Quaternionic Taub-NUT from the harmonic space approach

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Abstract

We use the harmonic space technique to construct explicitly a quaternionic extension of the Taub-NUT metric. It depends on two parameters, the first being the Taub-NUT 'mass' and the second one the cosmological constant.
1. An efficient way to explicitly construct hyper-Kähler and quaternionic-Kähler metrics is provided by the harmonic (super)space method [1] - [4]. It was firstly introduced in the context of $N = 2$ supersymmetry [1]. The basic idea was to extend the standard $N = 2$ superspace by a set of internal (‘harmonic’) variables $u^\pm_i, u^i_\pm u^-_i = 1$, parametrizing the automorphism group $SU(2)$ of $N = 2$ superalgebra. It was shown in [1] that all $N = 2$ theories admit a manifestly supersymmetric off-shell description in terms of unconstrained superfields given on an analytic subspace of the $N = 2$ harmonic superspace, harmonic analytic superfields. It was soon realized that the harmonics are also relevant to some purely bosonic geometric problems. As is shown in [3], the constraints defining the hyper-Kähler (HK) geometry can be given an interpretation of the integrability conditions for the existence of analytic fields in a $SU(2)$ harmonic extension of the original $4n$-dimensional HK manifold $\{x^{\mu}\}, (i = 1, 2; \mu = 1, ..., 2n)$. This time, the $SU(2)$ to be ‘harmonized’ is an extra $SU(2)$ rotating three complex structures of the HK manifold. The analytic subspace is spanned by the harmonic variables $u^\pm_i$ and half of the initial $x$-coordinates, $x^\mu_\pm$. The constraints of HK geometry can be solved via an unconstrained analytic HK potential $L^{+4}(x^\mu, u^\pm_i)$. It encodes (at least, locally) all the information about the associated metric. Remarkably, it allows one to explicitly construct the HK metrics by simple rules [3].

In [4], a generalization of this approach to the quaternionic-Kähler (QK) manifolds was given. These manifolds generalize the HK ones in that the extra $SU(2)$ which transforms complex structures becomes an essential part of the holonomy group. It was shown in [4], that the QK geometry constraints can be also solved in terms of some unconstrained potential $L^{+4}$ living on the analytic subspace parametrized by $SU(2)$ harmonics and half of the original coordinates. The specificity of the QK case is the presence of a non-zero constant $Sp(1)$ curvature on all steps of the way from $L^{+4}$ to the related metric. It is interesting to consider some examples in order to see in detail how the machinery proposed in [4] works. Only the simplest case of the homogeneous QK manifold $Sp(n + 1)/Sp(1) \times Sp(n)$ (corresponding to $L^{+4} = 0$) was considered in ref. [4].

The aim of this paper is to demonstrate the power of the harmonic geometric approach on the example of less trivial QK metric, a quaternionic generalization of the well-known four-dimensional Taub-NUT (TN) metric [5]. Like in the HK case [2], the computations are greatly simplified due to the $U(1)$ isometry of the quaternionic TN metric. The metric depends on two parameters, the TN ‘mass’ parameter and the constant $SU(2)$ curvature parameter which can be interpreted as the inverse ‘radius’ of the corresponding ‘flat’ QK background $\sim Sp(2)/Sp(1) \times Sp(1)$. We perform the identification of the metric with those known in literature and consider its few important particular limits.

2. We first recall some salient features of the construction of [3]. One starts with a $4n$-dimensional Riemann manifold parametrized by local coordinates $\{x^{\mu m}\}, \mu = 1, 2, ..., 2n; m = 1, 2$, and uses a vielbein formalism. The QK geometry can be defined as a restriction of the general Riemannian geometry in $4n$-dimensions, such that the holonomy group of the corresponding manifold is a subgroup of $Sp(1) \times Sp(n)$ [4]. Thus one can choose the tangent space group from the very beginning to be $Sp(1) \times Sp(n)$ and define the QK geometry via appropriate restrictions on the curvature tensor lifted to the tangent space.

