EXPLICIT APPLICATION OF WALDSPURGER’S THEOREM

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Abstract. For a given cusp form \( \phi \) of even integral weight satisfying certain hypotheses, Waldspurger’s Theorem relates the critical value of the \( L \)-function of the \( n \)-th quadratic twist of \( \phi \) to the \( n \)-th coefficient of a certain modular form of half-integral weight. Waldspurger’s recipes for these modular forms of half-integral weight are far from being explicit. In particular, they are expressed in the language of automorphic representations and (adelic) Hecke characters. We translate these recipes into congruence conditions involving easily computable values of Dirichlet characters. We illustrate the practicality of our ‘simplified Waldspurger’ by giving several examples.

1. Introduction

In 1983 Tunnell [22] gave a remarkable solution to the congruent number problem, assuming the celebrated Birch and Swinnerton-Dyer Conjecture. This ancient Diophantine question asks for the classification of congruent numbers, those positive integers which are the areas of right-angled triangles whose sides are rational numbers. For positive \( n \), write \( E_n : y^2 = x^3 - n^2x \); note that \( E_n \) is the \( n \)-th quadratic twist of \( E_1 \). It is straightforward to show that \( n \) is a congruent number if and only if the elliptic curve \( E_n / \mathbb{Q} \) has positive rank. Tunnell expresses the critical value of the \( L \)-function of \( E_n \) in terms of coefficients of certain modular forms of weight \( 3/2 \). These modular forms are in turn written in terms of theta-series of ternary quadratic forms. Applying the conjecture of Birch and Swinnerton-Dyer, Tunnell is then able to give a simple and elegant criterion for \( n \) to be a congruent number.

Tunnell’s Theorem is a highly non-trivial consequence of a theorem of Waldspurger [24 Théorème 1]. For a given cusp form \( \phi \) of even integral weight satisfying certain hypotheses, Waldspurger’s Theorem relates the critical value of the \( L \)-function of the \( n \)-th quadratic twist of \( \phi \) to the \( n \)-th coefficient of a certain modular form of half-integral weight. Waldspurger’s recipes for these modular forms of half-integral weight are far from being explicit. In particular, they are expressed in the language of automorphic representations and (adelic) Hecke characters. We translate these recipes into congruence conditions involving easily computable values of Dirichlet characters. We illustrate the practicality of our ‘simplified Waldspurger’ by giving several Tunnell-like examples, of which the following is the simplest.

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Proposition 1.1. Let $E$ be an elliptic curve of conductor 50 given by
\begin{equation}
E : Y^2 + XY + Y = X^3 + X^2 - 3X + 1.
\end{equation}
Let $Q_1, Q_2, Q_3, Q_4$ be the following positive-definite ternary quadratic forms,
\begin{align*}
Q_1 &= 25x^2 + 25y^2 + z^2, \\
Q_2 &= 14x^2 + 9y^2 + 6z^2 + 4yz + 6xz + 2xy, \\
Q_3 &= 25x^2 + 13y^2 + 2z^2 + 2yz, \\
Q_4 &= 17x^2 + 17y^2 + 3z^2 - 2yz - 2xz + 16xy.
\end{align*}
Let $n$ be a positive square-free number such that $5 \nmid n$. Then,
\begin{equation*}
L(E_{-n}, 1) = L(E_{-1}, 1) \sqrt{n} \cdot c_n^2
\end{equation*}
where $E_{-n}$ is the $-n$-th quadratic twist of $E$ and
\begin{equation*}
c_n = \sum_{i=1}^{4} \frac{(-1)^{i-1}}{2} \cdot \# \{(x, y, z) : Q_i(x, y, z) = n \}.
\end{equation*}

For the elliptic curve $E$ in (1) and $n$ a positive square-free integer such that $5 \nmid n$, we give a similar formula for $L(E_n, 1)$ that involves 38 quadratic forms.

The paper is arranged as follows. In Section 2 we review Shimura’s decomposition of the space of cusp forms of a certain level and half-integral weight into certain subspaces appearing in Waldspurger’s Theorem. In Section 3 we review the correspondence between Dirichlet characters and (adelic) Hecke characters and we prove a result that allows us to evaluate components of a Hecke character corresponding to a given Dirichlet character. Next, in Section 4 we review the correspondence between modular forms of even integral weight and automorphic representations and prove a result needed for simplifying the hypotheses of Waldspurger’s Theorem. In Section 5 we state Waldspurger’s Theorem in simplified form. To apply Waldspurger’s Theorem in conjunction with the Birch and Swinnerton-Dyer Conjectures it is convenient to express the period of the $n$-th twist of a given elliptic curve in terms of the period of the elliptic curve itself. We do this in Section 6. To apply Waldspurger’s recipes we need to be able to answer questions of the following form: for a given cusp form of half-integral weight $f = \sum a_n q^n$, and positive integers $a, M$, is $a_n = 0$ for all $n \equiv a \pmod{M}$? We give an algorithm for answering this question in Section 7. Finally, Section 8 is devoted to extensive examples which combine our algorithm for computing Shimura decomposition with Waldspurger’s Theorem as made explicit in this paper.

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often not the case for $k = 3$. More precisely, a generating set for $S_0(N, \chi)$ is given by

$$S = \{ \sum_{m=1}^{\infty} \psi(m)mq^{m^2} : 4r_2^2t \mid N \text{ and } \psi \text{ is a primitive odd character of conductor } r_2 \text{ such that } \chi = \left( \frac{-4t}{\cdot} \right) \psi \},$$

which in fact constitutes a basis for $S_0(N, \chi)$ as shown in [13]. The interesting part (from the point-of-view of Waldspurger’s Theorem) of the space $S_{k/2}(N, \chi)$ is the orthogonal complement of $S_0(N, \chi)$ with respect to the Petersson inner product, denoted by $S^\perp_{k/2}(N, \chi)$.

Let $N' = N/2$. For $M \mid N'$ such that $\text{Cond}(\chi^2) \mid M$ and a newform $\phi \in S^\text{new}_{k-1}(M, \chi^2)$ Shimura defines

$$S_{k/2}(N, \chi, \phi) = \{ f \in S^\perp_{k/2}(N, \chi) : T_p f = \lambda_p(\phi) f \text{ for almost all } p \nmid N \};$$

here $T_p(\phi) = \lambda_p(\phi)\phi$ and, gives the following decomposition theorem [17]:

**Theorem 1.** (Shimura) $S^\perp_{k/2}(N, \chi) = \bigoplus_\phi S_{k/2}(N, \chi, \phi)$ where $\phi$ runs through all newforms $\phi \in S^\text{new}_{k-1}(M, \chi^2)$ with $M \mid N'$ and $\text{Cond}(\chi^2) \mid M$.

We point out that the summands $S_{k/2}(N, \chi, \phi)$ occur in Waldspurger’s Theorem and their computation is necessary for explicit applications of that theorem. However the above theorem is not suitable for computation since for any particular prime $p \nmid N$, we do not know if it is included or excluded in the ‘almost all’ condition. In [13] we proved the above theorem with a more precise definition for the spaces $S_{k/2}(N, \chi, \phi)$:

$$S_{k/2}(N, \chi, \phi) = \{ f \in S^\perp_{k/2}(N, \chi) : T_p f = \lambda_p(\phi) f \text{ for all } p \nmid N \},$$

whilst showing that our definition is equivalent to Shimura’s definition. We also proved the following theorem that gives an algorithm for computing the Shimura decomposition.

**Theorem 2.** [13] Let $\phi$ be a newform of weight $k - 1$, level $M$ dividing $N'$, and character $\chi^2$. Let $p_1, \ldots, p_n$ be primes not dividing $N$ satisfying the following: for every newform $\phi' \neq \phi$ of weight $k - 1$, level dividing $N'$ and character $\chi^2$, there is some $p_i$ such that $\lambda_{p_i}(\phi') \neq \lambda_{p_i}(\phi)$, where $T_{p_i}(\phi) = \lambda_{p_i}(\phi) \cdot \phi$. Then

$$S_{k/2}(N, \chi, \phi) = \bigg\{ f \in S_{k/2}(N, \chi) : T_{p_i} f = \lambda_{p_i}(\phi) f \quad \text{for } i = 1, \ldots, n \bigg\}.$$

### 3. Correspondence between Dirichlet Characters and Hecke Characters on $\mathbb{A}_{\mathbb{Q}}^\times / \mathbb{Q}^\times$ of Finite Order

We shall need the correspondence between Dirichlet characters and Hecke characters on $\mathbb{A}_{\mathbb{Q}}^\times / \mathbb{Q}^\times$ of finite order. This material is in Tate’s thesis [5, Chapter XV], but we found the presentation in [3, Section 3.1] more useful.

**Proposition 3.1.** Let $\chi = (\chi_p)$ be a character on $\mathbb{A}_{\mathbb{Q}}^\times$. Then there exists a finite set $S$ of places, including all the Archimedean ones, such that if $p \notin S$, then $\chi_p$ is trivial on the unit group $\mathbb{Z}_p^\times$.

Recall that if $\chi_p$ is trivial on the unit group $\mathbb{Z}_p^\times$, then $\chi_p$ is unramified. Thus by the above proposition, $\chi_p$ is unramified for all but finitely many $p$. 
Theorem 3. (3, Proposition 3.1.2) Suppose $\chi = (\chi_p)$ is a character of finite order on $\mathbb{A}_Q^\times/Q^\times$. There exists an integer $N$ whose prime divisors are precisely the non-Archimedean primes $p$ such that $\chi_p$ is ramified, and a primitive Dirichlet character $\chi$ modulo $N$ such that if $p \nmid N$ is non-Archimedean then $\chi(p) = \chi_p(p)$. This correspondence $\chi \mapsto \chi$ is a bijection between characters of finite order of $\mathbb{A}_Q^\times/Q^\times$ and the primitive Dirichlet characters.

In our work, we shall need to start with a Dirichlet character $\chi$ of modulus $N$ and then do computations with the corresponding adelic character $\chi$. We collect here some facts that will help us with these computations.

Lemma 3.2. We keep the notation of Theorem 3.

(i) For any $\alpha \in \mathbb{Q}^\times$, $\prod \chi_p(\alpha) = 1$.

(ii) Suppose $p = \infty$ and $\alpha \in \mathbb{Q}_\infty^\times = \mathbb{R}^\times$. Then $\chi_\infty(\alpha) = 1$ if $\alpha > 0$, or if $\chi$ has odd order.

(iii) Let $p$ be a non-Archimedean prime such that $p \mid N$ and $\alpha, \beta \in \mathbb{Z}_p^\times$ be non-zero. Suppose that $\beta \equiv \alpha \pmod{\alpha \mathbb{Z}_p}$.

Then $\chi_p(\beta) = \chi_p(\alpha)$.

(iv) Let $p$ be non-Archimedean such that $p \nmid N$ then, $\chi_p$ is unramified.

Proposition 3.3. Let $\chi$ be a Dirichlet character modulo $N$ (not necessarily primitive) and let $\chi = (\chi_p)$ be the corresponding character on $\mathbb{A}_Q^\times/Q^\times$. Let $a \in \mathbb{Z}$ be non-zero.

(a) If $q \nmid N$ then $\chi_q(a) = \chi(q)^r$ where $r = \text{ord}_q(a)$.

(b) Suppose $q$ divides $N$ and let $q_1, \ldots, q_r$ be the other primes dividing $N$. Let $b$ be a positive integer satisfying

$$b \equiv \begin{cases} a \pmod{aN\mathbb{Z}_q} \\ 1 \pmod{N\mathbb{Z}_{q_i}} \end{cases} \quad i = 1, \ldots, r;$$

such $b$ can easily be constructed by the Chinese Remainder Theorem. Write

$$b = q^{\text{ord}_q(a)} \prod_{j=1}^{s} \ell_j^{\beta_j}$$

where the $\ell_j$ are distinct primes. Then

$$\chi_q(a) = \prod_{j=1}^{s} \chi(\ell_j)^{-\beta_j}.$$

Proof. Let $N'$ be the conductor of $\chi$ and note that $N' \mid N$. Now if $q \nmid N$ then, $\chi_q$ is unramified. Write $a = q'a'$ where $q \nmid a'$. Then $a' \in \mathbb{Z}_q^\times$. Thus by definition of unramified, $\chi_q(a') = 1$. Moreover, from Theorem 3 $\chi_q(q) = \chi(q)$. This proves (a).

Now suppose $q \mid N$ and let $q_1, \ldots, q_r$ be the other primes dividing $N$. Let $b$ be as in the proposition. Since $N' \mid N$, we have

$$b \equiv \begin{cases} a \pmod{aN'\mathbb{Z}_q} \\ 1 \pmod{N'\mathbb{Z}_{q_i}} \end{cases} \quad i = 1, \ldots, r.$$
Theorem 4. and Li \[1\].

\[ \chi_q(b) = \chi_q(a), \text{ and } \chi_{q_i}(b) = 1 \text{ for } i = 1, \ldots, r. \]

\[ \chi_q(a) = \chi_q(b) = \prod_{p \nmid q} \chi_p(b)^{-1} \quad \text{by (i) of Lemma 3.2} \]

\[ = \prod_{p \nmid N} \chi_p(b)^{-1} \quad \text{since } \chi_{q_i}(b) = 1, \]

\[ = \prod_{j=1}^s \chi(\ell_j)^{-\beta_j} \quad \text{using part (a)}. \]

This completes the proof. \[ \square \]

4. Correspondence between Modular Forms of Even Integer Weight and Automorphic Representations

Let \( k \) be a positive odd integer with \( k \geq 3 \). Let \( \phi = \sum_{n=1}^{\infty} a_n q^n \in S_{k-1}^{\text{new}}(N, \chi) \) be a newform of weight \( k - 1 \), level \( N \) and character \( \chi \).

