Leading-order analysis of the twist-3 space- and time-like cut vertices in QCD.

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Abstract

We review the recent theoretical progress in the construction and solution of the evolution equations which govern the scale dependence for the twist-three structure and fragmentation functions of the nucleon.
1 Introduction.

In the naive parton model [1] the cross sections for several inclusive processes can be expressed in terms of the density of probability of partons ($\phi$) in a hadron ($h$) $F_{\phi/h}(x)$ or hadrons in a parton $F_{h/\phi}(\zeta)$ with $x$, $1/\zeta$ being the momentum fraction of the final state particle with respect to the initial one. The probabilistic picture relies on the fact that the constituents in the high energy processes behave as a bunch of noninteracting quanta at small space-time separations. However, the description of hard reactions using this simple intuitive picture is very restrictive. The rigorous field theoretical basis at the leading twist and beyond comes from the asymptotically free QCD and the use of the factorization theorems [2] which give the possibility to separate the contributions responsible for physics of large and small distances involved in any hard reaction. At the lowest twist level, the parton model is trustworthy and can be mapped onto the language of operator product expansion approach (OPE) (for deep inelastic scattering). The contribution of large distances is parametrized by the distribution (fragmentation) functions mentioned above, or by parton correlators in a broader sense, which are uncalculable at the moment from the first principles of the theory. The second ones — hard-scattering subprocesses — can be dealt perturbatively. The parton distributions are defined in QCD by the target matrix elements of the light-cone correlators of field operators [4]. This representation allows for the estimation of these quantities by the non-perturbative methods presently available: like the QCD sum rules [5, 6], the effective chiral Lagrangians [7], the MIT bag model [8-11, 12, 13], [14].

After addressing for over decades mainly the spin averaged observables which have probed the nonpolarized substructure of the hadrons and have yielded an important information on the parton content of nucleon, the attention has been shifted towards more subtle dynamics underlying the polarized scattering. A renewed interest in the high energy spin physics in the last years has been concentrated on the transverse spin phenomena in hard processes. It revives many ideas developed over a decade ago. In particular, the notion of the twist-2 transversity distribution $h_1(x)$, first mentioned by Ralston and Soper [15] has been reinvented as well as its twist-3 counterparts have been addressed [16]. Due to chirality conservation $h_1(x)$ cannot appear in the inclusive deep inelastic scattering (DIS) but it can be measured, for instance, in Drell-Yan reactions through the collision of the transversely polarized hadrons [15, 16] and in semi-inclusive pion production in DIS on the nucleon [17]. In the last case, it enters into the cross section as a leading contribution together with the twist-3 chiral-odd spin-independent fragmentation function $I(\zeta)$ [17]. It is well known that there is considerable difference between the structure and fragmentation functions. Namely, the moments of the former are expressed in terms of reduced matrix elements of the tower of the local operators of definite twist. This property is established by exploiting the Wilson OPE for inclusive DIS. Although the light-cone expansion for the fragmentation processes is similar to DIS, the moments of the corresponding functions are not related to any short-distance limit. As a substitute for the local operators come the Mueller’s time-like cut vertices [18], which are essentially nonlocal in the coordinate space so that the analogy to the operator language is only useful mnemonic.

The transverse spin phenomena [19] attained by the inclusive DIS are associated with explicitly interaction-dependent, i.e. higher twist (twist-3), effects. To disentangle the underlying complicated dynamics, the unravelling of the twist-3 effects in the hard processes,
which manifest the quantum mechanical interference of partons in the interacting hadrons is needed. At the twist-3 level, the nucleon has three structure functions $g_2(x)$, $e(x)$ and $h_L(x)$ and the fragmentation functions corresponding to each introduced distribution. These new characteristics have been the subject of intensive theoretical study until recently since they open a new window to explore the nucleon content. The most important advantage of the twist-3 structure functions is that while being important for understanding of the long-range quark-gluon dynamics they contribute at the leading order in $1/Q$ ($Q$ being the momentum of the probe) to certain asymmetries [9, 10] and, therefore, are directly accessible by either the polarized (semi)inclusive DIS [20] or Drell-Yan or hadron production in $e^+e^-$-annihilation processes. To confront the theory with high-precision data, the knowledge of the size of the logarithmic violation of the Bjorken scaling by the QCD radiative corrections in the measurable quantities is highly required.

There exist two equivalent (and complementary) approaches to studying the $Q^2$-dependence of the structure functions in the leading logarithmic approximation (LLA) of the perturbation theory. The first is based on the use of the OPE for the product of currents. It makes possible the study of logarithmic violation of Bjorken scaling as well as of the power suppressed contributions (higher twists) responsible for many subtle phenomena in a polarized scattering. The former is achieved by the calculation of the anomalous dimensions of the corresponding local operators. However, there exists an alternative approach to the analysis of the corresponding quantities which is based on the evolution equations [21, 22]. In spite of the fact that the latter has some difficulties as compared to the former, in the study of higher twists (like the loose of the explicit gauge and Lorentz invariance of calculations and also the presence of the overcomplete set of correlation functions) it has an important advantage as being the closest to the intuitive physical parton-like picture. There is another advantage of the latter approach to studying the higher twist effects from the point of view of experimental capabilities since the OPE provides us with the moments of the structure functions, and in order to extract the former, one needs to measure the latter in the whole region of the momentum fraction very accurately. Obviously, it is a quite difficult task even for the next generation of colliders. While, with a set of evolution equations at hand, one can find, in principle, the $Q^2$-dependence of the cross section in question by putting the experimental cuts on the region of the attained momentum fractions. This approach can also be used in the situations when the OPE is no longer valid, i.e. the time-like processes.

Thus, the $Q^2$-evolution of the parton distributions [21, 22] can be predicted unambiguously by exploiting the powerful methods of renormalization group (RG) and QCD perturbation theory. As distinguished from the leading twist evolution, the twist-3 two-quark fragmentation functions receive contribution from the quark-gluon correlators even in the limit of asymptotically large momentum transfer. To solve the problem, one should correctly account for the mixing of correlators of the same twist and quantum numbers in the course of renormalization.

In the subsequent discussion we attempt to review the recent theoretical progress in the study of the twist-3 polarized and nonpolarized chiral-odd and -even structure and fragmentation functions in the framework of QCD. The presentation will be organized as follows: The first chapter is devoted to the consideration of the parton distribution functions. Here we address the issues of the sum rules, construction of the basis of independent correlators closed with respect to renormalization group evolution, the machinery for the calculation of
the evolution kernels, simplification which occurs in the limit of a large number of colours and the orbital momenta. The last features make possible the finding of the analytical solution of the approximated evolution equations. The second chapter concerns the case of fragmentation functions. We mainly cover the same subjects as in the first part of the paper.

2 Space-like cut vertices.

2.1 Correlation functions of nonleading twist.

As we have mentioned in the introduction, the parton distribution functions in QCD are defined by the Fourier transforms along the null-plane of the forward matrix element of the parton field operators product separated by an interval $\lambda$ on the light cone ($n^2 = 0$):

$$F(\lambda) \equiv F(\lambda, 0) = \phi^* (0) \Phi[0, \lambda n] \phi(\lambda n),$$

where $\phi$ denotes a quark $\psi$ or a gluon field $B_\mu$ and $\Phi$ is a path ordered exponential along the straight line which insures the gauge invariance of the parton distribution

$$\Phi[x,y] = P \exp \left( ig(x - y)_\mu \int_0^1 d\sigma B_\mu(y + \sigma(x - y)) \right).$$

We suppress the dependence on the renormalization scale $\mu_R$ in Eq. (1), necessary to render this quantity well defined in the field theory. The Fourier transformations from the coordinate to the momentum space and vice versa are given by

$$F(x) = \int \frac{d\lambda}{2\pi} e^{i\lambda x} \langle h|F(\lambda)|h\rangle, \quad \langle h|F(\lambda)|h\rangle = \int dx e^{-i\lambda x} F(x).$$

Both of these representations display the complementary aspects of the factorization. The light-cone position representation is suitable to make contact with the operator product expansion approach, while the light-cone fraction representation is appropriate for establishing the language of the parton model. Throughout the paper we will use the light-cone position and the light-cone fraction representations in parallel.

It is well known that in order to endow the theory with parton-like interpretation and to get much deeper insight into the corresponding perturbative calculations, it is necessary to use to ghost-free gauges. Owing to this fact we choose in what follows the light-cone gauge $B_+ = n^\mu B_\mu = 0$ for the boson field. The advantage of this gauge is that the gluon field operator $B_\rho$ is related to the field strength tensor $G_{\rho\sigma}$ via the simple relation

$$B_\mu(\lambda n) = \partial_+^{-1} G_{+\mu}(\lambda n) = \frac{1}{2} \int_{-\infty}^{\infty} dz \epsilon(\lambda - z) G_{+\mu}(z).$$

Here, the residual gauge degrees of freedom are fixed by imposing antisymmetric boundary conditions on the field, which allows a unique inversion. Thus, the gauge invariant result can be restored after all required calculations have been performed.

To trace the origin of the operator definition of the hadron’s parton density we sketch briefly below the factorization procedure of Ellis-Furmanski-Petronzio (EFP) for the
hadron matrix element of the $T$-product of the electromagnetic currents $T(P, q)$ whose imaginary part defines the familiar structure functions of DIS. To this end we use the Sudakov decomposition of the four-momentum of the active parton in transverse and longitudinal components

$$k^\mu = xp^\mu + \alpha n^\mu + k^\mu_\perp.$$  

(5)

Here $n$ is a light-cone vector $n^2 = 0$ normalized with respect to the four-vector $P = p + \frac{1}{2} M^2 n$ of the parent hadron $h$ of mass $M$, i.e. $nP = 1$, and $p$ is a null vector along the opposite tangent to the light cone such that $p^2 = 0$, $np = 0$.

The physical observable $T(P, q)$ can be factorized into the hard $H$ and soft $S$ blocks (we omit the Lorentz indices of the currents $J^\mu$)

$$T(P, q) = i \int d^4 z e^{i(qz)} \langle h| T \{ J(z) J(0) \} | h \rangle = \int \prod_i d^4 k_i [H(k_i, q^2)S(k_i, p, \Lambda^2)]$$

$$= \int \prod_i d^4 k_i d x_i \delta(x_i - (k_i n)) [(H(x_i p, q^2) + \ldots)S(k_i, p, \Lambda^2)]$$

$$= \int \prod_i dx_i [(H(x_i p, q^2) + \ldots)S(\lambda_i, p, \Lambda^2)] \equiv [H \otimes S].$$  

(6)

where we have used the collinear expansion of the momentum of the struck parton with respect to the large $+$-direction of the hadron momentum.

$$S(x_i, p, \Lambda^2) = \int \prod_i d^4 k_i \delta(x_i - (k_i n))S(k_i, p, \Lambda^2)$$  

(7)

with

$$S(k_i, p, \Lambda^2) = \int \prod_i d^4 z_i e^{i z_i k_i} \langle h| \phi(z_1) \phi(z_2) \ldots \phi(z_n) | h \rangle.$$  

(8)

Simple manipulations allow one to write the final answer for the soft part

$$S(x_i, p, \Lambda^2) = \int \prod_i \frac{d\lambda_i}{2\pi} e^{i \lambda_i x_i} \langle h| \phi(\lambda_1 n) \phi(\lambda_2 n) \ldots \phi(\lambda_n n) | h \rangle.$$

(9)

Note that $\Phi = 1$ in the gauge we have chosen. By exploiting the Poincaré invariance of the forward matrix element we can exclude the overall translation and, in a particular case, come to Eq. (1).

The multiparton distributions corresponding to the interference of higher Fock components in the hadron wave functions that emerge at the twist-3 level are the generalizations of (1) to the 3-parton fields and present already in (9)

$$F(\lambda, \mu) \equiv F(\lambda, 0, \mu) = \phi^*(\mu n)\phi(0)\phi(\lambda n).$$

(10)

We do not display the quantum numbers of the field operators since they are not of relevance at the moment. The direct and inverse Fourier transforms are

$$F(x, x') = \int \frac{d\lambda}{2\pi} \frac{d\mu}{2\pi} e^{i\lambda x - i\mu x'} \langle h| F(\lambda, \mu) | h \rangle, \quad \langle h| F(\lambda, \mu) | h \rangle = \int dx dx' e^{-i\lambda x + i\mu x'} F(x, x').$$

(11)
The variables $x$ and $x'$ are the momentum fractions of incoming $\phi$ and outgoing $\phi^*$ partons, respectively. The restrictions on their physically allowed values come from the support properties of the multiparton distribution functions discussed at length in Ref. \[24\], namely $F(x, x')$ vanishes unless $0 \leq x \leq 1$, $0 \leq x' \leq 1$.

Beyond the leading-twist level the intuitive parton-like picture is not so immediate, as one usually starts with an overcomplete set of correlation functions. However, the point is that the equations of motion for field operators imply several relations between correlators, and the problem of construction of a simpler operator basis is reduced to an appropriate exploitation of these equalities. The guiding line to disentangle the twist structure is clearly seen in the light-cone formalism of Kogut and Soper \[25\]. Consider, for instance, the correlators containing two quarks $\bar{\psi}\psi$. Then, decomposing the Dirac field into “good” and “bad” components with Hermitian projection operators $P_\pm = \frac{1}{2} \gamma_+ \gamma_\pm$: $\psi_\pm = P_\pm \psi$, we have three possible combinations $\psi_+^\dagger \psi_+$, $\psi_+^\dagger \psi_- \pm \psi_-^\dagger \psi_+$, and $\psi_-^\dagger \psi_-$, which are of twist 2, 3 and 4, respectively. The origin of this counting lies in the dynamical dependence of the “bad” components of the Dirac fermions

$$\psi_- = -\frac{i}{2} \partial_+^{-1} (i \slashed{D}_\perp + m) \gamma_+ \psi_+.$$  \hspace{1cm} (12)

These components depend on the underlying QCD dynamics, i.e. they implicitly involve extra partons and thus correspond to the generalized off-shell partons, which carry the transverse momentum. For this reason we come back to the on-shell massless collinear partons of the naive parton model, but supplemented with multiparton correlations through the constraint \[12\]. The operators constructed of the “good” components only were named quasi-partonic \[26\]. The advantage of handling them is that they endow the theory with a parton-like interpretation for higher twists.

The EFP approach is close to the OPE; this equivalence is established by identifying the moments of the parton correlation functions (S-block) with the reduced matrix elements of the local operators. The singularities of the product of the currents on the light-cone are absorbed in the coefficient functions $H$ in front of operators. In the momentum space they result in the inverse powers of the large momentum scale $Q$ at which the operators contribute to the cross section and it is controlled by their twist $\tau$ ($\tau = d_C - s$, where $d_C$ is a canonical dimension of an operator and $s$ is its Lorentz spin). The leading contribution comes from the operators of twist $\tau = 2$:

$$[\phi^*(0)\phi(z)]^{tw-2} = \sum_{n!} z_{\mu_1} z_{\mu_2} \ldots z_{\mu_n} \{\phi^*(0)\partial_{\mu_1} \partial_{\mu_2} \ldots \partial_{\mu_n} \phi(0) - \text{traces}\}.$$  \hspace{1cm} (13)

However, as has been established in Refs. \[27\,28\] one can give the definition of the twist without appealing to the concept of the local operators which is particularly useful in cases when the short distance expansion is no longer relevant, as it happens, for instance, for inclusive production of hadron in the $e^+e^-$-annihilation \textit{et cetera}. The point is that the nonlocal string operator given by Eq. \[13\] obeys the Laplace equation

$$\Box [\phi^*(0)\phi(z)]^{tw-2} = 0, \quad \Box \equiv \partial^2$$  \hspace{1cm} (14)

with the boundary condition on the light-cone

$$\phi^*(0)\phi(z) = \sum_{n!} z_{\mu_1} z_{\mu_2} \ldots z_{\mu_n} \{\phi^*(0)\partial_{\mu_1} \partial_{\mu_2} \ldots \partial_{\mu_n} \phi(0)\}, \quad \text{for } z^2 = 0.$$  \hspace{1cm} (15)
The solution can be written in the form

\[ [\phi^*(0)\phi(z)]^{\text{tw}-2} = \phi^*(0)\phi(z) + \sum_{n=0}^{\infty} \frac{1}{n!(n-1)!} \left(-\frac{z^2}{4}\right)^n \int_0^1 dv \left(\frac{d^{d/2-1}(\bar{v}v)}{n^1}\right)\phi^*(0)\phi(vz), \]

(16)

here \(d\) is a dimension of space-time and \(\bar{v} = 1 - v\). For the time-like processes the connection to the local operators is lost, so that we are left with (16) up to an arbitrary solution of (14) which vanishes on the light cone. To this accuracy we can define

\[ [\phi^*(0)\phi(z)]^{\text{tw}-2} = \phi^*(0)\phi(z) + \sum_{n=0}^{\infty} \frac{1}{n!(n-1)!} \left(-\frac{z^2}{4}\right)^n \int_{1}^{\infty} dv \left(\frac{d^{d/2-1}(\bar{v}v)}{n^1}\right)\phi^*(0)\phi(vz), \]

(17)

where the integration region mimics the physical domain of the parton momentum fraction of the annihilation channel and thus the matrix elements of this string operator possesses the correct support properties in the light-cone variables.

2.2 \(g_2(x)\).

