Exact Analytic Continuation with Respect to the Replica Number in the Discrete Random Energy Model of Finite System Size

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Abstract

An expression for the moment of partition function valid for any finite system size $N$ and complex power $n$ ($\text{Re}(n) > 0$) is obtained for a simple spin glass model termed the discrete random energy model (DREM). We investigate the behavior of the moment in the thermodynamic limit $N \to \infty$ using this expression, and find that a phase transition occurs at a certain real replica number when the temperature is sufficiently low, directly clarifying the scenario of replica symmetry breaking of DREM in the replica number space \textit{without using the replica trick}. The validity of the expression is numerically confirmed.

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§1. Introduction

The replica method (RM) is one of the few analytic schemes available for the research of disordered systems. In physics, this method has been well known since the 1970s and has been successfully applied to the analyses of spin-glass models, although the essential idea behind the method can be dated back to the end of 1920s where it appeared as a theorem for computing the average of logarithms. More recently, considerable attention has been paid to the similarity between statistical mechanics of disordered systems and the Bayesian method in problems related to information processing (IP). The number and variety of applications of RM to problems the IP research are increasing rapidly, including error-correcting codes, image restoration, neural networks, combinatorial problems, and so on.

Although only the limit value \( \frac{1}{N} \langle \ln Z \rangle \rightarrow \lim_{n \to 0} \left( \langle Z^n \rangle^{1/N} - 1 \right) / n \) is usually emphasized, RM can be considered to be a systematic procedure for calculating generalized moments \( \langle Z^n \rangle \) of the partition function \( Z \) in the case of \( N \to \infty \) when \( Z \) depends on a certain external randomness. Here, \( N \) characterizes the system size, \( n \in \mathbb{R} \) (or \( \mathbb{C} \)) is a real (or complex) number and \( \langle \cdots \rangle \) means the average over the external randomness. For most problems, a direct assessment of \( \langle Z^n \rangle \) is difficult for a general \( n \in \mathbb{R} \) (or \( \mathbb{C} \)), whereas such an assessment for natural numbers \( n = 1, 2, \ldots \) is possible in the thermodynamic limit \( N \to \infty \). Therefore, \( \langle Z^n \rangle \) is first computed for natural numbers and their analytic continuation is used to extend \( \langle Z^n \rangle \) to \( n \in \mathbb{R} \) (or \( \mathbb{C} \)). This is usually termed the replica trick.

However, the validity of the replica trick is doubtful. The most obvious analytic continuation, obtained under the replica symmetric (RS) ansatz, sometimes leads to the wrong results. The causes of these errors were actively debated in the 1970s until Parisi discovered the replica-symmetry-breaking (RSB) scheme for constructing reasonable solutions within the framework of RM. Since this discovery, there have been no known examples for which physically wrong results have been derived by RM, in conjunction with the Parisi scheme if necessary. Therefore, RM is now empirically recognized as a reliable procedure in physics, although the mathematical justification of the replica trick still remains open. However, this problem is now generating interest again, in particular, in the application of RM to IP problems. This is because most theories in IP research have conventionally been developed with mathematical soundness highly valued.

The purpose of this paper is to provide a method to approach the problems of RM. Specifically, we give a useful formula to compute \( \langle Z^n \rangle \) directly for \( n \in \mathbb{C} \) at finite \( N \) for a simple spin glass model, termed the discrete random energy model (DREM). This formula is numerically tractable, so one can directly observe how the system approaches the
thermodynamic limit with the aid of numerical calculation. Furthermore, this analytically clarifies the correct behavior for $N \to \infty$, making a direct examination of the validity of RM.

We have two main reasons for picking DREM from among the various disordered systems. First, this model is simple enough to handle analytically. It is already known that RM, in conjunction with the Parisi scheme, can evaluate the correct free energy for a family of random energy models (REM) including DREM at the limit of $n \to 0$.

However, the existing procedure seems at odds with a theorem concerning analytic continuation provided by Carlson, which holds for DREM of finite $N$ claiming the uniqueness of analytic continuation from natural numbers $n \in \mathbb{N}$ to complex numbers $n \in \mathbb{C}$, when the temperature is sufficiently low. For this, our approach shows that a phase transition occurs at a certain critical replica number $n_c \in [0, 1]$ in such cases, which clarifies that RM can be consistent with Carlson’s theorem. This may offer a useful discipline to perform analytic continuation from $n \in \mathbb{N}$ to $n \in \mathbb{R}$ (or $\mathbb{C}$) in RM. The second reason is a relationship between REM and certain problems of IP. Recent research on error-correcting codes has revealed that REM is closely related to a randomly constructed code. These codes are known to provide the best error correction performance in information theory, and the performance evaluation of such codes is similar to the computation of $\langle Z^n \rangle$ for $n \in \mathbb{R}$ (see appendix A). Therefore, the current investigation will indirectly justify the RM-based analysis of error-correcting codes performed previously.

This paper is organized as follows. In section 2, we introduce DREM and briefly review how RM has been conventionally employed in analyzing this system. Referring to Carlson’s theorem, we address how the conventional scenario for taking a limit $n \to 0$ seems controversial. In order to resolve this difficulty, we propose in section 3 a new scheme to directly evaluate $\langle Z^n \rangle$ for REM of finite $N$ and complex $n$ without using the replica trick. Taking the limit $N \to \infty$, we analytically clarify how $\langle Z^n \rangle$ behaves in the thermodynamic limit and numerically verify this behavior. In section 4, we show how RM can be consistent with the results obtained by the proposed scheme. Section 5 is a summary.

§2. Replica method in the discrete random energy model (DREM)

In order to clearly state the problem addressed in this paper, we first review how RM has been conventionally employed in analyzing REM. For convenience in the later analysis, we mainly concentrate on DREM, but the addressed problem is widely shared with other versions of REM as well.

A DREM is composed of $2^N$ states, the energies of which, $\epsilon_A$ ($A = 1, 2, \ldots, 2^N$), are
independently drawn from an identical distribution

\[ P(E_i) = 2^{-M} \left( \frac{M}{2} M + E_i \right), \quad \left( E_i = i - \frac{M}{2} \right), \quad (2.1) \]

over \( M + 1 \) energy levels \( E_i = -M/2, -M/2 + 1, \ldots, M/2 - 1, M/2 \). For each realization \( \{ \epsilon_A \} \), the partition function

\[ Z = \sum_{A=1}^{2^N} \exp(-\beta \epsilon_A), \quad (2.2) \]

and the free energy (density)

\[ F = -\frac{kT}{N} \log Z \quad (2.3) \]

can be used for computing various thermal averages. However, when the configurational average is required, one has to compute the averaged free energy \( \langle F \rangle = -\frac{kT}{N} \langle \log Z \rangle \), the direct evaluation of which is generally difficult. Here, \( \langle \cdots \rangle \) denotes the configurational average with respect to \( \{ \epsilon_A \} \). On the other hand, the moment of the partition function \( \langle Z^n \rangle \) can be easily calculated in various models for natural numbers \( n = 1, 2, \ldots \). Therefore, the replica method evaluates the averaged free energy using the replica trick

\[ \frac{1}{N} \langle \log Z \rangle = \lim_{n \to 0} \frac{\langle Z^n \rangle^{\frac{1}{n}} - 1}{n}, \quad (2.4) \]

analytically continuing the expression of \( \langle Z^n \rangle \) for \( n = 1, 2, \ldots \) to that for real (or complex) numbers \( n \).

For a given natural number \( n \), the moment of DREM is calculated as

\[
\langle Z^n \rangle = \sum_{A_1=1}^{2^N} \sum_{A_2=1}^{2^N} \cdots \sum_{A_n=1}^{2^N} \sum_{i_1=0}^{M} P \left( E_{i_1}^{(1)} \right) \sum_{i_2=0}^{M} P \left( E_{i_2}^{(2)} \right) \cdots \sum_{i_{2N}=0}^{M} P \left( E_{i_{2N}}^{(2N)} \right)
\]
\[
\exp \left( -\beta \sum_{B=1}^{2^N} E_{i_B}^{(B)} \sum_{\mu=1}^{n} \delta_{BA_{\mu}} \right)
\]
\[
e^{\frac{nM\alpha}{2}} \sum_{A_1=1}^{2^N} \sum_{A_2=1}^{2^N} \cdots \sum_{A_n=1}^{2^N} \prod_{B=1}^{2^N} \left( 1 + e^{-\beta \sum_{\mu=1}^{n} \delta_{BA_{\mu}}} \right)^M
\]
\[
= \sum_{A_1=1}^{2^N} \sum_{A_2=1}^{2^N} \cdots \sum_{A_n=1}^{2^N} \prod_{B=1}^{2^N} \exp \left( N \alpha I \left( \beta \sum_{\mu=1}^{n} \delta_{BA_{\mu}} \right) \right)\quad (2.5)
\]

where \( \alpha = M/N \), the function \( I(x) \) is defined as

\[ I(x) = \log \left( \cosh \frac{x}{2} \right). \quad (2.6) \]
An identity \( \prod_{B=1}^{2^N} \exp \left( -\frac{\beta}{2} \sum_{\mu=1}^{n} \delta_{BA\mu} \right) = \exp \left(-\frac{n^2\beta}{2}\right) \) was employed for obtaining the final expression of Eq. (2.5). From now on, we focus on the case \( \alpha > 1 \), for which the replica symmetry can be broken when the temperature is sufficiently low.

