Tests of Cosmic Censorship in the Ernst Spacetime

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Abstract

The Ernst spacetime is a solution of the Einstein-Maxwell equations describing two charged black holes accelerating apart in a uniform electric (or magnetic) field. As the field approaches a critical value, the black hole horizon appears to touch the acceleration horizon. We show that weak cosmic censorship cannot be violated by increasing the field past this critical value: The event horizon remains intact. On the other hand, strong cosmic censorship does appear to be violated in this spacetime: For a certain range of parameters, we find evidence that the inner horizon is classically stable.
1. Introduction

One of the most important open questions in classical general relativity is Penrose’s cosmic censorship hypothesis [1], which states that naked singularities cannot be created by realistic physical processes. An early test of cosmic censorship [2] involved charged black holes, which have two horizons; an inner Cauchy horizon as well as the event horizon. These horizons coincide in the extremal limit $Q = M$, and are absent for $Q > M$. Consider adding charged test particles with $q > m$ to a black hole with $Q < M$. If one could increase the black hole charge $Q$ faster than its mass $M$, one could exceed the extremal limit, and turn a black hole into a naked singularity. However Wald showed [2] that cosmic censorship could not be violated this way. For a nearly extremal black hole, in order for a $q > m$ test particle to reach the horizon, it must be sent in with sufficient kinetic energy so that the net increase in the mass of the black hole exceeds the increase in the charge. One can interpret this result as providing evidence that one cannot destroy the event horizon of a black hole by forcing it to meet an inner Cauchy horizon.

Now consider a spacetime with a horizon outside the black hole event horizon. One can ask if it is possible in this case to destroy the event horizon by causing it to meet the horizon outside. One example is a black hole in de Sitter space, which has a cosmological horizon outside the event horizon. In this case, there is a maximum size for a black hole set by the cosmological constant [3]. If one could increase the effective cosmological constant, one could easily violate cosmic censorship. All one would have to do is create a black hole and then increase the cosmological constant so that the black hole mass exceeds the maximum allowed value. Unfortunately, it is unlikely that one can increase the effective cosmological constant without violating a local energy condition.

Another example of a spacetime with a horizon outside the event horizon is one describing a charged black hole uniformly accelerating in a background magnetic field\footnote{One could equally well consider electric fields, but the form of this field is somewhat more complicated in the solutions we will consider.}. This spacetime has an acceleration horizon outside the event horizon. As one increases the magnetic field, one increases the acceleration which causes the event horizon to move closer to the acceleration horizon. Unlike the effective cosmological constant, it is possible in principle to physically increase the strength of a background magnetic field. Can one increase the field so high that the event horizon touches the acceleration horizon and destroys the black hole? This is the question we wish to address in this paper. Since the event hori-
zon has a finite size, it would appear that it would touch the acceleration horizon at a finite value of the acceleration and hence a finite value of the magnetic field. Exceeding this value should turn the black hole into a naked singularity, providing a rather ‘clean’ counterexample to cosmic censorship.

This question appears to be straightforward to answer since the solution to the Einstein-Maxwell equations describing a pair of oppositely charged black holes uniformly accelerating in a background magnetic field was found by Ernst [4] about twenty years ago. It was noticed in [5] that the event horizon and acceleration horizon appeared to touch at a finite value of the magnetic field. In section 2, we review this calculation and discuss further properties of the Ernst metric when the background field is large. It turns out that a proper understanding of the situation requires a special limit of the Ernst metric. This is performed in section 3, where it is shown that event horizon never actually meets the acceleration horizon.

In section 4 we consider a different test of cosmic censorship. A strong form of this hypothesis states that generic solutions should be globally hyperbolic, so that singularities are not visible even inside black holes. This appears to be supported by studies showing that the inner Cauchy horizon of the Reissner-Nordström solution is unstable [6]. However, it has been shown that for charged black holes in de-Sitter space, the situation is different: There are a range of parameters for which the inner horizon of the Reissner-Nordström de Sitter solution is stable [7]. We will argue that the same is true for accelerating black holes. Thus even without a cosmological constant, it appears that one can violate strong cosmic censorship. Section 5 contains some concluding comments.

2. The Ernst Solution with Large Magnetic Fields

The solution describing the background magnetic field was found by Melvin [8] and is given by

\[ ds^2 = \Lambda^2 \left[ -dt^2 + dz^2 + d\rho^2 \right] + \Lambda^{-2} \rho^2 d\phi^2 \]

\[ A_\phi = \frac{B \rho^2}{2\Lambda} \]  

(2.1)

\[ \Lambda = 1 + \frac{1}{4} B^2 \rho^2 . \]

\( A_\phi \) is the only nonzero component of the vector potential. This solution describes a static, cylindrically symmetric flux tube. The flux tube has a radius of order \( 1/B \) and field
strength of order $B$. The total flux passing through a $z =$ constant plane is $\Phi = 4\pi / B$. Notice that as $B$ increases, the total flux decreases.