We can associate to \( \phi \) an automorphic representation \( \rho \). Let \( \rho_p \) be the local component of \( \rho \) at a prime \( p \).

If \( \phi = \sum_{n=1}^{\infty} a_n q^n \) is an eigenform, then we define its twist by a character \( \mu \) to be the modular form \( \phi_\mu = \sum_{n=1}^{\infty} a_n \mu(n) q^n \).

Waldspurger works with the following different definition of twist: Let \( \phi \) be a newform of weight \( k - 1 \) and character \( \chi \). Let \( \mu \) a Dirichlet character. We denote by \( \phi \otimes \mu \) the (unique) newform of weight \( k - 1 \) with character \( \chi \mu^2 \) satisfying \( \lambda_p(\phi \otimes \mu) = \mu(p) \lambda_p(\phi) \) for almost all primes \( p \), where \( \lambda_p \) is the eigenvalue under \( T_p \).

Now fix a prime number \( p \). Let \( \xi_p \) be the set of primitive Dirichlet characters with \( p \)-power conductor. The following holds (see [14, Section III]):

(i) \( \rho_p \) is supercuspidal if and only if for all \( \mu \in \xi_p \), the level of \( \phi \otimes \mu \) is divisible by \( p \) and \( \lambda_p(\phi \otimes \mu) = 0 \).

(ii) \( \rho_p \) is an irreducible principal series if and only if either

(a) there exists a character \( \mu \) in \( \xi_p \) such that \( p \) does not divide the level of \( \phi \otimes \mu \); or,

(b) there exist two distinct characters \( \mu_1, \mu_2 \) in \( \xi_p \) such that \( \lambda_p(\phi \otimes \mu_1) \neq 0, \lambda_p(\phi \otimes \mu_2) \neq 0 \).

(iii) \( \rho_p \) is a special representation if and only if the following conditions hold:

(a) for all \( \mu \in \xi_p \), the level of \( \phi \otimes \mu \) is divisible by \( p \);

(b) there exists a unique \( \mu \) in \( \xi_p \) such that \( \lambda_p(\phi \otimes \mu) \neq 0 \).

We shall need the following theorem which is extracted from the paper of Atkin and Li [1].

**Theorem 4. (Atkin and Li)** Let \( \phi = \sum_{n=1}^{\infty} a_n q^n \) be a newform of weight \( k - 1 \), character \( \chi \) and level \( N \). Let \( \mu \) be a primitive character of conductor \( m \). Then

(a) If \( \gcd(m, N) = 1 \) then \( \phi \otimes \mu = \phi_\mu \), and it is a newform of weight \( k - 1 \), character \( \chi \mu^2 \) and level \( Nm^2 \) ([1, Introduction]).

(b) Suppose \( \mu \) is of \( q \)-power conductor where \( q \mid N \) and write \( N = q^s M \) where \( q \nmid M \). Then \( \phi \otimes \mu \) is a newform of weight \( k - 1 \), character \( \chi \mu^2 \) and level \( q^s M \) for some \( s' \geq 0 \). Moreover, \( \lambda_p(\phi \otimes \mu) = \mu(p) \lambda_p(\phi) \) for all primes \( p \nmid N \) ([1, Theorem 3.2]). In particular if \( s = 1 \) and \( \chi \) is trivial, then for
µ with conductor q^r, r ≥ 1, it turns out that φ ⊗ µ = φµ is a newform of level q^{2r}M and character µ^2 \[\text{[1 Corollary 4.1]}\).

(c) Let q \mid N. Suppose φ is q-primitive and a_2 = 0. Then for all characters µ of q-power conductor, φ ⊗ µ = φµ is a newform of level divisible by N (Recall that φ is q-primitive if φ is not a twist of any newform of level lower than N by a character of conductor equal to some power of q) \[\text{[1 Proposition 4.1]}\).

(d) Let N = q^sM where q \nmid M; let Q = q^s. Let χ_Q be the Q-part \[\text{[1]}\] of the character χ. If s is odd and cond χ_Q ≤ \sqrt{Q} then φ is q-primitive.

Now suppose q = 2. Then, if s = 2 then φ is always 2-primitive; if s is odd then φ is 2-primitive if and only if cond χ_Q < \sqrt{Q}; if s is even and s ≥ 4 then φ is 2-primitive if and only if cond χ_Q = \sqrt{Q} \[\text{[1 Theorem 4.4]}\).

We deduce the following corollaries which we will be using later.

**Corollary 4.1.** Let φ = ∑_{n=1}^{∞} a_nq^n ∈ S_{k-1}^{new}(N) be a newform with trivial character. Let ρ_2 be the local component at 2 of the corresponding automorphic representation. Suppose either

(i) N is odd; or

(ii) ν_2(N) = 1 and a_2 ≠ 0.

Then ρ_2 is not supercuspidal.

If ν_2(N) = 1 and a_2 = 0 then ρ_2 is supercuspidal. Further if ν_2(N) ≥ 2 and φ is 2-primitive then ρ_2 is supercuspidal, hence if either ν_2(N) = 2 or ν_2(N) > 1 is odd then ρ_2 is supercuspidal.

**Proof.** If N is odd, take µ to be the identity character. Thus µ ∈ ξ_2 and the level of φ ⊗ µ is odd and hence ρ_2 is not supercuspidal. If N = 2M such that M is odd and a_2 ≠ 0, again taking µ as identity character we get that λ_2(φ ⊗ µ) = a_2 ≠ 0 and thus ρ_2 is not supercuspidal.

Let ν_2(N) ≥ 2. Then a_2 = 0. If φ is 2-primitive then it follows using part (c) of Theorem \[\text{[1]}\] that for any µ ∈ ξ_2, φ ⊗ µ = φµ is newform of level divisible by 2. Write T_2(φµ) = ∑_{n=1}^{∞} b_nq^n. Then, b_n = a_{2n}µ(2n) + µ^2(2)2^{k-2}a_{n/2}µ(n/2) for all n. Thus T_2(φµ) = 0. Therefore, λ_2(φ ⊗ µ) = λ_2(φµ) = 0 and ρ_2 is supercuspidal. If ν_2(N) = 1 and a_2 = 0 then similarly conclude using part (b) that ρ_2 is supercuspidal. The final statement is a direct application of part (d) of Theorem \[\text{[1]}\]. \[\square\]

**Corollary 4.2.** Let φ be as in the above corollary.

(i) If N = pM with M coprime to p and a_p ≠ 0, then ρ_p is a special representation.

(ii) If p \nmid N, then ρ_p is an irreducible principal series.

**Proof.** We first prove (i). By part (b) of the Theorem \[\text{[1]}\] for any µ ∈ ξ_p, the level of φ ⊗ µ is divisible by p. Further if µ is the identity character then λ_p(φ ⊗ µ) = a_p ≠ 0; we claim that this is unique such character in ξ_p. Let µ ∈ ξ_p be such that µ is a character of conductor p^r, r ≥ 1. Then φ ⊗ µ = φµ is a newform in S_{k-1}(p^{2r}M, µ^2) such that λ_p(φµ) = a_pµ(p) = 0 and hence λ_p(φ ⊗ µ) = 0.

1 Let χ be a Dirichlet character with modulus p_1^{e_1} · · · p_s^{e_s} where the p_i are distinct primes. Then χ can be written uniquely as a product ∏ χ_{p_i^{e_i}} where χ_{p_i^{e_i}} has modulus p_i^{e_i}. See \[\text{[1]}\].
The proof of (ii) is obvious and does not require the condition that newform \( \phi \) has trivial character. \( \square \)

5. Waldspurger’s Theorem and Notation

In this section we will present Waldspurger’s Theorem. We will introduce and simplify the notation used in the theorem. This is needed in the following section where we will discuss how to use the theorem for elliptic curves and compute critical values of \( L \)-functions in terms of coefficients of corresponding half-integral weight forms. An important application is the computation of orders of the Tate-Shafarevich groups assuming the Birch and Swinnerton-Dyer Conjecture.

Let \( k \) be positive integers with \( k \geq 3 \) odd. Let \( \chi \) be an even Dirichlet character with modulus divisible by 4. Fix a newform \( \phi \) of level \( M_\phi \) in \( S^\text{new}_{k-1}(M_\phi, \chi^2) \). Let \( p \) be a prime number. Let \( \nu_p \) be the \( p \)-adic valuation on \( \mathbb{Q} \) and \( \mathbb{Q}_p \times \mathbb{Q}_p / \mathbb{Q}_p \). Let \( m_p = \nu_p(M_\phi \chi^2) \) and \( \lambda_p \) be the Hecke eigenvalue of \( \phi \) corresponding to the Hecke operator \( T_p \).

Let \( \rho \) be the automorphic representation associated to \( \phi \) and \( \rho_p \) be the local component of \( \rho \) at \( p \). Let \( S \) be the (finite) set of primes \( p \) such that \( \rho_p \) is not irreducible principal series. If \( p \notin S \), \( \rho_p \) is equivalent to \( \pi(\mu_1, p, \mu_2, p) \) where \( \mu_1, p \) and \( \mu_2, p \) are two continuous characters on \( \mathbb{Q}_p \) such that \( \mu_1, p \mu_2, p \neq |\cdot|^{1/2} \).

Let \( \mu_1, p(-1) = \mu_2, p(-1) = 1 \). Let \( (H1) \) be the following hypothesis:

\[ (H1) \quad \text{For } p \notin S, \mu_1, p(-1) = \mu_2, p(-1) = 1. \]

Theorem 5. (Flicker) There exists \( N \) such that \( S_{k/2}(N, \chi, \phi) \neq \{0\} \) if and only if the hypothesis \( (H1) \) holds.

Theorem 6. (Vigneras) Flicker’s condition \( (H1) \) always holds whenever \( \phi \) is a newform of even weight with trivial character.

Proof. For the proof refer to [23]. \( \square \)

From the theorems of Flicker and Vigneras we have the following easy corollary.

Corollary 5.1. Let \( \phi \) be a newform of weight \( k-1 \), level \( M_\phi \) and trivial character \( \chi_{\text{triv}} \). Let \( \chi \) be a Dirichlet character satisfying \( \chi^2 = \chi_{\text{triv}} \). Then there exists some \( N \) such that \( S_{k/2}(N, \chi, \phi) \neq \{0\} \).

Henceforth, we will always assume that \( \phi \) has trivial character and \( \chi \) is quadratic, thus the conclusion of the corollary holds. We will now introduce several pieces of notation used by Waldspurger [24 Section VIII] before stating his main theorem.

Let \( \chi_0 \) be the Dirichlet character associated to \( \chi \) given by

\[ \chi_0(n) := \chi(n) \left( \frac{-1}{n} \right)^{(k-1)/2}. \]

Let \( \chi_{0, p} \) be the local component of \( \chi_0 \) at prime \( p \). For each prime \( p \) we will later define non-negative integer \( n_p \) that depends only on the local components \( \rho_p \) and \( \chi_{0, p} \). Let \( \overline{N_\phi} \) be given by

\[ \overline{N_\phi} := \prod_p p^{\overline{n_p}}. \]

For prime \( p \) and natural number \( e \), we will later define a set \( U_p(e, \phi) \) which consists of some finite number of complex-valued functions on \( \mathbb{Q}_p^\times \) having support in \( \mathbb{Z}_p \cap \mathbb{Q}_p^\times \).

Let \( \mathbb{N}^\text{sc} \) be the set of positive square-free numbers and for \( n \in \mathbb{N} \), let \( n^\text{sc} \) be the square-free part of \( n \). Let \( A \) be a function on the set \( \mathbb{N}^\text{sc} \) having values in \( \mathbb{C} \) and \( E \)
be an integer such that \( \tilde{N}_\phi | E \). Denote \( e_p = \nu_p(E) \) for all prime numbers \( p \) and let \( \mathfrak{z} = (e_p) \) be any element of \( \prod_p U_p(e_p, \phi) \). Define

\[
f(\mathfrak{z}, A)(z) := \sum_{n=1}^{\infty} A(n^\infty) n^{(k-2)/4} \prod_p c_p(n)q^n, \quad z \in \mathbb{H}
\]

and let \( \mathcal{U}(E, \phi, A) \) be the space generated by these functions \( f(\mathfrak{z}, A) \) on \( \mathbb{H} \) where \( \mathfrak{z} \in \prod_p U_p(e_p, \phi) \).

With the above notation, we are now ready to state the main theorem of Waldspurger [24, Page 481].

**Theorem 7.** (Waldspurger) Let (H2) be the following hypothesis: One of the following holds:

1. the local component \( \rho_2 \) is not supercuspidal;
2. the conductor of \( \chi_0 \) is divisible by 16;
3. \( 16 \mid M_\phi \).

Let \( \chi \) be a Dirichlet character and \( \phi \) be a newform of weight \( k-1 \) and character \( \chi^2 \) such that (H1) and (H2) hold. Then there exists a function \( A_\phi \) on \( \mathbb{N}^\infty \) such that for \( t \in \mathbb{N}^\infty \):

\[
A_\phi(t)^2 := L(\phi \otimes \chi_0^{-1} \chi_t, 1) \cdot \epsilon(\chi_0^{-1} \chi_t, 1/2).
\]

Moreover, for \( N \geq 1 \),

\[
S_{k/2}(N, \chi, \phi) = \bigoplus \mathcal{U}(E, \phi, A_\phi)
\]

where the sum is over all \( E \geq 1 \) such that \( \tilde{N}_\phi | E | N \).

