The Gamow Functional.

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Abstract

We present a formalism that represents pure states, mixtures and generalized states as functionals on an algebra containing the observables of the system. Along these states, there are other functionals that decay exponentially at all times and therefore can be used to describe resonance phenomena.

1 Introduction.

Gamow vectors are vector states describing the exponentially decaying part of a resonance [1, 2]. Whether or not this exponentially decaying state is an element of the reality is controversial [3]. The question is if it can be properly considered as a quantum state.

If we assume that resonances are produced in a resonant scattering process [2, 4], produced by a “free” Hamiltonian $H_0$ and an interacting Hamiltonian $H = H_0 + V$, Gamow vectors are eigenvalues of $H$ with complex eigenvalues. This complex eigenvalues correspond to poles (that we are assuming here to be simple) of the analytic continuation of the $S$-matrix in the energy representation. These poles are located in the lower half plane in the second sheet of the Riemann surface associated to the transformation $|p| = \sqrt{E}$. As the total Hamiltonian is self adjoint, the Gamow vector cannot belong to a Hilbert space of states, but instead to the bigger space of a Gelfand triplet or rigged Hilbert space (RHS), $\Phi \subset \mathcal{H} \subset \Phi^\times$. As a consequence the Gamow vector, $|f_0\rangle$, cannot be normalized. Worse of all, there is no clear manner of defining the mean value of the energy on a Gamow vector [4].
Pure states as well as mixtures are represented by density operators. In order to study the properties of the Gamow “state”, we may try to write it as a density operator on an extended (or rigged) Liouville space. Here, new difficulties arise that suggest that a Gamow vector cannot give rise a reasonable quantum state [6].

Inspired by the methods of the Brussels school [7, 8], the group of Rosario has developed a formalism that allows us to make calculations using Gamow “states” [9, 10, 11, 12, 13]. However, the use that has been made in these papers of the Gamow object has been merely operational and no clear definition of it has been provided. It is the aim of the present paper to give a possible definition of the Gamow object as a functional on an algebra of states compatible with this formalism and give its most interesting properties.

2 Algebras of observables and the Gamow functional.

2.1 Observables.

Let us assume that $H_0$ has simple continuous spectrum equal to $\mathbb{R}^+ = [0, \infty)$. Then, for each $E \in \mathbb{R}^+$ there exists a generalized eigenvector $|E\rangle$ of $H_0$: $H_0|E\rangle = E|E\rangle$. The vector $|E\rangle$ lies in the bigger space of a RHS $\Phi \subset \mathcal{H} \subset \Phi^\times$. The set of vectors $|E\rangle$ is complete in the sense that

$$H_0 = \int_0^\infty E|E\rangle\langle E|dE$$

They also verify that $\langle E|E'\rangle = \delta(E - E')$, where the Dirac delta is refered to the integration from 0 to $\infty$.

We say that an operator $O$ is compatible with $H_0$ if it can be written in the form:

$$O = \int_0^\infty dEO_E|E\rangle\langle E| + \int_0^\infty dE \int_0^\infty dE' O_{EE'}|E\rangle\langle E'|$$

where $O_E$ is a function of the real variable $E$ and $O_{EE'}$ is a function of the two dimensional variable $(E, E')$.

\footnote{Here, the operator $O$ maps $\Phi$ into $\Phi^\times$. This is a generalization of the usual concept of operator that maps a subspace $\Phi$ of the Hilbert space $\mathcal{H}$ into $\mathcal{H}$.}
We want that the set of operators compatible with \( H_0 \) form an involutive algebra with identity. In order to do this, we need to make a choice on the functions \( O_E \) and \( O_{EE'} \). Let \( D \) be the space of infinitely differentiable functions, at all points, with compact support. This space is endowed with a locally convex topology [14]. Now, take the Fourier transform of all functions in \( D \). The resulting space, \( \mathcal{Z} \equiv \mathcal{F}(D) \), is a vector space of entire analytic functions with these two properties:

i.) The product of two functions \( f(z), g(z) \in \mathcal{Z} \), \( f(z)g(z) \), is also in \( \mathcal{Z} \).

ii.) If \( p(z) \) is a polynomial and \( f(z) \in \mathcal{Z} \), then, \( p(z)f(z) \) is also in \( \mathcal{Z} \).

