A LEFSCHETZ HYPERPLANE THEOREM FOR MORI DREAM SPACES

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Abstract. Let $X$ be a smooth Mori dream space of dimension $\geq 4$. We show that, if $X$ satisfies a suitable GIT condition which we call small unstable locus, then every smooth ample divisor $Y$ of $X$ is also a Mori dream space. Moreover, the restriction map identifies the Néron-Severi spaces of $X$ and $Y$, and under this identification every Mori chamber of $Y$ is a union of some Mori chambers of $X$, and the nef cone of $Y$ is the same as the nef cone of $X$. This Lefschetz-type theorem enables one to construct many examples of Mori dream spaces by taking “Mori dream hypersurfaces” of an ambient Mori dream space, provided that it satisfies the GIT condition. To facilitate this, we then show that the GIT condition is stable under taking products and taking the projective bundle of the direct sum of at least three line bundles, and in the case when $X$ is toric, we show that the condition is equivalent to the fan of $X$ being 2-neighborly.

Introduction

The main purpose of this paper is to prove an analogue of the Lefschetz hyperplane theorem for Mori dream spaces.

Let $X$ be a smooth complex projective variety, and let $N^1(X)$ be the group of numerical equivalence classes of line bundles on $X$. Recall from [HK00] that $X$ is called a Mori dream space if $\text{Pic}(X)_\mathbb{Q} = N^1(X)_\mathbb{Q}$ (equivalently $H^1(X, \mathcal{O}_X) = 0$), and $X$ has a finitely generated Cox ring (Definition 1.2). As the name might suggest, Mori dream spaces are very special varieties on which Mori theory works extremely well (see the nice survey article of Hu [Hu05]). On the other hand, not many classes of examples of them are known. It has been understood for a while that toric varieties are Mori dream spaces; indeed their Cox rings are polynomial rings, Cox’s homogeneous coordinate rings [Cox95]. Besides that, it was only proved very recently, in the spectacular paper of [BCHM], that (log) Fano varieties are also Mori dream spaces.

The Cox rings of certain Mori dream spaces have been the focus of much study: see, for example, [BP04], [STV07], [SS07], [CT06], [Cas07].

The most prominent feature of a Mori dream space discovered in [HK00] is the existence of a polyhedral chamber decomposition of its pseudo-effective cone; these chambers are known as the Mori chambers. Specifically if $L$ is a line bundle on a...
Mori dream space $X$, then its section ring

$$R(X, L) := \bigoplus_{n \in \mathbb{N}} H^0(X, L^\otimes n)$$

is finitely generated. Thus the rational map defined by the linear series $|L^\otimes n|$

$$\phi_{|L^\otimes n|} : X \dasharrow \mathbb{P} H^0(X, L^\otimes n)$$

stabilizes to some rational map

$$\phi_L : X \dasharrow \text{Proj } R(X, L)$$

for all large and sufficiently divisible $n$. Two line bundles $L_1$ and $L_2$ are said to be \textit{Mori equivalent} if $\phi_{L_1} = \phi_{L_2}$. This equivalence relation naturally extends to $\text{Pic}(X)_\mathbb{Q}$, and a Mori chamber is just the closure of an equivalence class in $N^1(X)_\mathbb{R}$ which has a nonempty interior. It was shown in [HK00] that these Mori chambers are polyhedral and in one-to-one correspondence with birational contractions of $X$ having $\mathbb{Q}$-factorial image.

In this paper, we first define the notion of a \textit{Mori dream hypersurface} of a Mori dream space. Since the chamber structure plays such a key role in the geometry of a Mori dream space, we propose that what deserved to be called a Mori dream hypersurface should not only be a Mori dream space itself, but should also respect the chamber structure in the following sense:

**Definition 1.** Let $X$ be a Mori dream space. A hypersurface $Y \subset X$ is called a \textit{Mori dream hypersurface} if it satisfies the following three requirements:

(i) $Y$ is a Mori dream space;
(ii) The restriction map determines an isomorphism between $N^1(X)_\mathbb{R}$ and $N^1(Y)_\mathbb{R}$;
(iii) After identifying $N^1(X)_\mathbb{R}$ and $N^1(Y)_\mathbb{R}$ via the restriction map, each Mori chamber of $Y$ is a union of some Mori chambers of $X$.

Note that the second requirement in the above definition is satisfied for any smooth projective variety $X$ of dimension $\geq 4$ and $Y \subset X$ a smooth ample divisor, thanks to the Lefschetz hyperplane theorem. On the other hand, if $X = \mathbb{P}^1 \times \mathbb{P}^{n-1}$, then ample divisors $Y \subset X$ are generally not Mori dream hypersurfaces. This leads to the question of finding suitable conditions under which a “Lefschetz-type” theorem would hold in the category of Mori dream spaces. We will give one such condition in this paper. Before stating our condition, we give some corollaries:

**Corollary 2.** Let $X$ be a smooth projective variety of dimension $\geq 4$. Suppose $X$ is a product of some Mori dream spaces, each having dimension $\geq 2$ and Picard number one. Then $X$ is a Mori dream space, and every smooth ample divisor $Y \subset X$ is a Mori dream hypersurface; moreover, the restriction map identifies $\text{Nef}(X)$, the nef cone of $X$, with $\text{Nef}(Y)$, the nef cone of $Y$. 
Corollary 3. Let $X$ be a smooth projective toric variety of dimension $\geq 4$ associated to a fan $\Delta$. Suppose that for any two rays in $\Delta$, the two-dimensional convex cone they span is also in $\Delta$. Then $X$ is a Mori dream space in which every smooth ample divisor $Y$ is a Mori dream hypersurface, and the restriction map identifies $\text{Nef}(X)$ with $\text{Nef}(Y)$.

In fact more examples satisfying the conclusion of the above two corollaries can be obtained by a suitable projective bundle construction: see Proposition 8.

