On the Price of Satisficing in Network User Equilibria

Mahdi Takalloo
Department of Industrial and Management Systems Engineering, University of South Florida, Tampa, FL 33620

Changhyun Kwon*
Department of Industrial and Management Systems Engineering, University of South Florida, Tampa, FL 33620

September 6, 2019

Abstract
When network users are satisficing decision-makers, the resulting traffic pattern attains a satisficing user equilibrium, which may deviate from the (perfectly rational) user equilibrium. In a satisficing user equilibrium traffic pattern, the total system travel time can be worse than in the case of the PRUE. We show how bad the worst-case satisficing user equilibrium traffic pattern can be, compared to the perfectly rational user equilibrium. We call the ratio between the total system travel times of the two traffic patterns the price of satisficing, for which we provide an analytical bound. We compare the analytical bound with numerical bounds for several transportation networks.

Keywords: bounded rationality; satisficing; user equilibrium

1 Introduction
Instead of assuming a perfectly rational person with a clear system of preferences and perfect knowledge of the surrounding decision-making environment, we can consider boundedly rational persons with (1) an ambiguous system of preferences and (2) lack of complete information, following Simon (1955). When decision makers are indifferent among alternatives within a certain threshold, they are called satisficing decision makers, opposed to optimizing decision makers. The notion of satisficing was first introduced by Simon (1955, 1956). Satisficing decision makers choose any alternative whose utility level is above a threshold, called an aspiration level, even when the alternative is not optimal. The satisficing behavior is related to the first source of boundedness—an ambiguous system of preferences.

In transportation research, modeling drivers’ route choice is an important task. While the travel-time minimization has been traditionally used as a basis for such modeling, sub-optimal route-choice behavior has gained attention. Since Mahmassani and Chang (1987), bounded rationality

*Corresponding author: chkwon@usf.edu
has gained attention in the transportation research literature (Szeto and Lo, 2006; Wu et al., 2013; Han et al., 2015; Szeto and Lo, 2006; Ge and Zhou, 2012; Di et al., 2014; Guo, 2013; Lou et al., 2010). Empirical evidence supports bounded rationality of drivers (Nakayama et al., 2001; Zhu and Levinson, 2010). The notion of bounded rationality has also been considered in the evaluation of value of times in connection to route-choice modeling (Xu et al., 2017), and in the model of behavior adjustment process (Ye and Yang, 2017). We refer readers to a review of Di and Liu (2016). In the non-transportation literature, the notion of bounded rationality and satisficing has also received much attention (Charnes and Cooper, 1963; Lam et al., 2013; Jaillot et al., 2016; Chen et al., 1997; Brown and Sim, 2009).

While the above-mentioned transportation research literature considers boundedly rational drivers, their discussion is limited to satisficing drivers without considering the second source of boundedness: lack of complete information on the decision environment. Sun et al. (2018) connect the first and the second sources of boundedness by considering both satisficing behavior and incomplete information, in the context of shortest-path finding in congestion-free networks. Sun et al. (2018) study the second source by considering errors in drivers’ perception of arc travel time, and conclude that their perception-error model can generally capture both sources of boundedness in rationality in a single unified modeling framework.

In the literature, the traditional network user equilibrium, Wardrop equilibrium in particular, is called the perfectly rational user equilibrium (PRUE), while a traffic pattern equilibrated among satisficing drivers is called a boundedly rational user equilibrium (BRUE). In this paper, we will use a new term satisficing user equilibrium (SatUE) instead of BRUE to emphasize that it only considers the first source of boundedness without considering drivers’ incomplete information on the decision environment. We believe that the term ‘BRUE’ should be used to describe a broader and more general class of models, including SatUE.

Note that SatUE differs from the stochastic user equilibrium (SUE) (Sheffi, 1985) in two important aspects. First, drivers are assumed to be optimizing decision makers in SUE, while they are satisficing in SatUE. Second, with appropriate probability distributions assumed in the random utility model in SUE, each path possesses a probability of being chosen; hence we can compute the expected traffic flow rate in each path. In SatUE, however, each satisficing path is acceptable to drivers, but it may or may not be chosen by drivers and we do not know its probability of being chosen. See further discussion in Di and Liu (2016).

The main contribution of this paper is the quantification of how bad the total system travel time in SatUE can be. In a SatUE traffic pattern, the total system travel time can be either greater than or less than that of PRUE. We define the price of satisficing (PoSat) as the ratio between the worst-case total system travel time of SatUE and the total system travel time of PRUE. This paper quantifies PoSat analytically and compares with numerical bounds.

The analytical quantification of PoSat is related to the price of anarchy (PoA) (Koutsoupias and Papadimitriou, 1999; Roughgarden and Tardos, 2002) that compares the performances of the system optimal solutions and the PRUE solutions. Using a similar idea, we can also compare the performance of the
perfectly rational user equilibrium traffic patterns and satisficing user equilibrium traffic patterns. While PoA quantifies how much system-wide performance we can lose by competing, PoSat quantifies how much we can lose by satisficing. Roughgarden and Tardos (2002) define and study the PoA of approximate Nash equilibria, which are essentially SatUE patterns. We develop our bounds for PoSat based on the bounds for PoA of approximate Nash equilibria (Christodoulou et al., 2011) and the ideas from the sensitivity analysis of traffic equilibria (Dafermos and Nagurney, 1984). Note that Perakis (2007) studies the PoA of the exact Nash equilibria with general nonlinear, asymmetric cost functions.

The notion of PoSat is also related to the price of risk aversion (Nikolova and Stier-Moses, 2015) and the deviation ratio (Kleer and Schäfer, 2016). When network users are risk-averse decision makers, the price of risk aversion compares the performances of the resulting equilibrium among risk-averse users and the (risk-neutral) PRUE. When network users’ cost functions are deviated from the true cost functions for some reasons, the deviation ratio compares the performances of the resulting equilibrium and the PRUE. Kleer and Schäfer (2016) show that the price of risk aversion is a special case of the deviation ratio. In both research articles, however, only cases with a common single origin node are considered. In this paper, we consider general cases with multiple origin nodes and multiple destination nodes, with asymmetric travel time functions.

This paper is organized as follows. In Section 2, we introduce the notation and define various concepts including user equilibrium, system optimum, satisficing behavior, price of anarchy, and price of satisficing. In Section 3, we define the user equilibrium with perception errors and make connections with satisficing user equilibrium. Our main result is introduced in Section 4, where we derive the analytical worst-case bound on the price of satisficing. In Section 5, we compare the analytical bound with numerical bounds. Section 6 concludes this paper.

2 Notation and Definitions

Since we will use path-based and arc-based flow variables and their corresponding functions and sets interchangeably, we need clear definitions of variables, sets, and functions. We use boldfaced lower-case letters for vector quantities as in \( \mathbf{v} \) and normal lower-case letters for their components as in \( v_a \); similarly, vector-valued functions like \( t(\cdot) \) and their components like \( t_a(\cdot) \). We use boldfaced upper-case letters for the set that they belong to, as in \( \mathbf{V} \). We use calligraphic capital letters for sets of indices as in \( \mathcal{N} \). The only exception is that \( Q \) (a bold-face capital letter instead of lower case) represents a vector of \( Q_w \), the demand for OD pair \( w \). The lower-case version \( q_i^w \) is instead the net amount of flow associated with OD pair \( w \) that enters or leaves node \( i \) in the next subsection.

2.1 Traffic Flow Variables and Feasible Sets

We consider a network with a set of origin and destination \( W \) that is represented by directed graph \( G(\mathcal{N}, \mathcal{A}) \), where \( \mathcal{N} \) is the set of nodes, and \( \mathcal{A} \) is the set of arcs. For each OD pair \( w \in W \), the travel demand is \( Q_w \) and the set of available paths is \( P_w \). The set of all available paths in the whole
network is defined as $\mathcal{P} = \bigcup_{w \in \mathcal{W}} \mathcal{P}_w$.

We also define the set of path flow variables $f$ as

$$\mathcal{F} = \left\{ f : \sum_{p \in \mathcal{P}_w} f_p = Q_w \quad \forall w \in \mathcal{W}, \quad f_p \geq 0 \quad \forall p \in \mathcal{P} \right\}$$

and the corresponding set of arc flow variables $v$ is defined as

$$\mathcal{V} = \left\{ v : v_a = \sum_{p \in \mathcal{P}_w} \delta_a^p f_p, \quad \forall a \in \mathcal{A}, \quad f \in \mathcal{F} \right\}$$

where $\delta_a^p = 1$ if path $p$ contains arc $a$ and $\delta_a^p = 0$ otherwise. Let $\mathcal{A}_i^+$ and $\mathcal{A}_i^-$ be the set of arcs whose tail node and head node are $i$, respectively. When we need to preserve OD information in arc flow variables, we use $x$ as follows:

$$\mathcal{X} = \left\{ x : x_a^w = \sum_{p \in \mathcal{P}_w} \delta_a^p f_p, \quad \forall a \in \mathcal{A}, w \in \mathcal{W}, \quad f \in \mathcal{F} \right\}$$

$$\mathcal{X} = \left\{ x : \sum_{a \in \mathcal{A}_i^+} x_a^w - \sum_{a \in \mathcal{A}_i^-} x_a^w = q_i^w \quad \forall w \in \mathcal{W}, i \in \mathcal{N} \right\}$$

where $q_i^w = -Q_w$ if $i = o(w)$, $q_i^w = Q_w$ if $i = d(w)$, and $q_i^w = 0$ otherwise.

We have $v_a = \sum_{p \in \mathcal{P}} \delta_a^p f_p$, $x_a^w = \sum_{p \in \mathcal{P}_w} \delta_a^p f_p$, and $v_a = \sum_{w \in \mathcal{W}} x_a^w$. Therefore, the transformations from $f$ to $v$, from $f$ to $x$, and from $x$ to $v$ are unique, which are denoted by $f \mapsto v$, $f \mapsto x$, and $x \mapsto v$, respectively. The inverse transformations are, however, not unique. In the rest of this paper, to emphasize the non-uniqueness of the transformation and refer to any result of such transformation, we use $\mapsto_{\text{any}}$; for example, with $v \mapsto_{\text{any}} f$, we consider any $f$ such that $v_a = \sum_{p \in \mathcal{P}} \delta_a^p f_p$.

We will use $v$, $f$, and $x$ interchangeably to describe the same traffic pattern. In particular, we define

- $f^*, v^*, x^*$: system optimal flow vectors (Section 2.2)
- $f^0, v^0, x^0$: perfectly rational user equilibrium flow vectors (Section 2.3)
- $f^\kappa, v^\kappa, x^\kappa$: (multiplicative) satisficing user equilibrium flow vectors with a multiplicative factor (to be defined subsequently) $\kappa$ (Section 2.4)

Note that when $\kappa = 0$, we have $f^\kappa = f^0$.

