CARDINAL INEQUALITIES FOR S(n)-SPACES

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Abstract. Hajnal and Juhaš proved that if \( X \) is a \( T_1 \)-space, then \(|X| \leq 2^{\chi(X)}\psi(X)\), and if \( X \) is a Hausdorff space, then \(|X| \leq 2^{\psi(X)}\chi(X)\) and \(|X| \leq 2^{s(X)}\). Schröder sharpened the first two estimations by showing that if \( X \) is a Hausdorff space, then \(|X| \leq 2^{U_s(X)}\psi_c(X)\), and if \( X \) is a Urysohn space, then \(|X| \leq 2^{U_s(X)}\chi(X)\).

In this paper, for any positive integer \( n \) and some topological spaces \( X \), we define the cardinal functions \( \chi_n(X) \), \( \psi_n(X) \), \( s_n(X) \), and \( c_n(X) \) called respectively \( S(n) \)-character, \( S(n) \)-pseudocharacter, \( S(n) \)-spread, and \( S(n) \)-cellularity and using these new cardinal functions we show that the above-mentioned inequalities could be extended to the class of \( S(n) \)-spaces. We recall that the \( S(1) \)-spaces are exactly the Hausdorff spaces and the \( S(2) \)-spaces are exactly the Urysohn spaces.

1. Introduction

Hajnal and Juhaš, in 1969 (see [9]) proved that if \( X \) is a \( T_1 \)-space, then \(|X| \leq 2^{\chi(X)}\psi(X)\), and if \( X \) is a Hausdorff space, then \(|X| \leq 2^{\psi(X)}\chi(X)\) and \(|X| \leq 2^{s(X)}\). Later Schröder in [15] proved that if \( X \) is a Hausdorff space, then the first inequality could be sharpened to \(|X| \leq 2^{U_s(X)}\psi_c(X)\), and if \( X \) is a Urysohn space, then the second estimation could be improved to \(|X| \leq 2^{U_c(X)}\chi(X)\). In [15, Theorem 12] the author also claimed that if \( X \) is a Urysohn space, then the third inequality could be improved to \(|X| \leq 2^{U_s(X)}\). The proof of his Theorem 12 is based on [15, Lemma 11] which states that if \( X \) is a Urysohn space, then \( \psi_c(X) \leq 2^{U_c(X)}\). Unfortunately, there is a gap in the proof of that Lemma 11 and therefore the validity of the last two claims are unknown. We were able to prove both claims only when \( X \) is an \( S(3) \)-space (see Corollaries 3.14 and 3.19).

Let \( n \) be a positive integer. In this paper, for some spaces \( X \), we define the cardinal functions \( \chi_n(X) \), \( \psi_n(X) \), \( s_n(X) \), and \( c_n(X) \) called respectively \( S(n) \)-character, \( S(n) \)-pseudocharacter, \( S(n) \)-spread, and \( S(n) \)-cellularity and using them we extend the above-mentioned inequalities to the class of \( S(n) \)-spaces (see Definition 2.1). In particular, we show that if \( X \) is an \( S(n) \)-space, then \(|X| \leq 2^{\chi_n(X)}\chi_n(X)\) (Theorems 3.3 and 3.4). When \( n = 1 \) we obtain Hajnal and Juhaš’ inequality \(|X| \leq 2^{\psi(X)}\chi(X)\) for Hausdorff spaces \( X \) and the case \( n = 2 \) is the second part of Alas and Kočinac’ Theorem 2 in [1], which sharpens Schröder’s inequality \(|X| \leq 2^{U_c(X)}\chi(X)\) for Urysohn spaces \( X \) (see Corollary 3.5).

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Some of the results in this paper were announced at the Spring Topology and Dynamical Systems Conference, Berry College, Mount Berry, GA, March 17–19, 2005.
Our Theorems 3.8 and 3.9 contain the generalizations to the class of \( S(n) \)-spaces of the corresponding inequalities of Hajnal and Juhász and Schröder mentioned above: Let \( k \) be a positive integer. If \( X \) is an \((k - 1)\)-space, then \(|X| \leq 2^{2s_{k-1}(X)} \) and if \( X \) is an \((k)\)-space, then \(|X| \leq 2^{s_k(X)} \).

At the end of the paper we generalize Hajnal and Juhász’ inequality that if \( X \) is a Hausdorff space, then \(|X| \leq 2^{2s_1(X)} \). For that end we find upper bounds for the cardinality of the \( S(n) \)-space of the corresponding inequalities of Hajnal and Juhász and Schröder mentioned above: Let \( k \) be a positive integer. If \( X \) is an \((k - 1)\)-space, then \(|X| \leq 2^{2s_{k-1}(X)} \) and if \( X \) is an \((k)\)-space, then \(|X| \leq 2^{s_k(X)} \).

In addition, we show that for every positive integer \( k \), the following are true:

(a) If \( X \) is an \((3k)\)-space, then \( \psi_{2k} \leq 2^{s_{2k}(X)} \) (Lemma 3.13) and \(|X| \leq 2^{s_{2k}(X)} \) (Theorem 3.18);

(b) If \( X \) is an \((3k - 2)\)-space, then \( \psi_{2k-1}(X) \leq 2^{s_{2k-1}(X)} \) (Lemma 3.14) and \(|X| \leq 2^{s_{2k-1}(X)} \) (Theorem 3.20);

(c) If \( X \) is an \((3k - 1)\)-space, then \( \psi_{2k}(X) \leq 2^{s_{2k}(X)} \) and \( \psi_{2k}(X) \leq 2^{s_{2k-1}(X)} \) (Lemma 3.16 and Lemma 3.17) and \(|X| \leq 2^{s_{2k-1}(X)}2^{s_{2k}(X)} \), hence \(|X| \leq 2^{s_{2k-1}(X)} \) (Theorem 3.21).

More results about the cardinality of \( S(n) \)-spaces involving the cardinal functions \( d_n(X) \), \( t_n(X) \) and \( b_n(X) \), called respectively \( S(n) \)-density, \( S(n) \)-tightness and \( S(n) \)-bitightness, are contained in our paper [8].

2. Preliminaries

Notations and terminology in this paper are standard as in [6], [11], and [10]. Unless otherwise indicated, all spaces are assumed to be at least \( T_1 \) and infinite. \( \alpha \), \( \beta \), \( \gamma \), and \( \delta \) are cardinal numbers, while \( \kappa \) denotes infinite cardinal; \( \kappa^+ \) is the successor cardinal of \( \kappa \). As usual, cardinals are assumed to be initial ordinals, while \( \kappa^+ \) is the successor cardinal of \( \kappa \). The set \( \mathcal{P}(X) \) and \(|X|^{\leq n} \) denote the power set of \( X \) and the collection of all subsets of \( X \) having cardinality \( \leq \kappa \), respectively.

Definition 2.1. Let \( X \) be a topological space, \( A \subset X \) and \( n \in \mathbb{N}^+ \). A point \( x \in X \) is \( S(n) \)-separated from \( A \) if there exist open sets \( U_1, U_2, \ldots, U_n \) such that \( x \in U_1 \cup U_2 \cup \cdots \cup U_n \) and \( U_1 \cap U_2 \cap \cdots \cap U_n \cap A = \emptyset \); \( x \) is \( S(n) \)-open from \( A \) if \( x \notin A \) and \( X \) is \( S(n) \)-closed if every two distinct points in \( X \) are \( S(n) \)-separated.

For \( h \in \mathbb{N} \). The set \( cl_{\theta^h}(A) = \{ x \in X : x \text{ is not } S(n) \text{-separated from } A \} \) is called \( \theta^h \)-closure of \( A \). A is \( \theta^h \)-closed if \( cl_{\theta^h}(A) = A \); \( U \subset X \) is \( \theta^h \)-open if \( X \setminus U \) is \( \theta^h \)-closed; and \( A \) is \( \theta^h \)-dense in \( X \) if \( cl_{\theta^h}(A) = X \).

It follows directly from Definition 2.1 that \( S(1) \) is the class of Hausdorff spaces and \( S(2) \) is the class of \( U \)-spaces. Since we are going to consider here only \( T_1 \)-spaces, \( S(0) \)-spaces will be exactly the \( T_1 \)-spaces. Also, \( cl_{\theta^0}(A) = \overline{A} \) and \( cl_{\theta^1}(A) = cl_{\theta}(A) \) - the so called \( \theta \)-closure of \( A \). We want to emphasize here that in general, when \( n \in \mathbb{N}^+ \), the \( \theta^n \)-closure operator is not idempotent, hence it is not a Kuratowski closure operator. In particular \( cl_{\theta}(cl_{\theta}(A)) \neq cl_{\theta^2}(A) \). Finally, we note that if \( X \) is a space and \( n \in \mathbb{N}^+ \), then \( cl_{\theta^n}(U) = cl_{\theta^{n-1}}(\overline{U}) \) whenever \( U \) is an open subset of \( X \) (see [3] Lemma 1.4(c)).
In this paper it will be more convenient for us to think about $S(n)$-spaces in more 'symmetric' way similar to the way how $S(n)$-spaces are defined in [4], [5] or [13] but here we are going to use different terminology and notation.

