Asymptotic equisingularity and topology of complex hypersurfaces

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Abstract

We consider an equisingularity problem for polynomial families of affine hypersurfaces $X_\tau \subset \mathbb{C}^n$ with (at worst) isolated singularities. We show that the constancy of the global polar invariants $\gamma^*(X_\tau)$ is equivalent to the $t$-equisingularity at infinity, an asymptotic-type equisingularity that we introduce. We prove that $\gamma^*$-constancy implies $C^\infty$-triviality in the neighbourhood of infinity. We show how the invariants $\gamma^*$ enter in the description of a CW-complex model of a hypersurface $X_\tau$ and therefore provide in particular new invariants at infinity for polynomial functions $f : \mathbb{C}^n \to \mathbb{C}$.

1 Introduction

Let $\{X_\tau\}_{\tau \in \mathbb{C}}$ be a one-parametre polynomial family of affine hypersurfaces $X_\tau \subset \mathbb{C}^n$. Let us suppose, for simplicity, that $X_\tau$ is nonsingular, for $\tau \in \mathbb{C}$. It is well known that the family may fail to be topologically trivial because of jumps in behavior at infinity. We pose and give an answer to the following natural problem.

Problem Define a notion of equisingularity within such a family in order to be controlled by numerical invariants defined in the affine space and to imply $C^\infty$-triviality of this family in the neighbourhood of infinity.

This problem was not considered up to now since the lack of tools adapted to the situation at infinity. This is different from the local case essentially because of genericity failure: general slices in the affine space are not general any more in the neighbourhood of infinity.

We define $t$-equisingularity at infinity, a type of equisingularity which depends a priori on the compactification of the family $\{X_\tau\}_{\tau \in \mathbb{C}}$ but does not refer to any stratification. This is inspired by the notion of “$t$-regularity” defined by Siersma and the author [ST] for polynomial functions $f : \mathbb{C}^n \to \mathbb{C}$. It was shown in [ST] and [Pa] that $t$-regularity is equivalent to an asymptotic condition known as the Malgrange condition, (see loc. cit.).

Next we define generic affine polar invariants $\gamma^*(X_\tau)$ within the affine space itself, where we have fixed a system of coordinates. We show (Theorem 3.6) that they
provide a CW-complex model for the hypersurfaces $X_\tau$. They also appear to be new invariants for polynomial functions $f : \mathbb{C}^n \rightarrow \mathbb{C}$, representing a far-reaching refinement of the “Milnor numbers at infinity” which were defined in [ST] for a special class of polynomials.

The leading idea of this paper is to show that $t$-equisingularity at infinity and constancy of $\gamma^*(X_\tau)$ are equivalent conditions.

1.1 Theorem Let $F : \mathbb{C} \times \mathbb{C}^n \rightarrow \mathbb{C}$ be a polynomial function defining a family of hypersurfaces $X_\tau = \{x \in \mathbb{C}^n \mid F_\tau(x) = F(\tau, x) = 0\}$. Let $\tau_0 \in \mathbb{C}$ and assume that there is a compact set $K \subset \mathbb{C}^n$ such that $\text{Sing} X_\tau \subset K$ for all $\tau \in D$, where $D \subset \mathbb{C}$ is some disc centered at $\tau_0$. Then the following are equivalent:

(a) The family $\{X_\tau\}_{\tau \in \mathbb{C}}$ is $t$-equisingular at infinity at $\tau_0$ with respect to the projective compactification $(\overline{t}, X, \mathbb{C} \times \mathbb{P}^n)$.

(b) The numbers $\gamma^*(X_\tau)$ are constant, for $\tau$ close enough to $\tau_0$.

The assumption about the compact set $K$ is equivalent to the following one (stated in Theorem 3.6): for $\tau$ close enough to $\tau_0$, the hypersurfaces $X_\tau$ have at most isolated singularities which do not tend to infinity as $\tau$ approaches $\tau_0$. This is the assumption we shall work with throughout the paper.

Our statement is a posteriori natural and similar to Teissier’s famous, 20 years old, local equisingularity result i.e., the formula characterising Whitney equisingularity. We refer to Teissier [Te-1], [Te-2] and Briançon-Speder [BS-2].

The proof is based on the interplay between global and local features and uses limits of hyperplanes techniques. Nevertheless it does not follow from known local equisingularity results: the problem is that, if one tries to use Whitney equisingularity along some stratum on the part at infinity $X^\infty$, then the generic local polar invariants might not be global invariants of our family of affine hypersurfaces, see Remark 2.9 and §3.

We completely answer our Problem by proving (in a larger context) that $t$-equisingularity implies $C^\infty$-triviality in the neighbourhood of infinity. This fact, combined with Theorem 1.1, yields the following result:

1.2 Theorem Let $\{X_\tau\}_{\tau \in \mathbb{C}}$ be a family of nonsingular affine hypersurfaces. If $\gamma^*(X_\tau)$ is constant at $\tau_0$, then this family is $C^\infty$ trivial at $\tau_0$ (i.e., $t : X \rightarrow \mathbb{C}$ is a $C^\infty$ trivial fibration at $\tau_0$).

2 $t$-equisingularity at infinity

Limits of tangent spaces appear naturally in the study at infinity of affine hypersurfaces and they were systematically employed in this context in [ST], [T1]. We introduce the notion of $t$-equisingularity at infinity with respect to some compactification of the family $\{X_\tau\}_{\tau \in \mathbb{C}}$ and prove that it implies $C^\infty$-triviality at infinity. Let us start with some preliminaries, following [T1].
2.1 Let $X = \{ F(\tau, x) = 0 \} \subset \mathbb{C} \times \mathbb{C}^n$ and let $t : X \to \mathbb{C}$ be the projection to the first coordinate.

