HIGHER ORDER SGFEM FOR ONE-DIMENSIONAL INTERFACE ELLIPTIC PROBLEMS WITH DISCONTINUOUS SOLUTIONS

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Abstract. We study a class of enriched unfitted finite element or generalized finite element methods (GFEM) to solve a larger class of interface problems, that is, 1D elliptic interface problems with discontinuous solutions, including those having implicit or Robin-type interface jump conditions. The major challenge of GFEM development is to construct enrichment functions that capture the imposed discontinuity of the solution while keeping the condition number from fast growth. The linear stable generalized finite element method (SGFEM) was recently developed using one enrichment function. We generalized it to an arbitrary degree using two simple discontinuous one-sided enrichment functions. Optimal order convergence in the $L^2$ and broken $H^1$-norms are established. So is the optimal order convergence at all nodes. To prove the efficiency of the SGFEM, the enriched linear, quadratic, and cubic elements are applied to a multi-layer wall model for drug-eluting stents in which zero-flux jump conditions and implicit concentration interface conditions are both present.

Key Words. generalized finite element method, elliptic interface, implicit interface jump condition, Robin interface jump condition, linear and quadratic finite elements.

1. Introduction

Consider the interface two-point boundary value problem

\begin{equation}
\begin{cases}
-(\beta(x)u'(x))' + w(x)u(x) = f(x), & x \in I = (a, b), \\
u(a) = u(b) = 0,
\end{cases}
\end{equation}

where $w(x) \geq 0$, and $0 < \beta \in C[a, \alpha] \cup C[\alpha, b]$ is discontinuous across the interface $\alpha$ with the jump conditions on $u$ and its flux $q := \beta u'$:

\begin{equation}
[u]_\alpha = \lambda F(q^+, q^-, [u']_\alpha), \quad \lambda \in \mathbb{R}, \quad F : [c, d] \to \mathbb{R},
\end{equation}

\begin{equation}
[\beta u']_\alpha = f_\alpha, \quad f_\alpha \in \mathbb{R},
\end{equation}

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where the jump quantity
\[ [s]_\alpha := s(\alpha^+) - s(\alpha^-), \quad \sigma^\pm := s(\alpha^\pm) := \lim_{\epsilon \to 0^\pm} s(\alpha \pm \epsilon). \]
The primary variable \( u \) may stand for the pressure, temperature, or concentration in a medium with certain physical properties and the derived quantity \( q := -\beta u' \) is the corresponding Darcy velocity, heat flux, or concentration flux, which is equally important. The piecewise continuous \( \beta \) reflects a nonuniform material or medium property (we do not require \( \beta \) to be piecewise constant). The function \( w(x) \) reflects the surroundings of the medium. The case of \( \lambda = 0 \) is widely studied, while the case of \( \lambda > 0 \) gives rise to a more difficult situation. For example, the case of rightward concentration flow \([27, 28, 29]\) imposes
\[
\begin{cases}
[u]_\alpha = \lambda (\beta u')(\alpha^-) \\
[\beta u']_\alpha = 0,
\end{cases}
\]
which generates an implicit condition since the left-sided derivative is unknown. Implicit interface conditions abound in higher dimensional applications \([1, 14, 17, 19]\). For definiteness, we will study a class of efficient enriched methods for problem \((1)\) under the jump conditions \((4)\), but our methods apply to problem \((1)\) subject to the general conditions \((2)-(3)\) with a well-posed weak formulation. After a simple calculation, it is easy to see that \((4)\) is equivalent to
\[
\begin{cases}
[u]_\alpha = \gamma [u']_\alpha, \\
[\beta u']_\alpha = 0,
\end{cases}
\]
which is indeed of the type \((2)-(3)\).

The model problem \((1)\), allowing the solution to be discontinuous at the interfaces, is a more general form of interface problem than the ones studied by Babuška et al. using SGFEM \([4, 3, 18]\) and by Li et al. using immersed finite element method (IFEM) \([20, 21, 23, 22]\). In these works, the interface problem is assumed to have a continuous solution at the interfaces. This problem is well-studied in literature. A large class of the methods is developed based on the unfitted meshes, which has been demonstrated to be more efficient than methods using fitted meshes, especially when the interface is moving \([15]\); see also \([7, 8, 12, 13, 16, 27]\).

The generalized FEMs (GFEM) were first introduced to capture certain known features of the solution to the crack problem \([5, 6, 11, 21]\). However, it has been shown that GFEM suffers from a lack of robustness with respect to the mesh configurations and bad conditioning. The conditioning of GFEM can be of order \(O(h^{-4})\) where \( h \) characterizes the mesh size, which is of magnitude worse than that of the standard FEM which is of order \(O(h^{-2})\). With these issues in mind, SGFEM has been developed \([3, 4, 18]\). In particular, SGFEM has two distinguishing features that outperform GFEM: (a) the condition number of SGFEM is not worse than that of the standard FEM; (b) the condition number growth is robust with respect to the mesh configurations. Further development of SGFEM has been active. For example, \([32]\) extends the linear SGFEM to quadratic-order; \([10]\) extends the method for eigenvalue interface problems, and \([31]\) generalizes the SGFEM idea to isogeometric analysis with B-spline basis functions.

