A class of integral operators from weighted integral transforms to Dirichlet spaces

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Abstract: We consider a class of integral operators from weighted integral transforms to Dirichlet spaces. The boundedness and compactness of these operators from weighted integral transforms to Dirichlet spaces are characterized. We also compute norm of integral operators acting between these spaces.

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1. Introduction

Let \( D \) be the open unit disk in the complex plane \( \mathbb{C} \), \( T \) its boundary, \( dA(z) = \frac{1}{\pi} dxdy \) (\( z = x + iy \)) the normalized area measure on \( D \), \( H^\infty \) the space of all bounded holomorphic functions on \( \bar{D} \), \( H(D) \) the class of all holomorphic functions on \( D \), and \( \mathcal{M} \) the space of all complex Borel measures on \( T \). For \( \alpha > 0 \), the family \( F_\alpha \) of weighted integral transforms is the collection of functions \( f \in H(D) \) which admits a representation of the form

\[
f(z) = \int_T K^\alpha_\mu(x) d\mu(x) \quad (z \in D),
\]

where the Cauchy kernel \( K^\alpha_\mu \) is given by

\[
K^\alpha_\mu(z) = \frac{1}{(1 - \bar{z}x)^\alpha}
\]

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PUBLIC INTEREST STATEMENT

One of the core problems in the study of operators on function spaces is to characterize the boundedness of integral operators between function spaces. Several criteria for compactness and a formula for exact norm of integral-type operators between weighted integral transforms to Dirichlet-type spaces are obtained.
and \( \mu \) varies over all measures in \( \mathcal{M} \). The principal branch is used in the power function in Equation (2) and throughout the rest of the paper. The space \( \mathcal{T}_a \) is a Banach space with respect to the norm

\[
\|f\|_{\mathcal{T}_a} = \inf \|\mu\|,
\]

where the infimum is taken over all complex Borel measures \( \mu \) satisfying (1) and \( \|\mu\| \) denotes the total variation of \( \mu \).

Let \( dA_\beta(z) = (1 + \beta)(1 - |z|^2)^\beta dA(z) \) be the probability measure on \( \mathbb{D} \), where \( \beta \in (-1, \infty) \). For \( 0 < p < \infty \) and \( \beta \in (-1, \infty) \), the weighted Bergman space \( \mathcal{A}_\beta^p \) consists of functions \( f \in \mathcal{H}(\mathbb{D}) \) such that

\[
\|f\|_{\mathcal{A}_\beta^p} = \left( \int_{\mathbb{D}} |f(z)|^p dA_\beta(z) \right)^{1/p} < \infty.
\]

Also recall that the Dirichlet space \( \mathcal{D}_\beta^p \) is the collection of functions \( f \in \mathcal{H}(\mathbb{D}) \) for which

\[
\|f\|_{\mathcal{D}_\beta^p} = \left( |f(0)|^p + \int_{\mathbb{D}} |f'(z)|^p dA_\beta(z) \right)^{1/p} < \infty.
\]

