Symmetry group classification of differential equations: case of free group actions

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Abstract

Based on an original classification of differential equations by types of regular Lie group actions, we offer a systematic procedure for describing partial differential equations with prescribed symmetry groups. Using a new powerful algebraic technique based on the so-called covariant form of a differential equation, we give an effective algorithm for constructing differential equations whose symmetry groups regularly and freely act on the space of dependent and independent variables. As an application, we derive a complete classification of quasi-linear scalar second-order partial differential equations with regular free symmetry groups of dimension no greater than three.

Introduction

Originally developed as a powerful tool for integrating ordinary differential equations, group analysis of differential equations has long been extended beyond this aim and extensively used in differential geometry and mathematical physics. Researchers in these fields generally deal with the following two interrelated problems. The first problem is finding the maximal symmetry group for a given differential equation, whereas the second is classifying all the differential equations admitting a prescribed symmetry group. The most traditional approach for solving these problems is to apply the classical infinitesimal technique developed by Sophus Lie and his followers [1–5].

Until now, numerous results have accumulated in the context of the first problem. In particular, in most cases, the maximal symmetry groups of physically interesting differential equations have already been obtained. The second problem (which is much more difficult than the first one) consists of an exhaustive listing of all the differential equations invariant under prescribed transformation group $G$ whose action is generated by a Lie algebra of vector fields. The solution of this classification task is not only of mathematical interest but also has physical significance, because it eventually leads to the possibility of establishing exact integrable models of physical theories.

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The main concept of solving the problem of group classification was formulated by Sophus Lie himself in his classic paper on the classification of ordinary differential equations admitting non-trivial symmetry groups [6]. Recall that the first step of this classification consists of listing all the inequivalent realizations of the Lie algebra of a prescribed symmetry group. Following this, the second step is to calculate the differential invariants for each inequivalent realization and then use them for constructing the classes of invariant differential equations. Following this procedure, Sophus Lie exhaustively listed all inequivalent Lie algebra realizations in vector fields on the complex plain and all the differential invariants for the corresponding local transformation groups [6] (see also [7] for a more modern viewpoint). Subsequently, an analogous classification was obtained for the Lie algebras of vector fields in $\mathbb{R}^2$ [8], by which Nesterenko constructed bases of the differential invariants and the operators of invariant differentiation for associated local Lie transformation groups [9,10]. We also mention here Mahomed’s paper [11], in which a survey of the principal results relating to the group classification of scalar ordinary differential equations (second-order and linear $n$th-order) is provided. It is important to note that solving the classification problem for the above cases was largely possible owing to the small dimensions of the spaces of independent and dependent variables as the computational difficulties of applying the infinitesimal technique grow rapidly with increasing these dimensions.

It also was Sophus Lie who successfully provided one of the first general solutions of the classification problem applied to partial differential equations. In his study [12], he performed a complete group classification of linear second-order partial differential equations in two independent variables. A further systematic development of these concepts was conducted by Ovsyannikov [2,13]. His approach, based on the use of equivalence transformations for differential equations and the concept of the so-called arbitrary element, made it possible to implement a large-scale program for the group classification of the basic equation of mathematical physics. In particular, Ovsyannikov himself has performed the complete group classification of a class of non-linear heat conductivity equations [13], and subsequently Ibragimov et al. [14] employed his approach for the classification of non-linear filtration equations. The concepts of Ovsyannikov have also formed the basis of the preliminary group classification method, which was first introduced in [15] and was made well-known by Ibragimov et al. [16]. Ibragimov and Torrisi applied this method to non-linear detonation equations [17] and non-linear hyperbolic equations [16], and subsequently, Torrisi and coauthors obtained the preliminary group classification of some classes of diffusion and heat conduction equations [18,19]. Furthermore, the preliminary group classification method was repeatedly modified, and the range of its applicability was constantly expanded (see, for example, [20,21]). In particular, we mention here a series of the papers [22,25], in which classification approaches based on a combination of Lie’s infinitesimal method, Ovsyannikov’s concept of the equivalence group, and the theory of Lie algebra realizations is suggested.

It should be noted that Ovsyannikov’s approach as well as the preliminary group classification method suggest that the class of differential equations which is of interest is restricted by some predefined ansatz. Generally, such an ansatz is some model equation (for instance, the wave equation or the heat conductivity equation), into which, based on a physical motivation, one artificially introduces arbitrary elements, i.e., undefined functions of the dependent and independent variables. It is remarkable that a similar problem statement leads to a partial group classification involving only equations that are obtained as “deformations” of the initial
model equation. (In fact, this is equivalent to the problem of constructing differential invariants of a particular form). At the same time, the original infinitesimal Lie’s method, if it is effective, allows performing the complete group classification of all the differential equations of arbitrary order.

As we already noted, the main difficulty in using Lie’s method is its computational complexity. Indeed, both steps of this method (the classification of the inequivalent realizations of a given Lie algebra and the construction of their differential invariants) include the integration of systems of partial differential equations, whose dimensions grow with increasing number of dependent and independent variables. Recently, however, some new results have been obtained that can be used for developing a comparatively more effective approach for group classification; this approach completely excludes the stages of integrating differential equations.

First, we would like to highlight the study in [26], in which Popovich and co-authors obtained the complete set of local inequivalent realizations of real Lie algebras of dimensions no larger than four on a space of an arbitrary number of variables using the technique of the so-called megaideals. Slightly subsequently, the authors of the present paper proposed a more effective method for constructing Lie algebra realizations, which is amenable for implementation on a computer using a symbolic computer algebra system [27]. Subsequently, Nesterenko applied this method to solve numerous physically important classification problems, such as the classification of the realizations of the Galilei and Poincaré algebras [28,29].

Some progress has been made recently in the theory of differential invariants. Olver [30] proposed a method to construct differential invariants without a direct integration of differential equations (see also [31]). Another result that we would like to mention is an algebraic approach suggested in [32] to calculate differential invariants and operators of invariant differentiation. As demonstrated in this study, if the dependent variables are invariant under some transitive Lie group of transformations, then the differential invariants are entirely generated by the invariant operators on the corresponding homogeneous space of the independent variables. In turn, the invariant operators on the homogeneous space can be effectively constructed by a purely algebraic approach [33].

The references cited above display that, as the first step of Lie’s classification program, the problem of listing all the inequivalent realizations of a prescribed Lie algebra can be effectively solved by only purely algebraic techniques. Therefore, in the present study, we basically focus on the second stage of Lie’s program: to find a maximally wide class of differential equations whose symmetry group is generated by a given Lie algebra realization. For this, we develop an algebraic approach based on the formalism outlined in [32]. Within this approach, the construction of differential invariants is performed in terms of the algebra of invariant differential operators, an object that is in a manner dual to the Lie algebra realization. As we have noted, the range of applicability of the formalism is initially limited by the group actions for which the dependent variables are invariant. To overcome this restriction, we introduce the so-called covariant form of a differential equation, in which dependent and independent variables \{x\} and \{u\} are treated equally (and are regarded as independent variables) and invariant auxiliary variable \(w\) is added. Note that this form of differential equations is used in a well-known method of integrating quasi-linear first-order partial differential equations (see, for example, [34]). Furthermore, it should be emphasized that our algebraic approach for constructing differential invariants essentially depends on the type of the given group action; in this relation, we propose an original classification of differential equations based on the types of prescribed
symmetry groups.

The article is organized as follows. In Section 1, we briefly recall the basic facts concerning differential equations and their symmetry groups in regard to differential geometry. In this section, we also suggest a classification of differential equations based on the types of symmetry group actions; this classification enables dividing differential equations in an invariant manner into five distinct types. In the second section, we introduce the concept of the covariant form of a differential equation and display that any scalar differential equation can be written in such a form. Moreover, we formulate the necessary and sufficient condition under which a given equation is the covariant form of some differential equation. In the third section, we present Theorems 3 and 4 that provide an effective algorithm for the solution of the group classification problem of differential equations with simply transitive symmetry groups. Applying this algorithm, we present an exhaustive classification of both second-order differential invariants and quasi-linear second-order differential equations for all inequivalent two- and three-dimensional simply transitive symmetry groups, in Section 4. Finally, the problem of the group classification of second-order differential equations whose symmetry groups act freely and have dimensions up to and including three is completely solved, in Section 5.

1 Problem statement

In this section, we recall the standard mathematical constructions and formulate the problems discussed in this study. For more details of the group analysis of differential equations, see classical literature [2–5]. The general information on Lie groups and their homogeneous spaces can be found, for example, in [35–37].

Let \( Z \subset \mathbb{R}^n \) be an open subset (regarded as the space of independent and dependent variables for some differential equation), and let \( \text{Diff}^n Z \) be the infinite-dimensional diffeomorphism group of \( Z \). The subset \( Z \) looks locally like the Cartesian product \( Z \cong U \times X \), where \( X \) and \( U \) are the spaces of independent and dependent variables, respectively. The space \( U \) can be uniquely extended to the \( k \)-jet space \( U \rightarrow U_k = U \times U \times \cdots \times U \) with local coordinates \((u^\alpha, u_i^\alpha, u_{ij}^\alpha, \ldots)\). A differential equation (or a system of differential equations) is a surface \( \mathcal{E} \) in the extended space \( Z_k \cong U_k \times X \):

\[
\mathcal{E} : \quad E_\mu(x, u, u_i, u_{ij}, \ldots) = 0, \quad \mu = 1, \ldots, s.
\]

Let \( r_\mu \) be the maximal order of derivatives on which the \( \mu \)-th equation in (1) depends. It is understandable that this number does not change under diffeomorphisms of \( Z \) and the choice of the “splitting” \( Z \cong U \times X \). Thus, the sequence of numbers \( \text{ord} \mathcal{E} = \{r_1, \ldots, r_s\} \) is a \( \text{Diff}^n Z \)-invariant characteristic of the differential equation \( \mathcal{E} \).

