EFFECTIVE CRITICAL EXPO NENTS FOR
DIMENSIONAL CROS SOVER AND QUANTUM SYSTEMS FROM
AN ENVIRONMENTALLY FRIENDLY RENORMALIZATION GROUP

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Abstract: Series for the Wilson functions of an “environmentally friendly” renormalization group are computed to two loops, for an \( O(N) \) vector model, in terms of the “floating coupling”, and resummed by the Padé method to yield crossover exponents for finite size and quantum systems. The resulting effective exponents obey all scaling laws, including hyperscaling in terms of an effective dimensionality, \( d_{\text{eff}} = 4 - \gamma_\lambda \), which represents the crossover in the leading irrelevant operator, and are in excellent agreement with known results.
Physical systems can exhibit different types of scaling behaviour in different asymptotic regimes. The crossover between such asymptotic regimes is important both theoretically, and experimentally. One may think of a crossover as being induced by some “environmental” variable. Two of the most interesting crossovers are: those induced by finite size effects [1], and those induced by quantum effects [2], [3]. For finite size systems the environmental variable is $L$, the system size, while in quantum systems it is the inverse absolute temperature, $\beta\hbar$. Other environmental factors that may induce a crossover are: long range interactions, anisotropic interactions, external fields, boundary conditions, etc..

The main difficulty in treating systems which exhibit a crossover is that the qualitative nature of the effective degrees of freedom (DOF), i.e. the fluctuations, changes significantly as a function of scale, being very sensitive to the environment in the crossover region. The renormalization group (RG) is our most powerful tool for investigating how physical systems change as a function of scale. If one views the RG transformation as a “coarse graining” procedure, one must ask whether a particular coarse graining is capturing the qualitative changes associated with the crossover. We call an RG which tracks the changing nature of the effective DOF — “environmentally friendly”. Momentum shell integration is, in principle, a naturally environmentally friendly form of renormalization, however, the associated RG’s are not easily computed beyond lowest order in perturbation theory. In addition one must be careful not to throw away possible important environment dependence by arguing that integrating out large momenta $k \gg g$, where $g$ is the characteristic scale set by the environment, should be $g$ independent, as this leads to RG flow equations which when propagated to scales $k < g$ coarse grain effective DOF which are a very poor representation of the system’s fluctuations at that scale.

The most accurate RG results for properties of critical systems, such as critical exponents, have been achieved by applying field theoretic techniques [4], and are in very impressive agreement with experiment [5]. Three approaches have been used: $\varepsilon$ expan-
sion [6], $\frac{1}{N}$ expansion [7], and fixed dimension perturbation theory [8]. The first fails in crossovers where the upper critical dimension changes, such as in finite size crossover. The second fails in crossovers where the order parameter can change its symmetry, such as bicritical crossover. We adopt the spirit of the fixed dimension approach, if not the letter, in the context of environmentally friendly renormalization.

Field theory historically, as distinct from momentum shell integration, has emphasized the role of ultraviolet (UV) divergences, which are independent of infrared scales and therefore environment insensitive. The result is an RG which does not track the changing nature of the effective DOF, and this leads, typically, to a breakdown of perturbation theory. More environmentally friendly RGs have been implemented in some contexts. Amit and Goldschmidt [9] introduced the concept of generalized minimal subtraction (GMS) when considering crossover at a bicritical point. Their results for $\gamma_{eff}$, however, differ from those found using momentum shell integration [10], the latter find a characteristic “dip” in the effective exponent curves. Our methodology applied to a bicritical crossover [11] gives results in agreement with the momentum shell approach. GMS was also applied to uniaxial dipolar ferromagnets in [12], the results of our analysis [13, 11] differ somewhat. Differences can be understood, since our renormalization procedure puts complete Feynman diagrams into the Wilson functions, GMS does not. When the RG equation is solved all the diagrammatic information in these functions is “exponentiated”, the remainder must be taken into account perturbatively, in some way. Other related work is that of Schmeltzer [14] who calculated $\gamma_{eff}$ to one loop for a three dimensional quantum ferroelectric and Lawrie [15] who considered dimensional crossover for $d$-dimensional quantal and $d + 1$ dimensional finite-sized Ising models for $3 < d < 4$ using an $\varepsilon$ expansion. Unlike our method the $\varepsilon$ expansion could not capture the crossover between two non-trivial fixed points as the upper critical dimension changes across the crossover. Field theoretic results for dimensional crossover in a fully finite geometry or a cylinder have been obtained [16]
but the techniques used have not been extended to the case of a system with more than
one fixed point.

