PERIODIC SHADOWING OF VECTOR FIELDS

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Abstract. A vector field has the periodic shadowing property if for any \( \varepsilon > 0 \) there is \( d > 0 \) such that, for any periodic \( d \)-pseudo orbit \( g \) there exists a periodic orbit or a singularity in which \( g \) is \( \varepsilon \)-shadowed. In this paper, we show that a vector field is in the \( C^{1} \) interior of the set of vector fields satisfying the periodic shadowing property if and only if it is \( \Omega \)-stable. More precisely, we prove that the \( C^{1} \) interior of the set of vector fields satisfying the orbital periodic shadowing property is a subset of the set of \( \Omega \)-stable vector fields.

1. Introduction. The structural stability of diffeomorphisms and vector fields is one of the most important topics in the global qualitative theory of dynamical systems in the last several decades. An important property of structurally stable systems is the shadowing property, which is also known as the pseudo orbit tracing property. Generally speaking, the shadowing property means that near a sufficiently precise approximate orbit of a dynamical system, there exists an exact orbit. For more details, we refer to the monographs [9, 11].

The theory of the shadowing property was originally introduced by Anosov [1] and Bowen [3]. From these two references one knows that a diffeomorphism has 2010 Mathematics Subject Classification. Primary: 37C50, 37C20; Secondary: 37J45.

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the shadowing property in a neighborhood of a hyperbolic set and a structurally stable diffeomorphism has the shadowing property on the whole manifold [7, 15, 17]. Moreover, in a neighborhood of a hyperbolic set, the shadowing property is Lipschitz.

Sakai [16] showed that the $C^1$ interior of the set of diffeomorphisms with the shadowing property coincides with the set of structurally stable diffeomorphisms. However, the structural stability is not equivalent to shadowing because there exist some counterexamples [12] of the diffeomorphism which is not structurally stable but has the shadowing property. Recently, it was proved in [12, 14] that the Lipschitz shadowing and the so-called variational shadowing are equivalent to the structural stability. Osipov, Pilyugin and Tikhomirov [8] studied the relationship between the periodic shadowing and the $\Omega$-stability. They demonstrated that the $C^1$ interior of the set of diffeomorphisms having the periodic shadowing property coincides with the set of $\Omega$-stable diffeomorphisms. Moreover, it was illustrated that the Lipschitz periodic shadowing property is equivalent to the $\Omega$-stability [14].

Similar questions were posed for vector fields. It is known that there are some essential differences between the cases of discrete dynamical systems generated by diffeomorphisms and systems with continuous time generated by vector fields. One of key differences between diffeomorphisms and vector fields is the possibility of accumulation of recurrent orbits to a singularity. Pilyugin and Tikhomirov [13] constructed an example of a vector field which contains singularities and is in the $C^1$ interior of the set of vector fields with the shadowing property, but it is not structurally stable. However, Lee and Sakai [6] proved that if a vector field without singularities is in the $C^1$ interior of the set of vector fields with the shadowing property, then it satisfies Axiom A and the strong transversality condition, and so it is structurally stable.

In a very recent paper [10], Palmer, Pilyugin and Tikhomirov studied the periodic shadowing property, which shows that periodic pseudo orbits can be approximated by exact periodic trajectories. They showed that a vector field satisfies the Lipschitz periodic shadowing property if and only if the vector field is $\Omega$-stable. It becomes clear that the Lipschitz periodic shadowing property is strictly weaker than the standard shadowing property [4, 10, 13].

In this paper, we are concerned with the vector field satisfying the (orbital) periodic shadowing property, and show that every vector field satisfying the $C^1$ robustly (orbital) periodic shadowing property is $\Omega$-stable. It is worth noting that Pilyugin [12] gave an example of a diffeomorphism that has the shadowing property but is not $\Omega$-stable. By analogy with this, one may construct a vector field that has the (orbital) periodic shadowing property but is not $\Omega$-stable either, and so it does not have the Lipschitz periodic shadowing property. Apparently, the (orbital) periodic shadowing property is strictly weaker than the Lipschitz periodic shadowing property.

The paper is organized as follows. In Section 2, we present some preliminaries and summarize main result of this paper. In Section 3, we present some results regarding star vector fields. Section 4 is dedicated to the local linear structure of a singularity with a homoclinic connection. Finally, we demonstrate the proof of our main result in Section 5.
2. Preliminaries and main result. In order to state our discussions in a straightforward way, in this section we present some basic definitions and the related terminologies which may help us better understand the obtained results and proofs described in Sections 4 and 5.

Let $M$ be a compact smooth Riemannian manifold without boundary. Denote by $\mathcal{X}^1(M)$ the set of $C^1$ vector fields on $M$. Let

$$\text{Sing}(X) = \{x \in M : X(x) = 0\}$$

be the set of singularities of $X$. For any $X \in \mathcal{X}^1(M)$, $X$ generates a $C^1$ flow

$$\phi_t = \phi_{X,t} : M \rightarrow M, \ t \in \mathbb{R}.$$

Denote by $\text{Orb}(x) = \phi_{[-\infty, +\infty]}(x)$ and $\text{Orb}^+(x) = \phi_{[0, +\infty]}(x)$ the orbit and forward orbit of $x$, respectively.

For any $d > 0$, a map $g : \mathbb{R} \rightarrow M$ (not necessarily continuous) is called a $d$-pseudo orbit, provided that

$$\text{dist}(g(t + \tau), \phi_\tau(g(t))) < d$$

for all $t \in \mathbb{R}$ and $\tau \in [0, 1]$. Moreover, we say a $d$-pseudo orbit to be $g$ periodic, if there exists a positive constant $T > 0$ such that

$$g(t + T) = g(t)$$

for all $t$.

We say that a vector field $X \in \mathcal{X}^1(M)$ has the periodic shadowing property if for any $\varepsilon > 0$, there exists a constant $d > 0$ such that for any periodic $d$-pseudo orbit $g$ of $X$, $g$ is $\varepsilon$-shadowed by a periodic orbit or a singularity of $X$. That is, there exists a point $x \in M$ which is a periodic point or a singularity of $X$ and is an increasing homeomorphism (reparametrization) $h(t)$ of the real line satisfying

$$h(0) = 0, \quad \left| \frac{h(t_2) - h(t_1)}{t_2 - t_1} - 1 \right| \leq \varepsilon$$

for $t_1 \neq t_2$, and

$$\text{dist}(g(t), \phi_{h(t)}(x)) \leq \varepsilon$$

for all $t$.

