Abstraction and Application in Adjunction

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Abstract

The postulates of comprehension and extensionality in set theory are based on an inversion principle connecting set-theoretic abstraction and the property of having a member. An exactly analogous inversion principle connects functional abstraction and application to an argument in the postulates of the lambda calculus. Such an inversion principle arises also in two adjoint situations involving a cartesian closed category and its polynomial extension. Composing these two adjunctions, which stem from the deduction theorem of logic, produces the adjunction connecting product and exponentiation, i.e. conjunction and implication.

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1 Introduction

If one bases set theory on two notions, one being abstraction of a set from a property, i.e. finding the extension of the property, and the other the property of having a member, then the fundamental postulates of comprehension and extensionality may be understood as stating that these two notions are inverse to each other. These two set-theoretical postulates are analogous to the postulates of beta and eta conversion in the lambda calculus, where the role of set abstraction is played by functional abstraction, and the role of having a member by application to an argument. Abstraction binds a variable and application to a variable introduces it.

An analogous inversion principle arises also in two adjoint situations involving a cartesian closed category and its polynomial extension. In one of these adjunctions we find for the functor that maps the original cartesian closed category to its image in the polynomial extension a left-adjoint functor based on product, and in the other we find for this functor a right-adjoint functor based
on exponentiation. These two adjunctions, which stem ultimately from the deduction theorem of logic, and which had been anticipated in combinatory logic, were first recognized by Lambek under the name *functional completeness* in his pioneering work in categorial proof theory (see [1]. Part I, and references therein). Functional completeness is presented quite explicitly as adjunction in [1] and [3].

After a preliminary section on matters pertaining to the inversion principle of the postulates of set theory and of the lambda calculus, we shall turn to categorial proof theory and cartesian closed categories. We shall review the construction of a polynomial extension of a cartesian closed category, because this construction, though not difficult, is usually not presented with sufficient accuracy and detail. Then we shall go through the main steps of the proof of the two adjunctions of functional completeness, one involving product and the other exponentiation. We shall see that when these two adjunctions are composed they give the usual adjunction connecting product and exponentiation in cartesian closed categories, which is well-known from Lawvere’s work [11].

We shall define precisely all we need, but we shall omit the calculations in proofs. These calculations are not entirely trivial, but they would not be new, and a reader with some previous experience with cartesian closed categories (which he may have acquired by reading, for example, [10]), or with categories in general (for which many rely on [12]), should be able to perform them.

## 2 Set-Theoretical Postulates and Lambda Conversion

The two grammatical categories of terms (i.e. individual terms) and of propositions are basic grammatical categories, with whose help other grammatical categories can be defined as functional categories: predicates map terms into propositions, functional expressions map terms into terms, and connectives and quantifiers map propositions into propositions.

The set-abstracting expression \{x : ...\} maps a proposition \(A\) into the term \(\{x : A\}\), where the variable \(x\) is bound. This term is significant in particular when \(x\) is free in \(A\), but it makes sense for any \(A\) too. The expression \(x \in ...\) is a unary predicate: it maps a term \(a\) into the proposition \(x \in a\). The ideal set theory would just assume that \(\{x : ...\}\) and \(x \in ...\) are inverse to each other, according to the following postulates:

- **Comprehension:** \(x \in \{x : A\} \iff A\),
- **Extensionality:** \(\{x : x \in a\} = a\),

provided \(x\) is not free in \(a\).

In the presence of replacement of equivalents and of Comprehension, Ext-
Extensionality is equivalent to the more usual extensionality postulate

\[ \forall x(x \in a_1 \leftrightarrow x \in a_2) \implies a_1 = a_2, \]

provided \( x \) is not free in \( a_1 \) and \( a_2 \). The replacement of equivalents needed here is the principle that from \( \forall x(A_1 \leftrightarrow A_2) \) we can infer \( \{ x : A_1 \} = \{ x : A_2 \} \), which can be understood as a principle of logic. That Extensionality entails Extensionality* is shown as follows. From the antecedent of Extensionality* with replacement of equivalents we obtain \( \{ x : x \in a_1 \} = \{ x : x \in a_2 \} \), which yields \( a_1 = a_2 \) by Extensionality. To show that, conversely, Extensionality* entails Extensionality, we have \( x \in \{ x : x \in a \} \leftrightarrow x \in a \) by Comprehension, from which we obtain Extensionality by universal generalization and Extensionality*.

If doubt is cast on the replacement of equivalents used above, note that this principle is implied by Comprehension and Extensionality*. From \( \forall x(A_1 \leftrightarrow A_2) \) by Comprehension we obtain \( \forall x(x \in \{ x : A_1 \} \leftrightarrow x \in \{ x : A_2 \} \) ), and then by Extensionality* we obtain \( \{ x : A_1 \} = \{ x : A_2 \} \).

