Cohomological Splitting of Coadjoint Orbits

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Abstract

The rational cohomology of a coadjoint orbit $O$ is expressed as tensor product of the cohomology of other coadjoint orbits $O_k$, with $\dim O_k < \dim O$.

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1 C-splitting of coadjoint orbits

The purpose of this note is to express the rational cohomology of a given coadjoint orbit of a compact Lie group in terms of the cohomology of “smaller” coadjoint orbits. Our result is based upon two facts: The coadjoint orbit hierarchy, and the cohomological splitting of certain Hamiltonian bundles.

The coadjoint orbit hierarchy.

Let $G$ be a compact and connected Lie group. We consider the coadjoint action of $G$ on $\mathfrak{g}^*$. By $X_A$ is denoted the vector field on $\mathfrak{g}^*$ generated by $A \in \mathfrak{g}$. If $\mu \in \mathfrak{g}^*$, we denote by $O = G \cdot \mu$ the coadjoint orbit of $\mu$. Then $O = G/G_\mu$, where $G_\mu$ is the subgroup of isotropy of $\mu$. The manifold $O$ possesses a natural symplectic structure defined by the 2-form $\omega$, with $\omega_\nu(X_A, X_B) = \nu([A, B])$, for any $\nu \in O$. If $l_g$ denotes the left multiplication by $g \in G$; that is, $l_g : \nu \in O \mapsto g \cdot \nu \in O$, then $l_g^* \omega = \omega$. Moreover $\iota_{X_A} \omega = d h_A$, with $h_A$ the function on $O$ defined by $h_A(\nu) = \nu(A)$. Therefore the action of $G$ on $O$ is Hamiltonian; that is, $G$ is a subgroup of the group $\text{Ham}(O)$ of Hamiltonian symplectomorphisms of $O$. And using Morse theory one can prove that $O$ is simply-connected.

On the other hand, if $\mu_1, \mu_2 \in \mathfrak{g}^*$ and $G_1 := G_{\mu_1} \subset G_{\mu_2} =: G_2$, then the orbits $O_j = G \cdot \mu_j$, $j = 1, 2$ are in the following hierarchy: There is a
symplectic fibration of $\mathcal{O}_1$ over $\mathcal{O}_2$. In fact $\mathcal{O}_1 = G \times_{G_2} (G_2 : \mu_1)$. So $\mathcal{O}_1$ is a fiber bundle over $G/G_2$ with fiber the orbit of $G_2 : \mu_1$ of $G_2$. Thus the fiber is in turn a symplectic manifold, and on it the group $G_2$ acts as a group of Hamiltonian symplectomorphisms, if $G_2$ is connected (for details see [2]).

Cohomological splitting of Hamiltonian bundles. Let $P \to B$ be a fiber bundle, with fiber a symplectic manifold $M$. This bundle is said to be Hamiltonian if its structural group reduces to the group $\text{Ham}(M)$ of Hamiltonian symplectomorphisms of $M$ [5]. Lemma 4.11 of [4] states that the rational cohomology of any Hamiltonian fiber bundle $M \to P \to \mathcal{O}$, whose base is a coadjoint orbit, splits additively as the tensor product of the cohomology of the fiber by the one of $\mathcal{O}$; that is, $H^*(P) \cong H^*(M) \otimes H^*(\mathcal{O})$.

If we apply the result of Lalonde and McDuff to our Hamiltonian fibration

\[ G_2 : \mu_1 \to \mathcal{O}_1 \to \mathcal{O}_2, \]

we obtain an additive isomorphism

\[ H^*(\mathcal{O}_1, \mathbb{Q}) \cong H^*(\mathcal{O}_2, \mathbb{Q}) \otimes H^*(G_2/G_1, \mathbb{Q}), \]

in other words

**Theorem 1** If $G_1 \subset G_2$ are stabilizers of the coadjoint action of the compact, connected Lie group $G$ and $G_2$ is connected, then there is an additive isomorphism

\[ H^*(G/G_1, \mathbb{Q}) \cong H^*(G/G_2, \mathbb{Q}) \otimes H^*(G_2/G_1, \mathbb{Q}). \]

**Corollary 2** If $\mu_1, \ldots, \mu_k$ are points of $g^*$, such that

\[ G_{\mu_1} \subset G_{\mu_2} \subset \ldots \subset G_{\mu_k} \neq G, \]

and the $G_{\mu_j}$ are connected, then

\[ H^*(G/G_{\mu_1}) \cong H^*(G/G_{\mu_k}) \otimes \bigotimes_{j=2}^{k} H^*(G_{\mu_j}/G_{\mu_{j-1}}). \]