A solution of the differential equation (1) is a surface \( S \) in \( Z \):

\[
S : \quad u^\alpha = f^\alpha(x), \quad \alpha = 1, \ldots, \dim U.
\]

The codimension, \( \text{codim} S \), of this surface coincides with the number of dependent variables, and is also a \( \text{Diff}^n Z \)-invariant characteristic of the differential equation \( \mathcal{E} \). The extended surface \( S_k \subset Z_k \),

\[
S_k : \quad u^\alpha_i = \frac{\partial f^\alpha(x)}{\partial x^i}, \quad u^\alpha_{ij} = \frac{\partial^2 f^\alpha(x)}{\partial x^i \partial x^j}, \quad \ldots ; \quad (x, u) \in S,
\]
which could be regarded as an integral submanifold of the distribution of Pfaffian forms
\[ du^\alpha - u_\alpha^i dx^i = 0, \quad du^\alpha_i - u_\alpha^{ij} dx^j = 0, \quad \ldots, \]
is a submanifold of the manifold \( \mathcal{E} \).

Denote by \( \text{Sol} \mathcal{E} \) the set of all the solutions of the differential equation (1). By definition, the symmetry group of this equation is a subgroup \( G = \text{Sym} \mathcal{E} \subset \text{Diff} \mathcal{Z} \) that transforms the solutions of \( \mathcal{E} \) to other solutions:
\[ g \in \text{Sym} \mathcal{E} \iff g(S) \in \text{Sol} \mathcal{E}, \quad \forall S \in \text{Sol} \mathcal{E}. \]

In this study, we restrict our attention to connected local Lie groups of symmetries \( G \), which are generated by the corresponding Lie algebras \( \mathfrak{g} \) of vector fields \( X_i \) on \( \mathcal{Z} \):
\[ X_i = \zeta_i^a(z) \frac{\partial}{\partial z^a} \in T_z \mathcal{Z}; \quad [X_i, X_j] = C_{ij}^k X_k. \]

Here, \( C_{ij}^k \) are the structure constants of the Lie algebra \( \mathfrak{g} \).

Remark 1. In the entire study, we assumed that the symmetry group \( G \) acts regularly on \( \mathcal{Z} \). Moreover, we also assume that the group act effectively; otherwise, we can simply replace \( G \) by the effectively acting quotient group \( G/G_0 \), where \( G_0 \) denotes the global isotropy subgroup of \( \mathcal{Z} \): \( G_0 z = z \) for all \( z \in \mathcal{Z} \).

Let \( G \) be a symmetry group of a given differential equation \( \mathcal{E} \) with infinitesimal generators \( X_i \), and let \( \mathfrak{g} \) be the corresponding Lie algebras \( \mathfrak{g} \) of vector fields \( X_i \) on \( \mathcal{Z} \):
\[ X_i = \zeta_i^a(x,y) \frac{\partial}{\partial x^a}. \]

Note that the isotropy subgroup \( H \) and its Lie algebra \( \mathfrak{h} \), in general, depend on the invariant variables \( y \in \mathcal{Y} \), i.e. \( H = H(y), \mathfrak{h} = \mathfrak{h}(y) \). From the geometric perspective, this suggests that when a point \( z_0 \in \mathcal{Z} \) moves to another point \( z_1 \) along a line transverse to the tangent space \( T_{z_0} M_0 \), the homogeneous space \( M_0 = G_0 z_0 \) is mapped into the homomorphic homogeneous space \( M_1 = G z_1 \). The algebraic properties of the latter can differ from those of the space \( M_0 \) owing to the inequation \( \mathfrak{h}(y_0) \neq \mathfrak{h}(y_1) \) (see Fig. 1). This, in turn, leads to the infinitesimal generators \( X_i \) whose coefficients depend on the invariant variables \( \{y\} \) in the coordinate chart \( \mathcal{Z} \simeq U \times X \),
\[ X_i = \zeta_i^a(x,y) \frac{\partial}{\partial x^a}. \]
Moreover, this dependence cannot be eliminated by coordinate transformations. In this regard, we introduce a numerical characteristic \( \text{ind}(X, Y) \), called the *coupling index*, whose definition will be provided in our subsequent papers.

In view of the above considerations, the (regular) actions of a connected Lie group \( G \) can be divided into several types.

**Type I.** *Simply transitive action of \( G \) on \( Z \).*

In this case, \( \text{rank} \, \zeta^a_i(x) = \dim \mathfrak{g} = \dim Z, \mathfrak{h} = \{0\} \), and there are no invariant variables \( \{y\} \).

**Type II.** *Free action of \( G \) on \( Z \).*

In this case, \( \text{rank} \, \zeta^a_i(x) = \dim \mathfrak{g} < \dim Z, \mathfrak{h} = \{0\} \), and there are invariant variables \( \{y\} \); however, coefficients \( \zeta^a_i(x) \) do not depend on their \( (\text{ind}(X, Y) = 0) \).

**Type III.** *Transitive action of \( G \) on \( Z \).*

In this case, \( \text{rank} \, \zeta^a_i(x) = \dim \mathfrak{g}/\mathfrak{h} = \dim Z, \dim \mathfrak{h} > 0 \), and there are no invariant variables \( \{y\} \).

**Type IV.** *Intransitive action of \( G \) on \( Z \) with the maximal coupling index.*

In this case, \( \text{rank} \, \zeta^a_i(x, y) = \dim \mathfrak{g}/\mathfrak{h} < \dim Z, \dim \mathfrak{h} > 0, \mathfrak{h} = \mathfrak{h}(y) \), and there are the invariant variables \( \{y\} \), \( \text{ind}(X, Y) = \dim Y \).

**Type V.** *Intransitive action of \( G \) on \( Z \) with a non-maximal coupling index.*

In this case, \( \text{rank} \, \zeta^a_i(x, y) = \dim \mathfrak{g}/\mathfrak{h} < \dim Z, \dim \mathfrak{h} > 0, \mathfrak{h} = \mathfrak{h}(y) \), and there are the invariant variables \( \{y\} \), \( \text{ind}(X, Y) < \dim Y \).

Here, each subsequent type contains in a sense all the previous types as special cases. The usefulness of this classification is because of the most effective methods for constructing invariant differential equations are essentially different for different types of group actions.

Thus, it is convenient to introduce the \( \text{Diff} \, Z \)-invariant classification of differential equations given in Table 1. In the table, symbols \( n, m, l, k, r_j, s, \) and \( t \) are arbitrary positive integers.

| Type | \( \dim Z \) | \( \text{ord} \mathfrak{E} \) | \( \text{codim} \mathcal{S} \) | \( \dim \mathfrak{g} \) | \( \dim \mathfrak{h} \) | \( \text{ind}(X, Y) \) |
|------|--------------|-----------------|------------------|----------------|----------------|----------------|
| I    | \( n \)     | \( \{r_1, \ldots, r_t\} \) | \( s \)          | \( n \)        | 0              | 0              |
| II   | \( n + l \) | \( \{r_1, \ldots, r_t\} \) | \( s \)          | \( n \)        | 0              | 0              |
| III  | \( n \)     | \( \{r_1, \ldots, r_t\} \) | \( s \)          | \( n + m \)    | \( m \)        | 0              |
| IV   | \( n + k \) | \( \{r_1, \ldots, r_t\} \) | \( s \)          | \( n + m \)    | \( m \)        | \( k \)        |
| V    | \( n + l + k \) | \( \{r_1, \ldots, r_t\} \) | \( s \)          | \( n + m \)    | \( m \)        | \( k \)        |

It is notable that the first position in Table 1, which indicates types of differential equations, is not informative and will frequently be omitted. Furthermore, note that instead of specifying the numbers \( \dim \mathfrak{g} \) and \( \dim \mathfrak{h} \) one may describe the structures of the Lie algebra \( \mathfrak{g} \) and its
subalgebra $\mathfrak{h}$ in more detail. For instance, this may be realized by specifying the commutation relations $[e_i, e_j] = C^k_{ij} e_k$ of the Lie algebra $\mathfrak{g}$ in some basis $\{e_i\}$ and by the explicit listing of the basis elements of the subalgebra: $\mathfrak{h} = \langle u^i(y) e_i \rangle$. It will be displayed in our subsequent studies that all the information is sufficient for the complete description of the class of differential equations of a given type.

**Remark 2.** The interested reader may find a discussion on Diff $Z$-invariant approaches for the group classification of differential equations in Ovsyannikov’s book [2]. In particular, Ovsyannikov himself suggested the classification to be conducted as follows. To each system of differential equations, we associate a quadruple of numbers $(\nu, \mu, \kappa, \sigma)$, where $\nu$ and $\mu$ are the numbers of independent and dependent variables, respectively. The number $\kappa$ is the maximal order of the derivatives that appear in the system, and $\sigma$ is the number of equations. Remarkably, the classification scheme outlined in Table 1 is a more subtle systematization of differential equations than Ovsyannikov’s one because it considers the structure of the symmetry group action. The relation between Ovsyannikov and our classifications is given as follows:

$$\nu + \mu = \dim Z, \quad \mu = \text{codim} \mathcal{S}, \quad \kappa = \max \text{ord} \mathcal{E}, \quad \sigma = \dim \text{ord} \mathcal{E} = t.$$

Our overall aim is to provide a complete description of all the non-linear partial differential equations whose types are presented in Table 1. In this study, we focus on the second-order scalar partial differential equations of types I ($\{n, \{2\}, 1, n, 0, 0\}$) and II ($\{n+m, \{2\}, 1, n, 0, 0\}$).

We note that the case of one invariant variable, which was declared as a dependent one, was considered in [32]. According to our classification, this case includes types $\{n+1, \{2\}, 1, n, 0, 0\}$ (the special case of type II) and $\{n+1, \{2\}, 1, n+m, m, 0\}$ (the special case of type V). Let us show that there is a fundamental difference between the equations of types $\{n, \{2\}, 1, n, 0, 0\}$ and $\{n+1, \{2\}, 1, n, 0, 0\}$.

First, we consider the last type, i.e., the case of a scalar second-order differential equation with $n+1$ dependent and independent variables. In this case, an $n$-dimensional symmetry group $G$ acts on an $n$-dimensional space simply transitively (since $\dim \mathfrak{h} = 0$), and there is one invariant variable $y_1$, which is declared, for convenience, as a dependent one, $u = y_1$. As the dimension of the $G$-orbits coincides with the dimension of the group $G$, we can assume that the group acts on itself by right multiplications and that the infinitesimal generators of this action coincide with the left-invariant vector fields on $G$:

$$X_i = \xi_i = \sum_{a=1}^{n} \xi^a_i (x) \frac{\partial}{\partial x^a}, \quad i = 1, \ldots, n = \dim \mathfrak{g}. \quad (3)$$

(Here and elsewhere, symbol $\xi_i$ denotes the left-invariant vector field on $G$ associated with the basis element $e_i \in \mathfrak{g}$).

