Dynamical instability of polytropic spheres in spacetimes with a cosmological constant

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ABSTRACT
The dynamical instability of relativistic polytropic spheres, embedded in a spacetime with a repulsive cosmological constant, is studied in the framework of general relativity. We apply the methods used in our preceding paper to study the trapping polytropic spheres with $\Lambda = 0$, namely, the critical point method and the infinitesimal and adiabatic radial perturbation method developed by Chandrasekhar. We compute numerically the critical adiabatic index, as a function of the parameter $\sigma = p_c/(\rho_c c^2)$, for several values of the cosmological parameter $\Lambda$ giving the ratio of the vacuum energy density to the central energy density of the polytrope. We also determine the critical values for the parameter $\sigma_c$, for the onset of instability, by using both approaches. We found that for large values of the parameter $\lambda$, the differences between the values of $\sigma_c$, calculated by the critical point method differ from those obtained via the radial perturbations method. Our results, given by both applied methods, indicate that large values of the cosmological parameter $\lambda$ have relevant effects on the dynamical stability of the polytropic configurations.

Key words: stars: interiors – stars: oscillations – cosmology: dark energy – methods: numerical

1 INTRODUCTION
There are indications that contrary to the inflationary era, in the recent era the dark energy could correspond to the vacuum energy related to a repulsive cosmological constant $\Lambda > 0$ implying important consequences in astrophysical phenomena (Stuchlík 2005; Stuchlík et al. 2020). The estimate of the so-called relic cosmological constant, governing the acceleration of the recent stage of the Universe expansion, reads $\Lambda \sim 10^{-52}$ m$^{-2}$ (Ade et al. 2016). Although the relic physical phenomena could be quite significant, being limited by the so-called static radius introduced by Stuchlík (1983) and discussed by Stuchlík (1984); Stuchlík & Hledík (1999); Stuchlík & Slaný (2004); Faraoni (2016); Stuchlík et al. (2018). The static radius represents an upper limit on the existence of both Keplerian (Stuchlík et al. 2005) and toroidal fluids (Stuchlík et al. 2000; Slaný & Stuchlík 2005; Stuchlík & Kovář 2008; Stuchlík et al. 2009), accretion disks, the limit on gravitationally bound galaxy systems (Stuchlík & Schee 2011; Stuchlík & Schee 2012; Schee et al. 2013), and even the limit on gravitationally bound polytropic configurations that could represent a model of dark matter halos (Stuchlík et al. 2016; Novotný et al. 2017; Stuchlík et al. 2017). Test fields around black holes in spacetimes with $\Lambda > 0$ were treated by Konoplya & Zhidenko (2011); Toshmatov & Stuchlík (2017), indicating possible instabilities.

In spacetimes with a cosmological constant $\Lambda$, the interior Schwarzschild solution with uniform distribution of energy density was found by Stuchlík (2000) for star-like configurations, and extended to more general situations by Boehmer (2004a,b). The role of the cosmological constant on the so-called ‘$\Lambda$-electrovacuum’ solutions was studied by Posada & Batic (2014).

In spacetimes with a positive cosmological constant, the polytropic spheres represented by a polytropic index $n$, a parameter $\sigma$, giving the ratio of pressure and energy density at the centre, and vacuum constant index $\lambda$, giving the ratio of the vacuum energy to the central energy density, were exhaustively discussed by Stuchlík et al. (2016); Novotný et al. (2017). It has been shown that in some special cases of polytropes with sufficiently large values of the polytropic index $n$, extremely extended configurations representing a dark matter halo could have gravitationally unstable central parts that could collapse leading to the formation of a supermassive black hole (Stuchlík et al. 2017).

The polytropic spheres are well known models of compact objects, as they represent extremely dense nuclear matter inside neutron or quark stars. For example, they describe the fluid configurations constituted from relativistic ($n = 4/3$) and non-relativistic ($n = 5/3$) Fermi gas (Shapiro & Teukolsky 1983), considered as basic approximations of neutron stars matter. Of course, in realistic models describing the neutron stars interior, the equations of state (EoS) of nuclear matter are considered. However, in a recently developed approach, such realistic EoS are represented by sequences of polytropes with appropriately tuned parameters (Alvarez-Castillo & Blaschke 2017).
The stability of the polytropic spheres can be addressed from two different approaches. One of them is related to the energetic considerations, or critical point method (Zel’dovich & Novikov 1973; Shapiro & Teukolsky 1983), developed by Tooper (1964). A second approach deals with the dynamical theory of infinitesimal radial oscillations pioneered in a seminal paper by Chandrasekhar (1964). Using his ‘pulsation equation’, Chandrasekhar established the conditions of stability, against radial oscillations, for homogenous stars and polytropic spheres. The main conclusion of this study is that the critical adiabatic index \( \gamma_{cr} \), for the onset of instability, increases due to relativistic effects from the Newtonian value \( \gamma = 4/3 \). The radial oscillations method has been widely used in different contexts (Gleiser & Watkins 1989; Moustakidis 2017; Posada & Chirenti 2019). The role of the cosmological constant on the radial stability of the uniform energy density stars, using Chandrasekhar’s method, was studied by Stuchlík & Hledík (2005); Boehmer & Harko (2005). These authors concluded that a large value of the vacuum constant index \( \lambda \) increases significantly the critical adiabatic index from its value with \( \lambda = 0 \).

