CHARACTERIZATION OF SMOOTH SOLUTIONS TO THE NAVIER-STOKES EQUATIONS IN A PIPE WITH TWO TYPES OF SLIP BOUNDARY CONDITIONS

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ABSTRACT

Smooth solutions of the stationary Navier-Stokes equations in an infinitely long pipe, equipped with the Navier-slip or Navier-Hodge-Lions boundary condition, are considered in this paper. Three main results are presented.

First, when equipped with the Navier-slip boundary condition, it is shown that, $W^{1,\infty}$ axially symmetric solutions with zero flux at one cross section, must be swirling solutions: $u = (-C x_2, C x_1, 0)$, and $x_3$-periodic solutions must be helical solutions: $u = (-C_1 x_2, C_1 x_1, C_2)$.

Second, also equipped with the Navier-slip boundary condition, if the swirl or vertical component of the axially symmetric solution is independent of the vertical variable $x_3$, solutions are also proven to be helical solutions. In the case of the vertical component being independent of $x_3$, the $W^{1,\infty}$ assumption is not needed. In the case of the swirl component being independent of $x_3$, the $W^{1,\infty}$ assumption can be relaxed extensively such that the horizontal radial component of the velocity, $u_r$, can grow exponentially with respect to the distance to the origin. Also, by constructing a counterexample, we show that the growing assumption on $u_r$ is optimal.

Third, when equipped with the Navier-Hodge-Lions boundary condition, we can show that if the gradient of the velocity grows sublinearly, then the solution, enjoying the Liouville-type theorem, is a trivial shear flow: $(0, 0, C)$.

KEYWORDS: Navier-Stokes system, Navier-slip boundary, Navier-Hodge-Lions boundary, axially symmetric, helical solutions.

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1 INTRODUCTION

The 3D stationary Navier-Stokes (NS) equations which describes the motion of stationary viscous incompressible fluids follows that

$$\begin{cases} u \cdot \nabla u + \nabla p - \Delta u = 0, \\ \nabla \cdot u = 0 \end{cases} \quad \text{in } \mathcal{D} \subset \mathbb{R}^3. \quad (1.1)$$

Here $u(x) \in \mathbb{R}^3$, $p(x) \in \mathbb{R}$ represents the velocity and the scalar pressure respectively. In this paper, we consider the domain $\mathcal{D}$ to be an infinitely long pipe, i.e.

$$\mathcal{D} = \{ x : |x_h| < 1, x_3 \in \mathbb{R} \}.$$
where \( x = (x_1, x_2, x_3), x_h = (x_1, x_2) \) and \(|x_h| = \sqrt{x_1^2 + x_2^2}\). The boundary condition will be equipped with the following:

**The total Navier-slip boundary condition:**

\[
\begin{align*}
(Su \cdot n)_r &= 0, \\
u \cdot n &= 0,
\end{align*}
\]

\(\forall x \in \partial D,\) \hfill (NSB)

or the **Navier-Hodge-Lions boundary condition:**

\[
\begin{align*}
(\nabla \times u) \times n &= 0, \\
u \cdot n &= 0,
\end{align*}
\]

\(\forall x \in \partial D,\) \hfill (NHLB)

Here \( Su = \frac{1}{2} (\nabla u + (\nabla u)^T) \) is the stress tensor, where \((\nabla u)^T\) is the transpose of the Jacobian matrix \( \nabla u \), and \( n \) is the unit outer normal vector of \( \partial D \). For a vector field \( v \), \( v_r \) stands for its tangential part: \( v_r := v - (v \cdot n)n \). The condition [NSB] is from the general Navier-slip boundary condition and impermeable boundary condition which was introduced by Claude-Louis Naiver in 1820s [28]:

\[
\begin{align*}
2(Su \cdot n)_r + \alpha u_t &= 0, \\
u \cdot n &= 0.
\end{align*}
\]

(1.2)

Here \( \alpha \geq 0 \) stands for the friction constant which may depend on various elements, such as the property of the boundary and the viscosity of the fluid. When \( \alpha = 0 \), boundary condition [1.2] turns to the total Navier-slip boundary [NSB], and when \( \alpha \to \infty \), boundary condition [1.2] degenerates into the no-slip boundary condition \( u \equiv 0 \) on the boundary.

The boundary condition [NHLB] is a special case in a family of boundary conditions proposed by Navier [28], which has been studied extensively in the literature and was attributed to different authors. The boundary condition was called the Navier-Hodge boundary condition in [25] and the Navier-Lions boundary condition in [23]. For this reason, we will call it the Navier-Hodge-Lions boundary condition in this paper.

We write \( D \) to be

\[ D = \Sigma \times \mathbb{R}, \]

where the cross section \( \Sigma \in \mathbb{R}^2 \) is a unit disc. The domain considered here is a high-degree simplification of the following “distorted cylinder”, i.e.

\[ \tilde{D} = \tilde{\Sigma} \times \mathbb{R}, \]

where \( \tilde{\Sigma} \in \mathbb{R}^2 \) is a simply connected bounded domain with smooth boundary.

Let \( \tilde{D}_0 \) be a simply connected bounded domain with smooth boundary in \( \mathbb{R}^3 \) and \( \tilde{D}_0 \cap \tilde{D} \neq \emptyset \). Existence problem of weak solutions in domain \( \tilde{D}_{\text{Union}} := \tilde{D} \cup \tilde{D}_0 \) with the Navier-slip boundary [NSB] was addressed in [17] and regularity of solutions was also implied there. On the other hand, if \( \tilde{D}_0 \subset \tilde{D} \) is an “obstacle” in \( \tilde{D} \), then the two dimensional existence problems and asymptotic behaviors of smooth solutions in domain \( \tilde{D}_{\text{Diff}} := \tilde{D} \setminus \tilde{D}_0 \) with the total Navier-slip boundary condition are obtained in [26, 27].
There have also been many pieces of literature in studying the existence, uniqueness and asymptotic behavior of the Navier-Stokes equations in a distorted pipe $\tilde{D}_{\text{Union}}$ or $\tilde{D}_{\text{Diff}}$ with no-slip boundary and with the Poiseuille flow as the asymptotic profile at infinity (Leray’s problem: Ladyzhenskaya [18] p. 77 and [19] p. 551). The first remarkable contribution on the solvability of Leray’s problem is due to Amick [1, 2], who reduced the solvability problem to the resolution of a variational problem related to the stability of the Poiseuille flow in a flat cylinder. However, uniqueness and existence of solutions with large flux are left open. Ladyzhenskaya and Solonnikov [20] gave a detailed analysis of this problem on existence, uniqueness and asymptotic behavior of small-flux solutions. One may refer to [3, 14, 30] and references for more details on well-posedness, decay and far-field asymptotic analysis of solutions for Leray’s problem and related topics. A systematic review and study of Leray’s problem can be found in [11, Chapter XIII]. Recently Wang-Xie in [33] studied uniform structural stability of Poiseuille flows for the 3D axially symmetric solutions in the 3D pipe $D$ where a force term appears on the right hand of equation (1.1).

Compared to the no-slip boundary condition, this model with the total Navier-slip or Navier-Hodge-Lions boundary condition has different physical interpretations and gives different mathematical properties. Literature [26, 17] addressed the existence and regularity problems of weak solutions for the total Navier-slip boundary condition in a pipe, but uniqueness was left open. Also readers can refer to, i.e., [4, 10, 23, 34] for some well-posedness results for the Navier-Hodge-Lions boundary condition in different domains. In this paper, we attempt to derive some uniqueness results for the Navier-Stokes equations with the total Navier-slip or Navier-Hodge-Lions boundary condition in the regular infinite pipe $D$. We emphasize that our results below do not require any smallness and decay assumptions.

In this paper, for the boundary condition [NSB], a family of smooth helical solutions will be given, and for the Navier-Hodge-Lions boundary condition [NHLB], the trivial shear flow is easy to be discovered. We concern on the characterization and uniqueness of these two types of smooth solutions in $D$.

Most of our proof will be carried out in the framework of cylindrical coordinates $(r, \theta, z)$ and some of our results are restricted to the axially symmetric solutions. Here we give the formulation of axially symmetric solutions in the cylindrical coordinates which enjoy the following relationship with 3D Euclidian coordinates:

$$x = (x_1, x_2, x_3) = (r \cos \theta, r \sin \theta, z).$$

A stationary axially symmetric solution of the incompressible Navier-Stokes equations is given as

$$u = u_r(r, z)e_r + u_\theta(r, z)e_\theta + u_z(r, z)e_z,$$

where the basis vectors $e_r, e_\theta, e_z$ are

$$e_r = \left(\frac{x_1}{r}, \frac{x_2}{r}, 0\right), \quad e_\theta = \left(-\frac{x_2}{r}, \frac{x_1}{r}, 0\right), \quad e_z = (0, 0, 1),$$
while the components \( u_r, u_\theta, u_z \), which are independent of \( \theta \), satisfy

\[
\begin{cases}
(u_r \partial_r + u_z \partial_z) u_r - \frac{(u_\theta)^2}{r} + \partial_r p = \left( \Delta - \frac{1}{r^2} \right) u_r, \\
(u_r \partial_r + u_z \partial_z) u_\theta + \frac{u_\theta u_r}{r} = \left( \Delta - \frac{1}{r^2} \right) u_\theta, \\
(u_r \partial_r + u_z \partial_z) u_z + \partial_z p = \Delta u_z, \\
\nabla \cdot b = \partial_r u_r + \frac{u_r}{r} + \partial_z u_z = 0,
\end{cases}
\]

(1.3)

where \( b = u_r e_r + u_z e_z \).

We can also compute the axi-symmetric vorticity \( \omega = \nabla \times u = \omega_r e_r + \omega_\theta e_\theta + \omega_z e_z \) as follows

\[
\omega_r = -\partial_z u_\theta, \quad \omega_\theta = \partial_z u_r - \partial_r u_z, \quad \omega_z = \left( \partial_r + \frac{1}{r} \right) u_\theta,
\]

which satisfies

\[
\begin{cases}
(u_r \partial_r + u_z \partial_z) \omega_r - \left( \Delta - \frac{1}{r^2} \right) \omega_r - (\omega_r \partial_r + \omega_z \partial_z) u_r = 0, \\
(u_r \partial_r + u_z \partial_z) \omega_\theta - \left( \Delta - \frac{1}{r^2} \right) \omega_\theta - \frac{u_r}{r} \omega_\theta - \frac{1}{r} \partial_z (u_\theta)^2 = 0, \\
(u_r \partial_r + u_z \partial_z) \omega_z - \Delta \omega_z - (\omega_r \partial_r + \omega_z \partial_z) u_z = 0.
\end{cases}
\]

(1.4)

In the cylindrical coordinates, the total Navier-slip boundary condition (NSB) is represented as

\[
\begin{cases}
\partial_r u_\theta - \frac{u_\theta}{r} = 0, \\
\partial_r u_z = 0, \quad \forall x \in \partial D, \\
u_r = 0,
\end{cases}
\]

(1.5)

while the Navier-Hodge-Lions boundary condition (NHLB) is given by

\[
\begin{cases}
\partial_r u_\theta + \frac{u_\theta}{r} = 0, \\
\partial_r u_z = 0, \quad \forall x \in \partial D, \\
u_r = 0,
\end{cases}
\]

(1.6)

whose computations are postponed to Appendix A.

Clearly direct calculation shows that, for arbitrary constants \( C_1 \) and \( C_2 \), the following type of helical solutions

\[
u = C_1 r e_\theta + C_2 e_z
\]

solves (1.3) with the boundary condition (1.5).
We further note that helical solutions (1.7), which is smooth in $D$, enjoys the following property: 

The solution itself and its gradient are uniformly bounded in $D$.  \((*)\)

Thus a natural question raises:

*Are helical solutions (1.7) the only smooth solutions of system (1.3) with the boundary condition (1.5) which enjoys property \((*)\)?*

Before answering this question, we recall that the flux $\Phi(z)$ at the cross section $\Sigma$, which is defined by

$$\Phi(z) := \int_{\Sigma} u(x_h, z) \cdot v dx_h,$$

is a constant. Here $v = e_z$ is the unit normal vector of $\Sigma$ pointing to the positive $z$ direction. Actually by using the divergence free condition of the velocity and the boundary condition \([\text{NSB}]_2\), we have

$$\frac{d}{dz} \Phi(z) = \int_{\Sigma} \frac{d}{dz} u(x_h, z) dx_h = - \int_{\Sigma} (\partial_{x_1} u_1 + \partial_{x_2} u_2)(x_h, z) dx_h$$

\[= - \int_{\partial \Sigma} (n_1 u_1 + n_2 u_2)(x_h, z) dS(x_h) \]  

\[= - \int_{\partial \Sigma} (u \cdot n)(x_h, z) dS(x_h) = 0,\]

where $n = (n_1, n_2, 0)$ is the unit outer normal vector of $\partial D$. Then for any $z \in \mathbb{R}$, we will denote $\Phi(z) = \Phi$.