Example 4. The simplest example of a space $X$ as in Corollary 2 is a product of general complete intersections in projective spaces. The simplest example of a space $X$ in Corollary 3 other than $\mathbb{P}^n$ is the blowup of $\mathbb{P}^n$ along a linear subspace $\mathbb{P}^m$ for $0 < m < n - 2$.

Remark 5. In Corollary 2 the part of the result about the preservation of nef cones has previously been obtained by Hassett-Lin-Wang [HLW02, Theorem 4.1], which they called “the weak Lefschetz principle for ample cones”. See also the results of Kollár [Bor91, Appendix] and Wiśniewski [Wis91, Theorem 2.1]. In the category of Mori dream spaces, however, our result applies to more spaces, such as those in Corollary 3, which are not covered by the results in [HLW02].

To explain the condition lying behind the above corollaries which allows a Mori dream space to enjoy this Lefschetz-type property for its nef cone and ample divisors, we need to recall the GIT construction of a Mori dream space [HK00, Proposition 2.9], which says roughly that every Mori dream space $X$ is naturally a GIT quotient of an affine variety under an algebraic torus action. More specifically, let $V = \text{Spec } R$ where $R$ is a Cox ring of $X$. Since $R$ is graded by a lattice $N$ in the Néron-Severi space of $X$, the algebraic torus $T = \text{Hom}(N, \mathbb{C}^*)$ naturally acts on the affine variety $V$. Let $\chi \in N$ be a character of $T$ which corresponds to an ample class in the Néron-Severi space of $X$. Then Hu and Keel showed that $X = V \sslash \chi$, the GIT quotient constructed with respect to the trivial line bundle on $V$ endowed with a $T$-linearization by $\chi$. Moreover, this GIT quotient is a good geometric quotient, and the unstable locus $V^\text{un}_\chi$ always has codimension $\geq 2$ in $V$. These considerations suggest the following theorem, which we will prove in Section 2.

Theorem 6. Let $X$ be a smooth Mori dream space of dimension $\geq 4$, and let $V$, $T$, and $\chi$ be as above. Assume further that the following condition (*) is satisfied:

\begin{equation}
(*) \quad \text{The unstable locus } V^\text{un}_\chi \text{ has codimension } \geq 3 \text{ in } V.
\end{equation}

Then every smooth ample divisor $Y \subset X$ is a Mori dream hypersurface, and the restriction map identifies $\text{Nef}(X)$ with $\text{Nef}(Y)$.

Definition 7. We will say that a Mori dream space $X$ has small unstable locus if the condition (*) above is satisfied.
From this theorem, Corollary 2 and 3 follow immediately once the following Proposition 8 and 10 are established in Section 3:

**Proposition 8.** Let $X$, $X_1$ and $X_2$ be Mori dream spaces.

(a) If $X$ has dimension at least two and Picard number equal to one, then $X$ has small unstable locus.

(b) If $X_1$ and $X_2$ both have small unstable locus, then $X_1 \times X_2$ is a Mori dream space which has small unstable locus.

(c) Suppose that $X$ has small unstable locus. Let $L_1, \ldots, L_k$ be line bundles on $X$, $k \geq 3$. Then the projective bundle $\mathbb{P}(\bigoplus_{i=1}^{k} L_i^{\otimes m})$ is a Mori dream space having small unstable locus for all sufficiently divisible integer $m$.

**Definition 9.** A fan $\Delta$ is called $m$-neighborly if for any $m$ rays in $\Delta$, the convex cone they span is also in $\Delta$.

**Proposition 10.** Let $X$ be a simplicial projective toric variety associated to a fan $\Delta$. Then $X$ has small unstable locus if and only if $\Delta$ is 2-neighborly.

**Remark 11.** Fans which are $m$-neighborly and give rise to complete smooth toric varieties have been the subject of interest in a couple of papers by Kleinschmidt, Sturmfels and others ([GKS90], [KSS91]). Our proof of Proposition 10 indeed shows that the fan $\Delta$ is $m$-neighborly if and only if in Cox’s GIT description of the corresponding toric variety $X$ [Cox95], the unstable locus has codimension at least $m + 1$. This reveals that the neighborliness property of the fan, which is of a combinatorial nature, has a nice GIT interpretation on the corresponding variety side.

**Remark 12.** We point out that using Proposition 8 (c), one can construct Mori dream spaces which have small unstable locus and also possess a nontrivial small $\mathbb{Q}$-factorial modification [HK00, Definition 1.8]. For example let $Z$ be the blowup of $\mathbb{P}^4$ along a line, and let $L_1$ be the line bundle corresponding to the exceptional divisor, and $L_2$ and $L_3$ be the line bundle corresponding to the pullback of a hyperplane section; then Proposition 8 (c) says that $X = \mathbb{P}(\bigoplus_{i=1}^{3} L_i^{\otimes m})$ is a Mori dream space which has small unstable locus if $m$ is sufficiently divisible, and one sees that $X$ has a nontrivial small $\mathbb{Q}$-factorial modification since the stable base locus of $\mathcal{O}_X(1)$ has codimension 3.

Finally, we remark that it would be interesting to clarify the relation between Wiśniewski’s [Wis91, Theorem 2.1] and our Theorem 6.

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1. **Mori dream space as a GIT quotient**

   In this section we collect some results and set some notations which will be used later, centering around the idea of representing a Mori dream space as a GIT quotient.
We also show in Proposition 1.12 that every Mori dream space has a normal Cox ring. We point out that the important Theorem 1.8 and Theorem 1.9 are taken from [HK00].

**Notation 1.1.** For an \( r \)-tuple of line bundles \( L = (L_1, \ldots, L_r) \) on a projective variety \( X \) and an \( r \)-tuple of integers \( m = (m_1, \ldots, m_r) \), we let
\[
L^m := L_1^\otimes m_1 \otimes L_2^\otimes m_2 \otimes \cdots \otimes L_r^\otimes m_r.
\]
Also we let \( N^1(X, L) \subset N^1(X) \) be the subgroup generated by \([L^m]\), the numerical class of \( L^m \), for all \( m \in \mathbb{Z}^r \), and we define \( T_L \) to be the algebraic torus
\[
T_L := \text{Hom}(N^1(X, L), \mathbb{C}^*)
\].

