VOLUMES OF THE SETS OF TRANSLATION SURFACES WITH SMALL SADDLE CONNECTIONS

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ABSTRACT. In this paper, we prove some estimates of the volumes of the subsets of translation surfaces and quadratic differentials having several small, disjoint saddle connections in strata and Prym eigendeterminant loci. The main idea of the proofs is based on an upper bound of some integrals on the spaces of translation surfaces with marked saddle connections which already appeared in [N12].

1. INTRODUCTION

1.1. Abelian differentials. Fix \( k = (k_1, \ldots, k_n), k_i \in \mathbb{N} \), and let \( \mathcal{H}(k) \) denote the moduli space of pairs \((X, \omega)\), where \( X \) is a compact Riemann surface, and \( \omega \) is a holomorphic 1-form which has exactly \( n \) zeros with orders \( k_1, \ldots, k_n \). The one-form \( \omega \) defines a flat metric structure on \( X \) whose transition maps are translations \( z \mapsto z + c \) of \( \mathbb{R}^2 \cong \mathbb{C} \). The singularities of this metric are the zeros of \( \omega \), the cone angle at a zero of order \( k \) is \( 2\pi(k+1) \). The space \( \mathcal{H}(k) \) is called a stratum of the moduli space of translation surfaces of genus \( g \), where \( g \) and \( k \) are related by the equation

\[ k_1 + \cdots + k_n = 2g - 2. \]

We denote by \( \mathcal{H}_1(k) \) be the subset of \( \mathcal{H}(k) \) consisting of surfaces of area one (with respect to the flat metric). It is sometimes useful to consider translation surfaces with marked regular points, in this situation, we consider the marked points as zeros of order 0 of \( \omega \). This convention does not affect the arguments in what follows.

It is well known that the space \( \mathcal{H}(k) \) is a complex algebraic orbifold, equipped with a natural volume form \( \mu \) coming from the period mapping (see Section 2.2). We have \( \dim_{\mathbb{C}} \mathcal{H}(k) = 2g + n - 1 \). Let \( \mu_1 \) be the volume form on \( \mathcal{H}_1(k) \) which is defined by the formula

\[ d\mu = d\mu_1 dA, \]

where \( A \) is the area function.

Given \((X, \omega) \in \mathcal{H}(k)\), a saddle connection on \( X \) is a geodesic segment for the flat metric whose endpoints are singularities (the endpoints are not necessarily distinct). We denote the length of a saddle connection \( \gamma \) by \( |\gamma| \).

Definition 1.1. Let \( \{\gamma_1, \ldots, \gamma_m\} \) be a family of saddle connections on \( X \). This family will be called an independent family if the following conditions are fulfilled

(i) \( \text{int}(\gamma_i) \cap \text{int}(\gamma_j) = \emptyset \) if \( i \neq j \).

(ii) the family \( \{\gamma_1, \ldots, [\gamma_m]\} \) is linearly independent in \( H_1(X, \{p_1, \ldots, p_n\} ; \mathbb{R}) \), where \( p_1, \ldots, p_n \) are zeros of \( \omega \).

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Given $\xi = (\epsilon_1, \ldots, \epsilon_m) \in (\mathbb{R}_{>0})^m$, let us denote by $\mathcal{H}_{1, \xi}^{(m)}(k)$ the subset of $\mathcal{H}_1(k)$ consisting of surfaces $(X, \omega) \in \mathcal{H}_1(k)$ on which there exists an ordered independent family of saddle connections $\{\gamma_1, \ldots, \gamma_m\}$ such that

$$|\gamma_j| < \epsilon_j, \; j = 1, \ldots, m.$$ 

When $\epsilon_j$ are small enough, such a family only exists if $m < \dim_{\mathbb{C}} \mathcal{H}(k) = 2g + n - 1$. In this paper, we will prove the following theorem, which provides an estimate of the volume of the set $\mathcal{H}_{1, \xi}^{(m)}(k)$

**Theorem 1.2.** For any $m < \dim_{\mathbb{C}} \mathcal{H}(k)$, there exists a constant $C = C(k, m)$ such that

$$\mu_1(\mathcal{H}_{1, \xi}^{(m)}(k)) < C\epsilon_1^2 \cdots \epsilon_m^2.$$ 

### 1.2 Quadratic differentials.

Let $X$ be a compact Riemann surface, and $\phi$ be a meromorphic quadratic differential on $X$. The quadratic differential $\phi$ defines on $X$ a flat metric structure whose transition maps are of the form $z \mapsto \pm z + c$, $c$ is constant. If $\int_X |\phi|^2$ is finite, which implies that all the poles of $\phi$ are simple, the area of this metric structure is finite, and its singularities are isolated cone points corresponding to the zeros and poles of $\phi$. The cone angle at a zero of order $k$ is $(k + 2)\pi$, where a zero of order $-1$ is a simple pole.

In the case $\phi = \omega^2$, where $\omega$ is a holomorphic one-form, the flat metrics defined by $\phi$ and $\omega$ are the same, thus we will be only interested in quadratic differentials which are not square of any holomorphic one-form. In this case, the corresponding flat surface is sometimes called a half-translation surface.

There is a natural stratification on the set of quadratic differentials by the orders of their zeros, we denote by $Q(d)$, $d = (d_1, \ldots, d_n), d_i \in \{-1, 0, 1, 2, \ldots\}$, the set of pairs $(X, \phi)$ where $\phi$ has exactly $n$ zeros, with orders given by $d_i$ and $\phi$ is not the square of a holomorphic one-form. By convention, a zero of order $0$ is a marked regular point of the flat metric. Remark that the genus of $X$ is determined by the formula

$$d_1 + \cdots + d_n = 4g - 4.$$ 

It is well known that $Q(d)$ is also a complex algebraic orbifold of dimension $2g + n - 2$, which is equipped with a natural Lebesgue volume form $\mu$. Let $Q_1(d)$ denote the subset of $Q(d)$ consisting of surfaces of area one, then we have a volume form $\mu_1$ on $Q_1(d)$ which is defined by

$$d\mu = dA d\mu_1,$$

where $A$ is the area function on $Q(d)$.

Given a meromorphic quadratic differential $(X, \phi)$, which is not a square of a holomorphic one-form, there exists a canonical (ramified) double covering $\tau: \hat{X} \to X$ such that the quadratic differential $\hat{\phi} = \tau^* \omega$ is the square of a holomorphic one-form $\hat{\omega}$ on $\hat{X}$. We have an involution $\tau: \hat{X} \to \hat{X}$ satisfying $q = \tau(p)$ if and only if $\pi(p) = \pi(q)$, and $\tau^* \omega = -\hat{\omega}$. Let $S$ denote the set of singularities of the flat metric structure defined by $\phi$ on $X$. We will call a geodesic segment $\gamma$ with endpoints in $S$ a saddle connection of $X$. The pre-image of $\gamma$ in $\hat{X}$ consists of two geodesic segments $\gamma_1, \gamma_2$ with endpoints in $\hat{S} = \pi^{-1}(S)$. We choose the orientation of $\gamma_1$ and $\gamma_2$ so that $\tau(\gamma_1) = -\gamma_2$ and $\tau(\gamma_2) = -\gamma_1$. We will denote by $[\gamma]$ the homology class in $H_1(\hat{X}, \hat{S}; \mathbb{Z})$ represented by $[\gamma_1] + [\gamma_2]$. We have a natural decomposition

$$H_1(\hat{X}, \hat{S}; \mathbb{R}) = H_1(\hat{X}, \hat{S}; \mathbb{R})^- \oplus H_1(\hat{X}, \hat{S}; \mathbb{R})^+,$$
where $H_1(\hat{X}, \hat{S}; \mathbb{R})^\pm$ are the eigenspaces of $\tau$ corresponding to the eigenvalues $\pm 1$ respectively. By definition, $[\hat{y}]$ belongs to $H_1(\hat{X}, \hat{S}; \mathbb{R})^-$, and $\hat{\omega}$ belongs to $H^1(\hat{X}, \hat{S}; \mathbb{C})^- = \text{Hom}_\mathbb{C}(H_1(\hat{X}, \hat{S}; \mathbb{R})^-, \mathbb{C})$.

**Definition 1.3.** Let $\{\gamma_1, \ldots, \gamma_m\}$ be a family of saddle connections on $X$. We will say that $\{\gamma_1, \ldots, \gamma_m\}$ is an independent family if the following conditions are fulfilled

(i) $\text{int}(\gamma_i) \cap \text{int}(\gamma_j) = \emptyset$ if $i \neq j$.

(ii) the family $\{[\gamma_1], \ldots, [\gamma_m]\}$ is independent in $H_1(\hat{X}, \hat{S}; \mathbb{R})$.

For all $\xi = (\epsilon_1, \ldots, \epsilon_m) \in (\mathbb{R}_{>0})^m$, where $m < \dim \mathbb{C} Q(\underline{d}) = 2g + n - 2$, let us denote by $Q_{\xi, \underline{d}}^{(m)}(\underline{d})$ the set of $(X, \phi) \in Q_{\xi, \underline{d}}(\underline{d})$ such that there exists an ordered independent family of saddle connections $\{\gamma_1, \ldots, \gamma_m\}$ on $X$ satisfying

$$\|\gamma_j\| < \epsilon_j, \ j = 1, \ldots, m.$$ 

Similar to the case of Abelian differentials, we will prove

**Theorem 1.4.** For any $m < \dim \mathbb{C} Q(\underline{d})$, there exists a constant $C = C(\underline{d}, m)$ such that for every $\xi \in (\mathbb{R}_{>0})^m$, we have

$$\mu_1(Q_{\xi, \underline{d}}^{(m)}(\underline{d})) < C \epsilon_1^2 \ldots \epsilon_m^2.$$ 

### 1.3. Prym eigenform locus

A Prym eigenform is a triple $(X, \rho, \omega)$ where

- $X$ is a Riemann surface, and $\rho : X \to X$ is an involution which satisfies $\dim \mathbb{C} \Omega(X, \rho)^- = 2$, where $\Omega(X, \rho)^- = \ker(\tau + \text{id}) \subset \Omega(X)$,
- $\omega$ is a holomorphic $1$-form on $X$ belonging to $\Omega(X, \rho)^-$,
- Define Prym$(X, \rho) = (\Omega(X, \rho)^-)^*/H_1(X, \mathbb{Z})^-$. There exist a quadratic order $O_D = \mathbb{Z}[x]/(x^2 + bx + c)$, where $b^2 - 4c = D$, and an injective ring morphism $i : O_D \to \text{End}(\text{Prym}(X, \rho))$ such that $i(O_D)$ is a self-adjoint proper subring of $\text{End}(\text{Prym}(X, \rho))$ for which $\omega$ is an eigenvector $(R \subset \text{End}(\text{Prym}(X, \rho))$ is a proper subring if for any $T \in \text{End}(\text{Prym}(X, \rho))$ if there exists $n \in \mathbb{Z}, n \neq 0$, such that $nT \in R$, then $T \in R$).

**Remark 1.5.**

1. Given $(X, \rho, \omega)$, the embedding $i$ is unique up to isomorphism of $O_D$, which means that the image $i(O_D)$ in $\text{End}(\text{Prym}(X, \rho))$ is uniquely determined.

2. The pair $(X, \omega)$ does not uniquely determine $\rho$ and $i$ in general, but by a slight abuse of language, we will call a translation surface $(X, \omega)$ a Prym eigenform if there exists an involution $\rho$ such that $(X, \rho, \omega)$ satisfies the conditions stated above.

We call a positive integer $D \equiv 0, 1 \mod 4$ a discriminant. Given a discriminant $D$, the set of Prym eigenforms for a self-adjoint proper subring isomorphic to the order $O_D$ is denoted by $\Omega E_D$. In [Mc00], McMullen shows that $\Omega E_D$ is close and invariant under the action of $\text{SL}(2, \mathbb{R})$. Thus the intersection of $\Omega E_D$ with any stratum $\mathcal{H}(\underline{k})$, denoted by $\Omega E_D(\underline{k})$, is a $\text{SL}(2, \mathbb{R})$-invariant suborbifold of $\mathcal{H}(\underline{k})$. In the same paper, McMullen also shows that $\Omega E_D$ only exists in genus at most 5, and the intersection of $\Omega E_D$ with the minimal strata $\mathcal{H}(2g - 2)$ consists of finitely many $\text{GL}^+(2, \mathbb{R})$ closed orbits. For $\underline{k} \in [(1, 1), (2, 2), (1, 1, 2)]$, we have $\dim \mathbb{C} \Omega E_D(\underline{k}) = 3$. Moreover,

**Theorem 1.6.**

1. If $\underline{k} = (1, 1)$, then $\Omega E_D(1, 1)$ is non-empty and connected for any $D \geq 5, D \equiv 0, 1 \mod 4$. 


(b) If \( k \in \{(2, 2), (1, 1, 2)\} \), then for \( D \geq 8 \), \( \Omega E_D(k) \) is empty if \( D \equiv 5 \) mod 8, and non-empty if \( D \equiv 0, 1, 4 \) mod 8. Moreover, \( \Omega E_D(k) \) is connected if \( D \equiv 0, 4 \) mod 8 and has two components if \( D \equiv 1 \) mod 8.

**Remark 1.7.** (a) is proven by McMullen \cite{Mc06, Mc07}, (b) is proven by Lanneau-Nguyen \cite{LN13c}.

To simplify the notations, we will call a connected component of \( \Omega E_D(k) \) a Prym eigenform locus.

### 1.4. Volume form on Prym eigenform loci.

Recall that in local coordinates defined by period mappings, a neighborhood of an element \((X, \omega) \in \mathcal{H}(k)\) is identified with an open of \( H^1(X, \Sigma; \mathbb{C}) \), where \( \Sigma \) is the set of zeros of \( \omega \). We can write \( H^1(X, \Sigma; \mathbb{C}) = H^1_{\text{abs}}(X, \Sigma; \mathbb{C}) \oplus H^1_{\text{rel}}(X, \Sigma; \mathbb{C}) \), where \( H^1_{\text{abs}}(X, \Sigma; \mathbb{C}) \) is the subspace generated by the cohomology classes which vanish on the relative cycles, and \( H^1_{\text{rel}}(X, \Sigma; \mathbb{C}) \) is generated by the relative cohomology classes which vanish on the subspace \( H_1(X, \mathbb{Z}) \) of \( H_1(X, \Sigma; \mathbb{Z}) \). Observe that \( H^1_{\text{abs}}(X, \Sigma; \mathbb{C}) \) is isomorphic to \( H^1(X, \mathbb{C}) \).

Locally, a Prym eigenform locus corresponds to a linear subspace \( V \) of \( H^1(X, \Sigma; \mathbb{C}) \) which is defined by some linear equations with real coefficients. We can write

\[
V = V_{\text{abs}} \oplus V_{\text{rel}}, \quad \text{where} \quad V_{\text{abs}} = V \cap H^1_{\text{abs}}(X, \Sigma; \mathbb{C}) \quad \text{and} \quad V_{\text{rel}} = V \cap H^1_{\text{rel}}(X, \Sigma; \mathbb{C}).
\]

We have \( \dim_{\mathbb{C}} V_{\text{abs}} = 2 \), and \( \dim_{\mathbb{C}} V_{\text{rel}} = 1 \). Let \([\text{Re}(\omega)]\) and \([\text{Im}(\omega)]\) denote the cohomology classes in \( H^1(X, \mathbb{R}) \) of \( \text{Re}(\omega) \) and \( \text{Im}(\omega) \) respectively. Let \( V_{\text{abs}}(\mathbb{R}) \) denote the subspace of \( H^1(X, \mathbb{R}) \) generated by \([\text{Re}(\omega)]\) and \([\text{Im}(\omega)]\). We then have

\[
V_{\text{abs}} = \mathbb{C} \cdot [\text{Re}(\omega)] \oplus \mathbb{C} \cdot [\text{Im}(\omega)] = V_{\text{abs}}(\mathbb{R}) \oplus \iota \cdot V_{\text{abs}}(\mathbb{R}).
\]

Let \( \Omega \) denote the symplectic form on \( H^1(X, \mathbb{R}) \) which is induced by the intersection form of \( H_1(X, \mathbb{R}) \). Remark that the restriction of \( \Omega \) to \( V_{\text{abs}}(\mathbb{R}) \) is non-degenerate, thus \( \Omega \) gives rise to a volume form on \( V_{\text{abs}}(\mathbb{R}) \). Consequently, we have a volume form on \( V_{\text{abs}} = V_{\text{abs}}(\mathbb{R}) \oplus \iota \cdot V_{\text{abs}}(\mathbb{R}) \) which will be denoted by \( \Omega^2 \).

Set \( V_{\text{rel}}(\mathbb{Z}) := V_{\text{rel}} \cap H^1(X, \Sigma; \mathbb{Z} \oplus \iota \mathbb{Z}) \). In the cases we consider, that is, \( \Omega E_D(k) \) with \( k \in \{(1, 1), (2, 2), (1, 1, 2)\} \), \( V_{\text{rel}} \) is defined by equations with integral coefficients, thus \( V_{\text{rel}}(\mathbb{Z}) \) is a lattice in \( V_{\text{rel}} \). There exists a unique volume form \( \lambda \) on \( V_{\text{rel}} \) proportional to the Lebesgue measure such that \( \lambda(V_{\text{rel}}/V_{\text{rel}}(\mathbb{Z})) = 1 \). Let us define a volume form \( \text{vol} \) on \( V \) by the following

\[
d\text{vol} = d\Omega^2 \times d\lambda.
\]

Observe that \( \text{vol} \) is proportional to the Lebesgue measure on \( V \) considered as a linear of subspace \( \mathbb{C}^d \cong H^1(X, \Sigma; \mathbb{C}) \), and that \( \text{vol} \) is preserved by the action of \( \text{SL}(2, \mathbb{R}) \).

Since transition maps for period mappings consist of changing the basis of \( H_1(X, \Sigma; \mathbb{Z}) \), we see that the lattice \( H^1(X, \Sigma; \mathbb{Z} \oplus \iota \mathbb{Z}) \) and the intersection form of \( H_1(X, \mathbb{R}) \) are preserved as one changes the local chart. Therefore, \( \Omega \) and \( \lambda \) are invariant by the transition maps, from which we deduce that the volume form \( \text{vol} \) is well-defined.

Let \( \mathcal{M} \) be a Prym eigenform locus in one of the following strata \( \mathcal{H}(1, 1), \mathcal{H}(2, 2), \mathcal{H}(1, 1, 2) \). We denote by \( \mathcal{M}_l \) the subset of surfaces of unit area in \( \mathcal{M} \). We also have volume form on \( \mathcal{M}_l \) which is defined by \( d\text{vol}/dA \), where \( A \) is the area function. By a slight abuse of notation, we denote by \( \text{vol} \) both the volume form on \( \mathcal{M} \) and the one induced on \( \mathcal{M}_l \).

