Hydrodynamic Modes in a Trapped Strongly Interacting Fermi Gases of Atoms
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The zero-temperature properties of a dilute two-component Fermi gas in the BCS-BEC crossover are investigated. On the basis of a generalization of the variational Schwinger method, we construct approximate semi-analytical formulae for collective frequencies of the radial and the axial breathing modes of the Fermi gas under harmonic confinement in the framework of the hydrodynamic theory. It is shown that the method gives nearly exact solutions.

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1. Introduction

The newly created ultracold trapped Fermi gases with tunable atomic scattering length [1-20] in the vicinity of a Feshbach resonance offer the possibility to study highly correlated many-body systems including the cross-over from the Bardeen-Cooper-Schrieffer (BCS) phase to the Bose-Einstein condensate (BEC) of molecules. Various investigations based on the hydrodynamic theory have appeared recently [21-31].

The purpose of this letter is to construct a simple, semi-analytical and nearly exact formulae for hydrodynamic frequencies. Since the collective frequencies can be measured with high precision, these formulae will provide a simple quantitative tool for the analysis of experimental data in the hydrodynamic regime.

2. Hydrodynamic theory

Our starting point is the quantum hydrodynamic theory [21-23] for a dilute two-component Fermi gas in a trap potential

$$ V_{ext}(\vec{r}) = \left( \frac{m}{2} \right) \left( \omega_{\perp}^2 (x^2 + y^2) + \omega_z^2 z^2 \right) $$

where

$$ V_{xc}(\vec{r}, t) = \left[ \frac{\partial}{\partial n} \left( \epsilon(n) \right) \right]_{n=\bar{n}(\vec{r}, t)}, $$

$\epsilon$ is the ground state energy per particle of the homogeneous system and $n$ is the density, $n(\vec{r}, t) = | \Psi(\vec{r}, t) |^2$, normalized to the total number of atoms, $\int n(\vec{r}, t) d^3r = N$. It is useful to rewrite Eq.(1) in a form

$$ \frac{\partial n}{\partial t} + \nabla (n\vec{v}) = 0, \quad (2) $$

$$ \frac{\partial \vec{v}}{\partial t} + \frac{1}{m} \nabla (V_{ext} + \frac{d(n\epsilon(n))}{dn}) + \frac{1}{2} m \vec{v}^2 - \frac{\hbar^2}{2m} \frac{1}{\sqrt{n}} \nabla^2 \sqrt{n} = 0, \quad (3) $$

where $\vec{v}$ is the velocity field, which for $\Psi = e^{i\phi(\vec{r}, t)} n^{1/2}(\vec{r}, t)$ can be written as $\vec{v} = (\hbar/m) \nabla \phi$.

It can be proved [21] that every solution of the Eqs.(2,3) is a stationary point corresponding to the Lagrangian density

$$ \mathcal{L}_0 = \hbar \dot{\phi} n + \frac{\hbar^2}{2m} (\nabla \sqrt{n})^2 + \frac{\hbar^2}{2m} n (\nabla \phi)^2 + \epsilon(n) n + V_{ext} n. \quad (4) $$

It was shown in Refs.[21-23] that for experimental conditions of Refs.[14,16,17] the quantum pressure term in Eqs.(3,4) can be neglected. For the reminder of this Letter we will use this
hydrodynamic approximation. For the harmonic trap a trial function in the scaling ansatz is taken as [21-23,26,28]

\[ \phi(\vec{r}, t) = \phi_0(t) + (m/(2\hbar))^3 \sum_{i=1}^{3} \beta_i(t)x_i^2, \]

\[ n(\vec{r}, t) = n_0(x_i/b_i(t))/\kappa(t), \]

where \( \kappa(t) = \prod_j b_j \) and the Hamilton principle, \( \delta \int dt \int L_0 d^3r = 0 \), gives the following equations for the scaling parameters \( b_i \) [21-22]

\[ \ddot{b}_i + \omega_i^2(t)b_i - \frac{\omega_i^2 f[n^2\epsilon(n)/dn]_{n=n_0(\vec{r})/\kappa(t)} d^3r}{f[n^2\epsilon(n)/dn]_{n=n_0(\vec{r})} d^3r} \kappa(t) = 0, \]  
\( \text{(5)} \)

