Boundary dynamics driven entanglement

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Abstract

We will show how to generate entangled states out of unentangled states on a bipartite system by means of dynamical boundary conditions. The auxiliary system is defined by a symmetric but not self-adjoint Hamiltonian, and we will also study the space of self-adjoint extensions of the bipartite system. We will show that only a small set of these extensions leads to separable dynamics, and we will characterize these extensions. Various simple examples illustrating this phenomenon are discussed; in particular, we will analyze the hybrid system consisting of a planar quantum rotor and a spin system under a wide class of boundary conditions.

Keywords: boundary control, boundary generated entanglement, non-separable boundary conditions, self-adjoint extensions

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(Some figures may appear in colour only in the online journal)

1. Introduction

There is an increasing interest in the physics associated with the ‘boundary’ of a given physical system. Because the boundary can be regarded as an effective way of describing the
interaction of the system with the external universe, its modeling could account for a number of significant physical phenomena.

It is impossible to cover the range of physics associated with boundary structures in a few sentences. We mention Casimir’s effect, which is arguably one of the most conspicuous physical phenomena associated with the presence of boundaries (see, for instance, [4–6] and references therein for an extensive account of the role of boundary conditions and vacuum structures), the quantum Hall effect [13], and the appearance of edge states [7]. We would also like to mention the possibility of describing topology change as a boundary effect. This idea was already considered in [8] and further elaborated in relation to specific boundary conditions in [3], but it has gained new impetus because of the recent contributions of Wilczek et al [18] to it.

In this paper, we will explore how the manipulation of boundary conditions of composite systems allows the generation of entangled states. More precisely, consider two systems, A and B, and assume that system B, which will be called the ‘bulk’ or controlled system, is complete; that is, its Hamiltonian, \( H_B \), is a Hermitean (self-adjoint) operator on a Hilbert space, \( \mathcal{H}_B \), and its evolution, \( U^B_t = \exp (i t H_B) \), is unitary. However, system A, which will be called the ‘auxiliary system,’ is defined by a merely symmetric operator, \( H_A \), on a Hilbert space, \( \mathcal{H}_A \). In other words, the evolution, \( U^A_t = \exp (i t H_A) \), will not be unitary until we have selected a self-adjoint extension of the operator \( H_A \), if it exists. It is worth pointing out that such situations will actually arise whenever system A is defined in a bounded domain \( \Omega_A \) in \( \mathbb{R}^n \) with boundary \( \partial \Omega_A \). In such cases, the Hilbert space \( \mathcal{H}_A \) is the space \( \mathcal{L}^2(\Omega) \) of square integrable complex-valued functions on \( \Omega_A \), and the Hamiltonian operator is

\[
H_A = -\frac{\hbar^2}{2m}\Delta_\eta + V,
\]

with \( \Delta_\eta \), the Laplace-Beltrami operator, defined by some metric \( \eta \) on \( \Omega_A \), and \( V \) as a potential function. Under such circumstances, it can be shown that the self-adjoint extensions of \( H_A \) are determined by boundary conditions satisfied by the functions on the corresponding domain [3].

The main, relevant observation for the purposes of this paper is that, if we consider the bipartite system defined on the Hilbert space, \( \mathcal{H}_A \otimes \mathcal{H}_B \), the family of self-adjoint extensions of the symmetric Hamiltonian, \( H = H_A \otimes 1 + I \otimes H_B \), is much larger than the family of self-adjoint extensions of the stand-alone symmetric Hamiltonian \( H_A \). As we will discussed throughout the paper, many of the possible self-adjoint extensions of the bipartite system generate entangled states out of separable states. In other words, the dynamics defined by many self-adjoint extensions of the composite system are not separable (i.e., they do not preserve separable states).

Separable dynamics for a class of hybrid composite systems will be characterized, and it will be shown that they correspond to boundary conditions defined by the tensor product of the operator defining the boundary conditions determining a self-adjoint extension of system A times the identity operator (see theorem 2). Thus, self-adjoint extensions corresponding to boundary conditions with a different structure will define non separable dynamics, and separable states will evolve into non separable ones. We will call this source of entangled states the boundary driven entanglement.

It will be illustrated, using a simple example, how by choosing a non trivial tensor product extension of a given self-adjoint extension of system A, we obtain non separable dynamics (see section 5). We will also show how by modifying the chosen self-adjoint extension, we can generate entangled states not only between auxiliary system A and system B, but even within system B itself (as long as it is a composite system).
Such instances will be discussed first by using an example consisting of the free particle moving on the half-line as auxiliary an system, and a two-level system as a bulk system. In this particular instance, we will shown that the ground state of the half-line (its only eigenstate) becomes entangled with the eigenstates of the bulk system, and how such an entangled state can be driven by modifying the boundary conditions compatible with this scenario. Finally, we will discuss a ‘quantum compass’ which is a planar rotor with a spin 1/2 system inside it. Now, two families of non trivial boundary conditions for this system, extending quasi-periodic boundary conditions for the planar rotor trivial boundary conditions [2], will be considered and their spectral properties will be discussed (see section 6).

2. Boundary conditions and self-adjoint extensions

We will start by briefly reviewing the most salient aspects of the relation ship between self-adjoint extensions and boundary conditions, using the Laplace-Beltrami operator as an illuminating example.

Given a symmetric operator, $T$, on a Hilbert space, $\mathcal{H}$, the operator $T$ has dense domain $D_0 \subset \mathcal{H}$ and $T \subset T^\dagger$; we may use von Neumann’s theorem [14] to describe all its self-adjoint extensions, if any. (See, for instance. [15] for an exhaustive account of the theory.) Namely, we first compute its deficiency spaces, $\mathcal{N}_\pm := \ker(T^\dagger \pm i\mathbb{I}) = \text{Ran}(T \pm i\mathbb{I})^\perp$. Then there is a one-to-one correspondence between self-adjoint extensions of $T$ and unitary operators $K: \mathcal{N}_+ \rightarrow \mathcal{N}_-$. Von Neumann’s theorem establishes that for any such unitary operator, $K$, one can associate the self-adjoint operator, $T_K$, with domain

$$D_K = D_0 \oplus (I + K)\mathcal{N}_+$$

and define by

$$T_K(\Phi_0 \oplus (I + K)\xi_+) = T\Phi_0 \oplus i(I - K)\xi_+, \quad \forall \Phi_0 \in D_0, \quad \xi_+ \in \mathcal{N}_+.$$

In many cases, the operator $T$ is a differential operator on a manifold, $\Omega$, with non empty boundary, $\partial \Omega$. Let us consider, as an example, a free particle moving on a curved manifold, $\Omega$, with Riemannian metric $\eta$. In this case, the Hamiltonian describing the geodesic motion is the negative Laplace-Beltrami operator, $-\Delta_\eta$, (i.e., we are in the situation of equation (1) with $V \equiv 0$), that in local coordinates $x^i, i = 1, \ldots, n = \dim \Omega$, takes the explicit form

$$\Delta_\eta = \frac{1}{|\eta|} \frac{\partial}{\partial x^i} |\eta|^{\frac{1}{2}} \eta^{ij} \frac{\partial}{\partial x^j},$$

with $|\eta| = \det(\eta_{ij})$. It is natural to start by defining this operator in the domain $C^\infty_c(\text{Int}(\Omega))$ (i.e., in the set of complex-valued functions with compact support contained in the interior of $\Omega$, which is a dense subspace of the Hilbert space, $H = L^2(\Omega)$, the space of square integrable functions with respect to the Riemannian volume defined by $\eta$). A simple integration by parts leads to

$$\langle \Phi, \Delta_\eta \Psi \rangle = \langle \Delta_\eta \Phi, \Psi \rangle \quad \forall \Phi, \Psi \in C^\infty_c(\text{Int}(\Omega)).$$

This shows that the operator, $\Delta_\eta$, defined in the previous domain is symmetric. The minimal closed extension of the operator $\Delta_\eta$ is defined on the domain $D_0 = H^2(\Omega)$, which is the closure of $C^\infty_c(\text{Int}(\Omega))$ with respect to the Sobolev norm, $\| \cdot \|_{2,2}$. The domain, $D_0$, is just the Sobolev space of order 2 with functions that vanish at the boundary, and such that their normal derivatives vanish, too.
The adjoint operator, $\Delta_\eta^*$, is the operator defined in the domain $D_0 = \{ \Phi \in L^2(\Omega) | \Delta_\eta \Phi \in L^2(\Omega) \}$. This operator is actually the maximal extension of $\Delta_\eta$, and certainly $\Delta_\eta \subset \Delta_\eta^*$.

