Dissipation in dynamos at low and high magnetic Prandtl numbers

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Using simulations of helically driven turbulence, it is shown that the ratio of kinetic to magnetic energy dissipation scales with the magnetic Prandtl number in power law fashion with an exponent of approximately 0.6. Over six orders of magnitude in the magnetic Prandtl number the magnetic field is found to be sustained by large-scale dynamo action of alpha-squared type. This work extends a similar finding for small magnetic Prandtl numbers to the regime of large magnetic Prandtl numbers. At large magnetic Prandtl numbers, most of the energy is dissipated viscously, lowering the amount of magnetic energy dissipation, which means that simulations can be performed at magnetic Reynolds numbers that are large compared to the usual limits imposed by a given resolution. This is analogous to an earlier finding that at small magnetic Prandtl numbers, most of the energy is dissipated resistively, lowering the amount of kinetic energy dissipation, so simulations can then be performed at much larger fluid Reynolds numbers than otherwise. The decrease in magnetic energy dissipation at large magnetic Prandtl numbers is discussed in the context of underluminous accretion found in some quasars.

1 Introduction

The magnetic fields in astrophysical bodies often have a pronounced large-scale component that is associated with large-scale dynamo action. Examples are the cyclic magnetic fields in late-type stars such as the Sun and the magnetic spirals in many galaxies, including even irregular galaxies; see Beck et al. (1996) for a review. In addition, all observed magnetic fields also have a significant small-scale component that may either be the result of turbulent motions distorting the large-scale field, or, alternatively, it could be the result of what is known as small-scale dynamo action (Cattaneo 1999).

Much of our knowledge about large-scale and small-scale dynamos has come from numerical simulations; see Brandenburg & Subramanian (2005) for a review. It is clear that, in order for simulations to approach an astrophysically interesting regime, one wants to make both the magnetic diffusivity and the kinematic viscosity as small as possible. This means that the magnetic and fluid Reynolds numbers should be as large as possible for a given numerical resolution, $N^3$. The relevant criterion for sufficient numerical resolution is that the kinetic and magnetic energy spectra should develop an exponentially decaying dissipative subrange at a wavenumber that is at least a factor of 10 below the Nyqquist frequency, $k_{Ny} = \pi N/L$. In practice, for example, with a simulation at a resolution of $512^3$ mesh points, one can hardly exceed values of the magnetic and fluid Reynolds number of about 500–700 (e.g., Brandenburg 2009). However, as will be discussed in more detail in this paper, this empirical constraint on the resolution really only applies if the ratio of magnetic and fluid Reynolds numbers is about unity. This ratio is also referred to as the magnetic Prandtl number, $Pr_M$, and there is hardly any system where this number is unity. In galaxies and galaxy clusters this number tends to be very large, while in stars and stellar accretion discs it is quite small. Also liquid metals used in laboratory experiments have small $Pr_M$. Therefore, much of what has been learnt from numerical simulations at $Pr_M \approx 1$ has to be re-examined in cases of low and high values of $Pr_M$.

The purpose of this paper is to focus on the relative importance of viscous and ohmic dissipation rates at different values of $Pr_M$. Often, viscous and ohmic dissipation are only treated “numerically” by making sure the code is stable. In such cases, viscosity and magnetic diffusivity are usually not even stated explicitly in the equations, suggesting that these terms are negligible and not important. This is of course not the case, as can be illustrated by considering the case of quasars that belong to the most luminous objects in the sky. The discovery of the first quasar, 3C 273, is nicely explained by Rhodes (1978) in a popular magazine. Indeed, 3C 273, has about $2 \times 10^{12}$ times the luminosity of the Sun and is indeed the brightest one in the sky. This quasar would not shine at all if it was not for the effect of microphysical viscosity that leads to viscous dissipation. But how important is viscous dissipation compared with ohmic dissipation? In order to address this problem we need to understand the effects of both viscosity and magnetic diffusivity in a turbulent system where the magnetic
field is self-sustained by dynamo action. In this paper we
review briefly some recent work on dynamos in the regime
of small $Pr_M$ and turn then to the investigation of large $Pr_M$.

