STUDY OF AN ELLIPTIC PARTIAL DIFFERENTIAL EQUATION MODELLING THE ANTARCTIC CIRCUMPOLAR CURRENT

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Abstract. We consider the ocean flow of the Antarctic Circumpolar Current. Using a recently-derived model for gyres in rotating spherical coordinates, and mapping the problem on the sphere onto the plane using the Mercator projection, we obtain a boundary-value problem for a semi-linear elliptic partial differential equation. For constant and linear oceanic vorticities, we investigate existence, regularity and uniqueness of solutions to this elliptic problem. We also provide some explicit solutions. Moreover, we examine the physical relevance of these results.

1. Introduction. The Antarctic Circumpolar Current (ACC), the world’s longest and strongest current, is situated between the 40th and 65th latitudes South. Unlike the Arctic, which is an ocean surrounded by land, the ACC is unobstructed by any landmasses, and encircles Antarctica, isolating it from warm subtropical waters and thus strongly contributing to the prevention of ice melting on the southern continent [20]. It is the only current on Earth which encircles the entire planet and it transports 140 million cubic meters of water per second over a distance of 24 thousand kilometers [25]. With an average speed of about 4 kilometers per hour, it takes the water about 8 years to make a complete trip around the Earth. It is on average about 2000 kilometers wide, with a depth varying between 2000 and 4000 meters ([21]). Although very difficult to measure, its average vorticity has been calculated to be of the order of about $10^{-6}$ radians per second ([14]).

Because of a lack of continental boundaries, the current can be organized into multiple filamentary quasi-zonal jets, each having relatively uniform properties. They have an average width of 40 kilometers and a typical velocity of 1 meter per second [6]. These jets can be seen in measurements of sea surface temperature and sea surface height from satellites (see images in [21]). Very complex, the ACC is the current we know the least about. However, it is of fundamental importance: also known as the “great ocean conveyor”, it is responsible for global ocean circulation, exchanging waters between the Atlantic, Pacific and Indian oceans. Since the formation of the waters of the ACC are due to interactions between the ocean, the atmosphere and sea ice, the ACC is therefore also responsible for renewing a substantial fraction of the world ocean volume. As a result, any changes in this

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formation rate will change the rate in which the ocean stores heat, carbon dioxide and fresh water (from sea ice), and thus greatly influence climate change ([21]).

Since the ACC flows in a full circle around the planet and is created by wind-friction and the Coriolis force, it has a very gyre-like flow structure (we refer the reader to [13], [17] and [19] for previous studies of the flow of the ACC). In addition, since its vertical velocity is negligible with respect to its horizontal one [6], we can consider it as a two-dimensional flow. We therefore study it using a recently-derived model for gyres. Using the Mercator projection to project this model from the sphere onto the plane, we obtain a semi-linear elliptic partial differential equation with Dirichlet boundary conditions. In this paper, we will investigate existence, regularity and uniqueness of solutions to this elliptic problem for constant and linear oceanic vorticity. We then give sufficient conditions on our linear vorticity for it to be physically relevant. We also provide some explicit solutions.

2. The Antarctic Circumpolar Current. Let \( \theta \in [0, \pi) \) be the polar angle, where \( \theta = 0 \) corresponds to the North Pole. Our latitude angle is therefore \( \theta - \frac{\pi}{2} \).

Let \( \phi \in [0, 2\pi) \) be the azimuthal (or longitude) angle. The polar and azimuthal velocity components of the horizontal flow on the spherical Earth are given by

\[
\frac{1}{\sin(\theta)} \psi_\phi, \quad -\psi_\theta,
\]

respectively, where \( \psi(\theta, \phi) \) represents the stream function in spherical coordinates.

In this context, let us point out that while the Earth has the shape of an oblate spheroid with mean radius 6378 km and an equatorial radius roughly 20 km longer than the polar radius, for large-scale ocean flows one idealizes this shape as a sphere (see the discussion in [24]). Moreover, since the vertical component of the velocity is by a factor of about \( 10^4 \) smaller than the horizontal components (see the discussion in [5]), we can concentrate on the horizontal flow, within the framework of shallow-water theory.

