A Note on Perturbative and Nonperturbative
Instabilities of Twisted Circles

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Abstract

In this short note we compare the endpoint of tachyon condensation of twisted circles with the endpoint of nonperturbative brane nucleation in the Kaluza-Klein Melvin spacetime.
1. Introduction

Twisted circles are constructed using an identification where a rotation in a plane (by a rational angle) is combined with a shift along an orthogonal real line. This is an interesting construction because on the one hand, the freely acting orbifold is smoothing out the conical singularity which would occur by an identification by rotation without a shift [[1]]. This therefore provides a nice arena to study localized tachyon condensation (see also [[1][2][3]]). On the other hand a Kuluza-Klein reduction produces a Melvin fluxbrane spacetime [[4]] which has gotten a lot of attention recently [[5][6][7][8][9][10][11][12][13][14][15][16]].

Methods of semi-classical quantum gravity can be used to analyze nonperturbative instabilities [[1][6][14]] corresponding to the nucleation of KK-branes. In this note we will compare what happens to the twisted circle under perturbative tachyon condensation [[3]] to what happens after nucleation of a spherical brane in a Melvin background [[17]]. Although the regimes where the analysis of the instabilities is valid are very different we find that the end result of the tachyon condensation and nucleation of branes is the same: the twisted circle untwists itself and the radius of the compact circle increases. This might be considered as some evidence that these two seemingly very different processes are actually related, a conjecture made in [[11][13]].

2. The nonperturbative instability

Starting with the $d$ dimensional flat space metric

$$ds^2 = -dt^2 + dx_1^2 + \cdots + dx_{d-4}^2 + dr^2 + r^2 d\varphi^2 + R^2 dy^2,$$

(2.1)

where $y$ is periodic with period $2\pi$, one reduces along the orbits of the Killing vector $\partial_y + q \partial_\varphi$, which means that a translation $y \rightarrow y + 2\pi$ is accompanied by a rotation $\varphi \rightarrow \varphi + 2\pi \gamma$, where $\gamma = qR$. It is useful to introduce a new single valued angular variable $\tilde{\varphi} = \varphi - qRy$ which has standard periodicity. In the new coordinates the metric becomes

$$ds^2 = -dt^2 + dx_1^2 + \cdots + dx_{d-4}^2 + dr^2 + r^2 (d\tilde{\varphi} + qR dy)^2 + R^2 dy^2.$$
Using the standard formulae for Kaluza-Klein reduction,

\[ ds^2_d = e^{\frac{4}{d-2}\phi} (dy + 2A_\mu dx^\mu)^2 + e^{-\frac{4}{(d-3)(d-2)}\phi} ds^2_{d-1}. \]  

(2.3)

Rescaling brings the metric into the following canonical form

\[ ds^2_{d-1} = (1 + b^2 r^2) \left( -dt^2 + dx_1^2 + \cdots + dx_{d-4}^2 + dr^2 + \frac{r^2}{1 + b^2 r^2} d\varphi^2 \right) \]

\[ e^{\frac{4}{d-2}\phi} = R^2 (1 + b^2 r^2), \quad A_{\varphi} = \frac{br^2}{2R^{d-2} (1 + b^2 r^2)}, \quad b = \frac{q}{R^{d-3}}. \]  

(2.4)

In [5] it was shown that the gravitational instanton mediating the creation of KK-branes in a Melvin background is given by the Euclidean Myers-Perry [18] black hole,

\[ ds^2 = \left( 1 - m r^{d-5} \Sigma \right) dx_d^2 - \frac{2mk \sin^2 \theta}{r^{d-5} \Sigma} dx_d dx_\varphi + \frac{\Sigma}{r^2 - k^2 - mr^{5-d}} dr^2 + \Sigma d\theta^2 \]

\[ + \sin^2 \theta \sum \left( r^2 - k^2 \right) \Sigma - \frac{m}{r^{d-5} k^2 \sin^2 \theta} d\varphi^2 + r^2 \cos^2 \theta d\Omega_{d-4}, \]

where \( \Sigma = r^2 - k^2 \cos^2 \theta. \) Under analytic continuation the horizon of the Minkowskian black hole becomes an Euclidean ‘bolt’, with radius \( r_+ \) defined by

\[ r_+^2 - k^2 - \frac{m}{r_+^{d-5}} = 0. \]  

