Inverse problems for nonlinear Helmholtz Schrödinger equations and time-harmonic Maxwell’s equations with partial data

Xuezhu Lu

July 1, 2022

Contents

1 Introduction and Main Results 2

2 Nonlinear Helmholtz Schrödinger equations 9
  2.1 Proof of Theorem 1.1 . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 9
  2.2 Proof of Theorem 1.2 . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 10
    2.2.1 Local results . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 11
    2.2.2 Extend to global results . . . . . . . . . . . . . . . . . . . . . . . . . . 13

3 Applications 15
  3.1 Simultaneous recovery of cavity and coefficients . . . . . . . . . . . . . . . 15
  3.2 Simultaneous recovery of boundary and coefficients . . . . . . . . . . . . . . 17

4 Nonlinear Maxwell equations 19
  4.1 Proof of Theorem . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 19
    4.1.1 Local result . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 20
    4.1.2 From local to global . . . . . . . . . . . . . . . . . . . . . . . . . . . . 23
  4.2 Proof of Theorem 1.4 . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 29
    4.2.1 Local result . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 30
    4.2.2 From local to global . . . . . . . . . . . . . . . . . . . . . . . . . . . . 34

A Appendix 36
  A.1 Well-posedness of nonlinear Helmholtz Schrödinger equation . . . . . . . . . 36
  A.2 Runge approximation . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 37
Abstract

We consider Calderón’s inverse boundary value problems for a class of nonlinear Helmholtz Schrödinger equations and Maxwell’s equations in a bounded domain in \( \mathbb{R}^n \). The main method is the higher order linearization of the Dirichlet-to-Neumann map of the corresponding equations. The local uniqueness of the linearized partial data Calderón’s inverse problem is obtained following [11]. The Runge approximation properties and unique continuation principle allow to extend to global situations. Simultaneous recovery of some unknown cavity/boundary and coefficients are given as some applications.

1 Introduction and Main Results

In this paper, we study the partial data inverse boundary value problems for a class of nonlinear Helmholtz Schrödinger equations and Maxwell’s equations in a bounded domain. We first consider the inverse boundary value problems of the following nonlinear Helmholtz Schrödinger equation

\[
\begin{align*}
-\Delta u(x) - k^2 u(x) + q(x)u^2(x) &= 0 \quad \text{in } \Omega, \\
u(x) &= f(x) \quad \text{on } \partial \Omega,
\end{align*}
\]

where \( f \in C^s(\partial \Omega) \), \( s > 1 \) with \( s \not\in \mathbb{Z} \). Here, \( \Omega \) is a bounded open set in \( \mathbb{R}^n \) with \( n \geq 2 \). Assume that the boundary \( \partial \Omega \) is smooth.

In Appendix A we show that except for a discrete set of real-valued \( k \), the Dirichlet problem \( (1) \) has a unique small solution \( u \in W^{2,p}(\Omega) \) for sufficiently small boundary data \( f \in W^{2-\frac{1}{p},p}(\partial \Omega) \), for \( p > n \). It is also shown in [36] that the boundary value problem admits a unique solution in \( C^s(\Omega) \) when \( \|f\|_{C^s(\partial \Omega)} \) is small in the case of \( k = 0 \); and then the full boundary Dirichlet-to-Neumann (DN) map \( \Lambda_q \) is well defined as follows:

\[
\Lambda_q : C^s(\partial \Omega) \rightarrow C^{s-1}(\partial \Omega), \quad f \mapsto \nu \cdot \nabla u_f |_{\partial \Omega},
\]

where \( u_f \) is the solution to \( (1) \) and \( \nu \) is the unit outer normal vector to the boundary \( \partial \Omega \).

Let \( \Gamma_1 \) and \( \Gamma_2 \subset \partial \Omega \) be two proper open subsets of the boundary. The \( W^{2,p} \) well-posedness guarantees that the partial boundary Dirichlet-to-Neumann map, denoted by \( \Lambda_{q,\Gamma_1,\Gamma_2} \), is well extended to

\[
\Lambda_{q,\Gamma_1,\Gamma_2}(f) := \Lambda_q(f)|_{\Gamma_2}, \quad \text{for all } f \in W^{2-\frac{1}{p},p}(\partial \Omega) \text{ with supp}(f) \subset \Gamma_1.
\]

We are interested in the unique determination of the nonlinear potential \( q(x) \) in \( \Omega \) from the partial DN map \( \Lambda_{q,\Gamma_1,\Gamma_2} \). Moreover, we solve an inverse problem of simultaneously determining the potential and an embedded obstacle or cavity in \( \Omega \) from the partial DN
map, similar to that in [37] where the problem was formulated for the semilinear elliptic equation $\Delta u + a(x, u) = 0$.

In the second part of the paper, we consider the inverse boundary value problems for two types of nonlinear Maxwell’s equations, that arise in potential applications using electromagnetic waves to explore and image the nonlinear properties of the media, using measurements of electromagnetic fields on part of the boundary. The first model of nonlinear electromagnetic medium we consider is for the Kerr-type, described by the boundary value problem

\[
\begin{aligned}
\nabla \wedge E - i\omega \mu_0 H &= i\omega p(x)|H|^2 H, \\
\nabla \wedge H + i\omega \varepsilon_0 E &= -i\omega q(x)|E|^2 E,
\end{aligned}
\]

in $\Omega \subseteq \mathbb{R}^3$, with $\nu \wedge E|_{\partial \Omega} = f$, where $\nu$ is the unit outer normal vector to the boundary $\partial \Omega$. The electrical permittivity $\varepsilon_0$ and the magnetic permeability $\mu_0$ are positive constants and $p(x), q(x) \in C^1(\overline{\Omega})$. Here, with some abuse of notations we still denote by $\Omega$ the domain for Maxwell’s equations in dimension three instead of arbitrary dimension as considered in the previous Helmholtz equation.

The above Maxwell’s equations of the Kerr-type are derived from the time-dependent Maxwell’s equations

\[
\nabla \wedge E + \partial_t B = 0, \quad \nabla \wedge H - \partial_t D = 0,
\]

for time harmonic solutions with a fixed frequency $\omega$

\[
E(x, t) = E(x)e^{-i\omega t} + \overline{E(x)}e^{i\omega t}, \quad H(x, t) = H(x, t)e^{-i\omega t} + \overline{H(x)}e^{i\omega t},
\]

assuming nonlinear constitutive relations of the displacement $D$ and induction $B$

\[
D = \varepsilon_0 E + P_{NL}(E), \quad B = \mu_0 \mathcal{H} + \mathcal{M}_{NL}(\mathcal{H}).
\]

Here $P_{NL}$ and $\mathcal{M}_{NL}$ are the nonlinear polarization and magnetization. In a Kerr-type electromagnetic medium, usually that of centrosymmetric structure, the nonlinear polarization is of the form

\[
P_{NL}(x, E(x, t)) = \chi_e(x, |E|^2)E(x, t),
\]

where $\chi_e$ is the scalar susceptibility depending on a memory term such as the time-average of the intensity of $E$. It is common to use $\chi_e(x, |E|^2) = p(x)|E|^2$. The reader is referred to [46, 51] for more details and other examples of electric nonlinear phenomena. Similarly, the nonlinearity can be generalized to magnetization by assuming

\[
\mathcal{M}_{NL}(x, H(x, t)) = \chi_m(x, |H|^2)H(x, t), \quad \chi_m(x, |H|^2) = q(x)|H|^2.
\]

Note that the time-averages of the intensities of $E$ and $H$ are $2|E|^2$ and $2|H|^2$ respectively.
Such nonlinear magnetizations appear in the study of certain metamaterials built by combining an array of wires and split-ring resonators embedded into a Kerr-type dielectric. See [41] for a numerical implementation of this nonlinear assumption.

The well-posedness of the forward problem of (2) is given in [1] that except for a discrete set of frequencies, there exists a unique solution \((E, H) \in W_{\text{Div}}^{1,p}(\Omega) \times W_{\text{Div}}^{1,p}(\Omega)\) for sufficiently small \(f\) in a \(L^p\) Sobolev space \(TW_{\text{Div}}^{1-1/p,p}(\partial \Omega)\) with \(3 < p \leq 6\) defined by

\[
W_{\text{Div}}^{1,p}(\Omega) = \{ u \in W^{1,p}(\Omega; \mathbb{C}^3) : \text{Div}(\nu \wedge u) \in W^{1-1/p,p}(\partial \Omega) \},
\]

\[
TW_{\text{Div}}^{1-1/p,p}(\partial \Omega) = \{ f \in W^{1-1/p,p}(\partial \Omega; \mathbb{C}^3) : \text{Div}(f) \in W^{1-1/p,p}(\partial \Omega) \},
\]

where \(\text{Div}\) is the surface divergence operator on \(\partial \Omega\). Thus the admittance map \(\Lambda_{p,q} : TW_{\text{Div}}^{1-1/p,p}(\partial \Omega) \to TW_{\text{Div}}^{1-1/p,p}(\partial \Omega)\) is well defined by

\[
\Lambda_{p,q}(f) := \nu \wedge H|_{\partial \Omega}.
\]

For the corresponding partial data inverse problem, we use the partial admittance map associated to two open non-empty open sets \(\Gamma_1\) and \(\Gamma_2\) of \(\partial \Omega\) as

\[
\Lambda_{p,q}^{\Gamma_1,\Gamma_2}(f) := \Lambda_{p,q}(f)|_{\Gamma_2} = \nu \wedge H|_{\Gamma_2}
\]

for all \(f \in C^3(\Gamma_1; \mathbb{C}^3) \cap TW_{\text{Div}}^{1-1/p,p}(\partial \Omega)\). We consider the inverse problem in determining the nonlinear coefficients \(p\) and \(q\) from this partial admittance map.

Another important nonlinear electromagnetic behavior of the materials is second harmonic generation (SHG). Applications of the nonlinear optical phenomena includes obtaining coherent radiation at a wavelength shorter than that of the incident laser, through the frequency doubling effect of SHG. Moreover, in the second-harmonic imaging microscopy (SHIM), a second-harmonic microscope obtains contrasts from variations in a specimen’s ability to generate second-harmonic light from the incident light while a conventional optical microscope obtains its contrast by detecting variations in optical density, path length, or refractive index of the specimen. The SHIM is also exploited in imaging flux residues (see the work in Chen Lab at the University of Michigan). Although nonlinear optical effects are in general very weak, the significant enhancement of SHG was shown using diffraction gratings or periodic structures.

We consider an inverse boundary value problem for Maxwell’s equations

\[
\begin{align*}
\nabla \wedge E^\omega - i\omega \mu_0 H^\omega &= 0 \\
\nabla \wedge H^\omega + i\omega \varepsilon_0 E^\omega &= -i\omega \chi^{(2)} E^\omega \cdot E^\omega \\
\n\nabla \wedge E^{2\omega} - i2\omega \mu_0 H^{2\omega} &= 0 \\
\n\nabla \wedge H^{2\omega} + i2\omega \varepsilon_0 E^{2\omega} &= -i2\omega \chi^{(2)} E^\omega \cdot E^\omega \\
\nu \wedge E^\omega|_{\partial \Omega} &= f^\omega \\
\nu \wedge E^{2\omega}|_{\partial \Omega} &= f^{2\omega}
\end{align*}
\]
where \( f^{k\omega} \in C_c^s(\partial\Omega) \cap TW_{\text{Div}}^{1-1/p,p}(\partial\Omega) \) and \( \text{supp} f^{k\omega} \subseteq \Gamma_1 \) for \( k = 1, 2 \). This models the phenomenon when a beam with time-harmonic electric field \( E(t, x) = E(x)e^{-i\omega t} + \text{c.c.} \) is incident upon an SHG medium (e.g., a noncentrosymmetric crystal), new waves are generated at zero frequency and at frequency \( 2\omega \), and assuming that the susceptibility parameter is isotropic \( \chi^{(2)} = (\chi_i^{(2)})_{i=1}^3 \). The corresponding output of the admittance map

\[
\Lambda_{\Gamma_1, \Gamma_2}^{\chi^{(2)}}(f^{k\omega}) = \nu \wedge H^{k\omega}|_{\Gamma_2}
\]

is measured on \( \Gamma_2 \) with \( \Gamma_1 \cap \Gamma_2 \neq \emptyset \). \( \chi^{(2)} \) is the second order susceptibility parameter. Then we are solving the problem of determining the susceptibility \( \chi^{(2)} \).

The type of inverse boundary value problems was first formulated by Calderón in [7] for a proposed imaging method, known as the electrical impedance tomography, in which one aims to determine the electrical conductivity, modeled by the function \( \sigma(x) \) for \( x \) in a body \( \Omega \), from the boundary measurements of the electric voltage and current, formulated using the DN map \( \Lambda_\sigma : u|_{\partial\Omega} \mapsto \nu \cdot \sigma \nabla u|_{\partial\Omega} \) for the linear elliptic equation \( \nabla \cdot (\sigma \nabla u) = 0 \) in \( \Omega \). The first global uniqueness result was proved by Sylvester and Uhlmann [52] for \( C^2 \)-conductivities \( \sigma(x) \) in dimensions \( n \geq 3 \) by solving the problem of determining an electric potential \( q(x) \) in the Schrödinger equation \( (-\Delta + q)u = 0 \) from the boundary DN map. Later the regularity condition of conductivities was relaxed in [3] and [49]. In two dimensions, Nachman in [13] and Astala-Päivärinta in [4] proved the global uniqueness result for \( C^2 \)-conductivities and for \( L^\infty \)-conductivities, respectively. Bukhgeim in [6] obtained the uniqueness result for \( L^p \)-potentials of Schrödinger equations from Cauchy data in two dimensions. Many further results for global uniqueness of inverse boundary value problems are available in the literature. We refer readers to [3, 53, 54, 59] for anisotropic conductivities in two and higher dimensions, [45, 50] for magnetic Schrödinger operators and [47, 48, 30] for Maxwell’s equations, and [10, 12, 13, 15] in Riemannian geometries. We also refer readers to the surveys [60, 61] and the references therein.

The study of the inverse problem for nonlinear elliptic equations goes back to the 1990’s. The method introduced by Isakov in [20] for the parabolic equations, shows that the linearization of the nonlinear DN map indeed gives full information about the DN map of the corresponding linear equation, and thus the uniqueness results of inverse problems for linear equations could be applicable. The method was then generalized to recover nonlinear coefficients for a class of semilinear equations [26, 25, 57] in two and higher dimensions and for a parabolic systems of semilinear equations in [21]. Also see [22] for an application of the linearized method to quasilinear equations to recover nonlinear
coefficients from partial data. Furthermore, a second order linearization was applied to a
class of quasilinear equations for the unique determination of quadratic terms in [27] and
determination of anisotropic conductivities in [56, 58]. Also see [11, 2, 15, 31, 32, 36, 37] for
an application of higher order linearization methods to nonlinear elliptic equations for the
recovery of power-type nonlinear terms. More details on inverse problems for nonlinear
equations can be found in the surveys [60, 61].

In the last few years, the higher order linearization method has become a powerful
tool in dealing with inverse problems for nonlinear hyperbolic equations including wave
equations in [14, 34, 39, 16, 62] and einstein equations in [40, 33]. In those works, the
nonlinearity is proven to be helpful in determining the information of the coefficients in the
operators, in some situations combined with microlocal analysis of the newly generated
singularities in order to solve the inverse problems whose corresponding version for linear
equations are otherwise still open.

Our first result for the nonlinear Helmholtz Schrödinger equation is given below.

Theorem 1.1 Let \( q_1 \) and \( q_2 \) ∈ \( L^\infty(\Omega) \), and the Dirichlet-Neumann map \( \Lambda_{q_j} \) satisfy that
\[
\Lambda_{q_1}(f)|_{\Gamma_2} = \Lambda_{q_2}(f)|_{\Gamma_2}, \quad \forall f \in C^s_c(\Gamma_1),
\]
with \( \|f\|_{C^s_c(\Gamma)} < \delta, \ s > 1 \) with \( s \notin \mathbb{Z} \), where \( \delta \) is sufficiently small. Then \( q_1 = q_2 \) in \( \Omega \).

Remark 1.1 Following Theorem A.1 in appendix, the well-posedness of nonlinear Helmholtz
equation holds for small boundary value \( f \in W^{2 - \frac{1}{p}, p}(\partial \Omega) \). Thus the Dirichlet-to-Neumann
map is well defined and uniquely determines \( q(x) \in L^\infty(\Omega) \). The well-posedness is also
given in [36] in Hölder spaces.

We remark here that our result for the nonlinear Helmholtz type Schrödinger equation
[1] is not contained in the cases discussed in [37], where the linearized equation is of
Laplace type. Due to the non-positivity of operator \( -\Delta - k^2 \), one needs modified density
arguments discussed below. On the other hand, the equation [1] can be viewed as a
fixed frequency time-harmonic equation for the wave operator \( \partial_t^2 - \Delta \). Therefore, our
result sheds some light on the inverse boundary problem for wave equations. In [12], the
elliptic DN map of Calderón problem of Schrödinger equation in an infinite cylinder is
reduced to the hyperbolic DN map of a wave equation. Based on the unique continuation
property for the reduced wave equation, the boundary control method is then applied to
recover the uniqueness. Our paper presents a direct proof of partial data inverse problem
for Helmholtz Schrödinger equation. We also remark a progress in [55], which reduces
Calderón problems to the injectivity of some geodesic X-ray transforms and and provides
a unified approach for inverse boundary problems for Laplace type, transport and wave equations.

In general, the higher order linearization method gives rise to an integral identity which involves an integral of some product of solutions to the linearized equations. The problem is then reduced to an integral geometry problem after plugging in proper linear solutions. When the linearized equation is the Laplace equation and the boundary DN-map is given as partial data \( \Lambda^{\Gamma_1, \Gamma_2} \), the solutions are usually chosen to be harmonic exponentials that vanish on part of the boundary. The density result of the product of two such exponentials was shown in [11] as an enlightening step of the study of Calderón problem for Schrödinger equation \(-\Delta u + qu = 0\). Their techniques are further exploited in [31] to show the density result of the set of the product of two gradients of some harmonic functions with some vanishing boundary values. Such density results were then directly or indirectly used in proving the uniqueness of inverse boundary problems for nonlinear elliptic equations in [31, 32, 35, 36, 37]. As a comparison, for the partial data problems of linear equations, two approaches are mainly proposed: one is a reflection argument in [23, 8] by assuming that the inaccessible boundary is either on a plane or on a sphere; the other is to establish the Carleman estimate as in [30, 10] where the partial data on the boundary are closely related with the constructed limiting Carleman weight. A combination of these two approaches is exploited in [28] to extend the uniqueness result to manifolds. These methods aim at constructing new complex geometrical optics (CGO) solutions similar to those in [52] for the linear equations and usually requires some extra assumptions on \( \Gamma_1 \) and \( \Gamma_2 \).

In our paper, following the techniques in [11] we first prove the following density result in order to obtain the global uniqueness of the coefficient for a nonlinear Helmholtz Schrödinger equation after higher order linearization.

**Theorem 1.2** Let \( \Omega \) be a bounded open set in \( \mathbb{R}^n \), \( n \geq 2 \) with smooth boundary \( \partial \Omega \), and \( \Gamma \subset \partial \Omega \) be a proper nonempty open subset of the boundary. Let \( q(x) \in L^\infty(\Omega) \). Suppose the cancellation equality

\[
\int_\Omega q(x)v_1(x)v_2(x)dx = 0
\]  

holds for any smooth solutions \( v_1(x), v_2(x) \in C^\infty(\bar{\Omega}) \) to the equation

\[
\begin{cases}
-\Delta v(x) - k^2 v(x) = 0 & \text{in } \Omega, \\
v(x) = 0 & \text{on } \partial \Omega \setminus \Gamma.
\end{cases}
\]

Then \( q(x) \) vanishes in \( \Omega \).

