Classically integrable field theories with defects

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ABSTRACT

Some ideas and remarks are presented concerning a possible Lagrangian approach to the study of internal boundary conditions relating integrable fields at the junction of two domains. The main example given in the article concerns single real scalar fields in each domain and it is found that these may be free, of Liouville type, or of sinh-Gordon type.

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1 Introduction

Recently, there has been some interest in the study of integrable classical or quantum field theories restricted to a half-line, or interval, by imposing integrable boundary conditions, see for example [1, 2, 3, 4, 5, 6, 7]. The simplest situation, which is also the best understood, contains a real self-interacting scalar field \( \phi \) with either a periodic (cos), or non-periodic (cosh) potential. The sinh-Gordon model can be restricted to the left half-line \(-\infty \leq x \leq 0\), without losing integrability, by imposing the boundary condition

\[
\partial_x \phi \big|_{x=0} = \frac{\sqrt{2}m}{\beta} \left( \varepsilon_0 e^{-\sqrt{2}/\phi(0,t)} - \varepsilon_1 e^{\sqrt{2}/\phi(0,t)} \right),
\]

where \( m \) and \( \beta \) are the bulk mass scale and coupling constant, respectively, and \( \varepsilon_0 \) and \( \varepsilon_1 \) are two additional parameters [2, 6]. This set of boundary conditions generally breaks the reflection symmetry \( \phi \rightarrow -\phi \) of the model although the symmetry is explicitly preserved when \( \varepsilon_0 = \varepsilon_1 \equiv \varepsilon \). The restriction of the sinh-Gordon model to a half-line is a considerable complication, and renders the model more interesting than it appears to be in the bulk. This is because there will in general be additional states in the spectrum associated with the boundary, together with a set of reflection factors compatible with the bulk S-matrix (see [2, 3, 5, 8]). The weak-strong coupling duality enjoyed by the bulk theory emerges in a new light [9, 10, 11].

In this article a slightly different situation is explored. There is no reason in principle why the point \( x = 0 \) should not be an internal boundary linking a field theory in the region \( x < 0 \) with a (possibly different) field theory in the region \( x > 0 \). The quantum version of this setup has been examined before and imposing the requirements of integrability was found to be highly restrictive. This sort of investigation was pioneered by Delfino, Mussardo and Simonetti some years ago [12], and there has also been some recent interest [13, 14]. However, the objective of this article is to explore a Lagrangian version of this question and derive the conditions linking the two field theories at their common boundary. This situation does not appear to have been discussed previously, although the results turn out to be interesting and reminiscent of some earlier work by Tarasov [15].

Internal boundary conditions will be referred to as ‘defect’ conditions.

Integrability in the bulk sinh-Gordon model requires the existence of conserved quantities labelled by odd spins \( s = \pm 1, \pm 3, \ldots \), and some of these should survive even in the presence of boundary conditions. Since boundary conditions typically violate translation invariance, it is expected that the ‘momentum-like’ combinations of conserved quantities will not be preserved. However, the ‘energy-like’ combinations, or some subset of them, might remain conserved, at least when suitably modified (see [2] for the paradigm). As was the case for the theory restricted to a half-line, the spin three charge already supplies the most general restrictions on the boundary condition. The Lax pair approach developed in [7] can be adapted to this new context and used to re-derive the boundary conditions, thereby demonstrating that the preservation of higher spin energy-like charges imposes no further restrictions on the boundary conditions. This will be discussed below in section 4.
The starting point for the discussion is the Lagrangian density for the pair of real scalar fields \( \phi_1, \phi_2 \):

\[
\mathcal{L} = \theta(-x) \left( \frac{1}{2} (\partial \phi_1)^2 - V_1(\phi_1) \right) + \theta(x) \left( \frac{1}{2} (\partial \phi_2)^2 - V_2(\phi_2) \right) + \delta(x) \left( \frac{1}{2} (\phi_1 \partial_t \phi_2 - \phi_2 \partial_t \phi_1) - B(\phi_1, \phi_2) \right),
\]

