CONNECTIVITY FUNCTIONS AND POLYMATROIDS

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Abstract. A connectivity function on a set $E$ is a function $\lambda : 2^E \to \mathbb{R}$ such that $\lambda(\emptyset) = 0$, that $\lambda(X) = \lambda(E - X)$ for all $X \subseteq E$ and that $\lambda(X \cap Y) + \lambda(X \cup Y) \leq \lambda(X) + \lambda(Y)$ for all $X, Y \subseteq E$. Graphs, matroids and, more generally, polymatroids have associated connectivity functions. We introduce a notion of duality for polymatroids and prove that every connectivity function is the connectivity function of a self-dual polymatroid. We also prove that every integral connectivity function is the connectivity function of a half-integral self-dual polymatroid.

1. Introduction

Let $E$ be a finite set, and $\lambda$ be a function from the power set of $E$ into the real numbers. Then $\lambda$ is symmetric if $\lambda(X) = \lambda(E - X)$ for all $X \subseteq E$; $\lambda$ is submodular if $\lambda(X \cap Y) + \lambda(X \cup Y) \leq \lambda(X) + \lambda(Y)$ for all $X, Y \subseteq E$; and $\lambda$ is normalised if $\lambda(\emptyset) = 0$. If $\lambda$ is symmetric, submodular and normalised, then we say that $\lambda$ is a connectivity function with ground set $E$. We also say that $\lambda$ is a connectivity function on $E$. If $\lambda$ is a connectivity function on $E$, then $\lambda$ is integer-valued if $\lambda(X) \in \mathbb{Z}$ for all $X \subseteq E$. The connectivity function $\lambda$ is unitary if $\lambda(\{x\}) \leq 1$ for all $x \in E$.

Graphs and matroids have natural associated connectivity functions. These auxiliary structures capture vital information. It turns out that a number of quite fundamental properties of graphs and matroids hold at the level of general connectivity functions. In particular this is the case for properties associated with branch-width and tangles of graphs and matroids. This is implicit — but clear on a close reading — in the paper of Robertson and Seymour [11]. More explicit results for connectivity functions are proved in Geelen, Gerards and Whittle [2], Clark and Whittle [1], Hundertmark [5], and Grohe and Schweitzer [3].

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Given that we can prove quite strong theorems for connectivity functions, the study of these structures is well motivated and this paper forms part of that study. The natural question arises as to just how general connectivity functions are. The main purpose of this paper is to give an answer to that question. Polymatroids are defined in the next section. We prove that every connectivity function is the connectivity function of an associated polymatroid, and every integral connectivity function is the connectivity function of an associated half-integral polymatroid. The proofs of these facts are quite simple — almost unnervingly so — but the results are apparently new and we believe that they are worth reporting. Moreover, our main result surprised at least one of us as a number of naturally arising connectivity functions seem to have little to do with polymatroids.

As well as proving the above results we introduce a new notion of duality for polymatroids. Via this duality we get stronger theorems. Every connectivity function is the connectivity function of a self-dual polymatroid. An interesting feature of this notion of duality is that, when restricted to the class of matroids, it gives a duality that is subtly different from usual matroid duality.

The results of this paper had their genesis in the MSc thesis of Mo [9] and were further developed in the MSc thesis of Jowett [6]. These theses also contain a number of other results on connectivity functions and their connection with polymatroids.

Since writing the first draft of this paper we have become aware of a paper of Matúš [8]. While our perspective and terminology is quite different from those of Matúš the fact is that a number of the results of this paper follow from results of his. In particular our Lemmas 4.2 and 4.3 (ii) and (iii) follow from Theorem 1 of [8]. At a deeper level it is clear that most of the key ideas for which this paper could claim originality are already present in [8]. The existence of Matúš’ paper came as a considerable surprise to as as we believed throughout that we were exploring perfectly new territory. On the other hand Matúš’ perspective is quite different from ours — he is motivated by problems in information theory and our primary motivation comes from matroid theory. Moreover the two papers have very different styles of exposition. The two papers should appeal to different audiences and we believe that there is a real advantage in having both papers in print.

