Fixed points, stability and intermittency in a shell model for advection of passive scalars

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(18 November 1999)

We investigate the fixed points of a shell model for the turbulent advection of passive scalars introduced in [2]. The passive scalar field is driven by the velocity field of the popular GOY shell model. The scaling behavior of the static solutions is found to differ significantly from Obukhov-Corrsin scaling $\theta_n \sim k_n^{-1/3}$ which is only recovered in the limit where the diffusivity vanishes, $D \to 0$. From the eigenvalue spectrum we show that any perturbation in the scalar will always damp out, i.e. the eigenvalues of the scalar are negative and are decoupled from the eigenvalues of the velocity. Furthermore we estimate Lyapunov exponents and the intermittency parameters using a definition proposed by Benzi et al. [3]. The full model is as chaotic as the GOY model, measured by the maximal Lyapunov exponent, but is more intermittent.

PACS numbers:

I. INTRODUCTION

In recent years much attention has been paid to the origin of intermittency in fully developed turbulence. Progress has been made in the somewhat simpler problem of the anomalous scaling in the passive advection of a scalar quantity (e.g. temperature or the density of a pollutant). Fundamental analytical results have been obtained when the advecting velocity field was assumed to be Gaussian and delta-correlated in time, the so-called Kraichnan model [3,4], and in the context of shell models [5]. Here we consider the perhaps more realistic situation where the passive scalar shell model is driven by a non-Gaussian velocity field with finite correlation time, namely the velocity field of the GOY model [6,7]. In this case the model can be analyzed using standard techniques of (low) dimensional dynamical systems, e.g. the study of bifurcations, eigenvalue spectra and Lyapunov exponents.

In absence of convective effects the passive scalar field $\Theta$ is governed by the equation:

$$\partial_t \Theta + (\vec{v} \cdot \nabla) \Theta = D \nabla^2 \Theta + F_\Theta$$

(1)

Here $v$ is the velocity field, $F_\Theta$ is an external forcing and $D$ is the diffusion coefficient.

According to the analogue of the K41 theory [5] for the passive scalar, developed by Obukhov and Corrsin [10], the structure functions

$$S_p(\vec{r}) = \langle |\Theta(\vec{x} + \vec{r}) - \Theta(\vec{x})|^p \rangle \sim l^H_p$$

(2)

(where $l = |\vec{r}|$) scale linearly with $p$, more precisely

$$H(p) = p/3.$$  

Experimentally, however, one observes for $p > 3$ [11] strong deviations from this. This is usually referred to as anomalous scaling or intermittency. The deviations seem to be even more pronounced than for the structure functions of the velocity: the passive scalar is said to be more intermittent.

This paper is organized as follows: after having introduced the model, we examine in section III the scaling of its fixed points and in section IV their stability. These studies have already been performed for the GOY model [12, 13], to which the passive scalar model is coupled. We will review these results for the sake of completeness. In section V we study the full dynamics of the model and investigate its chaotic and intermittent behavior.

II. SHELL MODEL FOR PASSIVE SCALARS

Shell models appear to capture many properties of fully developed turbulent flows but are easier to study than the Navier-Stokes equations (see [14] for a review). In this letter we study the passive scalar shell model proposed in Ref. [2]. The multiscaling of the model is in good agreement with experimental data [14]. The GOY model has been studied intensively [12, 13, 21, 22]: the passive scalar model has attracted somewhat less attention [23, 24, 25].

Both models are constructed in Fourier space, retaining only the complex modes $u_n$ and $\theta_n$ as a representative of all modes in the shell of wave number $k$ between $k_n = k_0 \lambda^n$ and $k_{n+1}$. One uses the following assumptions: (i) the dissipation resp. diffusion is represented by a linear term of the form: $-\nu k_n^2 u_n$ resp. $-D k_n^2 \theta_n$ (ii) the nonlinear terms of the form $k_n u_n u_n$ resp. $k_n \theta_n u_n$ with (iii) $n'$ and $n''$ among the nearest and next-nearest neighbors of $n$ and (iv) in absence of forcing and damping conservation of volume in phase space and conservation of $\sum_n |u_n|^2$ and $\sum_n |\theta_n|^2$. Moreover the scaling laws $u_n \sim k_n^{-1/3}$ and $\theta_n \sim k_n^{-1/3}$ form a fixed point of the inviscid unforced equations.