Now, let us consider an equation of type $\{n, \{2\}, 1, n, 0, 0\}$, i.e., the case of a scalar second-order differential equation with $n$ dependent and independent variables. In this case, an $n$-dimensional symmetry group $G$ acts simply transitively on an $n$-dimensional space $Z$ (since $\dim \mathfrak{h} = 0$) and there are no invariant variables $\{y\}$. Again, by virtue of the dimension of the orbits coinciding with the dimension of the group, it can be assumed that $G$ acts on itself by right multiplications and the infinitesimal generators $[2]$ of the action coincide with the left-invariant vector fields: $X_i = \xi_i = \xi_i^j (z) \partial_{z^j}$. All variables $z^j$ ($j = 1, \ldots, n$) are treated equally;
therefore, we can select the variable $z^n$ as an independent one and rewrite the generators $X_i$ in a relatively traditional form:

$$X_i = \sum_{a=1}^{n-1} \xi_i^a(x,u) \frac{\partial}{\partial x^a} + \eta_i(x,u) \frac{\partial}{\partial u}, \quad i = 1, \ldots, n = \dim g.$$  (4)

Here, $\eta_i(x,u) = \xi_i^n(z)$, $\xi_i^a(x,u) = \xi_i^a(z)$, $a = 1, \ldots, n-1$.

Comparing Eqs. (3) and (4), it can be seen that the case of $G$-action with the generators (3) considered in [32] is much easier than the case of $G$-action with the generators (4).

In the next section, we will display that an arbitrary differential equation can be written in the so-called covariant form, which preserves the original “equivalence” of the dependent and independent coordinates. Specifically, the procedure of “splitting” into the dependent and independent variables, $Z \simeq U \times X$, is not necessary. This will allow us to describe $G$-invariant differential equations with the generators (3) and (4) in a uniform manner.

2 Covariant form of differential equations

In this study, we explore the classes of differential equations that are orbits of the diffeomorphism group $\text{Diff } Z$. However, “splitting” the space $Z$ into dependent and independent variables is not a $\text{Diff } Z$-invariant procedure. To illustrate this, we consider the following example.

Let $(x,y)$ be local coordinates on a two-dimensional space $Z$, where $x$ is an independent variable and $y$ is a dependent one. Let $g \in \text{Diff } Z$ be the diffeomorphism $(x,y) \mapsto (p,q) = g(x,y) = (y + x/2, y - x)$. We note that in this case, we cannot answer the question: what is the image of a function $E = E(x,y,y_x,y_{xx})$ under the diffeomorphism $g \in \text{Diff } Z$? Indeed, it may be a function $\tilde{E}(p,q,p_q,p_{qq})$ or function $\tilde{E}(p,q,q_p,q_{pp})$, or some other one. In fact, to answer this question, we need to again “split” the coordinates on $Z$ into dependent and independent after acting the diffeomorphism.

It should be emphasized that if the diffeomorphism is an element of the one-parameter group generated by a vector field on $Z$, one generally accepts the rule that “splitting” into dependent and independent variables is defined by continuity. For instance, if the diffeomorphism $g \in \text{Diff } Z$ is generated by the vector field

$$X = \eta(x,y) \frac{\partial}{\partial y} + \zeta(x,y) \frac{\partial}{\partial x}, \quad g_a(x,y) = (x_a,y_a), \quad g = g_1;$$

$$\frac{\partial y_a}{\partial a} = \eta(x_a,y_a), \quad \frac{\partial x_a}{\partial a} = \zeta(x_a,y_a), \quad y_a|_{a=0} = y, \quad x_a|_{a=0} = x,$$

then, by continuity, we assume that $y_a$ are the dependent variables for all $a > 0$.

Let us show that any differential equation can be rewritten in a special form (the covariant form) that preserves the original symmetry between the dependent and independent variables.

First, we formulate the basic idea. Instead of the “splitting” $Z \simeq U \times X$, which is non-invariant under $\text{Diff } Z$, we introduce the invariant “splitting” $Z \times W$, where the extra variables $w = (w^a) \in W$ are invariant under the group $\text{Diff } Z$. Our task is to show that the differential equation (1) can be represented in the $\text{Diff } Z$-invariant form

$$\dot{E} : \quad \dot{E}_\mu(z,w_i,w_{ij},\ldots) = 0, \quad \mu = 1, \ldots, s.$$  (5)
where the functions $w^\alpha$ satisfy the additional conditions

$$w^\alpha = 0, \quad \alpha = 1, \ldots, \dim W.$$  \hfill (6)

**Example 1.** \(\triangleright\) As a simple illustration of this idea, we consider a quasi-linear first-order partial differential equation of type \(\{n, \{1\}, 1, \ast, \ast, \ast\}\):

$$E : \sum_{i=1}^{n-1} a^i(x, u) \frac{\partial u}{\partial x^i} + b(x, u) = 0.$$ \hfill (7)

The main trick of reducing (7) to a linear partial differential equation is well-known; instead of the equation (7), we consider the equation

$$\tilde{E} : \sum_{i=1}^{n-1} a^i(z) \frac{\partial w}{\partial z^i} - b(z) \frac{\partial w}{\partial z^n} = 0,$$ \hfill (8)

where $z^n = u, \; z^i = x^i$ for $i = 1, \ldots, n - 1$. If $w = w(z)$ is a solution of (8), then the equality $w(z) = 0$ determines a solution of (7). The differential equation (8) (together with the condition $w = 0$) is the covariant form of the differential equation (7). \(\triangleright\)

It is not difficult to generalize this trick for an arbitrary first-order partial differential equation.

**Proposition 1.** A first-order partial differential equation

$$E(x, u; u_a) = 0$$ \hfill (9)

can be represented in the covariant form

$$\tilde{E}(z; w) = 0,$$ \hfill (10)

$$w = 0.$$ \hfill (11)

**Proof.** \(\triangleright\) (Here and below we will use the notation: $n = \dim Z, \; u = z^n, \; x^a = z^a \; (a = 1, \ldots, n - 1)$). Assuming, by definition, that (11) holds, we obtain

$$dw = w_idz^i = w_adx^a + w_n du = 0.$$ 

By virtue of $du - u_a dx^a = 0$, we have

$$u_a = -w_a/w_n.$$ \hfill (12)

(We have obtained the well-known rule of implicit differentiation). Substituting (12) into (9), we deduce the equation

$$\tilde{E}(z; w) = E(z; w_a/w_n),$$

which, together with the condition (10), is the covariant form of the first-order partial differential equation (7). \(\triangleright\)

The following example illustrates the content of Proposition 1.
Example 2. Let us consider a first-order partial differential equation, which is linear with respect to the derivatives of $u$:

$$g^{ab}(x, u)u_a u_b + b(x, u) = 0.$$ 

Using (12), we can rewrite it as

$$g^{ab}(z)w_a w_b + w_n^2 b(z) = 0.$$ 

Together with the condition (10), this leads to the covariant form of the given first-order partial differential equation.

Recall that a function $f = f(x), x \in \mathbb{R}^n$ is said to be homogeneous of degree $k$, if $f(\lambda x) = \lambda^k f(x)$ for all real numbers $\lambda$.

Proposition 2. Let $\tilde{E}(z; w)$ be a function that is homogeneous with respect to the derivatives $w_i$. The differential equation (10) together with the condition (11) is the covariant form of the differential equation (9), where $E(x, u; u_a) = \tilde{E}(x, u; -u_a, 1)$.

Proof. Applying (12) and using the homogeneity of $\tilde{E}$, we obtain

$$\tilde{E}(z; w_i) = \tilde{E}(z; w_a, w_n) = \tilde{E}(z; -w_n u_a, w_n) = w_n^k \tilde{E}(x, u; -u_a, 1).$$

Thus, if $\tilde{E}(z; w)$ is a homogeneous function of the derivatives $w_i$, then the equation (10) together with (11) is the covariant form of the differential equation $\tilde{E}(x, u; -u_a, 1) = 0$.

Example 3. Let us consider a differential equation of the form (10), where $\tilde{E}(z, w_i)$ is a function that is homogeneous of degree two with respect to the second derivatives:

$$g^{ij}(z)w_i w_j = 0.$$ 

By virtue of (12), we can write this in the form

$$g^{na}(z)w_n w_a + g^{nn}(z)w_n^2 = 0 \iff w_n^2 \left( g^{ab}(z)u_a u_b - 2g^{na}(z)u_a + g^{nn}(z) \right) = 0.$$ 

which is equivalent to the differential equation

$$g^{ab}(x, u)u_a u_b - 2g^{na}(x, u)u_a + g^{nn}(x, u) = 0.$$ 

Now, we consider second-order partial differential equations.

Proposition 3. An arbitrary second-order partial differential equation

$$E(x, u; u_a; u_{ab}) = 0. \quad (13)$$

can be represented in the covariant form

$$\tilde{E}(z; w_1 w_2) = 0, \quad (14)$$

$$w = 0. \quad (15)$$
Proof. In view of the Pfaffian forms
\[ dw - w_idz^i = 0, \quad dw_i - w_iz^j = 0, \quad i, j = 1, \ldots, n; \]
\[ du - u_adx^a = 0, \quad du_a - u_abdx^b = 0, \quad a, b = 1, \ldots, n - 1, \]
we obtain the expressions for the second derivatives of the function \( u = u(x) \) determined implicitly by the relation \( w = 0 \):
\[ u_{ab} = \frac{-w_{ab}w_n + w_{na}w_{bn} - w_{an}w_{bn}}{w_n^2}. \] (16)
Substituting (12) and (16) into (13) leads to a differential equation of the form (14).