When the solution to the underlying interface problem is continuous, in 1-D the single enrichment function associated with the linear SGFEM can be used for high-order elements and the higher-order SGFEM can be developed straightforwardly.
However, for an interface problem whose solution is discontinuous at interfaces, the natural extension with one enrichment function at each interface fails. A more sophisticated construction of enrichment functions is called for, which motivates the present work.

In our enriched finite element method or GFEM, the approximation finite element space $V_h^{enr}$ takes the form:

$$V_h^{enr} := S_h + V_E = \{p_h + q_h \psi : p_h, q_h \in S_h, \psi \in F_{enr}\}$$

where $S_h$ is a standard finite element space (e.g., $\mathbb{P}_k$-conforming, $k \geq 1$), and the function $\psi$ is from the enrichment function space

$$F_{enr} := \text{span}\{\psi_0, \psi_1, \ldots, \psi_m\}, \dim(F_{enr}) = m + 1.$$

Here the basis functions $\psi_i$ are to reflect the interface condition(s) at $\alpha$, e.g., zero or nonzero jump of the function value across $\alpha$. For example, for continuous solution case, a single ($m = 0$) enrichment function suffices, whereas we show in this paper that for discontinuous solution case we need two enrichment functions ($m = 1$) defined in (10). There are some distinct features about $V_h^{enr}$ in this case. Firstly, the subspaces $S_h$ and $V_E$ have nonempty intersection, i.e., $S_h \cap V_E \neq \emptyset$. To construct the basis functions for the space $V_h^{enr}$, it requires weeding out the bubble basis functions associated with the elements containing interfaces from $S_h$ (cf. Theorem 2.3). Secondly, the local shape basis of IFEM utilizes information on discontinuous $\beta$ while GFEM does not. Thus, an enriched method does not require the discontinuous diffusion coefficient to be piecewise constant, which is an advantage. This difference affects the overhead and complexity of the convergence analysis. To carry out the error analysis in a conforming GFEM, we use the principle that the error in the finite element solution $u_h$ should be bounded by the approximation error in the finite element space $V_h^{enr}$:

$$||u - u_h|| \leq C \inf_{v \in V_h^{enr}} ||u - v||.$$

The optimal error estimate is established by demonstrating the existence of an optimal order approximate piecewise polynomial in $V_h^{enr}$. The approach of finding this interpolating polynomial was hinted from Deng and Calo [10] on problem (11) with a continuous solution ($[u]_\alpha = 0$). Therein, they demonstrated the convergence of GFEM solutions in all $\mathbb{P}_p$-conforming spaces ($p \geq 1$) enriched by the well-known hat function [3, 4, 5, 10] (cf. Eq. 51 below). On the other hand, Chou et. al. [9] used a single enrichment function to handle problem (11) with $[u]_\alpha \neq 0$ using the linear GFEM ($p = 1$). In this paper we use two one-sided enrichment functions for each interface and show the effectiveness and convergence of the associated GFEM of all orders, i.e., $p \geq 1$.

In addition to demonstrating optimal order convergence in the $L^2$ and broken $H^1$ norms, we also show superconvergence of nodal errors. More specifically, the organization of this paper is as follows. In Section 2, we state the weak formulation for the implicit interface condition problem (1) with (4), and define enrichment functions and spaces. In Theorem 2.6 we show optimal order convergence in the $L^2$ and broken $H^1$ norms. Furthermore, in Theorem 2.7 we show $2p$ order convergence at the nodes and the exactness of the approximate solution at nodes for the piecewise constant diffusion case. In Section 3, we provide numerical examples of a porous wall model to demonstrate the effectiveness of the present GFEM and confirm the convergence theory. Linear, quadratic, and cubic elements are tested.
Furthermore, following the viewpoint of the SGFEM [3] [4] [10], we compare the condition numbers of our (discontinuous solution) method with those in the continuous solution case [2], and numerically show that they are comparable for the same mesh sizes. Finally, in Section 4 we give some concluding remarks.

