INTRODUCTION

In any first order phase transition a quantity of fundamental importance is the rate at which bubbles of the equilibrium ("true vacuum") phase nucleate within the metastable ("false vacuum") phase. The bubble nucleation rate per unit volume, $\Gamma$, can be calculated by a method, due to Coleman, which is based on finding a "bounce" solution to the classical Euclidean field equations. Thus, for a theory with a single scalar field, one seeks a solution of

$$\partial_\mu \partial_\mu \phi \equiv \Box \phi = \frac{\partial V}{\partial \phi}$$

for which the $\phi$ is near its true vacuum value near the origin, but approaches its false vacuum value as any of the $x_\mu$ tend to $\pm \infty$.

A difficulty arises when one deals with theories where the symmetry breaking is a result of radiative corrections\textsuperscript{2} In such cases the vacuum structure is not determined by $V(\phi)$, but instead can be found only by examining the effective potential, $V_{\text{eff}}(\phi)$. The bounce equation (1) is clearly inappropriate — in fact, if $V(\phi)$ has only a single minimum there will not even be a bounce solution. An obvious solution is to replace $V(\phi)$ by $V_{\text{eff}}(\phi)$ in the bounce equation\textsuperscript{3}

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Although plausible, and clearly a step in the right direction, this procedure raises some questions. The one-loop radiative corrections generate an effective action which contains not only $V_{\text{eff}}$, but also terms involving derivatives (of all orders) of the fields. Can these terms be neglected when dealing with configurations, such as the bounce solution, which are not constant in (Euclidean) space-time? Even if this can be done in a first approximation, what are the nature and magnitude of the corrections which these terms generate? There are also questions relating to $V_{\text{eff}}$ itself. First, the effective potential obtained by perturbative calculations differs considerably from that defined by a Legendre transform (the latter must be convex, while the former is not). The latter clearly does not lead to an appropriate bounce, but how precisely does the formalism pick out the former? Further, the perturbative effective potential is known to be complex for certain values of the fields. How is the imaginary part of $V_{\text{eff}}$ to be handled?

In this talk I will describe a systematic calculational scheme\(^4\) which gives an answer to these questions. The general idea is to use the path integral approach of Callan and Coleman\(^5\) but to integrate out certain fields at the outset. This leads to a modified effective action that gives a correct description of the vacuum structure of the theory and has a bounce solution that can provide the basis for a tunneling calculation. To leading approximation, one obtains the same result for $\Gamma$ as would have been obtained by replacing $V$ by $V_{\text{eff}}$ in the standard procedure. The next-to-leading terms, which can be expressed in terms of the effective action, give calculable and significant corrections. The corrections beyond these, although well-defined, do not have a simple expression in terms of the effective action. In particular, the potentially complex terms in the effective potential do not appear directly, but only as part of more complicated functional determinants that can easily be shown to be real.

Throughout this talk I will confine myself to the case of bubble nucleation by quantum mechanical tunneling at zero temperature. Similar issues arise in connection with finite temperature bubble nucleation, even in theories where radiative corrections have little effect on the zero-temperature vacuum structure. It should be possible to deal with these by appropriate extensions of the methods described here.

**A SIMPLE MODEL**

Although radiative symmetry breaking is of greatest interest in the context of gauge theories, I will begin by using a somewhat simpler model to illustrate the method. This avoids some technical complications associated with gauge theories, which I discuss later. The model has two scalar fields, $\phi$ and $A$, and is governed by the Lagrangian

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi)^2 + \frac{1}{2} (\partial_\mu A)^2 - V(\phi, A)$$

with

$$V = \frac{1}{2} \mu^2 \phi^2 + \frac{\lambda}{4!} \phi^4 + \frac{1}{2} m^2 A^2 + \frac{f}{4!} A^4 + \frac{1}{2} g^2 \phi^2 A^2.$$  \(3\)

Both $\mu^2$ and $m^2$ are positive, so the tree-level potential has only a single symmetric minimum. We would like $A$-loop corrections to induce symmetry breaking with $\langle \phi \rangle = \sigma$ and
Figure 1. The two-loop graphs which contribute to the effective potential at order \( g^6 \). Solid and dashed lines represent \( \phi \) and \( A \) propagators, respectively.

