Quantum groups in higher genus and Drinfeld’s new realizations method ($sl_2$ case).

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Abstract. We define double (central and cocentral) extensions of Manin pairs introduced by Drinfeld, attached to curves and meromorphic differentials. We define “infinite twistings” of these pairs and quantize them in the $sl_2$ case, adapting Drinfeld’s “new realizations” technique. We study finite dimensional representations of these algebras in level 0, and some elliptic examples.

Introduction.

In [5], V. Drinfeld introduced examples of Manin pairs, attached to the data of a curve, a meromorphic differential on it, and a finite dimensional reductive Lie algebra. He remarked, that only in the cases where the curve had genus $\leq 1$, could these Manin pairs be given a structure of Manin triple; in these cases, the quantization of these Manin triples gives rise to well-known Hopf algebras (the Yangians, quantum affine algebras and algebras connected with Sklyanin algebras). He raised the question of quantizing these Manin pairs in the higher genus case, in the sense of quasi-Hopf algebras.

In this paper, we first present a double (central and cocentral) extension of these Manin pairs. The general definition of these extensions, in the case of Manin triples, is due to M. Semenov-Tian-Shansky ([13]). This leads us to the problem of the quantization of these extended Manin pairs.

We then remark, that this quantization problem can be approached in the spirit of the “new realizations” of Drinfeld (introduced in [4] and developed in [10], [3], [1]). This technique enabled Drinfeld to give a quantum analogue of the passage from the Serre to the loop presentations of an affine algebra; it can be presented as follows. The bialgebra structure corresponding to quantum affine algebras, is a double bialgebra structure. Conjugating the corresponding Manin triple by a double group element, the bialgebra structure of the double gets changed by a twisting (in the sense of [5]). Let us conjugate by affine Weyl group elements; when their length tends to infinity, we get a limit Manin triple which it is simple enough to quantize. The resulting Hopf algebra is then a twisting of the one obtained by quantization of the initial Manin triple (the Drinfeld-Jimbo Hopf algebra).

In the present situation, we introduce a Lagrangian supplementary in our Manin pair, and conjugate it by affine Weyl group elements as before. (We note, that a family of supplementaries is provided by a covering of the space of principal $G$-bundles over the curve $X$; we hint at a possible connection between the closedness of a 1-form on it, defined in terms of twisting, and a generalized classical Yang-Baxter identity, underlying the integrability of the Hitchin system. We hope to return to this question in [6].) In the limit, we obtain a Manin triple, whose quantization (in the $sl_2$ case) is the main goal of this paper. Let us describe more precisely its contents.

Let $X$ be a smooth compact complex curve, $\omega$ a meromorphic nonzero one-form on $X$, $\{x_i\} \subset X$ the set of its zeroes and poles. Let for each $i$, $k_{x_i}$ be the local
field at \(x_i, \mathcal{O}_{x_i}\), the local ring at this point, \(R \subset \oplus_i k_{x_i}\) the ring of functions regular outside \(\{x_i\}\). We choose a supplementary \(\Lambda\) to \(R\) in \(\oplus_i k_{x_i}\), Lagrangian for the scalar product defined by \(\omega\). To define a quantization of our Manin triple, we need operators \(A : R \to \oplus_i k_{x_i}\) and \(B : \Lambda \to \oplus_i k_{x_i}\), which serve to define \(h - e\) and \(h - f\) relations (by \(h - e\) relations, we understand relations between Fourier modes of the quantum analogues of fields \(h(z)\) and \(e(z)\), etc.; with \(e, h, f\) the Chevalley generators of \(sl_2\)). These operators also provide us with \(e - e\) and \(f - f\) relations, which appear in the form

\[
e(z)e(w) = a(z, w)e(w)e(z), \quad w \ll z;
\]

Our aim is to put these relations in the form

\[
(z - w + \sum_{i \geq 1} h^i \alpha_i(z, w))e(z)e(w) = (z - w + \sum_{i \geq 1} h^i \beta_i(z, w))e(w)e(z),
\]

\(\alpha_i, \beta_i\) formal series in \(z, w\), \((h\text{ is the quantization parameter})\) similar to the quantum affine algebra relations

\[
(qz - w)e(z)e(w) = (z - qw)e(w)e(z).
\]

It turns out that to achieve this task, essentially one possibility for the operators \(A\) and \(B\) remains. The proof that it indeed leads to \(e - e\) and \(f - f\) relations of the desired form, relies on a statement about derivatives of a Green function (prop. 1), which allows to give a universal treatment for all pairs \((X, \omega)\). The formal series \(\alpha_i\) and \(\beta_i\) are then obtained from formal solutions to certain differential equations (eqs. (3.7)), where the variable is \(\hbar\).

The quantization we propose depends both on a choice of \(\Lambda\), and of a certain element \(\tau \in R \otimes R\). We show that the various quantizations obtained are related to each other by twisting operations (in the sense of [5]).

We turn then to the problem of finite dimensional representations of our algebras, at level 0; these representations are indexed by points of formal discs. We construct a family of 2-dimensional representations. We expect that higher spin representations can be constructed as well and their tensor products have properties similar to those explained in [2].

We close the paper by making explicit some examples. In a certain elliptic case, we recover as \(e - e\) relations some elliptic \(\mathcal{W}\)-algebra relations discovered in [8]. We also propose a twisting of the “automorphic” Manin triples of [12], which could lead to a “quantum currents” description of the Hopf algebra arising in [15].

Further problems related to the present construction could be the following: applying the Reshetikhin-Semenov method for constructing central elements at the critical level ([11]); construction of level 1 representations as in [9], vertex operators and the corresponding quantum Knizhnik-Zamolodchikov (KZ) equations; generalization from \(sl_2\) to an arbitrary semisimple Lie algebra. The resulting quantum KZ equations at the critical level, might then be considered as \(q\)-deformations of the holonomic systems of equations occurring in the geometric Langlands program; working
out such $q$-deformations using quantum affine algebras was proposed by E. Frenkel and N. Reshetikhin. Finally, in [7], P. Etingof and D. Kazhdan showed how to attach to any associator, a quantization procedure for bialgebras. It would be interesting to understand, whether the construction presented here can be obtained from the KZ associator, as it is the case for finite dimensional Lie algebras.

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1. Manin triple

1. Drinfeld’s Manin pairs.

Let $X$ be a smooth compact complex curve, $\omega$ a meromorphic nonzero one-form on $X$, $\{x_i\} \subset X$ the set of its zeroes and poles. Let for each $i$, $k_{x_i}$ be the local field at $x_i$, $O_{x_i}$ the local ring at this point, $R \subset \oplus_i k_{x_i}$ the ring of functions regular outside $\{x_i\}$. Let $a$ be a simple complex Lie algebra, $\langle , \rangle_a$ its Killing form. Recall the Manin pair defined by Drinfeld in [5]: endow $g_0 = a(\oplus_i k_{x_i})$ with the bilinear form $\langle x_1, x_2 \rangle_0 = \sum_i \text{res}_{x_i}(\langle x_1, x_2 \rangle_a \omega)$; then $a(R)$ is a Lagrangian subalgebra of $g_0$; this defines Drinfeld’s Manin pair $(a(\oplus_i k_{x_i}), a(R))$.

2. Double extension.

We extend this Manin pair in the following way. Let $\partial$ be the derivation of $a(\oplus_i k_{x_i})$ defined by $\partial f = df/\omega$. We denote in the same way the derivation of $\oplus_i k_{x_i}$, defined by the same formula. Let $g$ be the skew product of $g_0$ by $\partial$; we have

$$g = g_0 \oplus C\tilde{D},$$

$g_0 \subset g$ is a Lie algebra homomorphism, and $[\tilde{D}, x] = \partial x$ for $x \in g_0$. Since $\partial$ preserves $a(R)$, $a(R) \oplus C\tilde{D}$ is a Lie subalgebra of $g$. Let $\hat{g}$ be the central extension of $g$ by $CK$ using the cocycle defined by $c(x, y) = \text{res}_x \langle x, dy \rangle_a K$, $c(x, \tilde{D}) = 0$ for $x \in g_0$. We have

$$\hat{g} = g \oplus CK,$$

with the usual commutation rules. Since $c$ vanishes on $a(R) \oplus C\tilde{D}$, this algebra has a section to $\hat{g}$, that we denote by $g_R$. Identifying $\hat{g}$ with $g \oplus CK$, $g_R$ is identified with

$$(a(R) \oplus C\tilde{D}) \times \{0\}.$$
$x_1, x_2 \in \mathfrak{g}_0$. Then $(\cdot, \cdot)$ is invariant, and $\mathfrak{g}_R$ is a subalgebra of $\hat{\mathfrak{g}}$. Lagrangian w.r.t. this scalar product.

\[(\hat{\mathfrak{g}}, \mathfrak{g}_R)\]

is a double extension (central and cocentral) of the above Manin pair.

