On Equivalence of Stochastic Block Model and Nonnegative Matrix Factorization

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Abstract

Community structures detection in complex network is important for understanding not only the topological structures of the network, but also the functions of it. Stochastic block model and nonnegative matrix factorization are two widely used methods for community detection, which are proposed from different perspectives. In this paper, the relations between them are studied. The logarithm of the likelihood function for the stochastic block model can be reformulated under the framework of nonnegative matrix factorization. Besides the model equivalence, the algorithms employed by the two methods are different. Preliminary numerical experiments are carried out to compare the behaviors of the algorithms.

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I. INTRODUCTION

One of the fundamental problems in network analysis is community structures detection ([1, 2]). In most cases, a community is a group of nodes which connect with each other tightly while connect loosely with the rest of the network. There are other types of communities as well. Communities often correspond to functional units. For example, in a social network, a community might correspond to a group of people brought together by a common interest. Detection of communities is very important for understanding not only the topological structures of the network, but also the functions of it, such as how the nodes communicate with each other, or how new ideas diffuse in the network ([3]), etc.

Many kinds of methods have been proposed to detect community structures in the literature recently. One group of methods is based on generative model. The basic motivation is that the network we observed is an instance generated by a set of hidden parameters, and we can detect the community structures in the network by revealing the parameters. Among them, the most representative one is the stochastic block model (SBM, [4–12]). SBM provides a well-founded principled approach for understanding the network structures, and is very flexible such that not only the traditional type of community structures, but also a wide variety of structures in networks can be formulated into the model and be detected ([11, 13]). In this paper, our focus is on the stochastic block models proposed by Karrer and Newman ([12]), and its variants ([13–16]).

Another group of methods for community detection is based on optimization of some global criteria over all possible network partitions, including graph partitioning ([17]), spectral clustering([18]), modularity maximization ([2, 19]) and nonnegative matrix factorization([20]), etc. Nonnegative matrix factorization (NMF) was originally proposed as a method for finding matrix factors with parts-of-whole interpretations ([20, 21]), and has become a powerful tool for data analysis with enhanced interpretability. By accommodating a variety of objective functions, NMF has been successfully applied to a lot of distinct areas ([22–27]). Specifically, NMF has been successfully adopted to community structures detection recently ([28–32]). Both spectral clustering and probabilistic latent semantic indexing can be reformulated under the framework of NMF ([33, 34]).

Although there were so many crucial researches of community detection based on SBM and NMF respectively, few attempt has been made to build the connections between them
while emphasizing their differences in community detection. In this paper we prove that the standard stochastic block models and its several variants can be reformulated under the NMF framework. To the best of our knowledge, this is the first time establishing the connections between SBM and NMF. Specifically, we show that the likelihood functions to be maximized for SBM is equivalent to the objective functions for nonnegative matrix factorization model. Empirical experiments are carried out to compare the difference between the algorithms employed by the two models.

The rest of the paper is organized as follows. From Section 2 to Section 7, we prove the equivalence of six types of blockmodels and nonnegative matrix factorizations, respectively. The six blockmodels are the standard SBM, the degree-corrected SBM, the directed SBM, the signed SBM, the bipartite SBM and the normal distributed SBM. Experimental results are given in Section 8. And Section 9 concludes.

II. EQUIVALENCE OF STANDARD STOCHASTIC BLOCK MODEL AND NON-NEGATIVE MATRIX FACTORIZATION

Let $G$ be an undirected multigraph with $n$ vertices, possibly including self-edges. Each vertex $i$ belongs to one of $c$ latent communities and assume the number of edges between each two nodes (or self-edges) to be independently Poisson distributed. Let $A$ be the adjacency matrix of $G$, with its element $A_{ij}$ denoting the number of edges between vertices $i$ and $j$. And denote $W$ to be a $c \times c$ matrix with its element $w_{rs}$ the expected value of $A_{ij}$ for vertices $i$ and $j$ lying in community $r$ and $s$ respectively. Also, we introduce community membership matrix $G$ to record the community assignment of vertices in the network $G$, where $G_{ir} = 1$ if vertex $i$ belongs to community $r$; 0 otherwise.

