fully heavy pentaquarks

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Very recently, the LHCb collaboration reported a fully charmed tetraquark state \(X(6900)\) in the invariant mass spectrum of \(J/\psi\) pairs. If one \(J/\psi\) meson is replaced with a fully charmed baryon, we obtain a fully charmed pentaquark candidate. In this work, we perform a systematical study on the mass spectra of the \(S\)-wave fully heavy pentaquark \(QQQQ\) in the framework of the chromomagnetic interaction model. Based on our results in two different schemes, we further investigate the decay behaviors for them. We hope that our study will be helpful to search for such types of the exotic pentaquark states in experiment in the future.

I. INTRODUCTION

At the birth of quark model \([1,3]\), Gell-Mann and Zweig indicated that hadronic states with the \(qqq\bar{q}\) and \(qqq\bar{q}q\) quark configurations should exist in nature. Such exotic states were further investigated with some phenomenological models soon afterwards. For example, Jaffe adopted the quark-bag model to study \(qqq\bar{q}\) hadrons, where the mass spectrum and dominant decay behavior were predicted \([4]\). In 1979, Strottman calculated the masses of \(q^2\bar{q}\) and \(q^5\bar{q}^2\) in the framework of MIT bag model \([5]\). The name \textit{pentaquark} was firstly proposed by Lipkin in 1987 \([6]\), where he studied anticharmed strange pentaquarks. Since 2003, with the accumulation of experimental data, more and more charmonium-like \(XYZ\) states were reported in experiment. Especially, the observation by the LHCb Collaboration confirms the existence of pentaquark \(P_c\) states \([7,9]\). In the past about twenty years, great progresses have been made on studying exotic multiquarks \([10,13]\).

Recently, the LHCb collaboration studied the invariant mass spectrum of \(J/\psi\) pairs, and they reported a narrow structure around 6.9 GeV and a broad structure in the mass range 6.2-6.8 GeV. The global significance for these two structures are larger than 5\(\sigma\). Such distinct structures are expected to be with the \(cc\bar{c}\bar{c}\) configuration \([14]\).

The \(cc\bar{c}\bar{c}\) tetraquark had been discussed a lot in the literature before the discovery from the LHCb collaboration. In Ref. \([15]\), a \(cc\bar{c}\bar{c}\) tetraquark was predicted at about 6.2 GeV. The \(cc\bar{c}\bar{c}\) tetraquark system was systematically investigated based on a quark-gluon model in Ref. \([16]\). There are other discussions about the properties of fully heavy tetraquarks \([17,33]\).

This important signal from the LHCb collaboration provides us a new ground to understand the non-perturbative behavior of QCD \([34,35]\). The mass of the observed signal around 6.9 GeV is consistent with a previous QCD sum rule predictions \([32]\). After the LHCb collaboration reported their results, the strong decay properties of \(S\)- and \(P\)-wave tetraquark states were further studied in Ref. \([37]\), and the observed structure at around 6.9 GeV is suggested to be with \(J^{PC} = 0^{-+}\) or \(1^{-+}\). Becchi et al have studied that the tetraquarks \(cc\bar{c}\bar{c}\) with \(J^{PC} = 0^{++}, 2^{++}\) decay into 4 mouns and into hidden-and open-charmed mesons and provided the decay widths of fully charmed tetraquark \([38]\). The inner structures of the fully charmed tetraquark state were studied \([39]\). The mass spectrum of \(cc\bar{c}\bar{c}\) tetraquarks were studied in an extended relativized quark model, QCD sum rule, chromomagnetic model and so on \([40,47]\). The production mechanism of \(cc\bar{c}\bar{c}\) was also studied in various schemes \([48,55]\).

The discoveries of fully heavy tetraquark states and \(P_c\) states make one speculate that the pentaquark state with fully heavy quarks \(QQQQ\) may also exist. If the mass is above the baryon-meson thresholds, the heavy pentaquark state may allow the strong decays into the corresponding two body. The study of the masses and decay properties would help to search for the heavy pentaquark states in experiment.

The strong interaction in a fully heavy multiquark state is not clear at present. The Chromomagnetic Interaction (CMI) model provides us a simply picture to quantitatively understand the spectrum of multiquark states. In the framework of CMI model \([56]\), the strong interaction between quarks via gluon-exchange force is parameterized into effective quark masses and quark coupling parameters. Despite its simple Hamiltonian, this model can catch the basic features of hadron spectra, since the mass splittings between hadrons reflect the basic symmetries of their inner structures \([27]\). This model has been widely adopted to study the mass spectra of multiquark systems \([47,57,74]\). In this work, we systematically study the S-
wave $QQQQ\bar{Q}$ pentaquark system within the framework of CMI model to calculate the mass spectra, the relative partial decay widths and find possible stable pentaquark states.

This paper is organized as follows. In Sec. [II] we introduce the CMI model and determine the relevant parameters used in the CMI model. The \textit{flavor} $\otimes$ \textit{color} $\otimes$ \textit{spin} wave functions are constructed and the CMI Hamiltonian elements are calculated for the $QQQQ\bar{Q}$ pentaquark system in Sec. [III]. In Sec. [IV] we present the mass spectra, the mass splittings, possible strong decay channels, and the relative partial decay widths, and also discuss the stability for the pentaquark states. A short summary is followed by Sec. [V]. Finally, some useful expressions are presented in Appendix. [A]

II. THE CHROMOMAGNETIC INTERACTION MODEL

The masses of the ground hadrons can be obtained by the effective Hamiltonian at quark level

\[ H = \sum_i m_i + H_{\text{CMI}} \]

\[ = \sum_i m_i - \sum_{i<j} C_{ij} \vec{\lambda}_i \cdot \vec{\lambda}_j \sigma_i \cdot \sigma_j, \tag{1} \]

where the $H_{\text{CMI}}$ denotes the Hamiltonian of the chromomagnetic interaction \cite{56}. $\sigma_i$ and $\lambda_i$ are the Pauli matrices and the Gell-Mann matrices, respectively. For antiquark, the $\lambda_i$ should be replaced with the $-\lambda_i^\ast$. $m_i$ is the effective mass of the $i$-th constituent quark. In the above Hamiltonian, the chromoelectric interaction and color confinement effect are also incorporated in the effective quark mass $m_i$. $C_{ij}$ is the effective coupling constant between the $i$-th quark and $j$-th quark. The effective quark mass $m_i$ and the coupling constant $C_{ij}$ can be determined from the experimental hadron masses.

