Problems on
Minkowski sums of convex lattice polytopes

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Throughout, we fix the notation $M := \mathbb{Z}^r$ and $M_\mathbb{R} := \mathbb{R}^r$.
Given convex lattice polytopes $P, P' \subset M_\mathbb{R}$, we have

$$(M \cap P) + (M \cap P') \subset M \cap (P + P'),$$

where $P + P'$ is the Minkowski sum of $P$ and $P'$, while the left hand side means $\{m + m' \mid m \in M \cap P, m' \in M \cap P'\}$.

**Problem 1** For convex lattice polytopes $P, P' \subset M_\mathbb{R}$ when do we have the equality

$$(M \cap P) + (M \cap P') = M \cap (P + P')?$$

We always have the equality if $r = 1$. This need not be the case, however, if $r \geq 2$ as the following example shows:
In this example, each of \( P \) and \( P' \) is nice (known as basic or unimodular), but their relative position is not.

We may regard the case \( P' = \nu P \) for a positive integer \( \nu > 0 \) as a special case of nice relative position. We have \( P + \nu P = (\nu + 1)P \), and

\[
(M \cap P) + (M \cap \nu P) \subset M \cap (\nu + 1)P.
\]

Problem 2 When do we have the equality

\[
(M \cap P) + (M \cap \nu P) = M \cap (\nu + 1)P
\]

for all \( \nu \in \mathbb{Z}_{>0} \)?

This problem is related to the projective normality of projective toric varieties.

We obviously have the equality if \( r = 1 \). Koelman [9] showed that the equality always holds if \( r = 2 \).

More generally, Sturmfels [10] and others showed that the equality holds if \( P \) has a basic (also known as unimodular) triangulation.

In view of toric geometry, the following could be a reasonable formulation of the problem as to when \( P \) and \( P' \) are in nice relative position: We fix an \( r \)-dimensional convex lattice polytope \( P \), and let \( P' \) to be obtained from \( P \) by independent parallel translation of facets (codimension one faces) of \( P \). The combinatorial face structure of \( P' \) might differ from that of \( P \).

By toric geometry, the convex lattice polytope \( P \) corresponds to an \( r \)-dimensional projective toric variety \( X \) over the complex number field \( \mathbb{C} \) together with an ample divisor \( D \) on \( X \), while \( P' \) gives rise to an effective divisor \( D' \) on \( X \). \( D' \) is ample if the combinatorial face structure of \( P' \) coincides with that of \( P \). When \( D' \) is merely nef, the combinatorial face structure of \( P' \) could be slightly degenerate.

Problem 3 If \( D' \) is nef, do we have the surjectivity of the canonical multiplication map

\[
H^0(X, \mathcal{O}_X(D)) \otimes \mathcal{O}_X(D') \rightarrow H^0(X, \mathcal{O}_X(D + D'))?
\]

We know that \( H^1(X, \mathcal{O}_X(D')) = 0 \) when \( D' \) is nef, hence

\[
H^1(X \times X, \mathcal{O}_{X \times X}(p_1^{-1}D + p_2^{-1}D')) = 0
\]

by K"unneth formula. Consequently, Problem 3 is equivalent to the following:

Problem 4 Let \( I \) be the \( \mathcal{O}_{X \times X} \)-ideal corresponding to the diagonal subvariety \( \Delta(X) \) of \( X \times X \). If \( D' \) is nef, do we have

\[
H^1(X \times X, I \otimes \mathcal{O}_{X \times X}(p_1^{-1}D_1 + p_2^{-1}D')) = 0?
\]
Hopefully, we might have an affirmative answer at least when $X$ is smooth and $D'$ is ample. There have been unsuccessful attempts in this direction by means of Frobenius splittings in characteristic $p > 0$.

Without assuming $X$ to be smooth nor $D'$ to be ample, let us give another formulation for the problem.

Let $N := \text{Hom}_Z(M, Z)$ with the canonical bilinear pairing $\langle \cdot, \cdot \rangle : M \times N \rightarrow Z$. Consider the finite complete fan $\Sigma$ for $N$ corresponding to $X$. As usual, denote by $\Sigma(1) = \{\rho_1, \rho_2, \ldots, \rho_l\}$ the set of one-dimensional cones in $\Sigma$, and let $n_j \in N$ be the primitive generator for $\rho_j \in \Sigma(1)$.