With the help of substitution for free variables, which is also a principle of pure logic, we derive the following form of Comprehension:

\[ \text{Comprehension*}: \exists y \in \{ x : A \} \leftrightarrow A^y_x, \]

where \( A^y_x \) is obtained by substituting uniformly \( y \) for free occurrences of \( x \) in \( A \), provided the usual provisos for substitution are satisfied. These provisos will be satisfied if \( y \) doesn't occur in the proposition \( A \) at all, neither free nor bound. For such a \( y \) we have by Extensionality

\[ \{ x : A \} = \{ y : y \in \{ x : A \} \}, \]

which with Comprehension* and replacement of equivalents gives \( \{ x : A \} = \{ y : A^y_y \} \).

We know, of course, that ideal set theory is inconsistent if in propositions we find negation, or at least implication. To get consistency, either \( \{ x : A \} \) will not always be defined, and we replace Comprehension by a number of restricted postulates, or we introduce types for terms.

Instead of \( \{ x : \ldots \} \) let us now write \( (\lambda x \ldots) \), and instead of \( x \in \ldots \) let us write \( (\ldots x) \). Then Comprehension and Extensionality become respectively

\[ ((\lambda x A)x) \leftrightarrow A, \]
\[ (\lambda x(ax)) = a. \]

If we take that \( (\lambda x \ldots) \) maps a term \( a \) into the term \( (\lambda x a) \), while \( (\ldots x) \) maps a term \( a \) into the term \( (ax) \), and if, furthermore, we replace equivalence by equality, and omit outermost parentheses, our two postulates become the following postulates of the lambda calculus:

- **\( \beta \)-equality**: \( (\lambda x a)x = a \),
- **\( \eta \)-equality**: \( \lambda x(ax) = a \),
provided $x$ is not free in $a$ in $\eta$-equality. The present form of $\beta$-equality yields the usual form

$$(\lambda x a)b = a_b^x$$

in the presence of substitution for free variables. The usual form of $\beta$-equality and $\eta$-equality imply the $\alpha$-equality $\lambda x a = \lambda y a_y^x$, provided $y$ doesn’t occur in $a$; we proceed as in the derivation of $\{x : A\} = \{y : A_y^x\}$ above. The fact that the lambda calculus based on $\beta$-equality and $\eta$-equality is consistent is due to the fact that the language has been restricted, either by preventing anything like negation or implication to occur in terms, or by introducing types. Without restrictions, in type-free illative theories, we regain inconsistency.

So the general pattern of Comprehension and Extensionality, on the one hand, and of $\beta$ and $\eta$-equality, on the other, is remarkably analogous. These postulates assert that a variable-binding, abstracting, expression $\Gamma x$ and application to a variable $\Phi x$ are inverse to each other, in the sense that $\Phi x \Gamma x \alpha$ and $\Gamma x \Phi x \alpha$ are either equivalent or equal to $\alpha$, depending on the grammatical category of $\alpha$. It is even more remarkable that theories so rich and important as set theory and the lambda calculus are based on such a simple inversion principle.

3 The Deduction Theorem in Categorial Proof Theory

To speak about deductions we may use labelled sequents of the form $f : \Gamma \vdash B$, where $\Gamma$ is a collection of propositions making the premises, the proposition $B$ is the conclusion, and the term $f$ records the rules justifying the deduction. If the premises can be collected into a single proposition, and this is indeed the case if $\Gamma$ is finite and we have a connective like conjunction, then we can restrict our attention to simple sequents of the form $f : A \vdash B$, where both $A$ and $B$ are propositions. We can take that $f : A \vdash B$ is an arrow in a category in which $A$ and $B$ are objects.

Special arrows in a category are axioms, and operations on arrows are rules of inference. Equalities of arrows are equalities of deductions. For that, categorial equalities between arrows have to make proof-theoretical sense, as indeed they do in many sorts of categories, where they follow closely reductions in a normalization or cut-elimination procedure. In particular, equalities between arrows in cartesian closed categories correspond to equivalence between deductions induced by normalization or cut-elimination in the implication-conjunction fragment of intutionistic logic.

Our purpose here is to show that in the context of deductions, as they are understood in categories, there is something analogous to the inversion principle we encountered before in set theory and the lambda calculus.

Take a category $\mathcal{K}$ with a terminal object $T$ (this object behaves like the constant true proposition), and take the polynomial category $\mathcal{K}[x]$ obtained by
extending $\mathcal{K}$ with an indeterminate arrow $x : T \vdash D$. Below, we shall explain precisely what this means, but let us introduce this matter in a preliminary manner. We obtain $\mathcal{K}[x]$ by adding to the graph of arrows of $\mathcal{K}$ a new arrow $x : T \vdash D$, and then by imposing on the new graph equalities required by the particular sort of category to which $\mathcal{K}$ belongs. Note that $\mathcal{K}[x]$ is not simply the free category of the required sort generated by the new graph, because the operations on objects and arrows of $\mathcal{K}[x]$ should coincide with those of $\mathcal{K}$ on the objects and arrows inherited from $\mathcal{K}$. We can conceive of $\mathcal{K}[x]$ as the extension of a deductive system $\mathcal{K}$ with a new axiom $D$.

