Application of wavelet analysis to the analysis of geomagnetic field variations

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Abstract. Wavelet analysis is becoming more popular in geophysics. It is used for numerous researches, including tropical convection, the El Nino-Southern Oscillation, atmospheric cold fronts, temperature variations, the dispersion of ocean waves, and coherent structures in turbulent flows, number of sunspots etc. In this paper we research how informative is the application of wavelet analysis to the analysis of geomagnetic field variations at the mid-latitude observatory "Mikhnevo" of Institute of Geosphere Dynamics of Russian Academy of Science. We review continuous wavelet transform by focusing attention on such aspects as choice of mother wavelet, choice of scales, cone of influence, visualization of results, reconstruction of time series from wavelet transform and its application to estimate the Holder exponents and singularity spectra. In our work we use Morlet wavelet with frequency parameter of 6. In so doing, the reconstruction of the time series from the wavelet transform has a mean square error of 3.4%. The application of wavelet analysis made it possible to distinguish pronounced periodicities of the geomagnetic field with periods of 27, 13.5, 9, 6 days. In solar quiet-day variations is dominated by the 24- 12-, 8-, and 6-hour period components. An analysis of the modulus of the wavelet transform coefficients qualitatively indicates a scaling (close to the monofractal) character of the variations of the geomagnetic field in the diurnal range. Moreover, the intensity of periodic variations of geomagnetic variation isn’t constant in time. The application of the method of wavelet transform modulus maxima confirmed the monofractal character of the diurnal variation for any solar activity. In contrast to the 1-day variation, the 27-day variation and its harmonics show a higher degree of multifractality during a maximum of solar activity in comparison with the minimum of solar activity.

1. Introduction

Wavelet analysis is becoming a common tool for analysing localized variations within a time series. By decomposing a time series into time-frequency space, one is able to determine both the dominant modes of variability and how those modes vary in time. Wavelet transform has a major advantage over classical signal analysis techniques such as the Fourier transform, which only uses a single-window analysis, resulting in time-averaged results that lose their temporal information [1, 2]. The main issue with the fixed window size used in the windowed Fourier transform is that it loses the time localization at high frequencies when the window is sliding along the time series because there are too many oscillations captured within the window. It also loses the frequency localization at low frequencies because there is only a few low-frequency oscillations included in the window [3]. The
wavelet transform can handle these issues by decomposing a one-dimensional signal into two-dimensional time–frequency domains at the same time [4]. Unlike sine waves, which are the main functions used in Fourier analysis, wavelets are usually irregular and asymmetric in shape. This property makes a wavelet ideal for analysing signals that contain sharp changes and discontinuities – a localized signal analysis [4]. Wavelet transforms use different window sizes, which are able to compress and stretch wavelets in different scales or widths; these are then used to decompose a time series [3].

The wavelet transform has been used for numerous studies in geophysics, including tropical convection [5], the El Nino-Southern Oscillation [6], atmospheric cold fronts [7], central England temperature [8], the dispersion of ocean waves [9], and coherent structures in turbulent flows [10].

The fractal analysis has been widely used in geophysics such as analysis of the time series, seismological studies, reservoir studies and other studies related to river discharge, rainfall, sunspot number, gravity covariance model etc. [11-13]. One of popular method for fractal analysis is wavelet transform modulus maxima based on the summation of the module of the wavelet coefficients on the chain of maxima, to calculate a new function which is called the function of partition, it covers only the part carrying information, in fractal theory the part of the signal associated with each position is called a "singularity", each singularity is characterized by a coefficient called Holder exponent [for example, 14].

In this paper we research how informative is the application of wavelet analysis to the analysis of geomagnetic field variations at the mid-latitude observatory "Mikhnevo" of Institute of Geosphere Dynamics of Russian Academy of Science.

In Sec. 2 we give an introduction to the continuous wavelet transform by focusing attention on such aspects as choice of mother wavelet, choice of scales, cone of influence, visualization of results, reconstruction of time series from wavelet transform. Sec. 3 reviews application wavelet transform to estimate the Holder exponents and singularity spectra. The data using in analysis are presented in Sec. 4. Finally, in Sec. 5 we give results and discuss our results.

2. Continuous wavelet transform

2.1 Continuous wavelet transform

The continuous wavelet transform is a time-frequency analysis method which differs from the more traditional short time Fourier transform by allowing arbitrarily high localization in time of high frequency signal features. The continuous wavelet transform does this by having a variable window width, which is related to the scale of observation a flexibility that allows for the isolation of the high frequency features. Another important distinction from the short time Fourier transform is that the continuous wavelet transform is not limited to using sinusoidal analysing functions [for example, 15].

