Quantum-state synthesis of multi-mode bosonic fields: Preparation of arbitrary states of 2-D vibrational motion of trapped ions

G. Drobný¹, B. Hladký¹,² and V. Bužek¹,²

¹ Institute of Physics, Slovak Academy of Sciences, Dúbravská cesta 9, 842 28 Bratislava, Slovakia
² Department of Optics, Comenius University, Mlynská dolina, 842 15 Bratislava, Slovakia

(April 22, 1998)

We present a universal algorithm for an efficient deterministic preparation of an arbitrary two-mode bosonic state. In particular, we discuss in detail preparation of entangled states of a two-dimensional vibrational motion of a trapped ion via a sequence of laser stimulated Raman transitions. Our formalism can be generalized for multi-mode bosonic fields. We examine stability of our algorithm with respect to a technical noise.

I. INTRODUCTION

The ability to control preparation and evolution of states of quantum systems opens new horizons in experimental physics (e.g. tests of fundamental concepts of quantum mechanics) as well as in potential technical applications (such as quantum information processing). Recent advances in quantum optics (micromasers, cavity QED) [1] and atomic physics (dynamics of trapped ions) [2] have demonstrated that a macroscopic observer can effectively control dynamics as well as to perform a complete measurement of states of microscopic quantum systems. In particular, preparation of an arbitrary quantum states of a single mode electromagnetic field in micromasers and 1-dimensional vibrational motion of trapped ions have been discussed in literature [3–9]. Experimental realizations of highly non-classical states, such as Fock states, squeezed states or Schrödinger-cat states of these single-mode bosonic fields have been reported [3–8]. The ability to synthesize an arbitrary motional state is a key prerequisite for a quantum measurement of an arbitrary motional observable of a trapped ion as proposed by Gardiner, Cirac and Zoller [10]. These authors have generalized the 1-D synthesis of motional states to 2-D and higher dimensions. In spite of the conceptual elegance, the method proposed by Gardiner, Cirac and Zoller is difficult to implement in general. The problem is that the number of laser operations required for a preparation of a two-mode target state |Ψtarget⟩ of the form

$$|\Psi_{\text{target}}⟩ = \sum_{m=0}^{M_{\text{max}}} \sum_{n=0}^{N_{\text{max}}} Q_{mn} |m, n⟩.$$  \hspace{1cm} (1.1)

depends exponentially on the dimensionality of the subspace of the Fock space in which the target state is embedded. In particular, if $N_{\text{max}} = M_{\text{max}}$ then the number of necessary operations is proportional to $2M_{\text{max}} \times 2^{M_{\text{max}}}$. This exponential dependence restricts applicability of the proposed method (see below). A novel approach which overcomes this obstacle was introduces very recently by Kneer and Law [11]. The authors have considered a photon–number–dependent interaction which is induced by a detuned standing–wave laser field in y direction. It keeps on resonance only transitions between Fock states $|m, n⟩$ with a fixed number $n = n_y$ of trap quanta in y direction while applying sequentially the 1-D preparation scheme \( \mathcal{H} \) in the x direction for each particular $n = n_y$.

In this paper we propose an alternative universal algorithm for a construction (synthesis) of an arbitrary quantum state of the two-mode bosonic field which can be straightforwardly generalized for the preparation of an arbitrary multi-mode system. In our algorithm the number of operations required for a preparation of the two-mode state (1.1) grows only polynomially as a function of $N_{\text{max}}$ and $M_{\text{max}}$. In particular, if $N_{\text{max}} = M_{\text{max}}$, then the number of operations is proportional to $8 M_{\text{max}}^2$.

To make our discussion as close as possible to experimental realization we consider the preparation of the two-dimensional vibrational motion of a trapped ion [2]. The choice of the system is motivated by the fact that dissipative effects can be significantly suppressed in ion traps which is important for a deterministic engineering of quantum states. Moreover, the quantized vibration motion of a trapped ion can be effectively controlled by a proper sequence of laser pulses tuned either to the atomic electronic transition or to resolved vibrational sidebands [2].

The simplest quantum state preparation is represented by a simple unitary evolution of an input state of the system governed by a specific (generally, nonlinear) Hamiltonian. Obviously, the fixed Hamiltonian restricts the family of target states which can be “generated” from available inputs. Another way how to prepare states of the system of interest (e.g. single-mode bosonic field) is to consider quantum interaction between this system and an adjoint quantum system (e.g. fermions). Dynamics of these two systems is governed by a specific interaction Hamiltonian. The desired quantum state engineering is then achieved by conditional measurements performed on the adjoint quantum system [1]. This non-unitary selection of specific quantum trajectories allows to synthesize...
essentially all quantum states of the system under consideration. But there is prize to pay - the probability of the outcome of the given conditional process can be extremely small. To overcome this problem one may consider a sequence of interactions between the original and the adjoint systems. These interactions (channels) are governed by different Hamiltonians with just one channel turned on at a given time. Coupling constants in these Hamiltonians and times of duration of given interactions are in this case free parameters which can be appropriately tuned so that at the output the system of interest is disentangled from the adjoint system (see below) and is prepared in the desired state. This approach has been recently utilized by Law and Eberly [8] who have shown that by a suitable switching between two channels of the atom-field interaction one can generate an arbitrary state of the single-mode cavity field. It is important to note that quantum states of 1-D vibrational motion of trapped ions are experimentally created also via a sequence of laser pulses tuned either to an electronic transition or an appropriate vibrational sideband [9], that is, sequential switching of different interaction Hamiltonians. The method can be generalized, e.g., for a trapped ion in a 3-D trap potential and other multi-mode bosonic systems. Even for higher dimensions the number of preparation operations scales polynomially with the dimensionality of the subspace of the Fock space in which the target state is embedded.