The standard approach used in the geophysical research literature is to argue that one can rely on the locally flat Cartesian coordinate system, by means of the \( f \)-plane or \( \beta \)-plane approximation. However, the \( f \)-plane equations do not capture any curvature effects, while the heuristic reasoning behind the \( \beta \)-plane approximation is that for a shallow fluid on a rotating sphere only the local normal component of the angular velocity vector matters and, for sufficiently small scales in the north-south direction, the only curvature effect that needs to be taken into account is the variation of the normal component of the angular velocity with latitude (see [24]). However, the \( \beta \)-plane approximation is only mathematically consistent in equatorial ocean regions (see the discussion in [10, 26]), and for this reason it should not be used in studies of the ACC, since in the context of this major current of the Southern Ocean the effects of the Earth’s curvature do matter. The recent gyre model in [5] relies on spherical coordinates and for this reason it is adequate for our purposes.

Let us also comment on the fact that we neglect waves. This is so because of the large-scales that are relevant for our study (of the order of tens of km), whereas the wave heights are of the order of tens of meters (at most). The study of wave-current interactions in the Southern Ocean is an active area of research (see the discussion in [7]). We retain for our purposes the fact that shearing is very important, thus anticipating the importance of oceanic vorticity (in addition to the vorticity induced by the Earth’s rotation). The basic sources of oceanic vorticity are wind force [15] and the tidal currents. Both these oceanic vorticities
are manifestations of a practically non-zero constant vorticity (see the discussions in \[8, 11, 23\]), with the sign (positive or negative) depending on the prevalent wind direction, and, respectively, on whether we are in an ebb or flood tidal mode – while tides refer to the vertical motion of water, the tidal current flood/ebb is the horizontal unidirectional movement of water associated with the rise and fall of the tide, respectively. Non-constant oceanic vorticities are also encountered and present a major challenge that we intend to address.

The governing equation for gyres in \[5\] is given by

\[
\frac{1}{\sin^2(\theta)}\Psi_{\varphi\varphi} + \Psi_\theta \cot(\theta) + \Psi_{\theta\theta} = F(\Psi - \omega \cos(\theta))
\]

(1)

where

\[
\Psi(\theta, \varphi) = \psi(\theta, \varphi) + \omega \cos(\theta)
\]

is associated with the vorticity of motion of the ocean relative to the Earth’s surface, \(2\omega \cos(\theta)\) is the spin vorticity due to the rotation of the Earth and \(F(\Psi - \omega \cos(\theta))\) is the oceanic vorticity, due to the motion of the ocean and specific to a particular ocean flow.

Equation (1) is in non-dimensional variables. The non-dimensionalization of the horizontal flow is

\[
(u', v') = U' (u, v)
\]

\[
r' = R' + H' z
\]

where \(z' = H' z\), \(R' \simeq 6378\text{km}\) is the radius of the Earth, \(H' \simeq 4\text{km}\), the length scale, is the average depth of the ocean, and \(U' \simeq 0.1\text{m.s}^{-1}\), the speed scale, is the typical horizontal velocity of large-scale ocean flows [24]. Moreover, the parameter \(\omega\) is defined by

\[
\omega = \frac{\Omega' R'}{U'}
\]

where \(\Omega' \simeq 7.29 \times 10^{-5}\text{rad.s}^{-1}\) is the constant rate of rotation of the Earth (see the discussion in [5]). Note that \(\omega \approx 4650\) is the typical value, so that this parameter (that accounts for the effects of the Earth’s rotation) is not small.

We would like to investigate this governing equation for the Antarctic Circumpolar Current. In particular, we will look at Drake’s passage, the passage between the southern tip of South America and Antarctica, which is located between the 56th and 60th parallel south. Just as for the rest of the current, the part of the ACC passing through this passage consists of several jets with a predominantly zonal flow.

For this analysis, we turn to the Mercator projection (see [9]). This conformal map, using the change of variable

\[
x = -\ln \left[ \tan \left( \frac{\theta}{2} \right) \right], \quad y = \varphi
\]

(2)

projects the sphere onto the plane which has the Equator as y-axis and the Central Meridian as x-axis. In particular, \(x = -\infty\) at the South Pole.

