The Conformal Equivalence of the half string
3-vertex and the Sciuto,
Caneschi-schwimmer-veneziano 3-vertex

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Abstract
In this paper we construct the operator connecting the three vertex in half string field theory and the dual model vertex of Sciuto, Caneschi Schwimmer and Veneziano (SCSV). This construction results in an explicit conformal transformation linking the two interactions in the matter sector at all levels. Thus establishing the equivalence in the matter sector between the two theories at least for N = 3.

1 Introduction
The work of references [1, 2, 3, 4] has generated much interest in the half string formulation of Witten’s theory of interacting open bosonic strings (thereafter referred to as Witten’s theory). Therefore we see it useful to extend our treatment of the half string approach to Witten’s theory [5, 6, 7, 8, 9]. One of the things we have not considered in [5, 6, 7, 8, 9] is the relationship between the half string vertex and that of the dual model vertex of Sciuto, Caneschi, Schwimmer and Veneziano [10]. In reference [12], an explicit operator transformation connecting the Witten’s 3-Vertex in the full string representation to the dual model vertex of SCSV was constructed. The fact that the half string approach to Witten’s theory gives the same vertex as the operator full string approach to Witten’s theory implies that there exist an operator transformation from the half string vertex directly to the dual model vertex of SCSV. The existence of

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this operator constitute a nontrivial check on the half string formulation of the open bosonic string and it offers a powerful computation tool. In this paper we will only deal with the matter sector of the theory and the ghost sector in its bosonized form. The treatment of the ghost sector in its original fermionic form will be considered in a separate paper [26].

2 Half string field theory

It was first suggested by Witten that there similarities between the product $\ast$ in his theory of open bosonic strings and the wedge product $\wedge$ of Lie-algebra valued form [18]. The multiplication $(S_L, S_R) \ast (T_L, T_R) = (S_L, T_R) \delta (S_R - T_L)$ is rather similar to matrix multiplication, $S_L$ and $S_R$ being like the left and right indices of a matrix $S_{ij}$. This interpretation of the product $\ast$ was first suggested by E. Witten and proved by Gervais et al, [13, 14, 5, 6, 7, 8, 9]. In Witten’s string theory of open bosonic strings, one singles out the mid-point $\sigma = \pi/2$, thus dropping reparametrization invariance. One splits the string coordinates $x^\mu(\sigma)$ and ghosts $c(\sigma), b(\sigma)$ into two sets. If we denote the two sets by $(\chi_L, c_L, b_L)$ and $(\chi_R, c_R, b_R)$, then any string functional say $A[x(\sigma), c(\sigma)]$ may be thought of as a functional $A[x, c_M; \chi_L, \chi_R]$, where $x = x^\mu(\pi/2)$ and $c = c(\pi/2)$. The string functional $A[x, \chi_L, \chi_R]$ resemble a matrix $M_{x_1x_2}(x)$, thus given two such functional $A[x_M, \chi_L, \chi_R]$ and $B[x, \chi_L, \chi_R]$, their $\ast$ product becomes a generalized matrix multiplication

$$A \ast B := \int dx \sum_{x_2x_3} \delta_{x_2x_3} M_{x_1x_2}(x) N_{x_3x_4}(x)$$

In this notation the $\ast$ product becomes a generalized matrix multiplication. Likewise given a functional $A[x, \chi_L, \chi_R]$ the $\int$ becomes a generalized trace

$$\int A := TrM = \int dx \sum_{x_1x_2} \delta_{x_1x_2} M_{x_1x_2}(x)$$

The string field is a ket vector $|\psi\rangle$ in the fock space of the string modes. For any given state $|\Lambda\rangle$ belonging to this space, the physical free field action is invariant under the transformation

$$\delta |\psi\rangle = L_n |\Lambda\rangle ; \quad n > 0$$

where $L_n$ are the Virasoro generators satisfying

$$[L_m, L_n] = (m - n) L_{m+n} + C \frac{(m^3 - m)}{12} \delta_{m+n}$$

Note that equation (3) is similar to the inhomogeneous term

$$\delta A_\mu = \frac{\partial A}{\partial x^\mu}$$
of a gauge transformation. The main difference is that one has the Virasoro generators $L_n$ which satisfy the Lie algebra, instead of the simple $\partial_\mu$ operator. Here we are going to review the generalization of Yang-Mills ideas which embodies this new feature. The present treatment follows closely that of reference [5, 6, 7]. The translation group of the Yang-Mills theory is replaced by a non-abelian group. In Witten’s string theory of open bosonic string we single out the mid-point, $\sigma = \pi/2$ of the string at the expense of dropping reparametrization invariance. Thus singling out the mid-point of the string, we divide the string coordinates $x^\mu(\sigma)$ and ghosts coordinates $c(\sigma), b(\sigma)$ into two sets. These sets are defined by

$$\chi^\mu_L(\sigma) = x^\mu(\sigma), \quad \chi^\mu_R(\sigma) = x^\mu(\pi - \sigma), \quad \sigma \in [0, \pi/2]$$

(6)

and similar expressions for the ghost coordinates $c(\sigma)$ and $b(\sigma)$. Suppressing (for simplicity of notation) the ghost part, a string functional $\psi$ in this approach will be a functional of only half a string; that is

$$\psi = \psi[x(\sigma)], \quad \sigma \in [0, \pi/2]$$

(7)

Similar to Yang-Mills theory we define an inner product between any two string functional as a sum of $\psi_1$ over all internal indices, thus

$$(\psi_1, \psi_2) = \int Dx(\sigma) \overline{\psi}_1[x(\sigma)] \psi_2[x(\sigma)]$$

(8)

where $\overline{\psi}$ is the complex conjugation of $\psi$. Furthermore we define the gauge group as the group of all unitary transformations

$$\psi_1[x_1(\sigma)] \rightarrow \overline{\psi}[x_1(\sigma)] = \int Dx_2(\sigma) U[x_1(\sigma), x_2(\sigma)] \psi[x_2(\sigma)],$$

(9)

which does preserve the inner product

$$\int Dx_2 U^+[x_1, x_2] U[[x_2, x_3] = \delta[x_1 - x_3]$$

(10)

where $U^+$ is the adjoint of $U$

$$U^+[x_1, x_2] = \overline{U}[x_2, x_1]$$

(11)

again here the bar ”” denotes complex conjugation. By analogy with the differentials $dx^\mu$ in standard Yang-Mills theories, the anticommuting differentials $\eta^\sigma$ are dual to the "translation" generators $L_\sigma$. These are the so-called BRST ghosts in [15, 16, 17, 18]. Hence the potential 1-form that correspond to $A = A_\mu dx^\mu$ in the standard Yang-Mills theories, here, is given by

$$A = \int_{-\pi/2}^{\pi/2} d\sigma A_\sigma \eta^\sigma$$

(12)
and the exterior derivative \( d = dx^\mu \partial / \partial x^\mu \) in the standard Yang-Mills theory which is nilpotent; that is \( d^2 = 0 \), gets modified to

\[
Q = \int_{-\pi/2}^{\pi/2} d\sigma \left( [L_\sigma] \eta^\sigma + 4i\pi \eta^\sigma \{ \eta^\sigma , \} \right) \quad (13)
\]

Notice that (13) has an extra term as compared to the exterior derivative \( \partial = [\partial_\mu ,] dx^\mu \) of standard Yang-Mills theory. This term is the consequence of replacing the base group by a non abelian one. Recall that the basic ingredients of Witten’s string field theory of open bosonic strings were the \( * \) product and integration \( \int \). These become the dot product of two matrices, say \( M \) and \( N \) and a trace. Hence,

\[
M \cdot N = \int Dx_2 D\phi_2 M[x_1, \phi_1; x_2, \phi_2]N[x_2, \phi_2; x_3, \phi_3] \quad (14)
\]

\[
Tr M = \int Dx D\phi e^{-im\phi(\pi/2)} M[x, \phi; x, \phi] \quad (15)
\]

In the above expressions \( \phi \) refers to the ghost coordinates in the bosonized representation of the ghost first stated by Witten \[18\]. Bosonization is carried out as the following:

\[
\eta^\sigma = \frac{1}{\sqrt{\pi}} : e^{\xi(\sigma)} : \quad , \quad \eta^\sigma = \frac{1}{\sqrt{\pi}} : e^{-\xi(\sigma)} :
\]

with

\[
\xi(\sigma) = \int_0^\sigma d\sigma' \left[ \frac{\delta}{\delta \phi(\sigma')} + i\pi \phi'(\sigma') \right] \quad (17)
\]

In this approach, the Chern-Simons 3-form reads

\[
A = Tr \left( A \cdot QA + \frac{2}{3} A \cdot A \cdot A \right) \quad (18)
\]

This action is by construction invariant under the gauge transformation

\[
\delta A = Q\epsilon + A \cdot \epsilon - \epsilon \cdot A \quad (19)
\]

for any infinitesimal Hermitian matrix \( \epsilon \). Equation (18) posses yet another invariance; that is the action in (18) is invariant under the variations

\[
\Delta A = i\nu^\sigma \left[ \hat{L}_\sigma, A \right] \quad (20)
\]

where

\[
\hat{L}_\sigma = L_\sigma + L_{gh}^\sigma \quad (21)
\]

In the above expression \( L_{gh}^\sigma \) is the corresponding generator (to \( L_\sigma \)) in the ghost candidate \( \phi \). It was shown in \[14\] that this formulation is equivalent to the standard formulation of the interacting open bosonic string theory developed
in [18]. However, the "space of functionals" is a rather vague concept and must be rendered precise. In the case of Witten’s theory this was done in references [20, 21, 19]. In [20, 21, 19] the Fock space of the free-string in flat space has been constructed and *, $\int$ and other functional operators were presented explicitly in terms of creation and annihilation operators. In the case of the half-string functional approach to open bosonic string, construction of the Fock space was done in [5, 6, 7, 8, 9].