\[ \begin{aligned} \text{Theorem 1.} & \quad \text{The differential equation} \quad (14) \text{together with the condition} \quad (15) \\
& \quad \text{is the covariant form of a differential equation of the form} \quad (13) \text{if and only if it is invariant under the infinite-dimensional group of the transformations:} \\
& \quad w \rightarrow s(z)w. \quad (17) \end{aligned} \]
Proof. Let us assume that the differential equation (14) together with the condition (15) is the covariant form of a differential equation of the form (13). In accordance with Proposition 3, (14) is obtained by substituting (12) and (16) \((u_a \rightarrow u_a[w], u_{ab} \rightarrow u_{ab}[w])\) into (13). It is quite easy to check that the right-hand sides of the substitutions (12) and (16) are invariant under the transformations (17):
\[ u_a[s(z)w] = -\frac{D_x(s(z)w)}{D_n(s(z)w)} = -\frac{s_a(z)w + s(z)w_n}{s_n(z)w + s(z)w_n} = -\frac{w_a}{w_n} = u_a[w]. \]
Here, we have used the condition (15). Analogously, it can be proved that equality \( u_{ab}[s(z)w] = u_{ab}[w] \) holds.
Now, suppose (14) and (15) are invariant under the transformations (17). Because the infinitesimal generator of these transformations has the form \( X = \sigma(z)w\partial_w \), the invariance of (14) is equivalent to the equalities
\[ \begin{aligned} X_2 \tilde{E}(z; w_1; w_2) |_{\tilde{E}=0} = & \ 0 \iff \left( \sigma(z)D + \sum_j \sigma_j(z)R_j \right) \tilde{E}(z; w_1; w_2) |_{\tilde{E}=0} = 0, \end{aligned} \]
where we have used the notation
\[ D = \sum_i w_i \frac{\partial}{\partial w_i} + \sum_{i \leq j} w_{ij} \frac{\partial}{\partial w_{ij}}, \quad R_j = \sum_i (1 + \delta_{ij})w_i \frac{\partial}{\partial w_{ij}}. \]
Thus, the invariance of (14) under the transformations (17) is equivalent to the equalities
\[ \begin{aligned} D\tilde{E}(z; w_1; w_2) = & \ k\tilde{E}(z; w_1; w_2), \quad (18) \\
R_j\tilde{E}(z; w_1; w_2) = & \ 0, \quad j = 1, \ldots, n. \quad (19) \end{aligned} \]
The equation (18) is the requirement that the function \( \tilde{E}(z; w; w) \) be homogeneous with respect to the first and second derivatives: \( \tilde{E}(z; \lambda w_1; \lambda w_2) = \lambda^2 \tilde{E}(z; w; w) \). Solutions of the system of equations (19) are the invariants:

\[
J_i = w_i, \quad 1 \leq i \leq n; \quad J_{ij} = w_i^2 w_{jj} + w_j^2 w_{ii} - 2 w_i w_j w_{ij}, \quad 1 \leq i < j \leq n.
\] (20)

Thus, up to a positive constant factor, the second-order differential equations invariant under the transformations (17) can be represented in the form

\[
\tilde{E}(z; w; w) = F(z, \tilde{J}_i, \tilde{J}_{ij}) = 0,
\] (21)

where \( \tilde{J}_i = w_i/w_n; \) \( \tilde{J}_{ij} = J_{ij}/w_n \), \( F(\cdot, \cdot, \cdot) \) is an arbitrary function.

Let \( z^n = u \) and \( x^a = z^a \), \( (a = 1, \ldots, n - 1) \) be the dependent and independent variables, respectively. Subsequently,

\[
u_a = -\tilde{J}_a, \quad \nu_{aa} = -\tilde{J}_{an}, \quad \nu_{ab} = (\tilde{J}_{ab} - \tilde{J}_a \tilde{J}_b - \tilde{J}_b \tilde{J}_a)/2\tilde{J}_a \tilde{J}_b.
\]

These expressions for the derivatives \( \nu_a, \nu_{ab} \) in terms of the invariants \( \tilde{J}_a, \tilde{J}_{ab} \) allow us to rewrite an arbitrary differential equation (13) in the form of (21).

To conclude this section, we formulate, without giving any proofs, the following theorem.

**Theorem 2.** A differential equation (a system of differential equations) of any order with dependent variables \( u = (u^1, \ldots, u^m) \) can be represented in the covariant form,

\[
E_\mu(z; w, w, \ldots, w) = 0, \quad \mu = 1, \ldots s; \tag{22}
\]

\[
w^\alpha = 0, \quad \alpha = 1, \ldots, m. \tag{23}
\]

The system of equations (22), (23) is invariant under the transformations

\[
w^\alpha \to s^\alpha_\beta(z) w^\beta, \quad \det ||s^\alpha_\beta(z)|| \neq 0.
\]

### 3 Differential equations of simply transitive type

Let us consider a differential equation (13) of type \( \{n, 2, 1, n, 0, 0\} \), i.e., the equation (13) is a scalar second-order partial differential equation admitting an \( n \)-dimensional group \( G = \text{Lie}(\mathfrak{g}) \) of point symmetries with infinitesimal generators (4). The group \( G \) acts simply transitively on an \( n \)-dimensional space \( Z \) of \( n - 1 \) independent variables \( x^a = z^a \), \( (a = 1, \ldots, n - 1) \) and one dependent variable \( u = z^n \). By virtue of the simple transitivity and local character of the problem, we can assume that the infinitesimal generators of the symmetry group form the realization of \( \mathfrak{g} \) by the left-invariant vector fields: \( X_i = \xi_i, i = 1, \ldots, n \).

Let \( \eta_i = \eta^i_\sigma(z) \partial_\sigma \) be the realization of \( \mathfrak{g} \) by the right-invariant vector fields:

\[
[\xi_i, \xi_j] = C^k_{ij} \xi_k, \quad [\eta_i, \eta_j] = -C^k_{ij} \eta_k, \quad [\xi_i, \eta_j] = 0.
\]

Below, we will also use the following notation:

\[
w(i) = \hat{\eta}_i w = \eta^i_\sigma(z) w_\sigma, \quad w(ij) = \frac{1}{2}(\hat{\eta}_i \hat{\eta}_j + \hat{\eta}_j \hat{\eta}_i) w = (\hat{\eta}_i \circ \hat{\eta}_j) w, \quad w(j_1 \ldots j_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \hat{\eta}_{j_{\sigma(1)}} \cdots \hat{\eta}_{j_{\sigma(k)}} w.
\] (24)
Here, \( \hat{\eta}_i = \eta_i^j D_j \), where \( D_j = \partial_{x_j} + w_j \partial_w + w_{jk} \partial_{w_k} + \cdots \) is the total derivative operator.

The following theorem was proved in [32].

**Theorem 3.** Let a differential equation of the form,

\[
E(z, w, \frac{w}{1}, \frac{w}{2}, \ldots, \frac{w}{k}) = 0
\]

with \( n \) independent variables \( \{z^i\} \) and one dependent variable \( w \) admit an \( n \)-dimensional symmetry group \( G \) with infinitesimal generators \( X_i = \xi_i = \xi_i^j(z) \partial_{x_j} \). Suppose \( G \) acts simply transitively in the space of the variables \( z \) (i.e., \( \det \|\xi_i^j\| \neq 0 \)). Then this equation can be represented in the form

\[
F(w, w(i), w(ij), \ldots, w(j_1 \ldots j_k), \ldots) = 0. \tag{25}
\]

Specifically, the set \( \{w, w(i), w(ij), \ldots, w(j_1 \ldots j_k), \ldots\} \) forms a basis of the differential invariants, and \( \hat{\eta}_i \) are the operators of the invariant differentiation.

Because for differential equations of the type under consideration the dependent variable \( \{u\} \) is not invariant, we cannot directly apply Theorem 3. However, the transition to the covariant form, in which the dependent variable is an invariant of the symmetry group, allows us to define a general form of the differential equations with simply transitive symmetry groups.

**Theorem 4.** The covariant form of any second-order partial differential equation with a simply transitive group of point symmetries has the form

\[
F(I(i), I(ij)) = 0, \quad w = 0, \tag{26}
\]

where

\[
I(i) = w(i)/w(n), \quad I(ij) = (w_i^j w_j) - 2w_i w_j w(ij) + w_i^j w(i)/w^3_n, \quad 1 \leq i < j \leq n. \tag{27}
\]

**Proof.** In accordance with Proposition 3 any second-order differential equation (13) can be written in the covariant form (14), (15), where the dependent variable \( w \) is an invariant of the symmetry group \( G \). Therefore, we can utilize Theorem 3 and represent (14) in the form (25) (for \( k = 2 \)). By virtue of Theorem 1, we also require invariance of (25) under the infinite-dimensional group of transformations (17). The second prolongation of the generators of this transformation group can be written as

\[
X_2 = \sigma(z) w \frac{\partial}{\partial w} + \sum_i D_{z_i} (\sigma(z) w) \frac{\partial}{\partial w_i} + \sum_{i \leq j} D_{z_i} D_{z_j} (\sigma(z) w) \frac{\partial}{\partial w_{ij}} =
\]

\[
= \sigma(z) w \frac{\partial}{\partial w} + \sum_i \hat{\eta}_i (\sigma(z) w) \frac{\partial}{\partial w(i)} + \sum_{i \leq j} (\hat{\eta}_i \circ \hat{\eta}_j)(\sigma(z) w) \frac{\partial}{\partial w(ij)}. \]

Then, considering (15), we have

\[
X F(w, w) \big|_{F=0} = 0 \iff \left( \sigma(z) D + \sum_j \sigma_{(j)} (z) R_{(j)} \right) F(w, w) \big|_{F=0} = 0.
\]

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Here, \( w^{(1)} = \{w_{(i)}\} \), \( w^{(2)} = \{w_{(ij)}\} \), \( \sigma(j)(z) = \hat{\eta}_j \sigma(z) \), and

\[
D = \sum_i w_{(i)} \frac{\partial}{\partial w_{(i)}} + \sum_{i<j} w_{(ij)} \frac{\partial}{\partial w_{(ij)}} , \quad R(j) = \sum_i (1 + \delta_{ij})w_{(i)} \frac{\partial}{\partial w_{(ij)}}
\]

By solving the system of differential equations, similar to (18) and (19), we obtain (26) and (27).

4 Differential equations admitting simply transitive symmetry groups of low dimensions

Let \( g^n \) be the \( n \)-dimensional Lie algebra of a local simply transitive symmetry Lie group. Let us provide an algorithm for calculating the second-order differential invariants of the symmetry group from the structure constants \( C_{ij}^k \) of the Lie algebra \( g^n \). This algorithm also allows us to output the representatives of the classes of differential equations invariant under the symmetry group and linear in the highest order derivatives. Any other equation admitting the Lie algebra \( g^n \) as a symmetry algebra can be reduced to one of the given representatives by changing the dependent and independent variables.

