Improved pyrotechnics:
Closer to the burning graph conjecture

Paul Bastide\textsuperscript{1}, Marthe Bonamy\textsuperscript{2}, Pierre Charbit\textsuperscript{3}, Théo Pierron\textsuperscript{4}, and Mikaël Rabie\textsuperscript{3}

\textsuperscript{1}École Normale Supérieure Rennes, Rennes, France
\textsuperscript{2}CNRS, LaBRI, Université de Bordeaux, Bordeaux, France.
\textsuperscript{3}IRIF, Paris, Paris, France.
\textsuperscript{4}LIRIS, Université de Lyon, Lyon, France.

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Abstract
Can every connected graph burn in $\lceil \sqrt{n} \rceil$ steps? While this conjecture remains open, we prove that it is asymptotically true when the graph is much larger than its growth, which is the maximal distance of a vertex to a well-chosen path in the graph. In fact, we prove that the conjecture for graphs of bounded growth boils down to a finite number of cases. Through an improved (but still weaker) bound for all trees, we argue that the conjecture almost holds for all graphs with minimum degree at least 3 and holds for all large enough graphs with minimum degree at least 4. The previous best lower bound was 23.

1 Introduction

How fast can a rumor propagate in a graph? One measure of that, introduced by Bonato, Janssen and Roshanbin \cite{BJR16}, is the burning number $b(G)$ of a graph $G$. At step 1, we set a vertex on fire. At every step $i \geq 2$, all neighbours of a vertex on fire catch fire themselves, and we set a new vertex on fire. If at the end of step $k$ the whole graph is on fire, then the graph is $k$-burnable. The burning number $b(G)$ of $G$ is defined to be the least $k$ such that $G$ is $k$-burnable.

A graph with $n$ isolated vertices is trivially not $(n-1)$-burnable. We focus on connected graphs. Paths are an interesting special case. For a path $P_n$ on $n$ vertices, it is not hard to check that $b(P_n) = \lceil \sqrt{n} \rceil$. When introducing the notion, Bonato et al. \cite{BJR16} conjectured that paths are, essentially, the worst case for the burning number of a graph.

Conjecture 1.1 (Bonato et al. \cite{BJR16}). Every connected graph $G$ satisfies $b(G) \leq \lceil \sqrt{|V(G)|} \rceil$.

Conjecture 1.1 is only known to hold with a constant factor. For any connected graph $G$ and any spanning tree $T$ of $G$, we have $b(G) \leq b(T)$. Therefore, it is sufficient to prove that the conjecture holds for trees. While Conjecture 1.1 is still open in general, it has been shown to hold for specific graph classes. Kamali, Miller and Zhang showed \cite{KMZ20} that the conjecture holds for graphs of minimum degree at least 23. Here, we observe that the conjecture almost holds for all connected graphs of minimum degree at least 3, and fully holds for those of minimum degree at least 4 that are large enough.
Theorem 1.2. For any connected graph $G$ on $n$ vertices, if every vertex in $G$ has degree at least 3 then $b(G) \leq \lceil \sqrt{n} \rceil + 2$.

Theorem 1.3. For any connected graph $G$ on $n$ vertices, if every vertex in $G$ has degree at least 4 and $n$ is large enough then $b(G) \leq \lceil \sqrt{n} \rceil$.

For trees, it has been proved by Bonato and Lidbetter [BL19] that spiders – trees containing a unique vertex of degree strictly greater than 2 – satisfy the conjecture. Hiller, Triesch and Koster [HTK19] were first to investigate the burning number of trees with fixed growth.

Definition 1.4. The growth of a connected graph $G$ is the smallest $k$ such that all vertices in $G$ are within distance $k$ of some path $P$ in $G$.

Note that a caterpillar has growth at most 1. The conjecture has only been confirmed for trees with growth at most 2 [HTK19]. Trees of growth $\leq k$ are sometimes referred to as $k$-caterpillars. We refer the reader to a nice recent survey on the topic for further details [Bon20]. Note that any result on trees of bounded growth immediately applies to graphs of bounded growth, by considering an appropriate spanning tree. In this paper we present the following theorem.

Theorem 1.5. For any tree $T$ on $n$ vertices, if $n$ is large enough compared to the growth of $T$, then $b(T) \leq \lceil \sqrt{n} \rceil + 1$.