2. Preliminaries

Recall that a polymatroid $P = (r, E)$ is a finite set $E$ together with a function $r : 2^E \to \mathbb{R}$ that is normalised, that is, $r(\emptyset) = 0$, submodular,
that is, $r(X \cap Y) + r(X \cup Y) \leq r(X) + r(Y)$ for all $X, Y \subseteq E$, and increasing, that is, $r(X) \leq r(Y)$ for all $X \subseteq Y \subseteq E$. The polymatroid $P$ is integer valued if $r(X) \in \mathbb{Z}$ for all $Z \subseteq E$. It is half-integral if $r(X) \in \{\frac{x}{2} : x \in \mathbb{Z}\}$ for all $X \subseteq E$. If $r(X) \leq k$ for all $X \subseteq E$, then $P$ is a $k$-polymatroid. We know of no case where $k$-polymatroids are of interest except when $k$ is a positive integer. We define the connectivity function $\lambda_P$ of the polymatroid $P$ by $\lambda_P(X) = r_P(X) + r_P(E-X) - r_P(E)$ for all $X \subseteq E$. It is well known and easily verified that, if $P$ is a polymatroid, then $\lambda_P$ is indeed a connectivity function.

Two special cases of polymatroids are of particular interest. Observe that a matroid $M$, when defined via its rank function is just an integer-valued polymatroid with the additional property that $r(\{e\}) \leq 1$ for all $e \in E$. In other words, a matroid is an integer-valued 1-polymatroid. Via this specialisation, the connectivity function $\lambda_M$ of the matroid $M$ as defined in, for example Oxley [10], is nothing more than the connectivity function we obtain when we regard $M$ as a polymatroid. Specifically, if $M$ is a matroid on $E$, then $\lambda_M(X) = r_M(X) + r_M(E-X) - r_M(E)$ for all $X \subseteq E$. Evidently connectivity functions of matroids are integer-valued and unitary.

Let $G = (V, E)$ be a graph. Then the connectivity function of $G$, denoted $\lambda_G$, is defined by $\lambda_G(X) = |V(X)| + |V(E-X)| - |V|$ for all $X \subseteq E$. Connectivity functions of graphs capture vertex connectivity. For each vertex cut of order $k$ in $G$ there is an associated partition $(X, E-X)$ of the edges such that $\lambda_G(X) = k$; in fact there may be more than one, so that the connectivity function of a graph gives more information than the vertex cuts of that graph. Of course connectivity functions of graphs are integer-valued but they are not usually unitary as, if $e$ is an edge of $G$, then $\lambda_G(\{e\}) = 2$ unless $e$ is a loop or is incident with a leaf. Associated with a graph is its cycle matroid $M(G)$. The connectivity function of $G$ is quite distinct from that of $M(G)$. Nonetheless, the two are related. For example, it is proved in [4] that, apart from essentially trivial exceptions, the branch-width of a graph and its cycle matroid are the same. In the language of connectivity functions this means that, apart from the same exceptions, the branch-width of $\lambda(G)$ is one greater than the branch width of $\lambda_{M(G)}$. In this paper when we refer to the connectivity function of a graph $G$ we will always mean $\lambda_G$ as defined above.

Associated with a graph $G = (V, E)$ we define an integer-valued set function $r_G$ on $E$ by setting $r_G(X) = |V(X)|$ for all $X \subseteq E$. We say that $r_G$ is the rank function of $G$. Evidently $r_G$ knows nothing about isolated vertices or vertex labels in $G$. Apart from that, the graph $G$ is determined by its rank function. Another feature of $r_G$ is that it is the
rank function of an integer-valued 2-polymatroid. The fact that graphs essentially correspond to a family of integer-valued 2-polymatroids is of some interest and we take the opportunity here of expanding a little on these relationships.

Let $M$ be a matroid on a set, say $V$, and let $E$ be a collection of subsets of $V$. Define the function $r_P$ on $E$ as follows:

$$r_P(X) = r_M(\bigcup_{x \in X})$$

for all $X \subseteq E$. It is well known, see for example [10, Theorem 11.1.9], that every integer-valued polymatroid can be obtained in this way. Let $M$ be a free matroid on $V$, that is $r(V) = |V|$. In essence, free matroids are trivial matroids. Let $E$ be a collection of subsets of $V$ of size at most 2. The polymatroids we construct from free matroids via this construction are precisely the 2-polymatroids we constructed from the edge sets of graphs in the previous paragraph. Note that this way of viewing graphs is nothing more than the time-honoured way of viewing a graph as a collection of lines generated by pairs of points of a simplex.

