Examples of subfactors from conformal field theory

Feng Xu

Abstract

Conformal field theory (CFT) in two dimensions provide a rich source of subfactors. The fact that there are so many subfactors coming from CFT have led people to conjecture that perhaps all finite depth subfactors are related to CFT. In this paper we examine classes of subfactors from known CFT. In particular we identify the so called $3^2 \times 2 \times 2$ subfactor with an intermediate subfactor from conformal inclusion, and construct new subfactors from recent work on holomorphic CFT with central charge 24.

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1 Introduction

Subfactor theory provides an entry point into a world of mathematics and physics containing large parts of conformal field theory, quantum algebras and low dimensional topology (cf. [22] and references therein). This paper is about the interactions between subfactors, algebraic conformal field theory (CFT) which have proved to be very fruitful lately (cf.

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A number of general results about cosets, orbifolds and other constructions have been obtained in the operator algebraic framework (cf. [30] [26] [16] [51] using subfactor techniques. These results have been conjectured for some time and have resisted all other attempts so far.

More recently there have been many new subfactors constructed by using Planar Algebras (cf. [23]) pioneered by V. Jones. For an incomplete list of references, see [8], [33] [35]. Also see [25] for a survey. M. Izumi (cf. [20] [21]) has constructed a large class of subfactors generalizing Haagerup subfactors using Cuntz algebras. T. Gannon and D. Evans (cf. [15]) have provided evidence suggesting that Haagerup subfactor may come from CFT. In [24] V. Jones has devised a renormalization program based on planar algebras as an attempt to show that all finite depth subfactors are related to CFT, i.e., the double of a finite depth subfactor should be related CFT. More generally, the program is the following: given a unitary Modular Tensor Category (MTC) $\mathcal{M}$, (cf. [12]), can we construct a CFT whose representation category is isomorphic to $\mathcal{M}$?

We shall call such a program “reconstruction program”, analogue to a similar program in higher dimensions by Doplicher-Roberts (cf. [13]). M. Bischoff has shown in [2] [3] that this can be done for all subfactors with index less than 4. In view of these recent developments, it is natural to examine subfactors from known CFT. In fact, it is already known that so called $2221$ subfactor is related to subfactor from conformal inclusions (cf. [8]), and it is an interesting question to see if any of these recently constructed subfactors are related to CFT. We do find a few more interesting examples in Section 3 after setting up basics in Section 2.

Another motivation for our work is that it is clear from [2] [8] that holomorphic CFT play an important role in the reconstruction program. In Section 4 we construct new subfactors from holomorphic CFT with central charge 24 based on recent work.

A major progress on reconstruction program would be to identify the origin of Haagerup subfactor in CFT. Despite the evidence in [15], this remains a challenging question. On the other hand, in Doplicher-Roberts Theorem a group is constructed first, and then a suitable local net is chosen for the group to act on. In other words the net is not constructed directly. It takes lots of efforts to construct conformal net or chiral algebra from MTC which are not related to groups (see [2] [3] [18] for recent results), even in concrete examples such as those in Section 3. Since conformal net seems to contain more than MTC, it is possible that a general reconstruction program may not work. See the end of Section 2.6 for a possible source of obstructions to the reconstruction program.

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## 2 Basics of Operator Algebraic Conformal Field Theory

### 2.1 Sectors

Given an infinite factor $M$, the sectors of $M$ are given by

$$\text{Sect}(M) = \text{End}(M) / \text{Inn}(M),$$

namely $\text{Sect}(M)$ is the quotient of the semigroup of the endomorphisms of $M$ modulo the equivalence relation: $\rho, \rho' \in \text{End}(M), \rho \sim \rho'$ if there is a unitary $u \in M$ such that $\rho'(x) = u \rho(x) u^*$ for all $x \in M$.

$\text{Sect}(M)$ is a *-semiring (there are an addition, a product and an involution $\rho \rightarrow \bar{\rho}$) equivalent to the Connes correspondences (bimodules) on $M$ up to unitary equivalence.
If $\rho$ is an element of $\text{End}(M)$ we shall denote by $[\rho]$ its class in $\text{Sect}(M)$. We define $\text{Hom}(\rho, \rho')$ between the objects $\rho, \rho' \in \text{End}(M)$ by

$$\text{Hom}(\rho, \rho') \equiv \{ a \in M : a \rho(x) = \rho'(x) a \ \forall x \in M \}.$$ 

We use $\langle \lambda, \mu \rangle$ to denote the dimension of $\text{Hom}(\lambda, \mu)$; it can be $\infty$, but it is finite if $\lambda, \mu$ have finite index. See [22] for the definition of index for type $II_1$ case which initiated the subject and [39] for the definition of index in general. Also see Section 2.3 of [26] for expositions.

$\langle \lambda, \mu \rangle$ depends only on $[\lambda]$ and $[\mu]$. Moreover we have if $\nu$ has finite index, then $\langle \nu \lambda, \mu \rangle = \langle \lambda, \nu \mu \rangle$, $\langle \lambda \nu, \mu \rangle = \langle \lambda, \mu \nu \rangle$ which follows from Frobenius duality. $\mu$ is a subsector of $\lambda$ if there is an isometry $v \in M$ such that $\mu(x) = v^* \lambda(x) v, \forall x \in M$. We will also use the following notation: if $\mu$ is a subsector of $\lambda$, we will write as $\mu \prec \lambda$ or $\lambda \succ \mu$.

A sector is said to be irreducible if it has only one subsector.

2.2 Local nets

In this section we recall the basic properties enjoyed by the family of the von Neumann algebras associated with a conformal Quantum Field Theory on $S^1$ (cf. [19]). This is an adaption of DHR analysis (cf. [14]) to chiral CFT which is most suitable for our purposes.

By an interval we shall always mean an open connected subset $I$ of $S^1$ such that $I$ and the interior $I'$ of its complement are non-empty. We shall denote by $I$ the set of intervals in $S^1$.

A M"obius covariant net $A$ of von Neumann algebras on the intervals of $S^1$ is a map

$$I \to A(I)$$

from $I$ to the von Neumann algebras on a Hilbert space $\mathcal{H}$ that verifies the following:

A. Isotony. If $I_1, I_2$ are intervals and $I_1 \subset I_2$, then

$$A(I_1) \subset A(I_2).$$

B. M"obius covariance. There is a nontrivial unitary representation $U$ of $G$ (the universal covering group of $PSL(2, \mathbb{R})$) on $\mathcal{H}$ such that

$$U(g)A(I)U(g)^* = A(gI), \quad g \in G, \quad I \in I.$$

The group $PSL(2, \mathbb{R})$ is identified with the M"obius group of $S^1$, i.e. the group of conformal transformations on the complex plane that preserve the orientation and leave the unit circle globally invariant. Therefore $G$ has a natural action on $S^1$.