Remark that we have

\[
A(X, \omega) = \int_X \text{Re}(\omega) \wedge \text{Im}(\omega) = \Omega([\text{Re}(\omega)], [\text{Im}(\omega)]).
\]
Therefore we can locally identify $M_1$ with $\text{SL}(2, \mathbb{R}) \rtimes \mathbb{R}^2$, and using this identification, we have $d\text{vol} = d\mu \times d\lambda$, where $d\mu$ is the Haar measure on $\text{SL}(2, \mathbb{R})$.

1.5. Statement of results. Our goal is to prove the following theorems

**Theorem 1.8.** For any $\epsilon > 0$, let $M_1(\epsilon)$ denote the subset of $M_1$ consisting of surfaces $(X, \omega)$ which have a saddle connection of length bounded by $\epsilon$. Then there exist some constants $C_1 > 0$, and $\epsilon_0 > 0$ such that for any $\epsilon < \epsilon_0$, we have

\[ \text{vol}(M_1(\epsilon)) < C_1 \epsilon^2 \]

**Theorem 1.9.** Given positive real numbers $\kappa$ and $\epsilon$, let $M_1(\kappa, \epsilon)$ denote the set of surfaces in $M_1$ which have a simple closed geodesic $c$ of length at most $\kappa$, and a saddle connection $s$ not parallel to $c$ of length at most $\epsilon$. Then there exist the constants $C_2, \kappa_0, \epsilon_0$ such that if $\kappa < \kappa_0$ and $\epsilon < \epsilon_0$, then

\[ \text{vol}(M_1(\kappa, \epsilon)) < C_2 \kappa^2 \epsilon^2. \]

**Remark 1.10.**

a) Theorem 1.2 and Theorem 1.4 can be proved by using the arguments of Masur-Smillie [MaS91]. Our aim here is to introduce another method of proving such estimates, which can be used in other contexts such as Prym eigeinform loci.

b) Theorem 1.8 implies in particular that $\text{vol}(M_1)$ is finite.

c) It follows from the work of Eskin-Mirzakhani-Mohammadi [EMi13, EMiMo13], that the Lebesgue measure $\text{vol}$ on $\Omega E_D(k)$ is ergodic and finite (see also [MinW02] and [Mc07]). As a consequence, Theorem 1.8 can be proven by using the Siegel-Veech formula (see [V98, EMa01, Theorem 2.2, and [EKZ11, Lemma 9.1]).

d) Theorem 1.9 implies that $\Omega E_D(k)$ are regular invariant submanifolds for the action of $\text{SL}(2, \mathbb{R})$ (see [EKZ11, Definition 1], it is an improvement for a particular case of a recent result by Avila-Matheus-Yoccoz [AMY13].

1.6. Outline. The main results of this paper are proven as consequences of some estimates of integrals over the spaces of (half-)translation surfaces with marked saddle connections. Roughly speaking, for each $\epsilon$ we will define a function $\tilde{\gamma}_\epsilon$ on the spaces of (half-)translation surfaces with $m$-marked saddle connections which involves the area of the surface and the lengths of the marked saddle connections. Theorems 1.2 and 1.4 then follow from the fact that the integral of $\tilde{\gamma}_\epsilon$ is bounded by $K \epsilon_1^2 \cdots \epsilon_m^2$ (see Theorems 2.3 and 6.1), where $K$ is a constant. The proof of these estimates uses construction of translation surfaces from triangles in $\mathbb{R}^2$, and similar ideas as in [N12]. To prove Theorem 1.8 and Theorem 1.9 we will use cylinder decompositions of Prym eigeinforms, and estimates of similar integrals.

This paper is inspired from the work [EMaZ03, MaZ08] of Eskin, Masur, and Zorich in which they carefully study the neighborhood of the boundary of moduli spaces of Abelian differentials and quadratic differentials to obtain formulae relating the Siegel-Veech constants and the volumes of the strata. One important ingredient of the proof of their results is a result of Masur-Smillie, which gives an estimate on the volume of the set of quadratic differentials which have two small non-parallel saddle connections (see [EMaZ03], Lemma 7.1 and [MaS91] Theorem 10.3). Actually Theorem 1.2 and Theorem 1.3 can be proved by using the arguments of Masur-Smillie in [MaS91], even though such statements did not appear in their paper.
2. Translation surface with marked saddle connections

2.1. Translation surfaces with marked saddle connections.

Definition 2.1. Let us denote by $\widehat{H}^{(m)}(k)$ the moduli space of triples $(X, \omega, \{\gamma_1, \ldots, \gamma_m\})$, where

- $(X, \omega)$ belongs to $H(k)$,
- $\{\gamma_1, \ldots, \gamma_m\}$ is an independent ordered family of saddle connections.

Let $M = (X, \omega, \{\gamma_1, \ldots, \gamma_m\})$ and $X' = (X', \omega', \{\gamma'_1, \ldots, \gamma'_m\})$ be two elements in $\widehat{H}^{(m)}(k)$. We say that $M$ and $X'$ have the same topological type if there exists a homeomorphism $\varphi : X \to X'$ which realizes a bijection between the sets of zeros of $\omega$ and $\omega'$ preserving the orders, such that $\varphi(\gamma_i) = \gamma'_i$.

For a fixed vector $k = (k_1, \ldots, k_n)$, there are only finitely many topological types for surfaces in $\widehat{H}^{(m)}(k)$.

2.2. Local chart and volume form. Let us first recall the definitions of local charts and volume form on $H(k)$ using the period mapping. Let $(X, \omega)$ be a point in $H(k)$. The zeros of $\omega$ are denoted by $p_1, \ldots, p_n$, and their orders by $k_i$ respectively. Let $\{c_1, \ldots, c_{2g+n-1}\}$ be a set of curves on $X$ which is a basis of $H_1(X, \{p_1, \ldots, p_n\}; \mathbb{Z})$. For any element $(X', \omega')$ close enough to $(X, \omega)$ in $H(k)$, we denote by $\{c'_1, \ldots, c'_{2g+n-1}\}$ the corresponding curves on $X'$. We can then define a map $\Phi$ from a neighborhood of $(X, \omega)$ into $\mathbb{C}^{2g+n-1}$, which sends a pair $(X', \omega')$ to the vector $(\int_{c'_1} \omega', \ldots, \int_{c'_{2g+n-1}} \omega')$. The map $\Phi$ is called a period mapping. It is well known that $\Phi$ is a local chart for $H(k)$, hence $H(k)$ has a complex orbifold structure of dimension $2g + n - 1$.

Let $\lambda(2g+n-1)$ denote the Lebesgue measure of $\mathbb{C}^{2g+n-1} \approx \mathbb{R}^{2g+n-1}$. Since the bases of $H_1(M, \{p_1, \ldots, p_n\}; \mathbb{Z})$ are related by elements of the group GL($2g + n - 1$, $\mathbb{Z}$), the volume form $\Phi^* \lambda(2g+n-1)$ is well defined on $H(k)$. We denote this volume form by $\mu$.

Let $H_1(k)$ denote the subspace of $H(k)$ consisting of pairs $(X, \omega)$ such that $\int_X |\omega|^2 = 1$. An element of $H_1(k)$ corresponds to a translation surface of area 1. The volume form $\mu$ induces a volume form $\mu_1$ on $H_1(k)$ by the following relation

$$d\mu = dA \mu_1,$$

where $A$ is the area function. It is proved by Masur [Ma82] and Veech [V90] that the total volume of $H_1(k)$ with respect to $\mu_1$ is finite.

There is an obvious projection $\varrho : \widehat{H}^{(m)}(k) \to H(k)$, which maps the triple $(X, \omega, \{\gamma_1, \ldots, \gamma_m\})$ to the pair $(X, \omega)$. If $(X', \omega') \in H(k)$ is close enough to $(X, \omega)$, then there also exists $m$ saddle connections $\gamma'_1, \ldots, \gamma'_m$ on $X'$ such that $M := (X, \omega, \{\gamma_1, \ldots, \gamma_m\})$ and $X' := (X', \omega', \{\gamma'_1, \ldots, \gamma'_m\})$ have the same topological type. Therefore, we can identify a neighborhood of $M$ in $\widehat{H}^{(m)}(k)$ with a neighborhood of $(X, \omega)$ in $H(k)$, and use $\Phi$ to define local charts for $\widehat{H}^{(m)}(k)$. Since $\{\gamma_1, \ldots, \gamma_m\}$ is an independent family in $H_1(X, \{p_1, \ldots, p_n\}; \mathbb{Z})$, we can choose a basis $\{c_1, \ldots, c_{2g+n-1}\}$ of $H_1(X, \{p_1, \ldots, p_n\}; \mathbb{Z})$ such that $c_1 = \gamma_1, \ldots, c_m = \gamma_m$. 
The projection $\varphi$ is clearly locally homeomorphic. Thus, we can pullback the volume form $\mu$ of $\mathcal{H}(k)$ to $\mathcal{H}^{(m)}(k)$. By a slight abuse of notation, we will also denote by $\mu$ this volume form on $\mathcal{H}^{(m)}(k)$.

Let us now resume what we have seen in the following

**Theorem 2.2.** $\mathcal{H}^{(m)}(k)$ admits a flat complex affine orbifold structure (that is, the transition maps are complex linear transformations) of dimension $2g + n - 1$, for which $\mu$ is a parallel volume form (i.e. invariant by transition maps of the flat complex affine structure).

### 2.3. Energy functions.

For every $\epsilon = (\epsilon_1, \ldots, \epsilon_m)$, with $\epsilon_j > 0$, we define a function $\bar{\bar{\gamma}}_{\epsilon} : \mathcal{H}^{(m)}(k) \rightarrow \mathbb{R}$ by the following formula

$$\bar{\bar{\gamma}}_{\epsilon} : (X, \omega, \{\gamma_1, \ldots, \gamma_m\}) \mapsto \exp\left(-\sum_{j=1}^{m} \frac{|\gamma_j|^2}{\epsilon_j^2} - A(X)\right).$$

Theorem 1.2 is a consequence of the following

**Theorem 2.3.** There exists a constant $K = K(k, m)$ such that

$$\int_{\mathcal{H}^{(m)}(k)} \bar{\bar{\gamma}}_{\epsilon} d\mu < K \epsilon_1^2 \ldots \epsilon_m^2.$$

The proof of this theorem will be given in Section 5.

### 2.4. Proof of Theorem 1.2.

**Proof.** Let us now give the proof of 1.2 using Theorem 2.3. Let $\mathcal{H}^{(m)}_{\epsilon}(k)$ be the subset of $\mathcal{H}(k)$ consisting of surfaces $(X, \omega)$ on which there exists an independent family of saddle connections $\{\gamma_1, \ldots, \gamma_m\}$ such that

$$|\gamma_j|^2 < \epsilon_j^2 A(X).$$

Note that $\mathcal{H}^{(m)}_{\epsilon}(k) = \mathbb{R}_+^* \cdot \mathcal{H}^{(m)}(k)$, where the action of $\mathbb{R}_+^*$ is given by $t \cdot (X, \omega) = (X, t \omega), \ t \in \mathbb{R}_+^*$. Let $\mathcal{H}^{(m)}_{\epsilon}(k) \subset \mathcal{H}^{(m)}(k)$ be the pre-image of $\mathcal{H}^{(m)}_{\epsilon}(k)$ by the natural projection $\varphi : \mathcal{H}^{(m)}(k) \rightarrow \mathcal{H}(k)$. Theorem 2.3 clearly implies

$$\int_{\mathcal{H}^{(m)}_{\epsilon}(k)} \bar{\bar{\gamma}}_{\epsilon} d\mu < K \epsilon_1^2 \ldots \epsilon_m^2.$$

Let us define the function $f_{\epsilon} : \mathcal{H}(k) \rightarrow \mathbb{R}$ as follows

$$f_{\epsilon} : (X, \omega) \mapsto e^{-A(X)} \sum_{\{\gamma_1, \ldots, \gamma_m\} \text{ independent family in } X} e^{-\frac{|\gamma_1|^2 + \ldots + |\gamma_m|^2}{\epsilon_1^2 + \ldots + \epsilon_m^2}}.$$

By definition, we have

$$\int_{\mathcal{H}^{(m)}_{\epsilon}(k)} \bar{\bar{\gamma}}_{\epsilon} d\mu = \int_{\mathcal{H}^{(m)}_{\epsilon}(k)} f_{\epsilon} d\mu.$$
But if \((X, \omega) \in \mathcal{H}_k^{(m)}\), then by definition there exists at least an independent family \(\{\gamma_1, \ldots, \gamma_m\}\) in \(X\) such that
\[
e^{-\frac{|x_1|^2}{\epsilon} + \ldots + \frac{|x_n|^2}{\epsilon}} > e^{-m \Lambda(X)}.
\]
Thus on \(\mathcal{H}_k^{(m)}\), we have
\[
f_\epsilon(.) > e^{-(m+1)\Lambda(.)},
\]
and (3) implies
\[
\int_{\mathcal{H}_k^{(m)}} e^{-(m+1)\Lambda} d\mu < K\epsilon_1^2 \ldots \epsilon_m^2.
\]
Recall that the space \(\mathcal{H}(k)\) can be locally identify with an open of \(\mathbb{R}^{2N}, N = 2g + n - 1\), so that \(\mu\) is the Lebesgue measure. By this identification, \(\mathcal{H}_1(k)\) corresponds to the hypersurface \(Q_1\) defined by the equation \(\Lambda = 1\). Let us make the following change of coordinates
\[
\Phi : Q_1 \times [0, +\infty] \rightarrow \mathbb{R}^{2N}, \quad (v, t) \mapsto t \cdot v.
\]
Note that we have \(\Phi^* \Lambda(v, t) = t^2\). Using the realtion \(d\mu = dA d\mu_1\) on \(Q_1\), one can easily check that
\[
\Phi^* d\mu = 2t^{2N-1} dtd\mu_1.
\]
Therefore,
\[
\int_{\mathcal{H}_k^{(m)}} e^{-(m+1)\Lambda} d\mu = \int_{\mathcal{H}_k^{(m)}} \left( \int_0^{+\infty} 2t^{2N-1} e^{-(m+1)t^2} dt \right) d\mu_1.
\]
It follows immediately from (4) that
\[
\mu_1(\mathcal{H}_k^{(m)}) < Ce_1^2 \ldots e_m^2.
\]
Theorem 1.2 is then proven.

3. TRIANGULATIONS AND LOCAL CHARTS

Let \(M := (X, \omega, \{\gamma_1, \ldots, \gamma_m\})\) be a point in \(\mathcal{H}_k^{(m)}\). It is a well known fact that there always exists a geodesic triangulation \(T\) of \(X\) whose vertices are the singular points such that the family \(\{\gamma_1, \ldots, \gamma_m\}\) is included in the 1-skeleton \(T^{(1)}\) of \(T\). We will call such a triangulation an admissible triangulation. Let \(N_1, N_2\) denote the number of edges and the number of triangles of \(T\) respectively. Note that \(N_1\) and \(N_2\) are completely determined by the genus \(g\) of \(X\) and the number \(n\) of zeros of \(\omega\), namely, we have
\[
N_1 = 3(2g + n - 2) \quad \text{and} \quad N_2 = 2(2g + n - 2).
\]
Choosing for each edge of \(T\) an orientation, we can then associate to each oriented edge \(e\) a complex number \(Z(e) = \int_e \omega\). Thus, the triangulation \(T\) provides us with a vector \(Z \in \mathbb{C}^{N_1}\). The coordinates of \(Z\) must satisfy the following condition: if \(e_i, e_j, e_k\) are three edges of a triangle of \(T\), then we have
\[
\pm Z(e_i) \pm Z(e_j) \pm Z(e_k) = 0.
\]
The signs of \(Z(e_i), Z(e_j), Z(e_k)\) depend on their orientation. We have \(N_2\) equations of type (5), each of which corresponds to a triangle in \(T^{(2)}\). Let \(S_T\) denote the system consisting of those linear equations, and \(V_T\) be the subspace of solutions of \(S_T\) in \(\mathbb{C}^{N_1}\).
Lemma 3.1. Let $Z'$ be a vector in $V_T$. If $Z'$ is close enough to $Z$ then there exists
- an element $M' = (X', \omega', \{\gamma'_1, \ldots, \gamma'_m\}) \in \tilde{H}^{(m)}(k)$ close to $M$,
- an admissible triangulation $T'$ of $X'$,
- a homeomorphism $\varphi : X \to X'$ which maps $T$ to $T'$,

such that $Z'$ is the vector associated to $T'$.

Proof. We consider $Z$ as a function from $T^{(1)}$ to $\mathbb{C}$ (where $T^{(k)}$ is the $k$-skeleton of $T$, $k = 0, 1, 2$). We can construct a flat surface from $Z' \in V_T$ as follows

(a) construct a triangulation from $Z'(e_j), Z'(e_k), Z'(e_l)$, whenever there exists a triangle in $T^{(2)}$ which contains $e_i, e_j, e_k$ ($(e_i, e_j, e_k)$ are distinct edges),
(b) glue two triangles together if the corresponding triangles in $T^{(2)}$ have a common edge.

This construction provides us with a surface $X'$ equipped with a flat metric structure with cone singularities. Since we only identify edges which have the same direction and the same length, the 1-form $dz$ on $\mathbb{C}$ is preserved by the gluings. Therefore, we also get a holomorphic 1-form $\omega'$ on $X'$, and hence a translation surface. As all the topological invariants (orientation, genus, number of singularities, cone angles) are discrete, it follows that if $Z'$ is close enough to $Z$ then $(X', \omega')$ has the same topological invariants as $(X, \omega)$, hence $(X', \omega') \in H^{(k)}$.

The surface $(X', \omega')$ comes equipped with a geodesic triangulation $T'$, the triangles of $T'$ are formed by triples of vectors $Z'(e_i), Z'(e_j), Z'(e_k)$ which satisfy an equation of type \ref{eq:5}. From the construction of $X'$, we have a homeomorphism $\varphi : M \to X'$ which sends a triangle of $T$ to a triangle of $T'$. Set $\gamma'_j = \varphi(\gamma_j)$, then $(X', \omega', \{\gamma'_1, \ldots, \gamma'_m\})$ is an element of $\tilde{H}^{(m)}(k)$ which has the same topological type as $(X, \omega, \{\gamma_1, \ldots, \gamma_m\})$. \qed

From Lemma $\ref{lemma:3.1}$ we see that one can define a map $\Psi_T : \mathcal{U} \to \tilde{H}^{(m)}(k)$, where $\mathcal{U}$ is an open subset of $V_T$ that contains $Z$. Actually, we have

Proposition 3.2. For $\mathcal{U}$ small enough, $\Psi_T$ is continuous and injective, it realizes an homeomorphism between $\mathcal{U}$ and its image.

Proof. Let us first show that $\dim_{\mathbb{C}} V_T = 2g + n - 1$. Since $\dim H_1(X, \{p_1, \ldots, p_n\}; \mathbb{Z}) = 2g + n - 1$, where $p_1, \ldots, p_n$ are the zeros of $\omega$, there always exists in $T^{(1)}$ a family of $2g + n - 1$ edges such that if we cut $X$ along this family the resulting surface is a union of topological disks. Let $\mathcal{B} = \{e_1, \ldots, e_{2g+n-1}\}$ be such a family.

