ON UNIVERSAL FAMILY OF HILBERT SCHEMES OF POINTS ON A
SURFACE

LEI SONG

ABSTRACT. For a smooth quasi-projective surface $X$ and an integer $n \geq 3$, we show that the universal family $Z^n$ over the Hilbert scheme $\text{Hilb}^n(X)$ of $n$ points has non $\mathbb{Q}$-Gorenstein, rational singularities, and that the Samuel multiplicity $\mu$ at a closed point on $Z^n$ can be computed in terms of the dimension of the socle. We also show that $\mu \leq n$.

1. Introduction

Let $X$ be a smooth quasi-projective surface over an algebraically closed field $k$ of characteristic 0. Let $\text{Hilb}^n(X)$ denote the Hilbert scheme of zero dimensional closed subschemes of $X$ of length $n$. Fogarty's fundamental result [6] claims that $\text{Hilb}^n(X)$ is a smooth, irreducible variety of dimension $2n$. When $X$ is complete, one may consider $\text{Hilb}^n(X)$ as a natural compactification of the space of $n$ unlabeled distinct points on $X$. For this reason, $\text{Hilb}^n(X)$ is a quite useful tool for dealing with infinitely near points on $X$.

The universal family $Z^n \subset \text{Hilb}^n(X) \times X$, with the induced projection $\pi : Z^n \to \text{Hilb}^n(X)$, is a flat, finite cover of degree $n$ over $\text{Hilb}^n(X)$. The importance of $Z^n$ lies in its role in inductive approaches for the study of Hilbert schemes, e.g. in deduction of the Picard scheme of $\text{Hilb}^n(X)$ [8], and in calculation of the Nakajima constants [5]. While $Z^2$ is simply the blow up of $X \times X$ along the diagonal, $Z^n$ is complicated in general. It is well known that $Z^n$ are irreducible, singular (except $n=2$), and Cohen-Macaulay. By [8], $Z^n$ is also normal with $R_3$ condition. More precisely, we prove in this note

Theorem 1.1. The universal family $Z^n$ is non $\mathbb{Q}$-Gorenstein, and has rational singularities. For a closed point $\zeta = (\xi, p) \in Z^n$, the Samuel multiplicity $\mu = \left(\frac{b_2+1}{2}\right)$, where $b_2 = b_2(O_{\xi, p})$ is the dimension of the socle of $O_{\xi, p}$.

We note that $b_2 + 1$ equals the minimal number of generators of the ideal $\mathcal{I}_{\xi, p} \subset O_{X, p}$ (see Lemma 2.1).

For $i > 0$, let $V^i = \{(\xi, p) \in Z^n \mid b_2(O_{\xi, p}) = i\}$. The locally closed $V^i$'s form a stratification of $Z^n$. A proposition due to Ellingsrud and Lehn [4] says that codim($V^i, \text{Hilb}^n(X) \times X$) $\geq 2i$. The kind of proposition is an ingredient in proving the irreducibility of $\text{Hilb}^n(X)$ and more generally, that of certain quot schemes. An immediate consequence of the codimension estimate is that $b_2 \leq n + 1$. In fact, we have

Theorem 1.2.

$$b_2 \leq \left\lfloor \frac{\sqrt{1 + 8n} - 1}{2} \right\rfloor.$$  

Moreover the bound is optimal. Consequently, one has $\mu \leq n$.

Key words and phrases. Hilbert scheme of points on a surface, universal family, rational singularities, Samuel multiplicity.
Haiman \cite{Hai} proves a similar result on multiplicity, namely \( \dim_k \mathcal{O}_{\xi, p} \geq \binom{b_2 + 1}{2} \), which implies that \( \mu \leq n \). We need to point out that \( \mu < \dim_k \mathcal{O}_{\xi, p} \) in general.

Acknowledgments: I am grateful to Lawrence Ein for valuable discussions, and to Steven Sam for bringing [3, Prop. 3.8] into my attention.