C. Positivity of the energy. The generator of the rotation subgroup $U(R)(\cdot)$ is positive.

Here $R(\vartheta)$ denotes the (lifting to $G$ of the) rotation by an angle $\vartheta$.

D. Locality. If $I_0, I$ are disjoint intervals then $A(I_0)$ and $A(I)$ commute.

The lattice symbol $\lor$ will denote ‘the von Neumann algebra generated by’.

E. Existence of the vacuum. There exists a unit vector $\Omega$ (vacuum vector) which is $U(G)$-invariant and cyclic for $\forall I \in I.A(I)$.

F. Uniqueness of the vacuum (or irreducibility). The only $U(G)$-invariant vectors are the scalar multiples of $\Omega$.

By a conformal net (or diffeomorphism covariant net) $A$ we shall mean a M"obius covariant net such that the following holds:
G. Conformal covariance. There exists a projective unitary representation $U$ of $Diff(S^1)$ on $\mathcal{H}$ extending the unitary representation of $G$ such that for all $I \in \mathcal{I}$ we have

$$U(g)\mathcal{A}(I)U(g)^* = \mathcal{A}(gI), \quad g \in Diff(S^1),$$

$$U(g)xU(g)^* = x, \quad x \in \mathcal{A}(I), \quad g \in Diff(I'),$$

where $Diff(S^1)$ denotes the group of smooth, positively oriented diffeomorphisms of $S^1$ and $Diff(I)$ the subgroup of diffeomorphisms $g$ such that $g(z) = z$ for all $z \in I'$.

Assume $\mathcal{A}$ is a Möbius covariant net. A Möbius covariant representation $\pi$ of $\mathcal{A}$ is a family of representations $\pi_I$ of the von Neumann algebras $\mathcal{A}(I)$, $I \in \mathcal{I}$, on a Hilbert space $\mathcal{H}_\pi$ and a unitary representation $U_\pi$ of the covering group $G = PSL(2, \mathbb{R})$, with positive energy, i.e. the generator of the rotation unitary subgroup has positive generator, such that the following properties hold:

$$I \supset \bar{I} \Rightarrow \pi_I|_{\mathcal{A}(I)} = \pi_{\bar{I}} \quad \text{(isotony)}$$

$$\text{ad}U_\pi(g) \cdot \pi_I = \pi_{gI} \cdot \text{ad}U_\pi(g) \quad \text{(covariance)}.$$  

A unitary equivalence class of Möbius covariant representations of $\mathcal{A}$ is called superselection sector.

The composition of two superselection sectors are known as Connes’s fusion (cf. [54]). The composition is manifestly unitary and associative, and this is one of the most important virtues of the above formulation. The main question is to study all superselection sectors of $\mathcal{A}$ and their compositions.

Let $\mathcal{A}$ be an irreducible conformal net on a Hilbert space $\mathcal{H}$ and let $G$ be a group. Let $V : G \to U(\mathcal{H})$ be a faithful unitary representation of $G$ on $\mathcal{H}$.

**Definition 2.1.** We say that $G$ acts properly on $\mathcal{A}$ if the following conditions are satisfied:

1. For each fixed interval $I$ and each $s \in G$, $\alpha_s(a) := V(s)aV(s)^* \in \mathcal{A}(I), \forall a \in \mathcal{A}(I)$;
2. For each $s \in G$, $V(s)\Omega = \Omega, \forall s \in G$. We will denote by $\text{Aut}(\mathcal{A})$ all automorphisms of $\mathcal{A}$ which are implemented by proper actions.

Define $\mathcal{A}^G(I) := B(I)P_0$ on $\mathcal{H}_0$, where $\mathcal{H}_0$ is the space of $G$ invariant vectors and $P_0$ is the projection onto $\mathcal{H}_0$. The unitary representation $U$ of $G$ on $\mathcal{H}$ restricts to an unitary representation (still denoted by $U$) of $\mathcal{G}$ on $\mathcal{H}_0$. Then (cf. [54]):

**Proposition 2.2.** The map $I \in \mathcal{I} \to \mathcal{A}^G(I)$ on $\mathcal{H}_0$ together with the unitary representation (still denoted by $U$) of $\mathcal{G}$ on $\mathcal{H}_0$ is an irreducible conformal net.

We say that $\mathcal{A}^G$ is obtained by orbifold construction from $\mathcal{A}$.

2.3 Complete rationality

We first recall some definitions from [30]. By an interval of the circle we mean an open connected proper subset of the circle. If $I$ is such an interval then $I'$ will denote the interior of the complement of $I$ in the circle. We will denote by $\mathcal{I}$ the set of such intervals. Let $I_1, I_2 \in \mathcal{I}$. We say that $I_1, I_2$ are disjoint if $\bar{I}_1 \cap \bar{I}_2 = \emptyset$, where $\bar{I}$ is the closure of $I$ in $S^1$. When $I_1, I_2$ are disjoint, $I_1 \cup I_2$ is called a 1-disconnected interval in $S^1$. Denote by $\mathcal{I}_2$ the set of unions of disjoint 2 elements in $\mathcal{I}$. Let $\mathcal{A}$ be an irreducible conformal net. For $E = I_1 \cup I_2 \in \mathcal{I}_2$, let $I_3, I_4$ be the interior of the complement of $I_1 \cup I_2$ in $S^1$ where $I_3, I_4$ are disjoint intervals. Let

$$\mathcal{A}(E) := \mathcal{A}(I_1) \lor \mathcal{A}(I_2), \mathcal{A}(E) := (\mathcal{A}(I_3) \lor \mathcal{A}(I_4))'.$$  

\[\text{If } V : G \to U(\mathcal{H}) \text{ is not faithful, we can take } G' := G/\text{ker}V \text{ and consider } G' \text{ instead}.\]
Note that $\mathcal{A}(E) \subset \hat{\mathcal{A}}(E)$. Recall that a net $\mathcal{A}$ is split if $\mathcal{A}(I_1) \vee \mathcal{A}(I_2)$ is naturally isomorphic to the tensor product of von Neumann algebras $\mathcal{A}(I_1) \otimes \mathcal{A}(I_2)$ for any intervals $I_1, I_2 \in \mathcal{I}$ whose closures are disjoint. $\mathcal{A}$ is strongly additive if $\mathcal{A}(I_1) \vee \mathcal{A}(I_2) = \mathcal{A}(I)$ where $I_1 \cup I_2$ is obtained by removing an interior point from $I$.

