Linearization of holomorphic germs with resonant linear part

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Abstract. We study the linearization problem of germs of holomorphic diffeomorphisms with resonant linear part. The formal linearization requires in general an infinite number of algebraic relations to be satisfied by the coefficients of the power series defining the holomorphic germ. For a fixed germ, the linearizing mappings are not unique, but the existence of a finite jet with a formal divergent linearization but no convergent one forces all the other linearizing mappings to be diverging. We consider also polynomial families formally conjugated to a fixed resonant linear part. Then all the elements of the family are holomorphically linearizable, or we are in the precedent diverging situation for all but an exceptional set of parameter values of zero $\Gamma$-capacity. In the case of maximal non-trivial resonance we prove the optimality of Bruno condition.

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Introduction.

The study of resonant germs of holomorphic diffeomorphisms $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$,

$$f(z) = Az + O(z^2)$$

is still in its infancy. The linear map $A \in GL_n(\mathbb{C})$ is the differential of $f$ at 0 and we can assume that it is given in Jordan normal form. We denote by $\{\lambda_1, \ldots, \lambda_n\}$ the set of eigenvalues of $A$ counted with multiplicity. The linear part $A$ is resonant if some relation of the form

$$\lambda_1^{i_1} \cdots \lambda_n^{i_n} - \lambda_j = 0$$

holds for some $i_1, \ldots, i_n \geq 0$, $|i_1| + \cdots + |i_n| \geq 2$, and $1 \leq j \leq n$.

A well studied resonant example (by H. Poincaré, G.D. Birkhoff, T.M. Cherry, C.L. Siegel, H. Rüssman, H. Eliasson,... [Po], [Bi], [Ch], [Si1], [Si2], [Si-Mo], [Ru], [El]) is the case of an elliptic fixed point at 0 for a symplectic holomorphic map $f$.

In the non-resonant case, when $A$ is on the Poincaré domain, i.e.

$$\max_i |\lambda_i|, \max_i |\lambda_i^{-1}| < 1,$$

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it is well known that the holomorphic germ $f$ is holomorphically linearizable, that is, there exists $h = \text{id} + O(z^2)$ a germ of holomorphic diffeomorphism, such that

$$h^{-1} \circ f \circ h = A.$$

The Bruno condition on $A$ (or the eigenvalues of $A$), for a resonant or non-resonant $A$, is defined as

$$-\sum_{k=1}^{+\infty} 2^{-k} \log \Omega(2^k) < +\infty$$

where

$$\Omega(m) = \inf_{2 \leq |i| \leq m, \lambda^i - \lambda_j \neq 0, 1 \leq j \leq n} |\lambda^i - \lambda_j| .$$

and the infimum is taken among all $i$ such that $2 \leq |i| \leq m$, $\lambda^i - \lambda_j \neq 0$, and $1 \leq j \leq n$.

By the theorem of C.L. Siegel [Si3] and A.D. Bruno [Br], $f$ is also holomorphically linearizable when $A$ is non-resonant, diagonal, and the eigenvalues satisfy Bruno’s diophantine condition stated above.

When $f$ is formally linearizable (i.e. there exists as a formal map, defined by formal power series, conjugating formally $f$ to its linear part), and the linear part satisfies Bruno’s condition, then $f$ is also holomorphically linearizable (see [Br], [Ru], [El], [E-V]). In this last situation the linearization is not unique.

When the formal linearization fails, one attempts to conjugate $f$ to a simpler normal form. In general the normalization mapping is diverging (this has been well studied for the Birkhoff’s normal form of a symplectic holomorphic germ with an elliptic fixed point). In this paper, except for some common algebraic preliminaries, we study the linearization. The general conjugation to an arbitrary normal form is treated in [PM2].

