ON AN EXTENSION OF THE BLASCHKE-SANTALÓ INEQUALITY

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Abstract. Let $K$ be a convex body and $K^\circ$ its polar body. Call $\phi(K) = \frac{1}{|K||K^\circ|} \int_K \int_{K^\circ} \langle x, y \rangle^2 dxdy$. It is conjectured that $\phi(K)$ is maximum when $K$ is the euclidean ball. In particular this statement implies the Blaschke-Santaló inequality. We verify this conjecture when $K$ is restricted to be a $p$–ball.

1. Introduction and notation

A convex body $K \subset \mathbb{R}^n$ is a compact convex set with non-empty interior. For every convex body, its polar set is defined

$$K^\circ = \{ x \in \mathbb{R}^n : \langle x, y \rangle \leq 1 \text{ for all } y \in K \}$$

where $\langle \cdot, \cdot \rangle$ denotes the standard scalar product in $\mathbb{R}^n$. Note that if $0 \in \text{int}K$ then $K^\circ$ is a convex body.

For $p \in [1, \infty]$, let us denote by $B^n_p$ the unit ball of the $p$–norm. It is:

$$B^n_p = \left\{ x \in \mathbb{R}^n : \sum_{i=1}^n |x_i|^p \leq 1 \right\}$$

$$B^n_\infty = \{ x \in \mathbb{R}^n : \max |x_i| \leq 1 \}.$$

It is well known that the polar body of $B^n_p$ is $B^n_q$ where $q$ is the dual exponent of $p \left( \frac{1}{p} + \frac{1}{q} = 1 \right)$. Along this paper $q$ will always denote the dual exponent of $p$.

Given two symmetric convex bodies $A \subset \mathbb{R}^n$, $B \subset \mathbb{R}^m$, for any $p \in [1, \infty]$ they define a symmetric convex body $A \times_p B \subset \mathbb{R}^{n+m}$ which is the unit ball of the norm given by

$$\|(x_1, x_2)^p_{A \times_p B} = \|x_1\|^p_A + \|x_2\|^p_B$$

$$\|(x_1, x_2)\|_{A \times \infty B} = \max\{|x_1| A, |x_2| B\}.$$\n
Note that the polar body of $A \times_p B$ is $A^\circ \times_q B^\circ$ and $B^n_p = B^n_{p^{-1}} \times_p [-1, 1]$.

A convex body $K$ is said to be in isotropic position if it has volume 1 and satisfies the following two conditions:

- $\int_K xdx = 0$ (center of mass at 0)
- $\int_K \langle x, \theta \rangle^2 dx = L^2_K \quad \forall \theta \in S^{n-1}$

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where $L_K$ is a constant independent of $\theta$, which is called the isotropy constant of $K$.

We will use the notation $\widetilde{K}$ for $|K|^{-\frac{1}{n}}K$.

Given a centrally symmetric convex body $K$, we call
\[
\phi(K) = \frac{1}{|K||K^\circ|} \int_K \int_{K^\circ} \langle x, y \rangle^2 dx dy.
\]
Note that $\phi(K) = \phi(TK)$ for all $T \in GL(n)$. It is conjectured in [5] that $\phi(K)$ is maximized by ellipsoids. It is, for every symmetric convex body $K \subset \mathbb{R}^n$
\[
\phi(K) \leq \phi(B_n^2) = \frac{n}{(n + 2)^2}.
\]

(1)

**Remark.** We can also define the functional $\phi$ when $K$ is not symmetric. When $K$ is a regular simplex with its center of mass at the origin, it is easy to compute that $\phi(K) = \phi(B_n^2)$.

