HOMOLOGY OF SPACES OF NON-RESULTANT POLYNOMIAL SYSTEMS IN $\mathbb{R}^2$ AND $\mathbb{C}^2$

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Abstract. The resultant variety in the space of systems of homogeneous polynomials of given degrees consists of such systems having non-trivial solutions. We calculate the integer cohomology groups of all spaces of non-resultant systems of polynomials $\mathbb{R}^2 \to \mathbb{R}$, and also the rational cohomology groups of similar systems in $\mathbb{C}^2$.

1. Introduction

Given $n$ natural numbers $d_1 \geq d_2 \geq \cdots \geq d_n$, consider the space of all real homogeneous polynomial systems

\[
\begin{aligned}
a_{1,0}x^{d_1} + a_{1,1}x^{d_1-1}y + \cdots + a_{1,d_1}y^{d_1} \\
\vdots \\
a_{n,0}x^{d_n} + a_{n,1}x^{d_n-1}y + \cdots + a_{n,d_n}y^{d_n}
\end{aligned}
\]

in two real variables $x, y$.

We will refer to this space as $\mathbb{R}^D$, $D = \sum_1^n (d_i + 1)$. The resultant variety $\Sigma \subset \mathbb{R}^D$ is the space of all systems having non-zero solutions. $\Sigma$ is a semialgebraic subvariety of codimension $n - 1$ in $\mathbb{R}^D$. Below we calculate the cohomology group of its complement, $H^*(\mathbb{R}^D \setminus \Sigma)$.

For the “affine” version of this problem (concerning the space of non-resultant systems of polynomials $\mathbb{R}^1 \to \mathbb{R}^1$ with leading terms $x^{d_i}$) see e.g. [5], [6], [4]. A similar calculation for spaces of real homogeneous polynomials in $\mathbb{R}^2$ without zeros of multiplicity $\geq m$ was done in [7].

Also, we calculate below the rational cohomology groups of the complex analogs $\mathbb{C}^D \setminus \Sigma_C$ of all spaces $\mathbb{R}^D \setminus \Sigma$.

2. Main results

2.1. Notation. For any $p = 1, 2, \ldots, d_1$ denote by $N(p)$ the sum of all numbers $d_i + 1$, $i = 1, \ldots, n$, which are smaller than or equal to $p$.

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plus $p$ times the number of those $d_i$ which are equal or greater than $p$. (In other words, $N(p)$ is the area of the part of Young diagram $(d_1 + 1, \ldots, d_n + 1)$ strictly to the left from the $(p + 1)$th column.) Let the index $\Upsilon(p)$ be equal to the number of even elements $d_i \geq p$ if $p$ is even, and to the number of odd elements $d_i \geq p$ if $p$ is odd. By $\widetilde{H}^*(X)$ we denote the cohomology group reduced modulo a point. $\overline{H}_*(X)$ denotes the Borel–Moore homology group, i.e. the homology group of the one-point compactification of $X$ reduced modulo the added point.

**Theorem 1.** If the space $\mathbb{R}^D \setminus \Sigma$ is non-empty (i.e. either $n > 1$ or $d_1$ is even) then the group $\widetilde{H}^*(\mathbb{R}^D \setminus \Sigma, \mathbb{Z})$ is equal to the direct sum of following groups:

A) For any $p = 1, \ldots, d_3$,

- if $\Upsilon(p)$ is even, then $\mathbb{Z}$ in dimension $N(p) - 2p$ and $\mathbb{Z}$ in dimension $N(p) - 2p + 1$,
- if $\Upsilon(p)$ is odd, then only one group $\mathbb{Z}_2$ in dimension $N(p) - 2p + 1$;

B) If $d_1 - d_2$ is odd then only one additional summand $\mathbb{Z}$ in dimension $D - d_1 - d_2 - 2$. If $d_1 - d_2$ is even, then additional summand $\mathbb{Z}^{d_2 - d_3 + 1}$ in dimension $D - d_1 - d_2 - 1$ and (if $d_2 \neq d_3$) summand $\mathbb{Z}^{d_2 - d_3}$ in dimension $D - d_1 - d_2 - 2$.