### 2.2 Travel Time Functions and System Optimum

We denote arc travel function with arc traffic volume $v$ by $t_a(v)$ for each arc $a \in \mathcal{A}$. We consider a performance function for each arc $a$ as

$$z_a(v) = t_a(v) v_a.$$
We denote the travel time function along path \( p \) with flow \( f \) by \( c_p(f) \). When written as functions of \( x \), the arc travel time is denoted as \( \tau_a(x) \) or \( \tau_w^a(x) \), where the latter is used to emphasize the focus on OD pair \( w \). Of course, \( \tau_w^a(x) = \tau_a(x) = t_a(v) \), where \( v = \sum_w x_w \). The performance function for path \( p \in P \) is as follows:

\[
z_p(f) = c_p(f)f_p.
\]

The following shows the relationship between path and arc travel times.

\[
c_p(f) = \sum_{a \in A} \delta_p^a \tau_a(v).
\]

The vector-valued function \( t(\cdot) \) is called monotone in \( V \) if

\[
[t(v^1) - t(v^2)]^\top (v^1 - v^2) \geq 0
\]

for all \( v^1, v^2 \in V \). If (1) holds as a strict inequality for all \( v^1 \neq v^2 \), it is said strictly monotone. The function \( t(\cdot) \) is called strongly monotone in \( V \) with modulus \( \alpha > 0 \) if

\[
[t(v^1) - t(v^2)]^\top (v^1 - v^2) \geq \alpha \|v^1 - v^2\|_V^2
\]

for all \( v^1, v^2 \in V \), where \( \|\cdot\|_V \) is the \( l^2 \)-norm in \( V \). The monotonicity of path-based travel time function \( c_p(\cdot) \) or its vector form \( c(\cdot) \) can be similarly defined. The path-based function \( c_p(\cdot) \), however, is not strongly monotone in general (e.g., see Example 3 in de Palma and Nesterov, 1998).

### 2.3 Perfectly Rational User Equilibrium

When network users are perfectly rational—they seek the shortest path—we attain the perfectly rational user equilibrium (PRUE) defined as follows:

**Definition 1** (Perfectly Rational User Equilibrium). A traffic pattern \( f^0 \) is called a perfectly rational user equilibrium (PRUE), if

\[
(f^0)_p > 0 \implies c_p(f^0) = \min_{p' \in P_w} c_{p'}(f^0)
\]
for all \( p \in \mathcal{P}_w \) and \( w \in \mathcal{W} \).

Using the arc travel function, the above condition can be restated as follows

\[
\delta \mu_p^0 (v^0) = \min_{p' \in \mathcal{P}_w} \sum_{a \in \mathcal{A}} \delta \mu_a^0 (v^0)
\]

for all \( p \in \mathcal{P}_w \) and \( w \in \mathcal{W} \).

It is well known that a solution to the following variational inequality problem is a user equilibrium traffic flow (Smith, 1979; Dafermos, 1980):

\[
\text{to find } f \in \mathcal{F}: \sum_{p \in \mathcal{P}} c_p(f_p - f_p') \geq 0 \quad \forall f \in \mathcal{F},
\]

which can be equivalently rewritten as:

\[
\text{to find } v \in \mathcal{V}: \sum_{a \in \mathcal{A}} t_a(v_a - v_a') \geq 0 \quad \forall v \in \mathcal{V},
\]

or

\[
\text{to find } x \in \mathcal{X}: \sum_{a \in \mathcal{A}} \sum_{w \in \mathcal{W}} \tau_a(v_a - v_a') \geq 0 \quad \forall x \in \mathcal{X}
\]

where \( \tau_a^w(x) = \tau_a(x) = v_a(x) \).

With strictly monotone functions \( t_a(\cdot) \), the solution \( \bar{v} \) to (6) is unique. While the transformations \( v \mapsto f \) and \( \bar{v} \mapsto \bar{x} \) are not unique, any such \( f \) and \( \bar{x} \) are solutions to (5) and (7), respectively; therefore, solutions to (5) and (7) are not unique in general.

When the travel time on arc \( a \) is a function of only \( v_a \), i.e., \( t_a = t_a(v_a) \), then it is called separable. With separable arc travel time functions, the variational inequality problem (6) admits an equivalent convex optimization problem as formulated by (Beckmann et al., 1956). In general, if the Jacobian matrix of the arc travel time function vector \( t(v) \) is symmetric, that is,

\[
\frac{\partial t_a(v)}{\partial v_e} = \frac{\partial t_e(v)}{\partial v_a} \quad \forall a, e \in \mathcal{A},
\]

for all \( v \in \mathcal{V} \), the variational inequality problem (6) can be reformulated as an equivalent Beckmann-type convex optimization problem (Patriksson, 2015; Friesz and Bernstein, 2016). When the Jacobian is asymmetric, no Beckmann-type convex optimization problem equivalent to (6) exists in general. In this case, the arc travel time functions is characterized as asymmetric and obtaining a PRUE flow requires solving a variational inequality problem.

2.4 Satisficing User Equilibrium

We introduce definitions of satisficing behavior and corresponding user equilibrium traffic patterns. In transportation research literature, boundedly rational user equilibrium (BRUE) is often defined with an additive term (see e.g., Lou et al., 2010; Di et al., 2013; Han et al., 2015). Herein, we refer
to BRUE in the literature as an ‘additive satisficing user equilibrium’ to (i) highlight its additive feature and (ii) limit user behavior to just satisficing. Bounded rationality includes behaviors other than satisficing as well.

**Definition 2 (Additive Satisficing).** A traffic pattern \( f \) is called an **additive satisficing user equilibrium (ASatUE)** with an additive factor \( E \), if

\[
(\text{ASatUE}) \quad f_p > 0 \implies c_p(f) \leq \min_{p' \in P_w} c_{p'}(f) + E
\]  

for all \( p \in P_w \) and \( w \in W \), where \( E \) is a positive constant.

We can also derive a similar definition using a multiplicative term. While the additive form in Definition 2 is popularly used in the transportation research literature, the multiplicative form in Definition 3 enables us to consider the satisficing level in disaggregate arc levels as we will observe in this paper. Multiplicative satisficing user equilibrium is also called approximate Nash equilibrium in the price of anarchy literature (Christodoulou et al., 2011).

**Definition 3 (Multiplicative Satisficing).** A traffic pattern \( f^\kappa \) is called a **multiplicative satisficing user equilibrium** with a multiplicative factor \( \kappa \), or \( \kappa \)-MSatUE, if

\[
(\text{MSatUE}) \quad f_p^\kappa > 0 \implies c_p(f^\kappa) \leq (1 + \kappa) \min_{p' \in P_w} c_{p'}(f^\kappa)
\]  

for all \( p \in P_w \) and \( w \in W \), where \( \kappa \geq 0 \) is a constant.

Note that the additive \((E)\) and multiplicative \((\kappa)\) factor in (8) and (9), respectively, may be defined for each OD pair \( w \). For example, \( E_w \) and \( \kappa_w \) can replace \( E \) and \( \kappa \) in (8) and (9), respectively, to allow for non-homogeneous satisficing thresholds. In such cases, however, we assume that travelers for the same OD pair are homogeneous with the same threshold \( E_w \) or \( \kappa_w \). In this paper, to describe the satisficing behavior, we focus only on MSatUE. Moreover, for simplicity, we use a single value of \( \kappa \) for all OD pairs.

### 2.5 Price of Satisficing

The price of anarchy (PoA) compares the performances of a satisficing user equilibrium \((C(f^\kappa))\) against that of a system optimum \((C(f^*))\). Among possibly multiple satisficing user equilibrium traffic patterns, we are interested in the worst-case. Let \( \Psi_\kappa(G, Q, t) \) be the set of all satisficing user equilibria with a multiplicative factor \( \kappa \) where \( G, Q \), and \( t \) denote the underling network, demand vector, and travel time function, respectively. Then, the PoA for the triplet \((G, Q, t)\) is defined as follows:

\[
\text{PoA}_\kappa(G, Q, t) = \max_{f^\kappa \in \Psi_\kappa(G, Q, t)} \frac{C(f^\kappa)}{C(f^*)},
\]  

(10)
where \( f^* \) is the system optimum flow for \((G, Q, t)\). We are usually interested in its upper bound over a set of triplets, \( \Omega \), i.e.

\[
\sup_{(G, Q, t) \in \Omega} \text{PoA}_\kappa(G, Q, t)
\]

In the context of bounded rationality and satisficing, we are more interested in comparing the performance of approximate Nash equilibrium \( C(f^\kappa) \) and the performance of the perfectly rational user equilibrium \( C(f^0) \). We define the price of satisficing (PoSat) of instance \((G, Q, t)\) as follows:

\[
\text{PoSat}_\kappa(G, Q, t) = \max_{f^\kappa \in \Psi_\kappa(G, Q, t)} \frac{C(f^\kappa)}{C(f^0)},
\]

and its upper bound over \( \Omega \) is

\[
\sup_{(G, Q, t) \in \Omega} \text{PoSat}_\kappa(G, Q, t)
\]

In this paper, \( \Omega \) is a set of all triplets where \( G \) is a directed graph with a finite number of nodes and arcs, \( Q \) is a vector of finite and positive constants, and \( t(\cdot) \) is a vector of polynomial functions with nonnegative coefficients and of order \( n \geq 0 \). To emphasize the latter, we also write \( \Omega(n) \) instead of \( \Omega \) when appropriate.

### 3 User Equilibrium with Perception Errors

Related to MSatUE is the user equilibrium with perception error (UE-PE) model. In this model, we assume that network users are optimizing, i.e. seeking the shortest path; however, we assume that users may have their own perception of the travel time function.

We let \( \varepsilon^w_a \) denote the perception error of travel time along arc \( a \) of users in OD pair \( w \). A vector \( \bar{x} \in X \) is a solution to the UE-PE model, if

\[
\sum_{a \in A} \sum_{w \in W} (t_a(\bar{v}) - \varepsilon^w_a)(x^w_a - \bar{x}_a^w) \geq 0 \quad \forall x \in X
\]

The above variational inequality assumes that \( \varepsilon \) is sufficiently small, i.e., \( 0 \leq \varepsilon^w_a < t_a(\bar{v}) \) for all \( a \in A, w \in W \). Under such an assumption, \( t_a(\bar{v}) - \varepsilon^w_a \) can be viewed as the \textit{perceived} travel time for arc \( a \) for drivers of OD pair \( w \).

The term \( \varepsilon^w_a \) represents the perception error for arc \( a \) and OD pair \( w \). In this model, we assume all drivers for each OD pair are homogeneous in their perception of arc travel time.