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The resulting equations are [1]:

\[
\begin{align*}
\frac{d}{dt} + \nu k_n^2 u_n &= i (a_n k_n u_n^{n+1} u_{n+2} + b_n k_n - u_{n+1} u_{n+1} + c_n k_{n-2} u_{n+2} u_{n-2}) + f \delta_{n,4} \\
\frac{d}{dt} + D k_n^2 \theta_n &= i (e_n (u_{n-1} \theta_{n+1} - u_{n+1} \theta_{n-1}) + g_n (u_{n-2} \theta_{n-1} + u_{n-1} \theta_{n-2}) + h_n (u_{n-1} \theta_{n+1} + u_{n+1} \theta_{n+1})) + f \delta_{n,4}
\end{align*}
\]

A possible choice for the coefficients is:

\[
\begin{align*}
& a_n = 1 \quad b_n = -\delta \quad c_n = -(1 - \delta) \\
& e_n = \frac{k_n}{2} \quad g_n = -\frac{k_{n-1}}{2} \quad h_n = \frac{k_{n+1}}{2}
\end{align*}
\]

The boundary conditions are:

\[
\begin{align*}
& b_1 = b_N = c_1 = c_2 = a_{N-1} = a_N = 0 \quad (7) \\
& e_1 = e_N = g_1 = g_2 = h_{N-1} = h_N = 0 \quad (8)
\end{align*}
\]

For the parameters, we choose for example the following standard values:

\[
\begin{align*}
& N = 19, \lambda = 2, k_0 = \lambda^{-1}, \nu = 10^{-6}, \\
& f = \bar{f} = 5 \cdot 10^{-3} \cdot (1 + i), D = 10^{-6}
\end{align*}
\]

The free parameter $\delta$ is related to a second quadratic invariant which for the canonical value $\delta = \frac{1}{3}$ is similar to the helicity [12].

These equations determine the evolution of the vector $(U, \Theta) = (\Re u_1, \Im u_1, \ldots, \Re u_N, \Im u_N, \Re \theta_1, \ldots, \Im \theta_N)$ and thus form a $4N$ dynamical system.

### III. SCALING OF FIXED POINTS

A first step towards a full understanding of the model consists of an investigation of its static properties. In this section we examine the scaling properties of the fixed points of equations (3)-(4). The major problem is to find the scaling properties of the fixed points of equations (3)-(4). The major problem is to find this section we examine the scaling properties of the fixed points of equations (3)-(4). The major problem is to find

It was found in [14] that (3) in the unforced inviscid limit allows three self-similar static solutions: the trivial fixed point $u_n = 0$, a “Kolmogorov” fixed point $u_n = k_n^{-1/3} g_1(n)$ and a “flux-less” fixed point $u_n = k_n^{-2} g_2(n)$, with $g_1(n)$ and $g_2(n)$ any function of period three in $n$ and $z = (-\ln(\delta - 1) + 1)/3$. The corresponding fixed points of (3) are: $\theta_n = 0$, $\theta_n = k_n^{-1/3} g_1(n)$ and $\theta_n = k_n^{-2} g_2(n)$. We will here focus on the Kolmogorov fixed point which is believed to be the most important for the dynamics [14] although it was suggested that also the trivial fixed point might play a major role [21].