Recently, the first experimental data for the measurement of the transverse spin structure function \(g_2(x)\) in the deep inelastic scattering of the longitudinally polarized muon beam on the transversely polarized proton target have been reported [20]. Although at present the statistics is too low to be able to extract its perturbative evolution it proves to be important to know the theoretical prediction for the latter from QCD. This issue has been extensively studied in the literature, therefore, we just outline the main results referring the interested reader to the original works [29, 30, 26, 31, 27, 32, 33] and reviews [34].

The function we are interested in appears as coefficient in the Lorentz decomposition of the antisymmetric part of the hadronic tensor, relevant to the polarized scattering, over the appropriate tensor structures:

\[ W_{\mu\nu} = \frac{1}{2\pi} \text{Im} \int dz e^{izqz} \langle h| T \left\{ J_{\mu}(z)J_{\nu}\right\} |h\rangle \]

\[ = \frac{i}{(pq)^2} T_{\mu\nu\rho\sigma} \left( s_{\sigma} g_1(x, Q^2) + g_2(x, Q^2) \left( s_{\sigma} - (s_{\sigma}^{pq})_{pq}\right) \right). \]

(18)

Using the nonlocal light-cone OPE we can express, as has been discussed in the preceding section, the polarized structure functions in terms of the hadronic matrix elements of Fourier transformed string operators

\[ S_+ g_1(x) = \frac{1}{2} \int \frac{d\lambda}{2\pi} e^{i\lambda x} \langle h| \bar{\psi}(0)\gamma_5\psi(\lambda n) |h\rangle, \]

\[ S_{\sigma} g_T(x) = \frac{1}{2} \int \frac{d\lambda}{2\pi} e^{i\lambda x} \langle h| \bar{\psi}(0)\gamma_5\gamma_\sigma\psi(\lambda n) |h\rangle, \]

(19)

where \(S_{\sigma}^\perp\) denotes the transverse polarization vector of the hadron \(h\) (\(S^2 = -M^2\)) and \(g_T = g_1 + g_2\).

As we have noted above, the higher-twist two-quark operators mix with multiparton correlators. Moreover, the operator corresponding to \(g_T\) does not possess a definite twist,
and as a consequence could not be renormalized multiplicatively. Taking into account the equation of motion for the quark field and equality arising from the use of the Lorentz invariance, one can find [30, 34]

\[ xg_T(x) - \bar{M}(x) - K(x) - \int dx' \bar{D}(x, x') = 0, \]  

(20)

\[ xg_1(x) = xg_T(x) - x \frac{\partial}{\partial x} K(x) - x \int dx' \frac{\bar{D}(x, x') + \bar{D}(x, x')}{(x' - x)}. \]  

(21)

Here, we have introduced the new correlation functions

\[ S_\sigma^\perp M(x) = \frac{1}{2} \int \frac{d\lambda}{2\pi} e^{i\lambda x} \langle h|\bar{\psi}(0)m\gamma_+\gamma^\perp_\sigma\gamma_5\psi(\lambda n)|h \rangle, \]  

(22)

\[ S_\sigma^\perp K(x) = \frac{1}{2} \int \frac{d\lambda}{2\pi} e^{i\lambda x} \langle h|\bar{\psi}(0)\gamma_+\partial_+^\perp\gamma_5\psi(\lambda n)|h \rangle, \]  

(23)

\[ S_\sigma^\perp D_1(x, x') = \frac{1}{2} \int \frac{d\lambda}{2\pi} \frac{d\mu}{2\pi} e^{i\lambda x - i\mu x'} \langle h|\bar{\psi}(\lambda n)g\gamma_+\gamma^\perp_\sigma\gamma_5\psi(\lambda n)|h \rangle, \]  

(24)

\[ S_\sigma^\perp D_2(x', x) = \frac{1}{2} \int \frac{d\lambda}{2\pi} \frac{d\mu}{2\pi} e^{i\mu x' - i\lambda x} \langle h|\bar{\psi}(\lambda n)g\gamma_+\gamma^\perp_\sigma\gamma_5\psi(\lambda n)|h \rangle, \]  

(25)

and

\[ \bar{D}(x, x') = \frac{1}{2} \left[ \bar{D}_1(x, x') + \bar{D}_2(x', x) \right] \]  

(26)

is a $C$-even combination of correlators which can enter into the cross section due to even photon state in the $t$-channel under the charge conjugation. The derivative in the correlation function $K(x)$ acts on the quark field before setting its argument on the light cone.

Solving the system of the differential equations (20) and (21) with respect to $g_T(x)$ the integration constant can be found from the support properties of the distribution: $g_T(x) = 0$ for $|x| \geq 1$. The solution provides us with the following relation between these functions:

\[ g_T(x) = \int_x^1 \frac{d\beta}{\beta} g_1(\beta) + \frac{1}{x} \bar{M}(x) - \int_x^1 \frac{d\beta}{\beta^2} \bar{M}(\beta) + \int_x^1 \frac{d\beta}{\beta} \int \frac{d\beta'}{\beta' - \beta} \left[ \frac{\partial}{\partial \beta} Y(\beta, \beta') + \frac{\partial}{\partial \beta'} Y(\beta', \beta) \right], \]  

(27)

where

\[ Y(x, x') = (x - x') \bar{D}(x, x'). \]  

(28)

Here $Y(x, x')$ is explicitly gauge invariant distribution since $\bar{D}$ is gauge variant provided we use a gauge other than the light-cone. To see this, we exploit the advantages of the light-cone gauge, where the gluon field is expressed in terms of the field strength tensor by Eq. (4) and take into account the relation

\[ \frac{1}{2} \int \frac{d\lambda}{2\pi} e^{\pm i\lambda x} \epsilon(\lambda - z) = \mp \frac{i}{2\pi} \text{PV} \int \frac{1}{x} e^{\pm i\lambda x}, \]  

(29)

we can easily obtain the expressions of the gauge-invariant quantities in terms of three-particle string operators which are nonlocal generalization of the Shuryak-Vainstein operators.
\[^{\pm}S_\sigma\]

Generically

\[S_\sigma^1 Y_1(x, x') = \frac{1}{2} \int \frac{d\lambda}{2\pi} e^{i\lambda x - i\mu x'} \langle h|^{\pm}S_\sigma(\lambda, 0, \mu)|h\rangle,\]

\[S_\sigma^1 Y_2(x', x) = \frac{1}{2} \int \frac{d\lambda}{2\pi} e^{i\mu x' - i\lambda x} \langle h|S_\sigma(\mu, 0, \lambda)|h\rangle,\]

with

\[^{\pm}S_\sigma(\lambda, 0, \mu) = \bar{\psi}(\mu) i g \gamma^+ [i\tilde{G}_{\sigma+}^\perp(0) \pm \gamma_5 G_{\sigma+}^\perp(0)] \psi(\lambda n),\]

where \(\tilde{G}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} G_{\rho\sigma}\) is the dual field strength tensor and we have used the relation \(\epsilon_{\rho\sigma}^\perp G_{\rho+}^\perp = \tilde{G}_{\sigma+}^\perp\) with the two-dimensional antisymmetric tensor \(\epsilon_{\rho\sigma}^\perp \equiv \epsilon_{\rho\sigma+}^\perp\).

Thus, the leading order analysis \([27]\) suggests that the structure function can be written as the following sum:

\[g_2(x) = g_{2W}(x) + \bar{g}_2(x),\]

where

\[g_{2W}(x) = -g_1(x) + \int_x^1 \frac{d\beta}{\beta} g_1(\beta)\]

is the twist-2 Wandzura-Wilczek contribution to the structure functions \([34]\), while \(\bar{g}_2(x)\) is a genuine twist-3 (explicitly interaction-dependent up to unessential quark-mass kinematical contribution) part which is expressed via the integral of the matrix element of the nonlocal operators which measure the quark-gluon correlation function in the target nucleon.

The distribution functions defined by Eqs. \([19], [22]-[28]\) form the redundant basis of operators closed under renormalization group evolution. Going over to the operators composed of the “good” components only, we are forced to consider the renormalization of the correlators \([17]\) \(M(x)\) and \(Y(x, x')\).

2.3 \(e(x)\).

In the unpolarized case we define the following redundant basis of the chiral-odd twist-3 correlations:

\[e(x) = x \int \frac{d\lambda}{2\pi} e^{i\lambda x} \langle h|\bar{\psi}(0)\psi(\lambda)|h\rangle,\]

\[M(x) = \frac{1}{2} \int \frac{d\lambda}{2\pi} e^{i\lambda x} \langle h|\bar{\psi}(0)m\gamma_+\psi(\lambda)|h\rangle,\]

\[D_1(x, x') = \frac{1}{2} \int \frac{d\lambda}{2\pi} \frac{d\mu}{2\pi} e^{i\lambda x - i\mu x'} \langle h|\bar{\psi}(\mu)g\gamma_+ B_{\sigma+}^\perp(0)\psi(\lambda)|h\rangle,\]

\[D_2(x', x) = \frac{1}{2} \int \frac{d\lambda}{2\pi} \frac{d\mu}{2\pi} e^{i\mu x' - i\lambda x} \langle h|\bar{\psi}(\lambda)g B_{\sigma+}^\perp(0)\gamma_+\psi(\mu)|h\rangle.\]

The functions \(D_1\) and \(D_2\) are related by complex conjugation \([D_1(x, x')]^* = D_2(x', x)\). The quantities determined by these equations form a closed set under renormalization; however,

\[\text{In this discussion, we restrict ourselves to consideration of the non-singlet channel only.}\]
they are not independent, since there is a relation between them due to the equation of motion for the Heisenberg fermion field operator:

\[ e(x) - M(x) - \int dx' D(x, x') = 0, \quad (39) \]

where again we have introduced the convention

\[ D(x, x') = \frac{1}{2} [D_1(x, x') + D_2(x', x)]. \quad (40) \]

This function is real-valued and antisymmetric with respect to the exchange of its arguments:

\[ [D(x, x')]^* = D(x, x'), \quad D(x, x') = -D(x', x). \quad (41) \]

Below, in section 2.8, as an illustration of the self-consistency of the whole approach to the higher twists we present a set of coupled RG equations for the correlation functions determined by Eqs. (35)-(38) derived in the abelian gauge theory. Relation (39) provides a strong check of our calculations\footnote{This fact follows from general renormalization properties of gauge-invariant operators as one expects that the counter term for the equation of motion operator can be given only by the operator itself. Its matrix element, being taken with respect to the physical state, decouples completely from the renormalization group evolution.}. It allows the reduction, as we have mentioned above, of the RG analysis to the study of scale dependence of the three-parton \( D \) and mass-dependent \( M \) correlators only.

Introducing the quantity

\[ Z(x, x') = (x - x') D(x, x'). \quad (42) \]

we can easily obtain from Eqs. (37) and (38) the definition of the gauge-invariant quantities \( Z \) in terms of three-particle string operators, namely

\[ Z(x, x') = \frac{1}{2} \int \frac{d\lambda}{2\pi} \frac{d\mu}{2\pi} e^{i\lambda x - i\mu x'} \langle h | Z(\lambda, \mu) + Z(-\mu, -\lambda) | h \rangle, \quad (43) \]

where

\[ Z(\lambda, \mu) \equiv Z(\lambda, 0, \mu) = \frac{1}{2} \bar{\psi}(0) G_{+\rho}(0) \sigma^\perp_{\rho+} \psi(\lambda n). \quad (44) \]

In the same way, for a mass-dependent non-local string operator

\[ \mathcal{M}^j(\lambda) \equiv \mathcal{M}^j(\lambda, 0) = \frac{m}{2} \bar{\psi}(0) \gamma_5 (iD_+(\lambda))^j \psi(\lambda n), \quad (45) \]

the Fourier transform is

\[ M^j(x) = x^j M(x) = \int \frac{d\lambda}{2\pi} e^{i\lambda x} \langle h | \mathcal{M}^j(\lambda) | h \rangle. \quad (46) \]

For the spin-dependent scattering discussed below, the only difference is that one should insert also a \( \gamma_5 \)-matrix between the fields in the definitions of the string operators (14), (15).
2.4 $h_L(x)$.

Analogously, the set of correlation functions for the polarized case is as follows:

\[ h_1(x) = \frac{1}{2} S_x^\perp \int \frac{d\lambda}{2\pi} e^{i\lambda x} \langle h|\bar{\psi}(0)i\sigma_+\gamma_5\psi(\lambda x)|h\rangle, \]  

\[ h_L(x) = \frac{x}{2} \int \frac{d\lambda}{2\pi} e^{i\lambda x} \langle h|ar{\psi}(0)i\sigma_+\gamma_5\psi(\lambda x)|h\rangle, \]  

\[ \widetilde{M}(x) = \frac{1}{2} \int \frac{d\lambda}{2\pi} e^{i\lambda x} \langle h|\bar{\psi}(0)m\gamma_+\gamma_5\psi(\lambda x)|h\rangle, \]  

\[ K(x) = \frac{1}{2} \int \frac{d\lambda}{2\pi} e^{i\lambda x} \langle h|\bar{\psi}(0)i\gamma_+\varphi_\perp\gamma_5\psi(\lambda x)|h\rangle, \]  

\[ \widetilde{D}_1(x, x') = \frac{1}{2} \int \frac{d\lambda}{2\pi} e^{i\lambda x} \langle h|\bar{\psi}(0)i\gamma_+\varphi_\perp(0)\gamma_5\psi(\lambda x)|h\rangle, \]  

\[ \widetilde{D}_2(x', x) = \frac{1}{2} \int \frac{d\lambda}{2\pi} e^{i\lambda x} \langle h|\bar{\psi}(0)\gamma_+\varphi_\perp(0)\gamma_5\psi(\mu n)|h\rangle, \]

Besides the identity arising from the equation of motion

\[ h_L(x) - \widetilde{M}(x) - K(x) - \int dx' \widetilde{D}(x, x') = 0, \]  

there is an equation provided by the Lorentz invariance

\[ 2xh_1(x) = 2h_L(x) - x \frac{\partial}{\partial x} K(x) - 2x \int dx' \widetilde{D}(x, x')(x' - x). \]  

It means that both parts of this equality are expressed in terms of matrix elements of different components of one and the same twist-2 tensor operator, and thus should possess the same anomalous dimensions. Again, we have introduced the $C$-even quantity $\widetilde{D}$, which has the properties

\[ [\widetilde{D}(x, x')]^* = \widetilde{D}(x', x), \quad \widetilde{D}(x, x') = \widetilde{D}(x', x). \]  

Following the same line as above, we come to the equation

\[ h_L(x) = 2x^2 \int_x^1 \frac{d\beta}{\beta^2} h_1(\beta) + \widetilde{M}(x) - 2x^2 \int_x^1 \frac{d\beta}{\beta^2} \widetilde{M}(\beta) \]  

\[ + x^2 \int_x^1 \frac{d\beta}{\beta^2} \int \frac{d\beta'}{\beta' - \beta} \left[ \frac{\partial}{\partial \beta} \frac{\partial}{\partial \beta'} \right] \bar{Z}(\beta, \beta'). \]  

A similar relation was found by Jaffe and Ji \[ \text{Ref. [10]} \] in Ref. \[ \text{[10]} \]. Here the dynamical twist-3 contribution is explicitly related to a particular integral of the three-parton correlation function $\bar{Z}$. In terms of local operators it looks like

\[ (n + 3)[h_L]_n = 2[h_1]_{n+1} + (n + 1)\widehat{M}_n + \sum_{l=1}^n (n - l + 1)\bar{Z}_l^l, \]  

and the definition of moments of distribution functions is given by Eq. \[ \text{[12]} \].

As before, excluding the functions \[ \text{[15]} \] and \[ \text{[34]} \], and using the relations \[ \text{[53]} \] and \[ \text{[54]} \], we can choose the basis of independent functions in the form: $h_1(x), \widetilde{M}(x), \widetilde{D}(x, x')$.

\[ \text{[3]} \text{The corresponding expressions in Ref. \[ \text{[10]} \] contain misprints.} \]
2.5 Construction of the evolution equations.

As long as \( z^2 \neq 0 \), the renormalization of the \( T \)-product of the operators \( T \{ \phi^*(0) \phi(z) \} \) is trivial and reduced to the familiar renormalization constants \( Z \) of the fundamental field operators entering into the Lagrangian density. However, if \( z = 0 \) or \( z^2 = 0 \) an additional divergence enters the game. This is a well-known fact from the renormalization theory since the product of (at least) two field operators entering into the same space-time point (or on the light cone) produces an ill-defined quantity from the point of view of the theory of distributions and the corresponding infinities have to be regularized and subtracted. In the momentum space this results in ultraviolet (UV) divergences of the momentum integrals in the perturbation theory, and as a by-product this causes the logarithmic dependence of the parton densities on the normalization point. This dependence is governed by the renormalization group. The evolution equations for the leading twist correlation functions determining their \( Q^2 \) dependence can be interpreted in terms of the kinetic equilibrium of partons inside a hadron (for distribution functions) or hadrons inside a parton (for fragmentation function) under the variation of the ultraviolet transverse momentum cut-off \([21]\). However, beyond the leading twist the probabilistic picture is lost due to a quantum mechanical interference and more general quantities emerge, \textit{i.e.} multiparticle parton correlation functions, whose scale dependence is determined by the Faddeev-type evolution equation with a pair-wise particle interaction \([26]\).