**Definition 2.3.** $\mathcal{A}$ is said to be completely rational, or $\mu$-rational, if $\mathcal{A}$ is split, strongly additive, and the index $[\hat{\mathcal{A}}(E) : \mathcal{A}(E)]$ is finite for some $E \in \mathcal{I}_2$. The value of the index $[\hat{\mathcal{A}}(E) : \mathcal{A}(E)]$ (it is independent of $E$ by Prop. 5 of [30]) is denoted by $\mu_{\mathcal{A}}$ and is called the $\mu$-index of $\mathcal{A}$. $\mathcal{A}$ is holomorphic if $\mu_{\mathcal{A}} = 1$.

The following theorem is proved in [54]:

**Theorem 2.4.** Let $\mathcal{A}$ be an irreducible conformal net and let $G$ be a finite group acting properly on $\mathcal{A}$. Suppose that $\mathcal{A}$ is completely rational. Then:

1. $\mathcal{A}^G$ is completely rational and $\mu_{\mathcal{A}^G} = |G|^2 \mu_{\mathcal{A}}$;
2. There are only a finite number of irreducible covariant representations of $\mathcal{A}^G$ and they give rise to a unitary modular category as defined in II.5 of [30].

For a modular tensor category $\mathcal{M}$, we define $\text{Aut}(\mathcal{M})$ to be the collection of automorphisms of $\mathcal{M}$. One can do equivariantization of $\mathcal{M}$ with elements of $\text{Aut}(\mathcal{M})$ (cf. [12]). We define $\text{Out}(\mathcal{M})$ to be $\text{Aut}(\mathcal{M})/N$ where $N$ is the normal subgroup consisting of these automorphisms fixing the isomorphism classes of each simple objects in $\mathcal{M}$. When $\mathcal{M}$ is the representation category of $\text{Rep}(\mathcal{A})$ for a complete rational net $\mathcal{A}$, it may happen that $\text{Out}(\mathcal{M})$ contain elements which do not come from $\text{Aut}(\mathcal{A})$. See the end of section 4.3 for an example.

Let $\mathcal{B}$ be a Möbius (resp. conformal) net. $\mathcal{B}$ is called a Möbius (resp. conformal) extension of $\mathcal{A}$ if there is a map

$$I \in \mathcal{I} \rightarrow \mathcal{A}(I) \subset \mathcal{B}(I)$$

that associates to each interval $I \in \mathcal{I}$ a von Neumann subalgebra $\mathcal{A}(I)$ of $\mathcal{B}(I)$, which is isotonic

$$\mathcal{A}(I_1) \subset \mathcal{A}(I_2), I_1 \subset I_2,$$

and Möbius (resp. diffeomorphism) covariant with respect to the representation $U$, namely

$$U(g)\mathcal{A}(I)U(g)^* = \mathcal{A}(g.I)$$

for all $g \in \text{PSL}(2, \mathbb{R})$ (resp. $g \in \text{Diff}(S^1)/(S^1)$) and $I \in \mathcal{I}$. $\mathcal{A}$ will be called a Möbius (resp. conformal) subnet of $\mathcal{B}$.

**Definition 2.5.** Let $\mathcal{A}$ be a Möbius covariant net. A Möbius covariant net $\mathcal{B}$ on a Hilbert space $\mathcal{H}$ is an extension of $\mathcal{A}$ if there is a DHR representation $\pi$ of $\mathcal{A}$ on $\mathcal{H}$ such that $\pi(\mathcal{A}) \subset \mathcal{B}$ is a Möbius subnet. The extension is irreducible if $\pi(\mathcal{A}(I)) \cap \mathcal{B}(I) = \mathbb{C}$ for some (and hence all) interval $I$, and if finite index if $\pi(\mathcal{A}(I)) \subset \mathcal{B}(I)$ has finite index for some (and hence all) interval $I$. The index will be called the index of the inclusion $\pi(\mathcal{A}) \subset \mathcal{B}$ and will be denoted by $[\mathcal{B} : \mathcal{A}]$. If $\pi$ as representation of $\mathcal{A}$ decomposes as $[\pi] = \sum \lambda m_{\lambda}[\lambda]$ where $m_{\lambda}$ are non-negative integers and $\lambda$ are irreducible DHR representations of $\mathcal{A}$, we say that $[\pi] = \sum \lambda m_{\lambda}[\lambda]$ is the spectrum of the extension. For simplicity we will write $\pi(\mathcal{A}) \subset \mathcal{B}$ simply as $\mathcal{A} \subset \mathcal{B}$.

**Lemma 2.6.** If $\mathcal{A}$ is completely rational, and a Möbius covariant net $\mathcal{B}$ is an irreducible extension of $\mathcal{A}$. Then $\mathcal{A} \subset \mathcal{B}$ has finite index, $\mathcal{B}$ is completely rational and

$$\mu_{\mathcal{A}} = \mu_{\mathcal{B}}[\mathcal{B} : \mathcal{A}]^2.$$

**Proof.** $\mathcal{A} \subset \mathcal{B}$ has finite index follows from Prop. 2.3 of [29], and the rest follows from Prop. 24 of [30].
2.4 Induction

Let $\mathcal{B}$ be a Möbius covariant net and $\mathcal{A}$ a subnet. We assume that $\mathcal{A}$ is strongly additive and $\mathcal{A} \subset \mathcal{B}$ has finite index. Fix an interval $I_0 \in \mathcal{I}$ and canonical endomorphism (cf. [36]) $\gamma$ associated with $\mathcal{A}(I_0) \subset \mathcal{B}(I_0)$. Then we can choose for each $I \subset \mathcal{I}$ with $I \supset I_0$ a canonical endomorphism $\gamma_I$ of $\mathcal{B}(I)$ into $\mathcal{A}(I)$ in such a way that the restriction of $\gamma_I$ on $\mathcal{B}(I_0)$ is $\gamma_{I_0}$ and $\rho_{I_1}$ is the identity on $\mathcal{A}(I_1)$ if $I_1 \subset I_0$ is disjoint from $I_0$, where $\rho_I \equiv \gamma_I$ restricted to $\mathcal{A}(I)$. Given a DHR endomorphism $\lambda$ of $\mathcal{A}$ localized in $I_0$, the inductions $\alpha_\lambda, \alpha_\overline{\lambda}$ of $\lambda$ are the endomorphisms of $\mathcal{B}(I_0)$ given by

$$
\alpha_\lambda \equiv \gamma^{-1} \cdot \text{Ad}(\varepsilon, \rho) \cdot \lambda \cdot \gamma, \quad \alpha_\overline{\lambda} \equiv \gamma^{-1} \cdot \text{Ad}(\varepsilon, \rho) \cdot \lambda \cdot \overline{\gamma}
$$

where $\varepsilon$ (resp. $\overline{\varepsilon}$) denotes the right braiding (resp. left braiding) (cf. Cor. 3.2 of [5]). In [50] a slightly different endomorphism was introduced and the relation between the two was given in Section 2.1 of [39].