(2.6)

The absence of a conical singularity at \( r = r_+ \) then determines the radius \( R \) of the Kaluza-Klein direction \( x_d. \) The second quantity characterizing the black hole solution is the (analytically continued) angular momentum \( \Omega. \) In terms of \( m \) and \( k, \) these are

\[ R = \frac{2mr^{d-5}}{(d-3)r_+^{d-5} - (d-5)k^2}, \quad \Omega = \frac{k^{d-5}}{m}. \]  

(2.7)

Note that the physical range of \( R, \Omega \) is restricted by \( |\Omega R| \leq 1. \) Since (2.5) is asymptotically flat one can embed the black hole in a Melvin fluxbrane by twisting\(^1\)

\[ q = \frac{\text{sgn}(\Omega)}{R}. \]  

(2.8)

\(^1\) As explained in [5] there is a second choice of twist corresponding to supersymmetry breaking boundary conditions on the compactification circle, however we will not discuss this case here.
Under the twisted identification the twist angle is given by

\[ \gamma = qR = \Omega R - sgn(\Omega) \]  \hspace{2cm} (2.9)

hence \( \Omega \) and \( \gamma \) are really periodic variables, which are identified modulo \( 2/R \) and 2 respectively.

We are interested in the Minkowskian evolution of the spacetime after the nucleation of a brane. To achieve this one analytically continues one of the angular variables of the \( d-4 \) sphere results into the time coordinate of the Minkowskian solution after nucleation.\(^2\) The Lorentzian metric post-nucleation is then given by

\[ ds^2 = \Lambda^{\frac{1}{n-3}} \left\{ \frac{\Sigma}{r^2 - k^2} dr^2 + \Sigma d\theta^2 + r^2 \cos^2 \theta (-dt^2 + \cosh^2 t \Omega^2 dr^2 - 5) \right\} + \frac{R^2}{\Lambda} \sin^2 \theta (r^2 - k^2 - mr^{d-5}) d\varphi^2 \] \hspace{2cm} (2.10)

Where \( \Lambda \) is given by

\[ \Lambda = R^2 \left( 1 - \frac{m}{r^{d-5}\Sigma} - \frac{2mk \sin^2 \theta}{r^{d-5}\Sigma} + \frac{q^2 \sin^2 \theta}{\Sigma} \left( (r^2 - k^2)\Sigma - mr^{d-5}k^2 \sin^2 \theta \right) \right) \] \hspace{2cm} (2.11)

As explained in \[5\] this metric for our choice \( q \) describes a spherical D6-brane expanding in a Flux 7-brane background. It is natural to ask what the 'leftover' spacetime after the D6-brane has moved to infinity looks like. This question was addressed in \[17\] and we refer the reader there for more details. The metric (2.10) has an acceleration horizon and only covers the region of spacetime inside it. To continue past it, it is useful to make some coordinate changes. Firstly define \( z = r \cos \theta \) and \( \tilde{r} = f(r) \sin \theta \), where

\[ \frac{1}{f} \frac{df}{dr} = \frac{r}{r^2 - k^2 - mr^{d-5}} \] \hspace{2cm} (2.12)

Secondly, define Rindler like coordinates \( X, T \) in terms of \( z, t \) by \( z = \sqrt{X^2 - T^2} \) and \( t = \arctanh(T/X) \). The exact form of the metric in the new coordinates is very complicated,\(^2\) Recently these spacetimes have been proposed as good laboratories for studying time dependence in string theory \[13\].
but we are only interested in the $T \to \infty$ limit where one can analyze the leading part of
the solution, dropping sub-leading terms of order $1/T$.

