On the local Lorentz invariance in $N = 1$ supergravity

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We discuss the local Lorentz invariance in the context of $N = 1$ supergravity and show that a previous attempt to find explicit solutions to the Lorentz constrain in terms of $\gamma$–matrices is not correct. We improve that solution by using a different representation of the Lorentz operators in terms of the generators of the rotation group, and show its compatibility with the matrix representation of the fermionic field. We find the most general wave functional that satisfies the Lorentz constraint in this representation.

I. INTRODUCTION

In the Lagrangian for $d = 4$, $N = 1$ supergravity

$$ L = \frac{1}{2} \sqrt{-g} R - \frac{i}{2} \epsilon^{\lambda \mu \nu \rho} \bar{\psi}_\lambda \gamma_5 \gamma_\mu D_\nu \psi_\rho , $$

$$ D_\nu = \partial_\nu + \frac{1}{2} \omega_\nu^{AB} \sigma^{AB} , $$

the fields are the local tetrad $e$ and the Rarita-Schwinger gravitino field $\psi$ (here we follow closely the conventions and notations of [1]). Here $\omega_\nu^{AB}$ is the connection associated with the local vierbein $e$ and

$$ \sigma^{AB} = \frac{1}{4} (\gamma^A \gamma^B - \gamma^B \gamma^A) $$

are the Lorentz generators in terms of the standard $\gamma$–matrices. The corresponding canonical $3 + 1$ decomposition of spacetime leads to a constrained system determined by the Hamiltonian

$$ H = e^A_0 \mathcal{H}_A + \frac{1}{2} \omega_0^{AB} \mathcal{J}_{AB} + \bar{\psi}_0 \mathcal{S} , $$

from which it follows that the constraint $\mathcal{H}_A \Psi[e, \psi] = 0$ is satisfied identically once the supersymmetric constraint $\mathcal{S} \Psi[e, \psi] = 0$ is fulfilled. Consequently, only the Lorentz and supersymmetric constraints need to be considered in order to obtain wave functionals.

In this work we will focus on the Lorentz constraint. The technical difficulties in the manipulation of this constraint are drastically reduced due to the fact that in an appropriate basis (see below) it does not depend explicitly on the bosonic variables. In principle, it is very easy to satisfy the Lorentz constraint since it simply states that the wave functional $\Psi[e, \psi]$ must be a Lorentz invariant, a condition that can be satisfied, for instance, by demanding that the wave functional contains the bosonic
and fermionic $\psi$ variables only in local Lorentz invariant combinations \[\mathbb{R}\]. In particular, one can take the wave functional as an expansion in terms of scalars constructed by means of bosonic and fermionic variables. Since the constraints of $N = 1$ supergravity are homogeneous in the gravitino field $\psi$, one can search for solutions containing homogeneous functionals of order $\psi^\alpha$. These states are called Grassmann number $n$ states. In this case, it is necessary to consider that there are no physical states that are purely bosonic or have fixed Grassmann number $n$. In fact, any physical state in $N = 1$ supergravity must contain fermionic degrees of freedom and must have infinite Grassmann number $n$. Notice that all this is valid only for the ground state of the wave functional. In order to analyze excited states, one needs to explore the Pauli-Lubanski operator and its eigenvalues. This would allow to investigate solutions with spin and/or helicity.

On the other hand, it has been claimed \[\mathbb{R}\] that the Lorentz constraint needs to be solved explicitly in completely general terms, without making any assumptions on the Lorentz behavior of wave functional. In \[\mathbb{R}\], different minisuperspace Bianchi models were analyzed, finding explicit solutions to the Lorentz constraint by using a specific matrix representation of the Lorentz operator. In \[\mathbb{R}\], the Lorentz constraint was solved by using a quite general matrix representation for the Bianchi IX cosmological model. The aim of this brief report is to improve an incorrect result obtained in \[\mathbb{R}\] and to comment on the meaning and importance of the Lorentz constraint in $N = 1$ supergravity.

