A DISSIPATIVE LOGARITHMIC-LAPLACIAN TYPE OF PLATE EQUATION: ASYMPTOTIC PROFILE AND DECAY RATES

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Abstract. We introduce a new model of the logarithmic type of wave like plate equation with a nonlocal logarithmic damping mechanism. We consider the Cauchy problem for this new model in \( \mathbb{R}^n \), and study the asymptotic profile and optimal decay rates of solutions as \( t \to \infty \) in \( L^2 \)-sense. The operator \( L \) considered in this paper was first introduced to dissipate the solutions of the wave equation in the paper studied by Charão-Ikehata [7]. We will discuss the asymptotic property of the solution as time goes to infinity to our Cauchy problem, and in particular, we classify the property of the solutions into three parts from the viewpoint of regularity of the initial data, that is, diffusion-like, wave-like, and both of them.

1. Introduction. We consider in this work a new model of evolution equations based on a nonlocal operator \( L \), that combines the composition of logarithm function with the Laplace operator as follows:

\[
\begin{align*}
  u_{tt} + Lu_{tt} + L^2 u + Lu + u_t &= 0, \quad (t, x) \in (0, \infty) \times \mathbb{R}^n, \\
  u(0, x) &= u_0(x), \quad u_t(0, x) = u_1(x), \quad x \in \mathbb{R}^n,
\end{align*}
\]

where the linear operator

\[
  L : D(L) \subset L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)
\]

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is defined by
\[ D(L) := \left\{ f \in L^2(\mathbb{R}^n) \mid \int_{\mathbb{R}^n} (\log(1 + |\xi|^2))^2 |\hat{f}(\xi)|^2 d\xi < +\infty \right\}, \]
and for \( f \in D(L) \),
\[ (Lf)(x) := \mathcal{F}_{\xi \to x}^{-1} \left( \log(1 + |\xi|^2) \hat{f}(\xi) \right)(x). \]
Here, one has just denoted the Fourier transform \( \mathcal{F}_{x \to \xi} \) of \( f(x) \) by
\[ \mathcal{F}_{x \to \xi}(f)(\xi) = \hat{f}(\xi) := \int_{\mathbb{R}^n} e^{-ix \cdot \xi} f(x) dx, \quad \xi \in \mathbb{R}^n, \]
as usual with \( i := \sqrt{-1} \), and \( \mathcal{F}_{\xi \to x}^{-1} \) expresses its inverse Fourier transform. Since the operator \( L \) is non-negative and self-adjoint in \( L^2(\mathbb{R}^n) \) (see [7]), the square root \( L^{1/2} : D(L^{1/2}) \subset L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n) \)
can be defined, and is also nonnegative and self-adjoint with its domain
\[ D(L^{1/2}) := \left\{ f \in L^2(\mathbb{R}^n) \mid \int_{\mathbb{R}^n} \log(1 + |\xi|^2) |\hat{f}(\xi)|^2 d\xi < +\infty \right\}. \]
Note that \( D(L^{1/2}) \) becomes Hilbert space with its graph norm
\[ \| v \|_{D(L^{1/2})} := \left( \| v \|^2 + \| L^{1/2} v \|^2 \right)^{1/2}, \]
where the \( L^2(\mathbb{R}^n) \)-norm is defined by
\[ \| \cdot \| := \| \cdot \|_{L^2(\mathbb{R})}. \]

It is easy to check that
\[ H^s(\mathbb{R}^n) \hookrightarrow D(L^{1/2}) \hookrightarrow L^2(\mathbb{R}^n) \]
for \( s > 0 \).
Symbolically writing, one can see
\[ L = \log(1 - \Delta), \]
where \( \Delta \) is the usual Laplace operator defined on \( H^2(\mathbb{R}^n) \).

As for the existence of the unique solution to problem (1)-(2) we discuss, on the
next section, by employing the standard Lumer-Phillips Theorem (see Theorem 2.1
below).

The associated energy inequality to the system (1)-(2) is
\[ E_u(t) \leq E_u(0), \quad t > 0, \tag{3} \]
where
\[ E_u(t) := \frac{1}{2} \left( \| u_t(t, \cdot) \|_{L^2}^2 + \| L^{1/2} u_t(t, \cdot) \|_{L^2}^2 + \| L u(t, \cdot) \|_{L^2}^2 \right). \]
The inequality (3) implies that the total energy is a non increasing function in
time because of the existence of some kind of dissipative term \( u_t \).

A main topic of this paper is to find an asymptotic profile of the solution in the
\( L^2 \) framework to problem (1)-(2), and to apply it to investigate the optimal rate
of decay of the solution in terms of the \( L^2 \)-norm. We study the equation (1) only
from the purely mathematical point of view. The model equation itself is strongly
inspired from the related paper due to Dharmawardane-Nakamura-Kawashima [20]
(see (4) for its model).
A motivation of this research has its origin in the study of the damped plate equation under effects of rotational inertia
\[ u_{tt} - \Delta u_{tt} - \Delta u + \Delta^2 u + u_t = 0 \]
that can be written as
\[ u_{tt} - \Delta u + (I - \Delta)^{-1} u_t = 0. \]  
(4)

In a work Charão-Espinoza-Ikehata [5] they study a more general model than (4) with a super damping. A pioneering work on the dissipative structure and nonlinear property of the more generalized systems including (4) is discussed in Dharmawar-dane [19], which is mentioning a regularity loss-structure of the equation. After [19], Fukushima-Ikehata-Michihisa [21] have studied the asymptotic profiles of the solution to (4), and such profiles are divided into two parts: one is the Gauss kernel like for high regularity initial data, and the other is related with the non-diffusive oscillating property for low regularity initial data from the viewpoint of regularity-loss structure. In this connection, such a regularity-loss structure has been first discovered and named by S. Kawashima through the analysis for the dissipative Timoshenko system (see e.g., [24]).

An analysis of the dissipative mechanism of (4) goes back to the two pioneering works of G. Ponce [37] and Y. Shibata [39], where they studied various \( L^p - L^q \) estimates of the solution to the Cauchy problem for the equation:
\[ u_{tt} - \Delta u - \Delta u_t = 0. \]  
(5)

After them, an asymptotic profile and the optimal estimates of the solution can be introduced in the papers [31], [26] and [28]. They investigated a singularity near 0-frequency region of the solution to (5) in terms of \( L^2 \)-norm of solutions. In this connection, in [1, 2] and [34] a higher order asymptotic expansion of the solution as \( t \to \infty \) to the equation (5) is precisely investigated.

On the other hand, the so-called critical exponent problem for semi-linear equations of (5) is first developed by D’Abbicco-Reissig [16], and this paper has been the beginning of a series of related papers studying structurally damped wave models with nonlinearity. Unfortunately, at present nobody knows the precise value of the critical exponent \( p^* \) of the equation (5) with power type nonlinearity \( |u|^p \). This study is based on the \( L^p-L^q \)-estimates derived in [39].

Recently, the equation (5) is generalized to the linear and semi-linear models, respectively:
\[ u_{tt} + (-\Delta)^\sigma u + (-\Delta)^\theta u_t = 0, \]  
(6)
\[ u_{tt} + (-\Delta)^\sigma u + (-\Delta)^\theta u_t = f(u, u_t). \]  
(7)
A study on asymptotic profile and \( L^p-L^q \) estimates to the equation (6) has been done in the papers [3], [12], [14], [15], [11], [35], and [30], and the corresponding critical exponent problems (mainly) to the equation (7) are treated in the papers [13], [17, 18], [32], and [36]. As for the related motivated topics concerning the asymptotic profiles for the higher order PDEs and the other types of linear equations one can also cite [8, 9], [38], [41], [40], and the references therein.

In [27] and [22], the so-called regularity-loss structure of the solution in the high frequency zone can be studied to (6) with \( \sigma = 1 \) and \( \theta > 1 \), and these researches are strongly inspired from the abstract theory due to [23].

Quite recently Charão-Ikehata [7] introduced a new type of damping term of logarithm type to the wave equation, and it is expressed in the Fourier space as
follows:
\[ \hat{u}_{tt} + |\xi|^2 \hat{u} + \log(1 + |\xi|^2) \hat{u}_t = 0. \]
Symbolically writing, one sees
\[ u_{tt} - \Delta u + \log(I - \Delta)u_t = 0. \]
In [4], (8) is more generalized to the equation such that
\[ u_{tt} - \Delta u + \log(I + (-\Delta)\theta)u_t = 0 \]
for \( \theta > 1/2 \).
On reconsidering our problem (1)-(2) in the Fourier space, our equation becomes
\[ (1 + \log(1 + |\xi|^2)) \hat{u}_{tt} + (1 + \log(1 + |\xi|^2)) \log(1 + |\xi|^2) \hat{u}_t = 0. \]
We should investigate characteristics roots of (9) to see where they are complex-valued or not, for small \( \xi \in \mathbb{R}^n \) and on large frequency zone.
In order to introduce our main results we should define function spaces with respect to the logarithmic Laplace operator \( L \) such that for \( \delta \geq 0 \)
\[ Y^\delta = \{ f \in L^2(\mathbb{R}^n); \int_{\mathbb{R}^n} (1 + \log(1 + |\xi|^2))^\delta |\hat{f}(\xi)|^2 d\xi < \infty \} \]
with its natural norm
\[ \| f \|_{Y^\delta} := \left( \int_{\mathbb{R}^n} (1 + \log(1 + |\xi|^2))^\delta |\hat{f}(\xi)|^2 d\xi \right)^{1/2}, \quad f \in Y^\delta, \]
and its corresponding inner product.

**Remark 1.** Due to the fact that \( \log(1 + |\xi|^2) \leq |\xi|^2 \) for all \( \xi \in \mathbb{R}^n \), one notices \( H^\delta(\mathbb{R}^n) \subset Y^\delta \subset L^2 \) for \( \delta \geq 0 \).
Furthermore, we set
\[ I_{0,l} := \sqrt{\|u_0\|_{L^1}^2_1 + \|u_1\|_{L^1}^2_1 + \|u_0\|_{Y^{l+1}}^2 + \|u_1\|_{Y^{l+1}}^2} \quad (l \geq 0), \]
and
\[ P_j := \int_{\mathbb{R}^n} u_j(x)dx \quad (j = 0, 1). \]
Our results read as follows.

**Theorem 1.1.** Let \( n \geq 1 \) and \( l \geq 1 \). If \((u_0, u_1) \in (L^{1,1}(\mathbb{R}^n) \cap Y^{l+1}) \times (L^{1,1}(\mathbb{R}^n) \cap Y^{l})\), then there exists a constant \( C > 0 \) which is independent of \( t, u_0, u_1 \) such that the weak solution \( u(t,x) \) to (1)-(2) satisfies
\[ \|u(t, \cdot) - F_{\xi=2}^{-1}(\phi_1(t, \xi))(\cdot)\|_2 \leq \begin{cases} C I_{0}^{n+2} t^{-\frac{n+2}{2}} & \text{if } l \geq 1 \text{ and } n \leq 2; \quad \text{if } n \geq 3 \text{ and } l \geq n/2, \\ C I_{0}^{n} t^{-\frac{n}{2}} + P_1 & \text{if } n \geq 4 \text{ and } n/2 - 1 < l \leq n/2; \quad \text{if } n = 3 \text{ and } 1 \leq l \leq 3/2, \end{cases} \]
for \( t \gg 1 \), where
\[ \phi_1(t, \xi) := e^{-t \log(1+|\xi|^2)(1+\log(1+|\xi|^2))} (P_0 + P_1). \]
Theorem 1.2. Let \( n \geq 5 \) and \( l \geq 1 \). If \( (u_0, u_1) \in (L^{l,1}(\mathbb{R}^n) \cap Y^{l+1}) \times (L^{l,1}(\mathbb{R}^n) \cap Y^l) \), then there exists a constant \( C > 0 \), which is independent of \( t, u_0, u_1 \) such that the weak solution \( u(t, x) \) to (1.1)-(2) satisfies

\[
\|u(t, \cdot) - \mathcal{F}_{\xi \to x}^{-1}(\varphi_2(t, \xi))(\cdot)\|_2 \leq \begin{cases} 
CI_{0,t}I_l t^{-\frac{n}{2}} & \text{if } 1 \leq l < n/2 - 1 \text{ and } 5 \leq n \leq 8 \\
CI_{0,t}I_l t^{-\frac{n+3}{2}} & \text{if } n > 8 \text{ and } n/2 - 3 < l < n/2 - 1,
\end{cases}
\]

for \( t \gg 1 \), where

\[
\varphi_2(t, \xi) := e^{-\frac{\xi^2}{2\log(1+|\xi|^2)t}} \left( \frac{\sin(\sqrt{\log(1+|\xi|^2)t})}{\sqrt{\log(1+|\xi|^2)t}} \right) u_1(\xi) + \cos(\sqrt{\log(1+|\xi|^2)t}) u_0(\xi).
\]

Theorem 1.3. Let \( n \geq 4 \) and \( l = \frac{n}{2} - 1 \). If \( (u_0, u_1) \in (L^{l,1}(\mathbb{R}^n) \cap Y^{l+1}) \times (L^{l,1}(\mathbb{R}^n) \cap Y^l) \), then there exists a constant \( C > 0 \), which is independent of \( t, u_0, u_1 \) such that the weak solution \( u(t, x) \) to (1.1)-(2) satisfies

\[
\|u(t, \cdot) - \mathcal{F}_{\xi \to x}^{-1}(\varphi(t, \xi))(\cdot)\|_2 \leq CI_{0,t}I_l t^{-\frac{n+3}{2}}
\]

for \( t \gg 1 \), where

\[
\varphi(t, \xi) := \varphi_1(t, \xi) + \varphi_2(t, \xi).
\]

Remark 2. In Theorems above one has assumed \( l \geq 1 \) because in this case the problem (1)-(2) has a unique weak solution in the class

\[
u \in C([0, +\infty), Y^2) \cap C^1([0, +\infty), Y^1) \cap C^2([0, +\infty), L^2(\mathbb{R}^n)).
\]

For details concerning the unique existence of solutions, we will discuss in Theorem 2.1 in subsection 2.1.

Remark 3. The value \( l^*(n) \) defined by \( l^*(n) := \frac{n}{2} - 1 \) expresses a kind of critical number on the regularity \( l \geq 1 \), which divides the property of the solution \( u(t, x) \) into three types: one is diffusive-like (Theorem 1.1), the other is wave-like (Theorem 1.2), and the remaining is both of them (Theorem 1.3).

Throughout the discussions to get Theorems 1.1, 1.2 and 1.3, one can obtain the following crucial results, which shows the optimality concerning the \( L^2 \)-decay rates of the solution \( u(t, x) \) to problem (1.1)-(1.2).

Theorem 1.4. Let \( n \geq 4 \) and \( 1 \leq l \leq \frac{n}{2} - 1 \). If \( (u_0, u_1) \in (L^{l,1}(\mathbb{R}^n) \cap Y^{l+1}) \times (L^{l,1}(\mathbb{R}^n) \cap Y^l) \), then the solution \( u(t, x) \) to problem (1.1)-(1.2) satisfies

\[
\|u(t, \cdot)\|_2 \leq CI_{0,t}I_l t^{-\frac{l+3}{2}}
\]

for \( t \gg 1 \), where \( C \) is a positive constant which depends only on \( n \).

Remark 4. The rate of decay obtained in Theorem 1.4 seems exactly optimal, however, one cannot obtain the lower bound of time-decay rate. This is still open.

Theorem 1.5. Let \( 1 \leq n \leq 3 \). If \( (u_0, u_1) \in (L^{1,1}(\mathbb{R}^n) \cap Y^2) \times (L^{1,1}(\mathbb{R}^n) \cap Y^1) \), then there exists constants \( C_1, C_2 > 0 \) independent of \( t \) such that the solution \( u(t, x) \) to problem (1.1)-(1.2) satisfies

\[
C_1|p_0 + P_1|t^{-\frac{2}{5}} \leq \|u(t, \cdot)\|_2 \leq C_2I_{0,t}I_l t^{-\frac{2}{5}}
\]

for all \( t \gg 1 \) provided that \( P_1 + P_0 \neq 0 \).
Theorem 1.6. Let $n \geq 4$ and $\varepsilon > 0$. If $(u_0, u_1) \in (L^{1,1}(\mathbb{R}^n) \cap Y^\frac{2}{n-2} + \varepsilon) \times (L^{1,1}(\mathbb{R}^n) \cap Y^\frac{2}{n} + \varepsilon)$, then there exists constants $C_1, C_2 > 0$ independent of $t$ such that the solution $u(t, x)$ to problem (1.1)-(1.2) satisfies

$$C_1 |P_0 + P_1| t^{-\frac{n}{2}} \leq \|u(t, \cdot)\|_2 \leq C_2 I_{0, \frac{n}{2} + \varepsilon - 1} t^{-\frac{n}{4}}$$

for all $t \gg 1$, provided that $P_1 + P_0 \neq 0$.

This paper is organized as follows. In Section 2 we prepare several important lemmas, which will be used later, and in particular, these lemmas are closely related with hypergeometric functions (see [7]). In the final part of Section 2 we discuss the unique existence of solutions to problem (1)-(2) using ideas from Luyo Sánchez [33]. In Section 3 we obtain decay estimates for the $L^2$-norm of solutions by using the multiplier method in the Fourier space (cf. [42]) only for $n \geq 3$ in the case when the initial data have a low regularity. The asymptotic profile and related estimates of solution are obtained in Section 4, and in particular, Theorems 1.1, 1.2 and 1.3 are proved in subsection 4.4, and in the final subsection 4.5 we give the proof of Theorems 1.4, 1.5 and 1.6.

Notation. Throughout this paper, $\| \cdot \|_q$ stands for the usual $L^q(\mathbb{R}^n)$-norm. For simplicity of notation, in particular, we use $\| \cdot \|$ instead of $\| \cdot \|_2$. We also introduce the following weighted functional spaces:

$$L^{1,\gamma}(\mathbb{R}^n) := \left\{ f \in L^1(\mathbb{R}^n) \mid \|f\|_{1,\gamma} := \int_{\mathbb{R}^n} (1 + |x|^\gamma)|f(x)|dx < +\infty \right\}.$$

Furthermore, we denote the surface area of the $n$-dimensional unit ball by $\omega_n := \int_{|\omega|=1} d\omega$.

2. Basic results and existence of solutions. In this section we shall collect important lemmas to derive precise estimates of various quantities related to the solution to problem (1)-(2). These are already studied and developed in our previous works (see [7, 4]).

The following main estimate to the function

$$I_p(t) = \int_0^1 (1 + r^2)^{-t} r^p dr$$

is a direct consequence of the cases $p \geq 0$ in Charão-Ikehata [7] and $-1 < p < 0$ in Charão-D’Abbicco-Ikehata [4]. In the following notation $A \approx B$ means that $c_1 A \leq B \leq c_2 A$ for some positive constants $c_1, c_2$.