3. Polymatroid Duality

Let $P = (r, E)$ be a polymatroid. For a set $X \subseteq E$, we let

$$||X||_r = \sum_{x \in X} (r(\{x\})).$$

We define the set function $r^*$ on $E$ by setting

$$r^*(X) = r(E - X) + ||X||_r - r(E)$$

for all $X \subseteq E$. We call the pair $P^* = (r^*, E)$, the dual of $P$. We will prove that $P^*$ is a polymatroid, but first note an elementary lemma.

Lemma 3.1. Let $P = (r, E)$ be a polymatroid and let $X$ and $Y$ be subsets with $X \subseteq Y \subseteq E$. Then $r(Y) - r(X) \leq ||Y - X||_r$.

Proof. By submodularity, $r(Y - X) + r(X) \geq r(Y)$, so that $r(Y) - r(X) \leq r(Y - X)$. Again, by submodularity, $r(Y - X) \leq \sum_{y \in Y - X} r(\{y\})$, so that $r(Y - X) \leq ||Y - X||_r$. Thus $r(Y) - r(X) \leq ||Y - X||_r$ as required. \(\square\)

Lemma 3.2. Let $P$ be a polymatroid on $E$. Then the dual $P^* = (r^*, E)$ is a polymatroid on $E$.

Proof. We need to show that $r^*$ is normalised, increasing and submodular. We have

$$r^*(\emptyset) = r(E) + ||\emptyset||_r - r(E) = 0,$$
so that \( r^* \) is normalised. Assume that \( X \subseteq Y \subseteq E \). Then
\[
r^*(Y) - r^*(X) = r(E - Y) + ||Y||_r - r(E) - r(E - X) - ||X||_r + r(E)
\]
\[
= r(E - Y) - r(E - X) + (||Y||_r - ||X||_r)
\]
\[
= ||Y - X||_r - (r(E - X) - r(E - Y)).
\]
However \( Y - X = (E - X) - (E - Y) \) so that it follows from Lemma 3.1 that \( r^*(Y) - r^*(x) \geq 0 \), that is, \( r^* \) is increasing.

Now say that \( X, Y \subseteq E \). Evidently \( ||X||_r + ||Y||_r = ||X \cup Y||_r + ||X \cap Y||_r \). Using this fact and submodularity we see that
\[
r^*(X) + r^*(Y)
\]
\[
=r(E - X) + ||X||_r - r(E) + r(E - Y) + ||Y||_r - r(E)
\]
\[
\geq r(E - (X \cup Y)) + r(E - (X \cap Y)) + ||X \cup Y||_r + ||X \cap Y||_r - 2r(E)
\]
\[
=r^*(X \cup Y) + r^*(X \cap Y).
\]
Thus \( r^* \) is submodular and the lemma follows.

An alternative notion of duality for integer-valued polymatroids was introduced in \[13\]. For a fixed positive integer \( k \), the set function \( r^*k \) is defined, for all \( X \subseteq E \), by
\[
r^*k(X) = r(E - X) + k||X||_r - r(E).
\]
In the case \( k = 1 \), we have the usual dual for matroids. The \( k \)-dual of an integer-valued \( k \)-polymatroid \( P \) is an integer-valued \( k \)-polymatroid, which we denote by \( P^{*k} \). Moreover \( k \)-duality enjoys two natural properties. First, \( k \)-duality is an involution on the class of \( k \)-polymatroids, that is, for any \( k \)-polymatroid \( P \), we have \( (P^{*k})^{*k} = P \). Second; \( k \)-duality interchanges deletion and contraction, that is, for any \( X \subseteq E \), we have \( (P \setminus X)^{*k} = P^{*k}/X \). Indeed, it is proved in \[13\] that \( k \)-duality is the only function on the class of \( k \)-polymatroids that enjoys both of these properties. The definition of duality for polymatroids we have given here is not restricted to \( k \)-polymatroids for any fixed \( k \) and is certainly different from \( k \)-duality, so something has to give. It turns out that our notion of duality is not in general an involution. Despite this, we shall see that the situation is not so dire. Indeed, it has its appeal.

