THE MIRROR LAGRANGIAN COBORDISM FOR THE EULER EXACT SEQUENCE

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ABSTRACT. For $X = \mathbb{P}^n$ the Euler sequence is given by

$$0 \to \Omega^1_{\mathbb{P}^n} \to \mathcal{O}^n_{\mathbb{P}^n}(-1) \to \mathcal{O}_{\mathbb{P}^n} \to 0$$

We describe the Lagrangian cobordism corresponding to this sequence via mirror symmetry, in the sense of Biran-Cornea. In particular, we describe the mirror Lagrangian of the cotangent sheaf $\Omega^1_{\mathbb{P}^n} \in D^b(\mathbb{P}^n)$ in the mirror Fukaya category $\text{Fuk}(U_\Delta)$.

1. Introduction and Summary of Main Results

For $X = \mathbb{P}^n$ the Euler exact sequence is given by

$$(1) \ 0 \to \Omega^1_{\mathbb{P}^n} \to \mathcal{O}^n_{\mathbb{P}^n}(-1) \to \mathcal{O}_{\mathbb{P}^n} \to 0,$$

see [24]. Let $\Delta$ be the moment polytope of $X$ with respect to the anti-canonical embedding. Homological mirror symmetry for toric Fano manifolds, postulates the equivalence of the categories $D^b(\mathbb{P}^n)$ and $D^b(Fuk(U_\Delta))$, where $U_\Delta := \text{Log}^{-1}|\Delta|$ is the inverse image of the polytope $\Delta$ of $X$ under the logarithm map $\text{Log}|\cdot| : (\mathbb{C}^*)^n \to \mathbb{R}^n$. By results of Abouzaid, mirrors of line bundles $\mathcal{O}_{\mathbb{P}^n}(k)$ are known to be given by tropical Lagrangian sections in $\text{Fuk}(U_\Delta)$, see [1, 2]. By results of Biran-Cornea, Lagrangian cobordisms in Fukaya categories are the mirror equivalent of triangles in $D^b(X)$, see [3, 6, 7, 8]. Our aim in this work is to introduce the Lagrangian cobordism corresponding to the Euler exact sequence $(1)$, to which we refer as the Lagrangian Euler mirror cobordism. Let us first consider the example of the projective line for $n = 1$:

Example 1.1 (Lagrangian Euler mirror cobordism for projective line). For $n = 1$ the cotangent bundle is given by $\Omega^1_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(-2)$. Hence, the Euler sequence is

$$(2) \quad 0 \to \mathcal{O}_{\mathbb{P}^1}(-2) \to \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \to \mathcal{O}_{\mathbb{P}^1} \to 0.$$ 

In this case, all three elements of the sequence are given in terms of line bundles $\mathcal{O}_{\mathbb{P}^1}(k)$. On the other hand, for the mirror, consider the annulus

$$(3) \quad U_{[-1,1]} := \text{Log}^{-1}([-1,1]) = \{e^{-1} \leq |z| \leq e\} \subset \mathbb{C}^*$$

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where the map $\log|\cdot| : \mathbb{C}^* \to \mathbb{R}$ is given by $z \mapsto \log|z|$. A Lagrangian $L$ (curve) in $U_{[-1,1]}$ is known to correspond to a line bundle $\mathcal{O}_{\mathbb{P}^1}(k)$ if it is a section of $\log|\cdot|$ beginning at $e$ and ending in $e^{-1}$. Specifically, $L$ corresponds to $\mathcal{O}_{\mathbb{P}^1}(k)$ if its lift in $\tilde{U}_{[-1,1]}$, the universal cover of $U_{[-1,1]}$, given by $[-1,1] \times \mathbb{R}$ begins at $(e,0)$ and ends at $(e^{-1},2\pi k)$, that is if it rotates around the origin $k \in \mathbb{Z}$ times. Applying Lagrangian surgery operations to two Lagrangians $L_1$ and $L_2$, in the sense of Polterovich [32], gives rise to a cobordism beginning in $L_1,L_2$ and ending in their surgery $L' = L_1 \# L_2$, see [6].

Fig 1 shows how the surgery of two Lagrangians $L_1$ (blue) and $L_2$ (red), which are mirror to $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ and $\mathcal{O}_{\mathbb{P}^1}$, results in a Lagrangian $L' = L_1 \# L_2$ mirror to $\mathcal{O}_{\mathbb{P}^1}(-2)$, giving rise to the Lagrangian Euler mirror cobordism in this case.

