The structure of projective indecomposable modules for \( A_n, n \leq 12 \)

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Abstract

This article gives the structure of all projective indecomposable modules in blocks with non-cyclic defect group for \( n \leq 11 \), and almost all in the case of \( n = 12 \), leaving four simple modules for \( p = 2 \) and one simple module for \( p = 3 \).

In the representation theory of the symmetric and alternating groups, blocks with abelian defect group are, in some sense, well understood. The decomposition numbers are not known in general, the \( \text{Ext}^1 \) matrices are not known in general, the structure of the projectives are not known in general, but things are better than when the defect group is non-abelian. A lack of explicit examples is of no help, of course, and the purpose of this short paper is to provide researchers with a few more examples of \( \text{Ext}^1 \)-matrices and the socle structures of the projective indecomposable modules for \( A_n \) for \( n \leq 12 \). This bound is largely imposed on the author by the fact that \( A_{12} \) is already difficult to do via computer without advanced methods. Indeed, it is not the construction of the PIMs but analysing their socle structure that is the real issue, taking many times longer than their construction.

Although we are mainly interested in non-abelian defect groups – so \( p = 2, 3 \) – we include the \( \text{Ext}^1 \)-matrices and PIM structures for \( p = 5 \) as well, since they might be of use to people. For \( p \geq 7 \) the defect group is cyclic and the structure of all blocks are known by the theory of Brauer trees, as is the case for \( p = 5 \) and \( n \leq 9 \). We omit the description of all projectives that lie in blocks of defect 0 or 1.

If we look through the literature, we can find the following cases where the projective indecomposable modules are already described. There might well be others, but I cannot find them at the moment.

| \( n \) | \( A_n, S_n, p = 2 \) | \( A_n, p = 3 \) | \( S_n, p = 3 \) |
|---|---|---|---|
| 7 | Erdmann [3] | Scopes [4] | Scopes [5] |
| 8 | Benson [1] | Scopes [4] | Scopes [5] |
| 9 | Benson [2] | Siegel [6] | Tan [8] |
| 10 | All here | Tan [8] | Tan [8] |
| 11 | All here | Three modules by Tan [7], rest here | Six modules by Tan [7], rest here |
| 12 | All but four here | All but one here | This paper |

Notice that for \( p = 2 \) we simply induce from \( A_n \) to \( S_n \), replacing a module \( M \) in the diagram for each projective by the self-extension of \( M \). (This also allows us to go backwards, of course.) For \( S_{10} \) and \( p = 3 \), the projectives are described in [8], and those for \( A_{10} \) can be obtained from them by restriction, although we reproduce them here for ease of reference. For \( p = 5 \) the blocks have weight 2, and are completely understood by work of Scopes [5]. We give the projectives here just to make life easier for people who want them.
1 \textit{A}_{10} \text{ and } \textit{S}_{10}, \text{ characteristic 2}

We firstly compute the projective indecomposable modules for the alternating group \textit{A}_{10}, and then induce them to \textit{S}_{10} to produce the projectives there.

There are seven simple modules in the principal block of \textit{S}_{10} (and \textit{A}_{10}) and they are below.

| \( \lambda \) | \( \dim(D^\lambda) \) | Factors of \( S^\lambda \) |
|--------------|-----------------|------------------|
| (10)         | 1               | 1                |
| (9,1)        | 8               | 8,1              |
| (6,4)        | 16              | 16, 48, 26       |
| (8,2)        | 26              | 26, 8, 1         |
| (7,3)        | 48              | 48, 26, 1        |
| (6,3,1)      | 198             | 198, 16, 48, 1, 26
| (5,3,2)      | 200             | 200, 198, 16, 26, 8, 1

The non-principal block that does not have defect zero has two characters, \( S^{(7,3,1)} = D^{(7,2,1)} \) of degree 160 and \( D^{(5,4,1)} \) of degree 128.