The purpose of this paper is to study in detail the role of the cosmological constant in the stability of the polytropic fluid spheres against radial oscillations, that is expected to be relevant for extremely extended non-compact configurations modeling galactic dark matter halos (Stuchlík et al. 2016). For that purpose we are reconsidering the analysis carried out by Stuchlík & Hledík (2005) in two ways: first, we will apply the methods introduced in our preceding paper (Hladik et al. 2020) to study the stability of polytropic spheres with \( \Lambda = 0 \), namely, the shooting method and trial functions to solve the Sturm–Liouville equation; and the critical point method based on the energetic approach. Second, we will extend the analysis to a larger family of spheres in the range of polytropic indexes \( 0.5 \leq n \leq 3 \), for several values of the vacuum constant index \( \lambda \in [10^{-3}, 10^{-1}] \).

The paper is organized as follows. In Sect. 2 we review the equations of structure of the relativistic polytropes in the presence of a cosmological constant. In Sect. 3 we summarise the general properties and gravitational energy for polytropic spheres with \( \Lambda \). In Sect. 4 Chandrasekhar’s procedure and the associated Sturm–Liouville eigenvalue problem, including the cosmological term, are outlined. In Sect. 5 we present our methods and results. In Sect. 6 we discuss our conclusions.

## 2 STRUCTURE EQUATIONS OF RELATIVISTIC POLYTROPIC SPHERES WITH A COSMOLOGICAL CONSTANT

We will consider throughout the paper perturbations which preserve spherical symmetry. This condition guarantees that motions along the radial direction will ensue. Thus, our starting point is a spherically symmetric spacetime in the standard Schwarzschild coordinates

\[
\text{d} s^2 = -e^{2\Phi} (c \text{d} t)^2 + e^{2\Psi} (\text{d} r)^2 + r^2 (\text{d} \theta^2 + \sin^2 \theta \text{d} \phi^2),
\]

where \( \Phi(r, t) \) and \( \Psi(r, t) \) are functions of \( t \) and the radial coordinate \( r \). The energy-momentum tensor for a spherically symmetric configuration takes the form

\[
T_{\mu}^\nu = (\epsilon + p) u_\mu u^\nu + p g_\mu^\nu,
\]

where \( \epsilon = \rho c^2 \) is the energy density (written as the product of the mass density \( \rho \) times the speed of light squared), \( p \) is the fluid pressure and \( u^\mu = \text{d} x^\mu / \text{d} \tau \) is its four-velocity.

Following our preceding paper (Hladik et al. 2020), we consider the models of static polytropic fluid spheres proposed by Tooper (1964), which are governed by the EoS

\[
p = K \rho^{1+(1/n)},
\]

where \( n \) is the polytropic index and \( K \) is a constant related to the characteristics of a specific configuration. It is conventional to introduce the parameter

\[
\sigma = \frac{p_c}{\rho_c c^2},
\]

which denotes the ratio of pressure to energy density at the centre of the configuration. The radial profiles of the mass density and pressure of the polytropic spheres are given by the relations

\[
\rho = \rho_c \theta^n, \quad p = p_c \theta^ {n+1},
\]

where \( \theta(x) \) is function of the dimensionless radius

\[
x = \frac{r}{L}, \quad L = \left[ \frac{\sigma(n+1)c^2}{4\pi G \rho_c} \right]^{\frac{1}{2}}.
\]

Here \( L \) is the characteristic length scale of the polytropic sphere and it is determined by the polytropic index \( n \), the parameter \( \sigma \), and the central density \( \rho_c \). From (5) we obtain immediately the boundary condition \( \theta(r = 0) = 1 \).

We consider a static configuration in equilibrium, immersed in the solution of Einstein’s equations for this problem for \( (t,t) \) and \( (r,r) \), which in presence of a cosmological constant \( \Lambda \) are given by (Tolman 1934)

\[
d \frac{\text{d} e^{2\Psi}}{\text{d} r} = 1 - \frac{8\pi G}{c^2} T_{00} - \Lambda r^2,
\]

\[
2\frac{\text{d} \Phi}{\text{d} r} = \frac{e^{2\Psi}}{r} - \frac{8\pi G}{c^2} T_{11} r - \Lambda r.
\]

Equation (7) can be recast into the form

\[
e^{2\Psi} = \left[ 1 - \frac{2Gm(r)}{c^2 r} - \frac{\Lambda}{3} r^2 \right]^{-1},
\]

where

\[
m(r) = 4\pi \int_0^r \rho(r) r^2 \text{d} r
\]

is the mass inside the radius \( r \). Using (9) and (10), we transform (8) into

\[
\frac{d \Phi}{d r} = \frac{(G/c^2)m(r) - \frac{A}{3} r^2 + (4\pi G/c^4)pr^3}{1 - \frac{2Gm(r)}{c^2 r} - \frac{\Lambda}{3} r^2}.
\]

Using the energy-momentum \( \rho \) conservation \( \dot{T}^\mu_{\nu;\nu} = 0 \), we can write (11) as a relation between pressure and energy density in the form (Boehmer & Harko 2005; Stuchlík 2000)

\[
\frac{dp}{dr} = - (\epsilon + p) \frac{(G/c^2)m(r) + (4\pi G/c^4)pr^3 - \frac{A}{3} r^2}{1 - \frac{2Gm(r)}{c^2 r} - \frac{\Lambda}{3} r^2},
\]

which is the Tolman–Oppenheimer–Volkoff (TOV) equation, including the effect of the cosmological constant.