As indicated in Ref. \cite{57,60}, the predicted hadron masses obtained from Eq. (1) are generally overestimated. The main reason is that the dynamical effects inside hadrons can not simply be absorbed into the effective quark masses. Thus, in order to take such effective interaction into account, we replace the sum of $m_i$ term in Eq. (1) with $M_{\text{ref}} - \langle H_{\text{CMI}} \rangle_{\text{ref}}$ where $M_{\text{ref}}$ is a reference mass scale and $\langle H_{\text{CMI}} \rangle$ is the corresponding CMI matrix element. The mass of ground hadron can thus be written as

\[ M = M_{\text{ref}} - \langle H_{\text{CMI}} \rangle_{\text{ref}} + \langle H_{\text{CMI}} \rangle. \tag{2} \]

In addition, we introduce another scheme to estimate the masses of pentaquark states. We separate the two-body chromoelectric effects out of the effective quark masses and generalize the chromomagnetic interaction model by writing the chromoelectric term explicitly \cite{47,72,74} i.e.,

\[ H = \left( \sum_i m_i^0 + H^0_{\text{CEI}} \right) + H^0_{\text{CMI}} \]

\[ = \sum_i m_i^0 - \sum_{i<j} A_{ij} \vec{\lambda}_i \cdot \vec{\lambda}_j - \sum_{i<j} v_{ij} \vec{\lambda}_i \cdot \vec{\lambda}_j \sigma_i \cdot \sigma_j, \]

\[ - \frac{3}{4} \sum_{i<j} m_{ij} \vec{\lambda}_i \cdot \vec{\lambda}_j - \sum_{i<j} v_{ij} \vec{\lambda}_i \cdot \vec{\lambda}_j \sigma_i \cdot \sigma_j + ... \tag{3} \]

where the omitted operator nullifies the color-singlet physical states, and

\[ m_{ij} = \frac{1}{4} (m_i^0 + m_j^0) + \frac{4}{3} A_{ij}. \tag{4} \]

This treatment has been successfully adopted in ref. \cite{28,47,72,76}. The parameters $m_{ij}$ and $v_{ij}$ are also determined from the experimental hadron masses. In this work, we label this method as the modified CMI model scheme.

To estimate the masses of the $QQQQ\bar{Q}$ pentaquark states, we need some hadron masses as input to fit the effective coupling parameters $C_{ij}$, $m_{ij}$, and $v_{ij}$ \cite{77}. These conventional hadrons are listed in Table [I]. Because some of the heavy flavor baryons are not yet observed, we introduce the theoretical results in Refs. \cite{57,73,74} as our input, and enclose the theoretical values of masses for these baryons with parentheses in Table [I]

Now we can fit the effective coupling parameters $C_{ij}$, $m_{ij}$, and $v_{ij}$ in the reference mass and modified CMI model schemes by applying Eq. (2) and Eq. (3) respectively. We present the obtained effective coupling parameters of $QQQQ\bar{Q}$ pentaquark states in Table [II]. One can refer to ref. \cite{13,73,74} for more details.

III. THE $QQQQ\bar{Q}$ PENTAQUARK WAVE FUNCTIONS AND THE CMI HAMILTONIAN

In order to study systematically the mass spectra of the $QQQQ\bar{Q}$ pentaquark system, we need construct the wave function of $QQQQ\bar{Q}$ pentaquark first. We exhaust all the possible color $\otimes$ spin wave functions of pentaquark states, and combine them with the corresponding flavor wave functions. The constructed pentaquark wave functions should be constrained appropriately by Pauli principle. After that, we can use these pentaquark wave functions to calculate the mass spectra of the corresponding pentaquark states.

The total wave function of S-wave $QQQQ\bar{Q}$ pentaquark can be described by the direct product of flavor, color, and spin wave functions

\[ \psi_{\text{tot}} = \psi_{\text{flavor}} \otimes \psi_{\text{color}} \otimes \psi_{\text{spin}}. \tag{5} \]
Based on Eq. (6), the first four quarks are in three color symmetries, i.e., (1) the first four quarks are identical: the color wave functions should be singlets due to the antisymmetric when exchanging identical quarks.

In the flavor space, we divide the pentaquark subsystems, (2) the first three quarks are identical: the pairs of identical quarks: the color wave functions for the antitriplet from anti-quark with the deduced three color triplets in Eq. (7), we obtain three color singlets for $QQQQ$ pentaquark system

$$|C_1\rangle = |[(12)_6 34)_3 \rangle \otimes (5)_3 \rangle = \frac{1}{4\sqrt{3}} \left( (2bbgr - 2bbrg + gbbr - gbbr + bgbr - bgbr) - rbgb + bbgb - brgb + brgb)\bar{b} + (2rrgb - 2rrgb + rgrb - grbr) - rbgr + bgbr - grbr - brgr)\bar{r} + (2ggbr - 2ggbg - rggg + rggg - grgg + grgg - grgg - grgg)\bar{y} \right).$$

(8)

Due to Pauli principle, this wave function should be fully antisymmetric when exchanging identical quarks.

In the flavor space, we divide the $QQQQ$ pentaquark system into three groups of subsystems according to their symmetries, i.e., (1) the first four quarks are identical: the $cccc$, $ccbb$, $bbbb$, and $bcbcb$ pentaquark subsystems, (2) the first three quarks are identical: the $cccb$, $ccbc$, $bcbc$, and $bcbcb$ pentaquark subsystems, (3) there are two pairs of identical quarks: the $ccbcb$ and $cccb$ pentaquark subsystems.

The color wave functions should be singlets due to the color confinement. The color wave functions for $QQQQ$ pentaquark system can be deduced from the following direct product

$$[3c \otimes 3c \otimes 3c \otimes 3c] \otimes \bar{3}_c = [(6c \otimes \bar{3}_c) \otimes 3_c \otimes 3_c] \otimes \bar{3}_c = [(10_c \otimes 8_c \otimes 8_c \otimes 1_c) \otimes 3_c] \otimes \bar{3}_c \rightarrow (8_c \otimes 3_c \otimes 3_c) \otimes (8_c \otimes 3_c \otimes 3_c) \otimes (1_c \otimes 3_c \otimes 3_c) = (3_c \otimes \bar{3}_c) \otimes (3_c \otimes \bar{3}_c) \otimes (3_c \otimes \bar{3}_c).$$

(6)

Based on Eq. (6), the first four quarks are in three color triplet, the corresponding Young tableau $[2,1,1]$ can be written as

$$[(12)_6 34)_3 \rangle = \frac{1}{4\sqrt{3}} \left( (2bbgr - 2bbrg + gbbr - gbbr + bgbr - bgbr) - rbgb + bbgb - brgb + brgb)\bar{b} + (2rrgb - 2rrgb + rgrb - grbr) - rbgr + bgbr - grbr - brgr)\bar{r} + (2ggbr - 2ggbg - rggg + rggg - grgg + rggg - grgg - grgg)\bar{y} \right).$$

Here, the subscript labels the irreducible representation of SU(3). Then, by combining the antitriplet from anti-quark with the deduced three color triplets in Eq. (7), we obtain three color singlets for $QQQQ$ pentaquark system

$$[C_1\rangle = |[(12)_6 34)_3 \rangle \otimes (5)_3 \rangle = \frac{1}{12} \left( (3grgb - 3grgb - 3grgb + 3grgb - 3grgb + 3grgb + 3grgb + 3grgb - 3grgb + 3grgb)\bar{g} + (2grgb - 2grgb + 2grgb - 2grgb + 2grgb - 2grgb + 2grgb - 2grgb + 2grgb - 2grgb)\bar{y} \right).$$

(9)

and

$$|C_3\rangle = |[(123)_1 4)_3 \rangle \otimes (5)_3 \rangle = \frac{1}{3\sqrt{2}} \left( (grbb - rgrb - grbr - brgr + grbr - grbr)\bar{b} + (grbr - rgrb - grbr - brgr - grbr)\bar{r} + (grbg - rbgg - rbgr - bggr - grbg - bggr)\bar{y} \right).$$

(10)
represented in terms of Young tableau can be written as

\[
\begin{array}{cc}
S \otimes S \otimes S \otimes S \otimes S \\
S \otimes S \otimes S \otimes S \otimes S
\end{array}
\]

For the pentaquark states with total spin \( J = 5/2 \), the spin state can be represented in terms of one-dimensional Young tableau \([5]\) as

\[
\begin{array}{c}
12345 \quad S_1
\end{array}
\]

(12)

For the pentaquark states with total spin \( J = 3/2 \), the spin state can be represented in terms of four-dimensional Young tableau \([4,1]\) as

\[
\begin{array}{cccc}
1234 & 1235 & 1245 & 1345 \\
S_1 & S_2 & S_3 & S_4 & S_5 \\
9 & 4 & 3 & 2 & 1
\end{array}
\]

(13)

Similarly, for the pentaquark states with total spin \( J = 1/2 \), the spin state can be represented in terms of five-dimensional Young tableau \([3,2]\) as

\[
\begin{array}{cccccc}
123 & 124 & 134 & 125 & 135 \\
S_1 & S_2 & S_3 & S_4 & S_5 \\
4 & 3 & 2 & 1 & 0
\end{array}
\]

(14)

Since the particle 5 is an antiquark, we can isolate this antiquark and discuss the symmetry property of the first four quarks 1, 2, 3, and 4 in color \( \otimes \) spin space.