The paper is organized as follows: in Section II we introduce our algorithm for 2-D quantum–state synthesis. In Section III we discuss a physical realization of the preparation scheme. In Section IV we examine the stability of our algorithm with respect to a technical noise. Fidelity between outputs and specific target states is evaluated. We finish our paper with conclusions.

II. QUANTUM-STATE SYNTHESIS

Without any loss of generality let us assume that we want to generate an arbitrary two–mode state given as a finite superposition of number (Fock) states \(| \ldots, 0 \rangle \). To describe the preparation algorithm we firstly split the whole Hilbert space into an appropriate set of finite-dimensional subspaces labeled by a specific quantum number. Then we introduce two sorts of interaction channels. The first set of Hamiltonians will be responsible for dynamics within a given subspace; while the second one will realize a “transfer” of probability amplitudes between different subspaces. In particular, let us consider vibrational states of a trapped ion, confined in 2-D harmonic potential. Excitations of two vibrational modes are described by creation and annihilation operators \( \hat{a}_\mu \) and \( \hat{a}_\mu^\dagger \) (\( \mu = x, y \)). The Hilbert space \( \mathcal{H}_{\text{ vib}} \) of all two–mode vibrational states can be divided to subspaces with constant total number of vibrational quanta, i.e., \( \mathcal{H}_{\text{ vib}} = \oplus \mathcal{H}_J \), \( J = 0, 1, \ldots, \infty \), where \( \mathcal{H}_J \) is spanned by two–mode Fock states \(|J, 0\rangle, |J - 1, 1\rangle, \ldots, |0, J\rangle \). The subspaces \( \mathcal{H}_J \) are dynamically independent (invariant) for those Hamiltonians which do not change the total number of vibrational quanta \( \hat{J} = \hat{a}_x^\dagger \hat{a}_x + \hat{a}_y^\dagger \hat{a}_y \). However, to realize controlled manipulations with states within a given subspace \( \mathcal{H}_J \) we need an interaction of the 2-D bosonic field with another quantum system. In the case of trapped ions it is natural to assume the coupling to internal energy levels of ions. In particular, we use three internal electronic states \(|i\rangle \) (in \( \Lambda \) or \( \Xi \) configurations) with energies \( \hbar \omega_i \) \( (i = a, b, c) \) which form the basis of the Hilbert subspace \( \mathcal{H}_{\text{in}} \). Due to quantum interaction between the vibrational and internal degrees of freedom the total state vector of the composite system with the given maximum number of vibrational quanta \( J_{\text{max}} = M_{\text{max}} + N_{\text{max}} \) at time \( t \) reads

\[
\begin{align*}
|\Psi(t)\rangle &= \sum_{m=0}^{M_{\text{max}}} \sum_{n=0}^{N_{\text{max}}} \sum_{i=a,b,c} Q_{m,n,i}(t) |m,n\rangle \otimes |i\rangle \\
&= \sum_{J=0}^{J_{\text{max}}} \sum_{k=0}^{J_{\text{max}}-J} \sum_{i=a,b,c} Q_{k,J-k,i}(t) |k, J - k\rangle \otimes |i\rangle ,
\end{align*}
\]

which reflects quantum-mechanical entanglement between the two subsystems. We assume that initially the ion is in an internal state \(|a\rangle \) and the vibrational motion is cooled to the ground state \(|0, 0\rangle \). That is, the initial state vector of the composed system \(|\Psi(t=0)\rangle = |0, 0\rangle \otimes |a\rangle \) is factorized. Our task is to find an unitary evolution \( \hat{U}^\dagger \), such that at some time \( t = T \) the state vector \(|\Psi(t=0)\rangle = |0, 0\rangle \otimes |a\rangle \) can again be factorized, while the vibrational state of the ion is described by the target vector \(|\tilde{p}\rangle \), i.e. \(|\Psi(t=T)\rangle = \hat{U}^\dagger |0, 0\rangle \otimes |\tilde{p}\rangle = |\Psi_{\text{target}}\rangle \otimes |a\rangle \).

