UPPER BOUND ON THE EDGE FOLKMAN NUMBER $F_e(3, 3, 3; 13)$

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Abstract

For a given graph $G$ let $V(G)$ and $E(G)$ denote the vertex and the edge set of $G$ respectively. The symbol $G \rightarrow (a_1, \ldots, a_r)$ means that in every $r$-coloring of $E(G)$ there exists a monochromatic $a_i$-clique of color $i$ for some $i \in \{1, \ldots, r\}$. The edge Folkman numbers are defined by the equality

$$F_e(a_1, \ldots, a_r; q) = \min \{|V(G)| : G \rightarrow (a_1, \ldots, a_r; q) \text{ and } cl(G) < q\}.$$

It is clear from the definition of edge Folkman numbers that they are a generalization of the classical Ramsey numbers. The problem of computation of edge Folkman numbers is extremely difficult and so far only eleven edge Folkman numbers are known. In this paper we prove the following upper bound on the number $F_e(3, 3, 3; 13)$, namely $F_e(3, 3, 3; 13) \leq 30$. So far it was only known that $F_e(3, 3, 3; 13) < \infty$.

Key words: Edge Folkman numbers

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1 Introduction

We consider only finite, non-oriented graphs without loops and multiple edges. We call a $p$-clique of the graph $G$ a set of $p$ vertices, each two of which are adjacent. The largest positive integer $p$, such that the graph $G$ contains a $p$-clique is called a clique number of $G$ and is denoted by $cl(G)$. We denote by

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$V(G)$ and $E(G)$ the vertex set and the edge set of the graph $G$ respectively. We shall also use the following notations

- $N(v), v \in V(G)$ is the set of all vertices of $G$ adjacent to $v$;
- $G[V], V \subseteq V(G)$ is the subgraph of $G$ induced by $V$;
- $G(v) = G[N(v)], v \in V(G)$, that is the subgraph induced by the vertices adjacent to $v$ in $G$;
- $K_n$ is the complete graph on $n$ vertices;
- $C_n$ is the cycle on $n$ vertices;
- if $U_1, U_2 \subseteq V(G)$ then $E(U_1, U_2)$ is the set of all edges in $E(G)$ connecting a vertex $U_1$ with a vertex of $U_2$.

The Zykov sum of two graphs $G_1 + G_2$ is the graph obtained from the graphs $G_1$ and $G_2$ when we connect each vertex from $G_1$ with each vertex from $G_2$.

**Definition 1.** Let $a_1, \ldots, a_r$ be positive integers. The symbol $G \rightarrow (a_1, \ldots, a_r)$ means that for each coloring of the edges of $G$ in $r$ colors ($r$-coloring) there is a monochromatic $a_i$-clique in the $i$-th color for some $i \in \{1, \ldots, r\}$.

The Ramsey number $R(a_1, \ldots, a_r)$ is defined as the least $n$ for which $K_n \rightarrow (a_1, \ldots, a_r)$.

The edge Folkman numbers are defined by the equality

$$F_e(a_1, \ldots, a_r; q) = \min\{|V(G)| : G \rightarrow (a_1, \ldots, a_r; q) \text{ and } cl(G) < q\}.$$ 

It is known that $F_e(a_1, \ldots, a_r; q)$ exists if and only if $q > \max\{a_1, \ldots, a_r\}$. This was proved for two colors by Folkman in [3] and in the general case by Nesertil and Rodl in [12]. It follows from the definition of $R(a_1, \ldots, a_r)$ that $F_e(a_1, \ldots, a_r; q) = R(a_1, \ldots, a_r)$ if $q > R(a_1, \ldots, a_r)$. In particular $F_e(3, 3; 6) = 6$ when $q \geq 6$ because $R(3, 3) = 6$. In 1967 P. Erdos posed the problems to compute $F_e(3, 3; q)$ when $q < 6$. In [4] Graham computed the number $F_e(3, 3; 6) = 8$. He established the upper bound proving that $K_3 + C_5 \rightarrow (3, 3)$. An example of a graph $G$ on 15 vertices with the properties $G \rightarrow (3, 3)$ and $cl(G) < 5$ was constructed by Nenov in [9] thus proving that $F_e(3, 3; 5) \leq 15$. In [13] Piwakowski, Radziszowski, Urbanski proved the opposite inequality $F_e(3, 3; 5) \geq 15$. Thus it was proved that $F_e(3, 3; 5) = 15$. The last of these Erdos's problems: to compute the number $F_e(3, 3; 4)$ is still open. In [2] Dudek and Rodl proved that $F_e(3, 3; 4) \leq 941$. The latest lower bound is $F_e(3, 3; 4) \geq 19$ established by S. Radziszowski and Xu Xiaodong in [14]. All these three Erdos problems were about edge Folkman numbers $F_e(3, 3; q)$ which are not equal to the Ramsey number $R(3, 3) = 6$. 