When the antiquark 5 is separated from the spin wave functions, the spin states represented in Young tableaux without the antiquark 5 can be directly obtained from Eqs. (12)-(14) as

\[
\begin{align*}
J = \frac{5}{2} : & \quad 1234 \quad S_1 \\
J = \frac{3}{2} : & \quad 1234 \quad S_1 \\
J = \frac{1}{2} : & \quad 123 \quad S_1 \\
\end{align*}
\]

(15)

We can identify the spin states in Eq. (15) with the Young-Yamanouchi bases for Young tableau \([4,1]\), \([3,2]\), and \([2,2]\).

With the above preparation, we can start to construct the flavor \( \otimes \) color \( \otimes \) spin wave functions of \( QQQQQ \) pentaquark states.

Firstly, we combine the Young tableaux \([2,1,1]\) of the color singlets in Eqs. (8)-(10) with Young tableaux \([4]\), \([3,1]\), \([2,2]\) of the spin states in Eq. (15) by the inner product of the permutation group, which is shown in Eq. (16). According to the isoscalar factors for \( S_1 \) and \( S_2 \) in Tables 6.2 and 6.3 of Ref. [79], we can extract the corresponding Clebsch-Gordan (CG) coefficient of \( S_4 \) \([16]\), and thus we can present the Young tableau of Eq. (16) explicitly in Eq. (A1) of the Appendix. With the help of Eq. (A1), one can write the wave functions of the irreducible representations in the color \( \otimes \) spin space.

\[
\begin{align*}
J = \frac{5}{2} : & \quad S_1 \quad c \otimes S \quad s \quad c S_5 \\
J = \frac{3}{2} : & \quad S_1 \quad c \otimes S \quad s \quad c S_4 \\
J = \frac{1}{2} : & \quad S_1 \quad c \otimes S \quad s \quad c S_3 \\
\end{align*}
\]

(16)

Then we combine the flavor wave functions with the color \( \otimes \) spin wave functions. We present the Young-Yamanouchi bases allowed by exchange symmetry in Table VI of Appendix A and the pentaquark wave functions satisfied Pauli Principle can then be obtained with the table.

Based on the possible \( \psi_{\text{flavor}} \otimes \psi_{\text{color}} \otimes \psi_{\text{spin}} \) bases of the \( QQQQQ \) pentaquark system, we calculate the CMI matrices for the corresponding pentaquark states. In Table VII of Appendix A we only present the expressions of CMI Hamiltonians for the \( cc \bar{c}c \bar{c}, \ ccc \bar{b}, \) and \( ccb \bar{b} \) pentaquark subsystems. The expressions of CMI matrices for the \( ccb, \ bbb \bar{b}, \ bbb \bar{c}, \) and \( ccb \bar{b} \) pentaquark subsystems can be obtained from those of the \( cc \bar{c}c \bar{c}, \ ccc \bar{b}, \) and \( ccb \bar{b} \) pentaquark subsystems according to their similar symmetry properties.

\section*{IV. MASS SPECTRA AND DECAY BEHAVIORS}

The interacting Hamiltonians can be diagonalized and one can thus obtain the eigenvalues as well as eigenvectors for the corresponding pentaquark systems. According to our results, we discuss the masses gaps, decay behaviors, and stabilities of all the \( QQQQQ \) pentaquark states.

Based on the two schemes proposed in Sec. II we present the mass spectra for all the \( QQQQQ \) pentaquark subsystems in Table III. Take the \( ccb \bar{b} \) pentaquark subsystem as an example. In the reference mass scheme, we use two types of baryon-meson reference systems \( \Omega_{cc \bar{c}} + \bar{B}_c \) and \( \Omega_{cc \bar{b}} + \bar{K}_c \) to estimate the masses of \( ccb \bar{b} \) states. Some results calculated from the two reference
systems differ by more than a hundred MeV for the \(cc\bar{c}\bar{b}\) pentaquark states. However, the gaps with different reference systems are still same. Thus, if one pentaquark state was observed, its partner states may be searched for with the relative positions presented in Table IV. Such a study can be used to test our calculation. Here, we need to emphasis that as a rough estimation, the dynamics and contributions from other terms in the interacting potential are not elaborately considered in Eq. (3) \[\text{[63]}.\]

The modified CMI model scheme takes the chromoelectric interaction explicitly compared to the reference mass scheme, and therefore we use the results in this scheme for the following analysis. According to the modified CMI model scheme, we present the masses of the \(QQQQ\) pentaquark states and the relevant baryon-meson thresholds in Fig. [1] In Fig. [1] we label the possible total angular momenta of the S-wave baryon-meson states. When the spin of an initial pentaquark state is equal to the total angular momentum of the channel below, it may decay into that baryon-meson channel through S wave.

Here we define the relatively 'stable' pentaquarks as those which cannot decay into the S wave baryon-meson states. We label these stable pentaquark states with \[^{\text{[4, 5, 47, 71, 73, 85]}}\]. Thus we calculate the overlaps of wave functions between a fully-heavy pentaquark states and the relevant baryon-meson thresholds of wave functions shown in Table IV. For the decay processes that we are interested in, \((k/m)^2\) is of \(O(10^{-2})\) or even smaller. Thus we only consider the S-wave decays.

\[\gamma_i\] represents other factors that contribute to the decay widths \(\Gamma_i\). For each process, \(\Gamma_i\) depends on also the spatial wave functions of the initial pentaquark state and final meson and baryon. In the quark model in the heavy quark limit, the spatial wave functions of the ground S-wave pseudoscalar and vector meson are the same \[\text{[73]}.\] As a rough estimation, we introduce the following approximations to calculate the relative partial decay widths of the pentaquark states.

