THE HIGHER STRUCTURE OF UNSTABLE HOMOTOPY GROUPS

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Abstract. We construct certain unstable higher order homotopy operations indexed by the simplex categories of \( \Delta^n \) for \( n \geq 2 \), and prove that all elements in the homotopy groups of a wedge of spheres are generated under such operations by Whitehead products and the group structure. This provides a stronger unstable analogue of Cohen’s theorem on the decomposition of stable homotopy.

0. Introduction

In [Co], Joel Cohen showed how all elements in the stable homotopy groups of the sphere spectrum can be generated under composition and higher order homotopy operations from certain atomic elements: the three Hopf classes and their odd-primary analogues. The main importance of this result is conceptual, as one of the few known global facts about the stable homotopy groups as a whole (see [Lin, N, Sc1, Sc2, BSS1, BSS2]). Our goal here is to show that a similar but stronger result holds for the unstable homotopy groups of wedges of spheres.

There are a number of competing definitions of higher homotopy operations, particularly in the unstable setting (see [Sp, BMa, Sh, Sa, CF, BJT2]), all of which ultimately involve obstructions to lifting appropriate diagrams from the homotopy category. For our purposes, the relevant indexing categories turn out to be finite subcategories of \( \Delta \) (see §0.5), yielding the notion of a higher order simplicial operation, defined in Section 1. This construction makes sense in any \((\infty, 1)\)-category, is in fact independent of the specific model of \( \infty \)-categories we use, and includes as a special case the long Toda brackets of [Wal, BJT3, BBS1]. Formally, it also includes the (iterated) suspension as a special case (see §5.1 below).

We are mainly concerned here with decomposability for the homotopy groups of spheres, since the decomposition for wedges of spheres follows from Hilton’s Theorem. This can be thought of as a higher order “ringoid” version of identifying the indecomposables for a graded algebra. However, following [Co, Theorem 4.5], we first address the “module” version: that is, the question of higher order decomposability for \( \pi_* X \) as a \( \Pi \)-algebra (a graded group equipped with an action of the primary homotopy operations – see [St, §1]), in the following sense:

0.1. Definition. Denote by \( \Pi_{>1} \) the graded set of all classes in \( \pi_* V \), for all finite wedges \( V \) of simply-connected spheres. For a simply-connected space \( X \in \text{Top}_s \), we say that \( \pi_* X \) is generated of higher order over \( \Pi_{>1} \) by a graded subset
A ⊂ \pi_\ast X \quad \text{if any } \gamma \in \pi_N X \quad \text{is a linear combination over } \mathbb{Z} \text{ of elements obtained}
from \ A \text{ recursively}

(i) By the action of a non-trivial primary operation – that is, a Whitehead product or precomposition with an element \( \alpha \in \pi_{N+1} S^N \) for some \( N \geq 2 \) (see [Wh, Ch. XI]); or

(ii) As the value of the higher order simplicial operation associated to an augmented restricted simplicial space \( W_\bullet \to X \) as in Section 2, with each \( W_n \) a wedge of simply-connected spheres.

Note that both types of constructions raise degrees.

0.2. Definition. The set \( A \) of atomic classes in \( \pi_\ast X \) consists of those \( [f] \in \pi_n X \) which, for some prime \( p \geq 0 \), induce a non-zero map \( f_*: H_n(S^k; F_p) \to H_n(X; F_p) \) (where \( F_0 := \mathbb{Q} \) by convention). These classes are determined up to multiplication by \( 1 < k < p \) when \( p > 0 \).

Theorem A. If \( X \) is simply-connected, the elements of \( \pi_\ast X \) are generated of higher order over \( \Pi_{>1} \) by the atomic maps.

See Theorem 3.1 below. Our main technical result is then:

Theorem B. If \( X \) is the \( k \)-connected cover of the \( k \)-sphere for \( k \geq 2 \), the atomic maps in \( \pi_\ast X \) consist of (the lifts of) the Hopf maps \( \eta_k \in \pi_{k+3} S^k \), \( \nu_k \in \pi_{k+3} S^k \), and \( \sigma_k \in \pi_{k+7} S^k \), and the first \( p \)-torsion classes \( \alpha_1(p) \in \pi_{k+2p-3} S^k \), for odd primes \( p \).

See Theorem 4.1 below. The “spheres of birth” for the various classes differ, of course:

\( k = 2 \) for \( \eta_k \), \( k = 4 \) for \( \nu_k \), \( k = 8 \) for \( \sigma_k \), and \( k = 3 \) for \( \alpha_1(p) \).

In this paper we are mainly concerned with the “ringoid” \( \Pi_{>1} \) itself, whose elements correspond to the primary operations on homotopy groups (as noted above).

0.3. Definition. We say that \( \Pi_{>1} \) is generated of higher order by a (graded) subset \( F_0 \) if there is an exhaustive increasing filtration \( F_0 \subseteq F_1 \subseteq F_2 \subseteq \ldots \) of \( \Pi_{>1} \), such that \( F_n \) is obtained from classes in lower filtration by group operations, composition (of elements in \( F_{<n} \)), or as the value of a higher order simplicial operation associated to a simplicial space \( W_\bullet \) with \( \|W_n\| \simeq \mathbb{V}_n \), where that each \( W_n \) is a finite wedge of spheres and each face map of \( W_\bullet \) is in \( F_{<n} \).

Theorem B then implies that \( \Pi_{>1} \) is generated of higher order by the atomic maps listed there, yielding an unstable version of [Co, Theorem 4.2] (see Corollary 4.13). A further analysis yields:

Theorem C. The homotopy groups of a finite wedge of simply-connected spaces are generated of higher order by the fundamental classes (i.e., inclusions of wedge summands) and their Whitehead products.

See Theorem 5.16 below.

In addition to the sample calculations of Section 5 used in the proof of Theorem C, in Section 6 we show what form it takes in practice by providing a rational higher operation description of the Hopf fibrations \( S^{2n+1} \to \mathbb{C}P^n \) for all complex projective spaces.
0.4. Remark. Cohen actually provides an algorithm for finding decompositions of stable classes, using spectral sequences (and conversely, see [Ada, WX] for examples of the use of such decompositions to calculate differentials in the Adams spectral sequence).

It seems unlikely that such an algorithm could be obtained unstably with the present state of our knowledge. However, in [BBS3] we provide a more detailed description of how the value of a previously defined higher order operation may be inserted in the diagram defining another such operation, with a view to defining an explicit notion of the “algebra of unstable higher homotopy operations”. This is perhaps more meaningful than Theorem C itself, since the abstract fact that a particular set \( \mathcal{F}_0 \) suffices to generate \( \Pi_{>1} \) is not very useful on its own.

0.5. Notation. Let \( \Delta \) denote the category of non-empty finite ordered sets and order-preserving maps (cf. [May, §2]), and \( \Delta_{\text{res}} \) the subcategory with the same objects but only monic maps. Similarly, \( \Delta_+ \) denotes the category of all finite ordered sets (including \( \emptyset \)), \( \Delta_{\text{res, +}} \) the corresponding subcategory of monic maps, and \( \Delta_{(n)} \) the subcategory of \( \Delta_{\text{res, +}} \) with consisting of ordered sets with at most \( n + 1 \) elements.

A simplicial object \( G_\bullet \) in a category \( C \) is a functor \( \Delta^{\text{op}} \to C \), a restricted simplicial (or semi-simplicial) object is a functor \( \Delta_{\text{res}}^{\text{op}} \to C \), an augmented simplicial object is a functor \( \Delta_+^{\text{op}} \to C \), a restricted augmented simplicial object is a functor \( \Delta_{\text{res, +}}^{\text{op}} \to C \), and an \( n \)-truncated restricted augmented simplicial object is a functor \( \Delta_{(n)}^{\text{op}} \to C \).

In each case write \( G_n \) for the value of \( G_\bullet \) at \( [n] = (0 < 1 < \ldots < n) \). There is a natural embedding \( c(-)_\bullet : C \to C^{\Delta^{\text{op}}} \), with \( c(A)_\bullet \) the constant simplicial object and similarly \( c_+(A)_\bullet \) for the constant augmented simplicial object. The inclusion of categories \( \Delta \to \Delta_+ \) induces the forgetful functor \( C^{\Delta_+^{\text{op}}} \to C^{\Delta^{\text{op}}} \).

The category of topological spaces will be denoted by \( \text{Top} \), that of pointed spaces by \( \text{Top}_* \), and that of pointed connected spaces by \( \text{Top}_0 \). The category of simplicial sets will be denoted by \( \mathcal{S} = \text{Set}^{\Delta^{\text{op}}} \), that of pointed simplicial sets by \( \mathcal{S}_* = \text{Set}_*^{\Delta^{\text{op}}} \), that of reduced simplicial sets by \( \mathcal{S}^{\text{red}} \) (see [GJ III, §3]), and that of simplicial groups by \( \mathcal{G} = \text{Gp}^{\Delta^{\text{op}}} \). Recall that a reduced simplicial set \( X \) is one that has a unique zero simplex \( X_0 = * \). Write \( \text{map}_*(X, Y) \) for the standard function complex in \( \mathcal{S}_*, \text{Top}_0, \) or \( \mathcal{G} \) (see [GJ I, §1.5]).

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1. Simplicial higher order operations

As noted in the introduction, many versions of higher order operations have appeared in the literature, starting with Adem’s secondary cohomology operations (see [Add]), Massey’s triple products (see [Mas]), and the Toda brackets of [TT]. The values of such operations usually appear as obstructions to lifting a homotopy-commutative diagram to a model category \( C \), and are characterized by the following properties:
(i) The operations are indeterminate, in the sense that more than one value may be obtained, depending on choices made in lifting;
(ii) An \( n \)-th order operation is defined only when all lower order operations (associated to partial diagrams) vanish, for some consistent set of choices.
(iii) The final values are expressible in \( \text{ho} \mathcal{C} \).

See [BMa, BJT2].

1.1. Models of \( \infty \)-categories. Although in this paper we are only concerned with topological spaces, the notion of higher operation we use here makes sense in a more general setting – namely, any model of \( \infty \)-category theory consisting of:

(a) A category \( \mathcal{C} \), with a distinguished full subcategory \( \mathcal{C}_0 \) of \( \infty \)-categories.
(b) A homotopy category functor \( \mathcal{P}: \mathcal{C}_0 \to \text{Cat} \), with a right adjoint \( B: \text{Cat} \to \mathcal{C}_0 \) called the nerve functor.
(c) A set of objects functor \( \text{Obj}: \mathcal{C} \to \text{Set} \), such that for each \( X \in \mathcal{C}_0 \) and \( x, y \in \text{Obj}(X) \), we have a Kan complex \( \text{Map}_X(x, y) \) with homotopy associative and unital composition, and \( \mathcal{P}(x, y) = \pi_0 \text{Map}_X(x, y) \). Morphisms \( F: X \to Y \) in \( \mathcal{C} \) induce \( \text{Map}_X(x, y) \to \text{Map}_Y(F(x), F(y)) \), which respects the composition operation up to homotopy.
(d) When dealing with any specific \( X \in \mathcal{C}_0 \), we shall assume that all necessary limits and colimits exist in \( X \), and that it is pointed (that is, the initial and final objects coincide).

Compare [RV] and [BMe, §2].

In Toën’s axiomatization (see [T, §4]), and in all examples of interest, \( \mathcal{C} \) is a model category and \( \mathcal{C}_0 \) consists of the fibrant objects in \( \mathcal{C} \). For example, if \( \mathcal{C} = \mathcal{S} \) with the Joyal model category, \( \mathcal{C}_0 \) consists of the quasi-categories, while if \( \mathcal{C} \) is simplicial categories, \( \mathcal{C}_0 \) consists of those enriched in Kan complexes. One then has a Quillen equivalence of \( \mathcal{C} \) to the complete Segal model structure of [R], and therefore to quasi-categories ([Lu]), simplicial categories ([B]), and the other standard models of \( (\infty, 1) \)-categories.

