The Lynden-Bell bar formation mechanism in simple and realistic galactic models

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ABSTRACT

Using the canonical Hamilton-Jacobi approach we study the Lynden-Bell (1979) concept of bar formation based on the idea of orbital trapping parallel to the long or short axes of the oval potential distortion. The concept considered a single parameter – a sign of the derivative of the precession rate over angular momentum, determining the orientation of the trapped orbits. We derived a perturbation Hamiltonian which includes two more parameters characterising the background disc and the perturbation, that just as important as the previously known. This allows us to link this old theory with the matrix approach in linear perturbation theory, the theory of weak bars, and explain some features of the nonlinear secular evolution observed in N-body simulations.

Key words: Keywords: galaxies: bar, galaxies: kinematics and dynamics

1 INTRODUCTION

A remarkable paper by Lynden-Bell (1979) had influenced not only bar formation in stellar discs but also the radial-orbit instability in spherical clusters (e.g. Polyachenko & Shukhman 2015). It considers a weak oval distortion of the potential (bar) rotating with pattern speed \( \Omega_p \). A substantial group of stars within the corotation radius obey a condition

\[
|\Omega - \Omega_p - \frac{1}{2} \kappa| \ll \Omega \quad (1.1)
\]

do which we will refer below as ‘the slowness condition’. Here \( \Omega \) and \( \kappa \) denote the angular speed and the epicyclic frequency of radial oscillations. In the reference frame of the bar, motion of these stars can be viewed as slow nodal precession of stellar orbits, as long as the fast motion of stars along the orbits can be averaged out. It turns out that dynamics of these orbits, in particular, ability to align parallel to the long/short axis of the potential thereby reinforcing/weakening the primordial oval perturbation can be viewed qualitatively in a very elegant way.

The orbits in such a weakly non-axisymmetric system possess a specific integral of motion \( J_f = L/2 + I \), while the angular momentum \( L \) and radial action \( I \) of the star is changing. The key insight of Lynden-Bell is that if the precession rate ceases/grows with \( L \) (at constant \( J_f \)), such orbits seek for stationary position perpendicular/parallel to the bar. The former orbits were declared as ‘normal’, whereas the latter declared ‘abnormal’ since they occupy only a small fraction of the phase space in the centre of the disc. Mathematically, the ‘normal’/‘abnormal’ orbits have negative/positive derivative of the precession rate over \( L \) at constant \( J_f \). Given its importance, we began to call it ‘LB-derivative’ (Polyachenko 2004).

Our matrix methods for study instability in the disc and spherical stellar systems show that sign of the precession rate is an important parameter, along with its LB-derivative. For instance, the loss cone instability (Polyachenko 1991; Tremaine 2005; Polyachenko et al. 2007, 2008) is sensitive to the sign of the precession rate itself, not to the sign of its derivative. On the other hand, Merritt (1985), and then Saha (1991); Weinberg (1991); Palmer (1994) used the Lynden-Bell idea to explain a mechanism of the radial-orbit instability (ROI) in spherical systems. This idea indeed can be justified in the case of extremely slow ROI, although generally it is invalid (see details in Polyachenko & Shukhman 2015). Besides, the Lynden-Bell theory is in some way contrary to the theory of (weak) bars (see 3.3.2 and 3.3.3 of Binney & Tremaine 2008, hereafter BT) that predicts orbits’ alignment parallel to the long axis of the potential in the region between (outer) inner Lindblad resonance and corotation.

The goal of this paper is to analyse the problem consistently using a standard rigorous technique of finding stationary point parenting families of trapped orbits. Section 2

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describes the technique in a short form. Section 3 contains two analytic examples (the power-law angular speed and the isochrone potential) and results of N-body simulations of a realistic Milky Way model. Finally, in Section 4 we discuss and summarise the results.

2 THE HAMILTONIAN-JACOBI APPROACH FOR STATIONARY POINTS

In this section, we employ the standard formalism to find families of orbits trapped by the bar potential. To this end, we find stationary points of the Hamiltonian equations that mark closed elliptical orbits parenting families of trapped orbits. These closed orbits are analogues of circular orbits in axisymmetric potentials. The stellar motion is considered in the epicyclic approximation, and the bar pattern speed $\Omega_p$ obeys the slowness condition (1.1).

