M Supergravity and Abelian Semigroups

Fernando Izaurieta, Eduardo Rodríguez and Patricio Salgado
Departamento de Física, Universidad de Concepción, Casilla 160-C, Concepción, Chile
E-mail: fizaurie@gmail.com, edurodriguez@udec.cl, pasalgad@udec.cl

Abstract. A gauge theory for the M algebra in eleven-dimensional spacetime is put forward. The gauge-invariant Lagrangian corresponds to a transgression form. This class of Lagrangians modifies Chern–Simons theory with the addition of a regularizing boundary term. The M algebra-invariant tensor required to define the theory comes from regarding the algebra as an abelian semigroup expansion of the orthosymplectic algebra \( \mathfrak{osp}(32|1) \). The explicit form of the Lagrangian is found by means of a transgression-specific subspace separation method. Dynamical properties are briefly analyzed through an example. The equations of motion are found to place severe constraints on the geometry, which might be partially alleviated by allowing for nonzero torsion.

1. Introduction
The gauge theory structure of standard eleven-dimensional Supergravity [1] has been a subject of much interest since its formulation (see, e.g., Refs. [2, 3, 4]). The lack of a clear geometrical formulation was already noted by Cremmer, Julia and Scherk (CJS), who hinted at the orthosymplectic algebra \( \mathfrak{osp}(32|1) \) as a possible underlying symmetry.

A different approach also pursued in the literature concerns starting with some superalgebra and building a gauge theory for it via some prescribed method [e.g. Chern–Simons (CS) Lagrangians; see Refs. [5, 6, 7, 8]]. The difficulty here lies on establishing a connection between this theory and standard CJS Supergravity.

Here we report on a new Poincaré-invariant Lagrangian with local supersymmetry in eleven-dimensional spacetime [9]. The construction follows the second of the approaches mentioned above; namely, we start with a definite superalgebra (the M algebra, see Ref. [10]) and write a gauge theory for it based on the general framework of transgression forms [11, 12, 13, 14, 15, 16].

A key ingredient in the construction of a transgression Lagrangian (and CS Lagrangians as well) is a symmetric invariant tensor. The one we use comes from regarding the M algebra as an abelian semigroup expansion [17] of \( \mathfrak{osp}(32|1) \). The advantage of this viewpoint lies on the number of nonzero components the resulting invariant tensor possesses, which largely surpasses that which can be obtained by more direct methods (e.g. the symmetrized supertrace).

The paper is organized as follows. In section 2 we briefly review the relationship between the M algebra and \( \mathfrak{osp}(32|1) \), and provide a recipe for an M algebra-invariant tensor based on one for \( \mathfrak{osp}(32|1) \). The M Supergravity Lagrangian is presented in section 3. In section 4 we discuss the dynamics produced by this Lagrangian via an explicit example. Section 5 contains our conclusions and an outlook for future work.
2. The M Algebra and $osp(32|1)$

Consider the semigroup $S_E^{(2)} = \{\lambda_0, \lambda_1, \lambda_2, \lambda_3\}$ defined by the multiplication law

$$\lambda_\alpha \lambda_\beta = \begin{cases} 
\lambda_{\alpha + \beta}, & \text{when } \alpha + \beta \leq 2 \\
\lambda_3, & \text{when } \alpha + \beta \geq 3
\end{cases}.$$  

(1)

This semigroup is clearly finite and abelian.

The M algebra can be obtained from $osp(32|1)$ in a three-step fashion, as described in detail in Refs. [9, 17] and depicted in a symbolic way in Fig. 1. Consider first the direct product $S_E^{(2)} \times osp(32|1)$. This product is by itself a Lie superalgebra, as can be easily checked. A subalgebra can be extracted by cleverly removing some sectors of this "$S_E^{(2)}$-expanded algebra," as shown in Fig. 1 (b). Last but not least, all sectors of the form $\lambda_0 \times osp(32|1)$ are mapped to zero, effectively removing them from the algebra and abelianizing all commutators formerly valued on this sector. The resulting superalgebra [see Fig. 1 (c)] is isomorphic to the M algebra.

Figure 1. The M algebra can be obtained from $osp(32|1)$ as an abelian semigroup expansion, as briefly explained in the text and in Refs. [9, 17]. Here $V_0$, $V_1$ and $V_2$ are distinct subspaces of $osp(32|1)$; $V_0$ corresponds to the Lorentz subalgebra, $V_1$ to the fermions and $V_2$ to the remaining bosonic generators.

The concept of expansion of Lie algebras was first proposed de Azcárraga et al. in Ref. [18] and generalized using abelian semigroups by these authors in Ref. [17].