The solution describing two oppositely charged black holes uniformly accelerating in this background was found by Ernst [4] and takes the form

$$ds^2 = \frac{\Lambda^2}{A^2 (x - y)^2} \left[ G(y) dt^2 - G^{-1}(y) dy^2 + G^{-1}(x) dx^2 \right] + \frac{G(x)}{A^2 (x - y)^2 \Lambda^2} d\phi^2$$

$$G(\xi) = (1 - \xi^2 - r_+ A \xi^3) (1 + r_- A \xi)$$

$$\Lambda (x, y) = \left[ 1 + \frac{1}{2} q B x \right]^2 + \frac{B^2 G(x)}{4 A^2 (x - y)^2}$$

$$A_\phi = - \frac{2}{B \Lambda} \left( 1 + \frac{1}{2} q B x \right).$$

with $q^2 = r_+ r_-$. The solution depends on four parameters $r_+, r_-, A, B$ which are related to the mass, charge, and acceleration of the black holes, and the background magnetic field.

The coordinate $y$ has the range $-\infty < y < x$. There is a curvature singularity at $y = -\infty$, and $y \approx x$ describes an asymptotic region that approaches the Melvin solution (2.1). It is convenient to introduce the following notation. Let $\xi_2 \leq \xi_3 < \xi_4$ be the three roots of the cubic in $G$. We also define $\xi_1 \equiv -1/(r_- A)$ and choose $r_-$ so that $\xi_1 \leq \xi_2$. The function $G(\xi)$ then takes the form

$$G(\xi) = -(r_+ A)(r_- A)(\xi - \xi_1)(\xi - \xi_2)(\xi - \xi_3)(\xi - \xi_4).$$

These roots are all real only if $0 < r_+ A \leq 2/(3\sqrt{3})$. In this case, the surface $y = \xi_1$ is the inner Cauchy horizon, $y = \xi_2$ is the black hole event horizon, and $y = \xi_3$ is the acceleration horizon. The limit $\xi_1 = \xi_2$ corresponds to an extreme black hole and was discussed in [3]. We are interested here in a different type of extremal limit: If $r_+ A = 2/(3\sqrt{3})$, $\xi_2 = \xi_3$, so the event horizon appears to coincide with the acceleration horizon. If $r_+ A > 2/(3\sqrt{3})$, the two roots $\xi_2, \xi_3$ are both complex and the spacetime has only one horizon at $y = \xi_1$. This spacetime describes two naked singularities accelerating apart. The question we wish to address is whether we can physically cause $r_+ A$ to increase past the critical value $2/(3\sqrt{3})$ by increasing the background magnetic field.

The roots $\xi_2, \xi_3$, and $\xi_4$ depend only on the single parameter $r_+ A$. As $r_+ A$ increases from 0 to $2/(3\sqrt{3})$, $\xi_4$ decreases monotonically from 1 to $\sqrt{3}/2$, $\xi_3$ decreases monotonically
from $-1$ to $-\sqrt{3}$, and $\xi_2$ increases monotonically from $-\infty$ to $-\sqrt{3}$. It is possible to determine $\xi_2$ and $\xi_3$ in terms of $\xi_4$ only, with the result

$$\xi_2 = \frac{-\xi_4 - \xi_4 (4\xi_4^2 - 3)^{1/2}}{2 (1 - \xi_4^2)}, \quad \xi_3 = \frac{-\xi_4 + \xi_4 (4\xi_4^2 - 3)^{1/2}}{2 (1 - \xi_4^2)}.$$ (2.4)

The coordinates $(x, \phi)$ in (2.2) are angular coordinates. To keep the signature of the metric fixed, the coordinate $x$ is restricted to the range $\xi_3 \leq x \leq \xi_4$ in which $G(x)$ is positive. One can always choose the range of $\phi$ so that there is no conical singularity at one pole $x = \xi_3$ or $x = \xi_4$, but for general choices of the parameters, there will be a singularity at the other. To ensure that the metric is free of conical singularities at both poles, we must require

$$-\frac{G'(\xi_3)}{G'(\xi_4)} = \left[ \frac{\Lambda(\xi_3)}{\Lambda(\xi_4)} \right]^2 = \left( \frac{1 + \frac{1}{2}qB\xi_3}{1 + \frac{1}{2}qB\xi_4} \right)^4$$ (2.5)

and set

$$\Delta \phi = \frac{4\pi}\Lambda^2 (\xi_3) \frac{G'(\xi_3)}{G'(\xi_4)}$$ (2.6)

where we have defined $\Lambda(\xi_i) \equiv \Lambda(x = \xi_i)$. Physically, (2.6) is the condition on the magnetic field which ensures that the applied force is consistent with the given acceleration.