The Jacobi integral for axisymmetric potential $\Phi_0(r)$ in the rotating frame can be written as

$$H_0(L, I) = \frac{1}{2}\Omega_0^2(R^2 + \Phi_0(R) - \Omega_0 L^2 + \kappa(R) I^2 + \beta(R) I^2),$$  

(2.1)

where $R = R(L)$ is the guiding centre radius. In order to obtain linear corrections $\mathcal{O}(I)$ for the angular speed $\Omega(R)$ and the epicyclic frequency $\kappa(R)$, we retain a small post-epicyclic term $\beta I^2$. An explicit form of $\beta$ can be found, e.g., in Shu (1969), Contopoulos (1975), Mark (1976) and Bertin (2014):

$$\beta = \frac{1}{8R^2}(3g - \frac{q}{3} - \frac{2}{3}R \frac{d\rho}{dR}), \quad q = \frac{d\ln(R^2)}{d\ln R}.$$  

(2.2)

From (2.1) we obtain:

$$\Omega_1(L, I) \equiv \frac{\partial H_0(L, I)}{\partial I} = \kappa + 2\beta I + \mathcal{O}(I^2),$$  

(2.3)

$$\Omega_2(L, I) \equiv \frac{\partial H_0(L, I)}{\partial L} = \Omega - \Omega_p + \frac{1}{2L} I + \mathcal{O}(I^2).$$  

(2.4)

The orbit precession rate in the rotating frame is

$$\Omega_{pr}(L, I) \equiv \Omega_2(L, I) - \Omega_1(L, I).$$  

(2.5)

Let’s $\delta \Phi$ be a weak oval distortion of the axisymmetric disc potential $\Phi_0(r)$ rotating with pattern speed $\Omega_p$,

$$\delta \Phi(r, \varphi) = A(r) \cos(2\varphi), \quad A < 0.$$  

(2.6)

This form suggests that troughs of the potential and crests of the perturbed surface density are oriented along the horizontal axis $OX$. A full Hamiltonian $H$ is then equal to a sum of the Jacobi integral (2.1) and the perturbed potential (2.6), $H = H_0 + \delta \Phi$.

Following Polyachenko (2004, 2005), we perform transformation of action-angle variables

$$I \to J_f = I + \frac{L^2}{2}, \quad w_2 \to \phi = w_2 - \frac{L}{2} w_1$$  

(2.7)

to benefit from having slowly varying angle variable $\phi$ compared to $w_1$, provided that $\Omega_p$ obeys (1.1). Averaging the Jacobi integral (2.1) over $w_1$ gives a new integral of motion $J_f$. Using the epicyclic approximation,

$$r = R - \rho \cos w_1, \quad \varphi = \phi + \frac{L}{2} w_1 + \frac{2\Omega_p}{R} \sin w_1,$$  

(2.8)

one can have for the averaged bar potential:

$$V(L, J_f, \phi) = \frac{1}{2\pi} \oint \delta \Phi \left( r(L, J_f, w_1), \varphi(L, J_f, w_1, \phi) \right) dw_1 = B(L, J_f) \cos(2\phi),$$  

(2.9)

where $\rho = (2L/\pi)^{1/2}$ is the epicyclic radius,

$$B(L, J_f) = -\frac{A(R)}{2} \left( \frac{\rho}{R} \right) \left( \frac{R}{x} \frac{dA}{dR} + \frac{4\Omega_p}{x} \right).$$  

(2.10)

From (2.7) and (2.8) we infer that orbit’s apocentre is parallel to the long axis of the potential if angle variable $\phi = \frac{1}{2}\pi$ or $\phi = \frac{1}{2}\pi$, and to the short axis if $\phi = 0$ or $\pi$.

Omitting the terms depending on $J_f$ only, one can end up with the following expression for the Hamiltonian averaged over the fast orbital motion:

$$H(J, I, \phi) = -2Q I + 2P I^2 + V(L, J_f, \phi) = -2Q I + 2P I^2 + b I^{1/2} \cos(2\phi),$$  

(2.11)

where $b$ is determined by relation $B(L, J_f) = b(L)^{1/2}$, i.e.

$$b(L) = -\frac{A(R)}{2} \left[ \frac{2}{\pi R^2} \right]^{1/2} \left( \frac{R}{x} \frac{dA}{dR} + \frac{4\Omega_p}{x} \right).$$  

(2.12)