Our first result in this paper gives a positive answer to the above question in the following two cases:

(i). the solution is axially symmetric and the flux $\Phi$ is zero (corresponding to $C_2 = 0$);

(ii). the solution is $z$-periodic.
Theorem 1.1. Let $u$ be a smooth solution of the Navier-Stokes equations in the infinite pipe $D$ subject to the total Navier-slip boundary condition NSB.

I. Suppose $u \in W^{1,\infty}(D)$ is axially symmetric and $\Phi = 0$, then $u$ must be the following type of swirling solutions:

$$u = C_1 r e_{\theta}, \quad p = \frac{C_1^2 r^2}{2}.$$  

II. Suppose $u$ is $z$-periodic, then $u$ must be the following type of helical solutions:

$$u = C_1 r e_{\theta} + C_2 e_z, \quad p(r, z) = \frac{C_1^2 r^2}{2}, \quad \forall C_1, C_2 \in \mathbb{R}. \quad (1.8)$$

Besides, we observe that solutions (1.7) enjoy the following property:

Its swirl component $u_\theta$ or vertical component $u_z$ is independent of $z$.

In the following theorem, we will conclude that if $u_\theta$ or $u_z$ is independent of $z$, then (1.7) are the only group of smooth solutions to (1.3) subject to the boundary condition (1.5).

In the case of $u_\theta$ being independent of $z$, the bounded $W^{1,\infty}$ assumption on the velocity will be extensively relaxed to the following:

$$\begin{cases} 
|u_r(r, z)| \leq C r e^{\gamma_0 |z|}; \\
|u_z(r, z)| \leq C |z|^{\bar{\delta}_0}; \quad \text{uniformly with } r \in [0, 1], \\
|\omega_\theta(r, z)| \leq C |z|^{M_0}, 
\end{cases} \quad (1.9)$$

for any $\gamma_0 < \alpha \approx 3.83171$, $\delta_0 < 1$, and $M_0 > 0$. Here $\alpha$ is the first positive root of the Bessel function $J_1$. We recall that $J_\beta$ are canonical solutions of Bessel’s ordinary differential equation

$$s^2 J_\beta''(s) + s J_\beta'(s) + (s^2 - \beta^2) J_\beta(s) = 0, \quad (1.10)$$

which can be expressed by the following series form:

$$J_\beta(s) = \sum_{n=0}^{\infty} (-1)^n \frac{(-s)^n}{n! \Gamma(n + \beta + 1)} \left(\frac{s}{2}\right)^{2n+\beta}. \quad (1.11)$$

In the case of $u_z$ being independent of $z$, no size assumptions such as (1.9) are imposed on the solution.

Remark 1.2. The reason why there is an $r$ on the righthand of (1.9) is that for a smooth solution $u$, in the cylindrical coordinates, $u_r$ vanishes at $r = 0$. When doing Taylor expansion of $u_r$ at $r = 0$ in the $r$ direction, the zero order derivative term is missing, so it is reasonable to assume a one order $r$ control on $u_r$ for $r \in [0, 1]$. 

□
Theorem 1.3. Let $u$ be a smooth solution of the axially symmetric Navier-Stokes equations (1.3) in the infinite pipe $\mathcal{D}$ subject to the total Navier-slip boundary condition (NSB). Then $u$ must be of helical solutions (1.8) if one of the following is satisfied.

I. $u$ satisfies (1.9) and $u_0$ is independent of $z$-variable;

II. $u_z$ is independent of $z$-variable.

Remark 1.4. We emphasize that the condition (1.9) above is sharp, because we have the following non-trivial counterexample which grows exactly as $Ce^{\alpha z}$ when $z \to \infty$:

$$
\begin{align*}
  u &= -\cosh(\alpha z)J_1(\alpha r)e_r + \sinh(\alpha z)J_0(\alpha r)e_z, \\
  p &= -\frac{1}{2} \left( \cosh^2(\alpha z)J_1^2(\alpha r) + \sinh^2(\alpha z)J_0^2(\alpha r) \right). 
\end{align*}
$$

(1.12)

Here $J_0$, $J_1$ are Bessel functions defined in (1.11), while $\alpha \approx 3.83171$ is the smallest positive root of $J_1$. One can verify (1.12) being the solution of (1.3) with the boundary condition (1.5) by direct calculations. Here we leave the details to the interested reader. Unfortunately, our example here can not reflect whether the growing assumptions in (1.9)_2 and (1.9)_3 are sharp.

If we switch the total Navier-slip boundary condition to the Navier-Hodge-Lions boundary condition (NHLB), one notices that a non-zero swirling solution does not enjoy (NHLB). In this situation, one can derive the following Liouville-type theorem:

Theorem 1.5. Let $u$ be a smooth solution of the Navier-Stokes equations (1.1) in the infinite pipe $\mathcal{D}$ subject to the Navier-Hodge-Lions boundary condition (NHLB). Suppose $\nabla u$ satisfies

$$
|\nabla u(x_h, z)| \leq C|z|^\beta
$$

for some $C > 0$ and $0 < \beta < 1$. Then $u = \frac{\Phi}{\pi} e_3$.



There has already been much literature studying Liouville-type results on the Navier-Stokes equations subject to various boundary conditions in various unbounded domains. Readers can refer to [8, 9, 31, 32, 7, 29] and references therein for more Liouville-type results on the stationary Navier-Stokes equations. Moreover, our results in the above Theorems can be extended from the stationary case to the case of ancient solutions (backward global solutions) under suitable assumptions. However, for simplification of idea presenting, we omit this extension here and leave it to further works. See [13] where the authors established a Liouville-type result for the ancient solution to the Navier-Stokes equations in the half plane with the no-slip boundary condition.

Liouville-type results of ancient solutions is connected to the regularity of solutions to the initial value problem of the non-stationary Navier-Stokes equations. Type I blow-up solutions of the Navier-Stokes initial value problem could not exist provided the Liouville-type result holds for bounded ancient solutions. See [15, 12].
Before ending our introduction, we briefly outline our strategy for proofs of Theorem 1.1, Theorem 1.3 and Theorem 1.5. The most important ingredient of proving Theorem 1.1 is to show that \( \mathcal{S} u \equiv 0 \). For Case I in Theorem 1.1, we need to first show that \( \mathcal{S} u \in L^2(\mathcal{D}) \), which is not trivial since the pipe is an unbounded domain. In this process, \( L^\infty \) oscillation boundedness of the pressure in \( \mathcal{D}_{2\pi} \setminus \mathcal{D}_2 \) (see (1.16) for the definition of \( \mathcal{D}_2 \)) is essential, which will be presented in Section 2.1.2. Then combining the square integrability of \( \mathcal{S} u \) and boundedness of the velocity together with its gradient, a trick of integration by parts and Poincaré inequality will indicate that \( u_\theta \) actually belongs to \( L^2(\mathcal{D}) \), which will result in the vanishing of \( \mathcal{S} u \). While vanishing of \( \mathcal{S} u \) in Case II of Theorem 1.1 is directly due to the \( x_3 \)-periodic property by performing the energy estimate in a single period. However, in the process, we must be careful of handling of the pressure, which may not be \( x_3 \) periodic. After analyzing ingredients of \( \mathcal{S} u \), we finally conclude the validity of Theorem 1.1.

The idea for proof of Theorem 1.3 is completely different from that of Theorem 1.1. Under the assumption of Case I in Theorem 1.3, we will see that the quantity \( \Omega := \omega_\theta/r \) satisfies a nice linear elliptic equation with an advection term. Under the growing assumption (1.9) in domain \( \mathcal{D} \), by using the Nash-Moser iteration, we can show that actually \( \Omega \equiv 0 \), which indicates that \( b = u_\theta e_r + u_\xi e_\xi \) must be harmonic in \( \mathcal{D} \). Then by constructing a barrier function, applying maximum principle and assumptions on \( b \), one derives \( u_\theta \equiv 0 \) and \( u_\xi \) must be a constant. From then on, (1.3) is reduced to a linear ordinary differential equation of \( u_\theta \), and we finally obtain \( u_\theta = C_1 r \). While under the assumption of Case II in Theorem 1.3, a rather direct computation implies that \( u_\theta \equiv 0 \) by using the divergence-free condition. Then by combining the equations of \( u_\theta \) and \( u_\xi \), independence of \( z \) variable for \( u_\theta \) can be obtained. Then the equation of \( u_\theta \) will degenerate to an ordinary differential equation with respect to \( r \), which will result in \( u_\theta = C_1 r \). At last, trivialness of \( u_\xi \) is achieved by solving a two-dimensional Laplacian equation with Neumann boundary condition.

Proof of Theorem 1.5 is to apply Lemma 4.1 which was originally announced in reference [20] as far as the authors know. Denote the energy integral in terms of \( v := u - \frac{\Phi}{\pi} e_3 \) as follows:

\[
Y(Z) := \int_{Z-1}^Z \int_{\mathcal{D}_r} |\nabla v|^2 dx d\zeta.
\]

A differential inequality of \( Y(Z) \), satisfying the assumption in Lemma 4.1 will be derived. In this process, boundary terms coming from integration by parts will be carefully addressed by using the boundary condition, which has a good sign compared with those from the Navier-slip boundary condition. At last, a direct application of Lemma 4.1 will imply the vanishing of \( Y(Z) \).

For the generalized Navier boundary condition (1.2) in \( \mathcal{D} \), one can derive that in cylindrical coordinates, (1.2) is equivalent to

\[
\begin{aligned}
\partial_r u_\theta - \frac{u_\theta}{r} + \alpha u_\theta &= 0, \\
\partial_r u_\xi + \alpha u_\xi &= 0, \\
u_\xi &= 0, \\
\forall x \in \partial \mathcal{D}.
\end{aligned}
\]  

For given flux \( \Phi := \int_{Z} u_\xi(x, \zeta) dx \zeta = \text{const.} \), we can find a family of bounded smooth solutions satisfying (1.3) with boundary condition (1.14) as follows:

\[
u = C_1 r \chi_{[\alpha=0]} e_\theta + \frac{2(\alpha + 2)\Phi}{(\alpha + 4)\pi} \left( 1 - \frac{\alpha}{\alpha + 2} r^2 \right) e_\xi, \quad p = \frac{C_1^2 r^2}{2} \chi_{[\alpha=0]} - \frac{8\alpha \Phi}{(\alpha + 4)\pi} \zeta, \]  

(1.15)
where $C_1$ is an arbitrary constant, and $\chi_{(a=0)}$ is the characteristic function on $\{a = 0\}$, which means
\[
\chi_{(a=0)} = \begin{cases} 
1, & \alpha = 0, \\
0, & \alpha > 0.
\end{cases}
\]

When $\alpha \to +\infty$, the boundary condition (1.14) becomes the no-slip boundary and the solution (1.15) corresponds to the Hagen-Poiseuille flow in $\mathcal{D}$. Uniqueness of Hagen-Poiseuille flow is still open for large flux $\Phi$. Our Theorem 1.1 states that in the case $\alpha = 0$ and $\Phi = 0$, we can show that (1.15) are the only bounded smooth solutions of (1.3) with the boundary condition (1.14). For general $0 \leq \alpha \leq +\infty$ and $\Phi$, we have the following conjecture.

**Conjecture 1.6.** Let $u$ be a smooth solution of the axially symmetric Navier-Stokes equations (1.3) in the infinite pipe $\mathcal{D}$ with the flux $\Phi$ and subject to the Navier-slip boundary condition (1.2) for any $0 \leq \alpha \leq +\infty$. Suppose $u$ and its gradient are uniformly bounded, then the solution $u$ must be of the form (1.15).

\[\square\]

**Remark 1.7.** From the authors’ recent paper [22], the uniqueness result there implies that Conjecture 1.6 is valid for the case $\alpha \in (0, +\infty)$ if the flux $\Phi$ is small.

\[\square\]

Throughout this paper, $C_{a,b,c,...}$ denotes a positive constant depending on $a$, $b$, $c$, ... which may be different from line to line. For two quantities $A_1$, $A_2$, we denote $A_1 \vee A_2 = \max\{A_1, A_2\}$. Meanwhile, for $Z > 1$, we denote
\[
\mathcal{D}_\zeta := \{ (r, \theta, z) : 0 \leq r < 1, 0 \leq \theta \leq 2\pi, -\zeta < z < \zeta \},
\]
the truncated pipe with the length of $2\zeta$. And notations $O_{x, \zeta}^\pm$ states
\[
O_{x, \zeta}^+ := (\mathcal{D}_\zeta - \mathcal{D}_{\zeta-1}) \cap \{ x \in \mathcal{D} : z > 0 \}, \quad O_{x, \zeta}^- := (\mathcal{D}_\zeta - \mathcal{D}_{\zeta-1}) \cap \{ x \in \mathcal{D} : z < 0 \},
\]
respectively. We also apply $A \preceq B$ to denote $A \leq CB$. Moreover, $A \simeq B$ means both $A \preceq B$ and $B \preceq A$.

