A note on the Petri loci

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Abstract

Let $M_g$ be the coarse moduli space of complex projective nonsingular curves of genus $g$. We prove that when the Brill-Noether number $\rho(g, r, n)$ is non-negative every component of the Petri locus $P^r_{g,n} \subset M_g$ whose general member is a curve $C$ such that $W^{r+1}_n(C) = \emptyset$, has codimension one in $M_g$.

1 Introduction

Let $C$ be a nonsingular irreducible projective curve of genus $g \geq 2$ defined over $\mathbb{C}$. A pair $(L, V)$ consisting of an invertible sheaf $L$ on $C$ and of an $(r + 1)$-dimensional vector subspace $V \subset H^0(L)$, $r \geq 0$, is called a linear series of dimension $r$ and degree $n = \deg(L)$, or a $g^r_n$. If $V = H^0(L)$ then the $g^r_n$ is said to be complete.

If $(L, V)$ is a $g^r_n$ then the Petri map for $(L, V)$ is the natural multiplication map

$$\mu_0(L, V) : V \otimes H^0(\omega_C L^{-1}) \longrightarrow H^0(\omega_C)$$

The Petri map for $L$ is

$$\mu_0(L) : H^0(L) \otimes H^0(\omega_C L^{-1}) \longrightarrow H^0(\omega_C)$$

Recall that $C$ is called a Petri curve if the Petri map $\mu_0(L)$ is injective for every invertible sheaf $L$ on $C$. By the Gieseker-Petri theorem \[5\] we know that in $M_g$, the coarse moduli space of nonsingular projective curves of genus $g$, the locus of curves which are not Petri is a proper closed subset $P_g$, called the Petri locus. This locus decomposes into several components, according to the numerical types and to other properties that linear series can have on a curve of genus $g$. We will say that $C$ is Petri with respect to $g^r_n$’s if the Petri map $\mu_0(L,V)$ is injective for every $g^r_n$ $(L, V)$ on $C$.

We denote by $P^r_{g,n} \subset M_g$ the locus of curves which are not Petri w.r. to $g^r_n$’s. Then

$$P_g = \bigcup_{r,n} P^r_{g,n}$$

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where the union is finite by obvious reasons. The structure of $P^r_{g,n}$ and of $P_g$ is not known in general: both might a priori have several components and not be equidimensional. In some special cases $P^r_{g,n}$ is known to be of pure codimension one (notably in the obvious case $\rho(g, r, n) = 0$, and for $r = 1$ and $n = g - 1$ [12]). If the Brill-Noether number

$$\rho(g, r, n) := g - (r + 1)(g - n + r)$$

is nonnegative then it is natural to conjecture that $P^r_{g,n}$ has pure codimension one if it is non-empty. The evidence is the fact that $P^r_{g,n}$ is the image in $M_g$ of a determinantal scheme $\tilde{P}^r_{g,n}$ inside the relative Brill-Noether scheme $W^r_n \to M_g$, and the expected dimension of $\tilde{P}^r_{g,n}$ is $3g - 4$. This is the point of view that we apply for the proof of our main theorem 1.1 (see below). One might ask if even $P_g$ has pure codimension one: there is not much evidence for this, except that it can be directly checked to be true for low values of $g$ (see the very recent preprint by M. Lelli-Chiesa [10]).

Before stating our result we recall what is known. Denote by $\overline{M}_g$ the moduli space of stable curves, and let

$$\overline{M}_g \setminus M_g = \Delta_0 \cup \cdots \cup \Delta_{\lfloor \frac{g}{2} \rfloor}$$

be its boundary, in standard notation. In [2] G. Farkas has proved the existence of at least one divisorial component of $P^1_{g,n}$ in case $\rho(g, 1, n) \geq 0$ and $n \leq g - 1$, using the theory of limit linear series. He found a divisorial component which has a nonempty intersection with $\Delta_1$. Another proof has been given in [1], by degeneration to a stable curve with $g$ elliptic tails. The method of [2] has been extended in [3] to arbitrary $r$. In this note without using any degeneration argument we prove the following result:

**Theorem 1.1** If $\rho(g, r, n) \geq 0$ then every component of $P^r_{g,n}$ whose general member is a curve $C$ such that $W^{r+1}_n(C) = \emptyset$, has codimension one in $M_g$.