The coefficients $Q$ and $P$ are the precession rate of the orbits in the rotating frame and the LB-derivatives of the precession rate in the limit of small $I$:

$$Q \equiv \Omega_{pr}(L, 0),$$  

(2.13)

$$P \equiv \frac{dQ}{dL} - \frac{1}{2} \frac{d\Omega_{pr}}{dL} + \frac{1}{2} \beta.$$  

(2.14)

Note that in fact $b$, $Q$ and $P$ are functions of invariants, so no derivation over $I$ is needed. These invariants can be substituted by $L$ in the used decomposition order. To justify this, one needs to consider a small perturbation of the angular momentum, $h \ll L - L_0$, near the angular momentum $L_0$ of the circular orbit on a given radius. The scaling accepted in this paper is the following: $h, I, Q = \mathcal{O}(\varepsilon^2)$ and $P = \mathcal{O}(1)$, where $\varepsilon$ is a small parameter characterising the oval distortion, i.e. $A = \mathcal{O}(\varepsilon)$. In doing so, we obtain $Q = \Omega_{pr}(J_f, 0)$ and $P = \mathcal{O}(L_0)$. Changing $J_f$ and $L_0$ in the arguments of these functions to $L$ gives additional terms of the order $\mathcal{O}(\varepsilon^2)$ which are smaller than all terms retained in the Hamiltonian ($\mathcal{O}(\varepsilon^4)$). The detailed derivation can be found in Polyachenko & Shukhman (2020).

Similar technique based on the averaged Jacobi Hamiltonian near IRL for spiral perturbations using the post-epicyclic approximation including the terms up to $I^{(1/2)}^4$ was elaborated in Contopoulos (1975), but it differs in some details. Apart from the different form of perturbation, there are distinctions in the derivation of the averaged Hamiltonian. In particular, Contopoulos considered $L_0$ as the angular momentum of stars exactly on ILR, while in our case, $L_0$ is the angular momentum of any orbit obeying (1.1); the ILR may be absent. Besides, two small parameters of the problem – the amplitude of the spiral potential $A$ and the epicyclic parameter $I(\sim h)$, were considered as independent ones, while in our case they are related by the scaling given above. The latter allows us to obtain the final results much easier.

Stationary points are derived from the equations:

$$\frac{\partial H_J}{\partial \phi} = 0, \quad \frac{\partial H_J}{\partial I} = 0,$$  

(2.15)
leading to conditions for the radial actions:

\[
f(I^{1/2}) = \begin{cases} \frac{-b}{4} & \text{(short axis: } \phi = 0, \pi), \\ \frac{-b}{2} & \text{(long axis: } \phi = \pi/2, 3\pi/2), \end{cases}
\]

(2.16)

\[
f(I^{1/2}) = \frac{-b}{2},
\]

(2.17)

where \( f(z) = \frac{Qz}{2P} - 3z^2 \).

The negative sign of \( b \) essentially occurs at the end of the bar, i.e. in the vicinity of the corotation, see discussion below. Thus we shall mainly assume \( b > 0 \); the opposite case will be treated separately.

Let’s consider first a more significant case \( Q < 0 \). Eq. (2.16) has no solutions for ‘abnormal’ orbits \( P > 0 \), and one solution \( I_1 \) for ‘normal’ orbits \( P < 0 \) (Fig. 1). This solution gives a closed orbit oriented parallel to the short axis of the potential (S-orbit). Depending on values of \( Q, P \) and \( b \), eq. (2.17) can have from zero to two solutions. For ‘normal’ orbits, if the amplitude of the potential \( b \) is smaller than the critical value

\[
b_{\text{crit}} \equiv \frac{8|Q|}{3} \left( \frac{Q}{6P} \right)^{1/2},
\]

(2.18)

the smaller solution \( I_1 \) corresponds to a closed orbit parallel to the long axis (L-orbit), while the larger solution \( I_2 \) corresponds to a saddle point. In the opposite case \( b > b_{\text{crit}} \), no solution exists. For ‘abnormal’ orbits \( P > 0 \), only one solution \( I_1 \) exists for arbitrary \( b \). It is often called the sequence \( x_1 \) (e.g., BT, sect. 3.3.2).

All possible types of phase portraits for \( Q < 0 \) are given in Fig. 2. The ‘abnormal’ orbits \( P > 0 \) allow for a family of L-orbits only (portrait L). This family exists when \( P \) decreases to zero and even below zero, when another stationary point and associated family of trapped S-orbits appears. We designate this portrait as ‘LS’, meaning that the first letter corresponds to a stationary point with smaller radial action \( I_1 \). This new family of S-orbits have much larger eccentricities.

