THE BRIDGE NUMBER OF SURFACE LINKS AND KEI COLORINGS

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ABSTRACT. Meier and Zupan introduced bridge trisections of surface links in $S^4$ as a 4-dimensional analogue to bridge decompositions of classical links, which gives a numerical invariant of surface links called the bridge number. We prove that there exist infinitely many surface knots with bridge number $n$ for any integer $n \geq 4$. To prove it, we use colorings of surface links by keis and give lower bounds for the bridge number of surface links.

1. Introduction

A surface link is a smoothly embedded closed surface in $S^4$, which may be disconnected or non-orientable. A surface link is called a surface knot if it is connected. In [8], Meier and Zupan introduced bridge trisections of surface links as a 4-dimensional analogue to bridge decompositions of classical links. Moreover, any bridge trisection is associated to a tri-plane diagram, which consists of three diagrams for trivial 1-dimensional tangles. Using these notions, Meier and Zupan defined the bridge number as a new geometrical complexity of surface links. Indeed, it is proved in [8] that if the bridge number of a surface link $\mathcal{K}$ is equal or less than 3, then $\mathcal{K}$ is one of trivially embedded $S^2$, $\mathbb{R}P^2$ and $T^2$.

One of fundamental problems about the bridge number is whether there exists a surface link with bridge number $n$ for any positive integer $n$. For the cases where $n \equiv 1 \text{ mod } 3$, this problem is resolved affirmatively in [8], while it remained open in general. The aim of this paper is to resolve this problem affirmatively for general cases.

Theorem 1.1. For any integer $n \geq 4$, there exist infinitely many distinct surface knots with bridge number $n$.

Here we remark that the bridge number of a surface link $\mathcal{K}$ is congruent to $-\chi(\mathcal{K})$ modulo 3, where $\chi(\mathcal{K})$ is the Euler characteristic of $\mathcal{K}$. Any of our examples for proving Theorem 1.1 is homeomorphic to one of $S^2$, $\mathbb{R}P^2$ and $T^2$.
To prove Theorem 1.1, we introduce kei colorings of tri-plane diagrams, and show that such colorings coincide with the original kei colorings of surface links. As its application, if we consider a finite kei, we have the following lower bound for the bridge number.

**Theorem 1.2.** For any surface link $K$ and finite kei $X$, we have the inequality

$$b(K) \geq 3 \log_{\#X}(\#\text{Col}_X(K)) - \chi(K),$$

where $b(K)$ is the bridge number of $K$ and $\#\text{Col}_X(K)$ the number of $X$-colorings of $K$.

For the $m$-twist-spinning $S_m(K)$ of a classical knot $K$, the relation between $\#\text{Col}_X(S_m(K))$ and $\#\text{Col}_X(K)$ is studied by Asami and Satoh [1]. Their study enables us to give a sufficient condition for the equality in Theorem 1.2 for $m$-twist spun knots and their stabilizations. Here, we denote by $P$ either one of the two trivially embedded $\mathbb{RP}^2$'s in $S^4$ (with normal Euler number $\pm 2$) and by $T$ a trivially embedded torus in $S^4$. For any integer $p \geq 2$, we denote by $R_p$ the dihedral kei of order $p$. (Note that the coloring by $R_p$ is coincident with the Fox $p$-coloring [5, 6] for classical links.)

**Theorem 1.3.** For a classical knot $K$, suppose that there exists an odd integer $p > 1$ with $b(K) = \log_{\#R_p}(\#\text{Col}_{R_p}(K))$, where $b(K)$ denotes the bridge number of $K$. Then for any $m \in \mathbb{Z}$, we have the following equalities:

- $b(S_{2m}(K)) = 3b(K) - 2$,
- $b(S_{2m}(K)\#P) = 3b(K) - 1$, and
- $b(S_{2m}(K)\#T) = 3b(K)$.

Theorem 1.1 immediately follows from Theorem 1.3.

**Proof of Theorem 1.1**

Since the $k$ times connected sum $#_k T_{2,q}$ of the $(2,q)$-torus knot satisfies

$$b(#k T_{2,q}) = \log_q(\#\text{Col}_{R_p}(#k T_{2,q})) = k + 1,$$

for any odd $q > 1$, Theorem 1.3 can be applied for $#k T_{2,q}$. Moreover, if an odd number $q'$ is less than $q$, then it follows from a direct calculation that $\#\text{Col}_{R_p}(#k T_{2,q'}) < q^{k+1}$. Hence their 0-twist spuns $\{S_0(#k T_{2,q})\}_{q>1}$ are mutually distinct, since their coloring numbers are the same as those of the original classical knots, respectively. 