\( \langle A \rangle = 0 \). This is done by choosing parameters so that \( \mu = O(g^2\sigma) \), \( m = O(g\sigma) \), \( f = O(g^2) \), and \( \lambda = O(g^4) \).

The one-particle-irreducible Green’s functions of this theory are generated by the effective action \( S_{\text{eff}} \), which can be expanded in the familiar derivative expansion

\[
S_{\text{eff}}(\phi, A) = \int d^4x \left[ V_{\text{eff}}(\phi, A) + \frac{1}{2}Z_\phi(\phi, A)(\partial_\mu \phi)^2 + \frac{1}{2}Z_A(\phi, A)(\partial_\mu A)^2 + \cdots \right]
\]

(4)

where the dots represent terms containing four or more derivatives. The leading contribution to the effective potential is of order \( g^4 \). It is given by the sum of the tree-level potential, all graphs with a single \( A \)-loop, and the corresponding counterterms and is equal to

\[
(V_{\text{eff}})_g^4(\phi) = \frac{1}{64\pi^2}(m^2 + g^2\phi^2)^2 \left[ \ln \left( \frac{m^2 + g^2\phi^2}{m^2 + g^2\sigma^2} \right) - \frac{1}{2} \right]
- \frac{1}{4} \left( \frac{\mu^2}{\sigma^2} + y_1 \right) (\phi^2 - \sigma^2)^2
\]

(5)

where \( y_1 \) is a constant of order \( g^2m^2/\sigma^2 \). The order \( g^6 \) contributions to \( V_{\text{eff}} \) arise from the two graphs shown in Fig. 1, together with several counterterm graphs. In these graphs the propagators are not simply those of the tree-level theory, but take into account the interaction with a background \( \phi \) field. Thus, the \( A \) propagator is given by

\[
G_A(k^2) = i(k^2 - m^2 - g^2\phi^2)^{-1}.
\]

(6)

The \( \phi \) propagator is a bit more complicated. Because the tree-level and one \( A \)-loop contributions are of the same magnitude, the effects of both must be included, giving

\[
G_\phi(k^2) = i \left[ k^2 - \mu^2 - (V_{\text{eff}})_g^4 \right]^{-1}.
\]

(7)

(The inclusion of one-loop effects in this propagator means that when calculating higher order corrections one must be careful to avoid double-counting of graphs; this will not be relevant for the calculations discussed here.) At order \( g^8 \) there are both three-loop graphs
containing $A$ propagators and graphs with a single $\phi$ loop. The latter graphs bring in terms proportional to $\ln(V_{\text{eff}})'^2$ that become complex for certain values of $\phi$.

This complex effective potential deserves some comment. The effective potential is often defined formally through a Legendre transform of the generating functional of connected Green’s functions. The properties of Legendre transforms then imply that it is everywhere convex, a property that the perturbatively calculated effective potential does not enjoy in theories with multiple stable or metastable vacua. To understand this, recall that the effective potential $V_{\text{Leg}}^{\text{eff}}(\hat{\phi})$ obtained by Legendre transform is equal to the minimum value of the energy density among all states $|\Psi\rangle$ such that $\langle\Psi|\phi(x)|\Psi\rangle = \hat{\phi}$. For values of $\hat{\phi}$ that lie between two vacua, the energy is minimized by states $|\Psi\rangle$ which are superpositions of the two vacua; these lead to a flat effective potential. The perturbative effective potential is most easily understood by decomposing $\phi(x,t)$ into a spatially uniform mode $\phi_0(t)$ and a part $\tilde{\phi}(x,t)$ whose spatial integral vanishes. In the infinite volume limit the former mode is essentially classical, and one can discuss states in which the wave functional is of the form $\delta(\phi_0 - \hat{\phi})\tilde{\Psi}[\tilde{\psi}(x)]$. The real part of the perturbative effective potential is the minimum expectation value of the energy density among states of this form subject to the additional condition that $\tilde{\Psi}[\tilde{\psi}(x)]$ be concentrated near $\tilde{\psi} = 0$. For values of $\hat{\phi}$ between two vacua these latter states are not eigenfunctions of the Hamiltonian and eventually decay to states whose wavefunctionals are concentrated on configurations with large fluctuations in $\tilde{\psi}(x)$. Being unstable, they have complex energies with the imaginary part of the energy related to their decay rate. In the region where the tree-level potential is concave the decay is essentially classical and is reflected in a perturbative imaginary part to the effective potential. Beyond this region decay proceeds by quantum tunneling and leads a nonperturbative imaginary part.