Stationary points for \( Q > 0 \) can be seen from Fig. 2 by changing \( P \to -P \) and horizontal shift of the portraits by \( \pi/2 \). Shifting the portrait (L) is topologically equivalent to (S). Below we shall refer to the shifted (LS) portrait as (SL) since the short axis family now appears with smaller radial action \( I_1 \). Changing of the sign of \( b \) results only in the horizontal shift of the portraits by \( \pi/2 \).

3 Examples

3.1 Power-law potentials

This type of potentials include motion in Keplerian and harmonic potentials, and the Mestel disc with the flat rotation curve. Let’s assume the angular speed in the form \( \Omega(R) = \Theta R^{-\alpha} \). It is easy to show that

\[
Q = \Theta R^{-\alpha} \cdot \left( 1 - \sqrt{1 - \alpha/2} \right) - \Omega_p,
\]

(3.1)

and

\[
P = \frac{\alpha}{2R^2} \left[ \left( \frac{8}{2 - \alpha} \right)^{1/2} - 3 \frac{\alpha}{4} - \frac{2}{2 - \alpha} \right].
\]

(3.2)

Curve \( R^2P \) versus \( \alpha \) is given in Fig. 3. It turns out that in power-law potentials, all near circular orbits could be either ‘normal’ if \( \alpha > 0,862 \), or ‘abnormal’ if \( \alpha < 0,862 \). Note that this boundary is close to \( \alpha_{\text{BW}} = 7/8 \) of the Bahcall & Wolf (1976) density profile (\( \propto r^{-\alpha} \)).

The ‘normal’ orbits naturally trap along the short axis of the potential inside the Lindblad resonance \( Q > 0 \), but they can be trapped along the long axis outside the Lindblad resonance if \( b \) is smaller than \( b_{\text{crit}} \). On the opposite, the ‘abnormal’ orbits naturally trap along the long axis of the potential outside the resonance but can be trapped along the short axis if \( b \) is small.

Figure 1. Solutions of eqs. (2.16, 2.17) for \( Q < 0, b > 0 \).

Figure 2. Phase portraits in (\( \phi - I \)) planes of the averaged Hamiltonian (2.11) for \( Q < 0, b > 0 \). The long axis of the potential corresponds to \( \phi = \pi/2, 3\pi/2 \).
3.2 The isochrone potential

Consider the isochrone potential

$$\Phi(r) = -\frac{GM}{a + (a^2 + r^2)^{1/2}},$$

(3.3)

for which the Jacobi integral reads:

$$H_0 = -2G^2 M^2 t^{-2} - \Omega_0 L,$$

(3.4)

where $t = 2J_I + s$, $s = (L^2 + 4GMa)^{1/2}$. The LB-derivative of the precession rate can be obtained explicitly for any orbit (see also Lynden-Bell 1979):

$$\frac{\partial \Omega_{pr}(L,J_I)}{\partial L} = 4G^2 M^2 \frac{1}{s^4} \left(4GMat - 3L^2 s\right).$$

(3.5)

In the limit of circular orbits (small $l$), one can use (2.14) or put $t = L + s$ in eq. (3.5).

Fig. 4a shows angular speed $\Omega$, $\Omega_l \equiv \Omega - \kappa/2$, and two pattern speeds $\Omega^{(1)}_b$ and $\Omega^{(2)}_b$; (b) LB-derivative (2.14); (c) stationary points $I_1$, $I_3$ for the pattern speeds in (a) in units of $\kappa R^2/2$ for model bar potential (3.6), $\varepsilon = 0.1$ (same colour coding). Solid/dotted lines in (c) show orbits parallel to the long/short axis. Blue/pink shades show where the slowness assumption breaks down for $\Omega^{(1)}_b/\Omega^{(2)}_b$. Ticks at 1.58, 3.73 and 4.25 mark maximum of $\Omega_l$ and zeros of $\mathcal{P}$ and $b$, correspondingly.

3.3 The Milky Way model

The model we use here was elaborated in detail in our previous paper (Polyachenko et al. 2016). It consists of three components: thin exponential disc, S´eric bulge and NFW halo. The disc is characterised by the radial scale $R_d = 2.9$ kpc, vertical scale $z_d = 300$ pc and mass $M_d = 4.2 \cdot 10^{10} \mathcal{M}_{\odot}$ (solar mass). The bulge has a weak cuspy density profile in the centre $\rho_b \propto r^{-1/2}$, and mass $M_b \approx 10^{10} \mathcal{M}_{\odot}$. The total circular velocity is bulge-dominated at radii $R \lesssim 2.5$ kpc, and halo-dominated at $R > 9$ kpc. At radius $R = 6$ kpc, where the disc contribution peaks, the force from the halo is about 2/3 of the force from the disc in the galactic plane.