**Definition 1.2.** Let \( X \) be a projective variety such that \( \text{Pic}(X)_{\mathbb{Q}} = N^1(X)_{\mathbb{Q}} \). By a Cox ring for \( X \) we mean the ring
\[
\text{Cox}(X, L) := \bigoplus_{m \in \mathbb{Z}^r} H^0(X, L^m)
\]
where \( L = (L_1, \ldots, L_r) \) are line bundles which form a basis of \( \text{Pic}(X)_{\mathbb{Q}} \). Note that the natural \( N^1(X, L) \)-grading on \( \text{Cox}(X, L) \) corresponds to a \( T_L \)-action on \( \text{Spec} \, \text{Cox}(X, L) \).

**Remark 1.3.** Although the definition of \( \text{Cox}(X, L) \) depends on a choice of basis \( L \), whether or not it is finitely generated is independent of this choice, due to the following well-known fact:

**Lemma 1.4.** Let \( R \) be a \( \mathbb{Z}^r \)-graded commutative ring with identity. For any \( m \in \mathbb{Z}^r \), we denote the subset of \( R \) consisting of all degree-\( m \) homogeneous elements and 0 as \( R_m \), and we define
\[
R^{(m)} := \bigoplus_{a \in \mathbb{Z}^r} R_{(a_1 m_1, \ldots, a_r m_r)}.
\]
If \( R \) is an integral domain, and the subring \( R_0 := R_{(0, \ldots, 0)} \) is Noetherian, then the following are equivalent:

(a) \( R \) is a finitely generated \( R_0 \)-algebra;
(b) There exists an \( m \in \mathbb{Z}^r \) such that \( R^{(m)} \) is a finitely generated \( R_0 \)-algebra;
(c) For any \( m \in \mathbb{Z}^r \), \( R^{(m)} \) is a finitely generated \( R_0 \)-algebra.

Moreover, when this is the case, then \( R \) is a finitely generated \( R^{(m)} \)-module for any \( m \in \mathbb{Z}^r \).

**Definition 1.5.** We will call \( X \) a Mori dream space if \( X \) is a normal \( \mathbb{Q} \)-factorial projective variety with \( \text{Pic}(X)_{\mathbb{Q}} = N^1(X)_{\mathbb{Q}} \) and a finitely generated Cox ring.

**Remark 1.6.** (a) The condition \( \text{Pic}(X)_{\mathbb{Q}} = N^1(X)_{\mathbb{Q}} \) in the above definition is equivalent to \( H^1(X, \mathcal{O}_X) = 0 \). Indeed, taking the cohomology of the exponential sequence \( 0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0 \), one obtains the following exact sequence...
\[
0 \longrightarrow \text{Pic}^0(X) \longrightarrow \text{Pic}(X) \longrightarrow \text{NS}(X) \longrightarrow 0
\]
\[
\text{H}^1(X, \mathcal{O}_X)/\text{H}^1(X, \mathbb{Z}) \bigcap \text{H}^2(X, \mathbb{Z})
\]

where \(\text{NS}(X)\) is the group of algebraic equivalence classes of line bundles on \(X\). By [Laz04, Remark 1.1.20], a class in \(\text{NS}(X)\) is numerically trivial if and only if it is torsion, in other words \(N^1(X) = \text{NS}(X)_{\text{t.f.}} := \text{NS}(X)/(\text{torsion})\). Thus if we tensor the above sequence with \(\mathbb{Q}\), we get
\[
0 \longrightarrow \text{Pic}^0(X)_\mathbb{Q} \longrightarrow \text{Pic}(X)_\mathbb{Q} \longrightarrow N^1(X)_\mathbb{Q} \longrightarrow 0,
\]
which shows that \(\text{Pic}(X)_\mathbb{Q} = N^1(X)_\mathbb{Q}\) if and only if \(H^1(X, \mathcal{O}_X) = 0\).

(b) Using the same observations in (a), one also sees that suitable conditions can imply the even stronger equality \(\text{Pic}(X) = N^1(X)\). For example:

- If \(\text{Pic}(X)\) is a free abelian group of finite rank (e.g. \(X\) is a toric variety [Ful93, §3.4, first proposition]), then \(\text{Pic}(X) = N^1(X)\).
- If \(X\) is smooth and \(H_1(X, \mathbb{Z}) = 0\) (e.g. Fano variety [Deb01, Corollary 4.29]), then \(\text{Pic}(X) = N^1(X)\). This is because \(H_1(X, \mathbb{Z}) = 0\) implies \(H^1(X, \mathcal{O}_X) = 0\) by Hodge theory, and also implies that \(H^2(X, \mathbb{Z})\) is torsion-free by the universal coefficient theorem [Mun84, Corollary 56.4].

**Notation 1.7.** For an affine variety \(V\) on which an algebraic torus \(T\) acts and a character \(\chi: T \rightarrow \mathbb{C}^*\), we will use \(V/\sslash \chi\) to denote the GIT quotient constructed from the \(T\)-linearized line bundle

\[
\mathcal{O}_V^\chi := \text{the trivial line bundle on } V, \text{ }T\text{-linearized by } \chi.
\]

We will also write \(V^\text{st}_\chi, V^\text{ss}_\chi, \text{ and } V^\text{un}_\chi\) to mean the stable, semi-stable, and unstable locus of this GIT quotient respectively.

The next two theorems are among the central results in [HK00]. The wording and notations we use are not exactly the same as the original.

**Theorem 1.8.** Let \(X\) be a Mori dream space. Let \(R = \text{Cox}(X, L)\) and let \(V\) be the affine variety \(\text{Spec } R\), with the natural action by the torus \(T := T_L\) as in Definition 1.2.