THE CALLAN-COLEMAN FORMALISM

In theories where the vacuum structure can be read off from the tree-level Lagrangian, $\Gamma$ can be calculated using a formula that was derived by Callan and Coleman\textsuperscript{5} using path integral techniques. The starting point is the quantity

$$G(T) = \langle \phi(x) = \phi_{\text{fv}} | e^{-HT} | \phi(x) = \phi_{\text{fv}} \rangle$$

$$= \int [d\phi] e^{-[S(\phi) + S_{\text{ct}}(\phi)]}$$

(8)

where the Euclidean action

$$S(\phi) = \int d^3x \int_{-T/2}^{T/2} dx_4 \left[ \frac{1}{2} (\partial_\mu \phi)^2 + V(\phi) \right]$$

(9)

and $S_{\text{ct}}$ contains the counterterms needed to make the theory finite.

The path integral is over all configurations such that $\phi$ takes its false vacuum value $\phi_{\text{fv}}$ at $x_4 = \pm T/2$ and at spatial infinity. In the limit $T \to \infty$, Eq. (8) is dominated by the lowest energy state with a non-vanishing contribution (i.e., the false vacuum) and is of the form

$$G(T) \approx e^{-\mathcal{E}T\Omega}$$

(10)
where \( \Omega \) is the volume of space and \( E \) may be interpreted as the energy density of the false vacuum state. Because this is an unstable state, \( E \) is complex with its imaginary part giving the decay rate, which in this case is simply the bubble nucleation rate. Dividing by \( \Omega \) gives the nucleation rate per unit volume,

\[
\Gamma = -2 \text{Im} E. \tag{11}
\]

The path integral may be approximated as the sum of the contributions about all of the stationary (or quasi-stationary) points of the Euclidean action \( S(\phi) \): the pure false vacuum, the bounce solution \( \phi_b \) with all possible locations in Euclidean space-time, and all multibounce configurations. In each case the contribution to the path integral is obtained by expanding the field about the classical solution \( \bar{\phi}(x) \):

\[
\phi(x) = \bar{\phi}(x) + \eta(x) \tag{12}
\]

and then integrating over \( \eta \). To leading approximation one keeps only the terms in the action which are quadratic in \( \eta \). Expanding these in terms of the normal modes of \( S''(\bar{\phi}) = -\Box + V''(\bar{\phi}) \) gives a product of Gaussian integrals. The evaluation of these integrals is completely straightforward in the case of the false vacuum, but about the bounce solution is complicated by the fact that \( S''(\phi_b) \) has four zero and one negative eigenvalues. The zero modes are treated by introducing collective coordinates; integrating over these gives a factor of \( \Omega T \). The negative mode is handled by deforming the contour of integration; aside from a factor of \( 1/2 \), this gives a contribution whose imaginary part is just that which would have been obtained from a naive application of the Gaussian integration formula. Finally, the contributions from the multibounce configurations are simply related to that from the single bounce. Summing over all of these stationary points gives

\[
G = G_0 + \Omega T G_0 G_b + \frac{1}{2}(\Omega T)^2 G_0 G_b^2 + \ldots = G_0 e^{\Omega T G_b}. \tag{13}
\]