**Example 1.** Let $n = 2$. If $d_1$ and $d_2$ are of the same parity, then $\mathbb{R}^D \setminus \Sigma$ consists of $d_2 + 1$ connected components, each of which has the homology of a circle. For the invariant separating systems from different components we can take the index of the induced map of the unit circle $S^1 \subset \mathbb{R}^2$ into $\mathbb{R}^3 \setminus 0$. This index can take all values of the same parity as $d_2$ from the segment $[-d_2, d_2]$. The 1-dimensional cohomology class inside any component is just the rotation index of the image of a fixed point (say, $(1, 0)$) around the origin.

If $d_1$ and $d_2$ are of different parities, then the space $\mathbb{R}^D \setminus \Sigma$ has the homology of a two-point set. The index separating its two connected components can be calculated as the parity of the number of zeros of the odd-degree polynomial of our non-resultant system, which lie in the (well-defined) domain in $\mathbb{R}P^1$ where the even-degree polynomial is positive.

Now, let $\mathbb{C}^D$ be the space of all polynomial systems \{1\} with complex coefficients $a_{i,j}$, and $\Sigma_\mathbb{C} \subset \mathbb{C}^D$ the set of systems having solutions in $\mathbb{C}^2 \setminus 0$.

**Theorem 2.** For any $n > 1$ the group $H^*(\mathbb{C}^D \setminus \Sigma_\mathbb{C}, \mathbb{Q})$ is isomorphic to $\mathbb{Q}$ in dimensions $0, 2n - 3, 2n - 1$ and $4n - 4$, and is trivial in all other dimensions.
Consider also the space $\mathbb{C}^{d+1}$ of all complex homogeneous polynomials

$$a_0 x^d + a_1 x^{d-1} y + \cdots + a_d y^d$$

and $m$-discriminant $\Sigma_m$ in it, consisting of all polynomials vanishing on some line with multiplicity $\geq m$.

**Theorem 3.** For any $m > 1$ and $d \geq 2m$, the group $H^*(\mathbb{C}^{d+1} \setminus \Sigma_m, \mathbb{Q})$ is isomorphic to $\mathbb{Q}$ in dimensions $0, 2m - 3, 2m - 1$ and $4m - 4$, and is trivial in all other dimensions. For any $m > 1$ and $d \in [m, 2m - 1]$ this group is isomorphic to $\mathbb{Q}$ in dimensions $0, 2m - 3, 2m - 1$ and $2d - 2$, and is trivial in all other dimensions.

3. **Some preliminary facts**

Denote by $B(M, p)$ the configuration space of subsets of cardinality $p$ in the topological space $M$.

**Lemma 1.** For any natural $p$, there is a locally trivial fibre bundle $B(S^1, p) \to S^1$, whose fiber is homeomorphic to $\mathbb{R}^{p-1}$. This fibre bundle is non-orientable if $p$ is even, and is orientable (and hence trivial) if $p$ is odd. □

Indeed, the projection of this fibre bundle can be realised as the product of $p$ points of the unit circle in $\mathbb{C}^1$. The fibre of this bundle can be identified in the terms of the universal covering $\mathbb{R}^p \to T^p$ with any connected component of a hyperplane $\{x_1 + \cdots + x_p = \text{const}\}$ from which all affine planes given by $x_i = x_j + 2\pi k$, $i \neq j$, $k \in \mathbb{Z}$, are removed. Such a component is convex and hence diffeomorphic to $\mathbb{R}^{p-1}$. The assertion on orientability can be checked immediately. □

Let us embed a manifold $M$ generically into the space $\mathbb{R}^T$ of a very large dimension, and denote by $M^r$ the union of all $(r-1)$-dimensional simplices in $\mathbb{R}^T$, whose vertices lie in this embedded manifold (and the “genericity” of the embedding means that if two such simplices have a common point in $\mathbb{R}^T$, then their minimal faces, containing this point, coincide).

**Proposition 1** (C. Caratheodory theorem, see also [6]). For any $r \geq 1$, the space $(S^1)^r$ is homeomorphic to $S^{2r-1}$. □

**Remark 1.** This homeomorphism can be realized as follows. Consider the space $\mathbb{R}^{2r+1}$ of all real homogeneous polynomials $\mathbb{R}^2 \to \mathbb{R}^1$ of degree $2r$, the convex cone in this space consisting of everywhere non-negative polynomials, and (also convex) dual cone in the dual space $\mathbb{R}^{2r+1}$ consisting of linear forms taking only positive values inside the previous
cone. The intersection of the boundary of this dual cone with the unit sphere in $\mathbb{R}^{2r+1}$ is naturally homeomorphic to $(S^1)^r$; on the other hand it is homeomorphic to the boundary of a convex 2r-dimensional domain.