Once an EoS \( p = p(\rho) \) is given, equations (10) and (12) can be integrated subject to the boundary conditions \( m(0) = 0 \) and \( p(R) = 0 \), where \( R \) is the radius of the star. In the exterior of the configuration (10) gives the total mass \( M = m(R) \), and the spacetime is described by the Kottler metric (Kottler 1918). The
mass relation (10) can be written in terms of the parameter \( \sigma \), as follows

\[
\sigma(n+1)\,d\theta = -(\sigma \theta + 1)\,d\Phi.
\]

Solving (13) with the condition that the interior and exterior metrics are smoothly matched at the surface \( r = R \), we have

\[
e^{2\Phi} = (1 + \sigma \theta)^{-2(n+1)} \left( 1 - \frac{2GM}{c^2 R} - \frac{\Lambda}{3} R^2 \right),
\]

which is a function of \( \theta \) and \( \sigma \). In order to find a relation for the function \( \theta \), we rewrite (13) in the form

\[
\frac{d\Phi}{dr} = \frac{\sigma(n+1)}{1 + \sigma \theta} \frac{d\theta}{dr}.
\]

Substituting (15) into (8) we have

\[
\frac{\sigma(n+1)\,r}{1 + \sigma \theta} \left[ 1 - \frac{2Gm(r)}{c^2 r} - \frac{\Lambda}{3} r^2 \right] \left( \frac{d\theta}{dr} \right) + \frac{Gm(r)}{c^2 r} - \frac{\Lambda}{3} r^2 = -G\sigma \theta \left( \frac{dm}{dr} \right).
\]

Similarly, the mass relation (10) can be written in terms of \( \theta \) as follows

\[
\frac{d\theta}{dr} = 4\pi r^2 \rho_v \theta^n.
\]

To facilitate the numerical computations, it is convenient to write (11) in a dimensionless form by using (6) and the quantities

\[
\nu(x) = \frac{m(r)}{4\pi L^3 \rho_v},
\]

\[
\lambda = \frac{\rho_v}{\rho_c},
\]

where \( M \) is a characteristic mass scale of the polytrope

\[
M = 4\pi L^3 \rho_v = \frac{c^2}{G} nL(n+1),
\]

and \( \lambda \) indicates the \( \Lambda \) vacuum constant index’ giving the ratio between the vacuum energy density and the central energy density of the polytropic sphere. The cosmological constant \( \Lambda \) is related to the energy density of the vacuum by

\[
\Lambda = \frac{8\pi G}{c^2} \rho_{vac}.
\]

Thus, the index \( \lambda \) (19) is connected with \( \Lambda \) through the relation

\[
\lambda = \frac{\Lambda c^4}{8\pi G \rho_c}.
\]

In terms of (6), (18) and (20), (11) takes the final form

\[
\frac{d\theta}{dx} = \frac{21}{3 - \sigma \theta^{n+1}} x - \frac{\nu}{x^2} \left( 1 + \sigma \theta \right) g_{rr},
\]

\[
\frac{dv}{dx} = x^2 \theta^n,
\]

where

\[
g_{rr} = \left[ 1 - 2\sigma(n+1) \left( \frac{\nu}{x} + \frac{\lambda}{3} x^2 \right) \right]^{-1}.
\]

1 There is a missprint in Eqn. (22) in (Stuchlik et al. 2016): the minus sign is missing.

2 There is a missprint in Eqn. (28) in (Stuchlik et al. 2016): the term \( 8\pi G/c^2 \) in the second term of the R.H.S should not appear.

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These equations, subject to the boundary conditions

\[
\theta(0) = 1, \quad \nu(0) = 0,
\]

can be solved numerically to give the radius \( x = x_1 \) of the configuration as the first solution \( \theta(x) = 0 \).

### 3 GENERAL PROPERTIES OF THE SOLUTIONS

#### 3.1 Structural parameters

Except for the case \( n = 0 \), corresponding to a constant density configuration (Stuchlik 2000; Boehner & Harko 2005), the structure equations (23) and (24) do not admit analytic solutions in a closed form for \( \sigma \neq 0 \). Thus, one must turn to numerical integration. Considering that a configuration in equilibrium has positive density and monotonically decreasing pressure (see however (Posada & Chirenti 2019)), we will concentrate in the range of values of \( \nu \) such that \( \theta > 0 \).

Assuming that \( \lambda, n, \sigma \) and \( \rho_c \) are given, we start the numerical integration at the centre of the sphere \( x = 0 \), where \( \theta(0) = 1 \) and \( \nu(0) = 1 \), and advance by small steps until the first zero \( \theta(x_1) = 0 \) is found at \( x_1 \), if it exists. Using this value in (6) we can determine the radius of the polytrope as

\[
R = L x_1.
\]

The mass of the configuration is determined by the solution of \( \nu(x) \) at the surface

\[
M = 4\pi L^3 \rho_v \nu(x_1) = \frac{c^2}{G} nL(n+1)\nu(x_1).
\]

From Eqs. (27) and (28) we can obtain the Mass–Radius (M–R) relation

\[
C \equiv \frac{GM}{c^2 R} = \frac{s(n+1)\nu(x_1)}{x_1},
\]

which gives the ratio of the \( \Delta \) gravitational radius \( r_g \equiv 2GM/c^2 \) to the coordinate radius \( R \), once \( \sigma \) has been specified. The \( g_{rr} \) and \( g_{rr} \) metric components can be written in terms of the function \( \theta \) and the parameter \( \sigma \) as

\[
e^{2\Phi} = \frac{1 - 2\sigma(n+1) \left( \frac{\nu(x_1)}{x_1} + \frac{\lambda}{3} x_1^2 \right)}{(1 + \sigma \theta)^{2(n+1)}},
\]

\[
e^{-2\Psi} = 1 - 2\sigma(n+1) \left[ \frac{\nu(x_1)}{x} + \frac{\lambda}{3} x^2 \right].
\]

