GRADIENT AND STABILITY ESTIMATES OF HEAT KERNELS FOR FRACTIONAL POWERS OF ELLIPTIC OPERATOR

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Abstract. Gradient and stability type estimates of heat kernel associated with fractional power of a uniformly elliptic operator are obtained. $L^p$-operator norm of semigroups associated with fractional power of two uniformly elliptic operators are also obtained.

Keywords. Gradient Estimates, Stability, Subordination, Fractional Powers.

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1. Introduction and main conclusions

Let $D$ be a domain in $\mathbb{R}^d$ and let $a : \mathbb{R}^d \to \mathbb{R}^{d^2}$ be a matrix valued function with $C^3$ or measurable entries. The operator $H = \nabla (a(x) \nabla)$ generated a semigroup $P_t$ which is given by $P_t f(x) = \int_{\mathbb{R}^d} p_t(x, y) f(y) dy$. Heat kernel, gradient and stability estimates associated with this semigroup are well-studied (see [2], [3], [10]). In this paper we are concerned with the similar estimates for the semigroup generated by the fractional powers of $H$, namely, $Q_t = e^{-t (-H)^{\alpha}}$, where $\alpha \in (0, 1)$ will be fixed throughout this paper. Our motivation is recent works on fractional diffusion in random environment (see [1], [6] and references therein) arisen from super and sub diffusion in random environment. However, we shall deal with this problem in separate project.

First let us recall a result. Using the classical Bromwich contour integral, Pollard in [7] obtained the following formula for the inverse Laplace transform of the function $e^{-u^\alpha}$.

\[ e^{-u^\alpha} = \int_{0}^{\infty} e^{-us} g(\alpha, s) ds, \quad u \geq 0. \] (1.1)

where

\[ g(\alpha, s) = \frac{1}{\pi} \int_{0}^{\infty} e^{-su} e^{-u^\alpha \cos \pi \alpha \sin (u^\alpha \sin \pi \alpha)} du, \quad s \geq 0. \] (1.2)

is a probability density function of $s \geq 0$. This class of density functions $g(\alpha, s)$ is called strictly $\alpha$-stable law which plays important role in the theory of probability.

Denote

\[ \lambda_t(ds) = g_t(\alpha, s) ds := t^{-1/\alpha} g(\alpha, t^{-\frac{1}{\alpha}}s) ds. \] (1.3)

Then

\[ e^{-u^\alpha t} = e^{-(ut^{1/\alpha})^\alpha} = \int_{0}^{\infty} e^{-ut^{1/\alpha} s} g(\alpha, s) ds = \int_{0}^{\infty} e^{-us} g_t(\alpha, s) ds \]
From this identity we can define the semigroup \( Q_t = e^{-t(-H)_{\alpha}} \) associated with \((-H)_{\alpha}\) as

\[
Q_t f(x) = \int_0^\infty P_s f(x) \lambda_s (ds) = \int_0^\infty P_s f(x) g_s(\alpha, s) ds, \quad f \in B. \tag{1.4}
\]

Then, \( \{Q_t, t \geq 0\} \) is also a strong continuous contraction semigroup on \( B \) and its infinitesimal generator satisfies that

\[
-(H)_{\alpha} f = M f = \int_0^\infty (P_s f - f) \rho(ds) = \frac{\alpha}{\Gamma(1 - \alpha)} \int_0^\infty \frac{P_s f - f}{s^{1+\alpha}} ds, \quad f \in D(L).
\]

Moreover, \( D(H) \) is a core of \( M \) which means that \( D(H) \subset D(M) \) and the closure of \( M|_{D(H)} \), the restriction of \( M \) to \( D(H) \), equals \( M \). In fact, \( H \) can be replaced by a more general operator.

The main results of the present paper are gradient and stability estimates of the heat kernels associated with the fractional power for uniformly elliptic operators.

