Consistent treatment for valence and nonvalence configurations in semileptonic weak decays

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Abstract

We discuss the semileptonic weak decays of $P \rightarrow P$ ($P$ denotes a pseudoscalar meson). In these timelike processes, the problem of the nonvalence contribution is solved systematically as well as the valence one. These contributions are related to the light-front quark model (LFQM), and the numerical results show the nonvalence contribution of the light-to-light transition is larger than of the heavy-to-light one. In addition, the relevant CKM matrix elements are calculated. They are consistent with the data of Particle Data Group.

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The study of exclusive semileptonic decays has attracted much interest for a long time. Heavy-to-heavy semileptonic decays, such as $B \rightarrow D l \nu$, provide an ideal testing ground for heavy-quark symmetry and heavy-quark effective theory (for a review, see [1]). On the other hand, heavy-to-light and light-to-light weak decays are much more complicated theoretically since there exists no guiding symmetry principle. Nevertheless, it is essential to understand the reaction mechanisms of these decay modes, because they are the main sources of information on the CKM mixing matrix between heavy and light quarks.

Hadronic matrix elements of weak $P \rightarrow P$ transition is described by two form factors. Phenomenologically, the hadronic form factors can be evaluated in various models, including the popular quark model. However, since usual quark-model wave functions best resemble meson states in the rest frame, or where the meson velocities are small, the form factors calculated in the non-relativistic quark model are therefore trustworthy only when the recoil momentum of the daughter meson relative to the parent meson is small. As the recoil momentum increases (corresponding to decreasing $q^2$), we have to consider relativistic effects seriously.

It is well known that the LFQM [2,3] is a relativistic quark model in which a consistent and fully relativistic treatment of quark spins and the center-of-mass motion can be carried out. This model has many advantages. For example, the light-front (LF) wave function is manifestly Lorentz invariant as it is expressed in terms of the momentum fraction variables (in “+” components) in analogy with the parton distributions in the infinite momentum frame. Moreover, hadron spin can also be correctly constructed using the so-called Melosh rotation. The kinematic subgroup of the LF formalism has the maximum number of interaction-free generators, including the boost operator which describes the center-of-mass motion of the bound state (for a review of LF dynamics, see [4]). The LFQM has been applied in the past to study the heavy-to-heavy and heavy-to-light weak decay form factors [5,6]. However, the weak form factors were calculated only for $q^2 \leq 0$ at the beginning, whereas physical decays occur in the time-like region $0 \leq q^2 \leq (M_i - M_f)^2$, with $M_i,f$ being the initial and final meson masses. Hence extra assumptions are needed to extrapolate the form factors to cover the entire range of momentum transfer [7,8]. Lately, the weak form factors for $P \rightarrow P$ transition were calculated in [9,10] for the first time for the entire range of $q^2$, so additional extrapolation assumptions are no longer required. This is based on the observation [11] that in the frame where the momentum transfer is purely longitudinal i.e. $q_\perp = 0, q^2 = q^+ q^-$ covers the entire range of momentum transfer. The price is that, besides the conventional valence-quark contribution, we must also consider the nonvalence configuration (or the so-called Z graph, see FIG. 1 (b)). The nonvalence contribution vanishes if $q^+ = 0$, but is supposed to be important for heavy-to-light transition near zero recoil [5,7,11,12]. Some methods for treating this nonvalence configuration exist: the authors of Ref. [10] considered the effective higher Fock state and calculated the effect in chiral perturbation theory. Ref. [13] follow a Schwinger-Dyson approach and related the nonvalence contribution to an ordinary LF wave function.

In this letter, we present a new way of handling the nonvalence contribution of $P \rightarrow P$ transition. For comparison, it will be instructive to analyze the known valence contribution in parallel. The main advantage of this way is that relativistic effects of the quark motion and spin are treated consistently in both valence and nonvalence configurations. We assume both normalization conditions of meson and quark states and a single interaction Hamiltonian to
obtain both the Melosh transformations of valence and nonvalence contributions. Combining these two contributions, we calculate completely the form factors of the semileptonic decay and the relevant CKM matrix elements.

