On Liouville-type theorems for the 2D stationary MHD equations

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Abstract
We establish new Liouville-type theorems for the two-dimensional stationary magneto-hydrodynamic incompressible system assuming that the velocity and magnetic field have bounded Dirichlet integral. The key tool in our proof is observing that the stream function associated to the magnetic field satisfies a simple drift–diffusion equation for which a maximum principle is available.

Keywords: Liouville theorem, incompressible magneto-hydrodynamics (MHD), stream function
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1. Introduction

1.1. Liouville-type results in fluid mechanics

We are interested in studying Liouville-type properties for the 2D incompressible stationary magneto-hydrodynamic (MHD) system on the whole plane $\mathbb{R}^2$:
In system (1.1), \( u, b : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) stand for the velocity and the magnetic field respectively, and \( \pi : \mathbb{R}^2 \rightarrow \mathbb{R} \) is the pressure; this system and its time-dependent analogue are used to model electrically conductive fluids such as plasma, liquid metals, electrolytes etc. For more physical background and mathematical theory, we refer to [13] and the references therein. Here, unlike what is done in the literature, we do not assume that the magnetic field \( b \) is divergence free (which comes from Gauss’ law for magnetism); we were able to show, instead, that this property can be obtained directly from the system (1.1) (see the end of this section for further discussion).

In this work, we are interested in Liouville-type results for solutions to (1.1) with finite Dirichlet energy, i.e.,

\[
\int_{\mathbb{R}^2} (|\nabla u|^2 + |\nabla b|^2) \, dx < \infty.
\] (1.2)

When \( b \equiv 0 \) in system (1.1), i.e., in the case of the incompressible stationary Navier–Stokes system

\[
\begin{aligned}
-\Delta u + u \cdot \nabla u + \nabla \pi &= 0, \\
\text{div } u &= 0,
\end{aligned}
\] (1.3)

the question of the triviality of \( u \) under the above condition was first solved in [7]; this triviality was also established later in [9] provided that the velocity \( u \) is just bounded. Those two results rely heavily on the fact that one has a nice equation for the vorticity \( w := \partial_1 u_2 - \partial_2 u_1 \), i.e.,

\[
-\Delta w + u \cdot \nabla w = 0.
\] (1.4)

In higher dimensions, we refer to the works [2, 6, 16, 17].

Note that, when the magnetic field \( b \) is not null, the equation (1.4) for the vorticity is no longer available, which adds to the difficulty of the Liouville problem for the MHD system. Indeed, the aforementioned Liouville-type results for the incompressible stationary Navier–Stokes system still remain unsolved for the 2D MHD system.

In recent years, many mathematicians attempted to bring a complete understanding to the Liouville problem for the MHD system; we cite the works of [1, 3, 14, 22] (and the references contained therein) for interesting results. We also point out the recent contribution of Wang and Wang in [20], where they proved a Liouville theorem for the system in question under some smallness condition on the \( L_1 \)-norm of \( b \).

Our work is more in the direction of this latter result, [20]. The novelty of the present paper is that we uncover a nice drift–diffusion equation for the stream function associated to the magnetic field, and for which a maximum principle is available. This allows us to bring new insights to this problem and to improve existing results; in particular, we do not need to assume any smallness condition in our work.

As mentioned above, it is very common to see in the literature an added incompressibility condition \( \text{‘div } b = 0 \) on the magnetic field; this comes from using Maxwell’s equation to model the dynamic of \( b \). However, by doing so, the system becomes, from a mathematical point of view, over-determined. One of the key observations in this work is that under condition (1.2), the incompressibility of the magnetic field \( b \) can be derived directly from our system (1.1)
(which should be interpreted within the classical Newtonian mechanics setup); this is done in proposition 3.1 below.

1.2. Preliminaries and notation

In this section, we recall the definition of various function spaces that will be central to our analysis and explain the notations we use throughout this work.

To begin with, we say that a function $f$ belongs to the space $\text{BMO}(\mathbb{R}^2)$ if the following quantity is finite:

$$\|f\|_{\text{BMO}(\mathbb{R}^2)} := \sup \left\{ \frac{1}{|B(0, r)|} \int_{B(x_0, r)} |f - [f]_{x_0, r}| \, dx : B(x_0, r) \subset \mathbb{R}^2 \right\},$$

where we denote the average of $f$ over the ball $B(x_0, r)$ centred at $x_0$ and with radius $r$ as

$$[f]_{x_0, r} := \frac{1}{|B(x_0, r)|} \int_{B(x_0, r)} f(x) \, dx.$$

More generally, the average of $f$ over a bounded subset $\Omega$ is denoted by $[f]_{\Omega} := \frac{1}{|\Omega|} \int_{\Omega} f \, dx$. See for instance [8, 19] for useful properties of the BMO space. In what follows, we denote the unit ball by $B$ and the ball with radius $R$ centred at the origin by $B(R)$.

We say that a vector field $v$ belongs to the space $\text{BMO}^{-1}(\mathbb{R}^2)$ if there exists a matrix $F \in \text{BMO}(\mathbb{R}^2) \cap \mathbb{R}^{2 \times 2}$ such that $v = \text{div} F$. This is the only property of this function space we need in this work; see for instance [10] for discussions regarding this space.

Next, we say that a function $f$ is in the Hardy space $\mathcal{H}^1(\mathbb{R}^2)$ if the following quantity is finite:

$$\|f\|_{\mathcal{H}^1(\mathbb{R}^2)} := \|Mf\|_{L^1(\mathbb{R}^2)},$$

where $Mf(x) := \sup_{\phi \in F} \sup_{\|\phi\| \leq 1} |(\phi \ast f)(x)|$, with

$$F := \{ \phi \in C^0_0(B) : |\nabla \phi| \leq 1 \}$$

and $\phi_t(x) := (\phi(x / t))$. See also [5, 8, 19] for important results concerning this function space.

We use $c$ or $C$ to denote an absolute constant and write $C(A, B, \ldots)$ when the constant depends on the parameters $A, B, \ldots$. This constant may change from line to line.

Finally, the notation ‘\('\perp'\)’ will be used to refer to the orthogonal of a vector field, i.e., for any $v = (v_1, v_2) \in \mathbb{R}^2$, its orthogonal is written $v^\perp = (v_2 - v_1)$. Likewise, for any scalar function $\phi$, we use the notation $\nabla^\perp \phi = (\partial_2 \phi - \partial_1 \phi)$ to refer to the orthogonal of its gradient. Furthermore, since we constrain ourselves to the two-dimensional case, the quantity curl $v$ for any vector field is a scalar function given by curl $v := \nabla \cdot v^\perp = \partial_1 v_2 - \partial_2 v_1$.

1.3. Outline of the paper

The paper is organised as follows. In section 2, we state our main Liouville-type theorems precisely. The key idea for proving our results is the observation that there exists a stream function $\psi$ for $b$ such that $b = \nabla^\perp \psi$ and the last equation in (1.1) can be rewritten as

$$\nabla^\perp (-\Delta \psi + u \cdot \nabla \psi) = 0$$

(which is also known by physicists as Faraday’s law). In section 3, we prove the incompressibility and existence of this stream function associated to the magnetic field and further properties. In section 4, we use these tools to prove our main results. Finally, in appendix A, we collect some technical lemmas used at various points in the paper.
2. Main results

We use the following definition of weak solution for (1.1).