2 Small magnetic Prandtl number dynamos

In the last 6 years the issue of low magnetic Prandtl
numbers, $Pr_M = \nu/\eta$, has become a frequently discussed topic
in the dynamo community. This is the regime where the
magnetic diffusivity $\eta$ is large compared with the kinematic
viscosity $\nu$. Already over a decade ago, Rogachevskii &
Kleerorin (1997) noticed that for small-scale dynamos the
critical value of the magnetic Reynolds number, $Re_M$, for
the onset of dynamo action should rise from a value around
35 at $Pr_M = 1$ to values around 400 for small values of
$Pr_M$. Here, $Re_M = u_{rms}/\eta k_1$ is defined with respect to the
wavenumber $k_1$ of the energy-carrying eddies and the
rms velocity, $u_{rms}$. However, the result of Rogachevskii &
Kleerorin was not widely recognized at the time. In 2004,
simulation began to address this point systematically. Sim-
ulations of Schekochihin et al. (2005) might have even sug-
gested those data, i.e. (Dobler et al. 2003)

$$E_{3D} = -dE_{1D}/d\ln k.$$  
(1)

Accepting thus the physical reality of the bottleneck effect, it
becomes plausible that the critical magnetic Reynolds number for the onset of small-scale dynamo action reaches
a maximum around $Pr_M = 0.1$, and that it decreases some-
what for smaller values of $Pr_M$. This is indeed what the sim-
ulations of Iskakov et al. (2007) suggest.

Let us now switch to large-scale dynamos. Their excit-
ation conditions are characterized by the dynamo number
which, for helical turbulence and in the absence of shear, is
just

$$C_\alpha = \frac{\alpha}{\eta \tau k_3} \approx \frac{\epsilon_f k_1}{k_1}.$$  
(2)

Here, $k_1 = 2\pi/L$ is the minimal wavenumber in the domain
of size $L$ and we have inserted standard approximations for
the $\alpha$ effect, $\alpha = 1/\tau \omega \cdot \mathbf{u}$, and the turbulent magnetic
diffusivity, $\eta_t = 1/\tau \omega^2$. Here, $\omega$, $\mathbf{U} - \mathbf{u}$ is the fluctuating velocity, i.e. the difference between the actual velocity $\mathbf{U}$ and the
mean velocity $\mathbf{U}$, $\tau \approx (u_{rms} k_1)^{-1}$ is the turnover time,
$\omega = \nabla \times \mathbf{u}$ is the fluctuating vorticity, $\epsilon_f \approx \omega \cdot \mathbf{u} / k_1^2$ is a measure for the relative helicity, and $\epsilon_f = 1 + 3/Re_M$
is a correction factor of order unity for sufficiently large
values of $Re_M$. It turns out that in all cases the spectra of
magnetic energy are at the largest scale approximately in-
dependent of $Re_M$ for $Pr_M$ between 1 and $10^{-3}$. This was
shown in Brandenburg (2009) and will here be extended to
$10 \leq Pr_M \leq 10^3$.

At larger wavenumbers there is a striking difference in the
magnetic energy spectra between $Pr_M = 1$ and $\ll 1$ in
that the resistive cutoff wavenumber moves toward smaller
values. At the same time, the kinetic energy spectrum be-
comes progressively steeper, leaving less kinetic energy to
dissipate. This has two important consequences. First of all,
the fractional kinetic energy dissipation decreases with de-
creasing $Pr_M$ proportional to $Pr_M^{-1/2}$ (Brandenburg 2009). On the other hand, the decrease of $\epsilon_k$ implies that the demand
for numerical resolution becomes less stringent. This, in
mean, means that one can increase the value of Re beyond
the normally established empirical limits. An important goal of
this present paper is the demonstration that the same is also
tru in the opposite limit of $Pr_M \gg 1$.

3 The model

Our model is similar to that presented in Brandenburg
(2001, 2009), where we solve the hydromagnetic equations
for velocity $U$, logarithmic density $\ln \rho$, and magnetic vector potential $A$ for an isothermal gas in the presence of an externally imposed helical forcing function $f$,

$$\frac{\partial U}{\partial t} = -U \cdot \nabla U - \varepsilon_A \nabla \ln \rho + f + (J \times B + \nabla \cdot 2\rho \mathbf{S})/\rho, \quad (3)$$

$$\frac{\partial \ln \rho}{\partial t} = -U \cdot \nabla \ln \rho - \nabla \cdot U, \quad (4)$$

