Chern-Simons And Twisted Supersymmetry in Various Dimensions

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We introduce special supersymmetric gauge theories in three, five, seven and nine dimensions, whose compactification on two-, four-, six- and eight-folds produces a supersymmetric quantum mechanics on moduli spaces of holomorphic bundles and/or solutions to the analogues of instanton equations in higher dimensions. The theories may occur on the worldvolumes of $D$-branes wrapping manifolds of special holonomy. We also discuss the theories with matter.

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1. Introduction

Recent advances in string duality and Matrix theory in particular suggest the existence of interesting theories in dimensions higher than four, whose effective description at low energies is that of a supersymmetric gauge theory. The standard lore says that the gauge theory in the space-time of dimension higher than four is either (infrared) trivial and/or non-renormalizable and therefore does not exist as a field theory. One may study the gauge theories whose ultraviolet description is provided by string theory. For example, the physics of $D$-branes is described at low energies by the supersymmetric gauge theory. This argument indicates that a restricted set of correlation functions of gauge theory can be defined even in the higher dimensional theories.

We attempt to describe three classes of such theories in this paper. The theories of the first type are the Cohomological Field Theories (CohFT) \[1\][2] describing intersection theory on a moduli space of solutions to some gauge covariant equations $\Phi_\alpha = 0$ for a $D$-dimensional gauge field $A_\mu$ and possibly scalar fields in adjoint representation. The space of gauge fields and possible scalars is denoted as $A_n$, where $n$ denotes topological sector. In gauge theory one usually sums over all topological sectors. Let $A = \amalg_n A_n$. The set $\Phi$ of equations $\Phi_\alpha$ which can be called “topological gauge conditions” define $G$-invariant submanifold of $A$, where $G = \amalg_n G_n$ is the gauge group. Suppose that the quotient $M_n$ of the space of solutions of the system $\Phi = 0$ by the gauge transformations is finite dimensional in each topological sector $n$. The space $M_n$ depends on the choice of data entering $\Phi_\alpha$, such as the space-time manifold $X^D$, metric and/or any other geometrical object on $X^D$. The theory is called $\mathcal{H}_D(A, \Phi, G)$. Assume that gauge theory (in string theory context) defines a compactification of $M_n$. The correlation functions in the theory $\mathcal{H}_D(A, \Phi, G)$ are the integrals of certain differential forms $\omega_i$ over the space $M_n$:

$$\langle O_1 \ldots O_p \rangle = \sum_n \int_{M_n} \omega_1 \wedge \ldots \wedge \omega_p$$ (1.1)

The definition of the operators $O_i$ and of the map $O_i \mapsto \omega_i$, is provided by the field theoretic realization of $\mathcal{H}_D(A, \Phi, G)$. Also, a class of Lagrangians is associated to $\mathcal{H}_D(A, \Phi, G)$.

Suppose the theory $\mathcal{H}_D(A, \Phi, G)$ is given. One may define the theory called $\mathcal{K}_D(A, \Phi, G; R)$. It is $D + 1$ dimensional field theory compactified on a circle of radius...
Its input is the same triple \((\mathcal{A}, \Phi, \mathcal{G})\) as of the theory \(\mathcal{H}_D\). The output is the set of correlation functions:

\[
\langle \hat{O}_1 \ldots \hat{O}_p \rangle = \int_{\mathcal{M}_n} \hat{A}_R(\mathcal{M}_n) \omega_1 \wedge \ldots \wedge \omega_p
\]

where the radius of the circle is the expansion parameter of the \(\hat{A}\) genus:

\[
\hat{A}_R(\mathcal{M}_n) = \prod_{i=1}^{\frac{1}{2}\dim\mathcal{M}_n} \frac{Rx_i/2}{\sinh(Rx_i/2)},
\]

where \(x_i\)'s are the Chern roots of the tangent bundle to \(\mathcal{M}_n\)\(^3\). Superficially the construction of \(\mathcal{K}_D\) is similar to that of \(\mathcal{H}_{D+1}(L\mathcal{A}, L\Phi, L\mathcal{G})\) where \(L\mathcal{A}, L\mathcal{G}\) denote respectively the loop space of \(\mathcal{A}\) and the loop group of \(\mathcal{G}\). The equations \(L\Phi\) are the same equations \(\Phi\), imposed at each point of a loop separately. The theory \(\mathcal{H}_{D+1}(L\mathcal{A}, L\Phi, L\mathcal{G})\) is sick since its moduli space is the loop space of the moduli space \(\mathcal{M}\). The theory \(\mathcal{K}_D\) is defined by enhancing the symmetry group to \(\mathcal{G} = L\mathcal{G} \rtimes U(1)\), where \(U(1)\) acts by rotations of loops. The difference with ordinary CohFT’s is that this \(U(1)\) is treated as a global symmetry. In particular, the ghost for ghost \(k = \frac{1}{R}\) associated to \(U(1)\) is fixed rather then integrated over as it is done in the ordinary case. This definition can be illustrated by the compactification of the theory on a \(D\)-fold which reduces the model to supersymmetric quantum mechanics on the moduli space \(\mathcal{M}_n\).

The last theory in our list is \(\mathcal{E}ll_D(\mathcal{A}, \Phi, \mathcal{G}; \rho, \tau)\) which is associated to the spaces of double loops, i.e. of the maps of torus \(E_{\rho, \tau}\) to \(\mathcal{A}\). It is \(D + 2\)-dimensional theory. The correlation function in \(\mathcal{E}ll_D\) are related to the elliptic genera of \(\mathcal{M}\)\(^3\).

We discuss the examples of \(\mathcal{K}_D\)-theories for \(D = 8, 6\) related to supersymmetric Yang-Mills theories in seven and nine dimensions. They are twisted versions of dimensional reductions of the \(\mathcal{N} = 1\ d = 10\) superYang-Mills theories. The \(d = 10\) theory is the theory of type \(\mathcal{E}ll_8\). We also describe briefly the theories \(\mathcal{K}_D\) in dimensions five and three. Superficially, \(9d\) and \(7d\) theories are related to the octonionic structure which prevails in 8 dimensions, \(5d\) is related to quaternions and \(3d\) to complex numbers.

\[\text{Some of the ideas presented here are explained in details in } \cite{3}\]

\[\text{It may seem that knowledge of (1.1) allows one to compute (1.2) immediately. In fact, the}
\text{subtleties with compactification of } \mathcal{M}_n \text{ make the problem unaccessible to current}
\text{techniques. Moreover it is not clear whether } A\text{-genus is among the observables of } D\text{-dimensional}
\text{theory. That is why we call it a new theory.}\]
One motivation for studying such theories is the following. Recently the higher dimensional analogues of Donaldson-Witten theory were studied \[4\]. It is natural to ask whether the analogues of Chern-Simons theory (as “theories of A. Schwarz type” \[5\]) exist. We will find them in the framework of $\mathcal{K}_D$ theories.

If the theory is defined in $D + 1$ dimensions one may try to study it on $\mathcal{X}^D \times I$, rather than $\mathcal{X}^D \times S^1$. We show that this leads to an interesting $WZW$-like theory in $D$ dimensions.

Concluding the introduction we must warn the reader that we do not discuss various subtleties related to our choice of regularization. In particular, not every gauge group may be realized in nine and eight dimensions within string theory, according to today’s knowledge. It is very interesting to see how the subtleties of higher dimensional physics are reflected in topology (anomalies) and geometry of the moduli spaces $\mathcal{M}$.

