Solving $\overline{\partial}$ with prescribed support on Hartogs triangles in $\mathbb{C}^2$ and $\mathbb{C}P^2$

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In this paper we consider the problem of solving the Cauchy-Riemann equation with prescribed support. More precisely, let $X$ be a complex manifold of complex dimension $n$ and $\Omega \subset X$ a subdomain of $X$. We ask the following questions:

Let $T$ be a $\overline{\partial}$-closed $(r,1)$-current, $0 \leq r \leq n$, on $X$ with support contained in $\overline{\Omega}$, does there exist a $(r,0)$-current on $X$, with support contained in $\overline{\Omega}$, such that $\overline{\partial}S = T$ ?

If moreover $T = f$ is a smooth form or a $C^k$ form or an $L^p_{\text{loc}}$ form, can we find $g$ with support contained in $\overline{\Omega}$ and with the same regularity as $f$ such that $\overline{\partial}g = f$ ?

This leads us to introduce the Dolbeault cohomology groups with prescribed support in $\Omega$. Let us denote by $H^{r,1}_{\overline{\Omega},\infty}(X)$ the quotient space

$$\left\{ f \in C^\infty_{r,1}(X) \mid \overline{\partial}f = 0, \text{supp} \, f \subset \overline{\Omega} \right\}/\overline{\partial}\left\{ f \in C^\infty_{r,0}(X) \mid \text{supp} \, f \subset \overline{\Omega} \right\}.$$ 

In the same way, we define $H^{r,1}_{\overline{\Omega},C^k}(X), H^{r,1}_{\overline{\Omega},L^p_{\text{loc}}}(X)$ and $H^{r,1}_{\overline{\Omega},\text{cur}}(X)$ for the $C^k$, $L^p_{\text{loc}}$ and the current category.

The cohomology groups $H^{r,1}_{\overline{\Omega},\infty}(X), H^{r,1}_{\overline{\Omega},C^k}(X), H^{r,1}_{\overline{\Omega},L^p_{\text{loc}}}(X)$ and $H^{r,1}_{\overline{\Omega},\text{cur}}(X)$ describe the obstruction to solve the Cauchy-Riemann equation with prescribed support in $\overline{\Omega}$, respectively in the smooth or $C^k$ or $L^p_{\text{loc}}$ or current category. Their vanishing is equivalent to the solvability of the Cauchy-Riemann equation with prescribed support in $\overline{\Omega}$ in the corresponding category (see section 2 in [11] and [10]).

Note that, if $\Omega$ is a relatively compact domain with Lipschitz boundary, by the Serre duality, the properties of the groups $H^{r,1}_{\overline{\Omega},\infty}(X), H^{r,1}_{\overline{\Omega},L^p_{\text{loc}}}(X)$ and $H^{r,1}_{\overline{\Omega},\text{cur}}(X)$ are directly related to the properties of the Dolbeault cohomology groups $\check{H}^{n-r,n-1}(\Omega), \check{H}^{n-r,n-1}_{L^p_{\text{loc}}}(\Omega)$ of Dolbeault cohomology for extendable currents, $L^p$ forms and of smooth forms up to the boundary.

If $\Phi$ is a family of supports in the complex manifold $X$, for example the family, usually denoted by $c$, of all compact subsets of $X$, we can consider the Dolbeault cohomology with support in $\Phi$. The group $H^{r,q}_{\Phi,\infty}(X)$ is the quotient of the space of $\overline{\partial}$-closed, smooth $(r,q)$-forms on $X$ with support in the family $\Phi$ by the range by $\overline{\partial}$ of the space of smooth $(r,q-1)$-forms on $X$ with support in the family $\Phi$. Similarly we can define the groups

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$H^{r,q}_{\phi,C_k}(X), H^{r,q}_{\phi,L^p_{loc}}(X)$ and $H^{r,q}_{\phi,cur}(X)$. It follows from Corollary 2.15 in [7] and Proposition 1.2 in [10], that the Dolbeault isomorphism holds for the Dolbeault cohomology with prescribed support. This means that all these groups are isomorphic and we denote them by $H^{p,q}_\phi(X)$. In this paper, we will show that such Dolbeault isomorphism no longer holds when we change the condition supported in a family of sets in $X$ to prescribed support. For Dolbeault cohomology groups with prescribed support, the following proposition is proved in Proposition 2.

**Proposition 0.1.** Let $X$ be a complex manifold and $\Omega \subset X$ a domain in $X$. For any integer $0 \leq r \leq \dim \mathbb{C}X$, the natural morphisms from $H^{r,1}_{\Omega,\infty}(X)$ (resp. $H^{r,1}_{\Omega,C_k}(X)$, $k \geq 0$, $H^{r,1}_{\Omega,L^p}(X)$, $1 \leq p \leq +\infty$) into $H^{r,1}_{\Omega,cur}(X)$ are injective. In particular, if $H^{r,1}_{\Omega,cur}(X) = 0$, then $H^{r,1}_{\Omega,\infty}(X) = 0$, $H^{r,1}_{\Omega,C_k}(X) = 0$, $k \geq 0$, and $H^{r,1}_{\Omega,L^p}(X) = 0$.

When $\Omega$ is a Hartogs triangle type set in $\mathbb{C}^2$ or $\mathbb{C}P^2$, we show that the Dolbeault isomorphisms fail to hold for the cohomology with prescribed support. When $\Omega$ is an unbounded Hartogs triangle in $\mathbb{C}^2$, we get

**Theorem 0.2.** If $X = \mathbb{C}^2$ and $\Omega = \{(z, w) \in \mathbb{C}^2 \mid |z| > |w|\}$, then $H^{0,1}_{\Omega,\infty}(X) = 0$, but $H^{0,1}_{\Omega,C_k}(X) \neq 0$, $k \geq 0$, $H^{0,1}_{\Omega,cur}(X) \neq 0$ and $H^{0,1}_{\Omega,L^2}(X) \neq 0$.

In the case when $\Omega$ is a Hartogs triangle in $\mathbb{C}P^2$, we prove

**Theorem 0.3.** If $X = \mathbb{C}P^2$ and $\Omega = \{[z_0, z_1, z_2] \in \mathbb{C}P^2 \mid |z_1| > |z_2|\}$, then $H^{0,1}_{\Omega,\infty}(X) = 0$ and $H^{0,1}_{\Omega,C_k}(X) = 0$, $k \geq 0$, but $H^{0,1}_{\Omega,cur}(X)$ and $H^{0,1}_{\Omega,L^2}(X) \neq 0$ are infinite dimensional and Hausdorff.

The non-vanishing of $H^{0,1}_{\Omega,L^2}(\mathbb{C}P^2)$ is especially interesting since it is in sharp contrast to the case of solving $\overline{\partial}$ with compact support for a bounded Hartogs triangle in $\mathbb{C}^2$ (see Remark 1 at the end of the paper). The infinite dimensionality of $H^{0,1}_{\Omega,L^2}(\mathbb{C}P^2)$ gives the following result. Let $\overline{\partial}_s$ be the strong $L^2$ closure $\overline{\partial}_s : L^2_{2,0}(\Omega) \to L^2_{2,1}(\Omega)$, i.e., the completion of $\overline{\partial}$ on smooth forms up to the boundary in the graph norm. Let $H^{2,1}_{\overline{\partial}_s,L^2}(\Omega)$ be the quotient of the kernel of $\overline{\partial}_s$ over the range of $\overline{\partial}_s$, i.e. the Dolbeault cohomology with respect to the operator $\overline{\partial}_s$.