The exterior region is described by the Kottler metric, or Schwarzschild-de Sitter (SdS) spacetime, which represents a Schwarzschild mass embedded into an asymptotically de Sitter spacetime. In the Schwarzschild coordinates, the exterior metric takes the form (Tolman 1934)

\[
e^{2\Phi} = e^{-2\Psi} = 1 - \frac{2GM}{c^2 r} - \frac{\Lambda}{3} r^2.
\]

#### 3.2 Gravitational energy

In the relativistic theory the total energy \( E \) of certain fluid sphere, which includes the internal energy and the gravitational potential energy, is \( Mc^2 \) where \( M \) corresponds to the mass producing the gravitational field

\[
E = Mc^2 = 4\pi \int_0^R r^2 c^2 \, dr.
\]
The proper energy and proper mass of a spherical gas is defined by

\[ E_{\text{bg}} = M_{\text{bg}} c^2 = 4\pi \int_0^R (\rho g_x c^2) e^\nu r^2 \, dr, \quad (34) \]

where \( M_{\text{bg}} \) equals (approximately) the rest mass density of baryons in the configuration and \( \rho g_x c^2 \) is the rest energy density of the gas particles. For polytropic gas the energy density can be written in terms of the total mass density as (Tooper 1964)

\[ \rho g = \frac{\rho c^2}{(1 + \sigma \theta)^\sigma}, \quad (35) \]

In our analysis of stability using the energy considerations, an important quantity is the ratio

\[ \frac{E_{\text{bg}}}{E} = \frac{1}{\nu(x_1)} \times \int_0^{x_1} \frac{\theta^n x^2}{(1 + \sigma \theta)^\sigma} \left[ 1 - 2\pi(n + 1) \left( \frac{n}{2} + \frac{1}{4} x^2 \right) \right]^{1/2} \, dx, \quad (36) \]

which gives the proper energy of the gas in units of the total energy \( E = M c^2 \). Note that (36) generalizes the expression given by Tooper (1964) to the case of a non-vanishing cosmological constant.

These two quantities define the binding energy of the system, namely, \( E_b = E_{\text{bg}} - E \), which corresponds to the difference in energy between an \( \alpha\bar{Y} \) initial state with zero internal energy and the particles that compose the system are dispersed, and an \( \alpha\bar{Y} \) final state where the particles are bound by gravitational interaction.

### 4 RADIAL OSCILLATIONS OF RELATIVISTIC SPHERES IN SPACETIMES WITH A COSMOLOGICAL CONSTANT

In this section we discuss the theory of infinitesimal, and adiabatic, radial oscillations of relativistic spheres developed by Chandrasekhar (1964), and its extension to the case of a non-vanishing cosmological constant (Stuchlík & Hledík 2005; Boehmer & Harko 2005).

We consider \( \alpha\bar{Y} \) pulsations which preserve spherical symmetry, therefore these do not affect the exterior gravitational field. In other words, there is no gravitational monopole radiation. Thus, we are considering a situation with the line-element (1), and a mass distribution described by the energy-momentum tensor (2).

The pulsation dynamics will be determined by the Einstein equations, including the cosmological constant, together with the energy-momentum conservation, baryon number conservation, and the laws of thermodynamics. The relevant components of the Einstein equations with \( \Lambda = 0 \) are given in (Tolman 1934; Boehmer & Harko 2005).

To obtain the radial pulsation equation for spherical fluids immersed into a cosmological background, the metric coefficients \( \Psi(r, t) \) and \( \Phi(t, r) \) together with the fluid variables \( \rho(r, t) \), \( p(r, t) \) and the number density of baryons \( n(r, t) \), as measured in fluid’s rest frame, are perturbed generally in the form

\[ q(r, t) = q_0(r) + \delta q(r, t), \quad (37) \]

where the canonical variable \( q \equiv (\Phi, \Psi, e, p, n) \) indicates the metric and physical quantities, and the subscript 0 refers to the variables at equilibrium. At first order in the perturbations, the components of the energy-momentum tensor (2) are given by

\[ T^0_0 = -\rho_0, \quad (38) \]

\[ T^i_j = p_i, \quad i = 1, 2, 3 \quad \text{(no summation)}, \quad (39) \]

\[ T^0_0 = -(\epsilon_0 + p_0) r^2, \quad (40) \]

\[ T^0_0 = (\epsilon_0 + p_0) v^2 (\Psi_0 - \Phi_0), \quad (41) \]

where \( v = dr/d\xi^0 \). The pulsation is represented by the radial, or \( \alpha\bar{Y} \) Lagrangian, displacement \( \xi \) of the fluid from the equilibrium position \( \xi = \xi(r, t) \), defined as

\[ u^r = \frac{\partial \xi}{\partial t} = \xi. \quad (42) \]

The derivation of the expressions for the linear perturbations in the quantities \( q(r, t) \) follows the same lines as for the case \( \Lambda = 0 \) discussed by Chandrasekhar (1964); Misner et al. (1973), therefore we just summarize the main results here. All the relevant equations must be linearized relative to the displacement from the static equilibrium configuration. We have to obtain the dynamic equation for evolution of the fluid displacement \( \xi(t, r) \), and a set of initial-value equations, expressing the perturbation functions \( \delta \Phi, \delta \Psi, \delta e, \delta p, \delta n \) in terms of the displacement function \( \xi(t, r) \).