Here are the projectives for the alternating group \textit{A}_{10}, which we can then induce to \textit{S}_{10}. For completeness we list them as well.
Here are the symmetric group ones.
|   | 1  | 8  | 26 | 200 |
|---|---|---|---|----|
|   | 1  | 1  | 8  | 26 |
|   | 1  | 1  | 1  | 8  |
|   | 1  | 1  | 8  | 16 |
|   | 1  | 1  | 8  | 26 |
|   | 1  | 1  | 1  | 8  |
|   | 1  | 1  | 8  | 16 |
|   | 1  | 1  | 8  | 26 |
|   | 1  | 1  | 1  | 8  |
|   | 1  | 1  | 8  | 16 |
|   | 1  | 1  | 8  | 26 |
|   | 1  | 1  | 1  | 8  |
|   | 1  | 1  | 8  | 16 |
|   | 1  | 1  | 8  | 26 |
|   | 1  | 1  | 1  | 8  |
|   | 1  | 1  | 8  | 16 |
|   | 1  | 1  | 8  | 26 |
|   | 1  | 1  | 1  | 8  |
|   | 1  | 1  | 8  | 16 |
|   | 1  | 1  | 8  | 26 |
|   | 1  | 1  | 1  | 8  |
|   | 1  | 1  | 8  | 16 |
|   | 1  | 1  | 8  | 26 |
|   | 1  | 1  | 1  | 8  |
|   | 1  | 1  | 8  | 16 |
|   | 1  | 1  | 8  | 26 |
|   | 1  | 1  | 1  | 8  |
|   | 1  | 1  | 8  | 16 |
|   | 1  | 1  | 8  | 26 |
|   | 1  | 1  | 1  | 8  |
|   | 1  | 1  | 8  | 16 |
|   | 1  | 1  | 8  | 26 | 200 |
|   | 1  | 1  | 8  | 16 |
|   | 1  | 1  | 8  | 26 |
|   | 1  | 1  | 8  |
|   | 1  | 8  | 16 |
|   | 8  | 16 | 200 |
|   |     |     |     |     |
This is the largest symmetric group where we are able to write the projectives like this, since the projective
cover of the trivial module is getting too wide for the page; thus we will just write down the projectives for the alternating groups.

## 2 $A_{11}$ and $S_{11}$, characteristic 2

We will only give here the projectives for the principal block of the alternating group, leaving as an exercise placing another copy of a number directly above all numbers to generate the projectives for the symmetric group.

There are seven simple modules in the principal block of $S_{11}$ (and $A_{11}$) and they are below.

\[
\begin{array}{cccc}
\lambda & \text{dim}(D^\lambda) & \text{Factors of } S^\lambda \\
(11) & 1 & 1 \\
(9,2) & 44 & 44 \\
(6,4,1) & 144 & 144, 164, 186, 198, 1 \\
(7,4) & 164 & 164, 1 \\
(8,2,1) & 186 & 186, 44, 1 \\
(7,3,1) & 198 & 198, 186, 164, 1^2 \\
(5,4,2) & 416 & 416, 144, 186, 198, 44, 1^2 \\
\end{array}
\]

The non-principal block has 2-core (2, 1), and has modules as follows for the alternating group.

\[
\begin{array}{cccc}
\lambda & \text{dim}(D^\lambda) & \text{Factors of } S^\lambda \\
(10,1) & 10 & 10 \\
(6,5) & 16_1 \oplus 16_2 & 32, 100 \\
(8,3) & 100 & 100, 10 \\
(6,3,2) & 848 & 848, 100, 32, 10 \\
(5,3,2,1) & 584_1 \oplus 584_2 & 1168, 848, 32^2, 100^2, 10^3 \\
\end{array}
\]

Here are the projectives for the principal 2-block of the alternating group $A_{11}$, which we can then induce to $S_{11}$, if that is what you need.
1
44 164 198 198 416
1 1 1 1 1 186 186
44 44 144 164 198 198 416 416
1 1 1 1 1 1 1 1 1 186 186 186
44 44 44 44 144 164 164 164 198 198 198 198 198 198 198 416 416
1 1 1 1 1 1 1 1 1 1 1 1 1 1 186 186 186 186
44 44 44 44 144 164 164 164 198 198 198 198 198 198 198 198 198 416 416
1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 144 164 186 186 186
1 1 1 1 1 1 1 1 1 1 1 1 144 164 186 186 186
1 1 1 1 1 1 1 1 1 1 1 1 144 164 186 198
1 1 1 1 1 1 1 144 164 186 198 198 198 416 416
1 1 1 1 1 1 1 1 144 164 186
44 164 186 198 198 416

44 144 164
1 186 1
164 416 198 44 198
1 1 186 1 1 186
44 44 44 198 198 164 186 416 144 164 164 416
1 1 1 1 1 1 1 1 1 1 1 1 1 1 186 186 186
144 144 164 164 416 416 44 44 198 44 44 198 198 198
1 1 1 1 1 1 1 1 1 1 1 1 186 44 44 198 198
1 144 1 144 1
144 144 164 164 416 416 1 44 198 1 44 44 198 198
1 1 1 1 1 1 1 1 1 1 1 1 1 1 144 164 186
1 1 1 1 1 1 1 1 1 1 144 164 186 198
1 1 1 1 1 1 1 144 164 186 198 198 416 416
1 1 1 1 1 1 1 1 144 164 186 416
1 1 1 1 1 1 1 1 144 164 186
164 186 416 1 144 198 1 44 198
1 44 186 416 1 164
44 144 164