Using the change of variable (2), \(x\) is negative in the Southern Hemisphere with

\[
\cos(\theta) = \tanh(x), \quad \sin(\theta) = \cosh^{-1}(x).
\]

Setting

\[
u(x, y) = \psi(\theta, \varphi),
\]
we can then rewrite the governing equation (1) as the following semilinear elliptic partial differential equation

\[ \Delta u(x, y) = \frac{F(u(x, y))}{\cosh^2(x)} + 2\omega \cdot \frac{\sinh(x)}{\cosh^3(x)} \]

(3)

The jets of the ACC are typically situated between two parallels, which we will denote by \( \theta_1 \) and \( \theta_2 \). We obtain a two-point boundary condition

\[ u(x_1, y) = u_1(y), \quad x = x_1 < 0 \]

(4)

\[ u(x_2, y) = u_2(y), \quad x = x_2 < 0 \]

(5)

provided that the boundary of our region on the sphere corresponds to the parallels

\[ \theta_1 = 2 \arctan(e^{-x_1}) \in \left( \frac{\pi}{2}, \pi \right) \]

and

\[ \theta_2 = 2 \arctan(e^{-x_2}) \in \left( \frac{\pi}{2}, \pi \right) \]

situated in the Southern Hemisphere. (see fig.1) In this paper, we will consider the homogeneous Dirichlet boundary conditions

\[ u_1(y) = u_2(y) = 0 \]

(6)

meaning that we regard the two parallels that delimit the jet region of the ACC as streamlines.

3. Constant vorticity. We first consider the case when we have constant relative vorticity, in other words, we consider the case \( F(u) = F_0 \). We therefore obtain the following Poisson equation

\[ \Delta u = f(x) \]

(7)

where

\[ f(x) = 2\omega \cdot \frac{\sinh(x)}{\cosh^3(x)} + \frac{F_0}{\cosh^2(x)} \]

Let us first adapt a standard result from the theory of elliptic partial differential equations to our context (see [4] for details).

**Proposition 1.** Let \( \Omega = [x_1, x_2] \times [0, 2\pi) \) be the domain in the plane. The Poisson equation (7) with boundary conditions (6) admits a unique solution \( u \in C^\infty(\overline{\Omega}) \).

**Proof.** We begin by rewriting (7) in its variational formulation. For any \( v \in H^1_0(\Omega) \):

\[ \int_{\Omega} \nabla u \cdot \nabla v = \int_{\Omega} f v \]

Applying the Lax-Milgram theorem in the Hilbert space \( H^1_0(\Omega) \) with the bilinear form

\[ \beta(u, v) = \int_{\Omega} \nabla u \cdot \nabla v \]

(8)

and the linear functional \( l : v \mapsto \int_{\Omega} f \cdot v \) we get the existence of a unique \( u \in H^1_0(\Omega) \) such that \( \beta(u, v) = l(v) \) for every \( v \in H^1_0(\Omega) \).

Although \( \Omega \) has corners, it is the periodicity box for smooth domains (in the \( y \)-direction), \( \{(x, y) : x_1 < x < x_2, \ y \in \mathbb{R}\} \), and for this reason the regularity results for smooth (compact) domains apply. Therefore, by the theory of regularity for elliptic Dirichlet problems, since \( f \in C^\infty(\overline{\Omega}) \), we have \( u \in C^\infty(\overline{\Omega}) \).
Finally, by the maximum principle for elliptic partial differential equations, we recover uniqueness for $u \in C^\infty(\Omega)$. 

4. **Linear vorticity.** Let us now consider the case of linear vorticity. We set

$$F(u) = au + b \quad \text{with} \quad a, b \in \mathbb{R}.$$ 

Equation (3) then takes the form:

$$-\Delta u + ag(x)u = f(x)$$ (9)

where

$$g(x) = \frac{1}{\cosh^2(x)}$$
and
\[ f(x) = \frac{b}{\cosh^2(x)} - 2a \frac{\sinh(x)}{\cosh^3(x)}. \]

**Theorem 4.1.** On the domain \( \Omega = [x_1, x_2] \times [0, 2\pi] \) the elliptic problem (9) with boundary conditions (6) has a unique solution \( u \in C^\infty(\overline{\Omega}) \) for all \( b \in \mathbb{R} \) and all \( a \in \mathbb{R} \) except for those for which the Legendre polynomial \( P_l(\tanh(x)) \) admits a zero on the interval \([x_1, x_2] \), where \( l \in \mathbb{N} \) is of the form \( l = \frac{-1 + \sqrt{1 - 4a}}{2} \).