**Definition 2.2.** Let $X$ be a topological space, $U \subseteq X$, $x \in U$ and $k \in \mathbb{N}^+$. We will say that $U$ is an $S(2k-1)$-neighborhood of $x$ if there exist open sets $U_i$, $i = 1, 2, ..., k$, such that $x \in U_1$, $U_i \subseteq U_{i+1}$, for $i = 1, 2, ..., k-1$, and $U_k \subseteq U$.

We will say that $U$ is an $S(2k)$-neighborhood of $x$ if there exist open sets $U_i$, $i = 1, 2, ..., k$, such that $x \in U_1$, $\overline{U}_i \subseteq U_{i+1}$, for $i = 1, 2, ..., k-1$, and $\overline{U}_k \subseteq U$.

Let $n \in \mathbb{N}^+$. When a set $U$ is an $S(n)$-neighborhood of a point $x$ and it is an open (closed) set in $X$, we will refer to it as open (closed) $S(n)$-neighborhood of $x$. A set $U$ will be called $S(n)$-open ($S(n)$-closed) if $U$ is open (closed) and there exists at least one point $x$ such that $U$ is an open (closed) $S(n)$-neighborhood of $x$.

**Remark 2.3.** We note that in what follows every $S(2k-1)$-open set $U$ in a space $X$, where $k \in \mathbb{N}^+$, will be considered as a fixed chain of $k$ nonempty sets $U_i$, $i = 1, 2, ..., k$, such that $U_i \subseteq U_{i+1}$, for $i = 1, 2, ..., k-1$, and $U_k \subseteq U$. (In fact, most of the time we will assume that $U_k = U$). Sometimes, when we need to refer to the first set $U_1$ in that chain we will use the notation $U(k)$, i.e. by definition $U(k) = U_1$.

Now, using the terminology and notation introduced in Definition 2.2 it is easy to see that the following propositions are true.

**Proposition 2.4.** Let $X$ be a topological space, $x \in X$ and $k \in \mathbb{N}^+$.
(a) Every closed $S(2k-1)$-neighborhood of $x$ is a closed $S(2k)$-neighborhood of $x$.
(b) Every $S(2k)$-neighborhood of $x$ contains a closed $S(2k)$-neighborhood of $x$; hence it contains a closed (and therefore an open) $S(2k-1)$-neighborhood of $x$. Thus, every $S(2k)$-neighborhood of $x$ is an $S(2k-1)$-neighborhood of $x$.
(c) Every $S(2k+1)$-neighborhood of $x$ contains an open $S(2k+1)$-neighborhood of $x$; hence it contains an open (and therefore a closed) $S(2k)$-neighborhood of $x$. Thus, every $S(2k+1)$-neighborhood of $x$ is an $S(2k)$-neighborhood of $x$.

**Proposition 2.5.** Let $X$ be a topological space and $k \in \mathbb{N}^+$.
(a) $X$ is an $S(2k-1)$-space if and only if every two distinct points of $X$ can be separated by disjoint (open) $S(2k-1)$-neighborhoods.
(b) $X$ is an $S(2k)$-space if and only if every two distinct points of $X$ can be separated by disjoint closed $S(2k-1)$-neighborhoods.
(c) $X$ is an $S(2k)$-space if and only if every two distinct points of $X$ can be separated by disjoint (closed) $S(2k)$-neighborhoods.
(d) $X$ is an $S(2k+1)$-space if and only if every two distinct points of $X$ can be separated by disjoint open $S(2k)$-neighborhoods.

**Definition 2.6.** Let $X$ be a topological space, $A \subseteq X$ and $k \in \mathbb{N}^+$. We will say that a point $x$ is in the $S(2k-1)$-neighborhood of $x$ intersects $A$ and we will say that a point $x$ is in the $S(2k)$-neighborhood of $A$ if and only if every (open) $S(2k-1)$-neighborhood of $x$ intersects $A$. For $n \in \mathbb{N}^+$, the $S(n)$-closure of $A$ will be denoted by $\theta_n(A)$. $A$ is $\theta_n$-closed if $\theta_n(A) = A$ and $U \subseteq X$ is $\theta_n$-open if $X \setminus U$ is $\theta_n$-closed, or equivalently, $U \subseteq X$ is $\theta_n$-open if $U$ is an $S(n)$-neighborhood of every $x \in U$. Finally, $A$ is $\theta_n$-dense in $X$ if $\theta_n(A) = X$. 
Clearly, for every $n \in \mathbb{N}^+$, every $\theta_n$-open set is open and every set of the form $\theta_n(A)$, where $A \subseteq X$, is a closed set. Also, it follows directly from Definition 2.4 that $\theta_1(A) = \text{cl}(A) = \overline{A}$ is the usual closure operator in $X$ and $\theta_2(A) = \text{cl}_0(A)$ is the $\theta$-closure operator introduced by Veličko [16]. We also note that, except for the case $n = 1$, for many $A \subset X$ we may have $\theta_n(\theta_n(A)) \neq \theta_n(A)$, or in other words, the $\theta_n$-closure operator is not idempotent. More information about the closure operator $\theta_n$ is contained in [3], [4] and [5].

**Definition 2.7.** Let $k \in \mathbb{N}^+$ and $X$ be a topological space.

(a) A family $\{U_\alpha : \alpha < \kappa\}$ of open $S(2k-1)$-neighborhoods of a point $x \in X$ will be called an open $S(2k-1)$-neighborhood base at the point $x$ if for every open $S(2k-1)$-neighborhood $U$ of $x$ there is $\alpha < \kappa$ such that $U_\alpha \subseteq U$.

(b) An $S(2k-1)$-space $X$ is of $S(2k-1)$-character $\kappa$, denoted by $\chi_{2k-1}(X)$, if $\kappa$ is the smallest infinite cardinal such that for each point $x \in X$ there exists an open $S(2k-1)$-neighborhood base at $x$ with cardinality at most $\kappa$. In the case $k = 1$ the $S(1)$-character $\chi_1(X)$ coincides with the usual character $\chi(X)$.

(c) An $S(2k)$-space $X$ is of $S(2k)$-character $\kappa$, denoted by $\chi_{2k}(X)$, if $\kappa$ is the smallest infinite cardinal such that for each point $x \in X$ there exists a family $\mathcal{V}_x$ of closed $S(2k-1)$-neighborhoods of $x$ such that $|\mathcal{V}_x| \leq \kappa$ and if $W$ is an open $S(2k-1)$-neighborhood of $x$, then $W$ contains a member of $\mathcal{V}_x$. In the case $k = 1$ the $S(2)$-character $\chi_2(X)$ coincides with the cardinal function $k(X)$ defined in [1].

(d) An $S(k-1)$-space $X$ is of $S(2k-1)$-pseudocharacter $\kappa$, denoted by $\psi_{2k-1}(X)$, if $\kappa$ is the smallest infinite cardinal such that for each point $x \in X$ there exists a family $\{U_\alpha : \alpha < \kappa\}$ of $S(2k-1)$-open neighborhoods of $x$ such that $\{x\} = \bigcap\{U_\alpha : \alpha < \kappa\}$. In the case $k = 1$ the pseudocharacter $\psi_1(X)$ coincides with the usual pseudocharacter $\psi(X)$.

(e) An $S(k)$-space $X$ is of $S(2k)$-pseudocharacter $\kappa$, denoted by $\psi_{2k}(X)$, if $\kappa$ is the smallest infinite cardinal such that for each point $x \in X$ there exists a family $\{U_\alpha : \alpha < \kappa\}$ of $S(2k-1)$-open neighborhoods of $x$ such that $\{x\} = \bigcap\{U_\alpha : \alpha < \kappa\}$. In the case $k = 1$ the pseudocharacter $\psi_2(X)$ coincides with the closed pseudocharacter $\psi_2(X)$.

In relation to Definition 2.7(c) we recall that for a topological space $X$, $k(X)$ is the smallest infinite cardinal $\kappa$ such that for each point $x \in X$, there is a collection $\mathcal{V}_x$ of closed neighborhoods of $x$ such that $|\mathcal{V}_x| \leq \kappa$ and if $W$ is a neighborhood of $x$, then $W$ contains a member of $\mathcal{V}_x$ [1]. Clearly, $k(X) \leq \chi(X)$. As it was noted in [1], $k(X)$ is equal to the character of the semiregularization of $X$. We also note that if $k \in \mathbb{N}^+$, then $\psi_{2k-1}(X) \leq \psi_{2k}(X) \leq \psi_{2k+1}(X) \leq \psi_{2k+2}(X)$ and $\chi_{2k}(X) \leq \chi_{2k-1}(X)$, whenever they are defined.