In what follows we prove that the unitary transformation \( \hat{U} \) can be represented by a sequence of five “elementary” unitary transformations \( \hat{U}^{(p)} \) \( (p = 1, \ldots, 5) \) which act in the product Hilbert space \( \mathcal{H}_{\text{ vib}} \otimes \mathcal{H}_{\text{in}} \) and which correspond to five interaction channels associated with interaction Hamiltonians \( \hat{H}^{(p)} \):

\[
\begin{align*}
\hat{U} &= \hat{A}_0 \{ \prod_{J=1}^{J_{\text{max}}} \hat{C}_J \hat{B}_{J-1} \hat{A}_J \}; & \quad \hat{A}_J &= \hat{U}^{(1)}_{J,0,b} \hat{U}^{(3)}_{J-1,1,a} \cdots \hat{U}^{(1)}_{2,1,b} \hat{U}^{(3)}_{1,1,a} \hat{U}^{(1)}_{1,1,b} \hat{U}^{(3)}_{0,0,a}; \\
\hat{B}_J &= \hat{U}^{(2)}_{J,0,c} \hat{U}^{(4)}_{J-1,1,b} \hat{U}^{(2)}_{J-1,1,c} \cdots \hat{U}^{(2)}_{1,1,b} \hat{U}^{(4)}_{1,1,c} \hat{U}^{(2)}_{0,0,b} \hat{U}^{(4)}_{0,0,c}; & \quad \hat{C}_J &= \hat{U}^{(5)}_{J,0,b} .
\end{align*}
\]
The operators $\hat{A}_j$, $\hat{B}_j$, $\hat{C}_j$ are build up from "elementary" unitary transformations $\hat{U}^{(p)}_{\{k,J-k,i\}}$ where superscript $p$ determines the interaction channel and subscript indicates that parameters of the interaction are adjusted to transfer completely the current population of the component state $|k,J-k\rangle \otimes |i\rangle$ to the "neighboring" state according to the given type of interaction (see below).

The set of the "elementary" unitary transformations $\hat{U}^{(p)}$ is sufficient (even though not unique) to evolve the target state into the vacuum state. We will perform this de-evolution via a systematic "transfer" of population from state vectors with higher to smaller number of vibrational quanta. That is, we choose the operators $\hat{U}^{(p)}$ so that at a given step of the unitary de-evolution a particular probability amplitude $Q_{k,J-k,i}$ is made equal to zero. In Fig.1, we visualize the "direction" of the action of operators $\hat{U}^{(p)}_{\{k,J-k,i\}}$. Namely, in Fig.1a we represent state vectors $|m,n\rangle \otimes |i\rangle$ as "ovals" located at the points of the lattice marked by parameters $m,n$ associated with vibrational state of an ion, while its internal state represented by a simple energy level diagram [population of a
given internal level is indicated by a bullet (•). Dashed lines indicate the subspaces $\mathcal{H}_J \otimes \mathcal{H}_{in}$ labeled by a constant number of trap quanta $J$.

Let us start the “de-evolution” procedure of the target state (3) into vacuum. After time $\tau = T - t$ the vector describing the vibrational and internal state of the ion is given by Eq. (2.1). We assume that the de-evolution is performed in such way that at this moment the state vector (2.1) is composed only of state vectors from subspaces $\mathcal{H}_J \otimes \mathcal{H}_{in}$ with $J \leq J$ ($\leq J_{max}$). Now we apply three sequences of operations described the operators $\hat{A}_J$, $\hat{B}_{J-1}$, and $\hat{C}_J$, respectively [see Eq. (2.2)].

The action of the operator $\hat{A}_J$ is illustrated in Fig. 1b. It describes a process in which the first “elementary” transformation $\hat{U}^{(3)}_{[0,J,a]}$ transfers completely the population of the state $|0, J \rangle \otimes |a \rangle$ to the state $|1, J \rangle \otimes |b \rangle$ (see the solid arrow in Fig. 1b; dashed arrows indicate simultaneous transitions which are not controlled at the given stage). This process can be realized by irradiating the ion with two external laser fields with tunable frequencies $\omega_x$ and $\omega_y$ in $x$ and $y$ directions. Adjusting the resonance conditions $\omega_y - \omega_x = \omega_y - \omega_x + \nu_x - \nu_y \equiv \omega_3$ the stimulated Raman transition [see the inset in Fig. 1b, $p = 3$] can be described in the Lamb–Dicke regime by the effective interaction Hamiltonian (we discuss physical conditions under which this and all subsequent Hamiltonians can be justified in Section III):

$$\hat{H}_3 = g_3 \hat{a}_x^\dagger \hat{a}_y^b |b\rangle \langle a| e^{-i\omega_3 t} + g_3^* \hat{a}_x \hat{a}_y^b |a\rangle \langle b| e^{i\omega_3 t};$$  

with the corresponding time evolution operator $\hat{U}^{(3)}_{[0,J,a]} = \exp(-i\hat{H}^{(3)}t)$ setting $\hbar = 1$. If the interaction constant $g_3 = |g_3|e^{i\phi_3}$ and the duration of the interaction $t$ are chosen to fulfill the condition