Now consider the variable-binding expression $\Gamma_x$ that assigns to every arrow term $f : A \vdash B$ of $\mathcal{K}[x]$ the arrow term $\Gamma_x f : A \vdash D \to B$ of $\mathcal{K}$, where $\to$, which corresponds to implication, is a binary total operation on the objects of $\mathcal{K}$ (in categories, $D \to B$ is more often written $B^D$). Passing from $f$ to $\Gamma_x f$ corresponds to the deduction theorem. Conversely, we have application to $x$, denoted by $\Phi_x$, which assigns to an arrow term $g : A \vdash D \to B$ of $\mathcal{K}$ the arrow term $\Phi_x g : A \vdash B$ of $\mathcal{K}[x]$. Now, passing from $g$ to $\Phi_x g$ corresponds to modus ponens.

If we require that

$$
(\beta) \quad \Phi_x \Gamma_x f = f,
$$

$$
(\eta) \quad \Gamma_x \Phi_x g = g,
$$

we obtain a bijection between the hom-sets $\mathcal{K}[x](A, B)$ and $\mathcal{K}(A, D \to B)$. If, moreover, we require that this bijection be natural in the arguments $A$ and $B$, we obtain an adjunction. The left-adjoint functor in this adjunction is the heritage functor from $\mathcal{K}$ to $\mathcal{K}[x]$, which assigns to objects and arrows of $\mathcal{K}$ their heirs in $\mathcal{K}[x]$, while the right-adjoint functor is a functor from $\mathcal{K}[x]$ to $\mathcal{K}$ that assigns to an object $B$ the object $D \to B$. We find such an adjunction in cartesian closed categories, whose arrows correspond to deductions of the implication-conjunction fragment of intuitionistic logic, and also in bicartesian closed categories, whose arrows correspond to deductions of the whole of intuitionistic propositional logic. (In bicartesian categories we have besides all finite products, including the empty product, i.e. terminal object, all finite coproducts, including the empty coproduct, i.e. initial object.)

In cartesian closed and bicartesian closed categories, as well as in cartesian categories tout court (namely, in categories with all finite products), we also have the adjunction given by the bijection between the hom-sets $\mathcal{K}(D \times A, B)$ and $\mathcal{K}[x](A, B)$. Here the heritage functor is right adjoint, and a functor from $\mathcal{K}[x]$ to $\mathcal{K}$ that assigns to an object $A$ the object $D \times A$ is left adjoint. The binary product operation on objects $\times$ corresponds to conjunction, both intuitionistic and classical, as $\to$ corresponds to intuitionistic implication.

Actually, in cartesian closed categories we don’t need the terminal object to express the adjunction involving $\to$. We could as well take an indeterminate $x : C \vdash D$, and show that there is a bijection between the hom-sets $\mathcal{K}[x](A, B)$
and κ(A, (C → D) → B), natural in the arguments A and B. Such an adjunction could also be demonstrated for categories that have only exponentiation and lack product. These categories, which correspond to the lambda calculus with only functional types, are not usually considered. This is probably because their axiomatization is not very transparent. It is similar to axiomatizations of systems of combinators à la Schönfinkel and Curry, where to catch extensionality we have some rather unwieldy equalities. The main difference with the axiomatizations of systems of combinators is that in categories composition replaces functional application, but otherwise these axiomatizations are analogous.

These adjunctions, which are a refinement of the deduction theorem, were first considered by Lambek under the name functional completeness (see references above; in his first paper on functional completeness [8] Lambek actually envisaged rather unwieldy combinatorially inspired equalities, like those we mentioned in the previous paragraph). Through the categorial equivalence of the typed lambda calculus with cartesian closed categories, which was discovered by Lambek in the same papers, our adjunctions are closely related to the so-called Curry-Howard correspondence between typed lambda terms and natural-deduction proofs. They shed much light on this correspondence.

4 Cartesian Closed Categories

Although it is assumed the reader has already some acquaintance with categories, and with cartesian closed categories in particular, to fix notation and terminology we have to go through some elementary definitions.

A graph is a pair of functions, called the source and target function, from a set whose members are called arrows to a set whose members are called objects. (We speak only of small graphs, and small categories later.) We use f, g, h, ..., possibly with indices, for arrows, and A, B, C, ..., possibly with indices, for objects. We write f : A ⊢ B to say that A is the source of f and B its target; A ⊢ B is the type of f. (We write the turnstile ⊢ instead of the more usual →, which we use below instead of exponentiation.)