2. Preliminaries

For a point \( p \in X \), \( m_p \) denotes the maximal ideal of the local ring \( \mathcal{O}_{X, p} \) and \( k(p) \) denotes the residue field. Let \( \xi \in X \) be a zero dimensional closed subscheme with the defining ideal \( \mathcal{I}_\xi \). The socle \( \text{Soc} \left( \mathcal{O}_{\xi, p} \right) \) is defined to be \( \text{Hom}_{\mathcal{O}_{\xi, p}}(k(p), \mathcal{O}_{\xi, p}) \). We denote the minimal number of generators of \( \mathcal{I}_\xi \) by \( e(\mathcal{I}_\xi, p) \). By Nakayama lemma \( e(\mathcal{I}_\xi, p) = \dim_k(p) \mathcal{I}_\xi, p \otimes k(p) \).

Since \( X \) is smooth and of dimension two, \( \mathcal{O}_{\xi, p} \) has a minimal free resolution

\[
0 \rightarrow b_1 \mathcal{O}_{X, p} \rightarrow b_2 \mathcal{O}_{X, p} \rightarrow \mathcal{O}_{X, p} \rightarrow \mathcal{O}_{\xi, p} \rightarrow 0, \tag{2.1}
\]

where obviously \( b_2 + 1 = b_1 \), and \( \varphi \) is represented by a matrix with entries in \( m_p \). The Hilbert-Burch theorem [3, Thm. 3.2] says that all the \( b_2 \times b_2 \) minors of \( \varphi \) generate the ideal \( \mathcal{I}_\xi \). A standard computation of homological algebra shows

**Lemma 2.1.** With the notations above, \( \dim_k(p) \text{Soc} \left( \mathcal{O}_{\xi, p} \right) = b_2 = e(\mathcal{I}_\xi, p) - 1. \)

We recall that the incidence correspondence Hilbert scheme \( \text{Hilb}^{n,n-1}(X) \) is defined as

\[ ((\xi, \xi') \mid \xi' \text{ is a closed subscheme of } \xi) \subset \text{Hilb}^n(X) \times \text{Hilb}^{n-1}(X) \]

For such \((\xi, \xi')\), one has a unique point \( p \) which fits into the exact sequence

\[ 0 \rightarrow \mathcal{I}_\xi \rightarrow \mathcal{I}_{\xi'} \rightarrow k(p) \rightarrow 0. \]

So naturally associated to \( \text{Hilb}^{n,n-1}(X) \), there are two morphisms

\[
\begin{array}{ccc}
\text{Hilb}^{n,n-1}(X) & \xrightarrow{\psi} & Z^n \\
& & \xrightarrow{\phi} \text{Hilb}^{n-1}(X) \times X \\
\end{array}
\]

by \( \psi : (\xi, \xi') \mapsto (\xi, p) \) and \( \phi : (\xi, \xi') \mapsto (\xi', p) \).

We collect a few facts about \( \text{Hilb}^{n,n-1}(X) \).

**Proposition 2.2** (\cite{Hai, Sam}). \( \text{Hilb}^{n,n-1}(X) \) is a smooth irreducible variety of dimension 2n.

**Proposition 2.3** (\cite{Sam}).

1. \( \text{Hilb}^{n,n-1}(X) \cong \mathcal{P}(\omega_{Z^n}) \), where \( \omega_{Z^n} \) is the dualizing sheaf. In particular, \( \text{Hilb}^{n,n-1}(X) \) is a resolution of singularities of \( Z^n \).
2. \( \text{Hilb}^{n,n-1}(X) \cong \mathcal{B}_{Z^n}(\text{Hilb}^{n-1}(X) \times X) \cong \mathcal{P}(\mathcal{I}_{Z^n} \cup \text{Hilb}^{n-1}(X) \times X). \)
3. For a closed point \((\xi, p) \in Z^n, \psi^{-1}((\xi, p)) \cong \mathbb{P}^{b_1-1}. \)

3. Universal Families

We deonte the discriminant divisor of the finite map \( \pi : Z^n \rightarrow \text{Hilb}^n(X) \) by \( B^n \), which is a prime divisor parameterizing all non-reduced subschemes of \( X \) of length \( n \).

