A GENERALIZATION OF HALL POLYNOMIALS TO ADE CASE

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Abstract. Certain computable polynomials are described whose leading coefficients are equal to multiplicities in the tensor product decomposition for representations of a Lie algebra of ADE type.

1. Hall theorem. Given a prime number $p$ and a positive integer $l$ let $F_p$ be a field with $p$ elements, $F_p[l]$ be an algebraic closure of $F_p$, and $F_p[l]^p$ be the subfield of $F_p[l]$ consisting of all $x \in F_p$ such that $x^p = x$. Thus $F_p[l]^p$ is a field with $p^l$ elements.

Let $k$ be a field, and let $k[[t]]$ be the ring of formal power series with coefficients in $k$ considered as a $k$-algebra. To a representation $\pi$ of $k[[t]]$ in a finite dimensional $k$-linear space $D$ one can associate a partition $\lambda$ as follows: $\lambda_i = \dim_k(\text{im}(\pi(t^{-i})/\text{im}(\pi(t^i)))$. This partition is called the type of the representation $\pi$ (in [Mac95] the conjugate partition is called called the type of $\pi$). Two representations are isomorphic if and only if they are of the same type.

Given a partition $\lambda$ let $\lambda_1 \geq \lambda_2 \geq \ldots$ be the parts of $\lambda$ and let $|\lambda| = \sum_i \lambda_i$. Let $\Omega_\lambda(k)$ be the set of all representations of $k[[t]]$ in $k^{[\lambda]}$ of type $\lambda$. If $k$ is algebraically closed then $\Omega_\lambda(k)$ is an irreducible smooth quasi-projective variety of dimension $\omega_\lambda = \sum_{i \neq j} \lambda_i \lambda_j$. Let $\omega_\lambda(p^l)$ be the cardinal of $\Omega_\lambda(F_p)$. It is known (cf. [Mac95] Chapter II) that $\omega_\lambda(p^l)$ is a monic polynomial function of $p^l$ of degree $\omega_\lambda$.

Let $\lambda$, $\mu^1$, $\ldots$, $\mu^n$, be partitions such that $|\lambda| = |\mu^1| + \ldots + |\mu^n|$. Let $\Omega_{\mu^1, \ldots, \mu^n}(k)$ be the set of all pairs $(\pi, \Pi)$ consisting of a representation $\pi$ of $k[[t]]$ in $k^{[\lambda]}$ of type $\lambda$ and an $n$-step filtration $\Pi = \{0\} = D^0 \subset D^1 \subset \ldots \subset D^n = k^{[\lambda]}$ of $k^{[\lambda]}$ by subrepresentations of $\pi$ such that $\pi|_{D^{a+1} / D^a}$ has type $\mu^a$ for $a = 1, \ldots, n$. If $k$ is algebraically closed then $\Omega_{\mu^1, \ldots, \mu^n}(k)$ is a quasi-projective variety over $k$. The set $\Omega_{\mu^1, \ldots, \mu^n}(F_p)$ can be identified with the set of $F_p$-rational points in the variety $\Omega_{\mu^1, \ldots, \mu^n}(F_p)$.

Steinitz [Ste01] and Hall [Hal59] observed that the sets $\Omega_{\mu^1, \ldots, \mu^n}(k)$ have some relation to the theory of symmetric functions, and thus to the representation theory of the general linear and symmetric groups. Given a partition $\lambda$ such that the number of non-zero parts of $\lambda$ is less than $(N + 1)$ let $L_\lambda$ denote the irreducible representation of $GL(N, \mathbb{C})$ with the highest weight $\lambda$, and given an arbitrary partition $\lambda$ let $\rho_\lambda$ be the irreducible representation of $S_{[\lambda]}$ over $\mathbb{C}$ associated to $\lambda$. Let $c_\lambda^{\mu_1, \ldots, \mu^n}$ be a non-negative integer defined in either of the following two (equivalent) ways:

\begin{align*}
(1.1) & \quad c_\lambda^{\mu_1, \ldots, \mu^n} = \dim_{\mathbb{C}} \text{Hom}_{GL(N, \mathbb{C})}(L_\lambda, L_{\mu_1} \otimes \ldots \otimes L_{\mu^n}), \text{ or } \\
(1.2) & \quad \sum_{\mu_1, \ldots, \mu^n} c_\lambda^{\mu_1, \ldots, \mu^n} = \dim_{\mathbb{C}} \text{Hom}_{\mathfrak{g}[\mu_1] \otimes \ldots \otimes \mathfrak{g}[\mu^n]}(\text{res}_{S_{[\lambda]}} S_{\mu_1} \otimes \ldots \otimes S_{\mu^n}) \rho_\lambda, \rho_{\mu_1} \otimes \ldots \otimes \rho_{\mu^n}) .
\end{align*}

In (1.1) it is assumed that the number of non-zero parts of each of the partitions $\lambda, \mu^1, \ldots, \mu^n$, is less than $(N + 1)$. 


Let \( v^1, \ldots, v^n \in \mathbb{Z}_{\geq 0}, u = \sum_{a=1}^n v^a \). Assume \( k \) is algebraically closed. Let 
\( f_{v^1, \ldots, v^n} = \sum_{a=0}^n v^a v^b \) be the dimension of the projective variety of all \( n \)-step partial flags in \( k^u \) with dimensions of the subfactors given by \( v^1, \ldots, v^n \).

**Theorem.**

1.a. If \( k \) is algebraically closed then the variety \( \mathfrak{M}^\lambda_{\mu^1 \ldots \mu^n}(k) \) is empty or has pure dimension
\[
\dim \mathfrak{M}^\lambda_{\mu^1 \ldots \mu^n}(k) = f_{|\mu^1| \ldots |\mu^n|}^{|\lambda|} + \frac{1}{2}(o_\lambda + o_{\mu^1} + \ldots + o_{\mu^n}).
\]

1.b. If \( k \) is algebraically closed then the number of irreducible components of 
\( \mathfrak{M}^\lambda_{\mu^1 \ldots \mu^n}(k) \) is equal to \( c^\lambda_{\mu^1 \ldots \mu^n} \).

1.c. There exists a computable polynomial \( n^\lambda_{\mu^1 \ldots \mu^n} \) with integer coefficients such that 
the cardinal of the set \( \mathfrak{M}^\lambda_{\mu^1 \ldots \mu^n}(\mathbb{F}_p) \) is equal to \( n^\lambda_{\mu^1 \ldots \mu^n}(p) \) for any prime number \( p \) and any positive integer \( l \).

1.d. The polynomial \( n^\lambda_{\mu^1 \ldots \mu^n} \) is identically equal to zero or has degree
\[
\deg n^\lambda_{\mu^1 \ldots \mu^n} = f_{|\mu^1| \ldots |\mu^n|}^{|\lambda|} + \frac{1}{2}(o_\lambda + o_{\mu^1} + \ldots + o_{\mu^n}).
\]

1.e. The leading coefficient of the polynomial \( n^\lambda_{\mu^1 \ldots \mu^n} \) is equal to \( c^\lambda_{\mu^1 \ldots \mu^n} \).

**Remark.** \( n^\lambda_{\mu^1 \ldots \mu^n}(x) = o_x(x) h^\lambda_{\mu^1 \ldots \mu^n}(x) \), where \( h^\lambda_{\mu^1 \ldots \mu^n} \) is the Hall polynomial (cf. [Mac95, Chapter II]).

**Proof.** Statement 1.a is proven by Spaltenstein [Spa82, II.5]; 1.c, 1.d, and 1.e are due to Steinitz [Ste01] and Hall [Hall54] (see also [Mac95, Chapter III]); 1.b follows from 1.a and 1.e.

