On the periodic Schrödinger-Boussinesq System

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Abstract

We study the local and global well-posedness of the periodic boundary value problem for the nonlinear Schrödinger-Boussinesq system. The existence of periodic pulses as well as the stability of such solutions are also considered.

1 Introduction

In this paper we consider the periodic Schrödinger-Boussinesq system (hereafter referred to as the SB-system)

\[
\begin{aligned}
\begin{cases}
iu_t + u_{xx} = \alpha vu, \\
v_{tt} - v_{xx} + v_{xxxx} = \beta (|u|^2)_{xx},
\end{cases}
\end{aligned}
\]

(1)

where \( t > 0, \ x \in [0, L], \) for some \( L > 0, \) and \( \alpha, \beta \) are real constants.

Here \( u \) and \( v \) are respectively a complex-valued and a real-valued function defined in space-time \([0, L] \times \mathbb{R}\). The SB-system is considered as a model of interactions between short and intermediate long waves, which is derived in describing the dynamics of Langmuir soliton formation and interaction.
in a plasma [28] and diatomic lattice system [32]. The short wave term
\( u(x,t) : [0,L] \times \mathbb{R} \to \mathbb{C} \) is described by a Schrödinger type equation with a potential \( v(x,t) : [0,L] \times \mathbb{R} \to \mathbb{R} \) satisfying some sort of Boussinesq equation and representing the intermediate long wave.

The nonlinear Schrödinger (NLS) equation models a wide range of physical phenomena including self-focusing of optical beams in nonlinear media, propagation of Langmuir waves in plasmas, etc. For an introduction in this topic, we refer the reader to [26]. Boussinesq equation as a model of long waves was originally derived by Boussinesq [8] in his study of nonlinear, dispersive wave propagation. We should remark that it was the first equation proposed in the literature to describe this kind of physical phenomena. This equation was also used by Zakharov [34] as a model of nonlinear string and by Falk et al [13] in their study of shape-memory alloys.

Our first aim here is to study the well-posedness of the periodic boundary value problem (BVP) for the SB-system (1), that is, we are interested in the solvability of system (1) subject to the initial conditions
\[
\begin{align*}
  u(x,0) &= u_0(x); \quad v(x,0) = v_0(x); \quad v_t(x,0) = (v_1)_x(x).
\end{align*}
\]

Concerning the local well-posedness question, some results has been obtained for the SB-system (1) in the continuous case. Linares and Navas [25] proved that (1) is locally well-posedness for initial data \( u_0 \in L^2(\mathbb{R}) \), \( v_0 \in L^2(\mathbb{R}) \), \( v_1 = h_x \) with \( h \in H^{-1}(\mathbb{R}) \) and \( u_0 \in H^1(\mathbb{R}) \), \( v_0 \in H^1(\mathbb{R}) \), \( v_1 = h_x \) with \( h \in L^2(\mathbb{R}) \). Moreover, by using some conservations laws, in the latter case the solutions can be extended globally. Yongqian [33] established a similar result when \( u_0 \in H^s(\mathbb{R}) \), \( v_0 \in H^s(\mathbb{R}) \), \( v_1 = h_{xx} \) with \( h \in H^s(\mathbb{R}) \) for \( s \geq 0 \) and assuming \( s \geq 1 \) these solutions are global. Finally, Farah [15] proved local well-posedness for initial data \( (u_0, v_0, v_1) \in H^k(\mathbb{R}) \times H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R}) \) provided
\[
\begin{align*}
  & (i) \quad |k| - 1/2 < s < 1/2 + 2k \text{ for } k \leq 0, \\
  & (ii) \quad k - 1/2 < s < 1/2 + k \text{ for } k > 0.
\end{align*}
\]

In particular, local well-posedness holds for initial data \( (u_0, v_0, v_1) \in H^s(\mathbb{R}) \times H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R}) \) with \( s > -1/4 \). Moreover when \( s = 0 \) the solution is global. We should mention that, in fact, it is possible to obtain global well-posedness for \( s \geq 0 \) in the continuous case. This can be proved using the arguments introduced by Bourgain [7] (see also Angulo et al. [4]). In the proof of Theorem 1.5 below we also apply these techniques for the periodic SB-system (1).
The local well-posedness for single dispersive equations with quadratic nonlinearities has been extensively studied in Sobolev spaces. The proof of these results are based in the Fourier restriction norm approach introduced by Bourgain [6] in his study of the nonlinear Schrödinger (NLS) equation \( iu_t + u_{xx} + u|u|^{p-2} = 0 \), with \( p \geq 3 \) and the Korteweg-de Vries (KdV) equation \( u_t + u_{xxx} + u_x u = 0 \). This method was further developed by Kenig, Ponce and Vega in [23] for the KdV equation and [24] for the quadratics nonlinear Schrödinger equations

\[
iu_t + u_{xx} + F_j(u, \bar{u}) = 0, \quad j = 1, 2, 3,
\]

where \( \bar{u} \) denotes the complex conjugate of \( u \) and \( F_1(u, \bar{u}) = u^2, F_2(u, \bar{u}) = u\bar{u}, F_3(u, \bar{u}) = \bar{u}^2 \) in one spatial dimension and in spatially continuous and periodic case.

The original Bourgain method makes extensive use of the Strichartz inequalities in order to derive the bilinear estimates corresponding to the nonlinearity. On the other hand, Kenig, Ponce and Vega simplified Bourgain’s proof and improved the bilinear estimates using only elementary techniques, such as Cauchy-Schwarz inequality and simple calculus inequalities.

This kind of technique was used by Farah [16] for the Boussinesq equation. However, we do not have good cancellations on the Boussinesq symbol. To overcome this difficulty, we observed that the dispersion in the Boussinesq case is given by the symbol \( \sqrt{\xi^2 + \xi^4} \) and this is, in some sense, related with the Schrödinger symbol (see Lemma 3.3 below). Therefore, we can modify the symbols and work only with the algebraic relations for the Schrödinger equation already used in Kenig, Ponce and Vega [24] in order to derive our relevant bilinear estimates.

To describe our results we define next the \( X^S_{s,b} \) and \( X^B_{s,b} \) spaces related respectively to the Schrödinger and Boussinesq equations. For the first equation, these spaces were introduced in [6]. In the case of Boussinesq equation, the \( X^B_{s,b} \) were first defined by Fang and Grillakis [14] for the Boussinesq-type equations in the periodic case. Using these spaces and following Bourgain’s argument introduced in [6] they proved local well-posedness for the BVP

\[
\begin{align*}
&u_{tt} - u_{xx} + u_{xxxx} + \partial_x^2 [f(u)] = 0, \\
&u_x(0) = u_0(x), \quad u_t(x, 0) = (u_1)_x(x),
\end{align*}
\]

where \( u_0 \in H^s_{\text{per}}, u_1 \in H^{-2+s}_{\text{per}}, \) with \( 0 \leq s \leq 1 \) and the nonlinearity \( f \) satisfying \( |f(u)| \leq c|u|^p \), with \( 1 < p < \frac{3 - 2s}{1 - 2s} \) if \( 0 \leq s < \frac{1}{2} \) and \( 1 < p < \infty \) if \( \frac{1}{2} \leq s \leq 1 \). Moreover, if \( u_0 \in H^1_{\text{per}}, u_1 \in H^{-1}_{\text{per}} \) and \( f(u) = \lambda |u|^{q-1}u - |u|^{p-1}u, \) with \( 1 < q < p \) and \( \lambda \in \mathbb{R} \) then the solution is global.
Next we give the precise definition of the $X_{s,b}^S$ and $X_{s,b}^B$ spaces used in the sequel.

**Definition 1.1** Let $\mathcal{Y}$ be the space of functions $F(\cdot)$ such that

(i) $F : [0, L] \times \mathbb{R} \to \mathbb{C}$.

(ii) $F(x, \cdot) \in S(\mathbb{R})$ for each $x \in [0, L]$.

(iii) $F(\cdot, t) \in C^\infty([0, L])$ for each $t \in \mathbb{R}$.

For $s, b \in \mathbb{R}$, $X_{s,b}^S$ and $X_{s,b}^B$ denotes, respectively, the completion of $\mathcal{Y}$ with respect to the norm

$$\|F\|_{X_{s,b}^S} = \|\langle \tau + (2\pi n/L)^2 \rangle^{s}\langle n \rangle^{s} \tilde{F}\|_{L^2}^2,$$

$$\|F\|_{X_{s,b}^B} = \|\langle |\tau| - \gamma_L(n) \rangle^{s} \langle n \rangle^{s} \tilde{F}\|_{L^2}^2,$$

where $\sim$ denotes the time-space Fourier transform, $\langle a \rangle \equiv 1 + |a|$ and $\gamma_L(n) \equiv (2\pi/L)^2 \sqrt{n^2 + n^4}$.

We will also need the localized $X_{s,b}$ spaces defined as follows:

**Definition 1.2** Let $I$ be a time interval. For $s, b \in \mathbb{R}$, $X_{s,b}^{S,I}$ and $X_{s,b}^{B,I}$ denotes the space endowed with the norm

$$\|w\|_{X_{s,b}^{S,I}} = \inf_{w \in X_{s,b}^{S,b}} \left\{ \|w\|_{X_{s,b}^{S,b}} : w(t) = u(t) \text{ on } I \right\},$$

$$\|w\|_{X_{s,b}^{B,I}} = \inf_{w \in X_{s,b}^{B,b}} \left\{ \|w\|_{X_{s,b}^{B,b}} : w(t) = u(t) \text{ on } I \right\}.$$

Now we state our main results concerning well-posedness.

**Theorem 1.1** Let $s \geq 0$ and $1/4 < a < 1/2 < b$. Then, there exists $c > 0$, depending only on $a, b, s$, such that

(i) $\|uv\|_{X_{s-a}^{S,b}} \leq c \|u\|_{X_{s,b}^{S,b}} \|v\|_{X_{s,b}^{B,b}}$.

(ii) $\|u_1\bar{u}_2\|_{X_{s-a}^{B,b}} \leq c \|u_1\|_{X_{s,b}^{S,b}} \|u_2\|_{X_{s,b}^{S,b}}$.

**Theorem 1.2** Let $s \geq 0$. Then for any $(u_0, v_0, v_1) \in H_{\text{per}}^s([0, L]) \times H_{\text{per}}^s([0, L]) \times H_{\text{per}}^{s-1}([0, L])$ there exist $T = T([\|u_0\|_{H_{\text{per}}^s}, \|v_0\|_{H_{\text{per}}^s}, \|v_1\|_{H_{\text{per}}^{s-1}}], b > 1/2$ and a unique solution $(u, v)$ of the BVP (1)–(2), satisfying

$u \in C([0, T] : H_{\text{per}}^s([0, L])) \cap X_{s,b}^{S,[0,T]}$ and $v \in C([0, T] : H_{\text{per}}^s([0, L])) \cap X_{s,b}^{B,[0,T]}$.

Moreover, the map $(u_0, v_0, v_1) \mapsto (u(t), v(t))$ is locally Lipschitz from $H_{\text{per}}^s([0, L]) \times H_{\text{per}}^s([0, L]) \times H_{\text{per}}^{s-1}([0, L])$ into $C([0, T] : H_{\text{per}}^s([0, L]) \times H_{\text{per}}^s([0, L]))$.  

4
We also obtain counter-examples for the bilinear estimates stated in Theorem 1.1.

**Theorem 1.3**

(i) The estimate

\[ \|uv\|_{X_{k,-a}^s} \leq c \|u\|_{X_{k,b}^s} \|v\|_{X_{s,b}^B} \]

holds only if \( k \leq s \).