Note that $\text{Hom}(\alpha_\lambda, \alpha_\mu) =: \{ x \in \mathcal{B}(I_0) | x\alpha_\lambda(y) = \alpha_\mu(y)x, \forall y \in \mathcal{B}(I_0) \}$ and $\text{Hom}(\lambda, \mu) =: \{ x \in \mathcal{A}(I_0) | x\lambda(y) = \mu(y)x, \forall y \in \mathcal{A}(I_0) \}$.

The following follows from Lemma 3.4 and Th. 3.3 of [50] (also cf. [5]):

**Theorem 2.7.** (1) $[\lambda] \rightarrow [\alpha_\lambda]$, are ring homomorphisms;

(2) $\langle \alpha_\lambda, \alpha_\mu \rangle = \langle \lambda, \mu \rangle$.

2.5 Normal inclusions and tensor equivalence

Let $\mathcal{B}$ be a completely rational net and $\mathcal{A} \subset \mathcal{B}$ be a subnet which is also completely rational.

**Definition 2.8.** Define a subnet $\tilde{\mathcal{A}} \subset \mathcal{B}$ by $\tilde{\mathcal{A}}(I) := \mathcal{A}(I)' \cap \mathcal{B}(I), \forall I \in \mathcal{I}$.

We note that since $\mathcal{A}$ is completely rational, it is strongly additive and so we have $\tilde{\mathcal{A}}(I) = (\bigvee_{J \in \mathcal{I}} \mathcal{A}(J))' \cap \mathcal{B}(I), \forall I \in \mathcal{I}$. The following lemma then follows directly from the definition:

**Lemma 2.9.** The restriction of $\tilde{\mathcal{A}}$ on the Hilbert space $\bigvee_I \mathcal{A}(I)\Omega$ is an irreducible Möbius covariant net.

The net $\tilde{\mathcal{A}}$ as in Lemma 2.9 will be called the *coset of $\mathcal{A} \subset \mathcal{B}$*. See [46] for a class of cosets from Loop groups.

The following definition generalizes the definition in Section 3 of [46]:

**Definition 2.10.** $\mathcal{A} \subset \mathcal{B}$ is called cofinite if the inclusion $\tilde{\mathcal{A}}(I) \bigvee \mathcal{A}(I) \subset \mathcal{B}(I)$ has finite index for some interval $I$.

The following is Prop. 3.4 of [55]:

**Proposition 2.11.** Let $\mathcal{B}$ be completely rational, and let $\mathcal{A} \subset \mathcal{B}$ be a Möbius subnet which is also completely rational. Then $\mathcal{A} \subset \mathcal{B}$ is cofinite if and only if $\tilde{\mathcal{A}}$ is completely rational.

Let $\mathcal{B}$ be completely rational, and let $\mathcal{A} \subset \mathcal{B}$ be a Möbius subnet which is also completely rational. Assume that $\mathcal{A} \subset \mathcal{B}$ is cofinite. We will use $\sigma_i, \sigma_j, ...$ (resp. $\lambda, \mu, ...$) to label irreducible DHR representations of $\mathcal{B}$ (resp. $\mathcal{A}$) localized on a fixed interval $I_0$. Since $\tilde{\mathcal{A}}$ is completely rational by Prop. 2.11, $\tilde{\mathcal{A}} \otimes \mathcal{A}$ is completely rational, and so every irreducible DHR representation $\sigma_i$ of $\mathcal{B}$, when restricting to $\tilde{\mathcal{A}} \otimes \mathcal{A}$, decomposes as direct sum of representations of $\tilde{\mathcal{A}} \otimes \mathcal{A}$ of the form $(i, \lambda) \otimes \lambda$ by Lemma 27 of [30]. Here $(i, \lambda)$ is a DHR representation of $\tilde{\mathcal{A}}$ which may not be irreducible and we use the tensor notation.
\[(i, \lambda) \otimes \lambda\) to represent a DHR representation of \(\tilde{A} \otimes A\) which is localized on \(I_0\) and defined by
\[(i, \lambda) \otimes \lambda(x_1 \otimes x_2) = (i, \lambda)(x_1) \otimes \lambda(x_2), \forall x_1 \otimes x_2 \in \tilde{A}(I_0) \otimes A(I_0).\]

We will also identify \(\tilde{A}\) and \(A\) as subnets of \(\tilde{A} \otimes A\) in the natural way. We note that when no confusion arises, we will use \(1\) to denote the vacuum representation of a net.

**Definition 2.12.** A Möbius subnet \(A \subset B\) is normal if \(\tilde{A}(I)’ \cap B(I) = A\) for some \(I\).

The following is an application of Lemma 2.24 in [55] (also cf. [38] and [2]):

**Theorem 2.13.** Assume \(A\) is completely rational, \(A \subset B\) is normal and cofinite, and let \(\sum_{\lambda \in \text{Exp}}[(\lambda, \bar{\lambda})]\) be the spectrum of \(A \otimes \tilde{A} \subset B\). Then \(\lambda \in \text{Exp}\) are simple objects of a closed fusion category of \(\text{Rep}(A)\) and there is an equivalence of braided tensor category \(F\) between the subcategory of \(\text{Rep}(A)\) generated by \(\lambda \in \text{Exp}\) and the subcategory of \(\text{Rep}(\tilde{A})\) \(\text{rev}\) generated by \(\lambda, \bar{\lambda} \in \text{Exp}\), where the braiding of \(\text{Rep}(\tilde{A})\) \(\text{rev}\) is the mirror image or reversed braiding of \(\text{Rep}(\tilde{A})\), and \(F\) maps \(\lambda\) to \(\bar{\lambda}\).

