Transition between self-focusing and self-defocusing in nonlocally nonlinear media

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We reveal the relevance between the nonlocality and the focusing/defocusing states in nonlocally nonlinear media, and predict a novel phenomenon that the self-focusing/self-defocusing property of the optical beam in the nonlocally nonlinear medium with a sine-oscillation response function depends on its degree of nonlocality. The transition from the focusing nonlinearity to the defocusing nonlinearity of the nonlocally refractive index will happen when the degree of nonlocality of the system goes across a critical value, and vice versa. Bright and dark soliton solutions are obtained, respectively, in the focusing state and in the defocusing state, and their stabilities are also discussed.

It is mentioned that such a phenomenon might be experimentally realized in the nematic liquid crystal with negative dielectric anisotropy or in the quadratic nonlinear medium.

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The optical Kerr effect (OKE) [1,2], as one of the most important effects in nonlinear optics, is a fundamental and widespread phenomenon in the nonlinear interactions of light with materials, such as semiconductors [3], polymers [4], liquid crystals [5,7], soft matters [5], photorefractive [5] and thermal [10–12] media. The equivalent OKE can also be found in optical quadratic nonlinear processes [13–15], and the other physical systems, such as Bose-Einstein condensates [16], quantum electron plasmas [17], and even on the surface of water [18]. The OKE refers to the light-intensity dependence of the refractive index n, that is, n = n0 + N, where n0 is its linear part and N is the light-induced nonlinear refractive index (NRI). Optical solitons [2,19] are the main phenomena resulting from the OKE.

The OKE is of two important intrinsic properties: the nonlocality and the focusing/defocusing. The NRI exhibits generally the nonlocality both in space and time [3]. In consideration of the spatial nonlocality in bulk materials, the NRI can be expressed phenomenologically as [3,20] N = n2 \int_{-\infty}^{+\infty} R(x-x')|E(x', z)|^2 dx', where n2 is the Kerr coefficient that is determined by material properties, the symmetric R(x) is the response function of the media and E the optical field. If R(x) becomes δ-function, then N = n2|E|^2, which is the well-known local OKE [1,2]. Otherwise, the nonlocality is non-negligible. Systematic study on the nonlocality began with the work by Snyder and Mitchell [21]. Their work has attracted lots of attentions [3,7,22,25], and experiments about the spatial nonlocality have been carried out in nematic liquid crystals [20], lead glasses [10], paraffin oils [11], and rhodamine aqueous solutions [12] for deeper and extensive investigations. On the other hand, the focusing/defocusing of the OKE refers to the phenomenon that the optical beam propagating in the bulk medium with the homogeneous n0 can focus or defocus itself by its induced NRI [1,2]. The material with n2 > 0 or n2 < 0 is called the self-focusing medium or the self-defocusing one, respectively [20]. It is commonly considered that the focusing/defocusing property is determined only by the medium properties, and has nothing to do with the nonlocality. In other words, the focusing/defocusing is irrelevant to the property of optical beams propagating in the medium.

In this letter, we will revisit the focusing/defocusing property of the media, and discover a dramatic relation between the focusing/defocusing and the nonlocality in the nonlocally nonlinear medium with a sine-oscillation response function, which was introduced in the study of quadratic solitons, first obtained by Nikolov et al., [14] and then mentioned in the other works [15,27]. By defining the focusing and defocusing states, we have found that in such a system there exist the focusing and defocusing states, and their inter-transition, which are related to the degree of nonlocality. Extensive discussions are also presented, including the bright and dark soliton solutions and their stability in the focusing and defocusing states, respectively.