A general result for operators commuting with conjugations shows the existence of self-adjoint extensions for $\Delta_\eta$, hence the existence of unitary operators $K : \mathcal{N}_+ \to \mathcal{N}_-$ and the applicability of Neumann’s theorem, as seen in equations (2) and (3).

Alternatively, we may argue as follows (see, for instance, [3] and references therein). Consider the restriction on the boundary $\partial \Omega$ of functions in $D_0^\eta$. Such restrictions will be denoted by $\varphi : \Phi \mid_{\partial \Omega}$. In the same way, we define the normal derivative, $\phi : = \partial \Phi / \partial \nu \mid_{\partial \Omega}$, as the outbound normal derivative along the boundary. We will consider that both $\varphi$, $\phi$ are in $L^2(\partial \Omega)$. Repeating the integration by parts for elements $\Phi, \Psi \in D_0^\eta$ we will obtain

$$\langle \Phi, \Delta_\eta \Psi \rangle - \langle \Delta_\eta \Phi, \Psi \rangle = \langle \varphi, \psi \rangle - \langle \psi, \varphi \rangle.$$ (6)

The inner product in the right-hand side (rhs) of the expression above is the one defined in $L^2(\partial \Omega)$: $\langle \varphi, \psi \rangle = \int_{\partial \Omega} \bar{\varphi}(x) \psi(x) d\mu_{\partial \Omega}(x)$, where $\mu_{\partial \Omega}$ is the measure associated to the Riemannian metric induced at the boundary $\partial \Omega$ by $\eta$.

Clearly, self-adjoint extensions of $\Delta_\eta$ will be determined by maximal subspaces of functions $\Phi$ in $D_0^\eta$ such that the bilinear form given by the rhs of equation (6) vanishes identically for the corresponding boundary values, $\varphi$ and $\phi$, of $\Phi$.

Such maximally isotropic spaces, $W$, of boundary values can be easily characterized by computing their Cayley transform; that is, we consider the linear isomorphism $C : L^2(\partial \Omega) \oplus L^2(\partial \Omega) \to L^2(\partial \Omega) \oplus L^2(\partial \Omega)$ defined by

$$C(\varphi, \phi) = \frac{1}{\sqrt{2}}(\varphi - i\phi, \varphi + i\phi).$$

The Cayley transform, $C$, maps a maximally isotropic subspace, $W$, onto the graph of a unitary operator, $U : L^2(\partial \Omega) \to L^2(\partial \Omega)$. More explicitly, $(\varphi, \phi) \in W$ iff there exists $U \in U(L^2(\partial \Omega))$ such that

$$\varphi - i\phi = U(\varphi + i\phi).$$ (7)

In this sense, the space of self-adjoint extensions of the Laplace-Beltrami operator can be naturally identified with the unitary group of the Hilbert space of square integrable functions at the boundary of $\Omega$. Equation (7) provides the explicit description of the corresponding domains. We will make extensive use of this characterization in the rest of this paper. Unfortunately, this description is complete only for one-dimensional Riemannian manifolds. The group of unitary operators acting on the Hilbert space of the boundary is in one-to-one correspondence with the space of maximally isotropic subspaces of the boundary form via the graph of the operator, $U$, as in equation (7), only if the Riemannian manifold is one-dimensional. In higher dimensions, one needs to assume some regularity conditions on $U$ in order for it to define a proper self-adjoint extension. This means that not every unitary $U$ defines a self-adjoint extension, because the space of boundary data is not the full Hilbert space of the boundary, $L^2(\partial \Omega)$, but a dense subspace, as characterized by the Lions’s trace theorem, cf [12]. In the one-dimensional case, the boundary is a set of points and the Hilbert space of the boundary is a finite dimensional Hilbert space. Hence, the previous difficulties do not arise. Rigorous characterization of unitaries that lead to self-adjoint extensions would, in our opinion, deflect attention from the true points of interest in this article. For more details on this characterization, we refer to [10].
3. Self-adjoint extensions of symmetric bipartite systems

Let us consider the case of a bipartite system, \( A \times B \), such that one of its subsystems is described by a symmetric operator. In particular, we consider system \( A \) to be defined, as in the previous section, by minus the Laplace-Beltrami operator on a Riemannian manifold \((\Omega_A, \eta_A)\); that is, \( A \) describes a free system on a manifold with a boundary. System \( B \) is defined by a self-adjoint operator, \( H_B \), on a Hilbert space, \( H_B \), with the dense domain \( \text{dom}(H_B) = D_B \). The Hilbert space, \( H_{AB} \), of pure states of the composite system is

\[
H_{AB} := H_A \otimes H_B = \mathcal{L}^2(\Omega_A) \otimes H_B \cong \mathcal{L}^2(\Omega_A; H_B).
\]

The last isomorphism can be defined explicitly on separable states as

\[
\mathcal{L}^2(\Omega_A) \otimes H_B \rightarrow \mathcal{L}^2(\Omega_A; H_B)
\]

\[
\phi \otimes \rho \rightarrow \phi_A(x) = \phi(x)\rho,
\]

where the multiplication on the rhs is just pointwise multiplication. It can then be extended by linearity to the full tensor product. Hence, pure states will be considered as square integrable maps, \( \Phi: \Omega_A \rightarrow H_B \), with inner product

\[
\langle \phi , \psi \rangle_{AB} = \int_{\Omega_A} \langle \phi(x) , \psi(x) \rangle_{H_B} \, d\mu_A(x).
\]

In what follows, we will use the latter identification when appropriate. The Hamiltonian operator of the composite system that we will consider is

\[
H = -\Delta_\eta \otimes 1 + 1 \otimes H_B
\]

acting on states \( \Phi \) as

\[
H\Phi = -\Delta_\eta \Phi + H_B \cdot \Phi,
\]

with \( \langle H_B \cdot \phi \rangle(x) = H_B(\phi(x)), x \in \Omega_A. \)

The natural symmetric domain, \( D_0 \), of the operator \( H \) is now \( D_0 = D_{A0} \otimes D_B \), where we now borrow the notation \( D_{A0} \) from section 2 to denote the minimal closed extension of the Laplace-Beltrami operator, defined on \( \Omega_A \).

Again, \( D_0 \) can be identified in a natural way with \( \mathcal{L}^2(\Omega_A; H_B) \), where the completion is taken with respect to the Sobolev norm of order 2. Notice that we could not previously consider the completion \( \otimes \) in the definition of \( D_0 \) because, since \( D_{A0} \) and \( D_B \) are dense, it would result that \( D_{A0} \otimes D_B = H_A \otimes H_B = H_{AB} \), but the operator \( H \) is not bounded.

The maximal extension of the operator, \( H \), is given by \( D_{A0} \otimes D_B \) using the notation of section 2 again. Notice that \( H_B \) is self-adjoint already, and then \( D_{A0} = D_B \). Computing the self-adjoint extensions of \( H \) is best done by using its boundary data structure (i.e., Green’s formula), as in the second part of section 2. In fact, by integrating by parts, we get the analogue of equation (6):

\[
\left\{ \phi , -\Delta_\eta \psi + H_B \cdot \psi \right\} - \left\{ -\Delta_\eta \phi + H_B \cdot \phi , \psi \right\} = \langle \phi , \psi \rangle - \langle \phi , \psi \rangle,
\]

where the inner product at the boundary appearing in the rhs of the previous equation is given simply by

\[
\langle \phi , \psi \rangle = \int_{\partial\Omega_A} \langle \phi(x) , \psi(x) \rangle_{H_B} \, d\mu_{\partial\Omega_A}(x),
\]

and \( \phi, \psi \) are defined as before. Then, \( \phi, \psi \) can be identified with functions on \( \partial\Omega \) with values in \( H_B \), and the space of boundary data is now \( \mathcal{L}^2(\partial\Omega_A; H_B) \equiv \mathcal{L}^2(\partial\Omega_A) \otimes H_B \),
Repeating the argument leading to equation (7), we obtain that the space of self-adjoint extensions of $H$ (i.e., the space of maximally isotropic, closed subspaces of the bilinear boundary form defined by the rhs of equation (11), is parametrized by unitary operators,

$$U: \mathcal{L}^2(\partial \Omega_A) \otimes H_B \to \mathcal{L}^2(\partial \Omega_A) \otimes H_B.$$  \hfill (13)

Thus, given a unitary operator as in equation (13), the domain, $D_U$, of the corresponding self-adjoint extension will consist of all functions, $\phi \in \mathcal{L}^2(\partial \Omega_A)$, such that

$$\phi - i\phi = U(\phi + i\phi), \quad \phi, \phi \in \mathcal{L}^2(\partial \Omega_A) \otimes H_B.$$  \hfill (14)

A similar result, but in a much more general situation, can certainly be obtained by using von Neumann’s theorem. Now, $H_A$ and $H_B$ are general, complex, separable Hilbert spaces and $H_A$, $H_B$ are operators on them.