Lemma 2.1. Let $p > -1$ be a real number. Then it holds that

$$I_p(t) \approx t^{-\frac{2+p}{2}}, \quad t \gg 1.$$

In order to deal with the high frequency part of estimates, one relies on the function again

$$J_p(t) = \int_1^\infty (1 + r^2)^{-t} r^p dr$$

for $p \in \mathbb{R}$.

Then the next lemma is important to get estimates on the zone of high frequency to problem (1)-(2). The proof appears in Charão-Ikehata [7].

Lemma 2.2. Let $p \in \mathbb{R}$. Then it holds that

$$J_p(t) \approx \frac{2^{-t}}{t-1}, \quad t \gg 1.$$
For later use we prepare the following simple lemma, which implies the exponential decay of the middle frequency part.

**Lemma 2.3.** Let \( p \in \mathbb{R} \), and \( \eta \in (0, 1] \). Then there is a constant \( C > 0 \) such that
\[
\int_\eta^1 (1 + r^2)^{-t} r^p dr \leq C(1 + \eta^2)^{-t}, \quad t \geq 0.
\]

**Remark 5.** We note that the proof of Lemma 2.1 is done by using simple differential calculus and the theory from hypergeometric functions (see Watson [43]).

**Lemma 2.4.** Let \( c, \nu \) be positive real numbers and \( a \in \mathbb{R} \). Then, there exists a constant \( C > 0 \) such that
\[
t^\nu e^{-c(1 + \log(1 + |\xi|^2))^{at}} \leq C(1 + \log(1 + |\xi|^2))^{-\nu a s}, \quad t \geq 0, \quad \xi \in \mathbb{R}^n.
\]

**Proof.** We set \( s := c(1 + \log(1 + |\xi|^2))^{at} \) for \( t \geq 0 \) and \( \xi \in \mathbb{R}^n \). Then \( t = c^{-1}(1 + \log(1 + |\xi|^2))^{-a s} \) and
\[
t^\nu = c^{-\nu}(1 + \log(1 + |\xi|^2))^{-\nu a s}.
\]
The definition of \( s \) implies
\[
t^\nu e^{-c(1 + \log(1 + |\xi|^2))^{at}} = c^{-\nu}(1 + \log(1 + |\xi|^2))^{-\nu a s} e^{-s}.
\]
Since the function \( \mathbb{R} \ni s \mapsto s^\nu e^{-s} \) is bounded, there exists \( C > 0 \) such that
\[
t^\nu e^{-c(1 + \log(1 + |\xi|^2))^{at}} \leq C(1 + \log(1 + |\xi|^2))^{-\nu a s}.
\]

**Lemma 2.5.** There exists a constant \( C > 0 \) such that
\[
\frac{\sinh x}{x} \leq C e^x
\]
for \( x > 0 \).

Let \( f \in L^1(\mathbb{R}^n) \). Then, we may decompose the Fourier transform of \( f \) as follows:
\[
\hat{f}(\xi) = A_f(\xi) - iB_f(\xi) + P_f,
\]
for all \( \xi \in \mathbb{R}^n \), where \( i := \sqrt{-1} \) and
- \( A_f(\xi) = \int_{\mathbb{R}^n} (\cos(x \cdot \xi) - 1)f(x)dx \),
- \( B_f(\xi) = \int_{\mathbb{R}^n} \sin(x \cdot \xi)f(x)dx \),
- \( P_f = \int_{\mathbb{R}^n} f(x)dx \).

We define the weighted \( L^1 \)-space \( L^{1, \kappa}(\mathbb{R}^n) \) by
\[
L^{1, \kappa}(\mathbb{R}^n) := \{ f \in L^1(\mathbb{R}^n) : \int_{\mathbb{R}^n} (1 + |x|^\kappa)|f(x)|dx < +\infty \}.
\]
The next lemma can be proved in a standard way (see [25]).

**Lemma 2.6.** i) If \( f \in L^1(\mathbb{R}^n) \), then for all \( \xi \in \mathbb{R}^n \) it is true that
\[
|A_f(\xi)| \leq L\|f\|_{L^1} \quad \text{and} \quad |B_f(\xi)| \leq N\|f\|_{L^1}.
\]
ii) If \(0 < \kappa \leq 1\) and \(f \in L^{1,\kappa}(\mathbb{R}^n)\), then for all \(\xi \in \mathbb{R}^n\) it is true that
\[
|A_f(\xi)| \leq K|\xi|^\kappa \|f\|_{L^{1,\kappa}} \quad \text{and} \quad |B_f(\xi)| \leq M|\xi|^\kappa \|f\|_{L^{1,\kappa}}
\]
with \(L, N, K\) and \(M\) positive constants depending only on the dimension \(n\) or \(n\) and \(\kappa\).

2.1. **Existence and uniqueness.** In this subsection we study the existence and uniqueness of solutions to the problem (1)-(2). For this purpose, we follow the work by Charão-Horbach [6] (see also [33]).

The Cauchy problem (1)-(2) rewritten as
\[
(I + L)u_{tt} + L(I + L)u + u_t = 0,
\]
\[
\begin{aligned}
&u(0, x) = u_0(x), \\
&u_t(0, x) = u_1(x).
\end{aligned}
\]

By taking the inner product of the equation in (12) by \(u_t\), we obtain
\[
\frac{1}{2} \frac{d}{dt} \left( \|u_t(t, \cdot)\|^2 + \|L^{1/2}u_t(t, \cdot)\|^2 + \|L^2u(t, \cdot)\|^2 \right) + \|u_t(t, \cdot)\|^2 = 0,
\]
for \(t > 0\). We define the total energy as
\[
E(t) := \|u_t(t, \cdot)\|^2 + \|L^{1/2}u_t(t, \cdot)\|^2 + \|L^2u(t, \cdot)\|^2.
\]

Then, we can observe that \(E(t)\) is a non-increasing function and it is well defined for weak solution of the problem (1)-(2) according to Theorem 2.10.

Associated to (1)-(2) one can choose the following energy space
\[
X = Y^2 \times Y^1.
\]

In this section, to study the existence of solutions is convenient to adopt the following norm
\[
\|f\|_{Y^2} = \left( \int_{\mathbb{R}^n_+} \left( 1 + \log(1 + |\xi|^2) + \log^2(1 + |\xi|^2) \right) |\hat{f}(\xi)|^2 d\xi \right)^{1/2}
\]
in the space \(Y^2\), which is equivalent to its natural norm defined in (10) with \(\delta = 2\).

The associated inner product to this norm is
\[
(u, v)_{Y^2} = \int_{\mathbb{R}^n_+} \left( 1 + \log(1 + |\xi|^2) + \log^2(1 + |\xi|^2) \right) \hat{u} \hat{v} d\xi, \quad u, v \in Y^2.
\]

Now, at least formally, from (12) one can write
\[
u_{tt} = -(I+L)^{-1}(L^2+L)u-(I+L)^{-1}u_t = -(I+L)^{-1}(L^2+L+I)u-(I+L)^{-1}(u_t-u).
\]

Then if we define \(v = u_t\) and \(U(t) = \begin{pmatrix} u \\ v \end{pmatrix}\) we can reduce the second order equation of (12) to a system of the first order as follows:
\[
\frac{dU}{dt} = \begin{pmatrix} u_t \\ 0 \end{pmatrix} = \begin{pmatrix} (-I-L)^{-1}(L^2+L+I)u-(I+L)^{-1}u_t + u_t \\ (I+L)^{-1}(u_t-u) \end{pmatrix}
\]
\[
\begin{pmatrix} 0 & I \\ -A & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} (I+L)^{-1}(u_v) \end{pmatrix}.
\]
Thus, the first order evolution equation to $U$ can be written as
\[
\frac{dU}{dt} = BU + JU, \quad U(0) = (u_0, u_1),
\]
where formally the operator $A$ is given by
\[
A := (I + L)^{-1}(L^2 + L + I) = L + (I + L)^{-1},
\]
and the operators $B, J$ are given by
\[
B = \begin{pmatrix} 0 & I \\ -A & 0 \end{pmatrix}, \quad JU = \begin{pmatrix} 0 \\ (I + L)^{-1}(u - v) \end{pmatrix}, \quad U = (u, v) \in D(B).
\]

We need to give a precise definition of the domains of operators $A$ and $B$. To do that we set
\[
D(A) = \{ u \in Y^2 : \text{there exists } y = y_u \in Y^1 \text{ such that} \}
\]
\[
(Lu, L\psi) + (L^{1/2}u, L^{1/2}\psi) + (u, \psi) = (y, \psi) + (L^{1/2}y, L^{1/2}\psi) \forall \psi \in Y^2 \}.
\]
We observe that $0 \in D(A)$, so $D(A) \neq \emptyset$ and we may define
\[
A : D(A) \to Y^1, \quad \text{by} \quad Au = y_u, \ u \in D(A).
\]
The suitable definition to the domain of $B$ is $D(B) := D(A) \times Y^2$. This definition implies that
\[
B : D(B) \to X = Y^2 \times Y^1,
\]
where $X$ is the energy space defined above.

The following result guarantees us that $A$ is well defined.

**Lemma 2.7.** For $u \in Y^2$, there exists at most one $y \in Y^1$ that satisfies
\[
(Lu, L\psi) + (L^{1/2}u, L^{1/2}\psi) + (u, \psi) = (y, \psi) + (L^{1/2}y, L^{1/2}\psi) \forall \psi \in Y^2.
\]

**Proof.** Suppose that $y_1, y_2 \in Y^1$ satisfy the above relation. Then $y = y_1 - y_2$ satisfies
\[
(y, \psi) + (L^{1/2}y, L^{1/2}\psi) = 0 \quad \text{for each } \psi \in Y^2.
\]
In particular,
\[
(y, \psi) + (L^{1/2}y, L^{1/2}\psi) = 0 \quad \text{for each } \psi \in C_0^\infty(\mathbb{R}^n).
\]
By density of $C_0^\infty(\mathbb{R}^n)$ in $Y^1$, there exists a sequence $\{y_n\} \subset C_0^\infty(\mathbb{R}^n)$ such that
\[
\|y_n - y\|_{Y^1} \to 0.
\]
This implies $\|y_n\|_{Y^1} \to \|y\|_{Y^1}$ and $\|y_n - y\|_{Y^1}^2 \to 0$. Thus we have
\[
\|y_n\|_{Y^1}^2 - 2(y_n, y)_{Y^1} + \|y\|_{Y^1}^2 = \|y_n - y\|_{Y^1}^2 \to 0,
\]
which implies
\[
\lim_{n \to \infty} (y_n, y)_{Y^1} = \|y\|_{Y^1}^2.
\]
On the other hand, by (16) and the density argument we get
\[
(y, y_n)_{Y^1} = (y, y) + (L^{1/2}y, L^{1/2}y_n) = 0,
\]
which implies $\lim_{n \to \infty} (y, y_n)_{Y^1} = 0$. Therefore, we can conclude that $\|y\|_{Y^1} = 0$. \qed

**Lemma 2.8.** $D(A) \subset Y^3$ and there exists $c > 0$ such that
\[
\|u\|_{Y^3} \leq c\|Au\|_{Y^1},
\]
for all $u \in D(A)$. 

Proof. Let $u \in D(A)$. Then there is $y \in Y^1$ that satisfies

$$(Lu, L\psi) + (L^{1/2}u, L^{1/2}\psi) + (u, \psi) = (y, \psi) + (L^{1/2}y, L^{1/2}\psi) \quad \forall \psi \in Y^2.$$ 

We define $F : Y^1 \to \mathbb{R}$ by

$$(F, \psi) = (y, \psi) + (L^{1/2}y, L^{1/2}\psi), \quad \psi \in Y^1.$$ 

Then $F$ is well defined, because $y, \psi, L^{1/2}y$ and $L^{1/2}\psi$ are in $L^2(\mathbb{R}^n)$ and $F$ is linear. Furthermore, $F$ is a continuous operator. In fact

$$
|\langle F, \psi \rangle| \leq |\langle y, \psi \rangle| + |\langle L^{1/2}y, L^{1/2}\psi \rangle| \leq \|y\|\|\psi\| + \|L^{1/2}y\|\|L^{1/2}\psi\|
= \|\hat{y}\|\hat{\psi}\| + \|\hat{y}d(1 + |\xi|^2)^{1/2}\|\|\hat{\psi}d(1 + |\xi|^2)^{1/2}\|
\leq 2(1 + \log(1 + |\xi|^2))^{1/2}\|\hat{y}\|\|\hat{\psi}\|\|1 + \log(1 + |\xi|^2)^{1/2}\|
= 2\|y\|_{Y^1}\|\psi\|_{Y^1}.
$$

Since $S(\mathbb{R}^n) \subset Y^2 \subset Y^1$, we have $(Y^1)' \subset (Y^2)' \subset S'(\mathbb{R}^n)$. In other words, $F$ can be seen as a tempered distribution and for all $\psi \in S(\mathbb{R}^n)$ it holds that

$$
\langle Lu, L\psi \rangle + (L^{1/2}u, L^{1/2}\psi) + (u, \psi) = \langle F, \psi \rangle.
$$

Thus

$$L^2u + Lu + u = F \quad \text{in} \quad S'(\mathbb{R}^n).$$

By applying the Fourier transform, the definition of $F$ and the operator $L$, we arrive that

$$[\log^2(1 + |\xi|^2) + \log(1 + |\xi|^2)] \hat{u} = \hat{F} = [1 + \log(1 + |\xi|^2)] \hat{y},$$

that is,

$$\hat{y} = \frac{\log^2(1 + |\xi|^2) + \log(1 + |\xi|^2) + 1}{1 + \log(1 + |\xi|^2)} \hat{u},$$

or

$$\sqrt{1 + \log(1 + |\xi|^2)^2} \hat{y} = \frac{\log^2(1 + |\xi|^2) + \log(1 + |\xi|^2) + 1}{\sqrt{1 + \log(1 + |\xi|^2)}} \hat{u}.$$

Then

$$\|y\|_{Y^1} = \int_{\mathbb{R}^n} (1 + \log(1 + |\xi|^2)) |\hat{y}|^2 d\xi
= \int_{\mathbb{R}^n} \left[\frac{1 + \log^2(1 + |\xi|^2) + \log(1 + |\xi|^2)}{1 + \log(1 + |\xi|^2)}\right] |\hat{u}|^2 d\xi.
$$

It is easy to note that

$$(1 + \log(1 + |\xi|^2))^{-1} (\log^2(1 + |\xi|^2) + \log(1 + |\xi|^2) + 1)^2 \approx (1 + \log(1 + |\xi|^2))^3.$$ 

Then

$$\|y\|_{Y^1} \approx \int_{\mathbb{R}^n} (1 + \log(1 + |\xi|^2)^3) |\hat{u}|^2 d\xi = \|u\|_{Y^2}^2.$$

Therefore, $u \in Y^3$ and, since $y = Au$, there exists constant $c > 0$ such that

$$\|u\|_{Y^3} \leq c\|Au\|_{Y^1}.$$

\[\Box\]

**Lemma 2.9.** $Y^3 \subset D(A)$. 

Proof. Initially, we observe that $Y^3 \subset Y^2$, because $(1 + \log(1 + |\xi|^2))^2 \leq (1 + \log(1 + |\xi|^2))^3$.

Now, let $u \in Y^3$, then $L^{3/2}u \in L^2(\mathbb{R}^n)$ and we need to show that there exists $y \in Y^1$ such that

$$(L^{3/2}u, L^{1/2}\psi) + (Lu, L\psi) + (u, \psi) = (y, \psi) + (L^{1/2}y, L^{1/2}\psi), \text{ for each } \psi \in Y^2.$$ 

We define $a : Y^1 \times Y^1 \rightarrow \mathbb{R}$ by

$$a(\psi, \phi) = (\psi, \phi) + (L^{1/2}\psi, L^{1/2}\phi).$$

This function is well-defined and is a symmetric bilinear form. Moreover,

- $a$ is continuous, in fact

$$|a(\psi, \phi)| \leq |(\psi, \phi)| + |(L^{1/2}\psi, L^{1/2}\phi)| \leq \|\psi\| \|\phi\| + \|L^{1/2}\psi\| \|L^{1/2}\phi\|$$

$$= \|\psi\| \|\phi\| + \|\log(1 + |\xi|^2)\| \|\log(1 + |\xi|^2)\|$$

$$\leq 2\|\sqrt{1 + \log(1 + |\xi|^2)}\| \|\sqrt{1 + \log(1 + |\xi|^2)}\|$$

$$= 2\|\psi\|_{Y^1} \|\phi\|_{Y^1}.$$ 

- $a$ is coercive, because

$$a(\phi, \phi) = (\phi, \phi) + (L^{1/2}\phi, L^{1/2}\phi) \geq \|\phi\|^2 + \|L^{1/2}\phi\|^2 = \|\phi\|^2_{Y^1}.$$ 

On the other hand, we define $G : Y^1 \rightarrow \mathbb{R}$ by

$$\langle G, \psi \rangle = (L^{3/2}u, L^{1/2}\psi) + (L^{1/2}u, L^{1/2}\psi) + (u, \psi).$$

This map is linear and continuous, in fact

$$| \langle G, \psi \rangle | = |(L^{3/2}u, L^{1/2}\psi)| + |(L^{1/2}u, L^{1/2}\psi)| + |(u, \psi)|$$

$$\leq \|L^{3/2}u\| \|L^{1/2}\psi\| + \|Lu\| \|\psi\| + \|u\| \|\psi\|$$

$$= \|\log(1 + |\xi|^2)^{3/2}\| \|\log(1 + |\xi|^2)^{1/2}\| + \|\log(1 + |\xi|^2)^{1/2}\| \|u\| \|\hat{\psi}\|$$

$$\leq \|\log(1 + |\xi|^2)^{3/2}\| \|\log(1 + |\xi|^2)^{1/2}\| + \|\log(1 + |\xi|^2)^{1/2}\|$$

$$+ \|\log(1 + |\xi|^2)^{1/2}\| \|u\| \|\hat{\psi}\|$$

$$= \left(\|\log(1 + |\xi|^2)^{3/2}\| + \|\log(1 + |\xi|^2)^{1/2}\| \right) \|u\| \|\hat{\psi}\|$$

$$= \|u\| \|\psi\|_{Y^1} \|\phi\|_{Y^1}.$$ 

Thus, we have a continuous linear functional $G$ and a coercive continuous bilinear form in the Hilbert Space $Y^1$. By the Lax-Milgram Theorem, there exists a unique $y \in Y^1$ such that

$$\langle G, \psi \rangle = a(y, \psi) \text{ for all } \psi \in Y^1.$$ 

In particular, this identity is valid for all $\psi \in D(\mathbb{R}^n)$, that is,

$$(L^{3/2}u, L^{1/2}\psi) + (L^{1/2}u, L^{1/2}\psi) + (u, \psi) = (y, \psi) + (L^{1/2}y, L^{1/2}\psi), \psi \in D(\mathbb{R}^n).$$

Since $D(\mathbb{R}^n)$ is dense in $Y^2$, such identity is valid for all $\psi \in Y^2$. Hence, this implies that $u \in D(A)$.