### 2.6 Subfactors from conformal nets

Given a conformal net \(A\), there are three general classes of subfactors:

(i) If \(\pi\) is a covariant representation of \(A\), then by locality we have the following subfactor \(\pi_1(\mathcal{A}(I)) \subset \pi_{I'}(\mathcal{A}(I'))\). These are known as Jones-Wassermann subfactors;

(ii) Let \(I_1, I_2 \in \mathcal{I}\). We say that \(I_1, I_2\) are disjoint if \(I_1 \cap I_2 = \emptyset\), where \(I\) is the closure of \(I\) in \(S^1\). Suppose that \(I_1, I_2\) are disjoint and let \(I_3 \cup I_4\) be the interior of the complement of \(I_1 \cup I_2\) in \(S^1\) where \(I_3, I_4\) are disjoint intervals. Let
\[
\mathcal{A}(E) := \mathcal{A}(I_1) \cup \mathcal{A}(I_2), \tilde{\mathcal{A}}(E) := (\mathcal{A}(I_3) \cup \mathcal{A}(I_4))'.
\]

Note that by locality we have subfactor \(\mathcal{A}(E) \subset \tilde{\mathcal{A}}(E)\). These are subfactors analyzed in [53] and [30]:

(iii) If \(B \subset A\) is a subnet, then we have subfactors \(\mathcal{B}(I) \subset \mathcal{A}(I), \forall I\), and under certain conditions we get irreducible finite index subfactors, and in such cases we can induce a representation of \(B\) to a soliton of \(A\): i.e., it is only a representation of net \(A\) restricted to a punctured circle which is isomorphic to the real line. By locality such solitons will also give subfactors (cf. [26] and [6]).

There are close relations between the index of subfactors in (i) and (ii). This is related to the notion of complete rationality in [30]. As for subfactors coming from (iii), a notable class of such examples come from conformal inclusions and simple current extensions. For an example, one can construct all subfactors of index less than 4 with principal graphs of type \(\mathcal{D}, \mathcal{E}\) from such constructions. Subfactors induced from simple current extensions also provide examples with interesting lattice of intermediate subfactors (cf. [27]).

Given a rational conformal net \(A\) and any finite group \(G\). Assume that \(G \leq S_n\) where \(S_n\) is a symmetric group on \(n\) letters. Note that \(S_n\) acts on \(A^{\otimes n}\) by permuting tensors, and by Th. [24] the fixed point algebra \((A^{\otimes n})^G\) is also rational. A particular interesting case is when \(G\) is generated by an anlyce, in this case one can relate the chiral quantities of \((A^{\otimes n})^G\) with that of \(A\), and this leads to many interesting equations (cf. [1], [51]) due to the rationality of \((A^{\otimes n})^G\) by Th. [24]. In fact P. Bantay proposes that any MTC \(\mathcal{M}\) will have an associated class of MTCs coming from the orbifolds as in the case of conformal nets. Following [1], we shall say such MTC verify Orbifold Covariance Principle. This suggests the following possible obstructions to reconstruction program: if one can find a MTC which does not verify Orbifold Covariance Principle, then such a MTC will not come from CFT. For an example, if one can find a MTC for which some of those identities in
are not verified, then such a MTC will not come from CFT. However it is not clear which identity to check, and in fact some properties of MTC such as certain equations among chiral identities which may follow from Orbifold Covariance Principle are in fact proved in [37] without constructing the associated class of orbifold MTCs.

3 Examples of subfactors from extensions and conformal inclusions

Let \( G = SU(n) \). We denote \( LG \) the group of smooth maps \( f : S^1 \to G \) under pointwise multiplication. The diffeomorphism group of the circle \( \text{Diff}S^1 \) is naturally a subgroup of \( \text{Aut}(LG) \) with the action given by reparametrization. In particular the group of rotations \( \text{Rot}S^1 \cong U(1) \) acts on \( LG \). We will be interested in the projective unitary representation \( \pi : LG \to U(H) \) that are both irreducible and have positive energy. This means that \( \pi \) should extend to \( LG \times \text{Rot}S^1 \) so that \( H = \oplus_{n \geq 0} H(n) \), where the \( H(n) \) are the eigenspace for the action of \( \text{Rot}S^1 \), i.e., \( r_{\theta}^n \xi = \exp(i n \theta) \) for \( \theta \in H(n) \) and \( \dim H(n) < \infty \) with \( H(0) \neq 0 \). It follows from [40] that for fixed level \( k \) which is a positive integer, there are only finite number of such irreducible representations indexed by the finite set

\[
P^k_{++} = \left\{ \lambda \in P \mid \lambda = \sum_{i=1,\ldots,n-1} \lambda_i \Lambda_i, \lambda_i \geq 0, \sum_{i=1,\ldots,n-1} \lambda_i \leq k \right\}
\]

where \( P \) is the weight lattice of \( SU(n) \) and \( \Lambda_i \) are the fundamental weights. We will write \( \lambda = (\lambda_1, \ldots, \lambda_{n-1}) \), \( \lambda_0 = k - \sum_{1 \leq i \leq n-1} \lambda_i \) and refer to \( \lambda_0, \ldots, \lambda_{n-1} \) as components of \( \lambda \).

We will use \( \Lambda_0 \) or simply \( 1 \) to denote the trivial representation of \( SU(n) \).

For \( \lambda, \mu, \nu \in P^k_{++} \), define \( N^\nu_{\mu} = \sum_{\delta \in P^k_{++}} S^\nu_{\lambda} S^{(\delta)}_{\mu} S^{(\delta^*)}_{\nu} / S^{(\delta)}_{\lambda_0} \) where \( S^\nu_{\lambda} \) is given by the Kac-Peterson formula:

\[
S^\nu_{\lambda} = c \sum_{w \in S_n} \varepsilon_w \exp(iw(\delta) \cdot \lambda 2\pi/n)
\]

where \( \varepsilon_w = \det(w) \) and \( c \) is a normalization constant fixed by the requirement that \( S^\nu_{\lambda} \) is an orthonormal system. It is shown in [28] P. 288 that \( N^\nu_{\mu} \) are non-negative integers. Moreover, define \( Gr(C_k) \) to be the ring whose basis are elements of \( P^k_{++} \) with structure constants \( N^\nu_{\lambda_0} \). The natural involution \( * \) on \( P^k_{++} \) is defined by \( \lambda \mapsto \lambda^* = \text{the conjugate of } \lambda \) as representation of \( SU(n) \).

We shall also denote \( S^{(\lambda)}_{\lambda_0} \) by \( S_1^{(\lambda)} \). Define \( d_\lambda = S^{(\lambda)}_{\lambda_0} / S_1^{(\lambda)} \). We shall call \( (S^\nu_{\lambda}^{(\delta)}) \) the S-matrix of \( LSU(n) \) at level \( k \).