As another application of the density result, our paper also yields the simultaneous recovery of unknown cavity or boundary and coefficients for the nonlinear Helmholtz
Schrödinger operator $-\Delta u + k^2 u + q(x)u^2$ following the discussion in [37]. In comparison, the solutions to the linearized equation in our case are no longer harmonic functions, and the tools available are restricted to the unique continuation principle.

Our main results for the nonlinear Maxwell’s equations are given below.

**Theorem 1.3 (Kerr-type)** Let $p_1, p_2, q_1$ and $q_2 \in L^\infty(\tilde{\Omega})$ and $\Gamma_1 \cap \Gamma_2 \neq \emptyset$. Suppose

$$\Lambda_{p_1,q_1}(f)|_{\Gamma_2} = \Lambda_{p_2,q_2}(f)|_{\Gamma_2}, \quad \forall f \in C^s_c(\Gamma_1; \mathbb{C}^3) \cap TW_{Div}^{1-1/p,p}(\partial\Omega),$$

with $\|f\|_{C^s_c(\Gamma)} < \delta$, $s > 1$ with $s \notin \mathbb{Z}$, where $\delta$ is sufficiently small. Then $p_1 = p_2$ and $q_1 = q_2$ in $\Omega$.

We also give the identifiability result of the partial data inverse problem of a class of nonlinear time-harmonic Maxwell’s system with the second harmonic generation on $\Omega$

**Theorem 1.4 (Second Harmonic Generation)** Let $\chi^{(2),1}$ and $\chi^{(2),2} \in (L^\infty(\tilde{\Omega}))^3$, the admittance map $\Lambda_{\chi^{(2)}}$ satisfy that

$$\Lambda_{\chi^{(2),1}}(f^{k\omega})|_{\Gamma_2} = \Lambda_{\chi^{(2),2}}(f^{k\omega})|_{\Gamma_2}, \quad \forall f^{k\omega} \in C^s_c(\Gamma_1) \cap TW_{Div}^{1-1/p,p}(\partial\Omega),$$

with $\|f^{k\omega}\|_{C^s_c(\Gamma)} < \delta$, $s > 1$ with $s \notin \mathbb{Z}$, where $\delta$ is sufficiently small for $k = 1, 2$. Then $\chi^{(2),1} = \chi^{(2),2}$ in $\Omega$.

For partial data inverse problems of Maxwell’s equations, the literature relies on geometrical assumptions on the inaccessible boundary $\Gamma_2^c$ to be either on a plane or on a sphere [8] or on the admissible manifold [9] with $\Gamma_1$ to be the global boundary. Our result applies the high order linearization method and $\Gamma_1$ and $\Gamma_2$ could be arbitrary.

We first extend the density result in [11] which states that the set of a product of harmonic functions with boundary value supported on part of the boundary of the domain is dense in $L^2(\Omega)$ to the set of a product of solutions to linear Helmholtz equations with boundary value supported on part of the boundary. Different from calculating symbols of the nonlinear interaction, we adopt integral identity and follow the microlocal analysis as in [11] to make the proof simple. Then we apply the result to recover the coefficients of a class of nonlinear Helmholtz Schrödinger and Maxwell’s equations with partial data by virtue of the higher order linearization method. We also give some simultaneous recovery results of coefficients and unknown cavity as applications.

The paper is organized as follows. In Section 2 we give the proofs of Theorem 1.1 and Theorem 1.2. Additionally, we provide some applications of simultaneous recovery of nonlinear Helmholtz Schrödinger equations in Section 3. Section 4 is devoted to the identifiability results of Maxwell equations, i.e., Theorem 4.1 and Theorem 1.4.
2 Nonlinear Helmholtz Schrödinger equations

2.1 Proof of Theorem 1.1

Following [36, 37], we adopt the higher order linearization of the DN-map to deal with the nonlinearity. Since the application of the high order linearization method here is standard, we provide key procedures and refer readers to [36, 37] for details.

Proof. Let $f = \epsilon_1 f_1 + \epsilon_2 f_2$, where $\epsilon_1, \epsilon_2 > 0$, and $f_1$ and $f_2 \in C^s_c(\Gamma_1)$. Consider the parameterized boundary value problem

$$
\begin{cases}
-\Delta u_j - k^2 u_j + q_j(x)u_j^2 = 0 & \text{in } \Omega, \\
u_j = \epsilon_1 f_1 + \epsilon_2 f_2 & \text{on } \partial \Omega,
\end{cases}
$$

(5)

with $j = 1, 2$. It follows from the well-posedness of the nonlinear Helmholtz equation given in Appendix A.1 that when $k$ is not an eigenvalue of the Laplace operator, there exist unique solutions $u_j := u_j(x; \epsilon_1, \epsilon_2)$ to (1) with $j = 1, 2$, provided $\epsilon_1, \epsilon_2$ are sufficiently small. Then the assumption $\Lambda_{q_1}(\epsilon_1 f_1 + \epsilon_2 f_2)|_{r_2} = \Lambda_{q_2}(\epsilon_1 f_1 + \epsilon_2 f_2)|_{r_2}$ gives

$$
\partial_\nu u_1|_{r_2} = \partial_\nu u_2|_{r_2}.
$$

(6)

Denote $v^{(l)} := \partial_\nu u_j|_{\epsilon_1=0}$ for $j = 1, 2$ and $l = 1, 2$. The differentiation of (5) w.r.t. $\epsilon_l$ at $\epsilon_1 = \epsilon_2 = 0$ and the initial value $u_j(x; 0, 0) = 0$ yield

$$
\begin{cases}
-\Delta v^{(l)}_j - k^2 v^{(l)}_j = 0 & \text{in } \Omega, \\
v^{(l)}_j = f_l & \text{on } \partial \Omega,
\end{cases}
$$

(7)

with $j = 1, 2$ and $l = 1, 2$. $v^{(l)}_1 = v^{(l)}_2$ follows from the uniqueness of Dirichlet boundary value problem of Helmholtz equation for each $l$. From now on, denote by $v^{(l)} := v^{(l)}_1 = v^{(l)}_2$, with $l = 1, 2$; thus $v^{(l)}$ is a solution to

$$
\begin{cases}
-\Delta v^{(l)} - k^2 v^{(l)} = 0 & \text{in } \Omega, \\
v^{(l)} = f_l & \text{on } \partial \Omega.
\end{cases}
$$

Denote $w_j := \partial_{\epsilon_1 \epsilon_2} u_j|_{\epsilon_1=\epsilon_2=0}$, with $j = 1, 2$. Then the second order linearization of (5) yields

$$
\begin{cases}
-\Delta w_j - k^2 w_j + 2q_j(x)v^{(1)}v^{(2)} = 0 & \text{in } \Omega, \\
w_j = 0 & \text{on } \partial \Omega.
\end{cases}
$$
Moreover, from (6) one has
\[
\partial_\nu w_1|\Gamma_2 = \partial_\nu w_2|\Gamma_2. 
\] (8)

In order to establish an integral identity we choose some proper solution \(v^{(0)}\) to compensate the unknown information on \(\Gamma_2\). Let \(v^{(0)}\) be a solution to
\[
\begin{cases} 
-\Delta v^{(0)} - k^2 v^{(0)} = 0 & \text{in } \Omega, \\
v^{(0)} = f_0 & \text{on } \Gamma_2,
\end{cases} 
\] (9)
where \(f_0 \in C^s_c(\Gamma_2)\) and recall that \(\Gamma_2 \subseteq \partial\Omega\) is an open nonempty subset. Thus,
\[
2 \int_{\Omega} (q_1(x) - q_2(x))v^{(1)}v^{(2)}v^{(0)}dx = 2 \int_{\Omega} \Delta(w_1 - w_2)v^{(0)} + k^2(w_1 - w_2)v^{(0)}dx = 2 \int_{\Omega} \Delta(w_1 - w_2)v^{(0)} - \Delta v^{(0)}(w_1 - w_2)dx = 2 \left( \int_{\partial\Omega} \partial_\nu v^{(0)}(w_1 - w_2) dS - \int_{\partial\Omega} \partial(w_1 - w_2)v^{(0)} dS \right)
\]
vanishes in view of (8) and (9). Hence the integral identity
\[
\int_{\Omega} (q_1(x) - q_2(x))v^{(1)}v^{(2)}v^{(0)}dx = 0, 
\] (10)
holds, where \(v^{(1)}, v^{(2)}\) are solutions to the linearized equation (7) and \(v^{(0)}\) satisfies the boundary value problem (9).

Apply Theorem 1.2 and one has
\[
(q_1(x) - q_2(x))v^{(0)}(x) = 0
\]
identically in \(\Omega\), where \(v^{(0)}\) is a solution to the boundary value problem (9). By constructing a solution \(v^{(0)}\) satisfying \(v^{(0)}(\tilde{x}_0) \neq 0\) for any fixed \(\tilde{x}_0 \in \Omega\), we are able to show \((q_1 - q_2)(\tilde{x}_0) = 0\). In fact, apply Proposition (A.1) with \(L = -\Delta - k^2\) and consider \(u(x) = e^{-ix\cdot\xi}\) with \(\xi \cdot \xi = k^2\) such that \(Lu = 0\). Moreover, \(\xi\) is chosen such that \(u(\tilde{x}_0) \neq 0\). Then Proposition (A.1) yields a solution \(v^{(0)}(x)\) close to \(u(x)\) and thus \(v^{(0)}(\tilde{x}_0) \neq 0\) by the continuity. Since \(\tilde{x}_0\) is arbitrary in \(\Omega\), \(q_1 = q_2\) in \(\Omega\). This completes the proof of Theorem 1.1.

\[ \square \]

2.2 Proof of Theorem 1.2

The proof of Theorem 1.2 consists of three parts. First we establish the new setting of the local problem and prove local results of Theorem 1.2 in section 2.2.1. Then section 2.2.2 contributes to the extension of local results to global results is given in section.
2.2.1 Local results

**Proposition 2.1** Let $\Omega$ be a bounded open set in $\mathbb{R}^n$, with $n \geq 2$ with smooth boundary $\partial \Omega$, $x_0 \in \partial \Omega$ a convex point on the boundary, and $\Gamma$ be a small neighborhood of $x_0$ on the boundary. Suppose that the cancellation equality (3) holds for any smooth solutions $v_1(x)$ and $v_2(x) \in C^\infty(\overline{\Omega})$ to the equation (4). Then there exists $\delta > 0$ such that $q(x)$ vanishes on $B(x_0, \delta) \cap \Omega$.

**Proof.** Choose a point $a \in \mathbb{R}^n \setminus \overline{\Omega}$ along the direction of the inward normal vector at $x_0 \in \partial \Omega$. As $x_0$ is convex and $\Omega$ is bounded, suppose that there exists a ball $B(a, r)$ containing $\Omega$ and $\partial B(a, r) \cap \overline{\Omega} = \{x_0\}$. Then a transform $\varphi(x)$ consisting of translation and rotation transforms such that $x_0$ is mapped to the origin and $a$ is mapped to $-e_1 = (-1, 0, \ldots, 0)$. Indeed, $\varphi$ can be defined as $x \in \Omega \mapsto y = \frac{1}{r} \cdot O(x - a) - e_1$, where $O$ is an orthogonal matrix such that $O(x_0 - a) = |x_0 - a| e_1 = re_1$. Thus $\hat{\Omega} = \varphi(\Omega)$ is contained in $B(-e_1, 1)$.

Note that the Laplacian $\Delta$ is invariant under $\varphi$ as a combination of translation and rotation transform. Then, denote $\tilde{q} = q \circ \varphi^{-1}$, $\tilde{u}_1(y) = u_1 \circ \varphi^{-1}$ and $\tilde{u}_2(y) = u_2 \circ \varphi^{-1}$. The cancellation (3) is changed to $\int_{\hat{\Omega}} \tilde{q} \tilde{u}_1 \tilde{u}_2 dy = 0$, where $\tilde{u}_1$ and $\tilde{u}_2$ satisfy $-\Delta u - k^2 r^2 u = 0$ in $\hat{\Omega}$ and vanish on the boundary $\hat{\Omega} \setminus \hat{\Gamma}$.

If $\tilde{q}$ vanishes near $y_0 = 0$, then $q$ also vanishes near $x_0$. Therefore, we start with a new cancellation equality. With some abuse of notations we omit the tilde sign and still use $x$-coordinate. In our new setting, denote $x_0 = 0$. Suppose that $|x + e_1| \leq 1$ for $x \in \Omega$, and $\partial \Omega \setminus \Gamma \subset \{x \in \partial \Omega \mid x_1 \leq -2c\}$. The following cancellation holds

$$
\int_{\Omega} qu_1 u_2 dx = 0,
$$

for any smooth solutions $u_1$ and $u_2$ to the equation

$$
\begin{cases}
-\Delta u - k^2 r^2 u = 0 & \text{in } \Omega, \\
 u = 0 & \text{on } \partial \Omega \setminus \Gamma.
\end{cases}
$$

The next proof of local results follows the literature [11]. Thus to shorten the length we only provide some critical procedures and estimates here. For more details of the calculation we refer to [11].

We first construct $u_1$ and $u_2$ by adding a correction term $w_1(x, \zeta)$ and $w_2(x, \eta)$ to eliminate the value of CGO solutions of the form $e^{-\frac{i}{2} x \cdot \zeta}$ and $e^{-\frac{i}{2} x \cdot \eta}$ on the boundary $\Omega \setminus \Gamma$, respectively, where $\zeta$ and $\eta$ are proper complex vectors.

Note that $p(\xi) = \xi^2$ is the principal symbol of the Laplacian on $\mathbb{R}^n$ and consider its complexification on $\mathbb{C}^n$. Let $\zeta$ and $\eta$ belong to the set

$$
p^{-1}(krh) = \{\zeta \in \mathbb{C}^n \mid \zeta \cdot \zeta = (krh)^2\}.
$$
Choose $\zeta_0 = (i, \sqrt{krh})^2 + 1, 0, \ldots, 0$ and $\eta_0 = (i, -\sqrt{krh})^2 + 1, 0, \ldots, 0$. Then $\zeta_0$ and $\eta_0$ belong to $p^{-1}(krh)$. Since the differential of the map $s : p^{-1}(krh) \times p^{-1}(krh) \to \mathbb{C}^n$, $(\zeta, \eta) \mapsto \zeta + \eta$ at $(\zeta_0, \eta_0)$ is surjective, any complex vector $z \in \mathbb{C}^n$ with $|z - 2aie_1| < 2a\varepsilon$ can be decomposed as a sum of the form

$$z = \zeta + \eta, \quad \zeta, \eta \in p^{-1}(krh), \quad |\zeta - a\zeta_0| < C\varepsilon, |\eta - a\eta_0| < C\varepsilon. \quad (11)$$

In the sequel, we use $C$ to denote different constants independent of $h$ in the inequalities for convenience.

Take a cutoff function $0 \leq \chi \leq 1 \in C_c^\infty(\mathbb{R}^n)$ which equals 1 on $x_1 \leq -2c$ containing $\partial \Omega \setminus \Gamma$ with compact support in $x_1 \leq -c$. There exists a solution $w_1(x, \zeta)$ to the Dirichlet problem

$$\begin{aligned}
-\Delta w - (kr)^2 w &= 0 \quad \text{in } \Omega, \\
w &= -\chi(x)e^{-1/kr} \quad \text{on } \partial \Omega.
\end{aligned} \quad (12)$$

Thus, fix $k$ outside a discrete set $\Sigma_r$, one has

$$\|w_1\|_{H^1(\Omega)} \leq C\|\chi(x)e^{-1/kr}\zeta\|_{H^{1/2}(\partial \Omega)},$$

and

$$\|w_1\|_{H^1(\Omega)} \leq C(1 + h^{-1}|\zeta|)^{3/2}e^{-1/kr}\Im \zeta_1 e^{1/kr}\Im \zeta^\prime}, \quad \text{if } \Im \zeta_1 \geq 0. \quad (13)$$

Let $u_1(x, \zeta) = e^{-1/kr} \zeta + w_1(x, \zeta)$, with $\zeta \in p^{-1}(krh)$. Then $u_1$ is a desired solution and vanishes on $\partial \Omega \setminus \Gamma$. Similarly, we construct $u_2(x, \eta) = e^{-1/kr} \eta + w_2(x, \eta)$, where $\eta \in p^{-1}(krh)$ and $w_2(x, \eta)$ is a correction term satisfying a similar estimate to (13) with $\zeta$ replaced by $\eta$.

Plug in $u_1(x, \zeta)$ and $u_2(x, \eta)$ into the cancellation and combine the estimates of $\|w_1\|_{H^1(\Omega)}$ and $\|w_2\|_{H^1(\Omega)}$. When $\Im \zeta_1 \geq 0$ and $\Im \eta_1 \geq 0$, the following estimate

$$\left| \int_\Omega q(x)e^{-1/kr}(\zeta + \eta) dx \right| \leq C\|q\|_{L^\infty(\Omega)}(1 + h^{-1}|\zeta|)^{3/2}(1 + h^{-1}|\eta|)^{3} \left( e^{-1/kr}\min\{\Im \zeta_1, \Im \eta_1\} e^{1/kr}(\Im \zeta^\prime + |\Im \eta^\prime|) \right),$$

holds. In particular, when $|\zeta - a\zeta_0| < C\varepsilon$ and $|\eta - a\eta_0| < C\varepsilon$ with $\varepsilon \leq \frac{1}{2C}$,

$$\left| \int_\Omega q(x)e^{-1/kr}(\zeta + \eta) dx \right| \leq Ch^{-3}\|q\|_{L^\infty(\Omega)}e^{-ca}e^{2C\varepsilon}. $$

Therefore, for all $z \in \mathbb{C}^n$ with $|z - 2aie_1| < 2a\varepsilon$, it follows from the decomposition (11) that

$$\left| \int_\Omega q(x)e^{-1/kr}z dx \right| \leq Ch^{-3}\|q\|_{L^\infty(\Omega)}e^{-ca}e^{2C\varepsilon}. $$
This is a similar estimate as in [11], which guarantees an exponential decay for the F.B.I. transform of $f$. Finally, an application of the watermelon approach as in [11] yields

$$q(x) = 0, \quad \forall x \in \Omega, \quad -\delta \leq x_1 \leq 0,$$

provided $\delta$ small enough. This completes the proof of Proposition 2.1.

\[\square\]

### 2.2.2 Extend to global results

In this section, we extend the local result (Proposition 2.1) globally to the whole $\Omega$. The following lemma is needed.

**Lemma 2.1** Let $\Omega_1 \subset \Omega_2$ with smooth boundary. Define $G_{\Omega_2}(x, y)$ to be the Green function of the following system

\[
\begin{aligned}
\begin{cases}
(-\Delta_y - k^2)G_{\Omega_2}(x, y) = \delta(x - y) & \text{in } \Omega_2, \\
G_{\Omega_2}(x, y) = 0 & \text{on } \partial\Omega_2.
\end{cases}
\end{aligned}
\]

The set

$$\mathcal{R} = \left\{ \int_{\Omega_2} G_{\Omega_2}(x, y) a(y) dy, \text{ for some } a(y) \in C^\infty(\Omega_2), \ supp \ a \subset \Omega_2 \setminus \Omega_1, x \in \Omega_2 \right\}$$

is dense for $L^2(\Omega_1)$ topology in the space $\mathcal{S} = \{ u(x) \in C^\infty(\Omega_1) : (-\Delta_x - k^2)u = 0, u|_{\partial\Omega_1 \cap \partial\Omega_2} = 0 \}$.

