Uniqueness of the bosonization of the $U_q(su(2)_k)$ quantum current algebra

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Abstract

Four apparently different bosonizations of the $U_q(su(2)_k)$ quantum current algebra for arbitrary level $k$ have recently been proposed in the literature. However, the relations among them have so far remained unclear except in one case. Assuming a special standard form for the $U_q(su(2)_k)$ quantum currents, we derive a set of general consistency equations that must be satisfied. As particular solutions of this set of equations, we recover two of the four bosonizations and we derive a new and simpler one. Moreover, we show that the latter three, and the remaining two bosonizations which cannot be derived directly from this set of equations since by construction they do not have the standard form, are all related to each other through some redefinitions of their Heisenberg boson oscillators.
1 Introduction

There is presently much interest in the field of quantum groups and algebras because of their rich applications in integrable models of quantum field theory and statistical systems, and conformal field theory. Quantum current (affine) algebras (QCA’s) are of special interest since they arise as both dynamical and infinite dimensional symmetries of some physical models, which are therefore highly constrained. The most standard example of these models is certainly the one-dimensional XXZ quantum spin chain in the thermodynamic limit. However, other examples of two-dimensional massive integrable models of quantum field theory like the sine-Gordon model are also known to possess quantum current symmetries. Another important motivation for studying the QCA’s is that one hopes that the great success of the usual current algebras in the context of conformal field theory will carry over also to the quantum case without many difficulties.

In particular, one hopes to extend the bosonization procedure (which is also known as the Feigin-Fuchs construction, Dotsenko-Fateev construction, and free field realization) of conformal field theory to the case of QCA’s. The reason is that this bosonization procedure makes the technical study and calculation of some relevant quantities like the correlation functions, the irreducible characters, the partition functions, the BRST-like cohomology structure and the fusion rules more transparent and accessible. This extension, if possible, provides us then with a promising alternative to the quantum inverse scattering method in computing the exact correlation functions of the massive integrable models.

One important ingredient of the bosonization recipe is the realization of the current algebra symmetries (or any other infinite-dimensional symmetries) of the integrable models in terms of free boson fields. This is because the free boson fields are generating functions of boson oscillators satisfying Heisenberg algebras, which are of course much simpler to handle. For example, the correlation functions of the vertex fields (which are the dynamical fields of an integrable model) are readily computed in terms of the much simpler two-point correlation functions of the free boson fields. This obviously assumes that these vertex fields can be realized in terms of the free boson fields. This is in fact a second important ingredient of
the bosonization recipe that we will not consider here.

The bosonization of the usual “classical” $su(2)_k$ current algebra with its rich consequences \cite{13, 14, 15, 16, 17} is well known and extensively studied in the literature. It depends on whether the level $k$ is equal to 1 or a generic positive integer number (only the unitary case is being considered). In the former case, it is called the Frenkel-Kac bosonization and requires one free boson field \cite{18}. In the latter case, after the bosonization of the pair of ghost fields first introduced in the Wakimoto construction, it is referred to as the Wakimoto bosonization and requires three boson fields \cite{19}. Then a natural question arises as to whether it is possible to extend this bosonization to the $U_q(su(2)_k)$ QCA case. The answer to this question turns out to be positive. As expected, it depends again on whether the level $k$ is equal to 1 or a generic positive integer number. In the former case, it is known as the Frenkel-Jing construction and requires one deformed Heisenberg algebra \cite{20}. In order to keep the analogy between the classical and quantum cases apparent, we will occasionally use the language of the Heisenberg algebras instead of that of the free boson fields. The reason is that in the quantum case several deformed boson fields, which are different generating functions built from the same Heisenberg algebra, might be required. More details clarifying this point will be given later. In the latter case, four apparently different bosonizations have recently been constructed independently by four different groups, namely, Abada-Bougourzi-El Gradechi (ABE) \cite{21, 22}, Matsuo \cite{23}, Shiraishi \cite{24, 25}, and Kimura \cite{26}. All these bosonizations require three independent (intercommuting) deformed Heisenberg algebras. By apparently different, we mean that deformations of the Heisenberg algebras, and therefore the quantum currents involved in one bosonization are different from those of the other three. However, except for the connection between the Matsuo and Kimura constructions \cite{26}, the relations among these bosonizations have so far only been raised but not resolved in the literature.

In this paper, we elucidate these relations. Strictly speaking, we derive a set of general equations by assuming a standard form for the quantum currents, i.e., they are expressed in terms of a “bosonized” parafermion plus a free boson. Each solution of this set of equations leads to a particular bosonization of the $U_q(su(2)_k)$ QCA. In particular, we recover the ABE and Matsuo bosonizations, which are by construction already in the standard form,
and we derive a new much simpler one as three different solutions of this same set of equations. Moreover, we show that these three bosonizations together with the Shiraishi and Kimura bosonizations, which cannot be recovered from this set of equations since they are not originally constructed in the standard form, are all related to each other through some redefinitions of the three sets of deformed Heisenberg boson oscillators.

This paper is organized as follows. In section 2, we briefly review the usual $su(2)_k$ current algebra, and its corresponding Frenkel-Kac and Wakimoto bosonizations. In section 3, we introduce the $U_q(su(2)_k)$ QCA. After reviewing the Frenkel-Jing bosonization in a slightly different language for the one used by Frenkel and Jing, we examine in detail the bosonization of the $su(2)_k$ QCA for a generic positive level $k$, which we refer to as “the deformation of the Wakimoto bosonization.” The analogy with the classical case is stressed all along. The aforementioned consistency equations are written down. Then, we show how the ABE and Matsuo bosonizations are recovered as special solutions of these equations, which in addition lead to a new compact bosonization that we refer to as “the fifth bosonization.” In section 4, we elucidate the relations among all these five bosonizations. More precisely, we show that each of the known four bosonizations can be obtained from the fifth one through some linear transformations relating their three sets of deformed Heisenberg boson oscillators. This is obviously sufficient to show that all of the five bosonizations can be obtained from one another in this manner. Up to these transformations, this result suggests then that the deformation of the Wakimoto bosonization is also unique as in the classical case. We devote section 5 to the conclusions.

2 The $su(2)_k$ current algebra

The $su(2)_k$ current algebra at level $k$ is generated by the currents $H(z)$ and $E^\pm(z)$ ($z$ being a complex variable), which in the Cartan-Weyl basis have the operator product expansions (OPE’s)

$$H(z).H(w) \sim \frac{k}{(z-w)^2}, \quad (2.1)$$
\[ H(z).E^\pm(w) \sim \pm \sqrt{2E^\pm(w)} \frac{z-w}{z-w}, \]  
\[ E^\pm(z).E^\mp(w) \sim \sqrt{2H(w)} \frac{z-w}{z-w} + \frac{k}{(z-w)^2}. \]  

where the symbol \( \sim \) as usual means an equality up to regular terms as \( z \) approaches \( w \).

Using standard techniques in conformal field theory to re-express the OPE’s of currents (generating functions) as commutators of their modes [27], we get the more familiar \( su(2)_k \) affine algebra [28]

\[
[H_n, H_m] = nk\delta_{n+m,0},
\]
\[
[H_n, E^\pm_m] = \pm \sqrt{2}E^\pm_{n+m},
\]
\[
[E^+_n, E^-_m] = \sqrt{2}H_{n+m} + nk\delta_{n+m,0},
\]

where

\[
H(z) = \sum_{n=-\infty}^{+\infty} H_n z^{-n-1},
\]
\[
E^\pm(z) = \sum_{n=-\infty}^{+\infty} E^\pm_n z^{-n-1}.
\]

Let us now briefly review the bosonization of the \( su(2)_k \) current algebra as given by (2.1)-(2.3). As mentioned in the introduction, this bosonization depends on whether the level \( k \) is equal to 1 or a generic positive integer number, and is called the Frenkel-Kac or Wakimoto bosonization respectively.