We first show that the external magnetic field remains finite when the two roots $\xi_2$ and $\xi_3$ coincide. In this case, $G(\xi)$ has a double zero, so $G'(\xi_3) = 0$. It then follows from (2.5) that $1 + \frac{1}{2}qB\xi_3 = 0$. Since $\xi_3 = -\sqrt{3}$ at the double zero, we obtain $qB = 2/\sqrt{3}$. So the product $qB$ remains finite in this limit. However, $q$ and $B$ are the physical black hole charge and asymptotic magnetic field only when the field is small. In general, the black hole charge is

$$\hat{q} = \frac{1}{4\pi} \int F = \frac{\Delta \phi}{4\pi} \left[ A_\phi(x = \xi_4) - A_\phi(x = \xi_3) \right]$$

$$= \frac{q\Delta \phi(\xi_4 - \xi_3)}{4\pi(1 + \frac{1}{2}qB\xi_3)(1 + \frac{1}{2}qB\xi_4)}$$ (2.7)

and the Ernst metric asymptotically approaches the Melvin solution with parameter [3][4]

$$\hat{B} = \frac{BG'(\xi_3)}{2\Lambda^{3/2}(\xi_3)}$$ (2.8)

Using (2.9), the product of the physical black hole charge and asymptotic magnetic field is thus

$$\hat{q}\hat{B} = \frac{qB(\xi_4 - \xi_3)}{2 + qB\xi_4}$$ (2.9)
Since $\xi_4 = \sqrt{3}/2$ when there is a double root, $\hat{q}\hat{B} = 1$. So the physical magnetic field is finite at the point when the two roots coincide! This seems to suggest that one could violate cosmic censorship by simply increasing the magnetic field past this finite value.

To understand this better, notice that (2.5) does not fix the magnetic field uniquely, but allows two solutions. This can be seen as follows. Define the left hand side of (2.5) to be $\alpha^4$. Then, when expressed in terms of the zeros of $G(\xi)$, $\alpha^4$ becomes

$$\alpha^4 = \frac{(\xi_3 - \xi_2)(\xi_3 - \xi_1)}{(\xi_4 - \xi_2)(\xi_4 - \xi_1)} \quad (2.10)$$

Note that $\alpha^4 < 1$ since $\xi_1, \xi_2 < \xi_3 < \xi_4$. Since $G(\xi)$ is independent of $B$, so is $\alpha$. We can now solve (2.5) for $qB$, and there are clearly two real solutions

$$qB = \frac{2(1 \mp \alpha)}{\pm \alpha \xi_4 - \xi_3} \quad (2.11)$$

with $\alpha \geq 0$. Substituting this into (2.9) yields the simple result

$$\hat{q}\hat{B} = 1 \mp \alpha \quad (2.12)$$

When $r_+ A \to 2/(3\sqrt{3})$ so $\xi_2 = \xi_3$, $\alpha = 0$ and the two solutions agree. However in the opposite limit when $r_+ A \to 0$, $\xi_1, \xi_2 \to -\infty$, $\xi_3 \to -1$, $\xi_4 \to 1$, and $\alpha \to 1$. Thus the upper sign corresponds to the expected result that $qB \to 0$, $\hat{q}\hat{B} \to 0$. However, the lower sign corresponds to another branch of solutions in which $qB \to \infty$ and $\hat{q}\hat{B} \to 2$.

The key to understanding these new solutions is to consider the flux of magnetic field crossing the acceleration horizon $y = \xi_3$, which is given by

$$\Phi = \Delta \phi \lim_{x \to \xi_3} \tilde{A}_\phi(x, y = \xi_3) = \frac{2\Delta \phi}{B(1 + \frac{1}{2}qB\xi_4)} \quad (2.13)$$

where $\tilde{A}_\phi = A_\phi - A_\phi(x = \xi_4)$ is the gauge equivalent vector potential which is regular on the axis $x = \xi_4$. For small $B$, the flux is $\Phi \approx 4\pi/B$ which agrees with the flux at infinity in the asymptotically Melvin region. This is expected since the black holes are very small compared to the size of the flux tubes. However, when $qB = O(1)$, the black holes are comparable to the size of the flux tube. They are also close enough so that their charges make a significant contribution to the flux. The limit of (2.13) when the two roots coincide can be calculated as follows. The range of $\phi$ (2.6) remains finite and nonzero in this limit since (2.5) implies $\Delta \phi = 4\pi |A^2(\xi_4)/G'(\xi_4)|$. Since $1 + \frac{1}{2}qB\xi_3 \to 0$ in the limit that the two

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roots coincide, and \( \Delta \phi \) remains finite, it follows from (2.7) that \( q \to 0 \). Since \( qB \to 2/\sqrt{3} \), the parameter \( B \) diverges. Thus, the magnetic flux crossing the acceleration horizon goes to zero: The flux between the two oppositely charged black holes completely cancels the external magnetic field.