This paper is arranged as follows: Section 2 is devoted to the proof of Theorem 1.1, and the proof of Theorem 1.3 could be found in Section 3. Proof of Theorem 1.5 will be presented in Section 4.

## 2 Proof of Theorem 1.1

In this section, we devote to proof of Theorem 1.1. Proof of Case I is shown in Section 2.1. In Section 2.1.1, we deduce a uniform bound of $\partial_z \omega_0$ by using classical energy estimate of (1.4) and the Moser’s iteration. Then it will be applied to derive the $L^\infty$ oscillation boundedness of the pressure in Section 2.1.2. Based on these preparations, we finish proving Case I of Theorem 1.1 in Section 2.1.3. Proof of Case II is directly derived in Section 2.2.
2.1 Proof of Case I

2.1.1 Uniform bound of $\partial_z \omega_\theta$

Denoting $g := \partial_z \omega_\theta$ and taking $z$-derivative on (1.4), one arrives

$$- \left( \Delta - \frac{1}{r^2} \right) g + b \cdot \nabla g = \nabla \cdot F,$$

where

$$F := -\omega_\theta \partial_z b + \left( \frac{u_r}{r} \omega_\theta + \frac{2}{r} \partial_z u_\theta \right) e_z.$$  \hspace{1cm} (2.1)

From (A.3), we see that $F \in L^\infty$ provided $u$ and $\nabla u$ are bounded. Meanwhile, we observe that from the boundary condition (1.5):

$$g \equiv 0, \text{ on } \partial D.$$

Now we are ready to state the desired lemma of this section, with its proof based on the Moser’s iteration and energy estimate.

**Lemma 2.1.** Let $(u_r, u_\theta, u_z)$ be a smooth solution of (1.3) in $D$, subject to Navier total slip boundary condition (1.5) and $\omega_\theta$ be the swirl component of its vorticity. Then $\partial_z \omega_\theta$ is uniformly bounded in $D$.

**Proof.** For $q \geq 1$, we multiply (2.1) by $q g^{q-1}$ to get

$$- q g^{q-1} \Delta g + \frac{q}{r^2} g^q + b \cdot \nabla g^q = q g^{q-1} \nabla \cdot F,$$  \hspace{1cm} (2.3)

Noting that $\Delta g^q = \text{div} \ (q g^{q-1} \nabla g) = q g^{q-1} \Delta g + q (q - 1) g^{q-2} |\nabla g|^2$, one derives from (2.3) that

$$- \Delta g^q + q (q - 1) g^{q-2} |\nabla g|^2 + \frac{q}{r^2} g^q + b \cdot \nabla g^q = q g^{q-1} \nabla \cdot F.$$  \hspace{1cm} (2.4)

Let $\phi$ be a smooth cut-off function in $z$ variable which is bounded up to its second-order derivatives, supported on $[L-1, L+1]$ for some $L \in \mathbb{R}$, which will be specified later. Using $g^q \phi^2$ as a test function to the equation (2.4) and noting that

$$q(q - 1) \int_D g^{2q-2} |\nabla g|^2 \phi^2 dx = \frac{q - 1}{q} \int_D |\nabla g|^2 \phi^2 dx \geq 0,$$

one deduces

$$\int_D \nabla g^q \cdot \nabla (g^q \phi^2) dx + q \int_D \frac{g^{2q} \phi^2}{r^2} dx + \int_D b \cdot \nabla g^q (g^q \phi^2) dx \leq q \int_D g^{2q-1} \nabla \cdot F \phi^2 dx.$$  \hspace{1cm} (2.5)

We further denote $f := g^q$ for convenience. First we see
\[ I_1 = \int_D |\nabla (f \phi)|^2 dx - \int_D f^2 |\nabla \phi|^2 dx. \]

Clearly, \( I_2 \geq 0 \). Using the divergence free property of \( b \), one finds \( I_3 \) satisfies
\[ I_3 = \frac{1}{2} \int_D b \cdot \nabla f^2 \phi^2 dx = - \int_D u_x \partial_x f^2 \phi^2 dx. \]

Applying integration by parts, one derives
\[ I_4 = -q(2q-1) \int_D g^{2q-2} \nabla g \cdot F \phi^2 dx = \int_D g^{2q-2} \nabla g \cdot F \phi^2 dx \]
\[ \leq \frac{1}{2} \int_D |\nabla (f \phi)|^2 dx + C q^2 \int_D |F|^2 |g|^{2q-2} \phi^2 dx + \int_D |g|^{2q-1} |F| \phi |\nabla \phi| dx. \]

Plugging estimates \( I_1-I_4 \) into (2.5), we conclude that
\[ \int_D |\nabla (f \phi)|^2 dx \leq C \left( \|\nabla \phi\|_{L^\infty(\Omega)} \left( \|u_1\|_{L^\infty(\Omega)} + \|\nabla \phi\|_{L^\infty(\Omega)} \right) + q^2 \right) \int_{\text{supp } \phi} \left( |g| \vee \|F\|_{L^\infty(\Omega)} \right)^{2q} dx. \]

Using the Sobolev imbedding and noting that \( \phi \) is supported on an interval whose length equals 2, one arrives
\[ \left( \int_{\{x: \phi = 1\}} \left( |g| \vee \|F\|_{L^\infty(\Omega)} \right)^{6q} dx \right)^{\frac{1}{6q}} \leq C \frac{1}{Z_1} \left( \|\nabla \phi\|_{L^\infty(\Omega)} \left( \|b\|_{L^\infty(\Omega)} + \|\nabla \phi\|_{L^\infty(\Omega)} \right) + q^2 \right)^{\frac{1}{Z_1}} \times \left( \int_{\text{supp } \phi} \left( |g| \vee \|F\|_{L^\infty(\Omega)} \right)^{2q} dx \right)^{\frac{1}{2q}}. \]

(2.6)

Let \( \frac{1}{2} \leq z_2 < z_1 \leq 1 \) and assume \( \phi \) is supported on the interval \([L - z_1, L + z_1]\), and \( \phi \equiv 1 \) on \([L - z_2, L + z_2]\). Meanwhile, the gradient of \( \phi \) satisfies the following estimate:
\[ \|\nabla \phi\|_{L^\infty} \leq \frac{C}{Z_1 - Z_2}. \]

Thus (2.6) indicates that
\[ \left( \int_{\Sigma \times [L - z_2, L + z_2]} \left( |g| \vee \|F\|_{L^\infty(\Omega)} \right)^{6q} dx \right)^{\frac{1}{6q}} \leq C \frac{1}{Z_1} \left( (z_1 - z_2)^{-2} + C \|b\|_{L^\infty(\Omega)} + q^2 \right)^{\frac{1}{Z_1}} \times \left( \int_{\Sigma \times [L - z_1, L + z_1]} \left( |g| \vee \|F\|_{L^\infty(\Omega)} \right)^{2q} dx \right)^{\frac{1}{2q}} \]

(2.7)

Now \( \forall k \in \mathbb{N} \cup \{0\} \), we choose \( q_k = 3^k \) and \( z_{1k} = \frac{1}{2} + \left( \frac{1}{2} \right)^{k+1}, z_{2k} = z_{1k+1} = \frac{1}{2} + \left( \frac{1}{2} \right)^{k+2} \), respectively.
Denoting
\[ \Psi_k := \left( \int_{\Sigma \times [L - z_{1k}, L + z_{1k}]} \left( |g| \vee \|F\|_{L^\infty(\Omega)} \right)^{2q_k} dx \right)^{\frac{1}{2q_k}}, \]
and iterating (3.14), it follows that

$$\Psi_{k+1} \leq C \frac{1}{3^k} \left( 4^{k+2} + C_{\|b\|_{L^\infty}} + 3^{2k} \right)^{\frac{1}{3^k}} \Psi_k \leq \cdots \leq \left( C_{\|b\|_{L^\infty}(D)} \right)^{\frac{1}{3^k}} 3^{\sum_{j=0}^{k-1} 3^{j}} \Psi_0 \leq C_{\|b\|_{L^\infty}(D)} \Psi_0.$$ 

Performing $k \to \infty$, the above Moser’s iteration implies

$$\|g\|_{L^\infty(\Sigma \times [L-1/2, L+1/2])} \leq C_{\|b\|_{L^\infty(D)}} \left( \|g\|_{L^2(\Sigma \times [L-1, L+1])} + \|F\|_{L^\infty(D)} \right). \tag{2.8}$$

Finally, define another cut off function of $z$-variable $\tilde{\phi}$ who has bounded derivatives up to order 2, supported on $[L-2, L+2]$ and $\tilde{\phi} \equiv 1$ in $[L-1, L+1]$. Multiplying (1.4) by $\omega_0 \tilde{\phi}^2$ and integrating on $D$, one deduces

$$\int_D |\nabla (\omega_0 \tilde{\phi})|^2 dx + \int_D \frac{\omega_0^2 \tilde{\phi}^2}{r^2} dx = \int_D \omega_0^2 |\nabla \tilde{\phi}|^2 dx - \int_D u_0 \partial_z \tilde{\phi} \omega_0^2 \tilde{\phi} dx - \int_D \frac{u_r}{r} \omega_0^2 \tilde{\phi}^2 dx - \int_D \frac{u_\theta}{r} \partial_\theta \omega_0 \omega_0 \tilde{\phi}^2 dx.$$

By the representation of $\nabla u$ (A.2), one derives that

$$\|\nabla \omega_0\|_{L^2(\Sigma \times [L-1, L+1])} \leq C_{\|u, \nabla u\|_{L^\infty(D)}}. \tag{2.9}$$

Meanwhile, expression of $F$ (2.2) also indicates that

$$\|F\|_{L^\infty(D)} \leq C_{\|u, \nabla u\|_{L^\infty(D)}}. \tag{2.10}$$

Substituting (2.9) and (2.10) in (2.8), one concludes that

$$\|g\|_{L^\infty(\Sigma \times [L-1/2, L+1/2])} \leq C_{\|u, \nabla u\|_{L^\infty(D)}}.$$

Noting that the right-hand side above is independent of $L$, thus we have derived the uniform boundedness of $g$ in $D$.

\[ \square \]

### 2.1.2 Boundedness of the pressure

Based on the boundedness of $\partial_z \omega_0$, the $L^\infty$ oscillation bound of the pressure $p$ in $D_{2Z} \setminus D_Z$ can be obtained. The lemma is stated as follows:

**Lemma 2.2.** Under the same assumptions of Theorem 1.1, $\forall Z > 1$, we have

$$\sup_{x \in D_{2Z} \setminus D_Z} |p(r, z) - p(0, Z)| \leq C, \tag{2.11}$$

where $C > 0$ is a uniform constant independent of $Z$.