### 2.1 The Frenkel-Kac bosonization

In this case \( (k = 1) \), the currents \( H(z) \) and \( E^\pm(z) \) are realized in terms of one free boson field \( \phi^1(z) \) as:

\[
H(z) = i\partial \phi^1(z),
\]
\[
E^\pm(z) = \exp\{\pm i\sqrt{2}\phi^1(z)\}.
\]

By free field, we mean that \( \phi^1(z) \) has the OPE \( \phi^1(z) \cdot \phi^1(w) \sim -\ln(z-w) \), which is the free Green function of the two-dimensional Laplace equation. This OPE can be translated into commutation relations if we write \( \phi^1(z) \) as a generating function, i.e.,

\[
\phi^1(z) = \phi^1 - i\phi^1_0 \ln z + i \sum_{n \neq 0} \frac{\phi^1_n}{n} z^{-n},
\]
In which case the set of boson oscillators \( \{ \phi^1, \phi^1_n \} \) satisfies the Heisenberg algebra

\[
\begin{align*}
[\phi^1_n, \phi^1_m] &= n \delta_{n+m,0}, \\
[\phi^1, \phi^1_0] &= i.
\end{align*}
\] (2.9)

In fact, this is valid only if we define the normal ordering of the boson oscillators, which is denoted by the symbol ::, such that

\[
\begin{align*}
: \phi^1_n \phi^1_m : &= \phi^1_m \phi^1_n, \quad n > 0, \\
: \phi^1_0 \phi^1 : &= \phi^1 \phi^1_0.
\end{align*}
\] (2.10)

The symbol :: is understood and therefore omitted from any field or product of fields defined at the same point \( z \). These definitions and conventions will be valid throughout the rest of this paper.

### 2.2 The Wakimoto bosonization

This is again valid for a generic positive level \( k \). Though the Wakimoto construction is based on one free boson field and a pair of ghost fields \([19]\), it is well known that it is equivalent to the following construction which involves only the free boson fields \( \phi^1(z), \phi^2(z) \) and \( \phi^3(z) \):

\[
\begin{align*}
H(z) &= i \sqrt{k} \partial \phi^1(z), \\
E^\pm(z) &= \left( \pm i \sqrt{\frac{k}{2}} \partial \phi^2(z) + i \sqrt{\frac{k+2}{2}} \partial \phi^3(z) \right) \exp\{ \pm i \frac{\sqrt{2}}{k} (\phi^2(z) + \phi^1(z)) \},
\end{align*}
\] (2.11, 2.12)

where \( \phi^1(z) \) is the same as the one used in the Frenkel-Kac bosonization. These three free boson fields are orthogonal to each other in the sense that if we write them as generating functions

\[
\phi^j(z) = \phi^j - i \phi^j_0 \ln z + i \sum_{n \neq 0} \frac{\phi^j_n}{n} z^{-n}, \quad j = 1, 2, 3,
\] (2.13)

the corresponding boson oscillators satisfy the three intercommuting Heisenberg algebras

\[
\begin{align*}
[\phi^j_n, \phi^\ell_m] &= (-1)^{j-1} n \delta^j_\ell \delta_{n+m,0}, \\
[\phi^j, \phi^\ell_0] &= (-1)^{j-1} i \delta^j_\ell,
\end{align*}
\] (2.14)

with all the other commutators being trivial. Note the presence of an extra minus sign for the set of oscillators \( \{ \phi^2, \phi^2_n \} \). It is primarily the above form of the currents \( E^\pm(z) \) that
we refer to as the standard form, i.e., a “bosonized” parafermion plus a free boson. This is because $E^\pm(z)$ can be rewritten as \[ \phi_1^\dagger(z) \exp\{\pm i\sqrt{2/k}\phi_1(z)\}\] (2.15), where $\phi_1(z)$ is a free boson field and $\psi^\pm(z) = \left(\pm i\sqrt{\frac{1}{2}}\partial\phi^2(z) + i\sqrt{\frac{k + 2}{2k}}\partial\phi^3(z)\right) \exp\{\pm i\sqrt{\frac{2}{k}}\phi^2(z)\}\] (2.16) are parafermions since they satisfy the OPE

$$\psi^+(z)\psi^-(w) \sim (z - w)^{2/k}.$$ (2.17)

This standard form amounts essentially to realizing the current $H(z)$ as a derivative of a single free boson field $\phi_1(z)$ as shown in (2.11). Note that if we do not include the parafermion in $E^\pm(z)$, we will obtain an almost trivial generalization (just a normalization factor $\sqrt{k}$ will arise) of the Frenkel-Kac bosonization. This is because the OPE’s (2.1) and (2.2) will be already satisfied. However, the remaining OPE (2.3) is not satisfied and no freedom is left over with only one free boson field $\phi_1(z)$ to satisfy it. Consequently, we need to add a parafermion, which is in turn “bosonized” in terms of two new independent free boson fields $\phi^2(z)$ and $\phi^3(z)$ as in (2.12). Note that the inclusion of this parafermion does not spoil the fact that the currents $E^\pm(z)$ still satisfy the OPE’s (2.1) and (2.2). The reason is that the current $H(z)$ depends only on $\phi_1(z)$ that is orthogonal to $\phi^2(z)$ and $\phi^3(z)$, which give rise to this parafermion. These remarks will be important when extending the Frenkel-Jing bosonization ($k = 1$) to the case of a generic positive level $k$. In fact, both the Frenkel-Jing and Wakimoto bosonizations will serve as a guiding tool to derive their respective quantum analogues.

3 The $U_q(su(2)_k)$ quantum current algebra

In order to underline the analogy with the classical case, let us first introduce this algebra in the language of commutation relations. It is generated by the operators $\{H_n, E^\pm_n, \ n \in \mathbb{Z}\}$
and reads in the Cartan-Weyl basis as [22, 30, 31]:

\[
[H_n, H_m] = \frac{[2n][nk]}{2n} \delta_{n+m,0}, \quad n \neq 0,
\]

\[
[H_0, H_m] = 0,
\]

\[
[H_n, E^+_m] = \pm \sqrt{2} q^{\mp|n|/2[2n]} E^+_n, q - q^{-1} \quad q \neq 0,
\]

\[
[H_0, E^+_m] = \pm \sqrt{2} E^+_m,
\]

\[
[E^+_n, E^-_m] = \frac{2^{k(n-m)/2} \Psi_{n+m} - q^{k(m-n)/2} \Phi_{n+m}}{q - q^{-1}},
\]

\[
E^+_{n+1} E^-_m - q^{\pm 2} E^+_m E^-_{n+1} = q^{\pm 2} E^+_n E^-_{m+1} - E^+_m E^-_n,
\]

where the familiar notation \([x] = (q^x - q^{-x})/(q - q^{-1})\) is used, \(q\) is the deformation parameter which is not a root of unity here, and \(\Psi_n\) and \(\Phi_n\) are the modes of the fields \(\Psi(z)\) and \(\Phi(z)\) that are defined by

\[
\Psi(z) = \sum_{n \geq 0} \Psi_n z^{-n} = q^{\sqrt{2} H_0} \exp\{\sqrt{2}(q - q^{-1}) \sum_{n > 0} H_n z^{-n}\},
\]

\[
\Phi(z) = \sum_{n \leq 0} \Phi_n z^{-n} = q^{-\sqrt{2} H_0} \exp\{-\sqrt{2}(q - q^{-1}) \sum_{n < 0} H_n z^{-n}\}.
\]