Theoretical model.— We consider the propagation of the optical beam along z axis in a nonlocally nonlinear medium described by the system of equations for dimensionless complex optical field amplitude φ(x, z) and non-
linear refractive index $\Delta n(x, z)$ given by
\begin{equation}
\frac{\partial \phi}{\partial z} + \frac{1}{2} \frac{\partial^2 \phi}{\partial x^2} + \Delta n \phi = 0, \quad \text{(1a)}
\end{equation}
\begin{equation}
w_m^2 \frac{d^2 \Delta n}{dx^2} + \Delta n - s|\phi|^2 = 0, \quad \text{(1b)}
\end{equation}
where $x \in (-\infty, \infty)$ and $z \in [0, \infty)$ stand, respectively, for the transverse and longitudinal coordinates scaled to a beam width and the Rayleigh distance, and $w_m$ is the nonlinear characteristic length (NCL) of the system, and $s = \pm 1$. When $w_m = 0$ (the local case), the NRI $\Delta n = s|\phi|^2$, and the system above is simplified into the well known nonlinear Schrödinger equation $i\partial_x \phi + (1/2)\partial^2_x \phi + s|\phi|^2 \phi = 0$, which has the stable sech-form bright soliton for $s = 1$ and the stable tanh-form dark soliton for $s = -1$, respectively [2, 28]. For the general case of nonzero $w_m$, the NRI in Eq. (??) can be obtained [14, 15, 27]
\begin{equation}
\Delta n(x, z) = s \int_{-\infty}^{\infty} R(x-x')|\phi(x', z)|^2 dx', \quad \text{(2)}
\end{equation}
where $R(x)$ is the response function with the oscillatory form being full of the infinite space of $x$
\begin{equation}
R(x) = \frac{1}{2w_m} \sin \left( \frac{|x|}{w_m} \right), \quad \text{(3)}
\end{equation}
which was first obtained by Nikolov et al. [14]. The generalized degree of nonlocality (GDN) of the system is defined as $\sigma = w_m/w_r$, where the beam width $w_r = (2\int_{-\infty}^{\infty} x^2|\phi|^2 dx/\int_{-\infty}^{\infty} |\phi|^2 dx)^{1/2}$ [29], since the NCL $w_m$ determines the oscillation period of $R(x)$ and does not represent any more the scale occupied by $R(x)$ like the case of the exponential-decay function [30].

Focus/defocusing states and their inter-transition.— To glimpse at the focusing/defocusing property of the nonlinear system described by Eqs. (??), we simulate the propagation of an initial Gaussian beam with the input power $P_0$
\begin{equation}
\phi_0(x) = \phi(x, z)|_{z=0} = \sqrt{\frac{P_0}{\sqrt{\pi} w_0}} \exp \left( -\frac{x^2}{2w_0^2} \right) \quad \text{(4)}
\end{equation}
in Eqs. (??) and (??), and show the typical results of the output beam width after propagating the distance $z = 1$ for the different initial GDN $\sigma_0 (= w_m/w_0)$ (the different $w_0$ and the fixed $w_m$) in Fig. (a). As can be seen in the figure, for a given $P_0$, the output beam widths will be larger or smaller, in the smaller or bigger sides of the $\sigma$-coordinate respectively, than linear case when $s = 1$; the inverse results will be got when $s = 1$. The higher $P_0$, the stronger the effect. It is well known that the optical beam sampling the defocusing nonlinearity expands faster than that of the linear case; and the focusing case follows an opposite trend. Obviously, the focusing/defocusing nonlinearity sampled by the optical beams depends on its GDN $\sigma$ dramatically for the nonlocal system (1). For the case that $s = -1$, the transition from the self-focusing to the self-defocusing will happen when $\sigma$ goes down cross the critical points that are the same and nothing to do with $P_0$, and vice versa. The case that $s = +1$ is on the contrary.

