Confidence Set for Group Membership* 

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Abstract

Our confidence set quantifies the statistical uncertainty from data-driven cluster assignment in clustered panel models. It covers the true cluster memberships jointly for all units with pre-specified probability and is constructed by inverting many simultaneous unit-specific one-sided tests for group membership. We justify our approach under $N, T \to \infty$ asymptotics using tools from high-dimensional statistics, some of which we extend or develop in this paper. We provide an empirical application as well as Monte Carlo evidence that the confidence set has adequate coverage in finite samples.

Keywords: Panel data, clustering, confidence set, joint one-sided test, high-dimensional statistics.

JEL codes: C23, C33, C38

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1. Introduction

Clustering units into discrete groups is one of the oldest problems in statistics (Pearson 1896). It have received increasing interests in the recent econometric literature (Lin and Ng 2012; Bonhomme and Manresa 2015; Sarafidis and Weber 2015; Ando and Bai 2016; Vogt and Linton 2017; Su, Shi, and Phillips 2016; Wang, Phillips, and Su 2018; Vogt and Schmid 2021; Gu and Volgushev 2019; Liu et al. 2020; Wang and Su 2021). As statistical procedures, data-driven clustering algorithms suffer from statistical error that has to be taken into account when interpreting the estimated clustering partition, i.e., the estimated assignments of units to groups. Still, even for modern clustering methods, inferential theory focuses on group characteristics and is underdeveloped for the assignment of units to groups (see e.g. McLachlan and Peel 2004). In this paper, we study clustered panel models and quantify the statistical uncertainty about the population partition. As far as we are aware, ours is the first contribution in the econometric and statistical literature to propose and justify a rigorous frequentist method to quantify clustering uncertainty.

In clustered panel models, individual regression curves are heterogeneous and exhibit a grouped pattern. All units that belong to the same latent group face the same regression curve. Estimation of clustered panel models and inference with respect to the group-specific regression curves are popular research topics in the recent econometric literature and are discussed in Lin and Ng (2012), Bonhomme and Manresa (2015), Sarafidis and Weber (2015), Ando and Bai (2016), Vogt and Linton (2017), Su, Shi, and Phillips (2016), Wang, Phillips, and Su (2018), Lu and Su (2017), Vogt and Schmid (2021), and Gu and Volgushev (2019).

Group memberships in clustered panel models are unobserved and have to be estimated by a clustering algorithm. Accordingly, estimated group memberships suffer from statistical uncertainty. The degree of statistical uncertainty can differ across units. It may occurs because there exists a strong degree of heteroskedasticity across units (Dzemski and Okui 2021). Another example are unbalanced panels, where for some units the group membership estimate is based on a lot of data, whereas for other units very little data is observed.

We propose a confidence set for group membership that quantifies clustering uncertainty jointly for all units in the panel. For a panel of $N$ units, an element of a joint confidence set is an $N$-dimensional vector that specifies a group assignment for every unit. Our confidence set gathers all $N$-vectors of group assignments that are not ruled out by the data and is guaranteed to contain the vector of true group memberships with a pre-specified probability, say 95%.

Footnote: Our usage of the word “clustered panel models” is different from an alternative usage which refers to situations in which error terms in a cluster are correlated with each other (Angrist and Pischke 2008, Chapter 8).
Based on our confidence set, empirical researchers can interpret the estimated clustering partition with rigorous statistical error control, leading to new economic and statistical insights. Our confidence set combines many unit-wise confidence intervals. This construction renders our joint confidence set easy to report and to interpret even though it covers a high-dimensional parameter. The confidence set admits an intuitive graphical representation or can be reported in table format.

Our empirical application illustrates how our confidence set provides new economic insights. We follow Wang, Phillips, and Su (2019) who estimate a panel model that allows the effect of a minimum wage on unemployment to vary between US states, depending on the assignment of each state to one of four latent clusters. Based on existing inferential results, Wang, Phillips, and Su (2019) conduct inference on the group-specific effects of the minimum wage. From their results, it is not possible to infer with statistical error control the effect for a given state. This is because the mapping of groups to states is potentially estimated with error. We use our confidence set to identify, up to a small pre-specified error probability, states for which the clustering algorithm recovers the true population clusters. For these states we can quantify with statistical error control the state-specific effect of the minimum wage.

Another potential use of our confidence set is to determine whether two subsets of units are assigned to distinct population clusters. Assignment to distinct clusters means that economic behavior differs between the two subsets. This use case requires a joint confidence set such as the one computed by our method.

Our confidence set can be easily visualized as demonstrated in our empirical application. In applications of clustered panel models, it is standard practice to provide a plot of the estimated clusters. This plot is provided even if the clustering structure is a nuisance parameter and not a structural parameter of interest. In such applications, the graphical representation of our confidence set gives additional diagnostic information about clustering uncertainty.

Quantifying clustering error in clustered panel models is important since misclassification occurs frequently in empirical practice. For example, Bonhomme and Manresa (2015) find substantial misclassification rates in Monte Carlo simulations that apply their clustering algorithm to a data generating process that is calibrated to an empirical application (see Table S.III in their supplemental appendix). For mathematical convenience, the theoretical analysis of clustered panel models proceeds often under assumptions that rule out any misclassification in the asymptotic limit (see e.g. Bonhomme and Manresa 2015, Vogt and

\[^2\]For example, see Figure 2 in Wang, Phillips, and Su (2019), Figure 2 in Bonhomme and Manresa (2015) and Figure 6 in Wang, Phillips, and Su (2018).
Dzemski and Okui (2021) analyze a simple clustered panel model under weaker assumptions that do not rule out misclassification and give sufficient conditions for consistent estimation of the group-specific regression curves.

An alternative to our approach based on frequentist confidence sets is Bayesian inference. Clustered panels that are fully parametric are finite mixtures models that can be estimated by the EM algorithm (Dempster, Laird, and Rubin 1977). Posterior probabilities for the group membership of individual units are computed in the E-step of the EM algorithm. These posterior probabilities are valid if the units in the panel are drawn independently from identical distributions that follow the assumed parametric specification. Our frequentist approach is valid under much weaker assumptions. We allow for cross-sectional dependence, potentially non-random patterns of heteroscedasticity and non-random group assignments. We do not assume a parametric distribution of the error term and show that our approach is valid for error distributions in a broad nonparametric class. The posterior probabilities computed by the EM algorithm address inference on the group membership of a single unit. It is not clear how the output from the EM algorithm can be used to accomplish our goal of joint inference with respect to the whole population partition.

Constructing a confidence set for true group memberships for all units in the panel is a problem of high-dimensional inference. The size of the vector of true group memberships for all units grows with the cross-sectional dimension of the panel. Based on recent advances in high-dimensional statistics, we provide an asymptotic justification of our procedure under the assumptions of either independent and weakly dependent time series. To construct our confidence set we invert a test for a high-dimensional parameter that is identified from many moment inequalities. This is the testing problem considered in Chernozhukov, Chetverikov, and Kato (2019). Their test is based on a single test statistic, whereas our test combines many simultaneous tests of group membership for individual units. The advantage of our approach is that it can be inverted without running a computationally infeasible exhaustive search over the space of all possible partitions. Each of the simultaneous unit-wise tests uses unit-specific and data-dependent critical values. To account for this feature of our approach in the theoretical analysis, we develop new mathematical tools, including a simultaneous anti-concentration inequality and a new comparison bound for the maximum of a Gaussian vector. We expect these new mathematical results to be useful in other applications of high-dimensional statistics.

The remainder of this paper is organized as follows. Section 2 introduces the clustered panel model. Section 3 defines two flavors of our confidence set for group membership based on two different strategies for computing unit-specific critical values. Section 4 describes a refinement of our method that can shrink the confidence set by adding an additional pre-
processing step that eliminates units for which the group membership is “obvious”. Section 5 gathers theoretical assumptions alongside asymptotic results that guarantee that all confidence sets described in this paper have correct coverage. We consider both independent and dependent time series. Section 6 provides an empirical application. Section 7 discusses Monte Carlo simulations that investigate the validity and power of our confidence set based on simulation designs inspired by our empirical application. Section 8 concludes.

An R package implementing the methods proposed in this paper is available from the authors.

2. Model

We observe panel data \((y_{it}, w_{it}, x_{it})\) for units \(i = 1, \ldots, N\) and time periods \(t = 1, \ldots, T\), where \(y_{it}\) is a scalar dependent variable and \(w_{it}\) and \(x_{it}\) are covariate vectors. Unit \(i\) belongs to group \(g_i^0 \in G = \{1, \ldots, G\}\). Group memberships are unobserved. The data is generated from the model

\[
y_{it} = w_{it}' \theta^w + x_{it}' \theta^g_i + \sigma_i v_{it},
\]

where \(v_{it}\) is a noise term with variance one. The slope coefficient \(\theta^w\) is common to all units. The slope coefficient on \(x_{it}\) is group-specific. For units \(i\) belonging to group \(g \in G\) it is given by \(\theta^g_i\). The heteroscedasticity parameter \(\sigma_i\) is unobserved.

Different estimation strategies for estimating the common and group-specific coefficients \((\theta^w, \theta^g)\) have been proposed in the literature. For example, Bonhomme and Manresa (2015) estimate slope coefficients \(\hat{\theta}^w, \hat{\theta}_1, \ldots, \hat{\theta}_G\) and group memberships \(\hat{g}_1, \ldots, \hat{g}_N\) by solving the least-squares problem

\[
(\hat{\theta}^w, \hat{\theta}_1, \ldots, \hat{\theta}_G, \hat{g}_1, \ldots, \hat{g}_N) = \underset{g_1, \ldots, g_N}{\text{argmin}} \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} (y_{it} - w_{it}' \theta^w - x_{it}' \theta_{g_i})^2
\]

via the the \textit{kmeans} algorithm. Su, Shi, and Phillips (2016) and Wang, Phillips, and Su (2018) propose estimators based on penalization.

Our procedures can be combined with any preliminary estimators of \(\theta^w\) and \(\theta^g\) that satisfy a weak rate condition stated in Assumption 1.

We assume that the number of groups \(G\) is either pre-specified or consistently estimated. Consistent estimates of \(G\) can be obtained, e.g., by applying the approach Bonhomme and Manresa (2015) based on an information criterion or the testing procedure proposed by Lu and Su (2017).
Remark 1. For ease of exposition we describe our procedures for balanced panels. The extension to unbalanced panels is straightforward.

Remark 2. Our procedures can be applied to panel data with unit fixed effects by interpreting the panel model (1) as representing the fixed-effect transformation that removes the unit fixed effect.

3. Confidence set for group membership

In this section, we present our method to construct a confidence sets for the group membership structure. First, we define an asymptotically valid confidence set for group membership. Second, we show that each group allocation corresponds to a set of moment inequalities and that a confidence set can be obtained by inverting a test of these inequalities. Then, we define the test statistic and critical values.

3.1. Definition

A joint confidence of group membership at confidence level $1 - \alpha$ is a random set $\hat{C}_\alpha$ of vectors in $\mathbb{G}^N$ that satisfies

$$\lim_{N,T \to \infty} \inf_{P \in \mathbb{P}_N} P \left( \{ g_i^0 \}_{1 \leq i \leq N} \in \hat{C}_\alpha \right) \geq 1 - \alpha,$$  (2)

where $\mathbb{P}_N$ is a set of data generating processes. If we observe $\{g_i\}_{1 \leq i \leq N} \in \hat{C}_\alpha$ then the partition with assignments $g_i^0 = g_i$ for $i = 1, \ldots, N$ is not ruled out by the data at confidence level $1 - \alpha$. Inequality (2) ensures that the confidence set is asymptotically valid in the sense that it rules out the population partition at most with probability $\alpha$.

We construct a joint confidence set by combining unit-wise confidence sets. This approach is computationally feasible. It can also be tabulated and visualized easily. A unit-wise confidence set for unit $i$ is computed by inverting a test for group membership

$$\hat{C}_{\alpha,N,i} = \left\{ g \in \mathbb{G} : \hat{T}_i(g) \leq \hat{c}_{\alpha,N,i}(g) \right\} \cup \{ \hat{g}_i \},$$

where $\hat{g}_i$ denotes an estimator of the group membership of unit $i$, $\hat{T}_i(g)$ is a test statistic and $\hat{c}_{\alpha,N,i}$ is a critical value. $\hat{T}_i(g)$ and $\hat{c}_{\alpha,N,i}$ are possibly unit-specific and data-dependent. The test statistic is defined in Section 3.3. Two different ways of computing the critical values are defined in Section 3.4. Our joint confidence set is given by the Cartesian product of the
unit-wise confidence sets:

\[ \hat{C}_\alpha = \bigotimes_{1 \leq i \leq N} \hat{C}_{\alpha,N,i}. \]

To make sure that the joint confidence set has the desired confidence level we use Bonferroni correction. Each of the unit-wise confidence set has nominal level \( 1 - \alpha/N \).

Our approach is computationally simple. In principle, it is possible to construct a joint confidence set directly without first computing unit-wise confidence sets. This can be accomplished by inverting a joint test for group membership. Such a direct approach avoids the possible power loss from Bonferroni correction but requires testing all \( G^N \) possible group memberships. This is computationally infeasible. Our approach carries out only \( GN \) tests.

An additional advantage of using Bonferroni correction here is that it produces confidence sets that are easy to report and to interpret. Since we consider the vector of all group memberships, interpretability of our confidence set is a potential concern. Our joint confidence set is completely defined from the marginal confidence sets for each unit. As we demonstrate in our application, this makes our confidence set straightforward to interpret and easy to report. In particular, to report the confidence set it is not necessary to enumerate all \( N \)-vectors contained in \( \hat{C}_\alpha \).

The Bonferroni correction renders our confidence set robust to any kind of cross-sectional dependence. If units are independent then this correction is only minimally conservative.

**Theorem 1.** Let \( 0 < \alpha < 1 \) and suppose that the unit-wise confidence sets satisfy

\[ P \left( g^0_i \in \hat{C}_{\alpha,N,i} \right) = 1 - \alpha/N \]

and that units are independent. Then

\[ \alpha - \frac{\alpha^2}{2} \leq P \left( \{g^0_i\}_{1 \leq i \leq N} \notin \hat{C}_\alpha \right) \leq \alpha - \frac{\alpha^2}{2} \left( 1 - \frac{\alpha}{3} + \frac{1}{N} \left( 1 - \frac{\alpha}{N} \right)^{-2} \right). \]

For example, Theorem 1 implies that for \( N \geq 8 \) independent units and nominal coverage level \( 1 - \alpha = 0.9 \), the Bonferroni correction inflates the joint coverage probability by only 0.5-0.55% if all unit-wise confidence sets have a coverage probability of \( 1 - \alpha/N \).

**Remark 3.** For applications where we are interested in inference on a single pre-specified unit \( i \), the unit-wise confidence set \( \hat{C}_{\alpha,1,i} \) covers the true group membership of \( i \) with approximately probability \( 1 - \alpha \). Our procedure can be also be adapted to produce joint confidence sets for pre-specified subsets of units.
3.2. Motivation of our test of group membership

The unit-wise confidence sets are constructed by inverting a test for group membership. We assume \( E[v_{it} | x_{it}, \sigma_i] = 0 \) and that \( \sum_{t=1}^{T} E[x_{it}x_{it}'] \) is positive definite. The null hypothesis \( H_0 : g_i^0 = g \) implies the moment inequalities

\[
\sum_{t=1}^{T} E[(y_{it} - w_{it}'\theta^w - x_{it}'\theta_g)^2] - \sum_{t=1}^{T} E[(y_{it} - w_{it}'\theta^w - x_{it}'\theta_h)^2] \leq 0, \text{ for all } h \in G \setminus \{g\}.
\]

It holds because the true value of the coefficient minimizes the squared population error and the true group membership allocation for \( i \) gives the true value of the coefficient for \( i \).

We base our test on a mean-adjusted version of these inequalities that removes possible slackness of the inequalities under the null hypothesis. The mean-adjusted inequalities are based on

\[
d_{it}(g, h) = \frac{1}{2} \left( (y_{it} - w_{it}'\theta^w - x_{it}'\theta_g)^2 - (y_{it} - w_{it}'\theta^w - x_{it}'\theta_h)^2 + (x_{it}'(\theta_g - \theta_h))^2 \right).
\]

The first two terms on the right-hand side are squared residuals. The third term applies moment re-centering and ensures that \( d_{it}(g, h) \) has mean zero under the null hypothesis. This can be seen by re-writing \( d_{it}(g, h) \) as

\[
d_{it}(g, h) = -\sigma_i v_{it} x_{it}' (\theta_g - \theta_h) + (\theta_g - \theta_h)' x_{it} x_{it}' (\theta_g - \theta_h).
\]

The first term on the right-hand side has mean zero. If the null hypothesis is true, i.e., if \( g_i^0 = g \), then the second term vanishes and \( \sum_{t=1}^{T} E[d_{it}(g, h)] = 0 \) for all \( h \in G \setminus \{g\} \). If the null hypothesis is false, i.e., if \( g_i^0 \neq g \), then there exists \( h \in G \setminus \{g\} \) such that \( \sum_{t=1}^{T} E[d_{it}(g, h)] > 0 \). To see this, note that choosing \( h = g_i^0 \in G \setminus \{g\} \) guarantees that the second term in \( d_{it}(g, h) \) is a quadratic form with strictly positive mean.

This establishes that \( H_0 : g_i^0 = g \) can be tested by testing

\[
H'_0 : \sum_{t=1}^{T} E[d_{it}(g, h)] = 0 \quad \text{for all } h \in G \setminus \{g\}
\]

against

\[
H'_1 : \text{there exists } h \in G \setminus \{g\} \text{ such that } \sum_{t=1}^{T} E[d_{it}(g, h)] > 0.
\]

This is a one-sided significance test for a vector of moments.
Remark 4. Under our assumptions \( \mathbb{E}[d_{it}(g^0_i, h) | x_{it}] = 0 \). In some applications, tests based on such conditional moment equalities are more powerful than tests based on unconditional moment equalities. In our setting, this would be the case if there was a function \( f \) such that the moment \( \mathbb{E}[d_{it}(g^0_i, h) f(x_{it})] \) reveals more evidence against a specific null hypothesis than the moment \( \mathbb{E}[d_{it}(g^0_i, h)] \). Our test based on the unconditional equality detects an alternative \( g \) if \( \mathbb{E}[(\theta_g - \theta^0_g)^' x_{it}]^2 \) is sufficiently large. It does not seem that using conditional equalities can improve on that.

3.3. Test statistic

Our test statistic is the maximum of \( t \)-statistics for the null hypothesis that \( \sum_{t=1}^{T} \mathbb{E}[d_{it}(g, h)] = 0 \) for \( i = 1, \ldots, N \) and \( h = \{ g \} \). A studentized sample counterpart of \( \sum_{t=1}^{T} \mathbb{E}[d_{it}(g, h)] \) is

\[
\hat{D}_i(g, h) = \frac{\sum_{t=1}^{T} \hat{d}_{it}(g, h)}{\sqrt{\sum_{t=1}^{T} \left( \hat{d}_{it}(g, h) - \bar{\hat{d}}_{it}(g, h) \right)^2}},
\]

where

\[
\hat{d}_{it}(g, h) = \frac{1}{2} \left( \left( y_{it} - w_{it}^\prime \hat{\theta}^w - x_{it}' \hat{\theta}_g \right)^2 - \left( y_{it} - w_{it}^\prime \hat{\theta}^2 - x_{it}' \hat{\theta}_h \right)^2 + \left( x_{it}^\prime (\hat{\theta}_g - \hat{\theta}_h) \right)^2 \right)
\]

and

\[
\bar{\hat{d}}_{it}(g, h) = \sum_{t=1}^{T} \hat{d}_{it}(g, h)/T.
\]

Our test statistic for the hypothesis \( H_0 : g^0_i = g \) is given by

\[
\hat{T}_i(g) = \max_{h \in G \setminus \{ g \}} \hat{D}_i(g, h).
\]

This statistic can be interpreted as a one-sided distance measure that measures positive deviations of the vector \( \{ \hat{D}_i(g, h) \}_{h \in G \setminus \{ g \}} \) from zero.

3.4. Critical values

We provide two different ways of computing critical values. SNS critical values are motivated by the theory of self-normalized sums (SNS), whereas MVT critical values are based on the multivariate \( t \)-distribution (MVT). We denote the confidence set using SNS critical values by \( \hat{C}_{\alpha}^{SNS} \) and the confidence set using MVT critical values by \( \hat{C}_{\alpha}^{MVT} \).
3.4.1. SNS critical values

The SNS critical value is given by

\[ c_{\alpha,N,i}^{\text{SNS}}(g) = c_{\alpha,N}^{\text{SNS}} = \sqrt{T/(T-1)} \left( 1 - \frac{\alpha}{(G-1)N} \right), \]

where \( t_{T-1}^{-1}(p) \) denotes the \( p \) quantile of a \( t \)-distribution with \( T - 1 \) degrees of freedom. The factor \( G - 1 \) carries out a Bonferroni correction to control for correlation among the within-unit moment inequalities, i.e., the \( G - 1 \) elements of the vector \( \{ \hat{D}_i(g,h) \}_{h \neq g} \).

The SNS critical values are easy to implement since, unlike the MVT critical values, they do not require the choice of a regularization parameter to regularize estimated covariance matrices.

In our asymptotic analysis, we show that SNS critical values can be justified under weaker assumptions than MVT critical values. In particular, SNS critical values require weaker restrictions on the relative rates of \( N \) and \( T \) which indicates that they may be more robust in panel applications with a relatively short time dimension.

Since the SNS approach does not adapt to the correlation of the within-unit moment inequalities it produces more conservative confidence sets than the MVT approach.

Remark 5. If \( \hat{d}_{it}(g,h) \) follows a normal distribution then the SNS critical value is equal to the finite sample \((1 - \alpha/(N(G-1)))\)-quantile of \( \tilde{D}_i(g,h) \).

3.4.2. MVT critical values

The MVT critical value is given by

\[ c_{\alpha,N,i}^{\text{MVT}}(g) = c_{\alpha,N}^{\text{MVT}} \left( \rho(\hat{\Omega}_i(g), \epsilon) \right) = \sqrt{T/(T-1)} \left( t_{\max,\rho(\hat{\Omega}_i(g), \epsilon),T-1}^{-1} \right) \left( 1 - \frac{\alpha}{N} \right), \]

where \( t_{\max,\Omega,T-1} \) denotes the distribution function of the maximal entry of a centered random vector with multivariate \( t \)-distribution with scale matrix \( \Omega \) and \( T - 1 \) degrees of freedom. \( \rho \) is the regularization function defined in Appendix A and \( \epsilon \) is a regularization parameter.

For \( i = 1, \ldots, N \) and \( g \in \mathbb{G} \), \( \hat{\Omega}_i(g) \) is the \((G-1) \times (G-1)\) matrix given by

\[ \left( \hat{\Omega}_i(g) \right)_{h,h'} = \frac{\sum_{t=1}^{T} \left( \hat{d}_{it}(g,h) - \bar{\hat{d}}_{it}(g,h) \right) \left( \hat{d}_{it}(g,h') - \bar{\hat{d}}_{it}(g,h') \right)}{\sqrt{\sum_{t=1}^{T} \left( \hat{d}_{it}(g,h) - \bar{\hat{d}}_{it}(g,h) \right)^2 \sum_{t=1}^{T} \left( \hat{d}_{it}(g,h') - \bar{\hat{d}}_{it}(g,h') \right)^2}}. \]

\(^3\) The distribution function of the multivariate \( t \)-distribution can be efficiently approximated by modern algorithms (Genz 1992). Implementations exist for Stata (Grayling and Mander 2016) and R (Azzalini and Genz 2016).
This matrix estimates the correlation of the within-unit moment inequalities, i.e., the correlation of the vector \( \{d_{it}(g, h)\}_{h \in G \setminus \{g\}} \). Since it adapts to the within-unit correlation, the MVT approach is less conservative than the SNS approach and asymptotically achieves the per-unit coverage probability required in Theorem 1.

The regularized correlation matrix \( \rho(\Omega, \epsilon) \) is obtained from \( \Omega \) by rounding pairwise correlations in \( \Omega \) that are less than \(-1 + \epsilon\) down to exactly minus one in a way that guarantees that \( \rho(\Omega, \epsilon) \) is a correlation matrix. For matrices \( \Omega \) that are bounded away from singularity, \( \rho \) is the identity map. In our simulations we find that our procedure is not very sensitive to the choice of \( \epsilon \). In our application we choose \( \epsilon = 0.01 \).

In our setting with many simultaneous tests, regularization is necessary to control the estimation error from estimating perfect negative pairwise correlations. For a finite number of simultaneous tests, regularization is not required. An example of a setting where within-unit inequalities are perfectly correlated is given by parallel groups, i.e., a setting where there are groups \( g \neq g' \) such that \( \theta_{g'} = a\theta_g \) for a scalar \( a \). Moreover, if \( G > p + 1 \), where \( p \) is the dimension of \( x_{it} \), then the population matrix is singular and may hence contain perfect negative correlations.

By computing critical values based on a multivariate t-distribution rather than its normal limit we obtain a slightly more conservative confidence set if \( T \) is small. This reduces the risk of undercoverage in short panels.

**Remark 6.** Other approaches for one-sided testing of vectors (e.g. those based on the quasi-likelihood ratio statistic as in Kudo (1963)) cannot be applied in settings with a singular population correlation matrix. The MAX statistic is only sensitive to issues arising from perfectly negative correlations.

**Remark 7.** For \( G = 2 \) groups the SNS and MVT critical values are identical.

### 4. Two-step procedure

We propose a two-step procedure that we call *unit selection*. In settings where some units are easy to classify correctly whereas other units are not, our confidence set can be conservative. Such settings arise if there is substantial heteroscedasticity, i.e., large variation in \( \sigma_i \). The two-step procedure can alleviate this problem.

The first step detects units that are easy to classify. For each of these units we report a singleton confidence set that contains only the unit’s estimated group membership. The remaining units are selected for additional processing in the second step. The second step computes our confidence sets from Section 3 with either MVT or SNS critical values on the
subsample of selected units. The two-step procedure controls the joint confidence level by slightly increasing the nominal size of the confidence sets that are computed in the second step.

Unit selection can increase the power of the confidence set because the second step carries out fewer simultaneous tests than the corresponding one-step procedure. For example, suppose that the first step eliminates \( N/3 \) units. The second step uses a Bonferroni correction to control the test error of simultaneous tests \( 2N/3 \), whereas the one-step procedure will use a Bonferroni correction for \( N \) simultaneous tests.

For our two-step procedure we assume that \( \hat{g}_i \) minimizes squared loss, i.e., we assume

\[
\sum_{t=1}^{T} \hat{d}_{it}^U(\hat{g}_i, h) \leq 0 \quad \text{for all } h \in G,
\]

where

\[
\hat{d}_{it}^U(g, h) = (y_{it} - w_{it}'\hat{\theta}w - x_{it}'\hat{\theta}g)^2 - (y_{it} - w_{it}'\hat{\theta}w - x_{it}'\hat{\theta}h)^2.
\]

This requirement is automatically satisfied if the grouped panel model is estimated by \textit{kmeans} clustering (e.g. Bonhomme and Manresa [2015]). Note that for our one-step procedures we place no assumptions on the estimator of group membership \( \hat{g}_i \).

The first step identifies a unit as easy to classify if it satisfies two conditions that we call \textit{moment selection} and \textit{hypothesis selection}. A unit \( i \) satisfies the moment selection criterion if we detect substantial slackness in inequality (3) for all \( h \neq \hat{g}_i \). A unit \( i \) satisfies the hypothesis selection criterion if all group memberships \( h \neq \hat{g}_i \) are rejected.

The first step is parameterized by \( \beta \in [0, \alpha/3) \). The larger \( \beta \), the more unit selection is carried out. Let

\[
\hat{D}_i^U(g, h) = \frac{\sum_{t=1}^{T} \hat{d}_{it}^U(g, h)}{\sqrt{\sum_{t=1}^{T} \left( \hat{d}_{it}^U(g, h) - \bar{d}_{i}^U(g, h) \right)^2}},
\]

where \( \bar{d}_{i}^U(g, h) = \sum_{t=1}^{T} \hat{d}_{it}^U(g, h)/T \). This is a counterpart to \( \hat{D}_i(g, h) \) that does not adjust for the mean under the null hypothesis.

The moment selection is given in terms of function \( \hat{M}_i(g) \) defined below. For \( g \in G \) and \( i = 1, \ldots, N \), let

\[
\hat{M}_i(g) = \left\{ h \in G \setminus \{g\} \mid \hat{D}_i^U(g, h) > -2c_{\beta,N}^{\text{SNS}} \right\}.
\]
This set gives the selected moment inequalities for the hypothesis $H_0 : g_0^i = g$. For units $i$ for which $\hat{M}_i(g)$ is empty we have strong evidence that $g_0^i = g$. These units satisfy the moment selection criterion for elimination in the first step. By (3), $\hat{M}_i(g)$ is never empty for $g \neq \hat{g}_i$. This ensures that moment selection does not eliminate misclassified units.

Hypothesis selection is carried out by an iterative algorithm. Let type = MVT, SNS. The algorithm proceeds as follows:

(A) Set $s = 0$ and $H_i(0) = \mathbb{G}$.

(B) Set $\hat{N}(s) = \sum_{i=1}^{N} \max_{g \in H_i(s)} 1\{\hat{M}_i(g) \neq \emptyset\}$.

(C) Set

$$H_i(s + 1) = \left\{ g \in \mathbb{G} \mid \hat{T}_i(g) \leq c_{\text{type}, \alpha - 2\beta, \hat{N}(s), \hat{g}_i}(g) \right\} \cup \{\hat{g}_i\}.$$

(D) If $H_i(s + 1) = H_i(s)$ for all $i$ then exit the algorithm. Otherwise set $s = s + 1$ and go to Step 2.

Step A initializes the algorithm by designating for each unit $i$ all possible group assignments as hypotheses that have to be tested. Step B of the algorithm counts the number $\hat{N}(s)$ of units that are not easy to classify. Unit $i$ is easy to classify if and only if $\hat{M}_i(\hat{g}_i)$ is empty (moment selection) and $H_i(s) = \{\hat{g}_i\}$ (hypothesis selection). Step C carries out hypothesis selection with critical values adjusted for $\hat{N}(s)$ simultaneous tests of group membership. $H_i(s + 1)$ gives a preliminary marginal confidence set for unit $i$ after $s$ iterations of hypothesis selection. Step D checks convergence of the algorithm. The algorithm always converges. This is because $\hat{N}(s)$ and the cardinality of $H_i(s)$ are decreasing in $s$.

The second step gives the final joint confidence set by

$$\hat{C}_{\text{sel}, \alpha, \beta}^{\text{type}} = \bigtimes_{1 \leq i \leq N} H_i(s^*).$$

where $s^*$ is the final value of $s$ in the algorithm. The second-step confidence set is calculated at nominal confidence level $1 - \alpha + 2\beta$ (see the construction of $H(s + 1)$ in Step C). The adjustment by $2\beta$ represents the cost of unit selection and controls for two possible errors at the first step. The first error is estimating an incorrect group membership for a unit that is easy to classify “in population”. The second error is erroneously declaring as easy to classify.

If sufficiently many units are eliminated in the first step, then the benefits of unit selection outweigh the cost and $\hat{C}_{\text{sel}, \alpha, \beta}^{\text{type}}$ is more powerful (“smaller”) than the corresponding one-step confidence set $\hat{C}_\alpha^{\text{type}}$. If insufficiently many units are eliminated, then a two-step confidence
set can be slightly more conservative (“larger”) than the corresponding one-step confidence set.

**Remark 8.** Let $\alpha_i$ denote the probability that the marginal confidence set for unit $i$ does not include $i$’s true group membership. The joint confidence set has a coverage probability of $1 - \alpha$ if $\sum_{i=1}^{N} \alpha_i \leq \alpha$. The one-step procedures allocate error probability evenly by setting $\alpha_i = \alpha/N$. For units that are easy to classify the singleton confidence set $\{\hat{g}_i\}$ covers the true group membership $g^0_i$ with probability strictly greater than $1 - \alpha/N$. This can render the joint confidence set conservative. Unit selection is a data-driven way of reshuffling allocated error probability from units that are easy to classify to units that are difficult to classify.

**Remark 9.** The term “moment selection” and also some arguments in our theoretical analysis are borrowed from the literature on testing moment inequalities (see e.g. Chernozhukov, Chetverikov, and Kato 2019; Andrews and Soares 2010; Andrews and Barwick 2012; Romano, Shaikh, and Wolf 2014; Canay and Shaikh 2017). However, our use of moment selection is different from previous applications that use moment selection to reduce the power loss from possibly slack moment inequalities. In these applications, moment selection identifies inequalities that are “obviously” satisfied and do not need to be tested. In contrast, our test strategy is such that we test inequalities that are always binding under the null hypothesis. We use moment selection to reduce the power loss from running many simultaneous tests. In our setting, moment selection identifies units for which $\hat{g}_i$ “obviously” estimates the true group membership. This allows us to ignore such units when constructing the joint confidence set.