We consider the group of formal diffeomorphisms of $(\mathbb{C}^n, 0)$ (vanishing at 0)

$$\hat{G} = \text{Diff}(\mathbb{C}^n, 0) \subset \mathbb{C}^\infty$$

We identify it to a subset of $\mathbb{C}^\infty$ by associating to $f \in \hat{G}$ the ordered set of the coefficients of its formal power series (choosing a monomial ordering). For $m \geq 1$, we consider the group $\hat{G}_m$ of $m$-jets of elements of $\hat{G}$, and the $m$-jet morphism

$$\pi_m : \hat{G} \to \hat{G}_m .$$

Let $d(m)$ be the number of coefficients of the $m$-jet. We identify $\hat{G}_m$ to a subset of $\mathbb{C}^{d(m)}$. The group $\hat{G}_m$ is an affine algebraic group. We consider also the normal subgroup $\hat{G}_0 \subset \hat{G}$ of elements $h \in \hat{G}$ with identity linear part, $D_0 h = I$.

We have the following preliminary algebraic result, where we denote by $N$ a fixed normal form:

**Theorem 1.** Let $N \in \hat{G}$ and consider the set $\hat{G}_N$ of elements $f \in \hat{G}$ such that $f$ is formally conjugated to $N$. The subset $\pi_m(\hat{G}_N) \subset \mathbb{C}^{d(m)}$ is an affine algebraic variety of
dimension $d(N, m)$. Moreover, we have an onto polynomial map of degree less or equal than $m$
\[
\varphi_{m,N} : \mathbb{C}^{d(m) - m} \to \pi_m(\hat{G}_N)
\]
whose fibers are isomorphic non-singular affine algebraic varieties.

We assume $N = A \in \text{GL}_n(\mathbb{C})$. For a given $f \in \hat{G}_A$, and any $m$, the $m$-jet projection of the set $\hat{G}_0(f) \subset \hat{G}_0$ of formal linearizations of $f$ is a linear sub-space of $\mathbb{C}^{d(m) - m}$ of dimension $\delta(A, m) = d(m) - m - d(A, m)$. We have a linear isomorphisms
\[
\psi_{m,A,f} : \mathbb{C}^{\delta(A, m)} \to \pi_m(\hat{G}_0(f)).
\]
and a universal linearization linear isomorphism (obtained as inductive limit)
\[
\psi_{\infty,A,f} : \mathbb{C}^{\delta(A)} \to \hat{G}_0(f).
\]
where $\delta(A) = \lim_{m \to +\infty} \delta(A, m)$ (can be infinite).

We will need also an algebraic preliminary for polynomial families.

**Theorem 2.** Let $(f_t)_{t \in \mathbb{C}^k} \subset \hat{G}_A$ be a polynomial family
\[
f_t(a) = Az + \sum_{0 \leq |i| \leq d_0} t^i f_i(z).
\]
We have a polynomial family of formal linearizations $(h_{t,s})_{(t,s) \in \mathbb{C}^k \times \mathbb{C}^{\delta(A)}} \subset \hat{G}_I$ with
\[
h_{t,s} = \psi_{\infty,A,f,t}(s).
\]
More precisely, the coefficients of monomials of order $m$ of $h_{t,s}$ are polynomials on $t$ of degree less or equal to $d_0m$ (and linear on $s$).

After this algebraic preliminaries, we are ready to state the following theorems. First for a single map.

**Theorem 3.** Let $f \in \hat{G}$ corresponding to a converging germ of holomorphic diffeomorphism of $(\mathbb{C}^n, 0)$. Denote $G_0(f) \subset \hat{G}_0(f)$ the set of converging linearizations of $f$.
We have the following dichotomy:
1) Up to finite order, all formal linearizations in $\hat{G}_0(f)$ are converging. More precisely, for any $m \geq 2$, we have
\[
\pi_m(\hat{G}_0(f)) = \pi_m(G_0(f)) \approx \mathbb{C}^{\delta(A,m)},
\]
and the set of converging linearizations is in one-to-one correspondence with the holomorphic centralizer of $A$. Moreover, there is a uniform radius of convergence $R_0 > 0$ such that for any $m \geq 2$ and $s_m \in \mathbb{C}^{\delta(A,m)}$ there exists a linearization $h$ of $f$ with $\pi_m(h) = s_m$ and radius of convergence at least $R_0$.
2) All formal linearizations in $G_0(f)$ are diverging.
After this theorem, we can talk without ambiguity about $f$ being linearizable or not. We say that $f$ is linearizable if it has a converging linearization, thus we are in case (1), and this notion is essentially independent of the linearization chosen.