$$\frac{\partial A}{\partial t} = U \times B - \mu_0 \eta J. \quad (5)$$

Here, $B = \nabla \times A$ is the magnetic field, $J = \nabla \times B/\mu_0$ is the current density, $\mu_0$ is the vacuum permeability, $c_s$ is the isothermal speed of sound, and $\mathbf{S}_{ij} = \frac{1}{2}(U_{i,j} + U_{j,i}) - \frac{i}{\pi} \delta_{ij} \nabla \cdot U$ is the traceless rate of strain tensor. We consider a triply periodic domain of size $L^3$, so the smallest wavenumber in the domain is $k_1 = 2\pi/L$. The forcing function consists of eigenfunctions of the curl operator with positive eigenvalues and is therefore fully helical with $f \cdot \nabla \times f = k f^2$, where $3.5 \leq k/k_1 \leq 4.5$ is the chosen wavenumber interval of the forcing function, whose average value is referred to as $k_f \approx 4 k_1$. The amplitude of $f$ is such that the Mach number is $u_{trans}/c_s \approx 0.1$, so compressive effects are negligible (Dobler et al. 2003). As in Brandenburg (2009), we choose as initial conditions a Beltrami field of low amplitude. The initial velocity is zero and the initial density is uniform with $\rho = \rho_0 = const$, so the volume-averaged density remains constant, i.e., $\langle \rho \rangle = \rho_0$.

In our simulations we change the values of magnetic and fluid Reynolds numbers,

$$R_{EM} = u_{trans}/\eta k_t, \quad R_e = u_{trans}/\nu k_t, \quad (6)$$

such that the ratio $R_{EM}/R_e = Pr_M$ has the desired value between $10^{-3}$ and $10^3$, and we monitor the resulting kinetic and magnetic energy dissipation rates per unit volume,

$$\epsilon_K = \langle 2\nu \mathbf{S}^2 \rangle, \quad \epsilon_M = \langle \eta \eta J^2 \rangle, \quad (7)$$

whose sum, $\epsilon_T = \epsilon_K + \epsilon_M$, will be used to define the fractional dissipation rates, $\bar{\epsilon}_K = \epsilon_K/\epsilon_T$ and $\bar{\epsilon}_M = \epsilon_M/\epsilon_T$.

We use the fully compressible PENCIL CODEootnote{http://www.pencil-code.googlecode.com} for all our calculations. We recall that, for the periodic boundary conditions under consideration, $\langle \mathbf{S}^2 \rangle = \langle \mathbf{W}^2 \rangle + \frac{1}{3} \langle (\nabla \cdot \mathbf{U})^2 \rangle$, highlighting thus the analogy between $W = \nabla \times U$ and $J$ in the incompressible case.

## 4 Results

In Table 1 we summarize the parameters of runs with $Pr_M$ between $10^{-3}$ and $10^3$. The runs with $10^{-2} \leq Pr_M \leq 1$ are those presented already in Brandenburg (2009) using $512^3$ mesh points, while those with $10 \leq Pr_M \leq 1000$ are new ones and have been performed using $256^3$ mesh points. In all cases, either Re or Re$_{Em}$ were close to the maximum possible limit at a given resolution. Indeed, for $Pr_M = 10^{-3}$ we were able to reach Re = 4400 (for $512^3$ mesh points) while for $Pr_M = 10^3$ we could go to Re$_{Em}$ = 1200 (for $256^3$ mesh points).

![Visualization of $U_y$ and $B_y$ on the periphery of the computational domain for $Pr_M$ ranging from 10 to 1000 at a resolution of $256^3$ mesh points.](image1)

We note that in all cases the total energy dissipation is approximately the same. This is perhaps not so surprising, because we keep the amplitude of the forcing function the same. However, the constancy of the energy dissipation rate implies that the rate of energy injection must also be always the same and thus independent of $Pr_M$. This means that the flow properties of the eddies at the energy-carrying scale must be essentially independent of $Pr_M$.

In Fig. 1 we present visualizations of the $y$ component of velocity and magnetic field at the periphery of the computational domain for $Pr_M$ ranging from 10 to 1000 at a resolution of $256^3$ mesh points.