3. Lagrangian supplementaries.

Consider on $\bigoplus_i k_{x_i}$, the scalar product defined by $(f_1, f_2)_{\bigoplus_i k_{x_i}} = \sum_i \text{res}_{x_i}(f_1 f_2 \omega)$. $R$ is a subspace of $\bigoplus_i k_{x_i}$, Lagrangian w.r.t. this scalar product. Fix a Lagrangian supplementary $\Lambda$ to $R$, commensurable with $\bigoplus_i \mathcal{O}_{x_i}$. Then

$$a \otimes \Lambda \oplus CK$$

is a Lagrangian supplementary to $\mathfrak{g}_R$ in $\hat{\mathfrak{g}}$.

We note here, that a family of such supplementaries can be defined in the following way. Let $\Lambda_0$ be a Lagrangian subspace of $\bigoplus_i k_{x_i}$, containing $\bigoplus_i \mathcal{O}_{x_i}$. Let $G$ be a Lie group, with Lie algebra $\mathfrak{a}$. For $g \in G(\bigoplus_i k_{x_i})$, let

$$g a(\Lambda_0) = g a(\Lambda_0) g^{-1},$$

for generic $g$ this defines a Lagrangian supplementary to $\mathfrak{g}_R$ in $\hat{\mathfrak{g}}$, which up to equivalence depends only on the class of $g$ in $G(R) \setminus G(\bigoplus_i k_{x_i})/\text{Stab} a(\Lambda_0)$. All the resulting bialgebra structures on $\mathfrak{g}_R$ are then associated by twisting. The twisting between two bialgebra structures associated to nearby points defines an element of $\wedge^2 a(R)$; we thus get a 1-form $\Phi \in \Omega^1(G(\bigoplus_i k_{x_i}), \wedge^2 a(R))$, equivariant w.r.t. left $G(R)$-translations. This 1-form is closed; we hope that the expression of this fact can be interpreted as the generalized Yang-Baxter identity for the dynamical $r$-matrices of the Hitchin system ([6]).

4. Infinite twisting.

Let us return to the triple formed by $\hat{\mathfrak{g}}$, $\mathfrak{g}_R$ and the Lagrangian supplementary $a \otimes \Lambda \oplus CK$, and conjugate it by affine Weyl group elements. In the limit, we obtain the Manin triple

\[(\hat{\mathfrak{g}}, \mathfrak{g}_+, \mathfrak{g}_-)\]

with

\[\mathfrak{g}_+ = h(R) \oplus \mathfrak{n}_+ (\bigoplus_i k_{x_i}) \oplus CD, \quad \mathfrak{g}_- = h \otimes \Lambda \oplus \mathfrak{n}_- (\bigoplus_i k_{x_i}) \oplus CK.\]

Here $h$, $\mathfrak{n}_+$ and $\mathfrak{n}_-$ are the Cartan and opposite nilpotent subalgebras of $\mathfrak{a}$. Our aim will be to give a quantization of the Lie bialgebras $(\mathfrak{g}_\pm, \delta_\pm)$, where $\delta_\pm$ are the cobrackets defined by the above Manin triple, in the case where $\mathfrak{a} = sl_2(\mathbb{C})$.

$\mathfrak{g}_+$ and $\mathfrak{g}_-$ can be presented as follows. $\mathfrak{g}_+$ has generators $D$, $h^+(r)$, $r \in R$, and $e(\varepsilon)$, $\varepsilon \in \bigoplus_i k_{x_i}$, with

\[(1.1) \quad h^+(\alpha_1 r_1 + \alpha_2 r_2) = \alpha_1 h^+(r_1) + \alpha_2 h^+(r_2), \quad \alpha_i \in \mathbb{C}, r_i \in R,\]

and

\[(1.2) \quad e(\alpha_1 \varepsilon_1 + \alpha_2 \varepsilon_2) = \alpha_1 e(\varepsilon_1) + \alpha_2 e(\varepsilon_2), \quad \alpha_i \in \mathbb{C}, \varepsilon_i \in \bigoplus_i k_{x_i} ;\]
\( \mathfrak{g}_- \) has generators \( K, h^-(\lambda), \lambda \in \Lambda \), and \( f(\varepsilon), \varepsilon \in \oplus_i k_{x_i} \), with

\[
(1.3) \quad h^-(\alpha_1 \lambda_1 + \alpha_2 \lambda_2) = \alpha_1 h^-(\lambda_1) + \alpha_2 h^-(\lambda_2), \quad \alpha_i \in \mathbb{C}, \lambda_i \in \Lambda,
\]

\[
(1.4) \quad f(\alpha_1 \varepsilon_1 + \alpha_2 \varepsilon_2) = \alpha_1 f(\varepsilon_1) + \alpha_2 f(\varepsilon_2), \quad \alpha_i \in \mathbb{C}, \varepsilon_i \in \oplus_i k_{x_i}.
\]

The relations are the following. Let \( e^i, e_i \) be dual bases of \( R \) and \( \Lambda \) and \( \sum \varepsilon^i \otimes \varepsilon_i = \sum e^i \otimes e_i + e_i \otimes e^i \). Let \( z = (z_i) \) be a system of local coordinates at each point \( x_i \). We define the formal series

\[
e(z) = \sum e(\varepsilon^i) \varepsilon_i(z), \quad f(z) = \sum f(\varepsilon^i) \varepsilon_i(z),
\]

\[
h^+(z) = \sum h^+(e^i)e_i(z), \quad h^-(z) = \sum h^-(e_i)e^i(z);
\]

then the Lie algebra relations for \( \mathfrak{g}_+ \) and \( \mathfrak{g}_- \) are respectively

\[
(1.6) \quad [h^+(r), h^+(r')] = 0, \quad [h^+(r), e(z)] = 2r(z)e(z), \quad r, r' \in R
\]

\[
[D, h^+(r)] = h^+(\partial r), \quad [D, e(z)] = -\partial_z e(z), \quad [e(z), e(w)] = 0
\]

and

\[
(1.7) \quad [h^-(\lambda), h^-(\lambda')] = 2\text{res}_x(\lambda d\lambda')K, \quad [h^-(\lambda), f(z)] = -2\lambda(z)f(z),
\]

\[
[f(z), f(w)] = 0, \quad [K, h^-(\lambda)] = [K, f(\lambda')] = 0, \quad \lambda, \lambda' \in \Lambda
\]

Equalities (1.6) and (1.7) give

\[
[h^+(z), e(w)] = 2a_0(w, z)e(w), \quad [h^-(z), e(w)] = 2a_0(z, w)e(w);
\]

we set \( a_0(z, w) = \sum_i e^i(z)e_i(w), \) \( e^i, e_i \) being dual bases of \( R \) and \( \Lambda \). We have

\[
(1.8) \quad a_0(z, w) = \left[ \frac{1}{r_0(w)} \sum_{i \in \mathbb{Z}} (z/w)^i \right] \rightarrow_R \quad (1.9)
\]

the index \( z \rightarrow R \) means acting by \( \pi \otimes 1, \pi \) the projection of \( \oplus_i k_{x_i} \) to \( R \) parallel to \( \Lambda \), in a completion of \( k \otimes k = \mathbb{C}(\langle z, w \rangle) \) and

\[
(1.9) \quad a_0(z, w) + a_0(w, z) = \frac{1}{r_0(z)} \sum_{i \in \mathbb{Z}} (z/w)^i,
\]

which can be proved viewing the l.h.s. as a kernel. Thanks to (1.9), we rewrite equalities (1.6.b,c), and (1.7.a,b) as

\[
(1.10) \quad [D, h^+(z)] = (-\partial_z h^+(z))_{z \rightarrow \Lambda} \quad [h^-(z), h^-(w)] = \tilde{\gamma}(z, w)K,
\]

\[
[h^+(z), e(w)] = 2a_0(w, z)e(w), \quad [h^-(z), f(w)] = -2a_0(z, w)f(w),
\]
We have then antisymmetric element $\partial e$. The Jacobi identity is ensured by (6) $(\bar{\gamma})_{\text{(6)}} = 0$. This identity is a consequence of (1.11), which is proven as follows. Pose $\partial e = \sum_j c^j e_j$, $\partial e_i = \sum_j c^j e_j + d_{ij} e^j$; $a^j + c^j = 0$ and $d_{ij} + d_{ji} = 0$ because $\partial$ is anti-self-adjoint. $-(\partial z + \partial w)a_0(z,w)$ is then equal to $\sum_{i,j} d_{ij} e^i \otimes e^j$, and so belongs to $R \otimes R$ and is antisymmetric; this proves (1.11). The pairing between $g_+$ and $g_-$ is given by

$$\langle D, K \rangle = 2, \quad \langle e(z), f(w) \rangle = \frac{1}{r_0(z)} \sum_{i \in \mathbb{Z}} (z/w)^i, \quad \langle h^+(z), h^-(w) \rangle = 2a_0(w, z)$$