3 Half-string Coordinates

Like in the operator formalism of Witten’s theory of the open bosonic string, in the half string formulation of Witten’s theory for open bosonic strings, the elements of the theory are defined by $\delta - function$ type overlaps

$$V_{3,0}^{HS,\phi} = e^{i \sum_{j=1}^{3} Q_j^\phi (\pi/2)} \prod_{j=1}^{3} \prod_{\sigma=0}^{\pi/2} [\delta (\chi_j^L (\sigma) - \chi_j^R (\sigma))] \delta (\varphi)$$ (22)

where the ghost overlaps $\delta (\varphi)$ having identical form to the matter coordinates. The factor $Q_j^\phi$ is the ghost number insertion at the mid-point which is needed for the BRST invariance of the theory [20, 6] and in this case $Q_1^\phi = Q_2^\phi = Q_3^\phi = 1/2$. We have also dropped the space time index $\mu$ to simplify the notation. But one should keep in mind that by $x$ one means $x^\mu$ with $\mu = 1, 2, ..., 25$. The half string ghost coordinates, $\varphi_j^{L,R} (\sigma)$, are defined in the usual way [6]. The string index $j = 1, 2, 3$ (it is to be understood that $j - 1 = 0 \equiv 3$). In the Hilbert space of the theory, the $\delta - functions$ translate into operator overlap equations which determine the precise form of the vertex. Here we are going to give a brief derivation of the transformation matrices between the half string coordinates and the full string coordinates needed for the construction of the half string interacting vertex in terms of the oscillator representation of the full string. For this we shall follow closely the discussion of reference [5, 6, 7, 8, 9]. To make this more concrete we recall the standard mode expansion for the open bosonic string coordinate

$$x^\mu (\sigma) = x_0^\mu + \sqrt{2} \sum_{n=1}^{\infty} x_n^\mu \cos(n \sigma), \quad \sigma \in \left[0, \frac{\pi}{2}\right]$$ (23)

where $\mu = 1, 2, ..., 26$ and $x_0^\mu (\sigma)$ correspond to the ghost part $\phi (\sigma)$. The half string coordinates $x_j^{L,R} (\sigma)$ for the left and right halves of the string are defined in the usual way

$$\chi_j^{L,R} (\sigma) = x_j^\mu (\sigma) - x_j^\mu (\pi/2), \quad \sigma \in \left[0, \frac{\pi}{2}\right]$$ (24)

where both $\chi_j^{L,R} (\sigma)$ satisfy the usual Neumann boundary conditions at $\sigma = 0$ and a Dirichlet boundary conditions $\sigma = \pi/2$. Thus they have mode
expansions

$$\chi^L,\mu(\sigma) = \sqrt{2} \sum_{n=1}^{\infty} \chi_n^{L,\mu} \cos(n\sigma)$$

$$\chi^R,\mu(\sigma) = \sqrt{2} \sum_{n=1}^{\infty} \chi_n^{R,\mu} \cos(n\sigma)$$ (25)

Comparing equation (23) and equation (25) we obtain an expression for the half string modes in terms of the full string modes

$$\chi_{n}^{L,\mu} = x_{2n-1}^\mu + \sum_{m=1}^{\infty} \sqrt{\frac{2m}{2n-1}} \left[ M_{mn}^1 + M_{mn}^2 \right] x_{2m}^\mu$$

$$\chi_{n}^{R,\mu} = -x_{2n-1}^\mu + \sum_{m=1}^{\infty} \sqrt{\frac{2m}{2n-1}} \left[ M_{mn}^1 + M_{mn}^2 \right] x_{2m}^\mu$$ (26)

where the change of representation matrices read

$$M_{mn}^1 = \frac{2}{\pi} \sqrt{\frac{2m}{2n-1}} \left( \frac{-1}{2n} \right)^{m+n}, \quad m, n = 1, 2, 3, ...$$ (27)

$$M_{mn}^2 = \frac{2}{\pi} \sqrt{\frac{2m}{2n-1}} \left( \frac{-1}{2n} \right)^{m+n}, \quad m, n = 1, 2, 3, ...$$ (28)

Since the transformation in (26) is non singular, one may invert the relations to obtain

$$x_{2n-1}^\mu = \frac{1}{2} \left( \chi_{n}^{L,\mu} - \chi_{n}^{R,\mu} \right)$$

$$x_{2n}^\mu = \frac{1}{2} \sum_{m=1}^{\infty} \sqrt{\frac{2m-1}{2n}} \left[ M_{mn}^1 - M_{mn}^2 \right] \left( \chi_{n}^{L,\mu} + \chi_{n}^{R,\mu} \right)$$ (29)

where $n = 1, 2, 3, ...$

In the decomposition of the string into right and left pieces in (24), we singled out the midpoint coordinate. Consequently the relationship between $x_n^\mu$ and $(\chi_n^{L,\mu}, \chi_n^{R,\mu})$ does not involve the zero mode $x_0^\mu$ of $x^\mu(\sigma)$. At $\sigma = \pi/2$, we have

$$x_M^\mu \equiv x^\mu \left( \frac{\pi}{2} \right) = x_0^\mu + \sqrt{2} \sum_{n=1}^{\infty} x_{2n}^\mu$$ (30)

and so the center of mass $x_0^\mu$ may be related to the half string coordinates and the midpoint coordinate

$$x_0^\mu = x_M^\mu - \frac{\sqrt{2}}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{2n-1} \left( \chi_n^{L,\mu} + \chi_n^{R,\mu} \right)$$ (31)

Equations (30) and (31) with equations (26) and (29) complete the equivalence between $x_n^\mu$, $n = 0, 1, 2, ...$, and $(\chi_n^{L,\mu}, \chi_n^{R,\mu}, x_M^\mu)$, $n = 1, 2, 3, ...$
For later use we also need the relationships between \{ψ_n^L, ψ_n^R, M\}_n=1^∞, the half string canonical momenta, and \{p_n^\mu\}_n=0^∞, the full string canonical momenta.

Using Dirac quantization procedure

\[ [\chi^r_n, ψ^s_m] = i\delta^{rs}\delta_{nm}, \]  

where the space-time index \( \mu \) is suppressed and \( r, s = 1, 2 = L, R \); we obtain

\[
ψ_n^L = \frac{1}{2}p_{2n-1} + \sum_{m=1}^{\infty} \sqrt{2m - 1} (M^1 - M^2)_{mn} p_{2m} - \frac{\sqrt{2}}{\pi} \frac{(-1)^n}{2n - 1} p_0 \]

\[
ψ_n^R = -\frac{1}{2}p_{2n-1} + \sum_{m=1}^{\infty} \sqrt{2m - 1} (M^1 - M^2)_{mn} p_{2m} - \frac{\sqrt{2}}{\pi} \frac{(-1)^n}{2n - 1} p_0 \]

and

\[ p_M = p_0 \]

To obtain the full string conjugate momenta in terms of the half string canonical momenta, we need to invert the above relations; skipping the technical details we find

\[
p_{2n-1} = ψ_n^L - ψ_n^R, \]

\[
p_{2n} = \sum_{m=1}^{\infty} \sqrt{2m - 1} (M^1 + M^2)_{mn} (ψ_m^L + ψ_m^R) + \sqrt{2} (-1)^n P_M \]

We notice that the existence of the one-to-one correspondence between the half string and the full string degrees of freedom guarantees the existence of the identification

\[ \mathcal{H} = \mathcal{H}_M \otimes \mathcal{H}_L \otimes \mathcal{H}_R \]

where \( \mathcal{H} \) stands for the completion of the full string Hilbert space and \( \mathcal{H}_L, \mathcal{H}_R, \mathcal{H}_M \) in the tensor product stand for the two half-string Hilbert spaces and the Hilbert space of functions of the mid-point, respectively.

### 4 The Half-String Overlaps

The half string three interaction of the open bosonic string (\( V^{HS}_x \)) had been addressed before \[5, 6, 7, 8, 9\]. Here we are interested in constructing the half string three interaction vertex in the full string representation so we are able to consider the relationship of \( V^{HS}_x \) to the dual model vertex of Sciuto, Caneschi, Schwimmer and Veneziano, \( V^{SCSV}_x \) \[10\]. Here we shall only consider the coordinate piece of the half string three interaction vertex. The ghost part of the vertex (\( V^{HS}_\phi \)) in the bosonic representation is identical to the coordinate piece apart from the ghost mid-point insertions \( 3i(\pi/2)/2 \) required for ghost number conservation at the mid-point. To simplify the calculation we introduce
a new set of coordinates and momenta based on a $Z_3$ Fourier transform \[ \begin{pmatrix} Q^r (\sigma) \\ Q^3, r (\sigma) \end{pmatrix} = \frac{1}{\sqrt{3}} \begin{pmatrix} e & e^\bar{e} & 1 \\ \bar{e} & e & 1 \end{pmatrix} \begin{pmatrix} \chi^{1, r} (\sigma) \\ \chi^{2, r} (\sigma) \\ \chi^{3, r} (\sigma) \end{pmatrix} \] (37)

where $e = \exp(2\pi i/3)$ and $r$ refers to the left ($L$) and right ($R$) parts of the string, respectively. The superscripts 1, 2 and 3 refers to string 1, string 2 and string 3, respectively. Similarly one obtains a new set for the momenta $P^r (\sigma), \bar{P}^s (\sigma)$ and $P^{3, s} (\sigma)$ as well as a new set for the creation-annihilation operators $\left( B^r_j, B^r_j \right)$. In the $Z_3$ Fourier space the degrees of freedom in the $\delta$-function overlaps equations decouple. Notice that in the $Z_3$ Fourier space the commutation relations are

$$[Q^r (\sigma), P^s (\sigma') ] = i\delta^{rs} \delta (\sigma - \sigma')$$ (38)

$$[Q^3, r (\sigma), P^{3, s} (\sigma') ] = i\delta^{rs} \delta (\sigma - \sigma')$$ (39)

$$[Q^{3, r} (\sigma), P^s (\sigma') ] = i\delta^{rs} \delta (\sigma - \sigma')$$ (40)

so $Q^r (\sigma)$ and $P^r (\sigma)$ are no longer canonical variables. The canonical variables in this case are $Q^r (\sigma)$ and $\bar{P}^s (\sigma)$. Thus the $Z_3$ Fourier transform does not conserve the original form of commutation relations. The variables $Q^{3, r} (\sigma)$ and $P^{3, s} (\sigma)$ are still canonical however. This is a small price to pay for decoupling string three in the $Z_3$ Fourier space from the other two strings as we shall see in the construction of the half string three interaction vertex. Recall that the overlap equations for the half string three interacting vertex are given by

$$\chi^{j, r} (\sigma) = \chi^{j-1, r-1} (\sigma), \quad 0 \leq \sigma \leq \pi/2$$ (41)

$$x^1_M = x^2_M = x^3_M$$ (42)

for the coordinates, where the mid-point coordinate $x_M$ is defined by $x_M \equiv x(\pi/2)$. The overlaps for the canonical momenta are given by

$$\wp^{j, r} (\sigma) = -\wp^{j-1, r-1} (\sigma), \quad 0 \leq \sigma \leq \pi/2$$ (43)

$$\wp^1_M + \wp^2_M + \wp^3_M = 0$$ (44)

where the mid-point momentum is defined in the usual way $\wp_M \equiv -i\partial/\partial x_M = -i\partial/\partial x_0 = p_0$. The half string coordinates and their canonical momenta obey the usual commutation relations

$$[\chi^{j, r} (\sigma), \wp^{j, s} (\sigma') ] = i\delta^{rs} \delta (\sigma - \sigma')$$ (45)
In $Z_3$ Fourier space of the comma, the overlap equations for the half string coordinates read

\[ Q^L (\sigma) = eQ^R (\sigma), \quad 0 \leq \sigma \leq \pi/2 \]  
\[ \overline{Q}^L (\sigma) = \overline{eQ}^R (\sigma), \quad 0 \leq \sigma \leq \pi/2 \]  
\[ Q_M = \Q_M = 0 \]  
\[ Q^{3,L} (\sigma) = Q^{3,R} (\sigma), \quad 0 \leq \sigma \leq \pi/2 \]  
\[ Q^{3}_M = 0 \]

where equation (48) is to be understood as an overlap equation (i.e., its action on the three vertex is zero). Similarly the conjugate momenta of the half string in the $Z_3$ Fourier space of the half string translate into

\[ P^L (\sigma) = -eP^R (\sigma), \quad 0 \leq \sigma \leq \pi/2 \]  
\[ \overline{P}^L (\sigma) = -\overline{eP}^R (\sigma), \quad 0 \leq \sigma \leq \pi/2 \]  
\[ P^{3,L} (\sigma) = -P^{3,R} (\sigma), \quad 0 \leq \sigma \leq \pi/2 \]  
\[ P^{3}_M = 0 \]