N-body simulations show bar instability producing a bar rotating with pattern speed $\Omega_b = 55$ km/s/kpc. A bar amplitude grows exponentially in time with a small growth rate $\gamma \sim 0.07 \Omega_b$ and saturates at the level 10...20 per cent of the axisymmetric background. After that, the amplitude stays nearly constant, but the bar pattern speed gradually decreases.

It is well known that the inner Lindblad resonance
(ILR) damps spiral waves (Mark 1974). Through this effect, the bar formation is suspended in flat disc galaxies. However, the bar can still be formed if ILR radius is comparable with or smaller than the disc vertical scale. Moreover, the bar pattern speed and the growth rate can be reproduced well from the linear perturbation theory for flat discs, if one uses an angular speed \( \Omega \) averaged over vertical axis \( z \), instead of in-plane \( \Omega \) calculated from the total axisymmetric potential (Polyachenko et al. 2016).

Fig. 5a shows the in-plane \( \Omega \) and \( \Omega_i \), \( z \)-averaged \( \Omega_i \), and the initial bar pattern speed \( \Omega_p \). A vertical dashed line at \( R = 0.55 \) kpc marks the maximum of \( \Omega_i \). Curve \( \mathcal{P}(R) \) on panel (b) is calculated using \( \Omega_i \). Similar to Fig. 4b, it is positive in the centre, and is slightly negative beyond \( R = 1.73 \) kpc.

Panel (c) presents stationary point curves for \( \Omega_i \) and the maximum bar amplitude \( b \) obtained from N-body snapshots (in particular, \( T \approx 1.3 \) Gyr). These curves are qualitatively similar to those shown on Fig. 4c. Remarkably, the curves come almost the same for the in-plane \( \Omega \) (dashed line for \( I_1 \); \( I_3 \) is not shown).

It is worth noting that power-law expression for \( \mathcal{P} \) (3.2) is a poor proxy for non-power-law angular velocities where \( -d \ln \Omega / d \ln R \) is used for \( \alpha \). E.g., for the isochrone potential in the centre this gives \( \alpha \approx R^2/a^2 \) and

\[
\mathcal{P} \approx \frac{1}{8} - \frac{1}{12} \frac{R^2}{a^2} \tag{3.7}
\]

(in units \( a^{-1/2} \)) while the correct expansion reads

\[
\mathcal{P} \approx \frac{1}{4} - \frac{3}{16} \frac{R^2}{a^2} \tag{3.8}
\]

Similarly for the realistic rotation curve, eq. (3.2) gives only qualitative shape of \( \mathcal{P} \). In particular, zero of the proxy is located at \( R = 1.04 \) instead of \( R = 1.73 \) for the true curve.

4 DISCUSSION AND SUMMARY

In our previous works on radial-orbit (Polyachenko et al. 2010a, 2015; Polyachenko & Shukhman 2015, 2017) and loss cone instabilities (Polyachenko et al. 2007, 2008, 2010b), we argue that the sign of the LB-derivative of the precession rate is less important than the sign of the precession rate itself. The latter was missed in the pioneer paper by Lynden-Bell (1979). Recall that cited paper emphasises the role of ‘abnormal’ orbits as design components for the bar in the isochrone potential. These orbits can be captured parallel to the long axis of the bar thereby reinforcing it. On the contrary, the ‘normal’ orbits (\( \mathcal{P} < 0 \)) fail to build the bar, because they are captured along the short axis thereby weakening the bar. Our theory reproduces these cases at the limit \( Q = 0 \), see a double arrow in the left panel of Fig. 6.

Meanwhile, taking into account the precession rate \( Q \) changes this picture considerably. In Fig. 4c, we see that \( I_3 \) families do not change their orientation at the critical point \( \mathcal{P} = 0 \) because of the presence of \( Q \)-term in eq. (2.17), i.e. orbits are still able to coalesce and add to the bar. Presence of \( Q \)-term essentially allows to put \( \mathcal{P} = 0 \) beyond the critical point disregarding the irrelevant short-axis branch \( I_3 \) because it falls into a region of radial actions free of stellar orbits.