With changes of variables \( \lambda^w_a t_a(v) = t_a(v) - \varepsilon^w_a \), the UE-PE model (12) can be restated as follows:

\[
(\text{UE-PE-X}) \quad \sum_{a \in A} \sum_{w \in W} \lambda^w_a t_a(\bar{v})(x^w_a - \bar{x}_a^w) \geq 0 \quad \forall x \in X
\]

for some \( \lambda \) such that \( \lambda^w_a \in (0, 1] \) for all \( w \in W \) and \( a \in A \). We observe that the UE-PE model generates a subset of MSatUE traffic flow patterns.
Lemma 1 (UE-PE-X$\iff$ MSatUE). Suppose $\overline{x}$ is a solution to UE-PE-X in (13) with some $\overline{\lambda}$ where $\overline{\lambda}_a^w \in \left[ \frac{1}{1+\kappa}, 1 \right]$ for all $w \in W$ and $a \in A$. Then any $\overline{f}$ with $\overline{x}$ any $\overline{f}$ is a $\kappa$-MSatUE flow.

Proof of Lemma 1. Given $\overline{f}$, let $\overline{v}$ be the arc flow vector from $\overline{f} \mapsto \overline{v}$. Let $\overline{e}$ be the perception error that makes $\overline{x}$ a solution to (12). Then, $\overline{x}$ is a user equilibrium flow with respect to arc travel time $\overline{\lambda}_a^w t_a(\cdot)$ and the following follows from (4):

$$\overline{f}_p > 0 \implies \sum_{a \in A} \delta^p_a \overline{\lambda}_a^w t_a(\overline{v}) = \min_{p' \in P_w} \sum_{a \in A} \delta^p_a \overline{\lambda}_a^w t_a(\overline{v})$$

for all $p \in P_w$ and $w \in W$. Since $\overline{\lambda}_a^w \in \left[ \frac{1}{1+\kappa}, 1 \right]$, the right-hand-side of (14) implies

$$\frac{1}{1+\kappa} \sum_{a \in A} \delta^p_a t_a(\overline{v}) \leq \min_{p' \in P_w} \sum_{a \in A} \delta^p_a \overline{\lambda}_a^w t_a(\overline{v}) \leq \min_{p' \in P_w} \sum_{a \in A} \delta^p_a t_a(\overline{v}),$$

which is equivalent to the following path flow form:

$$c_p(\overline{f}) \leq (1+\kappa) \min_{p' \in P_w} c_p(\overline{f}).$$

Therefore, we conclude that $\overline{f}$ is a $\kappa$-MSatUE traffic flow. 

We can also provide a path-based formulation of UE-PE:

$$(\text{UE-PE-F}) \quad \sum_{w \in W} \sum_{p \in P_w} \overline{c}_p^w(\overline{f})(f_p - \overline{f}_p) \geq 0 \quad \forall \overline{f} \in F$$

for the perceived path travel time functions $\overline{c}_p^w(\overline{f}) = \sum_{a \in A} \delta^p_a \overline{\lambda}_a^w t_a(\overline{v})$ with some $\overline{\lambda}$ such that $\overline{\lambda}_a^w \in (0,1]$ for all $a \in A, w \in W$.

Lemma 2 (UE-PE-F$\iff$ UE-PE-X). If $\overline{f}$ is a solution to UE-PE-F in (15) for some $\overline{\lambda}$ such that $\overline{\lambda}_a^w \in \left[ \frac{1}{1+\kappa}, 1 \right]$, then $\overline{x}$ with $\overline{f} \mapsto \overline{x}$ is a solution to UE-PE-X in (13). Conversely, if $\overline{x}$ is a solution to UE-PE-X in (13), then any $\overline{f}$ with $\overline{x}$ any $\overline{f}$ is a solution to UE-PE-F in (15).

Proof of Lemma 2. We can prove both directions by observing that

$$\sum_{w \in W} \sum_{p \in P_w} \overline{c}_p^w(\overline{f})(f_p - \overline{f}_p) = \sum_{w \in W} \sum_{p \in P_w} \sum_{a \in A} \delta^p_a \overline{\lambda}_a^w t_a(\overline{v})(f_p - \overline{f}_p)$$

$$= \sum_{w \in W} \sum_{a \in A} \overline{\lambda}_a^w t_a(\overline{v}) \left( \sum_{p \in P_w} \delta^p_a f_p - \sum_{p \in P_w} \delta^p_a \overline{f}_p \right)$$

$$= \sum_{w \in W} \sum_{a \in A} \overline{\lambda}_a^w t_a(\overline{v}) (x_a^w - \overline{x}_a^w).$$

Therefore, we conclude that $\overline{f}$ is a $\kappa$-MSatUE traffic flow. 

When the values of $\overline{\lambda}_a^w$ are the same across all $w \in W$, i.e. $\lambda_a = \overline{\lambda}_a^w$ for all $w \in W$, we can...
UE-PE-V $\Rightarrow$ UE-PE-X $\Rightarrow$ MSatUE $\Rightarrow$ (17)

\[
\text{UE-PE-F}
\]

Figure 1: Summary of Lemmas 1–4. The relation $X \Rightarrow Y$ means that any solution to $X$ yields a solution to $Y$.

simplify (13) as follows:

\[
(\text{UE-PE-V}) \sum_{a \in A} \lambda_a t_a(\overline{v})(v_a - \overline{v}_a) \geq 0 \quad \forall \overline{v} \in V
\] (16)

for some $\lambda$ such that $\lambda_a \in (0,1]$ for each $a \in A$. The simplified model (16) has been considered in the literature for approximate Nash equilibrium (Christodoulou et al., 2011) and Nash equilibrium with deviated travel time functions (Kleer and Schäfer, 2016). For the simplified model, we can state:

**Lemma 3** (UE-PE-V $\Rightarrow$ UE-PE-X). Suppose that $\overline{v} \in V$ is a solution to UE-PE-V in (16) for some $\lambda$ such that $\lambda_a \in [\frac{1}{1+\kappa}, 1]$ for all $a \in A$. Let $\overline{x}$ be any vector with $\overline{v} \mapsto \overline{x}$. Then $\overline{x}$ is a solution to UE-PE-X in (13).

While Lemmas 1, 2, and 3 provide sufficient conditions for a traffic flow pattern to be a $\kappa$-MSatUE, Theorem 1 of Christodoulou et al. (2011) provides a necessary condition. Although Christodoulou et al. (2011) assumed separable arc travel time functions, their proof is still valid for nonseparable travel time functions.

**Lemma 4** (A necessary condition of MSatUE). Let $f^\kappa \in F$ be a $\kappa$-MSatUE and $v^\kappa \in V$ be the corresponding arc flow vector with $f^\kappa \mapsto v^\kappa$. Then we have

\[
\sum_{a \in A} t_a(\overline{v}^\kappa)((1 + \kappa)v_a - v_a^\kappa) \geq 0 \quad \forall \overline{v} \in V.
\] (17)

Christodoulou et al. (2011) derive a tight bound on the price of anarchy on approximate Nash equilibria based on Lemma 4.

Figure 1 summarizes the relationships between problems addressed in Lemma 1 to 4. Note that this result does not mean that the notions of UE-PE and SatUE are equivalent in general. Rather, the specific modeling of UE-PE-X, UE-PE-F, and UE-PE-V with the perception error interval $[\frac{1}{1+\kappa}, 1]$, as defined in (13), (15), and (16), respectively, leads to the result in Figure 1. If we use a different definition of perception error sets, such a result may not hold.

4 Bounding the Price of Satisficing

We first provide analytical bounds of $C(f^\kappa)$ compared to $C(f^0)$.
4.1 Lessons from the Price of Anarchy

We first observe that \( \text{PoSat}_\kappa(G, Q, t) \leq \text{PoA}_\kappa(G, Q, t) \) for any network instance \((G, Q, t)\), since \( C(f^0) \geq C(f^*) \). This enables us to use the results from the price of anarchy literature for bounding PoSat. Theorem 2 of Christodoulou et al. (2011) bound the price of anarchy when arc travel-time functions are separable and polynomial with nonnegative coefficients and of degree \( n \) that leads immediately to the following result:

**Lemma 5.** Suppose \( f^\kappa \) is a \( \kappa \)-MSatUE flow, and \( t_a(\cdot) \) is polynomial with nonnegative coefficients and of degree \( n \). Define

\[
\zeta(\kappa, n) = \begin{cases} 
(1 + \kappa)^{n+1} & \text{if } \kappa \geq (n+1)^{1/n} - 1, \\
\left(1 + \frac{n}{(n+1)(n+1/n)}\right)^{-1} & \text{if } 0 \leq \kappa \leq (n+1)^{1/n} - 1.
\end{cases}
\]

Then we have

\[
C(f^*) \leq C(f^\kappa) \leq \zeta(\kappa, n) C(f^*) \leq \zeta(\kappa, n) C(f^0).
\]

(19)

That is, the PoSat is bounded above by \( \zeta(\kappa, n) \).

**Proof of Lemma 5.** From Theorem 2 of Christodoulou et al. (2011), we have

\[
C(f^\kappa) \leq \zeta(\kappa, n) C(f) \forall f \in F.
\]

(20)

Picking \( f = f^0 \) in (20), we obtain the upper bound on \( C(f^\kappa) \). Inequalities involving \( C(f^*) \) are from the fact \( C(f^*) \leq C(f) \) for all \( f \in F \).

The bound in Lemma 5 is not tight when \( \kappa \) is small. For example, when \( \kappa = 0 \), \( C(f^\kappa) = C(f^0) \). Thus, \( \frac{C(f^\kappa)}{C(f^0)} = 1 \). On the other hand, (19) yields the following:

\[
\frac{C(f^\kappa)}{C(f^0)} \leq \left(1 - \frac{n}{(n+1)^{n+1/n}}\right)^{-1}
\]

where the expression on the right is strictly larger than one. For example, the expression reduces to \( \frac{4}{3} \) when \( n = 1 \) and approaches infinity when \( n \) is large.

In Lemma 3 of Christodoulou et al. (2011), the existence of a network instance with \( C(f^\kappa) = (1 + \kappa)^{n+1}C(f^*) \) is shown for \( \kappa \geq (n+1)^{1/n} - 1 \) via a circular network example presented in Figure 2. The circular network includes \( m + l \) nodes where positive integers \( m \) and \( l \) are chosen so that \( \frac{m}{m+l} = 1 + \kappa \). All nodes lie in a circle and each node \( i \) is adjacent to two neighboring nodes via arc \((i, i+1)\) and \((i-1, i)\). The arc cost function for arc \( a \) is \( t_a(v_a) = (v_a)^n \), where \( v_a \) is the total arc flow in arc \( a \). There are \( m + l \) OD pairs \((i, i+m)\) for \( i = 1, 2, \ldots, m + l \), with unit demand from node \( i \) to node \( i + m \) (indices are taken cyclically). Note that the circular network can be easily converted to a directed network by replacing each undirected arc with two directed arcs with opposite direction.
The cost associated with arc $a$ will be $t_a(v_a, \hat{v}_a) = (v_a + \hat{v}_a)^n$ in this case where $\hat{v}_a$ is the flow in the arc with opposite direction.