Our goal at this section is to show that $B$ is an infinitesimal generator of a contraction $C^0$-semigroup. For this, we apply the Lumer-Phillips theorem. First, we note that $D(B) = Y^3 \times Y^2$ is dense in the energy space $X = Y^2 \times Y^1$.

In order to prove that $B$ is dissipative, we consider the Hilbert space $X = Y^2 \times Y^1$ with the following inner product:

$$\langle (u_1, v_1), (u_2, v_2) \rangle_X = (u_1, u_2)_{Y^2} + (v_1, v_2)_{Y^1}, \text{ for } u_1, u_2 \in Y^2, v_1, v_2 \in Y^1.$$
Thus, \( \text{Re} \left( \int_{\mathbb{R}^n} (1 + \log(1 + |\xi|^2)) \psi_1 \overline{\psi}_2 d\xi \right) \)

according to the corresponding definition of norm in \( Y^1 \) given by (10).

Now, for \( (u, v) \in D(B) \) one can observe that
\[
\langle B(u, v), (u, v) \rangle_X = \langle (v, -Au), (u, v) \rangle_X = \langle v, u \rangle_{Y^2} + \langle -Au, v \rangle_{Y^1}
\]
\[
= \int_{\mathbb{R}^n} (1 + \log(1 + |\xi|^2) + \log^2(1 + |\xi|^2)) \hat{\psi} \overline{\hat{v}} d\xi
\]
\[
- \int_{\mathbb{R}^n} (1 + \log(1 + |\xi|^2)) \hat{A} \hat{u} \overline{\hat{v}} d\xi
\]
\[
= \int_{\mathbb{R}^n} (1 + \log(1 + |\xi|^2) + \log^2(1 + |\xi|^2)) \hat{\psi} \overline{\hat{v}} d\xi
\]
\[
- \int_{\mathbb{R}^n} (1 + \log(1 + |\xi|^2)) \frac{1 + \log(1 + |\xi|^2) + \log^2(1 + |\xi|^2)}{1 + \log(1 + |\xi|^2)} \hat{u} \overline{\hat{v}} d\xi
\]
\[
= \int_{\mathbb{R}^n} (1 + \log(1 + |\xi|^2) + \log^2(1 + |\xi|^2)) (\hat{\psi} - \hat{u}) \overline{\hat{v}} d\xi
\]
\[
= 2i \int_{\mathbb{R}^n} (1 + \log(1 + |\xi|^2) + \log^2(1 + |\xi|^2)) \text{Im}(\hat{u} \overline{\hat{v}}) d\xi.
\]

Thus, \( \text{Re} \langle B(u, v), (u, v) \rangle_X = 0 \) for all \( (u, v) \in D(B) \) and \( B \) is dissipative.

Now we need to prove that \((I - B)(X) = X\). Clearly \((I - B)D(B) \subset X\). In its turn, let \((f, g) \in X = Y^2 \times Y^1\). Then, we will prove that there exists \((u, v) \in D(B)\) such that \((I - B)(u, v) = (f, g)\).

We define a mapping \( a : Y^2 \times Y^2 \to \mathbb{R} \) by
\[
a(\varphi, \psi) = (\varphi, \psi)_{Y^1} + (\varphi, \psi)_{Y^2}.
\]

Then \( a \) is a symmetrical bilinear form, which is
\begin{itemize}
  \item continuous:
    \[|a(\varphi, \psi)| \leq \|\varphi\|_{Y^1}\|\psi\|_{Y^1} + \|\varphi\|_{Y^2}\|\psi\|_{Y^2} \leq 2\|\varphi\|_{Y^2}\|\psi\|_{Y^2},\]
  \item coercive:
    \[a(\psi, \psi) = (\psi, \psi)_{Y^1} + (\psi, \psi)_{Y^2} = \|\psi\|^2_{Y^1} + \|\psi\|^2_{Y^2} \geq \|\psi\|^2_{Y^2}.
\end{itemize}

Furthermore, we consider the linear functional \( F : Y^2 \to \mathbb{R} \) given by
\[
\langle F, \psi \rangle = (f + g, \psi)_{Y^1},
\]
which is well-defined because of \( Y^2 \subset Y^1 \). Also, one has
\[
|\langle F, \psi \rangle| \leq \|f + g\|_{Y^1}\|\psi\|_{Y^1} \leq \|f + g\|_{Y^1}\|\psi\|_{Y^2},
\]
which just proves the continuity of \( F \).

Thus, we can apply the Lax-Milgram theorem to get the existence of a unique \( u \in Y^2 \) such that
\[
a(u, \psi) = F(\psi) \text{ for all } \psi \in Y^2.
\]

In other words,
\[
(u, \psi) + (L^{1/2}u, L^{1/2}\psi) + (u, \psi) + (L^{1/2}u, L^{1/2}\psi) + (Lu, L\psi)
\]
\[
= (f + g, \psi) + (L^{1/2}(f + g), L^{1/2}\psi)
\]
for all $\psi \in Y^2$. In particular, this equality is valid for all $\psi \in D(\mathbb{R}^n)$ and we have the following identity in $D'(\mathbb{R}^n)$

$$Au + u = f + g.$$ 

Finally, we observe that $u \in Y^3$. In fact, by applying the Fourier transform, we obtain

$$\hat{Au} + \hat{u} = \hat{f} + \hat{g}$$

and

$$\frac{1 + \log(1 + |\xi|^2) + \log^2(1 + |\xi|^2)}{\log(1 + |\xi|^2)} \hat{u} + \hat{u} = \hat{f} + \hat{g},$$

which is equivalent to

$$\frac{1 + \log(1 + |\xi|^2) + \log^2(1 + |\xi|^2)}{\sqrt{1 + \log(1 + |\xi|^2)}} \hat{u} = \sqrt{1 + \log(1 + |\xi|^2)}(\hat{f} + \hat{g} - \hat{u}).$$

From the fact that

$$\frac{1 + \log(1 + |\xi|^2) + \log^2(1 + |\xi|^2)}{\sqrt{1 + \log(1 + |\xi|^2)}} \approx (1 + \log(1 + |\xi|^2))^{3/2},$$

we have

$$(1 + \log(1 + |\xi|^2))^3|\hat{u}|^2 \approx (1 + \log(1 + |\xi|^2))|\hat{f} + \hat{g} - \hat{u}|^2.$$

Now, since $f, g, u \in Y^1$ we can conclude

$$\int_{\mathbb{R}^n} (1 + \log(1 + |\xi|^2))^3|\hat{u}(\xi)|^2d\xi < \infty.$$ 

This estimate proves that $u \in Y^3$. Then, $v = u - f \in Y^2$ because of $f \in Y^2$ and we have obtained that

$$(I - B)(u, v) = (f, g).$$

Hence, we have proved that $(I - B)D(B) = X$. Therefore, by the Lumer-Phillips theorem the operator $B$ is an infinitesimal generator of a $C^0$—semigroup of contractions.

Next we want to prove that $J : X \to X$ is a linear bounded operator. The linearity is simple. The boundedness is given by the following series of inequalities:

$$\|J(u, v)\|_{X}^2 = \left\| \sqrt{1 + \log(1 + |\xi|^2)} \frac{\hat{u} - \hat{v}}{1 + \log(1 + |\xi|^2)} \right\|^2 \leq 2 \int_{\mathbb{R}^n} \frac{|\hat{u}|^2}{1 + \log(1 + |\xi|^2)} d\xi + 2 \int_{\mathbb{R}^n} \frac{|\hat{v}|^2}{1 + \log(1 + |\xi|^2)} d\xi \leq 2 \int_{\mathbb{R}^n} |\hat{u}|^2 d\xi + 2 \int_{\mathbb{R}^n} (1 + \log(1 + |\xi|^2))|\hat{v}|^2 d\xi \leq 2\|u\|_{Y^2}^2 + 2\|v\|_{Y^1}^2 = 2\|(u, v)\|_{X}^2.$$ 

Therefore, $B + J$ is an infinitesimal generator of a $C^0$—semigroup in $X$ and we have arrived at the following result.

**Theorem 2.10.** Let $n \geq 1$ and $(u_0, u_1) \in Y^3 \times Y^2$. Then the problem (1)-(2) admits a unique solution in the class

$$u \in C([0, \infty), Y^3) \cap C^1([0, \infty), Y^2) \cap C^2([0, \infty), Y^1).$$
Moreover, for initial data \((u_0, u_1) \in X = Y^2 \times Y^1\) the problem (1)–(2) admits a unique solution in the class
\[ u \in C([0, \infty), Y^2) \cap C^1([0, \infty), Y^1) \cap C^2([0, \infty), L^2). \]

Remark 6. Note that for initial data \((u_0, u_1) \in D(A) \times Y^2 = Y^3 \times Y^2\) the solution given by Theorem 2.10 is a strong solution of the wave equation (according to (14)) with a weak dissipation term
\[ u_{tt} + Lu + (I + L)^{-1}u_t = 0, \]
but only a weak solution to the problem (1) with the same initial data.

3. Asymptotic behavior via multiplier method. We begin with this section by considering the Cauchy problem in the Fourier space associated to the problem (1)–(2) as follows
\[ (1 + \log(1 + |\xi|^2))\hat{u}_{tt} + \log(1 + |\xi|^2)(1 + \log(1 + |\xi|^2))\hat{u} + \hat{u}_t = 0, \quad t > 0, \xi \in \mathbb{R}^n \]
\[ \hat{u}(0, \xi) = \hat{u}_1(\xi), \quad \xi \in \mathbb{R}^n. \]

Multiplying the equation in (18) by \(\overline{\hat{u}}_t\) one can get the following pointwise energy identity
\[ \frac{d}{dt}E_0(t, \xi) + |\hat{u}_t(t, \xi)|^2 = 0, \quad t > 0, \xi \in \mathbb{R}^n \]
where \(E_0(t, \xi)\) is defined for \(t \geq 0\) and \(\xi \in \mathbb{R}^n\) by
\[ E_0(t, \xi) = \frac{(1 + \log(1 + |\xi|^2))|\hat{u}_t|^2}{2} + \frac{\log(1 + |\xi|^2)(1 + \log(1 + |\xi|^2))|\hat{u}|^2}{2}, \]
is the total density of the energy for the system (18). From (19) we see that \(E_0(t, \xi)\) is a non-increasing function of \(t\) for each \(\xi\).

Lemma 3.1. Consider the following three functions
\[ \varphi(\xi) = \log(1 + |\xi|^2)(1 + \log(1 + |\xi|^2)), \quad \psi(\xi) = \frac{1}{1 + \log(1 + |\xi|^2)}, \]
\[ \phi(\xi) = \sqrt{\log(1 + |\xi|^2)} \]
defined for \(\xi \in \mathbb{R}^n_+\). Then, there is a unique real number \(\delta_0 \in (0, 1)\) such that
(i). \(|\xi| \leq \delta_0 \Rightarrow \varphi(\xi) \leq \psi(\xi)\) and \(\varphi(\xi) \leq \phi(\xi)\),
(ii). \(|\xi| \geq \delta_0 \Rightarrow \psi(\xi) \leq \varphi(\xi)\) and \(\psi(\xi) \leq \phi(\xi)\).

Proof. Let \(\theta(r) = (1 + \log(1 + r^2))\sqrt{\log(1 + r^2)} - 1\) for \(r \geq 0\). We note that \(\theta(0) = -1\) and \(\theta(1) = (1 + \log 2)\sqrt{\log 2} - 1 > 0\). The continuity of this function implies there exists \(0 < \delta_0 < 1\) such that \(\theta(\delta_0) = 0\). Moreover, \(\theta(r)\) is an increasing function of \(r \geq 0\) and such fact implies that \(\delta_0 > 0\) is a unique number satisfying \(-1 \leq \theta(r) \leq 0\) for all \(0 \leq r \leq \delta_0\) and \(\theta(r) > 0\) for \(r > \delta_0\). That is, if \(0 \leq r \leq \delta_0\), then one has
\[ (1 + \log(1 + r^2))\sqrt{\log(1 + r^2)} \leq 1. \]
Thus, for \(0 \leq r \leq \delta_0\),
\[ (1 + \log(1 + r^2))(1 + r^2) \leq \frac{1}{1 + \log(1 + r^2)}. \]
Similarly, if \(r > \delta_0\), then
\[ (1 + \log(1 + r^2))\sqrt{\log(1 + r^2)} > 1. \]
Then, by adding (19) and (22), we get the following identity:

\[
(1 + \log(1 + r^2)) \log(1 + r^2) > \sqrt{\log(1 + r^2)} \quad \text{and} \quad \sqrt{\log(1 + r^2)} > \frac{1}{1 + \log(1 + r^2)},
\]

for \( r > \delta_0 \). These imply the desired estimates (i) and (ii). \( \square \)

For \( \delta_0 > 0 \) given in Lemma 3.1, we define the following function of \( \xi \in \mathbb{R}^n \) such that

\[
\rho(\xi) := \begin{cases} 
\frac{1}{2} \log(1 + |\xi|^2)(1 + \log(1 + |\xi|^2)), & \text{for } |\xi| \leq \delta_0, \\
\frac{1}{2(1 + \log(1 + |\xi|^2))}, & \text{for } |\xi| > \delta_0.
\end{cases}
\]  

As a consequence of Lemma 3.1, we have

\[
\rho(\xi) = \min \left\{ \frac{\log(1 + |\xi|^2)(1 + \log(1 + |\xi|^2))}{2}, \frac{1}{2(1 + \log(1 + |\xi|^2))}, \frac{\sqrt{\log(1 + |\xi|^2)}}{2} \right\}.
\]

By multiplying the equation (18) by \( \rho(\xi)\overline{u} \) we obtain the identity

\[
\rho(\xi)(1 + \log(1 + |\xi|^2))\ddot{u}_t \overline{u} + \rho(\xi) \log(1 + |\xi|^2)(1 + \log(1 + |\xi|^2))\dot{u}\overline{u} + \frac{\rho(\xi)}{2} \frac{d}{dt} |\dot{u}|^2 = 0
\]

for all \( t > 0 \) and \( \xi \in \mathbb{R}^n \). Taking the real part on the last identity we arrive at

\[
\frac{d}{dt} \left[ \rho(\xi)(1 + \log(1 + |\xi|^2))\Re(\dot{u}_t \overline{u}) + \frac{\rho(\xi)}{2} |\dot{u}|^2 \right] + \rho(\xi) \log(1 + |\xi|^2)(1 + \log(1 + |\xi|^2))|\dot{u}|^2 = \rho(\xi)(1 + \log(1 + |\xi|^2))|\ddot{u}_t|^2.
\]

To proceed further we need to define the following functions on \((0, \infty) \times \mathbb{R}_\xi^n\) such that

\[
E(t, \xi) = E_0(t, \xi) + \rho(\xi)(1 + \log(1 + |\xi|^2))|\dot{u}_t||\dot{u}| + \frac{\rho(\xi)}{2} |\dot{u}|^2,
\]

\[
F(t, \xi) = |\ddot{u}_t|^2 + \rho(\xi) \log(1 + |\xi|^2)(1 + \log(1 + |\xi|^2))|\dot{u}|^2,
\]

\[
R(t, \xi) = \rho(\xi)(1 + \log(1 + |\xi|^2))|\ddot{u}_t|^2.
\]

Then, by adding (19) and (22), we get the following identity

\[
\frac{d}{dt} E(t, \xi) + F(t, \xi) = R(t, \xi), \quad t > 0, \xi \in \mathbb{R}_\xi^n.
\]  

**Lemma 3.2.** It holds that

\[
\frac{1}{2} E_0(t, \xi) \leq E(t, \xi) \leq 3E_0(t, \xi), \quad t > 0, \xi \in \mathbb{R}_\xi^n.
\]

**Proof.** By Lemma 3.1 we have for \( t > 0 \) and \( \xi \in \mathbb{R}_\xi^n \),

\[
E(t, \xi) \leq E_0(t, \xi) + \rho(\xi)(1 + \log(1 + |\xi|^2))|\dot{u}_t||\dot{u}| + \frac{\rho(\xi)}{2} |\dot{u}|^2
\]

\[
\leq E_0(t, \xi) + \frac{1 + \log(1 + |\xi|^2)}{2} |\dot{u}_t|^2 + \frac{\rho(\xi)}{2(1 + \log(1 + |\xi|^2))} |\dot{u}|^2 + \frac{\rho(\xi)}{2} |\dot{u}|^2
\]

\[
\leq E_0(t, \xi) + \frac{1 + \log(1 + |\xi|^2)}{2} |\dot{u}_t|^2 + \frac{\log(1 + |\xi|^2)(1 + \log(1 + |\xi|^2))}{8} |\dot{u}|^2
\]

\[
+ \frac{\log(1 + |\xi|^2)(1 + \log(1 + |\xi|^2))}{4} |\dot{u}|^2
\]

\[
\leq 3E_0(t, \xi),
\]

according to the definition of \( E_0(t, \xi) \) in (19).
On the other hand, one has

\[-\rho(\xi)(1 + \log(1 + |\xi|^2))\Re(\hat{u}_t \bar{u}) \leq \rho(\xi)(1 + \log(1 + |\xi|^2))|\hat{u}_t||\hat{u}| \]

\[\leq \frac{1 + \log(1 + |\xi|^2)}{4} |\hat{u}_t|^2 + \rho(\xi)^2(1 + \log(1 + |\xi|^2))|\hat{u}|^2 \]

\[\leq \frac{1 + \log(1 + |\xi|^2)}{4} |\hat{u}_t|^2 \]

\[+ \frac{1}{4} \log(1 + |\xi|^2)(1 + \log(1 + |\xi|^2))|\hat{u}|^2,\]

for \( t > 0 \) and \( \xi \in \mathbb{R}_\xi^n \).

Thus, the last estimate implies

\[ E(t, \xi) = E_0(t, \xi) + \rho(\xi)(1 + \log(1 + |\xi|^2))\Re(\hat{u}_t \bar{u}) + \frac{\rho(\xi)}{2}|\hat{u}|^2 \]

\[\geq E_0(t, \xi) + \rho(\xi)(1 + \log(1 + |\xi|^2))\Re(\hat{u}_t \bar{u}) \]

\[\geq \left( \frac{1}{2} - \frac{1}{4} \right) (1 + \log(1 + |\xi|^2))|\hat{u}_t|^2 \]

\[+ \left( \frac{1}{2} - \frac{1}{4} \right) \log(1 + |\xi|^2)(1 + \log(1 + |\xi|^2))|\hat{u}|^2 \]

\[= \frac{1}{4} (1 + \log(1 + |\xi|^2))|\hat{u}_t|^2 + \frac{1}{4} \log(1 + |\xi|^2)(1 + \log(1 + |\xi|^2))|\hat{u}|^2 \]

\[= \frac{1}{2} E_0(t, \xi), \]

for \( t > 0 \) and \( \xi \in \mathbb{R}_\xi^n \). These imply the desired estimates. \( \square \)

Now, we need the next lemma.