*For the 4-dimensional case this definition has to be replaced by the requirement that the totally symmetric part of the $Sp(1)$ component of the curvature tensor lifted to the tangent space is vanishing.
(taking into account that the holonomy group is generated by this tensor). As explained in [4], for the QK manifold of generic dimension the defining constraints can be written as a restriction on the commutator of two covariant derivatives

\[ [\mathcal{D}_{\alpha(i)}, \mathcal{D}_{\beta(k)}] = -2\Omega_{\alpha\beta} R \Gamma_{(ik)} . \]  

(1)

Here

\[ \mathcal{D}_{\alpha i} = \epsilon^{\mu m}_{\alpha i}(x)\mathcal{D}_{\mu m} = \epsilon^{\mu m}_{\alpha i}(x) \frac{\partial}{\partial x^{\mu m}} + [Sp(1) \times Sp(n) - \text{connections}] , \]

(2)

\( \epsilon^{\mu m}_{\alpha i}(x) \) being the \( 4n \times 4n \) vielbein with the indices \( \alpha = 1, 2, ... 2n \) and \( i = 1, 2 \) rotated, respectively, by the tangent local \( Sp(n) \) and \( Sp(1) \) groups, \( \Omega_{\alpha\beta} \) is the \( Sp(n) \)-invariant skew-symmetric tensor serving to raise and lower the \( Sp(n) \) indices \( (\Omega_{\alpha\beta}\Omega^{\beta\gamma} = \delta^\gamma_\alpha) \), \( \Gamma_{(ik)} \) are the \( Sp(1) \) generators, and \( R \) is a constant, remnant of the \( Sp(1) \) component of the Riemann tensor (its constancy is a consequence of the QK geometry constraint and Bianchi identities). The scalar curvature coincides with \( R \) up to a positive numerical coefficient, so the cases \( R > 0 \) and \( R < 0 \) correspond to compact and non-compact manifolds, respectively. In the limit \( R = 0 \) eq. (1) is reduced to the constraint defining the HK geometry [3], in accord with the interpretation of HK manifolds as a degenerate subclass of the QK ones.

Like in the HK case [3], in order to explicitly figure out which kind of restrictions is imposed by (1) on the vielbein \( \epsilon^{\mu m}_{\alpha i}(x) \) and, hence, on the metric

\[ g^{\mu m \nu s} = \epsilon^{\mu m}_{\alpha i} e^{\nu s \alpha i} , \quad g_{\mu m \nu s} = \epsilon_{\mu m \alpha i} e_{\nu s \alpha i} , \]

(3)

one should solve the constraints (1) by regarding them as integrability conditions along some complex directions in a harmonic extension of the original manifold.

Due to the non-vanishing r.h.s. in (1), the road to such an interpretation in the QK case is more tricky. Modulo these peculiarities, the basic step still consists in extending \( \{x^{\mu m}\} \) by a set of some harmonic variables, \( \{x^{\mu m}, w^\pm_i\} \rightarrow \{x^{\mu m}, w^\pm_i\}, \ w^+i w^-i = 1 \). Then, following the general strategy [3], [4], one passes to a new (‘analytic’) basis in \( \{x^{\mu m}, w^\pm_i\} \)

\[ \{x^{\mu m}, w^\pm_i\} \rightarrow \{x^+_\alpha, x^-_\alpha, w^\pm_A\} \]

(4)

\[ x^+_\alpha = x^{\mu +}_i w^+_i + v^\pm i(x, w) , \quad w^+_i = w^+i - R v^{++}(x, w) w^-i , \quad w^-i = w^-i , \]

(5)

where the ‘bridges’ \( v^\pm i(x, w), v^{++}(x, w) \) are chosen so as to make the \( w^- \)-projection of \( \mathcal{D}_{\alpha i} \) in this basis to be proportional to the partial derivative with respect to \( x^-\mu \)

\[ \mathcal{D}_{\alpha}^+ \sim w^+i \mathcal{D}_{\alpha i} = w^+i \epsilon^{\mu m}_{\alpha i}(x)\partial_{\mu m} + \ldots = E^\mu_{\alpha}(x, w) \frac{\partial}{\partial x^-\mu} = E^\mu_{\alpha}(x, w)\partial^+_{\mu} \]

(6)

(simultaneously, one performs an appropriate \( Sp(n) \) rotation of the tangent space index \( \alpha \) by a matrix \( Sp(n)-\text{`bridge’} \)). The possibility to reduce \( \mathcal{D}_{\alpha}^+ \) to this ‘short’ form amounts to the possibility to define analytic fields living on the analytic subspace \( \{x^+_\alpha, w^+_A\} \). The original QK geometry constraints prove to be equivalent to the existence of such analytic fields and subspace [4]. An essential difference of the QK case from the HK case [3] is the necessity to shift the harmonic variables with the new bridge \( v^{++} \).

