Vanishing theorems for ample vector bundles

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February 1996

1 Introduction

Since the seminal paper published by P.A. Griffiths in 1969 [7], a whole series of vanishing theorems have been established for the Dolbeault cohomology of ample vector bundles on smooth projective varieties, mainly due to the efforts of J. Le Potier, M. Schneider, A. Sommese, J-P. Demailly, L. Ein and R. Lazarsfeld, the author, and more recently W. Nahm [2, 4, 11, 15, 16, 18, 19, 21].

This abundance of results and variants lead to a little confusion, in contrast with the case of ample line bundles for which the celebrated Kodaira-Akizuki-Nakano vanishing theorem [1] is the only general statement.

In this paper, we prove a vanishing theorem for the Dolbeault cohomology of a product of symmetric and skew-symmetric powers of an ample vector bundle, twisted by a suitable power of its determinant line bundle. This line bundle being ample, our theorem can be seen as an effective version, in that particular case, of the asymptotic vanishing theorem of Serre. It includes as special cases most of the above mentioned results.

Theorem A. Let $E$ be a holomorphic vector bundle of rank $e$, and $L$ a line bundle on a smooth projective complex variety $X$ of dimension $n$. Suppose that $E$ is ample and $L$ nef, or that $E$ is nef and $L$ ample. Then, for any sequences of integers $k_1, \ldots, k_l$ and $j_1, \ldots, j_m$,

$$H^{p,q}(X, S^{k_1}E \otimes \cdots \otimes S^{k_l}E \otimes \bigwedge^{j_1} E \otimes \cdots \otimes \bigwedge^{j_m} E \otimes \text{det}(E)^{l+n-p} \otimes L) = 0$$

as soon as $p + q > n + \sum_{s=1}^{m} (e - j_s)$.

The vanishing theorem of Griffiths, more precisely the version of this theorem given by M. Schneider, that is Theorem 2.4 in [19], corresponds to $p = n$, $m = 0$, $l = 1$. The extension of it due to J-P. Demailly, Theorem 0.2 in [4], corresponds to $p = n$ and $m = 0$, as will be made clear by Theorem A'.

The vanishing theorem of Le Potier, Proposition 2.3 in [11], corresponds to $p = n$, $l = 0$ and $m = 1$. It was generalized by L. Ein and R. Lazarsfeld : Proposition 1.7 of [6], at least when the vector bundles involved are identical, corresponds to $p = n$, $l = 0$ and $m$ arbitrary.
Vanishing theorems

For several ample bundles, say $E_1, \ldots, E_h$, Theorem A can easily be extended in the following way: the Dolbeault cohomology groups of a product of associated bundles, each of the form appearing in Theorem A, holds under the condition that $p + q$ be greater than $n$ plus the sum of the contributions of each term of the product. This does not follow from the trick used in [5] to deduce Proposition 1.7 from Le Potier’s theorem, but it is a straightforward exercise to modify our proof of Theorem A so as to handle the case of several vector bundles.

Theorem A was proven in [16] for $p$ close to $n$. The methods used in that paper were adapted from [3] and much more complicated than those we will use here. The main idea was to work on flag manifolds and compute suitable Dolbeault cohomology groups of well-chosen line bundles – unfortunately, there is no general method to do that and the argument was quite involved. Here, we restrict to relative products of projective bundles. On a projective space, the Dolbeault cohomology of a multiple of the hyperplane bundle is given by a special case of Bott’s theorem. Actually, this fortunate fact is a consequence of the irreducibility, as a homogeneous bundle, of the vector bundle of regular differential forms of a given degree on a projective space. On a grassmannian, these bundles of forms are no longer irreducible, but remain completely reducible – on a more general flag manifold this property definitely disappears.

Let us mention that our proof of Theorem A works mutatis mutandis in the case where $E$ or $L$, instead of being supposed ample, is only $k$-ample in the sense of Sommese: one then simply has to add $k$ to the numerical vanishing condition of the theorem to get a correct statement. Theorem A therefore also extends Proposition 1.14 of [21].

