Almost-normality of Isbell-Mrówka spaces

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Abstract

We explore almost-normality in Isbell-Mrówka spaces and some related concepts. We use forcing to provide an example of an almost-normal not normal almost disjoint family, explore the concept of semi-normality in Isbell-Mrówka spaces, define the concept of strongly $(\aleph_0, < c)$-separated almost disjoint families and prove the generic existence of completely separable strongly $(\aleph_0, < c)$-separated almost disjoint families assuming $s = c$ and $b = c$. We also provide an example of a Tychonoff almost-normal not normal pseudocompact space which is not countably compact, answering a question from P. Szeptycki and S. Garcia-Balan.

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1 Introduction

Isbell-Mrówka spaces are topological spaces associated to almost disjoint families. This class of spaces is used to provide examples and counter examples to numerous questions in General Topology, including questions that are initially not related to them. The topological properties of such spaces often depend on the combinatorial properties of the associated almost disjoint family. We cite the surveys [11] and [9] as references for this field of study.

If $N$ is a countable infinite set such that $N \cap [N]^\omega = \emptyset$, an almost disjoint family (over $N$) is an infinite collection $\mathcal{A}$ of infinite subsets of $N$ such that for all distinct $a, b \in \mathcal{A}$, $a \cap b$ is finite. A MAD family (maximal almost disjoint family) is an almost disjoint family which is not properly contained in any other almost disjoint family. By Zorn’s Lemma, every almost disjoint family can be extended to a MAD family and it is well known that there exist almost disjoint families of size $c$ [11]. The least cardinality of a MAD family is called $\frak{a}$, and it is well known that $\frak{a} \geq \omega_1$.

Given an almost disjoint family $\mathcal{A}$ over $N$, the Isbell-Mrówka space associated to $\mathcal{A}$, also known as psi-space of $\mathcal{A}$, and denoted by $\Psi(\mathcal{A})$ is the set $N \cup \mathcal{A}$ with the topology generated by $\{(n) : n \in N\} \cup \{\{a\} \cup (a \setminus F) : a \in \mathcal{A}, F \in [N]^{<\omega}\}$. It is immediate that $\mathcal{A}$ is a Hausdorff, locally compact (therefore Tychonoff) not countably compact zero dimensional separable topological space.

In general, $\Psi(\mathcal{A})$ does not need to be normal (e.g., if $|\mathcal{A}| = c$, $|\mathcal{A}|$ is a closed discrete subspace of size $c$ of the separable space $\Psi(\mathcal{A})$, so it is not normal by Jones’ Lemma) but it may be normal, since $\Psi(\mathcal{A})$ is
metrizable if \( \mathcal{A} \) is countable. The existence of an uncountable normal Isbell-Mrówka space is independent of the Axioms of ZFC, and is equivalent to the existence of a normal separable non-metrizable Moore space \[21\], \[10\], \[19\].

In this paper we study weakenings of normality on Isbell-Mrówka spaces. We say a topological space is normal iff every two closed disjoint subsets can be separated by open disjoint subsets. Various weakenings of normality have been proposed and studied, such as quasi-normality \[22\], almost-normality \[17\], mildly-normality \[16\] and semi-normality \[17\]. In this paper we will focus on the study of almost-normality and semi-normality on Isbell-Mrówka spaces. Recent results regarding the study of some weakenings of normality and Isbell-Mrówka spaces include \[20\] and \[1\].

Given a topological space \( X \), a regular closed set of \( X \) is a closed set \( F \) such that \( F = \text{cl}(\text{int}(F)) \), and an open set \( U \) is said to be regular open iff \( U = \text{int}(\text{cl}(U)) \). We say a topological space \( X \) is almost-normal iff whenever \( F \) is is a closed set and \( K \) is a regular closed set disjoint from \( F \), there exist disjoint open sets \( U, V \) such that \( F \subseteq U \), \( K \subseteq V \). We say that \( X \) is semi-normal iff for every closed set \( F \) and every open set \( U \) containing \( F \) there exists a regular open set \( V \) such that \( F \subseteq V \subseteq U \). The following proposition is from \[17\] and can be easily verified:

**Proposition 1.1** (\[18\]). A topological space is normal iff it is almost-normal and semi-normal.

In this paper, we say that an almost disjoint family \( \mathcal{A} \) is [semi, almost]-normal iff \( \Psi(\mathcal{A}) \) is [semi, almost]-normal.

In \[20\], P. Szeptycki, S. Garcia-Balan provided, among several other examples, an example in ZFC, of an almost disjoint family of true cardinality \( \mathfrak{c} \) which is not almost-normal. Moreover, they showed that the existence of a MAD family of true cardinality \( \mathfrak{c} \) implies the existence of a MAD family of true cardinality \( \mathfrak{c} \) which is not almost-normal. They asked the following question (Question 4.2 of \[20\]):

**Question 1.2.** Is there an almost-normal not normal almost disjoint family?

Recall \( \Psi(\mathcal{A}) \) is pseudocompact iff \( \mathcal{A} \) is MAD \[11\]. Thus, if \( \mathcal{A} \) is MAD, it cannot be normal since, as a consequence of Tietze’s theorem, pseudocompact normal spaces are countably compact. The authors of \[20\] also asked the following (Question 4.3 and 4.4 of \[20\]):

**Question 1.3.** Is there an almost-normal MAD family?

**Question 1.4.** Are almost-normal pseudocompact spaces countably compact?

In this paper, we use iterated forcing and a generalization of the notion of \( Q \)-set to provide a partial answer to Question 1.2 (consistently, yes) and answer negatively Question 1.4 in ZFC by providing a subspace of \( \beta\omega \) which serves as a counter example. Question 1.3 remains open.

In \[20\], they also define the concept of strongly \( \aleph_0 \)-separated almost disjoint family, which is related to almost-normal almost disjoint families, as follows: an almost disjoint family \( \mathcal{A} \) (over \( N \)) is said to be strongly \( \aleph_0 \)-separated iff for every two countable disjoint subsets \( \mathcal{B}, \mathcal{C} \) of \( \mathcal{A} \) there exists \( X \subseteq N \) such that:

1. For every \( a \in \mathcal{A} \), \( a \subseteq^* X \) or \( A \cap X =^* \emptyset \);
2. For every \( a \in \mathcal{B} \), \( a \subseteq^* X \);
3. For every \( a \in \mathcal{C} \), \( a \cap X =^* \emptyset \).
They showed that every almost-normal almost disjoint family is strongly $\aleph_0$-separated and showed that, under CH, there exist MAD families which are strongly $\aleph_0$-separated. In the last section of this paper we define a stronger concept we call strongly $(\aleph_0, < \mathfrak{c})$-separated almost disjoint family and prove that $\mathfrak{b} = \mathfrak{c}$ plus $\mathfrak{s} = \mathfrak{c}$ implies the generic existence of $(\aleph_0, < \mathfrak{c})$-separated completely separable MAD families.

