Sufficient variable screening via directional regression with censored response

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Abstract

We in this paper propose a directional regression based approach for ultrahigh dimensional sufficient variable screening with censored responses. The new method is designed in a model-free manner and thus can be adapted to various complex model structures. Under some commonly used assumptions, we show that the proposed method enjoys the sure screening property when the dimension \( p \) diverges at an exponential rate of the sample size \( n \). To improve the marginal screening method, the corresponding iterative screening algorithm and stability screening algorithm are further equipped. We demonstrate the effectiveness of the proposed method through simulation studies and a real data analysis.

Key Words: Sufficient dimension reduction, Sufficient Variable Selection; Sure independence screening; Ultrahigh dimensional covariates.
1 Introduction

Data sets collected in many contemporary scientific areas are ultrahigh dimensional and too complex to be analyzed through classical statistical methods. Consider data observed from a random sample of size $n$ from the distribution of $(Y, X)$, where $Y$ is a scalar response, $X = (x_1, \ldots, x_p)^T$ is a $p$-dimensional column vector of covariates, and the joint distribution of $(Y, X)$ is fully nonparametric. With an ultrahigh dimension $p >> n$, it is of great interest to identify $A \subset \{1, \ldots, p\}$ such that $X_A = \{x_k : k \in A\}$ is truly related to the response. To fulfill the goal of model-free variable selection based on the training data, Yin and Hilafu (2015) introduced the concept of sufficient variable selection as finding the smallest covariate set $X_A$ with $A \subset \{1, \ldots, p\}$ satisfying

$$Y \perp X \mid X_A,$$

where $\perp$ stands for independence and $\mid$ stands for conditioning. For convenience, in what follows we name both $I \subseteq \{1, \ldots, p\}$ and $X_I = \{x_k : k \in I\}$ as covariate set. If it is too hard to find the smallest covariate set $A$ satisfying (1.1) especially when $p >> n$, a weaker goal is to find a covariate set containing $A$ with size as small as possible, which is referred to as sufficient variable screening and is the focus of this paper.

Research on sufficient variable screening in ultrahigh dimensional setting has gained considerable momentum in recent years. Li et al. (2012b) and Shao and Zhang (2014) proposed to use marginal distance correlation and marginal martingale difference divergence for sufficient variable screening. Noticing the close relationship between sufficient variable selection and sufficient dimension reduction (Li, 1991; Cook, 1998), Yu et al. (2014), Yin and Hilafu (2015) and Yu et al. (2016) developed different dimension reduction based screening methods.

In many biomedical studies, the response are often censored rather than fully observed. We consider survival data in which $T$ is the true lifetime, $C$ is the censoring time and we only observe $T^o = \min\{T, C\}$ and the censoring indicator $\delta = I(T \leq C)$. Sufficient variable
selection with censored response is finding \( \mathcal{A} \) in (1.1) with \( Y \) replaced by \((T,C)\), i.e.,
\[
(T,C) \Perp \mathbf{X} \mid \mathbf{X}_{\mathcal{A}}.
\] (1.2)
While our focus is (1.2), we can only observe \((T^0, \delta)\), instead of \((T,C)\).

There exists very limited amount of work on model-free variable screening with censored responses. Assuming \( T \Perp C \mid \mathbf{X} \), the quantile adaptive sure independence screening procedure proposed by He et al. (2013) can be naturally extended to survival analysis. Li et al. (2016) proposed a survival impact index, which characterizes the impacts of a covariate on the distribution of true lifetime \( T \) by evaluating the absolute deviation of the covariate-stratified survival distribution from the unstratified survival distribution. The proposed survival impact index based screening seems to take some advantages over quantile adaptive sure independence screening when dealing with censored responses.

We in this paper give a modification of the directional regression (Li and Wang, 2007) capable for sufficient dimension reduction with censored response, and then characterize a suitable modified directional regression index for sufficient variable screening. The sure screening property is established in the ultrahigh dimensional setting, i.e., with probability tending to one, the smallest covariate set \( \mathcal{A} \) is contained in the set of covariates selected by our proposed procedure. We also discuss the limitations of such modified directional regression index and propose a refined iterative procedure of our screening approach. To further enhance the stability of variable screening, we follow Meinshausen and Bühlmann (2010) and He and Lin (2011) to integrate the resampling scheme into our proposal. After screening, the selected covariate set may contain some irrelevant covariates, but its size is much smaller than \( n \) so that we may apply variable selection or dimension reduction using an existing method to further reduce the size or dimension of the selected covariate set. Our approach are examined through simulation studies and an application to the diffuse large-B-cell lymphoma microarray data (Rosenwald et al., 2002).
2 Modified Directional Regression Index

To derive an index for covariate screening, we first reveal a relationship between sufficient variable selection and sufficient dimension reduction, another perspective in reducing covariate dimension. As a by-product, we extend one method in sufficient dimension reduction, the directional regression, to survival data with censoring, which leads to an index for sufficient variable screening.

Sufficient dimension reduction aims to identify a linear function of $X$ with dimension lower than $p$, without losing information. To be specific, we seek a $p \times d$ matrix $B$ with the smallest $d$ such that

$$(T, C) \perp \! \! \! \perp X \mid B^T X. \quad (2.1)$$

The linear space generated by columns of $B$ is called the central subspace and denoted as $S_{(T,C)\mid X}$. The following result reveals a deep connection between sufficient variable selection (1.2) and sufficient dimension reduction (2.1) for censored responses.

**Proposition 2.1.** Let $\beta_1, \ldots, \beta_d$ be columns of $B$ in (2.1) and $e_k$ be the $p \times 1$ vector whose $k$th element is 1 all other elements are 0. Then, $\sum_{j=1}^d |e_k^T \beta_j| > 0$ for $k \in A$ and $\sum_{j=1}^d |e_k^T \beta_j| = 0$ for $k \notin A$, where $A$ is given in (1.2).

This result tells us that $B$ in sufficient dimension reduction can be also used for sufficient variable selection. Inspired by this, in the following we first extend the directional regression (Li and Wang, 2007) to find the central space $S_{(T,C)\mid X}$ using survival data with censoring.

Let $Z = \Sigma^{-1/2} (X - \mu)$ be the standardized covariate, where $\mu = E(X)$ and $\Sigma = \text{Var}(X)$. Then, $S_{(T,C)\mid X} = \Sigma^{-1/2} S_{(T,C)\mid Z}$. A key result for the success of the directional regression in Li and Wang (2007) is that, if $(T, C)$ is observed, the column space of $E[2I_p - E\{(Z - \tilde{Z})(Z - \tilde{Z})^T \mid T, \tilde{T}, C, \tilde{C}\}]^2$ is equal to $S_{(T,C)\mid Z}$, where $I_p$ is the identity matrix of order $p$ and $(\tilde{Z}, \tilde{T}, \tilde{C})$ is an independent copy of $(Z, T, C)$. However, in survival analysis $(T, C)$ is unobservable; instead, we observe $(T^o, \delta)$. The next proposition extends the result in Li and Wang (2007) to the survival data with censoring.
Proposition 2.2. Let $M = E[2I_p - E\{(Z - \tilde{Z})(Z - \tilde{Z})^T \mid T^o, \tilde{T}^o, \delta, \tilde{\delta}\}]^2$, where $(\tilde{Z}, \tilde{T}^o, \tilde{\delta})$ is an independent copy of $(Z, T^o, \delta)$.

(i) Suppose that

(A1) For any $\nu \in \mathbb{R}^p$ and $\nu \perp S_{(T,C)|Z}$, $E(\nu^T Z \mid PZ)$ is a linear function of $Z$ for any projection $P$ onto $S_{(T,C)|Z}$;

(A2) For any $\nu \in \mathbb{R}^p$ and $\nu \perp S_{(T,C)|Z}$, $\text{Var}(\nu^T Z \mid PZ)$ is nonrandom for any projection $P$ onto $S_{(T,C)|Z}$.

Then column space of $M$ is contained in $S_{(T,C)|Z}$.

(ii) Suppose further that

(A3) For any $\psi \in S_{(T,C)|Z}$, $\psi \neq 0$, the random variable $E\{[\psi^T (Z - \tilde{Z})]^2 \mid T^o, \tilde{T}^o, \delta, \tilde{\delta}\}$ is not equal to a constant almost surely.

Then column space of $M$ is equal to $S_{(T,C)|Z}$.

Conditions (A1) and (A2) are known as linear conditional mean condition and constant conditional variance condition in the the sufficient dimension reduction literature; see Shao et al. (2007) and Li and Wang (2007) for more discussions. Condition (A3) is generally considered to be very mild. See Li et al. (2005) for more details.

Proposition 2.2 suggests that we can utilize $M$ for estimating $S_{(T,C)|Z}$. In applications we use $G$, a discretized version of $M$, to estimate $S_{(T,C)|Z}$. We partition the sample space of the uncensored observations with $\delta = 1$ into $H_1$ non-overlapping intervals $I_{11}, \ldots, I_{1H_1}$, and the sample space of censoring time $C$ with $\delta = 0$ into $H_0$ non-overlapping intervals $I_{01}, \ldots, I_{0H_0}$. Let $p_{ij} = E[I(\delta = l, T^o \in I_{ij})]$ and $D_{ijlm} = E[(Z - \tilde{Z})(Z - \tilde{Z})^T \mid \delta = i, T^o \in I_{ij}, \tilde{\delta} = l, \tilde{T}^o \in I_{lm}]$, where $(i, j, l, m) \in \{i, j, l, m : i, l = 0 \text{ or } 1, j = 1, \ldots, H_i, m = 1, \ldots, H_l\}$. Then $G$ is expressed as follows

$$G = \sum_{ij} \sum_{lm} p_{ij} p_{lm} (2I_p - D_{ijlm})^2. \quad \text{(2.2)}$$

We can recover $S_{(T,C)|Z}$ through the eigen-decomposition $G \eta_i = \lambda_i \eta_i$, where $\lambda_i$’s are scalars and $\eta_i$’s are $p \times 1$ vectors, and obtain $B = (\Sigma^{-1/2}\eta_1, \ldots, \Sigma^{-1/2}\eta_d)$. 


As $\Sigma^{-1/2}$ is involved in $M$, $G$, and $B$, the classical sufficient dimension reduction methods fail to work when $p > n$ unless we have a good estimator of $\Sigma^{-1/2}$.

For sufficient variable selection, we do not need the entire matrix $G$ in (2.2). Proposition 2.3 below shows that the following marginal utility of $G$,

$$g'_{k} = \sum_{ij} \sum_{lm} p_{ij} p_{lm} [e_{k}^{T} \Sigma^{-1/2}(2I_{p} - D_{ijlm}) \Sigma^{-1/2} e_{k}]^{2},$$  

(2.3)

is a perfect index for sufficient variable selection. Note that $e_{k}^{T} \Sigma^{-1/2}(2I_{p} - D_{ijlm}) \Sigma^{-1/2} e_{k}$ is not the $k$th diagonal element of the matrix $2I_{p} - D_{ijlm}$ in (2.2), but the $k$th diagonal element of the matrix $\Sigma^{-1/2}(2I_{p} - D_{ijlm}) \Sigma^{-1/2}$.