The Blaschke-Santaló inequality [6] says that for every symmetric convex body $K$
\[
|K||K^\circ| \leq |B_2^n|^2.
\]

The conjecture (1) is stronger than the Blaschke-Santaló inequality since
\[
\frac{n|K|\|K^\circ\|}{(n + 2)^2|B_2^n|^2} \leq \frac{1}{|K||K^\circ|} \int_K \int_{K^\circ} \langle x, y \rangle^2 dx dy.
\]
This fact is a consequence of Lemma 6 in [2]. In [3], Ball proved that for 1-unconditional bodies
\[
\int_K \int_{K^\circ} \langle x, y \rangle^2 dx dy \leq \frac{n|B_2^n|^2}{(n + 2)^2}
\]
and suggested that this inequality might be true for every convex body. This assertion is slightly weaker than the conjecture in [5], which is not known to be true even for 1-unconditional bodies. In section 2 we are going to prove that the conjecture is true if we restrict $K$ to be a $p$-ball, for some $p \geq 1$. We state this as a theorem:

**Theorem 1.1.** Among the $p$-balls, the functional $\phi$ is maximized for the euclidean ball.
\[
\max_{p \in [1, \infty]} \phi(B_p^n) = \phi(B_2^n) = \frac{n}{(n + 2)^2}.
\]

The conjecture (1) is also stronger than the hyperplane conjecture, which says that there exists an absolute constant $C$ such that for every isotropic convex body $L_K < C$. It can be proved that $\phi(K)$ is bounded from below by $\frac{c_1}{n}$, where $c_1$ is an absolute constant. If there exists an absolute constant $c_2$ such that $\phi(K) \leq \frac{c_2}{n}$, then the hyperplane conjecture would be true, since
\[
nL_K^2 L_K^2 \leq \frac{\phi(K)}{|K||K^\circ|} \leq c_2 \phi(K)
\]
where $c$ is an absolute constant.
In case that \( \widetilde{K} \) and \( \widetilde{K}^o \) are both isotropic then \( \phi(K) = n|K|\frac{\bar{K}}{|K^o|}L_K^2L_K^2 \) and the conjecture \( \phi(K) \sim \frac{1}{n} \) is equivalent to the hyperplane conjecture. This is the case of 1-symmetric bodies, for which the hyperplane conjecture is known to be true (A convex body is 1-symmetric if it is invariant under reflections in the coordinate hyperplanes and under permutations of the coordinates).

We will say that a symmetric convex body \( K \subset \mathbb{R}^n \) is a revolution body if there exists \( \theta \in S^{n-1} \) and a concave function \( r(t) \) such that for every \( t \in [-h_K(\theta), h_K(\theta)] \) \( K \cap (t\theta + \theta^\perp) = r(t)B_2^{n-1} \), where \( h_K(\theta) \) is the support function of \( K \):

\[
h_K(\theta) = \max\{\langle x, \theta \rangle : x \in K\}.
\]

In section 3 we will prove that there exists an absolute constant \( C \) such that whenever \( K \) is a symmetric convex body of revolution, \( \phi(K) \leq \frac{C}{n} \).

Along this paper, \( \psi \) will always denote the logarithmic derivative of the Gamma function. We will make use of the following identity on the derivatives of \( \psi \), known as polygamma functions:

\[
\psi^{(n)}(x) = (-1)^{n+1} \int_0^\infty t^n \frac{e^{-xt}}{1 - e^{-t}} dt.
\]

The letters \( C, c_1, c_2, \ldots \) will always denote absolute constants which do not depend on the dimension.

2. The \( p \)-balls

In this section we are going to prove theorem 1.1. We will obtain it as a consequence of the following

**Theorem 2.1.** For every \( A \subset \mathbb{R}^n, B \subset \mathbb{R}^m, p \in [1, \infty] \)

\[
\phi(A \times_p B) = f(n, n + m, p)\phi(A) + f(m, n + m, p)\phi(B)
\]

where

\[
f(y_1, y_2, p) = \begin{cases} 
\frac{(y_1+2)^2y_2^2\Gamma\left(\frac{y_1+2}{2}\right)\Gamma\left(\frac{y_2+2}{2}\right)\Gamma\left(\frac{y_1+2}{2}\right)}{y_1^2(y_2+2)^2\Gamma\left(\frac{y_1+2}{2}\right)\Gamma\left(\frac{y_2+2}{2}\right)\Gamma\left(\frac{1}{2}\right)} & p \neq 1, \infty \\
\frac{(y_1+2)y_2\Gamma\left(y_1+2\right)}{y_1(y_2+2)\Gamma\left(y_1+2\right)} & p = 1, \infty
\end{cases}
\]

attains its maximum when \( p = 2 \), for every \( 0 < y_1 < y_2 \).

