ON THE GONALITY, TREEWIDTH, AND ORIENTABLE GENUS OF A GRAPH

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Abstract. We examine connections between the gonality, treewidth, and orientable genus of a graph. Especially, we find that hyperelliptic graphs in the sense of Baker and Norine are planar. We give a notion of a bielliptic graph and show that each of these must embed into a closed orientable surface of genus one. We also find, for all \( g \geq 0 \), trigonal graphs of treewidth 3 and orientable genus \( g \), and give analogues for graphs of higher gonality.

The gonality of a graph can refer to many related notions inspired by the Brill-Noether theory of an algebraic curve. Baker and Norine were the first to define it as the least degree of a non-constant harmonic morphism of graphs \( G \rightarrow T \) where \( G \) is the graph of interest and \( T \) is a tree. Compare this to the definition of the gonality of an algebraic curve \( C \) being the least degree of a nonconstant morphism from \( C \) to \( \mathbb{P}^1 \). Several other notions of gonality have been defined by other authors, including Caporaso [3] and Cornelissen-Kato-Kool [4]. The last notion, stable gonality, is notable because it allows refinements formed by subdividing edges and adding leaves. This does not change the orientable genus of \( G \), or the least genus of a closed orientable surface into which \( G \) embeds. This stable gonality is also notable as it admits a spectral lower bound, i.e., in terms of the spectrum of the Laplacian of \( G \). This is particularly appealing because there is a certain type of graph which arises from algebraic curves called Shimura curves, and calculations suggest that only finitely many are planar, while nearly every other invariant of these graphs is spectral. Could it be that there is a connection between stable gonality and orientable genus? In the following we say that a graph is \( d \)-gonal if its stable gonality is \( d \). In the case \( d = 2 \), this ends up being equivalent to the notion of a hyperelliptic graph due to Baker and Norine when the (Euler) genus of \( G \) is at least 2 [1, §5].

**Theorem 1.** All hyperelliptic graphs are planar, and if \( d \geq 3 \) with \( d \not\equiv 2 \mod 4 \) then there exist 3-connected \( d \)-gonal graphs of all orientable genera at least \( (d/2 - 1)^2 \).

To make the above relation clear, recall that for a graph to be planar it is equivalent to having orientable genus zero. Similarly, we say a graph is toroidal if its orientable genus is at most 1. This is not the end of the story on the connection between gonality and orientable genus however, as there is much more from the Brill-Noether theory of curves to be adapted to the language of graphs. Consider for instance that an algebraic curve is called bielliptic if it admits a degree 2 morphism to an algebraic curve of genus one. Similarly, we let a graph \( G \) be bielliptic if it admits a degree 2 harmonic morphism to a graph \( G' \) of (Euler) genus one. We have the following.

**Theorem 2.** All bielliptic graphs are toroidal.

Since the utility graph \( K_{3,3} \) is bielliptic, this is the best that could be hoped for. In fact, we are led to the following question.

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Question 1. If $G$ is a graph which admits a degree 2 harmonic morphism to a graph $G'$ of (Euler) genus $g$, is the orientable genus of $G$ at most $g$?

An affirmative answer to this question would not be totally optimal - e.g., $K_5$ admits a degree 2 morphism to a genus 2 graph, but is toroidal. We know of no counterexamples to this statement and the proof of Theorem 2 suggests extensions but does not itself extend beyond the genus one case.

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1. Preliminaries on the involutions of graphs and hyperelliptic graphs

Although the notion of a hyperelliptic graph is well-established, we prefer to use the equivalent definition furnished by the hyperelliptic involution \cite{1}.

Definition. A \textit{mixing involution} on a graph $G$ is an order-two automorphism $\alpha : G \to G$ such that if $e$ is an edge between $x$ and $y$ fixed by $\alpha$ then $\alpha(x) = y$.

Note that a graph with a mixing involution $\alpha$ and without loops cannot have any edges $e$ fixed by $\alpha$ between $\alpha$-fixed vertices $x$ and $y$. Note that if $G$ is a graph with loops, then the graph $G'$ obtained by deleting those loops has the same orientable genus. We therefore make our first reduction.