**Lemma 3.3.** It is true that

\[ \frac{d}{dt} E(t, \xi) + \frac{\rho(\xi)}{2} E(t, \xi) \leq 0, \quad t > 0, \quad \xi \in \mathbb{R}_\xi^n. \]

**Proof.** By definitions of \( E(t, \xi) \), \( F(t, \xi) \), \( R(t, \xi) \), \( \rho(\xi) \) and (24), one obtains a series of inequalities

\[ \frac{d}{dt} E(t, \xi) + \frac{\rho(\xi)}{2} E(t, \xi) = R(t, \xi) - F(t, \xi) + \frac{\rho(\xi)}{2} E(t, \xi) \]

\[\leq R(t, \xi) - F(t, \xi) + \frac{3\rho(\xi)}{2} E_0(t, \xi) \]

\[= \left( \frac{7}{4} \rho(\xi)(1 + \log(1 + |\xi|^2)) - 1 \right) |\hat{u}_t|^2 \]

\[- \frac{1}{4} \rho(\xi) \log(1 + |\xi|^2)(1 + \log(1 + |\xi|^2))|\hat{u}|^2 \]

\[\leq \frac{1}{8} |\hat{u}_t|^2 - \frac{1}{4} \rho(\xi) \log(1 + |\xi|^2)(1 + \log(1 + |\xi|^2))|\hat{u}|^2 \]

\[\leq 0 \]

for \( t > 0 \) and \( \mathbb{R}_\xi^n \). The lemma is proved. \( \square \)

Now, it follows from Lemma 3.3 that

\[ E(t, \xi) \leq E(0, \xi) e^{-\frac{\rho(\xi)}{2} t}, \]
for $t > 0$ and $\xi \in \mathbb{R}^n_\varepsilon$. By combining the last estimate with Lemma 3.2, one can deduce the important pointwise estimate,

$$E_0(t, \xi) \leq 6E_0(0, \xi)e^{-\frac{\rho L^2}{2}t},$$

for $t > 0$ and $\xi \in \mathbb{R}^n_\varepsilon$.

The above estimate combined with the definition of $E_0(t, \xi)$ in (19) implies the following crucial pointwise estimate.

**Proposition 1.** It holds that

$$(1 + \log (1 + |\xi|^2)) |\hat{u}_t(t, \xi)|^2 + \log (1 + |\xi|^2) (1 + \log (1 + |\xi|^2)) |\hat{u}(t, \xi)|^2 \leq 6(1 + \log (1 + |\xi|^2)) e^{-\frac{\rho L^2}{2}t} |\hat{u}_1(\xi)|^2 + 6 \log (1 + |\xi|^2) (1 + \log (1 + |\xi|^2)) e^{-\frac{\rho L^2}{2}t} |\hat{u}_0(\xi)|^2$$

for all $t > 0$ and $\xi \in \mathbb{R}^n_\varepsilon$, and

$$|\hat{u}(t, \xi)|^2 \leq 6 e^{-\frac{\rho L^2}{2}t} |\hat{u}_1(\xi)|^2 + 6 e^{-\frac{\rho L^2}{2}t} |\hat{u}_0(\xi)|^2$$

for all $t > 0$ and $\xi \in \mathbb{R}^n_\varepsilon$, $\xi \neq 0$.

As a consequence of the second estimate in Proposition 1 one can get the following result.

**Proposition 2.** Let $n \geq 3$, and let $u(t, \xi)$ be the solution to problem (1)-(2). Suppose $u_0 \in L^1(\mathbb{R}^n) \cap Y^\frac{n-2}{2}$, $u_1 \in L^1(\mathbb{R}^n) \cap Y^\frac{n-2}{2}$. Then, the following estimate holds:

$$\int_{\mathbb{R}^n_\varepsilon} |u(t, x)|^2 dx \leq C_1 t^{-\frac{n-2}{2}} \left[ \|u_1\|_{L^2}^2 + \|u_0\|_{Y^\frac{n-2}{2}}^2 \right] + C_2 t^{-\frac{n}{2}} \left[ \|u_1\|_{Y^\frac{n-2}{2}}^2 + \|u_0\|_{L^1}^2 \right],$$

for $t > 0$, where $C_1$ and $C_2$ are positive constants depending only on $n$.

**Proof.** Let $\delta_0 > 0$ be a given real number obtained in Lemma 3.1. To prove the proposition above one needs to consider separately the zones of low and high frequency.

1) **Estimate on the zone $|\xi| \leq \delta_0$**

On this region one notices $\rho(\xi) = \frac{1}{2} \log (1 + |\xi|^2) (1 + \log (1 + |\xi|^2))$. Then, one can observe that $1 \leq 1 + \log (1 + |\xi|^2) \leq 1 + \log (1 + \delta_0^2)$ for $|\xi| \leq \delta_0$. Thus we get

$$\frac{1}{2} \log (1 + |\xi|^2) \leq \rho(\xi) \leq \frac{1 + \log (1 + \delta_0^2)}{2} \log (1 + \delta_0^2), \quad |\xi| \leq \delta_0.$$

Then, by applying the second estimate of Proposition 1, one obtains

$$\int_{|\xi| \leq \delta_0} |\hat{u}|^2 d\xi \leq 6 \int_{|\xi| \leq \delta_0} \frac{e^{-\rho L^2 t}}{\log (1 + |\xi|^2)} |\hat{u}_1|^2 d\xi + 6 \int_{|\xi| \leq \delta_0} e^{-\rho L^2 t} |\hat{u}_0|^2 d\xi$$

$$\leq 6 \int_{|\xi| \leq \delta_0} \frac{e^{-\frac{\log (1 + |\xi|^2)}{4}}}{\log (1 + |\xi|^2)} |\hat{u}_1|^2 d\xi + 6 \int_{|\xi| \leq \delta_0} e^{-\frac{\log (1 + |\xi|^2)}{4}} |\hat{u}_0|^2 d\xi$$

$$= 6 \int_{|\xi| \leq \delta_0} (1 + |\xi|^2)^{-\frac{1}{4}} \frac{1}{\log (1 + |\xi|^2)} |\hat{u}_1|^2 d\xi$$

$$+ 6 \int_{|\xi| \leq \delta_0} (1 + |\xi|^2)^{-\frac{1}{4}} |\hat{u}_0|^2 d\xi$$

$$\leq 6 \|\hat{u}_1\|_{L^1}^2 \int_{|\xi| \leq \delta_0} (1 + |\xi|^2)^{-\frac{1}{4}} \frac{1}{\log (1 + |\xi|^2)} d\xi$$
where we just applied Lemma 2.4 with \( \nu \) the definition of \( \rho \) with some constants \( C \).

2) Estimate on the zone \( |\xi| \geq \delta_0 \)

In this case, one notices \( \rho(\xi) = \frac{1}{2(1 + \log(1 + |\xi|^2))} \). By Proposition 1 and the definition of \( \rho(\xi) \) we have

\[
\int_{|\xi| \geq \delta_0} |\hat{u}|^2 d\xi \leq 6 \int_{|\xi| \geq \delta_0} \frac{e^{-\frac{\xi_1^2}{4(1 + |\xi|^2)}}}{\log(1 + |\xi|^2)} |\hat{u}_1|^2 d\xi + 6 \int_{|\xi| \geq \delta_0} e^{-\frac{\xi_1^2}{4(1 + |\xi|^2)}} |\hat{u}_0|^2 d\xi
\]

\[
= 6 \int_{|\xi| \geq \delta_0} \frac{e^{-\frac{\xi_1^2}{4(1 + |\xi|^2)}}}{\log(1 + |\xi|^2)} |\hat{u}_1|^2 d\xi + 6 \int_{|\xi| \geq \delta_0} e^{-\frac{\xi_1^2}{4(1 + |\xi|^2)}} |\hat{u}_0|^2 d\xi
\]

\[
\leq Ct^{-\nu} \int_{|\xi| \geq \delta_0} \frac{(1 + \log(1 + |\xi|^2))}{\log(1 + |\xi|^2)} |\hat{u}_1|^2 d\xi + C t^{-\nu} \int_{|\xi| \geq \delta_0} |\hat{u}_0|^2 d\xi
\]

\[
\leq Ct^{-\frac{n}{2}} \int_{|\xi| \geq \delta_0} \frac{(1 + \log(1 + |\xi|^2))}{\log(1 + |\xi|^2)} |\hat{u}_1|^2 d\xi + C t^{-\frac{n-2}{2}} \int_{|\xi| \geq \delta_0} (1 + \log(1 + |\xi|^2)) \frac{n-2}{2} |\hat{u}_0|^2 d\xi,
\]

where we just applied Lemma 2.4 with \( \nu = \frac{n}{2} \) and \( a = -1 \), and \( \nu' = \frac{n-2}{2} \) and \( a = -1 \) to the last two integrals.

Thus we may conclude that

\[
\int_{|\xi| \geq \delta_0} |\hat{u}|^2 d\xi = Ct^{-\frac{n}{2}} \int_{|\xi| \geq \delta_0} \frac{1 + \log(1 + |\xi|^2)}{\log(1 + |\xi|^2)} (1 + \log(1 + |\xi|^2)) \frac{n-2}{2} |\hat{u}_1|^2 d\xi
\]
+ \, C t^{-\frac{n-2}{2}} \int_{|\xi|\geq \delta_0} (1 + \log(1 + |\xi|^2))^{\frac{n-2}{2}} |\hat{u}_0|^2 d\xi \\
\leq \, C t^{-\frac{n}{2}} \int_{|\xi|\geq \delta_0} (1 + \log(1 + |\xi|^2))^{\frac{n}{2}} |\hat{u}_1|^2 d\xi \\
+ \, C t^{-\frac{n-2}{2}} \int_{|\xi|\geq \delta_0} (1 + \log(1 + |\xi|^2))^{\frac{n-2}{2}} |\hat{u}_0|^2 d\xi \\
\leq \, C t^{-\frac{n}{2}} \|u_1\|_{\dot{Y}^{n-2}}^2 + C t^{-\frac{n-2}{2}} \|u_0\|_{\dot{Y}^{n-2}}^2 ,

where one has just used the property
\[
\lim_{\sigma \to \infty} \frac{1 + \log(1 + \sigma)}{\log(1 + \sigma)} = 1.
\]

By adding the two estimates for low and high frequencies and using the Plancherel theorem, one can conclude the proof of proposition.

4. Asymptotic profile of solutions. Applying the Fourier transform on the problem (12) one obtain the associated problem in Fourier space:
\[
(1 + \log(1 + |\xi|^2)) \hat{u}_{tt} + \log(1 + |\xi|^2)(1 + \log(1 + |\xi|^2)) \hat{u} + \hat{u}_t = 0,
\]
\[
\hat{u}(0, \xi) = \hat{u}_0(\xi),
\]
\[
\hat{u}_t(0, \xi) = \hat{u}_1(\xi).
\]

The characteristic roots of the associated polynomial to the equation in (25) are given by
\[
\lambda_{\pm} = \frac{-1 \pm \sqrt{1 - 4 \log(1 + |\xi|^2)(1 + \log(1 + |\xi|^2))^2}}{2(1 + \log(1 + |\xi|^2))} .
\]

We observe that there exists a unique real number \( \delta > 0 \) such that
\[
1 - 4 \log(1 + |\xi|^2)(1 + \log(1 + |\xi|^2))^2 \begin{cases} 
    \geq 0 & \text{for } |\xi| \leq \delta , \\
    < 0 & \text{for } |\xi| > \delta .
\end{cases}
\]

In fact, the function \( f(r) = 1 - 4 \log(1 + r^2)(1 + \log(1 + r^2))^2 \) is decreasing for \( r \geq 0 \), continuous and
\[
f(0) = 1, \ f(1) = 1 - 4 \log 2 \cdot (1 + \log 2)^2 < 0 .
\]

Therefore, by the mean value theorem there exists a unique number \( \delta \in (0, 1) \) that satisfies (26). The same theorem guarantees us the existence of \( 0 < \eta < \delta \) such that
\[
\frac{1}{2} \leq \sqrt{1 - 4 \log(1 + |\xi|^2)(1 + \log(1 + |\xi|^2))^2} \leq 1 ,
\]
whenever \( |\xi| \leq \eta \).

The next lemma is very important to get sharp estimates. In the following notation \( A \approx B \) means that \( c_1 A \leq B \leq c_2 A \) for some positive constants \( c_1, c_2 \).

Lemma 4.1. It holds that
(i). \( \lambda_+ \approx - \log(1 + |\xi|^2) \),
(ii). \( \lambda_- \approx -1 \),
(iii). \( \lambda_+ + \lambda_- \approx -1 \),
whenever \( |\xi| \leq \delta \). And, in particular, in the case of \( |\xi| \leq \eta \), one has
(iv). \( \lambda_+ - \lambda_- \approx 1 \).
Remark 7. Note that the constants $c_1$ and $c_2$ appeared in Lemma 4.1 may depend on $\delta$ or $\eta$. We also note that the four items in Lemma 4.1 simultaneously hold on the zone $\{ |\xi| \leq \eta \}$ because of $\eta < \delta$.

Proof. (i). For $a > 0$ and $b \geq 0$, it holds that

$$-a + \sqrt{b} = \frac{b - a^2}{a + \sqrt{b}}.$$ 

This identity implies that

$$-1 + \sqrt{1 - 4 \log(1 + |\xi|^2)(1 + \log(1 + |\xi|^2))^2} = \frac{4 \log(1 + |\xi|^2)(1 + \log(1 + |\xi|^2))^2}{1 + \sqrt{1 - 4 \log(1 + |\xi|^2)(1 + \log(1 + |\xi|^2))^2}},$$

since $1 - 4 \log(1 + |\xi|^2)(1 + \log(1 + |\xi|^2))^2 \geq 0$ for $|\xi| \leq \delta$. Then,

$$\lambda_+ = -2 \log(1 + |\xi|^2) \frac{1 + \log(1 + |\xi|^2)}{1 + \sqrt{1 - 4 \log(1 + |\xi|^2)(1 + \log(1 + |\xi|^2))^2}}, \quad |\xi| \leq \delta.$$ 

Now, for $|\xi| \leq \delta$ we have $1 \leq 1 + \log(1 + |\xi|^2) \leq 1 + \log(1 + \delta^2) =: K_\delta$ and

$$1 \leq 1 + \sqrt{1 - 4 \log(1 + |\xi|^2)(1 + \log(1 + |\xi|^2))^2} \leq 2.$$

Therefore we may conclude that

$$-2 K_\delta \log(1 + |\xi|^2) \leq \lambda_+ \leq -\log(1 + |\xi|^2), \quad |\xi| \leq \delta.$$ 

This implies the desired statement of (i).

(ii). Since $0 \leq \sqrt{1 - 4 \log(1 + |\xi|^2)(1 + \log(1 + |\xi|^2))^2} \leq 1$ in the region $|\xi| \leq \delta$,

$$\frac{-1}{1 + \log(1 + |\xi|^2)} \leq \frac{-1 - \sqrt{1 - 4 \log(1 + |\xi|^2)(1 + \log(1 + |\xi|^2))^2}}{2(1 + \log(1 + |\xi|^2))} \leq \frac{-1}{2(1 + \log(1 + |\xi|^2))}.$$ 

Therefore,

$$-1 \leq \lambda_- \leq -\frac{1}{2K_\delta}.$$ 

(iii). To prove this item we observe that $\lambda_+ + \lambda_- = \frac{-1}{1 + \log(1 + |\xi|^2)}$. Hence,

$$-1 \leq \lambda_+ + \lambda_- \leq -\frac{1}{K_\delta},$$

for $|\xi| \leq \delta$. And we obtain the equivalence $\lambda_+ + \lambda_- \approx -1$.

(iv). We observe that we have chosen $\eta > 0$ in (27) such that

$$\frac{1}{2} \leq \sqrt{1 - 4 \log(1 + |\xi|^2)(1 + \log(1 + |\xi|^2))^2} \leq 1$$

for all $|\xi| \leq \eta$. Thus, one can deduce

$$\frac{1}{2K_\delta} \leq \frac{1}{2(1 + \log(1 + |\xi|^2))} \leq \frac{\sqrt{1 - 4 \log(1 + |\xi|^2)(1 + \log(1 + |\xi|^2))^2}}{1 + \log(1 + |\xi|^2)} \leq \frac{1}{1 + \log(1 + |\xi|^2)} \leq 1.$$ 

This estimate shows the desired equivalence $\lambda_+ - \lambda_- \approx 1$ on the region $|\xi| \leq \eta$. \qed
In the next subsection to use Lemma 4.1 we work on the zone \( \{ |\xi| \leq \eta \} \), where \( \eta \) is given in (27).