Let \( P = (r, E) \) be a polymatroid. Recall that we denoted the connectivity function of \( P \) by \( \lambda_P \). An element \( e \in E \) is compact if \( r(\{e\}) = \lambda_P(\{e\}) \). We say that the polymatroid \( P \) is compact if every element of \( P \) is compact. Intuitively compact elements are ones that do not “stick out” from the rest of the polymatroid. More formally, we have
Lemma 3.3. Let $P = (r, E)$ be a polymatroid. The element $e \in E$ is compact if and only if $r(E - \{e\}) = r(E)$.

Proof. The element $e$ is compact if and only if $r(\{e\}) = \lambda_P(\{e\}) = r(E - \{e\}) + r(\{e\}) - r(E)$. This holds if and only if $(E - \{e\}) = r(E)$.

A matroid is compact if and only if it has no coloops. The 2-polymatroid that we associate with a connected graph is compact if and only if the graph has no leaves. Given a polymatroid $P$, there is a natural compact polymatroid that we can associate with $P$ that has the same connectivity function as $P$. We consider this now.

Let $P = (E, r)$ be a polymatroid. Define the function $r^\flat : \mathbb{R} \to \mathbb{R}$ by

$$r^\flat(X) = r(X) + \sum_{x \in X} (\lambda(\{x\}) - r(\{x\}))$$

for all $X \subseteq E$. The pair $P^\flat = (E, r^\flat)$ is the compactification of $P$.

It turns out that $P^\flat$ is a polymatroid and $\lambda_{P^\flat} = \lambda_P$. These facts will follow from the connection with duality.

Lemma 3.4. Let $P = (E, r)$ be a polymatroid. Then the following hold.

(i) $\lambda_{P^*} = \lambda_P$.

(ii) $P^*$ is compact.

(iii) $(P^*)^* = P^\flat$.

Proof. Consider (i). For a set $X \subseteq E$, we have $||X||_r + ||E - X||_r = ||E||_r$. Using this fact and definitions we see that

$$\lambda_{P^*}(X) = r^*(X) + r^*(E - X) - r^*(E) = r(E - X) + ||X||_r - r(E) + r(X) + ||E - X||_r - r(E) - r(\emptyset) = \lambda_P(X).$$

Consider (ii). Say $e \in E$. Then

$$r^*(E - \{e\}) = r(\{e\}) + ||E - \{e\}||_r - r(E) = ||E||_r - r(E) = r(\emptyset) + ||E||_r - r(E) = r^*(E).$$

Therefore $e$ is compact in $P^*$, and (ii) follows.
Consider (iii). Say \( X \subseteq E \). Then
\[
(r^*)(X) = r^*(E - X) + ||X||r - r^*(E) \\
= r(E - (E - X)) + ||E - X||_r - r(E) + ||X||_r \\
- r(E - E) - ||E||_r + r(E) \\
= r(X) + \sum_{x \in X} (\lambda_P(\{x\}) - r(\{x\})) \\
= r^b(X).
\]
Therefore \((P^*)^* = P^b\) as required. \(\square\)

As an immediate consequence of Lemmas 3.2 and 3.4, we obtain

**Corollary 3.5.** Let \( P = (E, r) \) be a polymatroid. Then \( P^b \) is a polymatroid and \( \lambda_{P^b} = \lambda_P \).

Thus, while polymatroid duality is not an involution in general, it is an involution on the class of compact polymatroids. The situation is analogous to that of planar drawings of graphs. The planar dual is always connected so that planar duality is not an involution, but planar duality is an involution on the class of planar drawings of connected graphs.

Regarded as a polymatroid, a matroid is compact if and only if it has no coloops. Say \( e \) is a loop of the matroid \( M \). With the usual notion of matroid duality \( e \) becomes a coloop in the dual of \( M \). With the notion of duality given here, \( e \) remains a loop in the dual. Apart from that, the two notions of duality coincide for matroids.

We now consider the connection with minors. Let \( P = (r, E) \) be a polymatroid, and \( A \subseteq E \). The deletion of \( A \) from \( P \), denoted \( P \setminus A \), is defined, for all \( X \subseteq E - A \), by \( r_{P \setminus A}(X) = r_P(X) \). The contraction of \( A \) from \( P \), denoted \( P / A \), is defined for all \( X \subseteq E - A \), by \( r_{P/A}(X) = r_P(X \cup A) - r_P(A) \). These notions generalise familiar ones from matroid theory.