![Figure 1: Ladder diagram for deep inelastic scattering.](image)

There are two sources of the logarithmic dependence of the correlation functions. The first is the divergences of the transverse momentum integrals of the particles interacting with the vertex and forming the perturbative loop. Another source is the divergences due to the virtual radiative corrections. In the renormalizable field theory the latter are factorized into the renormalization constants \( Z \) of the corresponding Green functions. However, owing to the specific features of the renormalization in the light-like gauge, extensively reviewed in the next section, there is mixing of correlation functions due to the renormalization of field operators. The latter fact is closely related to the matrix nature of renormalization constants.
of the elementary Green functions in the axial gauge. For example, after renormalization of the fermionic propagator the matrix structure of the bare vertex could be changed, in general, since the renormalization matrix acts on the spinor indices of the vertex (see Eq. (189)).

In the Leading Logarithmic Approximation (LLA) there is a strong ordering \[21, 47\] of transverse particle momenta as well as their minus components, so that only the particles entering into the divergent virtual block \(\Sigma, \Gamma, \Pi\) or the particles adjacent to the vertex can achieve the maximum values of \(|k_\perp|\) and \(\alpha\) (see Fig. 1):

\[
|k_{n\perp}| \ll \ldots \ll |k_{2\perp}| \ll |k_{1\perp}| \ll \Lambda,
\]

\[
\alpha_n \ll \ldots \ll \alpha_2 \ll \alpha_1,
\]

while the plus components of the parton momenta are of the same order of magnitude

\[
x_n \sim \ldots \sim x_2 \sim x_1
\]

for a \(n\)-rank ladder-type diagram.

### 2.6 Renormalization in the light-cone gauge.

A peculiar feature of the light-like gauge is the presence of the spurious IR pole \(1/k_+\) in the density matrix of the gluon propagator

\[
D_{\mu\nu}(k) = \frac{d_{\mu\nu}(k)}{k^2 + i0}, \quad d_{\mu\nu} = g_{\mu\nu} - \frac{k_\mu n_\nu + k_\nu n_\mu}{k_+}.
\]

The central question is how to handle this unphysical pole when \(k_+ = 0\). There are two different ways to treat it which employ the Cauchy principal value (PV) and Mandelstam-Leibbrandt prescriptions (ML) \[36\]:

\[
PV \frac{1}{k_+} = \frac{1}{2} \left\{ \frac{1}{(kn) + i0} + \frac{1}{(kn) - i0} \right\},
\]

\[
ML \frac{1}{k_+} = \frac{(kn^*)}{(kn)(kn^*) + i0},
\]

with an arbitrary four-vector \(n^*\) satisfying \(n^{*2} = 0, nn^* = 1\) (without loss of generality, we can put it equal to \(p\)).

Here, we outline the one-loop renormalization program for the abelian gauge theory with PV prescription. In the next section, we show, using a simple example, the difference one encounters in dealing with the ML prescription.

Due to an additional power of the transverse momentum \(k_\perp\) in the numerator of the density matrix of the gluon propagator, there exist extra UV divergences of the Feynman graphs which are absent in the usual isotropic gauges. For our practical aims, we limit ourselves to the calculation of the one-loop expressions for the propagators and vertex functions. This is sufficient for reconstruction of the equations in LLA using the renormalization group invariance.
The unrenormalized fermion Green function is given by the expression

\[ G^{-1}(k) = k - m_0 - \Sigma(k), \]  

where \( \Sigma(k) \) is a self-energy operator. Calculating the latter to the one-loop accuracy we get the following result

\[ G(k) = (1 - \Sigma_1)U_1^{-1}(k)\frac{1}{k - m}U_1(k), \]  

where

\[ U_1(k) = 1 - \frac{m}{k_+}\Sigma_2(k)\gamma_+ - \frac{1}{k_+}(\Sigma_2(k) - \Sigma_1)\gamma_+k, \]  

\[ U_2(k) = 1 + \frac{m}{k_+}\Sigma_2(k)\gamma_+ + \frac{1}{k_+}(\Sigma_2(k) - \Sigma_1)\gamma_+, \]  

and

\[ \Sigma_1 = \alpha \frac{4}{\pi} \ln \Lambda^2, \quad \Sigma_2(k) = \alpha \frac{4}{\pi} \ln \Lambda^2 \int dz' \frac{z}{(z - z')} \Theta_0^{11}(z', z' - z). \]  

Here \( m \) is a renormalized fermion mass related to the bare quantity by the well-known relation

\[ m_0 = m(1 - 3\Sigma_1). \]  

The functions \( \Theta_{i_1i_2...i_n}^m \) used throughout the paper are given by the formula

\[ \Theta_{i_1i_2...i_n}^m(x_1, x_2, ..., x_n) = \int_{-\infty}^{\infty} \frac{d\alpha}{2\pi i} \alpha^m \prod_{k=1}^{n} (\alpha x_k - 1 + i0)^{-i_k}. \]  

For our purposes it is sufficient to have an explicit form of the simplest functions

\[ \Theta_0^0(x) = 0, \]  

\[ \Theta_0^1(x) = \delta(x), \]  

\[ \Theta_{11}^0(x_1, x_2) = \frac{\theta(x_1)\theta(-x_2) - \theta(x_2)\theta(-x_1)}{x_1 - x_2}, \]  

since the others can be expressed in their terms via the following relations:

\[ \Theta_{21}^0(x_1, x_2) = \frac{x_2}{x_1 - x_2}\Theta_{11}^0(x_1, x_2), \]  

\[ \Theta_{21}^1(x_1, x_2) = \frac{1}{x_1 - x_2}\Theta_{11}^0(x_1, x_2) - \frac{1}{x_1 - x_2}\Theta_2^0(x_1), \]  

\[ \Theta_{22}^0(x_1, x_2) = -\frac{2x_1x_2}{(x_1 - x_2)^2}\Theta_{11}^0(x_1, x_2), \]  

\[ \Theta_{111}^0(x_1, x_2, x_3) = \frac{x_2}{x_1 - x_2}\Theta_{11}^0(x_2, x_3) - \frac{x_1}{x_1 - x_2}\Theta_{11}^0(x_1, x_3), \]  

\[ \Theta_{111}^1(x_1, x_2, x_3) = \frac{1}{x_1 - x_2}\Theta_{11}^0(x_2, x_3) - \frac{1}{x_1 - x_2}\Theta_{11}^0(x_1, x_3). \]
As we can easily observe, the renormalization constants are not numbers any more but matrices acting on the spinor indices of fermion field operators. Moreover, the renormalization constants depend on the fractions \( k_i^+ \). The origin of this can be traced back to the lack of the rescaling invariance
\[ d_{\mu\nu}(\rho k) \neq d_{\mu\nu}(k), \tag{78} \]
obeyed by the unregularized propagator.

An abelian Ward identity leads to the equality of the renormalization constants of the gauge boson wave-function and a charge \( Z_3 = Z_g \) so that the corresponding logarithmic dependence on the UV cut-off cancels in the sum of these two contributions in the evolution equation and, therefore, we can neglect the fermion loop insertions into the boson line (in the QCD case this is no longer true).

One can easily calculate the vertex function to the same accuracy. The result is
\[ \Gamma_{\rho}(k_1, k_2) = (1 + \Sigma_1) U_{1}^{-1}(k_1) G_{\rho} U_{2}(k_2), \tag{79} \]
where
\[ G_{\rho} = \gamma_{\rho} - (k_1 - m) \mathcal{Q}_{\rho}(k_1, k_2) \gamma^+ - \gamma_+ \mathcal{Q}_{\rho}(k_1, k_2)(k_2 - m) \tag{80} \]
and
\[ \mathcal{Q}_{\rho}(k_1, k_2) = \Sigma_3(k_1) \gamma^- \gamma_+ \gamma_\rho + \Sigma_3(k_2) \gamma_\rho \gamma^+ \gamma^-, \tag{81} \]
here
\[ \Sigma_3(k_1) = \frac{\alpha}{8\pi} \ln \Lambda^2 \int dz' (z_i - z'_i) \Theta^{111}_{111}(z', z' - z_1, z' - z_2). \tag{82} \]

Apart from the graphs we accounted for, there exists an additional UV divergence of the virtual Compton scattering amplitude; however, we do not need its explicit expression for our practical purposes. This completes the consideration of virtual corrections which cause the logarithmic dependence on the UV momentum cut-off of the quantities in question.

### 2.7 Difference between the PV and ML prescriptions.

In the subsequent discussion we will use both the prescriptions for the gluon propagator in our practical calculations. The first one (PV) will be used in the momentum space \([30, 26, 37, 38]\), while the second one in the coordinate space formulation \([33, 38]\). As a by-product we verify that both of them do lead to the same result. However, it is worthwhile to realize the distinctive features one faces in the computation of the same quantities.

The most important difference of the second prescription is the presence of the additional absorptive part \([39]\) of the vector boson Green function, namely
\[ \text{Disc} \left\{ \text{ML} \frac{d_{\mu\nu}(k)}{k^2 + i\theta} \right\} = -2\pi i \theta(k_+) \left[ d_{\mu\nu}(k) \delta(k^2) - \frac{k_+ k_-}{k^2} (k_{\mu} n_{\nu} + k_{\nu} n_{\mu}) \delta(k_+ k_-) \right]. \tag{83} \]
The second contribution is of the "ghost" type since it has the wrong sign as compared to the conventional one. However, it is not an optional choice but it is an unavoidable consequence of equal time canonical quantization \([36]\). The consequence of this addendum can be easily recovered in the calculation of the one-loop evolution kernels. Let us consider,
for instance, the coordinate space formalism [33, 38]. Then, the operator vertices $V(\lambda_j)$ have only exponential dependence on the position on the light-cone

$$V(\lambda_j) = \Gamma_V e^{i \sum_{\lambda_j} k_j \lambda_j}.$$  \hspace{1cm} (84)

A simple calculation of the one-loop diagram for the leading twist density with $\Gamma_V = \gamma_+$ gives

$$\left[ \gamma_+ e^{i \lambda k_+} \right]_{\Lambda^2} = \frac{\alpha}{2\pi} \ln \Lambda^2 \int_0^1 dy \left[ \gamma_+ e^{i y \lambda k_+} \right] \left\{ \left[ \frac{2}{y} \right]_+ - 1 - y \right\},$$  \hspace{1cm} (85)

so that the well-defined $1/\bar{y}_+$-distribution appears already in the contribution with ”real-emission”. The self-energy insertions into external legs produce a familiar $\frac{3}{2\pi} \frac{\alpha}{2} \delta(\bar{y})$ term. Contrary to this, as we have seen in section 2.6, the renormalization constants of the field operators in the light-cone gauge with the PV prescription turn out to be momentum dependent, and as a by-product the plus-prescription fulfilling occurs only in the sum of the real and virtual corrections.

2.8 Evolution equations in the abelian gauge theory.

In this section, we present a pedagogical illustration of the renormalization group mixing problem for the redundant basis of unpolarized correlation functions defined by Eqs. (35)-(38), in the framework of the abelian gauge theory. Our aim here is to show the self-consistency of the whole approach we have used since the equations derived below satisfy the constraint equality given by Eq. (39), which are further employed to reduce the overcomplete set of correlators to the basis of independent functions.

![Diagrams](image)

Figure 2: One-loop radiative corrections to the two-particle correlators in the abelian gauge theory. The fermion propagator crossed with a bar on diagram (c) means the contraction of the corresponding line into the point.

The one-loop Feynman diagrams giving rise to the transition amplitudes of two-particle correlation functions into the two- and three-parton ones are shown in Fig. 2 (a,b). The last figure (c) on this picture is specific of the vertices having non-quasi-partonic form [26], that is for $e(x)$; it displays the addendum due to the contact term that results from the cancellation of the propagator adjacent to the quark-gluon and bare vertices. As an output
the vertex acquires the three-particle piece. The radiative corrections to the three-parton
correlators are presented in Fig. 3 (a,b,c).

Figure 3: The one-loop renormalization of the three-parton correlation functions. Self-
ergy insertions into external legs are implied.

A straightforward calculation yields the evolution equations for the spin-independent case
in the form [38]

$$
\dot{M}(x) = -\frac{\alpha}{2\pi} \int d\beta M(\beta) \left\{ 2 \left[ \frac{\beta}{(x-\beta)} \Theta_{11}^0(x, x-\beta) \right] + \frac{\beta + x}{\beta} \Theta_{11}^0(x, x-\beta) \right\}, \quad (86)
$$

$$
\dot{e}(x) = \frac{\alpha}{2\pi} \int d\beta \left( e(\beta) \left\{ \frac{x}{\beta} \Theta_{11}^0(x, x-\beta) + \frac{1}{2} \delta(\beta - x) \right\} 
- M(\beta) \left\{ 2 \left[ \frac{\beta}{(x-\beta)} \Theta_{11}^0(x, x-\beta) \right] + x \Theta_{21}^0(x, x-\beta) + 2 \Theta_{11}^0(x, x-\beta) \right\} 
- \int d\beta' D(\beta, \beta') \left\{ 2 \left[ \frac{\beta}{(x-\beta)} \Theta_{11}^0(x, x-\beta) \right] + \frac{x}{x-\beta} \Theta_{11}^0(x, x-\beta, x-\beta' + \beta') \right\} \right) \), \quad (87)
$$

$$
\dot{D}(x, x') = -\frac{\alpha}{2\pi} \left\{ \left[ \frac{x'}{x} e(x) - M(x) \right] \Theta_{11}^0(x', x'-x) - \left[ \frac{x'}{x'} e(x') - M(x') \right] \Theta_{11}^0(x, x-x') \right\} 
+ \int d\beta' \left( D(x, \beta')(\beta' - x + x') \Theta_{111}^0(x', x'-x, x'-x + \beta') \right) 
+ \frac{x'}{x'-\beta'} \left[ D(x-x'+\beta', \beta') - D(x, x')] \Theta_{111}^0(x', x'-\beta') \right) 
+ \int d\beta \left( D(\beta, x') \left( \frac{x'-x'}{x'-x} \right) \right) \Theta_{111}^0(x, x-x', x-x'+\beta) 
+ \frac{x}{x-\beta} \left[ D(\beta, x'-x+\beta) - D(x, x')] \Theta_{111}^0(x, x-\beta) \right) - \frac{3}{2} D(x, x'), \quad (88)
$$

where the dot denotes the derivative with respect to the UV cutoff $\Lambda = \Lambda^2 \partial / \partial \Lambda^2$ and the
plus-prescription is defined by the equation

\[
\left[ \frac{\beta}{x - \beta} \theta_{11}^0(x, x - \beta) \right]_+ = \frac{\beta}{x - \beta} \theta_{11}^0(x, x - \beta) - \delta(\beta - x) \int d\beta'' \frac{\beta}{(\beta'' - \beta)} \theta_{11}^0(\beta'', \beta'' - \beta).
\]

(89)

We have used also the equation

\[
\text{PV} \int d\beta \frac{x}{(x - \beta)} \left[ \theta_{11}^0(\beta, \beta - x) + \theta_{11}^0(x, x - \beta) \right] = 0.
\]

(90)

By exploiting the relation provided by the equation of motion, we can easily verify that the RG equations thus constructed are indeed correct and the renormalization program can be reduced to the study of logarithmic divergences of the three-parton \(Z(x, x')\) and quark mass \(M(x)\) correlators in perturbation theory.

### 2.9 QCD evolution of the twist-3 distributions.

For the non-abelian gauge theory the equality of the renormalization constants \(Z_g = Z_3\) implied above no longer holds; so we should account for the renormalization of the gluon wave-function as well as for the renormalization of charge explicitly. For these purposes, to complete the renormalization program outlined in the preceding section \(2.6\), we evaluate the gluon propagator to the same accuracy. The result can be written in the compact form

\[
D_{\mu\nu}(k) = \left( 1 + \Pi^{tr}(k) \right) U_{\mu\rho}(k) \frac{d_{\rho\sigma}(k)}{k^2 + i0} U_{\sigma\nu}(k),
\]

(91)

where

\[
U_{\mu\nu}(k) = g_{\mu\nu} - \frac{1}{2} \Pi^{add}(k) \frac{k_{\mu}n_{\nu} + k_{\nu}n_{\mu}}{k_+}
\]

(92)

and

\[
\Pi^{tr}(k) = \frac{2\alpha}{4\pi} \ln \Lambda^2 \left\{ C_A \int dz \frac{[z^2 - z\zeta + \zeta^2]^2}{z(z - \zeta)\zeta^2} \theta_{11}^0(z, z - \zeta) - \frac{N_f}{3} \right\},
\]

\[
\Pi^{add}(k) = \frac{\alpha}{4\pi} \ln \Lambda^2 C_A \int dz \frac{5z\zeta^2(z - \zeta) + 6z^2(z - \zeta)^2 + 2\zeta^4}{z(z - \zeta)\zeta^2} \theta_{11}^0(z, z - \zeta)
\]

(93)

are the transverse and longitudinal pieces of polarization operator. The renormalized charge is given by the well-known ”asymptotic freedom” formula

\[
g_0 = g \left[ 1 + \frac{\alpha}{4\pi} \ln \Lambda^2 \left( \frac{N_f}{3} - \frac{11}{6} C_A \right) \right].
\]

(94)

Now we are in a position to adduce the RG equations for the real QCD case. Just giving the final result (without intermediate steps) for the chiral-even distributions we then address in greater detail to the chiral-odd evolution.
2.9.1 Evolution of the chiral-even distributions.

