The limit shape of large alternating sign matrices

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Abstract. The problem of the limit shape of large alternating sign matrices (ASMs) is addressed by studying the emptiness formation probability (EFP) in the domain-wall six-vertex model. Assuming that the limit shape arises in correspondence to the ‘condensation’ of almost all solutions of the saddle-point equations for certain multiple integral representation for EFP, the limit shape of large ASMs is found. The case of 3-enumerated ASMs is also considered.

1. Introduction

An alternating sign matrix (ASM) is a matrix of 1’s, 0’s and −1’s such that in each row and column, all nonzero entries alternate in sign, and the first and the last nonzero entries are 1’s. In a weighted enumeration, or q-enumeration, ASMs are counted with a weight $q^k$, where $k$ is the total number of −1’s in each matrix. There are many nice results concerning ASMs, mainly devoted to their various enumerations; for a review see, e.g., book [1].

In this paper, we address the problem of the limit shape of large ASMs. The problem comes out from the fact that in ASMs their corner regions mostly contain 0’s while in the interior there are many nonzero entries. As the size of ASMs increases, the probability of finding entries equal to 1 (or −1) in the corner regions vanishes, while in the central region such probability remains finite. When considering very large ASMs in an appropriate scaling limit, (e.g., when large matrices are scaled to a unit square), it is common knowledge that such regions are sharply separated by a smooth curve.

In essence, such curve is an Arctic curve for ASMs, similar to the Arctic Circle of domino tilings of large Aztec diamonds [2]. Since the Arctic curve determines the shape of the internal, or “temperate”, region of ASMs, we call it here the limit shape of ASMs (cf. [3]). More generally, one can address this problem for q-enumerated ASMs. The case $q = 2$ corresponds to the Arctic Circle of the domino tilings. Here we obtain explicit expressions for the limit shapes of 1- and 3-enumerated ASMs.

To treat the problem, we exploit the one-to-one correspondence between ASMs and configurations of the six-vertex model, which has been found and efficiently used in papers [4–6]. This correspondence takes place when the six-vertex model is considered with the so-called domain wall boundary conditions (DWBC). The six-vertex model with these boundary conditions has been originally introduced and studied in [7–9].
Using the standard description of local states in terms of arrows (see, e.g., [10]), DWBC mean that the model is considered on a square lattice of $N$ vertical and $N$ horizontal lines, where the arrows on all external horizontal edges point outward, while the arrows on all external vertical edges point inward. Then, the 0’s in ASM’s entries correspond to the vertices of weights $a$ and $b$, while both 1’s or $-1$’s entries correspond to vertices of weight $c$, as shown on Figure 1. Figure 2 shows a possible configuration of the six-vertex model DWBC and the corresponding ASM.

To count ASMs, the weights $a$, $b$, $c$ are to be put all to the same value, e.g., $a = b = c = 1$. In $q$-enumeration, ASMs are counted by taking $a = b = 1$ and $c = \sqrt{q}$, since in each configuration the vertices of type five and six come in pairs, in addition to $N$ vertices of type six being always present due to DWBC.

The main tool which allows us to study limit shapes of the domain wall six-vertex model is a particular, non-local, correlation function, the so-called emptiness formation probability (EFP). In paper [11] some multiple integral representation for EFP has been derived. Here, we study the multiple integral representation for large size of the lattice (corresponding to large dimension of ASMs). Assuming that the limit shape occurs in correspondence to the peculiar situation where almost all roots of the saddle-point equations condense to the same value, we derive the Arctic curves in the two cases of values of weights matching 1- and 3-enumeration of ASMs. This ‘condensation hypothesis’ holds rigorously in the free-fermion case, considered in [12], where it allowed us to give an alternative derivation of the Arctic Circle (the limit shape of 2-enumerated ASMs).

Our paper is organized as follows. In the next Section we recall results of [11] on multiple integral representations. In Section 3 the condensation hypothesis is discussed, and a procedure for the derivation of limit shapes is explained. In Section 4, we derive the limit shape of 1- and 3-enumerated ASMs.
2. Emptiness formation probability

As in [11, 12], we call emptiness formation probability (EFP) and denote by $F^{(r,s)}_N$, where $r, s = 1, \ldots, N$, the probability of having all arrows on the first $s$ horizontal edges from the top of the lattice, located between $r$-th and $(r+1)$-th vertical lines (counted from the right), to be all pointing left.