J.-Ch. Yoccoz in [Yo] appendix 1 studies the linearization of germs with non-diagonalizable linear part. He proves that in this situation there always exists non-linearizable examples. More precisely, that in a one dimensional family with linear dependence on the parameter for almost all values of the parameter the dynamics is not linearizable. Yoccoz’s argument is incomplete in the resonant case. Next theorem (see also [PM1]) shows that the set of exceptional parameter values indeed has capacity 0 and the result includes the resonant case.

The next theorem is guided by the general principle stated in [PM1] that in generic families of dynamical systems presenting problems of small divisors:

**We have total convergence or general divergence except for a very small exceptional set in parameter space**

We use below the notion of $\Gamma$-capacity. We recall its definition in section 2 and refer to [Ro] for basic properties.

**Theorem 4.** Let $(f_t)_{t \in \mathbb{C}^k}$ be a polynomial family as above, and let $(h_{t,s})_{(t,s) \in \mathbb{C}^k \times \mathbb{C}^{\delta(A)}}$ be the universal family of uniformizations. We have the following dichotomy:

1) Up to finite order, all the formal linearizations $(h_{t,s})$ are converging, i.e. all $f_t$ are linearizable. More precisely, for any $m \geq 2$ and for any $(t, s_m) \in \mathbb{C}^k \times \mathbb{C}^{\delta(A,m)}$, there exists $s \in \mathbb{C}^{\delta(A)}$, with $s_m = \pi_m(s)$ and $h_{t,s}$ converging.

2) Except for an exceptional set $E$ in $\mathbb{C}^k$ of $\Gamma$-capacity 0, all formal linearizations in $h_{t,s}$, $t \notin E$, are diverging, i.e. all $f_t$, $t \notin E$, are non-linearizable.

In the last part of the article we study some examples. We give examples of resonant linear part $A$ with $\delta(A) = +\infty$.

A measure of the importance of resonance is the $\mathbb{Q}$-torsionless dimension of the $\mathbb{Q}$-vector space

$$V = V(\lambda_1, \ldots \lambda_n) = \mathbb{Q} \oplus \bigoplus_{j=1}^n \mathbb{Q} \left( \frac{1}{2\pi i} \log \lambda_j \right)$$

that is

$$l(A) = \dim_{\mathbb{Q}} V/\mathbb{Q}.$$  

The linearization problem in the case of minimal dimension $l(A) = 0$ and $A$ is diagonal is simple. Linearization is equivalent to finite order (for the composition) of the germ. The next non-trivial case corresponds to $l(A) = 1$. Then we have two cases, depending on whether

$$\mathbb{Q}/\mathbb{Z} \cap \bigoplus_{j=1}^n (\mathbb{Q}/\mathbb{Z}) \left( \frac{1}{2\pi i} \log \lambda_j \right) = \{0\}$$

(torsionless) or the intersection is not reduced to 0 (torsion).
There exists $\mu = e^{2\pi i \theta}$ and a $q = q(A)$-root of unity $\varepsilon$ ($\varepsilon = 1$ in the torsionless case, $q \geq 2$ in the torsion case) such that $\lambda_i = \varepsilon^{a_i} \mu^{b_i}$ for some integers $a_i$ and $b_i$. The following result seems to have been unnoticed (but see [Go] in for symplectic holomorphic maps in dimension 2):

**Theorem 5.** The eigenvalues $(\lambda_1, \ldots, \lambda_n)$ do satisfy Bruno condition if and only if $\mu$ satisfies the one dimensional Bruno condition (i.e. $\theta$ is a Bruno number).

The optimal arithmetic condition of holomorphic linearization of resonant formally linearizable holomorphic germs with linear part $A$ with $l(A) = 1$ is Bruno condition.

At this respect one should note that M. Herman has conjectured ([He] conjecture 1 p.147) in the non-resonant and diagonal case, that Bruno condition is optimal in all dimensions. Yoccoz has verified this in dimension 1 ([Yo]). The author conjectures that Herman’s conjecture is false in dimension larger than 2.

