A STABLE COHOMOTOPY REFINEMENT OF
SEIBERG-WITTEN INVARIANTS: II

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Abstract. A gluing theorem for the stable cohomotopy invariant defined in the first article in this series of two gives new results on diffeomorphism types of decomposable manifolds.

1. Introduction

A stable cohomotopy refinement of Seiberg-Witten invariants was defined in the first article [2] of this series of two. New invariants have to stand some acid tests: Do they give new insights? Can they be computed? The present article addresses these questions. It turns out that the stable cohomotopy invariants actually encode more information on four-manifolds than the integer valued Seiberg-Witten invariants. Recall from [2] that for a $K$-oriented, closed Riemannian 4-manifold the monopole map $\mu : A \to C$ is an $T$-equivariant fiber preserving map between Hilbert space bundles over $Pic^0(X) = H^1(X; \mathbb{R})/H^1(X; \mathbb{Z})$, which defines an element $[\mu]$ in an equivariant stable cohomotopy group $\pi^{b,T,H}(Pic^0(X); ind(D))$. Here $b = b^+_2(X)$ denotes the maximal dimension of a positive definite subspace of second de Rham cohomology of $X$ and $ind(D)$ denotes the virtual index bundle over $Pic^0(X)$ of the twisted Dirac operator. The subscript $H$ denotes a universe for the action of the group $T$ of complex numbers of unit length. This technical device is needed in the equivariant stable homotopy setting to specify the group action on the suspension coordinates. The key result of this article is a connected–sum theorem for this invariant. Recall that a $K$-theory orientation of a connected sum of 4-manifolds uniquely induces $K$-theory orientations on the respective summands.

Theorem 1.1. For a connected sum $X = X_0 \# X_1$ of 4-manifolds, the stable equivariant cohomotopy invariant is the smash product of the invariants of its summands

$$[\mu_X] = [\mu_{X_0}] \wedge [\mu_{X_1}].$$

Stating it loosely, the monopole map $\mu_X$ has the same stable cohomotopy invariant as the product $\mu_{X_0} \times \mu_{X_1}$.

Let’s consider some applications of this theorem. In these applications rudimentary facts about equivariant stable cohomotopy groups already gain geometrically significant old and new information about four-manifolds.
First some implications which are well known from Seiberg-Witten theory: If both sum-
mands have nonzero $b_2^+$, then by dimension reasons the smash product of the cohomotopy
invariants lies in the kernel of the homomorphism to the integers comparing the stable coho-
motopy invariants to the Seiberg-Witten invariants. Thus one regains the folklore theorem
stating that the Seiberg-Witten invariant for a connected sum of 4-manifolds vanishes if both
summands have nonzero $b_2^+$. The blowup formula for Seiberg-Witten invariants follows from
the above theorem by applying a well known result about the degree of $T$-equivariant self
maps of representation spheres.

If the manifold $X$ has vanishing first Betti number, then by forgetting the equi-
variance, the stable cohomotopy invariant gives rise to an element of the stable $k$-stem $\pi_k^s$ with
$k = \text{ind}_R(D) - b_2^+$. The above theorem then states that the invariant is multiplicative under
connected sums. If the $spin^c$-structure on $X$ comes from an almost complex structure, then
the invariant is an element in the group $\pi^s_1 \cong \mathbb{Z}/2$, with the Hopf map representing the
nontrivial element. It turns out that the invariant is the Hopf map if and only if both
the integer Seiberg-Witten invariant is odd and $b_2^+ \equiv 3 \mod 4$. The fact that the cube of
the Hopf map is nonvanishing, together with known computations of the Seiberg-Witten
invariants, lead to the following sample applications:

**Corollary 1.2.** The connected sum of two symplectic four-manifolds with vanishing first
Betti number and with $b_2^+ \equiv 3 \mod 4$ each does not split off a manifold with $b_2^+ \equiv 1 \mod 4$.

**Corollary 1.3.** Let $X$ be a connected sum of three symplectic four-manifolds with vanish-
ing first Betti number and with $b_2^+ \equiv 3 \mod 4$ for each summand. Suppose $X$ splits as a
connected sum $X \cong X_1 \# X_2$ with $b_2^+(X_1) \equiv 1 \mod 4$. Then the intersection form on $X_2$ is
negative definite.

**Corollary 1.4.** Let $K$ denote the $K3$-surface and suppose there is an oriented diffeo-
 morphism $f : X_1 \# K \# K \to X_2 \# K \# K$, where the $X_i$ are simply connected Kähler manifolds
with $b_2^+ = 3 \mod 4$. Then the integer Seiberg-Witten invariants of $X_1$ and $X_2$ are the same
mod 2. More precisely, let the basic set $B(X_i) \subset H^2(X_i; \mathbb{Z})$ consist of the characteristic
classes with odd Seiberg-Witten number. If one views $H^2(X_i; \mathbb{Z})$ as a direct summand in
$H^2(X_1 \# K \# K; \mathbb{Z})$, then $f$ preserves the basic sets, $f^* B(X_2) = B(X_1)$.

**Corollary 1.5.** Suppose the connected sum $\#_{i=1}^m E_i$ of simply connected minimal elliptic
surfaces of odd geometric genus is diffeomorphic to a connected sum $\#_{j=1}^n F_j$ of elliptic
surfaces. If $m < 4$, then $n = m$ and the $F_j$ and the $E_i$ are diffeomorphic up to permutation.

By the computation in Lemma 3.5 of [4], the connected sum of five almost complex manifolds
with vanishing first Betti number always has vanishing stable cohomotopy invariant. Mikio
Furuta suggested that in the case of four summands there still might exist nonvanishing
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invariants. Indeed, one has the following result, for which the equivariant setting is used in an essential way:

Corollary 1.6. Let $X = \#_{i=1}^4 X_i$ be the connected sum of four almost complex manifolds with vanishing first Betti numbers. Then the stable cohomotopy invariant is nonvanishing if and only if all of the following conditions hold: For every summand $X_i$, the integer Seiberg-Witten invariant is odd and \( b_2^+ \equiv 3 \) mod 4. For $X$, furthermore, the congruence \( b_2^+ \equiv 4 \) mod 8 holds.

In particular, the statements in 1.4 and 1.5 also hold for the connected sum of four manifolds, as long as the resulting manifold satisfies the congruence \( b_2^+ \equiv 4 \) mod 8.

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2. The setup

Let $X$ be the disjoint union of a finite number, say $n$, of closed connected Riemannian 4-manifolds $X_i$, each equipped with a $K$-theory orientation. Suppose each component contains a separating “long neck” $N(L)_i = [-L, L] \times S^3$. So it is a union

$$ X_i = X_i^- \cup X_i^+ $$

of closed submanifolds with common boundary $\partial X_i^\pm = \{0\} \times S^3$. The length $2L \gg 8$ of the long neck will be specified below, its radius is assumed to be equal in all components.

