Multiparameter Integrable QFT’s with $N$ bosons

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We introduce a new family of integrable theories with $N$ bosons and $N$ freely adjustable mass parameters. These theories restrict in particular limits to the “generalized supersymmetric” sine-Gordon models, as well as to the flavor anisotropic chiral Gross Neveu models (studied recently by N. Andrei and collaborators). The scattering theory involves scalar particles that are no bound states, and bears an intriguing resemblance with the results of a sharp cut-off analysis of the Thirring model carried out by Korepin in (1980). Various physical applications are discussed. In particular, we demonstrate that our theories are the appropriate continuum limit of integrable quantum spin chains with mixtures of spins.
1. Introduction

A variety of low dimensional experimental condensed matter systems have been studied recently, that involve field theories with several bosons. Examples include tunneling in quantum wires, where two bosons are necessary to describe the charge and spin degrees of freedom of the electrons [1], tunneling between multiple edges in fractional quantum Hall devices [2], nanotubes and two-leg ladders [3], etc. Properties of interest in these systems are usually non perturbative, and only a few techniques are available to obtain quantitatively reliable results, mostly conformal invariance and integrability. The search for integrable quantum field theories with several bosons is thus of some importance.

The problem is, that besides the sine-Gordon model, most known integrable bosonic theories are of little practical use: they are usually of Toda type, and involve real exponential of fields, that usually do not appear in a condensed matter context. Some exceptions to this unsatisfactory situation are known: for instance, the double sine-Gordon model turns out to be exactly solvable for some values of the couplings [4], [5], [6], with potential applications to quantum wires. Also, the “generalized supersymmetric” extensions of the sine-Gordon model [7] can be rebosonized using standard bosonization formulas for the parafermions [8]. These theories are useful in the discussion of the multichannel Kondo model [9], [10], [11]; the $\mathcal{N} = 1$ supersymmetric sine-Gordon model also appears in the context of quantum wires [12].

In this paper, we point out that there is a simple, integrable family of theories extending the generalized supersymmetric sine-Gordon models, that involve $N$ bosons and have $N$ adjustable mass parameters. This family can be considered as an extension of the flavor anisotropic Gross Neveu models that have been studied in the last few years by N.Andrei and collaborators( mostly in the context of the channel anisotropic Kondo model [13], [14], [15]), to the case where an anisotropy is introduced both in the color and flavor sectors. The case where the color anisotropy is at the special “Toulouse” value is of special interest for applications to quantum wires or dissipative brownian motion [16].

The models are presented in section 2, where integrability is proven and various limiting cases discussed. The scattering theory is discussed in section 3. The “classical” limit is analyzed in section 4, providing a general check of our approach. In section 5, the relation with quantum spin chains involving several species of spins is discussed. Some applications to impurity problems are discussed in section 6. Some final remarks are collected in section 7. The appendix contains numerous details on the numerical treatment both of the perturbation theory and of the TBA.
2. Generalities

2.1. The integrable theories

We consider a system of $N$ chiral bosons with propagators

$$
< \phi_j(z) \phi_j(w) > = -2 \frac{N-1}{N} \ln(z - w)
$$

$$
< \phi_j(z) \phi_k(w) > = \frac{2}{N} \ln(z - w), \ j \neq k,
$$

and introduce the following fields:

$$
\Psi^{(j)} = \frac{1}{\sqrt{N}} \left( \sum_{k=1}^{N} \omega^{jk} e^{i\phi_k} \right),
$$

where $\omega = e^{2i\pi/N}$. These fields provide different realizations of the fundamental parafermion $[8]$ of $Z_N$ type. The bosonic fields $\phi_j$ are not independent (one can set indeed $\phi_N = -\phi_1 - \ldots - \phi_{N-1}$); they can be expressed in terms of $N - 1$ independent fields $\Phi_j$ obeying

$$
< \Phi_j(z) \Phi_k(w) > = -2 \delta_{jk} \ln(z - w),
$$

by the transformation

$$
\phi_j = e_j \bullet \Phi, j = 1, \ldots, N;
$$

where the $\bullet$ denotes scalar product and the $e_j$ are weights of the fundamental representation of $SU(N)$

Introduce one additional bosonic field, which has trivial contractions with the preceding ones, and obeys

$$
< \Phi(z) \Phi(w) > = -\frac{1}{4\pi} \ln(z - w).
$$

Consider then the action (we assume all the $a_j$ are real positive numbers)

$$
A = \frac{1}{2} \int dxdy \sum_{j=1}^{N-1} \left[ \partial_\mu \left( \Phi_j + \Phi_j \right) \right]^2 + \left[ \partial_\mu \left( \Phi + \Phi \right) \right]^2
$$

$$
+ \left( \sum_{j=1}^{N} a_j \Psi^{(j)}(z) \right) \left( \sum_{j=1}^{N} a_j \bar{\Psi}^{(j)}(\bar{z}) \right) e^{i\beta[\Phi(z) + \Phi(\bar{z})]} + \text{conjugate.}
$$

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2 That is, $e_1 = \Lambda_1$, $e_2 = \Lambda_2 - \Lambda_1$, $\ldots$, $e_{N-1} = \Lambda_{N-1} - \Lambda_{N-2}$, $e_N = -\Lambda_{N-1}$, $\Lambda_i$ the fundamental weights of $SU(N)$ and $\Phi$ the $(N - 1)$ dimensional vector of coordinates $\Phi_1, \ldots, \Phi_{N-1}$.
In the case where all the coefficients $a_j$ but one vanish, this is the action of the “generalized supersymmetric” sine-Gordon model, which is known to be integrable \[7\]. We claim that this only a particular case of a more general integrable model, given by (2.6).

To establish this result, we first observe that the fields $\Psi^{(j)}$ obey the short distance expansions

$$\left[\Psi^{(j)}\right]^\dagger(z)\Psi^{(j)}(w) \approx \frac{1}{(z-w)^{2(N-1)}} \left[1 + 2\frac{N+2}{N} (z-w)^2 T^{(j)}(w) + \ldots\right], \quad (2.7)$$

where

$$T^{(j)}(z) = \frac{1}{N+2} \left[-\frac{1}{2} \sum_{k=1}^{N} (\partial_z \phi_k)^2 + \sum_{k \neq l} \omega^{j(k-l)} e^{i(\phi_k-\phi_l)}\right],$$

and

$$\left[\Psi^{(j)}\right]^\dagger(z)\Psi^{(k)}(w) \approx \frac{1}{(z-w)^{2N-1}} \left[(z-w)J^{(jk)}(w) + \ldots\right], \quad (2.8)$$

where

$$J^{(jk)} = -\frac{i}{N} \sum_{k=1}^{N} \omega^{-(k-j)} \partial_z \phi_k$$

We can then prove integrability, following [18], by establishing the existence of non local conserved currents. Introduce

$$\mathcal{J}^{-}(z) = \left(\sum_{j=1}^{N} b_j \left[\Psi^{(j)}\right]^\dagger(z)\right) \exp\left(-i\frac{4\pi}{\beta} \frac{2}{N} \Phi(z)\right). \quad (2.9)$$

then the short distance expansion of this current with the first term in the action reads, for the chiral part,

$$\approx \frac{1}{(z-w)^2} \left(\sum_{j=1}^{N} b_j a_j \left[\Psi^{(j)}\right]^\dagger(z)\right) \left(\sum_{k=1}^{N} a_k \Psi^{(j)}(w)\right) \exp\left(-i\frac{4\pi}{\beta} \frac{2}{N} \Phi(z)\right) \exp\left(i\beta \Phi(w)\right)$$

$$\approx \frac{1}{(z-w)^2} \left(\sum_{k=1}^{N} b_k a_k + (z-w) \sum_{k \neq l} b_k a_l J^{(kl)}(w) + \ldots\right) \exp\left(-\frac{8i\pi}{\beta N} \Phi(z) + i\beta \Phi(w)\right).$$

The residue of the simple pole with thus be a total derivative iff the factor of $(z-w)$ in the first bracket vanishes. This is equivalent to the condition

$$\sum_{k \neq l} b_k a_l \left[\omega^{(k-l)m} - 1\right] = 0, m = 1, \ldots, N - 1$$
which always has solutions, since it is a system of $N - 1$ equations with $N$ unknowns.

The short distance expansion of this current with the chiral part of the second term in the action has a leading term that goes as $(z - w)^{-2/N}(z - w)^{2/N}$, and thus no simple pole. Following the standard argument, the expansion of $\mathcal{J}^-$ having a simple pole whose residue is a total derivative, the non local charge $\int \mathcal{J}^-$ is conserved to first order in the perturbation. For generic value of $\beta$, one can then argue that this is true to any order in perturbation theory, and, presumably, non perturbatively as well.