$$ie^{i\phi_3} Q_{k,J-k} \cos(|g_3| t \sqrt{(k+1)(J-k)}) + Q_{k+1,J-K-k} \sin(|g_3| t \sqrt{(k+1)(J-k)}) = 0,$$  

then the population of the component state $|k, J-k \rangle \otimes |a \rangle$ is completely transferred to the state $|k+1, J-K-k \rangle \otimes |b \rangle$. (for instance, in the situation described by Fig. 1b the probability amplitudes $Q_{0,J,a}$ becomes equal to zero). Once this is done, then the transformation $\hat{U}^{(1)}_{[1,J-1,b]}$ is turned on. In this process the population of the component state $|1, J-1 \rangle \otimes |b \rangle$ is completely transferred to the state $|1, J \rangle \otimes |a \rangle$. The corresponding interaction channel is described by the Hamiltonian

$$\hat{H}_1 = g_1 |b\rangle \langle a| e^{-i\omega_1 t} + g_1^* |a\rangle \langle b| e^{i\omega_1 t}.$$  

Here we assume lasers to be tuned to the electronic transition, i.e., $\omega_y - \omega_x = \omega_b - \omega_a \equiv \omega_1$. To cancel the term $Q_{k,J-k} \in (2.1)$, the interaction constants $(|g_1| t, \theta_1)$ have to be chosen to satisfy the condition

$$Q_{k,J-k,a} \sin(|g_1| t) + ie^{-i\phi_1} Q_{k,J-k,b} \cos(|g_1| t) = 0.$$  

In particular, in Fig. 1b the operator $\hat{U}^{(1)}_{[1,J-1,a]}$ cancels $Q_{1,J-1,b}$ for $J = 3$. The successive action of the “elementary” transformations $\hat{U}^{(3)}$ and $\hat{U}^{(1)}$ which form the operator $A_J$ [see Eq. (2.2)] finally “shrinks” the population of the subspace $\mathcal{H}_J \otimes \mathcal{H}_{in}$ to a single state $|J, 0 \rangle \otimes |a \rangle$.

At this stage we start the process of cancellation of the contribution of component states $|k, J - 1 - k \rangle \otimes |b \rangle$ and $|k, J - 1 - k \rangle \otimes |c \rangle$ in the “neighboring” subspace $\mathcal{H}_{J-1} \otimes \mathcal{H}_{in}$ (see Fig. 1c). This intermediate procedure is required to prevent a reverse transfer of population from $\mathcal{H}_{J-1} \otimes \mathcal{H}_{in}$ to $\mathcal{H}_J \otimes \mathcal{H}_{in}$ (see below and Fig. 1d). For this purpose the operator $B_{J-1}$ is constructed from “elementary” operations $\hat{U}^{(2)}_{[0,J-1,c]}$ and $\hat{U}^{(4)}_{[0,J-1,b]}$ [see Eq. (2.2)]. Namely, the operator $\hat{U}^{(2)}_{[0,J-1,c]}$ describes the transfer of population of the state $|0, J-1 \rangle \otimes |c \rangle$ to $|0, J - 1 \rangle \otimes |b \rangle$. This transfer can be achieved with the help of lasers pulses tuned to the electronic transition between the levels $|b \rangle$ and $|c \rangle$. The corresponding interaction Hamiltonian $\hat{H}^{(3)}$ and the resonance condition are analogous to those for $\hat{H}^{(1)}$ (2.3) [we have to replace only $a \rightarrow c$ and $a \rightarrow b$]. Further the operation $\hat{U}^{(4)}_{[0,J-1,b]}$ cancels a contribution of the state $|0, J - 1 \rangle \otimes |b \rangle$ to the state vector (2.3). This interaction channel is described by the Hamiltonian $\hat{H}^{(4)}$ which is obtained from (2.3) by the substitution $b \rightarrow c$ and $a \rightarrow b$. We see that the operator $B_{J-1}$ acts like $A_J$ but instead of the stimulated Raman processes between $a \leftrightarrow b$ the transitions $b \leftrightarrow c$ are utilized. As the result of the action of the operator $B_{J-1}$ in the subspace $\mathcal{H}_{J-1} \otimes \mathcal{H}_{in}$ only the component states with the internal level $|a \rangle$ and the state $|J-1, 0 \rangle \otimes |b \rangle$ have non-zero amplitudes.

After the action of the operators $A_J$ and $B_{J-1}$ the unitary operator $\hat{C}_J$ is utilized to transfer the population of the component state $|J, 0 \rangle \otimes |a \rangle$ to the state $|J-1, 0 \rangle \otimes |b \rangle$ (see Fig. 1d for $J = 3$). The transformation $\hat{U}^{(5)}_{[J,0,a]}$ which performs transitions between subspaces with the number of trap quanta differed by one is realized by the process described by a single–mode interaction Hamiltonian

$$\hat{H}_5 = g_5 \hat{a}_x^b |b\rangle \langle a| e^{-i\omega_5 t} + g_5^* \hat{a}_x^b |a\rangle \langle b| e^{i\omega_5 t}$$  