**Remark 2.8.** Since the $\theta$-closure operator is not idempotent, for each positive integer $n$, a different type of pseudocharacter could be defined by requiring each point $x \in X$ to have a family of open neighborhoods $\{U_\alpha : \alpha < \kappa\}$ such that $\{x\} = \bigcap\{\text{cl}_n(\text{cl}_{n-1}(\ldots \text{cl}_0(U_\alpha) \ldots)) : \alpha < \kappa\}$, where the $\theta$-closure operator is repeated $n$ times. We used the notation $\psi_{\theta^n}(X)$ in [7] to denote that pseudocharacter and it is not difficult to see that $\psi_{\theta^n}(X)$ and the pseudocharacter $\psi_{\theta^n}(X)$ defined in Definition 2.7 are different.

**Definition 2.9.** Let $k \in \mathbb{N}^+$ and $X$ be a topological space.

(a) We shall call a subset $D$ of $X$ $S(2k-1)$-discrete if for every $x \in D$, there is an open $S(2k-1)$-neighborhood $U$ of $x$ such that $U \cap D = \{x\}$, and we define the
$S(2k-1)$-spread of $X$, denoted by $s_{2k-1}(X)$, to be $\sup\{|D|: D \text{ is a } S(2k-1)\text{-discrete subset of } X \} + \aleph_0$.

(b) We shall call a subset $D$ of $X$ $S(2k)$-discrete if for every $x \in D$, there is an open $S(2k-1)$-neighborhood $U$ of $x$ such that $\overline{U} \cap D = \{x\}$, and we define the $S(2k)$-spread of $X$, denoted by $s_{2k}(X)$, to be $\sup\{|D|: D \text{ is a } S(2k)\text{-discrete subset of } X \} + \aleph_0$.

It follows immediately from Definition 2.10 that a set $D$ in a topological space $X$ is discrete if and only if $D$ is $S(1)$-discrete and a set $D$ is Urysohn-discrete if and only if $D$ is $S(2)$-discrete. Therefore $s_1(X)$ is the usual spread $s(X)$ and $s_2(X)$ is the Urysohn spread $U_s(X)$ defined in [15].

**Definition 2.10.** Let $X$ be a topological space and $k \in \mathbb{N}^+$. 
(a) We shall call a family $\mathcal{U}$ of pairwise disjoint non-empty $S(2k-1)$-open subsets of $X$ $S(2k-1)$-cellular and we define the $S(2k-1)$-cellularity of $X$, denoted by $c_{2k-1}(X)$, to be $\sup\{|\mathcal{U}|: \mathcal{U} \text{ is an } S(2k-1)\text{-cellular family in } X \} + \aleph_0$.
(b) We shall call a family $\mathcal{U}$ of non-empty $S(2k-1)$-open subsets of $X$ $S(2k)$-cellular if for every distinct $U_1, U_2 \in \mathcal{U}$ we have $\overline{U_1} \cap \overline{U_2} = \emptyset$ and we define the $S(2k)$-cellularity of a space $X$, denoted by $c_{2k}(X)$, to be $\sup\{|\mathcal{U}|: \mathcal{U} \text{ is an } S(2k)\text{-cellular family in } X \} + \aleph_0$.

**Remark 2.11.** Let $X$ be a topological space.
(a) $c_1(X)$ coincides with the usual cellularity $c(X)$ of the space $X$;
(b) $c_2(X)$ coincides with the Urysohn cellularity $Uc(X)$ of $X$ introduced in [14];
(c) If $n, m \in \mathbb{N}^+$ and $n < m$, then $c_n(X) \geq c_m(X)$. For example of a space $X$ such that $c_1(X) > c_2(X)$ see [14, Example 4].
(d) If in Definition 2.10(b) we require $\mathcal{U}$ to be a family of pairwise disjoint non-empty $S(2k)$-open subsets of $X$ (similar to 2.10(a)), then, according to Proposition 2.4, we would get the definitions of $S(2k+1)$-cellular family and $S(2k+1)$-cellularity defined in 2.10(a).

### 3. Cardinal inequalities for $S(n)$-spaces

Hajnal and Juhasz proved that if $X$ is a $T_1$-space, then $|X| \leq 2^{s(X)_{\omega}(X)}$, and if $X$ is a Hausdorff space, then $|X| \leq 2^{c(X)_{\chi}(X)}$ and $|X| \leq 2^{\omega X}$ (see [9], [11] or [10]). Schröder in [15] sharpened the first two estimations by showing that if $X$ is a Hausdorff space, then $|X| \leq 2^{Uc(X)_{\omega}(X)}$, and if $X$ is a Urysohn space, then $|X| \leq 2^{Uc(X)_{\omega}(X)}$ and $|X| \leq 2^{C(X)_{\omega}(X)}$.

Below we formulate and prove the counterpart of the above inequalities for $S(n)$-spaces, where $n$ is any positive integer. We begin with extending the two inequalities $|X| \leq 2^{\omega X}$ and $|X| \leq 2^{\omega X}$ to $S(n)$-spaces.

The following lemma for $k = 1$ was proved by Charlesworth (see [2, Theorem 3.2] or [10, Proposition 3.4]).

**Lemma 3.1.** Let $k \in \mathbb{N}^+$, $X$ be a topological space and $\kappa = c_{2k-1}(X)$. If $\mathcal{U}$ is a family of $S(2k-1)$-open subsets of $X$, then there exists a subfamily $\mathcal{V}$ of $\mathcal{U}$ such that $|\mathcal{V}| \leq \kappa$ and $\bigcup\{U(k) : U \in \mathcal{U}\} \subseteq \Theta_{2k-1}(\bigcup\{V : V \in \mathcal{V}\})$.

**Proof.** Let $\mathcal{W} = \{W : W \neq \emptyset \text{ is } S(2k-1)\text{-open}, W \subseteq U, U \in \mathcal{U}\}$. Using Zorn’s lemma we can find a maximal $S(2k-1)$-cellular family $\mathcal{W}' \subseteq \mathcal{W}$. Then $|\mathcal{W}'| \leq \kappa$. For each $W \in \mathcal{W}'$ we fix $U_W \in \mathcal{U}$ such that $W \subseteq U_W$ and let $\mathcal{V} = \{U_W : W \in \mathcal{W}'\}$. Then $|\mathcal{V}| \leq \kappa$. Now, suppose that $\bigcup\{U(k) : U \in \mathcal{U}\} \subseteq \Theta_{2k-1}(\bigcup\{V : V \in \mathcal{V}\})$. Then
it is not difficult to see that there exist $U \in \mathcal{U}$, $x \in U(k) \setminus \theta_{2k-1}(\bigcup \{V : V \in \mathcal{V}\})$ and an $S(2k-1)$-open set $U_x$ such that $x \in U_x(k) \subseteq U(k)$, $U_x \subseteq U$ and $U_x \cap \bigcup \{V : V \in \mathcal{V}\} = \emptyset$. Clearly, $U_x \in \mathcal{W}$. Hence, $\{U_x\} \cup \mathcal{W}$ is an $S(2k-1)$-cellular family that properly contains $\mathcal{W}'$ and therefore $\mathcal{W}'$ is not maximal – contradiction. □

The next lemma for $k = 1$ was proved by Schröder (see [4] Lemma 7).

**Lemma 3.2.** Let $k \in \mathbb{N}^+$, $X$ be a topological space and $\kappa = c_{2k}(X)$. If $\mathcal{U}$ is a family of $S(2k-1)$-open subsets of $X$, then there exists a subfamily $\mathcal{V}$ of $\mathcal{U}$ such that $|\mathcal{V}| \leq \kappa$ and $\bigcup \{U(k) : U \in \mathcal{U}\} \subseteq \theta_{2k}(\bigcup \{V : V \in \mathcal{V}\})$.