The paper is organized as follows. The chapter 2 is devoted to the theories $\mathcal{H}_D$. The chapter 3 reviews some important constructions in supersymmetric quantum mechanics and then apply them in infinite-dimensional context. This application yields the theory $\mathcal{K}_D$. Then we discuss the examples of theories, related to octonions. The chapter 4 is devoted to observables. We find that Chern-Simons functionals can be promoted to the bona fide observables in the theory $\mathcal{K}_D$. The chapter 5 briefly described the theories in three and five dimensions, and also remarks on the theories with matter. The chapter 6 deals with $WZW$-like theories in higher dimensions. We present our conclusions in the chapter 7.

2. Constructions of the moduli spaces: $\mathcal{H}_D$ Theories

2.1. Cohomological Field Theories

Cohomological Field Theories allow to construct a theory of integration over a quotient $\mathcal{M} = \mathcal{N}/\mathcal{G}$ of submanifold $\mathcal{N}$ of a manifold $\mathcal{A}$ by the action of a group $\mathcal{G}$. The physically interesting cases related to gauge theories correspond to $\mathcal{A}$ being a space of gauge fields in some principal $G$-bundle $P$, $\mathcal{N}$ being the set of zeroes of a section $s$ of a certain infinite-dimensional vector bundle $\mathcal{V}$ over $\mathcal{A}$.

For example, in four dimensional gauge theory one may take:

$$\mathcal{V} = \Gamma \left( \Omega^{2,+}(\mathcal{X}^4) \otimes \mathfrak{g}_P \right),$$
where $g_P$ is the associated to $P$ adjoint bundle. The natural section $s$ is in this case $s = F^+_A$.

In general one sums over all topological types of $P$, hence the quotient $\mathcal{M}$ is the disjoint union of finite-dimensional (this is an assumption) manifolds $\mathcal{M}_n$:

$$\mathcal{M} = \Pi_n \mathcal{M}_n$$

The index $n$ stands for $\text{Ch}(P)$ and $w_+(P)$. Now assume that $\mathcal{G}$-equivariant bundle $\mathcal{V}$ over $\mathcal{A}$ is given, and choose a non-degenerate section $s$ of it. More precisely, the linearization of the equation $s = 0$ must be Fredholm on the complement to the tangent space to the gauge orbit. We think of the linearization of the equations as of the map: $ds : TA \to \mathcal{V}$ and it is required to have finite dimensional kernel and cokernel on the complement to the tangent space to gauge orbit. We endow $\mathcal{V}$ with a $\mathcal{G}$-invariant metric $g_{\alpha\beta}$. Sometimes we write the section $s$ in components: $s = \{\Phi_a\}$. In the introduction we denoted by $\Phi$ the set of equations $\Phi_\alpha$. It is more accurate to call $\Phi$ the pair $(\mathcal{V}, s)$ as it is this pair which enters the definition of our theory $\mathcal{H}_D(\mathcal{A}, \Phi, \mathcal{G})$.

We proceed by introducing the standard package of gauge CohFT \[\] . One has classical gauge fields $A_\mu$, ..., topological gauge conditions $\Phi_\alpha = 0$ and gauge symmetries $A_\mu \mapsto g^{-1}A_\mu g + g^{-1}\partial_\mu g$, ..., where ... denote possible additional fields and the action of the gauge group on them.

One has also fermions $\psi_\mu$ which represent the exterior derivatives of $A_\mu$ and the complex scalar field $\phi$ with values in the adjoint representation which represents the degree two generator in equivariant Cartan complex. In order to impose the equations $\Phi_\alpha = 0$ one needs a multiplet of fields with opposite statistics taking values in $\mathcal{V}^* (\chi^\alpha, H^\alpha)$. One also needs the projection multiplet $(\bar{\phi}, \eta)$ \[\] \[\] \[\]. All these fields fit in the context of the BRST technology, and the main process of building the QFT can be understood as a gauge fixing of the symmetries of relevant topological actions.

This leads one to introduce a generator $Q$, which is nilpotent \textit{up to the gauge transformations}:

$$QA_\mu = \psi_\mu, \quad Q\psi_\mu = D_\mu \phi \equiv \partial_\mu \phi + [A_\mu, \phi]$$

$$Q\phi = 0$$

$$Q\chi^\alpha = H^\alpha, \quad QH^\alpha = \phi^\alpha T^\alpha_{\alpha,\beta} \chi^\beta$$

$$Q\bar{\phi} = \eta, \quad Q\eta = [\phi, \bar{\phi}]$$

(2.1)
where $T_{\alpha,\beta}^\gamma$ represents the action of the gauge group in the fibers of $V$. In all our examples the action will be simply adjoint.

The property
\[ Q^2 = \text{gauge transformation with parameter } \phi \] (2.2)
allows to consider the following $Q$-invariant and gauge invariant Lagrangian:
\[ \mathcal{L} = \int_X \left\{ Q, i\chi^\alpha (\Phi_\alpha + ie^2 g_{\alpha\beta} H^\beta) + \text{Tr} \left( \bar{\phi} D^\mu \psi_\mu + \eta [\phi, \bar{\phi}] \right) \right\} \] (2.3)
The gauge fixing interpretation of the induced QFT is clear. Formally, the path integral
\[ \int \frac{DAD\psi D\eta D\bar{\phi} D\phi}{\text{Vol}(G)} e^{-L} \ldots \]
is $e^2$ independent and therefore reduces to the integral over $\mathcal{M} = \Phi^{-1}(0)/G$ provided that appropriate observables are inserted in $\ldots$. The integral is the infinite-dimensional version of Matthai-Quillen \textsuperscript{[8],[9],[10]} representative of the Euler class of $V$.

Now let us discuss the choices of $V$ and $s$. The curvature of the gauge field at a given point of space-time $\mathcal{X}$ is an element of $\Lambda^2 \otimes g$ - a $D(D-1)/2 \times \dim g$ dimensional vector space. Let $V$ be some rank $D-1$ vector bundle over $\mathcal{X}$ and choose a fiber-wise linear map $\Psi : \Omega^2(\mathcal{X}) \to V$. We try as the bundle $\mathcal{V}$ the space of sections of $V \otimes g_P$:
\[ \mathcal{V} = \Gamma(V \otimes g_P) \]
Consider the equations
\[ s = \Psi \cdot F_A = 0. \] (2.4)
These equations lead to nice CohFT’s, provided that the complex:
\[ \begin{array}{ccc}
\Omega^0 \otimes g_P & \xrightarrow{d_A} & \Omega^1 \otimes g_P \\
\Psi \cdot d_A & \xrightarrow{} & V \otimes g_P
\end{array} \] (2.5)
is elliptic. Its index equals the virtual dimension of $\mathcal{M}_n$.

Explicit examples of such equations are, e.g., flatness equations in $D = 2$ : $F = 0$, (anti-)self-duality in $D = 4$: $F^\pm = 0$ and complexified instantons and octonionic instantons in $D = 8$ and their dimensional reductions \textsuperscript{[11],[1],[12]}. Once the nice set of equations is obtained in dimension $D$ one may get the equations in lower number of space-time dimensions by performing dimensional reduction. In going to $D'$ dimensions the space $\mathcal{X}$ becomes a space of pairs $(A, H)$, where $A$ is the gauge field in a bundle $P'$ over $\mathcal{X}^{D'}$ and $H$ is the section of $g_{P'} \otimes E$. The bundle $E$ has rank $D - D'$. Its topology may be rather involved. Physics of Dirichlet-branes suggests that $E$ may be interpreted as a normal bundle to $\mathcal{X}^{D'}$ in some $D$-dimensional Calabi-Yau manifold \textsuperscript{[13]}.

In this way one gets Bogomolny equations in $D' = 3$, Hitchin equations in $D' = 2$ and balanced version of Donaldson-Uhlenbeck-Yau equations in $D' = 6$. 