With this definition, EFP measures the probability that all vertices in the top-left $(N - r) \times s$ sublattice have the same configuration of arrows, namely, with all arrows pointing left or downwards, or, equivalently, that these vertices are of type two (see Figure 1). This follows from the peculiarity of both the domain-wall boundary conditions and the six-vertex model rule of two incoming and two outgoing arrows at each lattice vertex (known also as the ‘ice-rule’).

In the language of ASMs the indicated property implies that all entries in the top-left $(N - r) \times s$ block are 0’s. The limit shape then arises in some appropriate scaling limit, and corresponds to some curve where EFP jumps from one to zero as the size of this $(N - r) \times s$ block increases. It is to be mentioned also that the study of the behaviour of EFP provides only the top-left ‘quarter’ of the ASM’s limit shape; the remaining three quarters follow from simple symmetry considerations.

In [11] several equivalent representations for EFP were given. For what follows we shall need two representations in terms of multiple integrals. To recall these formulae, we introduce some objects first.

An important role in our considerations is played by the function $h_N(z) = h_N(z; \Delta, t)$, where $\Delta$ and $t$ are to be regarded as parameters,

$$\Delta := \frac{a^2 + b^2 - c^2}{2ab}, \quad t := \frac{b}{a},$$

while $z$ is to be treated as a variable. This function, defined as a generating function, is a polynomial of degree $(N - 1)$ in $z$,

$$h_N(z) := \sum_{r=1}^{N} H^{(r)}_N z^{r-1}, \quad h_N(1) = 1.$$

Here the quantity $H^{(r)}_N = H^{(r)}_N(\Delta, t)$ is a boundary correlation function, introduced in [13]. Namely, it is the probability that the sole vertex of type six (having weight $c$), residing in the first row, appears at $r$-th position from the right. At $t = 1$ the function $h_N(z)$ has a special meaning as the generating function for refined $q$-enumerations of ASMs, with $q$ and $\Delta$ related by $\Delta = 1 - q/2$. Our derivation of the limit shapes below involves essentially the known explicit expressions for this generating function at some particular values of $q$.

For $s = 1, \ldots, N$, we define functions

$$h_{N,s}(z_1, \ldots, z_s) = \prod_{1 \leq j < k \leq s} (z_j - z_k)^{-1} \det_{1 \leq j, k \leq s} \left[ z_j^{k-1}(z_j - 1)^s - k h_{N-k+1}(z_j) \right].$$

These functions can be regarded as multi-variable generalizations of $h_N(z)$, in the sense that $h_{N,1}(z) = h_N(z)$. It can be easily checked that

$$h_{N,s+1}(z_1, \ldots, z_s, 1) = h_{N,s}(z_1, \ldots, z_s). \quad (1)$$

One also has

$$h_{N,s+1}(z_1, \ldots, z_s, 0) = h_N(0) h_{N-1,s}(z_1, \ldots, z_s). \quad (2)$$

Properties (1) and (2) are used in what follows.
We are now ready to turn to the multiple integral representations. In [11], the following multiple integral representation has been obtained

$$F_{N}^{(r,s)} = \frac{(-1)^s}{s!(2\pi i)^s} \oint_{C_0} \cdots \oint_{C_0} \prod_{j=1}^{s} \left[ \frac{(t^2 - 2t\Delta)z_j + 1}{z_j^j (z_j - 1)^{s-j+1}} \right] \times \prod_{1 \leq j < k \leq s} \frac{z_j - z_k}{t^2 z_j z_k - 2t\Delta z_j + 1} \times h_{N,s}(z_1, \ldots, z_s) \, dz_1 \cdots dz_s. \quad (3)$$

Here $C_0$ denotes some simple anticlockwise oriented contour surrounding the point $z = 0$ and no other singularity of the integrand. Formula (3) has been derived in [8, 9, 13]. This formula has also been discussed in [12] (see equation (3.6) therein).