We are interested in the matrix element which defines the weak form factors by

$$
\langle P'|Q'\gamma^{\mu}Q|P\rangle = f_+(q^2)(P + P')^\mu + f_-(q^2) q^\mu,
$$

(1)

where $q = P - P'$ is the momentum transfer. Assuming a vertex function $\Lambda_P$ [3] which is related to $Q\bar{q}$ bound state of $P$ meson, the quark-meson diagram depicted in FIG. 1 (a) yields

$$
\langle P'|Q'\gamma^{\mu}Q|P\rangle = - \int \frac{d^4p_1}{(2\pi)^4} \Lambda_P \Lambda_{P'} \text{Tr} \left[ \gamma_5 \frac{i(p_3 + m_3)}{p_3^2 - m_3^2 + i\epsilon} \gamma_5 \frac{i(p_2 + m_2)}{p_2^2 - m_2^2 + i\epsilon} \gamma^\mu \frac{i(p_1 + m_1)}{p_1^2 - m_1^2 + i\epsilon} \right],
$$

(2)

where $p_2 = p_1 - q$ and $p_3 = p_1 - P$. We consider the poles in denominators in terms of the LF coordinates $(p^-, p^+, p_\perp)$ and perform the integration over the LF “energy” $p^+_1$ in Eq. (2).

The result is then

$$
\langle P'|Q'\gamma^{\mu}Q|P\rangle = \int_0^q [d^3p_1] \frac{\Lambda_P}{p_3 - p_{3\text{on}}} (I^\mu_{p_{1\text{on}}}) \frac{\Lambda_{P'}}{p_3 - p_{3\text{on}} + p_2 - p_{2\text{on}}}
$$

$$
+ \int_0^q [d^3p_1] \frac{\Lambda_P}{p_1 - p_{1\text{on}}} (I^\mu_{p_{1\text{on}}}) \frac{\Lambda_{P'}}{p_2 - p_{2\text{on}}},
$$

(3)

where $i = 1, 2, 3$,

$$
[d^3p_1] = dp_1^+ dp_{1\perp}/(16\pi^3 \prod_i p_i^+) ,
$$

$$
I^\mu = \text{Tr} \left[ \gamma_5 (p_3 + m_3) \gamma_5 (p_2 + m_2) \gamma^\mu (p_1 + m_1) \right],
$$

$$
p_{3\text{on}}^- = m_1^2 + p_{1\perp}^2 + p_{1(3)}^+ = P_{3\text{on}}^- - p_{3(1)\text{on}}^-,
$$

(4)

and $p_{2\text{on}}^-$ equals respectively $p_{3\text{on}}^- - P_{3\text{on}}^-$ and $P_{3\text{on}}^- - p_{3\text{on}}$ in the first and second term of Eq. (3). It is worthwhile to mention every vertex function and its denominator corresponds exactly to the relevant meson bound state. This is clearer if we define $S_j \equiv p_j^- - p_{j\text{on}}^-$ and rewrite Eq. (3) in a more symmetrical form:

$$
\langle P'|Q'\gamma^{\mu}Q|P\rangle = \int_0^q [d^3p_1] \left[ \frac{\Lambda_P}{S_P + S_1 + S_3} I^\mu_{S_P + S_2 + S_3} \frac{\Lambda_{P'}}{S_P + S_1 + S_3} \right]_{S_{P,P',1}=0}
$$

$$
+ \int_q^P [d^3p_1] \left[ \frac{\Lambda_P}{S_P + S_1 + S_3} I^\mu_{S_P + S_2 + S_3} \frac{\Lambda_{P'}}{S_P + S_1 + S_3} \right]_{S_{P,P',3}=0}.
$$

(5)

