Derivatives with respect to the order of the Bessel function of the first kind

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Abstract
An explicit expression of the $k$-th derivative of the Bessel function $J_\nu(z)$, with respect to its order $\nu$, is given. Particularizations for the cases of positive or negative integer $\nu$ are considered.

1 Introduction
Along this paper we use the notation

$$ G^{(k)}(t) \equiv \frac{d^k}{dt^k} \frac{1}{\Gamma(t)} , \quad P^{(k)}_m(t) \equiv \frac{1}{k!} \frac{d^k}{dt^k} (t)_m , \quad Q^{(k)}_m(t) \equiv \frac{1}{k!} \frac{d^k}{dt^k} \frac{1}{(t)_m} , $$

(1)

to refer to the derivatives of the reciprocal gamma function and of the Pochhammer and reciprocal Pochhammer symbols.

Our purpose is to provide with a closed expression for the $k$-th derivative of the Bessel function $J_\nu(z)$ with respect to its order $\nu$, that we assume to be real. From the ascending series definition [7, Eq. 10.2.2]

$$ J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{m=0}^{\infty} \frac{(-z^2/4)^m}{m! \Gamma(\nu+1+m)} , $$

(2)

one obtains immediately, with the notation introduced in (1),

$$ \frac{\partial}{\partial \nu} J_\nu(z) = J_\nu(z) \ln(z/2) + \left(\frac{z}{2}\right)^\nu \sum_{m=0}^{\infty} G^{(1)}(\nu+1+m) \frac{(-z^2/4)^m}{m!} , $$

(3)

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an expression that can be found in all treatises dealing with Bessel functions. (See, for instance, [7 Eq. 10.15.1].) Derivation, \( k \)-1 times, with respect to \( \nu \) gives a recurrence relation

\[
\frac{\partial^k}{\partial \nu^k} J_{\nu}(z) = \ln(z/2) \left( \frac{\partial^{k-1}}{\partial \nu^{k-1}} J_{\nu}(z) \right) + (z/2)^\nu \sum_{m=0}^{\infty} \left[ \frac{(-z^2/4)^m}{m!} \right] \\
\times \sum_{l=1}^{k} \binom{k-1}{l-1} G^{(l)} (\nu+1+m) (\ln(z/2))^{k-l},
\] (4)

which would allow one to compute the successive derivatives to get the \( k \)-th one. We suggest, however, a procedure to obtain directly such \( k \)-th derivative, without need of computing the lower ones. As auxiliary for the main result, we consider, respectively in Sections 2, 3 and 4, explicit expressions for the symbols defined in Eqs. (1). Then, in Section 5, the \( k \)-th derivative of \( J_{\nu}(z) \) with respect to \( \nu \) is discussed. The possibility of extending the resulting expressions to the case of complex \( \nu \) is discussed in Section 6.

2 Derivatives of the reciprocal gamma function

We start with the series expansion [7 Eq. 5.7.1]

\[
\frac{1}{\Gamma(t)} = \sum_{j=1}^{\infty} c_j t^j,
\] (5)

convergent in the whole complex \( t \)-plane. Term by term derivation gives

\[
G^{(k)}(t) = \sum_{j=k}^{\infty} c_j \frac{j!}{(j-k)!} t^{j-k},
\] (6)

an expansion also convergent in the whole plane. Nevertheless, its convergence becomes slower and slower as \( k \) or \( |t| \) increase. It is not recommended, for numerical computation, unless \( |t| < 1 \). For large values of \( |t| \) it is preferable to use the asymptotic expansion obtained, in a former paper [1, Appendix B], by application of the saddle point method [10 Sec. 3.6.3] to the Hankel contour representation of the reciprocal Gamma function.

The expressions of the derivatives of the Bessel function, to be given below, contain \( G^{(k)}(1+\varepsilon) \), which can be calculated by using

\[
G^{(k)}(1+\varepsilon) = \sum_{j=0}^{\infty} c_{j+k+1} (j+1) k \varepsilon^j,
\] (7)

whenever \( |\varepsilon| < 1 \).

