Effective theory for the non-rigid rotor in an electromagnetic field: Toward accurate and precise calculations of $E2$ transitions in deformed nuclei

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We present a model-independent approach to electric quadrupole transitions of deformed nuclei. Based on an effective theory for axially symmetric systems, the leading interactions with electromagnetic fields enter as minimal couplings to gauge potentials, while subleading corrections employ gauge-invariant non-minimal couplings. This approach yields transition operators that are consistent with the Hamiltonian, and the power counting of the effective theory provides us with theoretical uncertainty estimates. We successfully test the effective theory in homonuclear molecules that exhibit a large separation of scales. For ground-state band transitions of rotational nuclei, the effective theory describes data well within theoretical uncertainties at leading order. In order to probe the theory at subleading order, data with higher precision would be valuable. For transitional nuclei, next-to-leading order calculations and the high-precision data are consistent within the theoretical uncertainty estimates. We also study the faint inter-band transitions within the effective theory and focus on the $E2$ transitions from the $0^+_1$ band (the “$\beta$ band”) to the ground-state band. Here, the predictions from the effective theory are consistent with data for several nuclei, thereby proposing a solution to a long-standing challenge.

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I. INTRODUCTION

Our understanding of deformed nuclei in the rare-earth and actinide regions of the nuclear chart is largely based on the geometric collective models [1–7], and the algebraic collective models [8, 9]. For even-even nuclei, these models employ quadrupole degrees of freedom (and an additional $s$ boson in the interacting boson model [10]). The collective models depend on a small numbers of parameters. They describe the key features of deformed nuclei, namely low-energy spectra consisting of rotational bands on top of vibrational band heads, with strong $E2$ intra-band transitions, and much weaker inter-band transitions. However, some finer details are not well described by the collective models, and the accurate description of inter-band electromagnetic transition strengths is a particular challenge. As an example we mention the overprediction (by factors of 2 to 10) of $E2$ transitions between the rotational band on top of the $0^+_2$ vibrational band head (historically called the “$\beta$ band”) and the ground-state band for well-deformed nuclei [7, 11, 12]. This situation is similar for transitional nuclei at the border between sphericity and deformation. Here, the models based on the $X(\bar{5})$ solution by Iachello [13] of the Bohr Hamiltonian describe energy spectra well but tend to overpredict electromagnetic inter-band transitions [14–17].

In recent years, computationally tractable approaches to collective models [18, 19] led to a better understanding of geometric models and their parameter space [20]. However, it seems that changes to the Bohr Hamiltonian, e.g. by studying non-separable potentials [21] or by considering other exact solutions [22], do not overcome the deficiencies for the inter-band transitions. We also note that a variety of approaches addressed other shortcomings of the collective models by focusing on tri-axial deformations [22], or inclusion of isovector modes [23, 24], see Ref. [26] for a review of present challenges.

Increasing the complexity of collective models, e.g. through the addition of more terms, can lead to an undesirable proliferation of parameters and a loss of predictive power. This unattractive feature of modeling can partly be overcome by effective field theories (EFTs). An EFT is based on symmetry principles alone and exploits a separation of scale for the systematic construction of Hamiltonians based on a power counting. In this way, an increase in the number of parameters (i.e. low-energy constants that need to be adjusted to data) goes hand in hand with an increase in precision, and thereby counters the loss of predictive power. Furthermore, this systematic increase in precision makes it possible to estimate theoretical uncertainties, see Furnstahl et al. [27] for a recent review. Finally, the EFT approach also helps us to identify inherent limitations that are due to the breakdown scale of the theory.

The successful reproduction of the low-energy spectra of deformed nuclei strongly suggests that the geometric collective model correctly captures key aspects such as relevant degrees of freedom and the interaction between them. This picture is also obtained in a model-independent approach to deformed nuclei based on EFT [28, 30].

The overprediction of the inter-band transition strengths in collective models thus leads us to scrutinize the operators that are employed in the calculations of $E2$ transition strengths. The Bohr Hamiltonian models the nucleus as an incompressible liquid drop with quadrupole surface oscillations. These corresponding five degrees of
freedom can be mapped onto three Euler angles (describing overall rotations of the nucleus) and two deformation parameters (describing vibrations in the body-fixed coordinate system). In this model, \( E2 \) transitions are computed from the quadrupole operator. This approach to electric transitions in deformed nuclei seems to be motivated by Siegert’s theorem [31], which allows one to employ the density instead of the current operator in the computation of some transition rates, see e.g. Ref. [32]. We recall that the derivation of Siegert’s theorem is based on gauge invariance and starts from gauging momentum operators [32]. Thus the applicability of Siegert’s theorem is not obvious for the collective models that employ quadrupole operators for momenta (as opposed to vectors).

The identification of the transition operator is even more challenging for the algebraic models because of the lack of a geometric picture. For the calculation of electromagnetic transition strengths, these models employ operators that couple the basic degrees of freedom to a spherical tensor whose rank equals the desired multipole order. For a recent analysis of this approach, we refer the reader to Ref. [33].

In this work we study the electromagnetic coupling of deformed nuclei within an effective theory motivated by similar approaches to other nuclear systems, see Refs. [35–40] for recent examples. In contrast to more phenomenological models, the consistent treatment of Hamiltonians and currents is a highlight of effective theories. As we will see, coupling the non-rigid rotor to electromagnetic fields in a model-independent way is an interesting problem in itself. Perhaps somewhat surprisingly, we are not aware of any literature addressing this problem. Our approach reproduces the strong intra-band transitions that are also described accurately by the collective models. For the weaker inter-band transitions, the effective theory approach yields a much improved description of data and thereby suggests steps toward overcoming some limitations of the geometric and algebraic collective models. Finally, the effective theory approach also permits us to give theoretical uncertainty estimates and thereby facilitates a meaningful comparison with data. As we will see, this comparison also suggests that data with higher precision for \( E2 \) transitions would be very valuable.

Ultimately, a microscopic theory of deformed nuclei must be based on fermionic constituents. Nuclear mean field and density functional theories (see Refs [11, 12] for reviews), are making impressive predictions of rotational bands and moments of inertia [13, 14], with new projection techniques being proposed [15]. In light \( p \)-shell nuclei, \emph{ab initio} approaches are now addressing the emergent behavior of rotational collective motion [16, 17]. Recently, fermionic approaches have also been used to constrain parameters of collective models [18].

This paper is organized as follows. In Sect. [II] we briefly review the effective theory for axially deformed nuclei. The electromagnetic coupling of the effective theory is described in Sect. [III] Section [IV] presents the results for intra-band \( E2 \) transitions and compares them to data on rotational and transitional nuclei. A somewhat surprising result is that much of the available data lacks the precision to challenge the effective theory. Sections [V] and [VI] include quadrupole degrees of freedom for the description of inter-band transitions. Comparison to data shows that the effective theory accounts well for these faint transitions. Finally, we present our summary.

II. EFFECTIVE THEORY FOR THE AXIALLY SYMMETRIC NON-RIGID ROTOR

In this Section we briefly review the effective theory for deformed nuclei [28, 30]. The presentation in this paper aims at being more intuitive and less technical, though. We first focus on the lowest-energy phenomena and thus on the axially symmetric non-rigid rotor. The coupling to vibrations is considered in Sect. [V].

A. Low-energy degrees of freedom

The effective theory is based on the emergent symmetry breaking from the rotational symmetry of the group \( G = \text{SO}(3) \) to axial symmetry of the subgroup \( H = \text{SO}(2) \). Thus, the Nambu-Goldstone modes parameterize the coset \( G/H = \text{SO}(3)/\text{SO}(2) \) which is isomorphic to the two-sphere. This agrees with our intuition: the orientation of an axially symmetric object is defined by two Euler angles or, equivalently, by the direction of its symmetry axis. In a finite system, the symmetry breaking has an emergent character, and (quantized) zero modes take the place of Nambu-Goldstone modes [39, 54, 57]. In our case, the polar and azimuthal angles \( \theta \) and \( \phi \) (also labeled compactly as \( \Omega \)) parameterize the two-sphere, i.e., the radial unit vector

\[
e_r \equiv \begin{pmatrix} \sin \theta \cos \phi \\ \sin \theta \sin \phi \\ \cos \theta \end{pmatrix}
\]

indicates the direction of the symmetry axis of the non-rigid rotor. Thus, the effective theory for this system is equivalent to that of a particle on the two-sphere.

The velocity of the orientation vector \( e_r \) is the time derivative

\[
d_r e_r = \dot{\theta} e_\theta + \dot{\phi} \sin \theta e_\phi \\
= v_\theta e_\theta + v_\phi e_\phi \equiv v.
\]

This vector lies in the plane tangent to the two-sphere at \( \Omega \). Here and in what follows, we employ dots to denote time derivatives. The low-energy Lagrangian is a scalar function of the velocity vector alone and does not depend on the vector \( e_r \) because of the emergent symmetry breaking. To make progress, we need to understand the behavior of \( v \) under rotations, and establish a power...
counting. With these two ingredients in hand, we then construct the most general Lagrangian that is consistent with rotational symmetry and at a given order of the power counting.

B. Rotational invariance

Under a rotation \( r \equiv r(\alpha, \beta, \gamma) = \exp(-i\alpha J_z) \exp(-i\beta J_y) \exp(-i\gamma J_z) \), parameterized by the three Euler angles \((\alpha, \beta, \gamma)\), the angles \( \theta \) and \( \phi \) transform non-linearly into \( \theta' \) and \( \phi' \). This constitutes a nonlinear realization of \( \text{SO}(3) \). This rotation transforms the velocity vector \( \mathbf{v}(\Omega) \) (or any vector in the tangent plane) into the vector \( \mathbf{v}'(\Omega') \) that lies in the tangent plane at \( \Omega' \). It is clear that the mapping from \( \mathbf{v} \) to \( \mathbf{v}' \) is equivalent to a \( \text{SO}(2) \) rotation in the tangent plane by an angle \( \chi = \chi(\alpha, \beta, \gamma; \Omega) \) that is a complicated function of the Euler angles and the original coordinates \( \Omega \). Details are given in Ref. \[28\].

At this point it is useful to introduce spherical components of the velocity inside the tangent plane as

\[
v_{\pm} \equiv \frac{1}{\sqrt{2}}(v_{\theta} \pm iv_{\phi}). \tag{3}\]

Under a rotation by the Euler angles \((\alpha, \beta, \gamma)\) the vector \( \mathbf{v} \) transforms as

\[
v_{\pm} \rightarrow e^{\mp i\chi} v_{\pm}. \tag{4}\]

Thus, under an \( \text{SO}(3) \) transformation, vectors in the tangent plane formally transform under an \( \text{SO}(2) \) transformation, and any Lagrangian build from elements in the tangent plane that is formally invariant under \( \text{SO}(2) \), is in fact invariant under \( \text{SO}(3) \).

For the general construction of invariant Lagrangians, we must also consider time derivatives of vectors in the tangent plane. The resulting vectors may not lie in the tangent plane. Thus, the ordinary time derivative needs to be replaced by the covariant derivative

\[
D_t \equiv d_t - i\phi \cos \theta \hat{J}_z, \tag{5}\]

which is the projection onto the tangent plane of the ordinary time derivative.