### 4.1. Estimates on the low frequency zone \( |\xi| \leq \delta \)

(i) Estimates on the low frequency zone \( |\xi| \leq \eta \):

We remember that \( \eta \) is defined in (27). In this case, the characteristics roots \( \lambda_{\pm} \) are real-valued, and the solution of (25) is explicitly given by

\[
\hat{u}(t, \xi) = \frac{\lambda_- \hat{u}_0(\xi) - \hat{u}_1(\xi)}{\lambda_- - \lambda_+} e^{t \lambda_+} + \frac{\hat{u}_1(\xi) - \lambda_+ \hat{u}_0(\xi)}{\lambda_- - \lambda_+} e^{t \lambda_-}.
\]  

(28)

We observe that

\[
\lambda_- = -\log(1 + |\xi|^2)(1 + \log(1 + |\xi|^2)) - (1 + \log(1 + |\xi|^2)) \lambda_+^2,
\]

\[
\lambda_+ = -\log(1 + |\xi|^2)(1 + \log(1 + |\xi|^2)) - (1 + \log(1 + |\xi|^2)) \lambda_-^2,
\]

for \( |\xi| \leq \delta \). Therefore we can rewrite \( \hat{u}(t, \xi) \) as follows

\[
\hat{u}(t, \xi) = e^{-t \log(1 + |\xi|^2)(1 + \log(1 + |\xi|^2))} \left( H_1(t, \xi) + H_2(t, \xi) \right),
\]

(29)

where

\[
H_1(t, \xi) = \frac{\lambda_- \hat{u}_0(\xi) - \hat{u}_1(\xi)}{\lambda_- - \lambda_+} e^{-t(1 + \log(1 + |\xi|^2)) \lambda_+^2},
\]

\[
H_2(t, \xi) = \frac{\hat{u}_1(\xi) - \lambda_+ \hat{u}_0(\xi)}{\lambda_- - \lambda_+} e^{-t(1 + \log(1 + |\xi|^2)) \lambda_-^2}.
\]

We can also use the Chill-Haraux [10] idea that has also been used in [29] to decompose \( H_1(t, \xi) \) as

\[
H_1(t, \xi) = \hat{u}_0 + \hat{u}_1 + \frac{\lambda_- - \lambda_+}{\lambda_- - \lambda_+} \hat{u}_0 - \frac{\lambda_+ - \lambda_-}{\lambda_+ - \lambda_-} \hat{u}_1 + H_1(t, \xi)
\]

\[
= \hat{u}_0 + \hat{u}_1 - \frac{\lambda_+}{\lambda_+ - \lambda_-} \hat{u}_0 + \frac{\lambda_-}{\lambda_- - \lambda_+} \hat{u}_1 \left( e^{-t(1 + \log(1 + |\xi|^2)) \lambda_+^2} - 1 \right) e^{-t(1 + \log(1 + |\xi|^2)) \lambda_-^2} - \frac{\lambda_+ - \lambda_-}{\lambda_+ - \lambda_-}
\]

By combining the last expression together with the decomposition \( \hat{u}_j(\xi) = A_j(\xi) - iB_j(\xi) + P_j \) for initial data given in (11) we can get the following expression for \( \hat{u}(t, \xi) \) which holds for \( |\xi| \leq \eta \):

\[
\hat{u}(t, \xi) = e^{-t \log(1 + |\xi|^2)(1 + \log(1 + |\xi|^2))} \left( A_0(\xi) - iB_0(\xi) + P_0 + A_1(\xi) - iB_1(\xi) + P_1 \right)
\]

\[
- e^{-t \log(1 + |\xi|^2)(1 + \log(1 + |\xi|^2))} \frac{\lambda_+}{\lambda_+ - \lambda_-} \hat{u}_0
\]

\[
+ e^{-t \log(1 + |\xi|^2)(1 + \log(1 + |\xi|^2))} \frac{\lambda_-}{\lambda_- - \lambda_+} \hat{u}_0 \left( e^{-t(1 + \log(1 + |\xi|^2)) \lambda_+^2} - 1 \right)
\]

\[
+ e^{-t \log(1 + |\xi|^2)(1 + \log(1 + |\xi|^2))} \frac{\lambda_+}{\lambda_+ - \lambda_-} \hat{u}_1 \left( e^{-t(1 + \log(1 + |\xi|^2)) \lambda_-^2} - (\lambda_+ - \lambda_-) \right)
\]

\[
+ e^{-t \log(1 + |\xi|^2)(1 + \log(1 + |\xi|^2))} H_2(t, \xi).
\]

(30)
Our main goal in this subsection is to introduce an asymptotic profile of the solution \( \hat{u}(t, \xi) \) in the low frequency region as \( t \to \infty \) in a simple form as
\[
\varphi_1(t, \xi) := (F_0 + F_1)e^{-t \log(1 + |\xi|^2)(1 + \log(1 + |\xi|^2))}.
\]

For this purpose, it is necessary to find suitable estimates for the other six terms of the expression (30) defined by the functions
\[
\begin{align*}
F_1(t, \xi) &= e^{-t \log(1 + |\xi|^2)(1 + \log(1 + |\xi|^2))} (A_0(\xi) - iB_0(\xi)), \\
F_2(t, \xi) &= e^{-t \log(1 + |\xi|^2)(1 + \log(1 + |\xi|^2))} (A_1(\xi) - iB_1(\xi)), \\
F_3(t, \xi) &= -e^{-t \log(1 + |\xi|^2)(1 + \log(1 + |\xi|^2))} \frac{\lambda_+}{\lambda_+ - \lambda_-} \hat{u}_0, \\
F_4(t, \xi) &= e^{-t \log(1 + |\xi|^2)(1 + \log(1 + |\xi|^2))} \frac{\lambda_-}{\lambda_+ - \lambda_-} \left( e^{-t(1 + \log(1 + |\xi|^2))\lambda^2_+ - 1} \right), \\
F_5(t, \xi) &= e^{-t \log(1 + |\xi|^2)(1 + \log(1 + |\xi|^2))} \frac{\hat{u}_1}{\lambda_+ - \lambda_-} \left( e^{-t(1 + \log(1 + |\xi|^2))\lambda^2_+ - (\lambda_+ - \lambda_-)} \right), \\
F_6(t, \xi) &= e^{-t \log(1 + |\xi|^2)(1 + \log(1 + |\xi|^2))} H_2(t, \xi).
\end{align*}
\]

Therefore, from (30) and (31), for \( |\xi| \leq \eta \) we have
\[
\hat{u}(t, \xi) - \varphi_1(t, \xi) = \sum_{j=1}^{6} F_j(t, \xi).
\]

In order to estimate the difference given by (32) on the zone of low frequency \( \{ |\xi| \leq \eta \} \) we shall develop the next computations based on Lemmas 2.6 and 4.1.

Now, we first observe that
\[
1 \leq 1 + \log(1 + |\xi|^2) \leq 1 + \log(1 + \eta^2) =: k_\eta, \quad |\xi| \leq \eta.
\]

Then, for \( j = 0, 1 \) by using Lemma 2.6 with \( \kappa = 1 \) one has
\[
\begin{align*}
\int_{|\xi| \leq \eta} e^{-2t \log(1 + |\xi|^2)(1 + \log(1 + |\xi|^2))} |A_j(\xi) - iB_j(\xi)|^2 \, d\xi \\
&\leq \int_{|\xi| \leq \eta} e^{-2t \log(1 + |\xi|^2)} |A_j(\xi) - iB_j(\xi)|^2 \, d\xi \\
&= \int_{|\xi| \leq \eta} (1 + |\xi|^2)^{-2t} |A_j(\xi) - iB_j(\xi)|^2 \, d\xi \\
&\leq (L + M)^2 \|u_j\|_{L^1}^2 \int_{|\xi| \leq \eta} (1 + r^2)^{-2t} r^{n+1} \, d\xi \\
&= \omega_n (L + M)^2 \|u_j\|_{L^1}^2 \int_0^1 (1 + r^2)^{-2t} r^{n+1} \, dr \\
&\leq \omega_n (L + M)^2 \|u_j\|_{L^1}^2 t^{-n+2} \\
&\leq C \|u_j\|_{L^1}^2 t^{-n+2}
\end{align*}
\]
for \( t \gg 1 \) due to Lemma 2.1. Consequently, for \( t \gg 1 \) we have
\[
\begin{align*}
\int_{|\xi| \leq \eta} |F_1(t, \xi)|^2 \, d\xi &\leq C \|u_0\|_{L^1}^2 t^{-n+2} \quad \text{and} \quad \int_{|\xi| \leq \eta} |F_2(t, \xi)|^2 \, d\xi &\leq C \|u_1\|_{L^1}^2 t^{-n+2}.
\end{align*}
\]
In order to get an estimate on the function $F_3(t, \xi)$, we observe that

$$\int_{|\xi| \leq \eta} |F_3(t, \xi)|^2 d\xi = \int_{|\xi| \leq \eta} e^{-2t \log(1 + |\xi|^2)(1 + \log(1 + |\xi|^2))} \left( \frac{\lambda_+}{\lambda_+ - \lambda_-} \right)^2 |\hat{u}_0|^2 d\xi$$

$$\leq C \int_{|\xi| \leq \eta} e^{-2t \log(1 + |\xi|^2) \log^2(1 + |\xi|^2)} |\hat{u}_0|^2 d\xi,$$

because, for $|\xi| \leq \eta$, we have

$$\frac{\lambda_+}{\lambda_+ - \lambda_-} \approx -\log(1 + |\xi|^2)$$

due to items (i) and (iv) of Lemma 4.1. We also observe that $\log(1 + r^2) \leq r^2$ for all $r \geq 0$ and we may use this simple inequality to conclude the $L^2$-estimate for $F_3(t, \xi)$ as follows.

$$\int_{|\xi| \leq \eta} |F_3(t, \xi)|^2 d\xi \leq C \|u_0\|^2 \int_{|\xi| \leq \eta} (1 + |\xi|^2)^{-2t} |\xi|^4 d\xi$$

$$= C\omega_n \|u_0\|^2 \int_0^\eta (1 + r^2)^{-2t} r^{n+3} dr$$

$$\leq C\omega_n \|u_0\|^2 \int_0^1 (1 + r^2)^{-2t} r^{n+3} dr$$

$$\leq C\omega_n \|u_0\|^2 t^{-\frac{n+4}{2}}, \ t \gg 1,$$

where we also used that $|\hat{u}_0(\xi)| \leq \|u_0\|_1$ for all $\xi \in \mathbb{R}^n$ and Lemma 2.1.

To estimate the function $F_4(t, \xi)$ we need the elementary inequality

$$|e^{-a} - 1| \leq |a|, \ a \geq 0.$$  \hspace{1cm} (33)

We also remember that

$$\left| \frac{\lambda_-}{\lambda_+ - \lambda_-} \right| \approx 1, \ |\xi| \leq \eta$$

from Lemma 4.1. Thus, we have

$$\int_{|\xi| \leq \eta} |F_4(t, \xi)|^2 d\xi \leq C \int_{|\xi| \leq \eta} e^{-2t \log(1 + |\xi|^2)(1 + \log(1 + |\xi|^2))} \left( e^{-t(1 + \log(1 + |\xi|^2))\lambda_+^2} - 1 \right)^2 |\hat{u}_0|^2 d\xi$$

$$\leq C t^2 \int_{|\xi| \leq \eta} e^{-2t \log(1 + |\xi|^2)(1 + \log(1 + |\xi|^2))}(1 + \log(1 + |\xi|^2))^2 \lambda_+^4 |\hat{u}_0|^2 d\xi$$

$$\leq C t^2 \|u_0\|^2 \int_{|\xi| \leq \eta} (1 + |\xi|^2)^{-2t} \log^4(1 + |\xi|^2) d\xi$$

$$\leq C t^2 \|u_0\|^2 \int_{|\xi| \leq \eta} (1 + |\xi|^2)^{-2t} |\xi|^8 d\xi$$

$$\leq \omega_n C t^2 \|u_0\|^2 t^{-\frac{n+8}{2}}, \ t \gg 1$$

because of $1 + \log(1 + |\xi|^2) \leq 1 + \log 2$, for $|\xi| \leq \eta < 1$, where we also used the fact that $|\lambda_+| \leq C \log(1 + |\xi|^2)$ for $|\xi| \leq \eta$. The constant $C_n > 0$ depends only on $n$.

In order to get estimates to the remainder function $F_5(t, \xi)$ on $|\xi| \leq \eta$, we can use the inequality $(a - b)^2 \leq 2a^2 + 2b^2$ to obtain the following estimate.
\[
\int_{|\xi| \leq \eta} |F_2(t, \xi)|^2 \, d\xi \\
= \int_{|\xi| \leq \eta} e^{-2t \log(1 + |\xi|^2)(1 + \log(1 + |\xi|^2))} \frac{(e^{-\tau_1(1 + \log(1 + |\xi|^2))(\lambda_+ - (\lambda_+ - \lambda_-))^2} (\lambda_+ - \lambda_-)^2 \, d\xi \\
\leq 2 \int_{|\xi| \leq \eta} e^{-2t \log(1 + |\xi|^2)(1 + \log(1 + |\xi|^2))} \frac{(e^{-\tau_1(1 + \log(1 + |\xi|^2))(\lambda_+ - (\lambda_+ - \lambda_-))^2} (\lambda_+ - \lambda_-)^2 \, d\xi \\
+ 2 \int_{|\xi| \leq \eta} e^{-2t \log(1 + |\xi|^2)(1 + \log(1 + |\xi|^2))} \frac{(1 - (\lambda_+ - \lambda_-))^2}{(\lambda_+ - \lambda_-)^2} \, d\xi. \quad (34)
\]

Now, let \( D = 1 - 4 \log(1 + |\xi|^2)(1 + \log(1 + |\xi|^2))^2 \). Then we observe that \( D \geq 0 \) for \( |\xi| \leq \eta \) and
\[
1 - (\lambda_+ - \lambda_-) = \frac{2 \log(1 + |\xi|^2) + \log(1 + |\xi|^2) + 4 \log(1 + |\xi|^2)(1 + \log(1 + |\xi|^2))^2}{(1 + \log(1 + |\xi|^2))(1 + \log(1 + |\xi|^2) + \sqrt{D})}.
\]

In particular, \( 1 - (\lambda_+ - \lambda_-) \) is positive and there exists a constant \( C_\eta > 0 \) such that
\[
|1 - (\lambda_+ - \lambda_-)| \leq C_\eta \log(1 + |\xi|^2) \quad (35)
\]
for all \( |\xi| \leq \eta \), where \( \eta \) is defined in (27). From (34), (33) and (35), we obtain the next estimate.
\[
\int_{|\xi| \leq \eta} |F_2(t, \xi)|^2 \, d\xi \leq C \int_{|\xi| \leq \eta} e^{-2t \log(1 + |\xi|^2)(1 + \log(1 + |\xi|^2))} \frac{\log^2(1 + |\xi|^2)(\lambda_+ - \lambda_-)^2 |\hat{u}_1|^2 \, d\xi \\
+ 2C_\eta \int_{|\xi| \leq \eta} e^{-2t \log(1 + |\xi|^2)(1 + \log(1 + |\xi|^2))} \log^2(1 + |\xi|^2)|\hat{u}_1|^2 \, d\xi \\
\leq C t^2 \int_{|\xi| \leq \eta} e^{-t \log(1 + |\xi|^2)} \log^4(1 + |\xi|^2)|u_1|^2 \, d\xi \\
+ 2C_\eta \int_{|\xi| \leq \eta} e^{-t \log(1 + |\xi|^2)} \log^2(1 + |\xi|^2)|u_1|^2 \, d\xi \\
\leq C t^2 \|u_1\|^2_1 \int_{|\xi| \leq 1} |1 + |\xi|^2|^{-t} |\xi|^4 \, d\xi + 2C_\eta \|u_1\|^2_1 \int_{|\xi| \leq \eta} |1 + |\xi|^2|^{-t} |\xi|^4 \, d\xi \\
\leq C_n \|u_1\|^2_2 t^{-\frac{n+4}{4}} + 2C_\eta, n \|u_1\|^2_2 t^{-\frac{n+4}{4}}, \quad t \gg 1.
\]

Finally, by (ii) of Lemma 4.1 one has \( \lambda_- \approx -1 \) on the region \( |\xi| \leq \eta \), so that there exists constants \( c_1, c_2 > 0 \) such that \( c_1 \leq 2(1 + \log(1 + |\xi|^2)) \lambda_2^2 \leq c_2 \) whenever \( |\xi| \leq \eta \). Therefore, it follows that
\[
\int_{|\xi| \leq \eta} |F_6(t, \xi)|^2 \, d\xi = \int_{|\xi| \leq \eta} e^{-2t \log(1 + |\xi|^2)(1 + \log(1 + |\xi|^2))} H_2(t, \xi) \, d\xi \\
\leq \int_{|\xi| \leq \eta} e^{-2t \log(1 + |\xi|^2)(1 + \log(1 + |\xi|^2))} e^{-2t \log(1 + |\xi|^2)(\lambda_+ - \lambda_-)^2 \, d\xi \\
+ \int_{|\xi| \leq \eta} e^{-2t \log(1 + |\xi|^2)(1 + \log(1 + |\xi|^2))} e^{-2t \log(1 + |\xi|^2)(\lambda_+ - \lambda_-)^2 \, d\xi \\
\leq C e^{-ct} \int_{|\xi| \leq \eta} |\hat{u}_1|^2 \, d\xi + C e^{-ct} \int_{|\xi| \leq \eta} e^{-t \log(1 + |\xi|^2)} \log^2(1 + |\xi|^2)|\hat{u}_0|^2 \, d\xi
\]
\[ \leq C e^{-c_2 t} \|u_1\|_1^2 \int_{|\xi| \leq \eta} (1 + |\xi|^2)^{-1} d\xi + C e^{-c_2 t} \|u_0\|_1^2 \int_{|\xi| \leq \eta} e^{-t \log(1 + |\xi|^2)} |\xi|^4 d\xi \]
\[ \leq C \|u_1\|_1^2 t^{-\frac{n+4}{2}} e^{-c_2 t} + C \|u_0\|_1^2 t^{-\frac{n+4}{2}} e^{-c_2 t}, \quad t \gg 1. \]

By combining the above estimates for \( F_j(t, \xi), \quad j = 1, \ldots, 6 \), with equation (32), we obtain that the solution \( \hat{u}(t, \xi) \) given by (30) has the following asymptotic property.

**Proposition 3.** Let \( n \geq 1 \) and \( \eta > 0 \) given by (27). For \( (u_0, u_1) \in L^{1,1}(\mathbb{R}^n) \times L^{1,1}(\mathbb{R}^n) \) the solution \( \hat{u}(t, \xi) \) to problem (25) satisfies

\[ \int_{|\xi| \leq \eta} |\hat{u}(t, \xi) - \varphi_1(t, \xi)|^2 d\xi \leq C \left( \|u_0\|_{1,1}^2 t^{-\frac{n+4}{2}} + \|u_1\|_{1,1}^2 t^{-\frac{n+4}{2}} + \|u_0\|_1^2 t^{-\frac{n+4}{2}} \right), \quad t \gg 1 \]

where \( \varphi_1(t, \xi) \) is defined in (31) and \( C \) is a positive constant that depends only on \( \eta \) and \( n \).

\( \square \)

(ii) **Estimates on the low-middle frequency zone** \( \eta \leq |\xi| \leq \delta \):

To obtain sharp estimates on the low-middle frequency zone \( \eta \leq |\xi| \leq \delta \) it should be noted that according to (26) the characteristics roots \( \lambda_\pm \) are still real-valued.