1.2. The homotopy spectral sequence of a simplicial object. Our definition of higher operations is inspired by the description of the differentials in the homotopy spectral sequence of a simplicial space in [BMe, §6], which we briefly recall:

For \( (\mathcal{C}, \mathcal{C}_0) \) and \( X \in \mathcal{C}_0 \) as in §1.1 let \( x_\bullet \) be a simplicial object in \( X \) (if \( \mathcal{C} = \mathcal{S} \)-Cat, then \( x_\bullet \) is just an \( \infty \)-homotopy commutative diagram in the sense of [DKSm, §2.3]). If \( y \) is a homotopy cogroup object in \( X \), then \( \widehat{W}_\bullet := \text{Map}_X(y, x_\bullet) \) is an \( \infty \)-homotopy commutative simplicial object in \( \mathcal{S} \) which may be made into a strict simplicial object \( W_\bullet \in \mathcal{S}^{\Delta^{op}} \) by [DKSm, Corollary 2.5] (or [BV, Theorem 4.49]). The homotopy spectral sequence of \( (x_\bullet, y) \) is then defined to be the Bousfield-Friedlander spectral sequence for \( W_\bullet \) (see [BF, Theorem B.5]), having

\[
E_2^{n,k} \cong \pi_n^{h} \pi^k W_\bullet \implies \pi_{n+k} \| W_\bullet \| ,
\]

where \( \| W_\bullet \| \), the diagonal of the bisimplicial set, is also its homotopy colimit (see [BK, XII, §2]), so it is weakly equivalent to \( \text{Map}_X(y, \text{colim}_{\Delta^{op}} x_\bullet) \) (assuming \( X \) has enough colimits).

The original version, for bisimplicial groups, is due to Quillen (see [Q1]). If we apply geometric realization to \( W_\bullet \) in each simplicial dimension, we obtain a simplicial
THE HIGHER STRUCTURE OF UNSTABLE HOMOTOPY GROUPS

topological space \( \hat{X}_\bullet \), with Reedy fibrant replacement \( X_\bullet \) (see [Hir §15.3]). Dwyer, Kan, and Stover constructed the spiral spectral sequence of \( X_\bullet \) (with each \( X_n \) connected), and showed in [DKSt, Proposition 8.3] that it is isomorphic to the above from the \( E^2 \)-term on.

In [BMc] we showed that this spectral sequence can be set up internally to the \( \infty \)-category \( X \), in terms of \( (x_\bullet, y) \) themselves: for this purpose, we represent any element in the \( E^2 \)-term of (1.3) by a map \( f : \Sigma^k y \to x_n \), and show that it survives to the \( E^r \)-term if and only if we can include \( f \) in a (homotopy coherent) diagram of the form

\[
\begin{array}{cccccc}
0 & : & 0 & : & \cdots & 0 \\
\Sigma^k y & \downarrow & \Sigma^k y & \downarrow & \cdots & \Sigma^k y \\
\vdots & \downarrow & \vdots & \downarrow & \cdots & \vdots \\
0 & \downarrow & d_0 & \downarrow & \cdots & d_0 \\
x_n & \downarrow & x_n & \downarrow & \cdots & x_n \\
\vdots & \downarrow & \vdots & \downarrow & \cdots & \vdots \\
d_n & \downarrow & d_{n-1} & \downarrow & \cdots & d_{n-1} \\
\end{array}
\]

(1.4)

in \( X \).

Moreover, as shown in [BMc] Theorem 6.8, the value of the differential \( \partial^r(f) \in E^r_{n-r,k+r-1} \) is represented in \( E^1_{n-r,k+r-1} \) by homotopy classes of various maps \( \alpha : \Sigma^{k+r-1} y \to x_{n-r} \), each obtained as a value of a certain \( r \)-th order homotopy operation associated to (1.4).

In particular, if \( [f] \in E^1_{n,k} \) is a permanent cycle, the element it represents in \( \pi_{n+k} \parallel W_\bullet \parallel \cong [\Sigma^{n+1} y, \text{colim } x_\bullet |_X] \) may be described in precisely the same way as the value of an \( (n+1) \)-st order operation.

1.5. Differentials in the spectral sequence. It is simplest to describe the differentials in \( C = S-Cat \), the category of simplicially enriched categories (or equivalently, simplicial categories with constant object sets), with \( C_0 \) thus consisting of categories enriched in Kan complexes. However, it is important to point out that the construction makes sense in any model of \( (\infty, 1) \)-categories satisfying the assumptions of §1.1. This follows from [BMc Corollary 6.11], and is illustrated for quasi-categories in [BMc §8].

Thus we work directly with a homotopy-coherent simplicial space \( Z_\bullet \). This means replacing the indexing category \( \Delta \) by a cofibrant replacement in \( S-Cat \), such as the Dwyer-Kan resolution \( DK(\Delta) \) (see [DK §2]). In fact, we can replace \( \Delta \) by \( \Delta_{\text{res}} \) (see §0.5), since our spectral sequence is actually determined by the restriction \( \hat{Z}_\bullet \) of \( Z_\bullet \) to \( \Delta_{\text{res}}^{op} \).

Now if \( \hat{Z}_\bullet \) is any strictification of \( Z_\bullet \), they have the same (homotopy) colimit in the \( (\infty, 1) \)-category of spaces, so by abuse of notation we may denote this colimit by \( \parallel Z_\bullet \parallel \) (since, as noted above, it may be identified with the diagonal \( \parallel \hat{Z}_\bullet \parallel \)). Moreover, \( \parallel Z_\bullet \parallel \simeq \text{colim}_{\Delta_{\text{res}}^{op}} Z_\bullet \) by [Sc Appendix A].

Recall that the \( n \)-permutohedron \( P^n \) is the convex hull of the \( (n+1)! \) points in \( \mathbb{R}^{n+1} \) obtained by permuting the coordinates of \( (x_0, \ldots, x_n) \) for \( n+1 \) distinct real numbers \( \{x_0, \ldots, x_n\} \). Thus the 1-permutohedron is a 1-simplex, and the
2-permutohedron is a hexagon:

$$\begin{align*}
(0,1) \times (2) & \rightarrow (1,2) \times (0,2) \\
(0,1,2) & \rightarrow (1,2,0) \\
(0,2,1) & \rightarrow (2,1,0) \\
(0,2) \times (1) & \rightarrow (2,0,1) \\
(0,1) & \rightarrow (1,0,2) \\
(1) & \rightarrow (0,2) \\
(2) & \rightarrow (0,1)
\end{align*}$$

By [BMe, Proposition 5.6], for every $-1 \leq j < m$ there is an isomorphism of simplicial sets

$$DK(\Delta_{\text{res},+}^\text{op})([m],[j]) \cong \coprod_{\theta: [j] \rightarrow [m]} P^{j-m-1}$$

between the mapping space in $DK(\Delta_{\text{res},+}^\text{op})$ and a disjoint union of (triangulations of) the $(j-m-1)$-permutohedron, indexed by the distinct maps $[j] \rightarrow [m]$ in $\Delta_{\text{res}}$.

This allows us to reduce the search for a diagram of the form (1.4) in our $\infty$-category $X$ to the case where $X = S\text{-Cat}$, $x_\bullet$ is replaced by the (strict) restricted simplicial space $W_\bullet$ realizing map$_X(y, x')$, and $\Sigma^n y$ in $X$ is replaced by a sphere $S^k$. However, the source (upper horizontal diagram in (1.4)) is still required to be cofibrant in $S\text{-Cat}$, and all the necessary higher homotopies needed to provide the coherence of (1.4) may be encoded by adjunction in a pointed map from the $r$-skeleton of (1.7) into map$_{S^r}(S^k, W_{n-r+1})$.

Moreover, by a more careful analysis of the spiral spectral sequence in the case where $W_\bullet$ is Reedy fibrant, one can show that only one of the components in the right hand side of (1.7) is needed (see [BMe, Theorem 6.8]). Using the fact that $P^n$ is a convex polytope in $\mathbb{R}^n$, so its boundary (or $(n-1)$-skeleton) is an $(n-1)$-sphere, by a further adjunction we obtain a single map $\alpha : S^{r+} \rightarrow W_{n-r+1}$, which represents $d^r(f)$ in $E^r_{n-r+1,i+r}$ by [BMe, Corollary 6.10].

As explained in [BMe, Corollary 6.11], the construction sketched above may be described in terms of a sequence of maps in the original $\infty$-category $X$, starting with the diagram (1.4) in the homotopy category of $X$, with each map determined by the universal property of a colimit on an appropriate subcategory of $\Delta_{\text{res}}$ in terms of the maps obtained inductively in earlier stages. For our purposes we shall not need the details of this construction, but only the following:

1.8. **Definition.** Let $X \in C_0$ be an $\infty$-category as in §1.1 $x_\bullet : B\Delta_{\text{res},+} \rightarrow X$ an augmented restricted simplicial object in $X$, and $y$ a cogroup object in $\text{ho} X$. For each $n \geq 1$, the associated simplicial $(n+1)$-st order operation is defined as follows

(a) The initial data for the operation consists of $x_\bullet$, together with the homotopy class of a map $f : \Sigma^n y \rightarrow x_n$ in $X$ with $d_j f \sim 0$ for $0 \leq j \leq n$. 

(b) **Full data** for the operation consists of a diagram in $X$:

\[
\begin{array}{ccccccc}
\Sigma^k y & & & & & & 0 \\
\downarrow f & & & & & & \downarrow 0 \\
x_n & \rightarrow & \cdots & \rightarrow & x_{n-1} & \rightarrow & \cdots & \rightarrow & x_0 & \rightarrow x_{-1} \\
& & d_0 & & & & & & & \\
& & 0 & & & & & & & \\
& & 0 & & & & & & & \\
\end{array}
\]

(1.9)

(implicitly involving choices for all higher coherences), if it exists.

(c) The **value** for the operation is the homotopy class of the map $\Sigma^{n+k} y \rightarrow x_{-1}$ determined by the full data, and the fact that the (homotopy) colimit of the top row of (1.9) is $\Sigma^{n+k} y$.

It is evident from the usual properties of spectral sequences that the simplicial higher order homotopy operations we have defined satisfy the three properties listed at the beginning of Section 1. Moreover, when $X$ is an $\infty$-category model for $\text{Top}$, $y$ is a sphere, and each $x_n$ is a wedge of spheres, we have in fact a higher homotopy operation **sensu stricto**, in as much as all the ingredients of the construction take values in homotopy groups.

1.10. **Remark.** Note that a diagram of the form (1.9) (or (1.4)) in $X$ or $\text{ho} X$ can be re-written as a single truncated restricted simplicial object

\[
\begin{array}{ccccccc}
\Sigma^k y & & & & & & 0 \\
\downarrow f & & & & & & \downarrow 0 \\
x_n & \rightarrow & \cdots & \rightarrow & x_{n-1} & \rightarrow & \cdots & \rightarrow & x_0 & \rightarrow x_{-1} \\
& & d_0 & & & & & & & \\
& & 0 & & & & & & & \\
& & 0 & & & & & & & \\
\end{array}
\]

(1.11)

extended one further degree to the left, with all face maps out of the $(n+1)$-slot (except $d_0$) equal to 0.

This was the original point of view in the construction of the spiral spectral sequence of a simplicial space $W_\bullet \in \text{Top}_{\Delta^{op}}^\circledast$. Dwyer, Kan, and Stover showed that if $W_\bullet$ is Reedy fibrant, there is a fibration sequence

\[
\begin{array}{ccccccc}
\Omega Z_{n-1} W_\bullet & \xrightarrow{d_{n-1}} & Z_n W_\bullet & \xrightarrow{j_n} & C_n W_\bullet & \xrightarrow{d_0} & Z_{n-1} W_\bullet \\
\end{array}
\]

(1.12) for each $n \geq 1$, where $C_n W_\bullet := \bigcap_{i=1}^n \text{Ker}(d_i)$ is the $n$-th Moore chains space, and $Z_n W_\bullet := \bigcap_{i=0}^n \text{Ker}(d_i)$ is the $n$-th Moore cycles space.

The spiral spectral sequence for $W_\bullet$ is that associated to the tower of fibrations (1.12) (see [DKS1] for further details). It is then clear that $[f] \in \pi_k C_n W_\bullet$ survives to $E^\infty_{n,k}$ if and only if it lifts to $Z_n W_\bullet$, thus strictifying (1.11) (or (1.9)). We deduce:

1.13. **Proposition.** An element $\gamma \in \pi_N \|W_\bullet\|$ is in filtration $n$ of the homotopy spectral sequence for $W_\bullet$ if and only if it is a canonical value for the $(n+1)$-st order simplicial operation associated to (1.9), for $N = n + k$.

