LETTER TO THE EDITOR

Walkers on the circle

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Abstract. We experimentally demonstrate that the statistical properties of distances between pedestrians which are hindered from avoiding each other are described by the Gaussian Unitary Ensemble of random matrices. The same result has recently been obtained for an $n$-tuple of non-intersecting (one-dimensional, unidirectional) random walks. Thus, the observed behavior of autonomous walkers conditioned not to cross their trajectories (or, in other words, to stay in strict order at any time) resembles non-intersecting random walks.

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The fact that non-intersecting one-dimensional random walks lead to universal system behavior has been known and discussed for at least 10 years [1], [2], [3]. It is also known that the results can be described in terms of random matrix theory - see for instance [4]. This fact is usually expressed in abstract mathematical theorems of universal validity, see for instance [5].

Our aim here is to use these abstract results in order to explain certain aspects of the observed behavior of pedestrians. A comprehensible application of the complicated mathematical theory is given e. g. in [6]. It analytically explains an experimental observation that the schedule of the city transport in Cuernavaca (Mexico) conforms to the predictions of the Gaussian Unitary Ensemble of random matrices (GUE). The reason for this interesting observation is the absence of a bus timetable, and primarily the fact that the buses do not overtake each other and hence their trajectories do not cross - see also [7] for the details. Our focus in this letter is pedestrians in a situation when they cannot avoid each other.

Pedestrian flow is a subject of intense study. This is understandable since the movement of large groups of people inevitably leads to injuries and deaths caused by trampling and by crowd-pressure. The consequences can be disastrous: for instance more than 1400 people were trampled to death during a stampede in Mecca in 1990. A proper understanding of the process how groups of people move is vital for taking effective precautions. The mathematical description is usually based on the pedestrian interactions denoted as "social forces" [8]. The exact character of these forces and of their cultural dependence remains unclear, however, and is a focus of recent discussions [9]. An evacuation dynamics of buildings is modeled in a similar way [10]. But not only panic situations are of importance. The comfortable and safe movement of people through corridors and on sidewalks is also of interest. Although people are autonomous individuals following their own destinations, they cannot move freely as soon as the pedestrian density exceeds a certain limit. For higher densities self-organizing phenomena occur. A typical example is the stratification of pavement walkers into layers for different direction [11].

We will discuss a unidirectional pedestrian motion in the range of intermediate density and in a narrow corridor that hinders mutual avoidance. Otherwise the people can move freely. Since to avoid another walker is not possible, the walker’s attention naturally focuses on the preceding fellow, in order not to collide with him. The situation resembles the assumptions of the model of vicious random walkers introduced by Fisher [12]. In a typical case, the vicious walkers move randomly on a one-dimensional discrete lattice. At each time step, that walker can move either to the left or to the right. The only constraint is that two walkers cannot occupy the same site at the same time. The model is easily modified to the situation when the motion is unidirectional (for instance right moving). In this case, the walkers are staying at the same site instead of moving to the left. The model has surprising relations with various fields of mathematics like combinatorics or random matrix theory [13]. The corresponding random matrix ensemble is, however, not fixed solely by the dynamics. It depends also on the particular
Our measurement is inspired by the paper [15] where the fundamental diagram (i.e. the dependence of the pedestrian flow on the pedestrian density) has been measured experimentally with volunteers walking in a circle. Fundamental diagrams are one of the basic tools used in car flow modeling and traffic jam prediction. In highway traffic, its shape is influenced by many factors like the road topology, the existence of a near slip road, and so on. The interesting question of how cultural differences influence the flow-density relation for pedestrians has been discussed in [16]. Beside the fundamental diagram, the work also discusses density fluctuations which are of vital interest for the stampede dynamics. In a crowd, it is sudden density changes that lead to the abrupt release of the local pressure and finally cause people to fall and be trampled [17]. In a one-dimensional system, the local density is inversely proportional to the local distance among the walkers (the pedestrian clearance). So we will discuss the statistical properties of the experimentally measured pedestrian distances.