**Proof.** We begin by rewriting (9) in its variational formulation. For any \( v \in H^1_0(\Omega) \), we have
\[ \int_\Omega \nabla u \cdot \nabla v + a \int_\Omega guv = \int_\Omega fv \] (10)

We set
\[ \beta(u, v) = \int_\Omega \nabla u \cdot \nabla v + a \int_\Omega guv \] (11)
as our bilinear form on \( H^1_0(\Omega) \times H^1_0(\Omega) \) and
\[ j(v) = \int_\Omega fv \]
as our linear form on \( H^1_0(\Omega) \). Clearly, \( \beta(u, v) \) and \( j(v) \) are continuous (for the bilinear form, this can be shown using the Cauchy-Schwarz inequality).

Using the Poincare inequality, the bilinear form \( \beta(u, v) \) is coercive for all \( a \geq 0 \). Therefore, from the Lax-Milgram theorem, we get that there exists a unique weak solution \( u \in H^1_0(\Omega) \) to (9) for all positive \( a \in \mathbb{R} \).

We now examine existence of solutions to (9) for negative vorticities. We define:
\[ L_a u := -\triangle u + agu = f \] (12)
and choose \( \lambda \gg |a| \), such that
\[ L_{a, \lambda} := -\triangle + ag + \lambda : H^2_0(\Omega) \rightarrow L^2(\Omega) \]
is invertible.

We now define:
\[ K := (-\triangle + ag + \lambda)^{-1} : L^2(\Omega) \rightarrow H^2_0(\Omega) \]

Because \( \Omega \) is bounded, by the theorem of Rellich-Kondrakov on compact Sobolev embeddings, \( K \) is compact in \( L^2(\Omega) \) and therefore \( L_{a, \lambda} \) is Fredholm with index 0.

Since \( L_{a, \lambda} u = L_a u + \lambda u \), we have \( \|u\|_{H^2(\Omega)} = C(\|L_a u\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}) \).

We now show that \( L_a \) is a semi-Fredholm operator, meaning that it has closed range and a finite dimensional kernel.

**Lemma 4.2.** \( L_a \) is a semi-Fredholm operator.

**Proof.** Clearly, \( L_a \) has a finite dimensional kernel. It therefore suffices to show that its range is closed.

Let \( u_n \in H^2_0(\Omega) \) such that \( L_a u_n = f_n \) converges to \( f \) in \( L^2(\Omega) \). It suffices to show that \( u_n \) has a subsequence converging in \( H^2_0(\Omega) \).

Since the kernel of \( L_a \) is finite dimensional, we write \( H^2_0(\Omega) = X \oplus \ker(L_a) \).

Without loss of generality, we assume that \( u_n \in X \) (since we are only concerned with the dimension of the range).
First let us suppose that $u_n$ is bounded in $H^2_0(Ω)$. By compactness of $H^2_0(Ω)$ in $L^2(Ω)$, we extract a convergent subsequence in $L^2(Ω)$. Since
\[ \|u_n - u_m\|_{H^2(Ω)} \leq C(\|f_n - f_m\|_{L^2(Ω)} + \|u_n - u_m\|_{L^2(Ω)}), \]
u_n converges in $X$.

Let us now suppose that $u_n$ is unbounded in $H^2_0(Ω)$. Then
\[ L_a u_n \|u_n\|_{H^2(Ω)} = f_n \|u_n\|_{H^2(Ω)} \to 0 \]
and therefore, after extraction, we get
\[ u_n \|u_n\|_{H^2(Ω)} \to v \in H^2_0(Ω) \]
However, then $v \in X$ has a norm $\|v\|_{H^2(Ω)}$ and $L_a v = 0$, which is a contradiction.

Therefore, since $L_a$ depends continuously on $a$, the index $\text{Ind}(L_a)$ is constant with respect to $a$. From previously, we know that for $a > 0$, the index of $L_a$ is 0 by coercivity. Therefore, $\text{Ind}(L_a) = 0$ for all $a \in \mathbb{R}$.