In general, the integrals in Eq. (5) are divergent if we treat the vertices as pointlike. Internal structures for these vertices are therefore necessary. In the LFQM, the internal structure [14,15,17] consists of $\phi$ which describes the momentum distribution of the constituents in the bound state, and $R_{\lambda_1,\lambda_2}^{S,S_z}$ which creates a state of definite spin $(S, S_z)$ out of LF helicity $(\lambda_1, \lambda_2)$ eigenstates and is related to the Melosh transformation [10]. Here we adopt a convenient approach relating these two parts. The interaction Hamiltonian is assumed to be $H_I = i \int d^3x \bar{\Psi} \gamma_5 \Psi \Phi$ where $\Psi$ is quark field and $\Phi$ is meson field containing $\phi$ and $P_{\lambda_1,\lambda_2}^{S,S_z}$. On the one hand, if we normalize the meson state depicted in FIG.2 (a) as [14]
\[
\langle M(P', S', S'_z) | H_1 H_1 | M(P, S, S'_z) \rangle = 2(2\pi)^3 P^+ \delta^3(P' - P) \delta_{SS'} \delta_{S_z S'_z},
\]
and the valence wave function \(\phi^v\) as
\[
\int \frac{d^3 p_1}{2(2\pi)^3} \frac{1}{P^+} |\phi^v|^2 = 1,
\]
where \(p_1\) and \(p_2\) are the on-mass-shell momenta; the valence configuration of \(R_{\lambda_1, \lambda_2}^{S, S_z}\) is
\[
R_{1,2}^v = \frac{\sqrt{P^+ p_1^+ p_2^+}}{2\sqrt{p_{1on} \cdot p_{2on} + m_1 m_2}}.
\]
On the other hand, if we normalize the quark state depicted in FIG.2 (b) as
\[
\langle Q(p'_3, s') | H_1 H_1 | Q(p_3, s) \rangle = 2(2\pi)^3 \delta^3(p'_3 - p_3) \delta_{s's},
\]
and the nonvalence wave function \(\phi^n\) as
\[
\int \frac{d^3 p_2}{2(2\pi)^3} \frac{1}{P^+_3} |\phi^n|^2 = 1;
\]
the nonvalence configuration of \(R_{\lambda_1, \lambda_2}^{S, S_z}\) is
\[
R_{2,3}^n = \frac{\sqrt{P^+ p_3^+}}{2\sqrt{p_{2on} \cdot p_{3on} - m_2 m_3}}.
\]
After taking the “good” component \(\mu = +\), the wave function and the Melosh transformation of the meson are related to the bound state vertex function \(\Lambda_P\) by
\[
\begin{align*}
\frac{\Lambda_P}{P + S' + S_3} |_{S_{p, p', 1} = 0} & \to R_{1,3}^v \phi^v_P, \\
\frac{\Lambda_{P'}}{P' + S_2 + S_3} |_{S_{p, p', 1} = 0} & \to R_{2,3}^n \phi^n_{P'}.
\end{align*}
\]
In the trace of \(I^+, p_1, p_2,\) and \(p_3\) must be on the mass shell for self-consistency. Hence the matrix element in LFQM is
\[
\langle P' | \bar{Q} \gamma^+ Q | P \rangle = \int_0^q [d^3 p_1] \left[ R_{1,3}^v \phi^v_P (I^+) \right] R_{2,3}^n \phi^n_{P'} |_{S_{1,2,3} = 0} + \int_0^q [d^3 p_1] \left[ R_{1,3}^v \phi^v_P (I^+) \right] R_{2,3}^n \phi^n_{P'} |_{S_{1,2,3} = 0}.
\]
We use the definitions of the LF momentum variables \((x, x', k, k'_1)\) and take a Lorentz frame where \(P_\perp = P'_\perp = 0\) amounts to having \(q_\perp = 0\) and \(k'_1 = k_1\). So from Eq. \(13\) we obtain

4
\[ H(r) = \int \frac{d^2k_\perp}{(2\pi)^3} \left\{ \int_0^r dx_0 \phi_P(x, k_\perp)\phi_P(x', k_\perp) \frac{\mathcal{A}\mathcal{A'} + k_\perp^2}{\sqrt{\mathcal{A}^2 + k_\perp^2}} \right. \\
+ \left. \int_r^1 dx_1 \phi_P(x, k_\perp)\phi_P(x', k_\perp) \frac{\mathcal{A}\mathcal{A'} + k_\perp^2}{\sqrt{\mathcal{A}^2 + k_\perp^2}} \right\} \] (14)

where

\[ \langle P'|\bar{Q}'\gamma^+ Q|P \rangle = 2P^+H(r), \] (15)

\[ \mathcal{A} = m_1x + m_3(1 - x), \quad \text{and} \quad \mathcal{A'} = m_2x' + m_3(1 - x'). \]  

\( x \) (\( x' \)) is the momentum fraction carried by the spectator antiquark in the initial (final) state in the first term of (14). However, \( x' \geq 1 \) the second term of (14), which shows that the momentum \( p_3^+ \) of the spectator quark is larger than the \( P^+ \) of the final meson.