We consider a compactification of $\{ X_\tau \}_{\tau \in \mathbb{C}}$, namely a triple $(\hat{t}, Y, Z)$ with the following properties:

(a) $Y$ is an algebraic variety such that $Y \setminus X$ is a Cartier divisor on $Y$. We denote $Y^\infty = Y \setminus X$ and call it the divisor at infinity.

(b) $Z$ is a connected smooth complex manifold containing $Y$.

(c) $\hat{t} : Y \to \mathbb{C}$ is an algebraic morphism and a proper extension of $t : X \to \mathbb{C}$.

2.2 Let $g : Y \cap U \to \mathbb{C}$ be a function which locally defines the reduced divisor $Y^\infty$, where $U \subset Z$ is an open set containing $p$. Let $T^*(Z)$ denote the cotangent bundle of $Z$. We consider the relative conormal $\{ T^*Z \}_{\tau \in \mathbb{C}}$, $T^*_g|_{Y \cap U} := \text{closure}\{ (y, \xi) \in T^*(Z) \mid y \in X^0 \cap U, \xi(T_g^{-1}(g(y))) = 0 \} \subset T^*(Z)|_{Y \cap U}$, where $T_g^{-1}(g(y))$ denotes the tangent space at $y$ to the hypersurface $g^{-1}(g(y)) \subset X^0 \cap U$ and $X^0 \cap U$ is the open dense subset of regular points of $X$ where $g$ is a submersion. One says that the relative conormal is conical since it has the property $(y, \xi) \in T^*_g|_{Y \cap U} \Rightarrow (y, \lambda \xi) \in T^*_g|_{Y \cap U}, \forall \lambda \in \mathbb{C}$.

Let $\pi : T^*(Z) \to Z$ be the canonical projection. We denote by $\mathbb{P}T^*(Z)$ the projectivised bundle i.e., $\mathbb{P}T^*(Z)$ is the quotient of $T^*(Z) \setminus T^*_Z$ by the $\mathbb{C}^*$-action $\lambda \cdot (y, \xi) = (y, \lambda \xi)$, where $T^*_Z$ denotes the zero section of $\pi : T^*(Z) \to Z$. The canonical projection $\overline{\pi} : \mathbb{P}T^*(Z) \to Z$ is then a proper map.

Let us denote $T^*_g|_{Y\cap U} \cap \pi^{-1}(p)$ by $(T^*_g|_{Y\cap U}p)$. We need the following result:

2.3 Lemma [13] Lemma 3.3 Let $(X, x) \subset (\mathbb{C}^n, x)$ be a germ of a complex analytic space and let $g : (X, x) \to (\mathbb{C}, 0)$ be a nonconstant analytic function germ. Let $h : X \to \mathbb{C}$ be analytic such that $h(x) \neq 0$ and denote by $W$ some small enough, open neighbourhood of $x$ in $\mathbb{C}^n$. Then $(T^*_g|_{X\cap W})_x = (T^*_h|_{X\cap W})_x$.

Let then $\{ (U_i, g_i) \}_{i \in I}$ be a family of pairs as the $(U, g)$ above such that $\cup_{i \in I} U_i \supset Y^\infty$ and each $U_i$ is included in a local chart of $Z$. By Lemma 2.3, the subspaces $T^*_g|_{Y\cap U_i}$ restricted to $Y^\infty$ can be glued together to yield an analytic subspace $\mathfrak{C}$ of the cotangent bundle $T^*(Z)$ of $Z$. This space is also conical.

2.4 Definition We call the subspace $\mathfrak{C}$ of $T^*(Z)$ constructed above the space of characteristic covectors at infinity. We shall denote by $\mathbb{P}\mathfrak{C}$ the image of $\mathfrak{C}$ by the $\mathbb{C}^*$-quotient projection. For some subset $S \subset Y^\infty$, we denote $\mathfrak{C}(S) := \mathfrak{C} \cap \pi^{-1}(S)$ and $\mathbb{P}\mathfrak{C}(S) := \mathbb{P}\mathfrak{C} \cap \pi^{-1}(S)$.

2.5 Definition $(t$-equisingularity at infinity$)$ We say that the family $\{ X_\tau \}_{\tau \in \mathbb{C}}$ is $t$-equisingular at infinity, at $c \in \mathbb{C}$, with respect to the compactification $(\hat{t}, Y, Z)$ if for all $p \in Y^\infty \cap \hat{t}^{-1}(c)$ there is an open neighbourhood $U_p \subset Z$ of $p$ such that $\mathbb{P}T^*_\hat{t}|_{Y\cap U_p} \cap \mathbb{P}\mathfrak{C}(p) = \emptyset$.

The condition $\mathbb{P}T^*_\hat{t}|_{Y\cap U_p} \cap \mathbb{P}\mathfrak{C}(p) = \emptyset$ is equivalent to the fact that the limit (whenever it exists) of the linear spaces $T_x X_\tau(x)$, as $x \to p$, in the appropriate Grassmannian, is transverse to $\mathfrak{C}(p)$, the dual of $\mathfrak{C}(p)$. 

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2.6 Definition We say that the family \( \{ X_\tau \}_{\tau \in \mathbb{C}} \) is \( C^\infty \)-trivial at infinity at \( c \in \mathbb{C} \) if there is a ball \( B_0 \subset \mathbb{C}^n \) centered at 0 and a disc \( D_c \subset \mathbb{C} \) centered at \( c \) such that the restriction \( t_1 : (X \setminus \mathbb{C} \times B) \cap t^{-1}(D_c) \to D_c \) is a \( C^\infty \)-trivial fibration, for any ball \( B \supset B_0 \) centered at 0.