The formulas for the cobracket of $g_+$ and $g_-$ are then respectively

$$\delta_+(e(z)) = e(z) \land h^+(z), \quad \delta_+(h^+) = 0$$

$$\delta_+(D) = \sum_{i,j} \text{res}_{z=x_i} \text{res}_{w=x_j} \gamma(z, w)(h^+(z) \land h^+(w)) \omega_z \omega_w$$

and

$$\delta_-(f(z)) = h^-(z) \land f(z) + K \land \partial_z f(z), \quad \delta_-(h^-) = K \land \partial_z h^-, \quad \delta_-(K) = 0$$

The fact (1.13) defines a cocycle can be checked directly using the identities (1.9) and

$$\sum_{i,j} \text{res}_{z=x_i} \text{res}_{w=x_j} (a(z', z)\gamma(z', w')(h^+(w') \omega_{z'} \omega_{w'})) = (\partial_z h^+(z))_{z \rightarrow R},$$

(as before, the index $z \rightarrow R$ has the meaning of applying $\pi$), which is equivalent to $\sum_i \text{res}_{w'=x_i} (\gamma(z, w')h^+(w') \omega_{w'}) = (\partial_z h^+(z))_{z \rightarrow R}$; since $h^+(z)$ belongs to a completion of $g \otimes \Lambda$, it is enough to check this identity replacing $h^+$ by any element of $\Lambda$, and in this case it follows from $(\partial e_i)_R = \sum_j d_{ij} e^j$.

Let us describe how a change of $\Lambda$ affects $a_0(z, w) \in R_z((w))$. Let $\Lambda'$ be another Lagrangian supplementary to $R$, commensurable with $O_x$; let $\pi'$ be the projection of $\otimes_i k_{x_i}$ onto $R$ parallel to $\Lambda'$, and let $a_0;\Lambda'(z, w) = \left(1 \otimes \pi' \right)\left(\frac{1}{r_0(w)} \sum_{i \in \mathbb{Z}} (z/w)^i \right)$, $\gamma_\Lambda'(z, w) = -(\partial z + \partial w)a_0;\Lambda'(z, w)$. The projection of $\Lambda$ on $R$ parallel to $\Lambda'$ gives some antisymmetric element $r_1 \in R \otimes R$, and to any such element corresponds such a $\Lambda'$. We have then

$$a_0;\Lambda'(z, w) = a(z, w) - r_1(z, w), \quad \gamma_\Lambda'(z, w) = \gamma(z, w) + (\partial z + \partial w) r_1(z, w).$$
2. Quantization of $g_+$ and $g_-$

Let $\hbar$ be a formal parameter and $T$ be the operator $\frac{\hbar h\partial}{\hbar^2}$ : $\oplus_i k_{x_i} \rightarrow (\oplus_i k_{x_i})[[\hbar]]$; since $T$ is symmetric for $\langle , \rangle_{\oplus_i k_{x_i}}$, the expression

$$
\sum_i T e^i \otimes e_i - e^i \otimes Te_i
$$

belongs to $S^2 R$. Let us fix $\tau \in R \otimes R[[\hbar]]$, such that

$$
(2.1) \quad \tau + \tilde{\tau} = \sum_i T e^i \otimes e_i - e^i \otimes Te_i,
$$

where we denote $\tilde{f}(z, w) = f(w, z)$. Let $U$ be the operator from $\Lambda$ to $R[[\hbar]]$, such that

$$
\tau = \sum_i U e_i \otimes e^i;
$$

$U$ verifies

$$
\sum (T + U) e_i \otimes e^i + e^i \otimes (T + U) e_i = \sum T e^i \otimes e_i + Te_i \otimes e^i.
$$

Let $U_{h^+}(0)$ (resp. $U_{h^-}$) be the algebra with generators $h^+(r), r \in R, e(\varepsilon), \varepsilon \in \oplus_i k_{x_i}$ (resp. $K$, $h^- (\lambda), \lambda \in \Lambda, f(\varepsilon), \varepsilon \in \oplus_i k_{x_i}$), subject to relations (1.1-4), organized in generating series (1.5), and subject to the relations

$$
(2.2) \quad [h^+(r), e(w)] = 2r(w) e(w), e(z) e(w) = e^{2h \sum_i (T+U) e_i (z)} e^i (w) e(w) e(z),
$$

$$
(2.3) \quad \Delta_+(h^+(r)) = h^+(r) \otimes 1 + 1 \otimes h^+(r), \Delta_+(e(z)) = e(z) \otimes \exp(\hbar ((T+U) h^+(z)) + 1 \otimes e(z);
$$

and

$$
(2.4) \quad K \text{ is central, } [h^-(z), f(w)] = -2q^K (\partial_z + \partial_w) \{ \sum e^i (z) (T+U) e_i (w) \} f(w),
$$

$$
(2.5) \quad \text{ or } [h^-(\lambda), f(w)] = -2q^K \partial ((T+U) (q^{-K} \partial) \Lambda) (w) f(w),
$$

$$
(2.6) \quad [h^-(z), h^-(w)] = \frac{2}{\hbar} (q^K (\partial_z + \partial_w) - q^{-K} (\partial_z + \partial_w)) \sum e^i (z) (T+U) e_i (w)
$$

$$
(2.7) \quad f(z) f(w) = q^K (\partial_z + \partial_w) \{ e^{2h \sum e^i ((T+U) e_i (w))} \} f(w) f(z),
$$

$$
(2.8) \quad \Delta_-(K) = K \otimes 1 + 1 \otimes K, \Delta_-(h^-(\lambda)) = h^-((q^{K_2} \partial) \Lambda) \otimes 1 + 1 \otimes h^-((q^{-K_1} \partial) \Lambda),
$$

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\[
\Delta_-(f(z)) = (q^{-K_2\partial} f)(z) \otimes \exp(-\hbar(q^{K_1\partial} h^-))(z) + 1 \otimes (q^{K_1\partial} f)(z),
\]

where, thanks to the formula for \(\Delta_-(K)\), we view \(\Delta_-\) as a system of maps from \((U_hg_-)_{K_1+K_2}\) to \((U_hg_-)_{K_1} \otimes (U_hg_-)_{K_2}\), for variable scalars \(K_i\), where \((U_hg_-)_k = U_hg_-/(K - k)\) for any scalar \(k\). (2.8.b) can also be written

\[
\Delta_- h^-(z) = (q^{-K_2\partial} h^-)(z) \otimes 1 + 1 \otimes (q^{K_1\partial} h^-)(z).
\]

We can also write the \(h^- - h^-\) commutator as

\[
[h^-(z), h^-(w)] = \frac{1}{\hbar}(T_z + T_w)(q^{K(\partial_z + \partial_w)} - q^{-K(\partial_z + \partial_w)})a_0(z, w)
+ \frac{2}{\hbar}(q^{K(\partial_z + \partial_w)} - q^{-K(\partial_z + \partial_w)}) \sum_i e^i \otimes Ue_i;
\]

recall that \(a_0(z, w) = \sum_i e^i(z)e_i(w)\) and that \((\partial_z + \partial_w) \geq 1 a_0(z, w) \in R_z \otimes R_w;\) this shows that the r.h.s. of the formula for \([h^-(z), h^-(w)]\) belongs to \(\wedge^2 R\), as it should be. (We will show later on how to put the \(e - e\) and \(f - f\) relations in a correct form.)

The skew antipodes have the form

\[
S_+'(h^+(r)) = -h^+(r), \quad S_+'(e(z)) = -\exp(-\hbar((T + U)h^+(z))e(z),
\]

\[
S'_-(K) = -K, \quad S'_-(h^-(\lambda)) = -h^-(\lambda), \quad S'_-(f(z)) = -\exp(hh^-(z))f(z).
\]

The pairing

\[
\langle e(z), f(w) \rangle = \delta(z/w), \quad \langle h^+(r), h^-(\lambda) \rangle = \frac{2}{\hbar}\langle r, \lambda \rangle\oplus_{i,kx_i}
\]

(with \(\delta(z/w) = \sum_{i \in \mathbb{Z}} (z/w)^i\)) extends to a Hopf algebra pairing between \(U_{h\mathfrak{g}_+}^{(0)}\) and \(U_{h\mathfrak{g}_-}\).

