Analytic resurgence in the O(4) model

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ABSTRACT: We study the perturbative expansion of the ground state energy in the presence of an external field coupled to a conserved charge in the integrable two-dimensional O(4) nonlinear sigma model. By solving Volin’s algebraic equations for the perturbative coefficients we study the large order asymptotic behaviour of the perturbative series analytically. We confirm the previously numerically found leading behaviour and study the nearest singularities of the Borel transformed series and the associated alien derivatives. We find a “resurgence” behaviour: the leading alien derivatives can be expressed in terms of the original perturbative series. A simplified ‘toy’ model is also considered: here the perturbative series can be found in a closed form and the resurgence properties are very similar to that found in the real problem.

KEYWORDS: Integrable Field Theories, Renormalization Regularization and Renormalons, Sigma Models

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1 Introduction

The two dimensional $O(N)$ $\sigma$ models provide an ideal playground for theoretical particle and statistical physics, where non-perturbative phenomena can be analyzed exactly without relying on approximate methods. These models share many features with QCD including asymptotic freedom in perturbation theory and the dynamical generation of a mass scale, which breaks the classical scale invariance in the quantum theory. These models were the first, where the exact S-matrix was determined [3]. By analysing the behaviour of the groundstate
energy density in a magnetic field [4] the perturbative scale was the first time exactly related to the dynamically generated mass [5, 6]. This energy density then can be expanded in perturbation theory and its asymptotic series revealed non-perturbative sectors [1, 7–9] related to renormalons, which exist also in QCD [10]. The theory which describes how the large order asymptotics coefficients “resurge” themselves into the non-perturbative contributions is called resurgence,\(^1\) which aims at constructing the full non-perturbative ambiguity free trans-series for the physical observables. The exact solvability of the \(O(N)\) \(\sigma\) models gives a hope to establish the complete resurgence structure exactly and to provide the sought trans-series, which could indicate similar behaviours for analogous realistic theories, such as for QCD.

The resurgence program has been analyzed in many two dimensional and related quantum field theories including also supersymmetric ones [14–21] and some of them focuses on the semiclassical domain of \(O(N)\) and principal chiral models [22–25]. Recently there have been active research and significant progress on relativistic integrable quantum field theories [7, 26, 27] and their non-relativistic statistical physical counterparts [28–30]. Resurgence proved to be a useful tool also in non-integrable quantum field theories, such as in the \(\phi^4\) theory [31, 32] and also in analysing the hydrodynamics of the Yang Mills plasma [33, 34]. There is no asymptotically free QFT, however, where the full resurgence structure was derived. In this paper we would like to make the first step into this direction by analysing the \(O(4)\) model, which is the simplest amongst the \(O(N)\) theories.

In the integrable \(O(N)\) \(\sigma\) models we can couple one of the \(O(N)\) conserved charges to a magnetic field and analyse the groundstate energy density in two different ways [5, 6]. On the one hand it can be calculated perturbatively in terms of \(\Lambda_{\text{MS}}\) for large magnetic fields. On the other hand, integrability enables to write an exact integral equation for the energy density, which can be expanded using Wiener-Hopf technics. The comparison of the first order terms provided the celebrated mass/coupling relation. Further perturbative terms turned to be decisive in the AdS/CFT correspondence, where the \(O(6)\) model plays a crucial role [35]. The calculation of higher order terms are cumbersome and the breakthrough came from the integrable side, where Volin could manage to translate the integral equation into a system of algebraic ones [1, 2]. These recursive equations could be solved for the first 20–50 terms analytically, which allowed to observe a factorial growth [1, 2, 7]. It also helped to identify the closest singularities on the Borel plane, which corresponds typically to UV and IR renormalons, diagrams in perturbation theory, where the asymptotic growths arise from integrating over UV or IR domains in specific renormalon diagrams [36]. In the large \(N\) limit the leading of such diagrams were identified and their contributions were calculated exactly, which agreed with the solution of the algebraic equations [26].

In our recent works [8, 9], we focused on the \(O(4)\) model, where the algebraic equations simplified drastically. We solved these equations numerically with very high precision, from which we conjectured analytical expressions for their leading and subleading factorial behaviours. We numerically identified logarithmic cuts on the Borel plane. We investigated the functions multiplying these cuts (alien derivatives) perturbatively from the asymptotics

\(^1\)See [11–13] for introductions and references.
of the original perturbative series. By using high precision numerics we could identify and
characterise further cuts on the real line and with their help we could manage to write the
first few terms of an ambiguity free non-perturbative trans-series, which we confronted with
the numerical solution of the original integral equation and we found complete agreement.
The full power of the resurgence theory can be used, however, when we can bridge the
various non-perturbative expansions to each other, as then we would have relations between
low order perturbative coefficients. Based on our numerics we conjectured such relations
for the nearest cuts on the Borel plane, i.e. for the leading non-perturbative corrections.
The aim of the present paper is to elevate these numerical observations to an analytically
proved level and to prepare the ground for systematic analytical studies, which could reveal
the full resurgence structure.

The paper is organized as follows: in the rest of the introduction we collect the needed
formulae from the literature. In particular, we introduce the exact linear integral equation
for the ground-state energy density and Volin’s approach for the corresponding resolvent,
which turns the integral equations into algebraic ones for the systematic perturbative
expansion. We simplify these recursive equations in the $O(4)$ case and emphasize that
the only input comes from the leading behaviour of the resolvent’s Laplace transformed,
which can be determined from the Wiener-Hopf solution. In the next section we summarize
the main results of this paper. Technically, we manage to formulate recursive equations
directly for the parametrizations of the Laplace transform of the resolvent without relying
on its coordinate space form. We then calculate the asymptotic form of the perturbative
expansions, which provides the analytical solution for the resolvent’s Laplace transform
around the origin. We then learn how to extend the asymptotic form selfconsistently,
which is valid on the unit circle and gives the complete analytical information on leading
singularities of the Borel plane, i.e. the leading alien derivates. We summarize the main
findings in the conclusion where we also present an outlook for further research. All the
technical details are relegated to appendices.

1.1 TBA calculation of the ground-state energy density

The central object of study in this paper is the ground-state energy in a magnetic field $h$ [4–6],
when the Hamiltonian is modified as $H(h) = H(0) - hQ_{12}$, where $H(0)$ is the original sigma
model Hamiltonian and $Q_{12}$ is the conserved charge associated to the (internal) rotation
symmetry in the 12 plane. Particles with largest $Q_{12}$ charge condense into the vacuum if $h$
is larger than $m$, the mass of the particles. Due to the integrability of the model this
condensate can be described by the Thermodynamic Bethe Ansatz (TBA) equation [4–6]

$$
\chi(\theta) - \int_{-B}^{B} \frac{d\theta'}{2\pi} K(\theta - \theta')\chi(\theta') = \cosh \theta
$$

(1.1)

where the kernel is determined by the scattering matrix $S(\theta)$ for the condensed particles:

$$
2\pi K(\theta) = -2\pi i \partial_{\theta} \log S(\theta) = \sum_{k=\{\frac{1}{2}, \Delta\}} \left\{ \Psi \left(k + \frac{1}{2} - \frac{i\theta}{2\pi}\right) - \Psi \left(k + \frac{i\theta}{2\pi}\right) \right\} + cc.
$$

(1.2)
Here $\Delta = \frac{1}{N - 2}$ and $\Psi$ is the digamma function: $\Psi(\theta) = \partial_\theta \log \Gamma(\theta)$. The particle density $mU$ and the energy density $m^2W$ are obtained simply from

$$U = \int_{-B}^{B} \frac{d\theta}{2\pi} \chi(\theta); \quad W = \int_{-B}^{B} \frac{d\theta}{2\pi} \cosh \theta \chi(\theta).$$

(1.3)

All quantities, the rapidity density $\chi(\theta)$, the energy density $m^2W$, the particle density $mU$, etc. depend on the parameter $B$. This dependence is understood throughout the paper and will not be indicated explicitly. In subsection 1.3 we will see that large $B$ corresponds to large $U$ and also large values of the external field $h$. We will study the large $B$ expansion of various quantities, but for technical reasons it is convenient to use the variable $z = 2B$ and we will consider $1/z$ expansions. For such expansions we will explicitly write out the argument $z$ to indicate that these are asymptotic series in the expansion variable $1/z$.

1.2 Perturbative expansion of the TBA equation, Volin’s method [1, 2]

The calculations can be simplified by introducing the resolvent

$$R(\theta) = \int_{-B}^{B} \frac{d\theta'}{2\pi} \frac{\chi(\theta')}{\theta - \theta'}$$

(1.4)

and its Laplace transform

$$\hat{R}(s) = \int_{-i\infty+0}^{i\infty+0} \frac{dz}{2\pi i} e^{sz} R(B + z/2).$$

(1.5)

The latter is related to the Fourier transform of $\chi(\theta)$:

$$\hat{R}(s) = 2e^{-2Bs} \tilde{\chi}(2is); \quad \tilde{\chi}(\omega) = \int_{-B}^{B} e^{-i\omega \theta} \chi(\theta) d\theta.$$  

(1.6)

Then, using the $\chi(\theta) = \chi(-\theta)$ symmetry

$$W = \int_{-B}^{B} \cosh \theta \chi(\theta) \frac{d\theta}{2\pi} = \int_{-B}^{B} e^\theta \chi(\theta) \frac{d\theta}{2\pi} = \frac{1}{2\pi} \tilde{\chi}(i) = \frac{e^B}{4\pi} \hat{R}(1/2).$$

(1.7)

We see that the discontinuity of the resolvent between $-B$ and $B$ is proportional to the rapidity density $\chi(\theta)$. Moreover, using the special properties of the $\Psi$ function appearing in the kernel we can derive the following periodicity property of the function $R(\theta)$:

$$R(\theta) = R(\theta - 2\pi i \Delta).$$

(1.8)

Here

$$R(\theta) = R_+(\theta + i\pi \gamma_1/2) + R_-(-\theta - i\pi \gamma_1/2), \quad \gamma_1 = 2\Delta - 1$$

(1.9)

and $R_\pm$ are the analytical continuation of the resolvent through the cut between $-B$ and $B$ from above/below. The simplest case is $N = 4$, since in this case $\Delta = 1/2$ and $\gamma_1 = 0$. In particular, the representation (1.10) takes a much more complicated form for generic $\Delta$.

Although it is possible to proceed for generic $O(N)$ models, in the following we restrict our attention to the $O(4)$ model since formulas are much simpler there.
Volin made the following two seminal observations. First, both $R(\theta)$ and $\hat{R}(s)$ can be expanded for large $B$ and the form of the two asymptotic expansions are determined by the TBA equation and the analytic properties of its solution. In rapidity space

$$R(\theta) = 2\hat{A}\sqrt{B} \sum_{n,m=0}^{\infty} \frac{c_{n,m}}{B^{m-n}(\theta^2 - B^2)^{n+1/2}}, \quad (1.10)$$

where the $c_{n,m}$ coefficients are numerical constants, but the overall constant $\hat{A}$ may depend on $B$. The density is obtained from the residue of the resolvent at infinity

$$U = \hat{A}\sqrt{B} \frac{1}{\pi} \sum_{m=0}^{\infty} c_{0,m} B^{-m}. \quad (1.11)$$

For the Laplace transform study of its analytic properties leads to the following ansatz

$$\hat{R}(s) = \frac{A}{\sqrt{s} \Gamma\left(\frac{1}{2} + s\right)} \left(\frac{1}{s + \frac{1}{2}} + \frac{1}{Bs} \sum_{n,m=0}^{\infty} \frac{Q_{n,m}}{B^{n+m}s^n}\right); \quad A = \frac{e^{B}\sqrt{\pi}}{2\sqrt{2}}. \quad (1.12)$$

with constant coefficients $Q_{n,m}$.

