n W_n \) has an identity element \( z \), necessarily a central idempotent in \( W_{n+1} \), it follows that \( I_{n+1} \) has a complementary ideal \( J_{n+1} \); moreover, \( J_{n+1} \cong H_{n+1} \), according to Lemma 4.4. By Theorem 4.6 and the algebraic independence of \( q, u_1, \ldots, u_r \) over \( \mathbb{Z} \), \( H_{n+1} \) is split semisimple. Thus \( W_{n+1} \) is split semisimple and \( W_{n+1} \cong I_{n+1} \oplus H_{n+1} \). This proves statement (3) for \( W_{n+1} \).

To prove statements (4), (5), and (6) for \( W_{n+1} \), we have to determine the branching diagram for \( W_n \subseteq W_{n+1} \). We know that the multiplicity matrix for the map \( x \mapsto xz \), from \( W_n \) to \( I_{n+1} \), is the transpose of the inclusion matrix for \( W_{n-1} \subseteq W_n \), by Theorem 2.17. The map \( x \mapsto x(1-z) \), from \( I_n \) to \( J_{n+1} \), is zero, since \( I_n \subseteq I_{n+1} \). The multiplicity matrix for the map \( x \mapsto x(1-z) \), from \( J_n \) to \( J_{n+1} \) is the same as that for the inclusion \( H_n \subseteq H_{n+1} \), which is given by Theorem 4.6. Statements (4), (5), and (6) follow from these observations.

It remains to prove statements (1), (2) and (7) for \( W_{n+1} \) and \( KT_{n+1} \).

Recall that for any up–down tableau \( T \) of length \( k \leq n + 1 \), \( T' \) denotes the truncation of \( T \) of length \( k - 1 \). If \( T \) is an up–down tableau of length \( k \) and shape \( \mu \), and if \( \lambda \in \Gamma_{k+1} \) satisfies \( \mu \leftrightarrow \lambda \), then \( T + \lambda \) denotes the extension of \( T \) of length \( k + 1 \) and shape \( \lambda \). Let \( p_T \) denote the path idempotent in \( W_k \) corresponding to an up–down tableau \( T \) of length \( k \), and let \( P_T = \varphi(p_T) \) be its image in \( KT_k \).
Lemma 4.9. Assume that assertions (1), (2), and (7) hold for $W_k$ and $KT_k$ for all $k \leq n$. If $T$ is an up-down tableau of length $n + 1$, then for all $k \leq n + 1$,

$$y_k p_T = p_T y_k = b(k, T) p_T.$$  

That is, assertion (7) also holds for $W_{n+1}$.

Proof of Lemma 4.9. Let $T$ be an up-down tableau of length $n + 1$. If $k \leq n$, then

$$y_k p_T = y_k p_T p_T = b(k, T') p_T = b(k, T) p_T,$$

and similarly for $p_T y_k$.

It remains to show that $y_{n+1} p_T = p_T y_{n+1} = b(n+1, T) p_T$.

Write $T = (\emptyset, \lambda^{(1)}, \ldots, \lambda^{(n-1)}, \mu, \lambda)$. In case $|\lambda| = n + 1$, $p_T$ belongs to the ideal $J_{n+1}$, which is isomorphic to the cyclotomic Hecke algebra $H_{n+1}$. The result then follows from formulas for the irreducible representations of the cyclotomic Hecke algebras, see [2, 1].

We assume now that $|\lambda| \leq n - 1$. Consider the minimal ideal $W_{n,\mu}$ of $W_n$ corresponding to $\mu$. This ideal is isomorphic to a full matrix algebra over $F$, and the path idempotents $p_U$ ($U \in T(n, \mu$)) are a complete family of mutually orthogonal minimal idempotents in $W_{n,\mu}$. Hence $W_{n,\mu}$ has a system of matrix units $e_{S,U}$ ($S, U \in T(n, \mu)$) such that $p_U = e_{U,S} p_S e_{S,U}$. Thus for any $S \in T(n, \mu)$, we can write