For each OD pair $(i, i + m)$, there are two paths, clockwise and counterclockwise. The former contains $m$ arcs and the latter has only $l$. Note that our choice requires that $\frac{m}{l} = (1 + \kappa)$ where $\kappa \geq 0$, i.e., $m \geq l$. Consider the all-or-nothing strategy that sends the unit demand for every OD pair along only one path, clockwise or counterclockwise. Using the clockwise strategy, each arc has $m$ units of flows and costs $m^n$. Thus, the clockwise path costs $m \cdot m^n$, while the cost of the counterclockwise one is $l \cdot m^n$. Because $\frac{m \cdot m^n}{l \cdot m^n} = \frac{m}{l} \geq 1$, the clockwise strategy is not in a user equilibrium unless $m = l$. For the counterclockwise strategy, each arc has $l$ units of flows instead and costs $l^n$. Similarly, the clockwise and counterclockwise path cost $m \cdot l^n$ and $l \cdot l^n$, respectively. Using the same reasoning as before, flow-bearing (or the counterclockwise) paths are less expensive than or the same as the unused ones. Thus, the counterclockwise strategy is in user equilibrium and yields $(m + l) \cdot l^{n+1}$ as the total travel cost. Consider MSatUE. Because it is PRUE, the counterclockwise strategy is automatically in MSatUE. But, the clockwise one is also in MSatUE because the cost of flow-bearing (or the clockwise) path is exactly $(1 + \kappa)$ time the cost of the shortest path, i.e., the counterclockwise one. Additionally, the total travel cost of the clockwise strategy is $(m + l) \cdot m^{n+1}$. Then, the PoSat of this circular network is

$$\frac{(m + l) \cdot m^{n+1}}{(m + l) \cdot l^{n+1}} = \left(\frac{m}{l}\right)^{n+1} = (1 + \kappa)^{n+1}.$$  

Thus, the bound is tight. Now consider the system problem for the circular network. The problem objective is to minimize $\sum_{a \in A} v_a t_a(v)$. Using the counterclockwise strategy, the partial derivative of the objective function with respect to $v_a$ is $(n+1)l^n$. Then, switching to the clockwise path would increase the total travel cost by $(m - l)(n + 1)l^n \geq 0$ per unit flow. Thus, the counterclockwise strategy is system optimal because switching to the unused path does not lead to a reduction in the total travel cost. Observe that the UE-PE-X model can capture the satisficing behavior of network users in the circular network adequately. Let us consider $\lambda$ as follow as (indices are taken
Worst-Case Price of Satisficing

\[ 4(1 + \kappa) \left( \frac{1}{3 - \kappa} \right) - \kappa (1 + \kappa) \]

(a) \( n = 1 \)

(b) \( n = 4 \)

Figure 3: The worst-case price of satisficing for \( n = 1 \) and \( n = 4 \). Note that when \( n = 1 \), the right-hand-side of (21) becomes \( \frac{4(1+\kappa)}{3-\kappa} \). For \( n = 1 \), when \( \kappa \geq 1 \), we know for sure that the worst-case price of satisficing is exactly the dotted line. When \( \kappa \leq 1 \), the worst case falls in the shaded interval between the solid line and dotted line. When \( n = 4 \), it is similar.

cyclically):

\[
\lambda^w_a = \begin{cases} 
\frac{1}{1+\kappa} & \text{for } a \in \{(j, j+1) : j = i, i+1, ..., i+m-1\} \\
1 & \text{for } a \in \{(j-1, j) : j = i, i-1, ..., i-l+1\} 
\end{cases}
\]

Under the above \( \lambda \), if all network users choose the clockwise path, the flow in each arc will be equal to \( m \), the path cost for each OD pair will be \( \frac{1}{1+\kappa}m^{n+1} = lm^n \), which is equal to the cost of the alternative path, and thus the clockwise path is a solution to UE-PE-X and PoSat,\( \kappa = (1 + \kappa)^{n+1} \).

In Section 5, we will compute the PoSat numerically for these examples to confirm that UE-PE-X is a useful model to find PoSat,\( \kappa \).

By Lemma 5, when travel time functions are polynomials of degree \( n \) with nonnegative coefficients, the PoSat is bounded as follows:

\[
\text{PoSat}_{\kappa}(G, Q, t) \leq \begin{cases} 
(1 + \kappa)^{n+1} & \text{if } \kappa \geq (n+1)^{1/n} - 1, \\
\left( \frac{1}{1+\kappa} - \frac{n}{(n+1)^{(n+1)/n}} \right)^{-1} & \text{if } 0 \leq \kappa \leq (n+1)^{1/n} - 1.
\end{cases}
\] (21)

for all \((G, Q, t) \in \Omega(n)\) and by the circular network example in Figure 2, we know that there indeed exists a network instance \((G, Q, t) \in \Omega(n)\) such that \(\text{PoSat}_{\kappa}(G, Q, t) = (1 + \kappa)^{n+1}\) for all \( \kappa \geq 0 \). Therefore when \( \kappa \geq (n+1)^{1/n} - 1 \), the bound in (21) is tight. Figure 3a shows the bounds in (21) when travel time functions are linear or when \( n = 1 \). For smaller \( \kappa \) values, the worst-case PoSat falls in the shaded interval, while for larger \( \kappa \) values, it is exactly \((1 + \kappa)^2\). Figure 3b shows the same bounds when \( n = 4 \) instead. When \( \kappa \) is zero, we have \( f^\kappa = f^0 \); hence, we must have the
PoSat approach to 1. With this observation, we naturally ask a question: Does \((1 + \kappa)^{n+1}\) provide a tight bound on \(\text{PoSat}_\kappa\) for all \(\kappa \geq 0\)? We present partial answers to this question in the following sections.

4.2 Increased Travel Demands and Travel Time Functions

We first define new sets of flow vectors. When the travel demand \(Q_w\) for each \(w \in W\) is multiplied by the factor \(1 + \kappa\), we define

\[
F_{1+\kappa} = \left\{ f : \sum_{p \in P} f_p = (1 + \kappa)Q_w \quad \forall w \in W, \quad f_p \geq 0 \quad \forall p \in P \right\},
\]

\[
V_{1+\kappa} = \left\{ v : v_a = \sum_{p \in P} \delta_a^p f_p \quad \forall a \in A, \quad f \in F_{1+\kappa} \right\},
\]

\[
X_{1+\kappa} = \left\{ x : x_a^w = \sum_{p \in P} \delta_a^w f_p \quad \forall a \in A, w \in W \quad f \in F_{1+\kappa} \right\}.
\]

The above three sets can equivalently be written as follows:

\[
F_{1+\kappa} = \{(1 + \kappa)f : f \in F\},
\]

\[
V_{1+\kappa} = \{(1 + \kappa)v : v \in V\},
\]

\[
X_{1+\kappa} = \{(1 + \kappa)x : x \in X\}.
\]

We will use ‘hat’ for flow vectors in these sets, for example, \(\hat{f}^\kappa \in F_{1+\kappa}\), while without hat in the original sets as in \(f^\kappa \in F\).

We consider cases when the travel time functions \(t_a(\cdot)\) are polynomials of order \(n\), in particular, the following form of asymmetric arc travel time function for each \(a \in A\):

\[
t_a(v) = \sum_{m=0}^{n} b_{am} \left( \sum_{e \in A} d_{aem} v_e \right)^m = \sum_{m=0}^{n} b_{am} \left( d_{am}^\top v \right)^m \quad (22)
\]

for some constants \(b_{am}\) for \(m = 0, 1, ..., n\) and \(d_{aem}\) for \(e \in A\) and \(m = 0, 1, ..., n\). Note that we use the vector form \(d_{am} = (d_{aem} : e \in A)\). The travel time function (22) is a general form of the travel time functions considered in the traffic equilibrium literature (Meng et al., 2014; Panicucci et al., 2007). If \(d_{am}\) is a unit vector such that \(d_{aem}\) is 1 if \(a = e\) and 0 otherwise, we have a separable polynomial arc travel time function that has been used in the literature popularly (Christodoulou et al., 2011; Roughgarden and Tardos, 2002):

\[
t_a(v_a) = \sum_{m=0}^{n} b_{am}(v_a)^m = b_{a0} + b_{a1}v_a + b_{a2}(v_a)^2 + \cdots + b_{an}(v_a)^n. \quad (23)
\]
Lemma 6. With the polynomial travel time function (22), for any \( f \in F \), we have

\[
C((1 + \kappa)f) \leq (1 + \kappa)^{n+1}C(f)
\]  

(24)

for all \( \kappa \geq 0 \) and \( n \geq 0 \).

Proof of Lemma 6. By simple comparison, we can show

\[
C((1 + \kappa)f) = Z((1 + \kappa)v) = \sum_{a \in A} \left( \sum_{m=0}^{n} b_{am} \left( (1 + \kappa)(d_{am}^T v)^m \right) \right) (1 + \kappa)v_a
\]

\[
\leq (1 + \kappa)^{n+1} \sum_{a \in A} \left( \sum_{m=0}^{n} b_{am} \left( d_{am}^T v \right)^m \right) v_a
\]

\[
= (1 + \kappa)^{n+1}Z(v)
\]

\[
= (1 + \kappa)^{n+1}C(f)
\]

where \( v \) is the arc flow vector from \( f \mapsto v \).

4.3 Cases with Separable, Monomial Arc Travel Time Functions

As a simple case, we consider separable, monomial functions of degree \( n \) for arc travel time of the following form:

\[
t_a(v_a) = b_a(v_a)^n
\]

(25)

with a positive scalar \( b_a \) for each \( a \in A \) and nonnegative constant \( n \).

It is well known (Beckmann et al., 1956) that \( v^0 \in F \) is a user equilibrium flow, if and only if it minimizes the following potential function

\[
\Phi(v) = \sum_{a \in A} \int_{0}^{v_a} t_a(u) \, du = \sum_{a \in A} \frac{b_a}{n+1} (v_a)^{n+1}
\]

when the arc travel time functions are separable, so that the integral is well defined. Similarly, \( v^\kappa \in V \) is a \( \kappa \)-MSatUE flow, if it is a solution to \( \text{UE-PE-V} \), or equivalently, if it minimizes the following potential function (Christodoulou et al., 2011)

\[
\Psi(v; \lambda) = \sum_{a \in A} \int_{0}^{v_a} \lambda_a t_a(u) \, du = \sum_{a \in A} \frac{\lambda_a b_a}{n+1} (v_a)^{n+1}
\]

for some \( \lambda_a \in [\frac{1}{1+\kappa}, 1] \) for each \( a \in A \).

When travel time functions are separable, we can show the following result (Englert et al., 2010; Takalloo and Kwon, 2018):

Lemma 7. When the arc travel time functions are in the form of (23), let \( f^0 \in F \) and \( \hat{f}^0 \in F_{1+\kappa} \)}
be the PRUE flows with the corresponding travel demands. We can show

\[ C(\hat{f}^0) \leq (1 + \kappa)^{n+1}C(f^0) \]  

for all \( \kappa \geq 0 \) and \( n \geq 0 \).

Although Englert et al. (2010) consider cases with a single OD pair only with interest in the changes in the path travel time, the same technique can be used to prove Lemma 7 for cases with multiple OD pairs. For completeness, we include the proof to Lemma 7 in the appendix.