Another conserved current is easily found by complex conjugation:

$$\mathcal{J}^+(z) = \left( \sum_{j=1}^{N} b_j^* \Psi^{(j)}(z) \right) \exp \left( \frac{4\pi}{\beta} \frac{2}{N} \Phi(z) \right).$$  \hfill (2.10)

The conservation of $\mathcal{J}^\pm$ then ensures integrability [18].

2.2. The case $N = 2$

Let us discuss in more details the simplest example where $N = 2$. In that case, the parafermions are self-conjugated (up to a sign). We set

$$\psi^{(1)} = -i\sqrt{2} \sin \phi_1 = i\chi$$
$$\psi^{(2)} = \sqrt{2} \cos \phi_1 = \psi,$$

where $\psi$ and $\chi$ are (real) Majorana fermions. The perturbative part of the action reads then

$$(a\psi + ib\chi)(a\bar{\psi} + ib\bar{\chi})e^{i\beta\Phi} + (a\psi - ib\chi)(a\bar{\psi} - ib\bar{\chi})e^{-i\beta\Phi},$$  \hfill (2.12)

that is, regrouping terms

$$2 \left[ a^2\psi\bar{\psi} - b^2\chi\bar{\chi} \right] \cos \beta\Phi + 2ab \left( \psi\bar{\chi} + \chi\bar{\psi} \right) \sin \beta\Phi.$$  \hfill (2.13)

The non local conserved currents read then

$$\mathcal{J}^+(z) = (a\psi - ib\chi) \exp \left( \frac{4\pi}{\beta} \frac{2}{N} \Phi \right),$$
$$\mathcal{J}^-(z) = (a\psi + ib\chi) \exp \left( -\frac{4\pi}{\beta} \frac{2}{N} \Phi \right).$$  \hfill (2.14)

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3 The solution is easily expressed using the matrix $N \times (N - 1)$ matrix $M$ whose elements are $M_{jk} = a_{j+k-1}$ by $b_j$ equal to the $j^{th}$ cofactor.
If \( a = 0 \) or \( b = 0 \), the action reduces to the one of the supersymmetric sine-Gordon model (with an additional, decoupled, Majorana fermion). If \( a = b \), the combinations appearing in the action become Dirac fermions, \( \psi + i\chi = \sqrt{2}e^{i\phi_1} \). We can thus reexponentiate them, to write the perturbing term as \( a \cos(\beta\Phi + \phi_1) \), so the model is equivalent to a sine-Gordon model, at a coupling constant \( \beta' \) with 
\[
\frac{(\beta')^2}{8\pi} = \frac{\beta^2}{8\pi} + \frac{1}{2}.
\]

The currents \( J^\pm \) have a fractional spin \( s = \frac{1}{\gamma} \), where 
\[
\gamma = \frac{2\beta^2}{4\pi - \beta^2}.
\]

In the case \( a = 0 \) or \( b = 0 \), the currents are generators of the algebra \( \widehat{sl(2)}_q \) \([18]\), with deformation parameter \( q = -e^{-i\pi/\gamma} \). In the general case however, they do not form a closed algebra.

To understand the situation a little better, it is useful to go to the \( SU(2) \) symmetric point \( \beta^2 = 4\pi \). There are two underlying level one algebras, with generators 
\[
J_1^+ = \frac{\psi + i\chi}{\sqrt{2}} e^{i\sqrt{4\pi}\Phi},
J_1^- = \frac{\psi - i\chi}{\sqrt{2}} e^{-i\sqrt{4\pi}\Phi},
J_3^1 = i \left( -\psi\chi + \sqrt{4\pi}\partial\Phi \right)
\]

and 
\[
J_2^+ = \frac{\psi - i\chi}{\sqrt{2}} e^{i\sqrt{4\pi}\Phi},
J_2^- = \frac{\psi + i\chi}{\sqrt{2}} e^{-i\sqrt{4\pi\Phi}},
J_3^2 = -i \left( \psi\chi + \sqrt{4\pi}\partial\Phi \right),
\]

and all short distance expansions between operators of different algebras are non singular. The sum \( J_1 + J_2 \) provides a level two representation. For general \( a, b \), the currents can be written as combinations of the \( J_1 \) and \( J_2 \). Setting \( a = \mu + \lambda \) and \( b = \mu - \lambda \), we have 
\[
J^+ = \lambda J_1^+ + \mu J_2^+,
J^- = \lambda J_1^- + \mu J_2^-.
\]

This is suggestive of a system where two flavors of fermionic currents are combined in a flavor anisotropic fashion. Indeed, introduce new bosons defined by 
\[
-\phi_1 + \sqrt{4\pi\Phi} = \varphi_1,
\phi_1 + \sqrt{4\pi\Phi} = \varphi_2,
\]

5
the perturbing term is then proportional to
\[ \lambda^2 \cos(\varphi_1 + \varphi_1) + \mu^2 \cos(\varphi_2 + \varphi_2) + \lambda \mu \cos(\varphi_1 + \varphi_2) + \lambda \mu \cos(\varphi_1 + \varphi_2) \]

This is the abelian bosonized form of a Gross Neveu type interaction with flavor anisotropy
\[ (\lambda J_1^x + \mu J_2^x) \times (\lambda \bar{J}_1^x + \mu \bar{J}_2^x) + (x \to y) \]

The zz term is missing in this interaction - it is well known that this term is generated under renormalization [5]. Away from the SU(2) point, one can similarly consider our model as a color and flavor anisotropic chiral Gross Neveu model (upon bosonization, this model gives rise to 4 independent fields, but only 2 of them appear in the interaction due to chirality, the other 2 ones contributing free parts to the action).

2.3. Arbitrary N

The previous discussion easily extends to other values of \( N \). The currents \( J^\pm \) have a fractional spin \( s = \frac{1}{2} \), where
\[ \gamma = \frac{N \beta^2/8\pi}{\frac{1}{N} - \frac{\beta^2}{8\pi}}. \] (2.19)

In the case \( a = 0 \) or \( b = 0 \), the currents are generators of the algebra \( \hat{sl}(2)_q \) with deformation parameter \( q = -e^{-i\pi/\gamma} \). In the general case, they do not form a closed algebra. In the limit \( \beta^2 = 4\pi \), they can be expressed as combinations of \( N \) generators belonging to \( N \) different realizations of a level 1 \( SU(2) \) algebra. The rebosonized action is a Gross Neveu models with two colors and \( N \) flavors, and flavor anisotropy, with an interaction term of the form
\[ \left( \sum_{j=1}^{N} \lambda_j J_j^x \right) \times \left( \sum_{j=1}^{N} \lambda_j \bar{J}_j^x \right) + (x \to y) \]

where the anisotropy coefficients \( \lambda_j \) are related with the terms in the original action by
\[ \lambda_j = \sum_{k=1}^{N} \omega^{jk} a_k. \] (2.20)

That the Gross Neveu model with flavor anisotropy is integrable has been pointed out several years ago in fact, in [13],[14]. Integrability is established there by direct diagonalization of the bare hamiltonian together with “dynamical fusion”. An intriguing feature is that the proof presented in [14], strictly speaking, works only for the case \( N = 2 \) (and some
subcases of special flavor anisotropy for larger $N$). The reason is, that in the approach of \cite{14}, the bare particles must have a bare flavor scattering matrix that is a solution of the Yang Baxter equation. In the flavor isotropic case, this $S$ matrix is the standard $SU(N)$ R-matrix; but, as far as we know, there is no way to deform this R matrix in a non trivial way by introducing $N - 1$ independent anisotropic parameters - all available single and multiparameters quantum group approaches still explore a very small subset of all the possible flavor anisotropies. On the other hand, from the point of view we have adopted (that deals directly with the renormalized action), all flavor anisotropies play equivalent roles, and integrability appears generally true. The argument also extends straightforwardly to the case of color anisotropy, not considered in \cite{13}.

