A simple question about a complicated object

The complicated object is the cohomological induction functor (for which the biblical reference is Knapp & Vogan [KV]). Here is the simple question. For definiteness sake set $G := PU(n,1)$, $H := PU(k,1)$ with $k \leq n$, and let $\mathcal{H}_\rho^n$ (resp. $\mathcal{H}_\rho^k$) be the category of Harish-Chandra modules with the generalized infinitesimal character of the trivial module attached to $G$ (resp. $H$). I’ll surprise nobody by claiming that $H$ is a subgroup of $G$. What’s far less obvious, but proved by Khoroshkin [K], is the existence of a full embedding $F$ of $\mathcal{H}_\rho^k$ into $\mathcal{H}_\rho^n$ (Fuser [F,Thm I.4.2] showed that $F$ is even Ext-full, that is compatible with Ext calculus), prompting the question: is there a geometric interpretation of the embedding $F$? The first candidate for $F$ is the (ordinary) induction functor; but this fails miserably — so let’s break the pseudo-suspense of this introduction by saying that I claim that $F$ is (isomorphic to) a certain cohomological induction functor, and conjecture that this phenomenon is general.

1. Statements

Let $G$ be a center free connected semisimple Lie group, $K \subset G$ a maximal compact subgroup, $\mathfrak{g} \supset \mathfrak{k}$ the respective complexified Lie algebras. Let’s start by recalling the notion of Harish-Chandra module. Say that a $\mathfrak{g}$-module $V$ is $\mathfrak{k}$-finite if it is a sum of finite dimensional sub-$\mathfrak{k}$-modules, and that $V$ is an $(\mathfrak{g}, \mathfrak{k})$-module if it is $\mathfrak{k}$-finite and $\mathfrak{k}$-semisimple. The category $\mathcal{H} = \mathcal{H}(\mathfrak{g}, K)$ of Harish-Chandra modules is the full subcategory of $\mathfrak{g}$-mod whose objects are those $(\mathfrak{g}, \mathfrak{k})$-modules of finite length $V$ such that for any finite dimensional $\mathfrak{k}$-invariant subspace $F \subset V$ the action of $\mathfrak{k}$ on $F$ exponentiates to $K$. The category $\mathcal{H}$ is a $\mathbb{C}$-category in the sense of Bass [B] page 57. Let $I$ be the annihilator of the trivial module in the center of $U(\mathfrak{g})$, let

$$\mathcal{H}_\rho = \mathcal{H}_\rho(\mathfrak{g}, K)$$

be the full sub-$\mathbb{C}$-category of $\mathcal{H}$ whose objects are annihilated by some power of $I$, let $\mathcal{I}$ be the set of isomorphism classes of simple objects of $\mathcal{H}_\rho$ [it is a finite set] ; for
each $i \in I$ choose a representative $V_i \in i$ and let $\ell(i)$ be the projective dimension
of $V_i$ [i.e. the supremum in $\mathbb{Z} \cup \{+\infty\}$ of the set \{\(n \in \mathbb{Z} \mid \text{Ext}^n(V_i, -) \neq 0\}\].

(1) Definition. The \(H_\rho\)-ordering is the smallest partial ordering $\leq$ on $I$ satisfying

\[
\begin{aligned}
& i, j \in I \\
& \ell(j) = \ell(i) + 1 < \infty \\
& \text{Ext}^1(V_j, V_i) \neq 0
\end{aligned} \quad \Rightarrow \quad i \leq j.
\]

(2) Definition. The sub-\(\mathcal{C}\)-category generated by the subset $J$ of $H_\rho$ is the
full sub-\(\mathcal{C}\)-category $\langle J \rangle_{H_\rho}$ of $H_\rho$ characterized by the condition that an object
$V$ of $H_\rho$ belongs to $\langle J \rangle_{H_\rho}$ iff each simple subquotient of $V$ is isomorphic to $V_j$
for some $j \in J$.

(3) Definition. Say that a full sub-\(\mathcal{C}\)-category $C$ of $H_\rho$ is Ext-full in $H_\rho$ if for all
$V, W \in C$ the natural morphism

$$\text{Ext}^\bullet_C(V, W) \to \text{Ext}^\bullet_{H_\rho}(V, W)$$

is an isomorphism.

For $i \in I$ put $J_i := \{j \in I \mid j \leq i\}$, let $\theta$ be the Cartan involution of $(\mathfrak{g}, K)$, denote
by $d$ the dimension of $G/K$, and consider the following
(4) **Property of** $G$. For each $i \in I$ such that $V_i$ is unitary the cohomology $H^{d-\ell(i)}(g, K; V_i)$ is nonzero and there is a $\theta$-stable parabolic subalgebra of $g$ with Levi subgroup $L = L_i$ (see Vogan [V2,4.1,4.2] for definitions) satisfying

(a) the corresponding cohomological induction functor $F$ (see [KV]) sets up an equivalence

$$H_\rho(l, L \cap K) \overset{\sim}{\rightarrow} \langle J_i \rangle_{H_\rho(g, K)};$$

(b) $F \mathbb{C} \simeq V_i$;

(c) $\langle J_i \rangle_{H_\rho(g, K)}$ is Ext-full in $H_\rho(g, K)$;

(d) if $a$ is nonzero vector of $H^{d-\ell(i)}(g, K; V_i)$ and $V$ a simple object of $H_\rho(l, L \cap K)$, then the map

$$H^\bullet(l, L \cap K; V) \rightarrow H^{d-\ell(i)+\bullet}(g, K; FV)$$

$$x \mapsto F(x) \cup a$$

[where $\cup$ denotes the cup-product] is an isomorphism [of $H^\bullet(l, L \cap K; \mathbb{C})$-modules];

(e) we have $2 \ell(i) = d + \dim L/(L \cap K)$.

Note once and for all that (e) follows from (a) by the well known argument which consists in setting $V := \mathbb{C}$ and using Poincaré duality.

(6) **Conjecture.** All center free connected semisimple Lie groups have Property (4).

A partial proof (with explicitly indicated gaps) of the fact that $PU(n, 1)$, $PSpin(n, 1)$ and $SL(3, \mathbb{R})$ have Property (4) is contained in the expanded version of this text, downloadable from

http://www.iecn.u-nancy.fr/~gaillard/Recherche/Ci/ci.html
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