Let us introduce a free $Z$-module $\tilde{N} := \bigoplus_{j=1}^l Z\tilde{n}_j$ with the basis consisting of the symbols $\{\tilde{n}_1, \ldots, \tilde{n}_l\}$ corresponding to $\Sigma(1)$, and the $Z$-linear map

$$\pi : \tilde{N} \rightarrow N \quad \text{with} \quad \pi(\tilde{n}_j) := n_j$$

for $j = 1, \ldots, l$.

Let $\tilde{M} := \text{Hom}_Z(\tilde{N}, Z)$ with the dual basis $\{\tilde{m}_1, \ldots, \tilde{m}_l\}$. Since $\pi$ has finite cokernel, the dual $Z$-linear map

$$\pi^* : M \rightarrow \tilde{M} \quad \text{with} \quad \pi^*(m) := \sum_{j=1}^l \langle m, n_j \rangle \tilde{m}_j$$

for any $m \in M$ is injective. Let $\mathcal{M} := \text{coker}(\pi^*)$ and denote by $\mu_j \in \mathcal{M}$ the image of $\tilde{m}_j \in \tilde{M}$. We call $(\mathcal{M}, \{\mu_1, \ldots, \mu_l\})$ the linear Gale transform of $(N, \{n_1, \ldots, n_l\})$.

$\mathcal{M}$ is canonically isomorphic to the Weil divisor class group of $X$ (modulo linear equivalence). Let

$$\tilde{M} \supset \tilde{M}_{\geq 0} := \sum_{j=1}^l Z_{\geq 0} \tilde{m}_j \quad \text{and} \quad \mathcal{M} \supset \mathcal{M}_{\geq 0} := \sum_{j=1}^l Z_{\geq 0} \mu_j.$$

$\tilde{M}_{\geq 0}$ is canonically isomorphic to the semigroup of torus-invariant effective Weil divisors on $X$. For $j = 1, 2, \ldots, l$ we will use $D_j$ and $\tilde{m}_j$ interchangeably to denote the torus-invariant irreducible Weil divisor corresponding to the one-dimensional cone $\rho_j$.

The homogeneous coordinate ring introduced by Cox, Audin, Delzant, et al. (cf. [4]) is the semigroup algebra

$$S := \mathbb{C}[\tilde{M}_{\geq 0}] = \mathbb{C}[x_1, x_2, \ldots, x_l] \quad \text{with} \quad x_j := e(\tilde{m}_j) \in S \text{ for } j = 1, \ldots, l.$$

We endow the polynomial ring $S$ with the $(\mathcal{M}_{\geq 0})$-grading defined by

$$\deg x_j := \mu_j \quad \text{for } j = 1, \ldots, l.$$

For $\alpha \in \mathcal{M}$, we denote by $S_\alpha$ the homogeneous part of degree $\alpha$.

Note that the $(\mathcal{M}_{\geq 0})$-graded ring $S$ depends only on the 1-skeleton $\Sigma(1)$ of $\Sigma$. Problem 1 is more or less equivalent to the following:
Problem 5 Given $\alpha, \beta \in M_{\geq 0}$, when is the multiplication map

$$S_\alpha \otimes C S_\beta \rightarrow S_{\alpha+\beta}$$

surjective?

The fan $\Sigma$ determines the polyhedral cone

$$C \subset M_\mathbb{R} := M \otimes \mathbb{Z} \mathbb{R}$$

spanned by nef divisor classes. The intersection $M \cap C^\circ$ of $M$ with the interior $C^\circ$ of $C$ is the semigroup of ample divisor classes on $X$. Then Problems 3 and 4 are almost equivalent to the following:

Problem 6 Is the multiplication map $S_\alpha \otimes C S_\beta \rightarrow S_{\alpha+\beta}$ surjective if $\alpha \in M \cap C^\circ$ and $\beta \in M \cap C$? What if $\alpha, \beta \in M \cap C^\circ$?

The study of the diagonal ideal sheaf $I \subset O_{X \times X}$ is important not only in connection with Problem 4 but in its own right. In explaining a possible approach to the study, let us follow the notation of Cox [4].

We denote

$$x^D := \prod_{j=1}^l x_j^{a_j} \in S \quad \text{for} \quad D = \sum_{j=1}^l a_j \tilde{m}_j \in \tilde{M}_{\geq 0}$$

and

$$\deg x^D := \sum_{j=1}^l a_j \mu_j =: [D].$$

By our convention $D_j = \tilde{m}_j$, we have

$$x^{D_j} = x_j \quad \text{and} \quad [D_j] = \mu_j \quad \text{for} \quad j = 1, 2, \ldots, l.$$ 

For each $\alpha \in M$ we denote by $O_X(\alpha)$ the $O_X$-module corresponding to the degree-shifted graded $S$-module $S(\alpha)$. We also need the following notation later: For each $m \in M$ we denote the zero and polar divisors of the character $e(m)$ of the torus regarded as a rational function on $X$ by

$$D^+(m) := \sum_{1 \leq j \leq l, \langle m, n_j \rangle > 0} \langle m, n_j \rangle D_j \quad \text{and} \quad D^-(m) := \sum_{1 \leq j \leq l, \langle m, n_j \rangle < 0} (-\langle m, n_j \rangle) D_j,$$

hence $\pi^*(m) = D^+(m) - D^-(m)$.