**Proof.** It suffices to show that $\langle v, \mathcal{S} \rangle = 0$ for any $v(x) \in L^2(\Omega_1)$ such that $\langle v, \mathcal{R} \rangle = 0$. Following Fubini Theorem and the assumption, the equality

$$\int_{\Omega_1} v(x) \int_{\Omega_1} a(y) G_{\Omega_2}(x, y) dy dx = \int_{\Omega_2} \int_{\Omega_1} v(x) G_{\Omega_2}(x, y) dx a(y) dy = 0,$$

holds for any $a(y) \in C^\infty_c(\Omega_2)$. The assumption $supp \ a \subset \Omega_2 \setminus \Omega_1$ yields

$$\int_{\Omega_1} v(x) G_{\Omega_2}(x, y) dx = 0, \quad \text{for } y \in \Omega_2 \setminus \Omega_1. \quad (14)$$

Denote $\omega(y) = \int_{\Omega_1} v(x) G_{\Omega_2}(x, y) dx$. Then $\omega(y) = 0$ for $y \in \Omega_2 \setminus \Omega_1$. Moreover, $\omega \in H^1_0(\Omega_2)$ in view of $G_{\Omega_2}(x, y)|_{y \in \partial\Omega_2} = 0$.

Note that $(-\Delta_y - k^2)\omega(y) = v(y)$ for $y \in \Omega_2$. Thus, for any $u \in \mathcal{S},$

\[
\int_{\Omega_1} u \omega dx = \int_{\Omega_1} u(-\Delta_x - k^2)\omega(x) dx = \int_{\Omega_1} (\Delta u \omega - u \Delta \omega) dx
\]

\[
= \int_{\partial\Omega_1 \cap \partial\Omega_2} (u \partial_\nu \omega - \omega \partial_\nu u) dS + \int_{\partial\Omega_1 \setminus \partial\Omega_2} (u \partial_\nu \omega - \omega \partial_\nu u) dS.
\]
The right hand side term vanishes by combining \( u|_{\partial \Omega_1 \cap \partial \Omega_2} = 0, \omega \in H^1_0(\Omega_2) \) and \( \omega = 0 \) in \( \tilde{\Omega}_2 \setminus \Omega_1 \). Hence \( \int_{\Omega_1} uv dx = 0 \) for any \( u \in S \).

We are ready to conclude the proof of Theorem 1.2. The approach goes through as in [11].

**Proof of Theorem 1.2** Fix a point \( x_1 \in \Omega \) and let \( \theta : [0,1] \rightarrow \bar{\Omega} \) be a \( C^1 \) curve joining \( x_0 \in \Gamma \) to \( x_1 \) such that \( \theta(0) = x_0 \) and \( \theta'(0) \) is the interior normal to \( \partial \Omega \) at \( x_0 \) and \( \theta(t) \in \Omega \) for all \( t \in (0,1) \). Let \( \Theta_\epsilon(t) = \{ x \in \bar{\Omega} \mid d(x, \theta([0,t])) \leq \epsilon \} \) be a closed neighborhood of the curve \( \theta(t) \), \( t \in [0,1] \), and set \( I = \{ t \in [0,1] \mid f = 0 \ \text{a.e. on} \ \Theta_\epsilon(t) \cap \Omega \} \). \( I \) is a closed set. From Proposition 2.1 \( I \) is non-empty provided that \( \epsilon \) is small enough. We would like to show that \( I \) is also open. Thus by connectivity \( I = [0,1] \). Then \( x_1 \notin \text{supp} \ f \). Since \( x_1 \) is arbitrary, \( f = 0 \) in \( \Omega \). Hence, it suffices to show that \( I \) is open.

Let \( t \in I \) and \( \epsilon \) small enough. Then we can assume that \( \partial \Theta_\epsilon(t) \cap \partial \Omega \subset \Gamma \), and \( \Omega \setminus \Theta_\epsilon(t) \) can be smoothed out into an open subset \( \Omega_1 \subset \Omega \) with smooth boundary such that \( \Omega \setminus \Theta_\epsilon(t) \subset \Omega_1 \), \( \partial \Omega \setminus \Gamma \subset \partial \Omega \cap \partial \Omega_1 \). Furthermore, we augment \( \Omega \) by smoothing out the set \( \Omega \cup B(x_0, \epsilon) \) into an open set \( \Omega_2 \) with smooth boundary such that \( \partial \Omega \setminus \Gamma \subset \partial \Omega \cap \partial \Omega_1 \subset \partial \Omega \cap \partial \Omega_2 \).

Denote by \( G_{\Omega_2} \) be the fundamental solution to the system

\[
\begin{cases}
(\Delta_y - k^2)G_{\Omega_2}(x,y) = \delta(x-y) & \text{in } \Omega_2, \\
G_{\Omega_2}(x,y) = 0 & \text{on } \partial \Omega_2,
\end{cases}
\]

and let

\[
H(x,t) = \int_{\Omega_1} f(y)G_{\Omega_2}(x,y)G_{\Omega_2}(t,y)dy, \quad \text{for } t, x \in \Omega_2 \setminus \Omega_1.
\]

Then \( H \) satisfies \( -\Delta_x H - k^2 H = 0 \) (\( \Delta_t H - k^2 H = 0 \)) when \( x \in \Omega_2 \setminus \Omega_1 \) \( (t \in \Omega_2 \setminus \Omega_1) \) respectively. Since \( f \) vanishes on \( \Omega_1 \setminus \Omega \),

\[
H(x,t) = \int_{\Omega_1} f(y)G_{\Omega_2}(x,y)G_{\Omega_2}(t,y)dy = \int_{\Omega} f(y)G_{\Omega_2}(x,y)G_{\Omega_2}(t,y)dy.
\]

Thus, \( H(x,t) = 0 \) follows from the assumption, for \( t, x \in \Omega_2 \setminus \Omega_1 \). Note that \( H \) also satisfies \( -(\Delta_x + \Delta_t) H - 2k^2 H = 0 \), when \( x, t \in \Omega_2 \setminus \Omega_1 \). The Unique Continuation Principle yields \( H(x,t) = 0 \) when \( t, x \in \Omega_2 \setminus \Omega_1 \); i.e.,

\[
\int_{\Omega_1} f(y)G_{\Omega_2}(x,y)G_{\Omega_2}(t,y)dy = 0, \quad \text{for } t, x \in \Omega_2 \setminus \Omega_1.
\]

Applying Lemma 2.1, we obtain the equality \( \int_{\Omega_1} uv = 0 \), where \( u, v \in C^\infty(\bar{\Omega}_1) \) are solutions to \( -\Delta u - k^2 u = 0 \) and vanish on \( \partial \Omega_1 \cap \partial \Omega_2 \). Combining Proposition 2.1 \( f \) vanishes on a neighborhood of \( \partial \Omega_1 \setminus (\partial \Omega_1 \cap \partial \Omega_2) \). This implies that \( f \) vanishes on a bigger neighborhood \( \Theta_\epsilon(t') t' > t \) of the curve. Hence \( I \) is open. This completes the proof of Theorem 1.2. \( \square \)
3 Applications

In this section, we give simultaneous recovery of the partial data inverse problems for a class of nonlinear Helmholtz Schrödinger equations as an application of Theorem 1.2.

In some applications, the discontinuity of the medium extends to the cases of obstacles or cavities embedded in the medium. Mathematically, equations are satisfied in the medium $\Omega$ minus the obstacle or cavity region $D$ and subject to certain boundary conditions at $\partial D$. Determining an unknown obstacle or cavity goes back to Schiffer’s work and is a substantial topic in inverse scattering theory, for example for sound waves modeled by the linear wave equation and electromagnetic waves modeled by the linear Maxwell’s equations. Among many existing methods (such as the probe method and so on), one method, known as the enclosure method, uses CGO-type solutions to determine obstacles, cavities and inclusions. The enclosure method was first introduced by Ikehata in [17, 18] to reconstruct a cavity $D$ inside a conductive medium $\Omega$ for Schrödinger operator when the surrounding potential is known a priori. See [19, 24] for more uniqueness results of obstacle problems when knowing the potential and [42] for a simultaneous reconstruction of both embedded obstacle and its surrounding potential.

3.1 Simultaneous recovery of cavity and coefficients

**Theorem 3.1** Let $\Omega$ be a bounded open set in $\mathbb{R}^n$, $n \geq 2$ with smooth boundary $\partial \Omega$. Let $D_1, D_2 \subset \subset \Omega$ be nonempty subsets with smooth boundaries such that $\Omega \setminus \bar{D}_j$ are connected. For $j = 1, 2$ let $q_j(x) \in C^\infty(\Omega \setminus \bar{D}_j)$ and consider the following nonlinear Helmholtz equation

$$
\begin{cases}
-\Delta u_j - k^2 u_j + q_j(x) u_j^2 = 0 & \text{in } \Omega \setminus \bar{D}_j, \\
u_j = 0 & \text{on } \partial D_j, \\
u_j = f & \text{on } \partial \Omega.
\end{cases}
$$

(15)

Assume that the Dirichlet-Neumann map $\Lambda_{q_j}^{D_j}$ satisfy that

$$
\Lambda_{q_1}^{D_1}(f) = \Lambda_{q_2}^{D_2}(f) \text{ on } \partial \Omega, \quad \forall f \in C^s(\partial \Omega) \text{ is sufficiently small},
$$

with $s > 1$ with $s \notin \mathbb{Z}$. Then $D_1 = D_2 = D$ and $q_1 = q_2$ in $\Omega \setminus \bar{D}$.

**Proof.** We apply the standard high order linearization method and and take the same notations as shown in section 2.1. Let $f = \epsilon_1 f_1 + \epsilon_2 f_2$ for the boundary value of (15), where $f_1, f_2 \in C^s(\partial \Omega)$, and $\epsilon_1, \epsilon_2$ are sufficiently small numbers. Thus, $v_j^{(l)} := \partial_{\epsilon_l} u_j|_{\epsilon_l=0}$
satisfy
\[
\begin{align*}
-\Delta v_j^{(l)} - k^2 v_j^{(l)} &= 0 \quad \text{in } \Omega \setminus \bar{D}_j, \\
v_j^{(l)} &= f_l \quad \text{on } \partial D_j, \\
v_j^{(l)} &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]
(16)
with \( j = 1, 2 \) and \( l = 1, 2 \).

Let \( G \) be the connected component of \( \Omega \setminus \bar{D}_1 \cup \bar{D}_2 \) of which the boundary contains \( \partial \Omega \).

Let \( \tilde{v}^{(l)} = v_1^{(l)} - v_2^{(l)} \) for each \( l \) in \( G \). Then \( \tilde{v}^{(l)} \) satisfies
\[
\begin{align*}
-\Delta \tilde{v}^{(l)} - k^2 \tilde{v}^{(l)} &= 0 \quad \text{in } G, \\
\tilde{v}^{(l)} &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]
with \( l = 1, 2 \). The unique continuation principle of solutions to Helmholtz equation yields \( \tilde{v}^{(l)} = 0 \) in \( G \). Therefore, for each \( l, v_1^{(l)} = v_2^{(l)} \) in \( G \). Hence \( v_2^{(l)} = 0 \) on \( \partial D_1 \setminus D_2 \) from the continuity.

To recovery the unknown boundary we need to choose \( f_l \). Let \( U = D_1 \setminus \bar{D}_2 \). Then \( \partial U = (\partial D_1 \setminus D_2) \cup (\partial D_2 \cap D_1) \). Let \( \varphi \in H^1(U) \) such that
\[
\begin{align*}
-\Delta \varphi - k^2 \varphi &= 0 \quad \text{in } U, \\
\varphi &= 0 \quad \text{on } \partial D_2 \cap D_1, \\
\varphi &= g \quad \text{on } \partial D_1 \setminus D_2,
\end{align*}
\]
where \( g \in H^1(U) \) which has nonvanishing trace supported in \( \partial D_1 \setminus D_2 \).

Apply Proposition (A.1) with \( U = D_1 \setminus \bar{D}_2 \) and \( M = \Omega \setminus D_2 \). Then \( M \setminus U = \Omega \setminus D_2 \cup D_2 \) is connected. Let \( \Gamma = \partial \Omega \subset \partial M \). Hence, there exists some \( f_l \) with \( \text{supp } f_l \subset \Gamma \) and a corresponding solution \( v_2^{(l)} \) to
\[
\begin{align*}
-\Delta v_2^{(l)} - k^2 v_2^{(l)} &= 0 \quad \text{in } M, \\
v_2^{(l)} &= f_l \quad \text{on } \partial M,
\end{align*}
\]
such that \( v_2^{(l)} \) is close to \( \varphi \) in \( H^1(U) \) and so is the trace of \( v_2^{(l)} \) and \( \varphi \) on \( \partial U \). Note that \( v_2^{(l)} \) should vanish on \( \partial D_1 \setminus D_2 \). This contradicts with the assumption that the trace of \( \varphi \) has nonvanishing support \( g \) on \( \partial D_1 \setminus D_2 \). This shows that \( D_1 = D_2 \). From now on, denote by \( D = D_1 = D_2 \). It follows that
\[
v^{(l)} := v_1^{(l)} = v_2^{(l)} \quad \text{in } \Omega \setminus \bar{D},
\]
(17)
for \( l = 1, 2 \); and \( v^{(l)} \) is a solution to (16).

To recover the coefficient, we choose \( v^{(0)} \) to be a solution to
\[
\begin{align*}
-\Delta v^{(0)} - k^2 v^{(0)} &= 0 \quad \text{in } \Omega \setminus \bar{D}, \\
v^{(0)} &= 0 \quad \text{on } \partial D, \\
v^{(0)} &= f_0 \quad \text{on } \partial \Omega,
\end{align*}
\]
(18)
where \( f_0 \) is not identically zero on \( \partial \Omega \).

Denoted by \( w_j = \partial^2_{\epsilon_1 \epsilon_2} u_j \big|_{\epsilon_1 = \epsilon_2 = 0} \). It follows from the second order linearization that \( w_j = 0 \) on \( \partial D \cup \partial \Omega \). Moreover, the DN map yields

\[
\partial_{\nu} w_1|_{\partial \Omega} = \partial_{\nu} w_2|_{\partial \Omega}.
\]

Thus, similar as in section 2.1, one has the integral identity

\[
\int_{\Omega \setminus \bar{D}} (q_1(x) - q_2(x)) v^{(1)}_1 v^{(2)}_2 v^{(0)} dx = 0,
\]

where \( v^{(1)}, v^{(2)} \) satisfy (16) and \( v^{(0)} \) satisfies (18). Apply Theorem 1.2 and one has

\[
(q_1(x) - q_2(x)) v^{(0)}(x) = 0
\]

identically in \( \Omega \setminus \bar{D} \). Similarly, for any \( \tilde{x}_0 \in \Omega \setminus \bar{D} \), we apply Proposition (A.1) and construct \( v^{(0)} \) a solution to (18) by choosing some \( f_0 \), with \( v^{(0)}(\tilde{x}_0) \neq 0 \). Hence \( (q_1 - q_2)(\tilde{x}_0) = 0 \). Since \( \tilde{x}_0 \) is arbitrary in \( \Omega \setminus \bar{D} \), then \( q_1 = q_2 \) in \( \Omega \setminus \bar{D} \). \( \square \)

### 3.2 Simultaneous recovery of boundary and coefficients

**Theorem 3.2** Let \( \Omega_j \) be a bounded open set in \( \mathbb{R}^n \), \( n \geq 2 \) with smooth boundary \( \partial \Omega \) for \( j = 1, 2 \). Consider \( \Gamma \subset \partial \Omega \) be a proper nonempty open subset of \( \partial \Omega_1 \cap \partial \Omega_2 \). For \( j = 1, 2 \) let \( q_j(x) \in C^\infty(\Omega_j) \) and consider the following nonlinear Helmholtz equation

\[
\begin{cases}
-\Delta u_j - k^2 u_j + q_j(x) u_j^2 = 0 & \text{in } \Omega_j, \\
u_j = 0 & \text{on } \partial \Omega_j \setminus \Gamma, \\
u_j = f & \text{on } \Gamma.
\end{cases}
\]

Assume that the Dirichlet-Neumann map \( \Lambda_{q_j}^{\Omega_j} \) satisfy that

\[
\Lambda_{q_1}^{\Omega_1}(f) = \Lambda_{q_2}^{\Omega_2}(f) \text{ on } \Gamma, \forall f \in C^s_c(\Gamma) \text{ is sufficiently small},
\]

with \( s > 1 \) with \( s \not\in \mathbb{Z} \). Then \( \Omega_1 = \Omega_2 = \Omega \) and \( q_1 = q_2 \) in \( \Omega \).

**Proof.** We apply the standard high order linearization method and the procedures and notations as in the proof of Theorem 3.1. Let \( f = \epsilon_1 f_1 + \epsilon_2 f_2 \). Let \( v_j^{(l)} = \partial_{\epsilon_l} u_j \big|_{\epsilon_l = 0} \) be solutions to the linearized Helmholtz equation with boundary value \( f_l \in C^s_c(\Gamma) \), for \( l = 1, 2 \) and \( j = 1, 2 \). Similarly as in the proof of Theorem 3.1, the unique continuation principle gives rise to \( v_1^{(l)} = v_2^{(l)} \) in \( G \) for each \( l \), where \( G \) is a connected component of \( \Omega_1 \cap \Omega_2 \) of which the boundary contains \( \Gamma \). Hence \( v_2^{(l)} = 0 \) on \( \partial \Omega_1 \cap \Omega_2 \) from the continuity.
To recovery the unknown boundary we need to choose the boundary values \( f_l \) for \( l = 1, 2 \). Let \( U = \Omega_2 \setminus \bar{\Omega}_1 \). Then \( \partial U = (\partial\Omega_2 \setminus \Omega_1) \cup (\partial\Omega_1 \cap \Omega_2) \). Let \( \varphi \in H^1(U) \) such that \[
\begin{cases}
-\Delta \varphi - k^2 \varphi = 0 & \text{in } U, \\
\varphi = 0 & \text{on } \partial\Omega_2 \setminus \Omega_1, \\
\varphi = g & \text{on } \partial\Omega_1 \cap \Omega_2,
\end{cases}
\]
where \( g \in H^1(U) \) which has nonvanishing trace supported in \( \partial\Omega_1 \cap \Omega_2 \).

Apply Proposition \( \text{(A.1)} \) with \( U = \Omega_2 \setminus \bar{\Omega}_1 \) and \( M = \Omega_2 \). Then \( M \setminus \bar{U} = \Omega_1 \cap \Omega_2 \) is connected with \( \Gamma = \Gamma_0 \subset \partial M \). Hence, there exists some \( f_l \) with \( \text{supp} f_l \subset \Gamma \) and a corresponding solution \( v_2^{(l)} \) to \[
\begin{cases}
-\Delta v_2^{(l)} - k^2 v_2^{(l)} = 0 & \text{in } M, \\
v_2^{(l)} = f_l & \text{on } \partial M
\end{cases}
\]
such that \( v_2^{(l)} \) is close to \( \varphi \) in \( H^1(U) \) and so is the trace of \( v_2^{(l)} \) and \( \varphi \) on \( \partial U \). Note that \( v_2^{(l)} \) should vanish on \( \partial\Omega_1 \cap \Omega_2 \). This contradicts with the assumption that the trace of \( \varphi \) has nonvanishing support \( g \) on \( \partial\Omega_1 \cap \Omega_2 \). This shows that \( \Omega_1 = \Omega_2 \).

From now on, denote by \( \Omega = \Omega_1 = \Omega_2 \). It follows that \( v^{(l)} := v_1^{(l)} = v_2^{(l)} \) in \( \Omega \), for \( l = 1, 2 \); and \( v^{(l)} \) are solutions to the linearized Helmholtz equation with boundary value \( f_l \) supported on \( \Gamma \) for \( l = 1, 2 \). The recovery of the coefficient is then similar to the proof of Theorem \( 3.1 \). By taking the second order linearization and choosing some \( v^{(0)} \) as in \( (18) \) with \( \partial D \) replaced by \( \partial \Gamma \), one is able to construct an integral identity similar to \( (19) \). An application of Theorem \( 1.2 \) and similar arguments complete the proof. \( \Box \)
4 Nonlinear Maxwell equations

In this section, $\Omega \subseteq \mathbb{R}^3$ is a bounded domain with smooth boundary. The proof of Theorem 4.1 and Theorem 1.4 consist of three parts. We first adopt the standard procedures of high order linearization of the admittance map to derive an integral identity. Similarly, we only provide key procedures of higher order linearization here and refer readers to [36, 37] for more details. Then we follow the literature [11] to give the uniqueness result locally in a small neighborhood of some point on the boundary. Lastly, we extend the local result to the global situation.