Reduction 1. Hereon, all graphs will be loopless.

We are now in the proper setting to consider harmonic morphisms of graphs \cite{1} §2.1, an example of which is given by the quotient of a graph $G$ by a mixing involution $\alpha$. Notably, the quotient $G/\alpha$ has vertices of the form $\{v, \alpha(v)\}$ such that $v$ is a vertex of $G$ and edges of the form $\{e, \alpha(e)\}$ such that the bounding vertices of $e$ are inequivalent under $\alpha$.

One defines a map $G \to G/\alpha$ by sending vertices to the obvious place, edges to the obvious place provided that their bounding vertices are non-equivalent under $\alpha$. If $e$ is an edge of the form $e(v, \alpha(v))$ then of course we must send $e$ to the quotient vertex $\{v, \alpha(v)\}$.

In the terminology of Baker-Norine, if $G$ has at least 3 vertices, this map is a harmonic morphism of degree 2. All such morphisms on graphs with at least 3 vertices arise this way \cite{1} Lemma 5.6. If $G$ has two vertices, then there is an obvious mixing involution and the quotient is a point, and thus a tree, and it is only because that map is constant that we do not say it has degree 2.

Definition. We say that a connected graph $G$ admitting a mixing involution $\iota : G \to G$ such that $G/\iota$ is a tree is \textit{hyperelliptic} and that $\iota$ is the corresponding \textit{hyperelliptic involution}.

This is a slightly nonstandard definition in that we don’t require the genus to be at least 2. Typically one stipulates that because when $G$ is 2-edge-connected and has genus $\geq 2$, such an involution must be unique \cite{1} Corollary 5.15. Thankfully we can reduce to the 2-edge-connected case without pain by contracting all its bridges \cite{1} Corollary 5.11. There are no 2-edge connected trees, and the only 2-edge connected genus one graphs are cycles, which are planar.

Reduction 2. Hereon, when we refer to the graph $G$, we will mean it to be 2-edge connected.
Note also that a graph with all its bridges contracted has the same orientable genus as the original graph. Of course we will allow other graphs to not be 2-edge connected. Indeed $G/\mathcal{I}$ will often be a tree in what follows. Before proceeding further, we review some examples.

2. Hyperelliptic graphs associated to Shimura curves

The literature on Shimura curves which is relevant to the task at hand is simply too large and too technical to introduce here in a meaningful way. Let it suffice to say that Ogg has determined all Shimura curves $X^D$ which are hyperelliptic over $\mathbb{Q}$ [9]. In particular, note that in each case $D$ is the product of two primes and so there are only two primes of bad reduction to explore. In each case, the dual graph is also hyperelliptic. The following code verifies that all of these dual graphs are planar.

```magma
Dlist := [26,35,38,39,51,55,57,58,62,69,74,82,86,87,93,94,95,111,119,134,146,159,194,206]; // Ogg’s list of Shimura curves hyperelliptic over QQbar del := function(x)
if x eq 0 then return 0;
else return 1;
end if;
end function;
ReducedDualGraph := function(p,q)
// Returns in magma format the dual graph of X^{pq} over FFpbar // Rather, the "reduced dual graph" with parallel edges collapsed
M := BrandtModule(q,1);
d := Dimension(M);
Mx := MatrixRing(Integers(),d);
Bx := Mx!HeckeOperator(M,p);
for i in [1..d] do for j in [1..d] do
Bx[i,j] := del(Bx[i,j]);
end for; end for;
return Graph<2*Dimension(M)|BlockMatrix(2,2,
[[Mx!0,Bx],[Bx,Mx!0]])>
end function;
for D in Dlist do
G1 := ReducedDualGraph(PrimeDivisors(D)[1],PrimeDivisors(D)[2]);
G2 := ReducedDualGraph(PrimeDivisors(D)[2],PrimeDivisors(D)[1]);
D,IsPlanar(G1),IsPlanar(G2);
end for;
```