Note that a necessary numerical condition for the existence of a curve $C$ as in the statement is that $\rho(g, r + 1, n) < 0$. This condition, together with $\rho(g, r, n) \geq 0$ gives:

$$0 \leq \rho(g, r, n) < g - n + 2(r + 1)$$

or, equivalently:

$$\frac{r}{r + 1}g + r \leq n < \frac{r + 1}{r + 2}g + r + 1$$

For the proof of the theorem we introduce a modular family $C \to B$ of curves of genus $g$ (see (i) below for the definition) and we use the determinantal description of the relative locus $W^r_n(C/B)$ over $B$ and of the naturally defined closed subscheme $\tilde{P}^r_{g,n} \subset W^r_n(C/B)$ whose image in $M_g$ is $P^r_{g,n}$. Since it is a determinantal locus, every component of $\tilde{P}^r_{g,n}$ has dimension $\geq 3g - 4$. Then a theorem of F. Steffen [11] ensures that every component of $P^r_{g,n}$ has dimension $\geq 3g - 4$ as well, thus proving the result.
In a forthcoming paper (in preparation) we will show the existence of a divisorial component of \( P^1_{g,n} \) which has a non-empty intersection with \( \Delta_0 \), when \( \rho(g,1,n) \geq 1 \).

2 Proof of Theorem 1.1

In this section we fix \( g,r,n \) such that \( \rho(g,r,n) \geq 0 \) and \( \rho(g,r+1,n) < 0 \). Consider the following diagram:

\[
\begin{array}{ccc}
J_n(C/B) \times_B C & \longrightarrow & C \\
\downarrow & & \downarrow f \\
J_n(C/B) & \longrightarrow & B
\end{array}
\]

where:

(i) \( f \) is a smooth modular family of curves of genus \( g \) parametrized by a non-singular quasi-projective algebraic variety \( B \) of dimension \( 3g - 3 \). This means that at each closed point \( b \in B \) the Kodaira-Spencer map \( \kappa_b : T_bB \rightarrow H^1(C(b), T_{C(b)}) \) is an isomorphism. In particular, the functorial morphism

\[ \beta : B \longrightarrow \mathcal{M}_g \]

is finite and dominant. The existence of \( f \) is a standard fact, see e.g. [7], Theorem 27.2.

(ii) \( J_n(C/B) \) is the relative Picard variety parametrizing invertible sheaves of degree \( n \) on the fibres of \( f \).

(iii) For all closed points \( b \in B \) the fibre \( C(b) \) satisfies \( W^{r+1}_n(C(b)) = \emptyset \). This condition can be satisfied modulo replacing \( B \) by an open subset if necessary, because the condition \( W^{r+1}_n(C(b)) = \emptyset \) is open w.r. to \( b \in B \).

(iv) We may even assume that any given specific curve \( C \) of genus \( g \) satisfying \( W^{r+1}_n(C) = \emptyset \) appears among the fibres of \( f \). In particular we may assume that the dense subset \( \text{Im} (\beta) \subset \mathcal{M}_g \) has a non-empty intersection with all irreducible components of \( P^r_{g,n} \) whose general element parametrizes a curve \( C \) such that \( W^{r+1}_n(C) = \emptyset \).

Let \( \mathcal{P} \) be a Poincaré invertible sheaf on \( J_n(C/B) \times_B C \). Using \( \mathcal{P} \) in a well-known way one constructs the relative Brill-Noether scheme

\[ W^{\nu}_n(C/B) \subset J_n(C/B) \]
Consider the restriction of diagram (1) over $W_n(C/B)$:

$$
\begin{array}{c}
W_n(C/B) \times_B C \\
p_1 \\
W_n(C/B)
\end{array}
\xrightarrow{p_2 \circ f} C
$$

Every irreducible component of $W_n(C/B)$ has dimension $\geq 3g - 3 + \rho(g, r, n)$ and, since $\rho(g, r, n) \geq 0$, there is a component which dominates $B$. A closed point $w \in W_n(C/B)$ represents an invertible sheaf $L_w$ on the curve $C(q(w))$ such that $h^0(L_w) \geq r + 1$. Denoting again by $P$ the restriction of $P$ to $W_n(C/B) \times_B C$, we have a homomorphism of coherent sheaves on $W_n(C/B)$, induced by multiplication of sections along the fibres of $p_1$:

$$
\mu_0(P) : p_1^* P \otimes p_1^*[p_2^*(\omega_{C/B}) \otimes P^{-1}] \rightarrow p_1^*[p_2^*\omega_{C/B}]
$$

By condition (iii) above, these sheaves are locally free, of ranks $(r + 1)(g - n + r)$ and $g$ respectively. Moreover, by definition, at each point $w \in W_n(C/B)$, the map $\mu_0(P)$ coincides with the Petri map $\mu_0(L_w) : H^0(C(q(w)), L_w) \otimes H^0(C(q(w)), \omega_{C(q(w))}L_w^{-1}) \rightarrow H^0(C(q(w)), \omega_{C(q(w))})$.