5. **Asymptotic validity of confidence sets**

We provide asymptotic justifications of our one-step confidence sets both for data that is independent across time and for data that is weakly time-dependent. Our analysis for the case of independent data shows that the SNS critical values can be justified under weaker moment assumptions and less restrictions on the relative rates of $N$ and $T$ than the MVT critical values. Time-independence is empirically implausible in many panel settings. Our analysis for the case of weak time dependence shows that our confidence sets are still valid if this assumption is relaxed. In addition, we provide an asymptotic justification of our two-step procedure.

Proofs for all results in this section are in the supplemental appendix.
5.1. Assumptions

Our asymptotic framework is of the long-panel variety and takes both the number of units \( N \) and the number of time periods \( T \) to infinity. In particular, we consider asymptotic sequences in which \( T = T(N) \), where \( T(\cdot) \) is increasing but its exact form is unspecified except for conditions given in the statement of the theorems. In many panel data sets, the number of units far exceeds the number of time periods. We replicate this feature along the asymptotic sequence by allowing \( N \) to diverge at a much faster rate than \( T \). The parameters \( \theta^w, \{\theta_g\}_{g \in G} \) and \( \sigma_i \) depend potentially on \( N \). For notational convenience we keep this dependence implicit.

We list the maintained assumptions below. We consider a sequence \( \mathbb{P}_N \) of classes of probability measures. All our theoretical results hold uniformly over the sequence \( \mathbb{P}_N \). For a probability measure \( P \), let \( \mathbb{E}_P \) denote the expectation operator that integrates with respect to the measure \( P \).

**Assumption 1.**

1. (Consistent estimation of number of groups) The estimator \( \hat{G}_N \) of the number of latent clusters satisfies
   \[
   \lim_{N \to \infty} \sup_{P \in \mathbb{P}_N} P \left( \hat{G}_N \neq G \right) = 0.
   \]

2. (Estimation of auxiliary parameters) There are vanishing sequences \( r_{N,\theta} \) and \( a_{N,\theta} \) such that
   \[
   \sup_{P \in \mathbb{P}_N} P \left( \| \hat{\theta}^w - \theta^w \| \vee \max_{g \in G} \| \hat{\theta}_g - \theta_g \| > r_{N,\theta} \right) \leq a_{N,\theta}.
   \]

3. (Full rank) Let \( \lambda_{\min}(\cdot) \) denote the smallest eigenvalue of its argument. There is a finite constant \( C_\lambda > 0 \) such that
   \[
   \inf_{P \in \mathbb{P}_N} \min_{1 \leq i \leq N} \lambda_{\min} \left( \frac{1}{T} \sum_{t=1}^{T} \mathbb{E}_P \left[ x_{it} x'_{it} v_i^2 \right] \right) \geq C_\lambda^{-1}.
   \]

4. (Group separation) \( \iota_N \equiv \min_{g \in G} \min_{h \in G \setminus \{g\}} \| \theta_g - \theta_h \| > 0. \)

Assumption 1 requires that the number of groups is consistently estimated. This is a weak assumption that can be guaranteed by using an appropriate procedure for choosing the number of groups (e.g. Lu and Su [2017]; Vogt and Schmid [2021]).

Assumption 1.2 requires that the estimators \( \hat{\theta}^w \) and \( \hat{\theta}_g \) are consistent for \( \theta^w \) and \( \theta_g, g \in G \) at a certain rate. If the group-specific coefficients are estimated from a training set with
observed group memberships for \( N_{\text{train}} \) units then we can take \( r_{N,\theta} = O\left(N_{\text{train}}^{-1/2}\right) \) under some regularity assumptions. In settings without training data, rate calculations can be based on the results in Bonhomme and Manresa (2015), Su, Shi, and Phillips (2016), and Wang, Phillips, and Su (2018). Their methods provide \( \sqrt{NT} \)-consistent estimators for time-invariant coefficients. Dzemski and Okui (2021) show that the \( \sqrt{NT} \) rate can be maintained even if some units are misclassified in the limit.

The full-rank condition in Assumption 1.3 ensures that the denominator of \( \hat{D}_i(g, h) \) does not divide by too close to zero. It restricts \( v_{it} \) (which has variance one) but does not limit the magnitude of the error term \( \sigma_i v_{it} \) in our panel regression (1).

Assumption 1.4 maintains that groups are unique in the sense that there are not two groups that share the same coefficient values. The minimal distance between two groups is measured by \( \iota_N \) and is allowed to vanish asymptotically, provided that it satisfies additional rate conditions stated below.

5.2. SNS critical values under independence

We provide the validity of the procedure with SNS critical values under independence across time. We make the following assumption.

Assumption 2 (Independent sampling). The vectors \( \{(x_{it}', w_{it}', v_{it})\}_{1 \leq i \leq N} \) are independent across time \( t = 1, \ldots, T \).

Assumption 2 rules out serial dependence. However, it does not restrict cross-sectional correlation nor does it impose stationarity.

The quality of the asymptotic approximation of the SNS confidence set depends on the asymptotic behavior of the moments:

\[
D_{N,3}^3 = \max_{1 \leq i \leq N} \left( \frac{1}{T} \sum_{t=1}^{T} E_P[|v_{it}|^3 \|x_{it}\|^3] \right),
\]

and

\[
B_{N,8}^8 = 1 \vee \sup_{P \in \mathcal{P}_N} E_P \left[ \max_{q_1+q_2+q_3=4} \max_{1 \leq i \leq N} |v_{it}|^{2q_1} \|x_{it}\|^{2q_2} \|w_{it}\|^{2q_3} \right]
\]

Assumption 3. (Moment conditions) There is a finite constant \( C_4 \) such that for all \( N = \)
1, 2, \ldots,

\begin{align*}
\max_{1 \leq i \leq N} \mathbb{E}_P \left[ \max_{q_1+q_2+q_3=4} \left| v_{it} \right|^{q_1} \left| x_{it} \right|^{q_2} \left| w_{it} \right|^{q_3} \right] & \leq C_4,
\end{align*}

and $B_{N,8}$ is finite for all is $N = 1, 2, \ldots$.

We allow $B_{N,8}$ to diverge to infinity at a controlled rate.

**Theorem 2.** Suppose that $P_N$ satisfies Assumptions\footnote{If the $\hat{d}_{it}(g^0_i, h)$ are normally distributed, then the $t$-distribution describes the exact finite sample distribution.} 1, 2, and 3. Suppose that

\begin{align*}
T^{-1/3} D_{N,8}^4 \log N & = o(1), \\
r_{\theta,N} \log N / (t_N \wedge \min_{1 \leq i \leq N} \sigma_i) & = o(1), \\
T^{-1/3} D_{N,3}^2 \log N & = o(1).
\end{align*}

Then,

\begin{align*}
\liminf_{N,T \to \infty} \inf_{P \in \mathbb{P}_N} P \left( \left\{ g^0_i \right\}_{1 \leq i \leq N} \in \hat{C}_{\alpha}^{SNS} \right) & \geq 1 - \alpha.
\end{align*}

This theorem states that the SNS confidence set contains the true group membership structure at least with probability $1 - \alpha$ asymptotically. The deviation from the nominal coverage does not depend on $P$. Hence, the convergence is uniform over $\mathbb{P}_N$.

The first two conditions, $T^{-1/3} B_{N,8}^4 \log^{3/2} N = o(1)$ and $r_{\theta,N} \log N / (t_N \wedge \min_{1 \leq i \leq N} \sigma_i) = o(1)$ bound the effect of parameter estimation on the coverage probability of the confidence set. Condition $T^{-1/3} D_{N,3}^2 \log N = o(1)$ is used to approximate the behavior of each of $(G - 1)N$ estimated moment inequalities jointly by scaled $t$-distributions.\footnote{If the $\hat{d}_{it}(g^0_i, h)$ are normally distributed, then the $t$-distribution describes the exact finite sample distribution.}

Our result establishes that the SNS confidence set is valid even if $T$ is very small compared to $N$. For example, consider the following situation: $t_N$ is a positive constant; $\min_{1 \leq i \leq N} \sigma_i$ is bounded away from zero; $r_{N,\theta} = (NT)^{-1/2} \log^{1/2} T$; $a_{N,\theta} = T^{-1}$; $B_{N,8} = O(\sqrt{\log N})$; and $D_{N,3} = O(1)$. In this case, the conditions of the theorem are met when $\log^{3} N / T^{1/3} \to 0$. $T$ is thus allowed to diverge to infinity at a much slower rate than $N$. We therefore expect that the confidence set performs well even if the panel is rather short.

Our distributional approximation relies on a Cramér-type moderate deviation inequality for self-normalized sums (Jing, Shao, and Wang 2003). Chernozhukov, Chetverikov, and Kato (2019, Theorem 4.1) were the first to use this kind of argument in the context of testing many moment inequalities.
Nonetheless, our result differs from that of Chernozhukov, Chetverikov, and Kato (2019) in several respects. First, our critical value based on the $t$-distribution is always computable. Their critical value is a transformation of normal quantiles that is undefined for small $T$. Second, they do not consider parameter uncertainty, whereas our results quantify the effect of estimating the group-specific parameters under low-level assumptions that are easy to interpret.

5.3. MVT critical values under independence

We provide a theoretical justification of the procedure with MVT critical values under the assumption of independence over time. The validity of the MVT critical values depends on the correlation structure of the $t$-statistics. Let $\Omega_i(g_0^i)$ denote the $(G - 1) \times (G - 1)$-matrix with entries

$$
(\Omega_i(g_0^i))_{h,h'} = \frac{(\theta_{g_0^i} - \theta_h)' \frac{1}{T} \sum_{t=1}^{T} \mathbb{E}_P [x_{it} x_{it}' v_{it}^2] (\theta_{g_0^i} - \theta_{h'})}{s_i(h)s_i(h')}.
$$

where

$$
s_i^2(h) = (\theta_{g_0^i} - \theta_h)' \left( \frac{1}{T} \sum_{t=1}^{T} \mathbb{E}_P [x_{it} x_{it}' v_{it}^2] \right) (\theta_{g_0^i} - \theta_h).
$$

This matrix $\Omega_i(g_0^i)$ is the population counterpart of the sample correlation matrix $\hat{\Omega}_i(g_0^i)$.

**Assumption 4.** (Regular correlation matrix) There is a positive constant $c_\omega$ such that for all $N = 1, 2, \ldots$, all $h, h' \in 1, \ldots, G - 1$, all $i = 1, \ldots, N$ and all $P \in \mathbb{P}_N$

$$(\Omega_i(g_0^i), h)_{h,h'} \in \{-1, 1\} \cup [-1 + c_\omega, 1 - c_\omega].$$

This assumption does not rule out singularity of a population correlation matrix. It is automatically satisfied if the time series for all units are sampled identically from a distribution that does not change along the asymptotic sequence.

Compared to our assumptions for the validity of the SNS procedure, we need a stronger moment condition on $v_{it} x_{it}$.

---

5 In our setting, the critical value in Chernozhukov, Chetverikov, and Kato (2019) is given by $\Phi^{-1}(1 - \alpha/((G - 1)N))/\sqrt{1 - \Phi^{-1}(1 - \alpha/((G - 1)N))^2/T}$. If $T$ is small, then the term inside of the square root can be negative.

6 Chernozhukov, Chetverikov, and Kato (2019) consider parameter uncertainty for their bootstrap procedures in their online supplement B.2, but not for their SNS procedures. For their bootstrap procedures they give a high-level assumption under which parameter uncertainty can be ignored.
Assumption 5. (Moment condition for MVT) There is a finite constant $C_8$ such that for all $N = 1, 2, \ldots$

$$\sup_{P \in \mathbb{P}_N} \max_{1 \leq i \leq N} \mathbb{E}_P \left[ |v_{it}|^4 \|x_{it}\|^4 \right] \leq C_8,$$

The following theorem establishes validity of the joint confidence set with MVT critical values.

**Theorem 3.** Suppose that $\mathbb{P}_N$ satisfies Assumptions 1, 2, 3, 4 and 5. Moreover, suppose that the regularization parameter satisfies $\epsilon = \epsilon_N = o(\log^{-1} N)$ and

$$T^{-2/5}B_{N,8}^4 \log N = o(\epsilon_N),$$

$$r_{\theta,N}/(\epsilon_N \wedge \min_{1 \leq i \leq N} \sigma_i) = o(\epsilon_N),$$

$$T^{-1/2}B_{N,8}^4 \log^3(NT) = o(1).$$

Then,

$$\lim_{N,T \to \infty} \inf_{P \in \mathbb{P}_N} P\left( \left\{ g_i^0 \right\}_{1 \leq i \leq N} \in \hat{C}^{\text{MVT}}_{\alpha} \right) \geq 1 - \alpha.$$ 

The theorem requires slightly stronger assumptions than Theorem 2. For example, the rate conditions require that the relative magnitudes of $N$ and $T$ satisfy at least $B_{N,8}^4(\log^2 N)/T^{2/5} \to 0$. Stronger assumptions are needed because, in contrast to the proof of Theorem 2, we eliminate the randomness in the denominator of $\hat{D}_i(g,h)$ before deriving a distributional approximation.

Some parts of our proof approach share similarities with the theoretical analysis of a bootstrap test for many moment inequalities in Chernozhukov, Chetverikov, and Kato (2019, Theorem 4.3), whereas other parts are entirely new. Chernozhukov, Chetverikov, and Kato (2019) consider a single test statistic for many moment inequalities. In contrast, we conduct many simultaneous tests with different hypothesis-specific critical values. This requires new arguments. Novel developments in our proof include a new anti-concentration inequality (Lemma E.8) and a new comparison bound for the multivariate normal distribution (Lemma E.9).

We use the anti-concentration inequality to evaluate the effect of parameter estimation on the test statistic and thus obtain a high-dimensional analogue of Slutsky’s lemma. The inequality uniformly controls the effect of perturbations in any of the simultaneous unit-wise tests around its respective unit-wise critical value. This renders it different from the inequality used in Chernozhukov, Chetverikov, and Kato (2019) and originally proved in
Chernozhukov, Chetverikov, and Kato (2015) that examines concentration of a single test around a single critical value.

The comparison bound is based on Li and Shao (2002) and is used to control estimation error in the data-dependent critical values. It bounds the ratio of the probabilities of two tail events, whereas most existing comparison bounds (e.g. Chernozhukov, Chetverikov, and Kato 2015) bound the difference such probabilities.

**Remark 10 (Sharpness of Theorem 3).** The proof of Theorem 3 can be combined with Theorem 7 to show that

$$\limsup_{N,T \to \infty} \sup_{P \in \mathcal{P}_N} \mathbb{P}\left( \{g_i^0\}_{1 \leq i \leq N} \in \hat{C}_\alpha^{\text{SNS}} \right) \leq 1 - \alpha + \frac{\alpha^2}{2}$$

under cross-sectional independence. In this case, the inequality in Theorem 3 is almost sharp and the power loss from using Bonferroni correction is minimal.

### 5.4. Validity of one-step procedures under dependent data

We now provide theoretical justifications of the SNS and MVT critical values under dependent data. We need stronger assumptions on the tail behavior of regressors and error terms.

**Assumption 6.** There exist constants $a$ and $d_1 > 1$ such that $\mathbb{P}(|Z| > z) < \exp\left(-\left(z/a\right)^{d_1}\right)$ for sufficiently large $z$ where $Z$ is any component of the random vector $\left(x_{it}', w_{it}', v_{it}\right)$.

This assumption states that the tails of the distributions of the regressors and the error terms decay exponentially. It is stronger than Assumption 5 because it ensures the existence of any order of moment (see Lemma E.18).

**Assumption 7.** $(x_{it}', w_{it}', v_{it})$ is an $\alpha$-mixing sequence (as a sequence indexed by $t$) with mixing coefficient $\sup_i \alpha_i[k] \leq \exp(1 - bk^{d_2})$ for some $b > 0$ and $d_2 > 0$.

This assumption restricts the dependence of the data in terms of mixing coefficients. The mixing coefficients decay exponentially. For example, we exclude long memory processes.

**Assumption 8.** $\{x_{it}v_{it}\}_{i=1}^N$ is serially uncorrelated over $t$.

We assume that $x_{it}v_{it}$ is serially uncorrelated but do not rule out dependence in the variance or other higher moments. Under this assumption, the denominator of $\hat{D}_i(g, h)$ is a correct standard error of the numerator. Without it, the denominator has to be replaced by the square root of an estimator of the long-run variance. The analysis of such a modified statistic requires a different and more involved theoretical analyses.
The following theorem provides the validity of the SNS and MVT critical values in the case of dependent data.

**Theorem 4.** Suppose that $P_N$ satisfies Assumptions 7, 8, 9 and 10. Suppose also that $NT^{-\delta} \to 0$ and $N(G-1) \geq T^\delta'$ for some $\delta, \delta' > 0$, and assume $r_{N,\theta} = o(1 \wedge t_N \wedge \min_{1 \leq i \leq N} \sigma_i)$. Then,

$$\lim_{N,T \to \infty} \inf_{P \in \mathbb{P}_N} \inf P \left( \{ g_i^0 \}_{1 \leq i \leq N} \in \hat{C}^{\text{SNS}}_\alpha \right) \geq 1 - \alpha$$

and

$$\lim_{N,T \to \infty} \inf_{P \in \mathbb{P}_N} \inf P \left( \{ g_i^0 \}_{1 \leq i \leq N} \in \hat{C}^{\text{MVT}}_\alpha \right) \geq 1 - \alpha.$$

The theorem is based on several recent developments in high-dimensional statistics for dependent data. In particular, we rely on the high-dimensional CLT for dependent data in Chang, Chen, and Wu (2021). Moreover, we control the difference between sample averages and population means of a high-dimensional random vector by using a result in Merlevède, Peligrad, and Rio (2011).

The rate condition on the relative magnitude of $N$ and $T$ is slightly stronger than those required in Theorem 2 or Theorem 3. Nonetheless, the condition is satisfied as long as $N$ is of geometric order of $T$ and is therefore arguably still weak and appropriate in many panel settings.

### 5.5. Validity of two-step procedures

We now examine the asymptotic properties of the two-step procedures. We focus on cases with independence across time periods.

**Assumption 9.**

1. The group membership estimator $\hat{g}_i$ satisfies (3) for all $i = 1, \ldots, N$.

2. The vector $(\theta^w, \{ \theta_g \}_{g \in \mathbb{G}})$ is contained in a compact parameter space $\Theta$.

Assumption 9.1 states that $\hat{g}_i$ minimizes squared loss and is discussed in Section 4. The compactness condition in Assumption 9.2 is standard in asymptotic analysis.

The following result establishes the validity of our two-step procedure.

**Theorem 5.** Let $\text{type} \in \{ \text{SNS}, \text{MVT} \}$. If $\text{type} = \text{SNS}$ then assume that $P_N$ satisfies the assumptions of Theorem 2. If $\text{type} = \text{MVT}$ then assume $P_N$ that satisfies the assumptions
of Theorem 3. In addition, suppose that Assumption 9 holds and there is $0 < c < 1/2$ such that

$$T^{-1-c/2}D_{N,3}^3 \log^{3/2}(N/\beta) = o(1),$$

$$T^{-1-c/2}B_{N,8}^4 \log^2(N/\beta) = o(1),$$

$$T^{1/2+c} \sqrt{\log(N/\beta)r_{N,\theta}/(\nu_N \wedge \min_{1 \leq i \leq N} \sigma_i)} = o(1).$$

Then, for all $0 < \alpha < 1$,

$$\liminf_{N,T \to \infty} \inf_{P \in \mathcal{P}_N} P \left( \left\{ g_i^0 \right\}_{1 \leq i \leq N} \in \mathcal{C}_{\text{sel},\alpha,\beta} \right) \geq 1 - \alpha.$$

This result requires stronger assumptions than the results for the one-step procedures. This is because we are using an auxiliary statistic in the first-step that summarizes moment inequalities that may be slack under the null hypothesis. In particular, (6) requires $r_{N,\theta} = o(T^{-1/2-c})$.

6. Application to heterogeneous effects of the minimum wage

In this section, we revisit the work by Wang, Phillips, and Su (2019) to illustrate our procedures. Wang, Phillips, and Su (2019) estimate a grouped panel model to study heterogeneous effects of a minimum wage in the restaurant sector. Their analysis builds on Dube, Lester, and Reich (2010) who employ a similar panel model but do not allow for effect heterogeneity. We assess the significance of the group memberships estimated in Wang, Phillips, and Su (2019) by computing confidence sets for group memberships.

We use the panel data described in Dube, Lester, and Reich (2010). It contains quarterly data for 1380 US counties, ranging from the first quarter of 1990 to the second quarter of 2006. The grouped panel model estimated in Wang, Phillips, and Su (2019) is given by

$$\log(\text{emp}_{it}) = \eta_{g_0(s_i)} \log(\text{mw}_{it}) + \gamma_{g_0(s_i)} \log(\text{pop}_{it}) + \delta_{g_0(s_i)} \log(\text{emp}_{TOT}) + \phi_i + \tau_t + \epsilon_{it}$$

for counties $i = 1, \ldots, 1380$ and time periods $t = 1990Q1, \ldots, 2006Q2$. Here, $\text{emp}_{it}$ is employment in the restaurant sector, $\text{mw}_{it}$ is the minimum wage, $\text{emp}_{TOT}$ is total employment in all sectors, $\phi_i$ is the county fixed effect, $\tau_t$ is the time fixed effect and $\epsilon_{it}$ is an idiosyncratic error term. County $i$ is located in state $s_i \in \mathcal{S} = \{\text{AL}, \ldots, \text{WY}\}$ and states exhibit a latent group structure that is described by the mapping $g^0 : \mathcal{S} \to \{1, \ldots, G\}$, where $G$ is the number of
| g | \(g_\eta\) | \(g_\gamma\) | \(g_\delta\) |
|---|---|---|---|
| 1 | 0.55 | 0.63 | 0.51 |
| 2 | -0.03 | 0.60 | 0.61 |
| 3 | 0.06 | 0.34 | 0.41 |
| 4 | -0.25 | 0.47 | 0.53 |

Table 1: Estimates for the group-specific slope coefficients.

Wang, Phillips, and Su (2019) determine \(G = 4\) and estimate the model by employing the CLasso estimator (Su, Shi, and Phillips 2016). Their estimates for the slope coefficients are given in Table 1. There are two groups with an estimated positive effect of the minimum wage on employment and two groups where the effect is negative.

Based on these estimates for the slope coefficients we estimate group memberships by running one update step of the \textit{kmeans} algorithm. This ensures that the estimated group memberships satisfy inequality (3) and produces estimated group memberships that are identical to the CLasso estimates in all but six cases. Estimated group memberships are displayed in Figure 1 and reported in Table B.1 in the supplemental appendix. Note that Figure 1 is slightly different from Figure 2 in Wang, Phillips, and Su (2019) due to the additional \textit{kmeans} step.

To translate the panel model to our framework we enumerate the three dimensions state, county and time by using the subscripts \(s\), \(c\) and \(t\), respectively, and identify the \(s\) dimension latent groups. County \(i\) faces slope coefficients that depend on the group membership of the state that \(i\) is located in.

Wang, Phillips, and Su (2019) determine \(G = 4\) and estimate the model by employing the CLasso estimator (Su, Shi, and Phillips 2016). Their estimates for the slope coefficients are given in Table 1. There are two groups with an estimated positive effect of the minimum wage on employment and two groups where the effect is negative.

Based on these estimates for the slope coefficients we estimate group memberships by running one update step of the \textit{kmeans} algorithm. This ensures that the estimated group memberships satisfy inequality (3) and produces estimated group memberships that are identical to the CLasso estimates in all but six cases. Estimated group memberships are displayed in Figure 1 and reported in Table B.1 in the supplemental appendix. Note that Figure 1 is slightly different from Figure 2 in Wang, Phillips, and Su (2019) due to the additional \textit{kmeans} step.

To translate the panel model to our framework we enumerate the three dimensions state, county and time by using the subscripts \(s\), \(c\) and \(t\), respectively, and identify the \(s\) dimension latent groups. County \(i\) faces slope coefficients that depend on the group membership of the state that \(i\) is located in.
Table 2: One-step confidence intervals at level $1 - \alpha = 0.95$ for the subset of states for the p-value of the estimated group membership (computed by the one-step SNS procedure) exceeds 1%. “p-val $\hat{g}_i$” is the p-value for the significance of the estimated group membership for the state. “CS cardinality” is the cardinality of the marginal confidence set for the state. ‘CS’ is the marginal confidence set for the state. Results for all states are reported in Table B.1 in the supplemental appendix.

| State             | $\hat{g}_i$ | MVT  | SNS  | CS cardinality | MVT | SNS |
|-------------------|-------------|------|------|----------------|-----|-----|
| Delaware          | 2           | 0.0908 | 0.1251 | 2, 2, 2         | 2, 2 |
| District of Columbia | 2        | 0.1591 | 0.3745 | 2, 2, 2, 2, 3, 4 | 3, 3 |
| Hawaii            | 2           | 0.4395 | 0.5274 | 2, 2, 2         | 2, 2 |
| Idaho             | 3           | 0.0641 | 0.0962 | 2, 2, 3, 4      | 3, 4 |
| Massachusetts     | 2           | 0.0227 | 0.0341 | 1, 1, 2         | 2, 2 |
| Nebraska          | 4           | 0.0582 | 0.0874 | 2, 2, 3, 4      | 3, 4 |

As the cross-sectional dimension and the $c$ and $t$ dimensions jointly as the second (“time”) dimension. We compute our confidence sets based on the fixed-effect transformation

$$\hat{\text{lemp}}_{sct} = \eta_{g_0} \hat{\text{lmw}}_{sct} + \gamma_{g_0} \hat{\text{pop}}_{sct} + \delta_{g_0} \hat{\text{lemp}}_{sct} + \hat{\tau}_t + \hat{\epsilon}_{sct}.$$  

Here, $\hat{\text{lemp}}_{sct} = \log(\text{lemp}_{sct}) - \frac{1}{T} \sum_{t'=1}^T \log(\text{lemp}_{sct'})$ and $\hat{\text{lmw}}_{sct}, \hat{\text{pop}}_{sct}$ and $\hat{\text{lemp}}_{sct}$ are defined similarly. Moreover, $\hat{\tau}_t = \tilde{\tau}_t - \frac{1}{T} \sum_{t'=1}^T \tau_{t'}$ and $\hat{\epsilon}_{sct} = \epsilon_{sct} - \frac{1}{T} \sum_{t'=1}^T \epsilon_{sct'}$.

We compute the one-step joint confidence set for the group structure for both MVT and SNS critical values and level $1 - \alpha = 0.95$. To compute the MVT critical values we set the regularization parameter to $\epsilon = 0.01$. This value works well in our simulations in Section 7. Table 2 presents results for selected states. Results for all states are reported in Table B.1 in the supplemental appendix.

Both choices for the critical value produce the same realized confidence set. The cardinality of the state-wise marginal confidence sets is three for one state (District of Columbia), two for four states (Delaware, Hawaii, Idaho, Nebraska) and one for the 46 remaining states. This implies that we are able to identify the true group memberships of 46 states at confidence level $1 - \alpha = 0.95$.

For the one-step procedures with MVT and SNS critical values we compute p-values for significance of the estimated group membership. We say that the estimated group membership $\hat{g}_i$ for state $i$ is significant at level $\alpha$ if the marginal confidence set for state $i$ contains only $\hat{g}_i$. Up to a failure probability of at most $\alpha$, significantly estimated group memberships reveal the true group membership and cannot be attributed to estimation error. The p-value for significance of $\hat{g}_i$ is the smallest value of $\alpha$ such that $\hat{g}_i$ is significant at level $\alpha$. Table 2 gives
results for the states with the largest p-values and Table B.1 in the supplemental appendix reports results for all states. The p-values based on MVT critical values are smaller than the p-values based on SNS critical values. This is explained by the fact that the test based on MVT critical values adapts to the within-unit moment inequalities whereas the test based on SNS critical values does not. The results in Table 2 demonstrate the empirical relevance of accounting for within-unit correlation. For example, the estimated group membership of Delaware (group 2) is significant at level $\alpha = 0.1$ under MVT critical values. Under SNS critical values, it cannot be ruled out that Delaware belongs to group 3 and its estimated group membership is not significant.

Displaying a visual representation of the estimated clusters as we do in Figure 1 is standard practice even in applications where the clustering structure is considered a nuisance parameter, not a parameter of interest. Visual inspection of the clusters is meant to confirm their economic plausibility and serves as an informal test of model specification. Based on our new procedures, such an informal analysis can be complemented by a graphical representation of clustering uncertainty. Figure 2 represents clustering uncertainty by shading US states according to the p-value of their respective estimated cluster memberships and by the cardinality of the marginal confidence sets ($1 - \alpha = 0.95$). It shows that the clustering structure is precisely estimated. If most of the maps in Figure 2 was shaded in a dark hue then this would point to a large degree of clustering uncertainty and might indicate that the clustering algorithm is overfitting on the sample, rather than picking up structural heterogeneity.

We compute the joint confidence set at confidence level $1 - \alpha = 0.95$ using our two-step procedure with MVT critical values in the second step. We set $\beta = \alpha/10 = 0.005$. This value works well in our simulations. In this application, the two-step procedure does not shrink the confidence set. Figure 3 summarizes how our unit-selection procedure determines the significance of the estimated group memberships. The first step eliminates eight units. This reduces the second-step p-values, since the Bonferroni adjustment has to account for only 43 rather than 51 simultaneous tests. Turning on unit selection lowers the threshold p-value at which we can conclude significance from $\alpha = 0.05$ to $\alpha - 2\beta = 0.04$. In this application, unit selection substantially lowers the second step p-values for some states but not by enough to push any state over the shifted threshold of significance.
Significance of estimated group membership

p-val ≤ 0.01 0.01 < p-val ≤ 0.05 0.05 < p-val ≤ 0.1 p-val > 0.1

Cardinality of state-wise confidence sets at α = 0.05

1 2 3

Figure 2: Visual representation of clustering uncertainty. The upper panel shows joint state-wise p-values for the respective estimated group memberships. The lower panel shows the cardinality of the state-wise marginal confidence sets at a joint confidence level of 1 − α = 0.9.
Figure 3: Second-step p-values for significance of the estimated group memberships with and without unit selection ($\alpha = 0.05$, $\beta = 0.005$). The dashed horizontal line indicates the threshold for significance without unit selection. The solid horizontal line indicates the threshold for significance with unit selection.

7. Simulations

Our simulation designs draw from our empirical application. The data generating process is given by

$$\tilde{\text{emp}}_{it} = \eta_{g_i} \tilde{\text{mw}}_{it} + \gamma_{g_i} \tilde{\text{pop}}_{it} + \delta_{g_i} \tilde{\text{emp}}_{it}^\text{TOT} + \sigma_i v_{it}$$

for $i = 1, \ldots, N$ and $t = 1, \ldots, T$. The joint distribution of the regressors $\tilde{\text{mw}}_{it}$, $\tilde{\text{pop}}_{it}$ and $\tilde{\text{emp}}_{it}^\text{TOT}$ is defined from the data used in our empirical application. In particular, $\tilde{\text{mw}}_{it}$, $\tilde{\text{pop}}_{it}$ and $\tilde{\text{emp}}_{it}^\text{TOT}$ are sampled from the pooled empirical distribution of the respective fixed-effect transformations of $\log(\text{mw}_{it})$, $\log(\text{pop}_{it})$ and $\log(\text{emp}_{it}^\text{TOT})$. The numerical values for the group-specific slope coefficients $\eta_g$, $\gamma_g$ and $\delta_g$ for groups $g = 1, \ldots, 4$ are given by the estimates from our empirical application (Table I). The error component $v_{it}$ is a standard normal noise term. In Table C.1 in the online appendix we report additional simulation results for a non-normal model where $v_{it}$ follows a normalized chi square distribution. Both specifications for the noise term yield qualitatively similar results.

All simulations are based on one thousand replications and a nominal level of $1 - \alpha = 0.95$.

Our first set of simulations studies how the finite sample behavior of our one-step procedures is affected by changes in the number of observed units $N$ and time periods $T$. To this end, we adapt a symmetric design with $N$ identical units that all belong to the same group...
\( g_i^0 = g \) and exhibit homoscedasticity, \( \sigma_i = \sigma \). We tie the variance of the error term to the number of observed time periods by setting \( \sigma = 0.01\sqrt{T} \). This adjustment counteracts the accumulation of information from observing more time periods. Without adjusting for \( T \), large \( T \) reveals the true group memberships with near certainty and the unit-wise confidence sets collapse to conservative singletons. The time adjustment ensures that the simulations are informative about the accuracy of our asymptotic approximation as \( T \) varies.

