A new and complete proof of the Landau condition for pinch singularities of Feynman graphs and other integrals

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The Landau equations give a physically useful criterion for how singularities arise in Feynman amplitudes. Furthermore, they are fundamental to the uses of perturbative QCD, by determining the important regions of momentum space for asymptotic problems. Generalizations are also useful. We will show that in existing treatments there are significant gaps in derivations, and in some cases implicit assumptions that will be shown here to be false in important cases like the massless Feynman graphs ubiquitous in QCD applications. In this paper is given a new proof that the Landau condition is both necessary and sufficient for physical-region pinches in the kinds of integral typified by Feynman graphs. The proof’s range is broad enough to include the modified Feynman graphs that are used in QCD applications. Unlike many existing derivations, there is no need to use the Feynman parameter method. Some possible further applications of the new proof and its subsidiary results are proposed.

I. INTRODUCTION

The subject of this paper is a set of related topics centered around the Landau analysis [1] of singularities of Feynman graphs. What makes this subject currently important is not merely the classic application to the locating of singularities of Feynman amplitudes, but its application by Libby and Sterman [2] to determine and analyze regions of low virtuality for amplitudes and cross sections in various asymptotic high-momentum limits. Their analysis shows that, in the loop-momentum space of Feynman graphs, these regions are determined by the locations of pinches in the massless limit (without any requirement that the full itself theory is massless) [3]. In addition, they formulate a power-counting analysis to determine which regions contribute at leading power in a given theory. To locate the pinches, they use the Landau criterion applied in a massless theory, in a form given by Coleman and Norton [4]. That form is that the pinches correspond to classically allowed processes; this is rather easy to apply in the massless limit. Libby and Sterman’s analysis results, among other things, in the well-known classification of momenta into hard, collinear and soft. It then underlies all results in factorization, which is an essential tool in most current QCD phenomenology. Moreover, as will be explained in more detail below, a number of extensions are needed to the Landau results for current and future work.

The primary outcome of the Landau analysis is a criterion for the existence of a pinch of the contour of integration in terms of what are called the Landau equations [4] when the objects of study are standard momentum-space Feynman graphs. To cover more general situations, I will use the term “Landau condition” (or criterion, depending on the shade of meaning needed).

However, given that the Landau criterion and the work of Coleman and Norton are foundational to most work on perturbative QCD (pQCD), it is very disconcerting that there are notable deficiencies in existing treatments of the Landau criterion, and that these become particularly noticeable in massless theories. I will review some of the problems in the next paragraphs, and then in more detail in Sec. [V]. One of the problems is that the actual proof by Coleman and Norton, of the Landau criterion for pinches, fails completely in the massless case. This is not simply a matter of a subtle issue in a high-order graph, but something that happens in a one-loop self-energy graph. It turns out that an implicit and apparently obvious and uncontroversial assumption is false. None of the deficiencies necessarily entail that the Landau criterion is incorrect. Indeed, a primary result of the present paper is a proof that does work and that is valid for the massless case, as well as for other cases needed in work on QCD, among others.

Nevertheless the problems indicate areas where some conceptual understanding has been missing. This can seriously impact efforts to use the methods in other situations. In addition, loopholes in the original arguments suggest the possibility of interesting new results.

The aim of Coleman and Norton’s derivation was to show that the Landau condition is both necessary and sufficient for physical-region pinches in the space of loop momenta. From it they then derived the well-known result that the location of a pinch corresponds to a classically allowed process [4]. They apply a Feynman paramet-

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1 See also many textbooks on QFT.
2 The exact wording of this sentence might appear to be somewhat at odds with what Libby and Sterman actually wrote. See Ch. 5 of Ref. [3] for my attempt to explain the logic.
3 In this paper we will solely be concerned with physical-region singularities and pinches (or their equivalent for more general integrals).
4 In a theory with no massless particles, this part of the proof is correct. But as pointed out by Ma [6], the proof needs extensions to make it work in the massless case.
ter representation and then perform the momentum integrals. Their proof is applied to the integral over Feynman parameters, with its single denominator. There is an unstated assumption that there is a pinch in the original momentum-space integral if and only if there is a pinch in the parameter integral. But in the massless case that implication simply fails, and it fails in the simplest graph, a one-loop self energy. As shown in the present paper in App. D1 the parameter integral for this graph has no pinch at all corresponding to the well-known collinear pinch in momentum space; this is quite unlike the situation for the normal threshold singularity in a massive theory.

Therefore, the first aim of this paper is to prove the necessity and sufficiency of the Landau criterion for a pinch directly in momentum space. The proof given here applies to a whole class of integrals, of which standard Feynman graphs are only one example. Unfortunately some restrictions are applied to make the proof work, but these are obeyed both for standard Feynman graphs and for various other kinds of graph that are commonly used in QCD. More work is needed to investigate more general cases.

It should be noted that there are important shifts of emphasis between the Landau analysis and the QCD applications. The Landau analysis was concerned mainly with where actual singularities of graphs occur as functions of their external parameters, and was almost entirely confined to the massive case. But the QCD applications are fundamentally concerned with locating regions where an integration contour is trapped by propagator singularities to be in a region of low virtuality compared with some large scale $Q^2$. These regions correspond to manifolds of exact pinches in the massless theory; the regions in a possibly massive theory are neighborhoods of the pinch singular surfaces in the massless theory.

Moreover, once one has a trap of the contour, one is also interested in what contour deformations are allowed and hence which subset of propagator singularities are involved in trapping the contour and which can be avoided.

Symptoms of problems in the available treatments are found in the classic book by Eden, Landshoff, Olive, and Polkinghorne (ELOP). In their Sec. 2.1, they give a general treatment of singularities of integrals over an arbitrary number of variables. They present derivations of the Landau condition for a singularity, in a form appropriate for a general integral, and not merely for those integrals that arise in a treatment of the Glauber region, and which are otherwise missing. In a sense, one basic problem with both references is that they try to be too general in the integrals they work with. For the results in which we are interested for Feynman graphs, the denominators are real for real values of their arguments and there is an $\epsilon i$ prescription, and we are interested in physical-region pinches. These properties are what enable the proof in the present paper to work.

Primary new results of the present paper are as follows:

1. A full proof is given that the Landau condition is both necessary and sufficient to determine the locations of physical-region pinch configurations in a class of integrals that includes momentum-space integrals for Feynman graphs. The applicability to Feynman graphs includes not only standard relativistic Feynman graphs, but also the various modified graphs that appear in factorization (notably including Wilson lines and the approximated graphs that arise in a treatment of the Glauber region as well as those containing Wilson lines).

The proof applies directly to the momentum-space integrals for Feynman graphs without any need to invoke the Feynman parameter method.

2. The proof is in two parts. One part is a detailed analysis of the conditions for a trapped contour in terms of constraints on the direction of contour deformation. Certain restrictions apply to this part of the proof — see item 4. The other part of the proof is embodied in a purely geometric theorem in arbitrarily high dimension on whether or not the constraints can be satisfied.

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5 See Ref. [8] for details, including in its Sec. 5.6 an analysis that uses the Landau criterion.
6 Undoubtedly the second part of the proof is closely related to the mathematical subjects of which an account is given in books by Gallier [10, 11]. But I have not yet found the result that is needed for the applications treated in this paper.
The presentation of the two parts is in the reverse order to the description just given. The geometric part comes first, since its results are used in analyzing properties of contour deformations.

3. A simple example is given, in App. [B1] to illustrate a difficulty that has to be overcome in the part of the proof analyzing contour deformations.

Based on experience in visualizable examples in integrals over one complex dimension, it is natural to assume that if a contour deformation avoids the singularity due to a zero of a denominator, then there must be a (non-zero) positive first-order shift in the imaginary part of the denominator, given the usual $i\epsilon$ prescription. The example in App. [B1] shows that this supposition is false; singularities can be avoided with a contour deformation that gives a zero first-order shift; I term this an “anomalous deformation”. Two or more complex dimensions are needed for an anomalous deformation to exist.

In the particular example given, the contour is not pinched and one can equally avoid the singularities by a non-anomalous deformation. But to make an satisfactory determination of the condition(s) for a pinch, it is essential to exclude the possibility of an anomalous deformation that avoids the singularities of the integrand when the Landau criterion is satisfied. This leads to considerable complications in the proof in this paper.

4. Overcoming the difficulties just mentioned, leads to a need to impose certain non-trivial restrictions on the denominators in the integral, in order for the methods of proof used here to succeed. The restrictions are that the denominators are at most quadratic in their arguments and that any quadratic terms obey a certain sign constraint — see the statement of Thm. [H].

Luckily the restrictions apply to the pure momentum-space form of standard Feynman graphs, including many of the modified graphs used in QCD factorization. However, it would be obviously be useful to find better proofs that would eliminate the restrictions as much as possible. The difficulties suggest areas for further investigation that appear not to have been properly considered in the original proofs.

5. A simple explanation is given that the Landau condition is necessary but not sufficient for a singularity as a function of external parameters (contrary to the situation concerning pinches of the contour of integration). A supplementary analysis is needed to determine whether or not there is an actual singularity given that the contour is trapped.

6. The proofs apply to more general situations than standard Feynman graphs. The range of applicability includes the modified and approximated Feynman graphs ubiquitous in QCD factorization. Others include systematic treatments of properties of Feynman graphs in coordinate space. Some illustrations are provided.

In coordinate space, only some restrictions on contour deformations arise from singularities of the integrand. Other restrictions arise when one has rapidly oscillating factors like $e^{ik\cdot x}$. These give strong cancellations in the integral, and to get a good analysis it is needed, if possible, to deform the contour in a direction that gives an exponential suppression. The geometrical part of this paper’s proofs applies directly to such cases to determine where such a deformation is not possible.

Many or all of the issues are elementary or even trivial in integrals over one complex variable. In that situation, issues about contour deformation are readily visualizable. But this is no longer the case in higher dimensions, as illustrated by the simple example in App. [B1].

Some areas for future extension and use of the methods and results in this paper are:

1. For a number of purposes, it would be useful to have a systematic and general determination for a given process of which regions of space-time for vertices dominate. For example, Brodsky et al. [12] have given an argument that the momentum sum rule is violated in deep inelastic scattering, in contradiction with standard results from the operator product expansion (OPE) and factorization. Their argument depends on properties of the regions of space-time involved. To assess their work completely, it is necessary to have systematic and fully deductive derivations of the space-time regions involved in processes such as those to which the OPE and standard factorization are applied.

Here we see examples of the situations mentioned above where one needs to determine where there is a lack of suppression in an integral containing multiple oscillating exponentials of the form $e^{ik\cdot x}$, and to be able to do this systematically to all orders of perturbation theory (at least).

2. Given that there is a pinch at some point in an integration for a Feynman graph, it is common that the pinch is restricted to a subset of the propagators. Then the contour of integration can be deformed to take the other propagators off shell, while not crossing the poles for the pinched propagators. How in general is one to characterize the allowed directions for such deformations and determine unambiguously which propagators are trapped and which not?

3. The methods in this paper could be very useful in calculations of hard scattering coefficients and other quantities, as is needed for much Standard Model phenomenology. Loop integrals are
encountered that are not readily amenable to analytic calculations, so that numerical calculations are needed. In the numerical implementation of integrals, it is very desirable to have algorithmic methods to deform integration contours away from non-pinch singularities of the integrand. In addition, where there is a pinch, it is important to deform the contour away from singularities that do not participate in the pinch.

Some important work in this area is by Gong, Nagy and Soper [13], and by Becker and Weinzierl [14, 15]. The geometric methods obtained in the present paper should be able to contribute to more general methods.

Although the results in this paper are in principle purely mathematical, the motivations and the situations considered arise from certain kinds of physics problem. Thus the presentation, the examples, and the terminology are strongly influenced by the physics applications.

A guide to the statements of the main results is as follows:

- The statement of the main result on pinches is Thm. [1].
- It applies to an integral of the form (2.4).
- It uses Definition 3 for a Landau point.
- The proof of the theorem uses a corresponding geometrical result, Thm. [2] and the relation to the notation for the integral is specified in Eq. (8.7).

II. PINCHES OF SINGULARITIES OF INTEGRAND IN MOMENTUM-SPACE FEYNMAN GRAPHS, ETC

In this section, I present the classic problem of determining where the contour of integration is trapped in the kind of situation exemplified by momentum-space Feynman graphs. The integrals are restricted to those such as occur in momentum-space Feynman graphs in the physical region. These restrictions are: (a) the external parameters are real; (b) the integration variables before contour deformation are real; (c) the denominators giving the singularities of the integrand are real for real values of their arguments; (d) the integral is defined by an \( \epsilon \) prescription.

Although much of the material is basically standard, the presentation here is needed to emphasize particular issues that are important in the sequel, and to define the notation to be used.

Motivation can be made, both for the general problem and for the geometrical formulation, from an elementary example. To this end, App. [C] gives the well-known example of a one-loop self-energy.

The general case is in an arbitrarily high dimension with arbitrarily many denominators whose zeros give singularities of the integrand. Considerable subtleties occur, as we will see. Hence to provide fully water-tight derivations, it is important to have precise operational definitions of the relevant concepts about a given contour deformation, about its compatibility or not with the integrand’s singularities, and about its avoidance or non-avoidance of the singularities. It is important that the definitions can be applied mechanically and essentially computationally, without the need for creativity or special insights.

The work in this section will motivate the geometric theorem to be proved in Secs. [IV–VII]. Only after that will be able to find a full proof that a necessary and sufficient condition for a pinch of the integration contour is that a particular Landau condition is obeyed.

This is the canonical application of the more abstract geometrical theorem, and it will influence the terminology used. Some further applications are summarized in Sec. [III].

A. Formulation of problem

1. Momentum-space Feynman graphs

The value of a momentum-space Feynman graph has the form

\[
I(p, m) = \lim_{\epsilon \to 0+} \int_{\Gamma_0} \prod_{j=1}^{N} \frac{X(k; p, m)}{f_j(k; p, m) + i\epsilon} \, \text{d}^d k.
\]  

(2.1)

Here \( p \) is the multi-dimensional array of variables for the external momenta, and \( m \) is the array of masses of the theory. The integration variable \( k \) is the array of all loop momenta, and has dimension \( d \), which may be arbitrarily high. We call each \( f_j + i\epsilon \) a denominator factor, and we call \( X \) the numerator factor. Each is a function of the integration variable \( k \) and the external parameters \( p \) and \( m \).

We restrict from now to situations where:

- The denominator factors \( f_j \) are real-valued when their arguments are real.
- The values of the external parameters \( p \) and \( m \) are real, and the initial contour of integration, denoted

\[\text{footnotemark}7\]

7 For the purposes of this paper, saying that a graph is in the physical region means that the external momenta are real and that before contour deformation the integral is calculated with internal loop momenta all real \[\text{footnotemark}8\]. There is no requirement that the external momenta be on-shell.

\[\text{footnotemark}8\] No restriction is placed on whether the masses are zero or non-zero.
by \( \Gamma_0 \), gives an integral over all real values of \( k \). This we will call the restriction to the physical region, and it implies a restriction solely to physical-region pinches and singularities.

- There is an \( i\epsilon \) with each denominator, and the end result is for the boundary value as \( \epsilon \) goes to zero from positive values.

- All the functions \( f_j \) and \( X \) are analytic functions of their arguments. In particular, \( X \) has no singularities for any finite value of \( X \). Then all singularities of the integrand are due to zeros in one or more of the denominator factors.

These properties evidently apply to a much wider class of integrands than those for standard relativistic Feynman graphs, but they are motivated by that situation. The methods we use could be easily applied to certain more general classes of integrand, but we will not do so. In the standard case, each \( f_j \) is a quadratic function of its arguments and the numerator \( X \) is polynomial in momenta and masses. But other possibilities can and do arise. Notably, denominators from straight Wilson lines (or eikonal lines) have linear dependence on momentum instead of quadratic.

Since the numerator factor \( X \) is non-singular as a function of \( k \), it does not affect the determination of where pinch singularities occur. In contrast, the numerator factor often affects power-counting analyses \([2]\) for quantifying the size of the contribution associated with a pinch, but that is not the concern of the present paper.

The exponent \( n_j \) of a denominator factor is typically unity; however, one regularly meets other cases. Our concern is with singularities of the integrand caused by zeros of one or more \( f_j \), and with whether the singularities obstruct contour deformations. The most general case is that each \( n_j \) is not zero or a negative integer, and this is what we will assume henceforth. (In the remaining cases, where an \( n_j \) is a negative integer or zero, a zero in \( f_j \) does not cause a singularity of the integrand, and then the factor \( 1/(f_j + i\epsilon)^{n_j} \) can be incorporated in the numerator factor \( X \).

Furthermore, it is possible that when a particular \( f_j \) is zero, the numerator factor is also zero in such a way as to remove the singularity of the integrand due to a factor \( 1/f_j^{n_j} \). In such cases the denominator factor can be removed and compensated by a corresponding change in the numerator factor. So we remove such cases from consideration.

For most values of its arguments, \( I(p, m) \) is an analytic function; this is shown by differentiating the integrand with respect to \( p \) and \( m \). However, that argument fails if one or more \( f_j \) is zero somewhere on the initial contour \( \Gamma_0 \). But if the contour can be deformed away from the singularity/ies of the integrand, then the differentiation argument can be applied on the deformed contour, and gives analyticity of \( I(p, m) \) at the values of \( p \) and \( m \) under consideration.

Hence our primary aim is to determine situations where such a deformation away from a singularity of the integrand is not possible. We term such a situation a pinch (by analogical generalization from corresponding situations in one-dimensional contour integrals).

A contour of integration is a surface that in terms of real variables has dimension \( d \). It is embedded in a space of \( d \) complex dimensions, i.e., of \( 2d \) real dimensions.

The only singularities of the integrand are at zeros of one or more denominators. When \( \epsilon \) is nonzero and when all of \( k \), \( p \), and \( m \) are real, the integrand is non-singular on all of the initial contour \( \Gamma_0 \), because the imaginary part of each denominator is non-zero. When \( \epsilon \to 0^+ \), i.e., \( \epsilon \) approaches zero from positive values, singularities may appear on \( \Gamma_0 \) at positions where one or more \( f_j \) is zero. In analyzing a candidate deformation to a contour \( \Gamma \), we wish first to know whether or not a singularity is encountered for positive \( \epsilon \) during the deformation and before \( \epsilon \) is finally taken to zero. Such a deformation is disallowed. Finally, for an allowed deformation we wish to know whether the deformed contour avoids a particular given singularity after \( \epsilon \to 0^+ \).

2. Feynman parameters

One technique for evaluating Feynman graphs is to use Feynman parameters. This converts the original formula \([2]\) to an integral with a single denominator factor. It has more integration variables, but the integrals over momenta can be performed analytically.

After Feynman parameterization, but before the integral over momenta, the integral becomes

\[
I(p, m) = \lim_{\epsilon \to 0^+} \int d^dk \prod_j \left( \int_0^1 d\alpha_j \right) \delta \left( \sum_j \alpha_j - 1 \right) \frac{\Gamma \left( \sum_j n_j \right)}{\prod_j \Gamma(n_j)} X(k; p, m) \prod \alpha_j^{-1} \frac{\prod_j \alpha_j f_j(k; p, m) + i\epsilon}{\prod_j \alpha_j^{n_j}}. \tag{2.2}
\]

The single denominator has an exponent that is the sum of the exponents in the original problem. The extra normalization factor with the Gamma-functions can be absorbed into a redefinition of the numerator factor. The same applies to the factor of powers of \( \alpha_j \), which can be singular only at endpoints of the integration, and then only if some \( n_j \)s are not positive integers.

After exchanging the order of the integrations and performing the momentum integrals, one gets an integral of the form

\[
I(p, m) = \lim_{\epsilon \to 0^+} \prod_j \left( \int_0^1 d\alpha_j \right) \delta \left( \sum_j \alpha_j - 1 \right) \frac{C(\alpha; p, m)}{(D(\alpha; p, m) + i\epsilon)\prod_j \alpha_j}. \tag{2.3}
\]
where $\alpha$ without a subscript denotes the array $j \mapsto \alpha_j$. The rules for obtaining the functions $C$ and $D$ can be found in textbooks, e.g., [6].

3. General situation

We can treat all of these integrals as special cases of the following form:

$$I(z) = \lim_{\epsilon \to 0^+} \int_0^\infty d^j w \frac{B(w; z)}{\prod_{j=1}^N (A_j(w; z) + i\epsilon)^{\nu_j}}, \quad (2.4)$$

with each $n_j$ not equal to zero or a negative integer. This is the most general form we will consider for our work. It is just like (2.1) except that we allow the contour of integration to have boundaries, and the values of $d$, $N$, and $n_j$ may not be the same as before. To indicate the more general notation, the notation has been changed: all external parameters are folded into a single multidimensional variable $z$, and the symbols are changed for the integration variables, the denominators, and the numerator.

The numerator $B$ and the denominators $A_j$ are analytic for all values of their arguments, and we restrict attention to the case that every $A_j$ is real when its arguments are real. The numerator factor $B$ need not be real when its arguments are real, and in fact $B$ will play no role in our work.

4. Singularities of integral

Landau’s original problem was to determine where the integral $I(p, m)$ is singular as a function of $p$ and/or $m$. Now we change to the more general notation of Eq. (2.4). By definition, $I(z)$ is analytic at some point when it is complex-differentiable in a neighborhood of the point. If it is analytic, then all derivatives exist, by standard theorems. Hence a function is singular at some point if and only if one or more derivatives fails to exist at that point, or arbitrarily close to it.

As already observed, we can apply derivatives with respect to $z$ inside the integration. Then a singularity can only arise in the dependence on the external variable $z$ if the initial contour of integration is pinched somewhere by a singularity caused by a zero of one or more $A_j$s. The integral might actually diverge if integrand is singular enough, or an actual divergence might only occur in a derivative or multiple derivative.

So in a general integral, like (2.4), we can only get a singularity as a function of external parameters if one of the following occurs (see [6] and other references):

1. The integration contour is trapped, i.e., pinched, by singularities of the integrand. That is, there is no contour deformation that avoids these singularities.

2. Singularities of the integrand occur on a boundary of the integration, and cannot be avoided by a deformation that preserves the boundary of the contour. This case does not occur for pure momentum-space Feynman integrals of a standard kind, but it can occur in more general situations (including Feynman graphs with the use of Feynman parameters).

In a general integral of the form of (2.4), it is also possible that for particular values of $z$, the integral acquires divergences from where some integration variables go to infinity; this requires that the integration range is infinite in those variables. This situation does not arise for Feynman graphs in a pure momentum-space formulation, since differentiation with respect to $p$ or $m$ in (2.1) always improves ultra-violet convergence. But it can occur when $I(p, m)$ is expressed as an integral over space-time coordinates of vertices, and divergences occur for large positions. Such cases do in fact appear to be able to be treated by elementary generalizations of the methods considered here, but we leave that for further work.

Notice that the above argument says that the existence of a singularity of $I(z)$ at a particular value of $z$ implies that the contour of integration is trapped at a singularity of the integrand considered as a function of the integration variable $w$. The converse is definitely not always valid, as shown in App. A with the aid of a trivial counterexample. Given that a pinch has been found, a separate calculation of the contribution to the integral from a neighborhood of the pinch point to determine the existence or non-existence of an actual singularity.

But, as already observed in the introduction, what is important to many modern applications is not the actual existence of a singularity of $I(z)$ as a function of the external parameter(s) $z$, but whether or not the singularity is trapped, and where.