Besides the opportunity to make \( \mathcal{D}_{\alpha}^+ \) short, the passing to the harmonic extension of \( \{x^{\mu m}\} \) and further to the analytic basis and frame (‘the \( \lambda \)-world’) allows one to exhibit the
fundamental unconstrained objects of the QK geometry, the QK potential. While in the original formulation (‘the \( \tau \)-world’) the basic geometric objects are the vielbeins \( e^{\mu\nu}_{\lambda}(x) \) properly constrained by eq. \( (\ref{eq:original_formulation}) \), in the analytic basis such objects are the harmonic vielbeins covariantizing with respect to the harmonic variables. In the original basis the harmonic derivatives are \( D^{\pm\pm} = \partial^{\pm\pm}_w = w^{\pm\pm}\partial/\partial w^{\pm\pm} \), \( D^0 = \partial^0_w = w^{+\pm}\partial/\partial w^{\pm} - w^{-\pm}\partial/\partial w^{-\pm} \), \( [\partial^{+\pm}_w, \partial^{-\pm}_w] = \partial^0 \), i.e. they contain no partial derivatives with respect to the variables \( x^{\pm\pm} \), because the harmonic space \( \{x^{\pm\pm}_m, w^{\pm\pm}_i\} \) has the structure of the direct product \( \{x^{\pm\pm}_m\} \otimes \{w^{\pm\pm}_i\} \). After passing to the analytic basis by eqs. \( (\ref{eq:passing_to_analytic_basis}) \), the derivatives \( D^{\pm\pm} \) acquire terms proportional to \( \partial^\pm_w \equiv \partial/\partial x^{\pm\pm}_i \). Besides, in \( D^{++} \) there emerges a term proportional to \( \partial^{-\pm}w^\pm \). These new terms appear with the appropriate vielbein components \( H^{+\pm\mu}, H^{-\pm\mu}, H^{++} \) which are related to the bridges as follows

\[
\begin{align*}
(\partial^{++}_w + R^{++}_w)x^{+\mu}_A &= H^{+\pm\mu}, \\
(\partial^{++}_w + R^{++}_w)v^{++} &= -H^{++}, \\
\frac{1}{1 - R^{\pm\pm}_w} \partial^{-\pm}_{-\pm} x^{\pm\mu}_A &= H^{-\pm\mu}.
\end{align*}
\tag{7}
\tag{8}
\tag{9}
\]

Note that \( x^{-\mu} \) is determined in terms of \( x^{+\mu} \) by the equation

\[
(\partial^{++}_w - R^{++}_w)x^{-\mu}_A = x^{+\mu}_A.
\tag{10}
\]

The original QK geometry constraints require \( H^{+\pm\mu}, H^{++} \) to be analytic

\[
\partial^\pm_H H^{+\pm\mu} = \partial^\pm_H H^{++} = 0 \quad \Rightarrow \quad H^{+\pm\mu}(x^+_A, w_A), \quad H^{++}(x^+_A, w_A),
\tag{11}
\]

and express \( H^{+\pm\mu} \) in terms of \( H^{++} \). Basically, the analytic harmonic vielbein \( H^{++} \) is just the unconstrained QK potential. To be more precise, the QK potential \( \mathcal{L}^{++} \), as it was defined in \( (\ref{eq:QK_potential}) \), is related to \( H^{++} \) as (after properly fixing the \( \lambda \)-world gauge freedom)

\[
H^{++}(x^+_A, w_A) = \mathcal{L}^{++}(x^+_A, w_A) + x^+_\mu H^{+\pm\mu}(x^+_A, w_A), \quad x^+_\mu \equiv \Omega_{\mu\nu} x^{++}\nu,
\tag{12}
\]

and

\[
H^{+\pm\nu} = \frac{1}{2} \Omega^{\mu\nu}_\lambda \partial^\pm_H \mathcal{L}^{++}, \quad \partial^{-\pm}_\mu \equiv \partial^{-\pm}_\mu + R x^+_{\mu} \partial^{-\pm}_A.
\tag{13}
\]