(ii) The estimate

\[ \|uv\|_{X_{k,-a}^s} \leq c \|u\|_{X_{k,b}^s} \|v\|_{X_{s,b}^B} \]

holds only if \( k + s \geq 0 \).

(iii) The estimate

\[ \|u_1 \bar{u}_2\|_{X_{k,-a}^s} \leq c \|u_1\|_{X_{k,b}^s} \|u_2\|_{X_{s,b}^s} \]

holds only if \( s \leq k \).

**Theorem 1.3** has an important consequence. It shows that our local well-posed result is sharp, in the sense that it cannot be improved using the spaces \( X_{s,b}^S \) and \( X_{s,b}^B \). This situation is very different from the continuous case obtained in Farah [15] where we have local well-posedness for initial data in different Sobolev spaces with negative indices.

Next we obtain bilinear estimates for the case \( s = 0 \) and \( b, b_1 < 1/2 \). These estimates will be useful to establish the existence of global solutions.

**Theorem 1.4**

Let \( a, a_1, b, b_1 > 1/4 \), then there exists \( c > 0 \) depending only on \( a, a_1, b, b_1 \) such that

(i) \[ \|uv\|_{X_{k,-a}^s} \leq c \|u\|_{X_{0,a}^s} \|v\|_{X_{0,b}^B} \]

(ii) \[ \|u_1 \bar{u}_2\|_{X_{0,-a}^s} \leq c \|u_1\|_{X_{0,a}^s} \|u_2\|_{X_{0,b}^s} \]

The bilinear estimates in **Theorem 1.4** are the essential tools to prove the global result. It asserts that the local solution given by **Theorem 1.2** is in fact a global one, for all \( s \geq 0 \).

**Theorem 1.5**

Let \( s \geq 0 \). Then, the BVP (1)–(2) is globally well-posed for data \( (u_0, v_0, v_1) \in H_{per}^s([0,L]) \times H_{per}^a([0,L]) \times H_{per}^{s-1}([0,L]) \). Moreover, the solution \((u, v)\) satisfies, for all \( t > 0 \),

\[ \|v(t)\|_{H_{per}^s} + \|(-\Delta)^{-1/2} v(t)\|_{H_{per}^{s-1}} \leq e^{(\ln 2)\|u_0\|_{H_{per}^s}^2} \max \{ \|v_0, v_1\|_{L^2}, \|u_0\|_{H_{per}^s} \} \]
The argument used to prove Theorem 1.5 follows the ideas introduced by Colliander, Holmer, Tzirakis [10] to deal with the Zakharov system. The intuition for this Theorem comes from the fact that the nonlinearity for the second equation of the SB-system (1) depends only on the first equation. Therefore, noting that the bilinear estimates given in Theorem 1.4 hold for \( a, a_1, b, b_1 < 1/2 \), it is possible to show that the time existence depends only on the \( \|u_0\|_{L^2_{per}} \). But since this norm is conserved by the flow, we obtain a global solution.

Our second aim is to study existence and orbital (nonlinear) stability of periodic traveling waves. These two questions are very important in the understanding of the dynamic of the system under consideration.

The stability study of traveling waves has been extensively studied for the whole Euclidean space case (solitary waves), whereas the study under periodic boundary conditions has been started quite recently and few works are available in the current literature. To cite a few important contributions, in [1] Angulo studied the orbital stability of \( dnoidal \) wave solutions for the cubic Schrödinger and modified Korteweg-de Vries equations; his method of proofs follows the pioneers ideas of Benjamin, Bona and Weinstein. In [2], Angulo \textit{et al.} gave a complete stability study of \( cnoidal \) wave solutions for the Korteweg-de Vries equation, adapting to the the periodic context the abstract theory developed in [18]. For others well-known equations and systems see e.g. [3], [4], [11], [20], [29] (and references therein).

One of the main reasons why the stability study in the periodic case has been received little attention, lies on the needed spectral theory associated with some linearized operator. Indeed, to fix ideas, suppose we have a Schrödinger type operator \( \mathcal{L} = -\frac{d^2}{dx^2} + q(x) \), where \( q(x) \) is a smooth real potential. If \( q(x) \) and \( \phi \) are rapidly decaying smooth functions such that \( \mathcal{L}\phi = 0 \) and assuming that \( \phi \) has exactly two zeros on the whole real line, then it follows immediately from Sturm-Liouville’s theory that zero is the third eigenvalue of operator \( \mathcal{L} \) and it is a simple eigenvalue. On the other hand, if \( q(x) \) is a periodic function with period \( L > 0 \) and \( \phi \) is also \( L \)-periodic such that \( \mathcal{L}\phi = 0 \) and has exactly two zeros on the interval \( [0, L] \) then from Floquet’s theory, the eigenvalue zero is the second or the third one (see e.g. [12]). In most cases, it is a hard task to decide when zero is the second or the third eigenvalue. As a consequence, most of the current papers deal with explicit solutions. This is the case of the present paper.

In general, the studied dispersive equations admits periodic explicit solutions depending on the Jacobian elliptic functions (\( dnoidal \), \( cnoidal \) and \( snoidal \) type). So, the main idea to obtain the spectral properties for the linearized operator is to reduce matter to some known periodic eigenvalue
problem. The most popular one deals with the periodic eigenvalue problem associated with the Lamé operator

\[ \mathcal{L}_{\text{Lame}} := -\frac{d^2}{dx^2} + n(n+1)sn^2(x; k), \]  

for some determined value of \( n \in \mathbb{N} \) (see e.g. [1], [2], [3], [29]).

Here, we will consider \( \alpha = \beta = -1 \) in (1) and look for solutions of the form

\[ u(x, t) = e^{jwt} \psi_\omega(x), \quad v(x, t) = \phi_\omega(x), \]  

where \( \omega \) is a real parameter and \( \psi_\omega, \phi_\omega : \mathbb{R} \to \mathbb{R} \) are \( L \)-periodic functions with a period \( L > 0 \). Then, substituting this waveform into the system and integrating twice the second equation in the obtained system, we have

\[ \begin{align*}
\psi''_\omega - \omega \psi_\omega + \psi_\omega \phi_\omega &= 0, \\
\phi''_\omega - \phi_\omega + \psi_\omega^2 &= 0.
\end{align*} \]  

To reduce system (9) to a single ordinary differential equation, we assume \( \omega = 1 \) and \( \psi_\omega = \phi_\omega = \psi \), so that it reduces to

\[ \psi'' - \psi + \psi^2 = 0. \]  

Before proceeding, we point out that existence and stability of hyperbolic-secant-type solitary waves were recently considered in [19]. The author has proved a orbital stability result by using the abstract theory contained in [18], taking the advantage of the spectral properties established in [27].

In the periodic approach, it is not difficult to prove that (10) has a periodic solution of cnoidal type, namely,

\[ \psi(x) = \beta_2 + (\beta_3 - \beta_2)cn^2 \left( \sqrt{\frac{\beta_3 - \beta_1}{6}} x; k \right), \quad k^2 = \frac{\beta_3 - \beta_2}{\beta_3 - \beta_1} \]  

where \( cn(\cdot, k) \) denotes the Jacobian elliptic function of dnoideal type and \( \beta_1, \beta_2, \beta_3 \) are real parameters.

Our main theorem concerned with the orbital stability of cnoidal waves reads as follows:

**Theorem 1.6** Let \( \psi \) be the cnoidal wave solution given in (11). Then, the periodic traveling wave \( (e^{jwt}, \psi) \) is orbitally stable in the energy space

\[ X = H^1_{\text{per}}([0, L]) \times H^1_{\text{per}}([0, L]) \times L^2_{\text{per}}([0, L]) \]  

by the flow of system (1).
To prove Theorem 1.6 we shall employ the classical theory developed by Grillakis, Shatah and Strauss [18]. To do so, we first observe that system (1) (with $\alpha = \beta = -1$) can be written in Hamiltonian form (see (52)). We point out that although the operator $J$ in (53) is not onto, along the lines of proofs in [18] the stability result still holds (see also [19], [31]).

Our strategy to get the needed spectral properties is to combine the results in [3], which are essentially proved from well-known results for the Lamé operator in (7), with the min-max principle for the eigenvalues characterization.

Finally, we also obtain periodic traveling waves for $\omega \neq 1$. Our idea is simple: once obtained the cnoidal solution for $\omega = 1$, we employ the Implicit Function Theorem combined with spectral properties related with the linearized operator to extend our range of parameters for $\omega$ near 1.

The plan of this paper is as follows: in Section 2, we introduce some notation and state important propositions that we will use throughout the paper. The proof of the bilinear estimates and the relevant counter examples are given in Sections 3 and 4, respectively. In Section 5 we prove Theorem 1.5. Finally, the stability questions are treated in Sections 6.

## 2 Notations and Preliminaries

In what follows we use $a \lesssim b$ to say that $a \leq Cb$ for some constant $C > 0$. Also, we denote $a \sim b$ when, $a \lesssim b$ and $b \lesssim a$. We write $a \ll b$ to denote an estimate of the form $a \leq cb$ for some small constant $c > 0$. In addition, $a^+ = a + \varepsilon$.

Let us recall some properties of $L$-periodic functions. For a detailed presentation of the spaces of periodic functions and its properties we refer the reader, for instance, to [21]. We define the Fourier transform of $f \in L^1([0, L])$ by

$$\hat{f}(n) = \frac{1}{L} \int_0^L e^{-2\pi i \frac{n}{L} x} f(x) dx.$$ 

For $f$ in an appropriate class of functions we have $f = (\hat{f})^\vee$, where for a sequence $s = \{s_n\}_{n \in \mathbb{Z}}$, the symbol $\vee$ denotes the inverse Fourier transform of $s$ given by

$$(s)^\vee(x) = \sum_{n \in \mathbb{Z}} e^{2\pi i \frac{n}{L} x} s_n$$

Moreover, we have the Plancherel identity

$$\|f\|_{L^2_{\text{per}}} = \|\hat{f}\|_{l^2_\mathbb{Z}}.$$
The periodic Sobolev space $H^s_{\text{per}}([0,L])$ is defined to be space of all periodic distributions such that
\[ \|f\|_{H^s_{\text{per}}} := \| \langle n \rangle^s \hat{f}(n) \|_2^2 < \infty. \]

Moreover, the operator $(-\Delta)^{-1/2}$ is defined, via Fourier transform, by
\[ [(-\Delta)^{-1/2} f]^\wedge(n) = |n|^{-1} \hat{f}(n) \quad n \neq 0. \]

Next, we recall some facts on the linear Schrödinger and Boussinesq equations. Consider the free Schrödinger equation
\[ iu_t + u_{xx} = 0. \tag{12} \]
It is easy to see that the solution of (12), with initial data $u(0) = u_0$, is given by the formula
\[ u(t) = U(t)u_0, \tag{13} \]
where
\[ U(t)u_0 = \left( e^{-(2\pi/L)^2 t n^2} \hat{u}_0(n) \right)^\wedge. \]

On the other hand, for the linear Boussinesq equation
\[ v_{tt} - v_{xx} + v_{xxxx} = 0 \tag{14} \]
it is well-known that the solution, with initial data $v(0) = v_0$ and $v_t(0) = (v_1)_x$, is given by
\[ u(t) = V_c(t)v_0 + V_s(t)(v_1)_x, \tag{15} \]
where
\[ V_c(t)v_0 = \left( \frac{e^{(2\pi/L)^2 t \sqrt{n^2+n^4}} + e^{-(2\pi/L)^2 t \sqrt{n^2+n^4}}}{2} \hat{v}_0(n) \right)^\wedge, \]
\[ V_s(t)(v_1)_x = \left( \frac{e^{(2\pi/L)^2 t \sqrt{n^2+n^4}} - e^{-(2\pi/L)^2 t \sqrt{n^2+n^4}}}{2i \sqrt{n^2+n^4}} (v_1)_x(n) \right)^\wedge. \]

As a consequence, by Duhamel’s Principle the solution of (11)–(12), is equivalent to
\[ u(t) = U(t)u_0 - i \int_0^t U(t-t')(\alpha vu)(t')dt' \]
\[ v(t) = V_c(t)v_0 + V_s(t)(v_1)_x + \int_0^t V_s(t-t')(\beta |u|^2)_{xx}(t')dt'. \tag{16} \]
Let $\theta$ be a cutoff function satisfying $\theta \in C_0^\infty(\mathbb{R})$, $0 \leq \theta \leq 1$, $\theta \equiv 1$ in $[-1,1]$, supp($\theta$) $\subseteq [-2,2]$ and for $0 < T \leq 1$ define $\theta_T(t) = \theta(t/T)$. In fact, to work in the $X^S_{s,b}$ and $X^B_{s,b}$ we consider another versions of (16), that is

$$u(t) = \theta(t)U(t)u_0 - i\theta_T(t) \int_0^t U(t-t')(\alpha vu)(t')dt'$$

$$(17)$$

and

$$u(t) = \theta_T(t)U(t)u_0 - i\theta_T(t) \int_0^t U(t-t')(\alpha vu)(t')dt'$$

$$(18)$$

We will use equation (17) (resp. (18)) to study the local (resp. global) well-posedness problem associated to (1)–(2).