Similar lists exist for, e.g., bielliptic Shimura curves, each of which has $D \leq 546$. Similar code to the above suggests that if $X^D$ has a dual graph (of its reduction modulo $p$ for $p \mid D$) which is planar and has at least six vertices, then for $D \geq 500$ the complete list of $(D,p)$ is
3. Planarity and Toroidality of Graphs with Involutions

Suppose now $G$ is a graph which is 2-edge-connected, loopless, and has a hyperelliptic involution $\iota$. A given vertex can be either fixed or moved by $\iota$. We let $F$ denote the set of vertices which are fixed by $\iota$. By definition, all other vertices are permuted, and there must be an even number of these. Let $A$ and $B$ be disjoint sets of permuted vertices: we let $a_1, \ldots, a_n$ be the elements of $A$, so $B = \{b_1 = \iota(a_1), \ldots, b_n = \iota(a_n)\}$.

The edges of $G$ must therefore fall into one of the following categories.

- The set $E_A$ of edges from $A$ to itself.
- The set $E_B = \iota(E_A)$ of edges from $B$ to itself.
- The set $E_F$ of edges from $F$ to itself.
- The “horizontal edges” $H$ from some $a_i$ to $b_i$.
- The “cross edges” $C$ from some $a_i$ to some $b_j$ such that $i \neq j$.
- The “transfer edges” $T_A$ and $T_B$ respectively from $F$ to $A$ and $F$ to $B$. Note that $T_B = \iota(T_A)$.

We note some properties of subgraphs of $G$.

**Lemma 1.** The involution $\iota$ maps the subgraph $(A, E_A)$ isomorphically onto $(B, E_B)$ and both are a finite disjoint union of trees.

*Proof.* The isomorphism between the two is simply given by restricting $\iota$ to $(A, E_A)$. We must therefore have an isomorphic copy of $(A, E_A)$ in the quotient $G/\iota$, which is a finite connected tree. Any subgraph of a tree must be a disjoint union of trees and so the result follows. □

**Lemma 2.** The connected components of the subgraph $(F, E_F)$ are either single vertices or chains of vertices $f_1, \ldots, f_r$ such that between $f_i$ and $f_{i+1}$ there are exactly two edges and between $f_i$ and $f_j$ there are no edges if $|i - j| > 1$.

*Proof.* Let $e \in E_F$ and let $f, f'$ be the bounding vertices of $e$. Since $\iota$ fixes $f, f'$ and $\iota$ is mixing, we must have $\iota(e) \neq e$. Therefore there are at least 2 edges between $f$ and $f'$. If we suppose to the contrary that there was a third edge $e'$ then $\iota(e')$ would be distinct from $e'$ again by the mixing property. But also since $e' \neq e$ and $e' \neq \iota(e)$ we must also have $\iota(e') \neq e$ and $e' \neq \iota(e)$. The quotient graph $G/\iota$ would then have a cycle $ee'$ and since the hyperelliptic involution is unique we have a contradiction.

Therefore between any two vertices $f, f'$ in our subgraph $(F, E_F)$ there are either zero or two edges. If $f, f', f''$ each have two edges between them, then in the quotient, we would have a cycle $e(f, f')e(f', f'')e(f'', f)$. The result follows. □
We see therefore that \((F, E_F)\) is planar, and although a given connected component may have a cycle, and for the purpose of orientable genus we may think of each one as a point. We can therefore make the following reduction by replacing \(F\) with the set of connected components of \(F\) and \(E_F\) by the empty set.

\[
\begin{align*}
\bullet f_1 & \quad \cdots \quad \bullet f_r \\
\end{align*}
\]

\textbf{Reduction 3.} We will assume \(E_F\) is empty.

In the same way, we can replace \((A, E_A)\) and \((B, E_B)\) by the connected components of each.

\textbf{Reduction 4.} We will assume \(E_A\) and \(E_B\) are empty.

Note that if we were to refine \(G\) by adding a point in the middle of each horizontal edge we would obtain a new graph. Embedding this refined graph into an orientable surface of genus \(g\) induces an embedding of \(G\) into the same surface. We therefore refine \(G\) by adding a new element each of \(F\), \(T_A\) and \(T_B\) as we eliminate \(H\).