According to Karrer and Newman ([12]), the probability $P(G|W, G)$ of graph $G$ given the parameters $W$ and the community assignment matrix $G$ is

$$P(G|W, G) = \frac{1}{\prod_{i<j}A_{ij}! \prod_i 2^{A_{ii}/2} (A_{ii}/2)!} \times \prod_{rs} w_{rs}^{m_{rs}/2} \exp\left(-\frac{1}{2}n_r n_s w_{rs}\right),$$

(1)

where $n_r$ is the number of vertices in community $r$ and $\sum_{r=1}^c n_r = n$, $m_{rs}$ is the total number of edges between community $r$ and $s$, or twice that number if $r = s$ and $\sum_{k=1}^c G_{ik} = 1$. The
goal is to maximize this probability (1) with respect to the unknown model parameters \( W \) and the community membership matrix \( G \). By neglecting constants which are independent of the parameter \( W \) and the community assignment matrix \( G \), the problem above is reduced to

\[
\max_{W,G} \log P(\mathcal{G}|W,G) = \max_{W,G} \sum_{rs} (m_{rs} \log w_{rs} - n_r n_s w_{rs}),
\]

which is equivalent to

\[
\min_{W,G} \sum_{rs} - (m_{rs} \log w_{rs} - n_r n_s w_{rs}).
\]

The first term in equation (2) is

\[
- \sum_{r,s} m_{rs} \log w_{rs} = - \sum_{r,s} \sum_{i,j} A_{ij} G_{ir} G_{js} \log w_{rs} \\
= - \sum_{i,j} A_{ij} (\sum_r G_{ir} \sum_s G_{js} \log w_{rs}) \\
= - \sum_{i,j} A_{ij} (G_{ir}^* G_{js}^* \log w_{r^* s^*}) \\
= - \sum_{i,j} A_{ij} (\log (G_{ir}^* G_{js}^* w_{r^* s^*})) \\
= - \sum_{i,j} A_{ij} \log (\sum_{r,s} G_{ir} G_{js} w_{rs}) \\
= - \sum_{i,j} A_{ij} \log (GWG^T)_{ij} \\
= \sum_{i,j} A_{ij} \log \frac{1}{(GWG^T)_{ij}},
\]

the third equality above holds because there is only one 1 in each row of matrix \( G \), which is denoted by \( G_{ir^*} \) and \( G_{js^*} \), respectively.

The second term in equation (2) is

\[
\sum_{r,s} n_r n_s w_{rs} = n^T W n = 1^T GWG^T 1 = \sum_{i,j} (GWG^T)_{ij},
\]

where \( n = (n_1, \ldots, n_c)^T \) is a \( c \times 1 \) column vector, and \( 1 \) is a \( c \times 1 \) vector with all elements 1. \( A^T \) is the transpose of a matrix (or a vector) \( A \).

Hence, the optimal problem (2) is equivalent to

\[
\min_{W,G} \left[ \sum_{i,j} A_{ij} \log \frac{1}{(GWG^T)_{ij}} + \sum_{i,j} (GWG^T)_{ij} \right].
\]
By adding some constants which are independent of $W$ and $G$, and relaxing the constraints on $G$ from binary to non-negativity, we have the following optimization problem

$$\begin{align*}
\min_{W,G} & \sum_{i,j} \left[ A_{ij} \log \frac{A_{ij}}{(GWG^T)_{ij}} + (GWG^T)_{ij} - A_{ij} \right] \\
\text{s.t.} & \sum_{r=1}^{c} G_{ir} = 1, \; i = 1, 2, \ldots, n. \\
& G_{ir}, w_{rs} \geq 0, \; i = 1, 2, \ldots, n; \; r, s = 1, 2, \ldots, c.
\end{align*}$$

(4)

which is typically a nonnegative matrix factorization model, and can be naturally extended to overlapping community structures detection.

III. EQUIVALENCE OF DEGREE-CORRECTED STOCHASTIC BLOCK MODEL AND NONNEGATIVE MATRIX FACTORIZATION

In the standard SBM mentioned in Sect. 2, vertices in the same community are identical, i.e., vertices in the same community have equal probability connecting to others, and thus are supposed to have the same degree distribution, which is not realistic, since real networks are often degree heterogeneity. Karrer and Newman ([12]) proposed the degree-corrected SBM to take degree heterogeneity into account when generating a network. In this section, we will prove the equivalence of degree-corrected SBM and NMF.