B. Deformations of contour

In the integral (2.4), the contour deformations to be considered are replacements of the real values of the integration variable $w$ by

$$w = w_R + i\lambda v(w_R). \quad (2.5)$$

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9 This restriction and the $i\epsilon$ prescription do not appear in the mathematical work in Refs. [7][9]. The restrictions lead here to more powerful results of physical relevance [2][4] for Feynman graphs in the physical region.

10 The condition of preserving the boundary of the contour is actually not quite what we need. In a multidimensional case, Cauchy’s theorem can allow deformations that preserve the value of an integral while moving the boundary. Since our concern in this paper is non-boundary singularities, we will not consider the ramifications of this remark. It can matter when a momentum-space integral is decomposed into sectors and contour deformations determined separately on each sector, as in applications of the numerical methods of Ref. [19].
Here $w_R$ is a real variable that ranges over all values on the original real contour $\Gamma_0$. The real variable $\lambda$ is in the range 0 to 1, and parameterizes the amount of deformation. For each $\lambda$ we have a particular contour $\Gamma_\lambda$, parameterized by $w_R$. The original contour is at $\lambda = 0$, and the final deformed contour is at $\lambda = 1$. The function $w_R \mapsto v(w_R)$ is from real values $w_R$ to real $d$-dimensional values, so that $i\lambda v(w_R)$ gives the imaginary part of $w$ on the deformed contour. The function $v$ must be continuous and piecewise differentiable (but only in the sense of differentiation with respect to real variables, not necessarily with respect to complex variables). We take it to be zero on any boundary of $\Gamma_0$, so that the boundaries of the contour are unchanged.

The value of the integral on the deformed contour is

$$I(z, \lambda, \epsilon) \overset{\text{def}}{=} \int_{\text{real } \Gamma_0} d^d w_R J(w_R, \lambda) \frac{B(w; z)}{\prod_{j=1}^N (A_j(w; z) + i\epsilon)^{n_j}}. \quad (2.6)$$

Here, the integral symbol is still equipped with the symbol for the undeformed contour $\Gamma_0$, but now it is the real variable of integration $w_R$ that is on $\Gamma_0$, and not the argument $w$ of the integrand. We have chosen not to take the limit $\epsilon \to 0^+$ yet. Here $w$ is given by (2.5), and $J$ is the Jacobian of the transformation from $w_R$ to $w$. According to Cauchy’s theorem\footnote{An accessible and elementary proof of the Cauchy theorem beyond the one-dimensional case is given by Soper\cite{Soper}. Note that it is possible to generalize the theorem to certain cases where the boundary changes, but we will not deal with that issue here.} the value of the integral is independent of $\lambda$ provided that the integrand has no singularities on the contour when $0 \leq \lambda \leq 1$, and hence that every $A_j(z, w_R + i\lambda v(w_R)) + i\epsilon$ is non-zero for all $w_R$ on $\Gamma_0$, for $0 \leq \lambda \leq 1$. Thus $I(z, 1, \epsilon) = I(z, 0, \epsilon)$.

The target value of the integral is the limit as $\epsilon$ decreases to zero of the integral on the undeformed contour, i.e., of $I(z, 0, 0^+)$. On the undeformed contour, $\lambda = 0$, there are typically singularities of the integrand for some values of $w_R$; these are where there are zeros of one or more denominators. To get the same value for the integral on the deformed contour, we must apply the condition of all $A_j$s being nonzero for all $\lambda$ in the range $0 \leq \lambda \leq 1$ and for all $\epsilon$ in a range $0 < \epsilon \leq \epsilon_0$, for some positive number $\epsilon_0$. Notice that the range of $\epsilon$ considered in the condition excludes zero. Then taking the limit $\epsilon \to 0^+$ gives $I(z, 1, 0^+) = I(z, 0, 0^+)$. Even if the contour does not avoid all singularities, it is useful to deform a contour to avoid as many singularities as is possible. For such a deformation to be useful, we require that no singularities of the integrand appear on the contour until the very last step of taking $\epsilon$ to zero with the contour fully deformed. We call such a deformation allowed, while any deformation that encounters a singularity before that step is not an allowed deformation.

Given some of the complications that arise in a complete analysis, it is useful to have very precise operational specifications of what is meant by an allowed deformation, and of what is meant by a singularity-avoiding deformation. In analyzing problems concerning possible deformations, it is also useful to define such concepts relative to particular subsets of denominators. In particular, we may be interested not only in whether a contour deformation avoids all singularities of the integrand, but also in whether it avoids singularities at particular values of $w_R$, and possibly only for those singularities caused by zeros in particular subsets of the $A_j$. One physical motivation is that in many applications in QCD, sets of singular propagators correspond to factors in a factorization theorem, each of which can be considered separately; given a kinematic configuration of momenta, only in a part of a graph that corresponds to a hard scattering factor can we deform away from propagator singularities.

In all of the following definitions, we will assume a particular value of the external parameter $z$.

**Definition 1.** Given a particular subset $\mathcal{A}$ of the $A_j$s, we define that at a particular value $z$ and $w_S$ for the external parameter and integration variable, a deformation is defined to be compatible with the denominators $\mathcal{A}$ if there exists a positive non-zero $\epsilon_0$ such that $A_j(w_R + i\lambda v(w_R); z) + i\epsilon$ is non-zero when $w_R$ is in a neighborhood of $w_S$ for all $A_j$ in $\mathcal{A}$, and for $0 \leq \lambda \leq 1$ and $0 < \epsilon \leq \epsilon_0$.

Note that the case $\epsilon = 0$ is specifically not included. The lack of singularities in the given range indicates that the zeros of the specified denominators do not give singularities that obstruct the use of Cauchy’s theorem. We specify that there is a lack of zeros not only at $w_R$ but in a neighborhood. The reason is that if there is a zero of $A_j(w)$ at $w = w_S$, then as $\lambda$ is increased from zero, the position of the zero usually migrates to nearby values of $w_R$; such a zero is equally effective at obstructing a contour deformation.

**Definition 2.** A locally allowed deformation at $z$ and $w_S$ is one that is compatible with all the denominators at $w_S$. An allowed deformation is one that is an allowed deformation at all $w_S$ in the range of integration.

As already observed, for an allowed deformation there is no obstruction to the contour deformation, so that $I(z, 0, \epsilon) = I(z, 1, \epsilon)$ when $0 < \epsilon \leq \epsilon_0$, and hence $I(z, 0, 0^+) = I(z, 1, 0^+)$. (It is sufficient to take a common nonzero maximum $\epsilon_0$ for all denominators.)

It should be noted that a trivial special case of an allowed deformation is when there is no change in the contour at all, i.e., when $v(w_R)$ is zero for all $w_R$.

We next have the definition of a deformation that avoids singularities:

**Definition 3.** Given a particular subset $\mathcal{A}$ of the $A_j$s, we define that at $z$ and $w_S$ a deformation avoids an $\mathcal{A}$-associated singularity if the deformation is allowed at $z$
and $w_S$, and if the non-zero condition on $A_j + i\epsilon$ also applies for the given $A_j \in A$ at $\epsilon = 0$ and $0 < \lambda \leq 1$, for $w_R$ in some neighborhood of $w_S$. By bringing in the definition of an allowed deformation, the condition for a singularity-avoiding deformation is that the only place where $A_j + i\epsilon$ is zero (with $A_j \in A$) is where $\epsilon$ and $\lambda$ are both zero. (Here it is taken for granted that the relevant ranges for $\epsilon$ and $\lambda$ are for non-negative values below $\epsilon_0$ and 1, respectively.)

Of course, no requirement is placed when $\lambda = \epsilon = 0$, because the interesting case is when one or more $A_j(w_S)$ is zero, and then we ask whether a particular deformation avoids the resulting singularity in the integrand.

We can define more global kinds of singularity avoidance:

**Definition 4.** A deformation avoids any singularity at $w_S$ if it is allowed and avoids $A_j$-related singularities for all $A_j$.

**Definition 5.** We define that at $z$ a deformation (globally) avoids any singularity if it is allowed and avoids $A_j$-related singularities for all $A_j$ and for all $w_S$ on $\Gamma_0$.

An illustration of how these definitions are used is given in Fig. 1. In the left-hand diagram of the $\lambda$-$\epsilon$ plane is illustrated the situation for a disallowed contour deformation. The dashed line indicates the sequence of values we wish to use to get from the target value of the integral, i.e., $I(z, 0, 0+)$, to the value on the deformed contour. The solid line is where the contour deformation first hits a zero of $A_j + i\epsilon$ somewhere on the contour of integration as $\lambda$ is increased from zero. On the right-hand horizontal axis, where $\lambda$ is zero and $\epsilon$ is positive, there is no singularity. As $\lambda$ is increased eventually a zero of an $A_j$ hits the contour, and that is indicated by

where the dashed line intersects the solid line. For larger $\lambda$ Cauchy’s theorem fails. The solid line goes all the way to the origin; otherwise, simply by restricting $\lambda$ to a smaller range, the zero(s) of $A_j + i\epsilon$ are avoided, and we can convert the deformation to the standard form by rescaling the deformation.

In contrast, for a singularity-avoiding deformation, any line or region of zeros of $A_j + i\epsilon$ does not come all the way to the origin, as in the middle diagram. There may be a singularity when both $\lambda$ and $\epsilon$ are zero; that is the case of interest, i.e., of a singularity on the undeformed contour. It is possible that there are singularities when large enough deformations are considered, shown in the middle diagram above $\lambda = 1$. Thus $v(w_R)$ has been scaled down enough to avoid encountering the singularity/ies.

If the deformation is allowed but doesn’t avoid singularities, then we have a line of singularity-encounters on the vertical axis, i.e., where $\epsilon = 0$ and $\lambda > 0$, and this line goes all the way to $\lambda = 0$, as in the right-hand diagram. Note that in the diagrams, we are concerned with singularities of the integrand anywhere on the integration over $w_R$. The solid lines correspond to the existence of an singularity somewhere in the integration range. A singularity that is at some point $w_R = w_S$ when $\epsilon = \lambda = 0$ often migrates to other values of $w_R$ as $\lambda$ is increased.

C. Conversion to a geometrical problem

Based on experience with simple examples, it is natural to suppose that one can determine whether or not a singularity due to a zero of $A_j(w_R)$ is avoided or collided with, by examining the sign of the (imaginary) first-order derivative of $A_j$ in $\lambda$. Suppose there is a zero of $A_j$ at $w = w_S$. Then a Taylor expansion in powers of $\lambda$ gives

$$A_j(w_S + i\lambda v(w_S)) + i\epsilon = i\lambda v(w_S) \cdot \partial A_j(w_S) + O(\lambda^2) + i\epsilon,$$

(2.7)

where $\partial_n A_j(w_R) = \partial A_j(w_R)/\partial w_R^n$. Then one would normally expect that the singularity is avoided if and only if $v(w_S) \cdot \partial A_j(w_S)$ is strictly positive, which is a geometric condition on the deformation vector $v(w_S)$ at the zero of $A_j$. In contrast $v(w_S) \cdot \partial A_j(w_S)$ would be zero if the deformation is allowed but doesn’t avoid the singularity, while if $v(w_S) \cdot \partial A_j(w_S)$ were negative, then the deformation would not be allowed. If these statements were all exactly correct, then applying the positivity condition on $v(w_S) \cdot \partial A_j(w_S)$ to all $A_j$ for which $A_j(w_S) = 0$ would give the condition that the deformation avoids any singularity at $w_S$.

As we will see, the Landau condition gives a necessary and sufficient criterion that these positivity conditions are incompatible and hence that the contour is trapped at $w_S$.

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12 It may be that for yet larger $\lambda$ there remains a zero of an $A_j$ on the contour. That is irrelevant to our considerations.
However, it is possible to arrange a contour deformation that avoids a singularity by use of second-order or higher-order terms in $\lambda$, as shown in App. B1. Now our interest is in the exact conditions under which contours are trapped or not trapped, i.e., we need a condition for a trap that is both necessary and sufficient. Therefore the result in App. B1 suggests that there could be an interesting loophole in the Landau analysis.

To exclude the loophole, we need a more detailed analysis, which will be given in Sec. VIII. The precise definitions given above will assist that analysis. In addition, the following observations concerning the $\lambda$ dependences of $A_j$ will also be useful. They are used in treating the zeros of $A_j$ as a function of $\lambda$ at a fixed value of the real part $w_R$ of the integration variable.

Define $f_{j,w_S}(\lambda) = A_j(w_S + i\lambda v(w_S))$. Since $A_j(w)$ is analytic as a function of $w$, $f_{j,w_S}(\lambda)$ is analytic as a function of the one-dimensional variable $\lambda$ with $w_S$ fixed.\(^{13}\)

Suppose that $A_j(w_S) = 0$. Then $f_{j,w_S}(0) = 0$. By standard properties of analytic functions, either this is an isolated zero of $f_{j,w_S}(\lambda)$ or $f_{j,w_S}(\lambda) = 0$ for all $\lambda$. In the first case, $f_{j,w_S}(\lambda)$ is non-zero for all sufficiently small non-zero $\lambda$. In the second case, we haven’t avoided the singularity by the contour deformation under consideration.

### D. Primary theorems

In order to provide context for later sections, I state here the main theorems to be proved.

First come a couple of convenient terminological definitions, of a Landau point and a Landau condition:

**Definition 6.** Let $(D_1, \ldots, D_N)$ be a list of dual vectors on a real vector space $V$. We define a Landau point for $(D_1, \ldots, D_N)$ to be a list of real numbers $\lambda_j$ $(1 \leq j \leq N)$ such that

- All the $\lambda_j$ are non-negative, and at least one is strictly positive,
- $\sum_{j=1}^N \lambda_j D_j = 0$.

**Definition 7.** We define the Landau condition for $(D_1, \ldots, D_N)$ to be obeyed if and only if there exists a Landau point for them.

Recall that a dual vector on a vector space $V$ is a linear map from $V$ to the scalars — e.g., real numbers — of a vector space. A standard example is the derivative of a function on $V$. Thus in the preceding subsection, the derivative of a function $A_j(w_R)$ is $\partial A_j$. It can be considered a dual vector $D_j$ by the mapping of vectors to scalars that is given by $D_j(v) = v \cdot \partial A_j = \sum_{\mu} v^{\mu} \partial A_j(w_R) / \partial w_R^{\mu}$.

---

\(^{13}\) Note that it is not necessary that the function $v(w_R)$ specifying the contour deformation be analytic as a function of $w_R$. Hence $f_{j,w_R}(\lambda)$ is not necessarily analytic as a function of $w_R$. Observe that in the case that there is only a single denominator $A_i$, i.e., $N = 1$, a Landau point is one where $A = 0$ and $D = 0$.

The main theorem to be proved concerning pinches is:

**Theorem 1.** Given an integral of the form (2.4), but subject to the extra restrictions stated below, consider a real (vector) valued point $w_S$ where a nonempty set of denominators is zero. Then the integration is trapped at $w_S$ if and only if a Landau point exists for the first derivatives of those denominators that are zero.

The extra restrictions are that (a) the denominators $A_j(w)$ are at most quadratic in $w$, and (b) the signs of the nonzero quadratic terms obey a condition which is stated below in (8.19) and the following paragraphs. This condition is obeyed for the denominators encountered in Feynman graphs, including cases with Wilson lines. It also applies to the modified Feynman graphs obtained by applying the typical approximations used in deriving factorization.

The theorem may well be true without these extra restrictions or with weaker restrictions. But the proof that we will give only applies when the restrictions are valid. Either better methods or much more work would be needed to give a proof of a less restricted theorem.

However, we do in all cases impose the reality conditions etc that were listed below Eq. (2.4).

Our proof of the main theorem is made by combining two subsidiary theorems.

The first subsidiary theorem relates the trapping or non-trapping of a contour to the positivity of the first-order (imaginary) shift in denominators:

**Theorem 2.** With the same hypotheses as in Thm. 4 the integration is not trapped at $w_S$ if and only if there is a direction $v$ such that $v \cdot \partial A_j(w_S)$ is strictly positive for every $A_j$ which is zero at $w_S$.

Notice that the theorem does not say that a contour deformation given by a function $v \mapsto v(w_R)$ avoids a singularity at $w_S$ if and only if $v(w_S) \cdot \partial A_j(w_S)$ is strictly positive for every $A_j$ which is zero at $w_S$. That property appears to be universally assumed in textbook proofs, but it is fact false, as shown by the example in App. B1. That is, it is possible to avoid singularities with a anomalous contour deformation, i.e., one for which $v \cdot \partial A_j(w_S)$ is zero instead of positive for one or more of the relevant denominators. Hence some trouble is needed to prove a correct theorem, as we will do later. What the theorem does enable one to say is that if there exists an anomalous deformation there is also a non-anomalous deformation that avoids the singularity.

The second subsidiary theorem is a purely geometrical result:

**Theorem 3.** Let $V$ be a real vector space, and let $(D_1, \ldots, D_N)$ be dual vectors on $V$. Then, there is a direction $v$ for which $D_j(v) > 0$ for all $j$, if and only if there is no Landau point for the $D_j$.\(^{13}\)
When \( v \) is the deformation direction of a contour, this theorem gives the condition under which the first order imaginary-direction shifts in denominators can be made all positive.

It will be convenient to prove these theorems in the opposite order to which they are stated, since the proof of the contour deformation theorem uses the geometrical theorem. But the motivation and relevance of the geometric theorem arises from considerations about contour deformations, so it was convenient to state the contour deformation theorems first.

## E. Elementary parts of proofs

Certain directions of implication in Thms. 2 and 3 are almost trivial to prove, as follows:

Suppose at some point \( w_S \) in the integration range, a set of denominators is zero and that a contour deformation gives positive first order shifts in these denominators. Without loss of generality, list the denominators that are zero as \( (A_1, \ldots, A_N) \). Let the derivatives be \( D_j = \partial A_j(w_S) \). The positivity of first-order shifts means that \( D_j(v(w_S)) > 0 \) for \( 1 \leq j \leq N \). Then we have already seen that the contour deformation avoids the integrand’s singularity at \( w_S \).

Now, under the same conditions, consider any array of real numbers \( \lambda_j \) \((1 \leq j \leq N)\) which are non-negative and for which at least one is positive. Then \( \sum_{j=1}^{N} \lambda_j D_j(v(w_S)) > 0 \) and hence the dual vector \( \sum_{j=1}^{N} \lambda_j D_j \) is nonzero. Therefore there is no Landau point.

In order to get the desired necessary and sufficient conditions, we also need to prove the reverse implications, which is quite non-trivial. It is interesting to observe, that even to get one of the directions of implication in the main theorem requires that we use of a non-trivial direction of implication in one or other of the subsidiary theorems.

### III. USEFUL GENERALIZATIONS

In this section, I gather a couple of illustrations of applications of the derived theorems to situations beyond the standard analyses of singularities of ordinary Feynman amplitudes. The standard application to momentum-space Feynman graphs is illustrated in App. C.1

#### A. Glauber region

One example of the need for a more general derivation of the Landau criterion is the general analysis of the Glauber region given in Sec. 5.6.3 of Ref. 3. In that situation, approximations have been made for a Feynman graph that are valid in a certain region of its loop momenta, with the momenta being classified into soft, collinear, and hard categories. It is desired to determine when there is a trap in the Glauber region; this is important because the approximations used for soft momenta fail when a soft momentum is of the kind called Glauber. The importance of this issue is that in some situations, there are uncanceled Glauber contributions, and these break standard formulations of factorization in interesting cases, e.g., 17.

An appropriate method [3, Sec. 5.6.3] to locate Glauber contributions uses a version of the Libby-Sterman argument, but applied to the approximated graph in which standard soft and collinear approximations have been made. If a contour deformation cannot be made to avoid the Glauber regions, then there is a corresponding exact pinch in the approximated graph. The use of the relevant Landau condition gives a necessary and sufficient condition for the Glauber pinch.

The importance of this analysis, with its systematic use of an improved Landau analysis, is that it can be used to locate in full generality where extra regions and scalings in momentum space are important beyond the usual classification into soft, collinear, and hard, with associated scalings of momentum components.

#### B. Coordinate space

Another example is the extraction of coordinate-space properties of amplitudes. For example, the Fourier transform of a free propagator is

\[
S_F(x) = \int \frac{d^n k}{(2\pi)^n} e^{-ikx} \frac{i}{k^2 - m^2 + i\epsilon},
\]

where \( n \) is the number of space-time dimensions, and the limit \( \epsilon \to 0 \) from positive values is implicit, as usual.

Suppose we are interested in how this integral behaves when \( x \) is scaled to large values: \( x \mapsto \kappa x \) with \( \kappa \to \infty \). Of course, in the particular case given, a solution can be found analytically, since the free propagator is a kind of Bessel function with known asymptotics. But it is important to have a method that can be applied much more generally without appealing to properties of known special functions. To do this, we observe that over much of the space of real \( k \), one can deform the contour of \( k \) so as to give \( k \cdot x \) a negative imaginary part. But near the pole at \( k^2 = m^2 \), we need to have the deformation compatible with the \( ik \) prescription in the denominator. If these two conditions on contour deformation are incompatible, then we must leave the contour on the real “axis” and get an unsuppressed contribution to the large \( \kappa \) asymptotics.

Let us specify the deformed contour as

\[
k = k_R + iv(k_R).
\]

Then the condition for an exponential suppression is

\[
- v(k_R) \cdot x > 0,
\]
while the condition for avoiding the propagator pole is
\[ v(k_R) \cdot k_R > 0 \quad \text{when} \quad k_R^2 = m^2. \] (3.4)

These conditions are incompatible when \( x \) is proportional to \( k_R \) with a positive coefficient and \( k_R \) is on-shell. If \( k_R \) has positive energy, then the relevant values of \( x \) are future pointing in the same direction as \( k_R \), while if \( k_R \) has negative energy, \( x \) is past pointing.

Given a value of \( x \), this observation determines which values (if any) of \( k_R \) give unsuppressed contributions to \( S_F(x) \). Here “unsuppressed” means “not exponentially suppressed”; this use of “unsuppressed” allows it to include merely “power suppressed”.

Now an on-shell value of \( k_R \) is time-like. Hence, when \( x \) is space-like, there is no value of \( k_R \) giving an unsuppressed contribution. Then there is no obstruction to deforming the contour, and an exponential suppression of \( S_F(x) \) is a consequence.

In contrast, when \( x \) is time-like, the deformation cannot be made, and that gives power-law behavior as \( x \) is scaled. The dominant contribution comes from near the pole in momentum-space, and the asymptote can be extracted by suitable approximation methods. These methods continue to apply if the free momentum-space propagator is replaced by the full propagator in an interacting theory, which has a more general dependence on momentum, but with its strongest singularity still being a pole at the physical particle mass.

It is worth noting that similar methods can also be applied to get from the behavior of a coordinate-space Green function to particular functions in momentum space. Thus one can determine for the vertices of a graph the dominant regions in coordinate space that contribute to a particular process. We leave the systematic codification of such results to future work.

Some relevant recent work is by Erdoğan and Sterman [18, 20].

\[ v(k_R) \cdot k_R > 0 \quad \text{when} \quad k_R^2 = m^2. \] (3.4)

IV. LITERATURE REVIEW

In this section I assess some of the classic literature about the Landau analysis. Since many of these works continue to be cited regularly as the primary sources for results on singularities and pinches of contours, it is useful to examine their arguments in detail. The review in this section extends observations already made in the introduction.

\[ v(k_R) \cdot k_R > 0 \quad \text{when} \quad k_R^2 = m^2. \] (3.4)

It should be observed that typical treatments rely on the use of Feynman parameters to combine the denominators into a single denominator. Then they examine the conditions for a pinch of the integration contour, rather than trying to create a more detailed geometrical argument that applies to the multiple-denominator situation. This rules out any easy application of the methods to more general situations, e.g., examining properties of integrals involving coordinate space properties, as in Sec. [1111] or the issues of algorithmic deformation of a contour for numerical integration of a Feynman graph in the pure momentum-space formulation.

a. Landau [1] Landau’s paper [1] gives the original treatment of the his criterion for singularities of a Feynman graph as a function of its external parameters.