4.1 Proof of Theorem

Proof. Let $f = \epsilon_1 f_1 + \epsilon_2 f_2 + \epsilon_3 f_3$. $f_l \in C^\infty(\partial \Omega)$ with supp $f_l \subseteq \Gamma_1$, for $l = 1, 2, 3$. Let $\epsilon = (\epsilon_1, \epsilon_2, \epsilon_3)$. Denote

$$W^{(l)}_j = \partial_{\epsilon_l} H_j|_{\epsilon=0}, \quad V^{(l)}_j = \partial_{\epsilon_l} E_j|_{\epsilon=0}, \quad W^{(123)}_j = \partial_{\epsilon} H_j|_{\epsilon=0}, \quad V^{(123)}_j = \partial_{\epsilon} E_j|_{\epsilon=0},$$

for $j = 1, 2$ and $l = 1, 2, 3$. Similarly as in section 2.1, one can show that $V^{(l)} := V^{(1)} = V^{(2)}$ and $W^{(l)} := W^{(1)} = W^{(2)}$ satisfy

$$\begin{cases}
\nabla \wedge V^{(l)} - i \omega \mu_0 W^{(l)} = 0, \\
\nabla \wedge W^{(l)} + i \omega \epsilon_0 V^{(l)} = 0,
\end{cases}$$

in $\Omega$ with $\nu \wedge V^{(l)}|_{\partial \Omega} = f_l$ on the boundary $\partial \Omega$, for $l = 1, 2, 3$. By direct computation one can show that $(E, H)$ of the following form is a solution to (21)

$$\begin{align*}
E &= \epsilon_0^{-\frac{1}{2}} e^{-\frac{i}{\hbar} x \cdot \xi} \left( \frac{1}{\hbar^2} (\xi \cdot a) \xi - \frac{k}{\hbar} \xi \times b - k^2 a \right), \\
H &= \mu_0^{-\frac{1}{2}} e^{-\frac{i}{\hbar} x \cdot \xi} \left( \frac{1}{\hbar^2} (\xi \cdot b) \xi + \frac{k}{\hbar} \xi \times a - k^2 b \right),
\end{align*}$$

(22)

where $a, b \in \mathbb{C}^3$ are complex vectors to be chosen later; and $\xi \in \mathbb{C}^3$ satisfies $\xi \cdot \xi = k^2 \hbar^2$.

The highest order linearization of the admittance map yields

$$\begin{cases}
\nabla \wedge V^{(123)}_j - i \omega \mu_0 W^{(123)}_j = i \omega p_j \sum_{(i_1 i_2 i_3) = \sigma(123)} W^{(i_1)}_j \cdot W^{(i_2)}_j \cdot W^{(i_3)}_j, \\
\nabla \wedge W^{(123)}_j - i \omega \epsilon_0 V^{(123)}_j = i \omega q_j \sum_{(i_1 i_2 i_3) = \sigma(123)} V^{(i_1)}_j \cdot V^{(i_2)}_j \cdot V^{(i_3)}_j,
\end{cases}$$

in $\Omega$ with $\nu \wedge V^{(123)}_j|_{\partial \Omega} = 0$. Here, $\sigma \in S_3$ denotes the permutation of the indices. Thus,
we obtain the integral identity

\[
\int_{\Omega} (p_1 - p_2) \sum_{(i_1,i_2,i_3) = \sigma(123)} W^{(i_1)} \cdot W^{(i_2)} W^{(i_3)} \cdot W^{(0)} dV, \tag{23}
\]

\[
+ \int_{\Omega} (q_1 - q_2) \sum_{(i_1,i_2,i_3) = \sigma(123)} V^{(i_1)} \cdot V^{(i_2)} V^{(i_3)} \cdot V^{(0)} dV = 0, \tag{24}
\]

by choosing \(W^{(0)}\) and \(V^{(0)}\) satisfying the following conjugate Maxwell’s equations

\[
\begin{cases}
\nabla \lhd V^{(0)} - i\omega \mu_0 W^{(0)} = 0, \\
\nabla \lhd W^{(0)} + i\omega \varepsilon_0 V^{(0)} = 0,
\end{cases}
\tag{25}
\]

in \(\Omega\) with boundary value \(\nu \lhd V^{(0)}|_{\partial \Omega} = f_0\). Here \(f_0 \in C^\infty(\partial \Omega)\) with \(\text{supp} f_0 \subseteq \Gamma_2\).

### 4.1.1 Local result

Below we are to prove a local version of Theorem 4.1, i.e., to show \(p_1 = p_2\) and \(q_1 = q_2\) locally in a small neighborhood of some \(x_0 \in \Gamma_1\). Without loss of generality, as in section 3, the domain can be set below the coordinate \(\{x_1 \leq 0\}\) by orthogonal transformation and translation, and \(x_0 = 0 \in \Gamma_1\).

**Proposition 4.1** Let \(\Omega\) be a bounded open set in \(\mathbb{R}^3\) with smooth boundary \(\partial \Omega\). Let \(x_0 \in \Gamma_1 \cap \Gamma_2 \subset \partial \Omega\) be a convex point on the boundary such that \(\Gamma \subset \Gamma_1 \cap \Gamma_2\). Let \(\Gamma\) be a small neighborhood of \(x_0\) on the boundary. Suppose that the cancellation \(23\) and \(24\) hold for any smooth solutions \((V^{(l)}, W^{(l)})\) with \(l = 1, 2, 3\), and \((V^{(0)}, W^{(0)})\) to the systems \(21\) and \(25\), respectively. Then there exists \(\delta > 0\) such that \(p_1 - p_2\) and \(q_1 - q_2\) vanish on \(B(x_0, \delta) \cap \Omega\).

**Proof.** Let \(p^{-1}(kh) = \{\xi \in C^3 | \xi \cdot \xi = k^2 h^2\}\), \(\xi_0 = (i, \sqrt{k^2 h^2 + 1}, 0) \in p^{-1}(kh)\) and \(\eta_0 = (i, -\sqrt{k^2 h^2 + 1}, 0) \in p^{-1}(kh)\). Suppose \(|\xi - \xi_0| < \epsilon, \xi \in p^{-1}(kh)\), and \(|\eta - \eta_0| < \epsilon, \eta \in p^{-1}(kh)\). Then \(\text{Im} (\xi_1) > 0\) and \(\text{Im} (\eta_1) > 0\).

Below we construct solutions to the systems \(21\) and \(25\), achieved by adding correction terms to the solutions satisfying the corresponding homogeneous Maxwell’s equations of the form \(22\). Choose \(b = 0\) in \(22\). Denote

\[
E_1 = \varepsilon_0^{-\frac{1}{2}} \left( \frac{1}{h^2} (\xi \cdot a_1) \xi - k^2 a_1 \right) e^{-\frac{i}{h} x \cdot \xi}, \quad H_1 = \mu_0^{-\frac{1}{2}} \frac{k}{h} (\xi \times a_1) e^{-\frac{i}{h} x \cdot \xi},
\]

\[
E_2 = \varepsilon_0^{-\frac{1}{2}} \left( \frac{1}{h^2} (\xi \cdot a_2) \xi - k^2 a_2 \right) e^{-\frac{i}{h} x \cdot \xi}, \quad H_2 = \mu_0^{-\frac{1}{2}} \frac{k}{h} (\xi \times a_2) e^{-\frac{i}{h} x \cdot \xi},
\]

\[
E_3 = \varepsilon_0^{-\frac{1}{2}} \left( \frac{1}{h^2} (\xi \cdot a_3) \xi - k^2 a_3 \right) e^{-\frac{i}{h} x \cdot \xi}, \quad H_3 = \mu_0^{-\frac{1}{2}} \frac{k}{h} (\xi \times a_3) e^{-\frac{i}{h} x \cdot \xi},
\]

20
and
\[
E_0 = \varepsilon_0^{-\frac{1}{2}} \left( \frac{1}{h^2} (\bar{\eta} \cdot \bar{a}_0) \bar{\eta} - k^2 \bar{a}_0 \right) e^{i k x \cdot \bar{\eta}}, \quad H_0 = -\mu_0^{-\frac{1}{2}} \frac{k}{h} (\bar{\eta} \times \bar{a}_0) e^{i k x \cdot \bar{\eta}},
\]

(27)

where the constant vectors are defined by
\[
a_1 = (1, 0, 1), \quad a_2 = a_3 = \bar{\xi} \times \bar{a}_1, \quad a_0 = \xi \times a_1.
\]

(28)

Take a cutoff function \(0 \leq \chi \leq 1 \in C_c^\infty\) which equals 1 on \(x_1 \leq -2c\) containing \(\partial \Omega \setminus \Gamma_1\) and \(\partial \Omega \setminus \Gamma_2\) and is compactly supported in \(x_1 \leq -c\). There exists \((r_1, t_1)\) to the Dirichlet problem
\[
\begin{aligned}
\nabla \wedge r_1 - i \omega \mu_0 t_1 &= 0, \\
\nabla \wedge t_1 + i \omega \varepsilon_0 r_1 &= 0,
\end{aligned}
\]

in \(\Omega\) with boundary value \(\nu \wedge r_1 \mid_{\partial \Omega} = -\chi(x) E_1\), satisfying the following estimates
\[
\|r_1\|_{H^1(\Omega)}, \|t_1\|_{H^1(\Omega)} \leq C (1 + h^{-1} |\xi|)^\frac{3}{2} e^{-\frac{\varepsilon}{c} \text{Im} \xi_1} e^{\frac{\varepsilon}{c} |\text{Im} \xi'|} \quad \text{if Im} \xi_1 \geq 0.
\]

(29)

Likewise, we have \((r_2, t_2), (r_3, t_3)\) and \((r_0, t_0)\) with similar estimates to (29). Thus,
\[
\begin{aligned}
V^{(1)} &= E_1 + r_1, \quad W^{(1)} = H_1 + t_1; \\
V^{(2)} &= E_2 + r_2, \quad W^{(2)} = H_2 + t_2; \\
V^{(3)} &= E_3 + r_3, \quad W^{(3)} = H_3 + t_3; \\
V^{(0)} &= E_0 + r_0, \quad W^{(0)} = H_0 + t_0,
\end{aligned}
\]

are solutions to the systems (21) and (25), respectively. Now, the derived cancellation (23) and (24) become
\[
\int_{\Omega} (p_1 - p_2) \sum_{(i_1 i_2 i_3) = \sigma(123)} (H_{i_1} + t_{i_1}) \cdot (H_{i_2} + t_{i_2}) (H_{i_3} + t_{i_3}) \cdot (H_0 + t_0) dV \quad \text{(30)}
\]

\[
+ \int_{\Omega} (q_1 - q_2) \sum_{(i_1 i_2 i_3) = \sigma(123)} (E_{i_1} + r_{i_1}) \cdot (E_{i_2} + r_{i_2}) (E_{i_3} + r_{i_3}) \cdot (E_0 + r_0) dV = 0. \quad \text{(31)}
\]

By virtue of (28), one has \(H_1 \cdot H_2 = H_1 \cdot \bar{H}_3 = H_1 \cdot \bar{H}_0 = 0\). Therefore, only the items involving some \(t_i\) survive in the expansion of the product (30), which are exponentially decaying in view of (29) when \(\text{Im} \xi_1 \geq 0\) and \(\text{Im} \xi' = 0\).

Next we calculate the items in the expansion of the product (31). Note that the items involving some \(r_i\) also have exponentially decaying properties from (29) when \(\text{Im} \xi_1 \geq 0\) and \(\text{Im} \xi' = 0\). We only need to calculate \((E_1 \cdot \bar{E}_2)(E_3 \cdot \bar{E}_0), \ (E_1 \cdot \bar{E}_3)(E_2 \cdot \bar{E}_0), \ (E_2 \cdot \bar{E}_3)(E_1 \cdot \bar{E}_0)\).
Let $\xi = \xi_0 + l_1$ and $\eta = \eta_0 + l_2$, where $|l_1|, |l_2| \leq \epsilon$.

$$h^8 e^{\frac{1}{\gamma}x \cdot (2\xi - \xi_0 + \eta)} (E_1 \cdot \bar{E}_2)(E_3 \cdot \bar{E}_0) = \left((\xi_0 \cdot a_1)\xi_0 - h^2 k^2 a_1 + ((l_1 \cdot a_1)\xi_0 + (\xi_0 \cdot a_1)l_1)\right) \cdot \left((\xi_0 \cdot a_2)\xi_0 - h^2 k^2 a_2 + ((l_1 \cdot a_2)\xi_0 + (\xi_0 \cdot a_2)l_1)\right) \cdot \left((\xi_0 \cdot a_3)\xi_0 - h^2 k^2 a_3 + ((l_1 \cdot a_3)\xi_0 + (\xi_0 \cdot a_3)l_1)\right) \cdot \left((\eta_0 \cdot a_0)\eta_0 - h^2 k^2 a_0 + ((l_2 \cdot a_0)\eta_0 + (\eta_0 \cdot a_0)l_2)\right) \geq ((\xi_0 \cdot a_1)(\xi_0 \cdot a_2)(\xi_0 \cdot a_3) + O(\epsilon) + O(h^2)) \cdot ((\xi_0 \cdot a_3)(\eta_0 \cdot a_0)(\xi_0 \cdot \eta_0) + O(\epsilon) + O(h^2)) \geq \left(4 + 24\epsilon a_1 a_2 a_3 \xi_0 \rho + O(h^2)\right) \cdot \left(8 + 25\epsilon a_2^2 \xi_0^3 + O(h^2)\right).

Therefore, $h^8 e^{\frac{1}{\gamma}x \cdot (2\xi - \xi_0 + \eta)} (E_1 \cdot \bar{E}_2)(E_3 \cdot \bar{E}_0)$ is greater than some positive constant, provided $\epsilon$ and $h$ are sufficiently small. Direct calculations yield similar estimates for $(E_1 \cdot \bar{E}_3)(E_2 \cdot \bar{E}_0), (E_3 \cdot \bar{E}_1)(E_2 \cdot \bar{E}_0), (E_2 \cdot \bar{E}_3)(E_1 \cdot \bar{E}_0)$ and $(E_2 \cdot \bar{E}_3)(E_1 \cdot \bar{E}_0)$. Hence, when $\epsilon$ and $h$ are sufficiently small, one has

$$\left|\int_{\Omega} (q_1 - q_2) \sum_{(i_1 i_2 i_3) = \sigma(123)} E_{i_1} \cdot \bar{E}_{i_2} E_{i_3} \cdot \bar{E}_{i_4} dV\right| \geq \left|\int_{\Omega} (q_1 - q_2) h^8 e^{-\frac{1}{\gamma}x \cdot (2\xi - \xi_0 + \eta)} dV\right| \cdot 4$$

and the R.H.S. of the inequality is controlled by items involving at least one $t_i$ or $r_i$. Combining the estimate (29) yields

$$\left|\int_{\Omega} (q_1 - q_2) e^{-\frac{1}{\gamma}x \cdot (2\xi - \xi_0 + \eta)} dV\right| \leq C h^8 \|q_1 - q_2\|_{L^\infty(\Omega)} (1 + h^{-1}|\xi|)^2 (1 + h^{-1}|\eta|)^2 e^{-\frac{\epsilon}{h} \min(\text{Im}\xi, \text{Im}\eta)} e^{\frac{1}{h}(3\text{Im}\xi' + |\text{Im}\eta'|)}$$

for $\text{Im}\xi_1 \geq 0$ and $\text{Im}\eta_1 \geq 0$. In particular, let

$$z = 2\xi - \bar{\xi} + \eta, \quad z_0 = (4i, 0, 0).$$

When $|\xi - a\xi_0| < Ca\varepsilon$ and $|\eta - a\eta_0| < Ca\varepsilon$ with $\varepsilon \leq \frac{1}{2C}$, then $|z - a z_0| < 4Ca\varepsilon$, and

$$\left|\int_{\Omega} (q_1 - q_2) e^{-\frac{1}{\gamma}x \cdot z} dV\right| \leq C h^2 \|q_1 - q_2\|_{L^\infty(\Omega)} e^{-\frac{ca}{2k} e^{-\frac{1}{\gamma}} \cdot \frac{4Ca\varepsilon}{h}}.$$

Conversely, for any $z \in \mathbb{C}^3$, $|z - a z_0| < 4Ca\varepsilon$, $z$ can be decomposed to

$$z = 2\xi - \bar{\xi} + \eta, \quad \xi, \eta \in p^{-1}(kh), \quad |\xi - a\xi_0| < Ca\varepsilon, |\eta - a\eta_0| < Ca\varepsilon.$$

Therefore, for all $z \in \mathbb{C}^n$, $|z - a z_0| < 4Ca\varepsilon$, the estimate holds

$$\left|\int_{\Omega} (q_1 - q_2) e^{-\frac{1}{\gamma}x \cdot z} dV\right| \leq C h^2 \|q\|_{L^\infty(\Omega)} e^{-\frac{ca}{2k} e^{-\frac{1}{\gamma}} \cdot \frac{2Ca\varepsilon}{h}}.$$

Hence we are able to obtain an exponential decay for the F.B.I. transform of $T(q_1 - q_2)$ in the neighborhood $|z - a z_0| < 4Ca\varepsilon$. Then apply the water melon approach similarly.
to show the uniqueness of $q(x)$ in a small neighborhood of $x_0 = 0$, i.e., there exists some $\delta_1 > 0$ such that

$$ q_1(x) = q_2(x), \quad \forall x \in \Omega, \quad -\delta_1 \leq x_1 \leq 0. $$

Now let $a = 0$ in (22) and choose $b_1, b_2, b_3, b_0$ similar as in (28). It is easy to see that the expressions of $E$ and $H$ interchange in (26)-(27) and the similar procedures follow. Hence in the same way we arrive at the uniqueness of $p(x)$ in a small neighborhood of $x_0 = 0$, i.e., there exists some $\delta_2 > 0$ such that

$$ p_1(x) = p_2(x), \quad \forall x \in \Omega, \quad -\delta_2 \leq x_1 \leq 0. $$

This completes the proof of Proposition 4.1. \qed

4.1.2 From local to global

To extend to global results we need some preparatory work. Lemma 4.1 describes a density result involving solutions to Maxwell’s equation as Lemma 4.1. Remark 4.1 and Remark 4.2 provide some properties of Green functions of Maxwell’s equations and Helmholtz equation, which will be needed in the sequel.

Lemma 4.1 Let $\Omega_1 \subset \Omega_2 \subset \mathbb{R}^3$ with smooth boundary and $\partial \Omega_1 \cap \partial \Omega_2 \neq \emptyset$. Let $\mathcal{R}$ be the set of $\{(v|_{\Omega_1}, w|_{\Omega_1}) \in C^\infty(\Omega_1, \mathbb{C}^3)^2\}$ such that

$$\begin{cases}
\nabla \wedge v - i\omega \mu_0 w = a, & \text{in } \Omega_2, \\
\nabla \wedge w + i\omega \varepsilon_0 v = \bar{a}, & \text{in } \Omega_2,
\end{cases}$$

with boundary value $\nu \wedge v|_{\partial \Omega_2} = 0$, for some $a \in C^\infty_c(\Omega_2, \mathbb{C}^3)$ with supp $a \subset \Omega_2 \setminus \Omega_1$. Let $\mathcal{S}$ be the set of $\{ (\tilde{v}, \tilde{w}) \in C^\infty(\Omega_1, \mathbb{C}^3)^2 \}$ such that

$$\begin{cases}
\nabla \wedge \tilde{v} - i\omega \mu_0 \tilde{w} = 0, & \text{in } \Omega_1, \\
\nabla \wedge \tilde{w} + i\omega \varepsilon_0 \tilde{v} = 0, & \text{in } \Omega_1,
\end{cases}$$

with boundary value $\nu \wedge \tilde{v}|_{\partial \Omega_1 \cap \partial \Omega_2} = 0$. Then $\mathcal{R}$ is dense in the space $\mathcal{S}$ with respect to the $L^2(\Omega_1, \mathbb{C}^3)^2$ topology.