If \( p < \beta + 1 \), then it is well known that \( \mathcal{D}_\beta^p = \mathcal{A}_\beta^p \). On the other hand if \( p > \beta + 1 \), then \( \mathcal{D}_\beta^p \subset \mathcal{H}^\infty \). Furthermore, \( \mathcal{D}_\beta^p \) is the classical Besov space and \( \mathcal{D}_\beta^{p-1} \) are closely related to the classical Hardy space \( \mathcal{H}_p \). In fact, it is well known that \( \mathcal{H}_p \subset \mathcal{D}_\beta^{p-1} \) if \( 2 \leq p < \infty \) and \( \mathcal{D}_\beta^{p-1} \subset \mathcal{H}_p \) if \( 0 < p < 2 \). Moreover, \( \mathcal{D}_\beta^1 = \mathcal{H}_2 \), but \( \mathcal{D}_\beta^{p-1} \neq \mathcal{H}_p \) for \( p \neq 2 \). For more about these type of spaces see [Girela and Peiaez (2006)] and references therein. If \( (X, \| \cdot \|_X) \) and \( (Y, \| \cdot \|_Y) \) are Banach spaces, and \( T \) is a linear operator from \( X \) to \( Y \), then \( T \) is bounded if there exists a positive constant \( C \) such that \( \|T(f)\|_Y \leq C\|f\|_X \) for all \( f \in X \) and the operator norm of \( T \) is defined as \( \|T\|_{X \rightarrow Y} = \inf \{ C \in \mathbb{R} : \|T(f)\|_Y \leq C\|f\|_X \} \). We denote by \( \mathcal{L}(X, Y) \) the set of all bounded linear operators from \( X \) to \( Y \). If \( T \in \mathcal{L}(X, Y) \), then we say that \( T \) is a compact operator from \( X \) to \( Y \) if the image of every bounded set of \( X \) is relatively compact (that is, has compact closure) in \( Y \). Equivalently, a linear operator \( T \) is a compact operator from \( X \) to \( Y \) if and only if every bounded sequence \( (f_n) \) in \( X \) has a convergent subsequence in \( Y \). We will denote by \( \mathcal{K}(X, Y) \) the compact linear operators from \( X \) into \( Y \).

In this paper, we consider a class of integral operators defined as follows

\[
J_{\varphi, \rho}^{(m)}(z) = \int_{\mathbb{D}} f^{(m)}(\varphi(\zeta))g(\zeta)d\zeta, \quad z \in \mathbb{D},
\]

where \( g \in \mathcal{H}(\mathbb{D}) \), \( \rho \) a holomorphic self-map of \( \mathbb{D} \) and \( n \in \mathbb{N} \cup \{0\} \) from \( \mathcal{T}_a \) to \( \mathcal{D}_\beta^p \). Operator (3) was first of all defined in Sharma and Sharma (2011) and is an extension of many operators appearing in the literature. If \( J_{\varphi, \rho}^{(m)} \) is so called generalized composition operator, which is a natural extension of the integral operator by Yoneda (2004) (see Cowen & MacCluer, 1995 for more about composition operators. In Hibschweiler and MacGregor (1989), Hibschweiler and MacGregor proved that if \( a \geq 1 \), then every holomorphic self-map \( \varphi \) of \( \mathbb{D} \) induces a bounded composition operator on \( \mathcal{T}_a \). In contrast with the situation when \( a \geq 1 \), a self-map \( \varphi \) of \( \mathbb{D} \) need not induce a bounded composition operator on \( \mathcal{T}_a \) when \( 0 < a < 1 \). In fact, the condition \( \varphi \in \mathcal{T}_a \) is necessary for \( \mathcal{C}_a \) to be bounded on \( \mathcal{T}_a \). Hibschweiler and MacGregor (1989), constructed a self-map \( \varphi \) of \( \mathbb{D} \) with \( \varphi \not\in \mathcal{T}_a \) \( (0 < a < 1) \). For some recent results in this area (see Choa & Kim, 2003, Hibschweiler, 1998, 2012; Sharma & Sharma, 2014; Stevic & Sharma, 2011). Motivated by work in the above-cited articles, here we provide characterizations of when \( J_{\varphi, \rho}^{(m)} \in \mathcal{L}(\mathcal{T}_a, \mathcal{D}_\beta^p) \) or \( J_{\varphi, \rho}^{(m)} \in \mathcal{K}(\mathcal{T}_a, \mathcal{D}_\beta^p) \). We also compute the norm of \( J_{\varphi, \rho}^{(m)} \) acting from \( \mathcal{T}_a \) to \( \mathcal{D}_\beta^p \). Throughout this paper constants are denoted by \( C \), they are positive and not necessarily the same at each occurrence.
2. Main results