We have a canonical homomorphism of $(M_{\geq 0} \times M_{\geq 0})$-graded $C$-algebras

$$S \otimes C S \rightarrow C[M_{\geq 0}] \otimes C S$$
defined by
\[ x^D \otimes x^E \mapsto e([D]) \otimes x^{D+E} \quad \text{for } D, E \in \tilde{M}_{\geq 0}. \]

The ideal \( I \subset O_{\Delta X} \) for the diagonal subvariety \( \Delta(X) \subset X \times X \) obviously corresponds to the \( (\mathcal{M}_{\geq 0} \times \mathcal{M}_{\geq 0}) \)-homogeneous ideal
\[ I := \ker(S \otimes C S \rightarrow C[\mathcal{M}_{\geq 0}] \otimes C S). \]

Problems 5 and 6 ask the surjectivity of the \((\alpha, \beta)\)-component
\[ S^\alpha \otimes C S^\beta \rightarrow e(\alpha) \otimes C S^{\alpha+\beta} \]
of this homomorphism under various conditions on \( \alpha, \beta \in \mathcal{M}_{\geq 0} \).

Identifying the logarithmic derivatives \( dx_j/x_j \) with \( \tilde{m}_j \) for \( j = 1, 2, \ldots, l \) as usual, we get a canonical injective homomorphism of graded \( S \)-modules
\[ \Omega_1^S \longrightarrow S \otimes \tilde{M}, \quad dx_j \mapsto x_j \otimes \tilde{m}_j \quad \text{for } j = 1, 2, \ldots, l. \]

Denote by
\[ \Omega := \ker(\Omega_1^S \rightarrow S \otimes \tilde{M} \rightarrow S \otimes \mathcal{M}) \]
the kernel of the composite of this homomorphism with the canonical projection \( S \otimes \tilde{M} \rightarrow S \otimes \mathcal{M} \). It is not hard to show that the sheaf \( \Omega_1^X \) of Zariski differential 1-forms (resp. \( \bigoplus_{j=1}^l O_X(-D_j) \)) is the \( O_X \)-module associated to the graded \( S \)-module \( \Omega \) (resp. \( \Omega_1^S \)). In this way, we get the following well-known result:

**Proposition 7** (Generalized Euler exact sequence. cf. Batyrev-Cox [1])

We have an exact sequence of \( O_X \)-modules
\[ 0 \rightarrow \Omega_1^X \rightarrow \bigoplus_{j=1}^l O_X(-D_j) \rightarrow O_X \otimes \mathcal{M} \rightarrow 0. \]

**Remark 8** The graded \( S \)-module \( \Omega \) is generated over \( S \) by
\[ \left\{ x^D dx^E - x^E dx^D \mid D, E \in \tilde{M}_{\geq 0}, \ D \sim E \right\}, \]
hence by
\[ \left\{ x^{D^+(m)} dx^{D^-(m)} - x^{D^-(m)} dx^{D^+(m)} \mid m \in \mathcal{M} \right\}. \]

The vectors \( n_1, n_2, \ldots, n_l \in \mathbb{N} \) give rise to an arrangement \( \mathcal{A} \) of hyperplanes \( \{n_j\}^\perp \subset M_\mathbb{R} \). A chamber \( \Gamma \) for \( \mathcal{A} \) is one of the top-dimensional polyhedral cones appearing in the partition of \( M_\mathbb{R} \) induced by the arrangement \( \mathcal{A} \). If we choose for each chamber \( \Gamma \) a set \( \Xi_\Gamma \) of generators of the semigroup \( M \cap \Gamma \), then \( \Omega \) is generated over \( S \) by
\[ \left\{ x^{D^+(m)} dx^{D^-(m)} - x^{D^-(m)} dx^{D^+(m)} \mid m \in \bigcup_{\text{chambers } \Gamma} \Xi_\Gamma \right\}. \]
Remark 9 As in Remark 8 we see that the “diagonal” ideal $I \subset S \otimes_C S$ is generated over $S \otimes_C S$ by

$$\left\{ x^D \otimes x^E - x^E \otimes x^D \mid D, E \in \tilde{M}_{\geq 0}, D \sim E \right\},$$

hence by

$$\left\{ x^{D^+(m)} \otimes x^{D^-(m)} - x^{D^-(m)} \otimes x^{D^+(m)} \mid m \in \bigcup_{\text{chambers } \Gamma} \Xi_\Gamma \right\}.$$ 

