b– Bistochastic Quadratic Stochastic Operators and Their Properties

Farrukh Mukhamedov\textsuperscript{1}, Ahmad Fadillah Embong\textsuperscript{2}

\textsuperscript{1} Department of Computational & Theoretical Sciences, Faculty of Science, International Islamic University Malaysia, P.O. Box, 25200, Kuantan, Pahang, Malaysia
\textsuperscript{2} Department of Computational & Theoretical Sciences, Faculty of Science, International Islamic University Malaysia, P.O. Box, 25200, Kuantan, Pahang, Malaysia
E-mail: farrukh_md@iium.edu.my, ahmadfadillah.90@gmail.com

Abstract. In the present paper, we consider a class of quadratic stochastic operators (q.s.o.) called b-bistochastic q.s.o. We study several descriptive properties of b-bistochastic q.s.o. It turns out that, upper triangular stochastic matrix defines a linear b-stochastic operator. This allowed us to find some sufficient conditions on cubic stochastic matrix to be a b-bistochastic q.s.o.

1. Introduction
The history of quadratic stochastic operators (q.s.o.) can be traced back to Bernstein’s work \cite{1} where such kind of operators appeared from the problems of population genetics (see also \cite{7}). Such kind of operators describe time evolution of variety species in biology are represented by so-called Lotka-Volterra(LV) systems \cite{20}. Nowadays, scientists are interested in these operators, since they have a lot of applications especially in modelings in many different fields such as biology \cite{5, 15}, physics \cite{17, 19}, economics and mathematics \cite{7, 15, 19}.

Let us recall how q.s.o. appears in biology. The time evolution of species in biology can be comprehended by the following situation. Let \(I = \{1, 2, \ldots, n\}\) be the \(n\) type of species (or traits) in a population and we denote \(x^{(0)} = (x_1^{(0)}, \ldots, x_n^{(0)})\) to be the probability distribution of the species in an early state of that population. By \(P_{ij,k}\) we denote the probability of an individual in the \(i^{th}\) species and \(j^{th}\) species to cross-fertilize and produce an individual from \(k^{th}\) species (trait). Given \(x^{(0)} = (x_1^{(0)}, \ldots, x_n^{(0)})\), we can find the probability distribution of the first generation, \(x^{(1)} = (x_1^{(1)}, \ldots, x_n^{(1)})\) by using a total probability, i.e.

\[
x_k^{(1)} = \sum_{i,j=1}^{n} P_{ij,k} x_i^{(0)} x_j^{(0)}, \quad k \in \{1, \ldots, n\}.
\]

This relation defines an operator which is denoted by \(V\) and it is called quadratic stochastic operator (q.s.o.). In other words, each q.s.o. describes the sequence of generations in terms of probabilities distribution, if the values of \(P_{ij,k}\) and the distribution of the current generation are given. The main problem in the nonlinear operator theory is to study the behavior of nonlinear operators. Presently, there is only a small number of studies on dynamical phenomena on higher
dimensional systems, even though they are very important (see for example, [6, 11, 12, 13, 18]). In case of q.s.o., the difficulty of the problem depends on the given cubic matrix \((P_{ijk})^{m}_{i,j,k=1}\). In [3, 10], it has given along self-contained exposition of the recent achievements and open problems in the theory of the q.s.o.

In [16] a new majorization was introduced, and it opened a path for the study to generalize the theory of majorization by Hardy, Littlewood and Polya [4]. The new majorization has an advantage as compared to the classical one, since it can be defined as a partial order on sequences. While the classical one is not an antisymmetric order (because any sequence is majorized by any of its permutations), it is only defined as a preorder on sequence [16]. Most of the works in the mentioned paper were devoted to the investigation of majorized linear operators (see [4, 16]). Therefore, it is natural to study nonlinear majorized operators.

In what follows, to differentiate between the terms majorization and classical majorization that was popularized by Hardy et al.[4], we call majorization as \(\leq b\) order (which is denoted as \(\leq b\)) while classical majorization as majorization (which is denoted as \(\prec\)) only. In [2] it was introduced and studied q.s.o. with a property \(V(x) \prec x\) for all \(x \in S^{n-1}\). Such an operator is called bistochastic. In [9], it was proposed to a definition of bistochastic q.s.o. in terms of \(b\)-order. Namely, a q.s.o. is called \(b\)-bistochastic if \(V(x) \leq b x\) for all \(x\) taken from the \(n-1\)-dimensional simplex. In this paper we continue our previous investigations on \(b\)-bistochastic operators. We study several descriptive properties of \(b\)-bistochastic q.s.o. It turns out that, upper triangular stochastic matrix defines a linear \(b\)-stochastic operator. This allowed us to find some sufficient conditions on cubic stochastic matrix to be a \(b\)-bistochastic q.s.o.