One can easily verify that as \(q\) approaches 1 this quantum algebra (3.18) reduces to the classical one (2.4). For the sake of bosonization, it is convenient to rewrite this quantum algebra as OPE’s. Now the role of the current \(H(z)\) in the classical case (2.11) will be played instead by the currents \(\Psi(z)\) and \(\Phi(z)\). The \(U_q(su(2)_k)\) QCA then reads [22, 23, 32]

\[
\Psi(z) \cdot \Phi(w) = \frac{(z - w q^{2+k})(z - w^{-2-k})}{(z - w q^{-2-k})(z - w^{-2+k})} \Phi(w) \cdot \Psi(z),
\]

\[
\Psi(z) \cdot E^\pm(w) = q^{\pm 2}(z - w q^{(2+k)/2}) \frac{E^\pm(w)}{z - w q^{(2-k)/2}} \Phi(w) \cdot \Psi(z),
\]

\[
\Phi(z) \cdot E^\pm(w) = q^{\pm 2}(z - w q^{(2-k)/2}) \frac{E^\pm(w)}{z - w q^{(2+k)/2}} \Psi(z) \cdot \Phi(z),
\]

\[
E^+(z) \cdot E^-(w) \sim \frac{1}{w(q - q^{-1})} \left\{ \frac{\Psi(w q^{k/2})}{z - w q^k} - \frac{\Phi(w q^{-k/2})}{z - w q^{-k}} \right\},
\]

\[
E^+(z) \cdot E^+(w) = \frac{(z q^{2-k} - w)}{z - w q^2} E^+(w) \cdot E^+(z),
\]

where \(E^\pm(z)\) are generating functions of the modes \(E^\pm_n\) as in (2.3) and the symbol \(\sim\) means that the regular terms as \(z\) approaches \(w q^{+k}\) are being omitted. To go from (3.18) to (3.20)-(3.24) and vice versa the identification \(\sum_{n \geq 0} z^n = (1 - z)^{-1}\) with \(|z| < 1\) is used. This will also be understood in the subsequent development. Let us now first review in this notation the quantum analogue of the Frenkel-Kac bosonization, namely, the Frenkel-Jing bosonization.
### 3.1 The Frenkel-Jing bosonization

This bosonization is again valid only for \( k = 1 \). The currents \( \Psi(z) \), \( \Phi(z) \) and \( E^\pm(z) \), which satisfy the OPE's (3.20)-(3.24) are now realized as follows [20, 22]:

\[
\Psi(z) = \exp \left\{ i \sqrt{\frac{2}{q}} \left( \varphi_1^1(zq^{1/2}) - \varphi_1^1(zq^{-1/2}) \right) \right\} \tag{3.25}
\]
\[
\Phi(z) = \exp \left\{ i \sqrt{\frac{2}{q}} \left( \varphi_1^1(zq^{-1/2}) - \varphi_1^1(zq^{1/2}) \right) \right\} \tag{3.26}
\]
\[
E^\pm(z) = \exp \left\{ \pm i \sqrt{2} \varphi_1^1(z) \right\}. \tag{3.27}
\]

Here \( \varphi_1^1(z) \) are two different deformations of the same free boson field \( \phi_1^1(z) \) introduced in the Frenkel-Kac bosonization, that is, they are two different generating functions of the boson oscillators \( \varphi_n^1 \) and \( \varphi_1 \) satisfying the same deformation of the Heisenberg algebra used in the Frenkel-Kac bosonization (2.9). This means they both reduce to \( \phi_1^1(z) \) as \( q \) tends to 1. In this regard, \( \varphi_n^1 \) and \( \varphi_1 \) are deformations (as suggested by the symbols) of \( \phi_n^1 \) and \( \phi_1 \) respectively. More specifically, the deformed free fields \( \varphi_1^1(z) \) are given by

\[
\varphi_1^1(z) = \varphi_1 - i \varphi_0^1 \ln z + i \sum_{n \neq 0} q^{\pm|n|/2} \frac{|n|}{n} \varphi_n^1 z^{-n}, \tag{3.28}
\]

with the deformed Heisenberg algebra being

\[
[\varphi_n^1, \varphi_m^1] = \frac{2n|n|}{2n} \delta_{n+m,0}, \quad [\varphi^1, \varphi_0^1] = i. \tag{3.29}
\]

It can easily be checked with the normal ordering defined in (2.10) that the quantum currents \( \Psi(z), \Phi(z) \) and \( E^\pm(z) \) as realized through (3.25)-(3.27) do indeed satisfy the \( U_q(su(2)_1) \) QCA (3.20)-(3.24).

### 3.2 Deformation of the Wakimoto bosonization

Let us now consider the extension of the previous bosonization to the \( U_q(su(2)_k) \) QCA case in a way which is parallel as much as possible to the manner the Wakimoto bosonization was
extended from the Frenkel-Kac one in the classical case. In particular, we will maintain the standard form for the quantum currents. Consequently, the generalization of the currents $\Psi(z)$, $\Phi(z)$ and $E^\pm(z)$ (3.25)-(3.27) so that the OPE's (3.20)-(3.22) are satisfied is given by

$$\Psi(z) = \exp\{i\sqrt{2k}\varphi^{1+}(zq^{k/2}) - \varphi^{1-}(zq^{-k/2})\} \quad \text{(3.30)}$$

$$\Phi(z) = \exp\{i\sqrt{2k}\varphi^{1+}(zq^{-k/2}) - \varphi^{1-}(zq^{k/2})\} \quad \text{(3.31)}$$

$$E^\pm(z) = \exp\{\pm i\sqrt{2k}\varphi^{1,\pm}(z)\} \quad \text{(3.32)}$$

where now we have

$$\varphi^{1,\pm}(z) = \varphi^1 - i\varphi_0^1 \ln z + ik\sum_{n\neq 0} q^{\pm|n|/2}^{nk/2} \varphi_n^{1}\varphi_n^{1}z^{-n}, \quad \text{(3.33)}$$

and

$$[\varphi_n^{1}, \varphi_m^{1}] = nI_1(n)\delta_{n+m,0},$$

$$[\varphi^1, \varphi_0^1] = i. \quad \text{(3.34)}$$

Here $I_1(n) = \frac{[2n][nk]}{2nk^2}$. The usefulness of the notation $I_1(n)$ will be clear shortly. By analogy to the classical case as explained below equation (2.17), this achieves fully the bosonization of the currents $\Psi(z)$ and $\Phi(z)$, but only partially that of the currents $E^\pm(z)$, that is, their $\varphi^{1,\pm}(z)$ exponential parts. To see this, $E^\pm(z)$ as given by (3.32) satisfy the following OPE's instead of the one given in (3.23):

$$E^\pm(z).E^\mp(w) = \exp\left\{\frac{2k}{\hbar}\langle\varphi^{1,\pm}(z)\varphi^{1,\mp}(w)\rangle\right\} : E^\pm(z).E^\mp(w) :,$$

$$E^\pm(z).E^\pm(w) = \exp\left\{-\frac{2k}{\hbar}\langle\varphi^{1,\pm}(z)\varphi^{1,\pm}(w)\rangle\right\} : E^\pm(z).E^\pm(w) :, \quad \text{(3.35)}$$

where the various two-point correlation functions that result from the normal ordering of the boson oscillators are given by

$$\langle\varphi^{1,\pm}(z)\varphi^{1,\mp}(w)\rangle = -\ln z + \frac{k}{2} \sum_{n>0} \frac{[2n]}{n[nk]} z^{-n}w^n,$$

$$\langle\varphi^{1,\pm}(z)\varphi^{1,\pm}(w)\rangle = -\ln z + \frac{k}{2} \sum_{n>0} \frac{q^n [2n]}{n[nk]} z^{-n}w^n. \quad \text{(3.36)}$$
To derive a correct and complete bosonization of $E^\pm(z)$ that is consistent with (3.23) we need to introduce two more deformed free boson fields $\varphi^2(z)$ and $\varphi^3(z)$ or rather two more sets of deformed boson oscillators $\{\varphi^2, \varphi^2_n\}$ and $\{\varphi^3, \varphi^3_n\}$ since we do not yet know how the former are generating functions of the latter. The sets $\{\varphi^2, \varphi^2_n\}$ and $\{\varphi^3, \varphi^3_n\}$ are respectively the deformations (quantum analogues) of the sets $\{\phi^2, \phi^2_n\}$ and $\{\phi^3, \phi^3_n\}$ introduced in the Wakimoto bosonization. They make up a deformed parafermion. The main purpose of this paper is to determine the deformation of the two Heisenberg algebras generated by $\{\varphi^2, \varphi^2_n\}$ and $\{\varphi^3, \varphi^3_n\}$ and use them to complete the bosonization of $E^\pm(z)$. It is mainly this part that distinguishes from one another the several bosonizations that have been recently proposed in the literature. The Wakimoto bosonization of $E^\pm(z)$ involves classical derivatives as shown in (2.12) and therefore its natural deformation must also involve some quantum derivatives. Furthermore, it must also coincide with the Wakimoto bosonization (2.11)-(2.12) or any of its equivalent forms through some field redefinitions as $q$ tends to 1. Finally, it must produce the two different simple poles $z = wq^\pm k$ in the OPE $E^+(z)E^-(w)$. Guided with these constraints, the general form of the bosonization of the quantum currents $E^\pm(z)$ reduces to