The first test for our new type of equisingularity would be if it implies topological triviality. We may in fact prove more than that: it implies \( C^\infty \) triviality.

2.7 Theorem If the family \( \{ X_\tau \}_{\tau \in \mathbb{C}} \) is \( t \)-equisingular at infinity at \( c \in \mathbb{C} \), then it is \( C^\infty \)-trivial at infinity at \( c \).

Proof The proof is essentially contained in [Tt, Theorem 4.3], [ST, Theorem 5.5]. We give just a brief account. We construct a \( C^\infty \) real non-negative function \( \phi \) on a neighbourhood of \( \hat{t}^{-1}(c) \cap Y^\infty \) by patching together the local equations of \( Y^\infty \), such that the map \( t \) is transversal to the positive levels of \( \phi \) within some open neighbourhood \( V \) of the compact set \( \hat{t}^{-1}(c) \cap Y^\infty \). Consequently, there exists a bounded \( C^\infty \) vector field \( \mathbf{v} \) which lifts the vector field \( \frac{\partial}{\partial t} \) over a small enough \( D_c \), to the open set \( V \cap \mathbb{X} \cap t^{-1}(D_c) \) and is tangent to the positive levels of \( \phi \).

Since the restriction \( t_1 : (D_c \times \partial B) \cap \mathbb{X} \to \mathbb{C} \) is a proper submersion, for big balls \( B \) and small enough \( D_c \), we may take a \( C^\infty \) vector field \( \mathbf{w} \) on \( (D_c \times (B' \setminus B)) \cap \mathbb{X} \) which lifts \( \frac{\partial}{\partial t} \) and is tangent to \( (D_c \times \partial B) \cap \mathbb{X} \). Then we glue it by a \( C^\infty \) partition of unity to the above defined vector field \( \mathbf{v} \) and by integrating this new vector field, we get the desired \( C^\infty \)-trivialization.

2.8 Corollary If a family \( \{ X_\tau \}_{\tau \in \mathbb{C}} \) of nonsingular affine hypersurfaces is \( t \)-equisingular at infinity at any \( c \in \mathbb{C} \), then it is a \( C^\infty \) locally trivial family (i.e., \( t : X \to \mathbb{C} \) is a \( C^\infty \)-locally trivial fibration).

Proof Using the notations of the above proof and the same argument as for constructing the vector field \( \mathbf{w} \) above, we construct this time a \( C^\infty \) vector field \( \mathbf{w}' \) on \( (D_c \times B) \cap \mathbb{X} \) which lifts \( \frac{\partial}{\partial t} \) and is tangent to \( (D_c \times \partial B) \cap \mathbb{X} \), then glue it by a \( C^\infty \) partition of unity to the previously defined vector field \( \mathbf{v} \).

We consider the projective compactification \( (\hat{t}, X, \mathbb{C} \times \mathbb{P}^n) \) of \( \{ X_\tau \}_{\tau \in \mathbb{C}} \), where \( X = \{ \hat{F}_\tau(x, x_0) = 0 \} \) is the closure of \( X \) in \( \mathbb{C} \times \mathbb{P}^n \), \( \hat{t} : X \to \mathbb{C} \) is the natural proper extension of \( t \) and \( \hat{F}_\tau \) denotes the polynomial obtained by homogenizing \( F_\tau \) by the new variable \( x_0 \).

2.9 Remark Suppose that the point \( p = (s, \tau_0) \in X^\infty \) has coordinate \( x_n \neq 0 \). Denote by \( F_n \) the function \( \hat{F}(x_1, \ldots, x_{n-1}, 1, x_0, \tau) \). Then, by [Te-I] and [BS-2], Whitney equisingularity along the line \( \{ s \} \times \mathbb{C} \) at \( p \) is equivalent to the integral closure criterion:

\[
\left( \frac{\partial F_n}{\partial t} \right) \in \mathfrak{m}_p\left( \frac{\partial F_n}{\partial x_1}, \ldots, \frac{\partial F_n}{\partial x_{n-1}}, \frac{\partial F_n}{\partial x_0} \right),
\]

where \( \mathfrak{m}_p \) is the maximal ideal of the analytic algebra \( \mathcal{O}_p \) at \( p \in X \).

In contrast, \( t \)-equisingularity at \( p \) is equivalent, by [ST] 5.5, Proof, to the following different integral closure condition:

\[
\left( \frac{\partial F_n}{\partial t} \right) \in \left( \frac{\partial F_n}{\partial x_1}, \ldots, \frac{\partial F_n}{\partial x_{n-1}} \right).
\]
3 The global $\gamma^*$-invariants

We define global polar invariants which control the $t$-equisingularity at infinity. They do not depend on the compactification triple $(\hat{t}, Y, Z)$ but on the chosen coordinate system on $\mathbb{C}^n$ (see Remarks 3.4).

Let our affine hypersurface $X \subset \mathbb{C} \times \mathbb{C}^n$ be stratified by its canonical (minimal) Whitney stratification $\mathcal{W}$, cf. [Te-2]. This is a finite stratification, having $X \setminus \text{Sing}X$ as a stratum. Let $\mathcal{P}^{n-1}$ denote the dual projective space of all hyperplanes of $\mathbb{P}^{n-1}$. The linear form $\mathbb{C}^n \to \mathbb{C}$ which defines the hyperplane $H \in \mathcal{P}^{n-1}$ will be denoted by $l_H$.

3.1 Definition For two complex analytic functions $f, g : X \to \mathbb{C}$, we define their polar locus with respect to $\mathcal{W}$ by:

\[ \Gamma_\mathcal{W}(f, g) := \text{closure}\{\text{Sing}_\mathcal{W}(f, g) \setminus (\text{Sing}_\mathcal{W}f \cup \text{Sing}_\mathcal{W}g)\}, \]

where \( \text{Sing}_\mathcal{W}f := \bigcup_{W_i \in \mathcal{W}} \text{Sing}_Wf|_W \) is the singular locus of $f$ with respect to $\mathcal{W}$.