For an even permutation $\tau \in A_n$, let $X^\tau$ be the manifold obtained from $X$ by interchanging the positive parts of its components, that is

$$ X_i^\tau = X_i^- \cup X_i^+_{\tau(i)}. $$

The $K$-theory orientation of $X$ induces by gluing a $K$-theory orientation of $X^\tau$. The main result of the present article compares the stable cohomotopy invariants of the two manifolds $X$ and $X^\tau$. The connected sum theorem will be an immediate consequence.

For the manifold $X$ the monopole map of [2] can be described as follows: Let $S^+$ and $S^-$ denote the Hermitian rank-2 bundles associated to the given $K$-orientation and let $A$ denote a $spin^c$-connection, which we may assume to induce the flat connection on $\text{det}(S^\pm)$ over the long neck. Fix once and for all identifications of the spinor bundles and the chosen $spin^c$-connections over the $n$ copies of $[-L, L] \times S^3$ in $X$.

The gauge group $G = \text{map}(X, \mathbb{T})$ acts on the space $\Gamma(S^\pm)$ of spinors via multiplication with $u : X \to \mathbb{T}$, on connections via addition of $i\, du^{-1}$ and trivially on forms. Consider the subgroup $G_\tau \subset G$ consisting of gauge transformations which are trivial over the “short neck” $N(1) = [-1, 1] \times \bigsqcup_{i=1}^n S^3$. The group $G_\tau$ decomposes into a product of gauge groups,
each corresponding to one of the manifolds $X_i^\pm$. The orbit of the action of $G_\tau$ on the space of spin$^c$-connections is of the form $(A + i\ker d_r)$, where $\ker d_r \subset \ker d$ is the space of closed 1-forms on $X$ vanishing identically on the short neck. Using the identification of the chosen spin$^c$-connections $A$ and $A^\tau$ over the short neck, the space $A + i\ker d_r \cong A^\tau + i\ker d_r$ can be viewed as a subspace of the space of spin$^c$-connections both over $X$ and $X^\tau$. After suitable Sobolev completion, $A + i\ker d_r/G_\tau$ identifies this way both with $Pic^0(X) = H^1(X; \mathbb{R})/H^1(X; \mathbb{Z})$ and $Pic^0(X^\tau)$.

Let $\mathcal{A}$ and $\mathcal{C}$ denote the quotients

$$
\mathcal{A} = (A + i\ker d_r) \times (\Gamma(S^+) \oplus \Omega^1(X))/G_r
$$

$$
\mathcal{C} = (A + i\ker d_r) \times (\Gamma(S^-) \oplus \Omega^2_+(X) \oplus H^1(X; \mathbb{R}) \oplus \Omega^0(X)/\mathbb{R})/G_r
$$

by the action of the gauge group. Both spaces are bundles over $Pic^0(X)$ and the monopole map

$$
\mu = \bar{\mu}/G_0 : \mathcal{A} \to \mathcal{C}
$$

is a fiber preserving, $T$-equivariant map over $Pic^0(X)$, defined by

$$(A', \phi, a) \mapsto (A', D_{A' + a}\phi, F_{A' + a}^+ - \sigma(\phi), pr(a), d^*a).$$

Here $\sigma(\phi)$ denotes the trace free endomorphism $\phi \otimes \phi^* - \frac{1}{4}|\phi|^2 \cdot id$ of $S^+$, considered as a self-dual 2-form on $X$. The map $pr$ is a linear map $\Omega^1(X) \to H^1(X; \mathbb{R})$, which is an isomorphism on the space $\ker(d^* + d^+)$ of harmonic one-forms. Any such map will do. For the purposes of this paper, we make a choice as follows: Fix $b_1$ closed smooths, smoothly embedded in the complement of the long neck, which form a basis of $H_1(X; \mathbb{R})$. For a continuous one-form $a$ on $X$, integration of $a$ along multiples of these cycles defines the element $pr(a) \in Hom(H_1(X; \mathbb{R}), \mathbb{R}) = H^1(X; \mathbb{R})$.

It is convenient to use fiberwise $L_k^1$ and $L_{k-1}^2$ Sobolev completions ($k \geq 4$) for $\mathcal{A}$ and $\mathcal{C}$, respectively. The fiberwise $T$-action on $\mathcal{A}$ and $\mathcal{C}$ is trivial on forms and is given by complex multiplication on spinors.

To compare the monopole maps $\mu : \mathcal{A} \to \mathcal{C}$ on $X$ and $\mu^\tau : \mathcal{A}^\tau \to \mathcal{C}^\tau$ on $X^\tau$, we will use gluing maps $\mathcal{A} \to \mathcal{A}^\tau$ and $\mathcal{C} \to \mathcal{C}^\tau$. To define these maps, we first need a smooth path

$$
\psi : [0, 1] \to SO(n)
$$

starting from the unit, i.e. $\psi(0) = id$, and ending at $\tau$, considered as the permutation matrix $(\delta_{i,\tau(j)})_{i,j} \in SO(n)$. A second ingredient in the construction is a smooth function

$$
\gamma : [-L, L] \times S^3 \to [0, 1],
$$

depending only on the first variable. This function $\gamma$ is supposed to vanish on the $[-L, -1]$-part of the neck and to be identical to 1 on the $[1, L]$-part.

For a section $e$ of a bundle $E$ over $X$, denote by $e_i$ its restriction to the bundle $E_i = E_i|_{X_i}$. Suppose the restrictions of $E_i$ to the long necks are identified with a bundle $\widetilde{F}$. Using
these identifications, the restrictions $E_{X_i} \tau$ glue together to give a bundle $E^\tau$ over $X^\tau$. Smooth sections of $E$ will be patched together to give smooth sections of $E^\tau$ as follows: The restrictions of $e_i$ to the complement of the long neck $N(L)$ remain unchanged. Over the long neck, the restrictions of the sections $e_i$ can be viewed as the components of a section

$$\vec{e} = \left( \begin{array}{c} e_1 \\ \vdots \\ e_n \end{array} \right)$$

of the bundle $\oplus_{i=1}^n F$ over $[-L, L] \times S^3$. The $i$-th component of the section

$$e^\tau_i = (\psi \circ \gamma) \cdot \vec{e}$$

now restricts to the section $e_i$ over the $[-L, -1]$-part and to the section $e_{\tau(i)}$ over the $[1, L]$-part. Patching parts together, we obtain a smooth section $e^\tau$ of the bundle $E^\tau$.

This gluing construction, applied to forms $\alpha$ and spinors $\phi$ on $X$, defines a linear map, for which we will use the shorthand notation

$$V : (\phi, \alpha) \mapsto (\psi \circ \gamma) \cdot (\phi, \alpha) = (\phi^\tau, \alpha^\tau).$$

In total we obtain bundle isomorphisms $A \to A^\tau$ and $C \to C^\tau$ of the Hilbert space bundles over the identification $Pic^0(X) \xrightarrow{\cong} Pic^0(X^\tau)$ detailed above. All of these isomorphisms will be denoted by $V$.