Though our general approach is applicable to a very wide class of crossovers [11,17], we
restrict our attention to finite size crossover and quantum/classical crossover. We begin
with the “microscopic” Landau-Ginzburg-Wilson Hamiltonian

$$H[\varphi_B] = \int_0^L \int d^d x \left[ \frac{1}{2} (\nabla \varphi_B)^2 + \frac{1}{2} m_B^2 \varphi_B^2 + \frac{1}{2} t_B(x) \varphi_B^2 + \frac{\lambda_B}{4!} \varphi_B^4 - H_B(x) \varphi_B \right]$$

(1)

which describes either: a layered \( d+1 \) dimensional system, of thickness \( L \); or a \( d \) dimen-
sional quantum system, with \( L = \beta \hbar, \beta \) being the inverse temperature. We will assume
the order parameter possesses an \( O(N) \) symmetry, the case \( N = 1 \) of quantum/classical
crossover represents an Ising model in a transverse magnetic field. In the finite size case
\( m_B^2 + t_B = T - T_0 \), and in the Ising model in a transverse field \( m_B^2 + t_B = \Gamma - \Gamma_0 \). Here,
\( T_0 \) and \( \Gamma_0 \) are the critical temperature and transverse field respectively, in the mean field
approximation.

An \( L \) dependent renormalization is necessary to obtain the desired environmentally
friendly RG. Use of \( L \) dependent normalization conditions achieves this and ensures that
all the Feynman diagrammatic information is exponentiated in the solution of the resulting
RG equation. The relation between the bare and renormalized vertex functions is

$$\Gamma_B^{(N,M)} = Z_\varphi^{-N} Z_{\varphi^2}^{-M} \Gamma^{(N,M)}$$

The renormalized dimensionful coupling is similarly related to the bare
one by \( \lambda_B = Z_\lambda^{-1} \lambda \) (see [4] for a discussion of the notation). We choose the normalization
conditions

\[
\begin{align*}
(i) \quad & \Gamma^{(2)}(k = 0, t = \kappa^2, \lambda, L, \kappa) = \kappa^2 \\
(ii) \quad & \frac{\partial \Gamma^{(2)}}{\partial k^2}(k, t = \kappa^2, \lambda, L, \kappa)|_{k = 0} = 1 \\
(iii) \quad & \Gamma^{(4)}(k = 0, t = \kappa^2, \lambda, L, \kappa) = \lambda \\
(iv) \quad & \Gamma^{(2,1)}(k, t = \kappa^2, \lambda, L, \kappa) = 1
\end{align*}
\]

(2)

which specify \( Z_\varphi, Z_{\varphi^2} \) and \( Z_\lambda \). Condition (i), together with the multiplicative renormal-
ization of \( t \), implies that \( t \) is proportional to \( T - T_c(L) \) for the finite size system, and
\( \Gamma - \Gamma_c(\beta) \) for the quantal Ising model, i.e. that one is measuring temperature/field deviations
relative to the \( L \) dependent critical point. We are assuming here that the system can
exhibit critical behaviour for any value of $L$, which restricts our attention to $d > 1$ in the case of $N > 1$, but in no way restricts the generality of our approach however. We have applied our methods successfully at one loop to dimensional crossover in a non-linear $\sigma$ model [18] and find results in qualitative agreement with those of [3] where a momentum shell integration approach was used, also at the one-loop level. We believe our methods are more easily extended to higher orders. If a normalization condition with $L = \infty$ had been used, then temperature/field deviations would be measured relative to $T_c(\infty)$ or $\Gamma_c(\infty)$, the critical temperature of the bulk system, or critical field of the $\beta = \infty$ quantum system respectively. Note that we are here using an RG which runs the renormalized temperature parameter in distinction to the Callan-Symanzik equation which runs the physical correlation length.

The RG equation can be viewed as a simple consequence of the fact that the bare theory is independent of the arbitrary renormalization scale $\kappa$ at which we choose to define our parameters, i.e. $\kappa \frac{d}{d\kappa} \Gamma^{(N)}_B = 0$. Using the relation between the bare and renormalized vertex functions and expressing things in terms of the renormalized parameters the infinitesimal form of the RG equation then becomes

$$\left( \frac{\kappa}{\partial \kappa} + \frac{\beta}{\partial \lambda} + \gamma_\phi \frac{\partial}{\partial \phi} - \frac{1}{2} \gamma_\phi \left[ N + \bar{\phi}_B \frac{\partial}{\partial \phi B} \right] \right) \Gamma^{(N)} = 0 \quad (3)$$

with

$$\gamma_\phi = \frac{1}{Z_\phi} \kappa \frac{dZ_\phi}{d\kappa}; \quad \gamma_{\phi^2} = \frac{1}{Z_\phi^2} \kappa \frac{dZ_{\phi^2}}{d\kappa}; \quad \text{and} \quad \frac{\beta(\lambda)}{\lambda} = \gamma_\lambda = \frac{1}{Z_\lambda} \kappa \frac{dZ_\lambda}{d\kappa} \quad (4)$$

The functions $\gamma_\phi$, $\gamma_{\phi^2}$ and $\gamma_\lambda$ are the Wilson functions. They are explicitly $L$ dependent due to the normalization conditions (2) and all the physics of the crossover can be gleaned from them.

