Ternary algebras and groups

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Ternary algebras and groups

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Abstract. We construct explicitly groups associated to specific ternary algebras which extend
the Lie (super)algebras (called Lie algebras of order three). It turns out that the natural variables
which appear in this construction are variables which generate the three-exterior algebra. An
explicit matrix representation of a group associated to a peculiar Lie algebra of order three is
constructed considering matrices with entry which belong to the three exterior algebra.

1. Introduction

Most of the laws of physics are based on algebras i.e. on mathematical structures equipped with
a binary product. That is, if one considers an algebra \( A \) with a product \( m_2 \) we have for any
\( a \otimes b \in A \otimes A, m_2(a \otimes b) \in A \). Among algebras the Lie algebras and the Lie superalgebras are of
special importance. The former allow a description of the space-time and internal symmetries
in particle physics although the latter lead to a supersymmetric extension of symmetries. In
both cases the Lie (super)algebras correspond to the symmetries of a physical system at the
infinitesimal level. However, in these two cases one is able to define symmetries at a finite level.
The corresponding symmetries are associated to Lie groups or Lie supergroups.

The ternary algebras have been considered in physics only occasionally (see for instance
[1, 2, 3, 4, 5, 6] and references therein). For some mathematical references one can see [7, 8, 9]. A
ternary algebra \( A \) is an algebraic structure equipped with a ternary product \( m_3 : A \otimes A \otimes A \rightarrow A \).

In [10, 11] an \( F \)-arry algebra which can be seen as a possible generalisation of Lie
(super)algebras have been considered and named Lie algebra of order \( F \). A Lie algebra of order
\( F \) admits a \( \mathbb{Z}_F \)-grading \( (F = 3 \text{ in this paper}), \) the zero-graded part being a Lie algebra. An
\( F \)-fold symmetric product (playing the role of the anticommutator in the case \( F = 2 \)) expresses
the zero graded part in terms of the non-zero graded part. This means that part of the algebra
is a (binary) algebra and part of the algebra is a ternary algebra (when \( F = 3 \)). Subsequently,
a specific Lie algebra or order 3, leading to a non-trivial extension of the Poincaré algebra, has
been studied together with its implementation in Quantum Field Theory [12, 13, 14]. A general
study of the possible non-trivial extensions of the Poincaré algebra in \((1 + 3)\)–dimensions has
been undertaken in [15, 16]. However, all these mathematical structures have been considered
at the level of algebras i.e. at the level of infinitesimal transformations and no groups associated
to Lie algebras of order 3 were considered. At a first glance these two structures seem to be
incompatible since for a Lie algebra of order 3 for some elements only the product of three
elements is defined although for a group the product of two elements is always defined.

In this paper we show that in fact a specific ternary group that we might call a Lie group
of order 3 can be associated to a Lie algebra of order 3. We show that one possible way to
associate a group to a given Lie algebra of order 3 is to endow the universal enveloping algebra of a Lie algebra of order 3 with a Hopf algebra structure. Then we identify the parameters of the transformation. It turns out that these parameters are strongly related to the 3–exterior algebra. The 3–exterior algebra being the algebra generated by canonical generators $\theta^1$ satisfying [17]

$$\theta^i \theta^j \theta^k + \theta^j \theta^k \theta^i + \theta^k \theta^i \theta^j + \theta^i \theta^j \theta^k + \theta^j \theta^k \theta^i + \theta^k \theta^i \theta^j = 0.$$ 

The content of this paper is the following. In section 2 we recall the definition of Lie algebras of order 3 and give some examples. We then define its universal enveloping algebra and endow it with a Hopf algebra structure. Then the consideration of the Hopf dual enable us to define the parameters of the transformations which turn out to be related to the 3–exterior algebra [17]. In section 3 we construct explicitly a group associated to a specific Lie algebra of order 3 by considering matrices with entries which belong to the three exterior algebra.

2. Lie algebras of order 3

2.1. Definition and examples

The general definition of Lie algebras of order $F$, was given in [10, 11] together with an inductive way to construct Lie algebras of order $F$ associated with any Lie algebra or Lie superalgebra. We recall here the main results useful for the sequel. Let $F$ be a positive integer and define $q = e^{2\pi i F}$. We consider $g$ a complex vector space and $\varepsilon$ an automorphism of $g$ satisfying $\varepsilon^F = 1$. Set $g_1 \subseteq g$ the eigenspace corresponding to the eigenvalue $q^i$ of $\varepsilon$. Then, we have $g = g_0 \oplus \cdots \oplus g_{F-1}$.

Definition 2.1 Let $F \in \mathbb{N}^+$. A $\mathbb{Z}_F$-graded $\mathbb{C}$–vector space $g = g_0 \oplus g_1 \oplus g_2 \cdots \oplus g_{F-1}$ is called a complex Lie algebra of order $F$ if

(i) $g_0$ is a complex Lie algebra.

(ii) For all $i = 1, \ldots, F-1$, $g_i$ is a representation of $g_0$. If $X \in g_0, Y \in g_i$ then $[X,Y]$ denotes the action of $X \in g_0$ on $Y \in g_i$ for all $i = 1, \cdots F - 1$.

(iii) For all $i = 1, \ldots, F-1$ there exists an $F$–linear, $g_0$–equivariant map

$$\{ \cdots \} : \mathcal{S}^F(g_i) \to g_0,$$

where $\mathcal{S}^F(g_i)$ denotes the $F$–fold symmetric product of $g_i$, satisfying the following (Jacobi) identity

$$\sum_{j=1}^{F+1} [Y_j, \{Y_1, \ldots, Y_{j-1}, Y_{j+1}, \ldots, Y_{F+1}\}] = 0,$$

for all $Y_j \in g_i$, $j = 1, \ldots, F + 1$.