The overlap conditions on $Q^r (\sigma)$ and $P^r (\sigma)$ determine the form of the half string three interaction vertex. Thus in the $Z_3$ Fourier space of the half string the overlap equations separate into two sets. The half string vertex $V^{HS}_x$, therefore separates into a product of two pieces one depending on $B^{3,r\dagger}$

\[ B^{3,r} = \frac{1}{\sqrt{3}} (b^{1,r} + b^{2,r} + b^{3,r}) , \quad r = L, R \]  
\[ B^{3,r\dagger} = \frac{1}{\sqrt{3}} (b^{1,r\dagger} + b^{2,r\dagger} + b^{3,r\dagger}) , \quad r = L, R \]

and the other one depending on $\left( B^r, \overline{B}^{\dagger} \right)$

\[ B^r = \frac{1}{\sqrt{3}} (eb^{1,r} + \overline{eb}^{2,r} + b^{3,r}) , \quad r = L, R \]  
\[ B^{r\dagger} = \frac{1}{\sqrt{3}} (eb^{1,r\dagger} + eb^{2,r\dagger} + b^{3,r\dagger}) , \quad r = L, R \]  
\[ \overline{B}^r = \frac{1}{\sqrt{3}} (\overline{eb}^{1,r} + eb^{2,r} + b^{3,r}) , \quad r = L, R \]  
\[ \overline{B}^{r\dagger} = \frac{1}{\sqrt{3}} (eb^{1,r\dagger} + \overline{eb}^{2,r\dagger} + b^{3,r\dagger}) , \quad r = L, R \]

Notice that in this notation we have $B^{r\dagger}_n = \overline{B}^r_{-n}$ and $\overline{B}^{r\dagger}_n = B^{r}_{-n}$ (where for the $b$'s the usual convention $b_{-n} = b^n$ applies). Observe that the first of these equations is identical to the overlap equation for the identity vertex. Hence, the
half string 3-Vertex takes the form
\[
|V_Q^{HS}| = \int dQ_M dQ_N dQ_P \delta(Q_M) \delta(Q_N) Q_M Q_N Q_P \times e^{-\frac{i}{2} (B^a |C| B^a) - (B^1 |H| B^1)} \prod_{r=L,R} |0 >^3_r |0 >^r |0 >^r
\]
where \( C \) and \( H \) are infinite dimensional matrices computed in [23] and the integration over \( Q_M^3 \) gives \( \delta(P_M^3) \). However, \( P_M^3 = P_0^3 \) (see [5]) and so \( \delta(P_M^3) \) is the statement of conservation of momentum at the center of mass of the three strings. Notice that the half string three interaction vertex separates into a product of two pieces as anticipated. The vacuum of the three strings, i.e., \( \prod_{j=1}^3 |0 >^j_L |0 >^j_R \), is however invariant under the \( Z_3 \)-Fourier transformation. Thus we have \( \prod_{r=1}^2 |0 >^{3,r} |0 >^r |0 >^r = \prod_{j=1}^3 |0 >^j_L |0 >^j_R \). If we choose to substitute the explicit values of the matrices \( C \) and \( H \), the above expression reduces to the simple form
\[
|V_x^{HS}| = \int \prod_{i=1}^3 dx_M^3 \delta(x_M^3 - x_M^{3-1}) \delta \left( \sum_{j=1}^3 p_M^3 \right) \\
\times e^{-\sum_{j=1}^3 \sum_{n=1}^\infty \delta(j \cdot L^3 + j^{3-1} \cdot R^3)} |0 >^r |0 >^r
\]
where \( |0 >^L,R \) denotes the vacuum in the left (right) product of the Hilbert space of the three strings. Here \( \delta(j \cdot L^3 + j^{3-1} \cdot R^3) \) denotes oscillators in the \( L(R) \) \( j \)th string Hilbert space. For simplicity the Lorentz index \((\mu = 0, ..., 25)\) and the Minkowski metric \( \eta_{\mu\nu} \) used to contract the Lorentz indices, have been suppressed in equation (61). We shall follow this convention throughout this paper.

Though the form of the half string 3-Vertex given in equation (61) is quite elegant, it is very cumbersome to relate it directly to the \( SCSV \) 3-Vertex due to the fact that connection between the vacuum in the half string theory and the vacuum in the \( SCSV \) is quite involved; see reference [5]. One also need to use the change of representation formulas to recast the quadratic form in the half string creation operators in terms of the full string creation-annihilation operators which adds more complications to an already a difficult problem [5]. Alternatively we could rewrite the \( SCSV \) vertex in the half string basis. Both ways are involved and lead to a considerable amount of algebra. On the other hand the task could be greatly simplified if we express the half string vertex in the full string basis. This may be achieved simply by reexpressing the half string overlaps in terms of overlaps in the full string basis. But before we do that, first, we are going to need to solve the half string overlap equations in (46), (49) and (51), (53) for the Fourier modes of the half string coordinates and momenta respectively. The overlap equations for the coordinates in (46) and (47)

\[
Q^r (\sigma) = Q^r (-\sigma) \quad \text{and} \quad Q^r (\sigma) = -Q^r (\pi - \sigma),
\]
\[
\overline{Q}^r (\sigma) = \overline{Q}^r (-\sigma) \quad \text{and} \quad \overline{Q}^r (\sigma) = -\overline{Q}^r (\pi - \sigma),
\]
\[
Q^{3,r} (\sigma) = Q^{3,r} (-\sigma) \quad \text{and} \quad Q^{3,r} (\sigma) = -Q^{3,r} (\pi - \sigma)
\]
and the condition imposed on the Fourier expansion of the half string coordinates, where \(0 \leq \sigma \leq \pi/2\), imply that their \(Z_3\) Fourier modes in half string basis satisfy
\[
Q^L_{2n-1} = eQ^R_{2n-1}, \quad (65)
\]
\[
Q^R_{2n-1} = eQ^L_{2n-1} \quad (66)
\]

From the overlap in (49) we obtain
\[
Q^3_{2n-1} = Q^3_{2n-1} \quad (67)
\]

For the Fourier modes of the conjugate momenta one obtains
\[
P^L_{2n-1} = -eP^R_{2n-1}, \quad (68)
\]
\[
P^R_{2n-1} = -eP^L_{2n-1} \quad (69)
\]
and
\[
P^{3,L}_{2n-1} = -P^{3,R}_{2n-1}, \quad (70)
\]

where \(n = 1, 2, 3, \ldots\). We see that the half string overlaps in the full string basis separates into a product of two pieces depending on
\[
A_n^{\dagger} = \frac{1}{\sqrt{3}} \left( a_n^{\dagger} + a_n^{2\dagger} + a_n^{3\dagger} \right) \quad (71)
\]
and on
\[
A_n = \frac{1}{\sqrt{3}} \left( e a_n^{\dagger} + \bar{e} a_n^{2\dagger} + a_n^{3\dagger} \right), \quad (72)
\]
\[
\bar{A}_n = A_n^{2\dagger} = \frac{1}{\sqrt{3}} \left( e a_n^{\dagger} + \bar{e} a_n^{2\dagger} + a_n^{3\dagger} \right), \quad (73)
\]
respectively, where the creation and annihilation operators \(A_n^{\dagger}\) and \(A_n\) in the \(Z_3\)-Fourier space are defined in the usual way
\[
Q_n = \frac{i}{2} \sqrt{\frac{2}{n}} \left( A_n - A_n^{\dagger} \right), \quad n = 1, 2, 3, \ldots \quad (74)
\]
\[
Q_0 = \frac{i}{2} \left( A_0 - A_0^{\dagger} \right) \quad (75)
\]
\[
P_n = -i \frac{\partial}{\partial Q_n} = \sqrt{\frac{n}{2}} \left( A_n + A_n^{\dagger} \right), \quad n = 1, 2, 3, \ldots \quad (76)
\]
\[
P_0 = -i \frac{\partial}{\partial Q_0} = \left( A_0 + A_0^{\dagger} \right) \quad (77)
\]
and similarly for \(A_n^{\dagger}, A_n, A_n^{3\dagger}, A_n^3\). Notice that in the \(Z_3\)-Fourier space, \(A_n^{\dagger} = \bar{A}_{-n}, A_n = A_{-n}\). For the matter sector, the half string 3-Vertex would be
represented as exponential of quadratic form in the creation operators $A_{n}^{\dagger}$, $A_{n}^{\dagger}$ and $\overline{A}_{m}$. Thus the half string 3-Vertex in the full string $Z_3$-Fourier space takes the form

$$|V_{Q}^{HS}\rangle = \int dQ_{M}d\overline{Q}_{M} \delta(Q_{M}) \delta(\overline{Q}_{M}) V_{Q}^{HS} \left(A_{n}^{\dagger}, A_{n}, \overline{A}_{n}\right) |0\rangle_{123}$$

(78)

where $|0\rangle_{123}$ denotes the matter part of the vacuum in the Hilbert space of the three strings and

$$V_{Q}^{HS} \left(A_{n}^{\dagger}, A_{n}, \overline{A}_{n}\right) = e^{-\frac{1}{2} \sum_{n,m=0}^{\infty} A_{n}^{\dagger} C_{nm} A_{m} + \sum_{n,m=0}^{\infty} A_{n}^{\dagger} H_{nm} \overline{A}_{m}}$$

(79)

The ghost piece of the 3-Vertex in the bosonized form has the same structure as the coordinate piece apart from the mid point insertions. In the $Z_3$-Fourier space $Q_{M}^{\phi} = \overline{Q}_{M} = 0$ and only $Q_{M}^{3} \neq 0$. Thus the mid-point insertion is given by $\frac{3}{2}iQ_{M}^{3}$. The effect of the insertion is to inject the ghost number into the vertex at its mid-point to conserve the ghost number at the string mid-point, where the conservation of ghost number is violated due to the concentration of the curvature at the mid-point. Thus the ghost part of the 3-Vertex takes the form

$$|V_{Q}^{HS}^{\phi}\rangle = e^{3iQ_{M}^{3}/2} V_{Q}^{HS} \left(A_{n}^{\dagger}, A_{n}, \overline{A}_{n}\right) |0\rangle_{123}^{\phi}$$

