SOME CORRELATION FUNCTIONS
OF MINIMAL SUPERCONFORMAL
MODELS COUPLED TO SUPERGRAVITY

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ABSTRACT

We compute general three-point functions of minimal superconformal models
coupled to supergravity in the Neveu-Schwarz sector for spherical topology thus
extending to the superconformal case the results of Goulian and Li and of Dotsenko.

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1. Introduction. The successes of the double scaling limit [1] and its connection with the KP-hierarchy [2] in the computation of correlation functions of minimal models coupled to 2D-gravity (see for example [3]) prompted a good deal of activity in trying to reproduce the same results directly in the continuum limit (for details and references see [4,5]).

The original approach to the coupling of Conformal Field Theories (CFT) to gravity appeared in [6,7]. Using the light-cone gauge these authors were able to exactly compute the gravitational dimensions of the gravitationally dressed primary fields. These results were obtained subsequently in the conformal gauge [8,9] which also allowed the generalization of some of the results to non-spherical topologies. Several methods have been suggested for the computation of correlation functions in Liouville theory coupled to minimal conformal models [10,11] in the continuum limit in order to reproduce the results of matrix models. The first proposal consisted of an analytic continuation in the value of the central charge of the Virasoro algebra [12]. This technique was further explored in [13,14]. The second proposal, closely related to the previous one, was a generalization of the Coulomb gas technique [15] to the Liouville case in order to include negative numbers of screening charges [16]. So far these techniques have only allowed a direct computation of generic one-, two-, and three-point functions and there is no clear procedure known on how to extend the same ideas to higher point functions except in some special cases. The second technique was used in [17] to clarify some issues concerning the fusion rules in the presence of 2D-gravity.

The basic technical problem in most of these computations is the evaluation of some integrals which also appear in the computation of the structure constants of the minimal conformal models which was done in [15]. In the case of $N = 1$ Superconformal Fields Theories [18,11,19] although there is available a Coulomb gas formulation [20], there are only partial results with respect to the structure constants of the operator algebra (see for example [21, 22, 23]). The generalization of the results in [15] to the minimal $N = 1$ models in the Neveu-Schwarz sector has been carried out in [24]. As a simple application of the results in this paper we
can calculate the one-, two- and three-point functions of $N = 1$ minimal models coupled to $2D$-supergravity on surfaces of spherical topology. The derivation of the supergravitational dressing and dimensions was carried out in the light-cone gauge in [25] and in the superconformal gauge in [26]. We follow the proposal in [16], although one could equally well extend the ideas in [12]. These computations have some interest because there is as yet no analogue of the matrix model formulation for superconformal theories coupled to supergravity (see nevertheless [27]).

2. Formulation of the problem. In the Coulomb gas formulation of the minimal superconformal models [20] the matter energy-momentum tensor is built in terms of a free massless scalar superfield $X(Z)$

$$T_M(Z) = -\frac{1}{2} : DX\partial X : + \frac{i}{2} \alpha_0 D\partial X$$

where $Z = (z, \theta)$ represents a point on the superplane. It is convenient to introduce two quantities $\alpha_+, \alpha_-$ satisfying $\alpha_+ + \alpha_- = \alpha_0, \alpha_+\alpha_- = -1$. Then the central charge and the screening charges take the form

$$\hat{c} = 1 - 2\alpha_0^2, \quad J_\pm = e^{i\alpha_\pm X(Z)}$$

The background charge $\alpha_0$ changes the dimension of a super-vertex operator $e^{i\alpha X}$ from $\alpha^2/2$ to $\alpha(\alpha - \alpha_0)/2$. The minimal superconformal models are obtained for special values of $\alpha_0, \alpha_+, \alpha_-$. Take two integers $p', p; p' > p$ such that 1) $p' - p \equiv 0(\text{mod} \ 2)$, 2) if they are odd, then they are coprime, or 3) if both are even then $p'/2, p/2$ are coprime. In the $N = 1$ minimal models we have

$$\alpha_+ = \sqrt{\frac{p'}{p}}, \quad \alpha_- = -\sqrt{\frac{p}{p'}}, \quad \hat{c} = 1 - 2\left(\frac{p' - p}{pp'}\right)^2$$

