Lower central subgroups of a free group and its subgroup

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Abstract. For a given free group $F$ of arbitrary rank (possibly infinite), and its subgroup $G$, we address the question whether a lower central subgroup of $G$ can contain a lower central subgroup of $F$. We show that the answer is no if $G$ does not normally generate $F$. The question comes from a study of Hirzebruch-type invariants from iterated $p$-covers for 3-dimensional homology cylinders.

1. Introduction

For a group $G$, denote by $G_m$ the $m$th term of the lower central series of $G$, defined inductively by $G_1 = G$, $G_{m+1} = [G_m, G]$ for each $m \geq 1$.

If $F$ is a free group and $G$ its subgroup, then it is obvious that $G_m$ is contained in $F_m$ for every $m \geq 1$. In this paper, we investigate the converse relation: whether some $F_m$ is contained in some $G_k$. Note that $\bigcap_m F_m = 0$. For $k = 1$, if $G$ is a normal subgroup of $F$ with abelian $F/G$, then $G_1$ contains $F_m$ for every $m \geq 2$. We can ask if a subgroup $G$ satisfies $G_2 \supseteq F_m$ for a certain large $m$. As an answer, we prove the following result:

**Theorem 1.1.** Let $F$ be a free group and $G$ a subgroup of $F$ whose normal closure is not $F$. Then $G_2$ never contains $F_m$ for any $m \in \mathbb{N}$.

This starts from a study of structures of geometric objects. Let $\Sigma_{g,n}$ be a compact oriented surface of genus $g$ with $n$ boundary components. A homology cylinder over $\Sigma_{g,n}$ is defined as a homology cobordism between two copies of $\Sigma_{g,n}$. The set $\mathcal{H}_{g,n}$ of homology cobordism classes of homology cylinders becomes a group under juxtaposition. The group was introduced as an enlargement of the mapping class group by Garoufalidis and Levine [GL05, Lev01]. It is also a generalization of the concordance group of framed string links.

In [So16], the author studied the structure of $\mathcal{H}_{g,n}$ by defining extended Milnor invariants and Hirzebruch-type invariants for homology cylinders. Throughout this paper, $p$ denotes a prime number. Hirzebruch-type intersection form defects associated to $p$-fold covers are defined by Cha in [Cha10] to study homology cobordism of closed 3-manifolds and concordance of links. Let $d$ be a power of $p$. For a CW-complex $X$, a pair of a cover $\tilde{X}$ obtained by taking $p$-covers repeatedly and a homomorphism $\pi_1(\tilde{X}) \to \mathbb{Z}_d$ is called a ($\mathbb{Z}_d$-valued) $p$-structure for $X$. Here, a $p$-cover means a cover of $p$-power degree. The invariant of a $p$-structure for a 3-manifold is the difference between the Witt classes of the $\mathbb{Q}(\zeta_d)$-valued intersection form and the ordinary intersection form of a 4-manifold bounded by $\tilde{X}$ over $\mathbb{Z}_d$, where $\zeta_d = \exp(2\pi\sqrt{-1}/d)$. This lives in the Witt group $L^0(\mathbb{Q}(\zeta_d))$ of nonsingular hermitian forms over $\mathbb{Q}(\zeta_d)$. 

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The invariants give rise to invariants of a subgroup of string link concordance group, consisting of $\hat{F}$-string links [Cha09]. We refer [So16, p.897] for the definition of $\hat{F}$-string link. Remark that $\hat{F}$-(string) links form the largest known class of (string) links with vanishing Milnor invariants; it is a big open problem in link theory whether all (string) links with vanishing Milnor invariants are $\hat{F}$-(string) links. It turned out that the Hirzebruch-type invariants are homomorphisms on the subgroup of $\hat{F}$-string links.