**Lemma 2** (see [8], Lemma 3). For any $r > 1$ the group $H_*(((S^2)^r, \mathbb{C})$ is trivial in all positive dimensions. □

Consider the “sign local system” $\pm \mathbb{Q}$ over $B(\mathbb{C}P^1, p)$, i.e. the local system of groups with fiber $\mathbb{Q}$, such that the elements of $\pi_1(B(\mathbb{C}P^1, p))$ defining odd (respectively, even) permutations of $p$ points in $\mathbb{C}P^1$ act in the fiber as multiplication by $-1$ (respectively, 1).

**Lemma 3** (see [8], Lemma 2). All groups $H_i(B(\mathbb{C}P^1, p; \pm \mathbb{Q})$ with $p \geq 1$ are trivial except only for $H_0(B(\mathbb{C}P^1, 1, \pm \mathbb{Q}) \sim H_2(B(\mathbb{C}P^1, 1), \pm \mathbb{Q}) \sim H_2(B(\mathbb{C}P^1, 2), \pm \mathbb{Q}) \sim \mathbb{Q}$. □

4. Proof of Theorem [1]

Following [1], we use the Alexander duality

\[ \tilde{H}'(\mathbb{R}^D \setminus \Sigma) \simeq \tilde{H}_{D-i-1}(\Sigma), \]

where $\tilde{H}_*$ denotes the Borel—Moore homology.

4.1. Simplicial resolution of $\Sigma$. To calculate the right-hand group in (2) we construct a resolution of the space $\Sigma$. Let $\chi : \mathbb{R}P^1 \to \mathbb{R}^T$ be a generic embedding, $T >> n$. For any system $\Phi = (f_1, \ldots, f_n) \in \Sigma$, not equal identically to zero, consider the simplex $\Delta(\Phi)$ in $\mathbb{R}^T$, spanned by the images $\chi(x_i)$ of all points $x_i \in \mathbb{R}P^1$, corresponding to all possible lines, on which the system $f$ has a common root. (The maximal possible number of such lines is obviously equal to $d_1$.)

Further consider a subset in the direct product $\mathbb{R}^D \times \mathbb{R}^T$, namely the union of all simplices of the form $\Phi \times \Delta(\Phi)$, $\Phi \in \Sigma \setminus 0$. This union is not closed: the set of its limit points, not belonging to it, is the product of the point $0 \in \mathbb{R}^D$ (corresponding to the zero system) and the union of all simplices in $\mathbb{R}^T$, spanned by the images of no more than $d_1$ different points of the line $\mathbb{R}P^1$. By the Caratheodory theorem, the latter union is homeomorphic to the sphere $S^{2d_1-1}$. We can assume that our embedding $\chi : \mathbb{R}P^1 \to \mathbb{R}^T$ is algebraic, and hence this sphere is semialgebraic. Take a generic $2d_1$-dimensional semialgebraic disc in $\mathbb{R}^T$ with boundary at this sphere (e.g., the union of segments connecting the points of this sphere with a generic point in $\mathbb{R}^T$) and add the product of the point $0 \in \mathbb{R}^D$ and this disc to the previous union of simplices in $\mathbb{R}^D \times \mathbb{R}^T$. The resulting set will be denoted by $\sigma$ and called a simplicial resolution of $\Sigma$. 
Lemma 4. The obvious projection $\sigma \to \Sigma$ (induced by the projection of $\mathbb{R}^D \times \mathbb{R}^T$ onto the first factor) is proper, and the induced map of one-point compactifications of these spaces is a homotopy equivalence.

This follows easily from the fact that this projection is a stratified map of semialgebraic spaces, and the preimage of any point $\bar{\Sigma}$ is contractible, cf. [5], [6]. □

So, we can (and will) calculate the group $\overline{H}_*(\sigma)$ instead of $\overline{H}_*(\Sigma)$.

