The Fuzzy Kähler Coset Space
by the Fedosov Formalism

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March 27, 2022

Abstract

We discuss deformation quantization of the Kähler coset space by using the Fedosov formalism. We show that the Killing potentials of the Kähler coset space satisfy the fuzzy algebrae, when the coset space is irreducible.
Quantum field theory on a non-commutative space has recently raised much interest mainly due to the connection with the string theory[1]. There appears non-commutativity of the flat target space via a constant $B$-field. The ordinary product of fields in the effective theory is replaced by the non-commutative one, called the Moyal product. A lot of works have been done on the physics caused by this non-commutativity of the flat target space[2]. It is of obvious interest to generalize the studies to the string theory with a non-constant $B$-field and a curved target space[3]. Then the Moyal product should be replaced by a more general non-commutative one, called the $\star$ product.

Generalization of the Moyal product has a long history[4] in the study of deformation quantization on symplectic manifolds or more generally Poisson manifolds. Several definitions of the $\star$ product exist in the literature. However it has been discussed in the physical context in few references. The work by Cattaneo and Felder[5] which gave a quantum field theoretical interpretation to Kontsevich’s $\star$ product[6] is one of them. The recent work by Kishimoto[7] was also done from motives for physical application. Namely he explicitly constructed the $\star$ product on the fuzzy sphere $S^2$ by using the Fedosov formalism for deformation quantization[8]. With the explicit form of the $\star$ product it was shown that there exist the fuzzy sphere algebrae[9], which may be useful for studying non-commutative non-linear $\sigma$-models.

In this paper we generalize his arguments to the fuzzy Kähler coset space $G/H$. The key observation is that the ordinary Kähler coset space has a set of the Killing potentials $M^A(z, \overline{z})$ satisfying

$$\sum_{A=1}^{\dim G} M^A(z, \overline{z})M^A(z, \overline{z}) = R. \tag{1}$$

Here $R$ is the Riemann scalar curvature of $G/H$, which is a constant. We study the $\star$ product by the Fedosov formalism on the Kähler coset space $G/H$ and show that the Killing potentials satisfy the fuzzy algebrae

$$M^A(z, \overline{z}) \star M^B(z, \overline{z}) = M^B(z, \overline{z}) \star M^A(z, \overline{z}) = R + c_2\hbar^2 + c_4\hbar^4 + \cdots, \tag{2}$$

when the coset space is irreducible. The coefficients $c_1, c_2, c_3, \cdots$ are numerical constants. As a demonstration, they will be calculated to order of $\hbar^4$ for the case of $CP^1$.

We start with a brief review on the Fedosov formalism for deformation quantization[8]. We consider a real $2N_0$-dimensional Riemann manifold $\mathcal{M}$ with local coordinates $x^a = (x^1, x^2, \cdots, x^{2N_0})$. The line element of the manifold is given by

$$ds^2 = g_{ij}dx^idx^j. \tag{3}$$

Suppose that it is endowed with a symplectic structure given by a non-degenerate 2-form

$$\omega = \frac{1}{2}\omega_{ij}dx^i \wedge dx^j. \tag{4}$$
We require \( \omega_{ij} \) to be covariantly constant:

\[
\omega_{ij;k} = 0. \tag{5}
\]

Then (4) is closed, \( d\omega = 0 \). \( \omega_{ij} \) is inverted by \( \omega^{ij} \):

\[
\omega_{ik}\omega^{kj} = \omega^{ik}\omega_{ki} = \delta_i^j. \tag{6}
\]

For a differential \( q \)-form on \( \mathcal{M} \)

\[
a_0 = \frac{1}{q!} a(x)_{j_1j_2\cdots j_q} \theta^{j_1}\theta^{j_2} \cdots \theta^{j_q}, \quad \theta^i \equiv dx^i,
\]

we consider a deformed \( q \)-form \[\]

\[
a = \sum_{p=0}^{\infty} \frac{1}{p!q!} a(x)_{i_1i_2\cdots i_pj_1j_2\cdots j_q} y^{i_1}y^{i_2} \cdots y^{i_p}\theta^{j_1}\theta^{j_2} \cdots \theta^{j_q},
\]

