The Edge-State Theory of Integer-Quantum-Hall-Effect to Insulator Transition

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Direct transitions, driven by disorder, from several integral quantum Hall states to an insulator have been observed in experiment. This finding is enigmatic in light of a theoretical phase diagram, based on rather general considerations, that predicts a sequence of transitions in which the integer \(n\) characterizing the Hall conductivity is reduced successively by unity, eventually going from \(n = 1\) into an insulator. In this work, we suggest that the direct transition occurs because, in certain parameter regime, the edge states of different Landau levels are strongly coupled and behave as a single edge state. It is indicated under what conditions successive transitions may be seen.

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The conventional scaling theory of localization \(^[1]\) predicts that all electrons in a two-dimensional system are localized in the absence of a magnetic field. In the presence of a strong magnetic field, the energy spectrum becomes a series of disorder broadened Landau bands. The phenomenon of the integral quantum Hall effect (IQHE) indicates the existence of extended states at the center of each Landau band, separated by localized states at other energies. The integrally quantized plateaus are observed when the Fermi level lies in the localized states, with the value of the Hall resistance, \(R_H = h/ne^2\), related to the number of extended bands \((n)\) below the Fermi energy. As a function of the magnetic field, the Hall resistance jumps from one quantized value to another when the Fermi energy crosses an extended-state level.

In order to reconcile the presence of extended states at finite magnetic fields with the lack thereof at zero magnetic field, Khmelnitskii \(^[2]\) and Laughlin \(^[3]\) came up with a picture in which the extended states associated with Landau levels levitate due to disorder, eventually pushed to very high energies, leaving an Anderson insulator behind. Based on this physics, Kivelson, Lee and Zhang \(^[4]\) (KLZ) proposed a topological phase diagram as shown in Fig. 1(a). A crucial prediction of this phase diagram is that an IQHE state \(n\) in general can only go into another IQHE states \(n \pm 1\), and that a transition into an insulating state is allowed only from the \(n = 1\) state.

This phase diagram motivated numerous theoretical and experimental studies. \(^[5,6,7,8,9,10,11,12,13]\). However, direct transitions from \(n = 1, n = 2, n = 3,\) and \(n = 6\) IQHE states to the insulator have been observed in recent experiments \(^[2,3,4,5,6,7,8,9,10,11,12,13]\). The transition from \(n = 2\) IQHE state to the Anderson insulator may perhaps be related to the spin degeneracy in the lowest Landau level \(^[4]\), but the other transitions are inconsistent with the KLZ phase diagram.

Recently, Liu, Xie and Niu \(^[5]\) studied numerically a tight-binding model of two-dimensional electrons in a magnetic field and a random potential. They found that the extended levels do not float up to infinity but are instead destroyed by strong disorder at a critical magnetic field. They calculated phase diagrams in the energy-field and energy-disorder planes, which may be combined to yield a non-nested phase diagram shown in Fig. 1(b).
(solid line). In this phase diagram transitions from any IQHE state to the insulating phase are allowed when the disorder is increased at fixed B. The non-nested phase diagram of Fig. 1(b) was confirmed by recent calculations with an important modification on the lowest Landau level $\mathbb{E}$ (dashed line), which is now in excellent agreement with experiments [12,13].

Thus far, none of the theoretical calculations in this context includes explicitly the edge states, whose consideration is crucial for transport in the QHE regime for the following reasons. First, as clarified by the Landauer approach, and confirmed in numerous experiments, the Hall current is carried to a large extend by the edge states. On the QHE plateaus, backscattering is exponentially suppressed due to a spatial separation of edge states carrying current in opposite directions, which is responsible for the spectacular accuracy of the quantization of the Hall resistance. A non-zero longitudinal resistance owes its origin to an inter-edge backscattering. An insulator is obtained when the edge channels are fully reflected backwards, with the forward transmission coefficient becoming exponentially small. Thus, an IQHE to insulator transition occurs when the edge channels shut off. Furthermore, Landau level mixing due to disorder, relevant for the issue at hand, is expected to be particularly strong at the edges, because here the states from different LLs are close not only in space but also in energy. The goal of the present work is to investigate the mixing of edge states on the same edge of the sample relative to the mixing of edge states on opposite edges, responsible for inter-edge scattering.

We will work within a continuum model. A numerical study of a two-dimensional continuum model is difficult due to the limitation in computational power. Therefore, we study a narrow strip and ask: How does a quantum Hall state evolves as the magnetic field decreases or as disorder increases?