Of course, a particular choice of anisotropy is when one of the coefficients $\lambda_j$ vanishes exactly, in which case the modele reduces to one with $N - 1$ flavors. In the case of general $\beta$, the same condition, say $\lambda_N = 0$ leads, using formula (2.2), to a problem where the field $\phi_N$ has disappeared from the action. We are then left with a set of $N - 1$ fields satisfying

$$<\phi_j(z)\phi_j(w)> = -2\frac{N - 1}{N} \ln(z - w)$$

$$= -2\frac{N - 2}{N - 1} \ln(z - w) - \frac{2}{N(N - 1)} \ln(z - w)$$

$$<\phi_j(z)\phi_k(w)> = \frac{2}{N} \ln(z - w)$$

$$= \frac{2}{N - 1} \ln(z - w) - \frac{2}{N(N - 1)} \ln(z - w), j \neq k$$

We can then write $\phi_j = \phi_j' + \Phi'$ where there are $N - 1$ fields $\phi_j'$ satisfying relations similar to (2.21) but with the replacement $N \rightarrow N - 1$. The problem is then equivalent to the case $N \rightarrow N - 1$, but with a shift of the leftover exponential, $\frac{\beta^2}{8\pi} \rightarrow \frac{\beta^2}{8\pi} + \frac{1}{N(N - 1)}$.

3. Conjectured scattering theory

3.1. The thermodynamic Bethe ansatz

Though there are arguments based on symmetry to infer what the scattering theory should look like, the approach we use is to first conjecture a set of thermodynamic Bethe ansatz equations (TBA) to compute the free energy of the 1+1 quantum field theory associated with the action (2.6) at temperature $T$. We parametrize

$$\frac{\beta^2}{8\pi} = \frac{\gamma}{N(N + \gamma)},$$

(3.1)
and consider the case $\gamma$ an integer. Our conjecture is as follows. Introduce the TBA diagram

![TBA Diagram](image)

with incidence matrix $N_{jk}$ such that $N_{jk} = 1$ if the nodes $j$ and $k$ are connected, and 0 otherwise (in particular $N_{jj} = 0$). With this diagram, we associate the set of pseudo energies (one for each node) solution of the system ($R = 1/T$, $T$ the temperature)

$$
\epsilon_j = \sum_{k=1}^{N} \delta_{jk} m_k R \cosh \theta - \sum_{k} N_{jk} \int \frac{d\theta'}{2\pi} \frac{1}{\cosh (\theta - \theta')} \ln \left( 1 + e^{-\epsilon_k(\theta')} \right).
$$

(3.2)

The free energy reads then

$$
F = -\frac{T}{2\pi} \sum_{k=1}^{N} \int \frac{d\theta}{2\pi} m_k \cosh \theta \ln \left( 1 + e^{-\epsilon_k(\theta)} \right).
$$

(3.3)

In the foregoing equations, the $m_k$ are a set of masses which depend on the couplings $a_k$ in the bare action. By dimensional analysis, $[a_k] = [\text{length}]^{\frac{2}{M^2} - \frac{2}{N}}$. Therefore, we have

$$
m_k = G_k(a_1, \ldots, a_N) = a_1^{N-k+1} F_k(a_2/a_1, \ldots, a_N/a_1).
$$

(3.4)

The $G_k$ are homogeneous functions of the couplings $a_k$. Some properties of these functions are known beforehand of course. They are symmetric functions of their arguments. If all the $a_k$ but one vanish, we know that the problem becomes equivalent to the $N^{th}$ supersymmetric sine-Gordon model, and therefore, from known results [18], [19], all the masses but the $N^{th}$ one must vanish. This means that $G_j = 0$, $j = 1, \ldots, N - 1$, when all the $a_k$ but one vanish. Also, we know that the $N^{th}$ mass vanishes when one of the fields decouples, i.e, when one of the coefficients (2.20) vanishes. More generally, the masses $m_k \ldots m_N$ vanish when $N - k + 1$ of these coefficients vanish. We will get back to the determination of the functions $G_k$ below.

The evidence for the TBA comes first from the compatibility with all limiting cases. Moreover, the analysis of the $Y$ system [19] associated with it shows that the dimension of the UV perturbing operator is always the same as in the generalized supersymmetric
case (it does not depend on the number of massive nodes), ie 
\[ h = \frac{\gamma + N - 1}{\gamma + N} = \frac{\beta^2}{8\pi} + \frac{N - 1}{N} \]
as desired. Finally, the central charge, in the generic case when all the \( m_k \) are non zero
is simply equal to the number of massive nodes, ie \( c = N \). This is easily checked. Using
standard formulas, the central charge is expressed in terms of the solutions of the system
of \( h \) as \( T \to 0 \) and \( T \to \infty \). In the first case, the \( N \) first \( \epsilon \)'s are all infinite, the others
follow from
\[ x_{j} = e^{-\epsilon_{j}} = (j + 1)^2 - 1, \quad j = N + 1, \ldots, N + \gamma - 2 \]
\[ x_{N+\gamma-1} = x_{N+\gamma} = \gamma - 1. \]
In the second case, one has to solve the same system with more nodes, ie
\[ y_{j} = e^{-\epsilon_{j}} = (j + 1)^2 - 1, \quad j = 1, \ldots, N + \gamma - 2 \]
\[ y_{N+\gamma-1} = y_{N+\gamma} = N + \gamma - 1. \]
The central charge is then (here \( L \) designates the Euler dilogarithm) \[19\]
\[ c = \frac{6}{\pi^2} \sum L \left( \frac{y}{1+y} \right) - L \left( \frac{x}{1+x} \right). \tag{3.5} \]
For a \( D \) diagram, as \( T \to \infty \), the sum \( \frac{6}{\pi^2} \sum L \left( \frac{y}{1+y} \right) \) is equal to the number of nodes
minus one, so the central charge is simply, from \( L(1) = \frac{\pi^2}{6} \), equal to the number of massive
nodes, ie \( c = N \) indeed.

3.2. Scattering theory

The scattering theory associated with this TBA is very simple. One introduces a
set of \( N - 1 \) scalar massive particles with masses \( m_1, \ldots, m_{N-1} \). One also introduces a
pair soliton/antisoliton with masses \( m_N \). The latter scatter with the usual sine-Gordon S
matrix that corresponds to the quantum group parameter introduced above, \( q = -e^{-i\pi/\gamma} \)
- it is the same as the S matrix of an ordinary sine-Gordon model at coupling \( \frac{\beta^2}{8\pi} = \frac{\gamma}{\gamma + 1} \).
The scalar particle of label \( k \) scatters trivially with all particles, except the ones of label
\( k \pm 1 \), with which it scatters with the CDD factor \( S = i \tanh \left( \frac{\beta}{2} - i\frac{\pi}{4} \right) \). When \( k = N - 1 \),
the particle scatters with the soliton and antisoliton with the same CDD factor. It is
important to stress that the sine-Gordon S-matrix considered here has no poles in the
physical strip: the scalar particles are not bound states of the soliton and antisoliton.

Remarkably, a scattering theory built with similar ingredients appears in a paper by
Korepin \[20\]. There, the author discusses the Thirring model in the repulsive regime, using
a sharp cut-off regularization. For a coupling corresponding to a sine-Gordon parameter
\( \frac{\beta^2}{8\pi} \in \left[ \frac{l}{l+1}, \frac{l+1}{l+2} \right] \), he finds, in addition to the soliton and antisoliton, a spectrum made of \((l-1)\) neutral particles, with same S-matrices as ours, but where all the masses are uniquely determined as a function of \(l\) (in particular, the soliton mass becomes infinite when \( \frac{\beta^2}{8\pi} = \frac{l+1}{l+2} \)). The soliton and antisoliton scatter through a sine-gordon S matrix with a renormalized \( \left( \frac{\beta^2}{8\pi} \right)_l = \frac{1-l(l-\beta^2/8\pi)}{1-(l-1)(1-\beta^2/8\pi)} \) parameter.

The relation with our problem, if any, is not clear. It is usually admitted that the in
the repulsive regime of SG, the quantum theory must be defined with care, and depends on the cut-off. For smoother cut-offs, as well as XXZ type regularizations, the results of [20] are not supposed to hold, and the standard description of [21] with only the soliton and antisoliton to be correct.

Observe that the sine-Gordon part of the S matrix commutes with the quantum algebra \( \hat{sl}(2)_q \). While the non local conserved charges \( \mathcal{Q}^\pm \) have commutation relations that do not close, they can be expressed as combinations of \(N\) basic charges generating \( sl(2)_q \). A simple representation of the algebra generated by the \( \mathcal{Q}^\pm \) is then obtained by identifying all these charges - it is realized on the multiparticle soliton antisoliton states as in the usual sine-Gordon model [18]. In the general case, there are no other conserved charges. If \( N = 2 \) for instance, another, local, conserved charge appears when \( \text{eg} \ b = 0 \), since the local currents \( \mathcal{G} = \psi \partial \Phi \) is conserved [18]. But away from this value (and \( a = 0 \) of course), this is not true. We must thus complete the S matrix by a sector with no apparent symmetries, that does not spoil the Yang Baxter equation: besides “vertex” and “RSOS” type solutions, the only available choice is a set of particles with diagonal scattering. Requiring the central charge to be \( N \), and the TBA to restrict to the known ones when some of the couplings vanish (and additional symmetries appear) seems to leave no choice but our result.