We write $X \in \text{PerSh}(M)$ if $X \in \mathcal{X}^1(M)$ has the periodic shadowing property. Moreover, we say that a vector field $X$ has the $C^1$ robustly periodic shadowing property if $X \in \text{Int}^1(\text{PerSh}(M))$, where $\text{Int}^1(\text{PerSh}(M))$ is the $C^1$ interior of $\text{PerSh}(M)$.

The following notions and terminologies are regarding periodic shadowing.

A vector field $X \in \mathcal{X}^1(M)$ is said to have the Lipschitz periodic shadowing property, if there exist two positive constants $d_0$ and $L$ such that if $g$ is a periodic $d$-pseudo orbit for some $d < d_0$, then $g$ is $Ld$-shadowed by a periodic orbit or a singularity of $X$. Denote by $\text{LipPerSh}(M)$ the set of vector fields satisfying the Lipschitz periodic shadowing property.

A vector field $X \in \mathcal{X}^1(M)$ is said to have the orbital periodic shadowing property if for any $\varepsilon > 0$ there exists a constant $d > 0$ such that for any periodic $d$-pseudo orbit $g$ of $X$, there exists a point $x \in M$ which is a periodic point or a singularity of $X$ satisfying

$$\text{dist}_H(g(\mathbb{R}), \text{Orb}(x)) < \varepsilon,$$

where $\text{dist}_H$ is the Hausdorff distance. Denote by $\text{OrbPerSh}(M)$ the set of vector fields satisfying the orbital periodic shadowing property. Moreover, we say that a vector field $X$ has the $C^1$ robustly orbital periodic shadowing property if $X \in \text{Int}^1(\text{OrbPerSh}(M))$. 
The orbital periodic shadowing property indicates that each periodic pseudo orbit can be approximated by an exact periodic orbit in the sense of the Hausdorff distance. In this case the (time) parameter \( t \) will not be concerned. It is clear to see that

\[
\text{LipPerSh}(M) \subset \text{PerSh}(M) \subset \text{OrbPerSh}(M). \tag{2.1}
\]

We summarize our main result as follows.

**Theorem 2.1.** Every vector field satisfying the \( C^1 \) robustly orbital periodic shadowing property is \( \Omega \)-stable.

It was proved in [10] that the set of vector fields having the Lipschitz periodic shadowing property coincides with the set of the \( \Omega \)-stable vector fields. Since the set of the \( \Omega \)-stable vector fields is open, the following Corollary is a direct consequence of Theorem 2.1 and (2.1).

**Corollary 1.** Let \( X \in \mathcal{X}^1(M) \) be a \( C^1 \) vector field. Then the following statements are equivalent to each other.

- \( X \) is \( \Omega \)-stable;
- \( X \) has the \( C^1 \) robustly orbital periodic shadowing property;
- \( X \) has the \( C^1 \) robustly periodic shadowing property;
- \( X \) has the Lipschitz periodic shadowing property.

### 3. Star vector fields.

To prove the \( \Omega \)-stability for each vector field satisfying the \( C^1 \) robustly orbital periodic shadowing property, we need to introduce some results of star vector fields. Recall that a vector field \( X \) is a star vector field on \( M \) if \( X \) has a \( C^1 \) neighborhood \( U \) in \( \mathcal{X}^1(M) \) such that, for every \( Y \in U \), every singularity of \( Y \) and every periodic orbit of \( Y \) is hyperbolic. Denote by \( \mathcal{X}^* (M) \) the set of star vector fields on \( M \).

By a standard argument, we can show that every vector field with the \( C^1 \) robustly orbital periodic shadowing property is a star vector field. Here we briefly illustrate the idea of the proof and refer to [13] for more details.

**Lemma 3.1.** Assume that \( X \in \text{Int}^1 (\text{OrbPerSh}(M)) \). Then \( X \) is a star vector field.

**Proof.** Suppose on the contrary that there exists a vector field \( X \in \text{Int}^1 (\text{OrbPerSh}(M)) \) which is not a star vector field. Note that the vector field is still in the interior of OrbPerSh(M) after a \( C^1 \) small perturbation. Thus we can suppose that \( X \) itself has non-hyperbolic singularities or non-hyperbolic periodic orbits. Let us explain the idea of the proof in a simple way: we assume that there exists a non-hyperbolic singularity \( \sigma \) of \( X \) such that, \( X \) is linear in a small ball \( B(\sigma, r) \) for some \( r > 0 \) and \( DX(\sigma) \) has a real eigenvalue \( \lambda_0 = 0 \), where \( DX(\sigma) \) denotes the Jacobi matrix of \( X(\sigma) \).

Let \( F \subset T_\sigma M \) be the 1-dimensional eigenspace with respect to \( \lambda_0 \) and \( \tilde{F} \) be the \((\dim M - 1)\)-dimensional eigenspace with respect to the other eigenvalues. Changing the Riemannian metric if necessary, we assume that \( F \) and \( \tilde{F} \) are orthogonal. Then \( F \oplus \tilde{F} \) gives a coordinate \( x = (x^F, x^{\tilde{F}}) \) in \( T_\sigma M \).

Taking a constant \( \varepsilon \in (0, r/8) \). Let \( d \) correspond to \( \varepsilon \) according to the orbital periodic shadowing property.

Let \( p = (r/2, 0) \) and \( q = (r/4, 0) \) be two points in \( B(\sigma, r) \). In particular, \( \text{dist}(p, q) = r/4 > 2\varepsilon \). Choose a positive integer \( N \) such that \( r/4N < d \). We
define a periodic $d$-pseudo orbit as follows:

$$g(t) = \begin{cases} 
(r/2 - ir/4N, 0), & t \in [2kN + i, 2kN + i + 1), \ 0 \leq i \leq N, \ k \in \mathbb{N}; \\
(r/4 + ir/4N, 0), & t \in [2kN + N + i, 2kN + N + i + 1), \ 1 \leq i \\
& \leq N - 1, \ k \in \mathbb{N}.
\end{cases}$$

Thus, there exists a periodic orbit or a singularity Orb$(x)$ of $X$ such that

$$\text{dist}_H(g(R), \text{Orb}(x)) < \varepsilon.$$ 

Since Orb$(x) \subset B(\sigma, r)$, we know that the $X^F$ coordinate of Orb$(x)$ of $F$ is a constant, denoted by $a$. It implies that dist$(p, g(t)) \geq |r/2 - a|$ and dist$(q, g(t)) \geq |r/4 - a|$ for any $t$. Note that $p, q \in g(t)$ and either $|r/2 - a|$ or $|r/4 - a|$ is greater than $\varepsilon$. This yields a contradiction. □

We say that a point $x \in M$ is preperiodic of $X$, if for any neighborhood $U$ of $x$ in $M$ and any $C^1$ neighborhood $U$ of $X$ in $X^1(M)$, there exists $Y \in U$ and $y \in U$ such that $y$ is a periodic point (not singularity) of $Y$. Denote by $\text{Per}_*(X)$ the set of preperiodic points of $X$.