2
3 Derivatives of the Pochhammer symbol \((t)_m\) with respect to its argument

As \((t)_m\) is a polynomial of degree \(m\) in \(t\), derivatives of order greater than \(m\) vanish,

\[
P^{(k)}_m(t) = 0 \quad \text{for} \quad k > m .
\]

For the nontrivial case of \(k \leq m\), the \(P^{(k)}_m(t)\) can be computed by means of the recurrence relation

\[
P^{(k)}_{m+1}(t) = (t + m) P^{(k)}_m(t) + P^{(k-1)}_m(t), \quad k > 0,
\]

with starting values

\[
P^{(k)}_0(t) = \delta_{k,0}, \quad P^{(0)}_m(t) = (t)_m.
\]

Explicit expressions for \(P^{(k)}_m(t)\) can be found easily. From the generating function of the Pochhammer symbols [6, Sec. 6.2.1, Eq. (2)]

\[
\sum_{m=0}^{\infty} (t)_m (-z)^m / m! \equiv {}_1F_0(t; -z) = (1 + z)^{-t}, \quad |z| < 1,
\]

one obtains by derivation, \(k\) times, with respect to \(t\)

\[
\sum_{m=0}^{\infty} k! P^{(k)}_m(t) (-z)^m / m! = (-1)^k (1 + z)^{-t} [\ln(1 + z)]^k, \quad |z| < 1.
\]

The term \(P^{(k)}_m(t)\) can be isolated in this way

\[
P^{(k)}_m(t) = \frac{(-1)^{k-m}}{k!} \frac{\partial^m}{\partial z^m} \left((1 + z)^{-t} [\ln(1 + z)]^k\right)\bigg|_{z=0}
\]

\[
= \frac{(-1)^{k-m}}{k!} \sum_{l=0}^{m} \binom{m}{l} \left(\frac{\partial^l}{\partial z^l} (1 + z)^{-t} \left(\frac{d^{m-l}}{d z^{m-l}} [\ln(1 + z)]^k\right)\right)\bigg|_{z=0} .
\]

Now we make use of the trivial result

\[
\frac{\partial^l}{\partial z^l} (1 + z)^{-t} \bigg|_{z=0} = (-1)^l (t)_l
\]

and of the generating relation of the Stirling numbers of the first kind [7] Eq. 26.8.8,

\[
[\ln(1 + z)]^k = k! \sum_{n=k}^{\infty} s(n, k) z^n / n!, \quad |z| < 1,
\]

to obtain the explicit expression

\[
P^{(k)}_m(t) = (-1)^{m-k} \sum_{l=0}^{m-k} (-1)^l \binom{m}{l} s(m-l, k) (t)_l \quad \text{for} \quad m \geq k .
\]
An alternative expression, in terms of generalized Bernoulli polynomials \[2, 3, 9\],

\[
P_m^{(k)}(t) = (-1)^{m-k} \binom{m}{k} B_{m-k}^{(m+1)}(1-t).
\]  

(17)

can be obtained from a recent paper by Coffey \[5, \text{Eq. (2.5)}\].

For the particular case of \( t = 0 \), Eq. (16) gives

\[
P_0^{(k)}(0) = \delta_{k,0}, \quad P_m^{(0)}(0) = \delta_{m,0} ,
\]  

(18)

\[
P_0^{(k)}(0) = (-1)^{m-k} s(m,k) \quad \text{for} \quad m \geq k > 0.
\]  

(19)

In the case of \( t = 1 \), use of the property \[7, \text{Eq. 26.8.20}\]

\[
s(n+1,k+1) = n! \sum_{j=k}^{n} \frac{(-1)^{n-j}}{j!} s(j,k)
\]  

(20)

allows one to obtain, from Eq. (16),

\[
P_m^{(k)}(1) = (-1)^{m-k} s(m+1,k+1).
\]  

(21)

4 Derivatives of the reciprocal Pochhammer symbol \(1/(t)_m\) with respect to its argument

For numerical implementation of the derivatives of the reciprocal Pochhammer symbol with respect to its variable, one may use the recurrence relations

\[
Q_{m+1}^{(k)}(t) = \left( Q_m^{(k)}(t) - Q_{m+1}^{(k-1)}(t) \right) / (t + m),
\]  

(22)

with initial values

\[
Q_0^{(k)}(t) = \delta_{k,0} , \quad Q_0^{(0)}(t) = 1/(t)_m ,
\]  

(23)

Very simple explicit expressions of the \(Q_m^{(k)}(t)\) can be easily obtained from the relation \[8, \text{Eq. 4.2.2.45}\]

\[
\frac{1}{(t)_m} = \sum_{l=0}^{m-1} \frac{(-1)^l}{l! (m - 1 - l)!} \frac{1}{t + l}, \quad m > 0 ,
\]  