The analysis solely uses the Feynman parameter representation in the form \( \langle 2, 2 \rangle \). It relies on the denominators being those of standard Feynman graphs. Then in Landau’s Eq. (4) the single denominator is written as \( \phi + K(k', l', \ldots) \), where \( \phi \) is a function only of the external parameters and \( K \) is a homogeneous quadratic form in a set of variables that are formed by a (parameter-dependent) linear transformation from the original loop momenta. This by itself rules out the case that some denominators have linear dependence on some (or all) loop momenta. Such cases arise in practice. For example, in QCD applications we have cases with Wilson denominators. In such a situation, the equivalent of \( K \) is not a homogeneous quadratic function.

The argument then continues to determine that a singularity of the integral (as a function of external parameters) occurs when there is a point in integration space where the denominator and its first derivative vanish. No detailed argument is given, the core parts of the argument being treated as “easy to verify”. However, a detailed derivation, in Sec. [1111] of the present paper, is not at all easy. In fact the proof fails whenever the matrix of second derivatives of the denominator has an eigenvector with zero eigenvalue. This situation does in fact sometimes arise in practice, as mentioned in a later paper by Coleman and Norton [4].

Moreover, Landau’s argument is rather difficult to apply as written if there are massless particles, as is essential in applications to QCD factorization. In contrast, the methods of the present paper do apply unchanged to such cases. They are also applied directly to the momentum space integral without an appeal to Feynman parameters.

A minor problem is that the \( i \epsilon \) prescription is not mentioned explicitly even though that is critical in determining whether or not there is a pinch.

b. Coleman and Norton [4] Coleman and Norton [4] again use a parametric representation. In the first part of the paper, they discuss the version with both momentum and parameter integrations. They state, rather like Landau, that to get pinch there needs to be either a coalescing pair of singularities or an end-point singularity. This immediately gives the Landau equations. However,

\[ v(k_R) \cdot k_R > 0 \quad \text{when} \quad k_R^2 = m^2. \] (3.4)

14 In this statement, we are assuming that avoiding the pole can always be done by a contour deformation that gives a positive first-order shift to the imaginary part of the denominator. The complications hidden in justifying this assumption have already been mentioned. Nevertheless, use of the methods of Sec. [1111] will show that an exponential suppression with a singularity-avoiding contour occurs if and only if there is a contour obeying Eqs. [3.3] and [3.4].
given this first part of the derivation, the Landau condition is clearly necessary but not sufficient, since it has not yet been determined whether or not coalescing singularities actually pinch the contour. It is also not really obvious what the term “coalescing singularities” means except in one dimension. In addition, it is not clear why attention is restricted to pairs of singularities.

To provide an actual proof of necessity and sufficiency, Coleman and Norton perform the momentum integrals analytically, and work with an integral solely over the parameters, i.e., an integral of the form \[ 2.3 \] and restore the \( \ii \). It is not actually clear why they switch to this kind of integral. The rest of their argument appears to apply to a general multidimensional integral (subject to certain conditions on the quadratic terms, as we will see). Thus their arguments appear to apply equally to the integral with both momentum and parameter integrations. But they clearly think that this approach would fail.

Then they examine the denominator in the neighborhood of a point where both the denominator and its first derivative are zero. This is a place where the Landau condition is satisfied, because of the zero first derivative. They expand the denominator to quadratic order in small deviations from the candidate pinch location, which gives a formula for the denominator of the form

\[
A = \frac{1}{2} \sum_{ij} E_{ij} \eta_i \eta_j.
\]  

(4.1)

The authors then state that it is easy to show that the contour is trapped, but only if none of eigenvalues of \( E_{ij} \) is zero. However, as will be seen later in the present paper, in Sec. VIII they are a form of the Landau condition. But no proof and not make explicit what is not rigorous.

That cases of zero eigenvalues arise in massive theories in reality is mentioned; they occur only at “very exceptional points”. The reader is referred to Ref. [27] for more details. But that paper appears not to contain a clear statement of whether such singularities can occur in the physical region. Considerable further work is apparently needed to resolve the issue.

In contrast, in a massless theory, a much simpler failure happens, as will be explained in this paper in App. D1 for the case of a one-loop self-energy with massless particles. This graph has a well-known collinear pinch when the external momentum is light-like. But it is found that in the parameter integral there is no pinch that corresponds to the collinear pinch in momentum space.

A further complication is found in App. D2 in an example graph where propagators are linear in a momentum component. For that graph the pure parameter integral has a pinch independently of whether there is a pinch in the momentum integral.

Evidently Coleman and Norton have assumed that a pinch in momentum space occurs if and only if a corresponding pinch occurs in parameter space, and that this is so obvious as to need neither mention nor proof.

The examples just mentioned show that the implication is not even correct, in general, even if it works in the case of standard massive Feynman graphs.

After giving their derivation of the Landau condition, Coleman and Norton derive their well-known result that a pinch configuration corresponds to a situation with classical particles propagating and scattering in spacetime with momenta corresponding to the on-shell momenta of the lines participating in the pinch.

It is important to remember that it is not the result that breaks down, but the proof. But the proof’s breakdown is a symptom of things that were not understood. For example, in Apps. D1 and D2 are given counterexamples that implicate a failure of Coleman and Norton’s proof. But in both cases the Landau condition correctly locates pinch(es) in the momentum-space integral. The general proof in the present paper applies perfectly well to those cases. Of course the new proof is much longer than those in the old papers.

After a clear discussion of the one-dimensional case, they come to the multidimensional case on p. 47. Their subject matter is a general integral over multiple complex variables, but without the further “physical region” restrictions inherent in our [24]; these are a reality property of the denominators and an \( \ii \) prescription. Theirs is therefore in principle a more general treatment. Their equations for singularity surfaces \( S_r = 0 \) correspond to the equations \( A_j = 0 \) for the zeros of our denominator factors.

The first problem is that they say that when a singularity surface advances on the contour of integration, they say that if the singularity is to be avoided, the contour should be distorted “in the direction of the normal” to the singularity surface. This appears to say that there is a unique direction in which to distort the contour. But we have seen that in fact there is a whole half-space of possible directions, and it is absolutely necessary to take this into account. In addition, the concept of an unambiguous “normal” to a surface only makes sense in a Euclidean space, which is not the case for multidimensional complex variables with which we are concerned.

In addition, they appear to assume as so obvious as not to need a proof that for a contour deformation to avoid a singularity surface it must give a nonzero first order shift in the denominator factors (or the equivalent in their more general integral). But this definitely not the case — see App. [27].

Then in Eq. (2.1.19) they assert the conditions for singularity surfaces to trap the integration contour. These are a form of the Landau condition. But no proof and no reference to a proof is given. It is as if they think the equation is obvious. But as we will see in Secs. VII, VIII.
the condition is rather non-trivial to derive. They continue to refer to normals to surfaces, but have evidently confused the concept with the relevant one of dual vectors, so that there is considerable conceptual confusion not conducive to adequate reasoning. It is not at all obvious whether they consider the conditions to be both necessary and sufficient, and why.

Finally, their statement (2.1.19b) of the condition for a version of a Landau point lacks the positivity constraint needed for the kind of “physical region” pinch we consider. Recall that the positivity constraint is that the \( \lambda_j \) parameters in Defn. 6 are non-negative, and that at least one is positive. While the more general version is appropriate for pinches outside the physical region, further conditions are needed to determine whether or not there is a pinch. This can be seen from the fact that their version of the condition is trivially satisfied whenever the number of singularity surfaces is larger than the dimension of the integration space, as the authors do indeed observe. Hence some stronger condition than (2.1.19b) is needed to provide sufficient conditions to determine that there is a pinch.

In stark contrast, for a physical region pinch, the Landau condition (with the positivity constraint) is both necessary and sufficient. Of course this only applies given both the reality conditions on our denominator factors \( \Lambda_j \) and the \( i\epsilon \) prescription; the relevant theorem is Thm. 4 and its very non-trivial proof appears in later sections. (Our proof also has some further restrictions, given in the statement of the theorem; these are obeyed by standard and by important non-standard Feynman graphs.) It is worth re-emphasizing that it is solely the physical-region pinches that are relevant to QCD applications, and the positivity constraints on the \( \lambda_j \) parameters in the Landau point definition are very important in delimiting collinear configurations of partons.

The positivity conditions do appear in the ELOP treatment for physical region pinches/singularities, but only when they consider Feynman graphs in a Feynman parametric representation. Then the positivity conditions arise from the range of the Feynman parameters. But they do not derive the same constraints when they derive the conditions for a pinch from the pure momentum-space formula for a Feynman graph. Moreover, working in parameter space leads to the issues explained in the analysis of the Coleman-Norton treatment.

V. THE GEOMETRICAL THEOREM: SET UP

In this section and the next two sections, we will prove the last of the theorems listed in Sec. II E, i.e., the purely geometric Thm. 3. It can be regarded as giving a compatibility condition for linear constraints on directions in a vector space.

Throughout the treatment of this theorem, we work with a finite-dimensional\(^{15}\) real vector space \( V \) of dimension \( d \), and we suppose given a list \( D \) of dual vectors \( D_j \) on \( V \) (\( 1 \leq j \leq N \)). By definition, each \( D_j \) is a real-valued linear function from \( V \) to the space of real numbers. The constraints on vectors with which we are concerned are written \( D_j(v) > 0 \). In component notation, we write

\[
D_j(v) = \sum_\alpha D_{j\alpha} v^\alpha,
\]

where \( v^\alpha \) denotes the components of \( v \) with respect to some basis. But we will use coordinate-independent notation much of the time. The space of all dual vectors is a vector space \( V^* \) of the same dimension as \( V \) (if \( V \) is finite dimensional). We do not assume that there is any metric given on \( V \) or \( V^* \).

Observe that although our original subject was integration in a complex space, the manipulations involved in analyzing possible directions of deformation, and hence of the constraints \( D_j(v) > 0 \), only concern a real vector space.

In the integration problem, we were concerned with whether or not a contour deformation exists that avoids a singularity of the integrand. In the geometric problem that we are addressing at the moment, a concept corresponding to singularity avoidance in integrations is what we call a “good direction”, defined by

Definition 8. A good direction for \( (D_1, \ldots, D_N) \) is defined to be a \( v \in V \) such that \( D_j(v) > 0 \) for all \( j \).

Throughout this and the next two sections, we use the terminology of Landau points and Landau conditions that was defined in Defns. 8 and 9 names motivated by the application to integrals. The theorem to be proved is that a good direction exists for \( (D_1, \ldots, D_N) \) if and only if there is no Landau point. Alternatively, there is no good direction if and only if there is at least one Landau point.

We have already observed, in Sec. II E, that if there is a good direction then there is no Landau point and hence the Landau condition holds. Equivalently, if the Landau condition holds, then there is no good direction.

To complete the proof of Thm. 3 we need to prove the converse, i.e., that if there is no good direction then there is a Landau point. What is needed is to exclude with full generality the possibility that there might fail to exist both a Landau point and a good direction.

In simple examples, it is not too hard to see that the theorem is valid, with both directions of implication; such examples can often be visualized. But in general the vector space \( V \) can be of arbitrarily high dimension, and the number of \( D_j \) can be arbitrarily large. Then visualizing the details of the proof is hard. Thus careful abstract

\(^{15}\) The assumption of finite dimensionality can be relaxed, but we will not need to do so.
VI. GEOMETRY OF POSITIVE REGIONS OF SETS OF DUAL VECTORS

A. Setting up the problem

We use the notation of the previous section, and define the positive region of a list $\mathcal{D} = (D_1, \ldots, D_N)$ of dual vectors by

Definition 9. We define $P_\mathcal{D}$ to be the region of $V$ in which all the $D_j$s in $\mathcal{D}$ are strictly positive:

$$P_\mathcal{D} \equiv \{ v \in V : \forall D_j \in \mathcal{D}, D_j(v) > 0 \}. \quad (6.1)$$

We call this the “positive region” of $\mathcal{D}$.

The overall issue we are addressing is the determination of whether or not $P_\mathcal{D}$ is empty.

In this section, we will examine the case that $P_\mathcal{D}$ is non-empty, and determine properties of its boundary that we will need later. Observe that if $P_\mathcal{D}$ is non-empty, then all the $D_j \in \mathcal{D}$ are necessarily non-zero.

Definition 10. The complement of $P_\mathcal{D}$ is notated as:

$$\hat{P}_\mathcal{D} \equiv V \setminus P_\mathcal{D} \equiv \{ v \in V : \exists D_j \in \mathcal{D} : D_j(v) \leq 0 \}. \quad (6.2)$$

We make a lot of use of the intersection of the kernels of $D_j$. So we define

Definition 11.

$$K_\mathcal{D} \equiv \{ v \in V : \forall D_j \in \mathcal{D}, D_j(v) = 0 \}. \quad (6.3)$$

Definition 12. Define $n_\mathcal{D}$ to be the codimension of $K_\mathcal{D}$ in $V$, i.e., $n_\mathcal{D} = d - \dim(K_\mathcal{D})$.

It is well-known that $K_\mathcal{D}$ is a vector subspace of $V$. When $P_\mathcal{D}$ is non-empty, $K_\mathcal{D}$ cannot be the whole of $V$, so that in this case its codimension obeys $n_\mathcal{D} \geq 1$.

We can decompose $V$ as a direct sum of the form

$$V = V_{\perp \mathcal{D}} \oplus K_\mathcal{D}. \quad (6.4)$$

The dimension of $V_{\perp \mathcal{D}}$ is $n_\mathcal{D}$. Note that $V_{\perp \mathcal{D}}$ is non-unique, since its basis vectors can be changed by the addition of elements of $K_\mathcal{D}$. If we are given that $P_\mathcal{D}$ is non-empty, then there must be a region of $V_{\perp \mathcal{D}}$ where the $D_j$ are positive.

The critical result that we are working towards in this section is Thm. 8 below, where we find a set of non-zero “edge vectors” $e_L$ for $P_\mathcal{D}$ such that every element of $v$ of $P_\mathcal{D}$ has the form $v = \sum L C_L e_L + v_K$, where all the $C_L$ are positive real numbers; $C_L > 0$, and $v_K \in K_\mathcal{D}$.

To derive Thm. 8, we will need a series of subsidiary results, many of which are very elementary, and are obvious in low-dimensional examples. But these results need to be explicitly stated in order to ensure that the main theorem is properly proved in a space of arbitrarily high dimension; their cumulative effect is quite non-trivial. Many of the subsidiary results are likely to be useful in themselves for applications, e.g., for searching for good directions to deform a contour when there is no pinch.

B. Elementary properties of $P_\mathcal{D}$

Theorem 4. $P_\mathcal{D}$ obeys

(a) It is convex, i.e., if $v_1, v_2 \in P_\mathcal{D}$ and $\kappa$ is any real number between 0 and 1 inclusive (i.e., $0 \leq \kappa \leq 1$), then $\kappa v_1 + (1 - \kappa) v_2 \in P_\mathcal{D}$.

(b) If $v \in P_\mathcal{D}$ then $\lambda v \in P_\mathcal{D}$ for any positive real $\lambda$.

(c) $P_\mathcal{D}$ is connected.

(d) It is an open set.

Proof. Suppose that $v_1, v_2 \in P_\mathcal{D}$, that $\lambda_1$ and $\lambda_2$ are real numbers, that both $\lambda_1, \lambda_2 \geq 0$, and that at least one is strictly positive. Then each $D_j(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 D_j(v_1) + \lambda_2 D_j(v_2)$ is strictly positive, and hence $\lambda_1 v_1 + \lambda_2 v_2 \in P_\mathcal{D}$. (This demonstrates that $P_\mathcal{D}$ is an example of a convex cone in mathematical terminology.)

Properties (a) and (b) immediately follow, and then so does (c) from (a).

To derive part (d), let $v \in P_\mathcal{D}$, and let $l = \min_{D_j \in \mathcal{D}} D_j(v) > 0$. Now let $\delta v$ be another element of $V$. Then

$$D_j(v + \delta v) = D_j(v) + D_j(\delta v) \geq l + D_j(\delta v). \quad (6.5)$$

For all small enough $\delta v$, we have $|D_j(\delta v)| < l$ for every $D_j \in \mathcal{D}$, and then $v + \delta v \in P_\mathcal{D}$. Hence $P_\mathcal{D}$ is open.

Since $P_\mathcal{D}$ is open, it is a manifold of the same dimension as $V$, i.e., $d$, provided only that it is non-empty.

From now on, we will assume that $P_\mathcal{D}$ is non-empty, unless explicitly stated, and will only reiterate this assumption when it seems particularly important.
C. Basic properties of the boundary of $P_D$

We now consider the boundary $\partial P_D$ of $P_D$, i.e., the set of points of $V$ that are limit points both of $P_D$ and its complement $\hat{P}_D$.

**Theorem 5.** If $P_D$ is non-empty, the boundary of $P_D$ is characterized by

$$\partial P_D = \{ v \in V : \forall D_j \in D, D_j(v) \geq 0 \text{ and } \exists D_j \in D : D_j(v) = 0 \}. \quad (6.6)$$

It follows that the boundary is contained in $\hat{P}_D$.

**Proof.** Suppose we have a point $v \in \partial P_D$. Then there is a sequence $v_n$ in $P_D$ whose limit is $v$. So for all $D_j \in D$

$$D_j(v) = \lim_{n \to \infty} D_j(v_n) \geq 0. \quad (6.7)$$

If $D_j(v)$ were also nonzero for all $D_j$, then it would be in $P_D$. Since $P_D$ is open, this would imply that $v$ is not in its boundary. Hence we must have $D_j(v) = 0$ for at least one $D_j$.

Conversely, suppose we have a point $v \in V$ for which all the $D_j(v)$ are positive or zero, and at least one of which is zero, i.e.,

$$\forall D_j \in D, D_j(v) \geq 0, \text{ and } \exists D_j \in D : D_j(v) = 0. \quad (6.8)$$

Then choose $\delta v \in P_D$. For every positive real number $\lambda$, $D_j(v+\lambda \delta v) = D_j(v) + \lambda D_j(\delta v) > 0$, so that $v+\lambda \delta v \in P_D$. Thus $v$ is a limit point of $P_D$. But it is not in $P_D$, so it must be in the complement $\hat{P}_D$. It follows that $v$ is trivially a limit point of $\hat{P}_D$.

**Theorem 6.** (a) The subspace where all the $D_j$s are zero is inside the boundary of $P_D$. I.e., $K_D \subseteq \partial P_D$.

(b) $\partial P_D$ is connected.

**Proof.** Every element $k$ of $K_D$ obeys $D_j(k) = 0$, for all $j$, and is thus in $\partial P_D$, by Thm. Hence $K_D \subseteq \partial P_D$.

Since the zero vector is in $K_D$ it is also in $\partial P_D$. For any $v$ in $\partial P_D$, $\lambda v$ is also in $\partial P_D$ whenever $\lambda \geq 0$. This gives a line connecting an arbitrary element of $\partial P_D$ to one particular element, i.e., the zero vector. Hence $\partial P_D$ is connected.

For our purposes, the interesting parts of $\partial P_D$ are those that are not in $\hat{P}_D$, i.e., where at least one $D_j(v)$ is strictly positive. Therefore we define

**Definition 13.** The non-trivial part of the boundary of $P_D$ is

$$\tilde{\partial} P_D \overset{\text{def}}{=} \partial P_D \setminus K_D. \quad (6.9)$$

The set $\tilde{\partial} P_D$ may be empty; our later work shows that this happens if and only if $n_D = 1$ (or, of course if $P_D$ itself is empty).

From Thm. it follows that the non-trivial part of the boundary obeys

$$\tilde{\partial} P_D = \{ v \in V : \forall D_j \in D, D_j(v) \geq 0 \text{ and } \exists D_j \in D : D_j(v) = 0 \}, \quad (6.10)$$

i.e., all the $D_j(v)$ are non-negative, at least one is zero, and at least one is positive.

**Definition 14.** Here we define some auxiliary objects at a point $w$ that is in the non-trivial part of the boundary, $w \in \tilde{\partial} P_D$.

(a) The sets of $D_j$ with zero and non-zero values are:

$$Z(w) \overset{\text{def}}{=} \{ D_j \in D : D_j(w) = 0 \}, \quad (6.11a)$$

$$\tilde{Z}(w) \overset{\text{def}}{=} \{ D_j \in D : D_j(w) > 0 \}. \quad (6.11b)$$

Given $w$, each $D_j$ is in exactly one of these sets, of course. Both sets are non-empty when $w$ is in the non-trivial part of the boundary.

(b) The minimum non-zero value of the $D_j(w)$s is:

$$m(w) \overset{\text{def}}{=} \min_{D_j \in \tilde{Z}(w)} D_j(w) > 0. \quad (6.12)$$

(c) Let $K(w)$ be the intersection of the kernels of those $D_j$ that are in $Z(w)$:

$$K(w) \overset{\text{def}}{=} \{ v \in V : \forall D_j \in Z(w) : D_j(v) = 0 \} = \cap_{D_j \in Z(w)} \ker(D_j) \quad (6.13)$$

(d) Let $n(w)$ be the codimension of $K(w)$, so that the dimension of $K(w)$ is $d - n(w)$.

Note that $w$ is one (non-zero) element of the subspace $K(w)$.

Since $P_D$ is non-empty, there are vectors $v$ for which $D_j(v) > 0$ for all $j$. It follows that $K(w)$ cannot be the whole of $V$. Hence

$$n(w) \geq 1. \quad (6.14)$$

D. Decomposition of the boundary of $P_D$

In this section, we show that the boundary of $P_D$ can be decomposed into a hierarchy of disjoint flat segments. On each of these one set of $D_j$s is strictly positive and the others are zero.

First, given $w \in \tilde{\partial} P_D$, we construct the boundary segment of which it is part. We define

$$B(w) \overset{\text{def}}{=} \{ v \in K(w) : \forall D_j \in \tilde{Z}(w), D_j(v) > 0 \} = \{ v \in V : \forall D_j \in Z(w), D_j(v) = 0; \text{ and } \forall D_j \in \tilde{Z}(w), D_j(v) > 0 \}. \quad (6.15)$$
Note that $B(w)$ is a subset of $K(w)$.

The boundary segments have the following elementary properties

**Theorem 7.** (a) $B(w)$ is convex.

(b) Whenever $v \in B(w)$, so is $\lambda v$ for positive $\lambda$.

(c) $B(w)$ is a flat connected manifold of the same dimension as $K(w)$, i.e., $d - n(w)$.

(d) For every point $v \in B(w)$,

$$Z(v) = Z(w), \quad \hat{Z}(v) = \hat{Z}(w),$$

and similarly for $K(v), B(w)$, and $n(w)$. \hfill (6.16)

(e) When $v \in B(w)$, we have $B(v) = B(w)$.

(f) For any $v$ and $w$ in $\partial P_D$, either $B(v)$ and $B(w)$ are non-intersecting or they are equal. It immediately follows that the boundary $P_D$ is decomposed into a set of disjoint flat segments.