Still another check comes from discussing the limit \( \beta \to 0 \), to which we turn now.

4. The “classical” limit.

We call here classical the limit where the sine-Gordon part of the scattering theory becomes identical with the classical one, that is \( \beta \to 0, \gamma \to 0 \).
4.1. The TBA at the reflectionless points

So far we wrote a TBA in the simplest case \( \gamma \) an integer, for which \( \frac{\beta^2}{8\pi} \geq \frac{1}{N(N+1)} \), corresponding to the SG component of the S-matrix being in the repulsive regime. To approach \( \beta = 0 \), we consider instead the cases \( \gamma = \frac{1}{n} \), that is \( \frac{\beta^2}{8\pi} = \frac{1}{N(nN+1)} \): the SG S-matrix is now in the attractive regime, at the so called reflectionless points. This means that the spectrum has to be completed by the bound states of solitons and antisolitons, the \( n - 1 \) breathers of masses \( 2m \sin \left( \frac{j \pi}{2n} \right) \), where \( m \) is the soliton mass. We denote \( 2m_1 \sin \left( \frac{\pi}{2n} \right), \ldots, 2m_{N-1} \sin \left( \frac{\pi}{2n} \right) \) the masses of the scalar particles (recall these are not bound states; the factor \( 2 \sin \left( \frac{\pi}{2n} \right) \) in their masses is introduced for convenience only). By building the complete scattering theory using bootstrap and fusion, one finds the following TBA equations.

One first has equations for the right hand side of the diagram, that look like the standard ones for the attractive regime of sine-Gordon

\[
\epsilon_j = \sum_k N_{jk} K * \ln \left( 1 + e^{\epsilon_k} \right), N + 1 \leq j \leq N + n. \tag{4.1}
\]

There is then a central part involving the nodes \( N - 1 \) and \( N \), with

\[
\epsilon_N = K * \left[ \ln \left( 1 + e^{\epsilon_{N+1}} \right) - \ln \left( 1 + e^{-\epsilon_N} \right) \right], \tag{4.2}
\]

and

\[
\epsilon_{N-1} = \left[ 2m_{N-1} \cos \left( \frac{\pi}{2n} \right) + m \right] \tan \left( \frac{\pi}{2n} \right) \frac{\cosh \theta}{T} \\
- K * \ln \left( 1 + e^{\epsilon_N} \right) - K'' * \left( 1 + e^{-\epsilon_{N-1}} \right) - K' * \left( 1 + e^{-\epsilon_{N-2}} \right). \tag{4.3}
\]

Finally, the left hand side of the diagram looks in turn like the usual repulsive one

\[
\epsilon_j = 2m_j \sin \left( \frac{\pi}{2n} \right) \frac{\cosh \theta}{T} - \sum_k N_{jk} K' * \left( 1 + e^{-\epsilon_k} \right), j \leq N - 2. \tag{4.4}
\]

These equations can be conveniently encoded in the diagram
The asymptotic conditions for the attractive part are

\[ \epsilon_j \approx 2m \sin \left( \frac{(j - N + 1)\pi}{2n} \right) \frac{\cosh \theta}{T}, \quad N \leq j \leq n + N - 2 \]

\[ \epsilon_{n+N-1} \approx \epsilon_{n+N} \approx m \frac{\cosh \theta}{T}. \]

Introducing the Fourier transform \( g(\theta) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega \theta} \tilde{g}(\omega) \), one has \( \tilde{K} = \frac{1}{2 \cosh(\pi \omega/2n)} \) (the standard attractive kernel), \( \tilde{K}' = \frac{1}{2 \cosh(\pi \omega/2)} \) (the standard repulsive kernel), \( \tilde{K}'' = \frac{\cosh(n-1)\pi \omega/2n}{2 \cosh(n \omega/2n \cosh \pi \omega/2)} \).

For \( T \to 0 \), we have \( x_j = 0 \), since now all nodes are massive. For \( T \to \infty \), we have

\[ e^{-\epsilon_j} = (j + 1)^2 - 1, \quad j \leq N - 1 \]

\[ e^{\epsilon_j} = \left( j - N + 1 + \frac{1}{N} \right)^2 - 1, \quad N \leq j \leq N + n - 2 \]

\[ e^{\epsilon_{N+n-1}} = e^{\epsilon_{N+n}} = n + \frac{1}{N} - 1. \]

The central charge is thus

\[ c = \frac{6}{\pi^2} \left\{ 2L \left( \frac{1}{n + \frac{1}{N}} \right) + \sum_{j=1}^{n-1} L \left[ \frac{1}{(j + \frac{1}{N})^2} \right] + \sum_{j=1}^{N-1} L \left[ 1 - \frac{1}{(j + 1)^2} \right] \right\} = N. \]  

4.2. The classical limit

The “classical limit” is obtained by letting \( \beta \to 0 \). In our TBA, this means \( n \to \infty \). To get non trivial results then, we scale the soliton mass \( m \) with \( n \), so the mass of the first breather remains finite. Similarly, we assume that the parameters \( m_j \) are also scaled with \( n \), so that the masses of the scalar particles remain finite. Our computation follows the general strategy of [22], [23]. In that limit, it is convenient to introduce the new notation \( \kappa_j = \epsilon_{j+N-1}, \quad j \geq 1 \). When \( n \to \infty \), the kernels \( K \) and \( K'' \) become delta functions, (we set \( K' = s \)) and one finds the general solution

\[ e^{\kappa_j} + 1 = \left( \frac{a A^j - a^{-1} A^{-j}}{A - A^{-1}} \right)^2 \]

The constant \( A \) follows from the knowledge of mass terms, \( A = e^{C/2T}, C = \frac{m \pi}{n} \cosh \theta \), while the constant \( a \) depends on \( \epsilon_{N-1} \):

\[ 1 + e^{-\epsilon_{N-1}} = \left( \frac{A - A^{-1}}{a - a^{-1}} \right)^2 \]
In that limit, the equation satisfied by $\epsilon_{N-1}$ is

$$\epsilon_{N-1} = -s \ln (1 + e^{-\epsilon_{N-2}^2}) - \frac{1}{2} \ln (1 + e^{\kappa_1}) - \frac{1}{2} \ln (1 + e^{-\epsilon_{N-1}}) + \left( \Lambda + \frac{1}{2} \right) \frac{C}{T}$$

where $\Lambda = \frac{m_{N-1}}{m}$.

Let us then define

$$\epsilon'_{N-1} = s \ln (1 + e^{\epsilon_{N-2}^2}) + \Lambda \frac{C}{T}.$$  

By simple algebra, one finds that the following holds

$$1 + e^{-\epsilon_{N-1}} = (1 + e^{-\epsilon_{N}})(1 + e^{-\epsilon'_{N-1}})$$

together with

$$\epsilon_N = (\Lambda + 1) \frac{C}{T} - s \ln (1 + e^{-\epsilon_{N-2}^2})$$

We can thus trade completely the right hand side of the diagram for an additional node, and get the equations (recall $s = K', \tilde{s} = \frac{1}{2 \cosh(\pi \omega/2)}$)

$$\epsilon_j = \frac{M_j \cosh \theta}{T} - \sum_k N_{jk} s \ln (1 + e^{-\epsilon_k}), \quad j = 1, \ldots, N,$$  

(4.8)  

where the TBA diagram is a D diagram with N nodes

\[ \begin{array}{cccccc}
    1 & 2 & \cdots & N-3 & N-1 \\
    \otimes & \otimes & \cdots & \otimes & \otimes \\
    & N-2 & \otimes & N & \end{array} \]

and masses, in that limit are $M_1 = m_1 \frac{\pi}{n}, \ldots, M_{N-1} = m_{N-1} \frac{\pi}{n}$, and $M_N = M_{N-1} + m \frac{\pi}{n}$.