The phenomenon observed above can be well understood through the relationship between the variations of the light intensity and its induced NRI distributions. We define the focusing/defocusing state of the NRI by the variation of the NRI distribution against that of the light intensity on the transverse perpendicular to the propagation direction $z$. The NRI has two states: focusing and defocusing. For the focusing state, the NRI changes uniformly with the intensity, that is, the NRI increases as the intensity increases, and vice versa; The defocusing state is on the contrary. According to the definition, the focusing/defocusing states depend on the convexity-concavity of the NRI curves, and the sufficient condition for the realization of the transition between the focusing/defocusing states is $d^2\Delta n/dx^2|_{x=0} = 0$. For the local OKE [1, 2] and the nonlocal OKE with non-oscillatory $R(x)$ [3, 7, 10, 22, 30], for example, the exponential-decay function [3, 30], the focusing/defocusing states are only determined by $\text{sgn}(s)$ (the Kerr coefficient $n_2$ in the actual physical system). The focusing state appears when $s = 1$ ($n_2 > 0$), and the defocusing state does when $s = -1$ ($n_2 < 0$). The focusing/defocusing property of those two cases is determined only by the medium properties, and has nothing to do with the nonlocality, that is, irrelevant to the property of optical beams propagating in the medium. No transition between them can happen in both cases because $d^2\Delta n/dx^2|_{x=0} \neq 0$.  

**FIG. 1:** (color online) (a) Normalized output beam widths $w_r/w_0$ at $z = 1$ (one Rayleigh distance) as the function of $\sigma_0$, and the dashed black line presents the linear case where the analytic result is $w_r(z)/w_0 = \sqrt{1 + z^2}$ [31]. $w_m = 2$ for all of the curves. (b) Gaussian beam $\phi_0$ (thick solid black line, $w_0 = 5$) and its induced NRI for different $\sigma$ when $s = -1$. The NRI for $\sigma = 10$ is multiplied by 30.
The system \( \Pi \) with a sine-oscillation \( R(x) \), however, can realize the transition because \( d^2\Delta n/dx^2|_{x=0} = 0 \) for some critical points of the GDN. Since the NRI depends on the specific profile of the beams, by taking the Gaussian beam given in Eq. (4), we obtain
\[
\frac{d^2\Delta n}{dx^2}|_{x=0} = -SP_0 \left( \frac{1}{2} \right) \sigma - \sigma / \sqrt{\pi w_0^3},
\]
where
\[
F(x) = \exp(-x^2) \int_0^\infty \exp(i\xi^2) d\xi \text{ is the Dawson function, and a critical GDN } \sigma_c = 0.54. \]
Fig. 4(b) shows the transition process for \( s = -1 \) by giving the different curves of the NRI for different \( \sigma \). For smaller \( \sigma \) such that \( \sigma < \sigma_c \), the NRI is defocusing, and the beam samples the defocusing index, then will be self-defocusing, as can be observed in Fig. 4(a). When \( \sigma \) exceeds \( \sigma_c \), the focusing state of the NRI appears in the center, and both the range and the amplitude of the bell-shaped NRI increases as \( \sigma \) increases. When \( \sigma \) is slightly larger than \( \sigma_c \), moreover, the NRI is partially focusing and partially defocusing such that the central part of the beam samples the focusing index and its edges “sees” the defocusing one. The optical beam as a whole will continue to exhibit the self-defocusing behavior until the effect of the focusing-index dominates. This can explain why \( \sigma_c \) is smaller than the critical points in Fig. 4(a). The exact inverse is the case that \( s = 1 \). The transition from the self-focusing to self-defocusing of the optical beam will happen when the GDN \( \sigma_0 \) goes up across a critical value.

