Counterfactual Concealed Telecomputation

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Distributed computing is a fastest growing field—enabling virtual computing, parallel computing, and distributed storage. By exploiting the counterfactual techniques, we devise a distributed blind quantum computation protocol to perform a universal two-qubit controlled unitary operation for any input state without using preshared entanglement and without exchanging physical particles between remote parties. This distributed protocol allows Bob to counterfactually apply an arbitrary unitary operator to Alice’s qubit in probabilistic fashion, without revealing the operator to her, using a control qubit—called the counterfactual concealed telecomputation (CCT). It is shown that the protocol is valid for general input states and that single-qubit unitary teleportation is a special case of CCT. The quantum circuit for CCT can be implemented using the quantum Zeno gates and the protocol becomes deterministic with simplified circuit implementation if the initial composite state of Alice and Bob is a Bell-type state.

A nonlocal controlled unitary operation (CUO) is one of the fundamental building blocks in distributed quantum computing and quantum communications. Recently, it has been shown that any bipartite nonlocal unitary operation on a $d_A \times d_B$ dimensional quantum system can be implemented using at most $4d_A - 5$ nonlocal controlled unitary operators, regardless of $d_B$, where $d_A$ and $d_B$ denote the dimensions of quantum systems possessed by the remote parties Alice and Bob, respectively. In particular, two-qubit nonlocal controlled unitary operator plays an important role in distributed quantum computing as any $n$-qubit nonlocal unitary operation can be decomposed into a product of two-qubit nonlocal CUOs and single-qubit operations. Two-qubit nonlocal CUOs has been implemented using entanglement-assisted local operations and classical communication (LOCC) and quantum communications. In general, a two-qubit CUO can be represented as $U_c = I \otimes |0\rangle \langle 0| + U \otimes |1\rangle \langle 1|$, where $I$ is the single-qubit identity operator and $U$ is an arbitrary single-qubit unitary operator. To devise a two-qubit nonlocal CUO, $U_c$ can be further decomposed to

$$U_c = (A_1 \otimes B_1) \left( \sum_{kl} e^{i k \phi \theta} |k\rangle \langle l| \right) (A_2 \otimes B_2), \quad (1)$$

where $i = \sqrt{-1}$, $|k\rangle$ and $|l\rangle$ denote the computational basis of target and control qubits; and $A_1, A_2, B_1$ and $B_2$ are the single-qubit local unitary operators that depend on $U$. To date, nonlocal CUOs have been implemented using LOCC when the following two conditions are met: i) a sufficient amount of preshared entanglement is available and ii) $U$ is known to both Alice and Bob.

Blind quantum computation using entanglement-assisted LOCC is a unique capability enabled by quantum mechanics, which allows one party to use quantum computational resources of a remote party without revealing the input, computation, and output. Counterfactual quantum communication is another unique capability enabled by quantum mechanics, which allows remote parties to communicate information without exchanging physical particles. The counterfactuality was first introduced based on interaction-free measurement and, in addition to quantum communication, it has been successfully used in quantum computation and quantum cryptography.

This letter put forth a new type of blind quantum computation without using preshared entanglement and without exchanging physical particles between remote parties. A fundamental building block of blind quantum computation is a protocol that enables Bob to counterfactually apply an arbitrary unitary operator $U$ to Alice’s qubit in a probabilistic fashion, without revealing $U$ to Alice, using a control qubit. This is accomplished by decomposing the two-qubit CUO that corresponding to $U$ into global controlled flipping operations and local operations at the remote parties. The key features of this protocol are that i) the global controlled flipping operations are implemented in counterfactual way, and ii) both the global operations and Alice’s local operations are implemented in a way that $U$ is concealed from Alice. This protocol is called counterfactual concealed telecomputation (CCT). The counterfactual implementation of global controlled flipping operations using the quantum Zeno gates and chained quantum Zeno gates is given in the supplementary material. The protocol is shown to be valid for general input states, and single-qubit unitary teleportation can be seen as a special case of CCT. If the composite state of Alice and Bob is a Bell-type state, the protocol becomes deterministic and the quantum circuit for CCT can be simplified significantly.

The protocol.— In general, a single-qubit unitary operator $U$ can be represented as $U = R_z(\phi) R_y(\theta) R_x(\varphi)$ where $\phi$, $\theta$, and $\varphi$ are the Euler angles, and the rotation
To demonstrate the implementation of CCT (see Fig. 1), consider the arbitrary pure input states of Alice’s target qubit \(|\psi_A\rangle\) and Bob’s control qubit \(|\psi_B\rangle\) as follows:

\[
\begin{align*}
|\psi_A\rangle &= \alpha |0\rangle_A + \beta |1\rangle_A, \\
|\psi_B\rangle &= \gamma |0\rangle_B + \delta |1\rangle_B,
\end{align*}
\]

with the complex coefficients \(\alpha\), \(\beta\), \(\gamma\), and \(\delta\) satisfying \(|\alpha|^2 + |\beta|^2 = 1\) and \(|\gamma|^2 + |\delta|^2 = 1\). Bob starts the protocol by performing the local CNOT operation to entangle his qubit with the ancillary qubit \(|0\rangle_C\). Then, Alice and Bob have the separable composite state \(|\psi_{ABC}\rangle\) as follows:

\[
|\psi_{ABC}\rangle = |\psi_A\rangle (\gamma |00\rangle_{BC} + \delta |11\rangle_{BC}),
\]

where the subscript \(C\) denotes the ancillary qubit. Alice and Bob counterfactually apply Toffoli gate \(T\) to entangle Alice’s qubit with Bob’s and ancillary qubits. Toffoli gate, with Alice’s and ancillary qubits as control and

