CHARACTERIZATIONS OF NONCOMMUTATIVE $H^\infty$

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Abstract. We transfer a large part of the circle of theorems characterizing the generalization of classical $H^\infty$ known as ‘weak* Dirichlet algebras’, to Arveson’s noncommutative setting of subalgebras of finite von Neumann algebras.

1. Introduction

Around the early 1960’s, it became apparent that a circle of famous theorems about the classical $H^\infty$ space of bounded analytic functions on the disk, could be generalized to the setting of abstract function algebras. In particular, in [7], Hoffman showed that many of these theorems were valid for function algebras satisfying the logmodular condition which he introduced in that paper. In [23], Srinivasan and Wang isolated a function algebra setting in which the conclusions of many of these theorems, were each equivalent to logmodularity, and hence equivalent to each other. They called the algebras satisfying these equivalent conditions weak* Dirichlet algebras. Just a few years later, Arveson introduced his ‘finite maximal subdiagonal algebras’ [1], which we will consistently refer to as noncommutative $H^\infty$ algebras in our paper, for the sake of simplicity and brevity. The setting, and we will fix this notation for the rest of our paper, is a von Neumann algebra $M$ possessing a faithful normal tracial state $\tau$. By a noncommutative $H^\infty$ algebra, we shall mean a subalgebra of $M$ satisfying certain conditions that we shall spell out momentarily. These were intended to be the noncommutative generalization of weak* Dirichlet algebras. One may then consider the possible noncommutative versions of the famous theorems about classical $H^\infty$, and ask which of the conclusions of these theorems are equivalent to the conditions defining the noncommutative $H^\infty$ algebras. This is the topic of the present paper.

In [3], we defined a subalgebra $A$ of a $C^*$-algebra $B$ to be logmodular if every strictly positive element $b \in B$ (that is, every selfadjoint $b$ for which there exists an $\epsilon > 0$ with $b \geq \epsilon 1$), is a uniform limit of terms of the form $a^*a$ where $a \in A^{-1}$. Here, $A^{-1}$ is the set of invertible elements of $A$. We say that $A$ has factorization, if each strictly positive $b \in B$ may be written as $a^*a$ for some $a \in A^{-1}$. In [3], we also defined a tracial subalgebra of the algebra $M$ above, to be a weak* closed unital subalgebra $A$ of $M$ for which there exists a projection $\Phi$ from $A$ onto $D \overset{\text{def}}{=} A \cap A^*$, such that $\Phi$ is also a homomorphism, and $\tau = \tau \circ \Phi$ on $A$. By Theorem 5.6 of [3], $\Phi$ is precisely the restriction to $A$ of the unique faithful normal conditional expectation $\Psi$ from $M$ onto $D$ such that $\tau = \tau \circ \Psi$. Hence we may continue to write $\Psi$ as $\Phi$, and we call this extension the conditional expectation onto $D$.

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If \( S \) is a set, we will write \( S^* \) for the set \( \{ a : a^* \in S \} \). One may define a noncommutative \( H^\infty \) algebra to be a tracial subalgebra of \( M \) for which \( A + A^* \) is weak* dense in \( M \). Arveson showed that noncommutative \( H^\infty \) algebras have factorization, and thus they are logmodular; in the present paper we will prove the converse. Also, Arveson formulated the classical Szegö theorem, and the related Jensen’s inequality, in terms of the Fuglede-Kadison determinant. This is a certain function \( \Delta : M \to [0, \infty) \), which we will describe explicitly in Section 1. Following Arveson, we say that a tracial subalgebra \( A \) satisfies

- **Jensen’s inequality**, if \( \Delta(\Phi(a)) \leq \Delta(a) \) for all \( a \in A \),
- **Jensen’s formula**, if \( \Delta(\Phi(a)) = \Delta(a) \) for all \( a \in A^{-1} \),
- **Szegö’s theorem**, if \( \Delta(h) = \inf \{ \tau(h|a + d|^2) : a \in A_0, d \in D^{-1}, \Delta(d) \geq 1 \} \),

for all \( h \in L^1(M)_+ \).

Here and in the rest of our paper, \( A_0 = A \cap \text{Ker}(\Phi) \), an ideal in \( A \); and \( L^1(M)_+ \) is the positive part of the predual of \( M \). We remark that our formulation of Szegö’s theorem may look different from Arveson’s formulation, but it is in fact equivalent (see the Remark before Lemma 2.2). Arveson had shown that for noncommutative \( H^\infty \) algebras, Jensen’s inequality, Jensen’s formula, and his version of Szegö’s theorem, are all equivalent. Recently, the second author showed in [13] that for noncommutative \( H^\infty \) these three conditions are true (this was a major problem left open in [1]). For a general tracial subalgebra of \( M \), these conditions are certainly not equivalent\(^1\), but at least Szegö’s theorem implies the others. We show that a tracial subalgebra satisfying Szegö’s theorem, also satisfies the noncommutative variant of another of the important equivalent characterizations of weak* Dirichlet algebras, namely:

\[(1.1) \text{If } g \in L^1(M)_+ \text{ and } \tau(fg) = \tau(f) \text{ for all } f \in A, \text{ then } g = 1.\]

This condition may be rephrased as saying that there is a unique normal state on \( M \) extending \( \tau|_A \). Thus, we say that an algebra has the unique normal state extension property if it satisfies (1.1). In the classical situation, (1.1) is exactly the key condition underpinning the paper [5]. By analogy with the work of Hoffman and Srinivasan and Wang, one would expect to be able to complete this circle; namely that any tracial subalgebra of \( M \) having the unique normal state extension property, is a noncommutative \( H^\infty \) algebra. We are able to show this under extra hypotheses, and in particular if \( A \) is \( \tau \)-maximal, by which we mean that

\[ A = \{ x \in M : \tau(xA_0) = 0 \}. \]

The main result of the present paper is the following:

**Theorem 1.1.** For a tracial subalgebra \( A \) of \( M \), the following conditions are equivalent:

1. \( A + A^* \) is weak* dense in \( M \) (that is, \( A \) is a noncommutative \( H^\infty \) algebra),
2. \( A \) has factorization,
3. \( A \) is logmodular,
4. \( A \) satisfies Szegö’s theorem,
5. \( A \) is \( \tau \)-maximal and has the unique normal state extension property,
6. \( A \) has the unique normal state extension property, and \( A + A^* \) is norm-dense in \( L^2(M) \) (the latter space is defined below).

\(^1\)One can see this from the fact that \( D \) satisfies Jensen’s inequality and Jensen’s formula, but not Szegö’s theorem in general.
At the end of our paper, we briefly discuss characterizations of noncommutative $H^\infty$ algebras in terms of an invariant subspace theorem, or a Beurling-Nevanlinna factorization result. These matters are essentially disjoint from the rest of our paper, and deserve a more detailed investigation at some point in the future.

We end this introduction with a few other notational conventions. We write $B^{-1}$ for the set of invertible elements of an algebra $B$. For a set $S$, we write $S_+$ for the set $\{x \in S : x \geq 0\}$, where the symbol ‘$\geq$’ will usually denote the natural ordering in a $C^*$-algebra, or in the predual of a von Neumann algebra, or more generally in the noncommutative $L^p$ spaces. We recall the definition of the latter: For our (finite) von Neumann algebra $M$, we define $\tilde{M}$ to be the set of closed densely defined operators $a$ affiliated to $M$. If $1 \leq p < \infty$, then $L^p(M, \tau) = \{a \in \tilde{M} : \tau(|a|^p) < \infty\}$, equipped with the norm $\| \cdot \|_p = \tau(\cdot^p)^{1/p}$. For brevity, we will in the following write $L^p$ or $L^p(M)$ for $L^p(M, \tau)$, and we recall that for $M$ as above, $L^p(M)$ may be defined to be the completion of $M$ in the norm $\tau(\cdot^p)^{1/p}$. The spaces $L^p$ are Banach spaces satisfying the usual duality relations and Hölder inequalities \cite{6, 18}. Now let $\Phi$ be a faithful normal conditional expectation from $M$ onto a von Neumann subalgebra $D$ satisfying $\tau \circ \Phi = \tau$. The argument employed in e.g. Proposition 3.9 of \cite{10}, then shows that for each $1 \leq p < \infty$, the map $\Phi$ continuously extends to a map which contractively maps $L^p(M)$ onto $L^p(D)$.

Note that $M^{-1} \cap M_+$ is precisely the set of strictly positive elements of $M$. We will use the fact that in a finite von Neumann algebra, $xy = 1$ implies that $yx = 1$. We write $[S]_p$ for the closure of a set $S$ in $L^p(M)$. If $A$ is a noncommutative $H^\infty$ algebra, then $[A]_p$ is often called the noncommutative Hardy space $H^p$.

