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A scaling proof for Walsh’s Brownian motion
extended arc-sine law

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Abstract

We present a new proof of the extended arc-sine law related to Walsh’s Brownian
motion, known also as Brownian spider. The main argument mimics the scaling
property used previously, in particular by D. Williams [12], in the 1-dimensional
Brownian case, which can be generalized to the multivariate case. A discussion
concerning the time spent positive by a skew Bessel process is also presented.

AMS 2010 subject classification: Primary: 60J60, 60J65;
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Key words: Arc-sine law, Brownian spider, Skew Bessel process, Stable variables, Sub-
ordinators, Walsh Brownian motion.

1 Introduction

a) Recently, some renewed interest has been shown (see e.g. [9]) in the study of the law
of the vector

\[ \overrightarrow{A_i} = \left( \int_0^1 1_{(W_s \in I_i)} ds; \ i = 1, 2, \ldots, n \right), \]

where \( (W_s) \) denotes a Walsh Brownian motion, also called Brownian spider (see [10] for
Walsh’s lyrical description) living on \( I = \bigcup_{i=1}^n I_i \), the union of \( n \) half-lines of the plane,
meeting at 0.

For the sake of simplicity, we assume \( p_1 = p_2 = \ldots = p_n = 1/n \), i.e.: when returning
to 0, Walsh’s Brownian motion chooses, loosely speaking, its "new" ray in a uniform way.

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In fact, excursion theory and/or the computation of the semi-group of Walsh’s Brownian motion (see [1]) allow to define the process rigorously.

Since \((d(0, W_s); s \geq 0)\), for \(d\) the Euclidian distance, is a reflecting Brownian motion, we denote by \((L_t, t \geq 0)\) the unique continuous increasing process such that:

\((d(0, W_s) - L_s; s \geq 0)\) is a \(\mathcal{W}_s = \sigma \{W_u, u \leq s\}\) Brownian motion.

Let

\[ \overrightarrow{A}_t = \left( A^{(1)}_t, A^{(2)}_t, \ldots, A^{(n)}_t \right) \]

denote the random vector of the times spent in the different rays. In Section 2 we will state and prove our main Theorem concerning the distribution of \(\overrightarrow{A}_t\) for a fixed time. Section 3 deals with the general case of stable variables, First, we recall some known results and then we state and prove our main Theorem. Finally, Section 4 is devoted to some remarks and comments.

\(b)\) **Reminder on the arc-sine law:**

A random variable \(A\) follows the arc-sine law if it admits the density:

\[
\frac{1}{\pi \sqrt{x(1-x)}} 1_{[0,1)}(x).
\]  

Some well known representations of an arc-sine variable are the following:

\[
A \xrightarrow{\text{law}} \frac{N^2}{N^2 + \hat{N}^2} \xrightarrow{\text{law}} \cos^2(U) \xrightarrow{\text{law}} \frac{T}{T + \hat{T}} \xrightarrow{\text{law}} \frac{1}{1 + C^2},
\]

where \(N, \hat{N} \sim \mathcal{N}(0, 1)\) and are independent, \(U\) is uniform on \([0, 2\pi]\), \(T\) and \(\hat{T}\) stand for two iid stable \((1/2)\) unilateral variables, and \(C\) is a standard Cauchy variable.

With \((B_t, t \geq 0)\) denoting a real Brownian motion, two well known examples of arc-sine distributed variables are:

\[
g_1 = \sup \{t < 1 : B_t = 0\}, \quad \text{and} \quad A^+_1 = \int_0^1 ds \ 1_{(B_s > 0)},
\]

a result that is due to Paul Lévy (see e.g. [6, 7, 13]).

\(c)\) This point gives some motivation for Section 3. From (2), one could think that more general studies of the time spent positive by diffusions may bring 2 independent gamma variables (this because \(N^2\) and \(\hat{N}^2\) are distributed like two independent gamma variables of parameter 1/2), or 2 independent stable \((\mu)\) variables. It turns out that it is the second case which seems to occur more naturally. We devote Section 3 to this case.