**Definition 2.1 (weak solutions).** We say that the pair \((u, b) \in D'(\mathbb{R}^2) \times D'(\mathbb{R}^2)\) (here \(D'(\mathbb{R}^2)\) denotes the space of distributions on \(\mathbb{R}^2\)) is a weak solution to the 2D MHD equation (1.1) in a domain \(\Omega \subset \mathbb{R}^2\) provided that:

(a) \(\nabla u, \nabla b \in L_{2,\text{loc}}(\Omega)\);
(b) \(\text{div } u = 0\) in the sense of distributions;
(c) the couple \((u, b)\) satisfies

\[
\int_{\Omega} \nabla u \cdot \nabla \varphi \, dx - \int_{\Omega} u \otimes u : \nabla \varphi \, dx = \int_{\Omega} (b \cdot \nabla b) \cdot \varphi \, dx,
\]

\[
\int_{\Omega} \nabla b \cdot \nabla \varphi \, dx - \int_{\Omega} b \otimes u : \nabla \varphi \, dx = \int_{\Omega} (b \cdot \nabla u) \cdot \varphi \, dx
\]

for all \(\varphi \in C_c^\infty(\Omega)\) with \(\varphi = (\varphi_1, \varphi_2)\) and \(\text{div } \varphi = 0\).

Note that the requirement that \(\nabla u, \nabla b \in L_{2,\text{loc}}(\mathbb{R}^2)\) implies \(u, b \in L_{2,\text{loc}}(\mathbb{R}^2)\) (cf [15, chapter 1, theorem 1.1]), so the previous definition is reasonable.

**Remark 2.1 (regularity of weak solutions).** Standard regularity results (see for instance [15, chapter 3.2]) yield that any pair \((u, b) \in D'(\mathbb{R}^2) \times D'(\mathbb{R}^2)\) of weak solutions to (1.1), say in \(\mathbb{R}^2\), is actually smooth, i.e.,

\[u, b \in C^\infty(\mathbb{R}^2).\]

Our first main result is stated as follows.

**Theorem 2.1.** Let \(u, b\) be weak solutions to system (1.1) in \(\mathbb{R}^2\) such that condition (1.2) holds. Assume in addition that one of the following conditions holds:

(a) \(b \in L_2(\mathbb{R}^2)\);

(b) \(u \in L_p(\mathbb{R}^2)\) and \(b \in L_q(\mathbb{R}^2)\) with \(p, q \in [2, \infty]\) such that

\[
\frac{2}{\max(p, q)} + \frac{1}{p} \geq \frac{1}{2};
\]

(c) \(u \in L_p(\mathbb{R}^2)\) and \(b \in L_q(\mathbb{R}^2)\) with \(1/p + 1/q = 1\) and \(p \in [1, 2]\).

Then, \(u\) and \(b\) are constants.

**Remark 2.2.** Under condition (1.2), we have \(b \in \text{BMO}(\mathbb{R}^2)\) and, thanks to the interpolation between \(L_p\) and \(\text{BMO}\) spaces (see e.g. [4, theorem 2]), we can replace the first point of theorem 2.1 by \(b \in L_p(\mathbb{R}^2)\), with \(p \in [1, 2]\).

**Remark 2.3.** Let us also point out that, regarding the second point of the previous theorem, a particular case was proved in [20] for \(p = q\) in \([1, 6]\) without requiring in addition the finiteness of the Dirichlet energy of \(u\) and \(b\) (i.e. hypothesis (1.2)). Having this in mind, it will be clear from the proof of the second point of the above theorem (see step 2 of the proof of theorem 2.1) that we can also get rid of hypothesis (1.2) and still achieve Wang and Wang result in [20], but the price to pay will be to assume, prior, that \(\text{div } b = 0\) in \(\mathbb{R}^2\) which, as we pointed out, makes the system (1.1) over-determined.
Our second result addresses the case where we allow the velocity \( u \) to grow or rapidly oscillate.

**Theorem 2.2.** Let \( u, b \) be weak solutions to system (1.1) in \( \mathbb{R}^2 \) such that condition (1.2) holds. Assume in addition that the following condition holds:

\[
u \in \text{BMO}^{-1}(\mathbb{R}^2).
\]

Then,

(a) if \( b \in L^q(\mathbb{R}^2) \) with \( q \in ]2, \infty[ \), we have \( u, b \equiv 0 \);

(b) or, alternatively, if \( b \) belongs also to \( \text{BMO}^{-1}(\mathbb{R}^2) \), we have that \( u, b \equiv 0 \).

**Remark 2.4.** The second point in the previous theorem was also proved in [3] in the 3D case; the proof of this result presented herein is simpler.

3. Properties of the stream function associated to the magnetic field

In this section, we establish the incompressibility of the magnetic field \( b \) and, as a result, the existence of a stream function associated to it.

**Proposition 3.1 (incompressibility and existence of the stream function associated to the magnetic field).** Let \( u, b \) weak solutions to system (1.1) such that condition (1.2) holds. Then, \( \text{div} b \equiv 0 \) and there exists a stream function \( \psi \) such that \( b = \nabla \perp \psi \).

In order to prove it, we need the following simple Liouville-type result.

**Lemma 3.1 (Liouville-type theorem for drift–diffusion equations).** Let \( \rho \in L^\infty(\mathbb{R}^2) \) and let \( u \in \dot{\text{L}}^1_{\text{loc}}(\mathbb{R}^2) \) (the closure of \( C^\infty_0(\mathbb{R}^2) \) with respect to the semi-norm \( \| \cdot \|_{L^2(\mathbb{R}^2)} \)) be a vector field such that \( \text{div}(\rho u) = 0 \). Let \( f \in L^2(\mathbb{R}^2) \) be a solution of

\[
-\Delta f + \text{div}(\rho u f) = 0 \quad \text{in} \quad D'(\mathbb{R}^2).
\]

Then \( f \equiv 0 \).

**Proof.** Let us start by noticing that classical regularity theory insures that \( f \in C^1(\mathbb{R}^2) \). Next, we introduce the following cut-off function: \( \varphi \in C^\infty_0(\mathbb{R}^2) \) such that \( 0 \leq \varphi \leq 1 \) with \( \varphi \equiv 1 \) in \( B(1/2) \) and \( \varphi \equiv 0 \) in \( B(3/4) \); let \( R > 1 \) and set \( \varphi_R(x) := \varphi(x/R) \). By testing equation (3.1) with \( f \varphi_R^2 \) and then integrating by parts twice, we get

\[
\int_{B(R)} |\nabla f|^2 \varphi_R^2 \, dx = \int_{B(R)} \frac{|f|^2}{2} \Delta \varphi_R^2 \, dx + \int_{B(R)} |f|^2 \varphi_R^2 \rho \cdot \nabla \varphi_R \, dx
\]

\[
\leq \frac{c(\varphi)}{R^2} \int_{B(R) \setminus B(R/2)} |f|^2 \, dx + \frac{c(\varphi)[|u|_L^\infty]\|\rho\|_{L^\infty}}{R} \int_{B(R) \setminus B(R/2)} |f|^2 \, dx
\]

\[
+ \frac{c(\varphi)[|u|_L^\infty]}{\sqrt{R}} \left( \int_{B(R)} |f|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^2} |f \varphi_R|^4 \, dx \right)^{\frac{1}{2}} \left( \int_{B(R)} |u - [u]_R|^4 \, dx \right)^{\frac{1}{2}}.
\]
Now we use Ladyzhenskaya’s inequality to obtain:
\[
\left( \int_{\mathbb{R}^2} |f \varphi_R|^4 \, dx \right)^{\frac{1}{4}} \leq c \left( \int_{\mathbb{R}^2} |f \varphi_R|^2 \, dx \right)^{\frac{1}{4}} \left( \int_{\mathbb{R}^2} |\nabla f + f \nabla \varphi_R|^2 \, dx \right)^{\frac{1}{4}} \\
\leq c \left( \int_{B(R)} |\nabla f|^2 \varphi_R^2 \, dx \right)^{\frac{1}{4}} \left( \int_{B(R)} |f|^2 \, dx \right)^{\frac{1}{4}} + \frac{c(\varphi)}{\sqrt{R}} \left( \int_{B(R)} |f|^2 \, dx \right)^{\frac{1}{4}}.
\]