We would like to say that, just as with matroids, deletion and contraction are interchanged under duality, but this cannot be, since compactness can be lost by deletion. However compactness cannot be lost by contraction.

**Lemma 3.6.** Let \( P = (E, r) \) be a compact polymatroid. Then \( P / A \) is compact for any \( A \subseteq E \).
Proof. Say \( e \in E - \{a\} \). By Lemma 3.3, \( r(E - \{e\}) = r(E) \). We then have

\[
\lambda_{P/A}(\{e\}) = r_{P/A}(\{e\}) + r_{P/A}(\{(E - A) - \{e\}) - r_{P/A}(E - A)
\]

\[
= r_P(A \cup \{e\}) - r_P(A) + r_P(E - \{e\}) - r_P(A) - (r_P(E) - r_P(A))
\]

\[
= r_P(A \cup \{e\}) - r_P(A)
\]

\[
= r_{P/A}(\{e\}),
\]

so that \( P/A \) is indeed compact. \( \square \)

Again the situation is analogous to that of plane graphs where connectivity can be lost by deletion, but not by contraction. Combined with compactification, we do obtain a nice relation under duality.

**Lemma 3.7.** Let \( P = (E, r) \) be a polymatroid and \( A \subseteq E \). Then \((P/A)^* = (P^* \setminus A)^*\).

**Proof.** First note that both \((P/A)^*\) and \((P^* \setminus A)^*\) are defined on the same set, that is \(E - A\). Consider \( r_{(P/A)^*}(X) \) for \( X \subseteq E - A \). We have the following chain of equalities:

\[
r_{(P/A)^*}(X) = (r_{P/A})(X) + \sum_{a \in X} [(\lambda_{P/A})(\{a\}) - (r_{P/A})(\{a\})]
\]

\[
= (r_{P/A})(X) + \sum_{a \in X} [(r_{P/A})(\{(E - A) - \{a\}) - (r_{P/A})(E - A)]
\]

\[
= r_{P^*}(X) + \sum_{a \in X} [r_{P^*}(\{(E - A) - \{a\}) - r_{P^*}(E - A)]
\]

\[
= r_P(E - X) + ||X||_r - r_P(E) + \sum_{a \in X} [r_P(A \cup \{a\}) - ||E - \{a\}||_r - r_P(A) + ||E - A||_r
\]

\[
- r_P(E)]
\]

\[
= r_P(E - X) + ||X||_r - r_P(E) + \sum_{a \in X} [r_P(A \cup \{a\}) - ||E||_r + ||A||_r
\]

\[
- r_P(E)]
\]

\[
= r_P(E - X) + ||X||_r - r_P(E) + \sum_{a \in X} [r_P(A \cup \{a\}) - r_P(\{a\})
\]

\[
- r_P(A)]
\]

\[
= r_P(E - X) - r_P(E) + \sum_{a \in X} [r_P(A \cup \{a\}) - r_P(A)].
\]
Now consider $r_{(P/A)^*}(X)$ and consider the following chain of equalities:

$$r_{(P/A)^*}(X) = (r_{P/A})(E - A) - X + \|(E - A) - X\|_{r_{P/A}} - r_{P/A}(E - A)$$

$$= r_P(E - X) - r_P(A) + \|X\|_{r_{P/A}} - r_P(E) + r_P(A)$$

$$= r_P(E - X) - r_P(E) + \sum_{a \in X} (r_{P/A}(\{a\}) - r_P(A))$$

$$= r_P(E - X) - r_P(E) + \sum_{a \in X} [r_P(A \cup \{a\}) - r_P(A)].$$

Therefore $(P/A)^* = (P^* \setminus A)^\delta$.

Given the above correspondences, it seems natural to operate within the universe of compact polymatroids. In this universe one could incorporate compactification in the definition of deletion. Given that a polymatroid and its compactification have the same connectivity function, no real loss is incurred by taking this approach if our interest is in polymatroid connectivity.