![Figure 1. Two Lagrangians corresponding to $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ (blue) and $\mathcal{O}_{\mathbb{P}^1}$ (red) transitioning to the Lagrangian corresponding to $\mathcal{O}_{\mathbb{P}^1}(-2)$](image)

For the higher dimensional projective spaces with $n \geq 2$ the cotangent bundle $\Omega^1_{\mathbb{P}^n}$ is no longer a line bundle but rather a vector bundle of rank $n$. In particular, the mirror of $\Omega^1_{\mathbb{P}^n}$ is not directly given by Abouzaid’s construction. The construction of the Lagrangian Euler mirror cobordism is based on choosing two specific mirror Lagrangians $L_1, L_2$, representing $\mathcal{O}^{n+1}_{\mathbb{P}^n}(-1)$ and $\mathcal{O}_{\mathbb{P}^n}$. Our main result is Theorem 3.4 which shows that the Lagrangian $L' = L_1 \# L_2$ is the mirror of the cotangent bundle $\Omega^1_{\mathbb{P}^n}$, when considered as a $T$-equivariant bundle in the sense of Klyachko [29].

The rest of the work is organized as follows: In section 2 we recall relevant results and definitions from ”both sides of the mirror”. In Section 3 we construct the Lagrangian Euler mirror cobordism for general $n \geq 1$. In Section 4 we discuss concluding remarks.

### 2. Relevant Results and Definitions from Both Sides of the Mirror

In this section we review relevant definitions and results used:
2.1. Projective space as a toric Fano manifold. A toric variety is an algebraic variety $X$ containing an algebraic torus $T \simeq (\mathbb{C}^*)^n$ as a dense subset such that the action of $T$ on itself extends to the whole variety, see [16, 20] for standard references. A compact toric variety $X$ is said to be Fano if its anti-canonical class $-K_X$ is Cartier and ample.

Let $N \simeq \mathbb{Z}^n$ be a lattice and let $M = N^\vee = Hom(N, \mathbb{Z})$ be the dual lattice. Denote by $N_R = N \otimes \mathbb{R}$ and $M_R = M \otimes \mathbb{R}$ the corresponding vector spaces. Let $\Delta$ be an integral polytope and let $L(\Delta) := \bigoplus_{m \in \Delta \cap M} \mathbb{C}z^m$ be the space of Laurent polynomials whose Newton polytope is $\Delta$. The polytope $\Delta$ determines an embedding

$$i_\Delta : (\mathbb{C}^*)^n \to \mathbb{P}(L(\Delta)^\vee)$$

$$z \mapsto [z^m | m \in \Delta \cap M]$$

The polarized toric variety corresponding to the polytope $\Delta \subset \mathbb{R}^n$ is defined to be

$$X_\Delta = \overline{i_\Delta((\mathbb{C}^*)^n) \subset \mathbb{P}(L(\Delta)^\vee)},$$

the compactification of the image of $i_\Delta$. The polar polytope of $\Delta \subset M_R$ is given by

$$\Delta^\circ = \{n | \langle m, n \rangle \geq -1 \text{ for every } m \in \Delta \} \subset N_R$$

The polytope $\Delta$ is said to be reflexive if $0 \in Int(\Delta)$ and $\Delta^\circ$ is an integral polytope. A reflexive polytope is said to be Fano if every facet $\Delta^\circ$ is the convex hall of a basis of $M_R$. Batyrev showed in [4] that $X_\Delta$ is a Fano variety if and only if $\Delta$ is reflexive and, in this case, the embedding $i_\Delta$ is the anti-canonical embedding. The Fano variety $X_\Delta$ is smooth if and only if $\Delta^\circ$ is a Fano polytope. The duality between the polytope $\Delta$ and its polar $\Delta^\circ$ serves as the basis of mirror symmetry for toric Fano manifolds (see 2.2).

Let $\Sigma = \Sigma(\Delta)$ be the fan determined by the polytope $\Delta$, see [16, 20]. We say that a function $\psi : N_R \to \mathbb{R}$ is a $\Sigma$-support function if it is continuous, linear when restricted to each maximal cone $\sigma \in \Sigma(n)$, and $\psi(n_\rho) \in \mathbb{Z}$ for any primitive generator $n_\rho \in N_R$ of a one-dimensional ray $\rho \in \Sigma(1)$. Denote by $SF(\Sigma)$ the group of $\Sigma$-support functions. When $X$ is smooth one has $\text{Div}_T(X) \simeq SF(\Sigma)$ by setting

$$D_\psi := \sum_{\rho \in \Sigma(1)} \psi(n_\rho)V(\rho).$$

Note that when $X$ is fano the vertices of the polar polytope $\Delta^\circ(0)$ are exactly the primitive generators of the one-dimensional rays of the fan $\Sigma(1)$. Denote by

$$m(\psi, \sigma) \in M = Hom(N, \mathbb{Z})$$
the element such that $\psi(n) = \langle n, m(\psi, \sigma) \rangle$ for $n \in \sigma$, where $\sigma \in \Sigma(n)$ is a maximal cone.