\[
\gamma_{cc} J/\psi = \gamma_{cc} \eta_c, \quad \gamma_{cc} B_s = \gamma_{cc} B_c, \\
\gamma_{b\bar{b}} B_s^* = \gamma_{b\bar{b}} B_c, \quad \gamma_{b\bar{b}} \Upsilon = \gamma_{b\bar{b}} \eta_b, \\
\gamma_{cc} B_s^* = \gamma_{cc} \eta_c, \quad \gamma_{cc} \Upsilon = \gamma_{cc} \eta_c, \\
\gamma_{b\bar{b}} l/\psi = \gamma_{b\bar{b}} \eta_b, \quad \gamma_{b\bar{b}} B_s^* = \gamma_{b\bar{b}} B_c, \\
\gamma_{cc} l/\psi = \gamma_{cc} \eta_c, \quad \gamma_{cc} \Upsilon = \gamma_{cc} \eta_c, \\
\gamma_{b\bar{b}} l/\psi = \gamma_{b\bar{b}} \eta_b, \quad \gamma_{b\bar{b}} B_s^* = \gamma_{b\bar{b}} B_c, \\
\gamma_{cc} l/\psi = \gamma_{cc} \eta_c, \quad \gamma_{cc} \Upsilon = \gamma_{cc} \eta_c, \\
\gamma_{b\bar{b}} l/\psi = \gamma_{b\bar{b}} \eta_b, \quad \gamma_{b\bar{b}} B_s^* = \gamma_{b\bar{b}} B_c, \\
\gamma_{cc} l/\psi = \gamma_{cc} \eta_c, \quad \gamma_{cc} \Upsilon = \gamma_{cc} \eta_c.
\]

We present \(k \cdot |c_i|^2\) for each decay process in Table V. From Table V one can roughly estimate the relative decay widths between different decay processes of different initial pentaquark states if neglecting the \(\gamma_i\) differences.

We divide the \(QQQQ\) pentaquark system into the following three groups:

A. The \(cc\bar{c}\bar{Q}\) and \(b\bar{b}\bar{b}\bar{Q}\) pentaquark subsystems;

B. The \(cc\bar{b}\bar{Q}\) and \(b\bar{b}c\bar{Q}\) pentaquark subsystems;

C. The \(cc\bar{b}\bar{Q}\) pentaquark subsystem.

We discuss the mass spectra and strong decay properties of \(QQQQ\) pentaquark system group by group. For simplicity, we use \(P_{\text{content}}(\text{Mass, } I, J^P)\) to label a specific pentaquark state.

A. The \(cc\bar{c}\bar{Q}\) and \(b\bar{b}\bar{Q}\) pentaquark states

We first discuss the fully heavy pentaquark states with \(cc\bar{c}\bar{c}\), \(cc\bar{b}\bar{b}\), \(bb\bar{b}\bar{c}\), and \(b\bar{b}b\bar{b}\) flavor configurations. The \(cc\bar{b}\bar{b}\) and \(b\bar{b}c\bar{b}\) states are the absolute exotic states which have the different flavor quantum numbers from the conventional baryons. Because of the strong symmetrical constraint from Pauli principle, i.e., fully antisymmetric among the first four charm quarks, we only find two \(cc\bar{c}\bar{c}\) states: an \(I(J^P) = 0(3/2^-)\) state: \(P_{x^c}(7864, 0, 3/2^-)\)
TABLE III. The estimated masses for the $QQQQ$ ($Q = c, b$) system in units of MeV. The eigenvalues of $H_{\text{CMI}}$ matrix are listed in the second column. The corresponding masses in the reference mass scheme, also labeled as the 1st scheme, are listed in third and/or fourth columns. The masses with the modified CMI model scheme, also labeled as the 2nd scheme, are presented in the last column.

|         | 1st scheme | 2nd scheme | 1st scheme | 2nd scheme |
|---------|------------|------------|------------|------------|
| $J^P$   | $\omega$   | Mass       | $J^P$   | $\omega$   | Mass       |
| $\frac{3}{2}^-$ | 59.0       | 11131      | $\frac{1}{2}$ | 41.9       | 14247      | 14172      | 14246      |
| $\frac{3}{2}^-$ | 59.0       | 11116      | $\frac{3}{2}$ | 47.1       | 14252      | 14178      | 14373      |
| $\frac{3}{2}^-$ | 23.6       | 16111      | $\frac{3}{2}$ | 32.1       | 14237      | 14163      | 14246      |
| $\frac{3}{2}^-$ | -45.2      | 11042      | -30.6      | 14174      | 14100      | 14182      |
| $\frac{3}{2}^-$ | 101.3      | 16118      | 83.7       | 14288      | 14214      | 14411      |
| $\frac{3}{2}^-$ | 61.2       | 11148      | 45.5       | 14250      | 14176      | 14357      |
| $\frac{3}{2}^-$ | -26.2      | 16106      | -8.8       | 14196      | 14122      | 14238      |
| $\frac{3}{2}^-$ | 42.7       | 17407      | 32.0       | 20655      | 20659      | 20648      |
| $\frac{3}{2}^-$ | 44.7       | 17409      | 34.0       | 20657      | 20661      | 20654      |
| $\frac{3}{2}^-$ | 15.0       | 17379      | 19.4       | 20642      | 20646      | 20644      |
| $\frac{3}{2}^-$ | -77.3      | 17287      | -47.0      | 20576      | 20580      | 20578      |
| $\frac{3}{2}^-$ | 76.5       | 17441      | 67.5       | 20691      | 20694      | 20691      |
| $\frac{3}{2}^-$ | 36.2       | 17400      | 29.6       | 20653      | 20657      | 20653      |
| $\frac{3}{2}^-$ | -18.3      | 17346      | -18.7      | 20604      | 20608      | 20607      |
| $\frac{3}{2}^-$ | 41.9       | 14372      | 35.5       | 17484      | 17545      | 17477      |
| $\frac{3}{2}^-$ | 52.2       | 14383      | 39.5       | 17488      | 17549      | 17554      |
| $\frac{3}{2}^-$ | 20.7       | 14351      | 24.0       | 17472      | 17533      | 17479      |
| $\frac{3}{2}^-$ | 16.9       | 14348      | 13.1       | 17461      | 17522      | 17457      |
| $\frac{3}{2}^-$ | -63.0      | 14268      | -40.1      | 17408      | 17469      | 17416      |
| $\frac{3}{2}^-$ | 87.4       | 14418      | 73.9       | 17522      | 17583      | 17576      |
| $\frac{3}{2}^-$ | 46.5       | 14377      | 35.8       | 17484      | 175454     | 17496      |
| $\frac{3}{2}^-$ | -19.1      | 14312      | -15.5      | 17433      | 17494      | 17437      |
| $\frac{3}{2}^-$ | -69.6      | 14261      | -49.0      | 17399      | 17460      | 17405      |
TABLE IV. The overlaps of wave functions between a fully-heavy pentaquark state and a particular baryon $\otimes$ meson state. The masses are all in units of MeV. See the caption of Fig. 1 for meanings of "$^\Psi$" and "$^\Psi^*$".