We shall encounter the \( Z_n \) group of automorphisms of this set of weights, generated by

\[
\sigma : \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{n-1}) \to \sigma(\lambda) = (k - 1 - \lambda_1 - \cdots \lambda_{n-1}, \lambda_1, \cdots, \lambda_{n-2})
\]

We will use \( \left( [\lambda_1, \lambda_2, \cdots, \lambda_{n-1}] \right) \) to denote the orbit of \( \text{col}(\lambda) = \sum_i (\lambda_i - 1)i \). \( \text{col}(\lambda) \) will be referred to as the color of \( \lambda \). The central element exp \( \frac{2\pi i}{n} \) of \( SU(n) \) acts on representation of \( SU(n) \) labeled by \( \lambda \) as \( \exp \left( \frac{2\pi i \text{col}(\lambda)}{n} \right) \). The irreducible positive energy representations of \( LSU(n) \) at level \( k \) give rise to an irreducible conformal net \( \mathcal{A} \) (cf. [26]) and its covariant representations. We will use \( \lambda = (\lambda_1, \ldots, \lambda_{n-1}) \) to denote irreducible representations of \( \mathcal{A} \) and also the corresponding endomorphism of \( M = \mathcal{A}(I) \).

All the sectors \( [\lambda] \) with \( \lambda \) irreducible generate the fusion ring of \( \mathcal{A} \). We will use \( \left( [\lambda_1, \lambda_2, \cdots, \lambda_{n-1}] \right) \) to denote the orbit of the sector \( [\lambda_1, \lambda_2, \cdots, \lambda_{n-1}] \) under the \( Z_n \) action above.

**Definition 3.1.** \( v := (1, 0, \ldots, 0), v_0 := (1, 0, \ldots, 0, 1) \).
3.1 SU(2)$_8$ and the second fish of Bisch-Haagerup

We will label the irreps of SU(2)$_8$ by half integers $i, 0 \leq i \leq 4$. Note that the simple current 4 has integer conformal dimension, and $A_{SU(2)_8}$ has a local $\mathbb{Z}_2$ extension $\mathcal{B}$ given by the simple current 4. By Th. 2.7 we have $[\alpha_2] = [b_1] + [b_2], d_{b_1} = d_{b_2} = \frac{1+i\sqrt{5}}{2},$ and $[b_1^2] = [b_1] + [1], b_1 b_2 = [\alpha_1]$. Denote by $\rho$ the index 2 subfactor for the inclusion $\mathcal{A} \subset \mathcal{B}$. From the fusion rules above one can easily determine the principal graph for $\rho b_1$ to be the second fish of Bisch-Haagerup (cf. [4]), and the even vertices are labeled by integers 0, 1, 2, 3, 4 which are irreps of $A_{SU(2)_8}$. The double of such a fusion category was considered in [16]. This answers a question in the introduction of [17].

3.2 Conformal inclusions SU(n)$_{n+2} \subset SU(\frac{n(n+1)}{2})$

This class of subfactors were discussed in [50]. We note that the centralizer algebras $\text{Hom}(\alpha_n^\rho, \alpha_n^\rho), n \geq 0$ containing Hecke algebras (cf. [50]) and additional generators. The generators and relations for these algebras are found as in [35] and [34]. Another class of conformal inclusions $SU(n+2)_n \subset SU(\frac{n(n+1)}{2})_1$ are mirror extensions of $SU(n+2)_{n+2} \subset SU(\frac{n(n+1)}{2})_1$ and the corresponding class of subfactors are closely related (34). Most of subfactors from such conformal inclusions are not near group. Here we consider a special case $n = 4$ to compare with the near group subfactors in [21].

Let $A_{SU(4)_6} \subset A_{SU(10)}$ be inclusion of nets correspond to conformal inclusion $SU(4)_6 \subset SU(10)_1$. The fusion graphs for the generators $\alpha_{\Lambda_1}, \alpha_{\Lambda_2}$ can be determined as in [50], and is already given by different method in [11]. We have the following:

$$[\alpha_{\Lambda_2} \alpha_{\Lambda_2}] = [1] + [\omega \alpha_{\Lambda_2}] + [\omega^3 \alpha_{\Lambda_2}] + [\omega^{-1} \alpha_{\Lambda_2}] + [\omega^{-3} \alpha_{\Lambda_2}]$$

where $[\omega^{10}] = [1], \omega$ is the vector rep of $SU(10)_1$.

In particular if we choose $A = \omega^5 \alpha_{\Lambda_2}$, then $A$ and $\omega^2 = \eta$ generates a fusion subring with

$$[\eta^5] = [1], [A^2] = [1] + [\eta A] + [\eta^2 A] + [\eta^3 A] + [\eta^4 A]$$

Such a fusion category is not near-group, but seems to be closely related to the near-group fusion category in [21].

We also note that complex conjugation acts on $SU(4)_6 \subset SU(10)_1$, and we get inclusions $A_{SU(4)_6} \subset A_{SU(10)}$ (cf. [34] for more general case). This is related to $\mathbb{Z}_2$ equivariantization of the fusion category above.

3.3 Conformal inclusions SU(n)$_n \subset Spin(n^2 - 1)_1$

Denote by $\sigma_1$ the vector representation of $Spin(n^2 - 1)$ and $v_0$ the adjoint representation of $SU(n)_n$. We note that by the branching rules of $SU(n)_n \subset Spin(n^2 - 1)_1$ we have $\alpha_{v_0} \supset \sigma_1$. It follows that $[\sigma_1 \alpha_v] = [\alpha_v]$ and $\alpha_v$ contains an intermediate subfactor of index 2.

We note that the centralizer algebras $\text{Hom}(\alpha_n^\rho, \alpha_n^\rho), n \geq 0$ containing Hecke algebras (cf. [50]) and additional generators. It is an interesting questions to find generators and relations for these algebras as in [35] and [34].

The case when $n = 4$ is analyzed in [50]. Here we consider the case $n = 5$.

The branching rules for the conformal inclusion $SU(5)_5 \subset Spin(24)_1$ are given by (cf. [27]):

$$[1] = ((0, 0, 0, 0)) + ((0, 1, 0, 2)), [\sigma_1] = ((1, 1, 1, 1)) + ((1, 0, 0, 1)), [\sigma_2] = [\sigma_3] = 2((1, 1, 1, 1))$$
where $\sigma_1, \sigma_2, \sigma_3$ denote the vector and spinor representations which form $\mathbb{Z}_2 \times \mathbb{Z}_2$ under fusion, and $\{((\lambda_1, \lambda_2, \lambda_3, \lambda_4))\}$ denotes the orbit of sector $\{((\lambda_1, \lambda_2, \lambda_3, \lambda_4))\}$ under the center $\mathbb{Z}_3$ of $SU(4)$. Here by slightly abusing notations the left hand side of the equations above are understood as the restrictions from representations of $Spin(24)_1$ to $SU(5)_5$. The fusion of the adjoint representation is given by (cf. [17])

$$[v_0^2] = [1] + 2[v_0] + [(2, 0, 0, 2)] + [(0, 1, 1, 0)] + [(0, 1, 0, 2)] + [(2, 0, 1, 0)].$$