In particular, projective space $X = \mathbb{P}^n$ is given as a toric Fano polytope $X_\Delta$ where $\Delta$ is the polytope whose polar is given by

\begin{equation}
\Delta^o = \text{Conv} \left( \left\{ -\sum_{i=1}^{n} e_i, e_1, \ldots, e_n \right\} \right) \subset \mathbb{R}^n.
\end{equation}

For example, for $n = 2$ the polytopes $\Delta^o$ and $\Delta$ are illustrated in Fig. 2:

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.png}
\caption{The polytopes $\Delta^o$ and $\Delta$ for $X = \mathbb{P}^2$.}
\end{figure}

2.2. Line bundles and Abouzaid’s tropical Lagrangian sections. Let $X$ be a polarized toric manifold given by a polytope $\Delta$. Denote by $\Delta(k)$ the set of $k$-dimensional faces of $\Delta$ and let $V_X(F)$ be the closure of the $T$-orbit corresponding to the face $F \in \Delta(k)$. The group of toric divisors is given by

\begin{equation}
\text{Div}_T(X) := \bigoplus_{F \in \Delta(n-1)} V_X(F) \cdot \mathbb{Z}.
\end{equation}

The Picard group of line bundles $\text{Pic}(X)$ on a smooth toric manifold $X$ is described by the exact sequence

\begin{equation}
0 \to M \to \text{Div}_T(X) \to \text{Pic}(X) \to 0.
\end{equation}

Assume $X$ is a toric Fano manifold and let $\Delta^o$ be the polar polytope of $\Delta$. In [2] Abouzaid described the mirror Lagrangian branes corresponding to elements of $\text{Pic}(X)$ as follows: Let $W \in L(\Delta^o)$ be a generic Laurent polynomial whose Newton polytope is $\Delta^o$. Denote by $M_W = W^{-1}(0) \subset (\mathbb{C}^*)^n$ the fibre of $W$ over $0 \in \mathbb{C}$. Consider

\begin{equation}
\widetilde{W}_{t,s}(z) := 1 + \sum_{n \in \Delta^o(0)} t^{-1}(1 - s \cdot \phi_n(\text{Log}|z|)) z^n
\end{equation}

where $\phi_n \in C^\infty(\mathbb{C}^n)$ for $n \in \Delta^o(0)$ are required to satisfy certain decay conditions. For $s = 0$ one has $\widetilde{W}_{t,0} \in L(\Delta^o)$ but for $s \neq 0$ the function $\widetilde{W}_{t,s}$ is no longer a Laurent polynomial. However, $M_{t,s} = \widetilde{W}_{t,s}^{-1}(0)$ are all symplectomorphic for generic values of $s$ for
Denote by \( M = M_{t,1} \) with \( 0 << t \) big enough. We refer to the pair \( ((\mathbb{C}^*)^n, M) \) as the tropical polarized mirror model of the toric Fano manifold \( X \).

**Definition 2.1.** A Lagrangian brane \( L \subset (\mathbb{C}^*)^n \) is an embedded compact graded Lagrangian submanifold, which is spin and exact. A Lagrangian brane is said to be admissible if \( \partial L \subset M \) and there exists a neighbourhood of \( \partial L \) in \( L \) which agrees with the parallel transport of \( \partial L \) along a segment \( \gamma \subset \mathbb{C} \) with respect to \( W \). A pair of admissible Lagrangian branes \( (L_1, L_2) \) is said to be positive if their corresponding segments \( \gamma_1, \gamma_2 \subset \mathbb{C} \) lie in the half plane and their tangent vectors are oriented counter clock-wise such that \( \text{Im}(\gamma_2(\theta)) < \text{Im}(\gamma_1(\theta)) \).

Admissible Lagrangian branes are objects of the Fukaya \( \mathcal{A}_\infty \) pre-category \( \text{Fuk}((\mathbb{C}^*)^n, M) \) which we denote \( \text{Fuk}(U_\Delta) \), see [2, 31, 34]. The space of morphisms between two positive transverse objects \( L_1, L_2 \in \text{Fuk}(U_\Delta) \) is given by the Floer complex \( (CF^*(L_1, L_2), \partial) \). In [2] Abouzaid introduced the \( \mathcal{A}_\infty \) sub-pre-category of tropical Lagrangian sections \( \text{Fuk}_{trop}(U_\Delta) \subset \text{Fuk}(U_\Delta) \) which he proved to be quasi-equivalent to the \( \mathcal{D}G \)-category of line bundles over \( X \).

Consider the map \( \log|\cdot| : (\mathbb{C}^*)^n \to \mathbb{R}^n \) and let \( \mathcal{A} = \frac{1}{i} \log|M| \subset \mathbb{R}^n \) be the amoeba of \( M \), see [23]. In [1, 2] Abouzaid shows that there exists a component \( \tilde{\Delta} \subset \mathbb{R}^n \setminus \mathcal{A} \) in the complement of \( \mathcal{A} \) which is contained in the polytope \( \Delta \subset \mathbb{R}^n \) and is \( C^0 \) close to it. Note that the map \( \log|\cdot| \) can be viewed as a fibration \( \mathcal{A} \) whose fibre is \( \mathbb{T}^n \) and whose zero section is the Lagrangian \( (\mathbb{R}^+)^n \subset (\mathbb{C}^*)^n \). Consider the following definition:

**Definition 2.2.** A tropical Lagrangian section in \( ((\mathbb{C}^*)^n, M) \) is an admissible Lagrangian brane \( L \) which is a section of the map \( \log|\cdot| \) restricted to \( \tilde{\Delta} \).