When considering the degree-corrected SBM, the generation of $G$ depends not only on the parameters introduced previously, but also on a new set of parameters $\theta_i, i = 1, 2, \ldots, n$, which is the degree weight of vertex $i$ satisfying the condition $\sum_{i=1}^{n} \theta_i G_{ir} = 1, \; r = 1, 2, \ldots, c$. The expected value of the edge number between two vertices $i$ and $j$ is $\theta_i \theta_j w_{rs}$, instead of $w_{rs}$, where $r, s$ are the communities that vertex $i$ and $j$ belong to respectively.

For the degree-corrected SBM, the goal is to maximize the probability (5) below with respect to the unknown parameters $\theta, W$ and the community membership matrix $G$, where $\theta = (\theta_1, \ldots, \theta_n)^T$:

$$P(G|\theta, W, G) = \frac{1}{\prod_{i<j} A_{ij}! \prod_i 2A_{ii}/2!} \times \prod_{i<j} (\theta_i \theta_j)^{A_{ij}} \prod_i (\theta_i^2)^{A_{ii}/2} \prod_{r,s} w_{rs}^{m_{rs}/2} \exp\left(-\frac{1}{2}w_{rs}\right),$$

(5)
or

\[
\log P(\mathcal{G}|\theta, W, G) = 2 \sum_{i<j} A_{ij} \log(\theta_i \theta_j) + 2 \sum_i \frac{A_{ii}}{2} \log \theta_i^2 + \sum_{r,s} (m_{rs} \log w_{rs} - w_{rs})
\]

\[
= \sum_{i,j} A_{ij} \log(\theta_i \theta_j) + \sum_{r,s} (m_{rs} \log w_{rs} - w_{rs}).
\]  

(6)

Hence the problem is

\[
\max_{\theta, W, G} \log P(\mathcal{G}|\theta, W, G) = \max_{\theta, W, G} \left[ \sum_{i,j} A_{ij} \log(\theta_i \theta_j) + \sum_{r,s} (m_{rs} \log w_{rs} - w_{rs}) \right],
\]  

(7)

which is equivalent to

\[
\min_{\theta, W, G} \left( \sum_{i,j} A_{ij} \log \frac{1}{\theta_i \theta_j} + \sum_{r,s} m_{rs} \log \frac{1}{w_{rs}} + \sum_{r,s} w_{rs} \right),
\]  

(8)

The first two terms in (8) can be combined together as

\[
\sum_{i,j} A_{ij} \log \frac{1}{\theta_i \theta_j} + \sum_{r,s} m_{rs} \log \frac{1}{w_{rs}}
\]

\[
= \sum_{i,j} A_{ij} \log \frac{1}{\theta_i \theta_j} + \sum_{i,j} A_{ij} \log \left( \frac{1}{(GWG^T)_{ij}} \right)
\]

\[
= \sum_{i,j} A_{ij} \log \frac{1}{\theta_i \theta_j (GWG^T)_{ij}}
\]

\[
= \sum_{i,j} A_{ij} \log \frac{1}{((\theta \theta^T) \otimes (GWG^T))_{ij}}.
\]  

(9)

The third term in (8) is

\[
\sum_{r,s} w_{rs} = 1^T W 1
\]

\[
= \theta^T GWG^T \theta
\]

\[
= \sum_{i,j} \theta_i \theta_j (GWG^T)_{ij}
\]

\[
= \sum_{i,j} \left( (\theta \theta^T) \otimes (GWG^T) \right)_{ij},
\]  

(10)

where \( \otimes \) denotes the dot product of two matrices (or vectors) with the same dimensions.

And the second equality above holds because sum of weights in one community equals 1, that is, \( \sum_{i=1}^{n} \theta_i G_{ir} = 1, \ r = 1, 2, \ldots, c. \)
By combining (9) and (10) together, the optimal problem (7) is then equivalent to

$$\begin{align*}
\min_{\theta, \mathbf{W}, \mathbf{G}} & \sum_{i,j} \left[ A_{ij} \log \frac{A_{ij}}{((\theta \theta^T) \otimes (\mathbf{G} \mathbf{W} \mathbf{G}^T))_{ij}} + ((\theta \theta^T) \otimes (\mathbf{G} \mathbf{W} \mathbf{G}^T))_{ij} - A_{ij} \right] \\
\text{s.t.} & \sum_{i=1}^{n} \theta_i G_{ir} = 1, \quad r = 1, 2, \ldots, c. \\
& \sum_{r=1}^{c} G_{ir} = 1, \quad i = 1, 2, \ldots, n. \\
& G_{ir}, \theta_i, w_{rs} \geq 0, \quad i = 1, 2, \ldots, n; \quad r, s = 1, 2, \ldots, c.
\end{align*}$$

That is, optimization problem over the degree-corrected SBM is equivalent to the weighted NMF model.