Bob’s qubit as a target, transforms the composite state \(|\psi_{ABC}\rangle\) as follows:

\[
\begin{align*}
|\psi_2\rangle_{ABC} &= \gamma |\psi_A\rangle_{ABC} |00\rangle_{BC} \\
&+ \delta \left( |\psi_A\rangle_{ABC} |01\rangle_{BC} + |\psi_A\rangle_{ABC} |10\rangle_{BC} \right) |1\rangle_C.
\end{align*}
\]

Bob applies the following local operation \(V_1(U)\) on his qubits:

\[
V_1(U) = V_{14}V_{13}V_{12}V_{11},
\]

where

\[
\begin{align*}
V_{11} &= I \otimes |0\rangle_C |0\rangle + (R_z(\phi) \mathbf{X}) \otimes |1\rangle_C |1\rangle \\
&+ I \otimes |2\rangle_C |2\rangle, \\
V_{12} &= |0\rangle_B |0\rangle \otimes I + |10\rangle_{BC} |10\rangle + |12\rangle_{BC} |11\rangle \\
&+ |11\rangle_{BC} |12\rangle, \\
V_{13} &= I \otimes |0\rangle_C |0\rangle + (R_z(\phi) R_y(\theta)) \otimes |1\rangle_C |1\rangle \\
&+ (R_z(\phi) R_y(\theta)) \otimes |2\rangle_C |2\rangle, \\
V_{14} &= I \otimes (|0\rangle_C |0\rangle + |1\rangle_C |1\rangle) + \mathbf{X} \otimes |2\rangle_C |2\rangle,
\end{align*}
\]

and \(\mathbf{X}\) denotes the Pauli \(x\) operator. Note that the dependence of \(V_1\) on \(U\) is through \(V_{11}\) and \(V_{13}\). It transforms the composite state \(|\psi_2\rangle_{ABC}\) as follows:

\[
\begin{align*}
|\psi_3\rangle_{ABC} &= \gamma |\psi_A\rangle_{ABC} |00\rangle_{BC} \\
+ \delta \alpha e^{-i(\phi + \gamma/2)} \cos (\theta/2) |01\rangle_{ABC} \\
+ \delta \alpha e^{-i(\phi - \gamma/2)} \sin (\theta/2) |01\rangle_{ABC} \\
+ \delta \beta e^{i(\phi + \gamma/2)} \cos (\theta/2) |10\rangle_{ABC} \\
- \delta \beta e^{i(\phi - \gamma/2)} \sin (\theta/2) |10\rangle_{ABC}.
\end{align*}
\]

Now, Alice and Bob counterfactually apply the following
FIG. 2. Deterministic CCT protocol for Bell-type states to counteractually apply an arbitrary unitary operator on the Alice’s qubit in concealed and controlled fashion without using additional preshared entanglement. Similar to CCT protocol followed by the Hadamard gate Bob applies the local operation \( \psi \) on his qubits:

\[
\begin{align*}
Q_1 &= I \otimes (|00\rangle_{BC}(00) + |01\rangle_{BC}(01)) \\
   &\quad + |10\rangle_{BC}(11) + |02\rangle_{BC}(02) \\
   &\quad + X \otimes (|11\rangle_{BC}(11) + |12\rangle_{BC}(12)) \\
Q_2 &= (|00\rangle_A(00) + |11\rangle_A(11)) \otimes I \otimes |01\rangle_C(01) \\
   &\quad + |11\rangle_A(11) \otimes I \otimes |11\rangle_C(11) \\
   &\quad + |00\rangle_A(00) \otimes X \otimes |12\rangle_C(12) \\
   &\quad + |11\rangle_A(11) \otimes X \otimes |22\rangle_C(22),
\end{align*}
\]

which transform the composite state \(|\psi_3\rangle_{ABC}\) as follows:

\[
\begin{align*}
|\psi_4\rangle_{ABC} &= \gamma |\psi\rangle_A |00\rangle_{BC} \\
   &\quad + \delta e^{-i(\varphi+\phi)/2} \cos(\theta/2) |01\rangle_{ABC} \\
   &\quad + \delta e^{-i(\varphi-\phi)/2} \sin(\theta/2) |11\rangle_{ABC} \\
   &\quad + \delta \beta e^{i(\varphi+\phi)/2} \cos(\theta/2) |12\rangle_{ABC} \\
   &\quad - \delta \beta e^{i(\varphi-\phi)/2} \sin(\theta/2) |02\rangle_{ABC}.
\end{align*}
\]