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Here we shall discuss the edge Folkman numbers $F_e(3, 3; q)$ which are not equal to the Ramsey number $R(3, 3, 3) = 17$. In the most restricted case we know only the general fact that $F_e(3, 3, 3; 4) < \infty$.

Recently Dudek, Frankl, Rodl [1] posed the following

**Problem** Is it true that $F_e(3, 3, 3; 4) \leq 3^{81}$?

Nenov as consequence of a more general result proved in [10] that $F_e(3, 3, 3; 4) \geq 40$. The only edge Folkman numbers that are not equal to $R(3, 3, 3) = 17$ which we know are: $F_e(3, 3, 3; 17) = 19$, [5]; $F_e(3, 3, 3; 16) = 21$ (lower bound in [5] and upper bound in [6]); $F_e(3, 3, 3; 15) = 23$, [7]; $F_e(3, 3, 3; 14) = 25$, [8].

Define the graph $H = C_5 + C_5 + C_5 + C_5 + C_5 + C_5$, that is $H$ is a Zykov sum of six copies of the 5-cycle $C_5$. The main goal of this paper is to prove the following results:

**Theorem** $H \not\rightarrow (3, 3, 3)$.

As $cl(H) = 12$ we obtain the following corollary from the theorem

**Corollary** $F_e(3, 3, 3; 13) \leq 30$.

So far it was only known $F_e(3, 3, 3; 13) < \infty$, which follows from the already cited general result by Nesertil and Rodl in [12] that guarantees the existence of edge Folkman numbers. The latest lower bound $27 \leq F_e(3, 3, 3; 13)$ was obtained by Nenov in [10].

## 2 Preliminary results

Except the graph $H$ that we defined before the theorem we shall also need the following graphs:

$$S = C_5 + C_5 + C_5 + C_5 + C_5 = H - C_5$$

$$T = K_4 + C_5 + C_5 + C_5 + C_5$$

$$L = \hat{K}_4 + C_5 + C_5 + C_5 + C_5,$$

where $\hat{K}_4$ denotes the graph $K_4$ with one edge deleted.

We shall use the following statement from [11].

**Lemma 1** Consider a given disjunct partition of $V(T) = V_1 \cup V_2 \cup V_3$, $V_i \cap V_j = \emptyset$, $i \neq j$, such that $V_i \cap K_4 \neq \emptyset$, for each $i = 1, 2, 3$. Then for some $i$ we have $T[V_i] \not\rightarrow (3, 3)$. 

Consider a coloring of the edges of an arbitrary graph $G$ in three colors (3-coloring). We shall call the colors first, second and third. For each vertex $v \in V(G)$ we denote by $N_1(v), N_2(v), N_3(v)$ its neighbors in first, second and third color respectively. We shall denote $G_i(v) = G[N_i(v)]$ for $i = 1, 2, 3$ and $G(v) = G[N(v)]$. Now we shall prove the following lemmas.

**Lemma 2** Consider a 3-coloring of the edges of an arbitrary graph $G$.

(a) If for some $v \in V(G)$ and for some $i = 1, 2, 3$ we have that $G_i(v) \rightarrow (3, 3)$, then there is a monochromatic triangle in this 3-coloring.

(b) If for some $v \in V(G)$ and for some $i = 1, 2, 3$ we have that $cl(G_i) \geq 6$, then there is a monochromatic triangle in this 3-coloring.

**Proof.** (a) Let for example $G_1(v) \rightarrow (3, 3)$. If some edge in $G_1$ is in first color, then this edge together with the vertex $v$ forms a monochromatic triangle in first color. Therefore all edges in $G_1(v)$ are colored in two colors only (second and third) and it follows from $G_1(v) \rightarrow (3, 3)$ that there is a monochromatic triangle.