**Proof.** First of all we are going to prove that for every fixed \( 0 < y_1 < y_2 \), the function defined on \([0, 1]\) like \( f_1(x) = f(y_1, y_2, \frac{1}{x}) \) attains its maximum in \( x = \frac{1}{2} \). It is easy to check that \( f_1(0) = f_1(1) < f_1\left(\frac{1}{2}\right) \). \( f_1 \) has got a maximum in \( x = \frac{1}{2} \) if and only if \( \log f_1 \) has got a maximum in \( x = \frac{1}{2} \).

Since \( f_1(x) = f_1(1-x) \), it is enough to prove that \( \log f_1 \) is increasing in \((0, \frac{1}{2})\).

Now, if we call

\[
F(x, y) = (y + 2)[\psi((y + 2)x) - \psi((y + 2)(1 - x))] - y[\psi(yx) - \psi(y(1-x))]
\]
we have that
\[(\log f_1)'(x) = (y_1 + 2)[\psi((y_1 + 2)x) - \psi((y_1 + 2)(1 - x))] - y_1[\psi(y_1x) - \psi(y_1(1 - x))] - (y_2 + 2)[\psi((y_2 + 2)x) - \psi((y_2 + 2)(1 - x))] + y_2[\psi(y_2x) - \psi(y_2(1 - x))]
= F(x, y_1) - F(x, y_2).
\]

So it is enough to prove that for every fixed \(x \in (0, \frac{1}{2})\), \(F(x, y)\) is decreasing in \(y \in (0, \infty)\). Hence we compute
\[
\frac{\partial F}{\partial y}(x, y) = \psi((y + 2)x) - \psi((y + 2)(1 - x)) - \psi(yx) + \psi(y(1 - x))
+ (y + 2)x\psi'((y + 2)x) - (y + 2)(1 - x)\psi'((y + 2)(1 - x))
- yx\psi'((yx) + y(1 - x)\psi'(y(1 - x)).
\]

We call this last quantity \(G(x, y)\) and we will see that \(G(x, y) < 0\) if \(x \in (0, \frac{1}{2})\) and \(G(x, y) > 0\) if \(x \in (\frac{1}{2}, 1)\). Notice that \(G(\frac{1}{2}, y) = 0\), so we just need to check that for every fixed \(y > 0\), \(G(x, y)\) is increasing in \(x\). Computing its derivative we obtain
\[
\frac{\partial G}{\partial x}(x, y) = 2(y + 2)[\psi'((y + 2)x) + \psi'((y + 2)(1 - x))]
+ (y + 2)2[x\psi''((y + 2)x) + (1 - x)\psi''((y + 2)(1 - x))] - 2y[\psi'(yx) + \psi'(y(1 - x))]
- y^2[\psi''(yx) + (1 - x)\psi''(y(1 - x))]
= H(x, y + 2) - H(x, y).
\]

where we have called \(H(x, y)\) the following function
\[H(x, y) = 2y[\psi'(yx) + \psi'(y(1 - x))] + y^2[x\psi''(yx) + (1 - x)\psi''(y(1 - x))].\]

Hence, if for every fixed \(x \in (0, 1)\) \(H(x, y)\) is increasing in \(y\), then so it is \(G(x, y)\) in \(x\) for fixed \(y\) and the theorem is proved. In order to prove this, we need the following result concerning the \(\psi\) function whose proof can be found in [1]. We will write it here for the sake of completeness:

**Proposition 2.1.** The function \(f(x) = x^2\psi'(x)\) is convex in the interval \((0, \infty)\).

**Proof.** The second derivative of \(f\) is
\[f''(x) = 2\psi'(x) + 4x\psi''(x) + x^2\psi'''(x).
\]

Using the integral representation of the derivatives of \(\psi\) this is equal to
\[
f''(x) = \int_0^\infty \frac{e^{-xt}}{1 - e^{-t}}(2t - 4xt^2 + x^2t^3)dt
= \int_0^\infty \frac{t}{1 - e^{-t}} \frac{d^2}{dt^2}(t^2 e^{-xt})dt
= \int_0^\infty \frac{d^2}{dt^2} \left(\frac{t}{1 - e^{-t}}\right) t^2 e^{-xt}dt
\]
which is positive since the function \(\frac{t}{1 - e^{-t}}\) is convex in the interval \((0, \infty)\). \(\Box\)
Now, for every $x \in (0,1)$, $y > 0$ we have that
\[
\frac{\partial H}{\partial y}(x, y) = 2\psi'(yx) + 4yx\psi''(yx) + y^2x^2\psi'''(yx)
+ 2\psi'(y(1-x)) + 4y(1-x)\psi''(y(1-x)) + y^2(1-x)^2\psi'''(y(1-x)) > 0
\]
as a consequence of proposition 2.1 and this proves that $f(y_1, y_2, p) \leq f(y_1, y_2, 2)$
when $0 < y_1 < y_2$.