**Proposition 2.3.** If conditions (A1)-(A3) hold, then $g'_{k} > 0$ if $k \in A$ and $g'_{k} = 0$ if $k \notin A$.

The next result gives an alternative expression of $g'_{k}$, which is useful for our derivation.

**Lemma 2.4.** Let $U_{ij} = E[ZI(\delta = l, T^{o} \in I_{ij})]$ and $V_{ij} = E[ZZ^{T}I(\delta = l, T^{o} \in I_{ij})]$. Then

$$g'_{k} = 2 \sum_{ij} p_{ij} [e_{k}^{T} \Sigma^{-1/2}(p_{ij}^{-1} V_{ij} - I_{p}) \Sigma^{-1/2} e_{k}]^{2} + 4 \left( \sum_{ij} p_{ij}^{-1} e_{k}^{T} \Sigma^{-1/2} U_{ij} U_{ij}^{T} \Sigma^{-1/2} e_{k} \right)^{2},$$  

(2.4)

However, $g'_{k}$ in (2.3) or (2.4) still involves $\Sigma^{-1/2}$ which is hard to estimate when $p$ is bigger than or comparable to $n$. We then follow the idea in independence variable screening (Fan and Lv, 2008, Li et al., 2012b, and Yu et al., 2016), i.e., we replace $\Sigma^{-1/2}$ in (2.4) by $I_{p}$ and obtain the following modified directional regression index,

$$g_{k} = 2 \sum_{ij} p_{ij} (V_{ijk}/p_{ij} - 1)^{2} + 4 \left( \sum_{ij} U_{ijk}^{2}/p_{ij} \right)^{2},$$  

(2.5)

where $U_{ijk} = E[z_{k}I(\delta = l, T^{o} \in I_{ij})]$ and $V_{ijk} = E[z_{k}^{2}I(\delta = l, T^{o} \in I_{ij})]$, $z_{k} = (x_{k} - \mu_{k})/\sigma_{k}$, $\mu_{k} = E(x_{k})$, and $\sigma_{k}^{2} = \text{Var}(x_{k})$. Although $g_{k}$ in (2.5) is not a prefect index for sufficient variable selection as $\Sigma^{-1/2}$ may be incorrectly treated as $I_{p}$, it is good enough for sufficient variable screening, i.e., finding a set containing $A$ in (1.2) under some conditions. The following result is an example, in which the conditions are similar to those in Mai and Zou (2015) and Yu et al. (2016).
Proposition 2.5. Assume conditions (A1)-(A3). Suppose also that Cov($x_i, x_j$) has the same sign for $i, j \in A$, and that there exists $h \in \{1, \ldots, d\}$ such that the $(j, h)$th element of $B$ in (2.1) have the same sign for all $j \in A$. Then $g_k > 0$ if $k \in A$.

3  Sure Independence Screening

In this section we show that variable screening by using the index $g_k$ in (2.5) holds some asymptotic properties under some conditions. Procedures with weaker conditions are considered in the next section. Let $(x_{ki}, t^o_i, \delta_i), i = 1, \ldots, n, k = 1, \ldots, p$, be observations from the random sample from $(X, T^o, \delta)$, $\hat{\mu}_k = \sum_{i=1}^{n} x_{ki}/n$, $\hat{\sigma}_k = (\sum_{i=1}^{n} (x_{ki} - \hat{\mu}_k)^2/n)^{1/2}$, $\hat{z}_{ki} = (x_{ki} - \hat{\mu}_k)/\hat{\sigma}_k$, $\hat{p}_{lj} = \sum_{i=1}^{n} I(\delta_i = l, t^o_i \in I_{lj})/n$, $\hat{U}_{ljk} = \sum_{i=1}^{n} \hat{z}_{ki} I(\delta_i = l, t^o_i \in I_{lj})/n$, and $\hat{V}_{ljk} = \sum_{i=1}^{n} \hat{z}_{ki}^2 I(\delta_i = l, t^o_i \in I_{lj})/n$. A sample estimator of $g_k$ in (2.5) is $\hat{g}_k$ defined by (2.5) with $p_{lj}, U_{ljk},$ and $V_{ljk}$ replaced by $\hat{p}_{lj}, \hat{U}_{ljk},$ and $\hat{V}_{ljk}$, respectively. We select the set of covariates such that $\hat{g}_k$ is large enough. Define

$$\hat{A} = \{k : \hat{g}_k \geq \gamma, 1 \leq k \leq p\},$$

(3.1)

where $\gamma$ is a threshold to be specified later. To study the theoretical property of $\hat{A}$ in (3.1), we consider the following conditions:

(C1) $p > n$ and $\log p = O(n^\xi)$ for some $\xi \in (0, 1 - 2\kappa)$, where $\kappa$ is given in condition (C3);

(C2) There exist some $0 < \varsigma < 1/4$ such that $E\{\exp(tz_k^2)\} \leq K_0$ for $1 \leq k \leq p$ and all $|t| \leq \varsigma$, where $K_0$ is a fixed constant;

(C3) $\min_{k \in A} g_k > 2c_0n^{-\kappa}$ for some constants $c_0 > 0$ and $0 \leq \kappa \leq 1/2$.

Condition (C1) was also used by Fan and Lv (2008) and Li et al. (2012a), which allows $p$ to be as large as an exponential of the sample size $n$. Condition (C2) assumes that all covariates have an exponential-type tails, which is a common technique condition in ultrahigh dimensional data analysis; see, for example, Cai et al. (2011). Condition (C3) is naturally motivated from Proposition 2.5, and requires that the index $g_k$ for $k \in A$ is not too small, which is also a common condition in the literature of sure independence screening (Fan and Lv, 2008; Li et al., 2012a,b).
The next theorem confirms the sure screening property of $\hat{A}$.

**Theorem 3.1.** (i) Assume conditions (C1) and (C2). Then
\[
\Pr \left\{ \max_{1 \leq k \leq p} |\hat{g}_k - g_k| \geq C_0 (\log p/n)^{1/2} \right\} \leq 72p^{-\tau-1},
\]
where $\tau > 0$ is a constant and $C_0$ is defined in (6.12) in the Appendix.

(ii) Additionally, if condition (C3) also holds and $\gamma \leq c_0 n^{-\kappa}$, then
\[
\Pr \left\{ A \subseteq \hat{A} \right\} \geq 1 - 72p^{-\tau-1}, \tag{3.2}
\]
where $\hat{A}$ is given by (3.1).

Since $g_k$ in (2.5) is a modified directional regression index and Theorem 3.1 indicates that the probability in (3.2) converges to one as $n$ diverges to infinity, we name the proposed covariate screening procedure as the modified directional regression-sure independence screening (MDR-SIS) method. Note that $\Sigma^{-1/2} = I_p$ is assumed in the derivation of $g_k$, but it is not needed in establishing the result in Theorem 3.1, as long as (C1)-(C3) hold true. In the next section we obtain some further results in the case where (C3) may be violated.

The threshold value $\gamma$ depends on constants $c_0$ and $\kappa$ in (C3), which is unknown in real applications. We follow the convention developed in Fan and Lv (2008) and define the screened covariate set as
\[
\hat{A}^* = \{ k : \hat{g}_k \geq \hat{g}_{d_n} \}, \tag{3.3}
\]
where $\hat{g}_{d_n}$ is the $d_n$th largest ranked index among all $\hat{g}_k$’s. Following Fan and Lv (2008), $d_n$ can be set as $\lfloor n/\log n \rfloor$, where $\lfloor a \rfloor$ denotes the integer part of $a$. Theorem 3.1 together with Theorem 1 in Fan and Lv (2008) guarantee $\Pr(A \subseteq \hat{A}^*)$ converges to one as $n \to \infty$.

Let $X_B = \{ x_k : k \in B \}$ be the smallest covariate set related to the life time $T$, i.e., $B$ satisfies $T \perp X | X_B$. Sometimes we are interested in identifying $B$ instead of $A$. For example, if we assume $T \perp C | X$, which is typically needed for many survival analysis methods although it is not needed for the asymptotic property of MDR-SIS, then $T \perp C | X$ and $T \perp X | X_B$ imply $T \perp C | X_B$ so that survival analysis can be carried out using $X_B$. However, identifying $A$ may result in a more efficient analysis if information on $C | X$ is
useful.

Since $B \subseteq A$, the sure screening property $\Pr(B \subseteq \hat{A}^*) \to 1$ can still be achieved based on Theorem 3.1. Unless $C \perp \perp X \mid X_B$, $B$ is a strict subset of $A$. Even if we focus on $B$ only, it is unnecessary to do covariate screening to find a $\hat{B}^*$ with $\Pr(B \subseteq \hat{B}^*) \to 1$, because both $\hat{B}^*$ and $\hat{A}^*$ are screening methods aimed to reduce the size of covariate set to a manageable number $< n$ and a further dimension reduction or variable selection can be applied to $\hat{A}^*$ as the size of $\hat{A}^*$ is much smaller than $n$, i.e., $d_n/n \to 0$.

4 Enhanced Screening with Iteration and Resampling

4.1 Iterative variable screening

Condition (C3) plays a key role for the sure independence screening property of MDR-SIS. However, (C3) may be violated since $g_k$ ignores information contained in $\Sigma$. The next result identifies a situation where (C3) does not hold.

Proposition 4.1. Let $\beta_1, \ldots, \beta_d$ be columns of $B$ in (2.1). For any $\Sigma$, if there exists $k \in A$ such that $\sum_{i=1}^d |e_k^T \Sigma \beta_i| = 0$, then $g_k = 0$ and, hence, (C3) is violated.

In the situation described by Proposition 4.1, the sure screening property can not be guaranteed. To circumvent this issue, we should handle the correlations among covariates and consider iterative screening. Suppose that we have already selected a covariate set $X_F = \{x_k : k \in F\}$, where $F \subset \{1, \ldots, p\}$. Define $\mu_F = E(X_F)$ and $\Sigma_F = \text{Var}(X_F)$. For any $e \notin F$, let $\Sigma_{F,e} = \text{Cov}(X_F, x_e)$ and $x_{e|F} = x_e - \Sigma_{F,e}^T \Sigma_F^{-1} X_F$ be the residual of $x_e$ regressed on $X_F$. Then, $\text{Cov}(x_{e|F}, X_F) = 0$, which suggests that we can adopt the marginal utility of modified directional regression based on $(x_{e|F}, T^\circ, \delta)$ as an index for iterative screening. Define $\mu_{e|F} = \mu_e - \Sigma_{F,e}^T \Sigma_F^{-1} \mu_F$, $\sigma_{e|F}^2 = \sigma_e^2 - \Sigma_{F,e}^T \Sigma_F^{-1} \Sigma_{F,e}$, and $z_{e|F} = (x_{e|F} - \mu_{e|F})/\sigma_{e|F}$ as the standardized version of $x_{e|F}$. Then we define the following iterative modified directional
regression index:

\[
ge_{e|\mathcal{F}} = 2 \sum_{lj} p_{lj} (V_{lje|\mathcal{F}}/p_{lj} - 1)^2 + 4 \left( \sum_{lj} U_{lje|\mathcal{F}}^2/p_{lj} \right)^2, \quad e \notin \mathcal{F},
\]

where \(U_{lje|\mathcal{F}} = E[z_{e|\mathcal{F}} I(\delta = l, T^o \in I_{lj})]\) and \(V_{lje|\mathcal{F}} = E[z_{e|\mathcal{F}}^2 I(\delta = l, T^o \in I_{lj})]\). The next proposition illustrates the advantage of the proposed iterative screening method.