Bright solitons.—In the higher range of the GDN \( \sigma \) (\( \sigma > \sigma_c \)) for the case that \( s = -1 \), optical beams sample the focusing index, and bright solitons can form when the nonlinear effects balance the diffractive effects precisely. By the imaginary-time method \( [32] \), the numerical soliton solutions, \( \phi = u(x) \exp(i\lambda z) \), of Eq. (5) plus (6) do be obtained in the range \( \sigma > \sigma_{bac}(\sigma_{bac} = 1.21)! \) No soliton solutions can be found in the lower range that \( \sigma < \sigma_{bac} \). Shown in Figs. 5 and 5 are the \( \sigma \)-spectrums of the critical power of the soliton \( P_c(= \int u^2 dx) \) and the soliton propagation constant \( \lambda \) (that is, the dependences of \( P_c \) and \( \lambda \) upon \( \sigma \)), and the distributions of the soliton, respectively. All of the bright solitons are shown to be stable \( [33] \) by the linear stability analysis \( [34] \), which also be confirmed by the simulations, as shown in Fig. 5 The bright solitons of the system described by Eqs. (1) [equivalently Eq. (5) plus (6)] with \( s = -1 \) have two abnormal properties. First, the soliton propagation constants \( \lambda \) are negative, which results from the negative \( \Delta n(x) \) \( [35] \). For the bright solitons obtained before in the local non-linear media \( [5, 10] \) and the nonlocally nonlinear media with the non-oscillatory response function \( [2, 7, 22, 23, 34, 37] \), however, their propagation constants are all positive. Second, the slope of the \( P_c(\lambda) \) is negative, as shown in Fig. 5 Therefore, their stability criterion obeys an inverted Vakhitov-Kolokolov stability criterion \( [39] \).

Dark solitons.—In the defocusing side of \( \sigma \) (\( \sigma < \sigma_c \)) for \( s = -1 \), the optical beam samples the defocusing index. The exact dark soliton solution is found to exist in the condition that \( w_m/w_0 \leq 1/2 \) \( [11] \)
\[
\phi(\xi) = \gamma_1 \sqrt{\tanh^2(\xi) + \gamma_2 \tanh^2(\xi) + \gamma_3 \exp(i\varphi)}, \quad (5)
\]
and \( \Delta n = 4w_m^2/w_0^2 + 2/w_0^2 - 1/2w_m^2 - 3 \tanh^2(\xi)/w_0^2 \), where the phase \( \varphi = \theta + (4w_m^4/w_0^4 - v^2/2 - 3/4w_m^2) \xi, \theta = \theta_0 + vx + C(T_+ + T_-) + C\xi(\kappa_+ - 1), \kappa = (x - vx)/w_0, \kappa_\pm = 2/3 - (1 \pm \sqrt{4w_m^4/w_0^6 - 3/(12w_m^2/w_0^2)}, T_+ = \arctan(\tan(\xi)/\sqrt{\kappa_+ - 1})/\sqrt{\kappa_+ - 1} \right), \gamma_1 = 3\sqrt{2w_m/w_0^3}, \gamma_2 = (w_0^2/w_m^2 - 8)/6, \gamma_3 = (4w_m^4/w_0^4 - 4w_m^4/w_0^2 + 1)/6w_m^4/w_0^4 \right). \) The parameter \( v \) denotes the tangent of an incident angle of the dark soliton. Without loss of generality, the case of the normal incidence (\( v = 0 \)) is given in Fig. 6. By defining the width of the dark soliton as \( w_l = \frac{1}{2} \int_0^{\infty} x^2(\xi) - |\phi(\xi)|^2 dx / \int_0^{\infty} (\xi^2 - |\phi(\xi)|^2 dx)^{1/2} \) with \( |\phi(\xi)| \) being the background amplitude, we have its GND \( \sigma = (\sqrt{2}w_m/w_0)/\sqrt{\pi^2/3 + 8(w_m/w_0)^2} \), and then

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**FIG. 2:** (color online) Critical power \( P_c \) and the soliton propagation constant \( \lambda \) versus the GDN \( \sigma \). The inset shows \( P_c \) versus \( \lambda \). All of the results are obtained for \( w_m = 10 \).