In [So16], a Hirzebruch-type invariant $\lambda_T$ is defined for homology cylinders with a $p$-structure $T$ for $\Sigma_{g,n}$, or equivalently for $\sqrt{2g+n-1} S^1$. The $p$-structures are classified in [So16]. Let $(\hat{X}, \phi; \pi_1(\hat{X}) \to \mathbb{Z}_d)$ be a $p$-structure for $X$. For the cover $\hat{X}$ induced by $\phi$, if $\pi_1\hat{X} \supseteq (\pi_1X)_m$, the $p$-structure is said to be of order $m$. Every $p$-structure of a finite $CW$-complex is of order $m$ for some finite $m$; For the proof, see [So16, Lemma 5.3]. We revealed that when the invariant is defined; For a $p$-structure $T$ of order $m$, the invariant $\lambda_T$ is defined for (the homology cobordism class of) a homology cylinder if and only if the homology cylinder has vanishing extended Milnor invariants of length $m$. Let $\mathcal{H}_{g,n}(m)$ be the subgroup of $\mathcal{H}_{g,n}$ consisting of homology cylinders with vanishing extended Milnor invariants of length $m$ in $\mathcal{H}_{g,n}$. For a $p$-structure $T$ for $\Sigma_{g,n}$ of order $m$, the Hirzebruch-type invariant
\[ \lambda_T : \mathcal{H}_{g,n}(m) \to L^0(Q(\zeta_d)) \]
is well-defined. A sufficient condition that $\lambda_T$ is additive is given in [So16, Theorem 5.12]. It follows that $\lambda_T$ is a homomorphism on $\bigcap_m \mathcal{H}_{g,n}(m)$ for any $p$-structure $T$. Using homomorphisms $\lambda_T$, it turned out that the abelianization of $\bigcap_m \mathcal{H}_{g,n}(m)$ contains a subgroup isomorphic to $\mathbb{Z}^\infty$ if $b_1(\Sigma_{g,n}) = 2g + n - 1 > 1$ [So16, Theorem 6.7].

If we find $m$ such that the $\lambda_T$ are homomorphisms on $\mathcal{H}_{g,n}(m)$, then we will obtain that $H_1(\mathcal{H}_{g,n}(m))$ also contains a subgroup isomorphic to $\mathbb{Z}^\infty$. To find $\lambda_T$ which is a homomorphism on $\mathcal{H}_{g,n}(m)$, the author extracted the following from the sufficient condition.

**Proposition 1.2.** [So16, Corollary 5.13] Let $\Sigma = \Sigma_{g,n}$. Suppose $\mathcal{T} = (\hat{\Sigma}, \pi_1\hat{\Sigma} \to \mathbb{Z}_d)$ is a $p$-structure for $\Sigma$ of order $m$. If $(\pi_1\hat{\Sigma})_2 \supseteq (\pi_1\Sigma)_m$ for the $\mathbb{Z}_d$-cover $\hat{\Sigma}$ of $\Sigma$ then $T$ gives a homomorphism $\lambda_T : \mathcal{H}_{g,n}(m) \to L^0(Q(\zeta_d))$.

This naturally poses the problem to find a $p$-structure $T$ for $\Sigma$ satisfying the assumption of the proposition. The problem can be interpreted algebraically as follows:

**Problem.** Suppose $F$ is a finitely generated free group. Find a proper subgroup $G$ of $F$ such that there is an ascending chain $G = F_1 \supseteq F_k \supseteq \cdots \supseteq F_1 \supseteq F_0 = F$ with each $F_i/F_{i+1}$ an abelian $p$-group and $G_2 \supseteq F_m$ for some $m$.

We can simplify the problem as follows:

**Problem.** (simple version) Suppose $F$ is a finitely generated free group. Find a proper normal subgroup $G$ such that $F/G$ is abelian and $G_2 \supseteq F_m$ for some $m$.

This is equivalent to the following geometric problem which is the core of the original problem:

**Problem.** Let $X$ be a CW-complex with $\pi_1X$ free. Find an abelian cover $\hat{X}$ of $X$ such that the natural map $\pi_1\hat{X}/(\pi_1X)_m \to H_1(\hat{X})$ factors through $\pi_1\hat{X}/(\pi_1X)_m$ for some $m \geq 2$.

But, we finally obtain non-existence for the above problems as Theorem 1.1 shows. That said, it does not mean that there is no homomorphism $\lambda_T$ on $\mathcal{H}_{g,n}(m)$ since
Proposition 1.2 follows from only a sufficient condition for $\lambda_\tau$ to be additive in [So10, Theorem 5.12].