**Proof.** Let $\mathcal{W} = \{W : W \neq \emptyset \text{ is } S(2k-1) \text{-open}, W \subseteq U, U \in \mathcal{U}\}$. Using Zorn’s lemma we can find a maximal $S(2k)$-cellular family $\mathcal{W}' \subseteq \mathcal{W}$. Then $|\mathcal{W}'| \leq \kappa$. For each $W \in \mathcal{W}'$ we fix $U_W \in \mathcal{U}$ such that $W \subseteq U_W$ and let $\mathcal{V} = \{U_W : W \in \mathcal{W}'\}$. Then $|\mathcal{V}| \leq \kappa$. Now, suppose that $\bigcup \{U(k) : U \in \mathcal{U}\} \not\subseteq \theta_{2k}(\bigcup \{V : V \in \mathcal{V}\})$. Then it is not difficult to see that there exist $U \in \mathcal{U}$, $x \in U(k) \setminus \theta_{2k}(\bigcup \{V : V \in \mathcal{V}\})$ and $\mathcal{V}' \subseteq \{V : V \in \mathcal{V}\}$ such that $x \in U_x(k) \subseteq U(k)$, $U_x \subseteq U$, and $\mathcal{V}_x \cap \bigcup \{V : V \in \mathcal{V}'\} = \emptyset$. Clearly, $U_x \in \mathcal{W}$, and hence, $\{U_x\} \cup \mathcal{W}'$ is an $S(2k)$-cellular family that properly contains $\mathcal{W}'$ and therefore $\mathcal{W}'$ is not maximal – contradiction. □

The following two theorems, which proof is based on Lemma 3.1 and Lemma 3.2 give an upper bound for the cardinality of an $S(n)$-space, as a function of the $S(n)$-cellularity and the character of the space. The inequality for Hausdorff spaces (the case $k = 1$ in Theorem 3.3) was proved by Hajnal and Juhász (see [9], [11], Theorem 2.15(b)) or [12], Theorem 2) and for Urysohn spaces (the case $k = 1$ in Theorem 3.3) was proved by Schröder (see [15], Theorem 9).

**Theorem 3.3.** Let $k \in \mathbb{N}^+$. If $X$ is an $S(2k-1)$-space, then $|X| \leq 2^{c_{2k-1}(X)}x_{2k-1}(X)$.

**Proof.** Let $k \in \mathbb{N}^+$, $\kappa = c_{2k-1}(X)\chi_{2k-1}(X)$ and for every $x \in X$ let $\mathcal{B}(x)$ be an open $S(2k-1)$-neighborhood base at $x$ such that $|\mathcal{B}(x)| \leq \kappa$. By transfinite recursion we construct an increasing sequence $(A_\alpha)_{\alpha < \kappa^+}$ of subsets of $X$ and a sequence of families of open sets $(\mathcal{U}_\alpha)_{\alpha < \kappa^+}$ as follows:

(a) $\mathcal{U}_0 = \emptyset$ and $A_0 = \{x_0\}$, where $x_0$ is an arbitrary point of $X$;
(b) $|A_\alpha| \leq 2^{\kappa}$ for every $\alpha < \kappa^+$;
(c) $\mathcal{U}_\alpha = \bigcup \{\mathcal{B}(x) : x \in \bigcup_{\beta < \alpha} A_\beta\}$, for every $0 < \alpha < \kappa^+$;
(e) For $0 < \alpha < \kappa^+$, if $\mathcal{W} = \{\mathcal{V}_\gamma : \gamma < \kappa\}$ is a family of subsets of $X$ such that each $\mathcal{V}_\gamma$ is the union of at most $\kappa$ many elements of $\mathcal{U}_\alpha$ and $\bigcup_{\gamma < \kappa} \theta_{2k-1}(\mathcal{V}_\gamma) \neq X$, then we pick a point $x_{\mathcal{W}} \in X \setminus \bigcup_{\gamma < \kappa} \theta_{2k-1}(\mathcal{V}_\gamma)$. Let $E$ be the set of all such points $x_{\mathcal{W}}$. Then we set $A_\alpha = \mathcal{U}_\alpha \cup \bigcup_{\beta < \alpha} A_\beta$. Since $|\bigcup_{\beta < \alpha} A_\beta| \leq 2^\kappa$, we have $|A_\alpha| \leq 2^{\kappa}$ and therefore $|E| \leq (2^{\kappa})^\kappa = 2^\kappa$.

Thus $|A_\alpha| \leq 2^{\kappa}$ and therefore (b) is satisfied.

Now, let $A = \bigcup_{\alpha < \kappa^+} A_\alpha$. Since $|A| \leq \kappa^+ \cdot 2^\kappa = 2^\kappa$, to finish the proof it is sufficient to show that $X = A$. Suppose that there exists a point $y \in X \setminus A$ and let $\mathcal{B}(y) = \{B_\delta : \delta < \kappa\}$. For each $\delta < \kappa$ let $\mathcal{V}_\delta = \{V : V \in \mathcal{B}(x), x \in A, B_\delta \cap V = \emptyset\}$. Since $X$ is an $S(2k-1)$-space and $\mathcal{B}(x)$, for each $x \in X$, is an open $S(2k-1)$-neighborhood base at $x$, $A \subseteq \bigcup_{\delta < \kappa} \{V(k) : V \in \mathcal{V}_\delta\}$. It follows from Lemma 3.3 that for every $\delta < \kappa$ we can find a subfamily $\mathcal{W}_\delta$ of $\mathcal{V}_\delta$ such that $|\mathcal{W}_\delta| \leq \kappa$ and $\bigcup \{V(k) : V \in \mathcal{W}_\delta\} \subseteq \theta_{2k-1}(\bigcup \mathcal{W}_\delta)$. Using the fact that $|\bigcup_{\delta < \kappa} \mathcal{W}_\delta| \leq \kappa$ and that $\kappa^+$ is a regular cardinal we can find $\alpha < \kappa$ such that $\bigcup_{\beta < \kappa} \mathcal{W}_\beta \subseteq A_\alpha$. Since for each $\delta < \kappa$ we have $B_\delta \cap \bigcup \{W : W \in \mathcal{W}_\delta\} = \emptyset$,
Lemma 3.2 that for every $\delta < \kappa$ of subsets of $S$ of open sets we have $\delta < \kappa$.

Proof. Let $n \in \mathbb{N}$, $\kappa = c \cdot n(X)\chi_{2k}(X)$ and for each $x \in X$ let $B(x)$ be a family of open $S(2k - 1)$-neighborhoods of $x$ witnessing the fact that $\chi_{2k}(X) \leq \kappa$ (hence, $|B(x)| \leq \kappa$). By transfinite recursion we construct an increasing sequence $(A_\alpha)_{\alpha < \kappa^+}$ of subsets of $X$ and a sequence of families of open sets $(U_\alpha)_{\alpha < \kappa^+}$ as follows:

1. $U_0 = \emptyset$ and $A_0 = \{x_0\}$, where $x_0$ is an arbitrary point of $X$;
2. $|A_\alpha| \leq 2^\alpha$ for every $\alpha < \kappa^+$;
3. $U_\alpha = \bigcup \{B(x) : x \in \bigcup_{\beta < \alpha} A_\beta\}$, for every $0 < \alpha < \kappa^+$;
4. For $0 < \alpha < \kappa^+$, if $W = \{V_\gamma : \gamma < \kappa\}$ is a family of subsets of $X$ such that each $V_\gamma$ is the union of the closures of at most $\kappa$ many elements of $U_\alpha$ and $\bigcup_{\gamma < \kappa} \theta_{2k}(V_\gamma) \neq X$, then we pick a point $x_W \in X \setminus \bigcup_{\gamma < \kappa} \theta_{2k}(V_\gamma)$. Let $E$ be the set of all such points $x_W$. Then we set $A_\alpha = E \cup \bigcup_{\beta < \alpha} A_\beta$. Since $|\bigcup_{\beta < \alpha} A_\beta| \leq 2^\alpha$, we have $|U_\alpha| \leq 2^\alpha$ and therefore $|E| \leq ((2^\alpha)^\kappa)^\alpha = 2^\alpha$. Thus $|A_\alpha| \leq 2^\alpha$ and therefore (b) is satisfied.

Now, let $A = \bigcup_{\alpha < \kappa^+} A_\alpha$. Since $|A| \leq \kappa^+ \cdot 2^\kappa = 2^\kappa$, to finish the proof it is sufficient to show that $X = A$. Suppose that there exists a point $y \in X \setminus A$ and let $B(y) = \{B_\delta : \delta < \kappa\}$. For each $\delta < \kappa$ let $V_\delta = \{V : V \in B(x), x \in A, \overline{B_\delta} \cap V = \emptyset\}$. Since $X$ is an $S(2k)$-space and $B(x)$, for each $x \in X$, is a closed $S(2k - 1)$-neighborhood base at $x$, $A \subseteq \bigcup_{\delta < \kappa} \bigcup \{V(k) : V \in V_\delta\}$. It follows from Lemma 3.2 that for every $\delta < \kappa$ we can find a subfamily $W_\delta$ of $V_\delta$ such that $|W_\delta| \leq \kappa$ and $|\bigcup \{V(k) : V \in W_\delta\}| \subseteq \theta_{2k}(\bigcup \{\overline{W} : W \in W_\delta\})$. Using the fact that $|\bigcup_{\delta < \kappa} W_\delta| \leq \kappa$ and that $\kappa^+$ is a regular cardinal we can find $\alpha < \kappa^+$ such that $\bigcup_{\delta < \kappa} W_\delta \subseteq U_\alpha$. Since for each $\delta < \kappa$ we have $\overline{B_\delta} \cap \bigcup \{\overline{W} : W \in W_\delta\} = \emptyset$, $y \notin \bigcup_{\delta < \kappa} \theta_{2k}(\bigcup \{\overline{W} : W \in W_\delta\})$. Then it follows from the construction of the set $A_\alpha$ that there exists $x \in A_\alpha \cap (X \setminus \bigcup_{\delta < \kappa} \theta_{2k}(\bigcup \{\overline{W} : W \in W_\delta\}))$. Thus $x \in A_\alpha \subseteq A \subseteq \bigcup_{\delta < \kappa} \bigcup \{V(k) : V \in V_\delta\} \subseteq \bigcup_{\delta < \kappa} \theta_{2k}(\bigcup \{\overline{W} : W \in W_\delta\})$ – contradiction. □

The case $k = 1$ in the above theorem improves Schröder’s inequality [13] Theorem 9 and is the second part of Theorem 2 given in [1] without a proof. We write it below with more familiar notation.