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3. Theories in $D + 1$ dimensions: $K_D$ theories

In this chapter we generalize the ideas introduced in [3], where the two and four dimensional topological Yang-Mills theories were related to three and five dimensional theories. We shall see the existence of supersymmetric theories in $D + 1$ and $D + 2$ dimensions which can be projected to the $D$ dimensional topological theories discussed so far. Eventually, we will see that they have other limits in $D$ dimensions, of the WZW type.

3.1. Supersymmetric quantum mechanics

In this section we remind a few important constructions: supersymmetric quantum mechanics with target $M$, the one with target being a submanifold $N \subset M$ and the one with target being a quotient $M/G$ by an action of a group $G$. These constructions can be viewed as a passage from $D = 0$ dimensional to $D = 0 + 1$ dimensional theory.

Supersymmetric $\mathcal{N} = \frac{1}{2}$ quantum mechanics is the way to describe spinors on a manifold $M$ in the first quantized formalism. One studies the integrals over the space of maps $(x^\mu(t), \psi^\mu(t))$ of the worldline to the $(m|m)$ dimensional superspace $\Pi TM$. The path integral measure $Dx D\psi$ is well-defined and invariant under any changes of the coordinates $x$, provided that they are accompanied by the corresponding change in $\psi$.

The worldline supersymmetry:

\[
\begin{align*}
\delta x^\mu &= \psi^\mu \\
\delta \psi^\mu &= k \partial_t x^\mu
\end{align*}
\]

squares to the time translation with parameter $k$: $\delta^2 = k \partial_t$. $\delta$ acts as nilpotent operator on the observables, invariant under the rotation of the parameter $t$ and it is possible to define a cohomology space. The symmetry $\delta$ has the following interpretation. Consider the space of parameterized loops $X = LM$. The differential forms on $X$ can be identified with the functionals of $x^\mu(t)$ and $\psi^\mu(t)$, where $\psi^\mu(t)$ corresponds to the differential $dx^\mu(t)$. The group $U(1)$ acts on $X$ by rotations of loops and $\delta$ is the equivariant derivative $d + k_\mu V$, with $V$ representing the vector field $\partial_t x^\mu \frac{\delta}{\delta x^\nu}$. The number $k$ which serves as a normalization constant is degree two generator in $U(1)$ equivariant cohomologies [14].

The universal $\delta$-exact action which exists for any Riemannian $M$ is:

\[
\begin{align*}
\beta_k &= \int dt g_{\mu\nu} \left( \psi^\mu \nabla_t \psi^\nu + k \partial_t x^\mu \partial_t x^\nu \right) \\
&\sim \delta \int dt \left( g_{\mu\nu} \psi^\mu \partial_t x^\nu \right)
\end{align*}
\]
where $\nabla_t$ is the pull-back of the Levi-Chivita connection on $TM$ to the circle. The advantage of having the fermions and symmetry $\delta$ is the possibility to use the localization principle for evaluation of the partition function. The fixed points of the group action are the constant loops. Thus, the partition function

$$\int DxD\psi \ e^{-\beta k}$$

(3.3)

can be expressed as the integral over the space of constant loops, i.e. $M$ itself. The integrand is given by the ratio of determinants one gets by expanding around the constant loop. It is well-known that the answer is the index of Dirac operator \[15\][\[16\]]. The partition function is formally independent of any $\delta$-exact terms one could add to the action as long as everything is invariant under the rotations of the circle (and consequently $\delta$ squares to zero).

Suppose $\omega = \frac{1}{2} \omega_{\mu\nu} dx^\mu \wedge dx^\nu$ is a closed two-form on a simply-connected manifold $M$ with integer periods. One can form another $\delta$-invariant action:

$$\alpha_k = \int_{S^1} dt (\frac{1}{2} \omega_{\mu\nu} \psi^\mu \psi^\nu + k \theta_\mu \partial_t x^\mu)$$

(3.4)

Here we introduced a one-form $\theta = \theta_\mu dx^\mu = d^{-1}\omega$. Of course, $\theta$ is only defined locally, but the equivariant form $e^{2\pi ip}\alpha$ (it is sometimes called Polyakov’s loop) is well-defined, provided $pk \in \mathbb{Z}$.

When (3.4) and (3.2) are taken together the answer for the partition function is the index of Dirac operator coupled to abelian gauge field, whose curvature is $2\pi i \omega$.

It is of interest of extending this formalism in two respects.\[4\] (we will need both): the action of a group $G$ on $M$ and the quantization of a submanifold $N$ of $M$. In the case of our interest the moduli space $\mathcal{M}$ is a quotient of a submanifold $N$ of a manifold $M$. Suppose that $N$ can be realized as a set of zeroes of a section $s = \{\Phi_\alpha\}$ of some vector bundle $V$ over $M$. To get a restriction onto submanifold $N$ one introduces a multiplet of Lagrange multipliers $H_\alpha$ and their superpartners $\chi_\alpha$ which are the sections of a pullback of the bundle $V^*$ to the loop. Let $B_\alpha^\beta = B_{\mu,\alpha}^\beta dx^\mu$ be a connection on $V$. We assume that

---

\[4\] The Dirac operator is the space-time interpretation of $\delta$

\[5\] For the extended supersymmetry the relevant construction was presented in \[17\], but here we need to treat the $\mathcal{N} = \frac{1}{2}$ version of the story.
V is endowed with a metric \( g^{\alpha \beta} \) and that \( B \) is compatible with it. Let \( \mathcal{F} = dB + B^2 \) be the curvature of \( B \). The supersymmetry \( \delta \) acts on \( \chi, H \) as follows:

\[
\begin{align*}
\delta \chi^\alpha &= H^\alpha - B^\beta_{\mu, \alpha} \psi^\mu \chi^\beta \\
\delta H^\alpha &= k \partial_t \chi^\alpha - B^\alpha_{\mu, \beta} (H^\beta \psi^\mu - \dot{x}^\mu \chi^\beta) - \frac{1}{2} \psi^\mu \psi^\nu (\mathcal{F}_{\mu \nu})^\alpha_\beta \chi^\beta
\end{align*}
\] (3.5)

Consider the following interactions:

\[
\gamma_k = \delta \left( i \int_{S^1} \chi^\alpha \Phi_\alpha \right) = i \int H^\alpha \Phi_\alpha + i \int \chi^\alpha \psi^\mu \nabla_\mu \Phi_\alpha,
\]

where \( \nabla_\mu \Phi_\alpha \) is:

\[
\nabla_\mu \Phi_\alpha = \frac{\partial \Phi_\alpha}{\partial x^\mu} + B^\beta_{\mu, \alpha} \Phi_\beta,
\]

and

\[
\delta_k = \delta \left( \int_{S^1} -\frac{1}{2} g_{\alpha \beta} \chi^\alpha H^\beta \right) = -\frac{1}{2} \int_{S^1} \left( H^\beta \chi^\alpha + \chi^\beta \mathcal{D}_t \chi^\beta + (\mathcal{F}_{\mu \nu})_{\alpha \beta} \psi^\mu \psi^\nu \chi^\beta \chi^\alpha \right),
\]

where the covariant derivative \( \mathcal{D}_t \) is defined with the help of the pullback of \( A \):

\[
\mathcal{D}_t \chi^\alpha = k \partial_t \chi^\alpha + B^\alpha_{\mu, \beta} \dot{x}^\mu \chi^\beta.
\]

The path integral

\[
\int Dx D\psi D\chi DH \exp(\beta_k + \gamma_k + e^2 \delta_k)
\]

reduces to the integral (3.3) for the submanifold \( N \), as can be seen by integrating out \( H \) and then taking the limit \( e^2 \rightarrow 0 \). The reason why the prescription with \( e^2 \neq 0 \) is better then \( e^2 = 0 \) is that it works even if the section \( s \) is not generic (e.g. \( s = 0 \)).