Remark 2. There is a more canonical construction of a simplicial resolution of $\Sigma$ in the terms of “Hilbert schemes”. Namely, let $I_p$ be the space of all ideals of codimension $p$ in the space of smooth functions $\mathbb{R}P^1 \to \mathbb{R}^1$, supplied with the natural “Grassmannian” topology. It is easy to see that $I_p$ is homeomorphic to the $p$th symmetric power $S^p(\mathbb{R}P^1) = (\mathbb{R}P^1)^p/S(p)$, in particular it contains the configuration space $B(\mathbb{R}P^1, p)$ as an open dense subset. Consider the disjoint union of these $d_1$ spaces $I_1, \ldots, I_{d_1}$, augmented with the one-point set $I_0$ symbolizing the zero ideal. The incidence of ideals makes this union a partially ordered set. Consider the continuous order complex $\Xi_{d_1}$ of this poset, i.e. the subset in the join $I_1 * \cdots * I_{d_1} * I_0$ consisting of those simplices, all whose vertices are incident to one another. For any polynomial system $\Phi = (f_1, \ldots, f_n) \in \mathbb{R}^D$ denote by $\Xi(\Phi)$ the subcomplex in $\Xi_{d_1}$ consisting of all simplices, all whose vertices correspond to ideals containing all polynomials $f_1, \ldots, f_n$. The simplicial resolution $\tilde{\sigma} \subset \Sigma \times \Xi_{d_1}$ is defined as the union of simplices $\Phi \times \Xi(\Phi)$ over all $\Phi \in \Sigma$.

This construction is homotopy equivalent to the previous one. In particular, the Caratheodory theorem has the following version: the continuous order complex of the poset of all ideals of codimension $\leq r$ in the space of functions $S^1 \to \mathbb{R}^1$ is homotopy equivalent to $S^{2r-1}$.

However, this construction is less convenient for our practical calculations.

The space $\sigma$ has a natural increasing filtration $F_1 \subset \cdots \subset F_{d_1+1} \equiv \sigma$: its term $F_p$, $p \leq d_1$, is the closure of the union of all simplices of the form $\Phi \times \Delta(\Phi)$ over all polynomial systems $\Phi$ having no more than $p$ lines of common zeros.

Lemma 5. For any $p = 1, \ldots, d_1$, the term $F_p \setminus F_{p-1}$ of our filtration is the space of a locally trivial fiber bundle over the configuration space $B(\mathbb{R}P^1, p)$, with fibers equal to the direct product of an $(p-1)$-dimensional open simplex and an $(D - N(p))$-dimensional real space. The corresponding bundle of open simplices is orientable if and only if
Fig. 1. $E^1$ for $n = 1$, $d_1$ even and $n = 1$, $d_1$ odd

$p$ is odd (i.e. exactly when the base configuration space is orientable), and the bundle of $(D - N(p))$-dimensional spaces is orientable if and only if the index $\Upsilon(p)$ is even.

The last term $F_{d_1+1} \setminus F_{d_1}$ of this filtration is homeomorphic to an open $2d_1$-dimensional disc.

Indeed, to any configuration $(x_1, \ldots, x_p) \in B(\mathbb{R}P^1, p)$, $p \leq d_1$, there corresponds the direct product of the interior part of the simplex in $\mathbb{R}^T$, spanned by the images $\chi(x_i)$ of points of this configuration, and the subspace in $\mathbb{R}^D$, consisting of polynomial systems, having solutions on corresponding $p$ lines in $\mathbb{R}^2$. The codimension of the latter subspace is equal exactly to $N(p)$. The assertion concerning the orientations can be checked elementary. The description of $F_{d_1+1} \setminus F_{d_1}$ follows immediately from the construction. $\square$

Consider the spectral sequence $E^r_{p,q}$, calculating the group $\overline{H}_*(\Sigma)$ and generated by this filtration. Its term $E^1_{p,q}$ is canonically isomorphic to the group $\overline{H}_{p+q}(F_p \setminus F_{p-1})$. By Lemma 6 its column $E^1_{p,*}$, $p \leq d_1$, is as follows. If $\Upsilon(p)$ is even, then it contains exactly two non-trivial terms $E^1_{p,q}$, both isomorphic to $\mathbb{Z}$, for $q$ equal to $D - N(p) + p - 1$ and $D - N(p) + p - 2$. If $\Upsilon(p)$ is odd, then it contains only one non-trivial term $E^1_{p,q}$, isomorphic to $\mathbb{Z}_2$, for $q = D - N(p) + p - 2$. Finally, the column $E^1_{d_1+1,*}$ contains only one non-trivial element $E^1_{d_1+1,d_1-1} \sim \mathbb{Z}$.