**Theorem 1.1.** Suppose that \( D \) is a bounded \( C^2 \) domain in \( \mathbb{R}^d \) and \( H = \nabla (a(x) \nabla) \) on \( D \) where the matrix \( a(x) \) has \( C^2 \), \( \beta \in (0, 1) \) entries, and there exists a constant \( \lambda > 1 \) such that \( \lambda^{-1} \text{Id}_d \leq a(x) \leq \lambda \text{Id}_d \). Then the heat kernel \( q(t, x, y) \) of the fractional power of \( H \), i.e., \( M = -(H)_{\alpha} \), exists and has the following gradient estimates:

\[
|\nabla_q q(t,x,y)| \leq c_1 \left( t^{-\frac{d+1}{2\alpha}} \wedge \frac{t}{|x-y|^{d+1+2\alpha}} \right), \quad \forall (t,x,y) \in (0,\infty) \times D \times D, \tag{1.5}
\]

where \( c_1 = c_1(D, d, \alpha, \beta) \) is a strictly positive constant.

**Theorem 1.2.** Suppose that \( D = \mathbb{R}^d \) and \( H = \nabla (a(x) \nabla) \), \( \tilde{H} = \nabla (\tilde{a}(x) \nabla) \) on \( D \) with measurable coefficients. If there exists a constant \( \lambda > 1 \) such that \( \lambda^{-1} \text{Id}_d \leq a(x) \leq \lambda \text{Id}_d \) and \( \lambda^{-1} \text{Id}_d \leq \tilde{a}(x) \leq \lambda \text{Id}_d \), then their subordinated semigroups \( \{Q_t\}, \{\tilde{Q}_t\} \) and the corresponding heat kernels \( q(t,x,y), \tilde{q}(t,x,y) \) satisfy the following stability estimate: there exist bounded, continuous functions \( F_1(t,z), F_2(t,z) \) on \( (0,\infty) \times (0,\infty) \) with \( \lim_{z \to 0} F_i(t,z) = 0, i = 1,2 \) for each \( t > 0 \),

\[
\|Q_t - \tilde{Q}_t\|_p \leq F_1(t, \|a - \tilde{a}\|_{L^p_{loc}}), \quad \forall p \in [0,\infty] \tag{1.6}
\]

\[
|q(t, x, y) - \tilde{q}(t, x, y)| \leq F_2(t, \|a - \tilde{a}\|_{L^p_{loc}}). \tag{1.7}
\]

where

\[
\|a - \tilde{a}\|_{L^2_{loc}} = \sup_{k \in \mathbb{Z}^d} \sum_{i,j=1}^d |a_{ij} - \tilde{a}_{ij}|_{L^2(D_k)},
\]

\[
D_k = \left\{ x \in \mathbb{R}^d : |x - k| < 2\sqrt{d} \right\}.
\]

See below (2.15) and (2.14) for the explicit expression of the functions \( F_1(t,z), F_2(t,z) \).

**Remark 1.3.** Similarly, we can show that the inequality (1.6) is still valid for \( D \) is a bounded \( C^1 \)-smooth domain in \( \mathbb{R}^d \).
2. Proof of the main theorems

2.1. Preliminaries: Heat kernel of the subordination semigroup. The asymptotic behaviors of \( g(\alpha, s) \) when \( s \to 0 \) and when \( s \to \infty \) have been known. See for example [8] Equality (14.35) for \( s \to 0 \) and Equality (14.37) for \( s \to \infty \). For the convenience of readers we recall these asymptotic formulae in following proposition.

**Proposition 2.1.** The function \( g(\alpha, s) \) has the following asymptotic formulae:

\[
g(\alpha, s) \sim Ks^{-\frac{2\alpha}{d+2\alpha}} \exp(-As^{-\frac{\alpha}{d}}), \quad \text{for} \quad s \to 0+, \quad (2.1)
g(\alpha, s) \sim B s^{-1-\alpha}, \quad \text{for} \quad s \to \infty. \quad (2.2)
\]

where \( A, K \) and \( B \) are constants only depending on \( \alpha \).

It is known that if \( \{P_t\} \) has a positive kernel then so does \( \{Q_t\} \), see for example Lemma 3.4.1 of [3] and Lemma 5.4 of [5]. We restate it as the following proposition.