We need the following global result, a variant of the Polar Curve Theorem, cf. [Ti, Lemma 2.4]:

3.2 Lemma [Ti] There is a Zariski-open set $\Omega_t \subset \mathcal{P}^{n-1}$ such that, for any $H \in \Omega_t$, the polar locus $\Gamma_\mathcal{W}(l_H, t)$ is a curve or it is empty.

Let $\Omega_t$ be the Zariski-open set from Lemma 3.2. We denote by $\Omega_{t,c}$ the Zariski-open set of hyperplanes $H \in \Omega_t$ which are transversal to the canonical Whitney stratification of the projective hypersurface $X_c \subset \mathbb{P}^n$. This extra condition insures that $\dim(\Gamma_\mathcal{W}(l_H, t) \cap X_c) \leq 0, \forall H \in \Omega_{t,c}$.

3.3 Definition ($\gamma^*$-invariants) Let $H \in \Omega_{t,c}$, where $\Omega_{t,c}$ is as above. For any $c \in \mathbb{C}$ such that $X_c$ has isolated singularities, we define the generic polar intersection multiplicity at $c \in \mathbb{C}$:

\[ \gamma^{n-1}_c = \gamma^{n-1}(X_c) = \text{int}(\Gamma_\mathcal{W}(l_H, t), X_c), \]

where $\text{int}(\Gamma_\mathcal{W}(l_H, t), X_c)$ denotes the sum of the local intersection multiplicities at each point of the finite set $\Gamma_\mathcal{W}(l_H, t) \cap X_c$. Next, we take a hyperplane $H \in \Omega_{t,c}$ and denote by $\gamma^{n-2}_c$ the generic polar intersection multiplicity at $c \in \mathbb{C}$ of the family of affine hypersurfaces $\{X_\tau \cap H\}_{\tau \in \mathbb{C}}$. By induction, we define in this way $\gamma^{n-i}_c$, for $1 \leq i \leq n-1$.

The sequence:

\[ \gamma^*_c := \langle \gamma^{n-1}_c, \ldots, \gamma^1_c, \gamma^0_c \rangle, \]

will be called the set of generic polar intersection multiplicities, where it is natural to put $\gamma^0_c := \deg X_c$, by definition.

3.4 Remarks By a standard connectivity argument, the set of polar intersection multiplicities is well-defined i.e., it does not depend on the choices of generic hyperplanes.
The sequence $\gamma^*_c$ is invariant up to linear changes of coordinates but not invariant to nonlinear changes of coordinates (since hyperplanes are involved in its definition).

The number $\gamma^*_c$ is constant on $\mathbb{C} \setminus \Lambda^i$, where $\Lambda^i$ is a finite set. For instance $\Lambda^{n-1} := \hat{t}(\Gamma_W(l_H, t) \cap Y^\infty)$, where $H \in \Omega_i$ and $\Gamma_W(l_H, t)$ denotes the closure of the polar curve $\Gamma_W(l_H, t)$ within $Y$. Therefore the “jump” of $\gamma^*_c$ is due, so to say, to the loss of intersection points to infinity.

3.5 Definition We call the following number:

$$\lambda^i_c := \gamma^i_a - \gamma^i_c$$

the $i$-defect at infinity, at $c \in \mathbb{C}$, where $i \in \{1, \ldots, n-1\}$, $c \in \Lambda^i$ and $u \notin \Lambda^i$.

We show how the invariants $\gamma^*$ and $\lambda^*$ contribute to the topology of the hypersurfaces $X_c$. As an application, one may consider the family $\{X_\tau\}_{\tau \in \mathbb{C}}$ of the fibres of a polynomial function $f : \mathbb{C}^n \to \mathbb{C}$ and $c \in \mathbb{C}$ such that $f^{-1}(c)$ has (at most) isolated singularities.

Let then $\mu(X_c)$ denote the sum of the Milnor numbers of the isolated singularities of $X_c$. Recall that $X_c$ is defined by the polynomial $F_c : \mathbb{C}^n \to \mathbb{C}$.

3.6 Theorem Let $\{X_\tau\}_{\tau \in \mathbb{C}}$ be a polynomial family and let $c \in \mathbb{C}$. Suppose that, for all $\tau$ in some disc $D \subset \mathbb{C}$ centered at $c$, the hypersurfaces $X_\tau$ have at most isolated singularities which do not tend to infinity as $\tau$ tends to $c$. Then:

(a) $X_c$ is homotopy equivalent to a generic hyperplane section $X_c \cap \mathcal{H}$ to which one attaches $\gamma^*_c = \mu(X_c)$ cells of dimension $n-1$.