This leads to

\[
\Gamma = 2 \text{Im} G_b = e^{-B J} K e^{-[S_{ct}(\phi_b) - S_{ct}(\phi_{fv})]} (1 + \ldots) \tag{14}
\]

where

\[
B = S(\phi_b) - S(\phi_{fv}) \tag{15}
\]

and

\[
K = \left| \frac{\det'[-\Box + V''(\phi_b)]}{\det[-\Box + V''(\phi_{fv})]} \right|^{-1/2}. \tag{16}
\]

Here \( \det' \) indicates that the translational zero-frequency modes are to be omitted when evaluating the determinant. The determinants are divergent, but these divergences are cancelled by the terms containing \( S_{ct} \). Finally, \( J \) is a Jacobean factor associated with the introduction of the collective coordinates; it can be shown to equal \( B^2/4\pi^2 \).

Scaling arguments can be used to estimate the magnitude of the various terms in the expression for \( \Gamma \). Let us assume that we can identify a small coupling \( \lambda \) and a dimensionful quantity \( \sigma \) such that the potential can be written in the form

\[
V(\phi) = \lambda \sigma^4 U(\psi) \tag{17}
\]
where $U$ involves no small couplings and the dimensionless field $\psi = \phi/\sigma$. The minima of the potential must then be located either at $\phi = 0$ or at values of $\phi$ of order $\sigma$.

By defining a dimensionless variable $s = \sqrt{\lambda}\sigma x$, we may write the field equations as

$$\Box s \psi = \frac{\partial U}{\partial \psi}. \quad (18)$$

From the assumptions made above, this equation involves no small parameters and so has a bounce solution in which $\psi$ is of order unity and differs from the false vacuum within a region of a spatial extent (measured in terms of $s$) which is also of order unity. In terms of the original variables, the bounce has $\phi$ of order $\sigma$ and extends over a range of $x$ of order $1/(\sqrt{\lambda}\sigma)$.

With the same change of variables, the action becomes

$$S = \frac{1}{\lambda} \int d^4 s \left[ \frac{1}{2} \left( \frac{\partial \psi}{\partial s^\mu} \right)^2 + U(\psi) \right]. \quad (19)$$

Since the integrand contains no small parameters, while the volume of the bounce restricts the integration to a region of order unity, the bounce action $B$ is of order $\lambda^{-1}$.

Similarly, the determinant factor $K$ becomes

$$K = \frac{1}{2} \frac{\det'[(\lambda \sigma^2)(-\Box_s + U''(\psi_b))]^{-1/2}}{\det[(\lambda \sigma^2)(-\Box_s + U''(\psi_{fv}))]} \quad (20)$$

where the explicit factor of $\lambda^2 \sigma^4$ on the second line arises because the $\det'$ factor involves four fewer modes than the $\det$ factor. With this factor extracted, the ratio of determinants is formally of order unity, although divergent. Finally, the Jacobean factor $J$ is proportional to $B^2 \sim \lambda^{-2}$. Putting all of these factors together, we see that the nucleation rate is of the form

$$\Gamma = a\sigma^4 e^{-b/\lambda} \quad (21)$$

with $a$ and $b$ both of order unity. In practice $b$ can be calculated rather accurately, since it is not difficult to numerically solve the differential equation for the bounce solution. On the other hand, an accurate determination of $a$, which involves a functional determinant, is exceedingly difficult.