Given a time series \( x(t) \in L^2 \), the continuous wavelet projection (or continuous wavelet transform, CWT) of this series over the wavelet \( \psi \) at the point \( n \) and the scale of observation \( s \) is denoted by \( W_{\psi n}(\tau, s) \), and it is defined by:

\[
W_{\psi n}(\tau, s) = \int dt \frac{1}{\sqrt{s}} \psi^*_s \left( \frac{t - \tau}{s} \right) x(t) = x(t) \otimes \psi_{s, \tau}(t),
\]

where \( \otimes \) stands for the convolution product. To ensure that the wavelet transforms at each scale \( s \) are directly comparable to each other and to the transforms of other time series, the wavelet function at
each scale $s$ is normalized to have unit energy. Starting with a mother wavelet $\psi$, a family $\psi_{\tau,s}(t)$ of “wavelet daughters” can be obtained by simply scaling and translating $\psi$:

$$\psi_{\tau,s}(t) = \frac{1}{\sqrt{|s|}} \psi\left(\frac{t - \tau}{s}\right), \quad s, \tau \in \mathbb{R}, \ s \neq 0,$$

(2)

where $s$ is a scaling or dilation factor that controls the width of the wavelet (the factor $1/\sqrt{|s|}$ being introduced to guarantee preservation of the energy) and $\tau$ is a translation parameter controlling the location of the wavelet. Scaling a wavelet simply means stretching it (if $|s| > 1$) or compressing it (if $|s| < 1$), while translating it simply means shifting its position in time. Therefore, the wavelet projection consist in convolving the signal with an appropriate “focus kernel” $\psi_{\tau,s}(t)$, which can be tuned with the scale parameter $s$ in order to zoom in and out the details surrounding each point $\tau$ under analysis. By varying $s$ and $\tau$, we can construct a picture showing both the amplitude of any features versus the scale and how this amplitude varies with time.

The continuous wavelet transform is a time-frequency, or more correctly, a time-scale representation. To demonstrate this, we derive a “frequency domain” formulation of the CWT as follows. Substituting the inverse Fourier transforms $X(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x(t) e^{i\omega t} dt$ and $\psi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Psi(\nu) e^{i\nu t} d\nu$ into the formula for $W_{\psi s}(\tau, s)$, integrating the resulting Dirac $\delta$-function, and simplifying, one obtains:

$$W_{\psi s}(\tau, s) = \frac{|s|}{2\pi} \int d\omega \Psi^* (s\omega) X(\omega) e^{i\omega \tau}.$$ 

We see that the CWT can be viewed as a frequency-domain filtering of the signal by the dilated filter $\frac{|s|}{2\pi} \Psi(s\omega)$.

The CWT can be computed by first finding the Fourier transforms of the time series and the normalized wavelet. The inverse Fourier transform of the product of $X(\omega)$ and the scaled wavelet $\psi^*(s\omega)$ yields one constant-scale slice of the transform.

2.2 Continuous wavelet transform of time series

Given a time series, $x_n$, with a time spacing $\delta t$ ($x_n = x(n \delta t)$), $n = 0, \ldots, N - 1$ with $N$ even ($X(\omega) \approx 0$ for $|\omega| > \frac{2\pi}{\delta t} = \frac{\pi}{\delta t}$) the wavelet transform of $x$ with respect $\psi$ can be discretized as:

$$W_{\psi s}(\tau, s) = \frac{s}{2\pi} \int_{-\pi/s}^{\pi/s} d\omega X(\omega) \Psi^*(\omega) e^{i\omega \tau} = \frac{s}{2\pi} \frac{2\pi}{N \delta t} \sum_{k=-(N/2)+1}^{N/2} X\left(\frac{2\pi k}{N \delta t}\right) \Psi^*\left(\frac{s2\pi k}{N \delta t}\right) e^{\frac{2\pi ik}{N \delta t}} =$$

$$= \frac{s}{N \delta t} \sum_{k=-(N/2)+1}^{N/2} X\left(\frac{2\pi k}{N \delta t}\right) \Psi^*\left(\frac{s2\pi k}{N \delta t}\right) e^{\frac{2\pi ik}{N \delta t}}$$

(3)
But $X\left(\frac{2\pi k}{N\delta t}\right) = \hat{x}_k$, where $\hat{x}_k = \sum_{n=0}^{N-1} x_n e^{-\frac{i2mk}{N}}$ is the $k$th element of the discrete Fourier transform of the $N$-vector $(x_0, \ldots, x_{N-1})$ [16-17]. Hence, using the periodicity $\hat{x}_k = \hat{x}_{k-N}$, we obtain a discretized form of the continuous wavelet transform of a discrete sequence $x_n$, with a time spacing $\delta t$ and $n = 0, \ldots, N-1$:

$$W_{\psi^\prime}(\tau, s) = \sqrt{S} \sum_{k=-\infty}^{\infty} \hat{x}_k \Psi^\prime\left(s \frac{2\pi(k-N)}{N\delta t}\right) e^{-\frac{i2\pi(k-N)\tau}{\delta t}} + \sqrt{S} \sum_{k=N/2+1}^{N} \hat{x}_k \Psi^\prime\left(s \frac{2\pi(k-N)}{N\delta t}\right) e^{-\frac{i2\pi(k-N)\tau}{\delta t}},$$