**Theorem 2.1** Let $\alpha \geq 0$, $\beta > -1$, $n \in \mathbb{N} \cup \{0\}$, $g \in H(D)$ and $\varphi$ be a holomorphic self-map of $D$. Then $J_{g,\nu}^{(n)} \in L(F_{\alpha}, D^p_\rho)$ if and only if

$$M = \sup_{x \in T} \int_{D} |K^{n+\alpha}_{x}(\varphi(z))|^p |g(z)|^p dA_{f}(z) < \infty.$$  \hspace{1cm} (4)

Moreover, if $J_{g,\nu}^{(n)} \in L(F_{\alpha}, D^p_\rho)$, then

$$\|J_{g,\nu}^{(n)}\|_{F_{\alpha} \rightarrow D^p_\rho}^p = \left( \prod_{j=0}^{n-1} (\alpha + j) \right) \rho M.$$  \hspace{1cm} (5)

**Proof** First suppose that (4) holds. If $f \in F_{\alpha}$, then there is $\mu \in \mathcal{M}$ with $\|\mu\| = \|f\|_{\alpha}$ such that

$$f(z) = \int_{T} K^{n}_{x}(z) d\mu(x).$$

Thus, we have

$$f^{(n)}(z) = \prod_{j=0}^{n-1} (\alpha + j) \int_{T} K^{n+\alpha}_{x}(z) d\mu(x), \quad n \in \mathbb{N}.$$  \hspace{1cm} (6)

Replacing $z$ in (6) by $\varphi(z)$, using Jensen’s inequality and multiplying such obtained inequality by $|g(z)|^p$, we obtain

$$|g(z)|^p |f^{(n)}(\varphi(z))|^p \leq \left( \prod_{j=0}^{n-1} (\alpha + j) \right)^\rho \|\mu\|^p \int_{T} |g(z)|^p |K^{n+\alpha}_{x}(\varphi(z))|^p d|\mu|(x),$$  \hspace{1cm} (7)

Integrating (7) with respect to $dA(z)$ and applying Fubini’s theorem yield

$$\int_{D} |g(z)|^p |f^{(n)}(\varphi(z))|^p dA_{f}(z) \leq \left( \prod_{j=0}^{n-1} (\alpha + j) \right)^\rho \|\mu\|^{p-1} \int_{T} \left[ \int_{D} |g(z)|^p |K^{n+\alpha}_{x}(\varphi(z))|^p dA_{f}(z) \right] d|\mu|(x).$$  \hspace{1cm} (8)

By (4), the inner integral in the second term of (8) is atmost $M$ and so

$$\|J_{g,\nu}^{(n)}f\|_{D^p_\rho}^p \leq \left( \prod_{j=0}^{n-1} (\alpha + j) \right)^\rho M \|\mu\|^{p-1} \int_{T} d|\mu|(x) = \left( \prod_{j=0}^{n-1} (\alpha + j) \right)^\rho \|\mu\|^p \left( \prod_{j=0}^{n-1} (\alpha + j) \right)^\rho M = \left( \prod_{j=0}^{n-1} (\alpha + j) \right)^\rho M \|f\|_{\alpha}^p.$$  \hspace{1cm} (9)

Thus, $J_{g,\nu}^{(n)} \in L(F_{\alpha}, D^p_\rho)$ and

$$\|J_{g,\nu}^{(n)}\|_{F_{\alpha} \rightarrow D^p_\rho}^p \leq \left( \prod_{j=0}^{n-1} (\alpha + j) \right)^\rho M.$$  \hspace{1cm} (10)

Conversely, suppose that $J_{g,\nu}^{(n)} \in L(F_{\alpha}, D^p_\rho)$. Then using the fact that $\|K^{n}_{x}\|_{\alpha} = 1$ for each $x \in T$. Thus, by the boundedness of $J_{g,\nu}^{(n)} : F_{\alpha} \rightarrow D^p_\rho$, we have that
\[
\left( \prod_{j=0}^{n-1} (\alpha + j) \right)^p M = \sup_{x \in T} \left( \prod_{j=0}^{n-1} (\alpha + j) \right)^p \int_{\mathbb{D}} |K_{x}^{n}(\varphi(z))|^p |g(z)|^p \, dA_j(z) \\
= \sup_{x \in T} \left\| \int_{\mathbb{D}} |K_{x}^{n}(\varphi(z))|^p \, dA_j(z) \right\|_{L_x^n} \leq \sup_{x \in T} \left\| \mathcal{J}_{(\varphi, K_{x}^{n})} \right\|_{\mathcal{N}_p} \left\| K_{x}^{n} \right\|_{p} \\
= \left\| \mathcal{J}_{(\varphi, K_{x}^{n})} \right\|_{\mathcal{N}_p} \left\| K_{x}^{n} \right\|_{p} 
\] (10)