We simulate \( N = 50, 60 \) units and \( T = 15, 30, 60, 120 \) observed time periods and all four possible group assignments \( g = 1, \ldots, 4 \). In addition to confidence sets based on MVT and SNS critical values we also simulate a naïve singleton confidence set that contains only the estimated group membership. To compute the MVT critical values we set the regularization parameter to \( \epsilon = 0.01 \). We have verified that other choices yield qualitatively similar results. Our simulation results are reported in Table 3.

The naïve confidence set has a joint coverage probability of close to zero for \( g = 2, 3, 4 \). This indicates frequent misclassification and demonstrates the empirical relevance of our procedures for computing confidence sets. In our designs, group \( g = 1 \) is sufficiently separated from the other groups to make misclassification unlikely and the naïve confidence set performs well, covering the population partition with a probability of 89-98%.

At a nominal level of \( 1 - \alpha = 0.95 \), the confidence set based on SNS critical values achieves an empirical coverage that ranges from 97% to almost 100%. The procedure with MVT critical values computes a smaller (as measured by expected cardinality) and less conservative joint confidence set. Its empirical coverage ranges from 95% to almost 100%. For \( T = 15 \) our procedures produce valid but very conservative confidence sets. As \( T \) increases the quality of our confidence sets increases. For the well-separated group \( g = 1 \) the confidence sets converge toward the naïve confidence set. For groups \( g = 2, 3, 4 \), the confidence sets become less conservative and approach a coverage probability of 95-96% in all but one design.

With a second set of simulations we study the validity of our two-step procedure. The two-step procedure is sensitive to the distribution of heteroscedasticity and group membership, i.e., to the joint distribution of \((\sigma_i, T_i, g_i^0)\). We determine this distribution from the data used in our empirical application by mapping each simulated unit \( i \) to one of the \( N = 51 \) units from our empirical application. We set \( \sigma_i \) equal to \( m_\sigma \) times the standard deviation of the empirical residuals for unit \( i \), \( g_i^0 \) equal to the estimated group membership of \( i \) and \( T_i \) equal to the number of observed “time periods” for unit \( i \) (i.e. counties times quarters). The parameter \( m_\sigma = 1/4, 1, 4 \) shifts the global level of uncertainty. We simulate different values of the first-step parameter \( \beta = \alpha/5, \alpha/10 = 0.01, 0.005 \) and SNS as well as MVT critical values with \( \epsilon = 0.01 \). The simulation results are reported in Table 4.
| N  | T  | MVT coverage | MVT card | SNS coverage | SNS card | naïve coverage | naïve card |
|----|----|--------------|----------|--------------|---------|---------------|----------|
|    |    | g = 1       |          | g = 2        |          | g = 3         |          |
|    |    |              |          |              |          | g = 4         |          |
| 50 | 15 | 0.992        | 3.433    | 0.998        | 3.572   | 0.931         | 1        |
|    |    | 0.994        | 1.887    | 0.996        | 2.096   | 0.959         | 1        |
|    |    | 0.991        | 1.081    | 0.996        | 1.113   | 0.972         | 1        |
|    |    | 0.988        | 1.005    | 0.993        | 1.008   | 0.982         | 1        |
| 100| 15 | 0.995        | 3.628    | 0.999        | 3.727   | 0.867         | 1        |
|    |    | 0.990        | 2.221    | 0.995        | 2.467   | 0.938         | 1        |
|    |    | 0.990        | 1.146    | 0.995        | 1.206   | 0.947         | 1        |
|    |    | 0.976        | 1.010    | 0.987        | 1.015   | 0.961         | 1        |
| 50 | 15 | 0.986        | 3.686    | 0.994        | 3.822   | 0.020         | 1        |
|    |    | 0.982        | 2.952    | 0.989        | 3.219   | 0.034         | 1        |
|    |    | 0.975        | 2.391    | 0.979        | 2.527   | 0.059         | 1        |
|    |    | 0.982        | 2.082    | 0.973        | 2.183   | 0.090         | 1        |
| 100| 15 | 0.989        | 3.804    | 0.994        | 3.894   | 0.000         | 1        |
|    |    | 0.986        | 3.180    | 0.991        | 3.443   | 0.000         | 1        |
|    |    | 0.977        | 2.559    | 0.986        | 2.703   | 0.005         | 1        |
|    |    | 0.962        | 2.241    | 0.975        | 2.338   | 0.006         | 1        |
| 50 | 15 | 0.992        | 3.664    | 0.993        | 3.803   | 0.001         | 1        |
|    |    | 0.981        | 3.034    | 0.991        | 3.209   | 0.001         | 1        |
|    |    | 0.972        | 2.748    | 0.985        | 2.822   | 0.008         | 1        |
|    |    | 0.955        | 2.496    | 0.977        | 2.593   | 0.007         | 1        |
| 100| 15 | 0.994        | 3.777    | 0.998        | 3.873   | 0.000         | 1        |
|    |    | 0.985        | 3.169    | 0.994        | 3.361   | 0.000         | 1        |
|    |    | 0.976        | 2.846    | 0.987        | 2.907   | 0.000         | 1        |
|    |    | 0.960        | 2.639    | 0.975        | 2.724   | 0.000         | 1        |
| 50 | 15 | 0.988        | 3.581    | 0.987        | 3.741   | 0.010         | 1        |
|    |    | 0.966        | 2.803    | 0.988        | 2.976   | 0.012         | 1        |
|    |    | 0.964        | 2.362    | 0.976        | 2.446   | 0.019         | 1        |
|    |    | 0.962        | 2.054    | 0.979        | 2.145   | 0.026         | 1        |
| 100| 15 | 0.990        | 3.714    | 0.987        | 3.825   | 0.000         | 1        |
|    |    | 0.974        | 2.958    | 0.987        | 3.167   | 0.000         | 1        |
|    |    | 0.979        | 2.513    | 0.984        | 2.604   | 0.001         | 1        |
|    |    | 0.954        | 2.204    | 0.967        | 2.303   | 0.002         | 1        |

Table 3: Simulation results for one-step procedures based on 1000 replication and a nominal confidence level of \(1 - \alpha = 0.95\). “Coverage” gives the simulated joint coverage probability of the joint confidence set. “Card” gives the simulated average cardinality of the marginal unit-wise confidence set.
| $m_\sigma$ | $\alpha/\beta$ | success insignif signif | failure insignif signif | card with sel insignif signif | card without sel insignif signif | $\hat{N}$ | coverage |
|---------|------------|----------------|----------------|----------------|----------------|--------|---------|
| **MVT** |            |                |                |                 |                 |        |         |
| 0.25    | 10         | 0.60 | 0 | 0.00 | 0.00 | 1.42 | 1.00 | 2.00 | 1 | 10.00 | 1.00 |
| 5       | 0.51       | 0 | 0.00 | 0.00 | 1.52 | 1.00 | 2.00 | 1 | 9.29 | 1.00 |
| 1.00    | 10         | 0.13 | 0 | 0.00 | 0.00 | 2.18 | 1.00 | 2.24 | 1 | 33.66 | 1.00 |
| 5       | 0.00       | 0 | 0.01 | 0.02 | 2.25 | 1.00 | 2.24 | 1 | 32.35 | 1.00 |
| 4.00    | 10         | 0.00 | 0 | 0.49 | 0.42 | 2.75 | 1.02 | 2.72 | 1 | 50.95 | 0.96 |
| 5       | 0.00       | 0 | 0.81 | 0.75 | 2.78 | 1.05 | 2.72 | 1 | 50.92 | 0.97 |
| **SNS** |            |                |                |                 |                 |        |         |
| 0.25    | 10         | 0.53 | 0 | 0.00 | 0.00 | 1.51 | 1.00 | 2.00 | 1 | 10.04 | 1.00 |
| 5       | 0.46       | 0 | 0.00 | 0.00 | 1.58 | 1.00 | 2.00 | 1 | 9.32 | 1.00 |
| 1.00    | 10         | 0.13 | 0 | 0.00 | 0.00 | 2.21 | 1.00 | 2.27 | 1 | 33.73 | 1.00 |
| 5       | 0.00       | 0 | 0.01 | 0.04 | 2.27 | 1.00 | 2.27 | 1 | 32.42 | 1.00 |
| 4.00    | 10         | 0.00 | 0 | 0.52 | 0.44 | 2.78 | 1.02 | 2.75 | 1 | 50.95 | 0.98 |
| 5       | 0.00       | 0 | 0.81 | 0.73 | 2.82 | 1.05 | 2.75 | 1 | 50.92 | 0.98 |

Table 4: Simulation results for the two-step procedures (unit selection) with MVT and SNS critical values based on 1000 replications. Columns labeled “insignif” give averages over units that are insignificant if no unit-selection is performed. Columns labeled “signif” give averages over units that are significant if no unit-selection is performed. A unit is labeled as a “success” (“failure”) if its marginal confidence set is strictly smaller (strictly larger) under unit-selection than under no unit-selection. “Card with sel” (“card without sel”) gives the cardinality of unit-wise marginal confidence sets if unit-selection is turned on (turned off). $\hat{N}$ gives the simulated expected number of units that survive unit selection ($N = 51$). “Coverage” gives the simulated joint coverage probability of the two-step joint confidence set (nominal level $1 - \alpha = 0.95$).
In all designs, unit selection produces a valid joint confidence set that covers the true group structure at the prescribed nominal level.

The aim of unit selection is to tighten the marginal confidence sets for units for which estimated group memberships are insignificant under a one-step procedure. Among such units, the expected proportion of units for which a two-step procedures tightens the marginal confidence set varies across the different designs. In the design with low error variances ($m_\sigma = 0.25$) this proportion ranges between 46% and 60%. This means that the two-step procedure improves the marginal confidence sets for roughly half of the units for which they can be improved. In the design with medium error variances ($m_\sigma = 1$) this proportion lies between 0% and 13%. In the design with high error variances ($m_\sigma = 4$) there are no improvements. This illustrates the fact that the two-step procedures can only be successful if the overall uncertainty is low but unequally distributed across units. If overall uncertainty is high then the first-step cannot deselect units and hence the second-step confidence intervals cannot be tightened.

The two-step procedures can cause the confidence set to become wider if insufficiently many units are eliminated in the first step. This happens in the designs with high error variance ($m_\sigma = 4$): hardly any units are eliminated in the first step and the size of the marginal confidence sets increases both for units with significant and units with insignificant group membership estimates under the one-step procedure.

The two-step procedure with MVT critical values is more powerful than the two-step procedure with SNS critical values. In our designs, both choices of critical values select a similar number of units for the second step. Therefore, the power gain from using MVT critical values is almost entirely due to more efficient testing in step two.

8. Conclusion

We have constructed a confidence set for group membership for clustered panel models with time-invariant group-specific regression curves. Our confidence set can be easily tabulated and visualized as demonstrated in an empirical application. Applied researchers engaging in clustering analyses would benefit from our method to quantify the statistical uncertainty behind the results of clustering analyses.

The principle of our method can be extended into other models with latent group structure. Our method is based on the idea that the true group membership yields the best fit. This principle can be applied in a wide variety of situations although the meaning of the best fit might differ across situations. Such extensions are interesting but require different and non-trivial theoretical works.
For example, our approach can be extended to models with time-varying coefficients $\theta_g = \theta_{g,t}$. We studied this extension in a previous version of this paper (Dzemiński and Okui 2019). Panel models with group-specific but time-invariant regression curves are relevant in practice and dominate the literature on estimation of clustered panel models (see e.g. Su, Shi, and Phillips 2016; Wang, Phillips, and Su 2018; Vogt and Linton 2017). By focusing on time-invariant parameters, we simplify the presentation and are able to express the assumptions for asymptotic validity of the confidence set in terms of interpretable low-level conditions on the data generating process.

Extensions to nonlinear models may also be possible. In parametric analyses of nonlinear models, we may measure fits by log-likelihood functions (Liu et al. 2020; Wang and Su 2021). We may be able to construct confidence sets by checking if a given group membership assignment maximizes the log-likelihood function. These are interesting future research topics.

Appendix
A. Algorithm for regularization of correlation matrix

Compute the Cholesky decomposition with pivoting \( \Omega = PLLP' \) for a pivot matrix \( P \) and a lower-triangular matrix \( L \)
\[
\hat{\Omega} \leftarrow LL'
\]
OPEN \( \leftarrow \) empty list
VISITED \( \leftarrow \) empty list
\( \tilde{L} \leftarrow L \)
while there exists \( i \notin VISITED \) such there exists \( j \) such that \( \hat{\Omega}_{ij} < -1 + \epsilon \) do
\[
i \leftarrow \text{smallest } i \notin VISITED \text{ such that there exists } j > i \text{ with } \hat{\Omega}_{ij} < -1 + \epsilon
\]
\[
J \leftarrow \{ j : \Omega_{ij} < -1 + \epsilon \}
\]
add \((i, j, 1)\) for \( j \in J \) to OPEN
while OPEN \( \neq \emptyset \) do
\[
v \leftarrow \text{last entry in OPEN}
\]
remove \( v \) from OPEN
replace the \( v_2 \)th row of \( \tilde{L} \) by \((-1)^{v_3}\) times the \( v_2 \)th row of \( L \)
if \( v_2 \notin VISITED \) then
\[
\text{add } v_2 \text{ to VISITED}
\]
\[
J \leftarrow \{ j : \Omega_{v_2,j} < -1 + \epsilon \}
\]
add \((v_1, j, v_3 + 1)\) for \( j \in J \) to OPEN
end
end
return \( \rho(\Omega, \epsilon) = P\tilde{L}\tilde{L}'P' \)

Algorithm 1: Algorithm for matrix regularization.

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Supplemental Appendix

for

Confidence Set for Group Membership

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B. Additional results for application
| State           | $\hat{g}_i$ | p-val $\hat{g}_i$ | CS cardinality | CS   |
|-----------------|------------|-------------------|----------------|------|
| Alabama         | 1          | 0.0000            | 1              | 1    |
| Alaska          | 3          | 0.0000            | 1              | 3    |
| Arizona         | 3          | 0.0000            | 1              | 3    |
| Arkansas        | 2          | 0.0000            | 1              | 2    |
| California      | 3          | 0.0000            | 1              | 3    |
| Colorado        | 4          | 0.0908            | 2              | 3, 4 |
| Connecticut     | 3          | 0.0000            | 1              | 3    |
| Delaware        | 2          | 0.0000            | 1              | 4    |
| District of Columbia | 2   | 0.1591, 0.3745   | 3              | 2, 3, 4 |
| Florida         | 4          | 0.0000            | 1              | 4    |
| Georgia         | 1          | 0.0000            | 1              | 1    |
| Hawaii          | 2          | 0.4395, 0.5274    | 2              | 3    |
| Idaho           | 3          | 0.0641, 0.962     | 2              | 3, 4 |
| Illinois        | 3          | 0.0000            | 1              | 3    |
| Indiana         | 3          | 0.0000            | 1              | 3    |
| Iowa            | 4          | 0.0000            | 1              | 4    |
| Kansas          | 4          | 0.0000            | 1              | 4    |
| Kentucky        | 3          | 0.0000            | 1              | 3    |
| Louisiana       | 1          | 0.0000            | 1              | 1    |
| Maine           | 2          | 0.0000            | 1              | 2    |
| Maryland        | 2          | 0.0000            | 1              | 2    |
| Massachusetts   | 2          | 0.0227, 0.0341    | 1              | 2    |
| Michigan        | 2          | 0.0000            | 1              | 2    |
| Minnesota       | 2          | 0.0000            | 1              | 2    |
| Mississippi     | 1          | 0.0000            | 1              | 1    |
| Missouri        | 2          | 0.0000            | 1              | 2    |
| Montana         | 3          | 0.0018, 0.0027    | 1              | 3    |
| Nebraska        | 4          | 0.0582, 0.0874    | 2              | 3, 4 |
| Nevada          | 2          | 0.0000            | 1              | 2    |
| New Hampshire   | 3          | 0.0000            | 1              | 3    |
| New Jersey      | 2          | 0.0000            | 1              | 2    |
| New Mexico      | 3          | 0.0000            | 1              | 3    |
| New York        | 2          | 0.0000            | 1              | 2    |
| North Carolina  | 2          | 0.0000            | 1              | 2    |
| North Dakota    | 4          | 0.0000            | 1              | 4    |
| Ohio            | 1          | 0.0000            | 1              | 1    |
| Oklahoma        | 3          | 0.0000            | 1              | 3    |
| Oregon          | 4          | 0.0000            | 1              | 4    |
| Pennsylvania    | 3          | 0.0000            | 1              | 3    |
| Rhode Island    | 2          | 0.0000            | 1              | 2    |
| South Carolina  | 1          | 0.0000            | 1              | 1    |
| South Dakota    | 4          | 0.0000            | 1              | 4    |
| Tennessee       | 2          | 0.0000            | 1              | 2    |
| Texas           | 1          | 0.0000            | 1              | 1    |
| Utah            | 4          | 0.0000            | 1              | 4    |
| Vermont         | 4          | 0.0000            | 1              | 4    |
| Virginia        | 2          | 0.0000            | 1              | 2    |
| Washington      | 4          | 0.0000            | 1              | 4    |
| West Virginia   | 3          | 0.0000            | 1              | 3    |
| Wisconsin       | 3          | 0.0000            | 1              | 3    |
| Wyoming         | 3          | 0.0003, 0.0004    | 1              | 3    |

Table B.1: Estimated group memberships and 1-step confidence intervals. The confidence set is computed at level $1 - \alpha = 0.95$. 
C. Additional simulation results
| $g$  | $N$  | $T$  | MVT coverage | MVT card | SNS coverage | SNS card | naïve coverage | naïve card |
|------|------|------|--------------|---------|--------------|---------|----------------|-----------|
| 1    | 50   | 15   | 0.996        | 3.436   | 0.996        | 3.577   | 0.886          | 1         |
|      | 30   |      | 0.996        | 1.883   | 1.000        | 2.097   | 0.936          | 1         |
|      | 60   |      | 0.996        | 1.094   | 1.000        | 1.128   | 0.970          | 1         |
|      | 120  |      | 0.990        | 1.010   | 1.000        | 1.014   | 0.982          | 1         |
|      | 100  | 15   | 0.996        | 3.635   | 0.998        | 3.730   | 0.774          | 1         |
|      | 30   |      | 0.998        | 2.219   | 0.992        | 1.159   | 0.940          | 1         |
|      | 60   |      | 0.992        | 1.159   | 0.990        | 1.018   | 0.966          | 1         |
| 2    | 50   | 15   | 0.994        | 3.675   | 0.994        | 3.807   | 0.014          | 1         |
|      | 30   |      | 0.988        | 2.922   | 0.986        | 3.193   | 0.034          | 1         |
|      | 60   |      | 0.982        | 2.374   | 0.986        | 2.492   | 0.054          | 1         |
|      | 120  |      | 0.976        | 2.058   | 0.986        | 2.158   | 0.062          | 1         |
|      | 100  | 15   | 0.998        | 3.791   | 0.994        | 3.881   | 0.000          | 1         |
|      | 30   |      | 0.982        | 3.158   | 0.994        | 3.412   | 0.000          | 1         |
|      | 60   |      | 0.978        | 2.536   | 0.994        | 2.673   | 0.002          | 1         |
|      | 120  |      | 0.988        | 2.219   | 0.982        | 2.310   | 0.006          | 1         |
| 3    | 50   | 15   | 0.996        | 3.657   | 0.996        | 3.794   | 0.000          | 1         |
|      | 30   |      | 0.984        | 3.027   | 0.988        | 3.196   | 0.004          | 1         |
|      | 60   |      | 0.966        | 2.716   | 0.988        | 2.797   | 0.006          | 1         |
|      | 120  |      | 0.964        | 2.460   | 0.972        | 2.556   | 0.004          | 1         |
|      | 100  | 15   | 0.982        | 3.769   | 0.998        | 3.867   | 0.000          | 1         |
|      | 30   |      | 0.982        | 3.163   | 0.998        | 3.354   | 0.000          | 1         |
|      | 60   |      | 0.976        | 2.825   | 0.996        | 2.889   | 0.000          | 1         |
|      | 120  |      | 0.970        | 2.603   | 0.988        | 2.688   | 0.000          | 1         |
| 4    | 50   | 15   | 0.974        | 3.565   | 0.992        | 3.732   | 0.024          | 1         |
|      | 30   |      | 0.976        | 2.766   | 0.988        | 2.936   | 0.024          | 1         |
|      | 60   |      | 0.964        | 2.310   | 0.988        | 2.421   | 0.026          | 1         |
|      | 120  |      | 0.948        | 2.023   | 0.980        | 2.114   | 0.024          | 1         |
|      | 100  | 15   | 0.990        | 3.698   | 0.996        | 3.819   | 0.000          | 1         |
|      | 30   |      | 0.980        | 2.932   | 0.984        | 3.141   | 0.000          | 1         |
|      | 60   |      | 0.964        | 2.475   | 0.986        | 2.568   | 0.000          | 1         |
|      | 120  |      | 0.952        | 2.180   | 0.970        | 2.278   | 0.000          | 1         |

Table C.1: Simulation results for one-step procedures with $v_{it} = \frac{\chi_{it}^2 - 6}{\sqrt{2 \times 6}}$, where $\chi_{it}$ are iid chi squared random variables with six degrees of freedom. All other parameters are as described in the main text. Results based on 1000 replication and a nominal confidence level of $1 - \alpha = 0.95$. “Coverage” gives the simulated joint coverage probability of the joint confidence set. “Card” gives the simulated average cardinality of the marginal unit-wise confidence set.
D. Mathematical proofs

In the proofs, we drop the $g$ argument for ease of notation and write, e.g., $d_{it}(h)$ instead of $d_{it}(g_i^0, h)$. The $g$ argument is made explicit in the statements of the lemmas. Note that

$$d_{it}(h) = d_{it}(g_i^0, h) = -\sigma_i v_i x_i'(\theta_{g_i^0} - \theta_h).$$

Let

$$\tilde{D}_i(h) = \tilde{D}_i(g_i^0, h) = \frac{\sum_{t=1}^T d_{it}(g_i^0, h)}{\sqrt{\sum_{t=1}^T (d_{it}(g_i^0, h) - \bar{d}_i(g_i^0, h))^2}}$$

and

$$D_i(h) = D_i(g_i^0, h) = \frac{T^{-1/2} \sum_{t=1}^T d_{it}(g_i^0, h)}{\sqrt{E(d_{it}(g_i^0, h)^2)}}.$$

For a non-singular covariance matrix $V$, let $\Phi_{\text{max}, V}$ denote the cumulative distribution function of the maximum of a centered multivariate normal vector with covariance matrix $V$.

D.1. Proofs of main results

Proof of Theorem 1. The result follows directly from Lemma E.1.

Proof of Theorem 2. Note first that by Assumption 1.1, for any sequence $P_N$ such that $P_N \in \mathcal{P}_N$,

$$P_N \left( \hat{G} \neq G \right) = o(1).$$

Therefore, it suffices to prove the theorem on the event $\{\hat{G} = G\}$.

We now evaluate the effect of estimation error from estimating the group-specific coefficients. Let

$$\delta_N = C \frac{T^{\gamma} \theta_{i, g_i^0} \sigma_i}{\log N},$$

where we take $C'$ as a value larger than $C$ and $C'$ in (24) of Lemma E.11.

Define the event

$$\mathcal{E}_{N, 1} = \left\{ \max_{1 \leq i \leq N} \max_{h \in \mathcal{G} \setminus \{g_i^0\}} \left| \hat{D}_i(h) - \tilde{D}_i(h) \right| \leq \delta_N \right\}.$$
Applying Lemma E.11 with $c = 1/3$, whose condition is satisfied under the condition of the theorem, yields

$$P(E_{N,1}) \leq N^{-1} + CT^{-1/3} + a_{N,\theta}.$$ 

Next, we discuss the contribution of the estimation error to the coverage level. Define $\alpha_N$ implicitly by

$$c_{\alpha,N}^S = c_{\alpha,N} - \delta_N.$$ 

To see that $\alpha_N$ is well-defined, note that since $c_{\alpha,N}^S \to \infty$ and $\delta_N \to 0$ the right-hand side of the equation is diverging, and therefore positive for large $N$. Moreover, $c_{p,N}^{\text{SNS}} \downarrow 0$ as $p \uparrow N/2$. This establishes the existence of $\alpha_N$. Uniqueness follows from the strict monotonicity of the distribution function of the $t$-distribution. Let $t_{T-1}$ denote the distribution function of a $t$-distributed random variable with $T-1$ degrees of freedom, and let $f_{T-1}^t$ denote its density function. Let $c(\alpha) = t_{T-1}^{-1}(1 - \alpha/((G - 1)N))$ and $\delta_N^* = \sqrt{(T - 1)/T}\delta_N$. By the mean-value theorem

$$\frac{\alpha_N}{(G - 1)N} - \frac{\alpha}{(G - 1)N} = t_{T-1}(c(\alpha)) - t_{T-1}(c(\alpha_N)) = t_{T-1}(c(\alpha)) - t_{T-1}(c(\alpha) - \delta_N) = f_{T-1}^t(c^*)\delta_N^*,$$

where $c^*$ is a value between $c(\alpha_N)$ and $c(\alpha)$. Noting that $c(\alpha_N) < c(\alpha)$ and that $f_{T-1}^t$ is decreasing on the positive axis, rearranging this equality yields

$$|\alpha_N - \alpha| \leq f_{T-1}^t(c(\alpha_N))(G - 1)N\delta_N^* \leq 2c(\alpha_N)(1 - t_{T-1}(c(\alpha_N))(G - 1)N\delta_N^* \leq 4\delta_N\alpha\sqrt{\log((G - 1)N/\alpha)\alpha} \leq 4\delta_N\sqrt{\log((G - 1)N/\alpha) + 4\delta_N|\alpha_N - \alpha|}\sqrt{\log((G - 1)N/\alpha)} \leq 4\delta_N\sqrt{\log((G - 1)N/\alpha) + o(|\alpha_N - \alpha|)} ,$$

where the second inequality follows from Lemma E.4, the third inequality follows from Lemma E.5 (with $\epsilon = 1$), and the fourth inequality follows from $\delta_N\sqrt{\log N} \to 0$. This recursion implies

$$|\alpha_N - \alpha| \leq 5\delta_N\sqrt{\log((G - 1)N/\alpha)}.$$
for \( N \) large enough.

We now derive an approximation based on the theory of self-normalized sums. We apply Lemma \[ \text{E.3} \] Let \( g_T : x \to x/\sqrt{1 + x^2/T} \) as defined in Lemma \[ \text{E.3} \] and

\[
\tilde{D}_{i,T,3}(h) = \left( \frac{1}{T} \sum_{t=1}^{T} \mathbb{E}_P \left| d_{it}(h)/(\mathbb{E}_P(d_{it}(h)^2))^{1/2} \right|^3 \right)^{1/3}.
\]

We apply Lemma \[ \text{E.3} \] with \( \xi_t = d_{it}(h)/(\mathbb{E}_P(d_{it}(h)^2)), \nu = 1, \) and \( x = g_T(c_{\alpha,N,N}^{\text{SNS}}). \) The lemma requires

\[
g_T(c_{\alpha,N,N}^{\text{SNS}}) \leq T^{1/6}/\tilde{D}_{i,T,3}(h), \tag{7}
\]

for \( N \) large enough, for all \( i = 1, \ldots, N \) and \( h \in G \setminus \{g_i^0\}. \) We observe

\[
\tilde{D}_{i,T,3}(h) = \left( \frac{1}{T} \sum_{t=1}^{T} \mathbb{E}_P \left| \frac{\sigma_t v_{it} x_{it}'(\theta_{g_i^0} - \theta_h)}{\mathbb{E}_P((\sigma_t^2 v_{it}^2(x_{it}'(\theta_{g_i^0} - \theta_h))^{1/2})} \right|^3 \right)^{1/3}
\]

\[
= \left( \frac{1}{T} \sum_{t=1}^{T} \mathbb{E}_P \left| \frac{v_{it} x_{it}'(\theta_{g_i^0} - \theta_h)/\|\theta_{g_i^0} - \theta_h\|}{\mathbb{E}_P((x_{it}'(\theta_{g_i^0} - \theta_h)/\|\theta_{g_i^0} - \theta_h\|)^{1/2})} \right|^3 \right)^{1/3}
\]

\[
\leq \left( \frac{1}{T} \sum_{t=1}^{T} \mathbb{E}_P \left| v_{it} \right|^3 \left| x_{it} \right|^3 \right)^{1/3} C \lambda \leq CD_{N,3}.
\]

It thus holds

\[
T^{-1/6}g(c_{\alpha,N,N}^{\text{SNS}})\tilde{D}_{N,T,i} \leq CT^{-1/6}a_{\alpha,N}^{\text{SNS}}D_{N,3} \leq C \sqrt{T/(T-1))T^{-1/6}2\sqrt{\log ((G-1)N/\alpha)}D_{N,3},
\]

where the second inequality follows by Lemma \[ \text{E.3} \] with \( \epsilon = 1. \) Under our assumptions the right-hand side vanishes and condition (7) is verified. Applying Lemma \[ \text{E.3} \] yields

\[
\left| P \left( \tilde{D}_i(h) > c_{\alpha,N,N}^{\text{SNS}} \right) - \frac{\alpha_N}{(G-1)N} \right| \leq KT^{-1/2} \left( \tilde{D}_{i,T,3}^3 + 2^{9/2} \pi^{-3/2} \right) \left( 1 + g_T(c_{\alpha,N,N}^{\text{SNS}}) \right)^3 \frac{\alpha_N}{(G-1)N}
\]

\[
+ \left( KT^{-1/2} \left( \tilde{D}_{i,T,3}^3 + 2^{9/2} \pi^{-3/2} \right) \left( 1 + g_T(c_{\alpha,N,N}^{\text{SNS}}) \right)^3 \right)^2 \left( 1 - \Phi \left( g_T(c_{\alpha,N,N}^{\text{SNS}}) \right) \right).
\]

\[
\leq C \left( 2T^{-1/6}D_{N,3} \sqrt{\log \left( \frac{(G-1)N}{\alpha} \right)} \right)^3 \left( \frac{\alpha_N}{(G-1)N} + o \left( \frac{\alpha_N}{(G-1)N} \right) \right).
\]
where $K$ is the constant from Lemma [E.3] noting that $1 - t_{T-1}(c_{\alpha N,N}^{\text{SNS}}) = \alpha_N/((G-1)N)$ and $1 - \Phi(g_T(c_{\alpha N,N}^{\text{SNS}})) = \alpha_N/((G-1)N) + o(\alpha_N/((G-1)N))$ by the repeated application of Lemma [E.3].

Summing up, we have

$$P\left(\max_{1 \leq i \leq N} \max_{h \in G\{g_i^0\}} \hat{D}_i(h) > c_{\alpha N,N}^{\text{SNS}}\right) \leq P\left(\max_{1 \leq i \leq N} \max_{h \in G\{g_i^0\}} \hat{D}_i(h) > c_{\alpha N,N}^{\text{SNS}} - \delta_N\right) + P\left(E_{N,T,1}^c\right)$$

$$= P\left(\max_{1 \leq i \leq N} \max_{h \in G\{g_i^0\}} \hat{D}_i(h) > c_{\alpha N,N}^{\text{SNS}}\right) + P\left(E_{N,T,1}^c\right)$$

$$\leq \alpha_N + C \left(2T^{-1/6}D_{N,3}^3 \sqrt{\log \left(\frac{(G-1)N}{\alpha}\right)}\right) + 1 - P\left(E_{N,T,1}\right)$$

$$\leq \alpha + C \left(\delta_N \sqrt{\log N} + T^{-1/2}D_{N,3}^3 \log^{3/2} N + CT^{-1/3} + a_{N,\theta}\right).$$

**Proof of Theorem 3.** Write $c_{\alpha N}(\Omega)$ for the $1 - \alpha/N$-quantile of a $N(0, \Omega)$ random variable. Abbreviate $\Omega_i = \Omega_i(g_i^0)$ and $\hat{\Omega}_i = \hat{\Omega}_i(g_i^0)$. Note first that by Assumption [E.1] for any sequence $P_N$ such that $P_N \in \mathcal{P}_N$,

$$P_N \left(\hat{G} \neq G\right) = o(1).$$

Therefore, it suffices to proof the theorem on the event $\{\hat{G} = G\}$.