Here \( \chi_t = (4) \) is a quadratic character with conductor \( |t| \) if \( t \equiv 1 \) (mod 4), otherwise with conductor \( |4t| \) if \( t \equiv 2, 3 \) (mod 4).

**Remark.** Note that the function \( A_\phi \) depends only on \( \chi \) and \( \phi \). However \( A_\phi \) is not deterministic, so we cannot use this theorem for computing the basis for the space \( S_{k/2}(N, \chi, \phi) \). However in Theorem 2 we have already given an algorithm to compute this space and if \( f(z) = \sum_{n=1}^{\infty} a_n q^n \) is one of the basis elements then we can express the critical value of the L-function of twist of the newform \( \phi \) with character \( \chi_0^{-1} \chi_t \), in terms of the square of the Fourier coefficient \( a_t \) and the factor \( \epsilon(\chi_0^{-1} \chi_t, 1/2) \) which depends on the local components of \( \phi \) and \( \chi_0 \).

It is to be noted that \( \epsilon(\chi, 1/2) \) for any Hecke character \( \chi \) can be computed as shown in Tate’s article [21]. In particular, when \( \chi \) is quadratic, \( \epsilon(\chi, 1/2) = 1 \). Since we will be only dealing with the quadratic characters, we can ignore the \( \epsilon \)-factor. Moreover, note that if \( \chi \) is quadratic, then the conductor of \( \chi_0 \) is at most divisible by 8, so we do not need to consider possibility (b) of the hypothesis (H2).

Further by Corollary 1.1 possibilities (a) and (c) of the hypothesis (H2) can be simply stated in terms of the level \( M_\phi \). Assuming \( \chi \) to be quadratic, Waldspurger’s Theorem is applicable whenever either \( M_\phi \) is odd; or \( \nu_2(M_\phi) = 1 \) and \( \lambda_2 \neq 0 \); or \( \nu_2(M_\phi) \geq 4 \). The last condition is the same as possibility (c) of (H2).

In what follows \( \epsilon(\cdot, \cdot)_p \) stands for the Hilbert symbol defined on \( \mathbb{Q}_p^\times \times \mathbb{Q}_p^\times \). Recall that (see for example, [3]) if \( p = 2 \) and \( a, b \) are odd then

\[
(2^a a, 2^b b)_2 = \left( \frac{2}{|a|} \right)^t \left( \frac{2}{|b|} \right)^s (-1)^{(a-1)(b-1)/4}.
\]
For an odd prime $p$ and $a, b$ coprime to $p$,

$$ (p^s a, p^s b)_p = \left( \frac{-1}{p} \right)^{st} \left( \frac{a}{p} \right)^t \left( \frac{b}{p} \right)^s. $$

In particular, for an odd $n$, $(n, -1)_2 = (-1)^{n^2 - 1}$ and $(2, n)_2 = (-1)^{\frac{n^2 - 1}{2}}$. Also, if $\nu_p(n) = 0$ then $(p, n)_p = \left( \frac{n}{p} \right)$ and, if $\nu_p(n) = 1$ and $n = pm^2$, then $(p, n)_p = \left( \frac{m}{p} \right)$.

We now write down explicitly the definitions of the integers $\hat{\nu}$ and the local factors $U(\epsilon, \phi)$ used in Waldspurger’s Theorem. It is to be noted that for Waldspurger’s Theorem, we would be only requiring the values of the functions in $\mathcal{U}(\epsilon, \phi)$ at square-free positive integers. We will first define a certain set of functions.

Case 1. $p$ odd.

Waldspurger considered the following set of functions which we will be denoting as $A_p$:

$$ A_p = \{ c_{p, \delta}^{(0)}, c_{p, \delta}^{(1)}, c_{p, \delta}^{(2)}, c_{p, \delta}^{(3)}, c_{p, \delta}^{(4)}, c_{p, \delta}^{(5)}, c_{p, \delta}^{(6)} : \delta \in \mathbb{C} \}. $$

We will be only interested in values of the functions in $A_p$ at square-free numbers in $\mathbb{Z}_p \setminus \{0\}$. Let $n \in \mathbb{Z}_p \setminus \{0\}$ be square-free, hence we have $\nu_p(n) = 0$ or $\nu_p(n) = 1$.

We get the following after simplification:

$$ c_{p, \delta}^{(0)}(n) = \begin{cases} 1 \quad \nu_p(n) = 0 \\ 1 - (p, n)_p \chi_0, p \nu_p(n) = 1. \end{cases} $$

$$ c_{p, \delta}^{(1)}(n) = \begin{cases} 1 \quad \nu_p(n) = 0 \\ \delta \quad \nu_p(n) = 1. \end{cases} $$

$$ c_{p, \delta}^{(2)}(n) = \begin{cases} 1 - (p, n)_p \chi_0, p \nu_p(n) = 0 \\ 1 \quad \nu_p(n) = 1. \end{cases} $$

$$ c_{p, \delta}^{(3)}(n) = \begin{cases} 1 \quad \nu_p(n) = 0 \\ \delta - (p, n)_p \chi_0, p \nu_p(n) = 1. \end{cases} $$

$$ c_{p, \delta}^{(4)}(n) = \begin{cases} 0 \quad \delta(p - 1) \nu_p(n) = 0 \\ \delta(p - 1) \nu_p(n) = 1. \end{cases} $$

$$ c_{p, \delta}^{(5)}(n) = \begin{cases} 2^{1/2} \quad \nu_p(n) = 0, \quad (p, n)_p = -p^{1/2} \chi_0, p(p^{-1}) \delta \\ 0 \quad \nu_p(n) = 0, \quad (p, n)_p = p^{1/2} \chi_0, p(p^{-1}) \delta \\ 1 \quad \nu_p(n) = 1. \end{cases} $$

$$ c_{p, \delta}^{(6)}(n) = \begin{cases} 1 \quad \nu_p(n) = 0 \\ 2^{1/2} \delta \quad \nu_p(n) = 1, \quad (p, n)_p = -p^{1/2} \chi_0, p(p^{-1}) \delta \\ 0 \quad \nu_p(n) = 1, \quad (p, n)_p = p^{1/2} \chi_0, p(p^{-1}) \delta. \end{cases} $$

Case 2. $p = 2$.

In this case Waldspurger consider the following set of functions which we will be denoting as $A_2$:

$$ A_2 = \{ c_{2, \delta}^{(0)}, c_{2, \delta}^{(1)}, c_{2, \delta}^{(2)}, c_{2, \delta}^{(3)}, c_{2, \delta}^{(4)}, c_{2, \delta}^{(5)}, c_{2, \delta}^{(6)} : \delta \in \mathbb{C} \}. $$
Let \( n \in \mathbb{Z}_2 \setminus \{0\} \) be square-free so that either \( \nu_2(n) = 0 \) or \( \nu_2(n) = 1 \). We have:

\[
c_{2,\delta}^{(0)}(n) = \begin{cases} 1 & \nu_2(n) = 0 \\ \delta & \nu_2(n) = 1, \end{cases}
\]

\[
c_{2,\delta}^{(1)}(n) = \begin{cases} \delta - (2, n)2\chi_{0,2}(2)2^{-1/2} & \nu_2(n) = 0, (n, -1)_2 = \chi_{0,2}(-1) \\ 1 & \nu_2(n) = 0, (n, -1)_2 = -\chi_{0,2}(-1) \\ 1 & \nu_2(n) = 1, \end{cases}
\]

\[
c_{2,\delta}^{(2)}(n) = \begin{cases} \delta & \nu_2(n) = 0, (n, -1)_2 = \chi_{0,2}(-1) \\ 0 & \nu_2(n) = 0, (n, -1)_2 = -\chi_{0,2}(-1) \\ 0 & \nu_2(n) = 1, \end{cases}
\]

\[
c_{2,\delta}^{(3)}(n) = \begin{cases} \delta^{-1} & \nu_2(n) = 0 \\ \delta - (2, n)2\chi_{0,2}(2)2^{-1/2} & \nu_2(n) = 1, (n, -1)_2 = \chi_{0,2}(-1) \\ 1 & \nu_2(n) = 1, (n, -1)_2 = -\chi_{0,2}(-1), \end{cases}
\]

\[
c_{2,\delta}^{(4)}(n) = \begin{cases} 0 & \nu_2(n) = 0, (n, -1)_2 = \chi_{0,2}(-1) \\ 2\delta - (2, n)2\chi_{0,2}(2)2^{-1/2} & \nu_2(n) = 1, (n, -1)_2 = \chi_{0,2}(-1) \\ 1 & \nu_2(n) = 1, (n, -1)_2 = -\chi_{0,2}(-1), \end{cases}
\]

\[
c_{2,\delta}^{(5)}(n) = \begin{cases} 2^{1/2}\delta & \nu_2(n) = 0, (n, -1)_2 = \chi_{0,2}(-1), (2, n)_2 = 2^{1/2}\chi_{0,2}(2^{-1})\delta \\ 1 & \nu_2(n) = 0, (n, -1)_2 = -\chi_{0,2}(-1) \\ 1 & \nu_2(n) = 1, \end{cases}
\]

\[
c_{2,\delta}^{(6)}(n) = \begin{cases} \delta^{-1} & \nu_2(n) = 0 \\ 0 & \nu_2(n) = 1, (n, -1)_2 = \chi_{0,2}(-1), (2, n)_2 = 2^{1/2}\chi_{0,2}(2^{-1})\delta \\ 2^{1/2}\delta & \nu_2(n) = 1, (n, -1)_2 = \chi_{0,2}(-1), (2, n)_2 = -2^{1/2}\chi_{0,2}(2^{-1})\delta \\ 1 & \nu_2(n) = 1, (n, -1)_2 = -\chi_{0,2}(-1). \end{cases}
\]

We will be interested in the above functions for only particular values of \( \delta \). We will specify and further simplify them later.

Recall that \( \lambda_p \) is the Hecke eigenvalue of \( \phi \) corresponding to the Hecke operator \( T_p \) for any prime \( p \), and \( m_p = \nu_p(M_\phi) \). Let \( \chi_p' = p^{1-k/2}\lambda_p \). For \( p \nmid M_\phi \) let \( \alpha_p \) and \( \alpha_p' \) be such that

\[
\alpha_p + \alpha_p' = \lambda_p, \\
\alpha_p \cdot \alpha_p' = 1.
\]

It is to be noted that if \( \phi \) is rational newform of weight 2 then \( \alpha_p \neq \alpha_p' \), since otherwise \( \chi_p^2 = 4p^{k-2} \), which is a contradiction as \( \lambda_p \) is rational (\( p \)-th Fourier coefficient of \( \phi \)).

Next, we need to consider a subset of \( \mathbb{Q}_p^\times / \mathbb{Q}_p^{\times 2} \), denoted by \( \Omega_p(\phi) \), which is defined as

\[
\Omega_p(\phi) = \{ \omega \in \mathbb{Q}_p^\times / \mathbb{Q}_p^{\times 2} : \exists f \in S_{k/2}(N, \chi, \phi) \text{ for some } N \text{ and } \exists n \geq 1 \text{ such that } i) \text{ image of } n \text{ in } \mathbb{Q}_p^\times / \mathbb{Q}_p^{\times 2} \text{ is } \omega ; ii) \text{ } n \text{th coefficient of } f \neq 0 \}.
\]

It is to be noted that the set \( \Omega_p(\phi) \) depends on the newform \( \phi \) and character \( \chi \) that we started with. Computation of this set is important in our applications and
we will see that we need this set only in the case when \( m_p \geq 1 \) and \( \lambda_p = 0 \). Since this set consists of at most eight elements when \( p = 2 \), and four when \( p \) is an odd prime, computation doesn’t seem to be difficult. Indeed, we can use the results of Section 7 and our algorithm in Theorem 2 to compute most of the elements.

Waldspurger defined another set of local functions on \( \mathbb{Q}_p^\times / \mathbb{Q}_p^\times 2 \) which takes values in \( \mathbb{Z} / 2\mathbb{Z} \) and denote this set by \( \Gamma_p \),

\[
\Gamma_p = \{ \gamma_{e,v} : e \in \mathbb{Z}, \ v \in \mathbb{Q}_p^\times / \mathbb{Q}_p^\times 2 \text{ such that } \nu_p(v) \equiv e \pmod{2} \},
\]

where

\[
\gamma_{e,v}(u) = \begin{cases} 
1 & u \in v\mathbb{Q}_p^\times, \ \nu_p(u) = e \\
0 & \text{else.} 
\end{cases}
\]

If \( p = 2 \), define

\[
\gamma'_{e,v} = \frac{1}{2} (\gamma_{e,v} + \gamma_{e,5v}),
\]

\[
\gamma''_{e}(u) = \begin{cases} 
1 & \nu_2(u) = e \\
0 & \text{else,}
\end{cases}
\]

and

\[
\gamma^0_{e}(u) = \begin{cases} 
1 & \nu_2(u) = e, \ (u,-1)_2 = -\chi_0,2(-1) \text{ or } \nu_2(u) = e + 1 \\
0 & \text{else.}
\end{cases}
\]

Now we are ready to define the local factors \( \tilde{n}_p \) and the set \( U_p(e, \phi) \) for \( e = \tilde{n}_p \).