Example 10 When $X = \mathbb{P}^r$ is the projective space, the corresponding polytope $P$ is a unimodular simplex, and $l = r + 1$. We have $\mathcal{M} = \mathbb{Z}$, and $S_{\alpha} \otimes S_{\beta} \to S_{\alpha + \beta}$ in Problem 5 is obviously surjective for all $\alpha, \beta \in \mathcal{M}_{\geq 0}$. Nevertheless, the description of the diagonal ideal $I \subset \mathcal{O}_{\times \times X}$ is nontrivial. By Beilinson [2] we have an exact sequence

$$0 \to \mathcal{O}_X(-r) \boxtimes \Omega^r_X(r) \to \cdots \to \mathcal{O}_X(-j) \boxtimes \Omega^j_X(j) \to \cdots \to \mathcal{O}_X(-1) \boxtimes \Omega^1_X(1) \to I \to 0,$$

where $\mathcal{F} \boxtimes \mathcal{F}' := (p_1^*\mathcal{F}) \otimes_{\mathcal{O}_{\times \times X}} (p_2^*\mathcal{F}')$ is the external tensor product on $X \times X$ of $\mathcal{O}_X$-modules $\mathcal{F}$ and $\mathcal{F}'$. (Thanks are due to Miles Reid for pointing out this result to the author.) One way of proving this is to note that the $S \otimes_C S$-module homomorphism

$$S \otimes_C \Omega^1_S \longrightarrow S \otimes_C S, \quad 1 \otimes dx_j \mapsto x_j \otimes 1 \quad \text{for } j = 1, 2, \ldots, r + 1$$

induces a surjection $S \otimes_C \Omega \to I$. Thus the Koszul complex arising out of the corresponding $\mathcal{O}_{\times \times X}$-homomorphism (cf., e.g., Eisenbud [6])

$$\mathcal{O}_X(-1) \boxtimes \Omega^1_X(1) \longrightarrow \mathcal{O}_X \otimes X,$$

whose cokernel is $\mathcal{O}_{\Delta(X)}$, gives the exact sequence above.

Similarly, the Koszul complex arising out of the $S$-homomorphism $\Omega^1_S \to S$ which sends $dx_j$ to $x_j$ for $j = 1, 2, \ldots, r + 1$ is nothing but the complex $(\Omega^r_S, d)$. By Remark 8 we see that $\Omega$ is the image of $d : \Omega^2_S \to \Omega^1_S$. Consequently, we get an exact sequence of $\mathcal{O}_X$-modules

$$0 \to \mathcal{O}_X(-r - 1)^{\oplus_{r+1}C_{r+1}} \longrightarrow \cdots \to \mathcal{O}_X(-j)^{\oplus_{r+1}C_j} \longrightarrow \cdots \to \mathcal{O}_X(-2)^{\oplus_{r+1}C_2} \to \Omega^1_X \to 0,$$

where

$$r+1C_j := \binom{r+1}{j}$$

are the binomial coefficients.

Alternatively, we could use the Eagon-Northcott complex (cf. Eagon-Northcott [5], Kirby [8] and Eisenbud [6]) for the $2 \times 2$-minors of

$$\begin{pmatrix} x_1 \otimes 1 & x_2 \otimes 1 & \ldots & x_{r+1} \otimes 1 \\ 1 \otimes x_1 & 1 \otimes x_2 & \ldots & 1 \otimes x_{r+1} \end{pmatrix}$$
to get an exact sequence

$$0 \to \mathcal{E}_r \to \cdots \to \mathcal{E}_p \to \cdots \to \mathcal{E}_1 \to \mathcal{I} \to 0,$$

where

$$\mathcal{E}_p := \bigoplus_{j+k=p+1}^p \wedge \tilde{M} \otimes \mathcal{O}_{X \times X}(-j,-k)$$

with

$$\mathcal{O}_{X \times X}(\alpha, \beta) := \mathcal{O}_X(\alpha) \otimes \mathcal{O}_X(\beta) \text{ for } \alpha, \beta \in \mathcal{M} = \mathbb{Z}.$$

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