in which the bulk potentials \( V_1, V_2 \) depend only on the fields \( \phi_1, \phi_2 \), respectively, while the boundary potential \( B \) depends on the values of both fields at the boundary \( x = 0 \). The part of the boundary term depending on the time derivatives of the fields is not the most general possibility. However, although excluding terms of higher order in time derivatives, it is sufficiently general for present purposes. The field equations and associated boundary conditions are

\[
\begin{align*}
\partial^2 \phi_1 &= -\frac{\partial V_1}{\partial \phi_1}, \quad x < 0 \quad (1.3) \\
\partial^2 \phi_2 &= -\frac{\partial V_2}{\partial \phi_2}, \quad x > 0 \quad (1.4) \\
\partial_x \phi_1 - \partial_t \phi_2 &= -\frac{\partial B}{\partial \phi_1}, \quad x = 0 \quad (1.5) \\
\partial_x \phi_2 - \partial_t \phi_1 &= \frac{\partial B}{\partial \phi_2}, \quad x = 0. \quad (1.6)
\end{align*}
\]

\section{Consequences of the spin three conservation law}

For a single real scalar field \( \phi \) in the bulk, the spin three densities satisfy

\[
\partial_{\mp} T_{\pm 4} = \partial_{\pm} \Theta_{\mp 2}, \quad (2.1)
\]

where

\[
\begin{align*}
T_{\pm 4} &= \lambda^2 (\partial_{\pm} \phi)^4 + (\partial_{\pm}^2 \phi)^2 \\
\Theta_{\pm 2} &= -\frac{1}{2} (\partial_{\pm} \phi)^2 \frac{\partial^2 V}{\partial \phi^2} \quad (2.2)
\end{align*}
\]

and (2.1) requires

\[
\frac{\partial^3 V}{\partial \phi^3} = 4\lambda^2 \frac{\partial V}{\partial \phi}. \quad (2.3)
\]

Thus, the only possibilities are a free massive field \((\lambda = 0, V = m^2 \phi^2/2)\), a free massless field \((V = 0, \lambda \neq 0)\), or

\[
V = A e^{2\lambda \phi} + B e^{-2\lambda \phi},
\]

2
with $A$, $B$ being arbitrary constants.

With two fields participating in different regions, the energy-like conserved quantities will be given by the following expressions:

$$E_s = \int_{-\infty}^{0} dx \left( T_{s+1}^{(1)} + T_{s-1}^{(1)} - \Theta_{s-1}^{(1)} - \Theta_{s+1}^{(1)} \right) + B_s,$$

$$+ \int_{0}^{\infty} dx \left( T_{s+1}^{(2)} + T_{s-1}^{(2)} - \Theta_{s-1}^{(2)} - \Theta_{s+1}^{(2)} \right) + B_s,$$

for a suitable boundary functional of $\phi$, $B_s$. The latter will be determined by requiring

$$\frac{dE}{dt} = \left( T_{s+1}^{(1)} - T_{s-1}^{(1)} + \Theta_{s-1}^{(1)} - \Theta_{s+1}^{(1)} \right)_{x=0}$$

$$+ \frac{dB_s}{dt} = 0.$$  \hspace{1cm} (2.4)

For the energy itself $B_1 \equiv B$. For other values of $s$ the argument is the familiar one from [2], in the sense that the existence of $B_s$ and making use of (1.3) places severe constraints on the boundary potential $B$. Thus, for $s = 3$, and after some algebra:

$$\lambda_1^2 - \lambda_2^2 = 0$$

$$2 \left[ \lambda_1 \left( \frac{\partial B}{\partial \phi_1} \right)^2 - \lambda_2 \left( \frac{\partial B}{\partial \phi_2} \right)^2 \right] - V_1'' + V_2'' = 0$$  \hspace{1cm} (2.6)

$$\frac{\partial^3 B}{\partial \phi_2^2 \partial \phi_1} - \lambda_1 \frac{\partial B}{\partial \phi_1} = \frac{\partial^3 B}{\partial \phi_2^2 \partial \phi_2} - \lambda_2 \frac{\partial B}{\partial \phi_2} = 0$$

$$\frac{\partial^3 \tilde{B}}{\partial \phi_2^3} - \lambda_2 \frac{\partial \tilde{B}}{\partial \phi_2} = \frac{\partial^3 B}{\partial \phi_1^3} - \lambda_1 \frac{\partial B}{\partial \phi_1} = 0.$$  \hspace{1cm} (2.7)