The primary fields are represented as vertex operators in the NS sector

$$\Psi_{m', m} = e^{i\alpha_{m', m} X}, \quad m' - m \equiv 0(\text{mod} \ 2)$$
\[ \alpha_{m',m} = \frac{1}{2}(1 - m')\alpha_+ + \frac{1}{2}(1 - m)\alpha_- \] (4)

and in the Ramond sector by

\[ \Psi_{m',m} = \sigma e^{i\alpha_{m',m}X} \quad m' - m \equiv 1 \mod 2 \] (5)

Where \( \sigma \) is a spin field of the matter sector. The superconformal dimensions are

\[ h_{m',m} = \frac{1}{8pp'}[(mp' - m)p]^2 - (p' - p)^2] + \frac{1}{32}(1 - (-1)^{m'-m}) \] (6)

We will be exclusively concerned with correlators of NS fields in this paper. The range of \( m', m \) is \( 1 \leq m' \leq p' - 1, 1 \leq m \leq p - 1, \quad mp' - m'p \geq 0. \)

In the super-Liouville sector, after gauge fixing in the superconformal gauge [26] we can describe the Liouville part of the theory locally with an energy-momentum tensor expressible in terms of a real scalar superfield \( \Phi \):

\[ T_L(Z) = -\frac{1}{2}D\Phi \partial \Phi + \frac{Q}{2}D\partial \Phi \] (7)

The superconformal dimension of a vertex operator in the NS sector \( e^{\beta \Phi} \) is \(-\beta(\beta - Q)/2\). The Liouville background charge is determined by \( \alpha_0 \) and the ghost contributions to be

\[ Q^2 = \frac{9 - \tilde{c}}{2} = 4 + \alpha_0^2 \] (8)

The dressing of a NS field \( e^{i\alpha X} \) is given by

\[ e^{i\alpha X} e^{\beta \Phi}, \quad \frac{1}{2}\alpha(\alpha - \alpha_0) - \frac{1}{2}\beta(\beta - Q) = \frac{1}{2} \] (9)

making the dressed field into a \((1/2, 1/2)\)-form which can be integrated over the surface without any reference to the conformal factor of the background metric.
The two solutions to the quadratic equation in (9) are

$$\beta = \frac{Q}{2} \pm |\alpha - \frac{\alpha_0}{2}|$$  \hspace{1cm} (10)

The microscopic branch (see [4] for details) is obtained by choosing the minus sign. With this sign the superconformal dimension agrees with the classical dimension in the classical limit $\hat{c} \to -\infty$. For (4) we obtain

$$\beta_{m',m} = \frac{p + p' - |mp' - m'p|}{2\sqrt{pp'}}$$  \hspace{1cm} (11)

As with the conformal case we can introduce two quantities, $\beta_{\pm}$ satisfying

$$\beta_+ + \beta_- = Q \quad \beta_+ \beta_- = 1$$  \hspace{1cm} (12)

They are related to $\alpha_+, \alpha_-$ by

$$\alpha_+ = \beta_+ \quad \alpha_- = -\beta_-$$  \hspace{1cm} (13)

It is useful to notice that in super-Liouville theory the cosmological constant couples to the operators $e^{\beta_+ \Phi}$. Formally we can introduce two screening charges in the Liouville sector

$$U_\pm = e^{\beta_\pm \Phi}$$  \hspace{1cm} (14)

If we are interested in general properties of $n$-point correlators we first note that the area constraint in the conformal case is replaced here by a ”length” constraint because we are counting volumes of $N = 1$ supersurfaces. For arbitrary values of $p', p$ the field of lowest superconformal dimension is not necessarily the identity. Let $\Psi_{\text{min}}$ be the field of lowest dimension, and let $\beta_{\text{min}}$ be the corresponding dressing exponent. The cosmological constant $\mu$ should be taken as the coefficient of the operator $\Psi_{\text{min}} e^{\beta_{\text{min}} \Phi}$. If we concentrate for the time being on the super-Liouville contribution to the $n$-point function, we can obtain a useful expression
for $\langle \prod_i e^{\beta_i \Phi(Z_i)} \rangle$ by following step by step the arguments in the conformal case ([12], see also [28],[5]). First we introduce a $\delta$-function constraint fixing the length and integrate over lengths from 0 to $\infty$. Second, we separate the constant piece from $\Phi(Z_i)$ to explicitly solve the $\delta$-function, $\Phi(Z_i) = \phi_0 + \tilde{\Phi}(Z_i)$. For a surface of genus $h$, the result is