2. Enrichment Functions and Spaces

2.1. Weak Formulation. Let \( I^- = (a, \alpha) \) and \( I^+ = (\alpha, b) \), and define

\[
H_{\alpha,0}^1(I) = \{ v \in L^2(I) : v \in H^1(I^-) \cap H^1(I^+), v(a) = v(b) = 0 \}.
\]

We use conventional Sobolev norm notation. For example, \( |u|_{1,I} \) denotes the usual \( H^1 \)-seminorm for \( u \in H^1(J) \), and \( ||u||_{L,0}^2 = ||u||_{L,-}^2 + ||u||_{L,+}^2, i = 1, 2 \) for \( u \in H_{\alpha}^2(I) \), where

\[
H_{\alpha}^2(I) = H^2(I^-) \cap H^2(I^+).
\]

The space \( H_{\alpha,0}^1(I) \) is endowed with the \( \| \cdot \|_{1,I-\cup I+} \) norm, and \( H_{\alpha}^2(I) \) with the \( \| \cdot \|_{2,I-\cup I+} \) norm. The higher order spaces \( H_{\alpha}^m(I) \) and \( H_{\alpha,0}^m(I), m \geq 3 \) are similarly defined. With this in mind, the weak formulation of the problem \( \{1\} \) under \( \{1\} \) is: Given \( f \in L^2(I) \), find \( u \in H_{\alpha,0}^1(I) \) such that

\[
a(u, v) = (f, v) \quad \forall v \in H_{\alpha,0}^1(I),
\]

where

\[
a(u, v) = \int_a^b \beta(x)u'(x)v'(x)dx + \int_a^b w(x)u(x)v(x)dx + \frac{[u]_\alpha[v]_\alpha}{\lambda},
\]

\[
(f, v) = \int_a^b f(x)v(x)dx.
\]

The above weak formulation can be easily derived by integration-by-parts with \( \{1\} \). Since \( \lambda > 0 \), the bilinear form \( a(\cdot, \cdot) \) is coercive and is bounded due to Poincaré inequality. By the Lax-Milgram theorem, a unique solution \( u \) exists. Throughout the paper, we assume that the functions \( \beta, f, \) and \( w \) are such that the solution \( u \in H_{\alpha}^p(I), p \geq 2 \).

2.2. Enrichment Functions. We now introduce an approximation space for the solution \( u \). Let \( a = x_0 < x_1 < \ldots < x_k < x_{k+1} < \ldots < x_N = b \) be a partition of \( I \) and the interface point \( \alpha \in (x_k, x_{k+1}) \) for some \( k \). As usual, the mesh size \( h := \max_i h_i, h_i = x_{i+1} - x_i, i = 0, \ldots, N - 1 \). Define the two one-sided enrichment functions

\[
\psi_0(x) := \begin{cases} 0 & x \in [a, x_k] \\ m_1(x-x_k) & x \in [x_k, \alpha] \end{cases} \quad \text{and} \quad \psi_1(x) := \begin{cases} 0 & x \in [\alpha, x) \\ m_2(x-x_{k+1}) & x \in (\alpha, x_{k+1}] \\ 0 & x \in (x_{k+1}, b] \end{cases}
\]

where

\[
m_1 = \frac{1}{\alpha - x_k}, \quad m_2 = \frac{1}{\alpha - x_{k+1}}.
\]

Remark 2.1.
• It is the enrichment function space \( F_{\text{enr}} := \text{span}\{\psi_0, \psi_1\} \) that matters. Any two basis functions of \( F_{\text{enr}} \) qualify, i.e., any pairs of nonzero \( m_1 \) and \( m_2 \) will guarantee convergence, as we shall show below. This space can also be spanned by one continuous and one one-sided function.

For example, the familiar continuous function

\[
\tilde{\psi}_0(x) := \begin{cases} 
0 & x \in [a, x_k] \\
\tilde{m}_1(x - x_k) & x \in [x_k, \alpha) \\
\tilde{m}_2(x - x_{k+1}) & x \in [\alpha, x_{k+1}] \\
0 & x \in (\alpha, b] 
\end{cases}
\]

and a discontinuous \( \tilde{\psi}_1(x) := \psi_1(x) \)

where

\[
\tilde{m}_1 = \frac{\alpha - x_{k+1}}{x_{k+1} - x_k}, \quad \tilde{m}_2 = \frac{(\alpha - x_k)}{(x_{k+1} - x_k)}.
\]

• We will adopt choice (11) since it makes the ensuing error analysis more transparent and simpler.

• Note that the slopes \( \tilde{m}_1, \tilde{m}_2 \) in choice (13) are uniformly bounded by 1, while the slopes \( m_1, m_2 \) in choice (11) are not. We show in Appendix A that the bounded type may lead to a system of out-of-scale finite element equations while choice (11) does not. However, after a diagonal scaling, their resulting preconditioned systems have comparable condition numbers (cf. Section 3).

Let us describe the enriched space associated with \( \psi_i, i = 0, 1 \). Let \( \bar{I} = \cup_{i=0}^{N-1} I_i, I_i = [x_i, x_{i+1}] \) and let \( S_h \) be the standard \( P_p, p \geq 1 \) conforming finite element space

\[
S_p^h = \{ v_h \in C(\bar{I}) : v_h|_{I_i} \in P_p, i = 0, \ldots, N-1, v_h(a) = v_h(b) = 0 \} = \text{span}\{\phi_j, j \in N_p^h\},
\]

where \( \phi_j \)'s are the Lagrange nodal basis functions of order \( p \), and where the nodal index set \( N_p^h := \{1, 2, \ldots, pN-1\} \).