There are several possible solutions to these constraints. The typical one, assuming neither of the fields is free and massive in its bulk domain, requires $\lambda_1 = \lambda_2 = \lambda \neq 0$. Consequently (ignoring an overall additive constant),

$$\tilde{B} = a e^{\lambda (\phi_1 + \phi_2)} + b e^{\lambda (\phi_1 - \phi_2)} + c e^{-\lambda (\phi_1 - \phi_2)} + d e^{-\lambda (\phi_1 + \phi_2)},$$

$$V_1 = A_1 e^{2 \lambda \phi_1} + B_1 e^{-2 \lambda \phi_1}$$

$$V_2 = A_2 e^{2 \lambda \phi_2} + B_2 e^{-2 \lambda \phi_2},$$

where $a, b, c, d$ are constants, and the bulk potentials are given by

$$A_1 = 2 \lambda^2 ab, \quad A_2 = 2 \lambda^2 ac, \quad B_1 = 2 \lambda^2 cd, \quad B_2 = 2 \lambda^2 bd.$$  \hspace{1cm} (2.10)

Notice that this case allows one of the bulk fields to be massless and free but the other need not necessarily be (for example, taking $c = d = 0$ leads to a free massless field in $x > 0$, with a Liouville field in $x < 0$).
The alternative is that both fields are free and massive, so that \( \lambda_1 = \lambda_2 = 0 \). In that case, the conditions on the boundary and bulk potentials require the two masses to be the same, with a boundary potential of the general quadratic form:

\[
B = a\phi_1^2 + b\phi_1\phi_2 + c\phi_2^2,
\]

where \( a, b, c \) are constants.

If one of the bulk fields is free and massless (say \( \phi_1 \)), the other field may either also be free and massless, in which case the boundary potential has the form

\[
B = ae^{(\phi_1 \pm \phi_2)} + de^{-(\phi_1 \pm \phi_2)},
\]

or the other field may be Liouville, in which case the boundary term has the form

\[
B = \frac{m}{\beta^2} e^{\beta \phi_2 / \sqrt{2}} \left( \sigma e^{\beta \phi_1 / \sqrt{2}} + \frac{1}{\sigma} e^{-\beta \phi_1 / \sqrt{2}} \right),
\]

Finally, both fields could be of Liouville type. For example, choosing \( d = 0 \) in (2.10) leads to \( B_1 = B_2 = 0, \ A_1 \neq 0, \ A_2 \neq 0 \) and both potentials are Liouville potentials.

Returning to the general case, the fields may be shifted by a constant in each bulk domain so that in (2.11) \( A_1 = B_1 \) and \( A_2 = B_2 \), the latter in turn implying \( c = \pm b, \ d = \pm a \). It is convenient to choose \( \lambda = \beta / \sqrt{2} \), in order to agree with standard conventions for the sinh-Gordon model, and then to let \( a = m\sigma/\beta^2, \ b = m/\beta^2 \sigma \). With those choices, the bulk and boundary potentials are:

\[
V_1 = \frac{m^2}{\beta^2} \left( e^{\beta \phi_1} + e^{-\beta \phi_1} \right)
\]

\[
V_2 = \pm \frac{m^2}{\beta^2} \left( e^{\beta \phi_2} + e^{-\beta \phi_2} \right)
\]

\[
B = \frac{m\sigma}{\beta^2} \left( e^{\beta (\phi_1 + \phi_2) / \sqrt{2}} \pm e^{-\beta (\phi_1 + \phi_2) / \sqrt{2}} \right) + \frac{m}{\beta^2 \sigma} \left( e^{\beta (\phi_1 - \phi_2) / \sqrt{2}} \pm e^{-\beta (\phi_1 - \phi_2) / \sqrt{2}} \right)
\]

where, in all of these the ‘\( \pm \)’ signs are strictly correlated. (In the sinh-Gordon model the relative signs cannot be adjusted by a real shift of one of the fields). There is a single free parameter \( \sigma \) in the defect condition, which is perhaps puzzling since the half-line boundary condition (2.10) allows two free parameters.