No nuclear reactions are assumed during small radial perturbations, therefore the dynamics of the energy density and the pressure perturbations is governed by the baryon conservation law

\[ (n\epsilon)_{,\theta} = 0. \quad (43) \]

Using (43) we obtain the initial value equation for the pressure perturbation

\[ \delta p = -\gamma p_0 \frac{\partial \Phi_0}{\partial r} \left( r^2 e^{-\Phi_0} \right)' - \xi(p_0)'. \quad (44) \]

where \( \gamma = \frac{\partial \Phi}{\partial p} \), and we introduce

\[ \gamma = \frac{\partial n}{\partial p}, \quad (45) \]

as the adiabatic index that governs the linear perturbations of pressure inside the star (Chandrasekhar 1964; Shapiro & Teukolsky 1983). In general, this \( \gamma \) is not necessarily the same as the adiabatic index associated to the EoS (see discussion in Hladič et al. 2020).

The initial-value equation for the Lagrangian perturbation of the energy density \( \delta \rho \), and the metric functions \( \delta \Phi \) and \( \delta \Psi \), take a similar form as for the case \( \Lambda = 0 \)

\[ \delta \epsilon = -\frac{\Phi_0}{r^2} (\epsilon_0 + p_0) \left( r^2 e^{-\Phi_0} \right)' - \xi(\epsilon_0)', \quad (46) \]

\[ \delta \Psi = -\xi (\Psi_0 + \Phi_0)', \quad (47) \]

\[ (\delta \Phi)' = \left[ -\frac{\partial p}{(\epsilon_0 + p_0)} \right] - (\Phi_0 + \frac{1}{r}) \xi (\Psi_0 + \Phi_0)', \quad (48) \]

It is conventional to assume that all the perturbations have a time-dependence of the form \( e^{i\omega t} \), where \( \omega \) is a characteristic frequency to be determined. Thus, using the previous initial-value equations for the perturbations, and introducing the \( \alpha\bar{Y} \) renormalized displacement function \( \zeta \) (Misner et al. 1973)

\[ \zeta = r^2 e^{-\Phi_0} \xi, \quad (49) \]

we obtain the Sturm–Liouville dynamic pulsation equation with a cosmological constant

\[ \frac{d}{dr} \left( r^2 \frac{d \xi}{dr} \right) + \left[ (Q + \omega^2 W) \right] \xi = 0. \quad (50) \]
where the functions $P(r)$, $Q(r)$ and $W(r)$ are defined as

$$P(r) = \frac{\gamma p_0}{r^2} e^{3\Phi_0 + \Psi_0},$$

$$Q(r) = \frac{e^{3\Phi_0 + \Psi_0}}{r^2} \left[ (p_0')^2/e_0 + p_0 - \frac{4p_0}{r} - \left( \frac{8\pi G}{c^4} \rho_0 - \Lambda \right) (e_0 + p_0) e^{2\Psi_0} \right],$$

$$W(r) = \frac{e_0 + p_0}{r^2} e^{3\Phi_0 + 3\Psi_0}.$$

The boundary conditions must guarantee that the displacement function is not resulting in a divergent behaviour of the energy density and pressure perturbations at the centre of the sphere. On the other hand, the variations of the pressure must satisfy the condition $p(R) = 0$ at the surface of the configuration. Therefore we have

$$\left( \frac{\gamma p_0 e^\Phi}{r^2} \right) \zeta'' \to 0 \quad \text{as} \quad r \to 0, \quad \left( \zeta \right)_{(\zeta)} \quad \text{as} \quad r \to R.$$

The Sturm–Liouville equation (50), together with the boundary conditions (54) and (55), determine the eigenvalues $\omega_i$ (frequencies) and the pulsation eigenfunctions $\xi_i(r)$ which satisfy

$$\int_0^R e^{3\Phi_0 + 3\Psi_0} \left( e_0 + p_0 \right) \xi_i \rho_r \, dr = 0, \quad i \neq j.$$

(65)

The Sturm–Liouville eigenvalue problem can be written in the variational form, as described by Misner et al. (1973), because the extremal values of

$$\omega^2 = \int_0^R \left( P e^{2\zeta^2} - W e^{2\zeta^2} \right) \, dr = \int_0^R W e^{2\zeta^2} \, dr = \gamma_\text{cr}$$

(57)

determine the eigenfrequencies $\omega_i$. The absolute minimum value of (57) corresponds to the squared frequency of the fundamental mode of the radial pulsations. If $\omega^2$ is positive (negative) the configuration is stable (unstable) against radial oscillations. Moreover, if the fundamental mode is stable ($\omega^2 > 0$), all higher radial modes will also be stable. For this reason, a sufficient condition for the dynamical instability is the vanishing of the right side of (57) for certain trial function satisfying the boundary conditions.