\textbf{Reduction 5.} We assume that \(G\) has no horizontal edges.

We are now ready to prove our main theorem on hyperelliptic graphs. If we can embed any connected graph \(G\) into the plane, then by adding a point at infinity, we give an embedding of this graph into the 2-sphere \(S^2\), and in fact that graph defines a CW-decomposition of \(S^2\). For instance, if \(G\) has genus \(g\) then this decomposition has \(V(G)\) vertices, \(E(G)\) edges, and \(g+1\) faces. By spherical inversion we can simply assume that any one pair \(\{a_j, b_j\}\) lies on the same face as \(\infty\), or that they lie on the “outside face.” We will freely perform this in the following.

\textbf{Theorem 3.} All hyperelliptic graphs are planar. Moreover there is an embedding \(\rho_G\) into \(\mathbb{R}^2\) under which any pair \(\{a_j, b_j\}\) exchanged by the hyperelliptic involution \(\iota\) lie on a common face.

\textbf{Proof.} We induct on the size of \(#A = #B\). The following will be our inductive assumption.

- \textbf{Ind}(n): All connected hyperelliptic graphs with \(#A = #B \leq n\) admit a piecewise smooth (considering \(G\) e.g., as a simplicial complex) embedding \(\rho_G : G \to \mathbb{R}^2\) such that

  1. If \(\rho(v) = (x, y)\) then \(\rho(\iota(v)) = (-x, y)\) and
  2. If \(\{a_i, b_i\}\) are exchanged by \(\iota\) then there is a face \(F\) of the CW decomposition of \(S^2\) induced by \(\rho_G\) such that \(a_i, b_i \in \partial F\).

Clearly \textbf{Ind}(0) holds as we have shown that a hyperelliptic graph which fixes each vertex is planar. Almost-as-clearly, \textbf{Ind}(1) holds because there are no crossing edges, and so all edges are transfer edges by our reductive step. Since \(G\) is connected, between each fixed point \(f\) there is at least one transfer edge between \(f\) and \(a_1\) as well as \(f\) and \(b_1\). There is also at most one such edge, because if there were two edges between \(f\) and \(a_1\) then there would
be a cycle in the quotient. It follows that after our reductions, \( G \) embeds into the plane as the banana graph with midpoints. Of course, both \( a_1 \) and \( b_1 \) lie on the outside face.

Now suppose that \( G \) has \( \#A = n \) and \( \text{Ind}(n - 1) \) is satsified. We let

- \( A(n - 1) = \{a_1, \ldots, a_{n-1}\} \) and \( B(n) = \iota(A(n - 1)) \)
- \( T_A(n-1) = \{ \text{edges from } F \text{ to } A(n-1) \} \) and \( T_B(n-1) = \iota(T_A(n-1)) \).
- \( C(n-1) = \{ \text{cross edges from } A(n-1) \text{ to } B(n-1) \} \).

We therefore let \( G(n-1) \) be the graph whose vertices are \( A(n - 1) \cup B(n - 1) \cup F \) and whose edges are \( T_A(n-1) \cup T_B(n-1) \cup C(n-1) \). As \( G(n - 1) / \iota \) is a subgraph of \( G / \iota \), it is a finite disjoint union of trees. Let \( \Gamma_1, \ldots, \Gamma_m \) be the horizontal connected components of \( G(n - 1) \), i.e. \( \Gamma_i \) is either connected or the union of two vertices exchanged by \( \iota \). All of the images of the \( \Gamma_i \) in the quotient are connected trees. We note that the connected \( \Gamma_i \) are hyperelliptic and so satisfy the conclusions of \( \text{Ind}(n - 1) \).

Since \( G \) is connected, for each \( i \) there is a pair of transfer edges or a pair of cross edges from \( \{a_n, b_n\} \) to \( \Gamma_i \). In fact, there can be either a cross edge \( c_i \) from \( a_n \) to some \( b_k \) in \( \Gamma_i \) or a transfer edge \( t_i \) from \( a_n \) to a fixed point \( f \) in \( \Gamma_i \) and not both. There cannot be more than one else there would be a cycle in the quotient.