(80)

where the $A_{n}^{\phi}$’s are the bosonic oscillators defined by the expansion of the bosonized ghost $(\Phi(\sigma), P^{\phi}(\sigma))$ fields. The vertex $V_{Q}^{HS} \left(A_{n}^{\dagger}, A_{n}, \overline{A}_{n}\right)$ is the exponential of the quadratic form in the ghost creation operators with the same structure as the coordinate piece of the vertex and $|0\rangle_{123}^{\phi}$ denotes the ghost part of the vacuum in the Hilbert space of the three strings. The mid-point insertion $3iQ_{M}^{3}/2$ in (80) may be written in terms of the creation-annihilation operators

$$Q_{M}^{3} = Q_{n}^{3} + i \sum_{n=even, n=2}^{\infty} \frac{(-1)^{n/2}}{\sqrt{n}} (A_{n} - A_{n}^{\dagger})$$

(81)

If we now commute the annihilation operators in the mid-point insertion through the exponential of the quadratic form in the creation operators $V_{Q}^{HS}^{\phi}$, the three-string ghost vertex in (80) takes the form

$$|V_{Q}^{HS}^{\phi}\rangle = e^{3iQ_{n}^{3}/2} e^{3 \sum_{n=even, n=2}^{\infty} \frac{(-1)^{n/2}}{\sqrt{n}} A_{n}^{\dagger}} V_{Q}^{HS} \left(A_{n}^{\dagger}, A_{n}, \overline{A}_{n}\right) |0\rangle_{123}^{\phi}$$

(82)

Thus commuting the annihilation operators in the mid-point insertion $3iQ_{M}^{3}/2$ through $V_{Q}^{HS} \left(A_{n}^{\dagger}, A_{n}, \overline{A}_{n}\right)$ results in the doubling of the creation operator in the mid-point insertion.

\footnote{In reference \cite{25}, the coupling constants are referred to as $F$’s.}
5 The Comma 3-Vertex in the full string basis

To proceed further we need to express the half string overlaps in the Hilbert space of the full string theory. The change of representation between the half string modes and the full string modes derived in [5] is given by

\[ Q_{2n-1}^r = (-1)^{r+1} Q_{2n-1} + \sum_{m=1}^{\infty} \sqrt{\frac{2m}{2n-1}} \left[ M_{mn}^1 + M_{mn}^2 \right] Q_{2m} \]  
(83)

\[ P_{2n-1}^r = \frac{(-1)^{r+1}}{2} P_{2n-1} + \frac{1}{2} \sum_{m=1}^{\infty} \sqrt{\frac{2n-1}{2m}} \left[ M_{mn}^1 - M_{mn}^2 \right] P_{2m} \]

\[ -\frac{\sqrt{2}}{\pi} \frac{(-1)^n}{2n-1} P_0 \]

where \( r = 1, 2 \equiv L, R; \) \( n = 1, 2, 3, \ldots \)

The overlap equations in (65), (68) and (48) become

\[ (1 + e) Q_{2n-1} = - (1 - e) \sum_{m=1}^{\infty} \sqrt{\frac{2m}{2n-1}} \left[ M_{mn}^1 + M_{mn}^2 \right] Q_{2m} \]  
(84)

\[ (1 - e) \frac{1}{2} P_{2n-1} = - (1 + e) \frac{1}{2} \sum_{m=1}^{\infty} \sqrt{\frac{2n-1}{2m}} \left[ M_{mn}^1 - M_{mn}^2 \right] P_{2m} \]

\[ + (1 + e) \frac{\sqrt{2}}{\pi} \frac{(-1)^n}{2n-1} P_0 \]  
(85)

\[ Q_M = Q_0 + \sqrt{2} \sum_{n=1}^{\infty} (-1)^n Q_{2n} = 0 \]  
(86)

respectively. The overlaps for the complex conjugate of the first two equation could be obtained simply by taking the complex conjugation. Similarly from the overlaps in (67), (70) and (54) we obtain

\[ \sum_{m=1}^{\infty} \sqrt{\frac{2n-1}{2m}} \left[ M_{mn}^1 - M_{mn}^2 \right] P_{2m} - \frac{2\sqrt{2}}{\pi} \frac{(-1)^n}{2n-1} P_0^3 = 0 \]  
(88)

\[ P_M^3 = 0 \]  
(89)

Using the fact that \( P_M^3 = P_0^3 \), the overlap conditions in (88) and (89) reduce to

\[ \sum_{m=1}^{\infty} \sqrt{\frac{2n-1}{2m}} \left[ M_{mn}^1 - M_{mn}^2 \right] P_{2m} = 0 \]  
(90)

\[ P_0^3 = 0 \]  
(91)

It is important to keep in mind that the equality sign appearing in equations (84) through (91) is an equality between action of the operators when acting on
the half string vertex except for equation (91) which is the conservation of the momentum carried by the third string in the $Z_3$ Fourier space.

The half string vertex $V^{HS}(A^3_n, A^1_n, A^3_n)$ in the full string basis now satisfies the half string overlaps in (84), (85), (86), (87), (90) and (90). First let us consider the overlaps in (84), (85) and (86), i.e.,

$$0 = \left\langle V^{HS}(A^3_n, A^1_n, A^3_n) \right| \left( (1 + e) Q_{2n-1} + (1 - e) \sum_{m=1}^{\infty} \frac{2m}{2n-1} (M^1_{m,n} + M^2_{m,n}) Q_{2m} \right) \right| V^{HS}(A^3_n, A^1_n, A^3_n) \right\rangle$$

(92)

$$\left\langle V^{HS}(A^3_n, A^1_n, A^3_n) \right| \left( (1 - e) \frac{1}{2} P_{2n-1} + (1 + e) \frac{1}{2} \sum_{m=1}^{\infty} \frac{2m - 1}{2m} (M^1_{m,n} - M^2_{m,n}) P_{2m} - (1 + e) \frac{\sqrt{2}}{\pi} \frac{(-1)^n}{2n-1} P_0 \right) \right| V^{HS}(A^3_n, A^1_n, A^3_n) \right\rangle = 0,$$

(93)

and

$$\left\langle V^{HS}(A^3_n, A^1_n, A^3_n) \right| \left( Q_0 + \sqrt{2} \sum_{k=1}^{\infty} (-1)^k Q_{2k} \right) \right| V^{HS}(A^3_n, A^1_n, A^3_n) \right\rangle = 0$$

(94)

(as well as their complex conjugates), where $n = 1, 2, 3, ...$ For the remaining overlaps, i.e., equations (87) and (90), we have

$$Q_{2n-1}^3 \left\langle V^{HS}(A^3_n, A^1_n, A^3_n) \right| V^{HS}(A^3_n, A^1_n, A^3_n) \right\rangle = 0$$

(95)

$$\sum_{m=1}^{\infty} \sqrt{\frac{2m - 1}{2m}} (M^1_{m,n} - M^2_{m,n}) P_{2m}^3 \left\langle V^{HS}(A^3_n, A^1_n, A^3_n) \right| V^{HS}(A^3_n, A^1_n, A^3_n) \right\rangle = 0$$

(96)

$$P_0^3 \left\langle V^{HS}(A^3_n, A^1_n, A^3_n) \right| V^{HS}(A^3_n, A^1_n, A^3_n) \right\rangle = 0$$

(97)

where $n = 1, 2, 3, ...$ We notice that these overlaps are identical to the overlap equations for the identity vertex [20, 21, 8, 9]. Thus

$$C_{nm} = (-1)^n \delta_{nm}, \quad n, m = 0, 1, 2, ...$$

(98)

The explicit form of the matrix $H$, may be obtained from the overlap equations given by (92), (93) and (94) as well as their complex conjugates. It will turn out that the matrix $H$ has the following properties

$$H = H^\dagger, \quad \overline{H} = CHC, \quad H^2 = 1$$

(99)

which are consistent with the properties of the coupling matrices in Witten’s theory of open bosonic strings [20, 21]. This, indeed, is a nontrivial check on the validity of the half string approach to the theory of open bosonic strings.

Substituting (78) into (92) and writing $Q_n$ in terms of $A^1_n$ and $A_n$, we obtain the first equation for the matrix $H$
\[ H_{2n-1 \, k} + \delta_{2n-1 \, k} - i\sqrt{3} \sum_{m=1}^{\infty} \left( M_{m \, n}^1 + M_{m \, n}^2 \right) (H_{2m \, k} + \delta_{2m \, k}) = 0 \]  

(100)

where \( k = 0, 1, 2, ..., n = 1, 2, 3, ... \). Next from the overlap equation in (93) we obtain a second condition on the \( H \) matrix

\[ 0 = (H_{2n-1 \, k} - \delta_{2n-1 \, k}) + \frac{1}{\sqrt{3}} \sum_{m=1}^{\infty} \left( M_{m \, n}^1 - M_{m \, n}^2 \right) (H_{2m \, k} - \delta_{2m \, k}) \]

\[ - \frac{4 \, i}{\pi} \sqrt{\frac{(-1)^n}{(2n-1)^{3/2}}} (H_{0 \, k} - \delta_{0 \, k}) \]  

(101)

where \( k = 0, 1, 2, ..., n = 1, 2, 3, ... \) The overlaps for the mid-point in (94) gives

\[ \left[ (H_{0m} + \delta_{0m}) + \sqrt{2} \sum_{k=1}^{\infty} (-1)^k \sqrt{\frac{2}{2k}} (H_{2km} + \delta_{2km}) \right] = 0, \quad m = 0, 1, 2, ... \]  

(102)

Solving equations (100) and (101), we have (Be careful in the equations below I have relabeled the indices on the matrices just so the indices are what is usually used to label matrix elements)

\[ H_{2n \, 0} = \frac{1}{\pi} (H_{00} - 1) \sum_{m=1}^{\infty} \left[ \left( M_1^T + \frac{1}{2} M_2^T \right) \right]_{nm}^{-1} \frac{(-)^m}{(2m - 1)^{3/2}} \]  

(103)

\[ H_{2n \, 2k} = \frac{1}{\pi} H_{0 \, 2k} \sum_{m=1}^{\infty} \left[ \left( M_1^T + \frac{1}{2} M_2^T \right) \right]_{nm}^{-1} \frac{(-)^m}{(2m - 1)^{3/2}} - \sum_{m=1}^{\infty} \left[ \left( M_1^T + \frac{1}{2} M_2^T \right) \right]_{nm}^{-1} \left[ \frac{1}{2} M_1^T + M_2^T \right]_{m \, k} \]  

(104)

\[ H_{2n \, 2k-1} = -\frac{i\sqrt{3}}{2} \left[ \left( M_1^T + \frac{1}{2} M_2^T \right) \right]_{nk}^{-1} + \frac{1}{\pi} H_{0 \, 2k-1} \]

\[ \times \sum_{m=1}^{\infty} \left[ \left( M_1^T + \frac{1}{2} M_2^T \right) \right]_{nm}^{-1} \frac{(-)^m}{(2m - 1)^{3/2}} \]  