**2 Main result**

Our aim is to prove the following:

**Theorem 2.1.** The random vectors \(\overrightarrow{A}_T/T\) for:

\((i)\) \(T = t;\)  \((ii)\) \(T = \alpha^{(j)}_s = \inf \{t : A^{(j)}_t > s\};\)  \((iii)\) \(T = \tau_1,\) the inverse local times,
have the same distribution. In particular, it is specified by the iid stable \((1/2)\) subordinators:
\[
\left((A^{(j)}_n, l \geq 0); 1 \leq j \leq n\right).
\]

Hence:
\[
\overrightarrow{A}_1 \overset{\text{law}}{=} \frac{\overrightarrow{A}_{\tau_1}}{\tau_1},
\]
which yields that:
\[
\overrightarrow{A}_1 \overset{\text{law}}{=} \left(\frac{T_j}{\sum_{i=1}^{n} T_i}; j \leq n\right),
\]
where \(T_j\) are iid, stable \((1/2)\) variables.

The law of the right-hand side of (3) is easily computed, and consequently so is its left-hand side. We refer the reader to [2] for explicit expressions of this law, which for \(n = 2\) reduces to the classical arc-sine law.

**Proof of Theorem 2.1.**

a) Clearly, \((ii)\) plays a kind of "bridge" between \((i)\) and \((iii)\).

b) We shall work with \(\left(\alpha^{(1)}_s, s \geq 0\right)\), the inverse of \(\left(A^{(1)}_t, t \geq 0\right)\). It is more convenient to use the notation \(\left(\alpha^{(+)}_s, s \geq 0\right)\) for \(\left(\alpha^{(1)}_s, s \geq 0\right)\). We then follow the main steps of [13] (Section 3.4, p. 42), which themselves are inspired by Williams [12]; see also Watanabe (Proposition 1 in [11]) and Mc Kean [8].

\(A^{(j)}_t\) denotes the time spent in \(I_j\), for any \(j \neq 1\). Since
\[
\begin{align*}
A^{(j)}_{\alpha^{(+)}_1} &= A^{(j)}_{\tau_1 L^{(+)}_{\alpha^{(+)}_1}} \overset{\text{law}}{=} \left(L^{(+)}_{\alpha^{(+)}_1}\right)^2 A^{(j)}_{\tau_1}, \\
\alpha^{(+)}_1 &= 1 + \sum_j A^{(j)}_{\alpha^{(+)}_1}, \\
\text{and} \\
\text{for every } u, t \geq 0, \quad \left(L^{(+)}_{\alpha^{(+)}_u} < t\right) = \left(u < A^{(1)}_{\tau_1 \sqrt{t}}\right),
\end{align*}
\]
and invoking the scaling property, we can write jointly for all \(j\)’s:
\[
\begin{align*}
\left(A^{(j)}_{\alpha^{(+)}_1}, L^{2}_{\alpha^{(+)}_1}, \alpha^{(+)}_1\right) \overset{\text{law}}{=} \left(L^{2}_{\alpha^{(+)}_1} A^{(j)}_{\tau_1}, L^{2}_{\alpha^{(+)}_1}, 1 + \sum_j L^{2}_{\alpha^{(+)}_1} A^{(j)}_{\tau_1}\right) \\
\overset{\text{law}}{=} \left(\frac{A^{(j)}_{\tau_1}}{A^{(1)}_{\tau_1}}, \frac{1}{A^{(1)}_{\tau_1}}, \frac{\tau_1}{A^{(1)}_{\tau_1}}\right).
\end{align*}
\]