It is shown in [2] that up to Hamiltonian isotopy tropical Lagrangian sections in \( ((\mathbb{C}^*)^n, M) \) are in one-to-one correspondence with elements of \( \text{Pic}(X) \). For instance, the class of the trivial bundle \( O_X \in \text{Pic}(X) \) corresponds to the trivial section \( L_0 \subset (\mathbb{C}^*)^n \), which is a tropical section of the polarized mirror model. The correspondence is based on the fact that any tropical Lagrangian section \( L \subset (\mathbb{C}^*)^n \) must coincide with \( L_0 \subset (\mathbb{C}^*)^n \) in a small neighbourhood of the fibre \( \log^{-1}(x) \simeq \mathbb{T}^n \) for any vertex \( x \in \Delta(0) \subset \mathbb{R}^n \). As \( \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n \), a lift \( \tilde{L} \subset (\mathbb{R}^+)^n \times \mathbb{R}^n \) of the tropical section \( L \) to the universal cover gives rise to elements \( m(\tilde{L}, x) \in \mathbb{Z}^n \) for any \( x \in \Delta(0) \). In particular, define \( \psi_{\tilde{L}} \in SF(\Sigma) \) to be the support function defined by

\[
m(\psi, \sigma) = m(\tilde{L}, x_\sigma),
\]

\footnote{In fact, it is considered as the dual SYZ fibration of the moment map \( \mu : X \to \Delta \), in this case.}
for $x_\sigma \in \Delta(0) \simeq \Sigma(n)$. As a lift depends on a choice of deck transformation $m \in \mathbb{Z}^n$ we get that $Pic(X) \simeq \text{Fuk}_{\text{trop}}(U_\Delta)/\sim$.

2.3. Klyachko’s description of $T$-equivariant bundles. In [29] Klyachko generalized the classical description of $Pic(X)$ of a toric manifold $X$, presented in 2.1 to $T$-equivariant vector bundles $p : E \to X$ of arbitrary rank $r = \text{rank}(E)$. According to Klyachko such a vector bundle is uniquely determined by a $r$-dimensional vector space $E$ equipped with filtrations $\{E_\rho(i)\}_{i \in \mathbb{Z}}$ for any $\rho \in \Sigma(1)$ such that the following compatibility condition holds: For any $\sigma \in \Sigma$ the filtrations $E_\rho(i)$ for $\rho \in \sigma(1)$ consist of coordinate subspaces of some basis of the space $E$.

For instance, note that when $E$ is a line bundle the vector space $E \simeq \mathbb{C}$ is a one-dimensional space. Hence, for any $\rho \in \Sigma(1)$ the filtration $E_\rho(i)$ is determined by the index $i_\rho \in \mathbb{Z}$ at which the filtration changes from $E$ to zero, which is the same as giving a $T$-equivariant divisor in $\text{Div}_T(X)$.

The compatibility condition implies that the filtrations determine, for any $\sigma \in \Sigma(n)$, a decomposition $E = \bigoplus_{m \in \mathbb{Z}^n} E^\sigma(m)$ such that

\begin{equation}
E_\rho(i) = \sum_{(m, \rho) \geq i} E^\sigma(m),
\end{equation}

for all $\rho \in \sigma(1)$. For instance, when $E = O(D)$ is a line bundle with $D \in \text{Div}_T(X)$ one has

\begin{equation}
E^\sigma(m) = \begin{cases} 
\mathbb{C} & m = m(\psi_D, \sigma) \\
0 & m \neq m(\psi_D, \sigma)
\end{cases}
\end{equation}

Hence, the system of decompositions $E = \bigoplus_{m \in \mathbb{Z}^n} E^\sigma(m)$ for $\sigma \in \Sigma(n)$ generalizes the description of line bundles in terms of support functions. For any $T$-equivariant bundle $E$ and maximal cone $\sigma \in \Sigma(n)$ let us define

\begin{equation}
W(E, \sigma) := \{m | E^\sigma(m) \neq 0\} \subset M,
\end{equation}

to be the set of weights of $E$ at the cone $\sigma$.