In the proof of \cite{Sam} Theorem 7.6], Fogarty shows that there are precisely two prime Weil divisors on \( Z^n \) dominating \( B^n \), which are described as

\[ E^n_1 = \{(\xi, p) \mid \xi \text{ has type } (2, 1, \cdots, 1), \text{ and } l_p(\xi) = 2\}. \]
with the reduced induced closed subscheme structure. Moreover he proves that neither $E^n_1$ nor $E^n_2$ is $\mathbb{Q}$-Cartier.

Let $r_i$ be the ramification index of $E_i^n$ over $B^n$, and $K(\ast)$ the field of functions of $\ast$. From the identity

$$r_1[K(E^n_1) : K(B^n)] + r_2[K(E^n_2) : K(B^n)] = n,$$

and $[K(E^n_1) : K(B^n)] = 1$, $[K(E^n_2) : K(B^n)] = n - 2$, we see that $r_1 = 2$ and $r_2 = 1$.

Then Riemann-Hurwitz formula for finite maps claims that

$$(3.1) \quad K_{Z^n} + (2c - 1)E^n_1 + cE^n_2 = \pi^*(K_{\text{Hilb}^n(X)} + cB^n),$$

for all $c \in \mathbb{Q}$. Thereby we deduce that $K_{Z^n}$ cannot be $\mathbb{Q}$-Cartier by setting $c = \frac{1}{2}$ or 0.

**Proof of Theorem 1.1.** It remains to show the rational singularities of $Z^n$ and compute the Samuel multiplicities. Let $R$ be the local ring $O_{\text{Hilb}^n(X) \times X, \xi}$. Since $Z^n$ is codimension two Cohen-Macaulay subscheme of $\text{Hilb}^n(X) \times X$, one has the minimal free resolution of $O_{Z^n, \xi}$

$$(3.2) \quad 0 \to b_2 \oplus R \to b_1 \oplus R \to R \to O_{Z^n, \xi} \to 0.$$

**Claim A:** $b_2$ in (3.2) equals $b_2(O_{\xi, p})$.

**Proof of Claim A.** Let $X \times \{\xi\}$ denote the fibre of $\text{Hilb}^n(X) \times X$ over $\xi$. Tensoring (3.2) with $O_{X \times \{\xi\}, p}$ yields the complex

$$(3.3) \quad 0 \to b_2 \oplus O_{X, p} \to b_1 \oplus O_{X, p} \to O_{X, p} \to O_{\xi, p} \to 0.$$

To show that the complex is exact, it suffices to prove that $\text{Tor}^R_i(O_{Z^n, \xi}, O_{X \times \{\xi\}, p}) = 0$ for $i = 1, 2$, which follows from

$$\text{Tor}^R_i(O_{Z^n, \xi}, O_{X \times \{\xi\}, p}) \cong \text{Tor}^R_i(O_{\text{Hilb}^n(X) \times X, \xi}, k(\xi))$$

$$\cong \text{Tor}^R_i(O_{\text{Hilb}^n(X) \times X, \xi}(O_{Z^n, \xi}, k(\xi)) = 0,$$

where the last step is by the flatness of $O_{Z^n, \xi}$ over $O_{\text{Hilb}^n(X), \xi}$.