Next, by using lemma A.1 to control the term \(||u||_R||\) and by introducing for simplicity the norm \(||u|| := ||\nabla u||_{L^2(\mathbb{R}^2)} + ||u_1||_{L^2(\mathbb{R}^2)}\), we find
\[
\int_{B(R)} |\nabla f|^2 \varphi_R^2 \, dx \leq c(\varphi) \left( R^{-2} + \||\rho||_{L^\infty(\mathbb{R}^2)} R^{-1} \sqrt{\log 2R} ||u|| \right) \int_{B(R)\setminus(B(R/2))} |f|^2 \, dx \\
+ \frac{c(\varphi)||\rho||_{L^\infty}}{\sqrt{R}} ||\varphi_R \nabla f||_{L^2(B(R))} ||f||_{L^2(B(R))} ||\nabla u||_{L^2(B(R))} \\
+ \frac{c(\varphi)||\rho||_{L^\infty}}{R} ||f||_{L^2(B(R))} ||\nabla u||_{L^2(B(R))} \\
\leq \frac{1}{2} \int_{B(R)} |\nabla f|^2 \varphi_R^2 \, dx + c(\varphi) \left( R^{-2} + \||\rho||_{L^\infty} R^{-1} \sqrt{\log 2R} ||u||^2 \right) ||f||_{L^2(\mathbb{R}^2)}^2 \\
+ c(\varphi) \left( R^{-\frac{3}{2}} ||\rho||_{L^\infty}^\frac{3}{2} ||\nabla u||_{L^2(\mathbb{R}^2)}^\frac{3}{2} + R^{-1} ||\rho||_{L^\infty} ||\nabla u||_{L^2(\mathbb{R}^2)} \right) ||f||_{L^2(\mathbb{R}^2)}^4;
\]
and it is clear from the latter inequality that \(\int_{\mathbb{R}^2} |\nabla f|^2 \, dx = 0\) and, \textit{a fortiori}, \(f \equiv 0\).

**Proof of proposition 3.1.** Taking the divergence in the second equation in (1.1), we get:
\[-\Delta (\text{div } b) + u \cdot \nabla (\text{div } b) = 0 \quad \text{in } \mathbb{R}^2.\]

Applying lemma 3.1 with \(f = \text{div } b, \rho = 1\), we get that \(\text{div } b \equiv 0\), and the existence of the stream function \(\psi\) comes from the solvability of the equation \(-\Delta \psi = \text{curl } b\), which is clear since the right-hand side of this latter equation belongs to \(L^2(\mathbb{R}^2)\) due to (1.2).

**Remark 3.1** (Liouville theorem for the incompressible Navier–Stokes system). Another direct consequence of lemma 3.1 is that, when \(b \equiv 0\) (i.e. for the incompressible Navier–Stokes system) and \(\nabla u \in L^2(\mathbb{R}^2)\), by noticing that the vorticity \(w := \text{curl } u\) satisfies (1.4), we get that \(w \equiv 0\); thus \(-\Delta u = 0\) and, consequently, \(u\) is constant. We have thus recovered, in a simpler way, the result of Gilbarg and Weinberger in [7].

Now, let \(\psi\) be as in proposition 3.1. Then, through explicit computations, the second equation in (1.1) may be rewritten precisely as
\[
\nabla^\perp (-\Delta \psi + u \cdot \nabla \psi) = 0.
\]
Consequently, we get that, for some (\textit{a priori} unknown) constant \(c_0 \in \mathbb{R}\),
\[
-\Delta \psi + u \cdot \nabla \psi = c_0,
\]
(3.2)
which expresses Ohm’s law.

The next proposition provides more information on the value of \(c_0\).

**Proposition 3.2.** Let \(u, b\) be solutions to system (1.1) such that condition (1.2) holds. If we assume in addition that \(u\) or \(b\) belongs to \(\text{BMO}^{-1}(\mathbb{R}^2)\), then \(c_0 = 0\) (with \(c_0\) as in (3.2)).
Remark 3.2. It will be clear from the proof that follows that the conclusion of this proposition still holds if we assume for instance \( u \in L_p(\mathbb{R}^2) \) and \( b \in L_q(\mathbb{R}^2) \) with \( p, q \in [1, \infty] \); of course, this condition can be weakened even further.

**Proof.** We will present only the case \( b \in \text{BMO}^{-1}(\mathbb{R}^2) \) since the case \( u \in \text{BMO}^{-1}(\mathbb{R}^2) \) can be dealt with in a similar manner.

Recall that the Riesz transform of a function \( f \) is defined as \( Rf := \nabla f((−\Delta)^{-1/2}) f \). Because the Riesz transform is a bounded operator from \( \text{BMO}(\mathbb{R}^n) \) to \( \text{BMO}(\mathbb{R}^n) \) (see for instance [19]), we have that the stream function \( \psi \) associated to \( b \) belongs to \( \text{BMO}(\mathbb{R}^2) \).

Recall our cut-off function \( \varphi \in C_0^\infty(\mathbb{R}^2) \) such that \( 0 \leq \varphi \leq 1 \) with \( \varphi \equiv 1 \) in \( B(1/2) \) and \( \varphi \equiv 0 \) in \( B(3/4) \). As before, for \( R > 1 \), set \( \varphi_R(x) := \varphi(x/R) \). We get from (3.2) that

\[
C_0 \int_{B(R)} \psi_R \, dx = - \int_{B(R)} \Delta \psi \varphi_R \, dx + \int_{B(R)} u \cdot \nabla (\psi - [\psi]_R) \varphi_R \, dx.
\]

This implies,

\[
|C_0| R^2 \int_B \psi \, dx \leq R \| \nabla b \|_{L_2} \| \varphi \|_{L_2} + \left| \int_{B(R)} (\psi - [\psi]_R) u \cdot \nabla \varphi_R \, dx \right| \leq c(\varphi) \left( R \| \nabla b \|_{L_2(\mathbb{R}^2)} + \| \psi \|_{\text{BMO}(\mathbb{R}^2)} \| u \|_{L_2(B(R))} \right).
\]

From lemma A.1, we get that

\[
|C_0| \leq c(\varphi, b, u) \left( \frac{1}{R} + \frac{\log 2R}{R} \right) \to 0 \quad \text{as} \quad R \to \infty.
\]

This concludes the proof. \( \square \)

4. Proof of the main theorems

We are ready to give the proof of theorem 2.1.

**Proof of theorem 2.1.** We divide the proof into four steps: the first two steps deal, respectively, with the first two points in the theorem, and the third point in the theorem is proved in the last two steps.

**Step 1.** Since \( b \in L_2(\mathbb{R}^2) \) implies \( b \in \text{BMO}^{-1}(\mathbb{R}^2) \), we have thanks to proposition 3.2 that

\[ -\Delta \psi + u \cdot \nabla \psi = 0 \quad \text{in} \quad \mathbb{R}^2. \]

By setting \( \psi^R(x) := \psi(Rx) \) \((R > 0)\), we see that \( \psi^R \) satisfies the conditions in lemma A.2, where we note that the maximum principle property is inherited from the previous drift–diffusion equation for \( \psi \). Consequently, by taking \( r = 1/2 \) in the oscillation lemma, we obtain:

\[
\sup_{x \in B(1/2)} |\psi^R(x) - \psi(0)| \leq c \| \nabla \psi^R \|_{L_2(B(1/2))},
\]

thus

\[
\sup_{x \in B(R/2)} |\psi(x) - \psi(0)| \leq c \| b \|_{L_2(B(R/2))} \to 0 \quad \text{as} \quad R \to \infty.
\]

This implies that \( \psi \) is constant and therefore \( b \equiv 0 \); consequently we have that

\[ -\Delta u + u \cdot \nabla u + \nabla \pi = 0, \quad \text{div} \ u = 0 \quad \text{in} \quad \mathbb{R}^2, \]

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with $\nabla u \in L_2(\mathbb{R}^2)$. This implies, thanks to remark 3.1, that $u$ is constant.