in which \( y^i = (y^1, y^2, \cdots, y^{2N_0}) \) are deformation variables. The coefficient \( a(x)_{i_1i_2\cdots i_pj_1j_2\cdots j_q} \)

is a covariant tensor which is symmetric in \( i_1, i_2, \cdots i_p \) and anti-symmetric in \( j_1, j_2, \cdots j_q \). We define various operations for the deformed differential form:

\[
a \circ b = \sum_{n=0}^{\infty} \frac{1}{n!} (-\frac{i\hbar}{2})^n \omega^{i_1j_1} \omega^{j_2j_2} \cdots \omega^{i_nj_n} \partial_{i_1} \partial_{j_1} \cdots \partial_{i_n} a \partial_{j_1} \partial_{j_2} \cdots \partial_{j_n} b, \tag{7}
\]

\[
\partial a = \sum_{p=0}^{\infty} \frac{1}{p!q!} \theta^i \nabla^x_i a(x)_{i_1i_2\cdots i_pj_1j_2\cdots j_q} y^{i_1}y^{i_2} \cdots y^{i_p}\theta^{j_1}\theta^{j_2} \cdots \theta^{j_q}, \tag{8}
\]

\[
\delta a = \theta^i \frac{\partial}{\partial y^i} a, \tag{9}
\]

\[
\delta^{-1} a = \sum_{p=0}^{\infty} \frac{1}{p+q} y^i \frac{\partial}{\partial \theta^i} \left[ \frac{1}{p!q!} a(x)_{i_1i_2\cdots i_pj_1j_2\cdots j_q} y^{i_1}y^{i_2} \cdots y^{i_p}\theta^{j_1}\theta^{j_2} \cdots \theta^{j_q} \right]. \tag{10}
\]

In the \( \circ \)-product (7) \( \hbar \) is a deformation parameter. To \( \hbar \) and \( y^i \) we assign degrees 2 and 1 respectively. Then the \( \circ \) product preserves the degree. In the Fedosov formalism the graded commutator is defined with this \( \circ \) product:

\[
[a, b] = a \circ b - (-1)^{q_aq_b} b \circ a,
\]

in which \( q_a \) and \( q_b \) are the rank of the differential forms \( a \) and \( b \) respectively. In (8) \( \nabla^x_i \)

is the covariant derivative with respect to \( x \). The graded differentials \( \partial, \delta \) and \( \delta^{-1} \) are shown to satisfy the relations

\[
\delta^2 = \delta^{-2} = 0,
\]

\[
(\delta^{-1}\delta + \delta\delta^{-1})a = a - a_0, \tag{11}
\]

\[
\delta\partial + \partial\delta = 0,
\]

\[
\partial^2 a = \frac{i}{\hbar}[\mathcal{R}, a], \tag{12}
\]

As in ref. [8] we may consider a more general deformed form summing over the rank \( q \). But we here make a simplification to avoid unnecessary complications for our scope.
with
\[ R = \frac{1}{4} R_{ij}^k \omega_m \theta^j \theta^i y^k y^l. \] (13)

Here \( R_{ij}^k \) is the Riemann curvature. \(^2\)

We define a more general derivative by
\[ Da = \partial a - \delta a + \frac{i}{\hbar} [r, a], \]
with an appropriate deformed 1-form \( r \). A simple calculation gives
\[ D^2 a = \frac{i}{\hbar} [\Omega, a], \]
with
\[ \Omega = -\omega + R - \delta r + \partial r + \frac{i}{2\hbar} [r, r]. \]

If \( r \) satisfies the equation
\[ \delta r = R + \partial r + \frac{i}{2\hbar} [r, r], \] (14)
we have \( \Omega = -\omega \) so that \( D^2 a = 0 \) for any \( a \). Eq. (14) has a unique solution \( r \) obeying the condition
\[ \delta^{-1} r = 0. \] (15)

Namely owing to (11) and (15), eq. (14) may be written in the form
\[ r = r_0 + \delta^{-1} (\partial r + \frac{i}{2\hbar} [r, r]), \] (16)
with \( r_0 = \delta^{-1} R. \) Note that \( \text{deg } r_0 = 3 \) and \( \delta^{-1} \) raises the degree at least by 1. Hence we can solve (16) by iteration, expanding \( r \) in series
\[ r = r_0 + r_1 + r_2 + \cdots, \] (17)
with \( \text{deg } r_n = n + 3. \) Conversely the iterative solution satisfy (15) by the construction. This solution \( r \) plays a particular role in the Fedosov formalism. Provided with it, we may impose the constraint on \( a, Da = 0 \), i.e.,
\[ \delta a = \partial a + \frac{i}{\hbar} [r, a]. \] (18)