IV. EQUIVALENCE OF DIRECTED STOCHASTIC BLOCK MODEL AND NON-NEGATIVE MATRIX FACTORIZATION

In this section, we turn to consider the directed SBM setting. In a directed SBM, an edge is an ordered pair of vertices, that is, an edge from vertex $i$ to $j$ is different with an edge from vertex $j$ to $i$. This differs from the standard (undirected) SBM introduced in Sect. 2, in that the latter is defined in terms of unordered pairs of vertices.

Notations given in Section 2 and Section 3 have to be redefined and some new notations are provided here. $\mathcal{G}$ is a directed graph with $n$ vertices. $A$ is still an $n \times n$ adjacency matrix but is not symmetric, $A_{ij} = 1$ if there is an edge from node $i$ to node $j$ and 0 otherwise, where $i$ is named tail node and $j$ is named head node. For weighted networks, $A_{ij}$ is generalized to represent the weight of the edge from $i$ to $j$. $W$ is a $c \times c$ matrix with $w_{rs}$ denoting the probability that a randomly selected edge, of which the tail node is from group $r$ and the head node is from group $s$. $F$ and $H$ are the community membership matrices, where $F$ is a $n \times c$ matrix with its element $F_{ir}$ denoting the probability that the tail node $i$ is from community $r$, and $H$ is a $n \times c$ matrix with its element $H_{js}$ denoting the probability that the head node $j$ is from $s$, respectively.

According to Shen, Cheng and Guo ([14]), the goal is to maximize the probability $P(\mathcal{G}|F, W, H)$, which is the profile likelihood of the observed network, or equally to maxi-
mize the logarithm of the probability, with respect to the parameter $W$, and the community membership matrices $F$ and $H$. That is,

$$\max_{F,W,H} \log P(G|F,W,H) = \max_{F,W,H} \sum_{i,j} A_{ij} \log \left( \sum_{r,s} w_{rs} F_{ir} H_{js} \right),$$

with respect to the constraints $\sum_{r,s=1}^c w_{rs} = 1, \sum_{i=1}^n F_{ir} = 1, \sum_{j=1}^n H_{js} = 1, r, s = 1, 2, \ldots, c$ (14), which is equivalent to

$$\begin{cases}
\min_{F,W,H} \sum_{i,j} A_{ij} \log \frac{1}{\sum_{r,s} w_{rs} F_{ir} H_{js}} \\
\text{s.t. } \sum_{r,s=1}^c w_{rs} = 1, \sum_{i=1}^n F_{ir} = 1, \sum_{j=1}^n H_{js} = 1, r, s = 1, 2, \ldots, c.
\end{cases}$$ (11)

Since

$$\sum_{i,j} \sum_{r,s} w_{rs} F_{ir} H_{js} = \sum_{i,j} (FWHT)_{ij}$$

and

$$\sum_{i,j} \sum_{r,s} w_{rs} F_{ir} H_{js} = \sum_{i,r,s} w_{rs} F_{ir} \sum_{j} H_{js} = \sum_{i,r,s} w_{rs} F_{ir} = \sum_{r,s} w_{rs} \sum_{i} F_{ir} = \sum_{r,s} w_{rs} = 1,$$

we have that the optimal problem (11) has the following equivalent non-negative matrix factorization form

$$\begin{cases}
\min_{F,W,H} \left[ \sum_{i,j} A_{ij} \log \frac{A_{ij}}{(FWHT)_{ij}} + \sum_{i,j} (FWHT)_{ij} - \sum_{i,j} A_{ij} \right] \\
\text{s.t. } \sum_{r,s=1}^c w_{rs} = 1, \sum_{i=1}^n F_{ir} = 1, \sum_{j=1}^n H_{js} = 1, r, s = 1, 2, \ldots, c.
\end{cases}$$

$$F_{ir}, H_{js}, w_{rs} \geq 0, i, j = 1, 2, \ldots, n; r, s = 1, 2, \ldots, c.$$ which is actually also equivalent to probabilistic latent semantic indexing (34).
V. EQUIVALENCE OF SIGNED STOCHASTIC BLOCK MODEL AND NON-NEGATIVE MATRIX FACTORIZATION

Networks possessing both positive and negative links are called signed networks. In signed networks, most edges within a community are positive links, and most edges across communities are negative links. Signed networks exist in many occasions. For example, in a social network, positive links may denote agreement whereas negative links may denote disagreement. Chen et al. (15) studied the signed SBM. In this section, we will prove the equivalence of signed SBM and NMF.