Unfortunately, performing the summation in Eq. (2.5) exactly is difficult. Instead, in conventional RM, Eq. (2.5) is represented by the most dominant contribution in the summation. This can be justified for natural numbers \( n \) in the limit \( N \to \infty \). Notice that the summation is invariant with respect to the permutation of the replica indices \( \mu = 1, 2, \ldots, n \).

This \textit{replica symmetry} restricts the candidates of the most dominant contribution to three possibilities, which are here referred to as solutions of the replica symmetric 1 (RS1), the replica symmetric 2 (RS2), and the 1-step replica symmetry breaking (1RSB).

- **RS1**

  In RS1, all of the \( n \) replicas are assumed to occupy \( n \) different states \( B(=1, 2, \ldots, 2^N) \). Therefore, for a given \( B \),

  \[
  \sum_{\mu=1}^{n} \delta_{BA\mu} = \begin{cases} 
  1 & \text{(when } B \text{ is one of the } n \text{ occupied states)}, \\
  0 & \text{(otherwise).} 
  \end{cases} \tag{2.7}
  \]

  The number of ways to assign \( n \) replicas to \( n \) out of \( 2^N \) different states is

  \[
  2^N \times (2^N - 1) \times \cdots \times (2^N - n + 1) \sim 2^n N. \tag{2.8}
  \]

  For each case, the configuration of this type contributes

  \[
  \exp \left( nN \alpha I(\beta) \right), \tag{2.9}
  \]

  in Eq. (2.5). This means that the contribution to the moment from RS1 becomes

  \[
  \langle Z^n \rangle = 2^n N \exp \left( nN \alpha I(\beta) \right) = \exp nN \left( \log 2 + \alpha \log \left( \cosh \frac{\beta}{2} \right) \right). \tag{2.10}
  \]

- **RS2**

  In RS2, all of the \( n \) replicas are assumed to occupy a particular state \( B \). Therefore, for a given \( B \),

  \[
  \sum_{\mu=1}^{n} \delta_{BA\mu} = \begin{cases} 
  n & \text{(when } B \text{ is the occupied state)}, \\
  0 & \text{(otherwise).} 
  \end{cases} \tag{2.11}
  \]

  The number of ways to choose one out of \( 2^N \) states is \( 2^N \). For each case, the configuration of this type contributes

  \[
  \exp \left( N \alpha I(n\beta) \right), \tag{2.12}
  \]
in Eq. (2.5), which indicates that the contribution from RS2 is

$$
\langle Z^n \rangle = 2^N \exp \left( nN\alpha I(n\beta) \right)
= \exp \left( \log 2 + \alpha \log \left( \cosh \frac{n\beta}{2} \right) \right).
$$

(2.13)

• 1RSB

In 1RSB, $n$ replicas are assumed to be equally assigned to $n/m$ states $B$, where $m$ is an aliquot of $n$. Therefore, for a given $B$,

$$
\sum_{\mu=1}^{n} \delta_{BA_{\mu}} = \begin{cases} m & (\text{when } B \text{ is one of the } n/m \text{ occupied states}), \\ 0 & (\text{otherwise}). \end{cases}
$$

(2.14)

The number of ways to select $n/m$ out of $2^N$ states equally assigning $n$ replicas to the $n/m$ states is

$$
\frac{(2^N)!}{(2^N-n/m)!} \times \frac{n!}{m^{n/m}} \sim 2^{\frac{n}{m}N}.
$$

(2.15)

For each case, the configuration of this type contributes

$$
\exp \left( \frac{n}{m} N\alpha I(m\beta) \right),
$$

(2.16)

in Eq. (2.5). Taking all the possible values of $m$ into account, the contribution from 1RSB can be summarized as

$$
\langle Z^n \rangle = \sum_{m} 2^{\frac{n}{m}N} \exp \left( \frac{n}{m} N\alpha I(m\beta) \right)
= \sum_{m} \exp \left( \frac{n}{m} N \left[ \log 2 + \alpha I(m\beta) \right] \right)
\sim \exp \left( \text{extr} \left\{ \frac{n}{m} N \left[ \log 2 + \alpha I(m\beta) \right] \right\} \right).
$$

(2.17)

In the last expression, we have replaced the summation over $m$ with the extremization with respect to $m$ (extr$_m\{\cdots\}$), which is hopefully valid for large $N$, analytically continuing the expression with respect to $m$ from natural numbers to real numbers. The extremization with respect to $m$ yields the condition

$$
\log 2 + \alpha I(m\beta) = \alpha m\beta I'(m\beta),
$$

(2.18)

which implies that the moment is expressed as

$$
\langle Z^n \rangle = \exp \left[ nN\alpha \beta I'(m_{c}\beta) \right],
$$

(2.19)
Fig. 1. The configuration of replicas for each solution. In RS1, \( n \) replicas are distributed in different states. In RS2, \( n \) replicas are concentrated in one state. In 1RSB, \( n \) replicas are equally assigned to \( n/m \) states. For 1RSB, however, this figure does not directly correspond to the solution because the critical value of \( m \) is not necessarily an integer and can be larger than \( n \) for \( \beta > \beta_c \).

where \( m_c \) is the solution of Eq. (2.18). Eq. (2.6) indicates that \( m_c \) can be represented as \( m_c = \beta_c / \beta \), where the critical inverse temperature \( \beta_c > 0 \) is determined by

\[
\log 2 + \alpha \left( \log \left( \cosh \frac{\beta_c}{2} \right) - \frac{\beta_c}{2} \tanh \frac{\beta_c}{2} \right) = 0. \tag{2.20}
\]

This provides a simple expression of the 1RSB solution as

\[
\langle Z^n \rangle = \exp \left( nN\alpha \beta I'(\beta_c) \right) = \exp \left( \frac{nN\alpha\beta}{2 \tanh \frac{\beta_c}{2}} \right). \tag{2.21}
\]

The configurations of the replicas assumed for RS1, RS2, and 1RSB are pictorially presented in Fig. 1. It should be emphasized here that the above three solutions are derived for \( n = 1, 2, \ldots \), assuming \( N \) is sufficiently large. However, the obtained expressions are likely to hold for real \( n \) as well. Therefore, in conventional analysis, the replica trick of Eq. (2.4) is carried out, selecting one possibly relevant solution of the three, which is hopefully valid for large \( N \).

The existing prescription for selecting the relevant solution is as follows. For small \( n \), RS1, RS2, and 1RSB are ordered \( RS2 > RS1 > 1RSB \) from the viewpoint of their amplitudes (Fig. 2). The contribution from RS2, however, converges to \( 2^N \) rather than unity for \( n \to 0 \), and therefore, the replica trick leads to divergence. Hence, this solution is discarded. After excluding this solution, the leading contribution always comes from RS1, which guarantees a finite limit in Eq. (2.4).

Actually, the answer obtained from this solution is correct in the case of high temperatures \( 0 < \beta < \beta_c \). However, this solution becomes invalid for low temperatures \( \beta > \beta_c \), for which the correct answer is provided by 1RSB. It may be worth noting that the value of \( m_c \), in this low temperature case, is placed in the interval \( n \leq m_c \leq 1 \) when \( n \to 0 \), which is out of the ordinary range \( 1 \leq m_c \leq n \) for \( n = 1, 2, \ldots \). This prescription for taking the \( n \to 0 \)
limit is empirically justified for a family of REMs because it reproduces the correct answers obtainable by other schemes at the limit of $n \to 0$.\[13\] However, the following two issues still need further investigation.