**Proof.** We only consider $(D_{2Z} \setminus D_Z) \cap \{x : z > 0\}$ since the rest part is essentially the same. Let us start with the oscillation of the pressure along the $r-$axis. From (1.3) and the identity

$$\left( \Delta - \frac{1}{r^2} \right) u_r = \partial_z \omega_0,$$
one sees that
\[ \partial_r p = \partial_t \omega_0 - (u_t \partial_r + u_z \partial_z)u_r + \frac{(u_\theta)^2}{r}. \] (2.12)

For any \( z \in \mathbb{R} \) and \( r_1, r_2 \in [0, 1] \), we integrate (2.12) with \( r \) on \([r_1, r_2] \) to derive
\[
p(r_2, z) - p(r_1, z) = \int_{r_1}^{r_2} \partial_z \omega_0 dr - \int_{r_1}^{r_2} \left[ (u_t \partial_r + u_z \partial_z)u_r - \frac{u_\theta^2}{r} \right] dr
\]
\[
= \int_{r_1}^{r_2} \partial_z \omega_0 (r, z) dr - \frac{1}{2} (u_z^2(r_2, z) - u_z^2(r_1, z)) - \int_{r_1}^{r_2} (u_t \partial_r u_r)(r, z) dr
\] (2.13)
\[
+ \int_{r_1}^{r_2} \frac{u_\theta^2}{r}(r, z) dr.
\]

Noting that
\[ |\nabla u| \approx |\partial_t u_r| + |\partial_r u_r| + \left| \frac{u_r}{r} \right| + |\partial_z u_r| + |\partial_z u_\theta| + \left| \frac{u_\theta}{r} \right| + |\partial_z u_z| + |\partial_z u_\theta|, \]
which follows from (A.2), by the boundedness assumption of \( u \) and \( \nabla u \), together with the uniform bound of \( \partial_z \omega_0 \) in Section 2.1.1, one derives the oscillation bound from (2.13):
\[
|p(r_2, z) - p(r_1, z)| \leq C(1 + \|\partial_z \omega_0\|_{L^\infty(D_{3z})}) \leq C < \infty, \quad \forall r_1, r_2 \in [0, 1], \quad z \in \mathbb{R}, \] (2.14)
where \( C \) is an absolute constant which is independent of \( r_1, r_2 \) and \( z \). This finishes the oscillation estimate of \( p(r, z) \) when \( z \) is fixed. Now we turn to the oscillation of \( p \) along the \( z \)-direction. (1.3) and identity
\[ -\Delta u_z = \frac{1}{r} \partial_r (r \omega_0) \]
indicate that
\[
\partial_z p = -\frac{1}{r} \partial_r (r \omega_0) - u_r \partial_r u_z - u_z \partial_z u_z.
\] (2.15)

Multiplying (2.15) by \( r \) and integrating it with respect to \( r \) on \((0, 1)\), one obtains
\[
\frac{d}{dz} \int_0^1 p(r, z) rdr = - \int_0^1 \partial_r (r \omega_0) dr - \int_0^1 (u_t \partial_r + u_z \partial_z)u_r rdr.
\] (2.16)

Recalling the boundary condition (1.5)\(_{2,3} \), we find \( \omega_0 \equiv 0 \) on \( \partial D \), which implies \( P_1 \equiv 0 \). On the other hand, using the divergence-free condition and integration by parts, we derive
\[
P_2 = - \int_0^1 \partial_t (ru_r) u_r dr + \int_0^1 u_z \partial_z u_z r dr
\]
\[
= \int_0^1 \partial_z (ru_r) u_r dr + \frac{1}{2} \frac{d}{dz} \int_0^1 u_z^2 r dr
\]
\[
= \frac{d}{dz} \int_0^1 u_z^2 r dr.
\]
(2.16) indicates
\[ \frac{d}{dz} \int_0^1 p(r, z) r dr = -\frac{d}{dz} \int_0^1 u_z^2(r, z) r dr. \] (2.17)

For any fixed \( z \in [Z, 2Z] \), we integrate the above identity from \( Z \) to \( z \). Then we have
\[ \left| \int_0^1 [p(r, z) - p(r, Z)] r dr \right| \leq \int_0^1 [u_z^2(r, z) - u_z^2(r, Z)] r dr \leq C. \] (2.18)

Recalling the mean value theorem, there exists \( r^* \in [0, 1] \) such that
\[ |p(r^*, z) - p(r^*, Z)| = \frac{\int_0^1 [p(r, z) - p(r, Z)] r dr}{\int_0^1 r dr} \leq C. \] (2.19)

This completes the oscillation of \( p \) parallel to the \( z \)-direction. To conclude the general oscillation of the pressure in the pipe, we apply the triangle inequality: for any \( r \in [0, 1] \), it follows that
\[ |p(r, z) - p(0, Z)| = |p(r, z) - p(r^*, z)| + |p(r^*, z) - p(r^*, Z)| + |p(r^*, Z) - p(r, Z)|. \]

Plugging (2.14) and (2.19) into the above inequality, we finally arrive at
\[ |p(r, z) - p(0, Z)| \leq C, \] (2.20)

where \( C \) is an absolute positive constant independent of \( r, z \) and \( Z \). Thus (2.11) is proved by taking the supremum of (2.20) over \( (r, z) \in [0, 1] \times (-2Z, -Z] \cup [Z, 2Z) \).

\[ \square \]

2.1.3 End of the proof

In this subsection, we will finish the proof of Theorem 1.1. Namely: If the flux \( \Phi \equiv 0 \), any smooth solution of (1.3) in an infinite pipe subject to the Navier total slip condition with the velocity and its first-order derivatives being bounded must be a swirling solution
\[ u = C_1 r e_\theta. \]

The proof is divided into three steps: First we show the stress tensor \( \mathbb{S}u = \frac{1}{2} (\nabla u + (\nabla u)^T) \) is globally \( L^2 \)-integrable. Using a 2D Poincaré inequality and one insightful observation motivated by [35], we then find that \( u_z \) also belongs to \( L^2(D) \). Finally, we arrive at the vanishing of the stress tensor, which indicates the desired result in Theorem 1.1.

\( L^2 \) boundedness of stress tensor

Let \( \psi : \mathbb{R} \to [0, 1] \) be a smooth cut-off function satisfying
\[ \psi(l) = \begin{cases} 1, & l \in [-1, 1], \\ 0, & |l| \geq 2, \end{cases} \]
with $\varphi'$ and $\varphi''$ being bounded. Set
\[\psi_Z(z) := \varphi \left(\frac{z}{Z}\right),\]
where $Z$ is a large positive number. Clearly the derivatives of the scaled cut-off function $\psi_Z$ enjoy
\[|\partial_z^n \psi_Z| \leq \frac{C}{Z^n}, \quad \text{for any } n \in \mathbb{N}. \tag{2.21}\]
Testing the equation
\[u \cdot \nabla u + \nabla p = \Delta u\]
with $u\psi_Z$, we have
\[\int_D \psi_Z u \Delta u \, dx = \int_D \psi_Z u \left(u \cdot \nabla u + \nabla (p - p(0, Z))\right) \, dx. \tag{2.22}\]
To proceed the further calculation in the cylindrical coordinates, we first note that the divergence free property of the velocity indicates
\[\sum_{i,j=1}^{3} \int_{D_{2Z}} \psi_Z u_i \partial_j u_j \, dx = \sum_{i,j=1}^{3} \int_{D_{2Z}} \psi_Z u_i \partial_j (\partial_j u_i + \partial_i u_j) \, dx. \tag{2.23}\]
Below, we use the Einstein summation convention for repeated indexes. Using integration by parts, we further derive
\[\int_{D_{2Z}} \psi_Z u_i \partial_j (\partial_j u_i + \partial_i u_j) \, dx = -\underbrace{\int_{D_{2Z}} \partial_j \psi_Z u_i (\partial_j u_i + \partial_i u_j) \, dx}_{T_1} - \underbrace{\int_{D_{2Z}} \psi_Z \partial_j u_i (\partial_j u_i + \partial_i u_j) \, dx}_{T_2} + \underbrace{\int_{\partial D_{2Z}} \psi_Z u_i n_j (\partial_j u_i + \partial_i u_j) \, dS}_{T_3},\]
where $n_j$ is the $j$-th component of the $n$ – the unit outward normal vector field on $\partial D_{2Z}$. Term $T_1$ could be split into two parts, the first half reads
\[\int_{D_{2Z}} \partial_j \psi_Z u_i \partial_j u_i \, dx = \frac{1}{2} \int_{D_{2Z}} \partial_j \psi_Z \partial_\zeta |u|^2 \, dx = \frac{1}{2} \int_{D_{2Z}} \partial_\zeta \psi_Z \partial_\zeta |u|^2 \, dx = -\frac{1}{2} \int_{D_{2Z}} \partial_\zeta^2 \psi_Z |u|^2 \, dx,\]
where we have used the fact that $\psi_Z$ is only $z$-dependent and supported in $[-2Z, 2Z]$. Similarly, the second half of $T_1$ follows that
\[\int_{D_{2Z}} \partial_j \psi_Z u_i \partial_i u_j \, dx = \int_{\partial D_{2Z}} (u \cdot \nabla \psi_Z)(u \cdot n) \, dS - \int_{D_{2Z}} \partial_\zeta^2 \psi_Z u_\zeta^2 \, dx.\]
Due to the impermeable condition, one sees the first term on the right hand of the above equality is zero. Thus we conclude that
\[T_1 = -\int_{D_{2Z}} \partial_\zeta^2 \psi_Z \left(\frac{1}{2} |u|^2 + u_\zeta^2\right) \, dx. \tag{2.24}\]
Recalling that the stress tensor is defined by
\[
\mathbb{S} u = \frac{1}{2} (\partial_j u_i + \partial_i u_j)_{1 \leq i, j \leq 3},
\]
and using its symmetry, we arrive that
\[
T_2 = \frac{1}{2} \sum_{i,j=1}^{3} \int_{D_{2Z}} \psi Z (\partial_j u_i + \partial_i u_j)^2 \, dx = 2 \int_{D_{2Z}} \psi Z |\mathbb{S} u|^2 \, dx.  \tag{2.25}
\]

Now applying the Navier-slip condition \[(NSB)_{1}\], one notes that
\[
n_j (\partial_j u_i + \partial_i u_j) = c(x) n_i,
\]
where \(c(x)\) is a scalar-valued function. Inserting this identity to \(T_3\), we find
\[
T_3 = \int_{\partial D_{2Z}} c \psi Z (u \cdot n) \, dS = 0. \tag{2.26}
\]

Next we come back to the right hand side of \( (2.22) \). Noting \( u \) is divergence-free, integration by parts shows
\[
\begin{align*}
\int_{D_{2Z}} u \psi Z (u \cdot \nabla (p - p(0, Z))) \, dx &= \int_{D_{2Z}} \psi Z u_i \partial_i \left( \frac{1}{2} |u|^2 + [p - p(0, Z)] \right) \, dx \\
&= \int_{\partial D_{2Z}} \psi Z (u \cdot n) \left( \frac{1}{2} |u|^2 + [p - p(0, Z)] \right) \, dS - \int_{D_{2Z}} \partial_z \psi Z u_z \left( \frac{1}{2} |u|^2 + [p - p(0, Z)] \right) \, dx. \tag{2.27}
\end{align*}
\]

Here \( T_4 \) above also vanishes by the stationary wall condition \((1.5)_3\). Therefore we arrive that by plugging \((2.24), (2.25), (2.26), (2.27)\) into \((2.22)\)
\[
2 \int_{D_{2Z}} \psi Z |\mathbb{S} u|^2 \, dx = \int_{D_{2Z}} \partial_z \psi Z \left( \frac{1}{2} |u|^2 + u_z^2 \right) \, dx + \int_{D_{2Z}} \partial_z \psi Z u_z \left( \frac{1}{2} |u|^2 + [p - p(0, Z)] \right) \, dx. \tag{2.28}
\]

Recalling \((2.21)\), the bounds on the derivatives of scaled cut-off function \(\psi Z\), and the boundedness of \( u \) and pressure, one derives from \((2.28)\) that
\[
\int_{D_{2Z}} \psi Z |\mathbb{S} u|^2 \, dx \leq C|D_{2Z}| (Z^{-2} + Z^{-1}) \leq C,
\]
where \( C \) is a universal constant depending only on the \( L^\infty \) bound of \( u \) and \( \nabla u \) given in the assumption. After letting \( Z \to \infty \), the above inequality shows the stress tensor is globally \( L^2 \)-integrable:
\[
\int_D |\mathbb{S} u|^2 \, dx \leq C < \infty. \tag{2.29}
\]
**$L^2$ boundedness of $u_z$**

First we observe that $\|u_z\|_{L^2(\Sigma)}$ can be controlled by $\|\partial_z u_z\|_{L^2(\Sigma)}$ under the assumption that the flux $\Phi = 0$. Noting that

$$\frac{1}{|\Sigma|} \int_{\Sigma} u_z(x_h, z) dx_h = \frac{1}{|\Sigma|} \Phi = 0,$$

then we apply the one dimensional Poincaré inequality to derive

$$\int_{\Sigma} |u_z(r, z)|^2 dx_h = \int_{\Sigma} \left| u_z(x_h, z) - \frac{1}{|\Sigma|} \int_{\Sigma} u_z(x_h, z) dx_h \right|^2 dx_h \leq S_0^2 \int_{\Sigma} |\nabla_h u_z(x_h, z)|^2 dx_h,$$

where $\nabla_h = (\partial_1, \partial_2)$ and $S_0$ is independent of $z \in \mathbb{R}$. Integrating with $z$-variable on $\mathbb{R}$, we arrive

$$\|u_z\|_{L^2(\Sigma)} \leq S_0 \|\partial_z u_z\|_{L^2(\Sigma)}, \quad (2.30)$$

However, we cannot get the $L^2$ boundedness of $\partial_z u_z$ directly from (2.29). In fact, by the expression of the stress tensor $(A.4)$, one only has the uniform $L^2$ bound of $(\partial_r u_r + \partial_z u_z)$. Nevertheless, one observes

$$\int_{D_{2Z}} (\partial_r u_z)^2 \, dx = \int_{D_{2Z}} (\partial_r u_r + \partial_z u_z)^2 \, dx - \int_{D_{2Z}} (\partial_z u_r)^2 \, dx - 2 \int_{D_{2Z}} \partial_r u_r \partial_z u_r \, dx \leq C + 2 \left| \int_{D_{2Z}} \partial_r u_r \partial_z u_r \, dx \right|.$$ 