1.14. **Toda brackets.** We have used the homotopy spectral sequence of a restricted simplicial object as a convenient shorthand for defining our higher order operations.
However, they have an explicit geometric-combinatorial construction which is independent of – and more general than – such spectral sequences (see [BJT2] and the references there for more details). We illustrate this for the oldest example of a secondary operation: the ordinary Toda bracket of [T1 T2].

Assume that we are given a sequence of maps
\[
X \xrightarrow{h} Y \xrightarrow{g} Z \xrightarrow{f} W
\]
in a pointed model category $C$, with $f \circ g \sim \ast$ and $g \circ h \sim \ast$. This can be rewritten as a 2-truncated restricted augmented simplicial object in $\text{ho}C$:
\[
\begin{array}{cccccc}
X & \xrightarrow{d_0 = h} & Y & \xrightarrow{d_1 = *} & Z & \xrightarrow{d_0 = g} & W \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Sigma X & \xrightarrow{\varepsilon = f} & \Sigma Y & \xrightarrow{\varepsilon = f} & \Sigma Z & \xrightarrow{\varepsilon = f} & \Sigma W
\end{array}
\]
where the space in each simplicial dimension, from left to right, is the wedge of the relevant column, and $\text{inc}_i : CX \hookrightarrow \Sigma X$ ($i = 1, 2$) denote the two inclusions of the upper and lower cones into $\Sigma X$ (the pushout of $CX \leftarrow X \rightarrow CX$). Here $(f, g, h)$ is by definition the value of the Toda bracket associated to the two choices of nullhomotopies $F : f \circ g \sim \ast$ and $G : g \circ h \sim \ast$ (uniquely defined by the requirement that (1.17) satisfies the simplicial identities on the nose).

We see that (1.17) realizes (1.16) up to homotopy if and only if the Toda bracket vanishes, so that we can choose a nullhomotopy $K : (f, g, h) \sim \ast$, allowing us to replace $\Sigma X$ by $C\Sigma X$, with $\varepsilon = K$ on the cone.

See [BBS1] for a detailed treatment of Toda brackets of arbitrary length in a similar spirit (and compare [BBG]).

2. Simplicial space approximations

We would like to have a procedure for decomposing elements in the homotopy groups $\pi_* X$ of an arbitrary space $X$ in terms of higher order homotopy operations. By the description in §1.5, we can do so by providing a suitable simplicial space $W_\bullet$ with an augmentation to $X$ – most simply, if $\lVert W_\bullet \rVert \simeq X$.

2.1. Constructing CW approximations. For simplicity we may assume $X$ is a $(k - 1)$-connected pointed space with $k \geq 2$. We define a sequential approximation to $X$ to be a sequence
\[
W_\bullet^{[k]} \xrightarrow{i^{[k]}} W_\bullet^{[k+1]} \xrightarrow{i^{[k+1]}} W_\bullet^{[k+2]} \rightarrow \ldots W_\bullet^{[n]} \xrightarrow{i^{[n]}} W_\bullet^{[n+1]} \rightarrow \ldots
\]
of simplicial spaces with augmentations $ε[n] : W[n] → X$ (commuting with the maps $i^{[n]}$), such that for each $n ≥ k$:

(a) $W[n]$ is $(n - k)$-skeletal (in the simplicial direction).

(b) For each $i ≥ 0$, $W_i[n]$ is homotopy equivalent to a wedge of $(k-1)$-connected spheres.

(c) The map $ε[n] : ∥W[n]∥ → X$ induced by $ε[n]$ (out of the geometric realization) is an $n$-equivalence: that is, it induces an isomorphism in $π_i$ for $i < n$ and an epimorphism for $i = n$. This implies that any $n$-skeletal of $∥W[n]∥$ is an $n$-skeletal CW approximation for $X$.

Recall that a space is of finite type if each homotopy group is finitely generated. For a simply-connected space, this condition is equivalent to each homology group being finitely generated.

2.3. Proposition. Every $(k - 1)$-connected pointed space $(k ≥ 2)$ has a sequential approximation; if $X$ is of finite type, we may assume each $W_i[n]$ is homotopy equivalent to a finite wedge of simply-connected spheres.

Proof. We construct (2.2) by induction on $n ≥ 0$:

Step 1: For $n = k$, start with $W_0[k]$ a wedge of $k$-spheres with a map $ε[k] : W_0[k] → X$ which induces a surjection in $π_k$. We choose it to be minimal with this property (that is, no proper sub-wedge has such a surjection). We let $W_1[k]$ be $c(W_0[k])$.

Step 2: For $n = k + 1$, choose $W_1[k+1]$ to be a (minimal) wedge of $k$-spheres having a map $d_0 : W_1[k+1] → W_0[k]$ inducing a surjection onto the kernel of $ε[k] : π_kW_0[k] → π_kX$. Let $F : CW_1[k+1] → X$ be a nullhomotopy for $ε[k] ∘ d_0$, and set $W_0[k+1] := W_0[k] ∨ CW_1[k+1]$ and $W_1[k+1] := W_1[k+1] ∨ W_0[k+1]$. We obtain a $1$-skeletal simplicial space $W_0[k+1]$ with an augmentation to $X$ given by $ε[k] ∩ F$. The degeneracy $s_0 : W_0[k+1] → W_1[k+1]$ is given by the inclusion. The map $ε[k+1] : ∥W_0[k+1]∥ → X$ thus induces an isomorphism in $π_k$.

Now choose a (minimal) wedge of $(k+1)$-spheres $W_0[k+1]$ with a map $e : W_0[k+1] → X$ mapping onto $π_{k+1}X \setminus \text{Im}(ε[k+1])$. Thus if we set $W_0[k+1] := W_0[k+1] ∨ W_0[k+1]$ and $W_1[k+1] := W_1[k+1] ∨ W_0[k+1]$, we obtain a $1$-skeletal simplicial space $W_0[k+1]$ with an augmentation to $X$ such that the induced map $ε[k+1] : ∥W_0[k+1]∥ → X$ induces a surjection in $π_{k+1}$ (and so is a $(k + 1)$-equivalence).

Step 3: Assume given $W_0[n-1]$ as above, and let $m ≥ n - 1$ be maximal such that the map $ε[n-1] : ∥W_0[n-1]∥ → X$ (induced by $ε[n-1] : W_0[n-1] → X$) is an $m$-equivalence. If $m ≥ n$, we set $W_0[n] := W_0[n-1]$; otherwise $m = n - 1$, and we construct $W_0[n]$ by an inner induction as follows:

Case 1. If $π_{n-1}ε : π_{n-1}∥W_0[n-1]∥ → π_{n-1}X$ is injective (and thus an isomorphism), necessarily $π_nε$ is not surjective. Choose a minimal wedge of $n$-spheres
Case 2. If $\pi W$ is equivalent to a wedge of simply-connected rational spheres. Then the homotopy spectral sequence of $\pi W$ collapses at the $E^2$-term.

**Remark.** 2.5. Using the differential graded Lie model of \cite{Q2} for $\pi W$, we see that each $W_n$ is coformal (that is, has a cofibrant model with 0 differential). This implies that every $[\alpha] \in E^2_{n,k}$ represented by a Moore cycle $\alpha \in Z_n \pi_k W_n$ is in fact represented by $\bar{f} : S^k \to Z_n W_\bullet$ — so that it fits into a diagram of the form \eqref{eq:19}.

2.5. **Remark.** Of course, when $X = ||W_\bullet||$ itself is not coformal, the induced map $f : \Sigma^r S^r \to X$ in \cite{13} can still be non-trivial, so that $[f] \in \pi_{r+s} X$ may be the value of a higher homotopy operation, such as a rational higher Whitehead product (see \cite{AA}).

### 3. Atomic maps

Now that we have a procedure for generating new elements in $\pi_* X$ by higher order operations, we would like to show which elements are indecomposable with respect to such operations.

3.1. **Theorem.** If $X$ is $k$-connected for $k \geq 1$, any homotopy class $\varphi \in \pi_n X$ is generated of higher order over $\Pi_{>1}$ \eqref{eq:10} by the atomic maps \eqref{eq:12}.

**Proof.** We may assume that all spaces are localized at a prime $p \geq 0$, and consider the sequential approximations $W^{[m]}_n$ ($k \leq m \leq n$) for $X$ constructed in the...
proof of Proposition 2.3 which we may also take to be $p$-local and $k$-connected in each simplicial dimension. Assume that $\varphi$ is not atomic, so that $\varphi_* : H_n(S^n; \mathbb{F}_p) \to H_n(X; \mathbb{F}_p)$ is trivial. By replacing $X$ by its $(n+1)$-skeleton, we may assume it is $(n+1)$-dimensional (without affecting $\pi_n(X)$).

When $p > 0$ and $\varphi_* : H_n(S^n; \mathbb{Z}(p)) \to H_n(X; \mathbb{Z}(p))$ is non-zero, then the Hurewicz image of $\varphi$ is necessarily divisible by $p^r$ for some $r \geq 1$.

Now if $Z \xrightarrow{\alpha} \hat{X} \xrightarrow{\tau} X \xrightarrow{\alpha} \Sigma Z$ is the cofibration sequence for some CW structure on $X$ with $n$-skeleton $\hat{X}$, with $Z := \bigvee_{i=1}^m S^n_i$ (here $(i)$ is just an index). We have the associated Hurewicz diagram:

$$
\begin{array}{c}
\pi_n(Z) \\
\approx \tau_* \\
\pi_n(X) \\
\pi_n(\Sigma Z)
\end{array}
\xrightarrow{
\begin{array}{c}
h_n \\
h_n \\
h_n
\end{array}
} 
\begin{array}{c}
\pi_n(\hat{X}) \\
\pi_n(X; \mathbb{Z}(p)) \\
\pi_n(X; \mathbb{Z}(p)) \\
\pi_n(\Sigma Z; \mathbb{Z}(p)) = 0
\end{array}
\xrightarrow{
\begin{array}{c}
h_n \\
h_n \\
h_n
\end{array}
} 
\begin{array}{c}
H_n(Z; \mathbb{Z}(p)) \\
H_n(\hat{X}; \mathbb{Z}(p)) \\
H_n(X; \mathbb{Z}(p)) \\
H_n(\Sigma Z; \mathbb{Z}(p)) = 0
\end{array}
\tag{3.2}
$$

Since $i_#$ is surjective, a representative $f$ for $\varphi$ lifts to $f' : S^n \to \hat{X}$ (cellular approximation), and since $h_n([f])$ is non-trivial by assumption, so is $h_n([f'])$. However, we assumed $h_n([f]) = p^r \cdot \hat{y}$ for some $\hat{y} \in H_n(\hat{X}; \mathbb{Z}(p))$ and $r \geq 1$, so there is $y \in H_n(\hat{X}; \mathbb{Z}(p))$ with $i_*(y) = \hat{y}$. This means that $p^r \cdot y - h_n([f']) \in \ker(i_*) = \im(\tau_*)$, so if we write this expression as $\tau_*(h_n(\alpha))$ for some $\alpha \in \pi_n(Z)$, we see that we may replace our choice $f'$ by $f'' := f' + \tau_*(\alpha)$ with $i_*(f'') = f$, but now $h_n(f'')$ itself divisible by $p^r$ in $H_n(X; \mathbb{Z}(p))$.