Note that the integral equations (17) and (18) are defined for all $(t,x) \in \mathbb{R}^2$. Moreover, if $(u,v)$ is a solution of (17) or (18) then $(\tilde{u},\tilde{v}) = (u|_{[0,T]},v|_{[0,T]})$ will be a solution of (16) in $[0,T]$.

Before proceeding to the group and integral estimates for (17) and (18) we introduce the norm

$$\|v_0,v_1\|_{\mathcal{B}}^2 \equiv \|v_0\|^2_{H^s_{\text{per}}([0,L])} + \|v_1\|^2_{H^{s-1}_{\text{per}}([0,L])}.$$ 

For simplicity we denote $\mathcal{B}^0$ by $\mathcal{B}$ and, for functions of $t$, we use the shorthand

$$\|v(t)\|_{\mathcal{B}}^2 \equiv \|v(t)\|^2_{H^s_{\text{per}}([0,L])} + \|(-\Delta)^{-1/2}v(t)\|^2_{H^{s-1}_{\text{per}}([0,L])}.$$ 

The following three lemmas are standard in this context. Although we are studying the periodic case, the proofs are essentially the same of the continuous setting. We refer the reader to Farah [15] for the details.

**Lemma 2.1 (Group estimates)** Let $L = 2\pi$ and $0 < T \leq 1$.

(a) Linear Schrödinger equation

(i) $\|U(t)u_0\|_{C(\mathbb{R};H^s_{\text{per}})} = \|u_0\|_{H^s_{\text{per}}}.$

(ii) If $0 \leq b_1 \leq 1$, then

$$\|\theta_T(t)U(t)u_0\|_{X^S_{s,b_1}} \lesssim T^{1/2-b_1}\|u_0\|_{H^s_{\text{per}}}.$$
(b) Linear Boussinesq equation

(i) \[ \| V(t) v_0 + V_s(t)(v_1)_x \|_{C(\mathbb{R}; H_{per}^s)} \leq \| v_0 \|_{H_{per}^s} + \| v_1 \|_{H_{per}^{s-1}}. \]

(ii) \[ \| V(t) v_0 + V_s(t)(v_1)_x \|_{C(\mathbb{R}; \mathfrak{B})} = \| v_0, v_1 \|_{\mathfrak{B}}. \]

(iii) If \( 0 \leq b \leq 1 \), then

\[ \| \theta_T(t) (V(t)v_0 + V_s(t)(v_1)_x) \|_{X_{s,b}^B} \lesssim T^{1/2 - b} \left( \| v_0 \|_{H_{per}^s} + \| v_1 \|_{H_{per}^{s-1}} \right). \]

Next we estimate the integral parts of (17).

Lemma 2.2 (Integral estimates) Let \( L = 2\pi \) and \( 0 < T \leq 1 \).

(a) Nonhomogeneous linear Schrödinger equation

(i) If \( 0 \leq a_1 < 1/2 \) then

\[ \left\| \int_0^t U(t - t') z(t') dt' \right\|_{C([0,T]; H_{per}^s)} \lesssim T^{1/2 - a_1} \| z \|_{X_{s,-a_1}^S}. \]

(ii) If \( 0 \leq a_1 < 1/2 \), \( 0 \leq b_1 \) and \( a_1 + b_1 \leq 1 \) then

\[ \left\| \theta_T(t) \int_0^t U(t - t') z(t') dt' \right\|_{X_{s,b_1}^S} \lesssim T^{1 - a_1 - b_1} \| z \|_{X_{s,-a_1}^S}. \]

(b) Nonhomogeneous linear Boussinesq equation

(i) If \( 0 \leq a < 1/2 \) then

\[ \left\| \int_0^t V_s(t - t') z_{xx}(t') dt' \right\|_{C([0,T]; \mathfrak{B}^+)} \lesssim T^{1/2 - a} \| z \|_{X_{s,-a}^B}. \]

(ii) If \( 0 \leq a < 1/2 \), \( 0 \leq b \) and \( a + b \leq 1 \) then

\[ \left\| \theta_T(t) \int_0^t V_s(t - t') z_{xx}(t') dt' \right\|_{X_{s,b}^B} \lesssim T^{1 - a - b} \| z \|_{X_{s,-a}^B}. \]

We also know the following embedding concerning the \( X_{s,b}^S \) and \( X_{s,b}^B \) spaces.

Lemma 2.3 Let \( b > \frac{1}{2} \). There exists \( c > 0 \), depending only on \( b \), such that

\[ \| u \|_{C(\mathbb{R}; H_{per}^s)} \leq c \| u \|_{X_{s,b}^B} \]

\[ \| u \|_{C(\mathbb{R}; \mathfrak{B}^+)} \leq c \| u \|_{X_{s,b}^S}. \]
We finish this section with the following standard Bourgain-Strichartz estimates.

**Lemma 2.4** Let \( u \in L^3_{x,t} \), therefore
\[
\|u\|_{L^3_{x,t}} \leq c \min\{\|u\|_{X^{\frac{3}{6},1/4}_0}, \|u\|_{X^{\frac{1}{2},1/4}_0}\}.
\]

**Proof.** This estimate is easily obtained by interpolating between
\[
\begin{align*}
\bullet & \quad \|u\|_{L^4_{x,t}} \leq c \min\{\|u\|_{X^{\frac{3}{4},\frac{3}{8}}_0}, \|u\|_{X^{\frac{1}{2},\frac{3}{8}}_0}\} \text{ (See Bougain [6] and Fang and Grillakis [14]).} \\
\bullet & \quad \|u\|_{L^2_{x,t}} = \|u\|_{X^{0,0}_0} = \|u\|_{X^{0,0}_B} \text{ (by definition).}
\end{align*}
\]

**Remark 2.1** To simplify our well-posedness analysis we will assume \( L = 2\pi \). We will return to an arbitrarily \( L > 0 \) in Section 6, where we study stability questions.

## 3 Bilinear estimates

First we state some elementary calculus inequalities that will be useful later.

**Lemma 3.1** For \( p,q > 0 \) and \( r = \min\{p,q,p+q-1\} \) with \( p + q > 1 \), there exists \( c > 0 \) such that
\[
\int_{-\infty}^{+\infty} \frac{dx}{(x-\alpha)^p(x-\beta)^q} \leq \frac{c}{(\alpha-\beta)^r}.
\]

**Proof.** See Lemma 4.2 in [17].

**Lemma 3.2** If \( \gamma > 1/2 \), then
\[
\sup_{(n,\tau) \in \mathbb{Z} \times \mathbb{R}} \sum_{n_1 \in \mathbb{Z}} \frac{1}{(1 + |\tau \pm n_1(n-n_1)|)^\gamma} < \infty.
\]

**Proof.** See Lemma 5.3 in [24].
Lemma 3.3 There exists \(c > 0\) such that
\[
\frac{1}{c} \leq \sup_{x,y \geq 0} \frac{1 + |x - y|}{1 + |x - \sqrt{y^2 + y}|} \leq c.
\] (21)

Proof. Since \(y \leq \sqrt{y^2 + y} \leq y + 1/2\) for all \(y \geq 0\) a simple computation shows the desired inequalities. \(\blacksquare\)

Remark 3.1 In view of the previous lemma we have an equivalent way to estimate the \(X_{s,b}^B\)-norm, that is

\[
\|u\|_{X_{s,b}^B} \sim \|\tau - n^2\|^{b} \langle n \rangle^{s} \|\tilde{u}(\tau, n)\|_{L_2^{a}}.
\]

This equivalence will be important in the proof of Theorem 1.1. As we said in the introduction, the Boussinesq symbol \(\sqrt{n^2 + n^4}\) does not have good cancellations to make use of Lemma 3.1. Therefore, we modify the symbols as above and work only with the algebraic relations for the Schrödinger equation.

Now we are in position to prove the bilinear estimates stated in Theorem 1.1.

Proof of Theorem 1.1

(i) For \(u \in X_{s,b}^S\) and \(v \in X_{s,b}^B\) we define \(f(\tau, n) \equiv \langle \tau + n^2 \rangle^b \langle n \rangle^s \tilde{u}(\tau, n)\) and \(g(\tau, n) \equiv \langle |\tau| - \gamma(n) \rangle^b \langle n \rangle^s \tilde{v}(\tau, n)\). By duality the desired inequality is equivalent to

\[
|W(f, g, \phi)| \leq c \|f\|_{L_2^{a} L_2^{b}} \|g\|_{L_2^{a} L_2^{b}} \|\phi\|_{L_2^{a} L_2^{b}}
\] (22)

where

\[
W(f, g, \phi) = \sum_{n, n_1} \int_{\mathbb{R}^2} \frac{\langle n \rangle^s}{\langle n_1 \rangle^s \langle n_2 \rangle^s} \frac{g(\tau_1, n_1) f(\tau_2, n_2) \tilde{\phi}(\sigma, n)}{\langle \sigma \rangle^a \langle \sigma_1 \rangle^b \langle \sigma_2 \rangle^b} d\tau d\tau_1
\]

and

\[
n_2 = n - n_1, \quad \tau_2 = \tau - \tau_1, \\
\sigma = \tau + n^2, \quad \sigma_1 = |\tau_1| - \gamma(n_1), \quad \sigma_2 = \tau_2 + n_2^2.
\] (23)
In view of Remark 3.1, we know that \( \langle |\tau_1| - \gamma(n_1) \rangle \sim \langle |\tau_1| - n_1^2 \rangle \). Therefore splitting the domain of integration into the regions \( \{(n, \tau, n_1, \tau_1) \in \mathbb{R}^4 : \tau_1 < 0\} \) and \( \{(n, \tau, n_1, \tau_1) \in \mathbb{R}^4 : \tau_1 \geq 0\} \), it is sufficient to prove inequality (22) with \( W_1(f, g, \phi) \) and \( W_2(f, g, \phi) \) instead of \( W(f, g, \phi) \), where

\[
W_1(f, g, \phi) = \sum_{n,n_1} \int_{\mathbb{R}^2} \frac{\langle n \rangle^s}{\langle n_1 \rangle^s \langle n_2 \rangle^s} \frac{g(\tau_1, n_1) f(\tau_2, n_2) \bar{\phi}(\tau, n)}{\langle \sigma \rangle^a \langle \tau_1 + n_1^2 \rangle^b \langle \sigma_2 \rangle^b} d\tau d\tau_1
\]

and

\[
W_2(f, g, \phi) = \sum_{n,n_1} \int_{\mathbb{R}^2} \frac{\langle n \rangle^s}{\langle n_1 \rangle^s \langle n_2 \rangle^s} \frac{g(\tau_1, n_1) f(\tau_2, n_2) \bar{\phi}(\tau, n)}{\langle \sigma \rangle^a \langle \tau_1 - n_1^2 \rangle^b \langle \sigma_2 \rangle^b} d\tau d\tau_1.
\]