In [11], the following equivalent representation has been also given:

$$F_{N}^{(r,s)} = \frac{(-1)^{s(s+1)/2} Z_s}{s!(2\pi i)^s a^{s-1} c^s} \oint_{C_0} \cdots \oint_{C_0} \prod_{j=1}^{s} \left[ \frac{(t^2 - 2t\Delta)z_j + 1}{z_j^j (z_j - 1)^{s-j}} \right] \times \prod_{j,k=1}^{s} \frac{1}{t^2 z_j z_k - 2t\Delta z_j + 1} \prod_{1 \leq j < k \leq s} (z_k - z_j)^2 \times h_{N,s}(z_1, \ldots, z_s)h_{s,s}(u_1, \ldots, u_s) \, dz_1 \cdots dz_s. \quad (4)$$

Here $Z_s$ denotes the partition function of the six-vertex model with domain wall boundary conditions on $s \times s$ lattice, and

$$u_j := -\frac{z_j - 1}{(t^2 - 2t\Delta)z_j + 1}. \quad (5)$$

For later use we emphasise that the integrand of (4) is just the symmetrized version of the integrand of (3), with respect to permutations of the integration variables $z_1, \ldots, z_s$. This follows through the symmetrization procedure explained in [16], and some additional identity proven in [11].

We are interested in the behaviour of EFP in the so-called scaling limit, that is in the limit where $r$, $s$ and $N$ are all large, with the ratios $r/N$ and $s/N$ kept finite (and smaller than 1). By applying standard arguments of saddle-point analysis to representation (4), we obtain the following system of coupled saddle-point equations

$$-\frac{s}{z_j - 1} + \frac{r}{z_j} + \sum_{k=1}^{s} \frac{2}{z_j - z_k} + \sum_{k=1}^{s} \left( \frac{2\Delta t - t^2 z_{jk}}{t^2 z_j z_k - 2t\Delta z_j + 1} - \frac{t^2 z_k}{t^2 z_j z_k - 2t\Delta z_k + 1} \right) + \frac{\partial}{\partial z_j} \ln h_{N,s}(z_1, \ldots, z_s)$$

$$-\frac{t^2 - 2\Delta t + 1}{[(t^2 - 2\Delta t)z_j + 1]^2} \frac{\partial}{\partial u_j} \ln h_{s,s}(u_1, \ldots, u_s) = 0. \quad (6)$$

In deriving these equations we have used the fact that quantities like $\ln h_{N,s}$ are of order $s^2$, and that their derivatives with respect to $z_j$’s are of order $s$; all sub-leading contributions (estimated as $o(s)$) are neglected.
3. Condensation hypothesis and ‘reduced’ saddle-point equation

To address the problem of limit shapes, one need, in principle, to be able to describe solutions of saddle point equations (6). This task however is a formidable one, at least because the last two terms in (6) are rather implicit in general. Nevertheless, it turns out possible to outflank the obstacle provided that some hypothesis holds true.

In [12], it has been shown that in the case of $\Delta = 0$ the limit shape corresponds to the situation where almost all roots of (6), as $s \to \infty$, condense at point $z = 1$. More precisely, the condensation has been assumed first, and next, after obtaining the limit shape, the validity of this assumption has been confirmed. Here we conjecture that the same correspondence of limit shape and condensation of roots can be put forward also in a more general situation, since the main ingredients appear not to be specific to the $\Delta = 0$ case. The only difference is that now such assumption cannot be verified a posteriori (at least by simple methods), so we call it here ‘condensation hypothesis’.

When assuming condensation, in [12] we have relied on two observations which can be easily extended beyond the free fermion case. The first one, based on the physical interpretation of EFP as a correlation function which tests ferroelectric order macroscopically, is that in the scaling limit EFP has a step-function behaviour. The second observation is that some multiple integral, with just the same integrand as that appearing in representation for EFP, but with integration contours surrounding the point $z = 1$, is equal to one exactly.