$$p_T = p_T^\alpha = e_{T',S} p_S e_{S,T'} = e_{T',S} p_S e_{S,T'}^{(n+1)},$$

Suppose that there exists some $S \in T(n, \mu)$ such that

$$y_{n+1} p_S + \lambda = p_S + \lambda y_{n+1} = b(\alpha, \mu) p_S + \lambda,$$

where $\alpha$ is the node added or removed in passing from $\mu$ to $\lambda$. Then we have

$$y_{n+1} p_T = y_{n+1} e_{T',S} p_S + \lambda e_{S,T'} = e_{T',S} y_{n+1} p_S + \lambda e_{S,T'},$$

and similarly for $p_T y_{n+1}$. Therefore, it suffices to prove the result when $T$ has the form $T = (\emptyset, \lambda^{(1)}, \ldots, \lambda, \mu, \lambda)$. Let $T'$ be the truncation of $T$ of length $n$ and $T''$ the truncation of length $n - 1$. Then $p_{T''}$ is a minimal idempotent in the minimal ideal $W_{n-1,\lambda}$ of $W_{n-1}$. Let $e = (1/\delta_0) e_n$, so that $e b e = \varepsilon_n(b)e$ for $b \in W_n$. Then by Remark 2.18, $p_{T''} e$ is a minimal idempotent in $W_{n+1,\lambda}$. 
Consider the element \( p_{T'} e p_{T'} \). We have
\[
(p_{T'} e p_{T'})^2 = p_{T'} \varepsilon_n(p_{T'}) p_{T'} = (\varepsilon(p_{T'})/\varepsilon(p_{T''})) p_{T'} e p_{T''} p_{T'}
\]
using Lemma 2.19, so \( p_{T'} e p_{T'} \) is an essential idempotent. Write \( \kappa = \varepsilon(p_{T'})/\varepsilon(p_{T''}) \) and \( f = \kappa^{-1} p_{T'} e p_{T'} \). (We are using the assumption that the trace is non-degenerate on \( W_n \) and \( W_{n-1} \), which follows from assertions (1) and (2) for these algebras.)

We have \( f = \kappa^{-1} p_{T'}(p_{T''} e) p_{T'} \), which shows that \( f \) is in \( W_{n+1, \lambda} \). But then
\[
f = z_{\lambda}^{(n+1)} f z_{\lambda}^{(n+1)} = (z_{\lambda}^{(n+1)} p_{T'}) f (p_{T'} z_{\lambda}^{(n+1)}) = p_T f p_T.
\]
Therefore \( f \) is a multiple of, and hence equal to, the minimal idempotent \( p_T \). Now we can compute:
\[
y_{n+1} p_T = \kappa^{-1} y_{n+1} p_{T'} e p_{T'} = \kappa^{-1} p_{T'} y_{n+1} e p_{T'}
\]
\[
= \kappa^{-1} p_{T'} y_n^{-1} e p_{T'} = \kappa^{-1} y_n^{-1} p_{T'} e p_{T'}
\]
\[
y_n^{-1} p_T = b(\alpha, \mu)p_T,
\]
and similarly for \( p_T y_{n+1} \). This completes the proof of Lemma 4.9.

Recall the quantities \( Z_{k+1}(t) \) and \( Q_{k+1}(t) \) from Section 3.5. They are formal power series with coefficients in the center of \( KT_k \). We want to find an expression for \( Q_{k+1}(t)p_T \), where \( T \) is an up-down tableau of length \( k \).

Let \( \lambda \) be any \( r \)-tuple of Young diagrams. Define
\[
\tilde{Q}(t, \lambda) = p A(t) \prod_{\alpha} \frac{t - b(\alpha, \lambda)}{t - b(\alpha, \lambda)}^{-1},
\]
where \( p = \prod_j u_j \), \( A(t) \) is as in (3.24), and \( \alpha \) runs over all addable and removable nodes of \( \lambda \). Define
\[
\tilde{Z}(t, \lambda) = \tilde{Q}(t, \lambda) + \rho^{-1}/(q^{-1} - q) + t^2/(t^2 - 1).
\]

Note that if \( \emptyset \) is the \( r \)-tuple of empty Young diagrams, then
\[
\tilde{Q}(t, \emptyset) = p A(t) \prod_j \frac{t - u_j^{-1}}{t - u_j} = A(t) G(t^{-1}) = Q_1(t),
\]
using (3.27) and thus \( \tilde{Z}(t, \emptyset) = Z_1(t) \).