**Theorem 2.1.** Gluing via the map $V$ induces for $b = b^+_2(X)$ an isomorphism

$$\pi^{b}_T H^{b}_0(Pic^0(X); ind(D)) \to \pi^{b}_T H^{b}_0(Pic^0(X^\tau); ind(D^\tau)),$$

which identifies the classes of the monopole maps of $X$ and $X^\tau$ for corresponding $K$-theory orientations.

This theorem, the proof of which will be given in the next paragraph, implies the gluing theorem stated in the introduction. This is a consequence of the next two propositions, applied to the case where at most one component of $X$ is not diffeomorphic to the standard four-sphere.

**Proposition 2.2.** The monopole map $\mu$ for a $K$-theory orientation on $X = \coprod_{i=1}^n X_i$ is the product of the monopole maps on the components of $X$

$$\mu = \prod_{i=1}^n \mu_i : \mathcal{A} = \prod_{i=1}^n A_i \to \prod_{i=1}^n C_i = \mathcal{C}.$$

Thus the associated stable equivariant cohomotopy element is the smash product

$$[\mu] = \wedge_{i=1}^n [\mu_i] \in \pi^{b^+_2(X)}_T \oplus_{H_i} (Pic^0(X); ind(D))$$

of the cohomotopy elements associated to the respective components. The action of the torus $\mathbb{T}^n$ on the sum $\oplus_{i=1}^n H_i$ is factorwise. \qed
This proposition is merely spelling out the obvious. The gluing map $V$ is $\mathbb{T}$-equivariant with respect to the action of the diagonal subgroup of the torus.

**Proposition 2.3.** The stable cohomotopy element associated to any spin$^c$-structure on a connected four dimensional manifold $X$ with vanishing Betti numbers $b_1 = b_2 = 0$ is the class of the identity map

$$[\mu] = [\text{id}] \in \pi^0_{\mathbb{T}, H}(\ast) \cong \mathbb{Z}.$$  

**Proof.** In this case, the equivariant index $\text{ind}(D) \in RO(\mathbb{T})$ of the Dirac operator is zero. Hence, the ring $\pi^0_{\mathbb{T}, H}(\ast)$ coincides with the Burnside ring $A(\mathbb{T}) \cong \mathbb{Z}$. This isomorphism can be described as the map

$$\pi^0_{\mathbb{T}, H}(\ast) \to \pi^0_{\ast}(\ast) \cong \mathbb{Z}$$

induced by restriction to fixed point sets (cf. [4], 133ff). However, on the $\mathbb{T}$-fixed point set, the monopole map is just the linear isomorphism

$$d + d^* : \Omega^1(X) \to \Omega^2_+(X) \oplus \Omega^0(X)/\mathbb{R}.$$  

\[ \square \]

### 3. Proof of the gluing theorem

Let $\mu$ and $\mu^\tau$ denote the monopole maps on the spin$^c$-manifolds $X$ and $X^\tau$, respectively. The diagram

$$\begin{array}{ccc}
A & \xrightarrow{\mu} & C \\
\downarrow & & \downarrow \nu \\
A^\tau & \xrightarrow{\mu^\tau} & C^\tau
\end{array}$$

does not commute. The theorem claims that it commutes up to suitable homotopy. What does suitable homotopy mean in this context? The homotopy of course should be a homotopy through Fredholm maps, i.e. (nonlinear) compact deformations of linear Fredholm maps. However, for the purposes in the present article these Fredholm maps need not satisfy the boundedness condition of [2] at all times. Instead, consider homotopies of Fredholm maps

$$\mu_t = l_t + c_t : A \to C,$$

starting from $\mu_0 = \mu$ and ending at $\mu_1 = V^{-1}\mu^\tau V$, of the following kind: There is a bounded disk bundle $D \subset A$ with bounding sphere bundle $S$ over $Pic^0(X)$ such that both at the start and at the end of the homotopy all solutions are contained in the disk bundle. During the homotopy no solution is allowed to cross the bounding sphere bundle. In more technical terms this means

$$\mu_t^{-1}(0) \subset D \quad \text{for} \quad t \in \{0, 1\},$$
When composing homotopies (as will be done below), the first condition, of course, need only be checked at the very beginning of the first homotopy and at the end of the last homotopy.

The reason why it suffices to consider this sort of homotopy was basically mentioned in [2]:

The suitable homotopy is a homotopy of Fredholm maps of pairs of spaces (over $\text{Pic}^0(X)$)

$$\mu_t : (D, S) \rightarrow (C, C \setminus \{0\}).$$

Upon restriction to finite dimensional subbundles, the inclusions

$$(D, S) \rightarrow (A^+, A^+ \setminus D^\circ) \leftarrow (A^+, \emptyset^+)$$

and

$$(C^+, \emptyset^+) \rightarrow (C^+, (C \setminus \{0\})^+) \leftarrow (C, C \setminus \{0\})$$

via homotopy equivalence and excision give isomorphisms of groups of pointed stable maps from Thom spaces to spheres on the one side and stable maps from pairs (disk bundle, sphere bundle) to pairs (vector space, pointed vector space) on the other side. Both groups describe the stable cohomotopy group $\pi^{b_2^+}(X^T, \text{H}(\text{Pic}^0(X); \text{ind}(D)))$.

In the proof the help of several homotopies in the above sense will be invoked: The first two homotopies from $\mu$ (resp. $\mu^\tau$) to an auxiliary map $P : A \rightarrow C$ (resp. $P^\tau : A^\tau \rightarrow C^\tau$) will tame the quadratic terms in the monopole map: As an operator on sections over $X$, the map $P$ differs from the monopole map $\mu$ only over the long neck. Over the short neck, the operators $P$ and $P^\tau$ are the linearisations of $\mu$ and $\mu^\tau$. The boundedness control during the homotopy is achieved by the use of Weitzenböck formulas for both the Dirac operator and the covariant derivative. Positivity of scalar and Ricci curvature, respectively, on the long neck provide sufficient control on the spinor and form components of solutions. The homotopy is set up so to keep the pivotal pointwise estimate on the spinor of a solution at all times. The estimates on the forms will follow. However, in order to tune the estimates on spinors and forms, it may be necessary to stretch the long neck even longer. The final homotopy then “rotates” the operator $P$ into the operator $V^{-1}P^\tau V$.