Clearly (3.3) is minimal, so it is a minimal free resolution of $O_{\xi, p}$, and we are done. \(\Box\)

By taking dual $\text{Hom}_R(\cdot, R)$ for (3.2), one has the exact sequence

$$0 \to \oplus R \to \oplus R \to \text{Ext}^2_R(O_{Z^n, \xi}, R) \to 0.$$

Since $R$ is regular, one has $R \cong \omega_{R}$, the canonical module. By duality, $\text{Ext}^2_R(O_{Z^n, \xi}, R) \cong \omega_{Z^n, \xi}$, and hence $\mathbb{P}(\omega_{Z^n, \xi}) \to \mathbb{P} := \mathbb{P}^{\mathbb{P}^{b_2} - 1}$. Writing $\rho$ for the projection $\mathbb{P} \to \text{spec } R$, one has the commutative diagram

$$\begin{array}{ccc}
\mathbb{P}(\omega_{Z^n, \xi}) & \to & \mathbb{P} \\
\rho \downarrow & & \rho \\
\text{spec } O_{Z^n, \xi} & \to & \text{spec } R
\end{array}$$

**Claim B:** $\mathbb{P}(\omega_{Z^n, \xi})$ is locally a complete intersection in $\mathbb{P}$. 
Proof of Claim B. We have the diagram

\[
\begin{array}{ccc}
O_2 & \xrightarrow{\sigma} & O_2 \\
\downarrow & & \downarrow \\
O_2(1) & & O_2(1)
\end{array}
\]

where \(O_2(1)\) is the tautological invertible sheaf and the column map is from the Euler sequence.

On the one hand, \(\mathbb{P}(\omega_{Z^n, \xi})\) is the locus where the map \(\sigma = 0\), and hence has \(b_1\) defining equations. On the other hand, \(\dim \mathbb{P}(\omega_{Z^n, \xi}) = 2n = \dim \mathbb{P} - b_1\), i.e. \(\text{codim}(\mathbb{P}(\omega_{Z^n, \xi}), \mathbb{P}) = b_1\) as required.

In view of Claim B, the Koszul complex

\[
0 \to \bigoplus_{i=0}^{b_1} O_2(-1) \to \cdots \to \bigoplus_{i=0}^{2} O_2(-1) \to O_2 \to O_{\mathbb{P}(\omega_{Z^n, \xi})} \to 0
\]

is exact.

Observing that for all \(j > 0\) and \(b_2 - 1 \geq i \geq 0\),

\[
R^j \rho_* \left( \bigoplus_{i=0}^{b_1} O_2(-1) \right) = R^j \rho_* \left( \bigoplus_{i=0}^{b_1} O_2(-i) \right) = 0,
\]

thus by splitting the above sequence into short ones, we obtain that for all \(j > 0\),

\[
R^j \rho_* O_{\mathbb{P}(\omega_{Z^n, \xi})} = 0.
\]

Consider the fibred product for all closed point \(\xi \in Z^n\)

\[
\begin{array}{ccc}
\mathbb{P}(\omega_{Z^n, \xi}) & \xrightarrow{\rho} & \text{Hilb}^{n,n-1}(X) \\
\downarrow & & \downarrow \\
\text{spec} O_{Z^n, \xi} & \xrightarrow{\psi} & Z^n
\end{array}
\]

Since \(\nu\) is flat,

\[
(3.4) \quad R^j \psi_* O_{\text{Hilb}^{n,n-1}(X)} \otimes_{O_{Z^n, \xi}} O_{Z^n, \xi} = 0,
\]

by base change theorem.

Since \(\text{Hilb}^{n,n-1}(X)\) is smooth, it follows from \([5,4]\) that \(Z^n\) has rational singularities.

As for multiplicity, we use its relation with Segre class \(s_1(\cdot)\) and the invariance of Segre class under a birational morphism (cf. \([1]\) Chap. 4]), which give

\[
\mu = s_0(\xi, Z^n) = s_{b_1-1}(t^{b_1-1}, \text{Hilb}^{n,n-1}(X)) = s_{b_1-1}(N),
\]

where \(N := N_{\mathbb{P}^{b_2-1}/\text{Hilb}^{n,n-1}(X)}\) and we note \(N \cong N_{\mathbb{P}^{b_2-1}/(\mathbb{P}(\omega_{Z^n, \xi}))}\).