**Corollary 0.4.** The space $H^{2,1}_{\overline{\partial}_s,L^2}(\Omega)$ is infinite dimensional.

It is not known if $\overline{\partial}_s$ agrees with the weak $L^2$ extension or if the range of $\overline{\partial}_s$ is closed. If the domain $\Omega$ is bounded and Lipschitz, then the weak and strong closure are the same from the Friedrichs’ lemma. The Hartogs triangle is a candidate that the weak and strong closure of $\overline{\partial}$ might not be the same.

The vanishing of the Dolbeault cohomology groups with prescribed support in $\overline{\Omega}$ in bidegree $(0,1)$ is directly related to the extension of holomorphic functions defined on the complement of $\Omega$. This implies the following result:
Proposition 0.5. Let $X$ be a complex manifold and $\Omega \subset X$ a domain in $X$. Assume $H^{0,1}_{\Omega,\infty}(X) = 0$, then $X \setminus \Omega$ is connected. If moreover $X$ is not compact, $H^{0,1}_{c}(X) = 0$ and $\Omega$ is relatively compact, then $H^{0,1}_{\Omega,\infty}(X) = 0$ if and only if $X \setminus \Omega$ is connected.

We also prove some characterization of pseudoconvexity in terms of Dolbeault cohomology with prescribed support.

Theorem 0.6. Let $D$ be a bounded domain in $\mathbb{C}^2$ with Lipschitz boundary. Then the following assertions are equivalent:

(i) $D$ is a pseudoconvex domain;

(ii) $H^{0,1}(D,\mathbb{C}^2) = 0$ and $H^{0,2}(D,\mathbb{C}^2)$ is Hausdorff.

The plan of this paper is as follows: In section 1, we recall some basic properties of the support and the uniqueness of the solution for $\overline{\partial}$. In section 2 we discuss solving $\overline{\partial}$ with prescribed support and its relations with the holomorphic extension of functions in various function spaces. In section 3, we study the non-vanishing of Dolbeault cohomology with prescribed support on the unbounded Hartogs triangle in $\mathbb{C}^2$. We analyse the Hartogs triangles in $\mathbb{C}P^2$ in section 4. Theorems 0.2 and 0.3 provide interesting examples which give the non-vanishing for the Dolbeault cohomology groups. This is in sharp contrast with the well-known results of solving $\overline{\partial}$ for $(0,1)$-forms with prescribed support for bounded domain in $\mathbb{C}^n$. We prove Corollary 0.4 using $L^2$ Serre duality. This gives us some insight about the intriguing problem on weak and strong extension of the $\overline{\partial}$ operator in the $L^2$ sense, when the domain is not Lipschitz. The unbounded Hartogs domain in $\mathbb{C}^2$ or Hartogs domains in $\mathbb{C}P^2$ provide us with new unexpected phenomena. Many open questions and remarks are given at the end of the paper.

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1 Properties of the support and uniqueness of the solution

Let $X$ be a complex manifold of complex dimension $n$ and $T$ be a $\overline{\partial}$-exact $(0,1)$-current on $X$. We will describe some relations between the support of the current $T$ and the support of the solution $S$ of the Cauchy-Riemann equation $\overline{\partial}S = T$.

Proposition 1.1. Let $X$ be a complex manifold of complex dimension $n$ and $T$ be a $\overline{\partial}$-exact $(0,1)$-current on $X$. If $\Omega^c$ denotes a connected component of $X \setminus \text{supp } T$ and if $S$ is a distribution on $X$ such that $\overline{\partial}S = T$, then either $\text{supp } S \cap \Omega^c = \emptyset$ or $\Omega^c \subset \text{supp } S$.

Proof. Note that, since $\overline{\partial}S = T$, $S$ is a holomorphic function on $X \setminus \text{supp } T$ and in particular on the connected set $\Omega^c$. Assume that the support of $S$ does not contain $\Omega^c$, then $S$ vanishes on an open subset of $\Omega^c$ and by analytic continuation $S$ vanishes on $\Omega^c$, which means that $\supp S \cap \Omega^c = \emptyset$. \qed
Corollary 1.2. Let $X$ be a complex manifold of complex dimension $n$ and $T$ be a $\overline{\partial}$-exact $(0,1)$-current on $X$. Assume that $X \setminus \text{supp } T$ is connected, then if $S$ is a distribution on $X$ such that $\overline{\partial}S = T$, then either $\text{supp } S = \text{supp } T$ or $\text{supp } S = X$.

Proof. The support of $T$ is always contained in the support of $S$. If $\text{supp } S \neq X$, then the other inclusion holds by Proposition 1.1 since $X \setminus \text{supp } T$ is connected. \hfill \Box

Note that the difference between two solutions of the equation $\overline{\partial}S = T$ is a holomorphic function on $X$. Then analytic continuation implies the following uniqueness result.

Proposition 1.3. Assume that the complex manifold $X$ is connected. Let $T$ be a $\overline{\partial}$-exact $(0,1)$-current on $X$ such that $X \setminus \text{supp } T \neq \emptyset$ and $S$ and $U$ two distributions such that

$$\overline{\partial}S = \overline{\partial}U = T$$

and the support of $S$ and the support of $U$ do not intersect on the same connected component $\Omega^{-}_p$ of $X \setminus \text{supp } T$, then $S = U$.

In particular, the equation $\overline{\partial}S = T$ admits at most one solution $S$ such that $\text{supp } S = \text{supp } T$.

Remark 1.4. The equation $\overline{\partial}S = T$ may have no solution $S$ with $\text{supp } S = \text{supp } T$. Consider for example a relatively compact domain $D$ with $C^\infty$-smooth boundary in a complex manifold $X$ and a function $F \in C^\infty(D)$ which is holomorphic in $D$. Denote by $f$ the restriction of $F$ to the boundary of $D$ and set $S = F\chi_D$, where $\chi_D$ is the characteristic function of the domain $D$. Then, by the Stokes formula, $\overline{\partial}S = f[\partial\overline{\partial}]^{0,1}$, where $[\partial\overline{\partial}]^{0,1}$ is the part of bidegree $(0,1)$ of the integration current over the boundary of $D$. Clearly the support of $T = f[\partial\overline{\partial}]^{0,1}$ is the boundary of $D$, but, by Proposition 1.3, $S$ is the unique solution of $\overline{\partial}S = T$ whose support is contained in $\partial D$. So there is no solution whose support is equal to the support of $T$.

Let us end this section by considering the regularity of the solutions.

Proposition 1.5. Let $X$ be a complex manifold and $f$ a $(0,1)$-form with coefficients in $\mathcal{C}^k(X)$, $0 \leq k \leq +\infty$ (resp. $L^p_{loc}(X)$, $1 \leq p \leq +\infty$), which is $\overline{\partial}$-exact in the sense of currents. Then any solution $g$ of the equation $\overline{\partial}g = f$ is in $\mathcal{C}^k(X)$, $0 \leq k \leq +\infty$ (resp. $L^p_{loc}(X)$, $1 \leq p \leq +\infty$).

Proof. By the regularity of the Cauchy-Riemann operator (injectivity of the Dolbeault isomorphism §11 and §13), if $f$ has coefficients in $\mathcal{C}^k(X)$, $0 \leq k \leq +\infty$ (resp. $L^p_{loc}(X)$, $1 \leq p \leq +\infty$), then, since $f$ is $\overline{\partial}$-exact in the sense of currents, the equation $\overline{\partial}S = f$ has a solution in $\mathcal{C}^k(X)$, $0 \leq k \leq +\infty$ (resp. $L^p_{loc}(X)$, $1 \leq p \leq +\infty$). The difference between two solutions of the equation $\overline{\partial}S = f$ being a holomorphic function on $X$, all the solutions have the same regularity. \hfill \Box

Associating Proposition 1.3 and Proposition 1.5 we get:

Corollary 1.6. Assume that the complex manifold $X$ is connected. If $f$ is a $(0,1)$-form such that $X \setminus \text{supp } f \neq \emptyset$, then the equation $\overline{\partial}g = f$ has at most one unique solution such that $\text{supp } g = \text{supp } f$ and this solution has the same regularity as $f$. 