**Theorem 1.** Let $H_A$ be a densely defined, symmetric operator on the Hilbert space, $H_A$, and let $H_B$ be a bounded, self-adjoint operator on a Hilbert space, $H_B$, with discrete spectrum; then the deficiency spaces, $\mathcal{N}_\pm$, of the symmetric operator $H = H_A \otimes 1 + 1 \otimes H_B$ are isomorphic to $\mathcal{N}_A \otimes H_B$.

**Proof.** Let us assume for simplicity that $H_B$ has a nondegenerate discrete spectrum, $\lambda_n$, with eigenvectors $\rho_n$, $H_B \rho_n = \lambda_n \rho_n$. Then, the normalized eigenvectors, $\rho_n$, define an orthonormal basis for $H_B$, and any vector $\Phi \in \mathcal{L}^2(\partial \Omega_A) \otimes H_B$ has a unique representation as

$$\Phi = \sum_n \Phi_n \otimes \rho_n,$$

$$\Phi_n \in \mathcal{L}^2(\partial \Omega_A).$$  \hfill (15)

Hence, we get for vectors $\Phi^\pm \in \mathcal{N}_\pm = \ker(H^\dagger \mp i)$:

$$(H^\dagger \mp i)\Phi^\pm = \left(-\Delta^\dagger_{\pm \mp} \otimes 1 + 1 \otimes H_B \right) \left(\sum_n \Phi_n^\pm \otimes \rho_n \right) + i\Phi^\pm =$$

$$= \sum_n \left(-\Delta^\dagger_{\pm \mp} \Phi_n^\pm + \lambda_n \Phi_n^\mp \mp i\Phi_n^\mp \mp i\Phi_n^\mp \right) \otimes \rho_n = 0$$

which implies

$$-\Delta^\dagger_{\pm \mp} \Phi_n^\pm = (i \mp \lambda_n) \Phi_n^\pm = 0.$$  \hfill (16)

Thus, $\Phi_n^\pm$ must belong to the generalized deficiency spaces

$$\mathcal{N}_{A,\pm} = \left\{ \Phi^\pm \in D_{\pm \mp}^\dagger \mid -\Delta^\dagger_{\pm \mp} \Phi^\pm = z_n \Phi^\pm \right\},$$  \hfill (17)

$$\mathcal{N}_{A,\pm} = \left\{ \Phi^\pm \in D_{\pm \mp}^\dagger \mid -\Delta^\dagger_{\pm \mp} \Phi^\pm = \bar{z}_n \Phi^\pm \right\},$$  \hfill (18)

with $z_n = (\mp \lambda_n + i)$. However, all generalized deficiency spaces of the form $\mathcal{N}_{A,\pm}$ with Im $z > 0$ are isomorphic; that is, dim$\mathcal{N}_{A,\pm}$ is constant in the upper complex half-plane. Similarly, if Im $z < 0$, then dim$\mathcal{N}_{A,\pm}$ is constant in the lower half-plane, then $\mathcal{N}_{A,\pm} = \mathcal{N}_{A,\pm}$ is isomorphic to $\mathcal{N}_{A,\pm}(\mp \lambda_n)$ (see, for instance, [1]). Let us denote a choice for such isomorphism by $\alpha^\pm$. $\mathcal{N}_{A,\pm}(\mp \lambda_n) \to \mathcal{N}_{A,\pm}$.

We have shown that the deficiency spaces, $\mathcal{N}_\pm$, of the operator $H$ consist of vectors of the form $\sum_n \Phi_n^\pm \otimes \rho_n$ with $\Phi_n^\pm \in \mathcal{N}_{A,\pm}(\mp \lambda_n)$. The isomorphism, $\alpha^\pm: \mathcal{N}_\pm \to \mathcal{N}_A \otimes H_B$, defined by
\[
\sum_{k=1}^{\infty} \Phi_k^A \otimes \rho_k^B = \sum_{k=1}^{\infty} \alpha_k^+ (\Phi_k^A) \otimes \rho_k^B = 0
\]

provides an explicit identification of \( \mathcal{N}_{\Lambda} \) with \( \mathcal{N}_{\Lambda, +} \otimes H_B \).

The previous argument generalizes to the case of a general self-adjoint operator, \( H_B \), by judicious use of its spectral representation.

Notice that as a consequence of the previous theorem, the space of self-adjoint extensions of the composite system defined by the Hamiltonian, \( H \), is given by the space of unitary operators, \( K : \mathcal{N}_{\Lambda, +} \otimes H_B \rightarrow \mathcal{N}_{\Lambda, -} \otimes H_B \), which is much larger than the space of self-adjoint extensions of system \( A \) alone. In particular, the self-adjoint extensions defined by unitary operators of the form \( K_A \otimes 1 \) are in one-to-one correspondence with self-adjoint extensions of system \( A \) alone, \( K_A : \mathcal{N}_{\Lambda, +} \rightarrow \mathcal{N}_{\Lambda, -} \).

4. Separable dynamics and separable extensions

It is clear that if we have two complete quantum systems, \( A \) and \( B \), with Hilbert spaces of state vectors, \( \mathcal{H}_A \) and \( \mathcal{H}_B \), and Hamiltonian operators \( H_A \) and \( H_B \), respectively, then the bipartite system with Hilbert space \( \mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \) and total Hamiltonian \( \mathcal{H} = H_A \otimes 1 + 1 \otimes H_B \) induces a unitary flow

\[
U_t = e^{iHt} = e^{i(H_A \otimes 1 + 1 \otimes H_B)} = e^{i(H_A \otimes 1)} e^{i(1 \otimes H_B)}
\]

where \( U^A_t \), \( U^B_t \) denote the individual unitary flows of subsystems \( A \) and \( B \). Then we may call a one-parameter family of unitary operators, \( U_t \), on \( \mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \), separable if there exist two one-parameter families of unitary operators, \( U^A_t \) and \( U^B_t \), on \( \mathcal{H}_A \) and \( \mathcal{H}_B \), respectively, such that

\[
U_t = U^A_t \otimes U^B_t.
\]

Notice that \( U_t \) is separable if and only if \( U_t \Phi \) is separable for any separable state, \( \Phi = \Phi_A \otimes \Phi_B \), for any \( t \). Additionally, it is important to check that separable dynamics do not change the Schmidt index of a given state in \( \mathcal{H}_A \otimes \mathcal{H}_B \).

Now, if we are given a system \( H \) on \( \mathcal{H}_A \otimes \mathcal{H}_B \), which is obtained by means of a self-adjoint extension of the product of a symmetric operator on \( \mathcal{H}_A \) and a self-adjoint operator on \( \mathcal{H}_B \), can we determine when we are going to obtain separable dynamics? In other words, if \( U \in \mathcal{U}(\mathcal{L}^2(\partial \Omega_A) \otimes \mathcal{H}_B) \) is the unitary operator defining the self-adjoint extension, cf equation (14), under what conditions will it characterize separable dynamics?