2. Preliminaries
In this section we recall necessary definitions and facts about \(b\)-bistochastic operators.
Throughout this paper we consider the simplex

\[ S^{n-1} = \left\{ x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid x_i \geq 0, \sum_{i=1}^{n} x_i = 1 \right\}. \tag{2.1} \]

For each \(k \in \{1, \ldots, n-1\}\) we define functional \(V_k : \mathbb{R}^n \to \mathbb{R}\) by

\[ V_k(x_1, \ldots, x_n) = \sum_{i=1}^{k} x_i. \tag{2.2} \]

Let \(x, y \in S^{n-1}\). We say that \(x\) is \(b\)-ordered or \(b\)-majorized by \(y\) (\(x \leq b y\)) if and only if \(V_k(x) \leq V_k(y)\), for all \(k \in \{1, \ldots, n-1\}\).

The introduced relation is partial order i.e. it satisfies the following conditions:

(i) For any \(x \in S^{n-1}\) one has \(x \leq b x\),
(ii) If \(x \leq b y\) and \(y \leq b x\) then \(x = y\),
(iii) If \(x \leq b y\), and \(y \leq b z\) then \(x \leq b z\).

Using the defined order, one can define the classical majorization [8]. First, recall that for any \(x = (x_1, x_2, \ldots, x_n) \in S^{n-1}\), by \(x[i] = (x[1], x[2], \ldots, x[n])\) one denotes the rearrangement of \(x\) in non-increasing order, i.e. \(x[1] \geq x[2] \geq \cdots \geq x[n]\). Take \(x, y \in S^{n-1}\), then it is said that an element \(x\) is majorized by \(y\) and denoted \(x \prec y\) if \(x [i] \leq y [i]\). We refer the reader to [8] for more information regarding to this topic.

Recall that any operator \(V\) with \(V(S^{n-1}) \subseteq S^{n-1}\) is called stochastic.

**Definition 2.1.** A stochastic operator \(V\) is called \(b\)-bistochastic if one has \(V(x) \leq b x\) for all \(x \in S^{n-1}\).
Note that, the simplest nonlinear operators are quadratic ones. Therefore, we restrict ourselves to such kind of operators. Namely, a stochastic operator $V : S^{n-1} \rightarrow S^{n-1}$ is called \textit{quadratic stochastic operator (q.s.o.)} if $V$ has the following form:

$$V(x)_k = \sum_{i,j=1}^{n} P_{ij,k} x_i x_j, \quad k = 1, 2, \ldots, n, \quad x = (x_1, x_2, \ldots, x_n) \in S^{n-1},$$

where $\{P_{ij,k}\}$ are the heredity coefficients with the following properties:

$$P_{ij,k} \geq 0, \quad P_{ij,k} = P_{ji,k}, \quad \sum_{k=1}^{n} P_{ij,k} = 1, \quad i, j, k = 1, 2, \ldots, n .$$

\textbf{Remark 2.2.} If a q.s.o. $V$ satisfies $V(x) \prec x$ for all $x \in S^{n-1}$, then it is called bistochastic [2]. In our definition, we are taking $b$-order instead of the majorization. Note that if one takes absolute continuity instead of the $b$-order, then $b$-bistochastic operator reduces to Volterra q.s.o. [14].

Let $V$ be a q.s.o., then one can define an associated matrix $T_n(x) = [T_{ik}(x)]_{i,k=1}^{n}, x \in S^{n-1}$ by

$$T_{ik}(x) = \sum_{j=1}^{n} P_{ij,k} x_j, \quad x = (x_i) \in S^{n-1},$$

where $\{P_{ij,k}\}$ are the heredity coefficients. One can see that $T_n(x)$ is a stochastic matrix. Moreover, one has $V(x) = x T_n(x)$, and $T_n(x) = \sum_{l=1}^{n} x_l T_n(e_l)$. Hence, each q.s.o. $V$ can be uniquely defined by stochastic matrices, i.e.

$$V = \{T_n(e_1), T_n(e_2), \ldots, T_n(e_n)\} .$$

\section{Description of $b$-bistochastic q.s.o.}

In this section, we are going to provide some general properties of $b$-bistochastic q.s.o. In [9], we have proved the following fact.