By using the above and Th. 2.7, we have

$$\langle \alpha_{v_0}, \alpha_{v_0} \rangle = 3.$$

It follows that $[\alpha_{v_0}] = [\sigma_1] + [A] + [\sigma_1A]$ where $A$ is irreducible. Since $[\sigma_1 \alpha_v] = [\alpha_v]$, $\alpha_v$ has an intermediate subfactor denoted by $\rho_1$ and $[\rho_1 \rho_1] = [1] + [A]$. We have the following fusion rules:

$$[A^2] = [1] + [A\sigma_1] + [A\sigma_1] + [A\sigma_2] + [A\sigma_3]$$

Let $A_1$ be simple current extensions of $A_{SU(5)_5}$. Note that $A_{SU(5)_5} \subset A_1 \subset A_{Spin(24)_1}$. Note that color 0 irreducible representations of $A_{SU(5)_5}$ induce to DHR representations of $A_1$. We enumerate these 10 irreducible representations of $A_1$ as follows: 1, $z_1, z_2, z_3$ ad, $b_i, 1 \leq i \leq 5$ where 1 is the vacuum representation, $z_1, z_2, z_3$ are induced from $(0, 0, 2, 1), (0, 1, 0, 2)$ and $(0, 1, 1, 0)$ respectively, ad is induced from $(1, 0, 0, 1)$, and $b_i, 1 \leq i \leq 5$ are irreducible components of the representation induced from $(1, 1, 1, 1)$. Let $g \in \mathbb{Z}_3$ be the generator of $Aut(A_1)$ due to the simple current extension. We have $[Ad_g b_i] = [b_{i+1}], 1 \leq i \leq 5$ and $Ad_g$ fix the rest 5 irreducible representations of $A_1$. We consider the induction from $A_1$ to $A_{Spin(24)_1}$. The branching rules of the inclusion $A_1 \subset A_{Spin(24)_1}$ are given by:

$$[\Lambda_0] = [1] + [z_2], [\sigma_1] = [b_1] + [ad], [\sigma_2] = [b_2] + [b_3], [\sigma_3] = [b_4] + [b_5]$$

The following can be determined from the fusion rules and Th. 2.7:

$$[\alpha_{ad}] = [\sigma_1] + [A] + [\sigma_1A], [\alpha_{b_1}] = [\sigma_1] + [\sigma_2A] + [\sigma_3A],$$

$$[\alpha_{b_2}] = [\sigma_2] + [A] + [\sigma_2A], [\alpha_{b_3}] = [\sigma_2] + [\sigma_1A] + [\sigma_3A],$$

$$[\alpha_{b_4}] = [\sigma_3] + [A] + [\sigma_3A], [\alpha_{b_5}] = [\sigma_3] + [\sigma_1A] + [\sigma_2A],$$

$$[\alpha_{z_1}] = [\alpha_{z_3}] = [A] + [A\sigma_1] + [A\sigma_2] + [A\sigma_3],$$

$$[\alpha_{z_2}] = [1] + [A] + [A\sigma_1] + [A\sigma_2] + [A\sigma_3].$$

We see the intermediate subfactor, denoted by $\rho_1$ above, is exactly $3^{\mathbb{Z}_2 \times \mathbb{Z}_2}$ subfactor constructed by Izumi (cf. [17]). The double of this subfactor is computed in [17]. By [7] we can now see the double is $RepA_1 \otimes RepB^{rev}$. Consider inclusions $B \otimes B \subset A_{Spin(48)_1} \subset B_1$ where $B_1$ is $\mathbb{Z}_2$ extension of $A_{Spin(48)_1}$, which is holomorphic. Inspecting the spectrum of $B \otimes B \subset B_1$ we see that the inclusion $B \subset B_1$ is normal, and by Th. 2.7 we conclude that $RepB^{rev} \simeq RepB$ as braided tensor categories. So we have shown the following theorem:

**Theorem 3.2.** The double of $3^{\mathbb{Z}_2 \times \mathbb{Z}_2}$ subfactor is the category of representations of $A_1 \otimes A_{Spin(24)_1}$, and verifies the Orbifold Covariance Principle.

In [17], a $\mathbb{Z}_3$ equivarization of $3^{\mathbb{Z}_2 \times \mathbb{Z}_2}$ is found. However there is no such $\mathbb{Z}_3$ in $Aut(A_1)$ which lifts to $\mathbb{Z}_3$ on $A_{Spin(24)_1}$ permuting $\sigma_i$. If reconstruction program works, there must be a conformal net $B_1$ with $\mathbb{Z}_3 \subset Aut(B_1)$ such that the category of representations of $B_1^{\mathbb{Z}_3}$ is braided equivalent to the category of representations of $A_1 \otimes A_{Spin(24)_1}$. It is also an interesting question to see if one can relate $A_1$ to the double of $D2D$ subfactor in [17].
3.4 Conformal inclusions $SU(n)_m \times SU(m)_n \subset Spin(nm)_1$

Such cases were studied in [48] and [38]. In particular, by applying Th. 2.13 we get a braided tensor equivalence between the fusion subcategory of $\text{Rep}A_{SU(n)_m}$ generated by the color 0 irreducible representations and the fusion subcategory of $\text{Rep}A_{SU(m)_n}^{\text{rev}}$ generated by the color 0 irreducible representations.

4 Subfactors from holomorphic CFT of central charge 24

In [43] A. Schellekens gave a conjectured list of 71 holomorphic CFT of central charge 24. Many examples on this list have been realized (cf. [31, 32, 11], [55]). We note that the examples in [31, 32, 11] are given in the language of lattice VOAs, their orbifolds, and affine Kac-Moody algebras. To translate these results into conformal nets, we need to know that irreducible representations of lattice VOAs give rise to irreducible representations of corresponding lattice conformal nets, and irreducible twisted representations of lattice VOAs give rise to irreducible solitons of corresponding lattice conformal nets. These are proved as Prop. 3.15 and Prop. 4.25 of [10], and all these representations are of finite index (cf. Cor. 4.31 [10]). The fact that irreducible representations of affine Kac-Moody algebra give rise to irreducible representation of the corresponding conformal net can be seen from the theorem on P. 488 of [45].