2 Solving $\overline{\partial}$ with prescribed support

Let $X$ be a connected, complex manifold and $\Omega$ a domain such that $\overline{\Omega}$ is strictly contained in $X$ and the interior of $\overline{\Omega}$ coincides with $\Omega$. We set $\Omega^c = X \setminus \overline{\Omega}$, it is a non-empty open subset of $X$.

Let us denote by $H^{0,1}_{\Omega,\infty}(X)$ (resp. $H^{0,1}_{\Omega,cur}(X)$, $H^{0,1}_{\Omega,C^k}(X)$, $H^{0,1}_{\Omega,L^p_{loc}}(X)$) the Dolbeault cohomology group of bidegree $(0,1)$ for smooth forms (resp. currents, $C^k$-forms, $k \geq 0$, $L^p_{loc}$-forms, $1 \leq p \leq +\infty$) with support in $\overline{\Omega}$. The vanishing of these groups means that one can solve the $\overline{\partial}$ equation with prescribed support in $\overline{\Omega}$ in the smooth category (resp. the space of currents, the space of $C^k$-forms, the space of $L^p_{loc}$-forms).

It follows from Proposition 2.1, Proposition 2.2 and from the Dolbeault isomorphism with support conditions (Corollary 2.15 in \cite{7} and Proposition 1.2 in \cite{10}) that

**Proposition 2.1.** The natural morphisms from $H^{0,1}_{\Omega,\infty}(X)$ (resp. $H^{0,1}_{\Omega,cur}(X)$, $H^{0,1}_{\Omega,C^k}(X)$, $H^{0,1}_{\Omega,L^p_{loc}}(X)$, $1 \leq p \leq +\infty$) into $H^{0,1}_{\Omega,cur}(X)$ are injective. In particular, if $H^{0,1}_{\Omega,cur}(X) = 0$, then $H^{0,1}_{\Omega,\infty}(X) = 0$, $H^{0,1}_{\Omega,C^k}(X) = 0$ and $H^{0,1}_{\Omega,L^p_{loc}}(X) = 0$.

In the next sections, examples are given proving that there exist domains in $\mathbb{C}^2$ and $\mathbb{C}P^2$ such that $H^{0,1}_{\Omega,\infty}(X) = 0$, but $H^{0,1}_{\Omega,cur}(X) \neq 0$.

We will now consider the link between the vanishing of the group $H^{0,1}_{\Omega,cur}(X)$ and the extension properties of some holomorphic functions in $\Omega^c$.

**Proposition 2.2.** Assume $H^{0,1}_{\Omega,cur}(X) = 0$, then any holomorphic function on $\Omega^c = X \setminus \overline{\Omega}$, which is the restriction to $\Omega^c$ of a distribution on $X$, extends as a holomorphic function to $X$.

**Proof.** Let $f \in \mathcal{O}(\Omega^c)$ and $S_f \in \mathcal{D}'(X)$ a distribution such that $S_f|_{\Omega^c} = f$. Consider the $(0,1)$-current $\overline{\partial}S_f$, it is closed and has support in $\overline{\Omega}$. Since $H^{0,1}_{\Omega,cur}(X) = 0$, there exists $U \in \mathcal{D}'(X)$, with support in $\overline{\Omega}$ such that $\overline{\partial}U = \overline{\partial}S_f$ in $X$. Set $h = S_f - U$, it is a holomorphic function on $X$ and $h|_{\Omega^c} = S_f|_{\Omega^c} = f$. \( \square \)

In the same way, we can prove

**Proposition 2.3.** Assume $H^{0,1}_{\Omega,L^p_{loc}}(X) = 0$, $p \geq 1$, then any holomorphic function on $\Omega^c = X \setminus \overline{\Omega}$, which is the restriction to $\Omega^c$ of a form with coefficients in $W^{1,p}_{loc}(X)$, extends as a holomorphic function to $X$.

**Proposition 2.4.** Assume $H^{0,1}_{\Omega,C^k}(X) = 0$, $k \geq 0$, then any holomorphic function on $\Omega^c = X \setminus \overline{\Omega}$, which is of class $C^{k+1}$ on $X \setminus \Omega = \Omega^c$, extends as a holomorphic function to $X$.

**Proposition 2.5.** Assume $H^{0,1}_{\Omega,\infty}(X) = 0$, then any holomorphic function on $\Omega^c = X \setminus \overline{\Omega}$, which is smooth on $X \setminus \Omega = \Omega^c$, extends as a holomorphic function to $X$.

**Corollary 2.6.** Assume $H^{0,1}_{\Omega,\infty}(X) = 0$, then $\Omega^c = X \setminus \overline{\Omega}$ is connected.
Proof. Assume $\Omega^c$ is not connected. Let $f$ be a holomorphic function which is constant equal to 1 in one connected component of $\Omega^c$ and vanishes identically on all the other ones. By analytic continuation $f$ cannot be the restriction to $\Omega^c$ of a holomorphic function on $X$, and by Proposition 2.5 we get $H^{0,1}_{\Omega,\infty}(X) \neq 0$. \hfill $\square$

Remark 2.7. Note that, by Proposition 1.1 \( H^{0,1}_{\Omega \text{cur}}(X) \neq 0 \) if and only if there exists at least one $\overline{\partial}$-exact $(0,1)$-current $T$ with support contained in $\overline{\Omega}$ such that the support of each solution of the equation $\overline{\partial}S = T$ contains at least a connected component of $\Omega^c$.

Let us give a partial converse to Corollary 2.6. Let $H^{0,1}_c(X)$ denote the Dolbeault cohomology group for $(0,1)$-forms with compact support in $X$.

**Proposition 2.8.** Assume $\Omega$ is relatively compact in a non-compact complex manifold $X$ such that $H^{0,1}_c(X) = 0$. If $\Omega^c = X \setminus \overline{\Omega}$ is connected, then

$$H^{0,1}_{\Omega,\text{cur}}(X) = H^{0,1}_{\Omega,\infty}(X) = H^{0,1}_{\Omega,\text{cur}}(X) = H^{0,1}_{\Omega,\text{loc}}(X) = 0.$$  

**Proof.** By Proposition 1.3 it suffices to prove that $H^{0,1}_{\Omega,\text{cur}}(X) = 0$. This vanishing result follows directly from Proposition 1.1. More precisely, if $T$ is a $\overline{\partial}$-closed current on $X$ with support contained in $\overline{\Omega}$, there exists a distribution $S$, with compact support such that $\overline{\partial}S = T$, since $H^{0,1}_c(X) = 0$. Then the support of $S$ cannot contain the connected set $\Omega^c$, otherwise $X = \overline{\Omega} \cup \text{supp} \ S$ would be compact, and hence $\text{supp} \ S$ is contained in $\overline{\Omega}$. \hfill $\square$

In particular, if $X$ is a Stein manifold with $\dim_{\mathbb{C}} X \geq 2$ and $\Omega$ a relatively compact domain in $X$, then

$$H^{0,1}_{\Omega,\text{cur}}(X) = H^{0,1}_{\Omega,\infty}(X) = H^{0,1}_{\Omega,\text{cur}}(X) = H^{0,1}_{\Omega,\text{loc}}(X) = 0 \iff \Omega^c \text{ is connected.}$$

An immediate corollary of Proposition 2.8 and Proposition 2.2 is the following:

**Corollary 2.9.** Let $X$ be a non-compact, connected complex manifold such that $H^{0,1}_c(X) = 0$, and $\Omega$ a relatively compact, open subset of $X$ with connected complement, then any holomorphic function on $\Omega^c$ extends as a holomorphic function to $X$.