(b) $X_c$ is homotopy equivalent to the CW-complex obtained by successively attaching to $\deg F_c$ points a number of $\gamma^1_c$ cells of dimension 1, then $\gamma^2_c$ cells of dimension 2, ..., $\gamma^{n-2}_c$ cells of dimension $n-2$ and finally $\gamma^{n-1}_c = \mu(X_c)$ cells of dimension $n-1$. In particular,

$$\chi(X_c) = (-1)^n \mu(X_c) + \sum_{i=0}^{n-1} (-1)^i \gamma^i_c.$$  

(c) Suppose in addition that $X_u$ is nonsingular, $\forall u \in D \setminus \{c\}$. Then:

$$\chi(X_u) - \chi(X_c) = (-1)^{n-1} \mu(X_c) + \sum_{i=0}^{n-1} (-1)^i \lambda^i_c.$$  

Proof (a): Let $\overline{X_c}$ denote here the closure of $X_c$ in $\mathbb{P}^n$, where $X_c \subset \mathbb{C}^n$. Let $\mathcal{H} = \{h(x) - \alpha = 0\}$, where $h : \mathbb{C}^n \to \mathbb{C}$ is linear. We may choose $\mathcal{H}$ so that the projective hyperplane $\{h = 0\} \subset \mathbb{P}^{n-1}$ is transverse to any stratum of the minimal Whitney stratification of $\overline{X_c}$. Let $\{a_1, \ldots, a_k\}$ be the singular points of the restriction $h|_{X_c} : X_c \to \mathbb{C}$. By a standard Lefschetz type argument, it follows that, homotopically, $X_c$ is obtained from $X_c \cap \mathcal{H}$ by attaching a certain number of cells of dimension $= \dim X_c - 1$. Namely, at each point one has to attach a number of cells equal to the $(n-2)^{th}$ Betti number of the complex link of $(X_c, a_i)$. But this is also equal to the multiplicity at $a_i$.
of the generic polar curve \( \Gamma(l_H, t) \). In turn, by a typical argument concerning isolated singularities, we have:

\[
\text{mult}_a \Gamma_W(l_H, t) = \int_n \left( \Gamma_W(l_H, t), X_e \right) - \mu(X_e, a_i).
\]

Summing up, we get the claimed result.

(b): \( X_c \cap \mathcal{H} \) is nonsingular. We further slice \( X_c \cap \mathcal{H} \) and we get, by induction and using (a) at each step, the desired model for \( X_c \) as CW-complex. Notice that, for a general line \( L \), the number of points of \( X_c \cap L \) is \( \deg F_c \).

(c): Easy to prove from (b), since \( X_u \) has no singularities and therefore \( \chi(X_u) = \deg F_u + \sum_{i=1}^{n-1} (-1)^i \gamma_i \). \( \diamond \)

3.7 Note Let \( \{X_\tau\}_{\tau \in \mathbb{C}} \) be the family of fibres of a polynomial \( f : \mathbb{C}^n \to \mathbb{C} \). Under the strong hypothesis of \( \text{"isolated W-singularities at infinity"} \), we have defined in [ST] Def. 4.4, Prop. 4.5] a number \( \lambda_p \geq 0 \) which measures the local defect at infinity, at some point \( p \in X^\infty \). In this case, one can prove, by using Theorem 3.6 and [ST] Corollary 3.5], that these numbers are related to our newly defined \( \lambda^* \) invariants as follows:

\[
\begin{cases}
\lambda^*_c = 0, & i \leq n-2, \\
\lambda^*_c = \sum_{p \in X_c \cap X^\infty} \lambda_p.
\end{cases}
\]

3.8 Example \( f : \mathbb{C}^3 \to \mathbb{C}, f(x, y, z) = x + x^2yz \).

We shall compute the generic polar intersection multiplicities and the defects at infinity in the neighbourhood of the value 0. As general linear form we may take \( l_H = x + y + z \). Then \( \Gamma(l_H, f) = \{x^2y - 2xy^2 - 1 = 0, y = z\} \) and this polar curve intersects transversely the fibre \( f^{-1}(0) \) in 3 points, hence \( \gamma^0_0 = 3 \). We use next that the Euler characteristic of any fibre of \( f \) is 1 (see Remark 1.5). Since we have \( \gamma^0_0 = \deg f = 4 \), by Definition 3.3 and since \( \gamma^0_0 - \gamma^0_1 + \gamma^0_2 = \chi(f^{-1}(0)) = 1 \), by Theorem 3.6(b), it follows that \( \gamma^1_0 = 6 \).

Now let \( t \neq 0 \). We have \( \gamma^1_0 = \deg f = 4 \) and we want to find \( \gamma^1_1 \). The function \( f \) restricted to the hyperplane \( H = \{x + y + z = 0\} \) becomes \( f|_H = x - x^3y - x^2y \). This is a polynomial in two variables of degree 4 and of degree 2 in \( y \). One can easily compute that the homotopy type of a general fibre \( f|_H^{-1}(t) \) is a bouquet of 5 circles, therefore \( \chi(f|_H^{-1}(t)) = -4 \). By using Theorem 3.6(a), we get \( \gamma^1_1 = 4 + 4 = 8 \), for general \( t \). Then by Theorem 3.6(b) again, we get \( \gamma^2_1 = 5 \).

The defects at infinity at the value 0 are therefore \( \lambda^0_0 = \lambda^2_0 = 2 \). Refering to Note 3.7, we may also deduce that the polynomial \( f \) has non-isolated W-singularities at infinity.

4. Proof of Theorem 1.1 and some consequences

By hypothesis, \( \text{Sing } X_D \subset D \times K \) i.e., the singularities of \( X_\tau \) do not tend to infinity as \( \tau \) varies in \( D \), where \( D \) is a (small) disc centered at a fixed point \( c := \tau_0 \in \mathbb{C} \). We call the critical locus at infinity of \( \hat{\iota} : Y \to \mathbb{C} \) the set: \( \text{Crt}^\infty \hat{\iota} = \{p \in Y^\infty | P^*_H Y_{r \cap U_p} \cap \mathbb{P} \mathcal{C}(p) \neq \emptyset\} \), which is a closed analytic subset of \( Y^\infty \). The family \( \{X_\tau\}_{\tau \in \mathbb{C}} \) is \( t \)-equisingular.
at infinity at \( c \), with respect to \( \hat{t}, \mathbf{Y}, \mathbf{Z} \) if and only if \( \text{Crt}^* \hat{t} \cap \hat{t}^{-1}(c) = \emptyset \). In the case of the projective compactification \( \hat{t}, \mathbf{X}, \mathbb{C} \times \mathbb{P}^n \), it is straightforward to see that \( \text{Crt}^* \hat{t} = \{ p \in \mathbb{X}^\infty | (p, t') \in \mathbb{P}C(p) \} \), where \( t' \) denotes here the projection \( \mathbb{C} \times \mathbb{P}^n \to \mathbb{C} \) on the coordinate \( \tau \) and \( t' \) is viewed as a point in \( \mathbb{P}T^*_{p}(\mathbb{C} \times \mathbb{P}^n) \). We shall write \( t' \) instead of \( t'(p) \) when the point \( p \) is clearly specified.