| $J^P$ | Mass | $\Omega_{cc\bar{c}}$/$\Omega_{cc\bar{c}}$ηc | $\Omega_{cc\bar{c}}$/$\Omega_{cc\bar{c}}$ηc | $\Omega_{cc\bar{c}}$/$\Omega_{cc\bar{c}}$ηc | $\Omega_{cc\bar{c}}$/$\Omega_{cc\bar{c}}$ηc | $\Omega_{cc\bar{c}}$/$\Omega_{cc\bar{c}}$ηc | $\Omega_{cc\bar{c}}$/$\Omega_{cc\bar{c}}$ηc |
|-------|------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|
| $\frac{1}{2}^-$ | 7864 | 0.456 -0.354 | 0.456 -0.354 | 20652 | 0.456 -0.354 | 23775 | 0.456 -0.354 |
| $\frac{1}{2}^-$ | 7949 | -0.577 | 11177 | 0.577 | 20699 | -0.577 | 23821 | 0.577 |
| $\frac{3}{2}^-$ | 11124 | 1.000 | 0.333 | 14246 | 1.000 | 0.333 | 14246 | 1.000 |
| $\frac{3}{2}^+$ | 11137 | 0.812 0.236 | -0.361 -0.008 0.275 | 14373 | -0.046 -0.120 0.521 -0.352 -0.209 | 14246 | 0.999 -0.016 0.030 0.229 -0.242 | 14246 |
| 11101 | 0.569 -0.524 | 0.396 -0.279 0.102 | 14246 | 0.999 -0.016 0.030 0.229 -0.242 | 14246 | 0.999 -0.016 0.030 0.229 -0.242 | 14246 |
| 11038 | 0.130 0.818 | 0.095 -0.380 -0.250 | 14182 | 0.011 0.993 | 0.154 0.214 0.215 | 14182 | 0.011 0.993 | 0.154 0.214 0.215 |
| $\frac{1}{2}^-$ | 11175 | -0.543 | -0.587 0.034 -0.001 14411 | 0.130 | -0.626 0.199 0.087 | 14411 | 0.130 | -0.626 0.199 0.087 |
| 11137 | -0.657 | 0.172 | -0.519 -0.039 14357 | -0.206 | 0.126 0.571 -0.297 | 14357 | -0.206 | 0.126 0.571 -0.297 |
| 11048 | 0.523 | 0.180 | 0.316 -0.470 14238 | 0.969 | 0.004 0.068 0.355 | 14238 | 0.969 | 0.004 0.068 0.355 |
| $\frac{3}{2}^+$ | 17407 | 0.005 | 0.996 | -0.168 -0.203 -0.211 | 20578 | 0.026 0.942 0.038 0.291 0.249 | 20578 | 0.026 0.942 0.038 0.291 0.249 |
| $\frac{1}{2}^-$ | 17532 | -0.187 | 0.092 | 0.580 -0.296 | 20653 | -0.662 | -0.156 | 0.522 | 0.033 |
| 17399 | 0.973 | 0.010 | 0.073 0.351 | 20607 | 0.554 | -0.158 | 0.311 0.470 |
| $\frac{3}{2}^+$ | 14295 | -0.577 | -0.577 | 14295 | -0.577 | -0.577 | 14295 | -0.577 | -0.577 |
| $\frac{1}{2}^-$ | 14375 | -0.070 -0.088 -0.019 | 0.592 -0.316 0.450 | 14298 | -0.088 -0.019 | 0.592 -0.316 0.450 | 14298 | -0.088 -0.019 | 0.592 -0.316 0.450 |
| 14298 | 0.631 -0.057 0.370 | -0.138 -0.325 0.056 | 14298 | 0.631 -0.057 0.370 | -0.138 -0.325 0.056 | 14298 | 0.631 -0.057 0.370 | -0.138 -0.325 0.056 |
| 14274 | -0.251 0.126 0.507 | -0.224 0.242 0.471 | 14274 | -0.251 0.126 0.507 | -0.224 0.242 0.471 | 14274 | -0.251 0.126 0.507 | -0.224 0.242 0.471 |
| 14197 | -0.074 0.624 -0.225 | 0.229 0.390 -0.128 | 14197 | -0.074 0.624 -0.225 | 0.229 0.390 -0.128 | 14197 | -0.074 0.624 -0.225 | 0.229 0.390 -0.128 |
| $\frac{3}{2}^+$ | 14406 | 0.202 | -0.074 -0.025 -0.722 | -0.200 -0.242 | 14406 | 0.202 | -0.074 -0.025 -0.722 | -0.200 -0.242 |
| 14318 | 0.666 | 0.277 0.033 -0.008 | -0.348 -0.156 | 14318 | 0.666 | 0.277 0.033 -0.008 | -0.348 -0.156 |
| 14253 | 0.225 | -0.505 0.120 | -0.012 | 0.273 | -0.505 0.120 | -0.012 | 0.273 | -0.505 0.120 | -0.012 |
| 14185 | 0.146 | 0.155 0.633 | 0.184 | -0.354 -0.248 | 14185 | 0.146 | 0.155 0.633 | 0.184 | -0.354 -0.248 |

and an $I(J^P) = 0(1/2^-)$ state: $P_{c\bar{c}}(7949, 0, 1/2^-)$. Similarly, there are also only two pentaquark states in the $ccc\bar{c}$, $bbc\bar{c}$, and $bbb\bar{b}$ subsystems. From Fig. (a)-(d), the $J^P = 3/2^-$ states generally have smaller masses than the $J^P = 1/2^-$ states in the $ccc\bar{c}$ and $bbb\bar{b}$ pentaquark subsystems. Meanwhile, from Fig. (a)-(d), the masses of all the $ccc\bar{c}$ and $bbb\bar{b}$ pentaquark states are larger than the thresholds of
FIG. 1. Relative positions (units: MeV) for the cccccc states whose wave function overlaps with that of one special baryon-meson state more than 90%, with "⋄" after their masses. We mark the relatively stable states (a) cccccc states, (b) cccccb states, (c) bbbcb states, (d) bbbccc states, (e) cccbc states, (f) cccbbb states, (g) bbbcbc states, (h) bbbccb states, (i) ccbcb states, and (j) ccbbb states.

We represent the possible total angular momenta of the channels. We mark the pentaquark states (a) ccccc states, (b) ccccb states, (c) bbbcb states, (d) bbbccc states, (e) cccbc states, (f) cccbbb states, (g) bbbcbc states, (h) bbbccb states, (i) ccbcb states, and (j) ccbbb states.

We mark the relatively stable pentaquarks, unable to decay into the S-wave baryon-meson states, with "⋆" after their masses. We mark the pentaquark whose wave function overlaps with that of one special baryon-meson state more than 90%, with "⋄" after their masses.
the lowest possible baryon-meson systems. Thus, no stable pentaquark state exists in the $ccccQ$ and $bbbbQ$ pentaquark subsystems. The lowest baryon-meson channels are their dominant decay modes. In the future, searching for exotic signals in these baryon-meson strong decay channels would be an interesting topic.

The $cccc\bar{c}$ subsystem has one decay mode: $ccc \otimes c\bar{c}$, which could be $\Omega_{ccc}J/\psi$ or $\Omega_{ccc}\eta_c$. However, each $cccc\bar{c}$ state has only one decay channel from Table IV. The $J^P = 1/2^-$ $cccc\bar{c}$ pentaquark state cannot decay into S-wave $\Omega_{ccc}\eta_c$ because of the constraint of angular conservation law, while the $J^P = 3/2^-$ one cannot decay into $\Omega_{ccc}J/\psi$ since the mass is below the threshold.

From Table IV the ratio of decay widths between branching channels for $P_{b\bar{c}}(20652, 0, 3/2^-)$ with the assumptions of Eq. (18) is

$$\Gamma_{\Omega_{ccc}B_c^*} : \Gamma_{\Omega_{ccc}B_c} = 0.4 : 1,$$

and for $P_{b\bar{c}b}(23775, 0, 3/2^-)$ is

$$\Gamma_{\Omega_{ccb}B_c^*} : \Gamma_{\Omega_{ccb}B_c} = 0.4 : 1.$$  

Our results suggest that $\Omega_{ccc}\eta_c$ is the dominant decay channel in $ccc - b\bar{b}$ decay mode. Moreover, the $P_{c\bar{c}}(14373, 0, 3/2^-)$ also has three decay channels in $ccb - \bar{c}b$ decay mode. Their relative partial decay widths are

$$\Gamma_{\Omega_{ccc}B_c^*} : \Gamma_{\Omega_{ccc}B_c} : \Gamma_{\Omega_{cc\bar{b}}B_c} = 3.3 : 3.7 : 1,$$

i.e., the partial decay widths of the $\Omega_{ccc}B_c^*$ and $\Omega_{cc\bar{b}}B_c$ channels are larger than that of the $\Omega_{ccc}B_c$.