$$E^\pm(z) = \exp\{\pm i\sqrt{\frac{2}{k}} \varphi^1(z)\} \left(\exp\{\pm i\sqrt{\frac{2}{k}} X_A^\pm(z)\} - \exp\{\pm i\sqrt{\frac{2}{k}} X_B^\pm(z)\}\right), \quad (3.37)$$

where the $\varphi^1(z)$ exponential part is already fixed by (3.32) and

$$X_A^\pm(z) = \varphi^2 - i\varphi^2_0 \ln z q^{A^\pm_2} - i\varphi^3_0 \ln q^{A^\pm_3} + i \sum_{n \neq 0} \{A^\pm_2(n)\varphi^2_n + A^\pm_3(n)\varphi^3_n\} \frac{z^{-n}}{n}. \quad (3.38)$$

$X_B^\pm(z)$ is given by a similar expression to (3.38) with $A$ being replaced by $B$. The boson oscillators $\{\varphi^2, \varphi^2_n\}$ and $\{\varphi^3, \varphi^3_n\}$ satisfy the following deformed Heisenberg algebras:

$$[\varphi_n^j, \varphi_m^\ell] = (-1)^{j-1} n I_j(n) \delta^{j,\ell} \delta_{n+m,0},$$

$$[\varphi^j, \varphi^j_0] = (-1)^{j-1} i \delta^{j,\ell}, \quad j, \ell = 2, 3. \quad (3.39)$$

Here no sum with respect to $j$ is meant. The question of the deformation of the Wakimoto bosonization translates now into fixing the unknown parameters $A^\pm_2(n)$, $A^\pm_3(n)$, $A^\pm_4$, $B^\pm_2(n)$, $B^\pm_3(n)$, $B^\pm_4$, $I_2(n)$ and $I_3(n)$ consistently with (3.23) and the above constraints.
First, the consistency with (3.23) requires the following relations to be satisfied:

$$\exp\{i\sqrt{\frac{q}{2}}X_B^+(z)\}.\exp\{-i\sqrt{\frac{q}{2}}X_A^-(w)\} = \exp\left\{ -\frac{2}{k}(\varphi^{1,+}(z)\varphi^{1,-}(w)) \right\} \times \exp\{i\sqrt{\frac{2}{k}}(X_B^+(z) - X_A^-(w))\} :,$$

$$X_B^+(wq^k) = X_A^-(w),$$

$$\exp\{i\sqrt{\frac{q}{2}}X_A^+(z)\}.\exp\{-i\sqrt{\frac{q}{2}}X_B^-(w)\} = \exp\left\{ -\frac{2}{k}(\varphi^{1,+}(z)\varphi^{1,-}(w)) \right\} \times \exp\{i\sqrt{\frac{2}{k}}(X_A^+(z) - X_B^-(w))\} :,$$

$$X_A^+(wq^{-k}) = X_B^-(w),$$

$$\exp\{i\sqrt{\frac{q}{2}}X_B^+(z)\}.\exp\{-i\sqrt{\frac{q}{2}}X_B^-(w)\} = q\exp\left\{ -\frac{2}{k}(\varphi^{1,+}(z)\varphi^{1,-}(w)) \right\} \times \exp\{i\sqrt{\frac{2}{k}}(X_B^+(z) - X_B^-(w))\} :,$$

$$\exp\{i\sqrt{\frac{q}{2}}X_I^+(z)\}.\exp\{-i\sqrt{\frac{q}{2}}X_I^-(w)\} = q^{-1}\exp\left\{ -\frac{2}{k}(\varphi^{1,+}(z)\varphi^{1,-}(w)) \right\} \times \exp\{i\sqrt{\frac{2}{k}}(X_I^+(z) - X_I^-(w))\} :.$$

These relations translate to these conditions on the unknown parameters:

$$A_2^\pm(n) = q^{-nk}B_2^\mp(n),$$

$$A_3^\pm(n) = q^{-nk}B_3^\mp(n),$$

$$A_2^\pm = -B_2^\pm = k/2,$$

$$A_3^\pm = B_3^\mp = \pm\sqrt{\frac{k^2}{2}},$$

and

$$A_2^\pm(n)A_2^\pm(-n)I_2(n) - A_3^\pm(n)A_3^\pm(-n)I_3(n) = \frac{k}{2}\left(\frac{[n(k+2)]}{[nk]} - 1\right), \quad n > 0,$$

$$A_2^\pm(n)A_2^\pm(-n)I_2(n) - A_3^\pm(n)A_3^\pm(-n)I_3(n) = \frac{k}{2}\left(\frac{[2n]}{[nk]}\right), \quad n > 0.$$

The relations (3.43) and (3.44) are in fact derived only from the consistency with the limit $q \to 1$. The parameters $A_2^\pm, A_3^\pm, B_2^\pm$ and $B_3^\pm$ are then completely fixed by the latter relations. Therefore only the parameters $A_2^\pm(n), A_3^\pm(n), B_2^\pm(n), B_3^\pm(n), I_2(n)$ and $I_3(n)$ are left to be determined. Furthermore, since $B_2^\pm(n)$ and $B_3^\pm(n)$ are related to $A_2^\pm(n)$ and $A_3^\pm(n)$ through (3.41) and (3.42), we will henceforth focus just on $A_2^\pm(n), A_3^\pm(n), I_2(n)$ and $I_3(n)$. These are restricted to satisfy the set of general “master” equations (3.45). Each solution of these equations yields a particular bosonization. In fact, this is how we will now recover both the bosonizations of ABE and Matsuo, and derive a new one called “the fifth” which also has the standard form defined previously.
3.2.1 The ABE bosonization

This bosonization corresponds to the following choice of the parameters $A_2^\pm(n)$, $A_3^\pm(n)$, $I_2(n)$ and $I_3(n)$, which do indeed satisfy the general equations (3.43):

$$
A_2^\pm(n) = q^{-nk/2},
A_3^\pm(n) = \pm \frac{1}{2} \sqrt{k^2 + 2} (q^{-nk} - 1),
I_2(n) = \frac{k}{4} \left[ n(k+2) + \frac{[2n] - [nk]}{[nk]} \right],
I_3(n) = \frac{k^2}{k+2} \left[ n(k+2)/2 \right].
$$

(3.46)

In order to distinguish later the ABE bosonization from the other ones let us introduce the following notations, which are specific just for this bosonization:

$$
\varphi_n^i \equiv \xi_n^i, \quad \varphi^i \equiv \xi^i, \quad i = 1, 2, 3.
$$

(3.47)

The ABE bosonization can then be summarized in this notation as [22]:

$$
\Psi(z) = \exp \left\{ i \sqrt{\frac{2}{k}} \left( \xi^1(zq^{k/2}) - \xi^1(zq^{-k/2}) \right) \right\}
= q^{\sqrt{2k}\delta_0} \exp \left\{ \sqrt{2k} (q - q^{-1}) \sum_{n>0} \xi_n^1 z^{-n} \right\},
\Phi(z) = \exp \left\{ i \sqrt{\frac{2}{k}} \left( \xi^1(zq^{k/2}) - \xi^1(zq^{-k/2}) \right) \right\}
= q^{-\sqrt{2k}\delta_0} \exp \left\{ -\sqrt{2k} (q - q^{-1}) \sum_{n<0} \xi_n^1 z^{-n} \right\},
E^\pm(z) = \frac{\exp \left\{ \pm i \sqrt{\frac{2}{k}} \xi^{1,\pm}(z) \right\}}{z(q-q^{-1})} \left( \exp \left\{ \pm i \sqrt{\frac{2}{k}} \xi^3 \left( zq^{k/2} \right) + i \sqrt{\frac{k+2}{2k}} \left( \xi^3(zq^{k/2}) - \xi^3(z) \right) \right\} \right.
- \exp \left\{ \pm i \sqrt{\frac{2}{k}} \xi^3 \left( zq^{-k/2} \right) + i \sqrt{\frac{k+2}{2k}} \left( \xi^3(zq^{-k/2}) - \xi^3(z) \right) \right\} \right),
$$

where the deformed free bosons $\xi^{1,\pm}(z)$, $\xi^3(z)$ and $\xi^3(z)$ are given by

$$
\xi^{1,\pm}(z) = \xi^1 - i\xi_n^1 \ln z + i k \sum_{n \neq 0} \frac{q^{\mp |n| k/2}}{[nk]} \xi_n^1 z^{-n},
\xi^j(z) = \xi^j - i\xi_n^j \ln z + i \sum_{n \neq 0} \frac{z^{-n}}{n} \xi_n^j, \quad j = 2, 3.
$$