Bob applies the local operation \( V_2 \) on his qubits:

\[
V_2 = |0\rangle_B \otimes I \\
   &\quad + |1\rangle_B (1 \otimes (|0\rangle_C(1) + |1\rangle_C(2) + |2\rangle_C(0))),
\]

followed by the Hadamard gate \( H \) on the ancillary qubit to disentangle the ancillary qubit from Alice’s and Bob’s qubits. At the end of the protocol, Bob performs the measurement on the ancillary qubit in the computational basis and the composite state \( |\psi_4\rangle_{ABC} \) collapses to

\[
|\psi_{5m}\rangle_{AB} = \gamma |\psi\rangle_A |00\rangle_B \\
   &\quad + \delta e^{-i(\varphi+\phi)/2} \cos(\theta/2) |01\rangle_{AB} \\
   &\quad + \delta e^{-i(\varphi-\phi)/2} \sin(\theta/2) |11\rangle_{AB} \\
   &\quad + (-1)^m \delta \beta e^{i(\varphi+\phi)/2} \cos(\theta/2) |12\rangle_{AB} \\
   &\quad + (-1)^{1-m} \delta \beta e^{i(\varphi-\phi)/2} \sin(\theta/2) |02\rangle_{AB},
\]

where \( m \in \{0, 1\} \) is a measurement outcome with equal probability. Bob announces the measurement result with classical communication and apply the unitary operation \( Q_3 \) in counterfactual way where

\[
Q_3 = \begin{cases} 
I, & \text{for } m = 0, \\
(\vec{Z} \otimes \vec{X}) Z_c (\vec{I} \otimes \vec{X}), & \text{for } m = 1,
\end{cases}
\]

and \( Z_c \) is the controlled-\( Z \) operation and \( Z \) denotes the Pauli \( z \) operator. It transforms the composite state \( |\psi_{5m}\rangle_{AB} \) as follows:

\[
|\psi_{6m}\rangle_{AB} = \gamma |\psi\rangle_A |00\rangle_B \\
   &\quad + \delta e^{-i(\varphi+\phi)/2} \cos(\theta/2) |01\rangle_{AB} \\
   &\quad + (-1)^m \delta e^{-i(\varphi-\phi)/2} \sin(\theta/2) |11\rangle_{AB} \\
   &\quad + \delta \beta e^{i(\varphi+\phi)/2} \cos(\theta/2) |12\rangle_{AB} \\
   &\quad + (-1)^{1-m} \delta \beta e^{i(\varphi-\phi)/2} \sin(\theta/2) |02\rangle_{AB}.
\]

The equation (23) shows that Bob has successfully performed the CCT on Alice’s qubit.

For the protocol, if Bob sets the initial state of his qubit \( |\psi\rangle_B = |1\rangle_B \), then

\[
|\psi_{6m}\rangle_{AB} = (U_m |\psi\rangle_A) \otimes |1\rangle_B.
\]

Equation (24) shows that at the end of the protocol, the qubits of Alice and Bob are in a separable state, resulting in the unitary transformation of the arbitrary input state \( |\psi\rangle_A \) of Alice—namely, unitary teleportation.

**Bell-type states.** Consider that initial states of Alice and Bob are Bell-type states (see Fig. 2). In general, the Bell-type states are given as

\[
|\psi_0\rangle_{AB} = \begin{cases} 
\alpha |00\rangle_{AB} + \beta |11\rangle_{AB}, & \text{Bell-type states,} \\
\beta |01\rangle_{AB} + \alpha |10\rangle_{AB}, & \text{Bell-type states}
\end{cases}
\]

where

\[
\alpha = \cos \theta, \quad \beta = \sin \theta
\]

and

\[
|m\rangle_B = \begin{cases} 
|0\rangle_B, & \text{if } m = 0, \\
|1\rangle_B, & \text{if } m = 1.
\end{cases}
\]
where $| \psi_{AB}^{\pm} \rangle_{AB}$, $\ell = 0, 1$, are called the $\ell$-class states. Assume that Bob knows either the input is an 0-class or 1-class state. Similar to the general scheme, Bob starts the protocol by entangling his qubit with the ancillary qubit. Bob directly applies

$$
\hat{V}_1(U) = I \otimes |0\rangle_C |0\rangle + (X^{1-\ell} U X^\ell) \otimes |1\rangle_C |1\rangle,
$$

(26)

for the $\ell$-class states. Now Alice and Bob counterfactually apply

$$
\hat{Q}_1 = I \otimes (|00\rangle_{BC} |0\rangle + |01\rangle_{BC} |1\rangle + |10\rangle_{BC} |0\rangle + |11\rangle_{BC} |1\rangle),
$$

(27)

followed by

$$
\hat{Q}_2 = (|0\rangle_A |0\rangle + |1\rangle_A |1\rangle) \otimes I \otimes |0\rangle_C |0\rangle + |1\rangle_A |1\rangle \otimes X^{1-\ell} |0\rangle_C |1\rangle + |0\rangle_A |0\rangle \otimes X^\ell |1\rangle_C |1\rangle,
$$

(28)

for the $\ell$-class states. Bob applies the local CNOT operation at his qubits to disentangle the ancillary qubit and the composite state is given as

$$
|\psi\rangle_{ABC} = |\psi\rangle_{AB} \otimes |0\rangle_C,
$$

(29)

where

$$
|\psi\rangle_{AB} = (I \otimes |0\rangle_B |0\rangle + U \otimes |1\rangle_B |1\rangle) |\psi\rangle_{AB}.
$$

(30)

As the ancillary qubit is already in a separable state with qubits $|\psi\rangle_{AB}$, Bob does not need to perform measurement on his ancillary qubit.

In summary, we devise a new protocol for distributed quantum computing that allows Alice and Bob to apply a two-qubit CUO on any input state in a probabilistic fashion without using preshared entanglement, without revealing the unitary operator to Alice, and without exchanging physical particles between remote parties. This protocol enables in some respect Bob to counterfactually perform teleportation on Alice’s qubit in a concealed fashion. While this paper do not discuss the security of the protocol, future work may consider suitable variations of the current protocol for the cryptography tasks.

