Pion observables calculated in Minkowski and Euclidean spaces from Ansatz quark propagators

D. Kekez1 and D. Klabučar2

1Rugjer Bošković Institute, Bijenička cesta 54, 10000 Zagreb, Croatia
2Physics Department, Faculty of Science, University of Zagreb, Bijenička cesta 32, 10000 Zagreb, Croatia

(Dated: June 4, 2020)

We study two quark–propagator meromorphic Ansätze that admit clear connection between calculations in Euclidean space and Minkowski spacetime. The connection is established through a modified Wick rotation in momentum space, where the integration contour along the imaginary axis is adequately deformed. The Ansätze were previously proposed in the literature and fitted to Euclidean lattice QCD data. The generalized impulse approximation is used to calculate the pion electromagnetic and transition form factors. The pion decay constant and distribution amplitude are also calculated. The latter is used to deduce the asymptotic behavior of the form factors. Such an asymptotic behavior is compared with those obtained directly from the generalized impulse approximation and the causes of differences are pointed out.

I. INTRODUCTION

Obtaining the properties of hadrons as quark and gluon bound states, from the underlying theory of strong interactions, QCD, has proven to be extremely challenging. Reproducing even relatively simple observables, such as decay constants, is difficult whenever the nonperturbative regime of QCD must be dealt with. However, powerful tools for this task have been developed over the last decades. These tools include lattice QCD [1, 2] and continuum functional methods. The latter is exemplified by Functional Renormalization Group (see, e.g., Refs. [3, 4] and references therein) and Schwinger–Dyson equations (see, e.g., Refs. [5] for reviews and Refs. [6, 7] for examples of calculations of some observables addressed also in the present paper).

Due to technical complications inherent to these two continuum functional approaches, most corresponding calculations are not done in physical Minkowski spacetime but in four-dimensional Euclidean space. Hereby one exploits a technical trick, the so-called Wick rotation, to map quantum field theory in Minkowski spacetime to Euclidean space. The situation with the Wick rotation relating Minkowski with Euclidean space must be under control, but this is highly nontrivial in the nonperturbative regime of QCD. In particular, it should be clarified whether nonperturbative QCD Green’s functions employed in a calculation permit Wick rotation.

On the formal level, Osterwalder–Schrader reconstruction theorem states that the Schwinger functions of some Euclidean field theory can be analytically extended to Wightman functions of the corresponding Minkowski space quantum field theory, providing that these Schwinger functions satisfy some set of constraints, the Osterwalder–Schrader axioms [8].

The widely used rainbow–ladder truncation to the coupled Schwinger–Dyson equation (SDE) for the dressed quark propagator (“gap equation”) and Bethe–Salpeter Equation (BSE) for a quark–antiquark bound state are usually formulated in the Euclidean space and equations are solved for spacelike momenta [9]. Although some physical quantities can be extracted from the results in Euclidean space alone, many others, like, e.g., decay properties, cannot be calculated with just real Euclidean four–momenta. In general, for solving BSE and calculation of processes, the knowledge is needed about the analytic behavior in the part of complex momentum–squared plane (see, e.g., Ref. [10]). In this respect, analytic continuation of auxiliary quantities like Green’s functions of the theory, notably the quark propagator, open up the possibility to provide an understanding of strong-interaction processes from results of lattice QCD and functional methods.

Only a limited number of papers deals with the quark propagator modeling, or solving its SDE, in Minkowski space. Sauri, Adam, and Bicudo [11] have explored the fermion–propagator SDE in Minkowski space. The interaction used is a meromorphic function of momentum transfer squared; it has two simple poles on the real axis, in the timelike region. Various spectral representations of the fermion propagator are employed. Ruiz Arriola and Broniowski [12] have proposed a spectral quark model based on a generalization of the Lehmann representation of the quark propagator and applied it to calculate some low–energy quantities. While their \( \sigma_v \) and \( \sigma_g \) functions, defined by Eq. (1), exhibit only cuts on the timelike part of the real axis, the quark dressing function \( A(z) \) [see Eq. (1)] has pairs of the complex–conjugate poles in the complex momentum plane. Sirigò [13, 14] has studied the analytic properties of gluon, ghost, and quark propagators in QCD, using a one–loop massive expansion in the Landau gauge. He studies spectral functions in Minkowski space, by analytic continuation from deep infrared, and finds complex conjugated poles for the gluon propagator but no complex poles for the quark propagator. A group of interconnected papers [15, 16] typically start from a consistently truncated system of SDE and BSE, or some algebraic Ansätze for the quark propagator and Bethe–Salpeter (BS) amplitude inspired by such a consistent system. They have calculated the electromag-
The function $M$ is shown as the blue solid line in Fig. I (This Ansatz form has been already used to fit lattice QCD data [31]. There, the parameter values $m_0$, $m$, and $\lambda$ are rather close to those used in Ref. [32] and in the present paper; nevertheless, the propagator of Ref. [34] exhibits one real and a pair of complex conjugated poles.) Asymptotic expansions of $M$ about $\infty$ and $0$ are

$$M(x) = m_0 + \frac{m^3}{x} - \frac{\lambda^2 m^3}{x^2} + O(\frac{1}{x^3}), \quad (3)$$

$$M(x) = \left( m_0 + \frac{m^3}{\lambda^2} \right) - \frac{m^3 x}{\lambda^4} + \frac{m^3 x^2}{\lambda^6} + O(x^3), \quad (4)$$

respectively. The functions $A$, $B$, $\sigma_V$, and $\sigma_S$ depend algebraically on $Z$ and $M$, and are defined for convenience. The quark dressing functions $\sigma_V$ and $\sigma_S$, introduced by Eq. (II), can be decomposed as

$$\sigma_V(x) = \sum_{j=1}^{3} \frac{b_{Vj}}{x + p_j}, \quad (5a)$$

$$\sigma_S(x) = \sum_{j=1}^{3} \frac{b_{Sj}}{x + p_j}, \quad (5b)$$

where the coefficients $p_j$, $b_{Vj}$, and $b_{Sj}$, ($j = 1, 2, 3$), are certain complicated algebraic functions of the parameters $m_0$, $m$, and $\lambda$. Obviously, $\sigma_{V,S}(x) \to 0$ for all $x \to \infty$.

### III. ADFM QUARK PROPAGATOR

The dressing functions $\sigma$ of the ADFM meromorphic Ansatz that have three real poles [33] are

$$\sigma_V(x) = \frac{1}{Z_2} \sum_{j=1}^{3} \frac{2r_j}{x + a_j^2}, \quad (6a)$$

$$\sigma_S(x) = \frac{1}{Z_2} \sum_{j=1}^{3} \frac{2r_j a_j}{x + a_j^2}, \quad (6b)$$

where $a_1 = 0.341$ GeV, $a_2 = -1.31$ GeV, $a_3 = -1.35919$ GeV, $r_1 = 0.365$, $r_2 = 1.2$, $r_3 = -1.065$, $Z_2 = 0.982731$ [33]. The coefficients $r_j$ and $a_j$ satisfy

$$\sum_{j=1}^{3} r_j = \frac{1}{2}, \quad \sum_{j=1}^{3} a_j r_j = 0. \quad (7)$$

The first of the above constraints follows from the consideration of the large-momentum limit of $\sigma_V(x)$; the second one arises from the requirement that $M(x)$ must vanish for large spacelike real momenta. The Ansatz (6) guarantees that the quark dressing functions $\sigma_{S,V}(z) \to 0$ as $z \to 0$ away from the chiral limit, the second sum would be equal to the renormalized quark mass.
for all \(|z| \to \infty\) in the complex \(z\) plane. For the given set of parameters the functions \(x \mapsto A(-x)\) and \(x \mapsto B(-x)\) have poles at \(x = 0.488784 \text{ GeV}^2\) and \(x = 2.65383 \text{ GeV}^2\). The corresponding quark mass function \(M\) is shown as the red dashed line in Fig. 1.