2. The determinant

For $a \in M$, the determinant $\Delta(a)$ is defined as follows: if $|a|$ is strictly positive, then we define $\Delta(a) = \exp \tau(\log |a|)$. Otherwise, we define $\Delta(a) = \inf \Delta(|a| + \epsilon 1)$, the infimum taken over all scalars $\epsilon > 0$.

We will use several basic properties of this determinant from \cite{3}. We will also need to extend the definition of this determinant to $L^1(M)$. Namely, for any $a \in L^1(M)$, we set $\Delta(a) = \exp \tau(\log |a|)$ if $|a| \geq 1$ for some scalar $\epsilon > 0$; and otherwise, we define $\Delta(a) = \inf \Delta(|a| + \epsilon 1)$, the infimum again taken over all scalars $\epsilon > 0$. Clearly $\Delta(|a| + \epsilon 1)$ decreases as $\epsilon \to 0$, with limit $\Delta(a)$.

We now explain why our definition of $\Delta(a)$ makes sense. If $0 < \epsilon < 1$, the function $\log t$ is bounded on $[\epsilon, 1]$. Also, $0 \leq \log t \leq t$ for $t \in [1, \infty)$. So given $h \in L^1(M)_+$ with $h \geq \epsilon 1$, it follows from the Borel functional calculus for positive unbounded operators that $(\log h)e_{[0,1]}$ is bounded, and that $0 \leq (\log h)e_{[1,\infty)} \leq he_{[1,\infty)} \leq h$. Here $e_{[0,\lambda]}$ denotes the spectral resolution for $h$. Thus $(\log h)e_{[0,1]}$ and $(\log h)e_{[1,\infty)}$ belong to $L^1(M)$. Hence $\log h \in L^1(M)$ in this case.

We will need the following variant of some formulae from \cite{3}.

Proposition 2.1. (Cf. 4.3.2 in \cite{3}) For any $h \in L^1(M)_+$, we have

$$\Delta(h) = \inf \{\tau(hb) : b \in M_+ \cap M^{-1}, \Delta(b) \geq 1\}. \tag{2.1}$$

Also, this infimum is attained on the von Neumann algebra generated by $h$ (see e.g. \cite{3} p. 349), which is a commutative subalgebra of $M$. For any $h \in L^1(M)$, we have

$$\Delta(h) = \inf \{\tau(|hb|) : b \in M_+ \cap M^{-1}, \Delta(b) \geq 1\}. \tag{2.2}$$
Proof. First assume that \( h \geq \epsilon 1 \) for some \( \epsilon > 0 \). As in the calculation above,
\[
0 \leq (\log h)e_{[n, \infty)} \leq he_{[n, \infty)} , \quad n \in \mathbb{N}.
\]
Since \( e_{[n, \infty)} \to 0 \) strongly, and since \( h \in L^1(M)_+ \), we have \( \lim_n \tau(h e_{[n, \infty)}) = 0 \). Thus \( \lim_n \tau((\log h)e_{[n, \infty)}) = 0 \). Hence \( \tau(\log h) = \lim_n \tau((\log h)e_{[0, n)}) \). That is,
\[
\Delta(h) = \exp \tau(\log h) = \lim_n \exp \tau((\log h)e_{[0, n)}).
\]

Given any \( b \in M_+ \), we have
\[
\tau(he_{[0, n)}b) = \tau(b^\frac{1}{2}he_{[0, n)}b^\frac{1}{2}) \leq \tau(b^\frac{1}{2}hb^\frac{1}{2}) = \tau(hb).
\]
Combining this fact with [1, 4.3.2], we have that
\[
\Delta(he_{[0, n)}) = \inf\{\tau(he_{[0, n)}b) : b \in M_+ \cap M^{-1}, \Delta(b) \geq 1\}
\leq \inf\{\tau(hb) : b \in M_+ \cap M^{-1}, \Delta(b) \geq 1\}.
\]
Now let \( n \to \infty \) to see that
\[
\Delta(h) \leq \inf\{\tau(hb) : b \in M_+ \cap M^{-1}, \Delta(b) \geq 1\}.
\]
To see that the infimum is precisely \( \Delta(h) \), and that it is attained on the minimal commutative von Neumann algebra \( M_0 \) generated by \( h \), it suffices to find, for each scalar \( \delta > 0 \), an element \( b_\delta \) in this von Neumann algebra, with \( \Delta(b_\delta) \geq 1 \), and
\[
\inf\{\tau(hb_\delta) \geq \delta > 0\} = \Delta(h).
\]
Since \( h \geq \epsilon 1 \), we have that \( h^{-1} \) is bounded, and is in \( M_+ \). Also \( \inf\{\tau(hb_\delta) \geq \delta > 0\} = \Delta(h) \), and \( \Delta(h) \geq 1 \). Then
\[
\Delta(b_\delta) = \Delta(h)\Delta(h^{-1} + \delta 1) = \exp(\tau(\log(h)))\exp(\tau(\log(h^{-1} + \delta 1)))
\leq \exp(\tau(\log(h)) + \log(h^{-1} + \delta 1)).
\]

By the Borel functional calculus for unbounded selfadjoint operators,
\[
0 \leq \log(1 + \delta h) = \log(h(h^{-1} + \delta 1)) = \log(h) + \log(h^{-1} + \delta 1),
\]
the sum and product here being the ‘strong’ ones (that is, we are taking closures of the operators involved). Hence,
\[
\Delta(b_\delta) = \exp(\tau(\log(1 + \delta h))) \geq 1.
\]

Next, if \( h \) is not \( \geq \epsilon 1 \) for any \( \epsilon > 0 \), then by the above we have
\[
\Delta(h) = \inf_{\epsilon > 0} \Delta(h + \epsilon 1) = \inf_{\epsilon > 0} \inf\{\tau((h + \epsilon 1)b) : b \in M_+ \cap M^{-1}, \Delta(b) \geq 1\}.
\]
Interchanging the infimums gives \( \mathbb{2} \). The infimum is again achieved on the von Neumann algebra generated by \( h \), since for any \( \epsilon > 0 \) this is the same as the von Neumann algebra generated by \( h + \epsilon 1 \).

To obtain \( \mathbb{2} \), first note that we may assume that \( h \geq 0 \). This is because if the result holds in the latter case, then in the general case,
\[
\Delta(h) = \Delta(|h|) = \inf\{\tau(|h|b) : b \in M_+ \cap M^{-1}, \Delta(b) \geq 1\}
= \inf\{\tau(|hb|) : b \in M_+ \cap M^{-1}, \Delta(b) \geq 1\},
\]
since \( |hb|^2 = bh^*hb = ||h||^2 \).
For $h \geq 0$, it then follows from the above that
\[
\Delta(h) = \inf \{ \tau(hb) : b \in M_+ \cap M^{-1}, \Delta(b) \geq 1 \} \\
\leq \inf \{ \tau(|hb|) : b \in M_+ \cap M^{-1}, \Delta(b) \geq 1 \}.
\]
To see that the last is an equality, we argue as in the corresponding part of the proof above. Indeed, if $\epsilon, \delta > 0$ then
\[
\Delta(h) \leq \tau(|h\Delta(h+\epsilon 1)((h+\epsilon 1)^{-1} + \delta 1)|) = \tau(h\Delta(h+\epsilon 1)((h+\epsilon 1)^{-1} + \delta 1)) \\
\leq \tau((h+\epsilon 1)\Delta(h+\epsilon 1)((h+\epsilon 1)^{-1} + \delta 1)) \rightarrow \Delta(h)
\]
as $\delta, \epsilon \to 0$ (similarly to the first part of the proof). This establishes (2.2). $\square$

**Remark.** In [1] Arveson defined the quantity $\Delta(\rho)$, for a normal state $\rho$ of $M$, to be $\inf \{ \rho(b) : b \in M_+ \cap M^{-1}, \Delta(b) \geq 1 \}$. However, normal states $\rho$ are in bijective correspondence with the norm-one elements $h$ of $L^1(M)_+$, via the map $h \mapsto \rho_h = \tau(h)$. We may therefore rephrase (2.1) as the statement that $\Delta(h) = \Delta(\rho_h)$ for all such $h$.