\textbf{Theorem 3.1.} [9] Let $V$ be a $b$-bistochastic q.s.o. defined on $S^{n-1}$, then the following statements hold:

(i) $\sum_{m=1}^{k} \sum_{j=1}^{n} P_{ij,m} \leq kn, \quad k \in \{1, \ldots, n\}$

(ii) $P_{ij,k} = 0$ for all $i, j \in \{k+1, \ldots, n\}$ where $k \in \{1, \ldots, n-1\}$

(iii) $P_{nn,n} = 1$

(iv) for every $x \in S^{n-1}$ one has

$$V(x)_k = \sum_{l=1}^{k} P_{ll,k} x_l^2 + \sum_{l=2}^{k} \sum_{j=l+1}^{n} P_{lj,k} x_l x_j \quad \text{where} \quad k = 1, n-1$$

$$V(x)_n = x_n^2 + \sum_{l=1}^{n-1} P_{ll,n} x_l^2 + \sum_{l=1}^{n-1} \sum_{j=l+1}^{n} P_{lj,n} x_l x_j .$$

(v) $P_{ij,l} \leq \frac{1}{2}$ for all $j \geq l+1, \quad l \in \{1, \ldots, n-1\}$. 


Example 3.2. Let $V : S^3 \rightarrow S^3$ be a q.s.o. given by the following heredity coefficients

\[
\begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & P_{4,1,3} & 1 - P_{4,1,3}
\end{bmatrix}
\begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 1/2 & P_{3,2,3} & 1/2 - P_{3,2,3} \\
0 & 1/2 & P_{4,2,3} & 1/2 - P_{4,2,3}
\end{bmatrix}
\begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 1/2 & P_{3,2,3} & 1/2 - P_{3,2,3} \\
0 & 1/2 & P_{4,2,3} & 1/2 - P_{4,2,3}
\end{bmatrix}
\begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 1/2 & P_{3,2,3} & 1/2 - P_{3,2,3} \\
0 & 0 & P_{4,3,3} & 1 - P_{4,3,3} \\
0 & 0 & P_{4,3,3} & 1 - P_{4,3,3}
\end{bmatrix}
\]

where $2P_{4,1,3} + 2P_{3,2,3} + 2P_{4,2,3} + P_{3,3,3} + 2P_{4,3,3} \leq 4$. Note that $P_{ij,k} = P_{ji,k}$. One can see that such an operator is not a $b$–bistochastic q.s.o., since for $x = (0.1, 0.1, 0.8, 0)$ we have

\[
V(x)_1 + V(x)_2 = P_{1,1,1}x_1^2 + 2P_{1,2,1}x_1x_2 + 2P_{1,3,1}x_1x_3 + 2P_{4,1,1}x_1 (1 - x_1 - x_2 - x_3) + P_{1,1,2}x_2^2 + 2P_{1,2,2}x_1x_2 + 2P_{1,3,2}x_1x_3 + 2P_{4,1,2}x_1 (1 - x_1 - x_2 - x_3) + P_{2,2,2}x_2^2 + 2P_{2,3,2}x_2x_3 + 2P_{4,2,2}x_2 (1 - x_1 - x_2 - x_3) = 0 + 0.28 \geq 0.1 + 0.1.
\]

Moreover, under the condition $2P_{4,1,3} + 2P_{3,2,3} + 2P_{4,2,3} + P_{3,3,3} + 2P_{4,3,3} \leq 4$, the property (i) in Theorem 3.1 holds. The other properties also satisfy accordingly due to the construction of the q.s.o.

In what follow, we want to recall a fascinating result on $b$–bistochastic linear operators. Let $T$ be a linear stochastic operator $T : S^{n-1} \rightarrow S^{n-1}$ such that

\[
T(x)_k = \sum_{i=1}^{n} t_{ik}x_i \quad \text{where} \quad t_{ik} \geq 0, \quad \sum_{k=1}^{n} t_{ik} = 1, \quad x = (x_1, \ldots, x_n) \in S^{n-1}.
\]

In [16], it was showed the simplified form of linear $b$–bistochastic operators. Namely,

**Theorem 3.3.** [16] Let $T$ be a linear stochastic operator defined on $S^{n-1}$. Then $T$ is a $b$–bistochastic if and only if $T$ is an upper triangular stochastic matrix, i.e.

\[
T = \begin{bmatrix}
t_{11} & t_{12} & t_{13} & \cdots & t_{1n} \\
0 & t_{22} & t_{23} & \cdots & t_{2n} \\
0 & 0 & t_{33} & \cdots & t_{3n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1
\end{bmatrix}.
\]

From this theorem, it is interesting to know the following question: if we take $\{T_n(e_k)\}$ (in the representation (2.6)) all the stochastic matrices to be upper triangular, then would $V$ be a $b$-stochastic q.s.o.?