**Lemma 4.10.** Suppose \( \mu \) and \( \lambda \) are \( r \)-tuples of Young diagrams and \( \lambda \) is obtained from \( \mu \) by adding a single node \( \alpha \). Let \( \tilde{c} = \tilde{c}(\alpha) \) be the multiplicative content of this node. Then
\[
\tilde{Q}(t, \lambda) = \tilde{Q}(t, \mu) \frac{(t - \tilde{c})^2(t - q^{-2} \tilde{c}^{-1})(t - q^2 \tilde{c}^{-1})}{(t - \tilde{c}^{-1})^2(t - q^2 \tilde{c})(t - q^2 \tilde{c})}.
\]
Proof of Lemma 4.10. Suppose $\alpha = (j, x, y)$, so $y - 1 = \mu_x^{(j)}$, the length of the $x$-th row of the $j$-th component of $\mu$. There are several cases to consider according to the relative size of $\mu_{x-1}^{(j)}$, $\mu_x^{(j)}$, and $\mu_{x+1}^{(j)}$. For example, suppose $\mu_x^{(j)} = \mu_{x+1}^{(j)}$, and $\mu_x^{(j)} + 1 < \mu_{x-1}^{(j)}$. Then adding the node $\alpha$:

1. eliminates the addable node at $\alpha$ and produces a removable node at $\alpha$,

and

2. produces two new addable nodes at $(j, x, y + 1)$ and $(j, x + 1, y)$.

The node at $\alpha$ contributes $((t - \tilde{c})/(t - \tilde{c}^{-1}))^2$ to the ratio $\tilde{Q}(t, \lambda)/\tilde{Q}(t, \mu)$. The two new addable nodes contribute

$$
\left(\frac{t - q^{-2}\tilde{c}^{-1}}{t - q^2\tilde{c}}\right) \left(\frac{t - q^2\tilde{c}^{-1}}{t - q^{-2}\tilde{c}}\right).
$$

Other cases can be analyzed similarly. This completes the proof of Lemma 4.10.

Lemma 4.11. Let $T$ be an up-down tableau of length $k \leq n + 1$ and shape $\lambda$. Then

$$Q_{k+1}(t) P_T = \tilde{Q}(t, \lambda) P_T.$$

and

$$Z_{k+1}(t) P_T = \tilde{Z}(t, \lambda) P_T.$$

Proof of Lemma 4.11. We do this by induction on $k$. There is a unique tableau of length 0, and the corresponding idempotent is the identity. So the statement for $k = 0$ reads $Q_1(t) = \tilde{Q}(t, \emptyset)$, which was observed in (4.3). Let $T$ be an up-down tableau of length $k \geq 1$ and suppose the assertion holds for all shorter up-down tableaux. Let $\lambda$ denote the shape of $T$ and $\mu$ the shape of $T'$. Then

$$Q_{k+1}(t) P_T = Q_k(t) \left(\frac{(t - Y_k)^2(t - q^{-2}Y_k^{-1})(t - q^2Y_k^{-1})}{(t - Y_k^{-1})^2(t - q^{-2}Y_k)(t - q^2Y_k)}\right) P_T P_T \equiv Q_k(t) P_{T'} \left(\frac{(t - Y_k)^2(t - q^{-2}Y_k^{-1})(t - q^2Y_k^{-1})}{(t - Y_k^{-1})^2(t - q^{-2}Y_k)(t - q^2Y_k)}\right) P_T.$$

By the induction hypothesis and Lemma 4.9, this is equal to

$$\tilde{Q}(t, \mu) P_{T'} \left(\frac{(t - b)^2(t - q^{-2}b^{-1})(t - q^2b^{-1})}{(t - b^{-1})^2(t - q^{-2}b)(t - q^2b)}\right) P_T \equiv \tilde{Q}(t, \mu) \left(\frac{(t - b)^2(t - q^{-2}b^{-1})(t - q^2b^{-1})}{(t - b^{-1})^2(t - q^{-2}b)(t - q^2b)}\right) P_T,$$
where \( b = b(\alpha, \mu) \). By Lemma 4.10, this equals \( \tilde{Q}(t, \lambda)P_T \). (In case \( \lambda \subseteq \mu \), reverse the roles of \( \lambda \) and \( \mu \) in the Lemma.) The conclusions of Lemma 4.11 follow from this.