The fact that our formulation of the noncommutative Szegö theorem in Section 1, is in line with that of Arveson, follows from this, and from the following Claim:

$$\inf \{ \tau(h|a + d|^2) : a \in A_0, d \in D^{-1}, \Delta(d) \geq 1 \} = \inf \{ \tau(h|a + d|^2) : a \in A_0, d \in D, \Delta(d) \geq 1 \},$$

for any $h \in L^1(M)_+$. Indeed, clearly the left side dominates the right. To prove the converse inequality, first note that the infimum on the right side is unchanged if one insists also that $d \geq 0$. For if $d \in D$, with $\Delta(d) \geq 1$, let $d = v|d|$ be the polar decomposition of $d$ in $D$. Set $\tilde{a} = v^*a \in A_0$. We have $\Delta(|d|) = \Delta(d) \geq 1$. Also $|a + d|^2 \geq |	ilde{a} + |d||^2$, so that $\tau(h|a + d|^2) \geq \tau(h|	ilde{a} + |d||^2)$.

If $d \in D_+$, then $d + \frac{1}{n} \in D^{-1}$, and $\Delta(d + \frac{1}{n}) \geq \Delta(d) \geq 1$, and

$$\tau(h|a + d|^2) = \lim_n \tau(h|a + d + \frac{1}{n}||^2) \geq \inf_n \tau(h|a + d + \frac{1}{n}||^2),$$

which dominates the left side of the equation in the Claim.

**Lemma 2.2.** Let $h \in L^1(M)_{sa}$. If for some $\delta > 0$ we have

$$\Delta(1 - th) \geq 1, \quad t \in (-\delta, \delta),$$

then $h = 0$.

**Proof.** Let $M_0$ be the von Neumann algebra generated by $h$ (see e.g. [9], p. 349), which is a commutative subalgebra of $M$. Let $\psi = \tau_{|M_0}$. Since $\psi$ is a faithful normal state on $M_0$, it is a simple consequence of the Riesz representation theorem applied to $\psi$, that $M_0 \cong L^\infty(\Omega, \mu_\tau)$ $\ast$-algebraically, for a measure space $\Omega$ and a Radon probability measure $\mu_\tau$. (This is a simpler case of the proof of 1.18.1 in [22]). We have $\tau(x) = \int_\Omega x \, d\mu_\tau$ for any $x \in M_0$, where we are abusing notation by writing $x$ for the corresponding element of $L^\infty(\Omega, \mu_\tau)$ too. Then $L^1(M_0)$, which is the completion of $M_0$ in the norm $\tau(|\cdot|)$, is isometric to $L^1(\Omega, \mu_\tau)$, the completion of $L^\infty(\Omega, \mu_\tau)$ in the $L^1$-norm. Clearly, $L^1(M_0) \subset L^1(M)$ isometrically, and the canonical extension of $\tau$ to $L^1(M)$, agrees with the canonical extension of $\psi$ to $L^1(M_0)$. In particular, for $h \in L^1(M_0)$, $\tau(h) = \int_\Omega h \, d\mu_\tau$, and $\tau(|h|)$ is the $L^1(\mu_\tau)$-norm of $h$. 

For \( q \) by the inductive hypothesis, and Hölder’s inequality, we have
\[
\Delta(1 - th) = \inf_{\epsilon > 0} \{ \exp \int_{\Omega} (\log |1 - th| + \epsilon 1) \, d\mu_x \} = \exp \int_{\Omega} \log |1 - th| \, d\mu_x.
\]
Thus the result we want follows from Hoffman’s lemma [7, Lemma 6.6].

3. Consequences of logmodularity

**Proposition 3.1.** For a tracial subalgebra, logmodularity implies Jensen’s formula.

**Proof.** This is a modification of the proof of the main result of [13]. We simply indicate the parts of the proof which need adjusting, beginning at the inductive step. We assume that
\[
\tau(|a|^{\frac{1}{q}}) \geq \tau(|\Phi(a)|^{\frac{1}{q}}), \quad a \in A^{-1},
\]
for an integer \( k \), and we want to prove the same inequality with \( k \) replaced by \( k + 1 \). Fix \( a \in A^{-1} \), and inductively define \((x_n) \subset M_+ \) by
\[
x_1 = |a|^{\frac{1}{q}}, \quad x_{n+1} = \frac{1}{2}(x_n + |a|^{\frac{1}{q}}x_n^{-1}).
\]
For each \( n \in \mathbb{N} \), select \((z_m^{(n)}) \subset A^{-1} \) with
\[
\lim_{m} |z_m^{(n)}| = x_n^{2k}, \quad n \in \mathbb{N}.
\]
For \( q = \frac{1}{2k} \), we have, as in [13], that
\[
(3.1) \quad \tau(|a(z_m^{(n)})^{-1}|^q) = \tau(|a||z_m^{(n)}|^{-2}|a|^{\frac{q}{2}}) \quad m \in \mathbb{N}.
\]
By the inductive hypothesis, and Hölder’s inequality, we have
\[
\frac{1}{2} \tau(|z_m^{(n)}|^q + |a(z_m^{(n)})^{-1}|^q) \geq \frac{1}{2} \tau(|\Phi(z_m^{(n)})|^q + |\Phi(a)\Phi(z_m^{(n)})^{-1}|^q)
\]
\[
\geq \frac{1}{2} \left\{ \tau(|\Phi(z_m^{(n)})|^q) + \frac{\tau(|\Phi(a)|^{\frac{q}{2k}})^2}{\tau(|\Phi(z_m^{(n)})|^q)} \right\}
\]
\[
\geq \tau(|\Phi(a)|^{\frac{1}{2k}}).
\]
Using the last inequality, and (3.1), we obtain
\[
\tau(|\Phi(a)|^{\frac{1}{2k}}) \leq \frac{1}{2} \tau(|z_m^{(n)}|^q + |a(z_m^{(n)})^{-1}|^q) = \frac{1}{2} \tau(|z_m^{(n)}|^q + (|a||z_m^{(n)}|^{-2}|a|^{\frac{1}{2k}}).
\]
The left side of this inequality does not depend on \( m \). Letting \( m \to \infty \) gives
\[
\tau(|\Phi(a)|^{\frac{1}{2k}}) \leq \frac{1}{2} \tau(x_n + x_n^{-1}|a|^{\frac{1}{q}}) = \tau(x_{n+1}).
\]
Since the \( x_{n+1} \)'s decrease monotonically to \(|a|^{\frac{1}{2k}} \), we have \( \tau(|\Phi(a)|^{\frac{1}{2k}}) \leq \tau(|a|^{\frac{1}{2k}}) \), as required.

**Proposition 3.2.** Let \( A \) be a logmodular tracial subalgebra of \( M \). For any \( h \in L^1(M) \), we have
\[
\Delta(h) = \inf \{ \tau(|ha|) : a \in A^{-1}, \Delta(a) \geq 1 \}.
\]
Proof. Since $|ha|^2 = a^*|h|^2a = ||h||^2$, we have
\[ \inf\{\tau(|ha|) : a \in A^{-1}, \Delta(a) \geq 1\} = \inf\{\tau(||h||a) : a \in A^{-1}, \Delta(a) \geq 1\}. \]
Since $\Delta(h) = \Delta(|h|)$, it suffices henceforth to assume that $h \geq 0$.

Given $a \in A$, we have that
\[ \tau(|ha|) = \tau(|a^*h|) = \tau(||a^*||) = \tau(|a^*|). \]
If $a \in A^{-1} \subset M^{-1}$, then $a^*$ and $|a^*|$ are in $M^{-1}$. Since $\Delta(a) = \Delta(a^*) = \Delta(|a^*|)$, by [12, 4.3.1], it follows from (2.1) that
\[ \inf\{\tau(|ha|) : a \in A^{-1}, \Delta(a) \geq 1\} = \inf\{\tau(|h||a^*||) : a \in A^{-1}, \Delta(a) \geq 1\} \geq \inf\{\tau(|hb| : b \in M_+ \cap M^{-1}, \Delta(b) \geq 1\} = \Delta(h). \]

Since $A$ is logmodular, given any $b_0 \in M_+ \cap M^{-1}$ with $\Delta(b_0) \geq 1$, we may select \{a_n\} $\subset A^{-1}$ so that $|a_n^a| \to b_0$ uniformly. By [12, 4.3.1 (iv)], we deduce that
\[ \Delta(a_n) = \Delta(a_n^a) = \Delta(|a_n^a|) \to \Delta(b_0). \]
Letting $\tilde{a}_n = \frac{\Delta(b_0)}{\Delta(a_n)}a_n \in A^{-1}$, we have $|\tilde{a}_n^a| = \frac{\Delta(b_0)}{\Delta(a_n)}|a_n^a| \to b_0$; and $\Delta(\tilde{a}_n) = \Delta(b_0) \geq 1$. Thus we clearly have, using also (2.1), that
\[ \tau(|h|b_0) = \lim_n \tau(|h|\tilde{a}_n^a) = \tau(|h|b_0) \geq \inf\{\tau(|ha|) : a \in A^{-1}, \Delta(a) \geq 1\}. \]
Since $b_0 \in M_+ \cap M^{-1}$ was arbitrary, the above combined with the earlier inequality in this proof, gives
\[ \Delta(h) = \inf\{\tau(|hb| : b \in M_+ \cap M^{-1}, \Delta(b) \geq 1\} = \inf\{\tau(|ha|) : a \in A^{-1}, \Delta(a) \geq 1\}, \]
which is the desired equality. □

Lemma 3.3. Let $A$ be a logmodular tracial subalgebra of $M$. If $h \in L^1(M)$ with $\tau(ha) = 0$ for every $a \in A$, then $\Delta(1 - h) \geq 1$.