Now assume $p \geq 0$, and let $\hat{X}$ be (an $n$-skeleton of) $\|W_\alpha^n\|$, by definition (see §2.1). Thus $\varphi$ factors through $f^{(n)} : S^n \to \|W_\alpha^n\|$, and we may assume that this is not atomic, either. The class $[f^{(n)}] \in \pi_n(\|W_\alpha^n\|)$ can be written (non-uniquely) as a sum of elements in various filtrations. All those in positive filtration are canonical values of higher order simplicial operations, by Proposition 1.13. The elements in filtration 0 factor through $W_0^{(n)}$, which is a wedge of spheres of dimensions $\leq n$. Thus by Hilton’s Theorem (see [Hil]), each such element is a sum of compositions of iterated Whitehead products on the fundamental classes of these spheres. All such summands are decomposable, except possibly for those which factor through the sub-wedge product $W$ consisting of the $n$-spheres. However, if $f^{(n)}_* : H_n(S^n; \mathbb{Z}(p)) \to H_n(\|W_\alpha^n\|; \mathbb{Z}(p))$ is non-zero and $p > 0$, then (because $f^{(n)}$ is not atomic) its Hurewicz image is $p$-divisible, and since the Hurewicz map for $W$ is an isomorphism in dimension $n$, this means the map $f^{(n)}$ is itself $p$-divisible, and thus decomposable. For $p = 0$, any non-zero map $f^{(n)}_* : H_n(S^n; \mathbb{Q}) \to H_n(\|W_\alpha^n\|; \mathbb{Q})$ is atomic. □

4. Atomic maps for spheres

There is little hope of describing all atomic maps for general $X$, since the Hurewicz image is not known even stably, in general (see [Co §2]). However, we can do so for spheres: since the construction of §2.1 is trivial for $S^k$ itself, more precisely, we consider a connected cover. All spaces in this section are localized at a prime $p$.

4.1. Theorem. For $k \geq 2$, let $X$ be the homotopy fiber of the fundamental class $\varepsilon_k : S^k \to K(\mathbb{Z}(p), k)$ (that is, the $(r-1)$-connected cover of the $k$-sphere for
r := k + 2p - 3), and assume that the lift \( f : S^n \to X \) of \( \varphi : S^n \to S^k \) is atomic for \( p \). Then the homotopy cofiber of \( \varphi \) supports a non-trivial mod \( p \) cohomology operation.

**Proof.** Our hypothesis is that the image of \( [f] \in \pi_n X \) under the Hurewicz homomorphism \( \pi_n X \to H_n(X; \mathbb{F}_p) \) is non-trivial. There is thus a class \( \lambda \in H^n(X; \mathbb{F}_p) \) with \( f^*(\lambda) \neq 0 \). Such a \( \lambda \) is necessarily indecomposable in the unstable cohomology algebra \( H^*(X; \mathbb{F}_p) \) (that is, it does not decompose in terms of any unstable cohomology operations, including the cup product).

We first observe that it suffices to show that any such indecomposable class \( \lambda \) is transgressive in the \( \mathbb{F}_p \)-cohomology Serre spectral sequence of the fibration sequence \( X \hookrightarrow S^k \to K(\mathbb{Z}_p, k) \).

If we write \( g : X \to S^k \) for the covering map, then \( \varphi := g \circ f \) fits into a commuting diagram with horizontal homotopy cofibration sequences, as follows:

\[
\begin{array}{ccc}
S^n & \xrightarrow{\varphi} & S^k \\
\downarrow f & & \downarrow r \\
X & \xrightarrow{g} & S^k \\
\end{array}
\begin{array}{ccc}
 & & \downarrow \delta \\
\text{Cof}(\varphi) & \xrightarrow{\delta} & S^n \\
\downarrow \Sigma f & & \\
\Sigma X & \xrightarrow{\rho} & K(\mathbb{Z}_p, k) \\
\end{array}
\]

where \( t \) represents \( \epsilon_k \), and the map \( \rho \) exists by the universal properties of \( \text{Cof}(g) \) and the fibration sequence \( X \xrightarrow{g} S^k \xrightarrow{\rho} K(\mathbb{Z}_p, k) \).

Since by assumption \( f^* \lambda = \epsilon_n \) (the fundamental class in \( H^n(S^n; \mathbb{F}_p) \)), under suspension we have \( \bar{\lambda} \in H^{n+1}(\Sigma X; \mathbb{F}_p) \) with \( (\Sigma f)^* \bar{\lambda} = \epsilon_{n+1} \in H^{n+1}(\Sigma S^{n+1}; \mathbb{F}_p) \).

If \( \lambda \) is transgressive, there is a class \( c \in H^{n+1}(K(\mathbb{Z}_p, k); \mathbb{F}_p) \) with \( \rho^* c = \bar{\delta}^* \bar{\lambda} \).

But \( c \) is necessarily of the form \( \theta(\iota_k) \) for \( \iota_k \in H^k(K(\mathbb{Z}_p, k); \mathbb{F}_p) \) the fundamental class, and a diagram chase in:

\[
\begin{array}{ccc}
H^k S^k & \ni \epsilon_k & = j^* a \in H^k \text{Cof}(\varphi) \\
\downarrow \iota^* & & \downarrow \epsilon^* \\
H^k S^k & \ni \epsilon_k & = c \in H^{n+1} K(\mathbb{Z}_p, k) \\
\end{array}
\begin{array}{ccc}
H^{n+1} \text{Cof}(\varphi) & \ni b & = \delta^* \epsilon_{n+1} = \delta^* \bar{\lambda} = \delta^* \bar{\iota_k} \\
\downarrow \iota_k & & \downarrow \rho^* \\
H^{n+1} \text{Cof}(g) & \ni \bar{\lambda} & = \bar{\iota_k} \in H^{n+1} \Sigma X \\
\end{array}
\begin{array}{ccc}
H^{n+1} \text{Cof}(\varphi) & \ni b & = \delta^* \epsilon_{n+1} = \delta^* \bar{\lambda} = \delta^* \bar{\iota_k} \\
\downarrow \iota_k & & \downarrow \rho^* \\
H^{n+1} \text{Cof}(g) & \ni \bar{\lambda} & = \bar{\iota_k} \in H^{n+1} \Sigma X \\
\end{array}
\begin{array}{ccc}
H^{n+1} \text{Cof}(\varphi) & \ni b & = \delta^* \epsilon_{n+1} = \delta^* \bar{\lambda} = \delta^* \bar{\iota_k} \\
\downarrow \iota_k & & \downarrow \rho^* \\
H^{n+1} \text{Cof}(g) & \ni \bar{\lambda} & = \bar{\iota_k} \in H^{n+1} \Sigma X \\
\end{array}
\]

shows that \( b = \theta(a) \), where \( a \in H^k(\text{Cof}(\varphi); \mathbb{F}_p) \) has \( j^* a = \epsilon_k \), and \( b \in H^{n+1}(\text{Cof}(\varphi); \mathbb{F}_p) \) has \( \delta^* \epsilon_{n+1} = b \). This \( \theta \) is the non-trivial cohomology operation required by our Theorem. We have thus reduced the proof of the Theorem to showing that any indecomposable class \( \lambda \) is transgressive in the Serre spectral sequence for \( X \to S^k \to K(\mathbb{Z}_p, k) \).

We compute \( H^*(X; \mathbb{F}_p) \), using the associated fibration sequence \( K(\mathbb{Z}_p, k - 1) \xrightarrow{\rho} X \to S^k \). Since the base is a sphere, the Serre spectral sequence reduces to the Wang
long exact sequence of \[\text{[Wan]}\]:
\[
\cdots H^n(\mathbf{K}(\mathbb{Z}_p), k - 1); \mathbb{F}_p) \xrightarrow{d_k} H^{n-k+1}(\mathbf{K}(\mathbb{Z}_p), k - 1); \mathbb{F}_p) \xrightarrow{\rho} H^{n+1}(\mathbf{X}; \mathbb{F}_p) \\
H^{n+1}(\mathbf{K}(\mathbb{Z}_p), k - 1); \mathbb{F}_p) \rightarrow \cdots
\]
We write \(E^0(p)\) for the image of \(H^*(\mathbf{X}; \mathbb{F}_p)\) in \(H^*(\mathbf{K}(\mathbb{Z}_p), k - 1); \mathbb{F}_p)\) — that is, \(E^0(p) := u^*H^*(\mathbf{X}; \mathbb{F}_p)\).

From here on we distinguish between two cases:

Case I: The prime 2.

For \(p = 2\), we have:
\[
H^*(\mathbf{K}(\mathbb{Z}_2), k); \mathbb{F}_2) \cong \mathbb{F}_2[\text{Sq}^I(t_k) \mid I \text{ admissible}, i_s \neq 1, \text{ex}(I) < k].
\]
(see [K, Proposition 3.5.8]). Here the multi-index \(I\) is of the form \((i_0, \ldots, i_s)\) with \(\text{Sq}^I := \text{Sq}^{i_0} \ldots \text{Sq}^{i_s}\), and \(\text{ex}(I)\) is the excess.

The exact sequence (4.4) is determined by the fact that \(d_k\) is an (anti-)derivation, which sends \(t_{k-1}\) to 1 and \(\text{Sq}^I(t_{k-1})\) to 0 for \(I \neq 0\). It follows that
\[
E^0(2) = \mathbb{F}_2[t_{k-1}^2, \text{Sq}^I(t_{k-1}) \mid 0 \neq I \text{ admissible}, i_s \neq 1, \text{ex}(I) < k - 1].
\]

We choose generators in \(H^*(\mathbf{X}; \mathbb{F}_2)\) corresponding to the polynomial algebra generators \(t_{k-1}^2\) and \(\text{Sq}^I t_{k-1}\), and also use the same notation for them. Thus \(H^*(\mathbf{X}; \mathbb{F}_2)\) is generated as an \(E^0(2)\)-module by 1 and \(\gamma_{2k-1} (= \rho(t_{k-1}))\), which lies in \(H^{2k-1}(\mathbf{X}; \mathbb{F}_2)\) under the exact sequence (4.4). Therefore, the indecomposable classes in \(H^*(\mathbf{X}; \mathbb{F}_2)\) consist of \(\gamma_{2k-1}\) and \(\text{Sq}^I(t_{k-1})\) for \(j \leq k - 1\). Note that the notation \(\text{Sq}^I(t_{k-1})\) for a class in \(H^*(\mathbf{X}; \mathbb{F}_2)\) which is sent to the usual \(\text{Sq}^I(t_{k-1})\) in \(H^*(\mathbf{K}(\mathbb{Z}_2), k); \mathbb{F}_2)\) is not meant to imply that it is in the image of a cohomology operation.

We now verify that all these classes are transgressive in the \(\mathbb{F}_2\)-cohomology Serre spectral sequence for \(\mathbf{X} \rightarrow \mathbf{S}^k \rightarrow \mathbf{K}(\mathbb{Z}_2), k)\), using the following map \(u\) of fibration sequences:
\[
\begin{align*}
\mathbf{K}(\mathbb{Z}_2), k - 1) & \xrightarrow{u} \mathbf{X} \\
\downarrow & \downarrow \\
PK(\mathbb{Z}_2), k & \xrightarrow{} \mathbf{S}^k \\
\downarrow & \downarrow \phi_k \\
\mathbf{K}(\mathbb{Z}_2), k & \xrightarrow{=} \mathbf{K}(\mathbb{Z}_2), k \\
\mathcal{F}^1 & \xrightarrow{u} \mathcal{F}^2 \\
\end{align*}
\]
where the left hand side is the path-loop fibration. This induces a map \(u^*\) of the corresponding Serre spectral sequences, with \(E_r(u^*) : E_r(\mathcal{F}^2) \rightarrow E_r(\mathcal{F}^1)\).

Observe that the first differential for \(\mathcal{F}^1\) is \(d_k\), determined by \(d_k(t_{k-1}) = t_k\). (See Figure 4.8).

The key step in the argument is the calculation of \(\mathcal{K}_r := \ker(E_r(u^*))\). For \(j \leq k\), observe that \(\mathcal{K}_r = \gamma_{2k-1} \cdot E_r(\mathcal{F}^2)\). For \(j = k + 1\), multiples of \(t_k\) are zero in
so that $\mathcal{K}_{k+1} = \gamma_{2k-1} \cdot E_{k+1}(\mathcal{F}^2) + \iota_k \cdot E_{k+1}(\mathcal{F}^2)$. Using our comparison map $u^* : E_r(\mathcal{F}^2) \to E_r(\mathcal{F}^1)$ and the fact that $Sq^j(\iota_{k-1})$ transgresses to $Sq^j(\iota_k)$ in $\mathcal{F}^1$, we deduce that either the class $Sq^j(\iota_{k-1})$ transgresses to $Sq^j(\iota_k)$ in $\mathcal{F}^2$, or else some differential carries $Sq^j(\iota_{k-1})$ to a class in $\mathcal{K}_r$ for some $r$. Now note that for $2 \leq j \leq k - 1$, $k + 1 \leq |Sq^j(\iota_{k-1})| \leq 2k - 2$, and that in these degrees $\mathcal{K}_r$ is zero. This forces $Sq^j(\iota_{k-1})$ to be transgressive.