Next, we establish

$$c_{\alpha N,N} (\Omega_i) \leq c_{\alpha N}^i (\rho(\hat{\Omega}_i, \epsilon_N)),$$

where $\alpha_N = \alpha \left(1 + 2G_{\log \frac{1}{2}} / c_N \log(N/\alpha)\right)$ and $C_{\log}$ is the constant from Lemma [E.9]. The proof of [9] proceeds in two steps based on Lemma [E.12] and Lemma [E.9] respectively. Let $t_{T-1}(\bullet)$ denote the cumulative distribution function of a $t$-distributed random variable with $(T-1)$-degrees of freedom and let $X$ denote a $(G-1)$ random vector distributed according to centered multivariate $t$-distribution with $T-1$ degrees of freedom and scale matrix $\rho(\hat{\Omega}_i(g_i^0), \epsilon_N)$. The marginal distribution of the first component of $X$, denoted by $X_1$,
is $X_1 \sim t_{T-1}$. Let $d_N = t_{T-1}^{-1}(1 - \alpha/N)$ and note that $d_N \to \infty$. Moreover,

$$\frac{\alpha}{N} = P(X_1 > d_N) \leq P\left(\max_{h \in 1, \ldots, G-1} X_h > d_N\right).$$

Therefore, $c_{\alpha,N}^t(\rho(\hat{\Omega}_i(g^0_i), \epsilon_N)) \geq \sqrt{T/(T-1)}d_N$ and for $N_0$ and $T_0$ independent of $\rho(\hat{\Omega}_i(g^0_i), \epsilon_N)$ and $t^*$ the constant defined in Lemma E.12 we can take

$$c_{\alpha,N}^t(\rho(\hat{\Omega}_i(g^0_i), \epsilon_N)) > t^*.$$ 

for all $N \geq N_0$, $T \geq T_0$. Therefore, the assumptions of Lemma E.12 are satisfied for $t = c_{\alpha,N}^t(\rho(\hat{\Omega}_i(g^0_i), \epsilon_N))$ and $N, T$ large enough and Lemma E.12 implies

$$1 - \frac{\alpha}{N} = t_{\max,\rho(\hat{\Omega}_i(g^0_i), \epsilon_N),T-1} \left(\sqrt{(T-1)/T}c_{\alpha,N}^t(\rho(\hat{\Omega}_i(g^0_i), \epsilon_N))\right) \leq \Phi_{\max,\rho(\hat{\Omega}_i(g^0_i), \epsilon_N)}\left(c_{\alpha,N}^t(\rho(\hat{\Omega}_i(g^0_i), \epsilon_N))\right)$$

and therefore $c_{\alpha,N}(\rho(\hat{\Omega}_i(g^0_i), \epsilon_N)) \leq c_{\alpha,N}(\rho(\hat{\Omega}_i, \epsilon_N))$. Next, we show that

$$c_{\alpha,N,N}(\Omega_i) \leq c_{\alpha,N}(\rho(\hat{\Omega}_i, \epsilon_N)).$$

Taking $c = 1/5$ in Lemma E.11 we conclude that

$$\|\hat{\Omega}_i - \Omega_i\|_{\max} \leq C\left(\frac{r_{\theta,N}}{\epsilon N \wedge \min_{1 \leq i \leq N} \sigma_i} + T^{-2/5} B^4_{N,8} \log N\right) \equiv \delta_N$$

on a set of probability at most $N^{-1} + CT^{-1/5} + a_N$. On this set, Lemma E.7 states that we can take $c_{\alpha,N}(\rho(\hat{\Omega}_i, \epsilon)) > \sqrt{\log(N/\alpha)} > \sqrt{2}$ for $N$ large enough. Under our assumptions about the rate at which $\epsilon_N$ vanishes we have $\delta_N/\epsilon_N \to 0$. In particular, we can take $\delta_N < \epsilon_N$. The conditions 1 and 2 in Lemma E.9 are satisfied by Lemma E.13. Since $c_\omega$ is a constant $c_\omega > 2\delta + 4(p-1)\sqrt{2}c$ is satisfied for $N$ large enough. This verifies all conditions of Lemma E.9 and we can conclude

$$c_{\alpha,N,N}(\Omega_i) \leq c_{\alpha,N}(\rho(\hat{\Omega}_i, \epsilon_N)).$$

By Lemma E.7 we can bound $c_{\alpha,N}(\rho(\hat{\Omega}_i, \epsilon_N)) \leq 2\sqrt{\log(N/\alpha)}$ for $N$ large enough. Since $c_{\alpha,N}(\cdot)$ is decreasing in $\alpha$, $c_{\alpha,N}(\Omega_i) \leq c_{\alpha',N}(\Omega_i)$ and [9] holds.
Applying Lemma [E.11] with $c = 1/5$ yields

$$
P \left( \max_{1 \leq i \leq N} \max_{h \in G \setminus \{g_i^0\}} \left| \widehat{D}_i(h) - D_i(h) \right| > C \sqrt{\log N \delta_N} \right) \leq N^{-1} + C'T^{-1/5} + a_{N,T}.
$$

Hence

$$
P \left( \exists i \in 1, \ldots, N \text{ such that } \widehat{D}_i(g_i^0) > c_{\alpha,N}(\widehat{D}_i) \right)
\leq P \left( \exists i \in 1, \ldots, N \text{ such that } \widehat{D}_i(g_i^0) > c_{\alpha,N}(\Omega_i) \right)
\leq P \left( \max_{1 \leq i \leq N} \max_{h \in G \setminus \{g_i^0\}} D_i(h) - c_{\alpha,N,N}(\Omega_i) + C \sqrt{\log N \delta_N} > 0 \right) + N^{-1} + C'T^{-1/5} + a_{N,T}
\leq P \left( \max_{1 \leq i \leq N} \max_{h \in G \setminus \{g_i^0\}} D_i(h) - c_{\alpha,N,N}(\Omega_i) + C \sqrt{\log N \delta_N} > 0 \right) + o(1).
$$

Lemma [E.6] yields a collection of jointly normal vectors $\{X_i\}_{1 \leq i \leq N}$ with mean zero and covariance matrices $E_P[X_iX_i^\top] = \Omega_i$ such that

$$
P \left( \max_{1 \leq i \leq N} \left( \max_{h \in G \setminus \{g_i^0\}} D_i(h) - c_{\alpha,N,N}(\Omega_i) \right) + C \sqrt{\log N \delta_N} > 0 \right)
- P \left( \max_{1 \leq i \leq N} \left( \max_{1 \leq h \leq G-1} X_{i,h} - c_{\alpha,N,N}(\Omega_i) \right) + C \sqrt{\log N \delta_N} > 0 \right)
\leq C \left( B_{N,T}^4 T^{-1/2} \log^3(NT) \right)^{1/3}.
$$

Now, by Lemma [E.8]

$$
P \left( \max_{1 \leq i \leq N} \max_{h \in G \setminus \{g_i^0\}} D_i(h) - c_{\alpha,N,N}(\Omega_i) + C \sqrt{\log N \delta_N} > 0 \right)
\leq P \left( \max_{1 \leq i \leq N} \left( \max_{1 \leq h \leq G-1} X_{i,h} - c_{\alpha,N,N}(\Omega_i) \right) + C \sqrt{\log N \delta_N} > 0 \right)
+ C \left( B_{N,T}^4 T^{-1/2} \log^3(NT) \right)^{1/3}
\leq P \left( \max_{1 \leq i \leq N} \left( \max_{1 \leq h \leq G-1} X_{i,h} - c_{\alpha,N,N}(\Omega_i) \right) > 0 \right)
+ P \left( \max_{1 \leq i \leq N} \left( \max_{1 \leq h \leq G-1} X_{i,h} - c_{\alpha,N,N}(\Omega_i) \right) \leq C \sqrt{\log N \delta_N} \right)
+ C \left( B_{N,T}^4 T^{-1/2} \log^3(NT) \right)^{1/3}
\leq P \left( \max_{1 \leq i \leq N} \left( \max_{1 \leq h \leq G-1} X_{i,h} - c_{\alpha,N,N}(\Omega_i) \right) > 0 \right)
+ C \left( \sqrt{\log N \delta_N \vee N^{-1}} \right) \sqrt{2 \log \left( N \sqrt{G - 1} \right)} + \left( T^{-1/2} B_{N,T}^4 \log^3(NT) \right)^{1/3}
$$

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\begin{align*}
&\leq \sum_{1 \leq i \leq N} P\left( \max_{1 \leq h \leq G-1} X_{i,h} - c_{\alpha_N,N}(\Omega_i) > 0 \right) + o(1) \\
&\leq \alpha_N + o(1) \\
&\leq \alpha + 2C_{E.9} \sqrt{\epsilon_N/c_{\omega} \log(N/\alpha)} + o(1) \to 0.
\end{align*}

\[ \square \]

**Proof of Theorem 4.** It suffices to prove the theorem for MVT critical values since the SNS critical values are always larger than the MVT critical values. Write $c_{\alpha,N}(\Omega)$ for the $1-\alpha/N$-quantile of a $N(0, \Omega)$ random variable. Abbreviate $\Omega_i = \Omega_i(g_i^0)$ and $\hat{\Omega}_i = \hat{\Omega}_i(g_i^0)$. Note first that by Assumption 1.1, for any sequence $P_N$ such that $P_N \in \mathcal{P}_N$, $P_N(\hat{G} \neq G) = o(1)$. Therefore, it suffices to prove the theorem on the event $\{\hat{G} = G\}$.

Next, we establish

$$c_{\alpha,N}(\Omega_i) \leq c'_{\alpha,N}(\rho(\hat{\Omega}_i, \epsilon_N)),$$

where $\alpha_N = \alpha \left(1 + 2C_{E.9} \sqrt{\epsilon_N/c_{\omega} \log(N/\alpha)} \right)$ and $C_{E.9}$ is the constant from Lemma E.9. The proof of (9) proceeds in two steps based on Lemma E.12 and Lemma E.9 respectively. Let $t_{T-1}(\bullet)$ denote the cumulative distribution function of a $t$-distributed random variable with $(T-1)$-degrees of freedom and let $X$ denote a $(G-1)$ random vector distributed according to centered multivariate $t$-distribution with $T-1$ degrees of freedom and scale matrix $\rho(\hat{\Omega}_i(g_i^0), \epsilon_N)$. The marginal distribution of the first component of $X$, denoted by $X_1$, is $X_1 \sim t_{T-1}$. Let $d_N = t_{T-1}^{-1}(1-\alpha/N)$ and note that $d_N \to \infty$. Moreover,

$$\alpha/N = P(X_1 > d_N) \leq P\left( \max_{h \in 1,\ldots,G-1} X_h > d_N \right).$$

Therefore, $c'_{\alpha,N}(\rho(\hat{\Omega}_i(g_i^0), \epsilon_N)) \geq \sqrt{T/(T-1)d_N}$ and for $N_0$ and $T_0$ independent of $\rho(\hat{\Omega}_i(g_i^0), \epsilon_N)$ and $t^*$ the constant defined in Lemma E.12 we can take

$$c'_{\alpha,N}(\rho(\hat{\Omega}_i(g_i^0), \epsilon_N)) > t^*.$$

for all $N \geq N_0, T \geq T_0$. Therefore, the assumptions of Lemma E.12 are satisfied for
The conditions 1 and 2 in Lemma E.9 are satisfied by Lemma E.13. Hence we can bound the rate at which ε vanishes we have δN/εN → 0. In particular, we can take δN < εN. The conditions 1 and 2 in Lemma E.9 are satisfied by Lemma E.12 and we can conclude

\[ c_{α,N}(Ω_i) \leq c_{α,N}(Ω_i), \]

By Lemma E.7, we can bound \( c_{α,N}(ρ(Ω_i, ε_N)) \leq 2\sqrt{\log(N/α)} \) for N large enough. Since \( c_ω \) is a constant \( c_ω > 2δ + 4(p-1)\sqrt{2}ε \) is satisfied for N large enough. This verifies all conditions of Lemma E.9 and we can conclude

\[ c_{α',N}(Ω_i) \leq c_{α,N}(Ω_i), \]

Applying Lemma E.22 yields

\[ P \left( \max_{1 \leq i \leq N} \max_{h \in \mathcal{G} \setminus \{g_i^0\}} |\hat{D}_i(h) - D_i(h)| > C \log N \delta_N \right) = o(1). \]

Hence

\[ P \left( \exists i \in 1, \ldots, N \text{ such that } \hat{T}_i(g_i^0) > c_{α,N}(Ω_i) \right) \]

\[ \leq P \left( \exists i \in 1, \ldots, N \text{ such that } \hat{T}_i(g_i^0) > c_{α,N}(Ω_i) \right) \]

\[ \leq P \left( \max_{1 \leq i \leq N} \max_{h \in \mathcal{G} \setminus \{g_i^0\}} D_i(h) - c_{α,N}(Ω_i) + C \log N \delta_N > 0 \right) + o(1). \]

Lemma E.19 yields a collection of jointly normal vectors \( \{X_i\}_{1 \leq i \leq N} \) with mean zero and
covariance matrices $\mathbb{E}_P[X_i X'_i] = \Omega$, such that

$$
\left| P \left( \max_{1 \leq i \leq N} \left( \max_{h \notin \mathcal{G} \setminus \{g_0^i\}} D_i(h) - c_{\alpha_N,N}(\Omega_i) \right) + C \log N\delta_N > 0 \right) - P \left( \max_{1 \leq i \leq N} \left( \max_{1 \leq h \leq G - 1} X_{i,h} - \alpha_{\alpha_N,N}(\Omega_i) \right) + C \log N\delta_N > 0 \right) \right| \leq o(1).
$$

Now, by Lemma E.8

$$
P \left( \max_{1 \leq i \leq N} \max_{h \notin \mathcal{G} \setminus \{g_0^i\}} D_i(h) - c_{\alpha_N,N}(\Omega_i) + C \log N\delta_N > 0 \right)
\leq P \left( \max_{1 \leq i \leq N} \left( \max_{1 \leq h \leq G - 1} X_{i,h} - \alpha_{\alpha_N,N}(\Omega_i) \right) + C \log N\delta_N > 0 \right) + o(1)
\leq P \left( \max_{1 \leq i \leq N} \left( \max_{1 \leq h \leq G - 1} X_{i,h} - \alpha_{\alpha_N,N}(\Omega_i) \right) > 0 \right)
+ P \left( \left| \max_{1 \leq i \leq N} \left( \max_{1 \leq h \leq G - 1} X_{i,h} - \alpha_{\alpha_N,N}(\Omega_i) \right) \right| \leq C \log N\delta_N + o(1) \right)
\leq P \left( \max_{1 \leq i \leq N} \left( \max_{1 \leq h \leq G - 1} X_{i,h} - \alpha_{\alpha_N,N}(\Omega_i) \right) > 0 \right)
+ C (\log N\delta_N \lor N^{-1}) \sqrt{2 \log (N\sqrt{G-1})} + o(1)
\leq \sum_{1 \leq i \leq N} P \left( \max_{1 \leq h \leq G - 1} X_{i,h} - \alpha_{\alpha_N,N}(\Omega_i) > 0 \right) + o(1)
\leq \alpha_N + o(1)
\leq \alpha + 2C^{E.8} \left( \epsilon_N/c_{\omega} \log(N/\alpha) + o(1) \right) \rightarrow \alpha.
$$

\[\square\]

**Proof of Theorem E.9.** We note that the hypothesis selection part of the procedure does not affect the theoretical analysis. This is because, here, we focus on size and thus need to consider only the behavior of the test statistics under $\{g_1^0\}_{i=1}^N$.

Let $J = \{(i,h) \mid i \in \{1, \ldots, N\}, h \in \mathcal{G} \setminus \{g_0^i\}\}$ and

$$
J_1 = \left\{ (i,h) \mid i \in \{1, \ldots, N\}, h \in \mathcal{G} \setminus \{g_0^i\}, \sqrt{T} \mathbb{E}_P(d_{i,T}^U(h)) \text{ s.t. } \frac{\sqrt{T} \mathbb{E}_P(d_{i,T}^U(h))}{s_{i,T}^U(h)} > -c_{\beta,N}^{SNS} \right\},
$$

where $(s_{i,T}^U(h))^2 = \sum_{t=1}^T Var(d_{i,T}^U(h))/T$. Roughly speaking, $J_1$ is the set of pairs of units and groups that are difficult to distinguish from true group membership.

In this proof, we set $c = 1/6$. 

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Step 1: We first prove that \( P\left( \max_{(i,h) \in J^c} \tilde{d}_i^U(h) \leq 0 \right) > 1 - \beta - N^{-1} - CT^{-c} - a_{N,\theta} \).

Note that \( \tilde{d}_i^U(h) > 0 \) for some \((i,h) \in J^c\) implies that
\[
\max_{(i,h) \in J} \frac{\sqrt{T}(\tilde{d}_i^U(h)) - \mathbb{E}_P(\tilde{d}_i^U(h))}{s_i^U(h)} > c_{\beta,N}^{\text{SNS}}.
\]

Let
\[
c_{SN}(\beta) = \frac{\Phi^{-1}(1 - \beta/((G - 1)N))}{\sqrt{1 - \Phi^{-1}(1 - \beta/((G - 1)N))^2/T}}.
\]

Let
\[
\epsilon_{N,1}^U = C \frac{r_{N,\theta}}{t_N + \min_{1 \leq i \leq N} \sigma_i}
\]

We have
\[
P\left( \max_{(i,h) \in J} \frac{\sqrt{T}(\tilde{d}_i^U(h)) - \mathbb{E}_P(\tilde{d}_i^U(h))}{s_i^U(h)} > c_{\beta,N}^{\text{SNS}} \right)
\leq P\left( \max_{(i,h) \in J} \frac{\sqrt{T}(\tilde{d}_i^U(h)) - \mathbb{E}_P(\tilde{d}_i^U(h))}{s_i^U(h)} > c_{\beta,N}^{\text{SNS}} - \epsilon_{N,1}^U \right)
+ P\left( \max_{(i,h) \in J} \left| \frac{\sqrt{T}(\tilde{d}_i^U(h)) - \mathbb{E}_P(\tilde{d}_i^U(h))}{s_i^U(h)} \right| > \epsilon_{N,1}^U \right).
\]

The second term on the right-hand side is bounded by \( N^{-1} + CT^{-c} + a_{N,\theta} \) by (55) in Lemma E.23. Let \( \beta_N \) solve \( c_{\beta_N,N}^{\text{SNS}} = c_{\beta,N}^{\text{SNS}} - \epsilon_{N,T,1}^U \). As in the proof of Theorem 2, we have
\[
|\beta_N - \beta| \leq 4\epsilon_{N,1}^U \sqrt{\log((G - 1)N/\beta)}.
\]

Thus we have
\[
P\left( \max_{(i,h) \in J} \frac{\sqrt{T}(\tilde{d}_i^U(h)) - \mathbb{E}_P(\tilde{d}_i^U(h))}{s_i^U(h)} > c_{\beta,N}^{\text{SNS}} \right)
\leq P\left( \max_{(i,h) \in J} \frac{\sqrt{T}(\tilde{d}_i^U(h)) - \mathbb{E}_P(\tilde{d}_i^U(h))}{s_i^U(h)} > c_{\beta_N,N}^{\text{SNS}} \right) + CT^{-c}
= P\left( \max_{(i,h) \in J} \frac{\sqrt{T}(\tilde{d}_i^U(h)) - \mathbb{E}_P(\tilde{d}_i^U(h))}{s_i^U(h)} > c_{SN}(c_{SN}^{-1}(c_{\beta_N,N}^{\text{SNS}})) \right) + CT^{-c}.
\]

Following essentially the same argument as that in Step 1 of the proof of Theorem 4.2 of
Chernozhukov, Chetverikov, and Kato (2019) shows that, under conditions (4) and (5),
\[
P\left( \max_{(i,h) \in J} \frac{\sqrt{T}(d_i^U(h) - \mathbb{E}_P(d_i^U(h)))}{s_i^U(h)} > c_{SN}(c_{SN}(c_{\beta_{N,N}})) \right) \leq c_{SN}(c_{\beta_{N,N}}) + CT^{-c}.
\]
Note that here we replace \( \hat{\sigma}_j \) and \( \sigma_j \) in the proof of Chernozhukov, Chetverikov, and Kato (2019) with \((T - 1)\sum_{t=1}^T (d_{it}^U(h) - \mathbb{E}_P(d_{it}^U(h)))^{1/2} \) and \( s_i^U(h) \). We have
\[
c_{SN}(c_{\beta_{N,N}}) = (G - 1)N \left( 1 - \Phi \left( \frac{c_{\beta_{N,N}}^{SNS}}{\sqrt{1 + (c_{\beta_{N,N}}^{SNS})^2/T}} \right) \right) = \beta + O \left( \frac{(c_{\beta_{N,N}}^{SNS})^3}{\sqrt{T}} \right).
\]
We thus have
\[
P\left( \max_{(i,h) \in J} \frac{\sqrt{T}(d_i^U(h) - \mathbb{E}_P(d_i^U(h)))}{s_i^U(h)} > \sqrt{T} \left( \frac{T}{T-1} \right)^{1-1} \left( 1 - \frac{\beta_N}{(G - 1)N} \right) \right) \leq \beta + O \left( \frac{(c_{\beta_{N,N}}^{SNS})^3}{\sqrt{T}} \right) + CT^{-c} \leq \beta + CT^{-c},
\]
where \((c_{\beta_{N,N}}^{SNS})^3/\sqrt{T} \leq CT^{-c}\) by that \((\log(N))^3/T \leq CT^{-c}\) which is implied by (4) together with \(D_{N,\beta} \geq 1\) and Lemma E.5 and \(\epsilon_{N,1} \sqrt{\log((G - 1)N/\beta)} \leq CT^{-c}\) by condition (6).

An implication of Step 1 is as follows. Let
\[
\mathbb{N} = \left\{ i \in \{1, \ldots, N\} \mid \max_{h \in G \setminus \{g_i^0\}} \frac{\sqrt{T}\mathbb{E}_P(d_i^U(h))}{s_i^U(h)} > -c_{\beta_{N,N}}^{SNS} \right\}.
\]
Then
\[
P\left( \max_{i \in \mathbb{N}} \max_{h \in G \setminus \{g_i^0\}} \tilde{d}_i^U(h) \leq 0 \right) > 1 - \beta - N^{-1} - CT^{-c} - a_{N,\beta}.
\]

Step 2: Next, we prove that \(P(\bigcap_{i=1}^N \hat{M}_i(g_i^0) \supseteq J_1) \geq 1 - \beta - CT^{-c}\). Here, we drop the \(g\) argument for simplicity of notation when arguments are \(g_i^0\) and \(h\).

We note that
\[
P\left( \bigcap_{i=1}^N \hat{M}_i(g_i^0) \nsubseteq J_1 \right) = P\left( \exists (i, h); \tilde{D}_i^U(h) \leq -2c_{\beta_{N,N}}^{SNS} \text{ and } \frac{\sqrt{T}\mathbb{E}_P(d_i^U(h))}{s_i^U(h)} > -c_{\beta_{N,N}}^{SNS} \right).
\]
\[
\leq P \left( \exists (i, h); \bar{d}^U_i(h) \leq -2c_{\beta,N}^{\text{SNS}} + \epsilon_{N, 2}^U + \epsilon_{N, T, 2}^U \right. \\
+ \left. P \left( \max_{1 \leq i \leq N} \max_{h \in G \setminus \{g_h^U\}} \left| \bar{d}^U_i(h) - \bar{d}^U_i(h) \right| > \epsilon_{N, 2}^U \right) \right),
\]

where

\[
\epsilon_{N, 2}^U = C r_{N, \theta} \left( \sqrt{T} + \sqrt{\log N} \right) / \min_{1 \leq i \leq N} \sigma_i.
\]

By (56) in Lemma E.23 noting that its condition is satisfied by (1), (5), and (10),

\[
P \left( \max_{1 \leq i \leq N} \max_{h \in G \setminus \{g_h^U\}} \left| \bar{d}^U_i(h) - \bar{d}^U_i(h) \right| > \epsilon_{N, 2}^U \right) < N^{-1} + CT^{-c} + a_{N, \theta}.
\]

We observe

\[
P \left( \exists (i, h); \bar{d}^U_i(h) \leq -2c_{\beta,N}^{\text{SNS}} + \epsilon_{N, T, 2}^U \right. \\
+ \left. P \left( \max_{1 \leq i \leq N} \max_{h \in G \setminus \{g_h^U\}} \left| \bar{d}^U_i(h) - \bar{d}^U_i(h) \right| > \epsilon_{N, T, 2}^U \right) \right) \\
\leq P \left( \max_{1 \leq i \leq N} \max_{h \in G \setminus \{g_h^U\}} \left[ \sqrt{T}(E(d^U_i(h)) - d^U_i(h)) - (2S^U_i(h) - s^U_i(h))c_{\beta,N}^{\text{SNS}} + 2S^U_{i, T}(h)\epsilon_{N, T, 2}^U \right] > 0 \right).
\]

Let

\[
(\bar{S}^U_{i, T}(h))^2 = \frac{1}{T} \sum_{t=1}^{T} (d^U_{i, t}(h) - \mathbb{E}_P(d^U_{i, t}(h)))^2 - \left( \frac{1}{T} \sum_{t=1}^{T} (d^U_{i, t}(h) - \mathbb{E}_P(d^U_{i, t}(h))) \right)^2.
\]

We observe that

\[
(S^U_{i, T}(h))^2 = \frac{1}{T} \sum_{t=1}^{T} (d^U_{i, t}(h) - \mathbb{E}_P(d^U_{i, t}(h)))^2 \\
+ \frac{2}{T} \sum_{t=1}^{T} (d^U_{i, t}(h) - \mathbb{E}_P(d^U_{i, t}(h))) (\mathbb{E}_P(d^U_{i, t}(h)) - \mathbb{E}_P(d^U_{i, t}(h))) \\
- \left( \frac{1}{T} \sum_{t=1}^{T} (d^U_{i, t}(h) - \mathbb{E}_P(d^U_{i, t}(h))) \right)^2 + \frac{1}{T} \sum_{t=1}^{T} (\mathbb{E}_P(d^U_{i, t}(h)) - \mathbb{E}_P(d^U_{i, t}(h)))^2 \\
\geq (\bar{S}^U_{i, T}(h))^2 + \frac{2}{T} \sum_{t=1}^{T} (d^U_{i, t}(h) - \mathbb{E}_P(d^U_{i, t}(h))) (\mathbb{E}_P(d^U_{i, t}(h)) - \mathbb{E}_P(d^U_{i, t}(h)))
\]
If \(1 - s_i^U(h)/\bar{S}_{i,T}^U(h) \geq -r/2\) and
\[
\frac{2}{T} \sum_{t=1}^{T} (d_{it}^U(h) - \mathbb{E}_P(d_{it}^U(h)))(\mathbb{E}_P(d_{it}^U(h)) - \mathbb{E}_P(\bar{d}_{it}^U(h))) \geq -(\bar{S}_{i,T}^U(h))^2 \left( \frac{r}{2} - \frac{r^2}{16} \right),
\]
for some \(0 < r < 1\), we have
\[
2S_{i,T}^U(h) - s_i^U(h) \geq (1 - r)\bar{S}_{i,T}^U(h)
\]
because
\[
2S_{i,T}^U(h) - s_i^U(h) \geq 2\bar{S}_{i,T}^U(h) \left( 1 - \frac{r}{2} - \frac{r^2}{16} \right)^{1/2} - s_i^U(h)
\]
\[
= \bar{\sigma}_{i,h} \left( 2 \left( 1 - \frac{r}{4} - \frac{s_i^U(h)}{S_{i,T}^U(h)} \right) \right) \geq (1 - r)\bar{S}_{i,T}^U(h).
\]

We thus have
\[
P \left( \max_{1 \leq i \leq N} \max_{h \in \mathcal{G}(g_i)} \left[ \sqrt{T}(E(d_{i}^U(h)) - \bar{d}_{i}^U(h)) - (2S_{i,T}^U(h) - s_i^U(h)) \epsilon_{\beta,N}^\text{SNS} + 2S_{i,T}^U(h) \epsilon_{N,T,2}^U \right] > 0 \right)
\]
\[
\leq P \left( \max_{1 \leq i \leq N} \max_{h \in \mathcal{G}(g_i)} \left[ \frac{\sqrt{T}(E(d_{i}^U(h)) - \bar{d}_{i}^U(h))}{\bar{S}_{i,T}^U(h)} > (1 - r)\epsilon_{\beta,N}^\text{SNS} - 2 \max_{1 \leq i \leq N} \max_{h \in \mathcal{G}(g_i)} \frac{S_{i,T}^U(h)}{\bar{S}_{i,T}^U(h)} \epsilon_{N,T,2}^U \right] \right) > (1 - r)\epsilon_{\beta,N}^\text{SNS} - 2 \max_{1 \leq i \leq N} \max_{h \in \mathcal{G}(g_i)} \frac{S_{i,T}^U(h)}{\bar{S}_{i,T}^U(h)} \epsilon_{N,T,2}^U
\]
\[
+ P \left( \max_{1 \leq i \leq N} \max_{h \in \mathcal{G}(g_i)} \left| \frac{2}{T} \sum_{t=1}^{T} \bar{a}_{it}(h) \right| > \frac{r}{2} - \frac{r^2}{16} \right) \label{eq:11}
\]
\[
+ P \left( \max_{1 \leq i \leq N} \max_{h \in \mathcal{G}(g_i)} \left| \frac{s_i^U(h)}{\bar{S}_{i,T}^U(h)} - 1 \right| > \frac{r}{2} \right) \label{eq:12},
\]
where
\[
\bar{a}_{it}(h) = 2(d_{it}^U(h) - \mathbb{E}_P(d_{it}^U(h)))(\mathbb{E}_P(d_{it}^U(h)) - \mathbb{E}_P(\bar{d}_{it}^U(h)))/(\bar{S}_{i,T}^U(h))^2.
\]

We now take \(r = T^{-(1-c)/2}B_{N,8}^4 \log((G - 1)N)\). The first term of (11) is
\[
P \left( \max_{1 \leq i \leq N} \max_{h \in \mathcal{G}(g_i)} \left[ \frac{\sqrt{T}(E(d_{i}^U(h)) - \bar{d}_{i}^U(h))}{\bar{S}_{i,T}^U(h)} > (1 - r)\epsilon_{\beta,N}^\text{SNS} - 2 \max_{1 \leq i \leq N} \max_{h \in \mathcal{G}(g_i)} \frac{S_{i,T}^U(h)}{\bar{S}_{i,T}^U(h)} \epsilon_{N,T,2}^U \right] \right)
\]
\[
\leq P \left( \max_{1 \leq i \leq N} \max_{h \in \mathcal{G} \setminus \{g(i)\}} \sqrt{T} \frac{(E(\overline{d}_i^U(h)) - \overline{d}_i^U(h))}{S_{i,T}^U(h)} > (1 - r)c_{\beta,N}^{s,N} - C\epsilon_{r,N,2}^U \right) \\
+ P \left( \max_{1 \leq i \leq N} \max_{h \in \mathcal{G} \setminus \{g(i)\}} \frac{S_{i,T}^U(h)}{S_i^U(h)} > \frac{1}{2} C \right).
\]

Note that we can take \( C > 2 \) and

\[
P \left( \max_{1 \leq i \leq N} \max_{h \in \mathcal{G} \setminus \{g(i)\}} \frac{S_{i,T}^U(h)}{S_i^U(h)} > \frac{1}{2} C \right) < CT^{-c}
\]

holds because

\[
\frac{S_{i,T}^U(h)}{S_i^U(h)} = \frac{S_{i,T}^U(h)}{s_i^U(h)} \frac{s_i^U(h)}{S_i^U(h)},
\]

following the same argument of Lemma D.5 of Chernozhukov, Chetverikov, and Kato (2019). Following the argument in the proof of Step 2 of Theorem 4.2 of Chernozhukov, Chetverikov, and Kato (2019) under (i), (5) and that (6) implies \( \epsilon_{N,2}^U \sqrt{\log((G - 1)/\beta)} \leq CT^{-c} \), it holds that

\[
P \left( \max_{1 \leq i \leq N} \max_{h \in \mathcal{G} \setminus \{g(i)\}} \sqrt{T} \frac{(E(\overline{d}_i^U(h)) - \overline{d}_i^U(h))}{S_{i,T}^U(h)} > (1 - r)c_{\beta,N}^{s,N} - C\epsilon_{r,N,2}^U \right) \leq \beta + CT^{-c}.
\]

For the second term (11), let \( a_{it}(h) = 2(d_{it}^U(h) - \overline{E}_P(d_{it}^U(h)))(\overline{E}_P(d_{it}^U(h)) - \overline{E}_P(\overline{d}_{i,t}^U(h)))/(s_i^U(h))^2 \). The second term is

\[
P \left( \max_{1 \leq i \leq N} \max_{h \in \mathcal{G} \setminus \{g(i)\}} \left| \frac{1}{T} \sum_{t=1}^T \overline{a}_{it}(h) \right| > \frac{r}{2} - \frac{r^2}{16} \right) \\
\leq P \left( \max_{1 \leq i \leq N} \max_{h \in \mathcal{G} \setminus \{g(i)\}} \left| \frac{1}{T} \sum_{t=1}^T a_{it}(h) \right| > \left( 1 - \frac{r}{2} \right) \left( \frac{r}{2} - \frac{r^2}{16} \right) \right) \\
+ P \left( \max_{1 \leq i \leq N} \max_{h \in \mathcal{G} \setminus \{g(i)\}} \left| \frac{(\overline{S}_{i,T}^U(h))^2}{(s_i^U(h))^2} - 1 \right| > \frac{r}{2} \right),
\]

where the inequality holds because \((\overline{S}_{i,T}^U(h))^2 \geq (1 - r/2)(s_i^U(h))^2\) if \(1 - (\overline{S}_{i,T}^U(h))^2/(s_i^U(h))^2 > r/2\). The second term is bounded by \( CT^{-c} \) by Lemma A.5 of Chernozhukov, Chetverikov, and Kato (2019) (Note that the statement of Lemma A.5 of Chernozhukov, Chetverikov, and Kato (2019) is about \( \hat{\sigma}_j / \sigma_j \) (in their notation) but their proof is based on \( \hat{\sigma}_j^2 / \sigma_j^2 \)). For
the first term, observe that
\[
\sum_{t=1}^{T} \mathbb{E}_P \left( (a_{it}(h)/T)^2 \right) \leq \frac{1}{T^2} \sum_{t=1}^{T} \frac{\text{var}(d_{it}^U(h))}{(s_{i}^U(h))^2} \left( \mathbb{E}_P (d_{it}^U(h)) - \mathbb{E}_P (\bar{d}_{i}^U(h)) \right)^2
\]
and
\[
\sum_{t=1}^{T} E \left( \max_{1 \leq i \leq N} \max_{h \in G \setminus \{g(i)\}} (a_{it}(h)/T)^2 \right) \leq \frac{1}{T^2} \sum_{t=1}^{T} \frac{\mathbb{E}_P (d_{it}^U(h)) - \mathbb{E}_P (\bar{d}_{i}^U(h))}{(s_{i}^U(h))^2} \leq \frac{1}{T} G B_{N,8}^4 C_4.
\]

By Lemma D.3 of Chernozhukov, Chetverikov, and Kato (2019), we have
\[
E \left( \max_{1 \leq i \leq N} \max_{h \in G \setminus \{g(i)\}} \left| \frac{1}{T} \sum_{t=1}^{T} a_{it}(h) \right| \right) \leq CD_{N,2} \left( \frac{\sqrt{\log((G-1)N)}}{\sqrt{T}} + B_{N,8}^4 \log((G-1)N)/T \right).
\]

By Lemma D.2 of Chernozhukov, Chetverikov, and Kato (2019), we thus have
\[
P \left( \max_{1 \leq i \leq N} \max_{h \in G \setminus \{g(i)\}} \left| \frac{1}{T} \sum_{t=1}^{T} a_{it}(h) \right| \geq C \left( \frac{\sqrt{\log((G-1)N)}}{\sqrt{T}} + B_{N,8} \log((G-1)N)/T \right) + t \right)
\leq e^{-t^2/(3(D_{N,2}^2/T)) + K \frac{1}{T^2} B_{N,8}^4},
\]
for any \( t > 0 \). Taking \( t = T^{-(1-c)/2} B_{N,8}^4 \) and arranging the terms, we have
\[
P \left( \max_{1 \leq i \leq N} \max_{h \in G \setminus \{g(i)\}} \left| \frac{1}{T} \sum_{t=1}^{T} a_{it}(h) \right| \geq C B_{N,8}^4 T^{-(1-c)/2} \log((G-1)N) \right) \leq CT^{-c}.
\]
We thus have
\[
P \left( \max_{1 \leq i \leq N} \max_{h \in G \setminus \{g(i)\}} \left| \frac{1}{T} \sum_{t=1}^{T} \tilde{a}_{it}(h) \right| > \frac{r}{2} - \frac{r^2}{16} \right) \leq CT^{-c},
\]
by (3).