Regarding notation, we define some of the set theoretical topological and cardinal characteristics concepts as we need them, for undefined concepts we refer (resp.) to [14], [5] and [2].

It is worth mentioning a stronger version of almost-normality, called $\pi$-normality, was proposed [13]: a subset of a topological space $X$ is said to be $\pi$-closed if it is a finite intersection of regular closed sets, and $X$ is said to be $\pi$-normal iff whenever $F \subseteq X$ is $\pi$-closed, $K \subseteq X$ is closed and $F \cap K = \emptyset$, there exists disjoint open sets separating $F$ from $K$. However, in [20] it was proven that almost-normality and $\pi$-normality are equivalent.

## 2 A Tychonoff, almost-normal, pseudocompact space which is not countably compact

In this section we give, in ZFC, a negative answer for Question 1.4 by constructing a suitable subspace of $\beta\omega$.

As noted by Kalantan in [12], extremely disconnected spaces are almost-normal since every regular closed set is a clopen, so it can be separated from any set disjoint from it. This fact will be useful to obtain our counterexample.

The following lemma is well known and can be easily proved by the reader. We refer [5].

**Lemma 2.1.** If $X$ is extremally disconnected and $D \subseteq X$ is a dense subset, then $D$ is also extremally disconnected.

The following Lemma is also known. We prove it for the sake of completeness.

**Lemma 2.2.** If $D \subseteq X$ is dense and every sequence in $D$ has an accumulation point in $X$, then $X$ is pseudocompact.

**Proof.** Suppose, by contradiction, that $X$ is not pseudocompact. There exists an unbounded continuous function $h : X \to [0, \infty) \subseteq \mathbb{R}$. For each $n \in \omega$, let $d(n) \in D \cap h^{-1}([n, \infty])$. Then $d : \omega \to D$ has no limit point $x$, for if it had, we would have $x \in \text{cl} (\{d(n) : n \geq m\})$ for every $m \in \omega$, thus, by continuity, $f(x) \geq m$ for every $m \in \omega$, a contradiction.

Now we present our example. For the construction, we identify $\beta\omega$ with the space of ultrafilters over $\omega$, where $U_n$ is the principal ultrafilter generated by $\{n\}$ for each $n \in \omega$ (and $n$ is identified with $U_n$). We write $N = \{U_n : n \in \omega\}$. $\omega^* \subseteq \beta\omega$ is the set of free ultrafilters over $\omega$. Given $A \subseteq \omega$, $\hat{A}$ is the basic clopen set $\{p \in \beta\omega : A \in p\}$.

**Example 2.3.** There exists a Tychonoff extremely disconnected (thus, almost normal) pseudocompact space which is not countably compact.

**Construction.** Let $(P_n : n \in \omega)$ be a partition of $\omega$ into pairwise disjoint infinite sets. For each $n \in \omega$, let $p_n$ be a free ultrafilter such that $P_n \in p_n$. Let $F = \{p_n : n \in \omega\}$. $F$ is infinite and discrete since given $n$, $\{p_n\} = F \cap \hat{P}_n$.

Given $A \in [\omega]^\omega$, let $q_A \in \omega^*$ be defined as follows:
(1) If there exists \( n \in \omega \) such that \( A \in p_n \), let \( q_A = p_n \), for any such \( n \) (e.g. the least such \( n \)), or

(2) if for all \( n \in \omega \) \( A \not\in p_n \), let \( q_A \in \omega^* \) be any free ultrafilter such that \( A \in q_A \).

In any case, \( A \in q_A \). Let \( X = N \cup \{q_A : A \in [\omega]^\omega\} \) and notice that, for each \( n \in \omega \), \( q_{P_n} = p_n \) by (1). Hence, \( F \subseteq X \).

\( X \) is a dense subspace of \( \beta\omega \) (since it contains \( N \)) and by Lemma 2.1, \( X \) is extremely disconnected. In particular, \( X \) is also almost normal.

\( X \) is pseudocompact: since \( N \) is dense in \( X \), it suffices to see that every sequence \( f : \omega \to N \) has an accumulation point. By passing to a subsequence, we can suppose \( f \) is either constant or injective. Constant sequences converge, so suppose \( f \) is injective. Let \( g : \omega \to \omega \) be such that \( f(n) = U_{g(n)} \). Let \( A = \text{ran}(g) \). We claim \( q_A \) is an accumulation point of \( f \). Given a basic nhood \( \hat{B} \supseteq q_A \), we know \( B \cap A \in q_A \) is infinite, so it follows that \( g^{-1}[A \cap B] \subseteq \{n \in \omega : f(n) \in \hat{B}\} \) is also infinite. Since \( B \) is arbitrary, the proof is complete.

\( X \) is not countably compact: we know \( F \) is an infinite discrete subspace of \( X \) (since it is in \( \beta\omega \)). Thus, it suffices to show that \( F \) is closed in \( X \). We show \( X \setminus F \) is open in \( X \). Clearly, every point of \( N \) is in the interior of \( X \setminus F \) since \( N \) is open. If \( A \in [\omega]^\omega \) and \( q_A \not\in F \), then (2) holds, so \( q_A \in \hat{A} \) and \( F \cap \hat{A} = \emptyset \), that is, \( q_A \in X \cap \hat{A} \subseteq X \setminus F \).

The space constructed in Example 2.3 answers negatively the Question 4.4 from [20]. It is worth mentioning that Isbell-Mrówek spaces are never extremally disconnected, so a similar strategy cannot be employed when trying to address Questions 4.2 and 4.3.

### 3 Equivalences for almost-normality in \( \Psi(A) \)

In this section we start to explore the notion of almost-normality in the realm of Isbell-Mrówek spaces. In particular, we aim to provide some characterizations for “\( A \) is almost-normal”. In order to do so, we will use the well known notion of a partitioner of an almost disjoint family. As in the introduction, \( N \) denotes an infinite countable set for which \( N \cap [N]^\omega = \emptyset \).

**Definition 3.1.** Let \( A \) be an almost disjoint family (over \( N \)). We say that \( X \subseteq N \) is a partitioner for \( A \) if for each \( a \in A \), \( a \subseteq^* X \) or \( a \cap X =^* \emptyset \).

We say that a partitioner \( X \) for \( A \) is a partitioner for \( \mathcal{B}, \mathcal{C} \subseteq A \) if \( b \subseteq^* X \) and \( c \cap X =^* \emptyset \) for each \( b \in \mathcal{B} \) and \( c \in \mathcal{C} \).

The main motivation for our equivalences is the following classical result. We give [11] and [10] as references.

**Proposition 3.2.** Let \( A \) be an almost disjoint family. Then \( A \) is normal if, and only if, for all \( \mathcal{B} \subseteq A \), \( B \) and \( A \setminus B \) can be separated by disjoint open sets of \( \psi(A) \).