Claim: the vector bundle

$$
[p_1^* P \otimes p_1^*[p_2^*(\omega_{C/B}) \otimes P^{-1}]]^\vee \otimes p_1^*[p_2^*\omega_{C/B}]
$$

is $q$-relatively ample.

Proof of the Claim. If we restrict diagram (2) over any $b \in B$ and we let $C = C(b)$, we obtain:

$$
\begin{array}{c}
W_n(C) \times C \\
\pi_1 \\
W_n(C)
\end{array}
\xrightarrow{\pi_2} C
$$

and the map $\mu_0(P)$ restricts over $W_n(C)$ to

$$
m_P : \pi_1^* P \otimes \pi_1^*[\pi_2^*\omega_C \otimes P^{-1}] \rightarrow H^0(C, \omega_C) \otimes \mathcal{O}_{W_n}
$$

where $P = P|_{W_n(C) \times C}$ is a Poincaré sheaf on $W_n(C) \times C$. The dual of the source of $m_P$ is an ample vector bundle (compare [3, §2]), while the target is a trivial vector bundle, and therefore

$$
[p_1^* P \otimes p_1^*[p_2^*(\omega_{C}) \otimes P^{-1}]]^\vee \otimes C H^0(C, \omega_C)
$$

is an ample vector bundle. This means that the vector bundle (3) restricts to an ample vector bundle on the fibres of $q$. This implies, by [6], Th. 4.7.1, applied
to the invertible sheaf $\mathcal{O}(1)$ on the projective bundle associated to (3), that (3) is $q$-relatively ample. This proves the Claim.

Consider the degeneracy scheme:

$$\tilde{P}_{g,n} := D_{(r+1)(g-n+r)-1}(\mu_0(\mathcal{P})) \subset W_n^r(C/B)$$

which is supported on the locus of $w \in W_n^r(C/B)$ such that $\mu_0(L_w)$ is not injective. By applying Theorem 0.3 of [11] to it we deduce that every irreducible component of $q(\tilde{P}_{g,n}) \subset B$ has dimension at least

$$\dim[W_n^r(C/B)] - [g - (r + 1)(g - n + r) + 1] = 3g - 4$$

Since $f$ is a modular family, it follows that every irreducible component of $\beta(q(\tilde{P}_{g,n})) \subset M_g$ has dimension $\geq 3g - 4$ as well. But $\beta(q(\tilde{P}_{g,n})) \subset P^r_{g,n} \neq M_g$, and therefore all the components of $\beta(q(\tilde{P}_{g,n}))$ are divisorial. Since, by (iv), $\beta(q(\tilde{P}_{g,n}))$ is the union of all the components of $P^r_{g,n}$, whose general element parametrizes a curve $C$ such that $W_n^{r+1}(C) = \emptyset$, the theorem is proved. $\square$

References

[1] A. Castorena, M. Teixidor i Bigas: Divisorial components of the Petri locus for pencils, J. Pure Appl. Algebra 212 (2008), 1500–1508.

[2] G. Farkas: Gaussian maps, Gieseker-Petri loci and large theta-characteristics, J. reine angew. Mathematik 581 (2005), 151-173.

[3] G. Farkas: Rational maps between moduli spaces of curves and Gieseker-Petri divisors, Journal of Algebraic Geometry 19 (2010), 243-284.

[4] W. Fulton - R. Lazarsfeld: On the connectedness of degeneracy loci and special divisors, Acta Math. 146 (1981), 271-283.

[5] D. Gieseker: Stable curves and special divisors, Inventiones Math. 66 (1982), 251-275.

[6] A. Grothendieck: Elements de Geometrie Algebrique III, 1. Inst. de Hautes Etudes Sci. Publ. Math. 11, 1961.

[7] R. Hartshorne: Deformation Theory, Springer GTM vol.257 (2010).

[8] Kempf G.: Schubert methods with an application to algebraic curves, Publication of Mathematisch Centrum, Amsterdam 1972.

[9] Kleiman S., Laksov D.: On the existence of special divisors, Amer. Math. J. 94 (1972), 431-436.

[10] M. Lelli-Chiesa: The Gieseker-Petri divisor in $M_g$ for $g \leq 13$. arXiv preprint n. 1012.3061v1.
[11] F. Steffen: A generalized principal ideal theorem with an application to Brill-Noether theory. *Inventiones Math.* 132 (1998), 73-89.

[12] M. Teixidor: The divisor of curves with a vanishing theta null, *Compositio Math.* 66 (1988), 15-22.

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