Now, in order to get a static metric in terms of the new coordinates, the old radial
coordinate has to behave as

$$r = r_+ + \left( \frac{\tilde{r}}{T} \right)^{1/c_h} = r_+ + \frac{1}{4c_h^2} \left( \frac{\tilde{r}}{T} \right)^2,$$  \hspace{1cm} (2.13)

where we have defined $c_h = R_{r_+}^{d-4}/(2\mu)$. It was shown in [17] that as $T \to \infty$, the metric
can again be brought into the canonical form (2.4) with parameters

$$b' = q\Omega^{\frac{1}{d-4}}, \hspace{0.5cm} e^\phi_0 = \Omega^{-\frac{d-2}{2}}.$$  \hspace{1cm} (2.14)

Where here and in the following the parameters characterizing the ’leftover’ spacetime are
primed.

$$q' = q, \hspace{0.5cm} R' = \frac{1}{|\Omega|}.$$  \hspace{1cm} (2.15)

From (2.8) it follows that

$$q = \Omega' - \frac{\sigma(\Omega')}{R'} = \Omega - \frac{\sigma(\Omega)}{R}$$  \hspace{1cm} (2.16)

and hence the new angular momentum and twist angle are given by

$$\Omega' = 2\Omega - \frac{\sigma(\Omega)}{R}, \hspace{0.5cm} \gamma' = qR = \Omega R - \sigma(\Omega).$$  \hspace{1cm} (2.17)

3. Relation between perturbative and non perturbative instabilities

In [3] a twisted circle in ten dimensional type II string theory was discussed. The
endpoints of tachyon condensation for twisted circles was analyzed (following [20],[21])
using a $N = 2$ gauged linear sigma model [22],[23],[24]. The fields of the GLSM consist of
a $U(1)$ gauge field, two chiral fields $\Phi_{-n}, \Phi_m$ of charge $(-n, m)$ and an ‘axion’ $P$ which
transforms by imaginary shifts under $U(1)$ gauge transformations. The gauged linear sigma
model has the following action

$$S = \frac{1}{2\pi} \int d^2\sigma d^4\theta \left[ \Phi_m e^{mV} \Phi_m + \Phi_{-n} e^{-nV} \Phi_{-n} + \frac{k}{4}(P + \bar{P} + V)^2 - \frac{1}{2\epsilon^2|\Sigma|^2} \right].$$  \hspace{1cm} (3.1)
Integrating out the gauge fields, the vacuum manifold is given by the solutions to the D-term condition (modulo the $U(1)$ gauge transformations which are responsible for the twisted identifications). In the low energy limit a nonlinear sigma model is defined by the massless fluctuations about the vacuum manifold.

\[ m|\varphi_1|^2 - n|\varphi_{-n}|^2 + kp_1 = 0. \] (3.2)

The role of the complex axion $P = p_1 + ip_2$ is twofold. The imaginary part $p_2$ is used to construct the circle whereas the real part $p_1$ is an auxiliary direction. In [3] it was proposed that motion along the auxiliary direction is equivalent to RG flow and this correspondence was used to determine the endpoint of the tachyon condensation. In the following we will compare the endpoints of perturbative tachyon condensation to the nonperturbative brane nucleation in several cases.