We note that the static solution for $u_n = u_n e^{i \phi_n}$ can be turned into real form by a change of phase. Following Schörgofer et al. [13] we choose the phases:

\[
\phi_n = \begin{cases} 
\frac{1}{3} \pi & \text{for } n = 1, 4, 7, \ldots \\
\frac{2}{3} \pi & \text{for } n = 2, 5, 8, \ldots \\
\frac{5}{3} \pi & \text{for } n = 3, 6, 9, \ldots
\end{cases}
\]

It can be shown that the static solution of $\theta$ picks up the same phase as that of $u$.

As has been observed [14], the dynamics of the system converges to the Kolmogorov fixed point for $\delta < 0.379634$. When increasing $\delta$ the system undergoes a series of Hopf bifurcations and becomes chaotic at $\delta = 0.38704$ [14,15,22]. In order to find the Kolmogorov fixed point one can thus vary $\delta$ in small steps and refine the solution with Newton’s method [13].

To study the scaling behavior of the static solutions, we apply a much larger number of shells, using the same parameter values as Kadanoff et al. [14]:

\[
N = 90, \lambda = 2, k_0 = \lambda^{-1}, \nu = D = 10^{-3}16^{-26}, f = \bar{f} = 1
\]

The forcing acts in this case on the first shell.

For the static solution of $u_n$ we obtain the same result as [14], see figure [1], where $\log_2(u_{n+1}/u_n)$ is plotted versus $n$. It is averaged over a period three to get rid of the well known period three oscillations. The solution is seen to follow Kolmogorov scaling.

![FIG. 1. Static solution of the GOY model at $\delta = 0.5$](image)

Surprisingly however, the fixed point for $\theta_n$ (with the fixed point of $u_n$ inserted) deviates strongly from the Obukhov scaling for a finite value of the diffusivity $D$, as can be seen in figure [2]. There is clearly not a well-defined power law scaling. This solution also displays the period three behavior. In the diffusive range the solution looks somewhat noisy, presumably due to the boundary conditions. It is clear, that there is a “slow” bending in the diffusive regime but as $D$ approaches zero, the curve becomes more and more flat and we recover the Obukhov scaling in the limit $D \to 0$. We thus note that in contrast
to the velocity-case where the viscosity only affects the viscous range, for the passive scalar the diffusion seems to act on the whole inertial range, at least for Prandtl numbers \( Pr = \frac{\nu}{D} \sim 1 \). This might have its origin in the linear character of the problem.

\[
\log_2(\frac{\theta_{n+1}}{\theta_n}) \quad (\text{averaged three consecutive shells})
\]

FIG. 2. Static solution, fixed point, of the passive scalar model where \( \log_2(\frac{\theta_{n+1}}{\theta_n}) \) (averaged three consecutive shells) is plotted versus the shell index \( n \). Here, the number of shells is \( N = 61 \).

IV. SCALING OF EIGENVALUE SPECTRA

Now the stability of the Kolmogorov-Obukhov fixed point is examined in terms of the eigenvalue spectra. The system eq. (3-4) is written as

\[
\begin{align*}
\dot{u} &= f(u) \\
\dot{\theta} &= g(u, \theta)
\end{align*}
\]

The Jacobian matrix is symbolically given by:

\[
J = \begin{pmatrix}
\frac{\partial f(u)}{\partial u} & 0 \\
\frac{\partial g(u, \theta)}{\partial u} & \frac{\partial g(u, \theta)}{\partial \theta}
\end{pmatrix}
\]

The matrix \( \frac{\partial g(u, \theta)}{\partial u} \) will not matter for the eigenvalues of \( J \). The eigenvalues of \( \frac{\partial f(u)}{\partial u} \) were studied in [14] and \[13\]. It is most convenient to look at disturbances of phase and modulus of the velocity variable \( u_n = \bar{u}_n e^{i\phi_n} \); \( \bar{u}_n = u_n^{(0)} + \delta u_n \) and \( \phi_n = \phi_n^{(0)} + \delta \phi_n \). One then obtains for the stability of the modulus:

\[
\delta \dot{u}_n = - \sum (D_{nm} + C_{nm}) \delta u_m
\]

The matrices \( D_{nm} \) and \( C_{nm} \) are the contributions of the dissipation and cascade terms, respectively. Their expressions are:

\[
D_{nm} = \nu k_n^2 \delta_{nm}
\]

and

\[
C_{nm} = \frac{\partial}{\partial u_m} (k_n u_{n+1} - \delta k_{n-1} u_{n-1} - (1 - \delta) k_n - 2 u_{n-1} u_{n+2}).
\]

where the index \( (0) \) has been dropped, for convenience.

