Spectral properties of horocycle flows for surfaces of constant negative curvature

R. Tiedra de Aldecoa*

Abstract

We consider flows, called $W^u$ flows, whose orbits are the unstable manifolds of a codimension one Anosov flow. Under some regularity assumptions, we give a short proof of the strong mixing property of $W^u$ flows and we show that $W^u$ flows have purely absolutely continuous spectrum in the orthocomplement of the constant functions. As an application, we obtain that time changes of the classical horocycle flows for compact surfaces of constant negative curvature have purely absolutely continuous spectrum in the orthocomplement of the constant functions for time changes in a regularity class slightly less than $C^2$. This generalises recent results on time changes of horocycle flows.

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1 Introduction

Horocycle flows for compact surfaces of constant negative curvature, and their generalisations, are a classical object of study in dynamical systems. Since the papers of G. A. Hedlund [8, 9] and E. Hopf [10] in the 1930’s, many properties of these flows have been put into evidence (far too many to be listed here). As for spectral properties, in particular the structure of the essential spectrum, not that many results are available.

In 1953, O. S. Parasyuk [19] has shown that the classical horocycle flows for compact surfaces of constant negative curvature have countable Lebesgue spectrum. In 1974, A. G. Kushnirenko [13] has shown that some small time changes of the classical horocycle flows for compact surfaces of constant negative curvature are mixing, and thus have purely continuous spectrum in the orthocomplement of the constant functions. In 1977, B. Marcus [15] has shown that a large class of minimal $W^u$ flows associated with codimension one Anosov flows (in particular, a large class of reparametrisations of the classical horocycle flows for compact surfaces of negative curvature) are mixing. Finally, more recently, G. Forni and C. Ulcigrai [6], and the author [23, 25] have shown that sufficiently smooth time changes of the classical horocycle flows for compact surfaces of constant negative curvature have purely absolutely continuous spectrum in the orthocomplement of the constant functions (in [6] Lebesgue spectrum is also obtained, and in [23] surfaces of finite volume are also considered).

The purpose of this paper is to extend these last results on the absolutely continuous spectrum of time changes of horocycle flows to a very general class of time changes, namely, time changes in a regularity class slightly less than $C^2$. Our approach is the following. We consider as B. Marcus a continuous

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minimal $W^u$ flow, that is, a continuous minimal flow whose orbits are the unstable manifolds of a $C^{1+\epsilon}$ codimension one Anosov flow on a compact connected Riemannian manifold (minimality is required for the $W^u$ flow to be uniquely ergodic). Under some regularity assumptions on the $W^u$ flow and the Anosov flow, we give a short proof of the strong mixing property of the $W^u$ flow. Then, under some additional regularity assumption, we construct a self-adjoint operator, called conjugate operator, satisfying a suitable positive commutator estimate with the self-adjoint generator of the $W^u$ flow. Finally, we deduce from this positive commutator estimate and commutator methods that the generator of the $W^u$ flow has purely absolutely continuous spectrum in the orthocomplement of the constant functions. As an application, we obtain that time changes of the classical horocycle flows for compact surfaces of constant negative curvature have purely absolutely continuous spectrum in the orthocomplement of the constant functions. As an application, we obtain that time changes of the classical horocycle flows for compact surfaces of constant negative curvature (Remark 3.9).

Here is a description of the content of the paper. In Section 2, we recall some definitions and results on codimension one Anosov flows and minimal $W^u$ flows, we introduce our regularity assumptions (Assumptions 2.1 and 2.4), and we give a proof of the strong mixing property of the $W^u$ flow (Theorem 2.6). In Section 3, we recall the needed facts on commutators of operators and regularity classes associated with them. Then, under some additional regularity assumption (Assumption 3.4), we construct the conjugate operator (Proposition 3.5), we prove the positive commutator estimate (Proposition 3.7), and we show that the generator of the $W^u$ flow has purely absolutely continuous spectrum in the orthocomplement of the constant functions (Theorem 3.8). Finally, we present the application of this result to time changes of the classical horocycle flows for compact surfaces of constant negative curvature (Remark 3.9).

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## 2 Strong mixing

In this section, we recall some definitions and results on codimension one Anosov flows and minimal $W^u$ flows, and we present a short proof of the strong mixing property of a class of minimal $W^u$ flows. We follow fairly closely the presentation and notations of B. Marcus [15], but we refer to the review papers [2, 17, 20] for additional information.

A $C^{1+\epsilon}$ Anosov flow on a compact connected Riemannian manifold $M$ with distance $d : M \times M \to [0, \infty)$ is a $C^{1+\epsilon}$ flow $\{f_t\}_{t \in \mathbb{R}}$ with $\epsilon > 0$, without fixed points, satisfying the following property: through each point $x \in M$ pass three submanifolds $W^u(x)$, $W^s(x)$, and $\text{Orb}(x)$ whose tangent spaces $E^u_x$, $E^s_x$, and $E_x$ (respectively) vary continuously with $x \in M$ and satisfy

$$T_x M = E^u_x \oplus E^s_x \oplus E_x.$$  

The regularity assumption on $\{f_t\}_{t \in \mathbb{R}}$ means that the function $\mathbb{R} \times M \ni (t,x) \mapsto f_t(x) \in M$ is of class $C^{1+\epsilon}$. So, $\{f_t\}_{t \in \mathbb{R}}$ has a vector field $X_f$ which is H"older continuous with exponent $\epsilon$. The submanifolds $W^u(x)$, $W^s(x)$, and $\text{Orb}(x)$ are called unstable manifolds, stable manifolds, and orbits (respectively), and

$$W^u(x) = \left\{ y \in M \mid \lim_{t \to -\infty} d(f_t(x), f_t(y)) = 0 \right\},$$

$$W^s(x) = \left\{ y \in M \mid \lim_{t \to +\infty} d(f_t(x), f_t(y)) = 0 \right\},$$

$$\text{Orb}(x) = \{ f_t(x) \mid t \in \mathbb{R} \}.$$  

The following two facts are well-known [2]:

$$\lim_{t \to -\infty} d(f_t(x), f_t(y)) = 0 \iff x \in W^u(y),$$

$$\lim_{t \to +\infty} d(f_t(x), f_t(y)) = 0 \iff x \in W^s(y).$$
The flow

\begin{equation}
\lambda \in \mathbb{R} \quad \text{is well-defined and belongs to } C(M; (0, \infty)), \text{ the flow } \{\phi_s\}_{s \in \mathbb{R}} \text{ has continuous vector field } \rho X \phi, \text{ and the measure } \tilde{\mu} \text{ satisfies [11, Prop. 3]:}
\end{equation}

\begin{equation}
\tilde{\mu} = \mu / \tilde{\rho} \quad \text{with} \quad \tilde{\rho} := \rho \int_M d\mu \rho^{-1}.
\end{equation}

Also, one verifies that the pullback operators in \( \mathcal{H} := L^2(M, \mu) \) associated with the flow \( \{\phi_s\}_{s \in \mathbb{R}} \),