To be specific, we consider the system shown in Fig. 2, in which there are two edge states at each edge of the sample. The states on a given edge of the sample are “chiral”, i.e., carry current only in one direction, determined by the $\vec{E} \times \vec{B}$ drift, with the electric field provided by the confinement potential. The edge states carry current in opposite directions on the opposite sides of the sample. Each edge carries a current of $\frac{e\hbar}{2e}$ per unit energy. The disorder causes couplings between the various edge states. The strength of the coupling depends on the overlap of the states, which increases with disorder. Coupling of the edge states on the same side by disorder gives only forward-scattering, which does not degrade the source-to-drain current. The only effect of such a scattering is an unimportant forward-scattering phase shift [10]. The current can be diminished only by scattering between two edge states on the opposite sides – namely, backscattering. When the reflection coefficient becomes unity, the edge states shut off, no current flows through, and an insulator is obtained.

Clearly, an understanding the transition from QHE states to the insulator is intimately related to the evolution of the edge states as a function of disorder. Two different scenarios are possible. (i) The edge channels shut off one by one. This would correspond to the nested KLZ phase diagram. (ii) The edge channels shut off all at once. This would correspond to the non-nested phase diagram of Fig. 1(b). Which scenario occurs depends on the relative strengths of the couplings of edge channels on the same side and of edge channels on the opposite sides. The former scenario is likely when the coupling between the edges states on the same side is weak, whereas the latter scenario is expected to occur when the coupling between the edge states on the same side is strong. This is the physical picture that we propose as an explanation of the direct transition between high integer IQHE states and an insulator. In the remainder of the paper, we confirm this through an explicit calculation.

![FIG. 2. Two edge channels in a sample. The channels on the top carry current from right to left, and those at the bottom carry the current in the opposite direction, as indicated by arrows. The backscattering is indicated by dashed lines.](image)

In Fig. 2, edge states 1 and 2 at bottom carry current from left to right, and edge states 1 and 2 on the top carry current in the opposite direction. Electrons can move from one side of the system to the other side through the four channels provided by the two edge states. Disectors may scatter electrons from one channel into another. Assume that $A_1$, $A_2$ and $B_1'$, $B_2'$ are the amplitudes of wavefunction in edge states 1 and 2 going into the sample from the left-hand and right-hand sides of the system, respectively, and similarly, that $B_1$, $B_2$ and $A_1'$, $A_2'$ are the amplitudes of wavefunction in edge states 1 and 2 coming out of the sample from the left-hand and right-hand sides of the system, respectively. These amplitudes are related by an S-matrix

$$
\begin{pmatrix}
B_1 \\
B_2 \\
A_2' \\
A_1'
\end{pmatrix}
= S
\begin{pmatrix}
B_1' \\
B_2' \\
A_2 \\
A_1
\end{pmatrix},
$$

or an M-matrix

$$
\begin{pmatrix}
B_1 \\
B_2 \\
A_2' \\
A_1'
\end{pmatrix}
= M
\begin{pmatrix}
B_1' \\
B_2' \\
A_2 \\
A_1
\end{pmatrix},
$$
The localization lengths of the two channels can be calculated by using the multiplicative property of the $M$-matrix. We first divide the system into many vertical slices. Each slice contains only very small number of disorders so that the partial $S$ matrix for each slice is close to the unit matrix. The backscattering is expected to be the strongest for the innermost edge channel. For simplicity, we assume that backward-scattering occurs only through edge channel 2, i.e. set $r_{11} = 0 = r'_{11}$. We further assume that randomness comes from the phase of each scattering process rather than its amplitude. Under these assumptions, the general unitary matrix $S$ becomes

$$S = \begin{pmatrix}
\sqrt{1-t^2} & te^{-i\phi_1} & 0 & 0 \\
-te^{-i\phi_1} & \sqrt{1-t^2} & 0 & 0 \\
0 & 0 & -te^{-i\phi_2} & \sqrt{1-t^2} \\
0 & 0 & \sqrt{1-r^2} & re^{-i\phi_3} \\
0 & 0 & 0 & \sqrt{1-r^2} \\
0 & 0 & 0 & 1
\end{pmatrix} \times \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \sqrt{1-r^2} & re^{-i\phi_3} & 0 \\
0 & 0 & \sqrt{1-r^2} & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},$$