Therefore, we have existence of solutions for all $a$ for which we can show that we have at most one solution to (9) for.

Before we turn to the question of uniqueness of solutions, which will enable us to determine for which $a$ we have existence, we would like to note, that till now we have only talked about existence of weak solutions. However, with similar considerations concerning $Ω$ as in the proof of Proposition 1, and since $f \in C^∞(Ω)$ and $g \in C^∞(Ω)$, by the theory on regularity for elliptic problems with smooth coefficients, we have $u \in C^∞(Ω)$.

We now examine for which $a$, (9) admits at most one solution. By the maximum principle for elliptic partial differential equations, since $g$ is bounded on the closure of our domain, we have uniqueness of solutions for all $a \in \mathbb{R}^+$. We now consider the case $a \in \mathbb{R}^-$. Since $g \in L^∞(Ω)$, by the generalized maximum principle for elliptic problems (see [18]), if we can find a function $σ > 0$ on $Ω$ such that $(\Delta + g)[σ] ≤ 0$ in $Ω$, then (9) with boundary conditions (6) has at most one solution.

A solution to the problem
\[ (L + g)[u] = 0 \]is the Legendre function $P_l(z)$ (see discussions in [1] and [16]) where $z = \tanh(x)$ and $a = -l(l+1)$ for all $l \in \mathbb{R} \setminus \mathbb{Z}$. We would like to note that the Legendre function in this case is simply equal to a hypergeometric function (see [1] for details):
\[ P_l(z) = {}_2F_1\left( -l, l + 1; 1; \frac{1-z}{2} \right) = \frac{\Gamma(1)}{\Gamma(l)\Gamma(l+1)} \sum_{n=0}^{\infty} \frac{\Gamma(-l+n)\Gamma(l+1+n)}{\Gamma(1+n)n!} \left( \frac{1-z}{2} \right)^n. \]
Since $Ω = [x_1, x_2] \times [0, 2\pi)$ with $-∞ < x_1 < x_2 < 0$, we have
\[ \frac{1 - \tanh(x)}{2} < 1 \]
on $\Omega$, and therefore
\[ s_n := \sum_{n=0}^{\infty} \frac{\Gamma(-l+n)\Gamma(l+1+n)}{\Gamma(1+n)n!} \left(\frac{1-z}{2}\right)^n \]
converges to some $M$ for all $l \in \mathbb{R} \setminus \mathbb{Z}$ (see [2]).

Without loss of generality, we assume that $M > 0$. We now choose:
\[ \sigma_1(x) = P_l(\tanh(x)) \]
for $l \in (2k-1, 2k)$, with $k \in \mathbb{N}^*$, and
\[ \sigma_2(x) = -P_l(\tanh(x)) \]
for $l \in (2k, 2k+1)$, with $k \in \mathbb{N}$.

We therefore have shown that we have at most one solution to (9) for all
\[ a \in (-4k^2 - 6k - 2, -4k^2 - 2k) \cup (-4k^2 - 2k, -4k^2 + 2k) \cup \mathbb{R}^+. \]
which is equivalent to all $a$ except those for which $l$ is a positive integer.

We would now like to investigate what happens when $l$ is a positive integer. A solution to (13) in the case when $l \in \mathbb{N}^*$, are the Legendre polynomials $P_l(z)$ where we once again have $z = \tanh(x)$ and $a = -l(l+1)$.
\[ P_l(z) = \frac{1}{2^l} \sum_{k=0}^{\lfloor l/2 \rfloor} (-1)^k \binom{l}{k} \left(\frac{2l-2k}{l}\right) z^{l-k}. \]
If we consider Drake's passage, which is situated between the 56th and 60th latitude south, from our Mercator projection, we have
\[ -\ln[\tan(75°)] = x_1 \leq x \leq x_2 = -\ln[\tan(73°)] \]
We therefore set
\[ \sigma_3(x) = P_l(\tanh(x)) \]
for all $l$ for which $P_l(\tanh(x)) > 0$ on the interval $[x_1, x_2]$, and
\[ \sigma_4(x) = -P_l(\tanh(x)) \]
for all $l$ for which $P_l(\tanh(x)) < 0$ on the interval $[x_1, x_2]$.