When \( a \) is a 0-form, one can similarly show that eq. (18) has a unique solution obeying the condition
\[ \delta^{-1} a = 0. \] (19)
To this end write eq. (18) in the form
\[ a = a_0 + \delta^{-1}(\partial a + \frac{i}{\hbar}[r, a]). \]  
(20)

With \( a_0 \) as an initial condition the solution in the form
\[ a = a_0 + a_1 + a_2 + \cdots, \]
with \( \text{deg } a_n = n \), may be obtained by iteration. It is obvious that the iterative solution satisfies the condition (19) because \( \delta^{-1}a_0 = 0 \). So the constraint (18) gives a unique way to deform the function \( a_0(x) \). For \( a \) and \( b \) thus deformed we define the \( \star \) product by
\[ a_0(x) \star b_0(x) = \sum_{n=0}^{\infty} \frac{i^n}{n!} \omega^{i_1j_1} \omega^{i_2j_2} \cdots \omega^{i_nj_n} \partial \frac{\partial}{\partial y^{i_1}} \partial \frac{\partial}{\partial y^{i_2}} \cdots \partial \frac{\partial}{\partial y^{i_n}} a \partial \frac{\partial}{\partial y^{j_1}} \partial \frac{\partial}{\partial y^{j_2}} \cdots \partial \frac{\partial}{\partial y^{j_n}} b \Bigg|_{y=0}. \]  
(21)

According to Fedosov\[8\] this \( \star \) product satisfies the associativity. It reduces to the Moyal product when the manifold \( \mathcal{M} \) is flat.

We shall apply the Fedosov formalism in the case where \( \mathcal{M} \) is the Kähler manifold. The Kähler manifold has local complex coordinates \( z^\alpha = (z^1, z^2, \ldots, z^{N_0}) \) and their complex conjugates. The line element (3) and the symplectic 2-form (4) respectively reduce to
\[ ds^2 = g_{\alpha\beta} dz^\alpha d\overline{z}^\beta, \]
\[ \omega = \frac{1}{2} \omega_{\alpha\beta} d\alpha \wedge d\overline{\beta}, \]  
(22)

with
\[ \omega_{\alpha\overline{\beta}} = ig_{\alpha\overline{\beta}}. \]  
(23)

From (3) we have
\[ \omega^{\alpha\overline{\beta}} = \frac{1}{2} \omega_{\gamma\alpha} \overline{g}^{\gamma\overline{\beta}}. \]  
(24)

In these complex coordinates the Riemann-Christophel symbols are given by
\[ \Gamma_{\alpha\beta}^\gamma = g^{\gamma\overline{\eta}} g_{\alpha\overline{\sigma}, \beta}, \quad \Gamma_{\overline{\alpha} \overline{\beta}}^\overline{\eta} = g^{\overline{\rho}} g_{\overline{\rho} \overline{\sigma}, \overline{\beta}}, \]
and other components are vanishing, due to \( d\omega = 0 \). The non-trivial components of the Riemann curvature take the forms
\[ R_{\alpha\overline{\beta} \gamma}^\delta = R_{\alpha\overline{\beta} \gamma}^\delta g^{\overline{\eta}} = \Gamma_{\alpha\beta}^\gamma \overline{\rho} \overline{\rho}, \]
and satisfy the symmetry
\[ R_{\alpha\beta \gamma}^\delta = -R_{\beta\alpha \gamma}^\delta = -R_{\overline{\alpha} \overline{\beta} \gamma}^\delta = R_{\gamma \alpha \overline{\beta}}^\delta = R_{\alpha \beta \gamma}^\delta. \]
Consequently eq. (13) reads
\[ R = \frac{1}{2} (R_{\alpha \beta \gamma}^\eta \omega_{\eta \theta} + R_{\alpha \beta \theta}^\eta \omega_{\eta \gamma}) \theta^\alpha \theta^\beta y^\gamma y^\delta \]
\[ = i R_{\alpha \beta \gamma}^\eta \theta^\alpha \theta^\beta y^\gamma y^\delta, \]
(25)
by (23). It is worth checking that the last formula indeed satisfies (12) with (24).