**FIG. 3:** (color online) The solitons (solid blue lines) for the small \( \sigma (=1.58) \) (a) and the large \( \sigma (=9.06) \) (b), which are corresponding, respectively, to the filled square and the filled circle in Fig. 4 and solid green lines represent the corresponding NRI. The numeral propagations of the solitons with 5% random noises are shown in (c) and (d), respectively.
find that the $\sigma$-range for the existence of the dark soliton solution is $\sigma \leq \sigma_{\text{dsc}} (= 0.31)$. When $\sigma = 0.31 (w_m/w_0 = 1/2)$, the dark soliton [Eq. (??)] is deduced to the bi-black soliton $|\phi| = |3 \text{sech}^2(x/2w_m) - 2|/2\sqrt{3}w_m$, and has two zero dips, which was obtained in Ref. [13]. When $\sigma < 0$, $0.29 (w_m/w_0 = \sqrt{1/5})$, 0.24, and 0.18 ($w_m/w_0 = 1/4$) from the bottom to the top (up) and from the top to the bottom (down), respectively.

$\sigma \leq 0.24 (w_m/w_0 \leq 1/2\sqrt{2})$, the dark soliton becomes the single gray soliton with one nonzero dip. The dark soliton in the range that $0.24 < \sigma < 0.31$ is the bi-gray soliton with two nonzero dips.

Modulation instability (MI) of system (11) has been discussed, and the main result is that the MI always occurs for the case that $s = -1$ and $w_m = 0$ [33]. Therefore, the dark soliton solutions (??) above are unstable when $w_m \neq 0$ due to the MI. When $w_m = 0$, however, the solution (??) does not work because of the term $w_m$ in denominator. As a result, the system described by Eq. (11) has two different kinds of dark soliton solutions in the range that $\sigma < 0.31$, which are the unstable solutions (??) when $w_m \neq 0$ and the stable solitons [3, 28] when $w_m = 0$, respectively.

The situation that $s = 1$.— As discussed above, optical beams sample, respectively, the defocusing and focusing beams when the boundary is involved, however, the single-hump bright soliton can sometimes exist for the case because the boundary can re-distribute the NRI of the system (11) such that $\Delta n$ can become focusing in the center region for some suitable conditions [27]. The dark solitons in the defocusing side ($\sigma > \sigma_c$), even if they exist, are unstable due to the existing MI. Therefore, we only take the bright soliton in the focusing side ($\sigma < \sigma_c$) into consideration. Because the scope of optical beams is much larger than $w_m$ for the small GND, to numerically search the soliton solution requires extremely high-precision sampling of the NRI, which leads to a huge need for computing sources. Due to the computer source limit, the (single-hump) bright soliton has not been found so far. We cannot be sure, therefore, whether the bright soliton exist in the small side ($\sigma < \sigma_c$) of the $\sigma$-coordinate, although the necessary condition is satisfied. What we can be sure, however, is there exists the sech-form bright soliton in Eqs. (11) in the smallest end that $w_m = 0$ [3, 28].

Possibilities of experimental realization.— The propagation of the (1+1)-dimensional optical beam $\Phi$ in the nematic liquid crystals (NLC) with negative dielectric anisotropy [41] can be described by the model (11) for $s = -1$. Following the procedure in Ref. [41], a set of equations satisfied by $\Phi$ (the optical beam) and $\Psi$ (the light-induced perturbation of the tilt angle of NLC molecules) can be obtained

$$W_m^2 \nabla_{XY}^2 N_L - N_L - n_0^2 |\Psi|^2 = 0, \quad (6a)$$

where

$$N_L = |\epsilon_0^e \sin(2\theta_0)\Psi/2n_0, \quad W_m = \{2\theta_0 K/\epsilon_0 |\epsilon_0^e \sin(2\theta_0) [1 - 2\theta_0 \cot(2\theta_0)] \}^{1/2}/E_{rf},$$

$$n_m^2 = (\epsilon_0^e)^2 \theta_0^2 \sin(2\theta_0)/4n_0 |\epsilon_0^e| E_0^2 [1 - 2\theta_0 \cot(2\theta_0)],$$