Applying Cauchy-Schwarz and Hölder inequalities it is easy to see that

\[
|W_1|^2 \leq \|f\|_{L^2_x(L^2_{\tau})}^2 \|g\|_{L^2_x(L^2_{\tau})}^2 \|\phi\|_{L^2_x(L^2_{\tau})}^2 \times \left| \sum_{n_1} \int_{\mathbb{R}^2} \frac{1}{\langle \sigma \rangle^{2a}} \frac{\langle n \rangle^s}{\langle n_1 \rangle^s \langle n_2 \rangle^s} \frac{d\tau_1}{\langle \sigma_2 \rangle^{2b}} \right|_{L^2_x(L^\infty_{\tau})}.
\]

Noting that \( s \geq 0 \) we have

\[
\frac{\langle n \rangle^s}{\langle n_1 \rangle^s \langle n_2 \rangle^s} \leq 1.
\] (24)

Therefore in view of Lemma 3.1 it suffices to get bounds for

\[
\sup_{n,\tau} \frac{1}{\langle \sigma \rangle^{2a}} \sum_{n_1} \frac{1}{\langle \tau + n^2 + 2n_1^2 - 2nn_1 \rangle^{2b}}.
\]

By Lemma 3.2 this expression is bounded provides \( a \geq 0 \) and \( b > 1/4 \).

Now we turn to the proof of inequality (22) with \( W_2(f, g, \phi) \). Using the Cauchy-Schwarz and Hölder inequalities and duality it is easy to see that

\[
|W_2|^2 \leq \|f\|_{L^2_x(L^2_{\tau})}^2 \|g\|_{L^2_x(L^2_{\tau})}^2 \|\phi\|_{L^2_x(L^2_{\tau})}^2 \times \left| \sum_{n_1} \int_{\mathbb{R}^2} \frac{1}{\langle n_1 + n_2 \rangle^{2s} \langle \sigma \rangle^{2b}} \frac{d\tau_1}{\langle n_1 \rangle^{2s} \langle \tau_1 - n_1^2 \rangle^{2b} \langle \sigma_2 \rangle^{2b}} \right|_{L^2_x(L^\infty_{\tau})}.
\]
Therefore in view of Lemma 3.1 and (21) it suffices to get bounds for 
\[ \sup_{n_2, \tau_2} \frac{1}{(\sigma_2)^{2b}} \sum_{n_1} \frac{1}{(\tau_2 + n_2^2 + 2n_1^2 + 2n_1n_2)^{2a}}. \]

By Lemma 3.2 this expression is bounded provides \( b \geq 0 \) and \( a > 1/4 \).

(ii) For \( u_1 \in X_{s,b}^S \) and \( u_2 \in X_{s,b}^S \) we define \( f(\tau, n) = \langle \tau + n^2 \rangle^{b/2} \langle n \rangle^{a} \tilde{u}_1(\tau, n) \) and \( g(\tau, n) = \langle \tau + n^2 \rangle^{b/2} \langle n \rangle^{a} \tilde{u}_2(\tau, n) \). By duality the desired inequality is equivalent to

\[ |Z(f, g, \phi)| \leq c \|f\|_{L^2} \|g\|_{L^2} \|\phi\|_{L^2} \]  

(25)

where

\[ Z(f, g, \phi) = \sum_{n, n_1} \int_{\mathbb{R}^2} \frac{\langle n \rangle^{a}}{\langle n_1 \rangle^{a/2}} \frac{h(\tau_1, n_1)f(\tau_2, n_2)\tilde{\phi}(\tau, n)}{\langle \sigma \rangle^a \langle \sigma_1 \rangle^b \langle \sigma_2 \rangle^b} d\tau d\tau_1 \]

and

\[ h(\tau_1, n_1) = \tilde{g}(\tau_1, n_1), \quad n_2 = n - n_1, \quad \tau_2 = \tau - \tau_1, \]

\[ \sigma = |\tau - \gamma(n)|, \quad \sigma_1 = \tau - \tau_1, \quad \sigma_2 = \tau_2 + n_2. \]

Therefore applying Lemma 3.3 and splitting the domain of integration according to the sign of \( \tau \) it is sufficient to prove inequality (25) with \( Z_1(f, g, \phi) \) and \( Z_2(f, g, \phi) \) instead of \( Z(f, g, \phi) \), where

\[ Z_1(f, g, \phi) = \sum_{n, n_1} \int_{\mathbb{R}^2} \frac{\langle n \rangle^{a}}{\langle n_1 \rangle^{a/2}} \frac{h(\tau_1, n_1)f(\tau_2, n_2)\tilde{\phi}(\tau, n)}{\langle \tau + n^2 \rangle^a \langle \sigma_1 \rangle^b \langle \sigma_2 \rangle^b} d\tau d\tau_1 \]

and

\[ Z_2(f, g, \phi) = \sum_{n, n_1} \int_{\mathbb{R}^2} \frac{\langle n \rangle^{a}}{\langle n_1 \rangle^{a/2}} \frac{h(\tau_1, n_1)f(\tau_2, n_2)\tilde{\phi}(\tau, n)}{\langle \tau - n^2 \rangle^a \langle \sigma_1 \rangle^b \langle \sigma_2 \rangle^b} d\tau d\tau_1. \]

The inequality (25) with \( Z_1(f, g, \phi) \) can be estimate by the same argument as the one used in the bound of \( W_2(f, g, \phi) \).

Now we prove inequality (25) with \( Z_2(f, g, \phi) \). First we make the change of variables \( \tau_2 = \tau - \tau_1, n_2 = n - n_1 \) to obtain

\[ Z_2(f, g, \phi) = \sum_{n, n_2} \int_{\mathbb{R}^2} \frac{\langle n \rangle^{a}}{\langle n_2 \rangle^{a/2}} \frac{h(\tau - \tau_2, n - n_2)f(\tau_2, n_2)\tilde{\phi}(\tau, n)}{\langle \tau - n^2 \rangle^a \langle (\tau - \tau_2) - (n - n_2)^2 \rangle^b \langle \tau_2 + n_2^2 \rangle^b} d\tau d\tau_2. \]
Then changing the variables \((n, \tau, n_2, \tau_2) \mapsto -(n, \tau, n_2, \tau_2)\) we can rewrite \(Z_2(f, g, \phi)\) as

\[
Z_2(f, g, \phi) = \sum_{n,n_2} \int_{\mathbb{R}^2} \frac{\langle n \rangle^s}{\langle n-n_2 \rangle^s \langle n_2 \rangle^s} \times \frac{k(\tau - \tau_2, n - n_2) \psi(\tau, n)}{(\tau + n^2)^a (\tau - \tau_2 + (n - n_2)^2)^b} \frac{d\tau d\tau_2}{(\tau - n_2)^b}
\]

where

\[
k(a, b) = h(-a, -b), \quad l(a, b) = f(-a, -b) \quad \text{and} \quad \psi(a, b) = \phi(-a, -b).
\]

Since the \(L^2\)-norm is preserved under the reflection operation the result follows from the estimate for \(Z_1(f, g, \phi)\).

\[\blacksquare\]

**Remark 3.2** Once the bilinear estimates in Theorem 1.4 are established, it is a standard matter to conclude the local well-posedness statement of Theorem 1.2. We refer the reader to the works [24], [5], [17] and [15] for further details.

Finally we should remark that Theorem 1.4 can be obtained easily using Lemma 2.3 (see Farah [15]). Before get to the end of this section we state a slightly modified bilinear estimates that will be useful in the proof of Theorem 1.5.

**Corollary 3.1** Let \(a, a_1, b, b_1 > 1/4\) and \(s \geq 0\), then there exists \(c > 0\) depending only on \(a, a_1, b, b_1, s\) such that

\[
(i) \quad \|uv\|_{X^{s,-a}_{x,b}} \lesssim \|u\|_{X^{s}_{x,b_1}} \|v\|_{X^{s}_{x,b}} + \|u\|_{X^{s}_{0,b_1}} \|v\|_{X_{x,b}},
\]

\[
(ii) \quad \|u \bar{u} \|_{X^{s}_{x,-a}} \lesssim \|u_1\|_{X^{s}_{x,b_1}} \|u_2\|_{X^{s}_{x,b_1}} + \|u_1\|_{X^{s}_{0,b_1}} \|u_2\|_{X^{s}_{x,b_1}}.
\]

**Proof.** The above estimates are direct consequence of Theorem 1.4 and the fact that, for all \(s > 0\), the following inequality holds

\[
\langle \xi \rangle^s \leq \langle \xi_1 \rangle^s + \langle \xi - \xi_1 \rangle^s.
\]

\[\blacksquare\]
4 Counterexample to the bilinear estimates

Proof of Theorem 1.3

(i) For $u \in X^{S}_{k,b}$ and $v \in X^{B}_{s,b}$ we define $f(\tau, n) \equiv \langle \tau + n^2 \rangle^{b} \langle n \rangle^{k} \tilde{u}(\tau, n)$ and $g(\tau, n) \equiv \langle |\tau| - \gamma(n) \rangle^{b} \langle n \rangle^{s} \tilde{v}(\tau, n)$. By Lemma 3.3 the inequality (5) is equivalent to

$$
\left\| \frac{\langle n \rangle^{k}}{\langle \sigma \rangle^{a}} \sum_{n_{1}} \int \frac{f(\tau_{1}, n_{1}) g(\tau_{2}, n_{1}) d\tau_{1}}{\langle n_{1} \rangle^{k} \langle \sigma_{1} \rangle^{b} \langle \sigma_{2} \rangle^{b}} \right\|_{L^{2}_{\tau_{1}}} \lesssim \| f \|_{L^{2}_{\tau_{1}}} \| g \|_{L^{2}_{\tau_{1}}}, \tag{26}
$$

where

$$
n_{2} = n - n_{1}, \quad \tau_{2} = \tau - \tau_{1},$$

$$
\sigma = \tau + n^{2}, \quad \sigma_{1} = \tau_{1} + n_{1}^{2}, \quad \sigma_{2} = |\tau_{2} - n_{2}^{2}|.
$$

For $N \in \mathbb{Z}$ define

$$
f_{N}(\tau, n) = a_{n} \chi((\tau + n^{2})/2), \text{ with } a_{n} = \begin{cases} 1, & n = 0, \\ 0, & \text{elsewhere.} \end{cases}
$$

and

$$
g_{N}(\tau, n) = b_{n} \chi((\tau + n^{2})/2), \text{ with } b_{n} = \begin{cases} 1, & n = N, \\ 0, & \text{elsewhere.} \end{cases}
$$

where $\chi(\cdot)$ denotes the characteristic function of the interval $[-1, 1]$. Thus

$$
a_{n_{1}} b_{n - n_{1}} \neq 0 \text{ if and only if } n_{1} = 0 \text{ and } n = N
$$

and consequently for $N$ large

$$
\int \chi((\tau_{1} + n_{1}^{2})/2) \chi((\tau - \tau_{1} + (n - n_{1})^{2})/2) \gtrsim \chi((\tau + (n - n_{1})^{2} + n_{1}^{2}))
$$

$$
\gtrsim \chi((\tau + N^{2})).
$$

Therefore, using the fact that $||\tau_{2} - n_{2}^{2}| \leq |\tau_{2} + n_{2}^{2}|$, inequality (26) implies

$$
1 \gtrsim \| N^{k-s} \chi((\tau + N^{2}) \|_{L^{2}_{\tau_{1}}} \gtrsim N^{k-s}.
$$

Letting $N \to \infty$, this inequality is possible only when $k \leq s$.