**Step 2.** By applying the divergence operator to the first equation in (1.1), we find that

$$-\Delta \pi = \text{div} \ (\text{div}(u \otimes u - b \otimes b)) \quad \text{in } \mathbb{R}^2;$$

(4.1)

this guarantees, if we set $r = \max(p, q)$, that

$$\|\pi\|_{L^r(\mathbb{R}^2)} \leq c \left(\|u\|_{L^p(\mathbb{R}^2)}^2 + \|b\|_{L^q(\mathbb{R}^2)}^2\right);$$

(4.2)

to obtain estimate (4.2), we used the hypothesis, the embedding of $u, b$ in BMO$(\mathbb{R}^2)$ (due to (1.2)) and the interpolation between $L_r$ and BMO spaces; see [4, theorem 2]. A detailed justification of this estimate (in a more general context) is given later in the document [see ‘step 1 (pressure estimates)’ of the proof of the first point in theorem 2.2].

Next, we rewrite the first equation in (1.1), in the following way:

$$-\Delta u + \nabla \left(\frac{|u|^2}{2} + \pi - \frac{|b|^2}{2}\right) - u \cdot \text{curl } u = -b^\perp \cdot \text{curl } b.$$ 

By taking the scalar product with $u$ in the previous line, we get that

$$-\Delta \frac{|u|^2}{2} + |\nabla u|^2 + u \cdot \nabla \left(\frac{|u|^2}{2} + \pi - \frac{|b|^2}{2}\right) = -u \cdot b^\perp \cdot \text{curl } b.$$ 

Meanwhile, in view of the equality $\nabla \psi = -b^\perp$ and thanks to (3.2) (together with remark 3.2), we have that

$$\text{curl } b = u \cdot b^\perp \quad \text{in } \mathbb{R}^2.$$ 

Consequently, returning to the previous elliptic equation, we have that

$$|\nabla u|^2 + |\text{curl } b|^2 = -\Delta \frac{|u|^2}{2} - u \cdot \nabla \left(\frac{|u|^2}{2} + \pi - \frac{|b|^2}{2}\right) \quad \text{in } \mathbb{R}^2;$$

(4.3)

For ease of notation, set $Q := |u|^2/2 + \pi - |b|^2/2 \in L_2(\mathbb{R}^2)$. By multiplying (4.3) by our usual rescaled cut-off function $\varphi_R$ and then integrating, we obtain that

$$\int_{B(R/2)} (|\nabla u|^2 + |\text{curl } b|^2) \, dx \leq \int_{B(R/2)} \frac{|u|^2}{2} \Delta \varphi_R \, dx + \int_{B(R/2)} u \cdot \nabla \varphi_R Q \, dx$$

$$\leq \frac{c(\varphi, p)}{R^7} \left(\int_{B(R/2)} |u|^p \, dx\right)^{\frac{2}{p}}$$

$$+ \frac{c(\varphi, p, q)}{R^{1+\frac{q}{2}}} \left(\int_{B(R/2)} |u|^p \, dx\right)^{\frac{2}{p}} \left(\int_{B(R/2)} |Q|^\frac{q}{2} \, dx\right)^{\frac{2}{q}},$$

with $1/s = 1 - 1/p - 2/r$. Thus, the right-hand side in last inequality above vanishes in the limit as $R \to \infty$ whenever $1 - 2/s \geq 0$, i.e., $1/p + 2/r \geq 1/2$. And the second point is proved.

**Step 3.** In this penultimate step, we prove the third point in the theorem for the non-critical cases $p \in [1, 2]$. Firstly, observe that, for this range of exponents, the div–curl lemma of
Coifman et al (cf [5, theorem 4]) applies. Due to the incompressibility condition, we deduce that \( u \cdot \nabla \psi \) belongs to the Hardy space \( \mathcal{H}^1(\mathbb{R}^2) \). Now, define \[
v := (-\Delta)^{-1}(-u \cdot \nabla \psi), \tag{4.4}\]i.e.,
\[
v(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \log |x-y|(u \cdot \nabla \psi)(y) \, dy, \quad \forall \, x \in \mathbb{R}^2.
\]
Observe that the above integral is well-defined since the function \( y \mapsto \log |y| \) belongs to the space \( \text{BMO}(\mathbb{R}^2) \), which is the dual of \( \mathcal{H}^1(\mathbb{R}^2) \); in fact, we readily deduce that \( v \in L^\infty(\mathbb{R}^2) \).

Moreover, an application of Lebesgue’s dominated convergence theorem shows that \[
\nabla v(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(x-y)}{|x-y|^2}(u \cdot \nabla \psi)(y) \, dy, \quad \forall \, x \in \mathbb{R}^2.
\]
The above and, for instance, [11, theorem 2.2] and [8, theorem 2.1.2], yield \( \nabla v \in L^2(\mathbb{R}^2) \).

Our goal is now to ‘transfer’ the square-integrability of \( \nabla v \) onto \( \nabla \psi \), since these satisfy similar equations. To this end, returning to the drift–diffusion equation for the stream function \( \psi \) and taking an additional derivative, we find that \[
-\Delta(\nabla \psi - \nabla v) = 0 \quad \text{in} \, \mathbb{R}^2.
\]
Since \( \nabla \psi - \nabla v \in L^p(\mathbb{R}^2) + L^2(\mathbb{R}^2) \), we deduce that \( \nabla \psi - \nabla v = 0 \) in \( \mathbb{R}^2 \). Hence, we get that \( b \in L^2(\mathbb{R}^2) \) and conclude as we did in ‘step 1’ of this proof.

**Step 4.** Finally, we prove the third point in the theorem for the critical case \( p = 1 \). In this case, we cannot apply the div–curl lemma of Coifman, Lions, Meyer and Semmes. Nevertheless, we can still construct a solution of the problem \( -\Delta v + u \cdot \nabla \psi = 0 \) with the desired properties, i.e., \( v \in L^\infty(\mathbb{R}^2) \) with \( \nabla v \in L^2(\mathbb{R}^2) \). This is encapsulated in lemma A.3, the proof of which is delayed until the appendix. Comparing the equation for \( v \) with the drift–diffusion equation for the stream function \( \psi \), we find once again \[
-\Delta(\psi - v) = 0 \quad \text{in} \, \mathbb{R}^2.
\]
By applying the operator ‘\( \nabla^\perp \)’ to the previous equation and by noticing that \( b - \nabla^\perp v \in L^\infty(\mathbb{R}^2) + L^2(\mathbb{R}^2) \), we obtain that there exists a constant \( a \in \mathbb{R}^2 \) such that \[
b = a + \nabla^\perp v \quad \text{in} \, \mathbb{R}^2.
\]
From the previous identity, we also get that \( \nabla v \in L^2(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2) \), which leads, by interpolation, to \[
\nabla v \in L^q(\mathbb{R}^2), \quad \forall \, q \in [2, \infty]. \tag{4.5}\]
We find, in this case, the following equation for the pressure:
\[
-\Delta \pi = \text{div div}(u \otimes u - a \otimes \nabla^\perp v - \nabla^\perp v \otimes a - \nabla^\perp v \otimes \nabla^\perp v),
\]
which leads, without loss of generality, to
\[ \| \pi \|_{L_r(\mathbb{R}^2)} \leq c \| u \otimes u - a \otimes \nabla^\perp v - \nabla^\perp v \otimes a - \nabla^\perp v \otimes \nabla^\perp v \|_{L_r(\mathbb{R}^2)}, \quad (4.6) \]
for all \( r \in [2, \infty] \) (more details can be found in ‘step 1 (pressure estimates)’ of the proof of the first point in theorem 2.2).