Recall that every equation in $S_T$ corresponds to a triangle in $T^{(2)}$, we write the equation associated to a triangle coherently with the orientation of its boundary. Since every edge $e$ of $T$ belongs to two triangles, $Z(e)$ appears in exactly two equations with opposite signs. Thus, the sum of all equations in $S_T$ is the trivial equation $0 = 0$, which implies that $\text{rk}(S_T) \leq N_2 - 1$, therefore we have

$$\dim V_T \geq N_1 - (N_2 - 1) = 2g + n - 1.\tag{6}$$

Assume that $e_1, \ldots, e_k$ are the $k$ edges of $T$ which bound an open embedded disk $D$ in $X$. The disk $D$ is divided into triangles in $T^{(2)}$, by taking the sum of the equations corresponding to the triangles contained in $D$, we get

$$\pm Z(e_1) \pm \cdots \pm Z(e_k) = 0.\tag{7}$$

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If \( e \in T^{(1)} \) is an edge the does not belong to the family \( \mathcal{B} \), then \( e \) and some edges in \( \mathcal{B} \) bound an embedded disk in \( X \). It follows from (7) that we can write \( Z(e) \) as a linear function of \( (Z(\tilde{e}_1), \ldots, Z(\tilde{e}_{2g+n-1})) \). Therefore, we have
\[
\dim \mathbb{C} V_T \leq 2g + n - 1.
\]
From (6) and (8), we can conclude that \( \dim \mathbb{C} V_T = 2g + n - 1 \), and the map
\[
F_T : V_T \rightarrow \mathbb{C}^{2g+n-1}
\]
\[
Z \mapsto (Z(\tilde{e}_1), \ldots, Z(\tilde{e}_{2g+n-1}))
\]
is an isomorphism.

There exists a neighborhood \( \mathcal{V} \) of \( (X, \omega, \gamma_1, \ldots, \gamma_m) \) in \( \mathcal{H}^{(m)}(\mathbb{k}) \), such that the period mapping \( \Phi : (X, \omega) \mapsto (\int_{\tilde{e}_1} \omega, \ldots, \int_{\tilde{e}_{2g+n-1}} \omega) \) is a local chart on \( \mathcal{V} \). Consider the map
\[
\Phi \circ \Psi_T : \Psi_T^{-1}(\mathcal{V}) \rightarrow \mathbb{C}^{2g+n-1}.
\]
By definition, we see that \( \Phi \circ \Psi_T(Z) = F_T(Z) = (Z(\tilde{e}_1), \ldots, Z(\tilde{e}_{2g+n-1})) \), therefore \( \Phi \circ \Psi_T \) is a restriction of the map \( F_T \). Since \( F_T \) is an isomorphism, \( \Phi \circ \Psi_T \) is continuous and injective, and the proposition follows. \( \square \)

4. Special triangulation

4.1. Construction. From Proposition 3.2, we know that there is a locally homeomorphic map from some open subset of \( V_T \) into \( \mathcal{H}^{(m)}(\mathbb{k}) \). Our goal in this section is to specify an open subset of \( V_T \) on which this map is one-to-one. For this purpose, we first construct for each surface in an open dense set of \( \mathcal{H}^{(m)}(\mathbb{k}) \) a particular triangulation which will be called special triangulation.

Let \( M := (X, \omega, \gamma_1, \ldots, \gamma_m) \) be a point in \( \mathcal{H}^{(m)}(\mathbb{k}) \), and \( p_1, \ldots, p_n \) be the zeros of \( \omega \). Let \( \xi \) be the vector field which is defined on \( X \setminus \{p_1, \ldots, p_n\} \) by the following condition, in any local chart of \( X \) such that \( \omega = dz \), we have \( \xi = \partial_z \). The orbits of the flow generated by \( \xi \) are called vertical geodesics in \( X \).

Let \( \mathcal{H}^{(m)}(\mathbb{k})^* \) denote the set of \( (X, \xi, \gamma_1, \ldots, \gamma_m) \) verifying

(a) none of the connections \( \gamma_1, \ldots, \gamma_m \) is parallel to the field \( \xi \),

(b) every geodesic ray emanating from the singular points of \( X \), which are called separatrices, in directions \( \pm \xi \) intersects the interior of one of the saddle connections \( \{\gamma_1, \ldots, \gamma_m\} \).

Lemma 4.1. \( \mathcal{H}^{(m)}(\mathbb{k})^* \) is an open dense subset of \( \mathcal{H}^{(m)}(\mathbb{k}) \).

Proof. It is not difficult to see that both conditions (a) and (b) are open conditions, thus \( \mathcal{H}^{(m)}(\mathbb{k})^* \) is an open in \( \mathcal{H}^{(m)}(\mathbb{k}) \).

To see that \( \mathcal{H}^{(m)}(\mathbb{k})^* \) is dense, let us consider an element \( M := (X, \omega, \gamma_1, \ldots, \gamma_m) \in \mathcal{H}^{(m)}(\mathbb{k}) \). By a classical result (see for example [MaT02]), we know that the set of \( \theta \in S^1 \) such that the vertical flow on the surface \( (X, e^{i\theta} \omega) \) is not minimal is countable, which implies that the set \( \Theta = \{\theta \in S^1 \mid \text{the flow in direction } \theta \text{ is minimal} \} \) is dense in \( S^1 \). For any \( \theta \in \Theta \), if \( \gamma \) is a saddle connection, then the direction of \( \gamma \) is different from \( \theta \), and every separatrix in direction \( \theta \) must intersect \( \gamma \) (since such a separatrix is dense in \( X \)). It follows in particular that we can find a sequence \( \{\theta_k\} \subset \Theta \) converging to 0 such that \( (X, e^{i\theta_k} \omega, \gamma_1, \ldots, \gamma_m) \) belongs to \( \mathcal{H}^{(m)}(\mathbb{k})^* \). Therefore \( X \) is contained in the closure of \( \mathcal{H}^{(m)}(\mathbb{k})^* \), and the lemma follows. \( \square \)
Suppose from now on that \( X \) belongs to \( \mathcal{H}^{(m)}(k)^* \). We truncate all the geodesic rays from the singularities of \( \Sigma \) in directions \( \pm \xi \) at their first intersection with the set \( \bigcup_{j=0}^{m} \text{int}(\gamma_j) \). Note that, for any geodesic segment \( \sigma \) in \( \Sigma \) which is not parallel to \( \xi \), by using the isometric embedding of a neighborhood of \( \sigma \) into \( \mathbb{R}^2 \) so that \( \xi \) becomes the vertical field \( \partial_y = (0, 1) \), we can specify the left and right endpoints of \( \sigma \). In what follows, all the geodesic segments are oriented from the left to the right.

Pick a saddle connection \( \gamma \) in the family \( \{\gamma_1, \ldots, \gamma_m\} \). If \( \eta \) is a vertical geodesic ray from a saddle point that intersects \( \text{int}(\gamma) \), we say that \( \eta \) is an upper ray (resp. a lower ray) at \( \gamma \) if \( \eta \) reaches \( \gamma \) from the upper side (resp. lower side). Equivalently, \( \eta \) is an upper side (resp. lower side) at \( \gamma \) if it starts from a singular point in direction \( -\xi \) (resp. \( \xi \)). Let \( A \) and \( B \) denote the left and right endpoints of \( \gamma \) respectively. We denote by \( \eta^+_1, \ldots, \eta^+_k \) the upper rays at \( \gamma \) from the left to the right. The endpoints of \( \eta^+_j \) will be denoted by \( \tilde{P}_j \) and \( Q_j \), where \( Q_j \in \text{int}(\gamma) \), and \( \tilde{P}_j \) is the singularity where the ray \( \eta^+_j \) starts. Note that we may have \( \tilde{P}_j = P_j \), even if \( j' \neq j \).

We will find a family of embedded triangles in \( X \) with disjoint interior whose union contains all the segments \( \eta^+_j \) by the following procedure: let \( j_0 \) be the smallest index such that \( |\eta^+_j| = \min(|\eta^+_j|, j = 1, \ldots, k) \). Using the developing map, we can realize \( \gamma \) and \( \eta^+_j \) as two segments \( \tilde{A}\tilde{B} \) and \( \tilde{P}_{j_0}\tilde{Q}_{j_0} \) respectively in the plane \( \mathbb{R}^2 \), where \( \tilde{Q}_{j_0} \in \tilde{A}\tilde{B} \), and \( \tilde{P}_{j_0}\tilde{Q}_{j_0} \) is vertical. Let \( \tilde{\Delta} \) denote the triangle in \( \mathbb{R}^2 \) with vertices \( \tilde{A}, \tilde{B}, \tilde{P}_{j_0} \).

Claim 4.2. There exists a locally isometric map \( \varphi : \tilde{\Delta} \rightarrow M \) which satisfies:

- \( \varphi(\tilde{A}) = A, \varphi(\tilde{B}) = B, \varphi(\tilde{P}_{j_0}) = P_{j_0} \),
- \( \varphi(\text{int}(\tilde{\Delta})) \) is disjoint from the family \( \{\gamma_1, \ldots, \gamma_m\} \),
- the restriction of \( \varphi \) to \( \text{int}(\tilde{\Delta}) \) is an embedding.

In particular, \( \Delta = \varphi(\tilde{\Delta}) \) is an embedded geodesic triangle in \( X \) whose vertices are singularities of the flat metric.

**Proof.** Let \( (\psi_t)_t \in \mathbb{R} \), denote the flow generated by \( \xi \) on \( X \). Given a point \( p \) in \( X \setminus \{p_1, \ldots, p_n\} \), if there exists \( s > 0 \) such that \( \psi_s(p) \in \bigcup_{j=1}^{m} \gamma_j \cup \{p_1, \ldots, p_n\} \), then for every \( t > s \), we consider by convention that \( \psi_t(p) = \psi_s(p) \).

Set \( t_0 = |\eta^+_{j_0}| \), and consider the set \( L = \bigcup_{0 \leq t \leq t_0} \psi_t(\gamma_j) \subset X \). Note that \( P_{j_0} \in \psi_{t_0}(\gamma_j) \). If \( \psi_t(\gamma_j) \) is disjoint from \( \bigcup_{j=1}^{m} \gamma_j \) for all \( t \leq t_0 \), then \( L \) is isometric to a parallelogram \( P \) in \( \mathbb{R}^2 \) with two vertical sides of length \( t_0 \), the lower side of this parallelogram is \( \tilde{A}\tilde{B} \), and the upper side contains \( \tilde{P}_{j_0} \). In general, \( L \) is isometric to a polygon \( \tilde{L} \) included in the parallelogram \( P \). The additional sides of \( \tilde{L} \) correspond to subsegments of \( \{\gamma_1, \ldots, \gamma_m\} \). The hypothesis \( |\eta^+_{j_0}| = \min(|\eta^+_j|, j = 1, \ldots, k) \) implies that \( \tilde{L} \) is convex. Therefore, \( \tilde{\Delta} \) is included in \( \tilde{L} \).

We have naturally a map \( \varphi : \tilde{\Delta} \rightarrow X \) which is locally isometric such that \( \varphi(\tilde{A}) = A, \varphi(\tilde{B}) = B, \varphi(\tilde{P}_{j_0}) = P_{j_0} \). Since the polygon \( \tilde{L} \) is convex, \( \varphi(\text{int}(\tilde{\Delta})) \cap \bigcup_{j=1}^{m} \gamma_j = \emptyset \).

Assume that \( \varphi(\text{int}(\tilde{\Delta})) \) is not an embedding, then there exists \( D, E \in \text{int}(\tilde{\Delta}) \) such that \( \varphi(D) = \varphi(E) \). Set \( \overrightarrow{D} = \overrightarrow{DE} \in \mathbb{R}^2 \). Since \( X \) is a translation surface, if \( P, Q \in \tilde{\Delta} \) satisfy \( \overrightarrow{PQ} = \pm \overrightarrow{D} \), then we have \( \varphi(P) = \varphi(Q) \). It is not difficult to see that there exists a point \( P \in \tilde{\Delta} \setminus \{A, B, P_{j_0}\} \) such that one of the
following holds

\[
\begin{align*}
  \overrightarrow{PP_{j_0}} &= \pm \nu, \\
  \overrightarrow{PA} &= \pm \nu, \\
  \overrightarrow{PB} &= \pm \nu.
\end{align*}
\]

Since \( \varphi(P) \) must be a regular point of the flat metric, we get a contradiction. Therefore, we can conclude that the restriction of \( \varphi \) to \( \text{int}(\tilde{A}) \) is an embedding. \( \square \)

Let \( \gamma' \) and \( \gamma'' \) denote the saddle connections corresponding to the sides \( AP_{j_0} \) and \( BP_{j_0} \) of \( \Delta \). By construction, \( \eta^+_j \) is contained in the triangle \( \Delta \), and all the other segments in the family \( \{\eta^+_j, j \neq j_0\} \) intersect either \( \gamma' \) or \( \gamma'' \). We can now apply the same arguments to \( \gamma' \) and \( \gamma'' \) and continue the procedure until we get a family of embedded triangles that covers all the segment \( \eta^+_j \).

**Remark 4.3.** Let us make the following important observation on the sides of the triangles we have constructed: if \( \delta \) is a side of one of the triangles above, \( \delta \) is not contained in the family \( \{\gamma_1, \ldots, \gamma_m\} \), then \( \delta \) is one side of an embedded vertical trapezoid \( P \) in \( X \), whose vertical sides corresponds to two vertical segments \( \eta^+_j, \eta^{-}_j \), and the opposite side of \( \delta \) in \( P \) is a subsegment \( Q_iQ_j \) of \( \gamma \). We will call \( Q_iQ_j \) the projection of \( \delta \) on \( \gamma \). Let \( \delta' \) be another side of one of the triangles we obtained, and \( Q_iQ_j' \) be its projection on \( \gamma \). By construction, if \( Q_iQ_j \) and \( Q_iQ_j' \) intersect then either \( Q_iQ_j \subset Q_iQ_j' \), or \( Q_iQ_j' \subset Q_iQ_j \).

By a symmetric procedure, we can find a family of embedded triangles, whose vertices are singularities of the flat metric, with disjoint interiors that covers all the vertical rays that intersect \( \gamma \) from the lower side. Note that for this case, we choose \( j_0 \) to be the largest index such that \( |\eta^-_{j_0}| = \min\{|\eta^-_j|, j = 1, \ldots, \ell\} \), where \( \eta^-_j \) are the lower rays at \( \gamma \).

Applying this construction to all of the saddle connections in the family \( \{\gamma_1, \ldots, \gamma_m\} \), we get a collection of embedded triangles \( \{\Delta_{\alpha}, \alpha \in I\} \) in \( X \) with vertices in the set \( \{p_1, \ldots, p_n\} \).

**Claim 4.4.** The family \( \{\Delta_{\alpha}, \alpha \in I\} \) is a triangulation of \( X \).

*Proof.* We first show that if \( \Delta_{\alpha_1} \) and \( \Delta_{\alpha_2} \) are two triangles in this family then \( \text{int}(\Delta_{\alpha_1}) \cap \text{int}(\Delta_{\alpha_2}) = \emptyset \). Assume that \( \text{int}(\Delta_{\alpha_1}) \cap \text{int}(\Delta_{\alpha_2}) \neq \emptyset \), since a triangle in \( \{\Delta_{\alpha}, \alpha \in I\} \) cannot be included in the other one, there must exist a side \( \delta_1 \) of \( \Delta_{\alpha_1} \), and a side \( \delta_2 \) of \( \Delta_{\alpha_2} \) such that \( \text{int}(\delta_1) \cap \text{int}(\delta_2) \neq \emptyset \). By construction (see Remark 4.3) \( \delta_i \) is one side of an embedded vertical trapezoid \( P_i \). Note that the endpoints of \( \delta_i \) are singularities of \( X \), thus they cannot be included in \( \text{int}(P_i) \). Recall that opposite side of \( \delta_i \) in \( P_i \) is a subsegment of a saddle connection in the family \( \{\gamma_1, \ldots, \gamma_m\} \). By construction, \( \delta_i \) and the vertical sides of \( P_i \) do not intersect the set \( \cup_{1 \leq i \leq m} \text{int}(\gamma_i) \). Using these properties, one can easily check that if \( \text{int}(\delta_1) \cap \text{int}(\delta_2) \neq \emptyset \) then we have a contradiction.

It remains to show that the union of all these triangles is \( X \). Let \( X' \) be the closure in \( X \) of the set \( X \setminus (\cup_{\alpha \in I} \Delta_{\alpha}) \). If \( X' \neq \emptyset \), then \( X' \) is a flat surface with piecewise geodesic boundary. Let \( V' \) be the finite subset of \( X' \) which arises from the singularities of \( X \). We can triangulate \( X' \) by geodesic segments with endpoints in \( V' \). Consider a triangle \( \Delta \) in this triangulation. We can embed isometrically \( \Delta \) into \( \mathbb{R}^2 \) such that \( \xi \) is mapped to the vertical vector field \( \partial_\xi \). Consider the vertical rays starting from the vertices of \( \Delta \), we see that there always exists a ray that intersects \( \text{int}(\Delta) \). This intersection corresponds to a subsegment of a separatrix in direction \( \pm \xi \). But by construction, such a subsegment must be contained in the union of \( \{\Delta_{\alpha}, \alpha \in I\} \), and we have a contradiction. The claim is then proven. \( \square \)
We will call this triangulation constructed above the \textit{special triangulation} of \((X, \omega, \{\gamma_1, \ldots, \gamma_m\})\). By construction, this triangulation is unique. Note that such a triangulation only exists for surfaces in \(\mathcal{H}^{(m)}(k)^*\).

4.2. Characterizing special triangulation. Let \(\Gamma\) be the dual graph of the special triangulation \(T\) of \((X, \omega, \{\gamma_1, \ldots, \gamma_m\})\), the vertices of \(\Gamma\) are the triangles in \(T^{(2)}\), and the (geometric) edges of \(\Gamma\) are the edges in \(T^{(1)}\). Remark that \(\Gamma\) is a trivalent graph, with \(N_2\) vertices and \(N_1\) edges. At each vertex of \(\Gamma\), we have a cyclic ordering on the set of edges incident to this vertex which is induced by the orientation of the surface.