**Proof.** It is sufficient to apply Proposition 2.8 and Proposition 2.2 to a neighborhood $D$ of $\overline{\Omega}$ with connected complement and to conclude by analytic continuation. \hfill $\square$

Corollary 2.9 is the classical Hartogs extension phenomenon. Note that all the previous results remain true if we replace the family of all compact subsets of a non-compact manifold by any family $\Phi$ of supports in a manifold $X$, different from the family of all closed subsets of $X$ (see e.g. [14] for the definition of a family of supports).

**Proposition 2.10.** Assume the complex manifold $X$ satisfies $H^{0,1}(X) = 0$. If any holomorphic function on $\Omega^c$, which is smooth on $X \setminus \Omega = \overline{\Omega}$, extends as a holomorphic function to $X$, then $H^{0,1}_{\Omega,\infty}(X) = 0$.

**Proof.** Let $f$ be a smooth $\overline{\partial}$-closed form in $X$ with support contained in $\overline{\Omega}$. Since $H^{0,1}(X) = 0$, there exists a function $g \in \mathcal{C}^\infty(X)$ such that $\overline{\partial}g = f$. Since the support of $f$ is contained in $\overline{\Omega}$, $g$ is holomorphic in $\Omega^c$ and by the extension property it extends as a holomorphic function $\tilde{g}$ to $X$. Set $h = g - \tilde{g}$, then the support of $h$ is contained in $\overline{\Omega}$ and $\overline{\partial}h = f$. \hfill $\square$
Similarly, since $H^{0,1}(X) = H^{0,1}_{\mathcal{C}^k}(X) = H^{0,1}_{\mathcal{L}_{loc}}(X) = H^{0,1}_{\mathcal{F}_{cur}}(X) = 0$ by the Dolbeault isomorphism, we have

**Proposition 2.11.** Assume the complex manifold $X$ satisfies $H^{0,1}(X) = 0$. If any holomorphic function on $\Omega^c$, which is of class $\mathcal{C}^k$, $k \geq 0$ on $X \setminus \Omega = \overline{\Omega}$, extends as a holomorphic function to $X$, then $H^{0,1}_{\mathcal{H}_{\mathcal{C}^k}}(X) = 0$.

**Proposition 2.12.** Assume the complex manifold $X$ satisfies $H^{0,1}(X) = 0$. If any holomorphic function on $\Omega^c = X \setminus \overline{\Omega}$, which is the restriction to $\Omega^c$ of a function $L_{loc}^p(X)$, $p \geq 1$, extends as a holomorphic function to $X$, then $H^{0,1}_{\mathcal{H}_{\mathcal{L}_{loc}}^p}(X) = 0$.

**Proposition 2.13.** Assume the complex manifold $X$ satisfies $H^{0,1}(X) = 0$. If any holomorphic function on $\Omega^c = X \setminus \overline{\Omega}$, which is the restriction to $\Omega^c$ of a distribution on $X$, extends as a holomorphic function to $X$, then $H^{0,1}_{\mathcal{H}_{\mathcal{F}_{cur}}}(X) = 0$.

Let us end this section by a characterization of pseudoconvexity in $\mathbb{C}^2$ by means of the Dolbeault cohomology with prescribed support.

**Theorem 2.14.** Let $D$ be a bounded domain in $\mathbb{C}^2$ with Lipschitz boundary. Then the following assertions are equivalent:

(i) $D$ is a pseudoconvex domain;
(ii) $H^{0,1}_{\mathcal{H}_{\mathcal{D}}}(\mathbb{C}^2) = 0$ and $H^{0,2}_{\mathcal{H}_{\mathcal{D}}}(\mathbb{C}^2)$ is Hausdorff.

**Proof.** By Serre duality (3 or Theorem 2.7 in [11]) assertion (ii) implies that $\tilde{H}^2,0(D)$ is Hausdorff, for all $1 \leq q \leq 2$ and moreover $\tilde{H}^{2,1}(D) = 0$ as the dual space to $H^{0,1}_{\mathcal{D}}(\mathbb{C}^2)$.

Let us prove now that the condition $\tilde{H}^{2,1}(D) = 0$ implies that $D$ is pseudoconvex. We will follow the methods used by Laufer [9] for the usual Dolbeault cohomology and prove by contradiction.

Assume $D$ is not pseudoconvex, then there exists a domain $\tilde{D}$ strictly containing $D$ such that any holomorphic function on $D$ extends holomorphically to $\tilde{D}$. Since $\text{interior}(\overline{\tilde{D}}) = D$, after a translation and a rotation we may assume that $0 \in \tilde{D} \setminus \overline{\tilde{D}}$ and there exists a point $z_0$ in the intersection of the plane $\{(z_1, z_2) \in \mathbb{C}^2 \mid z_1 = 0\}$ with $\tilde{D}$, which belongs to the same connected component of the intersection of that plane with $\tilde{D}$.

Let us denote by $B(z_1, z_2)$ the $(0, 1)$-form defined by

$$B(z_1, z_2) = \frac{\overline{z}_1}{|z|^4} dz_2 - \frac{\overline{z}_2}{|z|^4} dz_1 \wedge dz_1 \wedge dz_2.$$  

It is derived from the Bochner-Martinelli kernel in $\mathbb{C}^2$ and is a $\overline{\partial}$-closed form on $\mathbb{C}^2 \setminus \{0\}$. Then the $L^1$-form $\frac{\overline{z}_1}{|z|^2} \wedge dz_1 \wedge dz_2$ defines a distribution in $\mathbb{C}^2$ which satisfies

$$\overline{\partial}\left(\frac{\overline{z}_1}{|z|^2} dz_1 \wedge dz_2\right) = z_1 B(z_1, z_2) \quad \text{on} \quad \mathbb{C}^2 \setminus \{0\}.$$ 

On the other hand, if $\tilde{H}^{2,1}(D) = 0$, there exists an extendable $(2, 0)$-current $\nu$ such that $\overline{\partial} \nu = B$ on $D$ and by the regularity of $\overline{\partial}$ in bidegree $(2, 1)$, $\nu$ is smooth on $D$, since $B$ is smooth on $\mathbb{C}^2 \setminus \{0\}$. Set

$$F = z_1 \nu + \frac{\overline{z}_2}{|z|^2} \wedge dz_1 \wedge dz_2.$$
Then $F$ is a holomorphic $(2, 0)$-form on $D$, so its coefficient $F_1$ should extend holomorphically to $\bar{D}$, but we have $F_1(0, z_2) = \frac{1}{z_2}$ on $D \cap \{z_1 = 0\}$, which is holomorphic and singular at $z_2 = 0$. This gives the contradiction since $0 \in \bar{D} \setminus D$. This proves that (ii) $\Rightarrow$ (i).

For the converse, first note that if $D$ is a pseudoconvex domain in $\mathbb{C}^2$, then $\mathbb{C}^2 \setminus D$ is connected and by Proposition 2.8, we have $H_{\bar{D}, \infty}^{0, 1}(\mathbb{C}^2) = 0$. Then we apply Theorem 5 in [4] to get that if $D$ is pseudoconvex with Lipschitz boundary, then $H_{\infty}^{0, 1}(\mathbb{C}^2 \setminus D)$ is Hausdorff. Let us prove that if $H_{\infty}^{0, 1}(\mathbb{C}^2 \setminus D)$ is Hausdorff, then $H_{\bar{D}, \infty}^{0, 2}(\mathbb{C}^2)$ is Hausdorff.