Let us now state our main result in a slightly different form. Recall that if $V$ is some finite dimensional complex vector space, the dimension of which we denote by $v$, the irreducible polynomial representations of the reductive algebraic group $\text{Gl}(V)$ are in correspondence with partitions of length at most $v$, that is, non-increasing sequences of non-negative integers $\lambda = (\lambda_1 \geq \cdots \geq \lambda_v \geq 0)$. The $\lambda_i$ are the \textit{parts} of $\lambda$, while its \textit{length} $l(\lambda)$ is the number of positive parts. We denote the irreducible $Gl(V)$-module corresponding to the partition $\lambda$ by $S_{\lambda}V$, and call it the \textit{Schur power} of $V$ of exponent $\lambda$. As special cases, one of course recovers symmetric and skew-symmetric powers:

$$S^pV = S_{p, 0, \ldots, 0}V, \quad \wedge^qV = S_{1, \ldots, 1, 0, \ldots, 0}V,$$

where this last expression contains exactly $q$ ones. In particular, $S_{1, \ldots, 1}V = \det V$, and one has for each integer $m$ the factorization formula

$$S_{\lambda_1+m, \ldots, \lambda_v+m}V = S_{\lambda_1, \ldots, \lambda_v}V \otimes (\det V)^m.$$

One usually represents a partition $\lambda$ by its \textit{Ferrers diagram} $D(\lambda)$, which is a collection of left-justified rows of decreasing lengths from top to bottom, these lengths being given by the parts of $\lambda$. If we flip this diagram over its main diagonal, we get the Ferrers diagram of the \textit{conjugate partition} $\lambda^*$. The parts of this partition are
simply the lengths of the columns of the Ferrers diagram of $\lambda$. We then associate to each integer $l$, with $0 \leq l \leq v - 1$, the integer
\[ q_l(v, \lambda) = \sum_{j, \lambda_j^* > l} (v - \lambda_j^*). \]

Since a complex vector bundle $E$ of rank $e$ on $X$ has $Gl(e, \mathbb{C})$ for structure group, to each partition $\lambda$ of length at most $e$ corresponds an associated vector bundle $S_\lambda E$.

Theorem A' can then be restated in the following way:

**Theorem A'**. Let $E$ be a vector bundle of rank $e$, and $L$ a line bundle, on a smooth projective complex variety $X$ of dimension $n$. Suppose that $E$ is ample and $L$ nef, or that $E$ is nef and $L$ ample. Then, for each partition $\lambda$ of length at most $e$, and each integer $l$ between 0 and $e - 1$,
\[ H^{p,q}(X, S_\lambda E \otimes (\det E)^{l+n-p} \otimes L) = 0 \quad \text{for } p + q > n + q_l(e, \lambda). \]

Of course, $q_l(e, \lambda)$ is an decreasing function of $l$. For each Schur power of $E$, Theorem A' therefore determines a sufficient power of the determinant line bundle of $E$, to ensure the vanishing of the corresponding Dolbeault cohomology in a given degree.

The equivalence between Theorem A and Theorem A' is an easy consequence of Pieri’s rules for the tensor product of a given Schur power by some symmetric or skew-symmetric power. On the one hand, if $\lambda$ is a partition and $l$ an integer, we let $j_a = \lambda_a^*$, $1 \leq a \leq m$, if $\lambda_a^* > l$, and $k_b = \lambda_b - m$ if $b \leq l$. Then $S_\lambda E$ is a component of $S^{k_1} E \otimes \cdots \otimes S^{k_l} E \otimes \land^{j_1} E \otimes \cdots \otimes \land^{j_m} E$, and Theorem A thus implies Theorem A'. On the other hand, suppose that $S_\mu E$ is a component of $\land^{j_1} E \otimes \cdots \otimes \land^{j_m} E$. Its first part is then at most equal to $m$. Therefore, if $S_\lambda E$ is a component of $S_\mu E \otimes S^{k_1} E \otimes \cdots \otimes S^{k_l} E$, one has $\lambda_j \leq l$ if $j > m$, so that
\[ q_l(e, \lambda) \leq \sum_{j=1}^m (e - \lambda_j^*) \leq \sum_{j=1}^m (e - \mu_j) = \sum_{k=1}^m (e - j_k). \]

Theorem A' thus implies Theorem A.