Then, we choose \( O_E \) as the sum of a polynomial on \( E \) plus a function in \( \mathcal{Z} \) (one or the other or both may be eventually equal to zero). The two variable function \( O_{EE'} \) belongs to the algebraic tensor product \( \mathcal{Z} \otimes \mathcal{Z} \), i.e., it must be of the form

\[
O_{EE'} = \sum_{ij} \lambda_{ij} \psi_i(E) \phi_j(E')
\]  

(3)

where \( \psi_i(E), \phi_j(E') \in \mathcal{Z} \). The sum in (3) is finite. The set of operators compatible with \( H_0 \) forms an algebra where the sum of two operators and the product by scalars are given in an obvious manner. To multiply two operators, we must take into account that \( \langle E|E' \rangle = \delta(E - E') \) and i.) and ii.) as above. Then, this set form an algebra we call \( \mathcal{A}_0 \). This algebra can be endowed with a topology due to the fact that the space \( \mathcal{P} \) of polynomials can be endowed with a topology and \( \mathcal{Z} \) and \( \mathcal{Z} \otimes \mathcal{Z} \) have natural topologies [14, 16]. Thus, \( \mathcal{A}_0 \) is isomorphic (as a vector space) to the direct sum \( \mathcal{P} + \mathcal{Z} + \mathcal{Z} \otimes \mathcal{Z} \).

Next, let us assume that the choice of \( (H_0, H) \) (or \( V \)) is such that the Møller wave operators exist and the scattering is asymptotically complete. Then, we can define the vectors\(^2 \quad |E^{\pm}\rangle := \Omega_{\pm}|E\rangle \) that are generalized eigenvectors of the total Hamiltonian: \( H|E^{\pm}\rangle = E|E^{\pm}\rangle \). If \( O \) is an observable compatible with \( H_0 \), let us write:

\[
O^{\pm} := \Omega_{\pm} O \Omega_{\pm}^\dagger
\]

\(^2\text{Or the restriction to } \mathbb{R}^+ \text{ of a function in } \mathcal{Z}. \text{ Due to the fundamental theorem of analytic continuation (see [13]), this restriction fix uniquely the function in } \mathcal{Z}.\)

\(^3\text{These vectors belong to the bigger space in the RHS } \Phi^\pm \subset \mathcal{H} \subset (\Phi^\pm)^\times, \text{ where } \Phi^\pm = \Omega_{\pm} \Phi \) [17].
\[ = \int_0^\infty O_E |E^\pm\rangle\langle E^\pm| dE + \int_0^\infty dE \int_0^\infty dE' O_{EE'} |E^\pm\rangle\langle E^\pm'| \] (4)

It is natural to say that an operator is compatible with \( H \) if and only if it has one of the forms given in (4). Since

\[ \langle E^\pm | w^\pm \rangle = \langle E | \Omega^\dagger_\pm \Omega^\pm | w \rangle = \langle E | w \rangle = \delta(E - w) \] (5)

the set of operators of the form (4) form two algebras, \( A_\pm \), isomorphic to \( A_0 \). The topology on \( A_\pm \) is defined exactly as for \( A_0 \).

These algebras are involutive. If \( O \in A_0 \), its adjoint is

\[ O^\dagger := \int_0^\infty dE O_E^* |E\rangle\langle E| + \int_0^\infty dE \int_0^\infty dE' O_{EE'}^* |E'|\langle E| \] (6)

It is easy to show that this definition is consistent with the formula \( (\varphi, O \psi) = (O^\dagger \varphi, \psi) \), when \( \varphi, \psi \in \Phi \), where \( \Phi \) is a suitable space of test vector on which \( |E\rangle \) acts \([7]\). Similar definition applies for the adjoint of \( O^\pm \). An operator for which \( O = O^\dagger \) is called self adjoint. \( O \) (or \( O^\pm \)) is self adjoint if and only if \( O_E = O_E^* \) and \( O_{EE'} = O_{EE'}^* \). Self adjoint members of \( A_0 \) (\( A_\pm \)) are the observables of the system. If we write

\[ |E^\pm \rangle := |E^\pm\rangle\langle E^\pm| \quad ; \quad |EE'^\pm\rangle := |E^\pm\rangle\langle E'^\pm| \] (7)

then, \( O^\pm \) can be written as:

\[ O^\pm = \int_0^\infty dE O_E |E^\pm\rangle\langle E^\pm| + \int_0^\infty dE \int_0^\infty dE' O_{EE'} |E^\pm\rangle\langle E'^\pm| \] (8)

### 2.2 States.

Let us consider the duals, \( A_{\pm}^\times \), of the algebras \( A_\pm \). These are the vector spaces of continuous linear functionals (functionals are mappings from \( A_\pm \) into the set \( \mathbb{C} \) of complex numbers) on \( A_\pm \). Examples of vectors on \( A_{\pm}^\times \) are the mappings

\[ O^\pm \mapsto O_E \quad ; \quad O^\pm \mapsto O_{EE'} \] (9)

for given \( E \) and \( E' \) in \( \mathbb{C} \). When \( E \) and \( E' \) are real these mappings are called \( (E^\pm| \) and \( (EE'^\pm| \) respectively, so that

\[ (E^\pm|O^\pm) = O_E \quad ; \quad (EE'^\pm|O^\pm) = O_{EE'} \] (10)
If $\rho^\pm$ is a linear functional on $A_\pm$, we denote the action of $\rho^\pm$ on $O^\pm$ by $(\rho^\pm | O^\pm)$.