Let us begin with the transversely polarized structure function $g_2$. In the first papers on the logarithmic $Q^2$-variation of the moments of $\bar{g}_2$ a simple evolution has been derived:

$$\int_0^1 dx x^n \bar{g}_2(z, Q) = \left( \frac{\alpha(Q)}{\alpha(Q_0)} \right)^{\gamma_\sigma^n/\beta_0} \int_0^1 dx x^n \bar{g}_2(z, Q_0),$$

and the anomalous dimensions have been found by calculating the radiative corrections to the operator

$$\mathcal{O}^3_{\sigma \mu_1 \mu_2 ... \mu_n} = i^n \Gamma^\Delta \mathcal{S} \bar{\psi} \gamma_\sigma \gamma_5 D_{\mu_1} D_{\mu_2} ... D_{\mu_n} \psi$$

over the free quark states. As we have seen above, the conclusion of these works is erroneous since it mixes with other operators of the same twist and quantum numbers due to renormalization. The most significant departure from the naive expectation is that the number of operators involved in the RG evolution turns out to be increasing with the moment $\mathcal{S}$, and as a result it is impossible to write the equation of the DGLAP type which manages corresponding scale dependence for the $\bar{g}_2$. Moreover, the eigenvalues of the anomalous dimension matrix are not known analytically. However, an way out has been found in the two important limits: $N_c \to \infty$ and $x \to 1$, where evolution reduces to the DGLAP equation (95), though the anomalous dimension turns out to be different from that found in (96). The combining use of these asymptotics does provide an excellent approximation to the complete result.

To simplify the discussion, we neglect the quark-mass operator. The nonlocal string operators $\mathcal{S}_\rho$ introduced previously are related to each other by charge conjugation (so are the corresponding kernels which govern their evolution) and do not mix in the course of renormalization; therefore, we give only the result for $\mathcal{S}_\sigma$

\begin{align*}
-\mathcal{S}_\sigma(\mu, \lambda) &= \frac{\alpha}{2\pi} \int_0^1 dy \int dy' \mathcal{A} \left[ C_A \left[ \left[ N(y, z) \right]_+ - \frac{7}{4} \delta(y) \delta(z) \right] - \mathcal{S}_\sigma(\mu y, \lambda - \mu z) 
+ \left[ 2 \bar{z} + \left[ N(y, z) \right]_+ - \frac{7}{4} \delta(y) \delta(z) \right] - \mathcal{S}_\sigma(\mu - \lambda z, \lambda y) \right] 
+ \left( C_F - \frac{C_A}{2} \right) \left[ y \delta(z) \mathcal{S}_\sigma(-\mu y, \lambda - \mu y) - 2z \mathcal{S}_\sigma(\mu - \lambda z, -\lambda y) 
+ \left[ K(y, z) \right]_+ - \mathcal{S}_\sigma(\mu \bar{z} + \lambda z, \lambda \bar{y} + \mu y) \right] \right].
\end{align*}

To condense the notation we have used the dot as short-hand for the logarithmic derivative with respect to the renormalization scale $\dot{} = \mu^2_R \partial/\partial \mu^2_R$ and $-\mathcal{S}_\rho(\mu, \lambda) = -\mathcal{S}_\rho(\mu, 0, \lambda)$. The standard plus-prescription fulfilling is

$$[N(y, z)]_+ = N(y, z) - \delta(y) \delta(z) \int_0^1 dy' \int_0^y dz' N(y', z'), \quad N(y, z) = \delta(y - z) \frac{y^2}{y} + \delta(z) \frac{y}{y},$$

$$[K(y, z)]_+ = K(y, z) - \delta(y) \delta(z) \int_0^1 dy' \int_0^y dz' K(y', z'), \quad K(y, z) = 1 + \delta(y) \frac{\bar{z}}{z} + \delta(z) \frac{\bar{y}}{y}.$$

\footnote{This discussion follows Ref. [33]}

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To proceed further we construct a C-even operator from the Shuryak-Vainshtein ones

\[ Y_\sigma(\lambda, \mu) = +S^\rho(\lambda, \mu) + S^\rho(\mu, \lambda) \]  

and define a new distribution function as Fourier transform with respect to the variable \( \lambda \) only, so that

\[ y(x, u) S^\perp_{\sigma} = \frac{1}{2} \int \frac{d\lambda}{2\pi} e^{i\lambda x} \langle h | Y_\sigma(\bar{u} \lambda, -u \lambda) + (\lambda \to -\lambda) | h \rangle \]  

depends on the effective momentum fraction \( x \) and the gluon position on the light cone \( u \).

Skipping the details (to which we address in the following discussion of the chiral-odd distributions), we just note that the genuine twist-3 part of \( g_2 \) which can be expressed via the integral of the three-parton correlator

\[ \tilde{g}_2(x) = -\bar{g}_2(x) + \int_x^1 \frac{dy}{y} \bar{g}_2(y) \]  

with

\[ x\bar{g}_2(x) = -\frac{d}{dx} \int_0^1 du u y(x, u), \]  

satisfies the DGLAP evolution equation in the large-\( N_c \) limit (i.e. neglecting the terms \( \mathcal{O}(1/N_c^2) \))

\[ [x\bar{g}_2(x)] = \frac{\alpha_s}{4\pi} \int_x^1 \frac{dy}{y} \mathcal{P}_{gg} \left( \frac{x}{y} \right) \left[ y\bar{g}_2(y) \right], \]  

with the splitting function \[32, 33\]

\[ P_{gg}(y) = N_c \left\{ \left[ \frac{2}{y} \right]_+ - 2 - y + \frac{1}{2} \delta(y) \right\}. \]  

In the \( x \to 1 \) region the evolution equation remains of the DGLAP type even for the \( 1/N_c \)-suppressed contribution. The combining use of the Eq. (104) and an additional piece

\[ \Delta P_{gg}(y) = -\frac{1}{N_c} \left\{ \left[ \frac{2}{y} \right]_+ + 3 \delta(y) \right\}. \]  

for \( \mathcal{O}(1/N_c) \)-terms with \( x - 1 \ll 1 \) yields a good approximation for the exact evolution. Obviously, the function \( \bar{g}_2(x) \) obeys the same evolution equation as \( g_2 \). Actually, the splitting function (104) has been exploited in Ref. [13] to rescale the bag model predictions to values of \( Q^2 \) of the real experiment. It has been shown there that \( g_2^{WW}(x) \) and \( \bar{g}_2(x) \) enter into \( g_2(x) \) on equal footing at the model scale \( \mu_{bag}^2 \). Moreover, it has been figured out that this situation does not changed at higher \( Q^2 \) and \( \bar{g}_2(x) \) remains an important ingredient of \( g_2(x) \).

### 2.9.2 Evolution of the chiral-odd distributions

Turning to the case of the chiral-odd distributions, we have to note that in the leading logarithmic approximation the evolution equations that govern the \( Q^2 \)-dependence of the

\[ \mathcal{P}_{gg}(y) = N_c \left\{ \left[ \frac{2}{y} \right]_+ - 2 - y + \frac{1}{2} \delta(y) \right\}. \]  

\[ 1/N_c \]
three-particle correlation functions are the same, discarding the mixing with the quark mass operator. Therefore, we omit the "tilde" sign in what follows.

In the light-cone fraction representation we get for the correlation function $D(x, x')$

$$D(x, x') = -\frac{\alpha}{2\pi} \left\{ -C_F \frac{(x-x')}{x'x} \left[ x'M(x)\Theta_{11}(x', x' - x) \pm xM(x')\Theta_{11}^0(x, x-x') \right] 
+ \int d\beta \left( C_F D(\beta, x') \frac{x'}{x} \Theta_{11}(x, x-x') + \frac{C_A}{2} \left[ (D(\beta, x') - D(x, x')) \frac{x}{(x-x')} \Theta_{11}^0(x, x-x') \right] 
+ [D(\beta + x', x') - D(x, x')] \frac{(x-x')}{(x-x'-\beta)} \Theta_{11}^0(x-x', x-x'-\beta) 
+ \left( \frac{\beta + x - x'}{x'} \right) (D(\beta, x') \frac{x}{(x-x')} \Theta_{11}^0(x, x-x') + D(\beta + x', x') \Theta_{11}^0(x-x', x-x'-\beta)) \right) \right. 
\left. \left( C_F - \frac{C_A}{2} \right) \left( D(\beta, x') \frac{\beta + x - x'}{(x'-x)} \Theta_{11}(x-x', x-x' + \beta) 
+ [D(\beta, x' - x + \beta) - D(x, x')] \frac{x}{x-\beta} \Theta_{11}^0(x, x-x') \right) \right) 
+ \int d\beta' \left( C_F D(x, \beta') \frac{x'}{x} \Theta_{11}(x', x'-x) + \frac{C_A}{2} \left[ (D(x, \beta') - D(x, x')) \frac{x'}{(x'-\beta')} \Theta_{11}(x', x'-\beta') \right] 
+ [D(x, \beta' + x) - D(x, x')] \frac{(x'-x)}{(x'-x-\beta')} \Theta_{11}^0(x', x'-x-x'-\beta') 
+ \left( \frac{\beta' + x - x}{x} \right) (D(x, \beta') \frac{x}{(x-x')} \Theta_{11}^0(x', x'-\beta') + D(x, \beta' + x) \Theta_{11}^0(x'-x, x'-x'-\beta')) \right) \right. 
\left. \left( C_F - \frac{C_A}{2} \right) \left( D(x, \beta') \frac{\beta' + x - x}{(x'-x)} \Theta_{11}^0(x', x'-x-x' + \beta) 
+ [D(x - x' + \beta', \beta') - D(x, x')] \frac{x'}{x'-\beta'} \Theta_{11}^0(x', x'-\beta') \right) \right) \right) - \frac{3}{2} C_F D(x, x') \right\}, \tag{106}$$

and for the mass-dependent correlation function we have

$$\dot{M}(x) = -C_F \frac{\alpha}{2\pi} \int d\beta M(\beta) \left\{ 2 \left[ \frac{\beta}{(x-\beta)} \Theta_{11}^0(x, x-x') \right] \right. 
\left. + \frac{\beta + x}{\beta} \Theta_{11}^0(x, x-x') \right\}, \tag{107}$$

where we have used the standard plus-prescription fulfilling $\int dx[...]_+ = 0$ (for a definition see Eq. (89)). Throughout the paper the plus and minus signs in the mass-operator term correspond to the functions $D$ (for $e$) and $D$ (for $h_L$), respectively.

For the string operators we obtain the following compact RG equation:

$$\dot{Z}(\lambda, \mu) = \frac{\alpha}{2\pi} \int_0^1 dy \int_0^y dz \left\{ C_F \bar{y}^2 \delta(z) \left[ M^1(\lambda - \mu y) \pm M^1(\lambda y - \mu) \right] 
+ \frac{C_A}{2} \left[ 2\bar{z} + [N(y, z)]_+ - \frac{7}{4} \delta(y)\delta(z) \right] Z(\lambda y, \mu - \lambda z) + Z(\lambda - \mu z, \mu y) 
+ \left( C_F - \frac{C_A}{2} \right) \left[ \left[ L(y, z)]_+ - \frac{1}{2} \delta(y)\delta(z) \right] Z(\bar{y}z + \mu z, \mu \bar{y} + \lambda y) 
- 2\bar{z} \left[ Z(-\lambda y, \mu - \lambda \bar{z}) + Z(\lambda - \mu \bar{z}, -\mu y) \right] \right\}, \tag{108}$$
with the function $N(y, z)$ given by Eq. (98) and
\[ L(y, z) = \delta(y) \bar{z} \frac{\partial}{\partial z} + \delta(z) \bar{y} \frac{\partial}{\partial y}. \] (109)

The equations written so far should be supplemented by the following
\[ \dot{M}_1(\lambda) = \frac{\alpha}{2\pi} C_F \int_0^1 dy \left\{ \left[ \frac{2}{\bar{y}} \right]_{+} - 2 - y - y^2 \right\} M_1(\lambda y), \] (110)
\[ \dot{h}_1(\lambda) = \frac{\alpha}{2\pi} C_F \int_0^1 dy \left\{ \left[ \frac{2}{\bar{y}} \right]_{+} - 2 + \frac{3}{2} \delta(\bar{y}) \right\} h_1(\lambda y). \] (111)

The last one, when transformed to the momentum space using the formulae of the next section, coincides with the result obtained in Ref. [16].

2.10 Local anomalous dimensions.

Now we are able to pass from the evolution equations for correlators to the equations for their moments and find, in this way, the anomalous dimension matrix for local twist-3 quark-gluon operators.

We define the moments in following way:
\[ F_n = \int dx x^n F(x) \text{ for any two-particle correlator}, \]
\[ Z_n = \int dx dx' x^{n-l} x'^{l-1} Z(x, x'). \] (112)

In the language of operator product expansion these equalities specify the expansion of nonlocal string operators in towers of the local ones, namely
\[ Z_n^l = i^{n-1}(-1)^{l-1} \frac{\partial^{n-l}}{\partial \lambda^{n-l}} Z(\lambda, \mu)\big|_{\lambda=\mu=0} \]
\[ = \frac{1}{2} \bar{\psi}(0)(iD_+)^{l-1}gG_{\rho(0)}\sigma_{\rho+} \left( \begin{array}{c} I \\ \gamma_5 \end{array} \right) (iD_+)^{n-l}\psi(0), \]
\[ \mathcal{M}_n = i^n \frac{\partial^n}{\partial \lambda^n} \mathcal{M}(\lambda)\big|_{\lambda=0} = \frac{m}{2} \bar{\psi}(0)\gamma_+( \begin{array}{c} I \\ \gamma_5 \end{array} ) (iD_+)^n\psi(0). \] (113)

The inverse transformations to the nonlocal representation are given by
\[ Z(\lambda, \mu) = \sum_{n=0, m=0}^{\infty} (-i)^{n+m}(-1)^m m^n \frac{\lambda^n}{n!} Z_{n+m+1}^l, \quad \mathcal{M}(\lambda) = \sum_{n=0}^{\infty} (-i)^n \frac{\lambda^n}{n!} \mathcal{M}_n. \] (114)

Now it is a simple task to derive the algebraic equations for the mixing of local operators under the change of the renormalization scale from the evolution equations (106)-(110). They are
\[ \dot{M}_n = \frac{\alpha}{2\pi} \mathcal{M}_M\gamma^n M_n, \] (115)
\[ \dot{Z}_n^l = \frac{\alpha}{2\pi} \left\{ [z M_{n-l+1} M_n] + \sum_{k=1}^{n} z \gamma^k Z_{n}^l \right\}, \] (116)
where the anomalous dimensions are given by the compact expressions

\[ M M \gamma^n = -C_F (S_n + S_{n+2}), \]  
\[ Z M \gamma^n = \frac{3}{4} C_F \delta(l-k) + \frac{C_A}{2} \left\{ \theta(l-k-1) \frac{(k+1)(k+2)}{(l-k)(l+1)(l+2)} - \delta(l-k) [S_{k-1} + S_{k+2}] \right\} \]

\[ \begin{align*}
&+ \left( C_F - \frac{C_A}{2} \right) \left\{ \theta(l-k-1) \left[ \frac{2(-1)^k C^k_l}{l(l+1)(l+2)} + \frac{(-1)^{l-k} C^{k-1}_{n-l}}{C^k_n} \right] \right. \\
&\left. + \delta(l-k) \left[ \frac{2(-1)^k}{k(k+1)(k+2)} - S_k \right] \right\} + \left( k \to n - k + 1 \right). 
\end{align*} \]  
\[ (119) \]

Here, we have used the following step functions:

\[ \theta(i-j) = \begin{cases} 1, & i \geq j \\ 0, & i < j \end{cases}, \quad \delta(i-j) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}, \]  
\[ (120) \]

as well as the convention \( S_n = \sum_{k=1}^n \frac{1}{k} \) and the binomial coefficients \( C^m_n = \frac{n!}{m!(n-m)!} \). The results of this section have been independently derived in Ref. [41] using the standard approach based on the local operator product expansion.

\[ 2.11 \text{ Relating the evolution kernels in the light-cone position and light-cone fraction representations.} \]

Until recently the relation between different formulation of the evolution equations in the light-cone fraction [30, 26] and light-cone position [27, 42] representations has been obscure and has been thought to be difficult to realize [43]. However, having at hand the evolution equations in different representations for the same quantities, we are able to fill this gap and relate the kernels in both the cases [48]. Such a bridge can be easily established using the Fourier transformation for the parton distribution functions given by Eqs. (3) and (11).

To start with, we address ourselves to a simpler case of the two-particle correlation functions \( F \). The evolution equation in the light-cone position space is of the following generic form:

\[ \dot{F}(\lambda) = \int_0^1 dy K(y) F(\lambda y), \]  
\[ (121) \]

where \( K(y) \) is the corresponding evolution kernel. By exploiting the definitions (3) we can recast the Fourier transform on the language of two-particle evolution kernels. In this way, we find the direct transformation

\[ K(x, \beta) = \int_0^1 dy K(y) \delta(x - y\beta). \]  
\[ (122) \]

With the help of the general formula

\[ \int_0^1 dy f(y) \delta(x - y\beta) = f \left( \frac{x}{\beta} \right) \Theta_{11}^\theta(x, x - \beta). \]  
\[ (123) \]
we observe that the RG equations for two-parton correlators derived in the previous section are the same indeed. The inverse transformation can be done

$$\int \frac{dxd\beta}{2\pi} e^{-i\lambda x + i\mu \beta} K(x, \beta) = \int_0^1 dy K(y) \delta(\mu - y\lambda)$$

with the following result

$$\int \frac{dxd\beta}{2\pi} f \left( \frac{x}{\beta} \right) \Theta_{11}^0(x, x - \beta)e^{-i\lambda x + i\mu \beta} = \int_0^1 dy f(y) \delta(\mu - y\lambda).$$

The transformation for three-particle correlators is a little bit more involved. The general form of the evolution equation for the light-cone string operator $Z(\lambda, \mu)$ reads

$$\dot{Z}(\lambda, \mu) = \int_0^1 dy \int_0^y dz K(y, z) Z(\eta_{11} \lambda + \eta_{12} \mu, \eta_{21} \lambda + \eta_{22} \mu),$$

where $\eta_{ij}$ are linear functions of the variables $y, z$. In the momentum fraction representation the evolution equation looks like

$$\dot{Z}(x, x') = \int d\beta d\beta' K(x, x', \beta, \beta') Z(\beta, \beta').$$

Specifying the particular form of the functions $\eta_{ij}$, we display below an example for the conversion supplied with a general formula suitable for all practical cases of interest.