For $R \leq L$, let $\rho_R$ and $\rho^\tau_R$ be smooth cutoff functions, defined on $X$ and $X^\tau$ with values in $[0, 1]$ of the following form: The functions coincide, when restricted to the components $X^+_i$. On the middle part $N(R - 1)$ of length $2R - 2$ of the long neck, the functions vanish identically. Outside $N(R)$, the functions take the value 1. In the remaining part $N(R) \setminus N(R - 1)$, the function is constant on each sphere $\{r\} \times S^3$. The homotopies

$$\rho_{R,t} \overset{def}{=} (1 - t) + t\rho_R,$$

for the time parameter $t$ in the unit interval, describe a homotopy from the constant map 1 to the function $\rho_R$ on $X$ and on $X^\tau$. 
3.1. **The standard estimates.** The estimates in the proof of the main theorem are variations of the standard estimates used to show compactness of the moduli space. The argument from [3] (compare [2], 3.1.), gives a norm estimate for any point \((\phi, a)\) in the fibre over a point \(A\) in the Picard torus with \(\mu(A, \phi, a) = 0\). Let’s recall the three main steps of the argument:

**Step 1:** Applying the Weitzenböck formula for the Dirac operator associated to the connection \(A + a = A'\), one gets a pointwise estimate:

\[
\Delta |\phi|^2 \leq 2 < D_A^* D_{A'} \phi - \frac{s}{4} \phi + \frac{1}{2} F_{A'}^+ \phi, \phi >
\]

The equalities \(D_A' = 0\) and \(F_{A'}^+ = \sigma(\phi)\) thus imply an estimate

\[
\Delta |\phi|^2 + \frac{s}{2} |\phi|^2 + \frac{1}{2} |\phi|^4 \leq 0
\]

At the maximum of \(|\phi|^2\), its Laplacian is non-negative. So one obtains a pointwise estimate \(s|\phi|^2 + |\phi|^4 \leq 0\) for the norm of the spinor.

**Step 2:** The Sobolev estimate \(|a|_{C^0} \leq C_s |a|_{L^p_A}\) for some \(p > 4\) and the elliptic estimate \(|a|_{L^p_{A'}} \leq C_e (|d^a a|_{L^p_0} + |d^a a|_{L^p_0} + ||pr(a)||)\) combine with the equality \(d^a a = F^+_{A'} + \sigma(\phi)\) to an estimate

\[
|a|_{C^0} \leq C_s C_e (|F^+_{A'}|_{L^p_0} + ||\sigma(\phi)||_{L^p_0}).
\]

**Step 3:** For the bootstrapping assume for \(i \leq k\) inductively \(L^p_{i-1}\)-bounds on \((\phi, a)\) with \(p = 2^{k-i}\). To obtain \(L^p_{i-1}\)-bounds, compute:

\[
\|(\phi, a)|_{L^p_{i-1}} - \|(\phi, a)|_{L^p_{i-1}} = \|(D_A \phi, d^a a)|_{L^p_{i-1}} - \|(D_A \phi, F^+_{A'} + \sigma(\phi))|_{L^p_{i-1}}.
\]

The summands in the last expression are bounded by the assumed \(L^p_{i-1}\)-bounds on \((\phi, a)\).

3.2. **Varying the length of the long neck.** In the course of the proof it will be necessary to vary the length of the long neck. It is necessary to keep control on how the constants in step 2 of the argument do change. To formulate the result needed, let \(X' = X \setminus N(L - 1)\) denote the complement of the middle part of length \(2(L - 1)\) of the neck.

**Proposition 3.1.** There is a constant \(C_1\), independent of the length of the long neck, such that for any smooth partition \(\varphi_+, \varphi_-\) of unity on \(X\), such that \(\varphi_+\) is identical 1 on \(X^+ \cap X'\) and vanishes on \(X^- \cap X'\), the following elliptic estimate holds:

\[
|a|_{C^0(X')} \leq C_1 (|(d^s + d^a)\varphi_+ a|_{L^p} + |(d^s + d^a)\varphi_- a|_{L^p} + C||pr(a)||).
\]

**Proof.** The Sobolev constant \(C_s\) depends only on the local geometry of the manifold: Choose for each \(x \in X\) a bump function \(\beta_x : X \to [0, 1]\) with small support near \(x\) and \(\beta_x(x) = 1\). The bump functions \(\beta_x\) can be chosen such that their \(C^1\)-norm is bounded by a constant
from this elliptic estimate and the Sobolev estimate with constant the Laplace operator on the three-sphere. If to the compactly supported forms

\textbf{Lemma 3.2.}

The preimage

\begin{equation}
M \rightarrow \mathcal{C}
\end{equation}

on the three-sphere. As explained in [3], this operator \( \partial_{\alpha} \) induces a Fredholm map \( L^p_{\alpha}(Y^\pm) \rightarrow L^p_{\alpha}(Y^\pm) \) if \( \alpha \) is not in the spectrum of the operator \( L \), whose square is the Laplace operator on the three-sphere. If \( \alpha \) is negative and greater than the maximal negative eigenvalue of \( L \), kernel and cokernel of the operator \( \partial_{\alpha} + L \) on \( Y^\pm \) are isomorphic to \( H^1(Y^\pm; \mathbb{R}) \) and \( H^0(Y^\pm; \mathbb{R}) \oplus H^2_+(Y^\pm; \mathbb{R}) \), respectively. We may apply the estimate

\begin{equation}
||b||_{L^p_{\alpha}(Y^\pm)} \leq C^\pm ||(d^* + d^+)b||_{L^p_{\alpha}(Y^\pm)} + C||pr(b)||
\end{equation}

to the compactly supported forms \( b = \phi \pm a \) on \( Y^\pm \) which we get from a one-form \( a \) an \( X \). By construction both \( a \) and \( b \) have the same pointwise norm along \( X' \). The claim follows from this elliptic estimate and the Sobolev estimate with constant \( C_1 = C_{\alpha}max(C^\pm) \).

\textbf{3.3. The first homotopy.} Consider the homotopy \( \mu_t : A \rightarrow C \) defined by

\begin{equation}
\mu_t(A, \phi, a) = (A, D_{A + a} \phi, F_{A + a}^+ - \rho_{L, t} \sigma(\phi), pr(a), d^* a).
\end{equation}

\textbf{Lemma 3.2.} The preimage \( \mu_t^{-1}(0) \) is uniformly bounded for all times \( t \in [0, 1] \).
Proof. The standard argument applies with a minor change in step 1. At the maximum of $|\phi|^2$, the estimate
\[ s|\phi|^2 + \rho_{L,t}|\phi|^4 \leq 0 \]
holds for the spinor component of a solution. The scalar curvature is positive along the long neck. So the maximum is attained outside the long neck and is bounded by the norm of the scalar curvature. The norm of the term $\rho_{L,t}\sigma(\phi)$ in the rest of the argument is bounded by a multiple of the norm of $\sigma(\phi)$. These bounds are independent of the parameter $t$. \qed

3.4. The second homotopy. The next homotopy moderates the second quadratic term over $N(2)$:
\[ \mu_{t+1}(A,\phi,a) = (A,D^-(A+\rho_2,\alpha)\phi,F^+_{A+a} - \rho_L\sigma(\phi),pr(a),d^*a) \]
This homotopy starts at $\mu_1$ and ends at $\mu_2 = P$. Note that the latter differential operator is linear on the short neck. This second homotopy is more delicate than the first one. We need to restrict it to a bounded disk as explained in the beginning of this paragraph. In order to get the necessary bounds on the solutions during the homotopy, it may, moreover, become necessary to stretch the long neck even longer, like playing the trombone. So we have to make sure that the bounds we get for solutions $(\phi,a)$ to the operator $\mu_1$ are independent of the length of the long neck. The spinor component causes no problems as it satisfies a $C^0$-bound
\[ |\phi|^2 \leq s = \max(0,-s), \]
where $s$ denotes the scalar curvature of $X$ which is independent of $L$. The $C^0$-bound for the one-form component of a solution used in step 2 of the standard argument was of the form
\[ |a|_{C^0} \leq C_sC_e(||F^+_A||_{L^P} + ||\rho_L\sigma(\phi)||_{L^P}) \leq C_sC_e(||F^+_A||_{C^0} + S)\text{vol}(X') \]
When stretching the neck, the factor $C_e$ is problematic, since it depends a priori on the global geometry of $X$. So we will resort to make use of the less handy estimate 3.1.

