Parametric analysis of a fractional-order Newton-Leipnik system

Yuan Kang\textsuperscript{a*}, Kuang-Tai Lin\textsuperscript{a}, Juhn-Horng Chen\textsuperscript{b}, Long-Jye Sheu\textsuperscript{b}, and Hsien-Keng Chen\textsuperscript{c}

\textsuperscript{a} Department of Mechanical Engineering, Chun Yuan Christian University, 200 Chung Pei Rd., Chung Li 32023, Taiwan, R.O.C.

\textsuperscript{b} Department of Mechanical Engineering, Chung Hua University, Hsinchu 300, Taiwan, R.O.C.

\textsuperscript{c} Department of Mechanical Engineering, Hsiuping Institute of Technology, Taichung 41280, Taiwan, R.O.C.

E-mail address: yk@cycu.edu.tw

Abstract: In this paper, the influences of parameters on the dynamics of a fractional-order Newton-Leipnik system are numerically studied. The ranges of the parameters used in this study are relatively broad. The system displays comprehensive dynamic behaviors, such as fixed points, periodic motion (including periodic-3 motion), chaotic motion, and transient chaos. A period-doubling route to chaos is also found.

1. Introduction

Scientists and mathematicians have been working on problems of rigid body motion for over two centuries, and it has many practical engineering applications such as gyroscopes, satellites, spacecraft, and rockets. However, most existing analytical solutions to the general problems of rigid body motion are still far from complete due to the existence of chaos phenomena. Therefore, the research on chaos has becomes one of the hot issues in the field of nonlinear science. Chaos controls have great prospects for application in many fields, especially in electricity, communications, information science, medical sciences, etc. [1-5] Thus, choosing system parameters in engineering systems is very important. In 1981, Leipnik and Newton [6] found two strange attractors in rigid body motion which they presented in a pioneering report on the concept of chaotic motion in gyros. Very recently, this system was termed the “Newton-Leipnik system” by Wang and Tian [7]. They also studied the bifurcation of the Newton-Leipnik system and controlled the system using a simple linear controller. Since Leipnik and Newton’s work, the chaotic dynamics of rigid body motion have been intensively studied by many scientists [8-12]. Recently, in a study of the anti-control of chaos in rigid body motion, Chen and Lee [13] introduced a new chaotic system that can generate a two-scroll chaotic attractor. Richter [14] investigated the stability and chaos control of the Newton-Leipnik system with a static nonlinear feedback law based on Lyapunov’s direct method.

\textsuperscript{*} To whom any correspondence should be addressed.
Fractional calculus is one of the classical topics of mathematics. Applications of fractional-order systems are seen in many areas of science, including engineering systems [15-17]. In recent years, more attention has been focused on chaotic attractors in fractional-order systems [18-30]. Very recently, Sheu et al. [31] also studied the dynamics of a fractional-order Newton-Leipnik system, and the lowest order for this system to yield chaos was 2.82. In this paper in a further investigation of this system, the dynamics of the fractional-order Newton-Leipnik system was numerically studied by changing the system parameters with the fractional order specified as 2.82.

![Phase diagrams for $a=0.4$; (a) $b=0.15$ and (b) $b=0.43$.](image)

![Dynamic behaviors at $a=0.4$; and $b=0.3$ (a) Phase diagrams; and (b) Time history of $x(t)$](image)

2. The fractional-order Newton-Leipnik system
The Newton-Leipnik system [6] has been presented as
\[
\begin{align*}
\dot{x} &= -ax + y + 10yz \\
\dot{y} &= -x - 0.4y + 5xz \\
\dot{z} &= bz - 5xy
\end{align*}
\] (1)
where $x$, $y$, and $z$ are state variables and $a$ and $b$ are positive parameters. This is a chaotic system with two strange attractors, when $(a, b) = (0.4, 0.175)$, with the initial states of $(0.349, 0, -0.16)$ and $(0.349, 0, -0.18)$.

A fractional-order Newton-Leipnik system can be modeled by
\[
\begin{align*}
D^p x &= -ax + y + 10yz \\
D^q y &= -x - 0.4y + 5xz \\
D^r z &= bz - 5xy
\end{align*}
\] (2)
where $0 < p, q, r \leq 1$, and $D^\alpha$ is the Caputo fractional derivative operator of order $\alpha$ [32].