II. LOCAL LORENTZ INVARIANCE

The Lorentz constraint

$$J_{AB} \Psi[e, \psi] = 0$$

(7)

simply states that the ground state of the wave functional must be invariant with respect to local Lorentz transformations. Then, it is clear that any Lorentz invariant quantity is a solution to this constraint. The question is how to construct Lorentz invariant wave functionals. The first thing one should notice is that the Lorentz operator $J_{AB}$ acts only on geometric objects which include vierbein Lorentz indices $A, B, \ldots$. Then, any combinations of the variables $e$ and $\psi$ in which all the Lorentz indices are contracted represent scalar Lorentz invariants, i.e., solutions to the Lorentz constraint. This strategy has been used to find explicit wave functionals \[\mathbb{R}\] which also satisfy the supersymmetric constraint. In general, it has been argued that it is possible to construct the general solution of the Lorentz constraint by taking an expansion of the wave functional in bosonic and fermionic variables in such a way that in all the terms the Lorentz indices are contracted. However, this is obviously not true because it would imply that only scalar quantities could be Lorentz invariants. If we would attempt to find all the Lorentz invariants which depend on the field variables $e$ and $\psi$, we would have to consider also invariant tensors of higher order and would end up with an infinite number of possibilities, each one being a candidate to construct an expansion in bosonic and fermionic variables. We conclude that almost any “intelligent” combination of bosonic and fermionic variables is a Lorentz invariant. This is the reason why it is generally believed that the Lorentz constraint in $N = 1$ supergravity is straightforward to satisfy \[\mathbb{R}\].

On the other hand, if one insists on solving the Lorentz constraint explicitly, one first must find an appropriate representation for the Lorentz operator and the wave functional such that Eq.\[\mathbb{R}\] becomes plausible and suitable to be solved. For instance, in \[\mathbb{R}\] it was shown that the minimal matrix representation, which is in accordance with the algebra of the gravitino field, leads to a wave functional that must be represented by a 64-component vector, satisfying 384 coupled algebraic equations. Fortunately, this system of algebraic equations could be solved for the Bianchi IX minisuperspace model, and generated only two non-vanishing independent components for the wave functional. Obviously, this method could lead to the most general solution of the Lorentz constraint. In the relatively simple case of the diagonal Bianchi IX model the resulting system of coupled algebraic equations could be solved, but it is not clear whether less simple models would lead to a tractable set of equations. A different approach was used in \[\mathbb{R}\] which consists in choosing an $SO(3)$ basis for the vierbein $e$ in which the Lorentz constraint contains only fermionic degrees of freedom \[\mathbb{R}\]

$$J_{AB} = \frac{1}{2} \phi_A T \phi_B,$$

(8)

where $\phi_A$ are the densitized local components of the fermionic field $\phi = e^a e^\alpha \psi_\alpha$, with $e = (e^a)^{\alpha} = \det(e_\alpha^a)$. Here $a$ and $\alpha$ are spatial Lorentz and world indices, respectively. Moreover, since $\psi_\alpha$ is a Lagrange multiplier [cf. Eq.\[\mathbb{R}\]], one should choose $\phi_0$ accordingly. Then from Eq.\[\mathbb{R}\] it seems natural to write the Lorentz constraint as

$$J_{AB} \Psi[e, \psi] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & J_{12} & J_{13} \\ 0 & -J_{12} & 0 & J_{23} \\ 0 & -J_{13} & -J_{23} & 0 \end{pmatrix} \begin{pmatrix} \Psi_I \\ \Psi_{II} \\ \Psi_{III} \\ \Psi_{IV} \end{pmatrix} = 0.$$ 

(9)

Now the idea is to find a matrix representation for each of the components $J_{ab}$ and $\Psi_I$, etc. such that the solution of of the Lorentz constraint \[\mathbb{R}\] is compatible with the particular choice of a basis which lead to Eq.\[\mathbb{R}\]. Since in Eq.\[\mathbb{R}\] we used the product of $\gamma-$matrices to represent the Lorentz generators it seems reasonable to use them again to represent each of the non-vanishing components of the Lorentz operator $J_{AB}$. To see if this is possible we...
write explicitly the Lorentz constraint \((9)\) as

\[
\begin{align*}
\mathcal{J}_{12} \Psi_{I} &= -\mathcal{J}_{13} \Psi_{IV}, \\
\mathcal{J}_{12} \Psi_{I} &= \mathcal{J}_{23} \Psi_{IV}, \\
\mathcal{J}_{13} \Psi_{I} &= -\mathcal{J}_{23} \Psi_{I} ,
\end{align*}
\]

and assuming that none of the components \(\Psi_{II}, \Psi_{III}, \Psi_{IV}\) is zero, we obtain from Eqs. (10)–(12)

\[
\mathcal{J}_{13}(\mathcal{J}_{12})^{-1} \mathcal{J}_{23} = \mathcal{J}_{23}(\mathcal{J}_{12})^{-1} \mathcal{J}_{13}.
\]