The torsion case is irrelevant for purposes of linearization. Observe that

$$Q \cap \bigoplus_{j=1}^n \mathbb{Z} \left( \frac{1}{2\pi i} \log \lambda_j \right)$$

is a $\mathbb{Z}$-submodule of a free finitely generated $\mathbb{Z}$-module, thus it is finitely generated since $\mathbb{Z}$ is Noetherian. If there is torsion, the torsion $q = q(A)$ of $(\lambda_1, \ldots, \lambda_n)$ is the minimal positive integer $q \geq 2$ (which exists by the precedent observation) such that

$$\frac{1}{q} \mathbb{Z} \supset Q \cap \bigoplus_{j=1}^n \mathbb{Z} \left( \frac{1}{2\pi i} \log \lambda_j \right).$$

If $f$ is a resonant germ with linear part $A$ with torsion $q \geq 2$ then $f^q$ has linear part $A^q$ with no torsion. Moreover, it is easy to see that $f$ is linearizable if and only if $f^q$ is linearizable.

The case $l(A) \geq 2$ deserves further investigations that will be pursued in the future.
1) Algebraic preliminaries.

The study of conjugacy to a non-linear normal form is done in [PM2]. Part of the formal theory is the same as for the problem of linearization, this is why we include a general first part in theorem 1. We don’t develop here a complete algebraic study. For basic notions relative to algebraic groups we refer to [Sh]. Most of the work done here can be straightforward generalized to more general fields of coefficients than $C$.

Using the notations of the introduction, we consider a normal form $N \in \hat{G}$ with

$$D_0N = A$$

and its (formal) centralizer in $\hat{G}_0$,

$$\text{Cent} (N) = \{h \in \hat{G}_0; h \circ N = N \circ h\}.$$ 

Note that for each $m \geq 2$,

$$\pi_m(\text{Cent} (N)) \subset \pi_m(\hat{G}_0) \subset C^{d(m)}$$

is an affine algebraic subgroup. Thus it can be identified to a linear group.

It is natural to ask

**Question**: Which algebraic groups do arise as groups $\text{Cent} (N)$ for some $N$?

This question does not seem to have been explored at all.

Given $f \in \hat{G}$, we consider the set

$$\hat{G}_0(f, N) = \{h \in \hat{G}_0; h^{-1} \circ f \circ h = N\}.$$ 

of normalizations of $f$.

**Proposition 1.1.** For each $m \geq 2$,

$$\pi_m(\hat{G}_0(f, N)) \subset C^{d(m)}$$

is a smooth affine algebraic variety.

**Proof.** If the set $\hat{G}_0(f, N)$ is empty there is nothing to prove. Otherwise choose $h_0 \in \hat{G}_0(f, N)$. The map

$$\Phi : \pi_m(\text{Cent} (N)) \longrightarrow \pi_m(G_0(f, N))$$

$$\pi_m(h) \mapsto \pi_m(h \circ h_0)$$

is a linear isomorphism. The inverse is simply given by

$$\pi_m(h) \mapsto \pi_m(h \circ h_0^{-1}).$$

Now $\pi_m(\text{Cent} (N))$ is an affine algebraic group, thus it is non-singular. ☑
We consider now the set \( \hat{G}_N \) of elements \( f \in \hat{G} \) which are formally conjugated to \( N \),
\[
\hat{G}_N = \{ h^{-1} \circ N \circ h; h \in \hat{G}_0 \}
\]
If \( N \) has a resonant linear part \( A \) we have \( \hat{G}_N \neq \{ f \in \hat{G}; D_0 f = A \} \). Fixing \( m \geq 2 \), we can identify \( \pi_m(\hat{G}_0) \) to \( C^{d(m) - m} \). The map
\[
\varphi_{m,N} : C^{d(m) - m} \to \pi_m(\hat{G}_N)
\]
defined by \( \pi_m(h) \mapsto \pi_m(h^{-1} \circ N \circ h) \) is a polynomial map because the coefficients of the power series defining \( h^{-1} \) depend polynomially on those of \( h \). It is not difficult to check that each coordinate function is a polynomial in \( d(m) - m \) variables of degree less than \( m \). Each of its fibers is isomorphic as algebraic varieties to \( \pi_m(\text{Cent}(N)) \). Thus it is a non-singular affine variety. To end the proof of theorem 1 we still have to construct the map \( \psi_{m,A,f} \). We do this in the case where \( N = A \) is linear.