Thus the stability eigenvalues of modulus and phases differ only in a minus-sign in front of the cascade term. We obtain similar results as in [14]. For values of \( \delta < \delta_{bif} \approx 0.37 \) all the eigenvalues of both phase and modulus matrix have negative real part. Above this value, some of the real eigenvalues of the phase matrix turn complex (in pairs) and cross the imaginary axis. At \( \delta = 0.5 \) they become real again, but now positive. This situation is shown in figure 3. The eigenvalues of the modulus matrix eventually turn positive at \( \delta \approx 0.7 \). In figure 3 we have multiplied both real and imaginary part with a factor \( p \):

\[
p = \frac{\log_2(1 + 2^{10} |\sigma|)}{|\sigma|}
\]

Thus the phase remains unchanged, while the modulus is rescaled. In case the eigenvalues have constant ratios, they are evenly spaced on the plot. Thus we are able to visualize both the very small and very large eigenvalues.
It is quite trivial to generalize this method to the passive scalar in order to calculate the eigenvalues of \( \frac{\partial g(u, \theta)}{\partial \theta} \) and one finds:

\[
\delta \dot{\theta}_n = - \sum_m (D_{nm}^\theta + B_{nm}) \delta \theta_m
\]  

(19)

where \( D_{nm}^\theta \) is the dissipation term, very similar to the one before

\[
D_{nm}^\theta = Dk_n^2 \delta_{nm}
\]

(20)

and \( B_{nm} \) is the contribution of the cascade term given by:

\[
B_{nm} = \frac{1}{2} \frac{\partial}{\partial \theta_m} (k_n (u_{n-1} \theta_{n+1} - u_{n+1} \theta_{n-1}) - k_{n-1} (u_{n-2} \theta_{n-1} + u_{n-1} \theta_{n-2}) + \frac{1}{2} k_{n+1} (u_{n+2} \theta_{n+1} + u_{n+1} \theta_{n+2}))
\]

(21)

We see that \( B_{nm} \) does not depend on \( \theta \) as the \( \theta \)-variation will be differentiated out. For the phase disturbance \( \delta \psi_n \) one gets

\[
\dot{\delta \psi}_n = - \sum_m (D_{nm}^\theta - B_{nm}) \delta \psi_m
\]

(22)

We note that since \( B_{n,n+2} = \frac{1}{2} k_{n+1} u_{n+1} = -B_{n+2,n} \) and \( B_{n,n+1} = \frac{1}{2} (k_{n+1} u_{n+2} + k_n u_{n-1}) = -B_{n+1,n} \), the matrix \( B_{nm} \) is antisymmetric and its eigenvalues will be purely imaginary. The stability of phase and modulus will thus be the same. This is actually a consequence of the conservation of \( \sum \theta_n \). In the model any cascade term \( a_n u_{n+n'} \theta_{n+n''} \) has to be supplemented by a term \(-a_{n-n'} u_{n+n'-n''} \theta_{n-n''} \), since the conservation implies \( \sum \theta_n \theta_n = 0 \). The first term gives a contribution to the matrix \( B_{nm} \): \( B_{n,n+n'} = a_n u_{n+n'} \), the second \( B_{n,n-n''} = -a_{n-n''} u_{n+n'-n''} \) which corresponds to \( B_{n'+n',n} = -a_n u_{n+n'} \). The matrix is thus antisymmetric and this is not an artefact of the model since it stems from a conservation law also valid in a real system.