Expanding Eqs.(5) around equilibrium \( (b_i = 1) \) leads to the following result for the \( M = 0 \) modes frequencies, \( \omega^{(s)} \) in the scaling approximation

\[ \omega^{(s)} = \frac{\omega_s}{\sqrt{2}} [\eta_s \pm \sqrt{\eta_s^2 - 8\lambda^2(3\zeta_s + 5)}]^{1/2}, \]  
\( \text{(6)} \)

where \( \eta_s = 4 + 2\zeta_s + 3\lambda^2 + \zeta_s\lambda^2 \), \( \zeta_s = \int n^3_0 d\epsilon/(dn^2_0) d^3r/\int n^3_0 d\epsilon/(dn_0) d^3r \), \( \lambda = \omega_z/\omega_\perp \), and \( \pm \) refer to the transverse and axial mode, respectively.

The hydrodynamic equations after linearization take the form

\[ \frac{\partial^2}{\partial t^2} \delta n + \frac{1}{m} \nabla(n_0 \nabla(\frac{d^2(n_0\epsilon(n_0))}{dn^2_0} \delta n)) = 0, \]  
\( \text{(7)} \)

where \( \delta n(\vec{r}, t) \) is the change in the density profile with respect to the equilibrium configuration. If we consider oscillations with time dependence \( \delta n \propto \exp(i\omega t) \), Eq.(7) can be reduced to a Hermitian equation [27]

\[ \omega^2[\frac{d^2(n_0\epsilon(n_0))}{dn^2_0}]^{-1}|f > = L|f > \]  
\( \text{(8)} \)

where \( |f >= \frac{d^2(n_0\epsilon(n_0))}{dn^2_0}|\delta n >, L = -\frac{1}{m} \nabla n_0 \nabla \) and the equilibrium density, \( n_0 \), is given by equation

\[ \mu = V_{ext} + \frac{d(n_0\epsilon(n_0))}{dn_0}, \]  
\( \text{(9)} \)

where \( \mu \) is the chemical potential, in the region where \( n_0(\vec{r}) \) is positive and \( n_0(\vec{r}) = 0 \) outside this region.

We note here that \( d^2(n_0\epsilon(n_0))/dn^2_0 \) is positive, since the sound velocity for the homogeneous case is given by \( c^2 = (n_0/m)d^2(n_0\epsilon(n_0))/dn^2_0 \).
3. Equation of state

For the negative S-wave scattering length between the two fermionic species, \( a < 0 \), in the low-density regime, \( k_F |a| \ll 1 \), the ground state energy per particle, \( \epsilon(n) \), is well represented by an expansion in power of \( k_F |a| \) [32]

\[
\epsilon(n) = 2E_F[\frac{3}{10} - \frac{1}{3\pi} k_F |a| + 0.055661(k_F |a|)^2 - 0.00914(k_F |a|)^3 + ...],
\]

where \( E_F = \frac{\hbar^2 k_F^2}{2m} \) and \( k_F = \frac{3\pi^{2/3}}{\sqrt{n}} \). In the opposite regime, \( a \to -\infty \) (the Bertsch many-body problem, quoted in Refs.[33]), \( \epsilon(n) \) is proportional to that of the non-interacting Fermi gas

\[
\epsilon(n) = (1 + \beta) \frac{3 \hbar^2 k_F^2}{10m},
\]

where a universal parameter \( \beta \) [10] is estimated to be \( \beta = -0.56 \) [34]. The universal limit [10,34-37] is valid at least in the case where the width of the Feshbach resonance is large compared to the Fermi energy as in the cases of \( ^6Li \) and \( ^{40}K \).