When the Kähler manifold is a coset space $G/H$, the Fedosov formalism is further simplified. Here we summarize the useful properties of the Kähler coset space for our discussion. There exists a set of holomorphic Killing vectors
\[ R^A = (R_1^a, R_2^a, \ldots, R^D_a), \]
with $D = \dim G$, which represents the isometry $G$:
\[ R^{ab} R^{ca} - R^{bc} R^{a} = f^{ABC} R^C. \]
(26)
With $R^A = g_{a\beta} R^{A\beta}$ and $R^A = g_{a\alpha} R^{A\alpha}$ they satisfy the Killing equation
\[ R^A_{\alpha, \beta} + R^A_{\beta, \alpha} = 0. \]
(27)
They also satisfy
\[ R^A_{\alpha, \beta} = 0, \quad c.c., \]
(28)
from the holomorphic property. From (27) we may find real scalars $M^A(\phi, \bar{\phi})$, called Killing potentials\[10\], such that
\[ R^A_{\alpha} = i M^A_{,\alpha}, \quad R^A_{\alpha} = -i M^A_{,\alpha}. \]
(29)
These Killing potentials transform as the adjoint representation of the group $G$ by the Lie-variation
\[ L_{R^A} M^B = R^A M^B_{,\alpha} + R^{A\alpha} M^B_{,\alpha} = f^{ABC} M^C, \]
(30)
which may be written as
\[ R^A R^B_{,\alpha} - R^B R^A_{,\alpha} = i f^{ABC} M^C, \]
(31)
by (29). A manipulation of (30) with (29) leads us to write the Killing potentials in terms of the Killing vectors:
\[ M^A = -\frac{i}{N_{adj}} f^{ABC} R^{B\alpha} R^{C\beta} g_{\alpha\beta}. \]
(32)
Here we have used the normalization
\[ f^{ABC} f^{ABD} = 2 N_{adj} \delta^{CD}. \]
When the Kähler coset space $G/H$ is irreducible, there are relations between $M^A$ and the geometric quantities [1]:

- **Scalar curvature**: $M^A M^A = R$
  \[ (= - R_{\alpha, \beta, \gamma} g^{\alpha \beta} g^{\gamma \delta} = \text{const.}, \quad (33) \]

- **Metrics**: $M^A_{\alpha, \beta} M^A_{\gamma, \delta} = g_{\alpha \beta}$, \quad $M^A_{, \alpha} M^A_{, \beta} = 0$, \quad $M^A_{, \alpha, \beta} M^A = - g_{\alpha \beta}$, \quad c.c., \quad (34)

- **Riemann curvature**: $M^A_{, \alpha, \beta} M^A_{, \gamma, \delta} = - R_{\alpha \beta, \gamma}^\delta$. \quad (35)

In addition to these we later need the following formula:

\[ \nabla_\eta R_{\alpha, \beta, \gamma}^\delta = 0. \quad (36) \]

To show the formula note that

\[ \nabla_\eta M^A_{, \alpha, \beta} = \nabla_\eta \nabla_\beta M^A_{, \alpha} = R_{\eta \beta, \alpha}^\eta M^A_{, \alpha}, \]

by (28) and (29). Using this and (35) we have

\[ \nabla_\eta R_{\alpha, \beta, \gamma}^\delta = - R_{\eta \beta, \alpha}^\eta M^A_{, \alpha} M^A_{, \gamma, \delta} - R_{\eta \gamma, \alpha}^\eta M^A_{, \alpha, \beta} M^A_{, \gamma}. \]

Each term of the r.h.s. vanishes because

\[ M^A_{, \alpha} M^A_{, \gamma, \delta} = g_{\alpha \gamma} R^{A\bar{\gamma}} R^A_{\beta, \delta} = g_{\alpha \gamma} (R^{A\bar{\gamma}} R^A_{\beta, \delta})_{, \gamma} = 0, \]

by the holomorphic property of $R^{A\bar{\gamma}}$ and (34).