Now consider the solutions with \( \hat{q} \hat{B} > 1 \). This corresponds to the lower sign in (2.11) and hence \( 1 + \frac{1}{2} qB \xi^3 < 0 \). In order for the physical charge on the black hole to have the same sign as before, we must take the parameter \( q \) negative. Since \( qB \) is positive, the parameter \( B \) must also be negative. Thus the sign of the flux crossing the acceleration horizon is reversed. This means that for the same black hole parameters (mass, charge, and acceleration) the solution to the constraint (2.5) with \( \hat{q} \hat{B} > 1 \) has a smaller external magnetic flux than the one with \( \hat{q} \hat{B} < 1 \)! This is not a contradiction since \( \hat{B} \) is a measure of the strength of the magnetic field on the axis, and when this is large, spacetime is highly curved, yielding a smaller net flux. In fact, the limit \( \hat{q} \hat{B} \to 2 \) corresponds to the two black holes becoming infinitely far apart with zero acceleration. The flux through the acceleration horizon becomes equal in magnitude and opposite in sign to the one at infinity. For a magnetic field \( \hat{B} = 2/\hat{q} \) the asymptotic flux allowed by the Melvin solution is \( \Phi = 2\pi \hat{q} \). Combining this with the flux at the acceleration horizon (which is infinitely far away in the other direction) one has a net outward flux of \( \Phi = 4\pi \hat{q} \) which is just the flux of a charge \( \hat{q} \). In other words, the limit \( \hat{q} \hat{B} \to 2 \) corresponds to turning off the external magnetic field completely! One has compressed the magnetic flux from a charged black hole into two tubes so that the gravitational attraction exactly balances the magnetic pressure resulting in a static configuration. 

This shows that the branch of solutions with \( 1 < \hat{q} \hat{B} < 2 \) does not correspond to physically increasing the external magnetic flux. To understand whether one can violate cosmic censorship this way, we must study the limit \( \hat{q} \hat{B} \to 1 \) more closely. We will do this in the next section. For completeness, we will also study the limit \( \hat{q} \hat{B} \to 2 \).

3. Limiting Solutions

We wish to investigate the limit of the Ernst solution (2.2) as \( \hat{q} \hat{B} \to 1 \) and \( \hat{q} \hat{B} \to 2 \) keeping the black hole unchanged. It is not completely obvious what it means to say that black holes in two different spacetimes are the same. One cannot fix the mass, since there is no completely satisfactory definition of quasilocal mass for a nonspherical spacetime. Instead, we will keep the black hole charge and horizon area fixed. From (2.2) the horizon
Area is

\[ \text{Area} = \frac{\Delta \phi (\xi_4 - \xi_3)}{A^2 (\xi_3 - \xi_2)(\xi_4 - \xi_2)} \]  

(3.1)

with \(\Delta \phi\) given by (2.6). The limits \(\hat{q}\hat{B} \to 1\), \(\hat{q}\hat{B} \to 2\) must be taken carefully, since we have seen that the first requires \(q \to 0\), \(B \to \infty\), while the second requires \(qB \to \infty\).

To investigate the limit \(\hat{q}\hat{B} \to 1\), we set

\[ \xi_4 = \frac{\sqrt{3}}{2} + \epsilon^2. \]  

(3.2)

Using (2.4) one finds that to first order in \(\epsilon\), \(\xi_2\) and \(\xi_3\) are given by

\[ \xi_{2,3} = -\frac{\sqrt{3}}{2} \pm \frac{\gamma}{2} \epsilon \quad \gamma \equiv 3^{3/4} 4. \]  

(3.3)

The behavior of \(\xi_1\) is not determined since it depends only on \(r_- A\), which is independent of \(r_+ A\). Thus we are free to define its behavior for small \(\epsilon\) in terms of a parameter \(\beta \geq \gamma\):

\[ \xi_3 - \xi_1 = \beta \epsilon. \]  

(3.4)

With these definitions, \(\alpha\) can be determined from (2.10) to be \(\alpha = [4\beta \gamma / 27]^{1/4} \epsilon^{1/2}\). Thus, in this limit

\[ qB = \frac{2}{\sqrt{3}} - \left[ \frac{4\beta \gamma}{3} \right]^{1/4} \epsilon^{1/2}, \quad \hat{q}\hat{B} = 1 - \left[ \frac{4\beta \gamma}{27} \right]^{1/4} \epsilon^{1/2} \]  

(3.5)

where the sign has been chosen to correspond to weak magnetic field in the limit of small acceleration.