Euclidean formalism adopted in Ref. [32] avoids probing of the quark dressing functions near their poles, \(x = -a_j^2, j = 1,2,3\). As we want to analytically continue \(\sigma\)'s to the complex plane and use these functions for the calculation in Minkowski space, a prescription for the pole treatment ought to be defined. An obvious choice is Feynman’s \(i\varepsilon\) prescription, already used in MMF–Ansatz case, Eq. (2): we push the poles infinitesimally from the real axis: \(x = -a_j^2 + i\varepsilon, j = 1,2,3\). We use this prescription throughout this paper.

Functions \(A(x)\) and \(B(x)\) that follow from Eqs. (6) are also the rational functions, exhibiting real poles for \(x < 0\). For example, function \(B\), which will be used in further calculation, is of the form

\[
B(x) = -\frac{c}{b_1 - b_2} \frac{(x + a)}{(x + b_1)(x + b_2)} \left[\frac{(b_1 - a)}{x + b_1} + \frac{(a - b_2)}{x + b_2}\right], \tag{8}
\]

where the coefficients \(a, b_1, b_2,\) and \(c\) are some complicated algebraic functions of the original parameters \(Z_2, a_j,\) and \(r_j,\) appearing in Eqs. (6). Small \(i\varepsilon\) shift of \(\sigma_V\) and \(\sigma_S\) poles, \(x = -a_j^2 + i\varepsilon, j = 1,2,3,\) causes the similar shift of the \(B\) poles, \(x = -b_k + i\varepsilon', k = 1,2,\) in agreement with the Feynman prescription.

For \(z \in \mathbb{C}\) and large \(|z|\) we find that \(M(z) \propto 1/z\), but this asymptotic behavior is reached only at very high momenta squared, \(|z| \simeq 1000 \text{ GeV}^2\). The MMF quark propagator Ansatz shows the same asymptotics for \(m_0 = 0\), while \(M(z) \sim m_0\) for \(m_0 \neq 0\); see Eq. (3). A well-known QCD result [37,38] for the asymptotics of the quark mass function is

\[
M(z) \propto \left\{\begin{array}{ll}
[\log(z/\Lambda_{\text{QCD}}^2)]^{d-1}/z & \text{in the chiral limit} \\
[\log(z/\Lambda_{\text{QCD}}^2)]^{-d} & \text{otherwise}
\end{array}\right., \tag{9}
\]

where \(d = 12/(11N_c - 2N_f)\) is the anomalous mass dimension, \(N_c\) and \(N_f\) are the number of colors and flavors, respectively; \(\Lambda_{\text{QCD}} \sim 0.5 \text{ GeV}\) is the QCD scale. The simple meromorphic Ansätze, Eqs. (2) and (6), emulate the chiral–limit and away–from–the–chiral–limit behavior, respectively, of the quark mass function [9], up to the logarithmic corrections present in Eq. (9). The Ansätze are fitted to the respective lattice data: MMF quark propagator to lattice data of Ref. [36] and ADFM quark propagator to lattice data in the overlap [39,41] and Asqtad (tadpole improved staggered) [42] formulations.

**IV. PION DECAY CONSTANT**

The pion decay constant \(f_\pi\) is defined by the matrix element

\[
\langle 0 | d(x) \gamma^\mu \gamma_5 u(x) | \pi^+(P) \rangle = i \sqrt{2} f_\pi P^\mu e^{-iP \cdot x}, \tag{10}
\]

where \(u(x)\) and \(d(x)\) are the quark fields (see, e.g., Ref. [43]). This matrix element is the hadronic part of the amplitude for \(\pi^+ \to l^+ \nu_l\) decay, pictorially represented in Fig. 2. More explicitly, \(f_\pi\) can be expressed in terms of the BS vertex function \(\Gamma_\pi(q, P)\):

\[
f_\pi = \frac{N_c}{2M_\pi^2} \times \frac{1}{(2\pi)^4} \text{tr} \left( P\gamma_5 S(q + \frac{P}{2}) \Gamma_\pi(q, P) S(q - \frac{P}{2}) \right), \tag{11}
\]

where \(N_c = 3\) is the number of colors, and \(M_\pi\) is the pion mass. Dictated by dynamical chiral symmetry breaking the axial–vector Ward–Takahashi identity, taken in the chiral limit, gives us the quark–level Goldberger–Treiman relation for the BS vertex,

\[
\Gamma_\pi(q, P) \simeq -\frac{2B(-q^2)\epsilon_{c.l.}}{f_\pi} \gamma_5, \tag{12}
\]

which expresses \(\Gamma_\pi\) in terms of the chiral limit (c.l.) value of the quark dressing function \(B\); see, e.g., Ref. [16]. This approximation will be used throughout this paper.

The pion decay constant \(f_\pi\) corresponding to the MMF quark propagator model, Eq. (2), has been calculated in three different ways: (a) analytically using Mathematica packages FeynCalc 9.0 [44,45] and Package–X 2.0 [46,47], (b) numerical integration in the Euclidean space, and (c) Minkowski space integration utilizing light-cone momenta and analytic residua calculation. Let us explain them in more detail.
Numerically. Eventually, the resulting value of Ref. [32] is in agreement with our previous calculations. The result Eqs. (2a). The remaining two-dimensional integration is effectively two-dimensional, as the principal value vanishes. All three methods (a) and (b) mentioned above, and (d). The method (a), the trace appearing in Eq. (11) is evaluated using FeynCalc and LoopTools Mathematica packages, formally treating $B(x)$ as a sum of two propagators [see Eq. (5)].

(c) Alternatively, following the procedure used in Ref. [32], integral (11) is calculated introducing light-cone variables $q^+ = q^0 \pm q^3$. The integrand is a rational function in $q^+$, variable, with seven simple poles on the real $q^+$ axis. Cauchy’s residue theorem is used to calculate the integral over $q^+$, paying attention to the $i\epsilon$ rule for the displacement of poles, prescribed by Eqs. (20). The remaining two-dimensional integration over $q_+ \in [-M/2, M/2]$ and $(q_+)^2 + (q_-)^2$ is performed numerically. Eventually, the resulting $f_\pi = 87.5599$ MeV in agreement with our previous calculations. The result of Ref. [32] is $f_\pi = 90$ MeV, a little above our calculated value.

Regarding the ADFM Ansatz, $f_\pi$ is calculated using methods (a) and (b) mentioned above, and (d). The method (d) is the Minkowski space integration where the first integration, over $q^0$, boils down to residua calculation, as the principal value vanishes. All three methods give the same result, $f_\pi = 71.5611$ MeV. Regarding the method (a), the trace appearing in Eq. (11) is evaluated using FeynCalc and LoopTools Mathematica packages, formally treating $B(x)$ as a sum of two propagators [see Eq. (5)].