Corollary 3.5. If $X$ is a Urysohn space, then $|X| \leq 2^{\psi(X)\chi_{2k}(X)}$.

Now, we are going to formulate and prove the counterpart of the two inequalities $|X| \leq 2^{\psi(x)\psi(X)}$ and $|X| \leq 2^{s_2(x)\psi(x)}$ for $S(n)$-spaces.

The following lemma for the case $k = 1$ was proved by Šapirovskiǐ (see [10] Proposition 4.8).

Lemma 3.6. Let $k \in \mathbb{N}^+$, $X$ be a topological space, $\kappa = s_{2k-1}(X)$ and $C \subseteq X$. For each $x \in C$ let $U^x$ be an open $S(2k - 1)$-neighborhood of $x$ and let $\mathcal{U} = \{U^x : x \in C\}$. Then there exists an $S(2k - 1)$-discrete subset $A$ of $C$ such that $|A| \leq \kappa$ and $C \subseteq \bigcup_{A \in \mathcal{U}} (A \cup \{U^x : x \in A\})$.

Proof. Let $\{U_\alpha : \alpha < \beta\}$ be a well-ordering of the elements of $\mathcal{U}$ and let $x_0 \in C \cap U_0(k)$. Suppose that the points $\{x_\alpha : \gamma < \delta\}$ have already been chosen. Let $\alpha_\delta$ be the first ordinal greater than $\alpha_\gamma$, for each $\gamma < \delta$, such that there exists $x_{\alpha_\delta} \in \bigcup_{\alpha < \delta} \theta_{2k-1}(\bigcup \{U^x : x \in A\})$.
\[(C \cap U_\alpha(k)) \setminus \left( \bigcup_{\gamma < \delta} U_\alpha \right) \cup \theta_{2k-1}(\{x_\alpha : \gamma < \delta\}) \]. We finish this selection when such a point \(x_\alpha\) does not exist, i.e. when \(C \subseteq \left( \bigcup_{\gamma < \delta} U_\alpha \right) \cup \theta_{2k-1}(\{x_\alpha : \gamma < \delta\})\).

Therefore if \(A\) is the set of selected points, then \(C \subseteq \theta_{2k-1}(A) \cup \{U^x : x \in A\}\).

To finish the proof we need to show that \(A\) is an \(S(2k-1)\)-discrete set in \(X\) and therefore \(|A| \leq \kappa\). Let \(x_\alpha \in A\). Then \(x_\alpha \notin \theta_{2k-1}(\{x_\alpha : \gamma < \gamma_0}\) \). Hence, there exists an open \(S(2k-1)\)-neighborhood \(V_\alpha\) of \(x_\alpha\) such that \(V_\alpha \cap \{x_\alpha : \gamma < \gamma_0\} = \emptyset\). Now, suppose that there exists \(\gamma_1\) such that \(x_{\alpha_1} \in A \cap V_\alpha \cap U_\alpha\). Since \(x_{\alpha_1} \in V_\alpha\), we have \(\gamma_1 \geq \gamma_0\). From \(x_{\alpha_1} \in U_\alpha\) and \(x_{\alpha_1} \in U_\alpha\setminus (\bigcup_{\gamma < \gamma_1} U_\alpha \cap \theta_{2k-1}(\{x_\alpha : \gamma < \gamma_1\})\) conclude that \(U_\alpha \subseteq \bigcup_{\gamma < \gamma_1} U_\alpha\), thus \(\gamma_0 \geq \gamma_1\). Hence \(\gamma_1 = \gamma_0\) and therefore \(\{x_\alpha\} = A \cap U_\alpha \cap \bigcup_{\gamma < \gamma_1} U_\alpha\) i.e. \(A\) is an \(S(2k-1)\)-discrete set in \(X\).

The next lemma for the case \(n = 1\) was proved by Schröder (see [15] Lemma 6).

**Lemma 3.7.** Let \(k \in \mathbb{N}^+\), \(X\) be a topological space, \(\kappa = s_{2k}(X)\) and \(C \subseteq X\). For each \(x \in C\) let \(U^x\) be an open \(S(2k-1)\)-neighborhood of \(x\) and let \(U = \{U^x : x \in C\}\). Then there exist an \(S(2k)\)-discrete subset \(A\) of \(C\) such that \(|A| \leq \kappa\) and \(C \subseteq \theta_{2k}(A) \cup \bigcup \{U^x : x \in A\}\).

**Proof.** Let \(\{U_\alpha : \alpha < \beta\}\) be a well-ordering of the elements of \(U\) and let \(x_0 \in C \cap U_0(k)\). Suppose that the points \(\{x_\alpha : \gamma < \delta\}\) have already been chosen. Let \(\alpha_\delta\) be the first ordinal greater than \(\alpha_\gamma\), for each \(\gamma < \delta\), such that there exists \(x_\alpha \in (C \cap U_\alpha(k)) \setminus \left( \bigcup_{\gamma < \delta} U_\alpha \right) \cup \theta_{2k}(\{x_\alpha : \gamma < \delta\})\). We finish this selection when such a point \(x_\alpha\) does not exist, i.e. when \(C \subseteq \left( \bigcup_{\gamma < \delta} U_\alpha \right) \cup \theta_{2k}(\{x_\alpha : \gamma < \delta\})\).

Therefore if \(A\) is the set of selected points, then \(C \subseteq \theta_{2k}(A) \cup \bigcup \{U^x : x \in A\}\).

To finish the proof we need to show that \(A\) is an \(S(2k)\)-discrete set in \(X\) and therefore \(|A| \leq \kappa\). Let \(x_\alpha \in A\). Then \(x_\alpha \notin \theta_{2k}(\{x_\alpha : \gamma < \gamma_0\})\). Hence, there exists an open \(S(2k-1)\)-neighborhood \(V_\alpha\) of \(x_\alpha\) such that \(V_\alpha \cap \{x_\alpha : \gamma < \gamma_0\} = \emptyset\). Now, suppose that there exists \(\gamma_1\) such that \(x_{\alpha_1} \in A \cap \bigcap_{\gamma < \gamma_1} V_\alpha\). Since \(x_{\alpha_1} \in V_\alpha\), we have \(\gamma_1 \geq \gamma_0\). From \(x_{\alpha_1} \in \bigcap_{\gamma < \gamma_1} V_\alpha\) and \(x_{\alpha_1} \in U_{\alpha_1}(k) \setminus \left( \bigcup_{\gamma < \gamma_1} \bigcup_{\alpha_\gamma} \right) \cup \theta_{2k}(\{x_\alpha : \gamma < \gamma_1\})\) conclude that \(U_\alpha \subseteq \bigcup_{\gamma < \gamma_1} U_\alpha\), thus \(\gamma_0 \geq \gamma_1\). Hence \(\gamma_1 = \gamma_0\) and therefore \(\{x_\alpha\} = A \cap \bigcup_{\gamma < \gamma_1} U_\alpha\) i.e. \(A\) is an \(S(2k)\)-discrete set in \(X\).

The following theorem for the case \(k = 1\) was proved by Hajnal and Juhász [9].