Definitions [L.1] and [L.2] of \( c^\lambda_{\mu^1 \ldots \mu^n} \) are equivalent due to Schur-Weyl duality. However if one wants to generalize Theorem 1 to reductive groups other than \( GL(N) \) there are various directions. For example certain subvarieties of parabolic flag varieties play the role of \( \mathfrak{M}^\lambda_{\mu^1 \ldots \mu^n} \) for restriction multiplicities in the representation theory of Weyl groups. In this note some other varieties are considered which are relevant to tensor product multiplicities. They were independently described by Nakajima [Nak01], Varagnolo and Vasserot [VV01], and the author [Mal01].

2. A Lie algebra \( \mathfrak{g}' \). Let \( \mathfrak{g} \) be a simple simply laced Lie algebra over \( \mathbb{C} \) and \( \mathfrak{t} \) be a Cartan subalgebra of \( \mathfrak{g} \). Let \( \mathfrak{g}' = \mathfrak{g} \oplus \mathfrak{t} \) (a reductive Lie algebra) and let \( \mathfrak{t}' = \mathfrak{t} \oplus \mathfrak{t} \subset \mathfrak{g} \oplus \mathfrak{t} \) be a Cartan subalgebra of \( \mathfrak{g}' \). Let \( I \) be the set of vertices of the Dynkin graph of \( \mathfrak{g} \). Identify the weight lattice \( \mathcal{Q}_\mathfrak{g} \) of \( \mathfrak{g} \) with \( \mathbb{Z}[I] \) in such a way that \( i \in I \subset \mathbb{Z}[I] \) is the simple weight corresponding to the vertex \( i \). Let \( \mathcal{Q}^+_{\mathfrak{g}} = \mathbb{Z}_{\geq 0}[I] \) be the set of positive weights. Fix a lattice \( \mathcal{Q}_t \) in \( \mathfrak{t}' = \mathfrak{t} \oplus \mathfrak{t} \) and identify \( \mathcal{Q}_t \) with \( \mathbb{Z}[I] \) in some way. Let \( \mathcal{Q}^+_{\mathfrak{g}'} = \mathbb{Z}_{\geq 0}[I] \). In what follows representations of \( \mathfrak{g}' \) are assumed to have the action of \( \{0\} \oplus \mathfrak{t} \) given by elements of the lattice \( \mathcal{Q}_t \). One has an injective map \( \varphi : \mathbb{Z}[I] \oplus \mathbb{Z}[I] \to \mathcal{Q}_\mathfrak{g} \oplus \mathcal{Q}_t \) given by \( \varphi((u,v)) = (u-v, u-v+A v) \), where \( A \) is the Cartan matrix of \( \mathfrak{g} \). Let \( \mathcal{Q}_{\mathfrak{g}'} \) be the image of the map \( \varphi \). In what follows \( \mathcal{Q}_{\mathfrak{g}'} \) is called the weight lattice of \( \mathfrak{g}' \). For \( \xi \in \mathcal{Q}_{\mathfrak{g}'} \subset \mathcal{Q}_\mathfrak{g} \oplus \mathcal{Q}_t \) let \( |\xi| \) denote the \( \mathcal{Q}_t \)-component of \( \xi \). A weight \( \xi \in \mathcal{Q}_{\mathfrak{g}'} \) is called integrable if \( \varphi^{-1}(\xi) \in \mathbb{Z}_{\geq 0}[I] \oplus \mathbb{Z}_{\geq 0}[I] \subset \mathbb{Z}[I] \oplus \mathbb{Z}[I] \). Let \( \mathcal{Q}^+_{\mathfrak{g}'} \) be the set of integrable weights, and let \( \mathcal{Q}^+_{\mathfrak{g}'} = \mathcal{Q}^+_{\mathfrak{g}'} \cap (\mathcal{Q}^+_{\mathfrak{g}} \oplus \mathcal{Q}^+_{\mathfrak{t}}) \). Elements of \( \mathcal{Q}^+_{\mathfrak{g}'} \) are called positive integrable weights.