By definition when $F = 1$, $g = g_0$ is a Lie algebra and when $F = 2$, $g = g_0 \oplus g_1$ is a Lie superalgebra. Thus Lie algebra of order three can be seen as some possible generalisation of Lie (super)algebras. It can be noticed that for any $k = 1, \ldots, F - 1$, the $\mathbb{Z}_F$–graded vector spaces $g_0 \oplus g_k$ satisfy all the properties in Definition 2.1 and thus are Lie algebras of order $F$. We call these types of algebras elementary Lie algebras of order $F$.

Definition 2.2 A representation of an elementary Lie algebra of order $F$ is a linear map $\rho : g = g_0 \oplus g_1 \to \text{End}(V)$, such that (for all $X_i \in g_0, Y_j \in g_1$)
\[ \rho([X_1, X_2]) = \rho(X_1)\rho(X_2) - \rho(X_2)\rho(X_1) \]
\[ \rho([X_1, Y_2]) = \rho(X_1)\rho(Y_2) - \rho(Y_2)\rho(X_1) \]
\[ \rho([Y_1, \ldots, Y_F]) = \sum_{\sigma \in S_F} \rho(Y_{\sigma(1)}) \cdots \rho(Y_{\sigma(F)}) \]

\[ (S_F \text{ being the group of permutations of } F \text{ elements}). \]

By construction the vector space \( V \) is \( \mathbb{Z}_F \)-graded \( V = V_0 \oplus \cdots \oplus V_{F-1} \) and for all \( a \in \{0, \ldots, F-1\} \), \( V_a \) is a \( g_0 \)-module and we have \( \rho(g_1)(V_a) \subseteq V_{a+1} \).

From now on we only consider elementary Lie algebras of order 3. It can be observed that part of the algebra is binary and part of the algebra is ternary. Indeed we have by Definition 2.1 the following products: \( m_1 : g_0 \otimes g_0 \rightarrow g_0 \), \( m_2 : g_0 \otimes g_1 \rightarrow g_1 \), \( m_3 : g_1 \otimes g_1 \otimes g_1 \rightarrow g_0 \). Furthermore, there is essentially two types of Lie algebras of order 3. The first type is the Lie algebras of order 3 related to Lie algebras of order 3 obtained from a Lie (super)algebra through the inductive process [11]. One example of this type of algebras is the cubic extension of the Poincaré algebra.

**Example 2.3** Let \( g_0 = \langle L_{\mu\nu}, P_{\mu} \rangle \) be the Poincaré algebra in \( D \)-dimensions and \( g_1 = \langle V_\mu \rangle \) be the \( D \)-dimensional vector representation of \( g_0 \). The brackets

\[
\begin{align*}
[L_{\mu\nu}, L_{\rho\sigma}] &= \eta_{\rho\sigma} L_{\mu\nu} - \eta_{\mu\sigma} L_{\rho\nu} + \eta_{\nu\rho} L_{\mu\sigma} - \eta_{\mu\rho} L_{\nu\sigma}, \\
[L_{\mu\nu}, P_{\rho}] &= \eta_{\rho\nu} P_{\mu} - \eta_{\mu\rho} P_{\nu}, [L_{\mu\nu}, V_{\rho}] = \eta_{\rho\nu} V_{\mu} - \eta_{\mu\rho} V_{\nu}, P_{\mu}, V_{\nu} = 0,
\end{align*}
\]

with the metric \( \eta_{\mu\nu} = \text{diag}(1, -1, \ldots, -1) \), endow \( g = g_0 \oplus g_1 \) with an elementary Lie algebra of order 3 structure.

The second type of Lie algebras of order 3 are obtained as a sub-algebra of Lie algebras of order 3 defined by some matrix representation.

**Example 2.4** Let \( \mathfrak{gl}(m_1, m_2, m_3) \) and \( \mathfrak{gl}_c(m_1, m_2, m_3) \) be the set of \( (m_1 + m_2 + m_3) \times (m_1 + m_2 + m_3) \) matrices of the form

\[
\begin{aligned}
\mathfrak{gl}_c(m_1, m_2, m_3) &= \left\{ \begin{pmatrix} a_0 & b_1 & 0 \\ 0 & a_1 & b_2 \\ b_0 & 0 & a_2 \end{pmatrix} \right\}, \\
\mathfrak{gl}(m_1, m_2, m_3) &= \left\{ \begin{pmatrix} a_0 & b_1 & c_2 \\ c_0 & a_1 & b_2 \\ b_0 & c_1 & a_2 \end{pmatrix} \right\},
\end{aligned}
\]

with \( a_0 \in \mathfrak{gl}(m_1), a_1 \in \mathfrak{gl}(m_2), a_2 \in \mathfrak{gl}(m_3), b_1 \in \mathcal{M}_{m_1,m_2}(\mathbb{C}), b_2 \in \mathcal{M}_{m_2,m_3}(\mathbb{C}), b_0 \in \mathcal{M}_{m_1,m_3}(\mathbb{C}), c_0 \in \mathcal{M}_{m_2,m_1}(\mathbb{C}), c_1 \in \mathcal{M}_{m_3,m_2}(\mathbb{C}), c_2 \in \mathcal{M}_{m_1,m_3}(\mathbb{C}) \). A basis of these sets of matrices can be constructed as follow. Consider the \( (m_1 + m_2 + m_3)^2 \) canonical matrices \( e_{i}^{J}, 1 \leq i, J \leq m_1 + m_2 + m_3 \), \( 1 \leq i, J \leq m_1 + m_2 + 1 \leq i', J' \leq m_1 + m_2, m_1 + m_2 + 1 \leq i'', J'' \leq m_1 + m_2 + m_3 \), the generators are given by

\[
\begin{align*}
&\text{for } \mathfrak{gl}(m_1), \quad &\text{for } \mathfrak{gl}(m_2), &\text{for } \mathfrak{gl}(m_3), \\
&\text{for } \mathfrak{M}_{m_1,m_2}(\mathbb{C}), &\text{for } \mathfrak{M}_{m_2,m_3}(\mathbb{C}), &\text{for } \mathfrak{M}_{m_3,m_1}(\mathbb{C}), \\
&\text{for } \mathfrak{M}_{m_2,m_1}(\mathbb{C}), &\text{for } \mathfrak{M}_{m_3,m_2}(\mathbb{C}), &\text{for } \mathfrak{M}_{m_1,m_3}(\mathbb{C}).
\end{align*}
\]