The second seminal observation was that by re-expanding $R(\theta)$ and performing the Laplace transform, the result can be compared to $\hat{R}(s)$. This gives $\hat{A} = A$, and by matching the two parametrizations, all the unknown numerical coefficients $Q_{n,m}$ and $c_{n,m}$ can be determined completely algebraically in a recursive manner. This method was used by Volin to calculate the perturbative coefficients for generic $O(N)$ models. The calculations in the particular case of the $O(4)$ model take a simpler form.

### 1.3 $O(4)$ Volin equations

In this subsection we give all formulas necessary to formulate and solve Volin’s recursion relations (for the special case $N = 4$). For the origin and derivation of these equations we refer to Volin’s original papers [1, 2]. First, we have to introduce a few notations.

The input parameters of the recursion are the coefficients $a_n$ of the expansion

$$\sum_{n=0}^{\infty} a_n x^n = g(x) = \frac{e^{ax}}{1 + x} \frac{\Gamma^2(1 + x/2)}{\Gamma(1 + x)}, \quad a = \ln 2, \quad a_{-1} = 0, \quad (1.13)$$

and the rational constants

$$E_{n,k} = (-1)^k \frac{1}{2^k k!} \prod_{\ell=1}^{k} \left[n^2 - \left(\frac{2\ell - 1}{2}\right)^2\right], \quad E_{n,0} = 1. \quad (1.14)$$

Using the Gamma function representation

$$\Gamma(1 + z) = \exp\left\{-\gamma z + \sum_{k=2}^{\infty} \frac{\zeta_k}{k}(-z)^k\right\} \quad (1.15)$$

we can re-express the basic input function (1.13) in terms of Dirichlet’s eta coefficients defined by

$$\eta_k = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^k}. \quad (1.16)$$
They are simply related to the zeta coefficients by
\[ \eta_k = (1 - 2^{1-k})\zeta_k, \quad k \geq 2, \quad \eta_1 = \ln 2. \] (1.17)

Also using the expansion
\[ \ln(1 + x) = \sum_{k=1}^{\infty} (\frac{(-1)^{k-1}}{k})x^k \] (1.18)
g(x) can be recast in the form
\[ \sum_{n=0}^{\infty} a_n x^n = g(x) = \exp \left\{ \sum_{k=1}^{\infty} (\frac{(-1)^{k-1}}{k})(\eta_k - 1)x^k \right\}. \] (1.19)

For later convenience we introduce the new two-index variable \( p_{n,m} \) by
\[ Q_{k,m} = \frac{p_{m+k,k}}{2^{m+2k+1}}. \] (1.20)

We now (recursively in \( m \)) find the unknowns \( c_{n,m} (m = 0, 1, \ldots, n = 0, 1, \ldots) \) and \( p_{a,b} (a = 0, 1, \ldots, b = 0, 1, \ldots, a) \), starting from
\[ c_{n,0} = a_n. \] (1.21)

After having completed the recursion up to step \( m - 1 \), we first have to solve the set of equations \( r = 1, \ldots, m \)
\[ \sum_{k=r}^{m} E_{k-r,k} c_{k-r,m-k} = \frac{1}{2^m} \sum_{k=r}^{m} (a_{k-r} + a_{k-r-1})p_{m-1,k-1} \] (1.22)
for \( p_{m-1,k-1}, k = 1, \ldots, m \) and then use this solution to express
\[ c_{n,m} = -\sum_{k=1}^{m} E_{n+k,k} c_{n+k,m-k} + \frac{1}{2^m} \sum_{k=1}^{m} (a_{n+k} + a_{n+k-1})p_{m-1,k-1} \] (1.23)
n = 0, 1, \ldots To calculate the energy density \( W \) and the particle density \( U \) we only need \( c_{0,m} \) and
\[ W_n = \sum_{k=0}^{n} p_{n,k}. \] (1.24)

Explicitly, using \( z = 2B \),
\[ W = \frac{1}{2\pi} \int_{-B}^{B} d\theta \chi(\theta) \cosh \theta = \frac{e^{2B}}{16} \epsilon(z), \quad \epsilon(z) = 1 + 2 \sum_{n=1}^{\infty} \frac{W_{n-1}}{z^n}, \] (1.25)
\[ U = \frac{1}{2\pi} \int_{-B}^{B} d\theta \chi(\theta) = e^{B} \sqrt{\frac{B}{8\pi}} \rho(z), \quad \rho(z) = 1 + \sum_{n=1}^{\infty} \frac{2^n c_{0,n}}{z^n}. \] (1.26)

We see that large \( B \) implies large \( W \) and large \( U \). Moreover, we can consider the free energy,
\( \mathcal{F}(h) = m^2W - mhU \), where it is understood that \( B \) is expressed with \( h \) using the defining relation of the Legendre transformation,
\[ m^2 \frac{dW}{dB} = mh \frac{dU}{dB}. \] (1.27)
We see that large $B$ also implies large $h$. Finally, if we introduce the running coupling $\alpha(h)$ by

$$\frac{1}{\alpha(h)} + \Delta \ln \alpha(h) = \ln \frac{h}{\Lambda}, \quad (1.28)$$

where $\Lambda = \Lambda_{\overline{\text{MS}}}$, the perturbative scale parameter in the $\overline{\text{MS}}$ scheme, we see that the perturbative, small $\alpha(h)$ region corresponds to large $h$.

Later we will study the expansion of the generalized charges

$$W_p = \frac{1}{2\pi} \int_{-B}^B d\theta \chi(\theta) \cosh p\theta = \frac{e^{pB}}{4\pi} \hat{R}(p/2)$$

$$= \frac{e^{(p+1)B}}{4\pi \sqrt{p}} \frac{\Gamma^2(1+p/2)}{\Gamma(2+p)} \epsilon_p(z), \quad (1.29)$$

where

$$\epsilon_p(z) = 1 + \frac{p+1}{p} \sum_{n=1}^{\infty} \frac{H_n(p)}{(pz)^n}, \quad z = 2B, \quad (1.30)$$

and

$$H_n(p) = \sum_{k=1}^{n} p_{n-1,n-k} p^k. \quad (1.31)$$

For the original energy density

$$W = W_1, \quad \epsilon(z) = \epsilon_1(z), \quad W_{n-1} = H_n(1). \quad (1.32)$$

These generalized charges are defined for all values of $p$ and carry the complete information about the resolvent $\hat{R}(p)$. They are the analytical continuation of the Fourier transform of the rapidity density $\chi(\theta)$ for imaginary arguments. For positive integer values of $p$, they should be related to the higher spin conserved charges in the integrable description. Their direct physical meaning for generic $p$ in the Lagrangean formulation, however, is not obvious, neither is their calculation from a path integral point of view. In contrast, $W_1$ corresponds to the energy density while $W_0$ to the particle density and the ratio $W_1/W_0^2$ is related to the Legendre transform of the free energy itself, which can be calculated from the path integral.

An alternative form of $\epsilon_p(z)$, in terms of the variable $q = 1/p$, is

$$\epsilon_p(z) = 1 + (1 + q) \sum_{n=1}^{\infty} \frac{E_n(q)}{z^n}, \quad E_n(q) = \sum_{k=0}^{n-1} p_{n-1,k} q^k. \quad (1.33)$$

The input parameters $a_n$ are polynomial expressions (with rational coefficients) in $\zeta_2, \zeta_3, \ldots$ and $a = \ln 2$ of degree $n$ ($a = \ln 2$ counts as degree 1). Previously we observed that $W_n$ and $c_{0,m}$ do not depend on the even zeta functions $\zeta_2, \zeta_4, \ldots$. Since they are proportional to powers of $\pi$, we call this property $\pi$-independence. We now observed that not only the sum $W_n$ but all $p_{a,b}$ coefficients individually are $\pi$-independent. However, the even zeta functions do not completely disappear: the variables $c_{n,m}$, $n \geq 1$ do depend on them.

In [8, 9] we studied the solution of the recursive equations (1.22) and (1.23) with high precision numerics, up to 2000th order, studied the resurgence properties of the asymptotic
expansions, and compared the Borel resummation of the asymptotic series for both $U$ and $W$ to the results obtained from the numerical solution of the original TBA equation. In this paper we reproduce many of those previous results analytically. We hope that our methods may be used later in related similar models too.

2 Summary

In this section we summarize the analytic solution of the recursive equations and study the resurgence properties of the asymptotic expansions analytically. We not only confirm our previous findings but also discover some new properties of the asymptotic series.

2.1 Properties of the $\varepsilon_p(z)$ series

The second set of Volin’s equations, (1.23), can be solved for $c_{n,m}$ in closed form and using this solution in (1.22) we can obtain closed equations for the $p_{a,b}$ type variables. The details of the derivation are given in appendix A. The simplified equations are of the form

$$\mathcal{L}_r^{(M)} = \mathcal{R}_r^{(M)}, \quad M = 1, 2, \ldots, \quad r = 1, \ldots, M,$$

(2.1)

where

$$\mathcal{R}_r^{(M)} = \sum_{k=0}^{M-r} (a_k + a_{k-1})p_{M-1,k+r-1}$$

(2.2)

and

$$\mathcal{L}_r^{(M)} = \sum_{n=0}^{M-r} X_{r,n} Y_n^{(M-r)}.$$

(2.3)

Here $X_{r,n}$ are $M$-independent rational coefficients:

$$X_{r,n} = 2^r (-2)^n \sum_{p=0}^{n} (-1)^p E_p E_{p+1,n-p}$$

(2.4)

and the symbols $Y_n^{(M)}$ are

$$Y_n^{(M)} = \sum_{k=1}^{M-n} (a_{n+k} + a_{n+k-1})p_{M-n-1,k-1}, \quad n = 0, \ldots, M - 1, \quad Y_M^{(M)} = a_M.$$

(2.5)

The equations (2.1) can be solved recursively in $M$.

In appendix B, using this simplified form of Volin’s equations we prove the existence of the asymptotic (ASY) expansions

$$\frac{p_{N,N-j}}{\Gamma(N+1)} = \frac{1}{\pi} \left\{ \beta^{(j)} + \frac{\alpha_1^{(j)}}{N} + \frac{\alpha_2^{(j)}}{N(N-1)} + \frac{\alpha_3^{(j)}}{N(N-1)(N-2)} + \ldots \right\} \quad j = 0, 1, \ldots$$

(2.6)
Here the coefficients are given by generating functions:

\[
\sum_{j=0}^{\infty} \beta(j)p^{j+1} = u(p) \\
\sum_{j=0}^{\infty} \alpha_m(j)p^{j+1} = u(p)A_m(p) \quad m = 0, 1, \ldots
\]  

(2.7)

where

\[
u(p) = \frac{p}{1+p} \frac{g(-p)}{g(p)} \quad A_m(p) = (p-1) \sum_{k=0}^{m} p_{m,k}(-1)^k p^{m-k}
\]  

(2.8)

These results can be used to study the \( \epsilon_p(z) \) asymptotic series and related alien derivatives.\(^2\)

First we note that

\[
u(x) = \frac{x}{(1-x)^2} \frac{\Gamma(1+x) \Gamma^2(1-x/2)}{\Gamma(1-x) \Gamma^2(1+x/2)}
\]  

(2.9)

and using the representation of the Gamma function given by (1.15) we see that we have \( \pi \)-independence for the leading terms \( \beta(j) \) and also for \( \alpha_m(j) \) (at least for the first few \( m \) values, for which the \( \pi \)-independence of \( p_{m,k} \) is known by explicit construction). We also note that \( u(x) \) can alternatively be written as

\[
u(x) = \frac{x}{2^{2x+1}} \frac{\Gamma(1+x) \Gamma^2(1-x/2)}{\Gamma(2-x) \Gamma^2(1+x/2)}
\]  

(2.10)

hence for \(|x| < 2\) the only singular point of \( u(x) \) is \( x = -1 \):

\[(x \approx -1) : \quad u(x) \approx -\frac{1}{2} \frac{1}{1+x} = -\frac{1}{2} \sum_{j=0}^{\infty} (-x)^j.
\]  