We remark that Theorem 1.3 also holds even if we replace $R_p$ with any finite faithful kei. For more details, see Section 3. Here we also mention Meier-Zupan’s question [8, Question 5.2], which asks whether the equality $b(S_m(K)) = 3b(K) - 2$ holds for any $K$ and $m \neq \pm 1$. Theorem 1.3 gives a sufficient condition for the equality, while it can be applied to only the cases where $m$ is even.
Finally, we remark that if a given surface link is oriented, then we can define *quandle colorings* of its oriented tri-plane diagrams and can prove the coincidence of the colorings with the original quandle colorings. Such quandle colorings would be useful to approach Meier-Zupan’s question for odd \( m \).

2. Kei colorings of tri-plane diagrams

In this section, we introduce *kei colorings* of tri-plane diagrams, and show that such colorings coincide with original kei colorings of surface links. As a consequence of those arguments, we prove Theorem 1.2.

2.1. A review of tri-plane diagrams. We first recall bridge trisections, the bridge number and tri-plane diagrams of surface links.

A 0-trisection of \( S^4 \) is a decomposition \( S^4 = X_1 \cup X_2 \cup X_3 \), such that

1. \( X_i \) is a 4-ball,
2. \( B_{ij} = X_i \cap X_j = \partial X_i \cap \partial X_j \) is a 3-ball, and
3. \( \Sigma = X_1 \cap X_2 \cap X_3 = B_{12} \cap B_{23} \cap B_{31} \) is a 2-sphere.

A trivial \( c \)-disk system is a pair \((X, D)\) where \( X \) is a 4-ball and \( D \) is a collection of \( c \) properly embedded disks in \( X \) which are simultaneously isotopic into the boundary of the 4-ball \( X \). Using these terminologies, a bridge trisection of a surface link is defined as follows.

**Definition 2.1 ( [8, Definition 1.2] ).** A \((b; c_1, c_2, c_3)\)-bridge trisection \( T \) of a surface link \( K \subset S^4 \) is a decomposition of the form \((S^4, K) = (X_1, D_1) \cup (X_2, D_2) \cup (X_3, D_3)\) such that

1. \( S^4 = X_1 \cup X_2 \cup X_3 \) is a 0-trisection of \( S^4 \),
2. \((X_i, D_i)\) is a trivial \( c_i \)-disk system, and
3. \((B_{ij}, \alpha_{ij}) = (X_i, D_i) \cap (X_j, D_j)\) is a \( b \)-strand trivial tangle.

When appropriate, we simply refer to \( T \) as a \( b \)-bridge trisection.

It is proved in [8] that every surface link admits a bridge trisection. The *bridge number* \( b(K) \) of a surface link \( K \) is defined by

\[
b(K) = \min \{ b \mid K \text{ admits a } b \text{-bridge trisection} \}.
\]

Next we recall tri-plane diagrams. For a \((b; c_1, c_2, c_3)\)-bridge trisection \( T \) of a surface link \( K \), choose the orientation of \( B_{ij} \) as a submanifold of \( \partial X_i \). For each pair of indices, let \( E_{ij} \subset B_{ij} \) be an embedded disk with the property that \( e = \partial E_{12} = \partial E_{23} = \partial E_{31} \). We call the union \( E_{12} \cup E_{23} \cup E_{31} \) a *tri-plane*. We also suppose that the points \( p = K \cap \Sigma \) lie in the curve \( e = E_{12} \cap E_{23} \cap E_{31} \).

Choose an orientation of each \( E_{ij} \) so that the \( E_{ij} \) induce the same orientation of their common boundary \( e \). Then, a *tri-plane diagram* \( P \) of \( T \) is a triple of planar tangle diagrams \( P = (P_{12}, P_{23}, P_{31}) \) such that \( P_{ij} \) is obtained as a diagram of \( \alpha_{ij} \) in \( E_{ij} \). By definition, there is a canonical identification among \( \partial P_{ij} \), and if \( P_{ki} \) denote the mirror image of \( P_{ki} \), then \( P_{ij} \cup P_{ki} \) is a classical
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4 component trivial classical link, where the diagram lies in \( S^2 = E_{ij} \cup (-E_{ki}) \). Refer to [8] for more details.