Recall there is a one to one correspondence between the clopen subsets of \( \Psi(A) \) and the partitioners of \( A \) which can be defined as follows: for each \( X \subseteq N \) consider \( \mathcal{B}_X = \{a \in A : a \subseteq^* X\} \) and \( \mathcal{C}_X = \{a \in A : a \cap X =^* \emptyset\} \). It follows that:

- \( \mathcal{B}_X \cup X \) and \( \mathcal{C}_X \cup (N \setminus X) \) are disjoint open subsets of \( \Psi(A) \);
- If \( X \) is a partitioner for \( \Psi(A) \), then \( A = \mathcal{B}_X \cup \mathcal{C}_X \) and \( \Psi(A) = (\mathcal{B}_X \cup X) \cup (\mathcal{C}_X \cup (N \setminus X)) \) is union of clopen subsets.
Lemma 3.3. If $\mathcal{A}$ is an almost disjoint family, then $F : \text{Clop}(\Psi(\mathcal{A})) \to \{X \subseteq N : X \text{ is a partitioner for } \mathcal{A}\}$, defined by $F(W) = W \cap N$, is a bijective function, with inverse given by $F^{-1}(X) = \mathcal{B}_X \cup X$.

The regular closed subsets of $\Psi(\mathcal{A})$ are easily characterized by the following proposition:

Lemma 3.4. Let $\mathcal{A}$ be an almost disjoint family. Then $F \subseteq \Psi(\mathcal{A})$ is a regular closed set iff there exists $W \subseteq N$ such that $F = \text{cl}(W) = W \cup \{a \in \mathcal{A} : |a \cap W| = \omega\}$.

Proof. First, notice that given a subset $W$ of $N$, $W \subseteq int(\text{cl}(W))$, therefore $\text{cl}(W) \subseteq \text{cl}(\text{int}(\text{cl}(W)))$, concluding that $\text{cl}(\text{int}(\text{cl}(W))) = \text{cl}(W)$ since $\text{cl}(W)$ is closed. Also, it is easy to see that $\text{cl}(W) = W \cup \{a \in \mathcal{A} : |a \cap W| = \omega\}$. This proves the “if” clause.

To prove the “only if”, suppose $F$ is a regular closed set. Let $W = F \cap N$. Since $F$ is closed, $\text{cl}(W) \subseteq F$.

It remains to see that $F \subseteq \text{cl}(W)$, so fix $x \in F$.

- If $x \in F \cap N$, then $x \in W \subseteq \text{cl} W$. If $x \in \mathcal{A} \cap F$, we show that every basic nhood of $x$ intersects $W$. Let $L \in [N]^{<\omega}$ and $V = \{x\} \cup (x \setminus L)$. Since $F$ is a regular closed set, $V \cap \text{int}(F) \neq \emptyset$, so either $x \in \text{int}(F)$ or $(x \setminus L) \cap \text{int}(F) \neq \emptyset$.

- If $x \in \text{int}(F)$, then there exists a finite $L' \supset L$ such that $x \setminus L' \subseteq F$, so, by letting $a \in x \setminus L'$, we conclude $a \in V \cap F \cap N = V \cap W$.

- If $(x \setminus L) \cap \text{int}(F) \neq \emptyset$, given $a$ in this set (which is a subset of $N$), $a \in V \cap F \cap N = V \cap W$. \hfill $\Box$

Lemma 3.5. If $\Psi(\mathcal{A})$ is almost-normal, then for all $B, C \subseteq \mathcal{A}$, $B \cap C = \emptyset$, the following holds:

$B$ and $C$ are separated by open sets $\iff B$ and $C$ are separated by clopens.

Proof. Suppose that $B$ and $C$ are separated by open sets. Let $U_B$ and $U_C$ disjoint open sets such that $B \subseteq U_B$ and $C \subseteq U_C$. Then $F = cl_{\Psi(\mathcal{A})}(U_B \cap N)$ is a regular closed set and $\mathcal{A} \setminus F$ is closed.

Since $\mathcal{A}$ is almost-normal, there exists $V, W$ disjoint open subsets of $\Psi(\mathcal{A})$ such that $F \subseteq V$ and $\mathcal{A} \setminus F \subseteq W$.

Claim: $X = V \cap N$ is a partitioner for $\mathcal{A}$:

Indeed, let $a \in \mathcal{A}$, if $a \in F$, then $a \subseteq^* V \cap M = X$. On the other hand, if $a \in \mathcal{A} \setminus F$, $a \subseteq^* W \cap N$, thus $a \cap X =^* \emptyset$.

Then, by Lemma 3.3 $\mathcal{B}_X \cup X$ and its complement are the desired clopens:

- If $b \in B$, then $b \subseteq^* U_B \cap M \subseteq F \cap N \subseteq V \cap M = X$. If $c \in C$, $|c \cap U_B| < \omega$, thus $c \in \mathcal{A} \setminus F$, $c \subseteq^* W$ and it follows that $c \cap X =^* \emptyset$. \hfill $\Box$

From this lemma, it easily follows that $\Psi(\mathcal{A})$ is normal iff every two disjoint subsets $B, C \subseteq \mathcal{A}$ are separated by clopens. Now we are ready to characterize the almost-normality of Isbell-Mrówka space by using partitioners and clopens.

Theorem 3.6. If $\mathcal{A}$ is an almost disjoint family then the following are equivalent:

1. $\Psi(\mathcal{A})$ is almost-normal;
2. For each $F$ regular closed set, there exists a partitioner $X$ for $F \cap \mathcal{A}$ and $\mathcal{A} \setminus F$;
3. For each $F$ regular closed set, there exists a clopen $C$ such that $F \cap \mathcal{A} \subseteq C$ and $\mathcal{A} \setminus F \subseteq \Psi(\mathcal{A}) \setminus C$;
4. For each $F$ regular closed set, there exists a clopen $C$ such that $F \subseteq C$ and $\mathcal{A} \setminus F \subseteq \Psi(\mathcal{A}) \setminus C$;
(5) Closed sets are separated from regular closed sets by clopens.

Proof. (1) $\implies$ (3): If $F$ is a regular closed, then there exists $W \subseteq N$ such that $F = \text{cl}(W)$. Since $\Psi(A)$ is almost-normal, there exists disjoint open sets $U, V$ such that $F \subseteq U$ and $A \setminus F \subseteq V$. By Lemma 3.5 $F \cap A$ and $A \setminus F$ are separated by clopens.

(2) $\iff$ (3): This equivalence is clear by using the bijection between Clop($\Psi(A)$) and Sep($A$) from Lemma 3.3.

(3) $\implies$ (4): If $F$ is regular closed set of $\Psi(A)$, let $C$ be a clopen such that $F \cap A \subseteq C$ and $A \setminus F \subseteq \Psi(A) \setminus C$, it follows that:

$$F \subseteq C \cup ((F \cap N) \setminus C) = C \cup (F \cap N)$$

It is clear that $Y = (F \cap N) \setminus C$ is open since it is a subset of $N$. We prove that it is closed. Let $a \in \text{cl}(Y)$. If $a \in N$, then $a \in Y$ since $\{a\}$ is open. Suppose by contradiction that $a \in A$. Then $|a \cap Y| = \omega$ and it follows that $a \in \text{cl}(\Psi(A) \setminus C) = \Psi(A) \setminus C$, hence $a \subseteq^* \Psi(A) \setminus C$. On the other hand, $a \in \text{cl}(F \cap N) \subseteq F$, then $a \in F \cap A \subseteq C$, thus $a \subseteq^* C$, a contradiction.