Let \( \mathbf{L} \) denote a rotationally invariant Lagrangian in the velocities \( v_{\pm} \). The application of Noether’s theorem yields the angular momentum \( \mathbf{I} \) as the conserved quantity \[28\]. Its spherical components \( I_+ , I_0 \) and \( I_- \) are

\[
I_{+} = -\frac{1}{\sqrt{2}}e^{i\phi}(ip_{\theta} - p_{\phi} \cot \theta) \\
I_{0} = p_{\phi} \\
I_{-} = -\frac{1}{\sqrt{2}}e^{-i\phi}(ip_{\theta} + p_{\phi} \cot \theta). \tag{6}\]

Here

\[
p_{\theta} \equiv \partial_{\theta} L \quad p_{\phi} \equiv \partial_{\phi} L \tag{7}\]
denotes the canonical momenta. The squared angular momentum is

\[
\mathbf{I}^2 = p_{\theta}^2 + \frac{p_{\phi}^2}{\sin^2 \theta}. \tag{8}\]

This construction of the Lagrangian is particularly useful when further degrees of freedom are coupled to the axially symmetric rotor.

C. Power counting and the rotational Hamiltonian

The leading-order (LO) rotationally invariant Lagrangian

\[
L_{LO} = C_0 v_{+1} v_{-1} = \frac{C_0}{2}(\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta) \tag{9}\]
is quadratic in the velocities \( v_{\pm} \). It is equivalent to that of a particle restricted to move on the two-sphere, or to that of a rigid rotor. Here, \( C_0 \) is a low-energy constant and corresponds to the effective moment of inertia. This parameter of our theory must be fixed by data.

For the power counting, we need to introduce relevant energy scales. Let \( \xi \) denote the low-energy scale associated with rotations. Then, \( \xi \approx 80 \text{ keV} \) and \( \xi \approx 40 \text{ keV} \) for deformed rare-earth nuclei and actinides respectively. The breakdown scale \( \omega \) of the effective theory coincides with the onset of vibrational excitations and is of the order of \( 1 \text{ MeV} \) and \( 0.6 \text{ MeV} \) for rare-earth nuclei and actinides respectively. Thus, \( \xi/\omega \approx 1/10 \) is a conservative estimate.

The LO Lagrangian and the time derivatives (such as the velocities \( v_{\pm} \)) are of order \( \xi \). Thus,

\[
v_{\pm} \sim \dot{\phi} \sim \dot{\theta} \sim \xi \quad C_0 \sim \xi^{-1}. \tag{10}\]

A Legendre transformation of the LO Lagrangian yields the LO Hamiltonian

\[
H_{LO} = \frac{1}{2C_0} \left( p_{\theta}^2 + \frac{p_{\phi}^2}{\sin^2 \theta} \right) = \frac{1}{2C_0}\mathbf{I}^2. \tag{11}\]

The quantization is standard, and the angular momentum \( \mathbf{I} \) becomes the angular momentum operator \( \hat{\mathbf{I}} \) with spherical components \[28\]

\[
\hat{I}_{+1} = -\frac{1}{\sqrt{2}}e^{i\phi}(\partial_{\theta} + i \cot \theta \partial_{\phi}) \\
\hat{I}_{0} = -i\partial_{\phi} \\
\hat{I}_{-1} = -\frac{1}{\sqrt{2}}e^{-i\phi}(\partial_{\theta} - i \cot \theta \partial_{\phi}). \tag{12}\]

The squared angular momentum is

\[
\hat{\mathbf{I}}^2 = \hat{I}_{0}^2 - \hat{I}_{+1} \hat{I}_{-1} - \hat{I}_{-1} \hat{I}_{+1}. \tag{13}\]

We also recall that

\[
\hat{\mathbf{I}} = \mathbf{e}_r \times (-i\nabla_{\Omega}), \tag{14}\]
with
\[ \nabla \Omega = e_\theta \partial_\theta + e_\phi \frac{1}{\sin \theta} \partial_\phi \]  \hspace{1cm} (15)
being the angular derivative in the tangent plane \( \mathbb{T} \).

We note that \( -i \nabla \Omega \) is not an Hermitian operator.

The eigenfunctions of the Hamiltonian (11) are spherical harmonics \( Y_{IM}(\Omega) \) with eigenvalues
\[ \hat{H}_{\text{LO}} Y_{IM}(\theta, \phi) = \frac{I(I+1)}{2C_0} Y_{IM}(\theta, \phi) \]  \hspace{1cm} (16)
Higher-order corrections to the LO Lagrangian (9) include terms with higher powers of \( \mathbb{T} \). At next-to-leading order (NLO) the Lagrangian becomes \( L_{\text{LO}} + L_{\text{NLO}} \) with
\[ L_{\text{NLO}} = -\frac{C_2}{2} (\mathbb{T})^2. \]  \hspace{1cm} (17)
Thus, the corresponding Hamiltonian is \( H_{\text{LO}} + H_{\text{NLO}} \) with
\[ H_{\text{NLO}} = -\frac{C_2}{C_0} (H_{\text{LO}})^2 = -\frac{C_2}{4C_0^3} (\mathbb{T})^2, \]  \hspace{1cm} (18)
and the spectrum becomes
\[ E(I) = \frac{I(I+1)}{2C_0} - \frac{C_2}{4C_0^3} (I(I+1))^2. \]  \hspace{1cm} (19)
This deviation from the rigid-rotor behavior is due to omitted physics at the energy scale \( \omega \) of vibrations. From the expression for the NLO Hamiltonian (18) it is clear that \( C_2 \) has units of energy\(^{-3} \). The scaling is \( 28 \)
\[ C_2 \sim C_0 / \omega^2, \]  \hspace{1cm} (20)
and consequently, the ratio of the NLO correction to the LO contribution of the energy scales as
\[ \frac{\langle H_{\text{NLO}} \rangle}{\langle H_{\text{LO}} \rangle} \sim \left( \frac{\xi}{\omega} \right)^2 I(I+1). \]  \hspace{1cm} (21)
Thus, the effective theory of the axially symmetric non-rigid rotor is identical to the variable-moment-of-inertia model 59, 60, and the spectrum consists of increasing powers of \( I(I+1) \). It is important to notice that according to Eq. (21), the effective theory is expected to break down at spins of magnitude \( \sim \omega / \xi \), i.e. when the second term in Eq. (11) becomes as large as the first term.

Table I below shows values \( C_0 \xi, (C_2/C_0) \omega^2, (\xi/\omega)^2 \), and \( C_2/C_0^3 \) from the description of the ground-state bands of the homonuclear molecules \( \text{N}_2 \) and \( \text{H}_2 \), the rotational nuclei \( ^{236}\text{U}, ^{174}\text{Yb}, 166,168\text{Er} \), and \( ^{162}\text{DY} \), and the transitional nuclei \( 188\text{Os}, 154\text{Gd}, 152\text{Sm}, \) and \( 150\text{Nd} \), respectively. Here, \( \xi \) is the excitation energy of the lowest \( 2^+ \) state and \( \omega \) is the excitation energy of the lowest vibrational state. The values of \( C_0 \) and \( C_2 \) are obtained from a simultaneous fit to the lowest \( 2^+ \) and \( 4^+ \) levels to Eq. (19), respectively. For a rigid rotor, \( C_0 \xi = 3 \), \( \xi/\omega = 0 \), and \( C_2/C_0^3 = 0 \). Table II shows that the ratios \( (C_2/C_0)\omega^2 \) are of natural size, i.e. of order one, for the considered molecules and nuclei, and that the ratios \( C_2/C_0^3 \) are consistent with (but systematically smaller than) the scaling estimates \( (\xi/\omega)^2 \). Still, the values for the LEC \( C_2 \) are as expected from scaling estimates. Clearly the molecule \( \text{N}_2 \) is very close to the rigid-rotor limit. The comparison suggests that the molecule \( \text{H}_2 \) is as non-rigid a rotor as the nuclei \( ^{236}\text{U}, ^{174}\text{Yb} \) and \( 168\text{Er} \). The transitional nuclei \( 188\text{Os}, 154\text{Gd}, 152\text{Sm} \), and \( 150\text{Nd} \) exhibit even larger deviations from the rigid-rotor limit.

| System | \( C_0 \xi \) | \( C_0^2 \omega^2 \) | \( (\xi/\omega)^2 \) | \( C_2/C_0^3 \) | \( b/a \) |
|--------|--------------|-----------------|-----------------|-----------------|--------|
| \( \text{N}_2 \) | 3.00 | 2.1 | 0.000026 | 0.000006 | -0.000011 |
| \( \text{H}_2 \) | 2.99 | 2.2 | 0.0062 | 0.0015 | 0.0022 |
| \( ^{236}\text{U} \) | 2.99 | 2.3 | 0.0043 | 0.0011 |
| \( ^{174}\text{Yb} \) | 2.99 | 3.4 | 0.0026 | 0.0010 |
| \( ^{166}\text{Er} \) | 2.99 | 1.0 | 0.0094 | 0.0010 |
| \( ^{166}\text{Er} \) | 2.98 | 1.6 | 0.011 | 0.0020 |
| \( ^{162}\text{DY} \) | 2.98 | 1.9 | 0.0083 | 0.0017 |
| \( ^{188}\text{Os} \) | 2.91 | 1.5 | 0.06 | 0.012 | 0.008 |
| \( ^{154}\text{Gd} \) | 2.88 | 3.3 | 0.033 | 0.013 | 0.006 |
| \( ^{152}\text{Sm} \) | 2.88 | 3.5 | 0.032 | 0.013 | 0.003 |
| \( ^{150}\text{Nd} \) | 2.85 | 3.6 | 0.037 | 0.017 | 0.011 |

TABLE I. Dimensionless ratios of LECs and energy scales. The LECs \( C_0 \) and \( C_2 \) are obtained from the \( 2^+ \) and \( 4^+ \) levels of ground-state band for molecules and nuclei considered in this work. The ratio \( \xi/\omega \) measures the energy scales of rotations and vibrations. For a rigid rotor \( C_0 \xi = 3 \), \( \xi/\omega = 0 \), and \( C_2/C_0^3 = 0 \). The ratio \( b/a \) measures subleading corrections to transition quadrupole moments and similar in scale as the subleading energy correction \( C_2/C_0^3 \). A dash indicates that the experimental data is not precise enough to determine subleading corrections.

Within an effective field theory for emergent symmetry breaking in finite systems 50, vibrations enter as the quantized Nambu-Goldstone modes. The inclusion of vibrations into the theory pushes the breakdown scale \( \Lambda \) to higher energies. We have to distinguish two cases. In the first case, \( \Lambda \) is set by the appearance of new degrees of freedom. In nuclei, these are pairing effects, and \( \Lambda \approx 2 \) to 3 MeV. In molecules these are electronic excitations. The second case concerns the breakdown of the effective theory due to a restoration of spherical symmetry at large excitation energies. Indeed, for energies \( \Lambda \sim \omega^2 \xi \), the amplitude of vibrations approaches the scale of the static deformation \( \sim \xi^{-1/2} \). In nuclei \( \omega^2/\xi \approx 5 \) to 10 MeV, and the breakdown scale is thus given by the onset of new degrees of freedom.