**Step 1.** Given the structure constants of the Lie algebra \( g^n \), left- \( \{\xi_i = \xi_i^k(z)\partial_z^k\} \) and right-invariant \( \{\eta_i = \eta_i^k(z)\partial_z^k\} \) vector fields are constructed by applying the method proposed in [27].

**Step 2.** Introducing the notation \( x^a = z^a \) (\( a = 1, \ldots, n-1 \)), \( u = z^n \), the infinitesimal generators of the symmetry group are written as \( X_i = \xi_i^a(x,u)\partial_{x^a} + \xi_i^n(x,u)\partial_u \).

**Step 3.** In accordance with (24), quantities \( w_{(i)} \), \( w_{(ij)} \) as functions of variables \( (x,u,w_1,w_2) \) are expressed.

**Step 4.** Using (27), the invariants \( I(a) \), \( I(ij) \) are calculated as functions of the variables \( (x,u,w,\frac{1}{2}) \).

**Step 5.** From the formulae for the implicit derivatives (12) and (16), the inverse equalities are derived:

\[
w_a = -u_aw_n, \quad w_{ab} = -w_{an}u_{ab} - w_{bn}u_a - w_{an}u_b - u_au_bw_{mn}.
\]

Substituting these into the invariants \( I(a) \), \( I(ij) \), we obtain the first- and second-order differential invariants:

\[
v_a = v_a(x,u,\frac{1}{2}), \quad v_{ij} = v_{ij}(x,u,\frac{1}{2}).
\]

(The invariants \( I(a) \), \( I(ij) \) are arranged so that the “unnecessary” variables \( \{w_n, w_{an}, w_{mn}\} \) cancel and are not presented in the invariants \( v_a \), \( v_{ij} \)).

**Step 6.** The invariant equation linear in the second derivatives is constructed:

\[
E(x,u,\frac{1}{2}) = \sum_{1\leq i<j\leq n} a_{ij}^{ij}(v) v_{ij} + b(v) = 0.
\]
Step 7. The following results are summarized in a table: the non-zero commutation relations of the Lie algebra, \( g^n \), the infinitesimal generators \( X_i \) of the symmetry group, a basis of the first- and second-order differential invariants \( v_{ij} \), and the invariant differential equation linear in the second derivatives \( E(x, u, u_{1,2}) = 0 \).

As an illustration of the algorithm, we calculate the second order differential invariants and write down the general form of the invariant differential equation linear in the second derivatives for the three-dimensional Lie algebra \( \mathfrak{so}(3) \):

\[
[e_1, e_2] = e_3, \quad [e_2, e_3] = e_1, \quad [e_1, e_3] = -e_2.
\]

Before making this, we calculate the left- and right-invariant vector fields on the corresponding local Lie group. In the coordinates of the second kind \( g_z = e^{z_1 e_1} e^{z_2 e_2} e^{z_3 e_3} \), these can be constructed by applying the algebraic method described in [27]:

\[
\xi_1 = \partial_{z_1}, \quad \xi_2 = \sin z^1 \tan z^2 \partial_{z_1} + \cos z^1 \partial_{z_2}, \quad \xi_3 = \cos z^1 \tan z^2 \partial_{z_1} - \sin z^1 \partial_{z_2} + \frac{\cos z^1}{\cos z^2} \partial_{z_3};
\]

\[
\eta_1 = \frac{\cos z^3}{\cos z^2} \partial_{z_1} - \sin z^3 \partial_{z_2} + \cos z^3 \tan z^2 \partial_{z_3}, \quad \eta_2 = \frac{\sin z^3}{\cos z^2} \partial_{z_1} + \cos z^3 \partial_{z_2} + \sin z^3 \tan z^2 \partial_{z_3}, \quad \eta_3 = \partial_{z_3}.
\]

The infinitesimal generators \( X_i \) of the symmetry group are obtained from the left-invariant vector fields \( \xi_i \) by the formal substitute \( z^i \rightarrow x, z^2 \rightarrow y, z^3 \rightarrow u \):

\[
X_1 = \partial_x, \quad X_2 = \sin x \tan y \partial_x + \cos x \partial_y + \frac{\sin x}{\cos y} \partial_u, \quad X_3 = \cos x \tan y \partial_x - \sin x \partial_y + \frac{\cos x}{\cos y} \partial_u.
\]

The quantities \( w_{(i)} \) and \( w_{(ij)} \), which are defined by [24] and considered as functions of the variables \( x, y, u, w_1, w_2, w_{11}, w_{12}, w_{13}, w_{22}, w_{23}, w_{33} \), take the form

\[
w_{(1)} = w_1 \frac{\cos u}{\cos y} - w_2 \sin u + w_3 \cos u \tan y, \quad w_{(2)} = w_1 \frac{\sin u}{\cos y} + w_2 \cos u + w_3 \sin u \tan y, \quad w_{(3)} = w_3,
\]

\[
w_{(11)} = w_{11} \frac{\cos^2 u}{\cos^2 y} - w_{12} \frac{\sin 2u}{\cos y} + w_{13} \frac{\tan y \cos^2 u}{\cos y} + w_{22} \sin^2 u - w_{23} \sin 2u \tan y +
\]

\[
+ w_{33} \tan^2 y \cos^2 u - w_1 \frac{\sin 2u \tan y}{\cos y} - w_2 \cos^2 u \tan y - \frac{1}{2} w_3 (1 + 2 \tan^2 y) \sin 2u,
\]

\[
w_{(12)} = w_{11} \frac{\sin 2u}{2 \cos^2 y} + w_{12} \frac{\cos 2u}{\cos y} + w_{13} \frac{\sin 2u \tan y}{\cos y} - \frac{1}{2} w_{22} \sin 2u + w_{23} \tan y \cos 2u +
\]

\[
+ \frac{1}{2} w_{33} \sin 2u \tan^2 y + w_1 \frac{\tan y \cos 2u}{\cos y} - \frac{1}{2} w_2 \sin 2u \tan y + \frac{1}{2} w_3 (1 + 2 \tan^2 y) \cos 2u,
\]

\[
w_{(13)} = w_{13} \frac{\cos u}{\cos y} - w_{23} \sin u + w_{33} \cos u \tan y - \frac{1}{2} w_1 \frac{\sin u}{\cos y} - \frac{1}{2} w_2 \cos u - \frac{1}{2} w_3 \sin u \tan y,
\]

\[15\]
with (27). Substituting (28) into the invariants, we find

As a result, we have

Instead

\[
v \sim u = \cos 2u \cos y - u_y \sin y \sin u - u_x \cos y - u_u \sin y + u_x \cos y - u_y \sin y + \frac{1}{2} w_3 \sin 2u (1 + 2 \tan^2 y),
\]

Using the expressions for \( w_{(a)} \) and \( w_{(ij)} \), we obtain the invariants \( I_{(a)} \) and \( I_{(ij)} \) in accordance with (27). Substituting (28) into the invariants, we find

\[
v_1 = u_x \cos y - u_y \sin y - u_u \cos y + u_u \sin y, \quad v_2 = u_x \cos y + u_y \sin y, \quad v_3 = u_x \cos y + u_y \sin y,
\]

\[
v_{12} = \frac{1}{\cos^2 y} \left[ u_{xx} v_y^2 + 2 u_{xy} y + (\sin y - u_x) + u_{yy} (u_x - \sin y)^2 + \frac{1}{2} (1 - u_y^2) u_y \sin 2y - 3 u_x u_y \cos y - 2 (1 + u_y^2) u_y \sin y + \frac{4 u_x u_y}{\cos y} \right],
\]

\[
v_{13} = u_{xx} \cos^2 y - u_{xy} \sin 2y + u_{yy} \sin^2 y - u_y \tan y \sin^2 y + \frac{4 u_x u_y}{\cos y}
\]

\[
v_{23} = u_{xx} \sin^2 y + u_{xy} \sin 2y + u_{yy} \cos^2 y - u_x \cos 2y - u_y \tan y \sin^2 y + \frac{4 u_x u_y}{\cos y}
\]

Instead \( v_{12}, v_{13}, \) and \( v_{23}, \) it is convenient to choose the new set of invariants \( \tilde{v}_{12}, \tilde{v}_{13}, \tilde{w}_{23}, \) which is related to the old one by the relations

\[
\tilde{v}_{12} = \frac{v_{12}^2 v_{23} - v_{23}^2 v_{13}}{v_1 v_2}, \quad \tilde{v}_{13} = v_{13} - v_{23}, \quad \tilde{w}_{23} = v_{14} + v_{23}.
\]

As a result, we have

\[
\tilde{v}_{12} = \frac{u_{xx}}{\cos^2 y} + u_{yy} - u_y \tan y,
\]

\[
\tilde{v}_{13} = \cos 2u \left( \frac{u_{xx}}{\cos^2 y} - u_{yy} - \frac{2 u_{xy}}{\cos y} + u_y \tan y \right) + \sin 2u \left( -\frac{2 u_{xy}}{\cos y} + u_y^2 + u_x^2 \right),
\]

\[
\tilde{v}_{23} = -u_x \cos 2y \sin 2u \left( -\frac{2 u_{xy}}{\cos y} + u_y^2 + u_x^2 \right),
\]

16
In accordance with (29), the general form of the \( \mathfrak{so}(3) \)-invariant differential equation linear in the second derivatives can be written as

\[
\begin{align*}
\Delta u + [\tilde{a}^{13}(v_1, v_2) \cos 2u - \tilde{a}^{23}(v_1, v_2) \sin 2u] \left( \frac{u_{xx}}{\cos^2 y} - u_{yy} - \frac{2u_x u_y}{\cos y} + u_y \tan y \right) + \\
+ [\tilde{a}^{13}(v_1, v_2) \sin 2u + \tilde{a}^{23}(v_1, v_2) \cos 2u] \left( -\frac{2u_{xy}}{\cos y} + u_y^2 + \frac{1 - u_x^2}{\cos^2 y} \right) + \\
+ a^{12}(v_1, v_2) \left( \frac{u_{xx}}{\cos^2 y} + u_{yy} - u_y \tan y \right) + \tilde{b}(v_1, v_2) = 0, \tag{31}
\end{align*}
\]

where \( a^{12}, a^{13}, a^{23}, b \) are arbitrary smooth functions of their arguments, and \( v_1 \) and \( v_2 \) are defined by (30). We can yet further simplify this equation by dividing by \( a^{12} \) and re-designating the rest arbitrary functions:

\[
\Delta u + [\tilde{a}^{13}(v_1, v_2) \cos 2u - \tilde{a}^{23}(v_1, v_2) \sin 2u] \left( \frac{u_{xx}}{\cos^2 y} - u_{yy} - \frac{2u_x u_y}{\cos y} + u_y \tan y \right) + \\
+ [\tilde{a}^{13}(v_1, v_2) \sin 2u + \tilde{a}^{23}(v_1, v_2) \cos 2u] \left( -\frac{2u_{xy}}{\cos y} + u_y^2 + \frac{1 - u_x^2}{\cos^2 y} \right) + \tilde{b}(v_1, v_2) = 0. \tag{31}
\]