This result is surprising because it implies a relationship between the components of the Lorentz operator, whereas the idea of the Lorentz constraint is to impose conditions on the components of the wave functional. However, if we could find a representation for the components of \(J_{ab}\) such that the condition \((13)\) becomes an identity, the contradiction would be solved. In \(\mathbb{S}\) it was claimed that the choice

\[
\mathcal{J}_{12} = -\gamma^3 \gamma^0, \quad \mathcal{J}_{13} = -\gamma^1 \gamma^3, \quad \mathcal{J}_{23} = -\gamma^1 \gamma^0
\]

is the solution to this problem. However, a straightforward calculation shows that this choice is not a solution to the condition \((13)\), although it does lead to the choice

\[
\phi_1 = -i \gamma^3, \quad \phi_2 = -i \gamma^1, \quad \phi_3 = -i \gamma^0
\]

for the gravitino field which satisfies the definition \(\mathbb{S}\). By analyzing all possible products of \(\gamma\)-matrices one can show that the choice

\[
\mathcal{J}_{12} = -\gamma^1 \gamma^0, \quad \mathcal{J}_{13} = -\gamma^1 \gamma^3, \quad \mathcal{J}_{23} = -\gamma^2 \gamma^0
\]

is the only solution (modulo permutations) to the condition \((13)\). Unfortunately, this choice is not compatible with Eq. \((8)\), i.e. there is no way to choose the spatial components of the gravitino field \(\phi_a\) proportional to a \(\gamma\)-matrix so that \(\mathbb{S}\) is satisfied (the index \(a\) allows only three values whereas the last choice for \(J_{ab}\) involves four different \(\gamma\)-matrices). Thus, we have shown that for non-vanishing components of the wave functional the Lorentz constraint \((9)\) and the definition equation \((8)\) have no common solution which could be expressed in terms of \(\gamma\)-matrices. Unfortunately, this contradicts the results of \(\mathbb{S}\), where the incorrect conclusions are due to a computational error.

To solve this problem let us recall that equation \((8)\) is valid only when the spatial part of the local vierbein is an \(SO(3)\) basis. This suggests that the spatial components \(J_{ab}\) should be related with the generators of the rotation group. In fact, if we identify the three independent components of \(J_{ab}\) with the three generators \(J^a\) of the ordinary rotation group by means of the relationship

\[
\mathcal{J}_{ab} = i \varepsilon_{abc} J^c,
\]

with \(\varepsilon_{123} = 1\), and use standard conventions of \(\mathbb{7}\)

\[
\mathcal{J}_{23} = i J^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},
\]

\[
\mathcal{J}_{13} = -i J^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},
\]

\[
\mathcal{J}_{12} = i J^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix},
\]

the Lorentz constraint, represented in Eqs. \((10)–(12)\), can not be used to derive a condition similar to \((13)\), because the representation \((8)–(20)\) is singular and consequently the inverse of the Lorentz operator is not defined. Therefore, we must solve the Lorentz constraint \((10)–(12)\) explicitly as a system which imposes conditions on the components of the wave functional. So this representation is now in agreement with the idea of the Lorentz constraint, as mentioned above.