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[1] J. I. Cirac, A. K. Ekert, S. F. Huelga, and C. Macchiavello, Phys. Rev. A 59, 4249 (1999).
[2] W. Dai, T. Peng, and M. Z. Win, IEEE J. Sel. Areas Commun. 38, 540 (2020), special issue on Advances in Quantum Communications, Computing, Cryptography and Sensing.
[3] A. S. Cacciapuoti, M. Caleffi, F. Tafuri, F. S. Cataliotti, S. Gherardini, and G. Bianchi, IEEE Network 34, 137 (2019).
[4] J. Eisert, K. Jacobs, P. Papadopoulos, and M. B. Plenio, Phys. Rev. A 62, 052317 (2000).
[5] A. S. Fletcher, P. W. Shor, and M. Z. Win, IEEE Trans. Inf. Theory 54, 5705 (2008).
[6] M. Chiani, A. Conti, and M. Z. Win, Phys. Rev. A 102, 012410 (2020).
[7] L. Chen and L. Yu, Phys. Rev. A 91, 032308 (2015).
[8] A. Barenco, C. H. Bennett, R. Cleve, D. P. DiVincenzo, N. Margolus, P. Shor, T. Sleator, J. A. Smolin, and H. Weinfurter, Phys. Rev. A 52, 3457 (1995).
[9] L. Chen and L. Yu, Phys. Rev. A 89, 062326 (2014).
[10] A. Soeda, P. S. Turner, and M. Murao, Phys. Rev. Lett. 107, 180501 (2011).
[11] K. Lenn, K. Bartkiewicz, A. Černoch, M. Dušek, and J. Soubusta, Phys. Rev. Lett. 114, 153602 (2015).
[12] V. Giovannetti, L. Maccone, T. Morimae, and T. G. Rudolph, Phys. Rev. Lett. 111, 230501 (2013).
[13] C. A. Pérez-Delgado and J. F. Fitzsimons, Phys. Rev. Lett. 114, 220502 (2015).
[14] J. F. Fitzsimons, npj Quantum Inf. 3, 1 (2017).
[15] H. Salih, Z.-H. Li, M. Al-Amri, and M. S. Zubaïry, Phys. Rev. Lett. 110, 170502 (2013).
[16] Y. Aharonov and L. Vaidman, Phys. Rev. A 99, 010103 (2019).
[17] A. Elitzur and L. Vaidman, Found. Phys. 23, 987 (1993).
[18] P. Kwiat, H. Weinfurter, T. Herzog, A. Zeilinger, and M. A. Kasevich, Phys. Rev. Lett. 74, 4763 (1995).
[19] O. Hosten, M. T. Rakher, J. T. Barreiro, N. A. Peters, and P. G. Kwiat, Nature 439, 949 (2006).
[20] F. Kong, C. Ju, P. Huang, P. Wang, X. Kong, F. Shi, L. Jiang, and J. Du, Phys. Rev. Lett. 115, 080501 (2015).
[21] T.-G. Noh, Phys. Rev. Lett. 103, 230501 (2009).
[22] H. Salih, Phys. Rev. A 90, 012333 (2014).
[23] W. M. Itano, D. J. Heinzen, J. J. Bollinger, and D. J. Wineland, Phys. Rev. A 41, 2295 (1990).
[24] F. Zaman, Y. Jeong, and H. Shin, Sci. Rep. 8, 14641 (2018).
[25] F. Zaman, Y. Jeong, and H. Shin, Sci. Rep. 9, 11193 (2019).
[26] F. Zaman, H. Shin, and M. Z. Win, arXiv:1910.03200 (2019).
[27] See Supplementary Material for counterfactual implementation of global operations.
[28] Y.-F. Huang, X.-F. Ren, Y.-S. Zhang, L.-M. Duan, and G.-C. Guo, Phys. Rev. Lett. 93, 240501 (2004).
[29] S. F. Huelga, J. A. Vaccaro, A. Chefles, and M. B. Plenio, Phys. Rev. A 63, 042303 (2001).
Supplementary Material: Counterfactual Concealed Telecomputation

PRELIMINARIES

The counterfactual quantum communication [S1 S2] is based on the single-particle nonlocality and quantum measurement theory. A quantum state usually collapses back to its initial state if the time between repeated measurements is short enough [S3]. This quantum Zeno (QZ) effect has been demonstrated to achieve the interaction-free measurement (IFM) where the the state of a photon acts as an unstable quantum state corresponding to the presence of the absorptive object [S4]. We begin with a brief review on the overall actions of the QZ and chained QZ (CQZ) gates [S5 S6] that are invoked to devise CCT.