Let \(\chi \in N^1(X, L)\) be a character of \(T\) which corresponds to an ample class in \(N^1(X)\). Then \(V^\text{ss}_\chi\) does not depend on the choice of \(\chi\), \(V/\sslash \chi T = X\), and the following three properties hold:

(i) \(V^\text{un}_\chi\) has codimension at least 2 in \(V\);
(ii) \(V^\text{ss}_\chi = V^\text{st}_\chi\);
(iii) Both of the maps

\[
N^1(X)_\mathbb{Q} \rightarrow \text{Pic}^T(V^\text{ss}_\chi)_\mathbb{Q} \leftarrow \text{Pic}(X)_\mathbb{Q}
\]

are isomorphisms, where the left map sends a character \(\nu \in N^1(X, L)\) to \(\mathcal{O}_V^\nu\), and the right map is the pullback under the quotient map \(\pi: V^\text{ss}_\chi \rightarrow X\).
Moreover, one can choose the basis $L$ so that the action of $T$ on $V^{ss}_\chi$ is free. We will call such basis a preferred basis.

Proof. See the proof of [HK00, Proposition 2.9]. □

Theorem 1.9. Let $T$ be a torus acting on an affine variety $V$, and let $\chi$ be a character of $T$. If $X := V \sslash_T \chi$ is projective and $\mathbb{Q}$-factorial, and the conditions (i)–(iii) of Theorem 1.8 hold, then $X$ is a Mori dream space.

Proof. See the proof of [HK00, Theorem 2.3]. □

Lemma 1.10. Under the same setting as in Theorem 1.8, if $L$ is a preferred basis, then for any line bundle $L$ of the form $L = \mathbb{L}^m$, we have

$\pi^* L = \mathcal{O}^{[L]}_{V^{ss}_\chi}$

as $T$-linearized line bundles on $V^{ss}_\chi$, where $[L] \in N^1(X, \mathbb{L})$ is the numerical equivalence class of $L$.

Proof. Since $T$ acts freely on $V^{ss}_\chi$, any $T$-linearized line bundle on $V^{ss}_\chi$ descends to a (unique) line bundle on $X$; in particular $\mathcal{O}^{[L]}_{V^{ss}_\chi}$ descends to a line bundle $M$ on $X$. To identify which line bundle $M$ is, we look at the space of $T$-invariant sections of $\mathcal{O}^{[L]}_{V^{ss}_\chi}$: on the one hand, via $\pi^*$ we see that it is equal to $H^0(X, M)$; on the other hand, we claim that it is also equal to $R_{[L]}$, the space of degree-$[L]$ homogeneous elements in $R$. Since $R_{[L]} = H^0(X, L)$ by the definition of $R$, we have $M = L$.

It remains to prove the claim that

$\{T$-invariant sections of $\mathcal{O}^{[L]}_{V^{ss}_\chi}\} = R_{[L]}.$

Let $\{a_i \in H^0(X, \mathbb{L}^m)\}_{i=1}^\ell$ be a set of regular functions on $V$ whose common zero locus is $V^{un}_\chi$. A $T$-invariant section $s$ of $\mathcal{O}^{[L]}_{V^{ss}_\chi}$ is nothing but a degree-$[L]$ homogeneous regular function on $V^{ss}_\chi$, so we can represent $s$ as a compatible collection $\{b_i/a_i^p\}_{i=1}^\ell$ where $b_i \in H^0(X, L \otimes \mathbb{L}^m)$, and of course the compatibility means $b_i a_j^p = b_j a_i^p$, $\forall i, j \in \{1, \ldots, \ell\}$.

As divisors on $X$, we can write $\text{div}(b_i)$ and $\text{div}(a_i^p)$ as

$\text{div}(b_i) = D_i + B_i,$

$\text{div}(a_i^p) = D_i + A_i,$

where $D_i$, $B_i$ and $A_i$ are Weil divisors on $X$ such that $B_i$ and $A_i$ have no common component. Then the compatibility condition translates to the following equality of divisors on $X$:

$B_i + A_j = B_j + A_i,$ $\forall i, j \in \{1, \ldots, \ell\}$.
Since $B_i$ and $A_i$ have no common component, we must have $A_j \geq A_i$, and by symmetry $A_i \geq A_j$, thus all the $A_i$'s are the same divisor $A$. But then we must have $A = 0$, for otherwise if we take a sufficiently large integer $q$ such that $qA$ is Cartier and $\mathcal{O}_X(qA)$ is of the form $L^m$, then we see that the common zero locus of $\{a^p_i\}_{i=1}^\ell$ has codimension one in $V$, contradicting the property (i) of Theorem 1.8. Therefore $a^p_i$ divides $b_i$ in $R$ for all $i$, hence the section $s$ represented by $\{b_i/a^p_i\}_{i=1}^\ell$ is in $R[L]$. □

**Lemma 1.11.** Let $X$ be a Mori dream space, and let $L$ be a basis of $\text{Pic}(X)_\mathbb{Q}$. Suppose that the torus $T := T_L$ acts on some normal affine variety $V$, such that for some character $\chi \in N^1(X, L)$ we have $V/\chi T = X$ and the conditions (i)–(iii) of Theorem 1.8 hold. Moreover, suppose that $\pi^*L = \mathcal{O}_{V^{ss}_\chi}$ for any line bundle $L$ of the form $L = L^m$, where $\pi : V^{ss}_\chi \to X$ is the quotient map. Then $\chi$ corresponds to an ample class in $N^1(X)$, and the coordinate ring $R$ of $V$ is equal to $\text{Cox}(X, L)$.

**Proof.** By [Dol03, Theorem 8.1], there exists an ample line bundle on $X$ whose pullback under the quotient map $\pi : V^{ss}_\chi \to X$ equals some tensor power of the $T$-linearized line bundle which was used to construct the GIT quotient. It follows from this and the condition (iii) of Theorem 1.8 that $\chi$ corresponds to an ample class.