We will first solve the spectral problem for the self-adjoint extension of \( H \), defined by the boundary condition \( U = U_A \otimes 1 \). We will assume for simplicity that the spectrum of \( H_B \) is discrete and non degenerate. We denote the eigenvalues and eigenvectors of \( H_B \) by \( \lambda_k^B \) and \( \rho_k^B \), respectively; \( H_B \rho_k^B = \lambda_k^B \rho_k^B \), \( k = 1, 2, \ldots \). An arbitrary function, \( \Phi \in \mathcal{L}^2(\partial \Omega_A; \mathcal{H}_B) \), can be written uniquely as

\[
\Phi = \sum_{k=1}^{\infty} \Phi_k^A \otimes \rho_k^B, \quad (21)
\]
\[ \varphi = \sum_{k=1}^{\infty} \varphi_k^A \otimes \rho_k^B, \quad \psi = \sum_{k=1}^{\infty} \psi_k^A \otimes \rho_k^B. \]

with \( \varphi_k^A = \Phi_k^A \big|_{\Omega_{A_k}} \) and \( \varphi_k^A = d\Phi_k^A/du \big|_{\Omega_{A_k}} \), \( k = 1, 2, \ldots \). If \( U = \sum_{s=1}^{N} U_s^A \otimes U_s^B \) is a unitary operator acting on \( L^2(\Omega_{A_k}) \otimes \mathcal{H}_B \), we have \( U\varphi = \sum_{k=1}^{\infty} \sum_{s=1}^{N} U_s^A \varphi_k \otimes U_s^B \rho_k^B \) and so on. In the particular instance of \( U = U_A \otimes 1 \), we get

\[ U\varphi = \sum_{k=1}^{\infty} U_A \varphi_k^A \otimes \rho_k^B \]

and similarly for \( \psi \). If we denote by \( H_U \) the self-adjoint extension defined by \( U \), the spectral problem \( H_U \Phi = E\Phi \) becomes, after some trivial computations, the family of spectral problems

\[ H_U^A \Phi_k^A + \lambda_k^B \Phi_k^A = E\Phi_k^A, \quad k = 1, 2, \ldots \]  \hspace{1cm} (22)

and the boundary conditions defined by \( U \) become the family of boundary conditions

\[ \varphi_k^A - i\psi_k^A = U_A \left( \varphi_k^A + i\psi_k^A \right), \quad k = 1, 2, \ldots \]

Thus, for each \( k \) we have to solve the problem

\[ H_U^A \Phi^A = \left( E - \lambda_k^B \right) \Phi^A \]  \hspace{1cm} (23)

\[ \varphi^A - i\psi^A = U_A \left( \varphi^A + i\psi^A \right). \]  \hspace{1cm} (24)

Notice that if we denote by \( \Psi_l^A \) the eigenfunctions of the self-adjoint extension of the operator, \( H_A \), defined by \( U_A \), with the boundary conditions given in equation (23), we will have

\[ H_U^A \Psi_l^A = \lambda_l^B \Psi_l^A, \quad \psi_l^A - i\psi_l^A = U_A \left( \psi_l^A + i\psi_l^A \right), \quad l = 1, 2, \ldots \]

We will also assume, for simplicity, that the spectrum of the extension of \( H_A \) defined by \( U_A \) is discrete; this supposes no loss of generality for our purposes. In particular, for one-dimensional compact manifolds, the spectrum of any self-adjoint extension of the Laplace operator is discrete [17, theorem 8.18]. In general, using the spectral theorem [1], one can adapt this construction to the general case.

In what follows, we will denote the self-adjoint extensions of \( H_A \) determined by the unitary \( U_A \) by \( H_U \). We finally conclude that the spectrum of \( H_U \) is given by

\[ E = \lambda_l^A + \lambda_l^B, \quad k, l = 1, 2, \ldots, \]

with eigenvectors \( \Psi_l^A \otimes \rho_l^B \). Now, if \( \Phi \in L^2(\Omega_A; \mathcal{H}_B) \), we have

\[ \Phi = \sum_{k,l} c_{kl} \Psi_l^A \otimes \rho_l^B \]

with \( c_{kl} = \langle \Psi_l^A \otimes \rho_l^B, \Phi \rangle \); if \( \Phi \) is separable, \( \Phi = \Phi_A \otimes \rho_B \), we obtain

\[ c_{kl} = \langle \Psi_l^A, \Phi_A \rangle \langle \rho_l^B, \rho_B \rangle = a_l b_k, \]
with $\Phi_A = \sum a_i \Psi_i^A$ and $\rho_B = \sum b_k \rho_k^B$, respectively. Consequently, 
\[
e^{iH_U} (\Phi_A \otimes \rho_B) = \sum a_i b_k e^{iH_U} (\Psi_i^A \otimes \rho_k^B)
\] (25) 
\[
= \sum a_i b_k e^{i(a_i^* + b_k^* q)} (\Psi_i^A \otimes \rho_k^B)
\] (26) 
\[
= \left( \sum a_i e^{iH_U A} \Psi_i^A \right) \otimes \left( \sum b_k e^{iH_U B} \rho_k^B \right)
\] (27) 
\[
= \left( e^{iH_U A} \Phi_A \right) \otimes \left( e^{iH_U B} \rho_B \right)
\] (28) 
\[
= \left( e^{iH_U A} \otimes e^{iH_U B} \right) (\Phi_A \otimes \rho_B),
\] (29) 
which shows that the self-adjoint extension defined by the unitary matrix, $U = U_A \otimes \mathbb{1}$, is separable, as it was easy to presume.

Let us now discuss the boundary conditions of the simple form, 
\[
U = U_A \otimes U_B \in U^* (L^2(\partial \Omega_A) \otimes H_B),
\] (30) 
with $U_A \in U^* (L^2(\partial \Omega_A))$ and $U_B \in U^* (H_B)$, i.e. (decomposable elements in the unitary group, $U^* (L^2(\partial \Omega_A) \otimes H_B)$). Recall the form that the boundary conditions have in this case, equations (13) and (14). The unitary operator, $U_B$, should not be confused with the one-parameter group generated by the Hamiltonian, $H_B$ (i.e., $U_B \neq e^{itH_B}$ and besides the following condition, they are completely independent from each other). Consider that the unitary, $U_B$, defines a symmetry of the quantum system, $H_B$, which is $[H_B, U_B] = 0$. In this case, and contrary to a simple guess, the dynamics defined by the boundary condition in equation (14), where $U$ is of the form in equation (30), are non separable if $U_B \neq \mathbb{1}$.

The proof of this fact is as follows. Because $U_B$ is a unitary operator, it can be diagonalized, and the Hilbert space, $H_B$, decomposed as $H_B = \bigoplus_{s=1}^{\infty} W_s$, with $W_s$ orthogonal $U_B$-invariant subspaces such that can be 
\[
U_B \rho_s^B = e^{is\omega} \rho_s^B, \quad \forall \rho_s^B \in W_s.
\]
Contrary to what happens in equation (21), the orthonormal basis, $\{\rho_s^B\}$, is now the basis of the orthonormal vectors of $U_B$. Because $U_B$ commutes with $H_B$, $H_B$ will leave the subspaces $W_s$ invariant too, and we will use $H_{B,s}$, the restriction of $H_B$ to $W_s$, $s = 1, 2, \ldots$. Moreover, we have that 
\[
H = L^2(\Omega_A; H_B) = L^2(\Omega_A; \bigoplus_{s=1}^{\infty} W_s) \approx \bigoplus_{s=1}^{\infty} L^2(\Omega_A; W_s).
\] (31) 

Consider a separable state, $\Phi = \Phi_A \otimes \rho_B$. Our aim is to compute the evolution of such a state under the dynamics generated by $H_{U_A \otimes U_B}$. Hence, we need to solve the corresponding eigenvalue problem. Using the axiom of separation of variables, we consider the action of the Hamiltonian on states of the form $\Phi_A \otimes \rho_{s,i}^B$: 
\[
U = \sum_{s=1}^{\infty} O_{A,s} \otimes O_{B,i}.
\] (32) 
This is by no means the most general boundary condition allowed in this situation, which in general could be given by a unitary operator obtained by a linear combination of operators (not necessarily unitary) as follows:

$U = \sum_{s=1}^{\infty} O_{A,s} \otimes O_{B,i}$. 

\[
(H_A^s \otimes 1 + 1 \otimes H_B)\left( \Phi^A \otimes \rho^B_{s_i, i} \right) = E \left( \Phi^A \otimes \rho^B_{s_i, i} \right)
\] (32)
\[
\Phi^A \otimes \rho^B_{s_i, i} - i \dot{\Phi}^A \otimes \rho^B_{s_i, i} = \left( U_A \otimes U_B \right) \left( \Phi^A \otimes \rho^B_{s_i, i} - i \dot{\Phi}^A \otimes \rho^B_{s_i, i} \right).
\] (33)

This gives
\[
H_A^s \Phi^A \otimes \rho^B_{s_i, i} = \left( E - \lambda^B_{s, i} \right) \Phi^A \otimes \rho^B_{s_i, i}
\] (34)
\[
\left( \Phi^A - i \dot{\Phi}^A \right) \otimes \rho^B_{s_i, i} = e^{i \nu H_s} U_A \left( \Phi^A + i \dot{\Phi}^A \right) \otimes \rho^B_{s_i, i},
\] (35)

where \( \lambda^B_{s, i} \) are the eigenvalues of the Hamiltonian, \( H_B \). Hence, the system decouples with respect to the decomposition of equation (31). For each \( s \), we can solve the associated eigenvalue problem:
\[
H_A^s \Phi^A = \left( E - \lambda^A_{s, i} \right) \Phi^A
\] (36)
\[
\left( \Phi^A - i \dot{\Phi}^A \right) = e^{i \nu H_s} U_A \left( \Phi^A + i \dot{\Phi}^A \right).
\] (37)

Calling \( \lambda^A_{s, i} := E - \lambda^B_{s, i} \), we get that the eigenvalues of \( H_{U_s \otimes U_A} \) are
\[
E = \lambda^A_{s, i} + \lambda^B_{s, i}.
\]

Notice that the eigenbasis of this problem is not uniform in \( s \), as was the case in the previous situation. Hence, for each \( s \) we will have a different orthonormal basis of \( H_A \). We will denote this basis as \( \{ \psi^{A, s}_l \} \).