Let $f$ be a $\bar{\partial}$-closed $(0, 2)$-form on $\mathbb{C}^2$ with support contained in $\bar{D}$ such that for any $\bar{\partial}$-closed $(2, 0)$-current $T$ on $D$ extendable as a current to $\mathbb{C}^2$, we have $< T, f > = 0$. Since $H^{0, 2}(\mathbb{C}^2) = 0$, there exists a smooth $(0, 1)$-form $g$ on $\mathbb{C}^2$ such that $\bar{\partial}g = f$ on $\mathbb{C}^2$, in particular $\bar{\partial}g = 0$ on $\mathbb{C}^2 \setminus T$.

Let $S$ be any $\bar{\partial}$-closed $(2, 1)$-current on $\mathbb{C}^2$ with compact support in $\mathbb{C}^2 \setminus D$, then, since $H_{\infty}^{0, 1}(\mathbb{C}^2) = 0$, there exists a compactly supported $(2, 0)$-current $U$ on $\mathbb{C}^2$ such that $\bar{\partial}U = S$ and in particular $\bar{\partial}U = 0$ on $D$.

Thus

$$< S, g > = < \bar{\partial}U, g > = < U, \bar{\partial}g > = < U, f > = 0,$$

by hypothesis on $f$. Therefore the Hausdorff property of $H_{\infty}^{0, 1}(X \setminus D)$ implies there exists a smooth function $h$ on $X \setminus D$ such that $\bar{\partial}h = g$. Let $\tilde{h}$ be a smooth extension of $h$ to $\mathbb{C}^2$, then $u = g - \bar{\partial}\tilde{h}$ is a smooth form with support in $T$ and

$$\bar{\partial}u = \bar{\partial}(g - \bar{\partial}\tilde{h}) = \bar{\partial}g = f.$$

This proves that $H_{\bar{D}, \infty}^{0, 2}(\mathbb{C}^2)$ is Hausdorff, which proves that (i) $\Rightarrow$ (ii).

3 The case of the unbounded Hartogs triangle in $\mathbb{C}^2$

In $\mathbb{C}^2$, let us define the domains $\mathbb{H}^+$ and $\mathbb{H}^-$ by

$$\mathbb{H}^+ = \{(z, w) \in \mathbb{C}^2 \mid |z| < |w|\}$$
$$\mathbb{H}^- = \{(z, w) \in \mathbb{C}^2 \mid |z| > |w|\}$$

then $\mathbb{H}^+ \cap \mathbb{H}^- = \emptyset$ and $\mathbb{H}^+ \cup \mathbb{H}^- = \mathbb{C}^2$.

Let us denote by $H_{\mathbb{H}^+, \infty}^{0, 1}(\mathbb{C}^2)$ (resp. $H_{\mathbb{H}^+, \text{cur}}^{0, 1}(\mathbb{C}^2)$, $H_{\mathbb{H}^+, \text{loc}, \text{cur}}^{0, 1}(\mathbb{C}^2)$, $H_{\mathbb{H}^+, \text{loc}, \text{cur}}^{0, 1}(\mathbb{C}^2)$, $H_{\mathbb{H}^+, \text{loc}, \text{cur}}^{0, 1}(\mathbb{C}^2)$) the Dolbeault cohomology group of bidegree $(0, 1)$ for smooth forms (resp. currents, $L^2$-forms, $C^k$-forms) with support in $\mathbb{H}^+$.

The vanishing of these groups means that one can solve the $\bar{\partial}$ equation with prescribed support in $\mathbb{H}^-$ in the smooth category (resp. the space of currents, the space of $L^2$-forms, the space of $C^k$-forms).

We can apply Propositions 2.5 and 2.10 for $\Omega = \mathbb{H}^-$, since $H^{0, 1}(\mathbb{C}^2) = 0$, and we get

**Proposition 3.1.** We have $H_{\mathbb{H}^+, \infty}^{0, 1}(\mathbb{C}^2) = 0$ if and only if any holomorphic function on $\mathbb{H}^+$ which is smooth on $\mathbb{H}^+$ extends as a holomorphic function to $\mathbb{C}^2$. 
Proposition 3.2. Any holomorphic function on $\mathbb{H}^+$ which is smooth on $\mathbb{H}^+$ extends as a holomorphic function to $\mathbb{C}^2$.

Proof. Let $f \in C^\infty(\mathbb{H}^+) \cap O(\mathbb{H}^+)$. By Sibony’s result ([16], page 220), for any $R > 0$, the restriction of $f$ to $\mathbb{H}^+ \cap \Delta(0, R) \times \Delta(0, R)$ extends holomorphically to the bidisc $\Delta(0, R) \times \Delta(0, R)$ and then by analytic continuation $f$ extends holomorphically to $\mathbb{C}^2$. $\blacksquare$

It follows immediately from Proposition 3.1 and Proposition 3.2 that

Corollary 3.3. $H^{0,1}_{\mathbb{H},\infty}(\mathbb{C}^2) = 0$.

Let us consider now the case of currents. We can apply Proposition 2.4 to get

Proposition 3.4. Assume we have $H^{0,1}_{\mathbb{H},c^k}(\mathbb{C}^2) = 0$, $k \geq 0$ then any holomorphic function on $\mathbb{H}^+$, which is of class $C^{k+1}$ on $\mathbb{H}^+$, extends as a holomorphic function to $\mathbb{C}^2$.

Theorem 3.5. For any $k \geq 0$, $H^{0,1}_{\mathbb{H},c^k}(\mathbb{C}^2) \neq 0$, and $H^{0,1}_{\mathbb{H},cur}(\mathbb{C}^2) \neq 0$.

Proof. Let us consider the function $h$ define on $\mathbb{H}^+$ by $h(z, w) = z^l(\frac{z}{w})$, $l \geq 0$. It is of class $C^{k+1}$ on $\mathbb{H}^+$, if $l \geq k + 2$, but does not extend as a holomorphic function to $\mathbb{C}^2$. In fact if $h$ admits a holomorphic extension $\tilde{h}$ to $\mathbb{C}^2$, then we would have

$$\tilde{h}(z, w) = z^l(\frac{z}{w}) \quad \text{on} \quad \mathbb{C}^2 \setminus \{w = 0\},$$

which is not bounded nearby $\{(z, w) \in \mathbb{C}^2 \mid z \neq 0, w = 0\}$. By Proposition 3.4, we get $H^{0,1}_{\mathbb{H},c^k}(\mathbb{C}^2) \neq 0$. Then using Proposition 2.4, it follows $H^{0,1}_{\mathbb{H},cur}(\mathbb{C}^2) \neq 0$. $\blacksquare$

Proposition 3.1 still holds if we replace smooth forms by $W^1_{loc}$-forms (for $D \subset \mathbb{C}^2$, $W^1_{loc}(\mathbb{D})$ is the space of functions which are in $W^1(\mathbb{D} \cap B(0, R))$ for any $R > 0$) in the following way

Proposition 3.6. We have $H^{0,1}_{\mathbb{H},L^2_{loc}}(\mathbb{C}^2) = 0$ if and only if any function $f \in O(\mathbb{H}^+) \cap W^1_{loc}(\mathbb{H}^+)$, which is the restriction to $\mathbb{H}^+$ of a form with coefficients in $W^1_{loc}(\mathbb{C}^2)$, extends as a holomorphic function to $\mathbb{C}^2$.