2.2. Kei colorings of tri-plane diagrams. Now we introduce kei colorings of tri-plane diagrams.

A kei [1] is a non-empty set \( X \) equipped with a binary operation \((a, b) \mapsto a \ast b\) such that

(i) \( a \ast a = a \) for any \( a \in X \),
(ii) \( (a \ast b) \ast b = a \) for each \( a, b \in X \), and
(iii) \( (a \ast b) \ast c = (a \ast c) \ast (b \ast c) \) for any \( a, b, c \in X \).

For examples, the dihedral kei of order \( p \), denoted by \( R_p \), is a kei consisting of the set \{0, 1, \ldots, p - 1\} with the binary operation defined by \( i \ast j = 2j - i \) (mod \( p \)).

Here we recall kei colorings of tangle diagrams. For a given kei \( X \), an \( X \)-coloring of a tangle diagram \( T \) is an assignment of an element of \( X \) to each arc in \( T \) such that \( a \ast b = c \) holds at each crossing, where \( a \) and \( c \) (resp. \( b \)) are the colors of under-arcs (resp. the over-arc). Note that \( a \ast b = c \) is equivalent to \( c \ast b = a \) by the axiom (ii) of a kei. Any tangle diagram admits a trivial \( X \)-coloring where all the arcs are colored by the same element. We denote by \( \text{Col}_X(T) \) the set of all \( X \)-colorings of \( T \).

Definition 2.2. For a given kei \( X \), an \( X \)-coloring of a tri-plane diagram \( \mathcal{P} = (\mathcal{P}_{12}, \mathcal{P}_{23}, \mathcal{P}_{31}) \) is a triple of \( X \)-colorings \( C_{ij} \) of \( \mathcal{P}_{ij} \) such that for each end point \( p \in \partial \mathcal{P}_{ij} \), the arcs in \( \mathcal{P}_{ij} \) containing \( p \) are colored by the same element.

We denote by \( \text{Col}_X(\mathcal{P}) \) the set of all \( X \)-colorings of \( \mathcal{P} \).

Remark 2.3. If a given surface link \( K \) is oriented, then we can define an oriented tri-plane diagram of \( K \) so that the orientation of \( \mathcal{P}_{ij} \) is induced from \( \partial D_i \). Then, we can define quandle colorings of oriented tri-plane diagrams in a way similar to Definition 2.2.

2.3. Relationship to original kei colorings. Here we show that kei colorings of tri-plane diagrams can be identified with kei colorings of broken surface diagrams. As its application, we prove Theorem 1.2.

For a given surface link \( K \subset S^4 \), take a point \( p_{\infty} \in S^4 \setminus K \) and identify \( \mathbb{R}^4 \) with \( S^4 \setminus \{p_{\infty}\} \). A broken surface diagram of a surface link is a generic projection image in \( \mathbb{R}^3 \) where one of the two sheets near the double point curve is broken depending on the relative height. This convention is similar to classical knot diagrams. A broken surface diagram consists of broken sheets, that are mutually disjoint compact surfaces in \( \mathbb{R}^3 \). Any surface link admits a broken surface diagram, and any two broken surface diagrams of a surface link diagram of a c-component trivial classical link, where the diagram lies in \( S^2 = E_{ij} \cup (-E_{ki}) \). Refer to [8] for more details.
link are related by a finite sequence of Roseman moves. Refer to [2] for more details.

For a given kei $X$, an $X$-coloring of a broken surface diagram $\mathcal{D}$ is an assignment of an element of $X$ to each broken sheet in $\mathcal{D}$ such that $a * b = c$ holds along each double point curve, where $a$ and $c$ (resp. $b$) are the colors of under-sheets (resp. the over-sheet). Any diagram admits a trivial $X$-coloring where all the broken sheets are assigned the same element. We denote by $\text{Col}_X(\mathcal{D})$ the set of $X$-colorings of $\mathcal{D}$. We remark that each Roseman move from $\mathcal{D}_1$ to $\mathcal{D}_2$ induces a bijection from $\text{Col}_X(\mathcal{D}_1)$ to $\text{Col}_X(\mathcal{D}_2)$; see, for example, [3, 10]. In particular, for any finite kei $X$, the number of the $X$-colorings is an invariant of a surface link $K$. Hence we denote the value by $\#\text{Col}_X(K)$.