Let \( U \) be an up–down tableau of length \( n \). Note that \( P_U \) is a minimal idempotent in \( KT_n \) and \( \varepsilon(P_U) \neq 0 \) by claims (1) and (2) of the induction hypothesis. We have \( P_U = \sum T P_T \), where \( T \) ranges over all up–down tableaux of length \( n + 1 \) such that \( T' = U \). It follows that there exists some \( T \) such that \( \varepsilon(P_T) \neq 0 \). Fix such a \( T \).

Recall from Lemma 2.19 that for any up–down tableau \( S \) of length \( n + 1 \) such that \( S' = U \), we have

\[
\varepsilon_{n+1}(PS) = (\varepsilon(PS)/\varepsilon(P_U))P_U.
\]

Next, for any such up–down tableau \( S \), we claim

\[
\varepsilon(P_T E_{n+1} P_S E_{n+1} P_T) = \frac{\varepsilon(P_T) \varepsilon(PS)}{\varepsilon(P_U)}.
\]

In fact,

\[
\varepsilon(P_T E_{n+1} P_S E_{n+1} P_T) = \varepsilon(E_{n+1} P_S E_{n+1} P_T)
\]

\[
= \delta_0 \varepsilon(E_{n+1} \varepsilon_{n+1}(PS) P_T) = \delta_0 \frac{\varepsilon(PS)}{\varepsilon(P_U)} \varepsilon(E_{n+1} P_T P_T)
\]

\[
= \delta_0 \frac{\varepsilon(PS)}{\varepsilon(P_U)} \varepsilon(E_{n+1} P_T) = \frac{\varepsilon(P_T) \varepsilon(PS)}{\varepsilon(P_U)},
\]

where we have used Lemma 2.15 and (4.4).

Now consider the formal power series:

\[
E_{n+1}(t - Y_{n+1})^{-1}E_{n+1} = E_{n+1}(1/t) \sum_{a \geq 0} Y_{n+1}^a t^{-a} E_{n+1}
\]

\[
= (1/t) Z_{n+1}(t) E_{n+1}
\]

Let \( U \) and \( T \) be as above, so \( T' = U \) and \( \varepsilon(P_T) \neq 0 \). Denote the shape of \( U \) by \( \mu \). Multiply both sides of (4.6) by \( P_T \), and use Lemma 4.11 to get

\[
P_T E_{n+1}(t - Y_{n+1})^{-1}E_{n+1} P_T = (1/t) \tilde{Z}(t, \mu) P_T E_{n+1} P_T.
\]

Now take the trace on both sides,

\[
\varepsilon(P_T E_{n+1}(t - Y_{n+1})^{-1}E_{n+1} P_T) = (1/t) \tilde{Z}(t, \mu) \varepsilon(P_T) \delta_0^{-1} \varepsilon(P_T).
\]