Writing \( \mathfrak{gl}(m_1, m_2, m_3) = \mathfrak{gl}(m_1, m_2, m_3)_0 \oplus \mathfrak{gl}(m_1, m_2, m_3)_1 \oplus \mathfrak{gl}(m_1, m_2, m_3)_2 \) and \( \mathfrak{gl}_c(m_1, m_2, m_3) = \mathfrak{gl}_c(m_1, m_2, m_3)_0 \oplus \mathfrak{gl}_c(m_1, m_2, m_3)_1 \) we denote generically by \( X_i^J \) the generators of grade zero \( Y_i^J \) the generators of grade one and \( Z_i^J \) those of grade two, and the brackets read
[X_I^J, X_K^L] = \delta_J^L X_I^J - \delta_L^I X_K^J,
[X_I^J, Y_K^L] = \delta_J^L Y_I^J - \delta_L^I Y_K^J,
[X_I^J, Z_K^L] = \delta_J^L Z_I^J - \delta_L^I Z_K^J,
\{Y_I^J, Y_K^L, Y_M^N\} = \delta_J^K \delta_L^M X_I^N + \delta_N^I \delta_L^M X_K^J + \delta_L^M \delta_J^K X_I^N
+ \delta_M^N \delta_J^K X_I^L + \delta_N^I \delta_L^M X_K^J + \delta_L^M \delta_N^I X_K^J,
\{Z_I^J, Z_K^L, Z_M^N\} = \delta_J^K \delta_L^M X_I^N + \delta_N^I \delta_L^M X_K^J + \delta_L^M \delta_J^K X_I^N
+ \delta_M^N \delta_J^K X_I^L + \delta_N^I \delta_L^M X_K^J + \delta_L^M \delta_N^I X_K^J,

this shows that \( \mathfrak{gl}(m_1, m_2, m_3) \) (resp. \( \mathfrak{gl}_3(m_1, m_2, m_3) \)) is endowed with the structure of Lie algebra of order three (resp. the structure of elementary Lie algebra of order three).

The algebra of Example 2.3 was firstly introduced in [10] and its implementation in Quantum Field Theories was realised in 4–dimensions in [12, 13] and in \( D \)–dimensions in relations to generalised gauge fields or \( p \)-forms in [14]. A general classification of Lie algebra of order 3 extending non-trivially the Poincaré algebra in 4–dimensions was undertaken in [15, 16]. However, even if invariant Lagrangian were constructed all was done at the level of algebras i.e. of infinitesimal transformations and nothing were said at the level of groups. Indeed, by definition, a Lie algebra of order 3 is partially a ternary algebra since the product of three elements of \( \mathfrak{g}_1 \) is defined and not the product of two. This seem to be a priori in contradiction with groups since for a group a product of two elements is always defined. We will see in the next subsection how one can evade this contradiction and construct a group associated to the ternary algebras we are considering.

2.2. Hopf algebras associated to elementaries Lie algebras of order 3

In this subsection, we show that the introduction of Hopf algebras is a possible way to construct a group associated to a given Lie algebra of order 3. We also show that the parameters of the transformation associated to an element of \( \mathfrak{g}_1 \) belong to the 3–exterior algebra (see the appendix for definition).

Let \( \mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \) be a given Lie algebra of order 3. We denote generically by \( X \) the elements of \( \mathfrak{g}_0 \) and by \( Y \) the elements of \( \mathfrak{g}_1 \). To endow \( \mathfrak{g} \) with a Hopf algebra structure we need first to define \( \mathcal{U}(\mathfrak{g}) \) the enveloping algebra associated to \( \mathfrak{g} \). Consider the tensor algebra

\[ T(\mathfrak{g}) = \mathbb{C} \oplus \mathfrak{g} \oplus (\mathfrak{g} \otimes \mathfrak{g}) \oplus \cdots . \]

This is an associative algebra with multiplication given for any \( t_1, t_2 \in T(\mathfrak{g}) \) by \( t_1 \otimes t_2 \in T(\mathfrak{g}) \). Now, in order to assume the multiplication given in Definition 2.1 we define (for any \( X_1, X_2 \in \mathfrak{g}_0, Y_1, Y_2, Y_3 \in \mathfrak{g}_1 \)) the two-sided ideal \( I \) generated by

\[ X_1 \otimes X_2 - X_2 \otimes X_1 - [X_1, X_2], \]
\[ X_1 \otimes Y_2 - Y_2 \otimes X_1 - [X_1, Y_2], \]
\[ Y_1 \otimes Y_2 \otimes Y_3 + Y_2 \otimes Y_3 \otimes Y_1 + Y_3 \otimes Y_1 \otimes Y_2 + \]
\[ Y_1 \otimes Y_3 \otimes Y_2 + Y_2 \otimes Y_1 \otimes Y_3 + Y_3 \otimes Y_2 \otimes Y_1 - \{Y_1, Y_2, Y_3\} . \]

