Comments on the paper “Exact solutions of nonlinear diffusion-convection-reaction equation: A Lie symmetry approach”

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Abstract

This comment is devoted to the paper “Exact solutions of nonlinear diffusion-convection-reaction equation: A Lie symmetry approach” (CNSNS, 67 (2019), 253-263) in which several results are not new because were derived much earlier. Moreover, some results in the paper are incorrect or incomplete.

Keywords: diffusion-convection-reaction equation, Lie symmetry, exact solution, optimal system of sub-algebras.

1 Lie symmetry analysis

The recent paper [1] is devoted to search for Lie symmetries and exact solutions of the diffusion-convection-reaction equation

\[ u_t = (u^m)_{xx} + (b_0 u + b_1 u^{p+1})_x + (1 - u^p)(c_0 + c_1 u^p)u^{2-m}, \]  

where \( u(t, x) \) is unknown smooth function and all the parameters \( m, b_0, \ldots, p \) are real constants (Authors assume that \( m > 0 \) but the sign of \( m \) does not play any role for search of Lie symmetries).

It can be noted from the very beginning that the linear term \( b_0 u \) can be removed from Eq. (1) by the well-known transformation, the Galilei boost

\[ x^* = x + b_0 t. \]  

Hence Eq. (1) simplifies to the form

\[ u_t = (u^m)_{x^*x^*} + b_1 (u^{p+1})_{x^*} + (1 - u^p)(c_0 + c_1 u^p)u^{2-m}. \]  

In Lie symmetry analysis, especially for the Lie symmetry classification, such type of transformations plays important role and are called equivalence transformations (ETs) (see, e.g., monographs [2, 3]). If one ignores ETs then Lie symmetry analysis degenerates in a chaotic process with a lot of equivalent cases. Paper [1] is a typical example.

To the best of my knowledge, transformation (2) was firstly identified as an ET of the general diffusion-convection-reaction equation (not only Eq. (1)!) 

\[ u_t = [A(u)u_x]_x + B(u)u_x + C(u), \]  

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(here $A(u) \neq 0$, $B(u)$ and $C(u)$ are arbitrary smooth functions) in paper [4] (see formulae (2.12)-(2.13 therein). In [5], the full group of ETs for Eq. (1) was derived using a rigorous algorithm. The group has the form

$$t^* = \kappa_0 t + d_0, \quad x^* = \kappa_1 x + gt + d_1, \quad u^* = \kappa_2 u + d_2,$$

where all the parameters $\kappa_0, \ldots, d_2$ are arbitrary constants and $\kappa_0 \kappa_1 \kappa_2 \neq 0$. Of course, the group of ETs contains transformation (2) as a particular case (see the parameter $g$ in (5)).

Now we demonstrate that all the results derived in Section 2.2 of [1] are particular cases of those from [5] (actually they can be also deduced from earlier paper [4]).

Let us take Eq. (24) and Lie symmetries (25) from paper [1]. Setting $b_0 = 0$ (because ET(2) removes $b_0$) we obtain (star * is omitted for simplicity in what follows)

$$u_t = (u^m)_{xx} + b_1 (u^{p+1})_x$$

and

$$X_1 = \partial_t, \quad X_2 = \partial_x, \quad X_3 = (m - 2p - 1)t \partial_t + (m - p - 1)x \partial_x) - t \partial_x + u \partial_u$$

respectively. It can be easily checked that Eq. (6) and Lie algebra (7) follow as a particular case from Case 23 of Table 2 [5], when one sets $\lambda_8 = 0$ and makes renaming $k \rightarrow m - 1$ and $m \rightarrow p$ [5].

In a quite similar way, it can be checked that equation and Lie symmetries from (28) [1] can be essentially simplified by removing $b_0$. Having this done, one again sees that Case 24 of Table 2 [5] contains the above result. It should be noted that the parameter $c_1$ in (28) [1] can be reduced to $\pm 1$ by an appropriate ET.

Equations and Lie symmetries from (29) and (31) [1] after removing $b_0$ also follow as particular cases from Case 23 of Table 2 [5], if one sets $m = k$ and $m = k/2$ [5], respectively.

Finally, equation and Lie symmetries from (30) [1] are known at least 70 years because after removing $b_0$, one arrives at the classical result published by Ovsiannikov in 1959 (see his book [2]). In [5], this result is presented in Case 10 of Table 2.

Thus, all the equations and Lie symmetries presented in [1] were discovered many years ago in the papers [4] and [5]. Moreover, the Lie symmetry classification of Eq. (11) is incomplete. For example, there is a special case, Eq. (11) with $p + 1 = m$ and $c_0 = 0$. It can be noted that the equation with the same structure arises in Case 13 of Table 2 [5] and admits four-dimensional Lie algebra. This special case is missed in [1].