**Proposition 2.2.** Suppose that \( D = \mathbb{R}^d \) and \( H = \nabla (a(x) \nabla) \), \( \tilde{H} = \nabla (\tilde{a}(x) \nabla) \) on \( D \) with measurable coefficients and suppose there exists a constant \( \lambda \geq 1 \) such that \( \lambda^{-1} \text{id}_d \leq a(x) \leq \lambda \text{id}_d \) and \( \lambda^{-1} \text{id}_d \leq \tilde{a}(x) \leq \lambda \text{id}_d \). Then \( \{Q_t\} \) also has a positive kernel \( q(t, x, y) \) on \( (0, \infty) \times D \times D \) such that

\[
q(t, x, y) = \int_0^\infty p(s, x, y) \lambda_t(ds) = \int_0^\infty p(t^{\frac{1}{\alpha}} s, x, y) g(\alpha, s) ds. \quad (2.3)
\]

**Proof.** Since \( p(t, x, y) \) is the positive kernel of \( \{P_t\} \), it follows from Eq.(1.4)

\[
q(t, x, y) = \int_0^\infty p(s, x, y) \lambda_t(ds) \\
= t^{-\frac{1}{\alpha}} \int_0^\infty p(s, x, y) g(\alpha, t^{-\frac{1}{\alpha}} s) ds \quad \text{(by Eq.(1.3))} \\
= \int_0^\infty p(t^{\frac{1}{\alpha}} s', x, y) g(\alpha, s') ds'.
\]

This completes the proof. \( \square \)

As a direct corollary of the above two propositions, we have the following results.

**Theorem 2.3.** Suppose that \( D \) is a domain in \( \mathbb{R}^d \) and \( H = \nabla (a(x) \nabla) \) on \( D \) with measurable coefficients. If there exists a constant \( \lambda \geq 1 \) such that \( \lambda^{-1} \text{id}_d \leq a(x) \leq \lambda \text{id}_d \), then the heat kernel \( q(t, x, y) \) of the fractional power of \( H \), i.e., \( \mathcal{M} = (-H)^{\alpha} \), has the following Nash’s Hölder estimates: there are constants \( c = c(d, \lambda, \alpha) > 1 \) and \( \gamma = \gamma(d, \lambda) \in (0, 1) \) such that

\[
\begin{align*}
&\left\{ \frac{1}{c} (t^{\frac{\alpha}{d}} \wedge \frac{t}{|y-x|^{d+2\alpha}}) \leq q(t, x, y) \leq c(t^{\frac{\alpha}{d}} \wedge \frac{t}{|y-x|^{d+2\alpha}}) \right. \\
&\left| q(t, x, y) - q(t, x_1, y_1) \right| \leq ct^{\frac{2d\gamma}{d+2\gamma}} \left( |x-x_1| \vee |y-y_1| \right)^\gamma,
\end{align*}
\]

for all \( t > 0 \) and \( (x, y), (x_1, y_1) \in D \times D \).
Proof. The first result is already known, see [5, Lemma 5.4]. It is a consequence of (2.3) and the following estimates
\[
\frac{1}{M} t^{-\frac{d}{2}} \exp \left\{ -\frac{M |y-x|^2}{t} \right\} \leq p(t, x, y) \leq Mt^{-\frac{d}{2}} \exp \left\{ -\frac{|y|^2}{Mt} \right\}.
\] (2.6)

The second inequality follows from Eq.(1.3) of [2]. We shall not provide details since it will be similar to the proof that we present below. □

2.2. Proof of the main theorems. Theorem 1.1 is a corollary of the following proposition.

**Proposition 2.4.** Suppose that \( D \) is a domain in \( \mathbb{R}^d \). If there are two constants \( M > 0 \) and \( \ell \geq 0 \) such that \( |\nabla_x p(t, x, y)| \) has an upper bound
\[
|\nabla_x p(t, x, y)| \leq Mt^{-\frac{d}{2}} \exp \left\{ -\frac{|y-x|^2}{Mt} \right\}, \quad \forall (t, x, y) \in (0, \infty) \times D \times D,
\] (2.7)
then there is a strictly positive constant \( c_1 = c_1(M, \ell, \alpha) \) such that
\[
|\nabla_x q(t, x, y)| \leq c_1 \left( t^{-\frac{d}{2}} \wedge \frac{t}{|x-y|^{d+2\alpha}} \right), \quad \forall (t, x, y) \in (0, \infty) \times D \times D.
\] (2.8)