Thus, (4) holds. Moreover, combining (9) and (10), we get (5), as desired.

The next theorem is an easy consequence of the Theorem 2.1. We omit the details.

**Theorem 2.2** Let \( n \in \mathbb{N} \cup \{0\} \), \( g \in H(\mathbb{D}) \) and \( \varphi \) be a holomorphic self-map of \( \mathbb{D} \). Then \( \mathcal{J}_{(\varphi)}^n \in \mathcal{L}(\mathcal{F}_u, D_\beta^p) \) if and only if \( (g \cdot K_{x}^{n+1}) \) is a norm bounded subset of \( D_\beta^p \).

The next lemma can be found in Hibschweiler (2012).

**Lemma 2.3** Let \( \alpha > 0 \) and \( f \in H(\mathbb{D}) \). Then for \( f \in \mathcal{F}_u \) and \( z \in \mathbb{D} \), we have that

\[
|f(z)| \leq \frac{\left\| f \right\|_{\mathcal{F}_u}}{(1 - |z|)^{\alpha}}. \quad (11)
\]

By Equation (11) of Lemma 2.3, we have that the unit ball of \( \mathcal{F}_u \) is a normal family of holomorphic functions. A standard normal family argument then yields the proof of the the following lemma. See Proposition 3.11 of Cowen and MacCluer (1995) for details.

**Lemma 2.4** Let \( n \in \mathbb{N} \cup \{0\} \), \( g \in H(\mathbb{D}) \) and \( \varphi \) be a holomorphic self-map of \( \mathbb{D} \). Then \( \mathcal{J}_{(\varphi, \varphi)}^n \in \mathcal{K}(\mathcal{F}_u, D_\beta^p) \) if and only if for any sequence \( \{f_j\} \) in \( \mathcal{F}_u \) with \( \left\| f_j \right\|_{\mathcal{F}_u} \leq L \) and which converges to zero locally uniformly, we have \( \lim_{j \to \infty} \left\| \mathcal{J}_{(\varphi, \varphi)}^n f_j \right\|_{\mathcal{K}} = 0 \).

**Theorem 2.5** Let \( \alpha \geq 1 \), \( \beta > -1 \), \( n \in \mathbb{N} \cup \{0\} \), \( g \in H(\mathbb{D}) \) and \( \varphi \) be a holomorphic self-map of \( \mathbb{D} \). Suppose that \( \mathcal{J}_{(\varphi, \varphi)}^n \in \mathcal{L}(\mathcal{F}_u, D_\beta^p) \). Then the following statements are equivalent:

1. \( \mathcal{J}_{(\varphi, \varphi)}^n \in \mathcal{K}(\mathcal{F}_u, D_\beta^p) \).
2. The integral

\[
\int_{\mathbb{D}} |K_{x}^{(n+1)}(\varphi(z))|^p |g(z)|^p \, dA_j(z)
\]

is a continuous function of \( x \in \mathbb{T} \).

3. The family of measures \( \{ \nu_x : x \in \mathbb{T} \} \) defined by

\[
\nu_x(E) = \int_E |K_{x}^{(n+1)}(\varphi(z))|^p |g(z)|^p \, dA_j(z)
\]

is equi-absolutely continuous. That is, given \( \epsilon > 0 \), there exists a \( \delta > 0 \) such that \( \nu_x(E) < \epsilon \) for all \( x \in \mathbb{T} \) whenever \( A(E) < \delta \).