**A MODIFIED FORMALISM**

If we were to apply this formalism to our scalar field model, the first step would be to find a bounce solution to the Euclidean field equations implied by the Lagrangian (2). The problem is that, because the tree level potential has only the symmetric minimum, there
is no bounce solution. We can circumvent this problem by integrating out the $A$ field from
the start, and writing
\[
G(T) = \int [d\phi][dA]e^{-S(\phi,A)} \equiv \int [d\phi]e^{-W(\phi)}.
\] (22)

$W$ may be thought of as a kind of effective action, although it is not the same as the more
usual $S_{\text{eff}}$ of Eq. (4). Graphically, the two actions differ in that $W$ receives contributions
only from graphs with only internal $A$-lines and external $\phi$-lines. Also, $W$ is divergent
while $S_{\text{eff}}$, since it generates renormalized Green’s functions, must be finite. Performing
the path integral over $A$ fields, one obtains
\[
W(\phi) = \int d^4x \left[ \frac{1}{2} (\partial_\mu \phi)^2 + \frac{1}{2} \mu^2 \phi^2 + \frac{\lambda}{4} \phi^4 + \frac{1}{2} \langle x | \ln \left[ -\Box + M^2(\phi) \right] | x \rangle \right] + \text{counterterms} + \cdots
\] (23)

where $M^2(\phi) = m^2 + g^2 \phi^2$ and the dots represent the two-loop and higher order corrections.

Although $W$, in contrast with the tree-level action, does reflect the true vacuum
structure of the theory, we cannot simply proceed by solving its field equations to obtain
a bounce. There are two difficulties here. First, we only have a perturbative expansion
for $W$. Second, and more importantly, $W$ is nonlocal and will therefore lead to quite
complicated field equations.

Although nonlocal, $W$ can be approximated by a local functional if $\phi$ is sufficiently
slowly varying. (For the terms shown explicitly in Eq. (23), the requirement is that the
change in $\phi$ over a distance of order $M(\phi)^{-1}$ be small.) This local functional takes the
form of a derivative expansion
\[
W(\phi) = \int d^4x \left[ \hat{V}(\phi) + \frac{1}{2} \hat{Z}(\phi)(\partial_\mu \phi)^2 + \cdots \right].
\] (24)

Although this is similar in form to the expansion for $S_{\text{eff}}$, the functions entering the two
expansions are not the same. In line with the remarks made previously, $\hat{V}(\phi)$ and $\hat{Z}(\phi)$
differ from $V_{\text{eff}}(\phi,0)$ and $Z_{\phi}(\phi,0)$ by the omission of graphs with internal $\phi$-lines. In
particular, to $O(g^4)$ we have
\[
\hat{V}_g^4 = (V_{\text{eff}})_g^4
\] (25)
but at the next order
\[
\hat{V}_g^6 \neq (V_{\text{eff}})_g^6
\] (26)
because the second graph in Fig. 1 contributes to $V_{\text{eff}}$ but not to $\hat{V}$.

Returning to the bubble nucleation problem, let us define an action
\[
W_0(\phi) = \int d^4x \left[ \frac{1}{2} (\partial_\mu \phi)^2 + \hat{V}_g^4(\phi) \right]
\] (27)

that does display the correct vacuum structure and that, at least for slowly varying $\phi$, is
a good approximation to $W$. We can now attempt to evaluate the path integral over $\phi$ in
Eq. (22) by expanding about the stationary points $\bar{\phi}$ of $W_0$. These include the homogeneous false vacuum and a bounce solution obeying

$$\Box \phi = \partial^4 V_{g^4}. \quad (28)$$

Defining $\eta(x) = \phi(x) - \bar{\phi}(x)$, we obtain

$$W(\phi) = W(\bar{\phi}) + \int d^4z W'(\bar{\phi}; z) \eta(z) + \frac{1}{2} \int d^4z d^4z' W''(\bar{\phi}; z, z') \eta(z) \eta(z') + O(\eta^3) \quad (29)$$

where primes denote variational derivatives. Note that $W'(\bar{\phi}; z)$ does not vanish, since $\bar{\phi}$ is a stationary point of $W_0$ but not of $W$. Note also that we cannot use the derivative expansion of $W$ inside the path integral, since the integral includes rapidly varying field configurations. Inserting Eq. (29) into the path integral and then proceeding as in the standard case, we eventually obtain