(105)

\[ H_{2n-1 \, 2k-1} = \frac{2i}{\sqrt{3}} \sum_{m=1}^{\infty} \left[ \frac{1}{2} M_1^T + M_2^T \right]_{nm} H_{2m \, 2k-1} + \frac{2i}{\pi\sqrt{3}} \frac{(-)^n}{(2n-1)^{3/2}} H_{0 \, 2k-1} \]  

(106)

\[ H_{2n-1 \, 2k} = \frac{2i}{\sqrt{3}} \sum_{m=1}^{\infty} \left[ \frac{1}{2} M_1^T + M_2^T \right]_{nm} H_{2m \, 2k} + \frac{2i}{\pi\sqrt{3}} \left[ \frac{1}{2} M_1^T + M_2^T \right]_{nk} \]

\[ + \frac{2i}{\pi\sqrt{3}} \frac{(-)^n}{(2n-1)^{3/2}} H_{0 \, 2k} \]  

(107)
\[ H_{2n-1} \theta = \frac{2i}{\sqrt{3}} \sum_{m=1}^{\infty} \left[ \frac{1}{2} M_1^T + M_2^T \right]_{nm} H_{2m} \theta + \frac{2i}{\pi \sqrt{3} (2n-1)^{3/2}} (-)^n (H_{00} - 1) \]

(108)

where all \( n, k = 1, 2, 3, \ldots \). Finally equation (102) leads to

\[ (H_{00} + 1) = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{2n}} H_{2n} \theta \]

(109)

\[ H_{02m} = 2 \frac{(-1)^{m+1}}{\sqrt{2m}} + 2 \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{2k}} H_{2k} \theta \]

(110)

\[ H_{02m-1} = 2 \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{2k}} H_{2k} \theta \]

(111)

where \( m = 1, 2, 3, \ldots \). The explicit form of the \( H \) matrix is now completely given by the set of equations (103), (104), (105), (106), (107), (108), (109), (110) and (111) provided that the inverse of the \((M_1^T + \frac{1}{2} M_2^T)\) exist. The inverse \((M_1^T + \frac{1}{2} M_2^T)^{-1}\) has been computed before [24]

\[ \left[ \left( M_1^T + \frac{1}{2} M_2^T \right)^{-1} \right]_{nm} = \frac{(-)^{n+m}}{\sqrt{2n}\sqrt{2m}} \left[ \frac{a_{2n}b_{2m-1} + b_{2n}a_{2m-1}}{2n - (2m - 1)} - \frac{a_{2n}b_{2m-1} - b_{2n}a_{2m-1}}{2n + (2m - 1)} \right] \]

(112)

where the coefficients \(a_k\) and \(b_k\) are the modes appearing in the Taylor expansion of the functions \((\frac{1+z}{1-z})^{1/3}\) and \((\frac{1+z}{1-z})^{2/3}\) respectively. We have verified by direct computation that matrix \(H\) has the desired properties stated in [24], which is consistent with the properties of the coupling matrices in Witten’s theory of open bosonic strings [20, 21]. This, indeed, is a nontrivial check on the validity of the half string approach to the theory of open bosonic strings. We are now in a position to construct the operator connecting the half string vertex and that of the SCSV of the dual model.

6 The Operator Connecting the SCSV 3-Vertex and the Comma 3-Vertex

In [12], the explicit operator connecting the covariant and the dual vertices was constructed. The existence of the conformal operator was ensured by the fact that the construction of Witten’s covariant string theory was related to the dual model through a conformal mapping. The existence of such transformations, guarantees that all physical couplings of the vertex operators are identical, and

\[ ^2 \text{The properties of the } M \text{ matrix are discussed in ref. [20].} \]
moreover gives an alternative nontrivial computational tool as it has been shown in [12]. We have seen in [23, 8, 9, 1, 2, 3, 4], that the half string theory offers an alternative way of formulating Witten’s covariant theory in terms of the half-string degrees of freedom. The equivalence between the half string theory and Witten’s theory was discussed in [23, 8, 9], where most of the technical details were overcome, but for a few conceptual points regarding the uniqueness of the Witten interaction, and the role played by the mid-point of the string. To help understand these delicate points further, it is important that we find a way of relating the comma theory to the dual model of Sciuto, Caneschi, Schwimmer and Veneziano [10] [11]. In this section, we shall construct explicitly an operator connecting the half string theory and the dual model. The general procedure described here follows closely that employed in [12]. Furthermore here we will only concentrate on the matter part of the vertex, and a similar approach will be used for the ghost part of the vertex, which will be presented in [26].

The Sciuto-Caneschi-Schwimmer-Veneziano vertex, \( V^{SCSV}_x \), has the explicit form [10] [11]

\[
< V^{SCSV}_x | = < 0, 0, 0 | \delta \left( \sum_{i=1}^{3} p_i^0 \right) \exp \left[ \frac{1}{2} \sum_{i,j=1}^{3} \sum_{n=1,m=0}^{\infty} \alpha^i_n M^{ij}_{nm} \alpha^j_m \right] \]  

(113)

where

\[
M^{12}_{nm} = M^{23}_{nm} = M^{31}_{nm} = \frac{(-1)^n}{n} \binom{n}{m} 
\]  

(114)

and all other \( M \)'s vanish. The vertex in (113) satisfies the overlap equations

\[
< V^{SCSV}_x | \left[ \alpha^i_{-n} - \sum_{j=1}^{3} \sum_{m=0}^{\infty} nM^{ij}_{nm} \alpha^j_m \right] = 0 
\]  

(115)

where \( i = 1, 2, 3 \) and \( n = 1, 2, 3, \ldots \). Likewise, from equation (79), we can derive the equation defining the half string vertex

\[
< V^{HS}_x | \left[ \alpha^i_{-n} - \sum_{j=1}^{3} \sum_{m=0}^{\infty} nH^{ij}_{nm} \alpha^j_m \right] = 0 
\]  

(116)

where \( i = 1, 2, 3 \) and \( n = 1, 2, 3, \ldots \). Here the string indices \( i, j = 1, 2, 3 \) and the mode indices \( n, m = 0, 1, 2, \ldots \). It is important to notice that the relations in (116), are equivalent to the overlap equations for defining the half string three vertex in the full string basis in the sense that they determine the matter part of the vertex, but they are more convenient to work with.

Since physical on-shell states for the open bosonic string are given by,

\[
(L^i_n - \delta_{nm}) |phys> = 0, \quad n = 0, 1, 2, \ldots
\]  

(117)

therefore, the fact that both vertices lead to the same couplings for all physical on shell states guarantees the existence of an operator connecting both vertices.
\[ \hat{O} = \exp \sum_{i=1}^{3} \sum_{n=0}^{\infty} A^i_n \left( L^i_n - \delta_{n0} \right) \]  

(118)

such that

\[ \langle V^{SCSV}_x | \hat{O} = \langle V^{HS}_x | \]  

(119)

Here we shall construct such an operator. Guided by the work of reference [12], we will look for an operator of the form in (118)

\[ \hat{O} = \exp \sum_{i=1}^{3} \sum_{n=0}^{\infty} \Lambda^i_n \left( L^i_n - \delta_{n0} \right) \]  

(120)

such that

\[ \langle V^{SCSV}_x | \hat{O} = \langle V^{HS}_x | \]  

(121)

Equation (115) now gives

\[ 0 = \langle V^{HS}_x | \hat{O}^{-1} \left[ \alpha^i_{-n} - \sum_{j=1}^{3} \sum_{m=0}^{\infty} nM_{nm} \alpha^j_m \right] \hat{O} \]  

(122)

The completeness of (116), grants that this is some linear combination of relations (115) for different values of the indices \( i \) and \( n \). To determine the values of the coefficients \( \Lambda^i_n \), we need to compare (116) and (122). To do this successfully, we first need to compute the basic commutator

\[ \left[ \alpha^i_n, e^{\sum_{m=0}^{\infty} \Lambda^i_m \left( L^i_m - \delta_{m0} \right)} \right] \]  

(123)

The Virasoro generators are given in terms of the \( \alpha^i \)'s by

\[ L^i_m = \frac{1}{2} \sum_{k=-\infty}^{\infty} : \alpha^i_{m-k} \alpha^i_k : \]  

(124)

where, the normal ordering \( : : \) is with respect to the full string vacuum. The commutator in (123) is some function of the commutators \([L^i_m, \alpha^i_n]\), commutators of these commutators, etc. Thus equation (123) gives

\[ \left[ \alpha^i_n, e^{\xi \sum_{m=0}^{\infty} \Lambda^i_m L^i_m} \right] = e^{\xi \sum_{m=0}^{\infty} \Lambda^i_m L^i_m} \sum_k f^{(i,n)}_k (\xi) \alpha^i_k \]  

(125)

where we have introduced a parameter \( \xi \). Notice that this result reduces to the commutator in (123) by setting \( \xi = 1 \). Our original conformal operator in (120) is related to \( \hat{O} (\xi) \) through the relation \( \hat{O} (\xi) \mid_{\xi=1} = \hat{O} (\xi) \). By differentiating with respect to \( \xi \), one obtains a set of differential equations for the functions \( f^{(i,n)}_k (\xi) \)

\(^3\)We have ignored the constant term \( \Lambda^i_0 \delta_{00} \) in the exponential \( \sum_{n=0}^{\infty} \Lambda^i_n \left( L^i_n - \delta_{n0} \right) \) since this piece is a \( C - number \) and so it commutes with the \( \alpha^i \)'s.
which can be solved exactly as we shall see shortly. Differentiating both sides of the above expression and using the commutation relation

\[ [L^i_m, \alpha^j_n] = -m\alpha^i_{n+m} \]  

(126)
to pull the \( L \)'s past the \( \alpha \)'s, we find

\[
\sum_k df_k^{(i,n)}(\xi) \alpha_k^i = \sum_k k f_k^{(i,n)}(\xi) \sum_{m=0}^{\infty} \Lambda^i_m \alpha^i_{m+k} + \sum_{m=0}^{\infty} n\Lambda^i_m \alpha^i_{m+n} 
\]  

(127)

If we now write \( \alpha^i_{m+k} = \sum_l \alpha^i_l \delta_l m+k \) and \( \alpha^i_{m+n} = \sum_k \alpha^i_k \delta_k m+n \) and exchange the dummy indices \( l \) and \( k \) as needed, the above expression becomes

\[
\sum_k df_k^{(i,n)}(\xi) \alpha_k^i = \sum_k \left[ \sum_{m=0}^{\infty} (k-m) \Lambda^i_m f_k^{(i,n)}(\xi) + \sum_{m=0}^{\infty} (k-m) \Lambda^i_m \delta_{k-m} n \right] \alpha_k^i 
\]  

(128)

Since the \( \alpha \)'s are all linearly independent, it follows that

\[
\frac{df_k^{(i,n)}(\xi)}{d\xi} = \sum_{m=0}^{\infty} (k-m) \Lambda^i_m \left[ f_k^{(i,n)}(\xi) + \delta_{k-m} n \right] 
\]  

(129)