The first issue is the reason for expurgating RS2. Carlson’s theorem, which guarantees the uniqueness of analytic continuation from natural $n \in \mathbb{N}$ to complex $n \in \mathbb{C}$, might be useful for resolving this problem.\[22, 23]\] Unlike other systems such as the Sherrington-Kirkpatrick model\[24] and the original REM\[19] a modified moment of the current DREM $\langle (e^{-M/2}Z)^n \rangle^{1/N}$, which is extended from $n \in \mathbb{N}$ to $n \in \mathbb{C}$, satisfies an inequality,

$$\left| \langle (e^{-M/2}Z)^n \rangle^{1/N} \right| \leq \left( \langle e^{-M/2}Z \rangle^{\text{Re}(n)} \right)^{1/N} = \left( \sum_{A=1}^{2N} \exp \left[ -\beta (\epsilon_A + M/2) \right] \right)^{\text{Re}(n)}^{1/N} \leq \left( \sum_{A=1}^{2N} \right)^{\text{Re}(n)}^{1/N} = 2^{\text{Re}(n)} < O \left( \exp \left[ \pi |n| \right] \right), \quad (2.22)$$

for $\text{Re}(n) \geq 0$ and $\forall N = 1, 2, \ldots$, since $\epsilon_A + M/2$ is lower bounded by 0. Suppose that we could construct another extension $\psi(n; N)$ ($n \in \mathbb{C}$), which satisfies the the growth condition, $\psi(n; N) < O \left( \exp \left[ \pi |n| \right] \right)^*$, and agrees with $\langle (e^{-M/2}Z)^n \rangle^{1/N}$ at all the natural numbers $n = 1, 2, \ldots$. This indicates that a similar inequality $\left| \psi(n; N) - \langle (e^{-M/2}Z)^n \rangle^{1/N} \right| \leq \left| \psi(n; N) \right| + \left| \langle (e^{-M/2}Z)^n \rangle^{1/N} \right| < O \left( \exp[\pi |n|] \right)$ holds for $\text{Re}(n) \geq 0$ and the difference $\left| \psi(n; N) - \langle (e^{-M/2}Z)^n \rangle^{1/N} \right|$ vanishes at all the natural numbers $n = 1, 2, \ldots$, as $\psi(n; N)$ and $\langle (e^{-M/2}Z)^n \rangle^{1/N}$ completely coincide for $n \in \mathbb{N}$. Then, Carlson’s theorem (Theorem 5.81 in page 186 of Ref. [22]) ensures that $\left| \psi(n; N) - \langle (e^{-M/2}Z)^n \rangle^{1/N} \right|$ is identical to 0, implying that $\psi(n; N)$ and $\langle (e^{-M/2}Z)^n \rangle^{1/N}$ are identical and, therefore, analytic continuation of $\langle (e^{-M/2}Z)^n \rangle^{1/N}$ from natural $n \in \mathbb{N}$ to complex $n \in \mathbb{C}$ can be uniquely determined, unless analyticity of $\langle (e^{-M/2}Z)^n \rangle^{1/N}$ is lost in the limit $N \to \infty$. Since $e^{-M/2}$ is a non-vanishing constant, this means that analytic continuation of the moment $\langle Z^n \rangle^{1/N}$ is also unique as long as $\langle Z^n \rangle^{1/N}$ remains analytic with respect to $n$ in the limit $N \to \infty$. Hence, the dominant solution for $n = 1, 2, \ldots$, rather than for $0 < n < 1$, should be selected as the relevant solution for $n \to 0$. This recipe successfully reproduces the correct answer for the high temperature region $0 < \beta < \beta_c$. However, this is still not fully satisfactory because RS2 becomes dominant for $n = 1, 2, \ldots$ in the case of $\beta > \beta_c$ and, therefore, should be selected as

\[\text{This condition is necessary to exclude a trivial multiplicity caused by addition of certain analytic functions which vanish at all the natural numbers } n = 1, 2, \ldots, \text{ such as } \sin(\pi n).\]
Fig. 2. Solutions obtained under the RS1, RS2 and 1RSB assumptions. RS1 and RS2 agree at $n = 1$ while RS2 contacts 1RSB at a certain point of positive $n$. (a) For $\beta < \beta_c$, the contact point is placed in the region of $n > 1$. (b) For $\beta > \beta_c$, on the other hand, it is located in $0 < n < 1$.

The relevant solution for $n \to 0$ if analyticity is preserved for $N \to \infty$, which, unfortunately, leads to a wrong answer. Hence, a certain phase transition must occur at a critical replica number $0 < n_c < 1$ for $\beta > \beta_c$. However, to the knowledge of the authors, such a phase transition with respect to $n$ has not yet been fully examined for most disordered systems. This might be because so far the greatest attention has been paid only to the final results at the limit $n \to 0$. However, detailed analysis of phase transitions of this type may soon be needed as the replica calculation for non-vanishing $n$ has recently begun to be employed in problems related to IP and analysis of certain dynamics involved with multiple time scales.

The second question is the origin of 1RSB. In conventional analysis at the limit $n \to 0$, this solution is introduced by modifying RS1 to keep the entropy of the correct solution non-negative for $\beta > \beta_c$. It would appear that 1RSB originates from RS1. However, at least for positive $n$, this association seems unlikely because the two solutions cross each other only at $n = 0$ (see Fig. 2). Therefore, it is impossible to relate the origin of 1RSB to RS1 at large $n$, from which the solutions of smaller $n$ are extrapolated. On the other hand, 1RSB contacts RS2 at a certain point of positive $n$, implying that 1RSB bifurcates from RS2. As the contact point is located in $0 < n < 1$ for $\beta > \beta_c$, this seems consistent with the aforementioned possible phase transition at $n = n_c$ (see Fig. 2 (b)). Nevertheless, such a scenario cannot be so easily accepted since RS2 still dominates 1RSB even below the contact point; a mere contact does not change the dominance between the two solutions.

It may help in resolving these problems to analyze DREM employing a completely different methodology. In the next section, we provide a scheme to calculate the moment of DREM as a step towards clarifying the mysteries of RM. The proposed method is powerful
enough to evaluate the moment in the right half of the complex plane $\text{Re}(n) > 0$ for any arbitrary finite $N$ without using the replica trick, which supplies a model answer for RM.

§3. Direct calculation of the moment for grand canonical DREM (GCDREM)

3.1. General formula

In order to introduce a novel scheme for calculating moments of the partition function of DREM, we first rewrite the partition function using the occupation numbers $n_i$ ($i = 0, 1, 2, \ldots, M$) as

$$Z = \sum_{A=1}^{2^N} \exp(-\beta \epsilon_A) = \sum_{i=0}^{M} n_i \exp(-\beta E_i) = \omega^{M} \sum_{i=0}^{M} n_i \omega^i \ (\omega \equiv e^{-\beta}). \quad (3.1)$$

This means that the partition function of DREM can be completely determined by a set of occupation numbers $\{n_i\}$, in which details of the energy configuration $\{\epsilon_A\}$ are ignored. This makes it possible to assess the moments $\langle Z^n \rangle$ directly from $\{n_i\}$ without referring to the full energy configuration $\{\epsilon_A\}$. This significantly reduces the necessary cost for computing the partition function when the calculation is numerically performed.

Two methods are known for generating $\{n_i\}$. The straightforward method is to count $n_i$ independently, drawing the $2^N$ energy states from Eq. (2.1). The system obtained by this is referred to as the canonical discrete random energy model (CDREM). Although this yields a rigorously correct realization of DREM that satisfies the constraint $\sum_{i=0}^{M} n_i = 2^N$, it takes $2^N$ steps to count $\{n_i\}$ and hence, is computationally difficult. In order to resolve this difficulty in numerical experiments, Moukarzel and Parga\cite{20,21} have proposed the grand canonical version of the discrete random energy model (GCDREM)\footnote{Employment of GCDREM is not essential for reducing the numerical cost. We have discovered a scheme for generating CDREM in a time scale similar to that for GCDREM. It is shown in appendix C}. In GCDREM, the occupation numbers are independently determined using the Poisson distribution

$$P(n_i) = e^{-\gamma_i} \frac{\gamma_i^{n_i}}{n_i!}, \quad (\gamma_i = 2^N P(E_i)), \quad (3.2)$$

where $\gamma_i$ is the average occupation number. The greatest advantage of GCDREM is that one can drastically reduce the necessary computational cost for generating $\{n_i\}$ from $2^N$ to $M + 1$. One possible drawback is that the constraint $\sum_{i=0}^{M} n_i = 2^N$ is only satisfied in average, $\langle \sum_{i=0}^{M} n_i \rangle = 2^N$, which implies that this method does not correspond strictly to the original model. However, the RM-based calculation indicates that thermodynamical properties of GCDREM become identical to those of CDREM for $N \to \infty$, which is provided in appendix B and one can show that the difference rapidly vanishes as $N$ becomes large.
and is almost indistinguishable even for \( N = 3 \), verified in appendix \( \text{C} \). In addition, this version of DREM has another advantage in analytic calculations as the summation can be carried out independently, as is shown below.

In GCDREM, the moment is expressed as

\[
\langle Z_n \rangle = \sum_{n_0=0}^{\infty} \sum_{n_1=0}^{\infty} \cdots \sum_{n_M=0}^{\infty} P(n_0)P(n_1)\cdots P(n_M)Z^n
\]

\[
= \omega^{-\frac{m}{2}} \lim_{\epsilon \to 0} \sum_{n_0=0}^{\infty} \sum_{n_1=0}^{\infty} \cdots \sum_{n_M=0}^{\infty} P(n_0)P(n_1)\cdots P(n_M) \left( \sum_{i=0}^{M} n_i \omega^i + \epsilon \right)^n, 
\]

where an infinitesimal constant \( \epsilon > 0 \) is introduced in order to keep \( Z \) positive even when all the occupation numbers vanish. This makes it possible to employ an identity for the positive number \( c \)

\[
c^n = \int_{H} (-\rho)^{-n-1} e^{-c\rho} \frac{d\rho}{\Gamma(-n)}, \quad (c > 0, \Gamma(n) \equiv -2i \sin n\pi \Gamma(n)),
\]

where the integration contour is shown in Fig. \( \text{K} \) for assessing the moment as

\[
\langle Z^n \rangle = \omega^{-\frac{m}{2}} \lim_{\epsilon \to 0} \int_{H} (-\rho)^{-n-1} e^{-\left( \sum_{i=0}^{M} n_i \omega^i + \epsilon \right)\rho} d\rho
\]

\[
= \omega^{-\frac{m}{2}} \lim_{\epsilon \to 0} \int_{H} (-\rho)^{-n-1} e^{-c\rho} \left( \sum_{n_0=0}^{\infty} P(n_0)e^{-n_0\rho} \right) \left( \sum_{n_1=0}^{\infty} P(n_1)e^{-n_1\omega\rho} \right) \cdots \left( \sum_{n_M=0}^{\infty} P(n_M)e^{-n_M\omega^M\rho} \right) d\rho.
\]