### 4.1 Proof of (b) \( \Rightarrow \) (a).

Suppose that the family \( \{ X_r \}_{r \in \mathbb{C}} \) is not \( t \)-equisingular at infinity, at some \( c \in \mathbb{C} \), with respect to the projective compactification \( \hat{t}, \mathbf{X}, \mathbb{C} \times \mathbb{P}^n \). This means that the compact analytic set \( \Sigma_c := \text{Crt}^* \hat{t} \cap \hat{t}^{-1}(c) \) is nonempty.

We shall prove that, if \( \gamma^*(X_r) \) is constant at \( c \), then \( \Sigma_c = \emptyset \), which is a contradiction.

**Step 1.** Reduction to the case \( \dim \Sigma_c = 0 \).

For any hyperplane \( H \in \mathbb{C}^n \), we have \( \dim \Sigma_c \geq \dim \Sigma_c \cap \overline{H} \geq \dim \Sigma_c - 1 \), where \( \overline{H} \) is the closure of \( H \) in \( \mathbb{P}^n \). Let us remark that \( \Sigma_c \cap \overline{H} \) is contained in the critical locus at infinity \( \text{Crt}^* \overline{H} \), where \( t_1 := t|_{\mathbb{C} \times H} \) is the projection from \( X^1 := X \cap (\mathbb{C} \times H) \) to \( \mathbb{C} \) and \( (t_1, X^1, \mathbb{C} \times \overline{H}) \) is the projective compactification of the family \( \{ X_r \cap H \}_{r \in \mathbb{C}} \). We identify \( \overline{H} \) with \( \mathbb{P}^{n-1} \) and continue the slicing process. A natural consequence which we want to single out is that, by hyperplane slicing, the dimension of the critical locus at infinity \( \Sigma_c \) cannot drop by more than one, at each step.

On the other hand, the dimension of the critical locus at infinity has to drop until zero, by repeating a finite number of times the slicing with generic hyperplanes. This is indeed so, by the following argument. After \( n - 2 \) times slicing, we get a family of plane curves \( \{ X_r^{n-2} \}_{r \in \mathbb{C}} \). If one considers generic slicing, then these curves are reduced. In this case, it is not difficult to see that \( \dim \text{Crt}^* \overline{p^{n-2}} \leq 0 \), since the divisor at infinity \( (X^{n-2})^\infty \) is of dimension one. Namely, there exists a Whitney stratification of \( X^{n-2} \) which has \( X^{n-2} \setminus \text{Sing} X^{n-2} \) as a stratum and has a finite number of point-strata on \( (X^{n-2})^\infty \), hence, since \( \deg F_r \) is constant, there is a finite number of points where \( \overline{t}^{n-2} : (X^{n-2})^\infty \to \mathbb{C} \) is not a stratified submersion. The critical locus at infinity is then included in this finite set, since in the other points at infinity \( p \in (X^{n-2})^\infty \) we have \( t' \notin \mathbb{P}C(p) \). This is so because, by [BMM, Théorème 4.2.1] or [Ti, Theorem 2.9], our Whitney stratification is Thom \((a_{x_0})\)-regular, where \( x_0 = 0 \) is an equation of \( (X^{n-2})^\infty \) at \( p \). Our argument is now complete.

The final conclusion is that, after a number \( s \) of times (at least 1, at most \( n - 2 \)) of generically slicing the family \( \{ X_r \}_{r \in \mathbb{C}} \), the critical locus at infinity at \( c \) has dimension precisely zero, not more and not less. The Step 2. of our proof will then show that this contradicts our hypothesis \( \lambda_c^{n-s-1} = 0 \).

**Step 2.** The case \( \dim \Sigma_c = 0 \). We need the following lemma:

### 4.2 Lemma

In the notations above, let \( p = (c, y) \in \mathbb{X}^\infty \cap \hat{t}^{-1}(c) \) and let \( \text{Crt}_p(\hat{t}, x_0) \) denote the critical locus of the map germ \( (\hat{t}, x_0) : (\mathbf{X}, p) \to (\mathbb{C}^2, (c, 0)) \) with respect to some Whitney stratification of \( \mathbf{X} \) having \( \mathbf{X} \setminus \text{Sing} \mathbf{X} \) as a stratum, where \( x_0 = 0 \) is the equation of the hyperplane at infinity \( \mathbb{P}^{n-1} \subset \mathbb{P}^n \). Then, for any hyperplane \( H \in \Omega_{t,c} \), there exists a neighbourhood \( U \) of \( p \) such that \( (\text{Crt}_p(\hat{t}, x_0) \cap \mathbb{X}^\infty) \cap U = \Gamma_W(l_H, t) \cap U \).