We denote the usual \( P_p \)-interpolation operator by \( I_h : C(\bar{I}) \rightarrow S_p^h \),

\[
P_p g = \sum_{i=1}^{pN-1} g(t_i) \phi_i,
\]

where \( t_i, i \in N_p^h \) are the nodes such that \( \phi_i(t_j) = \delta_{ij} \). For each element \( \tau = I_i \), the local interpolation operator \( P^\tau \) is

\[
P^\tau g := P_p g \bigg|_{\tau} = \sum_{j \in N^\tau_p} g(t_j) \phi_j,
\]

and \( N^\tau_p \subset N_p^h \) is the set of nodes associated with \( \tau \). Define the enriched finite element space

\[
\bar{S}_p^h = S_p^h + S_{h,E} = \{ v_h + w_h, v_h \in S_p^h, w_h \in S_{h,E} \}
\]

and

\[
S_{h,E} := \text{span}\{v\psi_0, v\psi_1\},
\]
where \( v \in S_h^p \) and \( v \) is nonzero over the interface element \([x_k, x_{k+1}]\) that contains the interface. It will be shown later that algebraic sum ‘+’ in (16) cannot be a direct sum ‘\( \oplus \)’ since the intersection space, \( S_h^p \cap S_{h,E} \), is not empty.

The GFEM for problem (11) under (12) is to find \( u_h \in S_h^p \subset H^1_{\alpha,\delta} \) such that

\[
a(u_h, v_h) = (f, v_h) \quad \forall v_h \in S_h^p.
\]

To derive the corresponding linear algebraic system of equations, we need to form a basis for the enriched space \( S_h^p \), which we characterize as below.

2.3. Structure of the enrichment finite element space. It suffices to analyze the structure of the enriched space restricted to the interface element \([x_k, x_{k+1}]\). To this end, we first concentrate on the master element \([0,1] \) with an interface point \( \hat{\alpha} \).

Let \( \xi_i = i/p, i = 0, \ldots, p \), be the evenly distributed nodes and let the corresponding Lagrange interpolating polynomials be

\[
q_i(x) = \prod_{j \neq i} \frac{(x - \xi_j)}{(\xi_i - \xi_j)}.
\]

The local enriched space is

\[
V_h^{\text{enr}} := V + V_E := \text{span}\{q_0, q_1, \ldots, q_p\} + \text{span}\{q_i \psi_0, q_i \psi_1\}_{i=0}^p.
\]

For \( p \geq 2 \), we will refer to the \( p-1 \) functions \( \{q_i\}_{i=1}^{p-1} \) as the bubble functions since they vanish at the end points. Note that \( q_0(0) = 1 \) and \( q_p(1) = 1 \). We show in Lemma 2.2 that the bubble functions belong to \( V_E \), and consequently

\[
\dim(V_h^{\text{enr}}) = \dim(V + V_E) = 2 + 2(p + 1).
\]

Remark 2.1. Note that there is no bubble function for the linear case \((p = 1) \). Moreover, there holds \( V \cap V_E = \emptyset \). Thus, \( V_h^{\text{enr}} := V \oplus V_E \) and the dimension is \( \dim(V_h^{\text{enr}}) = \dim(V + V_E) = 2 + 2(p + 1) = 6 \).

Lemma 2.2 (Local linear dependence). Let \( p \geq 2 \). For any bubble function \( q_i, 1 \leq i \leq p-1 \), we have

\[
q_i \in V_E := \text{span}\{q_i \psi_0, q_i \psi_1\}_{i=0}^p,
\]

i.e., there exist \( s_{mj} \), \( 0 \leq m \leq p, j = 0,1 \), such that

\[
q_i = \sum_{j=0}^1 \sum_{l=0}^p s_{lj} \psi_j q_l.
\]

Proof. We only prove the theorem for \( p = 2 \) since the proof for general \( p \) follows closely this case but with more complicated indices. Note that there is only one bubble function \( q_1 \) for \( p = 2 \). On \([0, \hat{\alpha}] \), \( \psi_1 = 0 \) and we need to show the existence of \( s_{i0} \) such that \( q_1 = \sum_{i=0}^2 s_{i0} \psi_0 q_i \). Since \( m_1 \) can be absorbed into \( s_{i0} \), we can assume \( m_1 = 1 \) in the following calculation. Expressing \( q_i \) as a Taylor polynomial

\[
q_i = q_i(0) + q_i'(0)x + \frac{q_i''(0)}{2}x^2
\]

and comparing the coefficients of \( x, x^2, x^3 \) of

\[
q_1(0) + q_1'(0)x + q_1''(0)x^2 = \sum_{i=0}^2 s_{i0} \left( q_i(0)x + q_i'(0)x^2 + \frac{q_i''(0)}{2}x^3 \right)
\]

\[
q_1(0) + q_1'(0)x + q_1''(0)x^2 = \sum_{i=0}^2 s_{i0} \left( q_i(0)x + q_i'(0)x^2 + \frac{q_i''(0)}{2}x^3 \right)
\]