Here, \( \tilde{a}^{13} = a^{13}/a^{12} \), \( \tilde{a}^{23} = a^{23}/a^{12} \), \( \tilde{b} = b/a^{12} \), \( \Delta \) is the Laplace–Beltrami operator on the 2-sphere:

\[
\Delta u \equiv \frac{u_{xx}}{\cos^2 y} + u_{yy} - u_y \tan y.
\]

It is interesting to note that if the functions \( \tilde{a}^{13} \) and \( \tilde{a}^{23} \) are constants, then the coefficients of \( u_{xx}, u_{xy}, \) and \( u_{yy} \) in (31) do not depend on \( u_x \) and \( u_y \). In particular, if \( \tilde{a}^{13} = \tilde{a}^{23} = 0 \), then the coefficients do not also depend on \( u \) and can be regarded as the components of the standard metric on the 2-sphere \( S^2 \). In this case, the differential equation (31) has the form:

\[
\Delta u + \tilde{b}(v_1, v_2) = 0.
\]

Below, we list the second-order differential invariants and the invariant quasi-linear second-order differential equations for all two- and three-dimensional local Lie groups. To denote the Lie algebras, we use the notation \( \mathfrak{g}_n^k \), where \( n \) is the dimension of a Lie algebra and \( k \) is its number in the list. In addition to the invariants and differential equations, the tables also contain simply transitive realizations of the Lie algebras in the “splitting” coordinates \( z = (x, u) \).

### 4.1 Two-dimensional symmetry groups (type \( \{2, \{2\}, 1, 2, 0, 0\} \))

In the two-dimensional case, there exist only two non-isomorphic Lie algebras: the commutative Lie algebra \( 2\mathfrak{g}^1 \) and the non-commutative Lie algebra \( \mathfrak{g}^2 \).

In Table 2 \( b \) denotes an arbitrary smooth function of its argument.

### 4.2 Three-dimensional symmetry groups (type \( \{3, \{2\}, 1, 3, 0, 0\} \))

We use the classification of non-isomorphic three-dimensional Lie algebras suggested by Mubarakzhanov [38].
Table 2: Second-order differential invariants of two-dimensional symmetry groups and quasi-linear differential equations of type \{2, (2), 1, 2, 0, 0\}

| Lie algebra | Generators $X_i$ | Invariants $v_a, v_{ij}$ | Quasi-linear second-order differential equation |
|-------------|----------------|---------------------|-----------------------------------------------|
| $2g^1$      | $X_1 = \partial_x, X_2 = \partial_u$ | $u_x, u_{xx}$ | $u_{xx} + b(u_x) = 0$ |
| $g^2$       | $X_1 = \partial_x, X_2 = x\partial_x + \partial_u$ | $e^u u_x, e^{2u} u_{xx}$ | $u_{xx} + e^{-2u} b(e^u u_x) = 0$ |

Table 3: Second-order differential invariants of three-dimensional symmetry groups and quasi-linear differential equations of type \{3, \{2\}, 1, 3, 0, 0\}

| Lie algebra | Generators $X_i$ | Invariants $v_a, v_{ij}$ | Quasi-linear second-order differential equation |
|-------------|----------------|---------------------|-----------------------------------------------|
| $3g^1$      | $X_1 = \partial_x, X_2 = \partial_y, X_3 = \partial_u$ | $u_x, u_y, u_{xx}, u_{xy}$ | $u_{xx} + a_1(u_x, u_y)u_{xy} + a_2(u_x, u_y)u_{yy} + b(u_x, u_y) = 0$ |
| $g^1 + g^2$ | $[e_1, e_2] = e_1$ | $e^u u_x, u_y, e^{2u} u_{xx}, e^{u} u_{xy}$ | $u_{xx} + e^{-y} a_1(e^y u_x, u_y)u_{xy} + e^{-2y} a_2(e^y u_x, u_y)u_{yy} + e^{-2y} b(e^y u_x, u_y) = 0$ |
| $g^3$       | $[e_1, e_3] = e_1$ | $u_x - y, u_y, u_{xx}, u_{xy}, u_{yy}$ | $u_{xx} + a_1(u_x - y, u_y)u_{xy} + a_2(u_x - y, u_y)u_{yy} + b(u_x - y, u_y) = 0$ |
| $g^3$       | $[e_2, e_3] = e_1$ | $e^{-x} u_x, e^{-y} u_y, e^{x} u_{xx}, e^{y} u_{xy}$ | $e^{-x} u_{xx} + a_1(e^{-x} u_x, e^{-y} u_y)u_{xy} + a_2(e^{-x} u_x, e^{-y} u_y)u_{yy} + b(e^{-x} u_x, e^{-y} u_y) = 0$ |
| $g^3$       | $[e_1 + e_2, e_3] = e_1$ | $e^u u_x, e^u u_y, e^{2u} u_{xx}, e^{u} u_{xy}$ | $u_{xx} + a_1(e^u u_x, e^u u_y) + a_2(e^u u_x, e^u u_y)u_{yy} + e^{-2u} b(e^u u_x, e^u u_y) = 0$ |
| $g^3$       | $[e_1, e_3] = e_1$ | $e^u u_x, e^h u_y, e^{2u} u_{xx}, e^{h u} u_{xy}$ | $u_{xx} + e^{(h-1)u} a_1(e^u u_x, e^{h u} u_y) + e^{(h-1)u} a_2(e^u u_x, e^{h u} u_y)u_{yy} + e^{-2u} b(e^u u_x, e^{h u} u_y) = 0$ |
| $g^3$       | $[e_1, e_3] = e_1$ | $u_x + vy, u_{xx}, u_{xy}$ | $u_{xx} + u_{yy} + e^{-2u} b(u_x, u_y) + a_1(v_1^5, v_2^5)u_{xx} - u_{yy} \cos 2u + a_2(v_1^5, v_2^5)u_{xx} - u_{yy} \sin 2u + 2a_2(v_1^5, v_2^5)u_{xx} \cos 2u = 0$ |
| $g^3$       | $[e_2 + e_1, e_3] = e_1$ | $v_{12}^5, v_{13}^5, v_{23}^5$ | $a_1(v_1^5, v_2^5)(u_{xx} - u_{yy}) \cos 2u + a_2(v_1^5, v_2^5)(u_{xx} - u_{yy}) \sin 2u + 2a_2(v_1^5, v_2^5)u_{xx} \cos 2u = 0$ |
\[ \mathcal{X}_I = \partial_x, \]

\[ X_2 = \sin x \tan y \partial_x + \cos x \partial_y + \sin x \partial_u, \]

\[ X_3 = \cos x \tan y \partial_x - \sin x \partial_y + \cos x \partial_u \]

\[ v_1^7, v_2^7, v_7^7, v_{12}^7, v_{13}^7, v_{23}^7 \]

\[ \frac{u_{yy} - u_y \tan y}{\cos^2 y} + b(v_1^7, v_2^7) + \left[ a_1(v_1^7, v_2^7) \cos 2u - a_2(v_1^7, v_2^7) \sin 2u \right] \times \left( \frac{u_{xx}}{\cos^2 y} - \frac{u_{xy}}{\cos y} + u_y \tan y \right) + \left[ a_1(v_1^7, v_2^7) \sin 2u + a_2(v_1^7, v_2^7) \cos 2u \right] \times \left( -\frac{2u_{xy}}{\cos y} + u_y^2 + \frac{1 - u_x^2}{\cos^2 y} \right) = 0 \]

In Table 3, \( a_1, a_2, \) and \( b \) denote arbitrary functions of their arguments. Furthermore, we use the following notation:

\[ v_1^5 = e^{pu}(u_x \cos u - u_y \sin u), \quad v_2^5 = e^{pu}(u_x \sin u + u_y \cos u), \quad v_1^5 = e^{2pu}(u_{xx} + u_{yy}), \]

\[ v_{13}^5 = e^{2pu}((u_{xx} - u_{yy}) \cos 2u - 2u_{xy} \sin 2u), \quad v_{23}^5 = e^{2pu}((u_{xx} - u_{yy}) \sin 2u + 2u_{xy} \sin 2u); \]

\[ v_6^6 = e^u u_x + u_y^2, \quad v_2^6 = u_x - u, \quad v_1^6 = u_y - u, \]

\[ v_{13}^6 = e^u u_{xy} + 2u u_{yy} - u^2, \quad v_{23}^6 = e^u u_{xx} + 2u [u(2u_{yy} + u_x - u) + e^y(u_x + 2u_{xy})]; \]

\[ v_1^7 = u_x \frac{\cos u}{\cos y} - u_y \sin u - u \cos u \tan y, \quad v_2^7 = u_x \frac{\sin u}{\cos y} u_x + u_y \cos u - u \sin u \tan y, \]

\[ v_{12}^7 = u_{xx} \frac{\cos u}{\cos^2 y} + u_{yy} - u \tan y, \]

\[ v_{13}^7 = \cos 2u \left( \frac{u_{xx}}{\cos^2 y} - u_{yy} - \frac{2u_x u_y}{\cos y} + u_y \tan y \right) + \sin 2u \left( -\frac{2u_{xy}}{\cos y} + u_y^2 + \frac{1 - u_x^2}{\cos^2 y} \right), \]

\[ v_{23}^7 = -\sin 2u \left( \frac{u_{xx}}{\cos^2 y} - u_{yy} - \frac{2u_x u_y}{\cos y} + u_y \tan y \right) + \cos 2u \left( -\frac{2u_{xy}}{\cos y} + u_y^2 + \frac{1 - u_x^2}{\cos^2 y} \right). \]

### 5 Differential equations with free symmetry groups

Suppose that a symmetry group acts on a space \( Z \) freely and intransitively (we have considered the case of a free and transitive action in the previous section). Locally, we have \( Z \cong X \times Y \), where \( X \) is an orbit of the group action, and \( Y \) is a subspace transverse to the orbit and coordinates on \( Y \) are invariants of the symmetry group. The infinitesimal generators of the group action have the form \( \mathcal{X}_I \) and coincide with the left-invariant vector fields on \( X \). In contrast to the scenario examined in the study in [32], here we consider a more general case when there are several invariant variables \( \{y^1, \ldots, y^m\} \).