The Sturm–Liouville pulsation equation can be used to determine the dynamical stability of spherical configurations of perfect fluid. Given certain EoS, the critical adiabatic index $\gamma_\text{cr}$, given by the marginally stable condition $\omega^2 = 0$, can be determined by integration of the Sturm–Liouville equation. Using (57) we can deduce a general formula to find the critical adiabatic index, which reads

$$\gamma_\text{cr} = \frac{\int_0^R Q(r) \rho_r \, dr}{\int_0^R W e^{3\Phi_0 + \Psi_0} \rho_r \, dr}. \quad \gamma_\text{cr} = \frac{1}{\gamma - 2} \gamma_\text{cr} = \frac{1}{\gamma - 2} \left( \frac{\gamma}{R} \right)^2,$$

Thus, for $\gamma < \gamma_\text{cr}$ dynamical instability will ensue and the configuration will collapse. For the case of a homogeneous star immersed in a cosmological background, Boehmer & Harko (2005) showed that the condition for radial stability reads

$$\gamma > \frac{\gamma_\text{cr}}{1 - \frac{1}{2} \gamma_\text{cr} + \frac{19}{42} \left( \frac{\gamma_\text{cr}}{R} \right)^2}.$$
where the functions $P$, $Q$ and $W$ given by (51), (52) and (53) are now
\begin{equation}
P(x) = \frac{(\gamma)\sigma c r_0^2 \theta^{\sigma + 1} \rho_0 e^{3\Phi + \Psi}}{L^2 x^2},
\end{equation}
\begin{equation}
Q(x) = \frac{\sigma c r_0^2 (n + 1) \rho_0 \theta^{\sigma + 1} e^{3\Phi + \Psi}}{L^2 x^2} \left[ \frac{\sigma(n + 1)}{1 + \sigma\theta} \left( \frac{d\theta}{dx} \right)^2 - \frac{4}{x} \left( \frac{d\theta}{dx} \right) - 2(1 + \sigma\theta) \left( \sigma \theta^{n + 1} - 1 \right) e^{2\Psi} \right],
\end{equation}
\begin{equation}
W(x) = \frac{\rho_0 c r_0^2 (1 + \sigma\theta) \theta^{\sigma + 3}\rho_0 e^{3\Psi}}{L^2 x^2}.
\end{equation}

In the next section we will discuss the methods we used to solve the eigenvalue problem (65), subject to the boundary conditions (54) and (55), in order to study the radial stability of polytropic spheres in the presence of a cosmological constant.

5 RADIAL STABILITY OF RELATIVISTIC POLYTROPES IN THE PRESENCE A COSMOLOGICAL CONSTANT

5.1 Numerical Methods

Clearly, for general polytropes, the critical value of the adiabatic index related to the dynamical stability can be determined by numerical integration only. Several methods to solve the eigenvalue problem (60) have been described in the literature (see e.g. Bardeen et al. (1966) and references therein). Following Chandrasekhar (1964), we computed the critical values of the adiabatic index $\gamma_{c}$, for the onset of instability, by integrating numerically (65) in the case $\omega^2 = 0$. For that purpose we followed two different methods: the shooting method and trial functions.

In the shooting method (Press et al. 1992) one integrates (65) from the centre up to the surface of the configuration with some trial value of $\gamma$. The value for which the solution satisfies (within a prescribed error) the boundary conditions (54) and (55) corresponds to the critical adiabatic index $\gamma_{c}$.

In order to apply the shooting method to (65), it is convenient to transform it to a set of two ordinary differential equations. We follow the convention used by Kokkotas & Ruoff (2001) where (65) can be split in the following form
\begin{equation}
\frac{d\zeta}{dx} = \frac{\eta}{P(x)},
\end{equation}
\begin{equation}
\frac{d\eta}{dx} = -L^2 \left[ \omega^2 W(x) + Q(x) \right] \zeta,
\end{equation}
which satisfy the following behaviour near the origin
\begin{equation}
\zeta(x) = \frac{\eta_0}{3 \rho(0)} x^3 + O(x^5),
\end{equation}
\begin{equation}
\eta(r) = \eta_0,
\end{equation}
where $\eta_0$ is an arbitrary constant, which we choose to be the unity.

The second method is based on using trial functions to integrate (60). Following Chandrasekhar (1964); Boehmer & Harko (2005) we chose the following functions
\begin{equation}
\xi_1 = x e^{\Phi/2}, \quad \xi_2 = x,
\end{equation}
yielding
\begin{equation}
\zeta_1 = x^3 e^{-\Phi/2}, \quad \zeta_2 = x^3 e^{-\Phi}.
\end{equation}

We perform a detailed study of the stability for the whole range of the polytropes subject to the condition of causality due to the restriction on the parameter $\sigma$ (Toozer 1964)
\begin{equation}
\sigma < \sigma_{\text{causal}} = \frac{n}{n + 1}.
\end{equation}

However, we will see that for certain combinations of the parameters $(n, \lambda)$ the range of $\sigma$ is also limited by the condition of having finite size configurations. The limits on the existence of relativistic polytropic spheres were discussed by Stuchlík et al. (2016), in dependence of the polytropic index $n$ and the parameter $\sigma$. Condition (75) is obtained from the relation
\begin{equation}
\nu_{\text{sc}} = c \left( \frac{\sigma(n + 1)}{n} \right)^{1/2},
\end{equation}
which corresponds to the speed of sound at the centre of the sphere. Thus it might seem that (76) implies the restriction (75). However, note that (76) gives the phase velocity which is not the same as the group velocity, therefore condition (75) might not be definitive.

Applying the methods described above, we have computed critical values of the adiabatic index $\gamma_{c}$ for polytropes with characteristic values of the index $n$, for several values of the cosmological parameter $\lambda$. Using these results for $\gamma_{c}$, we also determined constraints on the parameter $\sigma$ in order to construct stable configurations. We present our results in the next section.

5.2 Results obtained via Chandrasekhar’s pulsation equation

As a first step in our analysis, we solved numerically the equations of structure (23) and (24) for relativistic polytropes in the presence of a cosmological constant. The integrations were carried out using the adaptive Runge–Kutta–Fehlberg method (Press et al. 1992). We provide some profiles of the dimensionless radius $x_1$, as a function of $\sigma$, for several values of the vacuum constant index $\lambda$. We studied a whole family of polytropic spheres with index $n \leq 3$, and we restrict the values of $\sigma$ by the causality limit (75). For comparison, we have included in the same plot (dashed lines) the profiles with $\lambda = 0$.