(ii) Now we define

$$
f_{N}(\tau, n) = a_{n} \chi((\tau + n^{2})/2), \text{ with } a_{n} = \begin{cases} 1, & n = -N, \\ 0, & \text{elsewhere.} \end{cases}
$$

and

17
\[ g_N(\tau, n) = b_n \chi((\tau - n^2)/2), \text{ with } b_n = \begin{cases} 1, & n = N, \\ 0, & \text{elsewhere.} \end{cases} \]

Then
\[ a_{n_1} b_{n-n_1} \neq 0 \text{ if and only if } n_1 = 0 \text{ and } n = N \]

and for \( N \) large
\[
\int \chi((\tau_1 + n_2^2)/2) \chi((\tau - \tau_1 - (n - n_1)^2)/2) \gtrsim \chi((\tau + n^2 - 2nn_1)) \\
\gtrsim \chi((\tau)).
\]

Therefore, using the fact that \( |||\tau_2| - n_2^2| \leq |\tau_2 - n_2^2| \), inequality (26) implies
\[
1 \gtrsim \| N^{-(k+s)} \chi((\tau)) \|_{L_2^\tau} \gtrsim N^{-(k+s)}.
\]

Letting \( N \to \infty \), this inequality is possible only when \( k + s \geq 0 \).

(iii) For \( u_1 \in X_{k,b}^S \) and \( u_2 \in X_{k,b}^S \) we define \( f(\tau, n) \equiv \langle \tau + n^2/b(\langle n \rangle^k \tilde{u}_1(\tau, \xi) \rangle \) and \( g(\tau, n) \equiv \langle \tau + n^2/b(\langle n \rangle^k \tilde{u}_2(\tau, \xi) \rangle \). By Lemma 3.3 the inequality (6) is equivalent to
\[
\left\| \frac{\langle n \rangle^s}{\langle \sigma \rangle^a} \sum_{n_1} \int \frac{f(\tau_1, n_1)h(\tau_2, n_2)d\tau_1}{\langle n_1 \rangle^k \langle n_2 \rangle^k \langle \sigma_1 \rangle^b \langle \sigma_2 \rangle^b} \right\|_{\bar{L}_2^\tau} \lesssim \| f \|_{\bar{L}_2^\tau} \| g \|_{\bar{L}_2^\tau}, \tag{27}
\]
where
\[
h(\tau_2, n_2) = \bar{g}(\tau_2, -n_2), \quad n_2 = n - n_1, \quad \tau_2 = \tau - \tau_1,
\]
\[
\sigma = |\tau| - n^2, \quad \sigma_1 = \tau_1 + n_1^2, \quad \sigma_2 = \tau_2 - n_2^2.
\]
For \( N \in \mathbb{Z} \) define
\[
f_N(\tau, n) = a_n \chi((\tau + n^2)/2), \text{ with } a_n = \begin{cases} 1, & n = N, \\ 0, & \text{elsewhere.} \end{cases}
\]
and
\[
h_N(\tau, n) = b_n \chi((\tau - n^2)/2), \text{ with } b_n = \begin{cases} 1, & n = 0, \\ 0, & \text{elsewhere.} \end{cases}
\]
where \( \chi(\cdot) \) denotes the characteristic function of the interval \([-1, 1]\).

Thus
\[ a_{n_1} b_{n-n_1} \neq 0 \text{ if and only if } n_1 = N \text{ and } n = N \]
and
\[ \int \chi((\tau_1 + n_1^2)/2) \chi((\tau - \tau_1 - (n - n_1)^2)/2) \gtrsim \chi((\tau - (n - n_1)^2 + n_1^2)) \gtrsim \chi((\tau + N^2)) \].

Therefore, using the fact that \(||\tau - n^2|| \leq ||\tau + n^2||\), inequality (27) implies
\[ 1 \gtrsim \| N^{s-k} \chi((\tau + N^2)) \|_{L^2_T} \gtrsim N^{s-k} \].

Letting \( N \to \infty \), this inequality is possible only when \( s \leq k \).

\[ \blacksquare \]

5 Global Well-posedness

We divide our analysis in two cases. The proof of Theorem 1.5 for \( s = 0 \) follows the same lines as in Farah [15] Theorem 1.4. For the convenience of the reader we repeat the proof of this case below. The case \( s > 0 \) can be proved using the arguments introduced by Bourgain [7] for the Schrödinger equation and further developed by Angulo et al. [4] for the Schrödinger-Benjamin-Ono system.

**Proof of Theorem 1.5.**

**Case** \( s = 0 \):

Let \((u_0, v_0, v_1) \in L^2_{per}(0,1) \times L^2_{per}(0,1) \times H^{-1}_{per}(0,1)\) and \( 0 < T \leq 1 \). Based on the integral formulation (18), we define the integral operators
\[ G^S_T(u,v)(t) = \theta_T(t)U(t)u_0 - i\theta_T(t) \int_0^t U(t-t')(\alpha vu)(t')dt' \]
\[ G^B_T(u,v)(t) = \theta_T(t)(V_x(t)v_0 + V_x(t)v_1) + \theta_T(t) \int_0^t V_x(t-t')(\beta|u|^2)_{xx}(t')dt'. \] (28)

Therefore, applying Lemmas 2.1-2.2 and Theorem 1.3, we obtain
\[ \|G^S_T(u,v)\|_{X_{0,b_1}^S} \leq cT^{1/2-b_1} \|u_0\|_{L^2_{per}} + cT^{1-(a_1+b_1)} \|uv\|_{X_{0,-a_1}^S} \]
\[ \leq cT^{1/2-b_1} \|u_0\|_{L^2_{per}} + cT^{1-(a_1+b_1)} \|u\|_{X_{0,b_1}^S} \|v\|_{X_{0,b_1}^B}, \]
\[ \|G^B_T(u,v)\|_{X_{0,b}^B} \leq cT^{1/2-b} \|v_0, v_1\|_{B} + cT^{1-(a+b)} \|u\|_{X_{0,-a}^B} \]
\[ \leq cT^{1/2-b} \|v_0, v_1\|_{B} + cT^{1-(a+b)} \|u\|_{X_{0,b_1}^B}. \] (29)
and also
\[
\|G_T^S(u, v) - G_T^S(z, w)\|_{X_{0,b}^S} \leq cT^{1-(a_1+b_1)} \left( \|u\|_{X_{0,b1}^S} \|v - w\|_{X_{0,b}^B} + \|u - z\|_{X_{0,b1}^S} \|w\|_{X_{0,b}^B} \right),
\]
\[
\|G_T^B(u, v) - G_T^B(z, w)\|_{X_{0,b}^B} \leq cT^{1-(a+b)} \left( \|u\|_{X_{0,b1}^S} + \|z\|_{X_{0,b1}^S} \right) \times \|u - z\|_{X_{0,b1}^S}.
\]

We define
\[
X_{0,b1}^S(d_1) = \left\{ u \in X_{0,b1}^S : \|u\|_{X_{0,b1}^S} \leq d_1 \right\},
\]
\[
X_{0,b}^B(d) = \left\{ v \in X_{0,b}^B : \|v\|_{X_{0,b}^B} \leq d \right\},
\]
where \( d_1 = 2cT^{1/2-b_1}\|u_0\|_{L_{per}^2} \) and \( d = 2cT^{1/2-b}\|v_0, v_1\|_{\mathcal{B}}. \)

For \((G_T^S, G_T^B)\) to be a contraction in \(X_{0,b1}^S(d_1) \times X_{0,b}^B(d)\) it needs to satisfy
\[
d_1/2 + cT^{1-(a_1+b_1)d_1} \leq d_1 \iff T^{3/2-(a_1+b_1)}\|v_0, v_1\|_{\mathcal{B}} \leq 1,
\]
\[
d/2 + cT^{1-(a+b)d_1^2} \leq d \iff T^{3/2-(a+2b_1)}\|u_0\|_{L_{per}^2} \leq \|v_0, v_1\|_{\mathcal{B}},
\]
\[
2cT^{1-(a+b)d_1^2} \leq 1/2 \iff T^{3/2-(a+b+b_1)}\|u_0\|_{L_{per}^2} \leq 1,
\]
\[
2cT^{1-(a_1+b_1)d_1} \leq 1/2 \iff T^{3/2-(a_1+2b_1)}\|u_0\|_{L_{per}^2} \leq 1.
\]

Therefore, we conclude that there exists a solution \((u, v) \in X_{0,b1}^S \times X_{0,b}^B\) satisfying
\[
\|u\|_{X_{0,b1}^S[0,T]} \leq 2cT^{1/2-b_1}\|u_0\|_{L_{per}^2}, \quad \text{and} \quad \|v\|_{X_{0,b}^B[0,T]} \leq 2cT^{1/2-b}\|v_0, v_1\|_{\mathcal{B}}.
\]

On the other hand, applying Lemmas 2.1, 2.2 we have that, in fact, \((u, v) \in C([0,T] : L^2) \times C([0,T] : L^2).\) Moreover, since the \(L^2\)-norm of \(u\) is conserved by the flow we have \(\|u(T)\|_{L_{per}^2} = \|u_0\|_{L_{per}^2}.\)

Now, we need to control the growth of \(\|v(t)\|_{\mathcal{B}}\) in each time step. If, for all \(t > 0, \|v(t)\|_{\mathcal{B}} \leq \|u_0\|_{L_{per}^2}^2\) we can repeat the local well-posedness argument and extend the solution globally in time. Thus, without loss of
generality, we suppose that after some number of iterations we reach a time $t^* > 0$ where $\|v(t^*)\|_B \gg \|u_0\|_{L^2_{\text{per}}}^2$.

Hence, since $0 < T \leq 1$, condition (32) is automatically satisfied and conditions (31)-(34) imply that we can select a time increment of size

$$T \sim \frac{1}{3/2 - (a_1 + b_1 + b)}.$$  \hfill (36)

Therefore, applying Lemmas 2.1(b)-2.2(b) to $v = G_B^T(u, v)$ we have

$$\|v(t^* + T)\|_B \leq \|v(t^*)\|_B + cT^{3/2 - (a_1 + 2b_1)}(\|u_0\|_{L^2_{\text{per}}}^2 + 1).$$

Thus, we can carry out $m$ iterations on time intervals, each of length (36), before the quantity $\|v(t^*)\|_B$ doubles, where $m$ is given by

$$mT^{3/2 - (a_1 + 2b_1)}(\|u_0\|_{L^2_{\text{per}}}^2 + 1) \sim \|v(t^*)\|_B.$$  \hfill (37)

The total time of existence we obtain after these $m$ iterations is

$$\Delta T = mT \sim \frac{\|v(t^*)\|_B}{T^{1/2 - (a_1 + 2b_1)}(\|u_0\|_{L^2_{\text{per}}}^2 + 1)}$$

$$\sim \frac{\|v(t^*)\|_B^{1/2 - (a_1 + 2b_1)}/(3/2 - (a_1 + b_1 + b)}(\|u_0\|_{L^2_{\text{per}}}^2 + 1).$$

Taking $a, b, a_1, b_1$ such that

$$a + 2b_1 - 1/2 \over (3/2 - (a_1 + b_1 + b)) = 1,$$

(for instance, $a = b = a_1 = b_1 = 1/3$) we have that $\Delta T$ depends only on $\|u_0\|_{L^2_{\text{per}}}$, which is conserved by the flow. Hence we can repeat this entire argument and extend the solution $(u, v)$ globally in time.