By the exact sequence

\[
0 \to N \to N_{\mathbb{P}^{b_2-1}/\mathbb{P}} \to N_{\mathbb{P}(\omega_{Z^n, \xi})/\mathbb{P})} |_{\mathbb{P}^{b_2-1}} \to 0,
\]

and observing \(N_{\mathbb{P}^{b_2-1}/\mathbb{P}} = \bigoplus_{i=0}^{2n+b_2} O_{\mathbb{P}^{b_2-1}}\) and \(N_{\mathbb{P}(\omega_{Z^n, \xi})/\mathbb{P})} |_{\mathbb{P}^{b_2-1}} = b_1 \oplus O_{\mathbb{P}^{b_2-1}(1)}\), we get

\[
\mu = s_{b_1-1}(N) = c_{b_1-1} \left( b_1 \oplus O_{\mathbb{P}^{b_2-1}(1)}(1) \right) = \left( b_1 \right) \left( b_2 - 1 \right) = \left( b_2 + 1 \right) \left( b_2 - 1 \right),
\]

where \(c\) is the Chern class. \(\blacksquare\)
Remark 3.1. As a smooth surface varies, the universal families over Hilbert schemes of points easily patch together. More precisely, let \( f : X \to C \) be a smooth, projective morphism from a 3-fold \( X \) to a smooth curve \( C \) over \( k \). By [7, Thm. 2.9], the relative Hilbert scheme of \( n \) points \( \text{Hilb}^n(X/C) \) is smooth over \( C \). The composite morphism \( h : Z^m_{X/C} \to \text{Hilb}^n(X/C) \to C \) from the universal family is a flat morphism, so one gets a flat family of algebraic varieties with non \( \mathbb{Q} \)-Gorenstein and rational singularities.

4. AN UPPER BOUND OF \( b_2 \)

This section is devoted to the proof of Theorem 1.2. We reduce the problem to the case of monomial ideals of \( k[x, y] \) by flat degeneration of ideals and the upper semi-continuity of \( b_2 \). For the monomial case, \( n \) is visualized as the number of boxes in a staircase and \( b_2 \) as the number of gray nodes (see Figure 1), and the matrix \( \varphi \) in (2.1) can be written down explicitly. The entries of \( \varphi \) are regulated by the following lemma, which is a local version of [3, Prop. 3.8].

**Lemma 4.1.** Let \( (A, m) \) be a regular local Noetherian \( k \)-algebra of dimension 2, and \( m = (x, y) \). Set the degrees of \( x \) and \( y \) to be one. Let \( I \subseteq m \) be a monomial ideal with height 2. Then \( A/I \) has following minimal free resolution

\[
0 \to \bigoplus_{i=0}^{t} A \xrightarrow{\varphi} A^t \xrightarrow{i} A \to A/I \to 0,
\]

where \( \varphi \) is given by a \((t+1)\times t\) matrix with monomial entries in \( m \). Let \( a_1 \geq \cdots \geq a_{t+1} \) and \( b_1 \geq \cdots \geq b_t \) be degrees of a set of minimal generators of \( I \) and that of their first syzygies, and define \( e_i = b_i - a_i \) and \( f_i = b_i - a_{i+1} \) for \( 1 \leq i \leq t \). Then \( a_i, b_i, e_i, f_i \) are subject to the following constraints

1. \( e_i \geq 1, f_i \geq 1 \),
2. \( a_i = \sum_{j \leq i} e_j + \sum_{j \geq i} f_j \),
3. \( b_i = a_i + e_i \) for \( i = 1, \cdots, t \) and \( \sum_i b_i = \sum_{i+1}^t a_i \),
4. \( f_i \geq e_i \) and \( f_i \geq e_{i+1} \).