$$\langle \prod_i e^{\beta_i \Phi(Z_i)} \rangle = \frac{1}{\beta_{\min}} \mu^s \Gamma(-s) \langle \tilde{L}^s \prod_i e^{\beta_i \tilde{\Phi}(Z_i)} \rangle S_L(0) L(\tilde{\Phi})$$

and

$$s = -\frac{1}{\beta_{\min}^2} (\sum_i \beta_i - Q(1 - h)) \quad (15)$$

$\tilde{L}$ is the reference zweibein and the integration over supermoduli parameters is not explicitly exhibited. The expectation value in (15) is computed in terms of the free super-Liouville action with a term representing the background charge $Q \tilde{R} \tilde{\Phi}$ ($\tilde{R}$ is the curvature associated to the superframe $\tilde{E}$; see [26] for details). Note parenthetically that we can read off from the $\mu$-dependence in (15) the string susceptibility and the gravitational dimensions of the fields in the theory. If $s$ were a positive integer we could easily evaluate (15) using free field techniques. This is the method proposed in [12]. One first computes the integral for $s$ an integer and then one analytically continues to arbitrary values. This prescription could also be applied to our case. We choose however to follow [16]. The idea is to treat the Liouville and matter sectors in the same way using the Coulomb gas formulation. We should make a few remarks before we proceed. When the matter theory is unitary, the field of lowest dimension is the identity operator $\Psi_{\min} = 1$. In the computation we will describe below for the general case, we take the cosmological constant by definition to be the coefficient of the dressed identity operator. In the non-unitary case this corresponds to a fine tuning of the coupling of all the
operators of negative dimension. The derivation of (15) goes through again, with the only change that $\beta_{\text{min}}$ is replaced by $\beta_-$. We can introduce vertex operators similar to (4):

$$
L_{m',m} = e^{\beta_{m',m} \Phi} \\
\beta_{m',m} = \frac{1 - m'}{2} \beta_- + \frac{1 - m}{2} \beta_+
$$

Since the dressing exponent of (4) is

$$
\beta(\alpha_{m',m}) = \frac{Q}{2} - \frac{1}{2} \beta_-(m' - \rho m)
$$

we can represent $\beta(\alpha_{m',m})$ in terms of $\beta_{m',m}$. There are two cases to distinguish:

$$
m' > \rho m, \quad \beta(\alpha_{m',m}) = \frac{Q}{2} - \frac{1}{2} \beta_-(m' - \rho m) = \beta_{m',-m} \\
m' < \rho m, \quad \beta(\alpha_{m',m}) = \frac{Q}{2} - \frac{1}{2} \beta_-(m' + \rho m) = \beta_{-m',m}
$$

We can assign a chirality $\chi$ to a primary field $\Psi_{m',m}$. $\chi = 1$ if $m' > \rho m$; $\chi = -1$ if $m' < \rho m$. To guarantee the choice of the microscopic branch in (10) there are two possible dressings

$$
A^+_{m',m} = \Psi^+_{m',m} L_{m',-m} \quad \chi = 1 \\
A^-_{m',m} = \Psi^-_{m',m} L_{-m',m} \quad \chi = -1
$$

The dressed fields have dimensions $(1/2, 1/2)$. Ignoring for the moment chirality labels, we are interested in

$$
\langle A_S A_N A_M \rangle = \int \prod_{1}^{3} d^2 Z_1 \hat{E}(A_S(Z_3)A_N(Z_2)A_M(Z_1))
$$

using $SL(2|1)$ invariance we can fix $Z_1 = (\infty, \infty \eta), Z_2 = (1, 0), Z_3 = (0, 0)$. Dividing out the $SL(2|1)$ volume we are left with a single integration over the odd
variable $\eta$:

$$
\langle A_S A_N A_M \rangle = \int d^2 \eta \langle A_S(0,0)A_N(1,0)A_M(\infty,\infty \eta) = \\
\int d^2 \eta \langle \Psi_S(0,0)\Psi_N(1,0)\Psi_M(\infty,\infty \eta)\rangle \langle L_S(0,0)L_N(1,0)L_M(\infty,\infty \eta) \rangle
$$

(21)

The matter correlation function is evaluated using the physical structure constants [24]

$$
\langle \Psi_S(0,0)\Psi_N(1,0)\Psi_M(\infty,\infty \eta) \rangle = D_{SNM}^{even} + D_{SNM}^{odd} |\eta|^2
$$

(22)