It is customary to define a state on $A_\pm$ as a continuous linear functional $\rho^\pm$ on $A_\pm$ verifying the following conditions [18]:

i.) Positivity: $(\rho^\pm | (O^\pm)^\dagger O^\pm) \geq 0$ for all $O^\pm \in A_\pm$.

ii.) Normalization: $(\rho^\pm | I^\pm) = 1$, where $I^\pm$ are the identities on the algebras $A_\pm$. These identities can be written as:

\[ I^\pm = \int_0^\infty dE |E^\pm\rangle\langle E^\pm| = \int_0^\infty dE |E^\pm\rangle. \quad (11) \]

In general a functional on $A_\pm$ can be written as

\[ \rho^\pm = \int_0^\infty dE \rho_E (E^\pm) + \int_0^\infty dE \int_0^\infty dE' \rho_{EE'} (EE'^\pm) \quad (12) \]

where $\rho_E$ and $\rho_{EE'}$ can be either functions or distributions, so that when applied to $O^\pm$ the result is

\[ (\rho^\pm | O^\pm) = \int_0^\infty dE \rho_E O_E + \int_0^\infty dE \int_0^\infty dE' \rho_{EE'} O_{EE'} \quad (13) \]

observe that (13) implies the relations

\[ (E^\pm | w^\pm) = \delta(E - w) \quad ; \quad (EE'^\pm | ww'^\pm) = \delta(E - w) \delta(E' - w'). \quad (14) \]

The evolution of the states $\rho^\pm$ under the total Hamiltonian $H$ can be defined using the following expression:

\[ (\rho_t^\pm | O^\pm) = (\rho | e^{itH} O^\pm e^{-itH}) \]

\[ = \int_0^\infty dE \rho_E O_E + \int_0^\infty dE dE' e^{it(E-E')} \rho_{EE'} O_{EE'} \quad (15) \]

which gives the evolution of the mean values using the Heisenberg definition of the evolution for the observables. The Schrödinger evolution of the states then results from (15) and is

\[ \rho_t^\pm = \int_0^\infty dE \rho_E (E^\pm) + \int_0^\infty dE \int_0^\infty dE' e^{it(E-E')} \rho_{EE'} (EE'^\pm) \quad (16) \]

\[ \text{The operators } I^\pm \text{ are the canonical imbeddings of } \Phi^\pm \text{ into their respective dual spaces } (\Phi^\pm)^\times, \text{ i.e., } I^\pm \varphi^\pm = \varphi^\pm, \text{ see [19].} \]
We see that the first term in the right hand side of (16) (the diagonal part of $\rho^\pm$) is invariant and the second term (the nondiagonal part) evolves with time. In accordance with this result, diagonal and nondiagonal parts of (16) are often called the invariant and the fluctuating parts respectively [9, 13].

Normalized vectors on Hilbert space representing quantum pure states and positive trace one operators representing mixtures can be easily written in the form (12). If $\psi(E)$ represents the wave function of a pure state in the energy representation, the coefficients $\rho_E$ and $\rho_{EE'}$ in (12) are given by

$$\rho_E = |\psi(E)|^2 \quad \text{and} \quad \rho_{EE'} = \psi^*(E)\psi(E') \quad (17)$$

A mixture can be written in the form $\rho = \sum_i \lambda_i |\psi_i\rangle\langle\psi_i|$ with $\langle\psi_i|\psi_j\rangle = \delta_{ij}$ and $\sum_i \lambda_i = 1$. If $\psi_i(E)$ is the wave function of the state $\psi_i$ in the energy representation, we have in here:

$$\rho_E = \sum_i \lambda_i |\psi_i(E)|^2 \quad \text{and} \quad \rho_{EE'} = \sum_i \lambda_i \psi_i^*(E)\psi_i(E') \quad (18)$$

Observe that, in any case, $\rho_E = \rho_{EE'}$. In addition to pure states and mixtures, this formalism allows for another kind of states called the states with diagonal singular or generalized states [7] for which $\rho_E \neq \rho_{EE'}$.