$k = n_0 k_0, \beta = \omega/c$, and the other symbols not specified here are the same with those in Ref. [41]. A family of Equations (6) are the standard form of the equation for the Kerr-type nonlocal nonlinear process [3], and $N_L$, $n_m^2 (n_m^2 < 0, \because \epsilon_0^e < 0)$, and $W_m$ are, respectively, the NRI, the Kerr coefficient, and the NCL for the NLC with negative dielectric anisotropy. Introducing the dimensionless transform $\phi = \Phi/\Phi_0$, $\Delta n = N_L/N_{LO}$, $x = X/Y/W_0, z = Z/kW_0^2$, and $w_m = W_m/W_0$, where $\Phi_0 = \sqrt{n_0/n_n^2/w_m^2}/W_0$ and $N_{LO} = 1/k_0^2 W_0^2 n_0$, the system can be expressed in the dimensionless form as

$$i \partial_x \phi + (1/2) \nabla^2_{x y} \phi + \Delta n \phi = 0 \text{ and } w_m^2 \nabla_{x y} \Delta n + \Delta n + |\phi|^2 = 0, \quad (1 + 1) - D \text{ case of which corresponds to Eqs. (11) with } s = -1. \text{ As a matter of fact, the stable bright soliton has been experimentally observed in such a configuration [41]. Since } W_m \text{ can be controlled by the bias voltage [42], we might expect to realize the switching the focusing/defocusing states by the bias voltage in the NLC with negative dielectric anisotropy.}$

The second candidate to experimentally observe the new phenomenon discovered might be the interaction process between the fundamental wave $E_1(X, Z)$ and the second-harmonic wave $E_2(X, Z)$ in quadratic nonlinear media. For the type-I phase matching, the process is described by [14, 15], $i \partial_Z E_1 + \gamma_1 \partial_X^2 E_1 + \chi_1 E_2 E_1^* \exp(-i\Delta k Z) = 0, i \partial_Z E_2 + \gamma_2 \partial_X^2 E_2 + \chi_2 E_1^2 \exp(i\Delta k Z) = 0$, where all symbols are same with those in Ref. [15]. As suggested in Ref. [14, 15], it can be assumed that $E_2 = e_2(X, Z) \exp(i\Delta k Z)$,
and $\partial_{x}^{2}$ can be neglected comparing with the terms left [15, 22]. In this way, we can have the standard equations $i\partial_{z}E_{1} + s_{1}|\phi|^{2}E_{1} + k_{0}N_{Q}E_{1}^{*} = 0$, $s_{2}W_{mQ}\partial_{X}^{2}N_{Q} - N_{Q} + n_{Q}^{2}E_{2}^{2} = 0$, where $N_{Q} = \chi_{2}E_{2}/k_{0}$, $K_{Q} = \chi_{2}X/k_{0}\Delta k$, $W_{mQ} = \sqrt{2}\gamma_{2}/\Delta k$, $s_{1} = \text{sgn}(\gamma_{1})$, and $s_{2} = \text{sgn}(\Delta k\gamma_{2})$. $N_{Q}$, $n_{Q}^{2}$, and $W_{mQ}$ are the equivalent NRI, the equivalent Kerr coefficient, and the NCL for quadratic nonlinear process, respectively. By the dimensionless trasnform $\phi = E_{1}/\bar{E}_{0}, \Delta n = \tilde{N}_{Q}/N_{Q}, x = X/W_{0}, z = 2|\gamma_{1}|Z/W_{0}^{2}$, where $E_{10} = \sqrt{2}\gamma_{1}|k_{0}n_{Q}^{2}|/W_{0} = \sqrt{2}\Delta k\gamma_{1}/\gamma_{1}X/W_{0}, \tilde{N}_{Q} = 2|\gamma_{1}|k_{0}W_{0}^{2}$, the dimensionless system can be obtained

\begin{equation}
\frac{i}{\beta}\frac{\partial\phi}{\partial z} + \frac{s_{1}}{2} \frac{\partial^{2}\phi}{\partial x^{2}} + \Delta n\phi^{*} = 0,
\end{equation}