**Theorem 3.2.** [5] Let $X \in X^* (M)$ be a star vector field. If Sing$(X) \cap \text{Per}_*(X) = \emptyset$, then $X$ is $\Omega$-stable.

According to Lemma 3.1 and Theorem 3.2, to prove Theorem 2.1, we see that the remaining part is to prove the following theorem.

**Theorem 3.3.** Assume that $X \in \text{Int}^1(\text{OrbPerSh}(M))$. Then Sing$(X) \cap \text{Per}_*(X) = \emptyset$.

4. **Linear structure of a singularity exhibiting a homoclinic connection.**

Recall that a homoclinic connection with respect to a singularity $\sigma$ is the closure of an orbit of a regular point (not a singularity) which is contained in both the stable and the unstable manifolds of $\sigma$. In this section, we will analyze the structure around a singularity exhibiting a homoclinic connection when $X$ is locally linear near the singularity.

We shall prove that if the vector field satisfies some “adapted setting” (that is, assumptions A1–A6 given below are satisfied, and see Subsection 4.3 for the details), then the forward orbit of a point cannot be contained in an arbitrarily small neighborhood of the homoclinic connection when it is not in the stable manifold of the singularity. This implies that there is no periodic orbit contained in some small neighborhood of the homoclinic connection. Since we can create periodic $d$-pseudo orbits for arbitrary small $d > 0$ in the homoclinic connection, we find that the vector field satisfying the “adapted setting” does not have the periodic shadowing property. Assumptions A1–A6 imposed on the field vector are satisfied for a $C^1$ small perturbation of any vector field $X$ (or $-X$) having a preperiodic singularity.

We would like to emphasize that in this section we do not need to assume that the vector field has any shadowing property.

Assume that $X \in X^1(M)$ is a $C^1$ vector field on $M$, $\sigma \in \text{Sing}(X)$ is a hyperbolic saddle type singularity, and

$$\Gamma \subset W^s(\sigma) \cap W^u(\sigma)$$

is a homoclinic connection of $\sigma$.

Let

$$T_\sigma M = E^s_\sigma \oplus E^u_\sigma$$
be the hyperbolic splitting. We denote by

\[ \text{Re}(\lambda_m) \leq \cdots \leq \text{Re}(\lambda_2) \leq \text{Re}(\lambda_1) < 0 < \text{Re}(\gamma_1) \leq \cdots \leq \text{Re}(\gamma_n) \]

the eigenvalues of \( DX(\sigma) \), where \( m = \dim E^s_\sigma \), \( n = \dim E^u_\sigma \) and the saddle value of \( \sigma \) is

\[ \text{SV}(\sigma) = \text{Re}(\lambda_1) + \text{Re}(\gamma_1). \]

Furthermore, we assume that \( X \) satisfies the following conditions:

A1. \( \text{SV}(\sigma) > 0 \).
A2. \( X \) is linear in a small neighborhood \( U_\sigma \) of \( \sigma \).
A3. \( \text{Re}(\lambda_{i+1}) < \text{Re}(\lambda_i) \) if \( \lambda_{i+1} \) and \( \lambda_i \) are not conjugate non-real complex numbers.

Thus we have the dominated splitting

\[ E^s_\sigma = E^s_{m'} + \cdots + E^s_2 + E^s_1, \]

where \( m' \leq m \) and \( E^s_i \) are the eigenspaces with \( \dim E^s_i = 1 \) or 2. Similarly, we assume that \( \text{Re}(\gamma_i) < \text{Re}(\gamma_{i+1}) \) if \( \gamma_i \) and \( \gamma_{i+1} \) are not conjugate non-real complex numbers. Then we have the dominated splitting

\[ E^u_\sigma = E^u_1 \oplus E^u_2 \oplus \cdots \oplus E^u_{n'}, \]

where \( n' \leq n \) and \( E^u_i \) are the eigenspaces with \( \dim E^u_j = 1 \) or 2. Furthermore, we assume that \( \{E^s_i, E^u_i\} \) are mutually orthogonal.

A4. \( \Gamma \cap U_\sigma \subset E^s_1 \oplus E^u_1 \).

We build 1 or 2-dimensional (orthogonal) coordinate system in each eigenspace \( E^s_i \) and \( E^u_i \). Putting them together, we get a \( (\dim M) \)-dimensional orthogonal coordinate system in the linear neighborhood \( U_\sigma \). For any \( x \in U_\sigma \), we denote by \( \pi_\sigma(x) \) the coordinate of \( x \) in a subspace \( F \subset T_x M \) and by \( |\pi_\sigma(x)| \) the norm of \( \pi_\sigma(x) \). For simplicity, let

\[ x^s = \pi_{E^s_\sigma}(x) \text{ and } x^u = \pi_{E^u_\sigma}(x) \]

be the coordinates of \( x \) in \( E^s_\sigma \) and \( E^u_\sigma \) respectively, and

\[ x^c = \pi_{E^s_1}(x), \quad x^{ss} = \pi_{E^s_1 \oplus \cdots \oplus E^s_2}(x), \quad x^{1u} = \pi_{E^s_1}(x), \quad x^{uu} = \pi_{E^u_1 \oplus \cdots \oplus E^u_n}(x) \]

be the coordinates of \( x \) in \( E^s_1, E^s_1 \oplus \cdots \oplus E^s_2, E^u_1 \) and \( E^u_1 \oplus \cdots \oplus E^u_{n'} \), respectively.

We take two points \( O_s, O_u \in \Gamma \cap U_\sigma \setminus \{\sigma\} \) contained in \( E^s_1 \) and \( E^u_1 \), respectively, and choose two small cross sections \( \Sigma^s \) and \( \Sigma^u \) in \( U_\sigma \) which are orthogonal to \( \Gamma \) at \( O_s \) and \( O_u \), respectively.

Denote by \( Q : \Sigma^u \to \Sigma^s \) the Poincaré map from \( \Sigma^u \) to \( \Sigma^s \), and by \( \tau_0(y) \) the minimal positive \( t \) such that \( \phi_t(y) = Q(y) \). Reducing \( \Sigma^u \) if necessary, we can assume that \( Q(\Sigma^u) \subset \Sigma^s \) and \( \tau_0 \) is continuous in \( \Sigma^u \).