The class $\gamma_{2k-1}$ lies in $\mathcal{K}_r$ for all $r$. As the spectral sequence converges to the cohomology of $S^k$, it must support a differential. Therefore, a differential on it must also lie in $\mathcal{K}_r$, which leaves the only possibility as $d_{2k}(\gamma_{2k-1}) = \iota_k^2$, for degree reasons.
Case II: odd primes.

For an odd prime $p$, we have:

\[(4.10) \quad H^*(\mathbf{K}(\mathbb{Z}(p), k); \mathbb{F}_p) \cong F^{	ext{gr}}_{\mathbb{F}_p}[\mathcal{P}^I(i_k) \mid \text{admissible}, \epsilon_{s+1}(I) = 0, \text{ex}(I) < k - 1]\]

(see [K] Proposition 3.5.8), where now $I := (\epsilon_0, i_0, \cdots, \epsilon_s, i_s, \epsilon_{s+1})$ for $\epsilon_i \in \{0, 1\}$, and $\mathcal{P}^I = \beta^{i_0} \mathcal{P}^{i_0} \cdots \mathcal{P}^{i_s} \beta_s^{-1}$. Here $F^{	ext{gr}}_{\mathbb{F}_p}$ denotes the free graded commutative algebra – that is, exterior on the odd degree classes and polynomial on the even degree classes.
We now distinguish between two cases, depending on the parity of \( k \): when \( k \) is odd, we find
\[
E^0(p) = F_p^\oplus [t_{k-1}^p, \mathcal{P}^j(t_{k-1}) | I \neq 0 \text{ admissible}, \epsilon_{s+1}(I) = 0, \text{ex}(I) < k]
\]
Note that \( d_k(t_{k-1}^p) = 0 \) in the exact sequence \([4.4]\), while \( d_k(t_{k-1}^j) = j t_{k-1}^{j-1} \) for \( j \leq p - 1 \). As in the \( p = 2 \) case, we write down the corresponding generators in \( H^*(X; \mathbb{F}_p) \) using the same notation. It follows that \( H^*(X; \mathbb{F}_p) \) is generated as an \( E^0(p) \)-module by 1 and \( \gamma_{(k-1)p+1} \), which is the image of \( t_{k-1}^{p-1} \) in \( H^{(p-1)(k-1)+k}(X; \mathbb{F}_p) \) in the exact sequence \([4.4]\).

Therefore, the indecomposable classes in \( H^*(X; \mathbb{F}_p) \) are \( \gamma_{(k-1)p+1} \) and \( \mathcal{P}^j(t_{k-1}) \) for \( j \leq \frac{k-1}{2} \) (noting that \( t_{k-1}^p = \mathcal{P}^\frac{k-1}{2}(t_{k-1}) \)). We verify that all these classes are transgressive in the \( \mathbb{F}_p \)-cohomology Serre spectral sequence for the fibration sequence \( X \to S^k \to K(\mathbb{Z}(p), k) \).

This follows from a very similar argument to that for the case \( p = 2 \). Note, however, that for reasons of degree, the classes \( \mathcal{P}^i(t_{k-1}) \) must be transgressive, while the class \( \gamma_{(k-1)p+1} \) must transgress to the class \( \beta \mathcal{P}^{\frac{k-1}{2}}(t_k) \) (which is the differential on \( t_{k-1}^{p-1} \cdot t_k \) in the spectral sequence \( F^1 \) by the Kudo transgression theorem (see [K, Theorem 3.5.3]).

For \( k \) even, we have
\[
E^0(p) = F_p^\oplus [\mathcal{P}^j(t_{k-1}) | I \neq 0 \text{ admissible}, \epsilon_{s+1}(I) = 0, \text{ex}(I) < k - 1],
\]
As \( t_{k-1}^p = 0 \). It follows that \( H^*(X; \mathbb{F}_p) \) is generated as an \( E^0(p) \)-module by 1 and \( \gamma_{2k-1} \), which is the image of \( t_{k-1} \) in \( H^{2k-1}(X; \mathbb{F}_p) \) in the exact sequence \([4.4]\).

Therefore, the indecomposable classes in \( H^*(X; \mathbb{F}_p) \) are \( \gamma_{2k-1} \) and \( \mathcal{P}^j(t_{k-1}) \) for \( j < \frac{k-1}{2} \). As above by calculating \( K_r \), we see that the only class that \( d_r(\gamma_{2k-1}) \) can be is \( t_{k-1}^2 \), so it is transgressive.

Finally for the classes \( \mathcal{P}^j(t_{k-1}) \), observe that if these classes are not transgressive, the differential on these classes must lie in \( K_r \) for some \( r \) (as they are transgressive in the spectral sequence for \( F^1 \)). Observe that \( K_r = \{ \gamma_{2k-1} - E_r(F^2) \text{ for } r \leq k \}, \) while \( K_{k+1} = \gamma_{2k-1} \cdot E_{k+1}(F^2) + t_k \cdot E_{k+1}(F^2) \).

If \( d_r(\mathcal{P}^j(t_{k-1})) = 0 \) for some \( r < 2k \), we must have \( r > k \) and \( d_r(\mathcal{P}^j(t_{k-1})) = \gamma_{2k-1}^q \). Note that \( q \) must be in the \( r \)-th column, and hence is of the form \( \mathcal{P}^L(t_k)q \) with \( \text{ex}(L) < k - 1 \), where \( q' \) is in the 0-th column, since \( r < 2k \). Thus \( d_r(\mathcal{P}^L(t_k)q') \gamma_{2k-1}^{q'} = \gamma_{2k-1}^q \).

Observe that the classes \( \mathcal{P}^j(t_{k-1}) \) in the cohomology of \( X \) are defined only up to a multiple of \( \gamma_{2k-1} \), so we may change the representative in such a way that the differential \( d_r \) vanishes on it. Finally, note that \( K_{2k+1} \) is simply \( \mathbb{F}_p \{ t_k \} \), so from this page on the differentials are determined by those of the spectral sequence for \( F^1 \). The result follows.

4.13. Corollary. All homotopy classes of maps between wedges of simply-connected spheres are generated of higher order by the fundamental classes, the Hopf maps \( \eta_k \in \pi_{k+1}S^k \), \( \nu_k \in \pi_{k+3}S^k \), and \( \sigma_k \in \pi_{k+7}S^k \), and the maps \( \alpha_1(p) \in \pi_{k+2p-3}S^k \), for odd primes \( p \) and \( k \geq 2 \).

Proof. This follows from the stable analysis in [Ada] and [Lin]. Note that we may post-compose the augmentation obtained from a resolution \( W_\bullet \) of \( W_\bullet \simeq S^k(k) \)
with the covering map \( p : S^k(k) \to S^k \) to obtain the augmentation \( W_\ast \to X = S^k \) in Definition 0.3 implies the “algebra” decomposability in \( \pi_\ast S^k(k) \) in the sense of Definition 0.2. 

In other words, the “module” decomposability in \( \pi_\ast S^k(k) \) in the sense of Definition 0.1 implies the “algebra” decomposability in \( \pi_\ast S^k \) itself in the sense of Definition 0.3.

5. Suspensions

So far we have only considered the way elements in the homotopy groups of a single space \( Y \) can be decomposed. However, our methods also allow us to describe the elements generated by suspending \( Y \).

5.1. The suspension spectral sequence. Given a connected pointed space \( Y \), we denote by \( Y \otimes S^n \) the \( n \)-skeletal simplicial space with a single non-degenerate copy of \( Y \) in simplicial dimension \( n \), so that its geometric realization is weakly equivalent to \( \Sigma^n Y \).

In particular, the homotopy spectral sequence for \( W_\ast := Y \otimes S^1 \) will be called the suspension spectral sequence for \( Y \). This simplicial space may be described combinatorially in terms of the degeneracies of \( Y \):

\[
\ldots s_1 s_0 Y \vee s_2 s_0 Y \vee s_2 s_1 Y \xrightarrow{d_0} \xrightarrow{d_1} \xrightarrow{d_2} \xrightarrow{d_3} \ldots W_3 \quad W_2 \quad W_1 \quad W_0
\]

with the degeneracies as indicated, and all face maps determined by the simplicial identities.

Since \( \pi_\ast W_0 = 0 \), \( \pi_\ast \Sigma Y \) has no elements in filtration 0 in this spectral sequence, so all its elements are formally exhibited as values of higher simplicial homotopy operations.

5.3. Remark. The elements of \( \pi_\ast \Sigma Y \) in filtration 1 are evidently just the image of the suspension homomorphism \( E : \pi_\ast Y \to \pi_{\ast+1} \Sigma Y \). Note that it is natural to think of suspensions as values of a secondary homotopy operation, since any map \( f : S^n \to Y \) is null homotopic in \( \Sigma Y \) for two different reasons — namely, the two cones on the “equatorial” copy of \( Y \).

5.4. Example. In the spectral sequence for \( W_\ast := S^2 \otimes S^1 \), the 1-cycles \( \nu_2, \eta_2, \eta_3, \eta_3 \) represent the suspension classes \( \nu_2 \in \pi_2 W_1 \), \( \eta_2, \eta_3 \in \pi_3 W_1 \), and \( \eta_3 \eta_3 \in \pi_3 W_1 \) respectively.

The 2-chain \( [s_0 \nu_2, s_1 \nu_2] \in \pi_3 W_2 \) is not a cycle, since \( [\nu_2, \nu_2] \neq 0 \) in \( \pi_3 W_1 \). However, \( [s_0 \nu_2, s_1 \nu_2] \circ \eta_3 \in \pi_4 W_2 \) is a cycle, since \( [\nu_2, \nu_2] \circ \eta_3 = (2\eta_2) \circ \eta_3 = 0 \) in \( \pi_3 W_1 \). It has order 2, and is not a boundary, so it must represent \( \pm \nu' \in \pi_6 S^3 \), with \( 2\nu' = \eta_3 \eta_3 \eta_3 \) (a suspension, and thus in filtration 1).

5.5. Remark. More generally, we could start with any simplicial resolution \( W_\ast \) of \( Y \), and by letting \( V_\ast := W_\ast \otimes S^k \) we obtain a resolution of \( \Sigma^k Y \).
classes, we see that whenever $Y$ is a bisimplicial object and $i_p : S^p \to Y$ and $i_q : S^q \to Y$, so the spectral sequences for each summand split off from that for $Y$; we are only interested in the remaining cross-term part.

First assume that $p, q \geq 2$: by Hilton’s Theorem, we know that the lowest dimensional non-trivial cross-term in the simplicial abelian group $\pi_* W_\ast$ in filtration (=simplicial dimension) 2 necessarily has the form:

$$w_2 := [s_0 i_p, s_1 i_q] - [s_1 i_p, s_0 i_q] \in \pi_{p+q-1} W_2$$

(see (5.2)). For dimensional reasons this cycle cannot be hit by any differentials, and the class it represents in $\pi_{p+q+1} W_\ast = \pi_{p+q+1} (S^{p+1} \vee S^{q+1})$ is therefore the generator $[i_{p+1}, i_{q+1}]$ (up to sign) — which thus can be expressed as the value of a third order homotopy operation.

Likewise, the lowest dimensional cycles in filtration 3 will have the form

$$[s_1 s_0 i_p, [s_2 s_0 i_p, s_2 s_1 i_q]] - [s_1 s_0 i_p, [s_2 s_1 i_p, s_2 s_0 i_q]] + [s_2 s_0 i_p, [s_2 s_1 i_p, s_1 s_0 i_q]] - [s_2 s_1 i_p, [s_2 s_0 i_p, s_1 s_0 i_q]]$$

(5.8)

representing $[i_{p+1}, [i_{p+1}, i_{q+1}]]$ in $\pi_{2p+q+1} \in S^{p+1} \vee S^{q+1}$ (and similarly for the other Hall basis elements). Note that the 2-cycle $[s_0 i_p, [s_0 i_p, s_1 i_q]] - [s_1 i_p, [s_0 i_p, s_1 i_q]]$, in the cross-term of $\pi_{2p+q-2} W_2$, bounds $[s_2 s_0 i_p, [s_1 s_0 i_p, s_2 s_1 i_q]]$, and so on.