7
For the non-principal block of $A_{11}$, we need $\mathbb{F}_4$ for the two pairs of dual modules, of dimension 16 and 584. We start by giving the structure of the projectives over $\mathbb{F}_2$ for the modules 10, 100, 848, and the two irreducible but not absolutely irreducible modules of dimension 32 and 1168. We do this for ease of translation to the symmetric group case. We then will give the projective covers of 16 and 584 over $\mathbb{F}_4$, choosing this labelling so that $\text{Ext}^1(584,16)$ is non-zero (but $\text{Ext}^1(584,16^*) = 0)$.
3 $A_{12}$ and $S_{12}$, characteristic 2

This group is getting towards the edge of what can be easily constructed using today’s computers. What makes it even more annoying is that there are simple modules in the principal 2-block of $A_{12}$ that are not realizable over $\mathbb{F}_2$.

There are eleven simple modules in the principal block of $S_{12}$, three of which split into two dual modules upon restriction to $A_{12}$, and they are below.

| $\lambda$ | $\dim(D^{\lambda})$ | Factors of $S^{\lambda}$ | $\dim(P(D^{\lambda}))$ |
|-----------|----------------------|---------------------------|-------------------------|
| (12)      | 1                    | 1                         | 204288                  |
| (11, 1)   | 10                   | 10                        | 159232                  |
| (7, 5)    | $32 = 16 \oplus 16^*$ | 32, 164, 100, 1            | 145408                  |
| (10, 2)   | 44                   | 44, 10                     | 69120                   |
| (9, 3)    | 100                  | 100, 44, 10                | 116224                  |
| (8, 4)    | 164                  | 164, 100, 10, 1            | 59904                   |
| (6, 5, 1) | $288 = 144 \oplus 144^*$ | 288, 570, 164, 100, 32, 1  | 55296                   |
| (6, 4, 2) | 416                  | 416, 1046, 570, 288, 164, 100, 44, 32, 10, $1^3$ | 50688 |
| (8, 3, 1) | 570                  | 570, 164, 100, 44, 10, $1^3$ | 48640 |
| (7, 3, 2) | 1046                 | 1046, 570, 164, 100, 32, 10, $1^3$ | 45056 |
| (5, 4, 2, 1) | $2368 = 1184 \oplus 1184^*$ | 2368, 1046, 570, 416$^2$, 288, 164, 100$^2$, 44$^2$, 32$^2$, 10$^5$, $1^5$ | 45056 |

Each of the dual pairs amalgamate over $\mathbb{F}_2$. We will not give the structures of the projectives for the 1-, 10-, 16- and 100-dimensional simple modules because it takes too long to find their socle layers. However, we do give the Ext$^1$ matrix below.

The non-principal block not of defect 0 has 2-core $(3, 2, 1)$, and has three modules as follows for the alternating and symmetric groups.

| $\lambda$ | $\dim(D^{\lambda})$ | Factors of $S^{\lambda}$ | $\dim(P(D^{\lambda}))$ |
|-----------|----------------------|---------------------------|-------------------------|
| (9, 2, 1) | 320                  | 320                        | 7680                    |
| (7, 4, 1) | 1408                 | 1408                       | 6656                    |
| (5, 4, 3) | 1792                 | 1792, 320                 | 5632                    |

There is also a block with defect 1 for $S_{12}$ and of defect zero for $A_{12}$, with simple of dimension 5632 and given by the partition $(6, 3, 2, 1)$.

Here is the Ext$^1$-matrix for the principal block for $p = 2$. 

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10
We now give the socle layers of the projectives that we have constructed.

| 44  | 1 1 | 1 1 | 1 0 1 6 16 16* 44 100 144 144* 164 416 570 1046 1184 1184* |
|-----|-----|-----|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| 1   | 0 1 | 0 0 | 1 0 0 0 1 0 1 2 1 0 0 0 1 | 1 1 | 2 1 0 0 0 |
| 10  | 1 0 | 0 0 | 1 2 0 0 0 0 0 1 1 | 1 1 |
| 16  | 0 0 | 0 1 0 1 1 0 0 0 0 1 1 | 1 0 |
| 16* | 0 0 | 1 0 0 1 0 1 0 0 0 1 0 1 |
| 44  | 1 1 | 0 0 | 0 0 0 0 0 0 0 0 0 0 0 0 |
| 100 | 0 2 | 1 1 0 0 0 0 1 0 0 0 0 0 |
| 144 | 0 0 | 1 0 0 0 0 0 0 0 1 1 0 0 0 |
| 144*| 0 0 | 0 0 1 0 0 0 0 0 1 1 0 0 0 |
| 164 | 1 0 | 0 0 | 0 1 0 0 0 0 0 0 0 0 0 |
| 416 | 1 0 | 0 0 | 0 0 1 1 0 0 0 0 0 1 1 |
| 570 | 2 0 | 0 0 | 0 0 1 1 0 0 0 0 0 0 0 |
| 1046| 1 1 | 1 1 | 0 0 0 0 0 0 0 0 0 0 0 |
| 1184| 0 1 | 1 0 | 0 0 0 0 0 0 1 0 0 0 0 |
| 1184*| 0 1 | 0 1 | 0 0 0 0 0 1 0 0 0 0 |