Firstly, we update some notations. \( G \) is a signed network with \( n \) vertices, and \( A \) is the adjacency matrix. We use \( A^+ \) and \( A^- \) to denote the positive and negative parts in the signed network, respectively. That is, \( A^+_{ij} = A_{ij} \) if \( A_{ij} > 0 \), 0 otherwise; \( A^-_{ij} = -A_{ij} \) if \( A_{ij} < 0 \), 0 otherwise. Let \( H \) be the community membership matrix with its element \( H_{ir} \) denoting the probability that the node \( i \) is in the community \( r \). \( W \) is a \( c \times c \) matrix with its element \( w_{rs} \) denoting the probability of an edge choosing between community \( r \) and \( s \). The normalization constraints on \( H \) and \( W \) are \( \sum_i H_{ir} = 1 \) and \( \sum_{r,s} w_{rs} = 1 \). Let \( W_1 = \text{diag}(W) \), which is a diagonal matrix, with its diagonal elements corresponding to those of \( W \), and \( W_2 = W - W_1 \).

According to Chen et al. (15), the goal is to maximize the logarithm of the likelihood of the signed network below with respect to the unknown parameters \( W \) and the community membership matrix \( H \).

\[
\max_{H,W} \log P(G|H,W) \\
= \max_{H,W} \sum_{i,j=1}^n \left[ A^+_{ij} \log \left( \sum_{rr} w_{rr} H_{ir} H_{jr} \right) + A^-_{ij} \log \left( \sum_{rs(r \neq s)} w_{rs} H_{ir} H_{js} \right) \right],
\]

which is equivalent to

\[
\min_{H,W} \sum_{i,j=1}^n \left( A^+_{ij} \log \frac{1}{\sum_{rr} w_{rr} H_{ir} H_{js}} + A^-_{ij} \log \frac{1}{\sum_{rs(r \neq s)} w_{rs} H_{ir} H_{js}} \right). \tag{12}
\]
Firstly,

\[
\sum_{i,j=1}^{n} \left( A_{ij}^+ \log \frac{1}{\sum_{rr} w_{rr} H_{ir} H_{js}} + A_{ij}^- \log \frac{1}{\sum_{rs(r\neq s)} w_{rs} H_{ir} H_{js}} \right)
\]

\[
= \sum_{i,j=1}^{n} \left[ A_{ij}^+ \log \frac{1}{(HW_1 H^T)_{ij}} + A_{ij}^- \log \frac{1}{(HW_2 H^T)_{ij}} \right].
\]  

(13)

Also, note that

\[
\sum_{i,j} (HWH^T)_{ij} = \sum_{i,j} \sum_{r,s} w_{rs} H_{ir} H_{js}
\]

\[
= \sum_{r,s} w_{rs} \sum_i H_{ir} \sum_j H_{js} = \sum_{r,s} w_{rs} = 1,
\]  

(14)

and on the other side,

\[
\sum_{i,j} (HWH^T)_{ij} = \sum_{i,j} (H(W_1 + W_2)H^T)_{ij}
\]

\[
= \sum_{i,j} (HW_1 H^T)_{ij} + \sum_{i,j} (HW_2 H^T)_{ij}.
\]  

(15)

By combining (13), (14) and (15), and adding the constraints on \(H\) and \(W\), we have that the optimal problem (12) is equivalent to the following joint non-negative matrix factorization model:

\[
\begin{align*}
\min_{H, W} & \left\{ \sum_{i,j=1}^{n} \left[ A_{ij}^+ \log \frac{1}{(HW_1 H^T)_{ij}} + (HW_1 H^T)_{ij} - A_{ij}^+ \right] \
+ \sum_{i,j=1}^{n} \left[ A_{ij}^- \log \frac{1}{(HW_2 H^T)_{ij}} + (HW_2 H^T)_{ij} - A_{ij}^- \right] \right\} \\
\text{s.t.} & \sum_{i=1}^{n} H_{ir} = 1, \sum_{r,s} (W_1 + W_2)_{rs} = 1.
\end{align*}
\]

\(W_1\) is nonnegative and diagonal,

\(W_2\) is nonnegative and its diagonal elements are zeros.