(4) $\implies$ (5): Let $F, K \subseteq \Psi(A)$ be disjoint closed sets, where $F$ is regular closed. By (4), there exists a clopen set $C$ such that $F \subseteq C$ and $A \setminus F \subseteq \Psi(A) \setminus C$.

Let $C' = C \setminus (K \cap N)$. Clearly, $C'$ is a closed set containing $F$. $K$ is disjoint from $C'$ since $K \cap A \subseteq A \setminus F$ is disjoint from $C$. It remains to see $C'$ is open.

Since $N$ is open and discrete, $C' \cap N$ is contained in the interior of $C'$. Now suppose $a \in A \cap C'$. Since $C$ is open, there exists a finite set $L \subseteq N$ such that $a \setminus L \subseteq C$. We show that $a \cap K$ is finite, so $\{a\} \cup (a \setminus (L \cup (a \cap K)))$ is a open nbhood of $a$ contained in $C'$.

To see $a \cap K$ is finite, suppose it is infinite. Since $K$ is closed, it follows that $a \in K \cap A \subseteq A \setminus F$, so $a \notin C$, a contradiction.

(5) $\implies$ (1): Trivial. \hfill \Box

This characterization will be useful in the next section to provide an example of an almost disjoint family which is almost-normal but not normal (consistently).

4 An almost-normal family which is not normal

In this section we partially answer Question 1.2 by using iterated forcing to create a model for ZFC+CH which has an almost-normal almost disjoint family which is not normal. We will use the equivalence between (1) and (2) of Theorem 3.6 and a generalization of the notion of $Q$-set.

Given $X \subseteq 2^\omega$, the almost disjoint family over $N = 2^{<\omega}$ induced by $X$ is the family $A_X = \{A_x : x \in X\}$, were $A_x = \{x_n : n \in \omega\}$ for each $x \in X$. As in [10], we say that an uncountable $X \subseteq 2^\omega$ is a $Q$-set iff every subset of $X$ is an $F_\sigma$ of $X$. The following folklore result holds (a proof can be found in Proposition 2.2 of [10]):

Proposition 4.1 ([10]). Given an uncountable $X \subseteq 2^\omega$, $\Psi(A_X)$ is normal iff $X$ is a $Q$-set.

In what follows next we give a similar characterization for almost-normal almost disjoint families. For this purpose, we need the following:

Definition 4.2. An almost $Q$-set in $2^\omega$ is an uncountable subset $X \subseteq 2^\omega$ such that for every $W \subseteq 2^{<\omega}$, $|W|_X = \{x \in X : \forall m \in \omega \exists n \geq m (x|n \in W)\}$ (which is $\{x \in X : |A_x \cap W| = \omega\}$) is an $F_\sigma$ in $X$. 
We note that the definition of $[W]_X$ is absolute for transitive models of ZFC.

The next proof is a modification of the proof of Proposition [1] found in [10].

**Proposition 4.3.** Given an uncountable $X \subseteq 2^\omega$, $A_X$ is almost-normal iff $X$ is an almost $Q$-set.

**Proof.** ($\Rightarrow$) For $W \subseteq 2^{<\omega}$ fixed, consider the regular closed set $F = \text{cl}_{A_X}(W)$. Since $\Psi(A_X)$ is almost-normal, by Theorem 3.18 there exists a partitioner $J \subseteq 2^{<\omega}$ for $A_X \cap F$ and $A_X \setminus F$. It follows that:

$$[W]_X = \{ x \in X : |A_x \cap W| = \omega \} = \{ x \in X : A_x \subseteq F \} = \{ x \in X : A_x \subseteq \ast J \} = \bigcup_{m \in \omega} \bigcap_{n \geq m} \{ x \in X : x|_n \in J \}.$$

Hence, $X$ is an almost $Q$-set in $2^\omega$.

($\Leftarrow$) By Theorem 4.30 it suffices to show that for every regular closed set $F$, there exists a partitioner $J \subseteq 2^{<\omega}$ for $F \cap A_X$ and $A_X \setminus F$.

If $F$ is a regular closed set in $A_X$, there exists $W \subseteq 2^{<\omega}$ such that $F = \text{cl}_{A_X}(W)$. Notice that $[W]_X$ is a $G_\delta$ since:

$$[W]_X = \bigcap_{n \in \omega} \bigcup_{n \geq m} \{ x \in X : x|_n \in W \}.$$

Since $X$ is almost $Q$-set, it follows that both $[W]_X$ and $X \setminus [W]_X$ are $F_\sigma$ in $X$. Thus $[W]_X = \bigcup_{n \in \omega} F_n$ and $X \setminus [W]_X = \bigcup_{n \in \omega} G_n$, where $F_n$ and $G_n$ are closed in $X$. We proceed with a standard shoelace argument, defining $J_0 = \widehat{F}_0$, $K_0 = \widehat{G}_0 \setminus \widehat{F}_0$, $J_n = \widehat{F}_n \setminus (\bigcup_{i<n} \widehat{G}_i)$, $K_n = \widehat{G}_n \setminus (\bigcup_{i<n} \widehat{F}_i)$. Let $J = \bigcup_{n \in \omega} J_n$. It follows that $J \cap K_m = \emptyset$ for all $m \in \omega$ and we prove that $J$ is a partitioner for $F \cap A_X$ and $A_X \setminus F$.

If $A_x \subseteq F$, then $x \in [W]_X$ and there exists an $n \in \omega$ such that $x \in F_n$. Since $\bigcup_{i<n} G_i$ is closed, there exists $k \in \omega$ such that $\{ f \in 2^{<\omega} : f|_k \subseteq f \} \cap \bigcup_{i<n} G_i = \emptyset$. Hence, $A_x \subseteq \ast J_n \subseteq J$. Similarly, if $A_x \subseteq A_X \setminus F$, $A_x \cap J = \ast \emptyset$.  

Before providing the forcing example, notice that if $M, N$ are countable transitive models for ZFC and $M \subseteq N$, then for every $X, Y \subseteq 2^{<\omega}$ in $M$ with $Y \subseteq X$, $Y$ (in an $F_\sigma$ of $X$) $M \rightarrow (Y$ is an $F_\sigma$ of $X$) $N$ since countable sets of $M$ are countable sets of $N$, and since closed/open subsets of $X$ in $M$ are closed/open subsets of $X$ in $N$.

Now we are ready for the main result of this section.

**Example 4.4.** It is consistent that there exists an almost $Q$ set of cardinality $\mathfrak{c}$, so that, by the previous lemma, there exists an almost-normal almost disjoint family of size $\mathfrak{c}$ (which is not normal since it has cardinality $\mathfrak{c}$).