Equipped with these formulae let us study the $*$-product for the irreducible Kähler coset space, defined by (21), and prove our claims (1) and (2). First of all we solve eq. (16) for $r$. The first key observation is that due to (26) the iterative solution (17) does not get contribution from $\partial \bar{r}$. Hence it turns out to take the form

\[ r = \sum_{N=11}^{2N} \sum_{n=1}^{2N} h^{2N-n} [R \otimes \cdots \otimes R]_{\alpha_0 \alpha_2 \cdots \alpha_n \beta_1 \beta_2 \cdots \beta_n} \theta^\alpha_0 g^{\alpha_1} y^{\alpha_2} \cdots g^{\alpha_n} y^{\beta_1} y^{\beta_2} \cdots y^{\beta_n}, \]

\[ -c.c. \quad (37) \]

Here $[R \otimes \cdots \otimes R]_{\alpha_0 \alpha_2 \cdots \alpha_n \beta_1 \beta_2 \cdots \beta_n}$ is a symbolic notation for summing over all the types of the tensor product of $N$ Riemann curvatures, contracted $(2N - n)$ times by $g^{\alpha \beta}$, but with no self-contraction like $R_{\alpha, \beta, \gamma} g^{\delta \gamma}$. For the sum we understand appropriate coefficients obtained by iterating eq. (14). Next we deform $M^A$ according to (18) with the solution (37). That is, we solve eq. (24) for $a$ with the initial condition $a_0 = M^A$. Observe that due to (36) the iterative solution consists of three types of quantities:

**Type 1**: \[ h^{2N-n} [R \otimes \cdots \otimes R \otimes M^A]_{\alpha_1 \alpha_2 \cdots \alpha_n \beta_1 \beta_2 \cdots \beta_n} y^{\alpha_1} y^{\alpha_2} \cdots y^{\alpha_n} y^{\beta_1} y^{\beta_2} \cdots y^{\beta_n}, \]
Type 2:
\[ h^{2N-n} [R \otimes \cdots \otimes R \otimes \nabla M^A]_{\alpha_1 \alpha_2 \cdots \alpha_{n+1} \beta_1 \beta_2 \cdots \beta_n} \bar{y}^{\alpha_1} y^{\alpha_2} \cdots y^{\alpha_{n+1}} \bar{y}_1 \bar{y}_2 \cdots \bar{y}_n \]
\[ + \text{c.c.,} \]
(degree = 4N),

Type 3:
\[ h^{2N-n} [R \otimes \cdots \otimes R \otimes \nabla \nabla M^A]_{\alpha_1 \alpha_2 \cdots \alpha_{n+1} \beta_1 \beta_2 \cdots \beta_{n+1}} \bar{y}^{\alpha_1} y^{\alpha_2} \cdots y^{\alpha_{n+1}} \bar{y}_1 \bar{y}_2 \cdots \bar{y}_{n+1} \]
\[ (\text{degree} = 4N + 1), \]

The quantities with higher covariant derivatives of \( M^A \) always reduce to the Type 2, or zero, for instance
\[ \nabla_\alpha \nabla_\beta \nabla_\gamma M^A = R_{\alpha \beta \gamma}^\delta \nabla_\delta M^A, \quad \nabla_\alpha \nabla_\beta \nabla_\gamma M^A = 0, \]

by (28). By using the deformed Killing potentials we calculate the l.h.s of (1), denoted by \( [M^A, M^B]_\ast \). All tensor indices in the \( \odot \) product are contracted by \( g_{\alpha \beta} \). There is no contribution from an even number of contractions. So it suffices to calculate the \( \odot \) product of the quantities of Type 2:
\[ [M^A, M^B]_\ast = h^{2(N+N') + 1} \sum_{N,N'} [R \otimes \cdots \otimes R]_\beta^\alpha (R^{A\alpha} R^{B\beta} - R^{B\alpha} R^{A\beta}), \]

by (29). If we have a formula such that
\[ [R \otimes \cdots \otimes R]_\beta^\alpha = \text{const.} \delta_\beta^\alpha, \]

then the claim (1) is shown by (31). If given the formula (38), we can also prove the claim (2). This time an odd number of contractions do not contribute to \( M^A \ast M^A \). Therefore we have
\[ M^A \ast M^A = h^{2(N+N')} \sum_{N,N'} [R \otimes \cdots \otimes R \otimes M^A \otimes M^A]_{1\alpha}^\alpha \]
\[ + h^{2(N+N')+2} \sum_{N,N'} [R \otimes \cdots \otimes R \otimes \nabla \nabla M^A \otimes \nabla \nabla M^A]_{1\alpha}^\alpha. \]