\[
q_1(0) + q_1'(0)x + q_1''(0)x^2 = \sum_{i=0}^2 s_{i0} \left( q_i(0)x + q_i'(0)x^2 + \frac{q_i''(0)}{2}x^3 \right)
\]
we have
\begin{align}
\sum_{i=0}^{2} s_i q_i(0) &= q'_1(0), \\
\sum_{i=0}^{2} s_i q_i'(0) &= q''_1(0), \\
\sum_{i=0}^{2} s_i q_i''(0) &= 0,
\end{align}
whose coefficient matrix is the Wronskian matrix evaluated at \( x = 0 \)
\begin{equation}
\begin{pmatrix}
q_0(0) & q_1(0) & q_2(0) \\
q'_0(0) & q'_1(0) & q'_2(0) \\
q''_0(0) & q''_1(0) & q''_2(0)
\end{pmatrix}.
\end{equation}
Since the Wronskian is nonsingular, \( s_{00}, s_{10}, s_{20} \) exist. Similarly for \( [\hat{\alpha}, 1] \), we use the Taylor polynomials at \( x = 1 \). All the equations are the same except they are evaluated at \( x = 1 \). Thus \( s_{01}, s_{11}, s_{21} \) exist. For the general case, the matrix in (24) is a Wronskian matrix of order \( p \).

The approximability of \( V^{enr}_k \)-functions on the master element is given below

**Theorem 2.3 (Local approximation).** Let \( \chi_i, i = 1, 2 \) be the characteristic functions of \([0, \hat{\alpha}]\) and \([\hat{\alpha}, 1]\), respectively. Define the space
\begin{equation}
W = \{ w : w = w_1 \chi_1 + w_2 \chi_2, w_i \in \mathbb{P}_p, i = 1, 2 \}.
\end{equation}
Then
\[ W \subset V^{enr}_k. \]
In other words, given two polynomials
\[ u_1 = \sum_{i=0}^{p} a_i x^i \quad \text{and} \quad u_2 = \sum_{i=0}^{p} b_i (x - 1)^i, \]
there exist unique \( \zeta_0, \zeta_p; \{ \zeta_i \}_{i=p+1}^{2p+1}, \{ \zeta_{j+2p+2} \}_{j=2p+2}^{3p+2} \) such that
\begin{equation}
\sum_{i=1}^{2} \chi_i u_i = \sum_{j=0}^{p} \zeta_j q_j + \sum_{j=0}^{p} \zeta_{j+p+1} q_j \psi_0 + \sum_{j=0}^{p} \zeta_{j+2p+2} q_j \psi_1.
\end{equation}

**Proof.** To prove the uniqueness of the solution of the square system in the unknown \( \zeta_i \)'s, we proceed as follow:
\begin{align}
\sum_{i=0}^{p} \zeta_i q_i + \sum_{i=0}^{p} \zeta_{i+(p+1)} q_i \psi_0 &= \sum_{i=0}^{p} a_i x^i, \\
\sum_{i=0}^{p} \zeta_i q_i + \sum_{i=0}^{p} \zeta_{i+2(p+1)} q_i \psi_1 &= \sum_{i=0}^{p} b_i (x - 1)^i.
\end{align}
We can take \( m_1 = m_2 = 1 \) because they can be absorbed into coefficients. Switching the indices in
\[ \sum_{i=0}^{p} \zeta_i \left( \sum_{i=0}^{p} \frac{q_i^{(l)}(0)}{l!} x^l \right) + \sum_{i=0}^{p} \zeta_{i+(p+1)} \left( \sum_{i=0}^{p} \frac{q_i^{(l)}(0)}{l!} x^l \right) x = \sum_{i=0}^{p} a_i x^i, \]
Comparing the coefficients of \(x^l\), \(0 \leq l \leq p\), we arrive at
\[
\sum_{i=0}^{p} \zeta_i \frac{q_i^{(0)}}{l!} + \sum_{i=0}^{p} \zeta_i (l-1)! q_i^{(l-1)}(0) = a_l,
\]
\[
\sum_{i=0}^{p} \zeta_i \frac{q_i^{(1)}}{l!} + \sum_{i=0}^{p} \zeta_i (l-1)! q_i^{(l-1)}(1) = b_l,
\]
with the understanding that the second terms on the left drop out when \(l = 0\). Collecting terms, we see that the square system is
\[
\sum_{i=0}^{p} \zeta_i + (p+1) q_i^{(p)}(0) = 0, \tag{33}
\]
\[
\sum_{i=0}^{p} \zeta_i + (p+1) q_i^{(p)}(0) = a_l, \quad 0 \leq l \leq p, \tag{34}
\]
\[
\sum_{i=0}^{p} \zeta_i + (p+1) q_i^{(p)}(1) = 0, \tag{35}
\]
\[
\sum_{i=0}^{p} \zeta_i (l-1)! q_i^{(l-1)}(1) = b_l, \quad 0 \leq l \leq p. \tag{36}
\]
Since this system is square it suffices to show that all \(\zeta_i\)'s are zero when \(a_l = b_l = 0 \leq l \leq p\). First observe that \(\zeta_0 = \zeta_p = 0\). In fact, with \(l = 0\) using the second and
Using the boundedness and coercivity properties of the bilinear form $a(\cdot, \cdot)$, we have
\begin{align*}
\zeta_0 q_0(0) + \zeta_p q_p(0) &= 0, \\
\zeta_0 q_0(1) + \zeta_p q_p(1) &= 0.
\end{align*}