**Proposition 4.2.** Let \(\mathcal{F}\) be a nonempty subset of \(\{1, ..., p\}\). Suppose that

(C4) \(\min_{k \in A, i = 1, \ldots, d} |\beta_{ik}| > c_1 n^{-\theta}\) for some constants \(c_1 > 0\) and \(0 < \theta \leq 1/8\), where \(\beta_{ik}\) is the \((i, k)\)th element of \(B\) in (2.1);

(C5) \(\sigma_{e|\mathcal{F}}^2 \geq c_2\) for some constant \(c_2 > 0\), where \(e \notin \mathcal{F}\).

Then \(g_{e|\mathcal{F}} > 2c_0 n^{-\kappa}\) for \(e \in A\) with some constants \(c_0 > 0\) and \(0 \leq \kappa \leq 1/2\).

Condition (C4) is a mild condition previously used by Fan and Lv (2008). Condition (C5) means that the \(e\)th relevant covariate missed in the previous steps should not be expressed only by the set of covariates selected by previous steps, which is a general condition under iteration construction.

The result of this proposition illustrates that utilizing the index \(g_{e|\mathcal{F}}\) is able to identify the informative predictors missed by MDR-SIS. To illustrate, suppose that \(\hat{A}_1^* = \hat{A}^*\) as define by (3.3) is selected by MDR-SIS. Suppose that we carry out one iteration to obtain a covariate set \(\hat{A}_2 = \{e : e \notin \hat{A}_1^*, \hat{g}_{e|\hat{A}_1^*} \geq \hat{g}_{e|\hat{A}_1^*, q}\}\), where \(\hat{g}_{e|\hat{A}_1, q}\) is the \(q\)th largest ranked index among all \(\hat{g}_{e|\hat{A}_1}\)'s. By Proposition 4.1, \(\hat{A}_2\) may recover some relevant covariates missed by \(\hat{A}_1^*\) selected by MDR-SIS, with an appropriate choice of \(q\). The covariate set after iteration is \(\hat{A}_1^* \cup \hat{A}_2\). Numerical studies show that \(q\) can be much smaller than \(\lfloor n/\log n \rfloor\).

Although \(\hat{A}_1^* \cup \hat{A}_2\) is better than \(\hat{A}_1^* = \hat{A}^*\) in terms of containing relevant covariates, its size is always larger than the size of \(\hat{A}^*\). Hence, to apply the iterative variable screening, we do not have to start with \(\hat{A}_1^* = \hat{A}^*\), especially when we doubt about whether MDR-SIS can select all relevant covariates. Instead, we may start with a \(\hat{A}_1^*\) smaller than \(\hat{A}^*\) and set the size of final covariate sets selected after iterations to be the same as that of \(\hat{A}^*\). This leads to the following general iterative procedure for covariate screening.
Step 1. Based on \( \hat{g}_k \ (k = 1, \ldots, p) \), we select \( p_1 \) covariates by MDR-SIS. Denote the set of indices of selected covariates by \( \hat{A}_1^* \).

Step 2. For \( e \notin \hat{A}_1^* \), we estimate \( g_{e|\hat{A}_1^*} \) by a sample estimator \( \hat{g}_{e|\hat{A}_1^*} \). Based on \( \hat{g}_{e|\hat{A}_1^*} \), we select \( p_2 \) covariates by MDR-SIS with the resulting covariate set denoted by \( \hat{A}_2^* \).

Step 3. Repeat Step 2 until the total selected number of covariates is \( d_n \). The final selected covariate set is then \( \hat{A}_I = \hat{A}_1^* \cup \cdots \cup \hat{A}_S^* \), where \( \sum_{v=1}^S p_v = d_n \).

We name this iterative procedure as the modified directional regression-iterative sure independent screening (MDR-ISIS) method. Under conditions (C1)-(C2) and (C4)-(C5), it can be shown similarly to Theorem 3.1 that \( \Pr(\mathcal{A} \subset \hat{A}_I) \to 1 \) as \( n \to \infty \). Some simulation results are presented in Section 5 for the selection of \( p_v \)’s and the results show that \( S = 2 \) works well under our simulated models.

### 4.2 Stability Screening

While MDR-ISIS is used to improve MDR-SIS in including all relevant covariates, the stability selection approach introduced in Meinshausen and Bühlmann (2010) is designed to reduce the number of falsely selected covariates through combining resampling with high dimensional variable selection. He and Lin (2011) adapted this resampling mechanism to iterative sure independence screening for genome-wide association studies. Along with their developments, we further propose the following procedure to improve MDR-ISIS. The algorithm is based on \( B \) independent subsamples of size \( n_s < n \) without replacement from the training data set. For the \( b \)th subsample, we apply MDR-SIS to select a candidate covariate set \( \hat{A}_I^{(b)} \). The stability screened covariate set based on this procedure is

\[
\hat{A}_S = \{ k : \pi_k \geq \pi_0 \}, \quad \pi_k = \frac{1}{B} \sum_{i=1}^B I(k \in \hat{A}_I^{(i)}),
\]

where \( I(\cdot) \) is the indicator function. Following He and Lin (2011), we prespecify threshold value \( \pi_0 \) to be 0.3 or 0.4 in practical use. We name this procedure as the modified directional regression-stability sure independence screening (MDR-SSIS) method. In Section 5, we compare MDR-SSIS with MDR-ISIS in simulations.
5 Numerical Results

In this section, we assess the performance of the proposed MDR-SIS, MDR-ISIS and MDR-SSIS by Monte Carlo simulation. We further examine the proposed screening procedure with an empirical analysis of a real-data example.

5.1 Simulation study

The covariate vector $X$ is generated from the multivariate normal distribution with mean 0 and covariance matrix $\Sigma$ whose $(i,j)$th element is $\rho^{|i-j|}$ with $\rho = 0, 0.4, \text{or } 0.8$ throughout our simulations. Let $\epsilon \sim N(0,1)$ be an error term independent of $X$ and the censoring time $C$. We consider the following five models representing various types of covariate functions with different degree of nonlinearity, and multiple failure and censoring distributions.

M1. $T = (2X^T\beta_1)^2 + 12\sin(3X^T\beta_2/7) + 0.2\epsilon$, $C \sim N(0,4) - N(5,1) + N(15,1)$, where $\beta_1$ and $\beta_2$ are $p \times 1$ vectors with their first six components being $(1,0,1,0,0,0)^T$ and $(0,0,0,0,1,1)^T$, respectively, and rest components being zeros.

M2. $T = (2X^T\beta_1)^2 + |8X^T\beta_2| + 0.2\epsilon$, $C \sim N(0,4) - N(5,1) + N(30,1)$, where $\beta_1$ and $\beta_2$ are same as those in (M1).

M3. $T = 10\sin(X^T\beta_1/4) + 4|X\beta_2^T| + 0.2\epsilon$, $C \sim N(0,4) - N(5,1) + N(15,1)$, where $\beta_1$ and $\beta_2$ are same as those in (M1).

M4. $T = \exp(X^T\beta_1) + |(X^T\beta_2)^3| + 0.2\epsilon$, $C \sim N(0,4) - N(5,1) + 4N(30,1)$, where $\beta_1$ and $\beta_2$ are $p \times 1$ vectors with their first six components being $(-4,4,3,0,0,0)^T$ and $(0,0,1,0,1,0)^T$, respectively, and rest components being zeros.

M5. $T = 1.5(X^T\beta_1)^2 + \exp(X^T\beta_2) + 0.2\epsilon$, $C = X^T\beta_3 + 8$, where $\beta_1$, $\beta_2$ and $\beta_3$ are $p \times 1$ vectors with their first six components being $(1,0,0,0,0,0)^T$, $(1,2,2,0,0,0)^T$ and $(0,0,1,0,0,1)^T$, and rest components being zeros.
In all models, $T \not\perp C \mid X$. In M1-M3, 4 relevant covariates for $T$ are $x_1, x_3, x_5,$ and $x_6$. In M4, 4 relevant covariates for $T$ are $x_1, x_2, x_3,$ and $x_5$. In M5, 3 relevant covariates for $T$ are $x_1, x_2,$ and $x_3$, and 2 relevant covariates for $C$ are $x_3$ and $x_6$.

We first fixed the sample size $n$ to be 200 and the dimension $p$ to be 400, and compare our method MDR-SIS with SII (Li et al., 2016) and QaSIS (He et al., 2013). To evaluate the performance of the 3 methods, we ran 500 simulations and, for each of the 3 methods, we computed the proportion that an individual relevant predictor was selected and the proportion that all relevant predictors were selected. The simulation results reported in Table 1 show that our method is the best among all methods in most cases. And for all the cases in which the other two methods perform well, our method performs at least better than the other two methods.

Table 2 reports the average computing time of three method with $p = 200$ and different values of $n$, or $n = 200$ and various values of $p$. The computations are performed using R on ECNU IBM Platform Application Center 9.1.3. We can see that our method is the most computational efficient among the three methods and is increasingly more efficient as $n$ and $p$ are larger. Also, SII is computational intensive, which may lead problems in applications with large $n$ and/or $p$.

Next, we consider $n = 300$ and $p = 2000$. As SII is very time consuming for $p = 2000$ in simulation, we only compare MDR-SIS with QaSIS under this setting. To assess the effect of $\alpha$ in QaSIS, we obtain results for QaSIS with $\alpha = 0.5$ and 0.7. The results reported in Table 3 show that MDR-SIS overwhelms QaSIS regardless of the choices of $\alpha$. Moreover, the performance of QaSIS can be influenced by the choice of $\alpha$.

Under model M5, only $C$ is related with $X_6$, which is denoted as relevant covariate 4 in Tables 1 and 3. Neither SII nor QaSIS can capture $X_6$, whereas MDR-SIS selects $X_6$ with high probability. This is expected since SII and QaSIS are not developed to search covariates related with $C$.