A suitable coupling, with respect to which perturbation theory can be performed, is the floating coupling [19,11], $h$, which is chosen so as to make the quadratic term in $\beta(h)$ have unit coefficient. Our perturbation theory is then carried out at the level of the Wilson
functions in terms of $h$. The expressions obtained are, however, only the leading terms in an asymptotic expansion of the functions $\beta(h, z)$, $\gamma_\phi(h, z)$ and $\gamma_\phi^2(h, z)$. We use $[2,1]$ Padé approximants to resum these asymptotic series obtaining

$$\beta(h, z) = -\varepsilon(z) h + \frac{h^2}{1 + 4 \left( \frac{(5N+22)}{(N+8)^2} f_1(z) - \frac{(N+2)}{(N+8)} f_2(z) \right) h}$$

and

$$\gamma_\phi^2(h, z) = \frac{(N+2)}{(N+8)} \frac{h}{1 + 6 \left( \frac{1}{(N+8)} (f_1(z) - \frac{1}{3} f_2(z)) \right) h}$$

where the functions $\varepsilon$, $f_1$ and $f_2$ depend on $d$ and $z = \kappa L$ but are independent of $N$. The original non Padé resummed series can be recovered by expanding $1/(1 + xh) \sim 1 - xh$. We will take the solution of (5) as our perturbation parameter. After these equations are solved it is then inappropriate to do any further expansion.

The functions $\varepsilon(z)$, $f_1(z)$ and $f_2(z)$ are the basic building blocks, their specific functional form depending on the particular crossover in question. $\varepsilon(z)$ can be thought of as being a measure of the “effective dimensionality” of the system. The functions $f_1$ and $f_2$ for general $d$ and the crossovers of interest here can be found in [17]. For $d = 3$, the expressions become especially simple, we find $\varepsilon(z) = 1 - z \frac{d}{dz} \ln(\sum m^{-3})$

$$f_1(z) = 2 \sum_{n_1,n_2} \frac{\left( \frac{1}{m_1} \left( \frac{1}{M} - \frac{1}{2m_2} \right) + \frac{1}{m_1 M^2} \left( \frac{1}{m_1} + \frac{2}{m_2} \right) \right)}{\left( \sum \frac{1}{m^3} \right)^2}$$

and

$$f_2(z) = 4 \sum_{n_1,n_2} \frac{1}{M^{m_1}} \frac{1}{\left( \sum \frac{1}{m^3} \right)^2}$$

with $m_i = (1 + \frac{4\pi^2}{z^2} n_i^2)^{\frac{1}{2}}$, $m_{12} = (1 + \frac{4\pi^2}{z^2} (n_1 + n_2)^2)^{\frac{1}{2}}$, $M = m_1 + m_2 + m_{12}$. We plot $\varepsilon(z)$, $f_1(z)$ and $f_2(z)$ against $\ln(1/z)$ in Figure 1.

Effective critical exponents defined as logarithmic derivatives of the associated thermodynamic quantities with respect to $T - T_c(L)$ at fixed $L$ for the finite size crossover and with $\Gamma - \Gamma_c(\beta)$ for fixed $\beta$ in the quantum problem, using the above RG [11,20] can be shown to obey all the usual scaling relations including hyperscaling. The usual
dimension is replaced by the effective dimension $d_{eff} = 4 - \gamma_\lambda$ which reflects the changing importance of the leading irrelevant operator. As a consequence these exponents are related to the Wilson functions through: $\nu_{eff} = 1/(2 - \gamma_\phi^2)$, $\eta_{eff} = \gamma_\phi$, $\gamma_{eff} = (2 - \gamma_\phi)/(2 - \gamma_\phi^2)$, $\alpha_{eff} = (\gamma_\lambda - 2\gamma_\phi^2)/(2 - \gamma_\phi^2)$, $\beta_{eff} = (2 - \gamma_\lambda + \gamma_\phi)/(4 - 2\gamma_\phi^2)$ and $\delta_{eff} = (6 - \gamma_\lambda - \gamma_\phi)/(2 - \gamma_\lambda + \gamma_\phi)$. Analogous effective exponents associated with variations with respect to $L$ at fixed $T$, and $T$ at fixed $\Gamma$ can also be defined and computed.