It can be shown that the only constraint to be satisfied by \( \mathcal{L}^{++} \) is its analyticity, so this object encodes all the information about the relevant QK geometry and metrics, whence its name ‘QK potential’. Choosing one or another explicit \( \mathcal{L}^{++} \), and substituting \( (\ref{eq:QK_potential}) \), \( (\ref{eq:QK_potential}) \) into eqs. \( (\ref{eq:passing_to_analytic_basis}) \) - \( (\ref{eq:QK_potential}) \), one can solve the latter for \( x^{\pm\mu} \) and \( v^{++} \) as functions of harmonics and the \( \tau \)-world coordinates \( x^{\pm\mu} \). Having at hand the explicit form of the variable change \( \Omega \), it remains to find the appropriate expression of the \( \lambda \)-world vielbeins in terms of \( \mathcal{L}^{++} \) in order to be able to restore the \( \tau \)-world vielbein and hence the QK metric itself.

Skipping intermediate steps (they can be found in \( [4] \)), the non-vanishing components of the \( \lambda \)-world inverse QK metric are given by the following expressions

\[
g^{\mu\nu}_{(\lambda)} = g^{\mu\nu}_{(\lambda)} = \Omega^{\mu\rho}_\sigma (\partial^\Sigma H^{-1})^{-1}_\rho \nu, \quad g^{\mu\nu}_{(\lambda)} = -2 \Omega^{\mu\nu}_\sigma (\partial^\Sigma H)^{-1}_\sigma (\partial^\Sigma H^{-1})^{-1}_\rho \partial^\Sigma H^{-3\nu},
\tag{14}
\]

where

\[
(\partial^\Sigma H)^\mu_\nu \equiv \partial^\nu \Sigma^{++\mu}, \quad \Sigma^{-\pm\mu} \equiv \frac{1}{1 - R(x \cdot H)} H^{-\pm\mu}, \quad x \cdot H \equiv x^+_{\mu} H^{-\pm\mu}.
\tag{15}
\]

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Then the $\tau$-world metric can be obtained via the change of variables inverse to (14)
\[ g_{\mu\nu} = \hat{g}_{\lambda}^{\mu} \partial_{\sigma} x^{\mu} \partial_{\sigma} x^{\nu} + \hat{g}_{\lambda}^{\nu} \partial_{\sigma} x^{\mu} \partial_{\sigma} x^{\nu} \left( \hat{\partial}_{\omega} x^{\mu} \partial_{\sigma} x^{\nu} + \hat{\partial}_{\omega} x^{\mu} \partial_{\sigma} x^{\nu} \right) . \] (16)

In the case of 4-dimensional QK manifolds we will deal with in the sequel ($\mu, \nu = 1, 2$) the $\tau$-basis metric (16), after some algebra, can be put in the form
\[ g_{\mu\nu} = \frac{1}{\det(\partial H)} \frac{1}{[1 - R(x \cdot H)]} G_{\mu\nu} , \] (17)
\[ G_{\mu\nu} = \epsilon^{\rho\lambda} \left[ \partial_{\omega} - X^{+\mu\nu} X_{\rho}^{\nu} + (\mu\nu \leftrightarrow \nu\mu) \right] . \] (18)

Here
\[ X^{+\mu\nu} \equiv \partial_{\mu} x^{\nu} \] (19)
are solutions of the system of algebraic equations
\[ X^{+\mu\nu} \nabla_{\mu\nu} x^{\nu} = \partial_{\mu} x^{\nu} - \delta^{\nu}_{\nu} , \quad X^{+\mu\nu} \nabla_{\mu\nu} x^{\nu} = \partial_{\mu} x^{\nu} = 0 , \] (20)
\[ \nabla_{\mu\nu} \equiv \partial_{\mu\nu} + \frac{R}{1 - R \partial_{w}^{-} v^{++}}(\partial_{\mu\nu} v^{++}) \partial_{w}^{-} . \] (21)

As we see, the problem of calculating the QK metric (17), (18) is reduced to solving the differential equations (7), (10), (8) which define, by the known $L^{+4}$, $x^{\pm\mu}$, $w_{A}^{\pm}$, $x^{\pm\mu}$ and $v^{++}$ as functions of the $\tau$-basis coordinates $x^{\mu\nu}$ and $w^{\pm}$. In general, it is a difficult task. However, it is simplified for the QK metrics with isometries, like in the HK case [2]. We will demonstrate this on the example of the QK analog of the Taub-NUT metric.