Now we focus on the left side of the equation. Note that for \( S \) an up–down tableau of length \( n + 1 \), \( P_S E_{n+1} P_T = P_S P_S' P_T E_{n+1} P_T \), and this is non–zero.
only if \( S' = T' = U \). Therefore, the left side equals
\[
\sum_S \varepsilon(P_T E_{n+1}(t - Y_{n+1})^{-1} P_S E_{n+1} P_T),
\]
where the sum is over all \( S \) such that \( S' = U \). By Lemma 4.9, and (4.5) this equals
\[
\sum_S (t - b(n + 1, S))^{-1} \varepsilon(P_T E_{n+1} P_S E_{n+1} P_T)
= \sum_S (t - b(n + 1, S))^{-1} \frac{\varepsilon(P_S) \varepsilon(P_T)}{\varepsilon(P_U)}
= \sum_{\alpha} (t - b(\alpha, \mu))^{-1} \frac{\varepsilon(P_S(\alpha)) \varepsilon(P_T)}{\varepsilon(P_U)},
\]
where the last sum is over all addable and removable nodes of \( \mu \), and \( S(\alpha) \) is the extension of \( U \) by \( \alpha \); that is, if \( U = (\lambda^{(0)}, \ldots, \mu) \), then \( S(\alpha) = (\lambda^{(0)}, \ldots, \mu, \lambda(\alpha)) \), where \( \lambda(\alpha) \) is obtained from \( \mu \) by addition or removal of the node \( \alpha \). Comparing 4.8 and 4.9 and cancelling \( \varepsilon \) by addition or removal

(4.10) \[
(1/t)\tilde{Z}(t, \mu) = \delta_0 \sum_{\alpha} (t - b(\alpha, \mu))^{-1} \frac{\varepsilon(P_S(\alpha))}{\varepsilon(P_U)},
\]

Since \( q, u_1, \ldots, u_r \) are algebraically independent over \( \mathbb{Z} \), if \( \alpha \) and \( \beta \) are two different addable or removable nodes of \( \mu \), then \( \varepsilon(\alpha) \neq \varepsilon(\beta)^\pm 1 \); hence \( b(\alpha, \mu) \neq b(\beta, \mu) \). Let \( S \) be any particular extension of \( U \), \( S = S(\beta) \), where \( \beta \) is an addable or removable node of \( \mu \); let \( b = b(\beta, \mu) \). Take the residue at \( t = b \) on both sides of 4.10. On the left side, we get
\[
\text{Res}(1/t)\tilde{Z}(t, \mu)_{|t=b} = \text{Res}(1/t)\tilde{Q}(t, \mu)_{|t=b}
= \frac{pb^{-1} A(b)(b - b^{-1})}{b - b(\alpha, \mu)} \prod_{\alpha \neq \beta} \frac{b - b(\alpha, \mu)^{-1}}{b - b(\alpha, \mu)},
\]
while on the right side we get \( \delta_0 \varepsilon(P_S)/\varepsilon(P_U) \). Therefore,
\[
\varepsilon(P_S) = \delta_0^{-1} \frac{p b^{-1} A(b)(b - b^{-1})}{b - b(\alpha, \mu)} \prod_{\alpha \neq \beta} \frac{b - b(\alpha, \mu)^{-1}}{b - b(\alpha, \mu)} \varepsilon(P_U).
\]

If \( r \) is odd, \( A(b)(b - b^{-1}) = \frac{b - b^{-1}}{q - q^{-1}} + 1 \), and \( p = \rho \); see (3.24). Therefore,
\[
\varepsilon(P_S) = \delta_0^{-1} \rho b^{-1} \left( \frac{b - b^{-1}}{q - q^{-1}} + 1 \right) \prod_{\alpha \neq \beta} \frac{b - b(\alpha, \mu)^{-1}}{b - b(\alpha, \mu)} \varepsilon(P_U).
\]
If \( r \) is even, 
\[
A(b)(b - b^{-1}) = \frac{q^{-1}b - qb^{-1}}{q - q^{-1}}, \text{ and } p = q^{-1}p. \text{ Therefore, }
\]
\[
(4.14) \quad \varepsilon(P_S) = \delta_0^{-1} \rho q^{-1}b^{-1}(-\frac{q^{-1}b - qb^{-1}}{q - q^{-1}}) \prod_{\alpha \neq \beta} \frac{b - b(\alpha, \mu)^{-1}}{b - b(\alpha, \mu)} \varepsilon(P_U).
\]

The algebraic independence of \( q, u_1, \ldots, u_r \) over \( \mathbb{Z} \) implies that \( \varepsilon(P_S) \neq 0 \).

The computation of \( \varepsilon(P_S) \) given here was adapted from [5], Theorem 7.