We have therefore shown that we have at most one solution to (9) with boundary conditions (6) for all $a \in \mathbb{Z} \setminus \mathbb{N}$ for which $P_l(\tanh(x))$, with $a = -l(l+1)$, has no zeros on the interval $x \in [-\ln(\tan(75°)), -\ln(\tan(73°))]$.

Examples of $l \in \mathbb{N}$ for which this is not the case are $l \in \{4, 9, 10, 15, 20,...\}$, for which we have $a = -20, -90, -110, -240, -420...$ respectively.

Combining these uniqueness results with the arguments above, we obtain existence of unique smooth solutions to (9) with boundary conditions (6) for all $a \in \mathbb{R} \setminus \{-20, -90, -110, -240, -420, ...\}$. \hfill \Box

**Remark 1.** In order to find which polynomials have roots in our interval $[x_1, x_2]$, it is interesting to note that the roots of a Legendre polynomial $P_l(z)$, denoted by $z_{l,j}$ where $j = 1, 2, ..., l$ denote the distinct roots such that $z_{l,1} < z_{l,2} < ... < z_{l,l}$ satisfy the following inequality (see [22]):
\[ \cos \left[\frac{(l-j+1)\pi}{l+1}\right] < z_{l,j} < \cos \left[\frac{(l-j+\frac{3}{2})\pi}{l+\frac{1}{2}}\right] \quad (14) \]
for $j = 1, 2, ..., \left\lfloor \frac{l}{2} \right\rfloor$. For the other half we have $z_{l,l-j+1} = -z_{l,j}$.
5. Further considerations concerning uniqueness. We would now like to take the results we found on uniqueness in the previous section one step further. In particular, recall that for negative $a$ we used the generalized maximum principle for elliptic problems which states that if we can find a function $\sigma > 0$ on $\Omega$ such that $(\triangle + g)[\sigma] \leq 0$ in $\Omega$, then (9) with boundary conditions (6) has at most one solution.

We recall the maximum principle for sufficiently narrow domains:

**Theorem A.** [3] Suppose that $L$ is strictly elliptic. Then there is a $d > 0$ such that if $S \subset \{(x, y) \in \Omega : |x| < d\}$ and $u \in C^2(S) \cap C(\overline{S})$ satisfies $Lu \geq 0$ in $S$, then there exists $\sigma \in C^2(\overline{S})$ with $\sigma > 0$ and $L\sigma \leq 0$ on $\overline{S}$.

In other words, we can find such an $S \subset \Omega$ in which (9) with boundary conditions (6) admits unique solutions for all $a, b \in \mathbb{R}$.

In particular, if we set $\sigma(x) = \cos(\alpha x)$, with $|x| \leq \frac{\pi}{4\alpha} = d$, we have that $\sigma(x) > 0$ on $S \subset \Omega$. By simple calculations, we get

$$L\sigma \leq \left( -\alpha^2 + \frac{|a|}{\|\cosh^2(x)\|_{L^\infty(S)}} \right) \frac{\sqrt{2}}{2} \tag{15}$$

which is negative if and only if $|a| \leq \alpha^2 \|\cosh^2(x)\|_{L^\infty(S)}$.

Since $\alpha = \frac{\pi}{4\overline{d}}$, we can therefore conclude that the smaller $d$ is, the more uniqueness we get for negative $a$. In our case, we would have $d = \frac{2\pi - 2\overline{d}}{2\alpha}$.

In particular, at Drake’s passage, the width of the ACC (as arc length in spherical coordinates) is about $4^\circ$, or approximately 500km. On the plane, we have

$$d = \frac{|-\ln(\tan(73)) + \ln(\tan(75))|}{2} \simeq 0.07$$

from which we get

$$\|\cosh^2(x)\|_{L^\infty(S)} \simeq 1$$

and therefore, using Theorem A, uniqueness of solutions for $|a| \leq 125$.