with the resonance condition $\omega_y - \omega_x = \omega_b - \omega_a - \nu_x \equiv \omega_5$. The parameters of this interaction channel are determined by the constraint

$$ie^{i\phi_5} Q_{J,0,a} \cos(|g_5| t \sqrt{J}) + Q_{J-1,0,b} \sin(|g_5| t \sqrt{J}) = 0,$$  

for $J = \tilde{J}$. As the result of the action of the operators $A_J$, $B_{J-1}$, and $\hat{C}_J$, all coefficients $Q_{k,J-k} \in (2.3)$ are equal to zero. Moreover, the situation before (Fig. 1a) and after (Fig. 1d) the action of these operators is the same,
except we just “moved” from the subspace $H_j \otimes H_{in}$ to $H_{j-1} \otimes H_{in}$ with the number of trap quanta decreased by one. This means that the procedure can be recursively repeated. The given solution of the “de-evolution” gives immediately the recipe for the creation of the target vibrational state from the two–mode vacuum (with electronic level $|a\rangle$): We have to change by $\pi$ the phase shift between external laser fields and repeat the sequence in the opposite order, i.e. we apply $U^\dagger$ on $|0, 0\rangle \otimes |a\rangle$. In this way we can synthesize an arbitrary state. The number of “elementary” operations involved in this process is proportional to $2J_{max}^2$, which is important for an experimental realization of the proposed scheme, i.e., the number of necessary operations increases only polynomially with the increase of the size of the Hilbert space in which the target state is embedded.

In the following section a physical implementation of the preparation scheme is discussed in more details.

III. REALIZATION OF INTERACTION CHANNELS

The Hamiltonians which are eligible for the synthesis of two–mode bosonic states has been discussed recently by several authors [11, 13]. Some of the proposals are based on laser stimulated dipole transitions [10] and phonon-number–dependent interaction via detuned standing–wave [11]. For our purposes we have utilized the stimulated Raman processes discussed in detail by Steinbach, Twamley and Knight [13]. In this section following Ref. [13] we briefly derive the Hamiltonians which are used in our algorithm [see Eqs. (3.3), (2.14), (2.3)] and discuss the range of their applicability. Let us consider a trapped ion confined in a 2D harmonic potential characterized by the trap frequencies $\nu_x$ and $\nu_y$ in two orthogonal directions $x$ and $y$. The ion is irradiated along the $x$ and $y$ axes by two external laser fields with frequencies $\omega_x, \omega_y$ and wave vectors $k_x, k_y$. The laser fields stimulate Raman transitions between two internal energy levels $|a\rangle$ and $|b\rangle$ via an auxiliary electronic level $|r\rangle$ which is far off resonance. For concreteness, we consider $A$–configuration with the upper level $|r\rangle$ as outlined in the inset of Fig. 1b. The interaction Hamiltonian for the system under consideration can be written in the dipole and rotating–wave approximation (RWA at laser frequencies) in the form:

$$\hat{H}_{int} = \hat{g}_x e^{i(k_x x - \omega_x t)} |r\rangle \langle b| + \hat{g}_y e^{i(k_y y - \omega_y t)} |b\rangle \langle r| + \hat{g}_z e^{i(k_z z - \omega_z t)} |a\rangle \langle r|.$$ (3.1)

The coupling constant $g_x (g_y)$ is proportional to the intensity of the laser field in $x (y)$ direction and the dipole moment of the electronic transition $|b\rangle \leftrightarrow |r\rangle$ ($|a\rangle \leftrightarrow |r\rangle$). The upper off–resonant level $|r\rangle$ can be adiabatically eliminated provided that $\Delta_x, \Delta_y \gg g_x, g_y, |\Delta_x - \Delta_y|$, where laser detunings for dipole transitions $|b\rangle \leftrightarrow |r\rangle$ and $|a\rangle \leftrightarrow |r\rangle$ are denoted as $\Delta_x = (\omega_x - \omega_b) - \omega_x$ and $\Delta_y = (\omega_y - \omega_b) - \omega_y$, respectively. After adiabatic elimination the effective interaction Hamiltonians for the stimulated Raman transition $|a\rangle \leftrightarrow |b\rangle$ reads [13]

$$\hat{H}_{int}^{(eff)} = \hat{g}_x e^{-i(\omega_y - \omega_x t)} \hat{D}_x (-i \epsilon_x) \hat{D}_y (\epsilon_y) |b\rangle \langle a| + \hat{g}_y e^{i(\omega_x - \omega_y t)} \hat{D}_x (i \epsilon_x) \hat{D}_y (-i \epsilon_y) |a\rangle \langle b|.$$ (3.2)

Here $\hat{D}_x (\epsilon_x) = e^{i\epsilon_x (\hat{a}_x^\dagger + \hat{a}_x)}$ is the displacement operator ($q = x, y$); the Lamb–Dicke parameter $\epsilon_q$ is defined as $\epsilon_q^2 = \hbar^2 k_q^2 / (2m \hbar \nu_q)$ and the effective interaction constant $\hat{g} = g_x g_y \hat{\Delta} / (\hat{\Delta} + \hat{\Delta}^\dagger)$. Further, we assume that the energies of the electronic levels $|a\rangle$ and $|b\rangle$ are redefined to include Stark shifts due to the adiabatic elimination of the off–resonant energy level $|r\rangle$ [13].