The contribution will come from $D_{\text{even or odd}}$ depending on the number of screening charges needed in the evaluation of (22) in the Coulomb gas formulation. For $\langle A_S A_N A_M \rangle$ to be non-vanishing, the matter and Liouville parts must have opposite Grassmann parity. This will be verified later by counting screenings in both cases. Using capital letters to label pairs of indices $(M = (m', m), \text{etc})$, the physical structure constants $D$ are defined in terms of the symmetric structure constants in the Coulomb gas definition of the correlators

$$
C_{SNM} = \langle \Psi_S \Psi_N \Psi_M(\text{screenings}) \rangle
$$

(23)

Defining the conjugate fields $\Psi_{\overline{S}} \equiv \Psi_{-S}, (s', \overline{s}) = (-s', -s)$, the asymmetric structure constants are defined according to

$$
C_{SN}^{M} = \langle \Psi_S \Psi_N \Psi_{\overline{M}}(\text{screenings}) \rangle
$$

(24)

In analogy with [15] it is shown in [24] for the superconformal case that the physical structure constants $D_{SNM}$ are simply related to $C_{SNM}$. There are four equivalent ways of writing $D$ in terms of $C$:

$$
D_{SNM} = (C_{SN}^M C_{MN}^N a_N)^{1/2} \\
= (a_N a_S a_{-1}^M)^{1/2} C_{SN}^M \\
= (a_N a_M a_{-1}^S)^{1/2} C_{MN}^S \\
= \Omega^{-1} (a_N a_S a_{-1}^M)^{1/2} C_{SNM} \\
a_S = (C_{SS}^S)^{-1} \quad \Omega = \frac{\rho}{4} \Delta \left( \frac{\rho - 1}{2} \right) \Delta \left( \frac{\rho' + 1}{2} \right)
$$

(25)
Let \( Q = (q', q) \) be the number of screenings in the matter sector. \( q' \) (resp. \( q \)) counts the number of \( J_- \) (resp. \( J_+ \)) screening charges in the correlator. The parity of \( q' + q \) is the same for the four representations in (25). The number of screenings in the Liouville sector depends on \( q', q \). Consider first the case with all chiralities \( \chi = -1, \langle - - - \rangle \):

\[
\langle L_{-s',s}(0,0) L_{-m',m}(1,0) L_{-n',n}(\infty, \infty \eta) \rangle
\]

(26)

The Liouville screenings are

\[
q'_L = \frac{-s' - n' - m' - 1}{2} = -q' - 1
\]

\[
q_L = \frac{s + n + m - 1}{2} = q
\]

hence the parity of the Liouville sector

\[
q'_L + q_L = -q' - 1 + q \equiv q' + q - 1 \pmod{2}
\]

is opposite to the matter parity as required for the non-vanishing of three-point correlators. For other chiralities it is easy to show that the same conclusion holds. The three-point functions can be expressed as products of ratios of \( \Gamma \)-functions, with the products ranging up to \( q-1 \) or \( q'-1 \). For the matter sector these numbers are positive. In the Liouville case however one of them is negative. The analytic continuation advocated in [16] is to use the definition

\[
\prod_{i=1}^{-l'-1} f(i) = \prod_{i=0}^{l'} \frac{1}{f(-i)} \quad l' > 0
\]

(27)

This together with the results of [24] is all we need to write the explicit form of the three-point correlators. Before we write the results, we have to determine the \( \mu \)-dependence of the correlators. The coefficient of \( U_- \) in the action is \( \mu \). However, the coefficient of \( U_+ \) in the Liouville action is taken as in [16] to be determined by its gravitational dimension, and it is given by \( \mu^0 \). This guarantees that the power of \( \mu \) in front of the correlators is exactly \( s \) as defined in (15).
3. *Computations.* There are four cases depending on the chirality: \(\langle - - - \rangle\), \(\langle +- - \rangle\), \(\langle + + - \rangle\), \(\langle + + + \rangle\). The computations are very similar in all four cases. Although the matter and Liouville correlators are separately very cumbersome, when both of them are put together almost everything cancels leaving only a set of terms similar to Polyakov leg factors \([29]\). Depending on the chirality of the matter fields the cancellation is made more evident by choosing one of the representations \((25)\).