III. COUPLING TO ELECTROMAGNETIC FIELDS

In this Section, we couple the axially-symmetric non-rigid rotor to electromagnetic fields. In leading order,
minimal couplings of the gauge fields describe the electromagnetic interaction, and non-minimal couplings enter as subleading corrections. For the long-wavelength $E^2$ transitions we are interested in, our approach is more technical than, and differs from, the usual approach taken for the collective models. The usual approach is motivated by the result of Siegert’s theorem, that allows one to employ density operators instead of current operators in transition matrix elements, see Eisenberg and Greiner [61] for example. While it is not obvious how to derive this result for the quadrupole degrees of freedom of the collective models, Siegert’s theorem is expected to hold in leading order, i.e. for the strong intra-band transitions. We recall that Mikhailov [62, 63] employed the quadrupole operator in the computation of the electromagnetic transition strengths, and the resulting formulas are well known and widely used [1]. However, this approach fails to describe the order of magnitude for the faint inter-band transitions.

Thus, it is interesting to more formally develop the electromagnetic theory of the rotor. Within an effective theory one consistently relates currents to the underlying Hamiltonian. We also note that Siegert’s theorem does not apply to magnetic transitions [64]. The importance of M1 transitions is another motivation for carrying out the formal development.

Deriving the electromagnetic couplings for non-relativistic many-body systems from first principles is no easy task [65], see also Kämpfer et al. [54] for a related study within effective field theory. Here, we follow a simpler path (at the possible cost of additional LECs). Within an EFT one writes down all gauge-invariant couplings of a low-energy effective theory. Here, we follow a simpler path (at the possible cost of additional LECs). Thus, the interaction Hamiltonian between the electromagnetic field and the particle becomes

$$
\hat{H}(A) = i\frac{hq}{2m} \left( A \cdot \nabla + \nabla \cdot A \right)
= i\frac{hq}{2m} \left( \frac{1}{r} \nabla_{\Omega} + A \cdot e_r, \partial_r \right)
+ i\frac{hq}{2m} \left( -\nabla_{\Omega} \cdot A + e_r, \partial_r \cdot A \right).
$$

We are interested in the long-wavelength limit and assume that the wave length $\lambda$ of the electromagnetic field fulfills $\rho/\lambda \ll 1$. (Note that the systems we are interested in actually fulfill $R/\lambda \ll 1$.) Thus, the radial variation of $A$ can be neglected and we can simply evaluate this field at $r = R$. The matrix element that governs electromagnetic transitions between the initial state $|i\rangle \equiv |NI,M_i\rangle$ and final state $|f\rangle \equiv |NI',M_f\rangle$ within the band with radial quantum number $N$ is

$$
\langle f|\hat{H}(A)|i\rangle = i\frac{hq}{2m} \left( 2\langle I_f M_f|A \cdot e_r|I_i M_i\rangle \langle N|\partial_r|N\rangle 
+ \langle I_f M_f|(A \cdot \nabla_{\Omega} + \nabla_{\Omega} \cdot A)|I_i M_i\rangle \langle N|\frac{1}{r}|N\rangle \right). \quad (24)
$$

We have

$$
\langle N|\frac{1}{r}|N\rangle = \int_0^\infty dr \frac{u_N^2(r)}{r} \approx R^{-1} \quad (25)
$$

for wave functions that are localized to a small region $\rho \ll R$ around $r \approx R$. Corrections to this expression are of order $\rho/R$.

Likewise,

$$
\langle N|\partial_r|N\rangle = \int_0^\infty dr r^2 \frac{u_N(r)}{r} \partial_r \frac{u_N(r)}{r}
= \int_0^\infty dr \left( u_N(r)u_N'(r) - \frac{u_N^2(r)}{r} \right)
\approx -R^{-1}, \quad (26)
$$

because the first term vanishes due to $u_N(0) = 0 = u_N(\infty)$, and the second term again yields approximately $-1/R$. Again, corrections are of order $\rho/R$.

\section*{A. Instructive example}

Let us consider a particle of charge $q$ and mass $m$ in a spherically-symmetric potential $V(r)$ that effectively confines the particle to a region $\rho \ll R$ around $r \approx R$. The Hamiltonian is

$$
\hat{H} = -\frac{\hbar^2}{2m}\Delta + V(r), \quad (22)
$$

with eigenfunctions $\psi(r, \theta, \phi) = \langle r\theta\phi|NI M\rangle = [u_N(r)/r]Y_{I M}(\theta, \phi)$. The rotational excitations are of order $\hbar^2 l(l + 1)/(2mR^2)$, and much smaller than radial excitations, which are of order $\hbar^2/(2m\rho^2)$. Therefore, the low-energy spectrum is rotational bands on top of band heads from radial excitations, and the effective theory developed in the previous Section applies. In what follows, we couple electromagnetic fields to the Hamiltonian [22].

For transitions within the ground-state band, we can neglect radial excitations and thereby gain insights into the couplings of a low-energy effective theory.

We minimally couple $-i\hbar \nabla \rightarrow -i\hbar \nabla - qA$, and keep only the term linear in $A$. Thus, the interaction Hamiltonian between the electromagnetic field and the particle becomes

$$
\hat{H}(A) = i\frac{hq}{2m} (A \cdot \nabla + \nabla \cdot A)
= i\frac{hq}{2m} \left( \frac{1}{r} \nabla_{\Omega} + A \cdot e_r, \partial_r \right)
+ i\frac{hq}{2m} \left( -\nabla_{\Omega} \cdot A + e_r, \partial_r \cdot A \right).
$$

We are interested in the long-wavelength limit and assume that the wave length $\lambda$ of the electromagnetic field fulfills $\rho/\lambda \ll 1$. (Note that the systems we are interested in actually fulfill $R/\lambda \ll 1$.) Thus, the radial variation of $A$ can be neglected and we can simply evaluate this field at $r = R$. The matrix element that governs electromagnetic transitions between the initial state $|i\rangle \equiv |NI,M_i\rangle$ and final state $|f\rangle \equiv |NI',M_f\rangle$ within the band with radial quantum number $N$ is

$$
\langle f|\hat{H}(A)|i\rangle = i\frac{hq}{2m} \left( 2\langle I_f M_f|A \cdot e_r|I_i M_i\rangle \langle N|\partial_r|N\rangle 
+ \langle I_f M_f|(A \cdot \nabla_{\Omega} + \nabla_{\Omega} \cdot A)|I_i M_i\rangle \langle N|\frac{1}{r}|N\rangle \right). \quad (24)
$$

We have

$$
\langle N|\frac{1}{r}|N\rangle = \int_0^\infty dr \frac{u_N^2(r)}{r} \approx R^{-1} \quad (25)
$$

for wave functions that are localized to a small region $\rho \ll R$ around $r \approx R$. Corrections to this expression are of order $\rho/R$.

Likewise,

$$
\langle N|\partial_r|N\rangle = \int_0^\infty dr r^2 \frac{u_N(r)}{r} \partial_r \frac{u_N(r)}{r}
= \int_0^\infty dr \left( u_N(r)u_N'(r) - \frac{u_N^2(r)}{r} \right)
\approx -R^{-1}, \quad (26)
$$

because the first term vanishes due to $u_N(0) = 0 = u_N(\infty)$, and the second term again yields approximately $-1/R$. Again, corrections are of order $\rho/R$. 

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\[5\]
Thus, for intra-band transitions the matrix element that governs long-wavelength transitions becomes in leading order of $\rho/R$

$$\langle f | \hat{H}^{(A)} | i \rangle \approx i \frac{\hbar q}{2mR} \left( \langle I_f | M_f | (\mathbf{A} \cdot \nabla + \nabla \mathbf{A}) | I_i M_i \rangle + 2 \langle I_f | M_f | \mathbf{A} \cdot \mathbf{e}_r | I_i M_i \rangle \right).$$

(27)

We note that this leading-order expression is independent of the confining radial potential, and it becomes exact in the limit $\rho/R \to 0$. We also note that the right-hand side of Eq. (27) does not reference the radial wave function. However, the term $\mathbf{A} \cdot \mathbf{e}_r$ originates from the current associated with the radial zero-point motion. Thus, in a low-energy effective theory, electromagnetic transitions are induced by the operator

$$\hat{H}^{(A)}(\Omega) = -\frac{q \hbar}{2mR} \times (\mathbf{A} \cdot (-i \nabla_\Omega + i \mathbf{e}_r) + (-i \nabla_\Omega + i \mathbf{e}_r) \cdot \mathbf{A}),$$

(28)

and corrections are of order $\rho/R$. We note that the operator

$$-i \nabla_\Omega + i \mathbf{e}_r = \frac{i}{2} \left[ \mathbf{I}^2, \mathbf{e}_r \right]$$

(29)

(unlike the operator $-i \nabla_\Omega$) is also Hermitian under the usual integration measure $d\Omega \equiv d\phi d\theta \sin \theta$ of the sphere. The identity (29) can be proved by a direct computation.

On the first view it might be surprising that the operator (28), relevant for the coupling of the low-energy degrees of freedom (the angles $\Omega$), references the radial component of the electromagnetic field $\mathbf{A}$. Indeed, decomposing the vector potential

$$\mathbf{A} = A_r \mathbf{e}_r + \mathbf{A}_\Omega$$

(30)

$$\mathbf{A}_\Omega = A_\theta \mathbf{e}_\theta + A_\phi \mathbf{e}_\phi$$

(31)

into a radial component and the projection $\mathbf{A}_\Omega$ on the tangential plane, and using the identity

$$-i \nabla_\Omega \cdot \mathbf{A} = -i \nabla_\Omega \cdot \mathbf{A}_\Omega - i 2e_r \cdot \mathbf{A}$$

(32)

we can rewrite the interaction (28) as

$$\hat{H}^{(A)}(\Omega) = i \frac{\hbar q}{2mR} \left( \mathbf{A}_\Omega \cdot \nabla_\Omega + \nabla_\Omega \cdot \mathbf{A}_\Omega \right).$$

(33)

This result is in keeping with expectations that a low-energy effective theory only involves low-energy degrees of freedom. While this expression reflects that the physics is entirely in the tangential plane, it is not ideal because of the appearance of the non-Hermitian operator $-i \nabla_\Omega$. An equivalent expression involving only Hermitian operators can be obtained using the angular momentum operator (14). This yields

$$\hat{H}^{(A)}(\Omega) = -\frac{q \hbar}{2mR} \left[ (e_r \times \mathbf{A}_\Omega) \cdot \mathbf{I} + \mathbf{I} \cdot (e_r \times \mathbf{A}_\Omega) \right].$$

(34)

The interaction terms (33) and (34) thus suggest that the electromagnetic coupling is achieved by gauging

$$-i \nabla_\Omega \rightarrow -i \nabla_\Omega - q \mathbf{A}_\Omega$$

(35)

and, equivalently,

$$\mathbf{I} \rightarrow \mathbf{I} - q \mathbf{e}_r \times \mathbf{A}_\Omega.$$ (36)

The next Subsection confirms this picture.

B. Gauging the effective theory

Let us now turn to couple electromagnetic fields to the non-rigid rotor. The LO effective theory starts from the Hamiltonian (11). Requiring invariance under local gauge transformations $\psi(\Omega) \rightarrow \exp(i\lambda(\Omega))\psi(\Omega)$ of its eigenfunctions $\psi(\Omega)$ introduces gauge fields according to

$$\hat{I} \rightarrow \mathbf{I} - q \mathbf{e}_r \times \mathbf{A}_\Omega,$$ (37)

with

$$\mathbf{A}_\Omega = -\nabla_\Omega \lambda(\Omega).$$ (38)

Here, the effective charge $q$ is a LEC and needs to be adjusted to data. Thus, the requirement of local gauge invariance introduces gauge fields with components in the tangential planes spanned by the vectors $\mathbf{e}_\theta$ and $\mathbf{e}_\phi$. As $\mathbf{A}_\Omega \cdot \mathbf{e}_r = 0$, we have $\mathbf{e}_r \times \mathbf{A}_\Omega = \mathbf{e}_r \times \mathbf{A}$, and this can be employed in the minimal coupling (37).