Theorem 3.7. $H^{0,1}_{\mathbb{H},L^2_{loc}}(\mathbb{C}^2) \neq 0$

Proof. Let us consider the function $h$ defined on $\mathbb{H}^+$ by $h(z, w) = z^3(\frac{z}{w})$. It is of class $C^2$ on $\mathbb{H}^+$ and it is in $W^1_{loc}(\mathbb{H}^+)$ and extends as a $C^2$ funtion to $\mathbb{C}^2$ by the Whitney extension Theorem, but does not extend as a holomorphic function to $\mathbb{C}^2$. In fact if $h$ would admit a holomorphic extension $\tilde{h}$ to $\mathbb{C}^2$, then we would have

$$\tilde{h} = z^3(\frac{z}{w}) \quad \text{on} \quad \mathbb{C}^2 \setminus \{w = 0\},$$

which is not bounded nearby $\{(z, w) \in \mathbb{C}^2 \mid z \neq 0, w = 0\}$. By Proposition 3.6, we get $H^{0,1}_{\mathbb{H},L^2_{loc}}(\mathbb{C}^2) \neq 0$. $\blacksquare$
Remark: Note that if we replace $H^{-}$ by the classical Hartogs triangle $T^{-} = H^{-} \cap \Delta \times \Delta$, where $\Delta$ is the unit disc in $\mathbb{C}$, then by Proposition 2.8 we have

$$H^{0,1}_{\bar{T}^{-},L^2_{loc}}(\mathbb{C}^2) = H^{0,1}_{\bar{T}^{-},L^2_{loc}}(\mathbb{C}^2) = H^{0,1}_{\bar{T}^{-},\infty}(\mathbb{C}^2) = 0.$$ 

So for solving the $\bar{\partial}$-equation with prescribed support, it is quite different to consider a bounded domain or an unbounded domain as support.

4 The case of the Hartogs triangles in $\mathbb{CP}^2$

In $\mathbb{CP}^2$, we denote the homogeneous coordinates by $[z_0, z_1, z_2]$. On the domain where $z_0 \neq 0$, we set $z = \frac{z_1}{z_0}$ and $w = \frac{z_2}{z_0}$. Let us define the domains $H^+$ and $H^-$ by

$$H^+ = \{[z_0 : z_1 : z_2] \in \mathbb{CP}^2 \mid |z_1| < |z_2|\}$$
$$H^- = \{[z_0 : z_1 : z_2] \in \mathbb{CP}^2 \mid |z_1| > |z_2|\}$$

then $H^+ \cap H^- = \emptyset$ and $H^+ \cup H^- = \mathbb{CP}^2$. These domains are called Hartogs’ triangles in $\mathbb{CP}^2$. The Hartogs triangles provide examples of non-Lipschitz Levi-flat hypersurfaces (see [6]).

For $k \geq 0$ or $k = \infty$, we denote by $H^{0,1}_{\bar{\partial},C^k}(\mathbb{CP}^2)$ (resp. $H^{0,1}_{\bar{\partial},cur}(\mathbb{CP}^2)$, $H^{0,1}_{\bar{\partial},L^2}(\mathbb{CP}^2)$) the Dolbeault cohomology group of bidegree $(0, 1)$ for $C^k$-smooth forms (resp. currents, $L^2$-forms) with support in $\mathbb{CP}^2$.

Again the vanishing of these groups means that one can solve the $\bar{\partial}$ equation with prescribed support in $\mathbb{CP}^2$ in the $C^k$-smooth category (resp. the space of currents, the space of $L^2$-forms).

We can also apply Propositions 2.5 and 2.10 for $\Omega = H^-$, since $H^{0,1}(\mathbb{CP}^2) = 0$, and we get

**Proposition 4.1.** We have, for $k \geq 0$ and for $k = \infty$, $H^{0,1}_{\bar{\partial},C^k}(\mathbb{CP}^2) = 0$ if and only if any holomorphic function on $H^+$ which is $C^{k+1}$-smooth on $\mathbb{CP}^2$ extends as a holomorphic function to $\mathbb{CP}^2$.

**Proposition 4.2.** Any holomorphic function on $H^+$ which is continuous on $\mathbb{CP}^2$ is constant.

**Proof.** Let $f \in \mathcal{C}(\mathbb{HP}^+) \cap \mathcal{O}(\mathbb{H}^+)$. Notice that the boundary $bH^+$ of $H^+$ is foliated by a family of compact complex curves described in non-homogeneous coordinates by

$$S_\theta = \{z = e^{i\theta} w\}, \quad \theta \in \mathbb{R}. \quad (4.1)$$

Restricted to each fixed $\theta$, $f$ is a continuous $CR$ function on the compact Riemann surface $S_\theta$. Thus $f$ must be a constant on each $S_\theta$. Since every Riemann surface $S_\theta$ contains the point $(0,0)$, this implies $f$ must be constant on $bH^+$.

$\square$
Note that in the case of the unbounded Hartogs triangle in $\mathbb{C}^2$, the function $f$ needs to be of class $C^\infty$ on $\mathbb{H}^+$ to be extendable as a holomorphic function to $\mathbb{C}^2$ (see Proposition 3.1 and the beginning of the proof of Theorem 3.5). But in $\mathbb{CP}^2$, in contrary to $\mathbb{C}^2$ we get (compare to Corollary 3.3 and Theorem 3.5) from the previous propositions that

**Corollary 4.3.** For each $k \geq 0$, $H^{0,1}_{\mathbb{H}^+,\text{cur}}(\mathbb{CP}^2) = 0$ and $H^{0,1}_{\mathbb{H}^+,\text{cur}}(\mathbb{CP}^2) = 0$.

As in the case of $\mathbb{C}^2$, we get for extendable currents

**Proposition 4.4.** Suppose that $H^{0,1}_{\mathbb{H}^+,\text{cur}}(\mathbb{CP}^2) = 0$. Then any holomorphic function on $\mathbb{H}^+$, which is extendable in the sense of currents, is constant.

**Theorem 4.5.** $H^{0,1}_{\mathbb{H}^+,\text{cur}}(\mathbb{CP}^2)$ does not vanish and is Hausdorff.

**Proof.** Let us consider the function $h$ defined on the open subset $\mathbb{H}^+$ of $\mathbb{CP}^2$ by $h([z_0 : z_1 : z_2]) = \frac{1}{z_0}$. It is holomorphic and bounded and hence defines an extendable current, but it is not constant, so by Proposition 4.4, we get $H^{0,1}_{\mathbb{H}^+,\text{cur}}(\mathbb{CP}^2) \neq 0$. By the Serre duality, to prove that $H^{0,1}_{\mathbb{H}^+,\text{cur}}(\mathbb{CP}^2)$ is Hausdorff, it is sufficient to prove that $H^{2,2}_{\mathbb{H}^+}(\mathbb{H}^+) = 0$.

Let $f$ be a smooth $(2,2)$-form on $\mathbb{H}^-$ and $U$ be a neighborhood of $\mathbb{H}^-$, we can choose $U$ such that $\overline{U}$ is a connected proper subset of $\mathbb{CP}^2$. Then $f$ extends as a smooth $(2,2)$-form on $U$, called $\tilde{f}$. By Malgrange’s theorem, the top degree Dolbeault cohomology group $H^{2,2}(U)$ vanishes since $U$ is a non compact connected complex manifold. Thus there exists a smooth $(2,1)$-form $u$ on $U$ such that $\overline{\partial} u = \tilde{f}$ on $U$. Then $v = u_{\mathbb{H}^+}$ is a smooth form on $\mathbb{H}^-$ which satisfies $\overline{\partial} v = f$ on $\mathbb{H}^-$. \hfill $\square$

Let us now consider the $L^2$ Dolbeault cohomology with prescribed support in an Hartogs triangle in $\mathbb{CP}^2$. As usual we endow $\mathbb{H}^+$ with the restriction of the Fubini-Study metric of $\mathbb{CP}^2$. The following proposition is already proved in Proposition 6 in [4].