\(H_{ir} \geq 0, i = 1, 2, \ldots, n; r = 1, 2, \ldots, c.\)
VI. EQUIVALENCE OF BIPARTITE STOCHASTIC BLOCK MODEL AND NON-NEGATIVE MATRIX FACTORIZATION

In this section, we consider the equivalence of bipartite SBM and NMF. \( \mathcal{G} \) is an undirected bipartite multigraph with \( n \) vertices, possibly including self-edges. There are two types of vertices, i.e. type I and type II, and only vertices of different types may be connected. Each community contains vertices of a single type. The number of vertices in type I and type II are \( n_1 \) and \( n_2 \), respectively. Let \( B \) be a \( n_1 \times n_2 \) bipartite adjacency matrix with \( B_{ij} = 1 \) if there is an edge between \( i \) and \( j \) from type I and type II respectively; 0 otherwise, and \( A \) be a \( n \times n \) adjacency matrix related to \( B \) as

\[
A = \begin{pmatrix}
0 & B \\
B^T & 0
\end{pmatrix}.
\]

We assume that the number of edges between each pair of vertices (including self-edges) is independently Poisson distributed, similar to that in Section 2, and define \( w_{rs} \) to be the expected value of \( A_{ij} \) for vertices \( i \) and \( j \) lying in community \( r \) and \( s \) respectively. Since vertices of the same type cannot be connected, we have

\[
w_{rs} = 0 \quad \text{when} \quad T_{rs} = 0,
\]

where \( T \) is a \( c \times c \) matrix with its element \( T_{rs} = 1 \) if the types of community \( r \) and \( s \) are different; 0 otherwise. Other notations are defined in Section 2.

According to Larremore et al. (16), the probability of the network \( \mathcal{G} \) is that,

\[
P(\mathcal{G}|W,G,T) = \prod_{i<j, i \in r, j \in s, T_{rs}=1} \frac{(w_{rs})^{A_{ij}}}{A_{ij}!} \exp(-w_{rs}),
\]

where \( i \in r \) stands for vertex \( i \) belonging to community \( r \). After a small amount of manipulation, and neglecting constants, taking the logarithm, (16) is equivalent to

\[
\log P(\mathcal{G}|W,G,T) = \sum_{r,s, T_{rs}=1} (m_{rs} \log w_{rs} - n_r n_s w_{rs}).
\]

The goal is to maximize (17) with respect to \( W, G \) and \( T \).

Since

\[
\sum_{r,s, T_{rs}=1} n_r n_s w_{rs} = 1^T G (T \otimes W) G^T 1,
\]
and

\[ \sum_{r,s, T_{rs}=1} m_{rs} \log w_{rs} = - \sum_{r,s, T_{rs}=1} m_{rs} \log \frac{1}{w_{rs}} \]

\[ = - \sum_{r,s, T_{rs}=1} \sum_{i,j} A_{ij} G_{ir} G_{js} \log \frac{1}{w_{rs}} \]

\[ = - \sum_{i,j} T_{rs} \left( \sum_{r,s} A_{ij} G_{ir} G_{js} \right) \log \frac{1}{(T \otimes W)_{rs}} \]

\[ = - \sum_{i,j} A_{ij} \left( \sum_{r,s} T_{rs} G_{ir} G_{js} \log \frac{1}{(T \otimes W)_{rs}} \right) \]

\[ = - \sum_{i,j} A_{ij} \log \frac{T_{rs} G_{ir} G_{js}}{(T \otimes W)_{rs}} \]

\[ = \sum_{i,j} A_{ij} \log \sum_{r,s} (T_{rs} G_{ir} G_{js} (T \otimes W)_{rs}) \]

\[ = \sum_{i,j} A_{ij} \log \sum_{r,s} (G_{ir} G_{js} (T \otimes W)_{rs}) \]

\[ = \sum_{i,j} A_{ij} \log (G (T \otimes W) G^T)_{ij}, \]

the maximization of expression (17) is equivalent to the following weighted nonnegative matrix factorization model:

\[
\begin{aligned}
& \min_{H,W} \sum_{i,j} \left[ A_{ij} \log \frac{A_{ij}}{(G (T \otimes W) G^T)_{ij}} + (G (T \otimes W) G^T)_{ij} - A_{ij} \right] \\
\text{s.t.} & \sum_{r=1}^{c} G_{ir} = 1, \; i = 1, 2, \ldots, n. \\
& G_{ir}, w_{rs} \geq 0, \; i = 1, 2, \ldots, n; \; r, s = 1, 2, \ldots, c. \\
& T_{rs} = 0, 1, r, s = 1, 2, \ldots, c. 
\end{aligned}
\]

Similarly, for the bipartite degree-corrected SBM, it’s easy to obtain the equivalence to NMF, too.