\begin{equation}
-s_{2}W_{m}^{2}\frac{\partial^{2}\Delta n}{\partial x^{2}} + \Delta n - s_{2}\phi^{2} = 0,
\end{equation}

where $W_{m} = W_{mQ}/W_{0}$ and $s = \text{sgn}(n_{Q}^{2})$. For the spatial case (optical beams), it can be realized that $s = -1$ and $s_{2} = -1$ by choozing $\Delta k < 0$ because $\gamma_{1} \approx 2\gamma_{2} > 0$ in this case [14]; while for the temporal case (optical pulses), both cases of $s = \pm 1$ and $s_{2} = -1$ might be realized [13] by carefully choozing the signs of $\Delta k$, $\gamma_{2}$, and $\gamma_{2}$. Although a set of Equations (7) are slight different from Eqs. (4) in their two last terms ($\phi^{*}$ rather than $\phi$ and $\phi^{2}$ rather than $|\phi|^{2}$), Eqs. (7) and (11) are the same in the stationary state [13,15]. Of course, whether the quadratic nonlinear process can exhibit the transition between the focusing and the defocusing states is still an open euestion, to answer which more careful and deeper investigations should be needed.

Conclusion.— We have discussed the focusing/defocusing property in the nonlocally nonlinear medium with a sine-oscillation response function, and found it depends on generalized degree of nonlocality of the system $\sigma$. The transition between defocusing and focusing states of the nonlinear refractive index will occur when $\sigma$ goes cross a critical value $\sigma_{c}$, which is 0.54 for the Gaussian beam. In the case that $s = -1$, the bright and dark soliton solutions are found to exist, respectively, in the range that $\sigma > 1.21$ (the focusing side) and in the range that $\sigma < 0.31$ (the defocusing side) of the $\sigma$-coordinate. The bright solitons are stable, and the dark soliton solution are unstable due to the existing MI. The theoretical results pave the way to the experimental observation in the nematic liquid crystals with negative dielectric anisotropy and in the quadratic nonlinear media, and are further expected to introduce significant novelties in all-optical devices.