**Proof.** We shall divide the proof into several steps. The idea is similar to the proof of Lemma 5.4 of [5].

**Step 1.** It follows from Lebesgue’s dominated theorem that condition (2.7) implies that one can take the derivative under the integral sign in Eq.(2.3), i.e.,
\[
\nabla_x q(t, x, y) = \int_0^\infty \nabla_x p(t, x, y) g(\alpha, s) ds.
\]
Hence inequality (2.7) imply that for all \( (t, x, y) \in (0, \infty) \times D \times D, \)
\[
|\nabla_x q(t, x, y)| \leq \int_0^\infty \left| \nabla_x p(t, x, y) \right| g(\alpha, s) ds
\leq Mt^{-\frac{d}{2}} \int_0^\infty s^{-\frac{d}{2}} \exp \left\{ -\frac{|y-x|^2}{Mt^2} \right\} g(\alpha, s) ds.
\] (2.9)
Since the exponential function in the above integrand is less than one, we have that
\[
\int_0^\infty s^{-\frac{d}{2}} \exp \left\{ -\frac{|x-y|^2}{Mt^2} \right\} g(\alpha, s) ds \leq \int_0^\infty s^{-\frac{d}{2}} g(\alpha, s) ds
\leq \int_0^1 s^{-\frac{d}{2}} g(\alpha, s) ds + 1
\]
since \( g(\alpha, s) \) is positive and \( \int_0^\infty g(\alpha, s) ds = 1 \). Using the asymptotic formula Eq.(2.1) we also see that \( \int_0^1 s^{-\frac{d}{2}} g(\alpha, s) ds \) is finite. Thus we have
\[
|\nabla_x q(t, x, y)| \leq Mt^{-\frac{d}{2}}
\] (2.10)

**Step 2.** It is easy to see from Eqs (2.1)-(2.2) that there exists a constant \( \tilde{c} := \tilde{c}(\alpha) > 0 \) such that
\[
g(\alpha, s) \leq \tilde{c} s^{-1-\alpha}, \quad \forall s \in [0, \infty).
\] (2.11)
Substituting this inequality into Eq. (2.9), we obtain that
\[
|\nabla_x q(t, x, y)| \leq \tilde{c} M^{-\frac{\alpha}{2}} \int_0^\infty s^{-\frac{\alpha}{2} - 1 - \alpha} \exp \left\{ - \frac{|x - y|^2}{M t s} \right\} ds \\
= \tilde{c} M^{-\frac{\alpha}{2}} \int_0^\infty r^{\frac{\alpha}{2} + 1} \exp \left\{ - \frac{|x - y|^2}{M t r} \right\} dr \quad (\text{let } r = \frac{1}{s}) \\
= \tilde{c} \frac{\Gamma(\alpha + \frac{\alpha}{2})}{t} \frac{Mt}{(|x - y|^2 / M t^{\frac{\alpha}{2}})^{\alpha + \frac{\alpha}{2}}} \\
= \tilde{c} \frac{t}{|x - y|^{\alpha + 2\alpha}},
\tag{2.12}
\]
where \(\Gamma(\cdot)\) is the Gamma function. By putting the inequalities (2.10) and (2.12) together, we obtain the desired gradient estimate (2.8).

\[\square\]

**Proof of Theorem 1.1.** For the uniformly elliptic operator \(H\), it is known that its kernel has the following gradient estimate, see for example [1] or [2],
\[
|\nabla_x p(t, x, y)| \leq M^{-\frac{\alpha}{2}} \exp \left\{ - \frac{|x - y|^2}{M t} \right\}, \quad \forall (t, x, y) \in (0, \infty) \times D \times D.
\]
Hence it follows from Proposition 2.4 that Theorem 1.1 holds.

\[\square\]