The double of \(U_{h\mathfrak{g}_-}\) has then the additional relations

\[
[h^+(r), h^-(\lambda)] = \frac{2}{\hbar}\langle (q^{K\partial} - q^{-K\partial})r, \lambda \rangle_{kx},
\]

\[
[h^+(r), f(z)] = -2(q^{K\partial} r)(z)f(z),
\]

\[
[e(z), f(w)] = (q^{K\partial\omega}(z/w))q^{(T+U)h^+(z)} - (q^{-K\partial\omega}(z/w))q^{-h^-(w)},
\]

\[
[h^-(\lambda), e(w)] = 2[(T + U)((q^{K\partial}\lambda)_{\lambda}))(w)e(w).
\]
Let us define

\[(2.19) \quad \tilde{D}(h^+(r)) = h^+(\partial r),\]

\[(2.20) \quad \tilde{D}(e(z)) = -\partial z e(z) + \frac{\hbar}{2}[\partial(T + U)h^+ - (T + U)(\partial h^+)\lambda](z)e(z)\]

where \((\partial h^+)\lambda(z) = \sum h^+(e^i)(\partial e_i)\lambda_i\), and the index \(\Lambda\) denotes the projection of \(\oplus_i k_{x_i}\) on \(\Lambda\) parallel to \(R\); \(\tilde{D}\) extends to a derivation of \(U_{h\mathfrak{g}_+^{(0)}}\). Let \(U_{h\mathfrak{g}_+}\) be the algebra generated by \(U_{h\mathfrak{g}_+^{(0)}}\) and the element \(D\), such that

\[(2.21) \quad [D, x] = \tilde{D}(x)\]

for \(x \in U_{h\mathfrak{g}_+^{(0)}}\). Let us extend \(\Delta_+\) to \(U_{h\mathfrak{g}_+}\) by

\[(2.22) \quad \Delta_+(D) = D \otimes 1 + 1 \otimes D - \frac{\hbar}{4}\{h^+\{(T + U)e_i\}_R \otimes h^+(\partial e^i) + h^+\{(\partial(T + U)e_i)\}_R \otimes h^+(e^i)\}\}.

Here the index \(R\) denotes the projection of \(\oplus_i k_{x_i}\) on \(R\) parallel to \(\Lambda\). Then \(\Delta_+\) defines a Hopf algebra structure on \(U_{h\mathfrak{g}_+}\), dual to \(U_{h\mathfrak{g}_-}\) if we extend the pairing (2.14) by

\[\langle D, K \rangle = 1/2, \quad \langle D, h^-(\lambda) \rangle = \langle D, f(z) \rangle = 0;\]

the counit and skew antipode extend then to \(\epsilon(D) = 0\),

\[(2.23) \quad S'_+(D) = -D - \frac{\hbar}{4}\sum_i \{h^+(\partial e^i)h^+\{(T + U)e_i\}_R + h^+(e^i)h^+\{(\partial(T + U)e_i)\}_R\},\]

and the quantum double relations are

\[(2.24) \quad [D, h^-(z)] = -\partial h^-(z) + q^{-K\partial}\{\partial[(T + U^*)h^+]_R - [(T\partial h^+)_{\Lambda} + U^*(\partial h^+)\lambda]\}(z)
\quad + \frac{1}{2}(q^{K\partial} - q^{-K\partial})[\partial U^*h^+ - (\partial(T h^+)\lambda)_R - U^*(\partial h^+)\lambda](z),\]

where \(U^* : \Lambda \rightarrow R[[\hbar]]\) is the map dual to \(U : \Lambda \rightarrow R[[\hbar]]\), and

\[(2.25) \quad [D, f(z)] = -\partial f(z) + \frac{\hbar}{2}q^{K\partial}[\partial(T + U)h^+ - (T + U)(\partial h^+)\lambda](z)f(z).\]
3. $e-e$ and $f-f$ relations.

3.1. Construction of $\gamma$.

Let us compute the endomorphism of $R$, defined by

\begin{equation}
(3.1) \quad \rho(f)(z) = \text{res}_w a_0^2(z, w)f(w)\omega_w.
\end{equation}

$\rho$ is the restriction to $R$ of the endomorphism $\bar{\rho}$ of $k_x$, defined by

\begin{equation}
(3.2) \quad \bar{\rho}(f) = \text{res}_w (a_0^2 - \bar{a}_0^2)(z, w)f(w)\omega_w.
\end{equation}

Let $\alpha(z, w) = (z - w)a_0(z, w)$; we have $\alpha(z, w) = -(z - w)\bar{a}_0(z, w)$ and

\[ [z, \bar{\rho}](f) = \text{res}_w \alpha(z, w)(a_0 + \bar{a}_0)(z, w)f(w)\omega_w. \]

Now $a_0 + \bar{a}_0 = \delta(z/w)/r_0(z)$ so

\[ [z, \bar{\rho}](f)(z) = \alpha(z, z)f(z). \]

But $a_0(z, w) = (\sum_{i \leq -N}(z/w)^i + \sum_{i > -N}z^i \lambda_i(w))/r_0(z)$, with $\lambda_i \in \Lambda$, so that

\begin{align*}
(z - w)a_0(z, w) &= -(w/r_0(z))(z/w)^{-N} + (z - w) \sum_{i > -N} z^i \lambda_i(w)/r_0(z) \\
&\in -z/r_0(z) + (z - w)\mathcal{C}((z, w)),
\end{align*}

so $\alpha(z, z) = -z/r_0(z)$; so that $[\bar{\rho}, z] = [\partial, z]$; so we have, $\bar{\rho} = \partial$-function. Since $\Lambda$ is isotropic, $\bar{\rho}(1) = 0$; this shows $\bar{\rho} = \partial$.

Let us consider now $\partial_z a_0(z, w) - a_0(z, w)^2$; this expression belongs to $R_z \otimes k_w$ and the endomorphism of $R$ it defines is zero, so it belongs to $R \otimes R$. We have shown:

**Proposition 1.**— There exists $\gamma \in R \otimes R$, such that $\partial_z a_0(z, w) = a_0(z, w)^2 + \gamma(z, w)$.

3.2. The kernel of $\partial^n$, $n \geq 0$.

In this section we will study the expressions $\sum \partial^k e^i \otimes e_i$. For $k = 0$, this expression is equal to $a_0$; for $k = 1$, it is equal to $\partial_z a_0 = a_0^2 + \gamma$; for $k = 2$, it is equal to $\partial_z (a_0^2 + \gamma) = 2a_0(a_0^2 + \gamma) + \partial_z \gamma = 2a_0^3 + 2a_0\gamma + \partial_z \gamma$. More generally, we have:

**Proposition 2.**— Let $(P_k^{(n)})_{k \in \mathbb{Z}, n \geq 0}$ be the system of polynomials in $\mathcal{C}[\gamma_0, \gamma_1, ...]$ defined by $P_{<0}^{(0)} = 0$, $P_{1}^{(0)} = 1$, $P_{>0}^{(0)} = 0$, and $P_k^{(n+1)} = DP_k^{(n)} + (k - 1)P_{k-1}^{(n)} + \gamma_0(k + 1)P_{k+1}^{(n)}$, for $n \geq 0$, where $D = \sum_{i \geq 0} \gamma_{i+1}\partial/\partial \gamma_i$. Then

\begin{equation}
(3.3) \quad \sum_{k \geq 0} \partial^n e^i \otimes e_i = \sum_{k \geq 0} P_k^{(n)}(\gamma, \partial_z \gamma, ...)a_0^k,
\end{equation}

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\[ \sum e^i \otimes \partial^n e_i = (-1)^n \sum_{k \geq 0} P_k^{(n)}((-1)^i \partial^i \tilde{\gamma})a_0^k. \]

The proof is a simple induction. Then, we have \( \sum (\sum_{n \geq 1} \frac{\hbar^n}{n!} P_k^{(n-1)}(\gamma, \partial \gamma, \ldots)) a_0^k \); let us set
\[ u_k(\hbar, \gamma_0, \gamma_1, \ldots) = \sum_{n \geq 1} \frac{\hbar^n}{n!} P_k^{(n-1)}(\gamma_0, \gamma_1, \ldots). \]

We have, with \( t \) an auxiliary variable, and with \( u(\hbar, t, \gamma_i) = \sum_{k \geq 0} t^k u_k \), the equations
\[ \frac{\partial u}{\partial \hbar} = t + Du + (t^2 + \gamma_0) \frac{\partial u}{\partial t}, \quad u|_{\hbar=0} = 0; \]
by Cauchy's principle, they determine uniquely \( u \). Let \( \phi, \psi \in \hbar C[\gamma_i][[\hbar]] \) be the solutions to
\[ \frac{\partial \psi}{\partial \hbar} = D\psi - 1 - \gamma_0 \psi^2, \quad \frac{\partial \phi}{\partial \hbar} = D\phi - \gamma_0 \psi, \]
then the expansions of \( \phi \) and \( \psi \) are \( \psi = -\hbar + \ldots \), \( \phi = \hbar^2 \gamma_0 + \ldots \), and \( \phi, \psi \) have the properties
\[ \phi(-\hbar, (-1)^i \gamma_i) = \phi(\hbar, \gamma_i), \quad \psi(-\hbar, (-1)^i \gamma_i) = -\psi(\hbar, \gamma_i). \]
Moreover, \( \phi - \ln(1 + a_0 \psi) \) satisfies (3.6); this identifies this function with \( u \).