We will be dealing with several cases and subcases and in each of them we will be simplifying Waldspurger’s formulae and making them more explicit for our use.

Case 1. \( p \) odd and \( m_p \geq 1 \).

We consider the following subcases:

(a) \( \lambda_p = 0 \).

In this case we need to compute \( \Omega_p(\phi) \). We know that \( \mathbb{Q}_p^\times / \mathbb{Q}_p^\times 2 = \{1, \ p, \ u, \ pu\} \) where \( u \) is unit in \( \mathbb{Z}_p \) which is a non-square mod \( p \). If there exists a \( \omega \in \Omega_p(\phi) \) such that \( \nu_p(\omega) = 0 \) then \( \tilde{n}_p = m_p \), and \( U_p(\tilde{n}_p, \phi) = \{ \gamma_{0,\omega} : \omega \in \Omega_p(\phi) \text{ and } \nu_p(\omega) = 0 \} \). In this case, the set \( U_p(\tilde{n}_p, \phi) \) consists of at most the functions \( \gamma_{0,1} \) and \( \gamma_{0,n} \). Otherwise, for all \( \omega \in \Omega_p(\phi), \ \nu_p(\omega) = 1 \). In this case \( \tilde{n}_p = m_p + 1 \), and \( U_p(\tilde{n}_p, \phi) = \{ \gamma_{1,\omega} : \omega \in \Omega_p(\phi) \text{ and } \nu_p(\omega) = 1 \} \), hence \( U_p(\tilde{n}_p, \phi) \) consists of at most \( \gamma_{1,p} \) and \( \gamma_{1,pu} \). Note that \( \gamma_{0,1}, \ \gamma_{0,n}, \ \gamma_{1,p}, \ \gamma_{1,pu} \) are characteristic functions of \( 1, \ u, \ p, \ pu \) modulo \( \mathbb{Q}_p^\times 2 \) respectively.

(b) \( \lambda_p \neq 0 \).

In this case we must have \( m_p = 1 \), since \( m_p \geq 2 \) implies that \( \lambda_p = 0 \). Recall that \( S \) is the collection of primes \( p \) such that \( \rho_p \) is not irreducible principal series. We have further subcases:

(i) \( p \notin S \).

By Waldspurger, in this case \( \tilde{n}_p = m_p = 1 \). Let \( \beta_p \in \mathbb{C} \) such that \( \beta_p^2 = \lambda_p \). Then \( U_p(1, \phi) = \{ e^{(1)}_{p,\beta_p} \} \).

However we note that we do not need to consider this subcase since by Corollary 4.2, \( \rho_p \) is a special representation and hence not a principal irreducible series. Thus in this case we always have \( p \in S \).

(ii) \( p \in S \).

Here we have the following subcases:
(i) $\chi_{0,p}$ is unramified.
Here again $\nu_p = m_p = 1$ and $U_p(1, \phi) = \{e^{(5)}_{p,\lambda_p}, e^{(5)}_{p,\lambda_p'}\}$. We use the theory of newforms to simplify the function $e^{(5)}_{p,\lambda_p}$. Since $m_p = 1$ we get that $\lambda_p = -\omega_p p^{(k-3)/2}$ and $\lambda_p' = -\omega_p p^{-1/2}$. Here $\omega_p \in \{\pm 1\}$ is the eigenvalue under the Atkin-Lehner involution corresponding to prime $p$. Hence we have in this case,

$$e^{(5)}_{p,\lambda_p}(n) = \begin{cases} 
2^{1/2} \nu_p(n) = 0, \left(\frac{n}{p}\right) = \omega_p \chi_{0,p}(p^{-1}) \\
0 \nu_p(n) = 0, \left(\frac{n}{p}\right) = -\omega_p \chi_{0,p}(p^{-1}) \\
1 \nu_p(n) = 1.
\end{cases}$$

(ii) $\chi_{0,p}$ is ramified.

We have $\nu_p = m_p = 1$ and $U_p(1, \phi) = \{e^{(6)}_{p,\lambda_p}, e^{(6)}_{p,\lambda_p'}\}$. As in the above subcase, we get the following simplification:

$$e^{(6)}_{p,\lambda_p}(n) = \begin{cases} 
1 \nu_p(n) = 0 \\
-\omega_p 2^{1/2} p^{-1/2} \nu_p(n) = 1, (p,n) = \omega_p \chi_{0,p}(p^{-1}) \\
0 \nu_p(n) = 1, (p,n) = -\omega_p \chi_{0,p}(p^{-1}) 
\end{cases}$$

Case 2. $p$ odd and $m_p = 0$.
We have the following subcases:

(a) $\chi_{0,p}$ is unramified.

Here, $\nu_p = m_p = 0$ and $U_p(0, \phi) = \{e^{(0)}_{p,\lambda_p}\}$. It is to be noted that $e^{(0)}_{p,\lambda_p}$ takes the value 1 at any square-free $n$.

(b) $\chi_{0,p}$ is ramified.

We have $\nu_p = 1$ and $U_p(1, \phi) = \{e^{(3)}_{p,\alpha_p}, e^{(3)}_{p,\alpha_p'}\}$ if $\alpha_p \neq \alpha_p'$, else $U_p(1, \phi) = \{e^{(3)}_{p,\alpha_p}, e^{(4)}_{p,\alpha_p}\}$. We note that if $p$ does not divide modulus of $\chi$, then we do not need to consider this subcase because in this case $\chi_{0,p}$ is unramified by Lemma 3.2.

Case 3. $p = 2$ and $m_2 \geq 1$.

Consider the following subcases:

(a) $\lambda_2 = 0$.

We compute $\Omega_2(\phi)$. Note that $\mathbb{Q}_2^x / \mathbb{Q}_2^{x^2} = \{\pm 1, \pm 2, \pm 5, \pm 10\}$. If there exists a $\omega \in \Omega_2(\phi)$ such that $\nu_2(\omega) = 0$ then $\nu_2 = m_2 + 2$, and $U_2(\nu_2, \phi) = \{\gamma_{0,\omega} : \omega \in \Omega_2(\phi) \text{ and } \nu_2(\omega) = 0\}$. In this case, the set $U_2(\nu_2, \phi)$ consists of at most $\gamma_{0,1}, \gamma_{0,3}, \gamma_{0,5},$ and $\gamma_{0,7}$. Otherwise, for all $\omega \in \Omega_2(\phi)$, $\nu_2(\omega) = 1$ and then $\nu_2 = m_2 + 3$, and $U_2(\nu_2, \phi) = \{\gamma_{1,\omega} : \omega \in \Omega_2(\phi) \text{ and } \nu_2(\omega) = 1\}$, hence $U_2(\nu_2, \phi)$ consists of at most $\gamma_{1,2}, \gamma_{1,6}, \gamma_{1,10}$ and $\gamma_{1,14}$. As above, $\gamma_{0,i}$ for $i \in \{1, 3, 5, 7\}$ are the characteristic functions of an odd residue class modulo 8 and $\gamma_{1,j}$ for $j \in \{2, 6, 10, 14\}$ are the characteristic functions of even residue class modulo $\mathbb{Q}_2^{x^2}$.

(b) $\lambda_2 \neq 0$.

Again we must have $m_2 = 1$. We have the following subcases:

(i) $2 \not\in S$.

In this case $\nu_2 = m_2 + 1 = 2$. Let $\beta_2 \in \mathbb{C}$ such that $\beta_2^2 = \lambda_2$. Then $U_2(2, \phi) = \{e^{(0)}_{2,\beta_2}\}$. 

(ii) $2 \in S$.

In this case $\nu_2 = m_2 + 2$. Let $\beta_2 \in \mathbb{C}$ such that $\beta_2^2 = \lambda_2$. Then $U_2(2, \phi) = \{e^{(0)}_{2,\beta_2}\}$. 

(iii) $2 \equiv \alpha_2 \pmod{m_2}$ for some $\alpha_2$.

In this case $\nu_2 = m_2 + 3$. Let $\beta_2 \in \mathbb{C}$ such that $\beta_2^2 = \lambda_2$. Then $U_2(2, \phi) = \{e^{(0)}_{2,\beta_2}\}$. 

(iv) $2 \equiv \beta_2 \pmod{m_2}$ for some $\beta_2$.

In this case $\nu_2 = m_2 + 1$. Let $\beta_2 \in \mathbb{C}$ such that $\beta_2^2 = \lambda_2$. Then $U_2(2, \phi) = \{e^{(0)}_{2,\beta_2}\}$.
We note that this subcase does not arise since as before by Corollary \ref{cor1.2},  \( \rho_2 \) is a special representation and hence \( p \in S \).

(ii) \( 2 \in S \).

Then, we have the following subcases:

(i') \( \lambda_{0,2} \) is trivial on \( 1 + 4\mathbb{Z}_2 \).

Here \( \widetilde{\omega} = 2 \) and \( U_2(2, \phi) = \{ c_{2, \lambda_2}^{(5)} \} \). Since \( m_2 = 1 \) we get that \( \lambda_2 = -\omega_2 2^{(k-3)/2} \) and \( \lambda_2' = -\omega_2 2^{-1/2} \); \( \omega_2 \in \{ \pm 1 \} \) is the eigenvalue under the Atkin-Lehner involution corresponding to 2. Hence we have,

\[
\begin{align*}
  c_{2, \lambda_2}^{(5)}(n) = & \begin{cases} 
    0 & \nu_2(n) = 0, \quad (\omega_2)^{-1/2} = \chi_{0,2}(-1), \quad (\omega_2)^{1/2} = -\omega_2 \chi_{0,2}(2^{-1}) \\
    -\omega_2 & \nu_2(n) = 0, \quad (\omega_2)^{-1/2} = \chi_{0,2}(-1), \quad (\omega_2)^{1/2} = \omega_2 \chi_{0,2}(2^{-1}) \\
    1 & \nu_2(n) = 0, \quad (\omega_2)^{-1/2} = -\chi_{0,2}(-1) \\
    1 & \nu_2(n) = 1.
  \end{cases}
\end{align*}
\]

(ii') \( \chi_{0,2} \) is nontrivial on \( 1 + 4\mathbb{Z}_2 \).

Here \( \widetilde{\omega} = 3 \) and \( U_2(3, \phi) = \{ c_{p, \lambda_2}^{(6)}, \gamma''_0 \} \) and we get the following simplification:

\[
\begin{align*}
  c_{2, \lambda_2}^{(6)}(n) = & \begin{cases} 
    -\omega_2 2^{1/2} & \nu_2(n) = 0 \\
    0 & \nu_2(n) = 1, \quad (n, -1) = \chi_{0,2}(-1), \quad (2, n)_2 = -\omega_2 \chi_{0,2}(2^{-1}) \\
    -\omega_2 & \nu_2(n) = 1, \quad (n, -1) = \chi_{0,2}(-1), \quad (2, n)_2 = \omega_2 \chi_{0,2}(2^{-1}) \\
    1 & \nu_2(n) = 1, \quad (n, -1) = -\chi_{0,2}(-1)
  \end{cases}
\end{align*}
\]

Case 4. \( p = 2 \) and \( m_2 = 0 \).

We have the following subcases:

(a) \( \chi_{0,2} \) is trivial on \( 1 + 4\mathbb{Z}_2 \).

We have \( \widetilde{\omega} = 2 \) and \( U_2(2, \phi) = \{ c_{2, \alpha_2}^{(1)}, \ c_{2, \alpha_2}^{(1)} \} \) if \( \alpha_2 \neq \alpha_2' \), else \( U_2(2, \phi) = \{ c_{2, \alpha_2}^{(1)}, \ c_{2, \alpha_2}^{(2)} \} \).

(b) \( \chi_{0,2} \) is nontrivial on \( 1 + 4\mathbb{Z}_2 \).

Here \( \widetilde{\omega} = 3 \) and \( U_2(3, \phi) = \{ c_{2, \alpha_2}^{(3)}, \ c_{2, \alpha_2}^{(3)}, \gamma''_0 \} \) if \( \alpha_2 \neq \alpha_2' \), else \( U_2(3, \phi) = \{ c_{2, \alpha_2}^{(3)}, \ c_{2, \alpha_2}^{(4)}, \ c_{2, \alpha_2}^{(4)}, \gamma''_0 \} \).

We would like to point out the following useful lemma:

**Lemma 5.2.** Let \( \chi \) be a quadratic character modulo \( N \) such that \( \nu_2(N) \) is at most 2. Then, \( \chi_{0,2} \) is nontrivial on \( 1 + 4\mathbb{Z}_2 \).

**Proof.** Since \( \chi \) is a quadratic, \( \chi_0 \) is also quadratic with modulus \( \text{lcm}(4, N) = 4N' \), where \( N' \) is also a quadratic. Now the lemma follows from part (iii) of Lemma \ref{lem5.2}. \( \square \)

**Remark.** These simplifications along with our method to compute the basis for \( S_{k/2}(N, \chi, \phi) \) for suitable \( N \) and \( \chi \) lead to an algorithm for computing critical values of the L-functions of certain quadratic twists of \( \phi \). For example, if \( M_\phi = p^n \) for some odd prime \( p \), then the possible choices for \( \widetilde{N}_\phi \) are either \( 4p^n \) or \( 4p^{n+1} \), hence we compute bases for spaces \( S_{k/2}(4p^n, \chi_{\text{triv}}, \phi) \) and \( S_{k/2}(4p^{n+1}, \chi_{\text{triv}}, \phi) \) and the sets \( U_2(2, \phi), U_2(\alpha, \phi), U_2(\alpha + 1, \phi) \) to apply Theorem \ref{thm7} in order to get the desired results.