Two consequences of Theorem A-A' should be of particular interest. First note that $q_l(e, \lambda) = 0$ for $l \geq l(\lambda)$. This implies the following partial answer to the problem raised in [2] by J-P. Demailly:

**Corollary B.** Under the same hypotheses, for each partition $\lambda$ of length at most $e$,
\[ H^{p,q}(X, S_\lambda E \otimes (\det E)^{l(\lambda)+n-p} \otimes L) = 0 \quad \text{for } p + q > n. \]

In particular, for symmetric powers,
\[ H^{p,q}(X, S^k E \otimes (\det E)^{n-p+1} \otimes L) = 0 \quad \text{for } p + q > n, \]
while for tensor powers,
\[ H^{p,q}(X, E^\otimes k \otimes (\det E)^{\min(k,e-1)+n-p} \otimes L) = 0 \quad \text{for } p + q > n. \]

Because of the factorization formula, one may always suppose that \( l(\lambda) < e \) in the first part of this statement. Corollary B therefore improves and makes more precise Theorem 0.3 of [2]. Moreover, its second part extends Griffiths’ vanishing theorem to the whole range of the Dolbeault cohomology. Its last part follows from the existence of a decomposition

\[ E^\otimes k = \bigoplus_{|\lambda|=k} m_\lambda S_\lambda E, \]

where the size \(|\lambda|\) of the partition \( \lambda \) is defined as the sum of its parts (the multiplicity \( m_\lambda \) can be shown to be the degree of the representation of the symmetric group, usually called a Specht module, defined by \( \lambda \)).

It was suggested in [3] that the exponent \( l(\lambda) + n - p \) of \( \det E \) in Corollary B could possibly be replaced by \( l(\lambda) + \min(n-p,n-q) \), and that this exponent could be the optimal one. This was suggested by the observation that if the Borel-Le Potier spectral sequence degenerates at the \( E_2 \) level, then the exponent \( e - 1 + \min(n-p,n-q) \) of \( \det E \) is sufficient to get a vanishing theorem in degree \( p + q > n \).

Unfortunately, it was shown in [13] that this degeneracy does not occur in general. Worse, the Borel-Le Potier spectral sequence can be non-degenerate at an arbitrary high level. We see no way to recover the lost symmetry between \( p \) and \( q \).

**Corollary C.** Under the same hypotheses, for each partition \( \lambda \) of length at most \( e \),
\[ H^{p,q}(X, S_\lambda E \otimes (\det E)^{n-p} \otimes L) = 0 \quad \text{for } p + q > n + q_0(e, \lambda), \]
where \( q_0(e, \lambda) = \sum_i (\lambda_1 - \lambda_i) = e\lambda_1 - |\lambda| \). In particular,
\[ H^{p,q}(X, \bigwedge^k E \otimes (\det E)^{n-p} \otimes L) = 0 \quad \text{for } p + q > n + e - k. \]

Sommese conjectured in [21] that, under the preceding positivity hypotheses,
\[ H^{p,q}(X, \bigwedge^k E \otimes L) = 0 \quad \text{for } p + q > n + e - k, \]
the case where \( p = n \) being Le Potier’s theorem. This conjecture was independantly disproved in 1988 by J-P. Demailly [2] and M. Schneider, who gave counterexamples already for \( p = n - 1 \). Nevertheless, Corollary C shows that Sommese’s conjecture becomes correct after a twist by \( \det E \) to the power \( n-p \).

Apart from a few very special cases, we have no idea about the optimality of our vanishing theorems. Actually, it is a difficult problem to obtain interesting examples of non-vanishing cohomology groups. To our knowledge, the only case that has been investigated in detail is that of grassmannians [21, 16], on which Bott’s theorem allow to compute the Dolbeault cohomology of homogeneous bundles in terms of diagrams.
and the so-called hook lengths, which play an important role in the representation
theory of symmetric groups.

In a final section, we propose an application of Theorem A-A' to the geometry
of degeneracy loci. Let \( \phi : E^* \to F \otimes L \) be a morphism between vector bundles
over an \( n \)-dimensional smooth projective variety \( X \), where \( E \) and \( F \) have ranks \( e \)
and \( f \), and \( L \) is a line bundle. Denote by \( D_k(\phi) \) its \( k \)-th degeneracy locus. Under a
strong positivity assumption, we then show that if \( D_k(\phi) \) has the expected dimension
\( \rho = n - (e - k)(f - k) \), the restriction morphism

\[
H^q(X, \mathcal{O}_X) \to H^q(D_k(\phi), \mathcal{O}_{D_k(\phi)})
\]

is an isomorphism for \( q < \rho \), and is injective for \( q = \rho \). A similar result holds when
\( F = E \) and \( \phi \) is symmetric or skew-symmetric. In particular, \( D_k(\phi) \) is connected, if
\( X \) is, as soon as \( \rho \) is positive.