We can see that identity (4.3) still holds true in our current setting; notice now that
\[ \nabla^{|b|^2} = \nabla^{|\nabla v|^2} + \nabla (a \cdot \nabla^\perp v). \]

Thus, by setting \( \tilde{Q} := |u|^2/2 + \pi - |\nabla v|^2/2 - a \cdot \nabla^\perp v \in L_r(\mathbb{R}^2) \) with \( r \in [2, \infty] \), we have that
\[ |\nabla u|^2 + |\text{curl } b|^2 = \Delta |\nabla u|^2/2 - u \cdot \nabla \tilde{Q} \quad \text{in } \mathbb{R}^2. \]

Consequently, by testing the previous equation against our usual rescaled cut-off function \( \varphi_R \), we obtain
\begin{align*}
\int_{B(R/2)} (|\nabla u|^2 + |\text{curl } b|^2) \, dx &\leq \frac{\int_{B(R) \setminus B(R/2)} |u|^2 \Delta \varphi_R \, dx + \int_{B(R) \setminus B(R/2)} u \cdot \nabla \varphi_R \tilde{Q} \, dx}{2} \\
&\leq \frac{c(\varphi)}{R^2} \int_{B(R) \setminus B(R/2)} |u|^2 \, dx + \frac{c(\varphi)}{R} \left( \int_{B(R) \setminus B(R/2)} |u|^2 \, dx \right)^{1/2} \\
&\quad \times \left( \int_{B(R) \setminus B(R/2)} |\tilde{Q}|^2 \, dx \right)^{1/2} \\
&\rightarrow 0 \quad \text{as } R \rightarrow \infty.
\end{align*}

This proves the final point of the theorem. \( \square \)

Let us give now the proof of theorem 2.2. We start with the second assertion in the theorem.

**Proof of the second point in theorem 2.2.** Because \( u, b \in \text{BMO}^{-1}(\mathbb{R}^2) \), we have that \( u = \nabla^\perp \phi \) and \( b = \nabla^\perp \psi \) with \( \phi, \psi \in \text{BMO}(\mathbb{R}^2) \). Thanks, to proposition 3.1, we know that
\[ -\Delta \psi + u \cdot \nabla \psi = 0; \]
by multiplying the above equation by \( (\psi - [\psi]_R)\varphi_R^2 \) (where \( \varphi_R \) is our usual rescaled cut-off function), and then integrating by parts, we get that
\begin{align*}
\int_{B(R)} |\nabla \psi|^2 \varphi_R^2 \, dx &= \frac{1}{2} \int_{B(R)} |\psi - [\psi]_R|^2 \Delta \varphi_R^2 \, dx \\
&\quad + 2 \int_{B(R)} \varphi_R \nabla \psi \cdot \nabla^\perp \varphi_R (\psi - [\psi]_R)(\phi - [\phi]_R) \, dx;
\end{align*}
this implies, thanks to Young’s inequality,
\[
\int_{B(R/2)} |\nabla \varphi|^2 \, dx \leq c(\varphi) \int_{B(R)} |\psi - [\psi]_R|^2 \, dx
+ \frac{c(\varphi)}{R^2} \int_{B(R)} |\psi - [\psi]_R|^2 |\phi - [\phi]_R|^2 \, dx
\leq c(\varphi)(||\psi||_{BMO}^2 + ||\phi||_{BMO}^2).
\]
Thus, by taking the limit \( R \to \infty \), we get that \( \nabla \psi \in L_2(\mathbb{R}^2) \), i.e., \( b \in L_2(\mathbb{R}^2) \). By applying the first point of theorem 2.1, we conclude the proof.

We prove now the first assertion in theorem 2.2.

**Proof of the first point in theorem 2.2.** We divide the proof into two steps. Let us also recall that, since \( u \in BMO^{-1}(\mathbb{R}^2) \), we have that \( u = \nabla^+ \phi \) with \( \phi \in BMO(\mathbb{R}^2) \).

**Step 1 (pressure estimates).** Let us start by noticing that, under our hypothesis, we have that:
\[
-\Delta u + u \cdot \nabla u \in (L^1_2(\mathbb{R}^2))^* = L^1_2(\mathbb{R}^2),
\]
the dual of \( L^1_2(\mathbb{R}^2) \) (the closure of \( C_0^\infty(\mathbb{R}^2) \) with respect of the semi-norm \( ||\nabla \cdot ||_{L^1_2(\mathbb{R}^2)} \)). To see this, consider \( \varphi \in C_0^\infty(\mathbb{R}^2) \); then, we have that
\[
\langle -\Delta u + u \cdot \nabla u, \varphi \rangle = \int_{\mathbb{R}^2} \nabla u \cdot \nabla \varphi \, dx - \int_{\mathbb{R}^2} \phi \nabla u \cdot \nabla \varphi \, dx
\leq c (||\nabla u||_{L^2_2(\mathbb{R}^2)} + ||\phi||_{BMO(\mathbb{R}^2)} ||\nabla u||_{L^2_2(\mathbb{R}^2)} ||\nabla \varphi||_{L^2_2(\mathbb{R}^2)}) \quad (\forall i = 1, 2),
\]
where, for the last inequality, we used the duality between \( BMO \) and Hardy \( \mathcal{H}^1 \) spaces, and the \( \text{div-curl} \) lemma of Coifman et al (see for instance [5, theorem 4]). Consequently, we have that (see for instance [6, theorem II.8.2]) there exists \( F \in L_2(\mathbb{R}^2; \mathbb{R}^{3 \times 2}) \) such that
\[
-\Delta u + u \cdot \nabla u = \text{div}(F);
\tag{4.7}
\]
in particular, we deduce that
\[
\nabla \pi \in (L^1_2(\mathbb{R}^2))^* + L_0(\mathbb{R}^2),
\tag{4.8}
\]
with \( 1/s = 1/q + 1/2 \). Next, let us introduce \( \pi_1 \) and \( \pi_2 \) defined as follow:
\[
\pi_1 := (-\Delta)^{-1} \text{div} \text{div}(F) \quad \text{and} \quad \pi_2 := -(-\Delta)^{-1} \text{div div}(b \otimes b),
\]
where \( F \) is as in (4.7). We have that \( \pi_1 \in L_2(\mathbb{R}^2) \) and \( \pi_2 \in L_2(\mathbb{R}^2) \) if \( q \neq \infty \) or \( \pi_2 \in BMO(\mathbb{R}^2) \) if \( q = \infty \), and \( \nabla \pi_2 \in L_0(\mathbb{R}^2) \) (with \( 1/s = 1/q + 1/2 \)). Going back to the first equation in (1.1), we get that
\[
-\Delta(\pi - \pi_1 - \pi_2) = 0;
\]
but since \( \nabla(\pi - \pi_1 - \pi_2) \in (L^1_2(\mathbb{R}^2))^* + L_0(\mathbb{R}^2) \) (see (4.8) and definition of \( \pi_1 \) and \( \pi_2 \)), we have that \( \nabla(\pi - \pi_1 - \pi_2) \equiv 0 \); therefore, without loss of generality
\[
\pi = \pi_1 + \pi_2.
\tag{4.9}
**Step 2.** Similarly to what we did in ‘step 2’ of the proof of theorem 2.1, we obtain that (see (4.3)):

\[
|\nabla u|^2 + |\text{curl } b|^2 = \Delta \frac{|u|^2}{2} - u \cdot \nabla \left( \frac{|u|^2}{2} + \pi_1 + \pi_2 - \frac{|b|^2}{2} \right) \quad \text{in } \mathbb{R}^2. \quad (4.10)
\]