We are interested in single-photon transitions, and the LO Hamiltonian that describes the non-rigid rotor plus electromagnetic fields system becomes

$$\hat{H}_{\text{LO}} = \hat{H}_{\text{LO}} + \hat{H}_{\text{LO}}^{(A)}$$ (39)

with the interaction Hamiltonian given by

$$\hat{H}_{\text{LO}}^{(A)} = -\frac{q}{2\hbar C_0} \left( (e_r \times \mathbf{A}) \cdot \mathbf{I} + \mathbf{I} \cdot (e_r \times \mathbf{A}) \right)$$

$$= i \frac{q}{2\hbar C_0} \left[ \mathbf{I}^2, \mathbf{e}_r \right] \cdot \mathbf{A} + i \mathbf{A} \cdot \mathbf{e}_r.$$ (40)

This is essentially the operator (34). Thus, the gauging of the effective theory yields the same interaction Hamiltonian as the removal of a high-energy degree of freedom in the direct calculation presented in the previous Subsection. The direct use of the operator (14) in the computation of matrix elements is cumbersome. Instead we return to Eq. (34), use the identity (29), and find

$$\langle e_r \times \mathbf{A}_\Omega | \mathbf{I} = \frac{i}{2} [\mathbf{I}^2, e_r] - i \mathbf{A} \cdot e_r$$

$$\mathbf{I} \cdot (e_r \times \mathbf{A}_\Omega) = \frac{i}{2} [\mathbf{I}^2, e_r] \cdot \mathbf{A} + i \mathbf{A} \cdot e_r.$$ (41)

Thus, in the long wave length limit and in LO of the effective theory, the interaction Hamiltonian is

$$\hat{H}_{\text{LO}}^{(A)} = -\frac{iq}{4\hbar C_0} \left( \mathbf{A} \cdot [\mathbf{I}^2, e_r] + [\mathbf{I}^2, e_r] \cdot \mathbf{A} \right).$$ (42)
This LO interaction Hamiltonian \( \hat{H}^{(A)}_{\text{LO}} \) can be rewritten by employing the LO Hamiltonian \( \hat{H}^{(1)}_{\text{LO}} \) of the rigid rotor, yielding
\[
\hat{H}^{(A)}_{\text{LO}} = -\frac{iq}{2} \left( \mathbf{A} \cdot \left[ \hat{H}_{\text{LO}, r}, \mathbf{e}_r \right] + \left[ \hat{H}_{\text{LO}, r}, \mathbf{e}_r \right] \cdot \mathbf{A} \right). \tag{43}
\]

At NLO, we start from the Hamiltonian \( \hat{H}^{(1)}_{\text{LO}} \) and minimally couple it according to Eq. (37). Again, we only keep terms linearly in \( \mathbf{A} \) because we are interested in single-photon transitions. This yields
\[
\hat{H}^{(A)\text{NLO}}_{\text{EM}} = \hat{H}^{(1)}_{\text{LO}} + \hat{H}^{(A)}_{\text{LO}} + \hat{H}^{(A)\text{NLO}}\text{.} \tag{44}
\]

Here, the NLO interaction takes the form
\[
\hat{H}^{(A)\text{NLO}}_{\text{LO}} = \frac{qC_2}{4\xi_0} \left( (\mathbf{e}_r \times \mathbf{A}) \cdot \hat{\mathbf{r}} + \hat{\mathbf{r}} \cdot (\mathbf{e}_r \times \mathbf{A}) \right) \mathbf{I}^2 + \frac{qC_2}{4\xi_0} \mathbf{I}^2 \left( (\mathbf{e}_r \times \mathbf{A}) \cdot \hat{\mathbf{r}} + \hat{\mathbf{r}} \cdot (\mathbf{e}_r \times \mathbf{A}) \right)
+ \frac{C_2}{2\xi_0} \left( \hat{H}^{(A)\text{LO}}_{\text{LO}} \mathbf{I}^2 + \mathbf{I}^2 \hat{H}^{(A)\text{LO}}_{\text{LO}} \right). \tag{45}
\]

Note that the LECs of \( \hat{H}^{(A)\text{NLO}}_{\text{LO}} \) are determined entirely by the Hamiltonian \( \hat{H}^{(1)}_{\text{LO}} \) and the LO electromagnetic transitions. This is the consistency between currents and Hamiltonian offered within an effective theory. This term is a factor \( C_2/C_0^3 \sim \xi^2/\omega^2 \) smaller than \( \hat{H}^{(A)\text{LO}}_{\text{LO}} \). Let
\[
M_{\text{LO}}(i \rightarrow f) \equiv \langle f | \hat{H}^{(A)\text{LO}}_{\text{LO}} | i \rangle \tag{46}
\]
be the LO matrix element for electromagnetic transitions. Then
\[
M_{\text{NLO}}(i \rightarrow f) \equiv \langle f | \hat{H}^{(A)\text{NLO}}_{\text{LO}} | i \rangle = \frac{C_2}{2\xi_0} \left[ I_f(I_f + 1) + I_i(I_i + 1) \right] M_{\text{LO}}. \tag{47}
\]

We will employ a multipole expansion. This expansion is valid if the wavelength of the radiation is considerably larger than the linear dimension of the rotor. Let \( k \) be the wave number of the electromagnetic field. We have \( k \sim \xi \) for transitions in the ground-state band. For a rigid rotor with extension \( R \) and mass \( M \), \( C_0 \sim MR^2 \sim \xi^{-1} \). Thus, \( kR \sim \sqrt{\xi/M} \). To give quantitative estimates, we consider rare earth nuclei. Here, \( kR \approx 1/300 \). Thus, the multipole expansion is rapidly converging.

To make progress, we employ a plane wave
\[
\mathbf{A}(\mathbf{r}, t) = A \mathbf{e}_z e^{i(kr - wt)} \tag{48}
\]
with amplitude \( A \), polarization \( \mathbf{e}_z \) in the \( z \) direction and momentum \( k = ke_z \) in the \( x \) direction. Here \( w = k \) (recall that the speed of light \( c = 1 \)). Taylor expansion of the plane wave yields the leading-order quadrupole component contained in the term
\[
\mathbf{A}^{(2)} = A \mathbf{e}_z kr \cos \phi \sin \theta . \tag{49}
\]

In what follows, we neglect the subleading contribution of \( \mathbf{A}^{(2)} \) to dipole transitions. When inserted into the LO interaction Hamiltonian \( \hat{H}^{(A)\text{LO}}_{\text{LO}} \), we find
\[
\hat{H}^{(A)\text{LO}}_{\text{LO}} = -\frac{iq}{2} \left[ \hat{H}_{\text{LO}, \mathbf{A}} \cdot \mathbf{e}_r \right]. \tag{50}
\]

This form of the quadrupole interaction is particularly suited for the computation of the quadrupole transition matrix elements \( \langle f | \mathbf{A} \cdot \mathbf{e}_r | i \rangle \), and
\[
M_{\text{LO}}(E2, i \rightarrow f) = -\frac{iqw}{2} \langle f | A^{(2)} \cdot \mathbf{e}_r | i \rangle \tag{51}
\]
Here, \( w = E_f - E_i \) is the difference between the LO energies of the final and initial states. The corresponding NLO interaction Hamiltonian can be obtained directly by inserting the Hamiltonian \( \hat{H}^{(A)\text{LO}}_{\text{LO}} \) into Eq. (45). At NLO, the matrix element for electric quadrupole transitions is equivalent to that of Eq. (51), with \( w \) being the difference between the NLO energies of the final and initial states. In the evaluation of these matrix elements, we will set \( r = R \), and absorb the factor \( kr \) by re-defining \( qR \rightarrow q \).

\[\text{C. Non-minimal couplings}\]

Non-minimal couplings (i.e. interaction terms that include electric and magnetic fields) arise because the low-energy degrees of freedom we employ describe composite objects. Such terms are gauge-invariant scalars that are consistent with the symmetries of the effective theory. For electric transitions, we can couple the low-energy degrees of freedom to the electric field \( \mathbf{E} \), and the power counting is in derivatives on the electric field and low-energy degrees of freedom. In leading order we have
\[
\hat{H}^{(E)}_{\text{LO}} = d \mathbf{E} \cdot \mathbf{e}_r. \tag{52}
\]

Here, the dimensionless number \( d \) is a LEC and has to be adjusted to data. We note that \( \mathbf{E} \sim \xi \mathbf{A} \) for low-energy transitions and assume that \( d \sim O(1) \). Thus, the non-minimal term \( \langle f | \mathbf{A}^{(2)} \cdot \mathbf{e}_r | i \rangle \) is of the same order as \( \hat{H}^{(A)\text{LO}}_{\text{LO}} \) in Eq. (43).

For the \( E2 \) transitions considered in this work \( \mathbf{E}^{(2)} = iw \mathbf{A}^{(2)} \), and its is clear that the transition matrix element of the non-minimal interaction \( \hat{H}^{(E)\text{LO}}_{\text{LO}} \) is equivalent to the LO gauged interaction \( \hat{H}^{(A)\text{LO}}_{\text{LO}} \) after identifying the LECs \( d = q \). We thus see that Siegert’s theorem is valid for the \( \text{LO} \) transitions.

We turn to higher-order non-minimal couplings. In principle, every single term that is invariant under gauge transformations, rotations, parity and time reversal must be considered. However, the power counting establishes which terms are relevant at each order. The relevant NLO terms are quadratic in \( \mathbf{I} \)
\[
\hat{H}^{(E)\text{NLO}}_{\text{NLO}} = \frac{-qd_1}{4} \left( \mathbf{E} \cdot \mathbf{e}_r \mathbf{I}^2 + \mathbf{I}^2 \mathbf{E} \cdot \mathbf{e}_r \right)
- \frac{qd_2}{4} \left( \mathbf{E} \cdot \mathbf{I}^2 \mathbf{e}_r + \mathbf{e}_r \cdot \mathbf{I}^2 \mathbf{E} \right), \tag{53}
\]
where the factor $q/4$ is included for convenience. As a NLO correction, it is expected to fulfill a relation similar to that of Eq. (21)

$$
\langle f|\hat{H}_{\text{NLO}}^{(E)}|i\rangle \sim \left(\frac{\xi}{\omega}\right)^2 f(I_i, I_f), \quad (54)
$$

where $f(I_i, I_f)$ is a function of the angular momenta of the initial and final states. From here, it is expected that $d_1 \sim d_2 \sim (\xi/\omega)^2$. These LECs need to be fitted to data.

In this work, we are only interested in electric transitions. For magnetic transitions, other non-minimally coupled terms involving the magnetic field $B$ must be included.