For the $C_A/2$ part of the evolution equation, the Fourier transformation gives

$$K(x, x', \beta, \beta') = \delta(\beta' - x') \int_0^1 dy \int_0^y dz K(y, z) \delta(x - x' z - \beta y).$$

The particular contribution is

$$K(y, z) = z \xrightarrow{FT} K(x, x', \beta, \beta') = \delta(\beta' - x') \left\{ \frac{x - \beta}{x'} \Xi_1(x, x - x', x - \beta) + \frac{\beta}{x'} \Xi_2(x, x - x', x - \beta) \right\}.$$ 

Here, we have used (123) and the following general result:

$$\Xi_n(x, x - x', x - \beta) \equiv \int_0^1 dy y^n \Theta_{11}^0((x - \beta) + y\beta, (x - \beta) - y(x' - \beta))$$

$$= \frac{1}{n} \left[ 1 - \left( \frac{\beta - x}{\beta - x'} \right)^n \right] \Theta_{11}^0(x, x - x') + \frac{1}{n x'} \left[ \left( \frac{\beta - x}{\beta - x'} \right)^n - \left( \frac{\beta - x}{\beta} \right)^n \right] \Theta_{11}^0(x, x - \beta).$$

The complete list of transformations can be found in Ref. [38]. Using these results, we can easily verify that the evolution equations given by Eqs. (106) and (108) agree with each other. It should be noted that it is sufficient to have at hand Eqs. (123) and (130) to perform the conversion from one representation to another.
2.12 Generalized DGLAP equations for the three-parton correlators.

As we have seen above, the evolution equations in the momentum space (106) turn out to be very complicated, while Eq. (108) being compact is not suitable for an analysis since only its Fourier transform is related to the physical observables. Therefore, to obtain a simple and manageable equation which can be attempted to be diagonalized, we are forced to proceed further in the same line as suggested in section 2.9.1 for the chiral-even structure function. For this purpose we define a new function, Fourier-transformed with respect to the $\lambda$ variable only:

$$Z(x,u) = \frac{1}{2} \int \frac{d\lambda}{2\pi} e^{i\lambda x} \langle h | Z(\bar{u}\lambda, -u\lambda) \pm (u \rightarrow \bar{u}) | h \rangle,$$

which is even under charge conjugation and depends on the variables $x$ and $u$. The latter has the meaning of the relative position of the gluon field on the light cone. For $0 \leq u \leq 1$ the gluon field lies between the two quark fields. Because of the support property $|x| \leq \max(1, |2u - 1|)$, the variable $x$ is then restricted to $|x| \leq 1$ and can be interpreted as an effective momentum fraction.

The evolution equation for $Z(x,u)$ can be derived in a straightforward way from the RG equation (108) for the nonlocal string operator $Z$. It can be presented in the form of a generalized DGLAP-type equation in the mixed representation:

$$\dot{Z}(x,u) = \frac{\alpha}{2\pi} \int \frac{dy}{y} \int dv \left\{ P_{zz}(y,u,v)Z\left(\frac{x}{y},v\right) + P_{zm}(y,u,v)m\left(\frac{x}{v}\right) \right\},$$

$$\dot{m}(x) = \frac{\alpha}{2\pi} \int \frac{dy}{y} P_{mm}(y)m\left(\frac{x}{y}\right).$$

Here, $m(x) = xM(x)$ and the integration region is determined by both the support of $Z(x,u)$ and the kernels

$$P_{zz}(x,u,v) = \left( C_F - \frac{C_A}{2} \right) \left[ \Theta_1(x,u,v)[L(x,u,v)]_+ - \Theta_2(x,u,v)M(x,u,v) - \frac{1}{4}\delta(u-v)\delta(x) \right]$$

$$+ \frac{C_A}{2} \Theta_3(x,u,v) \left[ M(x,u,v) + [N(x,u,v)]_+ - \frac{7}{4}\delta(u-v)\delta(x) \right] + \left( u \rightarrow \bar{u} \right),$$

$$P_{zm}(x,u,v) = C_F \bar{x}^2 \theta(x)\theta(\bar{x}) \left[ \frac{\delta(v - \bar{u} - xu)}{v} \pm \delta(v - u - x\bar{u}) \right],$$

$$P_{mm}(x) = C_F \left[ \frac{2}{\bar{x}} - 2 - x - x^2 \right],$$

where the auxiliary functions are defined by:

$$\Theta_1(x,u,v) = \theta(x)\theta(u-xv)\theta(\bar{u} - x\bar{v}).$$

\footnote{This particular representation of the evolution equation for three-particles distributions has first been given in the second paper of Ref. [33].}
\[ \Theta_2(x, u, v) = \theta \left( -\frac{xv}{u} \right) \theta \left( \frac{1 - xv}{u} \right) \theta \left( \frac{x - u}{u} \right), \]

\[ \Theta_3(x, u, v) = \theta \left( \frac{x}{u} \right) \theta \left( \frac{xv}{u} \right) \theta \left( \frac{xv - u}{u} \right), \]

\[ L(x, u, v) = \frac{u^2}{v(v - u)} \delta(u - xv), \]

\[ M(x, u, v) = \frac{2x(1 - xv)}{\bar{u}^3} , \]

\[ N(x, u, v) = \frac{\bar{v} \epsilon(\bar{u})}{\bar{u}(v - u)} \left[ \frac{\bar{v}}{u} \delta(\bar{x}) + \frac{u^2}{v} \delta(u - xv) \right]. \]

The plus-prescription for the arbitrary function \( A \) is defined by the equation

\[ \Theta_i(x, u, v)[A(x, u, v)]_+ = \Theta_i(x, u, v)A(x, u, v) - \delta(\bar{x})\delta(u - v) \int_0^1 dx' \int dv' \Theta_i(x', u, v')A(x', u, v'). \] 

Note that, due to the evolution, the variable \( u \) is no longer restricted to the region \( 0 \leq u \leq 1 \).

### 2.13 Eigenvalues and eigenfunctions.

Going further, for simplicity, we restrict ourselves to the homogeneous case, i.e. we discard the quark-mass operator, which is certainly a justified assumption for the light \( u \)- and \( d \)-quark species. The diagonalization of the evolution equation (132) can be achieved by introducing the Mellin transforms

\[ z^n(u) = \int dx x^{n-1} z(x, u), \]

where \( n \) is the complex angular momentum. Operators with different \( n \) do not mix with each other and satisfy the following equation:

\[ \hat{z}^n(u) = \frac{\alpha}{2\pi} \int dv P^n_{zz}(u, v) z^n(v). \]

With the kernel given by

\[ P^n_{zz}(u, v) = \left( C_F - \frac{C_A}{2} \right) \left[ \Theta_1(u, v)[L^n(u, v)]_+ - \Theta_2(u, v)M^n_1(u, v) - \frac{1}{4} \delta(u - v) \right] \]

\[ + \frac{C_A}{2} \Theta_3(u, v) \left[ M^n_2(u, v) + [N^n(u, v)]_+ - \frac{7}{4} \delta(u - v) \right] + \left( u \rightarrow \bar{u} \right), \]

where the auxiliary functions read

\[ \Theta_1(u, v) = \theta(v - u), \quad \Theta_2(u, v) = \theta(-\bar{v})\theta(1 - vu), \quad \Theta_3(u, v) = \theta(\bar{v})\theta(v - u). \]

\[ L^n(u, v) = \frac{\epsilon(v)}{v - u} \left( \frac{u}{v} \right)^{n+1}, \]

\[ M^n_1(u, v) = \frac{2}{\bar{u}^3} \left\{ \frac{1}{n + 1} \left[ \frac{1}{v^{n+1}} - u^{n+1} \right] - \frac{v}{n + 2} \left[ \frac{1}{v^{n+2}} - u^{n+2} \right] \right\}, \]

\[ M^n_2(u, v) = \frac{2}{\bar{u}^3} \left\{ \frac{1}{n + 1} \left[ \frac{1}{v^{n+1}} - u^{n+1} \right] - \frac{v}{n + 2} \left[ \frac{1}{v^{n+2}} - u^{n+2} \right] \right\}. \]
\[ M_2^n(u, v) = \frac{2}{\bar{u}^3} \left\{ \frac{1}{n+1} \left[ 1 - \left( \frac{u}{v} \right)^{n+1} \right] - \frac{u}{n+2} \left[ 1 - \left( \frac{u}{v} \right)^{n+2} \right] \right\}, \]
\[ N^n(u, v) = \frac{\bar{v} \epsilon(\bar{u})}{\bar{u}(v - u)} \left\{ \frac{\bar{v}}{\bar{u}} + \epsilon(v) \left( \frac{u}{v} \right)^{n+1} \right\}. \] (143)

The plus-prescription is defined as
\[ \Theta_i(u, v)[A^n(u, v)]_+ = \Theta_i(u, v)A^n(u, v) - \delta(u - v) \int dv'\Theta_i(u, v')A^n(u, v'). \] (144)

Here, we can check our calculation once more since in multicolour limit, exact Eq. (140) is reduced to the approximated equation of Ref. [4].

To obtain the solution of the evolution equation (140), we choose \( n \) as a positive integer. In this case, as follows from the definition (131) of \( z^n(u) \) the \( n \)-th moment is actually given by the following linear combination of local operators \( Z_n^l \) (see Eq. (112)):

\[ z^n(u) = \sum_{l=1}^{n} C^{l-1}_{n-l} u^{n-l} \bar{u}^{l-1} Z_n^l, \] (145)

so that \( z^n(u) \) is a polynomial of degree \( n - 1 \) in \( u \). Thus, the kernel \( P^n_{zz}(u, v) \) possesses \( n \) polynomial eigenfunctions \( e_i^n(v) \)
\[ \int dv P^n_{zz}(u, v)e_i^n(v) = -\lambda_i^n e_i^n(u), \quad l = 1, \ldots, n, \] (146)
where \(-\lambda_i^n\) denotes the eigenvalues. The solution we are interested in is given in terms of the eigenvalues and eigenfunctions of the anomalous-dimension matrix \( zz\gamma_{ik}^n \) of the local operators \( Z_n^l \). The eigenvalue problem we have attacked has no analytical solution; however, the diagonalization can be done numerically for a moderately large orbital momentum \( n \), e.g. \( n \leq 100 \), which is quite sufficient for practical purposes. These eigenfunctions can be constructed by diagonalization
\[ C^{l-1}_{n-l} \int dv P^n_{zz}(u, v)v^{n-k} \bar{v}^{k-1} = \sum_{l=1}^{n} C^{l-1}_{n-l} z z \gamma_{ik}^n u^{n-l} \bar{u}^{l-1}, \] (147)
where the anomalous-dimension matrix \( z z \gamma_{ik}^n \) of the local operators is given by Eq. (119).

Actually, this is a purely algebraic task and we find
\[ e_i^n(u) = \sum_{l=1}^{n} C^{l-1}_{n-l} u^{n-l} \bar{u}^{l-1} E_{ik}^n, \quad \text{with} \quad \{ (E^n)_{zz}^{-1} z z \gamma^n E^n \}_{kl} = -\lambda_k^n \delta(k - l), \] (148)
where \( \delta(k - l) \) is a Kronecker symbol defined by Eq. (120). The spectrum of the eigenvalues \( \lambda_i^n \) up to \( n = 50 \) is shown in Fig. 4. The solution for the moments \( z^n(u) \) (in the massless case) is then expressed in terms of the eigenfunctions and eigenvalues we have found [38]
\[ z^n(u, Q^2) = \sum_{l=1}^{n} c_i^n (Q_0^2) e_i^n(u) \exp \left\{ - \int_{Q_0^2}^{Q^2} \frac{dt}{t} \frac{\alpha(t)}{2\pi} \lambda_i^n \right\}. \] (149)

The coefficients \( c_i^n (Q_0^2) \) at the reference momentum squared \( Q_0^2 \) have to be determined from the non-perturbative input.
Figure 4: The spectrum of the eigenvalues $\lambda^*_n$ for the evolution kernel $P^n_{zz}$ defined in (141).

Obviously, in order to find the evolution of the higher $n$-moments of the structure functions the whole information about the relative size of the reduced matrix elements of the $(l = 1, \ldots, n)$ local quark-gluon operators is needed. At the moments, in the lack of complete understanding of the yet unclear confinement mechanism this problem is not accessible by nonperturbative methods presently available. However, we should note that in the nearest future it will not be possible to distinguish experimentally between the terms with different anomalous dimensions even for the transverse structure function $g_2(x)$ to say nothing of $h_L(x)$ (not to mention $e(x)$).

Eq. (149) can be rewritten directly for the structure functions entering into the physical cross sections. To this end, we have to dispose the relations similar to the ones given by Eqs. (39) and (56) transformed to the mixed representation (131). Namely, we have

$$e(x) = -\frac{1}{2} \frac{d}{dx} \int_0^1 du \, z(x, u), \quad (150)$$

$$\bar{h}_L(x) = -\frac{1}{2} \frac{d}{dx} \int_0^1 du \, (1 - 2u) \bar{z}(x, u), \quad (151)$$

where we introduce for convenience a new function $\bar{h}_L(x)$, so that $h_L(x)$ reads:

$$h_L(x) = 2 \int_x^1 \frac{dy}{y} x^2 h_1(y) - x \frac{d}{dx} \int_x^1 \frac{dy}{y} x^2 \bar{h}_L(y). \quad (152)$$

The last term on the RHS coincides with the twist-3 part $\bar{h}_L$.

Thus we arrive to an explicitly diagonalized form for the structure functions $f(x)$ (schematically):

$$[f(Q^2)]_n = \int_0^1 du \mathcal{W}^f(u) z^n(u, Q^2) = \sum_{l=1}^n \mathcal{J}^l_n \sum_{k=1}^n (E_{ik})^{-1} \langle h | Z^k_n(Q_0) | h \rangle \left( \frac{\alpha(Q)}{\alpha(Q_0)} \right)^{-\lambda^*_n/\beta_0} \quad (153)$$

where $\mathcal{W}^f(u)$ is a weight function in Eq. (150) and (151) and $\mathcal{J}^l_n$ is the overlap integral

$$\mathcal{J}^l_n = \int_0^1 du \mathcal{W}^f(u) e^*_l(u). \quad (154)$$
2.14 Solution of the evolution equations in multicolour QCD.

The complicated form of the evolution \([153]\) of the twist-3 distributions compels one to look for the simple approximated solution which could work with reasonable accuracy. The resolution of this problem is based on the observation that in the large-\(N_c\) limit only the planar diagrams (Fig. 2 (a,d)) survive and the kernel \(P_{xz}(u, v)\) has two known dual eigenfunctions: 1 and \(1 - 2u\) \([14, 38]\), so that \(\int_0^1 du \ e^l(u) = \delta_{11} + \mathcal{O}(1/N_c)\) and \(\int_0^1 du \ (1 - 2u)e^l(u) = \delta_{12} + \mathcal{O}(1/N_c)\), where \(l = 1, 2\) correspond to the lowest two eigenvalues of the spectrum shown in Fig. 3 thus in the sum in Eq. \((153)\) only the lowest two terms survive. Then, a straightforward calculation gives the following DGLAP evolution kernels:

\[
\int_0^1 du \left\{ \frac{1}{1 - 2u} \right\} P_{xz}(x, u, v) = N_c \theta(\bar{x})\theta(x) \left\{ \frac{x^2}{2} + \frac{5}{4} \delta(\bar{x}) \right\} + \mathcal{O}\left(\frac{1}{N_c}\right). \tag{155}
\]

As we have observed previously in section 2.9.1 in the context of the chiral-even distribution \(g_2(x)\) \([32]\), similar equations hold true also for the \(\frac{1}{N_c}\)-suppressed terms in the \(x \to 1\) limit for flavour non-singlet twist-3 evolution kernels. In the present chiral-odd case, we find

\[
\int_0^1 du \left\{ \frac{1}{1 - 2u} \right\} P_{xz}(x, u, v) = -\frac{1}{N_c} \theta(\bar{x})\theta(x) \left\{ \frac{1}{\bar{x}} + \frac{5}{4} \delta(\bar{x}) + \mathcal{O}(\bar{x}^0) \right\} + N_c \cdots, \tag{156}
\]

where the \(N_c \cdots\) symbolize the \(x \to 1\) limit of Eq. \((155)\).

The eigenfunctions we have obtained coincide precisely with the coefficients \(W^f(u)\) that appear in the decomposition of \(e(x, Q^2)\) and \(\bar{h}_L(x, Q^2)\) in terms of three-particle correlation functions.