**Proof** Without loss of generality, let us assume that \( H \) is the zero locus of a coordinate
of \( \mathbb{C}^n \), say \( x_1 \). Then

\[
\Gamma_W(x_1, t) = \text{closure}\{ (\tau, x) \in X = \{ F = 0 \} \subset \mathbb{C} \times \mathbb{C}^n \mid \frac{\partial F}{\partial x_2} = \cdots = \frac{\partial F}{\partial x_n} = 0 \}.
\]

On the other hand, the germ at \( p \) of the polar locus \( \Gamma(\bar{t}, x_0) := \text{Crt}_p(\bar{t}, x_0) \setminus \mathbb{X}^\infty \subset X \), in the chart \( U_1 := \mathbb{C} \times \{ x_1 \neq 0 \} \) is the germ at \( p \) of the analytic set \( \overline{G_1} \subset X \), where

\[
G_1 = \{ (\tau, [x, x_0]) \in X \times \mathbb{X}^\infty \mid \frac{\partial F^{(1)}_\tau}{\partial x_2} = \cdots = \frac{\partial F^{(1)}_\tau}{\partial x_n} = 0 \},
\]

where \( F^{(1)}_\tau = F_\tau(x_0, 1, x_2, \ldots, x_n) \). We may choose generic coordinates on \( \mathbb{C}^n \) such that \( \{ x_1 = 0 \}, \ldots, \{ x_n = 0 \} \in \Omega_{t,c} \). One may then assume that \( \dim \Gamma_W(x_1, t) \cap \{ x_1 = 0 \} \leq 0 \). Now, on the intersection of charts \( \mathbb{C} \times (U_0 \cap U_1) \), the function \( \frac{\partial F^{(1)}_\tau}{\partial x_j} \) is equal to \( \frac{\partial F}{\partial x_j} \) modulo a nowhere zero factor, for any \( j \neq 0, 1 \). Therefore \( \overline{G_1} \) is equal to the closure in \( X \) of the set \( \{ (\tau, x) \in X \mid \frac{\partial F^{(1)}_\tau}{\partial x_2} = \cdots = \frac{\partial F^{(1)}_\tau}{\partial x_n} = 0 \} \), which is just \( \Gamma(x_1, t) \). It follows that \( \Gamma_W(\bar{t}, x_0) \setminus \{ p \} = \Gamma(l_H, t) \) within some neighbourhood of \( p \).

Taking a small enough neighbourhood \( U \) of \( p \), let us first remark that the conormal space \( \mathbb{P}T^{*}_{x_0}X \cap U \subset X \times \mathbb{P}^n \) has dimension \( n + 1 \), where \( \mathbb{P}^n \) denotes the set of hyperplanes in \( \mathbb{C}^{n+1} \) through the origin, and let us denote by \( \pi_2 \) the projection on \( \mathbb{P}^n \).

Let then \( p \) be a point of \( \Sigma_c \). By identifying \( \{ 0 \} \times \mathbb{C}^n \) with a hyperplane of \( \mathbb{C} \times \mathbb{C}^n \) through the origin, we conclude that \( \pi_2^{-1}(\{ 0 \} \times \mathbb{C}^n) \) cannot be empty, since it contains \( p \), and therefore has dimension at least 1. Moreover, the set \( \pi_1(\pi_2^{-1}(\{ 0 \} \times \mathbb{C}^n)) \cap X \) has the same dimension and is in fact the polar locus \( \Gamma(\bar{t}, x_0) \). By Lemma 4.2 above, this means that the polar locus \( \Gamma(x_1, t) \) is not empty, where \( x_1 \) is a generic coordinate. Then the polar locus \( \Gamma(x_1, t) \) is a curve, not contained into \( X_c \). Therefore its intersection multiplicity with \( X_\tau \), for some \( \tau \) close enough to \( c \), is a strictly positive number and there are points of \( X_\tau \cap \Gamma_W(x_1, t) \) which tend to infinity, as \( \tau \) tends to \( c \). This means that part of the intersection multiplicity \( \gamma^{n-1}(X_\tau) \) “vanishes” when \( \tau \) tends to \( c \). Therefore we get \( \lambda_c^{n-1} \neq 0 \), which gives a contradiction and ends our proof.

4.3 Proof of (a) \( \Rightarrow \) (b)

On the contrary, suppose that, for some \( H \in \Omega_{t,c} \) the closure of \( \Gamma_W(l_H, t) \) in \( \mathbb{C} \times \mathbb{P}^n \) contains some point \( p = (c, y) \in \mathbb{X}^\infty \cap \bar{t}^{-1}(c) \). Lemma 4.2 shows that, in this case, the local polar locus \( \Gamma(\bar{t}, x_0) \) at \( p \) is not empty. But this contradicts the assumption \( t' \notin \mathbb{P}(\mathbb{C}) \). This proves that \( \lambda_c^{n-1} = 0 \).

To continue our proof we shall, of course, slice again. This time we need to slice such that to preserve the condition \( t' \notin \mathbb{P}(\mathbb{C}) \). We shall do this by choosing a finite complex Whitney stratification at infinity of \( X \). Then take the restriction \( S \) of this stratification to \( \mathbb{X}^\infty \cap \bar{t}^{-1}(c) \). There exists a Zariski-open set \( \Omega \subset \mathbb{P}^{n-1} \) such that, if \( H \in \Omega \), then \( H \) is transversal to all strata of \( S \). Such hyperplane is also transversal to the hyperplane \( \{ t' = c \} \) and therefore, slicing by it will preserve the hypothesis \( t' \notin \mathbb{P}(\mathbb{C}) \), \( \forall p \in \mathbb{X}^\infty \cap \bar{t}^{-1}(c) \). In this way we prove inductively that \( \lambda_c^{n-i} = 0 \), \( \forall i \in \{ 2, \ldots, n - 1 \} \). We also need to prove \( \lambda_c^0 = 0 \). Suppose the contrary, which means that \( \deg F_\tau \) is not constant at \( c \). We may assume without loss of generality
that $c = 0$. But then $\tilde{\ell}^{-1}(0) \cap X^\infty$ contains the divisor at infinity $\mathbb{P}^{n-1}$ of $\mathbb{P}^n$ and $\tilde{F}_r$ is of the form $t h_r + x_0 g_r$, for some polynomial functions $h_r, g_r : \mathbb{C}^n \to \mathbb{C}$ such that $d = \deg h_r > \deg g_r = d - i$. It follows that $\mathbb{P}\mathcal{C}(p) \neq \emptyset$, for any $p \in \tilde{\ell}^{-1}(0) \cap X^\infty = \mathbb{P}^{n-1}$. Moreover, there exists a Zariski-open subset $G \subset \mathbb{P}^{n-1}$ such that $\mathbb{P}\mathcal{C}(p) \subset \mathcal{P}(\mathbb{P}^{n-1}, p)$, for all $p \in G$. This shows that the singular locus $\Sigma_0$ contains $G$, which gives a contradiction to the assumed $t$-equisingularity at infinity. ♦