Dividing now both sides by \(\alpha^{(+)}_1\) and remarking that: \(\alpha^{(+)}_1 A^{(1)}_{\tau_1} = \tau_1\), we deduce:
\[
\frac{1}{\alpha^{(+)}_1} \left(A^{(j)}_{\alpha^{(+)}_1}, L^{2}_{\alpha^{(+)}_1}\right) \overset{\text{law}}{=} \frac{1}{\tau_1} \left(A^{(j)}_{\tau_1}, 1\right).
\]
With the help of the scaling Lemma below, we obtain:

\[
E \left[1_{(W_1 \in I_1)}f(\overrightarrow{A}, L_1^2)\right] = E \left[\frac{1}{\alpha_1^{(+)}} f \left(\frac{\overrightarrow{A_{\alpha_1^{(+)}}}}{\alpha_1^{(+)}, \alpha_1^{(+)}}\right)\right] \]

from (5)

\[
= E \left[\frac{A_1^{(1)}}{\tau_1} f \left(\frac{A_1}{\tau_1}, \frac{1}{\tau_1}\right)\right].
\]

(7)

\(I_1\) may be replaced by \(I_m\), for any \(m \in \{2, \ldots, n\}\). Adding the \(m\) quantities found in (7) and remarking that:

\[
\tau_1 = \sum_{i=1}^{n} A_{\tau_1}^{(i)},
\]

we get:

\[
E \left[f(\overrightarrow{A}, L_1^2)\right] = E \left[f \left(\frac{\overrightarrow{A_1}}{\tau_1}, \frac{1}{\tau_1}\right)\right].
\]

which proves (3). Note that from (6), the latter also equals:

\[
E \left[f \left(\frac{\overrightarrow{A_{\alpha_1^{(+)}}}}{\alpha_1^{(+)}, \alpha_1^{(+)}}\right)\right].
\]

Equality in law (4) follows now easily. Indeed, we denote by \(\nu\) the Itô measure of the Brownian spider, and we have:

\[
\nu = \frac{1}{n} \sum_{j=1}^{n} \nu_j,
\]

where \(\nu_j\) is the canonical image of \(n\), the standard Itô measure of the space of the excursions of the standard Brownian motion, on the space of the excursions on \(I_j\). Hence, with \(\lambda_j, j = 1, \ldots, n\) denoting positive constants:

\[
E \left[\exp \left(-\sum_{j=1}^{n} \lambda_j A_{\tau_1}^{(j)}\right)\right] = \exp \left(-\frac{1}{n} \sum_{j=1}^{n} \nu_j(d\xi_j)(1 - e^{-\lambda_j \nu_j})\right)
\]

\[
= \exp \left(-\frac{1}{n} \sum_{j=1}^{n} \sqrt{2\lambda_j}\right),
\]

thus:

\[
\overrightarrow{A_{\tau_1}} = (A_{\tau_1}^{(j)} ; j \leq n) \overset{\text{law}}{=} \left(\frac{1}{n^2} T_j ; j \leq n\right).
\]

The latter, using (8) yields:

\[
\overrightarrow{A} = \overrightarrow{A_{\tau_1}} = \sum_{i=1}^{n} \frac{A_{\tau_1}^{(i)}}{A_{\tau_1}^{(i)}} \overset{\text{law}}{=} \left(\frac{T_j}{n^2 \sum_{i=1}^{n} n^{-2} T_i} ; j \leq n\right),
\]

4
which finishes the proof.

It now remains to state the scaling Lemma which played a role in (7), and which we lift from [13] (Corollary 1, p. 40) in a "reduced" form.

Lemma 2.2. (Scaling Lemma) Let \( U_t = \int_0^t ds \theta_s \), with the pair \( (W, \theta) \) satisfying:

\[
(W_{ct}, \theta_{ct}; t \geq 0) \overset{(law)}{=} (\sqrt{c}W_t, \theta_t; t \geq 0).
\]

Then,

\[
E[\{F(W_u, u \leq 1) \theta_1\}] = E\left[\frac{1}{\alpha_1} F\left(\frac{1}{\sqrt{\alpha_1}} W_{\alpha_1}, v \leq 1\right)\right],
\]

where \( \alpha_t = \inf\{s : U_s > t\} \).