It is to be noted that in the above we have discussed computation of \( U_p(e, \phi) \) only for \( e = \widetilde{N}_\phi \). But in certain cases working with the level \( \widetilde{N}_\phi \) is not sufficient to get the complete information and one might need to go to higher levels.
6. Period

Lemma 6.1. Let $E$ be an elliptic curve, given by a minimal Weierstrass model, and let $E_n$ be the minimal model of its twist by square-free positive integer $n$. Then there is a computable non-zero rational number $\alpha_n$ such that

$$\Omega(E_n) = \frac{\alpha_n\Omega(E)}{\sqrt{n}}.$$ 

The proof we give also explains how to compute $\alpha_n$.

Proof. Let $\omega = dx/(2y + a_1x + a_3)$ be the invariant differential for the model

$$E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$ 

By definition, the period

$$\Omega(E) = \int_{E(\mathbb{R})} |\omega|.$$ 

Recall [18, page 49] that a change of variable

$$x = u^2x' + r, \quad y = u^3y' + u^2sx' + t$$ 

leads to a model $E'$ with invariant differential $\omega' = u\omega$; thus the periods are related by $\Omega(E') = |u|\Omega(E)$. Completing the square in $y$ we obtain the model

$$E' : y'^2 = x'^3 + Ax'^2 + Bx' + C$$ 

where

$$A = \frac{b_3}{4}, \quad B = \frac{b_4}{2}, \quad C = \frac{b_6}{4}.$$ 

Since $u = 1$ in this change of variable, $\omega' = \omega$ and $\Omega(E') = \Omega(E)$. Now let the model $E''$ be the twist of $E'$ by $n$:

$$E'' : y''^2 = x''^3 + Anx''^2 + Bn^2x'' + Cn^3.$$ 

Note that these are related by the change of variable

$$y'' = n^{3/2}y', \quad x'' = nx'.$$ 

Thus the invariant differentials satisfy

$$\omega'' = \frac{dx''}{2y''} = \frac{\omega'}{\sqrt{n}}.$$ 

Thus

$$\Omega(E'') = \frac{\Omega(E')}{\sqrt{n}} = \frac{\Omega(E)}{\sqrt{n}}.$$ 

Now the model $E''$ is not necessarily minimal (nor even integral at 2), but by Tate’s algorithm there is a change of variables

$$x'' = u^2X + r, \quad y'' = u^3Y + u^2sX + t$$ 

with rational $u, s, t$ (and $u \neq 0$) such that the resulting model $E_n$ is minimal. By the above

$$\Omega(E_n) = u\Omega(E'') = \frac{|u|\Omega(E)}{\sqrt{n}}.$$ 

$\square$
Lemma 6.2. Let $E : Y^2 = X^3 + AX^2 + BX + C$ be an elliptic curve with $A, B, C \in \mathbb{Z}$. Suppose that the discriminant of this model is sixth-power free. Let $n$ be a square-free positive integer. Then a minimal model for the $n$-th twist is $E_n : Y^2 = X^3 + AnX^2 + Bn^2X + Cn^3$. Moreover, the periods are related by the formula
\[
\Omega(E_n) = \frac{\Omega(E)}{\sqrt{n}}.
\]

Proof. Let $\Delta$ be the discriminant of the model $E : Y^2 = X^3 + AX^2 + BX + C$. We are assuming that $\Delta$ is sixth-power free. Thus it is 12-th power free, and so $E$ is minimal. Now the model $E_n : Y^2 = X^3 + AnX^2 + Bn^2X + Cn^3$ has discriminant $\Delta_n = \Delta \cdot n^6$. Since $n$ is square-free this is 12-th power free. Thus the model for $E_n$ is minimal. The argument in the proof of Lemma 6.1 completes the proof. \qed

7. Modular Forms are Determined by Coefficients Modulo $n$

As usual $N$ is a positive integer divisible by 4, $\chi$ a Dirichlet character modulo $N$. Let $k$ be an odd integer. Let $\phi$ be a newform of weight $k - 1$, level dividing $N/2$ and character $\chi^2$. To apply Waldspurger’s Theorem, we need to know (see page 10) for certain primes $p$, certain $\omega \in \mathbb{Q}_p^\times/\mathbb{Q}_p^{\times 2}$ and certain forms $f = \sum a_nq^n \in S_{k/2}(N, \chi, \phi)$, whether there is some $n$ such that the image of $n$ in $\mathbb{Q}_p^\times/\mathbb{Q}_p^{\times 2}$ is $\omega$ and $a_n \neq 0$. Given such $p$, $f$ and $\omega$ we can write down the first few coefficients of $f$ and test whether the image of $n$ in $\mathbb{Q}_p^\times/\mathbb{Q}_p^{\times 2}$ is $\omega$ and $a_n \neq 0$. If there is such an $n$ then we should be able to find it by writing down enough coefficients. However, sometimes it appears that $a_n = 0$ for all $n$ that are equivalent in $\mathbb{Q}_p^\times/\mathbb{Q}_p^{\times 2}$ to $\omega$. To be able to prove that, we have developed the results in this section.

Theorem 8. Let $N$ be a positive integer such that $4 \mid N$ and $\chi$ be a Dirichlet character modulo $N$. Let $f(z) = \sum_{n=1} a_n q^n \in S_{k/2}(N, \chi)$. Let $a, M$ be integers such that $(a, M) = 1$. Let $R = \frac{k}{24} \left[ \text{SL}_2(\mathbb{Z}) : \Gamma_1(24M^2) \right]$. Suppose $a_n = 0$ whenever $n \not\equiv a \pmod M$ for all integers $n$ up to $R + 1$. Then $a_n = 0$ whenever $n \equiv a \pmod M$ for all $n$. Moreover if $M^2 \mid N$ then the above statement holds with
\[
R = \begin{cases} 
\frac{k}{24} \left[ \text{SL}_2(\mathbb{Z}) : \Gamma_1(N) \right] & \text{if } \frac{N}{M^2} \equiv 0 \pmod{4} \\
\frac{k}{24} \left[ \text{SL}_2(\mathbb{Z}) : \Gamma_1(2N) \right] & \text{if } \frac{N}{M^2} \equiv 2 \pmod{4}.
\end{cases}
\]

We will be requiring the analogue of the following theorem of Sturm in the case of half-integral weight forms.

Theorem 9. (Sturm [20], Page 276) Let $\Gamma$ be a congruence subgroup and $k$ be a positive integer. Let $f, g \in \mathcal{M}_k(\Gamma)$ such that $f$ and $g$ have coefficients in $\mathcal{O}_F$, the ring of integers of a number field $F$. Let $\lambda$ be a prime ideal of $\mathcal{O}_F$. If
\[
\text{ord}_\lambda(f - g) > \frac{k}{12} \left[ \text{SL}_2(\mathbb{Z}) : \Gamma \right],
\]
then $\text{ord}_\lambda(f - g) = \infty$, i.e., $f \equiv g \pmod{\lambda}$.

Recall that in the above statement if $f(z) = \sum_{n \geq 0} a_n q^n$ then $\text{ord}_\lambda(f) := \inf\{n : a_n \not\equiv \lambda\}$. If $a_n \in \lambda$ for all $n$, then we let $\text{ord}_\lambda(f) := \infty$.

Lemma 7.1. Suppose $\Gamma'$ be a congruence subgroup such that $\Gamma' \subseteq \Gamma_0(4)$ and $k'$ be a positive odd integer. Then, statement of Theorem 9 is valid for $\Gamma'$ and $k = k'/2$. 


Proof. Let $h := f - g \in S_{k'/2}(\Gamma')$. By assumption, $\text{ord}_\lambda(h) > \frac{k'}{24}[\text{SL}_2(\mathbb{Z}) : \Gamma']$. Let $h' = h^4$. Then $h' \in M_{2k'}(\Gamma')$. This is because for any $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma'$ and $z \in \mathbb{H}$, \[
h'(\gamma z) = h^4(\gamma z) = j(\gamma, z)^{4k'} h^4(z) = (cz + d)^{2k'} h'(z).
\]
Also, $\text{ord}_\lambda(h') = 4 \cdot \text{ord}_\lambda(h) > \frac{2k'}{12}[\text{SL}_2(\mathbb{Z}) : \Gamma']$. So we apply Theorem 9 to $h'$ to get that $\text{ord}_\lambda(h') = \infty$. Hence $\text{ord}_\lambda(h) = \infty$. \hfill \Box

We note that the above lemma still holds if $f, g \in M_{k'/2}(\Gamma_0(N), \chi)$; the above proof works by taking $h' = h^{4n}$ where $n$ is the order of Dirichlet character $\chi$.

We will need the following lemmas for the proof of Theorem 8.

Lemma 7.2. Let $M$ be a positive integer and $a \in \mathbb{Z}$ such that $(a, M) = 1$. Define \[
I_a(n) := \begin{cases} 1 & \text{if } n \equiv a \pmod{M} \\ 0 & \text{otherwise}. \end{cases}
\]
Then we have \[
I_a(n) = \sum_{\psi \in \chi(M)} \frac{\psi(a)^{-1}}{\varphi(M)} \psi(n)
\]
where $\chi(M)$ denotes the group of Dirichlet characters of modulus $M$ and $\varphi$ is Euler’s phi function.

Proof. For the proof see [15, Page 63, Chapter 6]. \hfill \Box

Lemma 7.3. Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N)$ and $m^2 \mid N$. Let $0 \leq \nu' < m$ and $\frac{cm'}{m} \equiv 0 \pmod{4}$. Then, \[
\left( \frac{c}{m} \right) = \left( \frac{\nu'}{m} \right) = \left( \frac{m'}{m} \right).
\]

The proof of the above lemma requires the following reciprocity law as stated in Cassels and Fröhlich [5, Page 350]:

Proposition 7.4. Let $P, Q$ be positive odd integers and $a$ be any non-zero integer with $a = 2^n a_0$, $a_0$ odd. Then, \[
\left( \frac{a}{P} \right) = \left( \frac{a}{Q} \right) \text{ if } P \equiv Q \pmod{8a_0}.
\]

Proposition 7.5. Let $k$ be a positive odd integer, $\chi$ be a Dirichlet character modulo $N$ where $4 \mid N$ and $f(z) = \sum_{n=0}^{\infty} a_n q^n \in M_{k/2}(N, \chi)$. Suppose $\psi$ is a Dirichlet character of conductor $m$ and $f_\psi(z) = \sum_{n=0}^{\infty} \psi(n) a_n q^n$. Then,

(i) $f_\psi \in M_{k/2}(Nm^2, \chi \psi^2)$.

(ii) If $m^2 \mid N$ and $\frac{\psi}{\chi} \equiv 0 \pmod{4}$ then $f_\psi \in M_{k/2}(N, \chi \psi^2)$.

(iii) If $m^2 \mid N$ and $\frac{\psi}{\chi} \equiv 2 \pmod{4}$ then $f_\psi \in M_{k/2}(2N, \chi \psi^2)$.

Moreover, if $f$ is a cusp form then so is $f_\psi$.

Proof. The proof essentially follows that of Proposition 17 in [11, Chapter III] for integral weight case with some necessary changes. We use Lemma 7.3 to obtain (ii) and (iii). \hfill \Box
Lemma 7.6. Let \( k, N \) be positive integers such that \( 4 \mid N \) and \( k \) odd. Suppose \( f(z) = \sum_{n=1}^{\infty} a_nq^n \in S_{k/2}(N, \chi) \). Let \( a, M \) be positive integers such that \( (a, M) = 1 \). Define

\[
g(z) := \sum_{n=1}^{\infty} I_{a}(n)a_nq^n.
\]

Then \( g \in S_{k/2}(\Gamma_1(NM^2)) \).

Proof. We have

\[
g(z) = \sum_{n=1}^{\infty} I_{a}(n)a_nq^n
\]

\[
= \sum_{n=1}^{\infty} \sum_{\psi \in X(M)} \frac{\psi(a)^{-1}}{\varphi(M)} \psi(n) a_nq^n
\]

\[
= \sum_{\psi \in X(M)} \alpha_{\psi} \sum_{n=1}^{\infty} \psi(n) a_nq^n
\]

\[
= \sum_{\psi \in X(M)} \alpha_{\psi} f_{\psi},
\]

where \( \alpha_{\psi} = \frac{\psi(a)^{-1}}{\varphi(M)} \). Using Proposition 7.5 for all \( \psi \in X(M) \) we have \( f_{\psi} \in S_{k/2}(\Gamma_1(NM^2)) \). Hence \( g \in S_{k/2}(\Gamma_1(NM^2)) \). \( \square \)

Now we are ready to prove Theorem 8.

Proof of Theorem 8. Let \( h = f - g \) where we take \( g \) as in the above lemma. Since \( f \in S_{k/2}(\Gamma_1(NM^2)) \), so does \( h \). It is clear that

\[
\text{coefficient of } q^n \text{ in } h = \begin{cases} 
a_n & \text{if } n \not\equiv a \pmod{M} 
0 & \text{otherwise.}
\end{cases}
\]

Thus, \( h(z) = \sum_{n \not\equiv a \pmod{M}} a_nq^n \in S_{k/2}(\Gamma_1(NM^2)) \). Since we have assumed \( a_n = 0 \) whenever \( n \not\equiv a \pmod{M} \) for all integers \( n \) up to \( R + 1 \), we apply Lemma 7.4 to get \( h = 0 \). If \( M^2 \mid N \) we apply parts (ii) and (iii) of Proposition 7.5 to Proposition 7.6 to Lemma 7.6. \( \square \)

As an example when \( N = 1984 \), \( k = 3 \) and \( M = 8 \), since all Dirichlet characters modulo 8 are quadratic we in fact get a new improved bound which is given by \( R = 32 \mid \text{SL}_2(\mathbb{Z} : \Gamma_0(1984)) \mid = 384 \).