\[ \langle \pi^+(P')|J^\mu(0)|\pi^+(P)\rangle = Q_\pi^+(P^\mu + P'^\mu)F_\pi(Q^2) \]
\[ = i(Q_u - Q_d)N_c \int \frac{d^4q}{(2\pi)^4} \text{tr}\left\{ \bar{\psi}_\pi(q - P/2) \right. \]
\[ \times S(q + \frac{1}{2}(P' - P))\Gamma^\mu(q + \frac{1}{2}(P' - P), q - \frac{1}{2}(P' - P)) \]
\[ \times S(q - \frac{1}{2}(P' - P))\Gamma^\mu(q - \frac{1}{2}(P' - P), q + \frac{1}{2}(P' + P)) \right\}, \]
\[ (13) \]

in the generalized impulse approximation (GIA) [50], for spacelike $Q^2$, and the momentum routing as depicted in Fig. 3. The electromagnetic current is $J^\mu(x)$; the quark charge $Q_u = 2/3$ and $Q_d = -1/3$. We use the following kinematics: $k = (0, 0, 0, \sqrt{Q^2})$, $P = (E_\pi, 0, 0, -\sqrt{Q^2}/2)$, and $P' = (E_\pi, 0, 0, \sqrt{Q^2}/2)$, where $E_\pi = \sqrt{M^2 + Q^2}/4$ and $Q^2 \geq 0$. The Ball–Chiu vertex [53, 54] is used for the quark–quark–photon coupling throughout this paper:

\[ \Gamma^\mu(p', p) = \frac{1}{2}[A(-p'^2) + A(-p^2)]\gamma^\mu + \frac{(p' + p)^\mu}{(p'^2 - p^2)} \]
\[ \times \left\{ [A(-p'^2) - A(-p^2)]\frac{(p' + p)^2}{2} - [B(-p'^2) - B(-p^2)] \right\}. \]
\[ (14) \]

This vertex can be expressed completely in terms of the quark–propagator dressing functions and it becomes particularly simple in the case of the MMF Ansatz:

\[ \Gamma^\mu(p', p) = \gamma^\mu - \frac{m^3(p'^2 + p^2)}{(p'^2 - \lambda^2)(p^2 - \lambda^2)} \]
\[ (15) \]

Similarly to the case of $f_\pi$ calculation, three methods are used to calculate $F_\pi(Q^2)$ using the MMF Ansatz:

FIG. 2. Diagram for $\pi^+ \to t^+\nu_l$ decay.

FIG. 3. Impulse approximation to the charged pion electromagnetic form factor $F_\pi(Q^2)$. V. ELECTROMAGNETIC FORM FACTOR

The charged pion EMFF $F_\pi(Q^2)$ is given by
(a) FeynCalc and Package–X Mathematica packages, (b) numerical integration in Euclidean space using adaptive quadrature, and (c) Minkowski space integration utilizing light–cone momenta momenta and analytic residual calculation. Let us discuss these methods in more detail.

(a) $F_\pi(Q^2)$, given by Eq. (13), is calculated using FeynCalc and Package–X Mathematica packages analogously to the $f_\pi$ calculation. The results are represented in Fig. 4.

(b) Numerical integration is performed using adaptive quadrature: expressing the space part of the four-vector $q$ in spherical coordinates, $q = (q^0, \xi \sin \vartheta \cos \varphi, \xi \sin \vartheta \sin \varphi, \xi \cos \varphi)$, the poles of the integrand, in variable $q^0$, are

$$
(q^0)_{1,2} = \mp \sqrt{M_\xi^2 + \xi^2 - \xi \sqrt{Q^2} \cos \vartheta + Q^2/4}, \quad (16a)
$$

$$
(q^0)_{3,4} = \mp \sqrt{M_\xi^2 + \xi^2 + \xi \sqrt{Q^2} \cos \vartheta + Q^2/4}, \quad (16b)
$$

$$
(q^0)_{5,6} = \frac{1}{2} \left(4M_\pi^2 + Q^2 \mp 2 \sqrt{M_\xi^2 + \xi^2} \right), \quad (16c)
$$

$$
(q^0)_{7,8} = \frac{1}{4} \left(4M_\pi^2 + Q^2 \pm \sqrt{16M_\xi^2 + 16\xi^2 + 8\xi \sqrt{Q^2} \cos \vartheta + Q^2} \right), \quad (16d)
$$

$$
(q^0)_{9,10} = \frac{1}{4} \left(4M_\pi^2 + Q^2 \mp \sqrt{16M_\xi^2 + 16\xi^2 - 8\xi \sqrt{Q^2} \cos \vartheta + Q^2} \right), \quad (16e)
$$

where $M_\xi^2 \in \{p_1, p_2, p_3, \lambda^2\}$. The numbers $(-M_\xi^2)$ are poles of the propagator functions (15) and (2a). Changing $M_\xi^2 \to M_\xi^2 - i\varepsilon$ pushes odd–indexed poles to the complex upper half–plane and even-indexed poles to the lower half–plane. We define two sets,

$$
A = \left\{(q^0)_j : j = 1, 3, 5, 7, 9 \land M_\xi^2 = p_1, p_2, p_3, \lambda^2 \right\}, \quad (17a)
$$

$$
B = \left\{(q^0)_j : j = 2, 4, 6, 8, 10 \land M_\xi^2 = p_1, p_2, p_3, \lambda^2 \right\}, \quad (17b)
$$

where $A$ and $B$ contain poles that must be bypassed from below and from above, respectively. Note that not all four values of $M_\xi^2$ produce poles of the integrand. For example, $(q^0)_{7,8}$ are poles of the integrand only for $M_\xi^2 = \lambda^2$: these two poles correspond to singular behavior of $\Gamma_\pi(q - P/2, P')$ and are defined by equation $(q - P/2)^2 = \lambda^2$. For simplicity of definition, the sets $A$ and $B$ are allowed to contain superfluous points, but this does not obstruct the analysis hereafter. Numerical examination shows that $\max\{|A|\} < \min\{|B|\}$ for the chosen model parameters, so we define

$$
(q^0)_c = \frac{1}{2} (\max\{|A|\} + \min\{|B|\}), \quad (18)
$$

which is a function of $\vartheta$ and $\xi$, but does not depend on $\varphi$ thanks to the symmetry. Figure 4 illustrates the $\xi$–dependence of $(q^0)_c$‘s and $(q^0)_c$ for a fixed value of $\vartheta$. Unlike the case of $f_\pi$ calculation, Eq. (11), where the first and third quadrants of $q^0$ complex plane is free of poles and the naive Wick rotation $q^0 = -iq_4$ ($q_4 \in \mathbb{R}$) is allowed, in the present case of $F_\pi(Q^2)$ calculation, the path of integration ought to be shifted to pass between poles contained in the sets $A$ and $B$:

$$
q^0 = (q^0)_c - iq_4, \quad (19)
$$

where $q_4 \in (-\infty, \infty)$. Eventually, the numerical integration over $q_4$, $\xi$, and $\vartheta$ is performed using the adaptive quadrature; see Fig. 4 for the final result.

(c) Minkowski space integration utilizing light–cone momenta is again performed analogously to the $f_\pi$ calculation. Now, there are eleven poles, in variable $q_-$, of the integrand of Eq. (13). The residues are calculated analytically and adaptive quadrature are used for the final three–dimensional integration.

To conclude about the EMFF obtained with the MMF Ansatz, there are only insignificant differences, of order $< 0.1\%$, between results for $F_\pi(Q^2)$ calculated using methods (a), (b), and (c). The differences are compatible with the precision of numerical integration that we prescribed in methods (b) and (c). However, there is a significant discrepancy between our results (blue dots) and those of Ref. [22] (black dashed line in our Fig. 4). The MMF Ansatz [22] is also used in Ref. [55], with the same model parameter values. While $Q^2_F(J^2) = \propto$ practically constant for $Q^2 \gtrsim 3$ GeV$^2$ in the former paper, it

![Fig. 4](image-url)
and 1. In respect we do not follow Ref. [32] that forces $F_1$ normalization condition [61]. We obtain value of $H_1$ by adjusting the normalization of BS vertex (20).

The approximate BS vertex (12) does not fulfill. The general form of pseudoscalar BS Ansatz

$$F = \frac{1}{q^2 - M^2} \left[ \Gamma(q^2) \right]$$

yields the same results. Black dashed line represents the result of Mello et al. [32]. Black solid line corresponds to the perturbative QCD result [27] with asymptotic PDA.