**Lemma 1.2.** Let \( m \geq 2 \). The \( m \)-jet of the centralizer of \( A \), \( \pi_m(\text{Cent}(A)) \subset \pi_m(\hat{G}_0) \subset C^{d(m) - m} \), is a linear sub-space of dimension \( \delta(A,m) \leq d(m) - m \).

**Proof.** Let \( h \in \hat{G}_0 \subset C^{d(m) - m} \). Identifying the coefficients of monomials in the equation
\[
\pi_m(h \circ A) = \pi_m(A \circ h)
\]
we get \( d(m) - m \) linear equations in the \( d(m) - m \) coefficients of \( \pi_m(h) \).

The maps \( \psi_{m,A} : C^{\delta(A,m)} \to \pi_m(\text{Cent}(A)) \) are defined choosing a bases for \( \text{Cent}(A) \). Finally the map \( \psi_{m,A,f} \) is defined by
\[
\psi_{m,A,f} = \Phi \circ \psi_{m,A}
\]
where \( \Phi \) is the linear map used in the proof of proposition 1.1. Note that the construction is not canonical since it depends on the choice of the bases of \( \text{Cent}(A) \) and a choice of \( h_0 \in \hat{G}_0(f,A) = \hat{G}_A \) for \( \Phi \).

Theorem 2 for a polynomial family of formal germs \( (f_t) \) follows from an elimination argument. Writting down the equations for the linearization of \( f_t \), at each order \( m \) the equations determine some coefficients of order \( \leq m \) of the linearization in terms of \( t \). We set the other coefficients equal to 0. The system of equations are compatible by assumption. In that way we obtain a polynomial family \( (h_t) \) of linearizations. One easily checks that the coefficient of a monomial of order \( m \) is a polynomial on \( t \) of degree at most \( md_0 \). Now for each \( t \) we have a map \( \Phi_t : \text{Cent}(A) \to \hat{G}_0(f_t,A), h \mapsto h \circ h_t \). We define
\[
\psi_{\infty,A,f_t} = \Phi_t \circ \psi_{\infty,A}
\]
and \( h_{t,s} = \psi_{\infty,A,f_t} \).

One can do the above constructions in a more "functorial" way, but this is useless for our purposes.
2) Proof of theorem 2 and 3.

Proof of theorem 2.

The proof follows from theorem 1. Assume that we have a converging linearization $h_0$. From theorem 1 we know that all other linearizations are of the form $h \circ h_0$ where $h$ runs over all elements of $\text{Cent}(A)$. In particular, choosing $h$ to be polynomial of order at most $m$ we see that

$$\pi_m(\hat{G}_0(f)) = \pi_m(G_0(f)) .$$

Proof of theorem 3.

We recall Berstein’s lemma in approximation theory (see [Ra] p.156): 

**Lemma (Bernstein).** Let $K \subset \mathbb{C}$ be a non-polar set, and $\Omega$ be the component of $\mathbb{C} - K$ containing $\infty$.

*If $P$ is a polynomial of degree $n$, then for $z \in \mathbb{C}$

$$|P(z)| \leq ||P||_{C^0(K)} e^{ng_\Omega(z,\infty)}$$

where $g_\Omega$ is the Green function of $\Omega$.

**\(\Gamma\)-capacity.**

We recall the definition of $\Gamma$-capacity and we refer to [Ro] for more properties. Let $E \subset \mathbb{C}^k$. The $\Gamma$-projection of $E$ on $\mathbb{C}^{k-1}$ is the set $\Gamma_{k-1}^k(E)$ of $z = (z_1, \ldots, z_{k-1}) \in \mathbb{C}^{k-1}$ such that

$$E \cap \{(z,w) \in \mathbb{C}^k\}$$

has positive capacity in the complex plane $\mathbb{C}_z = \{(z,w) \in \mathbb{C}^k\}$. We define

$$\Gamma_1^k(E) = \Gamma_2^3 \circ \Gamma_3^4 \circ \ldots \Gamma_{k-1}^k(E) .$$

Finally, the $\Gamma$-capacity is defined as

$$\Gamma\text{-Cap}(E) = \sup_{A \in U(k,\mathbb{C})} \text{Cap} \, \Gamma_1^k(A(E)) .$$

where $A$ runs over all unitary transformations of $\mathbb{C}^k$.