3. The QK counterpart of the TN manifold is characterized by the same $L^{+4}$
\[ L^{+4} = (2i\lambda x^{+} \overline{x}^{+})^{2} \equiv (\phi^{++})^{2} . \] (22)

Here we introduced the notation
\[ (x^{+1}, x^{+2}) = (x^{+}, -\overline{x}^{+}) , \quad \overline{x}^{+} = (\overline{x}^{+}) , \quad (\overline{x}^{+}) = -x^{+} . \] (23)

We also assume
\[ \lambda = \overline{\lambda} \quad \Rightarrow \quad \phi^{++} = \phi^{++} . \] (24)

The basic equations (8), (11), (14) for the given case take the form
\[ \partial^{++} v^{++} + R(v^{++})^{2} = (\phi^{++})^{2} , \] (25)
\[ (\partial^{++} + R v^{++}) x^{+} = 2i\lambda x^{+} \phi^{++} , \] (26)
\[ (\partial^{++} - R v^{++}) x^{-} = x^{+} \] (27)
(together with their conjugates). These equations are covariant under two rigid symmetries preserving the analytic subspace $\{ x^{+}, \overline{x}^{+}, w_{A}^{\pm} \}$: $U(1)$ Pauli-Gürsey (PG) symmetry
\[ x^{+} ' = e^{i\alpha} x^{+} , \quad \overline{x}^{+} ' = e^{-i\alpha} \overline{x}^{+} , \] (28)\footnote{We adopt the convention $\epsilon_{12} = -\epsilon_{12} = 1$. The complex conjugation is always understood as a generalized one, i.e. the product of the ordinary conjugation and Weyl reflection of harmonics [1].}
and $SU(2)$ symmetry which uniformly rotates the doublet indices of the harmonic variables ($x^\pm$ and $v^{++}$ are scalars with respect to this $SU(2)$). They constitute the $U(2)$ isometry group of the QK TN metric.

We will firstly solve eq. (25). Defining

$$v^{++} = \partial^{++} v, \quad \omega \equiv e^{Rv}, \quad \hat{x}^+ \equiv \omega x^+, \quad \hat{\phi}^{++} = 2i\lambda \hat{x}^+ \hat{x}^+ = \omega^2 \phi^{++},$$

(29)

we rewrite (23), (26) as

$$\partial^{++} \hat{x}^+ = 2i\lambda x^+ \hat{\phi}^{++} \equiv 2i\lambda x^+ \kappa^{++}.\quad (31)$$

From eq. (31) and the definition of $\hat{\phi}^{++}$ one immediately finds

$$\partial^{++} \hat{\phi}^{++} = 0 \Rightarrow \hat{\phi}^{++} = \hat{\phi}^{ik}(x)w_i^+ w_k^+ .\quad (32)$$

We observe that eq. (30) coincides with the pure harmonic part of the equation defining the Eguchi-Hanson metric in the harmonic superspace approach [8]. Its general solution was given in [8], it depends on four arbitrary integration constants, that is, in our case, on four arbitrary functions of $x^{\mu i}$. However, these harmonic constants turn out to be unessential due to four hidden gauge symmetries of the set of equations (25) - (27). One of them is the scale invariance $v' = v + \beta(x)$, while three remaining ones form an extra local $SU(2)$ symmetry [7]. Using this gauge freedom one can gauge away four integration constants in $\omega$ and write a solution to eq. (30) in the following simple form

$$\omega = \sqrt{1 + R\hat{\phi}^2} \Rightarrow v = \frac{1}{2R} \ln(1 + R\hat{\phi}^2),\quad (33)$$

$$v^{++} = \partial^{++} v = \frac{\hat{\phi} \hat{\phi}^{++}}{1 + R\hat{\phi}^2}, \quad \hat{\phi} \equiv \hat{\phi}^{ik}(x)w_i^+ w_k^-.\quad (34)$$

One can restore the general form of the solution as it was given in [8], acting on (33) by a finite form of the aforementioned hidden symmetry transformations. In [4] we demonstrate that the whole effect of the full gauge $SU(2)$ transformation is reduced to the rotation of the $\tau$-world metric corresponding to the fixed-gauge solution (34) by some harmonic-independent non-singular matrix which becomes identity upon restriction to $x$-independent $SU(2)$ transformations. Thus in what follows we can stick to this solution.