Klyachko gives the following description of the cotangent bundle $\Omega^1_X$ of a toric manifold $X$ as the vector bundle corresponding to the following system of filtrations

\begin{equation}
\Omega_\rho(i) = \begin{cases} 
M_\rho & i < 0 \\
\text{Ker}\rho & i = 0 \\
0 & i > 0
\end{cases}
\end{equation}

where $\text{ker}\rho = \{\omega | \langle \omega, \rho \rangle = 0\}$. The following example presents the corresponding decomposition determined by the filtrations $\Omega_\rho(i)$ for the case of projective space $X = \mathbb{P}^n$. 
Example 2.3 (The weights of the cotangent bundle $\Omega^1_{\mathbb{P}^n}$). Let $X = \mathbb{P}^n$ be realized as a toric Fano manifold by $\Delta^\circ(0) = \{e_0, ..., e_n\}$ with $e_0 := - \sum_{i=1}^n e_i$, see 2.1. The maximal cones are
\begin{equation}
\sigma_0 = \sum_{i=1}^n e_i \cdot \mathbb{R}_+ \quad ; \quad \sigma_k = e_0 \cdot \mathbb{R}_+ + \sum_{i \neq k} e_i \cdot \mathbb{R}_+
\end{equation}
for $1 \leq k \leq n$. Hence, the decompositions of $\Omega = \mathcal{M}_C$, with dual basis $\{e_1^*, ..., e_n^*\}$, corresponding to Klyachko’s filtrations are given by
\begin{equation}
\Omega^\sigma_0 = \bigoplus_{i=1}^n (-e_i^*) \cdot \mathbb{C}
\end{equation}
and
\begin{equation}
\Omega^\sigma_k = e_k^* \cdot \mathbb{C} \oplus \left( \bigoplus_{i \neq k} (e_k^* - e_i^*) \cdot \mathbb{C} \right).
\end{equation}
In particular, the weights of $\Omega^1_X$ are given by
\begin{equation}
W(\Omega^1_{\mathbb{P}^n}, \sigma_0) = \{-e_1^*, ..., -e_n^*\}
\end{equation}
and
\begin{equation}
W(\Omega^1_{\mathbb{P}^n}, \sigma_k) = \{e_k^*\} \cup \{e_k^* - e_i^* | k \neq i\}.
\end{equation}

Remark 2.4 (Relation to the anti-canonical divisor). If we sum the weights of the decompositions (23) and (24) we get
\begin{equation}
m_{\sigma_0} = -e_1^* - ... - e_n^*
\end{equation}
and
\begin{equation}
m_{\sigma_k} = n \cdot e_k^* - \sum_{i \neq k} e_i^*
\end{equation}
for $k = 1, ..., n$. Note that these are exactly the linear functionals defining the support function $\psi_{-K}$ corresponding to the anti-canonical divisor $-K_{\mathbb{P}^n} = - \sum_{i=0}^n V(\rho_i)$. That is for which $m_{\sigma_k} = m(\psi_{-K}, \sigma_k)$ in the sense of [9].

2.4. Cobordisms, surgeries and triangles in the Fukaya category. We refer the reader to the works of Biran-Cornea for the theory of Lagrangian cobordisms, see [5, 6, 7, 8]. Two Lagrangian submanifolds $L_1, L_2 \subset (M, \omega)$ are said to be Lagrangian cobordant to a third Lagrangian submanifold $L' \subset (M, \omega)$ if there exists $(V, L_1 \cup L_2, L')$, a smooth

\footnote{We will mainly be concerned with cobordisms with two negative ends and one positive end, as these are the cobordisms corresponding to exact sequences. In general, one can take any number of positive and negative ends.}
cobordism (see Fig. 3), and a Lagrangian embedding \( V \subset ([0, 1] \times \mathbb{R}) \times M \) such that for some \( \epsilon > 0 \) one has

\[
V|_{[0, \epsilon) \times \mathbb{R}} = \bigcup_{i=1}^2 ([0, \epsilon) \times \{i\} \times L_i) \quad ; \quad V|_{(1-\epsilon, 1] \times \mathbb{R}} = ([0, \epsilon) \times \{1\} \times L_1') .
\]

**Figure 3.** A cobordism \( V \) between \( L_1 \cup L_2 \) and \( L' \).

We will henceforth refer to such cobordisms as triangular. Triangular Lagrangian cobordisms could be constructed via the method of Lagrangian surgery due Polterovich, see [32]. Locally, assume \( M = \mathbb{C}^n \) and consider the two Lagrangians \( L_1 = \mathbb{R}^n \) and \( L_2 = i\mathbb{R}^n \), intersecting transversally at the origin. Let \( H: \mathbb{R} \to \mathbb{C} \) be any smooth curve of the form \( H(t) = a(t) + ib(t) \) such that

1. \( H(t) = t \) for \( t \leq -1 \)
2. \( H(t) = it \) for \( t \geq 1 \)
3. \( a'(t), b'(t) > 0 \) for \( -1 < t < 1 \).

We refer to \( H \) as the handle of the cobordism. Consider the map \( i_H : \mathbb{R} \times S^{n-1} \to \mathbb{C}^n \) given by

\[
(t, x) \mapsto (H(t)x_1, ..., H(t)x_n) ,
\]

where \( S^{n-1} \) is considered as embedded in \( \mathbb{R}^n \) with coordinates \( x = (x_1, ..., x_n) \). We refer to \( L_H := i_H(\mathbb{R} \times S^{n-1}) \) as the local surgery model of \( L_1 \) and \( L_2 \) and denote \( L_H = L_1#^H L_2 \). We will usually omit the choice of handle \( H \) and write \( L' = L_1#L_2 \).