QZ Gates

Fig. S1 shows the Michelson version of the QZ gate [S2] to perform IFM. The QZ gate is to ascertain the classical behavior of an absorptive object, i.e., to infer the absence state \(|0\rangle\) or the presence state \(|1\rangle\) of AO without interacting with it. The H(V)-QZ gate takes an H(V) polarized photon as input. The switchable mirror SMN is initially turned off to allow passing the photon and is turned on for N cycles once the photon is passed. After N cycles, SMN is turned off again allowing the photon to arrive at the detector. The polarization rotator PRNHV gives rotation to the input photon by an angle \(\theta_N = \pi/(2N)\) as follows:

\[
PR_N^{HV} = \begin{pmatrix}
|H(V)\rangle_p \to \cos \theta_N |H(V)\rangle_p + \sin \theta_N |V(H)\rangle_p, \\
|V(H)\rangle_p \to \cos \theta_N |V(H)\rangle_p - \sin \theta_N |H(V)\rangle_p.
\end{pmatrix}
\] (S1)

The photon state \(|\phi\rangle\) after \(PR_N^{HV}\) in the first cycle of the H(V)-QZ gate is given by

\[
|\phi\rangle = \cos \theta_N |H(V)\rangle_p + \sin \theta_N |V(H)\rangle_p. \quad (S2)
\]

Then, the polarizing beam splitter PBS separates the H and V components of the photon into two different optical paths: SM \(\to MR_1\) and SM \(\to MR_2\). The H (V) component goes towards MR1 and the V (H) component goes towards MR2. The photon component in the second optical path only interacts with AO (control terminal).

- AO = \(|0\rangle_{AO}\): In the absence of the absorptive object, the V (H) component of the photon is reflected by MR2 and is returned back to PBS. Hence, the photon state remains unchanged. After \(n (< N)\) cycles, the photon state is given by

\[
|\phi\rangle = \cos (n\theta_N) |H(V)\rangle_p + \sin (n\theta_N) |V(H)\rangle_p. \quad (S3)
\]

The photon will end up in the state \(|V(H)\rangle_p\) with certainty by \(\pi/2\) rotation after N cycles.

- AO = \(|1\rangle_{AO}\): In the presence of the absorptive object, the V (H) component is absorbed by AO if it is found in the control terminal. In each cycle, the probability of this absorption event is equal to \(\sin^2 \theta_N\). Unless the photon is absorbed, the photon state collapses to the initial state \(|H(V)\rangle_p\). After N cycles, the photon is not absorbed and ends up in the state \(|H(V)\rangle_p\) with probability \(\cos^{2N} \theta_N\) tending to one as \(N \to \infty\).

Table SII shows the overall action of the QZ gate. Note that the H(V)-QZ gate has the output \(|H(V)\rangle_p\) in the presence state \(|1\rangle_{AO}\) if the photon has not traveled over the control terminal (quantum channel). Hence, the QZ gate is counterfactual only for this measurement outcome.

CQZ Gates

Fig. S2 shows the nested version of the QZ gates with M outer and N inner cycles [S24]. The CQZ gate enables to ascertain the absence or presence of the absorptive object counterfactually for both the outcomes. The H(V)-CQZMN gate also takes an H (V) polarized photon as input. In each outer cycle, V (H) component of the photon enters the inner V(H)-QZN gate.

- AO = \(|0\rangle_{AO}\): In the absence of the absorptive object, the inner V(H)-QZ gate transforms the photon state \(|V(H)\rangle_p\) into \(|H(V)\rangle_p\) after N cycles. This component ends up at the detector D after PBS. Hence, the inner QZ gate acts as an absorptive object for the outer QZ gate in the absence state \(|0\rangle_{AO}\) where D serves to detect the event that the photon is found in the control terminal. In each outer cycle, unless the photon is discarded, the photon state collapses back to the initial state.
TABLE I. H(V)-QZ\textsubscript{N} and H(V)-CQZ\textsubscript{M,N} gates.

| Input   | Control   | QZ Gate | CQZ Gate |
|---------|-----------|---------|----------|
| | | Output | Probability | Counterfactuality | Output | Probability | Counterfactuality |
| | | H(V)\textsubscript{p} | | | H(V)\textsubscript{p} | \lambda_0 | Yes |
| | | \cos^2 N \theta_N | | | \cos^2 N \theta_N | | |
| | | | | | | |
| | | | | | | |

Note that the CQZ gate is counterfactual for both the outcomes and infers the absence or presence of the absorptive object (with probability \lambda_0 or \lambda_1) but no physical particle (photon) is found in the control terminal (see Table I).

CCT IMPLEMENTATION USING QZ AND CQZ GATES

As shown in Fig. S3, an electron as a quantum absorptive object takes superposition of two paths |↑\rangle_e and |↓\rangle_e where the subscript e denotes the electron. In type I (Fig. S3(a)), the electron state |↑\rangle_e or |↓\rangle_e acts as the presence state |1\rangle_{AO} or the absence state |0\rangle_{AO} of the absorptive object for the H(V)-QZ\textsubscript{N} gate. For the counterfactuality of the protocols, we setup four gates:

(i) counterfactual electron-photon interaction (CEPI) gates using the H(V)-QZ\textsubscript{N} gate;
(ii) dual CEPI (D-CEPI) gates using the dual QZ (DQZ) gate;
(iii) distributed controlled flipping operation (DCFO) gates using the CEPI gates;
(iv) dual DCFO (D-DCFO) gates using the D-CEPI gates.

From the above list of gates, the second (first) and forth (third) gates use to devise the CCT for general (Bell-type) input states.