3. Parametric analysis
There are several numerical techniques for solving fractional-order differential equations [33-35]. Based on an algorithm of the scheme developed by Diethelm et al. [33], an efficient method for solving the Newton-Leipnik system with Caputo derivatives is the predictor-correctors scheme, which is a generalization of the classical Adams-Bashforth-Moulton integrator that is well known for being a numerical solution of first-order problems. In the following simulations, these methods were used to integrate Eq. (2), and the details regarding the variation of parameters are described below.

![Fig. 3. Phase diagrams at a=0.4; (a) b=0.405; (b) b=0.4; and (c) b=0.3915.](image)

![Fig. 4. Phase diagrams a=0.4; (a) b=0.45; and (b) b=1.0.](image)

![Fig. 5. Phase diagrams a=0.4; (a) b=1.2; and (b) b=2.6.](image)

In an analysis of the influences of system parameters $a$ and $b$, the orders of the system are specified as $p = q = r = 0.94$, with the initial state (0.349, 0, -0.18) throughout the paper.

(1) It is fixed that $a = 0.4$, and $b$ is varied. The system is calculated numerically versus $b \in [0.1, 4.3]$ with the incremental value of 0.01. These simulation results demonstrate the comprehensive dynamic behaviors of the system. It was found that when $0.128 \leq b \leq 0.39$, except when $b = 0.30$, and $0.41 \leq b \leq 0.43$, chaotic behaviors were observed in the system, and the phase diagrams of the $x$-$y$ plane for $b = 0.15$, and 0.43 are also plotted in Fig. 1a and b, respectively. When $b = 0.30$, the system displays completely different motion named period-3, and the phase diagrams of the $x$-$y$ plane and the time history of $x(t)$ are plotted in Fig. 2a and b, respectively. Equation (2) is periodic when $0.39 \leq b < 0.41$. Furthermore, the system can display period-1, -2, and -4 motions as shown in Fig. 3a-c. Thus, Fig. 3 identifies a period-doubling route to chaos. It was found that when $0.44 \leq b \leq 1.1$, periodic behaviors were observed in the system, and the phase diagrams of the $x$-$y$ plane are plotted in Fig. 4a and b, respectively. Attractors seem to rotate by approximately $180^\circ$. In other words, another strange attractor is also present in this case. When $1.2 \leq b \leq 2.8$, the system exhibits different behaviors. The phase diagrams of the $x$-$y$ plane are shown in Fig. 5a and b for $b = 1.2$ and 2.6, respectively, which shows...
two attractors rotating by nearly 180°. From Fig. 6a, when $2.9 \leq b \leq 3.4$, the orbit of the attractor seems to rotate by 90° compared with $b \approx 2.6$. From Fig. 6b, when $3.5 \leq b \leq 4.3$, the attractor rotates again by 180°. When $0.1 \leq b \leq 0.127$, Eq. (2) is stabilized to a fixed point as shown in Fig. 7a and b.

![Fig. 6. Dynamic behaviors at $a=0.4$; (a) $b=2.9$ and (b) $b=3.5$](image1)

(2) It is fixed that $b = 0.175$, and $a$ is varied. The system is calculated numerically versus $a \in [0.1, 5.0]$ with the incremental value of 0.01. Equation (2) is stabilized to a fixed point when $0.1 \leq a \leq 0.24$, and $0.496 \leq a \leq 5.0$, as shown in Fig. 8a and b, respectively. When $0.25 \leq a \leq 0.46$, it is evident in Fig. 9a and b that Eq. (2) displays chaotic motion. When $0.465 \leq a \leq 0.495$, the observed phenomenon of the system demonstrates transient chaotic behavior as shown in Fig. 10a and b, and the responses eventually converge to a fixed point. In this case, the system undergoes a heteroclinic bifurcation.

4. Conclusions

The influences of the system parameters on a fractional-order Newton-Leipnik system were studied in this paper. The order used for this system was fixed at 2.82. In addition, due to the relatively broad range of parameters, the comprehensive dynamic behaviors, such as fixed points, periodic motions (including period-3), chaotic motions, and transient chaos could be observed, and the variations in the orbit of attractors were also exhibited with changes in the parametric values. Meanwhile, it is important to mention that period-3 has also been observed through time history. Therefore, it can be concluded that selecting appropriate values of the system parameters can be used to restrain chaos or generate chaos.
Fig. 9. Dynamic behaviors at $b=0.175$; (a) $a=0.25$; and (b) $a=0.46$.