We have already seen that the range of \(\phi\) (2.6) remains finite and nonzero in the limit \(\epsilon \to 0\). In order for the black hole area (3.1) to remain finite and nonzero, we clearly need to rescale the parameter \(A\)

\[ A = \frac{\hat{A}}{\epsilon^{1/2}}. \]  

(3.6)

Since \(r_+ A\) and \(r_- A\) both have nonzero limits, this implies \(q = \sqrt{r_+ - r_-} = O(\epsilon^{1/2})\), so that the physical charge (2.7) also remains finite.

In order to obtain a well defined metric in the limit \(\epsilon \to 0\), we clearly have to introduce new coordinates. These coordinates can be derived by examining the asymptotic form of the solutions.

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2 One can add higher order corrections to ensure that the black hole area and charge are actually constant for finite \(\epsilon\). The above discussion will be sufficient to determine the limiting solution.
the solution and using the fact that the Melvin metric (2.1) remains well defined for all magnetic fields \footnote{A different choice of coordinates which does not preserve the asymptotic form of the solution in the limit was given in [9].}. The result is
\begin{align}
    x &= \xi_3 + \epsilon \hat{x} \\
y &= \xi_3 + \epsilon \hat{y} \\
t &= \eta \epsilon^2
\end{align}
(3.7)

So the new coordinates \(\hat{x}, \hat{y},\) and \(\eta\) are just rescaled versions of the old ones. Since \(y \to x = \xi_3\) describes spatial infinity this rescaling “opens up” a small region about the symmetry axis pointing toward spatial infinity. In terms of these new coordinates, the metric functions become
\begin{align}
    G(x) &= \epsilon^3 \hat{G}(\hat{x}) \\
    \hat{G}(\hat{x}) &= \frac{1}{\sqrt{3}} \hat{x}(\hat{x} + \gamma)(\hat{x} + \beta) \\
    \Lambda(x,y) &= \epsilon \hat{\Lambda}(\hat{x}, \hat{y}) \\
    \hat{\Lambda}(\hat{x}, \hat{y}) &= \frac{1}{2} \sqrt{3} \beta \gamma + \frac{3}{2} \frac{\hat{G}(\hat{x})}{(\hat{x} - \hat{y})^2}
\end{align}
(3.8)

The Ernst metric (2.2) now has a well behaved limit as \(\epsilon \to 0\)
\begin{align}
    ds^2 &= \frac{\hat{\Lambda}^2}{A^2 (\hat{x} - \hat{y})^2} \left[ \hat{G}(\hat{y}) d\eta^2 - \hat{G}^{-1}(\hat{y}) d\hat{y}^2 + \hat{G}^{-1}(\hat{x}) d\hat{x}^2 \right] + \frac{\hat{G}(\hat{x}) d\phi^2}{A^2 (\hat{x} - \hat{y})^2 \hat{\Lambda}^2}
\end{align}
(3.9)

and the limiting vector potential is
\begin{align}
    A_\phi &= -\frac{(\beta \gamma/3)^{1/4}}{\hat{\Lambda}}
\end{align}
(3.10)

There are two free parameters \(\hat{A}\) and \(\beta\) (\(\gamma\) is a fixed constant (3.3)) which is what one expects, since the original Ernst solution had three free parameters and we have fixed the magnetic field. In this metric, \(0 \leq \hat{x} < \infty\) with one pole of the sphere at \(\hat{x} = 0\) and the other at \(\hat{x} = \infty\). It is clear from (3.7) that as \(\epsilon \to 0\), \(\hat{x}\) must approach infinity to make \(x \to \xi_4\) since \(\xi_4 - \xi_3\) is finite. There is no conical singularity at either pole of the sphere. The coordinate transformation \(r = \hat{x}^{-1/2}, \phi = \frac{3 \sqrt{3}}{2} \hat{\phi}\) applied to the asymptotic form of the \((\hat{x}, \phi)\) metric,
\begin{align}
    \frac{3 \sqrt{3}}{4 A^2} \frac{d \hat{x}^2}{\hat{x}^3} + \frac{4 d\phi^2}{3 \sqrt{3} A^2 \hat{x}}
\end{align}
(3.11)
gives
\begin{align}
    \frac{3 \sqrt{3}}{A^2} \left( dr^2 + r^2 d\hat{\phi}^2 \right)
\end{align}
(3.12)
which is regular as \( r \to 0 \).

The range of \( \hat{y} \) is \(-\infty < \hat{y} < \hat{x} \) as before. The surface \( \hat{y} = -\beta \) is the inner Cauchy horizon, \( \hat{y} = -\gamma \) is the event horizon, and \( \hat{y} = 0 \) is the acceleration horizon. Thus despite the fact that the roots \( \xi_2 \) and \( \xi_3 \) coalesce in the original coordinates, the black hole horizon and acceleration horizon do not coincide in the limiting spacetime. It was noticed earlier [9] that the proper distance between these two horizons does not go to zero when the two roots coincide. However, this is also true for the inner and outer horizons of the Reissner-Nordström solution. The point is that there are different ways to take the extremal limit. Physically, one wants to take the limit in such a way that preserves the asymptotic behavior of the solution. In the Reissner-Nordström case, this implies that the two horizons coincide in the extremal limit, while in the Ernst metric, we have found that they do not.