Proof. Let $g = (g_1, g_2) \in L^2(\Omega_1, \mathbb{C}^3)^2$ and $g$ is orthogonal to $\mathcal{R}$. It suffices to show that $g$ is also orthogonal to $\mathcal{S}$.

Let $\tilde{g}_1$ and $\tilde{g}_2$ be the $L^2(\Omega_2, \mathbb{C}^3)$ extension of $g_1$ and $g_2$ by taking value 0 in $\Omega_2 \setminus \Omega_1$. Suppose $(f, h)$ are the solutions to the following nonhomogeneous Maxwell’s equations

$$\begin{cases}
\nabla \wedge f + i\omega \mu_0 h = \tilde{g}_2, & \text{in } \Omega_2, \\
\nabla \wedge h - i\omega \varepsilon_0 f = \tilde{g}_1 & \text{in } \Omega_2,
\end{cases}$$

23
with boundary value \( \nu \wedge f \mid_{\partial \Omega_2} = 0 \). Hence, for every \((v, w) \in \mathcal{R}\),

\[
\langle g_1, v \rangle_{\Omega_1} + \langle g_2, w \rangle_{\Omega_1} = \langle \nabla \wedge h - i \omega \varepsilon_0 f, v \rangle_{\Omega_2} + \langle \nabla \wedge f + i \omega \mu_0 h, w \rangle_{\Omega_2} \\
= \int_{\Omega_2} h \cdot \nabla \wedge v dV + \int_{\partial \Omega_2} \nu \wedge h \cdot \nu dS - \int_{\Omega_2} i \omega \varepsilon_0 f \cdot \nu dV \\
+ \int_{\Omega_2} f \cdot \nabla \wedge w dV + \int_{\partial \Omega_2} \nu \wedge f \cdot \nu dS + \int_{\Omega_2} i \omega \mu_0 h \cdot \nu dV \\
= \int_{\Omega_2} h \cdot \nabla \wedge v - i \omega \mu_0 \nu dV + \int_{\Omega_2} f \cdot \nabla \wedge w + i \omega \varepsilon_0 \nu dV \\
= \int_{\Omega_2} h \cdot \nu dV + \int_{\Omega_2} f \cdot \nu dV
\]

(32) vanishes. Suppose \( a = c(x) + d(x)i \), with supp \( c(x), d(x) \subset \Omega_2 \setminus \Omega_1 \). Then (32) becomes

\[
0 = \int_{\Omega_2} h \cdot (c(x) - d(x)i) dV + \int_{\Omega_2} f \cdot (c(x) + d(x)i) dV \\
= \int_{\Omega_2} (f + h) \cdot c(x) dV + \int_{\Omega_2} (f - h) \cdot d(x) dV.
\]

Since \( c(x) \) and \( d(x) \) are arbitrary in \( L^2(\Omega_2, \mathbb{R}^3) \), \( f + h \) and \( f - h \) vanish on \( \Omega_2 \setminus \Omega_1 \), which implies \( f \) and \( h \) vanish on \( \Omega_2 \setminus \Omega_1 \). Then

\[
\nu \wedge f \mid_{\partial \Omega_1 \setminus \partial \Omega_2} = 0 \quad \text{and} \quad \nu \wedge h \mid_{\partial \Omega_1 \setminus \partial \Omega_2} = 0.
\]

(33)

Combining the assumption \( \nu \wedge f \mid_{\partial \Omega_2} = 0 \), one has

\[
\nu \wedge f \mid_{\partial \Omega_1} = 0.
\]

(34)

For any \((\tilde{v}, \tilde{w}) \in \mathcal{S}\),

\[
\langle g_1, \tilde{v} \rangle_{\Omega_1} + \langle g_2, \tilde{w} \rangle_{\Omega_1} = \langle \nabla \wedge h - i \omega \varepsilon_0 f, \tilde{v} \rangle_{\Omega_1} + \langle \nabla \wedge f + i \omega \mu_0 h, \tilde{w} \rangle_{\Omega_1} \\
= \int_{\Omega_1} h \cdot \nabla \wedge \tilde{v} - i \omega \mu_0 \tilde{v} dV + \int_{\Omega_1} f \cdot \nabla \wedge \tilde{w} + i \omega \varepsilon_0 \tilde{v} dV \\
+ \int_{\partial \Omega_1} \nu \wedge h \cdot \nu dS + \int_{\partial \Omega_1} \nu \wedge f \cdot \nu dS,
\]

vanishes by virtue of (33), (34) and \( \nu \wedge \tilde{v} \mid_{\partial \Omega_1 \cap \partial \Omega_2} = 0 \). Therefore, \( g \) is orthogonal to \( \mathcal{S} \). □

Remark 4.1 (i) The fundamental solution \( G^{\mu, \varepsilon} = \left( \begin{array}{c} G^E \\ G^H \end{array} \right) = \left( \begin{array}{c} E^{(1)} \\ E^{(2)} \\ H^{(1)} \\ H^{(2)} \end{array} \right) \) consists of two parts \((E^{(1)}, H^{(1)})\) and \((E^{(2)}, H^{(2)})\) satisfying the following Maxwell’s equations, respectively:

\[
\begin{align*}
\nabla \wedge E^{(1)}(x, y) - i \omega \mu_0 H^{(1)}(x, y) &= \delta(x - y) I_3, \\
\nabla \wedge H^{(1)}(x, y) + i \omega \varepsilon_0 E^{(1)}(x, y) &= 0,
\end{align*}
\]

\[
\begin{align*}
\nabla \wedge E^{(2)}(x, y) - i \omega \mu_0 H^{(2)}(x, y) &= \delta(x - y) I_3, \\
\nabla \wedge H^{(2)}(x, y) + i \omega \varepsilon_0 E^{(2)}(x, y) &= 0.
\end{align*}
\]
in $\Omega_2$ with boundary value $\nu \wedge E^{(2)}|_{\partial \Omega_2} = 0$; and
\[
\begin{aligned}
\begin{cases}
\nabla \wedge E^{(2)}(x,y) - i\omega \mu_0 H^{(2)}(x,y) &= 0, \\
\nabla \wedge H^{(2)}(x,y) + i\omega \varepsilon E^{(2)}(x,y) &= \delta(x-y)I_3,
\end{cases}
\end{aligned}
\]
in $\Omega_2$ with boundary value $\nu \wedge E^{(2)}|_{\partial \Omega_2} = 0$.

(ii) Similarly, the fundamental solution $\tilde{G}^{e,\mu} = \begin{pmatrix} \tilde{G}^E \\ \tilde{G}^H \end{pmatrix} = \begin{pmatrix} \tilde{E}^{(1)} \\ \tilde{H}^{(1)} \\ \tilde{E}^{(2)} \\ \tilde{H}^{(2)} \end{pmatrix}$ consists of two parts which satisfy the Maxwell’s equations
\[
\begin{aligned}
\begin{cases}
\nabla \wedge \tilde{E}^{(1)}(x,y) - i\omega \varepsilon \tilde{H}^{(1)}(x,y) &= \delta(x-y)I_3, \\
\nabla \wedge \tilde{H}^{(1)}(x,y) + i\omega \mu_0 \tilde{E}^{(1)}(x,y) &= 0,
\end{cases}
\end{aligned}
\]
in $\Omega_2$ with boundary value $\nu \wedge \tilde{H}^{(1)}|_{\partial \Omega_2} = 0$; and
\[
\begin{aligned}
\begin{cases}
\nabla \wedge \tilde{E}^{(2)}(x,y) - i\omega \varepsilon \tilde{H}^{(2)}(x,y) &= 0, \\
\nabla \wedge \tilde{H}^{(2)}(x,y) + i\omega \mu_0 \tilde{E}^{(2)}(x,y) &= \delta(x-y)I_3,
\end{cases}
\end{aligned}
\]
in $\Omega_2$ with boundary value $\nu \wedge \tilde{H}^{(2)}|_{\partial \Omega_2} = 0$.

Remark 4.2 If $(v, w)$ satisfies the boundary value problem
\[
\begin{aligned}
\begin{cases}
\nabla \wedge v - i\omega \mu_0 w &= a, \\
\nabla \wedge w + i\omega \varepsilon_0 v &= \bar{a},
\end{cases}
\end{aligned}
\]
in $\Omega_2$, with boundary value $\nu \wedge v|_{\partial \Omega_2} = 0$, by direct computation $(\bar{w}, \bar{v})$ is a solution to the conjugate Maxwell’s equations
\[
\begin{aligned}
\begin{cases}
\nabla \wedge \bar{w} - i\omega \varepsilon_0 \bar{v} &= a, \\
\nabla \wedge \bar{v} + i\omega \mu_0 \bar{w} &= \bar{a},
\end{cases}
\end{aligned}
\]
in $\Omega_2$, with boundary value $\nu \wedge \bar{v}|_{\partial \Omega_2} = 0$. This implies that the conjugate $\bar{w}$ plays a role of the electricity $E$ while the conjugate $\bar{v}$ works as the magnetism $H$.

Let $G_{\Omega_2}(x,y)$ be the matrix of Green function of the Helmholtz equation $(-\Delta_y - k^2)G_{\Omega_2}(x,y) = \delta(x-y)I_3$, in $\Omega_2$, where $I_3$ is the $3 \times 3$ identity matrix.

Proof of Theorem 4.1 Below we show the extension of local result (Proposition 4.1) to the global situation following the same process as in section 2.2.2.

Let $\tilde{p}(y) = p_1 - p_2$ and $\tilde{q}(y) = q_1 - q_2$. Fix a point $x_1 \in \Omega$ and let $\theta : [0,1] \to \Omega$ be a $C^1$ curve joining $x_0 \in \Gamma$ to $x_1$ such that $\theta(0) = x_0$ and $\theta'(0)$ is the interior normal to $\partial \Omega$ at $x_0$ and $\theta(t) \in \Omega$ for all $t \in (0,1]$. Suppose $\Theta_\epsilon(t)$ is a small neighborhood of the curve $\theta([0,t])$ in $\Omega$ and $I$ is the set of the time where $\tilde{p}$ and $\tilde{q}$ vanish on $\Theta_\epsilon(t)$. Similarly, $I$ is a
closed set and $I$ is also non-empty due to local results, provided that $\epsilon$ is small enough. It suffices to show that $I$ is open.

Following the same setting as in section 2.2.2, we have $\partial \Theta_\epsilon(t) \cap \partial \Omega \subset \Gamma$, and

$$\Omega \setminus \Theta_\epsilon(t) \subset \Omega_1, \quad \partial \Omega \cap \Gamma \subset \partial \Omega \cap \partial \Omega_1; \quad \partial \Omega \setminus \Gamma \subset \partial \Omega \cap \partial \Omega_1 \subset \partial \Omega \cap \partial \Omega_2.$$  

Adopting the notation in Remark 4.1, let

$$G_1(x_1, y) = \begin{pmatrix} G_1^E(x_1, y) \\ G_1^H(x_1, y) \end{pmatrix}, \quad \tilde{G}_1(x_1, y) = \begin{pmatrix} \tilde{G}_1^E(x_1, y) \\ \tilde{G}_1^H(x_1, y) \end{pmatrix},$$

for $x_1, y \in \Omega_2$. Similarly, denote $G_2(x_2, y), G_3(x_3, y)$ and $\tilde{G}_0(x_0, y), \tilde{G}_2(x_2, y), \tilde{G}_3(x_3, y)$ for $x_2, x_3, x_0, y \in \Omega_2$, respectively. When $x_1, x_2, x_3, x_0 \in \Omega_2 \setminus \Omega_1$, define

$$H(x_1, x_2, x_3, x_0) = \int_{\Omega_1} \tilde{q}(y)K^E(x_1, x_2, x_3, x_0, y) + \tilde{p}(y)K^H(x_1, x_2, x_3, x_0, y)dy$$

$$= \int_{\Omega_1} \tilde{q}(y) \cdot G_1^E(x_1, y) \otimes \tilde{G}_2^H(x_2, y) \cdot G_3^E(x_3, y) \otimes \tilde{G}_0^H(x_0, y)$$

$$+ \tilde{q}(y) \cdot G_2^E(x_2, y) \otimes \tilde{G}_1^H(x_1, y) \cdot G_3^E(x_3, y) \otimes \tilde{G}_0^H(x_0, y)$$

$$+ \tilde{q}(y) \cdot G_3^E(x_3, y) \otimes \tilde{G}_1^H(x_1, y) \cdot G_2^E(x_2, y) \otimes \tilde{G}_0^H(x_0, y)$$

$$+ \tilde{q}(y) \cdot G_2^E(x_2, y) \otimes \tilde{G}_3^H(x_3, y) \cdot G_1^E(x_1, y) \otimes \tilde{G}_0^H(x_0, y)$$

$$+ \tilde{p}(y) \cdot G_1^H(x_1, y) \otimes \tilde{G}_2^E(x_2, y) \cdot G_3^H(x_3, y) \otimes \tilde{G}_0^E(x_0, y)$$

$$+ \tilde{p}(y) \cdot G_2^H(x_2, y) \otimes \tilde{G}_1^E(x_1, y) \cdot G_3^H(x_3, y) \otimes \tilde{G}_0^E(x_0, y)$$

$$+ \tilde{p}(y) \cdot G_3^H(x_3, y) \otimes \tilde{G}_1^E(x_1, y) \cdot G_2^H(x_2, y) \otimes \tilde{G}_0^E(x_0, y)$$

$$+ \tilde{p}(y) \cdot G_2^H(x_2, y) \otimes \tilde{G}_3^E(x_3, y) \cdot G_1^H(x_1, y) \otimes \tilde{G}_0^E(x_0, y)$$

$$+ \tilde{p}(y) \cdot G_3^H(x_3, y) \otimes \tilde{G}_2^E(x_2, y) \cdot G_1^H(x_1, y) \otimes \tilde{G}_0^E(x_0, y)dy.$$  

Here the tensor sign $\otimes$ represents the tensor product of the matrices and is also consistent with the tensor of distributions when the matrices are viewed as distributions in the sense that

$$\int (u_1(x_1) \otimes u_2(x_2))(\phi_1(x_1) \otimes \phi_2(x_2))dx_1dx_2 = \int u_1\phi_1dx_1 \int u_2\phi_2dx_2,$$

for $u_1, u_2 \in D'(X_j)$ and $\phi_j \in C_0^\infty(X_j)$ where $X_j$ are open sets of $\mathbb{R}^3$.

The term involving $\tilde{p}(y)$ plays the role of a magnetic field, while the term involving $\tilde{q}(y)$ serves as an electric field in the coupled Maxwell’s equations with vanishing tangential
component on the boundary. Moreover, for $x_1, x_2, x_3, x_0 \in \Omega_2 \setminus \Omega_1$,

$$H(x_1, x_2, x_3, x_0) = \int_{\Omega_1} \tilde{q}(y)K^E(x_1, x_2, x_3, x_0, y) + \tilde{p}(y)K^H(x_1, x_2, x_3, x_0, y)dy$$

$$= \int_{\Omega} \tilde{q}(y)K^E(x_1, x_2, x_3, x_0, y) + \tilde{p}(y)K^H(x_1, x_2, x_3, x_0, y)dy.$$  

Note that each entry of $H$ is a sum of terms involving $\tilde{p}(y)$ and $\tilde{q}(y)$ and satisfies the Helmholtz equation $(-\Delta_y - k^2)u = 0$, for $x_1, x_2, x_3, x_0 \in \Omega_2 \setminus \Omega_1$. Therefore, for $x_1, x_2, x_3, x_0 \in \Omega_2 \setminus \tilde{\Omega}$, from the assumption we have

$$\int_{\Omega} \tilde{q}(y)K^E(x_1, x_2, x_3, x_0, y) + \tilde{p}(y)K^H(x_1, x_2, x_3, x_0, y)dy = 0.$$

It follows that $H(x_1, x_2, x_3, x_0) = 0$ when $x_1, x_2, x_3, x_0 \in \Omega_2 \setminus \tilde{\Omega}$.

When $x_1, x_2, x_3, x_0 \in \Omega_2 \setminus \Omega_1$, the entries of $H$ satisfy the Helmholtz equation $(-\Delta_y - k^2)u = 0$. Hence, it follows from the Unique Continuation Principle that

$$H(x_1, x_2, x_3, x_0) = 0 \text{ when } x_1, x_2, x_3, x_0 \in \tilde{\Omega}_2 \setminus \Omega_1;$$

i.e., for $x_1, x_2, x_3, x_0 \in \tilde{\Omega}_2 \setminus \Omega_1$,

$$\int_{\Omega_1} \tilde{q}(y)K^E(x_1, x_2, x_3, x_0, y) + \tilde{p}(y)K^H(x_1, x_2, x_3, x_0, y)dy = 0.$$

Let the left hand side of (36) act on $(a_1(x_1) \otimes a_2(x_2), a_3(x_3) \otimes a_4(x_4))$ for $x_1, x_2, x_3, x_0 \in \Omega_2$.

Here all $a_i$ are assumed to have the forms $a_i(x_i) = (l_i(x_i), \overline{l_i(x_i)})^T$, where each $l_i$ is a smooth vector function on $C^\infty(\Omega_2, \mathbb{C}^3)$ with support in $\Omega_2 \setminus \Omega_1$. For each $i$, let

$$v_i(y) = \int_{\Omega_2} G_i^E(x_i, y)a_i(x_i)dx_i, \quad w_i(y) = \int_{\Omega_2} G_i^H(x_i, y)a_i(x_i)dx_i,$$

$$\tilde{v}_i(y) = \int_{\Omega_2} \tilde{G}_i^E(x_i, y)a_i(x_i)dx_i, \quad \tilde{v}_i(y) = \int_{\Omega_2} \tilde{G}_i^H(x_i, y)a_i(x_i)dx_i.$$

Then (36) becomes

$$0 = \int_{\Omega_1} \tilde{q}(y)K^E(x_1, x_2, x_3, x_0, y)\tilde{a} + \tilde{p}(y)K^H(x_1, x_2, x_3, x_0, y)\tilde{a}dy$$

$$= \int_{\Omega_1} \tilde{q}(y)(v_1 \cdot \tilde{v}_2) \cdot (v_3 \cdot \tilde{v}_0) + \tilde{q}(y)(v_2 \cdot \tilde{v}_1) \cdot (v_3 \cdot \tilde{v}_0)$$

$$+ \tilde{q}(y)(v_1 \cdot \tilde{v}_3) \cdot (v_2 \cdot \tilde{v}_0) + \tilde{q}(y)(v_3 \cdot \tilde{v}_1) \cdot (v_2 \cdot \tilde{v}_0)$$

$$+ \tilde{q}(y)(v_2 \cdot \tilde{v}_3) \cdot (v_1 \cdot \tilde{v}_0) + \tilde{q}(y)(v_3 \cdot \tilde{v}_2) \cdot (v_1 \cdot \tilde{v}_0)$$

$$+ \tilde{p}(y)(w_1 \cdot \tilde{w}_2) \cdot (w_3 \cdot \tilde{w}_0) + \tilde{p}(y)(w_2 \cdot \tilde{w}_1) \cdot (w_3 \cdot \tilde{w}_0)$$

$$+ \tilde{p}(y)(w_1 \cdot \tilde{w}_3) \cdot (w_2 \cdot \tilde{w}_0) + \tilde{p}(y)(w_3 \cdot \tilde{w}_1) \cdot (w_2 \cdot \tilde{w}_0)$$

$$+ \tilde{p}(y)(w_2 \cdot \tilde{w}_3) \cdot (w_1 \cdot \tilde{w}_0) + \tilde{p}(y)(w_3 \cdot \tilde{w}_2) \cdot (w_1 \cdot \tilde{w}_0)dy \quad (37)$$
The fundamental solutions of Remark 4.1 imply that \((v_i, w_i)\) is a solution to the inhomogeneous Maxwell’s systems

\[
\begin{align*}
\nabla \wedge v_i - i\omega \mu_0 w_i &= l_i, \\
\nabla \wedge w_i + i\omega \epsilon_0 v_i &= \overline{l_i},
\end{align*}
\]

in \(\Omega_2\) with boundary value \(\nu \wedge v_i|_{\partial \Omega_2} = 0\), while \((\tilde{v}_i, \tilde{w}_i)\) satisfies

\[
\begin{align*}
\nabla \wedge \tilde{w}_i - i\omega \epsilon_0 \tilde{v}_i &= l_i, \\
\nabla \wedge \tilde{v}_i + i\omega \mu_0 \tilde{w}_i &= \overline{l_i},
\end{align*}
\]

in \(\Omega_2\) with boundary value \(\nu \wedge \tilde{v}_i|_{\partial \Omega_2} = 0\). Hence, the uniqueness of solutions gives \(\tilde{v}_i = v_i\) and \(\tilde{w}_i = w_i\) in \(\Omega_2\), for \(i = 0, 1, 2, 3\). Therefore, (37) yields

\[
0 = \int_{\Omega_1} (p_1 - p_2) \sum_{(i_1 i_2 i_3) = \sigma(123)} (w_{i_1} \cdot \overline{w_{i_2}})(w_{i_3} \cdot \overline{w_0})dy + \int_{\Omega_1} (q_1 - q_2) \sum_{(i_1 i_2 i_3) = \sigma(123)} (v_{i_1} \cdot \overline{v_{i_2}})(v_{i_3} \cdot \overline{v_0})dy,
\]

where \((v_i, w_i)\) satisfies (38). Since all \(a_i \in C^\infty(\Omega_2, C^2)\) are supported in \(\tilde{\Omega}_2 \setminus \Omega_1\), the set of all the solutions \((v_i, w_i)\) forms the set \(R\). Lemma 4.1 states that \(R\) is dense in the space \(S\) with respect to the \(L^2(\Omega_1, C^2)\) topology. Thus for any \(V_i\) and \(W_i\) satisfying inhomogeneous Maxwell’s systems on \(\Omega_1\) with \(\nu \wedge V_i\) vanishing on \(\partial \Omega_1 \cap \partial \Omega_2\), we can choose \(v_i^{(n)}\) and \(w_i^{(n)}\) converge to \(V_i\) and \(W_i\) in \(L^2(\Omega_1, C^2)\) respectively.