4. \( g \in \mathcal{A}_\beta^p \) and

\[
\lim_{k \to \infty} \sup_{x \in \mathbb{T}} \int_{|z| > r} |K_{x}^{(n+1)}(\varphi(z))|^p |g(z)|^p \, dA_j(z) = 0. \quad (12)
\]

**Proof** (1) \( \Rightarrow \) (2). Let \( x_k \in \mathbb{T} \) with \( x_k \to x \) as \( k \to \infty \), and let \( K_{x_k}^n \) be defined as in Equation (2). Then \( \left\| K_{x_k}^n \right\|_{\mathcal{F}_u} = 1 \) and \( K_{x_k}^n \to K_x^n \) uniformly on compact subsets of \( \mathbb{D} \). Since \( \mathcal{J}_{(\varphi, \varphi)}^n : \mathcal{F}_u \to D_\beta^p \) is compact. By Lemma (2.4), we have...
\[ ||J_{g\omega}^{(n)}K_x^* - J_{g\omega}^{(n)}K_x||_{L^p} \to 0 \]

as \( k \to \infty \). Since \( F_{g\omega}^{(n)} : T_\omega \to \mathcal{D}_\omega \) is bounded, there is a constant \( C > 0 \) such that \( ||J_{g\omega}^{(n)}K_x||_{L^p} \leq C ||K_x||_{L^p} = C \) for all \( x \in T \). Using this along with Hölder’s inequality, we have

\[
\int_0^x \left| (K_x^{(n)}(\varphi(z)))^p - (K_x^{(n)}(\varphi(z)))^p \right| \, dA_y(z) \\
\leq C \int_0^x \left| (K_x^{(n)}(\varphi(z)))^p - (K_x^{(n)}(\varphi(z)))^p \right| \, dA_y(z) \\
\leq C \left( \int_0^x \left| (K_x^{(n)}(\varphi(z)) - (K_x^{(n)}(\varphi(z)))^p \right| \, dA_y(z) \right)^{1/p} \\
= C ||J_{g\omega}^{(n)}K_x^* - J_{g\omega}^{(n)}K_x||_{L^p} \to 0 \quad \text{as} \quad k \to \infty.
\]

Thus,

\[
\int_0^x |K_x^{(n)}(\varphi(z))|^p \, dA_y(z) \to \int_0^x |K_x^{(n)}(\varphi(z))|^p \, dA_y(z),
\]

which shows the continuity of the integral in Equation (2).

(2) \( \Rightarrow \) (3). Suppose that (3) does not hold. Then there exists a sequence \( \{x_k\} \) in \( T \) with \( x_k \to x \) and a sequence of Borel sets \( \{E_k\} \) in \( \mathcal{D} \) such that \( A(E_k) \to 0 \) as \( k \to \infty \), but \( v_{x_k}(E_k) \geq C > 0 \) for all \( k \in \mathbb{N} \). Note that

\[ |v_{x_k}(E_k) - v_x(E_k)| \leq \int_{E_k} \left| |K_x^{(n)}(\varphi(z))|^p - |K_x^{(n)}(\varphi(z))|^p \right| \, dA_y(z) \]

Thus,

\[ v_{x_k}(E_k) \leq \int_{E_k} \left| |K_x^{(n)}(\varphi(z))|^p - |K_x^{(n)}(\varphi(z))|^p \right| \, dA_y(z) + v_x(E_k) \tag{13} \]