### IV. TRANSITIONS WITHIN THE GROUND BAND

In this Section, we study electric transitions within ground-state bands of molecules and atomic nuclei. Molecules are a perfect testing ground for the effective theory because the separation of scale between rotations and vibrations is several orders of magnitude. After a brief discussion of molecules we consider rotational nuclei in the rare-earth and actinide regions. For these, the separation of scale between rotations and vibrations is largest in atomic nuclei. Finally, we consider transitional nuclei where the separation of scale is smaller, and NLO corrections are more prominent. The ratio $\xi/\omega$ in the second column of Table II is a measure for the separation of scale.

| Rotor | $\xi/\omega$ | $E_{2+}/E_{2+}$ | $Q_0[\text{eb}]$ | $\alpha$ |
|-------|-------------|----------------|-----------------|---------|
| N$_2$ | 0.005       | 3.33           | 2               |         |
| H$_2$ | 0.08        | 3.30           | 3/2             |         |
| $^{236}$U | 0.05       | 3.30           | 3.26            | 1       |
| $^{174}$Yb | 0.05       | 3.31           | 2.37            | 1       |
| $^{168}$Er | 0.10        | 3.31           | 2.41            | 1       |
| $^{166}$Er | 0.10        | 3.29           | 2.41            | 1       |
| $^{162}$Dy | 0.09        | 3.29           | 2.29            | 1       |
| $^{188}$Os | 0.24        | 3.08           | 1.58            | 1/2     |
| $^{154}$Gd | 0.18        | 3.01           | 1.96            | 1/2     |
| $^{152}$Sm | 0.18        | 3.01           | 1.85            | 1/2     |
| $^{150}$Nd | 0.19        | 2.93           | 1.65            | 1/2     |

**Table II.** Ratio $\xi/\omega$ and ratio $E_{2+}/E_{2+}$ of energies $E_{J\pi}$ of states with spin $J$ and parity $\pi$ (as measures of the separation of scale), effective quadrupole moment $Q_0$, and values of $\alpha$ (for uncertainty estimates) for the rotors considered in this work. For a rigid rotor, $\xi/\omega = 0$, and $E_{2+}/E_{2+} = 10/3$.

#### A. Transition strengths

The reduced transition probabilities of electric radiation with multipolarity $\lambda$, i.e. the $B(E\lambda)$ values, are given by Fermi’s golden rule

$$
B(E\lambda, i \rightarrow f) = \frac{1}{2I_i+1} \left| \langle f|\hat{\mathcal{M}}(E\lambda)|i\rangle \right|^2, \quad (55)
$$

where $\hat{\mathcal{M}}(E\lambda) \equiv \hat{H}^{(A)} + \hat{H}^{(E)}/wA$. As we will see below, these transition strengths contain a simple geometrical factor that governs the leading angular-momentum dependence. To understand transition strengths within an effective theory, it is very useful to remove this trivial factor. For this reason we define the quadrupole transition moments as

$$
Q_{ij} = \frac{B(E2, i \rightarrow f)}{C_{I_0,020}^f}. \quad (56)
$$

Here $C_{I_1M_1I_2M_2}^{I_0M_0}$ is a Clebsch-Gordan coefficient and governs the leading angular-momentum dependence.

If the quadrupole components of the vector potential $A$ and the corresponding electric field $E$ are inserted into the transition operators $\hat{H}^{(A)}$ and $\hat{H}^{(E)}$, they induce $E2$ transitions. At NLO, the $B(E2)$ values for decays within the ground-state band are

$$
B(E2, i \rightarrow f) = \frac{(aqR)^2}{60} \left( C_{I_0,020}^{I_f} \right)^2 \left[ 1 + \frac{b}{a} I_i(I_i - 1) \right]. \quad (57)
$$

Here $a = 1 + d_1$ and $b = 2(d_1 + d_2)$ are combinations of LECs from the non-minimal couplings. Thus, the quadrupole transition moments for these decays are

$$
Q_{ij} = \frac{(aqR)^2}{60} \left[ 1 + \frac{b}{a} I_i(I_i - 1) \right] \quad (58)
$$

or

$$
Q_{ij} = Q_0 \left[ 1 + \frac{b}{a} I_i(I_i - 1) \right] \quad (59)
$$

where $Q_0 \equiv \sqrt{(aqR)^2/60}$ may be thought of as the effective quadrupole moment. **Table III** shows the values of $Q_0$ for the systems considered in this work. They are obtained from fits to data presented in the second half of this Section. In LO the quadrupole transition moment $Q_{ij}$ is constant, reflecting the behavior of a rigid rotor. The NLO corrections are deviations from this behavior. They depend on the initial and final spins of the system and scale as $d_1 + d_2 \sim (\xi/\omega)^2$.

According to Bohr and Mottelson [3], the reduced matrix elements for quadrupole decays within the ground band are

$$
\langle f|\hat{\mathcal{M}}(E2)|i\rangle = M_1(2I_i + 1)^{1/2} C_{I_0,020}^{I_f} \times \left[ 1 + 2 \frac{M_2}{M_1} + \frac{M_3}{M_1} + 2 \frac{M_2}{M_1} + \frac{M_3}{M_1} I_i(I_i - 1) \right]. \quad (60)
$$

We rewrite this formula as

$$
Q_{ij} = (a_{BH}M_i)^2 \left[ 1 + \frac{b_{BH}}{a_{BH}} I_i(I_i - 1) \right]. \quad (61)
$$
with \( a_{BH} = 1 + 2M_{21} + 4M_{31} \), \( b_{BH} = 4(M_{21} + 8M_{31}) \) and \( M_{ij} = M_i/M_j \). Thus, at this order, the effective theory and the geometric collective model give the same description for decays within the ground band.

B. Estimates of theoretical uncertainties

There are statistical and systematic uncertainties we have to consider. Statistical uncertainties arise due to the fitting of LECs to experimental data which themselves exhibit uncertainties. Energy levels are measured with a very high precision, and our LECs \( C_0 \) and \( C_2 \) therefore exhibit very small statistical uncertainties. Thus, we neglect statistical uncertainties for \( C_0 \) and \( C_2 \). The situation is, however, different for \( B(E2) \) transition strengths. Here, experimental uncertainties are significant and presently preclude us from making any meaningful subleading predictions for the rotational nuclei \( 236U, 174Yb, 166,168Er, \) and \( 162Dy \). The situation is better though for the transitional nuclei \( 188Os, 154Gd, 152Sm, \) and \( 150Nd \), where data with higher precision is available. To test the effective theory for physical systems close to the rigid-rotor limit, we therefore consider the homonuclear molecules \( H_2 \) and \( N_2 \).

Through its power counting, the effective theory allows us to give estimates for theoretical uncertainties. These are due to neglected higher-order corrections. In ground-state bands, subleading corrections to energies and quadrupole transition matrix elements scale as increasing powers of \( (\xi/\omega)^2 \), and this scaling factor thus is a dimensionless error estimate. Assuming that theoretical errors are of natural size, \( \alpha(\xi/\omega)^2 \) is an estimate for the relative theoretical error, with \( 1/3 \lesssim |\alpha| \lesssim 3 \). Unfortunately, we are unable to give \textit{a priori} values for \( \alpha \). Assuming, however, that the effective theory describes the data reasonably well, we must choose \( \alpha \) such that that LO error estimates overlap with existing data. This establishes an approximate value for \( \alpha \) that can then be used in NLO estimates of theoretical uncertainties.

The LO uncertainty estimates for the quadrupole transition moments are

\[
\Delta Q_{LO} = \alpha C_2/C_0 I_i(I_i - 1). \tag{62}
\]

For an estimate of the NLO error we thus use

\[
\Delta Q_{NLO} = (\Delta Q_{LO})^2/Q_{LO}. \tag{63}
\]

Table II shows the values of \( \alpha \) employed for the physical systems considered in this work. While we choose slightly different values for the homonuclear molecules, the rotational nuclei, and the transitional nuclei, they are all of natural size.

C. Linear molecules

Linear molecules provide an ideal testing ground for the effective theory, because they are axially symmetric in their ground states and close to the rigid rotor limit. For these molecules, the separation of scale is excellent, and a good agreement between the effective theory and experimental data must be achieved at low order.

Homonuclear molecules appear in two isomeric forms, depending on the alignment of the nuclear spins. For antiparallel spins, the system possesses a positive \( R \) parity as rotations of \( \pi \) around any axis perpendicular to the symmetry axis do not change the wave function of the system. This symmetry implies that only states with even spin \( I \) are allowed in the ground band. Thus, within the ground band, \( E2 \) transitions are the most relevant, and this property is shared with axially symmetric atomic nuclei.

The para \( N_2 \) molecule energy ratios are extremely close to those of a rigid rotor, see Table II. Figure II shows \( E2 \) transition strengths for decays within the ground-state band. The LO calculations are in agreement with experimental data within 1% for initial angular momenta \( I_i \lesssim 30 \). NLO calculations deviates from experimental data less than 0.1%. The theoretical uncertainty estimates at NLO seem consistent with the data.

The much lighter \( H_2 \) molecule is farther from the rigid-rotor limit, as shown in Fig. II. In this case, LO calculations are in agreement with data for transitions with \( I_i \lesssim 8 \), while NLO calculations are in agreement with most data. We note that the NLO error estimates are of
similar size as the LO error estimates for the largest spins shown. Thus, the breakdown scale is reached for spins around \( I \approx 20 \), and this is consistent with the estimate of \( \omega/\xi \approx 13 \). Indeed, vibrational levels of \( \text{H}_2 \) appear at about the energy \( \omega \approx 13\xi \). We note that the difference between the theoretical centroids at NLO and the data is considerably smaller than the theoretical uncertainty estimates. The size of the theoretical uncertainties reflect the proximity of the breakdown scale \( \omega \) in the effective theory. If one were to include the almost harmonic vibrations at multiples of the energy \( \omega \) into the theoretical description, the theoretical uncertainty estimates would change and reflect the increase in the breakdown scale from the onset of vibrations at \( \omega \) to the onset of electronic excitations at energies \( \Lambda \gg \omega \).

The last column of Table I lists the NLO values for the LECs that enter the quadrupole transition function for the homonuclear molecules. Their values are consistent with the NLO correction \( C_2/C_0^3 \) obtained from the rotational energy spectrum.

D. Rotational nuclei

Axially-symmetric deformed nuclei possess positive \( R \) parity, and only states with even angular momentum \( I \) are allowed in the ground-state band.

The energy spectra of many nuclei in the actinide region makes them good candidates to test the effective theory. Figure 3 shows the quadrupole transition strengths for decays within the ground band of \( \text{U}_{236} \) and compares them to the experimental data from Browne and Tuli [67]. The results from our LO calculations are in good agreement with these data. Also shown (as a cross) is the 2001 world average for the \( 2^+ \rightarrow 0^+ \) transition. The theoretical results strongly suggest that this data point is not consistent with the remaining data. Data of higher precision would be necessary to probe the theory at NLO.

The rare-earth nuclei are also useful to confront the effective theory, with results shown in Fig. 4. One of the best rigid-rotor candidates in this region is \( \text{Yb}_{174} \) due to its small ratio of \( \xi/\omega \). The LO results for this nucleus and our uncertainty estimates are consistent with the experimental data [69]. We note that the data points for the \( 4^+_g \rightarrow 2^+_g \) and \( 8^+_g \rightarrow 6^+_g \) transitions is about one experimental sigma away from the theoretical predictions. Again, data of higher precision would be desirable.