Finally, when $\tau = m \delta t$, $m = 0, \ldots, N-1$, we obtain:

$$W_{\psi^\prime}(m\delta t, s) = \sqrt{S} \sum_{k=0}^{N/2} \hat{x}_k \Psi^\prime\left(s \frac{2\pi(k-N)}{N\delta t}\right) e^{-\frac{i2\pi(m+N-k)\omega_l}{N\delta t}} + \sqrt{S} \sum_{k=N/2+1}^{N} \hat{x}_k \Psi^\prime\left(s \frac{2\pi(k-N)}{N\delta t}\right) e^{-\frac{i2\pi(m+N-k)\omega_l}{N\delta t}},$$

where the angular frequency is defined as:

$$\omega_l = \begin{cases} \frac{2\pi k}{N\delta t} & k = 0, \ldots, N/2 \\ \frac{2\pi(k-N)}{N\delta t} & k = N/2 + 1, \ldots, N-1 \\ \end{cases},$$

the discrete Fourier transform of $x_n$ signal is given by:

$$\hat{x}_k = \sum_{n=0}^{N-1} x_n e^{-\frac{i2mk}{N}} \quad k = 0, \ldots, N-1$$

$$\Psi^\prime(s \omega_l)$$ is the discrete Fourier transform of the wavelet $\psi$:

$$\Psi(s \omega_l) = \sum_{n=0}^{N-1} \psi(t) e^{-\frac{i2\pi n \omega_l}{N}} \quad k = 0, \ldots, N-1.$$

Using (6) and a standard Fourier transform routine, one can calculate the continuous calculate the continuous wavelet transform (for a given $s$) at all $\tau$ simultaneously and efficiently. The Fourier coefficients $\hat{x}_k$ can be calculated efficiently by using a fast Fourier transformation (FFT) algorithm.

The computation of the scaled spectrum, $\Psi(s \omega_l)$, is more difficult [18].

The wavelet transform is computed only for a selected set of scale values $s_l$, $l = 0, \ldots, L-1$ (corresponding to a certain choice of frequencies $\omega$). Hence, our computed wavelet spectrum of the discrete-time series $x$ will simply be a matrix $W_{\psi^\prime}$ (wavelet spectral matrix) whose $(l, m)$ element is given by:

$$W_{\psi^\prime}(l, m) = \sqrt{S_l} \sum_{k=0}^{N-1} \hat{x}_k \Psi^\prime(s_l \omega_l) e^{-\frac{i2\pi km}{N}}.$$
For each scale $s_l$, $l = 0, \ldots, L - 1$, its right-hand side is simply the inverse DFT of the sequence $\sqrt{s_l} \hat{\Psi}(s_l, \omega_l)$ and can, therefore, be calculated using an inverse FFT.

### 2.3 Choice of mother wavelet

There are several considerations in making the choice of a wavelet, for example, real versus complex wavelets, continuous or discrete wavelets, orthogonal versus redundant decompositions. Briefly, the continuous wavelets often yield a redundant decomposition (the information extracted from a given scale band slightly overlaps that extracted from neighbouring scales) but they are more robust to noise as compared with other decomposition schemes. Discrete wavelets have the advantage of fast implementation but generally the number of scales and the time invariant property (a filter is time invariant if shifting the input in time correspondingly shifts the output) strongly depend on the data length.

In our work all the presented results were obtained for the non-analytic Morlet wavelet which is a one-parameter family of functions, given by: $\psi_{\omega_0}(t) = Ke^{j\omega_0 t}e^{-\frac{t^2}{2}}$, where $\omega_0$ is the non-dimensional frequency parameter [19].

The Morlet wavelet is constructed by modulating a sinusoidal signal by a Gaussian shape. Strictly speaking, the above functions are not true wavelets, since they fail to satisfy the admissibility condition. For $\psi_{\omega_0}(t)$ to have unit energy, the normalizing constant $K$ must be chosen as $K = \pi^{-1/4}$, (51) which, from now on, we will always assume to be true. The Fourier transform of the normalized wavelet is given by: $\psi_{\omega_0}(\omega) = \sqrt{2\pi}^{1/4}e^{-\frac{1}{2}(\omega - \omega_0)^2}$, and, hence, $\psi_{\omega_0}(0) = \sqrt{2\pi}^{1/4}e^{-\frac{1}{2}\omega_0^2} \neq 0$. However, for sufficiently large $\omega_0$, e.g. $\omega_0 > 5$, the values of $\psi_{\omega_0}(\omega)$ for $\omega_0 \leq 0$ are so small that, for numerical purposes, $\psi_{\omega_0}$ can be considered as an analytic wavelet [20]. If $\omega_0 = 6$ the admissibility condition is satisfied [1, 10].