Denote by \( P : \text{Dom}(P) \subset \Sigma^s \to \Sigma^u \) the Poincaré map from \( \Sigma^s \) to \( \Sigma^u \) in \( U_\sigma \), where

\[ \text{Dom}(P) = \{x \in \Sigma^s : \exists T > 0 \text{ such that } \phi_T(x) \in \Sigma^u \text{ and } \phi_{[0,T]}(x) \subset U_\sigma\} \]

is the domain of \( P \). Denote by \( \tau_p(x) \) the minimal positive \( t \) such that \( \phi_t(x) = P(x) \).

We consider the following two cases.
4.1. **Case 1:** \( \dim E^s_\sigma = 1 \). In this subsection, we assume that \( \dim E^s_\sigma = 1 \). Since \( E^s_1 = E^s_\sigma \) and \( E^s_\sigma \) is orthogonal to \( \Sigma^s \), we know that \( \Sigma^s \) is contained in the hyperplane \( \{x^s = O^s\} \).

The orthogonal coordinate system in \( U_\sigma \) induces two \( (\dim M - 1) \)-dimensional orthogonal coordinate systems in \( \Sigma^s \) and \( \Sigma^u \). Denote by \( x_{\Sigma^s}, y_{\Sigma^u} \) the coordinates of \( x, y \) in \( \Sigma^s, \Sigma^u \), and by \( |x|_{\Sigma^s}, |y|_{\Sigma^u} \) the norm of \( x, y \) in \( \Sigma^s, \Sigma^u \), respectively. Since \( E^s_\sigma \) is 1-dimensional and is orthogonal to \( \Sigma^s \), we know that \( x_{\Sigma^s} = \pi_{E^u}(x) = x^u \).

If \( \dim E^u_1 = 1 \) (i.e., \( \gamma_1 \) is real), then the coordinate system in \( \Sigma^u \) is \( E^s_\sigma \perp E^u_2 \perp \cdots \perp E^u_n \). Thus, \( y_{\Sigma^u} = (y^s, y^{uu}) \). If \( \dim E^u_1 = 2 \), let \( E^u_1 = E^u_{1,1} \perp E^u_{1,2} \) be the coordinate system in \( E^u_1 \). Without loss of generality, we assume that \( E^u_{1,1} \) is orthogonal to \( \Sigma^u \). Then \( E^s_\sigma \perp E^u_{1,2} \perp E^u_{2} \perp \cdots \perp E^u_n \) is the coordinate system in \( \Sigma^u \). Thus, we have \( y_{\Sigma^u} = (y^s, \pi_{E^u_{1,2}}(y), y^{uu}) \) (See Figure 1).

**Lemma 4.1.** There exist a neighborhood \( \Sigma^u_1 \subset \Sigma^u \) of \( O_u \) in \( \Sigma^u \) and a positive constant \( B > 0 \) such that for any \( x \in \Sigma^u_1 \), we have
\[
|Q(x)|_{\Sigma^s} \geq B|x|_{\Sigma^u}.
\]

**Proof.** Note that \( Q(O_u) = O_s \) and \( \tau_Q \) is bounded on \( \Sigma^u \). By the differentiability we can choose a neighborhood \( \Sigma^u_1 \subset \Sigma^u \) of \( O_u \) in \( \Sigma^u \) and a positive constant \( B > 0 \) such that for any \( x \in \Sigma^u_1 \), we have
\[
\text{dist}(Q(x), O_s) \geq B\text{dist}(x, O_u).
\]

Hence, there holds
\[
|Q(x)|_{\Sigma^s} \geq B|x|_{\Sigma^u}.
\]

\( \Box \)

Let \( A \subset \Sigma^u \). Denote by
\[
P^{-1}(A) = \{x \in \Sigma^s : P(x) \in A, \phi_{0,\tau_p(x)}(x) \subset U_\sigma\}
\]
the inverse image set of \( A \) under \( P \).

Since the expanding rate in \( E^s_\sigma \) is larger than the contracting rate in \( E^s_\sigma \) (\( \text{SV}(\sigma) > 0 \)), we obtain the following lemma.
Lemma 4.2. Let $B$ be as in Lemma 4.1. There exists a neighborhood $\Sigma^*_1 \subset \Sigma^*$ of $O_*$ in $\Sigma^*$ such that for any $x \in \Sigma^*_1 \cap P^{-1}(\Sigma^*_1)$, there holds

$$|P(x)|_{\Sigma^*} \geq 2B^{-1}|x|_{\Sigma^*}.$$

Proof. Reducing $\Sigma^*$ if necessary, we may assume that there exists $\rho > 0$ such that for any $y \in \Sigma^*$, we have $|y^{1u}| = |\pi_{e^x}(y)| \in [\rho, 2\rho]$. For any $x \in \Sigma^* \cap P^{-1}(\Sigma^*_1)$ (note that $\pi_{e^x}(x) \neq 0$), we have

$$|(P(x))^{1u}| = |\pi_{e^x}(P(x))| = |x^{1u}| e^{\text{Re}(\gamma_1)\tau_{p}(x)} \in [\rho, 2\rho].$$

Thus, we find

$$\tau_{p}(x) \in \left[ \frac{1}{\text{Re}(\gamma_1)} \log \frac{\rho}{|x^{1u}|}, \frac{1}{\text{Re}(\gamma_1)} \log \frac{2\rho}{|x^{1u}|} \right].$$

In particular, we have

$$\tau_{p}(x) \to +\infty \text{ as } |x^{1u}| \to 0.$$

Considering the coordinate of $P(x)$ in $E^*_x$, by a straightforward calculation, we have

$$|(P(x))^{\tau_{p}(x)}| = |O^*_x e^{\lambda_1 \tau_{p}(x)}| \geq |O^*_x| \left( \frac{|x^{1u}|}{2\rho} \right)^{-\lambda_1/\text{Re}(\gamma_1)} = |x^{1u}| \cdot \frac{|O^*_x|}{2\rho} \left( \frac{|x^{1u}|}{2\rho} \right)^{-\lambda_1/\text{Re}(\gamma_1)} = B_1(x) |x^{1u}|.$$

Since $\lambda_1 + \text{Re}(\gamma_1) = \text{SV}(\sigma) > 0$ and $\text{Re}(\gamma_1) > 0$, we have $B_1(x) \to +\infty$ as $|x^{1u}| \to 0$. One can take $\Sigma^*_1 \subset \Sigma^*$ sufficiently small (so that $|x^{1u}|$ can be small enough) such that

$$B_1(x) > 2B^{-1} \text{ and } e^{\text{Re}(\gamma_1)\tau_{p}(x)} > 2B^{-1},$$

for any $x \in \Sigma^*_1 \cap P^{-1}(\Sigma^*_1)$. The second inequality implies that

$$|(P(x))^{1u}| \geq e^{\text{Re}(\gamma_1)\tau_{p}(x)} |x^{1u}| > 2B^{-1}|x^{1u}|.$$

Thus, we have

$$|P(x)|_{\Sigma^*} \geq \left| |(P(x))^{\tau_{p}(x)}, (P(x))^{1u}| |_{\Sigma^*} \right| \geq 2B^{-1} |(x^{1u}, x^{1u})|_{\Sigma^*} = 2B^{-1}|x|_{\Sigma^*}. \tag*{□}$$

The following result is a direct consequence of Lemmas 4.1 and 4.2.