From the observations we have made above, it follows that in the large-\(N_c\) as well as in the large-\(x\) limit the twist-3 distributions satisfy the DGLAP (ladder-type) evolution equations that hold for the twist-2 operators. By combining the large-\(N_c\) evolution with the large-\(x\) result for the \(\frac{1}{N_c}\)-suppressed terms, we can improve the accuracy of multicolour approximation within a factor 5-10 and reach the precision of a few per cent as compared with the evolution predicted using an exact equation \((140)\) but supplied with a model for the light-cone position distribution of gluons between the quark fields \([38]\). Thus, the functions \(e(x, Q^2)\) and \(\bar{h}_L(x, Q^2)\) obey the following improved evolution equations:

\[
\hat{f}(x) = \frac{\alpha}{2\pi} \int_x^1 \frac{dy}{y} P_f(y) f\left(\frac{x}{y}\right), \tag{157}
\]

with

\[
P_{ee}(y) = 2C_F \left[ \frac{y}{\bar{y}} \right]_+ + \frac{C_A}{2} y + \left( \frac{C_F}{2} - C_A \right) \delta(\bar{y}) + \mathcal{O}(\bar{y}^0/N_c),
\]

\[
P_{hh}(y) = 2C_F \left[ \frac{y}{\bar{y}} \right]_+ - \frac{3C_A}{2} y + \left( \frac{7C_F}{6} - \frac{4C_A}{3} \right) \delta(\bar{y}) + \mathcal{O}(\bar{y}^0/N_c). \tag{158}
\]

Note that \(\bar{h}_L(x)\) fulfills the same evolution equation as \(\hat{h}_L(x)\). The simplest way to verify this is to make the Mellin transform of the corresponding evolution equations.
2.15 Remarks on the momentum space formalism.

Let us add a few remarks on the momentum space formulation. As we have seen above the solution of the evolution equations in the asymptotic regimes is the most straightforward in the light-cone position representation. It is by no means trivial to observe the appearance of the DGLAP equations in momentum fraction representation. However, we know that the asymptotic solution, in coordinate space, is given by the convolution of the three-particle correlation function with the same weight function that enters in the decomposition of the two-parton correlators at tree level. With this in mind, we are able to check that the integrals

\[ e(x) = \int d\beta' D(x, \beta'), \quad (159) \]
\[ \bar{h}_L(x) = x^2 \int_x^1 \frac{d\beta}{\beta^2} \int \frac{d\beta'}{\beta' - \beta} \left\{ 2 + (\beta - \beta') \left[ \frac{\partial}{\partial \beta} - \frac{\partial}{\partial \beta'} \right] \right\} \bar{D}(\beta, \beta'), \quad (160) \]

taken from Eqs. (39) and (56) neglecting quark-mass as well as twist-2 effects, satisfy the DGLAP equations, namely

\[ \dot{e}(x) = -\frac{\alpha}{4\pi} N_c \int d\beta e(\beta) \left\{ 2 \left[ \frac{\beta}{(x - \beta)} \Theta_{11}^0(x, x - \beta) \right] + \left( 2 - \frac{x}{\beta} \right) \Theta_{11}^0(x, x - \beta) - \frac{1}{2} \delta(\beta - x) \right\}, \quad (161) \]
\[ \dot{\bar{h}}_L(x) = -\frac{\alpha}{4\pi} N_c \int d\beta \bar{h}_L(\beta) \left\{ 2 \left[ \frac{\beta}{(x - \beta)} \Theta_{11}^0(x, x - \beta) \right] + \left( 2 + 3 \frac{x}{\beta} \right) \Theta_{11}^0(x, x - \beta) - \frac{1}{2} \delta(\beta - x) \right\}. \quad (162) \]

The corresponding anomalous dimensions are

\[ \left[ \dot{e} \right]_n = \frac{\alpha}{4\pi} N_c \left\{ -2\psi(n + 2) - 2\gamma_E + \frac{1}{2} + \frac{1}{n + 2} \right\} \left[ e \right]_n, \quad (163) \]
\[ \left[ \dot{\bar{h}}_L \right]_n = \frac{\alpha}{4\pi} N_c \left\{ -2\psi(n + 2) - 2\gamma_E + \frac{1}{2} - \frac{3}{n + 2} \right\} \left[ \bar{h}_L \right]_n. \quad (164) \]

Which are exactly the anomalous dimensions \( \gamma_{\alpha}^\pm \) found in Ref. [44] for \( e \) and \( \bar{h}_L \), respectively, with the replacement \( n \to j - 1 \).

Concluding this section, we have to note that the simple DGLAP equation (162) has been used, recently, for prediction of the size of the structure function \( h_L(x) \) [14] at high \( Q^2 \) starting from its value in the low energy point found in the framework of the MIT bag model. The main conclusion of this work is that the twist-3 contribution to \( h_L(x) \) is significantly reduced in the course of the evolution in contrast to the corresponding situation in the case of \( g_2(x) \) structure function discussed above. This observation is the direct consequence of the larger anomalous dimensions for \( \bar{h}_L(x) \) at low values of \( n \)-spins as compared with \( g_2(x) \). This means that it will be extremely difficult to extract \( \bar{h}_L(x) \) at large momentum transferred.

\footnote{The difference in the anomalous dimensions is due to an extra power of the momentum fraction \( x \) included in the definition of the twist-3 correlation functions.}
However, if the latter will be sizable at high $Q^2$ in future experiments it will indicate that the naive bag model predictions could not be trusted for the calculations of the quark-gluon correlations presented in $h_L(x)$ (as well as in the other twist-3 distributions).

3 Time-like cut vertices.

3.1 Time-like processes and the Gribov-Lipatov reciprocity.

The deep inelastic scattering of lepton beam on the hadron target has proved to be the most effective experimental tool for studying the dynamics of hadron reactions on the parton level which has a firm basis in the quantum field theory provided by the light-cone OPE. As we have seen in the preceding sections it makes possible the investigation of the logarithmic violation of the Bjorken scaling as well as the power suppressed contributions responsible for polarized phenomena. However, we mainly use an equivalent approach for the analysis of the corresponding quantities which is based on the factorization theorems and the evolution equations since the latter can be applicable in the situations when the OPE is no longer valid. These are the inclusive production of the hadron in the $e^+e^-$-annihilation, semi-inclusive deep inelastic scattering, Drell-Yan lepton pair production et al.

There is continuous interest in the inclusive production of hadrons in hard reactions. These processes involve a quark fragmentation function to describe the hadron creation from the underlying hard parton scattering. However, they differ considerably from the DIS as the short distance expansion could not be employed although the given processes go near the light cone. The theoretical basis for strict analyses of the above phenomena is realized by the generalization of the OPE to the time-like region in terms of Mueller’s $\zeta$-space cut vertices [18]. As we have mentioned in the introduction an essential departure from the DIS is that the moments of the fragmentation functions are essentially nonlocal in the coordinate space. However, this approach has all attractive features of the OPE as it provides a consistent framework to account for the higher twist effects [28] as well as it allows to sum up the UV logarithms [45] by using the powerful methods of the renormalization group.

The semi-inclusive hadron production from a quark fragmentation is described in QCD by the specific nonperturbative correlation functions of quark and gluon field operators over the hadron states which can be identified with $\zeta$-space cut vertices. While the behaviour of the latters with respect to the fraction of the parton momentum carried by the hadron is determined by the nonperturbative strong interaction dynamics, the large $Q^2$-scale dependence is governed by perturbation theory only. Since the cross section we are interested in cannot be related to the imaginary part of some $T$-product of currents, therefore, we must deal with particular discontinuities of the Feynman diagrams from the very beginning.

Inasmuch as the twist-3 fragmentation functions enter into several cross sections on the same footing as the distributions, their scale dependence is of great interest. Apart from significance for phenomenology, it is important for theoretical reasons: while it is know that in the leading order of the coupling constant the splitting functions for the twist-2 fragmentation functions can be found from the corresponding space-like quantities via the
Gribov-Lipatov reciprocity relation $^{[17]}$

\[ P_{SL}(x) = P_{TL}\left(\frac{1}{x}\right), \]  

(165)

(note that both quantities are defined in the physical regions of the corresponding channels) no such equality is known for higher twists.

We begin our discussion with a study of twist-3 nonpolarized chiral-odd (NCO) fracture functions. This is the simplest case with respect to the number of correlation functions involved in mixing under renormalization group evolution. From the phenomenological point of view they appear, for example, in the cross section for semi-inclusive hadron ($H$) production in the process of measuring the nucleon’s ($h$) transversity distribution $h_1(x)$ from deep inelastic scattering $^{[17]}$:

\[ \frac{d^4\sigma}{dx dy d(1/\zeta) d\phi} = 4\alpha_{em}^2 \left( \cos \chi \left(1 - \frac{y}{2}\right) G_1(x, \zeta) \right. \]

\[ \left. + \cos \phi \sin \chi \sqrt{(\kappa - 1)(1 - y)} \left( G_T(x, \zeta) - G_1(x, \zeta) \left(1 - \frac{y}{2}\right) \right) \right]. \]  

(166)

Here $\kappa = 1 + 4x^2M^2/Q^2$, $y = 1 - E'/E$ and the cross section is expressed in a frame where the lepton beam with energy $E$ defines the $z$-axis and the $x-z$-plane contains the nucleon polarization vector, which has the polar angle $\chi$ and the scattered electron $E'$ has the polar angles $\theta$, $\phi$. The functions $G_1$ and $G_T$ are expressed in terms of the product of the familiar distribution and fragmentation functions in the following way

\[ G_1(x, \zeta) = \frac{1}{2} \sum_i Q_i^2 g_i^1(x) \mathcal{F}^i(\zeta), \]  

(167)

\[ G_T(x, \zeta) = \frac{1}{2} \sum_i Q_i^2 \left[ g_i^T(x) \mathcal{F}^i(\zeta) + \frac{1}{x} h_i^1(x) \mathcal{I}^i(\zeta) \right]. \]  

(168)

All of them have the expressions in QCD in terms of the light-cone Fourier transformation of correlation functions of fundamental quark and gluon fields over specific hadron states $^{[10,17]}$. Some of the definitions were given already by Eqs. (19), while the others are

\[ S_{\mu}^+ h_1(x) = \frac{1}{2} \int \frac{d\lambda}{2\pi} e^{i\lambda x} \langle h|\bar{\psi}(0) i\sigma_{\mu+}^\perp \gamma_5 \psi(\lambda n)|h\rangle, \]

\[ \mathcal{F}(\zeta) = \frac{1}{4\zeta} \int \frac{d\lambda}{2\pi} e^{i\zeta \lambda} \langle 0|\gamma_+ \psi(\lambda n)|H, X\rangle \langle H, X|\bar{\psi}(0)|0\rangle. \]  

(169)

Note that we have used slightly different definition of the function $h_1(x)$ which has another dimension in mass units as compared to previous one. This is done in order to make the product $h_1(x)\mathcal{I}(\zeta)$ dimensionless. Of course, they can be made dimensionless separately by introducing certain characteristic scale $m_{\text{char}}^2$ (which can be set equal to the mass $M$ of the appropriate hadron) into the definition of the corresponding correlators. $\mathcal{I}$ is given by equation (170). Note, that the physical regions are different for distribution and fragmentation functions: $0 \leq x \leq 1$ and $1 \leq \zeta < \infty$, respectively.

In the following sections we address ourselves to the problem of building of the master equation for the function $\mathcal{I}$. This problem is of the same complexity as for corresponding quantities in the space-like domain since we face the mixing with other cut vertices of the same twist and quantum numbers in the course of the renormalization group evolution.
3.2 Recombination functions and their support properties.

For our purposes it is much more suitable to deal with correlation functions listed below which are the generalization of the formulae given in section 2.3 to the fragmentation region.

\[ \mathcal{I}(\zeta) = \frac{1}{4} \int \frac{d\lambda}{2\pi} e^{i\lambda\zeta} \langle 0 | \bar{\psi}(\lambda n) | H, X \rangle \langle H, X | \bar{\psi}(0) | 0 \rangle, \]

\[ \mathcal{M}(\zeta) = \frac{1}{4\zeta} \int \frac{d\lambda}{2\pi} e^{i\lambda\zeta} \langle 0 | m \gamma^+ \psi(\lambda n) | H, X \rangle \langle H, X | \bar{\psi}(0) | 0 \rangle, \]

\[ Z_1^{(1)}(\zeta', \zeta) = \frac{1}{4\zeta} \int \frac{d\lambda}{2\pi} \frac{d\mu}{2\pi} e^{i\lambda\zeta - i\mu\zeta'} \langle 0 | g_\gamma^+ \gamma_\rho^+ B_\rho^+ (\mu n) \bar{\psi}(0) | H, X \rangle \langle H, X | \bar{\psi}(0) B_\rho^+ (\mu n) | 0 \rangle, \]

\[ Z_1^{(2)}(\zeta, \zeta') = \frac{1}{4\zeta} \int \frac{d\lambda}{2\pi} \frac{d\mu}{2\pi} e^{i\mu\zeta' - i\lambda \zeta} \langle 0 | \bar{\psi}(0) B_\rho^+ (\lambda n) \bar{\psi}(\lambda n) | H, X \rangle \langle H, X | \bar{\psi}(0) | 0 \rangle, \]

\[ Z_2^{(1)}(\zeta, \zeta') = \frac{1}{4\zeta} \int \frac{d\lambda}{2\pi} \frac{d\mu}{2\pi} e^{i\mu\zeta' - i\lambda \zeta} \langle 0 | \bar{\psi}(0) g_\gamma^+ \gamma_\rho^+ B_\rho^+ (\mu n) \bar{\psi}(\lambda n) | H, X \rangle \langle H, X | \bar{\psi}(0) | 0 \rangle, \]

\[ Z_2^{(2)}(\zeta', \zeta) = \frac{1}{4\zeta} \int \frac{d\lambda}{2\pi} \frac{d\mu}{2\pi} e^{i\lambda\zeta - i\mu\zeta'} \langle 0 | B_\rho^+ (\lambda n) \bar{\psi}(\lambda n) | H, X \rangle \langle H, X | \bar{\psi}(0) g_\gamma^+ \gamma_\rho^+ \bar{\psi}(\mu n) | 0 \rangle. \]

The summation over \( X \) is implicit and covers all possible hadronic final states populated by the quark fragmentation. Again we use the light-cone gauge \( B_+ = 0 \), otherwise a link factor should be inserted in between the quark fields to maintain the gauge invariance. The quantities determined by these equations form the closed set under renormalization, however, they are not independent since there is relation between them due to equation of motion for the Heisenberg fermion field operator

\[ \mathcal{I}(\zeta) - \mathcal{M}(\zeta) - \int d\zeta' Z_1(\zeta', \zeta) = 0. \]

Here and in the following discussion we introduce the convention

\[ Z_j(\zeta', \zeta) = \frac{1}{2} [ Z_j^{(1)}(\zeta', \zeta) + Z_j^{(2)}(\zeta, \zeta') ] . \]

While the former two functions \( \mathcal{I} \) and \( \mathcal{M} \) can be made explicitly gauge invariant by inserting the \( P \)-ordered exponential (which is unity in the gauge we have chosen) between the quark
fields, the latter can be written in the gauge invariant manner by introducing the following objects:

\[ R_1(\zeta', \zeta) = \zeta' Z_1(\zeta', \zeta), \quad R_2(\zeta, \zeta') = \zeta Z_2(\zeta, \zeta'). \]  

(178)

Taking into account Eq. (4) it is easy to verify, in the same way as before, that they are indeed expressed in terms of correlators involving the gluon field strength tensor. The functions \( Z^{(1)} \) and \( Z^{(2)} \) are related by complex conjugation

\[ \left[ Z^{(1)}_1(\zeta', \zeta) \right]^* = Z^{(2)}_1(\zeta, \zeta'), \quad \left[ Z^{(2)}_2(\zeta, \zeta') \right]^* = Z^{(1)}_2(\zeta', \zeta). \]  

(179)

Their support properties can be found by applying Jaffe’s recipe [24]. It has been shown that the field operators entering the definition of the correlation functions can be placed in the arbitrary order on the light cone with appropriate sign change according to their statistics. Then taking the particular ordering and saturating the correlation function by the complete set of the physical states we immediately obtain (for definiteness, we consider the function \( Z^{(1)}_1 \))

\[ Z^{(1)}_1(\zeta', \zeta) = \frac{1}{4\zeta} \sum_{X,Y} \delta(\zeta - 1 - \zeta_X) \delta(\zeta' - \zeta_Y) \langle 0 | \psi | H, X \rangle \langle H, X | \bar{\psi} | Y \rangle \langle B^\perp \rangle \langle 0 | \psi | H, X \rangle \langle H, X | B^\perp | Y \rangle \langle Y | \bar{\psi} | 0 \rangle, \]  

(180)

with \( \zeta_X, \zeta_Y \geq 0 \) and we omit the unessential Dirac matrix structure of the vertex. From these equations the restrictions emerge on the allowed values of the momentum fractions: \( 1 \leq \zeta < \infty, 0 \leq \zeta' \leq \zeta \). By analogy one can easily derive similar support properties for other functions.

### 3.3 Feynman rules for the discontinuities of the diagrams. Keldysh diagram technique.

It is well known that the time-like cross section could not be related to the imaginary part of any \( T \)-product of the currents. Contrary, it is given by the particular absorptive parts of the ladder-type diagrams. In the physical region of the annihilation channel these graphs possess additional discontinuities, which are not relevant for our purposes, since we are restricted to the cuts that separate the possible hadrons in the final state. To be able to extract the imaginary part we are interested in we have to label in some way the field operators in the amplitudes to the right and to the left of the cut. This can be suitably done with the help of the Keldysh diagram technique [46]. It allows to recast the program for the calculation of the particular discontinuities of the Feynman diagrams to the operator-like language.