The enveloping algebra of \( \mathfrak{g} \) (in fact it is universal but this will be shown elsewhere [18]) is then defined by \( \mathcal{U}(\mathfrak{g}) = T(\mathfrak{g})/I \). This means in particular that in the associative algebra \( \mathcal{U}(\mathfrak{g}) \) the relation given in Definition 2.1 are satisfied. Furthermore, as in the Lie (super)algebra cases, a
basis of $\mathcal{U}(\mathfrak{g})$ can be given. This is an analogous of the the Poincaré-Birkhoff-Witt theorem in this case. Indeed, it can be proven \cite{18} that we have the following isomorphism of vector space

$$\mathcal{U}(\mathfrak{g}) \cong S(\mathfrak{g}_0) \otimes \Lambda_3(\mathfrak{g}_1)$$  \hfill (4)

where $S(\mathfrak{g}_0)$ is the symmetric algebra over $\mathfrak{g}_0$ and $\Lambda_3(\mathfrak{g}_1)$ is the three-exterior algebra on $\mathfrak{g}_1$ (see the appendix for definition). Since the composition of the natural map $\mathfrak{g} \to T(\mathfrak{g})$ with the canonical projection $T(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$ gives $\mathfrak{g} \subset \mathcal{U}(\mathfrak{g})$ we identify $\mathfrak{g}$ with its image under this map. Thus, if we denote $\{X_a, 1 \leq a \leq \dim \mathfrak{g}_0 = n_0\}$ (resp. $\{Y_i, 1 \leq i \leq \dim \mathfrak{g}_i = n_1\}$) a basis of $\mathfrak{g}_0$ (resp. $\mathfrak{g}_1$) a basis of $\mathcal{U}(\mathfrak{g})$ is given by the elements of the form \cite{18}

$$g_\ell \epsilon = \frac{X_{a_1}^{a_1}}{a_1!} \cdots \frac{X_{a_{n_0}}^{a_{n_0}}}{a_{n_0}!} Y_{\ell}, \quad (a_1, \ldots, a_{n_0}) \in \mathbb{N}^{n_0}, \quad \ell \in \mathbb{N}, \quad \epsilon_{\ell} \in \{1, \ldots, n_1\}^\ell \setminus I_{\ell,n_1}. \hfill (5)$$

where $Y_{\ell} = Y_{(i_1,\ldots,i_\ell)} = Y_{i_1} \cdots Y_{i_\ell}$. (See the appendix for the definition of $I_{\ell,n_1}$.) An element $Y_{\ell}$ with $I_{\ell} \in \{1,\ldots,n_1\}^\ell \setminus I_{\ell,n_1}$ is called a Roby element.

As in the usual Lie (super)algebra cases, $\mathcal{U}(\mathfrak{g})$ can be endowed with a Hopf algebra structure. Although the algebra structure depends on the brackets defining the Lie algebra of order 3 we are considering, the co-product not. Since $\mathfrak{g}$ is a graded vector space it is natural to define the co-product $\Delta$ to be an homorphism from $\mathcal{U}(\mathfrak{g})$ to $\mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$, where $\otimes$ is the $\mathbb{Z}_3$-graded tensor product. This means in particular that for homogeneous elements $a_1, a_2, b_1, b_2 \in \mathcal{U}(\mathfrak{g})$ we have

$$(a_1 \otimes a_2)(b_1 \otimes b_2) = q^{gr(a_2)gr(b_2)}a_1 b_1 \otimes a_2 b_2,$$

with $gr(a)$ the grade of $a$. We define

$$\Delta 1 = 1 \otimes 1, \quad \Delta g = g \otimes 1 + 1 \otimes g, \quad \text{for any } g \in \mathfrak{g}. \hfill (6)$$

The co-product extends to any elements of $\mathcal{U}(\mathfrak{g})$ using (5) and $\Delta(g_1 g_2) = \Delta(g_1) \Delta(g_2)$. (\hfill (6)\hfill)\hfill i. e.\hfill we have

$$\{\Delta Y_1, \Delta Y_2, \Delta Y_3\} = \Delta \{Y_1, Y_2, Y_3\}. \hfill (7)$$

The antipode is defined by

$$S(1) = 1, \quad S(g) = -g, \quad \text{for any } g \in \mathfrak{g}, \hfill (8)$$

and the co-unit by

$$\epsilon(1) = 1, \quad \epsilon(g) = 0, \quad \text{for any } g \in \mathfrak{g}. \hfill (9)$$

It should be mentioned that a different Hopf algebra associated to Lie algebras of order three has been introduced in \cite{19}, where the co-product was defined by the usual tensorial product and the twist (necessary to ensure (7)) was generated by an additional element.

Now, if for $g \in \mathfrak{g}$ we consider the formal element $e^g$ of $\mathcal{U}(\mathfrak{g})$ (for the moment, we do not take care of any convergence problem and $e^g$ is defined as a formal series), then, one can show that $e^g$ is a group-like element and we have

$$\epsilon(1) = 1, \quad \epsilon(g) = 0, \quad \text{for any } g \in \mathfrak{g}. \hfill (9)$$

It should be mentioned that a different Hopf algebra associated to Lie algebras of order three has been introduced in \cite{19}, where the co-product was defined by the usual tensorial product and the twist (necessary to ensure (7)) was generated by an additional element.
\[ \Delta e^g = e^g \otimes e^g, \quad S(e^g) = e^{-g}, \quad \epsilon(e^g) = 1. \]  

(10)

This means that \( e^g \) is a good candidate to define the group associated to the Lie algebra of order three \( \mathfrak{g} \). However, before identifying \( e^g \) to an element of the group associated to \( \mathfrak{g} \), the parameters of the transformation should be firstly identified. It is well known that the parameters associated to a Lie superalgebra are commuting numbers for the even part and Grassmann numbers for the odd part. Similarly, we now show that here the parameters associated to \( \mathfrak{g}_\theta \) are commuting numbers although the parameters associated to \( \mathfrak{g}_1 \) belongs to the 3–exterior algebra.