Proposition 2.4. For any tri-plane diagram $\mathcal{P}$ of a surface link $K$, there exists a broken surface diagram $\mathcal{D}(\mathcal{P})$ of $K$ in $\mathbb{R}^4$ such that for any kei $X$, we have a bijection

$$\varphi: \text{Col}_X(\mathcal{P}) \to \text{Col}_X(\mathcal{D}(\mathcal{P})).$$

In particular, if $X$ is finite, then we have

$$\#\text{Col}_X(\mathcal{P}) = \#\text{Col}_X(K).$$

Proof. Suppose that $\mathcal{P}$ is a tri-plane diagram of a $(b; c_1, c_2, c_3)$-bridge trisection $\mathcal{T}$. Take a point $p_\infty$ in $e = E_{12} \cap E_{23} \cap E_{31}$ with $p_\infty \notin K$. By taking an orientation preserving diffeomorphism between $S^4 \setminus \{p_\infty\}$ and $\mathbb{R}^4$ suitably, we may assume that

$$X_j \setminus \{p_\infty\} = \{ (r \cos \theta, r \sin \theta, z, w) \mid r \geq 0, \frac{2\pi j}{3} \leq \theta \leq \frac{2\pi (j+1)}{3} \}$$

for $j = 1, 2, 3$, and

$$E_{ij} \setminus \{p_\infty\} = \mathbb{R}^{2+}_k := \{ (r \cos \frac{2\pi j}{3}, r \sin \frac{2\pi j}{3}, z, 0) \mid r \geq 0 \}$$

for any cyclic permutation, denoted by $(i, j, k)$, of $(1, 2, 3)$ without loss of generality. Then it also follows that

- $B_{ij} \setminus \{p_\infty\} = \{ (r \cos (2\pi j/3), r \sin (2\pi j/3), z, w) \mid r \geq 0 \}$,
- $\Sigma \setminus \{p_\infty\}$ is the $zw$-plane, and
- $e \setminus \{p_\infty\}$ is the $z$-axis.

In particular, $D_i := \mathcal{D}_{ij} \cup \overline{\mathcal{D}_{ki}}$ can be regarded as a classical link diagram of the trivial $c_i$-component classical link $\partial D_i$, where the diagram lies in the plane $\mathbb{R}^{2+}_k \cup (-\mathbb{R}^{2+}_k)$ and the orientation of $\mathbb{R}^{2+}_k$ is induced from $E_{ij}$. To obtain a broken surface diagram of $K$, it suffices to construct a broken surface diagram...
of the trivial $c_i$-disk system $D_i = K \cap X_i$ in the half space

$$(X_i \setminus \{p_\infty\}) \cap \mathbb{R}^3 = \left\{(r \cos \theta, r \sin \theta, z, 0) \mid r \geq 0, \frac{2\pi j}{3} \leq \theta \leq \frac{2\pi(j + 1)}{3}\right\}$$

$$\cong \mathbb{R}^2 \times [0, \infty)$$

such that $D_i \cap (\mathbb{R}^2_+ \cup (-\mathbb{R}^2_+)) = D_i$. Then we will conclude that $\mathcal{D}(P) := D_1 \cup D_2 \cup D_3$ is a broken surface diagram of $K = D_1 \cup D_2 \cup D_3$ such that $\mathcal{D}(P) \cap \mathbb{R}^2_+ = P_{ij}$.

To obtain such $D_i$, we first take a sequence of Reidemeister moves and ambient isotopies of the plane

$$D_i = D_{i,0} \xrightarrow{m_1} D_{i,1} \xrightarrow{m_2} \cdots \xrightarrow{m_n} D_{i,n} = D_i'$$

such that all components of $D_i'$ simultaneously bound disks in the plane. Then, after fixing the identification $(X_i \setminus \{p_\infty\}) \cap \mathbb{R}^3 \cong \mathbb{R}^2 \times [0, \infty)$, we associate to each move $m_k$ a broken surface diagram $\mathcal{D}_{i,k}$ in $\mathbb{R}^2 \times [k-1,k]$ as follows: If $m_k$ is an ambient isotopy $F_k : \mathbb{R}^2 \times [0,1] \to \mathbb{R}^2$, then we define $\mathcal{D}_{i,k}$ as the image of the map

$$D_{i,k-1} \times [0,1] \to \mathbb{R}^2 \times [k-1,k], \ (x,t) \mapsto (F_k(x,t), t + k - 1).$$