\( U_t^s \varphi := \varphi \circ \phi_s, \quad s, t \in \mathbb{R}, \quad \varphi \in \mathcal{H}, \)
define a strongly continuous 1-parameter group of unitary operators with \( U_t^\phi C^1(M) \subset C^1(M) \) for all \( s \in \mathbb{R} \). Thus, Nelson’s criterion \([21, \text{Thm. VIII.10}]\) implies that the generator of the group \( \{U_t^\phi\}_{s \in \mathbb{R}} \),

\[
H_\phi \varphi := \lim_{s \to 0} i s^{-1} (U_s^\phi - 1) \varphi, \quad \varphi \in D(H_\phi) := \left\{ \varphi \in \mathcal{H} \mid \lim_{s \to 0} |s|^{-1} \| (U_s^\phi - 1) \varphi \| < \infty \right\},
\]
is essentially self-adjoint on \( C^1(M) \) and given by

\[
H_\phi \varphi = i \mathcal{X}_\varphi \varphi, \quad \varphi \in C^1(M).
\]

On another hand, the pullback operators associated with the flow \( \{f_t\}_{t \in \mathbb{R}} \),

\[
U_t' \varphi := \varphi \circ f_t, \quad t \in \mathbb{R}, \ \varphi \in \mathcal{H},
\]

are not unitary if \( \rho \neq 1 \), but they define a strongly continuous 1-parameter group of bounded operators:

**Lemma 2.2.** Suppose that Assumption 2.1 is satisfied. Then, \( U'_t \in \mathcal{B}(\mathcal{H}) \) for all \( t \in \mathbb{R} \), \( U'_s U'_t = U'_{s+t} \) for all \( s, t \in \mathbb{R} \), \( U'_0 = 1 \), and \( \lim_{\epsilon \to 0} \| (U'_{t+\epsilon} - U'_t) \varphi \| = 0 \) for all \( t \in \mathbb{R} \) and \( \varphi \in \mathcal{H} \).

**Proof.** A direct calculation using (2.2) and the fact that \( \tilde{\mu} \) is invariant under \( \{f_t\}_{t \in \mathbb{R}} \) implies for \( t \in \mathbb{R} \) and \( \varphi \in \mathcal{H} \) that

\[
\| U'_t \varphi \|^2 = \int_M d\tilde{\mu} \tilde{\rho} \| \varphi \circ f_t \|^2 = \int_M d\mu (\tilde{\rho} \circ f_t) |\varphi|^2 = \int_M d\mu \frac{\rho \circ f_t}{\rho} |\varphi|^2 \leq \frac{\max(\rho)}{\min(\rho)} \| \varphi \|^2.
\]

Thus, \( U'_t \in \mathcal{B}(\mathcal{H}) \) with

\[
\| U'_t \| \leq \sqrt{\frac{\max(\rho)}{\min(\rho)}}.
\]

The group properties \( U'_s U'_t = U'_{s+t} \) for \( s, t \in \mathbb{R} \) and \( U'_0 = 1 \) are evident. To show the last property, take \( t \in \mathbb{R} \) and \( \varphi \in C(M) \). Then, the continuity of \( (\varphi \circ f_{t+\epsilon} - \varphi \circ f_t) \) and Lebesgue dominated convergence theorem imply that

\[
\lim_{\epsilon \to 0} \| (U'_{t+\epsilon} - U'_t) \varphi \|^2 = \int_M d\mu \lim_{\epsilon \to 0} |\varphi \circ f_{t+\epsilon} - \varphi \circ f_t|^2 = 0.
\]

Since \( C(M) \) is dense in \( \mathcal{H} \) and \( (U'_{t+\epsilon} - U'_t) \in \mathcal{B}(\mathcal{H}) \), this implies that \( \lim_{\epsilon \to 0} \| (U'_{t+\epsilon} - U'_t) \varphi \| = 0 \) for all \( t \in \mathbb{R} \) and \( \varphi \in \mathcal{H} \).

For the next lemma, we need the following result of B. Marcus:

**Lemma 2.3** (Lemma 3.1 of [15]). Let \( \{\phi_s\}_{s \in \mathbb{R}} \) be a minimal \( W^u \) flow on \( M \). Then,

\[
\lim_{s \to -\infty} s^{-1} s^*(t, s, x) = \lambda^t
\]

uniformly in \( x \in M \) and \( t \) in a given compact interval of \( \mathbb{R} \).

We also need to introduce as B. Marcus in [15, Sec. 4] the following regularity condition on the function \( s^* \):

**Assumption 2.4.** The derivative

\[
u_{t,s}(x) := (\partial_1 \partial_2 s^*)(t, s, x)
\]

exists and is continuous in \( s, t \in \mathbb{R} \) and \( x \in M \).
Lemma 2.5. Suppose that Assumption 2.4 is satisfied. Then,
\[ \lim_{s \to \infty} s^{-1} (\partial_1 s^*) (t, s, x) = \int_M d\mu u_{t,0} = \ln(\lambda) \lambda^t \] (2.4)
uniformly in \( x \in M \) and \( t \) in a given compact interval of \( \mathbb{R} \).

Proof. Let \( I \subset \mathbb{R} \) be a compact interval, and take \( r, s \in \mathbb{R} \), \( t \in I \) and \( x \in M \). Using successively the relation \((\partial_1 s^*) (t, 0, x) = 0 \), Assumption 2.4 and the cocycle equation
\[ s^* (t, r + s, x) = s^* (t, r, x) + s^* (t, s, \phi_r (x)) \] (2.5)
we obtain
\[ (\partial_1 s^*) (t, s, x) = \int_0^s \frac{d}{ds} (\partial_1 s^*) (t, r, x) = \int_0^s \frac{d}{dt} \frac{d}{ds} s^* (t, r + s, x) = \int_0^s \frac{d}{d\mu} u_{t,0} (\phi_r (x)). \]
Therefore, it follows by the unique ergodicity of \( \{ \phi_s \}_{s \in \mathbb{R}} \) that
\[ \lim_{s \to \infty} s^{-1} (\partial_1 s^*) (t, s, x) = \lim_{s \to \infty} s^{-1} \int_0^s \frac{d}{d\mu} u_{t,0} (\phi_r (x)) = \int_M d\mu u_{t,0} \] (2.6)
uniformly in \( x \in M \) and \( t \in I \) (the uniformity in \( t \) follows from the continuity of \( u_{t,0} \) in \( t \)). To show the second equality in (2.4), we note from Lemma 2.3 that
\[ \lim_{s \to \infty} s^{-1} s^* (t, s, x) = \lambda^t \]
uniformly in \( x \in M \) and \( t \in I \). We also note from (2.6) that \( s^{-1} (\partial_1 s^*) (t, s, x) \) converges to some function of \( t \) uniformly in \( x \in M \) and \( t \in I \). So, it follows by a uniform convergence argument that
\[ \lim_{s \to \infty} s^{-1} (\partial_1 s^*) (t, s, x) = \frac{d}{dt} \lim_{s \to \infty} s^{-1} s^* (t, s, x) = \frac{d}{dt} \lambda^t = \ln(\lambda) \lambda^t \]
uniformly in \( x \in M \) and \( t \in I \). \( \square \)

In the following theorem, we give a short proof, inspired by [24, Thm. 4.1], of the strong mixing property of the flow \( \{ \phi_s \}_{s \in \mathbb{R}} \). It can be viewed as a simplified version of the proof of B. Marcus [15, Sec. 4] in the case Assumption 2.1 is satisfied (the proof of B. Marcus works without Assumption 2.1).

Theorem 2.6 (Strong mixing). Suppose that Assumptions 2.1 and 2.4 are satisfied. Then, \( \lim_{s \to \infty} \langle \psi, U_t^s \varphi \rangle = 0 \) for all \( \psi \in H \) and \( \varphi \in \ker (H_\rho)^\perp \). In particular, the flow \( \{ \phi_s \}_{s \in \mathbb{R}} \) is strongly mixing with respect to the measure \( \mu \).