(2.11)

Therefore for large \( j \)

\[
\beta(j) \approx \frac{1}{2} (-1)^j \quad \alpha_m(j) \approx \frac{1}{2} A_m(-1)(-1)^j \quad A_m(-1) = 2(-1)^{m+1} W_m.
\]  

(2.12)

The central object of study is

\[
\epsilon_p(z) = 1 + \frac{p+1}{p} \sum_{m=0}^{\infty} \frac{H_{m+1}(p)}{(pz)^{m+1}}.
\]  

(2.13)

For \( -p \) we have

\[
\epsilon_{-p}(z) = 1 + \sum_{m=0}^{\infty} \frac{A_m(p)}{(pz)^{m+1}}.
\]  

(2.14)

We can write alternatively

\[
\epsilon_p(z) = 1 + \frac{p+1}{p} \sum_{m=0}^{\infty} \frac{\Gamma(m+1)c_m^{(p)}}{z^{m+1}p^{m+1}}, \quad c_m^{(p)} = \frac{H_{m+1}(p)}{\Gamma(m+1)}.
\]  

(2.15)

\(^2\)The definitions related to alien derivatives are summarized in appendix F.
These expansion coefficients define the Borel space function
\[ B(x) = \sum_{m=0}^{\infty} c_m^{(\epsilon_p)} x^m. \]  

The Borel transform of \( \epsilon_p \) is given by
\[ \hat{\epsilon}_p(t) = \frac{p + 1}{p^2} B \left( \frac{1}{p} \right). \]  

Now it is easy to establish the ASY expansion of the Borel transform coefficients (for \(|p| < 1\)).
\[ c_M^{(\epsilon_p)} = \frac{H_{M+1}(p)}{\Gamma(M+1)} = \sum_{j=0}^{M} \frac{p^{M-j}}{\Gamma(M+1)} p^{j+1} \approx \sum_{j=0}^{\infty} \frac{p^{j+1}}{\pi} \left\{ \beta(j) + \sum_{m=0}^{\infty} \frac{\alpha_m}{M[m+1]} \right\} \]  

We have extended the upper limit of the summation from \( M \) to \( \infty \). This is possible since the error we make by this extension is exponentially small for \(|p| < 1\). Above we used the notation
\[ M[r] = M(M-1)(\ldots)(M-r+1), \quad M[1] = M, \quad M[0] = 1. \]  

From here we can read off the singular part of the Borel space function \( B(x) \):
\[ B^{\text{sing}}(x) = \frac{u(p)}{\pi} \left\{ \frac{1}{1-x} - \ln(1-x) \sum_{m=0}^{\infty} \frac{A_m(p)}{m!} (x-1)^m \right\}. \]  

Expressed in terms of alien derivatives we have
\[ [\Delta_p \epsilon_p](z) = -2i \frac{p+1}{p} u(p) \left\{ 1 + \sum_{m=0}^{\infty} \frac{A_m(p)}{pz^{m+1}} \right\} = -2i \frac{p+1}{p} u(p) \epsilon_{-p}(z). \]  

We can make it look simpler by introducing the rescaled function\(^3\)
\[ \tilde{\epsilon}_p(z) = g(p) \epsilon_p(z). \]  

In terms of the rescaled functions the alien derivative is simply
\[ [\Delta_p \tilde{\epsilon}_p](z) = -2i \tilde{\epsilon}_{-p}(z). \]  

We also note that since
\[ \epsilon_{-1}(z) \equiv 1, \quad u(1) = 1 \]  

in the \( p \to 1 \) limit we have
\[ [\Delta_1 \epsilon](z) = -4i, \]  

which is what we found earlier numerically in [8, 9]. It means that the only singularity of \( \tilde{\epsilon}(t) \) around \( t = 1 \) is a simple pole with residue \(-2/\pi\). 

\(^3\)Note that \( W_p = e^{(p+1)t)/(4\pi\sqrt{p})} \tilde{\epsilon}_p(z) \). See (1.29).
2.2 “Derivation” of the naive ansatz

Using (2.12) we see that the asymptotics if \( j \) is also large simplifies to

\[
\frac{p_{N,N-j}}{\Gamma(N+1)} = \frac{(-1)^j}{2\pi} \left\{ 1 + \frac{A_0(-1)}{N} + \frac{A_1(-1)}{N(N-1)} + \frac{A_2(-1)}{N(N-1)(N-2)} + \cdots \right\} (2.27)
\]

The precise meaning of the above formula is that first we consider the large \( N \) asymptotics of \( p_{N,N-j} \) for fixed \( j \) and next we let the fixed \( j \) to be large. If both \( N \) and \( j \) are very large but finite, the requirement is that

\[
N \gg j \gg 1.
\] (2.28)

We will now make the following bold assumption: (2.27) still holds if we take \( j = N-k \) with fixed \( k \). \( N \) and \( j \) are both large in this case but the above requirement is not satisfied. Making this (illegal) substitution we obtain

\[
\frac{p_{N,k}}{\Gamma(N+1)} = \frac{(-1)^N}{2\pi} (-1)^k \left\{ b(k) + \frac{\tilde{R}_0^{(k)}}{N} + \frac{\tilde{R}_1^{(k)}}{N(N-1)} + \frac{2\tilde{R}_2^{(k)}}{N(N-1)(N-2)} + \cdots \right\} \quad k = 0, 1, \ldots (2.29)
\]

with

\[
\tilde{R}_m = \frac{2}{m!} (-1)^{m+1} W_m
\] (2.30)

We tried to check numerically if our bold assumption is correct. We started from a somewhat more general ansatz:

\[
\frac{p_{N,k}}{\Gamma(N+1)} = \frac{(-1)^N}{2\pi} (-1)^k \left\{ b^{(k)} + \frac{\tilde{R}_0^{(k)}}{N} + \frac{\tilde{R}_1^{(k)}}{N(N-1)} + \frac{2\tilde{R}_2^{(k)}}{N(N-1)(N-2)} + \cdots \right\} \quad k = 0, 1, \ldots (2.31)
\]

and found the first few coefficients numerically:

\[
\begin{align*}
   b^{(k)} &= 1 \quad k = 0, 1, \ldots \\
   \tilde{R}_0^{(0)} &= -\frac{3}{4} \quad \tilde{R}_0^{(k)} = \tilde{R}_0 = -\frac{1}{2} = -2W_0 \quad k = 1, 2, \ldots \\
   \tilde{R}_1^{(0)} &= \frac{49}{32} - \frac{3a}{2} \quad \tilde{R}_1^{(1)} = \frac{27}{32} - a \quad \tilde{R}_1^{(k)} = \tilde{R}_1 = \frac{9}{8} - a = 2W_1 \quad k = 2, 3, \ldots (2.32) \\
   \tilde{R}_2^{(0)} &= -\frac{85}{256} + \frac{49a}{16} - \frac{3a^2}{2} \quad \tilde{R}_2^{(1)} = -\frac{45}{32} + \frac{27a}{16} - a^2 \quad \tilde{R}_2^{(2)} = -\frac{531}{256} + \frac{9a}{4} - a^2 \\
   \tilde{R}_2^{(k)} &= \tilde{R}_2 = -\frac{57}{32} + \frac{9a}{4} - a^2 = -W_2 \quad k = 3, 4, \ldots
\end{align*}
\]

We see that the leading term is correctly given by our bold assumption. (2.29) is not completely correct but we found the following interesting structure. There are irregular and regular coefficients:

\[
\text{irregular} : \quad \tilde{R}_m^{(k)} \quad k = 0, \ldots, m \quad \text{regular} : \quad \tilde{R}_m = \tilde{R}_m = \frac{2}{m!} (-1)^{m+1} W_m \quad k \geq m + 1.
\] (2.33)

We see that with the exception of a few irregular coefficients the generic coefficients are correctly given by (2.29).
2.3 The correct ansatz

Motivated by the naive “derivation” and based on our numerical studies we now start from the ansatz (2.31). Unlike the case for our earlier ansatz (2.6) we cannot analytically prove the existence of the asymptotic expansion (2.31). However, taking this form (with undetermined expansion coefficients \(b^{(k)}, \tilde{R}^{(k)}\)) for granted, we can analytically prove the special structure we found numerically.

The starting point is (2.1) and using (2.31) and comparing the asymptotic expansions of the two sides of the equations first of all we can establish the generic structure

\[
\begin{align*}
    b^{(k)} &= b \\
    \tilde{R}^{(k)} &= \tilde{R}_m \quad k \geq m + 1
\end{align*}
\]

Moreover, all irregular \(\tilde{R}^{(k)}_m\) coefficients \((k = 0, \ldots, m)\) can be expressed linearly in terms of the generic parameters \(b, \tilde{R}_m\). In particular, up to NNNLO we find the result

\[
\begin{align*}
    \tilde{R}^{(0)}_0 &= \tilde{R}_0 - p_{0,0}b, \\
    \tilde{R}^{(0)}_1 &= \tilde{R}_1 - p_{0,0}\tilde{R}_0 + p_{1,0}b, \\
    \tilde{R}^{(1)}_1 &= \tilde{R}_1 - p_{1,1}b, \\
    \tilde{R}^{(0)}_2 &= \tilde{R}_2 - \frac{1}{2}p_{0,0}\tilde{R}_1 + \frac{1}{2}p_{1,0}\tilde{R}_0 - \frac{1}{2}p_{2,0}b, \\
    \tilde{R}^{(1)}_2 &= \tilde{R}_2 - \frac{1}{2}p_{1,1}\tilde{R}_0 + \frac{1}{2}p_{2,1}b, \\
    \tilde{R}^{(2)}_2 &= \tilde{R}_2 - \frac{1}{2}p_{2,2}b.
\end{align*}
\]

The proof of the generic structure (2.34) and the technical details of the calculation leading to (2.35) can be found in appendix C.

The above findings suggest a product form of \(\Delta_{-1}\epsilon_p\):

\[
[\Delta_{-1}\epsilon_p](z) = iX(z)\epsilon_p(z) \quad X(z) = b + \sum_{m=0}^{\infty} \frac{X_m}{z^{m+1}}.
\]

The consequences of this relation are discussed in appendix D. Writing out (2.36) explicitly we obtain

\[
\begin{align*}
    \tilde{R}^{(k)}_m &= \tilde{R}_m - \frac{(-1)^{m+1}}{m!}X_m, \quad k \geq m + 1, \\
    \tilde{R}^{(m)}_m &= \tilde{R}_m - \frac{p_{m,m}b}{m!}, \\
    \tilde{R}^{(k)}_m &= \tilde{R}_m - \frac{(-1)^{m+k+1}}{m!} \left\{ bp_{m,k} + \sum_{r=0}^{m-1-k} X_r p_{m-r-1,k} \right\}, \quad k \leq m - 1.
\end{align*}
\]

This gives the same structure as above and up to NNNLO explicitly reproduces (2.35) taking into account the identification (2.37). The “naive” identification (2.30) gives \(b = 1\) and \(X_m = 2W_m\), i.e. \(X = \epsilon\). Finally we get

\[
[\Delta_{-1}\epsilon_p](z) = i\epsilon(z)\epsilon_p(z).
\]
This is the generalization of the relation
\[
[\Delta_{-1}\epsilon](z) = i\epsilon^2(z)
\] (2.41)
found numerically in [8, 9].