We can now compute the free energy. Its general expression is

$$\frac{F}{T} = -\sum_{j=1}^{\infty} j \int \frac{d\theta}{2\pi} C(\theta) \ln (1 + e^{-\kappa_j}) - \sum_{j=1}^{N-1} \frac{m_j}{m} \int \frac{d\theta}{2\pi} C'(\theta) \ln (1 + e^{-\epsilon_j}).$$  

(4.9)  

By using the basic TBA equation

$$\kappa_j \approx \frac{jC}{T} + 2 \sum_{k=1}^{\infty} k \ln (1 + e^{-\kappa_k}) - \ln (1 + e^{-\epsilon_{N-1}}),$$

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one finds after a few simple manipulations

\[
\frac{F}{T} = - \sum_{j=1}^{N} \int \frac{d\theta}{2\pi} M_j \cosh \theta \ln \left( 1 + e^{-\epsilon_j} \right) + \int \frac{d\theta}{2\pi} m \cosh(\theta) \ln \left( 1 - e^{-M \cosh \theta / T} \right). \tag{4.10}
\]

Here, the new mass \( M_N = M_{N-1} + \frac{m\pi}{n} \) as before, and \( M = \frac{m\pi}{n} \).

The results of the classical limit are therefore equations (4.8) and (4.10). This means, the classical limit is made up of \( N \) particles scattering in a non trivial way, plus a decoupled free boson.

In the particular case of the generalized supersymmetric sine-Gordon model, all the masses \( M_j \) but the \( N^{th} \) one vanish: the system (4.8) reproduces the well known TBA for \( Z_N \) field theories perturbed by the parafermion field [24], as is expected from letting \( \beta \to 0 \) in the action.

In the particular case \( N = 2 \), there is no node \( N - 2 \), and the TBA system is trivial. The system decouples into two free fermions of masses \( \frac{m_1\pi}{n} \) and \( (m_1 + m) \frac{\pi}{n} \), and a free boson of mass \( \frac{m\pi}{n} \). This is again expected from the action, and a rather non trivial check from the point of view of the TBA. The mass for the free boson certainly arises from counter terms analogous to ones arising in the \( \mathcal{N} = 1 \) supersymmetric action [25] (for more discussion about this, see next section). It also follows that, in the classical limit, the correspondence between masses in the TBA and bare couplings goes, assuming \( a^2 \leq b^2 \), as

\[
m_1 \propto a^2
\]

\[
m \propto (b^2 - a^2), \tag{4.11}
\]

where \( m \) is the mass of the first breather, and \( m_1 \) the mass of the scalar particle. In the more general case, the decoupled free boson presumably gets its mass from a counter term still analogous to what happens in the supersymmetric case [23]. The rest of the TBA corresponds to a non trivial theory, with \( N \) species of \( Z_N \) parafermions interacting.

It is interesting to discuss in more details the case \( N = 3 \). There, the TBA is based on the diagram

\[
\begin{array}{c}
1 \quad 2 \quad 3 \\
\bigotimes & \bigotimes & \bigotimes
\end{array}
\]

with central charge \( c = 2 \). Observe that this value can be obtained by \( 2 = \frac{1}{2} + \frac{7}{10} + \frac{4}{5} \), corresponding to the sum of the central charges for the Ising, tricritical and tetracritical Ising model (or, alternatively, the Ising and tricritical Ising models, and the 3 state Potts
model). Remarkably, the weight \( h = \frac{2}{3} \) of our three parafermions \( \Psi^{(j)} \) can be recovered by using fields of these minimal conformal field theories. The three fields \( \Phi^{(4/5)}_{13}, \Phi^{(7/10)}_{13} \Phi^{(4/5)}_{33} \) and \( \Phi^{(1/2)}_{13} \Phi^{(7/10)}_{33} \Phi^{(4/5)}_{33} \) (where lower labels are labels of the Kac table, upper labels are the central charges) do have conformal weights \( h = \frac{2}{3} \). One might be tempted to infer that the foregoing TBA also describes the perturbation of this product of three theories by a combination of these three fields. This cannot be true however: it is easy to check that the operator algebra of these three fields cannot be reproduced by using our parafermion fields only, due for instance to the appearance of powers \( 1/15 \) and \( 19/15 \). More generally, a TBA like

\[
\begin{array}{ccc}
1 & 2 & M \\
\otimes & \otimes & - \otimes
\end{array}
\]

has a central charge that can be written as \( c = c_1 + c_2 + \ldots c_M \), where \( c_M = 1 - \frac{6}{(M+2)(M+3)} \). The conformal weight of the perturbing operator is \( h = \frac{M-1}{M} \), which can be reproduced by \( M \) fields of the form \( \Phi_{13}^{(c_m)} \Phi_{33}^{(c_{m+1})} \ldots \Phi_{33}^{(c_M)} \), \( m = 1, \ldots, M \). While it is tempting to speculate that the TBA does describe the product of \( M \) minimal models perturbed by a combination of these fields, this result does not seem to be true.

5. Numerical check

To check the validity of the TBA besides the qualitative features we just discussed, one needs to compare the result for the free energy (3.3) to perturbative computations. We restrict here to the simplest case \( N = 2 \). Because the action (2.13) has two free parameters, reflected in the existence of the two masses \( m_1 \) and \( m_2 \), a full consistence check requires at least going to the \( 6^{th} \) order (odd orders vanish) in perturbation - a really complicated task, as discussed in more details in the appendix. The second order does not give any check but fixes a global scale. The fourth order does contain some information: consistency determines uniquely the relation between the masses and \( a, b \). In fact, it is not obvious a priori that this relation will be physical: finding it involves solving some quadratic equations whose solutions might well be complex, establishing, in fact, that the TBA is not the right one. We have however always found solutions that are physical, indicating at least that the TBA is consistent to that order. Moreover, the general shape of the functions \( G \) we obtain can be argued to be the right one based on limiting cases, giving us some confidence in the TBA indeed.
As an example let us consider the first non trivial attractive reflectionless case, which corresponds to $\frac{\beta^2}{8\pi} = \frac{1}{10}$. We have numerically solved the TBA equations (4.1), (4.2), (4.3) with $N = 2$, $n = 2$ for different values of the mass ratio $m/m_1$. The coefficients of the analytic expansion of the running central charge

$$C(r, m/m_1) = 2 + Br^2 + \sum_{k=1}^{\infty} a_{2k} r^{k(2-g)}$$ (5.1)

are determined by fitting. As a check, the bulk terms of the (extrapolated) limiting cases $m/m_1 = 0$ and $m_1/m = 0$, i.e. the sine-Gordon and the supersymmetric sine-Gordon points, agree with the exact values $B = 3/\pi$ and $B = 0$ within an accuracy of 0.1%. In Fig.1 we give the result for the adimensional ratio of coefficients

$$I = 12 \left\{ \frac{\Gamma(1/2 + \frac{\beta^2}{8\pi})^2}{\Gamma(1/2 - \frac{\beta^2}{8\pi})} \frac{\Gamma(-\frac{\beta^2}{4\pi})}{\Gamma(1 + \frac{\beta^2}{4\pi})} \right\}^2 \frac{a_4}{a_2^2}$$ (5.2)

as a function of the mass ratio $m/m_1$.

![Fig. 1: TBA result for the ratio I](image)

This curve has to be compared with the same universal ratio determined in the appendix with perturbation theory as a function of $x = (k_-/k_+)^2$, shown in Fig.2.

The limiting cases are the sine-Gordon model at $x = 0$ and the supersymmetric sine-Gordon model at $x = 1$. 

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Fig. 2: Perturbative result for ratio $I$

Fig. 3: “Quasi-classical” behaviour of the ratio of coupling constants

The value of the universal ratio $I$ at the minimum point found with perturbation theory is $I_{\text{min}}^{\text{pert}} = -2.374$ in good agreement with the TBA value $I_{\text{min}}^{\text{TBA}} = -2.366$. Another important check is the value of the universal ratio at the supersymmetric sine-Gordon point: the perturbative result is $I_{\text{susy}}^{\text{pert}} = 29.45$ while the TBA value is $I_{\text{susy}}^{\text{TBA}} = 29.49$,
confirming therefore the standard analysis\footnote{That is, no counter term is necessary to make the action supersymmetric and integrable away from $\beta = 0$.} of this point.

By solving the second order equation in $(k_+/k_-)^2$ one can extract the dependence of the coupling constants on the mass ratio. In Fig. 3 we show that the quantity $R = (b^2 - a^2)/a^2$ is still, for our value $\beta^2/8\pi = 1/10$, almost a linear function of $m/m_1$, like in the classical limit $\beta^2/8\pi \to 0$ where $R = m/m_1$; moreover, the slope is very close to its classical value of $1/2$.