Proof. Take \( \varphi \in C^1 (M) \), \( s, t \in \mathbb{R} \) and \( x \in M \). Then, we know from (2.1) that
\[ (\varphi \circ f_0 \circ \phi_s) (x) = (\varphi \circ \phi_{s^* (t,s,x)} \circ f_t) (x). \]
Applying the derivative \( \frac{d}{dt} \big|_{t=0} \) and using the relation \( s^* (0, s, x) = s \), we obtain
\[ (X_t \varphi) (\phi_s (x)) \]
\[ = \frac{d}{dt} \big|_{t=0} (\varphi \circ \phi_{s^* (t,s,x)} \circ f_t) (x) \]
\[ = \lim_{\epsilon \to 0} \epsilon^{-1} \{ (\varphi \circ \phi_{s^* (t,s,x)} \circ f_{\epsilon}) (x) - (\varphi \circ \phi_{s^* (0,s,x)} \circ f_{t}) (x) \} \]
\[ + \lim_{\epsilon \to 0} \epsilon^{-1} \{ (\varphi \circ \phi_s \circ f_{\epsilon}) (x) - (\varphi \circ \phi_s \circ f_{t}) (x) \} \]
\[ = (\partial_1 s^*) (0, s, x) \cdot (X_t \varphi) (\phi_s (x)) + (X_t \varphi) (\phi_s (x)) \].
which is equivalent to
\[(\partial_t s^*)(0, s, \cdot) U^\phi_s H^\phi \varphi = iU^\phi_s X^s \varphi - iX^s U^\phi_s \varphi. \quad (2.7)\]

So, for \( \psi \in C^1(M) \), we infer from Lemma 2.5 that
\[
\lim_{s \to \infty} \langle \rho^{-1} \psi, U^\phi_s H^\phi \varphi \rangle = \ln(\lambda)^{-1} \lim_{s \to \infty} \langle \rho^{-1} \psi, \ln(\lambda) U^\phi_s H^\phi \varphi \rangle
\]
\[= \ln(\lambda)^{-1} \lim_{s \to \infty} \langle \rho^{-1} \psi, s^{-1}(\partial_t s^*)(0, s, \cdot) U^\phi_s H^\phi \varphi \rangle
\]
\[= \ln(\lambda)^{-1} \lim_{s \to \infty} s^{-1} \langle \rho^{-1} \psi, iU^\phi_s X^s \varphi \rangle - \ln(\lambda)^{-1} \lim_{s \to \infty} s^{-1} \langle \rho^{-1} \psi, iX^s U^\phi_s \varphi \rangle
\]
\[= 0 - \ln(\lambda)^{-1} \lim_{s \to \infty} s^{-1} \langle \psi, i\rho^{-1} X^s U^\phi_s \varphi \rangle.
\]

But, the operator \( i\rho^{-1} X^s \) is symmetric on \( C^1(M) \). So,
\[
\lim_{s \to \infty} s^{-1} \langle \psi, i\rho^{-1} X^s U^\phi_s \varphi \rangle = \lim_{s \to \infty} s^{-1} \langle i\rho^{-1} X^s \psi, U^\phi_s \varphi \rangle = 0,
\]
and thus
\[
\lim_{s \to \infty} \langle \rho^{-1} \psi, U^\phi_s H^\phi \varphi \rangle = 0. \quad (2.8)
\]

Moreover, the set \( \rho^{-1} C^1(M) \) is dense in \( H \) because \( C^1(M) \) is dense in \( H \) and \( \rho^{-1} : H \to H \) is an homeomorphism, and the set \( H_\rho C^1(M) \) is dense in \( \ker(H_\rho)^\perp \) because \( C^1(M) \) is a core for \( H_\rho \) and \( H_\rho D(H_\rho) \) is dense in \( \ker(H_\rho)^\perp \). Therefore, (2.8) implies that \( \lim_{s \to \infty} \langle \psi, U^\phi_s \varphi \rangle = 0 \) for all \( \psi \in H \) and \( \varphi \in \ker(H_\rho)^\perp \). \Box

Under Assumptions 2.1 and 2.4, the result of Theorem 2.6 applies in particular to the case of reparametrisations of classical horocycle flows on the unit tangent bundle of compact connected orientable surfaces of constant negative curvature (see [15, Cor. 4.2]).

3 Absolutely continuous spectrum

We know from Theorem 2.6 that, under Assumptions 2.1 and 2.4, the flow \( \{\phi_t\}_{t \in \mathbb{R}} \) is strongly mixing with respect to the measure \( \mu \). Therefore, its generator \( H_\phi \) has purely continuous spectrum in \( \mathbb{R} \setminus \{0\} \).

Our goal in this section is to show that the spectrum of \( H_\phi \) is even purely absolutely continuous in \( \mathbb{R} \setminus \{0\} \) under some additional regularity assumption. For this, we first need to recall some results on commutator methods borrowed from \([1, 22]\) (see also the original paper \([18]\) of É. Mourre).

3.1 Commutators and regularity classes

Let \( H \) be a Hilbert space with scalar product \( \langle \cdot, \cdot \rangle \) antilinear in the first argument, denote by \( \mathcal{B}(H) \) the set of bounded linear operators on \( H \), and write \( \| \cdot \| \) both for the norm on \( H \) and the norm on \( \mathcal{B}(H) \).

Let \( A \) be a self-adjoint operator in \( H \) with domain \( D(A) \), and take \( S \in \mathcal{B}(H) \). For any \( k \in \mathbb{N} \), we say that \( S \) belongs to \( C^k(A) \), with notation \( S \in C^k(A) \), if the map
\[
\mathbb{R} \ni t \mapsto e^{-itA} S e^{itA} \in \mathcal{B}(H)
\]
is strongly of class \( C^k \).

In the case \( k = 1 \), one has \( S \in C^1(A) \) if and only if the quadratic form
\[
D(A) \ni \varphi \mapsto \langle \varphi, SA \varphi \rangle - \langle A \varphi, S \varphi \rangle \in \mathbb{C}
\]
is continuous for the topology induced by \( H \) on \( D(A) \). We denote by \([S, A]\) the bounded operator associated with the continuous extension of this form, or equivalently \( -i \) times the strong derivative of the function (3.1) at \( t = 0 \).
If \( H \) is a self-adjoint operator in \( \mathcal{H} \) with domain \( \mathcal{D}(H) \) and spectrum \( \sigma(H) \), we say that \( H \) is of class \( C^k(A) \) if \( (H - z)^{-1} \in C^k(A) \) for some \( z \in \mathbb{C} \setminus \sigma(H) \). So, \( H \) is of class \( C^1(A) \) if and only if the quadratic form
\[
\mathcal{D}(A) \ni \varphi \mapsto \langle \varphi, (H - z)^{-1} A \varphi \rangle - \langle A \varphi, (H - z)^{-1} \varphi \rangle \in \mathbb{C}
\]
extends continuously to a bounded form defined by the operator \([ (H - z)^{-1}, A] \in \mathcal{B}(\mathcal{H}) \). In such a case, the set \( \mathcal{D}(H) \cap \mathcal{D}(A) \) is a core for \( H \) and the quadratic form
\[
\mathcal{D}(H) \cap \mathcal{D}(A) \ni \varphi \mapsto \langle H \varphi, A \varphi \rangle - \langle A \varphi, H \varphi \rangle \in \mathbb{C}
\]
is continuous in the topology of \( \mathcal{D}(H) \) \cite[Thm. 6.2.10(b)]{1}. This form then extends uniquely to a continuous quadratic form on \( \mathcal{D}(H) \) which can be identified with a continuous operator \([ H, A] \) from \( \mathcal{D}(H) \) to the adjoint space \( \mathcal{D}(H)^* \). In addition, the following relation holds in \( \mathcal{B}(\mathcal{H}) \):
\[
[(H - z)^{-1}, A] = -(H - z)^{-1}[H, A](H - z)^{-1}.
\]