Using the definition of $\Gamma$-capacity it is easy to see that we are reduced to prove theorem 3 for $k = 1$.

From the algebraic preliminaries it follows

**Lemma 2.1.** The coefficient vectors $h_i(t, s_i)$, $s_i = \pi_i(s)$ of the formal linearization

$$h_{t,s}(z) = z + \sum_{i = (i_1, \ldots, i_n)} h_i(t, s_i) z^i$$

for $i_1 + \cdots + i_n \geq 2$.
have coordinates that are linear in the coordinates of $s_i$ and polynomial in the parameter $t = (t_1, \ldots, t_k)$ of degree less than $d_0(i_1 + \ldots i_n)$.

Taking into account the definition of $\Gamma$-capacity given above, as observed before, we can assume that $k = 1$.

Let

$$E = \{ t \in \mathbb{C}; f_t \text{ is linearizable } \} .$$

Fix $m \geq 2$, $s_m \in \mathbb{C}^{\delta(A,m)}$ and $s = (s_m, 0, \ldots)$. We want to show that $E$ is polar or the whole complex plane. We have

$$E = \bigcup_{j \geq 1} E_j$$

where $E_j$ the set of parameters $t$ such that $h_{t,s}$ has radius of convergence larger or equal to $1/j$. Thus if $E$ is non-polar, we have that for some $j \geq 1, E_j$ is not polar. Thus there exists $\rho > 0$ such that for all $t \in E_j$,

$$\varphi(t) = \limsup_{|i| \to +\infty} ||h_i(t,s)||_{C^0} \rho_0^{-|i|} < +\infty .$$

The function $\varphi$ is lower semicontinuous, and

$$E_j = \bigcup_p L_p$$

where $L_p = \{ z \in E_j; \varphi(t) \leq p \}$ is closed. By Baire theorem for some $p$, $L_p$ has non-empty interior (with respect to $E_j$), thus this $L_p$ has positive capacity. Finally we found a compact set $C = L_p$ of positive capacity such that there exists $\rho > 0$ such that for any $t \in C$ and all $i \in \mathbb{N}^n$,

$$||h_i(t,s)||_{C^0(C)} \leq \rho_1^{[i]} .$$

Using Bernstein’s lemma and lemma 1.1 we get that for any compact set $K \subset \mathbb{C}$ we have

$$||h_i(t)||_{C^0(K)} \leq C(K)^{d[i]} \rho_1^{[i]} ,$$

for some constant $C(K)$ depending only on $K$. Thus $f_t$ is linearizable for any $t \in \mathbb{C}$. The constant $C(K)$ can be estimated by the precise form of Bernstein lemma as

$$C(K) = \exp(\sup_K g_{\Omega}(t, \infty))$$

where $\Omega$ is the connected component containing $\infty$ of the complement of $C$. The asymptotic

$$g_{\Omega}(t, \infty) \approx \log |t|$$

for $t \to \infty$ can be used to give a lower estimate on the radius of convergence.

3) Some examples.
a) Examples with $\delta(A) = +\infty$.

Elliptic point.
A well known example is an elliptic fixed point. For example, consider

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$$

with $\lambda \in \mathbb{C}^*$ and $\lambda$ not a root of unity.

Lemma 3.1. The holomorphic (resp. formal) centralizer in $G_0$ (resp. $\hat{G}_0$) of $A$ is composed by maps of the form

$$(z_1, z_2) \mapsto l(z_1, z_2) = (z_1 + z_1 \varphi_1(z_1z_2), z_2 + z_2 \varphi_2(z_1z_2))$$

where $\varphi_j(z) = \sum_{i=1}^{+\infty} \varphi_{j,i} z^i$ is a holomorphic (resp. formal) power series.

Proof. The proof is straightforward identifying coefficients in $l \circ A = A \circ l$. ◊

Non-trivial Jordan block.
We consider now another more elaborate example. Let

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Proposition 3.2. The holomorphic (resp. formal) centralizer in $G_0$ (resp. $\hat{G}_0$) of $A$ is composed by maps of the form

$$(z_1, z_2) \mapsto l(z_1, z_2) = (z_1 + k(z_2), z_2)$$

where $k(z) = \sum_{i=2}^{+\infty} k_i z^i$ is a holomorphic (resp. formal) power series.