An invariant tensor of rank $n + 1$ for the M algebra can be constructed from one for $osp(32|1)$ by the methods described in [17]. The nonzero components fall into three classes: first, there is one of the form $\langle J^{n+1}\rangle$, where $J$ denotes Lorentz rotations. Second, there are three of the form $\langle J^n B \rangle$, where $B$ is some bosonic generator different from $J$. The third and last class is of the form $\langle Q J^{n-1} \bar{Q} \rangle$, where $Q$ is a 32-component Majorana spinor charge. The explicit form for these reads

$$\langle J^{n+1}\rangle_M = \alpha_0 \langle J^{n+1}\rangle_{osp},$$  

(2)

$$\langle J^n B \rangle_M = \alpha_2 \langle J^n B \rangle_{osp},$$  

(3)

$$\langle Q J^{n-1} \bar{Q} \rangle_M = \alpha_2 \langle Q J^{n-1} \bar{Q} \rangle_{osp},$$  

(4)

where $\alpha_0$ and $\alpha_2$ are arbitrary coupling constants.

3. The M Supergravity Lagrangian

The gauge-invariant Lagrangian we propose has the form

$$L = H_\alpha e^\alpha + \frac{1}{2} H_{ab} t_{ab}^2 + \frac{1}{5!} H_{abcde} b_{abcde} - \frac{5}{2} \bar{\psi} \gamma^\mu \gamma_5 \mathcal{R} \psi,$$  

(5)
where \( e^a \) is the graviton, \( b_2^{ab} \) and \( b_5^{abcde} \) are antisymmetric Lorentz tensors and \( \psi \) is the gravitino. The tensors \( H_a, H_{ab}, H_{abcde} \) and \( R \) are defined in terms of the Lorentz curvature and the invariant tensor derived from the \( S \)-expansion.

The field equations associated with variations of \( e^a, b_2^{ab}, b_5^{abcde} \) and \( \psi \) read

\[
\begin{align*}
H_a &= 0, \\
H_{ab} &= 0, \\
H_{abcde} &= 0, \\
R D_\omega \psi &= 0,
\end{align*}
\]

while the one for the spin connection \( \omega_{ab} \) has the form

\[
L_{ab} - 10 \left( D_\omega \bar{\psi} \right) Z_{ab} (D_\omega \psi) + 5 H_{abc} (T^c + \frac{1}{16} \bar{\psi} \Gamma^c \psi) + \frac{5}{2} H_{abcd} (D_\omega b^{cd} - \frac{1}{16} \bar{\psi} \Gamma^{cd} \psi) + \frac{1}{24} H_{abc1 \cdots c_5} (D_\omega b^{c_1 \cdots c_5} + \frac{1}{16} \bar{\psi} \Gamma^{c_1 \cdots c_5} \psi) = 0,
\]

where \( L_{ab}, H_{abc}, H_{abcd}, H_{a1 \cdots a_7} \) and \( Z_{ab} \) are further Lorentz tensors. Dirac matrices are denoted by \( \Gamma_a \).

This Lagrangian and equations of motion define a Supergravity theory in eleven-dimensional spacetime whose relation to the standard CJS one remains to be deciphered. The strength of this theory lies on its clear geometrical formulation; all independent fields can be interpreted as components of a one-form gauge connection for the M algebra, with the Lagrangian being invariant under gauge transformations. The algebra satisfied by these transformations (the M algebra, by construction) closes off-shell without the need for auxiliary fields.

4. Dynamics
In this section we probe the M Supergravity introduced in section 3 by analyzing how a geometrical ansatz devised by Hassaïne et al. [19] for a related CS Supergravity fits within the present scheme.

Let spacetime \( M \) have the form \( M = X_{d+1} \times S^{10-d} \), where \( X_{d+1} \) is a warped product of \( \mathbb{R} \) and a \( d \)-dimensional manifold \( M_d \). The criterion of propagating degrees of freedom around this solution demands that we pick \( d = 4 \). This solves eq. (6) provided \( M_d \) is made to satisfy Einstein’s equations with a positive cosmological constant.

The remaining equations of motion place severe constraints on the geometry. Allowing for nonzero eleven-dimensional torsion while still keeping a torsionless four-dimensional world, the transversal components camouflage as some sort of “matter” from the point of view of a four-dimensional observer. These provide with some room for geometrical degrees of freedom to deviate from a trivial solution.

More work on this issue is clearly needed in order to fully understand the implications of M Supergravity on the four-dimensional world.