In the \( a \to +0 \) limit the system reduces to the dilute Bose gas of dimers

\[
\epsilon(n) = E_F(-1/(k_F a)^2 + a_m k_F/(6\pi) + ...),
\]

where \( a_m \) is the boson-boson scattering length, \( a_m \approx 0.6a \) [38].

A simple interpolation of the form \( \epsilon(n) \approx E_F P(k_F a) \) with a smooth function \( P(x) \) was considered in several papers. In Ref.[21] a \([2/2]\) Padé approximant has been proposed for the function \( P(x) \) for the the case of negative \( a \)

\[
P(x) = \frac{3}{5} - 2 \frac{\delta_1 |x| + \delta_2 x^2}{1 + \delta_3 |x| + \delta_4 x^2},
\]

where \( \delta_1 = 0.106103, \delta_2 = 0.187515, \delta_3 = 2.29188, \delta_4 = 1.11616 \). Eq.(13) is constructed to reproduce the first four terms of the expansion (10) in the low-density regime and also to exactly reproduce results of the recent Monte Carlo calculations [34], \( \beta = -0.56 \), in the unitary limit, \( k_F a \to -\infty \).

For the positive \( a \) case (the interaction is strong enough to form bound molecules with energy \( E_{mol} \)) we have considered in Ref.[22] a \([2/2]\) Padé approximant

\[
P(x) = \frac{E_{mol}}{2E_F} + \frac{\alpha_1 x + \alpha_2 x^2}{1 + \alpha_3 x + \alpha_4 x^2},
\]

where parameters \( \alpha_i \) are fixed by two continuity conditions at large \( x \), \( 1/x \to 0 \), and by two continuity conditions at small \( x \), \( \alpha_1 = 0.0316621, \alpha_2 = 0.0111816, \alpha_3 = 0.200149, \) and \( \alpha_4 = 0.0423545 \).
In Ref. [39] a Padé approximation has been considered for the chemical potential. Authors of Ref. [28] have used a model for $P(x)$, interpolating the Monte Carlo results of Ref. [36] across the unitary limit and limiting behaviors for small $|x|$. We note here also the BCS mean-field calculations of Ref. [28].

4. Schwinger variational principle

The Schwinger variational principle (SVP) [40-43] can be generalized to the case of Eq. (8).

Since the entire treatment is based on the equivalence of the SVP and the method of separable representation, we briefly describe this method. Let us consider the symbolic identity

$$L = LL^{-1}L = \sum_{i,j} L|i\rangle < i|L^{-1}|j\rangle < j|L,$$

where $|i\rangle$ is a complete set. Truncating the summation over the complete set we obtain a separable approximation

$$L(q) = \sum_{i,j} L|\chi_i > d_{ij}^{-1} < \chi_j|L,$$

where $d_{ij} = < \chi_i|L|\chi_j >$.

We note that Eq. (16) represents an interpolation process, since $L(q)|\chi_i >= L|\chi_i >$ and $< \chi_j|L(q) = < \chi_j|L$.

Substituting $L(q)$ from Eq. (16) into Eq. (8), we obtain

$$\omega^2[\frac{d^2(n_0\epsilon(n_0))}{dn_0^2}]^{-1}|f >= L(q)|f >= \sum_{i,j} L|\chi_i > d_{ij}^{-1} < \chi_j|L|f >.$$

We seek a solution of Eq. (17) in the form

$$|f >= \sum_{i} c_i \frac{d^2(n_0\epsilon(n_0))}{dn_0^2} L|\chi_i >,$$

then $c_i$’s are defined from equations

$$\sum_{k=1}^{q} B_{ik}(\omega^2)c_k = 0,$$

where

$$B_{ik}(\omega^2) = < \chi_i|(\omega^2L - L\frac{d^2(n_0\epsilon(n_0))}{dn_0^2})L|\chi_k >$$

and frequencies $\omega$ are determined from the condition of vanishing of the determinant of the matrix $B_{ik}(\omega^2)$.