where $r$ measures the strength of backward-scattering in channel 2, and $t$ is the strength of forward-scattering from channel 1 to channel 2, and vice versa. The quantities $t$ and $r$ are chosen to be small in order to ensure that scattering in each slice is weak. Throughout this study, we set $r = 0.0002$ and vary $t$. The ratio $\alpha = t/r$ measures the relative strength of backscattering and forward-scattering. $\phi_1$, $\phi_2$, and $\phi_3$ are random numbers. The random matrix determines the $S$ matrix of the system with disordered thin slice, which, in turn, gives a random $M$ matrix. The product of $L$ independent random matrices of this type is used to calculate localization lengths of edge states [7]. The numerical result of $\xi_2/\xi_1$ vs $\ln \alpha$ is plotted in Fig. 3.

The principal result is that $\xi_2$ changes from 0 to 1 when $\ln \alpha$ increases from -1 to 4. $\xi_2/\xi_1 = 0$ means that channel 2 shuts off when channel 1 is still propagating from source to drain (or vice versa). This confirms that for small $\alpha$, the edge channels are blocked off one by one through backscattering. In this case, transitions of IQHE states to the insulator should be governed by the phase diagram of KLZ. On the other hand, for large $\alpha$, both edge channels shut off at the same time. It is remarkable that this happens even though the backscattering occurs in our model only in channel 2. This implies a direct transition from an arbitrary IQHE state to an insulator, consistent with the non-nested phase diagram of Fig. 1(b).

We would like to make several remarks. 1) In our approach, we only consider phase randomness due to disorder scattering. As usual, the amplitude randomness is not expected change the results. The structure of random matrix $M$ is the same whether only phase randomness or both phase and amplitude randomness are taken.

![FIG. 3. $\xi_2/\xi_1$ vs $\ln \alpha$. $\xi_1$ and $\xi_2$ are the localization lengths of channels 1 and 2, respectively. $\alpha$ is the ratio of the backward scattering strength of channel 2 and forward scattering between channels 1 and 2.](image-url)
into account. 2) The parameter $\alpha$ depends on the disorder potential, $V(r)$, as well as the sample width, $W$. Both forward and backward scatterings are proportional to the matrix element of the disorder potential between the two states involved, $\langle \Psi_1 | V | \Psi_2 \rangle$. When $\Psi_1$ and $\Psi_2$ are the edge states on the same side of a sample, the matrix element, which characterizes the forward scattering, is proportional to $\exp(-d^2/l_B^2)$, where the distance $d$ between the centers of the two edge states is on the order of the magnetic length, $l_B = \sqrt{\hbar/eB}$. The gaussian expression originates from the magnetic confinement on the Landau level wavefunction. On the other hand, when $\Psi_1$ and $\Psi_2$ correspond to the edge states on the opposite sides of the sample, the matrix element for backscattering is of order $\exp(-W/\xi_L)$, where $\xi_L$ is a disorder dependent length characterizing the transverse extent of the edge state. (Even though the electronic wave function of a perfect system in a magnetic field is a gaussian, it has an exponential tail in the presence of disorder.) This exponential suppression of backscattering as a function of the width suggests that sufficiently wide samples are in the $\ln \alpha > 4$ regime, governed by the non-nested phase diagram of Fig. 1b. Narrow samples, however, may be in the opposite regime. The actual width at which the crossover occurs depends on the disorder potential, but may be expected to be on the order of several magnetic lengths.

According to the picture presented here, a direct transition from an arbitrary IQHE state to an insulating state is intimately connected to the Landau level mixing at the edges of the sample. At first sight, this physics appears to bear no relation to the direct transitions seen in earlier numerical studies, because these studies employed periodic boundary conditions, i.e., the numerical samples had no edges due to a geometrical confinement. However, we believe that the underlying physics may be identical. Even though there are no real edges, internal edges, corresponding to equipotential contours, appear in the presence of disorder. When the disorder is sufficiently strong, the internal edges carry, in general, edge channels from several LL’s. The IQHE to insulator transition is a percolation transition in this picture, which has to do with the coupling between the network of internal edges. Again, if the channels from different LL’s are strongly coupled and behave effectively as a single edge channel, there would be a direct transition from IQHE to insulator.

In summary, we have investigated the role of edge states in disorder driven IQHE to insulator transition. We find that a direct transition from a general IQHE state to an insulator is possible when the edge channels of different Landau levels are strongly coupled.

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