More precisely, for any tree $T$ on $n$ vertices, we have $b(T) \leq \lceil \sqrt{n + 20k^2} \rceil$, where $k$ is the growth of $T$. In fact, for any fixed $k$, in order to prove Conjecture 1.1 for trees of growth at most $k$, it suffices to verify it for a finite number of them. It would be interesting to find ways to go further and transform this into a practical way for Conjecture 1.1 to be verified for $k$-caterpillars (for say $k = 10$) – the statement currently suggests unreasonably many cases, which seems unnecessary.

Theorem 1.6. For any $k \in \mathbb{N}$, if Conjecture 1.1 holds for all trees of growth at most $k$ on at most $(2k^2 + 4k + 3)^2$ vertices, then it holds for all trees of growth at most $k$.

Finally, we give a short argument that any tree can burn in $\sqrt{\frac{2n}{3}} + O(1)$ steps, thus improving upon the $\sqrt{\frac{3}{2}n} + O(1)$ bound of Land and Lu.

Theorem 1.7. For any graph $G$,

$$b(G) \leq \left\lfloor \sqrt{\frac{4n}{3}} \right\rfloor + 1$$

Note that Theorem 1.7 was independently obtained through similar arguments by Bonato and Kamali [BK21].

2 Preliminary Observations

We first include a lemma which we believe is folklore but whose source we were unable to trace$^1$. This is a short argument why $b(G)$ is upper-bounded by a constant factor of $\sqrt{|V(G)|}$.

$^1$Please email us with the information if you have it.
Lemma 2.1. For any connected graph $G$ on $n$ vertices,

$$b(G) \leq \lceil \sqrt{2n} \rceil.$$  

Proof. Consider a spanning tree $T$ of $G$, and construct a path $P$ by unfolding a depth-first-search on $T$. Formally $P$ is the sequence of vertices visited (with repetition) in the Eulerian tour of $T$. Intuitively, one can see $P$ as the sequence of vertices obtained by outlining $T$ with a unique line. As the outline of $T$ uses each edge at most twice, the path $P$ contains at most $2n$ vertices. Trivially, a burning sequence of $P$ induces a burning sequence of $G$, therefore $b(G) \leq \lceil \sqrt{2n} \rceil$. \qed

One of the main obstacles toward proving Conjecture 1.1 resides in the fact that it is hard to strengthen or weaken it in a meaningful way. We find it convenient to relax the statement a little by discussing set burning.

Definition 2.2. We say that a graph $G$ is $B$-burnable for some set of $B = \{b_1, \ldots, b_j\} \subseteq \mathbb{N}$ if there exists a sequence $X = \{v_1, \ldots, v_j\}$ of vertices of $G$ such that $\bigcup_{i \in \{1, \ldots, j\}} N_{b_i}[v_i] = V(G)$.

Note that multisets are not eligible, which is crucial in later arguments. Conjecture 1.1 can now be rephrased.

Conjecture 2.3. [Burning Graph Conjecture [BJR16]] Every tree $T$ on $n$ vertices is $\{0, \ldots, \lceil \sqrt{n} \rceil - 1\}$-burnable.

In the case of trees of bounded growth, there is a specific kind of set which we find appropriate, as follows.

Definition 2.4. For $T$ a tree of growth $k$ and of order $n$, $B$ is called a burning set of $T$ if $\{0, 1, \ldots, k\} \subseteq B$ and $\sum_{i \in B} (2i + 1) \geq \sum_{i \in B} (2i + 1) \geq n$.

As can be expected from the terminology, we conjecture that any tree $T$ is $B$-burnable whenever $B$ is a burning set of $T$. We provide now a powerful tool for induction.

Lemma 2.5. In any tree $T$, if there is a path $P$ with two endpoints $u$ and $v$ such that for $T_1$ (resp. $T_2$) the connected component of $T \setminus (P \setminus \{u, v\})$ containing $u$ (resp. $v$), $b(T_1 \cup T_2 + uv) = p$, then $b(T) \leq p + 1$ unless $d(u, v) + 2\max_{x \in V(T_1 \cup T_2)} d(P, x) \geq 2p + 3$.

In fact, a stronger lemma using the framework of set burning holds.