When $p = q = 1$, $\pi_* W_\ast$ is concentrated in dimension 1, and the Whitehead products are commutators, so we cannot use a dimension argument to identify the cycles. In this case it is convenient to work with a simplicial group model for $S^1 \vee S^1$, namely, $F(S^0 \vee S^0) \sim$ a constant (free) simplicial group. Thus $W_\ast$ is a bisimplicial group which is equivalent to its own diagonal, which is thus a simplicial group model $F(S^1 \vee S^1)$ for $S^2 \vee S^2$, and the spectral sequence collapses at the $E^2$-term.

In this case we see that the 1-dimensional analogue of (5.7) is the product of two commutators:

$$\omega := (s_0 s_1 \beta s_0 \alpha^{-1} s_1 \beta^{-1}) \cdot (s_1 \alpha s_0 \beta s_1 \alpha^{-1} s_0 \beta^{-1})$$

(5.9)

where $\alpha \in \pi_1 S_{(1)}$ and $\beta \in \pi_1 S_{(2)}$ are the fundamental classes of the two wedge summands in $Y = S_{(1)} \vee S_{(2)}$. This is known to represent the Whitehead product in $S^2 \vee S^2$ (see [Cu, §11.11]). A similar statement holds for the case $p = 1 < q$.

Since $S^k \vee S^k$ is the universal space for Whitehead products of $k$-dimensional classes, we see that whenever $Y$ is $(k-1)$-connected and $\alpha, \beta \in \pi_k Y$, the Whitehead product $[E \alpha, E \beta]$ in $\pi_{2k+1} \Sigma Y$ is represented in the spectral sequence by

$$\gamma := [s_0 \alpha, s_1 \beta] - [s_1 \alpha, s_0 \beta],$$

(5.10)

modulo suspensions.

We can use the suspension spectral sequence to provide an elementary proof of the following well-known facts (cf. [HW] (3.5)):

5.11. Lemma. Let $Y$ be $(k - 1)$-connected $(k \geq 1)$, and $\alpha, \beta \in \pi_k Y$.  

(a) If $[\alpha, \beta] = 0$ in $\pi_{2k-1} Y$, then $[E \alpha, E \beta]$ is divisible by 2 in $\pi_{2k+1} \Sigma Y$, modulo elements in the image of the suspension $E : \pi_* Y \to \pi_* \Sigma Y$. 

(b) If $\alpha = \beta$ and $k$ is even, then $[E \alpha, E \beta]$ is a suspension.
Proof. By the universal example (5.7), the class \( E_\alpha, E_\beta \) cycle representing \[ (5.10) \] is a cycle representing \([E_\alpha, E_\beta]\) (modulo lower filtration, which is \( \text{Im}(E) \)).

(a) If \( [\alpha, \beta] = 0 \), then \( \delta := [s_0\alpha, s_1\beta] \) is another cycle which survives to \( \pi_* \Sigma Y \) whose double is \( \gamma \).

(b) When \( k \) is even, \( \gamma = 0 \), so \([E_\alpha, E_\beta]\) vanishes modulo suspensions. \( \Box \)

5.12. Examples. (a) When \( Y = S^1 \) and \( \alpha = \beta = \iota_1 \), then \([\alpha, \beta] = 0 \) in \( \pi_1 S^1 \) and the Lemma yields the Hopf map \( \eta_2 \in \pi_3 S^2 \), with \( 2\eta_2 = [\iota_2, \iota_2] \) (since there are no elements in filtration 0 in this dimension).

(b) When \( Y = S^3 \) and \( \alpha = \beta = \iota_3 \), again \([\alpha, \beta] = 0 \) in \( \pi_3 S^3 \) and the Lemma yields \( \nu_4 \in \pi_7 S^4 \), and in fact
\[
2\nu_4 = [\iota_4, \iota_4] + EV
\]
by [12] (5.8), although we cannot determine the extension using the spectral sequence alone.

(c) When \( Y = S^4 \) and \( \alpha = \beta = \iota_4 \), we have \([\iota_5, \iota_5] = \nu_5 \circ \eta_8 = E(\nu_4 \circ \eta_7) \), where the first equality is [12] (5.10).

(d) When \( Y = S^6 \) and \( \alpha = \beta = \iota_6 \), we have \([\iota_7, \iota_7] = 0 \).

(e) When \( Y = S^7 \) and \( \alpha = \beta = \iota_7 \), we have \([\alpha, \beta] = 0 \) and the Proposition yields \( \sigma_8 \in \pi_{13} S^8 \), with
\[
2\sigma_8 = [\iota_8, \iota_8] + E\sigma',
\]
by [12] (5.16).

(f) When \( Y = S^8 \) and \( \alpha = \beta = \iota_8 \), we have
\[
[\iota_9, \iota_9] = \sigma_9 \circ \eta_{16} + \tilde{\nu}_9 + \epsilon_9 = E(\sigma_8 \circ \eta_{15} + \tilde{\nu}_8 + \epsilon_8),
\]
where the first equality is [12] (7.1).

5.15. Remark. As noted above, in the spectral sequence for \( W := S^4 \otimes S^1 \), the 1-chain \( \delta_4 := [s_0 \epsilon_4, s_1 \epsilon_4] \in \pi_7 W_2 \) is not a cycle, since \( d_0 \delta_4 = [\iota_4, \iota_4] = 2\nu_4 \otimes EV' \) by (5.13). Since \( EV' \circ \eta_7 \neq 0 \) and \( EV' \circ \eta_7 \circ \eta_8 \neq 0 \), also \( \delta_4 \circ \eta_7 \) and \( \delta_4 \circ \eta_7 \circ \eta_8 \) are not cycles. Similarly,
\[
(2\nu_4 - EV') \circ (2\nu_7) = 4\nu_4 \circ \nu_7 - (2EV') \circ \nu_7 = 4\nu_4 \circ \nu_7 - \eta^3 \circ \nu_7 = 4\nu_4 \circ \nu_7 \neq 0,
\]
since \( 2\nu' = \eta^3 \) and already \( \eta_5 \circ \nu_6 = 0 \), while \( \nu_4 \circ \nu_7 \) has order 8 in \( \pi_{10} S^4 \). This implies that \( 2\delta_4 \circ \nu_4 \) and thus \( \delta_4 \circ \nu_4 \) are not cycles, but \( \delta_4 \circ \eta^3 \) is a cycle in \( \pi_{10} W_2 \). Since it cannot bound anything, it must therefore represent the generator \( \sigma'' \) of \( \pi_{12} S^5_{(2)} \cong \mathbb{Z}/2 \).

Similarly, in the spectral sequence for \( W := S^6 \otimes S^1 \), \( \delta_6 := [s_0 \epsilon_6, s_1 \epsilon_6] \in \pi_{11} W_2 \) is not a cycle, since \( d_0 \delta_6 = [\iota_6, \iota_6] \in \pi_{11} S^6 \cong \mathbb{Z} \) has infinite order. However, \( \delta_6 \circ \eta_{11} \) is the only possible representative for \( \sigma' \in \pi_{14} S^7 \), of order 8, since it is not a suspension (though \( 2\sigma' = E\sigma'' \)). Incidentally, this shows that \([\iota_6, \iota_6] \circ \eta_{11} = 0 \) in \( \pi_{12} S^6 \).

We conclude this section by showing

5.16. Theorem. All elements in the homotopy groups of a finite wedge of simply-connected spheres are generated of higher order by the fundamental classes and their Whitehead products.
Proof. Since any suspension is the value of a secondary homotopy operation (see §6.3), by Corollary 4.13 it suffices to show this for \( \eta_2 \in \pi_3 S^2 \), \( \nu_4 \in \pi_7 S^1 \), and \( \sigma_8 \in \pi_{15} S^8 \), and the maps \( \alpha_1(p) \in \pi_{2p} S^3 \) for odd primes \( p \).

Indeed, from (5.9) we see that by Lemma 5.11(a):

- (a) \( \eta_2 \) is represented by either half of \( \omega \) e.g., \( h := s_0 t_2 \cdot s_1 t_2 \cdot s_0 t_1 \cdot s_1 t_1^{-1} s_{1 t 1^{-1}} \) in \( \pi_1 W_2 \) for \( W_* = S^1 \otimes S^1 \);
- (b) \( \nu_4 \) is represented by \( [s_0 t_3, s_1 t_3] \in \pi_5 W_2 \) for \( W_* = S^3 \otimes S^1 \);
- (c) \( \sigma_8 \) is represented by \( [s_0 t_7, s_1 t_7] \in \pi_{13} W_2 \) for \( W_* = S^7 \otimes S^1 \).

Now for any odd prime \( p \) we have

\[
\pi_1 S^3_p := \begin{cases} 
Z_{(p)} \langle t_2 \rangle & \text{for } i = 2 \\
Z_{(p)} \langle [t_2, t_2] \rangle & \text{for } i = 3 \\
0 & \text{for } 4 \leq i < 2p
\end{cases}
\]

with \( [[t_2, t_2], t_2] = 0 \). Since \( \alpha_1(p) \in \pi_{2p} S^3 \) is not a suspension, it must appear in filtration \( k \geq 2 \) and in dimension \( i = 2p - k \) in the spectral sequence for \( W_* = S^3_p \otimes S^1 \), and all classes in \( \pi_k W_k \) are iterated Whitehead products, by (5.17) and Hilton’s Theorem.

5.18. Example. A Hall basis for the lowest dimension 2-cycles for \( W_* = S^3_2 \otimes S^1 \) is given by \( \varepsilon_1 := [s_0 t_2, s_1 t_2], s_0 t_2 \) and \( \varepsilon_2 := [s_0 t_2, s_1 t_2], s_1 t_2 \) in \( \pi_4 W_3 \); a Hall basis for the non-degenerate classes in \( \pi_4 W_3 \) is given by \( \theta_1 := [s_2 s_0 t_2, s_1 s_0 t_2], s_2 s_1 t_2 \) and \( \theta_2 := [s_2 s_1 t_2, s_1 s_0 t_2], s_2 s_0 t_2 \).

We have \( d_0 \theta_i = 0 \) and \( d_3 \theta_i = 0 \) for \( i = 1, 2 \). Since \( [s_1 t_2, s_1 t_2] = 2(s_1 t_2 \cdot \eta_2) \) and

\[
[s_0 t_2, s_1 t_2 \eta_2] = [s_0 t_2, s_1 t_2] \cdot \eta_2 - [s_0 t_2, s_1 t_2], s_1 t_2 \equiv -\varepsilon_2 \pmod{3}
\]

by [BH] Theorem 3.24, we see that \( d_1 \theta_1 = \varepsilon_2 \), and \( d_2 \theta_1 = -2\varepsilon_2 \), so \( [\varepsilon_2] \) has order 3 in \( E^2_{2,4} \). Thus the generator \( \alpha_1(3) \) for \( \pi_6 S^3_3 \approx \mathbb{Z}/3 \) is represented by either of these two classes.

6. Complex projective spaces

For each \( n \geq 1 \), the complex projective space \( \mathbb{C}P^n \) fits into a fibration sequence

\[
S^1 \hookrightarrow S^{2n+1} \overset{g_n}{\to} \mathbb{C}P^n,
\]

with \( g_n \) a Hopf map, as well as a homotopy cofibration sequence

\[
S^{2n+1} \overset{g_n}{\to} \mathbb{C}P^n \overset{j_n}{\to} \mathbb{C}P^{n+1},
\]

starting with \( \mathbb{C}P^1 = S^2 \) and \( g_1 = \eta_2 \). Thus \( g_n \) determines an isomorphism \( \pi_i S^{2n+1} \cong \pi_i \mathbb{C}P^n \) for \( i \neq 2 \), with \( \pi_2 \mathbb{C}P^n \cong \mathbb{Z} \) generated by \( \tau_2 = j_{n-1} \circ \ldots \circ j_1 \).

It is nevertheless illuminating to analyze how all elements in \( \pi_* \mathbb{C}P^n \) are generated by the unique indecomposable \( \tau_2 \) using (higher) homotopy operations, as follows:

6.3. The four dimensional complex projective space. Our approach is necessarily inductive, starting with \( \mathbb{C}P^2 \). From (6.2) we see that a (minimal) simplicial resolution \( W_* \) for \( \mathbb{C}P^2 \) as in Section 2 must start with \( W_0 = S^2 \) and
$W_i = S^3 \vee s_0 S^2$, with $d_0$ given by the Hopf map $\eta_2 = g_1 : S^3 \to \mathbb{CP}^1 = S^2$ on the summand $S^2$.