Now we let the separable state, \( \Phi = \Phi^A \otimes \rho^B \), evolve under the action of the Hamiltonian, \( H_{U^s \otimes U_A} \). For this we expand \( \Phi \) as follows:
\[
\Phi = \Phi^A \otimes \sum_{s, i} b_{s, i} \rho^B_{s_i, i}
\]
\[
= \sum_s \left( \Phi^A \otimes \sum_i b_{s, i} \rho^B_{s_i, i} \right)
\]
\[
= \sum_s \left( \sum_l a^l \Phi^A_{s, l} \right) \otimes \left( \sum_i b_{s, i} \rho^B_{s_i, i} \right)
\]
\[
= \sum_{s, l, i} a^l b_{s, i} \psi^{A, s}_{l} \otimes \rho^B_{s_i, i}.
\]

Now we repeat the calculations analogue to equations (25)–(29)
\[
e^{it H_{U_s \otimes U_A}} \left( \Phi^A \otimes \rho^B \right) = \sum_{s, l, i} a^l b_{s, i} \alpha_l \left( \psi^{A, s}_{l} \otimes \rho^B_{s_i, i} \right)
\] (38)
\[
= \sum_{s, l, i} a^l b_{s, i} e^{it \lambda^A_{s, l} + it \lambda^B_{s, i}} \left( \psi^{A, s}_{l} \otimes \rho^B_{s_i, i} \right)
\] (39)
\[
= \sum_s \left( \sum_l a^l b_{s, i} e^{it \lambda^A_{s, l}} \psi^{A, s}_{l} \right) \otimes \left( \sum_i b_{s, i} e^{it \lambda^B_{s, i}} \rho^B_{s_i, i} \right).
\] (40)

Contrary to what happens in equations (25)–(29) one cannot factor out the sum of the left factor because it depends on \( s \). This is because for each \( s \), the evolution on system \( A \) corresponds to different self-adjoint operators, \( H_A^s e^{it U_s} \). Thus, we conclude that if \( U_B \neq 1 \), the
dynamics, \( H_U \), are non separable. Notice that the case \( \nu = \nu, s = 1, 2 \ldots \) is equivalent to \( U = e^{it}U_A \otimes I \).

We can prove the following theorem.

**Theorem 2.** Let \( H_A \) be a densely defined symmetric operator on \( L^2(\mathbb{R}) \) and let \( H_B \) be a self-adjoint operator on the Hilbert space, \( H_B \). Let \( U_A \) be a unitary operator on \( L^2(\mathbb{R}) \) such that self-adjoint extensions of the operator, \( H_A \), defined by \( e^{it}U \), have a discrete spectrum for all \( 0 \leq \alpha \leq \pi \). Let \( H_B \) have a discrete spectrum. Then, the dynamics, \( H_U \), on the product Hilbert space, \( L^2(\Omega_A) \otimes H_B \), defined by the unitary operator, \( U = U_A \otimes U_B \in U(L^2(\partial\Omega_A) \otimes H_B) \), are separable iff \( U_B = I \).

**Proof.** Let us assume that the dynamics defined by \( H_U \) is separable. Then,

\[
e^{itA} = e^{itA} \otimes e^{itB},
\]

where, in principle, neither \( \hat{H}_A \) nor \( \hat{H}_B \) have to agree with \( H_A \) nor \( H_B \), respectively. However, we have that

\[
H_U = \hat{H}_A \otimes I + I \otimes \hat{H}_B
\]

with \( \hat{H}_A \) and \( \hat{H}_B \) as self-adjoint operators. It is also clear that the one-parameter group of unitary operators,

\[
V_t = e^{it}U_A
\]

defines a symmetry group of \( H_U \),

\[
[H_U, V_t] = 0, \quad \forall \ t.
\]

Moreover the group \( V_t \) acts unitarily on the boundary space, \( L^2(\partial\Omega; H_B) \). It can be shown, cf [11], that then, by necessity,

\[
\left[ U, I \otimes \hat{H}_B \right] = 0.
\]

Hence, \( [U_B, \hat{H}_B] = 0 \). But now we have a self-adjoint extension defined by a unitary matrix of the form \( U_A \otimes U_B \) with \( [U_B, \hat{H}_B] = 0 \), as in the discussion preceding this theorem. Then, repeating the previous arguments, we will obtain that the dynamics are non separable unless \( U_B = I \). \( \square \)

**5. A simple example: the half-line/half-spin bipartite system**

We will now discuss now what is conceivably the simplest, non trivial example of a bipartite system of the kind considered in section 3. Let auxiliary system A be a free particle moving on the half-line \( \mathbb{R}^+ \) (\( \Omega_A = \mathbb{R}^+, \partial\Omega_A = \{0\} \)). The Hilbert space of the system is \( H_A = L^2(\mathbb{R}^+, dx) \), and the dynamics of that system are governed by the free Hamiltonian \( H = \frac{1}{2} \frac{d^2}{dx^2} \). Bulk system B will be a two-level system—for instance, a spin 1/2 system whose Hilbert space is \( C^2 \). The dynamics is given by an arbitrary \( 2 \times 2 \) Hermitean matrix, \( H_B \). We assume that \( \sigma(H_B) = \{ \lambda_1 > \lambda_2 \} \) with eigenvectors \( \rho_1, \rho_2 \), respectively. The corresponding bipartite system, \( AXB \), is defined in the Hilbert space \( H = H_1 \otimes H_2 = L^2(\mathbb{R}^+) \otimes C^2 \simeq L^2(\mathbb{R}^+; C^2) \), whose state vectors, \( \Phi \in H \), will be written as
\[ \Phi = \Phi_1 \otimes \rho_1 + \Phi_2 \otimes \rho_2 \cong \begin{pmatrix} \Phi_1(x) \\ \Phi_2(x) \end{pmatrix} \quad \Phi_{\alpha}(x) \in L^2(\mathbb{R}^+), \quad \alpha = 1, 2 \quad (41) \]

where we have used the orthonormal basis \( \{\rho_1, \rho_2\} \) to write the component vectors.

As we showed before in theorem 1, the deficiency spaces are easy to compute, and we get \( \mathcal{N}_\pm = \mathcal{N}_{\lambda \pm} \otimes \mathbb{C}^2 \cong \mathbb{C}^2 \), because, as it is easy to check, \( \dim \mathcal{N}_{\lambda \pm} = 1 \), and therefore \( \mathcal{N}_{\lambda \pm} = \mathbb{C} \). However, we work directly with boundary values which will prove to be more efficient. Thus, given \( \Phi \in H \), the boundary values of \( \Phi \) will live in \( L^2(\partial \mathbb{R}^+) \otimes \mathbb{C}^2 \). In fact,

\[ \varphi := \Phi |_{i\partial \mathbb{R}^+} = \Phi_1 |_{i\partial \mathbb{R}^+} \otimes \rho_1 + \Phi_2 |_{i\partial \mathbb{R}^+} \otimes \rho_2 = \begin{pmatrix} \Phi_1(0) \\ \Phi_2(0) \end{pmatrix} = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \]

and similarly

\[ \dot{\varphi}_\alpha = -\frac{d\varphi}{dx} \bigg|_{i\partial \mathbb{R}^+} = \dot{\varphi}_1 \otimes \rho_1 + \dot{\varphi}_2 \otimes \rho_2 = \begin{pmatrix} \dot{\varphi}_1 \\ \dot{\varphi}_2 \end{pmatrix}; \quad \ddot{\varphi}_\alpha = -\frac{d^2\varphi}{dx^2} \bigg|_{x=0}, \quad \alpha = 1, 2. \]

Finally, we combine the boundary data as

\[ \varphi_{\pm} = \varphi \pm i\dot{\varphi} = \begin{pmatrix} \varphi_1 \pm i\dot{\varphi}_1 \\ \varphi_2 \pm i\dot{\varphi}_2 \end{pmatrix} \]

and the self-adjoint extensions of \( H = -\frac{d^2}{dx^2} \otimes 1 + 1 \otimes H_B \) are characterized by unitary operators \( U \in U'(L^2(\partial \mathbb{R}^+) \otimes \mathbb{C}^2) \cong U(2) \), defining the domains, \( \varphi_\alpha = U\varphi_\alpha \).