Using Lemma 7, we show that a solution to \( \text{UE-PE-V} \) is an MSatUE flow.

**Theorem 1.** When the arc travel time functions are of the form (25), let \( \bar{v} \in V \) be a solution to \( \text{UE-PE-V} \) and \( \bar{f} \in F \) is any corresponding path flow with \( \bar{v} \overset{\text{any}}{\rightarrow} \bar{f} \). We let \( \hat{f}^0 \in F_{1+\kappa} \) be the PRUE flows. Then we have \( C(f^\kappa) \leq C(\hat{f}^0) \), and consequently \( C(f^\kappa) \leq (1 + \kappa)^{n+1}C(f^0) \) for all \( \kappa \geq 0 \).

**Proof of Theorem 1.** Since \( \hat{f}^0 \in F_{1+\kappa} \) is an user equilibrium flow that minimizes \( \Phi(\cdot) \), we have

\[ \Phi(\hat{v}^0) \leq \Phi((1 + \kappa)\bar{v}), \]

which implies

\[ \sum_{a \in A} \frac{b_a(\bar{v}_a^0)^{n+1}}{n+1} \leq \sum_{a \in A} \frac{b_a((1 + \kappa)\bar{v}_a)^{n+1}}{n+1} = (1 + \kappa)^{n+1} \sum_{a \in A} \frac{b_a(\bar{v}_a)^{n+1}}{n+1}. \]  

(27)

Since \( \bar{v} \in V \) is a solution to \( \text{UE-PE-V} \), we have

\[ \Psi(\bar{v}; \lambda) \leq \Psi(\hat{v}^0; \lambda), \]

for some \( \lambda \). Therefore, we have

\[ \sum_{a \in A} \frac{\lambda_a b_a(\bar{v}_a^0)^{n+1}}{n+1} \leq \sum_{a \in A} \frac{\lambda_a b_a((v_a)^0)^{n+1}}{(n+1)(1 + \kappa)^{n+1}} = \frac{1}{(1 + \kappa)^{n+1}} \sum_{a \in A} \lambda_a b_a(\bar{v}_a^0)^{n+1}. \]

Since \( \lambda_a \in [\frac{1}{1+\kappa}, 1] \), we obtain

\[ \frac{1}{1 + \kappa} \sum_{a \in A} \frac{b_a(\bar{v}_a)^{n+1}}{n+1} \leq \frac{1}{(1 + \kappa)^{n+1}} \sum_{a \in A} \frac{b_a(\bar{v}_a^0)^{n+1}}{n+1} \]

which implies

\[ (1 + \kappa)^n \sum_{a \in A} \frac{b_a(\bar{v}_a)^{n+1}}{n+1} \leq \sum_{a \in A} \frac{b_a(\bar{v}_a^0)^{n+1}}{n+1} \]  

(28)
Let us assume that $C(\bar{f}) > C(\hat{f}^0)$, which is equivalent to

$$\sum_{a \in A} b_a (\bar{v}_a^0)^{n+1} < \sum_{a \in A} b_a (\bar{v}_a)^{n+1}$$

(29)

From $A \times (27) + B \times (28) + C \times (29)$ for any positive constants $A$, $B$ and $C$, we obtain

$$\theta_1 \sum_{a \in A} \frac{b_a (\bar{v}_a^0)^{n+1}}{n+1} < \theta_2 \sum_{a \in A} \frac{b_a (\bar{v}_a)^{n+1}}{n+1}$$

(30)

where

$$\theta_1 = A - B + C(n + 1)$$
$$\theta_2 = A(1 + \kappa)^{n+1} - B(1 + \kappa)^n + C(n + 1).$$

In particular, consider $A$, $B$ and $C$ as follows:

$$A = (n + 1)((1 + \kappa)^n - 1)$$
$$B = (n + 1)((1 + \kappa)^n - 1) + (n + 1)\kappa(1 + \kappa)^{n+1}$$
$$C = \kappa(1 + \kappa)^{n+1}$$

We observe that $A$, $B$ and $C$ are all positive and $\theta_1 = 0$. We also see that

$$\theta_2 = -(n + 1)\kappa^2(1 + \kappa)^n((1 + \kappa)^{n+1} - 1) \leq 0$$

for all $\kappa \geq 0$ and $n \geq 0$, which leads to a contradiction. Therefore, we have

$$C(\bar{f}) \leq C(\hat{f}^0) \leq (1 + \kappa)^{n+1}C(\bar{f}^0),$$

where the last inequality is from Lemma 7. This completes the proof. \hfill \Box

Note that the bound obtained in Theorem 1 relies on the sufficient condition, not a necessary condition. Therefore, the result is not applicable to all MSatUE flows, although it provides a useful bound in the framework of UE-PE models.

4.4 Cases with Separable Arc Travel Time Functions

We consider general polynomial, separable arc travel functions in the form of (23).

**Theorem 2.** Suppose that the arc travel time functions are in the form of (23). Let $f^* \in F$ be any $\kappa$-MSatUE and $\hat{f}^0 \in F_{1+\kappa}$ be the PRUE flow. Suppose that $\kappa \geq 0$ is sufficiently small, in
particular, so that
\[
\sum_{p \in P} [c_p(\hat{f}^0) - c_p(f^\kappa)](\hat{f}_p^0 - f_p^\kappa) \geq \kappa \sum_{p \in P} c_p(f^\kappa)\left|\hat{f}_p^0 - f_p^\kappa\right|
\]  
(31)

Then we have \( C(f^\kappa) \leq C(\hat{f}^0) \). Consequently \( C(f^\kappa) \leq (1 + \kappa)^{n+1}C(f^0) \), and

\[
\sup_{(G, Q, t) \in \Omega(n)} \text{PoSat}_\kappa(G, Q, t) = (1 + \kappa)^{n+1}.
\]

**Proof of Theorem 2.** By slightly modifying the proof of Theorem 3, we can show \( C(f^\kappa) \leq C(\hat{f}^0) \). By Lemmas 6 and 7, we complete the proof.

Theorem 2 depends on condition (31) and a similar condition appears in general asymmetric cases as in Theorem 3. We discuss this condition in Section 4.6.

### 4.5 General Cases with Asymmetric Arc Travel Time Functions

We consider asymmetric arc travel time functions (22), in which case Lemma 7 is not applicable. We first observe that the multiple of a PRUE flow, \((1 + \kappa)f^0\), provides a satisficing solution to the traffic equilibrium problem with the increased travel demand.

**Lemma 8.** Suppose \( t_a(\cdot) \) are polynomials of order \( n \) as defined (22). If \( f^0 \in F \) is a PRUE flow, then \((1 + \kappa)f^0\) is a \( \sigma \)-MSatUE flow with \( \sigma = (1 + \kappa)^n - 1 \) in \( F_{1+\kappa} \). When \( n = 1 \), we have \( \sigma = \kappa \).

**Proof.** Let \( \mathbf{f} = (1 + \kappa)f^0 \), and \( \mathbf{v} = (1 + \kappa)v^0 \) for the corresponding arc flow vectors. If the condition

\[
\sum_{a \in A} \left( \sum_{m=0}^{n} \lambda_{am} b_{am} \left( d_{am}^\top \mathbf{v} \right)^m \right) (v'_a - \overline{v}_a) \geq 0 \quad \forall \mathbf{v}' \in V_{1+\kappa}
\]

(32)

holds for some constants \( \lambda_{am} \in \left[ \frac{1}{1+\sigma}, 1 \right] \) for \( m = 0, 1, \ldots, n \) and \( a \in A \), then we can find \( \lambda_a \in \left[ \frac{1}{1+\sigma}, 1 \right] \) such that

\[
\lambda_a \sum_{m=0}^{n} b_{am} \left( d_{am}^\top \mathbf{v} \right)^m = \sum_{m=0}^{n} \lambda_{am} b_{am} \left( d_{am}^\top \mathbf{v} \right)^m
\]

for all \( a \in A \); consequently, by Lemmas 1 and 3, \( \mathbf{f} \) is a \( \sigma \)-MSatUE flow in \( F_{1+\kappa} \).

Since \( v^0 \) is PRUE for \( V \), we know that

\[
\sum_{a \in A} \left( \sum_{m=0}^{n} b_{am} \left( d_{am}^\top v^0 \right)^m \right) (v_a - v_a^0) \geq 0 \quad \forall \mathbf{v} \in V.
\]

Therefore

\[
\sum_{a \in A} \left( \sum_{m=0}^{n} \frac{1}{(1 + \kappa)^m} b_{am} \left( (1 + \kappa) d_{am}^\top v^0 \right)^m \right) ((1 + \kappa)v_a - (1 + \kappa)v_a^0) \geq 0 \quad \forall \mathbf{v} \in V.
\]
Letting for all \( a \in A \)
\[
\lambda_m = \frac{1}{(1 + \kappa)^m}, \quad m = 0, 1, \ldots, n
\]
\[
\nu_a = (1 + \kappa)v_a^0, \\
\nu_a' = (1 + \kappa)v_a,
\]
we observe that \( \lambda_m \in \left[ \frac{1}{1 + \sigma}, 1 \right] \) and we obtain (32); hence proof.

By introducing an additional condition, we compare MSatUE flows with the proportional travel demand increase, and obtain the worst-case bound of PoSat.