Now, we assess the performance of MDR-ISIS. From Tables 1 and 3, all the three methods
Table 1: Simulation proportions of each relevant covariate and all relevant covariates selected by MDR-SIS, SII, and QaSIS with $\alpha = 0.5$; $p = 400$, $n = 200$, $d_n = [n/\log n] = 37$, simulation replication 500

| model | $\rho$ | 1 | 2 | 3 | 4 | all | relevant covariate | 1 | 2 | 3 | 4 | all | relevant covariate | 1 | 2 | 3 | 4 | all | relevant covariate |
|-------|-------|---|---|---|---|-----|-------------------|---|---|---|---|-----|-------------------|---|---|---|---|-----|-------------------|
| M1    | 0     | 0.90 | 0.95 | 0.91 | 0.86 | 0.66 | 0.50 | 0.61 | 1.00 | 1.00 | 0.21 | 0.39 | 0.34 | 0.81 | 0.83 | 0.08 |
|       | 0.4   | 0.98 | 0.97 | 0.98 | 1.00 | 0.93 | 0.70 | 0.80 | 1.00 | 1.00 | 0.57 | 0.42 | 0.41 | 0.92 | 0.92 | 0.14 |
|       | 0.8   | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 0.80 | 0.83 | 0.90 | 0.90 | 0.57 |
|       | 0.4   | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 0.80 | 0.83 | 0.97 | 0.96 | 0.61 | 0.52 | 0.51 | 0.81 | 0.79 | 0.15 |
|       | 0.8   | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 | 0.82 | 0.93 | 0.98 | 0.95 | 0.74 |
| M2    | 0     | 0.90 | 0.88 | 0.88 | 0.89 | 0.62 | 1.00 | 1.00 | 0.60 | 0.60 | 0.36 | 0.98 | 1.00 | 0.68 | 0.70 | 0.44 |
|       | 0.4   | 0.91 | 0.88 | 1.00 | 1.00 | 0.80 | 1.00 | 1.00 | 0.98 | 0.95 | 0.94 | 0.98 | 0.97 | 0.92 | 0.91 | 0.80 |
|       | 0.8   | 0.99 | 1.00 | 1.00 | 1.00 | 0.99 | 1.00 | 1.00 | 1.00 | 1.00 | 0.96 | 0.97 | 1.00 | 0.99 | 0.93 |
| M3    | 0     | 0.97 | 0.97 | 0.87 | 0.77 | 0.63 | 1.00 | 1.00 | 1.00 | 0.10 | 0.10 | 0.58 | 0.59 | 0.81 | 0.87 | 0.26 |
|       | 0.4   | 0.25 | 0.88 | 1.00 | 0.95 | 0.19 | 0.96 | 1.00 | 0.10 | 0.29 | 0.28 | 0.33 | 0.51 | 0.96 | 0.98 | 0.16 |
|       | 0.8   | 0.52 | 1.00 | 1.00 | 1.00 | 0.52 | 0.57 | 1.00 | 1.00 | 1.00 | 0.57 | 0.66 | 0.97 | 1.00 | 1.00 | 0.65 |
| M4    | 0     | 1.00 | 0.99 | 1.000 | 0.99 | 0.98 | 1.00 | 1.00 | 0.22 | 0.22 | 0.22 | 0.96 | 0.53 | 0.91 | 0.50 | 0.25 |
|       | 0.4   | 1.00 | 1.00 | 1.000 | 1.00 | 0.10 | 1.00 | 1.00 | 0.15 | 0.15 | 0.15 | 0.96 | 0.68 | 0.85 | 0.53 | 0.33 |
|       | 0.8   | 1.00 | 1.00 | 1.000 | 1.00 | 0.08 | 1.00 | 1.00 | 0.88 | 0.88 | 0.88 | 0.84 | 0.93 | 0.91 | 0.53 | 0.43 |

Table 2: Computing time (in seconds) required by MDR-SIS, SII and QaSIS with $\alpha = 0.5$ under model M1

| $n$   | MDR-SIS | QaSIS | SII  | $n$   | MDR-SIS | QaSIS | SII  |
|-------|---------|-------|------|-------|---------|-------|------|
| 200   | 0.17    | 0.74  | 577.11 | 200   | 0.08    | 0.38  | 279.55 |
| 400   | 0.24    | 0.87  | 736.24 | 400   | 0.17    | 0.74  | 577.11 |
| 1000  | 0.37    | 1.58  | 1268.74| 1000  | 0.37    | 1.74  | 1439.10|
| 2000  | 0.51    | 2.98  | 2201.27| 2000  | 0.79    | 3.65  | 2975.08|
| 3000  | 0.85    | 5.35  | 3121.63| 3000  | 1.20    | 5.61  | 4324.52|
| 5000  | 0.81    | 11.16 | 4806.57| 5000  | 2.06    | 8.42  | 7033.50|
| 10000 | 1.32    | 34.41 | 9298.11| 10000 | 5.21    | 18.53 | 15855.51|
under consideration performed not well in model M4 with $\rho = 0.4$ and 0.8, where the first relevant covariate is missed by MDR-SIS with high frequency. Since MDR-SIS performed well in model M4 with $\rho = 0$, these results indicate that the phenomenon described in Proposition 4.1 occurs when $\rho \neq 0$. Thus, we run more simulations under M4 with $\rho = 0.4$ and 0.8 to show that MDR-ISIS picks up $X_1$ missed by MDR-SIS and hence improves the overall performance. To see the performance of MDR-ISIS when MDR-SIS already has a satisfactory performance, we include model M3 with $\rho = 0.8$. Furthermore, we check the influence of $p_1, \ldots, p_S$, the sizes of covariate sets in iteration steps, and $S$, the number of iterations. We include $S = 2$ and $S = 4$, nearly equal, increasing, and decreasing $p_j$'s, with $\sum_j p_j = d_n = \lfloor n / \log n \rfloor$, which is 37 when $n = 200$ and 52 when $n = 300$. The special case with $p_1 = d_n$ and $p_2 = p_3 = p_4 = 0$ is MDR-SIS without iteration.

The simulation proportions that all relevant covariates are selected are reported in Table 4. From Table 4, MDR-ISIS improves MDR-SIS when the latter does not performs well, and is slightly worse than MDR-SIS when MDR-SIS already has a satisfactory performance.
Table 4: Simulation proportions of all relevant covariates selected by MDR-SIS and MDR-ISIS with different sizes $p_j$’s; simulation replication 500

| sizes in iteration | $p_1$ | $p_2$ | $p_3$ | $p_4$ | M4, $\rho = 0.4$ | M4, $\rho = 0.8$ | M3, $\rho = 0.8$ |
|--------------------|-------|-------|-------|-------|-----------------|-----------------|-----------------|
| $n = 200, p = 400$ | 37    | 0     | 0     | 0     | 0.19            | 0.52            | 0.99            |
|                    | 26    | 11    | 0     | 0     | 0.69            | 0.80            | 0.97            |
|                    | 23    | 14    | 0     | 0     | 0.69            | 0.81            | 0.97            |
|                    | 19    | 18    | 0     | 0     | 0.65            | 0.84            | 0.96            |
|                    | 14    | 23    | 0     | 0     | 0.60            | 0.87            | 0.94            |
|                    | 11    | 26    | 0     | 0     | 0.53            | 0.86            | 0.88            |
|                    | 24    | 5     | 4     | 4     | 0.64            | 0.75            | 0.97            |
|                    | 17    | 7     | 7     | 6     | 0.59            | 0.78            | 0.95            |
|                    | 10    | 9     | 9     | 9     | 0.48            | 0.77            | 0.86            |
| $n = 300, p = 2000$| 52    | 0     | 0     | 0     | 0.12            | 0.49            | 1.00            |
|                    | 40    | 12    | 0     | 0     | 0.80            | 0.78            | 1.00            |
|                    | 32    | 20    | 0     | 0     | 0.80            | 0.83            | 1.00            |
|                    | 26    | 26    | 0     | 0     | 0.77            | 0.86            | 1.00            |
|                    | 20    | 32    | 0     | 0     | 0.74            | 0.87            | 0.99            |
|                    | 12    | 40    | 0     | 0     | 0.63            | 0.89            | 0.98            |
|                    | 31    | 7     | 7     | 7     | 0.76            | 0.77            | 1.00            |
|                    | 26    | 10    | 8     | 8     | 0.76            | 0.80            | 1.00            |
|                    | 13    | 13    | 13    | 13    | 0.64            | 0.80            | 0.99            |

Regarding the influence of different patterns of $p_1, \ldots, p_S$ and $S$ on MDR-ISIS, the results in Table 4 show that $S = 2$ with nearly equal $p_j$’s or a large $p_1$ have better performances and therefore are recommended.

Finally, we assess the performance of MDR-SSIS. From Table 4, the proportions that all relevant covariates are selected by MDR-ISIS are in a satisfactory range. Thus, it is of interest to see whether MDR-SSIS can reduce the size of screened covariate set without losing the power in selecting all relevant covariates. Under the setting in Table 4 with $S = 2$ and nearly equal $p_1$ and $p_2$, in Table 5 we list the proportions of selecting all relevant covariates by MDR-SSIS with $B = 100$ subsamples of size $n_s = \lceil 4n/5 \rceil$ without replacement and threshold value $\pi_0 = 0.3$ as suggested by He and Lin (2011). Similar results for $\pi_0 = 0.4$ are obtained but not shown here. The median size of screened covariate set by MDR-SSIS and the inter-quartile range of sizes are also reported in Table 5. The screened covariate size of MDR-SIS and MDR-ISIS is $d_n$, which is fixed when $n$ is fixed and included in Table 5. The results in Table 5 show that MDR-SSIS maintains a satisfactory level of selecting all relevant
Table 5: Simulation proportions of all relevant covariates selected by MDR-SIS, MDR-ISIS, and MDR-SSIS; sizes of screened covariate sets by MDR-SSIS; $S = 2$, $p_1 \approx p_2$ for MDR-ISIS; $B = 100$, $n_s = \lfloor 4n/5 \rfloor$, $\pi_0 = 0.3$ for MDR-SSIS; simulation replication 500

| model      | $d_n$ | prob selecting all relevant covariates | size of MDR-SSIS |
|------------|-------|----------------------------------------|------------------|
| $n = 200$, $p = 400$ |       | MDR-SIS | MDR-ISIS | MDR-SSIS | MED | IQR |
| M4, $\rho = 0.4$ | 37    | 0.19 | 0.65 | 0.67 | 26 | 3  |
| M4, $\rho = 0.8$ | 37    | 0.52 | 0.84 | 0.89 | 26 | 4  |
| M3, $\rho = 0.8$ | 37    | 0.99 | 0.96 | 0.96 | 25 | 4  |
| $n = 300$, $p = 2000$ |       | M4, $\rho = 0.4$ | 52 | 0.12 | 0.77 | 0.72 | 28 | 4  |
| M4, $\rho = 0.8$ | 52    | 0.49 | 0.86 | 0.86 | 29 | 4  |
| M3, $\rho = 0.8$ | 52    | 1.00 | 1.00 | 0.99 | 27 | 3  |

MED: the median size of screened covariate set by MDR-SSIS
IQR: the inter-quartile range of size of screened covariate set by MDR-SSIS

covariates, and decreases the size of screened covariate set by 29.7% to 48.1%.

5.2 A Real Data Application

We apply our proposed methods to the diffuse large-B-cell lymphoma microarray data in Rosenwald et al. (2002). This data set consists of measurements on $p = 7399$ genes from 240 patients. The censored survival time $T^*$ ranges from 0 to 21.8 years. Following Bair and Tibshirani (2004), we use data from $n = 160$ patients as the sample training data and data from the rest 80 patients as validation data.

Bair and Tibshirani (2004) applied a supervised principal components (PC) method using the training data to select 17 genes from $p$ genes. Then they used validation data to fit a Cox proportional hazards model in which the covariate effect is a linear combination of the 17 genes. Using training data, He et al. (2013) selected $\lfloor n / \log n \rfloor = \lfloor 160 / \log 160 \rfloor = 31$ genes by applying QaSIS with $\alpha = 0.4$. Using validation data, they also fitted a Cox proportional hazards model with a linear combination of the 31 selected genes as the covariate effect.