6. Relations with multispin integrable lattice models

The conjectured TBA appears very naturally in an a priori different context: the study of inhomogeneous integrable lattice models of XXZ type with a mixture of different representations. To explain this in a concise manner, we refer the reader to \cite{26}, and use similar notations (though the matter is quite standard). We consider thus an integrable model based on $sl_{q_0}(2)$ R-matrices, whose “vertical space” is an array with spins $s_1 = s_2 = 1, s_3 = s_4 = 2, \ldots, s_{2N-1} = s_{2N} = N$ and $s_{2N+i} = s_i$ otherwise, ie made of blocks representing the $N$ first values of $SU(2)$ spin. The associated spectral parameters alternate $u_1 = -u_2 = i\Lambda + \lambda_1/2, \ldots, u_{2N-1} = -u_{2N} = i\Lambda + \lambda_N/2$, and $u_{2N+i} = u_i$ otherwise. The anisotropy is determined by the quantum group parameter $q_0 = \exp\left(\frac{i\pi}{\gamma+\Lambda}\right)$. For hamiltonian we chose

$$H = -\frac{1}{t} \frac{d}{du} \ln \left\{ t^1 \left( i\frac{\Lambda + \lambda_1}{2} + u \right) \left[ t^1 \left( -i\frac{\Lambda + \lambda_1}{2} - u \right) \right]^{-1} \right. \right.$$  

$$\left. \ldots t^N \left( i\frac{\Lambda + \lambda_N}{2} + u \right) \left[ t^N \left( -i\frac{\Lambda + \lambda_N}{2} - u \right) \right]^{-1} \right\} \bigg|_{u=0}. \tag{6.1}$$

Here, $t^u$ denotes the transfer matrix based on the foregoing vertical space and a “horizontal space” is a representation of spin $s$. The whole geometry can be illustrated on the following picture

\begin{center}
\begin{tikzpicture}[scale=0.8]
  \draw (0,0) -- (4,0);
  \draw (0,1) -- (4,1);
  \node at (0,1.5) {$u_1 \ldots$};
  \node at (4,1.5) {$u_j \ldots$};
  \node at (2,1) {$u$};
  \node at (0,0) {$s_1 \ldots$};
  \node at (4,0) {$s_j \ldots$};
  \node at (2,0) {$s$};
\end{tikzpicture}
\end{center}
In the case of an array with a single type of spin $j$, the physical equations are well known to be encoded in a TBA identical to what we studied above for $\beta^2 = \frac{2\gamma}{N(\gamma+N)}$, but with a mass term on node $j$ only. In this more general case, it is straightforward to show that one gets now a mass term for each of the $N$ first nodes, with masses

$$m_j = M \exp \left(-\frac{(\gamma+N)}{2}\lambda_j\right),$$

where $M = \frac{4}{\Lambda} \exp\left(-\frac{(\gamma+N)}{2}\Lambda\right)$.

By taking the continuum limit, this local integrable lattice model will give rise to an integrable quantum field theory that has exactly the TBA conjectured in section 3. This indicates that this TBA is more than an abstract set of equations, but must be related to a genuine quantum field theory: we conjecture this field theory is nothing but \(2.6\).

It is interesting to observe that the same TBA would also be obtained by choosing a uniform spectral parameter (eg all $\lambda_j = 0$) but by putting different amounts of the various spins, with densities proportional to the masses $m_j$; see [27] for more details on this approach. In the latter reference in particular, S matrices are directly derived from the lattice regularization, and agree with the results of section 3. In the context of lattice models, central charges have also been computed with TBA similar to ours [28],[29].

7. Impurity problems

The same argument of perturbative integrability carries through in the case of impurity problems [30]. Consider thus the problem with free bosons $\Phi_i, \Phi$ in the bulk, and an interaction term at the boundary

$$H_{bd} = \left(\sum_{j=1}^N a_j \Psi^{(j)}(0)\right) S^{-} e^{i\beta\Phi(0)} + \text{conjugate.}$$

Here, $S^\pm$ are raising and lowering operators in a spin $j^\mathbb{R}$ representation of the quantum group $sl(2)_{q_0}$ (here the deformation parameter is not the same than the one appearing in the S-matrices earlier, but rather the one of the lattice model above, $q_0 = \exp\left(i\frac{\pi}{\gamma+N}\right)$). Note that only the right moving part of the fields appears in the action, but that the bosons $\phi, \Phi$ all have Neumann boundary conditions, ie their right and left moving components are proportional.

5 In our conventions, the fundamental has spin one.
identical at the boundary. Alternatively, one could thus express the boundary perturbation with the total fields, and exponentials of half the argument. A particular case is where the boundary spin is in a “cyclic” representation, and can be gauged away [31], [32]. One finds then

\[ H_{\text{bdr}} = \sum_{j=1}^{N} \lambda_j \exp(i(\phi_j + \beta\Phi)) + \text{conjugate}, \]  

(7.2)

with \( \lambda_j \) defined in (2.20).

In the simplest case of the repulsive regime, and for \( \gamma \) an integer, the boundary free energy for spin \( j \) with \( j \leq \gamma + N - 2 \) reads, as in the usual anisotropic Kondo problem,

\[ F_{\text{bdr}} = -T \int \frac{d\theta}{2\pi} \frac{1}{\cosh(\theta - \theta_B)} \ln \left(1 + e^{-\epsilon_j}\right). \]  

(7.3)

Here, the \( \epsilon_j \) are obtained by solving the TBA system (3.2) in the massless limit; this means, one sends the masses to zero and the rapidities to \( \infty \), such that only right moving particles with dispersion relation \( e = p \) remain. In the TBA, the source terms are obtained by the simple substitution \( \cosh \theta \to \frac{1}{2}e^{\theta} \). The “masses” are simply parameters with the physical dimension of an inverse length. The rapidity \( \theta_B \) is such that \( m_1 e^{\theta_B} \propto a_1^{\gamma + \gamma} \). For the case of the cyclic spin, the free energy reads as (7.3) but with \( j = \gamma + N \).

It is especially interesting to consider the free energy in the classical limit \(^6\). There, the same formula (7.3) holds for spin \( j \leq N - 2 \) and the TBA now given by (4.8). For the spin \( j = N - 1 \), we have, due to some of the foregoing changes of variables,

\[ F_{\text{bdr}} = -T \int \frac{d\theta}{2\pi} \frac{1}{\cosh(\theta - \theta_B)} \left[ \ln \left(1 + e^{-\epsilon_{N-1}}\right) + \ln \left(1 + e^{-\epsilon_{N}}\right) \right]. \]  

(7.4)

Finally, for the case of cyclic boundary spin.

\[ F_{\text{bdr}} = -T \int \frac{d\theta}{2\pi} \frac{1}{\cosh(\theta - \theta_B)} \ln \left(1 + e^{-\epsilon_{N}}\right), \]  

(7.5)

(in the last formulas, we have subtracted the trivial boundary free energy of the free boson \( \Phi \)).

In the case \( N = 2 \), the problem has been studied in [34]. Take an impurity of spin 1/2, and set \( S^+ = d^+, S^- = d \). Introducing the Dirac fermion \( \Upsilon = \psi + i\chi \), the boundary action reads

\[ \frac{a-b}{2} (\Upsilon^+ d^+ + \Upsilon d) + \frac{a+b}{2} (\Upsilon d^+ + \Upsilon^+ d) \]

\(^6\) This is usually called the “Toulouse” limit in the context of impurity problems [33].
According to (7.4), the free energy in the spin 1/2 case (ie \( j = 1 \) in our notations), reads, using notations of section 2,

\[
F_{bdr} = -T \int \frac{d\theta}{2\pi} \left[ \frac{1}{\cosh(\theta - \theta_B)} + \frac{1}{\cosh(\theta - \theta'_B)} \right] \ln \left( 1 + e^{-e^\theta/T} \right),
\]

where we have made a shift of the variable of integration, and thus \( e^{\theta_B}/e^{\theta'_B} \propto a^2 b^2 \). In terms of the \( \lambda, \mu \) variables describing the couplings to the different channels, this reads \( e^{\theta_B}/e^{\theta'_B} \propto \left( \frac{\lambda - \mu}{\lambda + \mu} \right)^2 \), in agreement with results of [34].

For general \( N \), and when all the masses \( m_k \) but \( m_N \) vanish (the standard generalized supersymmetric case) we obtain from (7.3) the ratio of degeneracy factors in the UV and IR \( \frac{g_{UV}}{g_{IR}} = \sqrt{N} \), corresponding to a flow from free to fixed boundary conditions in the \( Z_N \) model. In general, we have here the solution of a multiboson problem with arbitrary couplings; when \( N = 3 \) for instance, this means that the problem with boundary perturbation

\[
\lambda \cos \phi_1 + \mu ( \cos \phi_2 + \cos \phi_3)
\]

is integrable. Rexpressing the bosons \( \phi \) in terms of the independent bosons \( \Phi \), we obtain a two boson problem that is nothing but a quantum wire problem: see [16] for more details.