We therefore create a function \( \psi_G : \{1, \ldots, m\} \rightarrow \{0, 1\} \) where \( \psi(i) = 0 \) if there is a transfer edge \( a_n \) to \( \Gamma_i \) and \( 1 \) in the case of a cross edge. We roughly create \( \rho_G \) as follows: \( \text{Ind}(n - 1) \) gives us an embedding of each \( \Gamma_i \) into \( \mathbb{R}^2 \), but moreover we can scale down into \( [-1, 1]^2 \) and still be symmetric under \( \iota \). We stack each copy of \( [-1, 1]^2 \) vertically in \( \mathbb{R}^2 \), put \( a_n \) to the left of this column, \( b_n \) to the right, and either directly attach the transfer edge if \( \psi_G(i) = 0 \) or possibly first apply \( \iota \) to \( \Gamma_i \) before attaching the cross edge if \( \psi_G(i) = 1 \). Hidden in this is that if \( \psi_G(i) = 0 \) we need to make sure to perform spherical inversion to make sure that the fixed point \( f \) is on the outside face, and if \( \psi_G(i) = 1 \) we need to make sure that both \( a_k \) and \( b_k \) are on the outside face. This latter part explains the second condition of \( \text{Ind}(n) \) and the remainder of the proof is simply verifying the conditions of \( \text{Ind}(n) \) and making the construction explicit.

As noted, if \( \psi_G(i) = 0 \) then we may assume that our \( \rho_{\Gamma_i} \) has \( f \) on the outside face. By scaling and shifting up or down we may assume that \( \rho_{\Gamma_i} \) has image in the interior of \( [-1, 1]^2 \) which is symmetric about the \( y \)-axis and \( \rho_{\Gamma_i}(f) = (0, 0) \). We may therefore draw a symmetric pair of edges between \( (0, 0) \) and \( (\pm 1, 0) \) which do not intersect \( \Gamma_i \). Note that these two new edges split the outside face of \( [-1, 1]^2 \) into two, but that \( \Gamma_i \) lies entirely on one side of that divide, so adding these edges does not change whether \( \text{Ind}(n) \) is satisfied. If \( \psi_G \) is identically zero, we embed a refinement of \( G \) into \( \mathbb{R}^2 \) as follows: send \( a_n, b_n \) to \( (\pm 1, 0) \), use \( \rho_{\Gamma_i} \) to send \( \Gamma_i \) to \( \{(x, y) : -1 \leq x \leq 1, i - 1 \leq y \leq i + 1\} \). We can symmetrically draw edges between \( (\pm 1, 0) \) and \( (\pm 1, i) \) which are pairwise disjoint and this produces an embedding \( \rho_G \) which is symmetric under \( \iota \) and preserves the face condition of our inductive assumption for \( G \).

Now let’s assume there are some \( i \) such that \( \psi_G(i) = 1 \). We assume that \( \Gamma_i \) is connected, else it is the disjoint union of two vertices, and adding some cross edges does not change the face condition of \( \text{Ind}(n) \). Let \( \rho_i = \rho_{\Gamma_i} \) be an embedding so that \( \rho_i(a_k), \rho_i(b_k) \) lie on the outside face with respectively positive and negative \( x \)-values, and let \( d_i, d_i' \) respectively be paths \( (-1, 0) \) to \( \rho_i(b_k) \) and \( (1, 0) \) to \( \rho_i(a_k) \) such that \( d_i' = \iota(d_i) \). Could it be that \( d_i, d_i' \) put \( a_j \) and \( b_j \) on different faces?