Concerning the low–$Q^2$ behavior, the pion charge radius, $r_\pi = \sqrt{-6F_2(0)}$, is calculated to be $r_\pi = 0.632$ fm and 0.699 fm for MMF and ADFM Ansatz, respectively. The both values are reasonably near the experimental value of $r_\pi = (0.672 \pm 0.008)$ fm [57]. Simple constituent quark model formula $r_\pi = \sqrt{3/(2\pi f_\pi)}$ [58, 59] gives $r_\pi = 0.621$ fm and 0.760 fm for MMF and ADFM Ansatz, respectively. The approximate BS vertex (12) does not guarantee that the normalization condition $F_\pi(0) = 1$ will be fulfilled. The general form of pseudoscalar BS vertex is

$$F_\pi(q, P) = \gamma_5 \left( H_1(q, P) + \not{P} H_2(q, P) + g H_3(q, P) + [\not{P}, g] H_4(q, P) \right),$$

where $H_1$, $H_2$, $H_3$, and $H_4$ are Lorentz–scalar functions [60]. Keeping solely $H_1$ component and neglecting others, just as we do in Eq. (12), leads to deviation from $F_\pi(0) = 1$ normalization condition [61]. We obtain $F_\pi(0) = 0.950$ and 1.32 for MMF and ADFM Ansatz, respectively. In that respect we do not follow Ref. [32] that forces $F_\pi(0) = 1$ by adjusting the normalization of BS vertex [20].

The high–$Q^2$ asymptotics of the charged pion EMFF

falls with $Q^2$ very noticeably in the latter one. Hence, Ref. [55] agrees better with our EMFF, although it still falls more slowly than ours.

For the ADFM quark propagator, we have calculated $F_\pi(Q^2)$ using only one method out of three adopted for the MMF Ansatz; namely the method (b), the modified Wick rotation, defined by Eq. (19), and subsequent three–dimensional adaptive Monte Carlo integration. The results are depicted as red solid circles in Fig. 5.

Concerning the low–$Q^2$ behavior, the pion charge radius, $r_\pi = \sqrt{-6F_2(0)}$, is calculated to be $r_\pi = 0.632$ fm and 0.699 fm for MMF and ADFM Ansatz, respectively. The both values are reasonably near the experimental value of $r_\pi = (0.672 \pm 0.008)$ fm [57]. Simple constituent quark model formula $r_\pi = \sqrt{3/(2\pi f_\pi)}$ [58, 59] gives $r_\pi = 0.621$ fm and 0.760 fm for MMF and ADFM Ansatz, respectively. The approximate BS vertex (12) does not guarantee that the normalization condition $F_\pi(0) = 1$ will be fulfilled. The general form of pseudoscalar BS vertex is

$$F_\pi(q, P) = \gamma_5 \left( H_1(q, P) + \not{P} H_2(q, P) + g H_3(q, P) + [\not{P}, g] H_4(q, P) \right),$$

where $H_1$, $H_2$, $H_3$, and $H_4$ are Lorentz–scalar functions [60]. Keeping solely $H_1$ component and neglecting others, just as we do in Eq. (12), leads to deviation from $F_\pi(0) = 1$ normalization condition [61]. We obtain $F_\pi(0) = 0.950$ and 1.32 for MMF and ADFM Ansatz, respectively. In that respect we do not follow Ref. [32] that forces $F_\pi(0) = 1$ by adjusting the normalization of BS vertex [20].

The high–$Q^2$ asymptotics of the charged pion EMFF

is discussed in Sec. VII along with the asymptotics of the neutral pion TFF, which is introduced in the next section.

VI. TRANSITION FORM FACTOR

The two–photon amplitude $T(k^2, k'^2)$ that describes $\pi^0 \to \gamma \gamma^{(*)}$ processes, depicted in Fig. 6, is given by

$$T_{\mu\nu}(k, k') = e^{\mu\nu\lambda\sigma} k_{\lambda} k'_{\sigma} T(k^2, k'^2)$$

$$= -N_c \frac{Q^2 - Q_0^2}{2} \int \frac{d^4q}{(2\pi)^4} \text{tr} \{ \Gamma^\mu(q - \frac{P}{2}, k + q - \frac{P}{2}) \times \Gamma^\nu(k + q - \frac{P}{2}, q + \frac{P}{2}) S(q + \frac{P}{2}) \} \times \Gamma_\pi(q, P) S(q - \frac{P}{2}) \} + (k \leftrightarrow k', \mu \leftrightarrow \nu),$$

in the GIA [11, 22, 23], where $k$ and $k'$ are the external photon momenta, $P = k + k'$ is the neutral pion momentum, $P^2 = M_\pi^2$. The TFF is defined as

$$F_\pi(Q^2) = |T(-Q^2, 0)|,$$

such that the $\pi^0 \to \gamma \gamma$ decay width can be written as

$$\Gamma(\pi^0 \to \gamma \gamma) = \frac{\alpha^2 M_\pi^3}{4} F_\pi(0)^2.$$

In respect of the MMF Ansatz, FeynCalc package is used to express the loop integral in Eq. (21) as a sum of the Passarino–Veltman functions, while Package-X is used for the final numerical evaluation, in a close analogy to the $F_\pi(Q^2)$ calculation, Sec. VI method (a). The results of our calculation are pictorially represented by the blue dots in Fig. 7. The experimental results are shown as solid circles and diamonds (with error bars) in the same figure.

On the other hand, the case of the ADFM Ansatz is treated using solely method (b) described in Sec. VI. The
The following method was used to determine in reasonable agreement with the experimental values. Q calculated several (−0.3 GeV² ≤ Q² ≤ 0.3 GeV² and −0.2 GeV² ≤ Q² ≤ 0.2 GeV² for MMF and ADFM Ansätze, respectively. These points were fitted to \( F_{πγ}(Q^2) = A/(1 + Q^2/B^2) \) curve; the derivative \( F'_{πγ}(0) \) was computed from this fit. A simple quark triangle model \([69]\) gives \( a = M_2^2/(12M^2) \), where \( M_c \) is the constituent quark mass. Using \( M_π = 135 \text{ MeV} \) and \( M_0 = 280 \text{ MeV} \) (estimated from Fig. 3) gives \( a = 0.02 \), somewhat below the experimental values and our model results. The high–Q² asymptotics of \( F_{πγ} \) is addressed in the next section and is compared with those calculated from the PDA.

VII. PION DISTRIBUTION AMPLITUDE AND ASYMPTOTICS OF FORM FACTORS

The factorization property of the QCD hard scattering amplitudes enables us to express these amplitudes in terms of the pertinent distribution amplitudes. The PDA, relevant for the TFF and EMFF calculation at large Q², can be expressed as the light–cone projection,

\[
\phi_π(u) = \frac{i N_c}{8\pi f_π} \text{tr} \left( \gamma_+ \gamma_5 \int \frac{dq_-}{2\pi} \int \frac{d^2q_⊥}{(2\pi)^2} \chi_π(q, P) \right),
\]

of the BS amplitude

\[
χ_π(q, P) = S(q + \frac{P}{2})Γ_π(q, P)S(q - \frac{P}{2}),
\]

The variable \( q_+ \), which is implicit in the integrand of Eq. (25), is defined by \( u = 1/2 + q_+/P_+ \). The integral resembles those of the \( f_π \)–calculation, Eq. (11), and could be treated in the same way. For both propagator Ansätze we use the Euclidean space integration, referred as method (b) in Secs. IV and V. The resulting PDAs are displayed in Fig. 8.

![Fig. 7](image_url)

**Fig. 7.** (color online). Blue dots represent \( π^0 \) transition form factor calculated using the MMF quark propagator Ansätze, Eqs. (1) and (2). The red pluses are calculated using the ADFM quark propagator, Eqs. (6). The blue solid line and red dashed line represent the Brodsky–Lepage interpolation formula, Eq. (29), for the MMF quark propagator and ADFM quark propagator models, respectively. Solid circles and diamonds (with error bars) represent the measurements of BaBar [62] and Belle [63] collaboration, respectively.