Notice, the model might be restricted to a half-line if \( \phi_2 \) say, were to be set to a constant value. Then the boundary condition satisfied by \( \phi_1 \) would be of the general type (1.1). However, typically, the constant value of \( \phi_2 \) would not satisfy the equation of motion in \( x > 0 \). On the other hand, even if the equation of motion were to be satisfied with a constant \( \phi_2 \) in \( x > 0 \) (ie for real \( \beta, \ \phi_2 = 0 \)), the boundary condition for \( \phi_2 \) would generally not be satisfied at \( x = 0 \).

Notice too, in none of these cases is there any reason why \( \phi_1 = \phi_2 \) at \( x = 0 \). Given the usual definition of ‘topological’ charge:

\[
Q = \int_{-\infty}^{0} dx \partial_x \phi_1 + \int_{0}^{\infty} dx \partial_x \phi_2 = \phi_2|_\infty - \phi_1|_{-\infty} + \phi_1|_0 - \phi_2|_0,
\]
it is clear the difference $\phi_1 - \phi_2$ measures the strength of the defect at $x = 0$. In the sine-Gordon model, where similar considerations would apply, this would appear to indicate that topological charge need not be preserved, and hence that a defect need not necessarily preserve soliton number.

It is worth noting that if eqs(1.3) were satisfied simultaneously in the bulk then the ‘defect’ conditions with the choice of $B$ given by (2.10) would become a Bäcklund transformation [16] relating the two fields $\phi_1$ and $\phi_2$. Indeed, a sufficient condition for the Bäcklund transformation to work in the bulk would be:

$$\frac{\partial^2 B}{\partial \phi^2_1} = \frac{\partial^2 B}{\partial \phi^2_2} = \left( \frac{\partial B}{\partial \phi_1} \right)^2 - \left( \frac{\partial B}{\partial \phi_2} \right)^2 = 2(V_1 - V_2).$$

Clearly both of these are satisfied in all the cases mentioned above. In the present setup, the ‘Bäcklund transformation’ at $x = 0$ represents the boundary between two domains. This sheds an interesting new light on the Bäcklund transformation itself.

Generalising the idea, any number of defects may be represented similarly at domain boundaries $x = x_1, x_2, \ldots$, and at each boundary the defect conditions ought to retain the same form, albeit with different free parameters $\sigma_i$ at each. In the bulk, the Bäcklund transformation between two solutions of the sine-Gordon equation generally changes soliton number (typically adding or subtracting a soliton), which appears to corroborate the suggestion above that a defect could allow a change of topological charge. In addition, different domains may contain fields of different character provided they are compatible with the boundary condition.

One interesting further point. The canonical momentum density (which is not expected to be preserved because of the loss of translation invariance) is given by:

$$P = \int^{0}_{-\infty} dx \partial_t \phi_1 \partial_x \phi_1 + \int^{\infty}_{0} dx \partial_t \phi_2 \partial_x \phi_2.$$

Although $P$ is not conserved, using the defect conditions (1.3) it is not difficult to derive the following:

$$\frac{dP}{dt} = \left( -\partial_t \phi_2 \frac{\partial B}{\partial \phi_2} - \partial_t \phi_1 \frac{\partial B}{\partial \phi_1} + \frac{1}{2} \left( \frac{\partial B}{\partial \phi_1} \right)^2 - \left( \frac{\partial B}{\partial \phi_2} \right)^2 - V_1 + V_2 \right)_{x=0}. \quad (2.15)$$