2.4 Asymptotic expansion of $\rho$

From the inversion formula we obtain the coefficients of the Borel transform of $\rho$:
\[
c_N^{(\rho)} = \sum_{n=0}^{N+1} \omega_n \frac{Y_n^{(N+1)}}{\Gamma(N+1)}, \quad \omega_n = (-2)^n E_{1,n}.
\] (2.42)

For the asymptotic expansion of $c_N^{(\rho)}$ up to NNNLO we need the asymptotics of
\[
\frac{Y_n^{(N+1)}}{\Gamma(N+1)} \quad \text{for} \quad n = 0, 1, 2, 3.
\]

To calculate the above expansions the parameters
\[
b^{(k)}, \quad R_m^{(k)} \quad m = 0, 1, 2 \quad k = 0, 1, 2, 3
\]
are needed. We already expressed all these in terms of the generic coefficients. The alien derivative of $\rho$ to this order is calculated in appendix E. The result is
\[
[\Delta_{-1}\rho](z) = i \left\{ b + \frac{X_0 + \tilde{u}_1 b}{z} + \frac{X_1 + \tilde{u}_1 X_0 + \tilde{u}_2 b}{z^2} + \frac{X_2 + \tilde{u}_1 X_1 + \tilde{u}_2 X_0 + \tilde{u}_3 b}{z^3} + \ldots \right\}.
\] (2.43)

Here we use the variables
\[
\tilde{u}_M = 2^M c_{0,M},
\] (2.44)
which appear in the $\rho$ asymptotic series:
\[
\rho(z) = 1 + \sum_{M=1}^{\infty} \frac{c_{0,M}}{B^M} = \sum_{M=0}^{\infty} \frac{\tilde{u}_M}{z^M}.
\] (2.45)

The meaning of (2.43) is that (to this order)
\[
[\Delta_{-1}\rho](z) = i X(z) \rho(z) = i \epsilon(z) \rho(z).
\] (2.46)

3 Conclusion

In this paper we investigated the perturbative expansion of the ground-state energy $\epsilon$ and the generalized charges $\epsilon_p$ of the O(4) model in a magnetic field. We derived a closed system of equations for these perturbative coefficients, which we solved asymptotically by calculating the leading factorial growth together with all the subsequent subfactorial behaviour. From this analytical solution we could show that the asymptotic coefficients contain only odd zeta numbers, a result similar to what was observed before for the original perturbative coefficients by explicit calculation [1, 2]. Our results also show that the leading
singularity of the Borel transform of $\epsilon_p$ is a logarithmic cut and the corresponding alien
derivative resurses to the same set of charges as

$$\left[ \Delta_p \epsilon_p \right](z) = -2i \frac{p + 1}{p} u(p) \epsilon_{-p}(z)$$

(3.1)

with $u(p)$ an explicitly known function (2.9). This, in particular, implies our previous
finding $\Delta_1 \epsilon = -4i$ for the energy density [8, 9]. We could also selfconsistently determine
the behaviour around the next singularity on the Borel plane, which manifested itself as a
logarithmic cut starting at $-1$. The corresponding alien derivative again showed resurgence
in the form

$$\left[ \Delta_{-1} \epsilon_p \right](z) = i\epsilon(z)\epsilon_p(z),$$

(3.2)

which is the generalization of what we found earlier numerically: $\Delta_{-1} \epsilon = i\epsilon^2$. We extended
this analysis also for the density and showed that $\Delta_{-1} \rho = i\rho$, again confirming previous
numerical observation. By this we completed the investigations of the leading singularities.
Our approach, however, is quite general and is capable of investigating subsequent, expo-
nentially suppressed asymptotics, i.e. further cuts on the Borel plane. We are planning to
advance into this direction later.

Originally, we introduced the generalized charges for values $p < 1$ as a technical tool
to suppress the high index contributions of $p_{n,m}$ and to have an analytical control over the
resolvent at $\hat{R}(p)$. Clearly for $p < 1$ the alien derivative $\Delta_p \epsilon_p$ describes the closest
singularity of $\epsilon_p$ on the Borel plane signaling non-perturbative contributions of order $e^{-2pB}$.
Since these charges do not have a direct path integral realization, it is not clear at all
what is the origin of these non-perturbative effects. As their magnitudes are related to
the parameters of the charges, it is hard to imagine any semiclassical origin. Not even
uniton or fracton configurations [23, 37] are expected to play any role here as their action
is also quantized. In our case $p$ is however continuous and could be chosen arbitrarily small.
Nevertheless, it would be very interesting to understand the semiclassical origin of these
non-perturbative contributions, if there is any of them.

The recursive equations have their inputs from the asymptotic form of the Laplace
transform of the resolvent, $g(x)$, and the functions $E_{n,m}$ playing a crucial role in moving
between a function and its Laplace transform, and which itself shows a kind of resurgence
property (B.10). As the model-specific input comes only from $g(x)$ we analyzed a toy
model, where we replaced all $\eta_n$ (see (1.17)) with 1. This was also motivated by the fact
that numerically they are not far from each other. We could solve this model explicitly,
see appendix G, and observed a similar resurgence pattern to what we found for the real
problem. This sheds some light on their origins. This toy model then can be systematically
dressed up, by including corrections in $\eta_n - 1$. In particular, the exact dependence on $\eta_1$ is
found in appendix G.

In the present paper we analysed the density and the generalized charges as a function
of the TBA parameter $B$. The change for the running coupling is well understood and the
composition formula for alien derivatives [38] allows a direct translation [8, 9].

So far we analyzed only the O(4) model, but the technics we developed here can also
be extended for other O(N) models and their supersymmetric generalizations as well as for
the principal chiral and Gross-Neveu models, all of which were analysed recently in [39] by directly investigating the integral equation. They may be useful also for non-relativistic two dimensional integrable theories relevant for statistical physics such as [28–30].

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A Simplifying Volin’s equations

A.1 Properties of the \(a_n\) coefficients

The input \(a_n\) parameters are obtained from the Taylor expansion of (1.13). The first few coefficients are

\[
a_0 = 1, \quad a_1 = a - 1, \quad a_2 = 1 - a + \frac{a^2}{2} - \frac{\pi^2}{24}. \tag{A.1}
\]

\(g(x)\) can also be represented as

\[
g(x) = 2^{x+1} \frac{T^2(2 + x/2)}{\Gamma(3 + x)} \frac{1}{1 + x/2} = \frac{1}{2 + x} + \text{conv}_4(x), \tag{A.2}
\]

where the second term in the last formula is some function having a power series expansion convergent for \(|x| < 4\). This form implies that for large \(n\)

\[
a_n \sim \frac{(-1)^n}{2^{n+1}}. \tag{A.3}
\]

A.2 \(c_{n,m}\) equations

For later purposes we note that

\[
E_{1,k} = \frac{(-1)^k}{2^k} \omega_k, \tag{A.4}
\]

where

\[
\omega_k = \frac{1}{k!} \prod_{\ell=1-k}^k \left( \ell + \frac{1}{2} \right), \quad k \geq 1, \quad \omega_0 = 1. \tag{A.5}
\]

We introduce the notation

\[
c_{n,m} = w_{n,n+m}. \tag{A.6}
\]

With this notation the \(c_{n,m}\) equations for fixed \(M \geq 1\) and \(n = 0, 1, \ldots M - 1\) read

\[
\sum_{k=n}^{M} E_{k,k-n} w_{k,M} = 2^{n-M} Y^{(M)}_n, \tag{A.7}
\]

where

\[
Y^{(M)}_n = \sum_{k=1}^{M-n} (a_{n+k} + a_{n+k-1}) p_{M-n-1,k-1}, \quad n = 0, \ldots, M - 1. \tag{A.8}
\]
The unknowns in these equations are

\[ w_{j,M}, \quad j = 0, \ldots, M - 1 \]  

since for the last component

\[ w_{M,M} = c_{M,0} = a_M, \]  

which is an input. However, we can formally extend (A.7) for \( n = M \) as well, if we define

\[ Y^{(M)}_M = a_M. \]  

This way we have \( M + 1 \) equations for \( M + 1 \) unknowns and the equations can be written in matrix form:

\[ \sum_{k=0}^{M} \mathcal{M}_{n,k} w_{k,M} = 2^{n-M} Y^{(M)}_n, \]  

where

\[ \mathcal{M}_{n,k} = \begin{cases} E_{k,k-n} & k \geq n \\ 0 & k < n \end{cases}. \]  

The solution is given by the inverse matrix:

\[ w_{a,M} = \sum_{n=0}^{M} \mathcal{M}^{-1}_{a,n} 2^{n-M} Y^{(M)}_n, \]  

We are mostly interested in

\[ c_{0,M} = w_{0,M} = \sum_{n=0}^{M} \mathcal{M}^{-1}_{0,n} 2^{n-M} Y^{(M)}_n, \]  

because these are the ones appearing in the \( B \)-expansion of \( \rho(B) \).

The inverse matrix is also triangular and we found the simple result

\[ \mathcal{M}^{-1}_{a,n} = (-1)^{n-a} E_{a+1,n-a}, \quad (n \geq a). \]  

In particular,

\[ \mathcal{M}^{-1}_{0,n} = (-1)^n E_{1,n} = \frac{\omega_n}{2^n}. \]  

For some purposes it is useful to express the variables \( w_{a,m} \) from the second set of Volin equations and if we use

\[ w_{a,M} = \sum_{n=a}^{M} (-1)^{n+a} E_{a+1,n-a} 2^{n-M} Y^{(M)}_n \]  

we can obtain closed equations for the \( p_{a,b} \) type variables, which can be solved recursively in \( M \). These equations are given by (2.1)–(2.4).
B Analytic proof of the asymptotic expansion (2.6)

B.1 Some definitions, notations, relations

The definition of the expansion coefficients of the basic input function \( g(x) \) is extended to negative indices as

\[
a_{-n} = 0, \quad n = 1, 2, \ldots
\]

(B.1)

If a function \( F(M) \) has asymptotic series (ASY) for large \( M \), the coefficients of the individual terms in this expansion will be denoted by \( F[\cdot] \):

\[
F(M) = \sum_{r=0}^{\infty} \frac{F[r]}{M[r]} = F[0] + \frac{F[1]}{M} + \frac{F[2]}{M(M-1)} + \frac{F[3]}{M(M-1)(M-2)} + \ldots
\]

(B.2)

In this appendix \( M \) is always assumed to be asymptotically large. For simplicity, the \( M \) dependence will not always be explicit. For example, we will use the notations

\[
\Omega(k) = \frac{p_{M,M-k}}{\Gamma(M+1)}, \quad k = 0, 1, \ldots
\]

(B.3)

and

\[
\bar{R}_j = \frac{R_{M+1-j}}{\Gamma(M+1)}, \quad \bar{L}_j = \frac{L_{M+1-j}}{\Gamma(M+1)}, \quad j = 0, 1, \ldots
\]

(B.4)

This appendix is about the analytic proof of the following two statements.

**Statement 1.** \( \Omega^{(j)} \) \((j = 0, 1, \ldots)\) has ASY expansion

\[
\frac{p_{M,M-j}}{\Gamma(M+1)} = \frac{1}{\pi} \left\{ \beta^{(j)} + \frac{\alpha_0^{(j)}}{M} + \frac{\alpha_1^{(j)}}{M(M-1)} + \frac{\alpha_2^{(j)}}{M(M-1)(M-2)} + \ldots \right\}
\]

(B.5)

Here we state the existence of the ASY expansion and define the expansion coefficients.