![Fig. 8](image_url)

**Fig. 8.** (color online). Pion distribution amplitudes \( \phi_π(u) \). Blue solid line and red dashed line correspond to the MMF and ADFM Ansätze, respectively. Black dotted line represents the asymptotic form, \( \phi_π^{\infty}(u) = 6a(1 - u) \).
The leading twist pQCD results for the asymptotics of the pion form factor is \[ F_\pi(Q^2) \sim \frac{16\pi\alpha_s(Q^2)f_\pi^2}{Q^2} \left| \frac{1}{3} \int_0^1 du \frac{\phi_\pi(u)}{u} \right|^2 \] (27)
for \( Q^2 \to \infty \), where \( \alpha_s \) is the QCD running coupling constant: \( \alpha_s(Q^2) = \alpha_s(\mu^2)Q^2/\Lambda_{\text{QCD}}^2 \) at the one–loop order of perturbation theory. The renormalization scale \( (\mu) \) dependence of PDA is implicit here. The asymptotic form of PDA, \( \phi_{\alpha}^a(u) = \lim_{u \to 0} \phi_{\alpha}^a(u) = 6u(1-u) \), gives \( \frac{1}{3} \int_0^1 du \phi_{\alpha}^a(u)/u = 1 \), leading to \( F_\pi(Q^2) \sim 16\pi\alpha_s(Q^2)f_\pi^2/Q^2 \) asymptotic behavior. The PDAs \( \phi_{\alpha}^a(u) \), related to the models under consideration, do not deviate too much from the asymptotic \( \phi_{\alpha}^a(u) \) function; see Fig. 8. The actual values of integrals are \( \frac{1}{3} \int_0^1 du \phi_{\alpha}^a(u)/u = 1.15 \) and 1.02 for the MMF and ADFM models, respectively. This results in respective 32\% and 4\% enhancement of \( F_\pi(Q^2) \) relative to value obtained with \( \phi_{\alpha}^a(u) \).

The asymptotic form of EMFF, Eq. (27), being dependent on \( \alpha_s(Q) \), critically reflects the perturbative nature of high-energy QCD. Our simple meromorphic Ansätze, Eqs. (27) and (30), which do not comply with the exact QCD asymptotics, Eq. (9), is not expected to reproduce the UV logarithmic behavior of Eq. (27). We computed \( F_\pi(Q^2) \) up to \( Q^2 = 40 \) GeV\(^2\) and indeed found no evidence that the asymptotic behavior \( F_\pi(Q^2) \propto 1/(Q^2\ln(Q^2)) \) was reached, for either of our models. The presently available experimental data on \( F_\pi(Q^2) \) are anyway well above the pQCD predictions, Eq. (27), as discussed in Ref. [72] in more detail.

The same PDA (27) also determines the leading term of the light–cone expansion of form factor \( F_{\pi\gamma}(Q^2) \) [72, 80],

\[ F_{\pi\gamma}(Q^2) \sim \frac{2f_\pi}{3Q^2} \int_0^1 du \frac{\phi_{\pi\gamma}(u)}{(1-u)} . \] (28)

The asymptotic form of PDA leads to \( F_{\pi\gamma}(Q^2) \sim 2f_\pi/Q^2 \) for \( Q^2 \to \infty \) asymptotic behavior [72, 81]. The Brodsky–Lepage (BL) dipole formula [81],

\[ F_{\pi\gamma}(Q^2) = \frac{1}{4\pi^2f_\pi} \left( 1 + \frac{Q^2}{8\pi^2f_\pi^2} \right)^{-1} , \] (29)

interpolates between \( F_{\pi\gamma}(0) = 1/(4\pi^2f_\pi) \), the ABJ anomaly result [65, 66], and \( \lim_{Q^2 \to \infty} Q^2F_{\pi\gamma}(Q^2) = 2f_\pi \), the pQCD limit. The current experimental data [62, 63], reaching up to \( Q^2 \sim 35 \) GeV\(^2\), do not show agreement with this limit yet. On the theoretical side, recent SDE studies in Euclidean space are not unanimous: Raya et al. [27] are consistent with the hard scattering limit, but Eichmann et al. [52] claim that the BL limit is modified whenever the other external photon is near–on–shell, \( i.e. \, k'^2 \simeq 0 \).

As we can see from Fig. [7] and Tab. [1] the high–Q\(^2\) behavior of \( F_{\pi\gamma}(Q^2) \) calculated in the GIA, Eq. (27),

\[ \gamma^{\mu}(\nu) \to \frac{1}{2} \left( 1 + A \left( [q - (\sigma + P)^2/2] \right) \right) \gamma^{\mu}(\nu) , \] (31)

and the GIA limit is recovered [11, 83],

\[ \lim_{Q^2 \to \infty} Q^2F_{\pi\gamma}(1 + A_{\text{soft}}/2)(Q^2) = 0.225 \text{ GeV} ; \] (32)

\[ \lim_{Q^2 \to \infty} Q^2F_{\pi\gamma}(Q^2) = 2f_\pi/Q^2 \] (27)
for \( Q^2 \to \infty \), where \( \alpha_s \) is the QCD running coupling constant: \( \alpha_s(Q^2) = \alpha_s(\mu^2)Q^2/\Lambda_{\text{QCD}}^2 \) at the one–loop order of perturbation theory. The renormalization scale \( \mu \) dependence of PDA is implicit here. The asymptotic form of PDA, \( \phi_{\alpha}^a(u) = \lim_{u \to 0} \phi_{\alpha}^a(u) = 6u(1-u) \), gives \( \frac{1}{3} \int_0^1 du \phi_{\alpha}^a(u)/u = 1 \), leading to \( F_{\pi}(Q^2) \sim 16\pi\alpha_s(Q^2)f_\pi^2/Q^2 \) asymptotic behavior. The PDAs \( \phi_{\alpha}^a(u) \), related to the models under consideration, do not deviate too much from the asymptotic \( \phi_{\alpha}^a(u) \) function; see Fig. 8. The actual values of integrals are \( \frac{1}{3} \int_0^1 du \phi_{\alpha}^a(u)/u = 1.15 \) and 1.02 for the MMF and ADFM models, respectively. This results in respective 32\% and 4\% enhancement of \( F_{\pi}(Q^2) \) relative to value obtained with \( \phi_{\alpha}^a(u) \).

The asymptotic form of EMFF, Eq. (27), being dependent on \( \alpha_s(Q) \), critically reflects the perturbative nature of high-energy QCD. Our simple meromorphic Ansätze, Eqs. (27) and (30), which do not comply with the exact QCD asymptotics, Eq. (9), is not expected to reproduce the UV logarithmic behavior of Eq. (27). We computed \( F_{\pi}(Q^2) \) up to \( Q^2 = 40 \) GeV\(^2\) and indeed found no evidence that the asymptotic behavior \( F_{\pi}(Q^2) \propto 1/(Q^2\ln(Q^2)) \) was reached, for either of our models. The presently available experimental data on \( F_{\pi}(Q^2) \) are anyway well above the pQCD predictions, Eq. (27), as discussed in Ref. [72] in more detail.