Consider, for instance, certain \( S \)-matrix element which is given by the following functional integral

\[ M = \int D\phi \ (\phi_1 \phi_2 \ldots \phi_n) \exp \left\{ i \int dz L(\phi) \right\}, \]  

(181)

\[ ^8 \text{In this discussion we will follow Ref. [24].} \]
where \( \phi = \psi, \bar{\psi}, B \). The cross section \( \sigma = MM^\dagger \) of the process is

\[
MM^\dagger = \int D \phi D \bar{\phi} \left( ^{\dagger} \phi_1 ^{\dagger} \phi_2 \ldots ^{\dagger} \phi_n \right) \left( ^{-} \phi_1 ^{-} \phi_2 \ldots ^{-} \phi_n \right) \\
\exp \left\{ i \int dz \, ^{\dagger} \bar{\phi}(\phi) - i \int dz \, L(\bar{\phi}) \right\}.
\]  

(182)

Here the "plus" and "minus" superscripts label the fields from the direct and conjugated amplitudes. Following the original works \[46\] one can accept that they are the components of a unique operator \( \Phi(z,t) \) composed from the time- and anti-time-ordered fields, i.e. \( \phi^+ \) and \( \phi^- \), respectively. Now, the Green function for the "big" field is a \( 2 \times 2 \)-matrix constructed from the usual Feynman propagator, its conjugated analogue and its discontinuity via the Cutkosky rules for the lines connecting the direct and final amplitudes. Thus, the radiative corrections to the bare cut vertex can be calculated then by using the conventional Feynman rules with the following modifications:

- All propagators and vertices on the RHS of the cut are Hermitian conjugated to that on the LHS.
- Every time crossing the cut the propagator \( 1/(k^2 - m^2 + i0) \) has to be replaced by \( -2\pi i\delta(k^2 - m^2) \).
- For each propagator crossing the cut there is a \( \theta \)-function specifying that the energy flow from the LHS to the RHS is positive. (In the infinite momentum frame this is the plus component of the four-momentum.)

These statements complete the rules to handle the cut vertices. They can be summarized in the Fig. \[3\] where \( V_{\mu\nu\rho} = (k_1 - k_2)_\mu g_{\nu\rho} + (k_2 - k_3)_\rho g_{\mu\nu} + (k_3 - k_1)_\nu g_{\mu\rho} \).

### 3.4 Abelian evolution.

In this section accepting the diagram technique derived in the preceding section we show our machinery on a simple example of abelian evolution and generalize it afterwards to the Yang-Mills theory. As in \[2.8\] we start with overcomplete set of the cut vertices defined by Eqs. \[170\]-(\[175\]) and disregard for a moment the relation between them. Then Eq. \[176\] verifies that the evolution equations thus obtained are indeed correct. We have to note that since the observed particle \( H \) is always in the final state some cuts of Feynman diagrams are not allowed and, therefore, we could not obtain the evolution kernel for the cut vertex taking naively the discontinuity of uncut graph as we are restricted over the limited set of the cuts.

**3.4.1 Sample calculation of the evolution kernels.**

As we have noted previously the UV divergences occur in the transverse-momentum integrals of partons interacting with a bare cut vertex. To extract this dependence properly it is sufficient to separate the perturbative loop from correlation function in question. To this
end, the latter can be represented in the form of momentum integral in which the integrations over the fractional energies of the particles attached to the vertex are removed

\[ \left( \frac{I}{\mathcal{M}(\zeta)} \right) = \int \frac{d^4k}{(2\pi)^4} \delta(\zeta - z) \left( \frac{1}{\zeta m \gamma^+} \right) F(k), \] (183)

where

\[ F(k) = \int d^4xe^{i\mathbf{k}\cdot\mathbf{x}} \langle 0|\psi(x)|H, X\rangle \langle H, X|\bar{\psi}(0)|0 \rangle. \] (184)

In the same way we can easily write down the corresponding expressions for the three-particle correlation functions.

Let us consider, for definiteness, the fracture function \( \mathcal{I} \). Simple calculation of the one-loop diagram depicted in Fig. 7 (a) for the \( 2 \rightarrow 2 \) transition gives in the LLA

\[ \mathcal{I}(\zeta) = g^2 \int \frac{d^4k}{(2\pi)^4} F(k) \int \frac{d^4k''}{(2\pi)^4} \delta(\zeta - z'') \theta(z'' - z) \frac{\delta((k'')^2 - k^2)}{k''^4} \]
\[ \times \left\{ -d_{\mu\nu}(k'' - k) \gamma_\mu(k'' + m) I(k'' + m) \gamma_\nu \right\} \]
\[ = -\frac{\alpha}{2\pi^2} \int \frac{d^4k}{(2\pi)^4} F(k) \int dz'' \frac{\delta(z'' - \zeta) \theta(z'' - z)}{(z'' - z)} \int d^2k'' \int d\alpha'' \delta \left( \alpha'' + \frac{k''^2}{2(z'' - z)} \right) \]

Figure 6: Modified rules for the discontinuities of the Feynman diagrams.
Figure 7: One-loop diagrams contributing to the transition kernels of the two-particle correlation functions to the two- and three-parton ones.

\[
\times \frac{1}{[2\alpha''z''+k_{-1}^{\prime\prime}]^2} \left\{ [2\alpha''z''+k_{-1}^{\prime\prime}] - 2m_{\gamma} \left[ \alpha'' + \frac{[2\alpha''z''+k_{-1}^{\prime\prime}]}{(z''-z)} \right] \right\}
= -\frac{\alpha}{2\pi} \ln \Lambda^2 \int \frac{dz}{z} \theta(\zeta-z) \left[ \mathcal{I}(z) - \mathcal{M}(z) \left( 1 + 2 \frac{z}{(\zeta-z)} \right) \right].
\]

(185)

As long as the logarithmic contribution appears when \( |k_{\perp}/|k''_{\perp}| \ll 1 \) and \( \alpha/\alpha'' \ll 1 \) we expand the integrand in powers of these ratios keeping the terms that do produce the logarithmic divergence. Similarly, one can evaluate the transition amplitudes of \( \mathcal{I} \) to the three-particle correlation functions \( \mathcal{Z}_j \) (for the diagrammatic representation, see Figs. 7(b,c)):

\[
\mathcal{I}(\zeta)_{\Lambda^2} = \frac{\alpha}{2\pi} \ln \Lambda^2 \int dz dz' \theta(\zeta-z) \mathcal{Z}_1(z',z) \left[ \frac{2}{(\zeta-z)} + \frac{1}{z-z'} \right],
\]

(186)

\[
\mathcal{I}(\zeta)_{\Lambda^2} = \frac{\alpha}{2\pi} \ln \Lambda^2 \int dz dz' \theta(\zeta-z) \mathcal{Z}_2(z,z') \frac{(\zeta-z)}{(\zeta-z')(z-z')},
\]

(187)

Figure 8: Contact-type contribution to the evolution equation of the fragmentation function \( \mathcal{I}(z) \).

Due to the non-quasi-partonic [28] form of the vertex \( \mathcal{I} \) there exists an additional contribution to the evolution equation coming from the contact term (Fig. 8) that results from
the cancellation of the propagator adjacent to the quark-gluon and bare cut vertices. As a consequence the vertex acquires the three-particle piece

\[ \mathcal{I}(\zeta) = \int \frac{d^4k}{(2\pi)^4} \frac{d^4k'}{(2\pi)^4} Z_{1\rho}(k', k) iG_\rho(k' - k, k) iG(k) \delta(\zeta - z) + (c.c.) \quad (188) \]

As can be seen Eqs. (185) and (186) possess the IR divergences at \( z = \zeta \). They disappear after we account for the virtual radiative corrections (renormalization of the field operators) discussed in section 2.6. The net result looks like

\[ \Gamma^R = (1 - \Sigma_1) U_1 \Gamma U_2^{-1}, \quad \Gamma = \left( I, \frac{1}{\zeta} m \gamma_+, \, g\gamma_+ \gamma_\rho^\perp \right) \quad (189) \]

Assembling all these contributions we come to the evolution equation for \( \mathcal{I} \) given below by Eq. (191).

### 3.4.2 Evolution equations.

![Figure 9: One-loop (abelian) corrections to the three-point correlation function \( Z_1 \).](image)

Now following the procedure just described it is not difficult to construct the closed set of the evolution equations by calculating the one-loop diagrams shown on Figs. [9](#) [10](#)

\[ \dot{\mathcal{M}}(\zeta) = \frac{\alpha}{2\pi} \int \frac{dz}{z} \theta(\zeta - z) P_{MM} \left( \frac{\zeta}{z} \right) \mathcal{M}(z), \quad (190) \]
Figure 10: The same as in Fig. 9 but for the three-point correlation function $Z_2$.

\[ \dot{I}(\zeta) = \frac{\alpha}{2\pi} \int \frac{dz}{z} \theta(\zeta - z) \left\{ P_{\Pi} \left( \frac{\zeta}{z} \right) I(z) + P_{\Pi M} \left( \frac{\zeta}{z} \right) M(z) \right\} + \int dz' \left[ P_{Z_1 Z_1} \left( \frac{\zeta}{z}, \frac{z'}{z} \right) Z_1(z', z) + \frac{z(\zeta - z)}{(\zeta - z') (z - z')} Z_2(z, z') \right] \]  

\[ \dot{Z}_1(\zeta', \zeta) = \frac{\alpha}{2\pi} \left\{ \Theta_{11}^0 (\zeta', \zeta' - \zeta) \left[ M(\zeta) - \frac{(\zeta - \zeta')}{\zeta} I(\zeta) \right] \right. 
+ \left. \theta(\zeta') \left[ \frac{1}{\zeta} M(\zeta - \zeta') - \frac{1}{(\zeta - \zeta')} \frac{z(\zeta - z)^2}{\zeta' (\zeta') (\zeta - \zeta') (\zeta - z')} Z_2(z, \zeta - \zeta') \right] \right\} , \]

\[ \dot{Z}_2(\zeta, \zeta') = \frac{\alpha}{2\pi} \left\{ \Theta_{11}^0 (\zeta', \zeta' - \zeta) \left[ \frac{1}{\zeta} M(\zeta') - \frac{(\zeta' - \zeta)}{\zeta} I(\zeta') \right] \right. 
+ \left. \theta(\zeta') \left[ \frac{z(z - \zeta)}{\zeta' (\zeta' - \zeta)} Z_1(z, \zeta' - \zeta') \right] \right\} , \]

where the dot denotes the derivative with respect to the UV cutoff $\dot{} = \Lambda^2 \partial / \partial \Lambda^2$ and splitting.
functions are given by the following equations

\[
P_{\text{MM}}(z) = -\left[\frac{2}{z(1-z)}\right]_+ + \frac{1}{z} + 1,
\]

\[
P_{\text{II}}(z) = -\left[\frac{1}{2}\delta(z-1)\right],
\]

\[
P_{\text{ML}}(z) = -\left[\frac{2}{z(1-z)}\right]_+ + \frac{2}{z} + 1,
\]

\[
P_{\text{MZ}}(z,y) = -\left[\frac{2}{z(1-z)}\right]_+ + \frac{2}{z} + \frac{1}{1-zy} - \delta(z-1)\frac{1}{y}\ln(1-y),
\]

\[
P_{\text{Z1}}(z,y) = -\left[\frac{2}{z(1-z)}\right]_+ + \frac{2}{z} + \frac{y}{1-zy} + \delta(z-1)\left[\frac{3}{2} - \ln(1-y)\right].
\]

Now it is an easy task to verify the fulfillment of the equation of motion (176) for the correlation functions as a consistency check of our calculations. By exploiting this relation we exclude \(I\) from the above set of functions and reduce the system to the basis of independent gauge invariant quantities \(\{\mathcal{M} \text{ and } R_j\}\).

An important note is in order now. As distinguished from the evolution of the structure functions, the above Eq. (192) has the logarithmic dependence on the ratio of the parton momentum fractions\(^9\). The consequence of its presence is obvious. Taking into account the restrictions imposed by Eq. (180) we can define the moments of the correlation functions in the following way

\[
\mathcal{M}_n = \int_1^\infty \frac{d\zeta}{\zeta^n} \mathcal{M}(\zeta),
\]

\[
R^m_n = \int_1^\infty \frac{d\zeta}{\zeta^n} \int_0^\zeta d\zeta' \zeta'^m R(\zeta', \zeta).
\]

We find for two-particle cut vertex

\[
\dot{\mathcal{M}}_n = -\frac{\alpha}{2\pi} (S_n + S_{n-2}) \mathcal{M}_n.
\]

Comparing it with Eqs. (113) and (117) we notice the universality of the evolution kernels for the space- and time-like two-particles quasi-partonic cut vertices, \(i.e.\) the Gribov-Lipatov reciprocity (165) relation, which looks like

\[
\gamma_{n+2}^{\text{TL}} = \gamma_n^{\text{SL}}
\]

in terms of the corresponding anomalous dimensions, is fulfilled.

On the other hand, it is impossible to write down the finite system of equations for any given physical moment of the three-parton correlation functions as the logarithm of

---

\(^9\)This is consequence of the fact that while the support properties of the multi-parton recombination functions are different from the distributions, \(i.e.\) the region of attained momentum fractions is nonsymmetric in the former case in contrast to the latter, the perturbative one-loop renormalization of the field operators is not sensitive to this. Actually, it is given by the same equations as in the previous discussion of the deep inelastic scattering.
the ratio of the parton momentum fractions in the evolution kernels leads to the infinite series of the moments as distinguished from the deep inelastic scattering where the rank of the anomalous dimension matrix was finite and increases with the number of the moment. This is consequence of essential nonlocality of the cut vertices in the coordinate space since even if we start from the local cut vertex it will smeared along the light cone upon the renormalization. Therefore, it is not possible to solve the system of equations successively in terms of moments as well as we do not succeed in solving it analytically in a general form. However, in the next section when dealing with the QCD evolution we will find that the system of coupled equations (192) and (193) can be reduced to the single equation in the multicolour limit and its solution can be found analytically.

3.5 Non-abelian evolution.

In the QCD case we should add the diagrams with triple-boson interaction vertex (see Figs. 11, 12) and gluon self-energy insertions (not shown).

![Diagram 1](image1.png)

Figure 11: Non-abelian radiative corrections to the evolution kernels of the fragmentation function $Z_1$.

![Diagram 2](image2.png)

Figure 12: The same as in Fig. 11 but for the fragmentation function $Z_2$.

Gathering these contributions together with equations obtained in the previous section
\[ 
\dot{Z}_1(\zeta', \zeta) = \frac{\alpha}{2\pi} \left\{ C_F \left[ \Theta_{011}(\zeta', \zeta' - \zeta) \frac{\zeta'}{\zeta} \mathcal{M}(\zeta) - \theta(\zeta') \frac{\zeta'}{\zeta(\zeta - \zeta')} \mathcal{M}(\zeta - \zeta') \right] 
+ \int \frac{dz}{\zeta} \theta(\zeta - z) \left[ Z_1(\zeta, z) \left( C_F - \frac{C_A}{2} \right) \frac{z(\zeta - z)^2}{\zeta(\zeta - \zeta')} Z_2(z, \zeta - \zeta') \right] 
+ \frac{C_A}{2} \left[ -2z \frac{\partial}{\partial \zeta'} \int_0^1 dv Z_1(\zeta' - v(\zeta - z), z) + \left( \frac{z}{(z - \zeta + \zeta')} - \frac{z(z + \zeta')}{\zeta(\zeta - \zeta')} \right) Z_1(z - \zeta' + \zeta, z) \right] \right) 
+ \left( C_F - \frac{C_A}{2} \right) \left( \Theta_{011}(\zeta', \zeta' - \zeta, \zeta' - \zeta + z') \frac{(\zeta' - \zeta + z')}{\zeta'} Z_1(\zeta, \zeta) \right) \right\}, \tag{203} \]

\[ 
\dot{Z}_2(\zeta, \zeta') = \frac{\alpha}{2\pi} \left\{ -C_F \theta(\zeta - \zeta') \frac{1}{\zeta'} \mathcal{M}(\zeta') \right. 
+ \int \frac{dz}{\zeta} \theta(\zeta - z) \left[ Z_2(\zeta, z) \left( C_F - \frac{C_A}{2} \right) \frac{z(\zeta - \zeta)}{\zeta^2} Z_1(z, \zeta, z) \right] 
+ \frac{C_A}{2} \left[ -2z \frac{\partial}{\partial \zeta} \int_0^1 dv Z_2(\zeta' - v(\zeta - z), z) + \left( \frac{z}{(z - \zeta + \zeta')} - \frac{z(z + \zeta)}{\zeta^2} \right) Z_2(z, \zeta - \zeta') \right] \right) \right) 
- \left( C_F - \frac{C_A}{2} \right) \theta(\zeta - \zeta') \frac{\zeta - \zeta'}{\zeta} Z_1(\zeta', \zeta') \right\}, \tag{204} \]
which differs from its abelian analogue (190) only by the Casimir operator

These equations should be supplemented by the equation for the mass dependent correlator

One can easily observe the significant reduction of the above evolution equations if we neglect

where

These equations should be supplemented by the equation for the mass dependent correlator \( M \) which differs from its abelian analogue (190) only by the Casimir operator \( C_F \).

3.6 Asymptotic solution of the evolution equations.

One can easily observe the significant reduction of the above evolution equations if we neglect

the terms in the kernel of the order of magnitude \( \mathcal{O}(1/N_c^2) \). In this case an additional three-
parton correlator \( \mathcal{Z}_2 \sim \langle 0 | \bar{\psi} \psi | H, X \rangle \langle H, X | B^+ | 0 \rangle \), which appears only through the radiative corrections, decouples from the evolution equation for \( \mathcal{Z}_1 \). Therefore, discarding the quark mass cut vertex we obtain homogeneous equation which governs the \( Q^2 \)-dependence of the three-parton correlation function \( \mathcal{Z}_1 \).