CEPI Gates

Fig. S4 shows the H(V)-CEPI\textsubscript{N} gate where the quantum absorptive object is in the superposition state

\[ |\text{electron}\rangle_e = \alpha |\uparrow\rangle_e + \beta |\downarrow\rangle_e, \] (S7)

with |\alpha|^2 + |\beta|^2 = 1. The H(V)-CEPI\textsubscript{N} gate collapses this quantum state by entangling and disentangling the electron-photon pair

\[ |\eta_0\rangle_{ep} = |\text{electron}\rangle_e |\text{H (V)}\rangle_p \] (S8)
shows the dual form of the H(V)-CEPI gate. The electron state is in an erasure state orthogonal to |↑⟩ and |↓⟩. In type I, the electron states |↑⟩_e and |↓⟩_e act as the presence (absence) state |1⟩_AO and the absence (presence) state |0⟩_AO of the absorptive object for the H(V)-QZ gate, respectively. In type II, the electron states simply act as |↑⟩_e = |0⟩_AO and |↓⟩_e = |1⟩_AO for the CQZ gate. If the photon is absorbed by the electron, the electron state is in an erasure state orthogonal to |↑⟩_e and |↓⟩_e.

The second (first) term of |η1⟩_ep is the outcome corresponding to the electron in the absence state for the H(V)-QZ_N gate. Since this outcome is not counterfactual, it is discarded (absorbed) by the electron using the PBS_H(V) and the X operator. To discard the factual (non-counterfactual) outcome |V(H)⟩_p of the H(V)-QZ_N gate, PBS_H(V) redirects this photon component to the quantum absorptive object (followed by the X operator) to be absorbed by the electron. Hence, whenever the photon is found in the quantum channel, the electron absorbs it and becomes in an erasure state, leading the H(V)-CEPI_N gate to output no photon and electron (e.g., particles in the erasure state). This enables the protocol to abort nonlocally by discarding both the photon and the electron whenever its counterfactuality is broken.

**D-CEPI Gates**

Fig. S5 shows the dual form of the H(V)-CEPI_N gate in Fig. S4. For counterfactuality, this D-CEPI_N gate works similarly to the H(V)-CEPI_N gate. The only difference is that the superposition polarization state

\[ |\text{photon}\rangle_p = \gamma |H]\rangle_p + \delta |V]\rangle_p \]  

(S12)

of the input photon is entangled with the ancillary path state in the DQZ_N gate as follows:

\[ |\text{photon}\rangle_{pa} = \gamma |H0\rangle_{pa} + \delta |V1\rangle_{pa} \]  

(S13)

where the ancilla states |0⟩_a and |1⟩_a show the paths for the H- and V-QZ gates, respectively. The D-CEPI_N gate transforms the electron-photon pair

\[ |\eta\rangle_{epa} = |\text{electron}\rangle_e |\text{photon}\rangle_{pa} \]  

(S14)
Alice performs the second and third steps in the protocol, which allows the electron to be recombined after K successive H(V)-CEPI \(_{N}\) operations. Bob performs the B operator on the V component of the photon, which allows the electron to pass and detour the H component to be recombined. The first H(V)-CEPI \(_{N}\) gate transforms the electron-photon pair 

\[
|\eta_{0}\rangle_{ep} \rightarrow |\eta_{1}\rangle_{ep} = \alpha|\uparrow H0\rangle_{ep} + \beta|\downarrow V0\rangle_{ep}
\]

\[
+ \alpha\delta|\uparrow H1\rangle_{ep}
\]

\[
+ \beta\delta|\downarrow V1\rangle_{ep},
\]

(15)

unless the photon is absorbed by the electron with probability

\[
(1 - \nabla_{1} \sin^{2} \theta_{N})^{N} \nabla_{1},
\]

(17)

where

\[
\nabla_{1} = |\alpha\gamma|^{2} + |\beta\delta|^{2}
\]

is the probability that the electron is in the presence state for the QZ gates in both paths.

**DCFO Gates**

To devise the DCFO gate, K H(V)-CEPI \(_{N}\) gates are concatenated serially where Alice has the quantum absorptive object (electron) and Bob equips the QZ gates (see Fig. S5). To explain the operation of the DCFO \(_{K,N}\) gate, consider the Bell-type state of electron-photon pair

\[
|\eta_{0}\rangle_{ep} \rightarrow |\eta_{1}\rangle_{ep} = \alpha|\uparrow H0\rangle_{ep} + \beta|\downarrow V0\rangle_{ep},
\]

(18)

where \(|\eta_{0}\rangle_{ep}\) and \(|\eta_{1}\rangle_{ep}\) are called 0- and 1-class states. The H(V)-DCFO \(_{K,N}\) protocol for Bell-type states of 1(0)-class takes the following steps:

1. Bob starts the H(V)-DCFO \(_{K,N}\) protocol by throwing his photon towards PBS\(_{V}\), which allows the V component to pass and detour the H component to be recombined after K successive H(V)-CEPI \(_{N}\) operations. Bob performs the B operator on the V component of the photon where \(B = X^{T}\) for \(\ell\)-class states.

2. Alice performs \(R_{y}(2\theta_{K})\) on her qubit (electron) where \(\theta_{K} = \pi/(2K)\). The rotation gate \(R_{y}(2\theta_{K})\) transforms \(|\uparrow\rangle_{e}\) and \(|\downarrow\rangle_{e}\) as follows:

\[
|\uparrow\rangle_{e} \rightarrow \cos \theta_{K} |\uparrow\rangle_{e} + \sin \theta_{K} |\downarrow\rangle_{e},
\]

(20)

\[
|\downarrow\rangle_{e} \rightarrow \cos \theta_{K} |\downarrow\rangle_{e} - \sin \theta_{K} |\uparrow\rangle_{e}.
\]

(21)