The conclusion follows using $q_p(1) = 1, q_p(0) = 0, q_0(0) = 1, q_0(1) = 0$. As a consequence, (34) and (36) decouple and simplify to two linear systems
\begin{align}
\sum_{i=0}^{p} \zeta_{i+(p+1)} q_i^{(t-1)}(0) &= 0, \\
\sum_{i=0}^{p} \zeta_{i+2(p+1)} q_i^{(t-1)}(1) &= 0
\end{align}
whose coefficient matrices are nonsingular, being Wronskian matrices. Thus, all $\zeta_i = 0$.

Transforming the above results on $[0,1]$ to $[x_k, x_{k+1}]$, we have the following result.

**Lemma 2.4** (Local interpolant). Let $g \in C(\omega), \omega = (c, d)$ and $I_p^h, \omega g \in P_p$ be the interpolating polynomial of $g$ at $p + 1$ nodes $\xi_i = c + i[d-c]/p, i = 0, \ldots, p$. Then for $v \in C[x_k, \alpha] \cap C[\alpha, x_{k+1}]$, there exists a $\rho \in V_h^{renr}_{[x_k, x_{k+1}]}$ such that
\begin{equation}
\left( I_p^h, [x_k, \alpha] v \right) \chi_{[x_k, \alpha]} + \left( I_p^h, [\alpha, x_{k+1}] v \right) \chi_{[\alpha, x_{k+1}]} = \rho.
\end{equation}

Combining the classical results for non-interface elements and Lemma 2.4 for the interface element, we have

**Lemma 2.5** (Global interpolant). Define a global interpolant $J_{h,E} : C[\alpha, \alpha] \cap C(\alpha, b)) \to \mathbb{R}$ by $J_{h,E} v = I_p^h, \omega v, \omega = [x_k, \alpha], [\alpha, x_{k+1}], [x_i, x_{i+1}], i = 0, \ldots, k - 1, k + 1, \ldots, N - 1$. Then
\begin{equation}
|v - J_{h,E} v|_{1, I-\cup I^+} \leq C h^p |v|_{p+1, I-\cup I^+}.
\end{equation}

**Theorem 2.6** (Error estimate). Let $u$ be the exact solution and $u_h$ be the approximate solution of (9) and (17), respectively. Then there exists a constant $C > 0$ such that
\begin{equation}
\|u - u_h\|_{0, I-\cup I^+} + h \|u - u_h\|_{1, I-\cup I^+} \leq C h^{p+1} \|u\|_{p+1, I-\cup I^+}
\end{equation}
provided that the norm of the exact solution in the right side is finite. The constant $C$ does not depend independent of $h$ and $\alpha$ but depends on the ratio $\rho := \frac{\beta^*}{\rho}$ with $\beta^* = \sup_{x \in [a,b]} \beta(x)$ and $\beta_* = \inf_{x \in [a,b]} \beta(x)$.

**Proof.** Subtracting (9) from (17), we have
\begin{equation*}
a(u - u_h, q_h) = 0 \quad \forall q_h \in S_h.
\end{equation*}
Using the boundedness and coercivity properties of the bilinear form $a(\cdot, \cdot)$, we get
\begin{align*}
\beta_* |u - u_h|^2_{1, I} \leq a(u - u_h, u - u_h) &= a(u - u_h, u - q_h) \\
&\leq \beta^* |u - u_h|_{1, I} |u - q_h|_{1, I},
\end{align*}
where \( \beta^* = \sup_{x \in [a, b]} \beta(x) \) and \( \beta_* = \inf_{x \in [a, b]} \beta(x) \). Thus, by Cea’s lemma and Lemma 2.5,

\[
|u - u_h|_{1, I} \leq \frac{\beta^*}{\beta_*} \inf |u - q_h|_{1, I} \leq \frac{\beta^*}{\beta_*} |u - J_h,E u|_{1, I} \leq Ch^p \|u\|_{p+1, I^- \cup I^+}.
\]

Then the usual duality argument leads to

\[
\|u - u_h\|_{0, I} \leq Ch^{p+1} \|u\|_{p+1, I^- \cup I^+}.
\]

\( \Box \)

We note that the jump ratios \( \rho := \frac{\beta^*}{\beta_*} \) are of moderate size for the wall model in the next section.