Extending the domain of $\lambda_\tau$ as a homomorphism may help study the mapping class groups of surfaces. The restriction of $H_{g,n}(m)$ on the mapping class group is the Johnson filtration $M_{g,n}[m] := \text{Ker}(M_{g,n} \to \text{Aut}(F/F_m))$. In other words, $H_{g,n}(m) \cap M_{g,n} = M_{g,n}[m]$. The subgroups $M_{g,n}[2]$ and $M_{g,n}[3]$ are well known as the Torelli group and the Johnson kernel, respectively. In 1938, Dehn proved that $M_{g,n}$ is finitely generated [Den38]. In 1983, Johnson proved that $M_{g,1}[2]$ and its quotient $M_{g,0}[2]$ are also finitely generated for $g \geq 3$, but it is discovered that $M_{g,1}[2]$ and $M_{2,0}[2] = M_{2,0}[3]$ are infinitely generated by McCullough and Miller [MM86] in 1986. Thereby, the question whether $M_{g,n}[3]$ is finitely generated for $g \geq 3$ has received a lot of attention since 1990s. Just lately, for $n = 0, 1$, Ershov and He [EH17] showed that $M_{2,n}[3]$ is finitely generated if $g \geq 12$ and $H_1(M_{g,n}[m])$ is also finitely generated if $m \geq 3, g \geq 8m - 12$. Church, Ershov and Putman proved that also for $n = 0, 1, M_{g,n}[3]$ is finitely generated if $g \geq 4$ and $M_{g,n}[m]$ is finitely generated if $m \geq 4, g \geq 2m - 3$ in [CEP17]. It is still open whether $M_{g,n}[m]$ is finitely generated for general $g$ and $n$. The Hirzebruch-type invariants may be used to prove that the abelianization is infinitely generated if we find a homomorphism $\lambda_\tau$ on the higher order Johnson subgroup.

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2. Non-existence of subgroups

We denote $[x, y] := xyx^{-1}y^{-1}$ where $x$ means $x^{-1}$.

**Theorem 2.1.** Suppose $F$ is a finitely generated free group. Then there is no normal subgroup of $F$ of prime index whose commutator subgroup contains a term of the lower central series of $F$.

**Proof.** Suppose there is an index $p$ normal subgroup $G$ of $F$ such that the commutator subgroup $[G, G]$ contains $F_m$ for some $m \in \mathbb{N}$. Then $G$ can be considered as the kernel of a surjective homomorphism $F \to \mathbb{Z}_p$.

It is enough to show that if $F = \langle x, y \rangle$ and $G = \text{Ker}(F \to \mathbb{Z}_p)$ where $f(x) = 1, f(y) = 0$, then $G_2 \not\subseteq F_m$ for all $m$.

Let $\omega_n := [\ldots [[x, y], x], \ldots, x] = [x, y, x, \ldots, x] \in F_{n+2}$ for $n \geq 0$. We claim that $\omega_n \notin G_2$ for every $n \in \mathbb{N}$. Since $\omega_n$ is an element of $G$, our claim is equivalent that $[\omega_n] \neq 0$ in $G/G_2 = H_1(G)$.

The subgroup $G = \langle \langle x^k, y \rangle \rangle_F = \langle x^p, y, xyx^{-1}y^2, \ldots, x^{p-1}y^{p-1} \rangle$. Let $a := x^p$ and $b_k := x^{k-1}y^{2k-1}$ for $k = 1, \ldots, p$, then $G = \langle a, b_1, \ldots, b_p \rangle$. Denote by $S$ the free generating set $\{a, b_1, \ldots, b_p\}$.

For $\omega \in G$ and $k = 1, \ldots, p$, let $P_k(\omega)$ be the sum of the powers of $b_k$ in $\omega$ as a word expressed in $S$. In other words, $P_k(\omega)$ is the power of $[b_k]$ in $[\omega] \in H_1(G)$. We note that

$$\sum_{k=1}^{p} \omega(k) b_k = \omega^p$$

where $\omega := \langle \langle x, y \rangle \rangle_F$.

(1)\hspace{1cm} xa = b_1 x b_2, \ldots, x b_{p-1} x b_p x = ab_1 a.$
Thus, conjugating any element of $G$ by $x$ preserves the sum of powers of $a$ in a word in $S$.

Since $a$ does not appear in the reduced word of $\omega_0 = xy_1x$ – $b_2b_1$, $[\omega_n, 0] = 0 \in H_1(G)$ if and only if $P_k(\omega_n) = 0$ for all $k$. We observe $P_1(\omega_0) = -1$, $P_2(\omega_0) = 1$, $P_k(\omega_0) = 0$ for $k \geq 3$, and $P_k(\omega_{n+1}) = P_k([\omega_n, x]) = P_k(\omega_n) + P_k(x\omega_n) = P_k(\omega_n) - P_k(x\omega_n) = P_k(\omega_n) - P_{k-1}(\omega_n)$. The last equality comes from (1). Hence we obtain

$$
\begin{pmatrix}
P_1(\omega_n) \\
P_2(\omega_n) \\
\vdots \\
P_p(\omega_n)
\end{pmatrix} =
\begin{pmatrix}
1 & -1 \\
-1 & 1 \\
\vdots & \vdots \\
-1 & 1
\end{pmatrix}^n
\begin{pmatrix}
-1 \\
1 \\
0 \\
0
\end{pmatrix}
$$