Fig. 10. Dynamic behaviors at $b=0.175$; and $a=0.48$ (a) Phase diagrams; and (b) Time history of $x(t)$.

Acknowledgements
This research was supported by the National Science Council of R.O.C. under grant number NSC94-2212-E-033-018. The authors also thank the R&D Center for Membrane Technology of CYCU and the Center-of-Excellence Program supported by the Ministry of Education of the R.O.C.

References
[1] Liu S and Chen G 2004 Chaos Soliton. Fract. 22 35
[2] Wang X and Tian L 2004 Chaos Soliton. Fract. 21193
[3] Yu Y and Zhang S 2003 Chaos Soliton. Fract. 15 897
[4] Liu F, Ren y, Shan X and Qiu Z 2002 Chaos Soliton. Fract. 13 723
[5] Lü J, Zhou T and Zhang S 2002 Chaos Soliton. Fract. 14 529
[6] Leipnik RB and Newton TA 1981 Phys Lett A 86 63
[7] Wang X and Tian L 2005 Chaos Soliton. Fract. 27 31
[8] Ge ZM, Chen HK and Chen HH 1996 J Sound Vib. 198 131
[9] Ge ZM and Chen HK 1996 Jpn J Appl Phys 35 1954
[10] Tong X and Mrad N 2001 Trans ASME J Appl Mech 68 681
[11] Chen HK 2002 J Sound Vib. 255 719
[12] Chen HK and Lin TN 2003 ImechE J Mech Eng Sci 217 331
[13] Chen HK and Lee CI 2004 Chaos Soliton. Fract. 21 957
[14] Richter H 2002 Phys Lett A 300 182
[15] Podlubny I 1999 Fractional differential equations (New York: Academic Press)
[16] Hilfer R 2001 Applications of fractional calculus in physics (NJ: World Scientific)
[17] Jumarie G 2007 Chaos Soliton. Fract. 23 969
[18] Hartley TT, Lorenzo CF and Qammer HK 1995 IEEE Trans CAS-I 42 485
[19] Arena P, Caponetto R, Fortuna L and Porto D 1995 Chaos in a fractional order Duffing system In: Proc. ECCTD, Budapest pp.1259-1262
[20] Ahmad W, El-Khayati R and El-Wakli A 2001 Electr Lett 3 1110
[21] Ahmad WM and Sprott JC 2003 Chaos Soliton. Fract. 16 339
[22] Grigorenko I and Grigorenko E 2003 Phys Rev Lett 91 034101
[23] Arena P, Fortuna L and Porto D 2000 Phys Rev E 61 776
[24] Sheu LJ, Chen HK, Chen JH and Tam LM 2007 Chaos Soliton. Fract. 31 1203
[25] Ge ZM and Hsu MY 2007 Chaos Soliton. Fract. 33 1711
[26] Ge ZM and Jhuang WR 2007 *Chaos Soliton. Fract.* **33** 270
[27] Ge ZM and Zhang AR 2007 *Chaos Soliton. Fract.* **32** 1791
[28] Yan JP and Li CP 2007 *Chaos Soliton. Fract.* **32** 725
[29] Li CP and Yan JP 2007 *Chaos Soliton. Fract.* **32** 751
[30] Ge ZM and Yi CX 2007 *Chaos Soliton. Fract.* **32** 42
[31] Sheu LJ, Chen HK, Chen JH, Tam LM, Chen WC, Lin KT and Kang Y *Chaos Soliton. Fract.* Doi:10.1016/j.chaos.2006.06.013.
[32] Caputo M 1967 *Geophys J R Astron Soc* **13** 529
[33] Diethelm K, Ford NJ and Freed AD 2002 *Nonlinear Dyn* **29** 3
[34] Momani S and Odibat Z 2007 *Chaos Soliton. Fract.* **31** 1248
[35] Odibat ZM and Momani S 2006 *Int J Nonlinear Sci* **7** 27