Approximation of Generating Function Barcode for Hamiltonian Diffeomorphisms

Ofir Karin

Tel Aviv University

October 28, 2022

- Symplectic Zoominar -

Joint work with Pazit Haim-Kislev

**Goal:** define a conjugation invariant *barcode* of compactly supported Hamiltonian diffeo’ $\varphi = \varphi_N \circ \cdots \circ \varphi_1$ of $\mathbb{R}^{2n}$, that can be numerically approximated from samples of generating functions of $\varphi_1, \ldots, \varphi_N$. 
Generating functions

\[
\operatorname{Ham}_c(\mathbb{R}^{2n}) \cup \overline{\mathbb{R}^{2n} \times \mathbb{R}^{2n}} \cup T^*\mathbb{R}^{2n} \cup T^*\mathbb{S}^{2n}
\]

\[
\varphi \xrightarrow{\text{graph}} \text{gr}(\varphi) \xrightarrow{\text{Lagrangian}} L_\varphi \xrightarrow{\text{compactify}} \overline{L_\varphi}
\]

if \( \varphi \) is \( C^1 \)-close to \( \text{Id} \), then \( L_\varphi = \text{gr}(dS) \), for \( S : \mathbb{R}^{2n} \to \mathbb{R} \)

**Definition:** \( S : \mathbb{R}_{x}^{2n} \times \mathbb{R}_{\xi}^{d} \to \mathbb{R} \) is a generating function of \( L \subset T^*\mathbb{R}^{2n} \) if

\[
L = \left\{ \left( x, \frac{\partial S}{\partial x} \right) \in T^*\mathbb{R}^{2n}, \frac{\partial S}{\partial \xi} = 0 \right\}, \quad \frac{\partial S}{\partial \xi} \cap 0
\]
Generating function homology

\[
\left\{ \text{(non-deg') fixed points of } \varphi \right\} \leftrightarrow \left\{ \text{(transversal) intersection points of } L_\varphi \cap L_0 \right\} \leftrightarrow \left\{ \text{(non-deg') crit' points of } S \right\}
\]

- \(S\) is a GFQI if \(S(x, \xi) = Q(\xi)\) (non-degenerate quadratic form) outside a compact set \(B\). Extend to \(S : S^{2n} \times \mathbb{R}^d \to \mathbb{R}\)
- for \(L_0 \overset{\text{Ham'}}{\sim} L \subset T^*S^{2n}\), all GFQI are equivalent (Viterbo)
- for \(L_0 \overset{\text{Ham'}}{\sim} L \subset T^*S^{2n}\), there exists a GFQI (Laudenbach, Sikorav)

**Definition:** The GF homology of \(\varphi\) with GFQI \(S\), w.r.t \(t \notin \text{Crit}(S)\) is

\[
G^{(-\infty, t]}_*(\varphi) := H_{*+i}(\{S \leq t\}, \{S \leq a\}), \quad a < \min S|_B, \quad i := \text{ind}(Q).
\]

- \(G^{(-\infty, t]}_*(\varphi)\) are independent of the choice of \(S\) and invariant under conjugation by \(\psi \in \text{Symp}(\mathbb{R}^{2n})\) (Traynor)
Generating function barcode

- **barcode** is a finite collection of intervals $I_j = (a_j, b_j]$ such that $-\infty < a_j < b_j \leq \infty$, with multiplicities $m_j \in \mathbb{N}$.

- **metric**: $d_{bot} = \inf \delta > 0$ s.t. after **removing** short bars ($< 2\delta$), we match the others by **aligning** their endpoints with distance $< \delta$.

\[
\begin{align*}
V_0 &= H_* (f < t_0 + \varepsilon) \\
V_1 &= H_* (f < t_1 + \varepsilon) \\
V_m &= H_* (f < t_m + \varepsilon)
\end{align*}
\]

- **bars** $(t_i, t_j]$ for classes of critical pts that are born at $t_i$ and die at $t_j$.

- **rays** $(t_i, \infty)$ for classes of critical pts that are born at $t_i$ and never die.

\[
\begin{align*}
\pi^{i,j} : V_i \to V_j, \ (i \leq j) \\
a \text{ class } \alpha \in V_i \\
is \text{ born at } t_i \text{ if: } \alpha \notin \text{ Im } (\pi^{i-1,i}) \\
and \text{ dies at } t_j \text{ if: } \pi^{i,j-1} (\alpha) \notin \text{ Im } (\pi^{i-1,j-1}) \\
and \pi^{i,j} (\alpha) \in \text{ Im } (\pi^{i-1,j})
\end{align*}
\]
**Generating function barcode**

- **Barcode** is a finite collection of intervals $I_j = (a_j, b_j]$ such that $-\infty < a_j < b_j \leq \infty$, with multiplicities $m_j \in \mathbb{N}$.