2.3 Gamow functionals.

Gamow vectors are usually obtained as residues of poles in analytic continuations of the $S$ operator (in the energy or momentum representations) [2, 17] or a reduced resolvent [21, 22, 23]. In previous works [9, 13], Gamow vectors have been heuristically obtained using analytic continuations on the variables $E$ and $E'$ of the fluctuating part of $(\rho^\pm|O)$, i.e., the second term in the right hand side of (13) and then, using a limiting process. Only the fluctuating part is there used because this is the only relevant to time evolution. Then, the result obtained does not have diagonal part. This suggest that a good definition of the Gamow vector should have diagonal part equal to zero.

Now we are in the position of defining the decaying Gamow functional in the following form:

$$\rho_D := \int_0^\infty dE \int_0^\infty dE' \delta_{z^+} \otimes \delta_{z_0} (EE') \quad (19)$$

Observe that, in our definition, the component $\rho_E$ of $\rho_D$ vanishes and the component $\rho_{EE'}$ is equal to $\delta_{z^+} \otimes \delta_{z_0}$. When applied this distribution to a
given $O_{EE'} \in \mathcal{Z} \otimes \mathcal{Z}$, we obtain $O_{z^*_0 z_0}$, i.e., the value of the function $O_{EE'}$ at the point $(z^*_0, z_0)$. Obviously

$$(\rho_D|O^+) = O_{z^*_0 z_0}$$

(20)

The decaying Gamow functional has the following properties:

1.- It is a continuous linear functional on $\mathcal{A}_+$. 
2.- It is positive: $(\rho_D|(O^+)^\dagger O^+) \geq 0$. 
3.- It decays exponentially for all values of time. If we write $\rho_D(0) = \rho_D$, then,

$$\rho_D(t)|O^+) := (\rho_D|e^{itH} O^+ e^{-itH})$$

$$= \int_0^\infty dE \int_0^\infty dE' \delta_{z^*_0} \otimes \delta_{z_0} O_{EE'} e^{it(E-E')} = e^{it(z^*_0 - z_0)} O_{z^*_0 z_0}$$

$$= e^{-it} O_{z^*_0 z_0} = e^{-it} (\rho_D|O^+)^\dagger$$

(21)

It is not difficult to show, using the inverse Fourier transform, that $O_{EE'} e^{it(E-E')}$ is in $\mathcal{Z} \otimes \mathcal{Z}$ for all values of $t$. In fact, if $\varphi(E) \in \mathcal{Z}$, we have that

$$\mathcal{F}^{-1}(e^{itE} \varphi(E)) = \int_{-\infty}^{\infty} e^{itE} e^{itE} \varphi(E) dE = (\mathcal{F}^{-1} \varphi)(t + \tau)$$

(22)

As $(\mathcal{F}^{-1} \varphi)(\tau)$ is an infinitely differentiable function with compact support, so is (22).

The exponential decay of $\rho_D(t)$ for all values of time allows us the call it the decaying Gamow functional. Observe that its decay mode is not a semigroup and this avoids the difficulties in determining the instant $t = 0$ at which “the preparation of the quasistationary state is completed and starts to decay” [2, 20]. Nevertheless, a redefinition for $O_E$ and $O_{EE'}$ is possible so that $\rho_D(t)$ is defined and decays exponentially for $t \geq 0$ only [24].

4.- Cannot be normalized: $(\rho_D|I^+) = 0$. This means that $\rho_D$ does not fit with the traditional definition of state [18]. Although most of data on unstable particles seems to confirm the exponential decay [25], traditional quantum mechanics forsees deviations of the exponential decay law for very small and very large times [3]. Traditionally, Gamow vectors and functionals have been used to construct generalized spectral expansions [19], useful for calculations [3, 10, 11, 12, 13].
5.- It is also obvious that
\[
(\rho_D|H^n) = 0, \quad n = 0, 1, 2, \ldots
\]
where
\[
H^n = \int_0^\infty dE E^n |E^+\rangle
\]
For \( n = 0 \), we recover \((\rho_D|I^+) = 0\). For \( n = 1 \), we obtain \((\rho_D|H) = 0\).
This means that the mean value of the energy on the Gamow state is zero, which agrees with some predictions [26] but contradicts some others [5]. In our opinion, (23) is an argument against the option of considering \( \rho_D \) as a physical state, as all momenta of \( H \) on \( \rho_D \) vanish.

**Concluding remarks**

We have presented here a formalism in which pure states, mixtures and generalized states with singular diagonal [7] are positive, normalized continuous functionals on an algebra containing the observables of a quantum system. Along these functionals we have obtained a functional that decays exponentially, the Gamow functional, but for which no normalization exists. This functional is always present in decaying processes and is useful for calculations. However, the fact that the mean value of \( H^n \) for \( n = 0, 1, 2, \ldots \) in \( \rho_D \) is always zero, suggest that the Gamow functional cannot be taken seriously as a truly physical state.

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