VII. EQUIVALENCE OF NORMAL DISTRIBUTED EDGE-WEIGHTED STOCHASTIC BLOCK MODEL AND NONNEGATIVE MATRIX FACTORIZATION

Aicher, Jacobs and Clauset ([13]) studied the weighted stochastic block model, and the normal distributed edge weight, as one case, was provided. In this section, we will prove that it is also equivalent to NMF. The model is defined as follows. For an undirected multigraph $\mathcal{G}$ on $n$ vertices, the weight of edge between two vertices $i, j$ is supposed to drawn from a normal distribution $N(\mu_{rs}, \sigma_{rs}^2)$, where $r, s$ are the community assignments of vertices $i$ and $j$ respectively. In this case, the likelihood function is

$$P(\mathcal{G} | \mu, \sigma^2) = \prod_{i,j \in r, j \in s} \frac{1}{\sqrt{2\pi}\sigma_{rs}} \exp \left(-\frac{(A_{ij} - \mu_{rs})^2}{2\sigma_{rs}^2}\right), \quad (18)$$

with $i \in r$ standing for vertex $i$ lying in community $r$. The goal is to maximize $18$ with respect to parameters $\mu$ and $\sigma$. Fixing $\sigma$, the maximization of $18$ is equivalent to minimizing the following expression over $\mu_{rs}$,

$$\sum_{i,j \in r, j \in s} (A_{ij} - \mu_{rs})^2. \quad (19)$$

Meanwhile, note that the expectation

$$\mu_{rs} = \sum_{r,s} G_{ir} G_{js} w_{rs} = (GWG^T)_{ij}, \quad (20)$$

where $w_{rs}$ is the expected edge weight between two vertices belonging to community $r$ and $s$, respectively, and $G$ again is the community membership matrix with $G_{ir} = 1$ if vertex $i$ belongs to group $r$; 0 otherwise. The equation $20$ is very critical, because it reveals the relationship between parameter $\mu_{rs}$ and the community assignment $G$.

Then the optimization problem $19$ is reduced to the following NMF model:

$$\begin{align*}
\min_{G,W} & \sum_{i,j} (A_{ij} - (GWG^T)_{ij})^2 \\
\text{s.t.} & \sum_{r=1}^{c} G_{ir} = 1, \ i = 1, 2, \ldots, n. \\
& G_{ir}, w_{rs} \geq 0, \ i = 1, 2, \ldots, n; \ r, s = 1, 2, \ldots, c.
\end{align*} \quad (21)$$
VIII. EXPERIMENTAL RESULTS

Although the likelihood function of SBM can be reformulated as the objective function of NMF, their algorithms are different. In this section, we use synthetic networks to compare the effectiveness of the algorithms employed by SBM and NMF, respectively.

A. Algorithms for SBM and NMF

There is a package “blockmodels” ([35]) in R, which uses variational EM algorithm to estimate the parameters in SBM with some common probability distribution functions including Bernoulli distribution, Poisson distribution and Gaussian distribution ([36]), and explore the community number by the ICL criterion ([37]). We use the command BM.poisson in the package, and fix the community number by setting both the parameters explore.min and explore.max to be the true community number.

We designed the multiplicative update rules for the nonnegative matrix factorization model (4) and (21) ([21]), which are summarized in Algorithm 1 and Algorithm 2, respectively. We set the iteration number iter to 500 for each of the algorithms.