To identify the parameters of the transformation one needs first to define \( \mathcal{U}(\mathfrak{g})^* \) the dual of the Hopf algebra \( \mathcal{U}(\mathfrak{g}) \). We define the dual basis to be \( g^{\ell_1,\ell_2}, b \in \mathbb{N}^{n_0}, \ell \in \mathbb{N} \) and \( I_{\ell} \in \{1, \cdots, n_1\}^{\ell} \setminus I_{\ell,n_1} \) with the natural pairing:

\[ \langle g^{\ell_1,\ell_2}, g_{\delta_1,\delta_2} \rangle = \delta^{\delta_1}_{\delta_2} \delta^{\ell_1}_{\ell_2}. \]

(11)

Denote \( \alpha^a \) the dual of \( X_a \) and \( \theta^i \) the dual of \( Y_i \). The co-product (6) on \( \mathcal{U}(\mathfrak{g})^* \) induces the product on \( \mathcal{U}(\mathfrak{g})^* \). Recall that if \( e_i \) constitute a basis of a Hopf algebra \( H \) with co-product \( \Delta e_i = \delta_i^{jk} e_j \otimes e_k \), for the dual space \( H^* \) with dual basis \( e^i \) we have \( e^i e^k = \delta_i^{jk} e^i \). Using (6) it is easy to see that the variables \( \alpha^a \) commute between themselves and with the variables \( \theta^i \). Furthermore, a little algebra gives that the variables \( \theta \) generate the 3–exterior algebra and thus, as we now show, (A.1) is satisfied for any \( \theta^1, \theta^2, \theta^3 \).

If we denote \( \theta^1, \theta^{(1,1)} \) the dual variables of \( Y_1 \) and \( Y_1 Y_1 \) the co-product (6) gives

\[ \theta^1 \theta^1 = -q^{2} \theta^{(1,1)}, \quad \theta^1 \theta^{(1,1)} = \theta^{(1,1)} \theta^1 = 0, \quad \text{and thus} \quad (\theta^1)^3 = 0. \]

The case with two different variables, say \( \theta^1, \theta^2 \) is more involved. We introduce \( \theta^1, \theta^2, \theta^{(1,2)}, \theta^{(2,1)}, \theta^{(1,2,1)}, \theta^{(2,1,1)} \) the dual variables of \( Y_1 Y_2, Y_1 Y_2 Y_2 Y_1, Y_1 Y_2 Y_1, Y_2 Y_1, Y_1 Y_2 Y_1, Y_2 Y_2 Y_1 \) (since \( Y_1 Y_1 Y_2 \) is not a Roby element \( -1 \leq 1 \leq 2 \) – there is no need to introduce the element \( \theta^{(1,1,2)} \)). Using

\[ \Delta(Y_1 Y_2) = Y_1 Y_2 \otimes 1 + Y_1 \otimes Y_2 + q Y_2 \otimes Y_1 + 1 \otimes Y_1 Y_2, \]

we obtain

\[ \theta^1 \theta^2 = \theta^{(1,2)} + q \theta^{(2,1)}, \quad \theta^2 \theta^1 = \theta^{(2,1)} + q \theta^{(1,2)}. \]

Similarly,

\[ \Delta(Y_1 Y_2 Y_1) = Y_1 Y_2 Y_1 \otimes 1 + Y_1 Y_2 \otimes Y_1 + q Y_1 Y_2 \otimes Y_1 + q^2 Y_2 Y_1 \otimes Y_1 \]
\[ + Y_1 \otimes Y_2 Y_1 + q Y_2 \otimes Y_1 Y_2 + q^2 Y_1 \otimes Y_2 Y_1 + 1 \otimes Y_1 Y_2 Y_1, \]
\[ \Delta(Y_2 Y_1 Y_1) = Y_2 Y_1 Y_1 \otimes 1 - q^2 Y_2 Y_1 \otimes Y_1 + q^2 Y_1 Y_1 \otimes Y_2 \]
\[ + Y_2 \otimes Y_1 Y_1 - Y_1 \otimes Y_1 Y_2 + 1 \otimes Y_2 Y_1 Y_1, \]

give

\[ \theta^1 \theta^1 \theta^2 = -\theta^{(1,2,1)} - q \theta^{(2,1,1)} \]
\[ \theta^1 \theta^2 \theta^1 = 2 \theta^{(1,2,1)} - \theta^{(2,1,1)} \]
\[ \theta^2 \theta^1 \theta^1 = -\theta^{(1,2,1)} - q^2 \theta^{(2,1,1)} \]
leading to $\theta^1 \theta^1 \theta^2 + \theta^1 \theta^2 \theta^1 + \theta^2 \theta^1 \theta^1 = 0$.

A similar, but more tedious calculus, gives along the same lines that for $i \neq j \neq k$ we have the relation (A.1). Thus, as an algebra we have the following isomorphism:

$$U(\mathfrak{g})^* \cong \mathbb{C}[\alpha^1, \ldots, \alpha^{n_0}] \otimes \Lambda_3(\mathbb{C}^{n_1}).$$

This means in particular that if one wants to associate a group to the Lie algebra of order 3 $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ the parameters of the transformation should be commuting numbers for the elements of $\mathfrak{g}_0$ and should belong to the 3-exterior algebra for the elements of $\mathfrak{g}_1$. This means, in relation with (10) that one might take something like

$$e^{\alpha^a X_a + \theta^i Y_i},$$

(12)

(with $\alpha^a$ commuting numbers and $\theta^i$ the canonical generators of the 3-exterior algebra) to obtain a group. However, here some care as to be taken. Firstly developing $e^{\theta^i Y_i}$ above since $\Lambda_3(\mathbb{C}^{n_1})$ is infinitely generated (see the appendix) there is an infinite number of terms in the series, and we do not know whether or not the series converge since no convergence properties have been established on $\Lambda_3(\mathbb{C}^{n_1})$, even if $Y_a$ are represented by finite-dimensional matrices. Secondly since there is no Baker-Campbell-Hausdorff formulae for elements of $\Lambda_3(\mathbb{C}^{n_1})$. This means that the product of two elements of the type (12) is not of the same type. These problems have been solved in [18], but here we propose another way to construct a group associate to Lie algebras of order 3.

3. Group associated to Lie algebra of order 3 – the $\mathfrak{g}_{el}(m_1, m_2, m_3)$ case

In this section we are constructing explicitly a group associated to the Lie algebra of order 3 $\mathfrak{g}_{el}(m_1, m_2, m_3)$ in a straight analogy with the construction of the Lie supergroup associated to the Lie superalgebra $\mathfrak{gl}(m,n)$. Using the results of subsection 2.2, since the parameters of the transformation are strongly related with the 3-exterior algebra, we are considering matricial groups whose entries belong to the 3-exterior algebra.