Note that each \(v_i^{(n)}\), \(w_i^{(n)}\) and \(V_i, W_i\) are smooth in \(\Omega_2\). Then \(\|v_i^{(n)}\|_{L^\infty(\Omega_2)}\) and \(\|w_i^{(n)}\|_{L^\infty(\Omega_2)}\) are bounded for each fixed \(i\). Thus assume

\[
\|v_i^{(n)}\|_{L^\infty(\Omega_2)}, \|w_i^{(n)}\|_{L^\infty(\Omega_2)}, \|V_i\|_{L^\infty(\Omega_2)}, \|W_i\|_{L^\infty(\Omega_2)} \leq M.
\]

It is easy to show that \((v_1^{(n)} \cdot \tilde{v}_2^{(n)})(v_3^{(n)} \cdot \tilde{v}_0^{(n)})\) converges to \((V_1 \cdot \tilde{V}_2)(V_3 \cdot \tilde{V}_0)\) in \(L^1(\Omega_1)\) as \(n \to \infty\). The equality (39) extends to

\[
0 = \int_{\Omega_1} (p_1 - p_2) \sum_{(i_1 i_2 i_3) = \sigma(123)} (W_{i_1} \cdot \overline{W_{i_2}})(W_{i_3} \cdot \overline{W_0})dV + \int_{\Omega_1} (q_1 - q_2) \sum_{(i_1 i_2 i_3) = \sigma(123)} (V_{i_1} \cdot \overline{V_{i_2}})(V_{i_3} \cdot \overline{V_0})dV,
\]

where \((V_i, W_i) \in C^\infty(\Omega_1, C^2)^2\) is a solution to inhomogeneous Maxwell’s systems and vanishes on \(\partial \Omega_1 \cap \partial \Omega_2\). Applying Proposition 4.1 \(\tilde{p}\) and \(\tilde{q}\) vanish on a neighborhood of \(\partial \Omega_1 \setminus (\partial \Omega_1 \cap \partial \Omega_2)\). Then \(\tilde{p}\) and \(\tilde{q}\) vanish on a bigger neighborhood \(\Theta_\epsilon(t')\ t' > t\) of the curve. Hence \(I\) is open. \(\square\)
### 4.2 Proof of Theorem 1.4

Below we follow the standard procedures of high order linearization of the admittance map.

**Proof.** Let $f^{k\omega} = \epsilon_1 f_1^{k\omega} + \epsilon_2 f_2^{k\omega}$, $f_1^{k\omega}, f_2^{k\omega} \in C^\infty(\partial\Omega)$ with $\text{supp} f_1^{k\omega} \subseteq \Gamma_1$, for $k = 1, 2$ and $l = 1, 2$. Let $\epsilon = (\epsilon_1, \epsilon_2)$. Denote $(V_l^{k\omega,j}, W_l^{k\omega,j}) = (\partial \epsilon_l E^{k\omega,j}|_{\epsilon=0}, \partial \epsilon_l H^{k\omega,j}|_{\epsilon=0})$. Similarly, one can show that $V_l^{k\omega} := V_l^{k\omega,1} = V_l^{k\omega,2}$ and $W_l^{k\omega} := W_l^{k\omega,1} = W_l^{k\omega,2}$ satisfy

\[
\begin{cases}
\nabla \wedge V_l^{\omega,j} - i \omega \mu_0 W_l^{\omega,j} = 0, \\
\nabla \wedge W_l^{\omega} + i \omega \varepsilon_0 V_l^{\omega} = 0, \\
\nabla \wedge V_l^{2\omega,j} - i 2 \omega \mu_0 W_l^{2\omega,j} = 0, \\
\nabla \wedge W_l^{2\omega} + i 2 \omega \varepsilon_0 V_l^{2\omega} = 0,
\end{cases}
\]

in $\Omega$ with boundary values $\nu \wedge V_l^{k\omega}|_{\partial\Omega} = f_l^{k\omega}$ for $k = 1, 2$ and $l = 1, 2$. The second linearization yields that $(W_{12}^{k\omega,j}, V_{12}^{k\omega,j}) = (\partial^2 \epsilon_l H^{k\omega,j}|_{\epsilon=0}, \partial^2 \epsilon_l E^{k\omega,j}|_{\epsilon=0})$ satisfies

\[
\begin{cases}
\nabla \wedge V_{12}^{\omega,j} - i \omega \mu_0 W_{12}^{\omega,j} = 0, \\
\nabla \wedge W_{12}^{\omega} + i \omega \varepsilon_0 V_{12}^{\omega} = -i \omega \chi^{(2),j} (V_1^{\omega} \cdot V_2^{2\omega} + V_2^{\omega} \cdot V_1^{2\omega}), \\
\nabla \wedge V_{12}^{2\omega,j} - i 2 \omega \mu_0 W_{12}^{2\omega,j} = 0, \\
\nabla \wedge W_{12}^{2\omega} + i 2 \omega \varepsilon_0 V_{12}^{2\omega} = -i \omega \chi^{(2),j} V_1^{\omega} \cdot V_2^{\omega},
\end{cases}
\]

in $\Omega$ with boundary values $\nu \wedge V_{12}^{k\omega,j}|_{\partial\Omega} = 0$ with $k = 1, 2$ and $j = 1, 2$. We are able to derive the following integral identities

\[
\int_{\Omega} \left( \chi^{(2),1} - \chi^{(2),2} \right) \cdot \nabla V_0^{\omega} \left( V_1^{\omega} \cdot V_2^{2\omega} + V_2^{\omega} \cdot V_1^{2\omega} \right) dV = 0,
\]

and

\[
\int_{\Omega} \left( \chi^{(2),1} - \chi^{(2),2} \right) \cdot V_0^{2\omega} \left( V_1^{2\omega} \cdot V_2^{\omega} \right) dV = 0,
\]

by choosing $(V_0^{k\omega}, W_0^{k\omega})$ to be the solution to the following conjugate Maxwell’s equations

\[
\begin{cases}
\nabla \wedge V_0^{\omega} - i \omega \mu_0 W_0^{\omega} = 0, \\
\nabla \wedge W_0^{\omega} + i \omega \varepsilon_0 V_0^{\omega} = 0, \\
\nabla \wedge V_0^{2\omega} - i 2 \omega \mu_0 W_0^{2\omega} = 0, \\
\nabla \wedge W_0^{2\omega} + i 2 \omega \varepsilon_0 V_0^{2\omega} = 0,
\end{cases}
\]

in $\Omega$ with boundary values $\nu \wedge V_0^{k\omega}|_{\partial\Omega} = f_0^{k\omega}$, where $f_0^{k\omega} \in C^\infty(\partial\Omega)$ with supports on $\Gamma_2$, for each $k$. Next we construct $V_1^{\omega}$, $V_2^{\omega}$ and $V_0^{2\omega}$ satisfying (40) and (43) to plug in the integral identity (42).
Remark 4.3 It is easy to check that \((E, H)\) of the following form
\[
\begin{align*}
E &= \varepsilon_0^{-\frac{1}{2}} e^{-\frac{i}{h} x \cdot \xi} A_{\xi}, \quad \text{satisfying } A_{\xi} \cdot \xi = 0, \\
H &= \varepsilon_0^{-\frac{1}{2}} e^{-\frac{i}{h} x \cdot \xi} B_{\xi}, \quad \text{with } B_{\xi} = -\frac{1}{kh} \xi \times A_{\xi},
\end{align*}
\]
satisfies the linearized Maxwell equations by noting that
\[
A_{\xi} = -\frac{1}{k^2 h^2} \cdot ((\xi \cdot \xi) A_{\xi} - (\xi \cdot A_{\xi}) \xi) = -\frac{1}{k^2 h^2} \xi \times (\xi \times A_{\xi}) = \frac{1}{kh} \xi \times B_{\xi}.
\]

4.2.1 Local result

Below we are to prove a local version of Theorem 1.4, i.e., to show \(\chi^{(2), 1} = \chi^{(2), 2}\) locally in a small neighborhood of some \(x_0 \in \Gamma_1\). Without loss of generality we assume that the domain is below the coordinate \(\{ x_1 \leq 0 \}\) by orthogonal transformation and translation and \(x_0 = 0\).

Proposition 4.2 Let \(\Omega\) be a bounded open set in \(\mathbb{R}^3\) with smooth boundary \(\partial \Omega\). Let \(x_0 \in \Gamma_1 \cap \Gamma_2 \subset \partial \Omega\) be a convex point on the boundary such that \(\Gamma \subset \Gamma_1 \cap \Gamma_2\). Let \(\Gamma\) be a small neighborhood of \(x_0\) on the boundary. Suppose that the cancellation \(\frac{43}{43}\) hold for any smooth solutions \((V_1^\omega, W_1^\omega), (V_2^\omega, W_2^\omega), (V_0^\omega, W_0^\omega)\) to the systems \(\frac{40}{40}\) and \(\frac{43}{43}\). Then there exists \(\delta > 0\) such that \(\chi^{(2), 1} = \chi^{(2), 2}\) on \(B(x_0, \delta) \cap \Omega\).

Proof. (i) Define the set \(p^{-1}(kh) = \{ \xi \in \mathbb{C}^3 | \xi \cdot \xi = k^2 h^2 \}\) and choose
\[
\xi_0 = (i, \sqrt{1 + k^2 h^2}, 0), \quad \eta_0 = (i, 0, \sqrt{1 + k^2 h^2}) \in p^{-1}(kh).
\]
Suppose \(|\xi - \xi_0| < \epsilon, \xi \in p^{-1}(kh)\), and \(|\eta - \eta_0| < \epsilon, \eta \in p^{-1}(kh)\). Then \(\text{Im}(\xi_1) > 0\) and \(\text{Im}(\eta_1) > 0\). Below we construct pairs of solutions as in subsection 4.1. Here we technically construct \((E_0^{2\omega}, H_0^{2\omega})\) of the expressions \(\frac{44}{44}\), \(\frac{45}{45}\) and \((E_1^\omega, H_1^\omega), (E_2^\omega, H_2^\omega)\) of the expression \(\frac{22}{22}\).

As in Remark 4.3 choose \(E_0^{2\omega} = \varepsilon_0^{-\frac{1}{2}} e^{-\frac{i}{k} x \cdot \zeta} \tilde{A}_\zeta\), where
\[
\zeta = (\sqrt{2(1 - k^2 h^2)} i, \sqrt{1 + k^2 h^2}, \sqrt{1 + k^2 h^2}) \in p^{-1}(2kh),
\]
\[
\tilde{A}_\zeta = \left(\begin{array}{c}
-\sqrt{2(1 - k^2 h^2)} i, \\
(1 - k^2 h^2) \sqrt{1 + k^2 h^2} - (1 - k^2 h^2) \sqrt{1 + k^2 h^2}
\end{array}\right),
\]

Since \(\zeta \cdot \tilde{A}_\zeta = 0\), \(E_0^{2\omega}\) together with some determined \(H_0^{2\omega}\) are solutions to the linearized Maxwell’s equations following Remark 4.3.
Let \( b = 0 \), \( a_1 = a_2 = (1,0,0) \) in (22). Denote \( H_1^\omega = \mu_0^{-\frac{1}{2}} \frac{k}{\hbar} (\xi \times a_1) e^{-\frac{i}{\hbar} x \cdot \xi}, \) \( H_2^\omega = \mu_0^{-\frac{1}{2}} \frac{k}{\hbar} (\eta \times a_2) e^{-\frac{i}{\hbar} x \cdot \eta} \), and
\[
E_1^\omega = \varepsilon_0^{-\frac{1}{2}} e^{-\frac{i}{\hbar} x \cdot \xi} \tilde{A}_\xi = \varepsilon_0^{-\frac{1}{2}} \left( \frac{1}{\hbar^2} (\xi \cdot a_1) \xi - k^2 a_1 \right) e^{-\frac{i}{\hbar} x \cdot \xi},
\]
\[
E_2^\omega = \varepsilon_0^{-\frac{1}{2}} e^{-\frac{i}{\hbar} x \cdot \eta} \tilde{A}_\eta = \varepsilon_0^{-\frac{1}{2}} \left( \frac{1}{\hbar^2} (\eta \cdot a_2) \xi - k^2 a_2 \right) e^{-\frac{i}{\hbar} x \cdot \eta},
\] (47)

When \( |\xi - \xi_0| < \epsilon \),
\[
\tilde{A}_\xi = h^{-2} ((\xi \cdot a_1) \xi - k^2 h^2 a_1) = h^{-2} \xi \times (\xi \times a_1)
\]
\[
= h^{-2} (\xi_0 \times (\xi_0 \times a_1) + O(\epsilon))
\]
\[
= h^{-2} \left(-1 - k^2 h^2, \sqrt{1 + k^2 h^2} i, 0\right) + O(\epsilon),
\]
where \( O(\epsilon) \) is some terms independent of \( x \). Similarly, when \( |\eta - \eta_0| < \epsilon \),
\[
\tilde{A}_\eta = h^{-2} ((\eta \cdot a_2) \eta - k^2 h^2 a_2) = h^{-2} \eta \times (\eta \times a_2)
\]
\[
= h^{-2} (\eta_0 \times (\eta_0 \times a_2) + O(\epsilon))
\]
\[
= h^{-2} \left(-1 - k^2 h^2, 0, \sqrt{1 + k^2 h^2} i\right) + O(\epsilon),
\]
where \( O(\epsilon) \) is some terms independent of \( x \). Construct \((r_1^\omega, t_1^\omega), (r_2^\omega, t_2^\omega)\) and \((r_0^2, t_0^2\omega)\) with exponentially decaying estimates as in [29] such that
\[
V_1^\omega = E_1^\omega + r_1^\omega, \quad W_1^\omega = H_1^\omega + t_1^\omega; \quad V_2^\omega = E_2^\omega + r_2^\omega, \quad W_2^\omega = H_2^\omega + t_2^\omega;
\]
\[
V_0^{2\omega} = E_0^{2\omega} + r_0^{2\omega}, \quad W_0^{2\omega} = H_0^{2\omega} + t_0^{2\omega}.
\]
are solutions to (40) and (43), respectively. Thus, the integral identity (42) becomes
\[
\int_\Omega (\chi^{(2),1} - \chi^{(2),2}) \cdot (E_0^{2\omega} + r_0^{2\omega}) \cdot (E_1^\omega + r_1^\omega) \cdot (E_2^\omega + r_2^\omega) dV = 0.
\] (48)

Next we calculate the terms in the expansion of the product (48). Note that the terms involving some \( r_i \) have exponentially decaying property from [29] when \( \text{Im} \xi_1 \geq 0 \) and \( \text{Im} \xi' = 0 \). As a result,
\[
\int_\Omega (\chi^{(2),1} - \chi^{(2),2}) \cdot E_0^{2\omega} \cdot (E_1^\omega \cdot E_2^\omega) dV \text{ consists of terms involving some } r_i^{k\omega}.
\] (49)

For the L.H.S. of (49), one has
\[
\left| \int_\Omega \varepsilon_0^{-\frac{3}{2}} e^{-\frac{i}{\hbar} x \cdot (\xi + \eta - \varsigma)} (\chi^{(2),1} - \chi^{(2),2}) \cdot \tilde{A}_\xi \cdot (\tilde{A}_\xi \cdot \tilde{A}_\eta) dV \right|
\]
\[
= \left| \left(-1 - k^2 h^2, \sqrt{1 + k^2 h^2} i, 0\right) \cdot (-1 - k^2 h^2, 0, \sqrt{1 + k^2 h^2} i) + O(\epsilon) \right|
\]
\[
\cdot \left| \int_\Omega h^{-4} \varepsilon_0^{-\frac{3}{2}} e^{-\frac{i}{\hbar} x \cdot (\xi + \eta - \varsigma)} (\chi^{(2),1} - \chi^{(2),2}) \cdot \tilde{A}_\xi dV \right|
\]
\[
\geq Ch^{-4} \varepsilon_0^{-\frac{3}{2}} \left| \int_\Omega e^{-\frac{i}{\hbar} x \cdot (\xi + \eta - \varsigma)} (\chi^{(2),1} - \chi^{(2),2}) \cdot \tilde{A}_\xi dV \right|,
\]
when ϵ and h is sufficiently small. Denote by

\[ f(x; h) = (\chi^{(2),1} - \chi^{(2),2}) \cdot \hat{A}_c. \]  

(50)

Hence, when \( \text{Im} \xi_1 \geq 0 \) and \( \text{Im} \eta_1 \geq 0 \),

\[
\left| \int_{\Omega} e^{-\frac{i}{h}x(\xi + \eta - \zeta)} f(x; h) dV \right| \leq C h^4 \varepsilon_0^2 \| \chi^{(2),1} - \chi^{(2),2} \|_{L^\infty(\Omega)} (1 + h^{-1} |\xi|)^{\frac{2}{3}} (1 + h^{-1} |\eta|)^{\frac{2}{3}} (1 + h^{-1} |\zeta|)^{\frac{2}{3}} e^{-\frac{\varepsilon_0}{4h}} e^{-\frac{C \epsilon a}{2h}}.
\]

In particular, let

\[ z = \xi + \eta - \zeta, \quad z_0 = ((2 + \sqrt{2 - 2kh^2}) i, 0, 0). \]

When \( |\xi - a\xi_0| < C \varepsilon \) and \( |\eta - a\eta_0| < C \varepsilon \) with \( \varepsilon \leq \frac{1}{2C} \), \( |z - az_0| < 2C \varepsilon \), and

\[
\left| \int_{\Omega} f(x; h) e^{-\frac{i}{h}x \cdot z} dx \right| \leq C h^{-\frac{1}{2}} \varepsilon_0^2 \| \chi^{(2),1} - \chi^{(2),2} \|_{L^\infty(\Omega)} e^{-\frac{\varepsilon_0}{4}} e^{-\frac{4C \epsilon a}{h}}.
\]

Conversely, for any \( z \in \mathbb{C}^3 \), \( |z - az_0| < 2C \varepsilon \), \( z \) can be decomposed to

\[ z = \xi + \eta - \zeta, \quad \xi, \eta \in p^{-1}(kh), \quad |\xi - a\xi_0| < C \varepsilon, |\eta - a\eta_0| < C \varepsilon. \]

Thus, following the previous procedures, one can show the exponentially decaying property of the F.B.I. transform \( Tf \) for all \( x \in \Omega \), \( |x| \leq \delta_1 \), with \( \delta_1 \) sufficiently small. Hence, in view of (46) (50) and

\[
\lim_{h \to 0} (2\pi h)^{-\frac{3}{2}} Tf(x) = (\chi^{(2),1} - \chi^{(2),2}) \cdot (\sqrt{2}i, -1, -1) \quad \text{in} \quad L^p(\Omega),
\]

one has

\[
(\chi^{(2),1} - \chi^{(2),2}) \cdot (\sqrt{2}i, -1, -1) = 0, \quad x \in \Omega, -\delta_1 \leq x_1 \leq 0. \]  

(51)

(ii) Now choose

\[ \xi_0 = (i, -\sqrt{1 + k^2h^2}, 0), \quad \eta_0 = (i, 0, -\sqrt{1 + k^2h^2}) \in p^{-1}(kh). \]

Suppose \( |\xi - \xi_0| < \epsilon, \xi \in p^{-1}(kh) \), and \( |\eta - \eta_0| < \epsilon, \eta \in p^{-1}(kh) \). Then \( \text{Im} \xi_1 > 0 \) and \( \text{Im} \eta_1 > 0 \).