**Theorem 2.7 (2p-th order accuracy at nodes).** Suppose that \( \beta \in C^{p+1}(a, b) \cap C^{p+1}(\alpha, b), \ p \geq 1 \) and \( w(x) = 0 \). Let \( u \) be the exact solution and \( u_h \) be the approximate solution of (9) and (17), respectively. Then there exists a constant \( C > 0 \) such that

\[
|u(\xi) - u_h(\xi)| \leq Ch^{2p}\|u\|_{p+1, I^- \cup I^+}, \quad \xi = x_i, 1 \leq i \leq n - 1,
\]

where \( C \) depends on certain norms of the Green’s function at \( \xi \).

**Superconvergence.** Furthermore, if \( \beta \) is piecewise constant with respect to \([a, \alpha]\) and \([\alpha, b]\), then

\[
u(\xi) = u_h(\xi) \quad \forall \xi = x_i, 1 \leq i \leq n - 1.
\]

\( \text{Proof.} \) Let \( g = G(\cdot, \xi), \xi \neq \alpha \) be the Green’s function satisfying

\[
a(G(\cdot, \xi), v) = < \delta(x - \xi), v >, \quad v \in H^1_{0, \alpha}(a, b)
\]

whose existence is guaranteed by the Lax-Milgram theorem, since in 1D point evaluation is a bounded operator. We can find the Green’s function via the classical formulation (for simplicity let \([a, b] = [0, 1]\) and \( \xi < \alpha \)):

\[-(\beta g')' = \delta(x - \xi), 0 < x < 1, \quad g(0) = g(1) = 0, \]

\[
[g]_{\alpha} = g'_{\alpha}, \quad [\beta g']_{\alpha} = 0, \quad [g]_{\xi} = 0, \quad [\beta g']_{\xi} = 1.
\]

Define

\[
K(x) := \int_0^x \frac{1}{\beta(t)} dt, 0 \leq x \leq \alpha; \quad K^c(x) := \int_x^1 \frac{1}{\beta(t)} dt, \alpha < x \leq 1.
\]

Then, similar to the techniques in [3] we have

\[
G(x, \xi) = \begin{cases} c_1 K(x), & 0 \leq x \leq \xi \\ c_3 (K(x) - K(\xi)) + c_1 K(\xi), & \xi \leq x < \alpha \\ -c_2 K^c(x), & \alpha < x \leq 1, \end{cases}
\]

where with \( \gamma = \frac{\lambda \beta^+ - \beta^-}{(\beta^+ - \beta^-)} \)

\[
c_3 = \frac{K(\xi)}{-K^c(\alpha) - K(\alpha) - \lambda}, \quad c_2 = c_3, \quad c_1 = 1 + c_3.
\]
Thus, for $\xi < \alpha$, $g = G(\cdot, \xi) \in H^{p+1}(\Omega)$, for $\Omega = (a, \xi), (\xi, x_k), (x_k, \alpha), (\alpha, x_{k+1})$, and $(x_{k+1}, b)$. Similar regularity holds if $\xi > \alpha$. Using the local estimates in Lemma 2.5, we conclude that there exists $I_h g \in \tilde{S}^p_h$ such that

\[(45) \quad |g - I_h g|_{1, \Omega} \leq C h^p \|g\|_{p+1, \Omega} \]

for all the $\Omega$’s listed above. Now with $\xi = x_i$

\[e(x_i) = a(g, e) = a(g - I_h g, e)\]

implies that

\[|e(x_i)| \leq C h^p \|g\|_{p+1, \Omega} h^p \|u\|_{p+1, I^- \cup I^+} \leq C h^{2p} \|g\|_{p+1, \Omega} \|u\|_{p+1, I^- \cup I^+} \]

where $\|g\|_{p+1, \Omega} := \sum \|g\|_{p+1, \Omega}$, the summation being over all the $\Omega$’s listed above.

We next prove the assertion (42). Since $\beta$ is piecewise constant, Green’s function (44) is piecewise linear in $x$ with fixed $\xi_i = x_i, 1 \leq i \leq n - 1$. Hence, $g = G(\cdot, x_i)$ is in $\tilde{S}^p_h$ by Theorem 2.3 and consequently

\[e(x_i) = a(g, e) = 0.\]

We will use the preconditioned conjugate gradient method to solve the resulting finite element equation. In Appendix A, we show that the stiffness matrix $A$ is not well-scaled when the slopes in choice (11) take the bounded form of $m_1 = (\alpha - x_{k+1})/h, m_2 = (\alpha - x_k)/h$ instead of the unbounded form of $m_1 = 1/(\alpha - x_k), m_2 = 1/(\alpha - x_{k+1})$ by examining its diagonal entries. In practice, we should perform a diagonal scaling or use the matrix $D_A$, the diagonal part of $A$, as a preconditioner and solve iteratively a system of the form $D_A^{-1/2}A D_A^{-1/2} y = b$. The scaled condition number $\kappa_2(D_A^{-1}A) \\text{(SCN)}$ for the higher order method will be computed in the next section.