Moreover, since in each step of time $\Delta T$ the size of $\|v(t)\|_B$ will at most double it is easy to see that, for all $\tilde{T} > 0$

$$\|v(\tilde{T})\|_B \lesssim \exp ((\ln 2)\|u_0\|_{L^2_{\text{per}}}^2 \tilde{T}) \max \{\|v_0, v_1\|_B, \|u_0\|_{L^2_{\text{per}}}} \}.$$  \hfill (37)

**Case $s > 0$:**

Let $(u_0, v_0, v_1) \in H^s_{\text{per}} \times H^s_{\text{per}} \times H^{s-1}_{\text{per}}$. By the previous case, there exists a global solution $(u, v) \in C([0, +\infty); L^2_{\text{per}}^2) \times C([0, +\infty); L^2_{\text{per}})$. Moreover,
$(u, v)$ is a solution of the integral equation (28) in the time interval $[0, \Delta T]$, with $\Delta T \sim \frac{1}{\|u_0\|_{L_{per}^2}^2 + 1}$, satisfying
\[
\max \left\{ \|u\|_{X^{S,[0,\Delta T]}_{1,3}}, \|v\|_{X^{B,[0,\Delta T]}_{1,3}} \right\} \lesssim C(\|u_0\|_{L_{per}^2}, \|v_0, v_1\|_{28}),
\] (38)
where the constant $C(\|u_0\|_{L_{per}^2}, \|v_0, v_1\|_{28}) > 0$ depends only on $\|u_0\|_{L_{per}^2}$ and $\|v_0, v_1\|_{28}$.

We claim that the solution $(u, v)$, in fact, belongs to $X^{S,[0,T_0]}_{1,3} \times X^{B,[0,T_0]}_{1,3}$ for all $0 < T_0 \leq \Delta T$. Indeed, applying Lemmas 2.1-2.2 and Corollary 3.1 with $a = b = a_1 = b_1 = 1/3$, we obtain
\[
\|u\|_{X^{S,[0,T_0]}_{1,3}} \lesssim \|u_0\|_{H_{per}^3} + T_0^{1/3} \left( \|u\|_{X^{S,[0,T_0]}_{1,3}} \|v\|_{X^{B,[0,T_0]}_{1,3}} + \|u\|_{X^{S,[0,T_0]}_{1,3}} \|v\|_{X^{B,[0,T_0]}_{1,3}} \right)
\] (39)
and
\[
\|v\|_{X^{B,[0,T_0]}_{1,3}} \lesssim \|v_0, v_1\|_{28} + T_0^{1/3} \left( \|u\|_{X^{S,[0,T_0]}_{1,3}} \|u\|_{X^{B,[0,T_0]}_{1,3}} \right),
\] (40)
where $0 < T_0 \leq \Delta T$. Inserting the inequality (40) into (39) and using (38) we conclude
\[
\|u\|_{X^{S,[0,T_0]}_{1,3}} \lesssim \|u_0\|_{H_{per}^3} + C(\|u_0\|_{L_{per}^2}, \|v_0, v_1\|_{28}) \|v_0, v_1\|_{28} + T_0^{1/3} C(\|u_0\|_{L_{per}^2}, \|v_0, v_1\|_{28}) \|u\|_{X^{S,[0,T_0]}_{1,3}}.
\]

Set
\[
T_0 \sim \frac{1}{\left(1 + C(\|u_0\|_{L_{per}^2}, \|v_0, v_1\|_{28})\right)^3}.
\]

Hence, from the choice of $T_0$, we deduce the following \textit{a priori} estimates
\[
\|u\|_{X^{S,[0,T_0]}_{1,3}} \lesssim \|u_0\|_{H_{per}^3} + C(\|u_0\|_{L_{per}^2}, \|v_0, v_1\|_{28}) \|v_0, v_1\|_{28}
\]
and
\[
\|v\|_{X^{B,[0,T_0]}_{1,3}} \lesssim \|v_0, v_1\|_{28} + C(\|u_0\|_{L_{per}^2}, \|v_0, v_1\|_{28}) \left( \|v_0, v_1\|_{28} + \|u_0\|_{L_{per}^2} \right).
\]

Thus, applying Lemmas 2.1-2.2 we get that $(u, v) \in C([0,T_0]; H_{per}^3) \times C([0,T_0]; H_{per}^3)$. The preceding statement remains valid for any bounded
interval $[0, T]$, since $T_0$ depends only on $\|u_0\|_{L^2_{\text{per}}}$ and $\|v_0, v_1\|_{\mathcal{B}}$ and we can iterate the above argument a finite number of times to deduce that

$$(u, v) \in C([0, T]; H^s_{\text{per}}) \times C([0, T]; H^s_{\text{per}}), \text{ for all } T > 0,$$

which completes the proof of Theorem 1.5.

6 Stability of periodic traveling waves

As we said in the Introduction, here we will consider system (1) with $\alpha = \beta = -1$, that is, we consider the system

\[
\begin{align*}
    iu_t + u_{xx} + uv &= 0, \\
    v_{tt} - v_{xx} + v_{xxxx} + (|u|^2)_{xx} &= 0,
\end{align*}
\]

and look for traveling waves of the form

\[
\begin{align*}
    u(x, t) &= e^{i\omega t} \psi_\omega(x), \\
    v(x, t) &= \phi_\omega(x),
\end{align*}
\]

where $\omega$ is a real parameter (to be determined later) and $\psi_\omega, \phi_\omega : \mathbb{R} \to \mathbb{R}$ are smooth periodic functions with the same fixed period $L > 0$. Then, substituting (42) into (41); integrating twice the second equation in the obtained system and assuming that the integration constants are zero, we obtain the system

\[
\begin{align*}
    \psi''_\omega - \omega \psi_\omega + \psi_\omega \phi_\omega &= 0, \\
    \phi''_\omega - \phi_\omega + \psi^2_\omega &= 0.
\end{align*}
\]

In order to solve system (43) we assume $\omega = 1$ and $\psi_1 = \phi_1$, so that system (43) reduces to a single ordinary differential equation, namely,

\[
\psi''_1 - \psi_1 + \psi^2_1 = 0.
\]

As we will see later in our stability analysis, it is necessary to construct a smooth branch of periodic wave solutions (depending on $\omega$) passing through solution $\psi_1$ of (44). Then, we will consider the family of equations

\[
\psi''_\omega - \omega \psi_\omega + \psi^2_\omega = 0,
\]

so that at $\omega = 1$ we obtain a solution for (44).
6.1 Existence of traveling waves

Along this subsection, we review the theory of finding solutions for (45). Indeed, equation (45) can be solved by using the standard direct integration method (for details we refer to [3]). As a matter of fact, equation (45) has a strictly positive solution of the form

$$\psi_\omega(x) = \beta_2 + (\beta_3 - \beta_2) \text{cn}^2 \left( \frac{\sqrt{\beta_3 - \beta_1}}{6} x; k \right), \quad k^2 = \frac{\beta_3 - \beta_2}{\beta_3 - \beta_1},$$

where $\text{cn}(\cdot; k)$ denotes the Jacobian elliptic function of cnoidal type, $k$ is the elliptic modulus and $\beta_1, \beta_2, \beta_3$ are real constants satisfying

$$\frac{3\omega}{2} = \sum_{i=1}^{3} \beta_i, \quad 0 = \sum_{i<j} \beta_i \beta_j, \quad \beta_1 \beta_2 \beta_3 = 3A_\psi,$$

where $A_\psi$ is an integration constant. Moreover, it must be the case that

$$\beta_1 < 0 < \beta_2 < \omega < \beta_3 < \frac{3\omega}{2}.$$}

The first question concerning solution (46) is the following: Fixed $L > 0$, can we choose $\beta_1, \beta_2, \beta_3$ such that solution (46) has fundamental period $L$? The answer is yes. To prove so, one first note since $\text{cn}^2(\cdot; k)$ has fundamental period $2K(k)$, where $K$ is the complete elliptic integral of the first kind defined by (see e.g., [9])

$$K(k) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2 t^2)}},$$

function $\psi_\omega$ given in (46) has fundamental period

$$T_{\psi_\omega} = \frac{2\sqrt{6}}{\sqrt{\beta_3 - \beta_1}} K(k).$$

Next we observe that $T_{\psi_\omega}$ can be rewritten as a function depending only on $\beta_2$ (and $\omega > 0$ fixed). In fact, by defining $\omega_0 = \omega/2$, we readily see from (47) that

$$T_{\psi_\omega}(\beta_2; \omega_0) = \frac{2\sqrt{6}}{\sqrt{\rho(\beta_2; \omega_0)}} K(k(\beta_2; \omega_0)), \quad \rho(\beta_2; \omega_0) = \sqrt{9\omega_0^2 - 3\beta_2^2 + 6\omega_0\beta_2}, \quad k^2(\beta_2; \omega_0) = \frac{1}{2} + \frac{3(\omega_0 - \beta_2)}{2\rho(\beta_2; \omega_0)}.$$
Moreover, from (49) it is easy to see that $T_{\psi_\omega} \to +\infty$, as $\beta_2 \to 0$ and $T_{\psi_\omega} \to \sqrt{2\pi/\omega_0}$, as $\beta_2 \to 2\omega_0$. Since the function $\beta_2 \in (0, 2\omega_0) \to T_{\psi_\omega}(\beta_2; \omega_0)$ is strictly decreasing (this will be proved in the next theorem) we see that, fixed $L > 0$ and choosing $\omega_0 > 2\pi^2/L^2$, there exists a unique $\beta_2 \equiv \beta_2(\omega_0) \in (0, 2\omega_0)$ such that the corresponding cnoidal wave given by (46) has fundamental period $T_{\psi_\omega}(\beta_2; \omega_0) = L$.

In supplement to the above analysis, fixed $L > 0$, we can construct a smooth curve (depending on $\omega$) of cnoidal waves solutions for (45) such that each one of its elements have fundamental period $L$. This is the content of the next theorem.

**Theorem 6.1** Let $L > 2\pi$ be fixed. Choose arbitrarily $\omega_0 > 2\pi^2/L^2$ and consider the unique $\beta_{2,0} = \beta_2(\omega_0) \in (0, 2\omega_0)$ such that

$$L = \frac{2\sqrt{6}}{\sqrt{\rho(\beta_{2,0}; \omega_0)}} \text{K}(k(\beta_{2,0}; \omega_0)).$$

Then,

(i) there exist an interval $J_1(\omega_0)$ around $\omega_0$, an interval $J_2(\beta_{2,0})$ around $\beta_{2,0}$ and a unique smooth function $\Lambda : J_1(\omega_0) \to J_2(\beta_{2,0})$ such that $\Lambda(\omega_0) = \beta_{2,0}$ and

$$L = \frac{2\sqrt{6}}{\sqrt{\rho(\beta_{2,0}; \eta)}} \text{K}(k(\beta_{2,0}; \eta)),$$

where $\eta \in J_1(\omega_0)$, $\beta_2 = \Lambda(\eta)$ and $k(\beta_2; \eta)$, $\rho(\beta_2; \eta)$ are defined in (50) with $\omega_0$ replaced with $\eta$. Moreover, the interval $J_1(\omega_0)$ can be chosen to be the interval $I = (2\pi^2/L^2, +\infty)$ and the modulus $k = k(\eta)$, where

$$k^2(\eta) := \frac{1}{2} + \frac{3(\eta - \Lambda(\eta))}{2\rho(\Lambda(\eta); \eta)}.$$  \hfill(51)

is a strictly increasing function (on the parameter $\eta$).