A similar structure of trapped orbits takes place in the realistic model presented in Section 3.3. This model has a weak cusp in the centre \( \rho \propto r^{-1/2} \) and nearly flat ro-
In the inside zone, the positive sign of $\mathcal{P}$ plays a major role in determining the orientation of orbits along the potential well. However, if $\mathcal{Q} > 0$, our theory predicts a family of short-axis orbits (S-orbits) for small amplitudes $b$ (portrait SL). This presumably explains the well-known phenomenon (e.g., Combes & Elmegreen 1993) that bars in N-body simulations often have pattern speeds larger than the maximum of $\Omega_i$ (i.e. $\mathcal{Q} < 0$) because S-orbits in case of $\mathcal{Q} > 0$ immediately destroy low amplitude bar-like perturbations. Only perturbations with $\mathcal{Q} < 0$ can be reinforced by trapping the orbits along the potential well. Remarkably, the matured bar can sustain the pattern speed decrease below the maximum of $\Omega_i$ because for large amplitudes $b$ only L-orbits are possible.

In the outside region beyond $\mathcal{P} = 0$, orbits continue to add to the bar unless the bar pattern speed is too low and the orbits find themselves between two ILR’s, $\mathcal{Q} > 0$.

In the theory of weak bars (Sanders & Huntley 1976; Sellwood & Wilkinson 1993), the epicyclic approximation is used to derive orientations of nearly circular orbits. Below we follow sect. 3.3.3 of BT to compare their closed loop orbits with ours. Their ‘epicyclic radius’ is

$$ C_2 = -\frac{A}{R\Delta} \left( \frac{R \, dA}{A \, dR} + \frac{2\Omega}{\Omega - \Omega_p} \right), $$

where $\Delta = x^2 - 4(\Omega - \Omega_p)^2$. Near the resonance $x \approx 2(\Omega - \Omega_p)$, so $\Delta$ can be substituted by $-4Qx$. The corresponding radial action is then

$$ I = \frac{x}{2} C_2^2 \approx \frac{A^2}{16R^2Q^2x^2} \left( \frac{R \, dA}{A \, dR} + \frac{4\Omega}{\Omega - \Omega_p} \right)^2. $$

The last expression coincides with our stationary point $I_1$ for L-orbits obtained from (2.17) outside ILRs, provided $\mathcal{P} = 0$ (see also Goldreich & Tremaine 1981).

The ‘epicyclic radius’ $C_2$ formally changes its sign at the inner and outer ILRs due to $\Delta$, resulting in appearance of $x_2$ sequence of orbits perpendicular to the potential well ($x_1,x_2-x_1$ sequence in Fig. 3.20 of BT). From our theory it follows (Fig. 5c) that orbits’ orientation along the potential well is retained between the resonances for large bar amplitudes (c.f. Fig. 3.18 of BT). Note that in case of the weak bar, the L-orbit family continues smoothly across the resonances, but additional S-orbit family appears around smaller $I_1$.

The amplitude $b$ becomes negative at radius $R_b$ where the round bracket in (2.12) vanishes. The physical meaning of this radius is the last closed orbit of $x_1$ sequence, so it can be used as a clearly detectable proxy of the bar length (see also Martinez-Valpuesta et al. 2006). At $R_b$ the LB-derivative is likely to be nearly zero and the precession rate $\mathcal{Q} < 0$.

So, summarizing the above, we can note that the sign of LB-derivative of the precession rate $\mathcal{P}$ does not specify the direction of orbit’s trapping with respect to the potential well. We show that in addition to this parameter, signs of the precession rate $\mathcal{Q}$ and the average amplitude $b$ are important. Despite a variety of combinations of these three parameters, only a few are relevant to bar formation and evolution in galactic models (see pink highlights in Fig. 6). The left panel (for $b > 0$) shows capture predominance parallel to the long axis of the potential almost regardless of signs of $\mathcal{P}$ and $\mathcal{Q}$. Trapping along the short axis is theoretically possible for a very low bar amplitude (SL), but the trapped orbits will immediately destroy the bar. The only realistic possibility to trap orbits perpendicular to the potential well is to have $b < 0$ (right panel).

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**DATA AVAILABILITY**

Data underlying this article will be shared on reasonable request to the authors via epolyach@inasan.ru. Data related to the initial conditions may be reproduced via the publically available software Galac1ICS.

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