We now give the socle layers of the projectives that we have constructed.
144
416 570
1 1
16 44 164
1 1 100 144 144* 1046
10 16 16* 44 164 416 570
1 1 1 1 100 100 144*
10 16 16* 44 164 416 570 570
1 1 1 1 100 144 144* 1046
10 10 16 16* 16* 44 44 164 164 570 1184*
1 1 1 1 1 10 100 100 144 144* 1046
10 16 16* 44 100 100 164 164 416 416 570 570
1 1 1 1 1 1 10 10 10 16 16* 100 100
10 16 16 16* 44 44 44 164 570 1046 1184
1 1 1 1 10 10 100 100 144 144* 1046
10 16 16* 44 100 164 164 416 416 570
1 1 1 1 1 10 100 100 144 144
10 16 16* 44 164 416 570 570
1 1 10 16* 100 144 144*
16 44 164 416 1046
1 1 144 144* 144* 1184*
16 416 570
144
416
1
44
1 288 1046
10 32 164 416 570
1 1 1 1 100 100
10 32 44 164 416 570 2368
1 1 1 10 10 100 288 1046
10 32 44 100 100 164 416 570
1 1 1 1 1 10 10 10 32 100 288 1046
10 10 32 32 44 44 100 164 164 416 416 570 570 1046 2368
1 1 1 1 1 1 1 1 10 10 10 10 10 32 100 100 100
10 32 44 100 100 100 100 164 164 416 416 416 570 1046
1 1 1 1 1 1 1 1 10 10 10 10 10 10 10 32 100 100 100 100
10 32 44 164 164 164 164 416 416 416 416 570 570 1046 2368
1 1 1 1 1 1 1 1 10 10 10 10 10 32 100 100 100 100
10 32 44 44 44 44 164 164 416 416 570 570 1046 1046 2368 2368
1 1 1 1 1 1 1 1 10 10 10 10 10 10 10 32 100 100 100 100
10 32 44 100 100 100 100 100 164 164 164 416 416 416 570 570
1 1 1 1 1 1 1 1 10 10 10 10 10 10 10 32 100 100 100 100
10 32 44 44 44 44 164 570 1046 1046 1046 2368
1 1 1 1 10 10 10 10 10 32 100 288
100 100 100 164 416 416 416 570
1 1 1 1 10 10 32 288
10 32 44 416 416 1046
1 288 2368
416
570
1 1
44 164
1 1 100 288 1046
10 32 44 164 416 570 570
1 1 1 1 1 1 100
10 32 44 164 416 570 570
1 1 1 1 1 1 1 10 32 100 288 1046
10 10 32 44 164 416 570 1046
1 1 1 1 1 1 1 1 100 100 100 100 288 288 1046
10 10 10 32 32 32 44 44 100 164 164 164 416 416 570 570 570 570
1 1 1 1 1 1 1 1 1 1 1 1 1 10 10 100 100 100 100 288 1046
10 10 10 32 32 32 44 44 44 44 100 164 164 164 164 164 164 416 570 570 570 570 2368
1 1 1 1 1 1 1 1 1 1 10 10 10 10 32 100 100 100 100 288 288 1046 1046
10 10 32 32 44 44 44 100 100 100 100 100 100 164 164 164 164 164 164 416 416 416 416 570 570 570 570 1046 2368
1 1 1 1 1 1 1 1 1 1 1 1 1 1 10 10 10 10 10 32 100 100 100 100 100 100 288
10 10 32 32 44 44 44 100 164 164 164 164 164 164 164 164 416 416 416 416 570 570 570 1046 2368
1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 10 10 10 10 32 100 100 100 288 288
10 32 44 164 164 416 570 570
1 1 1 10 32 100
44 164 570 1046
1 1 288
570

15
We also have the non-principal block, with projectives as follows:

\[
\begin{array}{cccc}
320 & 1408 & 1792 \\
1408 & 1792 & 320 & 320 \\
320 & 320 & 1792 & 1408 \\
1408 & 1792 & 320 & 1408 & 320 \\
320 & 1408 & 1792
\end{array}
\]