One does not expect to find any bifurcations since the diffusion matrix \(-D_{nm}^\theta\) will cause the eigenvalues to have negative real parts, whatever the value of the driving velocity.

Indeed one finds that for all values of \( \delta \), the spectrum of eigenvalues of the phase matrix is like shown in figure 4, where the eigenvalues are again multiplied by \( p \). One observes evenly spaced eigenvalues organized in branches. The presence of three branches (in both upper and lower half of the complex plane) might be caused by the period three of \( u_n \) (in \( n \)). If one inserts in the matrix \( B_{nm} \) a fixed point of an imaginary model without period 3, one just finds one branch in each half plane.

Let’s us now consider what the above imply for the dynamics of the model. At time \( t = 0 \), starting from the particular state of the scalar \( \theta_n(0) \) we impose a small perturbation: \( \theta_n'(0) = \theta_n(0) + \delta \theta_n(0) \) and look how the perturbation evolves in time, subjected to the same velocity field. For modulus and phase of \( \delta \theta \) we have again equations (19) and (22), since the matrices \( D^\theta \) and \( B \) do not depend on \( \theta \). Because the eigenvalues of \( B - D^\theta \) have negative real parts, the perturbation will damp out, meaning that after some time \( \theta_n'(t) \sim \theta_n(t) \). This has been observed already by Crisanti et al. [23].

V. INTERMITTENCY

The term intermittency is used in different contexts and a precise mathematical definition does not exist. In turbulence one speaks of intermittency corrections to the Kolmogorov or Obukhov-Corrsin power law. In dynamical systems in general, intermittency means the presence of quiescent periods randomly interrupted by burst. It is believed that these phenomena are related: the scaling corrections should have their origin in intermittent behavior (in space and/or time) of velocity or energy dissipation. Here we ask ourselves the question whether the more pronounced deviations from classical scaling for the passive scalar are reflected by a more intermittent behavior (in time) of the passive scalar field. In order to test this we invoke a definition of intermittency for dynamical systems proposed by Benzi et al. [2].

Firstly, the response function \( R(t) \) is defined as the rate at which a disturbance vector \( \delta \bar{x}(t) \) of the system \( \dot{\bar{x}} = f(\bar{x}) \) has expanded after a time \( t \).

\[
R(t) = \frac{\left| \delta \bar{x}(t + \tau) \right|}{\left| \delta \bar{x}(t) \right|}
\]

(23)
In chaotic systems one typically observes that $|\delta \tilde{x}(t + \tau)| \sim |\delta \tilde{x}(t)| e^{\tau \lambda}$, where $\lambda$ is the maximum Lyapunov exponent. This maximum Lyapunov exponent can be calculated by averaging the instantaneous Lyapunov exponents $\ln |R(\tau)| / \tau$: $\lambda = \lim_{\tau \to \infty} \frac{\ln |R(\tau)|}{\tau}$.

Intermittency can be thought of as connected to the fluctuations of the instantaneous Lyapunov exponent. An intermittency parameter $\mu$ is thus defined as the variance of $\ln |R(\tau)|$:

$$\sum (\ln |R(\tau)|)^2 - \langle (\ln |R(\tau)|)^2 \rangle = \mu \tau$$

Numerically we proceed as follows: the equations of motion are numerically integrated along the equations for the disturbance vector $\delta \tilde{x} = J(\tilde{x}) \delta \tilde{x}$ where $(\tilde{x}) = (u, \theta)$ and $J$ is the Jacobian matrix. We then estimate the response function and the first two cumulants of its logarithm. From time to time the disturbance vector is renormalized in order to avoid numerical overflow.