Let $L$ be a line bundle on $X$ of the form $L^m$. Since $\pi^*L = \mathcal{O}_{V^{ss}_\chi}$ by assumption, $\pi^*$ induces a natural isomorphism

$$\{T\text{-invariant sections of } \mathcal{O}_{V^{ss}_\chi}^{[L]}\} = H^0(X, L).$$

On the other hand, since $V$ is normal and $V^{un}_\chi \subset V$ has codimension $\geq 2$, we have

$$\{T\text{-invariant sections of } \mathcal{O}_{V^{ss}_\chi}^{[L]}\} = \{\text{Degree-}[L] \text{ homogeneous regular functions on } V^{ss}_\chi\} = \{\text{Degree-}[L] \text{ homogeneous regular functions on } V\} = R[L].$$

Hence $R[L] = H^0(X, L)$, and thus

$$R = \bigoplus_{L = L^m} H^0(X, L) = \text{Cox}(X, L).$$

□

**Proposition 1.12.** With the same setting as in Theorem 1.8, if we choose $L$ to be a preferred basis, then $V = \text{Spec Cox}(X, L)$ is normal.

**Proof.** Let $\varphi : V' \to V$ be the $T$-equivariant normalization of $V$. Since $X$ is normal, it follows that $V^{ss}_\chi$ is also normal (cf. the second paragraph of the proof of [BH03, Proposition 6.3]), so $\varphi$ is an isomorphism over $V^{ss}_\chi$.

We want to show that $V^{\text{min}}_\chi = \varphi^{-1}(V^{\text{un}}_\chi)$. The inclusion $V^{\text{min}}_\chi \subset \varphi^{-1}(V^{\text{un}}_\chi)$ is obvious. For the reverse inclusion, we need to show that if $f$ is a homogeneous regular function
on $V'$ whose degree is a multiple of $\chi$, then $f$ must vanish on $\varphi^{-1}(V^\text{un}_\chi)$. By the definition of normalization, $f$ satisfies an integral equation

$$f^n + a_{n-1} f^{n-1} + \cdots + a_1 f + a_0 = 0,$$

where the $a_i$'s are homogeneous elements in $R$ whose degrees are multiples of $\chi$. So if we plug in a point $p \in \varphi^{-1}(V^\text{un}_\chi)$ into the above equation, we get $f(p)^n = 0$ since $a_0(p) = \cdots = a_{n-1}(p) = 0$, so $f$ vanishes at $p$ as desired.

Since $\varphi$ is an isomorphism over $V^\text{ss}_\chi$ and $V^\text{un}_\chi = \varphi^{-1}(V^\text{un}_\chi)$, we thus have $V^\text{ss}_\chi = V^\text{ss}_\chi$, so $V' \sslash \chi T = X$ and this GIT quotient satisfies the properties (i)–(iii) of Theorem 1.8. Moreover, since $L$ is a preferred basis, by Lemma 1.10, the coordinate ring of $V'$ is equal to $\text{Cox}(X, L)$, namely $V' = V$.

\[ \square \]

2. Proof of Theorem 6

Proof of Theorem 6. We need to verify that $Y \subset X$ satisfies the three requirements in Definition 1. First, by the Lefschetz hyperplane theorem [Laz04, Example 3.1.24 and Example 3.1.25], the restriction map determines canonical isomorphisms $N^1(X) = N^1(Y)$ and $\text{Pic}(X) = \text{Pic}(Y)$. In particular, the second requirement in Definition 1 is satisfied, and $\text{Pic}(Y)_Q = N^1(Y)_Q$ since $\text{Pic}(X)_Q = N^1(X)_Q$.

We pick the basis $L$ to contain the line bundle $\mathcal{O}_X(Y)$. Let $R = \text{Cox}(X, L)$, and let $s \in R$ be the unique (up to constant multiples) section of $\mathcal{O}_X(Y)$ whose zero locus is $Y$. Since $Y$ is irreducible, the ideal $sR \subset R$ is a prime ideal, so it defines an irreducible subvariety $W \subset V$. By Theorem 1.8, $V^\text{un}_\chi$ does not depend on the choice of ample $\chi \in N^1(X, L)$, and since $Y$ is ample, we have $V^\text{un}_\chi \subset W$. Therefore

$$W^\text{un}_\chi = V^\text{un}_\chi \cap W = V^\text{un}_\chi$$

has codimension $\geq 2$ in $W$ due to the condition (Ⅱ). Also by Theorem 1.8, $V^\text{ss}_\chi = V^\text{st}_\chi$ and $W \sslash \chi T = W^\text{st}_\chi / T = X$ is a good geometric quotient, so it follows that $W^\text{ss}_\chi = W^\text{st}_\chi$. Moreover, since $\varphi$ is an isomorphism by Kempf’s descent lemma [DN89, Théorème 2.3], and thus the map $\text{Pic}(W^\text{ss}_\chi)_Q \leftarrow \text{Pic}(Y)_Q$ is an isomorphism by Kempf’s descent lemma [DN89, Théorème 2.3], and thus the map $N^1(Y)_Q \rightarrow \text{Pic}(W^\text{ss}_\chi)_Q$ is also an isomorphism since $N^1(Y)_Q = \text{Pic}(Y)_Q$. Therefore $Y$ is a Mori dream space by Theorem 1.9, which verifies the first requirement in Definition 1.