It is worth noting that the summation in this expression can be independently carried out as

\[
\sum_{n_i=0}^{\infty} P(n_i)e^{-n_i\omega^i\rho} = e^{-\gamma_i} \sum_{n_i=0}^{\infty} 1/n_i! (\gamma_i e^{-\omega^i\rho})^{n_i} = \exp \left[ -(1 - e^{-\omega^i\rho})\gamma_i \right].
\]

Therefore, the moment can be summarized as

\[
\langle Z^n \rangle = \omega^{-\frac{m}{2}} \lim_{\epsilon \to 0} \int_{H} (-\rho)^{-n-1} \exp \left[ -\epsilon\rho - \sum_{i=0}^{M} (1 - e^{-\omega^i\rho})\gamma_i \right] d\rho.
\]

Since this is convergent for \( \text{Re}(n) > 0 \), the following expression gives the analytic continuation of the moment to the right half complex plane of \( n \),

\[
\langle Z^n \rangle = \frac{\omega^{-\frac{m}{2}}}{\Gamma(-n)} \int_{H} (-\rho)^{-n-1} \exp \left[ -\sum_{i=0}^{M} (1 - e^{-\omega^i\rho})\gamma_i \right] d\rho.
\]
3.2. Thermodynamic limit

Using Eq. (3.8), one can analytically examine the behavior of the moment in the thermodynamic limit $N, M \to \infty$, keeping $\alpha = M/N$ finite. For this, we first convert the contour integration in the expression to an integration on the real axis for $p-1 < \text{Re}(n) < p$, where $p$ is an arbitrary natural number. Further, we define a function

$$f(\rho) \equiv \exp \left[- \sum_{i=0}^{M} (1 - e^{-\omega^i \rho}) \gamma_i \right] \equiv \sum_{j=0}^{\infty} f_i \rho^i,$$  \hspace{1cm} (3.9)

and a series of truncated summations as

$$f^{(p)}(\rho) \equiv \sum_{j=0}^{p-1} f_i \rho^i,$$  \hspace{1cm} (3.10)

which satisfy the identity

$$\int_{H} (-\rho)^{-n-1} f^{(p)}(\rho) d\rho = \int_{H+I} (-\rho)^{-n-1} f^{(p)}(\rho) d\rho = 0,$$  \hspace{1cm} (3.11)

for $p-1 < \text{Re}(n)$, since the contribution from the contour $I$ vanishes. Using this identity, the moment can be rewritten as

$$\langle Z^n \rangle = \frac{\omega^{-\frac{nM}{2}}}{\tilde{\Gamma}(-n)} \int_{H} (-\rho)^{-n-1} [f(\rho) - f^{(p)}(\rho)] d\rho.$$  \hspace{1cm} (3.12)

Eq. (3.11) guarantees that the infrared divergence is removed for $\text{Re}(n) < p$ in this expression. Therefore, the moment can be rewritten for $p-1 < \text{Re}(n) < p$ as

$$\langle Z^n \rangle = \frac{\omega^{-\frac{nM}{2}}}{\tilde{\Gamma}(-n)} \int_{0}^{\infty} \rho^{-n-1} [f(\rho) - f^{(p)}(\rho)] d\rho,$$  \hspace{1cm} (3.13)

where the function $\tilde{\Gamma}$ is replaced by the ordinary gamma function $\Gamma$.  

![Fig. 3. The integration contours.](image-url)
As we have a particular interest in the case of \( p = 1 \), i.e. \( 0 < \text{Re}(n) < 1 \), let us focus on the behavior of the following expression,

\[
\langle Z^n \rangle = \omega \frac{\frac{nM}{2}}{\Gamma(-n)} \int_0^\infty \rho^{-n-1} [f(\rho) - 1] d\rho
\]

\[
= \frac{\omega \frac{nM}{2}}{\Gamma(-n)} \int_0^\infty \rho^{-n-1} \{ \exp \left\{ - \sum_{i=0}^M (1 - e^{-\omega \rho}) \gamma_i \right\} - 1 \} d\rho.
\] (3.14)

To examine the behavior of the thermodynamic limit, it is convenient to introduce new variables

\[
x \equiv \frac{i}{N\alpha}, \quad y \equiv \frac{1}{N\alpha\beta} \ln \rho,
\] (3.15)

instead of \( i \) and \( \rho \), yielding the following expression for the moment

\[
\langle Z^n \rangle = \omega \frac{\frac{nM}{2}}{\Gamma(-n)} \int_{-\infty}^{\infty} e^{Ng(y)} dy,
\] (3.16)

where

\[
\begin{aligned}
G(y) &= -n\alpha\beta y + \frac{1}{N} \log(1 - e^{-F(y)}), \\
F(y) &= \int_0^1 e^{NH(x,y)} dx, \\
H(x,y) &= (1 - \alpha) \log 2 + \alpha H(x) + \frac{1}{N} \log(1 - e^{-e^{N\alpha\beta(y-x)}}),
\end{aligned}
\] (3.17)

and

\[
H(x) = -x \log x - (1 - x) \log(1 - x).
\] (3.18)

Here, we have replaced the summation with an integration, which can be verified when both \( M \) and \( N \) are sufficiently large.

For further analysis, the identity

\[
g(u) \equiv \frac{1}{N} \log \left( 1 - e^{-e^{Nu}} \right) \rightarrow \left\{ \begin{array}{ll}
0 & (u \geq 0) \\
u & (u < 0)
\end{array} \right.
\] (3.19)

which holds for large \( N \), may be useful. The shape of this function is displayed in Fig. 4. As this becomes singular at \( u = 0 \), the phase transition may occur in \( n \) space for \( N \to \infty \). We examine this below.
Fig. 4. The shape of $g(u)$ for $N \to \infty$. This is not analytic at $u = 0$.

Eq. (3.19) means that the function $\mathcal{H}$ can be expressed as

$$\mathcal{H}(x, y) = (1 - \alpha) \log 2 + \alpha \mathcal{H}(x) + \alpha \beta (y - x) \theta(y - x)$$  \hspace{1cm} (3.20)

in the thermodynamic limit. As a function of $x$, this exhibits three behaviors depending on a given value of $y$, as shown in Fig. 4. Employing the saddle point method, the maximum of $\mathcal{H}(x, y)$ given $y$ provides $\mathcal{F}(y)$ for large $N$, which yields

$$\lim_{N \to \infty} \frac{1}{N} \log \mathcal{F}(y) = \begin{cases} 
\frac{1}{2} & \left( y > \frac{1}{2} \right), \\
(1 - \alpha) \log 2 + \alpha \mathcal{H}(x) & \left( x_c < y < \frac{1}{2} \right), \\
(1 - \alpha) \log 2 + \alpha \tilde{\mathcal{H}}(\beta) + \alpha \beta y & (y < x_c),
\end{cases}$$  \hspace{1cm} (3.21)

where $x_c$ is a solution of

$$H'(x_c) = \beta,$$  \hspace{1cm} (3.22)

the behavior of which is shown in Fig. 6. Here, the function $\tilde{\mathcal{H}}$ is the Legendre transformation of the function $\mathcal{H}$,

$$\tilde{\mathcal{H}}(\beta) = \log \left( 2 \cosh \frac{\beta}{2} \right) - \frac{\beta}{2}.$$  \hspace{1cm} (3.23)

The function $\mathcal{G}$ becomes

$$\mathcal{G}(y) = -n\alpha \beta y + \frac{1}{N} \log \mathcal{F}(y) \theta(-\mathcal{F}(y))$$  \hspace{1cm} (3.24)

for large $N$, which directly controls the behavior of the moment in Eq. (3.16). The behavior strongly depends on the relation between $x_c$ and $x^* = (1 - \tanh \frac{\beta}{2}) / 2$ which satisfies the condition

$$(1 - \alpha) \log 2 + \alpha \mathcal{H}(x^*) = 0.$$  \hspace{1cm} (3.25)
\[ H(x, y) \]

(i) \( y > 1/2 \)

(ii) \( 1/2 > y > x_c \)

(iii) \( x_c > y \)

Fig. 5. The schematic shape of \( H(x, y) \) for \( N \to \infty \). It depends on the value of \( y \).

\[ F(y) \]

\[ \frac{1}{N} \log F \]

\[ \frac{1}{2} \]

\[ x_c \]

\[ y_c \]

\[ y \]

Fig. 6. The schematic shape of \( F(y) \) for \( N \to \infty \) obtained from the maximal value of \( H(x, y) \).