**Theorem 3.** Let \( f^\kappa \in F \) be any \( \kappa \)-MSatUE and \( \hat{f}^\sigma \in F_{1+\kappa} \) be any \( \sigma \)-MSatUE flows with the corresponding travel demands, when \( \sigma = (1 + \kappa)^n - 1 \). Suppose that \( \kappa \geq 0 \) is sufficiently small, in particular, so that
\[
\sum_{p \in P} [c_p(\hat{f}^\sigma) - c_p(f^\kappa)](\hat{f}_p^\sigma - f_p^\kappa) \geq \sigma \sum_{p \in P} \max\{c_p(\hat{f}^\sigma), c_p(f^\kappa)\}(\hat{f}_p^\sigma - f_p^\kappa). \tag{33}
\]
Then we have \( C(f^\kappa) \leq C(\hat{f}^\sigma) \). Consequently \( C(f^\kappa) \leq (1 + \kappa)^{n+1}C(f^0) \), and
\[
\sup_{(G, Q, t) \in \Omega(n)} PoSat_{\kappa}(G, Q, t) = (1 + \kappa)^{n+1}
\]

**Proof of Theorem 3.** We decompose \( P \) for each OD pair \( w \) into the following four subsets:
\[
\begin{align*}
P_w^1 &= \{ p \in P_w : \hat{f}_p^\sigma > 0, f_p^\kappa > 0, \hat{f}_p^\sigma - f_p^\kappa \geq 0 \}, \\
P_w^2 &= \{ p \in P_w : \hat{f}_p^\sigma > 0, f_p^\kappa > 0, \hat{f}_p^\sigma - f_p^\kappa < 0 \}, \\
P_w^3 &= \{ p \in P_w : \hat{f}_p^\sigma > 0, f_p^\kappa = 0 \}, \\
P_w^4 &= \{ p \in P_w : \hat{f}_p^\sigma = 0, f_p^\kappa > 0 \}.
\end{align*}
\]
We ignore cases with \( \hat{f}_p^\sigma = 0 \) and \( f_p^\kappa = 0 \). Note that \( \hat{f}_p^\sigma - f_p^\kappa > 0 \) for \( p \in P_w^1 \) and \( \hat{f}_p^\sigma - f_p^\kappa < 0 \) for \( p \in P_w^4 \). From the definition of MSatUE flows, we have
\[
\begin{align*}
\hat{f}_p^\sigma > 0 & \implies c_p(\hat{f}^\sigma) \leq (1 + \sigma)\mu_w(\hat{f}^\sigma), \\
f_p^\kappa > 0 & \implies c_p(f^\kappa) \leq (1 + \kappa)\mu_w(f^\kappa),
\end{align*}
\]
for all \( p \in P, w \in \mathcal{W} \). In addition, \( \mu_w(\hat{f}^\sigma) \leq c_p(\hat{f}^\sigma) \) and \( \mu_w(f^\kappa) \leq c_p(f^\kappa) \) for all \( p \in P \) by definition. Therefore, we have
\[
\sum_{p \in P} [c_p(\hat{f}^\sigma) - c_p(f^\kappa)](\hat{f}_p^\sigma - f_p^\kappa)
\]

This is natural, since we consider more general classes of travel time functions. From (33), we obtain
\[
\begin{align*}
\leq & \sum_{w \in W} \left\{ \sum_{p \in P_w} \left[ (1 + \sigma)\mu_w(\hat{f}_p^\sigma) - \mu_w(f^\sigma) \right] (\hat{f}_p^\sigma - f_p^\sigma) + \sum_{p \in P_w} \left[ \mu_w(\hat{f}_p^\sigma) - (1 + \kappa)\mu_w(f^\sigma) \right] (\hat{f}_p^\sigma - f_p^\sigma) \right. \\
& + \left. \sum_{p \in P^\Delta_w} \left[ (1 + \sigma)\mu_w(\hat{f}_p^\sigma) - \mu_w(f^\sigma) \right] (\hat{f}_p^\sigma - f_p^\sigma) + \sum_{p \in P^\Delta_w} \left[ \mu_w(\hat{f}_p^\sigma) - (1 + \kappa)\mu_w(f^\sigma) \right] (\hat{f}_p^\sigma - f_p^\sigma) \right\} \\
= & \sum_{w \in W} \left\{ \sum_{p \in P_w} \left[ \mu_w(\hat{f}_p^\sigma) - \mu_w(f^\sigma) \right] (\hat{f}_p^\sigma - f_p^\sigma) + \sigma \sum_{p \in P^\Delta_w} \mu_w(\hat{f}_p^\sigma) (\hat{f}_p^\sigma - f_p^\sigma) \right. \\
& \left. - \kappa \sum_{p \in P^\Delta_w \cup P^\Delta_w} \mu_w(f^\sigma) (\hat{f}_p^\sigma - f_p^\sigma) \right\} \\
\leq & \sum_{w \in W} \left\{ \sum_{p \in P_w} \left[ \mu_w(\hat{f}_p^\sigma) - \mu_w(f^\sigma) \right] (\hat{f}_p^\sigma - f_p^\sigma) + \sigma \sum_{p \in P^\Delta_w} \max \{ \mu_w(\hat{f}_p^\sigma), \mu_w(f^\sigma) \} (\hat{f}_p^\sigma - f_p^\sigma) \right\} \\
\leq & \sum_{w \in W} \sum_{p \in P_w} \left[ \mu_w(\hat{f}_p^\sigma) - \mu_w(f^\sigma) \right] (\hat{f}_p^\sigma - f_p^\sigma) + \sigma \sum_{p \in P^\Delta_w} \max \{ c_p(\hat{f}_p^\sigma), c_p(f^\sigma) \} (\hat{f}_p^\sigma - f_p^\sigma).
\end{align*}
\]

From (33), we obtain
\[
0 \leq \sum_{w \in W} \sum_{p \in P} \left[ \mu_w(\hat{f}_p^\sigma) - \mu_w(f^\sigma) \right] (\hat{f}_p^\sigma - f_p^\sigma) \\
= \sum_{w \in W} \left[ \mu_w(\hat{f}_w^\sigma) - \mu_w(f^\sigma) \right] \left( \sum_{p \in P_w} \hat{f}_p^\sigma - \sum_{p \in P_w} f_p^\sigma \right) \\
= \sum_{w \in W} \left[ \mu_w(\hat{f}_w^\sigma) - \mu_w(f^\sigma) \right] (\hat{Q}_w - Q_w) \\
= \kappa \sum_{w \in W} \mu_w(\hat{f}_w^\sigma) Q_w - \kappa \sum_{w \in W} \mu_w(f^\sigma) Q_w \\
= \frac{\kappa}{1 + \kappa} \sum_{w \in W} \mu_w(\hat{f}_w^\sigma) \hat{Q}_w - \kappa \sum_{w \in W} \mu_w(f^\sigma) Q_w \\
\leq & \frac{\kappa}{1 + \kappa} \sum_{w \in W} \sum_{p \in P_w} c_p(\hat{f}_p^\sigma) \hat{f}_p^\sigma - \frac{\kappa}{1 + \kappa} \sum_{w \in W} \sum_{p \in P_w} c_p(f^\sigma) f_p^\sigma \\
= & \frac{\kappa}{1 + \kappa} C(\hat{f}^\sigma) - \frac{\kappa}{1 + \kappa} C(f^\sigma).
\]

\[\Box\]

Lemmas 6 and 8 complete the proof.

Note that condition (33) is stronger than condition (31) for separable travel time functions. This is natural, since we consider more general classes of travel time functions.

### 4.6 Illustrative Examples

For the illustration purpose, we consider two examples in Figure 4 with linear travel time functions, where \( n = 1 \). In Example 1, the travel time function in the first arc is not increasing. We can verify
that

$$\max C(f^\kappa) = \begin{cases} Q + \kappa^2 & \text{if } \kappa \leq Q, \\ (1 + Q)Q & \text{if } \kappa \geq Q, \end{cases} \quad \text{with } f^\kappa = (Q - \kappa, \kappa)$$

among all $\kappa$-MSatUE flows in $F$ and

$$C(\hat{f}^0) = (1 + \kappa)Q \quad \text{with } \hat{f}^0 = (1 + \kappa)f^0 = ((1 + \kappa)Q, 0).$$

among all $\kappa$-MSatUE flows in $F_{1+\kappa}$. Comparing the two quantities, we observe $C(f^\kappa) \leq C(\hat{f}^0)$ in both cases. To prove Theorem 3, condition (31) needs to hold only for these two flow vectors. Regardless of the value of $\kappa$, however, it is impossible to satisfy condition (31), although the worst-case PoSat bound $(1 + \kappa)^{n+1}$ still holds for all $\kappa \geq 0$. The price of satisficing is $1 + \frac{\kappa^2}{Q}$ if $\kappa < Q$ and $1 + Q$ if $\kappa \geq Q$ in this example, both of which are less than $(1 + \kappa)^2$.

On the other hand, in Example 2, we have strictly monotone travel time functions in both arcs. Similarly, we consider

$$\max C(v^\kappa) = \frac{2 + 2\kappa + \kappa^2}{(2 + \kappa)^2}Q \quad \text{with } f^\kappa = \left(\frac{Q}{2 + \kappa}, \frac{(1 + \kappa)Q}{2 + \kappa}\right)$$

and can verify that $C(f^\kappa) \leq C(\hat{f}^0)$ for all $\kappa \geq 0$. In Example 2, we note that (33) holds for $\kappa \leq 0.206$. In this example, we observe that the price of satisficing is $\frac{2(2 + 2\kappa + \kappa^2)}{(2 + \kappa)^2}$, which is no greater than $(1 + \kappa)^2$ for all $\kappa \geq 0$.

### 4.7 Other Approaches

When there is a single origin and multiple destinations, i.e., a single common origin node, in the network, Kleer and Schäfer (2016) introduces the notion of the deviation ratio that compares the system performances of the user equilibrium and the equilibrium with deviated travel time functions $\tilde{t}_a(\cdot)$. The notion of deviation may also be interpreted as perception in our definition. In a special case, the deviation ratio is reduced to the price of risk aversion (Nikolova and Stier-Moses, 2015) that compares the performances of equilibria among risk-averse and risk-neutral network users.
Kleer and Schäfer (2016) define the (separable) deviated travel time functions with the following bounds:

\[ t_a(v_a) + \alpha t_a(v_a) \leq \tilde{t}_a(v_a) \leq t_a(v_a) + \beta t_a(v_a) \] (34)

where \(-1 \leq \alpha \leq 0 \leq \beta\). The consideration of this deviated travel time function generalizes our UE-PE model where \(\alpha = -\frac{\kappa}{1+\kappa}\) and \(\beta = 0\). Kleer and Schäfer (2016) show that the worst-case deviation ratio with (34) is bounded by

\[ 1 + \frac{\beta - \alpha}{1 + \alpha} \left\lfloor \frac{|\mathcal{N}| - 1}{2} \right\rfloor Q. \]

Therefore, we obtain the following theorem:

**Theorem 4** (Kleer and Schäfer, 2016). Consider a directed graph with a single common origin node with the total travel demand \(Q\) and let \(|\mathcal{N}|\) be the number of nodes. Then we have

\[ \frac{Z(v^\kappa)}{Z(v^0)} \leq 1 + \kappa \left\lfloor \frac{|\mathcal{N}| - 1}{2} \right\rfloor Q \] (35)

where \(v^\kappa\) is a solution to UE-PE-V in (16).

Note that Theorem 4 only covers a subset of the entire MSatUE flows, as it is limited to the solutions UE-PE-V in (16) and is applicable to cases with a single common origin. When Theorem 4 is applied in the examples in Figure 4, the bound (35) becomes \(1 + \kappa Q\).

5 Numerical Bounds

To quantify PoSat in typical traffic networks and compare it with the analytical bound obtained in Theorem 3, we define the worst-case problem for the total system travel time under MSatUE as follows:

\[
\max_{v^\kappa} Z(v^\kappa) = \sum_{a \in A} z_a(v^\kappa_a) = \sum_{a \in A} t_a(v^\kappa) v^\kappa_a
\] (36)

subject to \( v^\kappa\) is an MSatUE flow with \(\kappa\)

To quantify the benefit of satisficing, instead of maximizing, we can minimize the objective function (36). Since MSatUE involves path-based definition and formulation, (36) is numerically more challenging to solve. Instead, we replace MSatUE by UE-PE-X. We know that the UE-PE-X models provide a subset of MSatUE traffic flow patterns as seen in Lemma 1; hence by using UE-PE-X models, we will obtain suboptimal solutions to (36).