**Statement 2.** The expansion coefficients are given with the help of generating functions as follows.

\[
\sum_{j=0}^{\infty} \beta^{(j)} p^{j+1} = u(p), \quad \sum_{j=0}^{\infty} \alpha_m^{(j)} p^{j+1} = u(p) A_m(p), \quad m = 0, 1, \ldots
\]

(B.6)

Here

\[
u(p) = \frac{p}{p+1} \frac{g(-p)}{g(p)}.
\]

(B.7)

B.2 \( E_{a,N} \) resurgence

For \( N > a \) an alternative representation of \( E_{a,N} \) is

\[
E_{a,N} = (-1)^a \frac{\Gamma(2N - 2a + 1)\Gamma(2N + 2a + 1)}{2^{5N}\Gamma(N+1)\Gamma(N-a+1)\Gamma(N+a+1)}.
\]

(B.8)
The following combination has ASY expansion for large $N$, which can be calculated using the Stirling series.

\[
\frac{\pi}{\Gamma(N)} (-1)^a 2^N E_{a,N} = \frac{\Gamma(2N - 2a + 1) \Gamma(2N + 2a + 1)}{24N \Gamma(N) \Gamma(N + 1) \Gamma(N - a + 1) \Gamma(N + a + 1)}
\]

\[
= 1 + \sum_{m=1}^{\infty} \frac{1}{m!} \prod_{\ell=1}^{m} \frac{a^2 - (\ell - 1/2)^2}{N - \ell}
\]

\[
= 1 + \sum_{m=1}^{\infty} \frac{(-2)^m E_{a,m}}{(N - 1)(\ldots)(N - m)}.
\]

For later purpose we write the above expansion as

\[
2^N E_{a,N} = \frac{(-1)^a \Gamma(N)}{\pi} \left\{ 1 + \sum_{m=1}^{\infty} \frac{(-2)^m E_{a,m}}{(N - 1)(\ldots)(N - m)} \right\},
\]

which can be interpreted as a kind of “resurgence” of the function $E_{a,N}$.

**B.3 Proof of the statements**

We start from (2.1) in the form

\[
\bar{L}_j = \bar{R}_j, \quad j = 0, 1, \ldots
\]

(B.11)

The two sides of the equations are

\[
\bar{R}_j = \sum_{k=0}^{j} (a_k + a_{k-1}) \Omega^{(j-k)}
\]

(B.12)

and

\[
\bar{L}_j = \sum_{n=0}^{j} \frac{X_{M+1-j,n}}{\Gamma(M+1)} Y^{(j)}_n.
\]

(B.13)

First we consider the large $M$ asymptotics of the left hand side.

\[
\frac{X_{M+1-j,n}}{\Gamma(M+1)} = 2^{M+1-j} (-2)^n \sum_{p=0}^{n} (-1)^p \frac{E_{p+1, M+1-j}}{\Gamma(M+1)} E_{p+1, n-p}
\]

\[
= (-2)^n \sum_{p=0}^{n} 2^{-p} E_{p+1, n-p} \frac{1}{\pi} \frac{1}{M_{[j-p]}} \left\{ 1 + \sum_{m=1}^{\infty} \frac{(-2)^m E_{p,m}}{(p + M - j)(\ldots)(p + M - j + 1 - m)} \right\}
\]

\[
= (-2)^n \sum_{p=0}^{n} 2^{-p} E_{p+1, n-p} \sum_{m=0}^{\infty} \frac{(-2)^m E_{p,m}}{M_{[j-p+m]}}
\]

(B.14)

The $r^{th}$ component of the ASY expansion is given by the $m = r + p - j$ term (which is nonzero if $n \geq p \geq j - r$):

\[
\left( \frac{X_{M+1-j,n}}{\Gamma(M+1)} \right)^{[r]} = \frac{(-2)^n}{\pi} \sum_{p=\text{Max}(0,j-r)}^{n} 2^{-p} E_{p+1, n-p} (-2)^{r+p-j} E_{p, r+p-j}.
\]

(B.15)
If $j \geq r$ we can recognize here the triangular matrix (A.13) and its inverse (A.16) and the above can be written as
\[
\frac{(-1)^{r-j}}{\pi} 2^{r-j+n} \sum_{p=j-r}^{n} M_{p,n}^{-1} M_{j-r,p}. \tag{B.16}
\]
Since
\[
M_{p,n}^{-1} = 0 \quad \text{if } p > n \quad \text{and} \quad M_{j-r,p} = 0 \quad \text{if } p < j - r, \tag{B.17}
\]
we can extend the summation limits to $\sum_{p=0}^{M}$ and the summation becomes the matrix product of the matrix and its inverse and the result is simply the Kronecker delta
\[
\delta_{n,j-r}. \tag{B.18}
\]
Using this in (B.13) we get
\[
\bar{L}_{j}^{[r]} = \frac{(-1)^{j-r}}{\pi} Y_{j-r}. \tag{B.19}
\]
Explicitly,
\[
\bar{L}_{j}^{[0]} = \frac{(-1)^{j}}{\pi} a_{j},
\]
\[
\bar{L}_{j}^{[r]} = \frac{(-1)^{r-j}}{\pi} \sum_{k=0}^{r-1} (a_{j-r+k+1} + a_{j-r+k}) p_{r-1,k}, \quad j \geq r \geq 1. \tag{B.20}
\]
For $j < r$ (B.15) gives
\[
\left( \frac{X_{M+1-j,n}}{\Gamma(M + 1)} \right)^{[r]} = \frac{(-2)^{n+r-j}}{\pi} \sum_{p=0}^{n} (-1)^{p} E_{p+1,n-p} E_{p,r+p-j} = \frac{(-1)^{r-j}}{\pi} X_{r-j,n}. \tag{B.21}
\]
Using this in (B.13) we get
\[
\bar{L}_{j}^{[r]} = \frac{(-1)^{r-j}}{\pi} \sum_{n=0}^{j} X_{r-j,n} Y_{n}^{(j)} = \frac{(-1)^{r-j}}{\pi} L_{r-j}^{(r)}. \tag{B.22}
\]
Using (2.1) once more in this case we have
\[
\bar{L}_{j}^{[r]} = \frac{(-1)^{j-r}}{\pi} R_{r-j}^{(r)} = \frac{(-1)^{j-r}}{\pi} \sum_{k=0}^{j} (a_{k} + a_{k-1}) p_{r-1,k+r-j-1}
\]
\[
= \frac{(-1)^{j-r}}{\pi} \sum_{k=r-j-1}^{r-1} (a_{k+j-r+1} + a_{k+j-r}) p_{r-1,k}. \tag{B.23}
\]
Finally, using the convention (B.1) the summation limits can be extended:
\[
\bar{L}_{j}^{[r]} = \frac{(-1)^{j-r}}{\pi} \sum_{k=0}^{r-1} (a_{j-r+k+1} + a_{j-r+k}) p_{r-1,k}, \tag{B.24}
\]
which is miraculously the same as the formula for $j \geq r$.\[\[\]
To summarize, the final result of the calculation is (for any $j = 0, 1, \ldots$)

$$
\bar{L}_j^0 = \frac{(-1)^j}{\pi} a_j,
$$

$$
\bar{L}_j^r = \frac{(-1)^{j-r}}{\pi} \sum_{k=0}^{r-1} (a_{j-r+k+1} + a_{j-r+k}) p_{r-1,k}, \quad r \geq 1.
$$

(B.25)

This way we have proven by explicit calculation that the l.h.s. of (B.11) has ASY expansion. But then also the r.h.s. has ASY expansion. In particular

$$
\Omega^{(0)} = \bar{R}_0 = \bar{L}_0
$$

(B.26)

has ASY expansion, then

$$
\Omega^{(1)} = -(1 + a_1)\Omega^{(0)} + \bar{L}_1
$$

(B.27)

has ASY expansion, and so on: we can prove recursively that all

$$
\Omega^{(j)} = - \sum_{k=1}^{j} (a_k + a_{k-1}) \Omega^{(j-k)} + \bar{L}_j
$$

(B.28)

have ASY expansion. Thus we have proven Statement 1.

The proof of Statement 2 is now easy. We calculate the infinite sums

$$
\sum_{j=0}^{\infty} p^j \bar{L}_j^0 = \frac{1}{\pi} \sum_{j=0}^{\infty} (-p)^j a_j = \frac{g(-p)}{\pi}
$$

(B.29)

and (for $r \geq 1$)

$$
\sum_{j=0}^{\infty} p^j \bar{L}_j^r = \frac{(-1)^r}{\pi} \sum_{j=0}^{\infty} (-p)^j \sum_{k=0}^{r-1} (a_{k+j-r+1} + a_{k+j-r}) p_{r-1,k}
$$

$$
= \frac{(-1)^r}{\pi} \sum_{k=0}^{r-1} p_{r-1,k} \sum_{i=0}^{\infty} a_i [(-p)^{i+r-k-1} + (-p)^{i+r-k}]
$$

(B.30)

$$
= \frac{(p-1)g(-p)}{\pi} \sum_{k=0}^{r-1} p_{r-1,k} (-1)^k p^{r-1-k} = \frac{g(-p)}{\pi} A_{r-1}(p).
$$

For the r.h.s.

$$
\Omega^{(j)} = \frac{1}{\pi} \alpha^{(j)}_{r-1},
$$

(B.31)

where we have temporarily introduced the short hand notation

$$
\alpha^{(j)}_{r-1} = \beta^{(j)}.
$$

(B.32)

The infinite summation for the r.h.s. gives

$$
\sum_{j=0}^{\infty} p^j \bar{R}_j^r = \sum_{j=0}^{\infty} p^j \sum_{k=0}^{j} (a_k + a_{k-1}) \frac{1}{\pi} \alpha^{(j-k)}_{r-1} = \frac{1 + p g(p)}{p} \sum_{s=0}^{\infty} p^{s+1} \alpha^{(s)}_{r-1}.
$$

(B.33)
Comparing the infinite sums at the two sides we obtain
\[ \sum_{s=0}^{\infty} p^{s+1} c^{(s)} = \frac{p}{1 + p} \frac{g(-p)}{g(p)} = u(p), \]
\[ \sum_{s=0}^{\infty} p^{s+1} \alpha_{r-1}^{(s)} = \frac{p}{1 + p} \frac{g(-p)}{g(p)} A_{r-1}(p) = u(p) A_{r-1}(p), \quad r \geq 1. \]

This is Statement 2.

C Consistency of the conjectured asymptotics

In this appendix our starting point is the ansatz (2.31). However, for technical reasons we will use its (equivalent) index-shifted version:
\[ \frac{P_{M-1,k}}{\Gamma(M)} = \frac{(-1)^M}{2\pi} (-1)^{k+1} \left\{ b^{(k)} + R_0^{(k)} M + \frac{R_1^{(k)}}{M(M-1)} + \frac{2R_2^{(k)}}{M(M-1)(M-2)} + \ldots \right\} \]

\[ k = 0, 1, \ldots \]

This asymptotic expansion is assumed valid for fixed \( k = 0, 1, \ldots \) in the large \( M \) limit. Using this ansatz we can calculate the asymptotic expansion of both sides of the Volin equations (2.1).