The time it took to construct the projectives on my computer is as follows. Note that there are preliminary things like constructing the Cartan matrix and finding good subgroups for condensation that, if one constructs several projectives in one instance of Magma, will not be repeated, so use these numbers as a guide only.
| $D^\lambda$ | time to construct $P(D^\lambda)$ (seconds) |
|-----------|---------------------------------|
| 1        | 64295                          |
| 10       | 9196                           |
| 16 (over $\mathbb{F}_4$) | 27314                      |
| 32       | 30555                          |
| 44       | 9750                           |
| 100      | 14141                          |
| 164      | 6570                           |
| 144 (over $\mathbb{F}_4$) | 2676                      |
| 288      | 785                            |
| 416      | 7097                           |
| 570      | 2328                           |
| 1046     | 7647                           |
| 1184 (over $\mathbb{F}_4$) | 2490                      |
| 2368     | 618                            |

4 $A_{10}$, characteristic 3

In $A_{10}$ in characteristic 3 there are two blocks: the principal block with 3-core (1), of defect 4, and the non-principal block with 3-core (3, 1).

$$
\begin{array}{cccc|cccc}
\lambda & \dim(D^\lambda) & \text{Factors of } S^\lambda & \lambda & \dim(D^\lambda) & \text{Factors of } S^\lambda \\
(10) & 1 & 1 & (9, 1) & 9 & 9 \\
(8, 2) & 34 & 34, 1 & (6, 4) & 90 & 90 \\
(7, 3) & 41 & 41, 34 & (6, 2^2) & 126 & 126, 90, 9 \\
(7, 2, 1) & 84 & 84, 41, 34, 1 & (4^2, 2) & 36 & 36, 126, 90 \\
(6, 2, 1^2) & 224 & 224, 84, 41, 1 & (3^2, 2, 1^2) & 279 & 279, 126, 36, 9 \\
\end{array}
$$

The non-principal block is Morita equivalent to the two non-principal blocks of $S_{10}$, which are labelled by the 3-cores (3, 1) and its conjugate (2, 1^2). These are Morita equivalent to the principal 3-block of $S_8$, by standard Scopes moves [4].

The Ext$^1$-matrices are easy to describe, and we do this now.

| Block 1 | 1 | 34 | 41 | 84 | 224 |
|---------|---|----|----|----|-----|
| 1       | 0 | 1  | 1  | 1  | 1   |
| 34      | 1 | 0  | 1  | 0  | 1   |
| 41      | 1 | 1  | 0  | 1  | 1   |
| 84      | 1 | 0  | 1  | 1  | 0   |
| 224     | 1 | 1  | 1  | 0  | 1   |

| Block 2 | 9 | 36 | 90 | 126 | 279 |
|---------|---|----|----|-----|-----|
| 9       | 0 | 0  | 0  | 1   | 1   |
| 36      | 0 | 0  | 0  | 1   | 1   |
| 90      | 0 | 0  | 0  | 1   | 0   |
| 126     | 1 | 1  | 1  | 0   | 0   |
| 279     | 1 | 1  | 0  | 0   | 0   |
Finally, here are the radical and socle layers of the projectives, which coincide in this case.

\[
\begin{array}{cccccc}
1 & 34 & 41 & 84 & 224 & 1 \\
34 & 41 & 84 & 224 & & 34 \\
1 & 34 & 41 & 84 & 224 & 224 \\
1 & 1 & 34 & 34 & 41 & 84 \\
1 & 1 & 34 & 34 & 41 & 84 \\
1 & 1 & 34 & 41 & 84 & 84 \\
1 & 1 & 34 & 41 & 84 & 224 \\
34 & 41 & 84 & 224 & & 1 \\
1 & & & & & 34 \\
41 & & & & & 84 \\
1 & 34 & 84 & 224 & & 1 \\
1 & 34 & 84 & 224 & & 34 \\
1 & 1 & 34 & 34 & 41 & 84 \\
1 & 1 & 34 & 34 & 41 & 84 \\
1 & 1 & 34 & 41 & 84 & 224 \\
1 & 1 & 34 & 41 & 84 & 224 \\
1 & 34 & 84 & 224 & & 1 \\
1 & 34 & 84 & 224 & & 34 \\
9 & 36 & 90 & 126 & 279 & \\
126 & 279 & 126 & 9 & 36 & 90 \\
126 & 279 & 126 & 9 & 36 & 90 \\
126 & 279 & 126 & 9 & 36 & 90 \\
9 & 36 & 90 & 126 & 279 & \\
9 & 36 & 90 & 126 & 279 & \\
\end{array}
\]

These were described for the symmetric group by Tan in \cite{8}, and restriction is simply a case of replacing the module for $S_{10}$ by its restriction.

5 $A_{11}$, characteristic 3

In $A_{11}$ in characteristic 3 there are two blocks, the principal block with 3-core $(2)$, of defect 4, the non-principal block with 3-core $(3, 1, 1)$ of defect 2, and a block with defect 1 and 3-core $(4, 2, 1^2)$.