Notice that in matrix form, the operator \( H \) has the form

\[ H = -\frac{d^2}{dx^2} \otimes 1 + 1 \otimes H_B = \begin{bmatrix} -\frac{d^2}{dx^2} + \lambda_1 & 0 \\ 0 & -\frac{d^2}{dx^2} + \lambda_2 \end{bmatrix}. \quad (42) \]

We recall now that the boundary data space is given by \( L^2(\partial \mathbb{R}^+) \otimes \mathbb{C}^2 \). Hence, according to theorem 2, separable dynamics will be given by unitary operators of the form \( U = U_A \otimes 1 \), where \( U_A : L^2(\mathbb{R}^+) \to L^2(\mathbb{R}^+) \). Therefore, \( U_A = e^{i\alpha} \) is just multiplication by a phase. Incidentally, we may recall that these are all the self-adjoint extensions of the system \( A \) in the half-line, and they correspond to boundary conditions of the form

\[ \varphi_A - i\dot{\varphi}_A = e^{i\alpha}(\varphi_A + i\dot{\varphi}_A), \quad (43) \]

or equivalently

\[ \dot{\varphi}_A = -\tan \left( \frac{\alpha}{2} \right) \varphi_A, \quad \alpha \neq \pi \quad (44) \]

\[ \varphi_A = 0, \quad \alpha = \pi. \quad (45) \]

Now, because the space of self-adjoint extensions for the bipartite system is actually \( U(2) \), as shown above, there are many self-adjoint extensions that will define non-separable dynamics. Note that because the spectrum of the Laplace operator in the half-line is not discrete, we cannot apply theorem 2. However, we will proceed by a direct computation of the ground state of the composite system under different self-adjoint extensions.
We will consider the particular instance of self-adjoint extensions defined by unitary matrices of the form

\[ U = U_A \otimes V , \quad U_A \in U(\mathcal{L}^2(\partial \mathbb{R}^+)) , \quad V \neq \mathbb{I} \in U(2). \]  

Despite their form, they determine non separable dynamical evolution. In fact, among this class, and because \( U_A \) is just multiplication by a complex number of modulus 1, we can consider, as the simplest, non trivial example, a matrix, \( V \), of the form

\[
V = \left[ \begin{array}{cc}
\alpha & 0 \\
0 & e^{i\alpha_2}
\end{array} \right] , \quad \text{with} \quad e^{i\alpha_1} \neq e^{i\alpha_2}.
\]  

(i.e., a matrix \( V \) belonging to a maximal torus inside \( U(2) \)). It is also noticeable that such a \( V \) is the most general matrix commuting with \( H_B \). Notice that if \( \varphi = \varphi_1 \otimes \rho_1 + \varphi_2 \otimes \rho_2 \in \mathcal{L}^2(\partial \mathbb{R}^+) \otimes H_B \), then

\[
(1 \otimes V) \varphi = \varphi_1 \otimes V\rho_1 + \rho_2 \otimes V\varphi_2 = \begin{bmatrix} V_{11}\varphi_1 + V_{12}\varphi_2 \\ V_{21}\varphi_1 + V_{22}\varphi_2 \end{bmatrix} = V \cdot \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix}.
\]  

(48)

It is easy to compute the point spectrum of the self-adjoint operator, \( H_U \), defined by the unitary operator, \( U = \mathbb{I} \otimes V \). Notice that equation (14) becomes

\[
\begin{bmatrix}
\varphi_1 - i\varphi_1 \\
\varphi_2 - i\varphi_2
\end{bmatrix} = \begin{bmatrix}
e^{i\alpha_1} & 0 \\
0 & e^{i\alpha_2}
\end{bmatrix} \begin{bmatrix} \varphi_1 + i\varphi_1 \\ \varphi_2 + i\varphi_2 \end{bmatrix};
\]  

(49)

this is, \( \varphi_1 = e^{i\alpha_1} \varphi_1 \), \( a=1,2 \) or, if both \( \alpha_1, \alpha_2 \neq \pi \),

\[ \varphi_1 = -\tan \left( \alpha_1/2 \right) \varphi_1 , \quad a = 1, 2 . \]  

(50)

Then, the eigenvalue problem, \( H_U \Phi = E\Phi \), becomes

\[
H_U \Phi = -\frac{d^2}{dx^2} \Phi_1 \otimes \rho_1 - \frac{d^2}{dx^2} \Phi_2 \otimes \rho_2 + \Phi_1 \otimes H_B \rho_1 + \Phi_2 \otimes H_B \rho_2
= \left( -\frac{d^2}{dx^2} + \lambda_1 \right) \Phi_1 \otimes \rho_1 + \left( -\frac{d^2}{dx^2} + \lambda_2 \right) \Phi_2 \otimes \rho_2.
\]

This eigenvalue problem is equivalent to

\[
-\frac{d^2}{dx^2} \Phi_1 (E - \lambda_a) \Phi_1 = -\tan \left( \alpha_a/2 \right) \varphi_1 , \quad a = 1, 2 .
\]

(51)

We may start solving

\[
-\frac{d^2}{dx^2} \Phi_1 = (E - \lambda_1) \Phi_1 ; \quad \varphi_1 = -\tan \left( \alpha_1/2 \right) \varphi_1 .
\]

(52)

We see immediately that if \( \lambda_1 \leq E \), the solutions to this problem are not in \( \mathcal{L}^2(\mathbb{R}^+) \). Thus, \( \lambda_1 > E \) and the corresponding eigenfunction is \( \Phi_1(x) = C_1 e^{-\sqrt{\lambda_1-E} x} \). Moreover, \( \varphi_1 = -\frac{d\varphi_1}{dx}_{x=0} = C_1 \sqrt{\lambda_1 - E} \); hence \( \sqrt{\lambda_1 - E} = \tan \left( \alpha_1/2 \right) \) or \( E = \lambda_1 - \tan^2 \left( \alpha_1/2 \right) \).

Notice that \( E \) is the unique discrete eigenvalue of the operator and that the rest of the spectrum is continuous.

We can proceed similarly for the other component \( (a=2) \), finding again that \( E = \lambda_2 - \tan^2 \left( \alpha_2/2 \right) \) if \( E < \lambda_2 \). In consequence, if \( E < \lambda_2 \), we obtain the compatibility
condition (recall that \( \lambda_1 > \lambda_2 \))

\[
\tan^2(\alpha_1/2) - \tan^2(\alpha_2/2) = \lambda_1 - \lambda_2 > 0,
\]

which must be satisfied for the existence of an eigenvector with eigenvalue \( \lambda_1 > \lambda_2 > E \).

Figure 1 shows the space of self-adjoint extensions \((\alpha_1, \alpha_2)\) with non degenerate ground state, \(E\), for various values of the spectral gap, \(\sigma := \lambda_1 - \lambda_2\), of the bulk system.

In such cases, the groundstate and unique eigenvector of this problem is

\[
\Phi(x) = c \left[ \Phi_1(x) \otimes \rho_1 + \Phi_2(x) \otimes \rho_2 \right] = c \left[ e^{-\tan(\alpha_1/2)x} \rho_1 + e^{-\tan(\alpha_2/2)x} \rho_2 \right],
\]

with \(c\) being a normalization constant.

If \( \lambda_2 \leq E < \lambda_1 \), \(E\) is an eigenvalue again, but this time the eigenvector is going to have only the \(a=1\) component. We want to stress that the compatibility condition of equation (53) is only necessary for the existence of a non-void point spectrum. If it is not satisfied, then the problem has no point spectrum. Nevertheless, the Hamiltonian is self-adjoint even if there is no point spectrum.

The curves defined by equation (53) and determined by the values of \(\sigma\) provide families of non separable, self-adjoint extensions of \(H\) compatible with the structure of \(H_B\). Suppose that we select as an initial state the eigenstate corresponding to the extension defined by \(\alpha_1 = \arctan \sqrt{\sigma}, \quad \alpha_2 = 0\), this is \(\Phi_0 = e^{-\sqrt{\sigma}/2x} \otimes \rho_1\).