(3.49)

They are given as various deformed generating functions of the boson oscillators $\{\xi^j, \xi^j_n, \quad j = 1, 2, 3\}$ which satisfy the three deformed Heisenberg algebras

$$
[\xi_n^j, \xi_m^\ell] = (-1)^{j-1} n I_j(n) \delta^j,\ell \delta_{n+m,0},
[\xi^j, \xi^\ell_n] = (-1)^{j-1} i \delta^{j,\ell} \quad j, \ell = 1, 2, 3.
$$

(3.50)
with no sum with respect to $j$ being meant, and

$$I_1(n) = \frac{[2n][nk]}{2kn^2},$$
$$I_2(n) = \frac{[n(n+2)+2n-nk]}{[nk]},$$
$$I_3(n) = \frac{k^2 [n(n+2)/2]^2}{[nk][nk/2]}.$$ (3.51)

As a check, note that as $q$ approaches 1 all three $I_j(n)$, $j = 1, 2, 3$ tend to 1, which is consistent with (2.14) in the classical case, and the deformed free bosons $\xi^1, (z)$, $\xi^2(z)$ and $\xi^3(z)$ reduce to $\phi^1(z)$, $\phi^2(z)$ and $\phi^3(z)$ (2.13) respectively. Furthermore, in this same limit the quantum currents $E^\pm(z)$ and $\{\Psi(z)−\Phi(z)\}/\sqrt{2(q−q^{-1})}$ tend to their classical analogues (2.12) and (2.11) respectively. Finally, note that the remaining OPE (3.24) of the $U_q(su(2)_k)$ QCA is automatically satisfied.

### 3.2.2 The Matsuo bosonization

This bosonization is characterized by the following choice of the parameters $A_2^\pm(n)$, $A_3^\pm(n)$, $I_2(n)$ and $I_3(n)$, which also satisfy the general equations (3.45):

$$A_2^\eta(n) = \frac{nkq^{-nk/2}}{[nk]},$$
$$A_2^{-\eta}(n) = \frac{nkq^{-nk/2}}{[2n]} \left( \frac{n(n+2)}{[nk]} − 1 \right),$$
$$A_3^\eta(n) = 0,$$
$$A_3^{-\eta}(n) = \eta \sqrt{k(k+2)nq^{-nk/2}(q−q^{-1})[n]} [2n],$$
$$I_2(n) = \frac{[2n][nk]}{2kn^2},$$
$$I_3(n) = \frac{[2n][n(n+2)]}{2n^2(k+2)}.$$ (3.52)

where $\eta$ is equal to $+$ or $−$ depending on whether $n > 0$ or $n < 0$ respectively. Again as a check, notice that $I_2(n)$ and $I_3(n)$ approach 1 as $q$ tends to 1. In fact, the original bosonization of Matsuo uses slightly different normalization factors from those in the above solution. To recover the Matsuo bosonization in its initial notations, which will then distinguish it from the others in what follows, let us make the identifications
\[ \varphi^1_n \equiv \frac{\alpha_n}{\sqrt{2k}}, \]
\[ \varphi^2_n \equiv \frac{\bar{\alpha}_n}{\sqrt{2k}}, \]
\[ \varphi^3_n \equiv \frac{\beta_n}{\sqrt{2(k+2)}}, \]
\[ \varphi^1 \equiv -i\sqrt{2k}\alpha, \]
\[ \varphi^2 \equiv -i\sqrt{2k}\bar{\alpha}, \]
\[ \varphi^3 \equiv -i\sqrt{2(k+2)}\beta, \]

which lead to the commutation relations
\[
\begin{align*}
[\alpha_n, \alpha_m] &= \frac{[2n][nk]}{n} \delta_{n+m,0}, \\
[\bar{\alpha}_n, \bar{\alpha}_m] &= -\frac{[2n][nk]}{n} \delta_{n+m,0}, \\
[\beta_n, \beta_m] &= \frac{[2n][n(k+2)]}{n} \delta_{n+m,0}, \\
[\alpha_0, \alpha] &= 1, \\
[\bar{\alpha}_0, \bar{\alpha}] &= -1, \\
[\beta_0, \beta] &= 1. 
\end{align*}
\]

These are the precisely the three Heisenberg algebras used in the Matsuo bosonization, which is summarized as follows \[23\]:

\[ \Psi(z) = q^{\alpha_0} \exp\{ (q - q^{-1}) \sum_{n>0} \alpha_n z^{-n} \}, \]
\[ \Phi(z) = q^{-\alpha_0} \exp\{ -(q - q^{-1}) \sum_{n<0} \alpha_n z^{-n} \}, \]
\[ E^\pm(z) = \frac{Y^\pm(z)}{z(q-q^{-1})} \left\{ Z^\pm(zq^{-\frac{k+2}{2}})W^\pm(zq^{\frac{k}{2}})^\pm1 - Z^\pm(zq^{-\frac{k+2}{2}})W^\pm(zq^{\frac{k}{2}})^\pm1 \right\}, \]

where
\[
\begin{align*}
Y^\pm(z) &= \exp\{ \pm2(\alpha + \bar{\alpha}) \pm \frac{(\alpha_0 + \bar{\alpha}_0)}{k} \ln z \mp \sum_{n \neq 0} \frac{q^{\pm[nk]/2}}{[nk]} - (\alpha_n + \bar{\alpha}_n) z^{-n} \}, \\
Z^\pm(z) &= q^{\mp\alpha_0/2} \exp\{ \mp(q - q^{-1}) \sum_{n>0} \left[ \frac{|n|}{2n} \bar{\alpha}_n z^{\mp n} \right] \}, \\
W^\pm(z) &= q^{\mp\beta_0/2} \exp\{ \mp(q - q^{-1}) \sum_{n>0} \left[ \frac{|n|}{2n} \beta_\pm n z^{\mp n} \right] \}.
\end{align*}
\]

It can also be verified that as \( q \) approaches 1 we recover the classical bosonization \[2.11\] and \[2.12\] with or without the same normalization factors if we use \[3.52\] or \[3.55\] respectively.

