Diagrams of Persistence Modules Over Finite Posets

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Abstract

Starting with a persistence module – a functor \(M : P \to \text{Vec}_k\) for some finite poset \(P\) – we seek to assign to \(M\) an invariant capturing meaningful information about the persistence module. This is often accomplished via applying a Möbius inversion to the rank function or birth-death function. In this paper we establish the relationship between the rank function and birth-death function by introducing a new invariant: the kernel function. The persistence diagram produced by the kernel function is equal to the diagram produced by the birth-death function off the diagonal and we prove a formula for converting between the persistence diagrams of the rank function and kernel function.

1 Introduction

The persistence modules we consider here were introduced in [1] as a generalization of 1-parameter persistence modules. Explicitly, in this paper, a persistence module is a functor \(M : P \to \text{Vec}_k\) where \(P\) is any finite poset. The main goal of multiparameter persistence is to assign to \(M\) a persistence diagram that is efficiently computable and carries meaningful information about the structure of \(M\). This has been accomplished with invariants such as the rank function [2, 5] and the birth-death function [3, 6].

Here we extend the birth-death function used in [6] from filtrations to persistence modules by using free presentations of persistence modules. This pipeline is functorial: a morphism between persistence modules induces a morphism between persistence diagrams. The edit distance stability theorem proved in [6] naturally extends as a direct result of functoriality.

In Section 5 we show that the persistence diagrams obtained from the birth-death function and the rank function are both related to the persistence diagram of a third invariant we introduce: the kernel function. We show that the persistence diagrams from the birth-death function and the kernel function are equal off the diagonal. We also prove a formula that allows one to translate between the persistence diagrams from the rank function and the kernel function.
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2 Preliminaries

Many of the results in this paper depend heavily on tools from order theory. In particular, we make extensive use of the M"obius inversion and Galois connections. Here we provide a brief introduction to these topics. For a more thorough overview of these topics, see [4].

2.1 Partially Ordered Sets

Definition 2.1: A partially ordered set, or poset, is a set \( P \) with a relation \( \leq \) satisfying

- Reflexivity: \( a \leq a \) for any \( a \in P \).
- Antisymmetry: if \( a \leq b \) and \( b \leq a \) then \( a = b \) for any \( a, b \in P \).
- Transitivity: if \( a \leq b \) and \( b \leq c \) then \( a \leq c \) for any \( a, b, c \in P \).

We assume that all of our posets \( P \) come equipped with a metric \( d_P \). The shortest path metric can always be placed on any poset but frequently the posets that arise in multiparameter persistence will have a more natural metric. See [6] for more details.

Example 2.2: A recurring example we use throughout this paper is the poset \( P = \{a, b, c, d\} \) with the partial order \( a \leq b, b \leq c \), and \( b \leq d \). See Figure 1 for the Hasse diagram of \( P \).

Definition 2.3: For any poset \( P \), a subposet \( A \subseteq P \) is an upset if \( a \in A \) and \( b \in P \) with \( a \leq b \) implies \( b \in A \). For any element \( a \in P \) we define the upset \( \uparrow a := \{ b \in P \mid a \leq b \} \).

The persistence diagram of a persistence module is a function capturing information that persists over intervals in a poset. In order to define the persistence diagram of a module, we first need to define posets of intervals. When studying kernel functions and birth-death
functions, it is useful to include intervals going off to infinity. To accomplish this, let $P \cup \{\infty\}$ be the poset with $a \leq \infty$ for any $a \in P$.

**Definition 2.4:** For any poset $P$, its **poset of intervals** is

$$\bar{P} = \{(a, b) \in P \times (P \cup \{\infty\}) \mid a \leq b\}$$

with the order $(a, b) \leq (c, d)$ if $a \leq c$ and $b \leq d$.