The $J^P = 1/2^-$ pentaquark states $P_{c\bar{c}b\bar{b}}(14411, 0, 1/2^-)$ and $P_{c\bar{c}b\bar{b}}(14357, 0, 1/2^-)$ have different decay behaviors. The $P_{c\bar{c}b\bar{b}}(14411, 0, 1/2^-)$ can decay into the $\Omega_{ccc}B_c$, while this channel is kinetically forbidden for the $P_{c\bar{c}b\bar{b}}(14357, 0, 1/2^-)$. Meanwhile, the partial decay widths for the $P_{c\bar{c}b\bar{b}}(14411, 0, 1/2^-)$ has

$$\Gamma_{\Omega_{ccc}B_c^*} : \Gamma_{\Omega_{ccc}B_c} : \Gamma_{\Omega_{cc\bar{b}}B_c} = 31 : 4 : 1,$$

and for the $P_{c\bar{c}b\bar{b}}(14357, 0, 1/2^-)$

$$\Gamma_{\Omega_{ccc}B_c^*} : \Gamma_{\Omega_{ccc}B_c} : \Gamma_{\Omega_{cc\bar{b}}B_c} = 0 : 2.1 : 1.$$  

C. The $cccb\bar{Q}$ pentaquark states

The $cccb\bar{Q}$ ($ccbb\bar{b}$) subsystem has 8 possible rearrangement decay channels, including $\Omega_{ccc}B_c^*(\Omega_{ccc}B_c^*\bar{c})$, $\Omega_{ccc}B_c^*(\Omega_{ccc}B_c)$, $\Omega_{ccc}B_c^*(\Omega_{ccc}B_c)$, $\Omega_{ccc}B_c^*(\Omega_{ccc}B_c)$, $\Omega_{cc\bar{b}}B_c^*$, $\Omega_{cc\bar{b}}B_c^*(\Omega_{cc\bar{b}}B_c)$, $\Omega_{cc\bar{b}}B_c^*(\Omega_{cc\bar{b}}B_c)$, and $\Omega_{cc\bar{b}}B_c^*(\Omega_{cc\bar{b}}B_c)$.

According to Fig. (j), the $P_{c\bar{c}b\bar{b}}(17477, 0, 5/2^-)$ does not have S-wave strong decay channels, and thus this state is expected to be narrow. It can still decay into the D-wave final states of $\Omega_{ccc}\eta_c$ and $\Omega_{cc\bar{b}}B_c$.

Meanwhile, the lowest $J^P = 3/2^-$ $cccb\bar{Q}$ pentaquark state $P_{c\bar{c}b\bar{b}}(17416, 0, 3/2^-)$ is below all allowed strong decay channels. It should decay through the electromagnetic and weak interactions rather than the strong interaction. Thus, this state is considered as a good stable pentaquark.

The lowest $J^P = 1/2^-$ state $P_{c\bar{c}b\bar{b}}(17405, 0, 1/2^-)$ can only decay into $\Omega_{ccb}\eta_c$, and its mass is slightly larger than the $\Omega_{ccc}\eta_c$ threshold. Thus, its width should be narrow due to the small decay phase space.

For the other three $J^P = 1/2^-$ pentaquark states, the $P_{c\bar{c}b\bar{b}}(17437, 0, 1/2^-)$ only decays into $\Omega_{ccb}\eta_c$ in two-body strong decay. The $P_{c\bar{c}b\bar{b}}(17576, 0, 1/2^-)$ and $P_{c\bar{c}b\bar{b}}(17496, 0, 1/2^-)$ both have two different decay modes: $\bar{c}bb - \bar{c}b$ and $\bar{c}cb - \bar{b}b$. They can decay freely into many allowed decay channels, and therefore, they both have broad widths. In particular, the $P_{c\bar{c}b\bar{b}}(17576, 0, 1/2^-)$ has

$$\Gamma_{\Omega_{ccc}B_c^*} : \Gamma_{\Omega_{ccc}B_c} : \Gamma_{\Omega_{cc\bar{b}}B_c} = 5.6 : 0.5 : 1,$$

and

$$\Gamma_{\Omega_{ccc}B_c^*} : \Gamma_{\Omega_{ccc}B_c} : \Gamma_{\Omega_{cc\bar{b}}B_c} = 48 : 7.3 : 1.$$  

TABLE V. The values of $k \cdot |c|^2$ for the $cccc\bar{c}$, $cccb\bar{c}$, $bbbb\bar{c}$, $bbbc\bar{c}$, $cccb\bar{b}$, $bbbc\bar{b}$, $ccbb\bar{b}$, and $ccbb\bar{b}$ pentaquark states. The masses are all in units of MeV. The decay channel is marked with “$\times$” if kinetically forbidden. See the caption of Fig. 1 for meanings of “$\diamond$” and “$\star$”. One can roughly estimate the relative decay widths between different decay processes of different initial pentaquark states with this table if neglecting the $\gamma_i$ differences.

| $J^P$ | $cccc\bar{c}$ Mass | $cccb\bar{b}$ Mass | $bbbc\bar{c}$ Mass | $ccbb\bar{b}$ Mass |
|-------|-------------------|-------------------|-------------------|-------------------|
| $J^P$ | $\Omega_{ccc\bar{c}}/\Omega_{ccc\bar{b}}$ | $\Omega_{ccc\bar{b}}/\Omega_{ccc\bar{b}}$ | $\Omega_{ccc\bar{c}}/\Omega_{ccc\bar{c}}$ | $\Omega_{ccc\bar{b}}/\Omega_{ccc\bar{b}}$ |
| $^{1/2-}$ | 7864 | $x$ | 74 | 11130 |
| $^{1/2-}$ | 7949 | 167 | 11117 | 181 |

| $J^P$ | $cccb\bar{b}$ Mass | $ccbb\bar{b}$ Mass |
|-------|-------------------|-------------------|
| $J^P$ | $\Omega_{cccb\bar{b}}/\Omega_{cccb\bar{b}}$ | $\Omega_{cccb\bar{b}}/\Omega_{cccb\bar{b}}$ |
| $^{1/2-}$ | 11124 | 15 | 14246 |
| $^{1/2-}$ | 11137 | 36 | 70 |
| $^{1/2-}$ | 11101 | 3 | 14246 |
| $^{1/2-}$ | 11038 | 54 | 14182 |

| $J^P$ | $bbbc\bar{c}$ Mass | $bbbc\bar{b}$ Mass |
|-------|-------------------|-------------------|
| $J^P$ | $\Omega_{bbbc\bar{c}}/\Omega_{bbbc\bar{c}}$ | $\Omega_{bbbc\bar{b}}/\Omega_{bbbc\bar{b}}$ |
| $^{1/2-}$ | 17407 | $x$ | 14246 |
| $^{1/2-}$ | 17535 | 24 | 14246 |
| $^{1/2-}$ | 17406 | $x$ | 14246 |
| $^{1/2-}$ | 17291 | 24 | 14246 |