Let \( E^H(\cdot) \) denote the spectral measure of the self-adjoint operator \( H \), and assume that \( H \) is of class \( C^1(A) \). If there exist a Borel set \( I \subset \mathbb{R} \), a number \( a > 0 \) and a compact operator \( K \in \mathcal{B}(\mathcal{H}) \) such that
\[
E^H(I)[iH, A]E^H(I) \geq aE^H(I) + K,
\]
then one says that \( H \) satisfies a Mourre estimate on \( I \) and that \( A \) is a conjugate operator for \( H \) on \( I \). Also, one says that \( H \) satisfies a strict Mourre estimate on \( I \) if \( (3.3) \) holds with \( K = 0 \). One of the consequences of a Mourre estimate is to imply spectral results for \( H \) on \( I \). We recall here these spectral results in the case where \( H \) is of class \( C^2(A) \) (see \cite[Sec. 7.1.2]{1} and \cite[Thm. 0.1]{2} for more details).

**Theorem 3.1.** Let \( H \) and \( A \) be self-adjoint operators in a Hilbert space \( \mathcal{H} \), with \( H \) of class \( C^2(A) \). Suppose there exist a bounded Borel set \( I \subset \mathbb{R} \), a number \( a > 0 \) and a compact operator \( K \in \mathcal{B}(\mathcal{H}) \) such that
\[
E^H(I)[iH, A]E^H(I) \geq aE^H(I) + K.
\]
Then, \( H \) has at most finitely many eigenvalues in \( I \), each one of finite multiplicity, and \( H \) has no singular continuous spectrum in \( I \). Furthermore, if \( (3.4) \) holds with \( K = 0 \), then \( H \) has only purely absolutely continuous spectrum in \( I \) (no singular spectrum).

### 3.2 Absolutely continuous spectrum

We show in this section that the spectrum of \( H_\theta \) is purely absolutely continuous in \( \mathbb{R} \setminus \{0\} \) under some additional regularity assumption. We start with two technical lemmas.

**Lemma 3.2.** Suppose that Assumptions 2.1 and 2.4 are satisfied.

(a) If \( s \in \mathbb{R} \) and \( \varphi \in C^1(M) \), then
\[
U^\theta_s X_r U^\theta_{-s} \varphi = X_r \varphi + \int_0^s dr \left( u_{0,0} \circ \phi_r \right) X_r \varphi.
\]
(b) If \( z \in \mathbb{C} \setminus \mathbb{R} \) and \( \varphi \in C^1(M) \), then
\[
X_r (H_\theta - z)^{-1} \varphi \in \mathcal{H}.
\]
(c) If \( X_r(\rho) \in C(M) \), then \( u_{0,0} = \ln(\lambda) + \rho^{-1}X_r(\rho) \).

**Proof.** (a) Take \( s \in \mathbb{R} \) and \( \varphi \in C^1(M) \). Then, (2.7), the identity \((\partial_1 s^*)(0,0,\cdot) \equiv 0 \) and Assumption 2.4 imply
\[
U^\theta_s X_r U^\theta_{-s} \varphi = X_r \varphi + (\partial_1 s^*)(0,s,\cdot)X_r \varphi = X_r \varphi + \int_0^s dr u_{0,0} X_r \varphi.
\]

On another hand, the cocycle equation (2.5) implies for all \( r, s, t \in \mathbb{R} \) that
\[
s^*(t, s, \cdot) = s^*(t, s - r, \phi_r(\cdot)) - s^*(t, -r, \phi_r(\cdot)) = U^\theta_s (s^*(t, s - r, \cdot) - s^*(t, -r, \cdot)) U^\theta_{-r}.
\]
Deriving with respect to \( t \) and \( s \), we thus obtain
\[
u_{t,s} = U^\phi_t u_{t,s-r} U^\phi_r = u_{t,s-r} \circ \phi_r.
\]
In particular, we have \( u_{0,r} = u_{0,0} \circ \phi_r \), and the claim follows from (3.5).

(b) We give the proof in the case \( \text{Im}(z) > 0 \) since the case \( \text{Im}(z) < 0 \) is analogous. Take \( \varphi \in C^1(M) \).

Then, the formula \( (H_\phi - z)^{-1} \varphi = i \int_0^\infty \frac{e^{irz}}{t} \varphi \) and point (a) imply
\[
X_f(H_\phi - z)^{-1} \varphi
= i \lim_{t \to 0} \int_0^\infty \frac{e^{irz}}{t} (U^\phi_t - 1) U^\phi_r \varphi
= i \lim_{t \to 0} \int_0^\infty \frac{e^{irz}}{t} \int_0^t ds U^\phi_s X_f U^\phi_r \varphi
= i \lim_{t \to 0} \int_0^\infty \frac{e^{irz}}{t} \int_0^t ds U^\phi_s U^\phi_r \left( X_f + \int_0^r dq \left( u_{0,0} \circ \phi_q \right) X_\phi \right) \varphi.
\]

Now, we know from (2.3) that \( \|U_s^\phi\| \leq \frac{\max(\rho)}{\min(\rho)} \) for all \( s \in \mathbb{R} \). Thus, we have
\[
\left\| e^{irz} \int_0^t ds U^\phi_s U^\phi_r \left( X_f + \int_0^r dq \left( u_{0,0} \circ \phi_q \right) X_\phi \right) \varphi \right\|
\leq e^{-r \text{Im}(z)} \frac{\max(\rho)}{\min(\rho)} \left( \|X_f \varphi\| + r \|u_{0,0}\| \|X_\phi \varphi\| \right)
\in L^1([0, \infty), dr),
\]
and we can apply Lebesgue dominated convergence theorem to (3.6) to obtain
\[
X_f(H_\phi - z)^{-1} \varphi
= i \int_0^\infty \frac{e^{irz}}{t} \lim_{t \to 0} \int_0^t ds U^\phi_s U^\phi_r \left( X_f + \int_0^r dq \left( u_{0,0} \circ \phi_q \right) X_\phi \right) \varphi
= i \int_0^\infty \frac{e^{irz}}{t} U^\phi_r \left( X_f + \int_0^r dq \left( u_{0,0} \circ \phi_q \right) X_\phi \right) \varphi
\in \mathcal{H}.
\]