Since \( F_{g\omega}^{(n)} : T_\omega \to \mathcal{D}_\omega \) is bounded, so (4) holds. Therefore, \( v_x(E_k) \to 0 \) as \( k \to \infty \). Moreover, as in first part first term in (13) is dominated by

\[
\int_{E_k} \left| (K_x^{(n)}(\varphi(z)) - (K_x^{(n)}(\varphi(z)))^p \right| \, dA_y(z) \to 0
\]

as \( k \to \infty \). Therefore, \( v_{x_k}(E_k) \to 0 \) as \( k \to \infty \). This contradiction shows that (2) \( \Rightarrow \) (3). (3) \( \Rightarrow \) (1). Let \( \{f_k\} \) be a sequence in \( T_\omega \) such that \( \sup_k ||f_k||_{L^p} \leq 1 \) and \( f_k \to 0 \) uniformly on compact subsets of \( \mathcal{D} \). We have to show that \( ||J_{g\omega}^{(n)}f_k||_{L^p} \to 0 \) as \( k \to \infty \). For each \( k \), we can find \( \mu_k \in \mathcal{M} \) with \( ||\mu_k|| = ||f_k||_{L^p} \) such that

\[ f_k(z) = \int_T K_x(z) \, d\mu_k(x). \]

Composing with \( \varphi \) and applying Jensen’s inequality, we have

\[
|f_k^{(n)}(\varphi(z))|^p \leq \left( \prod_{j=0}^{n-1} (\alpha + j) \right)^p ||\mu_k||^p \int_T |K_x^{(n)}(z)|^p \, d\mu_k(x) \]

\[ = \left( \prod_{j=0}^{n-1} (\alpha + j) \right)^p ||\mu_k||^p \int_T |K_x^{(n)}(z)|^p \, d\mu_k(x). \]

Multiplying both sides by \( |g(z)|^p \), integrating the resultant with respect to \( dA_y(z) \) and then applying Fubini’s theorem, we have

\[
\int_0^x |f_k^{(n)}(\varphi(z))|^p |g(z)|^p \, dA_y(z) \\
\leq \left( \prod_{j=0}^{n-1} (\alpha + j) \right)^p ||\mu_k||^p \int_0^x \left[ \int_0^x |K_x^{(n)}(z)|^p |g(z)|^p \, dA_y(z) \right] \, d\mu_k(x).
\]
Let $\epsilon > 0$ be given. Now choose a compact set $\Omega \subset \mathbb{D}$ such that $A(\mathbb{D} \setminus \Omega) < \delta$. Thus,

\[
\int_{\partial \Omega} |f^{(n)}(\varphi(z))|^p |g(z)|^p dA_y(z) \\
\leq \left( \sum_{j=0}^{n-1} (\alpha + j) \right)^p \| \mu_k \|^{p-1} \int_{\partial \Omega} |f^{(n)}(\varphi(z))|^p |g(z)|^p dA_y(z) d|\mu_k|(x) \\
\leq \epsilon \left( \sum_{j=0}^{n-1} (\alpha + j) \right)^p \| \mu_k \|^{p-1} \int d|\mu_k|(x) \\
= \epsilon \left( \sum_{j=0}^{n-1} (\alpha + j) \right)^p \| f_k \|_p^p < \epsilon \left( \sum_{j=0}^{n-1} (\alpha + j) \right)^p.
\]

(14)

On $F$, $|f^{(n)}(\varphi(z))|^p < \epsilon$ as $j \geq j_0$. Moreover, by taking $f(z) = z^n/n! \in \mathcal{F}_\alpha$, the boundedness of $J_{g_0}^{(n)} : \mathcal{F}_\alpha \to D^p_{\delta}$ gives

\[
\int_0^1 |g(z)|^p dA_y(z) \leq C.
\]

Thus,

\[
\int_0^1 |f^{(n)}(\varphi(z))|^p |g(z)|^p dA_y(z) < \epsilon C
\]

(15)

as $j \geq j_0$. Therefore, by (14) and (15), $\| J_{g_0}^{(n)} f_k \|_{L^p_{\delta}} \to 0$ as $k \to \infty$. Hence, by Lemma (2.4), (1) holds.