8. Conclusions

We feel there is more to understand in the theories we have addressed. The relation with the bare Bethe ansatz solutions of [13], [14] in the color isotropic case is poorly understood, as discussed in the text. The scattering theory we have proposed is rather mysterious, and we have not answered the question of what the scalar particles have to do with the flavor symmetry breaking. The problem of analytically determining the relation between the action parameters \( a_j \) and the masses \( m_j \) remains in general open. Finally, we have restricted to positive coefficients \( a_j \) in the problem, but it is clear that the perturbations will not always be massive if we allow some of these coefficients to be negative: where the theories flow to in that case is also an open question.

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9. Appendix

As a check of the correctness of our solution we now compute with conformal perturbation theory the free energy for the case \( N = 2 \). The result has to be compared with the free energy obtained by numerically solving the TBA equations. Let’s consider the system in the strip geometry \((R, L)\) defined by the action

\[
A = A_{\text{cft}} - \int_{\text{strip}} \Phi_{\text{int}},
\]

where the interaction is given by (2.6). The dimensionless running central charge

\[
C(R, a, b) = \lim_{L \to \infty} \frac{6R}{\pi L} \ln Z[R, L] = \frac{6R^2}{\pi} F
\]

becomes in perturbation theory

\[
C(R, a, b) = c_{UV} + 12 \lim_{L \to \infty} \frac{R}{2\pi L} \sum_{k=2}^{\infty} \frac{1}{k!} \int_{\text{strip}} d^2 w_1 \cdots d^2 w_k < \Phi_{\text{int}}(w_1) \cdots \Phi_{\text{int}}(w_k) >^c.
\]

The first correction \( k = 2 \) is ultraviolet divergent for any real value of \( \beta^2 \), the anomalous dimension of the perturbing operator being \( \Delta = 1/2 + \frac{\beta^2}{8\pi} \). Whatever regularization we choose, for example a radially ordered one [35], we obtain a divergent part, to be subtracted by a constant counterterm in the lagrangian, and a universal part. The counterterm contains a possible finite contribution giving rise to a non-universal bulk term which has to be fixed by a normalization condition. This condition is \( C(R, a, b) \to 0 \) when \( R \to \infty \).

In practice the bulk term will be determined by comparison with the TBA result, while in integrable theories with only one mass scale it can be computed analytically. Taking into account the first non trivial correction, the running central charge reads

\[
C(R, a, b) = c_{UV} + c_{\text{bulk}}(R, a, b) +
\]

\[
6(2\pi)^{\frac{\beta^2}{2\pi}} \left[ \frac{\Gamma(1/2 + \frac{\beta^2}{8\pi})}{\Gamma(1/2 - \frac{\beta^2}{8\pi})} \right]^2 \Gamma\left(1 + \frac{k_+}{4\pi}\right) \left[ \frac{a^2 + b^2}{m/m_1} \right] \left[ \frac{r^2 - \beta^2}{2\pi} \right].
\]

The theory depends on the pair of massive coupling constants \((a, b)\), or alternatively on the two masses \((m_1, m)\). Since we are going to compare our perturbative expansion with the TBA result it is useful to introduce the dimensionless coupling constants \((k_+, k_-)\) and to make explicit the dependence on the mass ratio: \( a^2 + b^2 = k_+(m/m_1) m^1 - \frac{\beta^2}{4\pi} \) and \( a^2 - b^2 = k_-(m/m_1) m^1 - \frac{\beta^2}{4\pi} \). Defining the dimensionless quantity \( r = mR \), the second order running central charge becomes

\[
C(r, m/m_1) = 2 + c_{\text{bulk}}(m/m_1) r^2 +
\]

\[
6 \left(2\pi\right)^{\frac{\beta^2}{2\pi}} \left[ \frac{\Gamma(1/2 + \frac{\beta^2}{8\pi})}{\Gamma(1/2 - \frac{\beta^2}{8\pi})} \right]^2 \Gamma\left(1 + \frac{k_+}{4\pi}\right) \left[ \frac{k_+(m/m_1)}{m^1 - \frac{\beta^2}{4\pi}} \right] r^2 - \frac{\beta^2}{2\pi}.
\]
Let’s consider the case $\beta^2 < 2\pi$, which includes the attractive regime. The only UV divergence of perturbation theory is the one that occurs at second order. All the other perturbative contributions are UV finite. The third order correction is zero because the unperturbed three points correlation function of the interaction is zero by charge neutrality. In order to compute the fourth order correction we map the $u$-strip onto the $z$-plane, $z = \exp(i2\pi w/R)$, and we express by Wick theorem the unperturbed four points correlation function

$$G = 2 \left[ \frac{(a^2 - b^2)^2}{z_{12}z_{34}} - \frac{(a^2 + b^2)^2}{z_{13}z_{24}} + \frac{(a^2 + b^2)^2}{z_{14}z_{23}} \right]^2 \left[ \frac{|z_{12}|}{|z_{13}|} \frac{|z_{34}|}{|z_{24}|} \right] \frac{\beta^2}{2\pi} + \text{Perm.} \quad (9.5)$$

to which we have to subtract the disconnected term

$$D = 4(a^2 + b^2)^4 \left\{ \left[ \frac{1}{|z_{12}|^2} \right]^{2+\frac{\beta^2}{2\pi}} + \left[ \frac{1}{|z_{13}|^2} \right]^{2+\frac{\beta^2}{2\pi}} + \left[ \frac{1}{|z_{14}|^2} \right]^{2+\frac{\beta^2}{2\pi}} \right\}. \quad (9.6)$$

It is useful to rewrite the correlation function as the sum of two pieces

$$G_1 = 2 \left[ \frac{(a^2 - b^2)^4}{|z_{12}|^2 |z_{34}|^2} + \frac{(a^2 + b^2)^4}{|z_{13}|^2 |z_{24}|^2} + \frac{(a^2 + b^2)^4}{|z_{14}|^2 |z_{23}|^2} \right] \left[ \frac{|z_{12}|}{|z_{13}|} \frac{|z_{34}|}{|z_{24}|} \right] \frac{\beta^2}{2\pi} + \text{Perm.} \quad (9.7)$$

and

$$G_2 = 2 \left[ \frac{(a^4 - b^4)^2}{z_{12}z_{34}z_{13}z_{24}} - \frac{(a^4 - b^4)^2}{z_{12}z_{34}z_{14}z_{23}^*} - \frac{(a^2 + b^2)^4}{z_{13}z_{24}z_{14}z_{23}} + \text{c.c.} \right] \left[ \frac{|z_{12}|}{|z_{13}|} \frac{|z_{34}|}{|z_{24}|} \right] \frac{\beta^2}{2\pi} + \text{Perm.} \quad (9.8)$$

and to compute the perturbation integral as the sum of the two corresponding integrals $\int (G_1 - D)$ and $\int G_2$, both separately UV-finite. As a result, the fourth order term is the sum $C^{(4)} = C_1^{(4)} + C_2^{(4)}$ where

$$C_1^{(4)} = 12 \lim_{L \to \infty} \frac{R}{2\pi L} \int_{\text{strip}} d^2w_1 \cdots d^2w_4 |G_1 - D|_{\text{strip}} \quad (9.9)$$

and

$$C_2^{(4)} = 12 \lim_{L \to \infty} \frac{R}{2\pi L} \int_{\text{strip}} d^2w_1 \cdots d^2w_4 G_2|_{\text{strip}}. \quad (9.10)$$

In the computation of the first integral we take the limit $L \to \infty$ first, and we cancel the overall volume of the strip $RL$, and then we map the infinite strip to the whole $z$-plane. We thus get

$$C_1^{(4)} = \frac{(2\pi)^{\beta^2}}{2} R^{4-\beta^2} \frac{\beta^2}{2\pi} \int \frac{d^2z_2}{2\pi} \frac{d^2z_3}{2\pi} \frac{d^2z_4}{2\pi} \left( |z_2||z_3||z_4| \right)^{\beta^2 - 1} |G_1 - D|_{z_1=1} \quad (9.11)$$
and by using the residual symmetry on the integration variables we are essentially reduced to two integrals

\[ C_1^{(4)} = 3(2\pi)^{3/2} R^{4 - \frac{3}{2}} \left[ (a^2 - b^2)^4 A_1 + (a^2 + b^2)^4 A_2 \right] \]  

(9.12)

where

\[ A_1 = \int \frac{d^2 z_2 \ d^2 z_3 \ d^2 z_4}{2\pi} \left( \frac{|z_2||z_3||z_4|}{|z_{12}|^2|z_{34}|^2} \right)^{\frac{\rho^2}{2\pi}} \left( \frac{1}{|z_{12}| |z_{14}| |z_{23}| |z_{24}|} \right)^{\frac{\rho^2}{2\pi}} \bigg|_{z_1 = 1} \]