The principal block of $A_{11}$ is Morita equivalent to that of $S_{11}$, and we label the modules from this block by 3-regular partitions with 3-core $(2)$, rather than $(1^2)$, so that the 10-dimensional simple module has label $(5^2, 1)$, rather than $(10, 1)$.

| $\lambda$ | $\dim(D^\lambda)$ | Factors of $S^\lambda$ |
|-----------|--------------------|------------------------|
| $(11)$    | 1                  | 1                      |
| $(8, 3)$  | 109                | 109, 1                 |
| $(8, 2, 1)$ | 120               | 120, 109               |
| $(7, 3, 1)$ | 320               | 320, 120, 109, 1       |
| $(6, 3, 1^2)$ | 791              | 791, 320, 120, 1       |
| $(5^2, 1)$ | 10                 | 10, 320                |
| $(5, 4, 1^2)$ | 34               | 34, 10, 791, 320, 1^2  |
| $(5, 3^2)$ | 210                | 210, 10, 320, 120     |
| $(5, 3, 2, 1)$ | 714              | 714, 210, 34, 10, 791, 320, 120, 109, 1^2 |
| $(4, 3, 2^2)$ | 131              | 131, 714, 210, 34, 120, 109, 1^2  |
We now give the $\text{Ext}^1$-matrix, for those readers who do not want to look at the projectives directly.

| Block 1 | 1 | 10 | 34 | 109 | 120 | 131 | 210 | 320 | 714 | 791 |
|---------|---|----|----|-----|-----|-----|-----|-----|-----|-----|
| 1       | 0 | 0  | 0  | 1   | 1   | 0   | 0   | 1   | 0   | 1   |
| 10      | 0 | 0  | 1  | 0   | 0   | 0   | 1   | 1   | 0   | 0   |
| 34      | 0 | 1  | 0  | 0   | 0   | 1   | 0   | 0   | 1   | 1   |
| 109     | 1 | 0  | 0  | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| 120     | 1 | 0  | 0  | 0   | 0   | 0   | 1   | 1   | 0   | 0   |
| 131     | 1 | 0  | 1  | 0   | 0   | 0   | 1   | 0   | 0   | 0   |
| 210     | 0 | 1  | 0  | 0   | 1   | 1   | 0   | 0   | 1   | 0   |
| 320     | 0 | 1  | 0  | 0   | 1   | 0   | 0   | 0   | 0   | 0   |
| 714     | 1 | 0  | 1  | 0   | 0   | 0   | 1   | 0   | 0   | 0   |
| 791     | 1 | 0  | 1  | 0   | 0   | 0   | 0   | 1   | 0   | 0   |

| Block 2 | 45 | 126 | 126* | 693 |
|---------|----|-----|------|-----|
| 45      | 0  | 1   | 1    | 0   |
| 126     | 1  | 0   | 0    | 1   |
| 126*    | 1  | 0   | 0    | 1   |
| 693     | 0  | 1   | 1    | 0   |

For those who do, here they are.
|       | 109       | 120       |
|-------|-----------|-----------|
| 34    | 1         | 1         |
| 714   | 120 714   | 109       |
| 1     | 1         | 1         |
| 10 109 131 791 | 109 109 | 120 120 714|
| 1 1 34 210 320 | 1 1 34 210 320 | 1 1 34 210 320 |
| 1 131 714 | 1 120 714 | 10 109 109 131 791 |
| 1 34 34 210 | 1 34 34 210 | 1 34 34 210 320 |
| 10 10 109 120 131 131 714 791 | 10 109 109 131 791 | 120 120 714 |
| 1 1 34 34 34 210 210 320 | 1 1 34 210 320 | 1 1 34 210 320 |
| 10 131 714 791 | 120 714 | 10 109 120 120 131 791 |
| 34    | 1         | 1         |
|       | 120 320   | 120       |
| 131   | 120 714   | 120       |
| 1 34 210 | 1         | 1         |
| 120 714 | 10 109 131 | 10 109 791 |
| 1     | 1 1 34 210 320 | 1 34 210 320 |
| 10 109 131 131 | 120 714 | 120 714 |
| 1 1 34 34 210 210 | 1 34 210 210 | 1 320 |
| 10 120 131 131 714 | 10 10 109 120 131 131 714 791 | 10 10 109 120 791 |
| 1 34 210 | 1 1 34 34 210 210 210 320 | 1 34 210 320 320 |
| 131   | 10 120 131 714 | 10 120 791 |
|       | 210       | 320       |
| 714   | 1         | 1         |
| 109   | 791       | 791       |
| 1 34 210 | 1 34 320 | 120 714 |
| 120 714 | 120 714   | 1         |
|       | 1         | 1         |
| 10 109 131 791 | 10 109 791 | 1 1 34 210 320 | 1 34 210 320 |
| 120 714 714 | 10 120 714 791 | 1 1 34 210 | 1 34 320 |
| 10 109 131 714 791 | 791 | 1 34 210 |
| 1 34 210 | 320       |
| 714   |           |           |
And the non-principal block with defect 2.

\[
\begin{array}{cccc}
45 & 126 & 126^* & 693 \\
126 & 126^* & 45 & 693 \\
45 & 693 & 693 & 126 \ 126^* \\
126 & 126^* & 45 & 693 \\
45 & 693 & 126 & 693 \\
\end{array}
\]

We should note that three of these, of dimensions 109, 120 and 131, were determined by Tan in [7].