2 Proof of Theorem A

2.1 On the Borel-Le Potier spectral sequence

Let again \( E \) be a complex vector bundle of rank \( e \), \( F \) another vector bundle, on a
smooth compact complex variety \( X \) of dimension \( n \). Let \( Y = \mathbb{P}(E^*) \) be the variety
of hyperplanes of \( E \), let \( \pi \) be its projection onto \( X \). We denote as usual by \( \mathcal{O}_E(1) \)
the universal quotient line bundle on \( Y \), and by \( \mathcal{O}_E(k) \) its \( k \)-th power, \( k \in \mathbb{Z} \).

To compute the Dolbeault cohomology of \( \mathcal{O}_E(k) \otimes \pi^* F \) on \( Y \), we use the fact that
the bundle \( \Omega^p_Y \) of regular \( p \)-forms on \( Y \) may be filtered according to their degree \( t \)
on \( X \), so that its restriction to each fiber of \( \pi \) is the bundle of \( m \)-forms on that fiber.

This implies, for each integer \( p \), the existence of a spectral sequence, which we
call after \( [2] \) the Borel-Le Potier spectral sequence \([13] \). Its \( E_1 \)-term is given by

\[
p E_1^{t,p} = H^{t+s}(Y, \Omega^p_{Y/X} \otimes \mathcal{O}_E(k) \otimes \pi^*(\Omega^t_X \otimes F)),
\]

The corresponding quotient are the vector bundles

\[
G^{t,p} = F^{t,p} / F^{t+1,p} = \pi^* \Omega^t_X \otimes \Omega^p_{Y/X},
\]

where \( \Omega^m_{Y/X} \) denotes the vector bundle of relative differential forms of degree \( m \) – so
that its restriction to each fiber of \( \pi \) is the bundle of \( m \)-forms on that fiber.

This implies, for each integer \( p \), the existence of a spectral sequence, which we
call after \( [2] \) the Borel-Le Potier spectral sequence \([13] \). Its \( E_1 \)-term is given by

\[
p E_1^{t,p} = H^{t+s}(Y, \Omega^p_{Y/X} \otimes \mathcal{O}_E(k) \otimes \pi^*(\Omega^t_X \otimes F)),
\]

The Borel-Le Potier spectral sequence converges to the Dolbeault cohomology groups
\( H^{p,q}(Y, \mathcal{O}_E(k) \otimes \pi^* F) \), that is, there is a naturally defined filtration of that complex
vector space, with associated graded quotient

\[
gr H^{p,q}(Y, \mathcal{O}_E(k) \otimes \pi^* F) = \bigoplus_{t+s=q} p E_\infty^{t,s}.
\]
Let us compute the $E_1$-terms of the Borel-Le Potier spectral sequence. We will denote by $(k|l)$, $k \geq 0$, $l > 0$, the partition of length $l$, whose first part is equal to $k+1$, while its other non zero parts are equal to one : its Ferrers diagram is a hook. The corresponding bundle $S_{(k|l)}E$ associated to $E$ is the kernel of the natural contraction map

$$S^{k+1}E \otimes \Lambda^{l-1}E \rightarrow S^{k+2}E \otimes E^* \otimes \Lambda^{l-1}E \rightarrow S^{k+2}E \otimes \Lambda^{l-2}E.$$  

In particular, $S_{(k|1)}E = S^{k-1}E$, $S_{(k|e)}E = S^kE \otimes \det E$, $S_{(0|j)}E = \Lambda^jE$. We keep the same notation when $k < 0$ or $l > e$, in which case $S_{(k|l)}E = 0$ (except for $k = -1$ and $l = 0$).