In Fig. 1 we present our results which are in very good agreement with those reported by Stuchlík et al. (2016). Note that for certain combinations of the parameters $(n, \sigma)$, the extension of the configuration increases as compared to its corresponding value in the case $\lambda = 0$. This is expected as a consequence of the repulsive effect of a positive cosmological constant. Moreover, the presence of $\lambda$ sets strong constraints on the existence of polytropic configurations. For instance, for the case $\lambda = 10^{-2}$, polytropic configurations with $n > 2.4$ do not exist. For the case $\lambda = 10^{-4}$, existing polytropes are restricted to $n < 1$.

We also found that the vacuum constant index $\lambda$ constrains the allowed values of the parameter $\sigma$ for certain configurations. For instance, for the combination $(n = 2, \lambda = 10^{-2})$, the maximum allowed value of the parameter $\sigma$ we found was $\sigma_{\text{max}} \approx 0.3563$. Note that this value is lower than the value restricted by causality given by (75). In Fig. 1 we show the results for various combinations of the parameters $(n, \sigma)$. For the case of $\lambda = 10^{-3}$, deviations from the $\lambda = 0$ case are negligible.

As a second step in our analysis, we determined the critical adiabatic index $\gamma_{c}$ for the onset of instability, as a function of $\sigma$, for several values of the indexes $(n, \lambda)$. We show our results in Fig. 2, which were obtained via the shooting method (see Sect. 5.1). For comparison we also plotted the values of $\gamma_{c}$ for each corresponding polytropic configuration with $\lambda = 0$. For large values of $\lambda$, for
instance, \( \lambda = 10^{-2} \) and \( \lambda = 10^{-1} \) we found that the values of \( \gamma_{cr} \) increase with respect to their values with \( \lambda = 0 \). These results indicate that large values of \( \lambda \) tend to destabilize the polytropic spheres. Of particular interest is the case \( \lambda = 10^{-1} \) which shows that in the Newtonian limit, when \( \sigma \to 0 \), the value of \( \gamma_{cr} \) deviates from the expected value \( \gamma_N = 4/3 \).

We will be interested in the differences of certain general quantities \( q \) with \( \lambda \neq 0 \), from its corresponding value with \( \lambda = 0 \)

\[
\Delta q \equiv q^{(\lambda)} - q^{(\lambda_0)},
\]

where \( \lambda \) denotes \( \lambda \in \{10^{-6}, 10^{-4}, 10^{-3}, 10^{-2}\} \), and \( \lambda_0 \) indicates \( \lambda = 0 \). In Fig. 3 we show our results for the differences of \( \gamma_{cr} \) for \( \lambda \in [10^{-6}, 10^{-2}] \), for the polytropes \( n \in \{1.0, 1.5, 2.0, 2.5\} \). We have \( \Delta \gamma \) the differences by dividing by the corresponding value of \( \lambda \). Note that the curves of the differences show the same qualitative behaviour.

The onset of instability, given by the condition \( \langle \gamma \rangle > \gamma_{cr} \), in dependence on the parameter \( \lambda \) is shown in Fig. 4 for several representative values of \( n \). The effective \( \langle \gamma \rangle \) was computed from (62) by using the results obtained from the shooting method. We also determined the critical values of \( \sigma \) by using the trial functions (73).

In this case, we computed \( \gamma_{cr} \) from (58) and then we determined the effective adiabatic index from (62). We summarise all of our results in Fig. 6.

5.3 Dynamical instability determined via the critical point method

We follow the standard approach to examine the stability of the polytropic spheres using the energy considerations, or critical point method (Tooper 1964). This analysis relies on the properties of static solutions to Einstein’s equations. It is worthwhile to mention that static methods to study the stability of configurations, might not be
conclusive. Instabilities arising due to thermal effects may not be predicted by these methods so one must turn to the full dynamical approach studied in the last section.

Substituting (4) and (6) in the expression (28) for the total mass $M$ we obtain

$$M = \frac{1}{4\pi} (\sigma + 1)^{3/2} K^{n/2} \sigma^{c/2} (3-n)^{2/2} \nu(x_1),$$

(78)

where $K$ and $n$ are the parameters characterizing the polytrope (see Sect. 2). In our analysis we are considering configurations with $K$ and $n$ constants, therefore the total mass $M$ is proportional to $\sigma^{(3-n)/2} \nu(x_1)$ and the rest mass is proportional to $\sigma^{(3-n)/2} \nu(x_1) (E_{\text{ efficiencies}} / E)$.

To study the stability, in Fig. 5 we plot the total gravitational mass $M$ and the rest mass of baryons $M_{ \text{ efficiencies}}$ (aÂ¥preassembly mass'), given by (34), against the parameter $\sigma$. A necessary, but not sufficient, condition for stability is

$$\frac{dM_{\text{eq}}}{d\sigma} > 0,$$

(79)

where $M_{\text{eq}}$ indicates the total mass at equilibrium. At the critical point, where

$$\frac{dM_{\text{eq}}}{d\sigma} = 0,$$

(80)

there is a change in stability due to the change in the sign of $\omega^2$ (see Sect. 4). Therefore, the critical point where the total mass $M$ has a maximum indicates the critical value $\sigma_{\text{cr}}$ for stability.