We conclude from this the first part of

**Proposition 3.**— With \( \phi \) and \( \psi \) the solutions of (3.7), we have
\[ \sum q^\partial \frac{\partial - 1}{\partial} e^i \otimes e_i = \phi(\hbar, \partial^i \gamma) - \ln(1 + a_0 \psi(\hbar, \partial^i \gamma)), \]
\[ \sum \frac{1-q^\partial}{\partial} e^i \otimes e_i = -\phi(-\hbar, \partial^i \gamma) + \ln(1 + a_0 \psi(-\hbar, \partial^i \gamma)), \]
\[ \sum e^i \otimes \frac{q^\partial - 1}{\partial} e_i = -\phi(h, \partial^i \tilde{\gamma}) + \ln(1 - a_0 \psi(h, \partial^i \tilde{\gamma})), \]
\[ \sum e^i \otimes \frac{q^\partial - 1}{\partial} e_i = -\phi(-h, \partial^i \tilde{\gamma}) + \ln(1 - a_0 \psi(-h, \partial^i \tilde{\gamma})). \]
The last part is proved using the last part of prop. 2. These results imply the following statement:

\[ \phi(h, \partial^i_\gamma) - \ln(1 + a_0 \psi(h, \partial^i_\gamma)) - \phi(-h, \partial^i_\gamma) + \ln(1 - a_0 \psi(-h, \partial^i_\gamma)) = f(\partial_z) - f(-\partial_w) \left( \gamma - \tilde{\gamma} \right), \]

(3.13)

with \( f(x) = \frac{e^x - 1}{x} \); this is proven by noting that the l.h.s. belongs to \( R \otimes R \), and that \((\partial_z + \partial_w)a_0 = \gamma - \tilde{\gamma}\). In particular, we have

\[ \phi(h, \partial^i_\gamma) - \phi(-h, \partial^i_\gamma) + \ln(-\psi(-h, \partial^i_\gamma)/\psi(h, \partial^i_\gamma)) = \frac{f(\partial_z) - f(-\partial_w)}{\partial_z + \partial_w} \left( \gamma - \tilde{\gamma} \right) + \nu_0, \]

\[ \nu_0 \in R \otimes R, \text{ vanishing on the diagonal}. \]

**Proposition 4.**— Recall that \( T = \frac{q^\partial - q^{-\partial}}{2\hbar \partial} \); let us set

\[ \psi_0 = \frac{1}{2\hbar} (\phi(h, \partial^i_\gamma) - \phi(-h, \partial^i_\gamma)), \quad \psi_+(\gamma_i) = \psi(-h, \gamma_i), \quad \psi_-(\gamma_i) = \psi(h, \gamma_i), \]

then

\[ \sum T e^i \otimes e_i = \psi_0(\gamma, \partial_\gamma, ...) + \frac{1}{2\hbar} \ln \left( \frac{1 + a_0 \psi_+(\gamma, \partial_\gamma, ...)}{1 + a_0 \psi_-(\gamma, \partial_\gamma, ...)} \right), \]

(3.15)

and

\[ \sum e^i \otimes T e_i = -\psi_0(\partial^i_\gamma) + \frac{1}{2\hbar} \ln \left( \frac{1 - a_0 \psi_-(\partial^i_\gamma)}{1 - a_0 \psi_+(\partial^i_\gamma)} \right). \]

(3.16)

From (3.13) follows

\[ \psi_0(\partial^i_\gamma) + \frac{1}{2\hbar} \ln \left( \frac{1 + a_0 \psi_+(\partial^i_\gamma)}{1 + a_0 \psi_-(\partial^i_\gamma)} \right) + \psi_0(\partial^i_\gamma) - \frac{1}{2\hbar} \ln \left( \frac{1 - a_0 \psi_-(\partial^i_\gamma)}{1 - a_0 \psi_+(\partial^i_\gamma)} \right) = T_z - T_w \frac{\partial_z - \partial_w}{\partial_z + \partial_w} (\gamma - \tilde{\gamma}). \]

(3.17)

Remark that

\[ \sum T e^i \otimes e_i - e^i \otimes T e_i = \frac{T_z - T_w}{\partial_z + \partial_w} (\gamma - \tilde{\gamma}); \]

(3.18)

recall that \( \tau = \sum U e_i \otimes e^i \) satisfies

\[ \tau + \tilde{\tau} = \frac{T_z - T_w}{\partial_z + \partial_w} (\gamma - \tilde{\gamma}). \]

(3.19)
Let us precise now the $e - e$ and $f - f$ relations. Let

\[(3.20)\]

$$A = \sum e^i \otimes (T + U)e_i,$$

then $A = \sum Te^i \otimes e_i + e^i \otimes [(Te_i)_R + Ue_i]$; so

\[(3.21)\]

$$A = -\tau + \psi_0(\gamma, \partial_z \gamma, ...) + \frac{1}{2\hbar} \ln \frac{1 + a_0\psi_+(\gamma, \partial_z \gamma, ...)}{1 + a_0\psi_-(\gamma, \partial_z \gamma, ...)}$$

\(= \tilde{\tau} - \psi_0(\tilde{\gamma}, \partial_w \tilde{\gamma}, ...) + \frac{1}{2\hbar} \ln \frac{1 - a_0\psi_-(\tilde{\gamma}, \partial_w \tilde{\gamma}, ...)}{1 - a_0\psi_+(\tilde{\gamma}, \partial_w \tilde{\gamma}, ...)}\).

The relations are then written

\[(3.22)\]

$$e^{2\hbar \psi_0(\gamma, \partial_z \gamma, ...)}[z - w + \alpha(z, w)\psi_+(\gamma, \partial_z \gamma, ...)]e(z)e(w) = e^{2\hbar \tau(z, w)}[z - w + \alpha(z, w)\psi_-(\gamma, \partial_z \gamma, ...)]e(w)e(z)$$

\[(3.23)\]

$$q^K(\partial_z + \partial_w)\{e^{2\hbar \tau(z, w)}[z - w + \alpha(z, w)\psi_-(\gamma, \partial_z \gamma, ...)]\}f(z)f(w) = q^K(\partial_z + \partial_w)\{e^{2\hbar \psi_0(\gamma, \partial_z \gamma, ...)}[z - w + \alpha(z, w)\psi_+(\gamma, \partial_z \gamma, ...)]\}f(w)f(z);$$

recall that $\alpha(z, w) = (z - w)a_0(z, w)$ belongs to $(\oplus_i k_{x_i})^{\otimes 2}$.

Let us show that the $e - e$ and $f - f$ relations define a flat deformation of the symmetric algebras in the $e(\varepsilon)$ and $f(\varepsilon)$, $\varepsilon \in \oplus_i k_{x_i}$. This statement is equivalent to

$$e^{2\hbar[-\psi_0(\gamma, \partial_z \gamma, ...) + \tau]}\frac{z - w + \alpha\psi_-(\gamma, \partial_z \gamma, ...)}{z - w + \alpha\psi_+(\gamma, \partial_z \gamma, ...)} \sim 1,$$

and this is in turn written

$$e^{2\hbar(-\psi_0 - \tilde{\psi}_0)}e^{2\hbar \frac{z - w + \alpha\psi_-(\gamma, \partial_z \gamma, ...)}{z - w + \alpha\psi_+(\gamma, \partial_z \gamma, ...)}} \frac{1 + a_0\psi_-(\gamma, \partial_z \gamma, ...)}{1 + a_0\psi_+(\gamma, \partial_z \gamma, ...)} \frac{1 - a_0\psi_-(\tilde{\gamma}, \partial_w \tilde{\gamma}, ...)}{1 + a_0\psi_+(\tilde{\gamma}, \partial_w \tilde{\gamma}, ...)} = 1,$$

which amounts to the statement (3.17) above.

To summarize, we have:

**Theorem 5.**— Let $\tau \in R \otimes R[[\hbar]]$ satisfy (2.1). The algebra $U_{h, \Lambda, \tau, \mathfrak{g}}$ defined by generators $K, D, h^+(r), h^-(\lambda), e(\varepsilon), f(\varepsilon), \lambda \in \Lambda, r \in R, \varepsilon \in \oplus_i k_{x_i}$, subject to relations (1.1-4) organized in generating series (1.5), subject to relations

$$[h^+(z), e(w)] = 2\sum_i e_i \otimes e^i e(w),$$

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\[ [h^-(z), e(w)] = 2\left( \sum_i q^{-K\partial} e^i \otimes (T + U)e_i \right) e(w), \]

\[ [h^+(z), f(w)] = -2\left( \sum_i e_i \otimes q^K e^i \right) f(w), \]

\[ [h^-(z), f(w)] = -2\left( \sum_i q^K e^i \otimes q^K (T + U)e_i \right) f(w), \]

\[ [h^+(z), h^+(w)] = 0, \quad [h^+(z), h^-(w)] = 2q^{-K\partial} e^i, \]

\[ [h^-(z), h^-(w)] = \frac{1}{\hbar} (q^K \otimes q^K - q^{K\partial} \otimes q^{-K\partial}) \sum_i e^i \otimes (T + U)e_i, \]

\[ [e(z), f(w)] = (q^K w \delta(z/w)) q^{(T + U)h^+(z)} - (q^{-K\partial} w \delta(z/w)) q^{-h^-}(w), \]

where the variables \( z \) and \( w \) are affected respectively to the first and second factor; \( K \) is central, (3.22), (3.23); (2.21), (2.19), (2.20), (2.24), (2.25); with coproduct defined by (2.3), (2.8), (2.9), (2.22), counit defined to be zero on all generators, and skew antipode defined by (2.12), (2.13), (2.23), is a Hopf algebra, quantizing the Manin triple of section 1.4.