First of all we introduce some set of matrices with coefficients in $\Lambda_3(\mathbb{C}^p)$:

$$M_k(\Lambda_0) = M_k(\mathbb{C}) \otimes \Lambda(\mathbb{C}^p)_0,$$

$$M_k,\ell(\Lambda_1) = M_k,\ell(\mathbb{C}) \otimes \Lambda(\mathbb{C}^p)_1,$$

$$M_k,\ell(\Lambda_2) = M_k,\ell(\mathbb{C}) \otimes \Lambda(\mathbb{C}^p)_2.$$ (13)

A matrix, e.g. in $M_k(\Lambda_0)$ is given by

$$A(\Lambda_0) = A(0) + A(3)_{ijk\ell} \theta^i \theta^j \theta^k + A(6)_{ijkmn} \theta^i \theta^j \theta^k \theta^m \theta^n + \cdots$$ (14)

where the sum above is finite and taken only over Roby elements.

**Proposition 3.1** The matrix given in (14) is invertible if and only if $A(0)$ is invertible.

**Proof:** We introduce $B(\Lambda_0) = B(0) + B(3)_{ijk} \theta^i \theta^j \theta^k + B(6)_{ijkmn} \theta^i \theta^j \theta^k \theta^m \theta^n + \cdots$, with the sum taken over Roby elements. Solving the equation $A(\Lambda_0)B(\Lambda_0) = 1$, gives the coefficients of $B(\Lambda_0)$ terms by terms. We thus get for the two first terms $B(0) = A(0)^{-1}$, $B(3)_{ijk} = -A(0)^{-1}A(3)_{ijk}A(0)^{-1}$. 


However, for the higher order terms some care as to be taken. Indeed, the terms of degree six
give:
\[
(A_0 B_{(6)ijk\ell mn} + A_{(6)ijk\ell mn}B_0)\theta^i\theta^j\theta^k\theta^m\theta^n + A_{(3)i'j'k'}B_{(3)i''j''k''}\theta^{i'}\theta^{j'}\theta^{k'}\theta^{i''}\theta^{j''}\theta^{k''} = 0.
\]
But even though \(\{\theta^i\theta^j\theta^k\theta^m\theta^n\}\) are by definition of \(A(\Lambda_0)\) and \(B(\Lambda_0)\) independent
elements of \(A_3(\mathbb{C}^p)\), in general it is not true for \(\{\theta^{i'}\theta^{j'}\theta^{k'}\theta^{i''}\theta^{j''}\theta^{k''}\}\). Indeed, if the
sequences \((i', j', k'), (i'', j'', k'')\) \(\in \{1, \ldots, p\}^3 \setminus I_{3,p}\) this does not mean that the sequence
\((i', j', k', i'', j'', k'')\) \(\in \{1, \ldots, p\}^6 \setminus I_{6,p}\), as can be seen with \((1, 2, 1)\) and \((1, 2, 1)\). Decomposing
this last terms in the basis of \(A_3(\mathbb{C}^p)\) we have
\[
\theta^{i'}\theta^j\theta^{k'}\theta^{i''}\theta^{j''}\theta^{k''} = C^{i'j'k'i''j''k''}_{ijk\ell mn}\theta^i\theta^j\theta^k\theta^m\theta^n
\]
(where the sum is taken over \((ijk\ell mn) \in \{1, \ldots, p\}^6 \setminus I_{6,3}\) ) and we get
\[
B_{(6)ijk\ell mn} = -A_0^{-1}A_{(6)ijk\ell mn}A_0^{-1} - C^{i'j'k'i''j''k''}_{ijk\ell mn}A_0^{-1}A_{(3)i'j'k'}A_0^{-1}A_{(3)i''j''k''}A_0^{-1}.
\]
This procedure extends simply order by order. QED.

We denote by \(GL(m_1, \Lambda_0)\) the set of invertible matrices of \(M_{m_1}(\mathbb{C}, \Lambda_0)\).

**Proposition 3.2** Let \(A_0 \in GL(m_1, \Lambda_0), A_1 \in GL(m_2, \Lambda_0), A_2 \in GL(m_3, \Lambda_0), B_0 \in M_{m_3,m_1}(\Lambda_1), B_1 \in M_{m_1,m_2}(\Lambda_1), B_2 \in M_{m_2,m_3}(\Lambda_1)\) and \(C_0 \in M_{m_2,m_1}(\Lambda_2), C_1 \in M_{m_3,m_2}(\Lambda_2), C_2 \in M_{m_1,m_3}(\Lambda_2)\), then the \(m_1 \times m_2 \times m_3\) matrix with coefficients in \(\Lambda_3(\mathbb{C}^p)\)
\[
M = \begin{pmatrix} A_0 & B_1 & C_2 \\ C_0 & A_1 & B_2 \\ B_0 & C_1 & A_2 \end{pmatrix},
\]
is invertible.