The large \( M \) asymptotics of \( Y_n^{(M)} \) can be written as
\[ \frac{Y_n^{(M)}}{\Gamma(M)} \approx \frac{(-1)^{M-n}(-1)^n}{2\pi(M-1)(\ldots)(M-n)} \sum_{k=0}^{M-n-1} (a_{n+k+1} + a_{n+k})(-1)^k \]
\[ \times \left\{ b^{(k)} + \frac{R_0^{(k)}}{M-n} + \frac{R_1^{(k)}}{(M-n)(M-n-1)} + \ldots \right\}. \]  

(C.1) is only valid for \( k \ll M \), but for large \( k \) the coefficients \( a_{n+k} \) are exponentially small and the asymptotic expansion in powers of \( 1/M \) is not affected by this condition. Using the same reasoning, we can extend the \( k \)-summation to infinity:
\[ \frac{Y_n^{(M)}}{\Gamma(M)} \approx \frac{(-1)^{M-n}(-1)^n}{2\pi(M-1)(\ldots)(M-n)} \sum_{k=0}^{\infty} (a_{n+k+1} + a_{n+k})(-1)^k \]
\[ \times \left\{ b^{(k)} + \frac{R_0^{(k)}}{M-n} + \frac{R_1^{(k)}}{(M-n)(M-n-1)} + \ldots \right\}. \]

(C.3) We work up to NNNLO and rearrange the summation as follows.
\[ \frac{Y_n^{(M)}}{\Gamma(M)} = \frac{(-1)^{M-n}}{2\pi(M-1)(\ldots)(M-n)} \left\{ -a_n \left[ b^{(0)} + \frac{R_0^{(0)}}{M-n} \right] \right. \]
\[ + \left. \frac{R_1^{(0)}}{(M-n)(M-n-1)} + \frac{2R_2^{(0)}}{(M-n)(M-n-1)(M-n-2)} \right. \]
\[ + \left. \sum_{k=1}^{\infty} a_{n+k} (-1)^k \left[ b^{(k-1)} - b^{(k)} + \frac{R_0^{(k-1)} - R_0^{(k)}}{M-n} \right. \right. \]
\[ + \left. \frac{R_1^{(k-1)} - R_1^{(k)}}{(M-n)(M-n-1)} + \frac{2[R_2^{(k-1)} - R_2^{(k)}]}{(M-n)(M-n-1)(M-n-2)} \right] + \ldots \right\}. \]
and similarly
\[
\frac{Y_n^{(M-r)}}{\Gamma(M)} = \frac{(-1)^{M-n-r}}{2\pi(M-1)(\ldots)(M-n-r)} \left\{ -a_n \left[ b^{(0)} + \frac{R_0^{(0)}}{M-n-r} \right] \right.
+ \frac{R_1^{(0)}}{(M-n-r)(M-n-r-1)} + \frac{2R_2^{(0)}}{(M-n-r)(M-n-r-1)(M-n-r-2)} \right.
+ \sum_{k=1}^{\infty} a_{n+k}(-1)^k \left[ b^{(k)} - b^{(k)} + \frac{R_0^{(k-1)} - R_0^{(k)}}{M-n-r} \right]
+ \frac{R_1^{(k-1)} - R_1^{(k)}}{M(M-1)} + \frac{2[R_2^{(k-1)} - R_2^{(k)}]}{M(M-1)(M-2)} \bigg\} \right\}.
\]

(C.5)

Proceeding analogously for the r.h.s., up to NNNLO we find
\[
\frac{\mathcal{R}^{(M)}}{\Gamma(M)} = \frac{(-1)^r}{2\pi} \sum_{k=0}^{\infty} a_k(-1)^k \left\{ b^{(k+r)} - b^{(k+r)} + \frac{R_0^{(k+r-1)} - R_0^{(k+r)}}{M} \right. \\
+ \frac{R_1^{(k+r-1)} - R_1^{(k+r)}}{M(M-1)} + \frac{2[R_2^{(k+r-1)} - R_2^{(k+r)}]}{M(M-1)(M-2)} \bigg\} + \ldots.
\]

(C.6)

The crucial observation is that
\[
\frac{Y_n^{(M-r)}}{\Gamma(M)} = O(M^{-n-r})
\]

and so
\[
\frac{\mathcal{L}^{(M)}}{\Gamma(M)} = O(M^{-r}),
\]

and consequently we must also have
\[
\frac{\mathcal{R}^{(M)}}{\Gamma(M)} = O(M^{-r}).
\]

(C.9)

But this is possible only if
\[
b^{(k)} = b, \quad k = 0, 1, \ldots
\]

(C.10)

and
\[
R_m^{(k)} = R_m, \quad k \geq m + 1.
\]

(C.11)

This is precisely the structure we observed numerically.

Using the above simplifications, to a given order in the asymptotic expansion, both sides of the equations contain a finite number of nonzero summands only. We work at NNNLO and we need the following expressions.
\[
\frac{\mathcal{R}^{(M)}}{\Gamma(M)} = \frac{(-1)^{M-1}}{2\pi} \left\{ \frac{R_0^{(0)} - R_0}{M} + \frac{R_1^{(0)} - R_1^{(1)}}{M(M-1)} + a_1[R_1^{(1)} - R_1^{(1)}] \right. \\
+ \frac{2[R_2^{(0)} - R_2^{(1)}] + 2a_1[R_2^{(2)} - R_2^{(1)}] + 2a_2[R_2^{(2)} - R_2]}{M(M-1)(M-2)} \bigg\}.
\]

(C.12)
\[ \frac{\mathcal{R}_2^{(M)}}{\Gamma(M)} = \frac{(-1)^M}{2\pi} \left\{ \frac{R_1^{(1)} - R_1}{M(M-1)} + \frac{2[R_2^{(1)} - R_2^{(2)}]}{M(M-1)(M-2)} + \ldots \right\}, \]  
\[ \frac{\mathcal{R}_3^{(M)}}{\Gamma(M)} = \frac{(-1)^{M-1}}{2\pi} \left\{ \frac{2[R_2^{(2)} - R_2^{(1)}]}{M(M-1)(M-2)} + \ldots \right\}, \]  
\[ \frac{Y_0^{(M-1)}}{\Gamma(M)} = \frac{(-1)^M}{2\pi} \left\{ \frac{b + R_0^{(0)} + a_1[R_0^{(0)} - R_0]}{M(M-1)} + \frac{R_0^{(0)} + R_1^{(0)} + a_1[R_0^{(0)} - R_0] + a_2[R_1^{(0)} - R_1^{(1)}] + a_2[R_1^{(1)} - R_1^{(2)}]}{M(M-1)(M-2)} + \ldots \right\}, \]  
\[ \frac{Y_1^{(M-1)}}{\Gamma(M)} = \frac{(-1)^M}{2\pi} \left\{ \frac{-ba_1}{M(M-1)} - \frac{2ba_1 + a_1R_0^{(0)} + a_2[R_0^{(0)} - R_0]}{M(M-1)(M-2)} + \ldots \right\}, \]  
\[ \frac{Y_2^{(M-1)}}{\Gamma(M)} = \frac{(-1)^M}{2\pi} \left\{ \frac{ba_2}{M(M-1)(M-2)} + \ldots \right\}, \]  
\[ \frac{Y_0^{(M-2)}}{\Gamma(M)} = \frac{(-1)^M}{2\pi} \left\{ \frac{-b + R_0^{(0)} + a_1[R_0^{(0)} - R_0]}{M(M-1)(M-2)} + \ldots \right\}, \]  
\[ \frac{Y_1^{(M-2)}}{\Gamma(M)} = \frac{(-1)^M}{2\pi} \left\{ \frac{ba_1}{M(M-1)(M-2)} + \ldots \right\}, \]  
\[ \frac{Y_0^{(M-3)}}{\Gamma(M)} = \frac{(-1)^M}{2\pi} \left\{ \frac{b}{M(M-1)(M-2)} + \ldots \right\}. \]

Working up to NNNLO order, we need the asymptotic expansion of the combinations

\[ \mathcal{L}_1^{(M)} = X_{1,0}Y_0^{(M-1)} + X_{1,1}Y_1^{(M-1)} + X_{1,2}Y_2^{(M-1)} + \ldots, \]
\[ \mathcal{L}_2^{(M)} = X_{2,0}Y_0^{(M-2)} + X_{2,1}Y_1^{(M-2)} + \ldots, \]
\[ \mathcal{L}_3^{(M)} = X_{3,0}Y_0^{(M-3)} + \ldots \]

Comparing these expansions to \( \mathcal{R}_1^{(M)} \), \( \mathcal{R}_2^{(M)} \), and \( \mathcal{R}_3^{(M)} \) and matching 3, 2, and 1 asymptotic expansion coefficients respectively, we can write down 6 equations. These can be used to express 6 leading coefficients, \( R_0^{(0)} \), \( R_1^{(0)} \), \( R_1^{(1)} \), \( R_2^{(0)} \), \( R_2^{(1)} \), and \( R_2^{(2)} \), in terms of the “generic” ones, \( b \), \( R_0 \), \( R_1 \), and \( R_2 \). The result is

\[ R_0^{(0)} = R_0 - \frac{b}{4}, \]
\[ R_1^{(0)} = R_1 - \frac{R_0}{4} + \left( \frac{1}{32} - \frac{a}{2} \right) b, \]
\[ R_1^{(1)} = R_1 - \frac{9b}{32}, \]
\[ R_2^{(0)} = R_2 - \frac{R_1}{8} + \left( \frac{1}{64} - \frac{a}{4} \right) R_0 + \left( -\frac{3}{256} + \frac{a}{16} - \frac{a^2}{2} \right) b, \]
\(R_2^{(1)} = R_2 - \frac{9R_0}{64} + \left( \frac{3}{128} - \frac{9a}{16} \right) b,\)

\(R_2^{(2)} = R_2 - \frac{75}{256} b.\)  \(\text{(C.22)}\)

Rewriting the above result in terms of the original “tilde” variables appearing in \((2.31)\) we obtain \((2.35)\).

**D Resurgence**

The results \((2.35)\) show that the first few small-index \(p_{a,b}\) quantities appear in the expression of the first few asymptotic coefficients and this structure strongly suggests a product form for the \(\Delta_{-1}\) alien derivative.

In this appendix we will use the alternative representation of \(\epsilon_p(z)\) given by \((1.33)\) and write the Taylor expansion coefficients of its Borel transform as

\[c_n^{(\epsilon_p)} = (1 + q)^{E_{n+1}(q)} \frac{E_{n+1}(q)}{\Gamma(n+1)}, \quad n = 0, 1, \ldots\]  \(\text{(D.1)}\)

We will assume that \(|q| < 1\). Using the ansatz \((2.31)\) we can write down the asymptotic expansion of the Borel transform coefficients:

\[c_n^{(\epsilon_p)} = (1 + q) \frac{(-1)^n}{2\pi} \sum_{k=0}^{\infty} (-q)^k \left\{ b^{(k)} + \frac{\tilde{R}_0^{(k)}}{n(n-1)} + \frac{\tilde{R}_1^{(k)}}{n(n-1)(n-2)} + \ldots \right\}.\]  \(\text{(D.2)}\)

Here, again, we extended the upper limit of the summation to infinity. Comparing this expansion with the generic forms \((F.3)\) and \((F.5)\) we get

\[\pi \tilde{B}_0 = (1 + q) \sum_{k=0}^{\infty} (-q)^k b^{(k)},\]  \(\text{(D.3)}\)

\[\pi \tilde{B}_0 \tilde{q}_m = (1 + q) \sum_{k=0}^{\infty} (-q)^k \tilde{R}_m^{(k)}, \quad m = 0, 1, \ldots,\]  \(\text{(D.4)}\)

\[\Delta_{-1} \epsilon_p(z) = i(1 + q) \sum_{k=0}^{\infty} (-q)^k b^{(k)} + i(1 + q) \sum_{m=0}^{\infty} \frac{m!(-1)^m}{z^{m+1}} \sum_{k=0}^{\infty} (-q)^k \tilde{R}_m^{(k)}.\]  \(\text{(D.5)}\)

We can compare this expression to the assumed product form

\[\Delta_{-1} \epsilon_p(z) = iX(z) \epsilon_p(z),\]  \(\text{(D.6)}\)

where

\[X(z) = b + \sum_{m=0}^{\infty} \frac{X_m}{z^{m+1}}.\]  \(\text{(D.7)}\)

Before making the comparison it is useful to recast \((D.6)\) in the form

\[X(z) \epsilon_p(z) = X(z) + (1 + q) \sum_{n=0}^{\infty} \frac{\phi_{n+1}(q)}{z^{n+1}},\]  \(\text{(D.8)}\)
where
\[ X(z) \sum_{n=0}^{\infty} \frac{E_{n+1}(q)}{z^{n+1}} = \sum_{n=0}^{\infty} \frac{\phi_{n+1}(q)}{z^{n+1}}. \] (D.9)

The comparison of (D.5) with (D.6) gives
\[ \sum_{k=0}^{\infty} (-q)^k b^{(k)} = b \sum_{k=0}^{\infty} (-q)^k, \] (D.10)
\[ \sum_{k=0}^{\infty} (-q)^k \tilde{R}_m^{(k)} = \left( -1 \right)^{m+1} \frac{1}{m!} \left\{ X_m + \phi_{m+1}(q) \right\}. \] (D.11)

The first of these gives
\[ b^{(k)} = b, \quad k = 0, 1, \ldots \] (D.12)