Algorithm 1 NMF with Kullback-Leibler Divergence, model (4)

Input: $A$, iter

Output: $G, W$

1: for $t = 1 : \text{iter}$ do

2: $G_{ij} := G_{ij} \frac{\sum_l (A_{il}(WG^T)_{jl}/(GWG^T)_{il})}{\sum_l (WG^T)_{jl}}$

3: $W_{ij} := W_{ij} \frac{A}{\sum_{kl} G_{kl}G_{ij}}$

4: $G_{ij} := \frac{G_{ij}}{\sum_j G_{ij}}$

5: end for
Algorithm 2 NMF with Least Squares Error, model [21]

Input: $A$, $iter$

Output: $U$

1: for $t = 1 : iter$ do
2: $G_{ij} := G_{ij} \frac{(AGW)_{ij}}{(GWG^TGW)_{ij}}$
3: $W_{ij} := W_{ij} \frac{(AGW)_{ij}}{(GWG^TGW)_{ij}}$
4: $G_{ij} := G_{ij} \frac{G_{ij}}{\sum_j G_{ij}}$
5: end for

B. Datasets Description

In this paper we use the computer-generated networks for comparison.

1. The Girvan-Newman benchmark network (GN, [1]). The GN network contains four communities with 32 vertices each. On average, the number of edges between two vertices from the same community is $Z_{in}$, and that from different communities is $Z_{out}$. As expected, the communities become less clear as $Z_{out}$ increases. Here $Z_{in} + Z_{out}$ is set to be 16.

2. The Lancichinetti-Fortunato-Radicchi benchmark network (LFR, [38]). The LFR network was proposed to cover most characteristics of real networks, such as size of the network and heterogeneous degree distribution, which the GN networks did not capture. In LFR benchmarks, distributions of both the degree and the community size obey power laws with exponents $\alpha$ and $\beta$ respectively. With probability $\mu$, a vertex connects to another vertex from different communities, and in its own community with probability $1 - \mu$.

In this paper, the parameters of the LFR benchmark are set as follows: The number of vertices is 1000, the maximum of degree is 50, the exponents are $\alpha = 2$, $\beta = 1$, the average degree of the nodes is 20 and the range of the mixing parameter $p$ is from 0.1 to 0.9.
C. Simulation Results

In this subsection, we compare the numerical results of the algorithms employed by SBM and NMF on GN and LFR networks. We use the normalized mutual information (NMI, [39]) to evaluate the quality of the results, which can be formulated as follows:

\[
I(M_1, M_2) = \frac{\sum_{i=1}^{k} \sum_{j=1}^{k} n_{ij} \log \frac{n_{ij}n}{n_{i}^{(1)}n_{j}^{(2)}}}{\sqrt{\left(\sum_{i=1}^{k} n_{i}^{(1)} \log \frac{n_{i}^{(1)}}{n}\right)\left(\sum_{j=1}^{k} n_{j}^{(2)} \log \frac{n_{j}^{(2)}}{n}\right)}},
\]

where \(M_1\) and \(M_2\) are the implanted community label and the computed community label, respectively; \(k\) is the true community number; \(n\) is the number of nodes; \(n_{ij}\) is the number of nodes in the implanted community \(i\) that are assigned to the computed community \(j\); \(n_{i}^{(1)}\) is the number of nodes in the implanted community \(i\); \(n_{j}^{(2)}\) is the number of nodes in the computed community \(j\); and \(\log\) here is the natural logarithm. The larger the NMI value, the better the community partition.

The results are averaged over ten trials and are shown in Fig. 1. From the figure, one can conclude the following: (1) The algorithms employed by SBM and NMF are different, although their models to be optimized are equivalent. (2) There is no single winner. SBM works slightly better when the community structures are clear, and NMF with Least Square Error performs better when the degree heterogeneity is included, especially when the community structures are fuzzy.

IX. CONCLUSION

In this paper, we give the detailed analysis on the equivalence of different stochastic block models and the nonnegative matrix factorization. The studied stochastic block models include the standard stochastic block model, the degree-corrected stochastic block model, the directed stochastic block model, the signed stochastic block model, the bipartite stochastic block model and normal distributed edge-weighted block model. Preliminary numerical experiments are also performed on synthetic networks to compare the difference between the algorithms for SBM and NMF.

Based on our works, there are several interesting problems for future work, including
FIG. 1: Averaged NMI with the standard deviation of SBM and NMF on (a) GN networks and (b) LFR networks. SBM stands for stochastic block model, LSE stands for nonnegative matrix factorization with Least Square Error, and KL stands for nonnegative matrix factorization with KL divergence.

the general relations between the generative models and NMF, a systematically comparison among the algorithms employed by SBM and NMF, how to combine the algorithms to make them profit from each other and make up for each other’s deficiencies.

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