To solve the system of differential equations, we need the boundary condition for each of the \( k \) functions. This we accomplish by setting \( \xi = 0 \) in equation (125).

\[ [\alpha^i_n, 1] = \sum_k f_k^{(i,n)}(0) \alpha_k^i \]  

(130)

Since the commutator on the left hand side is identically zero and the \( \alpha \)'s are all linearly independent, it follows that

\[ f_k^{(i,n)}(0) = 0 \]  

(131)

for all values of \( k \). The set of \( k \) equations in (131) are the desired \( k \) boundary conditions. Using (131) in (129), we find that

\[ \left. \frac{df_k^{(i,n)}(\xi)}{d\xi} \right|_{\xi=0} = 0 \]  

(132)

for all values of \( k < n \). Combining equations (131) and (132), we find that

\[ f_k^{(i,n)}(\xi) = 0 \]  

(133)

\footnote{In fact the function \( f_k^{(i,n)}(\xi) \) does not depend on the string index \( i \) due to the cyclic symmetry in the string indices. Moreover it does not depend on the particular mode \( n \) and so we could drop the string label and the mode label if we please. However, to avoid confusion we keep them here for bookkeeping.}
for all values of $k < n$. For $k \geq n$, we make the substitution $k = n + q$, $q = 0, 1, 2, \ldots$ in equation \ref{129}.

\[
\frac{df_{k}^{(i,n)}(\xi)}{d\xi} = \sum_{m=0}^{\infty} (n + q - m) \Lambda_{m}^{i} \left[ f_{n+q-m}^{(i,n)}(\xi) + \delta_{m,n} \right] \tag{134}
\]

Since $f_{k}^{(i,n)}(\xi) = 0$ for $k < n$, then the infinite sum over the first term on the right hand side of the above expression reduces to a finite sum, that is, $\sum_{m=0}^{\infty} (n + q - m) \Lambda_{m}^{i} f_{n+q-m}^{(i,n)}(\xi) = \sum_{k=0}^{n} (n + k) \Lambda_{q-k}^{i} f_{n+k}^{(i,n)}(\xi)$. Likewise the second infinite sum over the second term on the right hand side reduces to a finite sum, that is, $\sum_{m=0}^{\infty} (n + q - m) \Lambda_{m}^{i} \delta_{m,n} = \sum_{q=0}^{n} (n + k) \Lambda_{q-k}^{i} \delta_{k,0}$. Thus the above expression reduces to

\[
\frac{df_{n+q}^{(i,n)}(\xi)}{d\xi} = \sum_{k=0}^{n} (n + k) \Lambda_{k}^{i} f_{n+k}^{(i,n)}(\xi) + \delta_{k,0} \tag{135}
\]

Making the substitution $\tilde{f}_{n+m}^{(i,n)}(\xi) = f_{n+k}^{(i,n)}(\xi) + \delta_{k,0}$, the differential equation takes on a more elegant form

\[
\frac{d\tilde{f}_{n+q}^{(i,n)}(\xi)}{d\xi} = \sum_{k=0}^{n} (n + k) \Lambda_{k}^{i} \tilde{f}_{n+k}^{(i,n)}(\xi) \tag{136}
\]

To find the explicit form of $\tilde{f}_{n+q}^{(i,n)}(\xi)$, we need to solve equation \ref{136} for all values of $q$. This can be simply achieved by solving \ref{136} for the first few values of $q$ and then guessing the general form of the solution and prove that it is the right solution by mathematical induction. Therefore, let us first consider $q = 0$, so that equation \ref{136} becomes

\[
\frac{d\tilde{f}_{n}^{(i,n)}(\xi)}{d\xi} = n\Lambda_{0}^{i} \tilde{f}_{n}^{(i,n)}(\xi) \tag{137}
\]

which has the solution

\[
\tilde{f}_{n}^{(i,n)}(\xi) = C_{0} e^{n\Lambda_{0}^{i} \xi} \tag{138}
\]

where $C_{0}$ is the constant of integration. Next we consider $q = 1$. Setting $q = 1$ in equation \ref{136} and then eliminating $\tilde{f}_{n}^{(i,n)}(\xi)$ with the help of \ref{138}, we find

\[
\frac{d\tilde{f}_{n+1}^{(i,n)}(\xi)}{d\xi} = n\Lambda_{1}^{i} C_{0} e^{n\Lambda_{0}^{i} \xi} + (n + 1) \Lambda_{0}^{i} \tilde{f}_{n+1}^{(i,n)}(\xi) \tag{139}
\]

which has the well known solution

\[
\tilde{f}_{n+1}^{(i,n)}(\xi) = e^{(n+1)\Lambda_{0}^{i} \xi} \int^{\xi} n\Lambda_{1}^{i} C_{0} e^{-\Lambda_{0}^{i} \xi} d\xi + C_{1} e^{(n+1)\Lambda_{0}^{i} \xi} \tag{140}
\]

where $C_{1}$ is the constant of integration. Evaluating a rather simple integral, we obtain

\[
\tilde{f}_{n+1}^{(i,n)}(\xi) = -C_{0} n \left( \frac{\Lambda_{1}^{i}}{\Lambda_{0}^{i}} \right) e^{n\Lambda_{0}^{i} \xi} + C_{1} e^{(n+1)\Lambda_{0}^{i} \xi} \tag{141}
\]
At this point, it is not hard to see that the general solution has the form

$$\tilde{f}_{n+q}(\xi) = \sum_{k=0}^{q} C_{qk}^{[i,n]} e^{(n+k)\Lambda^i_0} \xi$$  \hspace{1cm} (142)$$

To determine the coefficients $C_{qk}^{[i,n]}$, we only need to substitute this result back in (136). Doing so we obtain

$$\sum_{l=0}^{q} C_{ql}^{[i,n]} e^{(n+l)\Lambda^i_0} (n+l) \Lambda^i_0 = \sum_{l=0}^{q-1} \sum_{k=0}^{l} (n+k) \Lambda^i_q C_{kl}^{[i,n]} e^{(n+l)\Lambda^i_0} \xi$$  \hspace{1cm} (143)$$

We notice that the diagonal part ($C_{qq}^{[i,n]}$) of the coefficient $C_{qk}^{[i,n]}$ has the same multiplicative factor on both sides of the equation and so it drops out and the above expression after a bit of rather straightforward algebra reduces to

$$\sum_{l=0}^{q-1} C_{ql}^{[i,n]} e^{(n+l)\Lambda^i_0} (k-q) \Lambda^i_q = \sum_{l=0}^{q-1} \sum_{k=0}^{l} (n+k) \Lambda^i_q C_{kl}^{[i,n]} e^{(n+l)\Lambda^i_0} \xi$$  \hspace{1cm} (144)$$

In obtaining the above results, we had to exchange the dummy indices $k \leftrightarrow l$ at some point of the calculation. If we expand the right hand side, we see at once that the double sum $\sum_{l=0}^{q-1} \sum_{k=0}^{l} (\cdots) = \sum_{k=0}^{q-1} \sum_{l=k}^{q-1} (\cdots)$, and so one obtains a recursion relation between the $C_{qk}^{[i,n]}$ coefficients

$$C_{qk}^{[i,n]} (k-q) \Lambda^i_q = \sum_{l=0}^{q-1} \sum_{k=0}^{l} (n+k) \Lambda^i_q C_{kl}^{[i,n]}$$  \hspace{1cm} (145)$$

This result gives the off diagonal elements of $C_{qk}^{[i,n]}$ in terms of the $C_{kk}^{[i,n]}$, $C_{k+1k}^{[i,n]}$, $C_{k+2k}^{[i,n]}$, ..., $C_{q-1k}^{[i,n]}$. Thus we have

$$C_{qk}^{[i,n]} = -\frac{1}{q-k} \sum_{l=k}^{q-1} (n+l) C_{kl}^{[i,n]} \Lambda^i_{q-l}$$  \hspace{1cm} (146)$$

where $\Lambda^i_{m} = \Lambda^i_{m}/\Lambda^i_0$. To obtain the diagonal elements of $C_{qk}^{[i,n]}$, one needs to set $\xi = 0$ in equation (142) for $q = 0, 1, 2, \ldots$, and use the fact that $\tilde{f}_{n+q}^{(i,n)}(0) = \delta_{n+q,0}$.

\footnote{We have proved using mathematical induction that indeed this is the right solution. The proof is quite straightforward and we see no need to include it here.}
(which follows from equation 131 and the definition of \( \tilde{f}^{(i,n)}(\xi) \)), to find

\[
C_{i0}^{[i,n]} = 1
\]

\[
C_{10}^{[i,n]} + C_{11}^{[i,n]} = 0
\]

\[
C_{20}^{[i,n]} + C_{21}^{[i,n]} + C_{22}^{[i,n]} = 0
\]

\[
\ldots
\]

\[
C_{q0}^{[i,n]} + C_{q1}^{[i,n]} + \ldots + C_{qq-1}^{[i,n]} + C_{qq}^{[i,n]} = 0
\]

which can be written in a more compact form, that is

\[
C_{qq}^{[i,n]} = \delta_{q0} - \sum_{k=0}^{q-1} C_{qk}^{[i,n]}, \quad q = 0, 1, 2, \ldots
\]

To evaluate equation 122, we need to move the operator \( \hat{O} \) to the left side of the square bracket on the right hand side of equation 122. To do this successfully, we first need to compute the action of \( \alpha \) on \( \hat{O} \), that is we need to compute \( \alpha \hat{O} \). Consider

\[
\alpha \hat{O} = \hat{O} \alpha + \left[ \alpha, \hat{O} \right]
\]

Using equation 125 to evaluate the commutator, the above expression becomes

\[
\alpha \hat{O} = \hat{O} \left( \alpha + \sum_{k} f_k^{(i,n)} (\xi) \right) |_{\xi = 1} \alpha
\]

Using the fact that

\[
f_k^{(i,n)} (1) = \begin{cases} 0, & k < n \\ f_{n+q}^{(i,n)} (1), & k = n + q, q = 0, 1, 2, \ldots \end{cases}
\]

then the sum in equation 149 becomes \( \sum_{q=0}^{\infty} f_{n+q}^{(i,n)} (1) \alpha_{n+q} \) and so equation 149 takes the form

\[
\alpha \hat{O} = \hat{O} \left( \alpha + \sum_{q=0}^{\infty} \tilde{f}_{n+q}^{(i,n)} (1) \alpha_{n+q} \right)
\]

If we now recall that \( \tilde{f}_{n+q}^{(i,n)} (1) = f_{n+q}^{(i,n)} (1) + \delta_{q0} \), then the above equation becomes

\[
\alpha \hat{O} = \hat{O} \sum_{q=0}^{\infty} \tilde{f}_{n+q}^{(i,n)} (1) \alpha_{n+q}
\]
Furthermore, since $\tilde{f}_{n+q}^{(i,n)}(1) = C_{i0}^{(i,n)^{\prime}} e^{n\Lambda_0^i}$ and $C_{i0}^{(i,n)^{\prime}} = 1$, we find

$$\alpha_n^i \hat{O} = \hat{O} \left( e^{n\Lambda_0^i} \alpha_n^i + \sum_{q=1}^{\infty} \tilde{f}_{n+q}^{(i,n)}(1) \alpha_n^{i+q} \right) \quad (153)$$