For each $j \in \{1, \ldots, n\}$ let $T_n(e_j)$ be a stochastic matrix given be

\[
T_n(e_j) = \left(e_{ik}^{(j)} \right)_{i,k=1}^{n} : j = 1, 2, \ldots, n.
\]
We define a quadratic operator $V : S^{n-1} \to S^{n-1}$ by

$$
V(x)_k = \sum_{i,j} a^{(j)}_{ik} x_i x_j; \quad k = 1, n, \quad x = (x_1, x_2, \ldots, x_n) \in S^{n-1}. 
$$

(3.3)

Let us assume that, $T_n(e_j)$ is taking upper triangular stochastic form i.e.

$$
T_n(e_j) = \begin{bmatrix}
    a^{(j)}_{11} & a^{(j)}_{12} & a^{(j)}_{13} & \cdots & a^{(j)}_{1n} \\
    0 & a^{(j)}_{22} & a^{(j)}_{23} & \cdots & a^{(j)}_{2n} \\
    0 & 0 & a^{(j)}_{33} & \cdots & a^{(j)}_{3n} \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & 0 & \cdots & 1
\end{bmatrix}
$$

(3.4)

for every $j = 1, 2, \ldots, n$.

**Theorem 3.4.** Let $\{T_n(e_j)\}, \ j = 1, 2, \ldots, n$ be a collection of upper triangular stochastic matrices and $V$ be the associated q.s.o. defined by (3.3). Then $V$ is a b-bistochastic operator.

**Proof.** Let $x = (x_1, x_2, \ldots, x_n)$. Using the fact

$$
V(x)_k = \sum_{i,j} a^{(j)}_{ik} x_i x_j = \sum_{i=1}^{k} \sum_{j=1}^{n} a^{(j)}_{ik} x_i x_j
$$

one gets

$$
\sum_{l=1}^{k} V(x)_l = \sum_{j=1}^{n} a^{(j)}_{11} x_1 x_j + \sum_{i=1}^{2} \sum_{j=1}^{n} a^{(j)}_{i2} x_i x_j + \cdots + \sum_{i=1}^{k} \sum_{j=1}^{n} a^{(j)}_{ik} x_i x_j
$$

$$
= x_1 \sum_{j=1}^{n} a^{(j)}_{11} x_j + x_1 \sum_{j=1}^{n} a^{(j)}_{12} x_j + x_2 \sum_{j=1}^{n} a^{(j)}_{22} x_j + \cdots + x_k \sum_{j=1}^{n} a^{(j)}_{kk} x_j
$$

$$
= x_1 \left( \sum_{l=1}^{k} \sum_{j=1}^{n} a^{(j)}_{1l} x_j \right) + x_2 \left( \sum_{l=2}^{k} \sum_{j=1}^{n} a^{(j)}_{2l} x_j \right) + \cdots + x_k \sum_{j=1}^{n} a^{(j)}_{kk} x_j
$$

$$
= \sum_{i=1}^{k} x_i \left( \sum_{j=1}^{n} a^{(j)}_{il} x_j \right).
$$

We know that $\sum_{l=1}^{k} a^{(j)}_{il} \leq 1$ for each $i$, hence

$$
\sum_{l=1}^{k} V(x)_l \leq \sum_{i=1}^{k} x_i \left( \sum_{j=1}^{n} x_j \right) = \sum_{i=1}^{k} x_i
$$

This completes the prove.
Remark 3.5. It is worth to note, in general the necessary and sufficient conditions of $b$–bistochastic quadratic operators may not need to be upper triangular stochastic matrices. Let us choose the following quadratic operators (indeed it is a q.s.o.): 

$$
\left\{ \begin{array}{ccc}
0 & 0 & 1 \\
\frac{1}{2} & 0 & \frac{1}{2} \\
0 & 0 & 1 \\
\end{array} \right\}, \quad \left\{ \begin{array}{ccc}
0 & 0 & 1 \\
\frac{1}{2} & 0 & \frac{1}{2} \\
0 & 0 & 1 \\
\end{array} \right\}, \quad \left\{ \begin{array}{ccc}
0 & 0 & 1 \\
0 & 0 & 1 \\
0 & 0 & 1 \\
\end{array} \right\}.
$$

(3.5)

Let $x = (x_1, x_2, x_3) \in S^2$, then one has $V(x)_1 = x_1 x_2$ and $V(x)_2 = 0$. In addition, the statement $x_2 \leq 1$ and $x_1 x_2 \leq x_1$ is always true for any $x = (x_1, x_2, x_3) \in S^2$. Therefore, one finds that

$$
V(x)_1 = x_1 x_2 \leq x_1, \quad V(x)_1 + V(x)_2 = x_1 x_2 \leq x_1 + x_2.
$$

which shows q.s.o. given by (3.5) is a $b$–bistochastic q.s.o.