If the group \( G \) acts on \( M \) then \( LM \) is acted on by the group \( \mathfrak{g} = LG \rtimes U(1) \), where \( U(1) \) acts on \( LG \) by rotations of the loops. To avoid possible confusions let us stress that it is only \( LG \) which is being gauged, not \( \mathfrak{g} \). The appropriate setting is the equivariant cohomology of \( LM \) with respect to \( \mathfrak{g} \). The action of the group is incorporated by making \( \delta \) the equivariant derivative with respect to \( \mathfrak{g} \). One introduces a field \( \phi^a(t) \), which takes values in the Lie algebra \( \mathfrak{g} \) of \( G \). The field \( \phi^a \) may be thought of a one-dimensional gauge field. Then new operator \( \delta \) is the sum of (3.1) and the operator which maps \( \psi^\mu \) to \( \phi^a(t) V^\mu_a(x(t)) \) and \( H^\alpha \) to \( \phi^a(t) T^\alpha_{a, \beta} \chi^\beta \), where \( V^\mu_a \) is a vector field on \( M \), representing the Lie algebra element \( T_a \), and \( T^\alpha_{a, \beta} \) represents the action of \( G \) on the normal bundle to \( N \).
(Of course, for the whole construction to work the submanifold $N$ must be $G$-invariant). The regulator \((3.2)\) is gauged:

$$\beta_k = \delta \left( \int dt g_{\mu\nu} \psi^\mu D_t x^\nu \right)$$

$$= \int dt \left( g_{\mu\nu} \psi^\mu (\nabla_t \psi^\nu + \phi^a \partial_\xi V_\alpha^a \psi^\xi) + k g_{\mu\nu} D_t x^\mu D_t x^\nu \right)$$

$$D_t x^\mu = \partial_t x^\mu + \phi^a V_\alpha^a$$ \hfill (3.9)

The form \((3.4)\) changes as follows. If $G$ preserves $\omega$ and $\mu_a$ is the moment map, then \((3.4)\) is replaced by:

$$\alpha_k = \int_{S^1} dt \left( \frac{1}{2} \omega_{\mu\nu} \psi^\mu \psi^\nu + k \theta_\mu \partial_t x^\mu + \phi^a \mu_a \right)$$ \hfill (3.10)

3.2. The theory $K_D$: fields and supercharge

We want to generalize these constructions to cover infinite-dimensional cases. More precisely, we wish to consider as $M$ the space $A$ of gauge fields (not specifying the instanton sector) on space-time manifold $\mathcal{X}$, the group $G$ is the gauge group and the submanifold $N$ is the space of solutions of some natural (in particular, local in space-time) equations $\Phi_\alpha(F_A) = 0$, where $F_A$ is the curvature of the gauge field. Starting with a theory $\mathcal{H}_D$ in $D$ dimensions, defined by equations \((2.1)\) and \((2.3)\), we define a related interesting supersymmetric theory in $D+1$ dimensions by means of the following procedure, motivated by the discussion of supersymmetric quantum mechanics. The generator $Q$ below is simply the operator $\delta$ generalized to the infinite-dimensional setting. For simplicity we set $k = 1$. It can be recovered by the rescaling of the radius of the $t$ circle. We also set the connection $B$ to zero.

One considers the same fields as the previous section with a dependence on an additional coordinate $t = x^{D+1}$, e.g., $A(x) \rightarrow A(x, t)$. Moreover, one introduces an extra component $A_t(x, t)$ in $A_\mu(x, t)$, and an anticommuting component $\psi_t(x, t)$ in $\psi_\mu(x, t)$. The fermion $\psi_t$ is playing the rôle of $\eta$ (see below). Instead of complex field $\phi$ we have a real $g$-valued scalar $\varphi$.

The operator $Q$ \((2.1)\) is generalized into:

$$QA_\mu = \psi_\mu, \quad Q\psi_\mu = -F_{\mu t} - i D_\mu \varphi$$

$$Q\chi^\alpha = H^\alpha, \quad QH^\alpha = D_t \chi^\alpha + i[\varphi, \chi^\alpha]$$

$$Q\varphi = i \psi_t, \quad Q\psi_t = -i D_t \varphi$$ \hfill (3.11)
(the index \(\mu\) now runs from 1 to \(D+1\).)

The important property of \(Q\) is:

\[ Q^2 = \partial_t + \text{gauge transformation with parameter } A_t + i\varphi \quad (3.12) \]

One can improve (3.11) by introducing a scalar ghost \(c\) (interpreted as an ordinary Faddeev-Popov ghost), which gives rise to a modified operator \(Q\) squaring to \(\partial_t\) only. This is equivalent to doing the standard Weil complex procedure \[18\], \[7\].

We modify the transformation laws: \(Q \rightarrow s\):

\[
\begin{align*}
    sA_\mu & = \psi_\mu + D_\mu c, & s\varphi & = -[c, \varphi] + i\psi_t \\
    s\psi_\mu & = F_{t\mu} - iD_\mu \varphi - [c, \psi_\mu], & s\chi_\alpha & = H_\alpha - [c, \chi_\alpha] \\
    sH_\alpha & = D_t \chi_\alpha + i[\varphi, \chi_\alpha] - [c, H_\alpha] \\
    sc & = A_t + i\varphi - \frac{1}{2}[c, c]
\end{align*}
\]

(Notice that \(Q(A_t + i\varphi) = 0\) and \(s(A_t + i\varphi) = \partial_t c + [A_t + i\varphi, c]\)). As a result:

\[
    s^2 = \partial_t, \quad (3.14)
\]

and one can properly deal with the gauge fixing of the ordinary gauge symmetry of the \(Q\)-invariant action, by adding a \(s\)-exact term.\(^6\)

The equations (3.11) (3.13) break \(SO(D+1)\) invariance, as it is explicit in the transformation law of \(\psi_\mu\). We will come back to this point when discussing observables.

It is useful to establish the dictionary according to which the transformation \(s\) in (3.13) encodes the usual nilpotent topological BRST operator of the \(D\)-dimensional Yang-Mills theory.

For this, one can decompose the equations (3.13) as \((1 \leq i \leq D)\):

\[
\begin{align*}
    sA_i & = \psi_i + D_i c \\
    sc & = A_t + i\varphi - \frac{1}{2}[c, c] \\
    s\psi_i & = \partial_i A_i - D_i(A_t + i\varphi) - [c, \psi_i] \\
    s(A_t + i\varphi) & = \partial_t c - [c, A_t + i\varphi] \\
    s(A_t - i\varphi) & = 2\psi_t + \partial_t c - [c, A_t - i\varphi] \\
    s(2\psi_t + \partial_t c) & = -[c, 2\psi_t + \partial_t c] + [A_t + i\varphi, A_t - i\varphi] \\
    s\chi_\alpha & = H_\alpha - [c, \chi_\alpha] \\
    sH_\alpha & = D_t \chi_\alpha + i[\varphi, \chi_\alpha] - [c, H_\alpha]
\end{align*}
\]

\(^6\) For example, one may use Landau type gauges.
The relation between the symmetries in $D$ and $D+1$ dimensions is that $\phi$ and $\bar{\phi}$ are replaced respectively by $\partial_t + A_t + i\varphi$ and $\partial_t + A_t - i\varphi$. Moreover, by performing the standard dimensional reduction in which all fields become $t$ independent and all terms involving $\partial_t$ drop out one immediately sees that $\phi = A_t + i\varphi$ can be interpreted in $D$ dimension as the ghost of ghost for $c$, with ghost number 2, and $\bar{\phi} = A_t - i\varphi$ is the antighost for antighost with ghost number $-2$, while $\eta = 2\psi_t + \partial_t c$ is the Lagrange multiplier with ghost number $-1$ for the gauge condition on the vector ghost fields $\psi_i$. Hence as compared to $D$ dimensional theory there is a violation of ghost number in the $D+1$ dimensional theory. Obviously, in the process of dimensional reduction, one gets $\mathfrak{s}^2 = 0$.