Based on the same training data set, we selected $\lfloor n / \log n \rfloor = 31$ genes by applying SII and the proposed MDR-SIS, MDR-ISIS, and MDR-SSIS, and then fitted a Cox proportional hazards model with a linear combination of the 31 selected genes as the covariate effect, based on the validation data set. For MDR-ISIS we used $S = 2$, $p_1 = 16$, and $p_2 = 15$. For
we used $B = 100$ subsamples with $n_s = n/2 = 80$ and $\pi_0 = 0.4$, which resulted in 30 selected genes.

Table 6 shows $R^2$ statistics of Cox proportional hazards models and the associated p-values of log-rank tests, calculated by using the models with covariates selected by these six methods. The $R^2$ statistic for each model measures the percentage of variation in survival time that is explained by the model. Thus, when comparing models, one would prefer the model with a large $R^2$ statistic. It’s clear that the three methods we proposed are better than the others in terms of $R^2$.

Table 6: $R^2$ statistics and p-values for six methods based on the diffuse large-B-cell lymphoma microarray data

| method          | $R^2$ | p-value |
|-----------------|-------|---------|
| Supervised PC   | 0.113 | 0.001   |
| QaSIS($\alpha = 0.4$) | 0.375 | 0.083   |
| SII             | 0.358 | 0.335   |
| MDR-SIS         | 0.506 | 0.008   |
| MDR-ISIS        | 0.511 | 0.046   |
| MDR-SSIS        | 0.502 | 0.015   |

We also evaluate the predictive performance of the proposed methods similarly with Bair and Tibshirani (2004) and Li and Yin (2008). A Cox proportional hazards model is fitted with these subsets of genes selected by the proposed method as the predictors. Three risk groups of patients, the low-risk patients, the intermediate-risk patients, and the high-risk patients, are defined according to the 33% and 66% quantiles of the estimated risk scores. Figure 1 is based on different subsets of genes selected by MDR-SIS, MDR-ISIS, and MDR-SSIS, respectively. Panel (a) in Figure 1 shows the Kaplan-Meier estimates of survival curves for the three risk groups of patients in the training data, whereas panel (b) shows the same curves based on the validation data.

Panel (a) of Figure 1 shows that all three methods achieved good separation of the three risk groups, which indicates a good model fit to the training data. The log-rank test of difference among three survival curves yielded the p-value of 0 for all cases, which confirms our visual examination. The first block in panel (b) of Figure 1 shows that the estimator
Figure 1: Kaplan-Meier estimates of survival curves for the low-risk patients group (solid), the intermediate-risk patients group (dash), and the high-risk patients group (small dash) based on MDR-SIS, MDR-ISIS, and MDR-SSIS; panel (a) is based on training data and panel (b) is based on validation data.
based on MDR-SIS separated the low-risk group with the intermediate and high-risk groups, resulting in a p-value of 0.0146. However, it did not achieve satisfactory separation between the intermediate and high-risk groups. Meanwhile, the second and third blocks of panel (b) of Figure 1 show that the estimators based on MDR-ISIS and MDR-SSIS achieved a better separation of the three risk groups with the validation data, resulting in a p-value nearly 0. Overall, our proposed methods in conjunction with a Cox proportional hazards model demonstrate competent variable screening and model fitting.

6 Appendix: Proofs

Proof of Proposition 2.1. Let $\mathcal{A}^c$ be the complement of $\mathcal{A}$ in $\{1, \ldots, p\}$ and $I_{\mathcal{A}^c} = \text{diag}\{d_1, \ldots, d_p\}$ be the $p \times p$ dimensional diagonal matrix with $d_i = 1$ for $i \in \mathcal{A}$ and $d_i = 0$ for $i \in \mathcal{A}^c$. Similarly define $I_{\mathcal{A}}$ such that $I_{\mathcal{A}^c} + I_{\mathcal{A}} = I_p$. This proposition can be proved if we prove the equivalent result that $i \in \mathcal{A}^c$ if and only if $e_i^T B = 0$. First consider the only if part. By definition, $(T, C) \perp X_{\mathcal{A}^c} | X_\mathcal{A}$. By the definition of $B$, we have $S_{(T,C)|X} = \text{Span}(B) \subseteq \text{Span}(I_{\mathcal{A}})$. It follows immediately that $I_{\mathcal{A}^c} B = 0$. For $i \in \mathcal{A}^c$, the $i$th row of $I_{\mathcal{A}^c}$ is $e_i^T$. Thus we have $e_i^T B = 0$. Now consider the if part. Take $I_{\{i\}} = \text{diag}\{e_i\}$. Then $e_i^T B = 0$ guarantees that $I_{\{i\}} B = 0$. Let $\varepsilon = \{1, \ldots, i - 1, i + 1, \ldots, p\}$. Then $B^T X = B^T I_{\varepsilon} X = B^T I_{\varepsilon} X$. From the definition of $S_{(T,C)|X}$, we have $(T, C) \perp X | B^T X$, which is $(T, C) \perp X | B^T I_{\varepsilon} X$. As $I_{\varepsilon} X$ involves only 0 and elements in $X_\varepsilon$, we have $S_{(T,C)|X_\varepsilon}$. By the definition of the active set $\mathcal{A}$, we know $\mathcal{A} \subseteq \varepsilon$ and $i \in \mathcal{A}^c$. \hfill \Box

Proof of Proposition 2.2. For part (i), denote $E[(Z - \tilde{Z})(Z - \tilde{Z})^T | T, \tilde{T}, C, \tilde{C}]$ by $A(T, \tilde{T}, C, \tilde{C})$ and $E[(Z - \tilde{Z})(Z - \tilde{Z})^T | T^o, \tilde{T}^o, \delta, \tilde{\delta}]$ by $D(T^o, \tilde{T}^o, \delta, \tilde{\delta})$. Let $\nu \in S_{(T,C)|Z}^2$ and denote the column space of a matrix $W$ by $\text{Span}(W)$. $\text{Span}[2I_p - D(T^o, \tilde{T}^o, \delta, \tilde{\delta})] \subset S_{(T,C)|Z}$ leads to the fact that $\text{Span}(M) \subset S_{(T,C)|Z}$ cause $\nu^T M \nu = E\{\nu^T [2I_p - D(T^o, \tilde{T}^o, \delta, \tilde{\delta})]^2 \nu\} = 0$. Thus, it suffices to prove $\text{Span}[2I_p - A(T, \tilde{T}, C, \tilde{C})] \subset S_{(T,C)|Z}$ for any given $(T, \tilde{T}, C, \tilde{C})$ under
conditions (A1) and (A2). By choice of \((\tilde{Z}, \tilde{T}, \tilde{C}), (Z, T, C) \Vert (\tilde{Z}, \tilde{T}, \tilde{C})\). Thus

\[
\begin{align*}
A(T, \tilde{T}, C, \tilde{C}) &= E[(Z - \tilde{Z})(Z - \tilde{Z})^T | T, \tilde{T}, C, \tilde{C}] \\
&= E(ZZ^T | T, C) - E(Z | T, C)E(\tilde{Z}^T | \tilde{T}, C) \\
&+ E(\tilde{Z}\tilde{Z}^T | \tilde{T}, C) - E(\tilde{Z} | \tilde{T}, C)E(Z^T | T, C).
\end{align*}
\] (6.1)

It suffices to show that \(S^p_{(T,C)} \subset \{\text{Span}[2I_p - A(T, \tilde{T}, C, \tilde{C})]\}^\perp\).

By assumption (A1), \(E(\nu^T Z | PZ) = \alpha^T PZ\) for some \(\alpha \in \mathbb{R}^p\). Because \(ZZ^T = I\) and \(\nu \perp PZ \in S^p_{(T,C)}\), we have

\[
0 = E(\nu^T P \alpha) = E\{E[\nu^T Z(\alpha^T PZ)^T | PZ]\} = E[\alpha^T PZ(\alpha^T PZ)^T] = E(\alpha^T P \alpha).
\]

Thus \(E(\nu^T Z | PZ) = (\alpha^T PZ)^2 = \alpha^T P \alpha = 0\). By assumption (A2),

\[
E[(\nu^T Z)^2 | PZ] = c + E(\nu^T Z | PZ) = c,
\]

where \(c\) is a constant. Take unconditional expectations on both sides to obtain \(c = \nu^T \nu\).

Thus \(E[(\nu^T Z)^2] = \nu^T \nu\). Because \((T, C) \Vert Z | PZ\), we have

\[
E(\nu^T Z | T, C) = E[E(\nu^T Z | PZ) | T, C] = 0,
\]

\[
E[(\nu^T Z)^2 | T, C] = E\{E[(\nu^T Z)^2 | PZ] | T, C\} = \nu^T \nu.
\]

Substitute these in to (6.1), then the fact that \((Z, T, C)\) and \((\tilde{Z}, \tilde{T}, \tilde{C})\) have the same distribution lead to \(\nu^T A(T, \tilde{T}, C, \tilde{C}) \nu = 2\nu^T \nu\), implying that

\[
\nu^T [2I_p - A(T, \tilde{T}, C, \tilde{C})] \nu = 0.
\]

Thus \(\text{Span}[2I_p - A(T, \tilde{T}, C, \tilde{C})] \subset S_{(T,C)}\). Finally by derivation of \(T^o\) and \(\delta\), we have

\[
E(Z | T^o, \delta) = E[E(Z | T, C) | T^o, \delta]\quad\text{and}\quad E(Z^2 | T^o, \delta) = E[E(Z^2 | T, C) | T^o, \delta].
\]

Thus \((\tilde{Z}, \tilde{T}, \tilde{C}) \Vert (Z, T, C)\) leads to \(E[A(T, \tilde{T}, C, \tilde{C}) | T^o, \tilde{T}^o, \delta, \tilde{\delta}] = D(T^o, \tilde{T}^o, \delta, \tilde{\delta})\). Taking conditional expectation on \(A(T, \tilde{T}, C, \tilde{C})\) given \((T^o, \tilde{T}^o, \delta, \tilde{\delta})\), then we have \(\text{Span}[2I_p - D(T^o, \tilde{T}^o, \delta, \tilde{\delta})] \subset S_{(T,C)}\), which leads to the result \(\text{Span}(M) \subset S_{(T,C)}\).

For part (ii), with similar argument in proof of Theorem 3 in Li and Wang (2007), if \(\text{Span}(M) \subset S_{(T,C)}\), \(M = M^T\) and \(M \succeq 0\), then

\[
\text{Span}(M) = S_{(T,C)} \quad\text{if and only if}\quad \psi^T M \psi > 0 \quad\text{for all} \quad \psi \in S_{(T,C)} \quad\text{with} \quad \psi \neq 0. \quad (6.2)
\]

Note that \(\text{Span}(M) \subset S_{(T,C)}\) is guaranteed by assumptions (A1) and (A2), and \(M = M^T\).
and \( M \geq 0 \) follow from the definition of \( M \).