| $Pr_M$ | Re | Re$_{Em}$ | $\bar{\epsilon}_K$ | $\bar{\epsilon}_M$ | $k_K$ | $k_M$ | Res. |
|--------|-----|----------|---------------------|---------------------|-------|-------|------|
| $10^{-3}$ | 4400 | 4 | 0.01 | 0.99 | 426 | 8 | $512^3$ |
| $10^{-2}$ | 2325 | 23 | 0.04 | 0.96 | 344 | 25 | $512^3$ |
| $10^{-1}$ | 1175 | 118 | 0.13 | 0.87 | 286 | 81 | $512^3$ |
| $10^0$ | 455 | 455 | 0.39 | 0.61 | 179 | 201 | $512^3$ |
| $10^1$ | 20 | 200 | 0.76 | 0.24 | 24 | 99 | $256^3$ |
| $10^2$ | 9 | 850 | 0.90 | 0.10 | 14 | 263 | $256^3$ |
| $10^3$ | 0 | 425 | 0.99 | 0.01 | 3 | 129 | $256^3$ |
| $10^4$ | 1 | 1175 | 0.99 | 0.01 | 5 | 234 | $256^3$ |

Table 1 Summary of input and output parameters for the runs reported in this paper.

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There are compensated by in the saturated regime for Pr.
Compensated kinetic and magnetic energy spectra
Fig. 2

\[ \mathbf{E} = (\cos k_1 z, \sin k_1 z, 0) \]
although its wavevector could have pointed in any of the other two coordinate directions,
\((0, \cos k_1 x, \sin k_1 x)\) and \((\sin k_1 y, 0, \cos k_1 y)\) would have been equally probably alternatives. We recall that all these fields are indeed the eigenfunctions of an \(a^2\) dynamo problem (e.g., Brandenburg & Subramanian 2005), and they also emerge as the dominant field in helically driven turbulence.

It is clear that in a triply periodic domain such as that considered here, these fields require a resistive time to reach full saturation. For all further details we refer to Brandenburg (2001), where such a system was studied in full detail.

Next, we consider the spectra of kinetic and magnetic energies in Fig. 2 which are normalized such that

\[ \int E(k) \, dk = \frac{1}{2} \rho U^2 \]

and

\[ \int M(k) \, dk = \frac{1}{2} (B^2 / \mu_0) \]

It is evident from the spectra that with increasing values of \(Pr_M\), the viscous dissipation wavenumber, \(k_\nu = (\epsilon_K / \nu^2)^{1/4}\), moves to smaller and smaller values. Analagously to the case of \(Pr_M \ll 1\), this implies that most of the injected energy gets dissipated by the shorter of the two cascades – leaving only a reduced amount of energy for the other cascade. This means that the corresponding diffusion coefficient can be decreased further, without creating numerical difficulties.

It appears that it is not only the energy input at the small wavenumber end of the relevant cascade that is decreased, but that there is possibly a continuous removal of energy along the cascade, making the spectral index slightly steeper than \(-5/3\). For example, for \(Pr_M = 10^{-3}\) the spectral slope of \(E(k)\) is about \(-2/2\), while for \(Pr_M = 10^3\) the spectral slope of \(M(k)\) is about \(-2/0\).

It is quite extraordinary that in all these cases the nature of the large-scale dynamo is virtually unchanged, even though \(Pr_M\) is varied by 6 orders of magnitude. The reason is that in all cases the dynamo number, \(C_\alpha\), exceeds the critical value for dynamo action, \(C_\alpha^{crit} = 1\). Looking at Eq. (2), we see that \(C_\alpha\) is dominated by the scale separation ratio, which is here \(k_1 / k_2 \approx 4\). Furthermore, because the turbulence is nearly fully helical, we have \(\epsilon_t \approx 1\), and since \(Re_M \gg 1\), we have \(\epsilon \approx 1\). Thus, we have \(C_\alpha > 1\) for all runs. We recall also that the saturation amplitude of the field is essential given by the square root of the scale separation ratio (Brandenburg 2001), which is about 2 in units of the equipartition field strength. This is in reasonable agreement with the simulation results; see Fig. 2 where we show the resulting spectra for all the runs.

Next, we plot in Fig. 3 the ratio of kinetic to magnetic energy dissipation rates. In agreement with Brandenburg (2009), we find that the ratio is approximately proportional to \(Pr_M^{1/4}\), although a better fit is now provided by \(\epsilon_K / \epsilon_M \approx 0.6 Pr_M^{0.6}\). The reason for such a scaling is unclear. However, from Eq. (7) one can see that in the ratio \(\epsilon_K / \epsilon_M\) there is an implicit proportionality with respect to computational domain for the new results with \(Pr_M \geq 10\) and in Fig. 2 we show spectra of kinetic and magnetic energies, \(E(k)\) and \(M(k)\), respectively, for all values of \(Pr_M\) between \(10^{-2}\) and \(10^3\). In the velocity pattern one can clearly make out the typical scale of the dominant eddies, whose wave length is about 1/4 of the size of the box.