The peak frequency, the energy frequency and the central instantaneous frequency of the Morlet wavelet are all equal and given by $\omega_{\text{PE}} = \omega_{\text{EE}} = \omega_{\text{f}} = \omega_0$, facilitating the conversion from scales to frequencies.

For the Morlet wavelet the relation between the frequency and the wavelet scale this equivalence is done by $\frac{1}{f} = \frac{4\pi}{\omega_0 + \sqrt{2 + \omega_0^2}}$, with $\omega_0$ the central angular frequency of the wavelet ($x_0 = 2\pi f_0$). Then with $x_0$ around $2\pi$, the wavelet scale $s$ is therefore inversely proportional to the central frequency of the wavelet, $\frac{1}{f} \approx s$. This greatly simplifies the interpretation of the wavelet analysis and one can replace, in all equations, the scale $a$ by the frequency $f$ or the period: $p = \frac{1}{f}$. The Morlet wavelet has the advantage of having both real and imaginary parts. This allows separation of the phase and the amplitude of the studied signal.
2.4 Choice of scales

Wavelet transform scales affect the finally accuracy. Due to presence of measurement error, the choice of scale is not as smaller as better. The scales are usually chosen as fractional powers of 2:

\[ s_l = s_0 2^{l}, \quad l = 0, \ldots, L - 1, \quad (11) \]

\[ J = \delta t^{-1} \log_2 \left( N \delta s / s_0 \right), \quad (12) \]

where \( s_0 = 2^\delta \) is the smallest resolvable scale and \( J \) is the largest scale. The letter \( N \) denotes the length of data, \( \delta t \) for sampling intervals and \( \delta s \) stands for spacing between the discrete scales. For Morlet wavelet, \( \delta s \) less than about 0.5 for adequate scale resolution, for \( \omega_0 \) factor for scale averaging is 0.60 [1].

2.5 Cone of influence

The continuous wavelet transform applied to a finite length time-series inevitably suffers from border distortions; this is due to the fact that the values of the transform at the beginning and the end of the time-series are always incorrectly computed, in the sense that they involve missing values of the series which are then artificially prescribed. When using the equation (6), a periodization of the data is assumed. However, before implementing equation (6), one usually pads the series with zeros, to avoid wrapping. Since the "effective support" of the wavelet at scale \( s \) is proportional to \( s \), these edge-effects also increase with \( s \). The region in which the transform suffers from these edge effects is called the cone of influence (COI) which is defined usually as the \( e \)-folding time for the autocorrelation of wavelet transform at each scale [1, 21]. Inside the COI the values of the wavelet coefficients \( W_{\psi} \) are reduced due to the zero padding. Note that for cyclic series there is no COI.

2.6 Visualization of results

In analogy with the terminology used in the Fourier case, the (local) wavelet power spectrum (sometimes called scalogram or wavelet periodogram) is defined as \( |W_{\psi} (\tau, s)|^2 \). The wavelet power spectrum may be averaged over time for comparison with classical spectral methods. When the average is taken over all times, we obtain the so-called global wavelet power spectrum:

\[ \int_\tau |W_{\psi} (\tau, s)|^2 d\tau. \]

When the wavelet \( \psi \) is complex-valued (in our work, Morlet wavelet), the corresponding wavelet transform \( W_{\psi} (\tau, s) \) is also complex-valued. In this case, the transform can be separated into its real part, \( \mathcal{R} \{ W_{\psi} (\tau, s) \} \), and imaginary part, \( \mathcal{I} \{ W_{\psi} (\tau, s) \} \), or in its amplitude, \( |W_{\psi} (\tau, s)| \), and phase (or phase-angle), \( W_{\psi} (\tau, s) = |W_{\psi} (\tau, s)| \text{e}^{i\varphi_{\psi} (\tau, s)} \). Recall that the phase-angle \( \varphi_{\psi} (\tau, s) \) of the complex number \( W_{\psi} (\tau, s) \) can be obtained from the formula:

\[ \varphi_{\psi} (\tau, s) = \arctan \left( \frac{\mathcal{R} \{ W_{\psi} (\tau, s) \}}{\mathcal{I} \{ W_{\psi} (\tau, s) \}} \right). \]

For real-valued wavelet functions, the imaginary part is constantly zero and the phase is, therefore, undefined. Hence, in order to separate the phase and amplitude information of a time-series, it is important to make use of complex wavelets.