Proof. Suppose that $\tau(ha) = 0$ for every $a \in A$. We continue to write $\Phi$ for the canonical ‘extension by continuity’ of $\Phi$ to a map from $L^1(M)$ to $L^1(D)$ (see e.g. [21] or 3.9 in [16]). By routine approximation arguments, it is easy to see that this extension is still a contractive ‘conditional expectation’: $\Phi(ha) = \Phi(h)d$ for $h \in L^1(M), d \in D$. Similarly, $\tau \circ \Phi = \tau$ on $L^1(M)$. Using these facts, given $a_0 \in A, d \in D$, we have
\[
\tau(\Phi((1 - h)a_0)d) = \tau(\Phi((1 - h)a_0)d) = \tau((1 - h)a_0)d
= \tau(a_0d) = \tau(\Phi(a_0d)) = \tau(\Phi(a_0)d).
\]
This implies that $\Phi((1 - h)a_0) = \Phi(a_0)$. Next, let $a_0 \in A^{-1}$ be given. Since $\Phi$ is contractive on $L^1$, we have, by (2.2), and the above, that
\[
\tau(|(1 - h)a_0|) \geq \tau(|\Phi((1 - h)a_0)|) = \tau(|\Phi(a_0)|)
\geq \inf\{\tau(|\Phi(a_0)|) : b \in M_+ \cap M^{-1}, \Delta(b) \geq 1\}
= \Delta(\Phi(a_0)).
\]
By Jensen’s formula, the latter quantity equals $\Delta(a_0)$. By Proposition 3.2, we conclude that $\Delta(1 - h) \geq 1$. □

Corollary 3.4. For a tracial subalgebra $A$ of $M$, the following are equivalent:

\[ \Delta(1 - h) \geq 1 \]
(a) $A + A^* = M$ (that is, $A$ is a noncommutative $H^\infty$ algebra),
(b) $A$ has factorization,
(c) $A$ is logmodular.

Proof. That (a) $\Rightarrow$ (b) was proved in [1, 4.2.1]. That (b) $\Rightarrow$ (c) is evident from the definition.
Let $h \in L^1(M)$ be given. We will write $h \in (A + A^*)_\perp$ for the claim that $(h, a) = \tau(ha^*) = 0$ for each $a \in A + A^*$. If $A$ is logmodular, to show that $A + A^*$ is weak* dense in $M$, it suffices to show that if $h \in (A + A^*)_\perp$, then $h = 0$. Since $A + A^*$ is selfadjoint, it is easy to see that $h \in (A + A^*)_\perp$ if and only if $h + h^* \in (A + A^*)_\perp$, and $\frac{1}{2}(h - h^*) \in (A + A^*)_\perp$. We may therefore assume that $h$ is selfadjoint. By Lemma 3.3, $\Delta(1 - th) \geq 1$ for every $t \in \mathbb{R}$. By Lemma 2.8.

It follows from Corollary 3.4 and [13], that a logmodular tracial subalgebra $A$ satisfies Jensen’s inequality and Szegő’s theorem. In fact, one can prove this directly. Indeed a more precise result is stated next, for which we will need the following definitions:

\[ S_1 = \{ b : b \in M^{-1} \cap M_+, \Delta(b) \geq 1 \}, \]
\[ S_2 = \{ a^* a : a \in A, \Delta(\Phi(a)) \geq 1 \}, \]
\[ S_3 = \{ a^* a : a \in A^{-1}, \Delta(a) \geq 1 \}. \]

Proposition 3.5. (a) For any tracial subalgebra, Szegő’s theorem implies Jensen’s inequality, and Jensen’s inequality implies Jensen’s formula. If $A$ is logmodular, then Szegő’s theorem holds.
(b) For any tracial subalgebra $A$, $S_3 \subset S_1$; and $A$ is logmodular if and only if $S_1 = S_3$.

Proof. (b) It is clear that $S_3 \subset S_1$. If $A$ is logmodular, and $b \in S_1$, there is a sequence $(c_m) \subset A^{-1}$ with $c_m^* c_m \to b$. By [1, 4.3.1], $\Delta(c_m)^2 = \Delta(c_m c_m^*) \to \Delta(b)$. Letting $d_m = \frac{\Delta(b)}{\Delta(c_m)} c_m$, we have $d_m^* d_m \to b$, and $\Delta(d_m) = \sqrt{\Delta(b)} \geq 1$. Thus $b \in S_3$. Thus $S_1 = S_3$. Conversely, if $S_1 = S_3$, then a simple approximation argument shows that $A$ is logmodular. Namely, note that if $b \in M^{-1} \cap M_+$, we have $\Delta(b) \geq \epsilon \Delta(1) > 0$, where $\epsilon > 0$ is chosen so that $b \geq \epsilon 1$. Then by scaling, we may assume that $\Delta(b) = 1$. Then $b \in S_1 \subset S_3$, giving the desired approximations. This completes the proof of (b).

(a) Our proof is based on [1, 4.4.3]. The fact that Szegő’s theorem implies Jensen’s inequality, follows from a slight modification of the argument Arveson used in [1, 4.4.3] to show that Szegő’s theorem implies Jensen’s formula. The main change is that we begin the proof by choosing $a \in A$ (as opposed to $A^{-1}$). We obtain the inequality
\[ \inf_T \rho(|D + T|^2) \geq \phi(|D\Phi(a)|^2), \]
in line 2 of p. 613 of [1], instead of an equality. Following the argument until line 6 of that page, we obtain $\Delta(\Phi(a)) \leq \Delta(a)$ as required.

To see that Jensen’s inequality implies Jensen’s formula, is the calculation on lines 8 and 9 of p. 613 of [1].

As noted on p. 612 of [1], Arveson’s Szegő theorem is
\[ \inf \{ \tau(hb) : b \in S_1 \} = \inf \{ \tau(hb) : b \in S_2 \}, \quad h \in L^1(M)_+. \]
We will show that if A is logmodular, then
\[ \overline{S}_1 = \overline{S}_2 = \overline{S}_3. \]
This, together with the fact that \( \tau(h \cdot) \) is norm-continuous on \( M \), will complete the proof of (a).

If A is logmodular, we have \( S_3 \subset S_2 \) by Proposition 3.1. In the light of (b), to complete the proof we need only show that \( S_2 \subset \overline{S}_3 \). To this end, let \( a \in A \) with \( \Delta(\Phi(a)) \geq 1 \). For fixed \( n \in \mathbb{N} \), by logmodularity there exists a sequence \( (c_m) \subset A^{-1} \) with \( a^*a + \frac{1}{n}1 = \lim_m c_m^* c_m \). Thus for any \( \varepsilon > 0 \) there exists an \( N_\varepsilon \) such that \( a^*a + \frac{1}{n}1 \leq (1 + \varepsilon)c_m^* c_m \) for all \( m \geq N_\varepsilon \). By passing to a subsequence, we can assume that
\[
(3.3) \quad a^*a + \frac{1}{n}1 \leq (1 + \frac{1}{m})c_m^* c_m, \quad m \in \mathbb{N}.
\]
Since \( \Phi \) is completely positive, we may use the Kadison-Schwarz inequality (i.e. \( \Phi(x)^* \Phi(x) \leq \Phi(x^*x) \)), to see that
\[
(3.4) \quad \Phi(c_m^{-1})^* \Phi(a)^* \Phi(\Phi(c_m^{-1}) = \Phi(ac_m^{-1})^* \Phi(ac_m^{-1}) \leq \Phi((c_m^{-1})^* (a^*a + \frac{1}{n}1)c_m). \]
Combining (3.3) and (3.4), we have
\[
\Phi(c_m^{-1})^* \Phi(a)^* \Phi(\Phi(c_m^{-1}) \leq \Phi((1 + \frac{1}{m})1) = (1 + \frac{1}{m})1.
\]
Left and right multiplying by \( \Phi(c_m)^* \) and \( \Phi(c_m) \) respectively, we have
\[
(3.5) \quad \Phi(a)^* \Phi(a) \leq (1 + \frac{1}{m})\Phi(c_m)^* \Phi(c_m), \quad m \in \mathbb{N}.
\]
It follows from Proposition 3.1 and Proposition 4.3.1, that
\[
\Delta(c_m^* c_m) = \Delta(c_m)^2 = \Delta(\Phi(c_m))^2 = \Delta(\Phi(c_m)^* \Phi(c_m)).
\]
From this and (3.5), and facts in Proposition 4.3.1, we deduce that
\[
\Delta(a^*a + \frac{1}{n}1) = \lim_m \Delta(c_m^* c_m) \geq \lim_m \frac{\Delta(\Phi(a)^* \Phi(a))}{1 + m^{-1}} = \Delta(\Phi(a))^2 \geq 1.
\]
Thus \( a^*a + \frac{1}{n}1 \in S_1 \subset \overline{S}_3 \). Taking the limit over \( n \), we find \( a^*a \in \overline{S}_3 \) as desired. \( \square \)