3. Bob inputs the H (V) component to the H(V)-CEPI \(_{N}\) gate. The first H(V)-CEPI \(_{N}\) gate transforms the \(|\eta_{0}\rangle_{ep}\) as follows:

\[
|\eta_{0}\rangle_{ep} \rightarrow \left\{\begin{array}{ll}
|\eta_{00}\rangle_{ep} & = \alpha|\uparrow H0\rangle_{ep} + \beta|\downarrow V0\rangle_{ep}, \\
|\eta_{01}\rangle_{ep} & = \gamma|\uparrow H1\rangle_{ep} + \delta|\downarrow V1\rangle_{ep},
\end{array}\right.
\]

(22)

unless the photon is absorbed by the electron with probability

\[
\nabla_{2} = (1 - \nabla \cos^{2} \theta_{K} \sin^{2} \theta_{N})^{N}
\]

(23)

where \(\nabla = |\gamma|^{2} (|\beta|^{2})\). Whenever the photon is found in the transmission channel between Alice and Bob, the electron absorbs it and the protocol declares an erasure.

4. Alice and Bob repeat the second and third steps for subsequent H(V)-CEPI \(_{N}\) gates. After K H(V)-CEPI \(_{N}\) gates, unless the photon is absorbed by the electron, the electron absorbs it and the protocol terminates.
Now, Alice performs

... 

Alice and Bob repeat the second step for subsequent D-CEPI gates. After K D-CEPI gates, unless the photon is absorbed by the electron with probability

\[ \nabla_5 = (1 - \nabla_4 \cos^2 \theta_K \sin^2 \theta_N)^N (1 - \nabla_4 \sin^2 \theta_K), \]

where \( \nabla_4 = |\beta|^2 + |\delta|^2 \).

3. Alice and Bob repeat the second step for subsequent D-CEPI gates. After K D-CEPI gates, unless the photon is absorbed by the electron with probability

\[ \nabla_6 = \nabla_5^K, \]

Bob performs the \( X \) operator on the component of the photon in path state \( |0\rangle_a \) and recombines the respective components of the photons. Then, the composite state of electron-photon pair transforms to

\[ |\eta_2\rangle_{epa} = \frac{1}{\sqrt{2}} (-\alpha |\downarrow H\rangle_{ep} + \beta |\uparrow V\rangle_{ep}) |0\rangle_a \]

\[\]
Bob applies the component of the photon in path 

\[ |0\rangle_B = |\uparrow\rangle_p, \]

\[ |1\rangle_B = |H\rangle_p, \]

\[ |0\rangle_B = |V\rangle_p. \]

To devise the CCT, Alice and Bob takes the following steps (see Fig. S3).

1. Bob starts the protocol by throwing his photon towards PBS\(^H\) to entangle the polarization (control qubit) state \(|\psi\rangle_B\) with path state \(|0\rangle_C\). Then Alice and Bob has the composite state \(|\psi_1\rangle_{ABC}\).

2. Bob detours the H component of the photon to recombine it at the end of the protocol and inputs the V component of the photon to the V-CQZ\(M,N\) gate. It transforms \(|\psi_1\rangle_{ABC}\) to \(|\psi_2\rangle_{ABC}\), unless the photon is absorbed by the electron or discarded at the detector in V-CQZ\(M,N\) gate with probability

\[
\lambda_2 = (1 - |\alpha\delta|^2 \sin^2 \theta_M)^M \prod_{i=1}^{M} [1 - |\beta\delta|^2 \sin^2 (i\theta_M) \sin^2 \theta_N]^N. \tag{S32}
\]

Tending to one as \(M, N \to \infty\).

3. Bob applies the \(B_1 = R_e(\phi) X\) operation followed by PBS\(^H\) on the component of the photon in path state \(|1\rangle_C\).

4. Bob applies \(B_2 = R_e(\theta) R_c(\phi) X B_2\) operators on the photon components in the path states \(|1\rangle_C\) and \(|2\rangle_C\), respectively. Then, the composite state of Alice and Bob transforms to \(|\psi_3\rangle_{ABC}\).

5. Bob inputs the components of the photon in the path states \(|1\rangle_C\) and \(|2\rangle_C\) to the D-DCFO\(K,N\) gate.

From (S26) (S30), the composite state of Alice and Bob transforms to

\[
|\psi_4\rangle_{ABC} = \gamma |\psi\rangle_B |00\rangle_{BC} + \delta e^{-i(\varphi - \phi)/2} \sin (\theta/2) |101\rangle_{ABC} + \delta e^{-i(\varphi + \phi)/2} \cos (\theta/2) |011\rangle_{ABC} - \delta e^{i(\varphi - \phi)/2} \sin (\theta/2) |112\rangle_{ABC}, \tag{S33}
\]

unless the photon is absorbed by the electron in D-DCFO\(K,N\) gate with probability

\[
\lambda_3 = (1 - |\delta|^2 \sin^2 (\theta/2) \cos \theta_K \sin \theta_N)^{KN} \tag{S34}
\]

\[
(1 - |\delta|^2 \sin^2 (\theta/2) \sin \theta_K)^{K}.
\]

6. Alice applies the \(X\) followed by the \(Z\) on her qubit and Bob performs \(XZ\) \(X\) operation on the path state \(|2\rangle_C\), respectively.