A deductive system is a graph in which for every object A we have a special arrow 1_A : A ⊢ A, called an identity arrow, and whose arrows are closed under the binary partial operation of composition:

\[
\begin{array}{c}
f : A \vdash B \\
g : B \vdash C
\end{array} \quad \Rightarrow \quad g \circ f : A \vdash C
\]

A category is a deductive system in which the following categorial equalities between arrows are satisfied:

\[
f \circ 1_A = 1_B \circ f = f,
\]
\[
h \circ (g \circ f) = (h \circ g) \circ f.
\]
A cartesian closed deductive system, or CC system for short, is a deductive system in which we have a special object $T$, and the objects are closed under the binary total operations on arrows $\times$ and $\rightarrow$; moreover, for all objects $A$, $A_1$, $A_2$, $B$ and $C$ we have the special arrows (i.e. nullary operations)

\[
\begin{align*}
&k_A : A \vdash T, \\
p_{i}^{A_1, A_2} : A_1 \times A_2 \vdash A_i, \text{ for } i \in \{1, 2\}, \\
&\varepsilon_{A, B} : A \times (A \rightarrow B) \vdash B,
\end{align*}
\]

and the partial operations on arrows

\[
\begin{align*}
&f_1 : C \vdash A_1, \quad f_2 : C \vdash A_2 \\
&\langle f_1, f_2 \rangle : C \vdash A_1 \times A_2 \\
&f : A \times A \vdash B
\end{align*}
\]

\[
\gamma_{A, C} f : C \vdash A \rightarrow B
\]

We shall find it handy to use the following abbreviations:

\[
\begin{align*}
&\text{for } f : A \vdash B \text{ and } g : C \vdash D, \\
&f \times g =_{def} \langle f \circ p_{A, C}^1, g \circ p_{A, C}^2 \rangle : A \times C \vdash B \times D, \\
&f \rightarrow g =_{def} \gamma_{A, B \rightarrow C}(g \circ \varepsilon_{B, C} \circ (f \times 1_{B \rightarrow C})) : B \rightarrow C \vdash A \rightarrow D, \\
&\text{for } g : C \vdash A \rightarrow B, \quad \varphi_{A, B} g =_{def} \varepsilon_{A, B} \circ (1_A \times g) : A \times C \vdash B,
\end{align*}
\]

\[
\begin{align*}
&\gamma_{B, A, C} =_{def} \langle p_{A, B}^1 \circ p_{A \times B, C}^1, p_{A, B}^2 \circ 1_C \rangle : (A \times B) \times C \vdash A \times (B \times C), \\
&\gamma_{B, A, C} =_{def} \langle 1_A \times p_{B, C}^1, p_{B, C}^2 \circ p_{A, B \times C}^1 \rangle : A \times (B \times C) \vdash A \times B \times C, \\
&\varphi_{A, B} g =_{def} \langle p_{A, B}^1 \circ h, p_{A, B}^2 \circ h \rangle : A \times B \vdash B \times A.
\end{align*}
\]

A cartesian closed category, or CC category for short, is a CC system in which besides the categorial equalities the following CC equalities hold:

\[
\begin{align*}
&(T\eta) \text{ for } f : A \vdash T, \quad f = k_A, \\
&(\times \beta) \quad p_{A_1, A_2}^1 \circ \langle f_1, f_2 \rangle = f_1, \\
&(\times \eta) \quad p_{A_1, A_2}^2 \circ (f_1, f_2) = f_2, \\
&(\rightarrow \beta) \quad \varphi_{A, B} \gamma_{A, C} f = f, \\
&(\rightarrow \eta) \quad \gamma_{A, C} \varphi_{A, B} g = g.
\end{align*}
\]

If $\mathcal{K}$ and $\mathcal{L}$ are CC categories, a strict cartesian closed functor, or for short CC functor, $F$ from $\mathcal{K}$ to $\mathcal{L}$ is a functor that satisfies the following equalities in $\mathcal{L}$:

\[
\begin{align*}
&FT = T, \quad F(A \times B) = FA \times FB, \text{ where } \times \text{ is } \times \text{ or } \rightarrow, \\
&Fk_A = k_{FA}, \quad Fp_{i}^{A, B} = p_{F(A), FB}^i, \quad F\varepsilon_{A, B} = \varepsilon_{FA, FB}, \\
&F\langle f_1, f_2 \rangle = \langle Ff_1, Ff_2 \rangle, \quad F\gamma_{A, C} f = \gamma_{FA, FC} Ff.
\end{align*}
\]
5 The Polynomial Cartesian Closed Category

Given a CC category $\mathcal{K}$, and an object $D$ of $\mathcal{K}$, we shall construct the polynomial CC category $\mathcal{K}[x]$ obtained by adjoining an indeterminate arrow $x : T \vdash D$ by first constructing a CC system $\mathcal{S}$ obtained by adjoining the indeterminate arrow $x$ to $\mathcal{K}$.