Consider now the Lyapunov exponents of the full system of velocity and passive scalar field, with the parameters. We distinguish how the perturbation, initially applied equally on both systems ($\delta \tilde{u} = \delta \theta$ constant for all $n$), evolves independently in each of the two parts of the system. Therefore we introduce response functions that measure the expansions on the velocity part $R^{u}(\tau)$ and on the scalar part $R^{\theta}(\tau)$. They are defined as:

$$R^{u}(\tau) = \frac{|\delta \tilde{u}(t + \tau)|}{|\delta \tilde{u}(t)|}$$
$$R^{\theta}(\tau) = \frac{|\delta \theta(t + \tau)|}{|\delta \theta(t)|}$$

The corresponding maximal Lyapunov exponents is defined as:

$$\lambda^{u} = \frac{\langle \ln R^{u}(\tau) \rangle}{\tau}$$

and the analogue for $\lambda^{\theta}$. This is reminiscent to the “Eulerian Lyapunov exponent” and “Lagrangian Lyapunov exponent” introduced by Crisanti et al. However the Lagrangian behavior of the particles is only equivalent to the Eulerian passive scalar field in the case of vanishing diffusion.

The intermittency parameters are related to the variance of $\ln R(\tau)$ is the same way as before. We find numerically:

$$\lambda^{u+\theta} \approx \lambda^{u} \approx \lambda^{\theta} = 0.165 \pm 0.002$$

where we have denoted the Lyapunov exponent for the full system as: $\lambda^{u+\theta}$. On expects theoretically:

$$\lambda^{u+\theta} = \max(\lambda^{u}, \lambda^{\theta})$$

since the disturbance vector will evolve towards the most expanding direction. Our result is obviously in agreement with this relation. For the Eulerian and Lagrangian Lyapunov exponent, one has numerically found the generic inequality $\lambda_{L} \geq \lambda_{E}$. Our result $\lambda^{u} \approx \lambda^{\theta}$ might be due to the presence of a finite value of diffusivity.

The intermittency parameters $\mu$ differ on the other hand significantly from each other. We find (for $\delta = 0.5$): $\mu^{\theta+u} = 0.127 \pm 0.003$, $\mu^{u} = 0.125 \pm 0.003$ and $\mu^{\theta} = 0.151 \pm 0.003$. Thus the passive scalar behaves equally chaotic, but more intermittent, than the velocity. In order to understand this we write the equation for $\theta$ as follows:

$$\dot{\theta} = A(u)\theta + f$$

where $A$ is a matrix depending on $u$, whose eigenvalues $\sigma$ fulfill the inequality $\Re \sigma < 0$. Thus, if $u$ is constant in time, $\theta$ will converge to its fixed point $\theta = -A(u)^{-1}f$. Alternatively, if $u$ fluctuates as for $\delta = 0.5$ in the GOY model, this “fixed point” will fluctuate as well. Moreover the convergence towards it will be irregular since the eigenvalues of $A(u)$ vary. It is therefore natural to expect that the behavior of $\theta$ is more intermittent than that of $u$.

**VI. CONCLUSIONS AND OUTLOOK**

We studied various properties of a shell model for the advection of a passive scalar. We calculate the eigenvalue spectrum of the Kolmogorov fixed point and show that the passive scalar part is stable against perturbations. This is in fact not very surprising since it follows from conservation of $\sum |\theta|^2$.

The fixed point does on the other hand not follow the Obukhov-Corrsin scaling. This suggests that the scaling properties of the passive scalar are much more sensitive, than the velocity, to non-inertial properties such as the dissipation and perhaps also the forcing. In experiments the scaling zones of the passive scalar are less clearly observed and those in the passive scalar model. Thus the passive scalar behaves more intermittently which is in agreement with its more pronounced deviations from linear scaling. It would be interesting to establish these relations from the equations of motion.

It could also be interesting to investigate the relation between coherent structures in the GOY model and those in the passive scalar model.

**VII. ACKNOWLEDGMENTS**

J.K. is grateful to the Niels Bohr Institute for warm hospitality. Discussions with Hugues Chaté, Ted Janssen, Anaël Lemaître and Paolo Muratore-Ginanneschi are kindly acknowledged. We thank Detlef Lohse for providing us with the code to extract the Kolmogorov fixed point.
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