**Theorem 3.8.** Let \(k \in \mathbb{N}^+\). If \(X\) is an \(S(k-1)\)-space, then \(|X| \leq 2^{\kappa_{k-1}(X)}\).

**Proof.** Let \(k \in \mathbb{N}^+, \kappa = s_{2k-1}(X)\psi_{2k-1}(X)\) and for every \(x \in X\) let \(\mathcal{B}(x) = \{B_\alpha^x : \alpha < \kappa\}\) be a family of open \(S(2k-1)\)-neighborhoods of \(x\) such that \(\{x\} = \bigcap B_\alpha^x : \alpha < \kappa\). By transfinite recursion we construct an increasing sequence \((A_\alpha)_{\alpha \leq \kappa^+}\) of subsets of \(X\) and a sequence of families of \(S(2k-1)\)-open sets \((U_\alpha)_{\alpha \leq \kappa^+}\) as follows:

(a) \(A_0 = \emptyset\) and \(A_0 = \{x_0\}\), where \(x_0\) is an arbitrary point of \(X\);
(b) \(|A_\alpha| \leq 2^{\alpha}\) for every \(\alpha < \kappa^+\);
(c) \(U_\alpha = \bigcup \{\mathcal{B}(x) : x \in \bigcup_{\beta < \alpha} A_\beta\}\), for every \(0 < \alpha < \kappa^+\);
(e) For $0 < \alpha < \kappa^+$, if $W$ is the union of at most $\kappa$ many elements of $\mathcal{U}_\alpha$ and $\{F_\gamma : \gamma < \kappa\}$ is a family of subsets of $\bigcup_{\beta < \alpha} A_\beta$ such that $|F_\gamma| \leq \kappa$ for every $\gamma < \kappa$, and $W \cup \bigcup_{\gamma < \kappa} \theta_{2k-1}(F_\gamma) \neq X$, then we pick a point $x_W \in X \setminus \left(W \cup \bigcup_{\gamma < \kappa} \theta_{2k-1}(F_\gamma)\right)$. Let $E$ be the set of all such points $x_W$. Then we set $A_\alpha = E \cup \bigcup_{\beta < \alpha} A_\beta$. Since $|\bigcup_{\beta < \alpha} A_\beta| \leq 2^\kappa$, we have $|\mathcal{U}_\alpha| \leq 2^\kappa$ and therefore $|E| \leq (2^\kappa)^\kappa = 2^\kappa$. Thus $|A_\alpha| \leq 2^\kappa$ and therefore (b) is satisfied.

Now, let $A = \bigcup_{\alpha < \kappa^+} A_\alpha$. Since $|A| \leq \kappa^+ \cdot 2^\kappa = 2^\kappa$, to finish the proof it is sufficient to show that $X = A$. Suppose that there exists a point $y \in X \setminus A$ and let $\mathcal{B}(y) = \{B_\delta : \delta < \kappa\}$. Then $X \setminus \{y\} = \bigcup\{C_\delta : \delta < \kappa\}$, where $C_\delta = X \setminus B_\delta$ whenever $\delta < \kappa$, and $y \notin \theta_{2k-1}(C_\delta)$ for each $\delta < \kappa$. For each $\delta < \kappa$ let $D_\delta = A \cap C_\delta$ and for each $x \in D_\delta$ let $B^x \in \mathcal{B}(x)$ be such that $y \notin B^x$. Now, we can apply Lemma 3.7 to each $D_\delta$ and the family $\{B^x : x \in D_\delta\}$ to obtain a subset $G_\delta$ of $D_\delta$ such that $|G_\delta| \leq \kappa$ and $D_\delta \subseteq \theta_{2k-1}(G_\delta) \cup \bigcup\{B^x : x \in G_\delta\}$. Let $W = \bigcup_{\delta < \kappa} \text{B}(x = G_\delta)$. Then $A \subseteq W \cup \bigcup_{\delta < \kappa} \theta_{2k-1}(G_\delta)$ and $y \notin W \cup \bigcup_{\delta < \kappa} \theta_{2k-1}(G_\delta)$. Using the fact that $|\bigcup_{\delta < \kappa} G_\delta| \leq \kappa$ and that $\kappa^+$ is a regular cardinal we can find $\alpha < \kappa^+$ such that $\bigcup_{\delta < \kappa} G_\delta \subseteq \mathcal{U}_\alpha$. Then it follows from the construction of the set $A_\alpha$ that there exists $x \in A_\alpha \cap \left(X \setminus \left(W \cup \bigcup_{\delta < \kappa} \theta_{2k-1}(G_\delta)\right)\right)$. Thus $x \in A_\alpha \subseteq A \subseteq W \cup \bigcup_{\delta < \kappa} \theta_{2k-1}(G_\delta)$ — contradiction. □

The next result for the case $k = 1$ was proved by Schröder (see [15] Theorem 8).

**Theorem 3.9.** Let $k \in \mathbb{N}^+$. If $X$ is an $S(k)$-space, then $|X| \leq 2^{2^{2k}(X)}$.

**Proof.** Let $k \in \mathbb{N}^+$, $\kappa = s_{2k}(X)\psi_{2k}(X)$ and for every $x \in X$ let $\mathcal{B}(x) = \{B_\alpha : \alpha < \kappa\}$ be a family of open $S(2k-1)$-neighborhoods of $x$ such that $\{x\} = \bigcap\{B_\alpha : \alpha < \kappa\}$. By transfinite recursion we construct an increasing sequence $(A_\alpha)_{\alpha < \kappa^+}$ of subsets of $X$ and a sequence of families of $S(2k-1)$-open sets $(\mathcal{U}_\alpha)_{\alpha < \kappa^+}$ as follows:

(a) $\mathcal{U}_0 = \emptyset$ and $A_0 = \{x_0\}$, where $x_0$ is an arbitrary point of $X$;
(b) $|A_\alpha| \leq 2^\kappa$ for every $\alpha < \kappa^+$;
(c) $\mathcal{U}_\alpha = \bigcup\{\mathcal{B}(x) : x \in \bigcup_{\delta < \alpha} A_\delta\}$, for every $0 < \alpha < \kappa^+$;
(e) For $0 < \alpha < \kappa^+$, if $W$ is the union of the closures of at most $\kappa$ many elements of $\mathcal{U}_\alpha$ and $\{F_\gamma : \gamma < \kappa\}$ is a family of subsets of $\bigcup_{\beta < \alpha} A_\beta$ such that $|F_\gamma| \leq \kappa$ for every $\gamma < \kappa$, and $W \cup \bigcup_{\gamma < \kappa} \theta_{2k}(F_\gamma) \neq X$, then we pick a point $x_W \in X \setminus \left(W \cup \bigcup_{\gamma < \kappa} \theta_{2k}(F_\gamma)\right)$. Let $E$ be the set of all such points $x_W$. Then we set $A_\alpha = E \cup \bigcup_{\beta < \alpha} A_\beta$. Since $|\bigcup_{\beta < \alpha} A_\beta| \leq 2^\kappa$, we have $|\mathcal{U}_\alpha| \leq 2^\kappa$ and therefore $|E| \leq (2^\kappa)^\kappa = 2^\kappa$. Thus $|A_\alpha| \leq 2^\kappa$ and therefore (b) is satisfied.

Now, let $A = \bigcup_{\alpha < \kappa^+} A_\alpha$. Since $|A| \leq \kappa^+ \cdot 2^\kappa = 2^\kappa$, to finish the proof it is sufficient to show that $X = A$. Suppose that there exists a point $y \in X \setminus A$ and let $\mathcal{B}(y) = \{B_\delta : \delta < \kappa\}$. Then $X \setminus \{y\} = \bigcup\{C_\delta : \delta < \kappa\}$, where $C_\delta = X \setminus B_\delta$ whenever $\delta < \kappa$, and $y \notin \theta_{2k}(C_\delta)$ for each $\delta < \kappa$. For each $\delta < \kappa$ let $D_\delta = A \cap C_\delta$ and for each $x \in D_\delta$ let $B^x \in \mathcal{B}(x)$ be such that $y \notin B^x$. Now, we can apply Lemma 3.7 to each $D_\delta$ and the family $\{B^x : x \in D_\delta\}$ to obtain a subset $G_\delta$ of $D_\delta$ such that $|G_\delta| \leq \kappa$ and $D_\delta \subseteq \theta_{2k}(G_\delta) \cup \bigcup\{B^x : x \in G_\delta\}$. Let $W = \bigcup_{\delta < \kappa} \{B^x : x \in G_\delta\}$. Then $A \subseteq W \cup \bigcup_{\delta < \kappa} \theta_{2k}(G_\delta)$ and $y \notin W \cup \bigcup_{\delta < \kappa} \theta_{2k}(G_\delta)$. Using the fact that $|\bigcup_{\delta < \kappa} G_\delta| \leq \kappa$ and that $\kappa^+$ is a regular cardinal we can find $\alpha < \kappa^+$ such that
Lemma 3.10. Let $\kappa = s_{2k-1}(X)$ and $x \in X$. For each $y \in X \setminus \{x\}$ let $U^y_x$ and $U_y$ be open $S(2k-1)$-neighborhoods of $x$ and $y$, respectively, such that $U^y_x \cap U_y = \emptyset$. Then for the set $C = X \setminus \{x\}$ and the family $U = \{U_y : y \in C\}$ we can use Lemma 3.7 to find an $S(2k-1)$-discrete subset $A$ of $C$ such that $|A| \leq \kappa$ and $C \subseteq \theta_{2k-1}(A) \cup \bigcup \{U_y : y \in A\}$.