Before going any further, let us introduce the following special mean function:

\[
[a]_{\varphi,R} := \int_{B(R)} u(x) \varphi R(x) dx \left( \int_{B(R)} \varphi R(x) dx \right)^{-1},
\]

where \( \varphi R \) is our usual rescaled cut-off function. Now, we multiply \( \varphi R \) and integrate over the ball \( B(R) \) to obtain

\[
\int_{B(R/2)} (|\nabla u|^2 + |\text{curl } b|^2) \, dx \leq \frac{1}{2} \int_{B(R)\setminus B(R/2)} |u|^2 \Delta \varphi R \, dx \\
+ \int_{B(R)\setminus B(R/2)} (\phi - [\phi]_R) \nabla \left( |u|^2 / 2 \right) \cdot \nabla \varphi R \, dx \\
+ \int_{B(R)\setminus B(R/2)} \pi_1 u \cdot \nabla \varphi R \, dx \\
+ \int_{B(R)\setminus B(R/2)} (\phi - [\phi]_R) \nabla \pi_2 \cdot \nabla \varphi R \, dx \\
+ \int_{B(R)\setminus B(R/2)} (\phi - [\phi]_R) \nabla (|b|^2 / 2) \cdot \nabla \varphi R \, dx
\]

\[= \sum_{i=1}^{5} I_i(R).\]

Our goal now is to estimate the \( I_i \) terms. We have

\[
I_1(R) = \frac{1}{2} \int_{B(R)\setminus B(R/2)} |u - [u]_{B(R)\setminus B(R/2)}|^2 \Delta \varphi R \, dx \\
+ \int_{B(R)\setminus B(R/2)} (u - [u]_{B(R)\setminus B(R/2)}) \Delta \varphi R \, dx, \\
\leq \frac{c(\epsilon, \varphi)}{R^2} \int_{B(R)\setminus B(R/2)} |u - [u]_{B(R)\setminus B(R/2)}|^2 \, dx \\
+ 2\epsilon (|u|_{B(R)\setminus B(R/2)} - [u]_{\varphi,R})^2 + [u]_{\varphi,R}^2 \\
\leq c(\epsilon, \varphi) \int_{B(R)\setminus B(R/2)} |\nabla u|^2 \, dx + \epsilon c(\varphi) \int_{B(R)} |\nabla u|^2 \, dx + \epsilon \frac{c(\varphi)}{R^2} \|\varphi\|_{\text{BMO}(\mathbb{R}^2)}^2.
\]
Let us also point out that, for this last inequality, we used

\[ |[u]_{B(R)\setminus B(R/2)} - [u]_{\frac{\partial}{\partial R}}|^2 \leq c(\varphi) \int_{B(R)} |u - [u]_{B(R)\setminus B(R/2)}|^2 \, dx \]

and we bound the term \([u]_{\frac{\partial}{\partial R}}\) as follows:

\[ [u]_{\frac{\partial}{\partial R}} = \left\lfloor \frac{1}{R^2} \int_{B(R)} \nabla(\phi - [\phi]_R) \varphi \, dx \right\rfloor \]

\[ \leq \frac{c(\varphi)}{R} \int_{B(R)} |\phi - [\phi]_R| \, dx \]

\[ \leq \frac{c(\varphi)}{R} ||\phi||_{BMO(R^2)} \cdot \]

Next, we have

\[ I_2(R) = \int_{B(R)\setminus B(R/2)} (\phi - [\phi]_R) \nabla(|u - [u]_{B(R)\setminus B(R/2)}|^2/2) \cdot \nabla \varphi \, dx + [u]_{B(R)\setminus B(R/2)} \int_{B(R)\setminus B(R/2)} (\phi - [\phi]_R) \nabla u \nabla \varphi \, dx \]

\[ \leq c(\varphi) \left( \int_{B(R)\setminus B(R/2)} |\nabla u|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{B(R)} |\phi - [\phi]_R|^2 \, dx \right)^{\frac{1}{2}} \]

\[ \times \left( \int_{B(R)\setminus B(R/2)} |u - [u]_{B(R)\setminus B(R/2)}|^4 \, dx \right)^{\frac{1}{4}} + c(\varphi) \left| [u]_{B(R)\setminus B(R/2)} \right| \left( \int_{B(R)\setminus B(R/2)} |\nabla u|^2 \, dx \right)^{\frac{1}{2}} \]

\[ \times \left( \int_{B(R)} |\phi - [\phi]_R|^2 \, dx \right)^{\frac{1}{2}} \]

\[ \leq c(\varphi)||\phi||_{BMO(R^2)} \left( \int_{B(R)\setminus B(R/2)} |\nabla u|^2 \, dx \right)^{\frac{1}{2}} + c(\varphi)||\phi||_{BMO(R^2)} \left| [u]_{B(R)\setminus B(R/2)} \right| \left( \int_{B(R)\setminus B(R/2)} |\nabla u|^2 \, dx \right)^{\frac{1}{2}} ; \]

but as before, we deal with the term \([u]_{B(R)\setminus B(R/2)}\) in the following manner:

\[ [u]_{B(R)\setminus B(R/2)} \leq [u]_{B(R)\setminus B(R/2)} - [u]_{\frac{\partial}{\partial R}} + [u]_{\frac{\partial}{\partial R}} \]

\[ \leq c(\varphi) \left( \int_{B(R)} |\nabla u|^2 \, dx \right)^{\frac{1}{2}} + \frac{c(\varphi)}{R} ||\phi||_{BMO(R^2)} \cdot \]

Consequently, we have

\[ I_2(R) \to 0 \quad \text{as} \; R \to \infty. \]
Next, we estimate
\[
I_3(R) \leq c(\varphi) \left( \int_{B(R) \setminus B(R/2)} |\pi_1|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{B(R) \setminus B(R/2)} |u - [u]_{B(R) \setminus B(R/2)}|^2 \, dx \right)^{\frac{1}{2}} + c(\varphi) |\pi_1|_{B(R) \setminus B(R/2)} \left( \int_{B(R) \setminus B(R/2)} |\pi_1|^2 \, dx \right)^{\frac{1}{2}};
\]
therefore, because \(\pi_1 \in L^2(\mathbb{R}^2)\) and thanks to (4.12), we get
\[
I_3(R) \to 0 \quad \text{as} \quad R \to \infty.
\]
Finally, since \(\nabla \pi_2, \nabla (|b|^2/2) \in L^s(\mathbb{R}^2)\) with \(1/s = 1/q + 1/2\), we have that \(I_4(R)\) and \(I_5(R)\) can be treated in the same way:
\[
I_4(R) \leq \frac{c(\varphi, q)}{R^2} \left( \int_{B(R) \setminus B(R/2)} |\nabla \pi_2|^s \, dx \right)^{\frac{1}{s}} \left( \int_{B(R)} |\phi - [\phi]_R|^s \, dx \right)^{\frac{1}{s}} \to 0 \quad \text{as} \quad R \to \infty.
\]
Summarising our efforts, we obtain that
\[
\int_{\mathbb{R}^2} (|\nabla u|^2 + |\text{curl } b|^2) \, dx \leq \varepsilon c(\varphi) \int_{\mathbb{R}^2} |\nabla u|^2 \, dx,
\]
for all \(\varepsilon > 0\); and the proof is done. \(\square\)

5. Concluding remarks

At this point in time, it is still unclear to the authors whether one has a Liouville-type theorem for system (1.1) assuming only condition (1.2) to hold or, as suggested by proposition 3.2, assuming in addition to condition (1.2) that either \(u \in \text{BMO}^{-1}(\mathbb{R}^2)\) or \(b \in \text{BMO}^{-1}(\mathbb{R}^2)\). This will be the object of future investigations.