We can rewrite the solution \( \hat{u}(t, \xi) \), for \( \eta \leq |\xi| < \delta \), as follows

\[
\hat{u}(t, \xi) = e^{-\frac{t}{2(1 + \log(1 + |\xi|^2)^2)}} \left[ \cosh(c(\xi)t) \hat{u}_0(\xi) + \frac{\sinh(c(\xi)t)}{c(\xi)} \hat{u}_1(\xi) \right]
+ e^{-\frac{t}{2(1 + \log(1 + |\xi|^2)^2)}} \frac{\sinh(c(\xi)t)}{2(1 + \log(1 + |\xi|^2)^2)c(\xi)} \hat{u}_0(\xi),
\]

where

\[
c(\xi) := \frac{\sqrt{1 - 4 \log(1 + |\xi|^2)(1 + \log(1 + |\xi|^2))^2}}{2(1 + \log(1 + |\xi|^2)^2)} > 0, \quad |\xi| < \delta.
\]

Let

\[
C_\delta = \frac{1}{2(1 + \log(1 + \delta^2))}.
\]

It is important to observe that (36) is not defined for \( |\xi| = \delta \), because \( c(\xi) = 0 \) in this case. However, we note that it is a removable singularity of \( \hat{u}(t, \xi) \). Moreover, for \( \xi \in \mathbb{R}^n \) such that \( |\xi| = \delta \), the eigenvalues \( \lambda_\pm \) are equal and the solution formula is given by

\[
\hat{u}(t, \xi) = e^{-C_\delta t} \hat{u}_0(\xi) + C_\delta t e^{-C_\delta t} \hat{u}_0(\xi) + t e^{-C_\delta t} \hat{u}_1(\xi), \quad |\xi| = \delta.
\]

We remember that \( \delta \) is given in (26) and our choice for \( \eta \) is such that

\[
\sqrt{1 - 4 \log(1 + |\xi|^2)(1 + \log(1 + |\xi|^2))^2} \geq \frac{1}{2}
\]

when \( |\xi| \leq \eta \) (see (27)) and this is a decreasing function on \( |\xi| \). Thus, in the case for \( \eta \leq |\xi| \leq \delta \), one has

\[
c(\xi) \leq \frac{1}{4(1 + \log(1 + |\xi|^2)^2)}.
\]
Similarly, by using Lemma 2.5, we may obtain

\[ \int_{|\xi| \leq \delta} |\hat{u}(t, \xi)|^2 d\xi \leq 4e^{-C_\delta t} \|u_0\|_2^2 + 4C_\delta t^2 e^{-C_\delta t} \|u_0\|_2^2 + 4t^2 e^{-C_\delta t} \|u_1\|_2^2 \]  

(38)

for \( t > 0 \), where \( C \) is a positive constant that depends on \( \eta \), and \( C_\delta \) is defined above \( \delta \).

Proof. Due to the fact that \( \cosh a \leq e^a \) for all \( a \geq 0 \) we may estimate for \( t > 0 \)

\[ e^{-\frac{t}{1+\log(1+|\xi|^2)}} \cosh^2(c(\xi)t) \leq e^{-\frac{t}{1+\log(1+|\xi|^2)}} e^{2c(\xi)t} \leq e^{-\frac{t}{1+\log(1+|\xi|^2)}} e^{2(1+\log(1+|\xi|^2))} \]

\[ = e^{-\frac{2t}{1+\log(1+|\xi|^2)}} \leq e^{-C_\delta t}, \quad \eta \leq |\xi| < \delta. \]  

(39)

Similarly, by using Lemma 2.5, we may obtain

\[ e^{-\frac{t}{1+\log(1+|\xi|^2)}} \sinh^2(c(\xi)t) \leq t^2 e^{-C_\delta t}, \quad \eta \leq |\xi| < \delta. \]  

(40)

From the two solution formula (36), (37) and estimates (39), (40) combining with Young’s inequality we have

\[ |\hat{u}(t, \xi)|^2 \leq 4e^{-C_\delta t}|\hat{u}_0(\xi)|^2 + 4C_\eta t e^{-C_\delta t}|\hat{u}_0(\xi)|^2 + 4t e^{-C_\delta t}|\hat{u}_1(\xi)|^2, \]

(41)

for \( \eta \leq |\xi| \leq \delta \), where

\[ C_\eta = \frac{1}{2(1 + \log(1 + \eta^2))}. \]

Therefore, we may obtain the desired estimate

\[ \int_{|\eta| \leq |\xi| \leq \delta} |\hat{u}(t, \xi)|^2 d\xi \leq 4e^{-C_\delta t} \int_{|\eta| \leq |\xi| \leq \delta} |\hat{u}_0(\xi)|^2 d\xi + 4C_\eta t^2 e^{-C_\delta t} \int_{|\eta| \leq |\xi| \leq \delta} |\hat{u}_0(\xi)|^2 d\xi \]

\[ + 4t^2 e^{-C_\delta t} \int_{|\eta| \leq |\xi| \leq \delta} |\hat{u}_1(\xi)|^2 d\xi \]

\[ = 4e^{-C_\delta t}\|u_0\|_2^2 + 4C_\eta t^2 e^{-C_\delta t}\|u_0\|_2^2 + 4t^2 e^{-C_\delta t}\|u_1\|_2^2, \quad t \geq 1. \]

\[ \square \]

4.2. Estimates on the high frequency zone \( |\xi| \geq \delta \). On the zone of high frequency the characteristics roots are complex-valued (see 26) and are given by

\[ \lambda_\pm = -a(\xi) \pm ib(\xi), \]

where

\[ a(\xi) = \frac{1}{2(1 + \log(1 + |\xi|^2))} \quad \text{and} \quad b(\xi) = \frac{\sqrt{4\log(1 + |\xi|^2)(1 + \log(1 + |\xi|^2))^2 - 1}}{2(1 + \log(1 + |\xi|^2))}. \]  

(42)

Then the solution \( \hat{u}(t, \xi) \) to problem (25) in the high frequency zone is explicitly given by

\[ \hat{u}(t, \xi) = e^{-a(\xi)t} \cos(b(\xi)t)\hat{u}_0 + \frac{a(\xi)}{b(\xi)} e^{-a(\xi)t} \sin(b(\xi)t)\hat{u}_0 + \frac{1}{b(\xi)} e^{-a(\xi)t} \sin(b(\xi)t)\hat{u}_1. \]

(i) Estimate on the high-middle frequency zone \( \delta \leq |\xi| \leq \sqrt{e-1} \).

In this region, the function \( a(\xi) \) is equivalent to a constant, that is \( \frac{1}{4} \leq a(\xi) \leq \frac{1}{2} \).
Moreover, we can see that \( \frac{1}{b(\xi)} \) converges to \(+\infty\) when \( |\xi| \to \delta^+ \) according to (26). However, we remember that \( \sin a \leq a \) for all \( a \geq 0 \). Thus
\[
\sin(b(\xi)t) \leq b(\xi)t
\]
for all \( \xi \in \mathbb{R}^n \) and \( t \geq 0 \). By combining these properties together with the solution formula (43) one can obtain the following estimate for \( t > 0 \), which implies the exponential decay in such region.
\[
\int_{|\xi| \leq \sqrt{e^{-t}}} |\hat{u}(t, \xi)|^2 d\xi \leq e^{-\frac{t}{2}} \|u_0\|^2 + \frac{1}{4} t^2 \|u_0\|^2 + t^2 e^{-\frac{t}{2}} \|u_1\|^2. \tag{44}
\]

(ii) Estimate on the high frequency zone \(|\xi| \geq \sqrt{e^{-1}}\)

The estimates on this zone are more delicate and the derivation is one of essential contributions in our paper. We first need to rewrite the solution formula given by (43) into a more suitable expression.

First we observe that for \(|\xi| \geq \delta\), in particular, for \(|\xi| \geq \sqrt{e^{-1}}\), it holds that
\[
b(\xi) \leq \sqrt{\log(1 + |\xi|^2)}.
\]
Then the mean value theorem implies, for \(|\xi| \geq \sqrt{e^{-1}}\), that
\[
\cos(b(\xi)t) = \cos(\sqrt{\log(1 + |\xi|^2)} t) - \sin(\theta(t, \xi)) \left[ b(\xi) - \sqrt{\log(1 + |\xi|^2)} \right] t,
\]
with \( \theta(t, \xi) = ab(\xi)t + (1 - \alpha) \sqrt{\log(1 + |\xi|^2)} t \) for some \( \alpha \in (0, 1) \). Similarly,
\[
\sin(b(\xi)t) = \sin(\sqrt{\log(1 + |\xi|^2)} t) + \cos(\eta(t, \xi)) \left[ b(\xi) - \sqrt{\log(1 + |\xi|^2)} \right] t,
\]
with \( \eta(t, \xi) = \gamma b(\xi)t + (1 - \gamma) \sqrt{\log(1 + |\xi|^2)} t \) for some \( \gamma \in (0, 1) \).

Thus, one can rewrite \( \hat{u}(t, \xi) \) as follows:
\[
\hat{u}(t, \xi) = e^{-a|\xi|t} \cos(\sqrt{\log(1 + |\xi|^2)} t) \hat{u}_0 + te^{-a|\xi|t} \sin(\theta(t, \xi)) \left[ \sqrt{\log(1 + |\xi|^2)} - b(\xi) \right] \hat{u}_0 + \frac{a(\xi)}{b(\xi)} e^{-a|\xi|t} \sin(b(\xi)t) \hat{u}_0 + \frac{1}{b(\xi)} e^{-a|\xi|t} \sin(\sqrt{\log(1 + |\xi|^2)} t) \hat{u}_1 + te^{-a|\xi|t} \cos(\eta(t, \xi)) \left[ b(\xi) - \sqrt{\log(1 + |\xi|^2)} \right] \hat{u}_1. \tag{45}
\]

We introduce an important function to be the asymptotic profile on the zone of high frequency for the solution \( \hat{u}(t, \xi) \) given by (45) as follows
\[
\varphi_2(t, \xi) := e^{-\frac{t}{2\log(1 + |\xi|^2)}} \left( \frac{\sin(\sqrt{\log(1 + |\xi|^2)} t)}{\sqrt{\log(1 + |\xi|^2)}} \hat{u}_1(\xi) + \cos(\sqrt{\log(1 + |\xi|^2)} t) \hat{u}_0(\xi) \right). \tag{46}
\]

In the following part, one will prove that the function \( \varphi_2(t, \xi) \) is asymptotic profile for the solution \( \hat{u}(t, \xi) \) in the high frequency region \(|\xi| \geq \sqrt{e^{-1}}\). Then we obtain the following difference between the solution and the profile
\[
\hat{u}(t, \xi) - \varphi_2(t, \xi) = \left( e^{-a(\xi)t} - e^{-\frac{t}{2\log(1 + |\xi|^2)}} \right) \cos(\sqrt{\log(1 + |\xi|^2)} t) \hat{u}_0 + \frac{a(\xi)}{b(\xi)} e^{-a(\xi)t} \sin(b(\xi)t) \hat{u}_0 + \frac{1}{b(\xi)} e^{-a(\xi)t} \sin(\sqrt{\log(1 + |\xi|^2)} t) \hat{u}_1 + te^{-a(\xi)t} \sin(\eta(\xi, t)) \left[ \sqrt{\log(1 + |\xi|^2)} - b(\xi) \right] \hat{u}_0 + \frac{a(\xi)}{b(\xi)} e^{-a(\xi)t} \sin(b(\xi)t) \hat{u}_0 + + \frac{1}{b(\xi)} e^{-a(\xi)t} \sin(\sqrt{\log(1 + |\xi|^2)} t) \hat{u}_1 + te^{-a(\xi)t} \cos(\eta(t, \xi)) \left[ b(\xi) - \sqrt{\log(1 + |\xi|^2)} \right] \hat{u}_1.
\]
\[
+ e^{-a(t)} \left( \frac{1}{b(\xi)} - \frac{1}{\sqrt{\log(1 + |\xi|^2)}} \right) \sin(\sqrt{\log(1 + |\xi|^2)} t) \hat{u}_1 \\
+ \frac{1}{\sqrt{\log(1 + |\xi|^2)}} \left( e^{-a(t)} - e^{-\frac{1}{2 \log(1 + |\xi|^2)}} \right) \sin(\sqrt{\log(1 + |\xi|^2)} t) \hat{u}_1 \\
+ te^{-a(t)} \frac{1}{b(\xi)} \cos(\eta(\xi, t)) \left[ b(\xi) - \sqrt{\log(1 + |\xi|^2)} \right] \hat{u}_1. \tag{47}
\]

Then, the following functions

\[G_1(t, \xi) = \left( e^{-a(t)} - e^{-\frac{1}{2 \log(1 + |\xi|^2)}} \right) \cos(\sqrt{\log(1 + |\xi|^2)} t) \hat{u}_0,\]

\[G_2(t, \xi) = te^{-a(t)} \sin(\theta(\xi, t)) \left[ \sqrt{\log(1 + |\xi|^2)} - b(\xi) \right] \hat{u}_0,\]

\[G_3(t, \xi) = \frac{a(\xi)}{b(\xi)} e^{-a(t)} \sin(b(\xi) t) \hat{u}_0,\]

\[G_4(t, \xi) = e^{-a(t)} \left( \frac{1}{b(\xi)} - \frac{1}{\sqrt{\log(1 + |\xi|^2)}} \right) \sin(\sqrt{\log(1 + |\xi|^2)} t) \hat{u}_1,\]

\[G_5(t, \xi) = \frac{1}{\sqrt{\log(1 + |\xi|^2)}} \left( e^{-a(t)} - e^{-\frac{1}{2 \log(1 + |\xi|^2)}} \right) \sin(\sqrt{\log(1 + |\xi|^2)} t) \hat{u}_1,\]

\[G_6(t, \xi) = te^{-a(t)} \frac{1}{b(\xi)} \cos(\eta(\xi, t)) \left[ b(\xi) - \sqrt{\log(1 + |\xi|^2)} \right] \hat{u}_1\]

are the remainder terms that appear in (47).

From now, let us estimates these 6-remainers in the following lines.

We note that on the region such that \( |\xi| \geq \sqrt{e - 1} \) one has \( 1 + \log(1 + |\xi|^2) \leq 2 \log(1 + |\xi|) \). Also, by Lemma 2.4 with \( c = 1 \) and \( a = -1 \) one has

\[
\frac{e^{1 + \log(1 + |\xi|^2)}}{(1 + \log(1 + |\xi|^2))^\nu} \leq Ct^{-\nu}, \quad t > 0, \quad \xi \in \mathbb{R}^n,
\]

for a fixed \( \nu > 0 \). The above two inequalities will be used to get the next series of estimates for the functions \( G_j(t, \xi), j = 1, \ldots, 6. \)

The first estimate is concerned with the function \( G_1(t, \xi) \) and it can be obtained from (33).

\[
\int_{|\xi| \geq \sqrt{e - 1}} |G_1(t, \xi)|^2 d\xi = \int_{|\xi| \geq \sqrt{e - 1}} \left( e^{-a(t)} - e^{-\frac{1}{2 \log(1 + |\xi|^2)}} \right)^2 (\sqrt{\log(1 + |\xi|^2)} t) |\hat{u}_0|^2 d\xi \\
\leq \int_{|\xi| \geq \sqrt{e - 1}} \left( e^{-a(t)} - e^{-\frac{1}{2 \log(1 + |\xi|^2)}} \right)^2 |\hat{u}_0|^2 d\xi \\
= \int_{|\xi| \geq \sqrt{e - 1}} e^{1 + \log(1 + |\xi|^2)} \left( 1 - e^{-\frac{1}{2 \log(1 + |\xi|^2)} (1 + \log(1 + |\xi|^2))} \right)^2 |\hat{u}_0|^2 d\xi \\
\leq t^2 \int_{|\xi| \geq \sqrt{e - 1}} e^{1 + \log(1 + |\xi|^2)} \log^2(1 + |\xi|^2)(1 + \log(1 + |\xi|^2))^2 |\hat{u}_0|^2 d\xi \\
\leq t^2 \int_{|\xi| \geq \sqrt{e - 1}} e^{1 + \log(1 + |\xi|^2)} |\hat{u}_0|^2 d\xi.
\]
Using (48) and the fact that

\[ \frac{e^{1+\log(1+|t|^2)}}{(1 + \log(1 + |t|^2))^4} = \frac{e^{1+\log(1+|\xi|^2)}}{(1 + \log(1 + |\xi|^2))^{4+T}}(1 + \log(1 + |\xi|^2))^{4+T}, \quad t \geq 0, \xi \in \mathbb{R}^n, \]

we may obtain the next estimate to the function \( G_1(t, \xi) \).

\[
\int_{|\xi| \geq \sqrt{e-T}} |G_1(t, \xi)|^2 d\xi \leq t^2 \int_{|\xi| \geq \sqrt{e-T}} (1 + \log(1 + |\xi|^2))^{4+T+1} \frac{e^{1+\log(1+|\xi|^2)}}{(1 + \log(1 + |\xi|^2))^{5+T}} |\hat{u}_0|^2 d\xi \\
\leq C t^{-(l+5)} \int_{|\xi| \geq \sqrt{e-T}} (1 + \log(1 + |\xi|^2))^{4+T+1} |\hat{u}_0|^2 d\xi \\
\leq C t^{-(l+3)} \| u_0 \|^2_{L_{T+1}^2},
\]

for all \( t > 0 \) and \( l \geq 0 \), where we have used the inequalities (33), (48) and the fact that \( \log(1 + |\xi|^2) \geq 1 \) on the high frequency zone.

For \( |\xi| \geq \delta \), we introduce the auxiliary function \( R(t, \xi) \) defined by

\[ R(t, \xi) = \sqrt{1 - \frac{1}{4(1 + \log(1 + |\xi|^2))(1 + \log(1 + |\xi|^2))^2}}, \quad (49) \]

which is well defined due to (26). Additionally, one notes that for \( |\xi| \geq \sqrt{e-T} \) we have the following estimate

\[ |\sqrt{\log(1 + |\xi|^2)} - b(\xi)| \leq \frac{1}{4(1 + \log(1 + |\xi|^2))^2 \sqrt{\log(1 + |\xi|^2)} (1 + R(t, \xi)}} \]

Thus for \( t > 0 \) and \( l \geq 0 \) we get

\[
\int_{|\xi| \geq \sqrt{e-T}} |G_2(t, \xi)|^2 d\xi = t^2 \int_{|\xi| \geq \sqrt{e-T}} e^{-2a(\xi)t \sin^2 (\theta(\xi, t)) \frac{1}{\sqrt{\log(1 + |\xi|^2)} - b(\xi)}} \left( \frac{1}{\sqrt{\log(1 + |\xi|^2)}} \right)^2 |\hat{u}_0|^2 d\xi \\
\leq t^2 \int_{|\xi| \geq \sqrt{e-T}} 16(1 + \log(1 + |\xi|^2))^{4+T} \log(1 + |\xi|^2) |\hat{u}_0|^2 d\xi \\
\leq t^2 \int_{|\xi| \geq \sqrt{e-T}} \frac{e^{-2a(\xi)t \sin^2 (\theta(\xi, t)) \frac{1}{\sqrt{\log(1 + |\xi|^2)} - b(\xi)}}}{8(1 + \log(1 + |\xi|^2))^2} |\hat{u}_0|^2 d\xi \\
\leq t^2 \int_{|\xi| \geq \sqrt{e-T}} \frac{(1 + \log(1 + |\xi|^2))^{4+T} e^{-2a(\xi)t \sin^2 (\theta(\xi, t)) \frac{1}{\sqrt{\log(1 + |\xi|^2)} - b(\xi)}}}{8(1 + \log(1 + |\xi|^2))^{4+T}} |\hat{u}_0|^2 d\xi \\
= C t^{-(l+6)} \int_{|\xi| \geq \sqrt{e-T}} (1 + \log(1 + |\xi|^2))^{4+T} |\hat{u}_0|^2 d\xi \\
\leq C t^{-(l+4)} \| u_0 \|^2_{L_{T+1}^2},
\]

where one has just used the fact that \( 1 + \log(1 + |\xi|^2) \geq 2 \) for \( |\xi| \geq \sqrt{e-T} \) and (48).