Lemma 4.3. For any $x \in \Sigma^*_1 \cap P^{-1}(\Sigma^*_1)$, we have

$$|Q \circ P(x)|_{\Sigma^*} \geq 2|x|_{\Sigma^*}.$$

Lemma 4.3 implies that for any point $x \notin W^s(\sigma)$ close to $\Gamma$ enough, its forward orbit will move away from $\Gamma$. Thus, we have a result as follows.

Lemma 4.4. There exists a neighborhood $U_\Gamma$ of $\Gamma$ such that for any $x \in U_\Gamma \setminus W^s(\sigma)$ there exists a point $y \in \text{Orb}^+(x)$ which is not contained in $U_\Gamma$. 
Proof. Let $\Sigma^s_1$ and $\Sigma^u_1$ be the cross sections of $\Gamma$ given by Lemmas 4.1 and 4.2, respectively. We denote

$$\Sigma^s_0 = \Sigma^s_1 \cap Q(\Sigma^u_1) \quad \text{and} \quad \Sigma^u_0 = Q^{-1}(\Sigma^s_0) \subset \Sigma^u_1.$$ 

Note that $\Sigma^s_0$ and $\Sigma^u_0$ are neighborhoods of $O_s$ and $O_u$ in $\Sigma^s$ and $\Sigma^u$, respectively. Denote by $\sigma O_s$, $\sigma O_u$ the part of $\Gamma$ in $U_\sigma$ between $\sigma$ and $O_s$, $\sigma$ and $O_u$, respectively. Take a small neighborhood $U_P \Gamma$ of $\tilde{\sigma}O_s \cup \tilde{\sigma}O_u$ satisfying

$$U_P \Gamma \cap \Sigma^s \subset \Sigma^s_0 \quad \text{and} \quad U_P \Gamma \cap \Sigma^u \subset \Sigma^u_0.$$ 

Let

$$U_Q \Gamma = \{ \phi_t(x) | x \in \Sigma^u_0, t \in [0, \tau_Q(x)] \}.$$ 

Then

$$U_\Gamma = U_Q \Gamma \cap U_P \Gamma$$

is a neighborhood of $\Gamma$ (See Figure 2).

![Figure 2](image-url)

Suppose to the contrary that there exists $x \in U_\Gamma$ such that $x \not\in W^s(\sigma)$ and $\text{Orb}^+(x) \subset U_\Gamma$. Without loss of generality, we can assume that $x \in U_P \Gamma$. Since $x \not\in W^s(\sigma)$, we have that $\text{Orb}^+(x) \not\subset U_\sigma$. Let

$$t_0 = \sup\{ t > 0 | \phi_{[0,t]}(x) \subset U_\sigma \} < +\infty \quad \text{and} \quad y = \phi_{t_0}(x).$$

Note that

$$U_\Gamma \cap \partial U_\sigma \subset U_Q \Gamma,$$

where $\partial U_\sigma$ is the boundary of $U_\sigma$. It implies that $y \in U_Q \Gamma$. Thus, there exists $t_1 \in (0, t_0)$ such that $y_0 = \phi_{t_1}(x) \in \Sigma^u_0$. In particular, $y_0 \in \text{Orb}^+(x)$.

Let $x_1 = Q(y_0) \in \Sigma^u_0$. Notice that $x_1 \not= 0$ since $x_1 \not\in W^s(\sigma)$. Using a similar argument, we know that there exists $y_1 \in \Sigma^s_0$ such that $y_1 = P(x_1)$. Namely, $x_1 \in P^{-1}(\Sigma^u_0)$. Let $x_2 = Q(y_1) \in \Sigma^s_0$. By Lemma 4.3, we have

$$|x_2|_{s^*} \geq 2|x_1|_{s^*}.$$
Continuing the process in this way, we can find a sequence of \( \{x_n\} \subset \Sigma^s \) such that \( x_n \in \text{Orb}^+(x) \) and

\[
|x_n|_{\Sigma^s} \geq 2|x_{n-1}|_{\Sigma^s} \geq \cdots \geq 2^{n-1}|x_1|_{\Sigma^s} \to +\infty,
\]

which is a contradiction with \( \text{Orb}^+(x) \subset U_1 \).

\[ \square \]

4.2. Case 2: \( \dim E^s_\sigma > 1 \). In this case, since the strong contracting subspace \( E^s_\sigma \) is non-trivial, we cannot be sure that \( |x| \) is expanded under \( Q \circ P \) for any \( x \in \Sigma^s \cap P^{-1}(\Sigma^u) \). But since \( E^{ss}_\sigma \)-coordinate is invariant under \( P \), it suffices to prove that \( |x| \) is expanded under \( Q \circ P \) in the subspace \( E^s_\sigma \oplus E^u_\sigma \).

Recall that \( \Gamma \) is a homoclinic connection of a singularity \( \sigma \) satisfying the assumptions A1–A4. Moreover, we assume that

A5. \( \lambda_1 \) is real. Thus there exists a dominated splitting

\[
E^s_\sigma = E^{ss}_\sigma \oplus E^c_\sigma,
\]

where \( E^{ss}_\sigma = E^1_\sigma \) is the 1-dimensional eigenspace with respect to \( \lambda_1 \).

Recall that \( x^c = \pi_{E^1_\sigma}(x) \) for \( x \in U_\sigma \). Since \( E^c_\sigma = E^1_\sigma \) is 1-dimensional and is orthogonal to \( \Sigma^s \), one can see that \( \Sigma^s_\sigma \) is contained in the hyperplane \( \{x^c = 0\} \).