Given \( \lambda \in \mathcal{Q}^+_{\mathfrak{g}'} \) let \( L_\lambda \) be an irreducible highest weight representation of \( \mathfrak{g}' \) with the highest weight \( \lambda \). The representation \( L_\lambda \) is finite dimensional and all its weights
A finite dimensional representation of \( \mathfrak{g}' \) is called integrable if all its irreducible components have positive integrable highest weights. The category of finite dimensional integrable representations is closed with respect to tensor product. Let \( d_{\eta_1, \ldots, \eta^n}^{\xi} \) be a tensor product multiplicity:

\[
d_{\eta_1, \ldots, \eta^n}^{\xi} = \dim_c \operatorname{Hom}_{\mathfrak{g}'}(L_{\xi}, L_{\eta^1} \otimes \cdots \otimes L_{\eta^n}) ,
\]

where \( \xi, \eta^1, \ldots, \eta^n \in Q_{\mathfrak{g}'}^+ \) and \( |\xi| = |\eta^1| + \cdots + |\eta^n| \).

3. An algebra \( \tilde{F} \). Recall that \( I \) is the set of vertices of the Dynkin graph of \( \mathfrak{g} \). Let \( H \) be the set of pairs consisting of an edge of the Dynkin graph together with an orientation of the edge. Given a field \( k \) let \( F(k) \) be the path algebra of the Dynkin graph over \( k \), and let \( \cdot \) denote the multiplication in \( F(k) \). Fix a map \( \varepsilon : H \to \{ \pm 1 \} \) such that \( \varepsilon(h) = -\varepsilon(\overline{h}) \), where \( \overline{h} \) denotes the same edge as \( h \) but with the opposite orientation. Let \( \vartheta = \sum_{h \in H} \varepsilon(h)h \cdot \overline{h} \in F(k) \), and let \( \tilde{F}(k) \) be an associative \( k \)-algebra defined as follows. As a \( k \)-linear space \( \tilde{F}(k) = F(k) \oplus \bigoplus_{i \in I} ku_i \), where \( u_i \) are some symbols, and the multiplication \( \circ \) in \( \tilde{F}(k) \) is given by

\[
f \circ f' = f \cdot \vartheta \cdot f', \quad f \circ u_i = f \cdot [i], \quad u_i \circ u_j = \delta_{ij} u_i, \quad u_i \circ f = [i] \cdot f,
\]

where \( f, f' \in F(k) \), \( i, j \in I \), \( [i] \in F(k) \) is the path of length zero beginning and ending at a vertex \( i \in I \), and \( \delta_{ij} \) is the Kronecker symbol. The algebra \( \tilde{F}(k) \) was introduced by Lusztig. It is finitely generated as a \( k \)-algebra (cf. \[Lus00\], 2.2).

Let \( k^I \) be a semisimple \( k \)-algebra isomorphic to the direct sum of \( |I| \) copies of \( k \). In what follows \( \tilde{F}(k) \) and \( \tilde{F}(k) \) are considered as \( k^I \)-algebras with the embedding of the set of idempotents of \( k^I \) into \( \tilde{F}(k) \) (resp. \( \tilde{F}(k) \)) given by \( \{ [i] \} \) (resp. \( \{ u_i \} \)).