8. Applications of Waldspurger’s Theorem

In this section we will present a few examples explaining how to use Waldspurger’s Theorem. The idea of using Waldspurger’s Theorem for an elliptic curve is motivated by Tunnell’s famous work on the congruent number problem. We will see however that our case needs many more computations to get any desired result. In the examples that follow we will first use our algorithm in Theorem 2 to compute the space of cusp forms that are Shimura equivalent to the given elliptic curve and then use Waldspurger’s Theorem to get some interesting results. We will follow the notation adopted in the previous section.
8.1. **A First Example.** Our first example will be the elliptic curve $E$ over $\mathbb{Q}$ given by

$$E : Y^2 = X^3 + X + 1.$$  

The conductor of $E$ is $496 = 16 \times 31$ and $E$ does not have complex multiplication. Let $\phi \in S_{2}^{\text{new}}(496, \chi_{\text{triv}})$ be the corresponding newform given by the Modularity Theorem; $\phi$ has the following $q$-expansion,

$$\phi(z) = q - 3q^5 + 3q^7 - 3q^9 - 2q^{11} - 4q^{13} - q^{19} + O(q^{20}).$$

It is to be noted that $\phi$ satisfies the hypothesis (H1)—this follows by Theorem 6 and since $16 \mid M_\phi$, $\phi$ satisfies (H2). Let $\chi$ be a Dirichlet character with $\chi^2 = \chi_{\text{triv}}$. Hence by Theorem 5 there exists $N$ such that $S_{3/2}(N, \chi, \phi) \neq \{0\}$. Note that we must have $496 \mid (N/2)$.

In order to apply Waldspurger’s Theorem we would like to compute an eigenbasis for the summand $S_{3/2}(N, \chi, \phi)$ for a suitable $N$ and $\chi$. We will assume $\chi$ to be the trivial character $\chi_{\text{triv}}$. We use Theorem 2 to find out that $S_{3/2}(992, \chi, \phi) = \{0\}$. However at level $1984$ we get that the space $S_{3/2}(1984, \chi, \phi)$ has a basis $\{f_1, f_2, f_3\}$ where $f_1, f_2$ and $f_3$ have the following $q$-expansions:

$$f_1(z) = q^3 + q^{13} + 2q^{75} + 2q^{83} + q^{91} + 3q^{115} - 3q^{123} + O(q^{145}) := \sum_{n=1}^{\infty} a_n q^n$$

$$f_2(z) = q^{15} + q^{23} - q^{31} + 2q^{55} + q^{79} - 3q^{119} + O(q^{145}) := \sum_{n=1}^{\infty} b_n q^n$$

$$f_3(z) = q^{17} + q^{57} + q^{65} + 2q^{73} - q^{89} - q^{105} + q^{137} + O(q^{145}) := \sum_{n=1}^{\infty} c_n q^n.$$

We note that the space $S_{3/2}(1984, \chi)$ is 119-dimensional.

By Waldspurger’s Theorem 4 there exists a function $A_\phi$ on square-free positive integers $n$ such that

$$A_\phi(n)^2 = L(E_{-n}, 1)$$

and

$$S_{3/2}(1984, \chi, \phi) = \bigoplus \mathbb{U}(E, \phi, A_\phi),$$

where the sum is over all $E \geq 1$ such that $\mathcal{N}_\phi \mid E \mid 1984$. We already know the left-hand side of the above identity. Henceforth we will be interested in computing the right-hand side. We will first compute $\mathcal{N}_\phi$ and then $\mathcal{U}(E, \phi, A_\phi)$ for $\mathcal{N}_\phi \mid E \mid 1984$.

We need to compute local components $\mathcal{N}_p$ for each prime $p$. We consider the following cases.

**Case 1.** $p$ odd and $p \neq 31$.

In this case $m_p = 0$ and since $p \nmid N$ the local character $\chi_{0,p}$ is unramified. Hence we get that $\mathcal{N}_p = 0$.

**Case 2.** $p = 31$.

Here $m_{31} = 1$. Since $\lambda_{31} \neq 0$ using Corollary 4.2 it follows that the local component $\rho_{31}$ is a special representation of $\text{GL}_2(\mathbb{Q}_{31})$ and so $31 \in S$. Also, note that $\mathbb{Z}_{2}^{31}/\mathbb{Z}_{31}^{2}$ is generated by $11 \mod \mathbb{Z}_{31}^{2}$ and using Proposition 3.3 we can show that $\chi_{0,31}(11) = 1$. Thus $\chi_{0,31}$ is unramified and so, $\mathcal{N}_{31} = 1$. 

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Case 3. $p = 2$.

In this case $m_2 = 4$ and it is clear from the q-expansion of $\phi$ that $
_2 = 0$. We need some information about the set $\Omega_2(\phi)$ (see Equation 2).

In our case, looking at $f_1, f_2$ and $f_3$, we get that $\{1, 3, 7\} \subseteq \Omega_2(\phi)$. Since $\nu_2(1) = \nu_2(3) = \nu_2(7) = 0$, we get $n_2 = m_2 + 2 = 6$.

Hence

$$\tilde{N}_\phi = 31 \times 2^6 = 1984.$$

Thus we have $E = \tilde{N}_\phi = 1984$ and we would like to know how the space $\tilde{U}(1984, \phi, A_\phi)$ looks. For that the next immediate task will be to compute $U_p(c_p, \phi)$ where $c_p = \nu_p(1984)$. We consider the following cases:

Case 1. $p$ odd and $p \neq 31$.

Here, $e_p = 0$ and $U_p(0, \phi)$ consists of only one function $c^{(0)}_{p, \lambda}$ defined on $\mathbb{Q}^\times$. Recall that $c^{(0)}_{p, \lambda}(n) = 1$ for $n$ square-free.

Case 2. $p = 31$.

In this case $c_{31} = 1$ and as already seen, $31 \in S$ and $\chi_{0, 31}$ is unramified. So, $U_31(1, \phi) = \{c^{(5)}_{31, \lambda_{31}}\}$. Note that $\lambda_{31} = -1$ and hence $\lambda'_{31} = (31)^{-1/2}\lambda_{31} = -(31)^{-1/2}$. Again using Proposition 5.3 we can show that $\chi_{0, 31}(31^{-1}) = -1$. Also note that $(31, n)_{31} = (\frac{n}{31})$. So for $n$ square-free we have,

$$c^{(5)}_{31, \lambda'}(n) = \begin{cases} 2^{1/2} & \nu_{31}(n) = 0, \left(\frac{31}{n}\right) = -1 \\ 0 & \nu_{31}(n) = 0, \left(\frac{31}{n}\right) = 1 \\ 1 & \nu_{31}(n) = 1. \end{cases}$$

Case 3. $p = 2$.

Here $e_2 = 6$. Since $\lambda_2 = 0$ and $\{1, 3, 7\} \subseteq \Omega_2(\phi)$, we see that $U_2(6, \phi)$ consists of $\chi_{0, 1}, \chi_{0, 3}, \chi_{0, 7}$ which are the characteristic functions of residue classes of $1, 3, 7$ modulo $8$ respectively. By our methods so far we do not know whether 5 belongs to $\Omega_2(\phi)$ or not.

Recall that $\tilde{U}(E, \phi, A_\phi)$ is the space generated by the functions $f(\mathfrak{c}, A_\phi)$ where $\mathfrak{c} \in \prod_p U_p(c_p, \phi)$. Thus in our case $\mathfrak{c} = (c_p)_p$ where, for odd primes $p \neq 31$ we have $c_p = c_{p, \lambda'}$, $c_{31} = c^{(5)}_{31, \lambda'}$ and for $c_2$ the possible choices are $\chi_{0, 1}, \chi_{0, 3}, \chi_{0, 5}$ and $\chi_{0, 7}$.

By using Waldspurger’s Theorem 7 we have

$$S_{3/2}(1984, \chi, \phi) = \tilde{U}(1984, \phi, A_\phi)$$

and so every cusp form in the space on the left-hand side can be written in terms of

$$f(\mathfrak{c}, A_\phi)(z) := \sum_{n=1}^\infty A_\phi(n^{sc})n^{1/4} \prod_p c_p(n)q^n$$

for some $\mathfrak{c} = (c_p) \in \prod U_p(c_p, \phi)$.

We use Theorem 8 to conclude that $f_1$ have non-zero $n$-th coefficients only for $n \equiv 3 \pmod{8}$, $f_2$ have non-zero coefficients only for $n \equiv 7 \pmod{8}$ and $f_3$ have non-zero coefficients only for $n \equiv 1 \pmod{8}$. 
Since $f_1$ have non-zero $a_n$ only for $n \equiv 3 \pmod{8}$, taking $c$ as above with $c_2 = \gamma_{0,3}$ we get that for $n$ square-free,

\[ a_n = \beta_1 A_\phi(n)n^{1/4}c_2(n)c_31(n) = \begin{cases} 
2^{1/2}\beta_1 A_\phi(n)n^{1/4} & \nu_{31}(n) = 0, \left(\frac{n}{31}\right) = -1, n \equiv 3 \pmod{8} \\
\beta_1 A_\phi(n)n^{1/4} & \nu_{31}(n) = 1, n \equiv 3 \pmod{8} \\
0 & \text{otherwise,}
\end{cases} \]

for some complex constant $\beta_1$. Similarly, taking $c_2 = \gamma_{0,7}$ for $f_2$ and $c_2 = \gamma_{0,1}$ for $f_3$ respectively we get that

\[ b_n = \beta_2 A_\phi(n)n^{1/4}c_2(n)c_31(n) = \begin{cases} 
2^{1/2}\beta_2 A_\phi(n)n^{1/4} & \nu_{31}(n) = 0, \left(\frac{n}{31}\right) = -1, n \equiv 7 \pmod{8} \\
\beta_2 A_\phi(n)n^{1/4} & \nu_{31}(n) = 1, n \equiv 7 \pmod{8} \\
0 & \text{otherwise,}
\end{cases} \]

for some complex constant $\beta_2$ and

\[ c_n = \beta_3 A_\phi(n)n^{1/4}c_2(n)c_31(n) = \begin{cases} 
2^{1/2}\beta_3 A_\phi(n)n^{1/4} & \nu_{31}(n) = 0, \left(\frac{n}{31}\right) = -1, n \equiv 1 \pmod{8} \\
\beta_3 A_\phi(n)n^{1/4} & \nu_{31}(n) = 1, n \equiv 1 \pmod{8} \\
0 & \text{otherwise,}
\end{cases} \]

for some complex constant $\beta_3$.

We have the following proposition which allows us to calculate the critical values of the $L$-functions of $E_{-n}$, the $(-n)$-th quadratic twists of $E$.

**Proposition 8.1.** Let $E$ be as above and $n$ be a positive square-free integer.

(i) If $\nu_{31}(n) = 0, n \equiv 3 \pmod{8}$ and $\left(\frac{n}{31}\right) = -1$, then,

\[ L(E_{-n}, 1) = \frac{a_n^2}{2\beta_1^2 \sqrt{n}}. \]

(ii) If $\nu_{31}(n) = 1, n \equiv 3 \pmod{8}$ then,

\[ L(E_{-n}, 1) = \frac{a_n^2}{\beta_1 \sqrt{n}}. \]

(iii) If $\nu_{31}(n) = 0, n \equiv 7 \pmod{8}$ and $\left(\frac{n}{31}\right) = -1$, then,

\[ L(E_{-n}, 1) = \frac{b_n^2}{2\beta_2^2 \sqrt{n}}. \]

(iv) If $\nu_{31}(n) = 1, n \equiv 7 \pmod{8}$ then,

\[ L(E_{-n}, 1) = \frac{b_n^2}{\beta_2 \sqrt{n}}. \]
(v) If $\nu_{31}(n) = 0$, $n \equiv 1 \pmod{8}$ and $\left(\frac{n}{31}\right) = -1$ then,
\[ L(E_{-n}, 1) = \frac{c_n^2}{2\beta_3^2 \sqrt{n}}. \]

(vi) If $\nu_{31}(n) = 1$, $n \equiv 1 \pmod{8}$ then,
\[ L(E_{-n}, 1) = \frac{c_n^2}{\beta_3^2 \sqrt{n}}. \]

Proof. Using Waldspurger’s Theorem [4] we know the existence of a function $A_\phi$ on square-free numbers such that $A_\phi(n)^2 = L(E_{-n}, 1)$. The proof follows now using Equations (3), (4) and (5). □

We will show now how we use the above to calculate the order of the Tate-Shafarevich group $\Sha(E_{-n}/\mathbb{Q})$. We will be assuming the Birch and Swinnerton-Dyer Conjecture for rank zero elliptic curves:
\[ L(E_{-n}, 1) = |\Sha(E_{-n}/\mathbb{Q})| \cdot \Omega_{E_{-n}} \cdot \prod_p c_p \]
where $\Omega_{E_{-n}}$ stands for the real period of $E_{-n}$ (since $E_{-n}(\mathbb{R})$ is connected), $c_p$ for the $p$-th Tamagawa number of $E_{-n}$ and $E_{-n,\text{tor}}$ stands for the torsion group of $E_{-n}$, all of which are easily computable.