Now it remains to derive the boundedness of $T_6$. With idea motivated by [35], after using the divergence free of $u$ and integration by parts, we deduce

$$\int_{D_{2Z}} \partial_r u_z \partial_z u_r \, dx = -2\pi \int_{-2Z}^{2Z} \int_0^1 u_z \partial_z^2(r u_r) dr dz - 2\pi \int_{-2Z}^{2Z} \int_0^1 u_z \partial_z^2(r u_z) dr dz$$

$$\quad = -\left( \int_{D_{2Z}} (\partial_z u_r)^2 \, dx + 2\pi \left( \int_0^1 u_z(2Z) \partial_z u_z(2Z) r dr - \int_0^1 u_z(-2Z) \partial_z u_z(-2Z) r dr \right) \right).$$

Here $T_7$ can be bounded by the $L^2$ norm of stress tensor (2.29), while $T_8$ is controlled by the $L^\infty$ bounds of $u$ and $\nabla u$. Noting that $T_6$ is estimated uniformly with respect to $Z$. This, together with (2.30) implies

$$\|u_z\|_{L^2(\Sigma)} \leq C < \infty.$$

**Vanishing of $\int_{D} |\mathbb{A} u|^2$ and finishing of the proof**
Based on the $L^2$ bound of $u_z$, now we can estimate $T_5$ in (2.28) in an alternative approach, by using Hölder inequality:

$$|T_5| \leq \sup_{x \in \mathcal{D}_{2Z}} \left| \frac{1}{2} |u|^2 + \left[ p - p(0,Z) \right] \right| C \frac{\|u_z\|_{L^1(\mathcal{D}_{2Z})}}{Z} \mathcal{D}_{2Z}^{1/2} \leq CZ^{-1/2}.$$ 

Thus we deduce from (2.28)

$$\int_{\mathcal{D}_{2Z}} \psi_Z |\mathbb{S} u|^2 dx \leq C \mathcal{D}_{2Z} Z^{-2} + CZ^{-1/2} \to 0, \quad \text{as } Z \to +\infty,$$

which indicates that

$$\int_{\mathcal{D}} |\mathbb{S} u|^2 dx = 0 \quad (2.31)$$

by letting $Z \to \infty$. By the expression of $\mathbb{S} u$ (A.4), one finds

$$u_r \equiv \partial_z u_\theta \equiv \partial_z u_z \equiv \partial_r u_z \equiv 0, \quad \partial_r u_\theta = \frac{u_\theta}{r}.$$ 

The above estimates, together with the vanishing flux ($\Phi = 0$), indicate

$$u_z \equiv 0, \quad \text{and} \quad u_\theta = Cr.$$ 

Thus we conclude that $u = C r e_\theta$, which is a swirling solution.

\[\square\]

### 2.2 Proof of Case II

If $u$ is $z$-periodic with the minimal period $L > 0$, then we can drop the restriction $\Phi = 0$ in Case I. The proof is straightforward. Set $w = u - \frac{\Phi}{\pi} e_z$, then $(w, p)$ satisfies the following system:

$$\begin{cases}
(w + \frac{\Phi}{\pi} e_z) \cdot \nabla w + \nabla p - \Delta w = 0, & \text{in } \mathcal{D}, \\
\nabla \cdot w = 0, & \text{in } \mathcal{D}, \\
w(x_h, z) = w(x_h, z + L), & \text{in } \mathcal{D}, \\
(\mathbb{S} w \cdot n)_r = 0, \quad w \cdot n = 0, & \text{on } \partial \mathcal{D}.
\end{cases} \quad (2.32)$$

We first claim that, the pressure $p(x_h, z)$ has the following decomposition:

$$p(x_h, z) = az + \hat{p}(x_h, z), \quad (2.33)$$

where $a$ is a constant, and $\hat{p}$ is $z$-periodic with the minimal period $L > 0$. Set

$$a_0(x_h) = \frac{1}{L} \int_0^L (\partial_z p)(x_h, z) dz,$$

we decompose $\partial_z p$ as

$$(\partial_z p)(x_h, z) = a_0(x_h) + ((\partial_z p)(x_h, z) - a_0(x_h)) := a_0(x_h) + p_1(x_h, z). \quad (2.34)$$
By using the equation and $z$–periodicity of the solution, it is easy to check that $p_1(x_h, z)$ is $L$-periodic with respect to $z$ and that

$$\int_0^L p_1(x_h, \tilde{z}) \, d\tilde{z} = 0. \tag{2.35}$$

Integrating (2.34) with $z$ on $[0, z]$, one derives that

$$p(x_h, z) = p(x_h, 0) + a_0(x_h)z + \int_0^z p_1(x_h, \tilde{z}) \, d\tilde{z}. \tag{2.36}$$

It is worth noting that $\int_0^z p_1(x_h, \tilde{z}) \, d\tilde{z}$ is periodic in the $z$-direction due to (2.35) and periodicity of $p_1$. Hence,

$$\tilde{p}(x_h, z) = p(x_h, 0) + \int_0^z p_1(x_h, \tilde{z}) \, d\tilde{z}$$

is periodic in the $z$-direction, and

$$p(x_h, z) = a_0(x_h)z + \tilde{p}(x_h, z).$$

Finally, also from the equations and $z$-periodicity of the solution, we deduce $\nabla_h p = (\nabla_h a_0)z + (\nabla_h \tilde{p})$ is periodic with respect to $z$. Thus we get

$$a_0(x_h) = \text{constant} := a$$

since $\nabla_h a_0$ must be zero. Therefore we conclude

$$p(x_h, z) = az + \tilde{p}(x_h, z).$$

This finishes the proof of the claim.

Next, multiplying $w$ on both sides of (2.32)₁, and integrating on $\Sigma \times [0, L]$, one has

$$\int_{\Sigma \times [0, L]} w \cdot \Delta w \, dx = \int_{\Sigma \times [0, L]} w \cdot \left( w + \frac{\Phi}{\pi} e_z \right) \cdot \nabla w + \nabla p \right) \, dx. \tag{2.37}$$

It follows from (2.32)₂–(2.32)₄ that

$$\int_{\Sigma \times [0, L]} w \cdot \Delta w \, dx = -2 \int_{\Sigma \times [0, L]} |\mathbb{S}w|^2 \, dx,$$

where we have used the technique from (2.23) to (2.25) and $z$-periodicity of $w$ to do integration by parts. There are no boundary terms generated. Also

$$\int_{\Sigma \times [0, L]} w \cdot \left( w + \frac{\Phi}{\pi} e_z \right) \cdot \nabla w + \nabla p \right) \, dx = 0,$$

where we have used the decomposition (2.33) and $\int_S w_3 \, dx_h = 0$ to deal with the pressure term. Hence, we have that

$$\int_{\Sigma \times [0, L]} |\mathbb{S}w|^2 \, dx = 0,$$
which deduces that $\mathcal{S} w = 0$. It is well known that if $\mathcal{S} w = 0$, then $w$ has the form $w = A x + B$ (see [16 §6]), where $A$ is a skew-symmetric matrix with constant entries and $B$ is a constant vector, that is,

$$w = \begin{pmatrix} 0 & -a_1 & -a_2 \\ a_1 & 0 & -a_3 \\ a_2 & a_3 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ z \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} -a_1 x_2 - a_2 z + b_1 \\ a_1 x_1 - a_3 z + b_2 \\ a_2 x_1 + a_3 x_2 + b_3 \end{pmatrix},$$

where $a_i, b_i (i = 1, 2, 3)$ are some constants. Note that $w$ is periodic with respect to $z$, we have that $a_2 = a_3 = 0$. Next, from $\int_{\Sigma} w_3 \, dx_h = 0$, we obtain that $b_3 = 0$. Finally, note that $w_r = 0$ on $r = 1$, we get that $b_1 = b_2 = 0$. Thus, we have proved that

$$w = \begin{pmatrix} -a_1 x_2 \\ a_1 x_1 \\ 0 \end{pmatrix} = a_1 r e_\theta.$$

Therefore, $u = w + \frac{\Phi}{\pi} e_z = a_1 r e_\theta + \frac{\Phi}{\pi} e_z$.

Let us give some discussions of Theorem 1.1 here. Based on our previous proof in this section, we naturally believe that if the vanishing of $\Phi$ is abandoned, then an axially symmetric solution must be a helical solution:

$$u = C_1 r e_\theta + C_2 e_z \tag{2.36}$$

even without the $z$-periodic condition. However, our method in this paper fails when we handle solutions with the flux $\Phi \neq 0$, because we can no longer apply the Poincaré inequality in Section 2.1.3 to derive the $L^2$ integrability of $u_z$. Meanwhile, if we denote

$$c_0 := \frac{1}{|\Sigma|} \int_{\Sigma} u_z(x_h, z) \, dx_h = \frac{1}{|\Sigma|} \Phi,$$

then $u_z - c_0$ enjoys a similar Poincaré inequality as (2.30):

$$\|u_z - c_0\|_{L^2(D)} \leq S_0 \|\partial_r u_z\|_{L^2(D)},$$

which guarantees the $L^2$ boundedness of $u_z - c_0$. However, one additional term appears in $T_5'$ of (2.28), which is:

$$T_5' := c_0 \int_D \partial_z \psi_z \left( \frac{1}{2} |u|^2 + [p - p(0, Z)] \right) \, dx.$$

Without any integrability of the head pressure $\frac{1}{2} |u|^2 + [p - p(0, Z)]$, we can only show $T_5'$ is bounded, which results in

$$\int_D |\mathcal{S} u|^2 \, dx < C < \infty.$$

However, we are unable to conclude $T_5' \to 0$ as $Z \to \infty$, thus vanishing of $\int_D |\mathcal{S} u|^2 \, dx$ can not be obtained. In fact, using integration by parts on $z$ in $T_5'$, we have

$$T_5' = -c_0 \int_D \psi_z \partial_z \left( \frac{1}{2} |u|^2 + p \right) \, dx.$$
By following the argument in Section 2, one derives
\[ \int_D |\mathbb{S}u|^2 \, dx = -\lim_{Z \to \infty} \frac{C_0}{2} \int_D \psi_Z \partial_z \left( \frac{1}{2} |u|^2 + p \right) \, dx \]
instead of (2.31). Recalling (2.17), one deduces that
\[ \int_D |\mathbb{S}u|^2 \, dx = -\lim_{Z \to \infty} \frac{C_0}{4} \int_D \psi_Z \partial_z \left( u_r^2 + u_\theta^2 - u_z^2 \right) \, dx. \]  
(2.37)
Thus if \( \partial_z \left( u_r^2 + u_\theta^2 - u_z^2 \right) \in L^1(D) \) (or \( \partial_z \left( u_r^2 + u_\theta^2 - u_z^2 \right) \) has a fixed sign), one concludes the following identity by Lebesgue’s dominated convergence theorem (or monotone convergence theorem):
\[ \int_D |\mathbb{S}u|^2 \, dx + \frac{C_0}{4} \int_D \partial_z \left( u_r^2 + u_\theta^2 - u_z^2 \right) \, dx = 0. \]  
(2.38)

At the moment, even with identities (2.37) and (2.38) for bounded (up to first-order derivatives) smooth axisymmetric solutions of stationary Navier-Stokes equations in \( D \) subject to the total Navier-slip boundary condition in hand, we neither show the trivialness of \( \mathbb{S}u \), nor find a nontrivial bounded solution apart from (2.36) which satisfies conditions of Theorem 1.1. Indeed, we leave characterization of the non-zero flux solutions in Conjecture 1.6.

Nevertheless, a direct observation of (2.38) indicates that: If \( u \) is independent of \( z \), then the right hand side of (2.38) vanishes and we can conclude \( \mathbb{S}u \equiv 0 \), and thus conclude that \( u = C_1 r e_\theta + C_2 e_z \) as we desire. In the next section, we see that only \( u_\theta \) or \( u_z \) being independent of \( z \) is adequate for us to derive Theorem 1.3. Besides, the asymptotic assumptions of \( u \) and its derivatives can be largely loosened.

3 Proof of Theorem 1.3

3.1 Proof of Case I

Let us outline the proof at the beginning of this section: Under the assumptions of Case I in Theorem 1.3 our first step is showing \( \omega_\theta \equiv 0 \), which indicates \( b = u_\theta e_r + u_r e_z \) must be harmonic in \( D \). Then by applying the boundary condition and the asymptotic behavior of \( b \), one derives \( u_r \equiv 0 \) and \( u_z \) must be a constant. From then on (1.3) turns to a linear ordinary differential equation of \( u_\theta \), and we finally prove \( u_\theta = C_1 r \).