For visualization of results of wavelet transform we constructed a contour plot. Such plot shows, where magnitude of the wavelet transform dominates in time-frequency plane [22].
2.7 Reconstruction of time series

Since the wavelet transform is a bandpass filter with a known response function (the wavelet function), it is possible to reconstruct the original time series using either deconvolution or the inverse filter. The energy of the original function \(x(t)\) is preserved by the wavelet transform, i.e., the following Parseval-type relation holds:

\[
\int_{-\infty}^{\infty}|x(t)|^2 dt = \frac{1}{C_\psi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |W_{\psi}(\tau, s)|^2 \frac{d \tau ds}{s^2},
\]

which, in turn, ensures the possibility of recovering \(x(t)\) from its wavelet transform. If the \(x(t)\) is a \(L^2(\mathbb{R})\) time series, the inverse wavelet transform is:

\[
x(t) = \frac{2}{C_\psi} \int_{0}^{\infty} \left[ \int_{-\infty}^{\infty} W_{\psi}(\tau, s) \psi_{\tau, \tau}^* (t) d \tau \right] ds
\]

showing that no information is lost if we restrict the computation of the transform only to positive values of the scaling parameter \(s\), which is a usual requirement, in practice.

where \(C_\psi\) is defined as

\[
C_\psi = \int_{0}^{\infty} \frac{|\Psi(\omega)|^2}{\omega} d\omega < \infty
\]

\(\Psi(\omega)\) is the Fourier transform of the mother wavelet \(\psi(t)\). This equation is also called the admissibility condition.

In fact, due to the high redundancy of this transform (recall that a function of one variable is mapped into a bivariate function), many reconstruction formulas are available. We choose a delta \((\delta)\) function as a kernel for reconstruction of time series. In this case, the reconstructed time series is just the sum of the real part of the wavelet transform over all scales [10]:

\[
x_i = \frac{\partial \delta \hat{x}^{1/2} \sum_{L=0}^{-L} \mathcal{R}[W_{\psi}(s_j)]}{C_\delta \psi_0 (0) s_j^{1/2}}.
\]

The factor \(\psi_0 (0)\) removes the energy scaling, while the \(s_j^{1/2}\) converts the wavelet transform to an energy density.

The reconstruction factor \(C_\delta\) comes from the reconstruction of a \(\delta\) function from its wavelet transform using the function \(\psi_0\). We can define \(C_\delta\) for a wavelet function, for \(x_\delta (t) = \delta_{\delta_{\omega}}\) Fourier transform is

\[
\hat{x}_\delta = 1 / \mathbb{N}\text{ for all k, the wavelet transform becomes } W_\delta (s) = \frac{1}{\mathbb{N}} \sum_{k=0}^{\mathbb{N}} \Psi^*(s \omega_k) .
\]

The reconstruction (16) then gives

\[
C_\delta = \frac{\partial \delta \hat{x}^{1/2} \sum_{L=0}^{-L} \mathcal{R}[W_{\psi}(s_j)]}{\psi_0 (0) s_j^{1/2}}.
\]

The \(C_\delta\) is scale independent and is a constant for each wavelet function (for Morlet wavelet \(\omega_k = 6\) it equal 0.776, \(\psi_0 (0) = \pi^{-1/4}\)). The reconstruction of the time series from the wavelet transform has a mean square error of 3.4%.

3. Wavelet Transform Modulus Maxima

From the Taylor series approximation of a time series \(x(t)\) about some point \(t_i\) in the vicinity of \(t\), we obtain:
\[ f(t) = a_0 + a_1(t-t_i) + a_2(t-t_i)^2 + \ldots + a_h(t-t_i)^h, \]

where \( h_i \) is a non-integer exponent. As the real-time signals are noisy, the Taylor series approximation of it involves more number of terms. This means that the exponents also assume non-integer values or in other words the signal includes non-integer terms as given in (17). These correspond to the singularities present in the noisy signal that exist at a particular time. These singularities appear as step-like features or cup-like features called fractals [23]. The above expansion for \( x(t) \) can also be written as [24-26]:

\[ |f(t) - P_n(t-t_i)| \leq a_h |t-t_i|^h, \]

such that the polynomial \( P_n(t-t_i) \) has a degree, \( n \), \( h(t_i) \) is the Holder exponent or the Hurst exponent, which is defined as the largest \( h_i \) that satisfies the above condition. It accounts for the singularity strength at the location \( t_i \). The local degree of singularity around the point \( t_i \) is obtained from such largest non-integer exponent, \( h_i \). Thus in order to detect these singularities, we have to determine the exact location, \( t_i \) and the exponent, \( h(t_i) \) associated in each term. If we get same values for \( h(t_i) \) all the time, the time series is said to exhibit monofractality and if distinct values are obtained for \( h(t_i) \), the signal is multifractal in nature [24].