| $J^P$ | $ccbb\bar{c}$ Mass | $ccbb\bar{b}$ Mass |
|-------|-------------------|-------------------|
| $J^P$ | $\Omega_{ccbb\bar{c}}/\Omega_{ccbb\bar{c}}$ | $\Omega_{ccbb\bar{c}}/\Omega_{ccbb\bar{c}}$ |
| $^{1/2-}$ | 14295 | 3 | 115 |
| $^{1/2-}$ | 14375 | 2 | 115 |
| $^{1/2-}$ | 14298 | 2 | 115 |
| $^{1/2-}$ | 14274 | 2 | 115 |

| $J^P$ | $ccbb\bar{b}$ Mass |
|-------|-------------------|
| $J^P$ | $\Omega_{ccbb\bar{c}}/\Omega_{ccbb\bar{c}}$ |
| $^{1/2-}$ | 14406 | 3 | 115 |
| $^{1/2-}$ | 14318 | 2 | 115 |
| $^{1/2-}$ | 14246 | 2 | 115 |

| $J^P$ | $ccbb\bar{b}$ Mass |
|-------|-------------------|
| $J^P$ | $\Omega_{ccbb\bar{c}}/\Omega_{ccbb\bar{c}}$ |
| $^{1/2-}$ | 17477 | $x$ | 14246 |
| $^{1/2-}$ | 17554 | 2 | 115 |
| $^{1/2-}$ | 17479 | 2 | 115 |
| $^{1/2-}$ | 17457 | 2 | 115 |

| $J^P$ | $ccbb\bar{b}$ Mass |
|-------|-------------------|
| $J^P$ | $\Omega_{ccbb\bar{c}}/\Omega_{ccbb\bar{c}}$ |
| $^{1/2-}$ | 17576 | 3 | 115 |
| $^{1/2-}$ | 17496 | 2 | 115 |
| $^{1/2-}$ | 17437 | 2 | 115 |
| $^{1/2-}$ | 17405 | 2 | 115 |
Moreover, all the \( J^P = 1/2^- \ ccb\bar{b} \) pentaquark states can decay into \( \Omega_{ccb}\bar{b} \) final states, and this decay channel is crucial to find \( J^P = 1/2^- \ ccb\bar{b} \) pentaquark states.

For three unstable \( J^P = 3/2^- \) pentaquark states, the \( P_{c\bar{b}b\bar{b}}(17457, 0, 3/2^-) \) only has \( c\bar{b} - \bar{b}b \) decay mode and the relative decay widths are

\[
\Gamma_{\Omega_{ccb}\bar{b}}/\Gamma_{\Omega_{ccb}} = 0.3 : 1. \quad (27)
\]

The other two \( J^P = 3/2^- \) pentaquark states both have two decay modes: \( c\bar{b} - \bar{c}b \) and \( c\bar{b} - \bar{b}b \). The \( P_{c\bar{b}b\bar{b}}(17479, 0, 3/2^-) \) only decay into \( \Omega_{ccb}\bar{b} \) final state in \( bbc - \bar{c}\bar{b} \) decay mode, while in \( c\bar{b} - \bar{b}b \) decay mode it has

\[
\Gamma_{\Omega_{ccb}\bar{b}}/\Gamma_{\Omega_{ccb}} = 0.4 : 1. \quad (28)
\]

The heaviest \( J^P = 3/2^- \) state \( P_{c\bar{b}b\bar{b}}(17554, 0, 3/2^-) \) can easily decay into many two-body baryon-meson channels due to its large decay phase space.

\[\text{V. SUMMARY}\]

More and more exotic multiquark candidates are lastingly discovered in experiment these years \([10–13]\). The \( P_c(4312), P_c(4440), \) and \( P_c(4457) \) states and the fully-charmed tetraquark candidate \( X(6900) \) reported from the LHCb collaboration motivate us to discuss the possible pentaquark states with \( QQQQ \) configuration in the framework of CMI model.

In this work, we firstly construct the wave functions \( \psi_{flavor} \otimes \psi_{color} \otimes \psi_{spin} \) based on the \( SU(2) \) and \( SU(3) \) symmetry and Pauli Principle. Then, we extract the effective coupling constants from the conventional hadrons. After that, we systematically calculate the CMI Hamiltonian for the \( QQQQ \) pentaquark states and obtain the corresponding mass spectra in the reference system scheme. In the modified CMI scheme, the effect of chromoelectric interaction is explicitly added.

The mass spectra is studied for the \( QQQQ \) pentaquark system. In addition, we also provide the eigenvectors to extract useful information about the decay properties for the \( QQQQ \) pentaquark systems. The overlaps for the pentaquark state with a particular baryon \( \otimes \) meson state are obtained. Finally, we analyze the stability, possible quark rearrangement decay channels, and relative partial decay widths for all the \( QQQQ \) pentaquark states.

According to our calculations and analysis, we only find two \( ccc\bar{c}c \) states due to the constraint from Pauli principle: a \( J^P = 3/2^- \) state \( P_{c\bar{c}c}(7864, 0, 3/2^-) \) and a \( J^P = 1/2^- \) state \( P_{c\bar{c}c}(7949, 0, 1/2^-) \), and there exists no ground \( J^P = 5/2^- \) \( ccc\bar{c}c \) pentaquark state. The same situation also happen in the \( cccc, bbbb, \) and \( bbb\bar{b} \) subsystems. From the obtained tables and figures for the \( QQQQ \) pentaquark system, we find one good stable candidates: the \( P_{c\bar{b}b\bar{b}}(17416, 0, 3/2^-) \). It lies only below the allowable decay channel \( \Omega_{ccb}\bar{b} \) 4 MeV, and thus can only decay through electromagnetic or weak interactions. Meanwhile, the \( P_{c\bar{b}b\bar{b}}(17477, 0, 5/2^-) \) is also a relatively stable pentaquark since it is lower than all possible S-wave strong decay channels. It can still decay into \( \Omega_{ccb}\bar{b} \) and \( \Omega_{ccb}\bar{b} \) final states via \( D \)-wave.

Our systematical study can provide some understanding toward these pentaquark systems. We find some fully heavy pentaquark states can be very narrow and stable. If they do exist, identifying them may not be difficult from their exotic quantum numbers and masses. The \( X(6900) \) is found in the invariant mass spectrum of \( J/\psi \) pairs, where two pairs of \( c\bar{c} \) are produced. In our calculation, the lowest fully-heavy pentaquark state is the \( J^P = 3/2^- \ ccc\bar{c}c \) state. To produce the lightest \( ccc\bar{c}c \) pentaquark state, one needs to simultaneously produce at least four pairs of \( c\bar{c} \), this seems to be a difficult task in experiment. More detailed dynamical investigations on \( QQQQ \) pentaquark systems are still needed. We hope that our study may inspire experimentalists to pay attention to this kind of pentaquark system.