**Proof.** We will proceed by iterated forcing. For the forcing notation, we adopt the countable transitive approach, where $M$ is a fixed ctm for ZFC+CH.

Fix, in the ground model $M$, your favorite uncountable subset $X$ of $2^\omega$. We will construct a ccc forcing extension where CH also holds and $X$ is an almost $Q$-set.

First we study the basic step of the iteration which may be found in [9]. Given $A \subseteq X$ in $M$, let $P(A, X)$ be the sets of all finite $r \in [\omega \times (2^{<\omega} \cup A)]^{<\omega}$ such that for all $n \in \omega$, $x \in A$ and $s \in 2^{<\omega}$, if $(n, x) \in r$ and $(n, s) \in r$, then $s \not\subseteq x$. We order $P(A, X)$ by $r \leq r'$ ($r$ is stronger than $r'$) iff $r' \subseteq r$. $P(A, X)$ is $\sigma$-centered (thus, c.c.c.) since for all $r, r'$, if $r \cap (\omega \times 2^{<\omega}) = r' \cap (\omega \times 2^{<\omega})$, $r \cup r' \in P(A, X)$ is a common extension. Also, notice that in $M$, $|P(A, X)| \leq \max(|X|, \omega) = \omega_1$.

If $G$ is $P(A, X)$ generic over $M$, consider, for each $n$, the set $U_n = \{ x \in X : \forall s \in G \exists s \in 2^{<\omega} (n, s) \in r \text{ and } s \subseteq x \} \in M[G]$. Clearly, $U_n$ is an open subset of $X$. Then $A = \bigcup_{n \in \omega} X \setminus U_n$ since the sets $D_r = \{ r \in$
\(P(A, X) : \exists n \in \omega \,(n, y) \in r\) and \(E^c_\omega = \{r \in P(A, X) : \exists s \in 2^{<\omega} s \subseteq x \text{ and } (n, s) \in r\}\) for \(x \in X \setminus A, y \in A\) and \(n \in \omega\) are all dense. Thus, \(A\) is a \(F_\sigma\) of \(X\) in \(M[G]\).

Now we recursively construct, working in \(M\) a finite support \(\omega_1\)-stage iterated forcing construction \(((P_{\xi_1}, \leq_{\xi_1}, \xi_1) : \xi \leq \omega_1), ((Q_{\xi_1}, \leq_{\xi_1}, \bar{\xi}_1) : \xi < \omega_1))\). As in \([14]\), if \(\zeta, \xi \leq \omega_1, i^*_\xi\) is the usual complete embedding from \(P_{\zeta}\) to \(P_{\xi}\). Moreover, if \(i : P \rightarrow Q\) is a complete embedding between forcing posets and \(\tau\) is a \(P\)-name, \(i_\tau(\tau)\) is the \(Q\)-name recursively defined as \(\{i_\tau(\sigma), i_\tau(p)) : (\sigma, p) \in \tau\}\).

Fix a function \(f\) from \(\omega_1\) onto \(\omega_1 \times \omega_1\) such that if \(f(\zeta) \leq (\zeta, \mu)\), then \(\zeta \leq \xi\). We will use \(f\) as a bookkeeping device. Each \(\hat{Q}_\xi\) will have size \(\omega_1\) and will be forced by \(P_\xi\) to have the ccc, therefore for each \(\xi, P_\xi\) will have cardinality at most \(\omega_1\) and will have the ccc as well.

Suppose we have constructed \(((P_{\zeta}, \leq_{\zeta}, \bar{\zeta}) : \zeta \leq \xi), ((Q_{\xi}, \leq_{\xi}, \bar{\xi}) : \xi < \xi))\) for some \(\xi < \omega_1\). We must determine \((\hat{Q}_{\xi}, \leq_{\xi}, \bar{\xi}_{\xi})\). Suppose that for each stage \(\zeta < \xi\) we have also listed all \(P_{\zeta}\)-nice names for subsets of \(\hat{\omega}\) as \((\tau_{\xi, \zeta}^\mu : \mu < \omega_1)\). This is possible by CH since \(|P_{\zeta}| \leq \omega_1\) and has the countable chain condition. List all \(P_{\zeta}\)-nice names for subsets of \(\hat{\omega}\) as \((\tau_{\xi, \zeta}^\mu : \mu < \omega_1)\) as well.

Let \(f(\xi) = (\zeta, \mu)\). Since \(\zeta \leq \xi\), the name \((\tilde{\tau}_{\xi, \xi}^\zeta(\tau_{\xi, \xi}^\mu))\) is a nice \(P_{\xi}\)-name for a subset of \(\hat{\omega}\). Let \((\hat{Q}_{\xi}, \leq_{\xi}, \bar{\xi}_{\xi})\) be such that \(\hat{1}_\xi \Vdash_o (\hat{Q}_{\xi}, \leq_{\xi}, \bar{\xi}_{\xi}) \approx P\left([\tilde{\tau}_{\xi, \xi}^\zeta(\tau_{\xi, \xi}^\mu)]_X\right)\) and \(|\hat{Q}_{\xi}| \leq \omega_1\) (which is possible since \(|P(A, X)| = \omega_1\).

For instance, we may take \(\hat{Q}_{\xi}\) to be \(\omega_1\).

Let \(P = P_{\omega_1}\). \(P\) has the ccc and \(|P| = \omega_1\); therefore CH holds in any extension by \(P\) (by counting nice names). Let \(G\) be \(P\)-generic over \(M\). We claim \(X\) is an almost-\(Q\)-set in \(M[G]\). It is uncountable since \(P\) preserves cardinals. Now let \(W\) be a subset of \(\omega\) in \(M[G]\). There exists \(\zeta < \omega_1\) such that \(W \subseteq M[G_{\zeta}]\), where \(G_{\zeta} = (\tau_{\zeta}^{\omega_1})^{-1}[G]\). There exists \(\zeta < \omega_1\) such that \(W = \text{val}(\tau_{\zeta}^{\omega_1}, G_{\zeta})\). Let \(\xi\) be such that \(f(\xi) = (\zeta, \mu)\). Then, since \(W = \text{val}(\tau_{\zeta}^\mu(\tau_{\zeta}^{\omega_1}), G_{\zeta})\). Hence, by the choice of \(\hat{Q}_{\xi}\), \(M[G_{\zeta+1}]\) contains a \(P(\text{val}(W)_X, X)\)-generic filter over \(M[G_{\zeta}]\), so, in \(M[G_{\zeta+1}]\), \([W]_X\) is an \(F_\sigma\)-subset of \(X\), hence, the same happens in \(M[G]\).

Thus, it is consistent that there exists an almost-normal almost disjoint family of cardinality \(\omega\) (therefore, not normal), which gives a partial answer to Question \([1,2]\). The almost disjoint families we constructed are not MAD since no \(\mathcal{A}_X\) is a MAD family (since it can be extended by an infinite antichain of \(2^{<\omega}\)), so Question \([1,3]\) remains fully open.