We now consider the second limit \( \hat{q} \hat{B} \to 2 \). This corresponds to the limit of small \( r_+ A \), but with the lower sign in (2.12). Since the analysis is similar to that above, our discussion will be brief. Define \( \delta < 0 \) and \( \beta \geq 1 \) by

\[
\delta = 1/\xi_2, \quad \xi_1 = \beta \xi_2 = \beta/\delta
\]

We are interested in the regime \( |\delta| \ll 1 \), where \( \delta \approx -r_+ A \). Then to first order in \( \delta \), \( \xi_3 = -1 + \frac{1}{2} \delta \) and \( \xi_4 = 1 + \frac{1}{2} \delta \) so that

\[
qB = \frac{-8\beta}{\delta (1 + 3\beta)}.
\]

Requiring that the solution approach the Melvin metric asymptotically, and the physical charge and the area of the event horizon remain finite and nonzero as \( \delta \to 0 \) motivates the following definitions:

\[
c \equiv \frac{1}{2} qB\delta, \quad A = -\frac{c^2}{\delta} \hat{A},
\]

\[
y = \frac{\hat{y}}{\delta}, \quad t = -\delta \hat{t}, \quad x = -\cos \theta, \quad \phi = c^4 \phi/\delta^4
\]

Since \( \xi_3 \leq x \leq \xi_4 \), in the limit \( r_+ A \to 0 \), we have \( G(x) = 1 - x^2 \). The other metric functions become

\[
\Lambda = \frac{c^2}{\delta^2} \hat{\Lambda}, \quad \hat{\Lambda} = \cos^2 \theta + \frac{\beta}{\hat{y}^2} \sin^2 \theta,
\]

\[
G(y) = \frac{1}{\delta^2} \hat{G}(\hat{y}), \quad \hat{G}(\hat{y}) = -\frac{1}{\beta} \hat{y}^2 (\hat{y} - \beta) (\hat{y} - 1)
\]
Defining \( r = 1/\hat{y} \) and \( G(r) = (1 - \frac{1}{\hat{y}}) \left( 1 - \frac{1}{\beta r} \right) \) so that \( G(\hat{y})/\hat{y}^2 \to -G(r) \), the limit of the Ernst solution as \( \delta \to 0 \) is

\[
\begin{align*}
\frac{\hat{A}^2}{\Lambda^2} & \left[ -G(r) \frac{df^2}{dr^2} + \frac{dr^2}{G(r)} + r^2 d\theta^2 + \frac{r^2 \sin^2 \theta}{\Lambda^4} d\phi^2 \right] \\
A_\phi &= -\frac{\cos \theta}{\hat{A}\sqrt{\beta \Lambda}}
\end{align*}
\]

(3.17)

Near the poles, \( \hat{\Lambda} \simeq 1 \), so there are no conical singularities. There is a curvature singularity at \( r = 0 \), an inner horizon at \( r = 1/\beta \), and an event horizon at \( r = 1 \). The acceleration horizon \( (\hat{y} = 0) \) has moved off to infinity \( (r = \infty) \). The solution (3.17) describes a single magnetically charged black hole whose magnetic flux is confined to two flux tubes. It can alternatively be obtained as a limit of another class of solutions describing a single black hole in a background magnetic field \([10]\).

4. Violation of Strong Cosmic Censorship in Charged C-metric

We now turn to another test of cosmic censorship involving the stability of inner horizons. Since the background magnetic field will not play an essential role here, we will set it to zero. The Ernst solution (2.2) with \( B = 0 \) is just the charged C-metric. We will apply a general argument which indicates the stability of Cauchy horizons \([11]\) to this spacetime. Consider an ingoing flux of null radiation with a finite energy density at the acceleration horizon, as measured by a freely falling observer. If the energy density of the radiation at the Cauchy horizon, again measured by a freely-falling observer, remains finite, then the Cauchy horizon is likely to be stable. (This has been confirmed in the case of Reissner-Nordström black holes in de Sitter space \([11]\).) This condition will lead to the conclusion that the Cauchy horizon is probably stable whenever the surface gravity of the Cauchy horizon, \( \kappa_c \), is less than the surface gravity of the acceleration horizon, \( \kappa_a \). We will see that for the charged C-metric solution, ranges of the parameters exist for which this condition is satisfied.