(b) The statement of (b) follows directly from (a) and the fact $K_6 \rightarrow (3, 3)$.

**Lemma 3** Consider the graph $Q = K_1 + L = K_1 + \bar{K}_4 + C_5 + C_5 + C_5 + C_5$. We denote by $w$ the only vertex in $K_1$ and by $a$ and $b$ the only non-adjacent vertices in $\bar{K}_4$. Consider a 3-coloring of the edges of the graph $Q$, such that $E(w, V(\bar{K}_4))$ contains edges in all the three colors and the edges $wa$ and $wb$ are in different colors. Then there is a monochromatic triangle in this 3-coloring.

**Proof** The coloring of the edges of $K_1 + L$ into three colors induces in a natural way a disjunct partition of the vertices of $L$ into three sets $V_1, V_2, V_3$, namely: if the edge $wx$ is in color $i$, then the vertex $x$ is in $V_i, i = 1, 2, 3$. We add the edge $ab$. This completes the graph $L$ to the graph $T$. Then we have from Lemma 1 that $T[V_i] \rightarrow (3, 3)$, for some $i = 1, 2, 3$. As the edges $wa$ and $wb$ are in different colors then the vertices $a$ and $b$ are in different sets $V_i$. Thus $T[V_i] = L[V_i] = Q_i(w)$. So $Q_i(w) \rightarrow (3, 3)$ and Lemma 3 follows from Lemma 2(a).

**Lemma 4** Let $v \in V(H)$ and $S$ be the subgraph of $H$, induced by the five 5-cycles of $H$ not containing $v$. Assume that there is a 3-coloring of the edges of $H$ without monochromatic triangles. Then for every such coloring and for each color $i$ we have $N_i(v) \cap V(S) \neq \emptyset$.

**Proof** Assume the opposite. Let for example $N_1(v) \cap V(S) = \emptyset$. Then for each of the five 5-cycles $C_5$ in $S$ there is an edge of the graph $H$ either in $N_2(v) \cap V(C_5)$ or $N_3(v) \cap V(C_5)$. Thus either $N_2(v)$ or $N_3(v)$ contains $K_6$. Thus
which contradicts Lemma 2 (b).

3 Proof of the theorem

Assume the opposite. Consider a coloring of the edges of H in three colors without a monochromatic triangle.

We shall denote the 5-cycles in the graph H by $C_5^{(1)}$, $C_5^{(2)}$, $C_5^{(3)}$, $C_5^{(4)}$, $C_5^{(5)}$, $C_5^{(6)}$. We shall first prove the following claims.

Claim 1 Each $C_5^{(i)}$ is a monochromatic subgraph of H in the considered coloring.

Proof. Assume the opposite and let for example $C_5^{(1)} = v_1, v_2, v_3, v_4, v_5, v_1$ is not a monochromatic subgraph, and the edge $v_1v_2$ is in first color, and the edge $v_1v_5$ in second color. By Lemma 4 we have that there is a vertex $u_1$ belonging to some of the other five 5-cycles, such that the edge $v_1u_1$ is in third color. Without loss of generality we may assume that $u_1 \in C_5^{(2)}$. Let $u_2$ be a neighbor of $u_1$ in $C_5^{(2)}$. We apply Lemma 3 for $K_1 = \{v_1\}$ and the subgraph L induced by the vertices $v_2, v_3, u_1, u_2$ and the 5-cycles $C_5^{(3)}, C_5^{(4)}, C_5^{(5)}, C_5^{(6)}$ (the conditions of Lemma 3 are fulfilled because the edges $v_1v_2, v_1v_5, v_1u_1$ are in three different colors and the vertices $v_2$ and $v_5$ are not adjacent). According to Lemma 3 there is a monochromatic triangle, which is a contradiction.

Claim 2 Let $v \in C_5^{(i)}$. If $i \neq j$ then $E(v, V(C_5^{(j)}))$ cannot contain edges in the both colors different from the color of $C_5^{(i)}$.