**Proof.** Parts (a) and (b) follow by the same method used to prove the corresponding properties for $P_D$.

It immediately follows that $B(w)$ is connected and flat. As to the dimension, first note that from its definition, $B(w)$ is contained in the kernel space $K(w)$, so that its dimension is at most that of $K(w)$, i.e., $d - n(w)$. Furthermore, let $\delta w$ be any element of $K(w)$. For all small enough $\delta w$, $w + \delta w$ is in $B(w)$. This is because when $D_j \in Z(w)$, $D_j(w + \delta w) = D_j(w) + D_j(\delta w) = 0$, and because when $D_j \in \hat{Z}(w)$ and $\delta w$ is small enough the value of $D_j(\delta w)$ cannot compensate the positive value of $D_j(w)$. Hence the dimension of $B(w)$ is at least $d - n(w)$.

Property (e) now follows.

Next suppose $v \in B(w)$. By the definition of $B(w)$, $D_j(v) = 0$ for every $D_j$ in $Z(w)$, and $D_j(v) > 0$ for every $D_j$, so $Z(v)$ is the same as $Z(w)$, since it is the set of $D_j$ for which $D_j$ is zero at $v$. From this follows all of (6.16).

It then follows from the definition of $B(v)$ that $B(v) = B(w)$ whenever $v \in B(w)$, which thereby proves property (f).

Now consider $B(v)$ and $B(w)$ for two points $v$ and $w$. Either they do not intersect or they intersect. In the second case, pick $k$ in the intersection. From the previous result it follows that $B(k) = B(v)$ and $B(k) = B(w)$, and hence that $B(v) = B(w)$. This proves property (f). \hfill \square

Since the sets $Z(v)$, $\hat{Z}(v)$, and $K(v)$, and the number $n(v)$ are constant on any given boundary segment $B$, we can say that $Z$ etc are determined by the set of points $B$. Thus we can write

$$Z(B) \overset{\text{def}}{=} \{ D_j : D_j(v) = 0 \text{ for every } v \in B \} = Z(v) \text{ for every } v \in B,$$

and similarly for $\hat{Z}(B), K(B)$ and $n(B)$.

Notice that subspace $K(B)$ contains the common kernel subspace $K_D$ of all the $D_j$. For a non-trivial boundary segment $B$ this implies that the subspace $K(B)$ is strictly larger than $K_D$. This is because in this case there are points of $K(B)$ where at least one of $D_j$ is non-zero; these points cannot be in $K_D$. Hence the codimensions obey $n(B) \leq n_D$, with $n_D$ being the codimension of the smallest (trivial) boundary segment, i.e., the common kernel of all the $D_j$, and with equality only for the trivial boundary segment.

Observe that each boundary segment $B$ obeys all of the properties of positive regions, but with respect to $K(B)$ instead of the whole space $V$, and with respect to $\hat{Z}(B)$ instead of $D$. In particular, it is an open and convex set in $K(B)$. Moreover, the same arguments as given above for $P_D$ show that each $B$ itself has a boundary consisting of boundary segments, which are also boundary segments of $P_D$ itself, with all the associated properties.

There is in fact a hierarchy of boundary segments, for which it is possible to prove the following results:

1. The unique lowest dimension boundary segment is the subspace $K_D$, of dimension $d - n_D$.

2. There are boundary segments of every dimension between the minimum dimension $d - n_D$ and the maximum dimension $d - 1$, inclusive.

3. Each boundary segment of non-maximal dimension is a boundary segment of a boundary segment of one dimension higher. If it has the maximal dimension $d - 1$, it is a boundary segment only of $P_D$ itself.

4. $P_D$ and non-minimal boundary segments have one or more boundary segments of one dimension lower.

In visualizable examples, the existence of this hierarchy and many of its properties are quite obvious. But the general case needs a proof, which is non-trivial. For the purposes of this paper, we will not need the whole collection of properties of the hierarchy, so we will not make all the proofs.

What we do need are the boundary segments of one dimension higher than the minimal dimension, whose existence we will prove. Projected onto a subspace $V_{L\bar{D}}$ that gives a decomposition of the form in Eq. (6.4), the next-to-minimal boundary segments become line segments. This leads us to the concept of edge vectors specifying the directions of the next-to-minimal boundary segments. The edge vectors play a critical role in our later analysis.

**E. Edge vectors $e_L$**

Now we construct what we call the edge vectors $e_L$ of $P_D$. Each edge vector has a label $L$, whose meaning will be given below. There are two cases (with $P_D$ being non-empty, as we are assuming): One is where the subspace $V_{L\bar{D}}$ in Eq. (6.4) has dimension $n_{D_L} = 1$ and the other is where it has a higher dimension.
1. Case $n_{\mathcal{D}} = 1$

First is the case $n_{\mathcal{D}} = 1$, i.e., that the subspace $V_{\perp \mathcal{D}}$ defined in Eq. (3.4) has dimension one. As observed below that equation, there is a region of $V_{\perp \mathcal{D}}$ where all the $D_j$ are positive. We choose any vector in this region to be the single edge vector $e$ for $P_{\mathcal{D}}$; no more will be needed. For every $D_j$, $D_j(e) > 0$.

Then every vector $v \in V$ is of the form $v = Ce + k$ for some $k \in K$ and some real number $C$. Then

$$D_j(v) = CD_j(e) + D_j(k) = CD_j(e).$$  \hspace{1cm} (6.18)

So the condition that $v \in P_{\mathcal{D}}$ is simply that $C > 0$. Then

$$P_{\mathcal{D}} = \{Ce + k : C > 0 \text{ and } k \in K\}. \hspace{1cm} (6.19)$$

Note that $e$ is non-unique, but only up to a scaling by a positive factor and the addition of an element of $K$.

Any single choice of $e$ is sufficient for our purposes.

From Eq. (6.18) it follows that $D_j = \frac{D_j(e)}{D_1(e)} D_1$ and hence that all the $D_j$ are proportional to each other, with positive coefficients.

2. Case $n_{\mathcal{D}} \geq 2$

For all the higher co-dimension cases, we will see that $P_{\mathcal{D}}$ has non-trivial boundary segments, with lower dimension. These in turn have boundary segments, etc. At each stage of taking boundaries, one has a strictly lower dimension.

The minimum possible dimension for a non-trivial boundary segment is $d - n_{\mathcal{D}} + 1$. Later, we will prove results about the existence and properties such next-to-minimal boundary segments. Here we will simply provide a definition of corresponding edge vectors, i.e., a vector $e_L$ for each next-to-minimum dimension boundary segment $L$.

Let $L$ be one such boundary segment. We apply to it the argument of Sec. [VI] but applied for $L$ with respect to $K(L)$ instead of $P_{\mathcal{D}}$ with respect to $V$, and with the set $Z(L)$ instead of $D$. We then choose a corresponding vector $e_L$ in the boundary segment. A general $v$ in $K(L)$ is $\lambda e_L + k$ where $\lambda$ is real and $k \in K$. We find the conditions for $v$ to be in $L$ as follows: For $D_j \in Z(L)$, $D_j(e_L) = 0$ by the construction of $e_L$, so $D_j(v) = 0$. For $D_j \in \hat{Z}(L)$, $D_j(v) = \lambda D_j(e_L)$. Hence

$$L = \{\lambda e_L + k : \lambda > 0 \text{ and } k \in K\}. \hspace{1cm} (6.20)$$

3. Overall definition of set of edge vectors

If $n_{\mathcal{D}} \geq 2$, we define the set of edge vectors to be all the $e_L$ found in Sec. [VI] for each boundary segment $L$ that obeys $n(L) = n_{\mathcal{D}} - 1$.

If $n_{\mathcal{D}} = 1$, the set of edge vectors is simply the set consisting of the one element $e$ constructed in Sec. [VI].

The name “edge vector” is appropriate when $n_{\mathcal{D}} \geq 2$, since each $e_L$ then corresponds to a projection of boundary segment $L$ onto a line in $V_{\perp \mathcal{D}}$, a projection onto a segment of a line. But “edge vector” is a bit of a misnomer in the case that $V_{\perp \mathcal{D}}$ is one-dimensional, i.e., $n_{\mathcal{D}} = 1$.

F. The main decomposition theorem

We are now ready to prove the following theorem:

**Theorem 8.** Every element of $v$ of $P_{\mathcal{D}}$ can be written in the form

$$v = \sum L C_L e_L + v_K, \hspace{1cm} (6.21)$$

where all the $C_L$ are positive real numbers, $C_L > 0$, and $v_K \in K$, and where the set of $e_L$ is a set of edge vectors, as defined in Sec. [VI]. Conversely, every $v$ of the form Eq. (6.21) with positive $C_L$ is in $P_{\mathcal{D}}$.

Thus $P_{\mathcal{D}}$ is exactly the set of vectors of the form (6.21) with the stated restrictions.

Before proving the theorem, we make the following comments:

- The values $C_L$ need not be unique, since it may happen that the number of $e_L$ is larger than the dimension $n_{\mathcal{D}}$ of $V_{\perp \mathcal{D}}$. In that case, the $e_L$ are over-complete as a spanning set. If we removed the extra $e_L$ compared with those needed to make a basis for $V_{\perp \mathcal{D}}$, we could still express $v$ in the form (6.21), but some of the coefficients might need to be negative for some values of $v$.

- The edge vectors $e_L$ are not actually in $P_{\mathcal{D}}$ except in the almost trivial case that $n_{\mathcal{D}} = 1$. In other cases, they are always on the boundary of $P_{\mathcal{D}}$, as we saw.

1. Examples

Before treating the general case, we examine examples with effective dimension one and two, i.e., $n_{\mathcal{D}} = 1$ and $n_{\mathcal{D}} = 2$. Then the derivation of the corresponding specializations of the theorem will be elementary. The trick for the general case is to find a way of successively reducing the dimension of the problem by repeated application of the two-dimensional version.

In setting up the examples in a fairly general context, it is useful to recall the following theorem of linear algebra:

Let $E = \{E_1, \ldots, E_A\}$ be dual vectors on a vector space $V$, and let $K_E$ be the intersection of their kernels, as defined earlier. Let $F$ be another dual vector. Then $F$ is a linear combination of $E_1, \ldots, E_A$ if and only if the kernel of $F$ contains $K_E$, i.e., $\ker F \supseteq K_E$. 

\[ \text{FIG. 2. Positivity constraint in } V_{⊥D} \text{ for the case that it is one-dimensional, i.e., } n_D = 1. \text{ that } n_D = 1. \text{ All the } D_j \text{ are necessarily proportional. The solid line is where } D_j(v) > 0, \text{ i.e., it is } P_D \text{ projected onto } V_{⊥D}. \text{ The dotted line is where } D_j(v) \leq 0. \]

\[ \text{FIG. 3. Positivity constraints in } V_{⊥D} \text{ for a case where it is two-dimensional, i.e., } n_D = 2. \text{ The diagram depicts the case that there are three different } D_j \text{ involved. The diagonal lines are the locations of the kernels of the } D_j, \text{ and the shaded parts point to the negative regions of the } D_j. \]

The example of \( n_D = 1 \) was already treated in Sec. [VI.E.1]. Observe that the common kernel \( K_D \) of the \( D_j \) has its maximum possible dimension \( d - 1 \), and is equal to the kernel of every \( D_j \), and that all the \( D_j \) are all proportional to each other (with positive coefficients so that \( P_D \) is non-empty). We constructed an instance of the single edge vector needed for the problem, and obtained the decomposition Eq. (6.19). Positivity constraints can be obtained by examining values of \( D_j \) on the space \( V_{⊥D} \), and the results visualized because it is one-dimensional, as in Fig. [2].

In the case \( n_D = 2 \), \( V_{⊥D} \) is a two-dimensional space illustrated in Fig. [3]. Each of the \( D_j \) has a positive space delimited by its kernel. Let us parameterize vectors in \( V_{⊥D} \) by polar coordinates \((r, θ)\) with respect to some axes. Then the positive region for each \( D_j \) is a range \( r > 0 \) with \( θ \) in a continuous range of size \( π \). The kernel of each \( D_j \) is a line of fixed \( θ \). The common positive region is of the form \( α < θ < β \), where \( 0 < β - α < π \). The most limiting directions are given by two distinct \( D_j \) whose kernels are the lines of angles \( α \) and \( β \): we use vectors in these directions for the edge vectors \( e_L \), and it is evident that the common positive region \( P_D \) is the set of all linear combinations of the two \( e_L \) with positive coefficients. In polar coordinates, the edge vectors can be chosen as unit vectors with angles \( α \) and \( β \); they are linearly independent because \( 0 < β - α < π \).

If we had made a mistake in stating the situation, and in fact all the \( D_j \) were proportional to each other, then all the kernels would lie on top of each other, and we would get the situation shown in Fig. [4]. Then the positive range is \( α < θ < β \), but now with \( β - α = π \), so that the would-be \( e_L \) vectors from Fig. [3] at angles \( α \) and \( β \), are exactly opposite to each other, and are therefore linearly dependent. These now span one dimension of the kernel space instead of the positive manifold. The kernel space has its dimension increased by one, and correspondingly \( V_{⊥D} \) has its dimension reduced by one. To get an exemplar of the single edge vector that is needed, we choose a vector pointing in a direction intermediate between angles \( α \) and \( β \). To get the results in terms of \( V_{⊥D} \), we simply project onto a one-dimensional space in the direction of the edge vector, after which we recover a version of Fig. [2].

2. General case

If \( n_D = 1 \), we already proved the appropriate specialization of Thm. [8] in Sec. [VI.E.1] with \( L \) having one value and the associated \( e_L \) being the \( e \) of that section.

We now provide a method to deal with all the remaining cases \( n_D ≥ 2 \) (including the already treated case of \( n_D = 2 \)). Necessarily, at least two of the \( D_j \) are linearly independent. Otherwise all of them would be proportional to each other (with positive coefficients to allow \( P_D \neq ∅ \), and then the positive space is the positive space for one \( D_j \), so that we get \( n_D = 1 \).

Let \( v \) be any vector in \( P_D \). Then to prove that it is of the form Eq. (6.21), we adopt the following recursive strategy

1. Construct an expression for \( v \) as a linear combination of two vectors on non-trivial boundary segments. This we will do quite easily, by a simple generalization of the two-dimensional case that was illustrated in Fig. [3].

2. For each of these vectors:

   (a) Either its boundary segment is of the lowest possible dimension for a non-trivial boundary segment, i.e., \( d - n_D + 1 \), and we can write the vector as a positive coefficient times the chosen edge vector for the segment, plus a contribution from a vector in the kernel \( K_P \).
FIG. 5. The dotted line is the circle explored to express \( v \) in terms of boundary vectors, defined to be where the circle first hits the kernel of a \( D_j \). Here are seen the intersections of ker \( D_a \) and ker \( D_b \) with the two dimensional space spanned by \( v \) and \( w \). Note that there is not necessarily any metric specified on the space \( V \), and even if there were there would be no guaranteed constraint on the angle between \( v \) and \( w \). Nevertheless, it is always possible to change the coordinate system by applying a linear transformation. One can do this to go from a situation where \( v \) and \( w \) are in general directions to one where they are drawn at right angles, as is the case here. With this choice of coordinates, the loop of vectors in Eq. (6.22) becomes a circle.

(b) Or the boundary segment has a higher dimension, in which case we repeat the procedure to express the vector in terms of vectors on non-trivial boundary segments of yet lower dimension.

3. All of this terminates when we get to the lowest dimension non-trivial boundary segments. This gives the desired expansion.

To implement this strategy, given a vector \( v \) in \( P_D \), we first pick two independent \( D_j \) in \( D \), and call them \( D_a \) and \( D_b \). Then pick any vector \( \delta v \) in the kernel of \( D_b \) such that \( D_a(\delta v) > 0 \), and make it small enough that \( v + \delta v \) is still in \( P_D \). Then let \( w = v + \delta v \). The geometry of this situation in the two-dimensional space spanned by \( v \) and \( w \) is shown in Fig. 5. The vectors \( v \) and \( w \) are linearly independent, so that they do in fact span a two-dimensional space.

Now \( D_j(v) \) and \( D_j(w) \) are positive, for all \( j \) including \( j = a \) and \( j = b \), and in addition

\[
\begin{align*}
D_a(w) &= D_a(v) + D_a(\delta v) > D_a(v), \\
D_b(w) &= D_b(v) + D_b(\delta v) = D_b(v).
\end{align*}
\]  

(6.22)  

(6.23)

Let \( r = D_a(\delta v)/D_a(v) > 0 \), so that \( D_a(w) = (1 + r)D_a(v) \). Then consider the following loop of vectors in the plane of \( v \) and \( w \), parameterized by an angle \( \theta \):

\[
\begin{align*}
u(\theta) &\overset{\text{def}}{=} v \cos \theta + w \sin \theta,
\end{align*}
\]  

(6.24)

on which for a general \( D_j \)

\[
D_j(u(\theta)) = D_j(v) \cos \theta + D_j(w) \sin \theta.
\]  

(6.25)

Since both of \( D_j(v) \) and \( D_j(w) \) are positive, \( D_j(u(\theta)) \) is positive in the range \( 0 < \theta < \pi/2 \), and also somewhat beyond this range. Now define

\[
\theta_j \overset{\text{def}}{=} \arctan \frac{D_j(v)}{D_j(w)},
\]  

(6.26)

which is in the range \( 0 < \theta_j < \pi/2 \). The zeros of \( D_j(u(\theta)) \) are at \( \theta = -\theta_j \) and \( \theta = \pi - \theta_j \), so that \( D_j(u(\theta)) \) is positive when \(-\theta_j < \theta < \pi - \theta_j \).

For the specific cases of \( D_a \) and \( D_b \)

\[
\begin{align*}
D_a(u(\theta)) &= D_a(v)[\cos \theta + (1 + r) \sin \theta], \\
D_b(u(\theta)) &= D_b(v)[\cos \theta + \sin \theta],
\end{align*}
\]  

(6.27)  

(6.28)

so that

\[
\theta_a = \arctan \frac{1}{1 + r} < \frac{\pi}{4}, \quad \theta_b = \frac{\pi}{4}.
\]  

(6.29)

Now define the minimum and maximum values of the \( \theta_j \):

\[
\alpha \overset{\text{def}}{=} \min_j \theta_j, \quad \beta \overset{\text{def}}{=} \max_j \theta_j.
\]  

(6.30)

Then for \(-\alpha < \theta < \pi - \beta \), all \( D_j(u(\theta)) \) are positive, so \( u(\theta) \in P_D \). But at each of \( \theta = -\alpha \) and \( \theta = \pi - \beta \), at least one \( D_j(u(\theta)) \) is zero, so that \( u(-\alpha) \) and \( u(\pi - \beta) \) are on the boundary of \( P_D \). They are in fact on the non-trivial part of the boundary of \( P_D \) and are linearly independent. To see this, we first observe that from Eq. (6.29) and from \( 0 < \theta_j < \pi/2 \), it follows that \( 0 < \alpha \leq \theta_a < \pi/4 \), while \( \pi/2 > \beta \geq \theta_b = \pi/4 \). It follows that \( D_b(u(-\alpha)) \) and \( D_a(u(\pi - \beta)) \) are both positive, which puts the vectors \( u(-\alpha) \) and \( u(\pi - \beta) \) on the non-trivial part of the boundary of \( P_D \), where at least one \( D_j \) is positive. Furthermore, from the same bounds, it follows that \(-\alpha \) and \( \pi - \beta \) are not opposite angles, and hence that \( u(-\alpha) \) and \( u(\pi - \beta) \) are linearly independent. See Fig. 6 for an illustration of how another \( D_j \) can impose a more restrictive bound on where \( u_j(\theta) \in P_D \) than is given by \( D_a \) and \( D_b \) alone.

Since \( u(-\alpha) \) and \( u(\pi - \beta) \) are on the non-trivial part of the boundary of \( P_D \), we have incidentally proved that for the case we are treating, \( n_D \geq 2 \), there are in fact non-trivial boundary segments.

We can now express \( v \) in terms of non-trivial boundary vectors:

\[
v = v_1 \frac{\sin \beta}{\sin(\beta - \alpha)} + v_2 \frac{\sin \alpha}{\sin(\beta - \alpha)}
\]  

(6.31)

where

\[
v_1 = u(-\alpha), \quad v_2 = u(\pi - \beta).
\]  

(6.32)

The coefficients in Eq. (6.31) are positive, so we have accomplished our aim of expressing \( v \) in terms of vectors on the non-trivial part of boundary of \( P_D \) with positive
coefficients. Let the boundary segments in which \(v_1\) and \(v_2\) lie be \(B_1\) and \(B_2\)

First consider the case that \(v_1\)'s (non-trivial) boundary segment has the minimum dimension \(d - n(B_1) = d - n_D + 1\). Then there is an edge vector for that segment, as defined in Sec. VII E 2, and \(v_1\) can be expressed in terms of the edge vector, with a positive coefficient, plus an element of the common kernel \(K_D\).

The other case is that \(v_1\)'s boundary segment \(B_1\) is of higher dimension. Then we apply the whole argument of this section to \(v_1\), but now instead of \(D\) and \(V\), we apply the argument to the dual vectors \(\bar{Z}(B_1)\) that are non-zero at \(v_1\) and work in the space \(K(B_1)\). The argument needs to be extended only by the observation that all the vectors involved give zero for any \(D_j \in Z(B_1)\), i.e., for any \(D_j\) that is zero at \(v_1\) and hence on \(B_1\).

The result is to express \(v_1\) in terms of vectors in yet lower dimension boundary segments.

The same argument applies equally to \(v_2\).

Iterating the argument eventually stops when all the vectors obtained are proportional to edge vectors (plus elements of \(K_D\)), with positive coefficients. Thus any element \(v \in P_D\) is a linear combination of edge vectors with positive coefficients, plus a vector in \(K_D\).

Hence any vector in \(P_D\) is of the form (6.21) given in the statement of Thm. 8.

To complete the proof of Thm. 8 we need to show that any vector of the form \(6.21\) is in the positive manifold \(P_D\), as opposed to being in its boundary. So let \(v\) be any vector of the form \(6.21\). For each \(D_j\), at least one \(D_j(e_L)\) is positive, and so \(D_j(v)\) is strictly positive. Hence \(v \in P_D\).

VII. THE LANDAU THEOREM FOR DUAL VECTORS

Now we come to the already-stated Thm. 3 relating the Landau condition to the non-existence of good directions, i.e., to the non-existence of a \(v\) for which all of \(D_j(v)\) are positive. To prove the theorem, we consider the cases that there is and that there is not a good direction.

We already saw in Sec. VII E that if there is a good direction, then there can be no Landau point. It remains to show that if there is no good direction, then a Landau point exists. Given that there fails to be a good direction for \((D_1, \ldots, D_N)\), we will construct a set of \(\lambda_j\)'s that instantiates a Landau point.

It might be that one or more of the \(D_j\)'s is zero. In that case, let \(D_{j_0}\) be one of the zero dual vectors. Then set \(\lambda_{j_0} = 1\) and set the remaining \(\lambda_j\) to zero, and we have a Landau point.

So we only need further to consider the case that every \(D_j\) is non-zero.

Consider the following subsets of \(D_j\)'s, where we start with \(D_1\), and successively add an extra \(D_j\): \(S_1 = (D_1)\), \(S_2 = (D_1, D_2)\), \(\ldots\), \(S_N = (D_1, \ldots, D_N)\). Since \(D_1 \neq 0\), we can find a vector \(v \in V\) with \(D_1(v) = 1\), and so there exists a good direction for \(S_1\). But by hypothesis there is no good direction for \(S_N\). Therefore there is a last one in this sequence, \(S_{n_0}\), for which there is a good direction; for the next set, \(S_{n_0+1}\), there is no good direction.