This formula expresses the rational cohomology of the orbit $\mathcal{O}_1 = G/G_{\mu_1}$ in terms of the cohomology of orbits whose dimensions are less than $\dim \mathcal{O}_1$. 2
2 Cohomological splitting of flag manifolds

A partition $p$ of an integer $n$ is an unordered sequence $i_1, \ldots, i_s$ of positive integers with sum $n$. This partition of $n$ determines the subgroup

$$G_p := U(i_1) \times \ldots \times U(i_s)$$

of $U(n)$. Moreover this subgroup is a stabilizer for the coadjoint action of $U(n)$. The partitions $(11 \ldots 1), (1 \ldots 12), \ldots, (1n - 1)$ of $n$ determine a tower of subgroups

$$G_1 \subset G_2 \subset \ldots \subset G_{n-1}$$

of $U(n)$. The quotient $U(n)/G_1$ is the flag manifold $F_n$, i.e. the manifold of complete flags in $\mathbb{C}^n$, and $G_j/G_{j-1} \simeq U(j)/(U(1) \times U(j-1)) = \mathbb{C}P^{j-1}$.

From Corollary 2 we deduce

**Corollary 3** If $F_n$ denotes the flag manifold in $\mathbb{C}^n$, then

$$H^*(F_n, \mathbb{Q}) \simeq \bigotimes_{j=1}^{n-1} H^*(\mathbb{C}P^j, \mathbb{Q}).$$

As particular case we consider the group $G := U(4)$ and its subgroups

$$G_1 := U(1) \times \ldots \times U(1) \subset U(2) \times U(2) =: G_2.$$

Then by Theorem 1

$$H^*(F_4) \simeq H^*(G_{2,2}(\mathbb{C})) \otimes H^*(G_2/G_1) \simeq H^*(G_{2,2}(\mathbb{C})) \otimes H^*(\mathbb{C}P^1) \otimes H^*(\mathbb{C}P^1),$$

where $G_{2,2}(\mathbb{C})$ is the corresponding Grassman manifold in $\mathbb{C}^4$. So by Corollary 3

$$H^*(G_{2,2}(\mathbb{C})) \otimes H^*(\mathbb{C}P^1) \otimes H^*(\mathbb{C}P^1) \simeq H^*(\mathbb{C}P^1) \otimes H^*(\mathbb{C}P^2) \otimes H^*(\mathbb{C}P^3).$$

The existence of this isomorphism can be checked directly. The cohomology $H^*(G_{2,2})$ is generated by $\{c_1, c_2\}$, where $c_i$ is the corresponding Chern class of the 2-plane universal bundle over $G_{2,2}(\mathbb{C})$ (see 1). Moreover $c_1, c_2$ are algebraically independent up to dimension 4. So $\dim H^4(G_{2,2}) = 2$ and $\dim H^{2j}(G_{2,2}) = 1$, for $j \neq 2, 0 \leq j \leq 4$. Therefore it is possible to identify the graded vector spaces $H^*(G_{2,2}) \otimes H^*(\mathbb{C}P^1)$ and $H^*(\mathbb{C}P^3) \otimes H^*(\mathbb{C}P^2)$. This identification allows us to construct the isomorphism (2).

In general, a partition $p$ of $n$ determines the manifold of partial flags $F_p = U(n)/G_p$. The following corollary is a consequence of Theorem 1.
Corollary 4 If $p = \{i_1, \ldots, i_s\}$ and $p' = \{j_1, \ldots, j_r\}$ are partitions of $n$ with $r < s$ and $G_p \subset G_{p'}$, then

$$H^*(\mathcal{F}_p, \mathbb{Q}) = H^*(\mathcal{F}_{p'}, \mathbb{Q}) \otimes H^*(G_{p'}/G_p, \mathbb{Q}).$$

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References

[1] A. Dold, *Lectures on Algebraic Topology*, Springer, Berlin. (1980)

[2] V. Guillemin, E. Lerman, S. Sternberg, *Symplectic Fibrations and Multiplicity Diagrams*, Cambridge U. P., Cambridge. (1996)

[3] A. A. Kirilov *Elements of the Theory of Representations*, Springer-Verlag, Berlin. (1976)

[4] F. Lalonde, D. McDuff, *Symplectic Structures on Fiber Bundles* Topology 42(2) (2002) 309-347

[5] D. McDuff, D. Salamon, *Introduction to Symplectic Topology*, Clarenton Press, Oxford. (1998)

[6] L. Polterovich, *The Geometry of the Group of Symplectic Diffeomorphisms*, Birkhäuser, Basel. (2001)