Let \( E_2^{2\omega} = \varepsilon_0^{-\frac{1}{2}} e^{-\frac{i}{h}x \cdot \zeta} \hat{A}_c \) as in Remark 4.3 where

\[
\zeta = (\sqrt{2}(1 - k^2h^2)i, -\sqrt{1 + k^2h^2}, -\sqrt{1 + k^2h^2}) \in p^{-1}(2kh),
\]

\[
\hat{A}_c = \begin{pmatrix}
-\sqrt{2}(1 - k^2h^2)i & (1 - k^2h^2)(1 - k^2h^2) \\
1 + k^2h^2 & \sqrt{1 + k^2h^2}
\end{pmatrix}
\]

(52)
Again $E_{0}^{2\omega}$ with some determined $H_{0}^{2\omega}$ are solutions to the linearized Maxwell’s equations by noting $\zeta \cdot \mathbf{A}_c = 0$. $E_{1}^{\omega}$ and $E_{2}^{\omega}$ are defined as in (i) of the form (22), where $b = 0$, $a_1 = a_2 = (1, 0, 0)$. Let

$$z = \xi + \eta - \zeta, \quad z_0 = ((2 + \sqrt{2 - 2k^2h^2})i, 0, 0).$$

Similarly, the amplitudes of $E_{1}^{\omega}$ and $E_{2}^{\omega}$ have the estimates

$$\tilde{A}_x = h^{-2}(\xi_0 \times (\xi_0 \times a_1) + O(\epsilon))
= h^{-2}\left((-1 - k^2h^2, -\sqrt{1 + k^2h^2}i, 0) + O(\epsilon)\right),$$

and

$$\tilde{A}_\eta = h^{-2}\left((-1 - k^2h^2, 0, -\sqrt{1 + k^2h^2}i) + O(\epsilon)\right),$$

where $O(\epsilon)$ is some terms independent of $x$, when $|\xi - \xi_0| < \epsilon$ and $|\eta - \eta_0| < \epsilon$, respectively.

After going through similar procedures as before, we arrive at an exponential decaying property of $Tf$, where $f$ is defined in (50). Recall the expression (52) of $\tilde{A}_c$. Then,

$$\lim_{h \to 0} (2\pi h)^{-\frac{n}{2}} Tf(x) = (\chi^{(2),1} - \chi^{(2),2}) \cdot \left(\sqrt{2}i, 1, 1\right) \text{ in } L^p(\Omega).$$

Therefore, one has

$$(\chi^{(2),1} - \chi^{(2),2}) \cdot \left(\sqrt{2}i, 1, 1\right) = 0, \quad x \in \Omega, -\delta_2 \leq x_1 \leq 0. \quad (53)$$

(iii) We choose

$$\xi_0 = (i, \sqrt{1 + k^2h^2}, 0), \quad \eta_0 = (i, 0, -\sqrt{1 + k^2h^2}) \in p^{-1}(kh).$$

Suppose $|\xi - \xi_0| < \epsilon, \xi \in p^{-1}(kh)$, and $|\eta - \eta_0| < \epsilon, \eta \in p^{-1}(kh)$. Then $\text{Im} (\xi_1) > 0$ and $\text{Im} (\eta_1) > 0$.

Let $E_{0}^{2\omega} = \xi_0^{-\frac{1}{2}} e^{-\frac{1}{\pi} x \cdot \mathbf{c}} \tilde{A}_c$, where

$$\zeta = (\sqrt{2(1 - k^2h^2)i}, \sqrt{1 + k^2h^2}, -\sqrt{1 + k^2h^2}) \in p^{-1}(2kh),$$

$$\tilde{A}_c = \left(-\sqrt{2(1 - k^2h^2)i}, -\frac{1 - k^2h^2}{\sqrt{1 + k^2h^2}}, \frac{1 - k^2h^2}{\sqrt{1 + k^2h^2}}\right). \quad (54)$$

Again $E_{0}^{2\omega}$ together with some determined $H_{0}^{2\omega}$ are solutions to the linearized Maxwell equations by noting $\zeta \cdot \tilde{A}_c = 0$ and Remark 1.3.

Similarly, take $a_1 = a_2 = (1, 0, 0)$. When $|\xi - \xi_0| < \epsilon,$

$$\tilde{A}_x = h^{-2}(\xi_0 \times (\xi_0 \times a_1) + O(\epsilon))
= h^{-2}\left((-1 - k^2h^2, \sqrt{1 + k^2h^2}i, 0) + O(\epsilon)\right),$$

$$\tilde{A}_\eta = h^{-2}\left((-1 + k^2h^2, 0, -\sqrt{1 + k^2h^2}i) + O(\epsilon)\right),$$
where $O(\epsilon)$ is some terms independent of $x$; and when $|\eta - \eta_0| < \epsilon$, 

$$\tilde{A}_\eta = h^{-2} \left( (-1 - k^2 h^2, 0, -\sqrt{1 + k^2 h^2} i) + O(\epsilon) \right),$$

where $O(\epsilon)$ is some terms independent of $x$. Let 

$$z = \xi + \eta - \bar{\varsigma}, \quad z_0 = \left( (2 + \sqrt{2} - 2k^2 h^2) i, 0, 0 \right).$$

Following similar procedures as before and noting the expressions (50) and (54) for $f$ and $\tilde{A}_\varsigma$, we arrive at 

$$\lim_{h \to 0} (2\pi h)^{-\frac{n}{2}} Tf(x) = (\chi^{(2),1} - \chi^{(2),2}) \cdot \left( \sqrt{2}i, -1, 1 \right) \text{ in } L^p(\Omega).$$

It follows that 

$$(\chi^{(2),1} - \chi^{(2),2}) \cdot \left( \sqrt{2}i, -1, 1 \right) = 0, \quad x \in \Omega, -\delta_3 \leq x_1 \leq 0. \quad (55)$$

In summary of (51) (53) (55), one has 

$$\chi^{(2),1} - \chi^{(2),2} = 0, \quad x \in \Omega, -\delta \leq x_1 \leq 0,$$

for some positive $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. This completes the proof of Proposition 4.2.1.

### 4.2.2 From local to global

**Proof of Theorem 1.4** Below we show the extension of local result (Proposition 4.2.1) to the global situation following the same process as in section 2.2.2 and 4.1.2.

Let $\bar{\chi}(y) = \chi^{(2),1} - \chi^{(2),2}$. Fix a point $x_1 \in \Omega$ and let $\theta(t)$ be a parameterized curve in $\Omega$ joining $x_0 \in \Gamma$ to $x_1$ such that $\theta(0) = x_0$ and $\theta'(0)$ is the interior normal to $\partial \Omega$ at $x_0$. Suppose $\Theta_e(t)$ is a small neighborhood of the curve $\theta([0, t])$ in $\Omega$ and $I$ is the set of the time where $\bar{\chi}$ vanishes on $\Theta_e(t)$. The set $\Omega \cup B(x_0, \epsilon')$ is smoothed out into an open set $\Omega_2$ with smooth boundary such that 

$$\partial \Omega \setminus \Gamma \subset \partial \Omega \cap \partial \Omega_1 \subset \partial \Omega \cap \partial \Omega_2,$$

and it suffices to show the set $I$ is open.

Following the notations in Section 4.1.2, define 

$$H(x_1, x_2, x_0) = \int_{\Omega_1} \bar{\chi}(y) \otimes G_0^H(x_0, y) \cdot G_1^E(x_1, y) \otimes G_2^E(x_2, y) dy,$$

for $x_1, x_2, x_0 \in \Omega_2 \setminus \Omega_1$. Moreover, 

$$H(x_1, x_2, x_0) = \int_{\Omega_1} \bar{\chi}(y) \otimes \tilde{G}_0^H(x_0, y) \cdot G_1^E(x_1, y) \otimes G_2^E(x_2, y). \quad (56)$$
When \( x_1, x_2, x_0 \in \Omega_2 \setminus \Omega \), from the assumption we have \( H(x_1, x_2, x_3, x_0) = 0 \), as all the entries of \( H \) play the role of an electric field which satisfies the homogeneous Maxwell’s equations with vanishing tangential components on the boundary for \( x_1, x_2, x_0 \in \Omega_2 \setminus \Omega_1 \).

Note that all the entries of \( H \) satisfy the Helmholtz equation \((-\Delta_y - k^2)u = 0\), for \( x_1, x_2, x_0 \in \Omega_2 \setminus \Omega_1 \). Hence, from the unique continuation principle,

\[
H(x_1, x_2, x_0) = 0 \quad \text{when} \quad x_1, x_2, x_0 \in \Omega_2 \setminus \Omega_1;
\]
i.e., for \( x_1, x_2, x_3, x_0 \in \Omega_2 \setminus \Omega_1 \),

\[
\int_{\Omega_1} \bar{\chi}(y) \otimes G^H_0(x_0, y) \cdot G^E_1(x_1, y) \otimes G^E_2(x_2, y) dy = 0.
\]  

(57)

Let the operator at the left hand side of (57) act on

\[
(1 \otimes a_0(x_0), a_1(x_1) \otimes a_2(x_2))
\]

for \( x_0, x_1, x_2 \in \Omega_2 \). Here all \( a_i \) are of the forms \( a_i(x_i) = (l_i(x_i), l_i(x_i))^T \), and all \( l_i \) are smooth vector functions on \( C^\infty(\Omega_2, \mathbb{C}^3) \) with supports contained in \( \Omega_2 \setminus \Omega_1 \). Denote by

\[
v_i(y) = \int_{\Omega_2} G^E_i(x_1, y)a_i(x_i) dx_i, \quad \text{and} \quad \bar{v}_0(y) = \int_{\Omega_2} G^H_0(x_0, y)a_0(x_0) dx_0,
\]

for \( i = 1, 2 \). Thus \( v_i \) and \( w_i \) are solutions to the inhomogeneous Maxwell’s systems \((38)\) and similarly, one has \( \bar{v}_0 = \bar{v}_0 \) and \( \bar{w}_0 = \bar{w}_0 \) in \( \Omega_2 \), where \((v_0, w_0)\) satisfies similar Maxwell’s equations \((38)\) with frequency replaced by \( 2\omega \). Therefore, (57) becomes

\[
0 = \int_{\Omega_1} \bar{\chi} \cdot \bar{v}_0 \cdot v_1 \cdot v_2 dy,
\]

where \((v_i, w_i)\) satisfy the system \((38)\). Since all \( a_i \in C^\infty(\Omega_2, \mathbb{C}^3) \) with supports contained in \( \Omega_2 \setminus \Omega_1 \), the solutions \((v_i, w_i)\) form the set \( R \).

Lemma 4.1 states that \( R \) is dense in the space \( S \) with respect to the \( L^2(\Omega_1, \mathbb{C}^3)^2 \) topology. In the same way as in Section 4.1.2 one can show that the equality \((39)\) extends to

\[
0 = \int_{\Omega_1} \bar{\chi} \cdot \bar{V}_0 \cdot V_1 \cdot V_2 dV
\]

holds for every \( V_i, W_i \in C^\infty(\Omega_1, \mathbb{C}^3) \) which satisfy inhomogeneous Maxwell systems and of which tangential components vanish on \( \partial\Omega_1 \cap \partial\Omega_2 \). Hence by virtue of Proposition 2.1 the cancellation states that \( \bar{\chi} \) vanishes on a neighborhood of \( \partial\Omega_1 \setminus (\partial\Omega_1 \cap \partial\Omega_2) \). This implies that \( \bar{\chi} \) vanishes on a bigger neighborhood \( \Theta_{e}(t') \ t' > t \) of the curve. Hence \( I \) is open.
A Appendix

A.1 Well-posedness of nonlinear Helmholtz Schrödinger equation

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with smooth boundary. Consider the boundary value problem for the nonlinear Helmholtz equation with small boundary value
\[
\begin{aligned}
-\Delta u - k^2 u + q(x, u) &= 0 \quad \text{in } \Omega, \\
u &= f \quad \text{on } \partial \Omega.
\end{aligned}
\] (58)

Here $q(x, u)$ is smooth in $u$ and $\partial_k^k q(x, u_0) \in L^\infty(\Omega \times \mathbb{R})$ for $k \in \mathbb{N}$. Moreover, $q(x, 0) = \partial_u q(x, 0) = 0$.

**Theorem A.1** Suppose that $p > \frac{n}{2}$. There exists a discrete subset $\Sigma \subseteq \mathbb{R}$ such that for every $k \notin \Sigma$, the boundary value problem (58) has a unique solution $u(x)$ satisfying
\[
\|u\|_{W^{2,p}(\Omega)} \leq c \|f\|_{W^{2-\frac{1}{p},p}(\partial \Omega)},
\]
whenever $\|f\|_{W^{2-\frac{1}{p},p}(\partial \Omega)} < \delta$ is sufficiently small. Here, $c > 0$ is a positive constant depending only on $p, k, \Omega$ and the coefficients of $q$.

**Proof.** Given $F \in L^p(\Omega)$, from Fredholm theory, there exists a discrete subset $\Sigma \subseteq \mathbb{R}$ such that for any $k \notin \Sigma$, the Dirichlet boundary problem
\[
\begin{aligned}
-\Delta u - k^2 u &= F \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]
has a unique solution $u \in W_0^{2,p}(\Omega)$ satisfying $\|u\|_{W_0^{2,p}(\Omega)} \leq c \|F\|_{L^p(\Omega)}$, the constant $c$ depending only on $p, k$ and $\Omega$. In other words, for any $k$ outside $\Sigma$, the operator $\mathcal{L} = -\Delta - k^2 : W_0^{2,p}(\Omega) \to L^p(\Omega)$ has a continuous inverse.

As a result, one can show that, for any $k$ outside $\Sigma$ and $f \in W^{2-\frac{1}{p},p}(\partial \Omega)$, the linearized equation
\[
\begin{aligned}
-\Delta u - k^2 u &= 0 \quad \text{in } \Omega, \\
u &= f \quad \text{on } \partial \Omega,
\end{aligned}
\]
of (58) has a solution $u_0 \in W^{2,p}(\Omega)$ satisfying $\|u_0\|_{W^{2,p}(\Omega)} \leq c \|f\|_{W^{2-\frac{1}{p},p}(\partial \Omega)} \leq c \delta$. Hence, $u = u_0 + v$ is a solution to the boundary value problem (58), if $v$ satisfies
\[
\begin{aligned}
-\Delta v - k^2 v &= -q(x, u_0 + v) \quad \text{in } \Omega, \\
v &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\]
Denote by \( \mathcal{F}(v) = -q(x, u_0 + v) \). Since \( q(x, 0) = \partial_u q(x, 0) = 0 \), then \( q(x, u) = q_r(x, u)u^2 \), where \( q_r(x, u) = \int_0^1 \partial_u \delta q(x, tu) \, (1-t)dt \) is bounded. Furthermore, \( q_r(x, u) \) is also Lipschitz in \( u \) noting \( \partial_u^k q(x, u) \in L^\infty(\Omega \times \mathbb{R}) \) for \( k \in \mathbb{N} \). Combining that \( W^{2,p}(\Omega) \hookrightarrow C(\overline{\Omega}) \) by Sobolev embedding theorem for \( p > \frac{n}{2} \), for \( v_1 \) and \( v_2 \) in \( X_\delta \) defined as

\[
X_\varepsilon = \{ v \in W^{2,p}(\Omega) \mid \| v \|_{W^{2,p}(\Omega)} \leq c\varepsilon \},
\]

one has \( \| v_1 \|_{L^\infty(\Omega)}, \| v_2 \|_{L^\infty(\Omega)} \leq c\varepsilon \), and \( \| u \|_{L^\infty(\Omega)} \leq c\delta \). Thus,

\[
\begin{align*}
\| \mathcal{L}^{-1} \circ \mathcal{F}(v_1) - \mathcal{L}^{-1} \circ \mathcal{F}(v_2) \|_{L^p(\Omega)} &
\leq \| \mathcal{F}(v_1) - \mathcal{F}(v_2) \|_{L^p(\Omega)} \\
&= \| (q_r(x, u_0 + v_1) - q_r(x, u_0 + v_2))(u_0 + v_2)^2 \|_{L^p(\Omega)} \\
&\quad + \| q_r(x, u_0 + v_1)((u_0 + v_2)^2 - (u_0 + v_1)^2) \|_{L^p(\Omega)} \\
&\leq c\| v_1 - v_2 \|_{L^p(\Omega)}(\| u_0 + v_2 \|_{L^\infty(\Omega)}^2 + c\| u_0 + v_1 + v_2 \|_{L^\infty(\Omega)}\| v_1 - v_2 \|_{L^p(\Omega)}) \\
&\leq c(\delta^2 + \varepsilon^2 + c\delta + \varepsilon + \delta)\| v_1 - v_2 \|_{L^p(\Omega)}.
\end{align*}
\]

The constant is independent of \( \varepsilon \) and \( \delta \). This implies that \( \mathcal{L}^{-1} \circ \mathcal{F} \) is a contraction on \( X_\delta \) provided that \( \varepsilon \) and \( \delta \) are sufficiently small. It follows that there exists some \( v \in X_\delta \) such that \( v = \mathcal{L}^{-1} \circ \mathcal{F}(v) \). Furthermore,

\[
\| v \|_{W^{2,p}(\Omega)} \leq c\| \mathcal{F}(v) \|_{L^p(\Omega)} \leq c\| u_0 \|_{L^\infty(\Omega)}^2 + \| v \|_{L^\infty(\Omega)}^2 \\
\quad \leq c(\delta \| f \|_{W^{2,\frac{1}{p},p}(\partial \Omega)} + \varepsilon \| v \|_{W^{2,p}(\Omega)}).
\]

Thus one has \( \| v \|_{W^{2,p}(\Omega)} \leq \| f \|_{W^{2,\frac{1}{p},p}(\partial \Omega)} \) provided \( \delta \) and \( \varepsilon \) sufficiently small. In summary, \( u = u_0 + v \in W^{2,p}(\Omega) \) is a solution to the nonlinear Helmholtz equation with small boundary value satisfying the required estimate. \( \square \)

### A.2 Runge approximation

We need the following Runge approximation property in the previous sections.