The objects of $\mathcal{S}$ will be the same as the objects of $\mathcal{K}$. We provide a mathematical object $x$, which is not an arrow of $\mathcal{K}$, and different mathematical objects denoted by $\langle, \rangle^S$, $\gamma^S_{A,B}$, for every pair $(A, B)$ of objects of $\mathcal{K}$. Then we define inductively the arrows of $\mathcal{S}$:

1. $x$ is an arrow of $\mathcal{S}$ of type $T \vdash D$.
2. Every arrow of $\mathcal{K}$ is an arrow of $\mathcal{S}$, with the same type it has in $\mathcal{K}$.
3. If $f : A \vdash B$ and $g : B \vdash C$ are arrows of $\mathcal{S}$, then the ordered triple $(\circ^S, f, g)$ is an arrow of $\mathcal{S}$ of type $A \vdash C$.
4. If $f : A \times C \vdash B$ is an arrow of $\mathcal{S}$, then the ordered pair $(\gamma^S_{A,C}, f)$ is an arrow of $\mathcal{S}$ of type $C \vdash A \rightarrow B$.

We denote $(\circ^S, f, g)$, $(\langle, \rangle^S, f_1, f_2)$ and $(\gamma^S_{A,C}, f)$ by $g \circ^S f$, $(f_1, f_2)^S$ and $\gamma^S_{A,C}f$ respectively.

The CC category $\mathcal{K}[x]$ will have the same objects as $\mathcal{K}$ and $\mathcal{S}$, while its arrows will be obtained by factoring the arrows of $\mathcal{S}$ through a suitable equivalence relation. Consider the equivalence relations $\equiv$ on the arrows of $\mathcal{S}$ that satisfy the congruence law

$$\text{if } f_1 \equiv f_2 \text{ and } g_1 \equiv g_2, \text{ then } g_1 \circ^S f_1 \equiv g_2 \circ^S f_2$$

(provided the types of the arrows on the two sides of $\equiv$ are equal, and are such that $g_1 \circ^S f_1$ is an arrow of $\mathcal{S}$). Moreover, these equivalence relations satisfy analogous congruence laws for $\langle, \rangle^S$ and $\gamma^S_{A,C}$, and they satisfy basic equivalences obtained from the categorial and CC equalities by replacing the equality sign $=$ by $\equiv$, and by superscribing $S$ on $\circ$, $\langle, \rangle$ and $\gamma$. Finally, our equivalence relations satisfy the following basic equivalences for $f$, $g$, $f_1$ and $f_2$ arrows of $\mathcal{K}$ of the appropriate types:

$$g \circ^S f \equiv g \circ f,$$

$$\langle f_1, f_2 \rangle^S \equiv \langle f_1, f_2 \rangle,$$

$$\gamma^S_{A,C}f \equiv \gamma_{A,C}f,$$

where the operations on the right-hand sides are those of $\mathcal{K}$. Let us call equivalence relations that satisfy all that CC equivalence relations on the arrows of $\mathcal{S}$.
It is clear that the intersection of all CC equivalence relations on the arrows of $S$ is again a CC equivalence relation on the arrows of $S$—the smallest such relation—, which we denote by $\equiv_\cap$. Then for every arrow $f$ of $S$ take the equivalence class $[f]$ made of all the arrows $f'$ of $S$ such that $f \equiv_\cap f'$.

The objects of $K[x]$ are the objects of $K$, and its arrows are the equivalence classes $[f]$, the type of $[f]$ in $K[x]$ being the same as the type of $f$ in $S$ (all arrows in the same equivalence class have the same type in $S$). With the definitions

$$1_A = \text{def } [1_A],$$
$$[g] \circ [f] = \text{def } [g \circ_S f],$$

and other analogous definitions, it is clear that $K[x]$ is a CC category.

Note that $K[x]$ is not the same as the free CC category generated by the graph of $K$ extended with $x$. To pass from this free CC category to $K[x]$ involves further factoring of objects and arrows through suitable equivalence relations, so as to ensure that the new operations on objects and arrows coincide with the old operations on the objects and arrows of $K$. However, the extension of $K$ to $K[x]$ is free in a certain sense, which we shall explicate in the next section.

### 6 The Heritage Functor

We shall now define a CC functor $H$ from $K$ to $K[x]$, which is called the heritage functor. On objects $H$ is the identity function, while on arrows it is defined by

$$Hf = \text{def } [f].$$

This function on arrows is clearly not onto, because of the arrow $x$ and other arrows of $K[x]$ involving $x$. It is also in general not one-one. Conditions that ensure that $H$ is one-one on arrows are investigated in [4], [1] (I.5) and, especially, [1]. A necessary and sufficient condition, found in this last paper, is that the object $D$ of $x : T \vdash D$ be “nonempty”, nonemptiness being expressed in a categorial manner by requiring that the arrow $k_D : D \vdash T$ be epi, i.e. cancellable on the right-hand side of compositions. (A functor such as $H$, which is a bijection on objects, is full if and only if it is onto on arrows, and it is faithful if and only if it is one-one on arrows.)