The Lagrangian of the $D+1$ dimensional theory is a straightforward generalization of (2.3):

$$\mathcal{L} = \int_{\mathcal{X}^D \times \mathbb{R}^1} \{ Q, i\chi^\alpha (\Phi_\alpha + ie^2 g^{\alpha\beta} H_\beta) + \text{Tr} \psi^\mu (F_{t\mu} + iD_\mu \varphi) \}$$

(3.16)

3.3. The theory in $D+2$ dimensions: $\mathcal{E}ll_D$.

In turn, one can go one more dimension higher, that is to $D+2$, where the field $\varphi$ becomes the component $A_{D+2}$ of the $D+2$ dimensional gauge field $A$. Straightforward computations indicate that one obtains a theory with a supersymmetry charge satisfying:\n
$$\mathfrak{s}^2 = \partial_{D+1} + i\partial_{D+2} = \partial_z, \quad z = \frac{1}{2}(x^{D+1} + ix^{D+2})$$

(3.17)

The theory in $D+2$ dimensions is likely to be untwisted to an ordinary $\mathcal{N} = 1$ supersymmetric theory, provided one can arrange all anticommuting ghosts in a relevant spinorial representation space. One needs the equations $\Phi_\alpha$ to contain gauge fields only (no scalars). In this unifying theory, all reminders to ghost number assignments disappear. Summarizing, the theory containing scalars like $\phi$ or $\varphi$ can be pushed up in dimensions, making the scalars the remaining components of the gauge field. This is completely parallel to the way $T$-duality works for the theories on $D$-branes [19].

In the next section, we will apply these general statements to specific theories, and discuss the observables.

---

7 As it has been noticed in the discussions with G. Moore and S. Shatashvili there are difficulties with defining Chern-Simons like observables in this $D+2$ dimensional theory.
3.4. Examples: Octonionic theories

The nine dimensional theory As an explicit example, let us look at the 9-dimensional version of what was done in the previous section, that is, the case \( D = 8 \). In what follows, for the sake of notational clarity, greek indices \( \mu, \nu, \ldots \) run from 1 to 9, latin indices \( i, j, \ldots \) run from 1 to 8 and greek indices \( \alpha, \beta, \ldots \) run from 1 to 7.

The 8 dimensional theory relies on the seven independent constraints

\[
\Phi_\alpha = F_{8\alpha} - c_{\alpha\beta\gamma} F^{\beta\gamma}
\]

where the \( c_{\alpha\beta\gamma} \) are the structure coefficients of octonions \(^1\). Its field content is that of \( \mathcal{N} = 1, d = 10 \) theory dimensionally reduced down to nine dimensions. The fermions are \( \psi_\mu \) in 9 of \( SO(9) \) and \( \chi_\alpha \) in 7 of \( Spin(7) \subset SO(8) \subset SO(9) \) \(^1\).

Assuming that space-time nine-fold splits: \( M^9 = S^1 \times X^8 \), with \( X^8 \) being \( Spin(7) \) manifold\(^8\), the Lagrangian found by the standard procedure is:

\[
L = \{Q, \int \text{Tr} \left( i\chi^\alpha (F_{8\alpha} - c_{\alpha\beta\gamma} F^{\beta\gamma}) + \frac{i e^2}{2} H^\alpha \right) - \frac{1}{e^2} \psi_\mu (F_{\mu 9} - iD_\mu \varphi) \} \tag{3.19}
\]

where we omit the space-time metric. The expression of the action of \( Q \) is defined in (2.1).

This \( Q \) invariant action can be expanded after standard elimination of \( H \) as:

\[
L = -\frac{1}{e^2} \int \text{Tr} (|F_{8\alpha}|^2 + |F_{\alpha\beta}|^2 + |F_{\mu 9}|^2 + |D_\mu \varphi|^2 + \ldots) \tag{3.20}
\]

that is

\[
L = -\frac{1}{e^2} \int \text{Tr} (|F_{\mu\nu}|^2 + |D_\mu \varphi|^2 + \ldots) \tag{3.21}
\]

where \( \ldots \) stand for \( d \)-exact terms, topological terms and fermions.

The bosonic part of the action (3.20) is invariant under \( SO(9) \) rotations. One easily finds that by the dimensional reduction and the field redefinitions detailed in the previous section, that (3.21) can be dimensionally reduced to the 8 dimensional CohFT action built in \(^4\) (for the \( J \) case, corresponding to \( Spin(7) \) holonomy).

The link between this 9-dimensional theory and the 10-dimensional super Yang-Mills theory is quite transparent: \( |F_{\mu\nu}|^2 + |D_\mu \varphi|^2 \) is the dimensional reduction of the 10 dimensional Yang Mills action and the number of fermions in the 9 dimensional theory, that is, \( 8 \) fermions. The requirement that \( X^8 \) is \( Spin(7) \) manifold can be relaxed, see below
9 components for the $\psi_{\mu}$ and 7 components for the $\chi_{\alpha}$, counts the 16 independent components of the Majorana-Weyl spinor which is the $\mathcal{N} = 1$ supersymmetric partner of the Yang-Mills field in 10 dimensions.

Notice that the space of allowed eightfolds $\mathcal{X}^8$ is not bounded by Riemannian manifolds of $\text{Spin}(7)$ holonomy. One may replace the equations (3.18) by their deformations, which may involve other geometrical structures. The link to supersymmetric Yang-Mills theory is more involved in this case.

Seven dimensional theory. In [4] an eight dimensional CohFT is constructed using the gauge fixing of the topological invariant

$$\int_{\mathcal{X}^8} \Omega^4 \wedge \text{Tr} \, F \wedge F.$$  (3.22)

Therefore, a meaningful question is whether a seven dimensional QFT exists, which is directly defined from the Chern-Simons action associated to previous action

$$\int_{M^7} \Omega^4 \wedge \text{Tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A),$$  (3.23)

that is,

$$\int_{M^7} c_{\alpha\beta\gamma} \text{Tr}(A_\alpha \partial_\beta A_\gamma + \frac{2}{3} A_\alpha A_\beta A_\gamma).$$  (3.24)

The quantization of this action is however unclear, in contrast with the 3-dimensional case, for Gauss law in the $A_7 = 0$ gauge is not enough to consistently solve the theory.

It is therefore more adequate to consider a simpler action, assuming that $M^7 \sim S^1 \times \mathcal{X}^6$, with $\mathcal{X}^6$ being Calabi-Yau threefold. As expected, the theory will be related to the CohFT relying on balanced version of Donaldson-Uhlenbeck-Yau equations, which are dimensionally reduced octonionic instanton equations (see [4]). The effective six-dimensional theory turns out to be a six-dimensional version of gauged WZW model.