This is what we obtained already from the consistency of asymptotic expansions. Before using the second comparison (D.11) it is useful to write out \( \phi_{m+1}(q) \) explicitly:
\[ \phi_{m+1}(q) = bE_{m+1}(q) + \sum_{r=0}^{m-1} X_r E_{m-r}(q) = b \sum_{k=0}^{m} p_{m,k} q^k + \sum_{k=0}^{m-1} \sum_{r=0}^{m-1-k} X_r p_{m-k-1,r} q^k. \] (D.13)

We can now read off from (D.11) the relations (2.37)–(2.39). The structure (2.37) is precisely the same as we already found. Writing out (2.38) and (2.39) explicitly for \( m = 0, 1, 2 \) we get
\[ \tilde{R}_0^{(0)} = \tilde{R}_0 - p_{0,0} b, \]
\[ \tilde{R}_1^{(0)} = \tilde{R}_1 + bp_{1,0} + X_0 p_{0,0} = \tilde{R}_1 - p_{0,0} \tilde{R}_0 + p_{1,0} b, \]
\[ \tilde{R}_1^{(1)} = \tilde{R}_1 - p_{1,1} b, \]
\[ \tilde{R}_2^{(0)} = \tilde{R}_2 - \frac{1}{2} \left\{ bp_{2,0} + X_0 p_{1,0} + X_1 p_{0,0} \right\} \]
\[ = \tilde{R}_2 - \frac{1}{2} p_{0,0} \tilde{R}_1 + \frac{1}{2} p_{1,0} \tilde{R}_0 - \frac{1}{2} p_{2,0} b, \]
\[ \tilde{R}_2^{(1)} = \tilde{R}_2 + \frac{1}{2} \left\{ bp_{2,1} + X_0 p_{1,1} \right\} = \tilde{R}_2 - \frac{1}{2} p_{1,1} \tilde{R}_0 + \frac{1}{2} p_{2,1} b, \]
\[ \tilde{R}_2^{(2)} = \tilde{R}_2 - \frac{1}{2} p_{2,2} b. \] (D.14)

Here we used the identifications (2.37). We see that (D.14) exactly reproduces the earlier result (2.35).

The actual values of the generic coefficients are given by (2.33):
\[ X_m = 2W_m, \quad m = 0, 1, \ldots \] (D.15)

Thus the coefficient function is
\[ X(z) = 1 + \sum_{m=0}^{\infty} \frac{2W_m}{z^{m+1}} = \epsilon \] (D.16)
and
\[ \left[ \Delta_{-1} \epsilon_p \right](z) = i \epsilon(z) \epsilon_p(z). \] (D.17)
E Asymptotic expansion of $\rho$

The asymptotic expansion (C.4) can also be used to obtain the asymptotic series for $\rho$. Using the inversion formula

$$w_{0,M} = \sum_{n=0}^{M} (-1)^n E_{1,n} 2^{n-M} Y_{n}^{(M)}$$  \hspace{1cm}  (E.1)

we have

$$\tilde{u}_M = 2^M c_{0,M} = 2^M w_{0,M} = \sum_{n=0}^{M} \omega_n Y_{n}^{(M)}, \quad \omega_n = (-2)^n E_{1,n}.$$  \hspace{1cm}  (E.2)

Here are the first few $\omega_n$ coefficients:

$$\omega_0 = 1, \quad \omega_1 = \frac{3}{4}, \quad \omega_2 = -\frac{15}{32}, \quad \omega_3 = \frac{105}{128}.$$  \hspace{1cm}  (E.3)

and the first few $\tilde{u}_n$ values:

$$\tilde{u}_0 = 1, \quad \tilde{u}_1 = a - \frac{3}{4}, \quad \tilde{u}_2 = \frac{15}{32} + \frac{3a}{4} - \frac{a^2}{2},$$  \hspace{1cm}  (E.4)

$$\tilde{u}_3 = \frac{105}{128} + \frac{45a}{32} - \frac{9a^2}{8} + \frac{a^3}{2} + \frac{3}{8} \zeta_3.$$  \hspace{1cm}  (E.5)

We also recall the first few input parameters:

$$a_0 = 1, \quad a_1 = a - 1, \quad a_2 = 1 - a + \frac{a^2}{2} - \frac{1}{4} \zeta_2,$$  \hspace{1cm}  (E.6)

$$a_3 = a - 1 - \frac{a^2}{2} + \frac{a^3}{6} + \frac{1 - a}{4} \zeta_2 + \frac{1}{4} \zeta_3.$$  \hspace{1cm}  (E.7)

The Taylor coefficients of the Borel transform of $\rho$ are

$$c_{M-1}^{(\rho)} = \frac{\tilde{u}_M}{\Gamma(M)} = \sum_{n=0}^{M} \omega_n Y_{n}^{(M)} \Gamma(M).$$  \hspace{1cm}  (E.8)

To a given order of the asymptotic expansion (here we again work up to NNNLO) only a finite number of terms contribute to (C.4):

$$Y_{n}^{(M)} \Gamma(M) = \frac{(-1)^{M-n}}{2\pi(M-1)\ldots(M-n)} \left\{ -a_n \left[ b + \frac{R_0^{(0)}}{M-n} + \frac{2R_2^{(0)}}{(M-n)(M-n-1)} \right] + \frac{R_0^{(0)}}{M-n} + \frac{R_1^{(0)}}{(M-n)(M-n-1)} \right\} - a_{n+1} \left[ \frac{R_0^{(0)} - R_0}{M-n} + \frac{R_1^{(0)} - R_1}{(M-n)(M-n-1)} \right] + \frac{2[R_2^{(0)} - R_2^{(1)}]}{(M-n)(M-n-1)(M-n-2)} + \frac{[R_1^{(1)} - R_1]}{(M-n)(M-n-1)(M-n-2)} + \frac{2[R_2^{(1)} - R_2^{(2)}]}{(M-n)(M-n-1)(M-n-1)(M-n-2)} - a_{n+2} \right\} + \ldots.$$  \hspace{1cm}  (E.9)
To NNNLO we will need the $n = 0, 1, 2, 3$ cases and taking into account the shift of the index $M$ by 1 unit we have

\[
\frac{Y^{(M+1)}_0}{\Gamma(M+1)} = \frac{(-1)^M}{2\pi} \left\{ b + \frac{(1 + a_1)R_0^{(0)} - a_1 R_0}{M} \right. \\
+ \frac{1}{M(M-1)} [(1 + a_1)R_1^{(0)} - R_0^{(0)}] - (a_1 + a_2)R_1^{(1)} + a_1 R_0 + a_2 R_1 \\
+ \frac{2}{M(M-1)(M-2)} [(1 + a_1)R_2^{(0)} - R_1^{(0)} + R_0^{(0)}] - (a_1 + a_2)(R_2^{(1)} - R_1^{(1)}) \\
+ (a_2 + a_3)R_2^{(2)} - a_1 R_0 - a_2 R_1 - a_3 R_2 + \ldots \}, \\
\text{(E.10)}
\]

\[
\frac{Y^{(M+1)}_1}{\Gamma(M+1)} = \frac{(-1)^M}{2\pi} \left\{ \frac{-ba_1}{M} + \frac{a_2 R_0 - (a_1 + a_2)R_0^{(0)}}{M(M-1)} \\
+ \frac{(a_1 + a_2)(R_1^{(0)} - R_1^{(0)}) + (a_2 + a_3)R_1^{(1)} - a_2 R_0 - a_3 R_1}{M(M-1)(M-2)} + \ldots \}, \\
\text{(E.11)}
\]

\[
\frac{Y^{(M+1)}_2}{\Gamma(M+1)} = \frac{(-1)^M}{2\pi} \left\{ \frac{-ba_2}{M(M-1)} + \frac{(a_2 + a_3)R_0^{(0)} - a_3 R_0}{M(M-1)(M-2)} + \ldots \}, \\
\text{(E.12)}
\]

\[
\frac{Y^{(M+1)}_3}{\Gamma(M+1)} = \frac{(-1)^M}{2\pi} \left\{ -\frac{ba_3}{M(M-1)(M-2)} + \ldots \}. \\
\text{(E.13)}
\]

Comparing the expansion of $c^{(\rho)}_n$ to the generic case (F.3) we find for the constants determining the alien derivative of $\rho$:

\[
\pi \tilde{B}_o = b, \\
\pi \tilde{B}_o \tilde{q}_0 = (1 + a_1)R_0^{(0)} - a_1 R_0 - ba_1 \omega_1, \\
\pi \tilde{B}_o \tilde{q}_1 = (1 + a_1) \left( R_1^{(0)} - R_0^{(0)} \right) - (a_1 + a_2) R_1^{(1)} + a_1 R_0 + a_2 R_1 \\
+ \omega_1 \left[ a_2 R_0 - (a_1 + a_2) R_0^{(0)} \right] + ba_2 \omega_2, \\
2\pi \tilde{B}_o \tilde{q}_2 = 2 \left[ (1 + a_1) \left( R_2^{(0)} - R_1^{(0)} + R_0^{(0)} \right) - (a_1 + a_2) \left( R_2^{(1)} - R_1^{(1)} \right) \right] \\
+ (a_2 + a_3) R_2^{(2)} - a_1 R_0 - a_2 R_1 - a_3 R_2 \\
+ \omega_1 \left[ (a_1 + a_2) R_0^{(0)} - R_1^{(0)} \right] + (a_2 + a_3) R_1^{(1)} - a_2 R_0 - a_3 R_1 \\
+ \omega_2 \left[ (a_2 + a_3) R_0^{(0)} - a_3 R_0 \right] - ba_3 \omega_3. \\
\text{(E.14)}
\]

Using the solution (C.22) this simplifies to\(^4\)

\[
\pi \tilde{B}_o \tilde{q}_0 = \tilde{R}_0 + \left( \frac{3}{4} - a \right) b = -X_0 - \tilde{u}_1 b, \\
\]

\(^4\)Note that $\tilde{R}_0 = R_0$, $\tilde{R}_1 = R_1 - R_0$, $\tilde{R}_2 = R_2 - R_1 + R_0$. 

\[\text{-- 27 --}\]
\[ \pi B_0 \tilde{q}_1 = \tilde{R}_1 + \left( \frac{3}{4} - a \right) \tilde{R}_0 + \left( -\frac{15}{32} + \frac{3a}{4} - \frac{a^2}{2} \right) b = X_1 + \tilde{u}_1 X_0 + \tilde{u}_2 b, \]

\[ 2\pi B_0 \tilde{q}_2 = 2\tilde{R}_2 + \left( \frac{3}{4} - a \right) \tilde{R}_1 + \left( -\frac{15}{32} + \frac{3a}{4} - \frac{a^2}{2} \right) \tilde{R}_0 + \left( \frac{105}{128} - \frac{45a}{32} + \frac{9a^2}{8} - \frac{3}{8} a^3 - \frac{3}{8} \zeta_3 \right) b \]

\[ = -X_2 - \tilde{u}_1 X_1 - \tilde{u}_2 X_0 - \tilde{u}_3 b. \]  

(E.15)

From here we can read off the alien derivative of \( \rho \), given by (2.43).

### F Alien derivatives

In this appendix we summarize our conventions for alien derivatives. For details, see [11–13].