4. Connectivity functions and polymatroids

We say that a connectivity function $\lambda$ on $E$ is matroidal if there exists a matroid $M$ such that $\lambda = \lambda_M$. We say that $\lambda$ is connected if $\lambda(X) > 0$ whenever $X$ is a proper nonempty subset of $E$. Assume that $\lambda$ is a connected matroidal connectivity function, say $\lambda = \lambda_M$. Another matroid with the same connectivity function is $M^*$. It follows from work of Seymour [12] and Lemos [7] that, if $r(M) \neq r(M^*)$, then these are the only matroids whose connectivity functions are equal to $\lambda$. When $r(M) = r(M^*)$, there are cases where other matroids can have the same connectivity function. The situation is certainly highly structured, but, even in the case where the only matroids with connectivity function $\lambda$ are $M$ and its dual, it is by no means straightforward to find the rank function of $M$ or $M^*$ from $\lambda$.

Let $M$ and $N$ be matroids on $E$. Define the function $r$ by $r(X) = r_M(X) + r_N(X)$ for all subsets of $E$. It is well known and easily seen that $r$ is the rank function of a 2-polymatroid on $E$. In particular, this holds when $N = M^*$. To eliminate ambiguity caused by the distinction between the two types of duality for matroids, assume that $M$ is a loopless matroid. Then it is elementary to check that for all $X \subseteq E$, we have

$$r_M(X) + r_{M^*}(X) = \lambda(X) + |X|.$$

Thus we can canonically construct a 2-polymatroid $P$ from $\lambda$. Moreover, it is easily checked that $\lambda_P = 2\lambda$, so that, up to a scaling factor, we have constructed a 2-polymatroid whose connectivity function is
equal to $\lambda$. Alternatively, we could observe that the fractional polymatroid $P/2$ has connectivity function equal to $\lambda$. In themselves, these observations are not particularly interesting, except for the fact that they generalise as we now show.

Let $\lambda$ be a connectivity function on $E$. For $X \subseteq E$, we define $||X||_\lambda$ by

$$||X||_\lambda = \sum_{x \in X} \lambda(\{x\}).$$

**Lemma 4.1.** Let $\lambda$ be a connectivity function on $E$ and let $A$ and $B$ be subsets of $E$ with $A \subseteq B$. Then

$$\lambda(B) - \lambda(A) \leq ||B - A||_\lambda.$$

**Proof.** It follows from submodularity that, if $Z \subseteq E$, then $\lambda(Z) \leq ||Z||_\lambda$. It also follows from submodularity that $\lambda(B) - \lambda(A) \leq \lambda(B - A)$. Combining these two observations gives the result. □

**Lemma 4.2.** Let $\lambda$ be a connectivity function on $E$ and define the set function $r$ on $E$ by

$$r(X) = \lambda(X) + ||X||_\lambda$$

for all $X \subseteq E$. Then $r$ is the rank function of a polymatroid on $E$.

**Proof.** Evidently $r(\emptyset) = 0$. Assume that $X \subseteq Y \subseteq E$. Then we have

$$r(Y) - r(X) = \lambda(Y) + ||Y||_\lambda - \lambda(X) - ||X||_\lambda$$

$$= ||Y - X||_\lambda + \lambda(Y) - \lambda(X)$$

$$= ||(E - X) - (E - Y)||_\lambda - (\lambda(E - X) - \lambda(E - Y)).$$

It follows from Lemma 4.1 that $||(E - X) - (E - Y)||_\lambda - (\lambda(E - X) - \lambda(E - Y)) \geq 0$. Hence $r$ is increasing.

Assume that $X, Y \subseteq E$. Observe that $||X||_\lambda + ||Y||_\lambda = ||X \cup Y||_\lambda + ||X \cap Y||_\lambda$. Using this fact and the submodularity of $\lambda$, we have

$$r(X) + r(Y) = \lambda(X) + \lambda(Y) + ||X||_\lambda + ||Y||_\lambda$$

$$\geq \lambda(X \cup Y) + \lambda(X \cap Y) + ||X \cup Y||_\lambda + ||X \cap Y||_\lambda$$

$$= r(X \cup Y) + r(X \cap Y).$$

Hence $r$ is submodular and $r$ is the rank function of a polymatroid as claimed. □

We say that the polymatroid constructed via Lemma 4.2 is the polymatroid *induced* by $\lambda$.

**Lemma 4.3.** Let $\lambda$ be a connectivity function on $E$ and let $P = (E, r)$ denote the polymatroid induced by $\lambda$. Then the following hold.