The charged C-metric in retarded coordinates is \([12]\)

\[
\begin{align*}
\frac{\hat{A}^2}{\Lambda^2} & \left[ -G(r) \frac{df^2}{dr^2} + \frac{dr^2}{G(r)} + r^2 d\theta^2 + \frac{r^2 \sin^2 \theta}{\Lambda^4} d\phi^2 \right] \\
A_\phi &= -\frac{\cos \theta}{\hat{A}\sqrt{\beta \Lambda}}
\end{align*}
\]

(4.1)

\[
\begin{align*}
ds^2 &= H du^2 + 2 du dr + 2 Ar^2 dudx - r^2 \left( G^{-1} dx^2 + G d\phi^2 \right) \\
H &= -A^2 r^2 G \left( x - \frac{1}{Ar} \right) \\
G &= 1 - x^2 - 2mA x^3 - e^2 A^2 x^4.
\end{align*}
\]
Note that the form for $G(x)$ is different than in the previous sections; this is merely a coordinate gauge freedom. There are still four roots of $G$: $\xi_1$, $\xi_2$, $\xi_3$, and $\xi_4$. Again, $x$ must be restricted to lie between $\xi_3$ and $\xi_4$ to keep the correct signature of the metric. $u$ is a null coordinate ranging from $-\infty$ to $\infty$, and $r$ is a radial coordinate ranging from 0 to $\infty$. As before, the horizons occur at the zeros of $G$, with the Cauchy horizon at $\xi_1$, the black hole horizon at $\xi_2$ and the acceleration horizon at $\xi_3$. At these horizons, $H$ (the norm of the boost killing field $\partial/\partial u$) vanishes. In addition, the null surface at the boundary of this coordinate system, $u \to \infty$, consists of part of the acceleration horizon and part of the Cauchy horizon.

Consider an ingoing null flux of radiation with the stress-energy tensor

$$T_{\alpha\beta} = \left[ L(u) / (4\pi r^2) \right] l_\alpha l_\beta$$

where $l_\alpha$ is tangent to ingoing null geodesics. The general ingoing null geodesic can be shown to be [12]

$$l^\alpha = \left( H^{-1} (E + R), -R - AP, P/r^2, J_z/r^2 G \right)$$

$$R = (E^2 - J_z^2 H/r^2)^{1/2}$$

$$P = (GJ_z^2 - J_z^2)^{1/2}$$

(4.3)

where $E$, $J_z$ and $J$ are constants of the motion. Since we are interested in radial null geodesics, we set $J$ and $J_z$ to zero in (4.3) to give

$$l^\alpha = (2E/H, -E, 0, 0).$$

(4.4)

The energy density seen by a freely-falling observer with four-velocity $v^\alpha$ near the acceleration horizon is

$$\rho_a = T_{\alpha\beta} v^\alpha v^\beta = \left[ L(u) / (4\pi r^2) \right] (l_\alpha v^\alpha)^2.$$  

A family of timelike geodesics near a horizon is given by

$$v^\alpha (\lambda) = ( \hat{u}(\lambda), \hat{r}(\lambda), \hat{x}(\lambda), 0)$$

$$\hat{u}(\lambda) \approx \frac{1}{\kappa_\xi \lambda}, \quad \hat{r}(\lambda) \approx r_0 + r_1 \lambda, \quad \hat{x}(\lambda) \approx x_0 + x_1 \lambda$$

(4.5)

where $r_1, x_0, x_1$ label the different geodesics, $\lambda$ is an affine parameter which goes to zero on the horizon, $\kappa_\xi$ is the surface gravity of the horizon, and $r_0$ is determined by $x_0 - \frac{1}{Ax_0} = \xi$, a zero of $G$. These conditions only fix $\lambda$ up to a constant rescaling. The surface gravity is also ambiguous up to a constant rescaling since it depends on the timelike Killing field.
which does not have a canonical normalization at infinity. We will take the Killing field to be simply \( \psi^\alpha = (1, 0, 0, 0) \) in our coordinates. Then the surface gravity of the horizon associated with root \( \xi \) of \( G \), is

\[
\kappa_\xi = \frac{1}{2} A \left. \frac{dG(a)}{da} \right|_{a=\xi}.
\] (4.6)

We will only need the ratio of two different surface gravities which is independent of the ambiguity in their definition.

\((\lambda, r_1, x_0, x_1)\) are good coordinates near the horizon. In terms of these coordinates, \( H \) and its derivatives take the form:

\[
H \approx -2\kappa \left( Ar_0^2 x_1 + r_1 \right) \lambda, \quad \frac{\partial H}{\partial r} \approx -2\kappa, \quad \frac{\partial H}{\partial x} \approx -2A\kappa r_0^2
\] (4.7)

One can easily verify that \( v_\alpha v^\alpha = -1 \) on the horizons and that the form for \( v^\alpha \) given in (4.5) is consistent with the geodesic equations near a horizon.