3.1.1 Vanishing of \( \omega_\theta \)

Noting that \( u_\theta \) is independent of \( z \), we find (1.4) now turns to
\[ (u_r \partial_r + u_z \partial_z) \omega_\theta - \left( \Delta - \frac{1}{r^2} \right) \omega_\theta - \frac{u_r}{r} \omega_\theta = 0. \]

From the Navier-slip boundary condition (1.5), one has
\[ \omega_\theta = \partial_z u_r - \partial_r u_z = 0, \quad \text{on } \partial D. \]
Denoting $\Omega := \frac{\omega}{r}$, direct calculation shows

\[
\begin{cases}
(u_r \partial_r + u_z \partial_z) \Omega - \left(\Delta + \frac{2}{r} \partial_r\right) \Omega = 0, & \text{in } \mathcal{D}; \\
\Omega = 0, & \text{on } \partial\mathcal{D}.
\end{cases}
\]

In the following, we first provide a mean value inequality of $\Omega$ deduced by Moser’s iteration.

**Lemma 3.1.** Assume $b = u_r e_r + u_z e_z$ is a smooth divergence-free axially symmetric vector field. Then any weak solution $\Omega$ of boundary value problem (3.1) satisfies the following mean value inequality:

\[
\sup_{x \in \mathcal{D}} |\Omega| \leq C_q (\tau_1 - \tau_2)^{-\frac{q}{2}} \left(1 + \|u_z\|_{L^\infty(D_z \setminus D_z/2)}\right)^{\frac{q}{2}} Z^{-\frac{q}{2}} \left(\int_{\mathcal{D}_1} |\Omega|^2 dx\right)^{\frac{1}{2}},
\]

for any $q > 2$, $Z > 1$, and $\frac{1}{2} \leq \tau_2 < \tau_1 \leq 1$.

**Proof.** We only prove (3.2) with $\tau_1 = 1$, $\tau_2 = \frac{1}{2}$ for simplicity, since the general case could be derived by a direct scaling strategy. For any real number $l \geq 1$, we find $h := \Omega^l$ satisfies

\[
\Delta h - l(l - 1)\Omega^{l-2} |\nabla \Omega|^2 + \frac{2}{r} \partial_r h - b \cdot \nabla h = 0.
\]

Set $\frac{1}{2} \leq \sigma_2 < \sigma_1 \leq 1$ and choose $\zeta = \zeta(z)$ to be a smooth cut-off function satisfying

\[
\begin{cases}
\text{supp } \zeta \subset [-\sigma_1, \sigma_1], & \zeta = 1 \text{ in } [-\sigma_2, \sigma_2], \\
0 \leq \zeta \leq 1, & \frac{1}{\sigma_1 - \sigma_2} \leq |\zeta'|.
\end{cases}
\]

Denoting $\zeta_Z(z) := \zeta(\frac{z}{Z})$ and testing (3.3) with $\zeta_Z^2 h$, noting that

\[
l(l - 1) \int_{\mathcal{D}_1} \Omega^{2l-2} |\nabla \Omega|^2 \zeta_Z^2 dx = \frac{l - 1}{l} \int_{\mathcal{D}_1} |\nabla \Omega|^2 \zeta_Z^2 dx \geq 0,
\]

we arrive

\[
\int_{\mathcal{D}_1} \Delta h \zeta_Z^2 h dx + \int_{\mathcal{D}_1} \frac{2}{r} \partial_r h \zeta_Z^2 h dx - \int_{\mathcal{D}_1} b \cdot \nabla h \zeta_Z^2 h dx \geq 0.
\]

Next we handle $M_1$–$M_3$ term by term. Using integration by parts and direct calculations, we first see

\[
M_1 = - \int_{\mathcal{D}_1} \nabla h \cdot \nabla (\zeta_Z^2 h) dx = - \int_{\mathcal{D}_1} |\nabla (h \zeta_Z)|^2 dx + \int_{\mathcal{D}_1} h^2 |\zeta_Z'|^2 dx.
\]
Here the boundary term of the cylindrical surface is cancelled because \( h = 0 \) on \( \partial \mathcal{D} \), while those coming from the cross sections \( D \cap \{ z = \pm \sigma_1 Z \} \) vanish due to the cut off function \( \zeta_Z \) is compactly supported. On the other hand, using axisymmetry of the solution

\[
M_2 = 2\pi \int_{\mathbb{R}} \int_0^1 \partial_r (h^2 \zeta_Z^2) dr dz = -2\pi \int_{\mathbb{R}} h^2 (0, z) \zeta_Z^2 (z) dz \leq 0. \quad (3.6)
\]

Before we bound \( M_3 \), let us introduce the *stream function* of axisymmetric velocity field \( b = u, e_r + u, e_z \). By the divergence-free property \( \partial_r (ru_r) + \partial_z (ru_z) = 0 \), there exists a scalar function \( L_\theta = L_\theta (r, z) \) such that

\[
- \partial_r L_\theta = u_r, \quad \text{and} \quad \frac{1}{r} \partial_r (rL_\theta) = u_z. \quad (3.7)
\]

Using integration by parts again together with boundary condition \( h = 0 \) on \( \partial \mathcal{D} \), we derive that

\[
M_3 = \frac{1}{2} \int_{D_{r_1} \Sigma} b \cdot \nabla h^2 \zeta_Z^2 dx = - \int_{D_{r_1} \Sigma} u_z \zeta_Z^2 h^2 dx = -2\pi \int_{\mathbb{R}} \int_0^1 \partial_r (rL_\theta) \zeta_Z \zeta_Z^\prime h^2 dr dz = 4\pi \int_{\mathbb{R}} \int_0^1 (rL_\theta) \partial_r (h \zeta_Z) h \zeta_Z^\prime dr dz.
\]

By the mean value theorem and (3.7), there exists \( \tilde{r} \in (0, r) \) such that

\[
rL_\theta (r, z) = \tilde{r} u_z (\tilde{r}, z) r,
\]

thus we can further bound \( M_3 \) by

\[
|M_3| \leq 4\pi ||u_z||_{L^\infty (D_{r_1} \Sigma)} \int_{\mathbb{R}} \int_0^1 |\nabla (h \zeta_Z) h \zeta_Z^\prime | r dr dz \
\leq \frac{1}{2} \int_{D_{r_1} \Sigma} |\nabla (h \zeta_Z)|^2 dx + 2 ||u_z||_{L^\infty (D_{r_1} \Sigma)} \int_{D_{r_1} \Sigma} h^2 \zeta_Z^\prime |^2 dx. \quad (3.8)
\]

Now substituting (3.5), (3.6), and (3.8) in (3.4), taking the maximum of \( \zeta_z \), it follows that

\[
\int_{D_{r_1} \Sigma} |\nabla (h \zeta_Z)|^2 dx + 2\pi \int_{\mathbb{R}} h^2 (0, z) \zeta_Z^2 (z) dz \leq C \left( 1 + ||u_z||_{L^\infty (D_{r_1} \Sigma)}^2 \right) \int_{D_{r_1} \Sigma} h^2 dx. \quad (3.9)
\]

Recalling \( h = 0 \) on \( \partial \mathcal{D} \), for any fixed \( z \in \mathbb{R} \), the following 2D Poincaré inequality holds:

\[
||h(\cdot, z) \zeta_Z (z)||_{L^2 (\Sigma)}^2 \leq C ||\partial_r [h(\cdot, z) \zeta_Z (z)]||_{L^2 (\Sigma)}^2,
\]

where \( C > 0 \) here is independent of \( z \). Integrating with \( z \) on \( \mathbb{R} \) and taking the square root, one has the following 3D Poincaré inequality

\[
||h \zeta_Z ||_{L^2 (D_{r_1} \Sigma)} \leq C ||\partial_r (h \zeta_Z)||_{L^2 (D_{r_1} \Sigma)}. \quad (3.10)
\]
For any \( q \in (2, 6) \), Interpolation, Sobolev inequality and (3.10) imply that
\[
\| h \xi_z \|_{L^q(\mathcal{D}_{\sigma_1 z})} \leq \| h \xi_z \|_{L^6(\mathcal{D}_{\sigma_1 z})} \| h \xi_z \|_{L^{1/4}(\mathcal{D}_{\sigma_1 z})} \leq C \| \nabla (h \xi_z) \|_{L^2(\mathcal{D}_{\sigma_1 z})} \| h \xi_z \|_{L^{1/4}(\mathcal{D}_{\sigma_1 z})} \\
\leq C \| \nabla (h \xi_z) \|_{L^2(\mathcal{D}_{\sigma_1 z})} \| \partial_z (h \xi_z) \|_{L^{1/4}(\mathcal{D}_{\sigma_1 z})} \leq C \| \nabla (h \xi_z) \|_{L^2(\mathcal{D}_{\sigma_1 z})}.
\]
(3.11)

Here \( s \in (0, 1) \) depends on \( q \). Combining (3.9) and (3.11), we derive
\[
\| h \|_{L^q(\mathcal{D}_{\sigma_1 z})} \leq \frac{C}{(\sigma_1 - \sigma_2)^Z} \| h \|_{L^2(\mathcal{D}_{\sigma_1 z})},
\]
which is equivalent to
\[
\left( \int_{\mathcal{D}_{\sigma_1 z}} |\Omega|^\phi dx \right)^{\frac{1}{\phi}} \leq \frac{C^{1/\phi} (1 + \| u_z \|_{L^\infty(\mathcal{D}_{\sigma_1 z})})^{1/\phi}}{(\sigma_1 - \sigma_2)^Z} \left( \int_{\mathcal{D}_{\sigma_1 z}} |\Omega|^\phi dx \right)^{\frac{1}{\phi}}.
\]
(3.12)

Now for any \( k = 0, 1, 2, \ldots \), we choose \( l_k = \left( \frac{k}{2} \right)^k \) and \( \sigma_{1k} = \frac{1}{2} + \left( \frac{1}{2} \right)^{k+1} \), \( \sigma_{2k} = \frac{1}{2} + \left( \frac{1}{2} \right)^{k+2} \), respectively.

Denoting
\[
\Psi_k := \left( \int_{\mathcal{D}_{\sigma_{1k} z}} |\Omega|^{2l_k} dx \right)^{\frac{1}{2l_k}},
\]
and noting that
\[
\mathcal{D}_{\sigma_{1k} z} \setminus \mathcal{D}_{\sigma_{2k} z} \subset \mathcal{D}_z \setminus \mathcal{D}_{Z/2}, \quad \forall k = 0, 1, 2, \ldots,
\]
then (3.12) follows that
\[
\Psi_{k+1} \leq C \left( \frac{1}{2} \right)^{k} \left( 1 + \| u_z \|_{L^\infty(\mathcal{D}_{\sigma_{1k} z})} \right) \left( \frac{1}{2} \right)^{k} \left( \frac{1}{2} \right)^k \Psi_k \\
\leq \cdots
\]
\[
\leq C \sum_{j=0}^{\infty} \left( \frac{1}{2} \right)^{j} \left( 1 + \| u_z \|_{L^\infty(\mathcal{D}_{\sigma_{1j} z})} \right) \sum_{j=0}^{\infty} \left( \frac{1}{2} \right)^{j} \left( \frac{1}{2} \right)^j \Psi_0.
\]
(3.13)

Performing \( k \to \infty \), then iteration (3.13) implies a mean value inequality of \( \Omega \), that is
\[
\sup_{x \in \mathcal{D}_{Z/2}} |\Omega|^q \leq C_q \left( 1 + \| u_z \|_{L^\infty(\mathcal{D}_{\sigma_{1k} z})} \right) \left( \frac{1}{2} \right)^k Z^{-\frac{q}{2k}} \left( \int_{\mathcal{D}_z} |\Omega|^\phi dx \right)^{\frac{1}{\phi}},
\]
for any \( q > 2 \). This completes the proof of Lemma 2.2.