- **Metric**: $d_{\text{bot}} = \inf \delta > 0$ s.t after **removing** short bars ($< 2\delta$), we match the others by aligning their endpoints with distance $< \delta$.

**Definition:** The **GF barcode** $B(\varphi)$ is the barcode associated to $G_{*}^{(-\infty,t]}(\varphi)$.

**Goal:** define a conjugation invariant **barcode** of compactly supported Hamiltonian diffeomorphism $\varphi = \varphi_N \circ \cdots \circ \varphi_1$ of $\mathbb{R}^{2n}$, that can be numerically approximated from samples of generating functions of $\varphi_1, \ldots, \varphi_N$. 

\[ V_0 = G_{*}^{(-\infty,t_0+\varepsilon]}(\varphi) \]
\[ V_1 = G_{*}^{(-\infty,t_1+\varepsilon]}(\varphi) \]
\[ \vdots \]
\[ V_m = G_{*}^{(-\infty,t_m+\varepsilon]}(\varphi) \]
Approximation algorithm

- \( S_1, \ldots, S_N : \mathbb{R}^{2n} \to \mathbb{R} \), generating \( \varphi_1, \ldots, \varphi_N \in \text{Ham}_c (\mathbb{R}^{2n}) \), such that \( \text{supp} (\varphi) \subseteq \mathbb{B}_R \) and \( \| \varphi_j - \text{Id} \|_{C^1} < T \)

composition formula (Chekanov, Chaperon)

- \( S : \mathbb{R}^{2nN} \to \mathbb{R} \) a GFQI of \( \varphi = \varphi_N \circ \cdots \circ \varphi_1 \), equal to \( Q \) outside \( B \)

- construct a pair \( (K, L) \) of finite CW complexes with mesh \( \frac{1}{m} \), homotopic to \( (S^{2n} \times \mathbb{R}^d, \{ S \leq a \}) \) with \( a < \min S|_B \)

sample \( S|_{B \cap \frac{1}{m} \mathbb{Z}^{2nN}} \) to get a filtered complex \( (K^t, L) \)

- boundary matrices \( \partial_j \) with rows and columns ordered by the filtration

matrix reduction (Edelsbrunner, Letscher, Zomorodian)

- approximated barcode \( B \)

**Theorem:** \( d_{bot} (B, B (\varphi)) \leq C (R) \cdot \frac{\sqrt{nTN^2}}{m} =: \varepsilon \), polynomial time complexity in \( m \), super-exponential in \( N \) (for fixed \( \varepsilon \))
Computational experiment

- A MATLAB program was implemented with input:

\[
\begin{align*}
\{0 < R_j\}_{j=1}^N & \quad \text{support radii} \\
\{c_j\}_{j=1}^N & \quad \text{center points} \\
\{h_j \in C^\infty ([0, R_j^2/2])\}_{j=1}^N & \quad \text{profile functions}
\end{align*}
\]

and parameters \( T, T' > 0, m \in \mathbb{N} \) such that \( |h_j'| \leq T, |h_j''| \leq T' \).

- approximates samples of \( \{S_j : \mathbb{R}^2 \to \mathbb{R}\}_{j=1}^N \) generating \( \varphi^1_{H_j} \) where \( H_j (x) = h_j \left( \frac{|x-c_j|^2}{2} \right) \), using the Hamilton-Jacobi equation.

- returns a barcode \( \mathcal{B} \) such that

\[
d_{\text{bot}} (\mathcal{B}, \mathcal{B}(\varphi)) \leq C(R) \frac{TN^2}{m} + C_1 (T, T', R) \frac{N}{m} + C_2 (T, T', R) \cdot N \sqrt{E},
\]

where \( E \) is a numerical error term.
Computational experiment

- Consider the profile $h : \left[ 0, \frac{1}{2} \right] \rightarrow \mathbb{R}$ given by

- for $a > 0$, denote center points $c_1 = (-a, 0)$, $c_2 = (a, 0) \in \mathbb{R}^2$
Computational experiment

Conjectured barcode:

\[ [\max H_1] + [\max H_2] + [2] \]

\[ [3] \]

\[ [1] \]

\[ [\min H] \]

\[ \min H \quad \mathcal{A}([1]) = \mathcal{A}([3]) \quad \max H \quad \mathcal{A}([2]) \]

Computation results:

| \( a \) | Estimated longest finite bar/\( T \) | \( d_{\text{bot}}(\mathcal{B}(\psi \circ \varphi), \mathcal{B})/T \) |
|--------|----------------------------------|--------------------------------------------------|
| 0.70   | \( \sim 1.57 \times 10^{-2} \)   | \( \sim 2.03 \times 10^{-6} \)                  |

Thank you!