Let us prove now that
\[
\phi(A \times_p B) = f(n, n + m, p)\phi(A) + f(m, n + m, p)\phi(B).
\]
Assume that $p \neq 1, \infty$. We compute the volume of $A \times_p B$:
\[
|A \times_p B| = \int_A (1 - \|x_1\|^p_A)^{\frac{m}{p}}|B|dx_1 = \int_A \int_{\|x_1\|^p_A}^{1} \frac{m}{p} (1 - t)^{\frac{m}{p} - 1} dt |B| dx_1
= \int_{0}^{1} \int_{A} \frac{m}{p} (1 - t)^{\frac{m}{p} - 1} |B| dx_1 dt = \frac{m}{p} |A||B| \beta \left( \frac{m}{p}, 1, \frac{n}{p} \right)
= \frac{nm}{p(n + m)} |A||B| \beta \left( \frac{m}{q}, \frac{n}{q} \right).
\]

Since $(A \times_p B)^o = A^o \times_q B^o$, we have that
\[
|(A \times_p B)^o| = \frac{nm}{q(n + m)} |A^o||B^o| \beta \left( \frac{m}{q}, \frac{n}{q} \right).
\]

From the symmetry of $A$ and $B$ we obtain that
\[
\int_K \int_{K^o} (x_1, x_2, (y_1, y_2))^2 dydx = \int_K \int_{K^o} (x_1, y_1)^2 dydx + \int_K \int_{K^o} (x_2, y_2)^2 dydx
\]
where we have called $K = A \times_p B$.

Let us compute these integrals:
\[
\int_{K^o} (x_1, y_1)^2 dydx
= \int_A \int_{A^o} (x_1, y_1)^2 (1 - \|x_1\|^p_A)^{\frac{m}{p}} (1 - \|y_1\|^q_{A^o})^{\frac{m}{q}} |B||B^o| dy_1 dx_1
= |B||B^o| \int_{A} \int_{A^o} (x_1, y_1)^2 \int_{\|x_1\|^p_A}^{1} \frac{m}{p} (1 - t)^{\frac{m}{p} - 1} dt \int_{\|y_1\|^q_{A^o}}^{1} \frac{m}{q} (1 - s)^{\frac{m}{q} - 1} ds dy_1 dx_1
= |B||B^o| \frac{m^2}{pq} \int_{0}^{1} \int_{A} \int_{A^o} \frac{m}{p} \left( \frac{n}{p} - 1 \right) \left( \frac{m}{q} - 1 \right) \beta \left( \frac{m}{p}, n + 2, \frac{n + 2}{p} \right) \beta \left( \frac{m}{q}, n + 2, \frac{n + 2}{q} \right) (x_1, y_1)^2 dy_1 dx_1
= |B||B^o| \frac{m^2(n + 2)^2}{pq} \left( \frac{m}{p} \right) \left( \frac{n + 2}{p} \right) \beta \left( \frac{m}{q}, n + 2, \frac{n + 2}{q} \right) \int_A \int_{A^o} (x_1, y_1)^2 dy_1 dx_1
\]
and in the same way
\[
\int_K (x_2, y_2)^2 dydx =
\]
Theorem 3.1. There exists an absolute constant $C$ such that for every symmetric convex body of revolution $K$, $\phi(K) < \frac{C}{n}$.

This is not a new result since A. Giannopoulos proved it in his PhD thesis but it was left unpublished. I would like to thank him for allowing me to add this result to this paper.

Proof. Since $\phi(TK) = \phi(K)$ for every $T \in GL(n)$, we can assume that

$$K = \{ \bar{x} = (t, x) \in \mathbb{R}^n : t \in [-1, 1], |x| \leq r_1(t) \}$$

where $r_1(t)$ is a concave function such that $r_1(0) = 1$.