Consider now the (time-dependent) Hamiltonian, \(H\), for the bipartite system given by equation (42) and the domain defined by the one-parameter family of self-adjoint extensions defined by the unitary matrices

\[
U_{s(t)} = \begin{bmatrix}
    e^{2s(t)} & 0 \\
    0 & e^{2s'(t)}
\end{bmatrix}
\]

with \(s(t), s'(t)\) such that \(\tan^2 s - \tan^2 s' = \sigma\); this is \(s' = \arctan \sqrt{\tan^2 s - \sigma}\). That is, the time-dependence of the evolution of the system is not in the form of the infinitesimal generator, but rather on its domain, which changes with time according to equation (7), because the unitary operator, \(U_{s(t)}\), that defines the domain depends on \(t\).

Suppose that we modify the self-adjoint extension adiabatically. For that, we may choose the parametrization \(s = s(t)\), with \(t\) being the physical time, in such a way that \(0 < ds/dt \ll 1\). Then, in the adiabatic approximation, the eigenstate, \(\Phi_0\), will change with \(t\), but it will remain close to the ground state of the self-adjoint extension, \(H_{s(t)}\) (its (unique) eigenstate), and it will be given by

\[
\Phi(t) = C_1 e^{-\tan(s^*)} \otimes \rho_1 + C_2 e^{-\tan(s')} \otimes \rho_2, \quad 0 < s < \pi/2,
\]

cf equation (54). This state, \(\Phi(s)\), is generically an entangled state in \(\mathcal{H}_A \otimes \mathcal{H}_B\). Notice that the phase diagram of the self-adjoint extensions constructed in this way is periodic, as seen in figure 2. The crossings with the axes correspond to separable states of either form \(e^{-\sqrt{\xi}/2x} \otimes \rho_1\) or \(e^{-\sqrt{\xi}/2x} \otimes \rho_2\).

The half-line/multipartite spin 1/2 system

We can elaborate on the previous example by considering a system, \(B\), that is already a composite system (i.e., \(H_B = H_{B_1} \otimes H_{B_2}\) with \(\dim H_{B_1} = n_{a_1}, \alpha = 1, 2\)). The self-adjoint operators, \(H_{B_\alpha}, \alpha = 1, 2\), have eigenvalues \(\lambda_{k_{a_\alpha}}, k_{a_\alpha} = 1, ..., n_{a_\alpha}\) and a basis of eigenvectors of the operator \(H_{B_\alpha} \otimes 1 + 1 \otimes H_{B_\alpha}\) is given by
where $\rho_{k_1,k_2}^{(a)}$ are eigenvectors with eigenvalues $\lambda_{k_1}^{(a)}$. The eigenvalue corresponding to the eigenvector $\rho_{k_1,k_2}$ is just $\lambda_{k_1}^{(1)} + \lambda_{k_2}^{(2)}$. Now we compute the system $A \times B$ to get
\[ H = H_A \otimes H_B \cong L^2(\mathbb{R}^+; H_{B_1} \otimes H_{B_2}). \]  

(58)

and we expand \( \Phi \in H \) as

\[ \Phi = \sum_{1 \leq k_x \leq n_x} \Phi_{k_x,k_2} \otimes \rho_{k_x,k_2}. \]  

(59)

In the same way, \( \varphi = \sum_{1 \leq k_x \leq n_x} \varphi_{k_x,k_2} \otimes \rho_{k_x,k_2} \) and \( \psi = \sum_{1 \leq k_x \leq n_x} \psi_{k_x,k_2} \otimes \rho_{k_x,k_2} \) with \( \varphi_{k_x,k_2} = \Phi(0)_{k_x,k_2} \) and \( \psi_{k_x,k_2} = -\frac{d\Phi_{k_x,k_2}}{dx} \) when \( k_x = 0 \). Finally, we notice that the space of self-adjoint extensions of the composite symmetric operator, \( H \), is given by \( U'(L^2(\mathbb{R}^+; H_{B_1} \otimes H_{B_2})) \); that is,

\[ M_{AB} = U(n_1 \cdot n_2) = U(N), \quad N = n_1 \cdot n_2, \quad n_a = \text{dim} H_{B_a}, \quad \alpha = 1, 2. \]  

(60)

Notice that the assumptions of theorem 2 do not hold in this case. However, the intuition provided by theorem 2 makes us expect that separable dynamics will correspond to \( U = U_\alpha \times U_\beta \), with \( U_\alpha = e^{i\theta} \). Hence, let us choose boundary conditions leading to non-separable dynamics in the composite system \( AB \), and in \( B \) itself.

We consider, for instance, \( U = I \times V \) with \( V \in U'(H_{B_1} \otimes H_{B_2}) \). Again we choose a simplifying hypothesis and assume that the spectrum is non-degenerate. Consider the ordered spectrum of the Hamiltonian, \( H_B \) (i.e., \( \Lambda_1 = \max \{ \lambda_k^{(1)} + \lambda_k^{(2)} \} = \lambda_k^{(1)} + \lambda_k^{(2)} \), \( \geq \Lambda_2 = \lambda_k^{(1)} + \lambda_k^{(2)} \), \( \geq \cdots \geq \Lambda_N = \min \{ \lambda_k^{(1)} + \lambda_k^{(2)} \} \)), and let \( \Pi_1, \ldots, \Pi_N \) be the corresponding eigenvectors. Then, \( H_B \Pi_i = \lambda_i \Pi_i \). We choose the matrix, \( V \), to be diagonal in this basis,

\[ V = \text{diag}(e^{im_1}, e^{im_2}, \ldots, e^{im_N}), \]

and repeating the computations performed in the previous example we will get that the point spectrum of the operator \( H \) is given by \( E = \Lambda_l - \tan^2 \alpha_l, \quad l = 1, \ldots, N \), which imposes \( N - 1 \) conditions on the parameters, \( \alpha_l \), of the form

\[ \Lambda_l - \tan^2 \alpha_l = \Lambda_{l+1} - \tan^2 \alpha_{l+1}, \quad l = 1, \ldots, N - 1. \]  

(61)

Equation 61 defines a curve in the \( N \)-dimensional maximal compact abelian subgroup of \( U(N) \) similar to those exhibited in figure 2. Again, a similar analysis is found in the example of a single-spin 1/2 system, which allows us to conclude that an adiabatic deformation of the system along this curve will take a separable state, such as \( \Phi_{11} \otimes \rho_{11} = \Phi_{11} \otimes \rho_{11}^{(1)} \otimes \rho_{11}^{(2)} \), into (maximally) non-separable states.

6. The quantum planar rotor-spin system

We consider as a final example the interesting case of an hybrid system that captures some properties of electron–nucleus systems described recently (see [16]). System \( A \) will now be a particle moving in the interval \( \Omega_A = [0, 1] \) with measure \( dx \) (i.e., \( H_A = L^2([0, 1], dx) \)). Unlike in the previous case, now the boundary of system \( A \) has two points, and therefore the self-adjoint extensions of system \( A \) alone are going to be parametrized by matrices in \( U(2) \). All of them have a discrete spectrum ([17]), so that now we are going to be under the conditions of theorem 2. Actually, we are going to consider a planar rotor with quasi-periodic boundary conditions [2] (i.e. the previous system with self-adjoint extensions determined by the unitary matrix

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that corresponds to boundary conditions $\Phi(0) = e^{ib}\Phi'(1)$ and $\Phi'(0) = e^{ib}\Phi'(1))$. Now we will consider bulk system $B$ to be a two-level system, such as a spin $1/2$ system, with dynamics given by $H_B = \mu \sigma_z$, where $\sigma_z$ is the diagonal Pauli matrix

$$\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

and $\mu$ is a constant that accounts for both the coupling constant of the magnetic field with the spin $1/2$ system and the strength of the magnetic field. Then, $H = L^2(S^1; \mathbb{C}) \otimes \mathbb{C}^2 = L^2(S^1; \mathbb{C}^2)$ is the state space of the total system and we consider

$$H = -\frac{d^2}{dx^2} \otimes 1 + 1 \otimes H_B$$

as the total Hamiltonian. For this particular example we turn to the standard notation for spin systems and write the eigenstates corresponding to $H_B$ as $|\uparrow\downarrow\rangle$. Therefore, $H_B |\uparrow\downarrow\rangle = \pm \mu |\uparrow\downarrow\rangle$, and a particular element $\Phi$ of the composite system $H = H_A \otimes H_B$ will admit the decomposition $\Phi = \Phi^1 \otimes |\uparrow\rangle + \Phi^1 \otimes |\downarrow\rangle$. As boundary conditions we choose $U \in U(C^1_A) \otimes U(C^2_B)$ of the form $U = U_{A,\delta} \otimes U_B$, with $U_{A,\delta}$ as in equation (62).