(c) The proof is inspired by a result of L. W. Green in the case of the classical horocycle flows on the unit tangent bundle of compact connected orientable surfaces of negative curvature (see [7, Eq. (3.3) & Lemma 3.3]). Take \( \psi, \varphi \in C^1(M) \). Then, the facts that \( \{\phi_s\}_{s \in \mathbb{R}} \) has vector field \( \rho X_\phi \), that \( \{\phi_s\}_{s \in \mathbb{R}} \) and \( \{f_t\}_{t \in \mathbb{R}} \) preserve the measure \( \mu = \mu/\rho \) and that \( f_t \circ \phi_s = \phi_{-r \lambda s} \circ f_t \) imply
\[
\left\langle X_f \psi, X_f \varphi \right\rangle = \frac{d}{dt} \left. \frac{d}{ds} \right|_{s=0} \left( \rho^{-1}(\psi \circ \phi_s), \varphi \circ f_t \right)
= \frac{d}{dt} \left. \frac{d}{ds} \right|_{s=0} \left( \rho^{-1}(\psi \circ f_t \circ \phi_s \circ f_{-s}) \right)
= \frac{d}{dt} \left. \frac{d}{ds} \right|_{s=0} \left( \rho^{-1}(\psi, \varphi \circ \phi_{-r \lambda s} \circ f_t) \right)
= -\frac{d}{dt} \left. \frac{d}{ds} \right|_{s=0} \left( \rho^{-1}(\psi, \lambda f_t X_\phi \varphi) \right)
= \left\langle X_f \psi, X_f \varphi \right\rangle - \left( \psi, \ln(\lambda) X_\phi \varphi \right).
\]
On another hand, point (a) and the symmetricity of $i\rho^{-1}X_f$ on $C^1(M)$ imply

$$\langle \psi, u_{0,0}X_f\phi \rangle = \frac{d}{ds}\bigg|_{s=0} \langle \rho(U_s^*\psi, \rho^{-1}X_fU_s\phi) \rangle = -\langle X_f\psi, X_f\phi \rangle + \langle \rho^{-1}X_f\rho\psi, X_f\phi \rangle = -\langle X_f\psi, X_f\phi \rangle + \langle X_f\psi, X_f\phi \rangle + \langle \psi, \rho^{-1}X_f(\rho)X_f\phi \rangle.$$ 

Combining the two relations, we thus obtain

$$\langle \psi, (u_{0,0} - \ln(\lambda) - \rho^{-1}X_f(\rho))X_f\phi \rangle = 0,$$

and we infer from the density of $C^1(M)$ in $H$ and the density of $X_fC^1(M)$ in $\ker(H_\phi)^{\perp}$ that

$$\langle \psi, (u_{0,0} - \ln(\lambda) - \rho^{-1}X_f(\rho))\phi \rangle = 0$$

for all $\psi \in H$ and $\phi \in \ker(H_\phi)^{\perp}$. Now, Lemma 2.5, the fact that $\{f_t\}_{t \in \mathbb{R}}$ preserves $\bar{\mu} = \mu/\rho$, and the ergodicity of $\{\phi_s\}_{s \in \mathbb{R}}$ imply that

$$u_{0,0} - \ln(\lambda) - \rho^{-1}X_f(\rho) \in \left\{ \phi \in H \mid \int_M d\mu \phi = 0 \right\} = \ker(H_\phi)^{\perp}.$$ 

So, we can set $\psi = 1$ and $\phi = u_{0,0} - \ln(\lambda) - \rho^{-1}X_f(\rho)$ in (3.7) to get

$$\int_M d\mu (u_{0,0} - \ln(\lambda) - \rho^{-1}X_f(\rho))^2 = 0,$$

and then infer that $u_{0,0} - \ln(\lambda) - \rho^{-1}X_f(\rho) \equiv 0$ by the continuity of $u_{0,0}$ and $\rho^{-1}X_f(\rho)$.

**Lemma 3.3.** Let $X$ be a $C^1$ vector field on $M$ and $g \in C^1(M; \mathbb{R})$. Assume that the operator

$$A\phi := i(X + g)\phi, \quad \phi \in C^1(M),$$

is symmetric in $H$. Then, $A$ is essentially self-adjoint in $H$.

**Proof.** Since $X$ is of class $C^1$, $X$ admits a $C^1$ flow $\{s\}_{s \in \mathbb{R}}$ [12, Thm. 3.43]. Thus, the operators

$$V_s\phi := e^{i\int_0^s dr (\phi \circ \zeta)} \phi \circ \zeta, \quad s \in \mathbb{R}, \quad \phi \in C^1(M),$$

are well-defined operators in $H$. Simple calculations show that $V_sV_t\phi = V_{s+t}\phi$ and $V_0\phi = \phi$ for $s, t \in \mathbb{R}$ and $\phi \in C^1(M)$, that $\lim_{s \to 0} \|V_{s+t} - V_s\phi\| = 0$ for $s \in \mathbb{R}$ and $\phi \in C^1(M)$, that $V_sC^1(M) \subset C^1(M)$ for $s \in \mathbb{R}$, and that $\frac{d}{ds} V_s\phi = -iAV_s\phi$ for $s \in \mathbb{R}$ and $\phi \in C^1(M)$. Furthermore, we have for $s \in \mathbb{R}$ and $\psi, \phi \in C^1(M)$ the equalities

$$\langle V_s\psi, V_s\phi \rangle - \langle \psi, \phi \rangle = \int_0^s \frac{d}{dr} \langle V_r\psi, V_r\phi \rangle = i \int_0^s dr \left( \langle AV_r\psi, V_r\phi \rangle - \langle V_r\psi, AV_r\phi \rangle \right) = 0,$$

due to the symmetricity of $A$. Therefore, the family $\{V_s\}_{s \in \mathbb{R}}$ satisfies on $C^1(M)$ the properties of a strongly continuous 1-parameter group of isometric operators in $H$ with $V_sC^1(M) \subset C^1(M)$ for all $s \in \mathbb{R}$, and with generator equal to $A$ on $C^1(M)$. Since $C^1(M)$ is dense in $H$, it follows that $\{V_s\}_{s \in \mathbb{R}}$ extends to a strongly continuous 1-parameter group of isometric (and thus unitary) operators in $H$ with $V_sC^1(M) \subset C^1(M)$ for all $s \in \mathbb{R}$, and with generator equal to $A$ on $C^1(M)$. Thus, Nelson’s criterion [21, Thm. VIII.10] implies that $A$ is essentially self-adjoint in $H$.

In the rest of the paper, in addition to Assumptions 2.1 and 2.4, we assume the following:
Assumption 3.4. The vector fields $X_f$ and $X_\phi$ are of class $C^1$, $X_f(\rho) \in C(M)$ and $\rho^{-1} X_f(\rho) \in C^1(M)$.

The conditions $X_f(\rho) \in C(M)$ and $\rho^{-1} X_f(\rho) \in C^1(M)$ are equivalent to the condition $X_f(\ln(\rho)) \in C^1(M)$. So, if one prefers, one can replace the two conditions $X_f(\rho) \in C(M)$ and $\rho^{-1} X_f(\rho) \in C^1(M)$ in Assumption 3.4 by the single condition $X_f(\ln(\rho)) \in C^1(M)$.

In the next proposition we define and prove the self-adjointness of the conjugate operator. Intuitively, the conjugate operator is constructed as follows. First, we take the sum of the vector field $2iX_f$ and its “divergence” $i\rho^{-1} X_f(\rho)$ to get a symmetric operator on $C^1(M)$. Then, we take the Birkhoff average of the resulting operator along the flow $\{\phi_s\}_{s \in \mathbb{R}}$ to take into account the unique ergodicity of $\{\phi_s\}_{s \in \mathbb{R}}$.

Proposition 3.5 (Conjugate operator). Suppose that Assumptions 2.1, 2.4, and 3.4 are satisfied. Then, the operator

$$A_t \psi := \frac{1}{t} \int_0^t ds \, U^s_t (2iX_f + i\rho^{-1} X_f(\rho)) U^{-s}_t \psi, \quad t > 0, \; \psi \in C^1(M),$$

is essentially self-adjoint in $\mathcal{H}$ (and its closure is denoted by the same symbol).