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Appendix A. Technical lemmas

For the reader’s convenience, we present three technical lemmas needed in the proof our main results.

First, we recall an inequality from [12, appendix B, equations (B.9), (B.10) and (B.12)].
Lemma A.1. Let \( f \in L_{2,\text{loc}}(\mathbb{R}^2) \) with finite Dirichlet energy
\[
\int_{\mathbb{R}^2} |\nabla f|^2 \, dx < \infty.
\]

Then
\[
\int_{B(R)} |f|^m \, dx \leq c R^2 (\log 2R)^{m/2} \left( \int_{\mathbb{R}^2} |\nabla f|^2 \, dx + \int_{B(1)} |f|^2 \, dx \right)^{m/2},
\]
for all \( m, R \geq 1 \) and \( c > 0 \) and absolute constant.

Secondly, we state an oscillation lemma inspired by [18, theorem 4.2]; although our statement of this result is slightly different, the idea of the proof is the same. With the aim of keeping the paper self-contained, we also provide a proof.

Lemma A.2 (oscillation lemma). Let \( v \in C(\overline{B}) \cap H^1(B) \) such that for any \( r \in ]0,1[ \) the following maximum principle holds in \( B(r) \):
\[
\max_{B(r)} v = \max_{\partial B(r)} v, \quad \min_{B(r)} v = \min_{\partial B(r)} v.
\]

Then,
\[
\sup_{x \in B(r)} |v(x) - v(0)| \leq \frac{c}{\log r} \|\nabla v\|_{L^2(B)} \sqrt{2\pi},
\]
for all \( r \in ]0,1[ \) and with \( c > 0 \) an absolute constant.

Proof. For \( r \in ]0,1[ \), using radial and angular coordinates, we write
\[
\int_{B(r)} |\nabla v|^2 \, dx = \int_r^1 \int_0^{2\pi} |\nabla v(s, \theta)|^2 s \, d\theta \, ds
\]
\[
= \int_r^1 \int_0^{2\pi} \left[ |\partial_s v|^2 + \frac{1}{s^2} |\partial_\theta v|^2 \right] s \, d\theta \, ds
\]
\[
\geq \int_r^1 \frac{1}{s} \int_0^{2\pi} |\partial_\theta v(s, \theta)|^2 \, d\theta \, ds. \tag{A.1}
\]

On the other hand, by the fundamental theorem of calculus,
\[
v(s, \theta_1) - v(s, \theta_2) = \int_{\theta_1}^{\theta_2} \partial_\theta v(s, \theta) \, d\theta,
\]
which implies (using the Cauchy–Schwarz inequality)
\[
|v(s, \theta_1) - v(s, \theta_2)| \leq \int_0^{2\pi} |\partial_\theta v(s, \theta)| \, d\theta
\]
\[
\leq \sqrt{2\pi} \left( \int_0^{2\pi} |\partial_\theta v(s, \theta)|^2 \, d\theta \right)^{1/2}.
\]
Hence
\[ \max_{\theta_1 \in [0, 2\pi]} v(s, \theta_1) - \min_{\theta_2 \in [0, 2\pi]} v(s, \theta_2) \leq \sqrt{2\pi} \left( \int_0^{2\pi} |\partial_\theta v(s, \theta)|^2 d\theta \right)^{1/2}, \]
which yields
\[ \text{osc}_{\partial B(s)} v \leq \sqrt{2\pi} \left( \int_0^{2\pi} |\partial_\theta v(s, \theta)|^2 d\theta \right)^{1/2}. \]

Using (A.1), we deduce
\[ \int_r^1 (\text{osc}_{\partial B(s)} v)^2 \frac{ds}{2\pi s} \leq \int_{B(1)/r} |\nabla v|^2 \, dx. \]

We observe that \( \text{osc}_{\partial B(s)} v \geq \text{osc}_{\partial B(r)} v \geq 0 \) for \( s \in [r, 1] \). As a result,
\[ \int_{B(1)/r} |\nabla v|^2 \, dx \geq \frac{1}{2\pi} \int_r^1 s (\text{osc}_{\partial B(s)} v)^2 \, ds \geq \frac{(\text{osc}_{\partial B(r)} v)^2}{2\pi} \int_r^1 \frac{1}{s} \, ds = \frac{1}{2\pi} (-\log r)(\text{osc}_{\partial B(r)} v)^2. \]

Once again, by the maximum principle, we have that \( \text{osc}_{B(r)} v = \text{osc}_{\partial B(r)} v \), so
\[ \text{osc}_{B(r)} v \leq \sqrt{2\pi} \frac{\|\nabla v\|_{L^2(B(1)/r)}}{\sqrt{-\log r}}. \]
which concludes the proof. \( \square \)

Finally, we provide a technical lemma which was needed in the proof of theorem 2.1.

**Lemma A.3.** Suppose \( \psi_1, \psi_2 \) are smooth functions such that \( \nabla \psi_1 \in L^1(\mathbb{R}^2) \) and \( \nabla \psi_2 \in L^\infty(\mathbb{R}^2) \). Then there exists a distributional solution \( v \in L^\infty(\mathbb{R}^2) \) with \( \nabla v \in L^2(\mathbb{R}^2) \) to the equation
\[ -\Delta v = \nabla^\perp \psi_1 \cdot \nabla \psi_2 \quad \text{in } \mathbb{R}^2. \quad (A.2) \]

Moreover, there exists a positive constant \( c \) such that
\[ \|v\|_{L^\infty(\mathbb{R}^2)} + \|\nabla v\|_{L^2(\mathbb{R}^2)} \leq c \|\nabla \psi_1\|_{L^1(\mathbb{R}^2)} \|\nabla \psi_2\|_{L^\infty(\mathbb{R}^2)}. \quad (A.3) \]

**Proof.** Recall our usual cut-off function \( \varphi \in C_0^\infty(\mathbb{R}^2) \) such that \( 0 \leq \varphi \leq 1 \) with \( \varphi \equiv 1 \) in \( B(1/2) \) and \( \varphi \equiv 0 \) in \( B\setminus B(3/4) \). Now define, for each \( n \in \mathbb{N} \), the rescaled cut-off function \( \varphi_n(x) := \varphi(x/n) \), and
\[ \tilde{\psi}_1^n := (\psi_1 - [\psi_1]_n)\varphi_n, \quad \tilde{\psi}_2^n := (\psi_2 - [\psi_2]_n)\varphi_n. \]
We have that
\[ v^n := (-\Delta)^{-1}(\nabla^\perp \tilde{\psi}_1^n \cdot \nabla \tilde{\psi}_2^n) \]
is well-defined and
\[-\Delta v^n = \nabla^L \hat{\psi}_1^n \cdot \nabla \hat{\psi}_2^n \quad \text{in } \mathbb{R}^2. \tag{A.4}\]

Our aim is now to obtain estimates on $v^n$ that are independent of $n$. We begin by observing that
\[
\|\nabla \hat{\psi}_1^n\|_{L_1(\mathbb{R}^2)} \leq \|\nabla \psi_1\|_{L_1(\mathbb{R}^2)} + \frac{c(\phi)}{n} \int_{B_1} |\psi_1 - [\psi_1]_\alpha| \, dx
\leq c(\phi) \|\nabla \psi_1\|_{L_1(\mathbb{R}^2)},
\]
where we used the triangle inequality and the boundedness of $\phi$ and its derivative to obtain the first inequality, and the Poincaré–Wirtinger inequality on a ball to obtain the second inequality.