In the following, two different vector spaces come into play. One is the space \(V\) on which the \(D_j\)'s act, with an important role played by its submanifold where all the \(D_j\)'s are positive. The other space is a space \(\Lambda\) of the coefficients \(\lambda\) used in linear combinations of the form \(\sum_{j=1}^{n_0} \lambda_j D_j + D_{n_0+1}\), with its definition in Eq. (7.9) below.

A. The positive hyperplane \(P\)

Let \(P\) be the set of good directions for \(S_{n_0}\), i.e., \(P\) is the positive space for the corresponding \(D_j\)'s:

\[
P = P_{S_{n_0}} = \{ v \in V : D_j(v) > 0 \text{ whenever } 1 \leq j \leq n_0 \}.
\]

Then the lack of a good direction for \(S_{n_0+1}\) immediately shows that \(D_{n_0+1}(v) \leq 0\) for every \(v \in P\). In fact, strict inequality holds:

Lemma 1.

\[
D_{n_0+1}(v) < 0 \text{ for every } v \in P .
\]

Proof. We use the fact, following from Thm. 4 that \(P\) is an open set. Suppose that the strict inequality did not hold. Then there would be a \(v \in P\) for which \(D_{n_0+1}(v) = 0\). Since \(D_{n_0+1}\) is non-zero, we can find a \(w \in V\) for which \(D_{n_0+1}(w) = 1\). Then for every \(\kappa > 0\), \(D_{n_0+1}(v + \kappa w) = \kappa > 0\). Since \(P\) is an open set, \(v + \kappa w \in P\) for all small enough \(\kappa\), and we would therefore find a vector in \(P\) on which \(D_{n_0+1}\) is positive. That is, we would find a good direction for the set \(S_{n_0+1}\). This is contrary to hypothesis, so we need the strict inequality (7.2).
B. Kernels of $D_j$ (0 $\leq$ $j$ $\leq$ $n_0 + 1$)

Next, let $K$ be the intersection of the kernels of $D_1$, $\ldots$, $D_{n_0}$:

$$K \overset{\text{def}}{=} \{ v \in V : D_j(v) = 0 \text{ for } 1 \leq j \leq n_0 \}. \quad (7.3)$$

It is a vector subspace of $V$.

Now the other $D_j$ we consider, i.e., $D_{n_0+1}$, is also zero on $K$. To see this, suppose otherwise, and we will prove a contradiction. Thus, suppose that there is a $u \in K$ such that $D_{n_0+1}(u) \neq 0$. By scaling $u$, we can arrange $D_{n_0+1}(u) = 1$, while maintaining $D_j(u) = 0$ for the other $D_j$. Pick any $v \in P$, so that $D_j(v) > 0$ for every $1 \leq j \leq n_0$. Then for every positive real number $\kappa > 0$

$$D_{n_0+1}(\kappa u + v) = \kappa + D_{n_0+1}(v), \quad (7.4)$$

while

$$D_j(\kappa u + v) = \kappa D_j(u) + D_j(v) = D_j(v) > 0 \quad \text{for } 1 \leq j \leq n_0. \quad (7.5)$$

It follows that $\kappa u + v$ is also in $P$. But by choosing $\kappa$ large enough, we can make $D_{n_0+1}(\kappa u + v)$ positive, which would give us a good direction for the set $S_{n_0+1}$. We only avoid this by having $D_{n_0+1}(u) = 0$ for every element $u \in K$.

Thus the kernel of $D_{n_0+1}$ contains the intersection of the kernel of the other $D_j$s. It follows, by a standard theorem of linear algebra, that $D_{n_0+1}$ is a linear combination of the other $D_j$s. But we do not need to use this. In fact, we will prove a stronger result that a linear combination can be found where all the coefficients are negative or zero.

C. Spanning vectors of $P$

We now recall results from Sec. VI but applied with the set $D$ instead of the original set of dual vectors. The space $V$ can be decomposed as a direct sum $V = K \oplus V_\perp$. Then there is a set of non-zero edge vectors $e_L$ that give one-dimensional edges for $P \cap V_\perp$, and the general form for a vector $v \in P$ is

$$v = k + \sum L \, C_L e_L, \quad (7.6)$$

where $k \in K$ and all the real-valued coefficients $C_L$ are strictly positive: $C_L > 0$. The vectors $e_L$ span $V_\perp$, but they could be an over-complete set; the extra elements are needed to maintain the positivity property on the $C_L$s for every $v \in P$.

All the edge vectors obey $D_j e_L \geq 0$ for $1 \leq j \leq n_0$ and any $L$. For every $j$ in the range $1 \leq j \leq n_0$, there is at least one value of $L$ for which $D_j e_L$ is strictly positive. Similarly for every $L$ there is at least one value of $j$ in the range $1 \leq j \leq n_0$ for which $D_j e_L$ is strictly positive.

From the properties that $D_{n_0+1}(v) < 0$ for every $v \in P$ and that $D_{n_0+1}(v) = 0$ for every $v \in K$, it follows that

Lemma 2.

For all $L$, $D_{n_0+1}(e_L) \leq 0$, \quad (7.7)

and at least one $D_{n_0+1}(e_L)$ is strictly negative.

D. Linear combinations of $D_j$s; the regions $\Lambda$, $M$, and $\hat{M}$

Our aim is to find a set of $\lambda_j$ for which $\sum_{j=1}^{n_0+1} \lambda_j D_j = 0$, with all $\lambda_j \geq 0$, and with at least one non-zero (positive) $\lambda_j$. To obtain this, it is necessary that the last $\lambda$ is non-zero, i.e., $\lambda_{n_0+1} > 0$. This is because if it were zero, we would have the Landau point for $S_{n_0} = (D_1, \ldots, D_{n_0})$, i.e., we would have $\sum_{j=1}^{n_0} \lambda_j D_j = 0$ (with the sum up to $j = n_0$). But the definition of $n_0$ is that there is a good direction for $S_{n_0}$, which implies that there is no Landau point for $S_{n_0}$.

Therefore, to avoid a contradiction, any Landau point for $S_{n_0+1}$ must have $\lambda_{n_0+1}$ strictly greater than zero. We can now scale all the $\lambda_j$’s to make $\lambda_{n_0+1} = 1$, and still have a Landau point. So we will work with

$$D(\lambda) \overset{\text{def}}{=} \sum_{j=1}^{n_0} \lambda_j D_j + D_{n_0+1}, \quad (7.8)$$

where we use boldface notation $\lambda = (\lambda_1, \ldots, \lambda_{n_0})$ to denote a vector of only the first $n_0$ values, and we simply require allowed values to obey $\lambda_j \geq 0$. Then our aim is to find an allowed $\lambda$ for which $D(\lambda) = 0$.

Define $\Lambda$ to be the set of allowed $\lambda$:

$$\Lambda \overset{\text{def}}{=} \{ \lambda : \lambda_j \geq 0 \text{ for all } j \}, \quad (7.9)$$

and define the following subset of $\Lambda$:

$$M \overset{\text{def}}{=} \{ \lambda \in \Lambda : D(\lambda)(v) \leq 0 \text{ for all } v \in P \}, \quad (7.10)$$

i.e., $M$ is the set of all $\lambda$ in $\Lambda$ for which $D(\lambda)$ is negative or zero for every vector that makes all of $D_1, \ldots, D_{n_0}$ positive. Its complement in $\Lambda$ is the set $M$ for which we have a positive value for $D(\lambda)$ somewhere in $P$:

$$\hat{M} \overset{\text{def}}{=} \Lambda \setminus M = \{ \lambda \in \Lambda : \exists v \in V \text{ such that } D(\lambda)(v) > 0, \text{ and } D_j(v) > 0 \text{ for } 1 \leq j \leq n_0 \}. \quad (7.11)$$

We will find a Landau point at a certain corner or edge of the set $M$.

To visualize the kind of set that $M$ is, it is useful to refer to the simple example given in App. [C2] below. It results in a region for $M$ that is illustrated in Fig. 7. Notice that $M$ is convex and is a closed set. The boundaries are segments of straight lines. The value of $\lambda$ giving a Landau point is at the upper right-hand corner.
FIG. 7. Set $M$ for the example given in App. C.2

E. Properties of $M$

We first derive some elementary properties of $M$ and $\hat{M}$ for the general case:

1. The zero vector $0$ is in $M$, so that $M$ is non-empty. This is simply because $D(0) = D_{n_0+1}$, and $D_{n_0+1}(v)$ is negative for all vectors in $P$, Eq. (7.2).

2. $M$ is convex. Suppose that $\lambda_0$ and $\lambda_1$ are any 2 elements of $M$ and that $t$ is any real number obeying $0 \leq t \leq 1$. Then for every $v \in P$

$$D(t\lambda_0 + (1-t)\lambda_1)(v) = tD(\lambda_0)(v) + (1-t)D(\lambda_1)(v),$$

(7.12)

which is zero or negative because each term is. It follows that $t\lambda_0 + (1-t)\lambda_1 \in M$. Hence $M$ is convex.

3. $M$ is a closed set. Let $\lambda_n$ be any sequence of elements of $M$ that converges to some element $\lambda$ of $\Lambda$. To show that $M$ is closed, we need to show that the limit point $\lambda$ is actually in $M$. To do this, we observe that for every $v \in P$, all the $\lambda_n$ obey $D(\lambda_n)(v) \leq 0$, by the definition of $M$. Hence

$$D(\lambda)(v) = \lim_{n \to \infty} D(\lambda_n)(v) \leq 0,$$

(7.13)

by the continuity of linear functions. Hence $\lambda \in M$.

4. Consider an arbitrary non-zero $\lambda \in \Lambda$, and consider an arbitrarily scaled value $\kappa\lambda$, where $\kappa$ is a positive real number. Then for large enough $\kappa$, $D(\kappa\lambda) \in \hat{M}$, but not $M$

Proof: For any $v \in P$

$$D(\kappa\lambda)(v) = \kappa \lambda \cdot D(v) + D_{n_0+1}(v).$$

(7.14)

Since $v \in P$, at least one $\lambda_j > 0$, and the others are non-negative, $\lambda \cdot D(v)$ is positive, so for large enough $\kappa$, the quantity in (7.14) is positive, and hence $D(\kappa\lambda) \in \hat{M}$, but not $M$. The line of $\kappa\lambda$ intersects the boundary of $\hat{M}$ at some point, which may in degenerate situations be at 0.

It is easily checked that these properties are obeyed in the example shown in Fig. 7.

F. Characterization of $M$ in terms of properties of edge vectors $e_L$

We have seen in Eq. (7.9) that any vector in $P$ can be written as a sum of edge vectors with strictly positive coefficients plus an element of the kernel $K$, i.e., $v = k + \sum L C_L e_L$. Since $D_j(k) = 0$ for $1 \leq j \leq n_0+1$, it follows that

$$D(\lambda)(v) = \sum L C_L f(L, \lambda),$$

(7.15)

where

$$f(L, \lambda) \equiv \sum_{j=1}^{n_0} \lambda_j D_j(e_L) + D_{n_0+1}(e_L).$$

(7.16)

Therefore for any given $\lambda \in \Lambda$, we can characterize whether $\lambda$ is in $M$ or $\hat{M}$, and whether it is in the boundary of $\hat{M}$ by the following exclusive criteria:

1. Either at least one $f(L, \lambda)$ is strictly positive. In this case $\lambda \in \hat{M}$.

   The last statement is proved by letting $L_0$ be one of the cases for which $f(L_0, \lambda) > 0$, and we set $C_L = \delta_{L_0, \lambda} + \kappa$, with $\kappa > 0$. Then $v = \sum L C_L e_L \in P$ and

$$D(\lambda)(v) = f(L_0, \lambda) + \kappa \sum L f(L, \lambda).$$

(7.17)

   By making $\kappa$ small enough (but non-zero), we can make this positive. Hence $\lambda \in \hat{M}$.

2. Or all of $f(L, \lambda)$ are strictly negative. Then $\lambda \in M$ and $\lambda$ is in the interior of $M$, not on its boundary with $\hat{M}$.

   First, we observe that the negativity of $f(L, \lambda)$ implies that $D(\lambda)(v)$ is negative for all $v \in P$, so that $\lambda \in M$. Then we consider a nearby point $\lambda_n = \lambda + \delta \lambda$ that is still in $\Lambda$ (i.e., the components obey $\lambda_{n_0} \geq 0$), and we let a general element of $P$ be $v = k + \sum L C_L e_L$. We let

$$-F = \max_L f(L, \lambda) < 0,$$

(7.18)

$$G = \max_{j \neq L} D_j(e_L) > 0.$$  

(7.19)

Then we can bound

$$D(\lambda + \delta \lambda)(v) = \sum L C_L \left[ f(L, \lambda) + \sum_{j=1}^{n_0} \delta \lambda_j D_j(e_L) \right]$$

$$\leq \sum L C_L \left[ -F + \sum_{j=1}^{n_0} \delta \lambda_j G \right].$$

(7.20)

Now take $\sum_{j=1}^{n_0} |\delta \lambda_j| < F/G$. Then $D(\lambda + \delta \lambda)$ is negative on the whole of $P$, and so $\lambda + \delta \lambda$ is in $M$. Hence all points sufficient close to $\lambda$ are themselves in $M$, and not in $\hat{M}$. Therefore $\lambda$ is not on the boundary with $\hat{M}$. 

3. Or for all \( L \), \( f(L, \lambda) \leq 0 \), and at least one is zero. Then \( \lambda \in M \) and \( \lambda \) is on its boundary with \( \tilde{M} \).

Given that none of \( f(L, \lambda) \) is positive, \( \lambda \) must be in \( M \), not \( \tilde{M} \). It remains to show that it is on the boundary.

So pick \( L_0 \) such that \( f(L_0, \lambda) = 0 \), and pick \( \delta \lambda \) such that all the \( \delta \lambda_j \) are strictly positive, \( \delta \lambda_j > 0 \), for \( 1 \leq j \leq n_0 \). We will show that \( \lambda + \delta \lambda \) is in \( \tilde{M} \) no matter how small \( \delta \lambda \) is. First, \( \lambda + \delta \lambda \) is in \( \Lambda \), because each component of the vector is non-negative. Let

\[
-A = \min_L f(L, \lambda) \leq 0, \tag{7.21}
\]

and choose an element of \( P \) by \( v = \kappa e_{L_0} + \sum_L e_L \), with \( \kappa > 0 \). Then

\[
D(\lambda + \delta \lambda)(v) \geq -A\#(L) + \kappa \sum_j \delta \lambda_j D_j(e_{L_0}), \tag{7.22}
\]

with \( \#(L) \) being the number of \( e_L \) vectors. Hence, by choosing \( \kappa \) large enough, we make \( D(\lambda + \delta \lambda)(v) \) positive. Therefore \( D(\lambda + \delta \lambda) \) is in \( \tilde{M} \) no matter how small the non-zero \( \delta \lambda \) is, and so \( \lambda \) is on the boundary between \( M \) and \( \tilde{M} \).

G. Moving along boundary between \( M \) and \( \tilde{M} \)

Now consider a point \( \lambda_0 \) on the boundary between \( M \) and \( \tilde{M} \). Such a point exists. At it, \( f(L, \lambda_0) \) is zero for some number of edges \( L \) of the positive region \( P \), and negative for any others. We will now show that we can move from \( \lambda_0 \) along the boundary of \( M \) in such a way that we find a place where there is an increase in the number of \( L \) for which \( f(L, \lambda) = 0 \). We keep going, repeating this process, which terminates only when all are zero. At that point \( D(\lambda) \) is itself zero and we have a Landau point, as we aimed to find.

Let\(^16\) \( Z(\lambda_0) \) and \( \tilde{Z}(\lambda_0) \) be the set of \( L \) for which \( f(L, \lambda_0) \) is zero and non-zero (necessarily negative):

\[
Z(\lambda_0) = \{ L : f(L, \lambda_0) = 0 \}, \tag{7.23}
\]

\[
\tilde{Z}(\lambda_0) = \{ L : f(L, \lambda_0) < 0 \}. \tag{7.24}
\]

This is a partition of the set of all edges of \( P \).

Let \( v \) be a general element of \( P \), decomposed as in Eq.\(^7.6\). Then

\[
D(\lambda_0)(v) = \sum_{L \in Z(\lambda_0)} C_L f(L, \lambda_0) = \sum_{L \in \tilde{Z}(\lambda_0)} C_L f(L, \lambda_0). \tag{7.25}
\]

It is possible that all the \( F(L, \lambda) \) are zero, so that \( D(\lambda_0)(v) = 0 \) for every \( v \in P \). Since \( P \) is a manifold of the same dimension as the whole space \( V \), it follows that \( D(\lambda_0) \) itself is zero, so that we have a Landau point, and we need go no further.

Otherwise at least one \( F(L, \lambda) \) is nonzero and negative. To deal with this case, our method will be to first prove that the set of \( e_L \) with \( L \in Z(\lambda_0) \) spans a boundary segment of \( P \), rather than the whole of \( P \), and then that there is a \( j_0 \) for which

\[
D_{j_0}(e_L) = 0 \quad \text{for all} \quad L \in Z(\lambda_0). \tag{7.26}
\]

We will use this to provide a direction \( \delta \lambda \) in which to move while staying on the boundary of \( M \) and then eventually find a point where yet another \( f(L, \lambda) \) is zero.

The following simple results are useful in the sequel:

**Lemma 3.** \( D(\lambda_0)(v) \) is negative for every \( v \) in \( P \).

**Proof.** The \( F(L, \lambda_0) \) in Eq.\(^7.25\) are negative or zero, and at least is nonzero. Hence \( D(\lambda_0)(v) \) is negative for every \( v \) in \( P \), since all the \( C_L \) are strictly positive.

**Lemma 4.** There is a null intersection between the kernel of \( D(\lambda_0) \) and the positive region \( P \):

\[
\ker D(\lambda_0) \cap P = \emptyset. \tag{7.27}
\]

**Proof.** This follows from Lemma 3, since \( D(\lambda_0) \) is nonzero on the whole of \( P \).

**Lemma 5.** For every \( v \) in both \( P \) and its boundary, \( D(\lambda_0)(v) \leq 0 \).

**Proof.** We know that \( D(\lambda_0) \) is negative for every element of \( P \). A boundary point is obtained by taking a limit of points in \( P \). Therefore \( D(\lambda_0) \) is negative or zero on the boundary of \( P \).

**Lemma 6.** For every \( k \in K \) and any \( \lambda \),

\[
D(\lambda)(k) = 0. \tag{7.28}
\]

**Proof.** Since \( k \in K \), every \( D_j(k) = 0 \), for \( 1 \leq j \leq n_0 \). We have also seen in Sec. VII 13 that \( \ker D_{n_0+1} \subseteq K \), so \( D_{n_0+1}(k) = 0 \). Equation 7.28 follows.

Now consider vectors of the form

\[
w = k + \sum_{L \in Z(\lambda_0)} C_L e_L, \tag{7.29}
\]

where \( C_L > 0, k \in K \), and we have only used the subset of \( e_L \) for which \( f(L, \lambda_0) = 0 \). The vector \( w \) is in the kernel of \( D(\lambda_0) \), i.e., \( D(\lambda_0)(w) = 0 \). So by Lemma 4 it cannot be in \( P \). But by adding a term \( \kappa \sum_{L \in \tilde{Z}(\lambda_0)} C_L e_L \), with \( \kappa \) non-zero and positive, but arbitrarily small, we get a vector in \( P \) itself. Hence the given \( w \) is in a non-trivial boundary segment \( B \) of \( P \).

Now from Thm. 8 applied to \( P \) instead of \( P_B \), we know that on the boundary segment \( B \), there is a value \( j_0 \) for which \( D_{j_0} \) is zero on \( B \), and hence \( D_{j_0}(w) = 0 \).

---

\(^{16}\) Note that \( Z \) is now used with a different meaning and type of argument than before.
Now, in Eq. \ref{7.29}, for all \( j \) in the range \( 1 \leq j \leq n_0 \), \( D_0(k) = 0 \) and for all \( L D_j(\epsilon L) \geq 0 \). Hence from the zero value of \( D_{0}(w) \) it follows that \( D_{0}(eL) \) is zero for all \( L \in Z(\lambda_0) \), i.e., for all those \( L \) for which \( D(\lambda_0)(eL) \) is zero rather than negative. (At least one such \( L \) exists, since \( \lambda_0 \) is on the boundary of \( M \).

Now let us make an increment \( \delta \lambda \) to \( \lambda_0 \) defined by
\[
\delta \lambda_j = \delta_j \kappa,
\]
where \( \kappa \geq 0 \). All its components are non-negative, so \( \lambda_0 + \delta \lambda \) remains in \( \Lambda \). To determine its location with regards to \( M \) and \( \tilde{M} \), we calculate
\[
f(L, \lambda_0 + \delta \lambda) = f(L, \lambda_0) + \kappa D_{0}(eL).
\]
When \( L \in Z(\lambda_0) \), this is zero. When, instead, \( L \in \tilde{Z}(\lambda_0) \), this starts out negative and either stays at the same value or increases with \( \kappa \), depending on whether \( D_{0}(eL) \) is zero or not. (Recall that every \( D_j(eL) \) is positive or zero.)

Thus for small enough \( \kappa \), \( \lambda_0 + \delta \lambda \) is still on the boundary of \( M \). But given \( j_0 \), there is at least one \( eL \) for which \( D_{0}(eL) \) is strictly positive; this \( eL \) is necessarily one of those corresponding to \( \tilde{Z}(\lambda_0) \). So at least one of the initially negative \( f(L, \lambda_0 + \delta \lambda) \) increases. There is a least \( \kappa \) for which one (or more) of these reaches zero. Let \( \lambda_1 \) be the resulting position on the boundary of \( M \).

At this point, we go back to the start of this Sec. VII G and replace \( \lambda_0 \) by \( \lambda_1 \). We keep iterating this procedure, getting a sequence of boundary points \( \lambda_i \), with at each stage getting an increased number of \( L \) for which \( f(L, \lambda_i) \) = 0. Eventually this procedure has to stop because we run out of values of \( L \), and the only way this happens in the argument is that there are no values of \( \delta \lambda \) for which \( F(L, \lambda_{\max}) \) is negative, i.e., that all the \( F(L, \lambda_{\max}) \) are zero, it follows that \( D(\lambda_{\max}) = 0 \), i.e., we have a Landau point.

This completes our proof of Thm. 3.

### VIII. CONTOUR DEFORMATIONS AND PINCHES

We now return to the determination of the conditions for the existence of a pinch at a particular value \( w_S \) of the integration variable in an integral of the form given in Eq. \ref{2.4}. We will complete the proof of Thm. \ref{thm:7.1} that there is a pinch if and only if the corresponding Landau condition holds. To do this, we need to prove Thm. \ref{thm:7.2} relating a pinch to the non-existence of an allowed deformation with positive first-order shifts in the imaginary parts of the relevant denominators. Once that is proved, the already-proven geometric Thm. \ref{thm:7.3} gives Thm. \ref{thm:7.1} as an immediate consequence.

We saw in Sec. \ref{sec:7.4} that if an allowed deformation exists with positive first-order shifts in the relevant denominators, then the integration is not trapped. So it remains to show that if there is no such deformation, then the integral is trapped. Now in the one-dimensional case, this is easy to show, because a contour deformation avoids a singularity due to a zero of a denominator if and only if the first order shift in the denominator is positive — see App. \ref{app:b} for explicit details. But in higher dimensions, the example in App. \ref{app:c} shows that a singularity can be avoided while having a first-order shift that is zero, i.e., the direction of contour deformation can be tangent to the singularity surface. Hence a more detailed argument is needed for the general case, and it will turn out to be annoyingly difficult for such an apparently elementary result.