We also consider \( \text{Er}_{168} \) and \( \text{Er}_{166} \), two of the most studied rotational nuclei [73–77], see Fig. 4. For \( \text{Er}_{166} \), the LO results and error estimates are consistent with the experimental data. For \( \text{Er}_{168} \), the same statement is true, except for the experimental value for the \( 6^+_g \rightarrow 4^+_g \) transition, which is about three experimental sigmas away from the theoretical prediction. This value is difficult to understand in the effective theory, which is expected to be valid at such low spins. We note that the theoretical uncertainties for \( \text{Er}_{166} \) are about twice as large as for \( \text{Er}_{168} \), and this reflects the size of the ratios \( C_2/C_0^3 \) in Table I. Finally, we turn to \( \text{Dy}_{162} \), another well studied rotational nucleus [78, 79], see Fig. 4. Also here, our LO predictions are consistent with the data.
Summarizing the situation for rotational nuclei, our LO theory predictions are largely consistent with data given the experimental and theoretical uncertainties. A few transition strengths deviate more than expected from the effective theory, and one would like to see these data points to be measured with a higher precision. To probe NLO corrections for rotational nuclei would require data with considerably higher precision. It is somewhat surprising that the 1975 words of Bohr and Mottelson [4] “The accuracy of the present measurements of E2-matrix elements in the ground-state bands of even even nuclei is in most cases barely sufficient to detect deviations from the leading-order intensity relations” are still applicable today. The noted deviations, and the possibility to compare data with more precise NLO predictions, would make it very interesting to measure transition strengths in these nuclei with an increased precision.

E. Transitional nuclei

Transitional nuclei are characterized by energy spectra that deviate considerably from the rotational behavior. Ratios $E_{4+}/E_{2+} \approx 3$ identify these non-rigid rotors, and the separation of scale is less pronounced than for the rotational nuclei. The increased $\xi/\omega$ ratio implies that
NLO corrections are more relevant and also more visible. Fortunately, these nuclei data of sufficiently high precision exists. Thus, the systematic improvement of the effective theory can be checked in these nuclei.

Figure 5 shows data for quadrupole decays in $^{188}$Os (top left), $^{154}$Gd (top right), $^{152}$Sm (bottom left) and $^{150}$Nd (bottom right), and compares them to theoretical results from the effective theory, the triaxial rotor model (TRM) (for $^{188}$Os), and the $X(5)$ approximation of the Bohr Hamiltonian [32] (for the other nuclei). Overall, our NLO predictions agree well with data. The size of the NLO error bars becomes comparable to LO error bars around spins $I \approx 10$, consistent with expectations from the breakdown scale of the effective theory.

The last column of Table II shows the NLO values of LECs that enter the transition strengths for transitional nuclei. We note that the LECs are consistent with the expectation from the breakdown scale of the effective theory.

We have

$$\Psi = (\Psi_+^0, 0, \Psi_0, 0, \Psi_-^0).$$

It is convenient to rewrite the components of the field as

$$\Psi_0 = \zeta + \psi_0 \quad \Psi_{\pm} = \psi_2 e^{\pm i\gamma}.$$  

Here $\zeta$ is the constant vacuum expectation value of the zero mode (with $|\psi_0| \ll \zeta$), and the factor 2 in the phase $2\gamma$ has been introduced for convenience. Because of this factor $\gamma$ ranges from 0 to $\pi$.

The field in the laboratory frame can be written as an appropriate rotation of the field in the intrinsic frame

$$\Phi = g(\phi, \theta) \Psi,$$

which implies that under the SO(3) rotation $r(\alpha, \beta, \gamma)$

$$\psi_0 \rightarrow \psi_0 \quad \psi_2 \rightarrow \psi_2 \quad \gamma \rightarrow \gamma + \chi.$$  

Here, $\chi = \chi(\alpha, \beta, \gamma; \theta, \phi)$ is a complicated function of the rotation angles and the orientation angles $(\theta, \phi)$; the rotational symmetry is realized nonlinearly.

The kinetic terms in the quadrupole degrees of freedom are obtained by acting with the covariant derivatives onto $\Psi$. Thus, any Lagrangian $L$ in $v_\pm, \Psi_0, \Psi_{\pm}, D_i \Psi_0, D_i \Psi_{\pm}$ that is formally invariant under SO(2), is invariant under SO(3) due to the nonlinear realization of the rotational symmetry. The application of Noether’s theorem to such Lagrangian yields the total angular momentum $J$ with spherical components

$$J_{+1} = -\frac{1}{\sqrt{2}}e^{i\phi}(ip_\theta - p_\phi \cot \theta) - \frac{1}{\sqrt{2}}e^{i\phi} \frac{p_\gamma}{\sin \theta},$$

$$J_0 = p_\phi,$$

$$J_{-1} = -\frac{1}{\sqrt{2}}e^{-i\phi}(ip_\theta + p_\phi \cot \theta) + \frac{1}{\sqrt{2}}e^{-i\phi} \frac{p_\gamma}{\sin \theta},$$

as the conserved quantity. Here

$$p_\theta \equiv \partial_\theta L \quad p_\phi \equiv \partial_\phi L \quad p_\gamma \equiv \partial_\gamma L,$$

and the total angular momentum squared is

$$J^2 = p_\theta^2 + \left(\frac{p_\phi - p_\gamma \cos \theta}{\sin \theta}\right)^2 + p_\gamma^2.$$

We denote the total angular momentum as $J$, because its definition differs from Eq. (60) due to the newly introduced vibrational degrees of freedom.

Let us briefly compare the degrees of freedom in the effective theory to those of the Bohr Hamiltonian. In the effective theory, the angles $(\phi, \theta, \gamma)$ can be viewed as Euler angles, with the “slow” degrees of freedom $\Omega = (\theta, \phi)$ describing the orientation of the symmetry axis and the “fast” degree of freedom $\gamma$ describing rotations around the symmetry axis. The $\psi_0$ degree of freedom is a (fast) vibration that breaks the axial symmetry, while $\psi_2$ is a (fast) vibration that breaks the axial symmetry. The

V. ROTATIONS AND VIBRATIONS

In this Section, we are interested in inter-band transitions. These transitions are much weaker than the intraband transitions considered in the previous Section, and the accurate description of these faint transitions poses a challenge. For the description of rotational bands beyond the ground-state band, we need to include additional degrees of freedom into the effective theory. For even-even nuclei, these degrees of freedom represent higher-energetic vibrations of the nucleus with an energy scale $\omega$ below the breaking energy scale. These vibrations are the true remnants of Nambu-Goldstone modes in finite systems with emergent symmetry breaking [30]. The effective theory for this case has in parts been developed in Refs. [23, 29]. In leading order, the theory describes uncoupled vibrational states. At NLO, the vibrational states become heads of rotational bands. At NNLO, couplings between vibrational band heads enter. In the next Subsection, we briefly introduce quadrupole degrees of freedom. We then develop the Hamiltonian up to NNLO, for the ensuing description of inter-band transitions.

A. Quadrupole degrees of freedom

In even-even nuclei, the quantized Nambu-Goldstone modes due to the emergent symmetry breaking from SO(3) to SO(2) can be represented by a quadrupole field with two of its components replaced by the low-energy degrees of freedom $\theta$ and $\phi$. Note that these degrees of freedom have the quantum numbers of quadrupole modes, but they are not Bohr’s surface oscillations. In our treatment, the quadrupole degrees of freedom also realize the rotational symmetry nonlinearly; they are in the co-rotating coordinate system and can be viewed as being attached to the particle moving on the two-sphere.
Bohr Hamiltonian employs three Euler angles and two deformation parameters \([6]\). The two deformation parameters (usually labeled as \(\beta\) and \(\gamma\), respectively) describe the amplitude of the total deformations \((\beta)\), and the deformation breaks axial symmetry for \(\gamma \neq 0\). The variable \(\beta\) can be viewed as a hyper radius in five-dimensional space of the quadrupole degrees of freedom, while \(\gamma\) is a hyper angle in addition to the three Euler angles.

B. Power counting and Hamiltonian at NNLO

In addition to the power counting estimates \((10)\) we have \((28)\)

\[
\begin{align*}
\omega_0 \sim & \omega_2 \sim \dot{\gamma} \sim \omega \\
\psi_0 \sim & \psi_2 \sim \omega^{1/2} \\
\zeta \sim & \xi^{-1/2} \\
\psi_0 \sim & \psi_2 \sim \omega^{-1/2}.
\end{align*}
\]

For an understanding of this scaling we recall that the angles \(\theta, \phi\) and \(\gamma\) are dimensionless, and that a time derivative on these degrees of freedom must scale as the excitation energy of the motion they generate. The scaling of \(\psi_1, i = 0, 2\), is such that \(\psi_i^2 \sim \omega\). The expectation
value $\zeta$ is associated with the emergent symmetry breaking and must scale as $\xi^{-1/2}$.

The Lagrangian of this effective theory is $L_{LO} + L_{NLO} + L_{NNLO}$ where the leading-order Lagrangian

$$L_{LO} = \frac{1}{2} \dot{\psi}_0^2 + \dot{\psi}_1^2 + 4\gamma^2 \dot{\psi}_1^2 + \frac{\omega_0^2}{2} \psi_0^2 - \frac{\omega_3^2}{4} \psi_1^2$$

(72)
describes vibrations at the high-energy scale $\omega$, the NLO correction

$$L_{NLO} = \frac{C_0}{2} \left( \dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta \right) + 4\gamma \dot{\psi}_1 \dot{\psi}_1 \cos \theta$$

(73)
couples rotations at the low-energy scale $\gamma$ to vibrations via the $\gamma$ degree of freedom, and the NNLO correction

$$L_{NNLO} = \frac{C_\beta}{2} \psi_0 \left( \dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta \right) + \frac{C_\gamma}{2} \psi_2 \left( \dot{\theta}^2 - \dot{\phi}^2 \sin^2 \theta \right) \cos 2\gamma$$

(74)
$$+ C_\gamma \psi_2 \dot{\phi} \sin 2\gamma \sin \theta,$$

is treated as a perturbation that scales as $\xi(\xi/\omega)^{1/2}$. According to the power counting [17], this implies

$$C_\beta \sim C_\gamma \sim \xi^{-1/2}.$$ 

(75)

Note that $\gamma$ is a cyclic variable of the LO and NLO Lagrangians. Thus, in these orders, the projection of the angular momentum $\mathbf{J}$ onto the intrinsic symmetry axis $p_\gamma$ is a conserved quantity in addition to the total angular momentum $\hat{\Omega}$. A Legendre transformation of the Lagrangian yields the Hamiltonian $H_{LO} + H_{NLO} + H_{NNLO}$. Here

$$H_{LO} = \frac{p_0^2}{2} + \frac{\omega_0^2}{2} \psi_0^2 + \frac{p_2^2}{4} + \frac{1}{4\gamma^2} \left( \frac{p_2}{2} \right)^2 + \frac{\omega_3^2}{4} \psi_1^2$$

(76)
is the Hamiltonian of a harmonic oscillator with frequency $\omega_0$ coupled to a two-dimensional harmonic oscillator with frequency $\omega_2$. The quantization is standard

$$\hat{p}_0 = -i\partial_{\psi_0}, \quad \hat{p}_2 = -i\partial_{\psi_2}, \quad \hat{p}_\gamma = -i\partial_{\gamma}.$$ 

(77)

We denote the eigenstates of the LO Hamiltonian as $|n_0 n_2 K/2\rangle$, with integer $n_0$ and $n_2$ and even $K$. Here $n_0$, $n_2$ and $K/2$ are the number of phonons of the modes $\psi_0$, $\psi_2$, and $\gamma$, respectively. These states can be written as $|n_0 n_2 K/2\rangle$, where $|n_0\rangle$ are the states of the harmonic oscillator, and $|n_2\rangle$ are the radial wave functions of the two-dimensional harmonic oscillator.