Another important property is that

\[ \frac{1}{4(1 + \log(1 + |\xi|^2))} \]

is a decreasing function of \( |\xi| \), and it converges to zero as \( |\xi| \to +\infty \). Hence, it follows that

\[ 1 \leq \frac{1}{1 - \frac{1}{4(1 + \log(1 + |\xi|^2))}} = \frac{1}{R(t, \xi)} \leq \frac{16}{15}, \quad (50) \]
for $|\xi| \geq \sqrt{e-1}$.

From the above inequality one can obtain estimates of the $L^2$-norms of each functions $G_3(t, \cdot)$, $G_4(t, \cdot)$ and $G_5(t, \cdot)$ for $t > 0$. In fact, (50) implies that

$$
\int |G_3(t, \xi)|^2 d\xi = \int |G_3(t, \xi)|^2 d\xi = \int \left(\frac{a(\xi)}{b(\xi)}\right)^2 e^{-2a(\xi)t} \sin^2(b(\xi)t)|\hat{u}_0|^2 d\xi
$$

Thus, we may conclude the estimate for $G_3(t, \xi)$ as follows

$$
\int |G_3(t, \xi)|^2 d\xi \leq \frac{8}{15} \int |G_3(t, \xi)|^2 d\xi \leq \frac{8}{15} \int \left(1 + \log(1 + |\xi|^2)\right)^{t+1}|\hat{u}_0|^2 d\xi
$$

Further, for $|\xi| \geq \sqrt{e-1}$, we have $1 \leq \log(1 + |\xi|^2)$ and then $1 + \log(1 + |\xi|^2) \leq 2 \log(1 + |\xi|^2)$. Therefore,

$$
\frac{1}{\log(1 + |\xi|^2)} \leq \frac{2}{1 + \log(1 + |\xi|^2)}, \quad |\xi| \geq \sqrt{e-1}.
$$

Thus, we may conclude the estimate for $G_3(t, \xi)$ as follows

$$
\int |G_3(t, \xi)|^2 d\xi \leq \frac{8}{15} \int |G_3(t, \xi)|^2 d\xi \leq \frac{8}{15} \int \left(1 + \log(1 + |\xi|^2)\right)^{t+1}|\hat{u}_0|^2 d\xi
$$

To get an estimate for the $L^2$-norm of $G_4(t, \cdot)$ we first observe the following identity:

$$
\frac{1}{b(\xi)} - \frac{1}{\sqrt{\log(1 + |\xi|^2)}} = \frac{1}{4 \log^3(1 + |\xi|^2)(1 + \log(1 + |\xi|^2))^2 R(t, \xi)(1 + R(t, \xi))}
$$

holds for $|\xi| \geq \delta$, where $R(t, \xi)$ is given by (49).

By using the above identity, the estimate (50) and the fact that $1 + \log(1 + |\xi|^2) \leq 2 \log(1 + |\xi|^2)$ for $|\xi| \leq \sqrt{e-1}$, we can arrive at the following $L^2$-estimate to the function $G_4(t, \cdot)$:

$$
\int |G_4(t, \xi)|^2 d\xi = \int |G_4(t, \xi)|^2 d\xi \leq \frac{8}{15} \int |G_4(t, \xi)|^2 d\xi \leq \frac{8}{15} \int \left(1 + \log(1 + |\xi|^2)^{t+1}|\hat{u}_1|^2 d\xi
$$

Similarly to the previous estimate for $G_1(t, \cdot)$ one obtains

$$
\int |G_5(t, \xi)|^2 d\xi
$$
there exists a constant \( C > 0 \).

Lemma 4.3. Profile as defined for \( \xi \) in Proposition 4.

Let the following result.

For a positive constant \( C \), special functions:

\[ \phi \]

Estimates on the whole space \( \mathbb{R}^n \).

Hence, from the definition of \( G_0(t, \xi) \), we have

\[
\int_{|\xi| \geq \sqrt{e-1}} |G_0(t, \xi)|^2 d\xi \leq \frac{1}{15} t^2 \int_{|\xi| \geq \sqrt{e-1}} \frac{e^{-\frac{t}{4 \log(1 + |\xi|^2)} \log(1 + |\xi|^2)}}{(1 + |\xi|^2)^2} |\hat{u}_1|^2 d\xi \\
\leq \frac{4}{15} t^2 \int_{|\xi| \geq \sqrt{e-1}} \frac{e^{-\frac{t}{4 \log(1 + |\xi|^2)} \log(1 + |\xi|^2)}}{(1 + |\xi|^2)^6} |\hat{u}_1|^2 d\xi \\
\leq C t^{-(l+4)} \|u_1\|_{Y_1}^2, \quad t > 0.
\]

The estimates for \( G_j(t, \xi) \) (for \( j = 1, \ldots, 6 \)) together with the identity (47) provide the following result.

Proposition 4. Let \( n \geq 1 \), \( l \geq 0 \) and \((u_0, u_1) \in Y^{l+1} \times Y^l \). Then there exists a positive constant \( C \), which is independent of \( t \), u_0 and u_1, such that

\[
\int_{|\xi| \geq \sqrt{e-1}} |\hat{u}(t, \xi) - \varphi_2(t, \xi)|^2 d\xi \leq C (\|u_0\|^2_{Y^{l+1}} + \|u_1\|^2_{Y^l}) t^{-(l+3)}
\]

for \( t > 0 \), where \( \varphi_2(t, \xi) \) is given by (46).

\[ \Box \]

4.3. Estimates on the whole space \( \mathbb{R}^n \). In this subsection, we consider three special functions: \( \varphi_1(t, \xi), \varphi_2(t, \xi) \), which are given by (31) and (46), and

\[ \varphi(t, \xi) = \varphi_1(t, \xi) + \varphi_2(t, \xi) \]

defined for \( \xi \in \mathbb{R}^n \).

We will prove that, under certain conditions, each of them is an asymptotic profile as \( t \to \infty \) of the solution \( \hat{u}(t, \xi) \) in \( \mathbb{R}^n \).

Lemma 4.3. Let \( n \geq 1 \) and \((u_0, u_1) \in (L^{1,1}(\mathbb{R}^n) \cap Y^{l+1}) \times (L^{1,1}(\mathbb{R}^n) \cap Y^l) \). Then there exists a constant \( C > 0 \), which is independent of \( t \), u_0, u_1 such that

\[
\int_{\mathbb{R}^n} |\hat{u}(t, \xi) - \varphi(t, \xi)|^2 d\xi \leq C \left( t^{\frac{n+2}{2}} + t^{-(l+3)} \right) I_{0,1}^2
\]
for $t \gg 1$, where

$$I_{0,t} := \sqrt{\|u_0\|^2_t + \|u_1\|^2_t + \|u_0\|^2_{t+1} + \|u_1\|^2_{t+1}}. \quad (51)$$

**Proof.** On the region $|\xi| \leq \sqrt{e^{-t}-1}$, the function $\log(1 + |\xi|^2)$ is positive and bounded by 1, then for $t \geq 0$ it holds that

$$\frac{-t}{\log(1 + |\xi|^2)} \leq -t.$$

We also have $\sin a \leq a$ for all $a \geq 0$. Having this in mind we can get, for $t \geq 0$, the estimates

$$\int_{|\xi| \leq \sqrt{e^{-t}-1}} |\varphi_2(t, \xi)|^2 d\xi \leq 2 \int_{|\xi| \leq |\xi|e^{-1}} e^{-\frac{t}{\log(1 + |\xi|^2)}} \sin^2 \left(\frac{\sqrt{\log(1 + |\xi|^2)} t}{\log(1 + |\xi|^2)}\right) |\hat{u}_1|^2 d\xi$$

$$+ 2 \int_{|\xi| \leq 1} e^{-\frac{1}{\log(1 + |\xi|^2)}} \cos^2 \left(\frac{\sqrt{\log(1 + |\xi|^2)} t}{\log(1 + |\xi|^2)}\right) |\hat{u}_0|^2 d\xi$$

$$\leq 2t^2 e^{-t} \int_{|\xi| \leq |\xi|e^{-1}} |\hat{u}_1|^2 d\xi + 2e^{-t} \int_{|\xi| \leq 1} |\hat{u}_0|^2 d\xi$$

$$\leq 2t^2 e^{-t} |u_1|^2_2 + 2e^{-t} |u_0|^2_2. \quad (52)$$

On the other hand, one knows that

$$e^{-2t \log(1 + |\xi|^2)(1+\log(1 + |\xi|^2))} \leq e^{-2t \log(1 + |\xi|^2)},$$

because of $1 + \log(1 + |\xi|^2) \geq 1$. Then

$$\int_{|\xi| \leq |\xi|} |\varphi_1(t, \xi)|^2 d\xi = |P_0 + P_1|^2 \int_{|\xi| \geq \eta} e^{-2t \log(1 + |\xi|^2)(1+\log(1 + |\xi|^2))} d\xi$$

$$\leq |P_0 + P_1|^2 \int_{|\xi| \geq \eta} (1 + |\xi|^2)^{-2t} d\xi$$

$$= |P_0 + P_1|^2 \int_{\eta \leq |\xi| \leq 1} (1 + |\xi|^2)^{-2t} d\xi + |P_0 + P_1|^2 \int_{|\xi| \geq 1} (1 + |\xi|^2)^{-2t} d\xi$$

$$= \omega_n |P_0 + P_1|^2 \int_{\eta} (1 + r^2)^{-2t} r^{n-1} dr + \omega_n |P_0 + P_1|^2 \int_{1}^{\infty} (1 + r^2)^{-2t} r^{n-1} dr$$

$$\leq C|P_0 + P_1|^2 \left( (1 + \eta^2)^{-2t} + \frac{2 - t}{t - 1} \right)$$

$$\leq C \left( \|u_0\|^2_t + \|u_1\|^2_t \right) \left( (1 + \eta^2)^{-2t} + \frac{2 - t}{t - 1} \right)$$

with a generous constant $C > 0$ for $t \gg 1$ due to Lemmas 2.2 and 2.3. We also note that both above estimates are of exponential type.

Under these preparations we can get the desired estimate in the statement. At first, one notices that

$$|\hat{u}(t, \xi) - \varphi(t, \xi)| = |\hat{u}(t, \xi) - \varphi_1(t, \xi) - \varphi_2(t, \xi)| \leq |\hat{u}(t, \xi) - \varphi_1(t, \xi)| + |\varphi_2(t, \xi)|.$$

So, it holds that

$$|\hat{u}(t, \xi) - \varphi(t, \xi)|^2 \leq 2|\hat{u}(t, \xi) - \varphi_1(t, \xi)|^2 + 2|\varphi_2(t, \xi)|^2. \quad (53)$$

Similarly,

$$|\hat{u}(t, \xi) - \varphi(t, \xi)|^2 \leq 2|\hat{u}(t, \xi) - \varphi_2(t, \xi)|^2 + 2|\varphi_1(t, \xi)|^2. \quad (54)$$
Also, one has
\[ |\hat{u}(t, \xi) - \varphi(t, \xi)| \leq |\hat{u}(t, \xi)| + |\varphi_1(t, \xi)| + |\varphi_2(t, \xi)|. \]

By applying Young’s inequality, we obtain
\[ |\hat{u}(t, \xi) - \varphi(t, \xi)|^2 \leq 2|\hat{u}(t, \xi)|^2 + 4|\varphi_1(t, \xi)|^2 + 4|\varphi_2(t, \xi)|^2. \]

(55)

Let us apply the estimates (53) on the region $|\xi| \leq \eta$, (54) on the region $|\xi| \geq \sqrt{e - 1}$ and (55) on the middle frequency region $\eta \leq |\xi| \leq \sqrt{e - 1}$, respectively. Then one can proceed the estimates as follows.

\[
\int_{\mathbb{R}^n} |\hat{u}(t, \xi) - \varphi(t, \xi)|^2 d\xi = \int_{|\xi| \leq \eta} |\hat{u}(t, \xi) - \varphi(t, \xi)|^2 d\xi + \int_{\eta \leq |\xi| \leq \sqrt{e - 1}} |\hat{u}(t, \xi) - \varphi(t, \xi)|^2 d\xi + \int_{|\xi| \geq \sqrt{e - 1}} |\hat{u}(t, \xi) - \varphi(t, \xi)|^2 d\xi
\]

\[
\leq 2 \int_{|\xi| \leq \eta} |\hat{u}(t, \xi) - \varphi_1(t, \xi)|^2 d\xi + 2 \int_{|\xi| \leq \eta} |\varphi_2(t, \xi)|^2 d\xi + 2 \int_{|\xi| \leq \eta} |\varphi_1(t, \xi)|^2 d\xi + 4 \int_{|\xi| \geq \sqrt{e - 1}} |\hat{u}(t, \xi) - \varphi_2(t, \xi)|^2 d\xi
\]

\[
+ 4 \int_{|\xi| \leq \eta} |\varphi_2(t, \xi)|^2 d\xi + 2 \int_{|\xi| \geq \sqrt{e - 1}} |\varphi_2(t, \xi)|^2 d\xi + 4 \int_{|\xi| \leq \eta} |\varphi_1(t, \xi)|^2 d\xi
\]

\[
+ 4 \int_{|\xi| \leq \eta} |\varphi_1(t, \xi)|^2 d\xi + 4 \int_{|\xi| \geq \sqrt{e - 1}} |\varphi_1(t, \xi)|^2 d\xi
\]

\[
= 2 \int_{|\xi| \leq \eta} |\hat{u}(t, \xi) - \varphi_1(t, \xi)|^2 d\xi + 2 \int_{|\xi| \geq \sqrt{e - 1}} |\hat{u}(t, \xi) - \varphi_2(t, \xi)|^2 d\xi
\]

\[
+ 4 \int_{|\xi| \leq \eta} |\varphi_2(t, \xi)|^2 d\xi + 4 \int_{|\xi| \geq \sqrt{e - 1}} |\varphi_1(t, \xi)|^2 d\xi
\]

\[
+ 2 \int_{|\xi| \leq \eta} |\hat{u}(t, \xi)|^2 d\xi. \tag{56}
\]

Furthermore, Propositions 4.1 and 4.2 tell us that
\[
\int_{|\xi| \leq \eta} |\hat{u}(t, \xi) - \varphi_1(t, \xi)|^2 d\xi \leq C T_0^2 t^{-\frac{n+2}{2}},
\]

\[
\int_{|\xi| \geq \sqrt{e - 1}} |\hat{u}(t, \xi) - \varphi_2(t, \xi)|^2 d\xi \leq C T_0^2 t^{-(l+3)}
\]
for $t \gg 1$. By combining the previous two estimates with Propositions 3 and 4 and Lemma 4.2 we can conclude the desired estimate
\[
\int_{\mathbb{R}^n} |\hat{u}(t, \xi) - \varphi(t, \xi)|^2 d\xi \leq C T_0^2 (t^{-\frac{n+2}{2}} + t^{-(l+3)}) \quad t \gg 1.
\]

\[\square\]

**Lemma 4.4.** Let $u_0, u_1 \in L^1(\mathbb{R}^n)$ and the function $\varphi_1(t, \xi)$ is defined in (31). Then there exists positive constants $C_1, C_2$, depending only on dimension $n$, such that
\[
C_1 |P_0 + P_1|^2 t^{-\frac{n}{2}} \leq \int_{\mathbb{R}^n_+} |\varphi_1(t, \xi)|^2 d\xi \leq C_2 (\|u_0\|_1^2 + \|u_1\|_1^2) t^{-\frac{n}{2}}
\]
for $t \gg 1$.

**Proof.** The function $\varphi_1(t, \xi)$ satisfies
\[
|\varphi_1(t, \xi)| \leq |P_0 + P_1|(1 + |\xi|^2)^{-t}
\]
for $\xi \in \mathbb{R}^n_+$, since $1 + \log(1 + |\xi|^2) \geq 1$. By using Lemmas 2.1 and 2.2, we immediately concluded that
\[
\int_{\mathbb{R}^n_+} |\varphi_1(t, \xi)|^2 d\xi \leq |P_0 + P_1|^2 \int_{\mathbb{R}^n_+} (1 + |\xi|^2)^{-2t} d\xi
\]
\[
= \omega_n |P_0 + P_1|^2 \int_0^1 (1 + r^2)^{-2t} r^{n-1} dr + \omega_n |P_0 + P_1|^2 \int_1^{\infty} (1 + r^2)^{-2t} r^{n-1} dr
\]
\[
\leq C \omega_n |P_0 + P_1|^2 \left( t^{-\frac{n}{2}} + \frac{2^{-t}}{t-1} \right)
\]
\[
\leq C \omega_n (\|u_0\|_1^2 + \|u_1\|_1^2) t^{-\frac{n}{2}}
\]
for $t \gg 1$.

On the other hand, for $|\xi| \leq \eta$, we have $1 + \log(1 + |\xi|^2) \leq 1 + \log(1 + |\eta|^2) = k_\eta$. Thus,
\[
|\varphi(t, \xi)| \geq |P_0 + P_1|(1 + |\xi|^2)^{-k_\eta t}
\]
for $|\xi| \leq \eta$. First, we choose $t_0 > 0$ such that, for all $t > t_0$ it holds that $t^{-\frac{n}{2}} \leq \eta$, and
\[
\frac{1}{e^{2k_\eta t}} \leq \left( 1 + \frac{1}{t} \right)^{-2k_\eta t} \leq 1.
\]
Such $t_0$ exists, because one has
\[
\lim_{t \to \infty} \left( 1 + \frac{1}{t} \right)^{-2k_\eta t} = \frac{1}{e^{2k_\eta}}.
\]

For this choice, we can compute as follows:
\[
\int_{\mathbb{R}^n_+} |\varphi_1(t, \xi)|^2 d\xi \geq \int_{|\xi| \leq \eta} |\varphi_1(t, \xi)|^2 d\xi \geq \omega_n |P_0 + P_1|^2 \int_0^{\eta} (1 + r^2)^{-2k_\eta t} r^{n-1} dr
\]
\[
\geq \omega_n |P_0 + P_1|^2 \int_0^{t^{-\frac{n}{2}}} (1 + r^2)^{-2k_\eta t} r^{n-1} dr
\]
\[
\geq \omega_n |P_0 + P_1|^2 \left( 1 + \frac{1}{t} \right)^{-2k_\eta t} \int_0^{t^{-\frac{n}{2}}} r^{n-1} dr
\]
\[
\frac{\omega_n}{n} |P_0 + P_1|^2 \left(1 + \frac{1}{t}\right)^{-2k_0} t^{-\frac{n}{2}} \geq \frac{\omega_n e^{-4k_0}}{n} |P_0 + P_1|^2 t^{-\frac{n}{2}}
\]

for \( t > t_0 \).