(ii) For $\omega \in (4\pi^2/L^2, +\infty)$ and $\eta(\omega) = \omega/2$, the cnoidal wave solution $\psi_\omega(\cdot) = \psi_{\eta(\omega)}(\cdot; \beta_2(\eta(\omega)))$ has fundamental period $L$ and satisfies (45). In addition, the mapping

$$\omega \in \left(\frac{4\pi^2}{L^2}, +\infty\right) \mapsto \psi_\omega \in H_{\text{per}}^k([0, L]), \quad k = 0, 1, \ldots$$

is a smooth function.
Sketch of the proof. The proof is an application of the Implicit Function Theorem. Here we give only the main steps (for details see [3]). Define
\[ \Omega = \{ (\beta_2, \eta) \in \mathbb{R}^2; \eta > 2\pi^2/L^2, \beta_2 \in (0, 2\eta) \} \] and \( \Gamma : \Omega \to \mathbb{R} \) by
\[ \Gamma(\beta_2, \eta) = \frac{2\sqrt{6}}{\sqrt{\rho(\beta_2; \eta)}} K(k(\beta_2; \eta)) - L. \]

By our assumptions, we have \( \Gamma(\beta_2, 0, \omega_0) = 0 \). Moreover, taking into account the properties of the complete elliptic integrals and the definitions of \( k \) and \( \rho \) one infers that \( \partial \Gamma / \partial \beta_2 < 0 \) for all \( (\beta_2, \eta) \in \Omega \). So, an application of the Implicit Function Theorem gives us the desired. The fact that \( J_1(\omega_0) \) can be chosen to be \( \mathcal{I} \) follows from the fact that \( \omega_0 \) can be arbitrarily chosen in \( \mathcal{I} \) and the uniqueness of the function arising in the Implicit Function Theorem.

To see that \( k(\eta) \) is a strictly increasing function one just take the derivative with respect to \( \eta \) in (51) and note that \( dk/d\eta > 0 \).

\[ \square \]

Remark 6.1 We have assumed \( L > 2\pi \) in Theorem 6.1 because we want to get a smooth curve of cnoidal wave (defined in an open interval) passing through \( \omega = 1 \). Otherwise, that is, if \( L \leq 2\pi \) then such a curve does not exist.

6.2 Spectral Analysis

To obtain our stability results, we will use the Grillakis, Shatah and Strauss theory [18]. As it is well-known in such approach we need to study the spectrum of some linearized operators.

First, we note that introducing a new variable \( w \) defined by \( v_t = w_x \), system (41) can be written as an Hamiltonian system of the form
\[ \frac{d}{dt}U(t) = JE'(U(t)), \quad (52) \]
where \( U = (P, v, Q, w) \), \( P = \text{Re}(u) \), \( Q = \text{Im}(u) \), \( J \) is the skew-symmetric matrix
\[ J = \begin{pmatrix} 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & \partial_x \\ -1/2 & 0 & 0 & 0 \\ 0 & \partial_x & 0 & 0 \end{pmatrix} \]
(53)
and $\mathcal{E}$ is the energy functional given by

$$
\mathcal{E}(U) = \int_0^L \left\{ P_x^2 + Q_x^2 + \frac{v^2}{2} + \frac{w^2}{2} - v(P^2 + Q^2) \right\} dx.
$$

(54)

Next we will consider the linearized operator we need to study. We first remind that system (41) preserves the $L^2$ norm of $u$ and so, in the above notation,

$$
\mathcal{F}(U) = \int_0^L \{ P^2 + Q^2 \} dx
$$

is a conserved quantity of system (41).

To simplify our exposition, we denote $\Psi_\omega = (\psi_\omega, \psi_\omega, 0, 0)$, where $\psi_\omega$ is a cnoidal wave given in Theorem 6.1. By direct computation we see that $\Psi_\omega$ is a critical point of the functional $\mathcal{E} + \omega \mathcal{F}$ at $\omega = 1$, that is,

$$
\mathcal{E}'(\Psi_1) + \mathcal{F}'(\Psi_1) = 0.
$$

(55)

Now consider the operator

$$
\mathcal{A} := \mathcal{E}''(\Psi_1) + \mathcal{F}''(\Psi_1) = \begin{pmatrix} \mathcal{A}_R & 0 \\ 0 & \mathcal{A}_I \end{pmatrix},
$$

(56)

where $\mathcal{A}_R$ and $\mathcal{A}_I$ are the self-adjoint $2 \times 2$ matrix differential operators defined by

$$
\mathcal{A}_R = \begin{pmatrix} 2(-\partial_x^2 + 1 - \psi_1) & -2\psi_1 \\ -2\psi_1 & -\partial_x^2 + 1 \end{pmatrix}
$$

(57)

and

$$
\mathcal{A}_I = \begin{pmatrix} 2(-\partial_x^2 + 1 - \psi_1) & 0 \\ 0 & 1 \end{pmatrix}.
$$

(58)

Let us study the spectrum of operator $\mathcal{A}$. In what follows, we use the notation $\sigma(\mathcal{L})$ to represent the spectrum of the linear operator $\mathcal{L}$. We first remind that if $\sigma_{\text{ess}}(\mathcal{L})$ and $\sigma_{\text{disc}}(\mathcal{L})$ denote, respectively, the essential and discrete spectra of $\mathcal{L}$, then $\sigma(\mathcal{L}) = \sigma_{\text{ess}}(\mathcal{L}) \cup \sigma_{\text{disc}}(\mathcal{L})$.

To begin our analysis, we observe that since $\mathcal{A}$ is a diagonal operator we have $\sigma(\mathcal{A}) = \sigma(\mathcal{A}_R) \cup \sigma(\mathcal{A}_I)$. Moreover, since $\mathcal{A}$ has a compact resolvent we obtain $\sigma(\mathcal{A}) = \sigma_{\text{disc}}(\mathcal{A})$ (see e.g., [30])

Before studying the spectrum of operators $\mathcal{A}_R$ and $\mathcal{A}_I$, we recall the following lemma

27
Lemma 6.1 Let \( \psi = \psi_1 \) be the cnoidal wave given by Theorem 6.1. Then the following spectral properties hold

(i) Operator
\[
L_1 := -\partial_x^2 + 1 - 2\psi
\]
defined in \( L^2_{\text{per}}([0, L]) \) with domain \( H^2_{\text{per}}([0, L]) \) has exactly one negative eigenvalue which is simple; zero is an eigenvalue which is simple with eigenfunction \( \psi' \). Moreover, the remainder of the spectrum is constituted by a discrete set of eigenvalues.

(ii) Operator
\[
L_2 := -\partial_x^2 + 1 - \psi
\]
defined in \( L^2_{\text{per}}([0, L]) \) with domain \( H^2_{\text{per}}([0, L]) \) has no negative eigenvalues; zero is an eigenvalue, simple with eigenfunction \( \psi \). Moreover, the remainder of the spectrum is constituted by a discrete set of eigenvalues.

Proof. For the first part, see Theorem 4.1 in [3]. The second part follows immediately from Floquet’s theory. Indeed, in view of (44) we have that 0 is an eigenvalue for \( L_2 \) with eigenfunction \( \psi \). Moreover, since \( \psi \) has no zeros in the interval \([0, L]\), 0 must be the first eigenvalue (see e.g. [12, Chapter 3]). ■

With Lemma 6.1 at hands, we are able to prove some spectral properties for operators \( A_R \) and \( A_I \).

Theorem 6.2 Let \( \psi = \psi_1 \) be the cnoidal wave solution given by Theorem 6.1. Then,

(i) operator \( A_R \) in (57) defined in \( L^2_{\text{per}}([0, L]) \times L^2_{\text{per}}([0, L]) \) with domain \( H^2_{\text{per}}([0, L]) \times H^2_{\text{per}}([0, L]) \) has its first three eigenvalues simple, being the eigenvalue zero the second one with eigenfunction \( (\psi', \psi') \). Moreover, the remainder of the spectrum is constituted by a discrete set of eigenvalues.

(ii) Operator \( A_I \) in (58) defined in \( L^2_{\text{per}}([0, L]) \times L^2_{\text{per}}([0, L]) \) with domain \( H^2_{\text{per}}([0, L]) \times L^2_{\text{per}}([0, L]) \) has no negative eigenvalues; zero is the first eigenvalue which is simple with eigenfunction \( (\psi, 0) \). Moreover, the remainder of the spectrum is constituted by a discrete set of eigenvalues.
Proof. (i) First we observe that from (45) it is easy to see that zero is an eigenvalue with eigenfunction \((\psi', \psi')\). Now we consider the quadratic form associated with \(A_R\). Let \(Y = H^1_{\text{per}}((0, L]) \times H^1_{\text{per}}((0, L])\). Then, for \((f, g) \in Y\),

\[
Q_R(f, g) := \langle A_R(f, g), (f, g) \rangle
\]

\[
= \int^L_0 \{2(-\partial^2_x + 1 - \psi)f^2 - 4\psi fg + (-\partial^2_x + 1)g^2\} \, dx
\]

\[
= 2\langle L_1 f, f \rangle + \langle L_1 g, g \rangle + 2\int^L_0 \psi(f - g)^2 \, dx.
\]

(59)

In order to prove that \(A_R\) has at least one negative eigenvalue, let us prove that there exists a pair \((f, g) \in Y\) such that \(Q_R(f, g) < 0\). Indeed, from Lemma 6.1 there exist \(\mu_0 < 0\) and \(f_0 \in H^2_{\text{per}}([0, L])\) satisfying \(L_1 f_0 = \mu_0 f_0\) and so that \(\langle L_1 f_0, f_0 \rangle < 0\). Thus, by choosing \(f = g = f_0\), we obtain from (59),

\[
Q_R(f_0, f_0) = 3\langle L_1 f_0, f_0 \rangle < 0.
\]

This implies that the first eigenvalue of \(A_R\), say \(\lambda_1\), is negative. Now we will prove that the next eigenvalue is the zero one. To do so, we will use the min-max characterization of eigenvalues (see e.g., [30, Theorem XIII.1]). Thus, if \(\lambda_2\) denotes the second eigenvalue of \(A_R\), we have

\[
\lambda_2 = \max_{\langle \phi_1, \phi_2 \rangle \in Y} \min_{\substack{(f, g) \in Y \setminus \{(0, 0)\} \atop f \perp \phi_1, g \perp \phi_2}} \frac{Q_R(f, g)}{\| (f, g) \|_Y^2}.
\]

(60)

By taking \(\phi_1 = \phi_2 = f_0\), we see that

\[
\lambda_2 \geq \min_{\substack{(f, g) \in Y \setminus \{(0, 0)\} \atop f \perp f_0, g \perp f_0}} \frac{Q_R(f, g)}{\| (f, g) \|_Y^2}.
\]

Now, if \(f \perp f_0\) and \(g \perp f_0\) we obtain \(\langle L_1 f, f \rangle + \langle L_1 g, g \rangle \geq 0\) (recall that Lemma 6.1 implies that \(L_1\) has a unique negative eigenvalue). Moreover, since \(\psi\) is a strictly positive function (and thus, the last integral in (59) is non-negative) we obtain \(Q_R(f, g) \geq 0\), which implies \(\lambda_2 \geq 0\).