**Proof:** By definition, it is obvious to see that if \(M, N\) are two matrices of the type above
then the product \(MN\) is also as above. We now show that the matrix \(M\) is invertible. Define
\[
N = \begin{pmatrix} A'_0 & B'_1 & C'_2 \\ C'_0 & A'_1 & B'_2 \\ B'_0 & C'_1 & A'_2 \end{pmatrix},
\]
and solving \(MN = 1\), we get
\[ A'_0 = \left\{ A_0 - C_2(A_2 - C_1A_1^{-1}B_2)^{-1}(B_0 - C_1A_1^{-1}C_0) \right. \\
- B_1A_1^{-1}\left[ C_0 - B_2(A_2 - C_1A_1^{-1}B_2)^{-1}(B_0 - C_1A_1^{-1}C_0) \right] \}^{-1}, \]
\[ B'_0 = -(A_2 - C_1A_1^{-1}B_2)^{-1}(B_0 - C_1A_1^{-1}C_0)A'_0, \]
\[ C'_0 = -A_1^{-1}\left[ C_0 - B_2(A_2 - C_1A_1^{-1}B_2)^{-1}(B_0 - C_1A_1^{-1}C_0) \right]A'_0, \]
\[ A'_1 = \left\{ A_1 - C_0(A_0 - C_2A_2^{-1}B_0)^{-1}(B_1 - C_2A_2^{-1}C_1) \right. \\
- B_2A_2^{-1}\left[ C_1 - B_0(A_0 - C_2A_2^{-1}B_0)^{-1}(B_1 - C_2A_2^{-1}C_1) \right] \}^{-1}, \]
\[ B'_1 = -(A_0 - C_2A_2^{-1}B_0)^{-1}(B_1 - C_2A_2^{-1}C_1)A'_1, \]
\[ C'_1 = -A_2^{-1}\left[ C_1 - B_0(A_0 - C_2A_2^{-1}B_0)^{-1}(B_1 - C_2A_2^{-1}C_1) \right]A'_1, \]
\[ A'_2 = \left\{ A_2 - C_1(A_1 - C_0A_0^{-1}B_1)^{-1}(B_2 - C_0A_0^{-1}C_2) \right. \\
- B_0A_0^{-1}\left[ C_2 - B_1(A_1 - C_0A_0^{-1}B_1)^{-1}(B_2 - C_0A_0^{-1}C_2) \right] \}^{-1}, \]
\[ B'_2 = -(A_1 - C_0A_0^{-1}B_1)^{-1}(B_2 - C_0A_0^{-1}C_2)A'_2, \]
\[ C'_2 = -A_0^{-1}\left[ C_2 - B_1(A_1 - C_0A_0^{-1}B_1)^{-1}(B_2 - C_0A_0^{-1}C_2) \right]A'_2. \]  

The formula above make sense because of Proposition 3.1. QED

However, it can happen that the sum defining the inverse matrix of a given matrix \( A(\Lambda_0) \) is infinite even if \( A(\Lambda_0) \) is defined by a finite sum. Indeed, we have \( (A_0 + A_1\theta^3\theta^2\theta^1)^{-1} = \sum_{k\geq 0}(-A_0^{-1}A_1)^kA_0^{-1}(\theta^3\theta^2\theta^1)^k \), we thus restrict ourselves to the set of matrices such that \( A(\Lambda_0) \) and its inverse \( A^{-1}(\Lambda_0) \) are given by finite sums. Looking to the sum above, \( A(\Lambda_0)^{-1} \) becomes a finite sum if there is some nilpotent elements. For instance this happens if \( A_0^{-1}A_1 \) is nilpotent. It can also happen that some power of elements in \( \Lambda_3(C^p) \) vanishes. For instance, since \( (\theta^2\theta^2\theta^1)^2 = -\theta^2\theta^2(\theta^2\theta^2\theta^2+\theta^2\theta^2\theta^1) \theta^1 = 0 \), we have \( (A_0 + A_1\theta^3\theta^2\theta^1)^{-1} = A_0^{-1} - A_0^{-1}A_1A_0^{-1}\theta^2\theta^1 \).

Thus in order that the formul\(e \)e above make sense we consider the set of matrix (15) such that the various matrices together with their inverse (see (16)) are given by finite sum.

**Definition 3.3** The set of matrices of the forme

\[ M = \begin{pmatrix} A_0 & B_1 & C_2 \\ C_0 & A_1 & B_2 \\ B_0 & C_1 & A_2 \end{pmatrix}, \]  

\[ (A_0, B_1, C_2, C_0, A_1, B_2, B_0, C_1, A_2) \]  

where the matrices \( A_0 \in GL(m_1, \Lambda_0), A_1 \in GL(m_2, \Lambda_0), A_2 \in GL(m_3, \Lambda_0), B_0 \in \mathcal{M}_{m_3,m_1}(\Lambda_1), B_1 \in \mathcal{M}_{m_1,m_2}(\Lambda_1), B_2 \in \mathcal{M}_{m_2,m_3}(\Lambda_1), C_0 \in \mathcal{M}_{m_2,m_1}(\Lambda_2), C_1 \in \mathcal{M}_{m_3,m_2}(\Lambda_2), C_2 \in \mathcal{M}_{m_1,m_3}(\Lambda_2), \) together with the matrices \( A'_0 \in GL(m_1, \Lambda_0), A'_1 \in GL(m_2, \Lambda_0), A'_2 \in GL(m_3, \Lambda_0), B'_0 \in \mathcal{M}_{m_3,m_1}(\Lambda_1), B'_1 \in \mathcal{M}_{m_1,m_2}(\Lambda_1), B'_2 \in \mathcal{M}_{m_2,m_3}(\Lambda_1), C'_0 \in \mathcal{M}_{m_2,m_1}(\Lambda_2), C'_1 \in \mathcal{M}_{m_3,m_2}(\Lambda_2), C'_2 \in \mathcal{M}_{m_1,m_3}(\Lambda_2), \) (see (16)) are given by finite sums is a group we denote \( GL_f(m_1, m_2, m_3, \Lambda) \).

We now show, in relation with the previous section, that this group in non-empty. Indeed, if we take
with $A$ an arbitrary matrix of $\mathfrak{gl}(m_1) \times \mathfrak{gl}(m_2) \times \mathfrak{gl}(m_3)$ and $B_k$ an arbitrary nilpotent matrix of $\mathfrak{gl}(m_1, m_2, m_3) \otimes A_3(\mathbb{C})$, then $g \in GL_f(m_1, m_2, m_3, \Lambda)$. There is several type of matrix $B_k$.

For instance one can take $B_k = YP(\theta)$ with (i) $Y$ a nilpotent matrix of $\mathfrak{gl}(m_1, m_2, m_3_1)$ and $P$ an arbitrary polynomial of degree $1$ mod $3$ in $\theta$ or (ii) $Y$ an arbitrary element of $\mathfrak{gl}(m_1, m_2, m_3_1)$ and $P$ a nilpotent polynomial of degree $1$ mod $3$ in $\theta$ (some power of $P$ vanishes). More details will be given in [18].