- By \( \text{Ind}(n - 1) \), \( \rho_i(a_j) \) and \( \rho_i(b_j) \) share a face, and we need only worry if it is the outside face.
- If \( a_j = a_k \) then \( a_j \) still lies on the same face as \( b_j \) even after adding \( d_i \) and \( d_i' \).
So we assume \( a_j \) and \( b_j \) lie on the outside face, let \( \gamma_j^+ \) and \( \gamma_j^- \) be smooth symmetric paths from \( \rho_i(a_j) \) to \( \rho_i(b_j) \) which lie above and below \( \rho_i(\Gamma_i) \), meeting only at \( \rho_i(a_j) \) and \( \rho_i(b_j) \). As such, \( \gamma_j^+ \cup \gamma_j^- \) forms a simple Jordan curve, which has an inside and outside defined by the mod 2 intersection number \([\mathbf{2}, \S 3.3]\). Since \( a_j \neq a_k \), the path \( d_i \) has an odd number of transverse intersection points with \( \gamma_j^+ \cup \gamma_j^- \) up to multiplicity. If there is just one, we are done, as it has to lie on precisely one of \( \gamma_j^+ \) and \( \gamma_j^- \). The non-intersecting path lies within the face we desire. If there are three or more, we may pick an \( \varepsilon > 0 \) less than the distance from \( \rho_i(\Gamma_i) \) to any of the points of \( d_i \cap (\gamma_j^+ \cup \gamma_j^-) \). There is thus a smooth path between \( \rho_i(b_j) \) and \((-1,0)\) which agrees with \( d_i \) at distance less than \( \varepsilon \) from \( \rho_i(\Gamma_i) \), which is homotopic to \( d_i \), and which has precisely one point of intersection with \( \gamma_j^+ \cup \gamma_j^- \). By replacing \( d_i \) with this path and \( d_i' \) by the image under \( i \) we have reduced to the previous case. We conclude that \( \text{Ind}(n) \) holds and the proof of our Theorem is complete. \( \square \)

For good measure, we give a second proof of the planarity of hyperelliptic graphs.

\textbf{Proof.} By work of de Bruyn and Gijswijt \([\mathbf{5}]\), we know that for all graphs \( G \), the stable gonality of \( G \) is bounded below by the treewidth of \( G \). We know that \( G \) is hyperelliptic if and only if the stable gonality is 2. Since \( G \) is hyperelliptic, we find that it has treewidth 2, and therefore is a subgraph of a series-parallel graph \([\mathbf{2}]\), and is therefore planar. \( \square \)

There is also a third proof of this result due to Spencer Backman which characterizes the ear decomposition of a hyperelliptic graph and which predates work of de Bruyn and Gijswijt but was not written up. While it may not seem so, these proofs work out to being very similar. Since \( G \) is hyperelliptic, \( G/i \) is a tree. We may think of the inductive proof as rooting that tree and thus producing an embedding into a series-parallel graph. Note that our embedding \( \rho_G \) gives \( G/i \) as \( \rho_G(G) \cap \{ (x,y) : x \leq 0 \} \), so the source and sink vertices are respectively \( a_n \) and \( b_n \). The advantage of working so explicitly is that some natural improvements present themselves.

\textbf{Lemma 3.} On any hyperelliptic graph \( G \) with two pairs of vertices \( a_i \neq b_i \) and \( a_j \neq b_j \) exchanged by the hyperelliptic involution, we can find an embedding \( \rho_{i,j} \) of \( G \) into \( \mathbb{R}^2 \) such that \( a_i, b_i, a_j, b_j \) all lie on the boundary of a face. Moreover, the same is true when \( a_i \) and \( b_i \) are replaced by a hyperelliptic fixed vertex.

\textbf{Proof.} We proceed by induction in the same way as in the proof of Theorem \([\mathbf{3}]\). In fact, if \( a_i = a_j \) then our Lemma holds by appealing to Theorem \([\mathbf{3}]\). Therefore we suppose that \( a_i \neq a_j \) and thus \( b_i \neq b_j \). We make all necessary reductions to retain the notation of \( V(G) = A \cup B \cup F \) and \( E(G) = C \cup T \). We know therefore that \( \#A = \#B \geq 2 \). In the case of equality, \( G \) is outerplanar. If we do not have equality, we reorder \( A \) and \( B \) so that \( j = n \) and let \( \Gamma_1, \ldots, \Gamma_m \) be the horizontal connected components of \( G(n-1) \) as in the proof of the Theorem.