\(^1\)The currents \( E^\pm(z) \) are in fact equal to those of Matsuo up to an overall minus sign, which is irrelevant because it just reflects a \( Z_2 \) automorphism of the \( U_q(su(2)_k) \) QCA.
3.2.3 The fifth bosonization

This new fifth bosonization is specified by the following parameters, which also satisfy the general equations (3.45):

\[
A_2^\pm(n) = \sqrt{\frac{k+\pm nk}{2nk}} q^{(\pm 1-2)k/2}, \quad n > 0,
\]
\[
A_2^\pm(n) = \sqrt{\frac{k+\pm nk}{2nk}} q^{n+nk/2}, \quad n < 0,
\]
\[
A_3^+(n) = -\sqrt{2k} \frac{n}{[2n]} q^{nf-k/2},
\]
\[
A_3^-(n) = -\sqrt{2k} \frac{n}{[2n]} q^{nf-2-3k/2},
\]
\[
I_2(n) = \frac{[nk][n(k+2)]}{4n^2} q^{nk},
\]
\[
I_3(n) = \frac{[2n]^2}{4nk}.
\]

Here $f$ is a free parameter whose usefulness will become clear shortly. Again, in order to distinguish the fifth bosonization from the other ones and to write it in a more compact form, let us specify it by the following identifications:

\[
\varphi_n^i \equiv \chi_n^i, \quad n \neq 0, \quad i = 1, 2, 3,
\]
\[
\varphi_0^1 \equiv \chi_0^1,
\]
\[
\varphi_0^2 \equiv \sqrt{\frac{2+k}{2}} \chi_0^2 - \sqrt{\frac{k}{2}} \chi_0^3,
\]
\[
\varphi_0^3 \equiv -\sqrt{\frac{k}{2}} \chi_0^2 + \sqrt{\frac{2+k}{2}} \chi_0^3,
\]
\[
\varphi_1^1 \equiv \chi_1^1,
\]
\[
\varphi_2^2 \equiv \sqrt{\frac{2+k}{2}} \chi_2^2 - \sqrt{\frac{k}{2}} \chi_2^3,
\]
\[
\varphi_3^3 \equiv -\sqrt{\frac{k}{2}} \chi_3^2 + \sqrt{\frac{2+k}{2}} \chi_3^3.
\]

In this notation, this fifth bosonization reads simply as:

\[
\Psi(z) = \exp \left\{ i \sqrt{\frac{2k}{k}} \left( \chi_1^1(zq^{k/2}) - \chi_1^1(\sqrt{zq^{-k/2}}) \right) \right\}
\]
\[
= q^{\sqrt{2k} \chi_0^1} \exp \left\{ \sqrt{2k} (q - q^{-1}) \sum_{n>0} \chi_n^1 z^{-n} \right\},
\]
\[
\Phi(z) = \exp \left\{ i \sqrt{\frac{2k}{k}} \left( \chi_1^1(zq^{-k/2}) - \chi_1^1(\sqrt{zq^{k/2}}) \right) \right\}
\]
\[
= q^{-\sqrt{2k} \chi_0^1} \exp \left\{ -\sqrt{2k} (q - q^{-1}) \sum_{n<0} \chi_n^1 z^{-n} \right\},
\]
\[
E^+(z) = \frac{\exp \left\{ i \sqrt{\frac{2k}{k}} \chi_1^1(z) + i \sqrt{2k} \chi_1^2(z) \right\} \exp \left\{ -i \chi_1^3(zq^{-1}) - \exp \left\{ -i \chi_3^3(zq) \right\} \right\}}{z(q^{-1})},
\]
\[
E^-(z) = \frac{\exp \left\{ -i \sqrt{\frac{2k}{k}} \chi_1^1(z) - i \sqrt{2k} \chi_1^2(z) \right\} \left( \exp \left\{ -i \sqrt{2k} \chi_1^2(zq) + i \chi_3^3(zq^{1+k}) \right\} \right.}{\left. z(q^{-1}) \right)} - \exp \left\{ -i \sqrt{2k} \chi_1^2(zq^{-k}) + i \chi_3^3(zq^{-1-k}) \right\},
\]
where the deformed free boson fields $\chi^{1,\pm}(z)$, $\chi^2(z)$ and $\chi^3(z)$ are given by

\[
\begin{align*}
\chi^{1,\pm}(z) &= \chi^1 - i\chi_0^1 \ln z + ik \sum_{n \neq 0} \frac{q^{\pm i|n|/2}}{[nk]} \chi^1_n z^{-n}, \\
\chi^2(z) &= \chi^2 - i\chi_0^2 \ln z + ik \sum_{n \neq 0} \frac{q^{-|n|/2}}{[nk]} \chi^2_n z^{-n}, \\
\chi^3(z) &= \chi^3 - i\chi_0^3 \ln z + 2i \sum_{n \neq 0} \frac{q^{(f-1-k/2)}}{[2n]} \chi^3_n z^{-n}.
\end{align*}
\]

These fields are given as various deformed generating functions of the boson oscillators which satisfy the three deformed Heisenberg algebras

\[
\begin{align*}
[\chi^j_n, \chi^\ell_m] &= (-1)^{j-1} n I_j(n) \delta^{j,\ell} \delta_{n+m,0}, \\
[\chi^j, \chi^0_0] &= (-1)^{j-1} i \delta^{j,\ell},
\end{align*}
\]

with

\[
\begin{align*}
I_1(n) &= \frac{[2n][nk]}{2kn^2}, \\
I_2(n) &= \frac{[nk][n(2+k)]}{n^2 k(2+k) q^n} q^{nk}, \\
I_3(n) &= \frac{[2n]^2}{4n^2}.
\end{align*}
\]

Note that this bosonization not only satisfies the QCA but is also given in a very simple compact form. In fact, it is the simplest of all the bosonizations considered in this paper.

### 4 Relations among all the bosonizations of the $U_q(\mathfrak{su}(2)_k)$ QCA

In this section, we unravel the relations among the previous three bosonizations and those of both Shiraishi and Kimura. As we will see shortly, the latter two cannot be recovered directly from the general equations (3.45) because by construction they are not in the standard form. In fact, we will only show that the fifth bosonization is at the center of all these bosonizations in the sense that it can be related to each of the other four through some redefinitions of the three sets of deformed boson oscillators. This means that in this way they are all related to each other.
4.1 The relation between the ABE and the fifth bosonizations

One can easily check that the ABE bosonization (3.48) can be obtained from the fifth one (3.59) if their boson oscillators are related through the following linear transformations:

\[ \xi_{\pm n}^1 = \chi_{\pm n}^1, \quad n > 0, \]
\[ \xi_{\pm n}^2 = \frac{n k \sqrt{2+n}}{2 \sqrt{2+n|k|/2}} \chi_{\pm n}^2 - \frac{n \sqrt{k(2+k)(1+q^{(2+k)q^{n}})^2}}{\sqrt{2+n|k|/2}} \chi_{\pm n}^3, \quad n > 0, \]
\[ \xi_{\pm n}^3 = -\frac{n k \sqrt{2-n}}{2 \sqrt{2+n|k|/2}} \chi_{\pm n}^2 + \frac{n \sqrt{k(2+k)q^{2n(1-k/2)}}}{\sqrt{2+n|k|/2}} \chi_{\pm n}^3, \quad n > 0, \]

and \( \{\xi_0^j, \xi^j, \quad j = 1, 2, 3\} \) are related to \( \{\chi_0^j, \chi^j, \quad j = 1, 2, 3\} \) in the same way as in (3.58) with \( \{\varphi_0^j, \varphi^j, \quad j = 1, 2, 3\} \) being substituted by \( \{\xi_0^j, \xi^j, \quad j = 1, 2, 3\} \).

4.2 The relation between the Matsuo and the fifth bosonizations

These two bosonizations can also be related to each other through these transformations:

\[ \alpha_{\pm n} = \sqrt{2+k} \chi_{\pm n}^1, \quad n > 0, \]
\[ \bar{\alpha}_{\pm n} = \sqrt{k(2+k)} \chi_{\pm n}^3 - \frac{2[nkq^{(2+k)(1+q^{n})/2}]}{2n|k|/2} \chi_{\pm n}^3, \quad n > 0, \]
\[ \beta_{\pm n} = -\sqrt{k(2+k)} q^n \chi_{\pm n}^2 + \frac{2[n(2+k)q^{n(2+k)q^{(1+q^{n})/2}}]}{2n|k|/2} \chi_{\pm n}^3, \quad n > 0, \]

and

\[ \alpha_0 = \sqrt{2+k} \chi_0^1, \]
\[ \bar{\alpha}_0 = \sqrt{k(2+k)} \chi_0^3 - k \chi_0^3, \]
\[ \beta_0 = -\sqrt{2+k} \chi_0^2 + (2+k) \chi_0^3, \]
\[ \alpha = \frac{i}{2} \chi_1^1, \]
\[ \bar{\alpha} = \frac{i}{2} \sqrt{2+k} \chi_2^2 - \frac{i}{2} \chi_3^3, \]
\[ \beta = -\frac{i}{2} \sqrt{2+k} \chi_2^2 + \frac{i}{2} \chi_3^3. \]