Replacing $\tilde{f}_{n+q}^{(i,n)}(1)$ by its value from equation (112), the above expression becomes

$$\alpha_n^i \hat{O} = \hat{O} e^{n\Lambda_0^i} \left( \alpha_n^i + \sum_{q=1}^{\infty} \sum_{k=0}^{q} C_{qk}^{[i,n]} e^{k\Lambda_0^i} \alpha_n^{i+q} \right) \quad (154)$$

and so the desired identity for $\hat{O} \alpha^i_n \hat{O}$ follows at once

$$\hat{O}^{-1} \alpha^i_n \hat{O} = e^{n\Lambda_0^i} \left( \alpha_n^i + \sum_{q=1}^{\infty} \sum_{k=0}^{q} C_{qk}^{[i,n]} e^{k\Lambda_0^i} \alpha_n^{i+q} \right) \quad (155)$$

This result is valid for all integral values of $n$ including zero. Using this result in (122), we find

$$0 = <V_x^{HS} \left[ \alpha_{-m}^i + \sum_{q=1}^{\infty} \sum_{k=0}^{q} C_{qk}^{[i,n]} e^{k\Lambda_0^i} \alpha_{-n+q} \right.$$  

$$- \sum_{j=1}^{3} \sum_{m=0}^{\infty} nM_{nm}^{ij} e^{(m+n)\Lambda_0^i} \left( \alpha_m^j + \sum_{q=1}^{\infty} \sum_{k=0}^{q} C_{qk}^{[m,n]} e^{k\Lambda_0^i} \alpha_{m+q} \right) \left] \right> \quad (156)$$

Where we have dropped the superscript from the $\Lambda_0^i$ and $C_{qk}^{[i,n]}$ constants which follow from the observation that the $H$ and $M$ matrices have cyclic symmetry in the string indices and thus $\Lambda_k \equiv \Lambda_k^1 = \Lambda_k^2 = \Lambda_k^3$ and $C_{qk}^{[n]} \equiv C_{qk}^{[1,n]} = C_{qk}^{[2,n]} = C_{qk}^{[3,n]}$.

We notice that this result is a linear combination of the first $n$ equations in (122). To fix the $\Lambda$’s, we need to compare the coefficients of the operators $\alpha$’s in equation (156) to those in equation (116) for all the $n$ modes. The simplest relation we can retrieve from equations (116) and (156) is for the $n = 1$ mode. It will turn out that this equation is sufficient to fix all the $\Lambda$’s to any order and we will be able to determine the operator $\hat{O}$ completely just from this relation. Relations for higher modes that follow from equating the coefficients of the operators $\alpha$’s in equations (116) and (156) will then become consistency conditions. Moreover due to the cyclic symmetry in the string indices, we can set $i = 1$ without loss of generality. Thus setting $n = 1$ and $i = 1$ in both equations (156) and (116), we find

$$- \sum_{j=1}^{3} \sum_{m=0}^{\infty} G_{1m}^{ij} \alpha_m^j = \left[ \sum_{q=1}^{\infty} \sum_{k=0}^{q} C_{qk}^{[-1]} e^{k\Lambda_0^1} \alpha_{-1+q} \right. - \sum_{j=1}^{3} \sum_{m=0}^{\infty} M_{1m}^{ij} e^{(m+1)\Lambda_0^1}$$  

$$\times \left( \alpha_m^j + \sum_{q=1}^{\infty} \sum_{k=0}^{q} C_{qk}^{[m]} e^{k\Lambda_0^1} \alpha_{m+q} \right) \right] \quad (157)$$
If we compare the terms involving $\alpha_0^0$ in (157), we find

$$- \sum_{j=1}^{3} C_{j0}^{1} e^{j\Lambda_0} \alpha_0^0 = \sum_{k=0}^{3} C_{k0}^{[-1]} e^{k\Lambda_0} \alpha_0^0 - \sum_{j=1}^{3} M_{j0}^{1} e^{\Lambda_0} \alpha_0^j$$  \hspace{1cm} (158)$$

It is important to notice here that $\alpha_0^1$, $\alpha_0^2$ and $\alpha_0^3$ are not linearly independent. The redundancy can be removed using conservation of momentum. Expanding the sums in the above expression and then using the conservation of momentum $\alpha_0^1 + \alpha_0^2 + \alpha_0^3 = 0$ to eliminate $\alpha_0^3$, we obtain

$$- \left[ (H_{10}^{11} - H_{10}^{13}) \alpha_0^1 + (H_{10}^{12} - H_{10}^{13}) \alpha_0^2 \right]$$

$$= \left[ C_{10}^{[-1]} + C_{11}^{[-1]} e^{\Lambda_0} - e^{\Lambda_0} (M_{10}^{11} - M_{10}^{13}) \right] \alpha_0^1$$

$$- e^{\Lambda_0} (M_{12}^{10} - M_{13}^{10}) \alpha_0^2$$  \hspace{1cm} (159)$$

Since $\alpha_0^1$ and $\alpha_0^2$ are linearly independent, it follows that

$$e^{\Lambda_0} (M_{11}^{10} - M_{13}^{10}) - (H_{10}^{11} - H_{10}^{13}) = C_{10}^{[-1]} + C_{11}^{[-1]} e^{\Lambda_0}$$  \hspace{1cm} (161)$$

The second equation gives the value of $\Lambda_0$ in terms of the coupling matrices $H$ and $M$

$$\Lambda_0 = \ln \left( H_{10}^{12} - H_{10}^{13} \right)$$  \hspace{1cm} (163)$$

In arriving at the above equation, we used the fact that $M_{10}^{13} = 0$ and $M_{10}^{12} = 1$. The numerical values of the matrix elements $H_{10}^{12}$ and $H_{10}^{13}$ are computed in reference [25]

$$- H_{2n+1}^{13} = H_{2n+1}^{12} = \frac{1}{\sqrt{3}} \frac{a_{2n+1}}{2n+1}$$  \hspace{1cm} (164)$$

and so for $n = 0$, we have

$$- H_{10}^{13} = H_{10}^{12} = \frac{1}{\sqrt{3}} a_1 = \frac{2}{3\sqrt{3}}$$  \hspace{1cm} (165)$$

Putting these values in (163), we obtain

$$\Lambda_0 = \ln \frac{2^2}{3\sqrt{3}}$$  \hspace{1cm} (166)$$

which is precisely the result obtained in [12].

Equation (161) gives the value of $\Lambda_1 (= \Lambda_1/\Lambda_0)$. To see this, we first need to find the values of the constants $C_{10}^{[-1]}$ and $C_{11}^{[-1]}$. From equation (147), it follows that $C_{11}^{[-1]} = -C_{10}^{[-1]}$ and from equation (146) we find that $C_{10}^{[-1]} = C_{00}^{[-1]} \Lambda_1$ but $C_{00}^{[-1]} = 1$ by equation (147) and so equation (161) gives

$$- (H_{10}^{11} - H_{10}^{13}) = \Lambda_1 (1 - e^{\Lambda_0})$$  \hspace{1cm} (167)$$
where again the last step in obtaining the above result follows from the fact that $M_{10}^{11} = M_{10}^{13} = 0$. Solving the above equation for $\Lambda_1$ and then using (163) to eliminate $\Lambda_0$, we find

$$\Lambda_1 = -\frac{H_{10}^{11} - H_{10}^{13}}{1 - (H_{10}^{12} - H_{10}^{13})}$$

(168)

where we have used $M_{10}^{12} = 1$. Making use of the explicit values of the matrix elements $H_{10}^{11}$ and $H_{10}^{13}$

$$H_{2n+1,0}^{11} = 0, \quad n = 0, 1, 3, \ldots$$

(169)

$$-H_{2n+1,0}^{13} = H_{2n+1,0}^{12} = \frac{1}{\sqrt{3}} 2n + 1, \quad n = 0, 1, 3, \ldots$$

(170)

equation (168) becomes

$$\Lambda_1 = -\frac{6\sqrt{3}}{11} - \frac{8}{11}$$

(171)

Once more, this is the same result obtained in [12] using the full string formulation of Witten’s string theory of open bosonic strings. Thus at least at the first level, the half string theory gives the same physics as the SCSV. We will see that this conclusion in fact holds at any level.

Now we are in the position to derive a recursion relations for the $\Lambda'$s. Using the fact that $M_{nm}^{11} = M_{nm}^{13} = 0$, equation (157) becomes

$$\sum_{j=1}^{3} \sum_{m=0}^{\infty} H_{1m}^{1j} \alpha_m^j = \left[ \sum_{q=1}^{\infty} \sum_{k=0}^{\infty} C_{qk}^{-1} e^{k\Lambda_0} \alpha_{-1+q}^1 - \sum_{m=0}^{\infty} M_{1m}^{12} e^{(m+1)\Lambda_0} \right] \times \left[ e^{\alpha_m^2} + \sum_{q=1}^{\infty} \sum_{k=0}^{\infty} C_{qk}^{[m]} e^{k\Lambda_0} \alpha_{m+q}^2 \right]$$

(172)

If we now compare the coefficients for $\alpha_m^1$, we find

$$-H_{1m}^{11} = \sum_{k=0}^{m+1} C_{m+1k}^{-1} e^{k\Lambda_0}$$

(173)

valid for $m = 1, 2, 3, \ldots$ Using equations (147) to eliminate $C_{m+1m+1}^{-1}$, the above expression becomes

$$-H_{1m}^{11} = \sum_{k=0}^{m} C_{m+1k}^{-1} e^{k\Lambda_0} - \sum_{k=0}^{m} C_{m+1k}^{-1} e^{(m+1)\Lambda_0}$$

(174)

or alternatively

$$-H_{1m}^{11} = \left( 1 - e^{(m+1)\Lambda_0} \right) C_{m+1,0}^{-1} 0 + \sum_{k=1}^{m} C_{m+1k}^{-1} \left( e^{k\Lambda_0} - e^{(m+1)\Lambda_0} \right)$$