Let \( \psi \in S_{(T,C)}|Z \) and \( \psi \neq 0 \). By (6.2), it suffices to show that \( \psi^T M \psi > 0 \). Without loss of generality, assume that \( \| \psi \| = 1 \). Write \( D(T^o, \tilde{T}^o, \delta, \tilde{\delta}) - 2I_p \) as \( C(T^o, \tilde{T}^o, \delta, \tilde{\delta}) \). Then

\[
\psi^T M \psi = \psi^T E[C(T^o, \tilde{T}^o, \delta, \tilde{\delta})(I_p - \psi \psi^T)C(T^o, \tilde{T}^o, \delta, \tilde{\delta})] \psi + E[\psi^T C(T^o, \tilde{T}^o, \delta, \tilde{\delta}) \psi].
\]

Because \( I_p - \psi \psi^T \geq 0 \), the first term on the right is nonnegative. By assumption (A3), \( \psi^T D(T^o, \tilde{T}^o, \delta, \tilde{\delta}) \psi \) is nondegenerate; thus \( \psi^T C(T^o, \tilde{T}^o, \delta, \tilde{\delta}) \psi \) is nondegenerate. By Jensen’s inequality, \( E[(\psi^T C(T^o, \tilde{T}^o, \delta, \tilde{\delta}) \psi)^2] > [E(\psi^T C(T^o, \tilde{T}^o, \delta, \tilde{\delta}) \psi)]^2 = 0 \), where the equality holds because \( EC(T^o, \tilde{T}^o, \delta, \tilde{\delta}) = 0 \).

\[ \square \]

**Proof of Proposition 2.3.** Denote \( \text{Span}\{h(D_{ijlm}) : ijlm\} \) by \( \text{Span}\{h(D_{ijlm}) : i, l = 0 \text{ or } 1, j = 1, \ldots, H_i, m = 1, \ldots, H_l\} \). Note that

\[
g_k^* = \sum_{ij} \sum_{lm} p_{ijlm} [e_k^T \Sigma^{-1/2} (2I_p - D_{ijlm}) \Sigma^{-1/2} e_k]^2 \quad \text{and} \quad G = \sum_{ij} \sum_{lm} p_{ijlm} (2I_p - D_{ijlm})^2.
\]

By Proposition 2.2, (A1), (A2) and (A3) guarantee \( \text{Span}(G) = S_{(T,C)}|Z \). By the invariance law of the central space, we have \( \text{Span}(\Sigma^{-1/2} G \Sigma^{-1/2}) = S_{(T,C)}|X \). If \( k \in A^c \), we know from Lemma A.2 in Yu and Dong (2016) that \( e_k^T \Sigma^{-1/2} G \Sigma^{-1/2} e_k = 0 \). Because \( p_{i,j,m} > 0 \), \( e_k^T \Sigma^{-1/2} (2I_p - D_{ijlm}) = 0 \) for any \((i,j)\) and \((l,m)\). From the expression of \( g_k^* \), we have \( g_k^* = 0 \) if \( k \in A^c \). Condition (A3) guarantees that \( \text{Span}\{(2I_p - D_{ijlm})^2 : ijlm\} = S_{(T,C)}|Z \), which in turn implies \( \text{Span}\{2I_p - D_{ijlm} : ijlm\} = S_{(T,C)}|Z \). From the invariance law of the central space, \( e_k^T \Sigma^{-1/2} (2I_p - D_{ijlm}) \Sigma^{-1/2} e_k > 0 \) for at least one set of \((i,j)\) and \((l,m)\) if \( k \in A \). Otherwise we get a contradiction to the only if part of Lemma A.2 in Yu and Dong (2016). Thus, we have \( g_k^* \geq p_{ijlm} [e_k^T \Sigma^{-1/2} (2I_p - D_{ijlm}) \Sigma^{-1/2} e_k]^2 > 0 \) if \( k \in A \).

\[ \square \]

**Proof of Lemma 2.4.** Note that \( \sum_{ij} \sum_{lm} p_{ijlm} [e_k^T \Sigma^{-1/2} (2I_p - D_{ijlm}) \Sigma^{-1/2} e_k]^2 \) is the discretized version of \( E[(e_k^T \Sigma^{-1/2} (2I_p - D(T^o, \tilde{T}^o, \delta, \tilde{\delta})) \Sigma^{-1/2} e_k)^2] \), \( 2 \sum_{ij} \sum_{lm} p_{ijlm} [e_k^T \Sigma^{-1/2} (p_{ijlm}^{-1} V_{ij} - I_p) \Sigma^{-1/2} e_k]^2 \) is the discretized version of \( 2E[(e_k^T \Sigma^{-1/2} (E(ZZ^T \mid T^o, \delta) - I_p) \Sigma^{-1/2} e_k)^2](E(e_k^T \Sigma^{-1/2} E(Z \mid T^o, \delta))^2) \) and \( 4 \left( \sum_{ij} \sum_{lm} p_{ijlm}^{-1} e_k^T \Sigma^{-1/2} U_{ij} \Sigma^{-1/2} e_k \right)^2 \) is the discretized version of \( 4(E(e_k^T \Sigma^{-1/2} E(Z \mid T^o, \delta))^2) \). Let \( a_k(T^o, \tilde{T}^o, \delta, \tilde{\delta}) = e_k^T \Sigma^{-1/2} (2I_p - D(T^o, \tilde{T}^o, \delta, \tilde{\delta})) \Sigma^{-1/2} e_k \), all
we need to prove is that

\[
E[a_k^2(T^o, \tilde{T}^o, \delta, \tilde{\delta})] = 2E[\{e_k^T\Sigma^{-1/2}[E(ZZ^T | T^o, \delta) - I_p]\Sigma^{-1/2}e_k\}^2] \\
+ 4 \left( E[e_k^T\Sigma^{-1/2}E(Z | T^o, \delta)E(Z | T^o, \delta)^T\Sigma^{-1/2}e_k] \right)^2 \\
= 2E[\{e_k^T\Sigma^{-1/2}E(ZZ^T | T^o, \delta)\Sigma^{-1/2}e_k\}^2] - 2(e_k^T\Sigma^{-1/2}e_k)^2 \\
+ 4 \left( E[e_k^T\Sigma^{-1/2}E(Z | T^o, \delta)E(Z | T^o, \delta)^T\Sigma^{-1/2}e_k] \right)^2.
\]  (6.3)

Let \(d_k(T^o, \tilde{T}^o, \delta, \tilde{\delta}) = e_k^T\Sigma^{-1/2}D(T^o, \tilde{T}^o, \delta, \tilde{\delta})\Sigma^{-1/2}e_k\). Then

\[
E[a_k^2(T^o, \tilde{T}^o, \delta, \tilde{\delta})] = E[d_k^2(T^o, \tilde{T}^o, \delta, \tilde{\delta})] - 4(e_k^T\Sigma^{-1}e_k)^2.
\]  (6.4)

With similar argument of Proposition 2.2 (i), we have \(d_k^2(T^o, \tilde{T}^o, \delta, \tilde{\delta}) = c_k(T^o, \tilde{T}^o, \delta, \tilde{\delta}) + c_k(\tilde{T}^o, T^o, \delta, \tilde{\delta})\), where

\[
c_k(T^o, \tilde{T}^o, \delta, \tilde{\delta}) = e_k^T\Sigma^{-1/2}E(ZZ^T | T^o, \delta)\Sigma^{-1/2}e_k - e_k^T\Sigma^{-1/2}E(Z | T^o, \delta)E^T(\tilde{Z} | \tilde{T}^o, \tilde{\delta})\Sigma^{-1/2}e_k;
\]

\[
c_k(\tilde{T}^o, T^o, \delta, \tilde{\delta}) = e_k^T\Sigma^{-1/2}E(\tilde{Z}Z^T | \tilde{T}^o, \tilde{\delta})\Sigma^{-1/2}e_k - e_k^T\Sigma^{-1/2}E(\tilde{Z} | \tilde{T}^o, \tilde{\delta})E^T(Z | T^o, \delta)\Sigma^{-1/2}e_k.
\]

Plug them into (6.3), it follows that

\[
E[a_k^2(T^o, \tilde{T}^o, \delta, \tilde{\delta})] = 2E[c_k^2(T^o, \tilde{T}^o, \delta, \tilde{\delta})] + 2E[c_k(T^o, \tilde{T}^o, \delta, \tilde{\delta})c_k(\tilde{T}^o, T^o, \delta, \tilde{\delta})] - 4(e_k^T\Sigma^{-1}e_k)^2.
\]  (6.5)

By calculation, we have \(E[c_k^2(T^o, \tilde{T}^o, \delta, \tilde{\delta})] = c_{1k} + c_{2k} - c_{3k} - c_{4k}\), where

\[
c_{1k} = E[\{e_k^T\Sigma^{-1/2}E(ZZ^T | T^o, \delta)\Sigma^{-1/2}e_k\}^2];
\]

\[
c_{2k} = E[e_k^T\Sigma^{-1/2}E(Z | T^o, \delta)E^T(\tilde{Z} | \tilde{T}^o, \tilde{\delta})\Sigma^{-1/2}e_k e_k^T \Sigma^{-1/2}E(Z | T^o, \delta)E^T(\tilde{Z} | \tilde{T}^o, \tilde{\delta})\Sigma^{-1/2}e_k];
\]

\[
c_{3k} = E[e_k^T\Sigma^{-1/2}E(Z | T^o, \delta)E^T(\tilde{Z} | \tilde{T}^o, \tilde{\delta})\Sigma^{-1/2}e_k e_k^T \Sigma^{-1/2}E(ZZ^T | T^o, \delta)\Sigma^{-1/2}e_k];
\]

\[
c_{4k} = E[e_k^T\Sigma^{-1/2}E(ZZ^T | T^o, \delta)\Sigma^{-1/2}e_k e_k^T \Sigma^{-1/2}E(Z | T^o, \delta)^T(\tilde{Z} | \tilde{T}^o, \tilde{\delta})\Sigma^{-1/2}e_k].
\]

Because \((Z, T^o, \delta) \perp (\tilde{Z}, \tilde{T}^o, \tilde{\delta})\) and \(E(Z) = 0\), we have \(c_{3k} = c_{4k} = 0\), \(c_{2k} = E[e_k^T\Sigma^{-1/2}E(Z | T^o, \delta)E^T(Z | T^o, \delta)\Sigma^{-1/2}e_k]\) and \(E[c_k(T^o, \tilde{T}^o, \delta, \tilde{\delta})c_k(\tilde{T}^o, T^o, \delta, \tilde{\delta})] = (e_k^T\Sigma^{-1}e_k)^2 + c_{2k}\). Plug them into (6.5), it follows to (6.3) that complete the proof.

\[
\square
\]