The NLO correction

$$H_{NLO} = \frac{1}{2C_0} p_\gamma^2 = \frac{1}{2C_0} (J^2 - p_\gamma^2)$$

(78)
is the Hamiltonian of a symmetric top [81]. Here, the momentum in the tangential plane is

$$\mathbf{p}_\Omega = e_\theta p_\theta + e_\phi p_\phi,$$ 

(79)

with

$$p_{\theta\gamma} \equiv \frac{p_\phi - p_\gamma \cos \theta}{\sin \theta}.$$ 

(80)

We also have

$$\mathbf{J} = e_r \times \mathbf{p}_\Omega + e_r p_\gamma.$$ 

(81)

This form of the angular momentum agrees with the intuition. In particular, rotations around the symmetry axis $e_r$ yield a contribution to the angular momentum in the direction of this axis.

The quantization

$$\hat{p}_\Omega = -ie_\theta \partial_{\theta} - ie_\phi \frac{\partial_{\phi} - \partial_{\gamma} \cos \theta}{\sin \theta}$$

(82)
is standard. In what follows, we denote the differential operator corresponding to the momentum operator $\mathbf{p}_\Omega$ also as

$$-i\nabla_{\Omega} \equiv \mathbf{p}_\Omega.$$ 

(83)

The Hamiltonian eigenvalue problem becomes

$$\hat{H}_{NLO} |IMK\rangle = \frac{1}{2C_0} \left( I(I + 1) - K^2 \right) |IMK\rangle.$$ 

(84)

Here, we continued to denote the eigenvalues of the total angular momentum by the quantum number $I$. The wave functions are linear combinations of Wigner $D$ functions, consistent with the positive $R$ parity of the system, i.e.

$$\langle \Omega | IM0 \rangle = \sqrt{\frac{2I + 1}{4\pi^2}} D_{I0}^I(\Omega, \gamma) = (-1)^m \sqrt{\frac{m}{\pi}} Y_{I-M}(\Omega).$$ 

(86)

The Wigner D-functions $D_{IMK}^I(\Omega)$ fulfill the relations [58]

$$\dot{J}_z D_{MK}^I(\Omega) = -MD_{MK}^I(\Omega)$$

(87)

The complete Hamiltonian at NLO can be diagonalized as

$$\left( \hat{H}_{LO} + \hat{H}_{NLO} \right) |n_0 n_2 IMK\rangle = \left[ \omega_0 \left( n_0 + \frac{1}{2} \right) \right.$$}

(88)
$$+ \frac{\omega_2}{2} \left( 2n_2 + \frac{K}{2} + 1 \right) + \frac{I(I+1)-K^2}{2C_0} \right] |n_0 n_2 IMK\rangle.$$

Thus, at this order, the spectrum consists of rotational bands with rotational constant $1/(2C_0)$ on top of harmonic vibrations. The vibrational phonons determine


the band head, and the ground-state band has no vibrational phonons. Because $0 \leq \gamma \leq \pi$, the wave function in $\gamma$ must exhibit periodic boundary conditions at the domain boundaries. This limits $K$ to even values. Historically, the band head with $(n_0 = 1, n_2 = 0, K = 0)$, and the band head with $(n_0 = 0, n_2 = 0, K = 2)$ determine the “$\beta$ band” and the “$\gamma$ band” respectively. In what follows, we continue to use these labels.

The NNLO correction to the Hamiltonian is

$$H_{NNLO} = -\frac{1}{2C_0} \left( C_\beta \psi_0 \mathbf{p}_\Omega^2 + C_\gamma \psi_2 \mathbf{p}_\Omega^T, \hat{\Gamma}_\Omega \mathbf{p}_\Omega \right).$$  

(89)

Here,

$$\hat{\Gamma} \equiv \begin{bmatrix} \cos 2\gamma & \sin 2\gamma \\ \sin 2\gamma & -\cos 2\gamma \end{bmatrix}$$

acts on vectors in the tangent plane. The operator $H_{NNLO}$ is off-diagonal in the eigenstates of the NLO Hamiltonian. Thus, it is only effective in second-order perturbation theory, i.e. at order $N^3LO$. At that order, corrections to the rotational constant (or the effective moment of inertia) linear in the number of phonons are corrections to the rotational constant (or the effective moment of inertia) linear in the number of phonons are introduced $^{29}$. These corrections arise due to omitted physics at the breakdown scale $\Lambda \sim 3$ MeV, where pair-breaking effects need to be taken into account $^{30}$. Thus, deviations from the harmonic behavior of the band heads is expected to scale as $\omega/\Lambda \approx 1/3$ for nuclei in the rare-earth and actinide regions. In the following Section, we will determine the LECs $C_\beta$ and $C_\gamma$ by fit to inter-band transitions.

VI. INTER-BAND TRANSITIONS

In this Section, we couple electromagnetic fields to the Hamiltonian, and focus on the inter-band transitions. These transitions are much fainter than the strong intra-band transitions discussed in the Sect. [V]. The transitions from the $\beta$ band to the ground-state band are not understood very well (see Ref. [11] for a recent review), because the traditional models overpredict them by up to an order of magnitude. Furthermore, these transitions vary by about two orders of magnitude in well-deformed and transitional nuclei $^{12}$. Below we will see that the transitions pose no challenge to the effective theory. For the LO description of these transitions, we only need to gauge the NNLO Hamiltonian.

A. Transition operators

The NNLO Hamiltonian of the previous section can be coupled to an electromagnetic field employing the gauging

$$\hat{\mathbf{p}}_{\Omega,\gamma} \rightarrow \hat{\mathbf{p}}_{\Omega,\gamma} - q \mathbf{A}_\Omega = -i \nabla_{\Omega,\gamma} - q \mathbf{A}_\Omega \; .$$

(91)

This is equivalent to

$$\mathbf{J} \rightarrow \mathbf{J} - q \mathbf{e}_r \times \mathbf{A}_\Omega \; ,$$

(92)

and in full analogy to Eq. (37).

Thus, the angles $\theta, \phi,$ and $\gamma$ are gauged. Assuming that the vibrational degrees of freedom $\psi_0$ and $\psi_2$ also carry a charge, we could also couple these to the radial component $\mathbf{A} \cdot \mathbf{e}_r$ to obtain a rotationally invariant and gauge-invariant Hamiltonian. As discussed below, the corresponding terms do not yield independent contributions for the intra-band transitions considered in this paper, and they are therefore neglected.

The gauging of the NNLO contribution $^{39}$ to the Hamiltonian

$$\hat{H}_{NNLO}^{(A)} = \frac{i q}{2C_0} \frac{C_\beta}{C_0} \psi_0 (A \cdot \nabla_{\Omega,\gamma} + \nabla_{\Omega,\gamma} \cdot A)$$

$$+ \frac{i q}{2C_0} \frac{C_\gamma}{C_0} \psi_2 (A^T \nabla_{\Omega,\gamma} + \nabla_{\Omega,\gamma}^T A)$$

(93)

induces LO inter-band transitions. As the inter-band transitions originate from a small correction to the Hamiltonian, they are expected to be an order of magnitude weaker (in the power counting) than the intra-band transitions. Gauging of the fields $\psi_0$ and $\psi_2$ would add terms $q_0 A_r p_0$ and $q_2 A_r p_2$ to the Hamiltonian. Here $A_r = A \cdot \mathbf{e}_r$. These operators do not yield transition matrix elements that differ from those of the operators in the Hamiltonian $^{39}$. They are therefore neglected.

Following Eq. (55) we compute the transition strength as

$$B(E\lambda, i \rightarrow f) = \frac{1}{2I_i + 1} \frac{\langle f || \hat{M} || i \rangle^2}{\omega} \; ,$$

(94)

where $\hat{M}(E\lambda) \equiv \hat{H}^{(A)}/(wA)$, $w \equiv |I_f(I_f + 1) - I_i(I_i + 1) + K_f^2|/2C_0$, and $k$ is the energy (or momentum) of the photon involved in the transition.

The LO inter-band $B(E2)$ values for transitions from the $\beta$ band to the ground band are

$$B(E2, i_\beta \rightarrow f_g) = \frac{C_\beta^2}{2C_0^2 \omega_0} \frac{q^2}{60} \left( C_{I_f,0}^{I_i,0} \right)^2 \; ,$$

(95)

while LO $B(E2)$ values for transitions from the $\gamma$ band to the ground band are

$$B(E2, i_\gamma \rightarrow f_g) = \frac{3C_\gamma^2}{2C_0^2 \omega_2} \frac{q^2}{60} \left( C_{I_f,22}^{I_i,22} \right)^2 \; .$$

(96)

We can generalize the definition of the quadrupole transition moments to

$$Q_{if} = \frac{B(E2, i \rightarrow f)}{(C_{I_f,K_f}^{I_i,K_i})^2} \; ,$$

(97)

then

$$Q_{i\beta f} = \frac{C_\beta^2}{2C_0^2 \omega_0} Q^2 \; , \quad Q_{i\gamma f} = \frac{3C_\gamma^2}{2C_0^2 \omega_2} Q^2 \; .$$

(98)
where $Q \equiv \sqrt{q^2/60}$. We note that the strengths of transitions from the $\beta$ band are similar to those of the $\gamma$ band for similarly sized LECs $C_\beta$ and $C_\gamma$.

We note that – within the effective theory – the intra-band transitions depend on the LECs $C_\beta$ and $C_\gamma$. We recall that these LECs enter at the Hamiltonian at NNLO as off-diagonal corrections to the Hamiltonian, which prevent us from adjusting them to spectra at this order. As more terms enter the Hamiltonian at NNLO, it seems attractive to determine $C_\beta$ and $C_\gamma$ instead by intra-band transitions. In what follows, we adjust these coefficients to the description of one inter-band transition from the respective band to the ground band. Other inter-band transitions then are predictions.

In the traditional collective models, no new parameters enter the computation of the inter-band transitions. As a result, these faint transitions are overpredicted substantially. For example, the inter-band $B(E2)$ values according to the adiabatic Bohr model are (See, e.g., Ref. [71])

$$B(E2, i_\beta \rightarrow f_\beta) = \frac{\xi}{2\omega_0} \left( \frac{Ze\beta_0}{A} \right)^2 (C_{i\beta 0})^2,$$

$$B(E2, i_\gamma \rightarrow f_\gamma) = \frac{2\xi}{\omega_2} \left( \frac{Ze\beta_0}{A} \right)^2 (C_{i\gamma 0})^2,$$

implying inter-band transitions from both the $\beta$ are only a factor two weaker than those from the $\gamma$ bands. Here, $\beta_0$ is a deformation parameter. Thus, the effective theory is richer in structure (through two additional parameters). This more complex structure will not only allow us to describe inter-band transitions much more accurately, it also is a consequence of a theory that is based on symmetry principles alone.

\section*{B. Comparison with experimental data}

We test the expressions (98) and (100) by confronting them to data for inter-band transitions in $^{166,168}$Er and $^{154}$Sm. These isotopes of erbium are considered good rotors, while the samarium isotope is between rotors and transitional nuclei.