Similar to the above discussions, the orthogonal coordinate system in \( U_\sigma \) induces two (\( \dim M - 1 \))-dimensional orthogonal coordinate systems in \( \Sigma^s \) and \( \Sigma^u \). Denote by \( x_{\Sigma^s}, y_{\Sigma^u}, \) the coordinates of \( x, y \) in \( \Sigma^s, \Sigma^u \) and \( |x|_{\Sigma^s}, |y|_{\Sigma^u} \) the norm of \( x, y \) in \( \Sigma^s, \Sigma^u \), respectively. Since \( E^c_\sigma \) is 1-dimensional and orthogonal to \( \Sigma^s \), we know that

\[
x_{\Sigma^s} = (\pi_{E^1_\sigma}(x), \pi_{E^1_\sigma}(x)) = (x^{ss}, x^u).
\]

If \( \dim E^u_\sigma = 1 \) (\( \gamma_1 \) is real), the coordinate system in \( \Sigma^u \) is \( E^{ss}_\sigma \oplus E^c_\sigma \oplus E^u_2 \oplus \cdots \oplus E^u_n \). Thus, there holds

\[
y_{\Sigma^u} = (\pi_{E^1_\sigma}(y), \pi_{E^1_\sigma}(y), \pi_{E^2_\sigma} \oplus \cdots \oplus E^u_n(y)) = (y^{ss}, y^c, y^{uu}).
\]

We denote by \( y^{cu}_{\Sigma^u} = (y^c, y^{uu}) \) the \( E^c_\sigma \oplus E^{uu}_\sigma \)-coordinate of \( y \) in \( \Sigma^u \).

If \( \dim E^u_\sigma = 2 \), let \( E^u_\sigma = E^{1,1}_\sigma \oplus E^{1,2}_\sigma \) be the coordinate system in \( E^u_\sigma \). Without loss of generality, we assume that \( E^{1,1}_\sigma \) is orthogonal to \( \Sigma^u \). Then

\[
E^{ss}_\sigma \oplus E^c_\sigma \oplus E^{1,1}_\sigma \oplus E^{1,2}_2 \oplus \cdots \oplus E^u_n,
\]

is the coordinate system in \( \Sigma^u \). Thus, we have

\[
y^{cu}_{\Sigma^u} = (y^{ss}, y^c, \pi_{E^{1,1}_\sigma}(y), y^{uu}).
\]

In this case, we denote by \( y^{cu}_{\Sigma^u} = (y^c, \pi_{E^{1,1}_\sigma}(y), y^{uu}) \) the \( E^c_\sigma \oplus E^{1,2}_\sigma \oplus E^{uu}_\sigma \)-coordinate of \( y \) in \( \Sigma^u \).

In each case we have \( y^{cu}_{\Sigma^u} = (y^{ss}, y^{cu}_{\Sigma^u}) \). Recall that \( P \) is the Poincaré map from \( \Sigma^s \) to \( \Sigma^u \) which is linear in \( U_\sigma \). Then we have

\[
P(x) = y = (y^{ss}, y^{cu}_{\Sigma^u}) = (P(x^{ss}), P(x^u)),
\]

for any \( x \in \Sigma^s \cap P^{-1}(\Sigma^u) \).

Let

\[
L^s = \{x \in \Sigma^s | x^u = 0\}
\]

be an \( (m-1) \)-dimensional disc, and

\[
L^u = \{y \in \Sigma^u | y^{ss} = 0\}
\]
be an $n$-dimensional disc. Note that
\[ \dim \Sigma^s = \dim M - 1 = n + m - 1. \]

We further assume that
A6. $Q(L^s)$ is transverse to $L^s$ at $O_s$ in $\Sigma^s$.

For any $\eta \geq 0$, we define two cones as follows:
\[ C^s_\eta = \{ x \in \Sigma^s | ||x^s|_{\Sigma^s} \leq \eta ||x^u|_{\Sigma^u} \} \]
\[ C^u_\eta = \{ y \in \Sigma^u | ||y^s|_{\Sigma^s} \leq \eta ||y^u|_{\Sigma^u} \}. \]

We remark that A6 is used to avoid that the coordinate of $E^c_\sigma \oplus E^u_\sigma$ decreases quickly under the map $Q$. To see this, we present the following two technical lemmas.

\textbf{Figure 3.}

\textbf{Lemma 4.5.} There exists a positive constant $\beta > 0$ small enough such that
\[ Q(C^s_\beta) \cap C^u_\beta = \{ O_s \}. \]

\textit{Proof.} Note that $C^u_0 = L^u$, $C^s_0 = L^s$ and $Q(L^s) \cap L^c$ is transverse on $O_s$, thus we can get that $\beta > 0$ by the continuity (See Figure 3). $\square$

\textbf{Lemma 4.6.} There exist a neighborhood $\Sigma^u_1 \subset \Sigma^u$ of $O_u$ in $\Sigma^u$ and a positive constant $B > 0$ such that for any $x \in C^u_\beta \cap \Sigma^u_1$, we have
\[ ||(Q(x))^u|_{\Sigma^u} \geq B ||x^u|_{\Sigma^u}. \]

\textit{Proof.} Similar to the proof of Lemma 4.1, one can choose a neighborhood $\Sigma^u_1 \subset \Sigma^u$ of $O_u$ and a positive constant $B' > 0$ because of the continuity such that for any $x \in C^u_\beta \cap \Sigma^u_1$, we have
\[ \text{dist}(Q(x), O_u) \geq B' \text{dist}(x, O_u). \]
Lemma 4.9. For any \( x \in \Sigma^s \), we have \( |(Q(x))^u|_{\Sigma^s} \geq \frac{\beta}{\sqrt{1 + \beta^2}} \text{dist}(Q(x), O_s) \).

On the other hand, making use of Lemma 4.5 we have \( Q(\Sigma^u) \cap \Sigma^u = \{ O_s \} \), which implies that
\[
|\langle Q(x) \rangle^u|_{\Sigma^s} \geq \frac{\beta}{\sqrt{1 + \beta^2}} \text{dist}(Q(x), O_s).
\]

Since \( x \in C^u_\beta \), we have
\[
|\langle x \rangle^u|_{\Sigma^s} \leq \frac{1}{\sqrt{1 + \beta^2}} \text{dist}(x, O_u).
\]

Thus, there holds
\[
|\langle x \rangle^u|_{\Sigma^s} \geq \beta B^u |\langle x \rangle^u|_{\Sigma^s} = B |\langle x \rangle^u|_{\Sigma^s}.
\]

Lemma 4.7. There exists a neighborhood \( \Sigma^1 \subset \Sigma^s \) of \( O_s \) in \( \Sigma^s \) such that for any \( x \in \Sigma^1 \cap P^{-1}(\Sigma^u) \), we have \( P(x) \in C^u_\beta \cap \Sigma^1 \).