Then, $K^o$ is another revolution body

$$K^o = \{ \bar{y} = (s, y) \in \mathbb{R}^n : ts + r_1(t)|y| \leq 1, \forall t \in [-1, 1] \}$$

where $r_2(s)$ is a concave function such that $r_2(0) = 1$.

Let us now compute $\phi(K)$:

$$\phi(K) = \frac{1}{|K||K^o|} \int_K \int_{K^o} (ts + (x, y))^2 d\bar{y} d\bar{x}$$

$$= \frac{1}{|K||K^o|} \int_K \int_{K^o} t^2 s^2 + (x, y)^2 d\bar{y} d\bar{x}$$

$$= \frac{1}{|K||K^o|} \int_K \int_{K^o} t^2 s^2 d\bar{y} d\bar{x}$$

$$+ \frac{1}{|K||K^o|} \int_{-1}^1 \int_{-1}^1 \int_{r_1(t)B_n^o}^{r_2(t)B_n^o} (x, y)^2 dy dx dt$$

$$= \frac{|K|}{|K^o|} \frac{\hat{\phi}}{\hat{\phi}} \int_{\hat{K}} \int_{\hat{K}^o} t^2 \bar{d} \bar{x} \int_{\hat{K}^o} s^2 \bar{d} \bar{y}$$

$$+ \frac{1}{\phi(B_n^o)} \int_{-1}^1 r_1(t)^{n+1} dt \int_{-1}^1 r_2(s)^{n+1} ds$$

$$\geq \int_{-1}^1 r_1(t)^{n-1} dt \int_{-1}^1 r_2(s)^{n-1} ds$$

Since $\max\{r_1(t), t \in [-1, 1]\} = r_1(0) = 1$ and $\max\{r_2(s), s \in [-1, 1]\} = r_2(0) = 1$, for every $t, s \in [-1, 1]$ we have that

- $r_1(t)^{n+1} \leq r_1(t)^{n-1}$
- $r_2(s)^{n+1} \leq r_2(s)^{n-1}$

Therefore, $\phi(K) < \frac{C}{n}$ and the theorem is proved in the same way. \qed

3. Revolution bodies

In this section we are going to prove the following:

**Theorem 3.1.** There exists an absolute constant $C$ such that for every symmetric convex body of revolution $K$, $\phi(K) < \frac{C}{n}$.

Now from the definition of $\phi$ and the identity $\beta(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ we obtain the result. When $p = 1, \infty$ the theorem is proved in the same way. \qed
and hence the second summand is bounded by \( \phi(B_2^{n-1}) = \frac{n-1}{(n+1)^2} \).

To bound the first summand we will use the following well known result by Hensley\[4\]:  

"There exist absolute constants \( c_1, c_2 \) such that for every symmetric convex body \( K \subset \mathbb{R}^n \) with volume 1 and for every \( \theta \in S^{n-1} \)

\[
\frac{c_1}{|K \cap \theta^\perp|} \leq \left( \int_K \langle x, \theta \rangle^2 \, dx \right)^{\frac{1}{2}} \leq \frac{c_2}{|K \cap \theta^\perp|}.
\]

Hence

- \( \int_K t^2 \, dx \leq \frac{c |K|^2}{|K \cap e_1|^2} = \frac{c |K|^2 - \frac{n}{n+1}}{|B_2^{n-1}|^2} \),
- \( \int_{K^c} s^2 \, dy \leq \frac{c |K^n|^2}{|K \cap e_1|^2} = \frac{c |K^n|^2 - \frac{n}{n+1}}{|B_2^n|^2} \).

So, by Blaschke-Santáló inequality, the first summand is bounded by

\[
\frac{c |K| |K^n|}{|B_2^{n-1}|^4} \leq \frac{c |B_2^n|^4}{|B_2^{n-1}|^4}.
\]

Now, using the fact that \( |B_2^n| = \frac{\pi^{\frac{n}{2}}}{\Gamma(1+\frac{n}{2})} \) and Stirling’s formula, we obtain that the first summand is bounded by \( \frac{c}{\pi^2} \) and hence the theorem is proved. \( \square \)

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