One of the special property of such holomorphic CFT $\mathcal{B}$ is that if the weight 1 subspace is non-zero, then (cf. [9]) it generates a Kac-Moody subnet $\mathcal{A} \subset \mathcal{B}$ such that the spectrum of $\mathcal{A} \subset \mathcal{B}$ is finite. Hence just like Section 3 we can consider subfactors associated with such $\mathcal{A} \subset \mathcal{B}$. To apply the results of Section 3 we will also need that the index $[\mathcal{B} : \mathcal{A}] < \infty$. This is expected to be true in general but the general case is only known for type $A$ Kac-Moody algebras and type $D$ at odd level (cf. [45, 44]). We select a few examples from [43] which have been constructed recently, in a similar order as in the previous section.

4.1 No. 20 in [43]

This case is constructed in [32]. We have $[\mathcal{B} : \mathcal{A}] < \infty$ by [11], and so we have a new finite index subfactor. If we take the commutant of the subnet generated by $SU(2)_5^2$, then we see that we get a local extension of $A_{Spin(12)_5} \subset \mathcal{B}_1$ whose spectrum is given by

$$[1] + [010002] + [010020] + [100111] + [002000] + [200100].$$

**Theorem 4.1.** There is a local extension $A_{Spin(12)_5} \subset \mathcal{B}_1$ whose spectrum is given by

$$[1] + [010002] + [010020] + [100111] + [002000] + [200100]$$

where we have used the same notation of [43] for representations of $Spin(12)_5$.

We can now consider the induced subfactor $\alpha_v$ where $v$ denotes the vector representation of $A_{Spin(12)_5}$. The centralizer algebras $\text{Hom}(\alpha_v^n, \alpha_v^n)$, $n \geq 0$ will contain BMW algebra as in [52]. We know that $d_v = 7.7396813$. It is an interesting question to analyze the nature of such algebras.

**Remark 4.2.** The above local extension of $A_{Spin(12)_5} \subset \mathcal{B}_1$ should be the mirror extension (cf. [50]) associated with the conformal inclusion $Spin(5)_1 \subset E_8$. This example is similar to No. 27 on Schelleken’s list, which is also constructed by using the mirror extensions of conformal inclusion $SU(3)_9 \subset E_6$ in [55].
4.2 No. 11 in [43]

This case is constructed by C. Lam by using the ideas of [32]. We have $|\mathcal{B} : \mathcal{A}| < \infty$ by [45], and so we have a finite index subfactor.

**Theorem 4.3.** *There is a local extension $\mathcal{A}_{SU(7)} \subset \mathcal{B}$ whose spectrum is given by*

\[ ([1]) + ([0, 0, 1, 3, 0, 1]) + ([0, 0, 3, 1, 0, 0]) + ([0, 1, 0, 2, 0, 3, 0]) + ([0, 1, 0, 4, 1, 2, 2]) + 3[(1, 1, 1, 1, 1, 1)] \]

We can now consider the induced subfactor $\alpha_v$ where $v$ denotes the vector representation of $SU(7)$. $\mathcal{F}$ and $d_v = \frac{1}{\sin(\pi v)} = 4.493959$. The centralizer algebras $\text{Hom}(\alpha^n_v, \alpha^n_v)$, $n \geq 0$ contain Hecke algebras as in [50]. It is an interesting question to analyze the nature of such algebras.

4.3 No. 9 in [43]

This case is constructed in [31] and [11]. By examining the spectrum, we can see that we have $\mathcal{A}_1 \otimes \mathcal{A}_1 \subset \mathcal{B}$ and the spectrum is given by

\[ [1] \otimes [1] + [z_2] \otimes [z_2] + [z_1] \otimes [z_3] + [z_3] \otimes [z_1] + [b_1] \otimes [ad] + [ad] \otimes [b_\tau(1)] + [b_2] \otimes [b_\tau(2)] + [b_3] \otimes [b_\tau(3)] + [b_4] \otimes [b_\tau(4)] + [b_5] \otimes [b_\tau(5)] \]

where $\mathcal{A}_1$ and its irreducible representations are as in section 3.3 and $\tau \in S_5$. So the inclusion $\mathcal{A}_1 \subset \mathcal{B}$ is normal, and we have a braided tensor category equivalence $F_1 : \text{Rep}(\mathcal{A}_1) \to \text{Rep}(\mathcal{A}_1)^{\text{rev}}$ such that $F_1(z_1) = z_3$, $F_1(z_3) = z_1$, $F_1(b_1) = \text{ad}$.

Now consider the conformal inclusions $SU(5) \times SU(5) \subset SU(25)_1 \subset \mathcal{B}_2$ where $\mathcal{B}_2$ denotes the holomorphic net corresponding to No. 67 on Schelleken’s list. By examining the spectrum we find that we have $\mathcal{A}_1 \otimes \mathcal{A}_1 \subset \mathcal{B}_2$ where the spectrum is given by

\[ [1] \otimes [1] + [z_2] \otimes [z_2] + [z_1] \otimes [z_3] + [z_3] \otimes [z_1] + [b_1] \otimes [ad] + [ad] \otimes [b_1] + [b_2] \otimes [b_2] + [b_3] \otimes [b_3] + [b_4] \otimes [b_4] + [b_5] \otimes [b_5] \]

So the inclusion $\mathcal{A}_1 \subset \mathcal{B}_2$ is normal, and we have a braided tensor category equivalence $F_2 : \text{Rep}(\mathcal{A}_1) \to \text{Rep}(\mathcal{A}_1)^{\text{rev}}$ such that $F_2(z_1) = z_3$, $F_2(z_3) = z_1$. We note that the generator $g \in \text{Aut}(\mathcal{A}_1)$ induces braided tensor category equivalence of $\text{Rep}(\mathcal{A}_1)$ of order 5. Composing $g$ with $F_1 F_2^{-1}$, and examining actions on the 6 element set of irreducible representations $ad, b_i, 1 \leq i \leq 5$, we see that these two elements generate a subgroup of $S_6$ which acts transitively on 6 letters. Such a group has at least order 60. We have therefore proved the following:

**Theorem 4.4.** $\text{Out}(\text{Rep}(\mathcal{A}_1))$ has at least order 60.

It remains an interesting question to determine the equivariantizations of $\text{Rep}(\mathcal{A}_1)$ with respect to the group elements in the above theorem. Except for the case of $g \in \text{Aut}(\mathcal{A}_1)$, where the orbifold net is simply $\mathcal{A}_{SU(5)}$, the rest of the cases are not known to be related to CFT, and this includes the $\mathbb{Z}_3$ case considered in section 3.3.

We note that as soon as central charge is greater than 24, there are a lot more holomorphic CFT since there are many more nonunimodular even positive definite lattices. It is an interesting question to to see if one can get more interesting subfactors related to holomorphic CFT with central charge greater than 24.

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Department of Mathematics
University of California at Riverside
E-mail: xufeng@math.ucr.edu