7. Bob inputs the component of the photon in path state \(|1\rangle_C\) to the H-CQZ\(M,N\) gate and \(|2\rangle_C\) to the V-CQZ\(M,N\) gate, respectively. Bob applies \(X\) operator on the component of photon in path state \(|1\rangle_C\) and the composite of Alice and Bob transforms to \(|\psi_4\rangle_{ABC}\), unless the photon is discarded in the CQZ\(M,N\) gates with probability \(S7\)

\[
\lambda_4 = (1 - \nabla_7 \sin^2 \theta_M)^M \prod_{i=1}^{M} [1 - \nabla_8 \sin^2 (i\theta_M) \sin^2 \theta_N]^N. \tag{S35}
\]

where \(\nabla_7\) and \(\nabla_8\) as defined as:

\[
\nabla_7 = |\lambda|^2 \cos^2 (\theta/2) + |\lambda|^2 \sin^2 (\theta/2), \tag{S36}
\]

\[
\nabla_8 = |\lambda|^2 \cos^2 (\theta/2) + |\lambda|^2 \sin^2 (\theta/2). \tag{S37}
\]
FIG. S8. A CCT protocol for general input states using QZ and CQZ gates. Here $B_1 = R_x(\phi) X$ and $B_2 = R_x(\phi) R_y(\theta)$ where $R_x(\phi)$ denotes the rotation around z-axis.

To disentangle the ancillary qubit, Bob applies the local operation $V_2$ followed by the Hadamard gate $H$ on the ancillary qubit and performs the measurement on the ancillary qubit in the computational basis.

9. Measurement outcome $m = 0$: Unless the photon is discarded in the complete process with probability

$$\lambda_5 = \lambda_2 \lambda_3 \lambda_4,$$

the composite state of Alice and Bob collapses to $|\psi_0\rangle_{AB}$.

10. Measurement outcome $m = 1$: Alice applies the $Z$ operator on her qubit and Bob applies $X$. Bob throws his photon towards PBS$^H$ as shown in Fig. S9. After the V-CQZ$_{2M,N}$ gate, Bob recombines the H and V components of the photon followed by applying the $X$ operator on his qubit. Unless the photon is discarded in the protocol with probability

$$\lambda_6 = \lambda_5 \left(1 - |\alpha \gamma|^2 \sin^2 \theta_M\right)^{2M} \prod_{i=1}^{2M} \left[1 - |\beta \gamma|^2 \sin^2 (i\theta_M) \sin^2 \theta_N\right]^N,$$

the composite state of the Alice and Bob transforms to $|\psi_{61}\rangle_{AB}$.

### CCT for Bell-Type Input States

To demonstrate the implementation of CCT for Bell-type states, consider that the composite input state of Alice and Bob is $|\psi_0\rangle_{AB}$. Similar to the general setup, Bob starts the protocol by entangling his qubit with the ancillary qubit $|0\rangle_C$ by throwing his photon towards PBS$^H$ as shown in Fig. S10. The CCT protocol for Bell-type states of 1(0)-class takes the following steps.

1. Bob applies $C_1$ on the component of the photon in
path state $|1\rangle_C$ where $C_1 = X^{1-\ell}U X^\ell$ for $\ell$-class states.

2. Bob inputs the component of the photon in path state $|1\rangle_C$ to the H(V)-DCFO$_{K,N}$ gate. From (S19)-(S25), the H(V)-DCFO$_{K,N}$ gate transforms the composite of Alice and Bob to

$$|\psi'_0\rangle_{ABC} = \alpha |100\rangle_{ABC} \pm \beta \left(-e^{i(\varphi+\phi)/2} \cos(\theta/2) |001\rangle_{ABC} - e^{i(\varphi-\phi)/2} \sin(\theta/2) |111\rangle_{ABC}\right),$$

for 1 (0)-class states

3. Alice applies the $X$ followed by the $Z$ on her qubit and Bob performs $C_2 = XZ^{1-\ell}X^{1-\ell}$ for $\ell$-class states on the component of the photon in path state $|1\rangle_C$, respectively.

4. Bob inputs the component of the photon in path state $|1\rangle_C$ to the V-CQZ$_{M,N}$ gate and recombines the component of the photon in path state $|0\rangle_C$ and $|1\rangle_C$ after the V-CQZ$_{M,N}$ gate. Unless the photon is discarded in the whole process with probability

$$\lambda_8 = \lambda_7 \left(1 - \nabla_{9(10)} \sin^2 \theta_M \right)^M \prod_{i=1}^M \left[1 - \nabla_{10(9)} \sin^2 (i\theta_M) \sin^2 \theta_N \right]^N$$

It transforms the composite state of Alice and Bob to $|\psi\rangle_{ABC}$.

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[S1] H. Salih, Z.-H. Li, M. Al-Amri, and M. S. Zubairy, Phys. Rev. Lett. 110, 170502 (2013).

[S2] Y. Aharonov and L. Vaidman, Phys. Rev. A 99, 010103 (2019).

[S3] W. M. Itano, D. J. Heinzen, J. J. Bollinger, and D. Wineland, Phys. Rev. A 41, 2295 (1990).

[S4] P. Kwiat, H. Weinfurter, T. Herzog, A. Zeilinger, and M. A. Kasevich, Phys. Rev. Lett. 74, 4763 (1995).

[S5] F. Zaman, Y. Jeong, and H. Shin, Sci. Rep. 8, 14641 (2018).

[S6] F. Zaman, Y. Jeong, and H. Shin, Sci. Rep. 9, 11193 (2019).

[S7] In the presence of the absorptive object, the H(V)-CQZ$_{M,N}$ gate transforms the V (H) polarized photon to -H (-V).