As explained above, we shall work in the frame where \( q_\perp = 0 \) so that \( q^2 \geq 0 \). Defining \( r \equiv P^+/P^+ \) gives \( q^2 = (1 - r)(M_P^2 - M_{P'}^2)/r \). Consequently, for a given \( q^2 \), the two solutions for \( r \) are given by

\[ r_\pm = \frac{1}{2M_P^2} [M_P^2 + M_{P'}^2 - q^2 \pm 2M_PQ(q^2)], \] (16)

where \( Q(q^2) = \sqrt{(M_P^2 + M_{P'}^2 - q^2)^2 - 4M_P^2M_{P'}^2}/2M_P \). The \( \pm \) signs in (16) correspond to the daughter meson recoiling in the \( \pm z \)-direction relative to the parent meson. The form factors \( f_\pm(q^2) \) of course should be independent of the reference frame chosen for the moving direction of the daughter meson. For a given \( q^2 \), it follows from (14) that

\[ f_\pm(q^2) = \pm \frac{(1 \mp r_-)H(r_+) - (1 \mp r_+)H(r_-)}{r_+ - r_-}. \] (17)

It is easily seen that \( f_\pm(q^2) \) are independent of the choice of reference frames, as it should be. The scalar form factor \( f_0(q^2) \) is related to \( f_\pm(q^2) \) by

\[ f_0(q^2) = f_+(q^2) + \frac{q^2}{M_P^2 - M_{P'}^2} f_-(q^2). \] (18)

The differential decay rate for \( P \rightarrow P \) is given by

\[ \frac{d\Gamma}{dq^2} = \frac{G_F^2}{24\pi^3} |V_{q_1q_2}|^2 Q(q^2)(1 - 2\hat{s})^2 \times \left\{ [Q(q^2)]^2 (1 + \hat{s})|f_+(q^2)|^2 + M_P^2 \left( 1 - \frac{M_{P'}^2}{M_P^2} \right)^2 \frac{3}{4} \hat{s}|f_0(q^2)|^2 \right\}, \] (19)

where \( G_F \) is the Fermi constant, \( \hat{s} = m_l^2/2q^2, \) \( m_l \) is the mass of lepton \( l \), and \( V_{q_1q_2} \) is the CKM matrix element.

In principle, the momentum distribution amplitude \( \phi(x, k_\perp) \) can be obtained by solving the LF QCD bound state equation \([4]\). However, before such first-principle solutions are available, we shall have to use phenomenological amplitudes. The simplest conjecture is related to the Melosh transformation effect; for example, \( \phi = N \exp[-(\mathcal{A}^2 + k_\perp^2)/(2\omega^2)], \)
where \( N \) is normalization constant and \( \omega \) is a scale parameter. However, the contributions of the end-point regions \((x \to 0, 1)\) for this wave function are nonvanishing. Here we make a slight modification to:

\[
\phi(x, k_\perp) = N[x(1-x)]^{1/n} \left[ \frac{\omega^2}{(A^2 + k_\perp^2) + \omega^2} \right]^n,
\]

where \( n \) is an integer. When \( n \) is large \((\sim 20)\), the form of this power-law wave function is almost the same as the previous exponential one except at the end-points. In addition, we do not treat \( n \) as a new parameter because the differences between wave functions for different large \( n \)'s are negligible. Thus the three parameters are \( m_1, m_2, \) and \( \omega \) in Eq. (20).