Proof. Similar to the proof of Lemma 4.2, reducing \( \Sigma^u \) if necessary, we assume that there is \( \rho > 0 \) such that for any \( y \in \Sigma^u \) we have \( y^1u = \pi_{g^y}(y) \in [\rho, 2\rho] \) and
\[
|\langle (P(x))^1u \rangle| = |\pi_{g^y}(P(x))| = |\langle x^1u \rangle e^{|\text{Re}(\lambda_1)\tau_\rho(x)}| \in [\rho, 2\rho]
\]
for any \( x \in \Sigma^s \cap P^{-1}(\Sigma^u) \). Thus, it gives
\[
\tau_\rho(x) \rightarrow +\infty \text{ as } |x^1u| \rightarrow 0.
\]

Since \( E^s_{ss} \) is a strong contracting subspace, we have
\[
\frac{|\langle (P(x))^u \rangle^s|_{\Sigma^u}}{|\langle (P(x))^u \rangle^s|_{\Sigma^u}} \leq \frac{|\langle (P(x))^s \rangle^u|_{\Sigma^u}}{|\langle (P(x))^s \rangle^u|_{\Sigma^u}} \leq \frac{|\langle x^s \rangle^u| e^{\text{Re}(\lambda_2)\tau_\rho(x)}}{|\langle O_s^u \rangle| e^{\text{Re}(\lambda_2)\tau_\rho(x)}} \rightarrow 0 \text{ as } \tau_\rho(x) \rightarrow +\infty,
\]
which implies that \( P(x) \in C^u_\beta \) as \( |x^1u| \) is small enough.

Consequently, there exists a neighborhood \( \Sigma^1 \subset \Sigma^s \) of \( O_s \) such that
\[
P(x) \in C^u_\beta
\]
for any \( x \in \Sigma^1 \cap P^{-1}(\Sigma^u) \).

Note that \( P(x^u) = (P(x))^\Sigma^u \), which indicates that the action on the coordinate of \( E^s_{ss} \oplus E^s \) under \( P \) is independent of the coordinate of \( E^s_{ss} \). Thus, by using a similar argument to the proof of Lemma 4.2, we can derive the following result.

Lemma 4.8. There exists a neighborhood \( \Sigma^1 \subset \Sigma^s \) of \( O_s \) in \( \Sigma^s \) such that for any \( x \in \Sigma^1 \cap P^{-1}(\Sigma^u) \), there holds
\[
|\langle (P(x))^u \rangle|_{\Sigma^s} \geq 2B^{-1} |x^u|_{\Sigma^s}.
\]

Combined Lemma 4.6 with Lemma 4.7, we can also obtain the following result.

Lemma 4.9. For any \( x \in \Sigma^1 \cap P^{-1}(\Sigma^u) \), we have
\[
|\langle (Q \circ P(x))^u \rangle|_{\Sigma^s} \geq 2|x^u|_{\Sigma^s}.
\]
Proof. Taking $x \in \Sigma_s^2 \cap P^{-1}(\Sigma_u^4)$, by Lemmas 4.7 and 4.8 we have
$$P(x) \in C_b^1 \cap \Sigma_1^1 \quad \text{and} \quad \left| (P(x))^{cu}_{\Sigma^u} \right|_{\Sigma^u} \geq 2B^{-1}|x^u|_{\Sigma^u}.$$By Lemma 4.6 we get
$$|(Q \circ P(x))^u|_{\Sigma^s} \geq B \left| (P(x))^{cu}_{\Sigma^u} \right|_{\Sigma^u} \geq 2|x^u|_{\Sigma^u}.$$\hfill$\square$

According to Lemma 4.9, one can see that if the forward orbit of a point $x \notin W^s(\sigma)$ is very close to $\Gamma$, it will move away from $\Gamma$. The proof of the following result is similar to that of Lemma 4.4, so we omit it.

Lemma 4.10. There exists a neighborhood $U_{\Gamma}$ of $\Gamma$ such that for any $x \in U_{\Gamma} \setminus W^s(\sigma)$, there exists a point $y \in \text{Orb}^+(x)$ which is not contained in $U_{\Gamma}$.

4.3. Adapted setting of a homoclinic connection. We say that a homoclinic connection $\Gamma$ of a hyperbolic singularity $\sigma$ satisfies the adapted setting if conditions A1–A4 are satisfied when $\dim E^s_\sigma = 1$, or conditions A1–A6 are satisfied when $\dim E^s_\sigma \geq 2$.

According to Lemmas 4.4 and 4.10, we obtain the following proposition.

Proposition 1. Let $X \in X^1(M)$ be a vector field and $\sigma \in \text{Sing}(X)$ be a hyperbolic singularity with a homoclinic connection $\Gamma$. If $\Gamma$ satisfies the adapted setting, then there exists a neighborhood $U_{\Gamma}$ of $\Gamma$ such that for any $x \in U_{\Gamma}$, one of the following two statements holds:
- $x \in W^s(\sigma)$; or
- there exists a point $y \in \text{Orb}^+(x)$ which is not contained in $U_{\Gamma}$.

5. Proof of Theorem 3.3. To present the proof of Theorem 3.3, first let us recall the following lemma regarding the dominated splitting.

Lemma 5.1. [2, 20] Let $X \in X^*(M)$ be a star vector field and $\sigma \in \text{Sing}(X)$ be a hyperbolic saddle type singularity exhibiting a homoclinic connection. If $SV(\sigma) > 0$ and $\dim E^s_\sigma \geq 2$, then there exists a dominated splitting
$$E^s_\sigma = E^{ss}_\sigma \oplus E^c_\sigma,$$where $\dim E^c_\sigma = 1$.

We are now ready to prove Theorem 3.3.

Proof of Theorem 3.3. On the contrary, suppose that there exists a vector field $X \in \text{Int}^1(\text{OrbPerSh}(M))$ satisfying the $C^1$ robustly orbital periodic shadowing property and a singularity $\sigma \in \text{Sing}(X) \cap \text{Per}_s(X)$. It will be deduced that there exists a vector field in $\text{Int}^1(\text{OrbPerSh}(M))$, $C^1$ close to $X$, exhibiting a homoclinic connection of $\sigma$ which satisfies the adapted setting accordingly.

By Lemma 3.1, we know that $X \in X^*(M)$ is a star vector field. In particular, $\sigma$ is a hyperbolic saddle type singularity.