Let us recall that we start from a formal asymptotic series

\[ \Psi(z) = 1 + \sum_{n=1}^{\infty} \frac{\Gamma(n)c_{n-1}}{z^n} \]  

and define its Borel transform as

\[ \hat{\Psi}(t) = \sum_{n=0}^{\infty} c_n t^n. \]  

(F.1)

Here we assume that the coefficients have the asymptotic expansion

\[ c_n = \frac{\bar{A}_o}{2} \left\{ 1 + \frac{\bar{p}_0}{n} + \frac{\bar{p}_1}{n(n-1)} + \frac{2\bar{p}_2}{n(n-1)(n-2)} + \ldots \right\} + (-1)^n \frac{\bar{B}_o}{2} \left\{ 1 + \frac{\bar{q}_0}{n} + \frac{\bar{q}_1}{n(n-1)} + \frac{2\bar{q}_2}{n(n-1)(n-2)} + \ldots \right\}. \]  

(F.3)

In this case the singular part of the Borel transform is

\[ \hat{\psi}^{\text{sing}}(t) = \frac{\bar{A}_o}{2} \left\{ \frac{1}{1-t} - \ln(1-t) \sum_{m=0}^{\infty} \bar{p}_m (t-1)^m \right\} + \frac{\bar{B}_o}{2} \left\{ \frac{1}{1+t} - \ln(1+t) \sum_{m=0}^{\infty} (-1)^m \bar{q}_m (t+1)^m \right\} \]  

(F.4)

and the alien derivatives are given by

\[ [\Delta_1 \Psi](z) = -i\pi \bar{A}_o \left\{ 1 + \sum_{m=0}^{\infty} \frac{m! \bar{p}_m}{z^{m+1}} \right\}, \]

\[ [\Delta_{-1} \Psi](z) = i\pi \bar{B}_o \left\{ 1 - \sum_{m=0}^{\infty} \frac{m! (-1)^m \bar{q}_m}{z^{m+1}} \right\}. \]  

(F.5)

Note that we understand the alien derivative as a mapping from the space of asymptotic series to the same space, although it is defined by the singularity properties of the Borel-transformed function (alternatively the asymptotic properties of the coefficients \( c_n \)).
The $\eta_k = 1$ model

Since all $\eta_k$ are algebraically independent, it makes sense to study the simplest model

$$\eta_k = 1, \quad k = 1, 2, \ldots$$ \hfill (G.1)

For this model

$$g(x) = 1, \quad a_n = \delta_{n,0}. \hfill (G.2)$$

Let us make the following definitions.

$$K_n = p_{n-1,0}, \quad n = 1, 2, \ldots, \quad K_0 = 1 \hfill (G.3)$$

and

$$S_n = 2^n E_{0,n} = \frac{1}{n!} \prod_{\ell=1}^n \left( \ell - \frac{1}{2} \right)^2, \quad n = 1, 2 \ldots, \quad S_0 = 1. \hfill (G.4)$$

In this simple model the building blocks simplify enormously:

$$Y_n^{(M)} = \delta_{n,0} K_M, \quad n = 0, \ldots, M, \hfill (G.5)$$

$$w_{a,M} = \delta_{a,0} 2^{-M} K_M, \quad M = 0, 1, \ldots, \quad a = 0, \ldots, M. \hfill (G.6)$$

The left and right sides of the Volin equations simplify accordingly to

$$R_1^{(1)} = p_{0,0} \hfill (G.7)$$

and

$$M \geq 2 \begin{cases} R_r^{(M)} = p_{M-1,r-1} + p_{M-1,r}, & r = 1, \ldots, M - 1, \\ R_M^{(M)} = p_{M-1,M-1}, \end{cases} \hfill (G.8)$$

$$L_r^{(M)} = S_r K_{M-r}. \hfill (G.9)$$

Equating the two sides we obtain

$$M \geq 1 \begin{cases} p_{M-1,M-1} = S_M, \hfill (G.10) \\ p_{M-1,r-1} + p_{M-1,r} = S_r K_{M-r}, & r = 1, \ldots, M - 1. \hfill (G.11) \end{cases}$$

It is easy to find the solution of this recursion:

$$p_{M-1,M-k} = \sum_{p=0}^{k-1} (-1)^{k-p+1} K_p S_{M-p}, \quad k = 1, \ldots, M. \hfill (G.12)$$

In particular, the most interesting unknowns $K_n$ are determined recursively as

$$K_{M+1} = \sum_{p=0}^M (-1)^{M-p} K_p S_{M+1-p}, \quad M = 0, 1, \ldots \hfill (G.13)$$
Summing up (G.11) from $r = 1$ to $r = M - 1$ we obtain

$$2W_{M-1} = \sum_{r=0}^{M} S_r K_{M-r},$$

which also holds for $M = 1$. If we define

$$\xi_M = \sum_{r=0}^{M} S_r K_{M-r},$$

we can write

$$\epsilon(x) = \sum_{n=0}^{\infty} \xi_n x^n.$$  \hspace{1cm} (G.16)

Here $x = 1/z$. Similarly

$$\rho(x) = \sum_{n=0}^{\infty} K_n x^n$$

and we also introduce

$$B(x) = \sum_{n=0}^{\infty} S_n x^n.$$  \hspace{1cm} (G.18)

The solution of the recursion (G.13) is summarized by

$$\rho(x)B(-x) = 1$$

and (G.15) is equivalent to

$$\epsilon(x) = B(x)\rho(x).$$

Let us introduce

$$B(x) = 1 + b(x), \quad b(x) = \sum_{n=1}^{\infty} S_n x^n.$$  \hspace{1cm} (G.21)

The Borel transform of $b(x)$ is

$$\hat{b}(t) = \sum_{n=0}^{\infty} \beta_n t^n,$$

where

$$\beta_n = \frac{S_{n+1}}{\Gamma(n+1)}, \quad n = 0, 1, \ldots$$

We also define

$$D(x) = B(-x) = 1 + d(x), \quad d(x) = b(-x)$$

and see that

$$\hat{d}(t) = -\hat{b}(-t).$$

By inspecting the Taylor expansion coefficients (G.23) we recognize that

$$\hat{b}(t) = \frac{1}{4} F_1 \left( \frac{3}{2}, \frac{3}{2}; 2; t \right).$$

\hspace{1cm} (G.26)
Using Mathematica we can verify that in the vicinity of $t = 1$

$$\hat{b}(t) = -\frac{1}{\pi(t-1)} - \frac{\ln(1-t)}{\pi} \hat{d}(t-1) + \Phi(t-1), \quad (G.27)$$

where $\Phi$ is regular around the origin. This can be written in the language of alien derivatives as

$$\Delta_1 B = -2i - 2id = -2iD. \quad (G.28)$$

Similarly we have

$$\Delta_{-1} D = -2iB. \quad (G.29)$$

From

$$\rho = \frac{1}{D}, \quad \epsilon = \frac{B}{D} \quad (G.30)$$

we calculate

$$\Delta_1 \rho = 0, \quad \Delta_1 \epsilon = \frac{-2iD}{D} = -2i, \quad (G.31)$$

$$\Delta_{-1} \rho = -\frac{1}{D^2} \Delta_{-1} D = \frac{2iB}{D^2} = 2i\rho, \quad (G.32)$$

$$\Delta_{-1} \epsilon = -\frac{B}{D^2} \Delta_{-1} D = \frac{2iB^2}{D^2} = 2i\epsilon. \quad (G.33)$$

This structure of alien derivatives is very similar to what was found in the main part of the paper for the full model, it is only that $\Delta_1$ is by a factor 2 too small and $\Delta_{-1}$ is by a factor 2 too large. However, the overall factors can be changed easily, as we will see in the next subsection.

**G.1 a-dependence**

The alien derivatives transform under a change of variables. If the original expansion parameter $z$ is changed to $x$, where

$$z = z(x), \quad (G.34)$$

then the asymptotic expansions $\gamma(z)$ are changed to $C(x)$ by the simple substitution rule

$$C(x) = \gamma(z(x)). \quad (G.35)$$

Let us denote the original alien derivative (obtained from the Borel transform of $\gamma(z)$) at $\omega$ by $\Delta_\omega \gamma$ and the alien derivative in the new variable (obtained from the Borel transform of $C(x)$) by $D_\omega C$. The transformation rule $\Delta_\omega \rightarrow D_\omega$ is given by [38]

$$D_\omega \gamma(z(x)) = e^{-\omega(z(x)-x)}(\Delta_\omega \gamma)(z(x)) + \gamma'(z(x))(D_\omega z)(x). \quad (G.36)$$

In the case of a simple shift by a constant $m$,

$$z(x) = x + m, \quad (G.37)$$

(G.36) reduces to a constant rescaling:

$$D_\omega C = e^{-m\omega} \Delta_\omega \gamma. \quad (G.38)$$
If we treat the symbol $a$ in (1.13) as a generic parameter (rather than a shorthand for its true numerical value $a = \ln 2$) then

$$
\epsilon(a, z) \quad \text{and} \quad s(a, z) = z\rho^2(a, z)
$$

have the following remarkable property:

$$
\begin{align*}
\epsilon(a, z - 2a) &= s_o(z), \\
s(a, z - 2a) &= \epsilon_o(z),
\end{align*}
$$

(G.39)

where $s_o(z)$ and $\epsilon_o(z)$ are independent of the parameter $a$.

In the simplified model we calculated $\epsilon(1, z)$ and $\rho(1, z)$. For example for $\epsilon$ we then have

$$
\epsilon(a, x) = \epsilon(1, x + 2a - 2)
$$

(G.41)

and

$$
D_\omega \epsilon = e^{(2-2a)\omega} \Delta_\omega \epsilon = \left(\frac{e^2}{2}\right)^{2\omega} \Delta_\omega \epsilon.
$$

(G.42)

The transformation for the alien derivatives of $\rho$ is the same. We see that after restoring $a = \ln 2$ the alien derivatives of $\epsilon$ change to

$$
D_1 \epsilon = -i(e^2/2), \quad D_{-1} \epsilon = i(8/e^2)e^2.
$$

(G.43)

The changes for the alien derivatives of $\rho$ are the same. It is remarkable that these coefficients are numerically quite close to the “true” values (4 and 1):

$$
D_1 \epsilon = -3.69i, \quad D_{-1} \epsilon = 1.08ie^2.
$$

(G.44)

G.2 The alien derivatives of $\epsilon_p$ for the $\eta_k = 1$ model

We calculate $\epsilon_p$ for $\eta_k = 1$ as follows. Let us multiply (G.11) by $q^r$ and sum over $r$ from 1 to $M - 1$. In this way we obtain the relation (for $M \geq 2$)

$$
\sum_{r=0}^{M} S_r K_{M-r} q^r = (1 + q)E_M(q).
$$

(G.45)

The above relation also holds for $M = 1$ and we can write

$$
\begin{align*}
\epsilon_p &= 1 + (1 + q) \sum_{M=1}^{\infty} \frac{E_M(q)}{z^M} = \sum_{M=0}^{\infty} \frac{1}{z^M} \sum_{r=0}^{M} S_r K_{M-r} q^r \\
&= \sum_{r=0}^{\infty} \sum_{k=0}^{\infty} x^{r+k} S_r K_k q^r = \rho(x)B(qx).
\end{align*}
$$

(G.46)

For clarity let us define

$$
P(x) = B(qx) = 1 + p(x) = 1 + \sum_{n=1}^{\infty} S_n q^n x^n.
$$

(G.47)

\footnote{These hold for arbitrary $\eta_k$, not just for the simplified model $\eta_k = 1$. We have verified this property perturbatively up to 16th order using Mathematica.}
The Borel transform of \( p(x) \) is

\[
\hat{p}(t) = \hat{q}b(qt)
\]  

(G.48)

and therefore its singular part, using (G.27), is

\[
\hat{p}(t) \approx -\frac{-q}{\pi (qt - 1)} - \frac{q \ln(1 - qt)}{\pi} \hat{d}(qt - 1) + \ldots
\]  

(G.49)

From here we can read off the alien derivative

\[
\Delta_p P = -2i - 2id(qx) = -2iD(qx).
\]  

(G.50)

Now we can calculate

\[
\Delta_p \epsilon_p = \rho(x)\Delta_p P(x) = -2i\rho(x)D(qx) = -2i\epsilon_{-p}
\]  

(G.51)

and

\[
\Delta_{-1}\epsilon_p = B(qx)\Delta_{-1}\rho = 2i\epsilon\rho B(qx) = 2i\epsilon\epsilon_p.
\]  

(G.52)

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