Lemma 3.3. There is a constant $U$ and a threshold length $L_0$ such that for any solution $(\phi,a)$ to the operator $\mu_1$ on the manifold $X(L)$ with neck of length $2L \geq 2L_0$ the one-form component satisfies the $C^0$-bound $|a| \leq U$.

Proof. The one-form component of a solution on $X$ is harmonic on the part $N(L-1)$ of the neck. Because of nonnegative Ricci curvature along the neck, the maximum principle holds for the norm of such a one-form along $N(L-1)$. Let $\varphi_+,\varphi_-\ be a partition of unity as in 3.1 with approximately constant slope $|d\varphi_\pm| < L^{-1}$ along the neck. From 3.1 we get the estimate
\[ |a|_{C^0(X')} \leq C_1(||(d^* + d^\pm)\varphi_\pm a||_{L^P} + ||(d^* + d^\pm)\varphi_- a||_{L^P} + C||pr(a)||. \]
Since $pr(a) = 0$, the latter summand does not contribute. We compute
\[ ||(d^* + d^\pm)\varphi_\pm a||_{L^P} \leq ||\varphi_\pm(d^* + d^\pm)a||_{L^P} + ||d\varphi_\pm||_{L^P}|a|_{C^0(N(L-1))}. \]
The vanishing of \((d^* + d^+)a\) along \(N(L - 1)\) and the maximum principle
\[ |a|_{C^0(N(L-1))} \leq |a|_{C^0(X')} = |a|_{C^0(X)} \]
thus lead to an estimate
\[ |a|_{C^0(X)} \leq C_1((F^+_A|_{C^0} + S)\text{vol}(X')) + C_1L^{\frac{1}{2} - 1}\text{vol}(S^3)|a|_{C^0(X)}. \]
The claim follows with \(L_0 = (2C_1\text{vol}(S^3))^\frac{1}{2}\) and \(U = 2C_1((F^+_A|_{C^0} + S)\text{vol}(X')).\)

**Lemma 3.4.** If the long neck of \(X\) is longer than the threshold length \(L_1 \geq L_0\), then the following holds: Any solution \((\phi, a) \in \mu_t^{-1}(0)\) for \(1 \leq t \leq 2\), which satisfies the \(C^0\)-bounds \(|\phi|^2 \leq 2S\) and \(|a| \leq 2U\), also satisfies the stricter \(C^0\)-bounds \(|\phi|^2 \leq S\) and \(|a| \leq U\).

**Proof.** Note that the one-form component of a solution on \(X\) is again harmonic on the part \(N(L - 1)\) of the neck and hence satisfies the maximum principle. Moreover, because of the product structure of the neck, such an harmonic one-form on the neck splits into a sum \(a = a_i + a_s\) of harmonic one-forms, according to the direct sum decomposition of the cotangent bundle. The harmonic summand \(a_s\) pointing in the sphere direction satisfies an inequality
\[ \Delta|a_s|^2 \leq -2 < \text{Ric}(a_s), a_s > \]
Since the Ricci tensor in direction of the sphere is positive definite,
\[ -2 < \text{Ric}(a_s), a_s > \leq -\delta^2|a_s|^2 \]
for some \(\delta > 0\). If \(\alpha\) denotes the sum \(\sum_{i=1}^n |a_{s,i}|^2\) over all components of the neck, then \(\alpha\) satisfies a differential inequality
\[ \frac{d^2\alpha}{dr^2} \geq -\Delta_X \alpha \geq \delta^2\alpha. \]
Thus \(|a_s|^2\) is bounded by the function
\[ \frac{4nU^2 \cosh(\delta r)}{\cosh(\delta(L - 1))}. \]
In particular, there is exponential decay of the norm of \(a_s\) towards the middle of the neck.

Now we are ready to obtain for the spinor component of a solution a sharper bound than the assumed one. The argument of step 1 works fine, if the spinor component attains its maximum outside \(N(L - 1)\). So it suffices to make sure that the maximum is not attained along \(N(L - 1)\). Let’s analyze, what may go wrong: If the spinor component of a solution during the homotopy attains its maximum in \(N(L - 1)\), then at that maximum, it satisfies an inequality
\[ 0 \leq \Delta|\phi|^2 \leq -\frac{s}{2}|\phi|^2 + (dp_2,t \wedge a)^+ \phi, \phi > . \]
Because of \(dp_2,t \wedge a = dp_2,t \wedge a_s\), the norm of the latter summand decays exponentially with the length of the long neck. If one stretches the long neck, the second summand will decrease so that finally the scalar curvature summand in the inequality will prevail. So if
the neck is long enough, the spinor component of a solution cannot attain its maximum in
\( N(L - 1) \).

To get the sharper bound on the one-form component, one can now use exactly the same
argument as in \( 3.3 \), since the main ingredients for the argument hold: The bound on the
spinor component is unchanged, as is the fact that \( a \) is harmonic along \( N(L - 1) \).

3.5. The third homotopy. To finish the proof of the gluing theorem it remains to con-
stuct a homotopy between \( P \) and \( V^{-1}P^r V \). Note that both operators differ only over
the short neck in \( X \). Because both differential operators are linear over the short neck, their
difference is a multiplication operator:

\[
V^{-1}P^r V = P + d\log(V)
\]

For \( t \in [0,1] \), consider the matrix valued function \( \psi \circ t\gamma : [-L, L] \times S^3 \to SO(n) \). Multiplica-
tion of spinors or forms with this matrix valued function defines a map \( V_t \) over the long neck:
Pairs of forms or spinors over the long neck are mapped to pairs of forms or spinors over the
long neck. Multiplication with \( \psi \circ t\gamma \) will not make sense outside the long neck. However,
conjugation \( V_t^{-1}PV_t = P + d\log(V_t) \) extends nicely to an operator \( P + d\log(V_t) : A \to C \)
over all of \( X \) for \( 0 \leq t \leq 1 \). These operators

\[
V_t^{-1}PV_t = P + d\log(V_t)
\]

provide the final homotopy in the argument.

**Lemma 3.5.** If the long neck of \( X \) is longer than a threshold length \( L_2 \geq L_1 \), then the
following holds: Any solution \( (\phi, a) \in (P + d\log(V_t))^{-1}(0) \) for \( t \in [0,1] \), which is bounded
by the \( C^0 \)-bounds \( |\phi|^2 \leq 2S \) and \( |a| \leq 2U \), even satisfies the stricter \( C^0 \)-estimates \( |\phi|^2 \leq S \)
and \( |a| \leq U \).