**Proof of Theorem 1.2.** We shall divide the proof into several steps.

Step 1. Proposition 2.4 can be rewritten as the following: for a positive function \(f(t, x, y)\), if there are two constants \(M > 0\) and \(\ell \geq 0\) such that \(f(t, x, y)\) has an upper bound
\[
f(t, x, y) \leq M t^{-\frac{\alpha}{2}} \exp \left\{ - \frac{|x - y|^2}{M t} \right\}, \quad \forall (t, x, y) \in (0, \infty) \times D \times D,
\]
then there is a strictly positive constant \(c_1 = c_1(M, \ell, \alpha)\) such that
\[
\int_0^\infty f(t^{\frac{\alpha}{2}} s, x, y) g(\alpha, s) ds \leq c_1 \left( t^{-\frac{\alpha}{2}} \wedge \frac{t}{|x - y|^{\alpha + 2\alpha}} \right), \quad \forall (t, x, y) \in (0, \infty) \times D \times D. \tag{2.13}
\]
It follows form Theorem 1.2 of [2] that
\[
|p(t, x, y) - \tilde{p}(t, x, y)| \leq c' t^{-\frac{\alpha}{2}} \exp \left\{ - \frac{|x - y|^2}{ct} \right\} \left( t \wedge 1 \right)^{-\gamma} \|a - \tilde{a}\|_{L^2_{loc}}^\delta \\
\leq c' t^{-\frac{\alpha}{2}} \exp \left\{ - \frac{|x - y|^2}{ct} \right\} \left( t^{-\gamma} + 1 \right) \|a - \tilde{a}\|_{L^2_{loc}}^\delta,
\]
where \(c > 0, c' > 1, \gamma \in (0, 1), \delta \in (0, 1)\) depend only on \(d\) and \(\lambda\). Hence it follows from (2.3) and the inequality (2.13) that
\[
|q(t, x, y) - \tilde{q}(t, x, y)| \leq \int_0^\infty \left| p(t^{\frac{\alpha}{2}} s, x, y) - \tilde{p}(t^{\frac{\alpha}{2}} s, x, y) \right| g(\alpha, s) ds \\
\leq \tilde{c} \|a - \tilde{a}\|_{L^2_{loc}}^\delta \left[ t^{-\frac{\alpha}{2}} \wedge \frac{t}{|x - y|^{\alpha + 2\alpha}} + t^{-\gamma} \wedge \frac{t}{|x - y|^{\alpha + 2\alpha}} \right].
\]
On the other hand, the first inequality in Theorem 2.3 gives the following bound:

$$|q(t, x, y) - \tilde{q}(t, x, y)| \leq c_1 \left( t^{-\frac{d}{2}} \wedge \frac{t}{|x-y|^{d+2\alpha}} \right), \quad \forall (t, x, y) \in (0, \infty) \times D \times D,$$

where $c_1 = c_1(d, \lambda, \alpha)$ is a constant.

Combining the above two bounds together, we prove (2.14) with the choice

$$F_2(t, z) = c \left( t^{-\frac{d}{2}} \wedge \frac{t}{|x-y|^{d+2\alpha}} \right) \min \left\{ 1, t^{-\frac{d}{2}} \wedge |x-y|^{-2\alpha} z^\delta \right\}. \quad (2.14)$$

Step 2. It follows from (1.4) that

$$Q_t f(x) - \tilde{Q}_t f(x) = \int_0^\infty (P_s f(x) - \tilde{P}_s f(x)) g_s(\alpha, s) ds.$$ 

Thus it follows from Minkowski’s integral inequality that for any $p \in [0, \infty],

$$\|Q_t - \tilde{Q}_t\|_p \leq \int_0^\infty \|P_s - \tilde{P}_s\|_p g_s(\alpha, s) ds.$$ 

It follows from Theorem 1.1 of [2] that

$$\|P_t - \tilde{P}_t\|_p \leq c' \cdot (t \wedge 1)^{-\gamma} \|a - \tilde{a}\|_{L^\infty_{loc}}^\delta \leq c' \cdot (t^{-\gamma} + 1) \|a - \tilde{a}\|_{L^\infty_{loc}}^\delta.$$ 

Hence we have that

$$\|Q_t - \tilde{Q}_t\|_p \leq c' \cdot \|a - \tilde{a}\|_{L^\infty_{loc}}^\delta \int_0^\infty \left( (t^{-\gamma} + 1) g(\alpha, s) ds \right)$$

$$\leq \tilde{c} \cdot \|a - \tilde{a}\|_{L^\infty_{loc}}^\delta \left( 1 + t^{-\frac{d}{2}} \right).$$

Since $Q_t$ is also a contraction semigroup, we prove (1.6) by the choice

$$F_1(t, z) = \min \left\{ 2, c(1 + t^{-\frac{d}{2}}) z^\delta \right\}. \quad (2.15)$$

This completes the proof of Theorem 2. \qed

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