To verify that $Y$ respects the chamber structure, we will use the fact that the Mori chambers coincide with the GIT chambers [HK00, Theorem 2.3]. Suppose $\chi_1, \chi_2 \in N^1(X)$ are in the interior of the same Mori chamber of $X$. Then $V^\text{un}_{\chi_1} = V^\text{un}_{\chi_2}$, so

$$W^\text{un}_{\chi_1} = V^\text{un}_{\chi_1} \cap W = V^\text{un}_{\chi_2} \cap W = W^\text{un}_{\chi_2}.$$  

Hence $\chi_1$ and $\chi_2$ are also in the same Mori chamber of $Y$.  

\[ \square \]
To show that $\text{Nef}(X) = \text{Nef}(Y)$, suppose on the contrary that $\text{Nef}(Y) \supsetneq \text{Nef}(X)$. Then there exists $\nu \in N^1(X)$ which is ample on $Y$ and lies in the interior of some Mori chamber of $X$ not equal to $\text{Nef}(X)$. Since $\nu$ and $\chi$ are both ample on $Y$, $W_{\nu}^\text{un} = W_{\chi}^\text{un}$. Recalling that $W_{\chi}^\text{un} = V_{\chi}^\text{un}$, we thus have

$$V_{\nu}^\text{un} \cap W = W_{\nu}^\text{un} = W_{\chi}^\text{un} = V_{\chi}^\text{un},$$

so $V_{\nu}^\text{un} \supset V_{\chi}^\text{un}$. Since $V_{\chi}^\text{ss} / T = X$ is a good geometric quotient, if $V_{\nu}^\text{un} \supsetneq V_{\chi}^\text{un}$ then $V_{\nu}^\text{ss} / T$ would be an open subset of $X$ and hence not projective, a contradiction. So $V_{\nu}^\text{un} = V_{\chi}^\text{un}$, and hence $V_{\nu}^\text{ss} / T = V_{\chi}^\text{ss} / T = X$, which means $\nu$ and $\chi$ are both in $\text{Nef}(X)$. \qed

3. Proof of Proposition 8 and 10

Proof of Proposition 8. (a) If the Picard number of $X$ is one, then we can pick a very ample line bundle $L$ on $X$ as a basis of $\text{Pic}(X)_{\mathbb{Q}}$ so that $R := \text{Cox}(X, L)$ is precisely the homogeneous coordinate ring of $X$ embedded into some projective space by the linear series $|L|$. Hence $V := \text{Spec} R$ is the corresponding affine cone over $X$, $V_{[L]}^\text{un}$ is the origin, and $T = \mathbb{C}^*$. The codimension of $V_{[L]}^\text{un}$ in $V$ is thus equal to $\dim V = \dim X + 1 \geq 3$.

(b) By Remark 1.6(a) we have $H^1(X_1, \mathcal{O}_{X_1}) = 0$, which implies $\text{Pic}(X_1 \times X_2) = \text{Pic}(X_1) \times \text{Pic}(X_2)$ [Har77, Chapter III Exercise 12.6], and hence also $N^1(X_1 \times X_2) = N^1(X_1) \times N^1(X_2)$. Let $L_i$ be a basis of $\text{Pic}(X_i)_{\mathbb{Q}}$, $R_i = \text{Cox}(X_i, L_i)$, $V_i = \text{Spec} R_i$, $T_i = \text{Hom}(N^1(X_i, L_i), \mathbb{C}^*)$, and $p_i : X_1 \times X_2 \to X_i$ be the projection map for $i = 1, 2$. Let $L$ be the basis of $\text{Pic}(X_1 \times X_2)_{\mathbb{Q}}$ which consists of $p_1^* L_1$ and $p_2^* L_2$. Then by the Künneth formula we have

$$R := \text{Cox}(X_1 \times X_2, L) = R_1 \otimes R_2.$$ 

Thus $V := \text{Spec} R = V_1 \times V_2$ and $T := \text{Hom}(N^1(X_1 \times X_2, L), \mathbb{C}^*) = T_1 \times T_2$. Furthermore, since $\chi = p_1^* \chi_1 + p_2^* \chi_2$ is ample if and only if $\chi_i \in N^1(X_i)$ are both ample for $i = 1, 2$, we thus have

$$V_{\chi}^\text{un} = p_1^{-1} V_{\chi_1}^\text{un} \cup p_2^{-1} V_{\chi_2}^\text{un},$$

hence the result follows.

(c) We will in fact show that if the line bundles $L_i$'s are all of the form $L_{m_i}$ where $L$ is a preferred basis, then $\tilde{X} := \mathbb{P}(\bigoplus_{i=1}^k L_i)$ is a Mori dream space which has small unstable locus. Let $R = \text{Cox}(X, L)$, $V = \text{Spec} R$, $T = \text{Hom}(N^1(X, L), \mathbb{C}^*)$, and let $\chi \in N^1(X, L)$ be a character of $T$ which corresponds to a sufficiently ample class, to the extent that

$$\chi + \sum_{i=1}^k d_i [L_i]$$
is ample for all $d_i \geq 0$ and $\sum_{i=1}^{k} d_i = 1$. Let $\overline{L}$ be the basis of Pic($\overline{X}$)$_{\mathbb{Q}}$ consisting of $p^{*}L$ and $O_{\overline{X}}(1)$, where $p : \overline{X} \to X$ is the projection map. Then

$$\overline{T} := \text{Hom}(N^{1}(\overline{X}, \overline{L}), \mathbb{C}^{*}) = T \times T_{1},$$

where $T_{1} = \mathbb{C}^{*}$ is the one-dimensional torus for which $[O_{\overline{X}}(1)]$ generates the group of characters.