Let \( \mathbf{D} \) be a \( k^I \)-module (i.e. \( \mathbb{Z}[I] \)-graded \( k \)-linear space). A representation of \( \tilde{F}(k) \) in \( \mathbf{D} \) is a \( k^I \)-algebra homomorphism \( \tilde{F}(k) \to \operatorname{End}_{k^I} \mathbf{D} \). Let \( \mathbf{D} = \tilde{F}(k) \otimes_{k^I} \mathbf{D} \) (cf. \[Lus98\], \[Lus00\]). \( \mathbf{D} \) is naturally a left \( F(k) \)-module. A representation \( \pi \) of \( \tilde{F}(k) \) in \( \mathbf{D} \) is given by \( \pi \in \operatorname{Hom}_{k^I}(\tilde{F}(k), \mathbf{D}) \) by given \( \pi(f \otimes d) = \pi(f)d \), and let \( \mathcal{K}_\pi \) be the maximal \( F(k) \)-submodule of \( \mathbf{D} \) contained in the kernel of \( \pi \). Let \( \xi \in Q_{\mathfrak{g}'}^+ \) be given by \( \xi = \pi(\mathbf{d} + \mathbf{v} - A\mathbf{v}) \), where \( \mathbf{d} = \dim_k \mathbf{D}, \mathbf{v} = \dim_k (\mathbf{D} / \mathcal{K}_\pi) \) (note that \( |\xi| = d \)). Such \( \xi \) is called the type of \( \pi \). Two representations of the same type are not necessarily isomorphic. Given \( \xi \in Q_{\mathfrak{g}'}^+ \) let \( \mathcal{X}_\xi(k) \) be the set of all representations of \( \tilde{F}(k) \) in \( k^{|\xi|} \) of type \( \xi \). It is known (cf. \[Nak98\], \[Lus98\], \[Lus00\], \[CB00\]) that \( \mathcal{X}_\xi(k) \) is non-empty if and only if \( \xi \in Q_{\mathfrak{g}'}^{++} \), and that if \( k \) is algebraically closed then \( \mathcal{X}_\xi(k) \) is empty or is an irreducible smooth quasi-projective variety over \( k \) of dimension \( 2 \sum_{i \in I} |\xi| - A\mathbf{v}_i \), where \( (\mathbf{u}, \mathbf{v}) = x^{-1}(\xi) \).

4. Generalized Hall theorem. Let \( \xi, \eta^1, \ldots, \eta^n \in Q_{\mathfrak{g}'}^+ \), |\( \xi| = |\eta^1| + \cdots + |\eta^n| \), and let \( \mathcal{P}_{\xi, \eta^1, \ldots, \eta^n}(k) \) be the set of all pairs \((\pi, \mathcal{D})\) consisting of a representation \( \pi \) of \( \mathcal{F}(k) \) in \( k^{|\xi|} \) of type \( \xi \) and an \( n \)-step filtration \( \mathcal{D} = (\{ 0 \} = D^0 \subset D^1 \subset \cdots \subset D^n = k^{|\xi|}) \) of \( k^{|\xi|} \) by subrepresentations of \( \pi \) such that \( \pi|_{P_a} \) has type \( \eta^a \) for \( a = 1, \ldots, n \). If \( k \) is algebraically closed then \( \mathcal{P}_{\xi, \eta^1, \ldots, \eta^n}(k) \) is a quasi-projective variety over \( k \). The variety \( \mathcal{P}_{\xi, \eta^1, \ldots, \eta^n}(k) \) is a variant of a “multiplicity variety” of \[Mal01\]. It is also implicitly contained in \[Nak01\] and \[V00\]. The set \( \mathcal{P}_{\xi, \eta^1, \ldots, \eta^n}(F_p) \) can be identified with the set of \( F_p \)-rational points in the variety \( \mathcal{P}_{\xi, \eta^1, \ldots, \eta^n}(\overline{F}_p) \).
Let $v^1, \ldots, v^n \in \mathbb{Z}_{\geq 0}[I]$, $u = \sum_{i=1}^n v^i$. Assume $k$ is algebraically closed. Let $g_{v_1, \ldots, v_n} = \sum_{a < b} v^a v^b$ be the dimension of the projective variety of all $n$-step filtrations of $k^u$ by $k^L$-submodules with dimensions of the subfactors given by $v^1, \ldots, v^n$.