Since \( u_z \) satisfies (1.9) in \( \mathcal{D}_z \), (3.2) indicates that
\[
\sup_{x \in \mathcal{D}_{Z/2}} |\Omega|^2 \leq C_q (\tau_1 - \tau_2)^{-\frac{2q}{\phi_2}} Z^{-\frac{2q}{\phi_2}} \left( \int_{\mathcal{D}_z} |\Omega|^\phi dx \right)^{\frac{1}{\phi}}.
\]
(3.14)

However, due to the lack of boundedness of the second-order derivatives of \( u \), we are unable to control \( |\Omega|^2 \) at the moment. Next we will use an algebraic trick to convert the \( L^2 \)-norm on the right hand side of (3.14) to an \( L^1 \)-norm. This trick comes from Li-Schoen [21]. Here goes the lemma:
Lemma 3.2 (modified mean value inequality). Suppose \( b = u_r e_r + u_z e_z \) is a smooth divergence-free axisymmetric vector field and \( \|u_r\|_{L^\infty(D_Z)} \leq Z^{\delta_0} \). Then any weak solution \( \Omega \) of boundary value problem (3.1) satisfies the following mean value inequality for any \( q > 2, Z > 1 \):

\[
\sup_{x \in D_{Z/2}} |\Omega| \leq C_q Z^{\frac{(\delta_0-1)q}{q-2}} \int_{D_Z} |\Omega| dx.
\] (3.15)

**Proof.** For any \( \frac{1}{2} \leq \tau_2 < \tau_1 \leq 1 \), (3.14) implies that

\[
\sup_{x \in D_{\tau_2 Z}} |\Omega|^2 \leq C_q (\tau_1 - \tau_2)^{-\frac{2q}{q-2}} Z^{\frac{2(\delta_0-1)q}{q-2}} \left( \sup_{x \in D_{\tau_1 Z}} |\Omega|^2 \right)^{1/2} \int_{D_Z} |\Omega| dx.
\]

Denoting \( \tau_{1k} = \tau_{2,k+1} = 1 - \left( \frac{1}{2} \right)^{k+1}, \tau_{2k} = 1 - \left( \frac{1}{2} \right)^{k+1} \), and \( \Phi_k := \sup_{x \in D_{\tau_1 Z}} |\Omega|^2 \), it follows that

\[
\Phi_k \leq C_q 2^{\frac{2k}{q-2}} Z^{\frac{2(\delta_0-1)q}{q-2}} \Phi_{k+1}^{1/2} \int_{D_Z} |\Omega| dx.
\] (3.16)

Iterating (3.16) from \( k = 0 \) to infinity, one arrives

\[
\sup_{x \in D_{Z/2}} |\Omega|^2 \leq C_q \sum_{j=0}^{\infty} 2^{-j} \sum_{j=0}^{\infty} \left( Z^{\frac{(\delta_0-1)q}{q-2}} \right)^j \left( \int_{D_Z} |\Omega| dx \right)^{1/j} \leq C_q Z^{\frac{2(\delta_0-1)q}{q-2}} \left( \int_{D_Z} |\Omega| dx \right)^2.
\]

which follows that

\[
\sup_{x \in D_{Z/2}} |\Omega| \leq C_q Z^{\frac{\delta_0-1}{q-2}} \int_{D_Z} |\Omega| dx.
\]

This completes the proof of Lemma 3.2

\[ \square \]

Finally, one notes that

\[
\int_{D_Z} |\Omega| dx \leq 2\pi \|\omega_0\|_{L^\infty(D_Z)} \int_{0}^{Z} \int_{-Z}^{1} r dr dz \leq Z^{M_0+1}.
\]

Therefore, as long as \( \omega_0 \) is of polynomial growth (see (1.9)) when \( z \to \infty \), we can infer from (3.15) that

\[
\sup_{x \in D_{Z/2}} |\Omega| \leq C_q Z^{\frac{\delta_0-1}{q-2}+1+M_0}.
\] (3.17)

For any fixed \( \delta_0 < 1 \) and \( M_0 > 0 \), we can always choose \( q \) which is larger than but close enough to 2 such that (3.17) indicates

\[
\sup_{x \in D_{Z/2}} |\Omega| \lesssim Z^{-\gamma}
\]

for some \( \gamma > 0 \). This proves \( \omega_0 \) vanishes in \( D \) by performing \( Z \to \infty \).
3.1.2 Vanishing of $u_r$ and constancy of $u_z$

Noting that $\nabla \times b = \omega_\theta e_\theta \equiv 0$ and the divergence-free property of $b$, we apply the Lagrange’s formula for del to deduce

$$-\Delta b = \nabla \times \nabla \times b - \nabla (\text{div } b) = 0,$$

which indicates

$$\left( \Delta - \frac{1}{r^2} \right) u_r = 0; \quad \Delta u_z = 0.$$

To prove vanishing of $u_r$, for $\delta > 0$ being small, we consider the auxiliary function $\eta_\delta$ which is defined by

$$\eta_\delta(x) := J_1((\alpha - \delta)r) \cosh((\alpha - \delta)z).$$

Here $J_1$ is the Bessel function which is defined in (1.11) and satisfies (1.10) with $\beta = 1$, while $\alpha$ is the smallest positive root of $J_1$. Direct calculation shows

$$\left( \Delta - \frac{1}{r^2} \right) \eta_\delta = \left( \partial_r^2 + \frac{1}{r} \partial_r + \partial_z^2 - \frac{1}{r^2} \right) \eta_\delta = 0.$$

Owing to $u_r$ is growing as (1.9), we choose $\delta << 1$ small enough such that $\gamma_0 < \alpha - 2\delta$. Using the concavity of $J_1((\alpha - \delta)r)$ on the subset of $\{r : 0 \leq r \leq 1\}$ where $J_1((\alpha - \delta)r)$ is increasing, one has

$$\eta_\delta \geq J_1(\alpha - \delta)r \cosh((\alpha - \delta)z) \geq C_\delta re^{(\gamma_0+\delta)|z|},$$

where $C_\delta > 0$ is a constant depends only on $\delta$. Then the condition (1.9) indicates that

$$\lim_{|z| \to \infty} \frac{|u_r(r, z)|}{\eta_\delta(r, z)} = 0, \quad \text{uniformly with } r = \sqrt{x_1^2 + x_2^2} \in [0, 1].$$

Therefore, for any fixed $\varepsilon > 0$ and $\delta$, there exists an $N_{\varepsilon, \delta} > 0$ such that

$$\begin{cases} \left( \Delta - \frac{1}{r^2} \right) (\varepsilon \eta_\delta \pm u_r) = 0, & \forall x \in D_M, \\ \varepsilon \eta_\delta \pm u_r \geq 0, & \forall x \in \partial D_M = [\partial D \cap \{-M \leq z \leq M\}] \cup [D \cap \{z = \pm M\}], \end{cases}$$

for any $M > N_{\varepsilon, \delta}$. The maximum principle indicates

$$|u_r(x)| \leq \varepsilon \eta_\delta(x), \quad \forall x \in D_M. \quad (3.18)$$

By performing $M \to \infty$, one finds the estimate (3.18) actually holds for all $x \in D$. Thus $u_r \equiv 0$ is proved by the arbitrariness of $\varepsilon > 0$.

Finally, the divergence-free of $u$ implies $\partial_z u_z = -\frac{1}{r} \partial_r (ru_r) \equiv 0$ in $D$. The vanishing of $\omega_\theta$ and $u_r$ indicates $\partial_z u_z \equiv 0$. Thus $u_z$ must be a constant. This consequently indicates

$$b = C_2 e_z \quad (3.19)$$

for some constant $C_2 \in \mathbb{R}$. 

3.1.3 End of the proof

Now substituting (3.19) in (1.3) and noting that \( u_\theta \) is independent of \( r \), one arrives the following ODE of \( u_\theta \)
\[
u''_\theta(r) + \frac{1}{r} u'_\theta(r) - \frac{1}{r^2} u_\theta(r) = 0.
\]
This ODE, which is of Eulerian type, is solved by
\[
u_\theta(r) = \frac{C_0}{r} + C_1 r,
\]
for any \( C_0, C_1 \in \mathbb{R} \). Smoothness of \( u_\theta \) forces that \( C_0 = 0 \). Thus we conclude that
\[u = u_\theta \theta + b = C_1 r e_\theta + C_2 e_z,
\]
which completes the proof of Theorem 1.3.

Remark 3.3. Unlike Theorem 1.1, Theorem 1.3 actually needs weaker assumptions, (1.9), on the boundedness of solutions. As stated in the introduction, assumption (1.9) is sharp due to the non-trivial counterexamples in (1.12) which grow no slower than \( C e^{\alpha|z|} \) as \( z \to \infty \). Meanwhile, the counterexample in (1.12) has zero vorticity and zero flux in the cross section \( \Sigma \). Identities (2.37) and (2.38) no longer hold for the solution in (1.12) since we have no boundedness of the head pressure \( H := \frac{1}{2} |u|^2 + p - p(0, Z) \) in \( D_\Sigma \setminus \mathcal{D}_Z \).

3.2 Proof of Case II

Actually, if \( u_z \) is independent of \( z \)-variable, by the divergence free condition (1.3)_4, we have
\[
\partial_z (ru_r) = -r \partial_z u_z = 0,
\]
which indicates that \( ru_r = f(z) \) for some smooth function \( f(z) \). Then using the boundary condition (1.5)_3, we deduce that \( f(z) = 0 \), which implies
\[u_r = 0.
\]
Next, it follows from (1.3)_1 and (1.3)_3 that
\[
\partial_z (u_\theta)^2 = r \partial_r \partial_z p = r \partial_r \left( \partial^2_z u_z + \frac{1}{r} \partial_r u_z \right) = g(r),
\]
which deduces that
\[\left(u_\theta\right)^2(r, z) = g(r)z + \left(u_\theta\right)^2(r, 0).
\]
Note that \( g(r) \equiv 0 \) for any \( r \in [0, 1] \). Actually, if for some \( r_0 \) such that \( g(r_0) \neq 0 \), we can obtain a contradiction by taking
\[z = \frac{-1 - \left(u_\theta\right)^2(r_0, 0)}{g(r_0)}.
\]
Thus, we have that $u_\theta$ is independent of $z$-variable. Following the argument in Section 3.1.3, one concludes that $u_\theta = C_1 r$.

Then if we go back to the (1.3), we see that $\partial_r p = C_2 r^2$, which indicates that $p = \frac{1}{2} C_2^2 r^2 + f(z)$, for some smooth function $f(z)$. From (1.3), by using that $u_z$ is independent of $z$, we can have that $\partial_z p = f'(z) = C$ for some constant $C$. At last we see that $u_z$ satisfies the following two dimensional Laplacian equation in $\Sigma$ with Neumann boundary condition:

$$\begin{cases}
\Delta h u_z = C, \\
\partial_n h u_z = 0,
\end{cases} \quad (3.20)$$

Integrating directly on $\Sigma$ for (3.20), and using the boundary condition, we can obtain $C = 0$. Then multiplying by $u_z$ and integrating on $\Sigma$, we can have $\nabla h u_z = 0$, which implies that $u_z \equiv C_2$ for some constant $C_2$.

\section{Proof of Theorem 1.5}

In this section we derive the proof of Theorem 1.5 which shows a solution to the Navier-Stokes equations (1.1) with the Navier–Hodge–Lions boundary condition (NHLB) in the pipe $D$ must be a parallel flow $\Phi \pi e_z$, without axisymmetric assumptions. Our method is motivated by [20] in which authors show the uniqueness result for problems with the homogeneous Dirichlet boundary condition. Before proving the theorem, we need the following lemma that states the asymptotic behavior of a function satisfies an ordinary differential inequality.

\textbf{Lemma 4.1.} Let $Y(\zeta) \not= 0$ be a nondecreasing nonnegative differentiable function satisfying

$$Y(\zeta) \leq \Psi(Y'(\zeta)), \quad \forall \zeta > 0.$$ 

Here $\Psi : [0, \infty) \to [0, \infty)$ is a monotonically increasing function with $\Psi(0) = 0$ and there exists $C, \tau_1 > 0, m > 1$, such that

$$\Psi(\tau) \leq C \tau^m, \quad \forall \tau > \tau_1.$$ 

Then

$$\liminf_{\zeta \to +\infty} \zeta^{-\frac{m}{m-1}} Y(\zeta) > 0.$$ 

The next lemma on solving the divergence problem in a truncated pipe will be applied to bound the term related to pressure in the further proof:

\textbf{Lemma 4.2} (See [5, 6], also [11], Chapter III). Let $D = \Sigma \times [0, 1]$, $f \in L^2(D)$ with

$$\int_D f dx = 0,$$
then there exists a vector valued function \( V : D \to \mathbb{R}^3 \) belongs to \( H_0^1(D) \) such that

\[
\nabla \cdot V = f, \quad \text{and} \quad \|\nabla V\|_{L^2(D)} \leq C\|f\|_{L^2(D)}.
\]

(4.1)

Here \( C > 0 \) is an absolute constant.

The following lemma gives a Poincaré inequality for vectors in \( H^1(\Sigma) \) with only vanishing normal direction on the boundary. Readers can find some hint of the proof in Galdi [11, Page 71, Exercise II.5.6].

**Lemma 4.3.** Let \( f = f_1e_1 + f_2e_2 \) be a two dimensional vector function with components in \( H^1(\Sigma) \), and \( f \cdot \bar{n} = 0 \) on \( \partial \Sigma \), where \( \bar{n} \) is the unit outer normal of \( \partial \Sigma \). Then the following Poincaré inequality holds

\[
\|f\|_{L^2(\Sigma)} \leq C\|\nabla_h f\|_{L^2(\Sigma)},
\]

where \( \nabla_h = (\partial x_1, \partial x_2) \) is the gradient operator on \( x_1 \) and \( x_2 \) direction.