In the interaction picture the effective interaction Hamiltonian $\hat{H}_{int}^{(eff)} = e^{i\hat{H}_{tot} t} \hat{H}_{int}^{(eff)} e^{-i\hat{H}_{tot} t}$ can be expressed as [the free Hamiltonian $\hat{H}_0$ induces transformations $\hat{a}_q \rightarrow \hat{a}_0 e^{-i\epsilon_q t}$, $|a\rangle \langle b| \rightarrow |a\rangle \langle b| e^{i(\omega_b - \omega_x) t}$]

$$\hat{H}_{int}^{(eff)} = \hat{g}_x e^{-i(\omega_y - \omega_x t)} \hat{D}_x (-i \epsilon_x) \hat{D}_y (\epsilon_y) |b\rangle \langle a| + \text{h.c.}$$ (3.3)

where

$$\hat{F} = e^{-i(2\epsilon_x + 2\epsilon_y)} \sum_{k,l} (-1)^{k+l} \hat{\epsilon}_x^{2k} \hat{\epsilon}_y^{2l} \hat{a}_x^k \hat{a}_x^{l\dagger} \hat{a}_y^l \hat{a}_y^{l\dagger}.$$ (3.4)

In the Lamb–Dicke regime $\epsilon_x, \epsilon_y \ll 1$ the operator $\hat{F}$ is close to the unity operator and the Hamiltonian (3.4) which is written in the interaction picture acquires in the Schrödinger picture exactly the form of the two–mode Hamiltonian $\hat{H}_3 (2.3)$. With proper laser tunings we can design within the Lamb–Dicke limit also remaining interaction Hamiltonians required for the quantum state
synthesis described in Section II. In particular, retaining only resonant terms in $\hat{H}_1$ for the laser-stimulated Raman process with $\Delta = 0$ the interaction Hamiltonian $\hat{H}_1$ is obtained in the Schrödinger picture. The process with $\Delta = -\nu_z$ is described by the one–mode interaction Hamiltonian $\hat{H}_2$. Let us also notice that beyond the Lamb–Dicke limit this process is analogous to a nonlinear Jaynes–Cummings dynamics discussed by Vogel and de Matos Filho [1].

Considered approximations impose limitations on the applicability of the interaction Hamiltonians. Driven electronic transition $\hat{H}_3$ and one–mode interaction $\hat{H}_4$ have been considered already for the 1-D quantum state synthesis [3]. A new tool in our approach represents the two–mode interaction Hamiltonian $\hat{H}_5$. Comprehensive analysis of the limitations for (2.3) was done in Ref. [13]. It turns out that the most subtle point is the second RWA at trap frequencies [see Eqs.(3.3),(3.4)] for the laser-stimulated Raman (3.4) in the Lamb–Dicke regime. This problem goes beyond the scope of the present paper in which we demonstrate the applicability of the interaction Hamiltonian $\hat{H}_5$ to obtain (3.4) from (3.3). Nonetheless, it turns out that the most subtle point is the second RWA at trap frequencies within the Lamb–Dicke regime (3.6). This problem goes beyond the scope of the present paper in which we demonstrate the potential of our algorithm in the Lamb–Dicke regime.

IV. STABILITY OF SYNTHESIS

Our algorithm works ideally when there are essentially no dissipations in the system. It is the reason why we have consider effectively dissipation-free dynamics of trapped ions. Nevertheless, we have to stress that the ideal synthesis of quantum motional states assumes also perfect control of interaction constants and switching times of particular interaction channels. Namely, the values of $g_{p,t}$ found via the “de-evolution” procedure [i.e., solutions of Eqs.(2.4,2.6,2.8)] have to be controlled as precisely as possible. In practice one cannot avoid some level of “technical” noise, for example, due to imperfect timing of switching between interaction channels. Therefore in what follows we will study the stability of the presented algorithm with respect to the “technical” noise. This noise is simulated as random fluctuations of the ideal values $g_{p,t}$. In particular, fluctuations are equally distributed around the ideal (complex) values $g_{p,t}$ [i.e., solutions of Eqs.(2.4,2.6,2.8)] within a fixed interval $(1 + i)\delta$. In general, the output state vectors $|\Psi_\delta\rangle$ of the composed system which are prepared in the presence of “technical” fluctuations take the form (2.1), i.e., the vibrational and internal degrees of freedom are not disentangled. The Fig. 2 shows the fidelity $f$ of the “imperfect” output states $|\Psi_\delta\rangle$ with respect to the desired state vector $|\Psi_{\text{target}}\rangle \otimes |a\rangle$. The fidelity is defined as the averaged squared scalar product of particular realizations $|\Psi_\delta\rangle$ of $|\Psi_{\text{target}}\rangle \otimes |a\rangle$, i.e.,

$$f = \langle \langle |\Psi_{\text{target}}\rangle |a\rangle |\Psi_\delta\rangle \rangle \delta.$$  