**Proposition 4.6.** Let $\mathbb{H}^+ \subset \mathbb{CP}^2$ be the Hartogs’ triangle. Then we have the following:

1. The Bergman space of $L^2$ holomorphic functions $L^2(\mathbb{H}^+) \cap \mathcal{O}(\mathbb{H}^+)$ on the domain $\mathbb{H}^+$ separates points in $\mathbb{H}^+$.

2. There exist nonconstant functions in the space $W^1(\mathbb{H}^+) \cap \mathcal{O}(\mathbb{H}^+)$. However, this space does not separate points in $\mathbb{H}^+$ and is not dense in the Bergman space $L^2(\mathbb{H}^+) \cap \mathcal{O}(\mathbb{H}^+)$.\[2.\]

3. Let $f \in W^2(\mathbb{H}^+) \cap \mathcal{O}(\mathbb{H}^+)$ be a holomorphic function on $\mathbb{H}^+$ which is in the Sobolev space $W^2(\mathbb{H}^+)$. Then $f$ is a constant.

**Proposition 4.7.** Let $\mathbb{H}^+ \subset \mathbb{CP}^2$ be the Hartogs’ triangle. Any function $f \in W^1(\mathbb{H}^+) \cap \mathcal{O}(\mathbb{H}^+)$ can be extended to a function in $W^1(\mathbb{CP}^2)$.

**Proof.** In the non-homogeneous holomorphic coordinates $(z, w)$ for $\mathbb{H}^+$, any function $f \in W^1(\mathbb{H}^+) \cap \mathcal{O}(\mathbb{H}^+)$ has the form (see Proposition 6 in [4])

$$f_k(z, w) = \left( \frac{z}{w} \right)^k, \quad k \in \mathbb{N}.$$
It suffices to prove the proposition for each $f_k(z, w)$.

Let $\chi(t) \in C^\infty(\mathbb{R})$ be a function defined by $\chi(t) = 0$ if $t \leq 0$ and $\chi(t) = 1$ if $t \geq 1$. Let $\tilde{f}_k$ be the function defined by

$$\tilde{f}_k(z, w) = \chi \left(1 + \frac{1}{3} \left(1 - \frac{|z|^2}{|w|^2}\right)\right) f_k(z, w). \quad (4.2)$$

On $|z| < |w|$, it is easy to see that $\tilde{f}_k = f_k$. Thus $\tilde{f}_k$ is an extension of $f_k$ to $\mathbb{C}P^2$.

To see that $\tilde{f}_k$ is in $W^1(\mathbb{C}P^2)$, we first note that the function

$$\chi \left(1 + \frac{1}{3} \left(1 - \frac{|z|^2}{|w|^2}\right)\right) = 0$$

when restricted to $\{|z| \geq 2|w|\}$. Thus it is supported in $\{|z| \leq 2|w|\}$. On its support, the function $\frac{|z|}{|w|}$ is bounded. Using this fact and differentiating under the chain rule, we have that

$$|\nabla \chi \left(1 + \frac{1}{3} \left(1 - \frac{|z|^2}{|w|^2}\right)\right)| \leq C(\sup |\chi'|) \frac{1}{|w|} \leq C \frac{1}{|w|}. \quad (4.3)$$

Repeating the arguments as before, we see that the function $\frac{1}{|w|}$ is in $L^2$ on $\{|z| \leq 2|w|\}$.

Since the function $f_k$ is bounded on the set $\{|z| \leq 2|w|\}$, we conclude from (4.3) that the derivatives of $\tilde{f}_k$ is in $L^2(\mathbb{C}P^2)$. Thus $\tilde{f}_k$ is an extension in $W^1(\mathbb{C}P^2)$ of $f_k$.

\[ \square \]

**Remark.** Suppose $D$ is a bounded domain with Lipschitz boundary, then any function $f \in W^1(D)$ extends as a function in $W^1(\mathbb{C}P^2)$. It is not known if this is true for the Hartogs triangle $\mathbb{H}^+$. In the proof of Proposition 4.7, we have used the fact that the function $f_k$ are in $W^1(\mathbb{H}^+)$ and bounded on $\mathbb{H}^+$.

**Theorem 4.8.** Let $\mathbb{H}^- \subset \mathbb{C}P^2$ be the Hartogs’ triangle. Then the cohomology group $H^{0, 1}_{\mathbb{H}^-, L^2} (\mathbb{C}P^2) \neq 0$ and is infinite dimensional.

**Proof.** We recall that $\mathbb{H}^+ = \mathbb{C}P^2 \setminus \mathbb{H}^-$. From Proposition 4.6, the space of holomorphic functions in $W^1(\mathbb{H}^+) \cap \mathcal{O}(\mathbb{H}^+)$ is infinite dimensional. In the non-homogeneous coordinates, consider the holomorphic functions of the type $f_k = (\frac{z}{w})^k$, $k \in \mathbb{N}$.

We define the operator $\overline{\partial}_c$ as the weak minimal realization of $\overline{\partial}$, then the domain of $\overline{\partial}_c$ is the space of $L^2$ forms $f$ in $\mathbb{C}P^2$ with support in $\mathbb{H}^-$ such that $\overline{\partial} f$ is also an $L^2$ form in $\mathbb{C}P^2$.

Using Proposition 4.7, each holomorphic function $f_k$ can be extended to a function $\tilde{f}_k \in W^1(\mathbb{C}P^2)$. Suppose that $H^{0, 1}_{\mathbb{H}^-, L^2} (\mathbb{C}P^2) = 0$. Then we can solve $\overline{\partial}_c u_k = \overline{\partial} f_k$ in $\mathbb{C}P^2$ with prescribed support for $u_k$ in $\mathbb{H}^-$. Let $H_k = \tilde{f}_k - u_k$. Then $H_k$ is a holomorphic function in $\mathbb{C}P^2$, hence a constant. But $H_k = f_k$ on $\mathbb{H}^+$, a contradiction. This implies that the space $H^{0, 1}_{\mathbb{H}^-, L^2} (\mathbb{C}P^2)$ is non-trivial.

Next we prove that $H^{0, 1}_{\mathbb{H}^-, L^2} (\mathbb{C}P^2)$ is infinite dimensional. Each function $\tilde{f}_k$ corresponds to a $(0, 1)$-form $\overline{\partial} \tilde{f}_k$. We set $g_k = \overline{\partial} \tilde{f}_k$. Then $g_k$ is in $\text{Dom}(\overline{\partial}_c)$ and satisfies $\overline{\partial}_c g_k = 0$. Thus it induces an element $[g_k]$ in $H^{0, 1}_{\mathbb{H}^-, L^2} (\mathbb{C}P^2)$. To see that $[g_k]$’s are linearly independent,
let \( N > 1 \) be a positive integer and \( F_N = \sum_{k=1}^N c_k f_k \), where \( c_k \) are constants. Set \( G_N = \sum_{k=1}^N c_k g_k \). Suppose that \([G_N]\) = 0, then we can solve \( \overline{\partial} u = G_N \) and the function \( F_N \) holomorphic in \( \mathbb{H}^+ \) extends holomorphically to \( \mathbb{C}P^2 \). Thus \( F_N \) must be a constant and \( c_1 = \cdots = c_N = 0 \). Thus \([g_k]\)'s are linearly independent. This proves that \( H^{0,1}_{\mathbb{H}^+} \mathbb{L}^2(\mathbb{C}P^2) \) is infinite dimensional. \( \square \)

**Remark.** It follows from Proposition 2.1 and Theorem 4.8 that \( H^{0,1}_{\mathbb{H}^+} \mathbb{L}^2(\mathbb{C}P^2) \) is also infinite dimensional.