Globally, let \((M, \omega)\) be a general symplectic manifold and let \( L_1, L_2 \) be two Lagrangian submanifolds which intersect transversally at \( L_1 \cap L_2 = \{p_1, ..., p_n\} \). One defines the surgery \( L' = L_1#L_2 \) to be a Lagrangian coinciding with \( L_1 \cup L_2 \) away from a small neighbourhood of the points \( p_i \) and with the local surgery around \( p_i \) for each \( i = 1, ..., n \), see [32].

It is known that a triangular Lagrangian cobordism \((V, L_1 \cup L_2, L')\) determines an exact triangle in \( D(Fuk(U_{\Delta})) \) as follows

\[
\begin{array}{ccc}
L_1 & \leftrightarrow & L' \\
\downarrow F & & \downarrow F' \\
L_2 & & L''
\end{array}
\]
where $\mathcal{F}$ is the cobordism $V$ and $\mathcal{F}', \mathcal{F}''$ are the cobordisms obtained by bending the ends of $V$ so as to turn $L'$ to the left end and $L_1$ or $L_2$, respectively, to the right end, for the general case see [6] and [18], for surgeries.

3. Construction of the Lagrangian Euler mirror cobordism

Before describing the construction of the Euler mirror cobordism in the general case $n \geq 1$, let us revisit again the construction of Example 1.1 in the case $n = 1$.

Example 3.1 (The case $n = 1$ via lift to the universal cover). Let $X = \mathbb{P}^1$ be the projective line and let $\Delta = [-1,1]$ be the corresponding polytope. Set

$$U_\Delta := \text{Log}^{-1}(\Delta) = \{ e^{-1} \leq |z| \leq e \} \subset \mathbb{C}^*.$$  

For any $k \in \mathbb{Z}$ let $\gamma_k : I \to U_\Delta$ be the curve

$$\gamma_k(t) = (e \cdot t + e^{-1}(1 - t)) e^{2\pi kti}.$$  

According to Abouzaid’s mirror symmetry functor, the Lagrangian

$$L(k) := \{ \gamma(t) \}_{t \in I} \subset U_\Delta$$  

is the mirror representative of the line bundle $\mathcal{O}(k) \in \text{Pic}(\mathbb{P}^1)$.

Note that $U_\Delta \simeq \Delta \times \mathbb{T}$ and hence the universal cover is given by $\tilde{U}_\Delta \simeq \Delta \times \mathbb{R}$. Consider the Lagrangians

$$L_1 = L(-1) \cup L(-1) \ ; \ L_2 = L(0).$$

Corresponding to $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ and $\mathcal{O}$ of the Euler sequence. Consider the lifts $\tilde{L}_1, \tilde{L}_2$ of $L_1, L_2$ to the universal covering $\tilde{U}_\Delta$ described in Fig. 4 (the choice would be explained below, for the general case):

**Figure 4.** The lift of $L_1 = L(1) \cup L(1)$ and $L_2 = L(0)$ to the universal cover $\tilde{U}_\Delta$

The surgery leading to $L' = L_1 \# L_2 \simeq L(-2)$ is described in Fig. 5.
Figure 5. The surgery $\tilde{L}_1 \# \tilde{L}_2$ conducted in the universal cover $\tilde{M}$ and its equivalence to $\tilde{L}(-2)$.

In Example 3.1 we see that the lifts of the two components of $\tilde{L}_1$ are chosen so that they each intersect the zeros section $\tilde{L}_2$ in two different points, which are actually $(1, 0)$ and $(-1, 0)$, corresponding to the two vertices of the polytope $\Delta = [-1, 1]$. In fact, the lift is chosen so that the components of $\tilde{L}_1$ represent $O(-D_1)$ and $O(-D_{-1})$ where $Div_T(\mathbb{P}^1) = \mathbb{Z} \cdot D_1 \oplus \mathbb{Z} \cdot D_{-1}$. After applying surgery to $\tilde{L}_1$ and $\tilde{L}_2$ along these two points we obtain the Lagrangian $\tilde{L}' = \tilde{L}_1 \# \tilde{L}_2$, which represents the line bundle whose support function $\psi$ is given by $m(\psi, \sigma_1) = -1$ and $m(\psi, \sigma_1) = 1$ (note that these are exactly the right and left heights of $\tilde{L}'$ with respect to the $\mathbb{R}$ coordinate, as shown in Fig. 5). Direct computation shows that this is exactly the line bundle corresponding to $-K_{\mathbb{P}^1} = D_1 + D_{-1}$.