The same PDA (27) also determines the leading term of the light–cone expansion of form factor \( F_{\pi\gamma}(Q^2) \) [72, 80],

\[ F_{\pi\gamma}(Q^2) \sim \frac{2f_\pi}{3Q^2} \int_0^1 du \frac{\phi_{\pi\gamma}(u)}{(1-u)} . \] (28)

The asymptotic form of PDA leads to \( F_{\pi\gamma}(Q^2) \sim 2f_\pi/Q^2 \) for \( Q^2 \to \infty \) asymptotic behavior [72, 81]. The Brodsky–Lepage (BL) dipole formula [81],

\[ F_{\pi\gamma}(Q^2) = \frac{1}{4\pi^2f_\pi} \left( 1 + \frac{Q^2}{8\pi^2f_\pi^2} \right)^{-1} , \] (29)

interpolates between \( F_{\pi\gamma}(0) = 1/(4\pi^2f_\pi) \), the ABJ anomaly result [65, 66], and \( \lim_{Q^2 \to \infty} Q^2F_{\pi\gamma}(Q^2) = 2f_\pi \), the pQCD limit. The current experimental data [62, 63], reaching up to \( Q^2 \sim 35 \) GeV\(^2\), do not show agreement with this limit yet. On the theoretical side, recent SDE studies in Euclidean space are not unanimous: Raya et al. [27] are consistent with the hard scattering limit, but Eichmann et al. [52] claim that the BL limit is modified whenever the other external photon is near–on–shell, \( i.e. \, k'^2 \simeq 0 \).

As we can see from Fig. [7] and Tab. [1] the high–Q\(^2\) behavior of \( F_{\pi\gamma}(Q^2) \) calculated in the GIA, Eq. (27),

\[ \gamma^{\mu}(\nu) \to \frac{1}{2} \left( 1 + A \left( [q - (\sigma + P)^2/2] \right) \right) \gamma^{\mu}(\nu) , \] (31)

and the GIA limit is recovered [11, 83],

\[ \lim_{Q^2 \to \infty} Q^2F_{\pi\gamma}(1 + A_{\text{soft}}/2)(Q^2) = 0.225 \text{ GeV} ; \] (32)

\[ \lim_{Q^2 \to \infty} Q^2F_{\pi\gamma}(Q^2) = 2f_\pi/Q^2 \] (27)

\[ \gamma^{\mu}(\nu) \to \frac{1}{2} \left( 1 + A \left( [q - (\sigma + P)^2/2] \right) \right) \gamma^{\mu}(\nu) , \] (31)

and the GIA limit is recovered [11, 83],

\[ \lim_{Q^2 \to \infty} Q^2F_{\pi\gamma}(1 + A_{\text{soft}}/2)(Q^2) = 0.225 \text{ GeV} ; \] (32)
the superscript “\((1 + A_{soft})/2\)” indicates that \(F_{\pi\gamma}\) is calculated using vertex \([31]\) instead of the Ball–Chiu one. Hence, the nontrivial infrared behavior of the wave function renormalization, \(Z(x) = 1/x(A(x) \neq 1)\), is responsible that the two calculations, the first one based on GIA Eq. \((21)\) and the second one based on bare Eq. \((30)\) produce unequal asymptotics of \(F_{\gamma\gamma}(Q^2)\). Of course, for the MMF Ansätz, where \(Z(x) = 1\), the both calculations give the same asymptotic limit.

The respective integral \(\frac{1}{2} \int dz d\bar{u} \phi_\pi(u)/u_corr\) values of 1.02 and 1.15 for the MMF and ADFM Ansätze, which influence the EMFF asymptotics, Eq. \((27)\), are reflected also in the asymptotic behavior of TFF calculated from Eq. \((28)\) and shown in Tab. \(I\), in the row denoted by BL-\(non-asymptotic\) (for it is not calculated using the asymptotic form of \(\phi(u)\) but the model calculated one).

To the end of this section we explain similarity between the bare and BL-\(non-asymptotic\) approximation. Light–cone expansion of the time–ordered product of two electromagnetic currents, \(T\{J^\mu(x), J^\nu(y)\}\), leads to the following approximate expression:

\[
T^{\mu\nu}(k, k') \simeq 2 \frac{Q_0^2 - Q_1^2}{\sqrt{2}} \frac{1}{2\pi^2} e^{\mu\nu\lambda\sigma} x_i^e dz e^{ikz} \langle \text{vac} | \bar{d}(0)\gamma_\mu\gamma_5 u(z) : |\pi^+(P), z = 0 \rangle .
\]

(See, e.g., Refs. [87, 88].) \(I\) The path–ordered “string operator”

\[
P \exp \left( ig \int_x^0 A^\alpha(y) dy_\alpha \right),
\]

must be included between the quark fields. This operator equals to unity in light–cone gauge; see, e.g., Ref. [71].

On the one hand, using the above \(\pi^+\to\text{vacuum}\) matrix element through the BS amplitude,

\[
\langle \text{vac} | \bar{u}(0)\gamma^\mu\gamma_5 u(z) - \bar{d}(0)\gamma^\mu\gamma_5 d(z) : |\pi^0(P) \rangle = -N_c e^{-iPz/2} \int \frac{d^4q}{(2\pi)^4} e^{-i\vec{q} \cdot \vec{r}} \langle \gamma^\mu\gamma_5 \lambda_\pi(q, P) \rangle,
\]

we reproduce bare Eq. \((30)\). On the other hand, the definition of the PDA,

\[
\frac{1}{2} \langle \text{vac} | \bar{u}(0)\gamma^\mu\gamma_5 u(z) - \bar{d}(0)\gamma^\mu\gamma_5 d(z) : |\pi^0(P) \rangle_{z_+ = z_- = 0} = i \delta^{ab} f_\pi P^\mu \int_0^1 du e^{-iuPz} \phi_\pi(u),
\]

leads eventually to BL-\(non-asymptotic\) approximation, Eq. \((28)\). To conclude, both Eqs. \((28)\) and \((30)\) follow from Eq. \((33)\), except Eq. \((28)\) is derived without \(z^2 = 0\) constraint, i.e., without light-cone projection of the non-local operator: \(\bar{\psi}(0)\gamma^\mu\gamma_5 \psi(z)\). It turns out that such a difference is of little influence, at least for the models under considerations.

\[\text{VIII. SUMMARY AND CONCLUSIONS}\]

In this paper, we have studied two meromorphic \(Ansätze\) for the dressed quark propagator, suggested in Refs. [32] and [33]. Thanks to the quark–level Goldberger–Treiman relation \((12)\), the pseudoscalar BS vertex can be related to the dynamically dressed momentum–dependent quark mass function \([10]\). Besides, by exploiting the Ball–Chiu vertex [53, 54] as an approximation for the fully dressed quark–quark–photon vertex, we are provided with all the necessary elements to calculate the pion decay constant, EMFF, TFF, and PDA. The related amplitudes were calculated using several methods in order to check the robustness of the results.

The used quark \(Ansätze\) as well as the pertaining vertices exhibit mass–like singularities on the real timelike momentum axis and do not obey the pQCD asymptotic behavior; hence, we can hardly expect that the correct perturbative asymptotic behavior of the electromagnetic form factor, \(F_\pi(Q^2) \propto 1/(Q^2 \ln(Q^2))\), will be attained. Indeed, our numerical evaluation of \(F_\pi(Q^2)\) up to \(Q^2 = 40\ \text{GeV}^2\) did not show evidence that either \(F_\pi(Q^2) \sim 1/(Q^2 \ln(Q^2))\) limit or simpler power–law \(F_\pi(Q^2) \sim Q^2\) limit is reached. However, it should be acknowledged that the exact asymptotic behavior is of purely academic interest here because (a) it is generally expected that the asymptotic regime probably starts at \(Q^2 \gtrsim 20\ \text{GeV}^2\), well above Jefferson Lab capability after proposed upgrade [89], (b) and even existing Cornell experimental data at \(Q^2 = 6.30\ \text{GeV}^2\) and 9.77 GeV^2 have large error bars [90]. For high \(Q^2\), our results for \(Q^2 F_\pi(Q^2)\) obviously deviates from those of Ref. [32].

The low–\(Q^2\) behavior of \(F_\pi(Q^2)\), encoded in the pion charge radius \(r_\pi\), was found to be in a reasonable agreement with experiment, given the simplicity of the model.