- (A) \( x^* < x_c \)

Since \( x_c \) and \( x^* \) are defined in Eq. (3.22) and Eq. (3.25), respectively, the condition \( x^* < x_c \) can be written as

\[
(1 - \alpha) \log 2 + \alpha H(x_c) > 0. \tag{3.26}
\]

This is satisfied for \( \beta < \beta_c \), i.e. the high temperature phase. Notice that for \( \alpha \leq 1 \), as Eq. (2.20) does not have a positive solution, this is always satisfied independently of \( \beta \). Then, Eq. (3.24) can be represented as

\[
G(y) = -n\alpha\beta y + [(1 - \alpha) \log 2 + \alpha \tilde{H}(\beta) + \alpha\beta y] \theta (y_c - y), \tag{3.27}
\]

which is shown in Fig. 4. Here, \( y_c \) is defined by \( F(y_c) = 0 \) (Fig. 3), which yields

\[
y_c = -\frac{1}{\alpha\beta} [(1 - \alpha) \log 2 + \alpha \tilde{H}(\beta)]. \tag{3.28}
\]

The moment is then calculated as

\[
\langle Z^n \rangle = -N\alpha\beta \frac{\omega^{-nM}}{F(-n)} \left[ \int_{-\infty}^{y_c} \exp (1 - \alpha) \log 2 + \alpha \tilde{H}(\beta) + \alpha\beta (1 - n) y dy + \int_{y_c}^{\infty} \exp (-n\alpha\beta y) dy \right] \\
\sim e^{\alpha\beta n N (\frac{1}{2} - y_c)} \left[ \frac{1}{\Gamma(1 - n)} + \frac{n}{\Gamma(2 - n)} \right]. \tag{3.29}
\]
Therefore, the asymptotic behavior of the moment is

\[
\lim_{N \to \infty} \frac{1}{N} \log \langle Z^n \rangle = \alpha \beta nN \left( \frac{1}{2} - y_c \right)
\]

\[
= n \left( \log 2 + \alpha \log \cosh \frac{\beta}{2} \right)
\]

in this phase. This is consistent with RS1, as can be seen from Eq. (2.10).

- (B) \( x^* > x_c \)

The condition \( x^* > x_c \) is satisfied for \( \beta > \beta_c \), which may correspond to the low temperature phase. The function \( G(y) \) has the form

\[
G(y) = -n\alpha\beta y + [(1 - \alpha) \log 2 + \alpha H(y)] \theta(x^* - y),
\]

the behavior of which is further classified into two cases depending on the replica number \( n \).

- (i) \( H'(x^*) > n\beta \)

Since \( x^* \) is located in \( x_c < x^* < \frac{1}{2} \), there exists a critical replica number \( 0 < n_c < 1 \), which is characterized by

\[
n_c \equiv \frac{1}{\beta} H'(x^*) = \frac{\beta_c}{\beta}.
\]

The condition \( H'(x^*) > n\beta \) is satisfied for \( n < n_c \). The function \( G(y) \), shown in Fig. 7 is maximized at \( y = x^* \). Therefore, the moment can be expressed as

\[
\lim_{N \to \infty} \frac{1}{N} \log \langle Z^n \rangle = n\alpha\beta \left( \frac{1}{2} - x^* \right)
\]

\[
= n\alpha\beta \frac{2 \tanh \beta}{\beta_c}.
\]

This behavior is identical to that of 1RSB predicted by RM, as can be seen in Eq. (2.21).
The behavior of the moment in the thermodynamic limit obtained from the exact expression (full lines). In the high temperature phase $\beta < \beta_c$, the behavior of the moment is simple and corresponds to the result obtained from RS1. In the low temperature phase $\beta > \beta_c$, the behavior of the moment has interesting properties. The moment corresponds to that obtained from RS2 for $n > n_c \equiv \beta/\beta_c$ and corresponds to that obtained from 1RSB for $n < n_c$. A phase transition occurs at $n = n_c$.

(ii) $H'(x^*) < n\beta$

The condition $H'(x^*) < n\beta$ is satisfied for $n > n_c$. The function $G(y)$ is maximized not at $y = x^*$ but at $y = y_c$. Therefore, the moment can be asymptotically expressed as

$$\lim_{N \to \infty} \frac{1}{N} \log \langle Z^n \rangle = \log 2 + \alpha \log \cosh \frac{n\beta}{2}. \quad (3.34)$$

This coincides with the behavior of RS2 obtained by RM, as can be seen in Eq. (2.13).

The results are summarized in Fig. 8. In the high temperature phase $\beta < \beta_c$, the behavior of the moment is simple, being expressed by RS1 of RM. In the low temperature phase $\beta > \beta_c$, the behavior of the moment has two possibilities depending on $n$. More specifically, in the limit $N \to \infty$, the moment approaches RS2 for $n > n_c$, whereas 1RSB represents the correct behavior for $n < n_c$. This means that there exists a phase transition in the space of replica number at $n = n_c$. In conclusion, these are consistent with the known results obtained by RM at the limit $n \to 0$. \cite{20, 21}

3.3. Numerical validation

Eq. (3.8) is formulated as a two dimensional summation with respect to $\rho \in \mathbb{C}$ and $i = 0, 1, \ldots, M$, which is numerically tractable. This means that one can utilize Eq. (3.8) or Eq. (3.16) to numerically examine the behavior of DREM for a finite system size $N$, and how fast the results obtained for $N \to \infty$ become relevant as $N$ grows large.
Fig. 9. The logarithm of $\langle Z^n \rangle$ calculated from Eq. (3.16) and data obtained from numerical experiments of Eq. (3.1) using 10, 1000 and 100000 samples. The system size is $N = 10$. The points indicated as “Exact” are the results obtained from our expression Eq. (3.16).

Fig. 9 shows the logarithm of $\langle Z^n \rangle$ calculated by Eq. (3.16) and $\langle Z^n \rangle$ numerically evaluated from 10, 1000 and 100000 experiments, with $N = 10$. One can see that the data from the numerical experiments converge to the results of Eq. (3.16). This verifies that our expression accurately provides the moment even for a finite system size.

We next compare the results from our expression and RM in Fig. 10. At high temperatures, our result is consistent with RS1 as expected. The difference is negligible for all of the range $0 < n < 1$ even at $N = 10$. At low temperatures, our result fits RS2 for larger $n > n_c$, while 1RSB exhibits excellent consistency for smaller $n < n_c$. There is a slight difference between our expression and 1RSB for $N = 10$. The difference, however, becomes indistinguishable for $N = 100$. This strongly indicates that there occurs a phase transition between RS2 and 1RSB at $n = n_c$ in the limit $N \to \infty$.

§4. Origin of 1RSB: Extreme value statistics

The preceding two sections indicate that DREM exhibits a phase transition with respect to the replica number $n$ at a certain critical point $n_c \in [0, 1]$ when the temperature is sufficiently low. In this section, we discuss how this transition can be understood in the framework of RM. A formalism previously introduced for examining the domain size distribution of multi-layer perceptrons is useful for this purpose.

Since the partition function of DREM typically scales exponentially with respect to $N$, we first express this dependence as

$$Z \sim \exp \left[ -N \alpha \beta \left( y - \frac{1}{2} \right) \right],$$

where $y - \frac{1}{2}$ represents the free energy normalized by the scale of the energy amplitude $M$.\[18\]
Fig. 10. The logarithm of $\langle Z^n \rangle$ obtained from our expression and RM. The points indicated as “Exact” are the results obtained from our expression Eq. (3.16). The lines indicated “RS1”, “RS2”, and “1RSB” are the results obtained from RM, using Eq. (2.10), Eq. (2.13), and Eq. (2.21), respectively. At high temperatures $\beta = \beta_c/3 < \beta_c$, our result is consistent with RS1. The difference is very small even for $N = 10$. At low temperatures $\beta = 3\beta_c > \beta$, our result fits RS2 for larger $n > n_c = \beta_c/\beta = 1/3$ while 1RSB provides good consistency for smaller $n < n_c$. There is a slight difference between the results from our expression and that from 1RSB at $N = 10$. The difference, however, becomes imperceptible for $N = 100$.

for a given realization $\{\epsilon_A\}$ ($A = 1, 2, \ldots, 2^N$) (or $\{n_i\}$ ($i = 0, 1, \ldots, M$)). Clearly, $y$ is a random variable. Let us assume that the probability distribution of $y$, $\mathcal{P}(y)$, scales as

$$\mathcal{P}(y) \sim \exp \left[-Nc(y)\right],$$

where $c(y) \sim O(1)$ for large $N$. Notice that an inequality

$$c(y) \geq 0,$$
must hold to make $\mathcal{P}(y)$ satisfy the normalization condition $\int dy \mathcal{P}(y) = 1$ for $N \to \infty$. Eq. (4.2) indicates that the moment of the partition function can be calculated as

$$
\langle Z^n \rangle \equiv \int dy \exp \left[ -Nn\alpha\beta \left( y - \frac{1}{2} \right) \right] \mathcal{P}(y)
\sim \exp \left[ N \operatorname{extr} \left\{ -n\alpha\beta \left( y - \frac{1}{2} \right) - c(y) \right\} \right].
$$

This formula, however, may not be useful for computing $\langle Z^n \rangle$, as directly assessing $c(y)$ is rather difficult. Instead, it can be utilized to evaluate $c(y)$ from $\langle Z^n \rangle$, since Eq. (4.4) implies that $c(y)$ can be obtained from

$$
c(y) = \operatorname{extr} \left\{ -n\alpha\beta \left( y - \frac{1}{2} \right) - \frac{1}{N} \ln \langle Z^n \rangle \right\},
$$

where $(1/N) \ln \langle Z^n \rangle$ can be computed by RM for any real number $n$. This, in conjunction with the normalization constraint in Eq. (4.3), offers a useful clue to identify the origin of the phase transition with the aid of RM.