$$detB_{ik}(\omega^2) = 0$$
Defining the Schwinger functional $I_{SV P}[\chi]$ by

$$I_{SV P}[\chi] = \frac{\langle \chi | L | \chi \rangle}{\langle \chi | L | \chi \rangle},$$

we get $I_{SV P}[f] = \omega^2$, where $f$ is the solution of Eq.(8). Introducing the function $\chi = \sum_{r=1}^{q} c_r \chi_r$, where $c_1, c_2, \ldots c_q$ are $q$ variable parameters, we see that the functional $I_{SV P}$ is stationary if $\partial I_{SV P}/\partial c_r = 0$. The later equations coincide with Eqs.(19), and therefore the approximate solution of the Eq.(8), which is based on the SVP, is equivalent to the exact solution of the Eq.(8) with separable $L^{(q)}$.

Operator $L$ is clearly positive, which means that $\langle u | L | u \rangle \geq 0$ for all $u$, but is not positive definite, since $\langle u | L | u \rangle = 0$ for some $u \neq 0$. It can easily be seen that

$$\langle u | (L - L^{(q)}) | u \rangle \geq 0,$$

for all $u$. Indeed,

$$J = \langle (u + \sum_{i=1}^{q} c_i \chi_i) | L | (u + \sum_{j=1}^{q} c_j \chi_j) \rangle \geq 0$$

for all $c_i$. We choose the $c_i$’s from the conditions $\partial J/\partial c_i = 0$, and then obtain

$$J = \langle u | (L - L^{(q)}) | u \rangle \geq 0$$

for all $q$.

Since the problem is solved by replacing $L$ by $L^{(q)}$ and since the operator $(L - L^{(q)})$ is positive, the SVP leads to the approximate lower bounds for $\omega$ up to the second order of $|L - L^{(q)}|$.

For the most interested case of $M = 0$ modes, we can put in Eq.(16) $q = 2$, $\chi_1 = (x^2 + y^2)$ and $\chi_2 = z^2$, which give

$$\omega^{SV P}_{\pm} = \frac{\omega_{\pm}}{\sqrt{2}} \left[ \eta_{SV P} \pm \sqrt{\eta_{SV P}^2 - 8\lambda^2(9\zeta_{SV P} - 1)} \right]^{1/2},$$

(24)

where $\eta_{SV P} = 6\zeta_{SV P} + \lambda^2(3\zeta_{SV P} + 1)$, $\zeta_{SV P} = f n_0^2 \tilde{x}^3 \frac{\partial}{\partial \tilde{x}} [-1] d\tilde{x} / f n_0 \tilde{x}^4 d\tilde{x}$, with $\tilde{x} = \sqrt{x^2 + y^2 + \lambda^2 z^2}$ and $\pm$ signs refer to the transverse and axial mode, respectively.

It is easy to show that Eq. (24) gives exact solutions for frequencies of the breathing modes for the polytropic equation of state, $\epsilon(n) \approx n^\gamma$. In Ref.[24], on the basis of a generalization of the Hylleraas-Undheim method, we have constructed rigorous upper bounds to
the collective frequencies for the radial and the axial breathing mode of the Fermi gas under harmonic confinement in the framework of the hydrodynamic theory

\[
\omega_{\pm}^{\text{upper}} = \frac{\omega_\perp}{\sqrt{5\zeta_{\text{upper}} - 9}} \left[ \eta_{\text{upper}} \pm \sqrt{\eta_{\text{upper}}^2 - 8\lambda^2\zeta_{\text{upper}}(5\zeta_{\text{upper}} - 9)} \right]^{1/2},
\]

where \( \eta_{\text{upper}} = (3 + 4\lambda^2)\zeta_{\text{upper}} - (3 + 6\lambda^2) \), \( \zeta_{\text{upper}} = I_0I_4/I_2^2 \), and \( I_I = \int \tilde{x}^In_0(\tilde{x})d\tilde{x} \).