In the construction we used CH only for concreteness. One could suppose (again, just for concreteness) that in the ground model we have \(2^\omega = \omega_2\), \(\omega_2 = \omega_3\), and \(X\) of size \(\omega_1\), and use \(\omega_2\) iterations to obtain a model \(M[G]\) of ZFC plus \(2^{<\omega} < 2^{\omega}\) where \(X\) becomes an almost-\(Q\)-set which is not a \(Q\)-set since there are no \(Q\)-sets under \(2^{<\omega} < 2^{\omega}\) (or, by Jones’s Lemma and Proposition \([14]\) we would have \(2^{\omega_1} \leq 2^{\text{val}(|A|)} \leq \omega\) where \(Y\) is a \(Q\)-set), so it generates an almost-normal not normal almost disjoint family of size \(\omega_1 < \omega_2 = \omega\). This yields the following corollary:

**Corollary 4.5.** The following are relatively consistent with ZFC:

1. There exists an almost-normal almost disjoint family which is not normal plus CH.
2. There exists an almost-normal almost disjoint family of size \(\omega_1 < \omega\).

## 5 Semi-normality in Isbell-Mrówka spaces

In the previous section we have constructed an almost disjoint family which is almost-normal but is not normal by using iterated forcing. We do not know if such an almost disjoint family exists in ZFC. Due to Proposition \([14]\) a semi-normal almost disjoint family \(A\) is normal iff \(A\) is semi-normal. Thus, the
study of semi-normality may come in handy when trying to address the problem of whether an not normal almost-normal almost disjoint family exist in ZFC.

Semi-normality can be translated in combinatorial terms for Isbell-Mrówka spaces. In the end, it follows that semi-normality is equivalent to a weaker form of separation, which was considered by Dow in [4] and Brendle in [3] and we state next:

**Definition 5.1.** Let $A$ an almost disjoint family (over $\omega$) and two subfamilies $B, C \subseteq A$, we say that a set $X \subseteq N$ weakly separates $B$ and $C$ if for all $b \in B$ and $c \in C$, $|X \cap b| < \omega$ and $|X \cap c| = \omega$.

We say that $A$ is weakly separated if for every $B \subseteq A$, the pair $B$ and $A \setminus B$ can be weakly separated.

Now we are ready to present the combinatorial characterization of semi-normality in Isbell-Mrówka spaces. In the end, it follows that semi-normality is equivalent to a weaker form of separation, which was considered by Dow in [4] and Brendle in [3] and we state next:

**Proposition 5.2.** Let $A$ be an almost disjoint family. The following are equivalent:

1. $\Psi(A)$ is semi-normal;
2. For each $B \subseteq A$ and each $W \subseteq N$ such that $b \subseteq^* W$ for all $b \in B$, there exists $W_0 \subseteq W$ satisfying the following:
   
   $$\text{for all } a \in A : \quad a \in B \iff a \subseteq^* W_0. \quad (5.1)$$
3. $A$ is weakly separated.

**Proof.** (1) $\implies$ (2): Fix $B \subseteq A$ and $W \subseteq N$ such that each $b \in B$, $b \subseteq^* W$. Since $B$ is closed and $B \cup W$ is open, there exists a regular open set such that $B \subseteq V \subseteq B \cup W$. We claim that (5.1) holds for $W_0 = V \cap N$:

If $b \in B$, then $b \in V$, so $b \subseteq^* W_0$. On the other hand, if $b \in A$ is such that $b \subseteq^* W_0$, then $b \in \text{cl}(W_0) = \text{cl}(V)$. Since $b \subseteq^* V$, it follows that $b \in \text{int}(\text{cl}(V)) = V \subseteq B \cup W$, hence $b \in B$.

(2) $\implies$ (3): Fix $B \subseteq A$. By hypothesis, there exists $W_0 \subseteq N$ satisfying (5.1). We claim that $X = N \setminus W_0$ weakly separates $B$ and $A \setminus B$. Indeed, if $b \in B$ then $b \subseteq^* W_0$, so $b \cap (N \setminus W_0) = b \cap X$ is finite. On the other hand, if $a \in A \setminus B$, then $a \setminus W_0 = a \cap X$ is infinite since $a \not\subseteq^* W_0$.

(3) $\implies$ (1): Let $F$ closed and $U$ open such that $F \subseteq U$, then $F = B \cup K$ where $B = F \cap A$ and $K = F \cap N$. By hypothesis, there exists $X \subseteq N$ such that for each $b \in B$, $|b \cap X| < \omega$ and for each $a \in A \setminus B$, $|a \cap X| = \omega$. Consider $W = U \cap N$ and let $V = B \cup K \cup (W \setminus X)$, then $F \subseteq V \subseteq U$. We claim that $V$ is a regular open set.

Clearly, $V$ is open. For the regularity, let $x \in \text{int}(\text{cl}(V))$. If $x \in N$, then $x \in K \cup (W \setminus X) \subseteq V$. If $x \in A$, then $x \subseteq^* \text{cl}(V)$ and it follows that $x \subseteq^* K \cup (W \setminus X)$. In the case of $x \cap K$ is infinite, $x \in F \subseteq V$ since $F$ is closed, otherwise $x \subseteq^* (W \setminus X)$ and it follows that $x \cap X$ is finite, thus $x \in B \subseteq V$. $\square$

Recall a subset $A$ of $[\omega]^\omega$ is centered iff every finite subset of $A$ has infinite intersection, and a pseudointersection of $A$ is an infinite set $X$ such that $X \subseteq^* A$ whenever $A \in A$. The pseudointersection number $\mathfrak{p}$ is defined as the least size of a centered subset of $[\omega]^\omega$ which does not admit a pseudointersection. It is well known that $\omega_1 \leq \mathfrak{p} \leq \aleph_1$ [2].

In [3], Brendle observes that $\mathfrak{p} \leq \aleph_{\mathfrak{p}}$, where $\aleph_{\mathfrak{p}}$ is defined as the smallest cardinal $\kappa$ for which there exists an almost disjoint family $A$ of size $|A| = \kappa$ that is not weakly separated. The reader can verify this inequality directly by applying the following famous classical result:

\[\text{The proof can be found in [15] Theorem 2.15} \] were it is defined a $\sigma$-centered order, so the hypothesis about $\mathcal{M}(\mathfrak{p})$ can be replaced by limiting the size of $\mathcal{C}$ and $\mathcal{D}$ by $\mathfrak{p}$.

\[\text{[4]}\]
Proposition 5.3. Given $\mathcal{C}, \mathcal{D} \subseteq \mathcal{P}(N)$ such that $\max\{|\mathcal{C}|, |\mathcal{D}|\} < p$ and for all $x \in \mathcal{D}$ and $F \in [\mathcal{C}]^{<\omega}$, $|x \setminus F| = \omega$. Then, there exists $d \subseteq N$, such that for each $x \in \mathcal{C}$, $|d \cap x| < \omega$ and for each $y \in \mathcal{D}$, $|d \cap y| = \omega$.