In our simulations we have performed averaging over 100 runs of state-synthesis sequences. In these runs each value $g_{p,t}$ associated with a given elementary operation $U^{(p)}$ acquires a random fluctuation within the interval $(1 + i)\delta$. We have consider two different target states, two–mode cat–like states $|\Psi\rangle = N_{\text{cat}}(|\alpha\rangle|\alpha\rangle + | -\alpha\rangle\langle -\alpha\rangle)$ and two–mode correlated states $|\Psi_{\text{corr}}\rangle = e^{-|\alpha|^2/2} \sum_m \sqrt{\frac{\bar{n}}{m}} a^m |m, m\rangle$ with $\bar{n} = 2$. For this value of the amplitude these two states have approximately the same number of vibrational quanta $\bar{n} \approx 8.0$. We have considered two cases when $M_{\text{max}} = 12$ and $M_{\text{max}} = 20$ (here $N_{\text{max}} = M_{\text{max}}$).

Fig. 2 clearly indicates the fact, that larger the value of $M_{\text{max}}$ more pronounced is the role of fluctuations (compare solid and dashed lines which correspond to $M_{\text{max}} = 12$ and $M_{\text{max}} = 20$, respectively). This observation is easy to explain: The total number of operations in our algorithm is proportional to $8M_{\text{max}}^2$ (for $M_{\text{max}} = N_{\text{max}}$) which means that the case $M_{\text{max}} = 20$ requires almost three times more operations compared to the case with $M_{\text{max}} = 12$. The noise is accumulated as a function of elementary operations, therefore to improve the fidelity of the preparation process it is important to choose carefully $M_{\text{max}}$. To be specific, for a given $\varepsilon$ we have to choose the minimal values of $M_{\text{max}}$ and $N_{\text{max}}$ such that...
\[ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} |Q_{mn}|^2 - \sum_{m=0}^{M_{\text{max}}} \sum_{n=0}^{N_{\text{max}}} |Q_{mn}|^2 \leq \varepsilon. \]  

(4.2)

We can also use Fig. 2 to illustrate the fact that for a given value of \( M_{\text{max}} \), the fidelity of the preparation may depend on the target state. To be specific, we have found that smaller the number of nonzero amplitudes \( Q_{mn} \) in the target state \( \langle \Psi | \Psi_{\text{cat}} \rangle \) the higher the fidelity is for a given value of the range of fluctuations \( \delta \). This behavior can be rather surprising as for synthesis of two states “localized” within the same region of the vibrational “lattice” \( m, n \) we need comparable number of elementary operations. Nevertheless, each nonzero \( Q_{mn} \) after the synthesis is biased by more or less the same error and, consequently, the states with smaller number of nonzero amplitudes \( Q_{mn} \) are less sensitive to fluctuations. As already stated, this observation concerns only states “localized” within the same region of the vibrational lattice \( m, n \) and with a comparable mean number of vibrational quanta.

Finally, we briefly compare our algorithm with the one proposed by Gardiner, Cirac and Zoller [11] in which the number of operations in the preparation sequence is growing exponentially as \( 2^{M_{\text{max}}} \times 2^{M_{\text{max}}} \). Therefore fluctuations for \( M_{\text{max}} \) large enough cause insurmountable problem. There is also another problem with this procedure. Namely, Gardiner, Cirac, and Zoller have utilized only two internal atomic levels \( \langle |a, b \rangle \) which results in the fact that their algorithm is based on manipulations with vibrational states which are out of the original Hilbert space specified by the cutoffs \( M_{\text{max}} \) and \( N_{\text{max}} \). In other words, manipulations with highly excited vibrational states are required for construction of states with relatively small number of vibrational quanta. This means that not only the number of operations is exponentially growing but also the number of vibrational quanta during the preparation procedure may transiently exponentially increase.

We stress that our algorithm is associated with manipulations only within the original subspace of the Hilbert space specified by \( M_{\text{max}} \) and \( N_{\text{max}} \). Moreover, the number of operations grows only polynomially as \( 8^{-M_{\text{max}}} \). The dimension of the two–mode Fock subspace from which the component states are does not increase during the preparation procedure. This great reduction of number of operations is due to the fact that we have employed the third atomic level \( |c \rangle \) in the preparation procedure. Very recently a new 2-D preparation scheme was introduced by Kneer and Law [12]. The standing–wave laser field in \( y \) direction induces a photon-number dependent interaction enabling thus to use 1-D schemes selectively for particular subspaces with constant number of trap quanta in \( y \) direction. The number of operations scales also polynomially as \( 2^{M_{\text{max}}} \) with only two electronic levels involved.

\[ \begin{align*}
\text{fidelity} \quad & 1.0 \\
\delta \quad & 0.0 \quad 0.005 \quad 0.01 \quad 0.015 \quad 0.02 \\
\end{align*} \]