The situation has the closer similarity with phenomenon found in the evolution equations
for chiral-even and -odd distribution functions discussed in the first part of this paper [32, 14, 38] where in the multicolour limit (\( N_c \to \infty \)) there was a very important simplification as the evolution kernels have been vanishing for contributions with interchanged order of partons on the light cone, i.e. the momentum fraction carried by gluon in the matrix element of quark-gluon correlator varies only among the quark ones and does not exceed the latters. This property allowed to find the solution of simplified equations exactly in the nonlocal form. In the present case the decoupling of \( \mathcal{Z}_1 \) has the same consequences.

In the large-\( N_c \) limit the RG equation takes the form

\[
\dot{\mathcal{Z}}_1(\zeta', \zeta) = \frac{\alpha}{4\pi} \int dz' \frac{dz}{z} \theta(\zeta - z) \mathcal{K}(z, \zeta', \zeta, \zeta') \mathcal{Z}_1(z', z)
\]

(206)

and the evolution kernel is given by the following expression:

\[
\frac{1}{N_c} \mathcal{K}(z, \zeta', \zeta, \zeta') = 2\frac{z}{\zeta} \delta(\zeta' - \zeta + z - z') - \delta(\zeta' - \zeta + z)
\]

(207)

\[
- \frac{2}{z} \delta(\zeta' - \zeta + z - z') + 2 \int_1^\infty \frac{dz''}{z''(1 - z'')} \delta \left( 1 - \frac{\zeta'}{z} \right) \delta(\zeta' - z')
\]

\[
+ \left[ \delta(\zeta' - \zeta + z - z') - \delta(\zeta' - \zeta + z) \right] \left[ \frac{z}{z'} - \frac{z(z' + \zeta)}{\zeta(z' - z + \zeta)} \right]
\]

\[
+ \delta \left( 1 - \frac{\zeta}{z} \right) \left\{ \frac{3}{2} \delta(\zeta' - z') - \ln \left( 1 - \frac{z'}{z} \right) \delta(\zeta' - z') - 2 \left[ \frac{z'}{z' - z'} \Theta_1^0(\zeta', \zeta' - z') \right]_+ 
\]

\[
+ \frac{z - \zeta'}{z' - z'} \left\{ \frac{\zeta'}{z - z'} + \frac{z' - \zeta'}{z' - z} \right\} \Theta_1^0(\zeta', \zeta' - z) + \left[ \frac{\zeta'}{z - z'} + \frac{z'}{z' - 1} \right] \Theta_1^0(\zeta', \zeta' - z') \right\}.
\]
Inspired by our knowledge acquired from the previous study of the twist-3 structure functions we are able to check that Eq. (176) (with \( M = 0 \)) satisfies the ladder-type evolution equation with the following splitting function:

\[
\frac{1}{N_c} \int d\zeta' \mathcal{K}(z, z', \zeta, \zeta') = -\frac{2}{\xi^2 (1 - \zeta^2)} + 2\frac{z}{\xi} - 1 + \frac{1}{2} \delta \left( 1 - \frac{\zeta}{z} \right).
\] (208)

Thus, for the moments we obtain the following solution of the RG equation \( (Q > Q_0) \):

\[
\int_1^\infty dz \frac{1}{z^n I(z, Q)} = \left( \frac{\alpha(Q)}{\alpha(Q_0)} \right)^{N_{\text{CO}}} \gamma_n \beta_0 \int_1^\infty dz \frac{1}{z^n I(z, Q_0)}.
\] (209)

and the corresponding anomalous dimensions equal

\[
N_{\text{CO}} \gamma_n = N_c \left\{ -2 \psi(n - 1) - 2 \gamma_E - \frac{3}{n - 1} + \frac{1}{2} \right\},
\] (210)

as usual \( \beta_0 = \frac{2}{3} N_f - \frac{11}{3} C_A \).

As we have previously mentioned, there exists an equation which states that in the leading log approximation the time-like (TL) and space-like (SL) kernels corresponding to the twist-2 parton densities are directly related by the Gribov-Lipatov equations (165) or (202). Comparing the result given by Eq. (210) with the large-\( N_c \) anomalous dimensions (163) we see the absence of the universality of the corresponding twist-3 evolution kernels, i.e. the Gribov-Lipatov reciprocity is violated.

### 3.7 Generalization to other fragmentation functions.

Now we can proceed further and demonstrate that the evolution kernels for the time-like two-quark densities can directly be found from their space-like analogues by exploiting the particular form of the evolution equations given by Eqs. (161) and (162). Since the analytic structure of the uncut diagram (see Fig. [3]) is completely characterized by the integral representation of the \( \Theta \)-function given by Eq. (19), we can just take its particular discontinuities, using the usual Cutkosky rules supplied with appropriate theta-function specifying the positivity of the energy flow from the right- to the left-hand side of the cut, in order to obtain the corresponding time-like kernel. Since the observed particle is always in the final state for the fragmentation process, we are restricted to the single cut across the horizontal rank of the ladder diagram in Fig. [3]. Namely, using the integral representation of the corresponding step function \( \Theta^0_{11} \), we have

\[
\Theta^0_{11}(x, x - \beta) = \int_{-\infty}^{\infty} \frac{d\alpha}{2\pi i} \frac{1}{\alpha x - 1 + i0} \left[ \alpha(x - \beta) - 1 + i0 \right] \text{disc} - \frac{\theta(x - \beta)}{\beta}.
\] (211)

The self-energy insertions are not affected by the cut since it does not cross the corresponding lines. Taking into account different kinematic definitions of the correlation functions in the space- and time-like regions, we are able to find the kernels. It is easy to verify that the evolution kernels constructed for the time-like twist-2 cut vertices using this recipe coincide
with the known results. In the same way, we may obtain the above equation (208) from Eq. (161).

Since there exist fragmentation functions corresponding to each distribution, apart from the specific ones appearing from the final state interaction, we are in a position to find large-$N_c$ anomalous dimensions, which govern their $Q^2$-dependence, from the results (162) and (104). Namely, the genuine twist-3 contributions $\mathcal{H}_L^{\text{tw-3}}$ (PCO) and $\mathcal{G}_T^{\text{tw-3}}$ (PCE) to the corresponding fragmentation functions

$$\mathcal{H}_L(\zeta) = \frac{1}{4} \int \frac{d\lambda}{2\pi} e^{i\lambda \zeta} \langle 0 | i\sigma_+ \gamma_5 \psi(\lambda n) | H, X \rangle \langle H, X | \bar{\psi}(0) | 0 \rangle,$$

$$\mathcal{G}_T(\zeta) = \frac{1}{4} \int \frac{d\lambda}{2\pi} e^{i\lambda \zeta} \langle 0 | \gamma_\perp \gamma_5 \psi(\lambda n) | H, X \rangle \langle H, X | \bar{\psi}(0) | 0 \rangle.$$

after subtracting out the twist-2 piece [28] obey the evolution equation (209) with the following anomalous dimensions:

$$\text{PCO}_{\gamma_n} = N_c \left\{ -2\psi(n-1) - 2\gamma_E + \frac{1}{n-1} + \frac{1}{2} \right\},$$

$$\text{PCE}_{\gamma_n} = N_c \left\{ -2\psi(n-1) - 2\gamma_E - \frac{1}{n-1} + \frac{1}{2} \right\}.$$

Of course, it is a trivial task to invert the moments and to find the DGLAP kernels themselves.

To summarize this section, we have found that in the multicolour limit of QCD the twist-3 fragmentation functions obey the ladder-type evolution equations and the corresponding anomalous dimensions are known analytically. The Gribov-Lipatov reciprocity is not the property of the twist-3 distributions but it is strongly violated already in the LLA of the perturbation theory.

## 4 Discussion and conclusion.

We review above the approach to an analysis of the logarithmic violation of the Bjorken scaling in the twist-3 distribution and fragmentation functions of the nucleon. It consists in
the studying of the one-loop renormalization of the multi-parton correlators which explicitly involve the gluon degrees of freedom and further reconstruction of the evolution equation for them in the LLA using the renormalization group invariance. For these purposes, we have used the techniques, which employ the light-like gauge for the gluon field. The physically transparent picture which appears in this gauge (which is an essential ingredient of our method) makes the calculations simple. Accepting different prescriptions on the spurious IR pole in the gluon propagator, we were able to verify that they do lead to the same results. We present an exact leading-order evolution for the correlators in the light-cone fraction as well as in the light-cone position representations, which display the complementary aspects of the factorization, and establish the bridge between different formulations of the QCD evolution.

From the calculational point of view the momentum space technique is much easier to treat. However, the coordinate space makes the involved symmetries apparent and, as a by-product, diagonalization of evolution kernels is easy to handle. The complicated form of exact master equations for the twist-3 functions compels one to look for the approximation which could work with reasonable accuracy. The solution of this problem has been found in the fact that in the multicolour limit of QCD they are reduced to the ladder-type evolution equations which generally have very good precision, at the level of few per cent.

Comparing the analytical expressions for the anomalous dimensions of the twist-3 structure and fragmentation functions we observe that the Gribov-Lipatov reciprocity relation, valid for low-twist parton densities in the LLA, is broken already in the leading-order of perturbation theory.

The result we have discussed here are important from the theoretical point of view since they enrich the theory of the higher-twist effects in the hadron reactions as well as for phenomenology and could be used to analyze the experimental data when these become available.

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References

[1] R.P. Feynman, *Photon Hadron Interactions*, (W.A. Benjamin, New York), 1971.

[2] A.H. Mueller, ed., *Perturbative Quantum Chromodynamics* (World Scientific, Singapore, 1989); J. Qui, G. Sterman, Nucl. Phys. B 353 (1991) 105; *ibid.* B 353 (1991) 137.

[3] K. Wilson, Phys. Rev. 179 (1969) 1499; R. Brandt, G. Preparata, Nucl. Phys. B 27 (1971) 541.

[4] J.C. Collins, D.E. Soper, Nucl. Phys. B 194 (1982) 445.

[5] V.M. Belyaev, B.L. Ioffe, Nucl. Phys. B 310 (1988) 548.
[6] V.M. Braun, P. Gornicki, L. Mankiewicz, Phys. Rev. D 51 (1995) 6036;
A.V. Belitsky, Phys. Lett. B 386 (1996) 359 and the references given therein.

[7] D.I. Diakonov, V. Petrov, P.V. Pobylitsa, M.V. Polyakov, C. Weiss, Nucl. Phys. B 480 (1996) 341;
M.V. Polyakov, P.V. Pobylitsa, Phys. Lett. B 389 (1996) 350.

[8] R.L. Jaffe, Phys. Rev. D 11 (1975) 1953.

[9] R.L. Jaffe, Comments Nucl. Part. Phys. 19 (1990) 239.

[10] R.L. Jaffe, X. Ji, Phys. Rev. Lett. 67 (1991) 552;
R.L. Jaffe, X. Ji, Nucl. Phys. B 375 (1992) 527.

[11] R.L. Jaffe, X. Ji, Phys. Rev. D 43 (1991) 724.

[12] A.W. Schreiber, A.I. Signal, A.W. Thomas, Phys. Rev. D 44 (1991) 2653 and the references given therein.

[13] M. Stratmann, Z. Phys. C 60 (1993) 763.

[14] Y. Kanazawa, Y. Koike, Bag model prediction for the nucleon's chiral-odd twist-3 distribution $h_L(x,Q^2)$ at high $Q^2$, hep-ph/9703324.

[15] J. Ralston, D.E. Soper, Nucl. Phys. B 152 (1979) 109.

[16] X. Artru, M. Mekhfi, Z. Phys. C 45 (1990) 669;
J.L. Cortes, B. Pire, J.P. Ralston, Z. Phys. C 55 (1992) 409.

[17] R.L. Jaffe, X. Ji, Phys. Rev. Lett. 71 (1993) 2547;
X. Ji, Phys. Rev. D 49 (1994) 114.

[18] A.H. Mueller, Phys. Rev. D 18 (1978) 3705;
A.H. Mueller, Phys. Rept. 73 (1981) 237.

[19] M. Anselmino, A.V. Efremov, E. Leader, Phys. Rept. 261 (1995) 1.

[20] K. Abe et al., Phys. Rev. Lett. 76 (1996) 587.

[21] V.N. Gribov, L.N. Lipatov, Sov. J. Nucl. Phys. 15 (1972) 438;
L.N. Lipatov, Sov. J. Nucl. Phys. 20 (1974) 94;
A.P. Bukhvostov, L.N. Lipatov, N.P. Popov, Sov. J. Nucl. Phys. 20 (1974) 287.

[22] G. Altarelli, G. Parisi, Nucl. Phys. B 126 (1977) 298.

[23] R.K. Ellis, W. Furmanski, R. Petronzio, Nucl. Phys. B 212 (1983) 29.

[24] R.L. Jaffe, Nucl. Phys. B 229 (1983) 205.

[25] J. Kogut, D.E. Soper, Phys. Rev. D 1 (1970) 2901.
[26] A.P. Bukhvostov, G.V. Frolov, L.N. Lipatov, E.A. Kuraev, Nucl. Phys. B 258 (1985) 601.

[27] I.I. Balitsky, V.M. Braun, Nucl. Phys. B 311 (1988/89) 541.

[28] I.I. Balitsky, V.M. Braun, Phys. Lett. B 222 (1989) 123;
   I.I. Balitsky, V.M. Braun, Nucl. Phys. B 361 (1991) 93.

[29] E.V. Shuryak, A.I. Vainshtein, Nucl. Phys. B 201 (1982) 141.

[30] A.P. Bukhvostov, E.A. Kuraev, L.N. Lipatov, Sov. J. Nucl. Phys. 38 (1983) 263; ibid.
   39 (1984) 121;
   A.P. Bukhvostov, E.A. Kuraev, L.N. Lipatov, JETP Lett. 37 (1983) 482; Sov. Phys.
   JETP 60 (1984) 22.

[31] P.G. Ratcliff, Nucl. Phys. B 264 (1986) 493;
   X. Ji, C. Chou, Phys. Rev. D 42 (1990) 3637;
   J. Kodaira, Y. Yasui, K. Tanaka, T. Uemastu, Phys. Lett. B 387 (1996) 855.

[32] A. Ali, V.M. Braun, G. Hiller, Phys. Lett. B 266 (1991) 117.

[33] B. Geyer, D. Müller, D. Robaschik, Nucl. Phys. Proc. Suppl. 51C (1996) 106;
   B. Geyer, D. Müller, D. Robaschik, The evolution of the nonsinglet twist-3 parton dis-
   tribution function, hep-ph/9611452;
   D. Müller, Calculation of higher-twist evolution kernels for polarized deep inelastic scat-
   tering, CERN-TH/97-4, hep-ph/9701338.

[34] A.P. Bukhvostov, E.A. Kuraev, L.N. Lipatov, in Proceedings of the XVIII LNPI Winter
   School, p. 86, Leningrad, 1983 (in Russian);
   A.P. Bukhvostov, E.A. Kuraev, L.N. Lipatov, in Proceedings of the XI ITEP Winter
   School, p. 39, Moscow, Energoatomizdat, 1984 (in Russian).

[35] S. Wandzura, F. Wilczek, Phys. Lett. B 72 (1977) 195.

[36] For a review, see G. Leibbrandt, Rev. Mod. Phys. 59 (1987) 1067;
   A. Bassetto, G. Nardelli, R. Soldati, Yang-Mills theories in the algebraic non covariant
   gauges (World Scientific, Singapore, 1991).

[37] A.V. Belitsky, E.A. Kuraev, Evolution of the chiral-odd spin-independent fracture func-
   tions in Quantum Chromodynamics, hep-ph/9612256.

[38] A.V. Belitsky, D. Müller, Scale dependence of the chiral-odd twist-3 distributions $h_L(x)$
   and $e(x)$, hep-ph/9702354.

[39] A. Bassetto, Nucl. Phys. Proc. Suppl. 51C (1996) 281.

[40] M.A. Ahmed, G.G. Ross, Nucl. Phys. B 111 (1976) 441;
   J. Kodaira et al., Phys. Rev. D 20 (1979) 627;
   J. Kodaira et al., Nucl. Phys. B 159 (1979) 99.
[41] Y. Koike, K. Tanaka, Phys. Rev. D 51 (1995) 6125;  
Y. Koike, N. Nishiyama, Phys. Rev. D55 (1997) 3068.

[42] D. Müller, D. Robaschik, B. Geyer, F. M. Dittes, J. Hořejší, Fortschr. Phys. 42 (1994) 101.

[43] Folklore.

[44] I. Balitsky, V. Braun, Y. Koike, K. Tanaka, Phys. Rev. Lett. 77 (1996) 3078.

[45] G. Curci, W. Furmanski, R. Petronzio, Nucl. Phys. B 175 (1980) 27.

[46] L.V. Keldysh, Zh. Eksp. Teor. Fiz. 47 (1964) 1515;  
J. Schwinger, J. Math. Phys. 2 (1961) 407.

[47] V.N. Gribov, L.N. Lipatov, Sov. J. Nucl. Phys. 15 (1972) 675.

[48] A.V. Belitsky, *Evolution of twist-3 fragmentation functions in multicolour QCD and the Gribov-Lipatov reciprocity*, hep-ph/9702356.