By (23) and (35) it reduces to
\[ M^A \ast M^A = h^{2(N+N')} \sum_{N,N'} [R \otimes \cdots \otimes R]_{1\alpha}^\alpha \cdot \text{const.} \]
\[ - h^{2(N+N')+1} \sum_{N,N'} [R \otimes \cdots \otimes R]_{1\alpha}^\alpha. \]
So the claim (2) is proved by the formula (38).

Let us now show the formula. To this end we remind of the method for constructing the Kähler coset space $G/H$ [12]. We consider the irreducible case. Then the group $G$ have generators $T^A = \{X_\alpha, \overline{X}^\beta, H^i, Y\}$ which satisfy the Lie-algebra

\[
[X_\alpha, \overline{X}^\beta] = p(\Gamma^i)_{\beta}^\alpha H^i + q\delta_\beta^\alpha Y,
[X_\alpha, X_\beta] = 0,
[X_\alpha, H^i] = (\Gamma^i)_\alpha X_\beta,
[X_\alpha, Y] = X_\alpha, \text{ c.c.,}
\]

with some constants $p$ and $q$ depending on the representation of $G$. Here $X_\alpha$ and $\overline{X}^\beta$ are coset generators. In the method of ref. [12] the local coordinates of $G/H$ are denoted by $z^\alpha$ and $\overline{z}^\beta$, where upper or lower indices stand for complex conjugation. Therefore raising or lowering tensor indices in the foregoing discussions is done by writing the metrics $g^\beta_\alpha$ or $(g^{-1})^\beta_\alpha$ explicitly. Simple algebra gives

\[
[X_\alpha, [X_\gamma, \overline{X}^\beta]] = \{p(\Gamma^i)_{\beta}^\alpha (\Gamma^i)_{\gamma}^\delta + q\delta_\alpha^\delta \delta_\gamma^\beta \}X_\delta 
\equiv M^{\beta\delta}_{\alpha\gamma} X_\delta
\]

(40)

The Killing vectors $R^{A\alpha}$ ($R^{A\overline{\gamma}}$) are nothing but non-linear realizations of the Lie-algebra (39) on $z^\alpha$ ($\overline{z}^\gamma$):

\[
R^{A\alpha} \equiv -i\{T^A, z_\alpha\}, \text{ c.c.}
\]

Then the Lie-algebra (26) is equivalent to the Jacobi identities

\[
[T^A, [T^B, z_\alpha]] - [T^B, [T^A, z_\alpha]] = [[T^A, T^B], z_\alpha].
\]

By examining these Jacobi identities the Killing vectors were shown to take the forms

\[
R^{\gamma}_{\alpha} = i\delta_{\alpha}^\gamma, \quad R_{\gamma\alpha} = iM^{\beta\delta}_{\alpha\gamma} z_\beta z_\delta,
R^{i}_{\alpha} = i(\Gamma^i)_{\alpha}^\gamma z_\beta, \quad R_{\alpha} = iz_\alpha,
\]

owing to the formulae

\[
M^{\beta\gamma}_{\alpha\gamma} = M^{\beta\delta}_{\gamma\alpha} = M^{\delta\gamma}_{\alpha\beta} = M^{\delta\beta}_{\gamma\alpha},
M^{\lambda(\alpha}_{\rho\sigma} M^{\gamma)\gamma)}_{\tau\lambda} = M^{\lambda(\alpha}_{\sigma\tau} M^{\beta\gamma)}_{\rho\lambda} = M^{\lambda(\alpha}_{\tau\rho} M^{\beta\gamma)}_{\sigma\lambda},
\]

where the round brackets denote complete symmetrization over all three indices enclosed. The quantity $M^{\beta\delta}_{\alpha\gamma}$ defined by (40) is a building block for constructing the manifold. The Riemann curvature takes the form [12]

\[
R^{\beta\gamma\delta}_{\alpha} = M^{\alpha\lambda}_{\alpha\gamma} g^{\beta}_{\lambda} g^{\delta}_{\lambda}.
\]