Proof. Assume the opposite and let $v_1 \in V(C_5^{(1)}), C_5^{(1)}$ is monochromatic in first color and $E(v_1, V(C_5^{(2)}))$ contains edges $v_1a$ and $v_1b$ which are in second and third color respectively. Let $C_5^{(1)} = v_1, v_2, v_3, v_4, v_5, v_1$ and $C_5^{(2)} = u_1, u_2, u_3, u_4, u_5, u_1$. We consider two cases.

First case. The vertices a and b are not adjacent. Assume that $a = u_1$ and $b = u_3$. Now the edge $v_1u_1$ is in second color and the edge $v_1u_3$ is in third color. We apply Lemma 3 for $K_1 = \{v_1\}$ and the subgraph L induced by the vertices $v_2, u_1, u_2, u_3$ and the 5-cycles $C_5^{(3)}, C_5^{(4)}, C_5^{(5)}, C_5^{(6)}$. According to Lemma 3 there is a monochromatic triangle, which is a contradiction.

Second case. The vertices a and b are adjacent. Let for example $a = u_1$ and $b = u_2$. Now the edge $v_1u_1$ is in second color and the edge $v_1u_2$ is in third color. We shall prove that

$v_1u_3$ is in the same color in which is the edge $v_1u_1$, i.e. in second.

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Indeed, if we assume that the edge $v_1u_3$ is in third color, then we are in the situation of the first case for the vertices $u_1$ and $u_3$. If we assume that the edge $v_1u_3$ is in first color, then we apply Lemma 3 for $K_1 = \{v_1\}$ and the subgraph $L$ induced by the vertices $v_2$, $u_1$, $u_2$, $u_3$, and the 5-cycles $C_5^{(3)}$, $C_5^{(4)}$, $C_5^{(5)}$, $C_5^{(6)}$. It follows from Lemma 3 that there is a monochromatic triangle which is a contradiction. Thus we proved that $v_1u_3$ is in second color.

Analogously we prove that $v_1u_5$ is in the same color as the edge $v_1u_2$, i.e. in third color. Now we apply the first case for the vertices $a = u_3$ and $b = u_5$ and thus Claim 2 is proved.

Claim 3 If $C_5^{(i)}$ and $C_5^{(j)}$ are in two different colors, then the edges in $E(V(C_5^{(i)}), V(C_5^{(j)}))$ are in the color different from the colors of $C_5^{(i)}$ and $C_5^{(j)}$.

Proof. Let for example $C_5^{(1)}$ is in first color and $C_5^{(2)}$ is in second. Let as above $C_5^{(1)} = v_1, v_3, v_4, v_5, v_1$ and $C_5^{(2)} = u_1, u_2, u_3, u_4, u_5, u_1$. Assume the opposite, i.e. $E(V(C_5^{(1)}), V(C_5^{(2)}))$ contains at least one edge in first or in second color. Without loss of generality we may consider that $E(V(C_5^{(1)}), V(C_5^{(2)}))$ contains an edge in second color and that this edge is $v_1u_1$. Then it follows from Claim 2 that $E(v_1, V(C_5^{(2)}))$ contains edges in the first and second color only. It is not possible $E(v_1, V(C_5^{(2)}))$ to contain three edges in second color (otherwise the vertex $v_1$ and an edge from $C_5^{(2)}$ form a monochromatic triangle in second color). Hence $E(v_1, V(C_5^{(2)}))$ contains at least three edges in first color. Now we consider the vertex $v_2$. Then according to Claim 2 two cases are possible.

First case. $E(v_2, V(C_5^{(2)}))$ does not contain edges in third color. Now $E(v_2, V(C_5^{(2)}))$ cannot contain three edges in second color, because $C_5^{(2)}$ is in second color and $v_2$ together with an edge in $C_5^{(2)}$ would form a monochromatic triangle. Therefore $E(v_2, V(C_5^{(2)}))$ contains at least three edges in first color. But $E(v_1, V(C_5^{(2)}))$ contains at least three edges in first color. Therefore one of the vertices of $C_5^{(2)}$ and the edge $v_1v_2$ form a monochromatic triangle in first color - a contradiction.