We have shown that for every up–down tableau \( S \) of length \( n + 1 \), \( \varepsilon(P_S) = \varepsilon(\varphi(p_S)) \). Hence \( \varepsilon \) is a non–degenerate trace on \( W_{n+1} \). Moreover, this shows that \( \varphi \) is not zero on any minimal ideal of \( W_{n+1} \), and hence \( \varphi \) is injective on \( W_{n+1} \). This proves points (1) and (2) for \( W_{n+1} \), and completes the proof of the theorem.

**Remark 4.12.** (A semisimplicity criterion) Let \( K \) be a field with \( u \)–admissible parameters. The inductive proof of the theorem applies to \( W_{n,K,r} \) as long as the weights of the Markov trace remain non–zero. For this, what is needed is that for all \( r \)–tuples of Young diagrams \( \mu \) of size \( \leq n \), and for all addable or removable nodes \( \alpha \) and \( \beta \) of \( \mu \), with \( b = b(\beta, \mu) \) and \( b' = b(\alpha, \mu) \), we have

\[
A(b)(b - b^{-1}) \neq 0, \text{ and } b \neq (b')^{\pm 1}.
\]

For this, it suffices that for all \( j \), \( u_j \neq \pm q^{\pm(2k-1)} \), and for all \( j, j' \), \( u_j/u_{j'} \) and \( u_j' \) are not equal to \( q^{2k} \) for \( k \leq n \). In particular, we have the following result:

**Corollary 4.13.** Let \( K \) be a field with \( u \)–admissible parameters. Suppose that for all \( j, j' \), none of the quantities \( \pm u_j, u_j/u_{j'} \) and \( u_j u_{j'} \) is equal to an integer power of \( q \). Then \( W_{n,K,r} \) is split semisimple for all \( n \).

5. The \( \mathbb{Z}_r \)–Brauer Algebra and the Dimension of \( W_{n,F,r} \).

Recall that for \( \lambda \in \Gamma_n \), the set of up–down tableaux of length \( n \) and shape \( \lambda \) is denoted by \( \mathcal{T}(n, \lambda) \). The dimension of the simple \( W_{n,F,r} \) module labelled by \( \lambda \in \Gamma_n \) is \( |\mathcal{T}(n, \lambda)| \), and the dimension of \( W_{n,F,r} \) over \( F \) is the \( \sum \lambda \in \Gamma_n |\mathcal{T}(n, \lambda)|^2 \).

It should possible to compute this quantity by an appropriate Shensted type correspondence, but it is more convenient to exhibit another sequence of split semisimple algebras \( D_0 \subseteq D_1 \subseteq \cdots \subseteq D_n \) (over some field) whose dimensions are known and which have the same branching diagram.

For this purpose, we introduce the \( \mathbb{Z}_r \)–Brauer algebras. The \( \mathbb{Z}_r \)– and \( \mathbb{Z} \)–Brauer algebras were considered in [12], [27], and [9]. The \( G \)–Brauer algebras for an arbitrary abelian group \( G \) were studied in [25].
We recall that a Brauer diagram is a tangle diagram in the plane, in which information about over- and under-crossings is ignored: An \((n,n)\)-Brauer diagram is a figure in the rectangle \(R = I \times I\) consisting of

1. \(n\) distinguished points \(\{1, \ldots, n\}\) on the upper boundary \(I \times \{1\}\) of \(R\) and \(n\) distinguished points \(\{\bar{1}, \ldots, \bar{n}\}\) on the lower boundary \(I \times \{0\}\) of \(R\).
2. \(n\) curves connecting the points \(\{1, \ldots, n, \bar{1}, \ldots, \bar{n}\}\) in pairs, with each curve intersecting the boundary of \(R\) only at its two endpoints.

A \(Z_r\)-Brauer diagram is a Brauer diagram in which each curve (strand) is endowed with an orientation and labeled by an element of the group \(Z_r\). Two labelings are regarded as the same if the orientation of a strand is reversed and the group element associated to the strand is inverted. Note that the number of \(Z_r\)-Brauer diagrams is \(r^n(2n - 1)!!\).