Corollary 5.4. If $\mathcal{A}$ is an almost disjoint family with $|\mathcal{A}| < p$, then $\mathcal{A}$ is semi-normal.

Proof. This follows from $p \leq ap$ and from Proposition 5.2.

Corollary 5.5. If $\mathcal{A}$ is an almost disjoint family with $|\mathcal{A}| < p$, then $\mathcal{A}$ is normal iff $\mathcal{A}$ is almost-normal.

This gives us (consistently) a family of uncountable almost disjoint families for which normality and almost-normality are the same. In particular, if $p > \omega_1$, then no Luzin family is almost-normal. One may ask if, consistently, every almost disjoint family is semi-normal, since if this was the case, every almost-normal almost disjoint family would be normal. However, this is false.

Proposition 5.6. If an almost disjoint family $\mathcal{A}$ is semi-normal, then $2^{|\mathcal{A}|} = c$. In particular, almost disjoint families of cardinality $c$ are not semi-normal.

Proof. If $\mathcal{A}$ is semi-normal, then $\mathcal{A}$ is weakly separated by Proposition 5.2. But then we may inject $\mathcal{P}(\mathcal{A})$ into $\mathcal{P}(N)$ by letting, for each $B \subseteq \mathcal{A}$, $X_B$ be a subset of $N$ such that, for all $a \in \mathcal{A}$, $|a \cap X_B| = \omega$ iff $a \in B$.

By Proposition 5.2, $ap$ is the least cardinality of a non semi-normal almost disjoint family. In [3], Brendle showed that $ap \leq \min\{\text{add}(\mathcal{M}), q\}$, where $q$ is defined as the least cardinality of a subset of $2^{<\omega}$ which is not a $Q$-set and $\text{add}(\mathcal{M})$ is the least cardinality of a collection of meager subsets of $\mathbb{R}$ whose union is not meager. Thus, there exists a non semi-normal almost disjoint family of size $\leq \min\{\text{add}(\mathcal{M}), q\}$. In particular, this discussion yields the following:

Corollary 5.7. If $\text{add}(\mathcal{M}) = \omega_1$, there exists a non semi-normal almost disjoint family of size $\omega_1$.

6 Generic existence of $(\aleph_0, < c)$-separated MAD families

In [20], it is defined the concept of strongly $\aleph_0$-separated almost disjoint family, which is related to almost-normality. They show that every almost-normal almost disjoint family is strongly $\aleph_0$-separated and that an almost-normal family exists under CH. Their paper does not say anything about the converse, which we are going to argue to be consistently false. In this section we modify their technique to weaken the CH hypothesis. First, we define a suitable separation concept.

Definition 6.1. We say that an almost disjoint family $\mathcal{A}$ is strongly $(\aleph_0, < c)$-separated iff for every two disjoint $\mathcal{B}, \mathcal{C} \subseteq \mathcal{A}$, with $\mathcal{B}$ countable and $|\mathcal{C}| < c$, there exists a partitioner $X \subseteq \omega$ for $\mathcal{A}$ and $\mathcal{B}$.

Clearly, every strongly $(\aleph_0, < c)$-separated almost disjoint family is strongly $\aleph_0$-separated and these concepts are equivalent under CH.

Now we recall the definitions of $b$ and $s$. If $f, g \in \omega^{<\omega}$, we say that $f <^* g$ iff the set $\{n \in \omega : f(n) \geq g(n)\}$ is finite. An unbounded family in $\omega^{<\omega}$ is a set $B \subseteq \omega^{<\omega}$ such that for every $f \in \omega^{<\omega}$ there exists $g \in B$ such that $g <^* f$. The bounding number $b$ is the smallest cardinality of an unbounded family.

We say $\mathcal{S} \subseteq \mathcal{P}(\omega)$ is a splitting family iff for every $X \in [\omega]^{\omega}$ there exists $A \in S$ such that both $X \setminus A$ and $X \cap A$ are infinite. The splitting $s$ is the least size of a splitting family.

It is well known that $p \leq s \leq c$ and that $p \leq b \leq a$, and all inequalities are consistent to be strict [2].

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Lemma 6.2. Let $A$ an almost disjoint family with $|A| < b$. If $B \subseteq A$ is a countable set, then there exists a partitioner for $B$ and $A \setminus B$. In particular, $A$ is strongly $(\aleph_0, < c)$-separated.

Proof. Let $B = \{b_n : n \in \omega \}$ list all elements of $B$. For each $a \in A \setminus B$, consider the function $f_a : \omega^\omega$ defined by $f_a(n) = \sup(a \cap b_n)$. Since $F = \{f_a : a \in A \setminus B\}$ is family of functions with $|F| < b$, there exists $g \in \omega^\omega$ such that $f_a <^* g$, for all $a \in A \setminus B$.

Let $X = \bigcup_{n \in \omega} (b_n \setminus g(n))$. We claim that $X$ is a separator for $B$ and $A \setminus B$.

Clearly, we have that $b_n \subseteq X$, for all $n \in \omega$. Given $a \in A \setminus B$, since $g >^* f_a$, there exists $k \in \omega$ such that $b_n \setminus g(n) = \emptyset$ for all $n \geq k$, thus $X \cap a =^* \emptyset$.

Corollary 6.3. If $p > \omega_1$, there exists a strongly $(\aleph_0, < c)$-separated almost disjoint family which is not almost-normal.

Proof. By Corollary 5.3 and Lemma 6.2 every non normal almost disjoint family of size $\omega_1$ is strongly $(\aleph_0, < c)$-separated and is not almost-normal. So consider any non-normal almost disjoint family of size $\omega_1$ (e.g. any Luzin family).

Given an almost disjoint family $A$. In what follows, $J^+(A) = \{X \in \omega : \{|a \in A : |a \cap X| = \omega\} \geq \omega\}$. An almost disjoint family is said to be completely separable iff for every $A \in J^+(A)$ there exists $a \in A$ such that $a \subseteq A$. Completely separable almost disjoint families exist in ZFC [7], however, we don’t know if completely separable MAD families exist in ZFC even thought we know they exist in most models [11]. A concept related to completely separability is the true cardinality $c$.

In [8] Definition 1.2, the authors introduce the definition of generic existence of a MAD family in terms of a given property $P$. More precisely, we say that MAD families with a property $P$ exist generically iff all almost disjoint families of size less than $c$ can be extended to a MAD family with the property $P$. In this sense, we have the following result:

Theorem 6.4 ($b = s = c$). Completely separable MAD families which are strongly $(\aleph_0, < c)$-separated exist generically.

Proof. Let $A'$ be an infinite almost disjoint family of size $\kappa < c$ and write $A' = \{a_\gamma : \gamma < \kappa\}$ so that $a_\gamma \neq a_\nu$ whenever $\gamma \neq \nu$.