Using UE-PE-X in (12), we formulate the worst-case problem as follows:

\[
\max_{\mathbf{v}, \mathbf{v}_c} Z(\mathbf{v}) = \sum_{a \in A} z_a(\mathbf{v}) = \sum_{a \in A} t_a(\mathbf{v}) v_a
\] (37)
subject to \[
\sum_{a \in A} \sum_{w \in W} (t_a(\nu) - \varepsilon^w_a)(x_a^w - \pi^w_a) \geq 0 \quad \forall x \in X \tag{38}
\]
\[
\bar{\nu}_a = \sum_{w \in W} \pi^w_a \quad \forall a \in A \tag{39}
\]
\[
\bar{x} \in X
\]
\[
0 \leq \varepsilon^w_a \leq \frac{\kappa}{1 + \kappa} t_a(\nu) \quad \forall a \in A \tag{41}
\]

Problem (37) is an instance of mathematical programs with equilibrium constraints (MPEC). We can replace the equilibrium condition (38) by the following KKT conditions to create a single-level optimization problem:

\[
t_a(\nu) - \varepsilon^w_a + \pi^w_i - \pi^w_j \geq 0 \quad \forall w \in W, a \in A \tag{42}
\]
\[
\pi^w_a (t_a(\nu) - \varepsilon^w_a + \pi^w_i - \pi^w_j) = 0 \quad \forall w \in W, a \in A \tag{43}
\]
\[
\sum_{a \in A^+_i} \pi^w_a - \sum_{a \in A^-_i} \pi^w_a = q^w_i \quad \forall w \in W, i \in N \tag{44}
\]

The resulting problem is a mathematical program with complementarity conditions (MPCC), which is nonlinear and nonconvex. Finding a global solution to MPCC problems is in general difficult, and Kleer and Schäfer (2016) has shown that solving the above MPCC optimally is NP-hard. In order to solve this problem, we use an interior point method by utilizing the Ipopt nonlinear solver (Wächter and Biegler, 2006) with multiple starting solutions.

5.1 Numerical Experiments

In this section we present some examples to compare the total travel times in MSatUE and PRUE numerically for both separable and asymmetric networks. We approximate MSatUE by UE-PE-X and solve it by the Ipopt nonlinear solver, after reformulating (36) as a single-level optimization problem using KKT conditions. We use the Julia Language and the JuMP package (Dunning et al., 2017) for modeling and interfacing with the Ipopt solver.

5.1.1 Simple Networks

To test the validity and the strength of UE-PE-X model, we first consider Examples 1 and 2 in Figure 4. Figure 5 compares the PoSat under UE-PE-X with the PoSat under MSatUE, obtained in Section 4.6. As Figure 5 shows, the PoSat under UE-PE-X is equal to the PoSat under MSatUE for both examples, which suggests that the UE-PE-X model is an effective model.

5.1.2 Circular Network of Christodoulou et al. (2011)

We also compute the PoSat under UE-PE-X model for the circular network of Christodoulou et al. (2011) presented in Figure 2. For numerical experiments, we assign \( m \) and \( l \) to the smallest positive integers such that \( \frac{m}{l} = (1 + \kappa)^5 \), and \( \kappa \in \{0, 0.1, 0.2, ..., 1\} \). We also set \( n = 4 \). As it can be seen in
Figure 5: PoSat for the simple networks in Figure 4

(a) Example 1 with $Q = 1$

(b) Example 2

Figure 6: PoSat for the circular network in Figure 2

Figure 6, we obtained identical results for circular network under UE-PE-X, using the Ipopt solver, which shows that the UE-PE-X model can obtain the upper bound provided in Lemma 5.

5.1.3 Larger Networks

We present some examples to compare the total travel times in MSatUE and PRUE numerically and compare the numerical worst-cases with the analytical bound given in Theorem 2 for larger networks with both separable and non-separable, asymmetric arc cost functions. As (36) is a non-convex problem, the Ipopt solver can produce a local minimum at best. To obtain a higher-quality local minimum, we solve the problem multiple times by using different initial solution. For generating different initial solutions for the network with separable arc cost functions, we utilize UE-PE-V model. We generate initial $\lambda$ randomly and use the Frank-Wolfe algorithm to obtain the corresponding $v$ and $x$. For the network with non-separable cost function, we can use the fixed
point method (Dafermos, 1980) with a randomized $\lambda$ to obtain an initial solution $x$. We randomly generate five initial starting points for each example and report the largest PoSat values.

We first consider the nine-node network presented in Hearn and Ramana (1998). The nine-node network consists of 9 nodes and 18 arcs, and the travel time functions are polynomials of order $n = 4$. We also create an asymmetric variant of the nine-node network as shown in Figure 9 in Appendix B. The asymmetric nine-node network has non-separable arc cost function in the form of (49). The comparison result is presented in Figure 7a. As Figure 7a represents $\text{PoSat}_\kappa$ increases with $\kappa$ for both symmetric and asymmetric nine-node network since $\text{PRUE}_\text{total travel time}$ is fixed with respect to $\kappa$, while the worst-case $\text{MSat}_\text{UE}_\text{total travel time}$ increases as $\kappa$ increases. Moreover, $\text{PoSat}_\kappa$ is smaller for symmetric nine-node network compared to the asymmetric nine-node network for smaller $\kappa$ values (0.1 and 0.2), but it is greater for larger $\kappa$ values ($\kappa \geq 0.3$). In general, the gap between $\text{PoSat}_\kappa$ for symmetric nine-node network and asymmetric nine-node network is small.

Figure 7b compares the numerical $\text{PoSat}_\kappa$ with the analytical bound provided in Theorem 2 for $\text{MSat}_\text{UE}$ for the nine-node network. We observe that there is a large gap between the analytical and numerical bounds which increases with $\kappa$. Although the analytical result certainly provides a valid bound, it is too large to be practically useful in realistic road networks. This indicates opportunities for empirical studies on the analytical bounds that depend on more network-specific information such as travel demands and travel time functions. The bound $(1 + \kappa)^{n+1}$ in Theorem 2 is independent from such network-specific information.

We also consider the Sioux Falls network presented in Suwansirikul et al. (1987), which consists of 24 nodes, 76 arcs, and 576 OD pairs. The arc travel cost function is the BPR function, which is a polynomial function with degree $n = 4$. We also consider an asymmetric variant of Sioux Falls network with arc cost function in the form of (49). As Figure 8a represents, $\text{PoSat}_\kappa$ increases with $\kappa$ for both symmetric and asymmetric Sioux Falls network, and it is greater compared to the nine-node network for both symmetric and asymmetric networks. Furthermore, $\text{PoSat}_\kappa$ is greater for
asymmetric Sioux Falls network compared with the symmetric Sioux Falls network for all positive $\kappa$ values, and the gap between $\text{PoSat}_\kappa$ for symmetric Sioux Falls network and $\text{PoSat}_\kappa$ for asymmetric Sioux Falls network increases with $\kappa$. Figure 8b compares the numerical $\text{PoSat}_\kappa$ with the analytical bound. The gap between the analytical and the numerical bound is tighter compared to the nine-node network, but it is still considerable.

6 Concluding Remarks

When network users are satisficing decision makers, the resulting satisficing user equilibria may degrade the system performance, compared to the perfectly rational user equilibrium. To quantify how much the performance can deteriorate, this paper has quantified the worst-case analytical bound on the price of satisficing. We also quantified the price of satisficing for several networks numerically and compare it to the analytical bound.

As we have seen in the numerical examples in this paper, there is a large gap between the worst-case analytical bound and the actual bound. Clearly, this is a limitation of our approach. In the literature of the price of anarchy have similar observations been reported (O’Hare et al., 2016; Monnot et al., 2017; Colini-Baldeschi et al., 2017). Likewise, the behavior of the price of satisficing in practice can be quite different from what we have observed in this paper. Deriving empirical or network-specific bounds can be meaningful contributions as a future research direction.

We suggest additional potential future research directions. For the proposed analytical bound, our result is based on the condition (33). By attempting to relax this condition, one may obtain a global bound for any value of $\kappa$. In deriving the analytical bound, we utilized a novel technique comparing equilibrium patterns before and after the travel demand is increased; namely $V$ and $V_{1+\kappa}$. Applying this technique in the context of the price of risk aversion and the deviation ratio would be an interesting research direction.
Acknowledgments

This research was partially supported by the National Science Foundation under grant CMMI-1351357. The authors are thankful for the anonymous reviewers whose constructive comments have greatly improved this manuscript.

References

Beckmann, M., C. McGuire, C. Winsten. 1956. *Studies in the Economics of Transportation*. Yale University Press, New Haven, Conn.

Brown, D. B., M. Sim. 2009. Satisficing Measures for Analysis of Risky Positions. *Management Science* 55(1) 71–84.

Charnes, A., W. W. Cooper. 1963. Deterministic equivalents for optimizing and satisficing under chance constraints. *Operations Research* 11(1) 18–39.

Chen, H.-C., J. W. Friedman, J.-F. Thisse. 1997. Boundedly rational Nash equilibrium: a probabilistic choice approach. *Games and Economic Behavior* 18(1) 32–54.

Christodoulou, G., E. Koutsoupias, P. G. Spirakis. 2011. On the performance of approximate equilibria in congestion games. *Algorithmica* 61(1) 116–140.

Colini-Baldeschi, R., R. Cominetti, P. Mertikopoulos, M. Scarosini. 2017. The asymptotic behavior of the price of anarchy. *International Conference on Web and Internet Economics*. Springer, 133–145.

Dafermos, S. 1980. Traffic equilibrium and variational inequalities. *Transportation Science* 14(1) 42–54.

Dafermos, S., A. Nagurney. 1984. Sensitivity analysis for the asymmetric network equilibrium problem. *Mathematical Programming* 28(2) 174–184.

de Palma, A., Y. Nesterov. 1998. Optimization formulations and static equilibrium in congested transportation networks. *CORE Discussion Paper* 9861, Université Catholique de Louvain, Louvain-la-Neuve. 12–17.

Di, X., X. He, X. Guo, H. X. Liu. 2014. Braess paradox under the boundedly rational user equilibria. *Transportation Research Part B: Methodological* 67 86–108.

Di, X., H. X. Liu. 2016. Boundedly rational route choice behavior: A review of models and methodologies. *Transportation Research Part B: Methodological* 85 142–179.

Di, X., H. X. Liu, J.-S. Pang, X. J. Ban. 2013. Boundedly rational user equilibria (BRUE): Mathematical formulation and solution sets. *Transportation Research Part B: Methodological* 57 300–313.
Dunning, I., J. Huchette, M. Lubin. 2017. JuMP: A modeling language for mathematical optimization. *SIAM Review* **59**(2) 295–320. doi: 10.1137/15M1020575.

Englert, M., T. Franke, L. Olbrich. 2010. Sensitivity of Wardrop equilibria. *Theory of Computing Systems* **47**(1) 3–14.