**Proof.** The proof makes use of the following observation: Let \( (\phi, a) \) be a solution to the
partial differential equation

\[
(P + d\log(V_t))(\phi, a) = V_t^{-1}PV_t(\phi, a) = 0.
\]

Then \( (\phi, a) \) is also a solution to the operator \( \mu_2 \) over the complement of the short neck,
since the \( P + d\log(V_t) \) and \( \mu_2 \) only differ over the short neck \( N(1) \). Over the long neck, on
the other hand, the rotation \( V_t^{-1}(\phi, a) \) of \( (\phi, a) \) is a solution to the operator \( P \).

The proof of the \( C^0 \)-bound for \( |\phi|^2 \) works almost as in \( 3.4 \). The Weitzenböck argument
can be applied to \( \phi \) on the complement of the short neck and to \( V_t^{-1}\phi \) along \( N(L - 1) \) the
same way, since \( |\phi|^2 = |V_t^{-1}\phi|^2 \). Again we need to satisfy the condition that the norm of
\( a_s \) along the short neck is small compared to the scalar curvature: Exponential decay of
\( |a_s| = |(V_t^{-1}a)_s| \) towards the middle of the neck follows as in \( 3.4 \) from the fact that \( V_t^{-1}a \) as
a solution to \( P = \mu_2 \) is harmonic along \( N(L - 1) \). Combined with the assumed \( C^0 \)-bound
\( |a| \leq 2U \), the condition again can be met by neckstretching.
Since $V_t^{-1}a$ is harmonic along $N(L - 1)$, its $C^0$-norm $|V_t^{-1}a| = |a|$ obeys the maximum principle along this part of the neck. Let $\beta_\pm$ denote a smooth cutoff function supported on $X^\pm \setminus N(1)$ with $\beta_\pm(x) = 1$ for $x$ in the complement of $N(L - 1) \cap X^\pm$ and with almost linear decay along the neck, i.e. $|d\beta_\pm| < \frac{2}{L}$. Then the function $\varphi = 1 - \beta_+ - \beta_-$ is supported on $N(L - 1)$. The elliptic estimate \[3.1\] gives a bound
\[
|a|_{C^0(X)} \leq C_1 \left( \| (d^* + d^+)^\beta_- a \|_{L^p_0} + \| (d^* + d^+)^\varphi V_t^{-1}a \|_{L^p_0} + \| (d^* + d^+)^\beta_+ a \|_{L^p_0} \right)
\leq C_1 \left( |F_+^c|_{C^0} + S \right) \text{vol}(X') + C_1 \left( \| d\beta_- \|_{L^p_0} + \| d\varphi \|_{L^p_0} + \| d\beta_+ \|_{L^p_0} \right) |a|_{C^0(X')}
\leq \frac{1}{2}U + 4C_1 \left( \frac{1}{2}L^{\frac{1}{2}} - 1 \right) \text{vol}(S^3)(2U).
\]
Again, the second summand can be made arbitrarily small by neckstretching. \hfill \Box

For a proof of the theorem, one finally has to compose homotopies: The first two homotopies combine to a homotopy $\mu \sim P$. The third homotopy is between $P$ and $V^{-1}P^\tau V$. Again the first two homotopies, conjugated by $V$, combine to the homotopy $V^{-1}\mu_\tau V$ from $V^{-1}P^\tau V$ to $V^{-1}\mu^\tau V$. The composition of these homotopies satisfies the suitability conditions harped on at the beginning of the section.

4. Applications and Problems

A $K$-theory orientation (or equivalently a $\text{spin}^c$-structure) on a four dimensional manifold is the same as a stably almost complex structure. This is because the natural map between the respective classifying spaces $BU \to B\text{Spin}^c$, has the appropriate connectivity. Such a stably almost complex structure comes with a first Chern class, which is the integer lift $c \in H^2(X; \mathbb{Z})$ of the second Stiefel-Whitney class. The index of the Dirac operator associated to the $\text{spin}^c$-structure has complex dimension
\[
d = \text{ind}_C(D) = \frac{c^2 - \text{sign}(X)}{8}.
\]
Via the Pontrjagin-Thom construction, the stable cohomotopy invariant associated to a $K$-oriented four-manifold ideally can be thought of as encoding the equivariant framed bordism class of an $\mathbb{T}$-equivariant manifold $\tilde{M}$. The quotient $M$ of this equivariant manifold by the $\mathbb{T}$-action then would be the moduli space considered in Seiberg-Witten theory. Actually, this “ideal” picture does not hold in general. The reason is that the transversality arguments used in the Pontrjagin-Thom construction fail to hold in an equivariant setting. As a consequence, the moduli spaces considered in Seiberg-Witten theory do come with singularities in those cases where equivariant transversality fails to hold. Nevertheless, the moduli space will have an “expected dimension”
\[
k = 2d - (b_2^+ - b_1 + 1).
\]
Let’s consider an example where equivariant transversality fails. The following fact about equivariant maps is well known and can be proved by the use of the equivariant $K$-theory mapping degree (cf. e.g. [4]):

**Proposition 4.1.** Let $f : (\mathbb{R}^n \oplus \mathbb{C}^m + d) \to (\mathbb{R}^n \oplus \mathbb{C}^m) +$ be an $\mathbb{T}$-equivariant map such that the restricted map on the fixed points has degree 1. Then $d \leq 0$ and $f$ is homotopic to the inclusion.

Such maps are not nullhomotopic, as their restriction to the fixed point set is not, but the “expected dimension” $2d - 1$ is negative. If equivariant transversality held in this situation, then we get a contradiction: The manifold associated via Pontriagin-Thom construction would be empty by dimension reasons. But the empty manifold is associated to the nullhomotopic map.

The stable equivariant maps of 4.1 actually arise as monopole maps associated to four-manifolds with $b_2^+ = b_1 = 0$, that is of manifolds with negative definite intersection form with vanishing first Betti number. Applying the main theorem 1.1 to this case results in a generalization of the well-known “blowing-up” theorem:

**Corollary 4.2.** Let $X$ and $N$ be closed oriented four-manifolds with the intersection form on $N$ negative definite and $b_1(N) = 0$. Fix spin$^c$-structures on both manifolds. The equivariant stable homotopy invariant of the connected sum is

$$\mu_{X \# N} = \mu_X \wedge \gamma(\mathbb{C})^{|d|}$$

with the diagonal $\mathbb{T}$-action, where $\gamma(\mathbb{C})$ is the one-point compactified $\mathbb{T}$-map $\mathbb{C}^0 \to \mathbb{C}^1$ and $d = \text{ind}_\mathbb{C}(D_N) \leq 0$. In particular, if $b_2^+(X) - b_1(X) > 1$, then the integer Seiberg-Witten invariants of $X$ and $X \# N$ are the same, if $2|d|$ is not greater than the “expected dimension” of the monopole moduli space associated to $X$.