To close this section, we now show that the algebra corresponding to the group $GL_f(m_1, m_2, m_3, \Lambda)$ is $\mathfrak{gl}_3(m_1, m_2, m_2)$. Indeed, if we take the infinitesimal version of $GL_f(m_1, m_2, m_3, \Lambda)$ i.e. we assume that the parameters of the transformation characterised by $\theta$ goes to zero we have

$$
A_0 = A_{0(0)} + A_{0(3)} \theta^i \theta^j \theta^k + \cdots \quad \rightarrow \quad A_{0(0)} + O((\theta)^2)
$$

$$
B_0 = B_{0(1)} \theta^i + A_{0(4)} \theta^i \theta^j \theta^k \theta^l + \cdots \quad \rightarrow \quad A_{0(1)} \theta^i + O((\theta)^2)
$$

$$
C_0 = C_{0(2)} \theta^i \theta^j + C_{0(5)} \theta^i \theta^j \theta^k \theta^l + \cdots \quad \rightarrow \quad 0 + O((\theta)^2)
$$

(A.1)

with $A_{0(0)} \in GL(m_1)$ (and similar relations for $A_i, B_i, C_i, i = 1, 2, 3$). Since $\mathfrak{gl}(m_1)$ is the Lie algebra of $GL(m_1)$, the relations (A.1) ensure that the algebra associated to $GL_f(m_1, m_2, m_2, \Lambda)$ is the Lie algebra of order $3 \mathfrak{gl}_3(m_1, m_2, m_2)$. In fact the Hopf algebra technics of the previous sub-section gives the converse correspondence i.e. enables us to establish that the group associated to $\mathfrak{gl}_3(m_1, m_2, m_2)$ is $GL_f(m_1, m_2, m_3, \Lambda)$ [18].

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Appendix A. The three-exterior algebra

The three-exterior algebra has been introduced by Roby [17] in 1970 as a possible (cubic) generalisation of the Grassmann algebra. The three-exterior algebra $\Lambda_3(\mathbb{K}^n)$ is the unitary (with unit denoted by $1$) $\mathbb{K}$–algebra ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$) generated by $n$ canonical generators $\theta^1, \cdots, \theta^n$ submitted to the relation

$$
\theta^i \theta^j \theta^k + \theta^i \theta^k \theta^j + \theta^j \theta^k \theta^i + \theta^i \theta^j \theta^k + \theta^k \theta^i \theta^j = 0.
$$

This algebra which can be seen as a possible generalisation of the Grassmann algebra is indeed very different. Because the algebra $\Lambda_3(\mathbb{K}^n)$ is defined through cubic relations the number of independent monomials increases with polynomial’s degree (for instance, $(\theta^1)^2 \theta^2$ and $\theta^1 \theta^2 \theta^3$ are independent). This means that we do not have enough constraints among the generators to order them in some fixed way and, as a consequence, $\Lambda(\mathbb{K}^n)$ turns out to be an infinite dimensional algebra. To characterise precisely $\Lambda(\mathbb{K}^n)$ it should be interesting to obtain a basis. This is complicated by the fact that we have some cubic relations among the generators and for instance the three element $(\theta^1)^2 \theta^2, \theta^1 \theta^2 \theta^3$ and $\theta^2 (\theta^1)^2$ are not independent since there sum is zero. To characterise the set of independent elements one needs a definition.

**Definition A1** The sequence $(i_1, i_2, \cdots, i_k) \in \{1, 2, \cdots, n\}^k, k \geq 3$ has a rise of length 3 if there exists $0 \leq \ell \leq k - 3$ such that $i_{\ell + 1} \leq i_{\ell + 2} \leq i_{\ell + 3}$. We denote by $I_{k,n}$ the set of $k$ indices which has a rise of length 3.
Theorem A2 (N. Roby [17]) The elements
\[ \left\{ \theta^{i_1} \theta^{i_2} \cdots \theta^{i_k}, k \in \mathbb{N}, (i_1, \ldots, i_k) \in \{1, \ldots, n\}^k \setminus I_{k,n} \right\}, \]
constitute a basis of \( \Lambda_3(\mathbb{K}^n) \).

For instance if \( n = 2 \) a basis of \( \Lambda_3(\mathbb{C}^2) \) is given by
\[
\begin{align*}
1, \\
\theta^i, & \quad i = 1, 2, \\
\theta^i \theta^j, & \quad i, j = 1, 2, \\
\theta^i (\theta^2 \theta^1)^n, & \quad (\theta^2 \theta^1)^n \theta^i, \quad i = 1, 2, \quad n \geq 1, \\
\theta^i (\theta^2 \theta^1)^{n-1} \theta^j, & \quad i, j = 1, 2, \quad (\theta^2 \theta^1)^n, \quad n \geq 2.
\end{align*}
\]

From now on we call Roby elements the elements \( \theta^{I_k} = \theta^{i_1} \cdots \theta^{i_k} \) such that \( I_k \in \{1, \ldots, n\}^k \setminus I_{k,n} \).

The 3-exterior algebra is a \( \mathbb{Z}_3 \)-graded algebra. Indeed, if we define \( \Lambda(\mathbb{K}^n)_i, i = 0, 1, 2 \) the set of elements of degree \( i \) mod 3 we have \( \Lambda_3(\mathbb{K}^n)_i \Lambda_3(\mathbb{K}^n)_j \subseteq \Lambda_3(\mathbb{K}^n)_{i+j} \) and \( \Lambda_3(\mathbb{K}^n) = \Lambda_3(\mathbb{K}^n)_0 \oplus \Lambda_3(\mathbb{K}^n)_1 \oplus \Lambda_3(\mathbb{K}^n)_2 \).

Finally let us mention that the results given in this appendix extend to any \( F > 2 \) [17].