$$\Gamma = e^{-C_1} e^{C_2} \left| \frac{\det W''(\phi_b)}{\det W''(\phi_{fv})} \right|^{-1/2} J(1 + \cdots) \quad (30)$$

where

$$C_1 = W(\phi_b) - W(\phi_{fv}) \quad (31)$$

and

$$C_2 = W'(\phi_b)[W''(\phi_b)]^{-1} W'(\phi_b) - (\phi_b \to \phi_{fv}). \quad (32)$$

Let us now examine the various terms in this expression, beginning with $C_1$. Scaling arguments of the type described previously show that the bounce solution to Eq. (28) has a characteristic spatial size of order $1/(g^2 \sigma)$ and is therefore slowly varying relative to the scale $M(\phi)$ entering the one-loop approximation to $W$, Eq. (23). This both allows us to use the derivative expansion (24) to evaluate $W(\phi_b)$ and also suppresses the higher-derivative terms in this expansion. Doing the derivative expansion, one finds that the leading contribution to $C_1$ is

$$\hat{B}_0 = \int d^4x \left[ \frac{1}{2} (\partial_\mu \phi_b)^2 + \hat{V}_{g^4}(\phi_b) - (\phi_b \to \phi_{fv}) \right] \sim \frac{1}{g^4}. \quad (33)$$

Recalling that $\hat{V}_{g^4} = (V_{eff})_{g^4}$, we see that this is just the result one would expect from simply replacing $V$ by $V_{eff}$ in the standard formalism. The next-to-leading contribution is

$$\hat{B}_1 = \int d^4x \left[ \frac{1}{2} \hat{Z}_{g^2}(\phi_b)(\partial_\mu \phi_b)^2 + \hat{V}_{g^6}(\phi_b) - (\phi_b \to \phi_{fv}) \right] \sim \frac{1}{g^2}. \quad (34)$$

Because $\hat{V}_{g^6} = (V_{eff})_{g^6}$ (although $\hat{Z}_{g^2}$ and $(Z_\phi)_{g^2}$ are equal in our model), this term is not simply the next correction to $S_{eff}$. The term beyond this, which is of order unity, involves the $O(g^8)$ contributions to $\hat{V}$ and the $O(g^4)$ contributions to $\hat{Z}$, as well as the leading four-derivative terms.

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For $C_2$ we need both $W'$ and $W''$. The derivative expansion can be used for the former; because $\delta \phi$ is a stationary point of $W_0$, the leading contribution here is from $\delta W' \equiv W' - W'_0$. Matters are less simple for $W''(\delta \phi; z, z')$. Although the derivative expansion can be used when $|z - z'|$ is large, the behavior for small $|z - z'|$ is sensitive to the high momentum modes and so the derivative expansion fails no matter how slowly varying $\phi$ is. However, the relation

$$W'' = W''_0 \left[ 1 + (W''_0)^{-1} \delta W'' \right] \quad (35)$$

can be used to obtain formal expansions for $(W'')^{-1}$ and $\det W''$ as power series in $(W''_0)^{-1} \delta W''$. The actual utility of these expansions depends on the size of the contribution from the region of small $|z - z'|$. In the calculation of $C_2$ this contribution is subdominant and $(W'')^{-1}$ can be approximated by $(W''_0)^{-1}$; one finds that $C_2$ is of order unity. For the determinant factor, on the other hand, more terms must be retained:

$$\det'[W''] = \det'[W''_0] \det[I + (W''_0)^{-1} \delta W'']$$
$$= \det'[W''_0] \exp \left\{ \text{tr} \ln[I + (W''_0)^{-1} \delta W''] \right\}$$
$$= \det'[W''_0] \exp \left\{ \text{tr}(W''_0)^{-1} \delta W'' + \frac{1}{2} \text{tr} [(W''_0)^{-1} \delta W'']^2 + \cdots \right\} \quad (36)$$