If $m_k$ is a Reidemeister move, then we take a small disk $\delta$ in $\mathbb{R}^2$ so that $D_{i,k-1} \setminus \delta$ coincides with $D_{i,k} \setminus \delta$, and define $\mathcal{D}_{i,k}$ as the union of $(D_{i,k} \setminus \delta) \times [k-1,k]$ with one of the surfaces shown in Figure 1. For both cases, we have

$\mathcal{D}_{i,k} \cap (\mathbb{R}^2 \times \{k-1\}) = D_{i,k-1} \times \{k-1\}$ and $\mathcal{D}_{i,k} \cap (\mathbb{R}^2 \times \{k\}) = D_{i,k} \times \{k\}$.

Finally, we define $\mathcal{D}_{i,n+1}$ as mutually distinct disks smoothly embedded in $\mathbb{R}^2 \times [n,n+1]$ with boundary $D_{i,n} \times \{n\} = D_i' \times \{n\}$. Now we have a broken surface diagram

$$\mathcal{D}_i := \mathcal{D}_{i,0} \cup \cdots \cup \mathcal{D}_{i,n+1}$$

with boundary $D_i$. From the construction of $\mathcal{D}_i$, it is obvious that an embedded disks $D_i'$ in $X_i \setminus \{p_\infty\}$ whose projection image gives $\mathcal{D}_i$ are simultaneously
isotopic into $\partial X_i \setminus \{p_\infty\}$, and hence $D_i'$ is a trivial $c_i$-disk system with boundary $\partial D_i$. As shown in [7], any two trivial disk systems with the same boundary are isotopic, and hence $D_i'$ is isotopic to $D_i$. In particular, $D_i$ is regarded as a broken surface diagram of $D_i$. Now, we have a broken surface diagram $\mathcal{D}(P) = D_1 \cup D_2 \cup D_3$ of $K = D_1 \cup D_2 \cup D_3$.

Next, we give bijections between $\text{Col}_X(P)$ and $\text{Col}_X(\mathcal{D}(P))$. We first define a map

$$\varphi: \text{Col}_X(P) \rightarrow \text{Col}_X(\mathcal{D}(P))$$

as follows. Let $C = (C_{12}, C_{23}, C_{31}) \in \text{Col}_X(P)$. Note that $C_{ki}$ naturally induces a coloring $C_{ki}$ of $P_{ki}$, and $C_{ij} \cup C_{ki}$ defines an $X$-coloring of the diagram $D_i = D_{i,0}$. Moreover, each move $m_k$ and $X$-coloring of $D_{i,k}$ defines an $X$-coloring of $D_{i,k}$. Hence we have $X$-colorings of $D_{i,k} \times \{k\}$. Here we also note that any broken sheet of $\mathcal{D}$, has non-empty intersection to $\mathbb{R}^2 \times \{k\}$ for some $0 \leq k \leq n$. Now, we define $\varphi(C)$ so that the coloring of each broken sheet is the same as that of its intersection to $\bigcup_{0 \leq k \leq n} \mathbb{R}^2 \times \{k\}$. As shown in Figure 2 we can see that this assignment is well-defined as an $X$-coloring of $\mathcal{D}(P)$.

**Figure 2.** The coloring of each broken sheet is uniquely induced from its boundary.

We next define a map

$$\psi: \text{Col}_X(\mathcal{D}(P)) \rightarrow \text{Col}_X(P)$$

as follows. Let $C \in \text{Col}_X(\mathcal{D}(P))$. Regard $P_{ij}$ as embedded in $\mathcal{D}_i$, and then each arc of $P_{ij}$ is lying in a broken sheet of $\mathcal{D}_i$. We define $\psi(C)_{ij}$ so that the coloring of each arc is the same as that of the broken sheet containing the arc. Then, we can see that for each end point $p \in \partial P_{ij}$, the arcs in the $P_{ij}$ containing $p$ are lying in the same broken sheet of $\mathcal{D}$, and hence colored by the same element. Moreover, for each crossing $c$ in $P_{ij}$, the arcs near $c$ are lying in the broken sheets near a double point curve, and under-over information is preserved. These imply that $\psi(C) := (\psi(C)_{12}, \psi(C)_{23}, \psi(C)_{31})$ is a well-defined $X$-coloring of $\mathcal{D}$. Now, it is easy to see that both $\psi \circ \varphi$ and $\varphi \circ \psi$ are the identities, and hence these are bijections. □
By the same arguments as the proof of Proposition 2.4, we can also prove the following proposition.