The main interest in using the wavelet transform for analyzing the regularity of a function lies in its ability to be blind to polynomial behavior by an appropriate choice of the analyzing wavelet \( \psi \).

By using an analyzing wavelet \( \psi \) which has \( n_\psi \) vanishing moments, the behavior of \( W_{\psi n}(t_i,s) \) as a function of scale, \( s \), as \( s \to 0^+ \) characterizes the local behavior \( x(t) \) in equation (18), i.e.,

\[ W_{\psi n}(t_i,s) \propto s^{h(t_i)} \]

provided \( n_\psi > h(t_i) \). In other words, the local singular behavior of \( x \) around \( t_i \) is characterized by a power law behavior (with an exponent \( h(t_i) \)) of the wavelet transform of \( x \) at the point \( t_i \), as a function of scale, \( s \), as \( s \to 0^+ \). If \( h(t_i) \) is positive and small (implying that the function is singular), then we can visualize this as a slow decay of wavelet coefficients with scale; if \( h(t_i) \) is large and positive (implying that the function is regular) [27-28], then the wavelet coefficients decay rapidly as a function of scale. If \( n_\psi < 0 \) then the wavelet coefficients, instead of decaying, increase with scale [29]. Thus the more singular the function at a given location, the larger the wavelet coefficients at that location at small scales.

Therefore one can extract the exponent \( h(t_i) \) as the slope of a log-log plot of the WT amplitude versus the scale \( s \). On the contrary, if one choses \( n_\psi < h(t_i) \), the WT still behaves as a power-law but with a scaling exponent which is \( n_\psi \):

\[ W_{\psi n}(t_i,s) \propto s^{n_\psi}. \]

Thus, around a given point \( t_i \), the faster the WT decreases when the scale goes to zero, the more regular \( x \) is around that point. WT scaling exponent is given by \( n_\psi \), i.e. a value which is dependent on the shape of the analyzing wavelet. Therefore to resolve all singularities present in a time series, the analyzing wavelet must be chosen to have \( n_\psi > h_{\text{max}} \), where \( h_{\text{max}} \) is the weakest singularity present in the time series. Since \( h_{\text{max}} \) is not known a priori, the most appropriate way to correctly estimate all singularities is to analyze the given function with wavelets of increasing order (i.e., increasing number
of vanishing moments). If for order $n_\psi$ and $n_\psi + 1$, one gets the same $D(h)$ curve, then it can be assured that $n_\psi$ is the right order of the wavelet.

The wavelet transform modulus maxima (WTMM) are defined at each scale $s$ as the local maxima of $|W_{\psi}(t,s)|$ considered as a function of $t$ [30-31]. These WTMM form connected curves, called maxima lines (skeleton) and contain information about the hierarchical structure of singularities. An important feature of these maxima lines is that, each time the analyzed time series as a local Hölder exponent $h(t_i) > n_\psi$, there is at least one maxima line pointing towards $t_i$ along which equation (19) holds [26, 32-35]. In the case of fractal signals, we thus expect that the number of maxima lines will diverge in the limit $s \to 0$. The branching structure of the WTMM skeleton in the $(t, s)$ half-plane, enlightens the hierarchical organization of the singularities.

One can define a "partition function" $Z(q,s)$ from the wavelet coefficients:

$$Z(q,s) = \int |W_{\psi}(t,s)| dt$$

which can then be modified such that it is computed only on the wavelet transform skeleton [25, 31, 34, 35] as:

$$Z(q,s) = \sum_{l \in L(\alpha)} \left( \sup_{(t,s) \in l} |W_{\psi}(t,s)| \right)^q$$

where $q \in \mathbb{R}$, $L(\alpha)$ is the set of all maxima lines $l$.

It is found [35-37] that even this definition is not fully equipped with the ability to provide reliable and complete estimates of the scaling exponents spectrum ($\tau(q)$) and the singularity spectrum ($D(h)$). The particular case where this aforementioned formulation can diverge is when the maximum value of the modulus of the wavelet transform is very small in value. To alleviate this concern, the definition of the partition function in equation (21) is reformulated by replacing the value of the wavelet transform modulus at each maximum by the supremum value along the corresponding maxima line at scales smaller than $a$ [34, 36-37]:

$$Z(q,s) = \sum_{l \in L(\alpha)} \left( \sup_{(t,s) \in l} |W_{\psi}(t,s)| \right)^q,$$

where $q \in \mathbb{R}$ and the sup can be regarded as a way to define a scale adaptive "Hausdorff-like" partition. One can define the exponent $\tau(q)$ from the power-law behavior of the partition function [35-36]:

$$Z(q,s) \propto s^{\tau(q)}.$$

and, by taking the Legendre transform of $\tau(q)$, the singularity spectrum $D(h)$:

$$D(h) = \min_q \left[ qh - \tau(q) \right].$$

Similarly, for the case of a continuously differentiable $\tau(q)$, the following relations hold [36]:

$$\left\{ \begin{array}{l}
q = dD / dh \\
\tau(q) = q \alpha - D(h)
\end{array} \right.$$  \hspace{1cm} (26)

and, equivalently

$$\left\{ \begin{array}{l}
h = d \tau / dq \\
D(h) = q \alpha - \tau(q)
\end{array} \right.$$  \hspace{1cm} (27)
4. Data