Namely, basing on (3), let us consider, for $r, s = 1, \ldots, N$, the integral

$$I_N^{(r,s)} := \frac{(-1)^s}{(2\pi i)^s} \oint_{C_1^-} \cdots \oint_{C_1^-} \prod_{j=1}^s \frac{[(t^2 - 2t \Delta)z_j + 1]^{s-j}}{z_j^n(z_j - 1)^{s-j+1}} \times \prod_{1 \leq j < k \leq s} \frac{z_j - z_k}{t^2 z_j z_k - 2t \Delta z_j + 1} h_{N,s}(z_1, \ldots, z_s) \, dz_1 \cdots dz_s.$$

Here $C_1^-$ is a closed contour in the complex plane enclosing point $z = 1$ and no other singularity of the integrand; the minus in the notation indicates negative (clockwise) orientation. We have

$$I_N^{(r,s)} = 1. \quad (7)$$

Indeed, performing integration in variable $z_s$, and taking into account (1), identity (7) follows immediately by induction. Note that relation (7) implies the same relation for the integral with the integrand of (4), which is just the symmetrized version of the integrand of (3).

Thus, we can explore again, just as in [12], the condensation of saddle-point solutions at the value $z = 1$, as a possible mechanism for the emergence of limit shapes. For this purpose we introduce here the concept of the ‘reduced’ saddle point equation, which allows one to treat the consequences of such condensation rather directly, and in the general setting of equations (6). Below we rely on results of papers [17, 18], where the scenario of condensation of roots of saddle-point equations has been described in the context of random matrix models with logarithmic potentials (Penner models, [19]).
To consider the scaling limit we define \( x \) and \( y \) such that
\[
x := \frac{N - r}{N}, \quad y := \frac{s}{N}, \quad x, y \in [0, 1].
\]
By ‘condensation’ we mean that, as \( s \to \infty \), almost all roots \( z_j \) of (6) have the value \( z = 1 \), in the sense that the amount of non-condensed roots is of order \( o(s) \). This implies that at \( s = \infty \) the density of roots is just the Dirac delta-function \( \delta(z - 1) \), or that the Green function is \( G(z) = (z - 1)^{-1} \). To implement this condensation in (6), we pick up \( j \)-th equation and just set \( z_k = 1 \) for \( k \neq j \). Writing simply \( z \) for this \( z_j \), we arrive at the following equation
\[
\frac{y}{z - 1} - \frac{1 - x}{z} - \frac{t^2 y}{t^2 z - 2t \Delta + 1} + \lim_{N \to \infty} \frac{1}{N} \frac{\partial}{\partial z} \ln h_N(z) = 0.
\]
(8)
In deriving this equation from (6), we have used properties (1) and (2), in particular, from (5) we have \( u_k = 0 \) as \( z_k = 1 \), and hence property (2) shows that the last term in (6) becomes just the derivative of a constant, thus vanishing.

We call equation (8) the reduced saddle-point equation. The solutions of this equation give the non-condensed roots of (6) at condensation. As explained in [18], a necessary condition for condensation of roots of saddle-point equations like (6), is the presence of two coinciding non-condensed roots. In our case these roots must lie on the real axis, in the interval \([1, \infty)\).

Summarizing, we arrive at the following recipe for the derivation of limit shapes: we require (8) to have two coinciding roots, which moreover must run over the interval \([1, \infty)\). Denoting the value of these two roots by \( \omega \), we shall see below that as \( \omega \) runs over the real axis from point \( z = 1 \) to point \( z \to \infty \), it parameterizes the top-left portion of the limit shape, from the top contact point (corresponding to \( \omega = 1 \)) to the left contact point (corresponding to \( \omega = \infty \)).

An ingredient which is necessary for this programme, is the knowledge of the last term in (8). Fortunately, function \( h_N(z) \) is explicitly known in some interesting cases. These are, for \( t = 1 \), the case of \( \Delta = 1/2 \), corresponding to usual enumeration of ASMs, and the case of \( \Delta = -1/2 \), corresponding to 3-enumerated ASMs; for generic \( t \) the function \( h_N(z) \) is also known for \( \Delta = 0 \), corresponding to the six-vertex model in the free-fermion case (specializing further \( t = 1 \), one gets 2-enumerated ASMs).