(175)
Setting \( q = m + 1, k = 0, i = 1 \) and \( n = -1 \) in (146), we have
\[
C_{m+1}^{[-1]} = -\frac{1}{m+1} \sum_{l=0}^{m} (-1 + l) C_{l}^{[-1]} \Lambda_{m+1-l} \tag{176}
\]
Using \( C_{00}^{[-1]} = 1 \), and multiplying both sides by \((m+1)\), the above expression takes the form
\[
(m+1) C_{m+1}^{[-1]} = \Lambda_{m+1} - \sum_{l=2}^{m} (-1 + l) C_{l}^{[-1]} \Lambda_{m+1-l} \tag{177}
\]
Combining equations (175) and (177), we find
\[
\Lambda_{m+1} = \frac{(m+1)}{1 - e^{(m+1) \Lambda_{0}}} \left[ H_{1m}^{11} + \sum_{k=1}^{m} C_{m+1}^{[-1]} \left( e^{k \Lambda_{0}} - e^{(m+1) \Lambda_{0}} \right) \right]
+ \sum_{l=2}^{m} (-1 + l) C_{l}^{[-1]} \Lambda_{m+1-l} \tag{178}
\]
valid for \( m = 1, 2, 3, \ldots \). According to equations (146), the right hand side of equation (178) is a function of \( \Lambda_{0}, \Lambda_{1}, \Lambda_{2}, \ldots, \Lambda_{m} \). Thus equation (178) with the explicit values of \( \Lambda_{0} \) and \( \Lambda_{1} \) obtained in (166) and (171) generate all values of \( \Lambda \)'s. To illustrate the use of the recursion relation in (178), we now proceed to compute the first few constants in the expansion of the conformal operator. All the possible values of \( H_{nm}^{ii} \) have been computed in reference [25]. For \( m = \text{odd} \neq 1 \), we have
\[
H_{1m}^{11} = -\frac{2}{3^2} \left( - \right)^{(m-1)/2} \left[ b_{m} + 2a_{m} \over 1 + m + b_{m} - 2a_{m} \over 1 - m \right] \tag{179}
\]
where we have used the explicit values \( a_{1} = 2/3 \) and \( b_{1} = 4/3 \). To Compute the value of \( \Lambda_{2} \), we need the explicit value of \( H_{11}^{11} \). The value of \( H_{11}^{11} \) is given in [25]
\[
H_{11}^{11} = -\frac{2}{3^3} - \frac{1}{\pi 3 \sqrt{3}} \left[ \widehat{E}_{1}^{b} - 2 \widehat{E}_{1}^{a} \right] \tag{180}
\]
where once again we used the fact that \( a_{1} = 2/3, b_{1} = 4/3 \). Using the explicit values of \( \widehat{E}_{1}^{a} \) and \( \widehat{E}_{1}^{b} \) obtained in [25]
\[
\widehat{E}_{n=1}^{a} = \pi \sqrt{\frac{1}{3}} \left( \ln \frac{3}{2} + \frac{1}{6} \right) \tag{181}
\]
and
\[
\widehat{E}_{n=1}^{b} = 2 \pi \sqrt{\frac{1}{3}} \left( \ln \frac{3}{2} - \frac{1}{12} \right) \tag{182}
\]
the above expression for \( G_{11}^{11} \) becomes
\[
G_{11}^{11} = -\frac{5}{3^3} \tag{183}
\]
Setting $m = 1$ in equation (178) and substituting the explicit value of $H_{11}^{11}$ obtained above, we get

$$\Lambda_2 = -\frac{2}{1 - e^{2\Lambda_0}} \left[ G_{11}^{11} + C_2^{-1} \left( e^{\Lambda_0} - e^{2\Lambda_0} \right) \right]$$  \hspace{1cm} (184)

From equation (146), it follows that $C_{21}^{-1} = 0$ and so substituting the explicit values of $H_{11}^{11}$ and $\Lambda_0$, the above expression becomes

$$\Lambda_2 = \frac{10}{11}$$  \hspace{1cm} (185)

This is precisely the value obtained in [12] and so the half string theory gives the same physics at the second level. For $m = 2$, equation (178) gives

$$\Lambda_3 = -\frac{3}{1 - e^{3\Lambda_0}} \left[ G_{11}^{11} + C_{31}^{-1} \left( e^{\Lambda_0} - e^{3\Lambda_0} \right) + C_{32}^{-1} \left( e^{2\Lambda_0} - e^{3\Lambda_0} \right) \right] + C_{20}^{[-1]} \Lambda_1$$  \hspace{1cm} (186)

The $H_{nm}^{i}$ vanish for $r = 1, 2, 3$, and $n + m = \text{odd}$ (see reference [25]). The explicit values of the coefficients $C_{31}^{-1}$, $C_{32}^{-1}$, $C_{20}^{[-1]}$ are given by equation (146). For $C_{20}^{[-1]}$, equation (146) gives

$$C_{20}^{[-1]} = \frac{1}{2} C_{21}^{[-1]} \Lambda_2 = \frac{1}{2} \Lambda_2$$  \hspace{1cm} (187)

where we used the fact that $C_{00}^{[0]} = 1$. For $C_{31}^{-1}$, equation (146) gives

$$C_{31}^{-1} = -\frac{1}{2} C_{21}^{[-1]} \Lambda_1 = 0$$  \hspace{1cm} (188)

where the value of $C_{21}^{[-1]} = 0$ follows at once from equation (146). Likewise one sees that

$$C_{32}^{[-1]} = -C_{22}^{[-1]} \Lambda_1 = \left[ C_{20}^{[-1]} + C_{21}^{[-1]} \right] \Lambda_1 = \frac{1}{2} \Lambda_2 \Lambda_1$$  \hspace{1cm} (189)

Putting all these results in (186), yields

$$\Lambda_3 = -\frac{3}{1 - e^{3\Lambda_0}} \left[ \frac{1}{2} \Lambda_2 \Lambda_1 \left( e^{2\Lambda_0} - e^{3\Lambda_0} \right) \right] + \frac{1}{2} \Lambda_2 \Lambda_1$$  \hspace{1cm} (190)

Substituting the explicit values of $\Lambda_0$, $\Lambda_1$ and $\Lambda_2$ in the above expression, we get

$$\Lambda_3 = -\frac{30\sqrt{3}}{1417} \frac{2360}{15587}$$  \hspace{1cm} (191)

which is the desired result. To compute $\Lambda_4$, we set $m = 3$ in equation (178)

$$\Lambda_4 = -\frac{4}{1 - e^{4\Lambda_0}} \left[ H_{11}^{11} + C_{41}^{-1} \left( e^{\Lambda_0} - e^{4\Lambda_0} \right) + C_{42}^{-1} \left( e^{2\Lambda_0} - e^{4\Lambda_0} \right) 
+ C_{43}^{-1} \left( e^{3\Lambda_0} - e^{4\Lambda_0} \right) \right] + C_{20}^{[-1]} \Lambda_2 + 2C_{30}^{[-1]} \Lambda_1$$  \hspace{1cm} (192)
Using the value of $H^{11}_{13}$

$$H^{11}_{13} = \frac{2^5}{3^6}$$

(193)

and the coefficients $C^{[-1]}_{20}, C^{[-1]}_{30}, C^{[-1]}_{41}, C^{[-1]}_{42}, C^{[-1]}_{43}$ (see table in appendix A).

Substituting these values in equation (192) and using the explicit values of $\Lambda_0$, $\Lambda_1$, $\Lambda_2$ and $\Lambda_3$ obtained earlier, we find

$$\Lambda_4 = \frac{-7680\sqrt{3}}{670241} - \frac{7546}{60931}$$

(194)

For $\Lambda_5$, we set $m = 4$ in equation (178) to find

$$\Lambda_5 = \frac{-5}{1 - e^{5\Lambda_0}} \left[ 0 + C^{[-1]}_{51} (e^{\Lambda_0} - e^{5\Lambda_0}) + C^{[-1]}_{52} (e^{2\Lambda_0} - e^{5\Lambda_0}) 
+ C^{[-1]}_{53} (e^{3\Lambda_0} - e^{5\Lambda_0}) + C^{[-1]}_{54} (e^{4\Lambda_0} - e^{5\Lambda_0}) \right] 
+ C^{[-1]}_{20} \Lambda_2 + 2C^{[-1]}_{30} \Lambda_3 + 3C^{[-1]}_{40} \Lambda_4$$

(195)

where we have used the fact that $H^{11}_{14} = 0$. Substituting the explicit values of $\Lambda_0$, $\Lambda_1$, $\Lambda_2$, $\Lambda_3$ and $\Lambda_4$, and the explicit values of $C^{[-1]}_{51}$, $C^{[-1]}_{52}$, $C^{[-1]}_{53}$, $C^{[-1]}_{54}$, $C^{[-1]}_{20}$, $C^{[-1]}_{30}$, $C^{[-1]}_{40}$ obtained in the appendix, the above expression becomes

$$\Lambda_5 = \frac{7867620 200 382\sqrt{3}}{98058698 647 481} - \frac{18 702 116 671 704}{98 058 698 647 481}$$

(196)

Continuing this way we can compute $\Lambda_n$ to any desired value of $n$ and so this procedure gives the desired operator required for the transformation between the half string 3-Vertex and the SCSV 3-Vertex to all levels. The fact that this operator turns out to be the same operator connecting Witten’s interacting three vertex and the SCSV three vertex is a non trivial check on the equivalence of the half string theory [5, 6, 7, 8, 9] and Witten’s theory of the open bosonic string [18]. In appendix A, we give the first few values of the coefficients $C^{[i,n]}_{q_k}$ as calculated from equations (146) and (147).
A The $C_{qk}^{[i,n]}$ Coefficients

The coefficients $C_{qk}^{[i,n]}$ can easily be calculated from the the recursion relations in (140) and (147). Here we give the first few values of the coefficients $C_{qk}^{[i]}$ as computed from equations (140) and (147).

\[
\begin{align*}
C_{11}^{[-1]} & = 1 \\
C_{21}^{[-1]} & = \frac{6\sqrt{3} - 8}{11}, \quad C_{21}^{[-1]} = \frac{6\sqrt{3} + 8}{11} \\
C_{31}^{[-1]} & = \frac{5}{11}, \quad C_{32}^{[-1]} = 0, \quad C_{33}^{[-1]} = \frac{5}{11} \\
C_{41}^{[-1]} & = \frac{12960\sqrt{3} + 10240}{171457}, \quad C_{42}^{[-1]} = 0, \\
C_{43}^{[-1]} & = \frac{-30\sqrt{3} - 40}{121}, \quad C_{44}^{[-1]} = \frac{29550\sqrt{3} + 46440}{171457} \\
C_{51}^{[-1]} & = \frac{3317760\sqrt{3} - 4112144}{81099161}, \quad C_{52}^{[-1]} = 0, \\
C_{53}^{[-1]} & = \frac{480\sqrt{3} - 585}{1331}, \quad C_{54}^{[-1]} = \frac{1030080\sqrt{3} + 1806840}{1886027} \\
C_{55}^{[-1]} & = \frac{-12960\sqrt{3} - 26773}{57233} \\
C_{61}^{[-1]} & = \frac{-8907155740608\sqrt{3} - 30264361046016}{1078645685122291}, \quad C_{62}^{[-1]} = 0, \\
C_{63}^{[-1]} & = \frac{-7364100\sqrt{3} - 15193840}{20746297} \\
C_{64}^{[-1]} & = \frac{25372020\sqrt{3} + 44385840}{20746297} \\
C_{65}^{[-1]} & = \frac{1090198368\sqrt{3} - 1711928864}{892090771} \\
C_{66}^{[-1]} & = \frac{1351278627978\sqrt{3} + 5945189917}{5393228425611}
\end{align*}
\]

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