We can use Eqs. (17), (18), (14), and (20) to calculate the form factors of the processes \( K^0 \to \pi^\pm l^\mp \nu_l(K_{\ell\ell}^0) \) and \( D^0(B^0) \to \pi^- l^+ \nu_l \) which correspond to the light-to-light and heavy-to-light decay modes, respectively. On the one hand, the parameters appearing in the wave functions \( \phi^v_{K, \pi} \) are fixed by assuming the quark masses \( m_u = m_d \) and fitting to the experimental values of the decay constants \( f_{K, \pi} \) [17] and the charged radii \( \langle r^2 \rangle_{K^+, \pi^+} \) [18,19]. On the other hand, we determine the parameter \( \omega \) in \( \phi^n_{\pi} \) by fitting the data in Ref. [21] and treat it as universal among the other decay modes. As for the \( D \) and \( B \) mesons, the parameters are determined by assuming the quark masses \( m_c = 1.3 \text{ GeV}, m_b = 4.5 \text{ GeV} \) and fitting to the lattice QCD values of the decay constant \( f_{D,B} \) [20]. These parameters are as listed below (in units of GeV):

\[
\begin{align*}
  m_{u,d} & = 0.2, \quad m_s = 0.32, \quad m_c = 1.3, \quad m_b = 4.5, \quad \omega^n_{\pi} = 0.3, \\
  \omega^v_K & = 2.34, \quad \omega^v_{K^+} = 2.66, \quad \omega^n_B = 3.19, \quad \omega^n_B = 4.71.
\end{align*}
\]

The numerical results of the form factor \( f_+ \) for various decay modes are plotted in FIG. 3, 4, 5. From these figures, we easily find, for the same final meson, that the nonvalence contributions are smaller when the initial mesons are heavier. In addition, the nonvalence contribution is important for heavy-to-light transition near zero recoil \((q^2 \sim q_{\text{max}}^2)\). This result is consistent with the prediction in [5,7,11,12].

Finally, we can use the Eqs. (17), (18), (14) and the experimental data of the relevant decay rates [17] to calculate the three CKM matrix elements \( V_{us}, V_{cd}, \) and \( V_{ub} \). These values from this work and Ref. [17] are listed in Table I. The error bars in this work come from the uncertainties of the decay widths. We find these values are consistent with [17].

|          | \( V_{us} \)       | \( V_{cd} \)       | \( V_{ub} \)       |
|----------|---------------------|---------------------|---------------------|
| This work| 0.2179 ± 0.0016     | 0.247 ± 0.021       | 0.0037 ± 0.0007     |
| P.D.G.[17]| 0.2196 ± 0.0023     | 0.224 ± 0.016       | 0.0036 ± 0.0012     |

Table 1: The values of some CKM matrix elements in this work and Ref. [17].

In conclusion, a new treatment for the nonvalence configuration have been shown. We emphasize that the vertex functions correspond to LF valence and nonvalence wave functions exactly. The relativistic effects of the quark motion and spin were also treated consistently in both valence and nonvalence configurations. Therefore, we are able to calculate the form factors of the semileptonic decay completely. The numerical results showed the nonvalence contribution of the heavy-to-light transition is smaller than that of the light-to-light one. In addition, the CKM matrix elements evaluated from these form factors were consistent with the data in Particle Data Group.
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FIGURE CAPTIONS

Fig. 1  (a) The Feynman triangle diagram. (b) corresponds to the LF nonvalence configuration and diagram (c) to the valence one. Filled and empty circles incidate vertex functions and LF wave functions respectively.

Fig. 2  The Feynman diagrams of the self-energy of (a) meson and (b) quark.

Fig. 3  The normalized form factor $F_+$ for $K_{e3}^0$ decay compared with the experimental data [21]. The definition of $F_+$ is $f_+ (q^2)/f_+ (0)$.

Fig. 4  The form factor $f_+$ for $D^0 \to \pi^- l^+ \nu_l$ compared with the lattice QCD data [22].

Fig. 5  The form factor $f_+$ for $B^0 \to \pi^- l^+ \nu_l$ compared with the lattice QCD data [23].
FIGURES

FIG. 1.

\[
\begin{align*}
\text{(a)} & \quad W = p_1 p_2 p_3 \\
\text{(b)} & \quad = p_1 p_3 + \text{W} p_2 \\
\text{(c)} & \quad + p_2 p_3 p_1
\end{align*}
\]
FIG. 2.
FIG. 3.
FIG. 4.
FIG. 5.