In order to generalize for any $n \geq 1$ consider the vector bundle $E = O_{\mathbb{P}^{n+1}}(-1)$ as the $T$-equivariant bundle

$$E = \bigoplus_{i=0}^{n} O(-V(\rho_i)),$$

where $V(\rho_i) \in Div_T(\mathbb{P}^n)$ is the $T$-equivariant divisor corresponding to the ray $\rho_i$ for $i = 0, ..., n$. The following Lemma gives, via direct computation, the support functions corresponding to $V(\rho_i)$:

**Lemma 3.2.** Let $\psi_i := \psi_{-V(\rho_i)} \in SF(\Sigma)$ be the support-function of $-V(\rho_i)$ for $i = 0, ..., n$ and let $\sigma_k$ be the maximal cones of $\Sigma$ for $k = 0, ..., n$, as in (20). Then:

1. For $i \neq 0$ the weights of $\psi_i$ are given by

$$m(\psi_i, \sigma_k) = e_k^* - e_i^*; \quad m(\psi_i, \sigma_0) = -e_i^*; \quad m(\psi_i, \sigma_i) = 0,$$

for $k = 1, ..., n$.

2. For $i = 0$ the weights of $\psi_0$ are given by

$$m(\psi_0, \sigma_k) = e_k^*; \quad m(\psi_0, \sigma_0) = 0,$$

for $k = 1, ..., n$. 


Recall that
\[ \Delta = \text{Conv} \left( \{ m \sigma_k | k = 0, \ldots, n \} \right) \subset M_{\mathbb{R}} ; \quad \Delta^o = \text{Conv} \left( \{ e_i | i = 0, \ldots, n \} \right) \subset N_{\mathbb{R}}. \]

Set
\[ U_\Delta = \text{Log}^{-1}(\Delta) \simeq \Delta \times \mathbb{T}^n \subset (\mathbb{C}^*)^n. \]

Let \( \tilde{U}_\Delta \simeq \Delta \times M_{\mathbb{R}} \) be the universal cover of \( U_\Delta \) and denote by \( p : \tilde{U}_\Delta \to U_\Delta \) is the covering map. Consider the two projection maps
\[ \tilde{U}_\Delta \xrightarrow{pr_1} \Delta \xleftarrow{pr_2} \mathbb{R}^n. \]

Let us define:

**Definition 3.3.** Let \( \tilde{L} \subset \tilde{U}_\Delta \) be a Lagrangian. We refer to
\[ W \left( \tilde{L}, m \right) := \text{pr}_2 \left( \tilde{L}' \cap \text{pr}_1^{-1}(m) \right) \subset \mathbb{R}^n \]
as the set of weights of the Lagrangian \( \tilde{L} \) at the vertex \( m \in \Delta(0) \).

For any \( i = 0, \ldots, n \) let us define
\[ \tilde{L}(-V(\rho_i)) := \text{Conv} \left( \{(m \sigma_k, m(\psi_{\sigma_k})) | k = 0, \ldots, n \} \right) \subset \tilde{U}_\Delta, \]
to be the linear Lagrangian embedding of \( \Delta \) in \( \tilde{U}_\Delta \) given by rising each vertex \( m \sigma \in \Delta(0) \) of \( \Delta \) to height \( m(\psi_{\sigma_k}) \in M \subset M_{\mathbb{R}} \). Finally, set
\[ \tilde{L}_1 = \bigcup_{i=0}^n \tilde{L}(-V(\rho_i)) ; \quad \tilde{L}_2 = \Delta \times \{ 0 \}, \]
whose projection to \( U_\Delta \) are mirror Lagrangians representing \( O_{\mathbb{P}^n}^{n+1}(-1) \) and \( O_{\mathbb{P}^n} \), respectively. We have:

**Theorem 3.4.** Let \( \tilde{L}' = \tilde{L}_1 \# \tilde{L}_2 \subset U_\Delta \) be the Lagrangian obtained by surgery of \( \tilde{L}_1 \) and \( \tilde{L}_2 \). Then
\[ W \left( \tilde{L}', m_\sigma \right) = W \left( \Omega_{\mathbb{P}^n}^1, \sigma \right), \]
for any maximal cone \( \sigma \in \Sigma(n) \). In particular, \( L' := p(\tilde{L}) \) is a mirror Lagrangian representing \( \Omega_{\mathbb{P}^n}^1 \).

\[ ^3 \text{A priori, for a general Lagrangian } \tilde{L}, \text{ this set is not necessarily finite, integral or non-empty.} \]
Proof. Note that each \( \tilde{L}(\rho_i) \) intersects \( \tilde{L}_2 \) uniquely in the corresponding vertex \( m_{\sigma_i} \), that is
\[
(43) \quad \tilde{L}(\rho_i) \cap \tilde{L}_2 = (m_{\sigma_i}, 0),
\]
for any \( i = 0, ..., n \). The application of surgery at this point removes from the fibre of \( \tilde{L}' \) over \( m_{\sigma_i} \) the point of height zero (compare Fig. 5). In particular, by Lemma 3.2 for any \( k = 1, ..., n \) one has
\[
W \left( \tilde{L}', m_{\sigma_k} \right) = \{ e_k^* \} \cup \{ e_k^* - e_i^* | i \neq k \},
\]
and for \( k = 0 \) one has
\[
W \left( \tilde{L}', m_0 \right) = \{ -e_i^* | i = 1, ..., n \}.
\]
which shows that the weights of \( \tilde{L}' \) as a Lagrangian in \( \tilde{U}_\Delta \) coincide with the weights of \( \Omega_{n, \mathbb{P}^n} \) computed in (24) and (23) of Example 2.3, as required.