For any \(\gamma_i\), the union of the triangles that covers the vertical separatrices that intersect \(\gamma_i\) from the upper side (resp. lower side) is dual to a subgraph of \(\Gamma\) which is a tree, we denote this tree by \(\Gamma_{i,+}\) (resp. \(\Gamma_{i,-}\)), and choose its root to be the vertex dual to the triangle containing \(\gamma_i\). By construction, all the trees \(\Gamma_{i,\pm}\) are disjoint, and any vertex of \(\Gamma\) belongs to one of those trees. Note that if no separatrix in direction \(-\xi\) (resp. \(\xi\)) reaches \(\text{int}(\gamma_i)\), then the tree \(\Gamma_{i,+}\) (resp. \(\Gamma_{i,-}\)) is empty. Let us define

\begin{definition}
Let \(\Gamma\) be a trivalent graph of \(N_2\) vertices and \(N_1\) edges, the set of edges incident to each vertex is equipped with a cyclic ordering. Let \(\Gamma_{i,\pm}, \ i \in \{1, \ldots, m\}, \ \varepsilon \in \{\pm\}\) be \(2m\) disjoint subgraphs of \(\Gamma\), each of which is a tree with a specified root (\(\Gamma_{i,\pm}\) can be empty). We will say that \((\Gamma, \{\Gamma_{i,\pm}\})\) is an admissible family of graphs if the following conditions are satisfied.

\begin{itemize}
  \item any vertex of \(\Gamma\) belongs to one of the trees \(\Gamma_{i,\pm}\),
  \item for each \(i \in \{1, \ldots, m\}\) the roots of \(\Gamma_{i,+}\) and \(\Gamma_{i,-}\) are connected by an edge of \(\Gamma\).
\end{itemize}
\end{definition}

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{special-triangulation.pdf}
\caption{Special triangulation and dual graphs of a surface in \(\mathcal{H}^{(2)}(2)\).}
\end{figure}

By definition if \((\Gamma, \{\Gamma_{i,\pm}\})\) arises from the special triangulation of \((X, \omega, \{\gamma_1, \ldots, \gamma_m\})\), then \((\Gamma, \{\Gamma_{i,\pm}\})\) is an admissible family of graphs. Let us now choose a numbering of the edges of \(\Gamma\) as follows:

\begin{enumerate}
  \item we assign the edges dual to \(\{\gamma_1, \ldots, \gamma_m\}\) to be \(\{e_1, \ldots, e_m\}\) respectively,
  \item we next number the edges in the trees \(\Gamma_{i,\pm}\) subsequently in such a way that, for each \(\Gamma_{i,\pm}\)
    \begin{enumerate}
      \item if \(e_{i_1}, e_{i_2}\) belong to \(\Gamma_{i,\pm}\), and the distance from \(e_{i_1}\) to the root is smaller than the distance from \(e_{i_2}\) to the root then \(i_1 < i_2\),
      \item if \(e_{i_1}, e_{i_2}, e_{i_3}\), with \(i_1 < i_2 < i_3\), belong to \(\Gamma_{i,\pm}\), and \(e_{i_1}, e_{i_2}, e_{i_3}\) correspond to three sides of the same triangle \(\Delta\), then the cyclic order at the vertex of \(\Gamma_{i,\pm}\) dual to \(\Delta\) is \((e_{i_1}, e_{i_2}, e_{i_3})\).
    \end{enumerate}
  \item we finally number the remaining edges arbitrarily.
\end{enumerate}
In particular, our numbering satisfies the following conditions

(i) the first \( m \) edges are the duals of \( \{ \gamma_1, \ldots, \gamma_m \} \).

(ii) if \( e_i \) and \( e_t \) belong to one of the trees \( \Gamma_{i,e} \) and \( e_i \) is contained in the path from \( e_t \) to the root, then \( i_1 < i_2 \).

(iii) if \( e_i \) belongs to one of the tree \( \Gamma_{i,e} \) and \( e_i \notin \{ e_1, \ldots, e_m \} \) does not belong to any tree, then \( i_1 < i_2 \).

We will call a numbering of the edges of \( \Gamma \) satisfying the conditions above a compatible numbering with respect to the trees \( \{ \Gamma_{i,e} \} \).

By a slight abuse of notation, in what follows we will denote by the same letter an edge of \( \Gamma \) and its dual in \( T^{(1)} \). For any edge \( e_i \) in \( T^{(1)} \), let \( z_i = x_i + iy_i \) be the complex number associated to \( e_i \) (by an embedding of a neighborhood of \( e_i \) in \( \mathbb{R}^2 \) such that \( x_i \) is mapped to \( \partial_i \)). Recall that we have \( \text{card}(T^{(1)}) = N_1 \) and \( \text{card}(T^{(2)}) = N_2 \). By assumption, we have \( x_i \neq 0 \) for all \( i \in \{ 1, \ldots, N_1 \} \). We always choose the orientation of \( e_i \) so that \( x_i > 0 \).

Set \( Z = (z_1, \ldots, z_{N_1}) \in \mathbb{C}^{N_1} \). Each triangle in \( T^{(2)} \) (vertices of \( \Gamma \)) gives an equation of the form \( \text{det}(Z) = 0 \). The collection of all of these equations gives us a system \( S_T \). Let \( V_T \) denote the subspace of \( \mathbb{C}^{N_1} \) consisting of solutions of \( S_T \). Recall that from Lemma 3.1, we know that \( \dim_{\mathbb{C}} V_T = 2g + n - 1 \). Clearly we have \( Z \in V_T \). We wish now to describe the additional conditions that \( Z \) must satisfy. For this, let us pick a triangle \( \Delta \) in \( T^{(2)} \), and let \( (e_i, e_2, e_3) \) be the sides of \( \Delta \) written in the cyclic order induced by the orientation of \( X \), where \( i_1 = \min \{ i_1, i_2, i_3 \} \). Let \( \Gamma_\alpha \) be the tree in the family \( \{ \Gamma_{i,e} \} \) that contains the vertex of \( \Gamma \) dual to \( \Delta \).

(A) If \( \Delta \) is the root of \( \Gamma_\alpha \), then \( e_{i_1} \in \{ e_1, \ldots, e_m \} \). If \( e_{i_1} \notin \{ e_1, \ldots, e_m \} \), then \( e_{i_1} \) is the unique edge of \( \Gamma_\alpha \) that is included in the path from \( \Delta \) to the root. By construction, in both cases we always have

\[
0 < x_{i_2} < x_{i_1} \quad \text{and} \quad 0 < x_{i_3} < x_{i_1}.
\]

We will call \( e_{i_1} \) the base of \( \Delta \).

(B) Given \( v = (x, y) \) and \( v' = (x', y') \) two vectors in \( \mathbb{R}^2 \), let us define

\[
 v \wedge v' = \det \begin{pmatrix} x & x' \\ y & y' \end{pmatrix} = xy' - x'y.
\]

Considering a complex number \( z \) as a vector in \( \mathbb{R}^2 \), we have

\[
 A(\Delta) = \frac{1}{2} z_{i_2} \wedge z_{i_1} = \frac{1}{2} \text{Im}(z_{i_1} \bar{z}_{i_2}).
\]

Thus, \( Z \) must satisfy

\[
 \text{Im}(z_{i_1} \bar{z}_{i_2}) = x_{i_2}y_{i_1} - x_{i_1}y_{i_2} > 0.
\]

(C) Removing \( e_{i_1} \) from the tree \( \Gamma_\alpha \), we get two subtrees \( \Gamma'_\alpha \) and \( \Gamma''_\alpha \), where \( \Gamma''_\alpha \) is the one that contains \( \Delta \). The union of triangles of \( T \) dual to the vertices of \( \Gamma'_{\alpha} \) can be identified with a polygon \( P \) in \( \mathbb{R}^2 \) which has a side corresponding to \( e_{i_1} \). Note that if \( \alpha = (+, +) \), then \( e_{i_1} \) is the lower side of \( P \) (that is all vertical downward rays from the vertices of \( P \) intersect \( e_{i_1} \)), and if \( \alpha = (+, -) \), then \( e_{i_1} \) is the upper side of \( P \) (all vertical upward rays from the vertices of \( P \) intersect \( e_{i_1} \)).

Without loss of generality, we can assume that \( e_{i_1} \) is the lower side of \( P \). The triangle \( \Delta \) is identified with a triangle included in \( P \) whose vertices are denoted by \( A, B, C \), where \( A \) and \( B \)
are the left and right endpoint of $e_i$ respectively. Note that by the convention on the cyclic ordering, we have $e_i = CB, e_i = AC$.

If $P$ is a vertex of $P$ different from $A$ and $B$, we denote by $\hat{P}$ the intersection of the vertical (downward) ray from $P$ with $\text{int}(AB)$, we will call $\hat{P}$ the projection of $P$ to $AB$. Let us denote by $h(P)$ the length of the segment $P\hat{P}$. We denote the vertices of $P$ whose projection to $AB$ is between $A$ and $\hat{C}$ by $D_1, \ldots, D_r$ so that $\hat{D}_i$ is between $A$ and $\hat{D}_{i+1}$. Similarly, we denote the vertices of $P$ whose projection to $AB$ is between $\hat{C}$ and $B$ by $E_1, \ldots, E_s$ in such a way that $\hat{E}_j$ is between $\hat{C}$ and $\hat{E}_{j+1}$.

Recall that by construction, we have
\begin{equation}
 h(C) < h(D_i), \forall i = 1, \ldots, r,
\end{equation}
and
\begin{equation}
 h(C) \leq h(E_j), \quad j = 1, \ldots, s.
\end{equation}

It is easy to check that the condition $h(C) < h(D_i)$ is equivalent to $\overrightarrow{D_i\hat{C}} \wedge \overrightarrow{AB} > 0$. As $\overrightarrow{D_i\hat{C}} = \overrightarrow{D_i\hat{D}_{i+1}} + \cdots + \overrightarrow{D_i\hat{C}}$, and the sides of $P$ are the edges of $T$, we can write $\overrightarrow{D_i\hat{C}}$ as a linear function $f_i$ of $Z$. Consequently, the condition on $h(D_i)$ becomes
\begin{equation}
 \text{Im}(z_i f_i) > 0, \forall i = 1, \ldots, r.
\end{equation}

Similarly, for $E_j$, $j = 1, \ldots, s$, we have $h(C) \leq h(E_j)$, and these conditions are equivalent two
\begin{equation}
 \text{Im}(z_i g_j) \geq 0, \forall j = 1, \ldots, s.
\end{equation}

where $g_j$ are some linear functions of $Z$. The functions $f_i, g_j$ are completely determined by the tree $\Gamma'$. Note also that, a priori, we cannot replace the inequality in (12) by a strict one.

Let $(\Gamma, \{\Gamma_i, i\})$ be an admissible family of graphs, and choose a compatible numbering of the edges of $\Gamma$. Let $V$ be the space of solutions in $\mathbb{C}^{N_1}$ of the system of linear equations associated to the vertices of $\Gamma$. Let $\mathcal{D}$ be the subset of $V$ which is defined by the inequations (9), (10), (11), (12). For every $Z = (z_1, \ldots, z_{N_1}) \in \mathcal{D}$, we define $\Psi(Z)$ to be the triple $(X, \omega, \gamma_1, \ldots, \gamma_m)$, where
\begin{itemize}
  \item $X$ is the surface obtained by gluing the triangles constructed from the coordinates of $Z$ (using the graph $\Gamma$).
  \item $\omega$ is the holomorphic one-form on $X$ satisfying $\omega = dz$ in each triangle.
  \item $\gamma_i$ is the saddle connection in $\Sigma$ corresponding to the edge of $\Gamma$ which connects the roots of $\Gamma_{i, r}$ and $\Gamma_{i, l}$.
\end{itemize}

**Proposition 4.6.** Let $\mathcal{D}_0$ be a component of $\mathcal{D}$. If $\Psi(Z) \in \overrightarrow{\mathcal{H}(\mathcal{M})}(\mathbb{C})$ for some $Z \in \mathcal{D}_0$, then $\Psi(\mathcal{D}_0) \subset \overrightarrow{\mathcal{H}(\mathcal{M})}(\mathbb{C})$, and $\Psi$ realizes a homeomorphism from $\mathcal{D}_0$ onto its image.

**Proof.** Since the topological invariants of $\Psi(Z)$ depends continuously on $Z$, they are constant on each connected component of $\mathcal{D}$.

If $(X, \omega, \gamma_1, \ldots, \gamma_m) = \Psi(Z)$, the surface $X$ can be constructed as follows: for each tree $\Gamma_{i, r}$, we construct the polygon $P_{i, r} \in \mathbb{R}^2$ by gluing the triangles corresponding to the vertices of $\Gamma_{i, r}$, we then glue the polygons $P_{i, r}$ to get $X$. Note that by construction, we obtain $X$ together with an admissible triangulation $T$ whose dual graph is $\Gamma$. Since $Z$ satisfies the inequations (9), (10), (11), (12), $T$ is the special triangulation of $(X, \omega, \gamma_1, \ldots, \gamma_m)$. If we have $\Psi(Z) = \Psi(Z')$ but $Z' \neq Z$, then we obtain an
element in \( \tilde{\mathcal{G}}^{(m)}(k) \) with two special triangulations, which is impossible. It is clear that \( \Psi \) is a locally homeomorphic. The proposition is then proven.

\[ \square \]

Since \( D \) is defined by finitely many inequations, it has only finitely many components. Remark that the action of the subgroup \( U_- = \left( \begin{smallmatrix} 1 & 0 \\ 0 & \lambda \end{smallmatrix} \right) \) of \( \text{GL}^+(2, \mathbb{R}) \) on \( \mathbb{C}^{N_1} \) preserves the inequations (9), (10), (11), (12). Therefore, we have shown the following

**Theorem 4.7.** There exists a partition of \( \widetilde{\mathcal{G}}^{(m)}(k)^* \) into finitely many subsets which are \( U_- \)-invariant. Each subset in this partition is homeomorphic to a component of the subset of a linear subspace of dimension \( 2g + n - 1 \) of \( \mathbb{C}^{N_1} \), which is determined by the inequations (9), (10), (11), (12) via the map \( \Psi \).

5. Proof of Theorem 2.3

5.1. Primary and auxiliary families of indices. Given an admissible family of graphs \( (\Gamma, \{\Gamma_{i, \pm}\}) \), where \( \Gamma \) is the dual graph of a special triangulation \( T \) of an element \( (X, \omega, \{\gamma_1, \ldots, \gamma_m\}) \in \mathcal{G}^{(m)}(k) \), let us choose a compatible numbering of the edges of \( \Gamma \). We denote by \( S_\Gamma \) the systems of linear equations of type (5) associated to \( \Gamma \), and by \( V_\Gamma \) the subspace of \( \mathbb{C}^{N_1} \) consisting of solution of \( S_\Gamma \).

In what follows, we will say that a family of coordinates \( \{x_i, i \in I\} \) is dependent with respect to \( S_\Gamma \) if there exists a vector \( (\lambda_i)_{i \in I} \in \mathbb{R}^I \) such that \( \sum_{i \in I} \lambda_i x_i = 0 \), for all \( \{x_1, \ldots, x_N\} \in V_\Gamma \). The family \( \{x_i, i \in I\} \) is said to be independent with respect to \( S_\Gamma \), if such a vector does not exist.

We have another interpretation of dependent and independent families as follows: let \( a_i, i = 1, \ldots, N_2 \), denote the row vectors corresponding to the equations of the system \( S_\Gamma \), we consider \( a_i \) as an element of the space \( (\mathbb{C}^{N_1})^* \) which is equivariant by conjugation. Set \( V_{\Gamma}^* = \text{Vect}_\mathbb{C}\{a_1, \ldots, a_{N_2}\} \subset (\mathbb{C}^{N_1})^* \). For any family \( I \subset \{1, \ldots, N_1\} \), set \( \mathbb{R}^I = \{ (\lambda_1, \ldots, \lambda_N) : \lambda_i = 0 \text{ if } i \notin I \} \subset \mathbb{R}^N \), we regard \( \mathbb{R}^I \) as a subspace of the space of \( \mathbb{C} \)-linear forms \( (\mathbb{C}^{N_1})^* \) equivariant by conjugation. If \( a = (\lambda_i)_{i \in I} \in \mathbb{R}^I \cap V_{\Gamma}^* \), then \( a(Z) = \sum_{i \in I} \lambda_i z_i = 0 \) for all \( Z = (z_1, \ldots, z_N) \in V_\Gamma \). Thus \( I \) is an independent family if and only if \( \mathbb{R}^I \cap V_{\Gamma}^* = \{0\} \).

**Remark 5.1.**

- Since the coefficients of \( a_i \) are in \( \{0, \pm 1\} \), if the family \( \{x_i, i \in I\} \) is dependent, we can choose the vector \( (\lambda_i)_{i \in I} \) such that \( \lambda_i \in \mathbb{Z} \), for all \( i \in I \).
- Let \( e_i \) denote the edge of \( T \) which corresponds to \( z_i \). A family \( I \) is independent with respect to \( S_\Gamma \) if and only if the family \( \{e_i, i \in I\} \) is independent in \( H_1(X, \{p_1, \ldots, p_n\}; \mathbb{Z}) \).

**Definition 5.2.** The primary family of indices of \( (\Gamma, \{\Gamma_{i, \pm}\}) \) is the unique ordered subset \( (i_1, \ldots, i_{2g+n-1}) \) of \( \{1, \ldots, N_1\} \) that satisfies the following properties

- \( i_k < i_{k+1} \),
- \( i_1, \ldots, i_{2g+n-1} \) is an independent family of indices with respect to \( S_\Gamma \),
- For all \( k \in \{1, \ldots, 2g+n-1\} \), if \( i < i_k \) then the family \( (i_1, \ldots, i_{k-1}, i) \) is dependent with respect to \( S_\Gamma \).

**Remark 5.3.** The primary family of indices can be found inductively by the following procedure: first we take \( i_1 = 1, \ldots, i_m = m \), recall that \( e_1 = \gamma_1, \ldots, e_m = \gamma_m \), and by assumption \( (\gamma_1, \ldots, \gamma_m) \) is independent in \( H_1(X, \{p_1, \ldots, p_n\}; \mathbb{Z}) \), hence the family \( \{i_1, \ldots, i_m\} \) is independent with respect to \( S_\Gamma \). Assume that we already have an independent family \( (i_1, \ldots, i_k) \), then \( i_{k+1} \) is the smallest index such that \( (i_1, \ldots, i_k, i_{k+1}) \) is independent.
Let us denote by $I$ the primary family of index of $(\Gamma, (\Gamma_{i,\pm}))$.

**Definition 5.4.** An auxiliary family of $I$ is an ordered family of indices $J = (j_{m+1}, \ldots, j_{2g+n-1})$ which satisfies the following condition: for any $k \in \{m + 1, \ldots, 2g + n - 1\}$, and $e_{j_k}$ is the base of one of the triangles in $T$ that contains $e_{i_k}$. In particular, we have $j_k < i_k$.

**Remark 5.5.** By definition, we have $j_k < i_k$, thus $(i_1, \ldots, i_{k-1}, j_k)$ is a dependent family. Since the family $(i_1, \ldots, i_{k-1})$ is independent, it follows that $z_{j_k}$ is a linear function of $(z_{i_1}, \ldots, z_{i_{k-1}})$ in $V\Gamma$.