The third term (12) can also be analyzed by following the argument in the proof of Step 2
of Theorem 4.2 of Chernozhukov, Chetverikov, and Kato (2019) and is bounded by $\beta + CT^{-c}$ under (4) and (5).

Summing up, we have

$$P \left( \bigcap_{i=1}^{N} \hat{M}_i(g_i^0) \not\subseteq J_1 \right) \leq \beta + CT^{-c} + N^{-1} + a_{N,\theta}. $$

An implication of Step 2 is as follows. Let

$$\hat{N} = \left\{ i \in \{1, \ldots, N\} \mid M_i(g_i^0) \neq \emptyset \right\}. $$

Then

$$P \left( \hat{N} \supseteq N \right) \geq 1 - \beta - CT^{-c} - N^{-1} - a_{N,\theta}. $$

**Step 3:** First, consider the case in which $J_1 = \emptyset$. In this case, the argument in Step 1 yields that

$$P(\hat{g}_i = g_i^0, \forall i) = P \left( \max_{1 \leq i \leq N} \max_{h \in G} \hat{D}_i^U(h) \leq 0 \right) > 1 - \beta - N^{-1} - CT^{-c} - a_{N,\theta}. $$

Because $\{\hat{g}(i)\}_{i=1}^{N}$ is always included in the confidence set, the probability of the confidence set not including $\{g_i^0\}_{i=1}^{N}$ is less than $\beta + N^{-1} + CT^{-c} + a_{N,\theta} < \alpha + N^{-1} + CT^{-c} + a_{N,\theta}$.

Next, consider the case in which $|J_1| \geq 1$. Here, we consider the case with type $= SNS$. The proof for the case with type $= MAX$ is essentially the same, and therefore is omitted. Observe that

$$P \left( \{g_i^0\}_{i=1}^{N} \not\subseteq \hat{C}_{Sel,\alpha,\beta}^{SNS} \right)$$

$$= P \left( \bigcup_{i=1}^{N} \left\{ \hat{T}_i^{MAX}(g_i^0) > c_{\alpha-2\beta,\bar{N}}^{SNS} \right\} \cap \left\{ \max_{h \in G \setminus \{g_i^0\}} \hat{D}_i^U(h) > 0 \right\} \right)$$

$$\leq P \left( \bigcup_{i \in \mathbb{N}} \left\{ \hat{T}_i^{MAX}(g_i^0) > c_{\alpha-2\beta,\bar{N}}^{SNS} \right\} \cup \bigcup_{i \in \mathbb{N}} \left\{ \max_{h \in G \setminus \{g_i^0\}} \hat{D}_i^U(h) > 0 \right\} \right)$$

$$\leq P \left( \bigcup_{i \in \mathbb{N}} \left\{ \hat{T}_i^{MAX}(g_i^0) > c_{\alpha-2\beta,\bar{N}}^{SNS} \right\} \right) + P \left( \bigcup_{i \in \mathbb{N}} \left\{ \max_{h \in G \setminus \{g_i^0\}} \hat{D}_i^U(h) > 0 \right\} \right).$$
By Step 1, we have
\[
P\left( \bigcup_{i \in \mathbb{N}} \left\{ \max_{h \in G \setminus \{g_i^0\}} \hat{D}_i^U(h) > 0 \right\} \right) \leq \beta + CT^{-c} + N^{-1} + a_{N,\theta}.
\]

By Step 2, we have
\[
P\left( \bigcup_{i \in \mathbb{N}} \left\{ \hat{T}^{\text{MAX}}_i(g_i^0) > c^{\text{SNS}}_{\alpha - 2\beta, \hat{N}} \right\} \right)
\leq P\left( \{ \hat{N} \supset \mathbb{N} \} \cap \bigcup_{i \in \mathbb{N}} \left\{ \hat{T}^{\text{MAX}}_i(g_i^0) > c^{\text{SNS}}_{\alpha - 2\beta, \hat{N}} \right\} \right) + P(\{ \hat{N} \not\supset \mathbb{N} \})
\]
\[
\leq P\left( \bigcup_{i \in \mathbb{N}} \left\{ \hat{T}^{\text{MAX}}_i(g_i^0) > c^{\text{SNS}}_{\alpha - 2\beta, |\mathbb{N}|} \right\} \right) + \beta + CT^{-c} + N^{-1} + a_{N,\theta}.
\]

Thus we have
\[
P\left( \{ g_i^0 \}_{i=1}^N \notin \hat{C}^{\text{SNS}}_{\text{Sel}, \alpha, \beta} \right) \leq P\left( \bigcup_{i \in \mathbb{N}} \left\{ \hat{T}^{\text{MAX}}_i(g_i^0) > c^{\text{SNS}}_{\alpha - 2\beta, |\mathbb{N}|} \right\} \right) + 2\beta + CT^{-c} + N^{-1}.
\]

Theorem 2 implies
\[
P\left( \{ g_i^0 \}_{i=1}^N \notin \hat{C}^{\text{SNS}}_{\text{Sel}, \alpha, \beta} \right) \leq \alpha + C(\epsilon_{N,3} + T^{-c} + N^{-1} + a_{N,\theta}),
\]

where
\[
\epsilon_{N,3} = \frac{r_{N,\theta}}{l_{N} \wedge \min_{1 \leq i \leq N} \sigma_i} \log N + T^{-1/2} D_{N,3}^3 \log^{3/2} N
\]

\[\square\]

E. Technical lemmas

Lemma E.1. Let \((\phi_i)_{i=1}^n\) denote a collection of independent, non-randomized tests and suppose that
\[
\alpha_i = n \mathbb{P}(\phi_i > 0)
\]
with $\alpha_{\text{max}} := \max_{i=1,\ldots,n} \alpha_i < 1$. Then

$$\alpha_{\text{min}} - \frac{\alpha_{\text{min}}^2}{2} \leq \Pr \left( \max_{i=1,\ldots,n} \phi_i > 0 \right) \leq \alpha_{\text{max}} - \frac{\alpha_{\text{max}}^2}{2} \left( 1 - \frac{\alpha_{\text{max}}}{3} + \frac{1}{n} \left( 1 - \frac{\alpha_{\text{max}}}{n} \right)^{-2} \right),$$

where $\alpha_{\text{min}} := \min_{i=1,\ldots,n} \alpha_i$.

Proof. For fixed $0 < x < 1$, let $\bar{x}$ denote a generic intermediate value between zero and $x$. By a Taylor expansion around $x = 0$,

$$\exp(-x) = 1 - x + \frac{1}{2} x^2 - \frac{1}{6} \exp(-\bar{x}) x^3 \geq 1 - x + x^2 \left( \frac{1}{2} - \frac{x}{6} \right). \quad (13)$$

Moreover,

$$\log (1 - x) = 0 - x - \frac{x^2}{2(1 - \bar{x})^2} \geq -x - \frac{x^2}{2(1 - x)^2}. \quad (14)$$

Now, for $0 < \alpha < 1$,

$$\left( 1 - \frac{\alpha}{n} \right)^n = \exp \left( n \log \left( 1 - \frac{\alpha}{n} \right) \right) \geq \exp(-\alpha) \exp \left( -\frac{\alpha^2}{2n} \left( 1 - \frac{\alpha}{n} \right)^{-2} \right) \geq \left( 1 - \alpha + \alpha^2 \left( \frac{1}{2} - \frac{\alpha}{6} \right) \right) \left( 1 - \frac{\alpha^2}{2n} \left( 1 - \frac{\alpha}{n} \right)^{-2} \right) \geq 1 - \alpha + \frac{\alpha^2}{2} \left( 1 - \frac{\alpha}{3} \right) - \frac{\alpha^2}{2n} \left( 1 - \frac{\alpha}{n} \right)^{-2},$$

where the first inequality uses (14), the second inequality uses (13) and the last inequality uses

$$1 - \alpha + \alpha^2 \left( \frac{1}{2} - \frac{\alpha}{6} \right) \leq 1.$$

We conclude that

$$\Pr \left( \max_{i=1,\ldots,n} \phi_i > 0 \right) = 1 - \Pr \left( \max_{i=1,\ldots,n} \phi_i = 0 \right) \leq 1 - \left( 1 - \frac{\alpha_{\text{max}}}{n} \right)^n \leq \alpha_{\text{max}} - \frac{\alpha_{\text{max}}^2}{2} \left( 1 - \frac{\alpha_{\text{max}}}{3} \right) + \frac{\alpha_{\text{max}}^2}{2n} \left( 1 - \frac{\alpha_{\text{max}}}{n} \right)^{-2}.$$
Next, note that
\[
(1 - \frac{\alpha}{n})^n \leq \exp(-\alpha) \leq 1 - \alpha + \frac{\alpha^2}{2}
\]
and therefore
\[
\mathbb{P}\left(\max_{i=1,\ldots,n} \phi_i > 0\right) = 1 - \mathbb{P}\left(\max_{i=1,\ldots,n} \phi_i = 0\right) \geq 1 - \left(1 - \frac{\alpha_{\min}}{n}\right)^n \geq \alpha_{\min} - \frac{\alpha_{\min}^2}{2}.
\]

Lemma E.2 (Jing, Shao, and Wang [2003]). Let \(\xi_1, \ldots, \xi_T\) be independent centered random variables with \(E(\xi_i^2) = 1\) and \(E(|\xi_i|^{2+\nu}) < \infty\) for all \(1 \leq t \leq T\) where \(0 < \nu \leq 1\). Let \(S_T = \sum_{t=1}^T \xi_t\), \(V_T^2 = \sum_{t=1}^T \xi_t^2\) and \(D_{T,\nu} = (T^{-1} \sum_{t=1}^T E(|\xi_t|^{2+\nu}))^{1/(2+\nu)}\). Then uniformly in \(0 \leq x \leq T^{\nu/(2(2+\nu))}/D_{T,\nu}\),
\[
\left| \frac{P(S_T/V_T \geq x)}{1 - \Phi(x)} - 1 \right| \leq KT^{-\nu/2}D_{T,\nu}^{2+\nu}(1 + x)^{2+\nu}.
\]

Proof. This lemma is first proved by Jing, Shao, and Wang [2003]. Here we use the version by Chernozhukov, Chetverikov, and Kato [2019, Lemma D.1], which is based on de la Peña, Lai, and Shao [2009, Theorem 7.4].

Lemma E.3. We the same notation as that in Lemma E.2. In addition, let \(g_T(x) : x \mapsto x/\sqrt{1 + x^2/T}\), and \(\tilde{S}_T = \sum_{t=1}^T \xi_t/\sqrt{\sum_{t=1}^T (\xi_t - \sum_{s=1}^T \xi_s/T)^2}\). Then uniformly in \(0 \leq g_T(x) \leq T^{\nu/(2(2+\nu))}/D_{T,\nu}\),
\[
\left| P(\tilde{S}_T \geq x) - (1 - t_{T-1}(x)) \right|
\leq KT^{-\nu/2}(D_{T,\nu}^{2+\nu} + (8/\pi)(2+\nu)/2)(1 + g_T(x))^{2+\nu}(1 - t_{T-1}(x))
+ (KT^{-\nu/2}(D_{T,\nu}^{2+\nu} + (8/\pi)(2+\nu)/2)(1 + g_T(x))^{2+\nu})^2(1 - \Phi(g_T(x))).
\]

Proof. Noting that \(g_T(\tilde{S}_T) = S_T/V_T\), we have \(P(\tilde{S}_T > x) = P(S_T/V_T > g_T(x))\). Lemma E.2 implies that
\[
|P(S_T/V_T > g_T(x)) - (1 - \Phi(g_T(x)))| \leq KT^{-\nu/2}D_{T,\nu}^{2+\nu}(1 + g_T(x))^{2+\nu}(1 - \Phi(g_T(x))).\quad (15)
\]
The above inequality also holds for a special case of \(\xi_t \sim N(0, 1)\). In this case, \(P(S_T/V_T > g_T(x)) = P(\tilde{S}_T > x) = 1 - t_{T-1}(x)\). This implies that
\[
|1 - t_{T-1}(x) - (1 - \Phi(g_T(x)))| \leq KT^{-\nu/2}(8/\pi)(2+\nu)/2(1 + g_T(x))^{2+\nu}(1 - \Phi(g_T(x)))\quad (16)
\]
because $D_{T,\nu} = \sqrt{8/\pi}$ when $\xi_t \sim N(0,1)$. Combining (15) and (16) gives

$$\left| P(\tilde{S}_T \geq x) - (1 - t_{T-1}(x)) \right| \leq KT^{-\nu/2}(D_{T,\nu}^2 + (8/\pi)^{(2+\nu)/2})(1 + g_T(x))^{2+n}(1 - \Phi(g_T(x))).$$

Applying (16) again to the right hand side the above inequality gives the desired result. \hfill \Box

**Lemma E.4.** For $\nu \geq 1$, let $t_\nu$ and $f_\nu^t$ denote the distribution and density function of a $t$-distributed random variable with $\nu$ degrees of freedom. For $x^2 > 2$

$$f_\nu^t(x) < 2x(1 - t_\nu(x)).$$

**Proof.** Applying Theorem 1 in Soms (1976) with $n = 2$ yields the inequality

$$1 - t_\nu(x) \geq (1 + x^2/\nu) \left(1 - \frac{\nu}{(\nu + 2)x^2}\right) f_\nu(x)/x.$$

Now, $x^2 > 2$ implies

$$1 - t_\nu(x) > \left(1 - \frac{1}{2}\right) f_\nu(x)/x.$$

\hfill \Box

**Lemma E.5.** Let $\nu(N) \geq 1$ denote a sequence that converges to infinity, and let $c_N(\alpha)$ denote the $(1 - \alpha/N)$-quantile of the $t$-distribution with $\nu(N)$ degrees of freedom. Suppose that $(\log N)/\nu(N) \to 0$. For each $\epsilon > 0$ and $0 < \alpha < 1$, there is a threshold $N_0$ such that for $N \geq N_0$

$$\sup_{\alpha \leq \alpha < 1} c_N(\alpha) \leq \sqrt{2(1+\epsilon)\log(N/\alpha)}.$$

**Proof.** For notational convenience, write $\nu = \nu(N)$. We prove the bound for $\alpha = \alpha$ and write $c_N = c_N(\alpha)$. Then, the uniformity follows from the monotonicity of the distribution function. Clearly, $c_N \to \infty$ so that we can take $c_N \geq 1$, provided that $N$ is large enough. The density function of the $t$-distribution with $\nu$ degrees of freedom is given by $f_\nu^t(x) = c(\nu) (1 + x^2/\nu)^{-\frac{\nu + 1}{2}}$, where

$$c(\nu) = \frac{\Gamma\left(\frac{\nu + 1}{2}\right)}{\sqrt{\nu\pi\Gamma\left(\frac{\nu}{2}\right)}} \to \frac{1}{\sqrt{2\pi}}$$

as $\nu \to \infty$. It follows that there is a universal constant $C$ such that $c(\nu) \leq C$. We first show that $c_N^2/\nu = O(1)$. The proof is by contradiction. Suppose that $\limsup_{N \to \infty} c_N^2/\nu = \infty$. 


Applying Theorem 1 in Soms (1976) with \( n = 1 \) yields

\[
1 - t_\nu(c_N) \leq f_\nu^t(c_N) \frac{1}{c_N} \left( 1 + \frac{c_N^2}{\nu} \right). \tag{17}
\]

This implies that

\[
\frac{\alpha}{N} \leq c(\nu) \left( 1 + \frac{c_N^2}{\nu} \right)^{-\frac{\nu+1}{2}} \left( 1 + \frac{c_N^2}{\nu} \right) \leq C \left( 1 + \frac{c_N^2}{\nu} \right)^{-\frac{\nu+1}{2}}.
\]

Taking logs and rearranging gives

\[
\frac{\log(N/\alpha)}{\nu} \geq \frac{1}{2} \frac{\nu - 1}{\nu} \left( \log \left( 1 + \frac{c_N^2}{\nu} \right) - C \right).
\]

The left-hand side of the inequality vanishes under the assumptions of the lemma, whereas a subsequence of the right-hand side diverges to infinity. This establishes that the inequality is impossible and therefore \( c_N^2/\nu = O(1) \). This implies that there exists a constant \( b \) such that

\[
1 < b \leq \left( 1 + \frac{c_N^2}{\nu} \right)^{\frac{\nu}{c_N}} \leq e,
\]

so that we can take

\[
\left( 1 + \frac{c_N^2}{\nu} \right)^{\frac{\nu}{c_N}} \leq e^{-\nu/(1+\epsilon^*)}
\]

for a positive \( \epsilon^* \). Then,

\[
f_\nu^t(c_N) \leq C \left[ \left( 1 + \frac{c_N^2}{\nu} \right)^{\frac{\nu}{c_N}} \right]^{-\frac{c_N^2}{\nu} \left[ \frac{\nu+1}{\nu} \right]} \leq C \exp \left( -\frac{c_N^2}{2} (1 + \epsilon^*/2)^{-1} \right).
\]

Take \( N \) large enough that

\[
\frac{1}{1 + \epsilon^*/2} - \frac{4 \log c_N}{c_N^2} > \frac{1}{1 + \epsilon^*},
\]

Then, the right-hand side of (17) can be bounded by

\[
C \exp \left( -\frac{c_N^2}{2} (1 + \epsilon^*/2)^{-1} \right) \left( 1 + \frac{c_N^2}{\nu} \right) \leq 2C \exp \left( -\frac{c_N^2}{2} \left( (1 + \epsilon^*/2)^{-1} - \frac{4 \log c_N}{c_N^2} \right) \right).
\]
\[ \leq 2C \exp \left( -\frac{c_N^2}{2} (1 + \epsilon^*)^{-1} \right). \]

Plugging in \( 1 - t_\nu(c_N) = \alpha/N \) and taking logs gives

\[ c_N^2 \leq (1 + \epsilon^*) \log (N/\alpha) + \log(2C) \]
\[ \leq 2(1 + \epsilon^*) \log (N/\alpha) \left( 1 + \frac{1}{2(1 + \epsilon^*)} \log(2C) \right). \]

Hence, there is a constant \( C \) such that \( c_N^2 \leq C \log(N/\alpha). \) Using this inequality, we can now verify that \( c_N^2/\nu \to 0 \) so that

\[ \left( 1 + \frac{c_N^2}{\nu} \right)^{\frac{\nu}{c_N^2}} \to e, \]

allowing us to take \( \epsilon^* = \epsilon/2 \) for sufficiently large \( N \). Taking \( N \) large enough that

\[ (1 + \epsilon/2) \left( 1 + \frac{1}{2(1 + \epsilon/2)} \log(2C) \right) \leq 1 + \epsilon \]

yields \( c_N^2 \leq 2(1 + \epsilon) \log(N/\alpha). \)

\[ \square \]

**Lemma E.6 (Large CLT for MAX statistic).** Let \( \mathbb{P}_N \) denote a family of probability measures satisfying Assumptions 4 and 5 and let \( P \in \mathbb{P}_N \). For \( i = 1, \ldots, N \), there are centered normal random vectors \( X_i \) that take values if \( \mathbb{R}^{G-1} \) and satisfy \( \mathbb{E}_P[X_i X'_i] = \Omega_i(g_i^0) \) and

\[ \sup_{(r_1, \ldots, r_N) \in \mathbb{R}_{N+}^N} \left| P \left( \max_{1 \leq i \leq N} \left( \max_{h \in G \setminus g_i^0} D_i(g_i^0, h) - r_i \right) > 0 \right) - P \left( \max_{1 \leq i \leq N} \left( \max_{1 \leq h \leq G-1} X_{i,h} - r_i \right) > 0 \right) \right| \leq CB_{N,8}^{1/3} T^{-1/6} \log((G - 1)NT), \]

where \( C \) is a universal constant.

Let \( \delta_i(h) = \theta_{g_i^0} - \theta_h \) and

\[ Z_{it}(h) = \frac{d_{it}(h)}{\sqrt{\mathbb{E}_P[d_{it}^2(h)]}} = -\frac{x_{it}'(\delta_i(h)/\|\delta_i(h)\|)}{2s_i(h)/(\sigma_i\|\delta_i(h)\|)} v_{it}, \]

where

\[ s_i(h) = \sqrt{\frac{1}{T} \sum_{t=1}^T d_{it}^2(h)} = \frac{1}{2} \sqrt{\delta_i(h)' \left( \frac{1}{T} \sum_{t=1}^T \mathbb{E}_P[x_{it}'x_{it}'u_{it}^2] \right) \delta_i(h)}. \]
By Assumption \([13]\) there exists \(C_Z > 0\) such that \(2s_i(h)/(\sigma_i^2(h))^3 \geq C_Z^{-1}\). Define the vector
\[
Z_t = ((Z_{1t}(h))_{h \in G \setminus \{g_i^0\}}, \ldots, (Z_{Nt}(h))_{h \in G \setminus \{g_i^0\}})'
\]
and note that for \(s \neq t\), \(Z_s\) and \(Z_t\) are independent by Assumption \([12]\). Let
\[
D_N = (1 \vee C_Z^3) \left( B_{N,8}^2 \vee C_8 \right).
\]

We now consider transforming a component of \(Z_t\) and then averaging over \(1 \leq t \leq T\). The components of \(Z_t\) are indexed by \(1 \leq i \leq N\) and \(h \in G \setminus \{g_i^0\}\) and we consider a generic component of associated with a fixed \(i\) and \(h\). We have
\[
\frac{1}{T} \sum_{t=1}^{T} \mathbb{E}_P \left[ |Z_{it}(h)|^3 \right] \leq C_Z^3 \mathbb{E}_P \left[ ||x_{it}||^3 |v_{it}|^3 \right] \leq C_Z^3 (1 \vee C_8) \leq D_N,
\]
\[
\frac{1}{T} \sum_{t=1}^{T} \mathbb{E}_P \left[ |Z_{it}(h)|^4 \right] \leq C_Z^4 \mathbb{E}_P \left[ ||x_{it}||^4 |v_{it}|^4 \right] \leq (1 \vee C_Z^2)^2 (1 \vee C_8)^2 \leq D_N^2.
\]

This verifies that the sequence \(D_N\) satisfies the assumptions (M.2) for the sequence \(B_n\) in Chernozhukov, Chetverikov, and Kato (2017). Checking their assumption (E.2) for \(q = 4\) and \(B_n = D_N\) requires
\[
\mathbb{E}_P \left( \max_{1 \leq i \leq N} \left( |Z_{ij}(h)|/D_N \right)^4 \right) \leq 2.
\]

This is satisfied since
\[
\mathbb{E}_P \left( \max_{1 \leq i \leq N} |Z_{it}(h)|^4 \right) \leq C_Z^4 \mathbb{E}_P \left( \max_{1 \leq i \leq N} ||x_{it}||^4 |v_{it}|^4 \right) \leq C_Z^4 B_{N,8}^4 \leq D_N^4.
\]

Let \(\tilde{X}_t = (\tilde{X}_{1t}, \ldots, \tilde{X}_{Nt})'\) with \(\text{dim}(\tilde{X}_{it}) = G - 1\) for \(i = 1, \ldots, N, \ t = 1, \ldots, T\) denote a normal random vector with the property that \(\tilde{X}_t\) and \(\tilde{X}_s\) are independent for \(t \neq s\) and \(\mathbb{E}_P[\tilde{X}_t(\tilde{X}_s)'] = \mathbb{E}_P[Z_t(\tilde{Z}_t)']\) for \(i = 1, \ldots, N, \ t = 1, \ldots, T\). Define \(X_i = \sum_{t=1}^{T} \tilde{X}_{it} \sqrt{T}\). Clearly, \(X_i\) is a normal random vector with covariance matrix \(\Omega_i\). Let \(a_i = -\infty\) and \(b_i = r_i\). Then we may write
\[
\sup_{(r_1, \ldots, r_N) \in \mathbb{R}^N_{++}} \left| P \left( \max_{1 \leq i \leq N} \left( \max_{h \in G \setminus \{g_i^0\}} D_i(h) - r_i \right) > 0 \right) - P \left( \max_{1 \leq i \leq N} \left( \max_{1 \leq h \leq G-1} X_{i,h} - r_i \right) > 0 \right) \right|
\]
\[
\begin{align*}
& \leq \sup_{(r_1, \ldots, r_N) \in \mathbb{R}^N_{++}} \left| P \left( \bigcap_{i=1}^{N} \bigcap_{h \in G \setminus \{g_i^0\}} \{ a_i < D_i(h) \leq b_i \} \right) - P \left( \bigcap_{i=1}^{N} \bigcap_{h=1}^{G-1} \{ a_i < X_{i,h} \leq b_i \} \right) \right| \\
& \leq \sup_{(r_1, \ldots, r_N) \in \mathbb{R}^N_{++}} \left| P \left( \bigcap_{i=1}^{N} \bigcap_{h \in G \setminus \{g_i^0\}} \{ a_i < \frac{1}{\sqrt{T}} \sum_{t=1}^{T} Z_d(h) \leq b_i \} \right) \\
& \quad - P \left( \bigcap_{i=1}^{N} \bigcap_{h=1}^{G-1} \{ a_i < \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \tilde{X}_{it,h} \leq b_i \} \right) \right| \\
& \leq K \left\{ \left( \frac{D^2_N \log^7((G-1)NT)}{T} \right)^{1/6} + \left( \frac{D^2_N \log^3((G-1)NT)}{\sqrt{T}} \right)^{1/3} \right\},
\end{align*}
\]
for a universal constant \( K \). Here, the last inequality holds by Proposition 2.1 in Chernozhukov, Chetverikov, and Kato (2017).