#### A. Elementary results

First we prove some elementary results that strongly restrict the kinds of contour deformation that do or do not avoid singularities.

Given a denominator \( A_j(w_R) \) that is zero at \( w_R = w_S \), define its derivative by \( D_j = \partial A_j(w_S) \). Then consider a candidate contour deformation specified by \( v(w_R) \), and let \( v_S = v(w_S) \), the direction of deformation at \( w_S \). We classify what happens by the sign of \( D_j(v_S) = v_S \cdot \partial A_j \), and codify the results in some theorems.

It is useful to define \( \delta w = w_R - w_S \) and \( \delta v(\delta w) = v(w_S + \delta w) - v_S, \) i.e., the deviations from values at \( w_S \).

First, for positive \( D_j(v_S) \):

**Theorem 9.** Suppose, with the notation and conditions just stated, that the deformation is allowed and that \( D_j(v_S) > 0 \). Then the deformation avoids the singularity at \( w_S \) associated with the zero of \( A_j \).

**Proof.** Although this result is elementary, we will give a detailed argument, since this will introduce techniques to be used in more difficult situations.

The denominator is \( A_j(w_S + \delta w + i \lambda (v_S + \delta v)) + \epsilon \). Now we always require that \( A_j(w) \) is analytic and that it is real when \( w \) is real. Therefore all the Taylor coefficients for an expansion about \( w_S \) are real. We expand the denominator \( A_j + i \epsilon \) in powers of \( \delta w \) and \( \lambda \).

The imaginary part comes solely from the odd terms in \( \lambda \), and hence
\[
\Im(A_j + i \epsilon) = \epsilon + \lambda \left[ D_j(v_S) + O(\lambda^2) + O(\delta w) \right].
\]

For small enough \( \delta w \) and \( \lambda \), the correction terms are smaller in size than the positive \( D_j(v_S) > 0 \) term, and we therefore have a sum of two non-negative terms. Therefore the imaginary part of \( A_j \) is zero only if both \( \epsilon \) and \( \lambda \) are zero.

Now a zero of the denominator occurs when both its real and imaginary parts are zero. Hence in a neighborhood of \( w_S \), the denominator is nonzero when \( \lambda \) is small and positive, and therefore the singularity due to \( A_j(w_S) = 0 \) is avoided by the contour deformation. 

\[
\square
\]
Next, if $D_j(v_S)$ is negative, the candidate deformation is not even allowed:

**Theorem 10.** Suppose that $v(w_R)$ specifies a candidate deformation, and that $D_j(v_S) < 0$. Then the deformation is not allowed.

*Proof.* We need to show that if $D_j(v_S)$ is negative then we have a situation like that shown in Fig. 1(a). That is, for all small $\lambda$, there is a zero of $A_j(w_S + \delta w + i\lambda(v_S + \delta v(w))) + i\epsilon$ for some small positive $\epsilon$ and some value(s) of $\delta w$. Furthermore (at least for these values of $\epsilon$ and $\delta w$ approach zero as $\lambda \to 0+$). This corresponds to the negation of the definition of an allowed deformation given by Defns. 1 and 2.

We start with $\epsilon$ slightly positive, increase $\lambda$ from zero, and then decrease $\epsilon$ to zero. We encounter a situation where the imaginary part of $A_j + i\epsilon$ is zero. This can be seen from Eq. (8.1) given that $D_j(v_S)$ is negative. The zero of the imaginary part occurs both when $\delta w = 0$ and for all nearby values of $\delta w$, and it occurs for all small positive $\lambda$.

But a zero in the imaginary part of the denominator does not itself show that the deformation encounters a singularity in a zero in $A_j + i\epsilon$, because to get a zero we also need the real part to be zero, and we need to show that such a zero occurs independently of any higher order terms in the Taylor expansion of $A_j$ in powers of small quantities $\lambda$, $\delta w$ and $\delta v$.

Choose $w_R = w_S + xv_S$, i.e., $\delta w = xv_S$. Thus $x$ parameterizes a particular line in the space of $w_R$. Then $\delta v = O(x)$ as $x \to 0$. Hence applying a Taylor expansion of $A_j(w)$ about $w = w_S$ gives

$$A_j(w_S + xv_S + i\lambda(v_S + \delta v)) + i\epsilon = i\epsilon + (x + i\lambda)D_j(v_S) + O(|x|^2, \lambda^2, |x|\lambda). \quad (8.2)$$

Recall that $v_S \cdot D_j$ is real, and is negative in the situation that we are currently considering. Define a complex variable

$$\zeta = x + i\lambda, \quad (8.3)$$

and consider values $\zeta = re^{i\theta}$ for positive $r$ and for $0 \leq \theta \leq \pi$, so that both $x$ and $\lambda$ are at most of size $r$, with $x = r \cos \theta$ and $\lambda = r \sin \theta$. As we increase $\theta$ from 0 to $\pi$, $(x + i\lambda)v_S \cdot D_j$ traces out the semicircle in the lower half plane shown in Fig. 8. It necessarily crosses the negative imaginary axis. The value of $A_j$ differs from $(x + i\lambda)v_S \cdot D_j$ by terms of order $r^2$, so for small enough $r$, they only slightly modify the path in Fig. 8. It still starts on the negative real axis and ends on the positive real axis, and crosses the negative imaginary axis. But it crosses the imaginary axis with a value of $x$ that is order $r^2$ (and hence of order $\lambda^2$), instead of exactly zero. We therefore get a zero of $A_j + i\epsilon$ for any small $\lambda$ for some small $\epsilon$, and have not avoided a singularity. This gives the situation shown in Fig. 1(a), which shows where, as we take $\epsilon$ and $\lambda$ through the values used to try to get a successful deformation, we first encounter a singularity.

Hence the deformation is not allowed. \qed

From Thm. 10 the following property of allowed deformations immediately follows:

**Theorem 11.** Suppose that $v(w_R)$ specifies an allowed deformation. Then $v(w_S) \cdot D_j \geq 0$, whenever $A_j(w_S) = 0$, for every $j$ and every real $w_S$ in the integration range.

The remaining case for $D_j(v_S)$ is that it is zero. One possibility is that $v_S$ itself is zero. In that case $A_j(w_S + i\lambda v_S) = A_j(w_S) = 0$, Hence

**Theorem 12.** Suppose that $v_S = 0$. Then the deformation does not avoid the singularity due to the zero of $A_j$ at $w_S$.

This leaves one situation to treat, that $D_j(v_S) = 0$ but $v_S \neq 0$, which we defer to Sec. VIII B.

The difficulties in its analysis concern the possibility of a non-constant dependence of $v(w_R)$ on $w_R$. So it is useful to prove the simple results that obtain if $v(w_R)$ is independent of $w_R$, at least in a neighborhood of $w_S$.

**Theorem 13.** Suppose that $v(w_R)$ is a candidate deformation that has no dependence on $w_R$ near $w_S$, and that $D_j(v(w_S)) = 0$ but $A_j(w_S + i\lambda v_S)$ is non-zero for some (non-zero) $\lambda$. Then the deformation is not allowed.

*Proof.* The non-zero value of $A_j(w_S + i\lambda v_S)$ implies that on the deformed contour we no longer need encounter a zero of $A_j + i\epsilon$ when we restrict attention to $w_R = w_S$. To see this, first observe that the analyticity of $A_j(w_S + i\lambda v_S)$ as a function of $\lambda$ and its nonzero value for some value of $\lambda$ imply that the zero of $A_j$ at $\lambda = 0$ is isolated. Then we can get a situation where no zero of $A_j(w_S + i\lambda v_S) + i\epsilon$ is encountered, for all small enough non-zero $\lambda$.

But there are, in fact, zeros at nearby values of $w_R$, and these obstruct the deformation, as we now show. Consider values $w_R = w_S + xv_S$, with $x$ real, so that on the deformed contour we have $A_j(w_S + (x + i\lambda)v_S)$. This is an analytic function of $x + i\lambda$. The function is zero when $x = \lambda = 0$, and by the hypothesis of the theorem is not zero for some values of $x + i\lambda$.\[\square\]
Therefore there is a first non-zero term in the Taylor expansion:

\[ A_j(w_S + (x+i\lambda)v_s) = C(x+i\lambda)^n + O(|x+i\lambda|^{n+1}), \]  

(8.4)

with \( n \geq 2 \) since \( v_S \cdot D_j = 0 \). Since \( A_j \) is real for real values of its argument, so is \( C \). Set \( x+i\lambda = re^{i\theta} \), with \( r \) positive. Allowed values have \( 0 \leq \theta \leq \pi \), and any small \( r \) is possible. The value of \( A_j \) is

\[ Cr^n e^{i\theta n} + O(r^{n+1}). \]  

(8.5)

The value is real when \( \theta = 0 \) or \( \pi \). As \( \theta \) is increased from \( 0 \) to \( \pi \), the value \( A_j \) must go round the origin \( n/2 \) times, i.e., at least once. So it crosses the negative imaginary axis. The order \( r^{n+1} \) term from higher terms in the Taylor expansion can affect the position of this crossing, but do not affect its existence, at least when \( r \) is small enough.

Hence for small \( \lambda \) we find a zero of \( A_j + i\epsilon \) during the contour deformation, which therefore encounters a singularity, as in Fig. 1(a). Hence the deformation was not allowed, contrary to hypothesis. \( \square \)

**Theorem 14.** Suppose that \( v(w_R) \) is also required to be an allowed deformation as well as having no dependence on \( w_R \) near \( w_S \), and that \( v(w_S) \cdot D_j = 0 \). Then \( A(w_S + i\lambda v_S) = 0 \) for all \( \lambda \), and so the singularity due to the zero in \( A_j \) is not avoided.

**Proof.** This is an immediate consequence of Thm. 13 \( \square \)

**Theorem 15.** Suppose that \( v(w_R) \) is required to be an allowed deformation as well as having no dependence on \( w_R \) near \( w_S \), and that \( A_j(w_S) = 0 \) for one or more \( A_j \), where \( w_S \) is real. Then the singularity due to the zero in \( A_j \) is avoided if and only if \( v(w_S) \cdot D_j \) is strictly positive, i.e., \( D_j(v(w_S)) > 0 \), for every one of the zero denominators.

**Proof.** This follows directly from the application of the last few theorems proved so far to multiple denominators, together with the results of Sec. 8.2. \( \square \)

This last theorem is Thm. 2 with a restriction on the \( w_R \) dependence of \( v(w_R) \), but without any of the extra restrictions on the denominator that appear in the statement of Thm. 14.

Combined with Thm. 3, it gives our primary Theorem 1 under the same conditions.

**B. Analysis of neighborhood of singularity of integrand**

Consider a point \( w_S \) in an integral where some denominators are zero.

First consider the case that there is a vector \( v_S \) such that \( D_j(v_S) > 0 \) for the derivatives of all the zero denominators. Then we choose a contour deformation function which at \( w_S \) is equal to \( v_S \) (or proportional to it with a positive coefficient). Then we saw in Sec. 8.1 that the deformation avoids the singularity at \( w_S \). This gives part of the result stated in Thm. 2.

To complete the proof of Thm. 2 (and hence of Thm. 1) we now show that if no such \( v_S \) exists, then the integration is trapped at \( w_S \), i.e., that no allowed deformation avoids the integrand’s singularity at \( w_S \).

We start by assuming that we have an allowed deformation, given by \( v(w_R) \), and that there is no \( v_S \) such that \( D_j(v_S) \) is strictly greater than zero for all those \( D_j \) that correspond to zero denominators. We will obtain constraints that \( v(w_R) \) must obey, and hence show that in all cases the deformation does not avoid the singularity, thereby completing the proof of the theorem. As given in the statement of the theorem, we will restrict attention to denominators that are at most quadratic in the integration variable, and the reason for the remaining restriction in the statement of the theorem will emerge in the course of making the proof.

To simplify the notation, we shift the integration variable so that \( w_S = 0 \). We define \( v_0 = v(0) \), the deformation at the singular point being examined.

Since we cannot make all the relevant \( D_j(v_0) \) positive, Thm. 3 shows that the array of \( D_j(w) \) has a Landau point, i.e., there are values \( \alpha_j \) such that

\[ \sum_j \alpha_j D_j = 0, \]  

(8.6)

with all the \( \alpha_j \)s being non-negative and at least one being positive. The denominators for which \( \alpha_j = 0 \) will play no role in the proof, and so we now focus attention on only those values of \( j \) with nonzero \( \alpha_j \). With this focus, the denominators \( A_j(w) \) in the retained set are zero at \( w = 0 \) and have strictly positive \( \alpha_j \) in Eq. 8.6.

Now for an allowed deformation, \( D_j(v_0) \geq 0 \). From Eq. 8.6, \( \sum_j \alpha_j D_j(v_0) = 0 \). So strict positivity of the \( \alpha_j \) implies that each \( D_j(v_0) \) is actually zero.

We expand each denominator in powers of \( w \):

\[ A_j(w) = \sum_a D_{j,a} w^a + \frac{1}{2} \sum_{a,b} w^a E_{j,ab} w^b \]

\[ = D_j \cdot w + \frac{1}{2} w \cdot E_j \cdot w, \]  

(8.7)

given that the \( A_j \) are at most quadratic in the integration variables.

On the deformed contour \( w = w_R + i\lambda v(w_R) \), as usual. For the particular case of \( w_R = 0 \), a denominator on the deformed contour is

\[ A_j(i\lambda v_0) = i\lambda D_j \cdot v_0 - \frac{\lambda^2}{2} v_0 E_j \cdot v_0 = -\frac{\lambda^2}{2} v_0 E_j \cdot v_0. \]  

(8.8)

If \( v_0 \cdot E_j \cdot v_0 \) is zero for at least one \( j \), then we have a zero of \( A_j \) on the deformed contour, so that the deformation has not avoided the singularity. Then we need go no further for proving the target result.
So we now restrict attention to the case that \( v_0 \cdot E_j \cdot v_0 \) is nonzero for all of the attended denominators. We now examine the denominators near the origin, to search for possible zeros and to determine whether or not they are avoided. Define \( \delta v \) by

\[
v(w_R) = v_0 + \delta v(w_R),
\]

so that \( \delta v(w_R) \) goes to zero as \( w_R \) goes to zero, i.e., \( \delta(w_R) = o(1) \) in this limit. Then

\[
A_j(w_R + i\lambda \delta v(w_R)) = D_j \cdot w_R + \frac{1}{2} w_R \cdot E_j \cdot w_R - \frac{\lambda^2}{2} v_0 \cdot E_j \cdot v_0 - \lambda^2 v_0 \cdot E_j \cdot \delta v + i\lambda [D_j \cdot \delta v + w_R \cdot E_j \cdot v_0 + w_R \cdot E_j \cdot \delta v],
\]

(8.10)

where the first two lines give the real part and the last line gives the imaginary part.

We now search for possible obstructions to the contour deformation, i.e., zeros of \( A_j + i\epsilon \) for small \( w_R, \lambda \) and \( \epsilon \). Avoiding these will give constraints on the functional form of \( \delta v(w_R) \). We do this by choosing a small value of \( w_R \), finding a value of \( \lambda \) for which the real part of \( A_j + i\epsilon \) is zero, and then investigating the imaginary part. A zero of the real part is obtained by setting \( \lambda = \lambda(w_R) \), where

\[
\lambda(w_R) = \sqrt{\frac{2D_j \cdot w_R + w_R \cdot E_j \cdot w_R}{v_0 \cdot E_j \cdot v_0 + 2v_0 \cdot E_j \cdot \delta v + \delta v \cdot E_j \cdot \delta v}},
\]

(8.11)

provided that the argument of the square root is positive.

Now let us consider the particular case that \( w_R \) is in the direction \( v_0 \) and set \( w_R = x v_0 \). Then

\[
\lambda(x v_0) = \sqrt{\frac{x^2 v_0 \cdot E_j \cdot v_0}{v_0 \cdot E_j \cdot v_0 + 2v_0 \cdot E_j \cdot \delta v + \delta v \cdot E_j \cdot \delta v}} = |x| \left( 1 + \frac{2v_0 \cdot E_j \cdot \delta v + \delta v \cdot E_j \cdot \delta v}{v_0 \cdot E_j \cdot v_0} \right)^{-1/2}.
\]

(8.12)

Then there is a zero of the real part of \( A_j \) for all small enough \( x \), both positive and negative, and the solution has \( \lambda(x v_0) \approx |x| \). There the value of \( A_j \) only arises from its imaginary part

\[
A_j(x v_0 + i\lambda(x v_0) v(x v_0)) = i\lambda(x v_0) [D_j \cdot \delta v + x v_0 \cdot E_j \cdot v_0 + x v_0 \cdot E_j \cdot \delta v] = \lambda(x v_0) [D_j \cdot \delta v + x v_0 \cdot E_j \cdot v_0 + o(x)].
\]

(8.13)

If at any point we were to get a negative imaginary part for all small \( x \), then a zero of \( A_j + i\epsilon \) would be encountered in the contour deformation, so that the deformation would not be allowed. We therefore ask what constraints apply to \( \delta v(w_R) \) to avoid such a negative imaginary part.

For the deformation to be allowed, we must have

\[
D_j \cdot \delta v(x v_0) + x v_0 \cdot E_j \cdot v_0 + x v_0 \cdot E_j \cdot \delta v(x v_0) \geq 0
\]

(8.14)

for all small \( x \).

First notice that \( x \) can have either sign, and that when \( x \) has the opposite sign to \( v_0 \cdot E_j \cdot v_0 \), the term \( x v_0 \cdot E_j \cdot v_0 \) is negative. The \( o(x) \) term is strictly smaller (in the limit \( x \to 0 \)).

There are two cases to consider, according to whether \( D_j \) itself is zero or not.

If \( D_j \) is zero, then \( D_j \cdot v = 0 \), and the negative term \( x v_0 \cdot E_j \cdot v_0 \) dominates; we have a negative imaginary part, and the contour deformation is not allowed, contrary to our initial assumption.

Therefore \( D_j \) must be nonzero. Then it is conceivable that the \( D_j \delta v \) term compensates the negativity of \( x v_0 \cdot E_j \cdot v_0 \).

As announced in the statement of the theorem, we now restrict\(^{17}\) attention only to cases with the property that all non-zero \( v_0 \cdot E_j \cdot v_0 \) have the same sign. Thus

Either for all “relevant” \( j \), \( v_0 \cdot E_j \cdot v_0 \geq 0 \);

or for all “relevant” \( j \), \( v_0 \cdot E_j \cdot v_0 \leq 0 \),

(8.15)

where a “relevant” \( j \) is one for which \( \alpha_j \) is non-zero in Eq. (8.6), and for which \( A_j \) is zero, \( D_j \) is nonzero, and \( D_j \cdot v_0 = 0 \), both at the point of integration space under consideration. As already mentioned, if \( v_0 \cdot E_j \cdot v_0 \) is zero for at least one relevant \( j \), then the contour is definitely trapped, and we only now examine the case where all the \( v_0 \cdot E_j \cdot v_0 \) are nonzero.

The restriction is obeyed for standard applications to Feynman graphs and certain generalizations. To see this, observe that the standard Feynman denominator for a line of a Feynman graph has the form \( A_j = k^2 - m^2 \), where \( k \) is the line’s momentum. It is zero when \( k^2 = m^2 \). Let the projection of an allowed deformation onto the momentum of the line be \( \hat{v}_0 \). We then have \( D_j \cdot v_0 = 2k \cdot \hat{v}_0 \) and \( \frac{1}{2} v_0 \cdot E_j \cdot v_0 = \hat{v}_0 \cdot \hat{v}_0 \). For a massive line (i.e., \( m \neq 0 \)), all deformations that obey \( D_j \cdot v_0 = 0 \) must have a space-like (or zero) \( \hat{v}_0 \), and hence \( v_0 \cdot E_j \cdot v_0 \leq 0 \). In the massless case with \( k^2 = 0 \) and \( k \) nonzero, \( v_0 \) is either space-like or null (or zero), and again \( v_0 \cdot E_j \cdot v_0 \leq 0 \). In the massless case with \( k = 0 \), i.e., a soft line, \( D_j = 2k = 0 \), so the denominator is not one of the relevant ones in Eq. (8.15).

Another important case is of a Wilson line, for which the denominator is simply linear: \( A_j = k \cdot n \) for some vector \( n \), and hence \( E_j \) itself is zero. A non-relativistic propagator, with denominator \( E - \frac{p^2}{2m} \) has the for the quadratic term as in the massive relativistic case. One other case that can be met in QCD is an approximation where a longitudinal light-front component of momentum is set

\(^{17}\) It would be desirable to make a proof without this restriction, but it would require a harder proof beyond the scope of this paper.
to zero, but transverse momenta are preserved. Then the quadratic terms involve only transverse momentum, and the quadratic terms obey the same sign condition as for an unapproximated denominator.

Hence in all of these cases, the restriction \((8.14)\) is obeyed.

Given this restriction (independently now of which sign occurs), the second term in \((8.14)\) is negative when we give \(x\) the opposite sign to \(v_0 \cdot E_j \cdot v_0\). Most importantly, the same value of \(x\) can be used for all the relevant denominators. The third term in the imaginary part is always smaller when the size of \(x\) is small enough. So the only hope for getting a non-negative imaginary part is for the first term, \(D_j \cdot \delta v(xv_0)\), to compensate by being sufficiently positive. Since \(\delta v\) is zero when \(x\) is zero, this compensation relies on \(x\)-dependence in \(v(xv_0)\), and hence on \(w_R\) dependence in \(v(w_R)\).

In the present case, there is a Landau point, so that \(\sum_j \alpha_j D_j \cdot \delta v(xv_0) = 0\). Hence at least one \(D_j \cdot \delta v(xv_0)\) is not positive. Hence, for at least one \(j\), the first term in \((8.14)\) cannot compensate the negative value of the sum of the second and third terms. Then there is a zero in \(A_j\) that causes an obstruction to the contour deformation, and the proposed deformation would not be allowed.

We have now covered all the cases, so that given the existence of a Landau point, we have shown that all allowed contour deformations fail to avoid the singularity. This completes the proof of Thm. 2 and hence of Thm. 1. Notice how we used the existence of a Landau point, which was shown by a use of the geometrical Thm. 3.

We now revisit the rationale for extra restriction \((8.15)\). If the restriction were not obeyed, then there would be at least one positive and one negative \(v_0 \cdot E_j \cdot v_0\). The negative values of the second term in in \((8.14)\) would occur for opposite signs of \(x\) for different denominators. Hence an appeal to \(\sum_j \alpha_j D_j \cdot \delta v(xv_0) = 0\) would not be sufficient to rule out a compensation of the negative terms by some choice of \(\delta v(xv_0)\). A better argument would be needed, but I have not found one that is watertight.

C. Anomalous deformations

All but the very last part of the derivation in the previous subsection gives a strategy for finding examples like that in App. B1, where a singularity is avoided by a deformation that has zero first-order shifts at the singular point(s). Let us call such a deformation an “anomalous deformation”, formally defined by:

**Definition 15.** An anomalous deformation means an allowed deformation that avoids the singularity due to a zero of one or more denominators \(A_j\), but where the first-order imaginary part is zero.

What we did in the previous section, was to exclude the possibility that when the Landau condition is obeyed an anomalous deformation could exist that avoids singularities due to all of the denominators. But the proof relied on the extra restrictions on the denominators stated in Thm. 1.