Lemma 2.6. In any tree $T$, for any set $B$ where $p = \max B$, if there is a path $P$ with two endpoints $u$ and $v$ such that for $T_1$ (resp. $T_2$) the connected component of $T \setminus (P \setminus \{u, v\})$ containing $u$ (resp. $v$), and $(T_1 \cup T_2 + uv)$ is $B - \{p\}$-burnable, then $T$ is $B$-burnable unless $d(u, v) + 2\max_{x \in V(T_1 \cup T_2)} d(P, x) \geq 2p + 3$.

Proof. Consider a path $P$ with endpoints $u$ and $v$, we denote by $T_1$ (resp. $T_2$) the connected components of $T \setminus P$ containing $u$ (resp. $v$), and let $T' = (T_1 \cup T_2 + uv)$ be $B - \{p\}$ burnable, and $d(u, v) + 2\max_{x \in V(T_1 \cup T_2)} d(P, x) \leq 2p + 2$.

We consider a sequence $\{x_j\}_{j \in B \setminus \{p\}}$ of vertices which burns $T'$. Let $i$ be such that $i = \arg \max_{j \in B \setminus \{p\}} (j - d(x_j, \{u, v\}))$ and $X_i$ the sub-tree burned by $x_i$ in $T'$. We show that we can always place the element $p$ on a vertex $x_p$ of the path $P$ such that the elements $i$ on $x_i$ and $p$ on $x_p$ burn $(X_i \cup T_p)$.

We consider w.l.o.g that $x_i = u$, and we denote the vertices of $P$ by $v_0, v_1, \ldots, v_d$ from $u$ to $v$ where $d = d(u, v)$, in particular $v_0 = u$, and $v_d = v$. We place the element $p$ on the vertex
First note that $d(x_p, v) = p - (i - 1)$ therefore $X_i \cap T_2$ is burned by the element $p$, of course $X_i \cap T_1$ is burned by the element $i$, therefore $X_i$ is entirely burned. We still need to show that $T_p$ is burned too. Consider a vertex $w$ of $T_p$, if $d(w, u) \leq i$ then the element $i$ burns $w$, thus we can consider $d(w, u) > i$, which implies that the vertex $v_r$ in $P$ such that $w$ is in the sub-tree rooted in $v_r$ is such that $r \geq i - m + 1$ where $m = \max_{x \in V(T_1 \cup T_2)} d(P, x)$, therefore

$$d(w, x_p) \leq d(w, v_r) + d(w, x_p) \leq m + (d - (p - i + 1) - (i - m + 1)) = 2m + d - p - 2$$

Since $2m + d - p - 2 \leq p \iff 2m + d \leq 2p + 2$, the lemma holds.

\[ \square \]

### 2.1 More involved induction

To avoid confusion and because it is more intuitive, we refer to elements of a burning set as sparks.

**Theorem 2.7.** For every $k \in \mathbb{N}$, either Conjecture 1.1 holds for every $k$-caterpillar, or it fails for some $k$-caterpillar $T$ with $|V(T)| \leq (4k^2 + 5k + 4)^2$.

**Proof.** Assume that Conjecture 1.1 fails for some $k$-caterpillars, and consider $T$ being a smallest such one. Let $n = |V(T)|$. Assume for a contradiction that $n \geq (4k^2 + 5k + 4)^2$.

Let $T_0 = T$ and $B_0 = [0, \lfloor \sqrt{n} \rfloor - 1]$. By choice of $T$, the tree $T_0$ is not $B_0$-burnable.

We construct a sequence $(T_i, B_i)$ for $0 \leq i \leq 2k^2 + 2$ where $T_i$ is a $k$-caterpillar and $B_i$ is a set of sparks for $T_i$, such that the sequence is strictly decreasing, i.e. $T_i \subseteq T_{i+1}$ and $B_i \subseteq B_{i+1}$. Before we describe how to construct those sequences, let us sketch how we will obtain the desired conclusion. We prove the following:

**Claim 2.8.** For every $0 \leq i \leq 2k^2 + 2$, the tree $T_i$ is not $B_i$-burnable, and $\sum_{j \in B_i} (2j + 1) \geq |V(T_i)|$.

**Lemma 2.9.** There is some $1 \leq j \leq 2k^2 + 2$ such that $B_j = [1, r]$ for some $r$.

Combining Claim 2.8 and Lemma 2.9 immediately contradicts the minimality of $T$.