Let us calculate the eigenvalues of $A$. Since $\det(A - \lambda I) = (1 - \lambda)^p - 1$, the eigenvalues $\lambda_j$ of $A$ are $1 - \zeta^j$ where $\zeta$ is the $p$-th root of unity $e^{2\pi i/p}$ and $j = 1, \ldots, p$. The corresponding eigenvector $x_j$ to the eigenvalue $\lambda_j$ is

$$
\begin{pmatrix}
1 \\
\zeta^{(p-1)j} \\
\vdots \\
\zeta^{2j} \\
\zeta^j
\end{pmatrix}
$$

Since the eigenvalues $\lambda_j$ are all distinct, $x_j$ are linearly independent. Thus, $v_0$ can be expressed as a linear combination of $x_j$. Let $v_0 = \sum_{i=1}^p \alpha_i x_j$. Note that $\alpha_j$ is nonzero for some $j \neq p$ since $v_0 \neq \alpha x_p$ for any $\alpha$. Therefore, $v_n = A^n v_0 = \sum_{i=1}^p \alpha_i \zeta^{ij} x_j$ is nonzero for any $n \geq 1$. In conclusion, $\omega_n$ is not an element of $G_2$, and it implies that $G_2$ does not contain any $F_m$. \hfill $\square$

Note that prime index does not guarantee normality. For instance, there is a non-normal subgroup $\langle a, b^3, bab^2, bab \rangle$ of index 3 in $\mathbb{Z} \ast \mathbb{Z} = \langle a, b \rangle$.

In fact, the same argument holds not only for $p$ prime, but also when $p$ is replaced by an arbitrary integer $> 1$. Hence the theorem also holds not only for index $p$ normal subgroups but also for normal subgroups with finite cyclic factor groups. Moreover, we can extend Theorem 2.1 as follows:

**Corollary 2.2.** Let $F$ be a (possibly infinitely generated) free group. Suppose $G$ is a subgroup of $F$ such that there are $H$ and $K$ with $G \leq K \triangleleft H \leq F$, a nontrivial abelian factor group $H/K$. Then $G_2$ does not contain $F_m$ for any $m \in \mathbb{N}$.

**Proof.** First we generalize Theorem 2.1 to a free group of arbitrary rank. Let $G$ be a normal subgroup of index $p$ where $p$ is a prime. We can assume that $\{x_i \mid i \in I\}$ is a free generating set of $F$ with an index set $I \supseteq 1, 2$ and $G = \ker(f: F \rightarrow \mathbb{Z}_p)$ with $f(x_1) = 1$, $f(x_j) = 0$ for $j \neq 1 \in I$. Suppose $G_2 \supseteq F_m$ for some $m$. Let $H = \langle x_1, x_2 \rangle$, a subgroup of $F$. Then, $H \cap G = \ker(f|_H: H \rightarrow \mathbb{Z}_p)$ is an index $p$ normal subgroup of $H$. But, $(H \cap G) = H \cap G_2 \supseteq H \cap F_m \supseteq H_m$. It contradicts Theorem 2.1.
Now let us extend $G$ to a subgroup of $F$ with $G \leq K \triangleleft H \leq F$ and nontrivial abelian $H/K$. Suppose $G_2 \supseteq F_m$ for some $m \in \mathbb{N}$. Then, $K_2 \supseteq G_2 \supseteq F_m \supseteq H_m$. There is a prime index normal subgroup $K'$ of $H$ which contains $K$ since there is an epimorphism of $H/K$ onto a cyclic group of prime order. We have $(K')_2 \supseteq K_2 \supseteq H_m$, which is a contradiction.

For instance, if $F/G$ is the alternating group $A_5$, it has abelian subgroups isomorphic to $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_5$, so $G$ satisfies the hypothesis of the above corollary.

Lastly, we give a proof of Theorem 1.1 stated in the introduction:

**Proof of Theorem 1.1.** Let $K$ be the normal closure of $G$. Every nontrivial group has a nontrivial abelian subgroup, so there is a nontrivial abelian subgroup $H/K$ of $F/K$. Then $G < K < H < F$ satisfies the hypothesis of Corollary 2.2. Consequently, the conclusion of the corollary holds for every subgroup whose normal closure is not $F$. Hence we obtain Theorem 1.1. □

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