**Proof of Proposition 2.5.** Let \(A = \sum_{ij} \sum_{lm} p_{ij} p_{lm} \Sigma^{1/2} (2I_p - D_{ijlm}) \Sigma^{1/2}\). By proposition 2.2, (A1), (A2) and (A3) guarantee \(\text{Span}(\Sigma^{-1/2}A\Sigma^{-1/2}) = S_{(T,C)}Z\). By the invariance law
of the central space, we have \( \text{Span}(\Sigma^{-1}\Lambda\Sigma^{-1}) = \mathcal{S}_{(T,C)|x} \). Let \( \{\beta_1, \ldots, \beta_d\} = B \) be a basis for \( \mathcal{S}_{(T,C)|x} \). Then \( \text{Span}(\Sigma^{1/2}(2I_p - D_{ijlm})\Sigma^{1/2}) = \text{Span}\{\zeta_1, \ldots, \zeta_d\} \), where \( \zeta_i = \Sigma\beta_i \) for \( i = 1, \ldots, d \). And \( \sigma_k^2 g_k = \sum_{ij} \sum_{lm} p_{ij} p_{lm}[e_k^T \Sigma^{1/2}(2I_p - D_{ijlm})\Sigma^{1/2}e_k]^2 \). Similar with the proof of Proposition 2.3, if \( e_k^T \zeta \neq 0 \) when \( k \in \mathcal{A} \), then \( e_k^T \Sigma^{1/2}(2I_p - D_{ijlm})\Sigma^{1/2}e_k > 0 \) for at least one set of \((i, j)\) and \((l, m)\). Thus, we have \( g_k \geq \sigma_k^4 p_{ij} p_{lm}[e_k^T \Sigma^{-1/2}(2I_p - D_{ijlm})\Sigma^{-1/2}e_k]^2 > 0 \). So all we need to show is \( \zeta_{ik} \neq 0 \) for at least one of \( i = 1, \ldots, d \). Note that \( \beta_h \in \mathcal{S}_{(T,C)|x} \). By the definition of the active set \( \mathcal{A} \) and the central space \( \mathcal{S}_{(T,C)|x} \), we have \( \beta_{hj} = 0 \) for \( j \in \mathcal{A}^c \). Thus, the \( k \)-th component of \( \zeta_h = \Sigma\beta_h \) becomes \( \zeta_{hk} = \sum_{j \in \mathcal{A}} \text{Cov}(x_k, x_j)\beta_{hj} \). If \( k \in \mathcal{A} \), then \( \text{Cov}(x_k, x_j)\beta_{hj} \) has the same sign for all \( j \in \mathcal{A} \). Thus we have \( \zeta_{hj} \neq 0 \) and \( g_k > 0 \) as a result. \( \Box \)

**Proof of Theorem 3.1.** For part (i), let \( C_1 = 2 + \tau + \varsigma^{-1}K_0^2 \) and \( C_2 = 2 + \tau + \varsigma^{-1}e^2K_0^2 \). By condition (C2), we see that \( E[\exp[tz_k^2I(\delta = l, T^c \in I_{lj})]] \leq E[\exp[tz_k^2]] \leq K_0 \) and \( E[\exp[tz_kI(\delta = l, T^c \in I_{lj})]] \leq E[\exp(|tz_k|)] \leq eK_0 \) for \( |t| \leq \varsigma \). Following similar arguments in the proof of Theorems 1(a) and 4(a) in Cai et al. (2011), we derive that

\[
\Pr\{|\hat{V}_{ij} - V_{ij}| \geq \varsigma^{-1}C_1(\log p/n)^{1/2}\} \leq 2p^{-\tau - 2}, \tag{6.6}
\]

\[
\Pr\{|\hat{U}_{ij} - U_{ij}| \geq \varsigma^{-1}C_2(\log p/n)^{1/2}\} \leq 2p^{-\tau - 2}. \tag{6.7}
\]

Let \( p_{\min} = \min\{p_{11}, \ldots, p_{1H_1}, p_{01}, \ldots, p_{0H_0}\} \). Note that \( |I(\delta_l = l, y_l \in I_{lj}) - p_{lj}| < 1, E[I(\delta_l = l, y_l \in I_{lj}) - p_{lj}] = 0 \) and \( E[I(\delta_l = l, y_l \in I_{lj}) - p_{lj}^2] = (1 - p_{lj})p_{lj} = 1/4 \). By the Bernstein inequality (Lemma 2.2.9, Van der Vaart and Wellner (1996)), we have

\[
\Pr\{|\tilde{p}_{lj} - p_{lj}| \geq (2 + \tau)(\log p/n)^{1/2}\} = \Pr\{\sum_{i=1}^n I(\delta_l = l, y_l \in I_{lj}) - p_{lj} \geq \varsigma^{-1}(2 + \tau)(\log p/n)^{1/2}\} \leq 2p^{-\tau - 2}.
\]

By condition (C1), we can assume that \( \log p/n \leq p_{\min}^2/(4 + 2\tau)^2 < 1/4 \). Then

\[
\Pr\{|\tilde{p}_{lj} - p_{lj}| \geq (4 + 2\tau)p_{\min}^2(\log p/n)^{1/2}\} \leq \Pr\{|\tilde{p}_{lj} - p_{lj}^2| \geq (2 + \tau)(\log p/n)^{1/2}\} + \Pr\{p_{lj}^2 \tilde{p}_{lj} \leq p_{\min}(p_{\min} - p_{\min}/2)\} \leq 2p^{-\tau - 2} + 2p^{-\tau - 2} = 4p^{-\tau - 2}. \tag{6.8}
\]

By condition (C2), \( E(z_k^2) = E(\varsigma z_k^2)/\varsigma \leq \varsigma^{-1}E[\exp(\varsigma z_k^2)] \leq \varsigma^{-1}K_0 \). By Jensen’s inequality,
\[ E|z_k| \leq [E(z_k^2)]^{1/2} \leq \varsigma^{-1/2}K_0^{1/2}. \] Thus \( \max_{1 \leq k \leq p} |U_{ljk}| \leq \max_{1 \leq k \leq p} E|z_k| \leq \varsigma^{-1/2}K_0^{1/2} \) and \( \max_{1 \leq k \leq p} V_{ljk} \leq \max_{1 \leq k \leq p} E(z_k^2) \leq \varsigma^{-1}K_0. \) Let \( C_3 = (4+2\tau)\varsigma^{-1}K_0p_{\min}^{-2} + 4p_{\min}^{-1}K_0^{1/2}\varsigma^{-3/2}C_2 + 3(4+2\tau)^{-1}\varsigma^{-2}C_2^2. \) By \( \log p/n \leq p_{\min}^2/(4 + 2\tau)^2. \) Then
Combining (6.7) and (6.8) together, we have

\[
C_3 = (4 + 2\tau)\varsigma^{-1}K_0p_{\min}^{-2} + [2p_{\min}^{-1} + 2(4 + 2\tau)p_{\min}^{-2}p_{\min}/(4 + 2\tau)]K_0^{1/2}\varsigma^{-3/2}C_2 \\
+ \{p_{\min}^{-1}p_{\min}/(4 + 2\tau) + (4 + 2\tau)p_{\min}^{-2}[p_{\min}/(4 + 2\tau)]\}^{2}\varsigma^{-2}C_2^2 \\
\geq (4 + 2\tau)\varsigma^{-1}K_0p_{\min}^{-2} + [2p_{\min}^{-1} + 2(4 + 2\tau)p_{\min}^{-2}(\log p/n)^{1/2}]K_0^{1/2}\varsigma^{-3/2}C_2 \\
+ [p_{\min}^{-1}(\log p/n)^{1/2} + (4 + 2\tau)p_{\min}^{-2}\log p/n]^{2}\varsigma^{-2}C_2^2.
\]

Combining (6.7) and (6.8) together, we have

\[
\Pr\{|\tilde{U}_{ijk}^2/\hat{p}_{ij} - U_{ijk}^2/p_{ij}| \geq C_3(\log p/n)^{1/2}\} \\
\leq \Pr\{|(\hat{p}_{ij}^{-1} - p_{ij}^{-1})U_{ijk}^2| \geq (4 + 2\tau)\varsigma^{-1}K_0p_{\min}^{-2}(\log p/n)^{1/2}\} \\
+ \Pr\{|p_{ij}^{-1}(\tilde{U}_{ijk} - U_{ijk})^2| \geq p_{\min}^{-1}\varsigma^{-2}C_2^2 \log p/n\} \\
+ \Pr\{|2\hat{p}_{ij}^{-1}U_{ijk}(\tilde{U}_{ijk} - U_{ijk})| \geq 2\varsigma^{-3/2}K_0^{1/2}p_{\min}^{-1}C_2(\log p/n)^{1/2}\} \\
+ \Pr\{|2(\tilde{p}_{ij}^{-1} - p_{ij}^{-1})(\tilde{U}_{ijk} - U_{ijk})U_{ij|k} | \geq 2\varsigma^{-3/2}K_0^{1/2}(4 + 2\tau)p_{\min}^{-2}C_2 \log p/n\} \\
+ \Pr\{|(\tilde{p}_{ij}^{-1} - p_{ij}^{-1})(\tilde{U}_{ijk} - U_{ijk})^2| \geq (4 + 2\tau)p_{\min}^{-2}\varsigma^{-2}C_2^2(\log p/n)^{3/2}\} \\
\leq 4p^{-\tau-2} + 2p^{-\tau-2} + 2p^{-\tau-2} + (4p^{-\tau-2} + 2p^{-\tau-2}) + (4p^{-\tau-2} + 2p^{-\tau-2}) \\
= 20p^{-\tau-2}.
\] (6.9)

Defines two positive constants 
\[
C_4 = (4 + 2\tau)\varsigma^{-2}K_0^2p_{\min}^{-2} + 4p_{\min}^{-1}K_0\varsigma^{-2}C_1 + 2(4 + 2\tau)^{-1}\varsigma^{-2}C_1^2
\]
and \[
C_5 = (4 + 2\tau)\varsigma^{-1}K_0p_{\min}^{-2} + [(4 + 2\tau)^{-1} + p_{\min}^{-1}]\varsigma^{-1}C_1. \] By \(\log p/n \leq [p_{\min}/(4 + 2\tau)]^2\),

\[
C_4 = (4 + 2\tau)\varsigma^{-2}K_0^2p_{\min}^{-2} + [2p_{\min}^{-1} + (8 + 4\tau)p_{\min}^{-2}p_{\min}/(4 + 2\tau)]K_0\varsigma^{-2}C_1 \\
+ \{p_{\min}^{-1}p_{\min}/(4 + 2\tau) + (4 + 2\tau)p_{\min}^{-2}[p_{\min}/(4 + 2\tau)]\}^{2}\varsigma^{-2}C_1^2 \\
\geq (4 + 2\tau)\varsigma^{-2}K_0^2p_{\min}^{-2} + [2p_{\min}^{-1} + (8 + 4\tau)p_{\min}^{-2}(\log p/n)^{1/2}]K_0\varsigma^{-2}C_1 \\
+ [p_{\min}^{-1}(\log p/n)^{1/2} + (4 + 2\tau)p_{\min}^{-2}\log p/n]^{2}\varsigma^{-2}C_1^2
\]
and

\[
C_5 = (4 + 2\tau)\varsigma^{-1}K_0p_{\min}^{-2} + p_{\min}^{-1}\varsigma^{-1}C_1 + (4 + 2\tau)p_{\min}^{-2}(p_{\min}/2)^2\varsigma^{-1}C_1 \\
\geq (4 + 2\tau)\varsigma^{-1}K_0p_{\min}^{-2} + p_{\min}^{-1}\varsigma^{-1}C_1 + (4 + 2\tau)p_{\min}^{-2}\varsigma^{-1}C_1 \log p/n.
\]
Similar to the derivation of (6.9), by combining (6.6) and (6.8) we obtain

\[
\Pr\{|\hat{V}_{ijk}/\hat{p}_{ij} - V_{ijk}/p_{ij,}\| \geq C_4 (\log p/n)^{1/2}\}
\]
\[
\leq \Pr\{|(\hat{p}_{ij}^{-1} - p_{ij}^{-1})V_{ijk}| \geq (4 + 2\tau)\varsigma^{-2}K_0p_{\min}^{-2}(\log p/n)^{1/2}\}
\]
\[
+ \Pr\{|p_{ij}^{-1}(\hat{V}_{ijk} - V_{ijk})^2| \geq p_{\min}^{-1}\varsigma^{-2}C_1^2 \log p/n\}
\]
\[
+ \Pr\{|2p_{ij}^{-1}V_{ijk}(\hat{V}_{ijk} - V_{ijk})| \geq 2\varsigma^{-2}K_0p_{\min}^{-1}C_1(\log p/n)^{1/2}\}
\]
\[
+ \Pr\{|(\hat{p}_{ij}^{-1} - p_{ij}^{-1})(\hat{V}_{ijk} - V_{ijk})^2| \geq (4 + 2\tau)p_{\min}^{-2}\varsigma^{-2}C_1^2 (\log p/n)^{3/2}\}
\]
\[
\leq 4p^{-\tau-2} + 2p^{-\tau-2} + 2p^{-\tau-2} + (4p^{-\tau-2} + 2p^{-\tau-2}) + (4p^{-\tau-2} + 2p^{-\tau-2})
\]
\[
= 20p^{-\tau-2}
\]  \hfill (6.10)