**Remark 2.5:** There is another poset of intervals sometimes used in persistent homology. This is the poset

$$\hat{P} = \{(a, b) \in P \times (P \cup \{\infty\}) \mid a \leq b\}$$

with the order $(a, b) \supseteq (c, d)$ if $a \leq c$ and $d \leq b$. This poset is the natural domain of the rank function.

### 2.2 Galois Connections

Galois connections play a fundamental role in Möbius inversion based persistence. They are the key component of morphisms between persistence modules as well as morphisms between persistence diagrams. For those familiar with category theory, Galois connections are adjoint functors between posets.

**Definition 2.6:** A **Galois connection** $f = (f_*, f^*)$ between two posets $P$ and $Q$ consists of order-preserving maps

$$P \xleftarrow{f_*} Q \quad Q \xrightarrow{f^*} P$$

satisfying $f_*(p) \leq q$ if and only if $p \leq f^*(q)$ for any $p \in P$ and $q \in Q$.

**Proposition 2.7:** If $f : P \to Q$ is a Galois connection then $f$ extends to a Galois connection $\bar{f} : \bar{P} \to \bar{Q}$ defined by $\bar{f}_* : (a, b) \mapsto (f_*(a), f_*(b))$ and $\bar{f}^* : (c, d) \mapsto (f^*(c), f^*(d))$.

**Proof.** The proposition follows from the following argument

$$\bar{f}_*(a, b) \leq (c, d) \iff (f_*(a), f_*(b)) \leq (c, d) \iff f_*(a) \leq c \text{ and } f_*(b) \leq d \iff a \leq f^*(c) \text{ and } b \leq f^*(d) \iff (a, b) \leq \bar{f}^*(c, d).$$

**Proposition 2.8:** If $f : P \to Q$ and $g : Q \to R$ are Galois connections then $g \circ f = (g_* \circ f_*, f^* \circ g^*) : P \to R$ is a Galois connection.

**Proof.** Let $p \in P$ and $r \in R$. Then

$$g_*(f_*(p)) \leq r \iff f_*(p) \leq g^*(r) \iff p \leq f^*(g^*(r)).$$
2.3 Möbius Inversion

The Möbius inversion is the core tool at the heart of this approach to persistence. It can be thought of as a combinatorial version of the derivative: for a function \( m : P \rightarrow \mathbb{Z} \), its Möbius inversion \( \partial m : P \rightarrow \mathbb{Z} \) captures where, and how, \( m \) changes. Here we cover only the necessary background for this paper and refer the reader to [7] for a more thorough overview.

**Definition 2.9:** Let \( P \) be a finite poset and \( m : P \rightarrow \mathbb{Z} \) be an integral function. The **Möbius inversion** of \( m \) is the unique function \( \partial m : P \rightarrow \mathbb{Z} \) satisfying

\[
m(b) = \sum_{a \leq b} \partial m(a)
\]

for all \( b \in P \). We refer to this equation as the Möbius inversion formula.

Every function \( m : P \rightarrow \mathbb{Z} \) can be identified with a vector in \( \mathbb{Z}^P \) in the natural way. Under this identification, the Möbius inversion of \( m \) can be calculated with a matrix multiplication. Let \( \zeta \) be the \( P \times P \)-indexed matrix defined by

\[
\zeta_{a, b} = \begin{cases} 1 & \text{if } a \leq b \\ 0 & \text{otherwise.} \end{cases}
\]

Multiplication by the \( \zeta \) matrix has the following effect on functions

\[
m\zeta(b) = \sum_{a \leq b} m(a). \tag{1}
\]

The Möbius inversion formula can now be restated as \( m = (\partial m)\zeta \). The matrix \( \zeta \) has an inverse matrix \( \mu \) [7]. Therefore the Möbius inversion is a linear map on the space \( \mathbb{Z}^P \), a property we will make use of.