An auxiliary family of $I$ can be found as follows: for each $i_k \in I$, $k > m$, consider the corresponding edge $e_{i_k}$ of $\Gamma$. We have two cases:

- $e_{i_k}$ belongs to some tree $\Gamma_{i_k}$. In this case, let $\Delta_k$ be the triangle of $T$ dual to the endpoint of $e_{i_k}$ which is closer to the root of $\Gamma_{i_k}$. Let $e'_{i_k}$ and $e''_{i_k}$ be the other sides of $\Delta_k$ where $e'_{i_k} > e''_{i_k}$. We have $i_k \neq \min\{i_k, i'_{k}, i''_{k}\}$ by the definition of compatible numbering. Thus $i_{j_k} = \min\{i_k, i'_{k}, i''_{k}\}$, which implies that $e_{i_{j_k}}$ is the base of $\Delta_k$, and we can choose $j_k$ to be $i''_{k}$.
- $e_{i_k}$ does not belong to any of the trees $\Gamma_{i_k}$. Let $\Delta_k$ be one of the two triangles in $T$ that contains $e_{i_k}$. Again let $e'_{i_k}$ and $e''_{i_k}$ be the other sides of $\Delta_k$ where $e'_{i_k} > e''_{i_k}$. By the definition of compatible numbering, we also have $i_k \neq \min\{i_k, i'_{k}, i''_{k}\}$, hence we can choose $j_k$ to be $i''_{k}$.

**5.2. Proof of Theorem 2.3.**

**Proof.** Fix an admissible family of graphs $(\Gamma, (\Gamma_{i,\pm}))$ together with a compatible numbering of the edges of $\Gamma$. Let $I = (1, \ldots, i_{2g+n-1})$ be the primary family of indices of $(\Gamma, (\Gamma_{i,\pm}))$ and $J = (j_{m+1}, \ldots, j_{2g+n-1})$ an auxiliary family of indices of $I$. Since $\text{card}(I) = \dim C V\Gamma$, there exists a linear isomorphism with rational coefficients

$$F: \mathbb{C}^{2g+n-1} \rightarrow \mathbb{V}_{\Gamma}$$

$$(z_1, \ldots, z_{2g+n-1}) \mapsto (f_1(z_1, \ldots, z_{2g+n-1}), \ldots, f_N(z_1, \ldots, z_{2g+n-1}))$$

which satisfies $f_i = z_i$, $\forall i \in I$. To simplify the notations, we write $f_k$ instead of $f_{i_k}$ for every $j_k \in J$.

Let $D$ be the domain of $V\Gamma$ which is defined by the inequations (9), (10), (11), (12). By a slight abuse of notations, we will denote also by $D$ the corresponding domain in $\mathbb{C}^{2g+n-1}$ (via $F$). For every $(z_1, \ldots, z_{2g+n-1}) \in D$, let $A((z_1, \ldots, z_{2g+n-1}))$ denote the area of the surface constructed from $F((z_1, \ldots, z_{2g+n-1}))$.

Let us write $z_i = x_i + iy_i$, with $x_i, y_i \in \mathbb{R}$, $i = 1, \ldots, 2g + n - 1$, and $f_k = a_k + ib_k$, $a_k, b_k \in \mathbb{R}$, $k = m + 1, \ldots, 2g + n - 1$. Remark that the coefficients of $f_k$ are real, therefore $a_k$ is a real linear function of $(x_1, \ldots, x_k-1)$, and $b_k$ is a real linear function of $(y_1, \ldots, y_k-1)$.

To prove the theorem, is enough to show that

$$\int_D e^{-\frac{i}{\epsilon_1^2 + \cdots + \epsilon_m^2 + \epsilon_n^2 - \epsilon_1^2 \epsilon_2^2 \cdots \epsilon_m^2}} dx_1 dy_1 \cdots dx_{2g+n-1} dy_{2g+n-1} < K \epsilon_1^2 \cdots \epsilon_m^2.$$  

To prove the inequality (13) we will only make use of the conditions (9), and (10). We first observe that the inequation (9) implies that for any $(x_1 + iy_1, \ldots, x_{2g+n-1} + iy_{2g+n-1}) \in D$, we have

$$0 < x_k < a_k, \ k = m + 1, \ldots, 2g + n - 1.$$  

Set

$$D^* = \{(z, x), z \in \mathbb{C}^m, x \in \mathbb{R}^{2g+n-1-m}, (z, x) \text{ satisfies (14)}\} \subset \mathbb{C}^m \times \mathbb{R}^{2g+n-1-m}.$$
For any \((z, x) \in \mathcal{D}\), set
\[
\mathcal{Y}(z, x) = \{ y = (y_{m+1}, \ldots, y_{2g+n-1}) \in \mathbb{R}^{2g+n-1-m} : (z, x + ty) \in \mathcal{D}\}.
\]
The inequality (13) now becomes
\[
(15) \quad I := \int_{\mathcal{Y}(z, x)} e^{-\lambda_1 |z_1|^2 - \lambda_2 |z_2|^2} \left( \int_{\mathcal{Y}(z, x)} e^{-\lambda y_{m+1} \ldots y_{2g+n-1}} d\eta_{2g+n-1-m} \right) d\eta_{2g+n-1} < K e_1^2 \ldots e_m^2.
\]
where \(\lambda_{2g+n-1-m} = dx_{m+1} \ldots dx_{2g+n-1}\) and \(d\eta_{2g+n-1} = dx_1 dy_1 \ldots dx_m dy_m\) are the Lebesgue measures of \(\mathbb{R}^{2g+n-1-m}\) and \(\mathbb{C}^m\) respectively.

For every \(k \in \{m+1, \ldots, 2g + n - 1\}\), let \(\Delta_k\) denote the triangle in \(T\) that contains \(e_{i_k}\) and \(e_{j_k}\).

**Claim 5.6.** If \(k \neq k'\), then \(\Delta_k \neq \Delta_{k'}\).

**Proof.** We can assume that \(k < k'\). If \(\Delta_k = \Delta_{k'}\), then \(e_{i_k}, e_{j_k}, e_{i_{k'}}\) are contained in the same triangle, which implies that \(z_{i_k}\) is a linear function of \((z_{i_k}, z_{j_k})\) in \(V_1\). By assumption, \(z_{j_k}\) is a linear function of \((z_{i_k}, \ldots, z_{i_{k'}})\), thus \(z_{i_k}\) is a linear function of \((z_{i_k}, \ldots, z_{i_{k'}})\) which is impossible since \((i_1, \ldots, i_{k'})\) is an independent family. \(\square\)

Let \(\eta_k\) be the area of \(\Delta_k\). The previous claim implies immediately that
\[
(16) \quad A > \sum_{k=m+1}^{2g+n-1} \eta_k.
\]
As functions on \(\mathbb{C}^{2g+n-1}\), \(\eta_k\) are given by
\[
\eta_k = \pm \frac{1}{2} \text{Im}(f_k \bar{z}_k) = \pm \frac{1}{2} (a_k y_k - x_k b_k).
\]
Since \(b_k\) is a linear function of \((y_1, \ldots, y_{k-1})\), we have
\[
\frac{\partial \eta_k}{\partial y_j} = \begin{cases} \frac{1}{2} x_j & \text{if } j < k, \\ \frac{1}{2} a_k & \text{if } j = k, \\ 0 & \text{if } j > k, \end{cases}
\]
and
\[
d\eta_{m+1} \ldots d\eta_{2g+n-1} = \frac{1}{2^{2g+n-1-m}} \det \begin{pmatrix} \pm a_{m+1} & 0 & \ldots & 0 \\ * & \pm a_{m+2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \ldots & \pm a_{2g+n-1} \end{pmatrix} |dy_{m+1} \ldots dy_{2g+n-1}|.
\]
Therefore, we have
\[
dy_{m+1} \ldots dy_{2g+n-1} = \frac{2^{2g+n-1-m}}{a_{m+1} \ldots a_{2g+n-1}} d\eta_{m+1} \ldots d\eta_{2g+n-1}.
\]
It follows from (16) that
\[
I(z, x) := \int_{\mathcal{Y}(z, x)} e^{-A} dy_{m+1} \ldots dy_{2g+n-1} < \frac{2^{2g+n-1-m}}{a_{m+1} \ldots a_{2g+n-1}} \int_{\mathcal{Y}(z, x)} e^{-\eta_{m+1} \ldots \eta_{2g+n-1}} d\eta_{m+1} \ldots d\eta_{2g+n-1}.
\]
The conditions (10) mean that the functions \( \eta_k \) are positive on \( \mathcal{Y}(z, \lambda) \). Thus we have

\[
I(z, \lambda) < \frac{2^{2g+n-1-m}}{a_{m+1} \cdots a_{2g+n-1}} \int_0^{\infty} e^{-\eta_{m+1}} d\eta_{m+1} \cdots \int_0^{\infty} e^{-\eta_{2g+n-1}} d\eta_{2g+n-1} = \frac{2^{2g+n-1-m}}{a_{m+1} \cdots a_{2g+n-1}},
\]

and

\[
I < 2^{2g+n-1-m} \int_{\mathcal{D}^*} e^{-\frac{|z_1|^2}{\epsilon_1^2} + \cdots + \frac{|z_m|^2}{\epsilon_m^2}} \left( \left( \int_0^{a_{m+1}} \left( \left( \int_0^{a_{2g+n-1}} \frac{1}{a_{m+1} \cdots a_{2g+n-1}} dx_{2g+n-1} \right) \cdots dx_{m+1} \right) \right) \right) dx_{2g+n-1} \cdots dx_{m+1} dh_{2m}.
\]

From the definition of \( \mathcal{D}^* \), the right hand side of (17) is equal to

\[
2^{2g+n-1-m} \int_{\mathcal{D}^*} e^{-\frac{|z_1|^2}{\epsilon_1^2} + \cdots + \frac{|z_m|^2}{\epsilon_m^2}} \left( \left( \int_0^{a_{m+1}} \left( \left( \int_0^{a_{2g+n-1}} \frac{1}{a_{m+1} \cdots a_{2g+n-1}} dx_{2g+n-1} \right) \cdots dx_{m+1} \right) \right) \right) dx_{2g+n-1} \cdots dx_{m+1} dh_{2m}.
\]

Since \( a_{m+1}, \ldots, a_k \) do not depend on \( x_k \), we have

\[
\int_0^{a_{m+1}} \left( \left( \int_0^{a_{2g+n-1}} \frac{1}{a_{m+1} \cdots a_{2g+n-1}} dx_{2g+n-1} \right) \cdots dx_{m+1} \right) = 1.
\]

Thus

\[
I < 2^{2g+n-1-m} \int_{\mathcal{D}^*} e^{-\frac{|z_1|^2}{\epsilon_1^2} + \cdots + \frac{|z_m|^2}{\epsilon_m^2}} dx_1 dy_1 \cdots dx_m dy_m = 2^{2g+n-1-m} \pi m^2 \epsilon_1^2 \cdots \epsilon_m^2.
\]

The inequality (15) is then proven. The proof of Theorem 2.3 is now complete. \( \square \)

6. Quadratic Differentials

6.1. Half-translation surface with marked saddle connections. Let us denote by \( \tilde{Q}^{(m)}(\mathcal{D}) \) the set of triple \( (M, \phi, \{\gamma_1, \ldots, \gamma_m\}) \), where

- \((X, \phi)\) is an element of \( Q(\mathcal{D}) \),
- \( \{\gamma_1, \ldots, \gamma_m\} \) be an independent family of saddle connections in \( X \).

There exists a natural projection \( q : \tilde{Q}^{(m)}(\mathcal{D}) \to Q(\mathcal{D}) \) consisting of forgetting the markings of the saddle connections. This projection is locally homeomorphic, hence we can pullback the Lebesgue measure \( d\mu \) of \( Q(\mathcal{D}) \) to \( \tilde{Q}^{(m)}(\mathcal{D}) \). We define a function \( \tilde{\delta}_x : \tilde{Q}^{(m)}(\mathcal{D}) \to \mathbb{R} \) as follows

\[
\tilde{\delta}_x : (M, \phi, \{\gamma_1, \ldots, \gamma_m\}) \mapsto \exp\left( - \sum_{j=1}^{m} \frac{|\gamma_j|}{\epsilon_j^2} - A(X) \right).
\]

Theorem 1.4 is a direct consequence of the following theorem

**Theorem 6.1.** There exists a constant \( K = K(d, m) \) such that

\[
\int_{\tilde{Q}^{(m)}(\mathcal{D})} \tilde{\delta}_x d\mu < K \epsilon_1^2 \cdots \epsilon_m^2.
\]
6.2. Proof of Theorem 1.4

Proof. Let \( Q_m(d) \) be the subset of \( Q(d) \) consisting of pairs \( (X, \phi) \) such that there exists an independent family of \( m \) saddle connections \( \{\gamma_1, \ldots, \gamma_m\} \) on \( X \) satisfying \( |\gamma_j|^2 < \epsilon_j^2 A(X) \). Let \( \tilde{Q}_m(d) \) be the pre-image of \( Q_m(d) \) by the natural projection \( \tilde{q} : \tilde{Q}_m(d) \to Q(d) \). Theorem 6.1 implies immediately that

\[
\int_{\tilde{Q}_m(d)} \tilde{\gamma}_d d\mu < K \epsilon_1^2 \ldots \epsilon_m^2.
\]

Set

\[
f_\epsilon : Q(d) \to \mathbb{R}, \quad (X, \phi) \mapsto e^{-A(X)} \sum_{\{\gamma_1, \ldots, \gamma_m\} \text{ independent family in } X} e^{-|\gamma_1|^2/\epsilon_1^2 - \cdots - |\gamma_m|^2/\epsilon_m^2}.
\]

We have

\[
\int_{\tilde{Q}_m(d)} \tilde{\gamma}_d d\mu = \int_{Q_m(d)} f_\epsilon d\mu.
\]

But by definition we have \( f_\epsilon(.) > e^{-(m+1)A(.)} \) on \( Q_m(d) \). Therefore

\[
\int_{Q_m(d)} e^{-(m+1)A(.)} d\mu < K \epsilon_1^2 \ldots \epsilon_m^2.
\]

Now, the space \( Q(d) \) can be locally identified with an open in \( \mathbb{R}^{2d} \), where \( d = \dim_C Q(d) \), so that \( d\mu \) is the Lebesgue measure. In this identification, \( Q_1(d) \) becomes the hypersurface \( Q_1 \) defined by the equation \( A = 1 \). Using the change of variables

\[
\Phi : Q_1 \times [0, +\infty[ \to \mathbb{R}^{2d}, \quad (v, t) \mapsto t \cdot v,
\]

we have \( \Phi^* A = t^2 \), and \( \Phi^* d\mu = 2t^{2d-1} dt d\mu_1 \). The inequality (19) then implies

\[
\int_{Q_1(d)} \int_0^{+\infty} 2t^{2d-1} e^{-(m+1)t^2} dt d\mu_1 < K \epsilon_1^2 \ldots \epsilon_m^2
\]

Thus,

\[
\mu_1(Q_1(d)) < C \epsilon_1^2 \ldots \epsilon_m^2.
\]

\( \square \)
6.3. **Symmetric special triangulation.** Given an element \((X, \phi, \{\gamma_1, \ldots, \gamma_m\})\) of \(\overline{Q}^{(m)}(d)\), let \(\pi : \hat{X} \to X\) be the canonical double covering of \(X\), and \(\tau\) be the covering involution of \(\hat{X}\). Let \(\hat{\omega}\) be the holomorphic one-form on \(\hat{X}\) such that \(\pi^* \phi = \hat{\omega}^2\). We first notice that \((\hat{X}, \hat{\omega})\) belongs to some stratum \(\mathcal{H}^*(\hat{L})\), where the vector \(\hat{L}\) depends only on the component of \(Q(d)\) that contains \((X, \phi)\). We denote by \(\hat{\gamma}_1, \hat{\gamma}_2\) the pre-images of \(\gamma_i\) by \(\pi\), which are saddle connections in \(\hat{X}\). Note that the orientation of \(\hat{\gamma}_i\) is chosen so that \(\tau(\hat{\gamma}_i) = -\hat{\gamma}_i\) (which implies \(\tau(\hat{\gamma}_i^c) = -\hat{\gamma}_i^c\)). By assumption, we have \(\text{int}(\hat{\gamma}_i) \cap \text{int}(\hat{\gamma}_i^c) = \emptyset\) unless \((i, s) = (i', s')\).

Given \(v \in S^1\), the direction \(v\) is not well defined on \(X\), but the notion of “parallel” is well defined. In what follows, we will call a geodesic segment (resp. separatrix, saddle connection) parallel to the vectors \((0, \pm 1)\) a **vertical** segment (resp. separatrix, saddle connection).

Let us recall briefly a way to define the local chart on a neighborhood of \((X, \phi, \{\gamma_1, \ldots, \gamma_m\})\). Set \(\hat{S} = \pi^{-1}(S)\), where \(S\) is the set of zeros and poles of \(\phi\). Let \(H_1(\hat{X}, \hat{S}; \mathbb{Z})^-\) be the set of cycles \(c \in H_1(\hat{X}, \hat{S}; \mathbb{Z})\) such that \(\pi(c) = -c\). A surface \((X', \phi')\) close to \((X, \phi)\) gives rise to a surface \((\hat{X}', \hat{\omega}')\) close to \((\hat{X}, \hat{\omega})\) together with an involution \(\tau\) in the same homotopy class as \(\tau\). Thus we can identify \(H_1(\hat{X}, \hat{S}; \mathbb{Z})^-\) with \(H_1(\hat{X}', \hat{S}; \mathbb{Z})^-\). Choose a basis \(\{c_1, \ldots, c_d\}\) of \(H_1(\hat{X}, \hat{S}; \mathbb{Z})^-\), then the period mapping \(\hat{\Phi} : (X', \phi') \mapsto (\int_{c_1} \hat{\omega}', \ldots, \int_{c_d} \hat{\omega}')\) is a local chart for \(Q(d)\), in which the volume form \(\mu\) is identified with the Lebesgue measure of \(\mathbb{C}^d\). If \((X', \phi')\) is close enough to \((X, \phi)\) in \(Q(d)\), then there exists a homeomorphism \(\varphi : X \to X'\) such that \(\varphi(\gamma_1), \ldots, \varphi(\gamma_m)\) is a family of saddle connections with disjoint interior. Therefore, \(\hat{\Phi}\) can also be viewed as a local chart for \(\overline{Q}^{(m)}(d)\) in a neighborhood of \((X, \phi, \{\gamma_1, \ldots, \gamma_m\})\).

Let \(\tilde{Q}^{(m)}(d)^*\) be the set of \((X, \phi, \{\gamma_1, \ldots, \gamma_m\}) \in \overline{Q}^{(m)}(d)\) which satisfy the following condition: every vertical separatrix intersects the set \(\bigcup_{1 \leq i \leq m} \text{int}(\gamma_i)\) before reaching a singular point. The following lemma follows from the same argument as Lemma 4.1.