**Lemma D.** Suppose that $k > 0$. Then

$$pE^{t,s-t}_1 = H^{t,s}(X, S_{(k-p+t-1|p-t+1)}E \otimes F).$$

**Proof.** We use a Leray spectral sequence to compute the cohomology group

$$pE^{t,s-t}_1 = H^{s}(Y, \Omega^{p-t}_{Y/X} \otimes \mathcal{O}_E(k) \otimes \pi^*(\Omega^t_X \otimes F)).$$

Let $T$ denote the tautological hyperplane bundle, of rank $e-1$ on $Y$. The bundle of relative one-forms is $\Omega^1_{Y/X} = \mathcal{O}_E(-1) \otimes T$, so that its wedge powers $\Omega^h_{Y/X} = \mathcal{O}_E(-h) \otimes \Lambda^h T$. Such a bundle is homogeneous and irreducible, and its cohomology is therefore given by Bott’s theorem ([4], see also [12]):

$$R^i_{\pi^*}(\Omega^{p-t}_{Y/X} \otimes \mathcal{O}_E(k)) = R^i_{\pi^*}(\mathcal{O}_E(k-p+t) \otimes \Lambda^{p-t}T) = \delta_{i,0} S_{(k-p+t-1|p-t+1)}E.$$

The Leray spectral sequence then degenerates at $E_2$, and the lemma follows. \hfill \square

More generally, let $Y_h$ be the relative product over $X$, of $h$ copies of $Y$, and denote the corresponding product of powers of exponents $k_1, \ldots, k_h$ of the universal quotient line bundle by $\mathcal{O}_E(k_1, \ldots, k_h)$. The Dolbeault cohomology of this line bundle, twisted by the pull-back of $F$, is again computed by a Borel-Le Potier spectral sequence, whose $E_1$ term is given by the following straightforward generalization of lemma D :

**Lemma D’.** Suppose that $k_1, \ldots, k_h > 0$. Then

$$pE^{t,s-t}_1 = \bigoplus_{p_1+\ldots+p_h=p-t} H^{t,s}(X, S_{(k_1-p_1+1|p_1+1)}E \otimes \cdots \otimes S_{(k_h-p_h+1|p_h+1)}E \otimes F).$$

Such a cohomology group can be non zero only if $p_i < \min(e, k_i)$ for all $i \leq h$, hence

$$pE^{t,s-t}_1 = 0 \quad \text{if } t < p - \sum_{i=1}^h \min(e - 1, k_i - 1).$$
2.2 The induction

We now proceed by induction on \( r = n - p \). For simplicity, we divide it in two steps. Here, \( E \) is a vector bundle of rank \( e \), and \( L \) a line bundle on \( X \). Moreover, we make the positivity hypothesis that \( E \) is nef and \( L \) ample, or that \( E \) is ample and \( L \) nef.

First step. Let us prove that for any \( r \geq 0 \), Theorem A for \( n - p = r \) implies Theorem A for \( n - p = r \) and \( l = 0 \) that is, Corollary C for \( n - p = r \). A collection of integers \( j_1, \ldots, j_m \) being given, with \( j_k \leq e \) for all \( k \), let us consider the Borel-Le Potier spectral sequence that computes the Dolbeault cohomology of \( \mathcal{O}_E(j_1, \ldots, j_m) \otimes \pi^*((\det E)^r \otimes L) \) on \( Y_m \). If \( p_0 = n - r + \sum_{k=1}^{m}(j_k - 1) \), our previous lemma implies that

\[
p_0 E_1^{n-r,s-n+r} = H^{n-r,s}(X, \bigwedge^{j_1} E \otimes \cdots \otimes \bigwedge^{j_m} E \otimes ((\det E)^r \otimes L)),
\]

which is the cohomology group we want to show to be zero. Moreover, for each positive integer \( u \), \( p_0 E_1^{n-r-u,s-n+r+u} = 0 \), while \( p_0 E_1^{n-r+u,s-n+r-u} \) is given by the direct sum

\[
\bigoplus_{u_1 + \cdots + u_m = u} H^{n-r+u,s}(X, S_{(u_1\vert j_1-u_1)} E \otimes \cdots \otimes S_{(u_m\vert j_m-u_m)} E \otimes (\det E)^r \otimes L).
\]

To show that such a group vanishes, we observe that for each \( k \), \( S_{(u_k\vert j_k-u_k)} E \) is a direct factor of \( S^{u_k} E \otimes \bigwedge^{j_k-u_k} E \), so that it suffices to prove that

\[
H^{n-r+u,s}(X, S^{u_1} E \otimes \cdots \otimes S^{u_m} E \otimes \bigwedge^{j_1-u_1} E \otimes \cdots \otimes \bigwedge^{j_m-u_m} E \otimes (\det E)^r \otimes L)
\]

vanishes. But at most \( u \) of the above symmetric powers can have positive exponents, so that we have enough copies of \( \det E \) to apply our induction hypothesis in degree \( n - r + u \). We obtain

\[
p_0 E_1^{n-r+u,s-n+r-u} = 0 \quad \text{for } s > r + \sum_{k=1}^{m}(e - j_k).
\]