The leading order pQCD expression for the high–\(Q^2\) behavior of the transition form factor, \(F_{\gamma\gamma}(Q^2) \sim 2 f_\pi/Q^2\), depends only on \(f_\pi\), the low–energy pion observable, which is pretty insensitive to the details of the high–energy dynamics. Hence, we could naively expect that our \(Ansätze\), despite not incorporating the exact perturbative regime behavior, should produce the correct perturbative limit of the pion transition form factor. However, in the generalized impulse approximation the electromagnetic vertices keep one quark leg soft, even for the high–\(Q^2\) external photon. As the result, this approximation gave \(Q^2 F_{\gamma\gamma}(Q^2)\) finite for \(Q^2 \to \infty\), but generally unequal to the pQCD limit of \(2 f_\pi\); see also Refs. [11, 83]. In relation to low–\(Q^2\) behavior, our results for the TFF slope parameter are 10%-15% below the experimental value.
The pion distribution amplitudes that were calculated using our Ansätze did not deviate appreciably from the asymptotic one. If we input these amplitudes (instead the asymptotic one) to the pQCD form–factor formulae, the result is enhanced up to 30%, depending on the form factor and Ansätze.

The simple analytic structure of quark–propagator Ansätze employed, together with suitable approximations for the required vertices, enabled us to keep control of the Wick rotation when calculating some processes; the pertinent amplitudes can be calculated equally in Minkowski and Euclidean space. Kindred studies are mostly restricted to the Euclidean space; their propagators and vertices are sensibly defined for spacelike external momenta, \( q^2 = (q^0)^2 - |q|^2 < 0 \), but their analytic properties (singularities in the first and third quadrants of the complex \( q^0 \) plane) preclude Wick rotation back to the Minkowski space. In principle, it is not difficult to impose the correct perturbative asymptotic behavior on gluon and quark propagator in such models. In the context of the coupled Schwinger–Dyson and Bethe–Salpeter equation, such an example is provided in Ref. [91–93]; a similar and widely used model is introduced in Refs. [94, 95]. and its application reviewed in Ref. [96]. Among the variety of quark–propagator Ansätze explored in Ref. [33] that exhibit correct pQCD behavior, none is suitable for the calculation methods presented in this work: the branch cut in propagator functions do not allow the usage of perturbative techniques while the complicated singularity structure prevents the Wick rotation.

The future work may include calculation of some other processes involving quark loops, e.g., \( \gamma^* \rightarrow 3\pi, \gamma\gamma \rightarrow \pi\pi, \) and \( \pi^0 \rightarrow e^-e^+ \). The most appealing improvement would be a quark propagator Ansatz that has the correct UV behavior and, at the same time, enough simple analytic structure that allow Wick rotation (in the sense used in this paper). But it is not evident to us whether such a task could be achieved.

ACKNOWLEDGMENTS

This work was supported in part by by STSM grants from COST Actions CA15213 THOR and CA16214 PHAROS. The Feynman diagrams were drawn with the help of Jazzodraw [97], based on AoxoDraw [98].

[1] S. Hashimoto, J. Laiho, and S.R. Sharpe, “Lattice Quantum Chromodynamics” review for the PDG, published in Ref. [57].
[2] I. Montvay and G. Münster, Quantum fields on a lattice: Cambridge Monographs on Mathematical Physics (Cambridge University Press, 1997).
[3] J. M. Pawlowski, Aspects of the functional renormalisation group, Annals Phys. 322, 2831 (2007) arXiv:hep-th/0512261 [hep-th].
[4] B.-J. Schaefer and J. Wambach, Renormalization group approach towards the QCD phase diagram, Helmholtz International Summer School on Dense Matter in Heavy Ion Collisions and Astrophysics, Dubna, Russia, August 21-September 1, 2006., Phys. Part. Nucl. 39, 1025 (2008) arXiv:hep-ph/0611191 [hep-ph].
[5] R. Alkofer and L. von Smekal, The Infrared behavior of QCD Green’s functions: Confinement dynamical symmetry breaking, and hadrons as relativistic bound states, Phys.Rept. 353, 281 (2001) arXiv:hep-ph/0007355 [hep-ph].
[6] C. D. Roberts and S. M. Schmidt, Dyson-Schwinger equations: Density, temperature and continuum strong QCD, Prog.Part.Nucl.Phys. 45, S1 (2000) arXiv:nucl-th/0005064 [nucl-th].
[7] C. S. Fischer, Infrared properties of QCD from Dyson-Schwinger equations, J. Phys. G32, R253 (2006) arXiv:hep-ph/0605173 [hep-ph].
[8] D. Kekez and D. Klabučar, Two photon processes of pseudoscalar mesons in a Bethe-Salpeter approach, Phys. Lett. B387, 14 (1996) arXiv:hep-ph/9605219 [hep-ph].
[9] D. Klabučar and D. Kekez, \( \eta \) and \( \eta’ \) in a coupled Schwinger-Dyson and Bethe-Salpeter approach, Phys. Rev. D 58, 096003 (1998) hep-ph/9710206.
[10] D. Kekez, B. Bistriović, and D. Klabučar, Application of Jain and Munczek’s bound-state approach to \( \gamma\gamma \)-processes of \( \pi^0, \eta, \) and \( \eta_8 \), Int.J.Mod.Phys. A14, 161 (1999) arXiv:hep-ph/9809245 [hep-ph].
[11] D. Kekez and D. Klabučar, \( \gamma^*\gamma \rightarrow \pi^0 \) transition and asymptotics of \( \gamma^*\gamma \) and \( \gamma^*\gamma’ \) transitions of other unflavored pseudoscalar mesons, Phys.Lett. B457, 339 (1999) arXiv:hep-ph/9812495 [hep-ph].
[12] D. Kekez and D. Klabučar, \( \eta \) and \( \eta’ \) in a coupled Schwinger-Dyson and Bethe-Salpeter approach. II. The \( \gamma^*\gamma \) transition form factors, Phys.Rev. D65, 057901 (2002) arXiv:hep-ph/0110019 [hep-ph].
[13] D. Kekez and D. Klabučar, Pseudoscalar \( q\bar{q} \) mesons and effective QCD coupling enhanced by \( (A^2) \) condensate, Phys. Rev. D 71, 014004 (2005) hep-ph/0307110.
[14] D. Kekez and D. Klabučar, \( \eta \) and \( \eta’ \) mesons and dimension 2 gluon condensate \( \langle A^2 \rangle \), Phys. Rev. D 73, 036002 (2006) hep-ph/0512064.
[15] K. Osterwalder and R. Schrader, Axioms for Euclidean Green’s functions, Commun.Math.Phys. 31, 83 (1973).
[16] C. D. Roberts and A. G. Williams, Dyson-Schwinger equations and their application to hadronic physics, Int.J.Mod.Phys. 33, 477 (1994) arXiv:hep-th/9402224 [hep-th].
[17] R. Alkofer, P. Watson, and H. Weigel, Mesons in a Poincaré covariant Bethe-Salpeter approach, Phys. Rev. D65, 094026 (2002) arXiv:hep-ph/0202053 [hep-ph].
[18] V. Sauli, J. Adam, Jr., and P. Bicudo, Dynamical chiral symmetry breaking with Minkowski space in-
[19] E. Ruiz Arriola and W. Broniowski, Spectral quark model and low-energy hadron phenomenology, Phys. Rev. D67, 074021 (2003)

[20] F. Siringo, Analytical study of Yang-Mills theory in the infrared from first principles, Nucl. Phys. B907, 572 (2016)

[21] F. Siringo, Analytical study of Yang-Mills theory in the infrared from first principles, Nucl. Phys. B907, 572 (2016)

[22] C. D. Roberts, Electromagnetic pion form-factor and neutral pion decay width, Nucl. Phys. A605, 475 (1996)