**Lemma 6.2.** \(\tilde{Q}^{(m)}(d)^*\) is an open dense subset, hence of full measure, of \(\overline{Q}^{(m)}(d)\).

Suppose that \((X, \phi, \{\gamma_1, \ldots, \gamma_m\})\) belongs to \(\overline{Q}^{(m)}(d)^*\), then \((\hat{X}, \hat{\omega}, \{\hat{\gamma}_1, \hat{\gamma}_2, \ldots, \hat{\gamma}_m, \hat{\gamma}_m^c\})\) belongs to \(\tilde{H}^{(2m)}(\hat{L})^*\). Let \(\hat{T}\) be the special triangulation of \(\hat{X}\) with respect to the family \(\{\hat{\gamma}_i\}\). Since we have \(\tau^2 \hat{\omega} = -\hat{\omega}\), it follows that the involution \(\tau\) sends a separatrix direction \((0, 1)\) to a separatrix direction \((0, -1)\) and vice versa. By definition, the family of saddle connections \(\{\hat{\gamma}_i\}\) is invariant under \(\tau\), therefore, \(\tau\) induces a permutation on set of vertical separatrices joining singular points of \(\hat{X}\) to the segments \(\hat{\gamma}_i\). As a consequence, the triangulation \(\hat{T}\) is invariant under \(\tau\). We deduce in particular that \(\tau\) induces an involution on the set \(\hat{T}^{(1)}\).

Let \(\hat{N}_1\) and \(\hat{N}_2\) be the number of edges and of triangles in \(\hat{T}\) respectively. We choose the orientations of the edges of \(\hat{T}\) so that \(\hat{\omega}(t(e)) = -\hat{\omega}(e)\) for all \(e \in \hat{T}^{(1)}\). We consider a vector \(Z \in \mathbb{C}^{\hat{N}_1}\) as a function from \(\hat{T}^{(1)}\) to \(\mathbb{C}\). We have a system \(S_{\hat{T}}\) of \(\hat{N}_2\) linear equations of the form \(\mathbb{C}\), each of which corresponding to a triangle in \(\hat{T}^{(2)}\). We add to this system the equations

\[
Z(e') = Z(e)
\]

where \(e, e' \in \hat{T}^{(1)}\) such that \(\tau(e) = -e'\). Let \(S_{\hat{T}}\) denote the resulting system, and \(\hat{V}_{\hat{T}} \subset \mathbb{C}^{\hat{N}_1}\) denote the space of solutions of \(S_{\hat{T}}\). Clearly, the integrals of \(\hat{\omega}\) along the edges of \(\hat{T}\) gives us a vector in \(\hat{V}_{\hat{T}}\). Given \(Z' \in \hat{V}_{\hat{T}}\) close to \(Z\), we can construct a surface \((X', \hat{\omega}')\) together with \(2m\) saddle connections \(\hat{\gamma}_i^c\), and an involution \(\tau' : X' \to X'\) satisfying \(\tau'(\hat{\gamma}_i^c) = -\hat{\gamma}_i^c\). Thus we have a continuous mapping \(\Psi_{\hat{T}}\) defined in a neighborhood of \(Z\) in \(\hat{V}_{\hat{T}}\) to \(\overline{Q}^{(m)}(d)\). It is not difficult to check that \(\dim_{\mathbb{C}} \hat{V}_{\hat{T}} = \dim_{\mathbb{C}} \overline{Q}^{(m)}(d) = \hat{N}_1\).
2g + n − 2, and in a local chart of $\overline{Q}(d)$, $Ψ$ is a linear isomorphism between complex vector spaces. Let $D$ be the domain in $\hat{V}_1$ which is defined by the inequations (9), (10), (11), (12), and $D_0$ be the component of $D$ that contains $Z$. The following proposition follows from the same argument as Proposition 4.6

**Proposition 6.3.** The map $Ψ$ is well defined and injective in $D_0$, it realizes a homeomorphism between $D_0$ and its image in $\overline{Q}(d)$.

Let us now define

**Definition 6.4.** Let $Γ$ be a trivalent graph with $N_2$ vertices and $N_1$ edges. Let $Γ^s_{i,s}$, $i \in \{1, \ldots, m\}$, $s \in \{1, 2\}$, be a family of disjoint subgraphs of $Γ$ which are trees. We will say that $(Γ, (Γ^s_{i,s}))$ is an admissible family of graphs if $(Γ, (Γ^s_{i,s}))$ is an admissible family of graphs (see Definition 4.5), and there exists an involution $τ$ of $Γ$ which satisfies $τ(Γ^s_{i,s}) = Γ^{s+1}_{i-ε}$, where we use the conventions $3 ≡ 1$, “−−” ≡ “++”, “−+” ≡ “−−”.

Let $Γ$ be the dual graph of $\hat{T}$. For each $\hat{Γ}^s_i$, let $Γ^s_{i,s}$ (resp. $Γ^s_{i,-s}$) be the tree in $Γ$ which is dual to the union of triangles in $\hat{T}(2)$ that cover the vertical separatrices reaching $\text{int}(\hat{Γ}^s_i)$ from the upper side (resp. lower side). By construction, $(Γ, (Γ^s_{i,s}))$ is a symmetric admissible family of graphs. The following theorem follows from the same arguments as Theorem 4.7

**Theorem 6.5.** There exists a partition of $\overline{Q}(d)$ into finitely many subset, each of which corresponds to a component of the subset determined by the inequations (9), (10), (11), (12) of a subspace $\hat{V}$ of dimension $2g + n − 2$ of $C^{N_1}$. The space $\hat{V}$ itself is determined by a symmetric admissible family of graphs.

### 6.4. Proof of Theorem 6.1

**Proof.** Let $(Γ, (Γ^s_{i,s}))$ be a symmetric admissible family of graphs. Choose a compatible numbering of the edges of $Γ$. Without loss of generality, we can assume that $e_i$ is the root of $Γ^s_{i,s}$, $i \in \{1, \ldots, m\}$. Let $S_T$ denote the linear system associated to $(Γ, (Γ^s_{i,s}))$, and $\hat{S}_T$ the system obtained by adding to $S_T$ the equations of type (21). Let $V$ be the space of solutions of $\hat{S}_T$ in $C^{N_1}$. Let $D$ be the open subset of $V$ determined by the inequations (9), (10), (11), (12). We consider a vector $Z = (z_1, \ldots, z_{N_1}) ∈ C^{N_1}$ as a function from the edges of $Γ$ to $C$, where $z_j = Z(e_i)$. By construction, there exists a continuous map $Ψ : D → \overline{Q}(d)$ which is injective. The pullback of the function $\hat{S}_Z$ by $Ψ$ is given by

$$Ψ^* \hat{S}_Z(Z) = \exp(-\frac{|z_1|^2}{ε_1^2} + \cdots + \frac{|z_m|^2}{ε_m^2} - A(Ψ(Z))).$$

From Theorem 6.5 it suffices to show that

$$\int_D e^{-(|z_1|^2/ε_1^2 + \cdots + |z_m|^2/ε_m^2) - A} dμ < K ε_1^2 \cdots ε_m^2.$$

For this purpose, let us first prove

**Lemma 6.6.** The family of coordinates $(z_1, \ldots, z_m)$ is independent in $\hat{V}$. 

Proof. Assume that there exists \((\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m\) such that 
\[ \lambda_1 Z(e_1) + \cdots + \lambda_m Z(e_m) = 0 \]
for all \(Z \in \hat{V}\). Let \(e_k = \tau(e_k), \ k = 1, \ldots, m\). Since \(Z\) satisfies the equations (21), we also have
\[ \lambda_1 Z (e_1) + Z(e_k) + \cdots + \lambda_m Z(e_m) = 0, \forall Z \in \hat{V}. \]
Let \((\hat{X}, \hat{\omega}, \{\hat{\gamma}_1, \hat{\gamma}_2, \ldots, \hat{\gamma}_m\})\) be the canonical double covering of \(\Psi(Z)\), then \((e_k, e_k)\) corresponds to the pair of saddle connections \((\hat{\gamma}_k, \hat{\gamma}_k)\) that are exchanged by the involution \(\tau\) of \(\hat{X}\). Note that we have an isomorphism between \(\hat{V}\) and \(H^1(\hat{X}, \hat{\omega}; \mathbb{C})^{-} = \{\alpha \in H^1(\hat{X}, \hat{\omega}; \mathbb{C}) : \tau^*\alpha = -\alpha\}\). Condition (22) implies that
\[ \alpha(\lambda_1 [\hat{\gamma}_1] + \cdots + \lambda_m [\hat{\gamma}_m]) = 0, \forall \alpha \in H^1(\hat{X}, \hat{\omega}; \mathbb{C})^{-}, \]
where \([\hat{\gamma}_k] = [\hat{\gamma}_k] + [\hat{\gamma}_k].\) Since \(H^1(\hat{X}, \hat{\omega}; \mathbb{C})^{-} = \text{Hom}_\mathbb{R}(\hat{\omega}(\hat{X}, \hat{\omega}; \mathbb{R})^{-}, \mathbb{C})\), and \([\hat{\gamma}_k] \in H_1(\hat{X}, \hat{\omega}; \mathbb{R})^{-}, k = 1, \ldots, m,\) it follows that
\[ \lambda_1 [\hat{\gamma}_1] + \cdots + \lambda_m [\hat{\gamma}_m] = 0 \in H_1(\hat{X}, \hat{\omega}; \mathbb{R}). \]
Therefore, we have a contradiction to the hypothesis that the family \([\hat{\gamma}_1], \ldots, [\hat{\gamma}_m]\) is independent in \(H_1(\hat{X}, \hat{\omega}; \mathbb{R})\). The lemma is then proven. \(\square\)

The remaining of the proof of the theorem follows the same lines as Theorem 2.3. \(\square\)

7. LOCAL CHARTS FOR \(\Omega \mathcal{E}_D(k)\)

7.1. Cylinder decomposition. On a translation surface \((X, \omega)\), a cylinder \(C\) is a subset isometric to \((\mathbb{R} \times (0, h))/\mathbb{Z}\), where the action of \(\mathbb{Z}\) is generated by \((x, y) \mapsto (x + \ell, y)\), and \(C\) is maximal with respect to this property. The parameters \(h\) and \(\ell\) are respectively called the height and width of \(C\). Using this identification, we will call any simple closed geodesic corresponding to \(\mathbb{R} \times \{y\}/\mathbb{Z}, \ y \in (0, h)\), a core curve of \(C\).

By definition, we have an isometric immersion \(\varphi : \mathbb{R} \times (0, h) \to X\) such that \(\varphi(\mathbb{R} \times (0, h)) = C\). We can extend \(\varphi\) by continuity to a map \(\hat{\varphi} : \mathbb{R} \times [0, h] \to X\). The images of \(\mathbb{R} \times \{0\}\) and \(\mathbb{R} \times \{h\}\) by \(\hat{\varphi}\) are called the bottom and top borders of \(C\) respectively. Both borders of \(C\) are unions of saddle connections which are not necessarily disjoint.

Recall that a direction \(\theta \in S^1\) is said to be periodic on a translation surface \((X, \omega)\) if every trajectory of the directional flow in this direction is either a closed geodesic or a saddle connection. Note that a direction is periodic if and only if the surface admits a decomposition into (finitely many) cylinders in this direction. Translation surfaces that are Prym eigenform have some particular properties, namely

**Theorem 7.1.** Any Prym eigenform is completely periodic in the sense of Calta, that is, the direction of any simple closed geodesic is periodic.

In genus two, Theorem 7.1 follows from the work of Calta [C04] and McMullen [Mc07], a proof of this theorem for the cases where \(\dim_{\mathbb{C}} \Omega \mathcal{E}_D(k) = 3\) is given in [LN13], and the general case is proved in [W13].

Assume that a surface \((X, \omega)\) in some stratum \(\mathcal{H}(k)\) is decomposed into \(r\) cylinders in the horizontal direction. Since every horizontal ray emanating from a zero of \(\omega\) must end at a zero, we deduce that there are exactly \(|k| + 2g - 2\) horizontal saddle connections on \(X\). Since each saddle connection is contained the top (resp. bottom) border of a unique cylinder, it follows that we have two partitions of the set of horizontal saddle connections into \(r\) subsets, every subset in each partition has a cyclic ordering, and we also have a pairing between subsets of the two partitions. We will call all of these data the combinatorial data or topological model of the cylinder decompositions. Note that one can
encode those data in a diagram. Clearly, the set of topological model for cylinder decompositions in a given stratum is finite.

Let \( C \) be a cylinder. A crossing saddle connection in \( C \) is a saddle connection in \( \overline{C} \) intersecting all the core curves once. We will identify crossing saddle connections that are related by Dehn twists in \( C \). Observe that up to Dehn twists, the set of crossing saddle connections is finite. Assume that \((X, \omega)\) admits a cylinder decomposition in the horizontal direction, for which the combinatorial data are known. We pick a crossing saddle connection for each cylinders. Then \((X, \omega)\) is completely determined by the length of the horizontal saddle connections, and the vector associated to the crossing ones. We will call the period of a crossing saddle connection a crossing period.

7.2. Stable and unstable cylinder decompositions. Assume that \( \theta \in S^1 \) is a periodic direction on \((X, \omega)\), then the corresponding cylinder decomposition is said to be stable if every saddle connection in this direction joins a zero of \( \omega \) to itself, otherwise it is said to be unstable. Remark that in a stable decomposition, the top border (resp. bottom border) of every cylinder contains only one zero.

**Lemma 7.2.** For any Prym eigenform locus \( \mathcal{M} \), the set of surfaces admitting an unstable cylinder decomposition is zero.

**Proof.** In each stratum, the Prym eigenform locus are locally defined by a system of linear equations of period coordinates. For \( k \in \{(1, 1), (2, 2)\} \) those equations only involve absolute periods, for \( k = (1, 1, 2) \) there is an additional equation relating two relative periods, which is a consequence of the fact that the Prym involution fixes the double zero, and exchanges the simple ones.

If \((X, \omega)\) admits an unstable cylinder decomposition, then there are a relative period \( z \) and an absolute period \( w \) that satisfy

\[
\frac{z}{w} \in \mathbb{R} \iff \Im(z \bar{w}) = 0
\]

Since \([23]\) is not a consequence of the system defining \( \mathcal{M} \), the set of surfaces having an unstable cylinder decomposition has measure zero (with respect to the Lebesgue measure of \( \mathcal{M} \)). \( \square \)

**Lemma 7.3.** Let \((X, \omega) \in \mathcal{H}(k)\) be a translation surface for which the horizontal direction is periodic. For \( k = (1, 1), (2, 2), (1, 1, 2) \), if \( X \) has 3, 4, or 5 horizontal cylinders respectively, then the decomposition is stable.

**Proof.** Let \((X, \omega)\) be a surface in \( \mathcal{H}(k) \) for which the horizontal direction is periodic with \( k \) cylinders. It is a well-known result by Masur that \( k \leq c_{\text{max}}(k) = |k| + g - 1 \). For the reader’s convenience, we will sketch a proof of this result here below.

Let us denote by \( C_1, \ldots, C_k \) the horizontal cylinders of \( X \). For each \( j \in \{1, \ldots, k\} \) pick a core curve \( \alpha_j \) of \( C_j \). Observe that there exists a tree \( T \) connecting all the zeros of \( \omega \) consisting of saddle connections which either cross a unique horizontal cylinder, or are contained in the border of some cylinder.

Note that this tree has exactly \(|k| - 1\) edges. Let \( m \) be the number of cylinders crossed by the edges of \( T \). We can assume that those cylinders are \( C_1, \ldots, C_m \). We now claim that the family of curves \( \{\alpha_{m+1}, \ldots, \alpha_k\} \) is independent in \( H_1(X, \mathbb{Z}) \). Indeed, for each \( j \in \{m+1, \ldots, k\} \), let \( \beta_j \) be a saddle connection in \( C_j \) joining a zero in the top border to a zero in the bottom border. Then there exists a closed curve \( \gamma_j \) which is the union of \( \beta_j \) and a path in \( T \). We have

\[
\langle \alpha_j, \gamma_j \rangle = 1, \quad \text{and} \quad \langle \alpha_i, \gamma_j \rangle = 0 \quad \text{if} \quad i \neq j,
\]
where $\langle , \rangle$ is the intersection form of $H_1(X, \mathbb{Z})$, which implies immediately that the family \(\{a_{m+1}, \ldots, a_k\}\) is independent.

Since the curve $\alpha_j$ are pair-wise disjoint, the subspace of $H_1(X, \mathbb{R})$ generated by \(\{a_{m+1}, \ldots, a_k\}\) is a Lagrangian, thus its dimension is bounded by $g$. Consequently, we have

\[ k - m \leq g \Leftrightarrow k \leq g + m \leq g + |k| - 1 = c_{\text{max}}(k). \]

For $k = (1, 1), (2, 2), (1, 1, 2)$, we have $c_{\text{max}}(k) = 3, 4, 5$ respectively. It remains to show that when $c_{\text{max}}$ is attained, the cylinder decomposition is stable. Assume that $k = c_{\text{max}}$, and there exists a horizontal saddle connection that joins two distinct zeros of $\omega$. Then we can find a tree $T$ connecting all the zeros of $\omega$ which crosses at most $m = |k| - 2$ cylinders. Consequently, we have

\[ k \leq g + m \leq g + |k| - 2 < c_{\text{max}}. \]

Thus we get a contradiction. \(\square\)

**Remark 7.4.**

- For $\mathcal{H}(1, 1)$, all cylinder decompositions with less than $c_{\text{max}} = 3$ cylinders are unstable, and there exists only one topological model for cylinder decompositions with 3 cylinders.
- For $\mathcal{H}(2, 2)$ and $\mathcal{H}(1, 1, 2)$ there are several topological models for decompositions with maximal number of cylinders, and there are stable cylinder decompositions with less than $c_{\text{max}}$ cylinders.

### 7.3. Local charts by cylinder decompositions

The following proposition is crucial for our proof of Theorem \[7.9\]

**Proposition 7.5.** Let $M$ be a Prym eigenform locus in one of the strata in Theorem \[7.9\]. Let $(X, \omega)$ be a point in $M$ which admits a stable cylinder decomposition in the horizontal direction. Let $C_1, \ldots, C_k$ denote the horizontal cylinders of $X$. For each $C_j$ we pick a crossing saddle connection, and denote by $z_j$ its period. We always choose the representative of crossing saddle connections and its orientation so that $0 \leq \text{Re}(z_j) < \ell(C_j)$ and $\text{Im}(z_j) > 0$.

Let $a_1, \ldots, a_n$ be the horizontal saddle connections on $X$. Then there exist $k = (\lambda_1, \ldots, \lambda_n) \in (\mathbb{R}^*_+)^n$, $k$ pairs of real numbers $(\xi_j, \zeta_j) \neq (0, 0)$, $k$ real linear functions $f_j = f_j(|a_1|, \ldots, |a_n|)$ \((j = 1, \ldots, k)\), and two relative crossing periods $u, v$ such that

- (a) $\frac{|a_i|}{k_i} = |a_j| / k_j$, for all $i, j \in \{1, \ldots, n\}$.
- (b) $z_j = \xi_j u + \zeta_j v + f_j(|a_1|, \ldots, |a_n|)$.
- (c) There exists $(m, n) \in \mathbb{N}^2$, $m/n \in \{1/2, 1, 2\}$ such that $mu + nv$ is an absolute period.

Moreover, for a fixed discriminant $D$, given a topological model for the cylinder decomposition, there are only finitely many choices for $(\lambda_1, \ldots, \lambda_n)$ up to multiplication by $\mathbb{R}^*_+$, and finitely many choices for $(\xi_j, \zeta_j)$ and $f_j$.