Now we are prepared to solve eq. (26) (or (31)). This can be done in a full analogy with the hyper-Kähler TN case [2], based essentially upon the PG invariance (28). Using (34), we obtain

$$\kappa^{++} = \partial^{++} \kappa,\quad (35)$$

$$(1) \ R > 0, \ \kappa = \frac{1}{\sqrt{R}} \arctan \sqrt{R} \ \hat{\phi} \ ; \ (2) \ R < 0, \ \kappa = \frac{1}{\sqrt{|R|}} \arctanh \sqrt{|R|} \ \hat{\phi}.\quad (35)$$

For definiteness, in what follows we will choose the solution (1) in (35). Then, making the redefinition

$$\hat{x}^+ = \exp{2i\kappa} \hat{x}^+, \quad \bar{x}^+ = \exp{-2i\kappa} \bar{x}^+,\quad (36)$$

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we reduce \((31)\) to
\[
\partial^+ \bar{x}^+ = 0 \quad \Rightarrow \quad \bar{x}^+ = x^i w_i^+ , \quad \bar{x}^+ = \bar{x}_i w^i = -\bar{x}_i w_i^+, \quad \bar{\phi} = -2i\lambda x^{(i} x^{k)} w_i^+ w_k^- ,
\]
where, in expressing \(\bar{\phi}\), we essentially made use of the PG symmetry \((28)\).

Combining eqs. \((29)\), \((33)\), \((36)\) and \((37)\) we can now write the expressions for \(x^+ , \bar{x}^+\) in the following form
\[
x^+ = \frac{1}{\sqrt{1 + R\bar{\phi}^2}} \exp\{2i\kappa\} x^i w_i^+ , \quad \bar{x}^+ = -\frac{1}{\sqrt{1 + R\bar{\phi}^2}} \exp\{-2i\kappa\} \bar{x}_i w_i^+ ,
\]
where \(\kappa\) and \(\bar{\phi}\) are expressed through \(x^i , \bar{x}_i\) according to eqs. \((35)\), \((37)\). Comparing \((38)\) with the general definition of the \(x\)-bridges \((5)\), we can identify \(x^i , \bar{x}_i\) with the \(\tau\)-world coordinates, i.e. with the coordinates of the initial 4-dimensional QK manifold.

We still need to find \(x^- , \bar{x}^-\) as functions of \(x^i , \bar{x}_i\) and harmonics \(w^\pm_i\) by solving eq. \((27)\) and its conjugate. Dropping intermediate technical steps (they involve a number of redefinitions), its general solution can be presented in the following form
\[
x^- = \frac{1}{2\lambda} \sqrt{1 + R\bar{\phi}^2} \left[ e^{-2i\kappa(i\lambda s)} - e^{2i\kappa(\bar{\phi})} \right] \bar{x}^- ,
\]
\[
\bar{x}^- = \frac{1}{2\lambda} \sqrt{1 + R\bar{\phi}^2} \left[ e^{-2i\kappa(i\lambda s)} - e^{-2i\kappa(\bar{\phi})} \right] x^- .
\]
Here
\[
\bar{x}^- = x^i w_i^- , \quad \bar{x}^- = -\bar{x}_i w_i^- , \quad s = x^i \bar{x}_i , \quad \kappa(i\lambda s) \equiv \kappa_0 = \frac{i\lambda}{\sqrt{R}} \arctanh \sqrt{R}(\lambda s) .
\]

For what follows it will be convenient to define
\[
A(s) \equiv 1 - R\lambda^2 s^2 , \quad B(s) \equiv 1 + 4\lambda^2 s + R\lambda^2 s^2 , \quad C(s) \equiv 1 + R s + R\lambda^2 s^2 .
\]

Now we are ready to find explicit expressions for the two important quantities entering the general expression for the \(\tau\)-metric \((17)\), \((18)\):
\[
1 - R\partial^- v^{++} = A \frac{1 - R\bar{\phi}^2}{(1 + R\bar{\phi}^2)^2} , \quad 1 - R(x \cdot H) = \frac{C}{A} \frac{1 + R\bar{\phi}^2}{1 - R\bar{\phi}^2} .
\]