(1) $\Rightarrow$ (4). Since $J_{g_0}^{(n)} : \mathcal{F}_\alpha \to D^p_{\delta}$ is bounded, for $f(z) = z^n/n! \in \mathcal{F}_\alpha$, we have that $g \in \mathcal{A}_\delta^{(p)}$. Let $f_k(z) = z^k$, $k \in \mathbb{N}$. It is a norm bounded sequence in $\mathcal{F}_\alpha$ converging to zero uniformly on compacts of $\mathbb{D}$. Hence, by Lemma (2.4), it follows that $\| J_{g_0}^{(n)} f_k \|_{L^p_{\delta}} \to 0$ as $m \to \infty$. Thus, for every $\epsilon > 0$, there is an $k_0 \in \mathbb{N}$ such that for $k \geq k_0$, we have

\[
\left( \prod_{j=0}^{n-1} (k-j) \right)^p \int_0^1 |f^{(n)}(\varphi(z))|^p |g(z)|^p dA_y(z) < \epsilon.
\]

(16)

From (16), we have that for each $r \in (0, 1)$

\[
r^{n^2-n} \left( \prod_{j=0}^{n-1} (m-j) \right)^p \int_{|w| \leq r} |g(z)|^p dA_y(z) < \epsilon.
\]

(17)

Hence, for $r \in \left[ \prod_{j=0}^{n-1} (k-j) \right]^{1/ \frac{1}{1-r}}, 1)$, we have

\[
\int_{|w| \leq r} \left( \prod_{j=0}^{n-1} (k-j) \right)^p |g(z)|^p dA_y(z) < \epsilon.
\]

(18)

Let $f \in B_\alpha$ and $f_k(z) = f(tz)\delta < t < 1$. Then $\sup_{0 < t < 1} \| f \|_{L^p_{\delta}} \leq \| f \|_{L^p_{\delta}} f_k \in \mathcal{F}_\alpha, t \in (0, 1)$ and $f_k \to f$ uniformly on compacts of $\mathbb{D}$ as $k \to 1$. The compactness of $J_{g_0}^{(n)} : \mathcal{F}_\alpha \to D^p_{\delta}$ implies that

\[
\lim_{t \to 1} \| J_{g_0}^{(n)} f - J_{g_0}^{(n)} f_k \|_{L^p_{\delta}} = 0.
\]

Hence, for every $\epsilon > 0$, there is a $t \in (0, 1)$ such that

\[
\int_0^1 |f^{(n)}(\varphi(z)) - f^{(n)}(\varphi(z))|^p |g(z)|^p dA_y(z) < \epsilon.
\]

(19)

Inequalities (18) and (19), give
\[
\int_{|\omega(z)|<r} |f_t^{(n)}(\omega(z))|^p |g(z)|^p dA(z)
\leq 2^{p-1} \int_{D} |f_t^{(n)}(\omega(z)) - f_t^{(n)}(\omega(z))|^p |g(z)|^p dA(z)
+ 2^{p-1} \int_{|\omega(z)|>r} |f_t^{(n)}(\omega(z))|^p |g(z)|^p dA(z)
\leq 2^{p-1} \epsilon (1 + \|f_t^{(n)}\|_p^p).
\]

Hence, for every \( f \in B_{p} \), there is a \( \delta_{\epsilon} \in (0, 1) \), \( \delta_{\epsilon} = \delta_{\epsilon}(f, \epsilon) \), such that for \( \epsilon \in (\delta_{\epsilon}, 1) \)
\[
\int_{|\omega(z)|<r} |f_t^{(n)}(\omega(z))|^p |g(z)|^p dA(z) < \epsilon.
\tag{20}
\]

From the compactness of \( f_t^{(n)} : \mathcal{F}_a \to \mathcal{D}_r^p \), we have that for every \( \epsilon > 0 \) there is a finite collection of functions \( f_1, f_2, \ldots, f_k \in B_{p} \) such that for each \( f \in B_{p} \), there is an \( j \in \{1, 2, \ldots, k\} \) such that
\[
\int_{D} |f_t^{(n)}(\omega(z)) - f_j^{(n)}(\omega(z))|^p |g(z)|^p dA(z) < \epsilon.
\tag{21}
\]