3 Stable subordinators

3.1 Reminder and preliminaries on stable variables

In this Section, we consider \( S_\mu \) and \( S_\mu' \) two independent stable variables with exponent \( \mu \in (0,1) \), i.e. for every \( \lambda \geq 0 \), the Laplace transform of \( S_\mu \) is given by:

\[
E[\exp(-\lambda S_\mu)] = \exp(-\lambda^\mu).
\]

Concerning the law of \( S_\mu \), there is no simple expression for its density (except for the case \( \mu = 1/2 \); see e.g. Exercise 4.20 in [3]). However, we have that, for every \( s < 1 \) (see e.g. [15] or Exercise 4.19 in [3]):

\[
E[(S_\mu)^{\mu s}] = \frac{\Gamma(1-s)}{\Gamma(1-\mu s)}.
\]

We consider now the random variable of the ratio of two \( \mu \)-stable variables:

\[
X = \frac{S_\mu}{S_\mu'}.
\]

Following e.g. Exercise 4.23 in [3], we have respectively the following formulas for the Stieltjes and the Mellin transforms of \( X \):

\[
E\left[\frac{1}{1 + sX}\right] = \frac{1}{1 + s^\mu}, \quad s \geq 0,
\]

\[
E[X^s] = \frac{\sin(\pi s)}{\mu \sin(\frac{\pi s}{\mu})}, \quad 0 < s < \mu.
\]

Moreover, the density of the random variable \( X^\mu \) is given by (see e.g. [14, 5] or Exercise 4.23 in [3]):

\[
P(X^\mu \in dy) = \frac{\sin(\pi \mu)}{\pi \mu} \frac{dy}{y^2 + 2y \cos(\pi \mu) + 1}, \quad y \geq 0,
\]
or equivalently:
\[
\frac{S_\mu}{S'_\mu} = (C_\mu | C_\mu > 0),
\]
where, with \(C\) denoting a standard Cauchy variable and \(U\) a uniform variable in \([0, 2\pi]\),
\[
C_\mu = \sin(\pi \mu) C - \cos(\pi \mu) \overset{(law)}{=} \frac{\sin(\pi \mu - U)}{U}.
\]

\[\text{3.2 The case of 2 stable variables}\]

We turn now our study to the random variable:
\[
A = \frac{S'_\mu}{S'_\mu + S_\mu} = \frac{1}{1 + X},
\]

**Theorem 3.1.** The density function of the random variable \(A\) is given by:
\[
P(A \in dz) = \frac{\sin(\pi \mu)}{\pi} \frac{dz}{z(1 - z)\left[\left(\frac{1-z}{z}\right)^\mu + \left(\frac{z}{1-z}\right)^\mu + 2\cos(\pi \mu)\right]}, \quad z \in [0, 1].
\]

**Proof of Theorem 3.1.**

Identity (19) is equivalent to:
\[
X = \frac{1}{A} - 1.
\]

Hence, (15) yields:
\[
E\left[\frac{1}{1 + sX}\right] = E\left[\frac{A}{(1 - s)(A + s)}\right] = \frac{1}{1 + s^\mu}.
\]

We consider now a test function \(f\) and invoking the density (17) we have \((\nu = \frac{1}{\mu} > 1)\):
\[
E\left[f\left(\frac{1}{1 + X}\right)\right] = \frac{\sin(\pi \mu)}{\pi} \int_0^\infty \frac{dy}{y^2 + 2y \cos(\pi \mu) + 1} f\left(\frac{1}{1 + y^\nu}\right).
\]

Changing the variables \(z = \frac{1}{1 + y^\nu}\), we deduce:
\[
E[f(A)] = \frac{\sin(\pi \mu)}{\pi} \int_0^1 \frac{dz(1 - z)^{\mu - 1}}{z^{\mu + 1}} f(z) \Delta(z),
\]
where:
\[
\Delta(z) = \frac{1}{(z^{-1} - 1)^{2\mu} + 2(z^{-1} - 1)^\mu \cos(\pi \mu) + 1}
\]
\[
= \frac{1}{(1 - z)^{2\mu} + 2(1 - z)^\mu z^\mu \cos(\pi \mu) + z^{2\mu}},
\]
and (20) follows easily.