**Example 4.2.** Let \( A = \mathbb{C}[x, y] \) and \( I = (x^5, xy^4, x^3y^3, x^2y^2, x^3) \subset A \), see Figure 1. Generators correspond to black points. The length of \( A/I \) is 28. Degrees of generators are \((a_1, \cdots, a_5) = (9, 7, 6, 6, 5)\). One set of first syzygies are

\[
\begin{align*}
    x(x^5) - y^2(xy^4) &= 0, \\
    x^2(xy^4) - y(x^3y^3) &= 0, \\
    x^2(x^3y^3) - y(x^5y^2) &= 0, \\
    x^3(x^2y^2) - y^2(x^3) &= 0,
\end{align*}
\]

which correspond to gray points. \((b_1, b_2, b_3, b_4) = (11, 8, 7, 7)\). So \((e_1, e_2, e_3, e_4) = (2, 1, 1, 1), (f_1, f_2, f_3, f_4) = (4, 2, 1, 2)\).

![Figure 1. Staircase](image-url)
Therefore, is 0, which is a contradiction to Hilbert-Burch Theorem.

If \( e_j = \deg m_i \leq 0 \), then \( m_i = 0 \), since otherwise it contradicts the minimality of the resolution. It follows that \( m_{ik} = 0 \) for \( j \leq i, k \geq i \). In this case, the upper \( t \times t \) minor of \( M \) is 0, which is a contradiction to Hilbert-Burch Theorem.

(2) follows from (3). In fact, assuming \( \sum_{i=1}^{t} b_i = \sum_{i=1}^{t+1} a_i \), we have

\[
  a_i = \sum_{j=1}^{t} b_j - \sum_{j<i} a_j - \sum_{j\geq i+1} a_j = \sum_{j<i} (b_j - a_j) + \sum_{j\geq i} (b_j - a_{j+1}) = \sum_{j<i} e_j + \sum_{j\geq i} f_j.
\]

To prove (3), it suffices to note that syzygies of monomials \( F_1, \ldots, F_t \) are generated by the \( \frac{F_i}{\gcd(g, x, f, j)} F_i - \frac{F_j}{\gcd(g, x, f, j)} F_j \), and hence the minimal generators correspond to gray points as in Figure 1. The proof is then straightforward.

We also need the following easy lemma.

**Lemma 4.3.** As in Figure 1, assume there are \( t+1 \) black nodes with coordinates \( (a_0, b_0), \ldots, (a_t, b_t) \) and \( t \) gray nodes with coordinates \( (a', b'), \ldots, (a', b') \). Let \( n \) denote the number of boxes contained in the staircase. Then

\[
  2n = \sum_{i=1}^{t} (a_i + b_i)^2 - \sum_{i=0}^{t} (a_i + b_i)^2.
\]

**Proof of Theorem 1.2.** We may assume that \( \text{supp}(\xi) = \{p\} \). Let \( (A, m) \) denote the local ring \( O_{X, p} \) and \( I \) the defining ideal of \( \xi \). Since \( A/I \) is Artinian local \( k \)-algebra, one has \( A/I \cong \hat{A}/\hat{I} \) as \( k \)-algebras, where \( \hat{A} \cong k[[x, y]] \) is the \( m \)-adic completion of \( A \).

Let \( (R, n) \) be the local ring \( O_{X, 0} \) at the origin. For \( r \gg 0 \), one has the surjection \( R/\mathfrak{m}^r \cong \hat{A}/\mathfrak{m}^r \to \hat{A}/\hat{I} \to 0 \), so there exists an ideal \( J \subset R \) such that \( R/J \cong \hat{A}/\hat{I} \). Therefore it suffices to consider the case that \( X = \mathbb{A}^2 \) and \( \text{supp}(\xi) = \{O\} \).