The right hand side of (2.15) is a total time derivative provided that at $x = 0$

$$\left( \frac{\partial B}{\partial \phi_1} \right)^2 - \left( \frac{\partial B}{\partial \phi_2} \right)^2 - 2V_1 + 2V_2 = 0, \quad \text{and} \quad \frac{\partial^2 B}{\partial \phi^2_1} = \frac{\partial^2 B}{\partial \phi^2_2}. \quad (2.16)$$

These conditions are precisely satisfied by the boundary term indicated in (2.10) (and indeed coincide with the conditions in the bulk mentioned earlier for a working Bäcklund transformation). In other words, there exists a functional of $\phi_1, \phi_2$, call it $P_B$, so that $P + P_B$ is conserved. There appears to be a ‘total’ momentum which is preserved containing bulk and defect contributions. Thus, the fields can exchange both energy and momentum with the defect despite the lack of translation invariance. Clearly, there is a generalisation of this idea to a collection of defects situated at $x = x_1, x_2, \ldots$. 
3 The absence of reflection by defects

Consider the consequences of a linearised version of (2.13) by setting
\[ \phi_1 = e^{-i\omega t} \left( e^{ikx} + P(k) e^{-ikx} \right), \quad \phi_1 = e^{-i\omega t} T(k) e^{ikx}, \]
where \( R \) and \( T \) are reflection and transmission coefficients, respectively. (Strictly speaking the fields are the real parts of these expressions; however, in the linearised situation this is immaterial.) Imposing the defect conditions, it is convenient to set \( k = 2m \sinh \theta, \quad \omega = 2m \cosh \theta, \) and \( \sigma = e^\rho, \) to discover
\[
R(k) = 0, \quad T(k) = -\frac{2i \cosh \theta - (\sigma - 1/\sigma)}{2i \sinh \theta - (\sigma + 1/\sigma)} = -\frac{i \sinh \left( \frac{\theta + \rho}{2} - \frac{\pi}{4} \right)}{\sinh \left( \frac{\theta + \rho}{2} + \frac{\pi}{4} \right)}.
\] (3.1)

This is surprising, given the remarks about Bäcklund transformations, and should be compared with the results reported in [12]. There, the emphasis was different. The equations expressing the compatibility of reflection and transmission with the bulk factorisable S-matrix of a general model was found to be highly constraining and required the bulk S-matrix to satisfy \( S^2 = 1. \)

In the context introduced here the question would be to find all the defect transmission factors compatible with the sinh-Gordon S-matrix, given that most probably there can be no reflection.

To investigate what happens to a sine-Gordon soliton it is convenient to set \( \beta = 1/\sqrt{2}, \quad m = 1/2 \) and to write the bulk equations and defect condition as follows:
\[
\begin{align*}
x < 0: & \quad \partial^2 \phi_1 = -\sin \phi_1, \\
x > 0: & \quad \partial^2 \phi_2 = -\sin \phi_2, \\
x = 0: & \quad \partial_x \phi_1 - \partial_t \phi_2 = -\sigma \sin \left( \frac{\phi_1 + \phi_2}{2} \right) - \frac{1}{\sigma} \sin \left( \frac{\phi_1 - \phi_2}{2} \right), \\
& \quad \partial_x \phi_2 - \partial_t \phi_1 = \sigma \sin \left( \frac{\phi_1 + \phi_2}{2} \right) - \frac{1}{\sigma} \sin \left( \frac{\phi_1 - \phi_2}{2} \right).
\end{align*}
\] (3.4)