**Lemma 4.9.** The range of the strong \( \mathbb{L}^2 \) closure of \( \overline{\partial} \)

\[
\overline{\partial}_s : L^2_{2,1}(\mathbb{H}^-) \rightarrow L^2_{2,2}(\mathbb{H}^-)
\]

is closed and equal to \( L^2_{2,2}(\mathbb{H}^-) \).

**Proof.** It is clear that \( \overline{\partial} \) has closed range in the top degree and the range is \( L^2_{2,2}(\mathbb{H}^-) \). Let \( f \in L^2_{2,2}(\mathbb{H}^-) \). We extend \( f \) to be zero outside \( \mathbb{H}^- \). Let \( U \) be an open neighbourhood of \( \mathbb{H}^- \), then \( f \) is in \( L^2_{2,2}(U) \). We can choose \( U \) such that \( \overline{U} \) is a proper subset of \( \mathbb{C}P^2 \) and \( U \) has Lipschitz boundary. Since one can solve the \( \overline{\partial} \) equation for top degree forms on \( U \), there exists \( u \in L^2_{2,1}(U) \) such that

\[
\overline{\partial} u = f
\]

in the weak sense.

It suffices to show that \( f \) is in the range of \( \overline{\partial}_s \). Since \( U \) has Lipschitz boundary, using Friedrichs' lemma, there exists a sequence \( u_\nu \in C^\infty(\overline{U}) \) such that \( u_\nu \to u \) and \( \overline{\partial} u_\nu \to f \) in \( L^2_{2,2}(U) \). Restricting \( u_\nu \) to \( \mathbb{H}^- \), we have that \( u \) is in the domain of \( \overline{\partial}_s \) and

\[
\overline{\partial}_s u = f.
\]

Thus the range of \( \overline{\partial}_s \) is equal to \( L^2_{2,2}(\mathbb{H}^-) \). The lemma is proved. \( \square \)

**Corollary 4.10.** The cohomology group \( H^{0,1}_{\mathbb{H}^+} \mathbb{L}^2(\mathbb{C}P^2) \) is Hausdorff and infinite dimensional.

**Theorem 4.11.** Let us consider the Hartogs' triangle \( \mathbb{H}^- \subset \mathbb{C}P^2 \). Then the cohomology group \( H^{2,1}_{\overline{\partial}_s} \mathbb{L}^2(\mathbb{H}^-) \) is infinite dimensional.

**Proof.** Suppose that \( \overline{\partial}_s : L^2_{2,0}(\mathbb{H}^-) \rightarrow L^2_{2,1}(\mathbb{H}^-) \) does not have closed range. Then \( H^{2,1}_{\overline{\partial}_s} \mathbb{L}^2(\mathbb{H}^-) \) is non-Hausdorff, hence infinite dimensional.

Suppose that \( \overline{\partial}_s : L^2_{2,0}(\mathbb{H}^-) \rightarrow L^2_{2,1}(\mathbb{H}^-) \) has closed range. Using Lemma 4.9, \( \overline{\partial}_s : L^2_{2,0}(\mathbb{H}^-) \rightarrow L^2_{2,1}(\mathbb{H}^-) \) has closed range. From the \( L^2 \) Serre duality, \( \overline{\partial}_E : L^2(\mathbb{H}^-) \rightarrow L^2_{0,1}(\mathbb{H}^-) \) and \( \overline{\partial}_E : L^2_{0,1}(\mathbb{H}^-) \rightarrow L^2_{0,2}(\mathbb{H}^-) \) both have closed range. Furthermore,

\[
H^{2,1}_{\overline{\partial}_s} \mathbb{L}^2(\mathbb{H}^-) \cong H^{0,1}_{\mathbb{H}^+} \mathbb{L}^2(\mathbb{C}P^2).
\]

Thus from Theorem 4.8 it is infinite dimensional. \( \square \)
Remarks:

1. Let $T = \{ (z_1, z_2) \in \mathbb{C}^2 \mid |z_2| < |z_1| < 1 \}$ be the Hartogs triangle in $\mathbb{C}^2$. Then by Proposition 2.8, $H^{0,1}_{\bar{\partial}_c, L^2}(T) = H^{0,1}_{\bar{\partial}, L^2}(\mathbb{C}^2) = 0$.

This is in sharp contrast to Corollary 4.10.

It is well-known that $H^{0,1}_{\bar{\partial}_c}(T) = 0$ since $T$ is pseudoconvex, but $H^{0,1}_{\bar{\partial}, L^2}(T)$ (cohomology with forms smooth up to the boundary) is infinite dimensional (see [16]). In fact, $H^{0,1}_{\bar{\partial}, L^2}(T)$ is even non-Hausdorff (see [12]). We also refer the reader to the recent survey paper on the Hartogs triangle [15].

2. If $D$ is a domain in $\mathbb{C}\mathbb{P}^n$ with $C^2$ boundary, then we have $L^2$ existence theorems for $\bar{\partial}$ on $D$ for all degrees (see [1] [6], [2]). This follows from the existence of bounded plurisubharmonic functions on pseudoconvex domains in $\mathbb{C}\mathbb{P}^n$ with $C^2$ boundary (see [13]). This is even true if $D$ has only Lipschitz boundary (see [5]).

3. Suppose that $D$ is a pseudoconvex domain in $\mathbb{C}\mathbb{P}^n$ with Lipschitz boundary, we have $H^{p, q}_{L^2}(D) = 0$ for all $q > 0$. By the $L^2$ Serre duality (see [4]), we have $H^{0,1}_{\bar{\partial}_c, L^2}(D) = H^{0,1}_{\bar{\partial}, L^2}(\mathbb{C}\mathbb{P}^n) = 0$. Corollary 4.10 shows that the Lipschitz condition cannot be removed.

4. From a result of Takeuchi [17], $\mathbb{H}^-$ is Stein. It is well-known that for any $p$, $0 \leq p \leq 2$, $\bar{\partial} : L^2_{p,0}(\mathbb{H}^-, \text{loc}) \to L^2_{p,1}(\mathbb{H}^-, \text{loc})$ has closed range (see [8]) and the cohomology $H^{p,1}_{L^2_{\text{loc}}}(\mathbb{H}^-)$ in the Frechet space $L^2_{0,1}(\mathbb{H}^-, \text{loc})$ is trivial.

5. The (weak) $L^2$ theory holds for any pseudoconvex domain without any regularity assumption on the boundary for $(0, 1)$-forms. The (weak) $L^2$ Cauchy-Riemann operator $\bar{\partial} : L^2(\mathbb{H}^-) \to L^2_{0,1}(\mathbb{H}^-, \text{loc})$ has closed range and $H^{0,1}_{L^2}(\mathbb{H}^-) = 0$ (see [6] or [2]).

6. For $p = 1$ or $p = 2$, it is not known if the Cauchy-Riemann operator $\bar{\partial} : L^2_{p,0}(\mathbb{H}^-) \to L^2_{p,1}(\mathbb{H}^-)$ has closed range. It is also not known if $\bar{\partial}$ in the weak sense is equal to $\bar{\partial}_s$.

7. It is not known if the strong $L^2$ Cauchy-Riemann operator $\bar{\partial}_s : L^2_{2,0}(\mathbb{H}^-) \to L^2_{2,1}(\mathbb{H}^-)$ has closed range.

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Solving with prescribed support on Hartogs triangles

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