Since $\sigma$ is preperiodic, there exists a vector field $X'$ arbitrarily $C^1$ close to $X$ and a regular periodic orbit of $X'$ whose orbit is arbitrarily close to the local stable and local unstable manifolds of $\sigma_{x'}$, where $\sigma_{x'}$ is the continuation of $\sigma$. Then by using a classical argument of the uniform $C^1$ connecting lemma (see [18, 19]; for instance), we can get a vector field $Y$ exhibiting a homoclinic connection
$$\Gamma \subset W^s(Y(\sigma_y)) \cap W^u(Y(\sigma_y)).$$
Note that this perturbation can be made arbitrarily $C^1$ small. Thus we may still have $Y \in \text{Int}^1(\text{OrbPerSh}(M))$.

Denote by

$$\text{Re}(\lambda_m) \leq \cdots \leq \text{Re}(\lambda_2) \leq \text{Re}(\lambda_1) < 0 < \text{Re}(\gamma_1) \leq \text{Re}(\gamma_2) \leq \cdots \leq \text{Re}(\gamma_n)$$

the eigenvalues of $D_Y(\sigma_v)$.  

Up to a small perturbation, we may assume that $Y$ is linear in a small neighborhood $U'_{\sigma_v}$ of $\sigma_v$ on a proper chart, which still exhibits a homoclinic connection. It satisfies $\text{Re}(\lambda_i) < \text{Re}(\lambda_i)$ if $\lambda_i > 0$ and $\lambda_i$ are not conjugate non-real complex numbers. Moreover, one can assume that the saddle value $\text{SV}(\sigma_v) > 0$. Without loss of generality, we assume that $\text{SV}(\sigma_v) > 0$.

Then we have the dominated splitting of eigenspaces

$$E^s_{\sigma_v} = E^s_{m'} \oplus \cdots \oplus E^s_2 \oplus E^s_1,$$

and

$$E^u_{\sigma_v} = E^u_1 \oplus E^u_2 \oplus \cdots \oplus E^u_{n'},$$

where $E^s_j$ and $E^u_j$ are the eigenspaces with $\dim E^s_j = 1$ or 2, respectively. Changing the Riemannian metric if necessary, we assume that $\{E^s_j, E^u_j\}$ are mutually orthogonal. Denote by

$$E^{ss}_{\sigma_v} = E^s_{m'} \oplus \cdots \oplus E^s_2$$

and

$$E^{uu}_{\sigma_v} = E^u_1 \oplus \cdots \oplus E^u_{n'},$$

the strong contracting and strong expanding subspaces in $T_{\sigma_v} M$, respectively.

Up to a small perturbation out of $U'_{\sigma_v}$, we may assume that $\Gamma \cap W^s_{\text{loc}}(\sigma_v)$ is not contained in $E^{ss}_{\sigma_v}$, and $\Gamma \cap W^u_{\text{loc}}(\sigma_v)$ is not contained in $E^{uu}_{\sigma_v}$. Hence for any $x \in \Gamma \cap W^s_{\text{loc}}(\sigma_v) \setminus \{\sigma_v\}$, the vector $Y(x)$ is not contained in $E^{ss}_{\sigma_v}$. Since $E^{ss}_{\sigma_v}$ is the strong contracting subspace, it is clear that

$$<Y(\phi_{Y,t}(x))> = <D_Y(Y(x))> \rightarrow E^s_1,$$

where $<Y(x)>$ represents the subspace generated by the vector $Y(x)$, and $\rightarrow$ represents the upper limit which means that every limit of $<Y(\phi_{Y,t}(x))>$ is a 1-dimensional subspace of $E^s_1$. Moreover, we have

$$\frac{\text{dist}(\phi_{Y,t}(x), E^1_1)}{\text{dist}(\phi_{Y,t}(x), \sigma_v)} \rightarrow 0 \quad \text{as} \quad t \rightarrow +\infty.$$

By a small pushing into a small neighborhood of $\phi_{Y,t}(x)$, we can connect $\Gamma$ and some orbit in $E^s_1$. In particular, this perturbation is arbitrarily $C^1$ small when $t$ is arbitrarily large (See Figure 4).

Similarly, by another small perturbation, we may get $\Gamma$ into $E^u_i$. Thus we can find a vector field $Z \in \text{Int}^1(\text{OrbPerSh}(M))$ exhibiting a homoclinic connection which is linearizable in a neighborhood $U_{\sigma_v} \subset U'_{\sigma_v}$ of $\sigma_v$ and satisfies all of conditions A1–A4.

For simplicity, we may assume that $X$ satisfies A1–A4. So $\Gamma$ satisfies the adapted setting when $\dim E^s_1 = 1$.

In the case of $\dim E^s_1 = 2$, by Lemma 5.1, we have that $\lambda_1$ is real and $E^s_1 = E^c_\sigma$ is 1-dimensional. Choose the cross sections $\Sigma^s, \Sigma^u$ and the coordinate system as in Subsection 4.2. Let

$$L^s = \{x \in \Sigma^s | x^u = 0\} \quad \text{and} \quad L^u = \{y \in \Sigma^u | y^{*s} = 0\}.$$
Note that $L^s$ is an $(m-1)$-dimensional disc, $L^u$ is an $n$-dimensional disc, and $\Sigma^s$ is an $(n+m-1)$-dimensional hyperplane. Up to a $C^1$ small perturbation out of $U_\sigma$ (in a small neighborhood of $\Gamma \setminus U_\sigma$), we may assume that $Q(L^u)$ is transverse to $L^s$ in $U_\sigma$. Thus, one can see that both assumptions A5 and A6 are satisfied, and $\Gamma$ satisfies the adapted setting when $\dim E^s_\sigma \geq 2$.

Let $U_\Gamma$ be the neighborhood of $\Gamma$ given by Proposition 1. Let

$$\varepsilon = \dist(\Gamma, \partial U_\Gamma) > 0,$$

where $\partial U_\Gamma$ is the boundary of $U_\Gamma$. Let $d$ correspond to $\varepsilon$ with the orbital periodic shadowing property.

Let $p \in W^u_{loc} \setminus \{\sigma\}$ and $q = \phi_T(p) \in W^u_{loc} \cap \Gamma$ be two points $d/2-$close to $\sigma$. Then

$$g(t) = \phi_T(p), \quad t' = t - nT \in [0, T), \quad n \in \mathbb{N},$$

is a periodic $d$-pseudo orbit. Thus there exists a periodic point $x \in M$ such that

$$\text{dist}_H(g(x), \text{Orb}(x)) < \varepsilon.$$

In particular, we have that $x \in U_\Gamma$. However, by Proposition 1, we know that either $x \in W^s(\sigma)$ or there exists a point $y \in \text{Orb}^+(x) \setminus U_\Gamma$ with $\dist(y, \Gamma) \geq \varepsilon$. Both cases contradict with the orbital periodic shadowing property. Consequently, we have completed the proof. □

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