For $\beta > \beta_c$, RS2 provides the dominant solution for natural numbers $n = 1, 2, \ldots$ in the thermodynamic limit. Carlson’s theorem implies that this should offer a unique analytic continuation if no phase transition occurs. Therefore, let us first insert this solution, from Eq. (2.13), into Eq. (4.5), giving

$$
c(y) = \operatorname{extr} \left\{ -n\alpha\beta \left( y - \frac{1}{2} \right) - \log 2 - \alpha \log \left( \cosh \frac{n\beta}{2} \right) \right\} = (\alpha - 1) \log 2 - \alpha H(y).
$$

Notice that this does not satisfy the necessary condition given in Eq. (4.3) for $y > x^*$, and the critical value $y = x^*$ corresponds to $n = n_c = H'(x^*)/\beta = \beta_c/\beta$, which signals the occurrence of a phase transition at $n = n_c$ within the framework of RM.

The following consideration about the energy configuration provides a plausible scenario for this transition. As have already been pointed out, the partition function $Z$ in the low temperature region $\beta > \beta_c$, can be considered to be dominated by only the minimum energy $\epsilon_{\text{min}}$ of a given energy configuration $\{\epsilon_A\}$, such as $Z \sim \exp [-\beta \epsilon_{\text{min}}]$. This implies that $\mathcal{P}(y)$ of the normalized free energy $y - \frac{1}{2} = -(\log Z)/(N\alpha\beta)$ is given by the distribution of $\epsilon_{\text{min}}/M$ under the assumption that each energy level $\epsilon_A (A = 1, 2, \ldots, 2^N)$ is independently drawn from an identical distribution, given in Eq. (2.1), which has been already examined in the research of extreme value statistics (EVS). In the current case, the average occupation number of an energy level $E_i = i - \frac{1}{2} M = M \left( x - \frac{1}{2} \right)$, $\gamma_i = 2^{N-M} \left( \frac{M}{M/2 + E_i} \right) = \exp N \left( (1 - \alpha) \log 2 + \alpha H(x) \right)$ ($i = 0, 1, \ldots, M$ and $x = i/M \in [0, 1]$), grows exponentially
large with respect to \( N \) for \( x > x^* \). This indicates that \( \mathcal{P}(y) \) is very small for \( y > x^* \) because \( x > x^* \) does not provide \( \epsilon_{\text{min}}/M \) for a given energy configuration \( \{ \epsilon_A \} \) except for very rare cases, as energy levels lower than \( E_i = M \left( x - \frac{1}{2} \right) \) are included in the configuration with a very high probability. On the other hand, \( \gamma_i \) becomes exponentially small for \( 0 < x < x^* \), which means that the probability of having an energy level labelled by \( x \) of this interval in the energy configuration is low. Therefore, \( \mathcal{P}(y) \) also becomes small for \( 0 < y < x^* \). However, the functional form of \( \mathcal{P}(y) \) is not symmetric between \( y > x^* \) and \( 0 < y < x^* \). Actually, detailed analysis of EVS\(^{29, 32}\) shows that \( \mathcal{P}(y) \) exhibits the asymmetric scalings

\[
\mathcal{P}(y) \sim \begin{cases} 
\exp \left[ -\exp \left[ -N \left( \left( \alpha - 1 \right) \log 2 - \alpha H(y) \right) \right] \right], & (x^* < y), \\
\exp \left[ -N \left( \left( \alpha - 1 \right) \log 2 - \alpha H(y) \right) \right], & (0 < y < x^*),
\end{cases}
\tag{4.7}
\]

for large \( N \), which provide a singularity at \( y = x^* \). It may be worth noting that \( 1 - \exp [-\mathcal{F}(y)] \), which comes out in Eq. (3.16) after inserting Eq. (3.17) in the analysis presented section 3, corresponds to the cumulative distribution \( \int_{-\infty}^{y} d\tilde{y} \mathcal{P}(\tilde{y}) \). Eq. (4.7) implies that the extremum point in Eq. (4.4) for large \( n (> n_c) \) is in \( 0 < y < x^* \), which leads to RS2 and consistently provides the exponent of Eq. (4.6). On the other hand, the constant \( y = x^* \) offers the extremum for all small \( n (< n_c) \), which corresponds to 1RSB in the framework of RM. The intuitive implication of this is that \( \langle Z^n \rangle \) is dominated by an atypically low minimum energy generated with a small probability for \( n > n_c \), whereas the typical minimum energy \( \epsilon_{\text{min}} = M \left( x^* - \frac{1}{2} \right) \) provides the major contribution to \( \langle Z^n \rangle \) for \( n < n_c \). Thus, the origin of the phase transition can be attributed to the singularity of the free energy distribution \( \mathcal{P}(y) \), exhibited in the case of low temperatures \( \beta > \beta_c \).

Our analysis also illustrates that Carlson’s theorem is not necessarily useful for validating RM because an analytic continuation given for finite \( N \) can exhibit a singularity in the limit of \( N \to \infty \) and, therefore, taking the limit \( N \to \infty \) prior to \( n \to 0 \) for determining the continuation on the basis of expressions for \( n = 1, 2, \ldots \), which is usually performed in RM, sometimes yields a wrong answer for \( n < 1 \). However, in the current system, this drawback can be resolved by taking a constraint for the free energy distribution (i.e. Eq. (4.3)) into account, which leads to the conventional 1RSB. Although the importance of the notion of EVS in RM has been already addressed at the limit \( n \to 0 \)\(^{29}\) to our knowledge, the current analysis is the first work that directly clarifies how the property of EVS relates to a corruption of the analytic continuation in RM, expressed as a phase transition with respect to the replica number \( n \).
§5. Summary

In summary, we have offered an exact expression, Eq. (3.8), of the moment of the partition function \( \langle Z^n \rangle \) for GCDREM, without employing the replica trick, which is valid for any arbitrary system size \( N \) and complex number \( n \) (\( \text{Re}(n) > 0 \)). Simplifying the expression for \( 0 < n < 1 \), we have shown that a phase transition with respect to the number of replicas \( n \) occurs at a certain critical number \( n_c \in [0, 1] \) in the thermodynamic limit \( N \to \infty \) when the temperature \( \beta^{-1} \) is sufficiently low, if the ratio \( \alpha = M/N \) is greater than unity. This implies that Carlson’s theorem, which guarantees the uniqueness of the analytic continuation from the expressions for \( n = 1, 2, \ldots \) to those for \( n \in \mathbb{C} \), is not necessarily useful for validating the replica method, because taking the thermodynamic limit \( N \to \infty \) prior to the continuation fails to derive a correct result when the analyticity is broken for \( n < 1 \) in the case of infinite \( N \).

However, it has also been shown that this drawback can be overcome by taking the statistical property of the minimum energy level into account, clarifying how 1RSB originates from a replica symmetric solution, which has been conventionally discarded, at the critical replica number \( n_c \). We hope that the results obtained here offer a useful insight to unlock the remaining mysteries of RM.

Since Eq. (3.8) is valid throughout the right half complex plane \( \text{Re}(n) > 0 \) for any finite \( N \), one can directly observe how the singularities of \( \langle Z^n \rangle \) approach the real axis of \( n \) as \( N \to \infty \). This is sometimes referred to as Lee-Yang’s scenario of the phase transition. Analysis along this direction is currently under way.

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Appendix A

Link of DREM to error-correcting codes

As was shown in Ref. 9, certain types of spin glass models can be related to error-correcting codes. We here show that a known evaluation scheme of the error correcting ability of a random code ensemble can be reduced to a calculation of the general moment of the partition function with respect to DREM. To the knowledge of authors, a remark on this relationship has already been made in Ref. 37.

In a general scenario of error correcting codes, an \( N \)-dimensional binary vector \( s = (s_1, s_2, \ldots, s_N) \) \( (s_i = 0, 1, \ i = 1, 2, \ldots, N) \) is encoded to a codeword \( t = (t_1, t_2, \ldots, t_M) \) \( (t_i = 0, 1, \ i = 1, 2, \ldots, M) \) of an \( M(> N) \)-dimensional binary vector and transmitted via
a noisy channel. We here concentrate on a binary symmetric channel (BSC) in which each component of the codeword \( t_i \) is independently flipped to the opposite letter in the alphabet (0 or 1) with a probability \( 0 < p < 1/2 \). This implies that the vector \( r = t + n \) (mod 2) is received at the other terminal of the channel, where \( n \) is the noise vector, the component of which becomes 1 independently with probability \( p \) and 0 otherwise. However, as the codeword is represented somewhat redundantly, decoding \( r \) provides the correct message \( s \) with a high probability for sufficiently small \( p \).