During the public action called "Let us use our heads to play" (an event serving to popularize physics among schoolchildren), we prepared a circular corridor of a diameter of 4.5 m, built with chairs and ropes, see Figure 1. It was placed on a grass plot in front of the university building. As various school classes participated in this activity, we asked them to walk in the corridor for a time period of 3 minutes. The motion of walkers was registered by two light gates placed in a fixed distance of less than 1 m. The times when the light of the gates was interrupted by the walker were recorded by

![Figure 1. The measurement.](image-url)
a computer at a rate of 50 samples per second. It means that at the mean walking speed of 1.25 m/s, the device resolution was approximately 0.025 m. The interruption onset of the gate was taken as the pedestrian arrival. In this way, we obtained the record of 26 groups consisting of 12 to 20 walkers. Using these data, it is easy to obtain the pedestrian headway and velocity, and combining these quantities, the pedestrian clearance is obtained straightforwardly. We will, however, focus directly on the headway statistics, i.e. the statistics of the time intervals between two subsequent walkers. To avoid global density effects (like small jams inside the circle when one walker slows down unexpectedly) the headway data of each particular measurement were unfolded and scaled to a mean headway equal to one. The theory published in [6] predicts that the headway of autonomous (random) pedestrians hindered from avoiding each other is described by the level spacing distribution of the Gaussian unitary ensemble of random matrices. In other words, the headway statistics are universal and given by the Wigner formula for GUE:

$$P(s) = \frac{32}{\pi^2} s^2 \exp \left( -\frac{4s^2}{\pi} \right)$$

The results obtained for the circular corridor with 13-15 years old children from 26 different groups is plotted in the Figure 2.

As already mentioned, understanding the local fluctuations of the density is of particular interest since the unexpected density changes are suspected to be the main cause for people falling and being trampled [17]. In the one-dimensional pedestrian flow,
the density fluctuations were investigated in [15], [16]. The authors argued conjecturally that the fluctuations display a small dependence on the cultural backgrounds of the walkers (Indian and German pedestrians were compared). On the other hand, the mathematical theory developed for non-crossing trajectories predicts a universal behavior pattern. This means that the cultural factors should be statistically irrelevant. The question is how these two points can be reconciled. To investigate it, we will study the number of people passing a given point within certain time interval. Let \( n(T) \) be the number of walkers passing the measuring point within the time interval \( T \). The fluctuation \( \Sigma(T) \) is defined as

\[
\Sigma(T) = \langle (n(T) - \langle n(T) \rangle)^2 \rangle
\]

where \( \langle ... \rangle \) means the system average. Traditionally, this quantity is called the number variance, and its behavior is well understood. For uncorrelated events (Poisson process) \( \sigma(T) = T \) for large \( T \). For the headway governed by the GUE ensemble, we get \( \Sigma(T) \approx (ln(2\pi T) + 1.5772..)/\pi^2 \) and the fluctuation of the number of walkers passing a given point increases only logarithmically with the time. Generally, point processes leading to \( \Sigma(T) < T \) for large \( T \) are denoted as superhomogeneous (see for instance [18], [19]).

We used the measured data to evaluate the number variance for the pedestrians inside the circular corridor with foregoing restrictions. The results are plotted in Figure 3. We see that \( \Sigma \) follows the prediction of GUE up to \( T \approx 3 \). For larger \( T \) the increase is higher than the logarithmic prediction of GUE. It remains, however, substantially below the line \( \Sigma(T) = T \) obtained for the uncorrelated events. So the headway sequence is superhomogeneous and follows the GUE result for \( T \lesssim 3 \). The cultural differences reported in [16] can be related with the non-universal increase of \( \Sigma \) for \( T \gtrsim 3 \). Similar behavior has been also observed for bus transport [7].

The interactions between pedestrians are obviously asymmetrical - the walker is observing the walker ahead of him (so as not to collide with him) and has much less information on the walker behind him. Thus the Newton’s law of “actio = reaction” does not hold for pedestrians. According to our results, the distance distribution, however, agrees with classical many-particle systems, such as Dyson’s gas, where the interactions are symmetrical. The statistical properties of one-dimensional many-particle systems violating the “actio=reactio” law were studied in [20]. They have analytically shown that such systems exhibit the same statistics as classical Newtonian systems.

To summarize: we have presented experimental data on pedestrians walking in a circular corridor when they are hindered from avoiding each other. The results are in agreement with the prediction obtained for the mathematical models of one-dimensional vicious random walkers. Our results can also be regarded as (another) experimental demonstration of the theoretical results of [20].
Figure 3. The number variance \( \Sigma(T) \) evaluated with data measured for 26 different pedestrian groups (crosses) is compared with the prediction of GUE (full line) and with the prediction for a random headway sequence (dashed line).

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