**Theorem.**

4.a. If $k$ is algebraically closed then the variety $\mathfrak{F}_{v_1, \ldots, v_n}^\xi(k)$ is empty or has pure dimension

$$\dim \mathfrak{F}_{v_1, \ldots, v_n}^\xi(k) = g_{v_1, \ldots, v_n}^{|\xi|} + \frac{1}{2}(x_{v_1} + x_{v_2} + \ldots + x_{v_n}).$$

4.b. If $k$ is algebraically closed then the number of irreducible components of $\mathfrak{F}_{v_1, \ldots, v_n}^\xi(k)$ is equal to $d_{v_1, \ldots, v_n}^\xi$.

4.c. There exists a computable polynomial $p^\xi_{\eta_1, \ldots, \eta_n}$ with integer coefficients such that the cardinal of the set $\mathfrak{F}_{v_1, \ldots, v_n}^\xi(F_p)$ is equal to $p^\xi_{\eta_1, \ldots, \eta_n}(p^i)$ for any prime number $p$ and any positive integer $i$.

4.d. The polynomial $p^\xi_{\eta_1, \ldots, \eta_n}$ is identically equal to zero or has degree

$$\deg p^\xi_{\eta_1, \ldots, \eta_n} = g_{v_1, \ldots, v_n}^{|\xi|} + \frac{1}{2}(x_{v_1} + x_{v_2} + \ldots + x_{v_n}).$$

4.e. The leading coefficient of the polynomial $p^\xi_{\eta_1, \ldots, \eta_n}$ is equal to $d_{v_1, \ldots, v_n}^\xi$.

**Proof.** Statement 4.a is proven in [Nak01] and [Mal01]. In these papers the base field $k$ is assumed to be $\mathbb{C}$, but the proofs work for arbitrary algebraically closed field. 4.b is proven in [Mal01] (it is also a corollary of [Nak01, Lemma 4.4] or [VV01, Theorem 5.3]). Statements 4.d and 4.e follow from 4.a, 4.b, and 4.c.

To prove 4.c one can use the inductive argument in [Lus00, 6.6] replacing the variety (notation of [Lus00] $\Lambda^\ast_{\mathfrak{D}, \mathfrak{V}}$ (resp. $\Lambda^\ast_{\mathfrak{D}, \mathfrak{V}}$) by $\mathfrak{F}^{\mathfrak{h}_{\mu_1, \ldots, \mu_n}}_{\eta_1, \ldots, \eta_n}$ (resp. a tensor product variety – cf. [Mal01]). To prove that the number of $\mathbb{F}_p$-rational points of a tensor product variety depends polynomially on $p$ note (cf. [Mal01]) that a tensor product variety is a finite union of disjoint subsets each of which is a vector bundle over a fibration with the base equal to a $\mathbb{Z}[I]$-graded partial flag variety and fibers isomorphic to $\Lambda^\ast_{\mathfrak{D}_1, \mathfrak{V}_1, \mathfrak{U}_1} \times \cdots \times \Lambda^\ast_{\mathfrak{D}_n, \mathfrak{V}_n, \mathfrak{U}_n}$ for some $\mathfrak{D}^1, \ldots, \mathfrak{D}^n, \mathfrak{V}_1, \ldots, \mathfrak{V}_n, \mathfrak{U}_1, \ldots, \mathfrak{U}_n$ (again notation of [Lus00]). Thus it follows from [Lus00, Section 6] that the number of $\mathbb{F}_p$-rational points of a tensor product variety is given by a computable polynomial in $p$.

**Remark.** The Hall polynomials $h^\mu_{\nu_1, \nu_2}$ (cf. Section 1) give structure constants of the Hecke algebra of $GL(N)$ in the basis of characteristic functions of the double cosets. The author does not know what connection (if any) exists between the polynomials $p^\xi_{\eta_1, \ldots, \eta_n}$ and the Hecke algebra (of an extension of the algebraic group associated to $\mathfrak{g}$ by its Cartan subgroup).