Proof. Since $\rho^{-1} X_f(\rho) = \tilde{p}^{-1} X_f(\tilde{p})$, the operator $(2iX_f + i\rho^{-1} X_f(\rho))$ is symmetric on $C^1(M)$. Therefore, the operator $A_t$ is also symmetric on $C^1(M)$ because $U^s_t C^1(M) \subset C^1(M)$ for all $s \in \mathbb{R}$. To show that $A_t$ is essentially self-adjoint on $C^1(M)$, we take $\psi, \varphi \in C^1(M)$. Then, Lemma 3.2(a) implies that

$$\langle \psi, A_t \varphi \rangle = \left\langle \psi, \frac{1}{t} \int_0^t ds \left( 2iX_f + 2i \int_0^s dr \, (u_{0,0} \circ \phi_r) X_\phi + i(\rho^{-1} X_f(\rho)) \circ \phi_s \right) \varphi \right\rangle$$

$$= \left\langle \psi, i(2X_f + a_t X_\phi + b_t) \varphi \right\rangle$$

(3.8)

with

$$a_t := \frac{2}{t} \int_0^t ds \int_0^s dr \, (u_{0,0} \circ \phi_r) \quad \text{and} \quad b_t := \frac{1}{t} \int_0^t ds \, (\rho^{-1} X_f(\rho)) \circ \phi_s.$$

Furthermore, Assumption 3.4, Lemma 3.2(c) and Leibniz integral rule imply that $X_f$, $X_\phi$, $a_t$ and $b_t$ are of class $C^1$. Therefore, we can apply Lemma 3.3 with $X := 2X_f + a_t X_\phi$ and $g := b_t$ to conclude that $A_t$ is essentially self-adjoint in $\mathcal{H}$. \qed

Lemma 3.6. Suppose that Assumptions 2.1, 2.4, and 3.4 are satisfied. Then, we have for $t > 0$

(a) $(H_\phi - i)^{-1} \in C^1(A_t)$ with

$$[i(H_\phi - i)^{-1}, A_t] = 2(H_\phi - i)^{-1} c_t H_\phi (H_\phi - i)^{-1} - [(H_\phi - i)^{-1}, c_t] \quad \text{and} \quad c_t := \frac{1}{t} \int_0^t ds \, (u_{0,0} \circ \phi_s).$$

(b) $(H_\phi - i)^{-1} \in C^2(A_t)$.

Proof. (a) Set $A := (2iX_f + i\rho^{-1} X_f(\rho))$ on $C^1(M)$ and take $\varphi \in C^1(M)$. Then, we know from Lemma
3.2(b)-(c) that
\[
\langle (H_\phi + i)^{-1} \varphi, A \varphi \rangle - \langle A \varphi, (H_\phi - i)^{-1} \varphi \rangle \\
= \langle (2iX_\phi + i\rho^{-1} X_\phi(\rho))(H_\phi + i)^{-1} \varphi, (H_\phi - i)^{-1} (H_\phi - i) \varphi \rangle \\
- \langle (H_\phi + i)^{-1} (H_\phi + i) \varphi, (2iX_\phi + i\rho^{-1} X_\phi(\rho))(H_\phi - i)^{-1} \varphi \rangle \\
= i \frac{d}{ds} \bigg|_{s=0} \{ \langle (2iX_\phi + i\rho^{-1} X_\phi(\rho))(H_\phi + i)^{-1} \varphi, (H_\phi - i)^{-1} (U_\phi^s - s) \varphi \rangle \\
- \langle (H_\phi + i)^{-1} (U_\phi^s - s) \varphi, (2iX_\phi + i\rho^{-1} X_\phi(\rho))(H_\phi - i)^{-1} \varphi \rangle \} \\
= - \frac{d}{ds} \bigg|_{s=0} \{ \langle (H_\phi + i)^{-1} \varphi, (2X_\phi + u_{0,0}) (H_\phi - i)^{-1} U_\phi^s \varphi \rangle \\
- \langle (H_\phi + i)^{-1} \varphi, U_\phi^s (2X_\phi + u_{0,0}) (H_\phi - i)^{-1} \varphi \rangle \} \\
= -2 \left( \varphi, \frac{d}{ds} \bigg|_{s=0} (H_\phi - i)^{-1} \{ X_\phi (H_\phi - i)^{-1} U_\phi^s - U_\phi^s X_\phi (H_\phi - i)^{-1} \} \varphi \right) \\
- \langle \varphi, \frac{d}{ds} \bigg|_{s=0} (H_\phi - i)^{-1} \{ u_{0,0} (H_\phi - i)^{-1} U_\phi^s - U_\phi^s u_{0,0} (H_\phi - i)^{-1} \} \varphi \rangle.
\]

For the first term, the equation \((H_\phi - i)^{-1} = \int_0^\infty e^{-t} U_\phi^t\) (valid in the strong sense) and Lemma 3.2(a) imply
\[
\frac{d}{ds} \bigg|_{s=0} (H_\phi - i)^{-1} \{ X_\phi (H_\phi - i)^{-1} U_\phi^s - U_\phi^s X_\phi (H_\phi - i)^{-1} \} \varphi \\
= i \frac{d}{ds} \bigg|_{s=0} \int_0^\infty dt \ e^{-t} \ U_\phi^t (H_\phi - i)^{-1} (U_\phi^s X_\phi U_\phi^t - X_\phi) U_\phi^s \varphi \\
= i \frac{d}{ds} \bigg|_{s=0} \int_0^\infty dt \ e^{-t} \ U_\phi^t (H_\phi - i)^{-1} \int_0^s dt \ (u_{0,0} \circ \phi_t) U_\phi^t X_\phi \varphi \\
= -i \int_0^\infty dt \ e^{-t} \ (H_\phi - i)^{-1} u_{0,0} U_\phi^t X_\phi \varphi \\
= i (H_\phi - i)^{-1} u_{0,0} H_\phi (H_\phi - i)^{-1} \varphi.
\]

For the second term, a direct calculation gives
\[
\frac{d}{ds} \bigg|_{s=0} (H_\phi - i)^{-1} \{ u_{0,0} (H_\phi - i)^{-1} U_\phi^s - U_\phi^s u_{0,0} (H_\phi - i)^{-1} \} \varphi = -i \[(H_\phi - i)^{-1}, u_{0,0}] \varphi.
\]

So, putting together the last equations, we get
\[
\langle (H_\phi + i)^{-1} \varphi, A \varphi \rangle - \langle A \varphi, (H_\phi - i)^{-1} \varphi \rangle \\
= \langle \varphi, \{ -2i (H_\phi - i)^{-1} u_{0,0} H_\phi (H_\phi - i)^{-1} + i [(H_\phi - i)^{-1}, u_{0,0}] \} \rangle.
\]