(24)

provided \( t \) is different from a nonpositive integer, \(-n\), such that \( 0 \leq n < m \). Direct derivation with respect to \( t \) in this equation gives

\[
Q_0^{(k)}(t) = \delta_{k,0}, \quad Q_m^{(k)}(t) = (-1)^k \sum_{l=0}^{m-1} \frac{(-1)^l}{l! (m - 1 - l)!} \frac{1}{(t + l)^{k+1}}.
\]  

(25)
For the particular case of $t = 1$, this expression admits a more concise form in terms of modified generalized harmonic numbers, $\hat{H}^{(k)}_m$, defined by

$$\hat{H}^{(k)}_0 \equiv \delta_{k,0}, \quad \hat{H}^{(k)}_m \equiv \sum_{j=1}^{m} (-1)^{j-1} \binom{m}{j} \frac{1}{j^k}, \quad m \geq 1, \quad (26)$$

not to be confused with the generalized harmonic numbers,

$$H^{(k)}_m \equiv \sum_{j=1}^{m} \frac{1}{j^k}, \quad m \geq 1, \quad (27)$$

though

$$\hat{H}^{(1)}_m = H^{(1)}_m \equiv H_m \quad \text{for} \quad m \geq 1. \quad (28)$$

Besides the explicit expression (26), the recurrence relation

$$\hat{H}^{(k)}_{m+1} = \hat{H}^{(k)}_m + \frac{1}{m+1} \hat{H}^{(k-1)}_m, \quad m \geq 0, \quad k \geq 1, \quad (29)$$

with the starting values

$$\hat{H}^{(k)}_0 = \delta_{k,0}, \quad \hat{H}^{(0)}_m = 1, \quad (30)$$

may be used to calculate the $\hat{H}^{(k)}_m$. With that notation, Eq. (25) gives

$$Q^{(k)}_m(1) = \frac{(-1)^k}{m!} \hat{H}^{(k)}_m. \quad (31)$$

5 Derivatives of $J_\nu(z)$ with respect to $\nu$

We proceed to obtain our expression for the $k$-th derivative of $J_\nu(z)$ with respect to $\nu$. To avoid unnecessary complications in the resulting formulas, we assume $k \neq 0, 1$, i.e., $k = 1, 2, \ldots$.

Let us denote by $N$ the nearest integer to $\nu$, and define $\varepsilon$ by

$$\nu = N + \varepsilon, \quad |\varepsilon| \leq 1/2. \quad (32)$$

We distinguish two possible ranges of values of $N$.

5.1 $N \geq 0$

The ascending series in Eq. (2) can be written in the form

$$J_\nu(z) = (z/2)^\nu \frac{1}{\Gamma(1+\varepsilon)} \sum_{m=0}^{\infty} \frac{(-z^2/4)^m}{m!(1+\varepsilon)^m m + N}. \quad (33)$$
Derivation, \( k \) times, with respect to \( \nu \) gives, with the notation introduced in (1),

\[
\frac{\partial^k}{\partial \nu^k} J_\nu(z) = k! \left( \frac{z}{2} \right)^\nu \sum_{m=0}^{\infty} \frac{(-z^2/4)^m}{m!} \times \sum_{k_1=0}^{k} \frac{[\ln(z/2)]^{k_1}}{k_1!} \sum_{k_2=0}^{k-k_1} \frac{G^{(k_2)}(1+\epsilon)}{k_2!} Q^{(k-k_1-k_2)}_{m+N}(1+\epsilon),
\]

where \( G^{(k_2)}(1+\epsilon) \) is given in Eq. (7) and, according to Eq (25),

\[
Q_0^{(k)}(1+\epsilon) = \delta_{k,0}, \quad Q^{(k)}_{m+N}(1+\epsilon) = \sum_{j=1}^{m+N} \frac{(-1)^{k+j-1}}{(j-1)! (m+N-j)!} \frac{1}{(\epsilon+j)^{k+1}}.
\]