## 6 $A_{12}$, characteristic 3

And we are back up to the largest case you can feel comfortable with using current computers, and here we do not produce the socle layers of the projective cover of the trivial module.

There are four blocks for $A_{12}$ in characteristic 3, of defects 5, 2, 1 and 0. The block of defect zero has a module of dimension 2673, and the block of defect 1 has two modules of dimensions 891 and 3564.

| $\lambda$    | $\dim(D^\lambda)$ | Factors of $S^\lambda$ |
|--------------|--------------------|------------------------|
| (12)         | 1                  | 1                      |
| (11, 1)      | 10                 | 10, 1                  |
| (10, 1, 2)   | 45                 | 45, 10                 |
| (9, 3)       | 143                | 143, 10, 1             |
| (9, 2, 1)    | 120                | 120, 143, 45, 10, 1^2 |
| (8, 4)       | 131                | 131, 143, 1            |
| (8, 2, 1)    | 210                | 210, 131, 120, 143, 10, 1^2 |
| (7, 4, 1)    | 1013               | 1013, 131, 120, 143, 1 |
| (7, 3, 2)    | $126 \oplus 126^*$ | 252, 1013, 210, 131, 120, 143, 45, 10, 1 |
| (6, 4, 1, 2) | 1936               | 1936, 1013, 120, 10, 1 |
| (6, 3, 2, 1) | $714 \oplus 714^*$ | 1428, 1936, 252, 1013, 210^2, 131, 120^2, 143, 45, 10^2, 1^4 |

The principal block has thirteen simple modules. Here we give the Ext$^1$ matrix for these modules, since the projectives take several pages to write out, and we do not include $P(1)$, although of course this only means that Ext$^1(1, 1)$ is not given here, and this is ero for all groups $G$ with $O^p(G) = G$. 

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We now give the socle layers of the projective modules in the principal block except for the projective cover of the trivial module.

\[
\begin{array}{cccccccccccccc}
\text{Block 1} & 1 & 10 & 45 & 120 & 126 & 126^* & 131 & 143 & 210 & 714 & 714^* & 1013 & 1936 \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\
10 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
45 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
120 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 1 & 0 \\
126 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
126^* & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
131 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\
143 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
210 & 0 & 1 & 0 & 2 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
714 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
714^* & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1013 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1936 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

10
1 1936
1 45 143 210
1 10 10 120 120 120 714 714* 1013
1 1 1 10 120 120 131 714 714*
1 1 1 1 10 126 126* 131 143 143 143 1936
1 1 1 1 1 1 45 126 126* 143 143 210 210 210 1013 1013 1936 1936
1 1 1 10 10 45 45 120 120 120 143 143 210 210 210 210 714 714 714* 714* 1013
1 1 1 1 1 1 10 10 10 10 10 10 120 120 120 131 131 131 714 714 714* 714* 714*
1 1 1 1 1 1 1 10 120 120 120 126 126* 131 143 143 131 143 1936 1936
1 1 1 1 1 1 1 1 45 45 45 126 126 126* 143 143 143 210 210 210 210 1013 1013 1013 1936 1936
1 1 1 10 10 10 10 10 120 120 120 120 120 120 120 120 120 120 714 714 714* 714* 714* 1013
1 1 1 1 1 1 1 45 45 45 45 120 120 120 210 210 714 714 714* 1013 1013 1013 1013
1 10 10 10 10 120 120 126 126 126* 126* 131 131 131 143 210 1936
1 45 210 1013 1936
10
143
1
120 714 714*
1 1 1 10 131
143 143 1936 1936
1 1 1 1 1 45 210 210 210 1013
10 10 10 120 120 120 120 120 120 143 714 714 714 714 714 714 714
1 1 1 1 1 1 1 1 1 10 10 131 131 131 131
10 120 126 126 126* 126* 143 143 143 143 143 143 1936 1936
1 1 1 1 1 1 1 45 45 45 210 210 210 210 210 210 210 210 1013, 1013
10 10 10 120 120 120 120 120 120 143 714 714 714 714 714 714* 714* 714*
1 1 1 1 1 1 1 10 10 131 131 131 131 1013
10 120 126 126 126* 126* 131 143 143 143 1936 1936
1 1 1 1 45 210 210 210 1013
10 120 143 714 714*
1 131
143