We have the following lemma.

Lemma 8.2. Let $E : Y^2 = X^3 + X + 1$. Then $E_{n,\text{tor}} = 0$ for all square-free integers $n$.

Proof. Let $K = \mathbb{Q}(\sqrt{n})$. It is well-known that the map $E_n(\mathbb{Q}) \to E(K)$ given by
\[ O \mapsto O, \quad (X, Y) \mapsto \left(\frac{X}{n}, \frac{Y}{n\sqrt{n}}\right) \]
is an injective group homomorphism[3]. Thus it is sufficient to show that $E(K)$ has trivial torsion subgroup. Recall that the discriminant of $E$ is $-496 = -16 \times 31$. Let $p \neq 2, 31$ be a rational prime and let $\mathfrak{P}$ be a prime ideal of $K$ dividing $p$. Then $E$ has good reduction at $\mathfrak{P}$. Moreover, if $\epsilon_{\mathfrak{P}} < p - 1$ then the reduction map $E(K)_{\text{tor}} \to E(\mathbb{F}_p)$ is injective [10, page 501], where $\epsilon_{\mathfrak{P}}$ is the ramification index for $\mathfrak{P}$ and $\mathbb{F}_p$ denotes the residue field of $\mathfrak{P}$. Thus if $p \geq 5$ and $p \neq 31$ then this map is injective. Now we take $p = 5, 7$, so $E(\mathbb{F}_p)$ is a subgroup of $E(\mathbb{F}_{25})$ and $E(\mathbb{F}_{49})$ respectively. Using MAGMA we find
\[ E(\mathbb{F}_{25}) \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/9\mathbb{Z}, \quad E(\mathbb{F}_{49}) \cong \mathbb{Z}/55\mathbb{Z}. \]
Since these two groups have coprime orders, it follows that $E(K)_{\text{tor}} = 0$ and so $E_{n,\text{tor}} = 0$. □

[3] As the map simply scales the variables, it takes lines to lines and so must define a homomorphism of Mordell-Weil groups.
Further, since the discriminant of $E_{-1}$ is $-496 = 2^4 \times 31$, by Lemma 6.2 we know that $\Omega(E_{-1}) = \Omega(E_{-1})/\sqrt{31}$.

It is clear that the quantity $\frac{L(E_{-1})}{\Omega(E_{-1})}$ is an integer. Using MAGMA we compute this integer for $n \in \{3, 15, 17\}$ and using Lemma 6.2 one gets that

\begin{equation}
\Omega_{E_{-1}} = \frac{1}{4\beta_1^2} = \frac{1}{4\beta_2^2} = \frac{1}{8\beta_3^2}.
\end{equation}

Now recall that $W(E_{-n}/\mathbb{Q})$ denotes the root number for elliptic curve $E_{-n}$ over rational numbers. We have the following proposition. The methods used here to compute the root numbers are well-known and we refer to [2].

**Proposition 8.3.** For $E$ as above and $n$ positive square-free the following holds.

(i) If $\nu_{31}(n) = 0$ then,

$$W(E_{-n}/\mathbb{Q}) = \begin{cases} -1 & n \equiv 1, 3, 7 \pmod{8}, \left(\frac{n}{31}\right) = 1 \text{ or } \\ n \equiv 5 \pmod{8}, \left(\frac{n}{31}\right) = -1 \text{ or } \\ n \text{ even, } \left(\frac{n}{31}\right) = -1; \end{cases}$$

(ii) If $\nu_{31}(n) = 1$ then,

$$W(E_{-n}/\mathbb{Q}) = \begin{cases} -1 & n \equiv 5 \pmod{8} \text{ or } \\ n \text{ even; } \end{cases}$$

Before computing the order of the Tate-Shafarevich group $\text{III}(E_{-n}/\mathbb{Q})$, we have the following refinement of Theorem 8.1.

**Theorem 10.** Let $E : Y^2 = X^3 + X + 1$ and $f = f_1 + f_2 + \sqrt{2}f_3 = \sum d_nq^n$. Then, for positive square-free $n \equiv 1, 3, 7 \pmod{8}$

$$L(E_{-n}, 1) = \frac{2^{(\nu_{31}(n)+1)}\Omega_{E_{-1}} \cdot d_n^2}{\sqrt{n}}.$$

**Proof.** Note that $d_n = a_n + b_n + \sqrt{2}c_n$. It is important for the proof to note that $a_n = 0$ for $n \equiv 3 \pmod{8}$, and $b_n = 0$ for $n \equiv 7 \pmod{8}$, and $c_n = 0$ for $n \equiv 1 \pmod{8}$; we proved this by applying Theorem 8. It follows from equations 8.1 and 8.2 that $d_n = 0$ whenever $n \equiv 1, 3, 7 \pmod{8}$ and the Kronecker symbol $\left(\frac{n}{31}\right) = 1$. Further by Proposition 8.3 if $n \equiv 1, 3, 7 \pmod{8}$ and $\left(\frac{n}{31}\right) = 1$ then $W(E_{-n}, \mathbb{Q}) = -1$ and so $L(E_{-n}, 1) = 0$. Thus the theorem follows when $\left(\frac{n}{31}\right) = 1$.

In the case when $\left(\frac{n}{31}\right) = -1$, the refinement follows by using Equation 7.3 in Theorem 8.1.

We have now the following corollary which computes the order of the Tate-Shafarevich group $\text{III}(E_{-n}/\mathbb{Q})$.

**Corollary 8.4.** Let $E : Y^2 = X^3 + X + 1$ and $f = f_1 + f_2 + \sqrt{2}f_3 = \sum d_nq^n$. Let $n$ be positive square-free number such that $n \equiv 1, 3, 7 \pmod{8}$ and $E_{-n}$ has rank
The corresponding newform \( \phi \) of conductor 144.

Proof. From Lemma 8.2 we have \( \text{Birch and Swinnerton-Dyer Conjecture,} \)

\[ \text{Second Example.} \]

Now follows using Theorem 10.

By Proposition 8.3, if \( \text{Corollary 8.5.} \)

Suppose \( \text{that at the level } \)

Thus the analytic rank is even, and so by BSD, the rank is even. The corollary now follows using Theorem 10.

We use Tate’s algorithm (see [19, Pages 364–368]) to compute the Tamagawa numbers \( c_p \).

We have the following easy corollary to Theorem 10.

**Corollary 8.5.** Suppose \( n \equiv 1, 3, 7 \, (\text{mod 8}) \) and \( \left( \frac{n}{31} \right) = -1 \). Then assuming the Birch and Swinnerton-Dyer Conjecture,

\[ \text{Rank}(E_n) \geq 2 \iff d_n = 0. \]

Proof. By Proposition 8.3 if \( n \equiv 1, 3, 7 \, (\text{mod 8}) \) and \( \left( \frac{n}{31} \right) = -1 \) then \( W(E_n/Q) = 1 \). Thus the analytic rank is even, and so by BSD, the rank is even. The corollary now follows using Theorem 10.

8.2. Second Example. Our second example will be the rational elliptic curve \( E \) of conductor 144 given by

\[ E : Y^2 = X^3 - 1. \]

The corresponding newform \( \phi \) is given by

\[ \phi(z) = q + 4q^7 + 2q^{13} - 8q^{19} - 5q^{25} + 4q^{31} - 10q^{37} - 8q^{43} + 9q^{49} + O(q^{50}). \]

Here \( M_\phi = 144 \). Using Theorem 2 for computing Shimura’s decomposition, we find that at the level 576, the space \( S_{3/2}(576, \chi, \phi) \neq \{0\} \); and this space has a basis \( \{f_1, f_2, f_3, f_4\} \) where \( f_1, f_2, f_3 \) and \( f_4 \) have the following \( q \)-expansion:

\[ f_1(z) = q - q^{25} + 5q^{39} - 6q^{73} - 6q^{97} + O(q^{100}) := \sum_{n=1}^{\infty} a_n q^n \]

\[ f_2(z) = q^5 + q^{29} - q^{53} - 2q^{77} + O(q^{100}) := \sum_{n=1}^{\infty} b_n q^n \]

\[ f_3(z) = q^{13} - 2q^{61} + q^{85} + O(q^{100}) := \sum_{n=1}^{\infty} c_n q^n \]

\[ f_4(z) = q^{17} - q^{41} - q^{89} + O(q^{100}) := \sum_{n=1}^{\infty} d_n q^n. \]
Doing similar calculations as in the previous example we have the following result.

**Theorem 11.** Let $E : Y^2 = X^3 - 1$. Let

$$f = f_1/2 + f_2 + \sqrt{2}f_3 + \sqrt{3}f_4 := \sum_{n=1}^{\infty} e_n q^n.$$ 

Let $n \neq 1$ be a positive square-free integer such that $n \equiv 1, 2$ (mod 3). Then,

$$L(E_{-n}, 1) = \frac{\Omega_{E_{-1}}}{\sqrt{n}} \cdot e_n^2.$$ 

Further assuming BSD, if $E_{-n}$ has rank zero then,

$$|\text{III}(E_{-n}/\mathbb{Q})| = \frac{4}{\prod_p c_p} \cdot e_n^2$$

where the Tamagawa numbers $c_2 = 3$ if $n \equiv 1$ (mod 8), $c_2 = 1$ if $n \equiv 3, 5, 7$ (mod 8), $c_3 = 2$; $c_p = #E_{-1}(\mathbb{F}_p)[2]$ for $p | n$, $p \neq 3$; and $c_p = 1$ for all other primes.

**8.3. Example with a Non-Rational Newform.** In this example we start with a non-rational newform $\psi$ and show that we can get similar formulae as before for the critical values of $L$-functions of $\psi \otimes \chi_{-n}$.

Let $\psi \in S_2^{\text{new}}(62, \chi_{\text{triv}})$ be a newform of weight 2, level 62 and trivial character given by the following $q$-expansion,

$$\psi(z) = q - q^2 + aq^3 + q^4 + (-2a + 2)q^5 - aq^6 + 2q^7 - q^8 + (2a - 1)q^9 + O(q^{10})$$

where $a$ has minimal polynomial $x^2 - 2x - 2$.

As before using Theorem 2 we get that the space $S_{3/2}(124, \chi_{\text{triv}}, \psi) = \langle f \rangle$ where $f$ has the following $q$-expansion,

$$f(z) = q + (a + 1)q^2 - q^3 - 2aq^5 - aq^6 + (-a - 1)q^7 + (a + 1)q^8 - 2q^9 + O(q^{12}).$$

Note that Waldspurger’s theorem is applicable for the newform $\psi$ as the local automorphic representation of $\psi$ at 2 is not supercuspidal; this follows since $\nu_2(62) = 1$ and the second coefficient of $\psi$ is non-zero (see Corollary 4.1).

We have the following proposition.

**Proposition 8.6.** Let $\psi$ and $f := \sum_{n=1}^{\infty} a_n q^n$ be as above. Let $n$ be square-free such that $n \neq 3$ (mod 8) and $(\frac{4}{n}) \neq -1$. Then

$$L(\psi \otimes \chi_{-n}, 1) = \begin{cases} \frac{\beta \cdot a_n^2}{\sqrt{n}} & \text{if } \nu_3(n) = 1 \\ \frac{\beta \cdot a_n^2}{2 \sqrt{n}} & \text{if } \nu_3(n) = 0 \end{cases}$$

where $\beta = 2 \cdot L(\psi \otimes \chi_{-1}, 1)$.

**Proof.** The proof follows by the similar calculations as shown in the previous examples. □

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3In the case $n = 1$ we still have $L(E_{-1}, 1) = \frac{\Omega_{E_{-1}}}{\sqrt{n}} \cdot e_n^2$, but since $|E_{-1, \text{tor}}| = 6$ we get that $|\text{III}(E_{-n}/\mathbb{Q})| = \frac{36}{\prod_p c_p} \cdot e_n^2$. 

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8.4. **Ternary Quadratic Forms and Tunnell-like Formulae.** For a positive-definite integral quadratic form \( Q(x_1, \ldots, x_m) \) we define its theta-series by

\[
\theta_Q(z) = \sum_{n=0}^{\infty} \# \{ a \in \mathbb{Z}^m : Q(a) = n \} \cdot q^n; \quad q = \exp(2\pi iz).
\]

Siegel [13] showed that if \( Q_1 \) and \( Q_2 \) are positive-definite integral ternary quadratic forms both having level \( N \), character \( \chi_d \), and belonging to the same genus, then \( \theta_{Q_1} - \theta_{Q_2} \in S_{3/2}(N, \chi_d) \). Denote by \( S_q(N, \chi_d) \) the subspace of \( S_{3/2}(N, \chi_d) \) generated by all such differences of theta-series.

It is interesting, when applying Waldspurger’s Theorem to a weight 2 cuspidal \( \phi \) to ask if the relevant modular form of weight 3/2 belongs to \( S_q(N, \chi_d) \); in this case we would obtain a Tunnell-like formula expressing the critical values of the \( L \)-functions of twists of \( \phi \) in terms of ternary quadratic forms. We will illustrate this below by presenting several examples. We point out however that this is not always possible. In particular for the elliptic curve in our first example, \( E : Y^2 = X^3 + X + 1 \), the space \( S_{3/2}(1984, \chi_{\text{triv}}, \phi_E) \) has trivial intersection with the subspace \( S_q(1984, \chi_{\text{triv}}) \).