We formulate the theory for $\mathcal{X}^6$ Kähler threefold (not necessary Calabi-Yau). The local coordinates on $\mathcal{X}^6$ will be denoted as $z^i, \bar{z}^i$. The field content is the twisted version of the field content of dimensionally reduced 9-dimensional theory (3.19). In other words the bosonic fields are (0, 1) and (1, 0) forms $A_i, \bar{A}_i$ as gauge fields, (3, 0) and (0, 3) forms $\phi, \bar{\phi}$ as Higgs fields in the adjoint representation and the time-like component $A_t$ of the gauge field and its friend real scalar $\varphi$. The fermions are: (0, 0) forms $\chi_0, \psi_t \equiv \gamma$, (2, 0) and (0, 2) forms $\chi_{ij} dz^i \wedge dz^j$ and its conjugate, the (3, 0) and (0, 3) forms $\psi_{\phi}$ and its conjugate and (1, 0) and (0, 1) forms $\psi_t dz^t$ and its conjugate. This collection of fields is suitable for posing the moduli problem [4].
The six-dimensional equations are:

\[ F^{0,2} = \bar{\partial} \bar{A} \phi, \quad \omega^2 \wedge F^{1,1} \equiv (g^{i\bar{i}F_{i\bar{i}}}) \omega^3 = [\phi, \bar{\phi}], \]  

(3.25)

where \( \omega = \omega_{ij} dz^i \wedge d\bar{z}^j \) is the Kähler form on a six-fold \( X^6 \). The Lagrangian found by the procedure described above is:

\[
L = \{ Q \int \text{Tr} \left( \chi_0 (\omega^2 \wedge F + [\phi, \bar{\phi}] + \frac{ie^2}{2} H_0) \\
+ \chi_{ij} \left( \frac{ie^2}{2} H_{ij} - F_{ij} - \epsilon_{ijk} D_k \phi \right) + \text{c.c.} + \\
+ \left( \psi_\phi [D_t - i\varphi, \bar{\phi}] + \bar{\psi}_\phi [D_t - i\varphi, \phi] \right) + \psi^\mu (F_{t\mu} + iD_{t\mu} \varphi) \} \}
\]

(3.26)

where \( \mu = 1, \ldots, 7, \ i, j, k = 1, 2, 3 \), and we omit the metric \( g_{i\bar{k}} \) in all our formulæ. The action of \( Q \) on \( \phi, \bar{\phi} \) is obtained by dimensional reduction:

\[
Q\phi = \psi_\phi \quad Q\bar{\phi} = D_t \phi + i[\varphi, \phi] \\
Q\bar{\phi} = \bar{\psi}_\phi \quad Q\psi_\phi = D_t \bar{\phi} - i[\varphi, \bar{\phi}]
\]

(3.27)

After expansion of \( \{ Q, \ldots \} \) the action \((3.26)\) is similar in form to \((3.21)\).

4. Observables

4.1. Observables in \( H_D \) theory

One may construct the observables in the theory by means of the descend procedure: start with the operator \( \mathcal{O}_k^0 = \text{Tr} \phi^k \) and compute its exterior derivative \( d\mathcal{O}_0 \). It is \( Q \) of a one-form valued operator called \( \mathcal{S}_k^1 \). The integral of \( \mathcal{S}_k^1 \) along a closed curve \( C \) is therefore a \( Q \)-closed observable \( \mathcal{O}_k^1 \). Take \( d \) of \( \mathcal{S}_k^1 \) and so on:

\[
d\mathcal{S}_k^l = \{ Q, \mathcal{S}_k^{l+1} \}
\]

One gets in this way a chain of observables \( \mathcal{O}_k^p \):

\[
\mathcal{O}_k^p = \int_{C_p} \mathcal{S}_k^p
\]

where \( C_p \) is a \( p \)-cycle and a map

\[
\mu_k : H_*(X^D; \mathbb{R}) \to H^{2k-*}(\mathcal{M}; \mathbb{R})
\]

(4.1)

which in the field theoretic language is our desired map \( \mathcal{O}_i \mapsto \omega_i \). The form \( \omega_i \) represents a cohomology class of the moduli space \( \mathcal{M} = \Pi_n \mathcal{M}_n \). In this paper we ignore the subtleties associated to intersections of cycles \( C^l \), over which \( \mathcal{S}_k^l \) are integrated.
4.2. Observables in $K_D$ theory

One can ask whether there are in $D+1$ dimensions operators which represent non-trivial classes of $Q$-cohomology. The answer to this question is positive. This is a generalization of the 3 and 5 dimensional cases studied in [3]. Recall the property (3.17) that the supercharge $Q$ squares to the combination of the $t$ translation and the gauge transformation with parameter $A_t + i\varphi$. There are two types of observables in the theory.

The first type observables are obtained by the standard descend procedure applied to the gauge invariant zero-observables:

$$\mathcal{O}_k^0 = Tr g^k, \quad g = P \exp \oint (A_t + i\varphi) dt$$

which are the analogues of “vertical” Wilson loops.

The observables of second type contain Chern-Simons terms: start with

$$L_{CS} = \int_{M^{D+1}} T^{D-2} \wedge Tr(A \wedge dA + \frac{2}{3} A \wedge A \wedge A)$$

where $T^{D-2}$ is a closed $D-2$-form coming from $X^D$, i.e. it is a pullback of a form on $X^D$ with respect to the projection $M^{D+1} \to X^D$. The following operator turns out to be $Q$ invariant:

$$L_{CS,1} = \int_{M^{D+1}} T^{D-2} \wedge (CS_3(A) + 2Tr(i\varphi F + \psi \psi) \wedge dt)$$

This operator is actually a cousin of the term (3.4) in the SQM. As opposed to the $Q$ exact Lagrangian (3.11) which is the analogue of (3.2), $L_{CS}$ is not $SO(D+1)$ invariant. Rather it is only $SO(D)$ invariant, as it involves a form $T^{D-2}$ on $X^D$.

More generally, given a closed $D-2p$-form on $X^D$ one may construct the observable of the following type:

$$L_{CS,p} = \int_{M^{D+1}} T^{D-2p} \wedge \left( CS_{2p+1}(A) + (p+1)Tr(i\varphi F^p + \sum_{l=0}^{p-1} \psi F^{l} \psi F^{p-1-l}) \wedge dt \right)$$

with $CS_{2p+1}(A)$ being the standard Chern-Simons $2p+1$-form:

$$CS_3(A) = Tr \left( AdA + \frac{2}{3} A^3 \right)$$

$$CS_5(A) = Tr \left( A(dA)^2 + \frac{3}{2} A^3 dA + \frac{3}{5} A^5 \right)$$

$$CS_{2p+1}(A) = (p+1) \int_0^1 s^p ds Tr \left( A(dA + sA^2)^p \right)$$

All these operators share the property of being only $SO(D)$ invariant.
4.3. Quantization of $T^{D-2p}$.

The forms $T^{D-2p}$ are the higher dimensional analogues of the level $k$ in the three dimensional Chern-Simons theory. One expects an analogue of the quantization condition of $k$, like the quantization of Kahler form $\omega$ in [20]. Indeed, the presence of the term $T^{D-2p} \wedge CS_{2p+1}(A)$ implies that in order to preserve gauge invariance the forms $T^{D-2p}$ must represent integral cohomology classes of $\mathcal{X}^D$:

$$[T^{D-2p}] \in H^{D-2p}(\mathcal{X}^D; (2\pi i)^{p+2}\mathbb{Z})$$  \hspace{1cm} (4.7)

and the operators (4.5), (4.6) make sense only when they are in the exponential.

So, the actual observables of second type are:

$$O_p(T^{D-2p}) = \exp(\mathcal{L}_{CS,p}).$$  \hspace{1cm} (4.8)

4.4. Flow to “genuine” Chern-Simons theory

In $D+1$ dimension we systematically consider an action which is the sum of a Chern-Simons like action as in (4.4) and a $Q$ exact action as in (3.19). This leads to a well defined QFT. However one may wonder about the relation of this theory to a genuine Chern-Simons theory, without supersymmetric terms.

Getting rid of $\varphi$.