2 Reduction and exact solutions

There are three main techniques for constructing exact solutions by applying Lie symmetries (see the relevant discussion about their applicability in Chapter 1 of [3]). Many examples of their applications to a wide range of PDEs are presented in monograph [8]. The first technique is rather trivial. One takes the Lie group corresponding to the known Lie symmetry operator and a known solution of the equation in question. Applying the Lie group to the known solution, one obtains a set of solutions involving at least one free parameter. In the case of Lie
algebras of high dimensionality, very nontrivial formulae of multiplication of exact solutions can be derived, see, e.g. [8], [9].

The second technique is a direct application of a known Lie symmetry for deriving a special substitution, usually called ansatz, which reduces the given two-dimensional PDE to an ODE. If we start from the general linear combination of all the symmetries of a given PDE then all possible inequivalent reductions of this PDE to ODEs can be derived. In the case of Lie algebras of low dimensionality, it is a simple task (several examples can be found in [3]). However, the task transforms into a difficult problem if the equation in question admits Lie algebra of high dimensionality (say, 6 or higher) with a nontrivial structure.

The third technique is the most sophisticated one and was used in [1]. This technique is based on the so-called optimal systems of one-dimensional subalgebras and was suggested by Ovsiannikov [2]. Note that one needs to construct also optimal systems of two- and higher-dimensional subalgebras in the case of multidimensional PDEs. In the case of Lie algebras of low dimensionality, these optimal systems are known and were summarized in the classical paper [6] (incidentally this paper is not cited in [1]), while problems occurring for Lie algebras of high dimensionality is extensively discussed, e.g., in the recent book [10]. So, if the Lie algebra of invariance is two-, three-, or four-dimensional then it is enough to identify this algebra in [6] and the relevant optimal system of one-dimensional subalgebras can be readily written down. Notably, the authors of [6] use the notion 'system of non-conjugated subalgebras' instead of Ovsiannikov’s terminology 'optimal system of subalgebras'.

Although all the Lie algebras obtained in [1] are 3- or 4-dimensional, Authors decided to re-discover the optimal systems of one-dimensional subalgebras. Unfortunately, the results presented in Table 3 are incorrect, excepting Lie algebra (25). It may be noted that this algebra up to a constant multiplier coincides with one \( A_{35}^3 \) in Table I [6] and the relevant optimal systems of one-dimensional subalgebras do coincide. However, the optimal systems for Lie algebras (28), (29) and (31) are obviously wrong because they consist of 5 subalgebras. It can be easily seen from the last column of Table I [6] that the optimal system of any 3-dimensional Lie algebra consists of 4 (not 5!) subalgebras at maximum. Similarly, the optimal system of one-dimensional subalgebras for Lie algebra (30) is also incorrect because one consists of 8 subalgebras while one consists of 7 (see the algebra \( 2A_2 \) in Table II [6]). Moreover, the correct optimal system should contain a subalgebra with an arbitrary parameter (not only \( \delta = \pm 1 \) !).

Because the optimal systems derived in Table 3 [1] are incorrect (excepting the first line), the number of reduced equations listed on Page 12 [1] should be essentially smaller. Indeed, according to the general theory (see for details [2]), many solutions of those ODEs will lead to the same solutions of the corresponding nonlinear PDEs.

Finally, it should be noted that all the exact solutions obtained in Section 3.2 were known earlier. If one sets \( b_0 = 0 \) in the solutions derived then it can be easily identified. For example, the most complicated exact solution (47) with \( b_0 = 0 \) is nothing else but a particular case of that obtained in [7] (see formula (71) therein).
3 Conclusions

In this comment, it is shown that all the nonlinear PDEs and their Lie symmetries derived in the recent paper [1] are equivalent to those derived earlier in [2, 4] (the detailed proofs and several applications are presented in the recent monograph [3]). Moreover, the optimal systems of one-dimensional subalgebras of Lie algebras are incorrect and do not coincide (excepting a single case) with the results of the seminal work [6]. The exact solutions obtained in [1] also follows from earlier works.

In conclusion, I would like to stress that nowadays there are many papers devoted to the Lie symmetry analysis of nonlinear PDEs, in which the authors simplify the analysis to primitive calculations without knowing state-of-art (see, e.g., the recent books [3, 10, 12, 13] devoted to the symmetry-based methods and its direct applications). As a result, 'new' Lie symmetries and 'new' exact solutions are either equivalent to derived in earlier papers, or simply wrong, optimal systems of subalgebras are incorrect and real-world applications are absent (see, e.g., another typical example commented in [11]).

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