As a next step towards the QK Taub-NUT metric, one needs to find the entries of the matrix \(X^{\mu \nu} \equiv \partial^+ x^{\mu \nu}\) by solving the set of algebraic equations \((20)\). In the complex notation, this set is divided into the two mutually conjugated ones, each consisting of four equations. It is clearly enough to consider one such set, e.g.
\[
X^{+ \rho k} \nabla_{\rho k} x^- = 1 , \quad X^{+ \rho k} \nabla_{\rho k} x^+ = 0 , \quad X^{+ \rho k} \nabla_{\rho k} \bar{x}^\pm = 0 ,
\]
\[
A(s) \equiv 1 - R\lambda^2 s^2 , \quad B(s) \equiv 1 + 4\lambda^2 s + R\lambda^2 s^2 , \quad C(s) \equiv 1 + R s + R\lambda^2 s^2 .
\]
where $X^+\rho k \equiv X^{+\rho k}_1$, $(\bar{X}^+\rho k \equiv -X^{+\rho k}_2)$. It is convenient to work with

$$
\hat{X}^{+\rho k} = e^{R\nu} X^{+\rho k} = \sqrt{1 + R^2} X^{+\rho k}.
$$

(44)

It remains to calculate the transition matrix elements $\nabla_{\rho k} x^{\pm}$, $\nabla_{\rho k} x^{\pm}$ entering eqs. (20). This can be done straightforwardly, the corresponding expressions look rather involved and by this reason we do not quote them here explicitly (more details are given in [7]). Surprisingly, the expressions for $\hat{X}^{+\rho k}$ prove to be much simpler:

$$
\begin{align*}
\hat{X}^{+1k} &= \frac{1}{4} \left[ (3A + B)\epsilon^{kl} - 4\lambda^2 (A + C) x^{(k} x^{l)} \right] w^+_i e^{2\iota a}, \\
\hat{X}^{+2k} &= (\partial^+ x^k) = \lambda^2 (A + C) \ (\bar{x}^{k} x^l) w^+_i e^{2\iota a}
\end{align*}
$$

(45)

(the remaining components can be obtained by conjugation).

It will be convenient to rewrite the metric (17), (18) through $\hat{X}^{+\rho i}$

$$
g^{\rho i, \lambda k} = \frac{1}{C} \det(\partial \hat{H}) \ G^{\rho i, \lambda k},
$$

(46)

$$
\hat{G}^{\rho i, \lambda k} = (1 + R^2) G^{\rho i, \lambda k} = e^{\omega \beta} \ [ \partial^{--} \hat{X}^{+\rho i} \hat{X}^{+\lambda k} + (\rho i \leftrightarrow \lambda k) ].
$$

(47)

As the last step, one should compute $\det(\partial \hat{H})$. After some algebra, it can be represented in the following concise form

$$
\det(\partial \hat{H}) = -\frac{1}{2} \frac{A}{C^3} \left( 1 - R^2 \right) \left\{ e^{\alpha \beta} \partial^{--} \hat{X}^{+\rho k} \partial^{--} \hat{X}^{+\lambda l} \right\} \left\{ \nabla_{\rho k} x^{\pm} \nabla_{\lambda l} x^{\pm} \epsilon_{\mu \nu} \right\}.
$$

(48)

As a result of rather cumbersome, though straightforward computation one eventually gets the simple expression for $\det(\partial \hat{H})$

$$
\det(\partial \hat{H}) = A^2 \frac{B}{C^3} e^{4\iota a} = (1 - R\lambda^2 s^2)^2 \frac{1 + 2\lambda^2 s + \lambda^2 s (2 + s R)}{1 + R s + R \lambda^2 s^2} e^{4\iota a}.
$$

(49)

The harmonic dependence disappeared in $\det(\partial \hat{H})$, as it should be.

The calculation of this determinant is the most long part of the whole story. Once this has been done, the computation of the $\tau$ basis inverse metric amounts to the computation of entries of the matrix $G^{\rho i, \lambda l}$. The final answer for the metric tensor is as follows

$$
\begin{align*}
g_{1k,1t} &= \frac{D}{C^2 B} (\bar{x}_k \bar{x}_t), \\
g_{2k,2t} &= \frac{D}{C^2 B} (x_k x_t), \\
g_{1k,2t} &= \frac{1}{C^2 B} \left[ B^2 \epsilon_{kt} + D (\bar{x}_k x_t) \right],
\end{align*}
$$

(50)

\begin{align*}
A &= 1 - R\lambda^2 s^2, \\
B &= 1 + 4\lambda^2 s + R\lambda^2 s^2, \\
C &= 1 + R s + R \lambda^2 s^2.
\end{align*}