Let \( M \subset \mathbb{R}^n, n \geq 2 \) be a bounded open set with \( C^\infty \) boundary. Consider the dual space of \( H^1(M) \), given by

\[
\tilde{H}^{-1}(M) = \{ f \in H^{-1}(\mathbb{R}^n) \mid \text{supp}(f) \subset \overline{M} \}.
\]

For \( f \in \tilde{H}^{-1}(M) \) and \( g \in H^1(M) \), let \( \tilde{g} = \text{Ext}(g) \in H^1(\mathbb{R}^n) \) be an extension of \( g \).

Define the following duality pairing

\[
(f, g)_{\tilde{H}^{-1}(M), H^1(M)} = (f, \tilde{g})_{\tilde{H}^{-1}(\mathbb{R}^n), H^1(\mathbb{R}^n)} = \int_{\mathbb{R}^n} f\tilde{g}dx.
\]
Proposition A.1 Let \( U \subset M \) be a domain with \( C^\infty \) boundary such that \( M \setminus \bar{U} \) is connected. Assume that \( \partial U \cap \partial M = \bar{V} \), where \( V \subset \partial M \) is open with \( C^\infty \) boundary. Let \( \Gamma \) be a nonempty open subset of \( \partial M \), and \( \bar{\Gamma} \cap \bar{V} = \emptyset \). Then the set

\[
\mathcal{R} = \{ u|_U : -\Delta u - k^2 u = 0 \text{ in } M, u = g \text{ on } \partial M; g \in C^\infty_c(\Gamma) \}
\]

is dense in the space \( \mathcal{S} = \{ u \in H^1(U) : -\Delta u - k^2 u = 0 \text{ in } U, u = 0 \text{ on } \partial U \cap \partial M \} \) with respect to the \( H^1(U) \) norm.

Proof. Let \( f \in \tilde{H}^{-1}(U) \) be orthogonal to \( \mathcal{R} \), i.e., one has \( (f, u|_U)_{\tilde{H}^{-1}(U), H^1(U)} = 0 \) for any \( u|_U \) satisfying

\[
-\Delta u - k^2 u = 0 \text{ in } M, u = g \text{ on } \partial M; g \in C^\infty_c(\Gamma).
\]

From Hahn-Banach theorem it suffices to show that \( (f, v)_{\tilde{H}^{-1}(U), H^1(U)} = 0 \) for any \( v \in \mathcal{S} \).

Construct \( M_0 \) with smooth boundary such that \( M \subset M_0 \) and \( \partial M \setminus \bar{\Gamma} \subset \partial M \cap \partial M_0 \). Thus, \( \partial U \cap \partial M \subset \partial U \cap \partial M_0 \). For \( f \in \tilde{H}^{-1}(U) \), there exists a sequence \( \{f_j\} \subset C^\infty_0(U) \) such that \( f_j \to f \) in \( \tilde{H}^{-1}(U) \).

Let \( w \in H^1_0(M_0) \) and \( w_j \in C^\infty(\bar{M}_0) \cap H^1_0(M_0) \) be the unique solutions to the following Helmholtz equations

\[
\begin{cases}
-\Delta w - k^2 w = f & \text{in } M_0, \\
w = 0 & \text{on } \partial M_0,
\end{cases}
\]

and

\[
\begin{cases}
-\Delta w_j - k^2 w_j = f_j & \text{in } M_0, \\
w_j = 0 & \text{on } \partial M_0.
\end{cases}
\]

Since \( f_j \to f \) in \( \tilde{H}^{-1}(U) \), the regularity of solutions to the elliptic equation yields \( w_j \to w \) in \( H^1(M) \).

Let \( \varphi \in H^1(M) \) and \( \varphi_j \in C^\infty(\bar{M}) \cap H^1(M) \) be the unique solutions to the following Helmholtz equation

\[
\begin{cases}
-\Delta \varphi - k^2 \varphi = 0 & \text{in } M, \\
\varphi = w & \text{on } \partial M.
\end{cases}
\]

and

\[
\begin{cases}
-\Delta \varphi_j - k^2 \varphi_j = 0 & \text{in } M, \\
\varphi_j = w_j & \text{on } \partial M.
\end{cases}
\]

Note that \( w_j \to w \) in \( H^1(M) \), then \( \varphi_j \to \varphi \) in \( H^1(M) \).
Let \( \omega_j |_M = w_j - \varphi_j \) and \( \omega |_M = w - \varphi \). Then \( \omega \) and \( \omega_j \) satisfy
\[
\begin{aligned}
- \Delta \omega - k^2 \omega &= f \quad \text{in } M, \\
\omega &= 0 \quad \text{on } \partial M.
\end{aligned}
\]  \( (59) \)

and
\[
\begin{aligned}
- \Delta \omega_j - k^2 \omega_j &= f_j \quad \text{in } M, \\
w_j &= 0 \quad \text{on } \partial M,
\end{aligned}
\]
respectively. Thus, for \( u \in \mathcal{R} \), one has
\[
0 = (f, u |_U)_{H^{-1}(U), H^1(U)} = \lim_{j \to \infty} (f_j, u |_U)_{H^{-1}(U), H^1(U)} = \lim_{j \to \infty} \int_M - \Delta w_j u - k^2 w_j u dV
\]
\[
= \lim_{j \to \infty} \int_M (- \Delta u - k^2 u) w_j dV + \lim_{j \to \infty} \int_{\partial M} \partial \nu \omega_j u - \omega_j \partial \nu u dS
\]
\[
= \lim_{j \to \infty} \int_M \partial \nu \omega_j g dS = \int_M \partial \nu \omega g dS
\]
Hence, \( \partial \nu \omega |_{\Gamma} = 0 \). In summary, \( \omega \) satisfies
\[
\begin{aligned}
- \Delta \omega - k^2 \omega &= 0 \quad \text{in } M \setminus \bar{U}, \\
\omega &= 0 \quad \text{on } \partial M \\
\partial \nu \omega &= 0 \quad \text{on } \Gamma
\end{aligned}
\]
The unique continuation principle implies that
\[
\omega |_{M \setminus \bar{U}} = 0.
\]
Recall that \( \omega \in H_0^1(M) \), then \( \omega |_{\partial U \cap \partial M} = 0 \). Hence, \( \omega |_{\partial U} = 0 \).

Thus \( \omega \) may be identified with an element in \( H^1_0(U) \). There exists \( \psi_j \in C_0^\infty(U) \) such that \( \psi_j \to \omega \) in \( H^1(U) \). Hence \( - \Delta \psi_j - k^2 \psi_j \to - \Delta \omega - k^2 \omega \) in \( H^{-1}(\mathbb{R}^n) \).

For any \( v \in \mathcal{S} \), let \( \tilde{v} \) be the extension of \( v \) to \( \mathbb{R}^n \).
\[
(- \Delta \omega - k^2 \omega, \tilde{v})_{H^{-1}(\mathbb{R}^n), H^1(\mathbb{R}^n)}
\]
\[
= \lim_{j \to \infty} (- \Delta \psi_j - k^2 \psi_j, \tilde{v})_{H^{-1}(\mathbb{R}^n), H^1(\mathbb{R}^n)} = \lim_{j \to \infty} \int_M (- \Delta \psi_j - k^2 \psi_j) v dV
\]
\[
= \lim_{j \to \infty} \int_U (- \Delta v - k^2 v) \psi_j dV + \lim_{j \to \infty} \int_{\partial U} \partial \nu \psi_j v - \psi_j \partial \nu v dS = 0.
\]

Consider \( \hat{f} = f - (- \Delta \omega - k^2 \omega) \). \( \hat{f} \in H^{-1}(\mathbb{R}^n) \) and \( H^{-\frac{1}{2}}(\partial U) \) with \( \text{supp } \hat{f} \subseteq \partial U \). From \( \text{[8]} \),
\[
\hat{f} = \hat{f} \otimes \delta_{\partial U}, \quad \hat{f} \in H^{-\frac{1}{2}}(\partial U).
\]
Following \([59]\) and \(f \in \tilde{H}^{-1}(U)\) one has \(\text{supp} \hat{f} \subseteq \bar{V} = \partial U \cap \partial M\). Thus \(\text{supp} \tilde{f} \subseteq \bar{V}\). Therefore, there exists \(\tilde{f}_j \in C_0^\infty(V)\) such that \(\tilde{f}_j \to \tilde{f}\) in \(H^{-\frac{1}{2}}(\partial U)\). Then
\[
(\hat{f}, v|_U)_{\tilde{H}^{-1}(U), \tilde{H}^1(U)} = (\tilde{f}_j, v|_{\partial U})_{H^{-\frac{1}{2}}(\partial U), H^\frac{1}{2}(\partial U)} = \lim_{j \to \infty} \int_{\partial U} \tilde{f}_j v dS = 0,
\]
since \(\text{supp} \tilde{f}_j \subseteq V\) and \(v|_{\bar{V}} = 0\). Therefore,
\[
(f, v|_U)_{\tilde{H}^{-1}(U), \tilde{H}^1(U)} = (-\Delta \omega - k^2 \omega, \bar{v})_{H^{-1}(\mathbb{R}^n), H^1(\mathbb{R}^n)} + (\hat{f}, \bar{v})_{H^{-1}(\mathbb{R}^n), H^1(\mathbb{R}^n)} = 0.
\]
This completes the proof of Proposition \([A.1]\).

References

[1] Y. Assylbekov, T. Zhou, Direct and inverse problems for the nonlinear time-harmonic Maxwell equations in Kerr-type media, J. Spectr. Theory 11 (2021), no. 1, 138.

[2] Y. Assylbekov, T. Zhou, Inverse problems for nonlinear Maxwell’s equations with second harmonic generation. arXiv:2009.03467, 2020.

[3] K. Astala, M. Lassas, L. Päivärinta, Calderón’s inverse problem for anisotropic conductivity in the plane, Comm. Partial Differential Equations, 30 (2005), 207–224.

[4] K. Astala, L. Päivärinta, Calderón’s inverse conductivity problem in the plane, Ann. of Math. 163 (2006), 265–299.

[5] R. Brown, R. Torres, Uniqueness in the inverse conductivity problem for conductivities with \(3/2\) derivatives in \(L^p, p > 2n\), J. Fourier Analysis Appl., 9 (2003), 1049–1056.

[6] A. Bukhgeim, Recovering the potential from Cauchy data in the two-dimensional case, J. Inverse Ill Posed Probl., 16 (2008), 19–33.

[7] A.P. Calderón, On an inverse boundary value problem, Seminar on Numerical Analysis and its Applications to Continuum Physics, Soc. Brasileira de Matemática, Rio de Janeiro, 1980.

[8] P. Caro, P. Ola, M. Salo, Inverse boundary value problem for Maxwell equations with local data, Comm. Partial Differential Equations 34 (2009), 1425–1464.

[9] Francis J. Chung, P. Ola, M. Salo, L. Tzou, Partial data inverse problems for Maxwell equations via Carleman estimates, Ann. Inst. H. Poincaré Anal. Non Linéaire 35 (2018), no. 3, 605–624.

[10] D. Dos Santos Ferreira, C.E. Kenig, M. Salo, G. Uhlmann, Limiting Carleman weights and anisotropic inverse problems, Invent. Math. 178 (2009), 119–171.

[11] D. Dos Santos Ferreira, C.E. Kenig, J. Sjöstrand, G. Uhlmann, On the linearized local Calderón problem, Math. Res. Lett. 16 (2009), 955–970.
[12] D. Dos Santos Ferreira, Y. Kurylev, M. Lassas, M. Salo. The Calderón problem in transversally anisotropic geometries. J. Eur. Math. Soc. 18(2016), 2579–2626.
[13] D. Dos Santos Ferreira, Y. Kurylev, M. Lassas, M. Salo, The linearized Calderón problem in transversally anisotropic geometries. Int. Math. Res. Not. 22 (2020), 8729–8765.
[14] A. Feizmohammadi, L. Oksanen, Recovery of zeroth order coefficients in non-linear wave equations, preprint, arXiv:1903.12636, 2019.
[15] A. Feizmohammadi, L. Oksanen, An inverse problem for a semi-linear elliptic equation in Riemannian geometries, J. Differential Equations, 269 (2020), 4683–4719.
[16] M. de Hoop, G. Uhlmann and Y. Wang, Nonlinear responses from the interaction of two progressing waves at an interface, Ann. Inst. H. Poincaré Anal. Non Linéaire, 36 (2019), 347–363.
[17] M. Ikehata, How to draw a picture of an unknown inclusion from boundary measurements: Two mathematical inversion algorithms, J. Inverse Ill-Posed Problems 7 (1999), 255–271.
[18] M. Ikehata, Reconstruction of the support function for inclusion from boundary measurements, J. Inv. Ill-Posed Problems 8 (2000), 367–378.
[19] V. Isakov, On uniqueness in the inverse transmission scattering problem, Comm. Partial Differential Equations 15 (1990), no. 11, 1565–1587.
[20] V. Isakov, On uniqueness in inverse problems for semilinear parabolic equations, Archive for Rational Mechanics and Analysis, 124(1993), 1–12.
[21] V. Isakov, Uniqueness of recovery of some systems of semilinear partial differential equations, Inverse Problems, 17 (2001), 607–618.
[22] V. Isakov, Uniqueness of recovery of some quasilinear partial differential equations, Comm. Partial Differential Equations, 26 (2001), 1947–1973.
[23] V. Isakov, On uniqueness in the inverse conductivity problem with local data, Inverse Probl. Imaging 1 (2007), 95–105.
[24] V. Isakov, Inverse obstacle problems, Inverse Problems 25 (2009), no. 12.
[25] V. Isakov, A. Nachman, Global uniqueness for a two-dimensional elliptic inverse problem, Transactions of the American Mathematical Society, 347 (1995), no. 9, 3375–3391.
[26] V. Isakov, J. Sylvester, Global uniqueness for a semilinear elliptic inverse problem, Communications on Pure and Applied Mathematics, 47(1994), no. 10, 1403–1410.
[27] H. Kang, G. Nakamura, Identification of nonlinearity in a conductivity equation via the Dirichlet-to Neumann map, Inverse Problems, 18 (2002), 1079–1088.
[28] C. E. Kenig, M. Salo, The Calderón problem with partial data on manifolds and applications, Anal. PDE 6 (2013), no. 8, 2003–2048.
[29] C. E. Kenig, M. Salo, Recent progress in the Calderón problem with partial data, Contemp. Math, 615(2014), 193–222.
[30] C. E. Kenig, J. Sjöstrand, G. Uhlmann, The Calderón problem with partial data, Ann. of Math. 165 (2007), 567–591.
[31] K. Krupchyk, G. Uhlmann, Partial data inverse problems for semilinear elliptic equations with gradient nonlinearities, Math. Res. Lett. 27 (2020), no. 6, 1801-1824.
[32] K. Krupchyk, G. Uhlmann, A remark on partial data inverse problems for semilinear elliptic equations, Proc. Amer. Math. Soc. 148 (2020), no. 2, 681-685.
[33] Y. Kurylev, M. Lassas, L. Oksanen, G. Uhlmann, Inverse problem for einstein-scalar field equations. preprint (2014) arXiv:1406.4776.
[34] Y. Kurylev, M. Lassas, G. Uhlmann, Inverse problems for Lorentzian manifolds and non-linear hyperbolic equations, Invent. Math. 212 (2018), no. 3, 781–857.
[35] R. Lai, T. Zhou, Partial data inverse problems for nonlinear magnetic schrödinger equations, arXiv:2007.02475
[36] M. Lassas, T. Liimatainen, Y. Lin, M. Salo, Inverse problems for elliptic equations with power type nonlinearities, J. Math. Pures Appl., 145(2021), 44–82.
[37] M. Lassas, T. Liimatainen, Y. Lin, M. Salo, Partial data inverse problems and simultaneous recovery of boundary and coefficients for semilinear elliptic equations, Rev. Mat. Iberoam. 37 (2021), no. 4, 1553-1580.
[38] M. Lassas, T. Liimatainen, M. Salo, The Poisson embedding approach to the Calderón problem. Math. Ann. 377 (2020), no. 1-2, 19–67.
[39] M. Lassas, G. Uhlmann, Y. Wang, Inverse problems for semilinear wave equations on Lorentzian manifolds, Comm. Math. Phys. 360 (2018), no. 2, 555–609.
[40] M. Lassas, G. Uhlmann, Y. Wang, Determination of vacuum space-times from the Einstein-Maxwell equations. preprint (2017) arXiv:1703.10704
[41] N. Lazarides, G. P. Tsironis, Coupled nonlinear Schrödinger field equations for electromagnetic wave propagation in nonlinear left-handed materials, Physical Review E, 71 (2005), 036614.
[42] H. Liu, X. Liu, Recovery of an embedded obstacle and its surrounding medium from formally determined scattering data, Inverse Problems, 33(2017) no.6.
[43] A. Nachman, Global uniqueness for a two-dimensional inverse boundary value problem, Ann. of Math. 143 (1996), 7–96.
[44] A. Nachman, B. Street, Reconstruction in the Calderón problem with partial data, Comm. PDE 35 (2010), 375–390.
[45] G. Nakamura, Z. Sun, G. Uhlmann, Global identifiability for an inverse problem for the Schrödinger equation in a magnetic field, Math. Ann. 303 (1995), no. 3, 377–388.
[46] W. Nie, Optical nonlinearity: phenomena, applications, and materials, Advanced Materials, 5 (1993), 520–545.

[47] P. Ola, L. Päivärinta, E. Somersalo, An inverse boundary value problem in electrodynamics. Duke Math. J. 70 (1993), 617–653.

[48] P. Ola, E. Somersalo, Electromagnetic inverse problems and generalized Sommerfeld potentials. SIAM J. Appl. Math. 56 (1996), 1129–1145.

[49] L. Päivärinta, A. Panchenko, G. Uhlmann, Complex geometrical optics for Lipschitz conductivities, Revista Matematica Iberoamericana, 19 (2003), 57–72.

[50] M. Salo, Semiclassical pseudodifferential calculus and the reconstruction of a magnetic field. Comm. Partial Differential Equations 31 (2006), no. 10-12, 1639–1666.

[51] C. A. Stuart, Self-trapping of an electromagnetic field and bifurcation from the essential spectrum, Arch. Rational Mech. Anal., 113 (1991), 65–96.

[52] J. Sylvester, G. Uhlmann, A global uniqueness theorem for an inverse boundary value problem, Ann. of Math. 125 (1987), 153–169.

[53] J. Sylvester, An anisotropic inverse boundary value problem, Commun. Pure Appl. Math. 43 (1990), no 2, 201–232.

[54] J. Sylvester, G. Uhlmann, Inverse problems in anisotropic media Contemp. Math. 122(1991) 105–117.

[55] L. Oksanen, M. Salo, P. Stefanov, G. Uhlmann, Inverse Problems For Real Principal Type Operators, arXiv:2001.07599v2.

[56] Z. Sun, On a quasilinear inverse boundary value problem. Math. Z., 221(1996), no. 2, 293–305.

[57] Z. Sun, An inverse boundary-value problem for semilinear elliptic equations. Electronic Journal of Differential Equations (EJDE)[electronic only], 37(2010), 1–5.

[58] Z. Sun, G. Uhlmann, Inverse problems in quasilinear anisotropic media. American Journal of Mathematics, 119 (1997), no. 4, 771–797.

[59] Z. Sun, G. Uhlmann, Anisotropic inverse problems in two dimensions Inverse Problems 19 (2003), no. 5, 1001-1010.

[60] G. Uhlmann, Electrical impedance tomography and Calderón’s problem, Inverse Problems 25 (2009), 123011.

[61] G. Uhlmann, Inverse problems: seeing the unseen. Bull. Math. Sci. 4 (2014), no. 2, 209–279.

[62] Y. Wang, T. Zhou, Inverse problems for quadratic derivative nonlinear wave equations, Comm. Partial Differential Equations, 44 (2019), 1140–1158.