**Definition 2.10:** Let \( P \) and \( Q \) be finite posets with a Galois connection \((f_*, f^*) : P \rightarrow Q\). If \( \ell : P \rightarrow \mathbb{Z} \), the **pushforward** of \( \ell \) along \( f_* \) is the function \( f_*\ell : Q \rightarrow \mathbb{Z} \) given by

\[
f_*\ell : q \mapsto \sum_{p \in f_*^{-1}(q)} \ell(p).
\]

The following lemma is the keystone of this paper; it shows how the Möbius inversion commutes with Galois connections. This immediately implies that the assignment of a persistence diagram to a persistence module is functorial which, in turn, implies a stability theorem [6].

**Lemma 2.11:** Let \( P \) and \( Q \) be finite posets with a Galois connection \((f_*, f^*) : P \rightarrow Q\). If \( m : P \rightarrow \mathbb{Z} \) and \( n : Q \rightarrow \mathbb{Z} \) are functions such that \( n = m \circ f^* \) then \( \partial n = f_*\partial m \).
Proof. Applying the Möbius inversion formula to both sides of the equation $n(q) = m(f^*(q))$ gives

$$\sum_{a \leq q} \partial n(a) = \sum_{p \leq f^*(q)} \partial m(p)$$

$$(\partial n) \zeta(q) = \sum_{p \in P, f_*(p) \leq q} \partial m(p)$$

$$= \sum_{a \leq q} \sum_{p \in f_*^{-1}(a)} \partial m(p)$$

$$= (f_* \partial m) \zeta(q)$$

for any $q \in Q$. Therefore $\partial n \zeta = (f_* \partial m) \zeta$ and, since $\zeta$ is invertible, $\partial n = f_* \partial m$ as desired. 

\[\Box\]

3 Persistence Modules

Here we introduce the category of persistence modules and free persistence modules.

**Definition 3.1:** A **persistence module** is a functor $M : P \to \text{Vec}_k$ for some finite poset $P$.

**Example 3.2:** For any poset $P$ and any upset $A \subseteq P$ we can define a persistence module $k^A : P \to \text{Vec}_k$ by

$$k^A(a) := \begin{cases} k & \text{if } a \in A \\ 0 & \text{otherwise} \end{cases}$$

and $k^A(a \leq b) = \text{id}_k$ if $a \in A$.

**Definition 3.3:** Let $M : P \to A$ and $N : Q \to A$ be persistence modules. A morphism $f : M \to N$ consists of a Galois connection $f_* : P \to Q$ and $f^* : Q \to P$ such that $M \circ f^* \cong N$. With this notion of morphism, persistence modules form a category $\text{PMod}$.

**Remark 3.4:** The definition of a morphism between persistence modules presented here is an extension of the definition in [6] as follows. Let $P$ and $Q$ be finite lattices, $M : P \to \text{Vec}_k$, and $N : P \to \text{Vec}_k$ be persistence modules. A morphism $\alpha : M \to N$ is a bounded lattice map $\alpha : P \to Q$ satisfying $M(q^*) \cong N(q)$ for all $q \in Q$ where $q^* = \max \alpha^{-1}([\bot, q])$. Every bounded lattice map $\alpha : P \to Q$ has a right adjoint $\alpha^* : Q \to P$ defined by $\alpha^* : q \to q^*$. The morphism condition now reduces to $M \circ \alpha^* \cong N$.

**Example 3.5:** Figure 2 shows a persistence module $M : P \to \text{Vec}_k$ where $P$ is the poset from Example 2.2 and the maps $\pi$ and $\theta$ are

$$\pi : (x, y, z, w) \mapsto (x, y, z)$$

$$\theta : (x, y, z) \mapsto x.$$
3.1 Free Persistence Modules

Free persistence modules are necessary to extend the birth-death function from filtrations to persistence modules. Here we introduce free persistence modules and show that every persistence module admits a free presentation.