210
10 120 120 714 714*
1 1 1 131
126 126* 143 143 143 1936 1936
1 1 1 1 1 1 45 45 210 210 210 210 210 1013 1013
10 10 10 120 120 120 120 120 120 120 120 714 714 714 714 714* 714* 714* 714*
1 1 1 1 1 1 1 1 1 10 10 10 10 131 131 131 131
126 126 126* 126* 126* 143 143 143 143 143 143 143 1936 1936, 1936
1 1 1 1 1 1 1 45 45 210 210 210 210 210 210 210 210 210 1013 1013
10 10 10 120 120 120 120 120 120 120 120 120 126 126 126 126 126* 126* 126* 126*
131 131 131 131 131 131 143 143 143 143 1936 1936
1 1 1 1 45 45 210 210 210 210 210 210 210 1013 1013
10 120 120 126 126* 131 714 714*
210

26
The non-principal block of defect 2 is much smaller of course, and the projectives are as follows.

| 54 | 297 | 945 | 1431 | 1728 |
|---|---|---|---|---|
| 1431 1728 | 1728 | 1431 1728 | 54 945 | 54 945 297 |
| 54 297 945 | 54 297 945 | 54 297 945 945 | 1431 1728 | 1431 1728 1728 |
| 1431 1728 | 1728 | 1431 1728 | 54 945 | 54 297 945 |
| 54 | 297 | 945 | 1431 | 1728 |

7 $A_{10}$, characteristic 5

Since these can all be understood from the work of Scopes [5] we simply give the structure of the projectives and the dimensions of the $D^\lambda$, with the factors of $S^\lambda$ for the symmetric group. Where the module for the symmetric group splits into two upon restriction to $A_n$, we note this.

| $\lambda$ | $\text{dim}(D^\lambda)$ | $\text{Factors of } S^\lambda$ |
|---|---|---|
| (10) | 1 | 1 |
| (9, 1) | 8 | 8, 1 |
| (8, 1^2) | 28 | 28, 8 |
| (7, 1^3) | 56 | 56, 28 |
| (6, 1^4) | $35_1 \oplus 35_2$ | 70, 56 |
| (5^2) | 34 | 34, 8 |
| (5, 4, 1) | 217 | 34, 28, 8, 1 |
| (5, 3, 1^2) | $133_1 \oplus 133_2$ | 266, 217, 56, 28 |
8 $A_{11}$, characteristic 5

Here two modules for $S_{11}$ from the principal block do not remain irreducible when restricted. One restricts to two self-dual modules and one restricts to two dual modules, as we see below.

\[
\begin{array}{c|c|c|c}
\lambda & \text{dim}(D^\lambda) & \text{Factors of } S^\lambda \\
\hline
(11) & 1 & 1 \\
(9, 2) & 43 & 43, 1 \\
(8, 2, 1) & 188 & 188, 43 \\
(7, 2, 1^2) & 406 & 406, 188 \\
(6, 5) & 89 & 89, 43 \\
(6, 4, 1) & 372 & 372, 89, 188, 43, 1 \\
(6, 3, 1^2) & 133_1 \oplus 133_2 & 266, 372, 406, 188 \\
(6, 2, 1^3) & 126 \oplus 126^* & 252, 266, 406 \\
\end{array}
\]

9 $A_{12}$, characteristic 5

Here the principal blocks of $A_{12}$ and $S_{12}$ are Morita equivalent, and the non-principal block of $S_{12}$ of full defect is dual to it. We give the projectives and the factors of $S^\lambda$ for the principal block only.
| \( \lambda \) | \( \dim(D^\lambda) \) | Factors of \( S^\lambda \) |
|-------|----------------|-----------------|
| (12)  | 1              | 1               |
| (9, 3) | 153            | 153, 1          |
| (8, 3, 1) | 738          | 738, 153        |
| (7, 5) | 144            | 144, 153        |
| (7, 4, 1) | 372          | 372, 144, 738, 153, 1 |
| (7, 3, 1^2) | 1266        | 1266, 372, 738 |
| (7, 2, 1^3) | 462         | 462, 1266       |
| (6, 3, 1^3) | 1596        | 1596, 462, 1266, 372 |
| (5, 3, 1^4) | 1506        | 1506, 462, 1266, 372 |
| (4^3) | 89             | 89, 372, 1      |
| (4, 3^2, 1^2) | 1957        | 1957, 462, 1266, 372 |
| (4, 3, 2, 1^3) | 573         | 573, 1266, 1596 |
| (3^3, 2, 1) | 11             | 11, 1596, 144  |
| (3^2, 2^2, 1^2) | 43          | 43, 11, 573, 1957, 89 |

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