By Lemma 1.10, we have $\pi^{*}L_{i} = O_{V^{ss}_{\chi}}^{[L_{i}]}$, $i = 1, \ldots, k$. Thus we consider the normal affine variety

$$\overline{V} := V \times \mathbb{C} \times \mathbb{C} \times \cdots \times \mathbb{C}$$

(there the normality of $V$ follows from Proposition 1.12) with the following $\overline{T}$-action: given any $(v, a_{1}, \ldots, a_{k}) \in \overline{V}$ and $(t, c) \in T \times \mathbb{C}^{*} = \overline{T}$, define

$$(t, c) \cdot (v, a_{1}, \ldots, a_{k}) = (t \cdot v, \langle [L_{1}], t \rangle^{-1} a_{1} c, \ldots, \langle [L_{k}], t \rangle^{-1} a_{k} c),$$

where $[[L_{i}], t]$ denotes the natural pairing. Let $\overline{\chi}$ be the character $(\chi, [O_{\overline{X}}(1)])$ of $\overline{T}$. We claim that

$$\overline{V}_{\chi}^{un} = p_{1}^{-1}V_{\chi}^{un} \cup p_{2}^{-1}\{(0, \ldots, 0)\},$$

where $p_{1} : \overline{V} \to V$ and $p_{2} : \overline{V} \to \mathbb{C} \times \cdots \times \mathbb{C}$ are the projection maps. To see this, let $x_{1}, \ldots, x_{k}$ be the coordinate functions of $\mathbb{C} \times \cdots \times \mathbb{C}$, and let $f \in R_{\nu}$ be a homogeneous degree-$\nu$ regular function on $V$ for some $\nu \in N^{1}(X, L)$. If $d_{1}, \ldots, d_{k}$ are nonnegative integers which sum up to $d$, then $f x_{1}^{d_{1}} \cdots x_{k}^{d_{k}}$ is a regular function on $\overline{V}$ which is homogeneous of degree

$$(\nu - \sum_{i=1}^{k} d_{i}[L_{i}], d[O_{\overline{X}}(1)]),$$

hence $\overline{V}_{\chi}^{un}$ is the common zero locus of all such functions $f x_{1}^{d_{1}} \cdots x_{k}^{d_{k}}$ for which

$$\nu = d\chi + \sum_{i=1}^{k} d_{i}[L_{i}].$$

Note that such $\nu$ corresponds to an ample class thanks to our choice of $\chi$ in the beginning, and from this it follows easily that

$$\overline{V}_{\chi}^{un} = p_{1}^{-1}V_{\chi}^{un} \cup p_{2}^{-1}\{(0, \ldots, 0)\}.$$ 

Hence $\overline{V}_{\chi}^{un}$ has codimension $\geq 3$ in $\overline{V}$, and $\overline{V} \sslash_{\overline{T}} \overline{\chi} = \mathbb{P}(\bigoplus_{i=1}^{k} L_{i})$, so in particular this GIT quotient satisfies the property (i) in Theorem 1.8. The action of $T$ on $V_{\chi}^{ss}$ is free since $L$ is a preferred basis, hence the action of $\overline{T}$ on $\overline{V}_{\chi}^{ss}$ is also free, so in particular
the GIT quotient $\tilde{V}/\chi\tilde{T}$ satisfies the property (ii) in Theorem 1.8, and the right map in the property (iii) is an isomorphism even before tensoring with $Q$; to show the left map is also an isomorphism, we will show that $O_{\tilde{V}^{ss}} = \tilde{\pi}^* \tilde{L}$, where $\tilde{\pi}: \tilde{V}^{ss} \to \tilde{X}$ is the quotient map, and $\tilde{L}$ is any line bundle on $\tilde{X}$ of the form $\tilde{L}^m$. Since we already know that the pullback map $\operatorname{Pic}(\tilde{V}^{ss}) \leftarrow \operatorname{Pic}(\tilde{X})$ is an isomorphism, the line bundle $O_{\tilde{V}^{ss}}$ descends to some line bundle $\tilde{M}$ on $\tilde{X}$ in any case, so we just need to identify which line bundle $\tilde{M}$ is. To do this we look at the space of $\tilde{T}$-invariant sections of $O_{\tilde{V}^{ss}}$: on the one hand it is equal to $H^0(\tilde{X}, \tilde{M})$ via $\tilde{\pi}^*$; on the other hand, it is equal to the space of homogeneous degree-$[\tilde{L}]$ regular functions on $\tilde{V}$ (since $\tilde{V}$ is normal), and if $\tilde{L} = p^* L \otimes O_{\tilde{X}}(d)$, then such regular functions are precisely linear combinations of functions of the form $f x_1^{d_1} \cdots x_k^{d_k}$ where the $d_i$’s are nonnegative integers summing up to $d$ and $f$ is a homogeneous regular function on $V$ of degree $[L] + \sum_{i=1}^k d_i [L_i]$. In other words,

$$\{\tilde{T}\text{-invariant sections of } O_{\tilde{V}^{ss}}\} = \bigoplus_{d_1, \ldots, d_k \geq 0} H^0(X, L \otimes L_1^{d_1} \otimes \cdots \otimes L_k^{d_k}).$$

But the right-hand side is exactly $H^0(\tilde{X}, \tilde{L})$: indeed since $\tilde{L} = p^* L \otimes O_{\tilde{X}}(d)$, by the projection formula

$$H^0(\tilde{X}, \tilde{L}) = H^0(X, p^* \tilde{L}) = H^0(X, L \otimes p_* O_{\tilde{X}}(d)) = H^0(X, L \otimes S^d(\bigoplus_{i=1}^k L_i))$$

$$= \bigoplus_{d_1, \ldots, d_k \geq 0} H^0(X, L \otimes L_1^{d_1} \otimes \cdots \otimes L_k^{d_k}).$$

Hence the line bundle on $\tilde{X}$ which $O_{\tilde{V}^{ss}}$ descends to must be $\tilde{L}$.