We present our results in graphical form in figures 2 through 5. In all graphs the horizontal axis is $\ln(\xi_L/L)$, the different curves correspond to $N = 0$ (polymers), $N = 1$ (Ising model), $N = 2$ (XY-model), $N = 3$ (Heisenberg model) and $N = \infty$ (spherical model). The curves represent both a four dimensional layered geometry of thickness $L$ and a three dimensional quantum model at $\beta = L$. The logarithmic corrections to scaling at the bulk end are clearly visible, their magnitude is as expected from four dimensional calculations. All curves are with the boundary condition $h = 1$ at $\ln(\xi_L/L) = -20$, the value of $h$ at the initial scale parameterizes different possible crossover curves but all curves asymptote to the same form. In Figure 2 we plot $\nu_{eff}$, the correlation length exponent, for $N = \infty$, $\nu_{eff} \equiv 1/(d_{eff} - 2)$ across the entire crossover. In Figure 3 we plot $\eta_{eff}$, the exponent which governs the fall off in critical correlations at $T = T_{c}(L)$ and $\Gamma = \Gamma_{c}(\beta)$ for finite size and quantum systems respectively. This exponent is not a monotonic function of $N$ but attains a maximum for some value between $N = -2$ and $N = \infty$, where it is identically zero. This is the least accurate of our exponents and the peak appears to be at $N = 1$, though more accurate values for this exponent suggest it occurs at higher values, probably $N = 3$. Figure 4 shows a plot of the effective specific heat exponent $\alpha_{eff}$ which measures how the singular part of the free energy changes as $\Gamma$ or $T$ varies. The extra case $N = -2$ is added here, since, in the case of dimensional crossover it is distinguishable from the Gaussian model due to the fact that $\gamma_\lambda$ for the latter is zero whereas for the former it is non-zero, being a measure of the changing effect of the leading irrelevant operator. Across
the entire crossover one has $\alpha_{eff} = 2 - \nu_{eff} d_{eff}$. Not only does one see the change in sign of the specific heat exponent as a function of $N$ but one also sees that the effective specific heat exponent can change sign as a function of $\xi_L/L$. This is quite pronounced for the XY model which starts off positive, increases then turns negative at $\xi_L \sim 100 L$. It would be interesting, based on the Harris criterion for the relevance or irrelevance of weak disorder, to see whether disorder could change from being irrelevant to relevant as a function of size, or temperature in the case of a quantum system. In Figure 5 we plot $\gamma_{\lambda} = 4 - d_{eff}$ which also gives information about the effective dimensionality of the system. Notice that $\gamma_{\lambda}$ is very robust to changes in $N$, varying very little across the entire range of $N$, $[-2, \infty]$. The other effective exponents can be determined from the effective exponent laws, which we have verified also by direct calculation. Asymptotic values of critical exponents and associated quantities are tabulated below.

| N     | $\gamma_{\phi}$ | $\gamma_{\phi^2}$ | $h$    | $\gamma_{eff}$ | $\nu_{eff}$ | $\alpha_{eff}$ |
|-------|------------------|---------------------|--------|----------------|-------------|---------------|
| $-2$  | $0^*$            | $0^*$               | 1.800  | 1*             | 0.5*        | 0.5*          |
| $-1$  | 0.0200           | 0.145               | 1.820  | 1.088          | 0.590       | 0.351         |
| 0     | 0.0295           | 0.277               | 1.785  | 1.175          | 0.646       | 0.211         |
| 1     | 0.0329           | 0.388               | 1.732  | 1.257          | 0.639       | 0.083         |
| 2     | 0.0332           | 0.479               | 1.675  | 1.300          | 0.676       | - 0.029       |
| 3     | 0.0322           | 0.552               | 1.621  | 1.395          | 0.709       | - 0.126       |
| 4     | 0.0305           | 0.611               | 1.573  | 1.451          | 0.737       | - 0.211       |
| $\infty$ | $0^*$          | 1*                  | 1*     | 2*             | 1*          | - 1*          |

* These values are exact.

All these values are in very good agreement with corresponding high temperature series and experimental results (see [21] and references therein). We believe the entire crossover curves are of similar accuracy.

In this paper, using two loop Padé resummed perturbation theory for an environmentally friendly RG, we presented effective critical exponents for dimensional crossover in a four dimensional layered system with periodic boundary conditions and quantum to classical crossover in three dimensions. We paid special attention to polymers, Ising model,
XY-model, Heisenberg model and the spherical model. Asymptotic values for the exponents of these systems were found to be in very good agreement with known results and experiment. Our general formalism is applicable to a wide class of crossover problems. Three and higher loop calculations, we believe, are quite feasible numerically by methods similar to that of Nickel [22]. These should yield effective critical exponents to the same degree of accuracy as standard critical exponents. There is merit in pursuing such calculations as our methods provide a direct and physical connection between exponents in different dimensions.

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