To illustrate how the recipe is working, let us consider the case of \( \Delta = 0 \). We set also \( t = 1 \) for simplicity, so that function \( h_N(z) = h_N(z; \Delta, t) \) in this case is just \( h_N(z; 0, 1) = [(z + 1)/2]^{N-1} \) (see, e.g., [12]). The reduced saddle-point equation reads
\[
\frac{y}{z - 1} - \frac{x}{z} + \frac{1 - x}{z + 1} = 0.
\]
Denoting by \( g(z) \) the function in the left-hand side, we require \( g(z) = (z - \omega)^2 \tilde{g}(\omega) \) where \( \tilde{g}(\omega) \neq 0 \). This can be implemented by solving the system of two equations \( g(\omega) = 0 \) and \( g'(\omega) = 0 \), where the prime denotes derivative, for unknowns \( x \) and \( y \). This gives
\[
x = \frac{1}{\omega^2 + 1}, \quad y = \frac{(\omega - 1)^2}{2(\omega^2 + 1)}, \quad \omega \in [1, \infty).
\]
Eliminating \( \omega \), we also have the equation for the limit shape
\[
4x(1 - x) + 4y(1 - y) = 1.
\]
(9)
Here $x$ and $y$ take values in interval $[0, 1/2]$; equation (9) describes the top-left portion of the Arctic Circle, which is the limit shape of 2-enumerated ASMs.

4. Limit shapes of 1- and 3-enumerated ASMs

We start with considering the case of 1-enumerated ASMs. In this case we have $\Delta = 1/2$ and $t = 1$, and the function $h_N(z; \frac{1}{2}, 1)$ is given by the formula (see, e.g., [20, 21])

$$h_N(z; \frac{1}{2}, 1) = {}_2F_1 \left( \frac{-N + 1}{2}, N \middle| 1 - z \right).$$

We write here the hypergeometric function in such a way that the third parameter is positive, and larger than the second one, so that the Euler integral representation for Gauss hypergeometric function can be used to study the large $N$ limit.

Explicitly, the Euler integral representation gives the following expression

$$h_N(z; \frac{1}{2}, 1) = \frac{\Gamma(2N)}{\Gamma(N)^2} \int_0^1 [\tau(1 - \tau)(1 - \tau + z\tau)]^{N-1} d\tau.$$

The large $N$ behaviour of this integral can be found via the standard saddle-point analysis, and we find that, as $N \to \infty$,

$$\ln h_N(z; \frac{1}{2}, 1) = N \ln [4v(1 - v)(1 - v + zv)] + o(N),$$

where

$$v := \frac{2 - z - \sqrt{z^2 - z + 1}}{3(1 - z)}.$$

The reduced saddle-point equation in this case reads

$$\frac{y}{z - 1} - \frac{1 - x + y}{z} + \frac{1 - \sqrt{z^2 - z + 1}}{z(1 - z)} = 0. \tag{10}$$

The requirement that equation (10) has two coinciding roots gives the following parametric solution for the limit shape of large ASMs:

$$x = 1 - \frac{2\omega - 1}{2\sqrt{\omega^2 - \omega + 1}}, \quad y = 1 - \frac{\omega + 1}{2\sqrt{\omega^2 - \omega + 1}}, \quad \omega \in [1, \infty).$$

Eliminating $\omega$ from this parametric solution we obtain that the limit shape is described by the equation

$$4x(1 - x) + 4y(1 - y) + 4xy = 1, \quad x, y \in \left[0, \frac{1}{2}\right]. \tag{11}$$

Formula (11) is the central result of our paper.

The most refined numerical simulations of large ASMs (up to $N = 1300$) have been performed recently by Wieland [22], who also provided pictures comparing these numerical data with equation (11). As the size of ASMs increases, convergence to this curve is observed.

Let us now consider the case of 3-enumerated ASMs ($\Delta = -1/2$ and $t = 1$). The function $h_N(z; -\frac{1}{2}, 1)$ has been obtained in [20] (see also [21]). The following formulae are valid

$$h_N(z; -\frac{1}{2}, 1) = \begin{cases} (1/2)(z + 1)B_{2m}(z) & \text{for } N = 2m + 2, \\ (1/9)(2z + 1)(z + 2)B_{2m}(z) & \text{for } N = 2m + 3, \end{cases}$$

where $B_{2m}(z)$ is a Bernoulli polynomial of order $2m$. The large $N$ limit of this function is described by the equation

$$4x(1 - x) + 4y(1 - y) + 4xy = 1, \quad x, y \in \left[0, \frac{1}{2}\right].$$

The reduced saddle-point equation in this case reads

$$\frac{y}{z - 1} - \frac{1 - x + y}{z} + \frac{1 - \sqrt{z^2 - z + 1}}{z(1 - z)} = 0. \tag{10}$$