Fix a lexicographic order \( > \) for monomials in \( k[x, y] \). Let \( in_{x, I} \) be the initial ideal of \( I \). By [2] Thm. 15.17, Prop. 15.16, there exists a flat family of length \( n \) subschemes of \( X \) over a curve such that the central fibre corresponds to \( k[x, y]/in_{x, I} \), and the other fibres correspond to rings isomorphic to \( k[x, y]/I \). Because \( b_2 \) is an upper semi-continuous function on the set of closed points on \( \mathbb{P}^n \), it suffices to treat the case when \( I \) is a monomial ideal.

Keeping the notations as above, and by Lemma 4.1 and 4.3 we have

\[
  2n = \sum_{i=1}^{t} b_i^2 - \sum_{i=1}^{t+1} a_i^2
  = \sum_{i=1}^{t} (a_i + e_i)^2 - \sum_{i=1}^{t+1} a_i^2
  = \sum_{i=1}^{t} e_i(2a_i + e_i) - \left( \sum_{i=1}^{t} e_i \right)^2
  = 2 \sum_{i=1}^{t} \left( \sum_{j<i} e_j \right) e_i + 2 \sum_{i=1}^{t} \left( \sum_{j\geq i} f_j \right) e_i - 2 \sum_{i\neq j} e_i e_j
\]

Therefore,

\[
  n = \sum_{j\geq i} f_j e_i - \sum_{j\geq i} e_j e_i \geq \frac{(t+1)t}{2}.
\]

We get the inequality \( b_2 \leq \frac{\sqrt{t^2+8t-1}}{2} \) by setting \( b_2 = t \).
If \( n = \frac{m(m+1)}{2} \) for some integer \( m > 0 \), then \( \sqrt{1+8n-1} = m \in \mathbb{Z} \) and the equality is achieved by \( I = m^m \). If \( n = \frac{m(m+1)}{2} + r, \ 0 < r \leq m \), then \( \left\lfloor \sqrt{1+8n-1} \right\rfloor = m \). The ideal \( I = (x^m, x^{m-1}y, \ldots, xy^{m-1}, y^m+r) \) has co-length \( n \) and \( m+1 \) minimal generators, hence the equality is achieved by \( I \). \( \square \)

References

[1] Jan Cheah, On the cohomology of Hilbert schemes of points, Journal of Algebraic Geometry 5 (1996), no. 3, 479–512.
[2] David Eisenbud, Commutative algebra: with a view toward algebraic geometry, vol. 150, Springer, 1995.
[3] , The geometry of syzygies: A second course in algebraic geometry and commutative algebra, vol. 229, Springer, 2005.
[4] Geir Ellingsrud and Manfred Lehn, Irreducibility of the punctual quotient scheme of a surface, Arkiv för Matematik 37 (1999), no. 2, 245–254.
[5] Geir Ellingsrud and Stein Strømme, An intersection number for the punctual Hilbert scheme of a surface, Transactions of the American Mathematical Society 350 (1998), no. 6, 2547–2552.
[6] John Fogarty, Algebraic families on an algebraic surface, American Journal of Mathematics 90 (1968), no. 2, 511–521.
[7] , Algebraic families on an algebraic surface, American Journal of Mathematics 90 (1968), no. 2, 511–521.
[8] , Algebraic families on an algebraic surface, ii, the Picard scheme of the punctual Hilbert scheme, American Journal of Mathematics 95 (1973), no. 3, 660–687.
[9] W. Fulton, Intersection theory, vol. 2, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), Springer-Verlag, Berlin, 1998.
[10] Mark Haiman, Hilbert schemes, polygraphs and the Macdonald positivity conjecture, Journal of the American Mathematical Society 14 (2001), no. 4, 941–1006.
[11] AS Tikhomirov, On Hilbert schemes and flag varieties of points on algebraic surfaces, preprint (1992).

Department of Mathematics, Statistics and Computer Science, University of Illinois at Chicago, Chicago, IL 60607

E-mail address: lsong4@uic.edu