Let \( r \) be such that \( a_i \in \Gamma_r \) and let \( k \) be such that there is a cross edge from \( a_n \) to \( b_k \). We apply our inductive hypothesis to \( \Gamma_r \) to find an embedding of \( \Gamma_r \) into \( \mathbb{R}^2 \) such that \( a_i \) and \( a_k \) share a face. We use spherical inversion to move that face to the outside, and thereby give an embedding of \( G \) into \( \mathbb{R}^2 \) such that \( a_i \) and \( a_n = a_j \) share a face.

If \( a_i \) and \( b_i \) are replaced by a fixed vertex \( f \), then we let \( r \) be such that \( f \in \Gamma_r \) and we use spherical inversion to find a planar embedding of \( \Gamma_r \) such that \( f \) lies on the outside face. The result follows in the same way. \( \square \)
With the above in mind, we recall that a bielliptic graph is one which admits a mixing involution $\alpha$ such that $G/\alpha$ has genus one. We therefore have the following.

**Theorem 4.** Bielliptic graphs are toroidal.

*Proof.* Without loss of generality, we assume $G$ is 2-edge connected, and that the genus of $G$ is at least 3, else $G$ is already planar.

Since $G/\alpha$ has genus one, there is an edge $\bar{e}$ of $G/\alpha$ such that $G/\alpha - \bar{e}$ is a tree. Let $e, e' = \alpha(\bar{e})$ be the preimages of $\bar{e}$ in $G$ and let $G_0 = G - \{e, e'\}$ with $\alpha_0$ the induced involution, whose quotient is $G/\alpha - \bar{e}$.

$$
\begin{align*}
G_0 & \quad \rightarrow \quad G \\
\downarrow \quad & \quad \downarrow \\
G_0/\alpha_0 & \quad \rightarrow \quad G/\alpha
\end{align*}
$$

We show first that $G_0$ is connected: if not, let $a, b$ be the endpoints of $e$ and $G_a, G_b$ the connected components of each in $G_0$. In which of these can we find $\alpha_0(a)$ and $\alpha_0(b)$? If there is a path $\gamma_a$ between $a$ and $\alpha_0(a)$ then $G_0$ is connected, as there is a unique simple path in $G_0/\alpha_0$ between $\alpha_0^{-1}(v_1)$ and $\alpha_0^{-1}(v_2)$ for any $v_1 \in G_a$ and $v_2 \in G_b$. This path lifts to a path $\gamma$ between either $v_1$ and $v_2$ (in which case $G_0$ is connected) or $v_1$ and $\alpha_0(v_2)$ (in which case $\gamma_a\alpha_0(\gamma)$ is a path between $v_1$ and $v_2$). Thus there is no such path $\gamma_a$ when $G_0$ is disconnected. In other words, when $G_0$ is disconnected, $\alpha_0(a) \notin G_a$. Since $\alpha_0$ is an isomorphism, it must exchange $G_a$ with $G_b$ so that $\alpha_0 : G_a \to G_b$. But then the quotient is a tree, so $G_a$ and $G_b$ are trees. This however contradicts the statement that the genus of $G$ is at least 3.

It follows then that $G_0$ is hyperelliptic, and therefore planar. Moreover the embedding is planar in such a way as to recognize $\alpha_0$ as reflection about the $y$-axis. Let $a, b$ be the endpoints of $e$ and $a', b'$ be the endpoints of $e'$, so moreover we can find a planar embedding of $G_0$ such that $a, a', b, b'$ all lie on the outside face. The boundary of this outside face is a Jordan curve containing $a, a', b, b'$ which is broken up into four paths between the four of these points. If one of these is a path $\delta$ between $a$ and $b$, then another must be a path $\delta'$ between $a'$ and $b'$. In this case, $G$ itself is planar. If not, there are paths from $a$ to $a'$ and $b'$ in the boundary, and we can therefore flip $\rho_{G_0}$ along the $x$ and $y$ axes so that $a$ lands in $\{(x, y) : x > 0, y > 0\}$ and thus $a'$ lands in $\{(x, y) : x < 0, y > 0\}$. By scaling, we may assume the image of $\rho_{G_0}$ lies in $[-1, 1]^2$. We may then draw edges between $\rho_{G_0}(a)$ and $(0, 1)$, $\rho_{G_0}(a')$ and $(-1, 0)$, $\rho_{G_0}(b)$ and $(0, -1)$, as well as $\rho_{G_0}(b')$ and $(1, 0)$, none of which intersect each other or any other point of $\rho_{G_0}(G_0)$.