As in $[5.6]$ the lowest dimensional 1-cycle in $\pi_1 W_i$ (for $i = 4$) is given by the Whitehead product $\gamma_2 = [t_3, s_0 t_2]$. We see that indeed

$$d_0(\gamma_2) = [\eta_2, t_2] = [t_2, t_2] \circ \eta_3 = (2\eta_2) \circ \eta_3 = 0,$$

and since this is a permanent cycle which cannot be hit by a differential (for dimension reasons), it must represent $g_2 : S^5 \to \mathbb{CP}^2$ (up to sign).

Note that $\gamma_2$ exhibits $g_2$ as a canonical value of a secondary operation in the sense of Proposition $[1.13]$ (though not as a Toda bracket in the usual sense). Moreover, this also defines a map $\hat{\gamma}$ from the simplicial space $W_\ast$ to the simplicial 1-skeleton of $W_\ast$ described above (and thus to $W_\ast$ itself), realized by $g_2$:

$$\cdots \xrightarrow{V_2} \xrightarrow{V_1} \xrightarrow{V_0} W_0$$

From the description in $[5.1]$ we see that for any map $f : S^N \to S^1$, precomposition with $f \otimes S^1 : S^N \otimes S^1 \to V_\ast$ induces precomposition with $\Sigma f$ in $\pi_3 \mathbb{CP}^2$. Thus $[\gamma_2 \circ f] \in \pi_N W_1$ represents $[g_2 \circ \Sigma f] \in \pi_{N+1} \mathbb{CP}^2$.

6.6. Remark. Any class $\alpha \in \pi_k S^5 \cong \pi_k \mathbb{CP}^2$ ($k > 5$) which is not a suspension is represented by an element in higher filtration in the spectral sequence for $V_\ast$. The map of simplicial spaces $\hat{\gamma} : V_\ast \to W_\ast$ then induces a map of spectral sequences $\hat{\gamma}_\ast$, yielding a representative for the composite $g_2 \circ \alpha$ (which in this case is always non-trivial since $\hat{\gamma}_2$ is an isomorphism).

6.7. Example. The first such class is $\sigma''' \in \pi_12 S^5 \cong \mathbb{Z}/30$, of order 2. By $[5.15]$ this is represented by the element $[s_0 t_4, s_1 t_4] \circ \eta^3$ in $E_{2,10}^{1} = \pi_{10} V_2$, which maps under $\hat{\gamma}_2$ to the iterated Whitehead product

$$[[[s_0 t_3, s_1 s_0 t_2], [s_1 t_3, s_1 s_0 t_2]] \circ \eta^3$$

$$= [[[s_1 t_3, s_1 s_0 t_2], s_1 s_0 t_2], s_0 t_3] \circ \eta^3 + [[[s_1 t_3, s_1 s_0 t_2], s_0 t_3], s_1 s_0 t_2] \circ \eta^3$$

in $\pi_{10} W_2$.

6.8. Higher projective spaces. A simplicial resolution for $\mathbb{CP}^3$ will start with $W_0 = S^2$ and $W_1 = S^3 \vee s_0 S^2$, as for $\mathbb{CP}^2$, but now we set $W_2 := S^4 \vee s_0 S^3 \vee s_1 S^3 \vee s_1 s_0 S^2$, with $d_0$ given by the representative for $g_2 : S^5 \to \mathbb{CP}^2$ – that is, by $\gamma_2 = [t_3, s_0 t_2]$ – on the summand $S^4$. 

\[ \cdots \xrightarrow{V_2} \xrightarrow{V_1} \xrightarrow{V_0} W_0 \]
As before, the lowest dimensional 2-cycle in \( \pi_2 W_* \) is given by
\[
(6.9) \quad \gamma_3 := [t_4, s_1 s_0 t_2] - [s_0 t_3, s_1 t_4] \in \pi_2 W_2.
\]
By the Jacobi identity
\[
[(t_3, s_0 t_2], s_0 t_2] + [(s_0 t_2, s_0 t_2], t_3] + [(s_0 t_2, t_3], s_0 t_2] = 0,
\]
so
\[
2[(t_3, s_0 t_2], s_0 t_2] = -[(s_0 t_2, s_0 t_2], t_3] = -2[s_0 t_2 \circ \eta_1, t_3] = 2[t_3, s_0 t_2 \circ \eta_2].
\]
Therefore,
\[
d_0(\gamma_3) = [(t_3, s_0 t_2], s_0 t_2] - [t_3, s_0 t_2 \circ \eta_2] = [t_3, s_0 t_2 \circ \eta_2] - [t_3, s_0 t_2 \circ \eta_2] = 0.
\]
Since \( d_1 \gamma_3 = [t_3, \gamma_3] = 0 \), and \( d_2 \gamma_3 = 0 \), too, this is a permanent cycle. It cannot be hit by a differential (for dimension reasons), so it must represent \( g_3 : S^7 \to \mathbb{C}P^3 \) as a third-order simplicial operation.

It also defines a map \( \bar{g}_3 \) from the simplicial space \( U_* := S^5 \otimes S^2 \to W_* \), realized by \( g_3 \). Once more, for any map \( f : S^N \to S^5 \), precomposition with \( f \otimes S^2 : S^N \otimes S^2 \to U_* \) shows that \( [\gamma_3 \circ f] \in \pi_N W_2 \) represents \( [g_3 \circ \Sigma^2 f] \in \pi_{N+2} \mathbb{C}P^3 \).

Any class in \( \pi_k S^7 \cong \pi_k \mathbb{C}P^3 \) \((k > 7)\) which is not a double suspension is represented by an element in higher filtration in the spectral sequence for \( V_* \). The first such class is \( \sigma' \in \pi_{14} S^7 \cong \mathbb{Z}/120 \) — in fact, \( \sigma' \) is not a suspension, and \( 2\sigma' = E_2\sigma'' \) is not a double suspension, but \( 4\sigma' = E^2\sigma'' \).

6.10. Some combinatorics.

In order to describe the general case, we require some combinatorial notions:

6.11. Definition. Given a \( k \)-tuple \( I = (i_1 < \ldots < i_k) \) of non-negative integers (in ascending order), let \( J \) denote the underlying unordered set (and conversely), and \( s_I = s_{i_k} s_{i_{k-1}} \ldots s_{i_1} s_{i_1} \) the corresponding iterated degeneracy map, so that
\[
(6.12) \quad \forall x \exists y s_I x = s_J y \quad \text{if and only if} \quad j \in I.
\]
If \( I \) and \( J \) are disjoint sets of natural numbers, \( I \cup J = J \cup I \) will denote the disjoint union \( I \sqcup J \) in ascending order.

Given two finite sets \( I \) and \( J \) of non-negative integers, each arranged in ascending order, their sign \( \text{sgn}(I, J) \) is defined as follows:

(i) If \( I \) and \( J \) form a partition of \( \{1, 2, \ldots , n\} \), we write \( \sigma_{(I, J)} \in S_n \) for the permutation obtained by concatenating \( I \) with \( J \), and let \( \text{sgn}(I, J) := \text{sgn}(\sigma_{(I, J)}) \).

(ii) If \( I \) and \( J \) are disjoint subsets of \( \{0, 1, 2, \ldots , N\} \), let \( \pi : I \sqcup J \to \{1, 2, \ldots , n\} \) be an order-preserving isomorphism, with \( \text{sgn}(I, J) := \text{sgn}(\pi[I], \pi[J]) \).

(iii) If \( I \) and \( J \) are any two finite sets \( I \) and \( J \) of non-negative integers, let \( I' := I \setminus J \) and \( J' := J \setminus I \) (each in ascending order), and set \( \text{sgn}(I, J) := \text{sgn}(I', J') \).

6.13. Example. For \( I = \{2, 4\} \) and \( J = \{1, 3, 5\} \), we have \( \sigma_{(I, J)} = (1 2 3 4 5) \), so \( \text{sgn}(I, J) = -1 \).

For \( I = \{1, 5\} \) and \( J = \{0, 2, 7\} \), the map \( \pi \) is given by \( \begin{pmatrix} 0 & 1 & 2 & 5 & 7 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix} \) and so \( \pi[I] = \{2, 4\} \) and \( \pi[J] = \{1, 3, 5\} \), and thus \( \text{sgn}(I, J) = -1 \).

For \( I = \{1, 3, 5, 6\} \) and \( J = \{0, 2, 3, 6, 7\} \), we have \( I' = \{1, 5\} \) and \( J' = \{0, 2, 7\} \), so again \( \text{sgn}(I, J) = -1 \).
6.14. Notation. Given $n \geq 2$, for each $0 \leq k < n - 1$, let $\mathcal{I}_k^n$ denote the collection of all $(k, n - k - 1)$-partitions $(I, J)$ of $\{0, \ldots, n - 2\}$. When $2k = n - 1$, we include in $\mathcal{I}_k^n$ only sets with $0 \in I$ (to avoid double counting).

6.15. Remark. If $(I', J')$ is obtained from $(I, J)$ by switching a pair of elements between $I$ and $J$ (while maintaining ascending order), then $\text{sgn}(I', J') = -\text{sgn}(I, J)$. As a result, if $I \setminus J$ has cardinality $k$ and $J \setminus I$ has cardinality $\ell$, then

\[
\text{sgn}(I, J) = (-1)^{k-\ell} \text{sgn}(J, I).
\]

Finally, if $M = (m_1, \ldots, m_k)$ is an ordered $k$-tuple of natural numbers (in ascending order), denote its underlying set by $M$. Conversely, for any decomposition $M \sqcup N \sqcup P = \{0, \ldots, n - 1\}$, the sign of corresponding three-fold shuffle $(M, N, P)$ satisfies:

\[
\text{sgn}(M, N \cup P) \cdot \text{sgn}(N, P) = \text{sgn}(M, N, P) = \text{sgn}(M \sqcup N, P) \cdot \text{sgn}(M, N).
\]

We can now state our main result for this section:

6.18. Proposition. For each $n \geq 1$, there is a rational sequential approximation $W_\bullet$ for $\mathbb{C}P^n$ with a single non-degenerate $(k + 2)$-sphere in $W_k$ for each $0 \leq k < n$. The Hopf map $g_n : S^{2n+1} \rightarrow \mathbb{C}P^n$ is represented in the homotopy spectral sequence for $W_\bullet$ by the $(n - 1)$-cycle

\[
\gamma_n := \sum_{j=2}^{n+1} \sum_{(I, J) \in \mathcal{I}_{n-2}^n} (-1)^{n-j} \text{sgn}(I, J) \cdot [s_{1\ell_{n-j+3}}, s_{1\ell_n}]
\]

in $\pi_{n+2}W_{n-1}$. This also serves as the face map $d_0$ on $d_n+1 \in \pi_{n+1}S^{n+1}$ in simplicial dimension $n - 1$ in the sequential approximation for $\mathbb{C}P^{n+1}$.

We shall refer to $(-1)^{n-j}$ as the global coefficient of $[s_{1\ell_{n-j+3}}, s_{1\ell_n}]$.

Proof. All iterated Whitehead products in $\pi_iW_j$ are degenerate for $i \leq j > n$, and $\gamma_n$ is the first non-trivial class that does not come from $\mathbb{C}P^{n-1}$. By Proposition 2.1, it thus suffices to show that $\gamma_n$ is a Moore cycle in the integral homotopy spectral sequence for $W_\bullet$.

Step 1: If $J := \Phi(I)$ and $r \geq 1$, then $d_r([s_{1\ell_{n-j+3}}, s_{1\ell_n}]) = 0$ unless one of $\{r, r + 1\}$ is in $I$ and one is in $J$. If $(I', J')$ is then obtained from $(I, J)$ by interchanging $r$ and $r + 1$, $\text{sgn}(I', J') = -\text{sgn}(I, J)$ by Remark 6.15 while $d_r[s_{1\ell_{n-j+3}}, s_{1\ell_n}] = d_r[s_{1\ell_{n-j+3}}, s_{1\ell_n}]$.