Let $\{B_\beta : \kappa \leq \beta < c\}$ list all countable subsets of $c$ such that for each $\beta$, $B_\beta \subseteq \beta$ and, for all $B \in [\kappa]^\omega$, $|\{\beta : B_\beta = B\}| = c$ and list $[\omega]^\omega = \{Y_\alpha : \kappa \leq \alpha < c\}$.

We will define recursively almost disjoint families $A_\alpha$, $a_\alpha \in [\omega]^\omega$ and $X_\alpha \subseteq \omega$, for $\kappa \leq \alpha < c$ such that:

1. $A_\beta = \{a_\gamma : \gamma < \beta\}$;
2. $A_\kappa = A'$.
3. $\forall \beta < c : \forall \gamma \in B_\beta, a_\gamma \subseteq^* X_\beta$;
4. $\forall \beta < c : \forall \gamma \in B_\beta \setminus B_\beta, a_\gamma \cap X_\beta =^* \emptyset$;
5. $\forall \gamma < c : \forall \beta \leq \gamma, a_\beta \subseteq^* X_\beta$ or $a_\gamma \cap X_\beta =^* \emptyset$;
6. if $Y_\beta \in J^+(A_\beta)$, $a_\beta \subseteq Y_\beta$ is an infinite subset.
7. $\forall \eta < \gamma < c : a_\eta \cap a_\gamma$ is finite.
Fix \( \alpha < \gamma \) and suppose that \( X_\beta \) and \( a_\beta \) are defined for \( \kappa \leq \beta < \alpha \). Since \( B_\alpha \) is countable and \( |\alpha| < \varepsilon = b \), using Lemma 6.2 let \( X_\alpha \subseteq \omega \) be a partitioner for \( \{ a_\xi : \xi \in B_\alpha \} \) and \( \{ a_\xi : \xi \in \alpha \setminus B_\alpha \} \).

To define \( a_\alpha \), notice that since \( |\alpha| < \varepsilon \) there exists an infinite \( Y \subseteq \omega \) almost disjoint from \( a_\beta \), for all \( \beta < \alpha \). In addition, if \( Y_\alpha \in J^+(A_\alpha) \), we can take \( Y \subseteq Y_\alpha \):

Indeed, if \( \{ \gamma < \alpha : |a_\gamma \cap Y_\alpha| = \omega \} \) is finite, take \( Y = Y_\alpha \setminus \bigcup \{ a_\gamma : \gamma < \alpha \wedge |a_\gamma \cap Y_\alpha| = \omega \} \). Otherwise, note that \( \mathcal{B} = \{ a_\gamma \cap Y_\alpha : \gamma < \alpha \wedge |a_\gamma \cap Y_\alpha| = \omega \} \) is an almost disjoint family in \( Y_\alpha \). Since \( |\mathcal{B}| < B \), there exists \( Y \subseteq Y_\alpha \) almost disjoint from each element of \( \mathcal{B} \).

Since \( s = \varepsilon \), \( \{ X_\gamma \cap Y : \gamma \leq \alpha \} \) is not a splitting family in \( Y \). Thus, there exists \( a_\alpha \subseteq Y \) such that for all \( \gamma \leq \alpha \), \( a_\alpha \cap X_\gamma =^* \emptyset \) or \( a_\alpha \subseteq^* X_\gamma \).

Notice that \( A \) is an almost disjoint family extending \( A' \) by (2) and (7).

We show that for every infinite \( Y \subseteq \omega \), either \( Y \notin J^+(A) \) or there exists \( a_\alpha \subseteq Y \), thus proving \( A \) is MAD and completely separable. If \( Y \in J^+(A) \), let \( \alpha \) be such that \( Y = Y_\alpha \). Then \( Y \in J^+(A_\alpha) \), thus, by 6, \( a_\alpha \subseteq Y_\alpha \).

Finally, we prove that \( A \) is \( (\aleph_0, < \varepsilon) \)-separated. Given an infinite countable set \( \mathcal{B} \subseteq A \) and \( \mathcal{C} \in [A]^{< \varepsilon} \) such that \( \mathcal{B} \cap \mathcal{C} = \emptyset \), let \( B = \{ a_\gamma : a_\gamma \in \mathcal{B} \} \) and \( C = \{ a_\gamma : a_\gamma \in \mathcal{C} \} \). Notice that \( B \) is infinite and countable, \( |\mathcal{C}| < \varepsilon \), and \( B \cap C = \emptyset \). Let \( a_0 = \sup C \), which is less than \( \varepsilon \) since \( \varepsilon = b \) is regular, and let \( \alpha > a_0 \) be such that \( B_\alpha = B \). In particular, \( B = B_\alpha \subseteq \alpha \) and \( C \subseteq \alpha \setminus B_\alpha \). By (3), for all \( b \in B, b \subseteq^* X_\alpha \).

By (4), for all \( c \in \mathcal{C}, c \cap X_\alpha =^* \emptyset \).

By (3) and (4) together by using \( \alpha \) in the place of \( \beta \), we see that for every \( \gamma < \alpha \), \( a_\gamma \subseteq^* X_\alpha \) or \( a_\gamma \cap X_\alpha =^* \emptyset \). If \( \gamma > \alpha \), we apply (5) for this \( \gamma \) and \( \alpha \) in the place of \( \beta \) to conclude that \( a_\gamma \subseteq^* X_\alpha \) or \( a_\gamma \cap X_\alpha =^* \emptyset \).

It is well known that the cardinal characteristic \( \text{par} \), as defined in [2], equals the minimum of \( b, s \). Thus, the previous theorem could have its hypothesis replaced by \( \text{par} = \varepsilon \).

## 7 Conclusion

We have answered Question 4.4 of [20] by providing a counter example in \( \beta \omega \) and partially answered Question 4.2 of [20] by providing an example by using forcing. We have shown that an almost disjoint family is semi-normal iff it is weakly separated, thus, for weakly separated almost disjoint families normality and almost-normality are the same. However, Question 4.2 remains open. We may define \( \text{an} \) as the least cardinality of an almost-normal almost disjoint family which is not normal. We don’t know if this number is well defined in ZFC, however, if there is such an almost disjoint family, it follows that \( \text{ap} \leq \text{an} \). We may refine Question 4.2. as follows:

**Question 7.1.** Is \( \text{an} \) well defined in ZFC? If there is an almost-normal not normal almost disjoint family, does \( \text{an} = \text{ap} \) hold?

Recall that in [20] it was proven that almost-normal almost disjoint families are strongly \( \aleph_0 \)-separated. Here we have defined the concept of strongly \( (\aleph_0, < \varepsilon) \)-almost disjoint families and we have proved that strongly \( (\aleph_0, < \varepsilon) \)-separation property does not hold for all almost-normal almost disjoint families, at least consistently. However, the relation between these concepts is not fully understood. Thus, we ask:

**Question 7.2.** Are almost-normal almost disjoint families strongly \( (\aleph_0, < \varepsilon) \)-separated?

**Question 7.3.** Does CH imply that strongly \( \aleph_0 \)-separated almost disjoint families are almost-normal?
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