We then consider, for \( u > 0 \), the following morphisms of our spectral sequence:

\[
p_0 E_u^{n-r-u,s-n+r-1+u} \rightarrow p_0 E_u^{n-r,s-n+r} \rightarrow p_0 E_u^{n-r+u,s-n+r+1-u}.
\]

The left hand side is zero because \( p_0 E_1^{n-r-u,s-n+r-1+u} \) is zero, and the right hand side vanishes, as we have just seen, as soon as \( s \geq r + \sum_{k=1}^{m}(e - j_k) \). Hence an isomorphism

\[
p_0 E_1^{n-r,s-n+r} \simeq p_0 E_{\infty}^{n-r,s-n+r}.
\]

But this last term is a graded piece of some filtration of

\[
H^{p_0,s}(Y_m, \mathcal{O}_E(j_1, \ldots, j_m) \otimes \pi^*((\det E)^r \otimes L)).
\]

Since our positivity assumption on \( E \) and \( L \), and the fact that \( j_1, \ldots, j_m \) are all positive, implies that the line bundle \( \mathcal{O}_E(j_1, \ldots, j_m) \otimes \pi^*((\det E)^r \otimes L) \) is ample, this
Dolbeault cohomology group vanishes, by the Kodaira-Akizuki-Nakano vanishing theorem, as soon as \( p_0 + s > \dim Y_m \), that is, \( s > r + \sum_{k=1}^{m} (e - j_k) \). This is precisely what we wanted to prove.

**Second step.** We now show that Theorem A follows for \( n - p = r \) and \( l \) arbitrary.

We fix two collections of integers \( k_1, \ldots, k_l \) and \( j_1, \ldots, j_m \), and consider the Borel-Le Potier spectral sequence that computes the Dolbeault cohomology of

\[
\mathcal{O}_E(k_1 + e, \ldots, k_l + e) \otimes \pi^*(\bigwedge^{j_1} E \otimes \cdots \otimes \bigwedge^{j_m} E \otimes (\det E)^r \otimes L)
\]
on \( Y_i \). If \( p_1 = n - r + l(e - 1) = \dim Y_i - r \), we have, again by the previous lemma,

\[
p_1 E_1^{n-r,s-n+r} = H^{n-r,s}(X, S^{k_1} E \otimes \cdots \otimes S^{k_l} E \otimes \bigwedge^{j_1} E \otimes \cdots \otimes \bigwedge^{j_m} E \otimes (\det E)^{l+r} \otimes L),
\]

and this is the group we want to show to be zero. Moreover, for each positive integer \( v \), we have \( p_1 E_1^{n-r,v,s-n+r+v} = 0 \), while \( p_1 E_1^{n-r+v,s-n+r-v} \) is the direct sum

\[
\bigoplus_{v_1 + \cdots + v_l = v} H^{n-r,v,s}(X, S_{(k_1+v_1|e-v_1)} E \otimes \cdots \otimes S_{(k_l+v_l|e-v_l)} E \otimes \bigwedge^{j_1} E \otimes \cdots \otimes \bigwedge^{j_m} E \otimes (\det E)^{l+r} \otimes L).
\]

But now, for each \( i \), \( S_{(k_i+v_i|e-v_i)} E \) is a direct factor of \( S^{k_i+v_i} E \otimes e-v_i E \), and at least \( l - v \) of these wedge powers are copies of \( \det E \). We therefore have enough copies of the determinant line bundle to use our induction hypothesis in degree \( n - r + v \), and we obtain

\[
p_1 E_1^{n-r+v,s-n+r-v} = 0 \quad \text{for} \quad s > r + \sum_{k=1}^{m} (e - j_k).
\]

Then we consider, for \( v > 0 \), the following morphisms of our spectral sequence:

\[
p_1 E_v^{n-r-v,s-n+r-1+v} \longrightarrow p_1 E_v^{n-r,s-n+r} \longrightarrow p_1 E_v^{n-r+v,s-n+r+1-v}.
\]