**Lemma E.7** (Bounds on large quantiles). Let \( X \) denote a standard normal vector with \( p \times p \) correlation matrix \( \Omega \) and let \( 0 \leq d < 2 \). Let \( c_{\alpha,N} \) denote the \( 1 - \alpha/N \) quantile of \( X \). Then there is a constant \( N_0 \) that depends only on \( \alpha \) and \( d \) such that for \( \alpha \leq \alpha \leq 1 \) and \( N \geq N_0 \)

\[ \sqrt{d \log(N/\alpha)} \leq c_{\alpha,N} \leq \sqrt{2 \log p} + \sqrt{2 \log(N/\alpha)}. \]

**Proof.** The upper bound is given in Lemma D.4 in Chernozhukov, Chetverikov, and Kato (2019). To prove the lower bound put \( a_N = \sqrt{d \log(N/\alpha)} \). Let \( \Phi \) denote the cumulative distribution function of a standard normal random variable, and let \( \phi \) denote its probability density function. Gordon’s lower bound (see, e.g., Duembgen (2010)) states that

\[ 1 - \Phi(x) > \frac{\phi(x)}{x(1 - 1/x^2)} \]

for \( x > 0 \) and thus \( 1 - \Phi(x) > \frac{1}{2} \phi(x)/x \) for \( x > \sqrt{2} \). Therefore

\[ P \left( \max_{j=1, \ldots, p} X_j > a_N \right) \geq P(X_1 > a_N) \]

\[ = 1 - \Phi(a_N) \]

\[ > \frac{\phi(a_N)}{2a_N} = \frac{\exp \left( -\frac{a_N^2}{2} \right)}{a_N \sqrt{8\pi}} = \frac{(\alpha/N)^{d/2}}{a_N \sqrt{8\pi}} = \alpha/N \left( \frac{(N/\alpha)^{1-d/2}}{a_N \sqrt{8\pi}} \right) \]

\[ \geq \alpha/N \left( \frac{N^{1-d/2}}{\sqrt{8d\pi \log(N/\alpha)}} \right) \geq \alpha/N, \]

where the last inequality holds for \( N \geq N_0 \) and \( N_0 \) is chosen such that \( N \geq N_0 \) implies that
The inequality $P(\max_{j=1,\ldots,p} X_j > a_N) > \alpha/N$ implies $c_{\alpha,N} \geq a_N$. \hfill \square

**Lemma E.8** (Simultaneous anti-concentration). Let $\{X_i\}_{1 \leq i \leq N}$ denote a collection of random vectors such that $X_i \sim N(0, \Omega_i)$ for $p \times p$ correlation matrix $\Omega_i$. Let $\alpha$ such that $0 < \underline{\alpha} < \alpha < \bar{\alpha}$. There is $N_0$ depending only on $G$, $\underline{\alpha}$ and $\bar{\alpha}$ such that for all $\epsilon > 0$ and $N \geq N_0$

$$P\left(\max_{1 \leq i \leq N} \left(\max_{1 \leq h \leq G-1} X_{i,h} - c_{\alpha,N}(\Omega_i)\right) \leq \epsilon\right) \leq C(\epsilon \lor N^{-1})\sqrt{2\log(N\sqrt{G-1})}.$$  

Proof. There is $N_0$ depending only on $G$ and $\bar{\alpha}$ such that for all $N \geq N_0$

$$\sqrt{2\log(G-1)} + \sqrt{2\log(N/\underline{\alpha})} = \sqrt{2\log(N/\underline{\alpha})} \left(1 + \sqrt{\frac{\log(G-1)}{\log(N/\underline{\alpha})}}\right) \leq 2\sqrt{\log(N/\underline{\alpha})}.$$  

In conjunction with Lemma E.7 this implies that we can choose $N_0$ depending only on $G$, $\underline{\alpha}$ and $\bar{\alpha}$ such that

$$\sqrt{\log N/\underline{\alpha}} \leq c_{\alpha,N}(\Omega_i) \leq 2\sqrt{\log(N/\underline{\alpha})}$$  

for $1 \leq i \leq N$. This implies that for $N \geq N_0$ we can write $c_{\alpha,N}(\Omega_i) = a_i\sqrt{\log(N/\underline{\alpha})}$ for $1 \leq a_i \leq 2$. Therefore

$$P\left(\max_{1 \leq i \leq N} \left(\max_{1 \leq h \leq G-1} X_{i,h} - c_{\alpha,N}(\Omega_i)\right) \leq \epsilon\right) \leq P\left(\max_{1 \leq i \leq N} a_i \left(\max_{1 \leq h \leq G-1} X_{i,h} - \frac{X_{i,h}}{a_i} - \sqrt{\log(N/\underline{\alpha})}\right) \leq \epsilon\right) \leq P\left(\max_{1 \leq i \leq N} a_i \left(\max_{1 \leq h \leq G-1} \frac{X_{i,h}}{a_i} - \sqrt{\log(N/\underline{\alpha})}\right) \leq \epsilon \lor N^{-1}\right) \leq \sup_{x \in \mathbb{R}} P\left(\max_{1 \leq i \leq N} a_i \left(\max_{1 \leq h \leq G-1} \frac{X_{i,h}}{a_i} - x\right) \leq \epsilon\right) \leq C'(\epsilon \lor N^{-1})\sqrt{1 + 2\log(N\sqrt{G-1})},$$

where $C'$ is a universal constant and the last inequality follows from Corollary 1 in Chernozhukov, Chetverikov, and Kato (2015). \hfill \square

**Lemma E.9** (Comparison bound for critical values with regularization). Let $\Omega$ and $\hat{\Omega}$ denote $p \times p$ correlation matrices and let $\epsilon$, $\delta$ and $c_\omega$ denote positive constants. Suppose that $c_\omega < 1/4$ and

$$\Omega_{ij} \in \{-1, 1\} \cup [-1 + c_\omega, 1 - c_\omega],$$
and

\[ \| \hat{\Omega} - \Omega \|_{\text{max}} \leq \delta \]

and that \( \delta < \epsilon \) and

\[ c_\omega > 2\delta + 4(p-1)\sqrt{2\epsilon} . \]

Let \( \rho(\bullet, \epsilon) : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}^p \times \mathbb{R}^p \) denote a function that maps correlation matrices into correlation matrices and satisfies the following conditions:

1. For \( i \neq j \) such that \( \Omega_{i,j} \leq -1 + \epsilon \), \( \rho(\Omega, \epsilon)_{i,j} = -1 \).
2. \( \| \Omega - \rho(\Omega, \epsilon) \|_{\text{max}} \leq 2(p-1)\sqrt{2\epsilon} \).

Let \( X \sim N(0, \Omega) \) and \( \hat{X}^\epsilon \sim N(0, \rho(\hat{\Omega}, \epsilon)) \). Then, there is a constant \( C \) depending only on \( p \) such that for all \( a > \sqrt{2} \)

\[
\frac{P(\max_{j=1, \ldots, p} X_j > a)}{P(\max_{j=1, \ldots, p} \hat{X}^\epsilon_j > a)} - 1 \leq Ca\sqrt{\epsilon/c_\omega}. 
\]

In particular, suppose that \( \hat{c}_{\alpha, N} \) is the \( 1 - \alpha/N \) quantile of \( \max_{j=1, \ldots, p} \hat{X}^\epsilon_j \) and let \( c_{\alpha N, N} \) denote the \( 1 - \alpha_N/N \) quantile of \( \max_{j=1, \ldots, p} X_j \), where

\[ \alpha_N = \alpha(1 + \hat{c}_{\alpha, N} C \sqrt{\epsilon/c_\omega}) . \]

If \( \hat{c}_{\alpha, N} > \sqrt{2} \) then \( \hat{c}_{\alpha, N} \geq c_{\alpha N, N} \).

Proof. First, we derive a bound on

\[ \Delta_{ij} = \left( \arcsin \left( \rho(\hat{\Omega}, \epsilon)_{ij} \right) - \arcsin (\Omega_{ij}) \right)^+ . \]

Since \( \arcsin(\cdot) \) is strictly increasing on \((0, 1)\), a necessary condition for \( \Delta_{ij} \neq 0 \) is \( \rho(\hat{\Omega}, \epsilon)_{ij} > \Omega_{ij} \). This requires \( \hat{\Omega}_{ij} \geq -1 + \epsilon \) and \( \Omega_{ij} \leq 1 - c_\omega \). We have

\[ \hat{\Omega}_{ij} \leq \Omega_{ij} + \| \hat{\Omega} - \Omega \|_{\text{max}} \leq \Omega_{ij} + \delta. \]

Since \( \delta < \epsilon \) and \( \hat{\Omega}_{ij} \geq -1 + \epsilon \) this rules our \( \Omega_{ij} = -1 \) so that \( \Omega_{ij} \geq -1 + c_\omega \) and since \( \delta \leq c_\omega/2 \) also

\[ \hat{\Omega}_{ij} \geq -1 + c_\omega - \| \hat{\Omega} - \Omega \|_{\text{max}} \geq -1 + c_\omega - \delta . \]
Therefore
\[
\rho(\hat{\Omega}_{ij}, \epsilon) \geq \hat{\Omega}_{ij} - \|\hat{\Omega} - \rho(\hat{\Omega}, \epsilon)\|_{\max} \geq -1 + c_\omega - \delta - 2(p - 1)\sqrt{2\epsilon} \geq -1 + c_\omega/4.
\]

Similarly, we can take
\[
\rho(\hat{\Omega}, \epsilon)_{ij} \leq \Omega_{ij} + \|\hat{\Omega} - \Omega\|_{\max} + \|\rho(\hat{\Omega}, \epsilon)_{ij} - \hat{\Omega}_{ij}\|_{\max} \leq 1 - c_\omega + \delta + 2(G - 1)\sqrt{2\epsilon} \leq 1 - c_\omega/4.
\]

There is an intermediate value \( \rho^* \) between \(-1 + c_\omega/4\) and \(1 - c_\omega/4\) such that
\[
\Delta_{ij} = \frac{\rho(\hat{\Omega}, \epsilon)_{ij} - \Omega_{ij}}{\sqrt{1 - (\rho^*)^2}} \leq \frac{\|\rho(\hat{\Omega}, \epsilon)\|_{\max} + \|\hat{\Omega} - \Omega\|_{\max}}{\sqrt{1 - (1 - c_\omega/4)^2}} \leq \frac{2(p - 1)\sqrt{2\epsilon} + \delta}{\sqrt{c_\omega/2(1 - c_\omega/2)}} \leq \frac{4(p - 1)}{\sqrt{c_\omega/2}} \sqrt{\epsilon}.
\]

Let \( \Phi \) denote the cumulative distribution function of a standard normal random variable, and let \( \phi \) denote its probability density function. Gordon’s lower bound (see, e.g., Duembgen (2010)) states that
\[
1 - \Phi(a) > \frac{\phi(a)}{a(1 - 1/a^2)}
\]
for \( a > 0 \) and thus \( 1 - \Phi(a) > \frac{1}{2} \phi(a)/a \) for \( a > \sqrt{2} \). Therefore
\[
P\left(\max_{j=1, \ldots, p} \hat{X}_j > a\right) \geq P\left(\hat{X}_1^\epsilon > a\right) = 1 - \Phi(a) > \frac{\phi(a)}{2a} = \frac{\exp\left(-\frac{a^2}{2}\right)}{a\sqrt{8\pi}}
\]

By Theorem 2.1 in Li and Shao (2002),
\[
P\left(\max_{j=1, \ldots, p} X_j > a\right) - P\left(\max_{j=1, \ldots, p} \hat{X}_j^\epsilon > a\right) = P\left(\max_{j=1, \ldots, p} \hat{X}_j^\epsilon \leq a\right) - P\left(\max_{j=1, \ldots, p} X_j \leq a\right)
\leq \frac{1}{2\pi} \exp\left(-\frac{a^2}{2}\right) \sum_{1 \leq i < j \leq p} \Delta_{ij} \leq \exp\left(-\frac{a^2}{2}\right) \frac{p(p - 1)(p - 1)}{\pi \sqrt{c_\omega/2}} \sqrt{\epsilon}.
\]

We may assume \( P(\max_{j=1, \ldots, p} X_j > a) > P(\max_{j=1, \ldots, p} \hat{X}_j^\epsilon > a) \) since the statement of the
theorem holds trivially otherwise. Then, combining the bounds derived above yields

$$\frac{P(\max_{j=1,\ldots,p} X_j > a) - P(\max_{j=1,\ldots,p} \hat{X}_j > a)}{P(\max_{j=1,\ldots,p} \hat{X}_j > a)} < \frac{4p(p-1)^2}{\sqrt{\pi}} a \sqrt{\epsilon/c_\omega}.$$  

To prove the second claim of lemma, note that the first claim of the lemma implies

$$P\left(\max_{j=1,\ldots,p} X_j > \hat{c}_{\alpha,N}\right) = P\left(\max_{j=1,\ldots,p} X_j > \hat{c}_{\alpha,N}\right) \leq \alpha/N(1 + \hat{c}_{\alpha,N}C\sqrt{\epsilon/c_\omega}) \leq \alpha/N.$$  

Lemma E.10. Suppose that $P_N$ satisfies Assumptions 1, 2, 3, and 5. Assume that for $0 < c < 1$

$$T^{-(1-c)/2} B_{N,8}^4 \log N = o(1).$$  

Then, there are constants $C$ and $C'$ depending only on $P_N$ such for $N,T$ large enough the following statements hold.

$$\sup_{P \in P_N} P\left(\max_{1 \leq i \leq N} \left| \frac{1}{T} \sum_{t=1}^T (\beta_g - \beta_h)' \left( \|v_i^2 x_{it} x'_{it} - \mathbb{E}_P [v_i^2 x_{it} x'_{it}]\right) \|\beta_g - \beta_h'\right| \right) \geq C T^{-(1-c)/2} B_{N,8}^4 (\log N) \leq 2T^{-c}. \quad (18)$$

$$\sup_{P \in P_N} P\left(\max_{1 \leq i \leq N} \left| \frac{1}{T} \sum_{t=1}^T v_i x_{it} \right| \geq C T^{-1/2} \sqrt{\log N} \right) \leq N^{-1} + C'T^{-c}. \quad (19)$$

For $q_1 = 0, 1, 2$, $q_2 = 0, 1, 2$ and $q_3 = 1, \ldots, 3$ such that $q_1 + q_2 + q_3 = 4$

$$\sup_{P \in P_N} P\left(\max_{1 \leq i \leq N} \left| \frac{1}{T} \sum_{t=1}^T \left( \|v_{it}\|^{q_1} \|x_{it}\|^{q_1} \|w_{it}\|^{q_2} \mathbb{E}_P \left[\|v_{it}\|^{q_1} \|x_{it}\|^{q_1} \|w_{it}\|^{q_2} \right] \right) \right| \geq C T^{-(1-c)/2} B_{N,8}^4 (\log N) \right) \leq 2T^{-c}. \quad (20)$$
For $q_1 = 0, 1$, $q_2 = 1, 2$, and $q_3 = 0, 1$ such that $q_1 + q_2 + q_3 = 2$

\[
\sup_{P \in \mathbb{P}_N} \mathbb{P} \left( \max_{1 \leq i \leq N} \left| \frac{1}{T} \sum_{t=1}^{T} (|v_{it}|^{q_1} \|x_{it}\|^{q_2} \|w_{it}\|^{q_3} - \mathbb{E}_P [v_{it}^{q_1} \|x_{it}\|^{q_2} \|w_{it}\|^{q_3}]) \right) \right) \geq C T^{-1/2} \sqrt{\log N} \leq N^{-1} + C'T^{-c}.
\]

**Proof.** We first prove (21). By Lemma D.3 in Chernozhukov, Chetverikov, and Kato (2019), we can choose $N$ large enough such that

\[
\mathbb{E}_P \left( \max_{1 \leq i \leq N} \left| \frac{1}{T} \sum_{t=1}^{T} (\beta_g - \beta_h)' \left( v_{it}^{q_2} x_{it} x'_{it} - \mathbb{E}_P [v_{it}^{q_2} x_{it} x'_{it}] \right) \left( \left| \beta_g - \beta_h \right| \left\| \beta_g - \beta_h \right\| \right) \right) \leq C B_{N,8}^4 \frac{\log N}{\sqrt{T}}
\]

for all $P \in \mathbb{P}_N$. Thus, by Lemma D.2 in Chernozhukov, Chetverikov, and Kato (2019), for every $r > 0$ and the universal constant $K_2$ from Lemma D.2 in Chernozhukov, Chetverikov, and Kato (2019)

\[
P \left( \max_{1 \leq i \leq N} \left| \frac{1}{T} \sum_{t=1}^{T} (\beta_g - \beta_h)' \left( v_{it}^{q_2} x_{it} x'_{it} - \mathbb{E}_P [v_{it}^{q_2} x_{it} x'_{it}] \right) \left( \left| \beta_g - \beta_h \right| \left\| \beta_g - \beta_h \right\| \right) \right) \geq 2C B_{N,8}^4 \frac{\log N}{\sqrt{T}} + r \right) \leq \exp \left( - \frac{r^2 T}{3 B_{N,4}^2} \right) + \frac{K_2 B_{N,8}^8}{r^2 T}.
\]

Taking $r$ proportional to $B_{N,8}^4 T^{-(1-c)/2}$ yields the conclusion. The proof of (20) is similar.

We now prove (21). By Lemma D.3 in Chernozhukov, Chetverikov, and Kato (2019), we can choose $N$ large enough such that

\[
\mathbb{E}_P \left( \max_{1 \leq i \leq N} \left| \frac{1}{T} \sum_{t=1}^{T} (|v_{it}|^{q_1} \|x_{it}\|^{q_2} \|w_{it}\|^{q_3} - \mathbb{E}_P [v_{it}^{q_1} \|x_{it}\|^{q_2} \|w_{it}\|^{q_3}]) \right) \right) \leq C \left( T^{-1/2} \sqrt{\log N} + T^{-3/4} B_{N,8}^2 \log N \right) \leq C T^{-1/2} \sqrt{\log N}
\]

uniformly in $P \in \mathbb{P}_N$, where we used the bounds

\[
\max_{1 \leq i \leq N} \sum_{t=1}^{T} \mathbb{E}_P \left[ T^{-2} |v_{it}|^{2q_1} \|x_{it}\|^{2q_2} \|w_{it}\|^{2q_3} \right] \leq C T^{-1}
\]

and

\[
\mathbb{E}_P \left[ \max_{1 \leq i \leq N} \sum_{t=1}^{T} \mathbb{E}_P \left[ T^{-2} |v_{it}|^{2q_1} \|x_{it}\|^{2q_2} \|w_{it}\|^{2q_3} \right] \right]
\]

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\[ \leq T^{-2} \left( \sum_{t=1}^{T} \mathbb{E}_{P} \left[ \max_{1 \leq i \leq N} |v_{it}| |x_{it}|^{q_1} \| w_{it} \|^{q_2} \right] \right)^{1/2} \leq T^{-3/2} B_{N,8}^{4}. \]

Thus, by Lemma D.2 in Chernozhukov, Chetverikov, and Kato (2019), for every \( r > 0 \) and the universal constant \( K_4 \) from Lemma D.2 in Chernozhukov, Chetverikov, and Kato (2019) for all \( P \in \mathbb{P} \)

\[ P \left( \max_{1 \leq i \leq N} \left| \frac{1}{T} \sum_{t=1}^{T} \left( |v_{it}| |x_{it}|^{q_1} \| w_{it} \|^{q_2} - \mathbb{E}_{P} [ |v_{it}| |x_{it}|^{q_1} \| w_{it} \|^{q_2}] \right) \right| \right) \geq 2CT^{-1/2} \sqrt{\log N} + r \leq \exp \left( -TC'' r^2 \right) + K_4 r^{-4} T^{-3} B_{N,8}^{8}. \]

We now obtain (21) by setting \( r = (C'')^{-1} T^{-1/2} \sqrt{\log N} \) and noting that for this choice of \( r \)

\[ \exp \left( -TC'' r^2 \right) + K_4 r^{-4} T^{-3} B_{N,8}^{8} \leq N^{-1} + C'' T^{-c} \frac{(T^{-(1-c)/2} B_{N,8}^{4} \log N)^2}{\log^4 N} \leq N^{-1} + C'' T^{-c} \]

under the assumptions of the lemma. The proof of (19) is similar. \( \square \)

**Lemma E.11.** Let \( \mathbb{P}_N \) denote a family of probability measures satisfying Assumptions 1, 2, and 3. For \( 0 < c \leq 1/2 \) let

\[ b_N = \frac{r_{N,\theta}}{t_N \wedge \min_{1 \leq i \leq N} \sigma_i} + T^{-(1-c)/2} B_{N,8}^{4} (\log N). \]

and assume \( r_{N,\theta} = o(1 \wedge t_N) \) and \( b_N = o(1) \). There are constants \( C \) and \( C' \) such that for \( N \) and \( T \) sufficiently large,

\[ \sup_{P \in \mathbb{P}_N} P \left( \max_{1 \leq i \leq N} \left| \left( \hat{\Delta}_i(g_{i}^{0}) \right)_{h,h^*} - (\Omega_i(g_{i}^{0}))_{h,h^*} \right| > C b_N \right) \leq N^{-1} + C' T^{-c} + a_{N,\theta} \quad (22) \]

\[ \sup_{P \in \mathbb{P}_N} P \left( \max_{1 \leq i \leq N} \left| \left( \hat{\Delta}_i(g_{i}^{0}) - D_i(g_{i}^{0}) \right)_{h} \right| > C b_N \sqrt{\log N} \right) \leq N^{-1} + C' T^{-c} + a_{N,\theta} \quad (23) \]

\[ \sup_{P \in \mathbb{P}_N} P \left( \max_{1 \leq i \leq N} \left| \left( \hat{\Delta}_i(g_{i}^{0}) - D_i(g_{i}^{0}) \right)_{h} \right| > C \frac{r_{N,\theta} \sqrt{\log N}}{t_N \wedge \min_{1 \leq i \leq N} \sigma_i} \right) \leq N^{-1} + C' T^{-c} + a_{N,\theta} \quad (24) \]

**Proof.** Throughout the proof let \( C, C' \) and \( C'' \) denote generic constants that do not depend on \( P \in \mathbb{P} \). Let \( \delta_i(h) = \theta_{g_{i}^{0}} - \theta_h \) and \( \hat{\delta}_i(h) = \hat{\theta}_{g_{i}^{0}} - \hat{\theta}_h \). Define

\[ \hat{\delta}_i(h,h^*) = \frac{1}{T} \sum_{t=1}^{T} \left( \hat{d}_{it}(h) - \hat{d}_{it}(h^*) \right) \left( \hat{d}_{it}(h^*) - \hat{d}_{it}(h^*) \right), \]

\[ \hat{\delta}_i(h,h^*) \]

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and \( s_i(h) = \sqrt{s_i(h, h)} \). We prove below that

\[
\sup_{P \in \mathcal{P}_N} \left( \max_{1 \leq i \leq N} \left| \frac{\hat{s}_i(h, h^*)}{s_i(h)s_i(h^*)} \right| \right)
\geq C \left( \frac{T_{N, \theta}}{t_N \wedge \min_{1 \leq i \leq N} \sigma_i} + T^{-(1-c)/2} B_{N, \theta}^4 \log(N) \right) \leq N^{-1} + C'T^{-c} + a_{N, \theta}.
\]  

(25)

In particular, evaluating for \( h = h^* \) and applying the inequality \(|\sqrt{a} - 1| \leq |a - 1|\) yields

\[
\sup_{P \in \mathcal{P}_N} \left( \max_{1 \leq i \leq N} \left| \frac{\hat{s}_i(h)}{s_i(h)} - 1 \right| \right)
\geq C \left( \frac{T_{N, \theta}}{t_N \wedge \min_{1 \leq i \leq N} \sigma_i} + T^{-(1-c)/2} B_{N, \theta}^4 \log(N) \right) \leq N^{-1} + C'T^{-c} + a_{N, \theta}.
\]  

(26)

Inequality (22) now follows from the decomposition

\[
(\hat{\Omega}_i)_{h, h^*} - (\Omega_i)_{h, h^*} = \frac{\hat{s}_{i,h,h^*}}{s_{i,h} s_{i,h^*}} - \frac{s_{i,h,h^*}}{s_{i,h} s_{i,h^*}} - \frac{\hat{s}_{i,h,h^*}}{s_{i,h} s_{i,h^*}} + \frac{s_{i,h,h^*}}{s_{i,h} s_{i,h^*}} \left( \frac{\hat{s}_{i,h,h^*}}{s_{i,h} s_{i,h^*}} - \frac{s_{i,h,h^*}}{s_{i,h} s_{i,h^*}} \right) + \left( \frac{s_{i,h}}{s_{i,h}} - 1 \right) \left( \frac{s_{i,h}}{s_{i,h}} - 1 \right) \frac{s_{i,h,h^*}}{s_{i,h} s_{i,h^*}} + \frac{\hat{s}_{i,h,h^*}}{s_{i,h} s_{i,h^*}} - \frac{s_{i,h,h^*}}{s_{i,h} s_{i,h^*}}.
\]

and applying (25) and (26). We now prove inequality (25). We write

\[
\frac{1}{T} \sum_{t=1}^{T} \frac{\left( \hat{d}_{it}(h) - \bar{d}_{it}(h) \right) \left( \hat{d}_{it}(h^*) - \bar{d}_{it}(h^*) \right)}{s_i(h)s_i(h^*)} - \frac{s_i(h, h^*)}{s_i(h)s_i(h^*)} = \frac{\sigma_i^2 \| \delta_i(h) \| \| \delta_i(h^*) \|}{s_i(h)s_i(h^*)} \left( \frac{1}{T} \sum_{t=1}^{T} \frac{\left( \hat{d}_{it}(h) - \bar{d}_{it}(h) \right) \left( \hat{d}_{it}(h^*) - \bar{d}_{it}(h^*) \right)}{\sigma_i^2 \| \delta_i(h) \| \| \delta_i(h^*) \|} \right) - \frac{1}{T} \sum_{t=1}^{T} \mathbb{E}_P \left[ d_{it}(h) d_{it}(h^*) \right] - \frac{1}{T} \sum_{t=1}^{T} \mathbb{E}_P \left[ d_{it}(h) d_{it}(h^*) \right].
\]
Here
\[
\frac{\sigma_i^2 \|\delta_i(h)\|}{s_i(h)} = \frac{\sigma_i^2 \|\delta_i(h)\|}{\sqrt{T} \sum_{t=1}^{T} \mathbb{E}_P [d_{it}^2(h)]} = 2 \left( (\delta_i(h)/\|\delta_i(h)\|) \left( \frac{1}{T} \sum_{t=1}^{T} \mathbb{E}_P [v_{it}^2 x_{it} x_{it}'] \right) (\delta_i(h)/\|\delta_i(h)\|) \right)^{-1/2}
\]

is bounded away from infinity by Assumption 13. In addition, it can be shown that

\[
\sup_{P \in \mathcal{P}_N} \left( \max_{1 \leq i \leq N} \left\{ \frac{\sigma_i^2 \|\delta_i(h)\|}{\sqrt{T} \sum_{t=1}^{T} \mathbb{E}_P [d_{it}^2(h)]} \right\} \right) \leq C \left( \frac{r_{N,\theta}}{t_N \wedge \min_{1 \leq i \leq N} \sigma_i} + T^{-(1-c)/2} B^4_{N,8} (\log N) \right) \leq N^{-1} + C'T^{-c} + a_{N,\theta}.
\]

This follows from the decomposition

\[
\frac{1}{T} \sum_{t=1}^{T} \left( \frac{d_{it}(h) - \bar{d}_{it}(h)}{\sigma_i \|\delta_i(h)\|} \right) \left( \frac{\bar{d}_{it}(h) - d_{it}(h)}{\sigma_i \|\delta_i(h)\|} \right) - \frac{1}{T} \sum_{t=1}^{T} \mathbb{E}_P [d_{it}(h) d_{it}(h^*)] - \frac{1}{T} \sum_{t=1}^{T} \mathbb{E}_P [d_{it}(h) d_{it}(h^*)]\]

and

\[
\sup_{P \in \mathcal{P}_N} \left( \max_{1 \leq i \leq N} \left\{ \frac{\sigma_i^2 \|\delta_i(h)\|}{\sqrt{T} \sum_{t=1}^{T} \mathbb{E}_P [d_{it}^2(h)]} \right\} \right) \leq 2T^{-c} + a_{N,\theta},
\]

\[
\sup_{P \in \mathcal{P}_N} \left( \max_{1 \leq i \leq N} \left\{ \frac{\sigma_i^2 \|\delta_i(h)\|}{\sqrt{T} \sum_{t=1}^{T} \mathbb{E}_P [d_{it}^2(h)]} \right\} \right) \leq 2T^{-2} + a_{N,\theta}.
\]
We prove (32) below. The bounds on $J$ in conjunction with (26) and (27). The numerator in $J$ lower bound on its denominator from (18), and applying (19) to bound $\|T^{-1/2} \sum_{t=1}^T \frac{\hat{d}_t(h) - d_t(h)}{s_i(h)/(\sigma_i \|\delta_i(h)\|)}\|s_i(h)/(\sigma_i \|\delta_i(h)\|)$ and (20) to bound $|1 - \hat{s}_i(h)/s_i(h)|$, (27) for a lower bound on $s_i(h)/(\sigma_i \|\delta_i(h)\|)$ and

$$\frac{\hat{s}_i(h)}{s_i(h)} \geq 1 - \frac{\hat{s}_i(h)}{s_i(h)} - 1$$

in conjunction with (26) to derive a lower bound on $\hat{s}_i(h)/s_i(h)$. To bound $J_2$, we derive a lower bound on its denominator from

$$\frac{\hat{s}_i(h)}{s_i(h)/(\sigma_i \|\delta_i(h)\|)} = \frac{s_i(h)}{s_i(h)/(\sigma_i \|\delta_i(h)\|)} \left\{ \left( \frac{\hat{s}_i(h)}{s_i(h)} - 1 \right) + 1 \right\}$$

in conjunction with (26) and (27). The numerator in $J_2$ is bounded by

$$\sup_{P \in \mathcal{P}_N} P \left( \max_{1 \leq i \leq N} \left| \frac{1}{T} \sum_{t=1}^T \hat{d}_t(h) - d_t(h) \right| \leq C \left( \frac{r_{N,\theta}}{\ell_N \wedge \min_{1 \leq i \leq N} \sigma_i} \right) \right) \leq N^{-1} + CT^{-c} + a_{N,\theta}.$$  

(32)

We prove (32) below. The bounds on $J_1$ and $J_2$ yield (23).
To prove (24), we follow an argument similar that for (23). We observe

\[
\hat{D}_i(h) - \tilde{D}_i(h) = \left( \frac{\tilde{s}_i(h)}{\bar{s}_i(h)} - 1 \right) D_i(h) + \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \left( \frac{\tilde{d}_{it}(h) - d_{it}(h)}{\sigma_i ||\delta_i(h)||} \right) = J_1' + J_2,
\]

where \( \tilde{s}_i(h)^2 = \sum_{t=1}^{T} (d_{it}(h) - \tilde{d}_i(h))^2 / T \). The bound on \( J_2 \) has been discussed above. We write \( J_1' \)

\[
\left| \left( \frac{\tilde{s}_i(h)}{\bar{s}_i(h)} - 1 \right) \tilde{D}_i(h) \right| \leq \left| \frac{\tilde{s}_i(h)}{\bar{s}_i(h)} - 1 \right| \left( \frac{1}{\sqrt{T}} \sum_{t=1}^{T} v_{it} x_i \right) \left( \frac{\tilde{s}_i(h)}{\sigma_i ||\delta_i(h)||} \right)^{-1}.
\]

We apply (19) to bound \( \|T^{-1/2} \sum_{t=1}^{T} v_{it} x_{it} \| \). We observe that

\[
\frac{\tilde{s}_i(h)^2 - \bar{s}_i(h)^2}{\sigma_i^2 ||\delta_i(h)||^2} = \frac{1}{T} \sum_{t=1}^{T} \left( \frac{\tilde{d}_{it}(h) - d_{it}(h)}{\sigma_i ||\delta_i(h)||} \right)^2 + 2 \frac{1}{T} \sum_{t=1}^{T} \frac{d_{it}(h)}{\sigma_i ||\delta_i(h)||} \left( \frac{\tilde{d}_{it}(h) - d_{it}(h)}{\sigma_i ||\delta_i(h)||} \right) - T \left( \frac{1}{T} \sum_{t=1}^{T} \frac{\tilde{d}_{it}(h) - d_{it}(h)}{\sigma_i ||\delta_i(h)||} \right)^2 \left( \frac{1}{T} \sum_{t=1}^{T} \frac{\tilde{d}_{it}(h)}{\sigma_i ||\delta_i(h)||} + \frac{1}{T} \sum_{t=1}^{T} \frac{d_{it}(h)}{\sigma_i ||\delta_i(h)||} \right).
\]

A lower bound on \( \tilde{s}_i(h)/(\sigma_i ||\delta_i(h)||) \) has been established above. Thus, by (29), (30), (31) and (32) together with the fact that \( \sqrt{a} - 1 \leq |a - 1| \), we have

\[
\sup_{P \in \mathcal{P}_N} \left( \max_{1 \leq i \leq N} \left| \frac{\tilde{s}_i(h)}{\bar{s}_i(h)} - 1 \right| \right) \geq C \left( \frac{T_{N,\theta}}{\vartheta_{N,\theta} \min_{1 \leq i \leq N} \sigma_i} \right) \leq N^{-1} + C^e T^{-e} + a_{N,\theta}.
\]

Lastly, we observe that

\[
\frac{\tilde{s}_i(h)}{\sigma_i ||\delta_i(h)||} = \frac{\tilde{s}_i(h)}{\sigma_i ||\delta_i(h)||} \left\{ \left( \frac{\tilde{s}_i(h)}{\bar{s}_i(h)} - 1 \right) + 1 \right\}.
\]

We now have a bound on \( J_1' \), and obtain (24).