(9.13)

and

\[ A_2 = 2 \int \frac{d^2 z_2 \ d^2 z_3 \ d^2 z_4}{2\pi} \left( \frac{|z_2||z_3||z_4|}{|z_{12}|^2|z_{34}|^2} \right)^{\frac{\rho^2}{2\pi}} \times \left\{ \frac{1}{|z_{12}| |z_{14}| |z_{23}| |z_{24}|} \right\}^{\frac{\rho^2}{2\pi}} \bigg|_{z_1 = 1} \]

(9.14)

These two integrals are of Dotsenko-Fateev type [36]. By deforming the integration contours they can be transformed into products of two factors. Each factor is the sum, with proper trigonometric coefficients, of one-dimensional integrals of this kind

\[ \int_0^1 \prod_{i=1}^3 \left[ dv_i v_i^{\alpha_i} (1 - v_i)^{\beta_i} \right] (1 - v_1 v_2)^{\gamma_1} (1 - v_2 v_3)^{\gamma_2} (1 - v_1 v_2 v_3)^{\gamma_3}. \]

(9.15)

They can be formally integrated by binomially expanding the last three factors of the integrand and using the fundamental integral \( \int_0^1 dv v^a (1 - v)^b = \Gamma(1 + a) \Gamma(1 + b) / \Gamma(2 + a + b) \). Therefore the two integrals \( A_1 \) and \( A_2 \) can be reduced to the computation of (products of) converging series of three indices. We don’t give the explicit expressions because of their algebraic heaviness. The method is really a straightforward generalization of the paper [37]. The result can now be evaluated numerically by extrapolating the finite sums.

The integral contributing to \( C_2^{(4)} \) is evaluated with the method developed in [38]. We first map the finite strip to the annulus \( \rho < |z| < 1 \), where \( \rho = \exp(-2\pi L/R) \), and then we compute the leading contribution of the integral in the limit \( \rho \to 0 \). Using the symmetry under permutation of the four integration variables we obtain

\[ C_2^{(4)} = 12(2\pi)^{3/2} R^{4 - \frac{3}{2}} \lim_{\rho \to 0} \frac{1}{\ln(\frac{1}{\rho})} \int_{\rho}^{1} dr_1 \int_{\rho}^{r_4} dr_3 \int_{\rho}^{r_3} dr_2 \int_{\rho}^{r_2} dr_1 \prod_{i=1}^{4} \int_0^{2\pi} \frac{d\theta_i}{2\pi} r_i^{\frac{\rho^2}{2\pi}} G_2. \]

(9.16)
Having ordered the integration variables it is now possible to binomially expand each factor $z_j^\gamma = r_j^\gamma \exp(i\theta_j\gamma)(1 - \exp(i(\theta_l - \theta_j)r_l/r_j)^\gamma$ for $j > l$. Then we obtain a series on 12 indices constrained by three independent conditions from the angular integrations, each term of the series being a product of binomial coefficients and four radially ordered integrals of powers of the $r_i$. The last integral, the one in $dr_4$, gives the necessary overall volume divergence $2\ln(1/\rho)$. The result is

$$C_2^{(4)} = 3(2\pi)^{5/2} R^{4-\frac{a^2}{\pi}} \left[ (a^4 - b^4)S_1 + (a^2 + b^2)^4 S_2 \right]$$

(9.17)

where we give as an example a term contributing to $S_2$

$$\sum_{n_1,...,m_1...} \frac{-2q_{n,m}}{(n_1 + n_2 + n_3 + \frac{1}{2} + \frac{\beta^2}{8\pi})(n_3 + n_5 + n_6 + \frac{1}{2} + \frac{\beta^2}{8\pi})(n_2 + n_3 + n_4 + n_5 + 1 + \frac{\beta^2}{2\pi})}$$

(9.18)

where the symbol $\sum'$ means the following conditions on the indices $n_1,...,n_6,m_1,...m_6$

$$n_1 + n_2 + n_3 = m_1 + m_2 + m_3 ; \quad n_1 - n_4 - n_5 = m_1 - m_4 - m_5$$

$$n_2 + n_4 - n_6 = m_2 + m_4 - m_6 ; \quad n_3 + n_5 + n_6 = m_3 + m_5 + m_6.$$  

(9.19)

The coefficient $q_{n,m}$ is

$$q_{n,m} = b_{n_1} \left( 1 + \frac{\beta^2}{4\pi} \right) b_{m_1} \left( 1 + \frac{\beta^2}{4\pi} \right) b_{n_2} \left( 1 - \frac{\beta^2}{4\pi} \right) b_{m_2} \left( 1 - \frac{\beta^2}{4\pi} \right) b_{n_3} \left( 1 - \frac{\beta^2}{4\pi} \right) b_{m_3} \left( \frac{\beta^2}{4\pi} \right)$$

$$b_{n_4} \left( 1 - \frac{\beta^2}{4\pi} \right) b_{m_4} \left( \frac{\beta^2}{4\pi} \right) b_{n_5} \left( 1 - \frac{\beta^2}{4\pi} \right) b_{m_5} \left( 1 - \frac{\beta^2}{4\pi} \right) b_{n_6} \left( 1 + \frac{\beta^2}{4\pi} \right) b_{m_6} \left( 1 + \frac{\beta^2}{4\pi} \right)$$

(9.20)

with $b_n(x) = \Gamma(n + 1 - x)/(n\Gamma(1 - x))$.

Summarizing, the fourth order correction for the running central charge is given by

$$C^{(4)} = 3(2\pi)^{5/2} \left[ (A_2 + S_2) k_+^4 + S_1 k_-^2 k_+^2 + A_1 k_-^4 \right] r^{4-\frac{a^2}{\pi}}$$

(9.21)

where the numbers $A_1,A_2,S_1,S_2$ depend on $\frac{\beta^2}{8\pi}$ and can be computed numerically by extrapolating the values of the respective finite sums. Unfortunately the above sums, especially the $A_i$ ones, are slow to converge affecting therefore the precision of the extrapolated values. For the most favourable case $\frac{\beta^2}{8\pi} = \frac{1}{10}$ the extrapolated values are

$$A_1 = 49.9, A_2 = 1.44$$

$$S_1 = -20.1, S_2 = -1.79.$$  

(9.22)
For the first two coefficients \( A_1 \) and \( A_2 \) we have used the VBS extrapolation method over the set of finite sums with \( 25 < N \leq 40 \), while the other coefficients \( S_1 \) and \( S_2 \) have been determined with the BST extrapolation method with convergence parameter \( \omega = 1 \) over finite sums with \( N \leq 7 \). The need for difference extrapolation methods is due to the difference in the rate of convergence of the series. With our choice the given extrapolated values are the most stable with respect to \( N \). A concise introduction to the above extrapolation methods can be found in [39].

The numerical integration of the TBA equations gives us the running central charge \( C(r,m/m_1) \) with high precision and therefore the coefficients of the expansion

\[
C(r,m/m_1) = 2 + Br^2 \sum_{k=1}^{\infty} a_{2k} r^{k(2-\frac{\beta^2}{2\pi})} \tag{9.23}
\]

can be determined with a standard fitting procedure. By matching the perturbative computation with the first two coefficients \( a_2 \) and \( a_4 \) we can determine now the two functions \( k_+(m/m_1), k_-(m/m_1) \) as solution of the following second order algebraic system

\[
a_2 = 6(2\pi)^{\frac{\beta^2}{2\pi}} \left[ \frac{\Gamma(1/2 + \frac{\beta^2}{8\pi})}{\Gamma(1/2 - \frac{\beta^2}{8\pi})} \right]^2 \frac{\Gamma(-\frac{\beta^2}{4\pi})}{\Gamma(1 + \frac{\beta^2}{4\pi})} k_+^2
\]

\[
a_2 = \frac{1}{12} \left\{ \frac{\Gamma(1/2 - \frac{\beta^2}{8\pi})}{\Gamma(1/2 + \frac{\beta^2}{8\pi})} \right\}^2 \left\{ \frac{\Gamma(1 + \frac{\beta^2}{4\pi})}{\Gamma(-\frac{\beta^2}{4\pi})} \right\}^2 \left\{ A_1 \left( \frac{k_-}{k_+} \right)^4 + S_1 \left( \frac{k_-}{k_+} \right)^2 + (A_2 + S_2) \right\}.
\tag{9.24}
\]
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