Proof of Theorem 1.5: Denoting \( v := u - \frac{\Phi}{\pi} e_z \), one deduces from (1.1) that

\[
v \cdot \nabla v + \frac{\Phi}{\pi} \partial_z v + \nabla p - \Delta v = 0.
\]

(4.2)

We multiply (4.2) by \( v \) and integrate on \( D_\zeta \), it follows that

\[
LHS = \int_{D_\zeta} v \cdot \nabla vdx = \int_{D_\zeta} v \left( v \cdot \nabla v + \frac{\Phi}{\pi} \partial_z v + \nabla p \right) dx.
\]

(4.3)

Using integration by parts, the left hand side of (4.3) follows that

\[
LHS = -\int_{D_\zeta} |\nabla v|^2 dx + \frac{1}{2} \int_{\partial D_\zeta} \frac{\partial |v|^2}{\partial n} dS,
\]

where \( n \) is the unit outer normal vector on \( \partial D_\zeta \). In the cylindrical coordinate, one writes

\[
v = v_r e_r + v_\theta e_\theta + v_z e_z.
\]

Thus one has

\[
B_1 = \int_{\partial D_{\zeta} \cap \partial D} \left( v_r \partial_r v_r + v_\theta \partial_\theta v_\theta + v_z \partial_z v_z \right) dS + \int_{D_{\zeta} \cap \{z=\zeta\}} \left( v_r \partial_z v_r + v_\theta \partial_z v_\theta + v_z \partial_z v_z \right) dx_h
\]

\[
- \int_{D_{\zeta} \cap \{z=-\zeta\}} \left( v_r \partial_z v_r + v_\theta \partial_z v_\theta + v_z \partial_z v_z \right) dx_h.
\]

(4.4)
Since \( v \) satisfies the boundary condition (1.6), one derives

\[
B_{11} = - \int_{\partial D \cap \partial D} v_0^2 dS.
\]

Substituting above in (4.4), one derives

\[
LHS \leq - \int_{D(x)} |\nabla v|^2 dx - \int_{\partial D \cap \partial D} v_0^2 dS + \int_{D \cap \{ z = \pm \zeta \}} |v||\partial_z v| dx_h. \tag{4.5}
\]

Now we turn to the right hand side of (4.3). Integrating by parts, one deduces that

\[
\int_{D(x)} (v \cdot \nabla v + \nabla p) dx = \int_{D \cap \{ z = \zeta \}} v_3 \left( \frac{1}{2} |v|^2 + p \right) dx_h - \int_{D \cap \{ z = -\zeta \}} v_3 \left( \frac{1}{2} |v|^2 + p \right) dx_h. \tag{4.6}
\]

Meanwhile, one notices

\[
\int_{D(x)} \Phi v \cdot \partial_z v dx = \Phi \left( \int_{D \cap \{ z = \zeta \}} |v|^2 dx_h - \int_{D \cap \{ z = -\zeta \}} |v|^2 dx_h \right). \tag{4.7}
\]

Substituting (4.5), (4.6) and (4.7) in (4.3), one arrives at

\[
\int_{D(x)} |\nabla v|^2 dx \leq C \int_{D \cap \{ z = \pm \zeta \}} |v|(|\nabla v| + |v| + |v|^2) dx_h - \int_{D \cap \{ z = -\zeta \}} v_3 p dx_h + \int_{D \cap \{ z = \zeta \}} v_3 p dx_h.
\]

Integrating with \( \zeta \) on \([Z - 1, Z]\), where \( Z \geq 1 \), it follows that

\[
\int_{Z - 1}^{Z} \int_{D(x)} |\nabla v|^2 dx d\zeta \leq C \left( \int_{O^+_Z \cup O^-_Z} |v|(|\nabla v| + |v| + |v|^2) dx_h + \int_{O^+_Z \cup O^-_Z} v_3 p dx_h \right). \tag{4.8}
\]

In the following we only work on integrations on \( O^+_Z \) since the rest part are similar. Using Cauchy-Schwarz inequality and Gagliardo-Nirenberg inequality, one deduces

\[
T_1 \leq C \left( \|v\|_{L^2(O^+_Z)} \|\nabla v\|_{L^2(O^+_Z)} + \|v\|_{L^2(O^+_Z)}^2 \|\nabla v\|_{L^2(O^+_Z)} \right). \tag{4.9}
\]

Applying Lemma 4.3 in each cross section of the pipe, one finds

\[
T_1 \leq C \left( \|v\|_{L^2(O^+_Z)}^2 + \|\nabla v\|_{L^2(O^+_Z)}^3 \right). \tag{4.9}
\]

Now it remains to bound the pressure term \( T_2 \) in (4.8). Noticing that

\[
\int_{D \cap \{ x_3 = \zeta \}} v_3(x_h, z) dx_h = 0, \quad \forall \zeta \in \mathbb{R},
\]

we deduce that

\[
\int_{O^+_Z} v_3 dx = 0, \quad \forall \zeta \geq 1.
\]
Using Lemma 4.2, one derives the existence of a vector field $V$ satisfying (4.1) with $f = v_3$. By the momentum equation (1.1), one arrives

$$\int_{O^2_+ Z} v_3 p dx = \int_{O^2_+ Z} \nabla p \cdot V dx = \int_{O^2_+ Z} \left( v \cdot \nabla v + \frac{\Phi}{\pi} \partial_z v - \Delta v \right) \cdot V dx.$$  

integration by parts, one derives

$$\int_{O^2_+ Z} v_3 p dx = \sum_{i,j=1}^{3} \int_{O^2_+ Z} \left( \partial_i v_j - v_i v_j - \frac{\Phi}{\pi} \delta_{i3} v_j \right) \partial_i V_j dx.$$  

Here $\delta_{ij}$ is the Kronecker symbol. By applying Hölder inequality and (4.1) in Lemma 4.2, one deduces that

$$\left| \int_{O^2_+ Z} v_3 p dx \right| \leq C \left( \|\nabla v\|_{L^2(O^2_+ Z)}^2 + \|v\|_{L^2(O^2_+ Z)}^2 + \Phi \|v\|_{L^2(O^2_+ Z)} \|v_3\|_{L^2(O^2_+ Z)} \right).$$

Similarly as we bound $T_1$, using Lemma 4.3 and Gagliardo-Nirenberg inequality, one concludes

$$T_2 \leq C \left( \|\nabla v\|_{L^2(O^2_+ Z)}^2 + \|v\|_{L^2(O^2_+ Z)}^3 \right).$$

Substituting (4.9) and (4.10) (together with their related estimates on $O^2_-$), one deduces

$$\int_{Z-1}^{Z} \int_{D_c} |\nabla v|^2 dx d\zeta \leq C \left( \|\nabla v\|_{L^2(O^2_+ Z)}^2 + \|v\|_{L^2(O^2_+ Z)}^3 \right), \quad \forall Z \geq 1.$$  

Therefore, letting

$$Y(Z) := \int_{Z-1}^{Z} \int_{D_c} |\nabla v|^2 dx d\zeta,$$

one deduces

$$Y(Z) \leq C \left( Y'(Z) + (Y'(Z))^{3/2} \right).$$

Applying Lemma 4.1 one concludes

$$Y(Z) = \int_{Z-1}^{Z} \int_{D_c} |\nabla v|^2 dx d\zeta \geq C_0 Z^3, \quad \forall Z \geq 1$$

for some $C_0 > 0$. However, this creates a paradox since $|\nabla u|$ satisfies (1.13). This proves $u = \frac{\Phi}{\pi} e_3$.  

□

APPENDIX  Computation of the boundary condition

Here we give a derivation of the boundary condition (NSB) and (NHLB) in the cylindrical coordinates. First, noting that

$$0 = u \cdot n = u_r.$$  

(A.1)
In cylindrical coordinates, the gradient operator is represented by
\[ \nabla = e_r \partial_r + e_\theta \frac{\partial}{r} + e_z \partial_z. \]

Then we can calculate the matrix \( \nabla u \) in cylindrical coordinates and write it as a form of tensor product as follows
\[
\nabla u = \partial_r u e_r \otimes e_r + \left( \frac{1}{r} \partial_\theta u_r - \frac{u_\theta}{r} \right) e_r \otimes e_\theta + \partial_z u e_r \otimes e_z \\
+ \partial_r u_\theta e_\theta \otimes e_r + \left( \frac{1}{r} \partial_\theta u_\theta + \frac{u_r}{r} \right) e_\theta \otimes e_\theta + \partial_z u_\theta e_\theta \otimes e_z \\
+ \partial_r u_z e_z \otimes e_r + \frac{1}{r} \partial_\theta u_z e_z \otimes e_\theta + \partial_z u_z e_z \otimes e_z. \tag{A.2}
\]

Equivalently
\[
\nabla u = \begin{pmatrix}
\partial_r u_r & \frac{1}{r} \partial_\theta u_r - \frac{1}{r} u_\theta & \partial_z u_r \\
\partial_r u_\theta & \frac{1}{r} \partial_\theta u_\theta + \frac{1}{r} u_r & \partial_z u_\theta \\
\partial_r u_z & \frac{1}{r} \partial_\theta u_z & \partial_z u_z
\end{pmatrix}
: \begin{pmatrix}
e_r \otimes e_r & e_r \otimes e_\theta & e_r \otimes e_z \\
e_\theta \otimes e_r & e_\theta \otimes e_\theta & e_\theta \otimes e_z \\
e_z \otimes e_r & e_z \otimes e_\theta & e_z \otimes e_z
\end{pmatrix} \tag{A.3}
\]

Also a direct computation shows that
\[
\nabla \times u = \left( \frac{1}{r} \partial_\theta u_z - \partial_z u_\theta \right) e_r + \left( \partial_r u_r - \partial_z u_z \right) e_\theta + \frac{1}{r} \left( \partial_r (ru_\theta) - \partial_\theta u_r \right) e_z.
\]

Then \( \nabla u \) under the base \( \mathcal{A} \) is represented by
\[
\mathbb{S} u = \begin{pmatrix}
\partial_r u_r & \frac{1}{2} \left( \frac{1}{r} \partial_\theta u_r + \partial_r u_\theta - \frac{1}{r} u_\theta \right) & \frac{1}{2} \left( \partial_r u_r + \partial_r u_z \right) \\
\frac{1}{2} \left( \partial_r u_r + \partial_\theta u_\theta - \frac{1}{r} u_\theta \right) & \frac{1}{2} \left( \partial_\theta u_\theta + \frac{1}{r} u_r \right) & \frac{1}{2} \left( \partial_\theta u_z + \partial_\theta u_z \right) \\
\frac{1}{2} \left( \partial_r u_z + \partial_\theta u_\theta \right) & \frac{1}{2} \left( \partial_\theta u_\theta + \frac{1}{r} u_r \right) & \partial_z u_z
\end{pmatrix} : \mathcal{A}. \tag{A.4}
\]

Since the outward normal vector \( n = e_r \), we have
\[
\mathbb{S} u \cdot n = \partial_r u_r e_r + \frac{1}{2} \left( \frac{1}{r} \partial_\theta u_r + \partial_r u_\theta - \frac{1}{r} u_\theta \right) e_\theta + \frac{1}{2} \left( \partial_r u_r + \partial_r u_z \right) e_z.
\]

Then in cylinder coordinates, one has
\[
(\mathbb{S} u \cdot n)_r = \frac{1}{2} \left( \frac{1}{r} \partial_\theta u_r + \partial_r u_\theta - \frac{1}{r} u_\theta \right) e_\theta + \frac{1}{2} \left( \partial_r u_r + \partial_r u_z \right) e_z,
\]
and
\[
\nabla \times u \times n = -\left( \partial_r u_r - \partial_z u_z \right) e_z + \frac{1}{r} \left( \partial_r (ru_\theta) - \partial_\theta u_r \right) e_\theta.
\]
This, together with (A.1), the boundary condition (NSB) and (NHLB) in the cylindrical coordinates read

\[
\begin{align*}
\partial_t u_\theta - \frac{u_\theta}{r} &= 0, \\
\partial_r u_\theta &= 0, \\
u_r &= 0,
\end{align*}
\]

and

\[
\begin{align*}
\partial_t u_\theta + \frac{u_\theta}{r} &= 0, \\
\partial_r u_\theta &= 0, \\
u_r &= 0,
\end{align*}
\]

where we have used the fact that \( \partial_\theta u_r = \partial_z u_r = 0 \) on the boundary.

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On behalf of all authors, the corresponding author states that there is no conflict of interest.

**DATA AVAILABILITY STATEMENT**

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