For $^{168}$Er, the relevant energies are $\xi \approx 79.8$ keV, $\omega_0 \approx 1217.2$ keV, and $\omega_2/2 \approx 821.2$ keV. In the spirit of the theory, all constants were fitted to low-energy data. Thus, the effective quadrupole moment was fitted via the $2^+_g \rightarrow 0^+_g$ transition, while the constants $C_\beta$ and $C_\gamma$ where fitted via $2^+_\beta \rightarrow 0^+_g$ and $2^+_\gamma \rightarrow 2^+_g$ transitions respectively. For the $2^+_\beta \rightarrow 0^+_g$ transition we found conflicting experimental data for $B(E2)$ values, namely 0.00044 e$^2$h$^2$ from Härtelein et al. [82] and an approximately five times larger value of 0.0022 e$^2$h$^2$ from Lehmann et al. [83]. For the former, $C_\beta \approx 0.036$ keV$^{-1/2}$, while for the latter $C_\beta \approx 0.081$ keV$^{-1/2}$. The LEC governing the decay from the $\gamma$ band is $C_\gamma \approx 0.203$ keV$^{-1/2}$. The values of all these LECs are natural in size when compared to the scale $\xi^{-1/2} \approx 0.112$ keV$^{-1/2}$. However, the ratio $C_\gamma/C_\beta \approx 5.6$ ($C_\gamma/C_\beta \approx 2.5$) for the data from Ref. [82] (Ref. [83]), and the former ratio is possibly at the limit of what one would assume for coupling constants of a natural size. Clearly, more precise data for transitions between the $\beta$ and ground bands is required to determine the size of $C_\beta$. Table III shows experimental and theoretical $B(E2)$ values for transitions within the ground-state band and inter-band transitions in $^{168}$Er. Overall, the effective theory describes the data well. The theoretical uncertainties presented in Table III for transitions in the ground-state band are based on the discussion in Subsection IV B. However, for the uncertainties of transitions from the $\beta$ band or the $\gamma$ band, we employed more conservative uncertainty estimates based on the larger ratio $(\omega/\Lambda)^2 \approx 0.25$ that is due to the proximity of the breakdown scale $\Lambda$.

\begin{table}[h]
\centering
\begin{tabular}{cccc}
\hline
$i \rightarrow f$ & $B(E2)_{\exp}$ & $B(E2)_{\ET}$ & $B(E2)_{\BH}$ \\
\hline
$2^+_g \rightarrow 0^+_g$ & 1.173(22) & 1.173 & 1.173 \\
$4^+_g \rightarrow 2^+_g$ & 1.756(49) & 1.676(12) & 1.677 \\
$6^+_g \rightarrow 4^+_g$ & 2.335(99) & 1.846(30) & 1.842 \\
$8^+_g \rightarrow 6^+_g$ & 1.949(72) & 1.932(57) & 1.935 \\
$2^+_g \rightarrow 0^+_g$ & 0.0258(9) & 0.0309(77) & 0.1126 \\
$2^+_g \rightarrow 2^+_g$ & 0.0442(38) & 0.0442 & 0.1610 \\
$2^+_g \rightarrow 4^+_g$ & 0.0034(2) & 0.0022(6) & 0.0080 \\
$2^+_g \rightarrow 0^+_g$ & 0.0022 & 0.0022 & 0.0387 \\
$2^+_g \rightarrow 2^+_g$ & 0.0027 & 0.0031(8) & 0.0553 \\
$2^+_g \rightarrow 4^+_g$ & 0.0121 & 0.0057(14) & 0.0995 \\
$2^+_g \rightarrow 0^+_g$ & 0.00044(5) & 0.00044(5) & 0.0387 \\
$2^+_g \rightarrow 2^+_g$ & 0.00063(16) & 0.00063(16) & 0.0553 \\
$2^+_g \rightarrow 4^+_g$ & 0.00113(38) & 0.00113(38) & 0.0995 \\
\hline
\end{tabular}
\caption{Transition strength for $^{168}$Er in units of e$^2$h$^2$. Experimental transitions strengths $B(E2)_{\exp}$ are compared to theoretical results $B(E2)_{\ET}$ from the effective theory and $B(E2)_{\BH}$ from the adiabatic Bohr Hamiltonian. Experimental values are taken from [70] unless otherwise specified. Values for the adiabatic Bohr Hamiltonian are taken from Rowe and Wood [71]. Parenthesis denote experimental errors and theoretical uncertainty estimates.}
\end{table}

For $^{166}$Er, the energy scales are $\xi \approx 80.6$ keV, $\omega_0 \approx 1460$ keV and $\omega_2/2 \approx 785.9$ keV. This yields $C_\beta \approx 0.111$ keV$^{-1/2}$ and $C_\gamma \approx 0.213$ keV$^{-1/2}$, and both values are natural in size when compared to $\xi^{-1/2} \approx 0.111$ keV$^{-1/2}$. Once again, more precise experimental $B(E2)$ values for transitions between the $\beta$ and ground bands would be valuable. Table IV shows experimental and theoretical $B(E2)$ values for intra-band and inter-band transitions in this nucleus. Theoretical uncertainties are given as discussed for 168Er. The experimental $B(E2)$ value for the $2^+_g \rightarrow 4^+_g$ transition is too large (one order of magnitude larger than decays from the $\gamma$ band to the ground band) to be understood within the effective
TABLE IV. Same as Table III but for \textsuperscript{166}Er. Experimental values are taken from \cite{74} unless otherwise specified.

| \( i \rightarrow f \) | \( B(E2)_{\text{exp}} \) | \( B(E2)_{\text{ET}} \) | \( B(E2)_{\text{CBS}} \) | \( B(E2)_{\text{BIH}} \) |
|-----------------|----------------|----------------|----------------|----------------|
| \( 2^+ \rightarrow 0^+ \) | 1.166(1) \( ^{\text{ab}} \) | 1.166 \( ^{\text{ab}} \) | 1.166 \( ^{\text{ab}} \) |
| \( 4^+ \rightarrow 2^+ \) | 1.637(10) | 1.666(4) | 1.667 |
| \( 6^+ \rightarrow 4^+ \) | 1.941(14) | 1.853(10) | 1.831 |
| \( 8^+ \rightarrow 6^+ \) | 1.957(14) | 1.920(20) | 1.924 |
| \( 2^+ \rightarrow 0^+ \) | 0.0285(12) | 0.0370(93) | 0.1189 |
| \( 2^+ \rightarrow 2^+ \) | 0.0529(33) | 0.0529\( ^{\text{b}} \) | 0.1701 |
| \( 2^+ \rightarrow 4^+ \) | 0.0042(2) | 0.0026(7) | 0.0085 |
| \( 2^+ \rightarrow 0^+ \) | 0.0036(4) | 0.0036\( ^{\text{b}} \) | 0.0322 |
| \( 2^+ \rightarrow 2^+ \) | 0.0051(13) | 0.0060 |
| \( 2^+ \rightarrow 4^+ \) | 0.2113(325) | 0.0093(23) | 0.0828 |

\( ^{\text{a}} \) From Raman et al. \cite{68}.
\( ^{\text{b}} \) Values employed to adjust the LECs of the effective theory.

We let us also attempt to describe a less rigid rotor. The region around \textsuperscript{152}Sm has been well studied \cite{14,15,84}, and absolute \( B(E2) \) values for some inter-band transitions in \textsuperscript{154}Sm were measured recently \cite{85}. For \textsuperscript{154}Sm, the LECs related to inter-band transitions are \( C_\beta \approx 0.092 \text{ keV}^{-1/2} \) and \( C_\gamma \approx 0.181 \text{ keV}^{-1/2} \), where \( C_\beta \) was fitted via the \( 2^+ \rightarrow 2^+ \) transition. Both values are natural in size when compared to \( \xi^{-1/2} \approx 0.110 \text{ keV}^{-1/2} \). Table \( \text{V} \) shows our LO results for this nucleus. The theoretical uncertainties are computed as discussed for \textsuperscript{168}Er.

TABLE V. Same as Table III but for \textsuperscript{154}Sm. Theoretical results from the confined \( \beta \) soft (CBS) model \cite{63}, taken from Ref. \cite{87}, are also included. Experimental values are taken from \cite{87} and \cite{88} for intra-band and inter-band transitions, respectively.

| \( i \rightarrow f \) | \( B(E2)_{\text{exp}} \) | \( B(E2)_{\text{ET}} \) | \( B(E2)_{\text{CBS}} \) | \( B(E2)_{\text{BIH}} \) |
|-----------------|----------------|----------------|----------------|----------------|
| \( 2^+ \rightarrow 0^+ \) | 0.863(5) | 0.863(5) \( ^{\text{a}} \) | 0.853 | 0.863 |
| \( 4^+ \rightarrow 2^+ \) | 1.201(29) | 1.233(41) | 1.231 | 1.234 |
| \( 6^+ \rightarrow 4^+ \) | 1.417(39) | 1.358(101) | 1.378 | 1.355 |
| \( 8^+ \rightarrow 6^+ \) | 1.564(83) | 1.421(189) | 1.471 | 1.424 |
| \( 2^+ \rightarrow 0^+ \) | 0.0093(10) | 0.0110(28) | 0.0492 |
| \( 2^+ \rightarrow 2^+ \) | 0.0157(15) | 0.0157\( ^{\text{a}} \) | 0.0703 |
| \( 2^+ \rightarrow 4^+ \) | 0.0018(2) | 0.0008(2) | 0.0050 |
| \( 2^+ \rightarrow 0^+ \) | 0.0016(2) | 0.0025(6) | 0.0024 | 0.0319 |
| \( 2^+ \rightarrow 2^+ \) | 0.0035(4) | 0.0035\( ^{\text{a}} \) | 0.0069 | 0.0456 |
| \( 2^+ \rightarrow 4^+ \) | 0.0065(7) | 0.0063(16) | 0.0348 | 0.0821 |

\( ^{\text{a}} \) Values employed to adjust the LECs of the effective theory.

VII. SUMMARY

We studied \( E2 \) transitions in deformed nuclei within a model-independent approach based on an effective theory. The effective theory is based on the emergent symmetry breaking of rotational symmetry to axial symmetry. Electromagnetic transitions result from gauging of the Hamiltonian, and from higher-order non-minimal couplings that are consistent with gauge invariance and the symmetry of the system under consideration. The estimate of theoretical uncertainties is one of the highlights of the effective theory approach.

Homonuclear molecules provide us with an ideal test case because they possess a very large separation of scale and therefore exhibit only small corrections to the rigid-rotor limit. The effective theory describes \( E2 \) transitions in the diatomic molecules \text{N}_2 and \text{H}_2 very well, and deviations are within the theoretical uncertainties.

The effective theory describes \( B(E2) \) transitions in the ground-state bands of well-deformed nuclei at leading order, and more precise experimental data are necessary to probe subleading effects. Our model-independent results also suggest that some low-lying transitions in these nuclei would probably merit a more precise re-measurement, because they can not be easily understood within the effective theory. For transitional nuclei, the existing data are sufficiently precise to probe the effective theory at next-to-leading order. Here, data and theoretical results are consistent within the theoretical uncertainties.

The effective theory also suggests that the electromagnetic structure of deformed nuclei is more complex than the collective models assume, regarding both the Hamiltonian and the transition operator. The magnitude of the faint inter-band transitions is described correctly within the effective theory, proposing a solution to a long-standing problem. These results also cast some doubt on the usage of the quadrupole operator to describe faint electromagnetic transitions, as this approach seems to be...
limited to the strong (leading order) transitions between states within a band.

This work shows that the effective theory for deformed nuclei reproduces the traditional collective models regarding leading-order aspects (spectra and transitions) of deformed nuclei. In contrast to the models, however, the effective theory also accounts for finer details, and it provides us with theoretical error estimates. This increased precision, however, comes at the expense of additional parameters. We would hope that the results presented in this work might stimulate more precise measurements of electromagnetic transitions in deformed nuclei.

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