Physically, this system can be interpreted as follows (see figure 3). There is a charged particle moving along a circle [2]. In the center of this orbit, there is a fixed spin that interacts with a magnetic field of strength $\mu$ perpendicular to the plane of the orbit. The component $U_B$ of the boundary condition shall be interpreted as a macroscopic interaction triggered when the orbiting charged particle traverses an ideally infinitesimal region of the orbit.

We are now going to consider two different meaningful situations (compare with equation (47)) for the boundary conditions corresponding to subsystem $B$. The first situation will correspond to select the unitary matrix, $U_B$, diagonal in the basis of $H_B$; namely

$$U_B = \begin{bmatrix} e^{ia} & 0 \\ 0 & e^{-ia} \end{bmatrix}$$

Figure 3. Quantum compass
The boundary conditions defined by these unitary matrices take the explicit form
\[ \Phi^1(0) + i\Phi^1(0) = e^{i(\alpha + \delta)}(\Phi^1(1) + i\Phi^1(1)) \]
\[ \Phi^1(1) - i\Phi^1(1) = e^{i(\alpha - \delta)}(\Phi^1(0) - i\Phi^1(0)) \]
\[ \Phi^1(0) + i\Phi^1(0) = e^{i(-\alpha + \delta)}(\Phi^1(1) + i\Phi^1(1)) \]
\[ \Phi^1(1) - i\Phi^1(1) = e^{i(-\alpha - \delta)}(\Phi^1(0) - i\Phi^1(0)) \]

One can proceed as in the previous examples and impose the above boundary conditions to the general solution of the spectral problem, equation (63), given by
\[ \Phi^1(x) = A e^{i\sqrt{E - \mu}x} + B e^{-i\sqrt{E - \mu}x} \]
\[ \Phi^1(x) = C e^{i\sqrt{E + \mu}x} + D e^{-i\sqrt{E + \mu}x} \]
to find the corresponding spectral function associated with the problem. In this case, one obtains the following spectral function:
\[ \sigma_a(E) = \left[ 2i \sin \left( \sqrt{E - \mu} \right) + 2iE \sin \left( \sqrt{E - \mu} \right) 2i\mu \sin \left( \sqrt{E - \mu} \right) \right. \]
\[ -8\sqrt{E - \mu} \cos (\delta)e^{i\alpha} + 8\sqrt{E - \mu} \cos (\delta)e^{-i\alpha} \]
\[ - 2i(E - \mu + 1) \sin(E - \mu)e^{2i\alpha} \times \]
\[ \left. x \left[ 2i \sin \left( \sqrt{E + \mu} \right) + 2iE \sin \left( \sqrt{E + \mu} \right) + 2i\mu \sin \left( \sqrt{E + \mu} \right) \right. \right. \]
\[ -8\sqrt{E + \mu} \cos (\delta)e^{-i\alpha} + 8\sqrt{E + \mu} \cos (\delta)e^{i\alpha} \]
\[ - 2i(E + \mu + 1) \sin(E + \mu)e^{-2i\alpha} \right] \]
whose zeros are the corresponding eigenvalues.

Finding the zeros of this transcendental function has to be done numerically. However, this task can be challenging, especially because \( \sigma_a(E) \) is very close to vanishing in some regions. Moreover, the information about the separability of the dynamical evolution depends on the eigenfunctions of the problem as, shown in previous sections. For all these reasons, in order to check that the above problem is not leading to separable dynamics, we will take the approach introduced in [9]. There, an algorithm based on the finite element method is introduced that is able to solve the spectral problem for any self-adjoint extension of a one-dimensional Schrödinger problem. Then, it is enough to use the isomorphism \( \mathcal{L}^2([0, 1]) \otimes \mathbb{C}^2 \cong \mathcal{L}^2([0, 1]) \oplus \mathcal{L}^2([0, 1]) \) to rewrite the problem given by equation (63) into a form that can be handled by this numerical procedure. Figure 4 shows the eigenfunctions corresponding to the six smallest energies returned by the algorithm for \( \mu = 10, \delta = \pi/2, \alpha = \pi/2 \). In each graph, the two components, \( \Phi^1(x) \) and \( \Phi^3(x) \), of the eigenfunction \( \Phi = (\Phi^1 \uparrow \uparrow + \Phi^3 \downarrow \downarrow) \) are simultaneously represented. The particular values of the energies are not shown because they are not relevant to the discussion.

As one can be appreciate, the eigenfunctions are separable states in this case. However, separability of the eigenfunctions is not enough to guarantee the separability of the dynamics. According to section 4, the eigenfunctions of the total Hamiltonian need to admit a factorization, \( \psi \otimes \rho \), in terms of the eigenfunctions \( \{\psi_l\} \) and \( \{\rho_b\} \) of the Hamiltonians of the parties \( H_A \) and \( H_B \), respectively. In other words, the indices \( l \) and \( b \) must be independent. As one can be appreciate when comparing the eigenfunctions \( \Phi_1 \) and \( \Phi_2 \) corresponding to the eigenvalues \( E_1 \) and \( E_3 \), respectively, they are not of the form \( \Phi_1 = \psi(x) \otimes \rho_1 \) and \( \psi(x) \otimes \rho_2 \).
for some function \( \psi \in L^2([0, 1]) \), which shows that the set \( \{ \psi_i \} \) is not independent of \( \{ \rho_b \} \). The same argument holds for the pairs \( E_2, E_4 \) and \( E_5, E_6 \). Hence we conclude that we have non-separable dynamics for this particular choice of boundary conditions.

Now we consider a different situation where the unitary matrix, \( U_B \), is taken anti-diagonal with respect to the given basis of \( H_B \), and given by

\[
U_B = \begin{bmatrix}
0 & e^{i\beta} \\
e^{-i\beta} & 0
\end{bmatrix}.
\]

In this case, the boundary conditions defining the system take the form

\[
\begin{align*}
\Phi^+(0) + i\Phi^-(0) &= e^{i(\beta + \delta)} \left( \Phi^+(1) + i\Phi^-(1) \right) \\
\Phi^+(1) - i\Phi^-(1) &= e^{i(\beta - \delta)} \left( \Phi^+(0) - i\Phi^-(0) \right) \\
\Phi^+(0) + i\Phi^-(0) &= e^{i(-\beta + \delta)} \left( \Phi^+(1) + i\Phi^-(1) \right) \\
\Phi^+(1) - i\Phi^-(1) &= e^{i(-\beta - \delta)} \left( \Phi^+(0) - i\Phi^-(0) \right).
\end{align*}
\]

Again, one can compute the spectral function associated to this problem and get

\[
\sigma_{\beta}(E) \propto \sqrt{E^2 - \mu^2} \cos \left( \sqrt{E - \mu} \right) \cos \left( \sqrt{E + \mu} \right) - E \sin \left( \sqrt{E - \mu} \right) \sin \left( \sqrt{E + \mu} \right) - \sqrt{E - \mu} \sqrt{E + \mu} \cos (2\delta).
\]

Surprisingly, the spectral function does not depend on parameter \( \beta \) in this case, but the eigenfunctions do. In figure 5, the eigenfunctions corresponding to the case \( \mu = 10, \delta = \pi/2, \beta = \pi/2 \) are plotted. One can appreciate that they are non separable, and therefore the dynamics characterized by this last set of boundary conditions is not separable.
7. Conclusions and discussion

In this article, we have shown that manipulating the boundary conditions for a class of bipartite systems allows us to evolve a separable state into an entangled one. We have shown that we can achieve this dynamically by changing the boundary conditions in a time-dependent way, as seen in section 5. This phenomenon also arises for fixed boundary conditions, as the examples in section 6 show. The reason for this phenomenon lies in the existence of many self-adjoint extensions of a bipartite, symmetric system that lead to non-separable dynamics. We have been able to characterize all boundary conditions leading to separable dynamics in a class of symmetric bipartite systems.

The systems exhibited are hybrid systems and one of the parties, either the control or auxiliary system, is symmetric but not self-adjoint. The most remarkable fact about this class of systems is that the space of the self-adjoint extensions is much larger than the space of extensions of the stand-alone control system, and it incorporates boundary data that simultaneously affect the control and the controlled, or bulk, system. The controlled system has unitary dynamics, but together with the control system it becomes non-separable. Hence, taking the partial trace with respect to system $A$ will not give us back the original dynamics, $U^a_t$. These ideas can be used to generate entangled states in a precise way, or to help to preserve entanglement without actually interacting with the ‘bulk’ of the controlled system. The relation of these ideas to recent work on adiabatic computation and robust entanglement in hybrid systems will be pursued in the future.

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