It is possible to map the observable (4.8) for $p = 1$ to more conventional Chern-Simons like action. The idea is the following. Suppose that the representative of $T^{D-2}$ is such that

$$T^{D-2} \wedge F \wedge dt = 0$$  \hspace{1cm} (4.9)

is actually one of the constraints, say $\Phi_r$. Then consider adding to the Lagrangian (3.16) the term

$$\Delta \mathcal{L} = \kappa \{Q, \int \text{Tr} (\chi_r \varphi) \text{vol}_g \}$$  \hspace{1cm} (4.10)

where $\kappa$ is an arbitrary coefficient. Since the constraint $\Phi_r$ is indirectly imposed by varying $A_t$ it may seem that taking the limit $\kappa \to \infty$ does not change the behavior of the theory. The advantage is that in the limit $\kappa = \infty$ we can forget about other terms in (3.16) where $\varphi$ appears and integrate out $\varphi, \psi_t, \chi_r$ and $H_r$ altogether. This argument is very similar to the explanation of the relation between the physical and topological Yang-Mills theories in two dimensions, proposed in [21]. In this way we end up with the Chern-Simons theory
together with second class constraints $\Phi_\alpha = 0, \alpha \neq r$. In fact, the constraints are imposed in the $e^2 \to 0$ limit.

For example, in dimensions $D = 2, 4, 8$ the equations can be written as follows: $\Phi_\alpha = F_{D\alpha} - \xi_{\alpha\beta\gamma} F^{\beta\gamma}$ where:

$$
\begin{align*}
D = 2 & \quad \xi_{\alpha\beta\gamma} = 0 \\
D = 4 & \quad \xi_{\alpha\beta\gamma} = \epsilon_{\alpha\beta\gamma} \\
D = 8 & \quad \xi_{\alpha\beta\gamma} = c_{\alpha\beta\gamma}
\end{align*}
$$

(4.11)

In this cases $r = D - 1$.

**Strong coupling limit.** Now let us formally study the opposite, strong coupling limit $e^2 \to \infty$. The following manipulations are not justified without proper regularization. They may serve as useful indications but not as a proof that one may be left with the Chern-Simons theory alone. Indeed, if the term $\{Q, \chi^\alpha \Phi_\alpha\}$ in (3.16) can be neglected then we end up with the action (provided that $\Phi_r = T^{D-2} \wedge F \wedge dt$):

$$
\int d^{D+1} x \left( \chi_r \cdot \phi + \frac{ie^2}{2} \sum_{\alpha \neq r} \chi^\alpha H^\alpha \right)
$$

(4.12)

where we introduce the multiplet $(\bar{c}, \lambda)$ of Faddeev-Popov anti-ghost and Lagrange multiplier for the gauge $A_t^\perp = 0$, where $A_t^\perp$ is the projection of $A_t$ onto the complement to the Cartan subalgebra plus the projection of the abelian part of $A_t$ onto the space of non-zero (in $t$) modes. The supercharge $\mathfrak{s}$ acts on them as follows:

$$
\mathfrak{s}\bar{c} = \lambda - [c, \bar{c}], \quad Q\lambda = D_t \bar{c} + i[\varphi, \bar{c}] - [c, \lambda]
$$

Expanding (4.12) one formally eliminates all the fields except $\chi_\alpha, \alpha \neq r, A_i, i = 1, \ldots, D,$ $\bar{c}$ and $c$ which have the action:

$$
\int dt \wedge (T^{D-2} \wedge \text{Tr}(A \wedge D_t A) + \text{Tr}(\chi^\alpha \star D_t \chi^\alpha) + \text{Tr} \bar{c} \star D_t c)
$$

(4.13)

with $D_t = \partial_t + [A_t^{(0)}, \cdot], A_t^{(0)}$ being the constant in $t$ Cartan-valued matrix. The Hodge star $\star$ is taken in $D$ dimensions. Formally the determinants cancel in topologically trivial backgrounds. Of course, non-trivial gauge backgrounds induce effective action and then it may be not possible to neglect the $\frac{1}{e^2} F_{\mu\nu}^2$ terms.
4.5. Discussion of breaking the SO(D + 1) Lorentz invariance

One may complain about the breaking of Lorentz invariance by the operators (4.5). In fact, as long as \(2p < D\), the operator (4.5) is an observable, moreover the form \(T^{D-2p}\) can be taken supported at some submanifold \(\Sigma^{2p}\). As such, it is not illegal for the observable to violate \(D + 1\) dimensional Lorentz invariance, since the choice of \(\Sigma^{2p}\) already breaks it. What is a little bit surprising is that the choice of supercharge \(s\) or \(Q\) which preserves the observable violates \(2p + 1\) dimensional Lorentz invariance, as it requires the choice of \(t\) direction. On the other hand, if \(2p = D = 4\) then the Chern-Simons part of the operator (4.5) is being integrated over whole 4 + 1 -dimensional space-time. It is possible to check that when the \(Q\)-exact regulators are included added we end up with Lorentz invariant action in \(\mathbb{R}^{4+1}\) corresponding to five dimensional super-Yang-Mills with running coupling \(\frac{1}{g^2} \sim \varphi\) discussed in [22]. Note that a proposal that infrared fixed points in five dimensional supersymmetric gauge theories are described by Chern-Simons theories has been made in [23].

In fact, all this puts the constructions of [24][25] discussing five-dimensional Chern-Simons field theories in a more solid context.

5. More examples

Since pure topological Yang Mills theories exist in 4 and 2 dimensions, we can study the associated five and three dimensional theories.

5.1. Five dimensional theory

This theory was studied in [20][3]. It has the field content of the partially twisted \(\mathcal{N} = 1\ d = 6\) theory, dimensionally reduced down to five dimensions.

5.2. Three dimensional theory and Verlinde formula

The interesting property of this theory is that it has the field content of the partially twisted \(\mathcal{N} = 1\ d = 4\) theory, dimensionally reduced down to three dimensions. The expectation value of the observable containing the three dimensional Chern-Simons action is the Verlinde formula for the number of conformal blocks in two dimensional WZW theory [3]. In this case the trick (4.10) with eliminating \(\varphi\) allows one to get rid of all fields except \(\mathcal{A}\) and establishes the equivalence to the ordinary three dimensional Chern-Simons theory.
5.3. Theories with matter

The theories in five and three dimensions which have the field content of \( \mathcal{N} = 1 \) with \( d = 6 \) and/or \( d = 4 \) super-Yang-Mills can be coupled to matter multiplets. In this way one gets a deformation of Higgs branches of theories by including one extra dimension compactified on a circle. The simplest example is the theory in \( D = 2 \) describing gauged linear sigma model. Its Higgs branch (more precisely, its effective low energy target space) \( \mathcal{M}_H \) must be a Kähler manifold due to supersymmetry. By using our trick of going one dimension higher we obtained a deformed theory, where certain correlation functions contain insertions of expansion \( \hat{A}(\mathcal{M}_H) \). Notice that the first non-trivial operator in the expansion of \( A \)-genus has dimension four. It implies that effectively the target space \( \mathcal{M}_H \) is changed in codimension four. It is very tempting to speculate that this codimension four change is related to recent proposals of incorporating stringy interactions in Matrix theory \([26][27][28]\).

6. Application: new theory in \( D \) dimensions

Once we have constructed a theory in \( D + 1 \) dimensions with supercharge, squaring to the translation in the \( D + 1 \)'st direction we may consider its compactifications.

The dimensional reduction gives us back the theory in \( D \) dimensions we started from, but compactification on a circle produces a deformation of the original theory, the radius \( R \) being the parameter of the deformation. As a guiding example let us start in \( D = 2 \). The theory describing moduli space of flat connections is 2\( d \) topological Yang-Mills theory. The corresponding theory in \( 2 + 1 \) dimensions is going to be twisted 3\( d \) SYM with the Chern-Simons action as an observable. The compactification on a circle of the latter produces gauged \( G/G \) WZW theory \([29][30]\), which coincides with 2\( d \) YM in the large \( k \) limit. The compactification on an interval gives rise to the WZW model itself. See \([31]\) for the similar construction in the bosonic \( D = 4 \) case and \([32][33][34][35][36][37]\) for bosonic \( D = 2 \) case.