For each $z \in \theta_{2k-1}(A)$ and for each open $S(2k-1)$-neighborhood $U$ of $z$ we can choose a point $z_U \in U \cap U_z \cap A$ and let us denote by $F_z$ the resulting set. Then $F_z \subseteq A$, $z \in \theta_{2k-1}(F_z) \subseteq \theta_{2k-1}(U_z)$ and since $U^z_x \cap U_z = \emptyset$ we have $x \notin \theta_{2k-1}(U_z)$. Thus $x \notin \theta_{2k-1}(F_z)$.

Let $F = \{F : F \subseteq A, x \notin \theta_{2k-1}(F)\}$ and for each $F \in F$ let $U^F_x = X \setminus \theta_{2k-1}(F)$. Then $|F| \leq 2^\kappa$ and $\theta_{2k-1}(A) \subseteq \bigcup \{\theta_{2k-1}(F) : F \in F\}$. Therefore $\{X \setminus U^F_y : y \in A\} \cup \{U^F_x : F \in F\}$ is an open pseudobase of $x$ with cardinality $\leq 2^\kappa$.

Lemma 3.11. Let $k \in \mathbb{N}^+$. For every $S(2k)$-space $X$, $\psi(X) \leq 2^{s_{2k}(X)}$.

Proof. Let $\kappa = s_{2k}(X)$ and $x \in X$. For each $y \in X \setminus \{x\}$ let $U^y_x$ and $U_y$ be open $S(2k)$-neighborhoods of $x$ and $y$, respectively, such that $U^y_x \cap U_y = \emptyset$. Then for the set $C = X \setminus \{x\}$ and the family $U = \{U_y : y \in C\}$ we can use Lemma 3.7 to find an $S(2k)$-discrete subset $A$ of $C$ such that $|A| \leq \kappa$ and $C \subseteq \theta_{2k}(A) \cup \bigcup \{U_y : y \in A\}$.

For each $z \in \theta_{2k}(A)$ and for each open $S(2k)$-neighborhood $U$ of $z$ we can choose a point $z_U \in U \cap U_z \cap A$ and let us denote by $F_z$ the resulting set. Then $F_z \subseteq A$, $z \in \theta_{2k}(F_z) \subseteq \theta_{2k}(U_z)$ and since $U^z_x \cap U_z = \emptyset$ we have $x \notin \theta_{2k}(U_z)$. Thus $x \notin \theta_{2k}(F_z)$.

Let $F = \{F : F \subseteq A, x \notin \theta_{2k}(F)\}$ and for each $F \in F$ let $U^F_x = X \setminus \theta_{2k}(F)$. Then $|F| \leq 2^\kappa$ and $\theta_{2k}(A) \subseteq \bigcup \{\theta_{2k}(F) : F \in F\}$. Therefore $\{X \setminus U^F_y : y \in A\} \cup \{U^F_x : F \in F\}$ is an open pseudobase of $x$ with cardinality $\leq 2^\kappa$.

As a corollary of the previous two lemmas we obtain the following theorem:

Theorem 3.12. Let $n \in \mathbb{N}^+$. If $X$ is an $S(n)$-space, then $|X| \leq 2^{\psi(X)}$.

Proof. The claim follows directly from the fact that every $S(n)$-space is a $T_1$-space, Hajnal and Juhász theorem (Theorem 3.8 for $k = 1$), Lemma 3.11 for $n$-odd and Lemma 3.11 for $n$-even.

When $n = 1$ in Theorem 3.12 we obtain Hajnal and Juhász theorem that if $X$ is a Hausdorff space, then $|X| \leq 2^{\psi(X)}$ and when $n = 2$, Theorem 3.12 states that if $X$ is a Urysohn space, then $|X| \leq 2^{\psi(X)} = 2^{\psi_2(X)}$.

We note that in [15, Lemma 11] the author claims that if $X$ is a Urysohn space, then $\psi(X) \leq 2^{U_\ast(X)}$, or in our notation, $\psi_2(X) \leq 2^{s_2(X)}$. Based on that lemma
the author concluded that for every Urysohn space we have \(|X| \leq 2^{\omega_1}|X|\).
Unfortunately, there is a gap in the proof of Lemma 11 in [15] and therefore the question whether or not either one of both of these claims is true is open. What we can prove in that relation, in addition to the case \(n = 2\) of Theorem 3.12 is the case \(k = 1\) of Lemma 3.13 and Theorem 3.18 (see Corollaries 3.14 and 3.19).

**Lemma 3.13.** Let \(k \in \mathbb{N}^+\). For every \(S(3k)\)-space \(X\), \(\psi_{2k}(X) \leq 2^{s_2k(X)}\).

**Proof.** Let \(\kappa = s_2k(X)\) and \(x \in X\). Since \(X\) is an \(S(3k)\)-space, for each \(y \in X \setminus \{x\}\) let \(U^y_1(x), \ldots, U^y_k(x), U_1(y), \ldots, U_{2k}(y)\) be open sets in \(X\) such that \(x \in U^y_1(x) \subseteq \bigcup U^y_1(x) \subseteq \ldots \subseteq U^y_k(x), y \in U_1(y) \subseteq \bigcup U_1(y) \subseteq \ldots \subseteq U_{2k}(y)\) and \(U^y_k(x) \cap U_{2k}(y) = \emptyset\).

Then for each \(y \in C\), \(U_k(y)\) is an open \(S(2k - 1)\)-neighborhood of \(y\). Therefore for the set \(C = X \setminus \{x\}\) and the family \(\mathcal{U} = \{U_k(y) : y \in C\}\) we can use Lemma 3.2.1 to find an \(S(2k)\)-discrete subset \(A\) of \(C\) such that \(|A| \leq \kappa\) and \(C \subseteq \theta_{2k}(A) \cup \bigcup \{U_k(y) : y \in A\}\).

For each \(z \in \theta_{2k}(A)\) and for each open \(S(2k - 1)\)-neighborhood \(U\) of \(z\) we can choose a point \(z_U \in U \cap U_k(z)\) and let us denote by \(F\) the resulting set. Then \(F \subseteq A\), \(z \in \theta_{2k}(F)\) and \(\bigcup F \cap \theta_{2k}(F) = \emptyset\). Hence \(F \subseteq \theta_{2k}(A)\) and \(F \subseteq \bigcup \{U_k(x) : x \in A\}\).

**Corollary 3.14.** For every \(S(3)\)-space \(X\), \(\psi_2(X) \leq 2^{s_2(X)}\).

For the case \(k = 1\) of the following lemma see [10] Proposition 4.11.

**Lemma 3.15.** Let \(k \in \mathbb{N}^+\). For every \(S(3k - 2)\)-space \(X\), \(\psi_{2k-1}(X) \leq 2^{s_{2k-1}(X)}\).

**Proof.** Let \(\kappa = s_{2k-1}(X)\) and \(x \in X\). Since \(X\) is an \(S(3k - 2)\)-space, for each \(y \in X \setminus \{x\}\) let \(U^y_1(x), \ldots, U^y_k(x), U_1(y), \ldots, U_{2k-1}(y)\) be open sets in \(X\) such that \(x \in U^y_1(x) \subseteq \bigcup U^y_1(x) \subseteq \ldots \subseteq U^y_k(x), y \in U_1(y) \subseteq \bigcup U_1(y) \subseteq \ldots \subseteq U_{2k-1}(y)\) and \(U^y_k(x) \cap U_{2k-1}(y) = \emptyset\).

Then for each \(y \in C\), \(U_k(y)\) is an open \(S(2k - 1)\)-neighborhood of \(y\). Therefore for the set \(C = X \setminus \{x\}\) and the family \(\mathcal{U} = \{U_k(y) : y \in C\}\) we can use Lemma 3.2.1 to find an \(S(2k - 1)\)-discrete subset \(A\) of \(C\) such that \(|A| \leq \kappa\) and \(C \subseteq \theta_{2k-1}(A) \cup \bigcup \{U_k(y) : y \in A\}\).

For each \(z \in \theta_{2k-1}(A)\) and for each open \(S(2k - 1)\)-neighborhood \(U\) of \(z\) we can choose a point \(z_U \in U \cap U_k(z)\) and let us denote by \(F\) the resulting set. Then \(F \subseteq A\), \(z \in \theta_{2k-1}(F)\) and \(\bigcup F \cap \theta_{2k-1}(F) = \emptyset\). Hence \(F \subseteq \theta_{2k-1}(A)\) and \(F \subseteq \bigcup \{U_k(x) : x \in A\}\).