It is easy to check that $H$ is a CC functor. For example, we have

$$H(g \circ f) = [g \circ f] = [g \circ_S f] = [g] \circ [f] = Hg \circ Hf,$$

and we proceed analogously in other cases.

The polynomial CC category $K[x]$ and the heritage functor $H$ satisfy the following universal property, which explains in what sense the extension of $K$ to $K[x]$ is free:
For every CC category L, every CC functor M from K to L and every arrow f : T ⊢ MD of L, there is a unique CC functor N from K[x] to L such that Nx = f and M = NH.

This property characterizes K[x] up to isomorphism. It is analogous to the universal property one finds in the construction of a polynomial ring K[x] by adding an indeterminate x to a commutative ring K (see [2], IV.4). The analogue of the heritage functor is the insertion homomorphism from K to K[x] (which, however, is one-one, whereas the heritage functor need not be faithful). This explains the epithet polynomial ascribed to K[x].

In general, we encounter the same kind of universal property in connection with variables. The indeterminate x is in fact a variable, and a variable is a free element, or a free nullary operation. If to the set of terms A of an algebra of a certain kind we add a variable x so as to obtain the set of polynomial terms A[x], we shall have the following universal property involving the heritage (or insertion) homomorphism h from A to A[x]:

For every algebra B of the same kind as A, every homomorphism m from A to B and every element b of B, there is a unique homomorphism n from A[x] to B such that n(x) = b and for every element a of A we have m(a) = n(h(a)).

7 The Heritage Functor has a Left Adjoint

Our aim is now to show that the heritage functor H from the CC category K to the polynomial CC category K[x] has a left adjoint.

The arrows of the CC system S were defined inductively, and we shall first define by induction on the complexity of the arrow f : A ⊢ B of S a function Φ′x that assigns to f the arrow Φ′x f : D × A ⊢ B of K:

(0) Φ′x x = pD,T,
(1) Φ′x f = f ◦ pD,A, for f an arrow of K,
(2) Φ′x (g ◦ S f) = Φ′x g ◦ (pD,A, Φ′x f),
(3) Φ′x ⟨f1, f2⟩S = ⟨Φ′x f1, Φ′x f2⟩,
(4) Φ′x γA,D,C f = γA,D×C(Φ′x f ◦ bD,A,C ◦ (cA,D × 1C) ◦ γA,D,C).

The equalities [f] = [g] of K[x] stem from the equivalences f ≡g, which can be derived as in a formal system from the reflexivity of ≡g and the basic equivalences assumed for CC equivalence relations with the help of replacement of equivalents. We can prove the following lemma by induction on the length of the derivation of f ≡g.

Lemma 1 If [f] = [g] in K[x], then Φ′x f = Φ′x g in K.

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If we put

$$\Phi'_x[f] =_{def} \Phi'_x f,$$

Lemma 1 guarantees that this defines indeed a function from the arrows of $K[x]$ to the arrows of $K$.

Then we define a function $\Gamma'_{x,A}$ that assigns to an arrow $f : D \times A \vdash B$ of $K$ the arrow $\Gamma'_{x,A} f : A \vdash B$ of $K[x]$:

$$\Gamma'_{x,A} f =_{def} [f] \circ ([x] \circ k_A, 1_A).$$

Note that here, contrary to what we had in Sections 2 and 3, the variable-binding function that corresponds to abstraction has $\Phi$ in its name, while the function that corresponds to application has $\Gamma$. Before, it was the other way round. We make this switch to conform to the notation for adjoint situations of [3], [4], and [5]. Conforming to this same notation, in the next section matters will return to what we had in Sections 2 and 3.

We can verify that $\Phi'_x$ and $\Gamma'_{x,A}$ establish a bijection between the hom-sets $K(D \times A, B)$ and $K[x](A, B)$.

**Lemma 2** For every $[f] : A \vdash B$ of $K[x]$ we have $\Gamma'_{x,A} \Phi'_x[f] = [f]$ in $K[x]$.

**Lemma 3** For every $f : D \times A \vdash B$ of $K$ we have $\Phi'_x \Gamma'_{x,A} f = f$ in $K$.

We prove Lemma 2 by induction on the complexity of the arrow $f$ of $S$ (which involves some not entirely trivial computations when $f$ is of the form $\gamma_{B_1,D \times A} f'$), while Lemma 3 is checked directly.