**Proof.** Case $k = (1, 1)$: in this case there is only one topological model for stable cylinder decomposition (see Figure 2). In this case, we have $n = 4$ and $k = 3$. We choose the numbering as shown in Figure 2. We first remark that $|a_1| = |a_3|$ and $|a_2| = |a_4|$.

For $j = 1, 2$, let $\alpha_j$ be the homology class represented by the core curves of $C_j$, and $\beta_j$ be the homology class represented by $z_j + z_3$. Then $(\alpha_1, \beta_1, \alpha_2, \beta_2)$ is a symplectic basis of $H_1(X, \mathbb{Z})$. Since $(X, \omega)$ is a Prym eigenform, there exists a unique generator $T$ of the quadratic order $O_D$ which is
written in the basis \((\alpha_1, \beta_1, \alpha_2, \beta_2)\) by the matrix \(\begin{pmatrix} e & 0 & a & b \\ 0 & e & c & d \\ -c & a & 0 & 0 \end{pmatrix}\) such that \(\omega \cdot T = \lambda \omega\), with \(\lambda > 0\), and 
\((a, b, c, d, e) \in \mathbb{Z}^5\) (see [Mc07], [BT10], [LN11]).

In this basis, \(\omega\) is given by \((|a_1|, z_1 + z_3, |a_2|, z_2 + z_3) = (|a_1|, x_1 + x_3, |a_2|, x_2 + x_3) + t(0, y_1 + y_3, 0, y_2 + y_3)\).
Note that by assumption \(0 \leq x_j < |a_j|, j = 1, 2,\) and \(0 \leq x_3 < \ell(C_3) = |a_1| + |a_2|,\) and \(y_j > 0, j = 1, 2, 3.\)
Set \(\delta_j = z_j + z_3 = x_j' + iy_j'\). Elementary calculations show that

\[
\omega \cdot T = \lambda \omega \quad \iff \quad \begin{cases} 
  c = 0, \ \lambda^2 = e \lambda + ad, \\
  \frac{|a_2|}{|a_1|} = \frac{a}{\lambda}, \\
  x_2' = \frac{d}{\lambda} x_1' + \frac{b}{\lambda} |a_1|, \\
  y_2' = \frac{d}{\lambda} y_1'.
\end{cases}
\]

Note that the last two conditions can be rewritten as

\[
(24) \quad z_2' = \frac{d}{\lambda} z_1' + \frac{b}{\lambda} |a_1| \quad \iff \quad z_2 = \frac{d}{\lambda} z_1 + (\frac{d}{\lambda} - 1) z_3 + \frac{b}{\lambda} |a_1|.
\]

Since \(T\) is the generator of \(O_D\) we have \(e^2 + 4ad = D\), and the equations above imply \(a > 0, d > 0, \lambda = e + \sqrt{D}\).
Therefore there are only finitely many choices for \((a, d, e)\) and \(\lambda\).

Recall that we have \(0 \leq x_1 < |a_1|, 0 \leq x_2 < |a_2|, 0 \leq x_3 < |a_1| + |a_2|.\) From the equation
\[x_2' = \frac{d}{\lambda} x_1' + \frac{b}{\lambda} |a_1|,\] we draw

\[
b = \frac{\lambda}{|a_1|} (x_2 + x_3) - \frac{d}{|a_1|} (x_1 + x_3)
\]
which implies

\[
-\frac{d}{|a_1|} (|a_1| + (|a_1| + |a_2|)) \leq b \leq \frac{\lambda}{|a_1|} (|a_2| + (|a_1| + |a_2|)) \quad \Rightarrow \quad -d(2 + \frac{a}{\lambda}) \leq b \leq 2a + \lambda.
\]
Thus the set of \(b\) is also finite.
VOLUMES OF THE SETS OF TRANSLATION SURFACES WITH SMALL SADDLE CONNECTIONS

Set \( u = z_1, v = z_3, \lambda = (\lambda, a, \lambda, a), (m, n) = (1, 1), \) one can easily check that all the assertions of the proposition are now proved.

**Cases** \( k \in \{(2, 2), (1, 1, 2)\}: \) the proof for these cases follow from similar arguments, it suffices to specify a symplectic basis \((\alpha_1, \beta_1, \alpha_2, \beta_2)\) for \( H_1(X, \mathbb{Z})^- \), where \( \alpha_j \) is a combination of the core curves of horizontal cylinders. We show the choice of such bases in Figures 3 and 4.

\[
\text{FIGURE 3. A model for stable cylinder decompositions in } \mathcal{H}(2, 2), \text{ cylinders which are fixed by the Prym involution are colored. We have } |a_1| = |a_2| = |a_3| = |a_5| = |a_6| = |a_1| + |a_2|. \text{ Set } \alpha_1 = \alpha_{11} + \alpha_{12}, \beta_1 = \beta_{11} + \beta_{12} \text{ then } \{\alpha_1, \beta_1, \alpha_2, \beta_2\} \text{ is a symplectic basis of } H_1(X, \mathbb{Z})^-.
\]

\[
\text{FIGURE 4. A model for stable cylinder decompositions in } \mathcal{H}(1, 1, 2), \text{ cylinders fixed by the Prym involution are colored. We have } |a_1| = |a_2| = |a_3| = |a_4|, |a_5| = |a_6| = 2|a_1|. \text{ Set } \beta_{11} = b_1 + b_2 + b_3 + b_4, \beta_{12} = b_2 + b_3' + b_4 + b_5, \alpha_1 = \alpha_{11} + \alpha_{12}, \beta_1 = \beta_{11} + \beta_{12}. \text{ Then } \{\alpha_1, \beta_1, \alpha_2, \beta_2\} \text{ is a symplectic basis of } H_1(X, \mathbb{Z})^-.
\]

We also have
Lemma 7.6. Let \( z_j, (\xi_j, \zeta_j), (m, n), \) and \( u, v \) be as in Proposition 7.5. If \( z_j \) is an absolute period then \( \xi_j > 0 \) and \( \zeta_j > 0 \).

Proof. For any \( t \in \mathbb{R}, \) small, one can move \((X, \omega)\) vertically in its kernel foliation leaf (see [C04, LN13]) to get another surface \((X_t, \omega_t) \in \mathcal{M}\) with parameters \( z_j(t), u(t), v(t) \) such that \( u(t) = u + \alpha t, v(t) = v + \iota \beta t, \) where \( \alpha \beta \neq 0. \) Note that moving in the kernel foliation leaves does not affect the absolute periods. Therefore,

\[
m u(t) + n v(t) = m u + n v \quad \Rightarrow \quad (m \alpha + n \beta)t = 0 \quad \Rightarrow \quad m \alpha + n \beta = 0.
\]

On the other hand, since \( z_j \) is an absolute period, we also have \( \xi_j \alpha + \zeta_j \beta = 0. \) Thus, \( \xi_j / \zeta_j = m / n > 0. \) By assumption, we have \( \text{Im}(z_j) > 0, \text{Im}(u) > 0, \text{Im}(v) > 0, \) and \( \text{Im}(z_j) = \xi_j \text{Im}(u) + \zeta_j \text{Im}(v), \) from which we can conclude that \( \xi_j > 0 \) and \( \zeta_j > 0. \) \( \square \)

Pick a topological model for stable cylinder decompositions in some Prym eigenfrom locus \( \mathcal{M}. \) Let \((X_0, \omega_0)\) be an element of \( \mathcal{M}\) which admits a stable cylinder decomposition with this topological model in the horizontal direction. Let \( a_1, \ldots, a_n \) be the horizontal saddle connections, and \( b_1, \ldots, b_k \) the chosen crossing saddle connections in \( X_0. \)

Let \( (\lambda_1, \ldots, \lambda_n) \in (\mathbb{R}_+^*)^n, \) \( (\xi_j, \eta_j), \) \( j = 1, \ldots, k, \) and \( (f_j, j = 1, \ldots, k) \) be as in the Proposition 7.5. We will always normalize (using \( \mathbb{R}_+^* \)) so that \( \lambda_1 + \cdots + \lambda_n = 1. \) Let \( \mathcal{U} \) be the subset of \( \mathbb{S}^1 \times \mathbb{R}_+^* \times \mathcal{U} \times \mathcal{U}, \) where \( \mathcal{U} = \{ z \in \mathbb{C} | \text{Im}(z) > 0 \}, \) consisting of the tuples \((\theta, r, u, v)\) satisfying

(a) \( \xi_j \text{Im}(u) + \zeta_j \text{Im}(v) > 0, \)

(b) \( 0 \leq \xi_j \text{Re}(u) + \zeta_j \text{Re}(v) + f_j(\lambda_1 r, \ldots, \lambda_n r) < \hat{\lambda}_j r, \) where \( \hat{\lambda}_j = \sum_{i \in I_j} \lambda_i, \) and \( I_j \) is the family of indices corresponding to the saddle connections contained in the bottom border of \( \mathcal{C}_j. \)

We then have a map \( \Phi : \mathcal{U} \to \mathcal{M} \) which sends a tuple \((\theta, r, u, v)\) to a surface \((X, \omega)\) which admits a stable cylinder decomposition with the chosen topological model in direction \( \theta \) such that \( \omega(a_j) = e^{\theta} \lambda_j r \) and \( \omega(b_j) = e^{\theta} z_j. \)

Remark that \( \Phi \) is a local homeomorphism, in particular \( \Phi(\mathcal{U}) \) is an open subset of \( \mathcal{M}. \) By a slight abuse of language, we will call \( \Phi \) a local chart by cylinder decompositions of \( \mathcal{M}. \) Since on any translation surface \((X, \omega) \in \mathcal{M}\) there always exist periodic directions, from Lemma 7.7 and Proposition 7.5 we have the following

Corollary 7.7. There is a full measure subset of \( \mathcal{M} \) which is covered by finitely many cylinder local charts

Recall that in the period coordinates, \( \mathcal{M} \) is locally identified with an affine subspace of \( H^1(X, \Sigma, \mathbb{C}) \) (\( \Sigma \) is the set of zeros of the holomorphic 1-form), and the volume form \( \text{vol} \) on \( \mathcal{M} \) is proportional to the Lebesgue measure on that subspace. The following lemma follows from elementary computations, its proof is left to the reader.

Lemma 7.8. Set \( u = x_1 + ry_1, v = x_2 + ry_2, (x_i, y_i) \in \mathbb{R}^2, \) then we have

\[
\Phi^* \text{dvol} = \text{const.} \text{d} \theta(rdr)(dx_1 dy_1)(dx_2 dy_2).
\]

7.4. The area function. Let \( A \) denote the area of a surface \((X, \omega) = \Phi(0, r, u, v). \) We have

\[
(25) \quad A = \sum_{j=1}^k \ell(C_j) h(C_j) = \sum_{j=1}^k \hat{\lambda}_j r(\xi_j y_1 + \zeta_j y_2) = c_1 ry_1 + c_2 ry_2
\]
where the constants are determined by the parameters of the map $\Phi$.

**Lemma 7.9.** Both constants $c_1$ and $c_2$ in (25) are strictly positive.

*Proof.* Again, we will only give the proof for the case $k = (1, 1)$ since the other cases follow from similar arguments. Recall that $y_1 = \text{Im}(\omega(b_j))$, $j = 1, 2$, where $b_j$ is a saddle connection joining two distinct zeros of $\omega$, and $b_1 + b_2$ represents an element of $H_1(X, \mathbb{Z})$.

It is well known the Prym eigenform loci are preserved by the kernel foliation (see [C04] and [LN13]). Recall that moving in a leaf of the kernel foliation only changes the relative periods, but not the absolute ones, therefore the area function is constant along the leaves of this foliation. In particular, we see that for any $t \in \mathbb{R}$ small enough, there exists a surface $(X_t, \omega_t)$ close to $(X, \omega)$ (in the same Prym eigenform locus) such that $\omega_t(b_1) = \omega(b_1) + it$, $\omega_t(b_2) = \omega(b_2) - it$ and $\omega_t(a_j) = \omega(a_j)$. Note that $(X_t, \omega_t) = \Phi(0, r, u_t, v_t)$, where $u_t = u + it$, $v = v - it$. The condition $A(X_t, \omega_t) = A(X, \omega)$ then implies

$$(c_1 - c_2)rt = 0 \Rightarrow c_1 = c_2.$$

Since $r, y_1, y_2$ are all positive, we must have $c_1 = c_2 > 0$. □

8. **Integration on local charts by cylinder decomposition**

Let us pick a local chart $\Phi : U \to M$ defined in the previous section. Recall that to define $\Phi$, we have to fix a topological model for stable cylinder decomposition.

The following propositions play a key role in our proof of Theorem 1.9.

**Proposition 8.1.** There exists a constant $C > 0$ such that for any $\kappa > 0$, we have

$$
\int_U e^{-\frac{r^2}{\kappa^2} - A \phi} \Phi^* \text{d}vol < C \kappa^2.
$$

*Proof.* Assume that $(X, \omega) = \Phi(\theta, r, u, v)$. Let $C_1, \ldots, C_k$ be the cylinders of $X$ in direction $\theta$, we denote by $h(C_j)$ and $\ell(C_j)$ the height and the width of $C_j$ respectively. By construction: $\ell(C_j) = \hat{h}_j r$, and $h(C_j) = \hat{x}_j \text{Im}(u) + \hat{y}_j \text{Im}(v)$. Without loss of generality, we can suppose that $u$ and $v$ are the periods of the crossing saddle connections in $C_1$ and $C_2$ respectively. By construction, we have

$$
\begin{align*}
0 \leq \text{Re}(u) & = x_1 < \ell(C_1), & \text{Im}(u) & = y_1 > 0, & A(C_1) & = \ell(C_1)y_1, \\
0 \leq \text{Re}(v) & = x_2 < \ell(C_2), & \text{Im}(v) & = y_2 > 0, & A(C_2) & = \ell(C_2)y_2.
\end{align*}
$$

It follows

$$
\int_U e^{-\frac{r^2}{\kappa^2} - A \phi} \Phi^* \text{d}vol < \int_U e^{-\frac{r^2}{\kappa^2} - A(C_1) - A(C_2)} \Phi^* \text{d}vol
$$

$$
< 2\pi \int_0^\infty re^{-\frac{r^2}{\kappa^2}} \left( \int_0^{\ell(C_1)} dx_1 \int_0^{\ell(C_1)y_1} dy_1 \right) \left( \int_0^{\ell(C_2)} dx_2 \int_0^{\ell(C_2)y_2} dy_2 \right) dr
$$

$$
< 2\pi \int_0^\infty re^{-\frac{r^2}{\kappa^2}} dr
$$

$$
< \pi \kappa^2.
$$

□
For any \( \epsilon > 0 \), let \( U(\epsilon) \subset U \) be the subset consisting of elements \((\theta, r, u, v)\) such that \((X, \omega) = \Phi(\theta, r, u, v)\) has a saddle connection \(\sigma\) not in direction \(\theta\) such that \(|\sigma| < \epsilon \sqrt{A(X)}\), where \(A\) is the area function.

**Proposition 8.2.** For any \( \kappa > 0 \), there exist some constants \(C(\kappa) > 0\) and \(\epsilon_0 > 0\) such that for any \(0 < \epsilon < \epsilon_0\), we have

\[
I := \int_{U(\epsilon)} e^{-\frac{r^2}{\kappa^2} - A \cdot \Phi \cdot \Phi^*} \, d\text{vol} < C(\kappa) \epsilon^2.
\]

Moreover, there exists a constant \(\kappa_0 > 0\) such that if \(0 < \kappa < \kappa_0\) then \(C(\kappa) < C\kappa^2\), where \(C\) is a constant, that is for any \((\kappa, \epsilon)\) such that \(0 < \kappa < \kappa_0\) and \(0 < \epsilon < \epsilon_0\), we have

\[
I := \int_{U(\epsilon)} e^{-\frac{r^2}{\kappa^2} - A \cdot \Phi \cdot \Phi^*} \, d\text{vol} < C\kappa^2 \epsilon^2.
\]

By assumption, the saddle connection \(\sigma\) on \((X, \omega) = \Phi(\theta, r, u, v)\) crosses some cylinders of the decomposition in direction \(\theta\). By rotating, using \(r_\theta = \left( \begin{array}{cc} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{array} \right)\), we can assume that \(\theta = 0\). We will prove the proposition case by case depending on the number of cylinders crossed by \(\sigma\).

**8.1. Case 1: \(\sigma\) crosses only one cylinder.**

**Proof.** Let \(C\) be the unique cylinder crossed by \(\sigma\), \(\beta\) be the chosen crossing saddle connection in \(C\), and \(z = \omega(\beta)\).

- **Case 1.a:** \(z\) is an absolute period, that is \(\beta\) joins a zero to itself. Note that in this case \(\sigma\) can cross a core curve of \(C\) several times (this happens when there is a saddle connection which is contained in both top and bottom borders of \(C\)), but we always have \(h(C) = \text{Im}(z) \leq |\sigma| < \epsilon \sqrt{A}\). Since \(\beta\) is an absolute period, we have \(h(C) = \xi y_1 + \zeta y_2\), with \(\xi > 0\) and \(\zeta > 0\) (see Lemma [7.6]). Therefore, there exists \(\hat{c} > 0\) such that \(y_j < \hat{c} \epsilon \sqrt{A(X)}\), \(j = 1, 2\).

Recall that we have

\[
A(X) = \sum_{j=1}^{k} h(C_j) \ell(C_j) = c_1 r y_1 + c_2 r y_2.
\]

where \(c_1 > 0\) and \(c_2 > 0\) (see Lemma [7.9]). Therefore

\[
A < \hat{c}(c_1 + c_2) \epsilon \sqrt{A} r \Rightarrow \sqrt{A} < c_3 \epsilon r.
\]

It follows

\[
y_2 < \hat{c} \epsilon \sqrt{A} < c \epsilon^2 r.
\]

Let us fix \(r\) and make the following change of variables \((x_1, y_1, x_2, y_2) \mapsto (x_1, A, x_2, y_2)\). Elementary calculations give

\[
dx_1 dy_1 dx_2 dy_2 = \frac{1}{c_1 r} dx_1 dA dx_2 dy_2.
\]
Recall that by definition we have $0 \leq x_1 < \hat{\kappa}_1 r$ and $0 \leq x_2 < \hat{\kappa}_2 r$.

\[
I < 2\pi \int_0^\infty r e^{-r^2/\kappa^2} \left( \int_0^{\hat{\kappa}_1 r} dx_1 \int_0^\infty e^{-A} dA \right) \left( \int_0^{\hat{\kappa}_2 r} dx_2 \int_0^\infty dy_2 \right) dr
\]

\[
< 2\pi \left( \frac{c}{c_1} \right) \hat{\kappa}_1 \hat{\kappa}_2 e^2 \int_0^\infty r^3 e^{-r^2/\kappa^2} dr
\]

\[
< \pi \left( \frac{c}{c_1} \right) \hat{\kappa}_1 \hat{\kappa}_2 e^2 \kappa^4.
\]

Therefore, $I < \text{const} \kappa^2 e^2$ if $\kappa$ and $\epsilon$ are small enough.