4. The unique normal state extension property

In [14], Lumer showed the importance of the ‘uniqueness of representing measure’ criterion to the generalized \( H^p \) theory. Shortly thereafter, Hoffman and Rossi showed that in the setting considered in [23], condition [14] characterized weak* Dirichlet algebras. Note that in their setting, \( \tau|_A = \Phi|_A \) is multiplicative. Lumer, on the other hand, required there to be a unique state on \( L^\infty \) (that is, a unique probability measure on the maximal ideal space of \( L^\infty \)) extending \( \tau|_A \).

In our noncommutative context, there are several conditions, besides [14], which present themselves as generalizations of the ‘unique extension’ properties mentioned in the last paragraph. For example, one could consider the condition that there be a unique completely positive extension of \( \Phi|_A \). A stronger condition yet, is that every completely contractive representation\(^2\) of \( A \) has a unique completely positive extension to \( M \). It is known that noncommutative \( H^\infty \) algebras satisfy this latter condition [13]. Although this condition does have some interesting consequences, we

\( ^2 \)See e.g. [1] for the definition of these terms, if needed.
Lemma 4.1. For a tracial subalgebra $A$ of $M$, the unique normal state extension property $(\text{I})$ is equivalent to:

\begin{equation}
\text{(4.1)} \quad \text{If } g \in L^1(M)_+ \text{ and } \tau(fg) = 0 \text{ for all } f \in A_0, \text{ then } g \in L^1(D).
\end{equation}

Proof. Suppose that $(\text{4.1})$ holds. If $h \in L^1(M)_+$ with $\tau(ha) = \tau(a)$ for all $a \in A$, then taking $a \in A_0$, we conclude that $h \in L^1(D)$. We also have $\tau(hd) = \tau(d)$ for all $d \in D$, which forces $h = 1$.

Conversely, suppose that $(\text{4.1})$ holds, and that we are given an $h \in L^1(M)_+$, such that $\tau(ha) = 0$ for all $a \in A_0$. We may suppose that $h \geq 1$, by replacing $h$ with $h+1$ if necessary. Then also $\Phi(h) \geq 1$. If $a \in A$, we then have

$$
\tau(h\Phi(a)) = \tau(\Phi(h\Phi(a))) = \tau(\Phi(h)\Phi(a)) = \tau(\Phi(h)a).
$$

In the last line we have used several properties of $\Phi$ which are obvious for $\Phi$ considered as a map on $M$, and which are easily verified for the extension of $\Phi$ to $L^1(M)$. Hence

$$
\tau(ha) = \tau(h\Phi(a)) + \tau(h(a - \Phi(a))) = \tau(h\Phi(a)) = \tau(\Phi(h)a).
$$

Since $\Phi(h) \geq 1$, $\Phi(h)^{-1} \in D_+$. We have $\Phi(h)^{-\frac{1}{2}}a\Phi(h)^{-\frac{1}{2}} \in A$ for every $a \in A$. Setting $\tilde{h} = \Phi(h)^{-\frac{1}{2}}h\Phi(h)^{-\frac{1}{2}}$, we have $\tilde{h} \in L^1(M)_+$, and by the previous centered equation,

$$
\tau(ha) = \tau(h(\Phi(h)^{-\frac{1}{2}}a\Phi(h)^{-\frac{1}{2}})) = \tau(\Phi(h)\Phi(h)^{-\frac{1}{2}}a\Phi(h)^{-\frac{1}{2}}) = \tau(a),
$$

for any $a \in A$. By $(\text{I})$, $\tilde{h} = 1$, so that $h = \Phi(h) \in L^1(D)$. \hfill $\square$

Theorem 4.2. Let $A$ be a tracial subalgebra of $M$ which satisfies Szegő’s theorem. Then $A$ has the unique normal state extension property $(\text{I})$.

Proof. Suppose that we are given an $h \in L^1(M)_+$, such that $\tau(ha) = \tau(a)$ for all $a \in A$. Then $\tau(ha) = 0$ for all $a \in A_0$, and hence also for all $a \in A_0^*$, since $\tau(ha^*) = \tau(ha)$.

If $a \in A_0$, and $d \in D$, then

$$
\tau(ha + d^2) = \tau(ha^2 + hd^*a + ha^*d + h|d|^2) = \tau(h^2a^2 + h^2d^2 + |d|^2) \geq \tau(|d|^2).
$$

Appealing to Szegő’s theorem, we deduce that

$$
\Delta(h) = \inf\{\tau(|d|^2) : d \in D^{-1}, \Delta(d) \geq 1\}.
$$

By $(\text{2.4})$, and by $(\text{2.3.1})$, we have

$$
\tau(|d|^2) \geq \Delta(|d|^2) = \Delta(|d|)^2 = \Delta(d)^2.
$$

It follows that

$$
\Delta(h) = \inf\{\tau(|d|^2) : d \in D^{-1}, \Delta(d) \geq 1\} = 1.
$$

By hypothesis, we also have $\tau(h) = \tau(1) = 1$. We now consider the von Neumann algebra $M_0$ generated by $h$. With notations as in the proof of Lemma $(\text{2.4})$, we have $\int_{\Omega} h \, d\mu_\tau = 1 = \exp(\int_{\Omega} \log h \, d\mu_\tau)$. It is an elementary exercise in real analysis to show that this forces $h = 1$. (Letting $k = \log h$, we have $\int k = 0$ and $\int e^k = 1$. If $r = e^k - k - 1$ then $r$ is a nonnegative function, but $\int r = 0$. Thus $r = 0$, forcing $k = 0$.) \hfill $\square$

Definition 4.3. We say that a tracial subalgebra $A$ of $M$ satisfies $L^2$-density, if $A + A^*$ is dense in $L^2(M)$ in the usual Hilbert space norm on that space.
In fact, it follows from basic functional analysis, that $L^2$-density holds automatically for noncommutative $H^\infty$ algebras. Indeed, if $y \in L^2(M)$ with $y \perp A + A'$, then $y \in L^1(M)$, which forces $y = 0$ by the definition of noncommutative $H^\infty$. We will see later that Szegö’s theorem implies $L^2$-density.

**Remark.** For a tracial subalgebra, $L^2$-density is not equivalent to $A$ being a noncommutative $H^\infty$, or satisfying Szegö’s theorem (e.g. see [8, Section 4]).

If $A$ is a tracial subalgebra of $M$, we will write $A_\infty$ for $[A]_2 \cap M$, where $[A]_2$ is the norm-closure of $A$ in $L^2(M)$.

**Theorem 4.4.** If $A$ is a tracial subalgebra of $M$, then so is $A_\infty$, with canonical conditional expectation extending that of $A$. Also, $A = A_\infty$, if $A$ is a noncommutative $H^\infty$ algebra.

**Proof.** To see that $A_\infty$ is weak* closed in $M$, suppose that $x \in M$ was in the weak* closure of $A_\infty$. If $x$ were not in $[A]_2$, then we could find $b \in L^2(M)$ with $b \perp [A]_2$, but $\tau(bx) \neq 0$. But then $b \in L^1(M)$ with $\tau(by) = 0$ for all $y \in A_\infty$, and consequently $\tau(bx) = 0$ since $x$ is in the weak* closure of $A_\infty$. This contradiction shows that $x \in [A]_2 \cap M = A_\infty$, so that $A_\infty$ is weak* closed.

To see that $A_\infty$ is an algebra, one first checks that if $a \in A, b \in A_\infty$, then $ab \in A_\infty$. Indeed, if $(b_n) \subset A$ with $b_n \to b$ in $L^2(M)$, then $ab_n \in A$, and $ab_n \to ab$ in $L^2(M)$. Thus $ab \in [A]_2 \cap M = A_\infty$. If $a \in A_\infty$, and if $(a_n) \subset A$ with $a_n \to a$ in $L^2(M)$, then $a_nb \in A_\infty$ by what we just proved, and $a_nb \to ab$ in $L^2(M)$. Thus again $ab \in [A]_2 \cap M = A_\infty$.