And

\[
\Pr\{|\hat{V}_{ijk}/\hat{p}_{ij} - V_{ijk}/p_{ij}| \geq C_5 (\log p/n)^{1/2}\}
\]
\[
\leq \Pr\{|(\hat{p}_{ij}^{-1} - p_{ij}^{-1})V_{ijk}| \geq (4 + 2\tau)\varsigma^{-1}K_0p_{\min}^{-2}(\log p/n)^{1/2}\}
\]
\[
+ \Pr\{|p_{ij}^{-1}(\hat{V}_{ijk} - V_{ijk})| \geq p_{\min}^{-1}\varsigma^{-1}C_1(\log p/n)^{1/2}\}
\]
\[
+ \Pr\{|(\hat{p}_{ij}^{-1} - p_{ij}^{-1})(\hat{V}_{ijk} - V_{ijk})^2| \geq (4 + 2\tau)p_{\min}^{-2}\varsigma^{-1}C_1(\log p/n)^{3/2}\}
\]
\[
\leq 4p^{-\tau-2} + 2p^{-\tau-2} + (4p^{-\tau-2} + 2p^{-\tau-2}) = 12p^{-\tau-2}.
\]  \hfill (6.11)

Let \(H = H_0 + H_1\). Define positive constant \(C_0\) as follows:
Note that \( \sum_{ij} U^2_{ijk}/p_{ij} = \text{Cov}[E(z_k \mid \delta, T^o)] \leq \text{Var}(z_k) = 1 \). By (6.9), (6.10), (6.11) and (6.12), we could derive that

\[
\Pr\{|\hat{g}_k - g_k| \geq C_0(\log p/n)^{1/2}\}
\]

\[
\leq \Pr\{|2\hat{V}^2_{ijk}/\hat{p}_{ij} - 2V^2_{ijk}/p_{ij}| \geq 2C_4(\log p/n)^{1/2}\}
\]

\[
+ \Pr\{|4\hat{V}^2_{ijk}/\hat{p}_{ij} - 4V^2_{ijk}/p_{ij}| \geq 4C_5(\log p/n)^{1/2}\}
\]

\[
+ \Pr\{|8(\sum_{ij} \hat{U}^2_{ijk}/\hat{p}_{ij} - \sum_{ij} U^2_{ijk}/p_{ij})| \geq 8C_3(\log p/n)^{1/2}\}
\]

\[
+ \Pr\{|4(\sum_{ij} \hat{U}^2_{ijk}/\hat{p}_{ij} - \sum_{ij} U^2_{ijk}/p_{ij})^2| \geq 4C_2 \log p/n\}
\]

\[
\leq 20p^{-\tau - 2} + 12p^{-\tau - 2} + 20p^{-\tau - 2} + 20p^{-\tau - 2} = 72p^{-\tau - 2}.
\]  

Thus

\[
\Pr\{\max_{1 \leq k \leq p} |\hat{g}_k - g_k| \geq C_0(\log p/n)^{1/2}\} \leq p \max_{1 \leq k \leq p} \Pr\{|\hat{g}_k - g_k| \geq C_0(\log p/n)^{1/2}\} = 72p^{-\tau - 1}.
\]

The proof of Theorem 3.1 (i) is completed. For part (ii), if \( A \not\subseteq \hat{A} \), then there must exist some \( k \in \mathcal{A} \) such that \( \hat{g}_k < c_0n^{-\kappa} \). It follows from condition (C3) that \( |\hat{g}_k - g_k| > c_0n^{-\kappa} \) for some \( k \in \mathcal{A} \). Let \( p_0 \) denotes the size of \( \mathcal{A} \). Thus

\[
\Pr(\mathcal{A} \subseteq \hat{A}) \geq 1 - \Pr\{|\hat{g}_k - g_k| > c_0n^{-\kappa} \text{ for some } k \in \mathcal{A}\}
\]

\[
\geq 1 - \sum_{k=1}^{p_0} \Pr\{|\hat{g}_k - g_k| > c_0n^{-\kappa}\}
\]

\[
\geq 1 - p_0 \max_{1 \leq k \leq p} \Pr\{|\hat{g}_k - g_k| > c_0n^{-\kappa}\}
\]

\[
\geq 1 - p_0 \max_{1 \leq k \leq p} \Pr\{|\hat{g}_k - g_k| > C_0(\log p/n)^{1/2}\},
\]  

where the last inequality follows from condition (C1). By (6.13), (6.14) and condition (C1), it follows that \( \Pr(\mathcal{A} \subseteq \hat{A}) \geq 1 - p_072p^{-\tau - 2} \). By the definition of \( \mathcal{A} \) in (1.2), we have \( p_0 < p \) that lead to the final result \( \Pr(\mathcal{A} \subseteq \hat{A}) \geq 1 - 72p^{-\tau - 1} \).

**Proof of Proposition 4.1.** Define \( H(T^o, \tilde{T}^o, \delta, \tilde{\delta}) = E[2\Sigma - (X - \tilde{X})(X - \tilde{X})^T \mid T^o, \tilde{T}^o, \delta, \tilde{\delta}] \) and let \( H_{ii}(T^o, \tilde{T}^o, \delta, \tilde{\delta}) \) be its \( i \)th diagonal element. Following Proposition 2.2, we can obtain
that
\[
\text{Span}[\mathbf{H}(T^o, \tilde{T}^o, \delta, \tilde{\delta})] = \text{Span}\{\Sigma^{\frac{1}{2}}[\mathbf{D} - \mathbf{D}(T^o, \tilde{T}^o, \delta, \tilde{\delta})]\Sigma^{\frac{1}{2}}\} = \text{Span}(\Sigma\beta_1, \ldots, \Sigma\beta_d).
\]

Thus \(\mathbf{H}(T^o, \tilde{T}^o, \delta, \tilde{\delta}) = \sum_{i=1}^{d} \lambda_i(\Sigma\beta_i)(\Sigma\beta_i)^T\). Thus \(H_{kk}(T^o, \tilde{T}^o, \delta, \tilde{\delta}) = 0\) because the kth row of \((\Sigma\beta_1, \ldots, \Sigma\beta_d)\) is \((0, \ldots, 0)\). Invoking Lemma 2.4, we can further derive that
\[
g_k = E[H_{kk}(T^o, \tilde{T}^o, \delta, \tilde{\delta})^2]/\sigma_k^2 = 0.
\]

**Proof of Proposition 4.2.** Define \(\mathbf{H}(T^o, \tilde{T}^o, \delta, \tilde{\delta}) = E[2\Sigma - (\mathbf{X} - \tilde{\mathbf{X}})(\mathbf{X} - \tilde{\mathbf{X}})^T | T^o, \tilde{T}^o, \delta, \tilde{\delta}]\). Without loss of generality, set \(e = 1\) and \(\mathcal{F} = \{2, \ldots, k_0\} \supseteq \mathcal{A}\). Define \(p \times 1\) vector \(\mathbf{a}_{1|\mathcal{F}} = (1, -\Sigma_{T_{1,1}}^{-1}, 0, \ldots, 0)^T\). Then we have \(g_{1|\mathcal{F}} = \sigma_{1|\mathcal{F}}^4 E[\mathbf{a}_{1|\mathcal{F}}^T \mathbf{H}(T^o, \tilde{T}^o, \delta, \tilde{\delta}) \mathbf{a}_{1|\mathcal{F}}] \) by applying Theorem 2 in (Li and Wang, 2007). Similar with the proof of Proposition 2.3, \(g_{1|\mathcal{F}} = \sum_{ij} \sum_{lm} p_{ij} p_{lm} [\mathbf{a}_{1|\mathcal{F}}^T \mathbf{H}_{ijlm} \mathbf{a}_{1|\mathcal{F}}] / \sigma_{1|\mathcal{F}}^4\), where \(\mathbf{H}_{ijlm} = E[2\Sigma - (\mathbf{X} - \tilde{\mathbf{X}})(\mathbf{X} - \tilde{\mathbf{X}})^T | \delta = i, T^o \in I_{ij}, \tilde{\delta} = l, \tilde{T}^o \in I_{lm}]\). When \(e = 1 \in \mathcal{A}\), condition (C4) guarantee there exists \(k \in \{1, \ldots, d\}\) that \(\beta_{ke} \neq 0\). Then \(\mathbf{a}_{1|\mathcal{F}}^T \Sigma \beta_k = \sigma_{1|\mathcal{F}}^2 \beta_{ke} \neq 0\). Since \(\text{Span}[\mathbf{H}_{ijlm}] = \text{Span}(\Sigma\beta_1, \ldots, \Sigma\beta_d)\), we have
\[
\text{Span}[\mathbf{a}_{1|\mathcal{F}}^T \mathbf{H}_{ijlm} \mathbf{a}_{1|\mathcal{F}}] = \text{Span}(\mathbf{a}_{1|\mathcal{F}}^T \Sigma \beta_1, \ldots, \mathbf{a}_{1|\mathcal{F}}^T \Sigma \beta_d).
\]

Thus there exists \(a > 0\) that \(\mathbf{a}_{1|\mathcal{F}}^T \mathbf{H}_{ijlm} \mathbf{a}_{1|\mathcal{F}} > a(\mathbf{a}_{1|\mathcal{F}}^T \Sigma \beta_k)^2 = a\sigma_{1|\mathcal{F}}^4 \beta_{ke}^2\) for at least one set of \((i, j)\) and \((l, m)\). Denote \(\min p_{ij}\) by \(p_{\text{min}}\). By condition (C5), we have \(g_{e|\mathcal{F}} = g_{1|\mathcal{F}} > p_{\text{min}}^2 [\mathbf{a}_{1|\mathcal{F}}^T \mathbf{H}_{ijlm} \mathbf{a}_{1|\mathcal{F}}] / \sigma_{1|\mathcal{F}}^4 > p_{\text{min}}^2 a^2 \sigma_{1|\mathcal{F}}^4 \beta_{ke}^4 > c_0 n^{-\kappa}\), where \(c_0 = p_{\text{min}}^2 a^2 c_2^4\) and \(\kappa = 4 \theta \leq 1/2\). It follows that \(g_{e|\mathcal{F}} > 2c_0 n^{-\kappa}\) while \(e \in \mathcal{A}\).

\[\Box\]

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