The left hand side is zero, because \( p_1 E_1^{n-r-v,s-n+r-1+v} \) is zero, and the right hand side vanishes, as we have just seen, for \( s \geq r + \sum_{k=1}^{m} (e - j_k) \). Hence an isomorphism

\[
p_1 E_1^{n-r,s-n+r} \cong p_1 E_{\infty}^{n-r,s-n+r}.
\]

But this last term is a graded piece of some filtration of

\[
H^{p_1,v}(Y_i, \bigwedge^{j_1} \pi^* E \otimes \cdots \otimes \bigwedge^{j_m} \pi^* E \otimes (\det \pi^* E)^r \otimes \mathcal{O}_E(k_1 + e, \ldots, k_l + e) \otimes \pi^* L).
\]

Since \( \pi^* E \) is nef and \( \mathcal{O}_E(k_1 + e, \ldots, k_l + e) \otimes \pi^* L \) is ample on \( Y_i \), our first step implies the vanishing of the previous cohomology group for \( s > r + \sum_{k=1}^{m} (e - j_k) \). Theorem A is proved.
3 An application to degeneracy loci

It was suggested in [9] that vanishing theorems could provide interesting information on the geometry of degeneracy loci. We shall illustrate this idea in that section. It mainly consists in making use of a resolution of the ideal sheaf of a given degeneracy locus. Then we apply vanishing theorems to the different vector bundles involved in this resolution, so as to get some control of the cohomology of that ideal. This allows for example to prove the connectedness of such a degeneracy loci, although under a rather strong positivity hypotheses, and to go a little further.

The simplest case is that of a morphism $\phi : E^* \to F$ between vector bundles of ranks $e$ and $f$, on a smooth projective variety $X$ of dimension $n$. The $k$-th degeneracy locus

$$D_k(\phi) = \{x \in X, \text{ rank } \phi_x \leq k\}$$

has a natural scheme structure, given in a trivialisation by the vanishing of minors of order $k + 1$. We make the assumption that $D_k(\phi)$ has the expected codimension, namely $(e - k)(f - k)$. Its ideal sheaf $I$ then has Lascoux’s complex $K^\bullet \to I \to 0$ for minimal free resolution [10], where

$$K^i = \bigoplus_{|\lambda|=i} S_{\lambda(k)} E^* \otimes S_{\lambda^*}(k) F^*.$$

The notation $\lambda(k)$ in that expression means the following : if $l$ is the rank of $\lambda$, that is the side of the largest square contained in its diagram, then the partition $\lambda(k)$ is obtained by adjoining $k$ parts equal to $l$ to $\lambda$. To get a nonzero Schur power, we therefore need that $\lambda_1 + k \leq f$ and $\lambda^*_1 + k \leq e$. We will write $\lambda = (l, \mu, \nu)$ if $\mu$ and $\nu$ are the partitions defined by $\mu_i = \lambda_i - l$, $1 \leq i \leq l$, and $\nu_j = \lambda_{j+l}$.

**Theorem E.** Let $\phi : E^* \to F \otimes L$ be a morphism between vector bundles on a smooth projective variety $X$ of dimension $n$, where $E$ has rank $e$, $F$ has rank $f$, and $L$ is a line bundle. Let $k < \min(e, f)$ be a positive integer. We make the following assumptions :

- $E$ is ample and $F$ is nef, or $E$ is nef and $F$ ample,
- $L^\otimes k \geq \det E \otimes \det F$,
- $D_k(\phi)$ has the expected dimension $\rho = n - (e - k)(f - k)$.

Then the natural restriction map $H^q(X, O_X) \to H^q(D_k(\phi), O_{D_k(\phi)})$ is an isomorphism for $0 \leq q < \rho$, and is still injective for $q = \rho$.

If $A$ and $B$ are line bundles, we mean by $A \geq B$ that $A \otimes B^*$ is nef.