The magnetic field also shows a turbulent component, but there is a much stronger large-scale component superposed. This is essentially the Beltrami field which is of the form

\[ B = (\cos k_1 z, \sin k_1 z, 0) \]

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we see that we compare (Pearson et al. 2004). In the second and third panels of Fig. 4 Dependence of the ratio of the dissipation rates on Pr$_M$. Assuming, for simplicity, $\langle 2S^2 \rangle \approx \langle W^2 \rangle \approx W_{\text{rms}}^2$, we see that $\frac{\epsilon_K}{\epsilon_M} \approx \rho \frac{\nu}{\eta} \frac{W_{\text{rms}}^2}{J_{\text{rms}}} \propto \text{Pr}_M^n$, so

$$W_{\text{rms}} \propto \text{Pr}_M^{(n-1)/2} \approx \text{Pr}_M^{-1/4} \ldots \text{Pr}_M^{-1/6},$$

where we have assumed that $n$ lies between 1/2 and 2/3, which bracket the results seen here and in Brandenburg (2009). These scalings are surprising in view of the usually expected individual scalings, namely $W_{\text{rms}} \propto \nu^{-1/2}$ and $J_{\text{rms}} \propto \eta^{-1/2}$ (cf. Brandenburg & Subramanian 2005).

In order to illuminate the issue further, we ask whether not only the ratio $\epsilon_K/\epsilon_M$ scales with Pr$_M$, but whether $\epsilon_K$ and $\epsilon_M$ are individually proportional to Re and Re$_M$, respectively. In Fig. 4 we plot $\epsilon_K$ versus Re (blue, solid symbols) and $\epsilon_M$ versus Re$_M$ (red, open symbols). The scatter is now much larger than in Fig. 3 and it seems that the scaling exponent might even be as large as $n = 2/3$. We mentioned earlier that the total dissipation rate, $\epsilon_T$, is nearly independent of Pr$_M$. However, this is only true when we look the the dimensional value of $\epsilon_T$. It is customary to consider the normalized dissipation rate,

$$C_\epsilon = \frac{\epsilon_T}{u_{1D}/L},$$

where $u_{1D} = u_{\text{rms}}/\sqrt{3}$ is the one-dimensional rms velocity and $L = 3\pi/4k_1$ is conventionally used as the integral scale (Pearson et al. 2004). In the second and third panels of Fig. 4 we compare $C_\epsilon$ with $C_{\epsilon 0}$, which is based on the maximum value of $u_{1D}$ in all the runs. The difference is caused by the fact that $u_{\text{rms}}$ drops to rather low values in the large-Pr$_M$ regime. Part of this goes into magnetic energy, but it is not enough to make up for this difference.

It is important to realize that, on average, $\epsilon_M$ is just the same as the rate of work done against the Lorentz force, $-(U \cdot (J \times B))$. This becomes evident when considering the flow of energy in our system:

$$\langle \rho U \cdot f \rangle \rightarrow \{-U \cdot (J \times B)\} \rightarrow \langle 2\rho u S^2 \rangle \rightarrow \langle \eta \mu_0 J^2 \rangle.$$

Here, $\langle \rho U \cdot f \rangle \approx \epsilon_T$ is the rate of energy injection into the system by the forcing term. Normally, in the hydrodynamic case, $\langle 2\rho u S^2 \rangle$, or $\langle \nu W^2 \rangle$ in the incompressible case, stay constant as $\nu$ is decreased. In the case with dynamo action, however, a decrease in $\nu$ allows the dynamo to tap more energy, so $-(U \cdot (J \times B))$ and $\epsilon_M$ increase at the expense of $\epsilon_K$. This is indicated by the fact $\epsilon_K/\epsilon_M$ is found to be proportional to $(\nu/\eta)^n$, so $\epsilon_K$ decreases as $\nu$ decreases. This decrease is weak in the sense that $n \approx 1/2 \ldots 2/3$ is less than unity, but it is certainly no longer independent of $\nu$ as it would be in the purely hydrodynamic case.