**Case 1.b:** $z$ is a relative period. We first notice that $\sigma$ crosses any core curve of $C$ only once. This is because the zeros contained in the top and bottom borders of $C$ are distinct, hence a saddle connection in the top border cannot be identified with another one in the bottom border.

Without loss of generality, and by using a change of variables if necessary, we can suppose $u = z_1$, $v = z_2$, and $C = C_2$. Note that we have $0 < y_2 \leq |\sigma| < \epsilon \sqrt{A(X)}$. We consider two subsets $U^1(\epsilon)$ and $U^2(\epsilon)$ of $U(\epsilon)$ which are defined by

\[
\begin{aligned}
&U^1(\epsilon) = \{(\theta, r, u, v) \in U(\epsilon) : \ell(C_2) = \hat{\kappa}_2 r < \epsilon \sqrt{A} \}, \\
&U^2(\epsilon) = \{(\theta, r, u, v) \in U(\epsilon) : \ell(C_2) = \hat{\kappa}_2 r \geq \epsilon \sqrt{A} \}.
\end{aligned}
\]

Set $I_i = \int_{U^i(\epsilon)} e^{-r^2/\kappa^2} e^{-A} \Phi^* \text{vol}$, $i = 1, 2$.

On $U^1(\epsilon)$, using the change of variables $(x_1, y_1, x_2, y_2) \mapsto (x_1, A, x_2, y_2)$ and the condition $\hat{\kappa}_2 r < \epsilon \sqrt{A}$, we get

\[
I_1 < 2\pi \int_0^\infty r^2 e^{-r^2/\kappa^2} \left( \int_0^{\hat{\kappa}_1 r} dx_1 \int_0^\infty e^{-A} \left( \int_0^{\hat{\kappa}_2 r} dx_2 \int_0^\infty dy_2 \right) dA \right) dr
\]

\[
< 2\pi \left( \frac{\hat{\kappa}_1}{c_1} \right) e^2 \int_0^\infty re^{-r^2/\kappa^2} \left( \int_0^\infty Ae^{-A} dA \right) dr
\]

\[
< 2\pi \left( \frac{\hat{\kappa}_1}{c_1} \right) e^2 \int_0^\infty re^{-r^2/\kappa^2} dr
\]

\[
< \pi \left( \frac{\hat{\kappa}_1}{c_1} \right) \kappa^2 e^2.
\]

Thus for fixed $\kappa$, $I_1 < \text{const} e^2$ for $\epsilon$ small enough, and if $\kappa$ and $\epsilon$ are small enough then $I_1 < \text{const} \kappa^2 e^2$.

Let us fix $(\theta, r, u)$ and consider the set $(x_2, y_2)$ such that $(\theta, r, u, x_2 + iy_2) \in U^2(\epsilon)$. By definition there is a saddle connection $\sigma$ in $C_2$ such that $|\sigma| < \epsilon \sqrt{A} \leq \ell(C_2)$, in particular $|\text{Re}_{\theta}(\sigma)| < \ell(C_2)$.

Recall that we have an equivalence relation on the set of crossing saddle connections in $C_2$, two saddle connections are in the same equivalence classes if there is a Dehn twist sending one
to the other. Clearly, the number of equivalence classes is finite (it is completely determined by
the combinatorial data of \( C_2 \)). If \( \sigma_1 \) and \( \sigma_2 \) are in
the same equivalence class then \( \Re \omega(\sigma_1) - \Re \omega(\sigma_2) \in \mathbb{Z} \ell(C_2) \).
Remark that as \( t_2 \) varies in \([0, \ell(C_2))\), \( \Re \omega(\sigma_t) - t_2 \) is constant (when \( r \) is
fixed), hence there are at most 5 saddle connections \( \sigma \) in each equivalence class such that
\[
\min \{ |\Re \omega(\sigma)|, t_2 \in [0, \ell(C_2)] \} \leq \ell(C_2).
\]
Clearly, for each saddle connection \( \sigma \), the values of \( t_2 \) such that
\( |\Re \omega(\sigma)| < \epsilon \sqrt{A} \) is an interval of length \( 2\epsilon \sqrt{A} \).
Using the change of variables \((x_1, y_1, x_2, y_2) \mapsto (x_1, A, x_2, y_2)\) and
the fact that \( y_2 = \Im \omega(\sigma) < \epsilon \sqrt{A} \), we get
\[
I_2 < 2\pi \int_0^\infty r e^{-r^2/\kappa^2} \left( \frac{1}{c_1 r} \int_0^\delta_{1,r} dx_1 \int_0^\infty e^{-A} \left( 10\epsilon \sqrt{A} \int_0^\epsilon \sqrt{A} dy_2 \right) dA \right) dr
\]
\[
< 20 \left( \frac{\delta_1}{c_1} \right) \pi \epsilon^2 \int_0^\infty r e^{-r^2/\kappa^2} \left( \int_0^\infty A e^{-A} dA \right) dr
\]
\[
< 20 \left( \frac{\delta_1}{c_1} \right) \pi \kappa^2 \epsilon^2.
\]
It follows immediately if \( \kappa \) and \( \epsilon \) are small enough then \( I_2 < \text{const.} \kappa^2 \epsilon^2 \).

8.2. Case 2: \( \sigma \) crosses two cylinders or more.

**Proof.** If the crossing saddle connection of one of the cylinders gives an absolute period, that is the
two borders of this cylinder contain the same zero, then we are done by Case 1.a.

If the \( \sigma \) crosses two cylinders \( C_1, C_2 \) whose crossing saddle connections give relative periods, and
\( C_1, C_2 \) are not exchanged by the Prym involution, then we can choose \((u, v)\) to be the crossing periods
of \( C_1 \) and \( C_2 \), and the proposition also follows from the same argument as in Case 1.a.

Thus we only need to consider the case where \( \sigma \) crosses two cylinders \( C_1, C_2 \) that are exchanged
by the Prym involution and no other cylinders, moreover the crossing saddle connections in those
cylinders must be relative periods. Clearly in this case the two cylinders must be adjacent, that is the
top border of \( C_1 \) and the bottom border of \( C_2 \) have a common saddle connection, hence the top border
of \( C_1 \) and the bottom border of \( C_2 \) contains the same zero of \( \omega \). Since the derivative of the Prym
involution is \(-1\), it maps the top border of \( C_1 \) to the bottom border of \( C_2 \), thus the unique zero in the
top border of \( C_1 \) is fixed by the Prym involution. Note that if \( \omega \) has two zeros, then the two zeros of \( \omega \)
are exchanged by the Prym involution. Therefore this case only occurs in the stratum \( \mathcal{H}(1, 1, 2) \).

From now on we assume \((X, \omega) = \Phi(0, r, u, v) \) belongs to a Prym eigenform locus \( M \) is \( \mathcal{H}(1, 1, 2) \).
Let \( \rho \) be the Prym involution of \( X \). We denote by \( P_1, P_2 \) the simple zeros, and by \( Q \) the double zero
of \( \omega \). Without loss of generality, we can assume that the bottom border of \( C_1 \) contains \( P_1 \) and the top
border of \( C_2 \) contains \( P_2 \). In particular, we see that \( \sigma \) crosses any core curve of \( C_1 \) and \( C_2 \) only once.
Let \( b_1, b_2 \) be the marked crossing saddle connections of \( C_1 \) and \( C_2 \) respectively, we can assume that
\( \rho(b_1) = -b_2 \), and \( \omega(b_1) = \omega(b_2) = v \). From what has been said above, we have \( \omega(\sigma) = 2v + \hat{\lambda}r \), where
\( \hat{\lambda} \) is the sum of some coefficients in \( \{ \lambda_1, \ldots, \lambda_n \} \), \( \hat{\lambda} \) is determined by relative homology class of \( \sigma \). In
particular,
\[
\Re \omega(\sigma) = 2x_2 + \hat{\lambda}r.
\]
Let $a_{b_0}$ be the horizontal saddle connection in the top border of $C_1$ which intersects $\sigma$. Let $\sigma_1$ (resp. $\sigma_2$) be the saddle connection in $C_1$ (resp. $C_2$) that joins the left endpoint of $a_{b_0}$ to the endpoint of $\sigma$ in the bottom border of $C_1$ (resp. in the top border of $C_2$), see Figure 5. Note that $\sigma$ is uniquely determined by $\sigma_1$ and $\sigma_2$.

Assume that $\ell(C_1) = \ell(C_2) \geq \epsilon \sqrt{A}$. By assumption, we have $|\text{Re} \omega(\sigma)| < \epsilon \sqrt{A} \leq \ell(C_1)$. Hence

$|\text{Re} \omega(\sigma_i)| < |\text{Re} \omega(\sigma)| + |\omega(a_{b_0})| < 2\ell(C_1)$, $i = 1, 2$.

Up to Dehn twists, there are finitely many crossing saddle connections in $C_i$. As $x_2$ varies in $[0, \ell(C_1))$, there are at most 10 saddle connections in each equivalence class that satisfy

$$\min \{|\text{Re} \omega(\sigma)|, \ x_2 \in [0, \ell(C_1))\} < 2\ell(C_1).$$

It follows that as $x_2$ varies in $[0, \ell(C_1))$, the number of crossing saddle connections in $C_i$ satisfying (29) is bounded by some constant depending only on the combinatorial data of $C_i$. Therefore, the number of saddle connections $\sigma$ crossing $C_1$ and $C_2$ and satisfy

$$\min \{|\text{Re} \omega(\sigma)|, \ x_2 \in [0, \ell(C_1))\} < \epsilon \sqrt{A}.$$ 

is uniformly bounded by some constant depending on the topology of $C_1$ and $C_2$. For such a saddle connection, the interval of values of $x_2$ such that $|\text{Re} \omega(\sigma)| < \epsilon \sqrt{A}$ has length bounded by $2\epsilon \sqrt{A}$. Thus we can conclude that the measure of set of values of $x_2$ in $[0, \ell(C_1))$ such that there exists a saddle connection $\sigma$ crossing $C_1, C_2$ and satisfies (30) is bounded by $c \epsilon \sqrt{A}$, where $c$ is a constant depending only on the topology of $C_1$ and $C_2$. The proof of the proposition for this case then follows from the same lines as Case 1.b. \hfill \Box

9. PROOF OF THEOREM 1.8

**Proof.** Let $M(\epsilon)$ denote the subset of $M$ consisting of surfaces $(X, \omega)$ which have a saddle connection of length bounded by $\epsilon \sqrt{A(X)}$. The inequality (1) is equivalent to

$$\int_{M(\epsilon)} e^{-\lambda} d\text{vol} < C_1 \epsilon^2. \quad (31)$$

We consider two subsets $M'(\epsilon)$ and $M''(\epsilon)$ of $M(\epsilon)$, where $M'(\epsilon)$ consists of surfaces $(X, \omega)$ which have a saddle connection of length bounded by $\epsilon \sqrt{A(X)}$ and parallel to the core curves of a cylinder, while $M''(\epsilon)$ is the subset of surfaces $(X, \omega)$ which have a saddle connection of length bounded by $\epsilon \sqrt{A(X)}$ and not parallel to any cylinder. Clearly, we have $M(\epsilon) = M'(\epsilon) \cup M''(\epsilon)$. 

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**Figure 5.** Saddle connection crossing two adjacent cylinders permuted by the Prym involution.
Let \((X, \omega)\) be a surface in \(M'(\epsilon)\). By definition, there exists a saddle connection \(\sigma\) and a cylinder \(C\) in \(X\) which are parallel such that \(|\sigma| < \epsilon \sqrt{A(X)}\). Since \((X, \omega)\) is completely periodic in the sense of Calta, it admits a cylinder decomposition in direction of \(\sigma\). Since the set of surfaces admitting an unstable cylinder decomposition has measure zero (Lemma 7.2), we can assume that the decomposition in direction of \(\sigma\) is stable, that is \((X, \omega)\) is contained in the image of a local chart by cylinder decompositions \(\Phi : \mathcal{U} \to \mathcal{M}\).

By construction of \(\Phi\), there exists a constant \(\lambda > 0\) depending on the homology class of \(\sigma\) such that if \((X, \omega) = \Phi(\theta, r, u, v)\) then \(|\sigma| = \lambda r\). Set \(\mathcal{U}'(\epsilon) := \Phi^{-1}(M'(\epsilon))\). By definition, there exists a constant \(\lambda'\) depending only on \(\mathcal{U}\) such that for any \((\theta, r, u, v) \in \mathcal{U}'(\epsilon)\) we have \(r^2 < \epsilon^2 A(X, \omega)/\lambda'^2\). Therefore

\[
\int_{\mathcal{U}'(\epsilon)} e^{-\lambda \phi} \Phi^* d\text{vol} < \int_{\mathcal{U}'(\epsilon)} e^{-\frac{2\lambda^2 r^2}{2\epsilon^2} \Phi} d\text{vol} < \int_{\mathcal{U}} e^{-\frac{2\lambda^2 r^2}{2\epsilon^2} \Phi} d\text{vol} < \text{const.} e^2
\]

where the last inequality follows from Proposition 8.1. Since the number of cylinder local charts is finite, we have

\[
(32) \quad \int_{\mathcal{M}'(\epsilon)} e^{-\lambda} d\text{vol} < C_1 e^2.
\]

For any \(\kappa > 0\), we denote by \(\mathcal{U}_\kappa\) the set \(\{(\theta, r, u, v) \in \mathcal{U}, r < \kappa \sqrt{A}\}\). Let \((X, \omega)\) be a surface in \(M''(\epsilon)\). It is well known result by Masur-Smillie that for each stratum, there exists a constant \(\hat{\kappa} > 0\) such that every translation surface in that stratum has a simple closed geodesic of length bounded by \(\hat{\kappa} \sqrt{A}\). In [Vo03, Vo05], Vorobets attributes this result to Masur, but to the author’s knowledge, the first (sketch of) proof of this only appeared in [S00]. Some explicit estimates of the constant \(\hat{\kappa}\) are obtained by Vorobets by using the strategy of Smillie (see [Vo03]).

Let \(\alpha\) be a simple closed geodesic in \(X\) such that \(|\alpha| < \hat{\kappa} \sqrt{A(X)}\). Again, by Lemma 7.2, we can assume that the cylinder decomposition of \(X\) in direction of \(\alpha\) is stable, which means that \((X, \omega)\) is contained in the image of some local chart \(\Phi : \mathcal{U} \to \mathcal{M}\). Assume that \((X, \omega) = \Phi(\theta, r, u, v)\), since \(\alpha\) is a core curve of a cylinder in direction \(\theta\), there exists a constant \(\lambda > 0\) depending only on \(\Phi\) such that \(|\alpha| > \lambda \epsilon\). It follows that \(\Phi^{-1}((X, \omega)) \in \mathcal{U}_\kappa\), where \(\kappa = \hat{\kappa}/\lambda\).

Set \(\mathcal{U}_\kappa(\epsilon) = \mathcal{U}_\kappa \cap \mathcal{U}(\epsilon)\), where \(\mathcal{U}(\epsilon)\) is defined as in Proposition 8.2. \(\mathcal{U}_\kappa(\epsilon)\) is the subset of \(\mathcal{U}\) consisting of \((\theta, r, u, v)\) such that \(r < \kappa \sqrt{A(X)}\) and there exists a saddle connection \(\sigma\) in \((X, \omega)\), not in direction \(\theta\), of length bounded by \(\epsilon \sqrt{A(X)}\), where \((X, \omega) = \Phi(\theta, r, u, v)\). By definition, the set \(M''(\epsilon)\) is covered by the family \(\{\Phi(U_{\kappa}(\epsilon))\}\), where \(\kappa_0 = \max(\hat{\kappa}/\lambda)\). We have

\[
\int_{\mathcal{U}_{\kappa_0}(\epsilon)} e^{-\lambda \phi} \Phi^* d\text{vol} < \int_{\mathcal{U}_{\kappa_0}(\epsilon)} e^{-\frac{2\lambda^2 r^2}{2\epsilon^2} \Phi} d\text{vol} < \int_{\mathcal{U}(\epsilon)} e^{-\frac{2\lambda^2 r^2}{2\epsilon^2} \Phi} d\text{vol} < \text{const.} e^2
\]
where the last inequality follows from Proposition 8.2. Therefore, we have
\[\int_{M'(e)} e^{-\Lambda} d\nu < C_2' e^2.\]

The inequality (31) clearly follows from (32) and (33). Theorem 1.8 is then proved. \qed

10. PROOF OF THEOREM 1.9

Proof. Let \( M(\kappa, \epsilon) \) denote the subset of \( M \) consisting of surfaces \((X, \omega)\) such that there exists a cylinder \( C \), and a saddle connection \( \sigma \) not parallel to \( C \) satisfying \( \ell^2(C) < \kappa^2 A(X) \) and \( |\sigma|^2 < \epsilon^2 A(X) \).

Elementary calculus shows that the inequality (2) is equivalent to
\[\int_{M(\kappa, \epsilon)} e^{-2\Lambda} d\nu < C \kappa^2 \epsilon^2.\]

By Lemma 7.2, we can assume that the cylinder decomposition in the direction of \( C \) is stable, hence \((X, \omega)\) is contained in the image of some local chart \( \Phi : \mathcal{U} \to M \). Note that if \((X, \omega) = \Phi(\theta, r, u, v)\), then \( \ell(C) = \lambda r \), where \( \lambda \) is the sum of some \( \lambda_j \) in the family \( (\lambda_1, \ldots, \lambda_n) \). Therefore, there exists a constant \( c \) depending only on \( \mathcal{U} \) such that \( r < c \kappa \sqrt{A} \).

Recall that for any \( \kappa > 0 \), we have defined the subset \( \mathcal{U}_\kappa(\epsilon) \subset \mathcal{U} \) (see Section 9). By definition, the set \( M(\kappa, \epsilon) \) is covered by the family \( \{\mathcal{U}_\kappa(\epsilon)(\Phi)\} \), where \( \kappa_0 = c \kappa \). Since the set of local charts by cylinder decompositions is finite, the inequality (34) will follow from
\[\int_{\mathcal{U}_\kappa(\epsilon)} e^{-2\Lambda \cdot \Phi^*} d\nu < C \kappa^2 \epsilon^2.\]

By definition, on \( \mathcal{U}_\kappa(\epsilon) \) we have \( r^2/(c^2 \kappa^2) < A \), with a constant \( c \) depending on \( \mathcal{U} \) (thus \( e^{-2\Lambda} < e^{-r^2/(c^2 \kappa^2) - A} \)), and clearly \( \mathcal{U}_\kappa(\epsilon) \subset \mathcal{U}(\epsilon) \), therefore
\[\int_{\mathcal{U}_\kappa(\epsilon)} e^{-2\Lambda \cdot \Phi^*} d\nu < \int_{\mathcal{U}(\epsilon)} e^{-r^2/(c^2 \kappa^2) - A} d\nu.\]

As a consequence, the inequality (35) follows from Proposition 8.2. Theorem 1.9 is then proved. \qed

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