As the initial data, the results of recording the Earth's magnetic field at the Geophysical observatory "Mikhnevo" of the Institute of Geospheres Dynamics (IDG RAS) (54.959º N, 37.766º E), carried out in the period 2010-2015 are used. The observatory is located far from the technogenic sources of electromagnetic fields, which ensures a stable registration magnetic field over a wide frequency range.

Three components of the magnetic field induction were registered using a LEMI-018 ferrosonde magnetometer (manufactured by Lviv Centre of Institute of Space Research, Ukraine) with own built-in digital datalogger. The measurement accuracy was 0.1 nT and the sampling rate was set at 1 Hz. Ternary registration (the frame of references is as follows: the $X$-axis is directed to the geographic north, the $Y$-axis is directed to the east, and the $Z$-axis is directed vertically downwards) of the geomagnetic field (with components $B_x$, $B_y$, and $B_z$) was carried out in a stationary pavilion equipped for geomagnetic observations. The data were transmitted to the computer and then to the main server of IDG RAS by means of the interface RS-232.

The initial data for the analysis of variations were represented by numerical series with discretization of 1 hour.

5. Results and discussion

5.1 Periodicities of geomagnetic variations

As an example, figure 1, figure 2 and figure 3 shows the results of wavelet analysis of geomagnetic variations (the black line is cone of influence). It is evident from the presented picture that several well-defined periodicities are distinguished in the variations of the Earth's magnetic field. Processing all available data shows that geomagnetic variations at the observatory "Mikhnevo" are characterized by semi-annual periodicity and 27 day periodicity with two harmonics of ~ 6 days, ~ 9 days and ~ 14 days, by daily variation with harmonic periods of a day, i.e., 24 hours, 12 hours, 8 hours, 6 hours.

To explain the observed semi-annual variation of the geomagnetic field one can draw one of the basic hypotheses currently existing at the present time: equinoctial [38-40], axial [41,42] or Russell and McPherron [43]. Of the distinguished periodicities of the geomagnetic field, two-week [44-46] and 27-day periodicity [44,45,47] in geomagnetic variations are of particular interest. With a high probability, both these periodicity are associated with movements in the Earth-Moon system. The presence of a 14-day periodicity is most likely due to the characteristic period of the motion of the Earth-Moon system around its center of mass. A distinct 27-day periodicity, which is traditionally associated with the revolution of the Sun around its axis, can also have a component that is determined by the period of the Moon's motion around the Earth (the sidereal lunar cycle or the draconic lunar cycle close to it).

As a mechanism for the influence of the lunar tide on geomagnetic variations on the Earth's surface, for example, the change of current systems in the Earth's crust as a result of the gravitational action of the Moon, which causes a change in the voidness of terrestrial matter in the tidal hump, can be considered. Moreover, it should be added that the effect is greatly enhanced in zones of tectonic disturbances, where the concentration of deformations of terrestrial matter occurs [48]. Observed variations with periods of 9 and 6 days are associated with fluctuations in the ring current of the magnetosphere and with the sector structure of the interplanetary magnetic field [49,50].

It is well understood that daily variations are a consequence of electric currents flowing in the dynamo region of the ionosphere (between approximately 90 and 130 km), where the neutral wind drives an electromotive force through the ionospheric wind dynamo mechanism [51, 52]. Tidal winds move the ions across the Earth’s magnetic field producing electro-magnetic forces. Those forces drive electric
currents in the conducting $E$ region which give rise to daily variations in the magnetic field measured at the ground level. Through these mechanisms, two vortices of currents are induced, one in the northern Hemisphere (clock-wise) and another in the southern Hemisphere (counterclock-wise). At the same time, a strong eastward electric jet is formed throughout the equatorial region [53]. During high geomagnetic activity periods, magnetic signatures tend to be dominated by the effect of other currents such as magnetospheric ring currents, but the ionospheric wind-dynamo current system persists to produce solar quiet daily variations [54,55]. This breaks the structure of the wavelet transform pattern, which in solar quiet days demonstrates the monofractal behavior of the diurnal variation of the geomagnetic field (figure 2 and figure 3).