**Proposition 2.5.** For any oriented tri-plane diagram \( P \) of an oriented surface link \( K \), there exists an oriented broken surface diagram \( \mathcal{B}(P) \) of \( K \) in \( \mathbb{R}^4 \) such that for any quandle \( Q \), we have a bijection

\[
\varphi: \text{Col}_Q(P) \rightarrow \text{Col}_Q(\mathcal{B}(P)).
\]

In particular, if \( Q \) is finite, then we have

\[
\#\text{Col}_Q(P) = \#\text{Col}_Q(K).
\]

2.4. **Proof of Theorem 1.2.** To prove Theorem 1.2, we use the following lemma.

**Lemma 2.6.** If a surface link \( K \) admits a \((b; c_1, c_2, c_3)\)-trisection \( T \), then for any finite kei \( X \), we have

\[
\min\{c_1, c_2, c_3\} \geq \log_{\#X}(\#\text{Col}_X(K)).
\]

**Proof.** We first note that for the \( c \)-component trivial classical link \( L_c \) and the finite kei \( X \), we have

\[
\#\text{Col}_X(L_c) = (\#X)^c.
\]

Therefore, it suffices to prove that the inequality

\[
\#\text{Col}_X(L_{c_i}) \geq \#\text{Col}_X(K)
\]

holds for any \( i \in \{1, 2, 3\} \). Let \( P = (P_{12}, P_{23}, P_{31}) \) be a tri-plane diagram of \( T \). Then, we can define a map \( \iota: \text{Col}_X(P) \rightarrow \text{Col}_X(L_{c_i}) \) by using the colorings \( \{C_{ij}, C_{ki}\} \) of the tangle diagrams \( \{P_{ij}, P_{ki}\} \) to color the classical link diagram \( P_{ij} \cup P_{ki} \). Here we note that any coloring of a tangle diagram of a trivial tangle is characterized by the colorings of the end points. In particular, for any \( C = (C_{12}, C_{23}, C_{31}) \in \text{Col}_X(P) \), the coloring \( C_{jk} \) of \( P_{jk} \) is uniquely determined by \( \{C_{ij}, C_{ki}\} \). This implies that \( \iota \) is injective, and hence the desired inequality holds.

**Proof of Theorem 1.2.** Suppose that \( K \) admits a \((b; c_1, c_2, c_3)\)-trisection. Then \( K \) has a cell decomposition consisting of \( 2b \) vertices, \( 3b \) edges and \( (c_1 + c_2 + c_3) \) faces. In particular, we have

\[
\chi(K) = 2b - 3b + (c_1 + c_2 + c_3) \geq 3\min\{c_1, c_2, c_3\} - b.
\]

Now, the desired inequality immediately follows from Lemma 2.6. \( \square \)
3. Kei colorings of twist spun knots

In this section, we consider the equality of Theorem 1.2 for the cases of twist spun knots and their stabilizations.

A 1-tangle diagram $T_K$ of a classical knot $K \subset S^3$ is a diagram of a tangle $K \setminus B \subset S^3 \setminus B$, where $B \subset S^3$ is a 3-ball such that $K \cap B$ is a trivial 1-tangle. In [1], Asami and Satoh gave a broken surface diagram $D_m(K)$ of the $m$-twist-spinning $S_m(K)$ of $K$ such that there is an embedding $\iota_m: T_K \to D_m(K).$

Then, for any kei $X$, the induced map $\iota_m^*: \text{Col}_X(D_m(K)) \to \text{Col}_X(T_K)$ is defined so that the coloring of each arc of $T_K$ is the same as that of the broken sheet containing the arc. Now, the next lemma directly follows from [1, Lemma 5.1]. Here we choose an end point of $T_K$ and call the arc containing the end point the terminal arc of $T_K$. (Note that since we consider kei colorings and $T_K$ is unoriented, we can choose any end point of $T_K$ for defining the terminal arc.)