By our assumption in §6.14 it remains to deal with $d_1([s_{1\ell_{k+2}}, s_{1\ell_{k+2}}])$, for $n = 2k + 1$ and $|I| = |J| = k$ with $0 \in I$. So $I = (0 = i_1 < i_2 \ldots < i_k)$ and $J = (1 = j_1 < j_2 \ldots < j_k)$ (the case $k = 1$ is dealt with in (6.9)ff.). If $(I', J')$ is obtained from $(I, J)$ by interchanging $i_j$ with $j_i$ for all $2 \leq i \leq k$ (leaving $0 \in I'$ and $1 \in J'$), then $\text{sgn}(I, J) = (-1)^{(k-1)(k-1)} \text{sgn}(I', J')$, while

\[
d_1([s_{1\ell_{k+2}}, s_{1\ell_{k+2}}]) = (-1)^{(k+2)(k+2)} \cdot d_1([s_{1\ell_{k+2}}, s_{1\ell_{k+2}}])
\]

by [Wh] (7.5), so the two appear in $d_1\gamma_n$ with opposite signs. Thus we see that $d_r\gamma_n = 0$ for all $1 \leq r \leq n$. 
Step 2: By the Jacobi identity for any \( n \geq 2 \) and iterated degeneracy \( s_I \) we have 
\[
[[t_n, s_I t_2], s_I t_2] + [[s_I t_2, s_I t_2], t_n] + [[s_I t_2, t_n], s_I t_2] = 0,
\]
so
\[
2[[t_n, s_I t_2], s_I t_2] = -[s_I [t_2, t_2], t_n] = -2[s_I t_2^\#, t_n] = (-1)^{n+2}[t_n, s_I t_2^\#]
\]
and thus
\[
[[t_n, s_I t_2], s_I t_2] = (-1)^{n+1}[t_n, s_I t_2^\#]
\]
(since all summands in Hilton’s Theorem in this dimension are infinite cyclic). Thus
\[
d_0([s_0 t_n, s_J t_3]) = [t_n, s_J \gamma_1] = [t_n, s_J t_2^\#]
\]
(where \( J = \{1, \ldots, n-2\} \) and \( J' = \{0, \ldots, n-3\} \) – the only summand of \( d_0 \gamma_n \) not a three-fold Whitehead product – equals the first summand of \( d_0([t_n+1, s_J t_2]) = [\gamma_{n-1}, s_J t_2] \) for \( J'' = J' \cup \{n-2\} \) – i.e., \([t_n, s_J t_2], s_J t_2\], with opposite sign (due to the sign in (6.20) and the coefficient \((-1)^{n-j}\) in (6.19), where \( j = 3 \in \) (6.21)).

Step 3: To show that \( \gamma_n \) is a Moore cycle, consider triple Whitehead products summands of \( d_0 \gamma_n \) of the form
\[
A = [[s_I t_p, s_J t_q], s_K \gamma_r] = [[s_I t_p, s_J t_q], s_K t_r]
\]
associated to a partition \( \{0, 1, \ldots, n-2\} = L \sqcup M \sqcup N \), with cardinalities \( |N| = q-2 \), \( |M| = p-2 \), and \( |L| = r-2 \) (so \( p+q+r = n+4 \)), such that \( I = L \sqcup N \), \( J = L \sqcup M \), and \( K = M \sqcup N \) (and thus \( |I| = n-p \), \( |J| = n-q \), and \( |K| = n-r \)). By (6.19) and (6.14) we must assume \( p+q-1 \geq r \) and \( p \geq q \), and if \( p = q \), then \( \min(N) < \min(M) \). Given \( A \), set
\[
B = [[s_J t_q, s_K t_r], s_I t_p] \quad \text{and} \quad C = [[s_K t_r, s_I t_p], s_J t_q]
\]
with
\[
(-1)^{pq} A + (-1)^{qp} B + (-1)^{qr} C = 0
\]
by the Jacobi identity (see [Wh] X, (7.14)). Thus to show that \( d_0 \gamma_n = 0 \), we must show that each of \( A, B, \) and \( C \) appears exactly once in the expansion of \( d_0 \gamma_n \), with the appropriate sign.

By (6.19), \( A \) only appears in \( d_0([s_I t_p q-1, s_K t_r]) = [s_L d_0 \gamma_{p+q-3}, s_K t_r] \), corresponding to \( [s_M t_p, s_M t_q] \) in \( \gamma_{p+q-3} \), with \( \widehat{L} \) obtained from \( L \) by adding one to each index, and \( \widehat{K} \) obtained from \( K \) by adding one to each index and adjoining 0, so
\[
\text{sgn}(\widehat{K}, \widehat{L}) = \text{sgn}(K, L) \quad \text{and} \quad \text{sgn}(\widehat{L}, \widehat{K}) = (-1)^r \text{sgn}(L, K)
\]
By (6.12), \( s_I = s_L \circ s_N \) and \( s_J = s_L \circ s_M \), and by Definition (6.11)ii)-(iii):
\[
\text{sgn}(I, J) = \text{sgn}(N, M), \quad \text{sgn}(J, K) = \text{sgn}(L, N), \quad \text{sgn}(I, K) = \text{sgn}(L, M).
\]
Finally, by (6.17), we have
\[
\text{sgn}(L, K) = \text{sgn}(L, M \sqcup N) = \text{sgn}(L, M) \cdot \text{sgn}(M, N) \cdot \text{sgn}(J, N)
\]
\[
= \text{sgn}(L, N \sqcup M) = \text{sgn}(L, N) \cdot \text{sgn}(N, M) \cdot \text{sgn}(I, M)
\]
Step 4: We see that by (6.25), the coefficient of \( [s_I t_{p+q-1}, s_K t_r] \) in \( \gamma_n \) is
\[
(-1)^{qr} \text{sgn}(\widehat{L}, \widehat{K}) = (-1)^{nr+r} \text{sgn}(L, K) = (-1)^{(p+q)r} \text{sgn}(L, M) \text{sgn}(M, N) \text{sgn}(J, N)
\]
while the sign of \([s_{N tp}, s_{Mtq}]\) in \(\gamma_{p+q-3}\) is
\[
(-1)^{(p+q-3)q} \text{sgn}(N, M) = (-1)^p \text{sgn}(N, M) = \text{sgn}(M, N).
\]

Therefore, if we set
\[
(6.28) \quad \Theta := (-1)^q \text{sgn}(L, M) \cdot \text{sgn}(J, N),
\]
the sign of \(A\) in \(d_0(\gamma_n)\) is \((-1)^p \cdot \Theta\), so multiplication by \((-1)^p\) as in \((6.24)\) yields \(\Theta\) as the “corrected” coefficient of \(A\).

**Step 5:** We now assume that \(q > r\) and \(q + r - 1 > p\) (or \(q + r - 1 = p\) and \(0 \not\in I\)).

We see that \(B\) of \((6.23)\) as a summand in \(d_0(\hat{B}) = [s_{M d_0(t_{q+r-1}, s_{Itp})}]\) for the summand \(\hat{B} := [s_{M t_{q+r-1}, s_{Itq}}]\) in \((6.19)\), which has coefficient
\[
(-1)^p \text{sgn}(\hat{M}, \hat{I}) = (-1)^{np+p} \text{sgn}(M, I) = (-1)^{(q+r)p} \text{sgn}(M, L \sqcup N)
= (-1)^{(q+r)p} \text{sgn}(M, L) \text{sgn}(N, J) = (-1)^{pq} \text{sgn}(L, M) \text{sgn}(L, N) \text{sgn}(J, N)
\]
in \(\gamma_n\) by \((6.25)\) and \((6.17)\).

On the other hand, the summand \([s_{N tq}, s_{L t}]\) has coefficient \((-1)^{(q+r-3)r} \text{sgn}(N, L)\) in \(\gamma_{q+r-3}\), so altogether the coefficient of \(B\) in \(d_0(\gamma_n)\) is
\[
(-1)^{pq+qr} \text{sgn}(L, M) \text{sgn}(J, N) = (-1)^p \cdot \Theta,
\]
so multiplication by \((-1)^q\) as in \((6.24)\) yields \(\Theta\) as the corrected coefficient of \(B\).

Note that if \(q + r - 1 < p\), instead of \(B\) we would have \(B' := [s_{Itp}, [s_{Itq}, s_{Ktq}]]\) in \(d_0(\hat{B}') = [s_{Itp}, s_{M d_0(t_{q+r-1})}]\) for the summand \(\hat{B}' := [s_{Itp}, s_{M t_{q+r-1}}]\) of \((6.19)\).

By \([Wh]\) \((7.5)\), \(B' = (-1)^{p+q-1}B\), while the global coefficient \((-1)^{n(q+r-1)}\) of \(\hat{B}'\) agrees with \((-1)^{np}\) for \(\hat{B}\), since \((-1)^{n(n-1)} = +1\). On the other hand, by \((6.16)\)
\[
\text{sgn}(\hat{I}, \hat{M}) = (-1)^{(p-2)(q+r-5)} \text{sgn}(\hat{M}, \hat{I}) = (-1)^{(q+r-1)} \text{sgn}(\hat{M}, \hat{I}),
\]
so if we replace \(B'\) by \(B\) the total sign is unchanged.

**Step 6:** Since \(r < p\), \(C\) of \((6.23)\) must be replaced by \(C' := [[s_{Itp}, s_{Ktq}], s_{Itq}]\) (assuming \(p + r - 1 \geq q\)), with \(C' = (-1)^p C\) by \([Wh]\) \((7.5)\).

As above, \(C'\) appears as a summand in \(d_0(\hat{C'}) = [s_{N d_0(t_{q+r-1}), s_{Itq}] \text{ in } (6.19), with coefficient}\)
\[
(-1)^q \text{sgn}(\hat{N}, \hat{J}) = (-1)^n(-1)^{(n+1)} \text{sgn}(\hat{J}, \hat{N}) = \text{sgn}(J, N)
\]
in \(\gamma_n\), by \((6.25)\) and \((6.16)\).

On the other hand, the summand \([s_{Itp}, s_{Ktq}]\) has coefficient
\[
(-1)^{(p+r-3)p} \text{sgn}(J, K) = (-1)^p \text{sgn}(L, M)
\]
in \(\gamma_{q+r-3}\), by \((6.26)\). Thus the coefficient of \(C'\) in \(d_0(\gamma_n)\) is \((-1)^{(p+q)r} \cdot \Theta\), which means that of \(C\) is \((-1)^q \cdot \Theta\), and thus multiplication by \((-1)^q\) again yields \(\Theta\) as the corrected coefficient of \(C\). This shows that \((6.24)\) indeed holds, so these three terms in the boundary sum to 0. Thus \(\gamma_n\) is a Moore cycle, as required. □
6.29. Example. The next cycles after $\gamma_3$ of (6.9) are
\[ \gamma_4 = [t_5, s_2s_1s_0t_2] + [s_0t_4, s_2s_1t_3] - [s_1t_4, s_2s_0t_3] + [s_2t_4, s_1s_0t_3] \]
in $\pi_6 W_3$, with
\[ d_0(\gamma_4) = [[t_4, s_1s_0t_2], s_1s_0t_2] - [[s_0t_3, s_1t_3], s_1s_0t_2] + [t_4, s_1s_0t_3] \]
\[ - [[s_0t_3, s_1s_0t_2], s_1t_3] + [[s_1t_3, s_1s_0t_2], s_0t_3] , \]
which vanishes by combining (6.20) with (6.24), and
\[ \gamma_5 = [t_6, s_3s_2s_1s_0t_2] - [s_0t_5, s_3s_2s_1t_3] + [s_1t_5, s_3s_2s_0t_3] - [s_2t_5, s_3s_1s_0t_3] \]
\[ + [s_3t_5, s_2s_1s_0t_3] + [s_1s_0t_4, s_2s_4t_4] - [s_2s_0t_4, s_3s_1t_4] + [s_3s_0t_4, s_2s_1t_4] . \]

6.30. Remark. By Proposition 2.4, it implies that one can construct an integral simplicial resolution $W_\bullet$ of $\mathbb{CP}^n$ such that any element $[\alpha] \in E^2_{r,s}$ of infinite order in the spectral sequence has a multiple which is a permanent cycle. We hope to show in [BBS3] that the description in Proposition 6.18 is in fact valid integrally.

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