Moreover, the ACC consists of jets, each of which has an average width of 40km on the sphere, or an arc length in spherical coordinates of about $0.1^\circ$. Consequently, similarly as above, we get $d \simeq 1.6 \times 10^{-3}$ and therefore from Theorem A, uniqueness of solutions for $|a| \leq 2.5 \times 10^5$.

**Remark 2.** It is interesting to examine which vorticity values would be physically relevant. The typical vorticity values for the ACC are of order $10^{-6}$ s$^{-1}$ ([14]). Using the non-dimensionalization described in Section 2, we get that $F(u) = au + b$ is of order $10^{-5}$.

For simplicity of notation, we write $|F(u)| \leq C_v$ where $C_v > 0$ is a constant of order $10^{-5}$.

Using the variational formulation of (3), we obtain

$$\int_\Omega |\nabla u|^2 \leq C_v \int_\Omega |gu| + \int_\Omega |fu| \tag{16}$$

where

$$g(x) = -\frac{1}{\cosh^2(x)}$$
and

$$f(x) = -2\omega \frac{\sinh(x)}{\cosh^3(x)}.$$  

Since we have homogeneous Dirichlet boundary conditions (6), by the maximum principle for elliptic partial differential equations (see [18]), the maximum $\tilde{M}$ of $u$ is inside the strip. Let us set $\tilde{M} = u(x_0, y_0)$. We therefore have

$$|\tilde{M} - 0| = |u(x_0, y_0)| = \left| \int_{x_2}^{x_0} u_y^2 \, dy \right|^{\frac{1}{2}} \cdot |x_0 - x_2| \leq \left( \int_{x_2}^{x_0} u_y^2 \, dy \right)^{\frac{1}{2}} \cdot |x_1 - x_2| \cdot \frac{1}{2}$$

using Hölder’s inequality in the second line. Consequently, we obtain

$$\|u\|_{L^\infty(\Omega)} \leq \|\nabla u\|_{L^2(\Omega)} C_p$$ (17)

where $C_p \leq \frac{|x_1 - x_2|}{2}$ is the Poincaré constant associated with our domain $\Omega$.

However, from (16), using the Poincaré inequality and Hölder’s inequality, we have:

$$\|\nabla u\|_{L^2(\Omega)} \leq C_p \| (C_v |g| + |f|) \|_{L^\infty(\Omega)}$$ (18)

Combining (18) with (17), we get the following estimate

$$\|u\|_{L^\infty(\Omega)} \leq C_p^2 \| (C_v |g| + |f|) \|_{L^\infty(\Omega)}$$ (19)

with

$$C_p^2 \leq \left( \frac{\ln(tan(73)) + \ln(tan(75))}{2} \right)^2 \approx 4.4 \times 10^{-3}$$

and

$$\| (C_v |g| + |f|) \|_{L^\infty(\Omega)} \leq (C_v + 0.77\omega) \approx 3.6 \times 10^3$$

since we know from the non-dimensionalization described in Section 2 of this paper that $\omega \approx 4.7 \times 10^3$.

Therefore, using (19), we can estimate that physically relevant $a$ have to satisfy

$$a \leq 0.02b + O(10^{-6}).$$

Since $b$ is an arbitrary constant we can choose it so that all $a$ for which we have no uniqueness have no physical relevance.

6. Some more general linear vorticities. So far, we have dealt with linear vorticities of the form $F(u) = au + b$ where $a$ and $b$ are real numbers. We would like to take a brief look at the case when $a$ and $b$ are functions and in particular when $a, b \in C^\infty(\Omega)$. (3) then takes the form:

$$-\triangle u + a(x, y)g(x)u = f(x, y)$$ (20)

where

$$g(x) = \frac{1}{\cosh^2(x)}$$
and

\[ f(x, y) = \frac{b(x, y)}{\cosh^2(x)} - 2\omega \frac{\sinh(x)}{\cosh^3(x)}. \]

Similarly as in the proof of Theorem 4.1, from the variational formulation of (20), we set

\[ \beta(u, v) = \int_{\Omega} \nabla u \cdot \nabla v + \int_{\Omega} aguv \]  

(21)
as our bilinear form on \( H^1_0(\Omega) \times H^1_0(\Omega) \) and

\[ j(v) = \int_{\Omega} fv \]
as our linear form on \( H^1_0(\Omega) \). As before, \( \beta(u, v) \) and \( j(v) \) are clearly continuous.