By using this form of the Riemann curvature we obtain

\[
[R \otimes \cdots \otimes R]_{\alpha_1 \alpha_2 \cdots \alpha_n}^{\gamma_1 \gamma_2 \cdots \gamma_n} (g^{-1})_{\gamma_1}^{\beta_1} (g^{-1})_{\gamma_2}^{\beta_2} \cdots (g^{-1})_{\gamma_n}^{\beta_n}
= [[[M \otimes \cdots \otimes M]]_{\alpha_1 \alpha_2 \cdots \alpha_n}^{\beta_1 \beta_2 \cdots \beta_n}. \quad (41)
\]
Here the symbol \([ ]\) stands for the same contraction as \([\ ]\), but by the Kronecker delta \(\delta_\beta^\alpha\), in place of the metric \((g^{-1})_\alpha^\beta\). In the case of \(n = 1\) we calculate the r.h.s., using \(M_\alpha^\beta\), defined by \((10)\), \(\Gamma^i\Gamma^i = const.\) and the Lie-algebra of \(\Gamma^i\), to show the formula \((38)\).

Finally we apply the formalism to the case of \(CP^1\) as a demonstration. For \(CP^1\) with the line element given by

\[
ds^2 = \frac{1}{(1 + \frac{1}{2}|z|^2)^2}dzd\bar{z},
\]

we have the non-trivial components of the Riemann curvature

\[
R_{z\bar{z}z\bar{z}} = -\frac{1}{(1 + \frac{1}{2}|z|^2)^4},
\]

and the Killing potentials \(M^A\):

\[
M^+ = \frac{z}{1 + \frac{1}{2}|z|^2}, \quad M^- = \frac{\bar{z}}{1 + \frac{1}{2}|z|^2}, \quad M^0 = \frac{1 - \frac{1}{2}|z|^2}{1 + \frac{1}{2}|z|^2}.
\]

Then the solution \((17)\) is given by

\[
r = -i\frac{\bar{h}}{4(1 + \frac{1}{2}|z|^2)^4}|y|^2(y\bar{\theta} - \bar{y}\theta) + \frac{i\bar{h}^2}{64(1 + \frac{1}{2}|z|^2)^2}(|y|^2\bar{\theta} - \bar{y}\theta) + \ldots.
\]

With this \(r\) we find the deformed Killing potentials \(\hat{M}^A\):

\[
\hat{M}^A = M^A + [(1 - \frac{13}{192}\bar{h}^2)y + \frac{5}{12}(1 + \frac{1}{2}|z|^2)^2|y|^2\ldots]M^A,z + c.c.]
\]

\[
+ [(1 - \frac{1}{8}\bar{h}^2)|y|^2 + \frac{1}{3}(1 + \frac{1}{2}|z|^2)^2|y|^4\ldots]M^A,z\bar{z}.
\]

The fuzzy algebrae \((1)\) and \((2)\) take the forms

\[
[M^A, M^B] = -i[h - \frac{2}{9}\bar{h}^3 + O(\bar{h}^5)]\varepsilon^{ABC}M^C,
\]

\[
M^A \star M^A = 1 - \frac{1}{4}\bar{h}^2 + \frac{13}{144}\bar{h}^4 + O(\bar{h}^6).
\]

The coefficients differ from those obtained in ref. \(7\). Using the spherical coordinates, they found that \(c_n = 0\) for \(n \geq 3\). Apparently the coefficients of the fuzzy algebrae depend on the reparametrization of the coset space.

In this paper we have shown the fuzzy algebrae on the irreducible Kähler coset space by Fedosov’s \(\star\) product. It is desirable to extend the study to the reducible Kähler coset space by using the formalism in ref. \(13\). The calculations of the deformed Kähler potentials \(\hat{M}^A\) were rather complicated even in the case of \(CP^1\). They would be simplified by using a modified version of the \(\star\) product, which was discussed for the Kähler manifold in \(14\).
It is interesting to develop a diagramatical method which enables us to calculate the $\star$ product systematically. The study in this direction is on the course.

**Acknowledgements**

The work of T.M. was supported in part by JSPS Research Fellowships for Young Scientists.

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