Finally, let us mention that the Euler sequence is related to mutation operations of exceptional collections, see \cite{13, 14, 22, 21, 33}:

**Remark 3.5 (Relation to mutation operations).** Let \( \mathcal{B} = \{ E_1, E_2, E_3 \} \subset \mathcal{D}^b(\mathbb{P}^2) \) be a full strongly exceptional collection. The left mutation of \( \mathcal{B} \) is given by
\[
(46) \quad L(\mathcal{B}) = \{ E_1, L_{E_2}E_3, E_2 \},
\]
where the left mutation \( L_EF \) of an exceptional object \( F \) by an exceptional object \( E \) is defined by the triangle
\[
(47) \quad L_EF \to \text{Hom}(E, F) \otimes E \to F \to L_EF[1].
\]
According to Beilinson, the following two collections
\[
(48) \quad \mathcal{B}_1 := \{ \mathcal{O}_{\mathbb{P}^2}(-1), \mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}(1) \} \quad \text{and} \quad \mathcal{B}_2 := \{ \mathcal{O}_{\mathbb{P}^2}(-1), \Omega^1_{\mathbb{P}^2}(1), \mathcal{O}_{\mathbb{P}^2} \}
\]
in \( \mathcal{D}^b_{\mathbb{P}^2}(\mathbb{P}^n) \) are full strongly exceptional, see \cite{11}. In particular, for \( E = \mathcal{O}_{\mathbb{P}^2} \) and \( F = \mathcal{O}_{\mathbb{P}^2}(1) \) the left mutation \( L_{\mathcal{O}_{\mathbb{P}^2}}(\mathcal{O}_{\mathbb{P}^2}(1)) \) is defined by the exact triangle
\[
(49) \quad L_{\mathcal{O}_{\mathbb{P}^2}}(\mathcal{O}_{\mathbb{P}^2}(1)) \to \mathcal{O}_{\mathbb{P}^2}^{n+1} \to \mathcal{O}_{\mathbb{P}^2}(1) \to L_{\mathcal{O}_{\mathbb{P}^2}}(\mathcal{O}_{\mathbb{P}^2}(1))[1],
\]
which after tensoring by \( \mathcal{O}_{\mathbb{P}^2}(-1) \) is exactly the Euler sequence. Hence, we have by the Euler sequence
\[
(50) \quad L_{\mathcal{O}_{\mathbb{P}^2}}(\mathcal{O}_{\mathbb{P}^2}(1)) = \Omega^1_{\mathbb{P}^2}(1),
\]
and \( \mathcal{B}_1 = L(\mathcal{B}_2) \), that is the collection \( \mathcal{B}_2 \) is the left mutation of the collection \( \mathcal{B}_1 \). In this sense, our results can be viewed as the description of the mirror to this mutation operation.
in the Fukaya category $\text{Fuk}(U_\Delta)$, we refer the reader to [27, 28] where we studied the collection $\mathcal{B}_1$ from a mirror symmetry point of view.

4. Summary and Concluding Remarks

Homological mirror symmetry for toric Fano manifolds suggests the equivalence of the categories $\mathcal{D}^b(X)$ and $\mathcal{D}^b(\text{Fuk}(U_\Delta))$, a "dictionary" of sorts between the two categories. Currently, only a limited amount of "entries" in this dictionary are known in practice. For instance, Abouzaid’s description of the mirrors of line bundles $\mathcal{O}_X(D)$ as tropical Lagrangian sections, see [1, 2], and the description of the mirrors of structure sheaves $\mathcal{O}_\Sigma$ of hypersurfaces $\Sigma \subset X$ as tropical Lagrangians, see [25, 26]. In this work we have described the Lagrangian cobordism mirror to the Euler short exact sequence of projective space $X = \mathbb{P}^n$. In particular, we obtained a description of $L'$, the mirror of the cotangent sheaf $\Omega^1_{\mathbb{P}^n}$, arising as the result of a surgery operation. This could be viewed as an extension of the "mirror dictionary" to a new type of entry, an example of the mirror of a vector bundle of rank $n$. The results actually suggest the possibility of a general framework for mirrors of (T-equivariant) vector bundles, which we hope to pursue in future work. Furthermore, the results are related to mutation operations on $\mathcal{D}^b(X)$, as explained in Remark 3.5.

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