Furthermore, due to Theorem 3.1, we introduce a stochastic cubic matrix such that

$$
\begin{array}{cccc}
(a_{11}^{(1)} & a_{12}^{(1)} & \cdots & a_{1n}^{(1)} \\
a_{11}^{(2)} & a_{12}^{(2)} & \cdots & a_{1n}^{(2)} \\
\vdots & \vdots & \ddots & \vdots \\
a_{11}^{(n)} & a_{12}^{(n)} & \cdots & a_{1n}^{(n)}
\end{array}
$$

and let the corresponding collection $\{T_n(e_j)\}$ satisfy the constraints (3.6). If

$$
\sum_{i=1}^{n-1} (a_{i,j}^{(j)} + a_{j,i}^{(i)}) \leq 1 \text{ for every } j = i+1, n \text{ where } i = 1, n-1,
$$

then $V$ is a $b$-bistochastic operator.
Proof. Clearly from (3.7) one gets

\[ V(x)_1 = a_{1,1}^1 x_1^2 + \sum_{j=2}^n \left( a_{1,j}^{(j)} + a_{j,1}^{(1)} \right) x_1 x_j \]

\[ V(x)_2 = a_{1,2}^1 x_1^2 + a_{2,2}^1 x_2^2 + \sum_{j=2}^n \left( a_{1,j}^{(j)} + a_{j,2}^{(1)} + a_{j,2}^{(2)} \right) x_1 x_j + \sum_{j=3}^n \left( a_{j,3}^{(j)} + a_{j,3}^{(2)} \right) x_2 x_j \]

\[ \vdots \]

\[ V(x)_k = a_{1,k}^1 x_1^2 + \cdots + a_{k,k}^1 x_k^2 + \sum_{j=2}^n \left( a_{1,k}^{(j)} + a_{j,k}^{(1)} \right) x_1 x_j + \cdots + \sum_{j=k+1}^n \left( a_{k,k}^{(j)} + a_{j,k}^{(k)} \right) x_k x_j \]

Therefore, one finds that

\[ \sum_{l=1}^k V(x)_l = x_1 \left( x_1 \left( \sum_{l=1}^k a_{1,l}^{(1)} \right) + \sum_{j=2}^n \left( \sum_{l=1}^k a_{1,l}^{(j)} + a_{j,1}^{(1)} \right) x_j \right) + x_2 \left( x_2 \left( \sum_{l=2}^k a_{2,l}^{(2)} \right) + \sum_{j=3}^n \left( \sum_{l=2}^k a_{2,l}^{(j)} + a_{j,2}^{(2)} \right) x_j \right) + \cdots + x_k \left( x_k \left( a_{k,k}^{(k)} \right) + \sum_{j=k+1}^n \left( a_{k,k}^{(j)} + a_{j,k}^{(k)} \right) x_j \right), \]

for any \( k = 1, 2, \ldots, n - 1 \).

Due to stochasticity, then we know \( \sum_{l=1}^k a_{i,l}^{(i)} \leq 1 \) for every \( i = \overline{1,k} \). Moreover, based on the conditions (3.9), we have

\[ x_t \left( x_t \left( \sum_{l=t}^k a_{t,l}^{(l)} \right) + \sum_{j=t+1}^n \left( \sum_{l=t}^k a_{l,j}^{(l)} + a_{j,j}^{(l)} \right) x_j \right) \leq x_t \left( x_t + \sum_{j=2}^n x_j \right) \]

for any \( t \in \{1, \ldots, k\} \). By virtue of the last inequality, it will imply that

\[ \sum_{l=1}^k V(x)_l \leq x_1 \left( x_1 + \sum_{j=2}^n x_j \right) + x_2 \left( x_2 + \sum_{j=3}^n x_j \right) + \cdots + x_k \left( x_k + \sum_{j=k+1}^n x_j \right) \]

\[ \leq \sum_{l=1}^k x_l \]

This completes the proof. \( \square \)

Remind that, the reverse is not true. For instance, a quadratic operator (again it is a q.s.o.)

\[ \left\{ \begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \right\}, \quad (3.10) \]
which is a $b-$bistochastic operator i.e.

$$V(x)_1 = x_1x_2 \leq x_1 \text{ and } V(x)_1 + V(x)_2 = x_1x_2 + x_1x_2 \leq x_1 + x_2.$$ 

One can see that if $i = 1$ and $j = 2$, then $2a^{(2)}_{1,1} + 2a^{(1)}_{2,2} = 2 > 1$.

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