Let us now construct the sequence $(T_i, B_i)$, namely how to build for any $i$ the pair $(T_{i+1}, B_{i+1})$ from $(T_i, B_i)$. A key observation for the below construction is that $|B_0| \geq 4k^2 + 5k + 4$ and $|B_{i+1}| = |B_i| - 1$ (except possibly when $i = 2k^2 + 1$), so every $B_i$ satisfies $|B_i| \geq 4k + 3$ (except possibly when $i = 2k^2 + 2$), and the construction is well-defined.

We construct $T_{i+1}$ as a sub-tree of $T_i$ by placing the "largest possible" spark on an extremity of the spine $S_i$ of $T_i$. Throughout these operations we maintain the property that the spine "stays straight" (i.e. $S_{i+1} \subseteq S_i$ for every $i$, where $S_0$ is an arbitrary spine of $T_0 = T$). We denote $v_0^i, v_1^i, \ldots, v_{n^i}^i$ the vertices of the spine $S_i$ of $T_i$, from left to write. We drop the superscript when it is clear from context.

The case of $T_{2k^2 + 2}$ is handled a bit differently. We first discuss smaller cases. To construct $T_{i+1}$, we use the largest spark $p$ of $B_i$ such that $T_i \setminus N^p[v_p]$ has the following properties, which we denote $P$:

- $T_i \setminus N^p[v_p]$ is connected
- There exists a spine $S_{i+1}$ of $T_i \setminus N^p(v_p)$ such that $S_{i+1} \subseteq S_i$

Naturally, we then define $T_{i+1} = T_i \setminus N^p(v_p)$ and $B_{i+1} = B_i \setminus \{p\}$. Claim 2.8 is trivially maintained, as if $|S| \geq 2p + 1$ then $|N^p[v_p]| \geq 2p + 1$, otherwise $|T_{i+1}| = 0$.

The key is to argue that $p$ indeed exists.
Claim 2.10. For any $T'$ and $B'$, let $p_1, \ldots, p_c$ be the $c$ largest sparks in $B'$ in decreasing order. If $p_i$ does not satisfy $P$ for any $i \leq c$, then $\sum_{i \in [2p_{i-k}; 2p_{i+k}]} l(i) \geq c$, where $l(i)$ denotes the depth of the sub-tree rooted on the $i + 1$th leftmost vertex on the spine.

Proof. Let $(v_0, v_1, \ldots)$ denote the nodes on a spine $S'$ of $T'$, from "left" to "right". For every $i$, $l(i)$ denotes the depth of the sub-tree rooted in $v_i$. Suppose the $c$ largest sparks of $B'$, denoted decreasingly $p_1, \ldots, p_c$, do not satisfy $P$, it implies that for every $i \in [1, c]$, at least one of the two conditions holds:

\begin{align}
\exists 0 \leq r_i \leq k - 1, l(2p_i - r_i) &\geq r_i + 1 \tag{1} \\
\exists 0 \leq s_i \leq k - 1, l(2p_i + s_i + 1) &\geq s_i + 1 \tag{2}
\end{align}

Let $R$ (resp. $S$) denotes the set of the every $p_i$ for $i \in [1, c]$ such that $p_i$ satisfies 1 (resp. 2), let also $R_I = \{2a_i - r_i : i \in R\}$ and $S_I = \{2a_i + s_i + 1 : i \in S\}$.

We now prove that $\sum_{i \in R_I \cup S_I} l(i) \geq c$. For any $x \in \mathbb{N}$ consider $R_x = \{i \in R : 2p_i - r_i = x\}$ and $S_x = \{i \in S : 2p_i + s_i + 1 = x\}$. Note first that since every $p_i$ are distinct, it implies that for every $i, i' \in R_x$ (resp. $i, i' \in S_x$) $r_i$ and $r_i'$ (resp. $s_i$ and $s_i'$) are distinct. Note also that for every $i, j \in R_x, S_x$ the parity of $r_i$ is different from the parity of $s_i$ since $2p_i + 1$ and $2p_j$ have different parity. It yields that $l(x) \geq \max_{i, j \in R_x, S_x}(\max(r_i, s_i) + 1) \geq |R_x \cup S_x|$ which implies

$$\sum_{i \in R_I \cup S_I} l(i) \geq c \text{ since } ||[1, c]|| = c.$$ \hfill \Box

A corollary of Claim 2.10 is that for any $i \geq 1$, $T_i$ is well-defined as long as $|B_{i-1}| \geq 4k + 2$, applying the claim on the $2k$ largest sparks of $B_{i-1}$ one can state that either there exists one that satisfying $P$ and therefore $T_i$ is well defined, or there exists an interval $I \in V(T)$ of size smaller than $2(\max(B) - 2k + 2k - 2 \max(B) - 2k)$ such that $\sum_{i \in I} l(i) \geq 2k$ which contradict the minimality of $T$ thanks to Lemma 2.6. By supposition $|B_0| \geq (4k^2 + 5k + 4)$, and, by construction, for any $i \geq 1$, $|B_i| = |B_{i-1}| - 1$, therefore $T_i$ is well defined for $1 \leq i \leq 2k^2 + 1$.