5. Conclusions and Outlook
We have introduced a new Lagrangian for a Supergravity theory in eleven-dimensional spacetime. The theory possesses a firm geometrical interpretation as a gauge system for the M algebra. The Lagrangian is a transgression form, thus guaranteeing from the outset the invariance under gauge
transformations. The symmetric invariant tensor needed in order to define the theory has been
derived through the novel technique of abelian semigroup expansion.

Demanding propagating degrees of freedom around a tentative “vacuum” solution, a four-
dimensional universe satisfying Einstein’s equations with a positive cosmological constant is seen
to emerge. The full solution of the system may however prove too restrictive to yield realistic
dynamics. Torsion has been argued to provide a way to break some of the chains imposed by
the field equations on the geometry.

The interpretation of the M algebra as the S-expansion of \(osp(32|1)\) opens up a new web
of possibilities for finding a truly fundamental symmetry. The choice of abelian semigroup as
well as the tuning of the internal knobs within the S-expansion method cast the M algebra as
but one from a family of superalgebras sharing a host of properties whose physical consequences
remain to be explored.

Acknowledgments
F. I. and E. R. wish to thank P. Minning for having introduced them to so many beautiful
topics, especially that of semigroups. They are also grateful to D. Lüst for his kind hospitality
at the Arnold Sommerfeld Center for Theoretical Physics in Munich, where part of this work
was done. F. I. and E. R. were supported by grants from the German Academic Exchange
Service (DAAD) and from the Universidad de Concepción (Chile). P. S. was supported by
Fondo Nacional de Desarrollo Científico y Tecnológico (FONDECYT) Grant 1040624 and by
Universidad de Concepción through Semilla Grants 205.011.036-1S and 205.011.037-1S.

References
[1] Cremmer E, Julia B and Scherk J 1978 Phys. Lett. B 76 409.
[2] D’Auria R and Fré P 1982 Nucl. Phys. B 201 101, Erratum-ibid. B 1982 206 496
[3] Bandos I A, de Azcárraga J A, Izquierdo J M, Picón M and Varela O 2004 Phys. Lett. B 596 145 (Preprint arXiv: hep-th/0406020)
[4] Bandos I A, de Azcárraga J A, Picón M and Varela O 2005 Annals Phys. 317 238 (Preprint arXiv: hep-th/0409100)
[5] Achúcarro A and Townsend P K 1986 Phys. Lett. B 180 89
[6] Chamedélline A H 1990 Nucl. Phys. B 346 213
[7] Bañados M, Troncoso R and Zanelli J 1996 Phys. Rev. D 54 2605 (Preprint arXiv: gr-qc/9601003)
[8] Edelstein J D and Zanelli J 2006 J. Phys.: Conf. Series 33 83, (Preprint arXiv: hep-th/0605186)
[9] Izaurieta F, Rodríguez E and Salgado P 2006 Eleven-Dimensional Gauge Theory for the M Algebra as an
Abelian Semigroup Expansion of \(osp(32|1)\) (Preprint arXiv: hep-th/0606225)
[10] Townsend P K 1995 P-Brane Democracy (Preprint arXiv: hep-th/9507048)
[11] Borowiec A, Ferraris M and Francaviglia M 2003 J. Phys. A 36 2589 (Preprint arXiv: hep-th/0301146)
[12] Borowiec A, Fatibene L, Ferraris M and Francaviglia M 2006 Int. J. Geom. Meth. Mod. Phys. 3 755 (Preprint
arXiv: hep-th/0511060)
[13] Izaurieta F, Rodríguez E and Salgado P 2005 On Transgression Forms and Chern–Simons (Super)Gravity
Preprint arXiv: hep-th/0512014.
[14] Izaurieta F, Rodríguez E and Salgado P 2006 The Extended Cartan Homotopy Formula and a Subspace
Separation Method for Chern–Simons Supergravity Preprint arXiv: hep-th/0603061
[15] Mora P 2003 Transgression Forms as Unifying Principle in Field Theory Ph.D. Thesis, Universidad de la
República (Preprint arXiv: hep-th/051255)
[16] Mora P, Olea R, Troncoso R and Zanelli J 2006 JHEP 0602 067 (Preprint arXiv: hep-th/0601081)
[17] Izaurieta F, Rodríguez F and Salgado P 2006 Expanding Lie (Super)Algebras through Abelian Semigroups
Preprint arXiv: hep-th/0606215
[18] de Azcárraga J A, Izquierdo J M, Picón M and Varela O 2003 Nucl. Phys. B 662 185 Preprint arXiv: hep-
th/0212347
[19] Hassaïne M, Troncoso M and Zanelli J 2004 Phys. Lett. B 596 132, Preprint arXiv: hep-th/0306258