We expect that the difference between \( \omega_{\pm}^{\text{SVP}} \) and \( \omega_{\pm}^{\text{upper}} \) characterizes the error and is not very sensitive to the functional form of \( \epsilon(n) \). From Table 1 one can see that this difference is order of \( 10^{-5} \). We note that the comparison with the scaling approximation, Eq.(6), shows that the absolute precision of the scaling approximation is about \( 10^{-3} \) that agrees with Ref.[31].

In Fig.1, we have compared the hydrodynamic predictions for \( \omega_+ \) with experimental data [17]. There is a very good agreement with experimental data [17] near the unitary limit. We note here that two experimental results [17] and [16] (not shown in Fig.1) for \( \omega_+ \) are still about 10% in disagreement with each other, which is not fully understood yet.

To calculate \( \zeta \)'s, we have used the very fast converged expansion of Ref.[24]

\[
n_0(\vec{r}) \approx (1 - \beta V_{ext}(\vec{r}))^{1/(2-p)} \sum_{i=0}^{l-1} c_i[V_{ext}(\vec{r})]^i,
\]

where parameters \( \beta, p \) and \( c_i \) are fixed by requiring that \( n_0(\vec{r}) \) must satisfy a variational principle \( \delta \int n_0(V_{ext} + \epsilon(n_0))d^3r = 0 \) with a subsidiary condition \( \int n_0d^3r = N \).

4. Summary

We have generalized the Schwinger variational method for the trapped strongly interacting atoms in hydrodynamic regime and we have constructed semi-analytical and extremely accurate formulae for hydrodynamic collective frequencies. These formulae are very useful since they provide an easy and simple quantitative tool for the analysis of experimental data for trapped condensed gases without relying on complex and extensive computations.
Table 1. The transverse and axial frequencies in units of $\omega_{\perp}$ and $\omega_z$, respectively, in the BCS region as a function of the dimensional parameter $X = (N^{1/6}a/a_{ho})^{-1}$. The trap parameter $\lambda$ is assumed to be 0.045613. The [2/2] Padé approximation of Refs.[21,22] is used for the energy per particle $\epsilon(n)$.

| $X$ | $\omega_{\perp}^{upper}$ | $\omega_{\perp}^{SVP}$ | $\omega_{\perp}^{lower}$ | $\omega_{\perp}^{SVP}$ | $\omega_{\mp}$ | $\omega_{\pm}$ |
|-----|----------------|----------------|----------------|----------------|-------------|-------------|
| -0.1 | 1.8160 | 1.8160 | 1.5470 | 1.5470 | 1.8193 | 1.5477 |
| -0.3 | 1.8015 | 1.8015 | 1.5438 | 1.5438 | 1.8082 | 1.5453 |
| -0.5 | 1.7931 | 1.7931 | 1.5419 | 1.5419 | 1.8002 | 1.5435 |
| -0.7 | 1.7886 | 1.7886 | 1.5409 | 1.5409 | 1.7947 | 1.5423 |
| -0.9 | 1.7867 | 1.7867 | 1.5405 | 1.5405 | 1.7910 | 1.5414 |
| -1.1 | 1.7861 | 1.7861 | 1.5403 | 1.5403 | 1.7886 | 1.5409 |
| -1.3 | 1.7865 | 1.7865 | 1.5404 | 1.5404 | 1.7871 | 1.5406 |
| -1.5 | 1.7873 | 1.7873 | 1.5406 | 1.5406 | 1.7863 | 1.5404 |
| -1.7 | 1.7884 | 1.7884 | 1.5409 | 1.5409 | 1.7860 | 1.5403 |
| -2.0 | 1.7902 | 1.7902 | 1.5413 | 1.5413 | 1.7861 | 1.5403 |
Fig. 1. Radial breathing mode frequency ω+ in the BCS region as a function of the dimensional parameter $X = (N^{1/6}a/\alpha_h)^{-1}$ (the solid line). The dashed line represents the scaling approximation, Eq.(6). The circular dots with error bars are the experimental results given by the Duke University group [17]. Everything is measured in units of $\omega_\perp$. 
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