The requirement that equation (10) has two coinciding roots gives the following parametric solution for the limit shape of large ASMs:

$$x = 1 - \frac{2\omega - 1}{2\sqrt{\omega^2 - \omega + 1}}, \quad y = 1 - \frac{\omega + 1}{2\sqrt{\omega^2 - \omega + 1}}, \quad \omega \in [1, \infty).$$

Eliminating $\omega$ from this parametric solution we obtain that the limit shape is described by the equation

$$4x(1 - x) + 4y(1 - y) + 4xy = 1, \quad x, y \in \left[0, \frac{1}{2}\right]. \tag{11}$$

Formula (11) is the central result of our paper.

The most refined numerical simulations of large ASMs (up to $N = 1300$) have been performed recently by Wieland [22], who also provided pictures comparing these numerical data with equation (11). As the size of ASMs increases, convergence to this curve is observed.
where $B_{2m}(z)$ is the following polynomial of degree $2m$:

$$B_{2m}(z) = \frac{m+1}{3^{m-1}(2m+3)} z^m (z+2)^m \binom{-m, m+2}{2m+4} \frac{z^2-1}{z(z+2)}$$

$$- \frac{m}{3^{m-1}(2m+3)} z^m (z+2)^{m-1} \binom{-m+1, m+2}{2m+4} \frac{z^2-1}{z(z+2)}.$$

Again, in comparison with [21], we have written here the hypergeometric polynomials in such a way that the Euler integral formula can be directly applied.

Evaluating the integrals through the saddle-point method, we obtain that, as $N \to \infty$,

$$\ln h_N(z; -\frac{1}{2}, 1) = N \ln \left[ \frac{2(2z+1)(z+2)}{9(z+1)} \right] + o(N).$$

The reduced saddle-point equation reads:

$$\frac{y}{z-1} - \frac{1-x}{z} - \frac{y}{2+z} + \frac{2z^2 + 4z + 3}{(1+z)(2+z)(1+2z)} = 0. \quad (12)$$

The requirement that equation (12) has two coinciding roots gives us the following parametric solution for the limit shape of 3-enumerated ASMs:

$$x = \frac{7\omega^4 + 14\omega^3 + 19\omega^2 + 12\omega + 2}{(\omega^2 + 2)(2\omega + 1)^2(\omega + 1)^2},$$

$$y = \frac{(\omega - 1)^2(6\omega^4 + 16\omega^3 + 19\omega^2 + 16\omega + 6)}{3(\omega^2 + 2)(2\omega + 1)^2(\omega + 1)^2}, \quad \omega \in [1, \infty). \quad (13)$$

Equivalently, one can search for the equation connecting $x$ and $y$. In this way we obtain that the limit shape of 3-enumerated ASMs is described by the following
sextic equation:

\[
324 x^6 + 1620 x^5 y + 3429 x^4 y^2 + 4254 x^3 y^3 + 3429 x^2 y^4 + 1620 xy^5 + 324 y^6 \\
- 972 x^5 - 1458 x^4 y - 2970 x^3 y^2 - 2970 x^2 y^3 - 1458 xy^4 - 972 y^5 \\
- 6147 x^4 - 9150 x^3 y - 17462 x^2 y^2 - 9150 xy^3 - 6147 y^4 \\
+ 13914 x^3 + 24086 x^2 y + 24086 xy^2 + 13914 y^3 \\
- 11511 x^2 - 17258 xy - 11511 y^2 \\
+ 4392 x + 4392 y - 648 = 0, \quad x, y \in [0, \frac{1}{2}]. \tag{14}
\]

One can verify directly that this equation is indeed satisfied by \( x \) and \( y \) given by (13).

Figure 3 shows plots of the limit shapes for 1-, 2- and 3-enumerated ASMs. The area of the temperate region decreases as \( q \) increases, in agreement with both analytical and numerical considerations [23–25].

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