In the last line, the first term in the exponent is of order $1/g^2$ while the second is of order unity. In fact, the former term is given by the second graph of Fig. 1 which, it will be recalled, contributes to $V_{\text{eff}}$ but not to $\tilde{V}$. It combines with the terms in $\tilde{B}_1$ to give the full $O(g^6)$ contribution to the effective potential. Combining all other factors together in a prefactor $A$ of order unity, we obtain

$$\Gamma = A \sigma^4 e^{-(B_0 + B_1)} \quad (37)$$

where

$$B_0 = \tilde{B}_0 = \int d^4 x \left\{ (V_{\text{eff}})_g^4 (\phi_b) + \frac{1}{2} (\partial_\mu \phi_b)^2 \right\} \sim \frac{1}{g^4} \quad (38)$$

$$B_1 = \int d^4 x \left\{ (V_{\text{eff}})_g^0 (\phi_b) + \frac{1}{2} (Z_\phi)_g^2 (\phi_b) (\partial_\mu \phi_b)^2 \right\} - (\phi_b \to \phi_{fv}) \sim \frac{1}{g^2}. \quad (39)$$

After seeing how the expansion of the determinant combines with the exponent factors to reconstruct $S_{\text{eff}}$ to leading and next-to-leading orders, it is natural to speculate that this process might continue to higher orders. It does not. In particular, it is at the next order that we would encounter the potentially complex contributions to $V_{\text{eff}}$ which arise from graphs with a single $\phi$ loop. These would appear in a derivative expansion of the determinant factor

$$\frac{\left| \det' W''_0 (\phi_b) \right|^{-1/2}}{\det W''_0 (\phi_{fv})} \quad (40)$$

However, a derivative expansion of $\det W''_0$ is valid only for fields which vary slowly relative to $(V_{\text{eff}})_g^4$. The bounce solution does not satisfy this condition, and so the complex terms in $V_{\text{eff}}$ do not appear. (One can make a derivative expansion of the determinant in the denominator, but $(V_{\text{eff}})_g^4 (\phi_{fv})$ is real.)
These methods can be applied to gauge theories with only minor modifications. To be specific, consider the case of scalar electrodynamics with the quartic scalar self-coupling taken to be \(O(e^4)\) so that the vacuum structure is determined by the one-loop corrections to the effective potential. Directly following the approach used for the scalar field example, one would integrate out the photon field at the start to obtain an effective action for the complex scalar field. The classical equations following from this action have a bounce solution of the desired type which gives the leading approximation to \(\Gamma\). However, the calculation of the next order terms turns out to be much more complicated than in the scalar case. These complications can be avoided by an alternative approach. Since the bounce solution can be chosen to be entirely real, it is possible to integrate out both the photon field and the imaginary part of the scalar field to give an action \(W(\phi)\) which depends only on a single real scalar field. The analysis then proceeds very much as before. The \(O(e^4)\) terms in \(V_{\text{eff}}\) and \(\hat{V}\) are identical and, together with the dominant gradient term, lead to a contribution to the exponent proportional to \(e^{-4}\). At order \(e^6\) there are two graphs contributing to \(V_{\text{eff}}\), neither of which appears in \(\hat{V}\). Both are recovered from the expansion of \(\det W''\) and combine with \((Z_0)_{e^2} = (\hat{Z})_{e^2}\) to give a contribution to the exponent which is \(O(e^{-2})\). \(V_{\text{eff}}\) becomes complex at order \(e^8\) but, as before, the failure of the derivative expansion of \(\det W''\) about the bounce solution prevents the offending terms from entering the nucleation rate calculation. In fact, the appearance of infrared divergences at small \(\phi\) means that the derivative expansion cannot be carried out beyond the four-derivative terms. Instead, one must extract the potential and two-derivative terms, leaving a remainder which gives an \(O(1)\) contribution to \(\Gamma\) that can be absorbed in the prefactor.