We do not give details of how to compute \( S_q(N, \chi_d) \) or the intersection \( S_q(N, \chi_d) \cap S_{3/2}(N, \chi_d, \phi) \). We merely point out that it is straightforward to compute a basis for the space \( S_q(N, \chi_d) \) with the help of an algorithm of Dickson [3, 12] for computing quadratic forms of a given level and character up to equivalence. Computing the intersection with \( S_{3/2}(N, \chi_d, \phi) \) is easy using a suitable adaptation of our Theorem [2] and a result of Bungert [1] Proposition 4 for computing Hecke action on theta-series.

**Notation.** We will denote by \([a, b, c, r, s, t]\), the ternary quadratic form given by \( ax^2 + by^2 + cz^2 + ryz + sxz + txy \).

**Example 1.** Let \( E \) be an elliptic curve of conductor 50 as in Proposition [14]. Let \( \phi \) be the newform corresponding to \( E \),

\[
\phi : q + q^2 - q^3 + q^4 - q^6 - 2q^7 + q^8 - 2q^9 - 3q^{11} + O(q^{12}).
\]

Note that \( \nu_2(50) = 1 \) and second coefficient of \( \phi \) is non-zero, hence \( \rho_2 \) is not supercuspidal and so we can apply Waldspurger’s Theorem.

We get that \( \tilde{N} \phi = 100 \) and \( S_{3/2}(100, \chi_{\text{triv}}, \phi) \) has a basis consisting of \( f_1 \) and \( f_2 \) where

\[
f_1 = q + q^4 - q^6 - q^{11} - 2q^{14} + O(q^{15}) := \sum_{n=1}^{\infty} a_n q^n
\]

\[
f_2 = q^2 - q^3 + q^8 - q^{12} + 2q^{13} + O(q^{15}) := \sum_{n=1}^{\infty} b_n q^n.
\]

In fact it turns out that \( f_1 = (\theta_{Q_1} - \theta_{Q_2})/2 \) and \( f_2 = (\theta_{Q_3} - \theta_{Q_4})/2 \) where \( Q_i \)'s are the quadratic ternary forms as in Proposition [14] which can now be proved on the similar lines as Theorem [10].

Again we can compute the order of \( \text{III}(E_{-n}/Q) \) assuming the BSD. For example, we get that

\[
|\text{III}(E_{-9318}/Q)| = 33^2 = 1089.
\]
We can further consider the real quadratic twists $E_n$. For this we work with the elliptic curve $E_{-1}$ of conductor 400,

$$E_{-1} : Y^2 = X^3 + X^2 - 48X - 172.$$ 

We can show that if $5 \nmid n$ then,

$$L(E_n, 1) = \begin{cases} \frac{L(E_{-1}, 1) \cdot c_n^2}{\sqrt{n}} \quad \left( \frac{n}{5} = 1 \right) \\ \frac{L(E_{17}, 1) \cdot c_n^2}{\sqrt{65}} \quad \left( \frac{n}{5} = -1 \right) \end{cases},$$

where $c_n$ is the $n$-th coefficient of the following linear combination of theta-series of weight $3/2$ and level 1600 coming from the ternary quadratic forms:

$$-\frac{1}{5} \cdot \theta_{[5,5,17,-2,-4]} + \frac{1}{5} \cdot \theta_{[5,9,10,2,2,4]} + \frac{1}{10} \cdot \theta_{[1,4,400,0,0,0]} - \frac{1}{10} \cdot \theta_{[5,17,20,-8,0,-2]}$$
$$-\frac{1}{10} \cdot \theta_{[5,17,20,4,4,2]} + \frac{1}{10} \cdot \theta_{[8,13,20,12,8,4]} - \frac{1}{5} \cdot \theta_{[1,32,52,-16,0,0]} + \frac{1}{5} \cdot \theta_{[8,13,17,6,4,4]}$$
$$+ \frac{1}{10} \cdot \theta_{[4,5,400,0,0,-4]} - \frac{1}{10} \cdot \theta_{[4,16,101,0,-4,0]} + \frac{1}{10} \cdot \theta_{[400,100,1,0,0,0]}$$
$$-\frac{1}{10} \cdot \theta_{[125,100,4,0,0,100]} + \frac{1}{5} \cdot \theta_{[89,56,9,-4,-2,-44]} - \frac{1}{5} \cdot \theta_{[49,36,29,24,22,16]}$$
$$-\frac{1}{2} \cdot \theta_{[400,13,8,4,0,0]} - \frac{1}{10} \cdot \theta_{[100,25,17,10,0,0]} + \frac{1}{10} \cdot \theta_{[52,32,25,0,0,16]}$$
$$+ \frac{1}{2} \cdot \theta_{[53,33,25,-10,-10,-14]} + \frac{1}{2} \cdot \theta_{[400,400,1,0,0,0]} + \frac{9}{10} \cdot \theta_{[400,25,16,0,0,0]}$$
$$-\frac{1}{2} \cdot \theta_{[201,201,4,4,4,2]} + \frac{1}{10} \cdot \theta_{[224,89,9,-2,-8,-88]} - \frac{1}{10} \cdot \theta_{[209,36,25,20,10,36]}$$
$$-\frac{9}{10} \cdot \theta_{[129,100,16,0,-16,-100]} - \frac{4}{5} \cdot \theta_{[84,81,25,10,20,4]} + \frac{4}{5} \cdot \theta_{[89,49,41,-6,-14,-38]}$$
$$-\frac{1}{5} \cdot \theta_{[400,29,16,16,0,0]} + \frac{5}{3} \cdot \theta_{[125,100,16,0,0,100]} - \frac{2}{5} \cdot \theta_{[100,96,21,8,20,80]}$$
$$+ \frac{2}{5} \cdot \theta_{[84,69,29,2,12,28]} - \frac{3}{5} \cdot \theta_{[400,32,13,8,0,0]} + \frac{2}{5} \cdot \theta_{[171,52,32,-16,-24,-44]}$$
$$+ \frac{1}{5} \cdot \theta_{[400,25,17,10,0,0]} + \frac{1}{5} \cdot \theta_{[212,48,17,8,4,48]} + \frac{1}{10} \cdot \theta_{[208,32,25,0,0,32]}$$
$$-\frac{1}{5} \cdot \theta_{[212,33,25,-10,-20,-28]} - \frac{1}{10} \cdot \theta_{[208,33,32,32,32,16]} - \frac{1}{5} \cdot \theta_{[113,52,32,16,8,52]}.$$

Further using the root number arguments, we get that $L(E_{-5n}, 1) = 0$ whenever $n \equiv 3 \pmod{8}$ and $L(E_{5n}, 1) = 0$ whenever $n \equiv 5 \pmod{8}$. For the remaining cases, we look at the space $\mathcal{S}_{3/2}(8000, \phi)$.

**Example 2.** This example formulates Theorem 1 in terms of ternary quadratic forms. Let $E : Y^2 = X^3 - 1$. Let $n$ be positive square-free integer such that $n \equiv 1, 2 \pmod{3}$. Then

$$L(E_{-n}, 1) = \frac{\Omega_{E_{-1}}}{\sqrt{n}} \cdot a_n^2.$$
where \( a_n \) is the \( n \)-th coefficient of the cusp form \( f \) of weight 3/2 and level 576 that can be written as follows as a linear combination theta series:

\[
f = \sum_{n=1}^{\infty} a_n q^n = \\
\quad + \frac{1}{6} \cdot \theta_{[1,4,144,0,0,0]} - \frac{1}{6} \cdot \theta_{[4,4,37,0,-4,0]} + \frac{1}{6} \cdot \theta_{[4,5,36,0,0,-4]} - \frac{1}{6} \cdot \theta_{[13,13,-10,0,0]} \\
\quad + \frac{1}{3} \cdot \theta_{[1,20,32,-16,0,0]} + \frac{1}{6} \cdot \theta_{[4,5,29,-2,0,0]} - \frac{1}{2} \cdot \theta_{[4,9,17,-6,0,0]} + \frac{1}{2} \cdot \theta_{[1,36,45,-36,0,0]} \\
\quad - \frac{1}{2} \cdot \theta_{[4,9,37,0,-4,0]} + \frac{1}{6} \cdot \theta_{[144,16,1,0,0,0]} - \frac{1}{6} \cdot \theta_{[16,16,9,0,0,0]} - \frac{1}{3} \cdot \theta_{[144,5,4,0,0,0]} \\
\quad + \frac{1}{6} \cdot \theta_{[37,16,4,0,4,0]} + \frac{1}{6} \cdot \theta_{[16,13,13,10,0,0]} + \frac{1}{6} \cdot \theta_{[32,21,4,-4,0,-16]} - \frac{1}{6} \cdot \theta_{[29,16,5,0,2,0]} \\
\quad - \frac{1}{2} \cdot \theta_{[144,36,1,0,0,0]} + \frac{1}{6} \cdot \theta_{[144,9,4,0,0,0]} - \frac{1}{2} \cdot \theta_{[45,36,4,0,0,36]} - \frac{1}{6} \cdot \theta_{[144,144,1,0,0,0]} \\
\quad - \frac{1}{2} \cdot \theta_{[144,16,9,0,0,0]} + \frac{2}{3} \cdot \theta_{[49,36,16,0,-16,-36]} + \frac{1}{4} \cdot \theta_{[144,13,13,10,0,0]} \\
\quad - \frac{1}{4} \cdot \theta_{[45,36,16,0,0,36]} + \frac{1}{2} \cdot \theta_{[144,29,5,2,0,0]} - \frac{1}{2} \cdot \theta_{[32,29,29,22,16,16]} \\
\quad - \frac{1}{6} \cdot \theta_{[80,32,9,0,0,32]} + \frac{1}{2} \cdot \theta_{[80,17,17,-2,-16,-16]} - \frac{1}{3} \cdot \theta_{[41,32,20,16,20,8]}.
\]

**Example 3.** Let \( E : Y^2 + Y = X^3 - 7 \) be an elliptic curve of conductor 27 and let \( \phi \) be the corresponding newform. Using Corollary 4.1, we get that the local component of \( \phi \) at 2 is not supercuspidal and hence we can apply Waldspurger’s Theorem. We have the following proposition.

**Proposition 8.7.** With \( E \) as above let \( n \) be a square-free integer.

(i) Suppose \( n \equiv 1 \) (mod 3). Let \( f \) be given by

\[
f = \sum_{n=1}^{\infty} a_n q^n = \frac{1}{2} \cdot \theta_{[1,6,15,-6,0,0]} + \frac{1}{2} \cdot \theta_{[4,4,7,4,4,2]} + \theta_{[27,27,1,0,0,0]} \\
- \theta_{[28,27,4,0,4,0]} - \frac{1}{2} \cdot \theta_{[49,36,4,2,0,0]} - \frac{1}{2} \cdot \theta_{[16,9,7,-6,-4,-6]} + \theta_{[31,16,7,4,2,16]}.
\]

If either \( \nu_2(n) = 1 \) or, \( \nu_2(n) = 0 \) and \( n \equiv 1, 5 \) (mod 8) then

\[
\text{L}(E_{-n}, 1) = \frac{\text{L}(E_{-1}, 1)}{\sqrt{n}} \cdot a_n^2.
\]

Otherwise,

\[
\text{L}(E_{-n}, 1) = \frac{\kappa}{\sqrt{n}} \cdot a_n^2
\]

where \( \kappa = \sqrt{19} \cdot \text{L}(E_{-19}, 1) \) if \( n \equiv 3 \) (mod 8) and \( \kappa = \sqrt{7} \cdot \text{L}(E_{-7}, 1) \) if \( n \equiv 7 \) (mod 8).

(ii) Suppose \( n \equiv 0 \) (mod 3) and let \( n = 3m \). Let \( h \in S_{3/2}(324, \chi_{\text{triv}}, \phi) \) be the cusp form having the following \( q \)-expansion

\[
h = q^3 - q^{21} + 2q^{30} - q^{39} - 2q^{48} - q^{57} - 2q^{66} + q^{75} + O(q^{80}) := \sum_{n=1}^{\infty} b_n q^n.
\]
Further suppose \((\frac{m}{3}) = 1\). If either \(\nu_2(n) = 1\) or \(\nu_2(n) = 0\) and \(n \equiv 1, 5\) (mod 8) then
\[
L(E_{-n}, 1) = L(E_{-21}, 1) \cdot \sqrt{\frac{21}{n}} \cdot b_n^2.
\]
If \(n \equiv 3, 7\) (mod 8) then
\[
L(E_{-n}, 1) = \frac{\kappa}{\sqrt{n}} \cdot b_n^2
\]
where \(\kappa = \sqrt{3} \cdot L(E_{-3}, 1)\) if \(n \equiv 3\) (mod 8) and \(\kappa = \sqrt{39} \cdot L(E_{-39}, 1)\) if \(n \equiv 7\) (mod 8).

(iii) If \(n = 3m\) and \((\frac{m}{3}) = -1\) then \(L(E_{-n}, 1) = 0\).
(iv) If \(n \equiv 2\) (mod 3) then \(L(E_{-n}, 1) = 0\).

The proof of (i) and (ii) follows as in the previous examples, while for (iii) and (iv) one can use root number arguments. We point out that the cusp form \(h\) which appears in (ii) does not come from ternary quadratic forms. Moreover since \(E\) is isogenous to \(E_{-3}\), for \(n\) positive square-free \(L(E_n, 1) = L(E_{-3n}, 1)\). Thus using above proposition we are able to compute the critical values \(L(E_n, 1)\) for all \(n\) square-free.

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