We continue to write $\Phi$ for the canonical conditional expectation from $M$ to $D$ extending the projection from $A$ onto $D$ (e.g. see [3, Theorem 5.6]), and for the further extension to $L^p(M)$ (e.g. see [16, 3.9]). We claim that $\Phi(ab) = \Phi(a)\Phi(b)$ for all $a, b \in [A]_2$. Indeed, if $a_n, b_n \in A$ with $a_n \to a$ and $b_n \to b$ in $L^2(M)$, then $a_nb_n \to ab$ in $L^2(M)$, so that

$$
\Phi(ab) = \lim_n \Phi(a_nb_n) = \lim_n \Phi(a_n)\Phi(b_n) = \Phi(a)\Phi(b),
$$

by the continuity of $\Phi$ on each $L^p(M)$. In particular, $\Phi$ is a homomorphism on $A_\infty$.

By the argument in the second paragraph of the proof of [11, Proposition 2.1.4], $A_\infty \cap A_\infty^\perp = D$.

If $A$ is a noncommutative $H^\infty$, then it is well known that $A_\infty = A$. This follows, for example, from the fact that noncommutative $H^\infty$ algebras are $\tau$-maximal (see [11, Section 2]).

**Remark.** It is interesting that if $A$ satisfies Jensen’s inequality, then $A_\infty$ satisfies Jensen’s formula. To see this, note that by lines 8 and 9 of p. 613 of [11], we need only show that $\Delta(a) \geq \Delta(\Phi(a))$ for every $a \in A_\infty^{-1}$. To prove this, by replacing $a$ by $a\Phi(a)^{-1}$ if necessary, it suffices to show that $\Delta(a) \geq 1$ for every invertible $a \in A_\infty$ with $\Phi(a) = 1$. It is easy to approximate such $a$ in the $L^2$-norm, and hence in $L^1$-norm, by a sequence in $A$ with $\Phi(a_n) = 1$ for each $n$. By [24III.4.10], $(a_n) \to [a]$ in $L^1$-norm. If $\epsilon > 0$ is given, by Proposition 2.4 there exists a $b \in M^{-1} \cap M_+$ with $\Delta(b) \geq 1$, such that

$$
\Delta(a) + \epsilon \geq \tau(b[a]) = \lim_n \tau(b[a_n]) \geq \limsup_n \Delta(a_n).
$$

Thus, $\Delta(a) \geq \limsup_n \Delta(a_n)$, and since Jensen’s inequality holds for $A$, we have

$$
\Delta(a) \geq \limsup_n \Delta(a_n) \geq \limsup_n \Delta(\Phi(a_n)) = \Delta(1) = 1
$$
as required.

**Definition 4.5.** We shall say that a tracial subalgebra of $M$ has *partial factorization*, if whenever $b \in M_+ \cap M^{-1}$, we have $b = |a| = |c|$ for some elements $a, c \in A \cap M^{-1}$, with $\Phi(a)\Phi(a^{-1}) = \Phi(c)\Phi(c^{-1}) = 1$.

We remark that this notion is distinct from the one-sided partial factorization considered in [13].

**Theorem 4.6.** Let $A$ be a tracial subalgebra of $M$.

(a) If $A$ has the unique normal state extension property, then $A_\infty$ has partial factorization. If, further, $A$ satisfies $L^2$-density, then $A_\infty$ has factorization, and is a noncommutative $H^\infty$ algebra.

(b) If $A$ satisfies Szegö’s theorem, then $A$ satisfies $L^2$-density, and hence $A_\infty$ is a noncommutative $H^\infty$ algebra.

**Proof.** (a) Let $b \in M_+ \cap M^{-1}$ be given, and define an inner product on $L^2(M)$ by

$$
\langle f, g \rangle_b = \tau(b^{1/2}f^*gb^{1/2}), \quad f, g \in L^2(M).
$$

The norm $\| \cdot \|_b$ induced by this inner product is equivalent to the usual one. Indeed,

$$
\|f\|_b^2 = \tau(b|f|^2) \leq \|b\|\tau(|f|^2) \leq \|b\|\|b^{-1}\|\|f\|^2 = \|b\|\|b^{-1}\|\|f\|^2.
$$

Let $p$ be the orthogonal projection of 1 into the subspace $[A_0]_2$, taken with respect to the inner product $\langle \cdot, \cdot \rangle_b$. (Here $[A_0]_2$ is the $L^2$-closure of $A_0$ in $L^2(M)$.)

Now $1 - p \in [A_2]$, since $p \in [A_0]_2 \subset [A]_2$. Therefore, by an obvious approximation argument, $a_0(1 - p) \in [A_0]_2$ for all $a_0 \in A_0$. Since $1 - p \perp [A_0]_2$, we have

$$
0 = \langle a_0(1 - p), 1 - p \rangle_b = \tau(b^{1/2}(1 - p^*)a_0(1 - p)b^{1/2}) = \tau(a_0(1 - p)b(1 - p^*)) = \tau(\Phi((1 - p)b(1 - p^*))).
$$

By the hypothesis, combined with Lemma 4.11, we deduce that $(1 - p)b(1 - p^*) \in L^1(D)_+$. Let $\epsilon > 0$ be given such that $b \geq \epsilon 1$. If $d \in D_+$, then

$$
\tau((1 - p)b(1 - p^*)d) = \tau(d^{1/2}(1 - p)b(1 - p^*)d^{1/2}) \geq \epsilon \tau(d^{1/2}(1 - p)(1 - p^*)d^{1/2}).
$$

By the $L^2$-contractivity of $\Phi$ we deduce that

$$
\tau((1 - p)b(1 - p^*)d) \geq \epsilon \tau(|d^{1/2}(1 - p)^2|) \geq \epsilon \tau(|d^{1/2}\Phi(1 - p)|) = \epsilon \tau(d).
$$

This implies that $(1 - p)b(1 - p^*) \geq \epsilon 1$ in $L^1(D)$. Thus this element has a bounded inverse $((1 - p)b(1 - p^*))^{-1} \in D$. Set $e = ((1 - p)b(1 - p^*))^{-1/2} \in D$, and let $a = e(1 - p) \in [A]_2$. Since

$$
1 = e(1 - p)b(1 - p^*)e = |b^{1/2}(1 - p^*)e|^2,
$$

we deduce that $b^{1/2}(1 - p^*)e$, and consequently also $(1 - p^*)e$ and $a = e(1 - p)$, are bounded. Since they belong to $L^2(M)$, we deduce that $a \in M$. Hence $a \in M \cap [A]_2 = A_\infty$. Since $1 = aba^*$, and since $M$ is a finite von Neumann algebra, we also have $1 = ba^*a$, so that $b^{-1} = |a|^2$.

Note that $\Phi(a) = e \Phi(1 - p) = e$, since $\Phi$ is $L^2$-continuous, and $p \in [A_0]_2$. We have $a^{-1} = b(1 - p^*)e$. To see that $\Phi(a)\Phi(a^{-1}) = 1$, first note that for any $d \in D$,

$$
\tau(\Phi(pb(1 - p^*))d) = \tau(pb(1 - p^*)d) = \langle dp, 1 - p \rangle_b = 0,
$$

since $dp$ is in $[A_0]_2$, and $1 - p \perp [A_0]_2$. Since this is true for all $d \in D$, we have $\Phi(pb(1 - p^*)) = 0$. Thus

$$
1 = e \Phi((1 - p)b(1 - p^*))e = e \Phi(a^{-1}) = \Phi(a)\Phi(a^{-1}).
$$
Notice that we now have that $e^{-1}$ is bounded, and this in fact forces $p$ to be bounded. Therefore $p \in M$. However we do not see any use for the latter fact at this point.

An analogous argument, using the inner product $\langle f, g \rangle^b = \tau(b^*fg^*b^*)$, gives $b^{-1} = |c|^2$, for some $c \in A_\infty$, and $1 = \Phi(c)\Phi(e^{-1})$.

Now suppose that $A + A^*$ is norm-dense in $L^2(M)$. Then by an obvious argument, $L^2(M) = [A]_2 \oplus [A_0]_2$, where $[A_0]_2$ is the norm-closure of $A_0$ in $L^2(M)$. For any $a_0 \in A_0$, we have

$$\tau(a^{-1}a_0) = \tau(b(1-p^*)ea_0) = \langle ea_0, 1-p \rangle_b = 0,$$

since $ea_0 \in [A_0]_2$, and $1 - p \perp [A_0]_2$. Thus

$$a^{-1} \in M \cap (L^2(M) \ominus [A_0]_2) = M \cap [A]_2 = A_\infty.$$

We deduce that $A_\infty$ has factorization, and is therefore a noncommutative $H^\infty$ algebra by Corollary 3.4.