(i) $\lambda_P(X) = 2\lambda(X)$ for all $X \subseteq E$. 

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(ii) $P$ is compact.
(iii) $P$ is self dual.

Proof. Observe that $r(E) = \lambda(E) + \|E\|_\lambda = \|E\|_\lambda$. We use this fact several times. Say $X \subseteq E$. Then

$$\lambda_P(X) = r(X) + r(E - X) - r(E)$$

$$= \lambda(X) + \|X\|_\lambda + \lambda(E - X) + \|E - X\|_\lambda - \|E\|_\lambda$$

$$= \lambda(X) + \lambda(E - X)$$

$$= 2\lambda(X).$$

Hence (i) holds. Consider (ii). Say $e \in E$.

$$r(E) - r(E - \{e\}) = \|E\|_\lambda - \lambda(E - \{e\}) - \|E - \{e\}\|_\lambda$$

$$= \||\{e\}\|_\lambda - \lambda(\{e\})$$

$$= 0.$$

Hence $P$ is compact, so that (ii) holds. Consider (iii). Let $r^*$ denote the rank function of $P^*$. Then, for $X \subseteq E$,

$$r^*(X) = r(E - X) + \|X\|_r - r(E).$$

By definition, $\|X\|_r = 2\|X\|_\lambda$. Also $r(E - X) = \lambda(E - X) + \|E - X\|_\lambda$, and $\lambda(E - X) = \lambda(X)$. Hence we have

$$r^*(X) = \lambda(X) + \|E - X\|_\lambda + 2\|X\|_\lambda - \|E\|_\lambda$$

$$= \lambda(X) + \|X\|_\lambda$$

$$= r(X).$$

Hence $P$ is self dual. \[\square\]

From the above lemmas we obtain

**Theorem 4.4.** Every connectivity function is the connectivity function of a compact, self-dual polymatroid.

**Proof.** Let $\lambda$ be a connectivity function on $E$ and let $P$ be the polynomial induced by $\lambda$. Define $P/2$ by $P/2(X) = P(X)/2$ for all $X \subseteq E$. It is easily checked that $P/2$ satisfies the conditions of the theorem. \[\square\]

Specialising to the integer-valued case we obtain

**Corollary 4.5.** Every integer-valued connectivity function is the connectivity function of a half-integral self-dual polymatroid.

The case of unitary integer-valued connectivity functions is of particular interest. Up to a scaling factor, these are captured by self-dual integral 2-polymatroids. This does suggest that such 2-polymatroids are worth studying in their own right.
References

[1] B. Clark and G. Whittle, Tangles, trees, and flowers, J. Combin. Theory Ser. B 103 (2013) 385-407.
[2] J. Geelen, B. Gerards, and G. Whittle, Tangles, tree decompositions and grids in matroids, J. Combin. Theory Ser. B 99 (2009) 657-667.
[3] M. Grohe and P. Schweitzer, Computing with tangles, arXiv:1503.00190v1 [cs.DM] 2015.
[4] I. Hicks, The branchwidth of graphs and their cycle matroids, J. Combin. Theory Ser. B 97 (2007) 681-692.
[5] F. Hundertmark, Profiles. An algebraic approach to combinatorial connectivity, ArXiv, arXiv:1110.6207v1 [math.CO] 2011.
[6] S. Jowett, Recognition Problems for Connectivity Functions, MSc Thesis, Victoria University thesis, under examination.
[7] M. Lemos, Matroids Having The Same Connectivity Function, Discrete Math 131 (1994) 153-161.
[8] F. Matúš, Polymatroids and polyquantoids, Proceedings of WUPES 2012 (2012) 126–136.
[9] S. Mo, The Structure of Connectivity Functions, MSc Thesis, Victoria University of Wellington.
[10] J. Oxley, Matroid Theory, Oxford University Press, New York, 2011.
[11] N. Robertson and P. Seymour, Graph minors. X. Obstructions to tree decompositions, J. Combin. Theory Ser. B 52 (1991) 153-190.
[12] Seymour, P., On the Connectivity Function of a Matroid, J. Combin. Theory Ser. B 45 (1988) 25-30.
[13] G. Whittle, Duality in Polymatroids and Set Functions, Combinatorics, Probability and Computing 1 (1992) 275-280.

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