4. Dependence in \( \tau \) and \( \Lambda \).

1. Dependence in \( \tau \).

Let us study the dependence of the algebra \( U_{h,\Lambda,\tau} \mathbf{g} \) defined in thm. 5, with respect to \( \tau \). Let \( \tau' = \tau + v, v \in \wedge^2 R[[h]] \). Let us denote with a prime all quantities corresponding to the algebra \( U_{h,\Lambda,\tau} \mathbf{g} \). Let us denote by \( u : \Lambda \to R[[h]] \) the linear map defined by \( u(\lambda) = (v, 1 \otimes \lambda)_{\otimes i k_{x_i}} \). We have

\[
(4.1) \quad u = U' - U, \quad \tau' - \tau = v = \sum_i (U' - U)e_i \otimes e^i.
\]

Then:

Proposition 6.— The formulae

\[
(4.2) \quad i(e'(z)) = e^\frac{h}{4} u(h^+(z)) e(z), \quad i(f'(z)) = f(z) e^\frac{h}{4} (q^K e^z) u(h^+(z)),
\]

\[
(4.3) \quad i(h^{\prime+}z) = h^{+}(z), \quad i(h^{-}\prime z) = h^{-}(z) - \left( \frac{q^{K\partial} + q^{-K\partial}}{2} uh^{+}\right)(z),
\]

\( i(K') = K, i(D') = D, \) define an algebra isomorphism \( i : U_{h,\Lambda,\tau} \mathbf{g} \to U_{h,\Lambda,\tau} \mathbf{g} \). Moreover, we have

\[
(4.4) \quad \Delta(i(x)) = \text{Adexp} \left( \frac{h}{4} (h^+ \otimes h^+) v \right) \{(i \otimes i) \Delta'(x)\}, \quad \forall x \in U_{h,\Lambda,\tau} \mathbf{g},
\]

so that both Hopf algebra structures are isomorphic up to a twisting operation.
Proof. $i$ is well-defined, because $uh^+(z)$ is expressed as $\sum_{i \geq 0} h^i \sum_j h^+(r_j^{(i)})(r_j^{(i)}(z))$, $r_j^{(i)}, r_j^{(i)} \in R$, the sums $\sum_j h^+(r_j^{(i)})(r_j^{(i)}(z))$ being finite. To prove e.g. that $i$ is an algebra morphism, we make use (while checking the $e-f$ relations) of the following sequence of identities:

$$(q^{-K\partial} \delta(z/w)) e_{\frac{1}{2}(uh^+)}(z) q^{-h^{-}(w)} e_{\frac{1}{2}(q^{K\partial}uh^+)}(w) =
$$
$$= (q^{-K\partial} \delta(z/w)) e_{\frac{1}{2}(uh^+)}(z) q^{-h^{-}(w)} e_{\frac{1}{2}(q^{2K\partial}uh^+)}(z)
$$
$$= (q^{-K\partial} \delta(z/w)) e_{\frac{1}{2}((1-q^{2K\partial})uh^+)(z),h^{-}(w)} q^{-h^{-}(w)+\frac{1}{2}((1+q^{-2K\partial})uh^+)(z)}
$$
$$= (q^{-K\partial} \delta(z/w)) e_{\frac{1}{2}((1-q^{2K\partial})\otimes q^{K\partial-q-K\partial})(\sum u e\otimes e')} q^{-h^{-}(w)+\frac{1}{2}((q^{K\partial}+q^{-K\partial})uh^+)(w)}
$$
$$= (q^{-K\partial} \delta(z/w)) q^{-h^{-}(w)+\frac{1}{2}((q^{K\partial}+q^{-K\partial})uh^+)(w)}
$$

(the first identity follows from $\delta(z/w)f(z) = \delta(z/w)f(w)$, the second from $e^a e^b = e^b e^a e^{[a,b]}$ if $[a, b]$ is scalar, the last one from the fact that $v$ is antisymmetric, so that $(q^{2K\partial}-1) \otimes (q^{2K\partial}-1) v$ vanishes on the diagonal). The other identities are easily checked. While checking the twisting identity for $f'(z)$, we use also the fact that

$$[(q^{K\partial} - q^{-K\partial})uh^+)(z),h^{-}(z)] = \sum_i (q^{K\partial} - q^{-K\partial})v_i(z)[h^+(v'_i), h^{-}(z)]
$$
$$= \sum_i (q^{K\partial} - q^{-K\partial})v_i(z) \frac{2}{h}(q^{K\partial} - q^{-K\partial})v'_i(z)
$$
$$= 0,$$

with $v = \sum_i v_i \otimes v'_i$, because $\{q^{K\partial} - q^{-K\partial}\} \otimes (q^{K\partial} - q^{-K\partial}) v \in \wedge^2 R[[h]]$. ■

Proposition 7.— The formulae

$$(4.5) \quad i'(e'(z)) = e(z) e_{\frac{1}{2}h u(h^+)}(z), \quad i'(f'(z)) = e_{\frac{1}{2}(q^{K\partial}u)(h^+)}(z) f(z),
$$

$$(4.6) \quad i'(h^+(z)) = h^+(z), \quad i'(h^-(z)) = h^-(z) - \left(\frac{q^{K\partial} + q^{-K\partial}}{2}\right)uh^+(z),
$$

$i'(K') = K, i'(D') = D$, also define an algebra isomorphism $i' : U_{h,\Lambda,\tau} \hat{g} \to U_{h,\Lambda,\tau} \hat{g}$, satisfying

$$(4.7) \quad \Delta(i'(x)) = \text{Ad} \exp \left( \frac{h}{4}(h^+ \otimes h^+)v \right) \{(i' \otimes i') \Delta(x) \}, \quad \forall x \in U_{h,\tau} \hat{g}.
$$

It follows, that $i^{-1} \circ i$ is a Hopf algebra automorphism of $U_{h,\tau} \hat{g}$.  

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2. Dependence in $\Lambda$.

Let $\Lambda$ and $\bar{\Lambda}$ be two Lagrangian supplementaries to $R$. Then we have, $\bar{\Lambda} = (1 + r)\Lambda$, with $r : \Lambda \to R$, given by

\begin{equation}
(4.8) \quad r(\lambda) = (r_0, 1 \otimes \lambda), \quad r_0 \in \wedge^2 R.
\end{equation}

Dual bases for $R$ and $\bar{\Lambda}$ are then $(e^i)$ and $(\bar{e}_i)$, with $\bar{e}_i = (1 + r)e_i$. Let us set

\begin{equation}
(4.9) \quad \bar{\tau} = \sum_i U \bar{e}_i \otimes e^i = \tau - \sum_i T r e_i \otimes e^i;
\end{equation}

we have then

\begin{equation}
(4.10) \quad \sum_i (T + \bar{U}) \bar{e}_i \otimes e^i = \sum_i (T + U)e_i \otimes e^i.
\end{equation}

Let us consider the Hopf algebras $U_{h,\Lambda,\tau} \hat{g}$, $U_{h,\bar{\Lambda},\bar{\tau}} \hat{g}$ and let us denote with a bar the quantities occurring in the second.

Proposition 8.— The mapping

\[ j : U_{h,\Lambda,\tau} \hat{g} \to U_{h,\bar{\Lambda},\bar{\tau}} \hat{g} \]

defined by $j(\bar{e}(z)) = e(z)$, $j(\bar{f}(z)) = f(z)$, $j(\bar{h}^+(e^i)) = h^+(e^i)$, $j(\bar{h}^-(\bar{e}_i)) = h^-(e_i)$, $j(\bar{D}) = D$, $j(\bar{K}) = K$, defines a Hopf algebras isomorphism between $U_{h,\Lambda,\tau} \hat{g}$ and $U_{h,\bar{\Lambda},\bar{\tau}} \hat{g}$.