Therefore, for each \(t > 0\) we obtain
\[
\langle (H_\phi + i)^{-1} \varphi, A_t \varphi \rangle - \langle A_t \varphi, (H_\phi - i)^{-1} \varphi \rangle \\
= \frac{1}{t} \int_0^t ds \langle (H_\phi + i)^{-1} U_\phi^s \varphi, A U_\phi^s \varphi \rangle - \langle A U_\phi^s \varphi, (H_\phi - i)^{-1} U_\phi^s \varphi \rangle \\
= \frac{1}{t} \int_0^t ds \langle U_\phi^s \varphi, \{ -2i (H_\phi - i)^{-1} u_{0,0} H_\phi (H_\phi - i)^{-1} + i [(H_\phi - i)^{-1}, u_{0,0}] \} U_\phi^s \varphi \rangle \\
= \langle \varphi, \{ -2i (H_\phi - i)^{-1} c_t H_\phi (H_\phi - i)^{-1} + i [(H_\phi - i)^{-1}, c_t] \} \rangle
\]
with \(c_t = \frac{1}{t} \int_0^t ds \ (u_{0,0} \circ \phi_s)\), and the claim follows by the density of \(C^2(M)\) in \(\mathcal{D}(A_t)\).
(b) We know from point (a) that \((H_\phi - i)^{-1} \in C^1(A_t)\) with

\[
\begin{align*}
[i(H_\phi - i)^{-1}, A_t] &= 2(H_\phi - i)^{-1}c_t H_\phi (H_\phi - i)^{-1} - [(H_\phi - i)^{-1}, c_t] \\
&= 2(H_\phi - i)^{-1}c_t + 2i(H_\phi - i)^{-1}c_t (H_\phi - i)^{-1} - [(H_\phi - i)^{-1}, c_t].
\end{align*}
\]

So, it is sufficient to show that \(c_t \in C^1(A_t)\). For this, we note that \(c_t \in C^1(M)\) due to the assumption \(\rho^{-1}X_\rho(\rho) \in C^1(M)\), Lemma 3.2(c) and Leibniz integral rule. Then, we use (3.8) to get for \(\varphi \in C^1(M)\)

\[
\begin{align*}
\langle c_t \varphi, A_t \varphi \rangle - \langle A_t \varphi, c_t \varphi \rangle &= \langle \varphi, c_t (2X_t + a_t X_\phi + b_t) \varphi \rangle - \langle \varphi, (2X_t + a_t X_\phi + b_t)c_t \varphi \rangle \\
&= \langle \varphi, -i(2X_t(c_t) + a_t X_\phi(c_t)) \varphi \rangle
\end{align*}
\]

with \((2X_t(c_t) + a_t X_\phi(c_t))\) a bounded multiplication operator, and we note that this implies the claim because \(C^1(M)\) is dense in \(D(A_t)\).

In the next proposition, we show that \(H_\phi\) satisfies a strict Mourre estimate on compact subsets of \((0, \infty)\) and \((-\infty, 0)\). We use the operator \(-\ln(\lambda)A_t\) as conjugate operator on \((0, \infty)\) and the operator \(\ln(\lambda)A_t\) as conjugate operator on \((-\infty, 0)\).

**Proposition 3.7** (Strict Mourre estimate). Suppose that Assumptions 2.1, 2.4, and 3.4 are satisfied.

(a) If \(I \subset (0, \infty)\) is a compact set with \(I \cap \sigma(H_\phi) \neq \emptyset\), then there exist \(t > 0\) and \(a > 0\) such that

\[
E^{H_\phi}(I)[iH_\phi, -\ln(\lambda)A_t]E^{H_\phi}(I) \geq aE^{H_\phi}(I).
\]

(b) If \(J \subset (-\infty, 0)\) is a compact set with \(J \cap \sigma(H_\phi) \neq \emptyset\), then there exist \(t > 0\) and \(a > 0\) such that

\[
E^{H_\phi}(J)[iH_\phi, \ln(\lambda)A_t]E^{H_\phi}(J) \geq aE^{H_\phi}(J).
\]

Before the proof, we recall that the flow \(\{\phi_t\}_{t \in \mathbb{R}}\) is ergodic. Therefore, the spectrum of the operator \(H_\phi\) in \(\mathbb{R} \setminus \{0\}\) is not empty, and thus there exist compact sets \(I \subset (0, \infty)\) such that \(I \cap \sigma(H_\phi) \neq \emptyset\) and/or compacts sets \(J \subset (-\infty, 0)\) such that \(J \cap \sigma(H_\phi) \neq \emptyset\).

**Proof.** (a) Let \(t > 0\). Then, we know from Lemma 3.6(a) that \((H_\phi - i)^{-1} \in C^1(\mathbb{R})\) with

\[
[i(H_\phi - i)^{-1}, -\ln(\lambda)A_t] = -2\ln(\lambda)(H_\phi - i)^{-1}c_t H_\phi (H_\phi - i)^{-1} + \ln(\lambda)[(H_\phi - i)^{-1}, c_t].
\]

This, together with (3.2), implies that

\[
\begin{align*}
E^{H_\phi}(I)[iH_\phi, -\ln(\lambda)A_t]E^{H_\phi}(I) &= -(H_\phi - i)E^{H_\phi}(I)[i(H_\phi - i)^{-1}, -\ln(\lambda)A_t](H_\phi - i)E^{H_\phi}(I) \\
&= 2\ln(\lambda)E^{H_\phi}(I) c_t H_\phi E^{H_\phi}(I) - \ln(\lambda)(H_\phi - i)E^{H_\phi}(I) [(H_\phi - i)^{-1}, c_t] (H_\phi - i)E^{H_\phi}(I) \\
&= 2(\ln(\lambda))^2H_\phi E^{H_\phi}(I) + 2\ln(\lambda)E^{H_\phi}(I)(c_t - \ln(\lambda))H_\phi E^{H_\phi}(I) \\
&\quad - \ln(\lambda)(H_\phi - i)E^{H_\phi}(I) [(H_\phi - i)^{-1}, c_t - \ln(\lambda)] (H_\phi - i)E^{H_\phi}(I) \\
&\geq aE^{H_\phi}(I) + 2\ln(\lambda)E^{H_\phi}(I)(c_t - \ln(\lambda))H_\phi E^{H_\phi}(I) \\
&\quad - \ln(\lambda)(H_\phi - i)E^{H_\phi}(I) [(H_\phi - i)^{-1}, c_t - \ln(\lambda)] (H_\phi - i)E^{H_\phi}(I)
\end{align*}
\]
with \( a_t := 2(\ln(\lambda))^2 \inf(I) > 0 \). Furthermore, since \( \{\phi_s\}_{s \in \mathbb{R}} \) is uniquely ergodic, we obtain from Lemma 2.5 that
\[
\lim_{t \to -\infty} (c_t - \ln(\lambda)) = \int_M \frac{d\mu_0 \nu_0}{0} - \ln(\lambda) = 0
\]
uniformly on \( M \). Therefore, if \( t > 0 \) is large enough, there exists \( a \in (0, a_t) \) such that
\[
E^{H_{\phi}}(I)[iH_{\phi} - \ln(\lambda)A_1] E^{H_{\phi}}(I) \geq a E^{H_{\phi}}(I).
\]

(b) The proof is similar to that of point (a). \( \square \)

The next theorem is the main result of the paper.

**Theorem 3.8 (Absolutely continuous spectrum).** Suppose that Assumptions 2.1, 2.4, and 3.4 are satisfied. Then, \( H_{\phi} \) has purely absolutely continuous spectrum, except at 0, where it has a simple eigenvalue with eigenspace \( C \cdot 1 \).

**Proof.** We know from Lemma 3.6(b) that \( H_{\phi} \) is of class \( C^2(-\ln(\lambda)A_1) \) for all \( t > 0 \). Moreover, we know from Proposition 3.7(a) that for each compact set \( I \subset (0, \infty) \) with \( I \cap \sigma(H_{\phi}) \neq \emptyset \) there exist \( t > 0 \) and \( a > 0 \) such that
\[
E^{H_{\phi}}(I)[iH_{\phi} - \ln(\lambda)A_1] E^{H_{\phi}}(I) \geq a E^{H_{\phi}}(I).
\]
Therefore, it follows from Theorem 3.1 that \( H_{\phi} \) has purely absolutely continuous spectrum in \((0, \infty)\). Since the same holds for \((−\infty, 0)\), \( H_{\phi} \) has purely absolutely continuous spectrum, except at 0, where it has a simple eigenvalue with eigenspace \( C \cdot 1 \) due to the ergodicity of the flow \( \{\phi_s\}_{s \in \mathbb{R}} \).