Then, a single soliton solution in the two regions has the form (see for example [17])
\[
e^{i\phi_2/2} = \frac{1 - iE_a}{1 + iE_a}, \quad E_a = C_a e^{\alpha_a x + \beta_a t}, \quad \alpha_a^2 - \beta_a^2 = 1, \quad a = 1, 2,
\]
where \( C_a \) is real. In order to be able to satisfy the conditions (3.4) the time dependence must match in the two domains \( (\beta_1 = \beta_2) \) and the constants \( C_1, \) \( C_2 \) are related by
\[
C_2 = \left( \frac{e^{\theta} + \sigma}{e^{\theta} - \sigma} \right) C_1,
\]
where as before it is convenient to let \( \alpha_1 = \alpha_2 = \cosh \theta \) and \( \beta_1 = \beta_2 = \sinh \theta. \) Thus, the effect of the defect is to delay or advance the soliton as it passes through. One curious feature is that the defect can absorb or emit a soliton but only at a special value of rapidity. This is most easily seen by examining (3.6) and noting that \( C_2 \) vanishes for \( \sigma < 0 \) and \( e^\theta = |\sigma|, \) and \( C_2 \) is infinite for \( \sigma > 0 \) and \( e^\theta = \sigma. \) In either case, the implication is that \( \phi_2 = 0 \) and a soliton with this special rapidity, approaching the defect from the region \( x < 0, \) will be absorbed by it.
4 Defect Lax pairs

In this section the intention is to give an outline of the kind of approach one might adopt to set up Lax pairs in the presence of defects.

To construct Lax pairs along the lines suggested in [7] it is necessary to separate slightly the boundary conditions in the two regions $x < 0$ and $x > 0$, imposing the $\phi_1$ boundary condition at $x = a$, and the $\phi_2$ boundary condition at $x = b > a$, and to assume both fields are defined in the ‘overlap’ region $a \leq x \leq b$. Since the same framework applies to all the Toda, or affine Toda field theories this section will be quite general. Thus, with the same choices of coupling and mass scale as in the last section, the defect Lax pairs for models based on simply-laced root data are:

\[
\hat{a}_0^{(1)} = a_0^{(1)} - \frac{1}{2} \theta(x - a) \left( \partial_x \phi_1 - \partial_t \phi_2 + \frac{\partial B}{\partial \phi_1} \right) H
\]

\[
\hat{a}_1^{(1)} = \theta(a - x) a_1^{(1)}
\]

\[
\hat{a}_0^{(2)} = a_0^{(2)} - \frac{1}{2} \theta(b - x) \left( \partial_x \phi_2 - \partial_t \phi_1 - \frac{\partial B}{\partial \phi_2} \right) H
\]

\[
\hat{a}_1^{(2)} = \theta(x - b) a_1^{(2)},
\]

where for $p = 1, 2$,

\[
a_0^{(p)} = \frac{1}{2} \left[ \partial_x \phi_p \cdot H + \sum_i \sqrt{n_i} \epsilon^{\alpha_i \cdot \phi_p / 2} \left( \lambda E_{\alpha_i} - \frac{1}{\lambda} E_{-\alpha_i} \right) \right]
\]

\[
a_1^{(p)} = \frac{1}{2} \left[ \partial_t \phi_p \cdot H + \sum_i \sqrt{n_i} \epsilon^{\alpha_i \cdot \phi_p / 2} \left( \lambda E_{\alpha_i} + \frac{1}{\lambda} E_{-\alpha_i} \right) \right].
\]

Here $H$ are the generators in the Cartan subalgebra of the semi-simple Lie algebra whose simple roots are $\alpha_i$, $i = 1, \ldots, r$, and $E_{\pm \alpha_i}$ are the generators corresponding to the simple roots or their negatives. If the theory is affine then the lowest root $\alpha_0 = -\sum_i n_i \alpha_i$ is appended to the set of simple roots. In either case, affine or non-affine, the two expressions (4.2) are easily checked to be a Lax pair (for more details about this, and further references, see [18]).