These edges induce an embedding of $G$ into $\mathbb{R}^2/2\mathbb{Z}^2$ by identifying opposite edges of $[-1, 1]^2$. We therefore have shown that $G$ is toroidal in all cases. \( \square \)

One could imagine extending this to the case where $G/\alpha$ has genus $g$, but that would depend on finding a sequence of points interchanged by $\alpha$ which sequentially lie on common faces. This fails however, as we see in the following example where $G/\alpha$ has genus 2.
Nonetheless we note that this graph does indeed admit an embedding into a genus two surface! In particular, we get slightly lucky in that the above method shows how to embed this graph into the connected sum of two tori, albeit in a way that does not obviously generalize. Indeed, we do not know of an example of a graph with a mixing involution \( \alpha \) to a genus \( g \) graph which does not already embed into a genus \( g \) orientable surface. Sometimes as well, this construction is not optimal because different lifts of an edge need not cross: \( K_5 \) admits an essentially unique involution whose quotient has genus 2, but is well-known to be toroidal.

We conclude by noting that although our criterion for being toroidal has something to do with gonality, there is more that goes into the orientable genus than the gonality.

**Lemma 4.** There are trigonal graphs of all possible orientable genera. Moreover, there are \( d \)-gonal graphs which are either planar or of all possible orientable genera \( \geq \left( \frac{d}{2} - 1 \right)^2 \) whenever \( d \neq 2 \mod 4 \).

**Proof.** First we note that there are \( d \)-gonal planar graphs for all \( d \) - simply take \( n \geq d \) and note that the \( d \times n \) grid graph has gonality \( d \) [5, Example 3.3].

Then note that for \( 3 \leq d \leq n \), the complete bipartite graph has orientable genus \( \left\lceil \frac{(d-2)(n-2)}{4} \right\rceil \). If \( d \) is not \( 2 \mod 4 \) then this can be any integral value at least \( \left( \frac{d}{2} - 1 \right)^2 \).

On one hand, there is a clear degree \( d \) harmonic map from \( K_{d,n} \) to a tree given by simply identifying the vertices in the size \( d \) subset. Therefore the gonality of \( K_{d,n} \) is at most \( d \). On the other hand, the treewidth of \( K_{d,n} \) is \( d \), so this is a lower bound for gonality [5], and we find that \( K_{d,n} \) is \( d \)-gonal. \( \square \)

The use of the complete bipartite graph above was suggested by Spencer Backman and we thank him for the suggestion. We conclude by noting that in the above examples, gonality, stable gonality, and treewidth all coincide. It is conjectured for the hypercube graph \( Q_n \) that there is a gap between the two which increases along with \( n \) [5, §3]. In that case, the orientable genus is large and the conjectural least degree map to a tree is given by successive quotients by involutions \( Q_n \rightarrow Q_{n-1} \). It would be interesting to find other infinite families of
graphs with gaps between gonality and treewidth and see if those also have large orientable genus. It also still seems reasonable to wonder about a connection between the orientable genus of a graph and the spectrum of its Laplacian. After all, the spectrum of the $d \times n$ grid graph is very limited \[6\]: the eigenvalues can only be

$$\lambda_{j,k} = 4 \sin^2 \left( \frac{j \pi}{2n} \right) + 4 \sin^2 \left( \frac{k \pi}{2d} \right).$$

In particular, the spectral lower bound on gonality \[4\, \text{Theorem C}\] for this example tends to 0 as $n \to \infty$.

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