Second case. $E(v_2, V(C_5^{(2)}))$ contains at least one edge in third color. In this situation, according to Claim 2, $E(v_2, V(C_5^{(2)}))$ does not contain edges in second color. Therefore $E(v_2, V(C_5^{(2)}))$ contains either three edges in first color or three edges in third color. If $E(v_2, V(C_5^{(2)}))$ contains at least three edges in first color then having in mind that $E(v_1, V(C_5^{(2)}))$ contains at least three edges in first color then one of the vertices of $C_5^{(2)}$ and the edge $v_1v_2$ form a monochromatic triangle in first color-a contradiction. If $E(v_2, V(C_5^{(2)}))$ contains at least three edges in third color, as we proved that $E(v_1, V(C_5^{(2)}))$ contains at least three edges in first color, then there is a vertex $u$ in $C_5^{(2)}$, such that the edge $v_1u$ is in first color, and the edge $v_2u$ is in third color.
Now we apply Claim 2 for the vertex \( u \) and the cycle \( C_5^{(2)} \) and we obtain a contradiction. Now Claim 3 is proved.

According to Claim 1 there are three possible situations:

**First case.** There are three 5-cycles \( C_5^{(i)} \) of \( H \) that are monochromatic in three different colors. Let for example \( C_5^{(1)} \) is in first color, \( C_5^{(2)} \) is in second color and \( C_5^{(3)} \) is in third color. Without loss of generality we may assume that \( C_5^{(4)} \) is in third color. It follows from Claim 3 that the edges in \( E(V(C_5^{(1)}), V(C_5^{(3)})) \) and \( E(V(C_5^{(1)}), V(C_5^{(4)})) \) are in second color, and the edges of \( E(V(C_5^{(2)}), V(C_5^{(3)})) \) and \( E(V(C_5^{(2)}), V(C_5^{(4)})) \) are in first color. As there are no monochromatic triangles in first and second color, then \( E(V(C_5^{(3)}), V(C_5^{(4)})) \) contains edges in third color only. Then any two adjacent vertices in \( C_5^{(3)} \) and any two adjacent vertices in \( C_5^{(4)} \) induce even a monochromatic 4-clique in third color, which is a contradiction.

**Second case.** The 5-cycles of \( H \) are monochromatic in exactly two different colors. Then at least three of the 5-cycles are in one and the same color. Let for example \( C_5^{(1)} \) is in first color and \( C_5^{(2)}, C_5^{(3)}, C_5^{(4)} \) are in second color. Then it follows from Claim 3 that the edges in \( E(V(C_5^{(1)}), V(C_5^{(2)})) \), \( E(V(C_5^{(1)}), V(C_5^{(3)})) \) \( E(V(C_5^{(1)}), V(C_5^{(4)})) \) are in first color. If \( v_1 \in C_5^{(1)} \) then \( C_5^{(2)} + C_5^{(3)} + C_5^{(4)} \) is contained in \( H_3(v_1) \). Thus \( K_6 \subseteq H_3(v_1) \), which contradicts Lemma 2 (b).

**Third case.** All the 5-cycles of \( H \) are monochromatic in one and the same color, for example first. Let \( v_1 \in C_5^{(1)} \). Then it follows from Claim 2 that the edges in \( E(v_1, V(C_5^{(j)})), j = 2, \ldots, 6 \) are at most in two colors, one of which is first. As the edges of \( C_5^{(j)}, j = 2, \ldots, 6 \) are in first color then it is impossible \( E(v_1, V(C_5^{(j)})) \) to contain three edges in first color (otherwise an edge from \( C_5^{(j)} \) and the vertex \( v_1 \) would form a monochromatic triangle in first color). Then it follows from Claim 2 that \( E(v_1, V(C_5^{(j)})), j = 2, \ldots, 6 \) contains at least three edges in second color or at least three edges in third color. Then there are at least three 5-cycles among \( C_5^{(j)}, j = 2, \ldots, 6 \), such that \( E(v_1, V(C_5^{(j)})) \) contains either three edges in second color or three edges in third color. Thus at least three of the sets \( N_2(v_1) \cap V(C_5^{(j)}) \) contain an edge or at least three of the sets \( N_3(v_1) \cap V(C_5^{(j)}) \) contain an edge. Therefore \( cl(H_3(v_1)) \geq 6 \) or \( cl(H_2(v_1)) \geq 6 \), which contradicts Lemma 2 (b). This completes the proof of the theorem.

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