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[1] Y. R. Shen, Principles of Nonlinear Optics (Wiley, New York, 1984), 286-331 (Chapters 16-17).
[2] R. W. Boyd, Nonlinear Optics (Academic Press, Amsterdam, 2008), 3rd ed, 207-275 (Chapters 4-5).
[3] Q. Guo, D. Lu, and D. Deng, Nonlocal spatial optical solitons, Chapter 4 in Advances in Nonlinear Optics, edited by X. Chen et al., (De Gruyter, 2015), 227-305.
[4] M. Sheik-Bahae, D. J. Hagan, and E. W. Van Stryland, Phys. Rev. Lett. 65, 96 (1990).
[5] W. J. Blau, H. J. Byrne, and D. J. Cardin, Phys. Rev. Lett. 67, 1423 (1991).
[6] C. Conti, M. Peccianti, and G. Assanto, Phys. Rev. Lett. 91, 073901 (2003).
[7] G. Assanto, Nematicons: Spatial Optical Solitons in Nematic Liquid Crystals (John Wiley & Sons, Inc, 2013).
[8] C. Conti, G. Ruocco, and S. Trillo, Phys. Rev. Lett. 95, 183902 (2005).
[9] M. Segev et al., Phys. Rev. Lett. 68, 923 (1992).
[10] C. Rotschild et al., Phys. Rev. Lett. 95, 213904 (2005).
[11] A. Dreischuh et al., Phys. Rev. Lett. 96, 043901 (2006).
[12] N. Glofranida et al., Phys. Rev. Lett. 99, 043903 (2007).
[13] A. V. Buryak and Yu. S. Kivshar, Phys. Lett. A 197, 407 (1995).
[14] N. I. Nikolaev et al., Phys. Rev. E 68, 036614 (2003).
[15] B. K. Esbensen et al., Phys. Rev. A 86, 023849 (2012).
[16] P. Pedri and L. Santos, Phys. Rev. Lett. 95, 200404 (2005).
[17] P. K. Shukla and B. Eliasson, Phys. Rev. Lett. 96, 245001 (2006).
[18] A. Chabchoub et al., Phys. Rev. Lett. 110, 124101 (2013).
[19] Y. S. Kivshar and G. P. Agrawal, Optical Solitons: From Fibers to Photonic Crystals (Academic Press Inc, 2003).
[20] Q. Guo et al., Features of strongly nonlocal spatial solitons, Chapter 2 in Nematicons: Spatial Optical Solitons in Nematic Liquid Crystals, edited by G. Assanto, (John Wiley & Sons, Inc, 2013), 37-69.
[21] A. W. Snyder and D. J. Mitchell, Science 276, 1538 (1997).
[22] W. Krölkowski et al., J. Opt. B: Quantum Semiclass. Opt. 6, S288 (2004).
[23] M. Peccianti and G. Assanto, Phys. Rep. 516, 147 (2012).
[24] Q. Guo, Nonlocal spatial solitons and their interactions, in Optical Transmission, Switching, and Subsystems, edited by C. F. Lam et al. Proc. of SPIE 5281, 581-594 (Bellingham, WA, 2004).
[25] W. Krölkowski et al., Nonlocal solitons, in Optical Optics Applications, edited by M. A. Karpierz, A. D. Boardman, and G. I. Stegeman, Proc. of SPIE 5949, 59490B (2005).
[26] C. Conti, M. Peccianti, G. Assanto, Phys. Rev. Lett. 92, 113902 (2004).
[27] J. Wang et al., Opt. Lett. 39, 405 (2014).
By this definition, the beam width \( w_r \) of the Gaussian beam \( \phi_0(x) \) given by Eq. (3) is just right equal to \( w_0 \).

P. D. Rasmussen, O. Bang, W. Krolikowski, Phys. Rev. E 72, 066611 (2005).

H. A. Haus, Waves and Fields in Optoelectronics, (Prentice-Hall, Englewood Cliffs, NJ, 1984), 109-111, 159-160.

M. L. Chiofalo, S. Succi, and M. P. Tosi, Phys. Rev. E 62, 7438 (2000).

See Supplemental Material for the details of the linear stability analysis and the modulation instability analysis.

Z. Xu, Y. Kartashov, and L. Torner, Opt. Lett. 30, 3171 (2005).

Substitution of the soliton solution into Eq. (3) yields
\[
\frac{d^2 u}{dx^2} + 2[\Delta n(x) - \lambda] u = 0. \quad u(x) \text{ can be considered as the localized mode of the self-induced waveguide } \Delta n(x) \text{ at least in some range} \]
\[
\text{\lambda} < 0. \quad \Delta n(0) \text{ can be easily proved to be negative, which is the maximum of } \Delta n(x) \text{ in the scope of the optical beam. Therefore, } \Delta n(x) \text{ is all negative in the scope, as observed in Fig. 1(b) and Fig. 3. Thus we have } \lambda < 0.

Q. Guo, et al., Large phase shift of nonlocal optical spatial solitons, Phys. Rev. E 69, 016602 (2004).

Q. Shou, et. al., Large phase shift of spatial solitons in lead glass, Opt. Lett. 36, 4194 (2011).

A. W. Snyder, S. J. Hewlett., and D. J. Mitchell, Phys. Rev. Lett. 72, 1012 (1994).

H. Sakaguchi and B. A. Malomed, Phys. Rev. A 81, 013624 (2010).

About the detail, see: Y. Hu and S. Lou, Commun. Ther. Phys., submitted.

J. Wang et al., arXiv: 1403.2154v2.

W. Hu et al., Appl. Phys. Lett. 89, 071111 (2006).