To find the solution of the Lorentz constraint in the above representation we consider each of the components \(\Psi_I, \Psi_{II}, \Psi_{III}, \Psi_{IV}\) as a 4-component vector. Notice that the component \(\Psi_4\) is not affected by the action of the Lorentz constraint. For the remaining 12 components of \(\Psi_{II}, \Psi_{III}, \Psi_{IV}\) we obtain a set of 9 algebraic equations following from Eqs. \((10)–(12)\), namely

\[
\Psi^I_{II} = \Psi^I_{IV}, \quad \Psi^3_{II} = -\Psi^3_{IV},
\]

\[
\Psi^I_{II} = \Psi^I_{IV}, \quad \Psi^2_{II} = -\Psi^2_{IV},
\]

\[
\Psi^I_{II} = 0, \quad \Psi^2_{II} = -\Psi^2_{IV}.
\]

The right-hand side set of equations of this system implies that \(\Psi^I_{II} = \Psi^I_{III} = \Psi^I_{IV} = 0\), so that each of these vectors possesses one non-vanishing component only. Then, the final form of the wave functional can be written as

\[
\Psi [e, \psi] = \begin{pmatrix} \Psi^1_I \Psi^1_{II} \Psi^1_{III} \\ \Psi^2_I \Psi^2_{II} \Psi^2_{III} \\ \Psi^3_I \Psi^3_{II} \Psi^3_{III} \end{pmatrix}.
\]

This represents the most general solution of the Lorentz constraint, when the spatial part of the local vierbein \(e\) corresponds to an \(SO(3)\) basis. Notice that this final solution also explains why the condition \((13)\) for the components of the Lorentz operator is not valid. In fact, it was obtained under the assumption that all the components of \(\Psi_{II}, \Psi_{III}, \Psi_{IV}\) are non zero, a condition which is satisfied by only 3 of these 12 components.

We see that this final form for \(\Psi [e, \psi]\) contains 7 different components which furthermore must satisfy the supersymmetric constraint. In a previous work \(\mathbb{1}\) we
showed that in the special case of a midisuperspace described by a Gowdy cosmological model, the number of independent components can be reduced to two, after imposing the supersymmetric constraint. Incidentally, this final number of independent components of the wave functional coincides with the number obtained in [3] by applying a different approach for the minisuperspace described by the Bianchi IX cosmological model.

The general solution [24] contains three quantities $\Psi_I$, $\Psi_{II}$, and $\Psi_{IV}$ which are scalars. This is in agreement with the intuitive idea mentioned above about the character of the possible solutions for the Lorentz constraint. The four remaining degrees of freedom are contained in four-component vector $(\Psi_1, \Psi_2, \Psi_3, \Psi_4)$ which is not affected by the Lorentz operator. The particular solution in which only one degree of freedom remains $(\Psi_1, 0, 0, 0)$ and $\Psi_1$ is considered as a scalar, is called the “rest-frame” solution that has been intensively analyzed in the literature because of its simplicity.

Finally, we must show that the singular representation [13–20] is compatible with the definition of the components of the Lorentz operator in terms of the components of the gravitino field $\gamma$. This condition turns out to be identically satisfied. In fact, if we identify the gravitino components in terms of the generators of rotation in the form

$$\phi_a = 2iJ^a, \quad (25)$$

introduce this equation into the right-hand side of Eq. (8), and use the relationship (17) in the left-hand side of the same equation, we obtain [notice that \((J^a)^T = -J^a\)]

$$[J^a, J^b] = i\epsilon_{abc}J^c. \quad (26)$$

This last expression is simply the algebra of the generators of the rotation group in the representation $\gamma^-$. This is an interesting result showing that the components of the Lorentz operator and the gravitino field are completely determined by the generators of the rotation group in such a way that the corresponding algebra represents the compatibility condition for selecting this specific representation.

### III. CONCLUSIONS

Although it is very simple to find solutions to the Lorentz constraint of $N = 1$ supergravity in terms of scalar quantities constructed by means of bosonic and fermionic variables, one can try to solve it explicitly in order to find more general solutions. For the approach in which one tries to represent the Lorentz operator and the gravitino field in terms of the generators of the rotation group. In this case, the solution is simple as given in Eqs. (17) and (25), and the compatibility condition (8) coincides with the algebra satisfied by the generators of the rotation group.

For the present representation of the Lorentz operator to be valid it is necessary that the spatial part of the local vierbein coincides with an $SO(3)$ basis. In this case, the wave functional turns out to have at most 16 independent components. We solved the Lorentz constraint explicitly and found that 9 components vanish so that the general solution contains 7 degrees of freedom, four as the arbitrary components of a 4-vector and three in the form of Lorentz invariant scalars.

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