In order to apply the Lax-Milgram theorem, which will provide us with the existence of weak solutions to (20), it remains to show that our bilinear form \( \beta(u, v) \) is coercive.

Clearly, for every \( u \in H^1_0(\Omega) \):

\[ \beta(u, u) \geq \| \nabla u \|_{L^2(\Omega)}^2 - \int_{\Omega} |a| \cdot |g| u^2 \]

However:

\[ \int_{\Omega} |a| \cdot |g| u^2 \leq \| g \|_{L^\infty(\Omega)} \| a \|_{L^\infty(\Omega)} \| u \|_{L^2(\Omega)}^2 \leq \tilde{C} \| \nabla u \|_{L^2(\Omega)}^2. \]

The last step follows from the Poincare inequality and

\[ \tilde{C} \leq \| g \|_{L^\infty(\Omega)} \| a \|_{L^\infty(\Omega)} C_p \]

where \( C_p \leq \frac{|x_1 - x_2|}{2} \) is the Poincare constant associated with our domain \( \Omega \) (see Section 5).

We therefore have:

\[ \beta(u, u) \geq \| \nabla u \|_{L^2(\Omega)}^2 - \tilde{C} \| \nabla u \|_{L^2(\Omega)}^2 \]

from which we can conclude, using once more the Poincare inequality, that \( \beta \) is coercive if

\[ \| a \|_{L^\infty(\Omega)} < \frac{1}{\tilde{C} \| g \|_{L^\infty(\Omega)}} \approx 47.6. \]  

(22)

Since \( b(x, y) \in C^\infty(\Omega) \), just as in the proof of Theorem 4.1, for the solutions \( u \) to (20) (where \( a(x, y) \) satisfies the requirement (22)), with boundary condition (6), we have \( u \in C^\infty(\Omega) \) by the Theorem on regularity for elliptic partial differential equations (see [4]).

In Drake’s passage, from Theorem A and (15), we recover uniqueness of solutions, provided that

\[ \| a \|_{L^\infty(\Omega)} \leq \alpha^2 \| \cosh^2(x) \|_{L^\infty(S)} \leq 125. \]

This result is stronger than what we need since we only have existence for \( \| a \|_{L^\infty(\Omega)} < 47.6 \). In fact, from the considerations on physically relevant vorticites in Section 5, even condition (22) is more than what we need in order to study the ACC.
7. Explicit solutions for negative linear vorticities. Following the same process as in [12] for arctic gyres (where we had a different boundary-value problem, the Arctic being the geographical opposite of the Antarctic), and using the method of separation of variables, we find explicit solutions for the ACC for all negative vorticites of the form \( F(u) = au + b \), with \( a \in \mathbb{R}^- \) and \( b \in \mathbb{R} \).

**Theorem 7.1.** For \( a = -l(l+1) \), where \( l \in \mathbb{R} \) and for any \( b \in \mathbb{R} \),
\[
    u(x,y) = \sum_{k \in \mathbb{Z}^+ \mid |k| \leq l} \alpha_k [P_k^l(\tanh(x))]^{-1} P_k^l(\tanh(x)) e^{iky} \\
    + b[x + \ln(2 \cosh(x))] - \omega[1 + \tanh(x)]
\]
is the general solution to (9) with boundary conditions (4) and (5).

**Remark 3.** Here we have
\[
P_k^l(z) = \left[ 1 + z \right]^l \left[ 1 - z \right]^{-l} \frac{1}{\Gamma(-l)\Gamma(l+1)} \sum_{n=0}^{\infty} \frac{\Gamma(-l+n)\Gamma(l+1+n)}{\Gamma(1-k+n)\Gamma(2)} \frac{1}{2^n} (1-z)^n
\]
for all \( l \in \mathbb{R} \setminus \mathbb{Z} \), known as the associated Legendre functions [1],[2], and
\[
P_k^l(z) = (-1)^{l-k} \frac{1}{2^l l!} \frac{d^{l+1}}{dx^{l+1}} (z^2 - 1)^l
\]
for all \( l \in \mathbb{N} \), known as the associated Legendre polynomials [1].

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