Let \(A\) be an integral domain containing elements \(\vartheta_j\), for \(0 \leq j \leq \lfloor r \rfloor\) with \(\vartheta_0\) invertible. The \(Z_r\)-Brauer algebra \(D_{n,A,r}\) is defined as follows. As an \(A\)-module, \(D_{n,A,r}\) is the free \(A\)-module with basis the set of \(Z_r\)-Brauer diagrams with \(n\) strands. The product of two \(Z_r\)-Brauer diagrams is defined to be a certain multiple of another \(Z_r\)-Brauer diagram, determined as follows.

Given two \(Z_r\)-Brauer diagrams \(a, b\), first “stack” \(b\) over \(a\) (as for tangle diagrams and ordinary Brauer diagrams). In the resulting “tangle”, the strands are composed of one or more strands from \(a\) and \(b\). Give each composite strand \(s\) an orientation, arbitrarily. Make the orientations of the components of \(s\) from \(a\) and \(b\) agree with the orientation of \(s\), changing the signs of the \(Z_r\)-valued labels accordingly; the label of \(s\) is then the sum of the labels of its components from \(a\) and \(b\). For each \(j\) \((0 \leq j \leq \lfloor r \rfloor)\), let \(m_j\) be the number of closed loops with label equal to \(\pm[j]\). Let \(c\) be the \(Z_r\)-Brauer diagram obtained by removing all the closed curves. Then \(ab = (\prod_j \vartheta_j^{m_j}) \ c\). Extend the multiplication to a bilinear product on \(D_{n,A,r}\). One can easily check that the multiplication is associative.

Note that the subalgebra generated by \(Z_r\)-Brauer diagrams with only vertical strands is isomorphic to the group algebra of the wreath product \(Z_r \wr \mathfrak{S}_n\), so the \(Z_r\)-Brauer algebra can be considered as a sort of wreath product of \(Z_r\) with the ordinary Brauer algebra.

One has inclusion maps \(i : D_{n-1,A,r} \rightarrow D_{n,A,r}\) defined on the level of \(Z_r\)-Brauer diagrams by adding a vertical strand on the right labeled by 0. Moreover, one has a conditional expectation \(\varepsilon_n : D_{n,A,r} \rightarrow D_{n-1,A,r}\) defined as follows: First define a map \(\text{cl}_n\), from \(Z_r\)-Brauer diagrams with \(n\)-strands to \(Z_r\)-Brauer diagrams with
\((n-1)\)–strands, by joining the rightmost pair of vertices \(\bar{n}, n\) of a diagram \(d\) by a new strand, with label 0:

\[
\begin{align*}
\text{cl}_n : & \quad \begin{array}{c}
\text{original diagram}
\end{array} \quad \mapsto \quad \begin{array}{c}
\text{new diagram}
\end{array} \\
& \quad \begin{array}{c}
\text{closed loop}
\end{array}
\end{align*}
\]

Determine the \(\mathbb{Z}_r\)–valued label of any concatenated strand in the resulting diagram by the same rule as in the multiplication of \(\mathbb{Z}_r\)–Brauer diagrams. If the resulting diagram has a closed loop (which happens precisely if \(d\) contains a strand connecting \(n\) and \(\bar{n}\) with some label \(\pm [k]\)) then remove the closed loop and multiply the resulting \(\mathbb{Z}_r\)–Brauer diagram by \(\vartheta_k\). Finally, define \(\varepsilon_n : D_{n,A,r} \to D_{n-1,A,r}\) by

\[
\varepsilon_n(d) = \vartheta_0^{-1} \text{cl}_n(d).
\]

One can check that \(\varepsilon_n\) is a conditional expectation. Define

\[
\varepsilon = \varepsilon_1 \circ \cdots \circ \varepsilon_n : D_{n,A,r} \to D_{0,A,r} = A.
\]

Then \(\varepsilon\) is a trace.

**Lemma 5.1** ([25]). For each \(j\) (0 \(\leq j \leq \lfloor r \rfloor\)), let \(\vartheta_j\) be an indeterminant over \(\mathbb{C}\). Let

\[
K = \mathbb{C}(\vartheta_0^{\pm 1}, \vartheta_1, \ldots, \vartheta_{\lfloor r \rfloor}).
\]

The trace \(\varepsilon\) on \(D_{n,K,r}\) is non-degenerate.