Before calculating the differentials and further terms $E^r$, $r > 1$, let us consider several basic examples.

4.2. The case $n = 1$. If our system consists of only one polynomial of degree $d_1$, then the term $E^1$ of our spectral sequence looks as in Fig. 1, in particular all non-trivial groups $E^1_{p,q}$ lie in two rows $q = d_1$ and $d_1 - 1$.

Lemma 6. If $n = 1$ then in both cases of even or odd $d_1$, all possible horizontal differentials $\partial_1 : E^1_{p,d_1-1} \to E^1_{p-1,d_1-1}$ of the form $\mathbb{Z} \to \mathbb{Z}_2$,
Fig. 2. $E^1$ for $n = 2$, $d_1 - d_2$ odd and $n = 2$, $d_1 - d_2$ even

$p = d_1 + 1, d_1 - 1, d_1 - 3, \ldots$ are epimorphic, and all differentials $\partial_2 : E^2_{p,d_1-1} \to E^2_{p-2,d_1}$ of the form $\mathbb{Z} \to \mathbb{Z}$, $p = d_1 + 1, d_1 - 1, d_1 - 3, \ldots$ are isomorphisms. In particular, the unique surviving term $E^3_{p,q}$ for the “even” spectral sequence is $E^3_{1,d_1-1} \sim \mathbb{Z}$, and for the “odd” one it is $E^3_{2,d_1-1} \sim \mathbb{Z}$.

Indeed, in both cases we know the answer. In the “odd” case the discriminant coincides with entire $\mathbb{R}^D = \mathbb{R}^{d_1+1}$. In the “even” one its complement consists of two contractible components, so that $\overline{H}_*(\Sigma) = \mathbb{Z}$ in dimension $d_1$ and is trivial in all other dimensions. Therefore all terms $E_{p,q}$ with $p + q$ not equal to $d_1 + 1$ (respectively, $d_1$) in the odd (even) case should die on some step. $\square$

4.3. The case $n = 2$. There are two very different situations depending on the parity of $d_1 - d_2$. In Fig. we demonstrate these situations in two particular cases, $(d_1, d_2) = (6, 3)$ and $(7, 3)$. However, the general situation is essentially the same, namely the following is true.

If $n = 2$ and $d_1 - d_2$ is odd, then all indices $\Upsilon(p)$, $p = 1, \ldots, d_2 + 1$, are odd, and hence all non-trivial groups $E^1_{p,q}$ with such $p$ lie on the line $\{ p + q = d_1 + d_2 \}$ only and are equal to $\mathbb{Z}_2$.

If $n = 2$ and $d_1 - d_2$ is even, then all indices $\Upsilon(p)$, $p = 1, \ldots, d_2 + 1$, are even, and hence all non-trivial groups $E^1_{p,q}$ with such $p$ lie on two lines $\{ p + q = d_1 + d_2 \}$, $\{ p + q = d_1 + d_2 + 1 \}$, and are all equal to $\mathbb{Z}$.

In both cases, all groups $E^k_{p,q}$ with $p > d_2$ are the same as in the case $n = 1$ with the same $d_1$. Moreover, the differentials $\partial_1$ and $\partial_2$ between these groups also are the same as for $n = 1$, therefore all of
these groups die at the instant $E^3$ except for $E^3_{d_2+1,d_1-1} \sim \mathbb{Z}$ for even $d_1 - d_2$, and $E^3_{d_2+2,d_1-1} \sim \mathbb{Z}$ for odd $d_1 - d_2$.

In the case of even $d_1 - d_2$ all other differentials between the groups $E^r_{p,q}$ are trivial, because otherwise the group $\overline{H}^0(\mathbb{R}^d \setminus \Sigma)$ would be smaller than $\mathbb{Z}^{d_2}$, in contradiction to $d_2 + 1$ different components of this space indicated in Example 1.

On contrary, if $d_1 - d_2$ is odd, then all differentials $d^r_r : E^r_{d_2+2,d_1-1} \to E^r_{d_2+2-r,d_1-2+r}$, $r = 1, \ldots, d_1 - d_2 + 1$ are epimorphic just because the integer cohomology group of the topological space $\mathbb{R}^D \setminus \Sigma$ cannot have non-trivial torsion subgroup in dimension 1. Therefore the unique non-trivial group $E^\infty_{p,q}$ in this case is $E^\infty_{d_2+2,d_1-1} \sim \mathbb{Z}$.