The issue of gauge-dependence also arises. The nucleation rate is a measurable physical quantity and so should be independent of gauge. However, the formulas above express it in terms of the effective potential and other gauge-dependent quantities. The Nielsen identities indicate how this conflict can be resolved. In a class of gauges with photon propagator

\[
D_{\mu\nu}(k^2) = -g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} + \xi \frac{k_\mu k_\nu}{k^4}
\]

(41)

these identities take the form

\[
\xi \frac{\partial S_{\text{eff}}}{\partial \xi} = \int d^4 x C[\phi(x)] \frac{\delta S_{\text{eff}}}{\delta \phi}.
\]

(42)

(We do not need the explicit form of the functional \(C[\phi(x)]\), but only the fact that it is \(O(e^2)\).) For a uniform \(\phi\) field this reduces to

\[
\xi \frac{\partial V_{\text{eff}}}{\partial \xi} = C(\phi) \frac{\partial V_{\text{eff}}}{\partial \phi}.
\]

(43)

Now let us expand this in powers of \(e^2\) to yield a series of identities. The first of these states that \((V_{\text{eff}})_{e^4}\) is gauge-independent, as is easily verified. The next is

\[
\xi \frac{\partial (V_{\text{eff}})_{e^6}}{\partial \xi} = C_{e^2}(\phi) \frac{\partial (V_{\text{eff}})_{e^4}}{\partial \phi}.
\]

(44)
Further identities are obtained by making derivative expansions of both sides of Eq. (42) and expanding the various terms in powers of $e^2$. The first new identity obtained in this manner is

$$\xi \frac{\partial (Z\phi)e^2}{\partial \xi} = \frac{\partial C e^2}{\partial \phi}.$$  (45)

Let us now apply these identities to $\Gamma$. The leading ($O(1/e^4)$) term in the exponent involves $(V_{\text{eff}})e^4$, and is manifestly gauge-independent. The next term in the exponent, of order $1/e^2$, contains $(V_{\text{eff}})e^6$ and $(Z\phi)e^2$. Using the identities we have just obtained, we find that the gauge dependence of this term is given by

$$\xi \frac{\partial}{\partial \xi} \int d^4x \left[ (Z\phi)e^2(\partial_\mu \phi)^2 + (V_{\text{eff}})e^6 \right] = \int d^4x \left[ \frac{\partial C e^2}{\partial \phi} (\partial_\mu \phi)^2 + C e^2 \frac{\partial (V_{\text{eff}})e^4}{\partial \phi} \right] = \int d^4x C e^2 \left[ -\phi + \frac{\partial (V_{\text{eff}})e^4}{\partial \phi} \right].$$  (46)

Because $\phi_b$ is a solution of the Euclidean field equations, the last line vanishes.

**CONCLUSION**

In this talk I have described how the decay rate of a metastable vacuum can be calculated in a theory whose vacuum structure is determined by radiative corrections. As in the standard case, the result may be written as a dimensionful prefactor times the exponential of an action involving a bounce solution. To leading approximation this exponent is just the tree-level action supplemented by the dominant one-loop contribution to the effective potential. The first correction to the exponent arises from the next-to-leading contributions to the effective potential and the leading correction to the tree-level kinetic part of the effective action. Although smaller than the leading terms, these give an addition to the exponent which is larger than order unity and is thus more important than the prefactor. It does not appear that this correction need have any particular sign, but rather that it might increase the nucleation rate in some theories and reduce the rate in others.

All further corrections may be absorbed into the prefactor. Although some of these can be identified with particular terms in the effective action, this is not true of all higher corrections. Specifically, the graphs which give rise to complex terms in the effective potential cannot, when calculated in the background of the bounce, be expanded in a derivative expansion. Consequently, the imaginary part of the effective potential does not explicitly enter the bubble nucleation calculation and the problems of interpretation which it would entail are avoided.

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