Given this, it is fairly straightforward to apply Wenzl’s method from [29, 30] to the \(\mathbb{Z}_r\)–Brauer algebra. (In fact, this has been done more generally in [25], in the context of \(G\)–Brauer algebras, where \(G\) is a finite abelian group.) The result of this analysis is that the \(\mathbb{Z}_r\)–Brauer algebra \(D_n = D_{n,K,r}\) is split semisimple with simple modules labelled by \(\Gamma_n\) (i.e., the set of \(r\)–tuples of Young diagrams of total size \(\leq n\) and congruent to \(n\) mod 2) and the branching diagram for the sequence \(D_0 \subseteq D_1 \subseteq \cdots \subseteq D_n\) is exactly the same as that obtained for the sequence of cyclotomic BMW algebras in Theorem 4.8. Thus we have:

**Theorem 5.2.** Let \(F\) be as in Theorem 4.8. The dimension of \(W_{n,F,r}\) is \(r^n(2n - 1)!!\).

**Proof.** \(\dim_F(W_{n,F,r}) = \sum_{\lambda \in \Gamma_n} |T(n, \lambda)|^2 = \dim_K(D_{n,K,r}) = r^n(2n - 1)!!\). \(\square\)
6. Freeness and isomorphism with cyclotomic Kauffman tangle algebras.

We show in [10], Proposition 3.6, that every cyclotomic BMW algebra \( W_{n,S,r}(u_1, \ldots, u_r) \) is spanned over its ground ring \( S \) by a set \( \mathcal{A}'_r \) of cardinality \( r^n(2n - 1)!! \). Combining this result with Theorems 4.8 and 5.2, we obtain the following theorem (which has been proved independently by Wilcox and Yu.)

**Theorem 6.1** (Goodman and Hauschild–Mosley, Wilcox and Yu [33, 32]). Let \( S \) be an admissible integral domain. Then \( W_{n,S,r} \cong KT_{n,S,r} \), and \( W_{n,S,r} \) is a free \( S \)–module of rank \( r^n(2n - 1)!! \).

**Proof.** As noted above, \( KT_{n,S,r} \) has a spanning set \( \mathcal{A}'_r \) of cardinality \( r^n(2n - 1)!! \). To prove the theorem, it suffices to show that \( \varphi(\mathcal{A}'_r) = \mathcal{B}'_r \) is linearly independent in \( KT_{n,S,r} \). When \( S = F \), the field of fractions of the universal admissible integral domain \( \overline{R}_+ \), this follows from Theorem 4.8 and Theorem 5.2, since \( \mathcal{B}'_r \) is a spanning set whose cardinality equals the dimension of \( KT_{n,F,r} \). The map \( x \mapsto x \otimes 1 \) from \( KT_{n,\overline{R}_+ ,r} \) to \( KT_{n,\overline{R}_+ ,r} \otimes \overline{R}_+ \) is \( \overline{R}_+ \)–linear and maps \( \mathcal{B}'_r \) to a linearly independent set in \( KT_{n,F,r} \); hence \( \mathcal{B}'_r \) is linearly independent in \( KT_{n,\overline{R}_+ ,r} \). Finally, since any admissible integral domain \( S \) is a quotient of \( \overline{R}_+ \), and \( KT_{n,\overline{R}_+ ,r} \) is a free \( \overline{R}_+ \)–module with basis \( \mathcal{B}'_r \), it follows that \( KT_{n,S,r} \cong KT_{n,\overline{R}_+ ,r} \otimes \overline{R}_+ \) is a free \( S \)–module with basis \( \mathcal{B}'_r \).

**Corollary 6.2.** Suppose \( S \) is an admissible integral domain.

- (1) There is a trace–preserving conditional expectation \( \varepsilon_n : W_{n+1,S,r} \to W_{n,S,r} \).
- (2) The natural map \( \iota : W_{n,S,r} \to W_{n+1,S,r} \) is injective.

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