It remains to show (29), (30), (31) and (32). Let

\[
\hat{u}_{it} = y_{it} - x_{it}' \hat{\theta}_{\varphi_i} - w_{it}' \hat{\theta}_w
\]

so that

\[
\hat{u}_{it} - u_{it} = -x_{it}' \left( \hat{\theta}_{\varphi_i} - \theta_{\varphi_i} \right) - w_{it}' \left( \hat{\theta}_w - \theta_w \right).
\]
With this notation

\[ d_{it}(g_i^0, h) = d_{it}(h) = -\frac{1}{2}u_{it}x_{it}'\hat{\delta}_i(h), \]

\[ \hat{d}_{it}(g_i^0, h) = \hat{d}_{it}(h) = -\frac{1}{2}\hat{u}_{it}x_{it}'\hat{\delta}_i(h). \]

Decompose

\[
\frac{\hat{d}_{it}(h) - d_{it}(h)}{\sigma_i\|\hat{\delta}_i(h)\|} = -\frac{1}{2}\frac{\hat{u}_{it} - u_{it}}{\sigma_i}x_{it}'\left(\frac{\hat{\delta}_i(h)}{\|\hat{\delta}_i(h)\|}\right) - \frac{1}{2}v_{it}x_{it}'\left(\frac{\hat{\delta}_i(h) - \delta_i(h)}{\|\hat{\delta}_i(h)\|}\right)
\]

\[
= \frac{1}{2}x_{it}'\left(\hat{\theta}_i^0 - \theta_i^0\right) + w_{it}'\left(\hat{\theta}^w - \theta^w\right)x_{it}'\left(\frac{\hat{\delta}_i(h)}{\|\hat{\delta}_i(h)\|}\right) - \frac{1}{2}v_{it}x_{it}'\left(\frac{\hat{\delta}_i(h) - \delta_i(h)}{\|\hat{\delta}_i(h)\|}\right).
\]

In the following arguments we use that

\[
\|\hat{\delta}_i(h)\|/\|\delta_i(h)\| \leq 1 + \frac{\|\hat{\delta}_i(h) - \delta_i(h)\|}{\|\delta_i(h)\|}
\]

is bounded by the fact that \(r_{N,\theta} = o(1 \wedge t_N)\). We bound

\[
|\frac{\hat{d}_{it}(h) - d_{it}(h)}{\sigma_i\|\hat{\delta}_i(h)\|}| \leq C \left(\frac{\|x_{it}\|^2 + \|x_{it}\|\|w_{it}\|}{\min_{1 \leq i \leq N} \sigma_i} + \frac{|v_{it}|\|x_{it}\|}{\sigma_i}t_N\right) r_{N,\theta}, \tag{33}
\]

\[
\left(\frac{\hat{d}_{it}(h) - d_{it}(h)}{\sigma_i\|\hat{\delta}_i(h)\|}\right)^2 \leq C \left(\frac{\|x_{it}\|^4 + \|x_{it}\|^2\|w_{it}\|^2}{\min_{1 \leq i \leq N} \sigma_i^2} + \frac{|v_{it}|^2\|x_{it}\|^2}{\sigma_i^2}t_N^2\right) r_{N,\theta}^2, \tag{34}
\]

and

\[
\left|\frac{d_{it}(h^*)}{\sigma_i\|\hat{\delta}_i(h^*)\|} - \frac{\hat{d}_{it}(h) - d_{it}(h)}{\sigma_i\|\hat{\delta}_i(h)\|}\right| \leq C \left(\frac{|v_{it}|\|x_{it}\|^3 + |v_{it}|\|x_{it}\|^2\|w_{it}\|}{\sigma_i} + \frac{|v_{it}|^2\|x_{it}\|^2}{\sigma_i}t_N\right) r_{N,\theta} \tag{35}
\]

Combining (33), (21), Assumption 3 and the fact that we can take

\[ T^{-1/4}B_{N,8}^2\sqrt{\log N} \leq 1 \]

for \(N, T\) large enough yields (32). Combining (19) and (32) yields (31). Combining (34) and (21) yields (29). Combining (35) and (20) yields (30). \(\square\)

**Lemma E.12.** Let \(V\) denote a correlation matrix, which is possibly singular, and let \(\Phi_{\max,V}\) denote the distribution function of the maximum of a vector of multivariate normal random vector with covariance matrix \(V\). There is \(t^* \in \mathbb{R}\) independent of \(T\) and \(V\) such that for all
\[ t > t^* \]

\[ t_{\text{max},V,T^{-1}} \left( \sqrt{\frac{T - 1}{T}} t \right) \leq \Phi_{\text{max},V}(t). \]

**Proof.** Let \( x \) be a vector of random variables such that \( x \sim N(0,V) \). By the definitions of \( \Phi_{\text{max},V} \) and \( t_{\text{max},V,T^{-1}} \), we have

\[ \Phi_{\text{max},V}(t) = P(x \leq t) \]

and

\[ t_{\text{max},V,T^{-1}} \left( \sqrt{\frac{T - 1}{T}} t \right) = P \left( \frac{1}{\sqrt{z/(T-1)}} x \leq \sqrt{\frac{T - 1}{T}} t \right), \]

where an inequality such as \( x \leq t \) is understood in an element-wise way, and \( z \) is a \( \chi^2 \) random variable with degree of freedom \( T - 1 \) independent of \( x \).

Let \( r \) be the rank of \( V \). We have the following eigendecomposition of \( V \):

\[ V = U \Sigma U', \]

where \( \Sigma \) is a diagonal matrix with non-negative elements and \( U \) is a unitary matrix. We arrange the elements of \( U \) and \( \Sigma \) such that the first \( r \) diagonal elements of \( \Sigma \) are non-zero and its other diagonal elements are zero. Let \( \Sigma_r \) be the \( r \times r \) upper-left block of \( \Sigma \). Let

\[ x^* = U' x. \]

By construction, \( x^* \sim N(0,\Sigma) \). Because \( \Sigma \) is diagonal and only the first \( r \) diagonal elements are non-zero, the first \( r \) elements of \( x^* \) can be non-zero and its other elements are zero. Let \( x_r \) be the vector of the first \( r \) elements of \( x^* \). Note that by the definition of \( \Sigma_r \), \( x^* \sim N(0,\Sigma_r) \). This observation implies that

\[ x = U x^* = U_r x_r, \]

where \( U_r \) is the matrix that consists of the first \( r \) columns of \( U \).

We can then write

\[ \Phi_{\text{max},V}(t) = \int_{x \leq t} \phi_{\Sigma_r}(x_r) \, dx_r \]
and
\[
t_{\text{max},V,T-1}\left(\sqrt{\frac{T-1}{T}}t\right) = \int_{x \leq \sqrt{\frac{T-1}{T}}t} f^t_{\Sigma_r,T-1}(x_r) \, dx_r = \int_{x \leq t} f^t_{\Sigma_r,T-1}(x_r) \, dx_r,
\]
where
\[
\phi_{\Sigma_r}(x_r) = (2\pi)^{-r/2}(\det(\Sigma_r))^{-1/2} \exp \left( -\frac{1}{2} x_r' \Sigma_r^{-1} x_r \right),
\]
and
\[
f^t_{\Sigma_r,T-1}(x_r) = (\pi(T-1))^{-r/2}(\det(\Sigma_r))^{-1/2} \Gamma \left( \frac{T + r - 1}{2} \right) \left( \Gamma \left( \frac{T - 1}{2} \right) \right)^{-1}
\times \left( 1 + \frac{1}{T-1} x_r' \Sigma_r^{-1} x_r \right)^{-\frac{T+r-1}{2}},
\]
is the density of the multivariate \( t \) distribution with scale matrix \( V \) and \( T - 1 \) degrees of freedom, and
\[
f^{t*}_{\Sigma_r,T-1}(x_r) = (\pi T)^{-r/2}(\det(\Sigma_r))^{-1/2} \Gamma \left( \frac{T + r - 1}{2} \right) \left( \Gamma \left( \frac{T - 1}{2} \right) \right)^{-1}
\times \left( 1 + \frac{1}{T} x_r' \Sigma_r^{-1} x_r \right)^{-\frac{T+r-1}{2}}.
\]
We now identify a region in which \( f^{t*}_{\Sigma_r,T-1}(x_r) > \phi_{\Sigma_r}(x_r) \). We have
\[
\log f_{\Sigma_r,T-1}(x_r) - \log \phi_{\Sigma_r}(x_r) = A_T - \frac{T + r - 1}{2} \log \left( 1 + \frac{1}{T} x_r' \Sigma_r^{-1} x_r \right) + \frac{1}{2} x_r' \Sigma_r^{-1} x_r,
\]
where
\[
A_T = -\frac{r}{2} \log(T) + \log \Gamma \left( \frac{T + r - 1}{2} \right) - \log \Gamma \left( \frac{T - 1}{2} \right) + \frac{r}{2} \log(2).
\]
By the property of the logarithm function and the linear function, there is a unique value, denoted by \( x^*_T \), such that \( f^{t*}_{\Sigma_r,T-1}(x_r) \leq \phi_{\Sigma_r}(x_r) \) implies \( x_r' \Sigma_r^{-1} x_r \leq x^*_T \). To see this, we consider the two functions \( \log(1+y) \) and \( ay + b \), where \( a = T/(T + r - 1) \) and \( b = 2A_T/(T + r - 1) \). We want to find a value of \( y \), say \( y' \), such that if \( y \geq y' \) then \( \log(1+y) \leq ay + b \). Because \( \log(1+y) \) is increasing and concave and \( a > 0 \) there are two possibilities: 1) \( ay + b \geq \log(1+y) \) for any \( y \) and \( ay + b > \log(1+y) \) almost always; 2) the curves \( \log(1+y) \) and \( ay + b \) intersect with each other at two points, say \( y_1 \) and \( y_2 \) such that \( \log(1+y) < ay + b \)
for $y < y_1$, $\log(1 + y) \geq ay + b$ for $y_1 \leq y \leq y_2$, and $\log(1 + y) < ay + b$ for $y > y_2$. The first case does not apply to our situation, because if this was the case then $f_{\Sigma_r}^t(x_r) > \phi_{\Sigma_r}(x_r)$ almost always, contradicting the fact that both curves integrate to one. Thus, the second case applies. The values of $y_1$ and $y_2$ can be obtained by solving $\log(1 + y) = ay + b$. It holds $y_2 > 0$ because the slope of $\log(1 + y)$ at $y_2$ must be smaller than $a$ and $0 < a < 1$.

Choose $t$ large enough such that $x'_r \Sigma_r^{-1} x_r \leq x'^*_T$ implies $x \leq t$. This choice of $t$ depend on $T$ only through $x'^*_T$. In particular, if $x'^*_T = O(1)$ then $t$ can be chosen independently from $T$. To prove this set $t = \sqrt{x'^*_T \dim(x)}$. Since $V$ is a correlation matrix, its largest eigenvalue is bounded by $r$. This implies that and $x'_r \Sigma_r^{-1} x_r \geq \|x_r\|^2/r$. Because $x^*$ is a vector whose first $r$ elements are those of $x_r$ and other elements are zero, $\|x_r\|^2 = \|x^*\|^2$. By the definition of $x^*$, it holds that $\|x^*\|^2 = \|U'x\|^2 = \|x\|^2$, where the last equality uses the fact that $U$ is a unitary matrix. Observe that if $x \not\leq t$ so that an element of $x$ exceeds $t$, then $\|x\|^2 > t^2 \geq x'^*_T r$. This implies that $x'_r \Sigma_r^{-1} x_r \geq \|x\|^2/r > x'^*_T r/r = x'^*_T$.

We have

$$
\Phi_{\max,\Sigma_r}(t) - t_{\max,\Sigma_r,T-1}(\sqrt{\frac{T-1}{T}} t) = \int_{x \leq t} (\phi_{\Sigma_r}(x_r) - f_{\Sigma_r,T-1}^t(x_r)) \, dx_r
$$

$$
= \int_{x'_r \Sigma_r^{-1} x_r \leq x'^*_T} (\phi_{\Sigma_r}(x_r) - f_{\Sigma_r,T-1}^t(x_r)) \, dx_r
$$

$$
+ \int_{x \leq t, x'_r \Sigma_r^{-1} x_r > x'^*_T} (\phi_{\Sigma_r}(x_r) - f_{\Sigma_r,T-1}^t(x_r)) \, dx_r,
$$

where the first integral on the right hand side of the equation is taken over $x'_r \Sigma_r^{-1} x_r \leq a$ because $\{x_r : x'_r \Sigma_r^{-1} x_r \leq x'^*_T, x \leq t\} = \{x_r : x'_r \Sigma_r^{-1} x_r \leq x'^*_T\}$ by our choice of $t$. Because both $\phi_{\Sigma_r}(x_r)$ and $f_{\Sigma_r,T-1}^t(x_r)$ are densities and integrate to one, we have

$$
\int_{x'_r \Sigma_r^{-1} x_r \leq x'^*_T} (\phi_{\Sigma_r}(x_r) - f_{\Sigma_r,T-1}^t(x_r)) \, dx_r = -\int_{x'_r \Sigma_r^{-1} x_r > x'^*_T} (\phi_{\Sigma_r}(x_r) - f_{\Sigma_r,T-1}^t(x_r)) \, dx_r,
$$

Thus, for $t$ large enough such that $x'V^{-1}x \leq x'^*_T$ implies $x \leq t$, we have

$$
\Phi_{\max,\Sigma_r}(t) - t_{\max,\Sigma_r,T-1}(\sqrt{\frac{T-1}{T}} t)
$$

$$
= -\int_{x'_r \Sigma_r^{-1} x_r > x'^*_T} (\phi_{\Sigma_r}(x_r) - f_{\Sigma_r,T-1}^t(x_r)) \, dx_r
$$

$$
+ \int_{x \leq t, x'_r \Sigma_r^{-1} x_r > x'^*_T} (\phi_{\Sigma_r}(x_r) - f_{\Sigma_r,T-1}^t(x_r)) \, dx_r.
$$
= \int_{x \geq t, x' \Sigma^{-1} x > x^*_T} (\phi_{x;r}(x_r) - f^{t^*_r}_{V,T-1}(x_r)) \, dx_r \geq 0,

where the last inequality follows because $x' \Sigma^{-1} x > x^*_T$ implies $\phi_{x;r}(x_r) > f^{t^*_r}_{V,T-1}(x_r)$.

Next, we evaluate the order of $x^*_T$. Note that $x^*_T$ solves

$$\frac{1}{2} x^*_T + A_T = \frac{T + r - 1}{2} \log \left(1 + \frac{1}{T} x^*_T\right).$$

We first show that $A_T = O(1)$ where the order is taken with respect to $T$. To see this, we consider the cases of odd and even $G$ separately. Suppose that $r$ is even (we may assume $r \geq 2$). Then the recurrent relation of the Gamma function implies that

$$A_T = -\frac{r}{2} \log(T) + \sum_{j=0}^{r/2-1} \log \left(\frac{T - 1}{2} + j\right) + \frac{r}{2} \log(2) = \sum_{j=0}^{r/2-1} \log \left(\frac{T - 1 + 2j}{T}\right) = O(1)$$

as $T \to \infty$. Next, we consider cases in which $r$ is odd. For $r = 1$, $A_T = O(1)$ follows from

$$\sqrt{\frac{T}{2}} \frac{\Gamma\left(\frac{T-1}{2}\right)}{\Gamma\left(\frac{T}{2}\right)} \to 1. \quad (36)$$

For $r \geq 3$, by the recurrent relation of the Gamma function, we have

$$A_T = -\frac{r}{2} \log(T) + \sum_{j=0}^{r/2-1} \log \left(\frac{T}{2} + j\right) + \log \Gamma \left(\frac{T}{2}\right) - \log \Gamma \left(\frac{T - 1}{2}\right) + \frac{r}{2} \log(2) = \sum_{j=0}^{(r-1)/2-1} \log \left(\frac{T + 2j}{T}\right) + \frac{1}{2} \log \left(\frac{2}{T}\right) + \log \Gamma \left(\frac{T}{2}\right) - \log \Gamma \left(\frac{T - 1}{2}\right).$$

By (36)

$$\log \Gamma \left(\frac{T}{2}\right) - \log \left(\Gamma \left(\frac{T - 1}{2}\right) \left(\frac{T}{2}\right)^{1/2}\right) = O(1).$$

We have now established that $A_T = O(1)$ for all $r \geq 1$. To prove the lemma it now suffices to
prove \( x^*_T = O(1) \). Suppose the opposite is true. Then, there is a subsequence \( T_1, \ldots, T_k, \ldots \) such that \( x^*_{T_k} \) monotonically diverges to infinity. By the definition of \( x^*_T \) we have

\[
x^*_T + A_T = (T + r - 1) \log \left( 1 + \frac{1}{T} x^*_T \right).
\]

For sufficiently large \( y \), \( y/2 > \log(1 + y) \). Therefore, for sufficiently large \( k \), we have

\[
x^*_{T_k} + A_T < \frac{T + r - 1}{2T} x^*_{T_k}
\]

Rearranging terms yields

\[
\frac{T - r + 1}{2T} x^*_{T_k} + A_T < 0,
\]

contradicting that \( A_T = O(1) \) and \( x^*_{T_k} \) diverging to infinity can both be true. This proves \( x^*_T = O(1) \).

Lemma E.13. The regularization function \( \rho(\bullet, \epsilon) \) produced by Algorithm 1 satisfies conditions 1 and 2 of Lemma E.9.

Proof. Without loss of generality, assume that \( P \) is the identity matrix. Let \( \ell_j \) denote the \( j \)th column of \( L \) and \( \tilde{\ell}_j \) denote the \( j \)th column of \( \tilde{L} \). We first show the first property. Consider \( j < j^* \) such that \( \Omega_{j,j^*} < -1 + \epsilon \). Then the algorithm adds \((i, j^*, k)\) to the open list with \( i, k \in \mathbb{N} \) and \( i \leq j \). This implies \( \tilde{\ell}_j = (-1)^{k-1} \ell_i \) and \( \tilde{\ell}_{j^*} = (-1)^k \ell_i \), and therefore

\[
\langle \tilde{\ell}_j, \tilde{\ell}_{j^*} \rangle = -\langle \ell_i, \ell_i \rangle = -1.
\]

We now show the second property. Consider \( j \) such that \( \ell_j \neq \tilde{\ell}_j \). This implies that \((i, j, k)\), with \( i < j \) and \( k \geq 1 \) is added to the open list. It is added because there exists \( j^* \) that satisfies \( i \leq j^* < j \) such that \( \langle \ell^*_{j^*}, \ell_j \rangle < -1 + \epsilon \) and either \( k = 1 \) (and therefore \( i = j^* \)) or \((i, j^*, k - 1)\) was added to the open list. We now show by induction that this implies

\[
\|\ell_j - \tilde{\ell}_j\| \leq k\sqrt{2\epsilon}.
\]

Suppose first that \( k = 1 \). Then

\[
\|\ell_j - \tilde{\ell}_j\| = \sqrt{\langle \ell_j + \ell_i, \ell_j + \ell_i \rangle} = \sqrt{2(1 + \langle \ell_i, \ell_j \rangle)} < \sqrt{2\epsilon}.
\]
This proves the induction hypothesis for $k = 1$. Now suppose that $k > 1$ and

$$\|\ell_j - \tilde{\ell}_j\| \leq (k - 1)\sqrt{2\epsilon}.$$ 

This implies that $(i, j^*, k - 1)$ was added to the open list and

$$\|\ell_j - \tilde{\ell}_j\| = \|\ell_j - (-1)^k\tilde{\ell}_i\| = \|\ell_j + (-1)^{k-1}\tilde{\ell}_i\| = \|\ell_j + \tilde{\ell}_{j^*}\|$$

$$\leq \|\ell_j + \tilde{\ell}_{j^*}\| + \|\ell_j - \tilde{\ell}_{j^*}\|$$

$$= \sqrt{2(1 + \langle \tilde{\ell}_{j^*}, \ell_j \rangle)} + \|\ell_j - \tilde{\ell}_{j^*}\| \leq \sqrt{2\epsilon} + (k - 1)\sqrt{2\epsilon}.$$ 

This concludes the induction argument. An upper bound to $k$ is given by $G - 2$ so that

$$\|\ell_j - \tilde{\ell}_j\| \leq (G - 2)\sqrt{2\epsilon}.$$ 

Then, for any $i$ and $j$ the Cauchy-Schwarz inequality and noting that $\|\ell_i\| = \|\tilde{\ell}_{j^*}\| = 1$ yields

$$\left| \langle \ell_i, \ell_j \rangle - \langle \tilde{\ell}_i, \tilde{\ell}_j \rangle \right| \leq \left| \langle \ell_i - \tilde{\ell}_i, \ell_j \rangle \right| + \left| \langle \tilde{\ell}_i, \tilde{\ell}_j - \ell_j \rangle \right|$$

$$\leq \|\ell_i - \tilde{\ell}_i\|\|\ell_j\| + \|\ell_j - \tilde{\ell}_j\|\|\tilde{\ell}_i\| \leq 2(G - 2)\sqrt{2\epsilon}.$$ 

\[\square\]

**Lemma E.14.** Suppose that $\{\{X_{jt}\}_{j=1}^J\}_{t=1}^T$ is an $\alpha$-mixing sequence (as a sequence indexed by $t$) with mixing coefficients $\alpha(k)$. Suppose that $J \geq T^8$. Let $S_t = T^{-1/2}\sum_{s=1}^T(X_{st}, \ldots, X_{st})'$. Let $G \sim N(0, \Xi)$, where $\Xi$ is the long-run covariance matrix of $(X_{1t}, \ldots, X_{Jt})$. Assume the following three conditions:

1. There exist some universal constants $K_1 > 0$, $a > 0$ and $d_1 > 0$ such that $P(|X_{jt}| > x) < K_1 \exp(-(1/a)^{d_1}x^{d_1})$ for all $t \in \{1, \ldots, T\}$ and $j \in \{1, \ldots, J\}$.

2. There exist some universal constants $K_2 > 1$, $b > 0$ and $d_2 > 0$ such that $\alpha(k) \leq K_2 \exp(-bk^{d_2})$ for any $k \geq 1$.

3. There exists a universal constant $K_3 > 0$ such that $V_{T,j} \geq K_3$ for any $j \in \{1, \ldots, J\}$, where $V_{T,j} = \text{var}(\sum_{t=1}^T X_{jt}/\sqrt{T})$.

For $t \in \mathbb{R}^N$, define $A(t) = \{x \in \mathbb{R}^N : x_j \leq t_j$ for $j = 1, \ldots, J\}$. Let $A = \bigcup_{t \in \mathbb{R}^J} A(t)$. Then it holds that

$$\sup_{P \in \mathbb{P}} \sup_{A \in A} |P(S_N \in A) - P(G \in A)| \lesssim \frac{(\log J)^{(1+2d_2)/(3d_2)}}{T^{1/9}} + \frac{(\log J)^{7/6}}{T^{1/9}}.$$ 

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provided that \((\log J)^{3-d_2} = o(T^{d_2/3})\) and \(\mathbb{P}\) is a collection of probabilities measures under
which the above three conditions are satisfied with identical choices of \(K_1, K_2, a, b, d_1\)
and \(d_2\).

**Proof.** The lemma follows by Theorem 1 of Chang, Chen, and Wu (2021), noting the remark
in the beginning of Section 2.1 of Chang, Chen, and Wu (2021). Theorem 1 of Chang,
Chen, and Wu (2021) has three conditions and the second and third conditions are given
in the statement of the lemma. The first condition is “There exist a sequence of constants
\(B_J \geq 1\) and a universal constant \(d_1 \geq 1\) such that \(\|X_{jt}\|_{\psi_{d_1}} \leq B_J\) for all \(t \in \{1, \ldots, T\}\)
and \(j \in \{1, \ldots, J\}\), where \(\|\xi\|_{\psi_1} = \inf[\lambda > 0 : E(\psi_\lambda(\|\xi\|/\lambda)) \leq 1]\) for \(\psi_\lambda(x) = \exp(x^\lambda) - 1\) (the
Orlicz norm with \(\psi_\lambda\)). By Lemma 8.1 of Kosorok (2008), \(P(|X_{jt}| > x) < K_1 \exp(-a/(a)^{d_1}x^{d_1})\)
implies this condition by taking \(B_J = ((1 + K_1/(a)^{d_1}))^{1/d_1}\), which is constant if \(K_1, a\) and
\(d_1\) are constant.

**Lemma E.15.** Suppose that that \(X_{it}\) is a strongly mixing process with zero mean for each
\(i = 1, \ldots, N\) with tail probabilities \(\sup_{i=1,\ldots,N} P(|X_{it}| > x) \leq \exp(1 - (x/a)^{d_1})\) and with
strong mixing coefficients \(\sup_{i=1,\ldots,N} a_i[t] \leq \exp(-bt^{d_2})\), where \(a, b, d_1,\) and \(d_2\) are positive
constants. Let \(\mathbb{P}_N\) denote a sequence of sets of probability measures that satisfy the above
conditions with given values of \(a, b, d_1,\) and \(d_2\). Let

\[s_T^2 = \max_{i=1,\ldots,N} \max_{t=1,\ldots,T} \left( E(X_{it}^2) + 2 \sum_{s > t} |E(X_{it}X_{is})| \right).\]

Assume that \(s_T^2 < KN\) for a constant \(K\) which does not depend on \(N, T\) nor \(P\). Then, it
holds that for any constants \(C' > 0\) and \(0 < c < 1\), as \(N, T \to \infty\) with \(NT^{-\delta} \to 0\) for some
\(\delta > 0\),

\[\sup_{P \in \mathbb{P}_N} P \left( \max_{i=1,\ldots,N} \left| \frac{1}{T} \sum_{t=1}^{T} X_{it} \right| \geq C'T^{-1/2} \log N \right) \to 0.\]

**Proof.** By the Bonferroni inequality and inequality (1.7) in Merlevède, Peligrad, and Rio
(2011) which is an application of Rio (2017, Theorem 6.2) (the original French version was
published in 2000), we have

\[\sup_{P \in \mathbb{P}} P \left( \max_{i=1,\ldots,N} \left| \frac{1}{T} \sum_{t=1}^{T} X_{it} \right| \geq x \right) \leq \sup_{P \in \mathbb{P}} \sum_{i=1}^{N} P \left( \left| \frac{1}{T} \sum_{t=1}^{T} X_{it} \right| \geq x \right) \leq 4N \left( 1 + \frac{T(x/4)^{2}}{\tau s_T^2} \right)^{-r/2} + 4CN(x/4)^{-1} \exp \left( -a \frac{(Tx/4)^{d}}{b^d r^d} \right),\]

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where \( d = (d_1^{-1} + d_2^{-1})^{-1} \) and \( C \) is a positive constant. Thus for \( x = C'T^{-1/2} \log N \), it holds that
\[
\sup_{p \in P} P \left( \max_{i=1,...,N} \left| \frac{1}{T} \sum_{t=1}^{T} X_{it} \right| \leq C'T^{-1/2} \log N \right) \\
\leq 4N \left( 1 + \frac{(C')^2 \log^2 N}{16rs_T^2} \right)^{-r/2} + 16(C/C')NT^{1/2} \log^{-1} N \exp \left( -a \frac{(C'T^{1/2} \log N)^d}{4d'p^{1-d}} \right).
\]
Taking \( r = T^{1/2-c_1} \) for \( 0 < c_1 < 1/2 \), we have \( N^{r/2}T^c \log^2 N/(rs_T^2) \to \infty \) and \( T^{1/2} \log N/r = T^{c_1} \log N \). Thus under \( NT^{-\delta} \to 0 \) and \( s_T^2 < KN \), the right hand side converges to zero.
\[ \square \]

Lemma E.16. Suppose that two random variances \( X_1 \) and \( X_2 \) satisfy \( P(|X_a| > x) \leq K_a \exp(-b_a x^{d_a}) \) for \( a = 0, 1 \), then \( P(|X_1X_2| > x) \leq K \exp(-bx^d) \) for some positive constants \( K, b, \) and \( d_2 \), and \( P(|X_1 + X_2| > x) \leq K' \exp(-b'x^{d'}) \) for some constants \( K', b', \) and \( d' \).

Proof. The first statement follows because
\[
P(|X_1X_2| > x) \leq P(|X_1| > \sqrt{x}) + P(|X_2| > \sqrt{x}) \\
\leq K_1 \exp(-b_1 x^{d_1/2}) + K_2 \exp(-b_2 x^{d_2/2}) \\
\leq 2 \max(K_1, K_2) \exp(- \min(b_1, b_2) x^{\min(d_1, d_2)/2}).
\]
For the second statement, we have
\[
P(|X_1 + X_2| > x) \leq P(|X_1| > x/2) + P(|X_2| > x/2) \\
\leq K_1 \exp(-b_1/2^{d_1} x^{d_1}) + K_2 \exp(-b_2/2^{d_2} x^{d_2}) \\
\leq 2 \max(K_1, K_2) \exp(- \min(b_1/2^{d_1}, b_2/2^{d_2}) x^{\min(d_1, d_2)}).
\]
\[ \square \]

Lemma E.17. Suppose that \( (x_{it}, w_{it}, v_{it}) \) is a strong mixing sequence over \( t \) with mixing coefficients \( \sup_i a_i[t] \leq K \exp(-at^d) \) for some constant \( K, a \) and \( d \), then so is \( g(x_{it}, w_{it}, v_{it}) \) where \( g \) is a measurable function.

Proof. The proof follows the argument in the proof of Theorem 14.1 in Davidson \[1994\]. \[ \square \]

Lemma E.18. Suppose that a random variable, \( X \), satisfies that \( P(|X| > x) < K \exp(-(x/a)^d) \) for some \( K, a > 0 \) and \( d > 1 \). Then, \( E(|X|^p) < M \) for \( M \) which depends only on \( K, a, d \) and \( p \), where \( p \) is an integer.
Proof. By the argument given in Kosorok (2008, page 129) which is based on the series expansion of the exponential function, we have $(E(|X|^p))^{1/p} \leq p||X||_{\psi_1}$ where $|| \cdot ||_{\psi_1}$ is the Orlicz norm with $\psi_1(x) = \exp(x^a) - 1$ as defined in the proof of Lemma E.14. By Kosorok (2008, Lemma 8.1), $||X||_{\psi_1}$ is bounded by a constant which depends on $K, a$ and $d$.

\[\square\]

Lemma E.19 (Large CLT for MAX statistic with dependent data). Let $P_N$ denote a family of probability measures satisfying Assumptions 7, 8, 9 and 10. For any $P \in P_N$ and for $i = 1, \ldots, N$, there are centered normal random vectors $X_i$ that take values in $\mathbb{R}^{G-1}$ and satisfy $E_P[X_iX_i'] = \Omega_i(g_i^0)$ and

$$\sup_{(r_1, \ldots, r_N) \in \mathbb{R}^{N^2}} \left| P\left( \max_{1 \leq i \leq N} \left( \max_{h \in \mathcal{G}\setminus\{g_i^0\}} D_i(g_i^0, h) - r_i \right) > 0 \right) - P\left( \max_{1 \leq i \leq N} \left( \max_{1 \leq h \leq G-1} X_{i,h} - r_i \right) > 0 \right) \right| \leq \frac{\log N}{T^{1/9}} + \frac{\log N^{7/6}}{T^{1/9}} \hspace{1cm} (\text{provided that } (\log N)^3 - d_2 = o(T^{d_2/3}), \text{ where } C \text{ is a universal constant.})$$

Proof. Let $\delta_i(h) = \theta_{g_i^0} - \theta_h$ and

$$Z_{it}(h) = \frac{d_{it}(h)}{\sqrt{E_P[d_{it}(h)^2]}} = -\frac{x_{it}'(\delta_i(h)/\|\delta_i(h)\|)}{2s_i(h)/(\sigma_1||\delta_i(h)||)},$$

where

$$s_i(h) = \sqrt{\frac{1}{T} \sum_{t=1}^{T} d_{it}^2(h)} = \frac{1}{2} \left( \delta_i(h)' \left( \frac{1}{T} \sum_{t=1}^{T} E_P[x_{it}'x_{it}'d_{it}^2] \right) \delta_i(h) \right).$$

Define the vector

$$Z_t = ((Z_{1t}(h))_{h \in \mathcal{G}\setminus\{g_i^0\}}, \ldots, (Z_{Nt}(h))_{h \in \mathcal{G}\setminus\{g_i^0\}}').$$

To apply Lemma E.14 we verify the conditions. Note that $Z_t$ is a vector with $N(G - 1)$ elements and we set $J$ in Lemma E.14 as $N(G - 1)$. Assumption 8 and Lemma E.16 imply that $Z_t$ satisfies the first condition. The second condition is verified by Assumption 7 and Lemma E.17. By Assumption 13 there exists $C_Z > 0$ such that $2s_i(h)/(\sigma_1||\delta_i(h)||) \geq C_Z^{-1}$. Thus the third condition is satisfied by Assumption 8.