5. $A_{N-1}$ case. In this section it is explained why Theorem 4. The argument is quite standard (cf. [KP79]). Throughout the section $\mathfrak{g}$ is assumed to be $\mathfrak{sl}_N$, and the set of vertices $I$ of its Dynkin graph is identified with $\{1, 2, \ldots, N-1\}$ in such a way that two vertices are connected by an edge if and only if they are consequent integers.

Note that in the definition (5.3) of $c^\mu_{\nu_1, \ldots, \nu_n}$ the number of non-zero parts of each of the partitions $\lambda, \mu_1, \ldots, \mu_n$, is assumed to be less than $(N+1)$. Thus representations of $k[[t]]$ involved in the definition of $\mathfrak{F}^\lambda_{\mu_1, \ldots, \mu_n}(k)$ factor through $k[[t]]/k^{N}k[[t]]$. 


Recall that $[1] \in \mathcal{F}(k)$ denotes the path of length zero beginning and ending at the vertex 1.

**Proposition.** Let $\mathcal{I}$ be a two-sided ideal of $\tilde{\mathcal{F}}(k)$ generated by $u_2, u_3, \ldots, u_{N-1}$.

5.a. The factor algebra $\mathcal{F}(k)/\mathcal{I}$ is generated as a $k$-algebra by $(u_1+\mathcal{I})$ and $(1+\mathcal{I})$.

5.b. $(1 \circ 1 \circ \ldots \circ 1) \in \mathcal{I}$.

5.c. A $k$-algebra homomorphism $\varphi : k[[t]]/t^N k[[t]] \to \mathcal{F}(k)/\mathcal{I}$ uniquely defined by $\varphi(1+t^N k[[t]]) = u_1+\mathcal{I}$ and $\varphi(t+t^N k[[t]]) = [1]+\mathcal{I}$ is an isomorphism.

**Proof.** 5.a and 5.b are straightforward. Due to 5.a the map $\varphi$ is well-defined, and due to 5.b it is surjective. According to [KP79] there exists a faithful representation of $k[[t]]/t^N k[[t]]$ which factors through $\varphi$. Thus $\varphi$ is injective. 5.c follows.

Note that a representation of $\tilde{\mathcal{F}}(k)$ factors through $\mathcal{F}(k)/\mathcal{I}$ if and only if its type $\eta$ satisfies the condition: $|\eta|_i = 0$ for $i = 2, \ldots, N-1$. Given such $\eta$ let $(u, \nu) = \nu^{-1}(\eta)$ and let $\nu(\eta)$ be an $N$-tuple of integers given by $\nu(\eta)_i = |\eta|_i - \nu_1$, $\nu(\eta)_i = \nu_{i-1} - \nu_i$ for $i = 2, \ldots, N-1$, and $\nu(\eta)_N = \nu_{N-1}$. One has $\eta \in \mathcal{Q}_g^+$ if and only if $\nu(\eta)$ is an ordered partition (i.e. $0 \leq \nu(\eta)_i \leq \nu(\eta)_j$ for any $i \geq j$). Thus $\nu$ provides a bijection between types of representations of $\tilde{\mathcal{F}}(k)/\mathcal{I}$ and those of $k[[t]]/t^N k[[t]]$. Moreover if a representation $\pi$ of $\tilde{\mathcal{F}}(k)/\mathcal{I}$ is of type $\eta$ then the representation $\pi \varphi$ of $k[[t]]/t^N k[[t]]$ is of type $\nu(\eta)$, and one has the following bijection (isomorphism of varieties if $k$ is algebraically closed):

$$\mathcal{Q}_{\eta_1 \ldots \eta_n}(k) \simeq \mathcal{Q}_{\nu(\eta_1) \ldots \nu(\eta_n)}(k).$$

Therefore Theorem 1 follows from Theorem 4.

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