[23] M. R. Frank, K. L. Mitchell, C. D. Roberts, and P. C. Tandy, The off-shell axial anomaly via the $\gamma^*\pi^0 \to \gamma$ transition, Phys. Lett. B359, 17 (1995)

[24] H. L. L. Roberts, C. D. Roberts, A. Bashir, L. X. Gutierrez-Guerrero, and P. C. Tandy, Abelian anomaly and neutral pion production, Phys. Rev. C82, 065202 (2010)

[25] L. Chang, I. C. Clot, C. D. Roberts, S. M. Schmidt, and P. C. Tandy, Pion electromagnetic form factor at spacelike momenta, Phys. Rev. Lett. 111, 141802 (2013)

[26] C. Mezrag, L. Chang, H. Moutarde, C. D. Roberts, J. Rodriguez-Quintero, F. Sabati, and S. M. Schmidt, Sketching the pion’s valence-quark generalised parton distribution, Phys. Lett. B741, 190 (2015)

[27] R. Mertig, M. Bohm, and A. Denner, FEYN CALC: Computer-algebraic calculation of Feynman amplitudes, Comput. Phys. Commun. 64, 345 (1991)

[28] K. Raya, L. Chang, A. Bashir, J. J. Cobos-Martinez, L. X. Gutierrez-Guerrero, C. D. Roberts, and P. C. Tandy, Structure of the neutral pion and its electromagnetic transition form factor, Phys. Rev. D93, 074017 (2016)

[29] T. Horn and C. D. Roberts, The pion: an enigma within the Standard Model, J. Phys. G43, 073001 (2016)

[30] N. Nakashiti, Partial-Wave Bethe-Salpeter Equation, Phys. Rev. 130, 1230 (1963)

[31] N. Nakashiti, A General survey of the theory of the Bethe-Salpeter equation, Prog.Theor.Phys.Suppl. 43, 1 (1969)

[32] C. S. Mello, J. P. B. C. de Melo, and T. Frederico, Minkowski space pion model inspired by lattice QCD running quark mass, Phys. Lett. B766, 86 (2017)

[33] R. Alkofer, W. Detmold, C. Fischer, and P. Maris, Analytic properties of the Landau gauge gluon and quark propagators, Phys. Rev. D70, 041504 (2004)

[34] D. Duda, M. S. Gúimaraes, L. F. Palhares, and S. P. Sorella, Confinement and dynamical chiral symmetry breaking in a non-perturbative renormalizable quark model, Annals Phys. 365, 155 (2016)

[35] R. Oehme and W.-t. Xu, Asymptotic limits and sum rules for the quark propagator, Phys.Lett. B384, 269 (1996)

[36] M. B. Parappilly, P. O. Bowman, U. M. Heller, D. B. Leinweber, A. G. Williams, and J. B. Zhang, Scaling behavior of quark propagator in full QCD, Phys. Rev. D73, 054504 (2006)

[37] K. D. Lane, Asymptotic Freedom and Goldstone Realization of Chiral Symmetry, Phys.Rev. D10, 2605 (1974)

[38] H. D. Politzer, Effective Quark Masses in the Chiral Limit, Nucl.Phys. B117, 397 (1976)

[39] F. D. R. Bonnet, P. O. Bowman, D. B. Leinweber, A. G. Williams, and J.-b. Zhang (CSSM Lattice), Overlap quark propagator in Landau gauge, Phys. Rev. D65, 114503 (2002)

[40] J. B. Zhang, F. D. R. Bonnet, P. O. Bowman, D. B. Leinweber, and A. G. Williams, Towards the continuum limit of the overlap quark propagator in Landau gauge, Lattice field theory. Proceedings: 20th International Symposium, Lattice 2002, Cambridge, USA, Jun 24-29, 2002, Nucl. Phys. Proc. Suppl. 119, 831 (2003)

[41] J. B. Zhang, P. O. Bowman, D. B. Leinweber, A. G. Williams, and F. D. R. Bonnet (CSSM Lattice), Scaling behavior of the overlap quark propagator in Landau gauge, Phys. Rev. D70, 034505 (2004)

[42] P. O. Bowman, U. M. Heller, and A. G. Williams, Lattice quark propagator with staggered quarks in Landau and Laplacian gauges, Phys. Rev. D66, 014505 (2002)

[43] A. L. Yaukun, L. Oliver, O. Pene, and J.-C. Raynal, Hadron transitions in the quark model (Gordon and Brachence Publisher, New York, London, Paris, Montreux, Tokyo, Melbourne, 1988)

[44] R. Mertig, M. Bohm, and A. Denner, FEYN CALC: Computer-algebraic calculation of Feynman amplitudes, Comput. Phys. Commun. 64, 345 (1991)

[45] V. Shtabovenko, R. Mertig, and F. Orelana, New Developments in FeynCalc 9.0, Comput. Phys. Commun. 207, 432 (2016)

[46] H. H. Patel, Package-X 2.0: A Mathematica package for the analytic calculation of one-loop integrals, Comput. Phys. Commun. 197, 276 (2015)

[47] H. H. Patel, Package-X 2.0: A Mathematica package for the analytic calculation of one-loop integrals, Comput. Phys. Commun. 218, 66 (2017)

[48] G. Passarino and M. J. G. Veltman, One loop corrections for $e^+e^-$ annihilation into $\mu^+\mu^-$ in the Weinberg model, Nucl. Phys. B160, 151 (1979)

[49] T. Hahn and M. Perez-Victoria, Automated one loop calculations in four-dimensions and D-dimensions, Comput.Phys.Commun. 118, 153 (1999)

[50] H. Pagels and S. Stokar, Pion decay constant, electromagnetic form factor and quark electromagnetic self-energy in QCD, Phys. Rev. D20, 2947 (1979)

[51] C. D. Roberts, R. T. Cahill, M. E. Sevior, and N. Iannella, $\pi - \pi$ scattering in a QCD based model field theory, Phys. Rev. D49, 125 (1994)
kinematic and dynamical contributions, JHEP 01, 085 [arXiv:1111.6765 [hep-ph]]

[89] G. Huber et al., Measurement of the Charged Pion Form Factor to High $Q^2$, approved Jefferson Lab 12 GeV Experiment E12-06-101, 2006.

[90] C. J. Bebek et al., Electroproduction of single pions at low epsilon and a measurement of the pion form-factor up to $q^2 = 10$-GeV$^2$, Phys. Rev. D17, 1693 (1978)

[91] P. Jain and H. J. Munczek, Calculation of the pion decay constant in the framework of the Bethe-Salpeter equation, Phys.Rev. D44, 1873 (1991)

[92] H. J. Munczek and P. Jain, Relativistic pseudoscalar $q\bar{q}$ bound state: Results on Bethe-Salpeter wave functions and decay constants, Phys.Rev. D46, 438 (1992)

[93] P. Jain and H. J. Munczek, $q\bar{q}$ bound states in the Bethe-Salpeter formalism, Phys. Rev. D 48, 5403 (1993)

[94] P. Maris and C. D. Roberts, $\pi$- and $K$ meson Bethe-Salpeter amplitudes, Phys. Rev. C56, 3369 (1997) arXiv:nucl-th/9708029 [nucl-th]

[95] P. Maris and P. C. Tandy, Bethe-Salpeter study of vector meson masses and decay constants, Phys. Rev. C60, 055214 (1999) arXiv:nucl-th/9905056 [nucl-th]

[96] P. Maris and C. D. Roberts, Dyson-Schwinger equations: A tool for hadron physics, Int. J. Mod. Phys. E12, 297 (2003) nucl-th/0301049

[97] D. Binosi and L. Theussl, JaxoDraw: A Graphical user interface for drawing Feynman diagrams, Comput. Phys. Commun. 161, 76 (2004) arXiv:hep-ph/0309015 [hep-ph]

[98] J. A. M. Vermaseren, Axodraw, Comput. Phys. Commun. 83, 45 (1994)