**Definition 3.6**: A persistence module \( F : P \to \text{Vec}_k \) is **free** if it is a finite direct sum of persistence modules of the form \( k^A \) for some upsets \( A \subseteq P \). A **free presentation** of a persistence module \( M : P \to \text{Vec}_k \) consists of a free persistence module \( F : P \to \text{Vec}_k \) and a surjective natural transformation \( \varphi : F \to M \).

**Example 3.7**: Figure 3 shows a free presentation of the persistence module from Example 3.5. The natural transformation \( \varphi : F \to M \) is given by \( \varphi_a : (x_1, \ldots, x_5) \mapsto (x_1, \ldots, x_4) \) and \( \varphi_p = M(a \leq p) \circ \varphi_a \) for any other \( p \in P \).

**Proposition 3.8**: For any persistence module \( M : P \to \text{Vec}_k \) there is a free persistence module \( F : P \to \text{Vec}_k \) with a surjective natural transformation \( \varphi : F \to M \).

**Proof.** For any \( a \in P \), let \( n_a = \dim M(a) \) and \( F_a : P \to \text{Vec}_k \) be the free persistence \( F_a = \bigoplus_{i=1}^{n_a} k^{a_i} \).

Let \( \varphi_a(a) : F_a(a) \to M(a) \) be an isomorphism. Then \( \varphi_a(a) \) extends to a natural transformation \( \varphi_a : F_a \to M \) by

\[
\varphi_a(b) = \begin{cases} 
M(a \leq b) \circ \varphi_a(a) & \text{if } a \leq b \\
0 & \text{otherwise}.
\end{cases}
\]

We can construct \( F \) by \( F = \bigoplus_{a \in P} F_a \) and extend \( \{ \varphi_a \}_{a \in P} \) to a natural transformation \( \varphi : F \to M \) analogously. Each \( \varphi_a \) is surjective at \( a \) so \( \varphi \) is surjective everywhere.

**Remark 3.9**: There are often far smaller free presentations than the one constructed in the proof of Proposition 3.8. This only matters if one desires to compute persistence diagram with free presentations. However, if one is starting with a persistence module then it is easier to compute the persistence diagram with the kernel function instead. The kernel function sidesteps this issue and produces an equivalent persistence diagram; see Section 5.
4 Towards Persistence Diagrams

Persistence diagrams are typically constructed via an intermediary function such as the rank function or the birth-death function. Here we define a new, but closely related, option: the kernel function.

4.1 Kernel Functions

The kernel function of a persistence module captures what dies over an interval. It follows from the rank-nullity theorem that the kernel function carries the same information as the rank function; see Remark 4.7. Additionally, the kernel function produces a persistence diagram that is isomorphic to the persistence diagram defined by the birth-death function. In this way the kernel function acts as a bridge linking the rank function and the birth-death function.

**Definition 4.1:** For any persistence module $M : P \to \text{Vec}_k$ its **kernel function** is the function $\text{ker}_M : \bar{P} \to \mathbb{Z}$ given by

$$\text{ker}_M : (a, b) \mapsto \begin{cases} \dim (\ker M(a \leq b)) & \text{if } b \neq \infty \\ \dim (M(a)) & \text{if } b = \infty. \end{cases}$$

**Proposition 4.2:** If $M : P \to \text{Vec}_k$ and $N : Q \to \text{Vec}_k$ are persistence modules with a morphism $f : M \to N$ then $\text{ker}_M \circ f^* = \text{ker}_N$.

**Proof.** For any $(a, b) \in \bar{Q}$, we have the commutative diagram

$$
\begin{array}{c}
M(f^*(b)) \\
M(f^*(a) \leq f^*(b))
\end{array} \xleftarrow{=} \begin{array}{c}
N(b) \\
N(a \leq b)
\end{array}
\begin{array}{c}
M(f^*(a)) \\
M(f^*(a) \leq f^*(b))
\end{array} \xrightarrow{=} \begin{array}{c}
N(a)
\end{array}
$$

which implies $\text{ker}_M \circ f^*(a, b) = \text{ker}_N(a, b)$. \qed

**Example 4.3:** The kernel function of the persistence module depicted in Example 3.5 is shown in Table 1.
Table 1: The kernel function of the persistence module $M$ from Example 3.5. Only the intervals in the support of $\ker M$ are shown.