Friesz, T. L., D. Bernstein. 2016. *Foundations of network optimization and games*. Springer.

Ge, Y., X. Zhou. 2012. An alternative definition of dynamic user optimum on signalised road networks. *Journal of Advanced Transportation* **46**(3) 236–253.

Guo, X. 2013. Toll sequence operation to realize target flow pattern under bounded rationality. *Transportation Research Part B: Methodological* **56** 203–216.

Han, K., W. Szeto, T. L. Friesz. 2015. Formulation, existence, and computation of boundedly rational dynamic user equilibrium with fixed or endogenous user tolerance. *Transportation Research Part B: Methodological* **79** 16 – 49. doi: http://dx.doi.org/10.1016/j.trb.2015.05.002. URL http://www.sciencedirect.com/science/article/pii/S0191261515000971.

Hearn, D. W., M. V. Ramana. 1998. Solving congestion toll pricing models. *Equilibrium and Advanced Transportation Modelling*. Springer, 109–124.

Jaillet, P., S. D. Jena, T. S. Ng, M. Sim. 2016. Satisficing awakens: Models to mitigate uncertainty. *Working Paper* URL http://www.optimization-online.org/DB_HTML/2016/01/5310.html.

Kleer, P., G. Schäfer. 2016. The impact of worst-case deviations in non-atomic network routing games. *International Symposium on Algorithmic Game Theory*. Springer, 129–140.

Koutsoupias, E., C. Papadimitriou. 1999. Worst-case equilibria. *Annual Symposium on Theoretical Aspects of Computer Science*. Springer, 404–413.

Lam, S.-W., T. S. Ng, M. Sim, J.-H. Song. 2013. Multiple objectives satisficing under uncertainty. *Operations Research* **61**(1) 214–227.

Lou, Y., Y. Yin, S. Lawphongpanich. 2010. Robust congestion pricing under boundedly rational user equilibrium. *Transportation Research Part B: Methodological* **44**(1) 15–28.

Mahmassani, H. S., G.-L. Chang. 1987. On boundedly rational user equilibrium in transportation systems. *Transportation Science* **21**(2) 89–99.

Meng, Q., Z. Liu, S. Wang. 2014. Asymmetric stochastic user equilibrium problem with elastic demand and link capacity constraints. *Transportmetrica A: Transport Science* **10**(4) 304–326.

Monnot, B., F. Benita, G. Piliouras. 2017. How bad is selfish routing in practice? *arXiv preprint arXiv:1703.01599*.
Nakayama, S., R. Kitamura, S. Fujii. 2001. Drivers’ route choice rules and network behavior: Do drivers become rational and homogeneous through learning? Transportation Research Record: Journal of the Transportation Research Board 1752(1) 62–68.

Nikolova, E., N. E. Stier-Moses. 2015. The burden of risk aversion in mean-risk selfish routing. Proceedings of the Sixteenth ACM Conference on Economics and Computation. ACM, 489–506.

O’Hare, S. J., R. D. Connors, D. P. Watling. 2016. Mechanisms that govern how the price of anarchy varies with travel demand. Transportation Research Part B: Methodological 84 55–80.

Panicucci, B., M. Pappalardo, M. Passacantando. 2007. A path-based double projection method for solving the asymmetric traffic network equilibrium problem. Optimization Letters 1(2) 171–185.

Patriksson, M. 2015. The traffic assignment problem: models and methods. Courier Dover Publications.

Perakis, G. 2007. The “price of anarchy” under nonlinear and asymmetric costs. Mathematics of Operations Research 32(3) 614–628.

Roughgarden, T., É. Tardos. 2002. How bad is selfish routing? Journal of the ACM (JACM) 49(2) 236–259.

Sheffi, Y. 1985. Urban transportation networks: equilibrium analysis with mathematical programming methods.

Simon, H. A. 1955. A behavioral model of rational choice. The Quarterly Journal of Economics 99–118.

Simon, H. A. 1956. Rational choice and the structure of the environment. Psychological Review 63(2) 129.

Smith, M. J. 1979. The existence, uniqueness and stability of traffic equilibria. Transportation Research Part B: Methodological 13(4) 295–304.

Sun, L., M. H. Karwan, C. Kwon. 2018. Generalized bounded rationality and robust multi-commodity network design. Operations Research 66(1) 42–57.

Suwansirikul, C., T. L. Friesz, R. L. Tobin. 1987. Equilibrium decomposed optimization: a heuristic for the continuous equilibrium network design problem. Transportation Science 21(4) 254–263.

Szeto, W., H. K. Lo. 2006. Dynamic traffic assignment: properties and extensions. Transportmetrica 2(1) 31–52.

Takalloo, M., C. Kwon. 2018. Sensitivity of Wardrop equilibria: revisited. Working Paper URL http://www.chkwon.net/papers/takalloo_sensitivity.pdf.
Wächter, A., L. T. Biegler. 2006. On the implementation of an interior-point filter line-search algorithm for large-scale nonlinear programming. *Mathematical Programming* **106**(1) 25–57.

Wu, J., H. Sun, D. Z. Wang, M. Zhong, L. Han, Z. Gao. 2013. Bounded-rationality based day-to-day evolution model for travel behavior analysis of urban railway network. *Transportation Research Part C: Emerging Technologies* **31** 73–82.

Xu, H., H. Yang, J. Zhou, Y. Yin. 2017. A route choice model with context-dependent value of time. *Transportation Science* **51**(2) 536–548.

Ye, H., H. Yang. 2017. Rational behavior adjustment process with boundedly rational user equilibrium. *Transportation Science* **51**(3) 968–980.

Zhu, S., D. Levinson. 2010. Do people use the shortest path? An empirical test of Wardrop’s first principle. *91th Annual Meeting of the Transportation Research Board, Washington*, vol. 8.

**Appendices**

**A Proof of Lemma 7**

*Proof of Lemma 7.* This is a minor variant to the proof of Englert et al. (2010, Theorem 3). Since $v^0 \in F$ and $\tilde{v}^0 \in F_{1+\kappa}$ are PRUE flows that minimize $\Phi(\cdot)$ over their corresponding feasible sets, we have

$$\Phi(v^0) \leq \Phi\left(\frac{\tilde{v}^0}{1 + \kappa}\right) \quad \text{and} \quad \Phi(\tilde{v}^0) \leq \Phi((1 + \kappa)v^0),$$

which imply

$$ (1 + \kappa)^{n+1} \sum_{a \in A} \frac{b_a}{n + 1} (v_a^0)^{n+1} \leq \sum_{a \in A} \frac{b_a}{n + 1} (\tilde{v}_a^0)^{n+1} \quad (45) $$

and

$$ \sum_{a \in A} \frac{b_a}{n + 1} (\tilde{v}_a^0)^{n+1} \leq (1 + \kappa)^{n+1} \sum_{a \in A} \frac{b_a}{n + 1} (v_a^0)^{n+1} \quad (46) $$

respectively.

Let us assume that $C(f^0) > (1 + \kappa)^{n+1}C(f^0)$, which is equivalent to

$$ (1 + \kappa)^{n+1} \sum_{a \in A} b_a (v_a^0)^{n+1} < \sum_{a \in A} b_a (\tilde{v}_a^0)^{n+1}. \quad (47) $$

From $n \times (45) + ((n + 1)(1 + \kappa)^n - 1) \times (46) + ((1 + \kappa)^n - 1) \times (47)$, we obtain

$$ \theta_1 \sum_{a \in A} \frac{b_a (v_a^0)^{n+1}}{n + 1} < \theta_2 \sum_{a \in A} \frac{b_a (\tilde{v}_a^0)^{n+1}}{n + 1} \quad (48) $$

30
where
\[
\theta_1 = n \cdot \frac{(1 + \kappa)^{n+1}}{n + 1} - ((n + 1)(1 + \kappa)^n - 1) \cdot \frac{(1 + \kappa)^{n+1}}{n + 1} + \frac{(1 + \kappa)^n - 1}{n + 1} = 0
\]
\[
\theta_2 = n \cdot \frac{1}{n + 1} - ((n + 1)(1 + \kappa)^n - 1) \cdot \frac{1}{n + 1} + \frac{(1 + \kappa)^n - 1}{n + 1} = 0
\]
for all \( \kappa \geq 0 \). Therefore, (48) leads to \( 0 < 0 \), which is a contradiction. We conclude that \( C(\hat{f}^0) \leq (1 + \kappa)^{n+1}C(f^0) \).

\[\Box\]

**B Nine-node Asymmetric Networks**

In order to test the performance of **UE-PE-X** model in an asymmetric network, we create an asymmetric version of the nine-node network considered by Hearn and Ramana (1998). In the asymmetric nine-node network, which has been shown in Figure 9, we add a few additional arcs and assume that the arc travel cost function is:

\[
t_a(v) = A_a + B_a \left( \frac{0.5v_a + \hat{v}_a}{C_a} \right)^4
\]

(49)

where \( \hat{a} \) is the flow in the opposite arc. Thus, the arc travel function depends not only on the flow in that arc, but also on the flow in the arc in opposite direction. The values of parameters \( A_a, B_a \) and \( C_a \) are given in Table 1 for each arc.
Table 1: asymmetric nine-node network arc cost function parameters

|   |   |   |   |
|---|---|---|---|
| (1,5) | 12 | 1.80 | 5 |
| (1,6) | 18 | 2.70 | 6 |
| (2,5) | 35 | 5.25 | 3 |
| (2,6) | 35 | 5.25 | 9 |
| (5,6) | 20 | 3.00 | 9 |
| (5,7) | 11 | 1.65 | 2 |
| (5,9) | 26 | 3.90 | 8 |
| (6,8) | 33 | 4.95 | 6 |
| (6,9) | 30 | 4.50 | 8 |
| (7,3) | 25 | 3.75 | 3 |
| (7,4) | 24 | 3.60 | 6 |
| (7,8) | 19 | 2.85 | 2 |
| (8,3) | 39 | 5.85 | 8 |
| (8,4) | 43 | 6.45 | 6 |
| (9,7) | 26 | 3.90 | 4 |
| (9,8) | 30 | 4.50 | 8 |
| (5,1) | 12 | 1.80 | 5 |
| (6,1) | 18 | 2.70 | 6 |
| (5,2) | 35 | 5.25 | 3 |
| (6,2) | 35 | 5.25 | 9 |
| (6,5) | 20 | 3.00 | 9 |
| (7,5) | 11 | 1.65 | 2 |
| (9,5) | 26 | 3.90 | 8 |
| (8,6) | 33 | 4.95 | 6 |
| (9,6) | 30 | 4.50 | 8 |
| (3,7) | 25 | 3.75 | 3 |
| (4,7) | 24 | 3.60 | 6 |
| (8,7) | 19 | 2.85 | 2 |
| (3,8) | 39 | 5.85 | 8 |
| (4,8) | 43 | 6.45 | 6 |
| (7,9) | 26 | 3.90 | 4 |
| (8,9) | 30 | 4.50 | 8 |