(b) Suppose that $A$ satisfies Szeg"o's theorem. We will prove that $A + A^*$ is norm-dense in $L^2(M)$, and then the result follows from (a) and Theorem 2.2. In fact, if $A$ has the normal state extension property, then we will only need Szeg"o's theorem for $h \in M_+$, as opposed to $h \in L^1(M)_+$. For suppose that $k \in L^2(M)$ is such that $\tau(k(A + A^*)) = 0$. We show that the previously stated conditions are enough to then force $k = 0$. By the argument in the proof of Corollary 3.4, we may assume that $k = k^*$. Then $1 - k \in L^1(M)$, so that by (2.2), given $\epsilon > 0$ there exists an element $b \in M_+ \cap M^{-1}$ with $\Delta(b) \geq 1$ and $\tau(|(1-k)b|) < \Delta(1-k) + \epsilon$. By scaling, we may assume that $\Delta(b) = 1$, and hence $\Delta(b^{-2}) = 1$ by the multiplicativity of $\Delta$. By Szeg"o's theorem, there exists an $a_0 \in A_0$, and an invertible $d_0 \in D$ with $\Delta(d_0) \geq 1$, such that

$$1 - \epsilon = \Delta(b^{-2}) - \epsilon \geq \tau(b^{-2}|d_0 + a_0|^2).$$

We modify the argument at the beginning of the proof, but with $b$ replaced by $b^{-2}$. Thus we consider the inner product $\langle x, y \rangle_{b^{-2}} = \tau(b^{-2}y^*x)$ above, but now we let $p$ be the projection of $d_0$ onto the subspace $[A_0]_2$. By a simple modification of the earlier argument, $(d_0 - p)b^{-2}(d_0 - p)^* \in L^1(D)_+$, and there are constants $K_0, K_1 > 0$ such that for any $d \in D_+$,

$$\tau((d_0 - p)b^{-2}(d_0 - p)^*d) \geq K_0 \tau(d^{1/2}|d_0^*d|^{1/2}) \geq K_0 K_1 \tau(d).$$

We conclude, as before, that $(d_0 - p)b^{-2}(d_0 - p)^*$ has a bounded inverse in $D$, and we set $e = ((d_0 - p)b^{-2}(d_0 - p)^*)^{-\frac{1}{2}}$. As before, $a = e(d_0 - p) \in A_\infty$, $e^{-1} \in D$, and $b^2 = a^*a$. If $a_n \in A$ with $a_n \to a$ in $L^2$-norm, and if $d \in D$, then by Hölder's inequality we have that $a_n k d \to akd$ in $L^1$-norm. Thus $\tau(akd) = 0$, and hence it follows, by a routine argument that we have used several times already, that $\Phi(ak) = 0$. By the $L^1$-contractivity of $\Phi$, and (2.2), we have

$$\tau(|a - ak|) \geq \tau(||\Phi(a - ak)||) = \tau(||\Phi(a)||) = \tau(|ed_0|) \geq \Delta(ed_0) = \Delta(e) \Delta(d_0) \geq \Delta(e).$$

Hence

$$\tau(|(1-k)b|) = \tau(|b(1-k)|) = \tau(||a(1-k)||) = \tau(|a(1-k)|) \geq \Delta(e).$$

On the other hand, equation (4.2) informs us that $\|d_0 + a_0\|_{b^{-2}}^2 \leq 1 - \epsilon$, and so by definition of $p$ we have $\|d_0 - p\|_{b^{-2}}^2 \leq \|a_0 + d_0\|_{b^{-2}}^2 \leq 1 - \epsilon$. Thus, by (2.4) and the definition of $\| \cdot \|_{b^{-2}}$,

$$\Delta(e^{-2}) \leq \tau(e^{-2}) = \|d_0 - p\|_{b^{-2}}^2 \leq 1 - \epsilon.$$
By the multiplicativity of $\Delta$, we have $\Delta(e) \geq \frac{1}{\sqrt{1-\epsilon}}$. We conclude that

$$\Delta(1-k) \geq \tau(|(1-k)b|) - \epsilon \geq \Delta(e) - \epsilon \geq \frac{1}{\sqrt{1-\epsilon}} - \epsilon.$$ 

Since $\epsilon > 0$ is arbitrary, $\Delta(1-k) \geq 1$. As in the proof of Corollary 3.4, we conclude that $k = 0$. Thus $A + A^*$ is norm-dense in $L^2(M)$. □

Corollary 4.7. If $A$ is a $\tau$-maximal tracial subalgebra of $M$ satisfying the unique normal state extension property, then $A$ is a noncommutative $H^\infty$ algebra.

Proof. Let $A$ be a $\tau$-maximal tracial subalgebra of $M$. If $b \in A_\infty$, then there exists a sequence $(a_n) \subset A$ with $L^2$-limit $b$. If $c \in A_0$, then $a_n c \to bc$ in $L^2(M)$, and

$$\tau(bc) = \lim_n \tau(a_n c) = 0.$$ 

Thus $b \in A$. Therefore $A_\infty = A$. By the last part of the proof of Theorem 4.6, $\tau(a^{-1}c) = 0$ too, so that $a^{-1} \in A$. Thus $A$ has factorization, and is therefore a noncommutative $H^\infty$ algebra by Corollary 3.4. □

5. Maximal algebras

If $A$ is a tracial subalgebra of $M$, with canonical projection $\Phi : A \to D$, then as observed in [3], $\Phi$ extends canonically to a conditional expectation from $M$ onto $D$, which we continue to write as $\Phi$, such that $\tau \circ \Phi = \Phi$. We say that $A$ is maximal if there is no properly larger tracial subalgebra of $M$ with conditional expectation $\Phi$. Equivalently, there is no properly larger tracial subalgebra $B$ of $M$ whose conditional expectation onto $B \cap B^*$ extends the one on $A$. In this section, we will also consider tracial subalgebras which satisfy $L^2$-density (resp. the unique normal state extension property (1.1)). In this case, any larger tracial subalgebra clearly also has these properties, and so $A$ is maximal if and only if it is maximal among the tracial subalgebras with these properties and with conditional expectation $\Phi$.

Exel showed that noncommutative $H^\infty$ algebras are automatically maximal [5]. We will have to establish this for algebras satisfying some of our other conditions studied above.

Lemma 5.1. Let $A$ be a tracial subalgebra of $M$ with conditional expectation $\Phi$, which satisfies $L^2$-density. Then there is a unique largest tracial subalgebra of $M$ with conditional expectation $\Phi$ containing $A$, namely:

$$\{x \in M : \tau(xa) = 0 \text{ for all } a \in A_0\}.$$ 

In particular, if $A$ is maximal and satisfies $L^2$-density, then $A$ is $\tau$-maximal, that is $A = \{x \in M : \tau(xa) = 0 \text{ for all } a \in A_0\}$.

Proof. This follows by adapting the arguments in [11] Section 2.2. In particular, using the $L^2$-density as opposed to weak* density to simplify the argument from about the middle of p. 583 onwards. Note that $\tau(xa) = 0$ for all $a \in A_0$ if and only if $\tau(xad) = \tau(\Phi(xa)d) = 0$ for all $a \in A_0$, and if and only if $\Phi(xa) = 0$ for all $a \in A_0$. □