Similarly, for the case \(3k - 1\), \(k \in \mathbb{N}^+\), one can prove the following:

**Lemma 3.16.** Let \(k \in \mathbb{N}^+\). For every \(S(3k - 1)\)-space \(X\), \(\psi_{2k-1}(X) \leq 2^{s_{2k}(X)}\).
Proof. Let \( \kappa = s_{2k}(X) \) and \( x \in X \). Since \( X \) is an \( S(3k - 1) \)-space, for each \( y \in X \setminus \{x\} \) let \( U_1^y(x), \ldots, U_k^y(x), U_1(y), \ldots, U_{2k}(y) \) be open sets in \( X \) such that \( x \in U_1^y(x) \subseteq \overline{U_1^y(x)} \subseteq \ldots \subseteq U_k^y(x), y \in U_1(y) \subseteq \overline{U_1(y)} \subseteq \ldots \subseteq U_{2k}(y) \) and \( U_k^y(x) \cap U_{2k}(y) = \emptyset \). Then for each \( y \in C, U_y(k) \) is an open \( S(2k - 1) \)-neighborhood of \( y \). Therefore for the set \( C = X \setminus \{x\} \) and the family \( \mathcal{U} = \{U_y(k) : y \in C\} \) we can use Lemma 3.7 to find an \( S(2k) \)-discrete subset \( A \) of \( C \) such that \( |A| \leq \kappa \) and \( C \subseteq \theta_{2k}(A) \cup \bigcup \{U_y(k) : y \in A\} \).

For each \( z \in \theta_{2k}(A) \) and for each open \( S(2k - 1) \)-neighborhood \( U \) of \( z \) we can choose a point \( z_U \in U \cap \overline{U}(z) \cap A \) and let us denote by \( F_z \) the resulting set. Then \( F_z \subseteq A, z \in \theta_{2k}(F_z) \subseteq \theta_{2k}(U(z)) \subseteq U_{2k}(z) \) and since \( U_z^x(z) \cap U_{2k}(z) = \emptyset \) we have \( x \notin \theta_{2k-1}(U(z)) \), hence \( x \notin \theta_{2k-1}(F_z) \) and clearly, \( U_z^x(z) \cap \theta_{2k}(F_z) = \emptyset \).

Let \( \mathcal{F} = \{F : F \subseteq A, x \notin \theta_{2k-1}(F)\} \) and for each \( F \in \mathcal{F} \) let \( F \) be an open \( S(2k - 1) \)-neighborhood of \( x \) such that \( U_z^x(z) \cap \theta_{2k}(F) = \emptyset \). Then \( |\mathcal{F}| \leq 2^\kappa \) and \( \theta_{2k}(A) \subseteq \bigcup \{\theta_{2k}(F) : F \in \mathcal{F}\} \). Therefore the existence of the family \( \{U_z^x(z) : y \in A\} \cup \{U_z^x(z) : F \in \mathcal{F}\} \) shows that the \( S(2k - 1) \)-pseudocharacter at \( x \) has cardinality \( \leq 2^\kappa \).

Lemma 3.17. For every \( S(3k - 1) \)-space \( X, \psi_{2k}(X) \leq 2^{s_{2k-1}(X)} \).

Proof. Let \( \kappa = s_{2k-1}(X) \) and \( x \in X \). Since \( X \) is an \( S(3k - 1) \)-space, for each \( y \in X \setminus \{x\} \) let \( U_1^y(x), \ldots, U_k^y(x), U_1(y), \ldots, U_{2k}(y) \) be open sets in \( X \) such that \( x \in U_1^y(x) \subseteq \overline{U_1^y(x)} \subseteq \ldots \subseteq U_k^y(x), y \in U_1(y) \subseteq \overline{U_1(y)} \subseteq \ldots \subseteq U_{2k}(y) \) and \( U_k^y(x) \cap U_{2k}(y) = \emptyset \). Then for each \( y \in C, U_y(k) \) is an open \( S(2k - 1) \)-neighborhood of \( y \). Therefore for the set \( C = X \setminus \{x\} \) and the family \( \mathcal{U} = \{U_y(k) : y \in C\} \) we can use Lemma 3.6 to find an \( S(k) \)-discrete subset \( A \) of \( C \) such that \( |A| \leq \kappa \) and \( C \subseteq \theta_{2k}(A) \cup \bigcup \{U_y(k) : y \in A\} \).

Now, as corollaries of the previous four lemmas we obtain the following results:

Theorem 3.18. Let \( k \in \mathbb{N}^+ \). If \( X \) is an \( S(3k) \)-space, then \( |X| \leq 2^{s_{2k}(X)} \).

Proof. The claim follows directly from the fact that every \( S(3k) \)-space is an \( S(k) \)-space, Theorem 3.9 and Lemma 3.13.

Corollary 3.19. If \( X \) is an \( S(3) \)-space, then \( |X| \leq 2^{s_{2}(X)} \).

Theorem 3.20. Let \( k \in \mathbb{N}^+ \). If \( X \) is an \( S(3k - 2) \)-space, then \( |X| \leq 2^{s_{2k-1}(X)} \).

Proof. The claim follows directly from the fact that every \( S(3k - 2) \)-space is an \( S(k - 1) \)-space, Theorem 3.8 and Lemma 3.13.

When \( n = 1 \) in Theorem 3.20 we obtain again Hajnal and Juhász theorem that if \( X \) is a Hausdorff space, then \( |X| \leq 2^{s_{2}(X)} \).
Theorem 3.21. Let $k \in \mathbb{N}^+$. If $X$ is an $S(3k-1)$-space, then $|X| \leq 2^{s_{2k-1}(X)} - 2^{s_k(X)}$.

Proof. The claim follows directly from the fact that every $S(3k - 1)$-space is an $S(k - 1)$- and $S(k)$-space, Theorem 3.8, Theorem 3.9, Lemma 3.16, Lemma 3.17, and the fact that $s_{2k}(X) \leq s_{2k-1}(X)$ for every $k \in \mathbb{N}^+$.

References

[1] O. T. Alas and Lj. D. Kočinac, More cardinal inequalities on Urysohn spaces, Math. Balkanica (N.S.) 14 (2000), no. 3-4, 247–251.
[2] A. Charlesworth, On the cardinality of a topological space, Proc. Amer. Math. Soc. 66 (1977), no. 1, 138–142.
[3] D. Dikranjan and E. Giuli, $S(n)$-$\theta$-closed spaces, Topology Appl. 28 (1988), no. 1, 59–74.
[4] D. Dikranjan and W. Tholen, Categorical structure of closure operators. With applications to topology, algebra and discrete mathematics., Mathematics and its Applications, vol. 346, Kluwer Academic Publishers Group, Dordrecht, 1995.
[5] D. Dikranjan and S. Watson, The category of $S(\alpha)$-spaces is not cowellpowered, Topology Appl. 61 (1995), no. 2, 137–150.
[6] R. Engelking, General topology, second ed., Sigma Series in Pure Mathematics, vol. 6, Heldermann Verlag, Berlin, 1989 (Translated from the Polish by the author).
[7] I. S. Gotchev, Cardinal inequalities for Urysohn spaces involving variations of the almost Lindelöf degree, to appear in Serdica Math. J. 44 (2018).
[8] I. S. Gotchev and Lj. D. R. Kočinac, More on the cardinality of $S(n)$-spaces, to appear in Serdica Math. J. 44 (2018).
[9] A. Hajnal and I. Juhász, Discrete subspaces of topological spaces, Nederl. Akad. Wetensch. Proc. Ser. A 70=Indag. Math. 29 (1967), 343–356.
[10] R. E. Hodel, Cardinal functions. I, Handbook of set-theoretic topology, North-Holland, Amsterdam, 1984, pp. 1–61.
[11] I. Juhász, Cardinal functions in topology—ten years later, second ed., Mathematical Centre Tracts, vol. 123, Mathematisch Centrum, Amsterdam, 1980.
[12] R. Pol, Short proofs of two theorems on cardinality of topological spaces, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 22 (1974), 1245–1249.
[13] J. R. Porter and C. Votaw, $S(\alpha)$ spaces and regular Hausdorff extensions, Pacific J. Math. 45 (1973), 327–345.
[14] J. Schröder, Cardinality of $\theta$-closure, Questions Answers Gen. Topology 10 (1992), no. 1, 73–74.
[15] N. V. Veličko, Urysohn cellularity and Urysohn spread, Math. Japon. 38 (1993), no. 6, 1129–1133.
[16] N. V. Veličko, $H$-closed topological spaces, Mat. Sb. (N.S.) 70 (112) (1966), 98–112.
[17] G. Viglino, $T_\alpha$-spaces, Kyungpook Math. J. 11 (1971), 33–35.

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