4.3 The relation between the Kimura-Shiraishi and the fifth bosonizations

In this section, we have considered simultaneously the Shiraishi and the Kimura bosonizations because as we will show later they can be treated on the same basis as two special cases
of a more general single bosonization, which we refer to as the Kimura-Shiraishi bosonization and define by

$$
\Psi(z) = q^{L_0+M_0} \exp\{(q - q^{-1}) \sum_{n>0} (L_n + M_n) z^{-n}\},
$$

$$
\Phi(z) = q^{-(L_0+M_0)} \exp\{-(q - q^{-1}) \sum_{n<0} (q^{-ng} L_n + q^{-nh} M_n) z^{-n}\},
$$

$$
E^\pm(z) = \frac{1}{z(q-q^{-1})} \left( \exp\{M^\pm(z) + N^\pm(z)\} - \exp\{\bar{L}^\pm(z) + \bar{M}^\pm(z) + \bar{N}^\pm(z)\} \right),
$$

with

$$
M^+(z) = \bar{M}^+(z) = -\frac{M}{2} - \frac{M_0}{2} \ln z + \sum_{n<0} \frac{q^{n(h+2+k/2)}}{[2n]} M_n z^{-n} + \sum_{n>0} \frac{q^{n(2+k/2)}}{[2n]} M_n z^{-n},
$$

$$
N^+(z) = \bar{N}^+(z) = -\frac{N}{2} - \frac{N_0}{2} \ln z + \sum_{n\neq 0} \frac{q^{(f-k/2)}}{[2n]} N_n z^{-n},
$$

$$
L^-(z) = L_0 \ln q + (q - q^{-1}) \sum_{n>0} q^{-nk/2} L_n z^{-n},
$$

$$
\bar{L}^-(z) = -L_0 \ln q - (q - q^{-1}) \sum_{n<0} q^{n(k/2-g)} L_n z^{-n},
$$

$$
M^-(z) = \bar{M}^-(z) = -\frac{M}{2} + \frac{M_0}{2} \ln z q^{2+k} - \sum_{n<0} \frac{q^{n(h+2+3k/2)}}{[2n]} M_n z^{-n} - \sum_{n>0} \frac{q^{n(2+k/2)}}{[2n]} M_n z^{-n},
$$

$$
N^-(z) = \bar{N}^-(z) = \frac{N}{2} + \frac{N_0}{2} \ln z q^{1+k} + \sum_{n\neq 0} \frac{q^{n(f-2-3k/2)}}{[2n]} N_n z^{-n}.
$$

The sets of boson oscillators \(\{L, L_n\}, \{M, M_n\}\) and \(\{N, N_n\}\) satisfy the three deformed Heisenberg algebras

$$
[L_n, L_m] = \frac{[2n][n(2+k)]}{n} q^{-n(2+g)} \delta_{n+m,0},
$$

$$
[M_n, M_m] = -\frac{[2n]^2}{n} q^{-n(h+2+k)} \delta_{n+m,0},
$$

$$
[N_n, N_m] = \frac{[2n]^2}{n} \delta_{n+m,0},
$$

$$
[L_0, L] = 2(2+k),
$$

$$
[M_0, M] = -4,
$$

$$
[N_0, N] = 4.\tag{4.68}
$$

Here \(f\) is the same free parameter as that introduced in (3.54), and \(g\) and \(h\) are two new free parameters. It can be checked that despite the presence of these three free parameters, this Kimura-Shiraishi bosonization which is not in the standard form does indeed satisfy the \(U_q(su(2)_k)\) QCA \(\text{[3.20]-[3.24]}\).

The fifth bosonization can also be obtained from the Kimura-Shiraishi one through the following redefinitions of their respective boson oscillators:
\[\begin{align*}
L_n &= \frac{\sqrt{2k[2n]}q^{-2n}}{[nk]}\chi_1^n + \frac{\sqrt{k(2+k)[2n]}q^{-n(2+k)}}{[nk]}\chi_2^n, \\
M_n &= -\frac{\sqrt{2k[2n]}q^{-n(2+k)}}{[nk]}\chi_1^n - \frac{\sqrt{k(2+k)[2n]}q^{-n(2+k)}}{[nk]}\chi_2^n, \\
C_n &= 2\chi_3^n, \\
L_{-n} &= \frac{\sqrt{2k[2n]}q^{-n(2+k)}}{[nk]}\chi_1^{1-n} + \frac{\sqrt{k(2+k)[2n]}q^{-n(2+k)}}{[nk]}\chi_2^{2-n}, \\
M_{-n} &= -\frac{\sqrt{2k[2n]}q^{-n(2+k)}}{[nk]}\chi_1^{1-n} - \frac{\sqrt{k(2+k)[2n]}q^{-n(2+k)}}{[nk]}\chi_2^{2-n}, \\
C_{-n} &= 2\chi_3^{3-n},
\end{align*}\]

with \(n > 0\), and
\[\begin{align*}
\sqrt{\frac{k}{2}}L_0 &= (2 + k)\chi_1^1 + \sqrt{2(2 + k)}\chi_2^0, \\
\sqrt{\frac{k}{2}}M_0 &= -2\chi_1^0 - \sqrt{2(2 + k)}\chi_2^0, \\
\sqrt{\frac{k}{2}}N_0 &= \sqrt{2k}\chi_3^0.
\end{align*}\]

The set \(\sqrt{\frac{k}{2}}\{L, M, N\}\) is related to \(i\\{\chi^1, \chi^2, \chi^3\}\) in a similar manner as in (4.70).

Let us now show how both the Shiraishi and Kimura bosonizations can be recovered as two special cases from the Kimura-Shiraishi one. If we fix the three free parameters as:
\[\begin{align*}
f &= 3 + 3k/2, \\
g &= -4, \\
h &= -2k - 4,
\end{align*}\]

and make the following identifications:
\[\begin{align*}
L_n &\equiv a_n, \quad n \neq 0, \\
M_n &\equiv b_n, \quad n \neq 0, \\
N_n &\equiv c_n, \quad n \neq 0, \\
L_0 &\equiv \tilde{a}^0, \\
M_0 &\equiv \tilde{b}_0, \\
N_0 &\equiv \tilde{c}_0, \\
L &\equiv Q_a, \\
M &\equiv Q_b, \\
N &\equiv Q_c,
\end{align*}\]
we recover the Shiraishi bosonization in its original notations, that is, \[ \Psi(z) = q^{(\tilde{a}_0 + \tilde{b}_0)} \exp\{ (q - q^{-1}) \sum_{n>0} (a_n + b_n) z^{-n} \}, \]
\[ \Phi(z) = q^{-(\tilde{a}_0 + \tilde{b}_0)} \exp\{ -(q - q^{-1}) \sum_{n<0} (q^{4n} a_n + q^n (2k+4) b_n) z^{-n} \}, \]
\[ E^+ (z) = q^{-(\tilde{b}_0 + \tilde{c}_0)} \frac{(k+2)}{2} \exp\left\{ -b(2|q^{k-3}z^{-1}; 0) \right\} - \exp\left\{ -c(2|q^{-k-1}; 0) \right\}, \]
\[ E^- (z) = q^{(\tilde{b}_0 + \tilde{c}_0)} \frac{(k+2)}{2} \exp\left\{ -a(2|q^{k-3}z^{-1}; 0) \right\} - \exp\left\{ -b(2|q^{-k-1}; 0) \right\}. \]

(4.73)

Here the following deformed boson fields have been used:
\[ d(x|z; y) = \frac{Qd}{x} + \frac{\tilde{d}_0}{x} \ln z - \sum_{n\neq 0} \frac{q^{[n|x]} d_n}{[n,x]} z^{-n}, \quad d = a, b, c. \]  
(4.74)

Note that the extra term \[ q^{\pm (\tilde{b}_0 + \tilde{c}_0)(k+2)/2} \], which arises here but not in the Shiraishi bosonization, is irrelevant in the sense that it can be kept or omitted without affecting the correctness of this bosonization. This is because it commutes with the currents \( E^\pm (z), \Psi(z) \) and \( \Phi(z) \), and moreover it does not affect their limits as \( q \) approaches 1.