Lemma 3.3. Let $\psi(z_2)$ be a given formal power series of valuation $\geq 2$. We consider the equation

$$(\ast) \quad \varphi(z_1 + z_2, z_2) = \varphi(z_1, z_2) + \psi(z_2)$$

and seek solutions $\varphi(z_1, z_2)$ which are formal power series of valuation $\geq 2$.

If $\psi \equiv 0$ then the solutions of $(\ast)$ are the formal power series $\varphi(z_1, z_2) = \rho(z_2)$ independent of $z_1$.

If $\psi$ is not identically 0 there are no solutions to equation $(\ast)$.

Proof. We consider the linear operator

$$L : \varphi(z_1, z_2) \mapsto \varphi(z_1 + z_2, z_2) - \varphi(z_1, z_2)$$

defined in the vector space $E$ of formal power series in the two variables $(z_1, z_2)$ with valuation $\geq 2$. This linear operator leaves invariant the finite dimensional vector space
$E_n$, $n \geq 2$, of homogeneous polynomials of degree $n$. In order to solve $(\ast)$ we are reduced to solve the equation in $E_n$ for $n \geq 2$. Choosing the base $(z_1^n, z_1^{n-1}z_2, \ldots, z_2^n)$ of $E_n$, the matrix of $L_n = L/E_n$ is triangular with combinatorial coefficients $\binom{i}{j}$.

$$M = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 \\ 1 & 0 & 0 & \ldots & 0 \\ 1 & 2 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & n & \ldots & n & 0 \end{pmatrix}$$

The only eigenvalue is 0, the kernel is spanned by $(0, \ldots, 0, 1)$, and the image by the first $n - 1$ vectors of the bases. So the $n$-homogeneous part of $\psi(z_2)$ is not zero, then it doesn’t belong to the image of $L$, thus $(\ast)$ has no solution. Now if $\psi(z_2) \equiv 0$ then the homogeneous solutions of $(\ast)$ is the kernel of $L_n$. So a solution $\varphi$ of $(\ast)$ only depends on $z_2$.

**Proof of proposition 3.2.** Obviously any map of the above form commutes with $A$. Conversely, we write $l(z) = (z_1 + l_1(z), z_2 + l_2(z))$ with $l_i$ a formal power series in $(z_1, z_2)$ of valuation $\geq 2$. The second coordinate in the equation

$$l \circ A = A \circ l$$

gives (eliminating linear parts)

$$l_2(z_1 + z_2, z_2) = l_2(z_1, z_2).$$

This implies by the previous lemma that $l_2(z_1, z_2) = \psi(z_2)$. Now we look at the first coordinate of the commuting equation. We get

$$l_1(z_1 + z_2, z_2) = l_1(z_1, z_2) + l_2(z_1, z_2) = l_1(z_1, z_1) + \psi(z_2).$$

Using the lemma again we get that $\psi$ must be zero and $l_1(z_1, z_2) = k(z_2).$ q.e.d.

Along the same lines one can determine the centralizer of a non-trivial Jordan block

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & \ldots & \ldots & \ldots & 0 \\ 0 & 1 & 1 & 0 & \ldots & \ldots & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ldots & \ldots & \ldots & \ldots & 0 & 1 \\ 0 & \ldots & \ldots & \ldots & \ldots & 0 & 1 \end{pmatrix}$$

which is also infinite.
Proposition 3.4 If the matrix $A$ contains a non-trivial Jordan block with an eigenvalue 1 (or a root of unity) then $\delta(A) = +\infty$.

b) Linearization of resonant germs with $l(A) = 1$.

All eigenvalues $\lambda_j$ are non zero. We write $\lambda_j = e^{2\pi i \alpha_j}$ for some $\alpha_j \in \mathbb{C}$ determined up to an additive integer. We write

$$(\alpha_1, \ldots, \alpha_n) \in \mathcal{B}$$

to express that $(\lambda_1, \ldots, \lambda_n)$ satisfies the Bruno condition.