Similarly, we have that
\[
\|\nabla \hat{\psi}_2^n\|_{L_{\infty}(\mathbb{R}^2)} \leq \|\nabla \psi_2\|_{L_{\infty}(\mathbb{R}^2)} + \frac{c(\phi)}{n} \sup_{x \in B_1} |\psi_2(x) - \int_{B_1} \psi_2(y) \, dy|,
\]
where we used once again the triangle inequality and the boundedness of $\phi$ and its derivative. Then, due to the bound on the gradient of $\psi_2$, we have that
\[
\frac{1}{n} \sup_{x \in B_1} |\psi_2(x) - \int_{B_1} \psi_2(y) \, dy| \leq \frac{1}{n} \sup_{x \in B_1} \int_{B_1} |\psi_2(x) - \psi_2(y)| \, dy
\leq c \|\nabla \psi_2\|_{L_{\infty}(\mathbb{R}^2)}, \quad \forall \, n \in \mathbb{N},
\]
whence,
\[
\|\nabla \hat{\psi}_2^n\|_{L_{\infty}(\mathbb{R}^2)} \leq c(\phi) \|\nabla \psi_2\|_{L_{\infty}(\mathbb{R}^2)}, \quad \forall \, n \in \mathbb{N}. \tag{A.6}\]

Hence, the sequence $\{\nabla \hat{\psi}_n\}_{n \in \mathbb{N}}$ is uniformly bounded in $L_1(\mathbb{R}^2)$, and $\{\nabla \hat{\psi}_2^n\}_{n \in \mathbb{N}}$ is uniformly bounded in $L_{\infty}(\mathbb{R}^2)$.

In what follows, we directly estimate $v^n$ using its singular integral representation; these calculations are inspired by similar estimates due to Wente in [21]. By rewriting the right-hand side of (A.4) in polar coordinates $(s, \theta)$, we find that
\[
\nabla^L \hat{\psi}_1^n \cdot \nabla \hat{\psi}_2^n = \frac{1}{s} \partial_s (\hat{\psi}_1^n \partial_s \hat{\psi}_2^n) - \frac{1}{s} \partial_\theta (\hat{\psi}_1^n \partial_\theta \hat{\psi}_2^n),
\]
and hence, using the periodicity with respect to the angular variable, we get that the expression for $v^n(0)$ reads as
\[
v^n(0) = -\frac{1}{2\pi} \int_0^{2\pi} \left( \int_0^\infty \log s \partial_s (\hat{\psi}_1^n \partial_\theta \hat{\psi}_2^n) \, ds \right) \, d\theta,
\]
where we also applied the Fubini–Tonelli theorem to change the order of integration. In turn, after an integration by parts, this becomes
\[
v^n(0) = \frac{1}{2\pi} \int_0^{2\pi} \left( \int_0^\infty \hat{\psi}_1^n \frac{1}{s} \partial_\theta \hat{\psi}_2^n \, ds \right) \, d\theta;
\]

note that the boundary terms have vanished since
\[
|\log s \hat{\psi}_1^n \partial_\theta \hat{\psi}_2^n| \leq \|\hat{\psi}_1^n\|_{L_{\infty}(\mathbb{R}^2)} \|\nabla \hat{\psi}_2^n\|_{L_{\infty}(\mathbb{R}^2)} |s \log s| \to 0 \quad \text{as } s \to 0^+,
\]

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and because $\hat{\psi}_I^n$ and $\hat{\psi}_S^n$ are compactly supported. Using once again the Fubini–Tonelli theorem and the periodicity with respect to the angular variable in the $\partial_\theta \hat{\psi}_S^n$ portion of the integrand, the previous expression may be rewritten as

$$v^n(0) = \frac{1}{2\pi} \int_0^{2\pi} \left( \hat{\psi}_I^n - \frac{1}{2\pi} \int_0^{2\pi} \hat{\psi}_I^n(s, \alpha) \, d\alpha \right) \partial_\theta \hat{\psi}_S^n \, d\theta \, dx.$$ 

In turn,

$$|v^n(0)| \leq \frac{1}{2\pi} \| \nabla \hat{\psi}_I^n \|_{L^\infty(\mathbb{R}^2)} \int_0^{2\pi} \int_0^{2\pi} \left| \hat{\psi}_I^n - \frac{1}{2\pi} \int_0^{2\pi} \hat{\psi}_I^n(s, \alpha) \, d\alpha \right| \, dx \, d\theta$$

$$\leq c \| \nabla \hat{\psi}_I^n \|_{L^\infty(\mathbb{R}^2)} \left( \int_0^{2\pi} \int_0^{2\pi} |\partial_\theta \hat{\psi}_S^n| \, dx \, d\theta \right)$$

$$\leq c \| \nabla \hat{\psi}_I^n \|_{L^\infty(\mathbb{R}^2)} \| \nabla \hat{\psi}_I^n \|_{L^1(\mathbb{R}^2)},$$

where we used the Poincaré–Wirtinger inequality to obtain the second inequality. By translation and using the uniform estimates (A.5) and (A.6), we obtain

$$\| v^n \|_{L^\infty(\mathbb{R}^2)} \leq c \| \nabla \psi_2 \|_{L^\infty(\mathbb{R}^2)} \| \nabla \psi_1 \|_{L^1(\mathbb{R}^2)},$$

(A.7)

for some positive constant $c$, independent of $n$. Additionally, by testing (A.4) with $v^n \varphi_R$, where $\varphi_R$ is our usual rescaled cut-off function, we obtain that

$$\int_{\mathbb{R}^2} |\nabla v^n|^2 \varphi_R \, dx = \frac{1}{2} \int_{\mathbb{R}^2} |v^n|^2 \Delta \varphi_R \, dx + \int_{\mathbb{R}^2} (\nabla \cdot \nabla \hat{\psi}_I^n) \varphi_R \, dx.$$ 

We thereby deduce from the uniform estimates (A.5)–(A.7) that there exists a positive constant $c$ independent of $n$ such that

$$\| \nabla v^n \|_{L^2(\mathbb{R}^2)} \leq c \| \nabla \psi_2 \|_{L^\infty(\mathbb{R}^2)} \| \nabla \psi_1 \|_{L^1(\mathbb{R}^2)},$$

(A.8)

By the Banach–Alaoglu theorem, the uniform bounds (A.7) and (A.8) imply the existence of a subsequence (which we still label as $\{v^n\}_{n \in \mathbb{N}}$) and a function $v$ such that

$$v^n \rightharpoondown v \quad \text{in } L^\infty(\mathbb{R}^2), \quad \nabla v^n \rightharpoondown \nabla v \quad \text{in } L^2(\mathbb{R}^2).$$

(A.9)

Let us return now to the equation (A.4). We introduce the following test function $\zeta \in C^\infty_0(\mathbb{R}^2)$; there exists $R > 0$ such that $\text{supp } \zeta \subset B(R)$. We find that

$$\int_{\mathbb{R}^2} \nabla v^n \cdot \nabla \zeta \, dx = \int_{\mathbb{R}^2} (\nabla \hat{\psi}_1 \cdot \nabla \psi_2) \zeta \, dx$$

for all $n \geq 2R + 1$. We note that arguing in this way avoids having to directly consider the product of two weakly convergent sequences; on the right-hand side of the above.

Using (A.9) to pass to the limit in the previous equation, we find that

$$\int_{\mathbb{R}^2} \nabla v \cdot \nabla \zeta \, dx = \int_{\mathbb{R}^2} (\nabla \hat{\psi}_1 \cdot \nabla \psi_2) \zeta \, dx \quad \forall \zeta \in C^\infty_0(\mathbb{R}^2),$$

and hence $v$ solves (A.2) in the sense of distributions. Estimate (A.3) follows from the convergences (A.9), estimates (A.7) and (A.8) and the weak (resp. weak-*') lower semicontinuity of the norms. □
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