In Fig. 5 we show some profiles of total (and rest) mass, as a function of $\sigma$, for different polytropes in the range $0.5 < n < 3$ for several values of the index $\lambda \in [10^{-4}, 10^{-1}]$. In the plots we have also indicated the maximum of the curve $M$ which provides the critical parameter $\sigma_{\text{cr}}$, thus separating the stable from the unstable region.

The main result of our analysis is displayed in Fig. 6 where we determine the stable and unstable regions in the $n-\sigma$ parameter space, for several values of the vacuum constant index $\lambda \in [10^{-4}, 10^{-1}]$. In the plot we present the results for the critical values of the parameter $\sigma_{\text{cr}}$ as obtained from the critical point method (CP), and those obtained from Chandrasekhar’s method which were computed numerically using the shooting method (SM) (see Fig. 4) and the trial functions (73). For comparison, we have also included the $\sigma_{\text{cr}}$ values for the corresponding configurations with $\lambda = 0$.

A first thing to notice is that for large values of the index $\lambda$, in particular $\lambda = 10^{-2}$ and $\lambda = 10^{-1}$, the values of $\sigma_{\text{cr}}$ decrease relative to the case with vanishing $\lambda$. We show the corresponding differences, as defined in (77), in Fig. 7 for $\lambda \in [10^{-4}, 10^{-1}]$. Note that the differences are proportional, in order of magnitude, to the corresponding value of the index $\lambda$. These results are connected with those in Fig. 2 and the fact that large values of $\lambda$ tend to destabilise the polytropic spheres.

Remarkably we found that for large values of $\lambda$ the values of $\sigma_{\text{cr}}$ obtained by using the critical point method differ from those determined via the Chandrasekhar’s dynamical approach. In Fig. 8
we show the corresponding differences between both methods, as a function of \( n \), for \( \lambda \in [10^{-6}, 10^{-1}] \). Note that for the cases \( \lambda = 10^{-2} \) and \( \lambda = 10^{-1} \), the differences increase with \( n \). On the other hand, for lower values of \( \lambda \), for instance \( \lambda = 10^{-4} \), the bigger differences are found in the regime of small \( n \) and tend to zero as \( n \to 3 \). Note that these results are closely similar to those depicted in our preceding paper (Hladik et al. 2020, Fig. 8) for the case \( \lambda = 0 \).

Finally we would like to remark that the role of the vacuum energy on the radial stability of polytropic spheres becomes relevant for the parameter \( \lambda \) sufficiently large—it is negligible for \( \lambda \) smaller than \( 10^{-4} \), and becomes significant for \( \lambda \) comparable to \( 10^{-1} \).

ACKNOWLEDGEMENTS
The authors acknowledge the support of the Institute of Physics and its Research Centre for Theoretical Physics and Astrophysics at the Silesian University in Opava.

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Figure 4. The stability domain as determined by comparison of the effective adiabatic index (blue line) with the critical adiabatic index (black line), for polytropic spheres. The values of $\gamma_{cr}$ were computed via the shooting method. The red line separates the stable from the unstable region given by the condition $(\gamma) > \gamma_{cr}$. The intersection point indicates the maximum permitted value $\sigma_{cr}$ for stability. Note the role of the parameter $\lambda$ on $\sigma_{cr}$. 

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Figure 5. Profiles of the total mass (black line) and rest mass (blue line), as a function of $\sigma$, for some polytropic spheres for different values of the vacuum constant index $\lambda$. The maximum of the curve for the total mass determines the critical value of $\sigma$ for stability, thus it separates the stable and unstable regions.

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Figure 6. Critical values of the relativity parameter $\sigma_{cr}$, as a function of the polytropic index $n$, for the vacuum constant index $\lambda \in [10^{-4}, 10^{-1}]$. The forbidden region (grey background), corresponds to the region where the TOV equations do not give physically acceptable configurations for $\lambda \neq 0$. Here we show the results obtained by using the critical point method (CP) together with the results provided by Chandrasekhar’s approach via the shooting method (SM) and the trial functions $\xi_1$ and $\xi_2$. The dashed lines (same color) indicate the corresponding values of $\sigma_{cr}$ with $\lambda = 0$. Note that large values of $\lambda$, for instance $\lambda = 10^{-2}$ and $\lambda = 10^{-4}$, lowers the critical value $\sigma_{cr}$ with respect to the corresponding value with $\lambda = 0$. Moreover, for these same values of $\lambda$, the critical point method and Chandrasekhar’s dynamical approach predict different values of $\sigma_{cr}$. For values of $\lambda < 10^{-4}$, its influence on the radial stability is practically negligible.
Dynamical instability of polytropic spheres

Figure 7. Differences of the critical parameter $\sigma_{cr}$, as a function of $n$, between the values for $\lambda \in [10^{-4}, 10^{-1}]$ and their corresponding values for $\lambda = 0$. Note that as $\lambda$ increases the differences between the values predicted by the critical point (CP) method and Chandrasekhar’s approach also increases.

Figure 8. Differences $\delta\sigma_{cr}$ of the critical parameter $\sigma_{cr}$, as a function of the family of parameters $(n, \lambda)$, as determined via the critical point (CP) method ($\sigma_{cr}^{CP}$) and Chandrasekhar’s radial oscillations approach ($\sigma_{cr}^{Ch}$) via the shooting method (SM) and the trial functions $\xi_1$ and $\xi_2$. Note that for large values of $\lambda$ the differences grow as $n$ increases. For lower values of $\lambda$ the bigger differences are found in the low $n$ regime and tend to zero as $n \rightarrow 3$. 