4.4 Remark The fact that $\gamma^{-1}$ is constant does not imply $\gamma^*$ constant. This can be compared to the similar assertion in the local case, which has been proved by Briançon and Speder [BS-1]: a $\mu$-constant family of isolated hypersurface germs is not necessarily $\mu^*$-constant. A simple example which one may use in the global case is the following: $\{X_t\}_{t \in \mathbb{C}}$ is the family of fibres of the polynomial in 3 variables $f(x, y, z) = x + x^2 y$. One can easily see that $\gamma^2$ is constant, whereas $\gamma^1$ is not, since $\lambda^1_0 = 1$.

4.5 Remark Let now $\{X_t\}_{t \in \mathcal{D}}$ be a family of smooth hypersurfaces, where $\mathcal{D} \subset \mathbb{C}$ is some disc. Then, $\gamma^*$-constancy implies that this family is $C^\infty$ trivial over $\mathcal{D}$, if this disc is small enough, by Theorem 1.2. In particular, all the hypersurfaces have the same Euler characteristic. It is however not true that the invariance of the Euler characteristic implies the invariance of $\gamma^*$. One can show this by the example $f(x, y, z) = x + x^2 y$. Homotopically, the fibre $f^{-1}(0)$ is the disjoint union of $\mathbb{C}^2$ with a torus $\mathbb{C}^* \times \mathbb{C}^*$, whereas the fibre $f^{-1}(a)$, for $a \neq 0$, is the union of the torus $\mathbb{C}^* \times \mathbb{C}^*$ with $\{1\} \times \mathbb{C}$. Therefore the Euler characteristic of all fibres is equal to 1. On the other hand, by an easy computation, the defects at infinity at 0, namely $\lambda_0^2$ and $\lambda_0^1$, are both positive (see Example 3.8).

4.6 Theorem Under the hypothesis of Theorem 4.1, suppose in addition that $n = 2$ and that $\deg X_t$ -constant, for any $t \in \mathcal{D}$. Then the $\gamma^1$-constancy at $c := \tau_0 \in \mathbb{C}$ is equivalent to the topological triviality at infinity, at $c$, of the family $\{X_t\}_{t \in \mathbb{C}}$.

Proof To prove “$\Leftarrow$” we take up the arguments from 4.1. Step 1, case dim $X = 2$. There is a finite complex Whitney stratification of $X$ having $X \setminus \text{Sing} X$ as a stratum. Since $\deg F_\tau$ is constant at $c$, there is a finite number of points of $X^\infty$ where $\tilde{\ell} : X^\infty \to \mathbb{C}$ is not a stratified submersion. This implies that the variation of topology at infinity of the fibres of $t : X \to \mathbb{C}$ is localizable (in the sense of [Ts] Definition 4.1) exactly at those points at infinity $\{a_1, \ldots, a_k\}$. Then, by [Ts] Theorem 3.3, there is a big enough ball $B \subset \mathbb{C}^n$ centered at 0, there are small enough balls $B_i \subset \mathbb{C} \times \mathbb{P}^n$ centered at $a_i, i \in \{1, \ldots, k\}$, and there is a small enough disc $D_c \subset \mathbb{C}$ centered at $c$, such that the restriction $t_i : (X \setminus ((\mathbb{C} \times B) \cup \bigcup_{i=1}^k B_i)) \cap t^{-1}(D_c) \to D_c$ is a topologically trivial fibration. By excision, this implies that

$$\chi(X_c \setminus B) - \chi(X_{c+\epsilon} \setminus B) = \sum_{i=1}^k \chi(X_c \cap B_i) - \chi(X_{c+\epsilon} \cap B_i).$$

Now, by a local argument at each point $a_i$ (see e.g. [ST] Prop. 4.5), we have that the difference $\chi(X_c \cap B_i) - \chi(X_{c+\epsilon} \cap B_i)$ is just the (nongeneric!) polar number $\text{int}_{a_i}(\Gamma(t, x_0), \tilde{\ell}^{-1}(c))$. Moreover, by the definition of the 1-defect and by Lemma
the sum of the local polar numbers \( \sum_{i=1}^{\nu} \text{int}_{a_i}(\bar{\Gamma}(\bar{t}, x_0), \bar{t}^{-1}(c)) \) is equal to the defect \( \lambda^1_c \).

We may now conclude our proof as follows: if we suppose that \( \lambda^1_c \neq 0 \), then the relation (3) shows that topological triviality at infinity cannot hold.

We remark that the proof above works in higher dimensions for the following more general situation: \( \bar{f} \) has isolated stratified singularities at infinity with respect to some stratification of \( X \) which is a partial Thom stratification at infinity (see [Ti] Theorem 3.3 and loc.cit. for the terminology).

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