We now apply Lemma E.14. Let $\tilde{X}_i = (\tilde{X}_{1i}', \ldots, \tilde{X}_{Ni}')'$ with $\dim(\tilde{X}_i) = G - 1$ for $i = 1, \ldots, N, t = 1, \ldots, T$ denote a normal random vector with the property that $\tilde{X}_i$ and $\tilde{X}_s$ are independent for $i \neq s$ and $E_P[\tilde{X}_i(\tilde{X}_i)'] = E_P[Z_i(Z_i)']$ for $i = 1, \ldots, N, t = 1, \ldots, T$. Define $X_i = \sum_{t=1}^{T} \tilde{X}_{it}/\sqrt{T}$. Clearly, $X_i$ is a normal random vector with covariance matrix $\Omega_i$. Let
\[ a_i = -\infty \text{ and } b_i = r_i. \] Then we may write

\[
\begin{align*}
\sup_{(r_1, \ldots, r_N) \in \mathcal{R}^N_+} & \left| P \left( \max_{1 \leq i \leq N} \left( \max_{h \in G \setminus \{h_i^0\}} D_i(h) - r_i \right) > 0 \right) \right. \\
& - P \left( \max_{1 \leq i \leq N} \left( \max_{1 \leq h \leq G-1} X_{i,h} - r_i \right) > 0 \right) \\
\leq & \sup_{(r_1, \ldots, r_N) \in \mathcal{R}^N_+} \left| P \left( \bigcap_{i=1}^N \bigcap_{h \in G \setminus \{h_i^0\}} \{ a_i < D_i(h) \leq b_i \} \right) - P \left( \bigcap_{i=1}^N \bigcap_{h=1}^{G-1} \{ a_i < X_{i,h} \leq b_i \} \right) \right| \\
\leq & \sup_{(r_1, \ldots, r_N) \in \mathcal{R}^N_+} \left| P \left( \bigcap_{i=1}^N \bigcap_{h \in G \setminus \{h_i^0\}} \{ a_i < \frac{1}{\sqrt{T}} \sum_{t=1}^T Z_{it}(h) \leq b_i \} \right) - P \left( \bigcap_{i=1}^N \bigcap_{h=1}^{G-1} \{ a_i < \frac{1}{\sqrt{T}} \sum_{t=1}^T \tilde{X}_{it,h} \leq b_i \} \right) \right| \\
\leq & K \left( \frac{(\log N)^{(1+2d_1)/(3d_1)}}{T^{1/9}} + \frac{(\log N)^{7/6}}{T^{1/9}} \right),
\end{align*}
\]

where \( K \) is a universal constant. Here, the last inequality holds by Lemma \[\text{E.14}\].

\[\square\]

**Lemma E.20.** Suppose that probability measures in \( \mathbb{P}_N \) satisfy Assumptions \[\text{[I], [J], [K] and [L]}\]. Assume that \( NT^{-\delta} \to 0 \) for some \( \delta > 0 \). Then, there are constants \( C \) depending only on \( \mathbb{P} \) such for \( N, T \) large enough the following statements hold.

\[
\begin{align*}
\sup_{P \in \mathbb{P}} P \left( \max_{1 \leq i \leq N} \left[ \frac{1}{T} \sum_{t=1}^T \left( \frac{\beta_g - \beta_{h'}}{\|\beta_g - \beta_h\|} \left( v_{it}^2 x_{it} x_{it}' - \mathbb{E}_P \left[ v_{it}^2 x_{it} x_{it}' \right] \right) \right) \right \| \beta_g - \beta_{h'} \| \right) \\
\geq CT^{-1/2} \log N = o(1). \tag{37}
\end{align*}
\]

\[
\begin{align*}
\sup_{P \in \mathbb{P}} P \left( \max_{1 \leq i \leq N} \left[ \frac{1}{T} \sum_{t=1}^T v_{it} x_{it} \right] \right) \geq CT^{-1/2} \log N = o(1). \tag{38}
\end{align*}
\]

For \( q_1 = 0, 1, 2, q_2 = 0, 1, 2 \) and \( q_3 = 1, \ldots, 3 \) such that \( q_1 + q_2 + q_3 = 4 \)

\[
\begin{align*}
\sup_{P \in \mathbb{P}} P \left( \max_{1 \leq i \leq N} \left[ \frac{1}{T} \sum_{t=1}^T \left( \|v_{it}\|^{q_1}\|x_{it}\|^{q_1}\|w_{it}\|^{q_2} - \mathbb{E}_P \left[ \|v_{it}\|^{q_1}\|x_{it}\|^{q_1}\|w_{it}\|^{q_2} \right] \right) \right] \right) \\
\geq CT^{-1/2} \log N = o(1). \tag{39}
\end{align*}
\]
For $q_1 = 0, 1, q_2 = 1, 2$, and $q_3 = 0, 1$ such that $q_1 + q_2 + q_3 = 2$

$$
\sup_{P \in \mathcal{P}} \left( \max_{1 \leq i \leq N} \frac{1}{T} \sum_{t=1}^{T} \left[ |v_{it}|^{q_1} \|x_{it}\|^{q_2} \|w_{it}\|^{q_3} - \mathbb{E}_P \left[ |v_{it}|^{q_1} \|x_{it}\|^{q_2} \|w_{it}\|^{q_3} \right] \right] \right) \geq CT^{-1/2} \log N = o(1). \tag{40}
$$

**Proof.** These statements hold by Lemma E.15. Note that the conditions for Lemma E.15 are satisfied by Assumptions 6 and 7 together with Lemmas E.16 and E.17. □

**Lemma E.21.** Let $\mathcal{P}_N$ denote a family of probability measures satisfying Assumptions 1, 6, 7 and 8. Define

$$
\hat{s}_i(h, h^*) = \frac{1}{T} \sum_{t=1}^{T} \left( \hat{d}_{it}(h) - \bar{d}_{it}(h) \right) \left( \hat{d}_{it}(h^*) - \bar{d}_{it}(h^*) \right),
$$

$$
s_i(h, h^*) = \frac{1}{T} \sum_{t=1}^{T} \mathbb{E}_P \left[ d_{it}(h) d_{it}(h^*) \right]
$$

and $s_i(h) = \sqrt{s_i(h, h)}$. Suppose that

$$
\frac{r_{N, \theta}}{\iota_N \wedge \min_{1 \leq i \leq N} \sigma_i} + T^{-1/2} \log N \to 0.
$$

The following statements hold for some constant $C$.

$$
\sup_{P \in \mathcal{P}_N} \left( \max_{1 \leq i \leq N} \frac{1}{T} \sum_{t=1}^{T} \frac{\hat{d}_{it}(h) - d_{it}(h)}{\sigma_i \|\delta_i(h)\|} \right) \geq C \left( \frac{r_{N, \theta}}{\iota_N \wedge \min_{1 \leq i \leq N} \sigma_i} \right) = o(1). \tag{41}
$$

$$
\sup_{P \in \mathcal{P}_N} \left( \max_{1 \leq i \leq N} \frac{\hat{s}_i(h, h^*)}{s_i(h, h^*)} - \frac{s_i(h, h^*)}{s_i(h) s_i(h^*)} \right) \geq C \left( \frac{r_{N, \theta}}{\iota_N \wedge \min_{1 \leq i \leq N} \sigma_i} + T^{-1/2} \log N \right) = o(1). \tag{42}
$$

$$
\sup_{P \in \mathcal{P}_N} \left( \max_{1 \leq i \leq N} \frac{\hat{s}_i(h)}{s_i(h)} - 1 \right) \geq C \left( \frac{r_{N, \theta}}{\iota_N \wedge \min_{1 \leq i \leq N} \sigma_i} + T^{-1/2} \log N \right) = o(1). \tag{43}
$$
With this notation

In the following arguments we use that

Let

Proof. Let

\[ \hat{u}_{it} = y_{it} - x'_{it} \hat{\theta}_{\theta_i} - w'_it \hat{\theta}_w \]

so that

\[ \hat{u}_{it} - u_{it} = -x'_{it} \left( \hat{\theta}_{\theta_i} - \theta_i \right) - w'_it \left( \hat{\theta}_w - \theta_w \right). \]

With this notation

\[ d_{it}(g^0_i, h) = d_{it}(h) = -\frac{1}{2} u_{it} x'_{it} \delta_i(h), \]

\[ \hat{d}_{it}(g^0_i, h) = \hat{d}_{it}(h) = -\frac{1}{2} \hat{u}_{it} x'_{it} \hat{\delta}_i(h). \]

Decompose

\[
\frac{\hat{d}_{it}(h) - d_{it}(h)}{\sigma_i ||\delta_i(h)||} = -\frac{1}{2} \frac{\hat{u}_{it} - u_{it}}{\sigma_i} x'_{it} \left( \frac{\hat{\delta}_i(h)}{||\delta_i(h)||} \right) - \frac{1}{2} v_{it} x'_{it} \left( \frac{\hat{\delta}_i(h) - \delta_i(h)}{||\delta_i(h)||} \right) \frac{||\delta_i(h)||}{||\delta_i(h)||} \\
= -\frac{1}{2} \frac{x'_{it} (\hat{\theta}_{\theta_i} - \theta_i)}{\sigma_i} + w'_{it} (\hat{\theta}_w - \theta_w) \frac{x'_{it} (\hat{\delta}_i(h) - \delta_i(h))}{||\delta_i(h)||}.
\]

In the following arguments we use that

\[ ||\hat{\delta}_i(h)|| / ||\delta_i(h)|| \leq 1 + \frac{||\hat{\delta}_i(h) - \delta_i(h)||}{||\delta_i(h)||} \]

is bounded by the fact that \( r_{N, \theta} = o(1 \wedge \iota_N). \)

With probability at least 1 - \( a_{N, \theta} \), we bound

\[
\left| \frac{\hat{d}_{it}(h) - d_{it}(h)}{\sigma_i ||\delta_i(h)||} \right| \leq C \left( \frac{||x_{it}||^2 + ||x_{it}|| ||w_{it}||}{\min_{1 \leq i \leq N} \sigma_i} + \frac{||v_{it}|| ||x_{it}||}{\iota_N} \right) r_{N, \theta},
\]

(45)

\[
\left( \frac{\hat{d}_{it}(h) - d_{it}(h)}{\sigma_i ||\delta_i(h)||} \right)^2 \leq C \left( \frac{||x_{it}||^4 + ||x_{it}||^2 ||w_{it}||^2}{\min_{1 \leq i \leq N} \sigma_i^2} + \frac{||v_{it}||^2 ||x_{it}||^2}{\iota_N^2} \right) r_{N, \theta}^2.
\]

(46)
Combining (40), (46) and Lemma E.18 with Assumption 6 yields (41).

We now prove inequality (42). We write

\[
\frac{1}{T} \sum_{t=1}^{T} \frac{\hat{d}_{it}(h) - d_{it}(h)}{s_{i}(h) s_{i}(h^{*})} \leq \frac{\sigma_{i}^{2} \| \delta_{i}(h) \| \| \delta_{i}(h^{*}) \|}{s_{i}(h) s_{i}(h^{*})} \sqrt{\frac{T}{\sum_{t=1}^{T} \mathbb{E}_{P} [d_{it}^{2}(h)]}}
\]

Here

\[
\frac{\sigma_{i}^{2} \| \delta_{i}(h) \|}{s_{i}(h)} = \sqrt{\frac{T}{\sum_{t=1}^{T} \mathbb{E}_{P} [d_{it}^{2}(h)]}}
\]

We now prove inequality (42). We write

\[
\frac{1}{T} \sum_{t=1}^{T} \frac{\hat{d}_{it}(h) - \bar{d}_{it}(h)}{s_{i}(h) s_{i}(h^{*})} \left( \frac{\hat{d}_{it}(h^{*}) - \bar{d}_{it}(h^{*})}{s_{i}(h) s_{i}(h^{*})} \right)
\]

Combining (39), (47) and Lemma E.18 with Assumption 6 yields (49).
Thus, combining (48), (49) and (50) yields

\[
\frac{1}{T} \sum_{t=1}^{T} \left( \frac{\hat{d}_{it}(h) - \tilde{d}_{it}(h)}{\sigma_i \| \delta_i(h) \|} \right) \left( \frac{\hat{d}_{it}(h^*) - \tilde{d}_{it}(h^*)}{\sigma_i \| \delta_i(h^*) \|} \right) - \frac{1}{T} \sum_{t=1}^{T} \mathbb{E}_P \left[ d_{it}(h) d_{it}(h^*) \right] \frac{\sigma_i \| \delta_i(h) \|}{\| \delta_i(h^*) \|} \cdot \\
= \frac{1}{T} \sum_{t=1}^{T} \left( \frac{\hat{d}_{it}(h) - d_{it}(h)}{\sigma_i \| \delta_i(h) \|} \right) \left( \frac{\hat{d}_{it}(h^*) - d_{it}(h^*)}{\sigma_i \| \delta_i(h^*) \|} \right) + \frac{1}{T} \sum_{t=1}^{T} \frac{d_{it}(h^*)}{\sigma_i \| \delta_i(h^*) \|} \left( \frac{\hat{d}_{it}(h) - d_{it}(h)}{\sigma_i \| \delta_i(h) \|} \right) \\
- \left( \frac{1}{T} \sum_{t=1}^{T} \frac{d_{it}(h)}{\sigma_i \| \delta_i(h) \|} \right) \left( \frac{1}{T} \sum_{t=1}^{T} \frac{\hat{d}_{it}(h^*)}{\sigma_i \| \delta_i(h^*) \|} \right) \\
+ \frac{1}{T} \sum_{t=1}^{T} \frac{d_{it}(h) d_{it}(h^*)}{\sigma_i^2 \| \delta_i(h) \| \| \delta_i(h^*) \|} - \frac{1}{T} \sum_{t=1}^{T} \mathbb{E}_P \left[ d_{it}(h) d_{it}(h^*) \right] \frac{\sigma_i^2 \| \delta_i(h) \|}{\| \delta_i(h^*) \|} 
\]

Thus, combining (48), (49) and (50) yields

\[
\sup_{P \in \mathcal{P}} \mathbb{P} \left( \max_{1 \leq i \leq N} \left| \frac{1}{T} \sum_{t=1}^{T} \left( \frac{\hat{d}_{it}(h) - \tilde{d}_{it}(h)}{\sigma_i \| \delta_i(h) \|} \right) \left( \frac{\hat{d}_{it}(h^*) - \tilde{d}_{it}(h^*)}{\sigma_i \| \delta_i(h^*) \|} \right) - \frac{1}{T} \sum_{t=1}^{T} \frac{d_{it}(h) d_{it}(h^*)}{\| \delta_i(h) \| \| \delta_i(h^*) \|} \right| \right) 
\geq C \left( \frac{\sum_{i=N}^{\min_{1 \leq i \leq N} \sigma_i} \| \delta_i(h) \| \| \delta_i(h^*) \|}{\sigma_i} \right)^{t_{N,\theta}} + T^{-1/2} \log N \right) = o(1). 
\]

Hence, applying (37) implies (12).

The proof of (13) follows by evaluating (12) for \( h = h^* \) and applying the inequality \( |\sqrt{a} - 1| \leq |a - 1| \).

To show (44), observe that

\[
\frac{\hat{s}_i(h)^2 - \tilde{s}_i(h)^2}{\sigma_i^2 \| \delta_i(h) \|^2} = \frac{1}{T} \sum_{t=1}^{T} \left( \frac{\hat{d}_{it}(h) - d_{it}(h)}{\sigma_i \| \delta_i(h) \|} \right)^2 + \frac{1}{T} \sum_{t=1}^{T} \frac{d_{it}(h)}{\sigma_i \| \delta_i(h) \|} \frac{(\hat{d}_{it}(h) - d_{it}(h))}{\sigma_i \| \delta_i(h) \|} \\
- T \left( \frac{1}{T} \sum_{t=1}^{T} \frac{\hat{d}_{it}(h) - d_{it}(h)}{\sigma_i \| \delta_i(h) \|} \right) \left( \frac{1}{T} \sum_{t=1}^{T} \frac{d_{it}(h)}{\sigma_i \| \delta_i(h) \|} + \frac{1}{T} \sum_{t=1}^{T} \frac{d_{it}(h)}{\sigma_i \| \delta_i(h) \|} \right). 
\]

A lower bound on \( \hat{s}_i(h)/(\sigma_i \| \delta_i(h) \|) \) has been established above. Thus, by (41), (48), (49), and (50) together with the fact that \( |\sqrt{a} - 1| \leq |a - 1| \), (44) follows.

\[\square\]

**Lemma E.22.** Let \( \mathbb{P}_N \) denote a family of probability measures satisfying Assumptions 7, 8.
For $0 < c \leq 1/2$ let

$$b_N = \frac{r_{N,\theta}}{\iota_N \wedge \min_{1 \leq i \leq N} \sigma_i} + T^{-1/2} \log N,$$

and assume $r_{N,\theta} = o(1 \wedge \iota_N)$ and $b_N = o(1)$. There are constants $C$ and $C'$ such that

$$\sup_{P \in \mathbb{P}} \frac{\max_{1 \leq i \leq N} \left| (\hat{\Omega}_i(g^0_i))_{h,h^*} - (\Omega_i(g^0_i))_{h,h^*} \right|}{\hat{\sigma}_i(h) \hat{\sigma}_i(h^*)} = o(1),$$

$$\sup_{P \in \mathbb{P}} \frac{\max_{1 \leq i \leq N} \left| \hat{D}_i(g^0_i, h) - D_i(g^0_i, h) \right|}{\hat{\sigma}_i(h) \hat{\sigma}_i(h^*)} = o(1),$$

and

$$\sup_{P \in \mathbb{P}} \frac{\max_{1 \leq i \leq N} \left| \hat{D}_i(g^0_i, h) - \hat{D}_i(g^0_i, h) \right|}{\hat{\sigma}_i(h) \hat{\sigma}_i(h^*)} = o(1).$$

**Proof.** Throughout the proof let $C$, $C'$ and $C''$ denote generic constants that do not depend on $P \in \mathbb{P}$. Let $\delta_i(h) = \theta_{g^0_i} - \theta_h$ and $\hat{\delta}_i(h) = \hat{\theta}_{g^0_i} - \hat{\theta}_h$. Define

$$\hat{s}_i(h, h^*) = \frac{1}{T} \sum_{t=1}^{T} \left( \hat{d}_{it}(h) - \bar{d}_{it}(h) \right) \left( \hat{d}_{it}(h^*) - \bar{d}_{it}(h^*) \right),$$

$$s_i(h, h^*) = \frac{1}{T} \sum_{t=1}^{T} \mathbb{E}_P \left[ d_{it}(h) d_{it}(h^*) \right]$$

and $s_i(h) = \sqrt{s_i(h, h^*)}$.

Inequality (52) follows from the decomposition

$$\left( \hat{\Omega}_i(h,h^*) - (\Omega_i(h))_{h,h^*} \right) \equiv \left( \frac{\hat{s}_i(h, h^*)}{\hat{s}_i(h)} \right) \frac{s_i(h, h^*)}{s_i(h) s_i(h^*)} \equiv \left[ \left( \frac{\hat{s}_i(h)}{s_i(h)} \right) \left( \frac{s_i(h^*)}{\hat{s}_i(h^*)} \right) - 1 \right] \left( \frac{\hat{s}_i(h, h^*)}{s_i(h) s_i(h^*)} - \frac{s_i(h, h^*)}{s_i(h) s_i(h^*)} \right)$$

and applying (42) and (43).

To prove (53) we write

$$\hat{D}_i(h) - D_i(h) = \left( \frac{s_i(h)}{\hat{s}_i(h)} - 1 \right) D_i(h) + \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \left( \frac{\hat{d}_{it}(h) - d_{it}(h)}{s_i(h)/\|\delta_i(h)\|} \right)$$

and

$$\hat{D}_i(h) - D_i(h) = J_1 + J_2.$$
We bound $J_1$ by writing
\[
\left| \left( \frac{s_i(h)}{\hat{s}_i(h)} - 1 \right) D_i(h) \right| = \left| \frac{s_i(h)}{\hat{s}_i(h)} \left( 1 - \frac{\hat{s}_i(h)}{s_i(h)} \right) \frac{\sum_{t=1}^{T} v_{it} x'_t \delta_i(h)/\|\delta_i(h)\|}{s_i(h)/(\sigma_i \|\delta_i(h)\|)} \right| \\
\leq \frac{s_i(h)}{\hat{s}_i(h)} \left| 1 - \frac{\hat{s}_i(h)}{s_i(h)} \right| \left| \frac{1}{\sqrt{T}} \sum_{t=1}^{T} v_{it} x_i \right| \left( \frac{s_i(h)}{\sigma_i \|\delta_i(h)\|} \right)^{-1}
\]
and applying (38) to bound $\|T^{-1/2} \sum_{t=1}^{T} v_{it} x_{it}\|$, (43) to bound $\left| 1 - \frac{\hat{s}_i(h)}{s_i(h)} \right|$, (51) for a lower bound on $s_i(h)/(\sigma_i \|\delta_i(h)\|)$ and
\[
\frac{\hat{s}_i(h)}{s_i(h)} \geq 1 - \left| \frac{\hat{s}_i(h)}{s_i(h)} - 1 \right|
\]
in conjunction with (43) to derive a lower bound on $\hat{s}_i(h)/s_i(h)$. To bound $J_2$, we derive a lower bound on its denominator from
\[
\frac{\hat{s}_i(h)}{s_i(h)} = \frac{s_i(h)}{\sigma_i \|\delta_i(h)\|} \left\{ \left( \frac{\hat{s}_i(h)}{s_i(h)} - 1 \right) + 1 \right\}
\]
in conjunction with (43) and (51). The numerator in $J_2$ is bounded by (41). The bounds on $J_1$ and $J_2$ yield (53).

To prove (54), we follow an argument similar that for (53). We observe
\[
\hat{D}_i(h) - \bar{D}_i(h) = \left( \frac{\hat{s}_i(h)}{\bar{s}_i(h)} - 1 \right) D_i(h) + \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \left( \frac{\bar{d}_i(h) - d_i(h)}{\bar{s}_i(h)/(\sigma_i \|\delta_i(h)\|)} \right) = J'_1 + J_2,
\]
where $\bar{s}_i(h)^2 = \sum_{t=1}^{T} (d_i(h) - \bar{d}_i(h))^2/T$. The bound on $J_2$ has been discussed above. We write $J'_1$
\[
\left| \left( \frac{\hat{s}_i(h)}{\bar{s}_i(h)} - 1 \right) \bar{D}_i(h) \right| \leq \left| \frac{\hat{s}_i(h)}{\bar{s}_i(h)} - 1 \right| \left| \frac{1}{\sqrt{T}} \sum_{t=1}^{T} v_{it} x_i \right| \left( \frac{\hat{s}_i(h)}{\sigma_i \|\delta_i(h)\|} \right)^{-1}.
\]
We apply (38) and (44) to bound $\|T^{-1/2} \sum_{t=1}^{T} v_{it} x_{it}\|$ and $|\bar{s}_i(h)/\bar{s}_i(h) - 1|$, respectively. Lastly, we observe that
\[
\frac{\bar{s}_i(h)}{\sigma_i \|\delta_i(h)\|} = \frac{\hat{s}_i(h)}{\sigma_i \|\delta_i(h)\|} \left\{ \left( \frac{\hat{s}_i(h)}{\bar{s}_i(h)} - 1 \right) + 1 \right\}.
\]
We now have a bound on $J'_1$, and obtain (54).
Lemma E.23. Let $\mathbb{P}_N$ denote a family of probability measures satisfying Assumptions I, II, III and IV. For $0 < c \leq 1/2$, assume $r_{N,\theta} = o(1 \wedge \iota_N)$ and $T^{-(1-c)/2}B_{N,8}^4(\log N) = o(1)$. There are constants $C$ and $C'$ such that

\begin{equation}
\sup_{P \in \mathbb{P}_N} P \left( \max_{1 \leq i \leq N} \left| \frac{1}{T} \sum_{t=1}^{T} \frac{\hat{d}_{i,t}^U(h) - \hat{d}_{i,t}^V(h)}{\sigma_i \|\hat{\delta}_i(h)\|} \right| \geq C \frac{r_{N,\theta}}{\iota_N + \min_{1 \leq i \leq N} \sigma_i} \right) \leq N^{-1} + C'T^{-c} + a_{N,\theta}.
\end{equation}

\begin{equation}
\sup_{P \in \mathbb{P}_N} P \left( \max_{1 \leq i \leq N} \left| \hat{D}_{i}^U(g_i^0, h) - \hat{D}_{i}^V(g_i^0, h) \right| \geq C \frac{T^{1/2} \sqrt{N} \sigma_i}{\iota_N + \min_{1 \leq i \leq N} \sigma_i} \left( \sqrt{T} + \log N \right) \right) \leq N^{-1} + C'T^{-c} + a_{N,\theta}.
\end{equation}

Proof. Throughout the proof let $C$, $C'$ and $C''$ denote generic constants that do not depend on $P \in \mathbb{P}$. Let $\delta_i(h) = \theta_{g_i^0} - \theta_h$ and $\hat{\delta}_i(h) = \hat{\theta}_{g_i} - \hat{\theta}_h$. Note that

$$
\|\hat{\delta}_i(h)\|/\|\delta_i(h)\| \leq 1 + \frac{\|\hat{\delta}_i(h) - \delta_i(h)\|}{\|\delta_i(h)\|}
$$

is bounded by the fact that $r_{N,\theta} = o(1 \wedge \iota_N)$.

We observe

$$
\begin{align*}
\frac{1}{2} \left( (y_{it} - w_{it}'\hat{\theta}_w - x_{it}'\hat{\theta}_{g_i^0})^2 - (y_{it} - w_{it}'\hat{\theta}_w - x_{it}'\hat{\theta}_h)^2 \right) \\
&= -u_{it}x_{it}'\hat{\delta}_i(h) - \delta_i(h) + u_{it}'(\hat{\theta}_w - \theta_w)x_{it}'\hat{\delta}_i(h) + x_{it}'\delta_i(h)x_{it}'(\hat{\theta}_{g_i} - \theta_{g_i}) \\
&\quad - \frac{1}{2}(x_{it}'\hat{\delta}_i(h))^2 + \frac{1}{2}(x_{it}'\delta_i(h))^2.
\end{align*}
$$

Thus we have, with probability at least $1 - a_{N,\theta}$,

\begin{equation}
\left| \frac{d_{i,t}^U(h) - d_{i,t}^V(h)}{\sigma_i \|\delta_i(h)\|} \right| \leq C \left( \frac{\|x_{it}\|^2 + \|x_{it}\| \|w_{it}\|}{\min_{1 \leq i \leq N} \sigma_i} + \frac{\|v_{it}\| \|x_{it}\|}{\iota_N} \right) r_{N,\theta},
\end{equation}

\begin{equation}
\left( \frac{d_{i,t}^U(h) - d_{i,t}^V(h)}{\sigma_i \|\delta_i(h)\|} \right)^2 \leq C \left( \frac{\|x_{it}\|^4 + \|x_{it}\|^2 \|w_{it}\|^2}{\min_{1 \leq i \leq N} \sigma_i^2} + \frac{\|v_{it}\|^2 \|x_{it}\|^2}{\iota_N^2} \right) r_{N,\theta}^2,
\end{equation}

and

\begin{equation}
\left| \frac{d_{i,t}^U(h)}{\sigma_i \|\delta_i(h)\|} \left( \frac{d_{i,t}^U(h) - d_{i,t}^V(h)}{\sigma_i \|\delta_i(h)\|} \right) \right| \leq C \left( \frac{\|x_{it}\|^4 + \|v_{it}\| \|x_{it}\|^3 + \|v_{it}\| \|x_{it}\|^2 \|w_{it}\| + \|x_{it}\|^3 \|w_{it}\|}{\min_{1 \leq i \leq N} \sigma_i} \right) r_{N,\theta}.
\end{equation}
where (59) also rely on Assumption 9.

Combining (58), (21), Assumption 3 and the fact that we can take

$$T^{-1/4} B_{N,8}^2 \sqrt{\log N} \leq 1$$

for $N, T$ large enough yields (55).

Let

$$\hat{s}_i^U(h)^2 = \frac{1}{T} \sum_{t=1}^{T} \left( \hat{d}_i^U(h) - \hat{d}_i^U(h) \right)^2, \quad s_i^U(h)^2 = \frac{1}{T} \sum_{t=1}^{T} \left( d_i^U(h) - d_i^U(h) \right)^2$$

We observe that

$$\frac{\hat{s}_i^U(h)^2 - \tilde{s}_i^U(h)^2}{\sigma^2_i ||\hat{\delta}_i(h)||^2} = \frac{1}{T} \sum_{t=1}^{T} \left( \frac{\hat{d}_i^U(h) - d_i^U(h)}{\sigma_i ||\hat{\delta}_i(h)||} \right)^2 + 2 \frac{1}{T} \sum_{t=1}^{T} \frac{d_i^U(h)}{\sigma_i ||\hat{\delta}_i(h)||} \frac{(\hat{d}_i^U(h) - d_i^U(h))}{\sigma_i ||\hat{\delta}_i(h)||}$$

$$\quad - T \left( \frac{1}{T} \sum_{t=1}^{T} \frac{\hat{d}_i^U(h) - d_i^U(h)}{\sigma_i ||\hat{\delta}_i(h)||} \right) \left( \frac{1}{T} \sum_{t=1}^{T} \frac{d_i^U(h)}{\sigma_i ||\hat{\delta}_i(h)||} + \frac{2}{T} \sum_{t=1}^{T} \frac{d_i^U(h)}{\sigma_i ||\hat{\delta}_i(h)||} \right).$$

Note that

$$\left| \frac{1}{T} \sum_{t=1}^{T} \frac{d_i^U(h)}{\sigma_i ||\hat{\delta}_i(h)||} \right| \leq \left\| \frac{1}{T} \sum_{t=1}^{T} v_{it} x_{it} \right\| + \|\delta_i(h)|| \frac{1}{T} \sum_{t=1}^{T} \| x_{it} \|^2$$

By Assumptions 3 and 9 and (19), we have

$$\sup_{P \in P} \left( \max_{1 \leq i \leq N} \left| \frac{1}{T} \sum_{t=1}^{T} \frac{d_i^U(h)}{\sigma_i ||\hat{\delta}_i(h)||} \right| \geq C(1 + T^{-1/2} \sqrt{\log N}) \right) \leq N^{-1} + CT^{-c}. \quad (60)$$

Combining (20), (21), (57), (58), (59), and (60) yields

$$\sup_{P \in P} \left( \max_{1 \leq i \leq N} \left| \frac{\hat{s}_i^U(h)^2 - \tilde{s}_i^U(h)^2}{\sigma^2_i ||\hat{\delta}_i(h)||^2} \right| \geq C \left( \frac{r_{N,\theta}}{\ell_N \wedge \min_{1 \leq i \leq N} \sigma_i} \right) \right) \leq N^{-1} + CT^{-c} + a_{N,\theta}. \quad (61)$$

Let $s_i^U(h)^2 = \sum_{t=1}^{T} Var(d_i^U(h))/T$. Observing that $s_i^U(h) > s_i(h)$, $\sigma_i ||\delta_i(h)||/s_i^U(h)$ is bounded away from infinity by (27) under Assumption 113 This in turn implies that $\sigma_i ||\delta_i(h)||/s_i^U(h)$ is bounded away from infinity. We thus have

$$\sup_{P \in P} \left( \max_{1 \leq i \leq N} \left| \frac{s_i^U(h)}{\hat{s}_i^U(h)} - 1 \right| \geq C \left( \frac{r_{N,\theta}}{\ell_N \wedge \min_{1 \leq i \leq N} \sigma_i} \right) \right) \leq N^{-1} + CT^{-c} + a_{N,\theta}. \quad (62)$$
Lastly, we consider

\[
\hat{D}_i^U(h) - \tilde{D}_i^U(h) = \left( \frac{s_i^U(h)}{\hat{s}_i^U(h)} - 1 \right) D_i(h) + \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \left( \frac{d_{it}^U(h)}{s_i^U(h)} - d_{it}^W(h) \right) / (\sigma_i \| \delta_i(h) \|) = J_1^U + J_2^U.
\]

We bound \( J_1 \) by writing

\[
\left| \left( \frac{s_i^U(h)}{\hat{s}_i^U(h)} - 1 \right) \right| D_i(h) = \left| \frac{s_i^U(h)}{\hat{s}_i^U(h)} \right| \left( 1 - \frac{s_i^U(h)}{\hat{s}_i^U(h)} \right) \left| \frac{1}{\frac{\sum_{t=1}^{T} d_{it}^U(h)}{s_i^U(h)} / (\sigma_i \| \delta_i(h) \|)} \right|
\]

\[
\leq \frac{s_i^U(h)}{\hat{s}_i^U(h)} \left( 1 - \frac{s_i^U(h)}{\hat{s}_i^U(h)} \right) \left| \frac{1}{\sqrt{T}} \sum_{t=1}^{T} v_{it} x_i \right| + \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \| x_{it} \|^2 / (\sigma_i \| \delta_i(h) \|) \left( \frac{s_i^U(h)}{\sigma_i \| \delta_i(h) \|} \right)^{-1}
\]

and applying (19) to bound \( \| T^{-1/2} \sum_{t=1}^{T} v_{it} x_i \| \), (62) to bound \( | 1 - \hat{s}_i(h) / \hat{s}_i(h) | \), the discussion above (62) for a lower bound on \( s_i^U(h) / (\sigma_i \| \delta_i(h) \|) \) and

\[
\frac{s_i^U(h)}{\hat{s}_i^U(h)} \geq 1 - \frac{s_i^U(h)}{\hat{s}_i^U(h)} - 1
\]

in conjunction with (62) to derive a lower bound on \( s_i^U(h) / \hat{s}_i^U(h) \). To bound \( J_2^U \), we derive a lower bound on its denominator from

\[
\frac{s_i(h)}{\sigma_i \| \delta_i(h) \|} = \frac{\hat{s}_i(h)}{\sigma_i \| \delta_i(h) \|} \left\{ \left( \frac{s_i(h)}{\hat{s}_i(h)} - 1 \right) + 1 \right\}
\]

in conjunction with (26) and (27). The numerator in \( J_2^U \) is bounded by (55). The bounds on \( J_1^U \) and \( J_2^U \) yield (24).

\[\square\]

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