FIG. 2. We plot fidelity given by Eq.(4.1) as a function of the range of fluctuations \( \delta \). We consider two different target states, two–mode cat–like state \( \langle \Psi_{\text{cat}} | \rangle \) (denoted by \( \triangle \)) and two–mode correlated state \( \langle \Psi_{\text{corr}} | \rangle \) (denoted by \( \square \)) with \( \alpha = 2 \). We have considered two cases when \( M_{\text{max}} = 12 \) (solid lines) and \( M_{\text{max}} = 20 \) (dashed lines). Comparing two solid lines we see that smaller is the number of nonzero amplitudes \( Q_{mn} \) in the target state \( \langle \Psi_{\text{cat}} | \rangle \) the larger is the fidelity for a given value of \( \delta \).

\[ \begin{align*}
\text{V. CONCLUSION} \\
\end{align*} \]

In this paper we have presented a universal algorithm for an efficient deterministic preparation of an arbitrary two-mode bosonic state. We have adopted this algorithm as a computer program [14]. The proposed method can be generalized to 3-D trapping potential and three–mode vibrational states. In this case one would need four internal electronic levels and nine interaction channels coupled to the 3-D vibrational field. It can be shown that the number of operations required for quantum state synthesis scales polynomially \( (\sim M_{\text{max}}^3) \). Further generalization to multi-mode fields is possible [15].

\[ \begin{align*}
\text{ACKNOWLEDGMENTS} \\
\end{align*} \]

We thank Jason Twamley for helpful discussions.

[1] M. Brune, E. Hagley, J. Dreyer, X. Maitre, A. Maali, C. Wunderlich, J.M. Raimond, and S. Haroche, Phys. Rev. Lett. 76, 1800 (1996); M. Brune, F. Schmidtkailer, A. Maali, J. Dreyer, E. Hagley, J.M. Raimond, and S. Haroche, Phys. Rev. Lett. 77, 4887 (1996); S. Haroche, M. Brune, and J.M. Raimond, Phil. Trans. Roy. Soc. A 355, 2367 (1997).

[2] C. Monroe, D.M. Meekhof, B.E. King, S.R. Jeffers, W.M Itano, D.J. Wineland, and P. Gould, Phys. Rev. Lett. 75, 4011 (1995).
[3] See the special issue on Quantum State Preparation and Measurement, edited by W.P. Schleich and M.G. Raymer in J. Mod. Opt. 44, no. 11/12 (1997).
[4] K. Vogel, V.M. Akulin, and W.P. Schleich Phys. Rev. Lett. 71, 1816 (1993).
[5] A.S. Parkins, P. Marte, P. Zoller, and H.J. Kimble, Phys. Rev. Lett. 71, 3095 (1993).
[6] J.I. Cirac, A.S. Parkins, R. Blatt, and P. Zoller, Phys. Rev. Lett. 70, 556 (1993); J.I. Cirac, R. Blatt, A.S. Parkins, and P. Zoller, Phys. Rev. Lett. 70, 762 (1993); J.I. Cirac, A.S. Parkins, R. Blatt, and P. Zoller, Adv. At. Mol. Opt. Phys. 37, 237 (1996).
[7] W. Vogel and R.L. de Matos Filho, Phys. Rev. A 52, 4214 (1995); R.L. de Matos Filho and W. Vogel, Phys. Rev. Lett. 76, 608 (1996); Phys. Rev. Lett. 76, 4520 (1996); and references therein.
[8] C.K. Law and J.H. Eberly, Phys. Rev. Lett. 76, 1055 (1996).
[9] C. Monroe, D.M. Meekhof, B.E. King, and D.J. Wineland, Science 272, 1131 (1996); D.M. Meekhof, C. Monroe, B.E. King, W.M Itano, and D.J. Wineland, Phys. Rev. Lett. 76, 1796 (1996); D. Leibfried, D.M. Meekhof, B.E. King, C. Monroe, W.M Itano, and D.J. Wineland, Phys. Rev. Lett. 77, 4281 (1996); D. Leibfried, D.M. Meekhof, C. Monroe, B.E. King, W.M Itano, and D.J. Wineland, J. Mod. Opt. 44, 2485 (1997); and references therein.
[10] S.A. Gardiner, J.I. Cirac, and P. Zoller, Phys. Rev. A 55, 1683 (1997).
[11] B. Kneer and C.K. Law, Phys. Rev. A 57, 2096 (1998).
[12] S.-C. Gou and P.L. Knight, Phys. Rev. A 54, 1682 (1996); S.-C. Gou, J. Steinbach, and P.L. Knight, Phys. Rev. A 54, 4315 (1996).
[13] J. Steinbach, J. Twamley and P.L. Knight, Phys. Rev. A 56, 4815 (1997).
[14] We have developed a program with the help of which a sequence of manipulations for a construction of an arbitrary 2-D state is determined. This program is available at http://www.savba.sk/sav/inst/fyzi/qo.html.
[15] B. Hladký and G. Drobný, unpublished.