**Remark.** If $X$ is connected, $D_k(\phi)$ is connected as soon as $\rho > 0$ : this was proved in [4] for $L = O_X$ and $E \to F$ ample, without any of our extra hypotheses. But to our knowledge, the equality $h^{0,q}(D_k(\phi)) = h^{0,q}(X)$ for $q < \rho$ is new. Note that there is
in general no Barth-Lefschetz type isomorphism theorem for degeneracy loci (think for example to the Segre embedding of $\mathbb{P}^m \times \mathbb{P}^n$), although the map

$$H_q(D_k(\phi), \mathbb{Z}) \rightarrow H_q(X, \mathbb{Z})$$

is always surjective for $q \leq \rho$.[6]

**Proof.** Let again $I$ be the ideal sheaf of $D_k(\phi)$. Our statement is equivalent to the vanishing of $H^q(X, I)$ for $q \leq \rho$, so that it is enough to verify that

$$H^{q+i-1}(X, K^i) = 0 \quad \text{for } 0 < q \leq \rho, \ i > 0.$$

Now, by Serre duality, this is equivalent to the vanishing

$$H^{n,n-q-|\lambda|+1}(X, S_{\lambda(k)}E \otimes S_{\lambda^*(k)}F \otimes L^{\otimes |\lambda(k)|}) = 0$$

for $q \leq \rho$ and each non-empty partition $\lambda$. But if $\lambda$ has rank $l$, then $|\lambda(k)| \geq kl$ and $L^{\otimes |\lambda(k)|} \geq (\det E)^{\otimes l}(\det F)^{\otimes l}$. Theorem A', and the remarks following Theorem A, therefore imply that

$$H^{n-p}(X, S_{\lambda(k)}E \otimes S_{\lambda^*(k)}F \otimes L^{\otimes |\lambda(k)|}) = 0$$

for $p > q_l(\lambda(k)) = l(e - k - l) - |\nu| + l(f - k - l) - |\mu|$, with the notations of the previous picture. But for $p = n - q - |\lambda| + 1$, this condition is equivalent to

$$q \leq n - (e - k)(f - k) + (e - k - l)(f - k - l),$$

and the rightmost term of that inequality is non-negative. \(\square\)

The preceding approach is also suited to the case of a morphism $\phi : E^* \rightarrow E \otimes L$ which is supposed to be symmetric or skew-symmetric. The expected dimension of the degeneracy locus $D_k(\phi)$ is $\rho = n - \binom{e-k+1}{2}$ in the symmetric case, and $\rho = n - \binom{e-k}{2}$ in the skew-symmetric case. When this dimension is correct, the minimal resolution $K^i \rightarrow I \rightarrow 0$ of the ideal sheaf of $D_k(\phi)$ has been computed in [6]: if $\phi$ is symmetric,

$$K^i = \bigoplus_{\lambda=(l,\mu,\mu^*), \ i=|\mu|+l(l-1)/2} S_{\lambda(k-1)}E^*,$$

where $l$ has to be even, while if $\phi$ is skew-symmetric and $k$ is even,

$$K^i = \bigoplus_{\lambda=(l,\mu,\mu^*), \ i=|\mu|+l(l+1)/2} S_{\lambda(k+1)}E^*.$$

With the very same proof, Theorem E extends in the following way:

**Theorem F.** Let $\phi : E^* \rightarrow E \otimes L$ be a symmetric or skew-symmetric morphism on a smooth projective variety $X$ of dimension $n$, where $E$ has rank $e$ and $L$ is a line bundle. Let $k < e$ be a positive integer, with $k$ even if $\phi$ is skew-symmetric. We make the following assumptions:
• $E$ is ample, or simply nef if $L$ is ample,
• $L^\otimes k \geq \det E$,
• $D_k(\phi)$ has the expected dimension $\rho = n - (e-k+1)$ if $\phi$ is symmetric, and $\rho = n - (e-k)$ if $\phi$ is skew-symmetric.

Then the natural restriction map $H^q(X, \mathcal{O}_X) \to H^q(D_k(\phi), \mathcal{O}_{D_k(\phi)})$ is an isomorphism for $0 \leq q < \rho$, and is injective for $q = \rho$.

**Corollary G.** Under the same hypotheses, $D_k(\phi)$ is connected as soon as $\rho > 0$.

This connectedness was established in [22], without our extra hypotheses, in the skew-symmetric case, and in the symmetric case but only when the rank $e$ is even. Again we obtain extra information when the dimension of $D_k(\phi)$ is greater than one.

It would be interesting to know whether the conclusions of Theorems E and F hold true for a morphism $\phi : E^* \to F$ under the only assumption that $E \otimes F$ is ample for the former, and $\phi : E^* \to E \otimes L$ with $S^2 E \otimes L$ or $\wedge^2 E \otimes L$ ample for the latter, at least when the degeneracy locus has the expected dimension. We hope that suitable vanishing theorems will lead to this improvement of Fulton, Lazarsfeld and Tu’s results.

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