![Figure 1: Wavelet analysis of the mean daily values of geomagnetic variations](image1)

**Figure 1.** Wavelet analysis of the mean daily values of geomagnetic variations

![Figure 2: Wavelet analysis of the mean hourly values of geomagnetic variations for 2014](image2)

**Figure 2.** Wavelet analysis of the mean hourly values of geomagnetic variations for 2014

![Figure 3: Wavelet analysis of the mean hourly values of geomagnetic variations for 2016](image3)

**Figure 3.** Wavelet analysis of the mean hourly values of geomagnetic variations for 2016
5.2 Features of geomagnetic variations

The complexity of geomagnetic variations is clearly shown in figure 1, figure 2 and figure 3, which shows the results of the wavelet analysis of geomagnetic variations. It is important to emphasize that the intensity of periodic variations is not constant in time. The periods characterized by high-amplitude variations are interspersed with periods characterized by poorly expressed periodicity. This feature of geomagnetic variations is of some interest for further research, and it indicates complex processes of formation and degradation of sources of geomagnetic variations at middle latitudes.

Another feature of the observed periodicities of geomagnetic variations is their fractal character. Figure 4 demonstrates the hierarchical structure of the geomagnetic variations in the form of a cascade process. In this case, the lines of local maxima are well seen. The fragmentation of the scale is marked by the appearance of particular "forks" in the distribution of the coefficients, i.e., the bifurcation of local maxima: each of the lines characterizing the position of local maxima bifurcations, diverging into two independent local maxima. This is repeated with increasing scale, indicating the self-similarity and monofractality of the process [56]. The analysis of self-similar properties of the variation in geomagnetic field carried out on the basis of the wavelet coefficient pattern is mainly of a qualitative character.

More detail information of self-similar properties of variations in the geomagnetic field was obtained on the basis of the method of wavelet transform modulus maxima. The multifractal singularity spectra and the generalised Hurst exponents are the diagnostic information used to study the multifractal behaviour of geomagnetic field. The Hurst exponent, $h(q)$, in general, bears a non-linear relationship with the moments $q$. However, based on the nature of data under investigation, $h(q)$ may vary (remain constant) with $q$, in which case, the signal is considered to be multifractal (monofractal) in nature. Secondly, the width of the multifractal singularity spectra determines the degree of multifractality of a signal and thus the broader (narrower) the singularity spectrum, the stronger (weaker) the multifractal nature of the signal [57]. The $h(q)$ values estimated for observatory "Mikhnevo" clearly show a distinct behaviour for 27-day period and its harmonics and 1-day period. In case of former, the $h(q)$ values are greater than 1.0 and for the latter, they are less than 1.0 (table 1). This clearly supports our observation that the 27-day period and its harmonics display the presence of persistent slow-evolving fluctuations, while the 1-day period possesses persistent fast-evolving fluctuations. Persistence means that if during some time interval the magnitude of the Earth’s magnetic field increases, then we can expect it to increase during the subsequent interval of time of approximately the same duration. In contrast to the 1-day variation, the 27-day variation and its harmonics show a higher degree of
multifractality during a maximum of solar activity in comparison with the minimum of solar activity (table 1).

Table 1. Comparison of multifractal spectral widths for solar minimum year (2008) and solar maximum year (2014)

| Period (day) | 2008 | 2014 |
|-------------|------|------|
| 1           | 0.21 | 0.22 |
| 9           | 1.00 | 1.11 |
| 13.5        | 0.75 | 1.04 |
| 27          | 0.82 | 0.89 |

6. Conclusion

Wavelet analysis is a useful tool for analyzing time series with many different timescales or changes in variance. For correct wavelet analysis it is important to right choose a wavelet function and a set of scales, determine the cone of influence and to reconstruct of time series for estimating accuracy of applying wavelet transform.

Wavelet transform of geomagnetic variation with Morlet wavelet allows find a number of periodicities in the geomagnetic field, namely: semiannual variations, 27-day periodicity with harmonics 13.5, 9 and 6 days, daily variation with harmonics 24, 12, 8, 6 hours. The data obtained in this study prove that, in general, the amplitude of geomagnetic variations at middle latitudes (observatory "Mikhnevo") has a sporadic and scaling character. The discovered effect of the alternation of periods of increasing and degradation (intermittency) in the intensity of geomagnetic variation attracts interest. In general, geomagnetic field variations do not have a simple monofractal scaling behavior (only 1-day variation has monofractal character), which can be described by a single scaling exponent, but is characterized by a more complex scaling behavior, which can be described only by several scaling exponents (multifractal behavior). The 27-day variation and its harmonics show a higher degree of multifractality during a maximum of solar activity in comparison with the minimum of solar activity.

Further studies are necessary to determine that identified features are characteristic for all regions of the middle latitudes. In addition, the increase in the length of time series can also highlight additional features of geomagnetic variations.

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