The same procedure works in higher dimensions. One has to be careful, as the compactification works nicely only for the modes, annihilated by the supercharge \( s \). In particular, all constraints \( \Phi_\alpha \) but one must be imposed. In addition, one has to solve the equation

\[
F_{\mu t} + iD_\mu \varphi = 0
\]

which states that

\[
A|_{t=\tau} = (A|_{t=0})^{g(\tau)} \tag{6.1}
\]
where
\[ g(\tau) = P \exp \int_0^\tau (A_t + i\varphi)dt \] (6.2)

The \( CS_3 \) type observable gives rise to the \( WZW_2 \) like action. The observables corresponding to \( CS_5 \) and so on produce the generalized \( WZW \)-like actions, computed in [38].

One can also see these relations in the canonical approach. Let us consider for simplicity the case \( D = 8 \) with equations \( \Phi_\alpha = 0 \) being the complexified instanton equations (case \( H \) in the terminology of [4]). Then \( X \) is Kähler manifold. Let \( T^6 = \omega^3, \omega \) being the Kähler form. In the weak coupling limit (after eliminating \( \varphi, \chi_1 \) and \( H_1, \psi_t \)) the effective space \( B \) of fields is the space of solutions to equations \( \Phi_\alpha = 0, \alpha > 1 \). Suppose that we have inserted an operator \( O_1(T^6) \). The form \( T^6 \) defines a symplectic form on the space of gauge fields on \( X^D \):
\[ \Omega = \int_{X^8} T^6 \wedge \text{Tr} \delta A \wedge \delta A \] (6.3)
which is invariant under the gauge group action. Now we may attempt to quantize the symplectic quotient of \( B \) by the action of gauge group. The Gauss law:
\[ T^6 \wedge F = 0 \]
translates to the following property of the wavefunctional (we work in holomorphic polarization):
\[ \Psi(\bar{A}^g) = \exp \left( \int_{X^8} T^6 \wedge (S_{WZW_2}(g) + \text{Tr}(\bar{A}g^{-1}\partial g)) \right) \Psi(\bar{A}) \] (6.4)
where \( \bar{A} \) is the \((0,1)\) part of the gauge field. It must belong to \( B \), i.e. to obey \((0,2)\) part of the complexified instanton equations. One may find a formal solution to (6.4) as a path integral in \( WZW_8 \) theory in the background gauge field \( \bar{A} \) (this is completely parallel to [30][20][38]):
\[ \Psi(\bar{A}) \sim \int Dg \exp \left( \int_{X^8} T^6 \wedge (S_{WZW_2}(g) + \text{Tr}(\bar{A}g^{-1}\partial g)) \right) \] (6.5)

The condition \( \bar{A} \in B \) which must be imposed on the wave function by hand looks a little bit unnatural. It is possible to represent this condition in a different way, which makes the interesting use of the supersymmetric quantum mechanics which we described earlier.
Consider $D = 6$ case with the equations (3.25). Let us again separate them as $\Phi^{0,2} = F^{0,2} - \delta_A \bar{\phi}$ and "Gauss law" $\mu = \omega^2 \wedge F^{1,1} - [\phi, \bar{\phi}]$. Consider the canonical quantization of the theory with the Lagrangian (3.26). The bosonic fields in the theory form a configuration space $\mathcal{A}$ which is the space of gauge fields $A$ times the space of $(3,0)$ $\mathcal{G}$-valued forms $\phi$. The space $\mathcal{A}$ is an infinite-dimensional Kähler manifold, the Kähler form being

$$\Omega = \int_X \omega^2 \wedge \text{Tr} \delta A \wedge \delta \bar{A} + \text{Tr} \delta \phi \wedge \delta \bar{\phi}$$

(6.6)

It is preserved by the gauge group action and $\mu$ is the corresponding moment map. The fermionic kinetic term is roughly

$$\text{Tr} \left( \psi_i D_t \psi_i + \chi_{ij} D_t \chi_{ij} + \eta D_t \eta + \chi \eta \right)$$

(6.7)

where the metric is implicit. This kinetic term suggests that the wave functional can be factorized as follows:

$$\Psi = \Psi_0(A, \bar{A}, \phi, \bar{\phi}, \chi_{ij}, \psi_i) \otimes v$$

(6.8)

where $v$ is a vector in the two dimensional Hilbert space obtained by quantizing $\chi_0, \eta$ system. The piece $\Psi_0$ is naturally a section of $\Lambda^{0,*}(\mathcal{A}) \otimes \Lambda^* \mathcal{E}$ where $\mathcal{E}$ is the complex vector bundle over $\mathcal{A}$ whose fiber over $(A, \phi)$ is the space $\Omega^{0,2}$. The supercharge $\mathbf{s}$ can be represented as the equivariant version of the infinite-dimensional operator

$$\bar{\partial} + \bar{\partial}^\dagger + \delta + \delta^\dagger$$

(6.9)

where $\delta$ is the Koszul operator (see, [39]), which maps $\Gamma(\Lambda^p \mathcal{E}) \rightarrow \Gamma(\Lambda^{p+1} \mathcal{E})$ by exterior multiplication by $\Phi^{0,2} \in \Gamma(\mathcal{E})$.

The kernel of $\mathbf{s}$ is identified with the equivariant cohomology of the operator $\bar{\partial} + \delta + \varphi \nu$, where $\nu$ represents the complexified gauge group action. By the standard spectral sequence techniques one may first compute the cohomology of $\delta$ which is roughly speaking equivalent to imposing the $\Phi^{0,2} = 0$ constraint. Due to the infinite-dimensionality of the problem it seems more adequate to work with Koszul operator rather then with imposed non-linear equations $\Phi^{0,2} = 0$. The last remark concerns the role of $\mu$. It is known that taking the quotient with respect to the complexified group in the sense of theory of invariants is equivalent to imposing $\mu = 0$ constraint first and then taking the quotient with respect to the compact group. Cohomology-wise there is a surjective map from the equivariant cohomology of $\mathcal{A}$ to the cohomology of the symplectic quotient $\mathcal{A}//\mathcal{G}$ [40]. This explains why we need not include $\mu$ in the Koszul complex [41].

The analogous construction can be presented in other cases considered in this paper as well.
7. Discussion and conclusions

We have constructed twisted supersymmetric theories in higher dimensions. In fact, we only described one relevant supercharge $Q$. It is clear, though, that one may write down the superalgebra involving all supercharges. Of course, on a curved background $S^1 \times \mathcal{X}^D$ only one supercharge is conserved.

The interesting property of this supersymmetry is that it allows to construct $Q$-invariant observables of Chern-Simons type. Pure Chern-Simons theory is not very well defined and needs a regularization. The supersymmetric gauge theory provides such a regularization. The case of three dimensional theory is exceptional in the sense that one can get rid of all the regularizing fields by going to well justified strong coupling limit (see [35] for the formal derivation of the coupled Chern-Simons Yang-Mills system in three dimensions). This is not so in higher dimensional cases where the extra constraints are important.

Three dimensional Chern-Simons theory induces two dimensional WZW model. It turns out that a similar statement holds in higher dimensional case provided that one works in $Q$-cohomology. This may provide a further justification of higher dimensional WZW theories [20] [38].

The theories which we have studied in our paper may be of some relevance in the context of theories on $D$-branes. Indeed, Chern-Simons-like couplings to $RR$ fields are known to be present in the effective actions on the worldvolumes of $D$-branes [12]. Also, Chern-Simons terms appear on the worldvolumes of euclidean $D2$ branes, wrapping supersymmetric three-cycles [13].

Finally, the process of going one dimension up is similar to the action of $T$-duality on the worldvolume theory of a transverse $D$-brane.

As our paper was ready for publication we have learned about the recent preprint [44] which also drew attention to seven dimensional theory with Chern-Simons action.

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