On the tetracanonical map of varieties of general type and maximal Albanese dimension

Sofia Tirabassi

Abstract We prove the birationality of the 4-canonical map of smooth projective varieties of general type and maximal Albanese dimension.

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1 Introduction

An interesting problem in birational geometry is, given Z a projective variety of general type, to understand for which n the nth pluricanonical linear system \(|\omega^n_Z|\) yields a birational map.

For varieties of maximal Albanese dimension (i.e. the Albanese map \(Z \rightarrow \text{Alb}(Z)\) is generically finite), this study brought some surprising result, since Chen and Hacon [3] proved that, independently on the dimension, \(\omega^n_Z\) is birational for every \(n \geq 6\), while, for varieties with positive Euler characteristic, the tricanonical linear system is enough to induce a birational map. Similar results were obtained by Pareschi and Popa as a consequence of their M-regularity theory. More specifically in [9] they showed that the 3-canonical map is always birational for varieties of general type, maximal Albanese dimension and Albanese general type (i.e. with Albanese image not ruled by tori). The case of bicanonical maps was addressed in [1]. Recently it was shown by Jiang [5], applying ideas from [9], that, if Z is of general type and maximal Albanese dimension, the pentacanonical map \(\varphi_{|\omega^5_Z|}\) is birational. The main result of this paper, whose proof also uses M-regularity techniques introduced by Pareschi and Popa in [9], is an improvement of Chen-Hacon and Jiang’s theorems:

**Theorem A** If Z is a complex projective smooth variety of maximal Albanese dimension and general type, then the tetracanonical map \(\varphi_{|\omega^4_Z|}\) birational.
The argument consists in showing that reducible divisors $D_\alpha + D_{\alpha^\vee}$, with $\alpha \in \text{Pic}^0(Z)$, $D_\alpha \in |\omega^2_Z \otimes \alpha|$ and $D_{\alpha^\vee} \in |\omega^2_Z \otimes \alpha^\vee|$ separate points in a suitable open set of $Z$. The crucial point is that, for all $\alpha \in \text{Pic}^0(Z)$, the sections of $\omega^2_Z \otimes \alpha$ passing through a general point of $Z$ have good generation properties. In order to achieve this we use Pareschi–Popa theory of $M$-regularity and continuous global generation [8,9], joint with a theorem of Chen–Hacon [2] on the fact that the variety $V^0(\omega_Z)$ spans $\text{Pic}^0(Z)$.

It is worth mentioning that, in a former version of this paper, we conjectured that the result here presented was not sharp. In fact, shortly after this paper was finished a collaboration with Jiang and Lahoz started and in [6] we were able to prove that indeed the 3-canonical map of varieties of maximal Albanese dimension and of general type is always birational, this completing the analogy with curves. However the argument we present here, as it does not rely on induction on dimension, as the one in [6], is much more explicit. Furthermore it is certainly shorter and more transparent.

In what follows $Z$ will always be a smooth complex variety of general type and maximal Albanese dimension while $\omega_Z$ shall denote its dualizing sheaf. By $\text{Alb}(Z)$ we will mean the Albanese variety of $Z$. Given $\mathcal{L}$ an invertible sheaf on a projective variety $Y$, then $\mathcal{F}(|\mathcal{L}|)$ will be the asymptotic multiplier ideal sheaf associated to the complete linear series $|\mathcal{L}|$ (cf. [7]). Finally, if $\mathcal{F}$ is a coherent multiplier ideal sheaf associated to $Y$, then by $h^i(\mathcal{F})$ we will mean $\dim H^i(Y, \mathcal{F})$.

Given a linear system $V \subseteq |L|$, we will say that it is birational if the corresponding rational map $\varphi_V$ is birational.

### 2 Setup and definitions

In what follows we briefly recall some definitions about $M$-regular sheaves and their generation properties. Given a coherent sheaf $\mathcal{F}$ on a smooth projective variety $Y$ we denote by $\text{Bs}(\mathcal{F})$ the non-generation locus of $\mathcal{F}$, i.e., the support of the cokernel of the evaluation map $H^0(Y, \mathcal{F}) \otimes \mathcal{O}_Y \to \mathcal{F}$. If $\mathcal{F}$ is a line bundle, then $\text{Bs}(\mathcal{F})$ is just the base locus of $\mathcal{F}$.

The cohomological support loci of $\mathcal{F}$ are defined as:

$$V^i(Y, \mathcal{F}) := \{\alpha \in \text{Pic}^0(Y) \mid h^i(\mathcal{F} \otimes \alpha) > 0\} \subseteq \text{Pic}^0(Y).$$

As for cohomology groups, we will occasionally suppress $Y$ from the notation, simply writing $V^i(\mathcal{F})$.

**Definition 1** ([8, Definitions 2.1 and 2.10]) Let $\mathcal{F}$ a sheaf on a smooth projective variety $Y$.

1. It is said to be $GV_1$ if $\text{Codim} \ V^i(\mathcal{F}) > i$ for every $i > 0$. In the case $Y$ is an abelian variety, $GV_1$ sheaves are called $M$-regular.
2. Given $y \in Y$, we say that $\mathcal{F}$ is continuously globally generated at $y$ (in brief cgg) if the sum of the evaluation maps

$$\mathcal{E}_{U,y} : \bigoplus_{\alpha \in U} H^0(\mathcal{F} \otimes \alpha) \otimes \alpha^{-1} \to \mathcal{F} \otimes \mathbb{C}(y)$$

is surjective for every $U$ non empty open set of $\text{Pic}^0(Y)$.

### 3 Proof of Theorem A

We will prove this slightly more general statement:

**Theorem 1** Let $Z$ be a smooth complex projective variety of maximal Albanese dimension and of general type, then, for every $\alpha \in \text{Pic}^0(Z)$ the linear system $|\omega^2_Z \otimes \alpha|$ is birational.