The condition $b_2^+(X) - b_1(X) > 1$ can actually be removed by considering maps “up to equivariant homotopy modulo the fixed point set” as explained in [4].

**Proof.** Because of 4.1, the first part is just a restatement of the gluing theorem in this special case. The last statement is trivial in case the expected dimension $k$ of the monopole moduli space associated to $X$ is odd, since then the Seiberg-Witten invariants of both $X$ and $X \# N$ vanish by definition. Otherwise the claim follows by the very definition of the comparison map 3.3. from the stable cohomotopy invariants to the integers: Let $\mu_X \in \pi_1^b(\text{Pic}^0(X); \text{ind}(D))$ denote the stable cohomotopy invariant of $X$. Then $\mu_X \wedge \gamma(\mathbb{C})^{\frac{k}{2}}$ as an element in $\pi_1^b(\text{Pic}^0(X); \text{ind}(D) - \frac{k}{2} \mathbb{C}) \cong \mathbb{Z}$ is just the Seiberg-Witten invariant of $X$.

By a theorem of Hirzebruch and Hopf, the expected dimension is zero if and only if the stably almost complex structure on $X$ is actually an almost complex structure. Indeed, in
all currently known (at least to the author) examples of four dimensional spin\textsuperscript{c}-manifolds with nonvanishing integer valued Seiberg-Witten invariant, the spin\textsuperscript{c}-structure is associated to an almost complex structure.

4.3. Here are some general facts about Seiberg-Witten invariants:

- By a theorem of Taubes \[7\], the Seiberg-Witten invariant of a symplectic four-manifold is, up to sign convention, 1.
- If \(X\) is a four-dimensional manifold underlying a Kähler surface, then solutions to the monopole equations correspond to holomorphic sections of certain complex line bundles, compare \[5\]. As a consequence, the stable cohomotopy invariants and a fortiori the Seiberg-Witten invariants, are nonzero at most for such \(K\)-orientations which correspond to almost complex structures. In particular, the moduli space either has expected dimension 0 or is empty.

For simplicity, consider from now on only manifolds with vanishing first Betti number. If the \(K\)-orientation of \(X\) is associated to an almost complex structure, then the stable map \(\mu\) nonequivariantly is an element of the stable homotopy group \(\pi_{2}^{st}(S^{0})\). This group has two elements, the trivial map and the Hopf map \(\eta\).

**Proposition 4.4.** The cohomotopy invariant of an almost complex manifold with vanishing first Betti number nonequivariantly is the Hopf map if and only if both \(b_{2}^{+}\) is congruent \(3\) \(\text{mod} \ 4\) and the integer valued Seiberg-Witten invariant is odd.

**Proof.** As was shown in \[4\], the stable cohomotopy invariant for an almost complex manifold \(X\) is \(SW(X)\kappa_{d}\), where \(\kappa_{d}\) is a generator of the stable cohomotopy group \(\pi_{2}^{2d-2}(CP^{d-1}) \cong \mathbb{Z}\kappa_{d}\) and \(2d = b_{2}^{+} + 1\). The composition

\[
S^{2d-1} \to S^{2d-1}/\mathbb{T} = CP^{d-1}/\mathbb{T} \cong CP^{d-1} \kappa_{d} S^{2d-2},
\]

of \(\kappa_{d}\) with the quotient map of the free \(\mathbb{T}\)-action is the nontrivial Hopf element \(\eta \in \pi_{2}^{st}(S^{0})\) iff \(d\) is even. This can be seen as follows: The quotient map is the attaching map of the top cell in \(CP^{d}\). If \(z\) denotes the generator in \(H^{2}(CP^{d};\mathbb{F}_{2})\), then the Steenrod square \(Sq^{2}(z^{d-1})\) is nonzero if and only if \(d\) is even. But \(Sq^{2}\) detects the Hopf map.

By considering products of the \(\mathbb{T}\)-equivariant Hopf map, we get the following result, which in particular implies \[1.6\]

**Proposition 4.5.** Let \(X\) be a connected sum of \(n \geq 2\) almost complex manifolds \(X_{i}\) with vanishing first Betti numbers. The stable equivariant cohomotopy element of \(X\) is nonvanishing if and only if the following conditions are satisfied:

- For each summand, \(b_{2}^{+}(X_{i})\) is congruent \(3\) \(\text{mod} \ 4\).
- The integer valued Seiberg-Witten invariants are odd for each summand.
• If $n \geq 4$, then $n = 4$ and $b_2^+(X) \equiv 4 \mod 8$.

Proof. The result follows from the Atiyah-Hirzebruch spectral sequence, which computes the groups $\pi^{2d-1-n}(CP^{d-1})$, which contain the stable cohomotopy invariants in these cases. The results of this computation are stated in [2], 3.5.

First consider the case $n = 2$. The “if” part of the statement follows from the fact that the square of the Hopf map is the only nonzero element in the second stable stem.

On the other hand, the group $\pi^{2d-3}(CP^{d-1})$ is nonzero if and only if $d$ is even. In this case its only nontrivial element is the composition of the quotient map $CP^{d-1} \to CP^{d-1}/CP^{d-2} = S^{2d-2}$ with the Hopf map. Combined with [4], the result follows.

The result for $n = 3$ is immediate from the result for $n = 2$ and from the fact that nonequivariantly the cube of the Hopf map is nonzero.

If $n = 5$, the stable cohomotopy invariant is identified with a torsion element in the group $\pi^{2d-6}(CP^{d-1})$, which is torsion free. So in particular, the stable cohomotopy invariant is zero, if $n \geq 5$.

It remains to consider the case $n = 4$. This is more delicate, because the fourth power of the Hopf map is zero nonequivariantly. As is well known, the product in stable homotopy theory can be defined equivalently in two ways. One way is by smash product, the other is by composition. If we use the second description, we see that the first Hopf map factors through the quotient map $S^{2d-1} \to CP^{d-1}$ by the $T$-action, followed by the projection to the top cell. It remains to consider, whether projection to the top cell, followed by the cube of the nonequivariant Hopf map is stably nonzero as a map from $CP^{d-1}$ to $S^{2d-5}$. But this can be read off the Atiyah-Hirzebruch spectral sequence: This map is nonzero if and only if the torsion subgroup in $\pi^{2d-5}(CP^{d-1})$ is divisible by 8, which is the case if and only if $d$ is divisible by 8, which is the case if and only if $b_2^+(X)$ is congruent $4 \mod 8$.

Proof. (of 1.2) The proof follows from 4.5 and 4.3. The stable cohomotopy invariant of the connected sum $X$ of two symplectic manifolds with $b_2^+ \equiv 3 \mod 4$ nonequivariantly lies in the second stable stem $\pi^{2d}(S^0)$ and actually is the square of the Hopf map. If $X$ splits into a connected sum $X \cong X_1 \# X_2$ of manifolds with $b_2^+ \equiv 1 \mod 2$, then the stable cohomotopy invariants of the respective summands are elements of the $n_1$-th and $n_2$-th stable stem with $n_1$ both odd and $n_1 + n_2 = 2$. Since their product is the nonvanishing element in the second stable stem, both $n_i$ are equal 1. By the Hirzebruch-Hopf theorem both manifolds are almost complex. Now [4.3] applies again to yield the contradiction.