We list now each chirality case together with the number of screenings involved in the Liouville and matter sectors. Take \(k', k\) to be the number of \(-, +\) screening operators in the Liouville sector, and \(l', l\) in the matter sector. The four cases to be considered are

i). \(\langle - - - \rangle\)

\[
\begin{align*}
  k' &= \frac{1}{2}(-s' - n' - m' - 1) = -l_1' - 1 \\
  l_1' &= \frac{1}{2}(s' + n' + m' - 1) \\
  k &= \frac{1}{2}(s + n + m - 1) = l_1 \\
  l_1 &= \frac{1}{2}(s + m + n - 1)
\end{align*}
\]

(28)

and in performing the cancellation it is best to take \(D_{SNM} \sim C_{SNM} \).

ii). \(\langle + + - \rangle\)

\[
\begin{align*}
  k' &= \frac{1}{2}(s' + n' - m' - 1) = l_2' \\
  l_2' &= \frac{1}{2}(s' + n' - m' - 1) \\
  k &= \frac{1}{2}(-s - n + m - 1) = -l_2 - 1 \\
  l_2 &= \frac{1}{2}(s + m + n - 1)
\end{align*}
\]

(29)

and we take \(D_{SNM} \sim C_{MN}^M\).

iii). \(\langle + - - \rangle\)

\[
\begin{align*}
  k' &= \frac{1}{2}(s' - n' - m' - 1) = -l_3' - 1 \\
  l_3' &= \frac{1}{2}(-s' + n' + m' - 1) \\
  k &= \frac{1}{2}(-s + n + m - 1) = l_3 \\
  l_3 &= \frac{1}{2}(-s + m + n - 1)
\end{align*}
\]

(30)

and \(D_{SNM} \sim C_{NM}^S\).
iv). \( (+++) \)

\[
k' = \frac{1}{2}(s' + n' + m' - 1) = l'_1
\]

\[
k = \frac{1}{2}(-s - n - m - 1) = -l_1 - 1
\]

and \( D_{SNM} \sim C_{SNM}. \)

The matter three-point function up to irrelevant constants can be represented as a surface integral

\[
\lim_{R \to \infty} R^{4\Delta(\alpha_M)} \langle V_{\alpha_M}(R, R\eta) V_{\alpha_N}(1, 0) V_{\alpha_S}(0, 0) \int \prod_{i=1}^{l'} d^2 Z'_i V_{\alpha_{-}}(Z'_i, \overline{Z'_i}) \int \prod_{i=1}^{l} d^2 Z_i V_{\alpha_{+}}(Z_i, \overline{Z_i}) \rangle = \lim_{R \to \infty} R^{4\Delta(\alpha_M)} \int \prod_{i=1}^{l'} d^2 Z'_i \int \prod_{i=1}^{l} d^2 Z_i |R - 1|^{2\alpha_M\alpha_N} |R|^{2\alpha_M\alpha_S} |R - z_i - R\eta \theta_i|^2 \alpha_{M+} \\
|R - z'_i - R\eta \theta'_i|^{2\alpha_{M-}} \prod_{i=1}^{l'} |1 - z'_i|^{2\alpha_{N-}} |z'_i|^{2\alpha_{S-}} \prod_{i<j}^{l'} |z'_i - z'_j - \theta'_i \theta'_j|^2 \alpha_{-2} \\
\prod_{i=1}^{l} |1 - z_i|^{2\alpha_{N+}} |z_i|^{2\alpha_{S+}} \prod_{i<j}^{l} |z_i - z_j - \theta_i \theta_j|^{2\alpha_{-2}} \prod_{i,j}^{l,l'} |z_i - z'_j - \theta_i \theta'_j|^{-2}
\]

(32)

with the screening condition

\[
\alpha_M + \alpha_N + \alpha_S + l\alpha_+ + l'\alpha_- = \alpha_0 \quad \rho = \alpha_+^2 = \rho^{-1} = \alpha_{-2} \\
\alpha_S = \alpha_{s', s}, \ldots
\]

Simple manipulations in (32) yield

\[
\int \prod_{i=1}^{l'} d^2 Z'_i \int \prod_{i=1}^{l} d^2 Z_i \xi \prod_{i=1}^{l'} |1 - z'_i|^{2b} \prod_{i<j}^{l'} |z'_i - z'_j - \theta'_i \theta'_j|^{2\rho'} \\
\prod_{i=1}^{l} |1 - z_i|^{2a} \prod_{i<j}^{l} |z_i - z_j - \theta_i \theta_j|^{2\rho} \prod_{i,j}^{l,l'} |z_i - z'_j - \theta_i \theta'_j|^{-2}
\]