If, in contrast, there is no Landau point, then we can find a vector that gives positive first-order shifts in the denominators. Hence, in this situation of no Landau point, given the existence of an anomalous deformation we can find another that is not anomalous and is still singularity-avoiding.

D. Patching local deformations to global

The arguments in the preceding sections as to whether or not an integrand’s singularity can be avoided by a contour deformation were local. That is, the arguments were applied at each position \(w_S\) where there a singularity, and they involved determining (a) which directions of deformation at \(w_S\) are compatible with the integrand’s singularities, and most importantly (b) which directions avoid a singularity.

The question now arises as to whether such locally determined directions can be globally patched together consistently, so as to give a contour deformation \(v(w_R)\) for all \(w_R\) that has one of the determined directions at each point of singularity of the integrand, and that can be implemented without some kind of discontinuity.

As an indication of possible issues, the example of determining normal directions to a Möbius strip comes to mind. This is a situation in which the global topology of a surface prevents global patching of locally determined vectors. But the present situation is different. At each point on the initial integration contour, properties of the denominators determine a manifold of directions of singularity avoiding deformations (and similarly for allowed deformations that don’t avoid singularities). The boundary of the manifold of possible directions depends continuously on position in the manifold, and the denominators are single-valued functions of position. So we can steer the deformation to stay within the allowed manifold. Of course, at some parts of the original contour we may find that no deformation is avoids singularities.

If we take a tour of the initial integration contour going from some initial point back to the same point, then we have the same restrictions on the direction of deformation at the start and end, and no inconsistency.

This is an extremely simple-minded argument, and undoubtedly too weak to be fully persuasive. An improved argument would be useful.

Of course, if we changed our integral to one in which a denominator \(A_j\) had a branch cut on the initial integration contour, then the situation would be different. But that is not the case for the integrals that we consider in this paper. Now there are allowed to be non-integer exponents in Eq. (2.4), so that the integrand itself can have branch points and cut(s) that are on the initial integration contour. But that does not affect the possible directions of deformation, which are all determined by
the denominator functions themselves, $A_j$, which we require always to be analytic and single valued.

**E. Case of one denominator**

We now examine the situation when there is only one denominator. This is a common special case, because it occurs when Feynman parameters are used. Its analysis has some special features compared with the case of multiple denominators, so it is useful to treat this case specially. In particular, we will understand explicitly why Coleman and Norton [4] needed to put the restriction on their proof, that the matrix of second derivatives of the denominator has no zero eigenvalues.

Let the denominator be $A(w)$. The Landau criterion for a putative pinch at some point $w_S$ is simply that the denominator and its first derivative $D(w) \equiv dA(w)/dw$ are zero at $w_S$. The aim is to show, if possible, that the contour of integration is trapped at that point. Of course, given the zero derivative, any deformation that avoids the singularity has a zero first-order shift in the denominator, and hence is anomalous. If we don’t succeed in excluding the possibility of an anomalous deformation, then at least we can strongly constraint its properties and those of the denominator. Of course, if $A(w_S) = 0$ but the derivative were non-zero, then we can certainly avoid the singularity at $w_S$ by a deformation $w \mapsto w_R + i\lambda v(w_S)$ with $D(w_S) \cdot v(w_S) > 0$. So the case of a zero derivative is the only one to examine further.

If the denominator is quadratic in the integration variables, then the restrictions in Thm. 4 are obeyed, and the derivation in previous sections is valid. But the denominator from applying the Feynman parameter method to a standard Feynman graph is cubic if the momentum integrals are not performed.

If the momentum integrals are performed, as can be done analytically for standard Feynman graphs, then the denominator is a polynomial of order one plus the number of loops. It can therefore be of arbitrarily high order. But as we have already observed, a pinch in momentum space does not always entail a pinch in parameter space, so this case isn’t so generally useful.

Given the significance of the Feynman parameter representation of a Feynman graph before the momentum integrals are performed, we will restrict attention to the case that the denominator is at most cubic in the integration variables. As before, given a point $w_S$ where the denominator and its derivative are zero, we simplify the notation by shifting variables so that $w_S = 0$. Then the denominator has the form:

$$A(w) = \frac{1}{2} \sum_{a b} w^a E_{ab} w^b + \frac{1}{6} \sum_{a b c} w^a w^b w^c F_{abc}$$

where each of the arrays $E$ and $F$ is symmetric in its indices. At certain points it will be useful to follow Coleman and Norton, and diagonalize $E$ by a change of variable, to write

$$w \cdot E \cdot w = \sum_j c_j \eta_j^2,$$  \hspace{1cm} (8.17)

with each $\eta_j$ being a linear combination of $w^a$s. By rescaling the $\eta_a$, we can arrange that each non-zero $c_a$ has absolute value unity. Thus without loss of generality, we can arrange that each $c_a$ is either +1, −1 or 0.

There are several different cases to consider, so it is convenient to encapsulate in a lemma each of the separate cases, as well as several subsidiary results. The first lemma is elementary:

**Lemma 7.** For a contour deformation $w = w_R + i\lambda v(w_R)$ to avoid the singularity at caused by the denominator $\text{8.16}$ at $w = 0$, it is necessary (but not sufficient, as we will see), for $v_0 \cdot E \cdot v_0$ or $v_0 v_0 v_0 \cdot F$ (or both) to be nonzero. Here $v_0 = v(0)$.

Conversely, if both of $v_0 \cdot E \cdot v_0$ and $v_0 v_0 v_0 \cdot F$ are zero, the singularity is not avoided.

**Proof.** The trivial proof is to observe that $A(w_R + i\lambda v(w_R))$ needs to be nonzero at $w_R = 0$ if the singularity is to be avoided.

**Lemma 8.** For an allowed deformation, $v_0 \cdot E \cdot v_0$ must be zero.

**Proof.** The proof is a minor modification of the argument leading to Eq. (8.14). If $v_0 \cdot E \cdot v_0$ were nonzero, then we would have a zero of the real part of $A$ with $\lambda$ close to $|x|$. The higher-than quadratic terms that we now have do not affect that result. But in Eq. (8.14) we now longer have the single-derivative term, which is therefore not available to compensate the negative value of $x v_0 \cdot E \cdot v_0$ that occurs when $x$ has the opposite sign to $v_0 \cdot E \cdot v_0$. The cubic term does not affect that for small $\lambda$ (and $x$). So the constraint Eq. (8.14) for an allowed deformation cannot be obeyed.

This leaves only the case of zero $v_0 \cdot E \cdot v_0$ for an allowed deformation.

**Lemma 9.** For an allowed deformation, $v_0$ must be an eigenvector of $E$ with eigenvalue zero.

**Proof.** We now use the change of variables that gives the diagonalized form for the quadratic term, Eq. (8.17), with each $c_a$ being either +1, −1 or 0. Let $h$ be the result of applying the change of variables to $v_0$. Then

$$v_0 \cdot E \cdot v_0 = \sum_j c_j h_j^2 = 0.$$  \hspace{1cm} (8.18)

There are two cases to consider. One is where the only nonzero values of $h_j$ are with $c_j = 0$. Then $v_0$ is an eigenvector of $E$ with eigenvalue zero, so we are done.

The other case is where there is at least one $j$ with both of $h_j$ and $c_j$ nonzero. To get the zero value in Eq. (8.18), there must be at least one positive term and one
negative term. Permute the labels so that the $j = 1$ term is positive and the $j = 2$ term is negative: $v_0 \cdot E \cdot v_0 = h_1^2 - h_2^2 + \text{terms from other } j$, with both of $h_1$ and $h_2$ nonzero.

We now find a zero of the denominator that gives a deformation-obstructing singularity in the integrand. Choose $w_R$ to correspond to

$$\eta_j = x\delta_{j1} + y\delta_{j2}. \quad (8.19)$$

Then the denominator is

$$A(w_R(x, y) + i\lambda(v_0 + \delta v)) = \frac{1}{2}x^2 - \frac{1}{2}y^2 + \mathcal{O}(\lambda^2 \delta v^2) + \mathcal{O}(\lambda^3) + i\lambda \left[xh_1 - yh_2 + O(|x|\delta v) + O(|y|\delta v) + \mathcal{O}(\lambda^2) \right]. \quad (8.20)$$

If the first two terms in the real part had no corrections, then it would be zero whenever $|x| = |y|$, with all combinations of signs allowed. By taking $x$ to have the opposite sign to $h_1$ and $y$ to have the same sign as $h_2$, we get a negative value for the first two terms in the imaginary part. We now choose $x$ and $y$ to be of order $\lambda$, and take $\lambda$ to zero. Then the correction terms in the real part are smaller than the first two terms, and cause the position of the zero to move slightly, with the fractional change decreasing to zero as $\lambda \to 0$. The correction terms in the imaginary part are similarly small than the first two terms, and leave the imaginary part negative.

Hence when $\lambda$ and $\epsilon$ are decreased to zero, a zero of $A_j(w_R + i\lambda(v_R)) + i\epsilon$ is always encountered somewhere on the integration contour, and hence the deformation is obstructed by a singularity and is not allowed.

Hence the case that $v_0$ is not an eigenvector of $E$ of eigenvalue zero is ruled out, and the lemma is established.

**Lemma 10.** Suppose $w = 0$ is part of a manifold $M$ of points satisfying the Landau condition, i.e., $A = 0$ and $D = 0$. Then (a) any tangent vector $v$ to the manifold (at $w = 0$) is not an eigenvector of $E$ of eigenvalue zero; (b) $ttt \cdot F = 0$; (c) hence a contour deformation whose $v_0$ is tangent to $M$ at $w = 0$ does not avoid the singularity.

**Proof.** Consider a path within the manifold $M$, starting at the origin and with initial direction $t$. Let the path be parameterized by $P(x)$ where $x$ is real, $P(0) = 0$, and $P'(0) = t$, with $w'(x)$ denoting $dP(x) / dx$. Then for all $x$ for which $P(t) \in M$,

$$0 = A(w(x)) = \frac{1}{2}P(x) \cdot E \cdot P(x) + \frac{1}{6}P(x)P(x)P(x) \cdot F, \quad (8.21a)$$

$$0 = D(w(x)) = E \cdot P(x) + \frac{1}{2}P(x)P(x) \cdot F. \quad (8.21b)$$

with the second equation meaning

$$0 = D_a(w(x)) = \sum_b E_{ab}P^b(x) + \frac{1}{2} \sum_{bc} F_{abc}P^b(x)P^c(x). \quad (8.22)$$

Differentiate Eq. (8.21b) with respect to $x$ to get

$$0 = \frac{dD(w(x))}{dx} = E \cdot P'(x) + P'(x)P(x) \cdot F. \quad (8.23)$$

Set $x = 0$ to get $E \cdot t = 0$, i.e., $t$ is an eigenvector of $E$ of eigenvalue zero.

Now differentiate Eq. (8.21b) twice with respect to $x$ to get

$$0 = E \cdot P''(x) + P'(x)P''(x) \cdot F + P(x)P'''(x) \cdot F. \quad (8.24)$$

Setting $x = 0$, using $P(0) = 0$ and $P'(0) = t$, and contracting with $t$ gives

$$ttt \cdot F = 0, \quad (8.25)$$

which gives item (b) in the lemma.

Then because both $t \cdot E \cdot t$ and $ttt \cdot F$ are zero, a contour deformation with $v_0 \propto t$ gives a zero of $A$ at $w_R = 0$ on the deformed contour, so that the singularity is not avoided. This proves item (c).

**Lemma 11.** For any contour deformation that avoids the singularity (necessarily an anomalous deformation), $v_0$ has eigenvalue zero, but is not tangent to any manifold such as $M$ in the previous lemma.

**Proof.** This is immediate from the previous two lemmas.

**Lemma 12.** For there to exist a contour deformation that avoids the singularity, $E$ must have an eigenvector of eigenvalue zero that is not in the space of directions of manifolds of the form of $M$.

**Proof.** Immediate from the previous lemma.

We now see why Coleman and Norton needed their restriction on the eigenvalues of $E$. However, they did not explain why, and the argument in this section appears to show that the derivation is non-trivial. They refer to Ref. [21] for situations when the zero eigenvalue problem arises.

There is one common case of zero eigenvalues in a massless theory, and that is when there is a collinear...
region. In that case the corresponding pinch-singular-surface is not simply a point, being parameterized by longitudinal momentum fraction(s). But Lemmas 10 and 12 show that if the only zero eigenvalues are for tangents to the surface, the singularity is not avoided. There is perhaps an obscure reference to this in the third and fourth lines of p. 441 of Ref. [4].

Undoubtedly, it is possible to examine in more detail the case with a zero eigenvalue, and to find further constraints on allowed deformations. If the contour deformation direction \( \varepsilon(w_R) \) were required to be independent of \( w_R \), or sufficiently slowly varying, then Thms. 13 and 14 show that in full generality it is not possible to avoid the singularity due to a zero of the denominator and its derivative. The elementary proof simply uses the first non-zero term in the Taylor expansion of \( A(\lambda v) \) in powers of \( z \). In our case it would be the cubic term that is relevant.

However, the deformation direction can depend on \( w_R \). The non-trivial problem is there can then arise a nonzero contribution to the quadratic term involving \( \lambda \delta v \), and this is potentially capable of compensating the part of the cubic term that would otherwise result in an unavoidable singularity in the contour deformation.

IX. CONCLUSIONS AND IMPLICATIONS

In this paper, I have provided a complete proof of the necessity and sufficiency of the Landau condition for a pinch in the kind of integral typified by Feynman graphs in the physical region. The proof overcomes a number of deficiencies in existing work, and it can be applied directly to Feynman graphs in momentum space (unlike many previous proofs). The analysis of pinch singularities is foundational to perturbative QCD, so it is important not only to have a full explicit proof, but to have one whose domain of application, as here, includes Feynman graphs with massless propagators as well as massive ones, and also modified propagators such as the Wilson-line denominators that are common in QCD applications.

The methods and intermediate results in the proof have further implications, beyond simply determining where pinches occur. For example, an analysis of coordinate-space behavior can be made by deforming a contour of integration as much as possible to convert rapidly oscillating exponential factors into strongly decaying exponentials. Dominant regions are determined by the locations where such a deformation cannot be made. Allowed directions of deformation are constrained not only by the need to avoid singularities of the integrand, but also to avoid making the exponentials rapidly growing. Dominant regions of the integration variables are where the constraints cannot be satisfied, and it is useful to have an analysis that works at all orders of perturbation theory.

Another possible application, especially of the geometric results in Sec. V–VII, is to improve algorithmic methods for deforming contours in numerical calculations of Feynman graphs, as in Refs. [13–15].

In constructing the proof, some interesting subsidiary results were found. Some of these simply resulted from a close analysis of treatments in the classic literature (which give a strong inspiration to treatments in textbooks). Particular problems and even a demonstrably false assumption were found. Awareness of such issues is important to provide sound and properly persuasive pedagogical treatments.

Another notable case was to recognize the possibility of avoiding a singularity in the integrand by a contour deformation in a direction that is tangent to the singularity surface. Such a deformation I termed “anomalous”. With such a deformation, the first order shift of a denominator due to the contour deformation is zero. This contrasts with the natural intuition (engendered by experience in one dimensional cases) that, in order to avoid a singularity of the integrand, the contour deformation must give a positive first-order shift to the imaginary part of the denominator, matching the sign of the \( i \varepsilon \). A more general proof (or counterexample) would be obviously useful.

Here are some possible directions for further work.

1. It would be useful to apply the methods to give a fully systematic and general account in coordinate space of the large-\( Q \) behavior of amplitudes, such as appear in QCD factorization. This would extend, for example, the work of Erdoğan and Sterman [18–20].

2. Another direction is to determine from the geometrical considerations given in this paper the possible directions for allowed contour deformations at a pinch. Given the existence of a pinch at a particular point or at a points on some manifold, there is a certain set of denominator(s) whose corresponding singularity/ies of the integrand cannot be avoided. These effectively are the denominators that actually cause the pinch. But it is possible that other denominators are zero, but that the corresponding singularities can be avoided by a contour deformation that respects the constraints given by the pinching denominators. It would be useful to have a determination of the range of allowed directions.
Such issues were not important in the original application of the Landau analysis to determine singularities of an integral as a function of external parameters. But in QCD applications, the focus is rather on the momentum configurations at a pinch and their neighborhoods. The exact pinches of relevance in standard pQCD applications are in a massless theory; whereas the true theory is not massless; the massless version is simply a useful tool for locating relevant regions in the space of loop momenta. Moreover, a subtracted hard scattering coefficient, calculated in the massless limit as usual, is singular at zero mass, but the singularity is not strong enough to make the hard scattering actually divergent there, given the subtractions. (The same is not true of the derivatives of sufficiently high order with respect to mass at zero mass.)

3. Consider the Coleman-Norton result that locations where a Landau condition is obeyed correspond to possible classical processes. Their result is very useful for readily determining the well-known results on regions involved in asymptotic large $Q$ behavior, notably the classification into hard, collinear, and soft subgraphs. Coleman and Norton’s proof works in the massive case, but it becomes singular in the massless case and doesn’t fully capture what is actually needed in QCD applications. It would be useful to remedy this problem, perhaps in conjunction with a systematic treatment in coordinate space.

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Appendix A: Contour trapping without singularity

Consider the following dimensionless function:

$$I(Q/m) = m^2 Q \int_{-\infty}^{\infty} \frac{dk}{(k-m+i\epsilon)(k-i\epsilon)^2(k+Q-i\epsilon)}.$$  

(A1)

This is intended to be a simple analog of a QCD Feynman graph with a large momentum scale $Q$, a mass scale $m$, and with a certain numerator factor. The singularities of the integrand as a function of $k$ are shown in Fig. 9. There is a double pole at $k = 0$ just above the contour of integration, a pole at $k = m$ below the contour and a pole at $k = -Q$. When $m \to 0$, the contour is evidently trapped at $k = 0$.

For the integral itself, without the $m^2Q$ prefactor, elementary contour integration using the residue theorem shows that the value is $-2\pi i/[m^2(m + Q)]$. But with the explicit prefactor $m^2Q$, we find that the function $I(Q/m)$ equals $-2\pi i/(1 + m/Q)$, which has no singularity at $m = 0$.

The reason is for the lack of singularity is entirely trivial: The would-be power singularity is canceled by an explicit numerator factor $m^2$.

Since numerator factors can occur in Feynman graphs, their presence allows a potential violation of an absolute connection between the Landau condition and actual singularities of a Feynman graph as a function of external momenta or masses. Note that most singularities we treat in QCD are logarithmic, and hence are not so easily removed.

However, even though there is no singularity in the function $I(Q/m)$, there is a pinch in the integral in Eq. (A1), and the prefactor $m^2$ does not remove this pinch. Thus when $m/Q \to 0$, the computation of the integral is dominated by small values of $k$, of order $m$. The contribution of much larger values of $k$ is power suppressed because of the large number of denominator factors in the integrand.

If it were possible to deform the contour away from small values of $k$, i.e., values of order $m$, then $|k|$ would be of order $Q$ instead of being sometimes much smaller on the contour of integration. This could happen with a different choice of $i\epsilon$ prescriptions. In such a case, the result for $I$ have been of order $m^2/Q^2$ instead of order unity, as $Q/m \to \infty$. Thus the order unity result for $I$, in conjunction with power counting for the “ultra-violet” region of large $k$, is a symptom that the integration is trapped at small $k$.

The importance of this result in QCD is given by considering the statement by Libby and Sterman at the beginning of their paper [2]. They say that quantities in QCD with a large external scale $Q$ can be effectively computed provided that there are no mass divergences. Taken literally, this statement is falsified by examples like Eq. (A1). But Libby and Sterman’s statement becomes correct if the no-singularity property is replaced by a no-pinch property. In that case, the Landau condition is both necessary and sufficient. Moreover, it is, in fact, the presence or absence of pinches that is relevant for the QCD applications.
Appendix B: Singularity avoidance without a first-order shift in the denominator

1. Example of singularity avoidance with zero first-order shift in denominator

Consider an integral of the following form

$$I = \int \frac{dE dp}{E - p^2/(2m) + i\epsilon} f(E, p). \tag{B1}$$

The first factor has a singularity at $E = p^2/(2m)$, and we will consider contour deformations to avoid it. The other factor $f(E, p)$ generally has singularities. But for the purposes of constructing an example, we will assume they are far enough away not to concern us. We could choose a function like $f = 1/(E^2 + p^2 + Q^2)^2$, with $Q \gg m$; this factor has no singularities for real $E$ and $p$ and gives good convergence of the integral in the ultra-violet (i.e., at large $p$ and $E$).

The first factor is of the form of the propagator for a non-relativistic particle, indicating that this example is directly relevant to physics. Similar treatments to the one in this section can be applied in the relativistic case, but with more complication.

The derivative of the propagator with respect to the two-dimensional integration variable is

$$D = (1, -p/m). \tag{B2}$$

Let a contour deformation be made:

$$(E, p) = (E_R, p_R) + i\lambda(\eta, \xi), \tag{B3}$$

where $E_R, p_R, \eta$ and $\xi$ are real, and $\eta$ and $\xi$ are functions of $E_R$ and $p_R$. As usual, $0 \leq \lambda \leq 1$.

A natural and obvious candidate for a contour deformation to avoid the pole simply has $\eta$ positive and $\xi = 0$, so that

$$(\eta, 0) \cdot D = \eta > 0. \tag{B4}$$

for then the singular factor is

$$\frac{i}{E_R - \frac{p_R^2}{2m} + i\lambda \eta + i\epsilon} \tag{B5}$$

and we avoid encountering a singularity as we deform the contour.

We now construct two examples of contour deformation that avoid the pole, but which are anomalous, i.e., they obey

$$(\eta(E_R, p_R), \xi(E_R, p_R)) \cdot D = 0 \tag{B6}$$

in one or more situations where there is a singularity before deformation, i.e., where $E_R = p_R^2/(2m)$. The first example will obey this condition at $E_R = p_R = 0$. The second will obey it at all values of $p_R$. In both cases, the deformation direction is along a tangent to the surface of singularity.

To see how to construct such an example, consider the denominator on the deformed contour, in the case of a general deformation:

$$E - \frac{p^2}{2m} + i\epsilon = E_R - \frac{p_R^2}{2m} + \frac{\lambda^2 \xi^2}{2m} + i\lambda(\eta - \xi p_R) + i\epsilon. \tag{B7}$$

For a deformation to be allowed and to avoid the pole, we must arrange $\eta(E_R, p_R)$ and $\xi(E_R, p_R)$, such that the only zeros of the denominator occur at $\epsilon = \lambda = 0$.

In the first example, we choose $\eta(0, 0) = 0$ and $\xi(0, 0) = m$, so that the first order shift in the denominator, $(\eta(E_R, p_R), \xi(E_R, p_R)) \cdot D$, is zero at $E_R = p_R = 0$. Because of the non-zero $\lambda^2$ term in (B7), we no longer encounter a singularity on the deformed contour, for the case that $E_R = p_R = 0$. Now, if $\eta$ and $\xi$ were given no dependence on $E_R$ and $p_R$, the denominator (B7) would have a negative imaginary $\lambda$ term when $p_R$ is positive. We could make the real part of the denominator zero by choice of $E_R$, and then the whole denominator becomes zero at some point as we reduce $\epsilon$ to zero. In this case, the contour deformation crosses a singularity somewhere, and is not allowed.

But by choosing the $E_R$ and $p_R$ dependence of $\eta$ and $\xi$ appropriately, we can compensate the negative imaginary part.