Theorem 5.2. Let $A$ be a tracial subalgebra of $M$ which satisfies Szegö’s theorem (or simply $L^2$-density and the normal state extension property). Then $A$ is maximal, and $A$ is a noncommutative $H^\infty$ algebra.
Proof. It suffices, by Lemma 5.1 and Corollary 1.7 to show that $A$ is maximal. To do this we follow the proof of the main theorem in [6], indicating how it may be adapted to show that $A$ agrees with the maximal algebra guaranteed by Lemma 5.1. Specifically, we note that if $\xi \in L^2(M)$, and if $\langle x\xi, \xi \rangle = \langle \Phi(x)\xi, \xi \rangle$ for all $x \in A + A^*$, then $\tau(x\xi^*) = \langle x\xi, \xi \rangle = 0$ if $x \in A_0$. By (1.10), we deduce that $\xi^* \in L^1(D)$. It follows that for any $x \in M$, we have

$$\langle \Phi(x)\xi, \xi \rangle = \tau(\Phi(x)\xi \xi^*) = \tau(r(\Phi(x)\xi^*)) = \tau(x\xi^*) = \langle x\xi, \xi \rangle,$$

as needed for the proof in the middle of page 779 of [6] to proceed. The next obstacle one encounters is that the first centered equation on [6, p. 780] is not clear for the vector $\xi_1$ there. However, because $A$ satisfies $L^2$-density, this equation is easily seen to be equivalent to the next group of centered equations. In the notation of that paper, we have $u\overline{D\xi_1} \subset D\delta = L^2(D)$. Since $u$ commutes with $D$, it is easy to see that $u\overline{\xi_1}$ is a separating vector for the action of $D$ on $L^2(D)$. It follows, by [6, Exercise 9.6.2] for example, that $u\overline{\xi_1}$ is a cyclic vector for the $D$ action. Thus $Du\overline{\xi_1} = D\delta$. That is, $\delta = \lim_n d_n u\overline{\xi_1}$ for a sequence $(d_n) \subset D$. We have

$$u\overline{A_0\xi_1} \subset \overline{A_0u\overline{\xi_1}} \subset \overline{A_0D\delta \subset A_0\delta}.$$

Conversely,

$$a_0\delta = \lim_n a_0d_n u\overline{\xi_1} \in \overline{uA_0\xi_1}, \quad a_0 \in A_0.$$

Thus $u\overline{A_0\xi_1} = \overline{A_0\delta}$. Similarly, $u\overline{A_0\xi_1} = \overline{A_0\delta}$. As we said above, this gives the first centered equation on [6, p. 780]. The rest of the proof is unchanged. \(\square\)

6. Other characterizations of noncommutative $H^\infty$ algebras

We now turn to the remaining items in the list in [5, Section 3] of conditions equivalent to logmodularity. We recall that a simply (right) invariant subspace of $L^2(M)$, is a closed subspace $M$ of $L^2(M)$ such that $MA \subset M$, and the closure of $MA_0$ is properly contained in $M$. We have:

**Theorem 6.1.** Let $A$ be a tracial subalgebra of $M$, and consider the following properties:

(a) (Invariant subspace theorem) Every simply right invariant subspace of $L^2(M)$ is of the form $u[\mathcal{A}]_2$, for a unitary $u$ in $M$,

(b) (Beurling-Nevidjilina factorization) Whenever $f \in L^2(M)$ with $f \notin [fA_0]_2$, then $f = uh$, for a unitary $u$ in $M \cap [fA]_2$ and an $h$ with $[hA]_2 = [A]_2$,

(c) $A_\infty$ is a noncommutative $H^\infty$ algebra,

(d) $A$ is a noncommutative $H^\infty$ algebra.

Then (a) \(\Rightarrow\) (b) \(\Rightarrow\) (c). If $A$ is antisymmetric (that is, $D$ is one-dimensional), then (d) \(\Rightarrow\) (a). If $A$ has the unique normal state extension property, then (c) \(\Rightarrow\) (d).

**Proof.** That (a) \(\Rightarrow\) (b) \(\Rightarrow\) (c) follows just as in [5], for example, with insignificant modifications. One also needs to use the fact about invertibility in a finite algebra stated at the end of Section 1. The assertion about antisymmetric algebras is also essentially just as in [5] (see also [10]). Finally, if $A_\infty$ is a noncommutative $H^\infty$ algebra then it satisfies $L^2$-density. Therefore it is clear that $A$ satisfies $L^2$-density. Appealing to Theorem 5.2 we obtain (d). \(\square\)

As the last Theorem shows, it is of interest to know whether $A_\infty$ being a noncommutative $H^\infty$ algebra implies that $A$ is a noncommutative $H^\infty$ algebra. We end the paper with another sufficient condition under which this holds.
Lemma 6.2. Let $A$ be a tracial subalgebra of $M$. Then
\[
\{ x \in L^1(M) : \tau(xa) = 0 \text{ for all } a \in A \} = \{ x \in L^1(M) : \tau(xa) = 0 \text{ for all } a \in A_0 \} \cap \ker(\Phi).
\]
(Here we identify $\Phi$ with its extension to $L^1(M)$.) Moreover for any $y \in \{ x \in L^1(M) : \tau(xa) = 0 \text{ for all } a \in A_0 \}$, we have $\Phi(by) = \Phi(b)\Phi(y)$ for all $b \in A$.

Proof. Let $x \in L^1(M)$ be given. It is easy to see that $\tau(ax) = 0$ for all $a \in A$ if and only if $\tau(ax) = 0$ for all $a \in A_0$ and $\tau(dx) = 0$ for all $d \in \mathcal{D}$. Since for any $d \in \mathcal{D}$ we have $\tau(dx) = \tau(\Phi(dx)) = \tau(d\Phi(x))$ and since $\Phi(x) \in L^1(\mathcal{D})$, the first claim follows.

To see the second claim, let $y \in \{ x \in L^1(M) : \tau(xa) = 0 \text{ for all } a \in A_0 \}$ be given. For any $d \in \mathcal{D}$ and $a \in A_0$, we then have
\[
\tau(d\Phi(ay)) = \tau(\Phi(ady)) = \tau((da)y) = 0.
\]
Since $\Phi(ay) \in L^1(\mathcal{D})$, this suffices to show that $\Phi(ay) = 0$ for all $a \in A_0$. But then given any $b \in A$, we will surely have
\[
\Phi(by) = \Phi(b)\Phi(y) + \Phi(b - \Phi(b))y = \Phi(b)\Phi(y) = \Phi(b)\Phi(y).
\]
This is the desired identity. □

By analogy with the commutative context, the space $\tilde{M}$ mentioned at the end of the introduction may also be equipped with a topology of convergence in measure, in such a way that each $L^p(M, \tau)$ injects continuously into $\tilde{M}$ (see [20, 0, 25, 17] for details). With respect to this topology, $\tilde{M}$ becomes a complete Hausdorff topological $*$-algebra with respect to the ‘strong’ sum and product.

Corollary 6.3. Let the canonical extension of $\Phi$ to $L^1(M)$ be continuous with respect to the topology of convergence in measure. Then
\[
\{ x \in L^1(M) : \tau(xa) = 0 \text{ for all } a \in A_0 \} = \{ x \in L^1(M) : \tau(xa) = 0 \text{ for all } a \in (A_\infty)_0 \}.
\]

Proof. Since $A_0 \subset (A_\infty)_0$, the one inclusion is trivial. For the converse suppose that we are given $y \in \{ x \in L^1(M) : \tau(xa) = 0 \text{ for all } a \in A_0 \}$. By the lemma we then have that $\Phi(by) = \Phi(b)\Phi(y)$ for all $b \in A$. Given any $a_0 \in (A_\infty)_0$ we may now select $(a_n) \subset A_0$ so that $(a_n)$ converges to $a_0$ in $L^2$-norm. Thus $(a_n)$ converges to $a_0$ in the topology of convergence in measure (e.g. see [17, Theorem 5]). Since $\tilde{M}$ is a topological algebra in this topology [17, 20, $a_n \to a_0$ in this topology. Therefore if the extension of $\Phi$ to $L^1(M)$ is indeed continuous with respect to this topology, then
\[
0 = \lim_n \Phi(a_n)\Phi(y) = \lim_n \Phi(a_ny) = \Phi(a_0y).
\]
This clearly forces $0 = \tau(\Phi(a_0y)) = \tau(a_0y)$ as required. □

Theorem 6.4. Suppose that the canonical extension of $\Phi$ to $L^1(M)$ is continuous with respect to the topology of convergence in measure. Then $A_\infty$ is a noncommutative $H^\infty$ algebra if and only if any one of the equivalent conditions (a)–(f) in Theorem [17] holds. In particular, if $A_\infty$ is a noncommutative $H^\infty$ algebra, then $A = A_\infty$.

Proof. Suppose that $A_\infty + A_\infty^*$ is weak* dense in $M$, and that the extension of $\Phi$ to $L^1(M)$ is continuous in the topology of convergence in measure. Let $g \in L^1(M)$ be given with $g \perp A + A^*$. To prove the result, by Theorem [14] it is enough to
show that then \( g = 0 \). It clearly suffices to show that if \( g \perp A \) then \( g \perp A_\infty \). This is in turn a trivial consequence of the preceding results applied to \( g^* \).

\[ \square \]

**Closing Remark.** Although most results in Arveson’s paper \cite{1} are stated for finite subdiagonal subalgebras of von Neumann algebras with a faithful normal tracial state, he also considers subdiagonal subalgebras of general von Neumann algebras. It would be interesting if there was some way to extend some of our results to this context. See e.g. \cite{27} for recent work on the question of maximality for this larger class of algebras.

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