Note also that Claim 2.10 maintains that for $1 \leq i \leq 2k^2 + 1$, $\min(B_i) \geq \max(B_0) - (2k^2 + 1 + 2k) \geq (\max(B_0) + 2k)/2$. Informally this implies that the sparks place at step $i + 1$ is large enough to cover the subtree obtained by the Claim 2.10.

We now prove Lemma 2.9. We suppose that the $B_i$ is not of the form $[1, r]$ for $i \leq 2k^2 + 1$, and study the case $i = 2k^2 + 2$.

Let for any $i \leq 2k^2 + 2$, $P_i = B_i \setminus B_i$, informally $P_i$ is the set of sparks placed before obtaining $T_i$ and let us define the gain as the following sequence: For every $2 \leq i \leq 2k^2 + 1$, $g_i = |T \setminus (S \cup T_i)|$, and prove by induction on $i$ that $g_i \geq \max(B_0) - \min(P_i)$. This is equivalent to $g_i \geq \max(B_0) - (\max(P_i) - 1)$ since $g_i$ is a non decreasing sequence. Otherwise $p < \min(P_i)$, thus $p = \min(P_i)$, let $p' = \min(P_{i-1}) \neq p$, note that by supposition $T_{i-1}$ does not satisfy Lemma 2.9 therefore $B_{i-1} \neq B_{i-1}$ which implies that $p'$ is smaller or equal to the $p - p' + 1$ largest spark of $B_{i-1}$. Claim 2.10 implies $g_{i+1} \geq g_i + p - p' \geq \min(P_{i-1}) + p' - p = \min(P_i)$ which conclude the induction.

Applying the previous result on $g_{2k^2+1}$ we obtain the following inequality $g_{2k^2+1} \geq \max(B_0) - \min(P_{2k^2}) \geq 2k^2$, therefore $2k^2 + \sum_{p \in P_{2k^2+1}} 2p + 1 \geq |T \setminus T_{2k^2+1}|$. We can now construct $T_{2k^2+2}$ by placing the at most $2k$ sparks in $[\min(P_{2k^2+1}), \max(B_0)] \cap B_{2k^2+2}$ not yet placed. We place them such that it minimizes the smallest sub-graph of $T$ connected and
containing every non burned vertices. By doing so we ensure that a spark \( p \) placed like this burns at least \( 2p + 1 - k \) vertices by itself therefore

\[
|T_{2k^2 + 1}| \leq |T_{2k^2 + 1}| - \left( \sum_{p \in [\min(P_{2k^2 + 1}), \max(B_0)] \cap B_{2k^2 + 1}} 2p + 1 - k \right)
\]

\[
\leq |T_{2k^2 + 1}| - \left( \sum_{p \in [\min(P_{2k^2 + 1}), \max(B_0)] \cap B_{2k^2 + 1}} 2p + 1 \right) + 2k^2
\]

\[
\leq \left( \sum_{p \in B_{2k^2 + 1}} 2p + 1 \right) - \left( \sum_{p \in [\min(P_{2k^2 + 1}), \max(B_0)] \cap B_{2k^2 + 1}} 2p + 1 \right) + 2k^2 - 2k^2
\]

\[
\leq \sum_{p \in B_{2k^2 + 1} \setminus [\min(P_{2k^2 + 1}), \max(B_0)]} 2p + 1
\]

Since \( B_{2k^2 + 1} \setminus [\min(P_{2k^2 + 1}), \max(B_0)] = [0, \min(P_{2k^2 + 1}) - 1] \), Lemma 2.9 holds for \( i = 2k^2 + 2 \).

\[ \square \]

**Corollary 2.11.** For every \( k \)-caterpillar \( T \), we have \( b(T) \leq \sqrt{|V(T)| + 20k^2} \).

Indeed, for every \( k \)-caterpillar \( T \) on at most \((4k^2 + 5k + 4)^2\) vertices, we have \( b(T) \leq \sqrt{2 \cdot (4k^2 + 5k + 4)} \leq 20k^2 \) using Lemma 2.1 and a very coarse computation.