Proof. (of 1.3) If $X$ is the connected sum of three symplectic manifolds with $b_2^+ \equiv 3 \mod 4$, then the stable cohomotopy invariant is the cube of the Hopf map in the third stable stem (nonequivariantly). If $X$ splits into a connected sum $X \cong X_1 \# X_2$ with $b_2^+(X_1) \equiv 1 \mod 2$, then (because of [4.4]) the manifold $X_1$ cannot be almost complex. Its stable cohomotopy invariant lies in an odd stable stem, but not in the first. The stable cohomotopy invariant
of $X_2$ lies in an even stable stem. As their product is the nontrivial cube of the Hopf map in the third stable stem, only one possibility remains: The stable cohomotopy invariant of $X_2$ is an odd element in the zero’th stable stem $\pi_0^s(S^0) \cong \mathbb{Z}$. Now we have to apply equivariant homotopy theory \cite{equiv} again: The stable cohomotopy invariant of $X_2$ is represented by an $\mathbb{T}$-equivariant map

$$f : (\mathbb{R}^n \oplus \mathbb{C}^m + \frac{b}{2})^+ \to (\mathbb{R}^{n+b} \oplus \mathbb{C}^m)^+.$$ 

If $b = b_2^+(X_2)$ is nonzero, then the nonequivariant degree of such a map is necessarily zero. This finishes the proof.

**Proof.** (of 1.4) For the K3-surface the SW-invariants are completely known: They vanish except for the one $spin^c$-structure which lifts to a $Spin$-structure. For this the value is 1, up to sign convention. From the statements above it follows that for a connected sum $K\#K\#X$ or $K\#X$ for a simply connected Kähler surface $X$, the $spin^c$-structures supporting nontrivial stable cohomotopy invariants of the connected sum correspond to exactly those $spin^c$-structures on $X$ having odd SW-invariants (if $b_2^+$ is of correct modulus).

For simply connected manifolds, the $spin^c$-structures are detected by the first Chern classes of the associated spinor bundles. The argument above can be rephrased the following way: For a connected sum of Kähler surfaces, the nontrivial stable cohomotopy invariants detect the pairs (or triples) consisting of cohomology classes of the summands which support odd Seiberg-Witten invariants. This can be applied in special situations to recognize the summands:

**Proof.** (of 1.5) The proof is based on the known classification of elliptic surfaces (see e.g. \cite{elliptic}) via SW-invariants. To a simply connected minimal elliptic surface with given $b_2^+$ one can associate a pair $1 \leq m \leq n$ of coprime integers which, together with the geometric genus $p_g = b_2^+ - 1 - \frac{1}{2}$, classify the diffeomorphism type. Note that $b_2^+$ is congruent to 3 mod 4 iff the geometric genus is odd. The point in the proof is that one can recognize $m, n$ and $p_g$ for odd geometric genus from the pattern of the cohomology classes corresponding to odd SW-invariants, the “recognizable” classes. Here comes a description, how this can be accomplished combinatorially.

The cohomology classes corresponding to nontrivial SW-invariants are multiples of an indivisible element $f$ in the second cohomology with integer values of the elliptic surface. The multiplicities are of the form $(p_g - 1 - 2a)mn + (m - 2b - 1)n + (n - 2c - 1)m$ for nonnegative integers $a < p_g$, $b < m$ and $c < n$. The value of the SW-invariant for such a multiple of $f$ is $\binom{p_g - 1}{a}$.

Note that the distribution of basic classes is symmetric around the origin and the SW-invariant for the largest such multiple $k$ of $f$, where $a = b = c = 0$, is odd. So there are
at least two recognizable classes except in the case of a K3-surface $p_g = m = n = 1$, where there is exactly one recognizable class.

If there are no more than three recognizable classes, then either $m = n = 1$ or $n = 2m = 2p_g = 2$. In the latter case, the largest multiple is 1, in the former, it is $p_g - 1$, which is even.

In the case of at least four recognizable classes, consider the second but largest multiple. In case $n > 1$, the integer $m$, which is half the difference, is coprime to the largest multiple. If $m = n = 1$, then there is an $0 < 2a \leq p_g - 1$ with $\left(\frac{p_g - 1}{a}\right)$ odd. This integer $a$ has to be even, because otherwise $\left(\frac{p_g - 1}{a + 1}\right)$, which is obtained from it by multiplication with a rational number having $a + 1$ in its denominator, could not be an integer. In this case, half the difference cannot be coprime to the largest multiple; both are even. This makes it possible to distinguish the $m = n = 1$-cases.

Finally consider the largest multiple $k$ and the second largest multiple. It can be assumed that half the difference is coprime to $k$ and thus equals $m$. Consider the multiples $(k - 2\lambda m)f$ for $\lambda \geq 1$. The SW-invariants associated to these classes will be 1 for $\lambda < n$ and zero or $p_g - 1$, anyway even, for $\lambda = n$. This characterizes the second integer $n$. Knowing both integers this way, the geometric genus follows from the formula for $k$.

In the situation of a connected sum of no more than three elliptic surfaces, as in [1,3], the number of summands can be read off the dimension of the moduli spaces having nontrivial invariants. The cohomology classes associated to nontrivial invariants are situated in a bounded region in a sublattice of the second cohomology of rank at most 3. They form a box and the pattern characterizing the individual summands can be found on the respective edges of the box.

**4.6.** The monopole map $\mu$ may be pertubed quite a bit without changing the resulting stable cohomotopy invariant. For example the term $\sigma(\phi)$ may be replaced by some function $f(|\phi|)\sigma(\phi)$ as long as $f$ does not decay too fast at infinity. Any polynomial with positive leading coefficient will give different moduli spaces, but the same stable cohomotopy invariant.

Let’s end this article with two problems, which grow out of considering stable cohomotopy invariants of connected sums. The first problem is related to a well known theorem of C.T.C. Wall stating that any two homeomorphic simply connected four dimensional differentiable manifolds become diffeomorphic after taking connected sum with finitely many copies of $S^2 \times S^2$. Moreover, in many cases of algebraic surfaces it is known that it suffices to take connected sum with only one such copy of $S^2 \times S^2$. 

Problem 4.7. Suppose $X$ and $Y$ are homeomorphic, simply connected differentiable four-manifolds. Do they become diffeomorphic after taking connected sum with sufficiently many K3-surfaces?

Problem 4.8. Are there manifolds realizing stable cohomotopy elements in $\pi^4(\mathbb{C}P^{d-1})$ other than the powers of the Hopf map $\eta$, for example the element associated to the stable Hopf map $\nu : S^7 \to S^4$? Or more generally: Is there an indecomposable $K$-oriented four-manifold with nonvanishing stable cohomotopy invariant, which is not almost complex?

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