**Lemma 3.1** ([1, Lemma 5.1]). The map $\iota_m^*$ is injective. Moreover, $C \in \text{Col}_X(T_K)$ belongs to $\iota_m^*(\text{Col}_X(D_m(K)))$ if and only if $C \ast a_m = C$ holds, where $a_-$ is the color of the terminal arc of $T_K$.

As a corollary of Lemma 3.1 we have the following proposition.

**Proposition 3.2.** We have the equality

$$\iota_m^*(\text{Col}_X(D_m(K))) = \begin{cases} \text{Col}_X(T_K) & (m \text{ is even}) \\ \{\text{trivial } X\text{-colorings of } T_K\} & (m \text{ is odd}) \end{cases}.$$

**Proof.** For any $a \in X$, the map $\ast a^2: X \to X$ is the identity, and hence we have the desired equality for the cases where $m$ is even. Next, for any odd $m$, we see that

$$\iota_m^*(\text{Col}_X(D_m(K))) = \{C \in \text{Col}_X(T_K) \mid C \ast a_m = C\}$$

$$= \{C \in \text{Col}_X(T_K) \mid C \ast a_- = C\}$$

$$= \iota_1^*(\text{Col}_X(D_1(K))).$$

Here we recall that $D_1(K)$ is a broken surface diagram of a 1-twist spun $S_1(K)$, which is trivial for any $K$; see [12]. In particular, $D_1(K)$ admits only trivial $X$-colorings, and this fact implies the desired equality for any odd $m$. \qed

Now we prove the following theorem, which can be regarded as a generalization of Theorem 1.3. Here we recall that $P$ is either one of the two trivially embedded $\mathbb{R}P^2$'s in $S^4$ (with normal Euler number $\pm 2$) and $T$ a trivially embedded torus in $S^4$.

**Theorem 3.3.** For a 1-tangle diagram $T_K$ of a classical knot $K$, suppose that there exists a finite kei $X$ with $b(K) = \log_{\#X}(\#\text{Col}_X(T_K))$. Then, for any $m \in \mathbb{Z}$, we have the following equalities:
• $b(S_{2m}(K)) = 3b(K) - 2$,
• $b(S_{2m}(K) \# P) = 3b(K) - 1$, and
• $b(S_{2m}(K) \# T) = 3b(K)$.

Proof. Since a $(3b(K) - 2)$-bridge trisection of $S_{2m}(K)$ is given in [8], we have $b(S_{2m}(K)) \leq 3b(K) - 2$. Moreover, it is shown in Figure 15 and 17 of [8] that $b(P) = 2$ and $b(T) = 3$, and hence it follows from [8, Subsection 2.2] that
\[ b(S_{2m}(K) \# P) \leq b(S_{2m}(K)) + b(P) - 1 \leq 3b(K) - 1 \]
and
\[ b(S_{2m}(K) \# T) \leq b(S_{2m}(K)) + b(T) - 1 \leq 3b(K). \]

Next, since $P$ and $T$ have broken surface diagrams with a single broken sheet, it is obvious that
\[ \text{#Col}_X(S_{2m}(K)) = \text{#Col}_X(S_{2m}(K) \# P) = \text{Col}_X(S_{2m}(K) \# T). \]
Moreover, Proposition 3.2 gives
\[ \text{#Col}_X(S_{2m}(K)) = \text{Col}_X(T_K). \]
Combining these equalities with Theorem [1.2] we have the opposite inequalities. □

Finally, we show that Theorem [1.3] follows from Theorem [3.3] Here, we say that a kei $X$ is faithful if for any two distinct elements $a, b \in X$, the bijections $\ast a$ and $\ast b$ are different maps. Then the following lemma is shown in [9, 4].

Lemma 3.4 ([9, Lemma 5.6], [4, Lemma 4.4]). For any faithful kei $X$, we have $\text{Col}_X(T_K) = \text{Col}_X(K)$.

Proof of Theorem [1.3]. Note that $R_p$ is faithful for any odd $p > 1$. Therefore, by Lemma 3.4 the condition $b(K) = \log_{\# R_p} (\# \text{Col}_{R_p}(T_K))$ is equivalent to $b(K) = \log_{\# R_p} (\# \text{Col}_{R_p}(K))$. Now, Theorem [1.3] immediately follows from Theorem [3.3]. □

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