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Appendix A: Some expressions in detail

The possible Young-Yamakouchi color $\otimes$ spin bases of the Young tableaus in Eq. (16) are presented in Eq. (A1). We list the possible wave function satisfied by Pauli Principle in Table VI and some CMI Hamiltonians in Table VII.

\[ J = \frac{5}{2} : \]

\begin{align*}
C_{s_1} = & \left[ \begin{array}{c}
1 \ 3 \\
2 \\
4 \\
\end{array} \right] \\
C_{s_2} = & \left[ \begin{array}{c}
1 \ 4 \\
2 \\
3 \\
\end{array} \right] \\
\end{align*}

\[ J = \frac{3}{2} : \]

\begin{align*}
C_{s_1} = & \left[ \begin{array}{c}
1 \ 2 \ 3 \\
4 \\
\end{array} \right] \\
C_{s_2} = & \left[ \begin{array}{c}
1 \\
2 \ 3 \\
4 \\
\end{array} \right] \\
C_{s_3} = & \left[ \begin{array}{c}
1 \ 2 \\
3 \ 4 \\
\end{array} \right] \\
C_{s_4} = & \left[ \begin{array}{c}
1 \\
2 \ 4 \\
3 \\
\end{array} \right] \\
C_{s_5} = & \left[ \begin{array}{c}
1 \\
3 \\
2 \ 4 \\
\end{array} \right] \\
C_{s_6} = & \left[ \begin{array}{c}
1 \ 2 \\
3 \ 4 \\
\end{array} \right] \\
\end{align*}

\[ J = \frac{1}{2} : \]

\begin{align*}
C_{s_1} = & \left[ \begin{array}{c}
1 \ 2 \\
3 \ 4 \\
\end{array} \right] \\
C_{s_2} = & \left[ \begin{array}{c}
1 \ 3 \\
2 \\
4 \\
\end{array} \right] \\
C_{s_3} = & \left[ \begin{array}{c}
1 \\
2 \ 3 \\
4 \\
\end{array} \right] \\
C_{s_4} = & \left[ \begin{array}{c}
1 \\
2 \\
3 \ 4 \\
\end{array} \right] \\
C_{s_5} = & \left[ \begin{array}{c}
1 \\
3 \ 4 \\
2 \\
\end{array} \right] \\
C_{s_6} = & \left[ \begin{array}{c}
1 \\
2 \ 4 \\
3 \\
\end{array} \right] \\
\end{align*}

\[ (A1) \]
TABLE VI. All possible color $\otimes$ spin Young-Yamanouchi bases satisfied with Pauli Principle for a specific $QQQQ$ ($Q = c, b$) subsystems. The $J$ represents the spin of the pentaquark states.

| Flavor waves | $J$   | The color $\otimes$ spin Young-Yamanouchi bases |
|--------------|-------|-----------------------------------------------|
| $cccc\bar{Q}$ | $J = 3/2$ | 1 \quad 2 \quad 3 \quad 4 \quad CS_1 |
| $bbbdQ$ | $J = 1/2$ | 1 \quad 2 \quad 3 \quad 4 \quad CS_1 |
| $cccb\bar{Q}$ | $J = 5/2$ | 1 \quad 2 \quad 3 \quad CS_1 |
| $bbbb\bar{Q}$ | $J = 3/2$ | 1 \quad 2 \quad 3 \quad CS_1 \quad CS_2 \quad CS_5 |
| $cccc\bar{Q}$ | $J = 5/2$ | 1 \quad 2 \quad 3 \quad CS_1 |
| $bbbdQ$ | $J = 3/2$ | 1 \quad 2 \quad 3 \quad CS_1 \quad CS_2 \quad CS_5 |
| $cccb\bar{Q}$ | $J = 5/2$ | 1 \quad 2 \quad 3 \quad CS_1 |
| $cccc\bar{Q}$ | $J = 3/2$ | 1 \quad 2 \quad 3 \quad CS_1 \quad CS_2 \quad CS_5 |
| $bbbdQ$ | $J = 1/2$ | 1 \quad 2 \quad 3 \quad CS_1 \quad CS_2 \quad CS_5 |
| $cccb\bar{Q}$ | $J = 5/2$ | 1 \quad 2 \quad 3 \quad CS_1 |
| $cccc\bar{Q}$ | $J = 3/2$ | 1 \quad 2 \quad 3 \quad CS_1 \quad CS_2 \quad CS_5 |
| $bbbdQ$ | $J = 1/2$ | 1 \quad 2 \quad 3 \quad CS_1 \quad CS_2 \quad CS_5 |
TABLE VII. The expressions of CMI Hamiltonians for $cccc\bar{c}$, $cccb\bar{c}$, and $c\bar{c}bb\bar{c}$ pentaquark subsystems. The $J$ represents the spin of the pentaquark states.

| $J$ | The expressions of CMI Hamiltonian for $cccc\bar{c}$ subsystems |
|-----|---------------------------------------------------------------|
| $J = 3/2$ | $\frac{8}{3}C_{cc} - \frac{16}{3}C_{\bar{c}c}$ |
| $J = 1/2$ | $\frac{16}{3}C_{cc} + \frac{16}{3}C_{\bar{c}c}$ |
| $J$ | The expressions of CMI Hamiltonian for $cccb\bar{c}$ subsystems |
| $J = 5/2$ |  |
| $\left( \frac{28}{9}C_{cc} + \frac{28}{9}C_{cb} + \frac{112}{9}C_{\bar{c}c} \right) - \frac{8}{3}C_{cc} - \frac{8}{3}C_{\bar{c}c} - \frac{4}{9}\sqrt{2}C_{cc} - \frac{4}{9}\sqrt{2}C_{\bar{c}c} = - \frac{4}{3}\sqrt{2}C_{cc} - \frac{4}{3}\sqrt{2}C_{\bar{c}c} = \frac{16}{9}\sqrt{2}(C_{cc} - C_{\bar{c}c})$ |
| $J = 3/2$ |  |
| $\left( \frac{28}{9}C_{cc} + \frac{28}{9}C_{cb} + \frac{112}{9}C_{\bar{c}c} \right) - \frac{8}{3}C_{cc} - \frac{8}{3}C_{\bar{c}c} - \frac{4}{9}\sqrt{2}C_{cc} - \frac{4}{9}\sqrt{2}C_{\bar{c}c} = - \frac{4}{3}\sqrt{2}C_{cc} - \frac{4}{3}\sqrt{2}C_{\bar{c}c} = \frac{16}{9}\sqrt{2}(C_{cc} - C_{\bar{c}c})$ |

| $J$ | The expressions of CMI Hamiltonian for $c\bar{c}bb\bar{c}$ subsystems |
| $J = 5/2$ |  |
| $\left( \frac{28}{9}C_{cc} + \frac{28}{9}C_{bb} + \frac{112}{9}C_{\bar{c}\bar{c}} \right) - \frac{8}{3}C_{cc} - \frac{8}{3}C_{\bar{c}\bar{c}} - \frac{4}{9}\sqrt{2}C_{cc} - \frac{4}{9}\sqrt{2}C_{\bar{c}\bar{c}} = - \frac{4}{3}\sqrt{2}C_{cc} - \frac{4}{3}\sqrt{2}C_{\bar{c}\bar{c}} = \frac{16}{9}\sqrt{2}(C_{cc} - C_{\bar{c}\bar{c}}) $ |
| $J = 3/2$ |  |
| $\left( \frac{28}{9}C_{cc} + \frac{28}{9}C_{bb} + \frac{112}{9}C_{\bar{c}\bar{c}} \right) - \frac{8}{3}C_{cc} - \frac{8}{3}C_{\bar{c}\bar{c}} - \frac{4}{9}\sqrt{2}C_{cc} - \frac{4}{9}\sqrt{2}C_{\bar{c}\bar{c}} = - \frac{4}{3}\sqrt{2}C_{cc} - \frac{4}{3}\sqrt{2}C_{\bar{c}\bar{c}} = \frac{16}{9}\sqrt{2}(C_{cc} - C_{\bar{c}\bar{c}}) $ |