We define a functor $F$ from $K[x]$ to $K$ by

$$FA =_{def} D \times A,$$

$$F[f] =_{def} \Phi'_x(F'_{x,B} 1_{D \times B} \circ [f]) = (p'_{D,A}, \Phi'_x f).$$

To check that this is a functor left adjoint to the heritage functor $H$ it remains to establish

$$\Phi'_x([g] \circ [f]) = \Phi'_x[g] \circ F[f],$$

which was built into the definition of $\Phi'_x$, and

$$\Gamma'_{x,A}(f \circ \Phi'_x 1_A) = [f] = Hf$$

(see [3], § 3.1, [4], § 4.1.7, or [5], § 8).
8 The Heritage Functor has a Right Adjoint

To show that the heritage functor $H$ from the CC category $K$ to the polynomial CC category $K[x]$ has a right adjoint, we define first a function $\Gamma''_x$ that assigns to an arrow $[f] : A \vdash B$ of $K[x]$ the arrow $\Gamma''_x[f] : A \vdash D \rightarrow B$ of $K$:

$$\Gamma''_x[f] = \text{def} \gamma_{D,A}\Phi'_x[f].$$

Then we define a function $\Phi''_{x,B}$ that assigns to an arrow $g : A \vdash D \rightarrow B$ of $K$ the arrow $\Phi''_{x,B}g : A \vdash B$ of $K[x]$:

$$\Phi''_{x,B}g = \text{def} \Gamma'_{x,A}\varphi_{D,B}g.$$ 

It follows easily from Lemmata 2 and 3, together with the CC equalities $(\rightarrow \beta)$ and $(\rightarrow \eta)$, that $\Gamma''_x$ and $\Phi''_{x,B}$ establish a bijection between the hom-sets $K[x](A, B)$ and $K(A, D \rightarrow B)$. Namely,

- $$(\beta) \text{ for every } [f] : A \vdash B \text{ of } K[x] \text{ we have } \Phi''_{x,B} \Gamma''_x[f] = [f] \text{ in } K[x];$$
- $$(\eta) \text{ for every } g : A \vdash D \rightarrow B \text{ of } K \text{ we have } \Gamma''_x \Phi''_{x,B}g = g \text{ in } K.$$ 

We define a functor $G$ from $K[x]$ to $K$ by

$$GA = \text{def} D \rightarrow A,$$

$$G[f] = \text{def} \Gamma''_x([f] \circ \Phi''_{x,A}1_{D \rightarrow A}) = \gamma_{D,D \rightarrow A} (\Phi'_x f \circ (p^1_{D,D \rightarrow A}, \varepsilon_{D,A})).$$

To check that this is a functor right adjoint to the heritage functor $H$ it remains to establish either

$$\Gamma''_x([g] \circ [f]) = G[g] \circ \Gamma''_x[f]$$

or

$$\Phi''_{x,C}(g \circ f) = \Phi''_{x,C}g \circ [f],$$

together with

$$\Phi''_{x,B}(\Gamma''_x1_B \circ f) = [f] = Hf,$$

which can be done after some calculation.

Consider now the functors $FH$ and $GH$ from $K$ to $K$ obtained by composing the functors $F$ and $G$, respectively, with the heritage functor $H$. It is clear that the functors $FH$ and $GH$ make an adjoint situation in which $FH$ is left adjoint and $GH$ is right adjoint. This adjunction is the usual adjunction that ties $D \times$ and $D \rightarrow$ in CC categories, since we can verify that

$$FHf = 1_D \times f,$$

$$GHf = 1_D \rightarrow f.$$ 

The bijection, natural in the arguments $A$ and $B$, between $K(D \times A, B)$ and $K(A, D \rightarrow B)$ is given by the operations $\gamma_{D,A}$ and $\varphi_{D,B}$. (This bijection is actually natural in the argument $D$ too.)
9 Logical Constants and Adjunction

Adjointness phenomena pervade logic, as well as much of mathematics. An essential ingredient of the spirit of logic is to investigate inductively defined notions, and inductive definitions engender free structures, which are tied to adjointness. We find also in logic the important model-theoretical adjointness between syntax and semantics, behind theorems of the if and only if type called semantical completeness theorems. However, adjunction is present in logic most specifically through its connection with logical constants.

Lawvere put forward the remarkable thesis that all logical constants are characterized by adjoint functors (see [11]). Lawvere’s thesis about logical constants is just one part of what he claimed for adjunction, but it is a significant part.

For conjunction, i.e., binary product in cartesian categories, we have the adjunction between the diagonal functor from $\mathcal{K}$ to the product category $\mathcal{K} \times \mathcal{K}$ as left adjoint and the internal product bifunctor from $\mathcal{K} \times \mathcal{K}$ to $\mathcal{K}$ as right adjoint. Properties assumed for this bifunctor are not only sufficient to prove the adjunction, but they are also necessary—they can be deduced from the adjunction. Binary coproduct, which corresponds to disjunction, is analogously characterized as a left adjoint to the diagonal functor. The terminal and initial objects, which correspond respectively to the constant true proposition and the constant absurd proposition, may be conceived as empty product and empty coproduct. They are characterized by functors right and left-adjoint, respectively, to the constant functor into the trivial category with a single object and a single identity arrow.

In all that, one of the adjoint functors carries the logical constant to be characterized, i.e., it involves the corresponding operation on objects, and depends on the inner constitution of the category, while the other adjoint functor is a structural functor, which does not involve the inner operations of the category, and can be defined for any category (“structural” is here used as in the “structural rules” of Gentzen’s proof theory). The diagonal functor and the constant functor are clearly structural: they make sense for any kind of category.