**a)** The gauged linear sigma model with charges $(-n,1)$, where $n$ is an odd integer, interpolates between a twisted circle with twist $\gamma = -1 + 1/n$ and radius $R$ for $p_1 \to \infty$ and an untwisted circle $\gamma' = 0$ and radius $R' = nR$ for $p_1 \to -\infty$. For the Melvin background this twist can be realized by choosing

\[ R, \quad \Omega = \frac{1}{nR}, \quad \gamma = -1 + \frac{1}{n}. \] (3.3)

Using the formulas (2.13) and (2.17) for the radius and twist after the nucleated brane has accelerated away to infinity one finds

\[ R' = nR, \quad \Omega' = \frac{2}{nR} - \frac{1}{R} = \frac{2 - n}{nR}, \quad \gamma' = 3 - n. \] (3.4)

Since $n$ is odd, $\gamma'$ is even and by periodicity equivalent to $\gamma' = 0$. Hence the end point of the nucleation is an untwisted circle of $n$ times the original radius. Note that the action of the instanton diverges for the untwisted circle $\gamma = 0$ and there is no further nonperturbative instability. This agrees with the fact that the end point of the evolution is supersymmetric Type II theory compactified on a circle.
b) In the gauged linear sigma model with charges \((-n, n - 2)\), with \(n\) odd, it follows from (3.2) that one flows from twisted circle with twist \(\gamma = -2/n\) and radius \(R\) in the UV to a twisted circle with twist \(\gamma' = -2/(n - 2)\) and radius \(R' = nR/(n - 2)\). Note that such a flow corresponds to turning the lightest tachyon which does not completely untwist the circle. For the fluxbrane one chooses

\[
R, \quad \Omega = \frac{n - 2}{nR}, \quad \gamma = -\frac{2}{n},
\]

the spacetime after the brane has accelerated away has

\[
R' = \frac{nR}{n - 2}, \quad \Omega' = \frac{n - 4}{nR}, \quad \gamma = -\frac{2}{n - 2}.
\]

With exactly parallels the result for the perturbative tachyon condensation. Note that the resulting twisted circle theory contains itself tachyons or is nonperturbatively unstable. Using the analysis above it is easy to see that after \((n - 3)/2\) further bounces one ends up with (3.4), i.e the stable end point is supersymmetric type II theory on a circle of \(n\) times the original radius.

c) The two examples discussed above decay toward a supersymmetric end state. There are different systems which do not behave this way. However the underlying theory has a spin structure which breaks supersymmetry to start with. For example consider repeating the analysis for 1) but with \(n\) even. From (3.4) it follows that after the nucleation on ends up with \(R' = nR\) and \(\gamma = -1\). From (3.2) its easy to see this from the gauged linear sigma model too. For charge \((-n, 1)\) with \(n\) even, in the IR one has a twist \(-1\) (This depends on the extra \(-1\) in the twist we have to have for proper a proper GSO projection, see [2] for a discussion). This twist corresponds to supersymmetry breaking boundary conditions on the circle [25]. The nonperturbative instability is associated with Witten’s bubble of nothing [26]. Note however that for a twist \(\gamma = -1\) the bounce is given by a euclidean Schwarzschild black hole and the analysis of section 2 will not work since the leftover spacetime is not of the form of a Kaluza-Klein Melvin solution.
d) If the twist angle $\gamma$ is irrational, the gauged linear sigma model analysis cannot be applied. From the nonperturbative bounce one finds that the nucleation does not stop after a finite amount of steps and the radius increases monotonically and one ends up with (supersymmetric) theory at infinite circle. It is tempting to speculate that this would also be the case for the perturbative tachyon condensation on the twisted circles.

4. Discussion

In this note we have pointed out that for twisted circles the endpoint of tachyon condensation (using RG-flow which is believed to give the same result as on shell time evolution) and nonperturbative brane nucleation (where the nucleated brane accelerates off to infinity) are very similar. In particular in both cases the twist becomes smaller and the radius of the circle grows, even the quantitative features of the two endpoints agree. This might suggest that the twisted circle really wants to untwist itself, whether it takes a perturbative of a nonperturbative mechanism to do so. In this note we have considered the classical (tree level) perturbative tachyon and its condensation and the nonperturbative semi-classical instabilities. However we have not studied perturbative quantum instabilities coming from tadpoles at higher loops. The importance of those effects (in particular in comparison with the nonperturbative effects) is an important open question (see [27] and [28] for related discussion of this issue).

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