We first make the following choice:

$$\eta, \xi = (|E_R| + 2|p_R|, m), \tag{B8}$$

so that the denominator is

$$E - \frac{p^2}{2m} + i\epsilon = E_R - \frac{p_R^2}{2m} + \frac{\lambda^2 m}{2} + i\lambda(|E_R| + 2|p_R| - p_R) + i\epsilon. \tag{B9}$$

We ask when this is zero. On the deformed contour, i.e., with $\lambda$ positive, the term linear in $\lambda$ is positive and non-zero except for being zero at $E_R = p_R = 0$. Because $\lambda$ is non-zero, the real part is non-zero here. So in deforming the contour we encounter no poles; the deformation is allowed, despite the zero first-order shift.

Notice how the deformation has acquired a component in the direction of the natural deformation. At a general values of the integration variables, the first order shift is $i\lambda(|E_R| + 2|p_R| - p_R)$, which is always positive, except at $E_R = p_R = 0$. So the deformation is only anomalous at this one point.

The second example of a deformation is more striking because it avoids the singularity for all $p_R$, but is anomalous everywhere:

$$\eta, \xi = \left(p_R + \frac{p_R^2}{2m} - E_R, m\right), \tag{B10}$$

or
so that the denominator is
\[ E - \frac{p^2}{2m} + i\epsilon = E_R - \frac{p_R^2}{2m} + \frac{\lambda^2 m}{2} + i\lambda \left( \frac{p_R^2}{2m} - E_R \right) + i\epsilon. \]
(B11)

This looks more dangerous because the imaginary part in the \( i\lambda\) term is not always positive. However, to get a zero of the whole denominator, both the real and imaginary parts must be zero. A zero real part gives \( E\) the imaginary part of \( \lambda\) equals zero if and only if \( \lambda = \epsilon = 0\), and then the imaginary part is
\[ \frac{i\lambda^2 m}{2} + i\epsilon. \]
(B12)

This is zero if and only if \( \lambda = \epsilon = 0\), and again we have avoided the pole. Notice how the positive \( \lambda\)-dependent imaginary part in \ref{A.12} is of order \( \lambda^3 \) instead of its usual size \( \lambda\); the careful choice of deformation has canceled bigger terms.

The \( i\lambda\) term in \ref{A.11} does go negative, but only where the real part of the denominator is definitely non-zero.

2. In one dimension, singularity avoiding deformation requires positive first-order shift in denominator

To obtain a singularity-avoiding deformation with a zero first-order shift, the deformation vector \( v(w_R) \) at a singular point needed to be non-zero and tangent to the singularity surface. But in one dimension, singularities are at points, and there is no surface to which a tangent can be constructed. So we expect that a minimum example of an anomalous deformation must be in two dimensions. In this section we analyze the one-dimensional case in more detail.

The denominators are \( A_j(z) + i\epsilon \), with \( z \) being an ordinary complex number. As usual for this paper, \( A_j \) is real when \( z \) is real and is an analytic function of its argument. Suppose that a particular \( A_j(z) \) has a zero at a real value \( z = x_S \). To simplify the notation, shift the integration variable so that the zero is at \( z = 0 \). We now investigate the conditions under which the corresponding singularity is or is not avoided by a contour deformation \( x \mapsto x + i\lambda v(x) = x + i\lambda(v_0 + \delta v(x)) \). Here \( v_0 = v(0) \), the direction of contour deformation at \( x = 0 \), so that \( \delta v(0) = 0 \).

Let the Taylor expansion of \( A_j \) be
\[ A_j(z) = \sum_{n=1}^{\infty} a_n z^n, \]
(B13)

with all the \( a_n \) real. Then \( D_j = A_j'(0) = a_1 \).

If \( D_j v_0 > 0 \), then the deformed contour avoids singularities due to \( A_j \) in a neighborhood of \( x = 0 \), since the imaginary part of \( A_j \) on the deformed contour for all \( x \) near zero.

If \( D_j v_0 < 0 \), then the denominator is negative imaginary at \( x = 0 \). Then the deformation crosses a singularity, and hence is not allowed.

If \( v_0 = 0 \), then \( A_j(x + i\lambda v(x)) \) is zero at \( x = 0 \); the singularity is not avoided.

The above cases (and the trivial proofs) are no different than in the multi-dimensional case.

The remaining case is \( D_j v_0 = 0 \) with \( v_0 \) nonzero. This entails \( a_1 = 0 \). Here is the first difference between the one-dimensional case and higher dimensions. In higher dimensions \( D_j v_0 \) can be zero while having both \( D_j \) and \( v_0 \) be nonzero.

Let the lowest order non-zero \( a_n \) be for \( n = n_0 \) with \( n_0 \geq 2 \). Then
\[ A_j(z) = \sum_{n=n_0}^{\infty} a_n z^n, \]
(B14)

Redefine the real value \( x \) to be \( \tilde{x} v_0 \), so that
\[ A_j(x + i\lambda v(x)) = a_{n_0} v_0^{n_0} (\tilde{x} + i\lambda)^{n_0} + O(|x + i\lambda|^{n_0+1}). \]
(B15)

Set \( \tilde{x} + i\lambda = re^{i\theta} \). Then
\[ A_j(x + i\lambda v(x)) = a_{n_0} (r v_0)^{n_0} e^{i n_0 \theta} + o(r^{n_0}), \]
(B16)

where the \( o(r^{n_0}) \) notation means that this (“correction”) term divided by \( r^{n_0} \) goes to zero as \( r \to 0 \). When \( v_0 \) is positive, allowed (non-negative) values of \( \lambda \) correspond to \( 0 \leq \theta \leq \pi \). If \( v_0 \) is negative, the allowed values correspond to \( 0 \geq \theta \geq -\pi \). Because \( n_0 \geq 2 \), the first term on the right of Eq. \ref{A.16} goes in a circle round the origin at least once when \( \theta \) goes through its allowed values. Therefore, there is at least one allowed value of \( \theta \) where the real part is zero and the imaginary part is negative. The correction term modifies this position of this situation by only a small amount if \( r \) is small enough, but doesn’t affect its occurrence. Hence as the contour is deformed from \( \lambda = 0 \) and as \( \epsilon \) is taken to zero, a zero of \( A_j + i\epsilon \) is always encountered, and therefore the deformation is not allowed.

Combining all these cases shows that the contour deformation in one dimension avoids the singularity if and only if \( D_j v_0 \) is strictly positive. The singularity is not avoided if \( D_j v_0 = 0 \). It follows that anomalous deformations only exist in two or more dimensions.

Appendix C: Illustrative examples

1. Elementary example of Landau singularity

To provide elementary example of a Landau singularity, and to examine it in the light of the approach used in this paper, and also to be able to later pinpoint differences between the massive and massless cases, we consider the self-energy graph of Fig. 10. The integral for
it, with couplings, external propagators and symmetry factor omitted, is
\[
\Pi(p,m) = \int \frac{d^n k}{(2\pi)^n} \frac{1}{[k^2 - m^2 + i\epsilon][(p-k)^2 - m^2 + i\epsilon]},
\]
where we work in an n-dimensional space-time, and where \( p \) is the external momentum and \( m \) the mass for each line, which we will assume to be non-zero for the moment. Since the graph is ultra-violet (UV) divergent when \( n \geq 4 \), we will use \( n < 4 \). The modification to \( \Pi(p,m) \) to implement renormalization of UV divergences is in fact irrelevant\(^{18}\) for the pinch/singularity analysis, but it is easier not to have to bother with the issue.

Since our analysis concerns pinches and singularities in the physical-region, \( p \) is real. It is well-known that the only physical-region singularity in \( \Pi(p,m) \) is the normal threshold singularity at \( p^2 = 4m^2 \). In the Coleman-Norton analysis, this corresponds to a pinch where \( k = p/2 \), so that both lines are on-shell, and have equal momenta. The corresponding classical process corresponds to two particles of momentum \( p/2 \) starting at the same point. The particles propagate for an arbitrary time, with both of them having the same trajectory, so that they can then recombine again.

To derive this result explicitly, we first observe that the integrand is only singular at values of \( k \) where one or both lines are on-shell. There can be no pinch at other values of \( k \).

There are two cases. The first is that only one line is on-shell, for which we choose the first propagator. Let \( k = k_1 \) be the position of one zero of the denominator. Then we can avoid the corresponding singularity by a contour deformation \( k \to k_R + i\lambda v(k_R) \) such that \( v(k_1) \cdot k_1 > 0 \). It is always possible to find such a \( v(k_1) \) for a non-zero value of \( k_1 \), e.g., \( v(k_1) = k_1 \) for a massive on-shell momentum. Then the imaginary part of the denominator is positive and the singularity is avoided in a neighborhood of \( k_R = k_1 \).

The second case is when both propagators are on-shell, i.e., at the position \( k = k_1 \), so that \( k_1^2 = (p-k_1)^2 = m^2 \). The imaginary parts of the denominators are \( 2\lambda v(k_1) \cdot k_1 \) and \( 2\lambda v(k_1) \cdot (k_1 - p) \). Here \( 2k_1 \) and \( 2(k_1 - p) \) are the (non-zero) derivatives of the denominators. If the two derivatives are linearly independent, then \( v(k_1) \) can be chosen to make both imaginary parts positive, and hence the singularity is avoided, i.e., there is no pinch.

This cannot necessarily be done if the two vectors are proportional to each other, i.e., \( k_1 = c(k_1 - p) \) for some non-zero number \( c \). If \( c \) is positive, then by a choice of \( v(k_1) \), e.g., \( v(k_1) = k_1 \), we can make both imaginary parts positive. But if \( c \) is negative, then the signs of the imaginary parts of the denominators are always opposite. If one is positive, the other is negative and the singularities are not avoided.

It is conceivable that a cunningly chosen \( k \)-dependent deformation obeying \( v(k_1) \cdot k_1 = 0 \) could avoid both singularities, i.e., an anomalous deformation as treated in App. B\(^1\) for an unpinched situation. But when \( c \) is negative, this possibility is ruled out in the course of our proof in Sec. VIII.

Therefore the contour is pinched if and only if both denominators are zero and \( k_1 = c(k_1 - p) \) for negative \( c \). We can rewrite this in the standard form of a Landau condition as \( k_1 + a(k_1 - p) = 0 \) with \( a = -c > 0 \).

Solving this equation and the on-shell conditions gives \( a = 1 \) and \( k_1 = p - k = p/2 \). Hence \( p^2 = 4m^2 \) which is the standard normal threshold.

Observe that given \( p \), there is exactly a single pinch point.

Applying the Feynman parameter method gives
\[
\Pi(p,m) = \int_0^1 d\alpha \int \frac{d^n k}{(2\pi)^n} \frac{1}{(k^2 - 2k \cdot p + p^2\alpha - m^2 + i\epsilon)^2}.
\]
With a single denominator, the Landau condition for a pinch is just that the denominator and its first derivative are zero. This gives \( p^2 = 4m^2, \alpha = 1/2 \), and \( k = p - k = p/2 \), corresponding exactly to the previous determination.

Observe that there are endpoint zeros of the denominator, and hence endpoint singularities of the integrand. These occur when \( (\alpha = 0, k^2 = m^2) \) and when \( (\alpha = 1, (p-k) = m^2) \). While a contour deformation cannot change the value of \( \alpha \) at an endpoint, a deformation on \( k \) suffices to avoid the endpoint singularity of the denominator.

Finally, we can perform the \( k \) integral, using a standard formula, to get
\[
\Pi(p,m) = \frac{i\Gamma(2 - n/2)}{(4\pi)^{n/2}} \times \int_0^1 d\alpha \frac{1}{(p^2\alpha(1-\alpha) - m^2 + i\epsilon)^{2-n/2}}.
\]
Again there is a pinch when the single denominator and its first derivative are zero. This gives \( p^2 = 4m^2 \) and \( \alpha = 1/2 \), but without any direct indication of a suitable value of \( k \). Away from that case, the singularity of the integrand can be avoided by a contour deformation.

\(^{18}\) See Ref. [5] p. 5 and an unpublished paper by Hepp cited there.
The lack of direct information on a value of $k$ that corresponds to the pinch in the pure parameter integral indicates that the use of a pure parameter representation gives less information on the momenta concerned at a pinch compared with a representation with a momentum integral. This is important for many QCD applications, where the concern is not so much with singularities of an integral as a function of external parameter(s), but with the regions where the integration is trapped in a region of low virtuality for some lines.

2. Simple example for $M$

To understand and visualize the main ideas, in Sec. VII D and subsequent sections, let $V$ be a space of 2-dimensional real column vectors, and define

$$D_1 = (1, 0), \quad D_2 = (0, 1), \quad D_3 = (-a, -b),$$

where $a$ and $b$ are any chosen positive numbers. Of course, these obey

$$D(a, b) = aD_1 + bD_2 + D_3 = 0 \quad (C7)$$

i.e., the Landau condition is obeyed with $\lambda_1 = a$ and $\lambda_2 = b$. Here, we construct for this example the main objects used in our general proof that there is a Landau point given the knowledge that no “good direction” exists. These objects can be used as illustrations of the steps in the general proof.

The positive space of $D_1$ and $D_2$ is

$$P = \left\{ \left( \frac{\alpha}{\beta} \right) \text{ with } \alpha > 0, \beta > 0 \right\}. \quad (C8)$$

Now with $v = (\alpha, \beta)^T$, and $D(\lambda) = \lambda_1 D_1 + \lambda_2 D_2 + D_3$,

$$D(\lambda)(v) = \alpha(\lambda_1 - a) + \beta(\lambda_2 - b). \quad (C9)$$

In Eq. 7.10, we defined the set $M$ to be set of $\lambda$ for which $D(\lambda)$ is negative or zero on $P$. This is

$$M = \{(\lambda_1, \lambda_2) : 0 \leq \lambda_1 \leq a \text{ and } 0 \leq \lambda_2 \leq b\}, \quad (C10)$$

as illustrated in Fig. 7.

The Landau point is at the corner $(\lambda_1, \lambda_2) = (a, b)$. The construction in Sec. VII C starts from a point on the boundary between $M$ and $M$, and moves along the boundary in a direction where one of the $\lambda_j$ increases until no further increases are possible. We showed in general that the resulting extreme point is a Landau point.

The various results used in this process can be illustrated by using as a starting point $\lambda = (a, \lambda_2)$, with $0 \leq \lambda_2 < b$. There $D(\lambda) = (0, \lambda_2 - b)$.

Appendix D: Difficulties with the use of Feynman parameters

In standard derivations (e.g., Refs. [4, 6]) of the Landau criterion, a common technique is the use of Feynman parameters, as in Eqs. (2.2) and (2.3). This converts an integral with multiple denominators to an integral with one denominator. As regards non-endpoint singularities/pinches, the Landau condition for a pinch becomes simply the condition that the single denominator and its derivative with respect to every integration variable are zero. In this case, the justification of the condition as being both necessary and sufficient for a pinch no longer has any need for the complicated geometrical argument of Secs. VII C. The analysis of the single-denominator case can then be shortened to the treatment in Sec. VII D.

So it might be supposed that the use of Feynman parameters is a panacea for many of the difficulties exposed in this paper. However, this is not the case. In the first place, it is often useful to be able to analyze directly what happens in the original integral. For example, this applies to the region analysis so pervasive in QCD, and its further elaboration in treatments of Glauber-type regions; it also applies to the problem of finding a good algorithm for contour deformation in a numerical integral. Furthermore, methods for the extension to a coordinate-space analysis, such as is summarized in Sec. VII D, do not readily lend themselves to the use of Feynman parameters.

In this section, I will present some simple examples that show that certain further issues are a serious obstacle to a general-purpose use of Feynman parameters.

One issue is that it is not all clear that a pinch in a parametric form of an integral necessarily entails a pinch in the original integral, although this is implicitly assumed in essentially all the standard treatments. In the next Sec. D 1, I will show a counterexample where the assumption is actually wrong; the example is simply a one-loop massless self-energy graph.

A second issue is that for more general cases than standard relativistic Feynman graphs, it is not always the case that integrals involving Feynman parameters are sufficiently well-behaved to be treated by normal contour integration. In contrast, the Feynman parameter method was designed to be very useful for the standard quadratic denominators in normal relativistic Feynman graphs. For example, the momentum integrals can be calculated analytically, leaving an integral only over the Feynman parameters, with rules for the denominator being found in, for example, Sec. 1.5 of Ref. [6].

Other kinds of denominator do appear even in QCD, e.g., Wilson lines with their linear denominators. The general analysis in the present paper has no problems in

\[^{19}\text{The generalization to endpoint singularities requires modifications [6] to the analysis that are straightforward.}\]
the presence of linear denominators; indeed some parts become easier. But the example given in Sec. D2 below shows that the Feynman parameter method can become pathological in the presence of linear denominators.

1. Massless self-energy graph

Consider the massless version of the self energy graph that was treated in App. C1 for the massive case:

$$\Pi(p, 0) = \int \frac{d^n k}{(2\pi)^n} \frac{1}{[k^2 + i\epsilon] [(p - k)^2 + i\epsilon]}, \quad (D1)$$

and we search for conditions for a pinch.

a. Pure momentum representation

If both denominators in Eq. (D1) are non-zero, then there is no pinch, as before.

Next suppose the first denominator is zero, i.e., $k^2 = 0$, then either $k$ is non-zero and light-like, or it is zero. If it is light-like, then we can find a vector $v(k)$ such that $v(k) \cdot k > 0$, so that we can avoid the singularity by a contour deformation.

But if $k$ is zero, then both the denominator and its first derivative is zero, so we have a pinch there, independently of the value of $p$. Similarly the other denominator gives a pinch at $k = p$. Even though we have a pinch, there is no singularity of the value of the integral $\Pi(p)$ unless also $p^2 = 0$, a well-known property. The property of having a pinch on a massless line in a Feynman graph when the line’s momentum is zero evidently applies to all graphs with massless lines.

Finally, if both denominators are zero, i.e., $k^2 = (p - k)^2 = 0$, then the same approach as in App. C1 shows that when $p^2 = 0$, there is a line of collinear pinches, with

$$k = \alpha p \quad \text{with} \quad 0 \leq \alpha \leq 1. \quad (D2)$$

This also immediately follows from the Landau condition for the graph. The single-denominator pinches are at the endpoints of the collinear pinch line.

b. Mixed momentum-parameter representation

The mixed momentum-parameter form is

$$\Pi(p, m) = \int \frac{d^n k}{(2\pi)^n} \frac{1}{(k^2 - 2k \cdot m + m^2 + i\epsilon)^2}. \quad (D3)$$

First, there are zeros of the denominator at the endpoints in $\alpha$, at $(\alpha = 0, k = 0)$ and $(\alpha = 1, k = p)$. Unlike the massive case, there is a zero derivative of a denominator with respect to $k$ at these points. So the endpoint singularity can no longer be avoided, and we have a pinch.

This reproduces the first two configurations seen in the momentum representation.

For a non-endpoint value of $\alpha$, there is again a pinch when the denominator and its first derivative are zero, i.e., when

$$k^2 - 2k \cdot m + m^2 = 0, \quad k - m = 0, \quad \text{and} \quad -2k \cdot m + m^2 = 0. \quad (D4)$$

These have non-endpoint solutions only when $p^2 = 0$, and then the solution exists for all $\alpha$, with $k = \alpha m$, again reproducing the results in the momentum representation.

c. Pure parameter representation

The pure-parameter form is obtained by using a standard formula for the momentum integral:

$$\Pi(p, 0) = i\Gamma(2 - n/2) (p^2)^{n/2 - 2} (4\pi)^{n/2} \int_0^1 d\alpha \frac{1}{[\alpha(1 - \alpha) + i\epsilon]^{2-n/2}.} \quad (D5)$$

The momentum integration has given an overall factor that is a power of $p^2$ and that is singular at $p^2 = 0$ only. But there is no pinch at all in the $\alpha$ integral (which can in fact be performed analytically to give the textbook result

$$\Pi(p, 0) = \frac{i\Gamma(2 - n/2)[\Gamma(n/2 - 1)]^2 (p^2)^{n/2 - 2}}{(4\pi)^{n/2}\Gamma(n-2)} (p^2)^{n/2 - 2}. \quad (D6)$$

d. Results

We now see several differences compared with the massive case

- There is a pinch in the momentum integral for all $p$, even though $\Pi$ is nonsingular if $p^2 \neq 0$. This is another case beyond the rather trivial example in App. A where the existence of a pinch does not entail a singularity of the integral as a function of external parameters.

- In the massive case, there was a pinch at a single point of the integration variables. This applied in all three representations. But in the massless case, there is a whole line of collinear pinches, and this is visible in both the momentum and momentum-parameter representations.

- But in the pure parameter representation, there is no pinch correspond to the collinear singularity. Instead the singularity of the integral at $p^2 = 0$ is in the prefactor only.
It follows that the existence of a pinch in the momentum representation does not entail a pinch in the pure parameter representation, contrary to what is assumed as obvious in the standard literature. The example is not at all exotic; it gives the simplest possible example of a collinear pinch in a massless theory.

2. Denominators linear in some momentum components

In this section, I show explicitly that the Feynman-parameter method does not readily apply to situations with denominators that depend linearly on some or all components, e.g., with non-relativistic theories or with Wilson lines.

It suffices to consider one example, a non-relativistic analog of a self-energy graph in a 2-dimensional space-time:

\[ \Gamma(E, \rho) = \int d\omega \, dk \frac{1}{(\omega - \frac{k^2}{2m} + i\epsilon) \left(E - \omega - \frac{(p-k)^2}{2m} + i\epsilon\right)}. \]  

(D7)

Applying the Feynman parameter method and exchanging the order of the parameter and momentum integrals gives

\[ \Gamma(E, \rho) = \int_0^1 d\alpha \left\{ \int dk \, d\omega \frac{1}{[(1 - 2\alpha)\omega + E\alpha - \frac{1}{2im} \left(k^2 - 2\alpha pk + \alpha p^2\right) + i\epsilon]^2} \right\}. \]  

(D8)

By standard contour-integration methods, the integral over \( \omega \) is zero if \( \alpha \) is not equal to 1/2. To get a non-zero value, the result of integration over \( \omega \) has to be a non-trivial generalized function (distribution) localized at \( \alpha = 1/2 \). In fact, using the methods of Yan [22] gives the integral over \( \omega \):

\[ \int_{-\infty}^{\infty} d\omega \left\{ \frac{1}{[(1 - 2\alpha)\omega + E\alpha - \frac{1}{2im} \left(k^2 - 2\alpha pk + \alpha p^2\right) + i\epsilon]^2} \right\} = \frac{-4i\pi m}{2E - k^2 + 2pk - p^2 + i\epsilon} \delta\left(\alpha - \frac{1}{2}\right). \]  

(D9)

which implies that the result of performing the momentum integrals is not of the form of a normal integral such as [23]. (One could also say that if one tried restricting the integrals to conventional ones, then the exchange of order of integration is not allowed, contrary to the almost universally assumed situation for relativistic graphs.) In a sense, the existence of a \( \delta(\alpha - \frac{1}{2}) \) implies that the pure parameter integral (after performing the \( \omega \) integral and possibly the \( k \) integral) always has a pinch at \( \alpha = \frac{1}{2} \) independently of whether there is a pinch in the original momentum integral.

In contrast, if the analysis in the present paper is applied directly to the momentum space integral, one finds that there is a physical region pinch (and singularity) if and only if \( E = p^2 / (4m) \), i.e., the external energy-momentum corresponds to a particle of double the mass of that for the individual lines. This is easily verified by performing the momentum integral analytically. The pinch is at \( (\omega, k) = \frac{1}{2}(E, \rho) \).

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