We conclude with some remarks on Theorem 3.8.

**Remark 3.9.** (a) Under Assumptions 2.1, 2.4, and 3.4, the result of Theorem 3.8 applies, as in the case of Theorem 2.6, to reparametrisations of classical horocycle flows on the unit tangent bundle of compact connected orientable surfaces of constant negative curvature.

Our regularity assumptions on the function \( \rho \) are \( X_\tau(\rho) \in C(M) \) and \( \rho^{-1} X_\tau(\rho) \in C^1(M) \). Therefore, Theorem 3.8 extends the results of [6, 23, 25] on the absolutely continuous spectrum of time changes of the classical horocycle flows on the unit tangent bundle of compact orientable surfaces of constant negative curvature since the function corresponding to \( \rho \) in [6, 23, 25] is at least of class \( C^3 \) (in [23], the function \( \rho \) is of class \( C^2 \), but it satisfies another additional assumption). Also, the result of Theorem 3.8 answers the question on the regularity of the function \( \rho \) raised in [25, Rem. 3.4]. The result of [6] on Lebesgue spectrum and the result of [23] for surfaces of finite volume are of a different nature and are not covered by Theorem 3.8.

(b) In the case of the classical horocycle flows on the unit tangent bundle of compact connected orientable surfaces of negative curvature, L. W. Green has shown in [7, Thm. B] that the curvature of the surface is necessarily constant if the flow \( \{\phi_s\}_{s \in \mathbb{R}} \) admits a \( C^2 \) uniformly expanding reparametrisation \( \{\hat{\phi}_s\}_{s \in \mathbb{R}} \) (and thus a function \( \rho \) of class \( C^1 \)). Since it is possible that a similar argument also applies under our regularity assumptions (an anonymous referee signaled this to us), we prefer not to mention the case of compact surfaces of variable negative curvature. However, we hope that in some future the technics and the regularity assumptions presented in this manuscript could be improved in order to cover explicit examples of compact surfaces of variable negative curvature.

(c) In Lemma 3.6, Proposition 3.7 and Theorem 3.8, we stated and proved the results using only the operator \( H_{\phi} \) under study, and not its square \((H_{\phi})^2\) as we did in [23, 25]. This allowed us to simplify the exposition in comparison to [23, 25].
References

[1] W. O. Amrein, A. Boutet de Monvel, and V. Georgescu. \textit{C}_0\text{-groups, commutator methods and spectral theory of N-body Hamiltonians}, volume 135 of \textit{Progress in Mathematics}. Birkhäuser Verlag, Basel, 1996.

[2] D. V. Anosov. \textit{Geodesic flows on closed Riemann manifolds with negative curvature}. Proceedings of the Steklov Institute of Mathematics, No. 90 (1967). Translated from the Russian by S. Feder. American Mathematical Society, Providence, R.I., 1969.

[3] M. Beboutoff and W. Stepanoff. Sur la mesure invariante dans les systèmes dynamiques qui ne diffèrent que par le temps. \textit{Rec. Math. [Mat. Sbornik] N.S.}, 7 (49):143–166, 1940.

[4] M. B. Bekka and M. Mayer. \textit{Ergodic theory and topological dynamics of group actions on homogeneous spaces}, volume 269 of \textit{London Mathematical Society Lecture Note Series}. Cambridge University Press, Cambridge, 2000.

[5] R. Bowen and B. Marcus. Unique ergodicity for horocycle foliations. \textit{Israel J. Math.}, 26(1):43–67, 1977.

[6] G. Forni and C. Ulcigrai. Time-changes of horocycle flows. \textit{J. Mod. Dyn.}, 6(2):251–273, 2012.

[7] L. W. Green. Remarks on uniformly expanding horocycle parameterizations. \textit{J. Differential Geom.}, 13(2):263–271, 1978.

[8] G. A. Hedlund. Fuchsian groups and transitive horocycles. \textit{Duke Math. J.}, 2(3):530–542, 1936.

[9] G. A. Hedlund. Fuchsian groups and mixtures. \textit{Ann. of Math. (2)}, 40(2):370–383, 1939.

[10] E. Hopf. Statistik der geodätischen Linien in Mannigfaltigkeiten negativer Krümmung. \textit{Ber. Verh. Sächs. Akad. Wiss. Leipzig}, 91:261–304, 1939.

[11] P. D. Humphries. Change of velocity in dynamical systems. \textit{J. London Math. Soc. (2)}, 7:747–757, 1974.

[12] M. C. Irwin. \textit{Smooth dynamical systems}, volume 17 of \textit{Advanced Series in Nonlinear Dynamics}. World Scientific Publishing Co., Inc., River Edge, NJ, 2001. Reprint of the 1980 original, With a foreword by R. S. MacKay.

[13] A. G. Kushnirenko. Spectral properties of certain dynamical systems with polynomial dispersal. \textit{Moscow Univ. Math. Bull.}, 29(1):82–87, 1974.

[14] B. Marcus. Unique ergodicity of the horocycle flow: variable negative curvature case. \textit{Israel J. Math.}, 21(2-3):133–144, 1975. Conference on Ergodic Theory and Topological Dynamics (Kibbutz Lavi, 1974).

[15] B. Marcus. Ergodic properties of horocycle flows for surfaces of negative curvature. \textit{Ann. of Math. (2)}, 105(1):81–105, 1977.

[16] G. A. Margulis. Certain measures that are connected with U-flows on compact manifolds. \textit{Funkcional. Anal. i Priložen.}, 4(1):62–76, 1970.

[17] S. Matsumoto. \textit{Codimension one Anosov flows}, volume 27 of \textit{Lecture Notes Series}. Seoul National University, Research Institute of Mathematics, Global Analysis Research Center, Seoul, 1995.

[18] É. Mourre. Absence of singular continuous spectrum for certain selfadjoint operators. \textit{Comm. Math. Phys.}, 78(3):391–408, 1980/81.
[19] O. S. Parasyuk. Flows of horocycles on surfaces of constant negative curvature. *Uspekhi Matem. Nauk (N.S.)*, 8(3(55)):125–126, 1953.

[20] Y. B. Pesin. Geodesic flows with hyperbolic behavior of trajectories and objects connected with them. *Uspekhi Mat. Nauk*, 36(4(220)):3–51, 247, 1981.

[21] M. Reed and B. Simon. *Methods of modern mathematical physics. I*. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, second edition, 1980. Functional analysis.

[22] J. Sahbani. The conjugate operator method for locally regular Hamiltonians. *J. Operator Theory*, 38(2):297–322, 1997.

[23] R. Tiedra de Aldecoa. Spectral analysis of time changes of horocycle flows. *J. Mod. Dyn.*, 6(2):275–285, 2012.

[24] R. Tiedra de Aldecoa. The absolute continuous spectrum of skew products of compact lie groups. *Israel J. Math.*, 208(1):323–350, 2015.

[25] R. Tiedra de Aldecoa. Commutator methods for the spectral analysis of uniquely ergodic dynamical systems. *Ergodic Theory Dynam. Systems*, 35(3):944–967, 2015.