**Lemma 4.5.** Let \( n \geq 1, l \geq 0 \) and \((u_0, u_1) \in Y^{l+1} \times Y^l\). Then there exists a constant \( C > 0 \), which is independent of \( u_0, u_1 \) and \( t \), such that

\[
\int_{\mathbb{R}^n_t} |\varphi_2(t, \xi)|^2 d\xi \leq CI_0^2 t^{-(l+1)}
\]

for all \( t > 0 \), where \( I_{0,t} \) is given in (51).

**Proof.** By definition of \( \varphi_2(t, \xi) \) in (46), we have

\[
|\varphi_2(t, \xi)| \leq e^{-\frac{t}{2 \log(1 + |\xi|^2)} \sin(\sqrt{\log(1 + |\xi|^2) t}) |\hat{u}_1|}
+ e^{-\frac{t}{2 \log(1 + |\xi|^2)} \cos(\sqrt{\log(1 + |\xi|^2) t}) |\hat{u}_0|}.
\]

Hence, the Young's inequality enable us to get

\[
|\varphi_2(t, \xi)|^2 \leq 2e^{-\frac{t}{2 \log(1 + |\xi|^2)} \sin^2(\sqrt{\log(1 + |\xi|^2) t}) |\hat{u}_1|^2}
+ 2e^{-\frac{t}{2 \log(1 + |\xi|^2)} \cos^2(\sqrt{\log(1 + |\xi|^2) t}) |\hat{u}_0|^2}.
\]

It follows from (52), we get

\[
\int_{|\xi| \leq \sqrt{e-1}} |\varphi_2(t, \xi)|^2 d\xi \leq 2t^2 e^{-t} \|u_1\|_2^2 + 2e^{-t} \|u_0\|_2^2, \quad t > 0.
\]

On the high frequency zone \(|\xi| \geq \sqrt{e-1}\) it holds that

\[
\frac{1}{1 + \log(1 + |\xi|^2)} \leq \frac{1}{\log(1 + |\xi|^2)} \leq \frac{2}{1 + \log(1 + |\xi|^2)}.
\]

By using the inequality (60) and the estimates (48) and (58), one can obtain

\[
\int_{|\xi| \geq \sqrt{e-1}} |\varphi_2(t, \xi)|^2 d\xi \leq 2 \int_{|\xi| \geq \sqrt{e-1}} e^{-\frac{t}{2 \log(1 + |\xi|^2)} \sin^2(\sqrt{\log(1 + |\xi|^2) t}) |\hat{u}_1|^2} d\xi
+ 2 \int_{|\xi| \geq \sqrt{e-1}} e^{-\frac{t}{2 \log(1 + |\xi|^2)} \cos^2(\sqrt{\log(1 + |\xi|^2) t}) |\hat{u}_0|^2} d\xi
\]

\[
\leq 4 \int_{|\xi| \geq \sqrt{e-1}} e^{-\frac{t}{1 + \log(1 + |\xi|^2)}} \frac{1}{1 + \log(1 + |\xi|^2)} |\hat{u}_1|^2 d\xi
+ 2 \int_{|\xi| \geq \sqrt{e-1}} e^{-\frac{t}{1 + \log(1 + |\xi|^2)}} |\hat{u}_0|^2 d\xi
\]

\[
= 4 \int_{|\xi| \geq \sqrt{e-1}} e^{-\frac{t}{1 + \log(1 + |\xi|^2)}} (1 + \log(1 + |\xi|^2))^{l+1} (1 + \log(1 + |\xi|^2)) |\hat{u}_1|^2 d\xi
+ 2 \int_{|\xi| \geq \sqrt{e-1}} e^{-\frac{t}{1 + \log(1 + |\xi|^2)}} (1 + \log(1 + |\xi|^2))^{l+1} |\hat{u}_0|^2 d\xi
\]

\[
\leq 4Ct^{-(l+1)} \|u_1\|_{Y^{l+1}}^2 + 2Ct^{-(l+1)} \|u_0\|_{Y^{l+1}}^2, \quad t > 0.
\]
By combining estimates (59) and (61) one can conclude
\[
\int_{\mathbb{R}^n_+} |\varphi_2(t, \xi)|^2 d\xi \leq C t^{-l(l+1)} (\|u_1\|_{l+1}^2 + \|u_0\|_{l+1}^2), \quad t \gg 1
\]
for some generous constant \(C > 0\), independent of \(u_0, u_1\) and \(t\).

4.4. Proof of Theorems 1.1, 1.2 and 1.3. Now, let us prove our main Theorems 1.1, 1.2, and 1.3 at a stroke in the following paragraph.

First, it follows from Lemma 4.3 we have the estimate
\[
\int_{\mathbb{R}^n_+} |\dot{u}(t, \xi) - \varphi(t, \xi)|^2 d\xi \leq C I_0^2 \left( t^{-\frac{n+2}{2}} + t^{-(l+1)} \right) =: P_n(t),
\]
where \(I_0\) is defined by (51).

Next, since we can write as \(\dot{u}(t, \xi) - \varphi_1(t, \xi) = \dot{u}(t, \xi) - \varphi(t, \xi) + \varphi_2(t, \xi)\), and \(\dot{u}(t, \xi) - \varphi_2(t, \xi) = \dot{u}(t, \xi) - \varphi(t, \xi) + \varphi_1(t, \xi)\), from Lemmas 4.5 and 4.4 one has
\[
\int_{\mathbb{R}^n_+} |\dot{u}(t, \xi) - \varphi_1(t, \xi)|^2 d\xi
\leq 2 \int_{\mathbb{R}^n_+} |\dot{u}(t, \xi) - \varphi(t, \xi)|^2 d\xi + 2 \int_{\mathbb{R}^n_+} |\varphi_2(t, \xi)|^2 d\xi
\leq 2C I_0^2 \left( t^{-\frac{n+2}{2}} + t^{-(l+1)} + t^{-(l+1)} \right)
\leq 4C I_0^2 \left( t^{-\frac{n+2}{2}} + t^{-(l+1)} \right) =: M_n(t),
\]
and
\[
\int_{\mathbb{R}^n_+} |\dot{u}(t, \xi) - \varphi_2(t, \xi)|^2 d\xi
\leq 2 \int_{\mathbb{R}^n_+} |\dot{u}(t, \xi) - \varphi(t, \xi)|^2 d\xi + 2 \int_{\mathbb{R}^n_+} |\varphi_1(t, \xi)|^2 d\xi
\leq 2C I_0^2 \left( t^{-\frac{n+2}{2}} + t^{-(l+3)} + t^{-\frac{l+1}{2}} \right)
\leq 4C I_0^2 \left( t^{-(l+3)} + t^{-\frac{l+1}{2}} \right) =: Q_n(t).
\]

By an asymptotic profile we mean the part of \(\dot{u}(t, \xi)\) that decays with the slowest time rate. According to Lemmas 4.4 and 4.5 we know that
\[
\int_{\mathbb{R}^n_+} |\varphi_1(t, \xi)|^2 \leq C I_0^2 t^{-\frac{n+2}{2}}, \quad \int_{\mathbb{R}^n_+} |\varphi_2(t, \xi)|^2 \leq C I_0^2 t^{-(l+1)},
\]
\[
\int_{\mathbb{R}^n_+} |\varphi(t, \xi)|^2 \leq C I_0^2 (t^{-\frac{n+2}{2}} + t^{-(l+1)}).
\]

Therefore, the asymptotic profile of the solution \(\dot{u}(t, \xi)\) as \(t \to +\infty\) is
(i). \(\varphi_1(t, \xi)\) if \(\frac{n}{2} < l + 1\) (compare with (63)),
(ii). \(\varphi(t, \xi)\) if \(\frac{n}{2} = l + 1\) (compare with (62)),
(iii). \(\varphi_2(t, \xi)\) if \(\frac{n}{2} > l + 1\) (compare with (64)).

In the final part of this subsection we will discuss the decay rate of \(P_n(t), M_n(t)\) and \(Q_n(t)\) related to the differences between the solution \(\dot{u}(t, \xi)\) and the suitable asymptotic profiles.
Let \( l \geq 1 \) as this is necessary for the existence and uniqueness of the solution (see Theorem 2.2). For this purpose we introduce a (critical) value \( l^*(n) \) on the regularity \( l \geq 1 \) of the initial data such that

\[
l^*(n) := \frac{n}{2} - 1.
\]

Let us prove Theorems 1.1, 1.2 and 1.3 as follows.

(i). The case for \( l^*(n) < l \).

First, we consider \( l = 1 \). In this case, \( \varphi_1(t, \xi) \) is asymptotic profile for \( n < 4 \).

- If \( n \leq 2 \), then \( \frac{n+2}{2} \leq 2 = l + 1 \), and so \( M_n(t) \leq 8CI_0^2t^{-\frac{n+2}{2}} \).
- If \( n = 3 \), then \( 2 = l + 1 < \frac{n+3}{2} = \frac{5}{2} \). Thus \( M_n(t) \leq 8CI_0^2t^{-(l+1)} = 8CI_0^2t^{-2} \).

Now let us consider the case \( l > 1 \).

- In this case, the rate \( t^{-(l+1)} \) is better than \( t^{-l} \). Therefore, if \( n \leq 2 \), we have \( M_n(t) \leq 8CI_0^2t^{-\frac{n+2}{2}} \).
- If \( n > 2 \) and \( \frac{n}{2} \leq l \), then \( l > 1 \) and \( M_n(t) \leq 8CI_0^2t^{-\frac{n+2}{2}} \).
- If \( n \geq 4 \) and \( \frac{n}{2} - 1 < l < \frac{n}{2} \), we obtain \( l > 1 \) and \( M_n(t) \leq 8CI_0^2t^{-(l+1)} \).
- For \( n = 3 \), we need \( 1 < l \leq \frac{5}{2} \), in order that \( M_n(t) \leq 8CI_0^2t^{-(l+1)} \).

These observations together with the Plancherel Theorem imply the desired statement of Theorem 1.1.

(ii). The case for \( l^*(n) > l \).

If \( l = 1 \), \( \varphi_2(t, \xi) \) is asymptotic profile for \( n > 4 \).

- If \( 4 < n \leq 8 \), then \( \frac{n}{2} \leq 4 = l + 3 \). So \( Q_n(t) \leq 8CI_0^2t^{-\frac{n}{2}} \).
- For \( n > 8 \), we have \( \frac{n}{2} > 4 = l + 3 \) and \( Q_n(t) \leq 8CI_0^2t^{-(l+3)} = 8CI_0^2t^{-4} \).

By assuming \( l > 1 \), it is necessary that \( n > 2l + 2 > 4 \).

- If \( 4 < n \leq 8 \), then \( \frac{n}{2} \leq 4 < l + 3 \). Therefore, \( Q_n(t) \leq 8CI_0^2t^{-\frac{n}{2}} \).
- For \( n > 8 \) and \( \frac{n}{2} - 3 < l < \frac{n}{2} - 1 \), we have \( l > 1 \) and \( Q_n(t) \leq 8CI_0^2t^{-\frac{n}{2}} \).
- If \( n > 8 \) and \( 1 < l \leq \frac{n}{2} - 3 \), we obtain \( Q_n(t) \leq 8CI_0^2t^{-(l+3)} \).

These observations together with the Plancherel Theorem imply the desired statement of Theorem 1.2.

(iii). The case for \( l^*(n) = l \).

This condition implies that \( \frac{n+2}{2} = l + 2 < l + 3 \). Then we have \( P_n(t) \leq 2CI_0^2t^{-\frac{n+2}{2}} \). Due to \( l \geq 1 \), this estimate holds only for \( n \geq 4 \).

These observations together with the Plancherel Theorem imply the desired statement of Theorem 1.3.

4.5. Optimal decay rates of the solution. In this section we prove Theorems 1.4, 1.5 and 1.6 based on previously obtained decay estimates.

We have already used the decomposition such as

\[ \hat{u}(t, \xi) = \hat{u}(t, \xi) - \varphi(t, \xi) + \varphi_1(t, \xi) + \varphi_2(t, \xi), \]

where \( \varphi(t, \xi) = \varphi(t, \xi) + \varphi_2(t, \xi) \) with \( \varphi_1(t, \xi) \) and \( \varphi_2(t, \xi) \) are given by (31), (46). Since \( u_0 \) and \( u_1 \) have the required regularity in Lemmas 4.3, 4.4 and 4.5, one can get

\[
\int_{\mathbb{R}^3} |\hat{u}(t, \xi)|^2 \, dx \leq 4 \int_{\mathbb{R}^3} |\hat{u}(t, \xi) - \varphi(t, \xi)|^2 \, d\xi + 4 \int_{\mathbb{R}^3} |\varphi_1(t, \xi)|^2 \, d\xi + 4 \int_{\mathbb{R}^3} |\varphi_2(t, \xi)|^2 \, d\xi \\
\leq 4CI_0^2(t^{-\frac{n+2}{2}} + t^{-(l+3)} + t^{-\frac{n}{2}} + t^{-(l+1)}) (t \gg 1) \\
\leq KI_0^2(t^{-\frac{n}{2}} + t^{-(l+1)}) =: R_n(t)
\]

with some constant \( K > 0 \).
In the same way as we did in the previous subsection, we compare $\frac{n}{2}$ and $(l+1)$ in order to obtain the decay rate of the solution.

(i). If $l \geq 1$ and $n \leq 3$, then $\frac{n}{2} < 2 \leq l + 1$. So $R_n(t) \leq 2KI_0^2t^{-\frac{n}{2}}$.

(ii). If $n > 4$ and $l > \frac{n}{2} - 1$, we have $R_n(t) \leq 2KI_0^2t^{-\frac{n}{2}}$.

(iii). If $n \geq 4$ and $1 \leq l \leq \frac{n}{2} - 1$, then $l + 1 \leq \frac{n}{2}$. Thus $R_n(t) \leq 2KI_0^2t^{-(l+1)}$.

The last item (iii) combined with the expression (65) and Plancherel Theorem completes the proof of Theorem 1.4.

Proof. We first observe the case of $(u_0, u_1) \in Y^{l+1} \times Y^l$ with $l > \frac{n-2}{2}$. On the one hand,

$$\|\hat{u}(t, \xi)|^2 \leq (|\hat{u}(t, \xi) - \varphi(t, \xi)| + |\varphi(t, \xi)|)^2$$

$$\leq 2|\hat{u}(t, \xi) - \varphi(t, \xi)|^2 + 2|\varphi(t, \xi)|^2$$

$$\leq 4|\hat{u}(t, \xi) - \varphi(t, \xi)|^2 + 4|\varphi_1(t, \xi)|^2 + 4|\varphi_2(t, \xi)|^2.$$

So,

$$\int_{R^n_x} |\hat{u}(t, \xi)|^2 d\xi \leq 4 \int_{R^n_x} |\hat{u}(t, \xi) - \varphi(t, \xi)|^2 d\xi + 4 \int_{R^n_x} |\varphi_1(t, \xi)|^2 d\xi + 4 \int_{R^n_x} |\varphi_2(t, \xi)|^2 d\xi$$

$$\leq CI_0^2l \left( t^{-\frac{n+2}{2}} + t^{\frac{n}{2}} + t^{-(l+1)} \right)$$

$$\leq CI_0^2l^{-\frac{n}{2}},$$

since $l + 1 > \frac{n}{2}$, due to Lemmas 4.4, 4.5 and 4.3. Thus, the upper bound estimates in Theorems 1.5 and 1.6 can be proved by choosing $l = 1$ and $l = \frac{n-2}{2} + \varepsilon$, respectively.

On the other hand, since one has $|\varphi(t, \xi)| \leq |\varphi(t, \xi) - \hat{u}(t, \xi)| + |\hat{u}(t, \xi)|$ and $|\varphi_1(t, \xi)| \leq |\varphi_1(t, \xi) + \varphi_2(t, \xi)| + |\varphi_2(t, \xi)|$, by using Young’s inequality, we obtain

$$\|\hat{u}(t, \xi)|^2 \geq \frac{1}{2}|\varphi(t, \xi)|^2 - |\varphi(t, \xi) - \hat{u}(t, \xi)|^2$$

$$\geq \frac{1}{4}|\varphi_1(t, \xi)|^2 - \frac{1}{2}|\varphi_2(t, \xi)|^2 - |\varphi(t, \xi) - \hat{u}(t, \xi)|^2.$$

By using the estimates just obtained in Lemmas 4.4, 4.5 and 4.3, one can obtain the following expression:

$$\int_{R^n_x} |\hat{u}(t, \xi)|^2 d\xi \geq \frac{1}{4} \int_{R^n_x} |\varphi_1(t, \xi)|^2 d\xi - \frac{1}{2} \int_{R^n_x} |\varphi_2(t, \xi)|^2 d\xi - \int_{R^n_x} |\hat{u}(t, \xi) - \varphi(t, \xi)|^2 d\xi$$

$$\geq C_1|P_0 + P_1|^2 t^{-\frac{n}{2}} - CI_0^2l^{-\frac{n}{2}} - CI_0^2l^{-\frac{n}{2}} - CI_0^2l^{-\frac{n}{2}}$$

$$= t^{-\frac{n}{2}} \left( C_1|P_0 + P_1|^2 - CI_0^2l^{-\frac{2n}{2+n}} - CI_0^2l^{-1} - CI_0^2l^{-\frac{2n+6}{2+n}} \right).$$

(66)
If \( \frac{n}{2} < l + 1 \), then \( \frac{2l-n+2}{2} > 0 \) and \( \frac{2l-n+6}{2} > 0 \), because of \( l + 1 < l + 3 \). Hence, one has
\[
\lim_{t \to \infty} \left( CI_0^2 t^{-\frac{2l-n+6}{2}} + CI_0^2 t^{-1} + CI_0^2 t^{-\frac{2l-n+2}{2}} \right) = 0,
\]
so that there exists \( t_1 \gg 1 \) such that
\[
CI_0^2 t^{-\frac{2l-n+6}{2}} + CI_0^2 t^{-1} + CI_0^2 t^{-\frac{2l-n+2}{2}} \leq \frac{C_1}{2} |P_0 + P_1|^2
\]
for all \( t \geq t_1 \) in the case of \( |P_1 + P_0| \neq 0 \). That is, for \( t \geq t_1 \) it holds that
\[
C_1 |P_0 + P_1|^2 - CI_0^2 t^{-\frac{2l-n+6}{2}} - CI_0^2 t^{-1} - CI_0^2 t^{-\frac{2l-n+2}{2}} \geq \frac{C_1}{2} |P_0 + P_1|^2.
\]
Therefore, one can arrive at the crucial estimate
\[
\int_{\mathbb{R}^n} |\hat{u}(t, \xi)|^2 d\xi \geq \frac{C_1}{2} |P_0 + P_1|^2 t^{-\frac{n}{2}} \tag{67}
\]
for \( t \geq t_1 \) because of (66). By choosing \( l = 1 \) in Theorem 1.5, and \( l = \frac{n-2}{2} + \varepsilon \) in Theorem 1.6, one can get the desired estimates. \( \square \)

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