| Interval | $\ker M$ |
|----------|---------|
| $[a, b]$ | 1       |
| $[a, c]$ | 3       |
| $[a, d]$ | 4       |
| $[a, \infty]$ | 4     |
| $[b, c]$ | 2       |
| $[b, d]$ | 3       |
| $[b, \infty]$ | 3     |
| $[c, \infty]$ | 1     |

### 4.2 Birth-Death Functions

The birth-death functions used in [6] does not immediately extend to persistence modules. There the birth-death function is defined using the cycles and boundaries of a filtration. This information is lost when homology is applied to a filtration to obtain a persistence module. We overcome this by defining the birth-death function using free presentations. Since free presentations are not unique, birth-death functions of a persistence module also fail to be unique. However, after applying the Möbius inversion, different free presentations of a persistence module lead to isomorphic persistence diagrams – they differ only along the diagonal; see Corollary 5.5.

**Definition 4.4:** Let $M : \mathcal{P} \to \text{Vec}$ be a persistence module with a free presentation $\varphi : F \to M$. The **birth-death function** of $M$ induced by $\varphi$ is the map $m_{\varphi} : \overline{\mathcal{P}} \to \mathbb{Z}$ defined by

$$m_{\varphi} : (a, b) \mapsto \dim \left( F(a) \cap \ker(\varphi_{b}) \right)$$

where the intersection is given by the pullback

$$\begin{array}{ccc}
F(a) & \xrightarrow{\varphi_{a}} & F(b) \\
\downarrow & & \downarrow \\
F(a) \cap \ker(\varphi_{b}) & \longrightarrow & \ker(\varphi_{b}).
\end{array}$$

The following lemma relates the kernel function with birth-death functions.

**Lemma 4.5:** Let $M : \mathcal{P} \to \text{Vec}_{k}$ be a persistence module with a free presentation $\varphi : F \to M$. Then

$$\ker M(a, b) = m_{\varphi}(a, b) - m_{\varphi}(a, a)$$

for any $(a, b) \in \overline{\mathcal{P}}$.

*Proof.* Suppose $b \neq \infty$. The commutative diagram
Table 2: The birth-death function of the persistence module $M$ from Example 3.5 induced by the free presentation in Example 3.7. Only the non-zero rows are shown.

| Interval  | $m_\varphi$ | Interval | $m_\varphi$ |
|-----------|-------------|----------|-------------|
| $[a, a]$  | 1           | $[b, d]$ | 5           |
| $[a, b]$  | 2           | $[b, \infty]$ | 5 |
| $[a, c]$  | 4           | $[c, c]$ | 4           |
| $[a, d]$  | 5           | $[d, d]$ | 5           |
| $[a, \infty]$ | 5       | $[c, \infty]$ | 5 |
| $[b, b]$  | 2           | $[d, \infty]$ | 5 |
| $[b, c]$  | 4           |          |             |

\[
\begin{align*}
F(a) & \xrightarrow{F(a \leq b)} F(b) \\
\varphi_a & \downarrow \quad \downarrow \varphi_b \\
M(a) & \xrightarrow{M(a \leq b)} M(b)
\end{align*}
\]

implies

\[
\ker(M(a \leq b)) = \frac{F(a) \cap \ker(\varphi_b)}{\ker(\varphi_a)}
\]

and so

\[
\ker_M(a, b) = \dim(F(a) \cap \ker(\varphi_b)) - \dim(F(a) \cap \ker(\varphi_a)) = m_\varphi(a, b) - m_\varphi(a, a).
\]