Differential equations with such symmetry groups have type II: \( \{n+m, 2\}, 1, n, 0, 0 \}. Without the loss of generality, we can regard the invariant variable \( y_m \) as the dependent variable \( u \). Indeed, if the dependent variable is not invariant (for example, \( u = x^m \), a hodograph transformation can then be used to make the invariant variable \( y^m \) dependent. (We note that hodograph transformations have the property of mapping quasi-linear differential equations to quasi-linear ones).

In accordance with Theorem 3, \( y^m, u, u_{(i)}, u_{(ij)}, \ldots, \) are differential invariants. (Here, \( u_{(i)} = \hat{n}_i u, \ u_{(ij)} = (\hat{n}_i \circ \hat{n}_j)u, \ldots \). Moreover, it is easy to note that the total differentiation operators
\( D_{y^\mu} \) are operators of the invariant differentiation. Thus, to the above differential invariants, we add the invariants \( u_{y^\mu}, u_{y^\mu y^\nu}, u_{(i)y^\mu}, \ldots \). Let us formulate this result in the form of a theorem.

**Theorem 5.** A second-order partial differential equation whose symmetry group acts freely and intransitively has the form:

\[
F(y, u, u_{y^\mu}, u_{(i)}, u_{y^\mu y^\nu}, u_{(ij)}, u_{(i)y^\mu}) = 0.
\]

In particular, a differential equation linear in the highest derivatives is represented as follows:

\[
\sum_{\mu, \nu=1}^{m-1} a^{\mu\nu} u_{y^\mu y^\nu} + \sum_{i,j=1}^{n} b^{ij} u_{(ij)} + \sum_{\mu=1}^{m-1} n \sum_{i=1}^{n} c^{\mu i} u_{(i)y^\mu} + d = 0.
\]

(32)

Here, \( a^{\mu\nu}, b^{ij}, c^{\mu i}, d \) are arbitrary functions of \( y, u, u_{y^\mu}, u_{(i)} \).

Let us list all the inequivalent second-order differential equations of type II for local Lie groups of dimensions no larger than three. In the tables below, we indicate the symmetry Lie algebras, their infinitesimal generators, and all the second-order differential invariants. The invariant quasi-linear differential equations have the same structure as (32). Everywhere below we use the notation: \( y = \{y^1, \ldots, y^{m-1}\} \), \( u_\mu = u_{y^\mu} \), \( u_i = u_{x^i} \), \( u_{\mu\nu} = u_{y^\mu y^\nu} \), \( u_{ij} = u_{x^i x^j} \) (\( i, j = 1, \ldots, n; \mu, \nu = 1, \ldots, m-1 \)).

### 5.1 One-dimensional symmetry groups (type \{1 + m, \{2\}, 1, 1, 0, 0\})

Up to isomorphism, there is one one-dimensional Lie algebra \( g^1 \).

| Lie algebra | Generators \( X_i \) | Second-order differential invariants |
|-------------|------------------------|-------------------------------------|
| \( g^1 \)   | \( X_1 = \partial_{x^1} \) | \( y, u, u_1, u_{1\mu}, u_{11\mu}, u_{\mu\nu} \) |

### 5.2 Two-dimensional symmetry groups (type \{2 + m, \{2\}, 1, 2, 0, 0\})

As we have noted above, there are only two non-isomorphic two-dimensional Lie algebras: \( 2g^1 \) and \( g^2 \).

| Lie algebras | Generators \( X_i \) | Second-order differential invariants |
|--------------|------------------------|-------------------------------------|
| \( 2g^1 \)   | \( X_1 = \partial_{x^1}, X_2 = \partial_{x^2} \) | \( y, u, u_1, u_2, u_{\mu}, u_{11}, u_{12}, u_{22}, u_{1\mu}, u_{2\mu}, u_{\mu\nu} \) |
| \( g^2 \)    | \( [e_1, e_2] = e_1 \) | \( y, u, e^{x^1} u_1, u_2, u_{\mu}, e^{x^2} u_{11}, e^{x^2} \left( u_{12} + \frac{u_1}{2} \right), u_{22}, e^{x^2} u_{1\mu}, u_{2\mu}, u_{\mu\nu} \) |
5.3 Three-dimensional symmetry groups (type $\{3 + m, \{2\}, 1, 3, 0, 0\}$)

Here we also use the classification of three-dimensional Lie algebras suggested by Mubarakzyanov [38].

Table 6: Second-order differential invariants for three-dimensional Lie algebras

| Lie algebra     | Generators $X_i$                                                                 | Second-order differential invariants                                                                 |
|-----------------|---------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------|
| $\mathfrak{g}^3$ | $X_1 = \partial z_1, \ X_2 = \partial z_2, \ X_3 = \partial x_3$ | $y, \ u, \ u_1, \ u_2, \ u_3, \ u_\mu, \ u_{11}, \ u_{12}, \ u_{13}, \ u_{22}, \ u_{23}, \ u_{33}, \ u_{1\mu}, \ u_{2\mu}, \ u_{3\mu}, \ u_{\mu\nu}$ |
| $\mathfrak{g}^1 + \mathfrak{g}^2$ | $[e_1, e_2] = e_1$ | $X_1 = \partial z_1, \ X_2 = x^1 \partial x_1 + \partial x_2, \ X_3 = \partial x_3$ | $y, \ u, \ e^{x^1} u_1, \ u_2, \ u_3, \ u_\mu, \ e^{x^2} u_{11}, \ e^{x^2} (u_{12} + \frac{u_3}{2}), \ e^{x^2} u_{13}, \ e^{x^2} u_{22}, \ u_{23}, \ u_{33}, \ e^{x^2} u_{1\mu}, \ e^{x^2} u_{2\mu}, \ e^{x^2} u_{3\mu}, \ e^{x^2} u_{\mu\nu}$ |
| $\mathfrak{g}_1^3$ | $[e_2, e_3] = e_1$ | $X_1 = \partial z_1, \ X_2 = \partial z_2, \ X_3 = x^2 \partial x_1 + \partial x_3$ | $y, \ u, \ u_1, \ x^3 u_1 + u_2, \ u_3, \ u_\mu, \ u_{11}, \ x^3 u_{11} + u_{12}, \ u_{13}, \ (x^3)^2 u_{11} + 2 x^3 u_{12} + u_{22}, \ x^3 u_{13} + u_{23} + \frac{u_3}{2}, \ u_{33}, \ u_{1\mu}, x^3 u_{1\mu} + u_{2\mu}, \ u_{3\mu}, \ u_{\mu\nu}$ |
| $\mathfrak{g}_2^3$ | $[e_1, e_3] = e_1$ | $X_1 = \partial z_1, \ X_2 = \partial z_2, \ X_3 = (x^1 + x^2) \partial x_1 + + x^2 \partial x_2 + \partial x_3$ | $y, \ u, \ e^{x^1} u_1, \ e^{x^1} (x^1 u_1 + u_2), \ u_3, \ u_\mu, \ e^{2x^1} u_{11}, \ e^{2x^1} (x^1 u_{11} + u_{12}), \ e^{x^1} (u_{13} + \frac{u_2}{2}), \ u_{33}, \ e^{x^1} u_{1\mu}, \ e^{2x^1} \left((x^3)^2 u_{11} + 2 x^3 u_{12} + u_{22}\right), \ e^{x^1} x^3 (u_{13} + \frac{u_2}{2}) + e^{x^1} (u_{23} + \frac{u_1+u_2}{2}), \ e^{x^1} (x^3 u_{1\mu} + u_{2\mu}), \ u_{3\mu}, \ u_{\mu\nu}$ |
| $\mathfrak{g}_3^3$ | $[e_1, e_3] = e_1$ | $X_1 = \partial z_1, \ X_2 = \partial z_2, \ X_3 = x^1 \partial x_1 + x^2 \partial x_2 + \partial x_3$ | $y, \ u, \ e^{x^1} u_1, \ e^{x^1} u_2, \ u_3, \ u_\mu, \ e^{2x^1} u_{11}, \ e^{x^1} u_{12}, \ e^{x^1} u_{23}, \ u_{33}, \ e^{x^1} u_{1\mu}, \ e^{x^1} u_{2\mu}, \ u_{3\mu}, \ u_{\mu\nu}$ |
| $\mathfrak{g}_4^3$ | $[e_2, e_3] = he_2$ | $X_1 = \partial z_1, \ X_2 = \partial z_2, \ X_3 = x^1 \partial x_1 + \partial x_3$ | $y, \ u, \ e^{x^1} u_1, \ e^{x^1} u_2, \ u_3, \ u_\mu, \ e^{2x^1} u_{11}, \ e^{x^1} u_{12}, \ e^{x^1} u_{23}, \ u_{33}, \ e^{x^1} u_{1\mu}, \ e^{x^1} u_{2\mu}, \ u_{3\mu}, \ u_{\mu\nu}$ |
| $\mathfrak{g}_5^3$ | $[e_1, e_3] = p e_1 - e_2$ | $X_1 = \partial z_1, \ X_2 = \partial z_2, \ X_3 = (p x^1 + x^2) \partial x_1 + + (p x^2 - x^1) \partial x_2 + \partial x_3$ | $y, \ u, \ u_3, \ u_\mu, \ e^{2p x^1} (u_{11} + u_{22}), \ e^{p x^1} (u_1 \cos x^3 - u_2 \sin x^3), \ e^{p x^3} (u_1 \sin x^3 + u_3 \cos x^3), \ e^{2p x^1} [u_{11} - u_{22}] \cos 2x^3 - 2u_{12} \sin 2x^3, \ e^{2p x^3} (u_{11} - u_{22}) \sin 2x^3 + 2u_{12} \cos 2x^3, \ e^{p x^1} (u_{13} + \frac{p u_1 - u_2}{2}) \cos x^3 - \left(u_{23} + \frac{p u_2 + u_1}{2}\right) \sin x^3, \ e^{p x^3} (u_{13} + \frac{p u_1 - u_2}{2}) \sin x^3 + \left(u_{23} + \frac{p u_2 + u_1}{2}\right) \cos x^3, \ e^{p x^3} (u_{1\mu} \cos x^3 - u_{2\mu} \sin x^3), \ e^{p x^3} (u_{1\mu} \sin x^3 + u_{2\mu} \cos x^3), \ u_{33}, \ u_{3\mu}, \ u_{\mu\nu}$ |
| $\mathfrak{g}_6^3$ | $[e_1, e_2] = e_1$ | $X_1 = \partial z_1, \ X_2 = x^1 \partial x_1 + \partial x_2, \ X_3 = (x^1)^2 \partial x_1 + + 2x^1 \partial x_2 + e^{x^2} \partial x_3$ | $y, \ u, \ e^{x^1} u_1 + x^1 u_2 + (x^1)^2 u_3, \ u_2 + x^3 u_3, \ u_3, \ u_\mu, \ e^{2x^1} u_{11} + e^{x^1} x^3 \left(u_{12} + \frac{u_3}{2}\right) + + (x^3)^2 u_{23} + u_{22} + u_{12} \cdot x^3 u_3 + x^3 \left(u_{23} + \frac{u_3}{2}\right) + x^3 (2 u_{22} + u_2)$ |
and the subsequent construction of the differential equations that are invariant under each of