This suggests an amendment to Lawvere’s thesis: namely, the functor carrying the logical constant should be adjoint to a structural functor. This structural functor is presumably tied to some features of deduction that are independent of any particular constant we may have in our language, and are hence formal in the purest way. With this amendment the thesis might serve to separate the constants of formal logic from other expressions.

Lawvere’s way to characterize intuitionistic implication through adjunction is by relying on the bijection between $\mathcal{K}(D \times A, B)$ and $\mathcal{K}(A, D \to B)$ in cartesian closed categories, which can be obtained by composing the two adjunctions with the heritage functor, as we have seen in the previous section. The disadvantage of this characterization is that none of the adjoint functors $D \times$ and $D \to$ is structural.

Can the adjunctions of functional completeness serve to characterize con-
junction and intuitionistic implication? It would be nice if they could, because the heritage functor is structural. This is more important for intuitionistic implication than for conjunction, because for the latter we already have a characterization through an adjunction with a structural functor—namely, the adjunction with the diagonal functor. And it would be preferable if implication were characterized in the absence of conjunction, and of anything else, as in functional completeness with the categories that have only exponentiation and may lack product (which we mentioned in Section 3).

To define a polynomial category \( K[x] \) with an indeterminate \( x : C \vdash D \) we assume for \( K[x] \) that it has whatever it must have to make it a polynomial category of the required kind, to which \( K \) belongs. This is something, but it is nothing in particular. With an indeterminate \( x : T \vdash D \) we assume for the categories \( K \) and \( K[x] \) that they have also a special object \( T \). So, if intuitionistic implication could be characterized by the adjunction of functional completeness, this could be achieved even in the absence of \( T \), whereas the characterization of conjunction by the corresponding adjunction of functional completeness would depend on the presence of the terminal object \( T \). We would be able to characterize binary product only in the presence of empty product, i.e. \( T \). It seems all finite products go together. In any case, however, the definition of the heritage functor is structural. It will be the same for any kind of category.

A step towards showing that conjunction and intuitionistic implication can be characterized by the adjunctions of functional completeness was taken in [2], and, especially, [6]. What we need to show is that the assumptions made for cartesian, or cartesian closed categories, or categories that have only exponentiation and may lack product, are not only sufficient for demonstrating the appropriate adjunction of functional completeness, but they are also necessary. However, here the matter is not so clear-cut as when the assumptions concerning binary product are deduced from adjunction with the diagonal functor. It is shown in [6] (Section 5) that many of the assumptions for cartesian categories can be deduced from functional completeness, but still some assumptions stay simply postulated. One feels, however, that even these assumptions could be deduced if matters were formulated in the right way.

References

[1] D. Ćubrić, Embedding of a free cartesian closed category into the category of sets, J. Pure Appl. Algebra 126 (1998), 121-147.

[2] K. Došen, Modal logic as metalogic, J. Logic Lang. Inform. 1 (1992), 173-201

[3] K. Došen, Deductive completeness, Bull. Symbolic Logic 2 (1996), 243-283, Errata, Ibid. 523.
[4] K. Došen, *Cut Elimination in Categories*, Trends in Logic 6, Kluwer, Dordrecht, 1999.

[5] K. Došen, Definitions of adjunction, in: W.A. Carnielli and I.M.L. D’Ottaviano eds, *Advances in Contemporary Logic and Computer Science*, Contemp. Math. 235, American Mathematical Society, Providence, 1999, 113-126.

[6] K. Došen and Z. Petrič, Modal functional completeness, in: H. Wansing ed., *Proof Theory of Modal Logic*, Kluwer, Dordrecht, 1996, 167-211.

[7] C. Hermida and B. Jacobs, Fibrations with indeterminates: Contextual and functional completeness for polymorphic lambda calculi, *Math. Structures Comput. Sci.* 5 (1995), 501-531.

[8] J. Lambek, Deductive systems and categories III: Cartesian closed categories, intuitionist propositional calculus, and combinatory logic, in: F. W. Lawvere ed., *Toposes, Algebraic Geometry and Logic*, Lecture Notes in Math. 274, Springer, Berlin, 1972, 57-82.

[9] J. Lambek, Functional completeness of cartesian categories, *Ann. Math. Logic* 6 (1974), 259-292.

[10] J. Lambek and P.J. Scott, *Introduction to Higher-Order Categorical Logic*, Cambridge University Press, Cambridge, 1986.

[11] F.W. Lawvere, Adjointness in foundations, *Dialectica* 23 (1969), 281-296.

[12] S. Mac Lane, *Categories for the Working Mathematician*, Springer, Berlin, 1971.

[13] S. Mac Lane and G. Birkhoff, *Algebra*, second edition, MacMillan, New York, 1979.