With the notations of the introduction, we assume that $l(A) = 1$, that is

$$V/\mathbb{Q} = \mathbb{Q} \theta .$$

Then there is a root of unity $\varepsilon$ and integers $(a_k)$ and $(b_k)$ such that

$$\lambda_k = \varepsilon^{a_k} \mu^{b_k}$$

where $\mu = e^{2\pi i \theta}$. We prove

Proposition 3.5. We have

$$(\alpha_1, \ldots, \alpha_n) \in \mathcal{B}$$

if and only if $\theta \in \mathcal{B}$.

Lemma 3.6. Given a positive integer $q \geq 1$, we have

$$(\alpha_1, \ldots, \alpha_n) \in \mathcal{B}$$

if and only if

$$(q\alpha_1, \ldots, q\alpha_n) \in \mathcal{B} .$$

Proof. We observe that if $|a| = |b| = 1$ and $a^q - b^q$ is small then

$$|a^q - b^q| \approx q|a - b| .$$

Using this, we have that the small divisors for $\alpha$ and $q\alpha$ are the same, more precisely, for $m \geq 2$,

$$\Omega_{q\alpha}(m) \approx \Omega_\alpha(m)$$

and the result follows. ♦

Proof of proposition 3.5. Using the lemma we are reduced to prove the theorem in the case without torsion, i.e. $\varepsilon = 1$.

Observe that a small divisor for $(\lambda_1, \ldots, \lambda_n)$ can be written as

$$\lambda_1^{i_1} \ldots \lambda_n^{i_n} - \lambda_j = \mu \sum_k b_{ik}i_k - \mu^{b_j} = \mu^{b_j} \left( \mu \sum_k b_{ik} - b_j - 1 \right)$$
Also $|b, i| \leq ||b||_\infty |i|$, thus
\[
\Omega_\alpha(m) \geq C \Omega_\theta(||b||_\infty m).
\]
Also if we have a small divisor for $\mu, \mu^n - \mu$, and $b_j \neq 0$ then
\[
\lambda_j^n - \lambda_j = \mu^{nb_j} - \mu^{bj} \approx b_j(\mu^n - \mu)
\]
so we conclude that
\[
\Omega_\theta(m) \geq C \Omega_\alpha(m).
\]

\textbf{Proof of theorem 5.}

The first assertion has been proved above. The linearization under Bruno condition follows from the general theory. When Bruno condition is violated, then according to lemma 3.3 we have that all non-rational $\alpha_j$ do not satisfy Bruno condition. We just pick one such $\alpha_j$ and a non-linearizable holomorphic germ in one variable
\[
g(z) = e^{2\pi i \alpha_j} z + \varphi(z)
\]
with $\varphi(z) = O(z^2)$. We construct $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ by
\[
f(z_1, \ldots, z_n) = Az + (0, \ldots, 0, \varphi(z_j), 0, \ldots, 0)
\]
where we can assume that the $z_j$-axes is an eigendirection for $A$. Then $f$ is formally linearizable but not analytically linearizable.

We finish the proof of the last comments in the introduction about the irrelevance of the torsion for the purposes of linearization.

\textbf{Proposition 3.7.} Let $q \geq 1$. The germ $f$ is holomorphically (resp. formally) linearizable if and only if $f^q$ is holomorphically (resp. formally) linearizable.

\textbf{Proof.} We prove the non-trivial statement. If $f^q$ is holomorphically (resp. formally) linearizable there an element $k \in G_0$ (resp. $k \in \hat{G}_0$) such that
\[
k \circ f^q = A^q \circ k.
\]
Then
\[
k_0 = \frac{1}{q} \sum_{i=0}^{q-1} A^{-i} \circ k \circ f
\]
is an element of $G_I$ (resp. $\hat{G}_I$) that linearizes $f$.\hfill \Box

\textbf{Proposition 3.8.} If $A = D_0 f$, $l(A) = 0$ and $A$ is diagonal then $f$ is linearizable if and only if
\[
f^{q(A)} = \text{id}.
\]

\textbf{Proof.} We have $A^{q(A)} = I$ so if $f$ is linearizable we get $f^{q(A)} = \text{id}$. The converse results from the application of the previous proposition.\hfill \Box
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