In the particular case of \( \nu \) being a nonnegative integer, \( \nu = n \geq 0 \), Eq. (34) becomes, in terms of the modified generalized harmonic numbers defined in (26),

\[
\left. \frac{\partial^k}{\partial \nu^k} J_\nu(z) \right|_{\nu=n} = k! \left( \frac{z}{2} \right)^n \sum_{m=0}^{\infty} \frac{(-z^2/4)^m}{m! (m+n)!} \times \sum_{k_1=0}^{k} \frac{[\ln(z/2)]^{k_1}}{k_1!} \sum_{k_2=0}^{k-k_1} (-1)^{k-k_1-k_2} c_{k_2+1} H_{m+n}^{(k-k_1-k_2)}.
\]

Expressions for the first derivative can be found in the bibliography. Besides the familiar expressions given in, for instance, Sect. 10.15 of Ref. [7], alternative closed forms can be found in a paper by Brychkov and Geddes [4]. Our Eqs. (34) and (36) become, for \( k = 1 \),

\[
\frac{\partial}{\partial \nu} J_\nu(z) = \left( \ln(z/2) - \psi(1+\epsilon) \right) J_\nu(z) + \frac{(z/2)\nu}{\Gamma(1+\epsilon)} \sum_{m=0}^{\infty} \frac{(-z^2/4)^m}{m!} \sum_{j=1}^{m+N} \frac{(-1)^j}{(j-1)! (m+N-j)!} \frac{1}{(\epsilon+j)^2},
\]

where \( \psi \) represents the digamma function and the last sum is understood to be zero if \( m+N = 0 \). In the case of integer \( \nu = n \geq 0 \) we have

\[
\left. \frac{\partial}{\partial \nu} J_\nu(z) \right|_{\nu=n} = \left( \ln(z/2) + \gamma \right) J_n(z) - \left( z/2 \right)^n \sum_{m=0}^{\infty} \frac{(-z^2/4)^m}{m! (m+n)!} H_{m+n}^{(1)}.
\]

where \( \gamma \) represents the well known Euler-Mascheroni constant.
5.2 \( \bar{N} < 0 \)

Instead of Eq. (33) we have now

\[
J_\nu(z) = \frac{1}{\Gamma(1+\varepsilon)} \left[ \sum_{m=0}^{-N-1} \frac{(-z^2/4)^m}{m!} (-1)^{N-m} \frac{1}{(1+\varepsilon)^{m+N}} \right] \]  \hspace{1cm} (39)

Derivation with respect to \( \nu \) gives

\[
\frac{\partial^k}{\partial \nu^k} J_\nu(z) = k! \left( \frac{z}{2} \right)^\nu \sum_{k_1=0}^{k} \frac{[\ln(z/2)]^{k_1}}{k_1!} \sum_{k_2=0}^{k-k_1} \frac{G^{(k_2)}(1+\varepsilon)}{k_2!} \left[ \sum_{m=0}^{-N-1} \frac{(-z^2/4)^m}{m!} (-1)^{N-m+k-k_1-k_2} \mathcal{P}_{-N-m}^{(k-k_1-k_2)} (-\varepsilon) \right. \\
+ \left. \sum_{m=-N}^{\infty} \frac{(-z^2/4)^m}{m!} \mathcal{Q}_{m+N}^{(k-k_1-k_2)} (1+\varepsilon) \right], \]  \hspace{1cm} (40)

with \( \mathcal{P}_{-N-m}^{(k-k_1-k_2)} (-\varepsilon) \) given by Eqs. (16) or (17) and \( \mathcal{Q}_{m+N}^{(k-k_1-k_2)} (1+\varepsilon) \) by Eq. (35). In the particular case of \( \nu \) being a negative integer, \( \nu = -n, n > 0 \), this equation turns into

\[
\left. \frac{\partial^k}{\partial \nu^k} J_\nu(z) \right|_{\nu=-n} = k! \left( \frac{z}{2} \right)^{-n} \sum_{k_1=0}^{k} \frac{[\ln(z/2)]^{k_1}}{k_1!} \sum_{k_2=0}^{k-k_1} c_{k_2+1} \left[ \sum_{m=0}^{-n-1} \frac{(-z^2/4)^m}{m!} s(n-m, k-k_1-k_2) \right. \\
+ \left. (-1)^{k-k_1-k_2} \sum_{m=-n}^{\infty} \frac{(-z^2/4)^m}{m! (m-n)!} \hat{\beta}_{m-n}^{(k-k_1-k_2)} \right]. \]  \hspace{1cm} (41)