These considerations prove Main Theorem in the case $n = 2$.

4.4. The general case. Now suppose that our systems (1) consist of $n \geq 3$ polynomials. Let again $\sigma$ be the simplicial resolution of the corresponding resultant variety, constructed in §4.1 and $\sigma'$ be the simplicial resolution of the resultant variety for $n = 2$ and the same $d_1$ and $d_2$. The parts $\sigma \setminus F_{d_1}(\sigma)$ and $\sigma' \setminus F_{d_1}(\sigma')$ of these resolutions are canonically homeomorphic to one another as filtered spaces. In particular, $E^1_{p,q}(\sigma) = E^1_{p,q}(\sigma')$ if $p > d_3$, and $E^p_{p,q}(\sigma) = E^p_{p,q}(\sigma')$ if $p \geq d_3 + r$.

All non-trivial terms $E^p_{p,q}(\sigma)$ with $p \leq d_3$ are placed in such a way, that no non-trivial differentials $\partial_r$ can act between these terms, as well as no differentials can act to these terms from the cells $E^p_{p,q}$ with $p > d_3$, which have survived the differentials between these cells, described in the previous subsection.

Therefore the final term $E^\infty_{p,q}(\sigma)$ coincides with $E^1_{p,q}(\sigma)$ in the domain $\{p \leq d_3\}$, and coincides with the term $E^\infty_{p,q}(\sigma')$ of the truncated spectral sequence calculating the Borel-Moore homology of $\sigma' \setminus F_{d_1}(\sigma')$ in the domain $\{p > d_3\}$. This terminates the proof of Theorem 1.

5. Proof of Theorems 2, 3

The simplicial resolution $\sigma_\mathbb{C}$ of $\Sigma_\mathbb{C}$ appears in the same way as its real analog $\sigma$ in the previous section. It also has a natural filtration $F_1 \subset \cdots \subset F_{d_1+1} \equiv \sigma_\mathbb{C}$. For $p \in [1, d_1]$ its term $F_p \setminus F_{p-1}$ is fibered over the configuration space $B(\mathbb{C}P^1, p)$; its fiber over a configuration $(x_1, \ldots, x_p)$ is equal to the product of the space $\mathbb{C}^{D-N(p)}$ (consisting of all complex systems (1) vanishing at all lines corresponding to the points of this configuration) and the $(p-1)$-dimensional simplex, whose vertices correspond to the points of the configuration. In particular, our spectral sequence calculating rational Borel-Moore homology of $\sigma_\mathbb{C}$ has $E^1_{p,q} \sim \overline{H}^1_q(\mathbb{C}P^1; p; \pm \mathbb{Q})$ for such $p$. By Lemma
only the following such groups are non-trivial: $E_{1,2(D-n)-1}^1 \sim \mathbb{Q}$, $E_{1,2(D-n)+1}^1 \sim \mathbb{Q}$, and (if $d_1 > 1$) $E_{2,2(D-2n)+1}^1$.

The last term $F_{d_1+1} \setminus F_{d_1}$ is homeomorphic to the cone over the $d_1$-th self-join $(\mathbb{C}P^1)^{\ast d_1}$ with the base of this cone removed (as it belongs to $F_{d_1}$). Therefore by Lemma 2 the column $E_{d_1+1, \ast}^1$ is trivial if $d_1 > 1$ and contains unique non-trivial group $E_{2,1}^1 \sim \mathbb{Q}$ if $d_1 = 1$.

So, in any case the first leaf $E^1$ of our spectral sequence has only three non-trivial terms $E_{1,2(D-n)-1}^1 \sim \mathbb{Q}$, $E_{1,2(D-n)+1}^1 \sim \mathbb{Q}$, and $E_{2,2(D-2n)+1}^1$.

The differentials in it are obviously trivial, therefore the group $\overline{H}_s(\sigma)$ has three non-trivial terms in dimensions $2(D - n)$, $2(D - n) + 2$, and $2(D - 2n) + 3$. By Alexander duality in the space $\mathbb{C}D$ this gives us three groups $\overline{H}^{2n-3} \sim \mathbb{Q}$, $\overline{H}^{2n-1} \sim \mathbb{Q}$ and $\overline{H}^{4n-4} \sim \mathbb{Q}$ and zero in all other dimensions. □

Theorem 3 can be proved in exactly the same way. □

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