(33)
where

\[ \xi = |1 - \alpha_M \alpha_+ \sum_i \eta \theta_i - \alpha_M \alpha_- \sum_i \eta \theta_i'|^2 \]

\[ a' = \alpha_- \alpha s', s \quad a = \alpha_+ \alpha s', s \]

\[ b' = \alpha_- \alpha n', n \quad b = \alpha_+ \alpha n', n \]

\[ c' = \alpha_- \alpha m', m \quad c = \alpha_+ \alpha m', m \]

The integral (33) is given by the structure constants \( C_{SNM} = C_{SN}^{-M} \) which can be found in [24]:

\[
(const.) \rho^{2l'} \left( \frac{\rho'}{2} \right)^{2M_i'} \left( \frac{\rho}{2} \right)^{2M_i + 2M'_{l+1}} \]

\[
\prod_{1}^{l} \Delta(-l' + \frac{\rho' + 1}{2} i + M_i) \prod_{1}^{l'} \Delta(\frac{\rho' + 1}{2} i - M_i')
\]

\[
\prod_{0}^{l-1} \Delta(1 + a + \frac{\rho - 1}{2} i + M_i) \Delta(1 + b + \frac{\rho - 1}{2} i + M_i)
\]

\[
\prod_{0}^{l-1} \Delta(-a - b - \rho(l - 1) + l' + \frac{\rho - 1}{2} i + M_i)
\]

\[
\prod_{0}^{l-1} \Delta(1 + a' + \frac{\rho' - 1}{2} i + M_i') \Delta(1 + b' + \frac{\rho' - 1}{2} i + M_i')
\]

\[
\Delta(-a' - b' - \rho'(l' - 1) + l + \frac{\rho' - 1}{2} i + M_i')
\]

where

\[ M_i' = \lfloor \frac{i}{2} \rfloor \quad M_i = -l' + \lfloor \frac{l' + i}{2} \rfloor \]

with \( [x] = \) integer part of \( x \) and \( \Delta(x) = \Gamma(x)/\Gamma(1 - x) \). Furthermore from the screening conditions we have

\[ -a' - b' - \rho'(l' - 1) + l = 1 + c' \]

\[ -a - b - \rho(l - 1) + l' = 1 + c \]
The same techniques apply to the Liouville part

$$\lim_{R \to \infty} R^{4\Delta(M)} \langle V_{M}(R, R\eta) V_{S}(1, 0) V_{S}(0, 0) \rangle \int \prod_{i=1}^{k'} d^{2}Z_{i}^{'} \; V_{-}(Z_{i}', \overline{Z}_{i}') \int \prod_{i=1}^{k} d^{2}Z_{i} \; V_{+}(Z_{i}, \overline{Z}_{i})$$

$$= \int \prod_{i=1}^{k'} d^{2}Z_{i}^{'} \int \prod_{i=1}^{k} d^{2}Z_{i} \; \prod_{i=1}^{k'} |z_{i}'|^{2a} |1 - z_{i}'|^{2b} \prod_{i<j}^{k'} |z_{i}' - z_{j}' - \theta_{i}' \theta_{j}'|^{-2\rho'}$$

$$\prod_{i=1}^{k} |z_{i}|^{2a} |1 - z_{i}|^{2b} \prod_{i<j}^{k} |z_{i} - z_{j} - \theta_{i} \theta_{j}|^{-2\rho} \prod_{i,j}^{k,k'} |z_{i} - z_{j}' - \theta_{i}' \theta_{j}'|^{-2}$$

(35)

where now

$$a' = -\beta_{-}\beta_{S} \quad a = -\beta_{+}\beta_{S}$$
$$b' = -\beta_{-}\beta_{N} \quad b = -\beta_{+}\beta_{N}$$
$$c' = -\beta_{-}\beta_{M} \quad c = -\beta_{+}\beta_{M}$$

and $k', k$ depend on the chiralities. This integral is as in the matter sector except for the fact that $\rho, \rho'$ are changed into $-\rho, -\rho'$. We finally have

$$(\text{const.})(-\rho)^{2k + 2M_{\rho}'} \left(\frac{-\rho'}{2}\right)^{2M_{\rho}'} \left(\frac{-\rho}{2}\right)^{2N_{\rho} + 2M_{\rho}'+1}$$