For \( k = 1 \), Eq. (40) becomes

\[
\frac{\partial}{\partial \nu} J_\nu(z) = (\ln(z/2) - \psi(1+\varepsilon)) J_\nu(z) \\
+ \frac{(z/2)^\nu}{\Gamma(1+\varepsilon)} \left[ \sum_{m=0}^{-N-1} \frac{(-z^2/4)^m}{m!} (-N-m) B_{-N-m-1}^{(-N-m+1)} (1+\varepsilon) \right. \\
+ \left. \sum_{m=-N}^{\infty} \frac{(-z^2/4)^m}{m!} \sum_{j=1}^{m+N} \frac{(-1)^j}{(j-1)! (m+N-j)!} \frac{1}{(\varepsilon+j)^2} \right], \]  \hspace{1cm} (42)

7
where $B_n^{(α)}(x)$ represents the generalized Bernoulli polynomial [2, 3, 9]. It may be written in terms of Stirling numbers of the first kind by using the relation

$$(-N-m) B_{-N-m-1}^{(-N-m+1)}(1+ε) = \sum_{j=0}^{−N−m−1} (j+1) s(-N-m, j+1) ε^j. \quad (43)$$

In the case of $ν$ being a negative integer, Eq. (42) gives

$$\frac{∂}{∂ν} J_ν(z) \bigg|_{ν=-n} = (\ln(z/2) + \gamma) J_{-n}(z) - (z/2)^{-n} \left[ (-1)^n \sum_{m=0}^{n-1} \frac{(z^2/4)^m}{m! (n-m-1)!} + \sum_{m=n}^{∞} \frac{(-z^2/4)^m}{m! (m-n)!} \hat{H}_{m-n}^{(1)} \right]. \quad (44)$$

### 6 Extension to complex values of $ν$

The expressions of the derivatives of the reciprocal Gamma function and of the Pochhammer and reciprocal Pochhammer symbols given in sections 2 to 4 stay for complex values of their argument $t$. Therefore, our Eqs. (34), (37), (40) and (42) may be used safely for complex $ε$, i.e. complex $ν$, whenever $|\Im ν| \lesssim 1/2$.

As auxiliary integer $N$ one should consider again the nearest to $ν$ one, in such a way that, instead of (32), one would have

$$ν = N + ε, \quad |\Re ε| \leq 1/2. \quad (45)$$

For large values of $\Im ν$, the given expressions are correct, but they are not useful from a computational point of view. The reason, as pointed out in Sect. 2, is the slow convergence of the series in the right hand side of (6) for large values of $t$.

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### References

[1] J. Abad and J. Sesma, *Successive derivatives of Whittaker functions with respect to the first parameter*, Comput. Phys. Comm. 156 (2003), pp. 13–21.

[2] Yu.A. Brychkov, *On multiple sums of special functions*, Integral Transforms Spec. Funct. 21 (2010), pp. 877–884.

[3] Yu.A. Brychkov, *On some properties of the generalized Bernoulli and Euler polynomials*, Integral Transforms Spec. Funct. 23 (2012), pp. 723–735.
[4] Yu.A. Brychkov and K.O. Geddes, *On the derivatives of the Bessel and Struve functions with respect to the order*, Integral Transforms Spec. Funct. 16 (2005) 187–198.

[5] M.W. Coffey, *Series representations of the Riemann and Hurwitz zeta functions and series and integral representations of the first Stieltjes constant* [arXiv:1106.5146](http://arxiv.org/abs/1106.5146).

[6] Y.L. Luke, *The Special Functions and Their Approximations*, Academic Press, New York, 1969, Vol I.

[7] F.W.J. Olver, D.W. Lozier, R. Boisvert, and C.W. Clark, eds., *NIST Handbook of Mathematical Functions*, Cambridge Univ. Press, Cambridge, 2010. Available at [http://dlmf.nist.gov/](http://dlmf.nist.gov/).

[8] A.P. Prudnikov, Yu.A. Brychkov and O.I. Marichev, *Integrals and Series*, Gordon and Breach, New York, 1986, Vol 1.

[9] H.M. Srivastava and P.G. Todorov, *An explicit formula for the generalized Bernoulli polynomials*, J. Math. Anal. Appl. 130 (1988), pp. 509–513.

[10] N.M. Temme, *Special Functions*, John Wiley & Sons, New York, 1996.