Similarly, the Kimura bosonization is obtained from the Kimura-Shiraishi one if we set the three free parameters to
\[ f = k + 2, \]
\[ g = -2, \]
\[ h = -k - 2, \]

(4.75)

and make the identifications
\[ L_n \equiv b_n, \]
\[ M_n \equiv a_n, \]
\[ N_n \equiv \bar{a}_n, \]
\[ L \equiv b, \]
\[ M \equiv a, \]
\[ N \equiv \bar{a}. \]

(4.76)

We indeed retrieve in this way the Kimura bosonization as first proposed in [20], i.e.,
\[ \Psi(z) = q^{(a_0 + b_0)} \exp \left\{ (q - q^{-1}) \sum_{n>0} (a_n + b_n) z^{-n} \right\}, \]
\[ \Phi(z) = q^{-(a_0 + b_0)} \exp \left\{ -(q - q^{-1}) \sum_{n<0} (q^{n(k+2)} a_n + q^{2n} b_n) z^{-n} \right\}, \]
\[ E^+(z) = \frac{Y^+(z)}{z(q-q^{-1})} \{ Z_-(zq^{-k/2}) - Z_+(zq^{-k/2}) \}, \]
\[ E^-(z) = \frac{Y^-(z)}{z(q-q^{-1})} \{ Z_+(zq^{+k/2}) W_+(zq^{+k/2}) U_+(zq^{+k/2}) - Z_-(zq^{-k/2}) W_-(zq^{-k/2}) U_-(zq^{-k/2}) \}, \]

where
\[ Y^\pm(z) = \exp \{ \mp \frac{(a_0 + \bar{a}_0)}{2} \ln z \pm \sum_{n<0} \frac{q^k}{|2n|} (a_n + \bar{a}_n) z^{-n} \} \pm \sum_{n>0} \frac{q^{n(k+2)}}{|2n|} (a_n + \bar{a}_n) z^{-n}, \]
\[ Z^\pm(z) = q^{\pm a_0/2} \exp \{ \mp (q - q^{-1}) \sum_{n>0} \frac{|n|}{2|n|} q^{-n(k+2)(1+1)/2} a_{\pm n} z^{|n|} \}, \]
\[ W^\pm(z) = q^{\pm b_0} \exp \{ \pm (q - q^{-1}) \sum_{n>0} q^{n(k+1)+1} b_{\pm n} z^{|n|} \}, \]
\[ U^\pm(z) = q^{\pm(k+2)(a_0 + \bar{a}_0)/2} \exp \{ \pm (q - q^{-1}) \sum_{n>0} q^{n(k+1)+1} q^{-n(1+1)|n|} \frac{2n}{|2n|} (a_{\pm n} + \bar{a}_{\pm n}) z^{|n|} \}. \]

(4.77)

5 Conclusions

In this paper, we have proven that all of the four presently available and apparently different bosonizations of the $U_q(su(2)_k)$ quantum current algebra are in fact equivalent to each other and to a new simpler one through the redefinitions of the Heisenberg boson oscillators. Clearly, this result suggests that the deformation of the Wakimoto bosonization with three other and to a new simpler one through the redefinitions of the Heisenberg boson oscillators.

5 Conclusions

In this paper, we have proven that all of the four presently available and apparently different bosonizations of the $U_q(su(2)_k)$ quantum current algebra are in fact equivalent to each other and to a new simpler one through the redefinitions of the Heisenberg boson oscillators. Clearly, this result suggests that the deformation of the Wakimoto bosonization with three other and to a new simpler one through the redefinitions of the Heisenberg boson oscillators as in the classical case. Now that the question of the bosonization of the quantum currents which generate the $U_q(su(2)_k)$ quantum current algebra is clarified, the focus should be put on the other two important ingredients of the bosonization recipe, namely, the realization of the representations of this algebra or in the language of conformal field theory the quantum analogue of the primary fields, and the realization of the quantum analogue of the screening currents in terms of the same Heisenberg boson oscillators. The latter two ingredients are necessary for the calculation of many relevant quantities and in particular the correlation functions of the XXZ model. Moreover, we expect the fifth bosonization to simplify particularly the bosonization of the quantum analogue of the primary fields. We think that the quantum analogue of the parafermion algebra whose elementary generators can easily be
deduced from the currents $E^{\pm}(z)$ of the $U_q(su(2)_k)$ quantum current algebra also deserves more attention. These and other related questions are presently under investigation.

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References

[1] V. G. Drinfeld, Proc. ICM (Am. Math. Soc., Berkeley, CA, 1986).

[2] M. Jimbo, Commu. Math. Phys. 102 (1986) 537.

[3] S. L. Woronowicz, Comm. Math. Phys. 111 (1987) 613.

[4] L. Alvarez-Gaumé, C. Gomez and G. Sierra, *Topics in conformal field theory*, Knizhnik Memorial Volume (World Scientific, Singapore, 1990).

[5] H. J. De Vega, Int. J. Mod. Phys. A4 (1989) 2371.

[6] I. B. Frenkel and N. Yu Reshetikhin, Comm. Math. Phys. 146 (1992) 1.

[7] D. Bernard and A. Leclair, preprints CLNS-92/1147, SPhT-92-054; CLNS-90/1036, SPhT-90-173.

[8] B. Davies, O. Foda, M. Jimbo, T. Miwa and A. Nakayashiki, RIMS preprint No. 873 (1992).

[9] V. G. Knizhnic and A. B. Zamolodchikov, Nuc. Phys. B247 (1984) 83.

[10] B. L. Feigin and D. B. Fuchs, Funct. Anal. Appl. 16 (1982) 114; 17 (1983) 241.

[11] Vl. S. Dotsenko and V. A. Fateev, Nuc. Phys. B240 (1984) 312.

[12] O. Babelon and D. Bernard, preprints SPhT-92-062; LPTHE-92-20.

[13] K. Ito and Y. Kazama, Mod. Phys. Lett. A5 (1990) 215.

[14] A. Gerasimov, A. Marshakov, A. Morozov, M. Olshanetsky and S. Shatashvili, Int. J. Mod. Phys. A5 (1990) 2495.

[15] A. H. Bougourzi, Phys. Lett. B286 (1992) 279.

[16] D. Nemeschansky, Phys. Lett. B224 (1989) 566.

[17] T. Jayaraman, K. S. Narain and M. H. Sarmadi, Nucl. Phys. B343 (1990) 418.
[18] I. B. Frenkel and V. G. Kac, Invent. Math. 62 (1980) 23.

[19] M. Wakimoto, Comm. Math. Phys. 104 (1986) 605.

[20] I. B. Frenkel and N. H. Jing, Proc. Nat’l. Acad. Sci. (USA) 85 (1988) 9373.

[21] A.H. Bougourzi and M.A. El Gradechi, preprint CRM-1827 (1992), to appear in J. Group Theory Phys.

[22] A. Abada, A.H. Bougourzi and M.A. El Gradechi, preprint CRM-1829 (1992), to appear in Mod. Phys. Lett. A.

[23] A. Matsuo, Nagoya University preprints August (1992) and December (1992).

[24] J. Shiraishi, preprint UT-617 (1992).

[25] A. Kato, Y.-H. Quano and J. Shiraishi, preprint UT-618 (1992).

[26] K. Kimura, Kyoto University preprint (1992).

[27] P. Goddard and D. Olive, Int. J. Mod. Phys. A1 (1986) 303.

[28] V. G. Kac, Infinite dimensional Lie algebras, (Cambridge University Press, 1985).

[29] A. B. Zamolodchikov and V. A. Fateev, Sov. Phys. J.E.T.P. 62 (1985) 215.

[30] V. G. Drinfeld, Soviet Math. Doklady 32 (1985) 254; 36 (1988) 212.

[31] M. Jimbo, Lett. Math. Phys. 10 (1985) 63.

[32] D. Bernard, Lett. Math. Phys. 17 (1989) 239.