In view of the application to quasars, i.e. accretion discs in active galactic nuclei, it is relevant to consider the fraction of energy that goes into the heating of electrons. Indeed, such discs are known to be underluminous, which led to the standard paradigm of advection-dominated accretion (Narayan & Yi 1994; Abramowicz et al. 1995). Alterna-
tively, this might be associated with the small value of the ratio $\epsilon_M / \epsilon_T$, for which we find

$$\frac{\epsilon_M}{\epsilon_T} = \frac{\epsilon_M}{\epsilon_M + \epsilon_K} \approx \frac{1}{1 + \Pr_M^2}. \quad (12)$$

Using standard accretion disc theory, Balbus & Henri (2008) find that $\Pr_M$ depends on the distance $R$ from the black hole and is proportional to $R^{-3/8}$. In particular, they find that $\Pr_M$ exceeds unity within about 50 Schwarzschild radii. This would dramatically decrease $\epsilon_M$ in the inner parts and might be sufficient to explain underluminous accretion. However, this proposal hinges on several assumptions: (i) that the viscous heating heats the ions and not the electrons, (ii) that the resistive dissipation energizes electrons rather than ions, (iii) that the discs are essentially collisionless and, finally, (iv) that the magnetohydrodynamic approximation is then still applicable.

5 Conclusions

The present work has shown that the ratio of kinetic to magnetic energy dissipation follows one and the same relationship with $\Pr_M$ both for small and large values. An important additional condition obeyed by all our runs is, however, that the magnetic Reynolds number is large enough for dynamo action to occur. This constitutes an important difference between our current results for large-scale dynamos and those mentioned in the first section for small-scale dynamos. An important consequence of such scaling is the fact that at extreme values of the magnetic Prandtl number, larger Reynolds numbers can be tolerated by the numerical scheme at a resolution that would be insufficient if the magnetic Prandtl number were unity. This was shown previously for $\Pr_M = 10^{-3}$, in which case fluid Reynolds numbers of 4500 were possible at a resolution of 512$^3$ meshpoints, while for $\Pr_M = 1$ it was only possible to reach Reynolds numbers of less than 700. Both cases obeyed the empirical constraint that the spectral kinetic energy has developed a clear dissipative subrange with an exponential decay shortly before the Nyquist frequency. In the opposite case of large $\Pr_M$, here $\Pr_M = 10^3$, it was possible to reach magnetic Reynolds numbers of 1000 at 256$^3$ mesh points. In this case the magnetic energy spectrum has developed a dissipative subrange shortly before the Nyquist frequency, although it was less convincing for $\Pr_M = 10^2$.

The reason for the value of the exponent $n$ in the power law relation between the energy dissipation ratio $\epsilon_K/\epsilon_M$ and $\Pr_M$ remains unclear. At this point we cannot be certain that it is $n = 0.6$ and not, for example, 1/2 or 2/3. One source of error might come from the fact that at extreme values of $\Pr_M$ the effects of numerical viscosity associated with the advection operator are no longer negligible. For the third-order time step used in the PENCIL CODE, the numerical viscosity operator takes the form $-\nu^{CFL}_{2} \nabla \cdot \nabla$ where $\nu^{CFL}_{2} = u_{\text{rms}} \delta x^3 C^{CFL}_{2}/24$ is a numerical hyperviscosity\(^2\) that depends on the mesh size $\delta x$ and the Courant–Friedrich–Levy number $C^{CFL}_{2}$, whose default value is 0.4, but the code would still be numerically stable for $C^{CFL}_{2} = 0.9$. If such numerical effects do begin to play a role, we must expect that the effective values of $\Pr_M$ are less extreme, which means that the $n$ would have been underestimated and that $n$ might be 2/3 or even larger.

While earlier work focussed on the dependence of $\epsilon_M$ on $\Pr_M$ (Blackman & Field 2008), no clear conclusion about the dissipation ratio $\epsilon_K/\epsilon_M$ seems to have emerged. For example, if $\epsilon_K$ and $\epsilon_M$ were independent of viscosity and magnetic diffusivity, the ratio $\epsilon_K/\epsilon_M$ would have been constant. Instead, we find that $\epsilon_K$ decreases when $\text{Re}$ decreases, and likewise, $\epsilon_M$ decreases when $\text{Re}_M$ decreases. On the other hand, one must be cautious when applying results regarding the dependence on $\text{Re}_M/\text{Re} (= \Pr_M)$ for large values of $\text{Re}$ and $\text{Re}_M$, because we may still not be in an asymptotic parameter regime. It is therefore important to extend this work to larger values of $\text{Re}$ and $\text{Re}_M$ and to go to larger numerical resolution.

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