With this in mind, the same proof as for Theorem 1.6 yields that induction always goes through when the desired statement \( b(T) \leq \lceil \sqrt{n} \rceil \) is weakened to \( b(T) \leq \lfloor \sqrt{n} + 400k^2 \rfloor \).

The burning graph conjecture is asymptotically true for \( k \)-caterpillars. We can in fact prove that this extends to burning sets with similar arguments, though we do not go into details. We can also do a finer analysis to get the extra polynomial from \( O(k^4) \) to \( O(k^2) \), though it did not seem to us enough to justify an extra few pages of proofs.

### 2.2 Better Approximation

The role of this section is to prove Theorem 1.7. In order to do so we prove first the following lemma.

**Lemma 2.12.** For any tree \( T \) and \( B = \{b_0, \ldots, b_k\} \) with \( k \geq 3 \), either \( \text{diam}(T) \leq b_k \) or there exists \( i \in \lfloor [k/2]; k \rfloor \) and \( X \subset V(T) \) such that \( |X| \geq b_i + \lfloor \frac{k}{2} \rfloor + 1 \), the set \( X \) has diameter at most \( b_i \) in \( T \), and \( T \setminus X \) is connected.

**Proof.** By contradiction, let us consider a tree \( T \) and a set \( B \) that do not satisfy the Lemma 2.12. We root \( T \) in some arbitrary vertex \( r \), and consider \( P \) a longest path \((r_0, r_1, \ldots, r_{\ell})\) of \( T \), where \( r_\ell = r \). First note that if \( \ell \leq b_k \), then \( \text{diam}(T) \leq b_k \), which is one of the two desired outcomes. We assume from now on \( \ell > b_k \) and \( |V(T)| \geq 2b_k + 2 \).

For \( 0 \leq i \leq b_k \), we define \( T(i) \) to be the subtree of \( T \) rooted in \( r_i \), containing all descendants of \( r_i \) including itself. In particular, \( T(i) \subset T(i+1) \) for any \( i < \ell \). If \( i \in B \), we also define \( T'(i) \) to be the largest \( T(j) \) such that \( T(j) \subset B_i(r_i) \), and \( T''(i) \) to be \( T(b_k) \cap B_i(r_b) \). Note that for every \( i \in B \), the tree \( T'(i) \) satisfies all conditions of Lemma 2.12 except possibly for the size condition.
Every \( i \in \lfloor [k/2] ; k \rfloor \) is a smoulder if the spark \( b_i \) does not satisfy our induction purposes, i.e. \( |V(T'(b_i))| \leq b_i + \lfloor \frac{k}{2} \rfloor \). If any \( \lfloor \frac{k}{2} \rfloor \leq i \leq k \) is not a smoulder, we consider \( X = V(T'(b_i)) \) and note that all desired properties are satisfied. Therefore, we may assume that every \( \lfloor \frac{k}{2} \rfloor \leq i \leq k \) is a smoulder. We prove that consecutive smoulders give a lower bound on \( |T(b_k)| \).

**Claim 2.13.** For every \( \lfloor \frac{k}{2} \rfloor \leq i \leq k \), there is \( w_i \in T(b_k) \setminus (T(b_i) \cup P) \) such that \( d(w_i, r_{b_i}) = b_i + 1 \).

**Proof of Claim 2.13.** We proceed by induction on \( i \). Note that when \( i = \lfloor \frac{k}{2} \rfloor \), we have \( r_0, r_1, \ldots, r_{b_i + \lfloor \frac{k}{2} \rfloor} \in T''(i) \) as \( b_k \geq b_i + (k - i) \). Hence \( |T''(b_i)| \geq b_i + \lfloor \frac{k}{2} \rfloor + 1 \). Since \( i \) is a smoulder, we derive that \( T''(b_i) \subseteq T''(b_i) \). Therefore, the set \( T''(b_i) \) disconnects \( T \). Let \( w_i \) be a vertex in \( T(b_k) \setminus T''(b_i) \) that is at distance \( b_i + 1 \) of \( r_{b_i} \) and that is in the component that does not contain any vertex of \( P \) (such a vertex exists as every vertex in \( T(b_k) \) within distance \( b_i \) of \( r_{b_i} \) belongs to \( T''(b_i) \)). To handle the case \( i > \lfloor \frac{k}{2} \rfloor \), we observe that \( w_j \in T''(b_i) \) for every \( j < i \). Therefore, \( |T''(b_i)| \geq b_i + 1 + (k - i) + (i - \lfloor \frac{k}{2} \rfloor) \geq b_i + 1 + \lfloor \frac{k}{2} \rfloor \), and the same argument yields the existence of \( w_i \). \( \square \)