To ensure the Lax pair defined by (4.1) really corresponds to a zero curvature in the overlap of the two regions there should exist a group element $\mathcal{K}$ having the property

\[
\partial_t \mathcal{K} = \mathcal{K} \hat{a}_0^{(2)}(t, b) - \hat{a}_0^{(1)}(t, a) \mathcal{K}.
\]

Setting

\[
\mathcal{K} = e^{-\phi_2 H / 2} \tilde{\mathcal{K}} e^{\phi_1 H / 2}
\]

with $\partial_t \tilde{\mathcal{K}} \equiv 0$ has the effect of removing the time derivatives from the defect term in (4.1),

\[
\hat{a}_0^{(2)}(t, b) - \hat{a}_0^{(1)}(t, a) \mathcal{K} = 0.
\]
leading to
\[
\frac{\partial B}{\partial \phi_1} \cdot H \bar{K} + \bar{K} H \cdot \frac{\partial B}{\partial \phi_2} = \sum_i \sqrt{n_i} \left( -\lambda e^{\alpha_i (\phi_1 + \phi_2)/2} \left[ \bar{K}, E_{\alpha_i} \right] + \frac{1}{\lambda} \left( e^{\alpha_i (\phi_2 - \phi_1)/2} \bar{K} E_{-\alpha_i} - e^{\alpha_i (\phi_1 - \phi_2)/2} E_{-\alpha_i} \bar{K} \right) \right) \quad (4.4)
\]

which is effectively an equation for both \( B \) and \( \bar{K} \), see [7]. However, the structure of (4.4) is not the quite the same as that encountered previously. Nevertheless, a perturbative solution can be sought of the form
\[
\bar{K} = 1 + \frac{k_1}{\lambda} + \frac{k_2}{\lambda^2} + \ldots,
\]
the \( O(\lambda) \) terms are identically satisfied, and the other terms lead to the following set of expressions
\[
O(1): \quad \left( \frac{\partial B}{\partial \phi_1} + \frac{\partial B}{\partial \phi_2} \right) H = -\sum_i \sqrt{n_i} e^{\alpha_i (\phi_1 + \phi_2)/2} \left[ k_1, E_{\alpha_i} \right],
\]
\[
O(1/\lambda): \quad \frac{\partial B}{\partial \phi_1} \cdot H k_1 + k_1 H \cdot \frac{\partial B}{\partial \phi_2} = \sum_i \sqrt{n_i} \left( -e^{\alpha_i (\phi_1 + \phi_2)/2} \left[ k_2, E_{\alpha_i} \right] + E_{-\alpha_i} \left( e^{\alpha_i (\phi_2 - \phi_1)/2} - e^{\alpha_i (\phi_1 - \phi_2)/2} \right) \right),
\]
\[
\ldots
\]

The first of these can be satisfied for an arbitrary Toda model provided \( k_1 = \sum_i \rho_i E_{-\alpha_i} \) and
\[
B = \sum_i \sqrt{n_i} \rho_i e^{\alpha_i (\phi_1 + \phi_2)/2} + \tilde{B}(\phi_1 - \phi_2);
\]
however, the \( O(1/\lambda) \) equation does not appear to be compatible with all choices of simple roots. Indeed, most Toda models appear to be ruled out. For the simplest, based on the \( A_1 \) root system, a complete expression for \( \bar{K} \) is:
\[
\bar{K} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \rho \frac{1}{\lambda} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \rho_0 = \rho_1 = \rho, \quad (4.6)
\]
and
\[
\tilde{B}(\phi_1 - \phi_2) = \frac{1}{\rho} \left( e^{\alpha (\phi_1 - \phi_2)/2} + e^{\alpha (\phi_2 - \phi_1)/2} \right). \quad (4.7)
\]
Here, the conventions used are:
\[
\alpha_1 = \alpha = \sqrt{2} = -\alpha_0, \quad E_{\alpha} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = E_{-\alpha}^\dagger, \quad H = (1/\sqrt{2}) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]
In other words, it appears this style of Lax pair can really only work in the presence of a defect for the sinh-Gordon or Liouville models. This result is identical with results obtained by examining
the spin two conserved charges for the models based on the $A_n$ root systems for $n \geq 2$, although these will not be reported in detail here.

It was pointed out many years ago that the possible integrable boundary conditions are very constrained for all Toda models apart from the sinh-Gordon model in the sense that, in most cases, only a discrete set of parameters may be introduced at a boundary \cite{4,7}. It now appears that defects are still more strongly constrained and generally cannot exist in the Lagrangian form postulated in \cite{1,2}.

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