If $b = \infty$ then

\[
m_\varphi(a, \infty) - m_\varphi(a, a) = \dim F(a) - \dim \ker(\varphi_a) = \dim \frac{F(a)}{\ker(\varphi_a)} = \dim M(a) = \ker_M(a, \infty).
\]

4.3 Rank Functions

The typical way of constructing persistence diagrams is via the rank function. The disadvantage of the rank function is that its usual domain is the poset $\hat{P}$ from Remark 2.5 rather than $\bar{P}$. While a Galois connection $f_* : P \to Q$, $f^* : Q \to P$ induces a Galois connection between $\bar{P}$ and $\bar{Q}$, it fails to induce a Galois connection between $\hat{P}$ and $\hat{Q}$.

**Definition 4.6:** The **rank function** of a persistence module $M : P \to \text{Vec}_k$ is the map $r_k_M : \hat{P} \to \mathbb{Z}$ defined by

\[
r_k_M : (a, b) \mapsto \begin{cases} 
\dim(\text{im}(M(a \leq b))) & \text{if } b \neq \infty \\
0 & \text{if } b = \infty.
\end{cases}
\]
Table 3: The rank function of the persistence module $M$ from Example 3.5. Only the non-zero rows are shown.

| Interval | $rk_M$ |
|----------|--------|
| $[a, a]$ | 4 |
| $[a, b]$ | 3 |
| $[a, c]$ | 1 |
| $[b, b]$ | 3 |
| $[b, c]$ | 1 |
| $[c, c]$ | 1 |

Remark 4.7: For any persistence module $M$, its kernel function and rank function carry the same information. This follows from the rank-nullity theorem:

$$\dim M(a) = rk_M(a, b) + \ker M(a, b)$$

for any $a \leq b$. Using the fact that $\dim M(a) = rk_M(a, a) = \ker M(a, \infty)$ we get

$$rk_M(a, b) = \ker M(a, \infty) - \ker M(a, b) \quad \text{and} \quad \ker M(a, b) = rk_M(a, a) - rk_M(a, b).$$

5 Persistence Diagrams

In Section 4 we presented three different functions that can be used to obtain a persistence diagram. These functions each lead to their own notion of persistence diagram by applying the Möbius inversion. We show that the persistence diagrams obtained via the kernel function and birth-death function differ only along the diagonal. Additionally, there is a one-to-one correspondence between persistence diagrams obtained via the rank function and kernel function and we provide formulas for translating between them.

Definition 5.1: For a persistence module $M : P \rightarrow \text{Vec}$, there are three notions of persistence diagram we consider here. The first is the Möbius inversion of a birth-death function $\partial m_\varphi = m_\varphi \mu \leq$ with the product order. The second is the Möbius inversion of the rank function $\partial rk_M = rk_M \mu_2$ with the reverse containment order. The third is the Möbius inversion of the kernel function $\partial \ker M = \ker M \mu_\leq$ with the product order.

Each of these notions of persistence diagram has their own benefits. The diagram $\partial rk_M$ frequently has the smallest support of any of the diagrams while the diagrams $\partial m_\varphi$ and $\partial \ker M$ are stable with respect to the edit distance [6]. All three notions of persistence diagram are functions from $P \times (P \cup \{\infty\})$ to $\mathbb{Z}$ and therefore all three diagrams can be considered objects of the same category.

Definition 5.2: The category $\text{Dgm}$ of persistence diagrams has set functions $m : P \times (P \cup \infty) \rightarrow \mathbb{Z}$ over all finite posets $P$ as objects. A morphism between two diagrams $m : $
Table 4: The persistence diagrams of the persistence module $M$ from Example 3.5 computed with the kernel function, birth-death function, and rank function. Only the non-zero rows are shown.