We consider the set \( W = \{ w_{\lfloor \frac{k}{2} \rfloor}, \ldots, w_k \} \) obtained by applying Claim 2.13 on every eligible \( i \). Note that \( w_{i'} \) belongs to \( T''(i) \) whenever \( i > i' \). Consequently, the set \( W \) contains \( k - \lfloor \frac{k}{2} \rfloor + 1 = \lfloor \frac{k}{2} \rfloor + 1 \) distinct elements. By definition, \( w_i \in T(b_k) \setminus P \) for every \( i \). Hence \( |T(b_k)| \geq b_k + 1 + \lfloor \frac{k}{2} \rfloor + 1 = b_k + 2 \lfloor \frac{k}{2} \rfloor + b_k + k \), a contradiction to the fact that \( k \) is a smoulder. \( \square \)

**Proof of Theorem 1.7.** Let \( T \) be a tree on \( n \) vertices and let \( B = \lfloor 0; p \rfloor \). By repeatedly applying Lemma 2.12, the tree \( T \) is \( B \)-burnable unless \( n > \sum_{i=0}^{p} (i) + \sum_{j=0}^{p} (\lfloor \frac{j}{2} \rfloor + 1) \), i.e. \( n > \frac{p}{3} (p + 1) + \lfloor \frac{k}{2} \rfloor + 1 \). Since all elements involved are integers, we may assume \( n \geq \frac{p}{3} (p + 1) + \lfloor \frac{k}{2} \rfloor + 2 \).

Let \( p = \lfloor \sqrt{\frac{2}{3} n} \rfloor + 1 \), and write \( \sqrt{\frac{2}{3} n} = 2s + r \), where \( s \in \mathbb{N} \) and \( 0 \leq r < 2 \). Regardless of whether \( r = 0, 0 < r < 1, r = 1 \) or \( 1 < r < 2 \), this yields a contradiction. \( \square \)

### 2.3 Graphs of minimum degree 3

Using Theorem 1.7, we are now ready to argue that Conjecture 1.1 almost holds for graphs of minimum degree at least 3, and obtain Theorems 1.2 and 1.3. We use the following two very convenient theorems.

**Theorem 2.14.** [KW91] Every connected graph on \( n \) vertices each of degree at least 3 admits a spanning tree with at least \( \frac{n}{4} + 1 \) leaves.

**Theorem 2.15.** [GW92] Every connected graph on \( n \) vertices each of degree at least 4 admits a spanning tree with at least \( \frac{2n + 8}{5} \) leaves.

**Proof of Theorem 1.2.** Let \( G \) be a connected graph on \( n \) vertices each of degree at least 3. By Theorem 2.14, there is a spanning tree \( T \) with at least \( \frac{n}{4} + 1 \) leaves. Let \( T' \) be the tree obtained from \( T \) by deleting all leaves. Note that \( b(T) \leq b(T') + 1 \). By Theorem 1.7, we have \( b(T') \leq \lfloor \sqrt{\frac{2}{3} \cdot (\frac{2}{3} n - 1)} \rfloor + 1 \). We derive that \( b(G) \leq b(T) \leq \lfloor \sqrt{n} \rfloor + 2 \). \( \square \)

**Proof of Theorem 1.3.** Let \( G \) be a connected graph on \( n \) vertices each of degree at least 4. By Theorem 2.15, there is a spanning tree \( T \) with at least \( \frac{2n + 8}{5} \) leaves. Let \( T' \) be the tree obtained from \( T \) by deleting all leaves. Note that \( b(T) \leq b(T') + 1 \). By Theorem 1.7, we have \( b(T') \leq \lfloor \sqrt{\frac{2}{3} \cdot (\frac{2n + 2}{5})} \rfloor + 1 \). We derive that \( b(G) \leq b(T) \leq \lfloor \sqrt{n} \rfloor \) when \( n \) is large enough. \( \square \)
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