Finally, to prove that the third eigenvalue is strictly positive, we use the min-max principle again, taking into account that \(L_1\) has a unique negative eigenvalue and zero is a simple eigenvalue. This proves part (i).
In this case, if \( Q_I \) denotes the quadratic form associated with \( A_I \), we have
\[
Q_I(f, g) := \langle A_I(f, g), (f, g) \rangle
\]
\[
= \int_0^L \left\{ 2(-\partial_x^2 + 1 - \psi)f^2 + g^2 \right\} dx
\]
\[
= 2\langle L_2 f, f \rangle + \|g\|^2.
\]

Therefore, since \( L_2 \) has no negative eigenvalue (see Lemma 6.1) we have \( \langle L_2 f, f \rangle \geq 0 \) and then from (61) we deduce \( Q_I(f, g) \geq 0 \). This implies that \( A_I \) has no negative eigenvalue. Moreover, it is easy to see from (45) that zero is an eigenvalue with eigenfunction \((\psi, 0)\). This completes the proof of the theorem.

6.3 Orbital stability

In this subsection we prove our orbital stability result for the periodic wave \((e^{it}\psi, \psi)\), where \( \psi = \psi_1 \) is the cnoidal wave given in Theorem 6.1. To make clear our notion of orbital stability, we point out that system (41) has translation and phase symmetries, i.e., if \((u(x, t), v(x, t))\) is a solution for (41), so is
\[
(e^{i\theta}u(x + x_0, t), v(x + x_0, t)),
\]
for any \( \theta, x_0 \in \mathbb{R} \). Thus, our notion of orbital stability will be modulus such symmetries. To be more precise, we have the following definition

**Definition 6.1** A standing wave solutions for (41) of the form \((e^{i\omega t}\psi_\omega(x), \phi_\omega(x))\), is said to be orbitally stable in \( X = H^1_{\text{per}}([0, L]) \times H^1_{\text{per}}([0, L]) \times L^2_{\text{per}}([0, L]) \) if for any \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that if \((u_0, v_0, v_1)\) satisfies \( \| (u_0, v_0, v_1) - (\psi_\omega, \phi_\omega, 0) \|_X < \delta \), then the solution \( \tilde{u}(t) = (u, v, v_t) \) of (41) with \( \tilde{u}(0) = (u_0, v_0, v_1) \) exists for all \( t \) and satisfies
\[
\sup_{t \geq 0} \inf_{s, y \in \mathbb{R}} \| \tilde{u}(t) - (e^{is\psi_\omega(\cdot + y), \phi_\omega(\cdot + y), 0}) \|_X < \varepsilon.
\]

Otherwise, \((e^{i\omega t}\psi_\omega(x), \phi_\omega(x))\) is said to be orbitally unstable in \( X \).

From Theorem 6.2 we obtain the following properties

(i) The operator \( A \) has exactly one negative eigenvalue, that is, the negative eigenspace of \( A \), say \( N \), is one-dimensional.
(ii) For \( \overrightarrow{f} = (\psi', \psi', 0, 0) \) and \( \overrightarrow{g} = (0, 0, \psi, 0) \), the set \( \mathcal{Z} = \{ r_1 \overrightarrow{f} + r_2 \overrightarrow{g} \mid r_1, r_2 \in \mathbb{R} \} \) is the kernel of operator \( A \).

(iii) There exists a closed subspace, say \( \mathcal{P} \), such that \( \langle Au, u \rangle \geq \delta_0 \|u\|_X \) for all \( u \in \mathcal{P} \) and some \( \delta_0 > 0 \).

Therefore, from (i)–(iii) we obtain the following orthogonal decomposition for \( X_{\mathbb{R}} = \mathcal{N} \oplus \mathcal{Z} \oplus \mathcal{P} \):

\[
X_{\mathbb{R}} = \mathcal{N} \oplus \mathcal{Z} \oplus \mathcal{P}.
\] (63)

Next, for \( \omega \in \mathcal{I} = (4\pi^2/L^2, +\infty) \) and \( \psi_\omega \) the cnoidal wave given by Theorem 6.1 we define \( d : \mathcal{I} \to \mathbb{R} \) by

\[
d(\omega) = E(\Psi_\omega) + \omega F(\Psi_\omega),
\] (64)

where, as before, \( \Psi_\omega = (\psi_\omega, \psi_\omega, 0, 0) \).

In the present setting, our orbital stability result in Theorem 1.6 can be rephrased as follows

**Theorem 6.3** Let \( \psi = \psi_1 \) be the cnoidal wave given in Theorem 6.1. Then, the periodic traveling wave \((e^{it}\psi, \psi)\) is orbitally stable in space \( X \).

**Proof.** Since the initial value problem associated with (41) is globally well-posed in \( X \) (see Theorem 1.5), \( \Psi_1 = (\psi_1, \psi_1, 0, 0) \) satisfies (55), \( X_{\mathbb{R}} \) admits the decomposition (63) and \( \mathcal{N} \) is one-dimensional, the proof of the theorem follows from the Abstract Stability Theorem in Grillakis, Shatah and Strauss [18], provided we are able to show that \( d''(\omega) > 0 \), where \( d \) is the function defined in (64). This was essentially proved in [3], but for the sake of completeness we bring here the main steps. From direct computation, we obtain \( d''(\omega) = \mathcal{F}(\Psi_\omega) \). Thus,

\[
d''(\omega) = \frac{d}{d\omega} \left( \int_0^L \psi_\omega(x) dx \right).
\]

But integrating (45) over \([0, L]\), we get

\[
\int_0^L \psi_\omega^2(x) dx = \omega \int_0^L \psi_\omega(x) dx.
\]

Then, for the positivity of \( d''(\omega) \) it suffices to show that function

\[
G(\omega) = \omega \int_0^L \psi_\omega(x) dx
\]
is strictly increasing.

In what follows we replace (up to a multiplicative positive constant) \( \eta \) with \( \omega \) in the definition of \( k \) and \( \rho \) in Theorem 6.1. Using that

\[
\int_0^K \left[ \frac{E(k) - (1 - k^2)K(k)}{k^2} \right] \, dx,
\]

(where \( E(k) \) is the complete elliptic integral of the second kind) \( L = 2\sqrt{6}K/\sqrt{\beta_3 - \beta_1} \) and \( k^2 = (\beta_3 - \beta_2)/(\beta_3 - \beta_1) \), we deduce

\[
\int_0^L \psi_\omega(x) \, dx = \beta_2 L + 24 \frac{K}{L} \left[ E - (1 - k^2)K \right].
\]

Moreover, in view of the definitions of \( k \) and \( \rho \), we infer that

\[
\beta_2 = \frac{8K^2}{L} \left[ \sqrt{\kappa^4 - k^2} + 1 - 2k^2 \right].
\]

As a consequence,

\[
\int_0^L \psi_\omega(x) \, dx = \frac{8K^2}{L} \left[ \sqrt{\kappa^4 - k^2} + 1 - 2 + k^2 \right] + 24 \frac{KE}{L} \equiv H(k(\omega)).
\]

Finally,

\[
\frac{d}{d\omega} G(\omega) = \int_0^L \psi_\omega(x) \, dx + \omega \frac{dH}{dk} \frac{dk}{d\omega} > 0,
\]

where we have used that \( k \mapsto H(k) \) is a strictly increasing function and \( dk/d\omega > 0 \) (see Theorem 6.1). This completes the proof of the theorem.

6.4 Existence and Stability of non-explicit solutions

In Subsection 6.1 we proved that system (43) admits a periodic wave solution for \( \omega = 1 \) and \( \psi_\omega = \phi_\omega \), where \( \psi_\omega \) is given explicitly by the formula in (46). The advantage in that case is the reduction of system (43) to a single ordinary differential equation. However, one can naturally ask if the system also admits a periodic solution for \( \omega \neq 1 \). In this regard, we shall prove that for \( \omega \) sufficiently close to 1, system (43) does admit an even periodic solution such that at \( \omega = 1 \) this solution is the aforementioned one. We shall employ the Implicit Function Theorem combined with the spectral results given in Theorem 6.2.
Let $H^s_{\text{per},e}([0,L])$ be the subspace of $H^s_{\text{per}}([0,L])$ constituted by the even distributions and denote $X_e = H^2_{\text{per},e}([0,L]) \times H^2_{\text{per},e}([0,L])$ and $Y_e = L^2_{\text{per},e}([0,L]) \times L^2_{\text{per},e}([0,L])$. Define the function $\Phi : \mathbb{R} \times X_e \to Y_e$ by

$$\Phi(\omega, \psi, \phi) = (-\psi'' + \omega \psi - \psi \phi'' + \phi - \psi^2).$$

In view of Theorem 6.1 we deduce that $\Phi(1, \psi_1, \psi_1) = (0, 0)$. Moreover, if $\Phi_{(\psi, \phi)}$ denotes the Fréchet derivative of $\Phi$ at $(\psi, \phi)$, it is easy to check that

$$\Phi_{(\psi, \phi)}(\omega, \psi, \phi) = \begin{pmatrix} -\frac{\partial^2}{\partial x^2} + \omega - \phi & -\psi \\ -2\psi & -\frac{\partial^2}{\partial x^2} + 1 \end{pmatrix}.$$ 

Thus, at $\omega = 1$ and $\psi = \phi = \psi_1$, we obtain

$$\mathcal{B} := \Phi_{(\psi, \phi)}(1, \psi_1, \psi_1) = \begin{pmatrix} -\frac{\partial^2}{\partial x^2} + 1 - \psi_1 & -\psi_1 \\ -2\psi_1 & -\frac{\partial^2}{\partial x^2} + 1 \end{pmatrix}.$$ 

Let us prove that $\mathcal{B}$ is a bijection from $X_e$ into $Y_e$. In fact, it is sufficient to show that $0$ does not belong to $\sigma(\mathcal{B})$. An elementary calculation shows us that $(f, g) \in \text{Ker}(\mathcal{B})$ if and only if $(f, g) \in \text{Ker}(\mathcal{A}_R)$, where $\mathcal{A}_R$ is the operator given by (57). But, from Theorem 6.2 we have $\text{Ker}(\mathcal{A}_R) = [(\psi_1', \psi_1')]$ (as an operator on $L^2_{\text{per},e}([0,L]) \times L^2_{\text{per},e}([0,L])$). However, since $\psi_1$ is an even function, it follows that $\psi_1' \notin L^2_{\text{per},e}([0,L])$ and so $0 \notin \sigma(\mathcal{B})$ (as an operator on $Y_e$).

Consequently, from the Implicit Function Theorem there exist an $\varepsilon > 0$ and a unique smooth function $F : (1 - \varepsilon, 1 + \varepsilon) \to X_e$,

$$F(\omega) = (\psi_\omega, \phi_\omega),$$

such that $F(1) = (\psi_1, \psi_1)$ and $\Phi(\omega, F(\omega)) = (0, 0)$, for all $\omega \in (1 - \varepsilon, 1 + \varepsilon)$, that is, the pair $(\psi_\omega, \phi_\omega)$ is a solution of the system (43).

**Remark 6.2** The periodic solution we found here are also orbitally stable. This can be proved by using classical perturbation theory (see [22]) to show that the linearized operators arising in this context have the same spectral properties as those ones in Theorem 6.2 (for related references see e.g. [4], [29] and references therein).

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