5. Finite dimensional representations.

Let us fix $\Lambda$ and $\tau$, and denote by $U_{h\hat{g}}|_{K=0,\text{no} D}$ the algebra defined in thm. 5, without generator $D$ and with $K$ specialized to zero. We construct a morphism of algebras

\begin{equation}
(5.1) \quad \pi : U_{h\hat{g}}|_{K=0,\text{no} D} \to \text{End}(\mathbb{C}^2) \otimes (\oplus_i k_{x_i})[[h]],
\end{equation}

as follows: let us denote by $\zeta = (\zeta_i)$ the system of coordinates $(z_i)$, occurring in the r.h.s.; we define

\begin{equation}
(5.2) \quad \pi(h^+(r)) = r(\zeta) h + \rho^+(r)(\zeta) \text{Id}_{\mathbb{C}^2}, \pi(h^-(\lambda)) = (T + U)(\lambda)(\zeta) h + \rho^-(\lambda) \text{Id}_{\mathbb{C}^2},
\end{equation}

\begin{equation}
(5.3) \quad \pi(e(z)) = F(\zeta) \delta(z/\zeta) e, \quad \pi(f(z)) = \delta(z/\zeta) f,
\end{equation}

where the $e, f, h$ occurring in the r.h.s. are the matrices with nonzero coefficients $e_{12} = f_{21} = h_{11} = -h_{22} = 1$, and

\[ \rho^+ : R \to R[[h]], \rho^- : \Lambda \to (\oplus_i k_{x_i})[[h]], F(\zeta) \in (\oplus_i k_{x_i})[[h]] \]
are subject to the following conditions: recall that \( A(\zeta, z) = \sum e^i(\zeta)(T + U)(e_i)(z) \), and let

\[
\beta(\zeta, z) = \sum \rho^+(e^i)(\zeta)(T + U)(e_i)(z),
\gamma(\zeta, z) = \sum \rho^-(e_i)(\zeta)e^i(z);
\]

then

\[
q^{A+\beta} - q^{-A-\gamma} = F(z)\delta(z/\zeta), q^{-A+\beta} - q^{A-\gamma} = -F(z)\delta(z/\zeta).
\]

Let \( F(z)\delta(z/\zeta) = e^\sigma h(a_0 + \tilde{a}_0)(z, \zeta) \), with \( \sigma \in \mathbb{h} \otimes \mathbb{R}[[h]] \), then we have for some \( \rho_{1,2} \in \mathbb{h}^{-1} + \mathbb{R} \otimes \mathbb{R}[[h]] \),

\[
A = \frac{1}{2\mathbb{h}} \ln \frac{\rho_1 + a_0}{\rho_2 - a_0}, \beta = \frac{1}{2\mathbb{h}} \ln h^2(\rho_1 + a_0)(\rho_2 - a_0) + \sigma,
\]

\[
\tilde{A} = \frac{1}{2\mathbb{h}} \ln \frac{\rho_2 + \tilde{a}_0}{\rho_1 - \tilde{a}_0}, \gamma = -\frac{1}{2\mathbb{h}} \ln h^2(\rho_1 - \tilde{a}_0)(\rho_2 + \tilde{a}_0) - \sigma.
\]

Let us determine the possible \( \rho_{1,2} \) satisfying (5.6.a), (5.7.a). Comparing (5.6.a) and the second line of (3.21), it is enough to have

\[
\ln h(\rho_1 + a_0) = \ln(1 - a_0 \psi_-(\partial^i_w \tilde{\gamma})) + 2h\lambda
\]

\[
\ln h(\rho_2 - a_0) = \ln(1 - a_0 \psi_+(\partial^i_w \tilde{\gamma})) + 2h\bar{\lambda}
\]

with \( \lambda, \bar{\lambda} \in \mathbb{R} \otimes \mathbb{R}[[h]], \lambda - \bar{\lambda} = \tilde{\tau} - \psi_0(\partial^i_w \tilde{\gamma}) \); and comparing (5.7.a) and the first line of (3.21), it is enough to have

\[
\ln h(\tilde{\rho}_2 + a_0) = \ln(1 + a_0 \psi_+(\partial^i_z \tilde{\gamma})) + 2h\mu
\]

\[
\ln h(\tilde{\rho}_1 - a_0) = \ln(1 + a_0 \psi_-(\partial^i_z \tilde{\gamma})) + 2h\bar{\mu}
\]

with \( \mu, \bar{\mu} \in \mathbb{R} \otimes \mathbb{R} \), and \( \mu - \bar{\mu} = \psi_0(\partial^i_z \gamma) - \tau \).

(5.9-11) are equivalent to the fact that for certain \( \nu, \nu' \in \mathbb{R} \otimes \mathbb{R} \), equal to 0 on the diagonal,

\[
\tilde{\lambda} = \tilde{\mu} = \nu + \frac{1}{2h} \ln(h/\psi_+(\partial^i_w \tilde{\gamma})), \rho_2 = \frac{e^\nu}{\psi_+(\partial^i_w \tilde{\gamma})} + a_0(1 - e^\nu),
\]

and

\[
\lambda = \bar{\mu} = \nu' + \frac{1}{2h} \ln(-h/\psi_-(\partial^i_w \tilde{\gamma})), \rho_1 = -\frac{e^{\nu'}}{\psi_-(\partial^i_w \tilde{\gamma})} + a_0(e^{\nu'} - 1),
\]

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with the conditions on $\nu$ and $\nu'$

\begin{equation}
\nu' - \nu + \frac{1}{2\hbar} \ln(\psi_+(\partial^i_w \bar{\gamma})/\psi_-(\partial^i_w \bar{\gamma})) = \bar{\tau} - \psi_0(\partial^i_w \bar{\gamma}).
\end{equation}

Let us see now, how $\rho^{\pm}$ can be deduced from these equalities. The conditions on them are

\begin{equation}
\ln(1 - a_0 \psi_-(\partial^i_w \bar{\gamma})) + 2\hbar \lambda = h(1 + \rho^+)(e^i) \otimes (T + U)(e_i) - h\sigma,
\end{equation}

\begin{equation}
\ln(1 - a_0 \psi_+(\partial^i_w \bar{\gamma})) + 2\hbar \bar{\lambda} = h(\rho^+ - 1)(e^i) \otimes (T + U)(e_i) - h\sigma,
\end{equation}

\begin{equation}
\ln(1 + a_0 \psi_+(\partial^i_w \gamma)) + 2\hbar \mu = h\{e^i \otimes (T + U)(e_i) - e^i \otimes \rho^-(e_i)\} - h\bar{\sigma},
\end{equation}

\begin{equation}
\ln(1 + a_0 \psi_-(\partial^i_w \gamma)) + 2\hbar \bar{\mu} = -h\{e^i \otimes (T + U)(e_i) + e^i \otimes \rho^-(e_i)\} - h\bar{\sigma}.
\end{equation}

Let $T_+, T_-$ be the endomorphisms of $R$, defined by

\begin{equation}
T_{\pm}(r) = \langle \ln(1 - a_0 \psi_{\pm}(\partial^i_w \bar{\gamma})), 1 \otimes r \rangle;
\end{equation}
we have $T_+ = \frac{1 - q^{-\sigma}}{\sigma}, T_- = \frac{1 - q^{\sigma}}{\sigma}$. Since $T_{\pm} = h(\rho^+ \pm 1)T$, (recall that $T = \frac{q^{\sigma} - q^{-\sigma}}{2h\sigma}$),

\begin{equation}
\rho^+ = \frac{1 - q^{\sigma}}{1 + q^{\sigma}}.
\end{equation}

Due to (5.12) (resp. (5.13)), (5.15) and (5.16) (resp. (5.17) and (5.18)) are equivalent. (5.14) can be solved by posing

\begin{equation}
\tau = \frac{1}{2\hbar} f(\partial_z) - f(-\partial_w) (\gamma - \bar{\gamma}), \quad \nu' = 0, \nu = \bar{\nu}_0.
\end{equation}

then gives us

\begin{equation}
\hbar \sigma = \frac{1}{\partial_z + \partial_w} \left[ \frac{1}{1 + q^{\partial_z}} f(-\partial_z) + f(-\partial_w) \right] (\gamma - \bar{\gamma})
\end{equation}

\begin{equation}
+ \phi(h, \partial^i_w \bar{\gamma}) - \ln(-h/\psi_-(\partial^i_w \bar{\gamma})),
\end{equation}

and (5.18) gives then

\begin{equation}
\sum e^i \otimes \rho^-(e_i) = \sum e^i \otimes \left( \frac{(q^{\sigma} - 1)(1 - q^{-\sigma})}{2h\sigma} e_i - \frac{1}{\hbar}[\phi(-h, \partial^i_z \gamma) + \phi(h, \partial^i_z \gamma)] \right)
\end{equation}