It is known that the performance of correctly retrieving \( s \) from \( r \) becomes good when a code \( C \) (i.e., an invertible map \( C : s \leftrightarrow t \)) is randomly constructed\(^{35}\), providing a code ensemble. Let us evaluate the average probability of failing to correctly retrieve \( s \) in order to characterize the potential error correcting ability of the ensemble. We focus on the maximum likelihood (ML) decoding, which selects a codeword closest to the received vector \( r \) and returns a message vector corresponding to the codeword as an estimate \( \hat{s} \) of \( s \). ML decoding minimizes the decoding error probability \( P_E(C) \) when codeword vectors \( t \) are equally generated at the transmission terminal.\(^{35}\)

Notice that the message \( s \) can be correctly identified if its codeword \( t \) is provided, since the code is constructed as an invertible map. Therefore, for evaluating \( P_E(C) \), it is convenient to introduce an indicator function \( \Delta_{\text{ML}}(t, r | C) \) that returns 1 if \( r \) is not correctly decoded to the correct codeword \( t \) and 0 otherwise, which means the decoding error probability can be computed by

\[
P_E(C) = \sum_{t, r} P(t|C)P(r|t)\Delta_{\text{ML}}(t, r | C),
\]

and, hence, its average over the code ensemble can be represented as

\[
\langle P_E(C) \rangle_C = \sum_C P(C) \sum_{t \in C, r} P(t|C)P(r|t)\Delta_{\text{ML}}(t, r | C),
\]

where \( P(t|C) \) is the probability that the codeword vector \( t \) is generated given \( C \) and \( P(r|t) \) is the conditional probability that \( r \) is received when \( t \) is transmitted. \( P(C) \) is the probability that a code \( C \) is generated. \( \sum_{t \in C} \) denotes a summation over \( 2^N \) codeword vectors given \( C \).

We assume that the code \( C \) is designed by the source coding technique such that \( t \) is uniformly generated as \( P(t|C) = 1/2^N \).\(^{35}\) In addition, for BSC, the conditional probability can be represented as

\[
P(r|t) = \exp \left[ -F \sum_{i=1}^{M} \left( \frac{1}{2} - \delta_{r_i, t_i} \right) \right] \left( 2 \cosh \frac{F}{2} \right)^M,
\]

\(^{23}\)
where \( \delta_{x,y} \) returns 1 if \( x = y \) and 0, otherwise and \( F = \log [(1 - p)/p] \).

Unfortunately, expressing \( \Delta_{ML}(t, r|C) \) in a rigorously treatable form is difficult. However, Gallager’s inequality

\[
\Delta_{ML}(t, r|C) \leq \left( \sum_{t' \in C \setminus t} \left( \frac{P(r|t')}{P(r|t)} \right)^{\frac{1}{1+n}} \right)^n,
\]

(A.4)

which holds for arbitrary \( n \geq 0 \), offers a good upperbound for this.\(^{35,36}\) Here \( \sum_{t' \in C \setminus t} \) denotes a summation over \( 2^N - 1 \) codeword vectors \( t' \) of \( C \) excluding the possibility of the correct codeword \( t \). Inserting this into Eq. (A.2) provides

\[
\langle P_E(C) \rangle_C \leq \sum_C P(C) \sum_{t, r} P(t|C) P_{1+n}^{\frac{1}{1+n}}(r|t) \left( \sum_{t' \in C \setminus t} P_{1+n}^{\frac{1}{1+n}}(r|t') \right)^n,
\]

where we have performed the gauge transformation \( t \rightarrow 0, t' - t \rightarrow t' \) and \( r - t \rightarrow r \), assuming any code \( C \) contains the zero codeword \( t = 0 \).\(^{35} \) Minimizing the final expression with respect to \( n \geq 0 \), we can obtain the tightest bound of the average decoding error probability.

We here address that \( E(t) = \sum_{i=1}^{M} (\frac{1}{2} - \delta_{t_i, t_i}) \) in Eq. (A.5) obeys Eq. (2.1) independetly of \( r \) when each codeword \( t \) is equally generated in the ensemble such that each component is independently selected from an identical unbiased distribution \( P(t_i = 1) = P(t_i = 0) = 1/2 \), which holds for the random code ensemble,\(^{25,35} \) although non-trivial correlations of ‘energy’ \( E(t) \) between ‘states’ \( t \) arise making the energy distribution different from Eq. (2.1) in the case of practical linear codes. Therefore, for the current random code ensemble, Eq. (A.5) can be simplified as

\[
\langle P_E(C) \rangle_C \leq \left( \frac{\cosh \frac{F}{2(1+n)}}{\cosh \frac{F}{2}} \right)^M \left( \sum_{A=1}^{2N-1} \exp \left[ -\frac{F}{1+n} \epsilon_A \right] \right)^n,
\]

(A.6)

where \( \langle \cdots \rangle \) denotes the configurational average with respect to the ‘energy states’ \( \epsilon_A (A = 1, 2, \ldots, 2^N - 1) \) following Eq. (2.1). Regarding \( F/(1 + n) \) as the inverse temperature, this means that an assessment of the average decoding error probability can be linked to calculating the general moment of the ‘partition function’ \( Z = \sum_{A=1}^{2N-1} \exp \left[ -\frac{F}{1+n} \epsilon_A \right] \) of DREM.
Appendix B

The replica analysis of GCDREM

Section 3 indicates that the generalized moment of partition function can be evaluated without using RM for GCDREM even when the system size $N$ is finite. However, for direct comparison to the conventional analysis, it may be helpful to demonstrate the conventional RM-based analysis as well. Therefore, we here provide a brief sketch of the replica calculation of GCDREM.

From eq. (3.1), we first obtain an expression

$$
\langle Z^n \rangle = \omega^{-nM/2} \sum_{i_1=0}^{M} \sum_{i_2=0}^{M} \cdots \sum_{i_n=0}^{M} \left( \prod_{\mu=1}^{n} n_{i_{\mu}} \right) \omega^{\sum_{\mu=1}^{n} i_{\mu}}
$$

$$
= \omega^{-\frac{nM}{2}} \sum_{i_1=0}^{M} \sum_{i_2=0}^{M} \cdots \sum_{i_n=0}^{M} \prod_{i=1}^{M} \left( \left( \sum_{\mu=1}^{n} \delta_{i,i_{\mu}} \right) \omega^{\sum_{\mu=1}^{n} \delta_{i,i_{\mu}}} \right),
$$

(B.1)

for $n = 1, 2, \ldots$, where $\langle \cdots \rangle$ represents the average over the Poisson distribution, Eq. (3.2). Following the conventional recipe of RM, let us next evaluate the candidates of the most dominant contribution in the limit $N \to \infty$ under the RS1, RS2 and 1RSB assumptions following the conventional scheme of RM. Before proceeding further, it may be worth mentioning that the dynamical variables in GCDREM are not single states but energy levels $i = 0, 1, 2, \ldots, M$, each of which is composed of multiple states. This difference makes composition of the solutions slightly different from that of CDREM provided in section 2, although the final result is unchanged.

• RS1

In RS1, all the $n$ replica levels are assumed to be allocated to $n$ different levels $i(=0, 1, 2, \ldots, M)$. Therefore, for a given level $i$,

$$
\sum_{\mu=1}^{n} \delta_{i,i_{\mu}} = \begin{cases} 1 & \text{(when } i \text{ is one of the } n \text{ allocated energy levels),} \\ 0 & \text{(otherwise).} \end{cases}
$$

(B.2)

When $i$ is an allocated level,

$$
\langle n_{i}^{\sum_{\mu=1}^{n} \delta_{i,i_{\mu}}} \rangle = \gamma_{i} \to \exp \left[ N \left( (1 - \alpha) \log 2 + \alpha H(x) \right) \right],
$$

(B.3)

and

$$
\omega^{i \sum_{\mu=1}^{n} \delta_{i,i_{\mu}}} = \exp \left[ -\beta i \right] \to \exp \left[ -N \alpha \beta x \right],
$$

(B.4)

hold for large $N$, yielding the contribution

$$
\langle n_{i}^{\sum_{\mu=1}^{n} \delta_{i,i_{\mu}}} \rangle \omega^{i \sum_{\mu=1}^{n} \delta_{i,i_{\mu}}} \to \exp \left[ N \left( (1 - \alpha) \log 2 + \alpha (H(x) - \beta x) \right) \right],
$$

(B.5)
where \( x \equiv i/M = i/(\alpha N) \). This is maximized at \( x_c = (1 - \tanh \frac{\beta}{2}) \) to

\[
\exp N \left( \log 2 + \alpha \log \left( \cosh \frac{\beta}{2} \right) - \frac{\alpha \beta}{2} \right),
\]

which represents the contribution from a certain replica level \( i_\mu (= 1, 2, \ldots, n) \). Contributions from other replicas become smaller than Eq. (B.6) as any two replica levels must be located at different levels under the current assumption. However, the difference becomes negligible because there exist many levels in any vicinity of \( x = x_c \) in the limit \( N \to \infty \). Taking the prefactor \( \omega^{-nM} \) in front of the summation in Eq. (B.1) and the number of permutations over replica indices into account, this implies that the dominant contribution under the RS1 ansatz is evaluated as

\[
\langle Z^n \rangle = n! \times \exp nN \left( \log 2 + \alpha \log \left( \cosh \frac{\beta}{2} \right) \right) \\
 \sim \exp nN \left( \log 2 + \alpha \log \left( \cosh \frac{\beta}{2} \right) \right),
\]

which is identical to the RS1 solution of CDREM, Eq. (2.10), and equivalent to Eq. (3.30).