Here $D \equiv \lambda^2 (A + C)(A + B) = 2\lambda^2 (2 + R s)(1 + 2\lambda^2 s)$. One should observe that this final expression is valid for any sign of the parameter $R$ even if, in the intermediate steps (see relation (23) for instance), the sign choice plays a significant role.
4. To compare to the results in the literature one has to use

\[ dx^i = x^i \left( \frac{ds}{2s} + i \frac{\sigma_3}{2} \right) - \bar{x}^i \left( \frac{\sigma_2 - i \sigma_1}{2} \right), \quad d\sigma_i = \frac{1}{2} \epsilon_{ijk} \sigma_j \wedge \sigma_k. \]  

(51)

Using the notation \( A \cdot B \equiv A^i B_i \) relation (51) implies

\[-\bar{x} \cdot dx = \frac{ds}{2} + is \frac{\sigma_3}{2}, \quad dx \cdot d\bar{x} = \frac{ds^2}{4s} + \frac{s}{4} (\sigma_1^2 + \sigma_2^2 + \sigma_3^2), \quad s = x \cdot \bar{x}. \]  

(52)

The metric given by (50) becomes

\[ \frac{1}{2} \left[ \frac{B}{sC^2} ds^2 + \frac{sB}{C^2} (\sigma_1^2 + \sigma_2^2) + \frac{sA^2}{C^2 B} \sigma_3^2 \right]. \]  

(53)

The most general Bianchi IX euclidean Einstein metrics can be deduced from Carter’s results. A convenient standardization is the following

\[ dr^2 = l^2 \left\{ \frac{r^2 - 1}{\Delta(r)} (dr)^2 + 4 \frac{\Delta(r)}{r^2 - 1} \sigma_3^2 + (r^2 - 1)(\sigma_1^2 + \sigma_2^2) \right\}, \]  

(54)

with

\[ \Delta(r) = -\frac{\Lambda l^2}{3} r^4 + (1 + 2\Lambda l^2) r^2 - 2M r + 1 + \Lambda l^2. \]  

(55)

These metrics are Einstein, with Einstein constant \( \Lambda \) and isometry group \( U(2) \). If we take \( M = 4/3\Lambda l^2 + 1 \) the metric simplifies to

\[ d\tau^2(Q) = l^2 \left\{ \frac{r + 1}{r - 1} (dr)^2 + 4 \frac{r - 1}{r + 1} \Sigma(r) \sigma_3^2 + (r^2 - 1)(\sigma_1^2 + \sigma_2^2) \right\}, \]  

(56)

where now

\[ \Sigma(r) = 1 - \frac{\Lambda l^2}{3} (r - 1)(r + 3). \]  

(57)

The identifications

\[ \frac{r - 1}{2} = (4\lambda^2 - R) \frac{s}{1 + Rs + R\lambda^2 s^2}, \quad 4 \frac{\Lambda l^2}{3} = \frac{R}{4\lambda^2 - R}, \]  

(58)

give the relation

\[ 4(4\lambda^2 - R) \left[ \frac{B}{sC^2} ds^2 + \frac{sB}{C^2} (\sigma_1^2 + \sigma_2^2) + \frac{sA^2}{C^2 B} \sigma_3^2 \right] = \frac{d\tau^2(Q)}{l^2}. \]  

(59)

The quaternionic metric is complete for \( \Lambda < 0 \) and is asymptotically Anti de Sitter. It has been considered recently in under the name Taub-NUT-AdS metric and reveals itself a useful background for computing black-holes entropy.

5. In this paper we made the first practical use of the harmonic space formulation of the QK geometry to compute a non-trivial QK metric, the four-dimensional quaternionic Taub-NUT metric. As we were convinced, the harmonic space techniques, like in the HK case, allows one to get the explicit form of the QK metric starting from a
given QK potential and following a generic set of rules. It would be interesting to apply this approach to find the QK analogs of some other interesting 4- and higher-dimensional HK metrics, in particular, the quaternionic Eguchi-Hanson metric and the quaternionic generalization of the multicenter metrics of Gibbons and Hawking [13].

Finally, we note that the HK Taub-NUT metric plays an important role in the modern $p$-branes realm, yielding an essential part of one of the fundamental brane-like classical solutions of $D = 11$ supergravity, the so-called ‘Kaluza-Klein monopole’ (see, e.g. [15]). It would be of interest to reveal possible brane implications of the QK Taub-NUT metric constructed here.

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