| Interval | $\partial \ker M$ | $\partial m_\varphi$ | $\partial r_k M$ |
|----------|------------------|------------------|------------------|
| $[a, a]$ | 0                | 1                | 1                |
| $[a, b]$ | 1                | 1                | 2                |
| $[a, c]$ | 2                | 2                | 1                |
| $[a, d]$ | 3                | 3                | 0                |
| $[a, \infty]$ | -2         | -2               | 0                |
| $[b, b]$ | -1               | 0                | 0                |
| $[c, c]$ | -2               | 0                | 0                |
| $[d, d]$ | -3               | 0                | 0                |

$P \times (P \cup \infty) \to \mathbb{Z}$ and $n : Q \times (Q \cup \infty) \to \mathbb{Z}$ consists of a Galois connection $f_* : P \to Q$ and $f^* : Q \to P$ such that $n = f_\sharp m$. 

**Theorem 5.3:** The persistence diagrams obtained via the kernel function and the rank functions are related by the following formula

$$\partial r_k M = f_\sharp \partial \ker M \zeta \subseteq \mu_2 - \partial \ker M \zeta \subseteq \mu_2$$

where $f_\sharp$ is the pushforward of $f_* : \bar{P} \to \bar{P}$ defined by $f_* : (a, b) \mapsto (a, a)$.

**Proof.** We begin by establishing a Galois connection on $\bar{P}$. Let $f_* , f^* : \bar{P} \to \bar{P}$ be defined by $f_* : (a, b) \mapsto (a, a)$ and $f^* : (a, b) \mapsto (a, \infty)$. This is a Galois connection since $f_*(a, b) = (a, a) \leq (c, d)$ if and only if $(a, b) \leq (c, \infty) = f^*(c, d)$ for any $(a, b), (c, d) \in \bar{P}$.

Now from Remark 4.7 we have $r_k M = \ker M \circ f^* - \ker M$. Multiplying both sides on the right by $\mu_2$ gives

$$\partial r_k M = (\ker M \circ f^*) \mu_2 - \ker M \mu_2$$

$$= (\ker M \circ f^*) \mu_2 \leq \mu_2 - \ker M \mu_2$$

$$= (f_\sharp \partial \ker M) \mu_2$$

where $f_\sharp \partial \ker M \zeta \subseteq \mu_2 - (\partial \ker M) \zeta \subseteq \mu_2$.

**Theorem 5.4:** Let $M : P \to \text{Vec}_k$ be a persistence module with a free presentation $\varphi : F \to M$. Then $\partial m_\varphi$ and $\partial \ker M$ differ only along the diagonal.

**Proof.** By Lemma 4.5, $m_\varphi(a, b) - m_\varphi(a, a) = \ker M(a, b)$. Let $n : \bar{P} \to \mathbb{Z}$ be the function defined by $n : (a, b) \mapsto m_\varphi(a, a)$. Then $m_\varphi - n = \ker M$ and so the claim reduces to showing that $\partial n \equiv 0$.
Let $\ell : P \rightarrow \mathbb{Z}$ be the function $\ell : a \mapsto m_\varphi(a, a)$. Define $f_* : P \rightarrow \bar{P}$ by $f_*(a) := (a, a)$ and $f^* : \bar{P} \rightarrow P$ by $f^*(b, c) := b$. This is a Galois connection as $(a, a) \leq (b, c)$ if and only if $a \leq b$. Since $\ell \circ f^* = n$, by Lemma 2.11

$$\partial n(a, b) = f_* \partial \ell(a, b) = \sum_{c \in f_*^{-1}(a, b)} \partial \ell(c).$$

Now because $f_*^{-1}(a, b)$ is empty if $a \neq b$, it follows that $\partial n(a, b) = 0$ if $a \neq b$ and so $\partial n \cong 0$. \hfill \square

The following corollary is an immediate consequence of Theorem 5.4.

**Corollary 5.5:** If $M : P \rightarrow \text{Vec}_k$ is a persistence module with free presentations $\varphi : F \twoheadrightarrow M$ and $\psi : F \twoheadrightarrow M$ then $\partial m_\varphi \cong \partial m_\psi$. 
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