$$\prod_{i=1}^{k} \Delta(-k' - \frac{\rho - 1}{2} - N_{i}) \prod_{i=1}^{k'} \Delta(-\frac{\rho - 1}{2} + M_{i}')$$
$$\prod_{i=0}^{k-1} \Delta(1 + a - \frac{\rho + 1}{2} - i + N_{i}) \Delta(1 + b - \frac{\rho + 1}{2} - i + N_{i})$$
$$\Delta(-a - b + \rho(k - 1) + k' - \frac{\rho + 1}{2} - i + N_{i})$$
$$\prod_{i=0}^{k'-1} \Delta(1 + a' - \frac{\rho' + 1}{2} i + M_{i}') \Delta(1 + b' - \frac{\rho' + 1}{2} i + M_{i}')$$
$$\Delta(-a' - b' + \rho'(k' - 1) + k - \frac{\rho' + 1}{2} i + M_{i}')$$

(36)
Using (27) and standard properties of Γ-functions we obtain after some tedious computations

i). \( \langle - - - \rangle \)

\[
\langle \Psi_{-m',m}(\infty, \infty \eta)\Psi_{-n',n}(1, 0)\rangle \langle L_{-m',m}(\infty, \infty \eta)L_{-n',n}(1, 0)\rangle = \mu^s C_1 \Delta(1 - \frac{s}{2} + \frac{s'}{2} \rho') \Delta(1 - \frac{n}{2} + \frac{n'}{2} \rho') \Delta(1 - \frac{m}{2} + \frac{m'}{2} \rho')
\]

\[ (37) \]

ii). \( \langle + + - \rangle \)

\[
= \mu^s C_2 \Delta(1 - \frac{s'}{2} + \frac{s}{2} \rho) \Delta(1 - \frac{n'}{2} + \frac{n}{2} \rho) \Delta(1 + \frac{m'}{2} + \frac{m}{2} \rho)
\]

\[ (38) \]

iii). \( \langle + - - \rangle \)

\[
= \mu^s C_3 \Delta(1 + \frac{s}{2} - \frac{s'}{2} \rho') \Delta(1 - \frac{n}{2} + \frac{n'}{2} \rho') \Delta(1 - \frac{m}{2} + \frac{m'}{2} \rho')
\]

\[ (39) \]

iv). \( \langle + + + \rangle \)

\[
= \mu^s C_4 \Delta(1 - \frac{s'}{2} + \frac{s}{2} \rho) \Delta(1 - \frac{n'}{2} + \frac{n}{2} \rho) \Delta(1 - \frac{m'}{2} + \frac{m}{2} \rho)
\]

\[ (40) \]

There is a common infinite factor in \( C_i \) which can be absorbed in the normalization of the correlation function. Recall also that \( s \) is given by (15) but with \( \beta_{\min} \) replaced by \( \beta_- \). As a particular application, and to compare with the results in the conformal case [12, 16] we consider the unitary case \( p' = p + 2 \) and diagonal
fields $A_{m,m}$. It is convenient to evaluate a combination of correlators where the dependence on $\mu$ cancels

$$\frac{\langle A_{m_1,m_1}A_{m_2,m_2}A_{m_3,m_3}\rangle^2 Z}{\langle A_{m_1,m_1}\rangle\langle A_{m_2,m_2}\rangle\langle A_{m_3,m_3}\rangle} = \frac{m_1 m_2 m_3}{(1 + \rho) \rho (\rho - 1)}$$

In the case when $p' = p + 2 = 2(n + 1)$ these correlators cannot be distinguished from the same diagonal correlators in the $(n + 1, n)$ conformal theory. In this case not only the string susceptibility and the gravitational dimensions coincide, but also the special combination (41) of three-point functions. The arguments of the $\Gamma$-functions in (37)-(40) can be written in a simpler form if we combine the quantities $\alpha, \beta$ of a dressed vertex operator $e^{i\alpha X} e^{\beta \Phi}$ into a Minkowskian two-dimensional vector $p = (\beta, \alpha)$. Defining the background two-vector $b = (Q, \alpha_0)$, the dressing (on-shell) condition takes the form $(p - b/2)^2 = 0$, where for any two-vector $p^2 = \beta^2 - \alpha^2$. Then the argument $1 - m/2 + m' \rho/2$ appearing for example in (37) can be written as $\frac{1}{2} + \frac{p^2}{2}$. Finally the singularities of the factors in (37)-(40) appear in the boundary of the Kac table for values of $(m', m) = (0, m)$, with $m$ even in analogy with the conformal case [29]. These states should very likely have an interpretation as boundary operators as suggested for the conformal case in [30].