In Figure 1, we have plotted the density function \(g\) of \(A\), for several values of \(\mu\).

**Remark 3.2.** Similar discussions have been made in [4] in the framework of a skew Bessel process with dimension \(2 - 2\alpha\) and skewness parameter \(p\). Formula (20) is a particular case of formula in [4] for the density of the time spent positive (called \(f_{p,\alpha}\) in [4]).
3.3 The case of many stable \((1/2)\) variables

In this Subsection, we consider again \(n\) iid stable \((1/2)\) variables, i.e.: \(T_1, \ldots, T_n\), and we will study the distribution of:

\[
A^{(1)}_1 = \frac{T_1}{T_1 + \ldots + T_n}.
\]  

(21)

The following Theorem answers to an open question (and even in a more general sense) stated at the end of [9].

**Theorem 3.3.** The density function of the random variable \(A^{(1)}_1\) is given by:

\[
P\left(A^{(1)}_1 \in dz\right) = \frac{1}{\pi \sqrt{z(1-z)}} \frac{dz}{\sqrt{\frac{(n-1)z + \frac{1}{n-1}(1-z)}}}, \quad z \in [0, 1].
\]

(22)

**Proof of Theorem 3.3.**

We first remark that, with \(C\) denoting a standard Cauchy variable, using e.g. (2):

\[
A^{(1)}_1 \overset{(law)}{=} \frac{T_1}{T_1 + (n-1)^2T_2} \overset{(law)}{=} \frac{1}{1 + (n-1)^2C^2}.
\]

(23)

Hence, with \(f\) standing again for a test function, and invoking the density of a standard
Figure 2: The density function $h$ of $A_1^{(1)}$, for several values of $n$.

Cauchy variable, that is: for every $x \in \mathbb{R}$, $g(x) = \frac{1}{\pi(1+x^2)}$ we have:

$$E\left[f\left(A_1^{(1)}\right)\right] = E\left[f\left(\frac{1}{1+(n-1)^2C^2}\right)\right]$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{dx}{1+x^2} f\left(\frac{1}{1+(n-1)^2x^2}\right)$$

Changing the variables $z = \frac{1}{1+(n-1)^2y}$, we deduce:

$$E\left[f\left(A_1^{(1)}\right)\right] = \frac{1}{\pi} \int_0^1 \frac{dz}{(n-1)^2z^2} f\left(\frac{(n-1)\sqrt{z}}{\sqrt{z-1} \left(1+\frac{1}{(n-1)^2} \left(\frac{1}{2}-1\right)\right)}\right)$$

and (22) follows easily.

Figure 2 presents the plot of the density function $h$ of $A_1^{(1)}$, for several values of $n$.

**Corollary 3.4.** The following convergence in law holds:

$$n^2A_1^{(1)}(n) \xrightarrow{(law)} C^2.$$  (24)
Proof of Corollary 3.4.

It follows from Theorem 3.3 by simply remarking that $C^{(law)} = C^{-1}$. Hence:

$$n^2 A_1^{(1)}(n) = \frac{n^2}{1 + (n-1)^2 C^2} = \frac{1}{\frac{1}{n} + \left(\frac{n-1}{n}\right)^2 C^2} \xrightarrow{n \to \infty} \frac{1}{C^2} = C^{(law)}.$$

4 Conclusion and comments

We end up this article with some comments: usually, a scaling argument is "one-dimensional", as it involves a time-change. Exceptionally (or so it seems to the authors), here we could apply a scaling argument in a multivariate framework. We insist that the scaling Lemma plays a key role in our proof. The curious reader should also look at the totally different proof of this Theorem in [2], which mixes excursion theory and the Feynman-Kac method.

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