Integrating the equation for \( \dot{u} \) (4.5) shows that near the horizon \( \frac{1}{\lambda} = e^{\kappa u} \). Since \( l_\alpha v^\alpha \propto \frac{1}{\lambda} \), it follows that the energy density at the acceleration horizon is

\[
\rho_a \propto L(u) e^{2\kappa_a u}
\] (4.8)

and the energy density at the Cauchy horizon is

\[
\rho_c \propto L(u) e^{2\kappa_c u}
\] (4.9)

The requirement that \( \rho_a \) be finite gives the condition on \( L(u) \) that

\[
L(u) = K(u) e^{-2\kappa_a u}
\] (4.10)

where

\[
\lim_{u \to \infty} K(u) = K_\infty \neq 0.
\] (4.11)

Plugging this form for \( L(u) \) into the equation for \( \rho_c \) gives

\[
\rho_c \propto K(u) e^{(\kappa_c - \kappa_a)u}.
\] (4.12)

Clearly this expression is finite whenever \( \kappa_c < \kappa_a \). This condition can be met for a range of parameters. \( G(x) \) can be written in the form

\[
G(x) = -e^2 A^2 (x - \xi_1) (x - \xi_2) (x - \xi_3) (x - \xi_4).
\] (4.13)
The ratio of the surface gravities is then

\[
\frac{\kappa_a}{\kappa_c} = \frac{G'(\xi_3)}{G'(\xi_1)} = \frac{(\xi_3 - \xi_2)(\xi_4 - \xi_3)}{(\xi_2 - \xi_1)(\xi_4 - \xi_1)}.
\]

(4.14)

From the form of \(G(x)\) in (4.1) it is clear that by increasing \(e\), thus making the quartic term more negative, the maximum of \(G\) which occurs at a negative value of \(x\) must decrease. This will cause the lower two zeroes of \(G\), \(\xi_1\) and \(\xi_2\), to approach one another. Thus, by increasing \(e\) the ratio of surface gravities can become arbitrarily large. This corresponds to black holes which are near extremality. Therefore nearly extremal accelerating black holes are likely to have stable Cauchy horizons.

5. Discussion

We have considered two oppositely charged black holes uniformly accelerating in a background magnetic field. This situation is described by the Ernst solution of Einstein-Maxwell theory. To test cosmic censorship, we studied the effect of increasing the external magnetic field. In the standard coordinates, it appeared that the event horizon approached the acceleration horizon at a finite value of the background field. We showed that this is not the case: When the limit is taken carefully, one finds that these two horizons remain a finite distance apart. Thus, the situation is different from the Reissner-Nordström de Sitter metric, in which the event horizon can be made to coincide with the cosmological horizon at a finite value of the cosmological constant. We have also found another branch of Ernst solutions with larger values of the asymptotic magnetic field, but argued that this corresponds to decreasing the external flux and concentrating the flux due to the charged black holes.

Although weak cosmic censorship cannot be violated using the Ernst solutions, one is left with the physical question of what happens if one continues to increase the external magnetic flux. The result is apparently not described by any metric of the Ernst form. This is plausible if one recalls that the Ernst metric has zero energy relative to the asymptotic magnetic field. This is a consequence of the boost symmetry\(^4\). There must exist other solutions describing oppositely charged black holes in magnetic fields with both positive and negative energy. In some cases the black holes will collide, and in others they will

\(^4\) Not only does the usual energy associated with an asymptotic time translation vanish, but so does the boost energy associated with the timelike Killing field [13].
expand apart. In particular, there must exist a static solution analogous to the ones discussed in [14]. One will have to find and study the solutions where the black holes collide to determine if weak cosmic censorship can be violated in this case.

Within the context of the Ernst solutions, we have argued that strong cosmic censorship is violated. If one sends in a flux of null radiation which has finite energy density as measured by a freely-falling observer near the acceleration horizon, one finds that an open set of parameters exist for which the energy density as measured by freely-falling observers near the Cauchy horizon remains finite. Thus, unlike the Cauchy horizon of the Reissner-Nordström spacetime, the Cauchy horizon of the C-metric does not have a generic infinite blue-shift instability.

It may then be asked if cosmic censorship is violated semi-classically. Arguments have been made for the Reissner-Nordström-deSitter spacetime, which has a similar causal structure to that of Ernst, that while configurations with stable Cauchy horizons exist classically, only a set of measure zero exist semi-classically [15]. Near a horizon of the Ernst spacetime the scalar wave equation decouples in terms of Rindler modes and angular eigenfunctions [16], but in general the lack of spherical symmetry makes this problem difficult. It seems likely, however, that the Ernst Cauchy horizon will exhibit the same semiclassical instability as the Reissner-Nordström-deSitter solution.

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