We find it rather intriguing that in the case when $p' = p + 2$ is even (the case when the theory has a supersymmetric ground state before coupling to gravity), the zero-, one-, two- and three-point functions in the NS sector of the theory cannot be distinguished from the conformal case $(1 + \frac{p^2}{2}, \frac{p^2}{2})$.

While this paper was being typed we received three papers where similar issues are addressed [31,32,33].

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REFERENCES

1. E. Brezin and V. Kazakov: Phys. Lett. 236B (1990) 144; M. Douglas and S. Shenker: Nucl. Phys. B335 (1990) 635; D.J. Gross and A.A. Migdal: Phys. Rev. Lett. 64 (1990) 717.

2. M. Douglas: Phys. Lett. 238B (1990) 176.

3. P. Di Francesco and D. Kutasov: Nucl. Phys. B342 (1990) 589.

4. N. Seiberg: Notes on Quantum Liouville Theory and Quantum Gravity. RU-90-29. Talk presented at the 1990 Yukawa International Seminar.

5. L. Alvarez-Gaumé, C. Gomez: Topics in Liouville Theory. In Proceedings of the Trieste Spring School 1991. R. Dijkgraaf, S. Randjbar-Daemi and H. Verlinde eds. World Scientific (to appear). CERN-TH-6175/91.

6. A.M. Polyakov: Mod. Phys. Lett. A2 (1987) 893.

7. V.G. Knizhnik, A.M. Polyakov and A.B. Zamolodchikov: Mod. Phys. Lett. A3 (1988) 819.

8. J. Distler and H. Kawai: Nucl. Phys. B321 (1989) 509.

9. F. David: Mod. Phys. Lett. A3 (1988) 1651.

10. A.A. Belavin, A.M. Polyakov and A.B. Zamolodchikov: Nucl. Phys. B241 (1984) 333.

11. D. Friedan, Z. Qiu and S. Shenker: Phys. Rev. Lett. 51 (1984) 1575.

12. M. Goulian and M. Li: Phys. Rev. Lett. 66-16 (1991) 2051.

13. Y. Kitazawa: HUTP-91/A013.

14. P. Di Francesco and D. Kutasov: Phys. Lett. 261B (1991) 385.

15. V.S. Dotsenko and V.A. Fateev: Nucl. Phys. B240 (1984) 312; Nucl. Phys. B251 (1985) 691; Phys. Lett 154B (1985) 291.

16. V.S. Dotsenko: PAR-LPTHE 91-18.
17. L. Alvarez-Gaumé, J.L.F. Barbón and C. Gómez: *Fusion Rules in Two-Dimensional Gravity*. CERN-TH-6142/91. To appear in Nucl. Phys.B.

18. D. Friedan, Z. Qiu and S. Shenker: In *Vertex Operators in Mathematical Physics*. J. Lepowsky ed. Springer Verlag, 1984.

19. D. Friedan, Z. Qiu and S. Shenker: Phys. Lett. **151B** (1985) 37.

20. M. Bershadsky, V. Knizhnik and A. Teitelman: Phys. Lett. **151B** (1985) 31.

21. H. Eichenherr: Phys. Lett. **151B** (1985) 26.

22. Z. Qiu: Nucl. Phys. **B270** (1986) 205.

23. G. Mussardo, G. Sotkov and H. Stanishkov: Phys. Lett. **195B** (1987) 397; Nucl. Phys. **B305** (1988) 69.

24. L. Alvarez-Gaumé and Ph. Zaugg, CERN-TH-6242/91.

25. A.M. Polyakov, A.B. Zamolodchikov: Mod. Phys. Lett. **A3** (1988) 819.

26. J. Distler, Z. Hlousek and H. Kawai: Int. J. Mod. Phys. **A5** (1990) 391.

27. P. Di Francesco, J. Distler and D. Kutasov: PUPT-90-1189, 1990.

28. M. Bershadsky and I. Klebanov: PUPT-1241 (1991).

29. A.M. Polyakov: Mod. Phys. Lett. **A 6**(1991) 635.

30. E. Martinec, G. Moore and N. Seiberg, RU-14-91, YCTP-P10-91, EFI-91-14.

31. E. Abdalla, M.C.B. Abdalla, D. Dalwazi, and K. Harada, Print-91-0351 (Sao Paulo).

32. P. Di Francesco and D. Kutasov, PUPT-1276.

33. K. Aoki and E. D’Hoker; UCLA-91-TEP-33.