- **RS2**

In RS2, all the \( n \) replica levels are assumed to be allocated to a certain single level. Therefore, for a given level \( i \),

\[
\sum_{\mu=1}^{n} \delta_{i,i_\mu} = \begin{cases} n & \text{(when } i \text{ is the allocated energy level)}, \\
0 & \text{(otherwise)}. \end{cases}
\]

When \( i \) is the allocated level,

\[
\langle \sum_{\mu=1}^{n} \delta_{i,i_\mu} \rangle \sim \begin{cases} \gamma_i^n \to \exp \left[ nN \left( (1 - \alpha) \log 2 + \alpha H(x) \right) \right] & (x > x^*), \\
\gamma_i \to \exp \left[ N \left( (1 - \alpha) \log 2 + \alpha H(x) \right) \right] & (x < x^*), \end{cases}
\]

where \( x^* \) is determined by Eq. (3.25) and

\[
\omega^{i \sum_{\mu=1}^{n} \delta_{i,i_\mu}} = \exp \left[ -n\beta i \right] \to \exp \left[ -nN\alpha \beta x \right];
\]

hold for large \( N \). Notice that \( \langle \sum_{\mu=1}^{n} \delta_{i,i_\mu} \rangle \) behaves differently depending on whether \( x > x^* \) or not. Therefore, the behavior of the dominant contribution obtained by maximizing \( \omega^{i \sum_{\mu=1}^{n} \delta_{i,i_\mu}} \) is different depending on the relation between \( x^* \) and the position of the maximum \( x_c \). For \( \beta < \beta_c \) or \( \alpha \leq 1 \), \( x_c = (1 - \tanh \frac{\beta}{2})/2 \) becomes greater than \( x^* \), which yields the RS1 solution, Eq. (2.10), in spite that the RS2 ansatz is currently employed. This is because the RS2 ansatz for energy levels
does not necessarily mean that \( n \) replica states are identical, in particular, when the occupation number \( n_i \) is exponentially large, to which \( x_c > x^* \) corresponds. In such cases, even if \( n \) replica states are allocated to an identical energy level, they are typically distributed to \( n \) different states placed at the energy level, which corresponds to the RS1 ansatz of CDREM, and, therefore, Eq. (2.10) should be reproduced. On the other hand, for \( \beta > \beta_c \),

\[
\langle n_i \rangle = \exp \left( \log 2 + \alpha \log \left( \cosh \frac{m \beta}{2} \right) \right),
\]

which is identical to the RS2 solution of CDREM, Eq. (2.13), and equivalent to Eq. (3.34). As Eq. (2.13) can be dominant at \( n = 1, 2, \ldots \) only for \( \beta > \beta_c \), this is consistent with the result of the replica analysis of CDREM.

- 1RSB

In 1RSB, \( n \) replica levels are assumed to be equally allocated to \( n/m \) levels. Therefore, for a given level \( i \),

\[
\sum_{\mu=1}^{n} \delta_{i, \mu} = \begin{cases} m & \text{(when } i \text{ is the } n/m \text{ allocated energy levels)}, \\ 0 & \text{(otherwise)}. \end{cases}
\]

(B.12)

When \( i \) is the allocated level,

\[
\langle n_i \sum_{\mu=1}^{n} \delta_{i, \mu} \rangle \sim \begin{cases} \gamma^m_i \to \exp [mN ((1 - \alpha) \log 2 + \alpha H(x))] & (x > x^*), \\ \gamma^m_i \to \exp [N ((1 - \alpha) \log 2 + \alpha H(x))] & (x < x^*), \end{cases}
\]

and

\[
\omega^i \sum_{\mu=1}^{n} \delta_{i, \mu} = \exp [-m \beta i] \to \exp [-mN \alpha \beta x],
\]

hold for large \( N \). Similarly for the case of RS2, this reproduces the RS1 solution, Eq. (2.10), for \( \beta < \beta_c \) or \( \alpha \leq 1 \). So, we focus on the low temperature phase \( \beta > \beta_c \). Then, \( \langle n_i \sum_{\mu=1}^{n} \delta_{i, \mu} \rangle \omega^i \sum_{\mu=1}^{n} \delta_{i, \mu} \) is maximized at \( x_c = (1 - \tanh \frac{m \beta}{2})/2 < x^* \) to

\[
\exp N \left( \log 2 + \alpha \log \left( \cosh \frac{m \beta}{2} \right) - \frac{m \alpha \beta}{2} \right),
\]

which represents the contribution from one of the \( n/m \) allocated levels. The number of ways to select \( n/m \) out of \( M \) levels equally allocating \( n \) replicas is negligible for subsequent calculation. Taking contributions from all the \( n/m \) allocated levels and extremization with respect to \( m \) into account offers

\[
\langle Z^n \rangle \sim \exp N \left( \text{extr } \left\{ \frac{n}{m} \left( \log 2 + \alpha \log \left( \cosh \frac{m \beta}{2} \right) \right) \right\} \right)
\]
\[ = \exp N n_\alpha \beta \left( \frac{1}{2} - x^* \right) \]
\[ = \exp N \left( \frac{n_\alpha \beta}{2} \tanh \frac{\beta}{2} \right), \quad (B.16) \]

which is identical to the 1RSB solution of CDREM, Eq. (2.21), and equivalent to Eq. (3.33). As Eq. (2.21) can be dominant \( n = 1, 2, \ldots \) only for \( \beta > \beta_c \), this is consistent with the result of the replica analysis of CDREM.

Appendix C

--- Efficient sampling in CDREM ---

We here show that one can sample \( \{n_i\} \) with an \( O(M) \) computational cost in CDREM as well as in GCDREM. According to the probability distribution for each level

\[ P(E_i) = 2^{-M} \left( \frac{M}{\frac{1}{2} M + E_i} \right), \quad \left( E_i = i - \frac{M}{2} \right), \quad (C.1) \]

the probability to sample a configuration \( (n_0, n_1, \cdots, n_M) \) is

\[ \mathcal{P}(n_0, n_1, \cdots, n_M) = \{P(E_0)\}^{n_0} \{P(E_1)\}^{n_1} \cdots \{P(E_M)\}^{n_M} \frac{2^N!}{n_0! n_1! \cdots n_M!}. \quad (C.2) \]

To generate a sample \( (n_0, n_1, \cdots, n_M) \) in practice, we first determine \( n_0 \) according to the probability

\[ \mathcal{P}(n_0, \text{arbitrary}) = (p_0)^{n_0} (1 - p_0)^{2^N - n_0} \frac{2^N!}{n_0!(2^N - n_0)!} \quad (C.3) \]
\[ (p_0 \equiv P(E_0)). \quad (C.4) \]

We then determine \( n_1 \) according to the probability

\[ \mathcal{P}(n_0; n_1, \text{arbitrary}) = (p_1)^{n_1} (1 - p_1)^{2^N - n_0 - n_1} \frac{2^N!}{n_1!(2^N - n_0 - n_1)!} \quad (C.5) \]
\[ \left. \left( p_1 \equiv \frac{P(E_1)}{1 - P(E_0)} \right) \right) \quad (C.6) \]

Repeating this procedure to \( n_{M-1} \) as

\[ \mathcal{P}(n_0, n_1, \cdots; n_{M-1}, n_M) = (p_{M-1})^{n_1} (1 - p_{M-1})^{2^N - \sum_{i=0}^{M-1} n_i} \frac{2^N!}{n_{M-1}!(2^N - \sum_{i=0}^{M-1} n_i)!} \]
\[ \left. \left( p_{M-1} \equiv \frac{P(E_{M-1})}{1 - \sum_{i=0}^{M-2} P(E_i)} \right) \right), \quad (C.7) \]
we obtain a set of \((n_0, n_1, \cdots, n_M)\). This guarantees that the identity
\[
\sum_{i=0}^{M} n_i = 2^N,
\]
holds, which characterizes CDREM. Similarly for the case of GCDREM, this can be performed in \(O(M)\) computation.

A comparison of numerically evaluated moments of the partition function between CDREM and GCDREM is presented in Fig. 11. This indicates that the difference between the two models is almost indistinguishable even for \(N = 3\). Since the consistency becomes better as \(N\) grows larger\(^{20,21}\) working in GCDREM instead of CDREM is justified when \(N\) is large.

![Fig. 11. The free energies obtained from the sets \(\{n_i\}\) using CDREM and GCDREM. We find that CDREM and GCDREM give almost same results even for \(N = 3\).](image)

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