This procedure involves finding all the inequivalent realizations for a given abstract Lie algebra

approaches based on the preliminary group classification, we do not restrict the form of an
differential equations based on the regular actions of their symmetry groups. Unlike the traditional

Conclusion

In this paper, we have offered a systematic procedure for the group classification of differential equations based on the regular actions of their symmetry groups. Unlike the traditional approaches based on the preliminary group classification, we do not restrict the form of an equation by some ansatz and proceed rather similarly to Lie’s original two-step procedure. This procedure involves finding all the inequivalent realizations for a given abstract Lie algebra $\mathfrak{g}$ and the subsequent construction of the differential equations that are invariant under each of these realizations.
One of the main advantages of our method is that there is no need to explicitly integrate differential equations. This is possible if one considers the subtle features of the actions of the prescribed symmetry groups; in this relation, we have classified all the differential equations into five types, which are listed in Table 1. Another distinguishing feature of our method is its covariance, which is achieved by considering the dependent and independent variables equally. It allows us to describe differential equations invariant under different transformation groups in a uniform manner. For these purposes, we have introduced the so-called covariant form of a differential equation, i.e., equations (22) and (23). The transition to the typical form of the differential equation suggests the procedure of “splitting” of the variables into dependent and independent, and all the equations obtained by different approaches of such “splitting” are related by hodograph transformations.

In this study, we have restricted ourselves to scalar second-order differential equations of types \{n, 2, 1, n, 0, 0\} and \{n + m, \{2\}, 1, n, 0, 0\} in accordance with Table 1. The first type presupposes a symmetry group acting freely and transitively on the space of dependent and independent variables, whereas the symmetry groups for the equations of the second type are assumed to be free but intransitive, in general. We have given the most general form of the equations for these types (Theorems 4 and 5) and also formulated constructive algorithms of their group classification. Using these algorithms, we have obtained exhaustive classifications of the equations of types \{n, 2, 1, n, 0, 0\} and \{n + m, \{2\}, 1, n, 0, 0\} for all the local regular transformation groups of dimensions up to three (see Tables 2–6).

In future investigations, we plan to further develop our classification program; particularly, we will consider in detail the cases of differential equations of types III, IV, and V. There are also certain reasons to assume that the algebraic technique we have designed will constructively allow the calculation of operators of the invariant differentiation, which suggests the possibility of effectively constructing differential invariants of any order [2, 4].

References

[1] Sophus Lie. Theorie der transformationsgruppen I. *Mathematische Annalen*, 16(4):441–528, 1880.

[2] Lev V Ovsiannikov. *Group Analysis of Differential Equations*. New York: Academic Press, 1982.

[3] Nail H Ibragimov. *Transformation Groups Applied to Mathematical Physics*, volume 3. Springer Netherlands, 1985.

[4] Peter J Olver. *Applications of Lie groups to differential equations*, volume 107. Springer-Verlag New York, 1986.

[5] Peter J Olver. *Equivalence, Invariants and Symmetry*. Cambridge University Press, 1995.

[6] Sophus Lie. Classification und integration von gewöhnlichen differentialgleichungen zwischenxy, die eine gruppe von transformationen gestatten. *Mathematische Annalen*, 32(2):213–281, 1888.
[7] Peter J. Olver. Differential invariants and invariant differential equations. *Lie Groups and Their Applications 1*, (1):177–192, 1994.

[8] Artemio González-López, Niky Kamran, and Peter J Olver. Lie algebras of vector fields in the real plane. *Proceedings of the London Mathematical Society*, 3(2):339–368, 1992.

[9] Maryna O Nesterenko. Differential invariants of transformation groups on the real plane. *Proceedings of Institute of Mathematics of NAS of Ukraine*, 50:211–213, 2004.

[10] Maryna O Nesterenko. Transformation groups on real plane and their differential invariants. *International journal of mathematics and mathematical sciences*, 2006:1–17, 2006.

[11] Fazal Mahomed. Symmetry group classification of ordinary differential equations: survey of some results. *Mathematical Methods in the Applied Sciences*, 30(16):1995–2012, 2007.

[12] Sophus Lie. On integration of a class of linear partial differential equations by means of definite integrals. *CRC handbook of Lie group analysis of differential equations*, 2:473–508, 1881.

[13] Lev V Ovsiannikov. Group properties of heat conductivity equations. *Proc. Acad. Sci. USSR*, 125(3):492–495, 1959. (in Russian).

[14] Iskander Sh Akhatov, Rafail Kavyevich Gazizov, and Nail Hairullovich Ibragimov. Group classification of equations of nonlinear filtration. *Proc. Acad. Sci. USSR*, 293(5):1033–1035, 1987. (in Russian).

[15] Iskander Sh Akhatov, Rafail Kavyevich Gazizov, and Nail Hairullovich Ibragimov. Non-local symmetries. heuristic approach. *Journal of Soviet Mathematics*, 55(1):1401–1450, 1991.

[16] Nail Hairullovich Ibragimov, M Torrisi, and A Valenti. Preliminary group classification of equations $v_{tt} = f(x,v_x)v_{xx} + g(x,v_x)$. *Journal of Mathematical Physics*, 32(11):2988–2995, 1991.

[17] NH Ibragimov and M Torrisi. A simple method for group analysis and its application to a model of detonation. *Journal of mathematical physics*, 33(11):3931–3937, 1992.

[18] M Torrisi, R Tracina, and A Valenti. A group analysis approach for a nonlinear differential system arising in diffusion phenomena. *Journal of Mathematical Physics*, 37(9):4758–4767, 1996.

[19] M Torrisi and R Tracina. Equivalence transformations and symmetries for a heat conduction model. *International journal of non-linear mechanics*, 33(3):473–487, 1998.

[20] Alexander Bihlo, Elsa Dos Santos Cardoso-Bihlo, and Roman O Popovych. Complete group classification of a class of nonlinear wave equations. *Journal of mathematical physics*, 53(12):123515, 2012.
[21] Olena O Vaneeva, Alexander Bihlo, and Roman O Popovych. Equivalence groupoid and group classification of a class of nonlinear wave and elliptic equations. arXiv preprint arXiv:2002.08939, 2020.

[22] RZ Zhdanov and VI Lahno. Group classification of heat conductivity equations with a nonlinear source. Journal of Physics A: Mathematical and General, 32(42):7405, 1999.

[23] Peter Basarab-Horwath, V Lahno, and Renat Zhdanov. The structure of Lie algebras and the classification problem for partial differential equations. Acta Applicandae Mathematica, 69(1):43–94, 2001.

[24] V Lahno and R Zhdanov. Group classification of nonlinear wave equations. Journal of mathematical physics, 46(5):053301, 2005.

[25] V Lahno, R Zhdanov, and O Magda. Group classification and exact solutions of nonlinear wave equations. Acta Applicandae Mathematica, 91(3):253, 2006.

[26] Roman O Popovych, Vyacheslav M Boyko, Maryna O Nesterenko, and Maxim W Lutfullin. Realizations of real low-dimensional lie algebras. Journal of Physics A: Mathematical and General, 36(26):7337, 2003.

[27] Alexey A Magazev, Vitaly V Mikheyev, and Igor V Shirokov. Computation of composition functions and invariant vector fields in terms of structure constants of associated Lie algebras. SIGMA. Symmetry, Integrability and Geometry: Methods and Applications, 11:066, 2015.

[28] Maryna Nesterenko, Severin Pošta, and Olena Vaneeva. Realizations of Galilei algebras. Journal of Physics A: Mathematical and Theoretical, 49(11):115203, 2016.

[29] Maryna Nesterenko. The poincaré algebras $p(1,1)$ and $p(1,2)$: realizations and deformations. In Journal of Physics: Conference Series, volume 621, page 012009. IOP Publishing, 2015.

[30] Peter J Olver. Generating differential invariants. Journal of Mathematical Analysis and Applications, 333(1):450–471, 2007.

[31] Evelyne Hubert and Irina A Kogan. Smooth and algebraic invariants of a group action: local and global constructions. Foundations of Computational Mathematics, 7(4):455–493, 2007.

[32] Igor’Victorovich Shirokov. Differential invariants of the transformation group of a homogeneous space. Siberian Mathematical Journal, 48(6):1127–1140, 2007.

[33] Igor’Victorovich Shirokov. Identities and invariant operators on homogeneous spaces. Theoretical and Mathematical Physics, 126(3):326–338, 2001.

[34] Lokenath Debnath. Nonlinear Partial Differential Equations for Scientists and Engineers. Springer Science+Business Media, third edition edition, 2012.
[35] Frank W Warner. *Foundations of differentiable manifolds and Lie groups*, volume 94. Springer Science & Business Media, 2013.

[36] Claude Chevalley. *Theory of Lie groups*. Courier Dover Publications, 2018.

[37] Sigurdur Helgason. *Differential geometry, Lie groups, and symmetric spaces*. Academic press, 1979.

[38] Gamir M Mubarakzyanov. On solvable lie algebras. *Izvestiya Vysshikh Uchebnykh Zavedenii. Matematika*, (1):114–123, 1963.