\begin{equation}
- \frac{1}{\hbar} \frac{1}{\partial_z + \partial_w} \left[ \frac{1}{2} (f(\partial_w) - f(-\partial_z)) - \frac{1}{1 + q^{\partial_w}} (f(-\partial_w) + f(-\partial_z) + f(\partial_z)) \right] (\gamma - \bar{\gamma})
\end{equation}

and so

\begin{equation}
\rho^-(\lambda) = \frac{(q^{\sigma} - 1)(1 - q^{-\sigma})}{2h\sigma} \lambda - \frac{1}{\hbar}[\phi(-h, \partial^i_z \gamma) + \phi(h, \partial^i_z \gamma)] + \frac{1}{\partial_z + \partial_w} \left[ \frac{1}{2} (f(\partial_w)
\end{equation}

\begin{equation}
- f(-\partial_z)) - \frac{1}{1 + q^{\partial_w}} (f(-\partial_w) + f(-\partial_z) + f(\partial_z)) \right] (\gamma - \bar{\gamma}), \lambda \otimes 1)
\end{equation}

for $\lambda \in \Lambda$. So we have:
Proposition 9.— The formulae (5.2), (5.3) define a morphism of algebras

$$
\pi : U_{h} \mathcal{G}_{K=0,\text{no}D} \to \text{End}(\mathbb{C}^{2}) \otimes (\oplus_{i} k_{x_{i}})[[h]],
$$

provided \( \tau \) is chosen according to (5.21), with \( \rho^{\pm}, \sigma \) given by (5.20), (5.23) and (5.22).

Let us indicate how the formulae giving \( \rho^{\pm} \) and \( \sigma \) would be altered in the case of an arbitrary \( \tau \) (satisfying (2.1)). Let us denote with an exponent \((0)\) the quantities implied in prop. 9. The general form of a solution of (2.1) is

$$
\sigma = \sigma^{(0)} - ((1 + \rho^{+}) \otimes 1) \alpha, \quad \rho^{+} = \rho^{+(0)}, \quad \rho^{-}(\lambda) = \rho^{-(0)}(\lambda) - \rho^{+}(\langle \lambda \otimes 1, \alpha \rangle).
$$

6. Examples.

1. Trigonometric case.

Let \( X = \mathbb{CP}^{1} \), let \( z \) be a coordinate on \( X \), and let \( \omega = dz/z \). The set of marked points is \( \{0, \infty\} \). Let us pose

$$
\Lambda = \{(\lambda_{0}, \lambda_{\infty}) \in \mathbb{C}[[z]] \times \mathbb{C}[[z^{-1}]]|\lambda_{0}(0) + \lambda_{\infty}(\infty) = 0\}.
$$

Dual bases for \( R \) and \( \Lambda \) are \( e^{i} = z^{i} \) for \( i \in \mathbb{Z} \), and \( e_{i} = (z^{-i}, 0) \) for \( i < 0, -(0, z^{-i}) \) for \( i > 0, \frac{1}{2}(1, -1) \) for \( i = 0 \). We compute then

$$
\sum_{i} Te^{i} \otimes e_{i} = \frac{1}{2h} (\ln \frac{qz - w}{z - qw}, 0) - \frac{1}{2h} (0, \ln \frac{qw - z}{w - qz}),
$$

so that we can take \( U = 0 \); \( \exp(2h \sum_{i} Te^{i} \otimes e_{i}) = (\frac{qz - w}{z - qw}, \frac{qw - z}{z - qw}) \) and the \( e - e \) relation is

$$
(z - qw)e(z)e(w) = (qw - w)e(w)e(z),
$$

as it appeared first in [4].

2. Elliptic case.

Let \( X \) be the elliptic curve \( \mathbb{C}/\mathbb{Z} + \tau_{0}\mathbb{Z} \); let \( z \) be the coordinate on \( \mathbb{C} \), and let \( \omega = dz \). Let us consider the case \( x = 0 \). We choose \( \Lambda \) to be spanned by \( z^{-1}, z, z^{2}, z^{3}, ... \).

We define \( t = e^{2i\pi z}, q_{0} = e^{2i\pi \tau_{0}} \) (we assume \( |q_{0}| < 1 \));

$$
\theta(t) = \prod_{i \geq 0} (1 - q_{0}^{i} t) \prod_{i < 0} (1 - q_{0}^{i} t^{-1}), \quad \zeta = \frac{d}{dz}(\ln \theta).
$$

Let us compute the kernel of \( T \). We have \( a_{0}(z, w) = \zeta(z - w) - \zeta(z) + \zeta(w) \), so that for \( r \in R \),

$$
r(z) = \text{res}_{w=0}(\zeta(z - w) - \zeta(z) + \zeta(w))r(w)dw,
$$

and

$$
[(q^{\partial} - q^{-\partial})r](z) = \text{res}_{w=0}(\zeta(z - w + h) - \zeta(z + h) - \zeta(z - w - h) + \zeta(z - h))r(w)dw,
$$

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so

\[ \partial^{-1}(q^\partial - q^{-\partial}) e_i \in \ln \frac{\theta(z-w+h) \theta(z-h) \theta(-w-h)}{\theta(z-w-h) \theta(z+h) \theta(-w+h)} + R \otimes R[[\hbar]] \]

and so

\[ \sum e_i \otimes [-\partial^{-1}(q^\partial - q^{-\partial}) + U] e_i \in h + \ln \frac{\theta(z-w+h) \theta(z-h) \theta(-w-h)}{\theta(z-w-h) \theta(z+h) \theta(-w+h)} + \hbar^2 R \otimes R[[\hbar]]; \]

so that in the present case, the \( e - e \) relation takes the form

\[ e(z) e(w) = e^h \theta(z-w+h) \theta(z-h) \theta(-w-h) \theta(z-w-h) \theta(z+h) \theta(-w+h) e(w) e(z); \]

this relation is analogous to the relation (7.3) occurring in [8].

3. Double extensions and infinite twists of the Reyman-Semenov triples.

Let as above \( X \) be an elliptic curve, and \( X_n \) be the set of its \( n \)-division points. We fix an isomorphism of \( X_n \) with \( (\mathbb{Z}/n\mathbb{Z})^2 \), and denote by \( a \mapsto I_a \) the projective representation of \( (\mathbb{Z}/n\mathbb{Z})^2 \) on \( \bigoplus_{i \in \mathbb{Z}/n\mathbb{Z}} \mathbb{C} e_i \), defined by \( I_{(1,0)} e_i = \zeta^i e_i I_{(0,1)} e_i = e_{i+1} \), \( \zeta \) being a primitive \( n \)-th root of 1.

The following Manin triple was introduced in [12]. Let \( k_0, O_0 \) be the local field and ring at 0 \( \in X \), and let us define in \( g = sl_n(k_0) \) the scalar product \( \langle x, y \rangle_g = \text{res}_z \text{tr}(xy(z)) dz \). Let \( g_+ = sl_n(O_0) \) and \( g_- \) be the set of the expansions at 0, of the regular maps \( \sigma : X - X_n \to sl_n(C) \), such that \( \sigma(x+a) = \text{Ad} I_a \sigma(x) \), for \( a \in X_n \). Then \( (g, g_+, g_-) \) forms a Manin triple. Its quantization was treated in [15] in the \( sl_2 \) case, and is connected with Sklyanin algebras ([14]).

We propose the following double extension for this triple. Let \( \hat{g} = g \oplus CK \oplus CD \), and let us denote with an index 0 the Lie bracket in \( g \). We endow \( \hat{g} \) with the bracket \( [x, y] = [x, y]_0 + \text{res}_z \text{tr}(xy(z)) dz \), for \( x, y \in g, K \) is central. Let us also define a scalar product \( \langle , \rangle_{\hat{g}} \) on \( \hat{g} \) by \( \langle x, y \rangle_{\hat{g}} = \langle x, y \rangle_g \) for \( x, y \in g, \langle K, x \rangle_{\hat{g}} = \langle D, x \rangle_{\hat{g}} = 0 \) for \( x \in g, \langle K, D \rangle_{\hat{g}} = 2 \). Then \( \hat{g}_+ = g_+ \oplus CD \) and \( \hat{g}_- = g_- \oplus CK \) are Lagrangian subalgebras of \( \hat{g} \), so that \( (\hat{g}, \hat{g}_+, \hat{g}_-) \) forms a Manin triple.

We also propose the following twist for this triple. Let \( h \) and \( n_\pm \) be the diagonal and upper (resp. lower) triangular subalgebras of \( sl_n \), and let \( g_+ = h(O_0) \oplus n_+(k_0) \oplus CD \),

\[ g_- = \{ \sigma : X - X_n \to h(C) | \sigma(x+a) = \text{Ad} I_a \sigma(x), \text{ for } a \in X_n \} \oplus n_-(k_0) \oplus CK. \]

Then \( (\hat{g}, \hat{g}_+, \hat{g}_-) \) is a twist of the previous Manin triple, which can be quantized according to the techniques developed above in the case \( n = 2 \).
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