On the other hand, from (20) it follows that if \( \delta := \max_{1 \leq j \leq k} \delta_{\epsilon}(f_j, \epsilon) \), then for \( \epsilon \in (\delta, 1) \) and all \( j \in \{1, 2, \ldots, k\} \) we have
\[
\int_{|\omega(z)|<r} |f_t^{(n)}(\omega(z))|^p |g(z)|^p dA(z) < \epsilon.
\tag{22}
\]

From (21) and (22) we have that for \( \epsilon \in (\delta, 1) \) and every \( f \in B_{p} \)
\[
\int_{|\omega(z)|<r} |f_t^{(n)}(\omega(z))|^p |g(z)|^p dA(z) < 2^p \epsilon.
\tag{23}
\]

Applying (23) to the functions \( f_j(z) = 1/(1 - z)^{x} \), \( x \in \mathbb{T} \), we obtain
\[
\sup_{x \in \mathbb{T}} \int_{|\omega(z)|<r} |K_{x}^{x+j}(z)|^p |g(z)|^p dA(z) < 2^p \epsilon \left( \prod_{j=0}^{n-1} (\alpha + j) \right)^{-p},
\]
from which (12) follows.

(4) \( \Rightarrow \) (1). Assume that \( (f_k)_{k \in \mathbb{N}} \) is a bounded sequence in \( \mathcal{F}_a \), say by \( L \), converging to 0 uniformly on compacts of \( \mathbb{D} \) as \( k \to \infty \). Then by the Weierstrass theorem, \( f_k^{(n)} \) also converges to 0 uniformly on compacts of \( \mathbb{D} \), for each \( k \in \mathbb{N} \). By Lemma (2.4), we need to show that \( \|f_k^{(n)}\|_{\mathcal{D}_r^p} \to 0 \) as \( k \to \infty \).

For each \( k \in \mathbb{N} \), we can find a \( \mu_k \in \mathbb{R} \) with \( \|\mu_k\| = \|f_k\|_p \) such that
\[
f_k(z) = \int_{\beta} K_{x}^{x+j}(z) d\mu_k(x).
\tag{24}
\]

Differentiating (24) \( n \) times, composing such obtained equation by \( \varphi \), applying Jensen’s inequality, as well as the boundedness of sequence \( (f_k)_{k \in \mathbb{N}} \), we obtain
\[
|f_k^{(n)}(\varphi(z))|^p \leq L \left( \prod_{j=0}^{n-1} (\alpha + j) \right)^p \int_{\beta} |K_{x}^{x+j}(z)|^p d|\mu_k|(x).
\tag{25}
\]

By the condition (12) in (4), we have that for every \( \epsilon > 0 \), there is an \( r_1 \in (0, 1) \) such that for \( \epsilon \in (r_1, 1) \), we have
\[
\sup_{x \in \mathbb{T}} \int_{|\omega(z)|<r} |K_{x}^{x+j}(z)|^p |g(z)|^p dA(z) < \epsilon.
\tag{26}
\]

Now
\[ \|f^\nu \varphi\|_{L^p} = \left( \int_{|z|<r} |\varphi(z)|^p |f(z)|^p \, dA(z) \right)^{1/p}. \]

Using (25), (26), Fubini’s theorem and the fact that \( \sup_{|w|<\epsilon} |f^\nu(z)|^p < \epsilon \), for sufficiently large \( k \), say \( k \geq k_0 \), we have that for \( k \geq k_0 \)

\[ \|f^\nu \varphi\|_{L^p} \leq C \sup_{|w|<\epsilon} |f^\nu(z)|^p \int_{|z|<r} |g(z)|^p \, dA(z) \]

\[ + C \int_{|w|>\epsilon} |s^{\nu,\rho}(z)|^p |g(z)|^p \, dA(z) \, \mu_\rho(x) \]

\[ \leq C \left( M + \int \, d\mu_\rho(x) \right) \epsilon < C \epsilon. \]

Since \( \epsilon > 0 \) is an arbitrary, the implication follows.

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