Now we can use Theorem 1.9 to obtain that $\tilde{X}$ is a Mori dream space, and then use Lemma 1.11 to conclude that the affine variety $\tilde{V}$ is indeed $\operatorname{Spec} \operatorname{Cox}(\tilde{X}, \tilde{L})$, thus completing the proof. □

Proof of Proposition 10. It was shown in [Cox95] that in the toric case, the Cox ring $R$ and the unstable locus $V^{un}_\chi$ have the following explicit description. Let $\Delta(1)$ be the set of all one-dimensional cones of $\Delta$. For each $\rho \in \Delta(1)$, introduce a variable $x_\rho$. Then $R$ is the polynomial ring

$$R = \mathbb{C}[x_\rho : \rho \in \Delta(1)],$$
and the unstable locus is the zero locus of an ideal $I \subset R$ generated by squarefree monomials. To give these generators of $I$, let us introduce some notations. For a subset $A \subset \Delta(1)$, we denote the monomial $\prod_{\rho \in A} x_\rho$ as $x^A$, and we write $\hat{A}$ for the complement of $A$ in $\Delta(1)$. For a cone $\sigma \in \Delta$, we let $\sigma(1) = \{ \rho \in \Delta(1) : \rho \text{ is a face of } \sigma \}$. Then

$$I = \langle x^{\sigma(1)} : \sigma \in \Delta \rangle \subset R.$$ 

A squarefree monomial ideal such as $I$ can be very well understood by associating to it a simplicial complex, sometimes called the Stanley-Reisner complex. The Stanley-Reisner complex of $I$ is by definition an abstract simplicial complex $\Sigma$ on the vertex set $\Delta(1)$, whose faces are those subsets $A \subset \Delta(1)$ such that $x^A \notin I$. By [MS05, Theorem 1.7], the prime decomposition of $I$ is given by

$$I = \bigcap_{A \in \Sigma} \langle x_\rho : \rho \in \hat{A} \rangle.$$ 

Hence

The zero locus of $I$ has codimension $\geq 3 \iff |\hat{A}| \geq 3$ for all $A \in \Sigma$

$\iff$ If $A \subset \Delta(1)$ such that $|\hat{A}| \leq 2$, then $x^A \in I$

$\iff$ If $A \subset \Delta(1)$ such that $|A| \leq 2$, then $x^\hat{A} \in I$

$\iff$ If $A \subset \Delta(1)$ and $|A| \leq 2$, then $A = \sigma(1)$ for some $\sigma \in \Delta$.

\[ \square \]

REFERENCES

[BP04] Victor V. Batyrev and Oleg N. Popov, The Cox ring of a del Pezzo surface, Arithmetic of higher-dimensional algebraic varieties, Progress in Mathematics vol. 226, Boston: Birkhäuser, 2004, pp. 85–103.

[BH03] Florian Berchtold and Jürgen Hausen, Homogeneous coordinates for algebraic varieties, J. Algebra 266 (2003), 636–670.

[BCHM] Caucher Birkar, Paolo Cascini, Christopher D. Hacon, and James McKernan, Existence of minimal models for varieties of log general type, preprint, arXiv:math/0610203

[Bor91] Ciprian Borcea, Homogeneous vector bundles and families of Calabi-Yau threefolds, II, Several Complex Variables and Complex Geometry, Part 2, Proc. Symp. Pure Math., vol. 52, Amer. Math. Soc., Providence, RI, 1991, pp. 83–91.

[Cas07] Ana-Maria Castravet, The Cox ring of $\bar{M}_{0,6}$, to appear in Trans. Amer. Math. Soc., arXiv:0705.0070

[CT06] Ana-Maria Castravet and Jenia Tevelev, Hilbert’s 14-th problem and Cox rings, Compositio Math. 142 (2006), 1479–1498.

[Cox95] David A. Cox, The homogeneous coordinate ring of a toric variety, J. Alg. Geom. 4 (1995), 17–50.

[Deb01] Olivier Debarre, Higher-dimensional algebraic geometry, Universitext, New York: Springer, 2001.
[Dol03] Igor Dolgachev, *Lectures on invariant theory*, London Mathematical Society Lecture Note Series vol. 296, Cambridge University Press, 2003.

[DN89] J.-M. Drezet and M.S. Narasimhan, *Groupe de Picard des variétés de modules de fibrés semi-stables sur les courbes algébriques*, Invent. Math. 97 (1989), 53–94.

[Ful93] William Fulton, *Introduction to toric varieties*, Annals of Mathematics Studies vol. 131, Princeton University Press, Princeton, N.J., 1993.

[GKS90] Jörg Gretenkort, Peter Kleinschmidt and Bernd Sturmfels, *On the existence of certain smooth toric varieties*, Discrete Comput. Geom. 5 (1990), 255–262.

[Har77] Robin Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics vol. 52, New York: Springer, 1977.

[HLW02] Brendan Hassett, Hui-Wen Lin, and Chin-Lung Wang, *The weak Lefschetz principle is false for ample cones*, Asian J. Math. 6 (2002), no. 1, 95–100.

[Hu05] Yi Hu, *Geometric invariant theory and birational geometry*, arXiv:math.AG/0502462

[HK00] Yi Hu and Seán Keel, *Mori dream spaces and GIT*, Michigan Math. J. 48 (2000), 331–348.

[KSS91] Peter Kleinschmidt, Niels Schwartz, and Bernd Sturmfels, *Unimodular fans, linear codes, and toric manifolds*, Discrete and Computational Geometry: Papers from the DIMACS Special Year, American Math. Soc., Providence, 1991, pp. 179–186.

[Laz04] Robert Lazarsfeld, *Positivity in Algebraic Geometry I–II*, Ergeb. Math. Grenzgeb., vols. 48–49, Berlin: Springer, 2004.

[MS05] Ezra Miller and Bernd Sturmfels, *Combinatorial commutative algebra*, Graduate Texts in Mathematics vol. 227, New York: Springer, 2005.

[Mun84] James R. Munkres, *Elements of algebraic topology*, Addison–Wesley, Menlo Park, CA, 1984.

[SS07] Vera Serganova and Alexei Skorobogatov, *Del Pezzo surfaces and representation theory*, Algebra and Number Theory 1 (2007), 393–419.

[STV07] Mike Stillman, Damiano Testa, and Mauricio Velasco *Gröbner bases, monomial group actions, and the Cox rings of del Pezzo surfaces*, J. Algebra 316 (2007), no. 2, 777–801.

[Wiś91] Jaroslaw A. Wiśniewski, *On contractions of extremal rays of Fano manifolds*, J. Reine Angew. Math. 417 (1991), 141–157.