3. Numerical Examples

In this section, we present numerical examples to confirm the theoretic findings. In subsection 3.1, we verify optimal order convergence (11) and nodal exactness (12) in Theorem 2.7 using linear, quadratic, and cubic elements. In subsection 3.2, we test our methods on a much more complicated physical example of the multi-layer porous wall model for the drug-eluting stents [25] that has been studied using IFEM [27, 28, 29, 30].

3.1. Numerical Verification of Theorem 2.7. Problem 3.1. Consider

\[-(\beta u')' = f(x), \quad u(0) = u(1) = 0,\]

where

\[f(x) = \begin{cases} x^m & x \in [0, \alpha), \\ (x - 1)^m & x \in (\alpha, 1]. \end{cases}\]

$m$ is a nonnegative integer. The interface point is located at $\alpha$ and
\[ \beta(x) = \begin{cases} 
\beta^- & x \in [0, \alpha), \\
\beta^+ & x \in (\alpha, 1]. 
\end{cases} \]

The interface jump conditions are

\[ [u]_\alpha = \gamma [p']_\alpha \quad \text{and} \quad [\beta u']_\alpha = 0, \]

where \( \gamma = -\lambda \beta^+ \beta^+ / [\beta]_\alpha. \) In the numerical experiment, we set \( \lambda = 1, \beta^- = 100, \beta^+ = 1, \alpha = 1/\pi \) and \( m = 6. \)

The exact solution is

\[ (46) \quad p(x) = \begin{cases} 
1 & x \leq \alpha, \\
(x - 1)^{m+2} + c_1 x & x \geq \alpha,
\end{cases} \]

where

\[ R_1 = \frac{\alpha^{m+1}}{m+1} - \frac{(\alpha - 1)^{m+1}}{m+1} \]

and

\[ R_2 = \frac{\gamma \alpha^{m+1}}{(m+1)\beta^-} + \frac{(\alpha - 1)^{m+1}}{(m+1)(m+2)\beta^+} - \frac{\alpha^{m+2}}{(m+1)(m+2)\beta^-}. \]

In Tables 1-3, we list the test results of \( L^2, \) broken \( H^1, \) nodal errors, and SCN, using linear, quadratic, and cubic elements, respectively.

| Problem 3.1 | \( L^2 \) error | \( H^1 \) error | nodal error | SCN |
|------------|----------------|----------------|-------------|-----|
| \( h = 1/8 \) | 1.70269e-5 | 4.36431e-4 | 3.29597e-17 | 2.14500e+3 |
| \( h = 1/16 \) | 4.55438e-6 | 2.31316e-4 | 2.45897e-16 | 4.12316e+3 |
| \( h = 1/32 \) | 1.59149e-6 | 1.61163e-4 | 5.49907e-16 | 8.57700e+3 |
| \( h = 1/64 \) | 3.99445e-7 | 8.08569e-5 | 1.44893e-15 | 1.64576e+4 |
| \( h = 1/128 \) | 1.07953e-7 | 4.36984e-5 | 6.95841e-15 | 3.43050e+4 |
| \( h = 1/256 \) | 2.69949e-8 | 2.18537e-5 | 1.66360e-15 | 6.69246e+4 |
| \( h = 1/512 \) | 6.79220e-9 | 1.13806e-5 | 8.26292e-15 | 1.37217e+5 |
| \( h = 1/1024 \) | 1.71982e-9 | 5.56909e-6 | 1.42854e-15 | 2.58394e+5 |

| order | \( \approx 2 \) | \( \approx 1 \) | exact |

Table 1. \( L^2, \) broken \( H^1, \) nodal errors, and scaled condition numbers with discontinuous jump conditions for linear basis functions where \( \beta^- = 100, \beta^+ = 1 \) and \( m = 6. \)
Problem 3.1

| $h = 1/8$ | $L_2$ error | $H^1$ error | nodal error | SCN    |
|-----------|--------------|--------------|-------------|--------|
|           | 1.42416e-6   | 7.41072e-5   | 1.06252e-16 | 1.07214e+4 |
| $h = 1/16$| 1.88018e-7   | 1.95134e-5   | 8.67362e-17 | 1.54415e+4 |
| $h = 1/32$| 3.11738e-8   | 6.46623e-6   | 1.10155e-15 | 5.37850e+4 |
| $h = 1/64$| 3.90890e-9   | 1.62136e-6   | 2.05131e-16 | 6.7681e+4  |
| $h = 1/128$| 5.21883e-10 | 1.95134e-5   | 3.94303e-16 | 5.37850e+4 |
| $h = 1/256$| 6.52478e-11 | 1.95134e-5   | 4.99600e-16 | 7.68010e+4 |
| $h = 1/512$| 7.82325e-12 | 1.95134e-5   | 7.41072e-16 | 1.07214e+4 |
| $h = 1/1024$| 9.28938e-12| 7.05922e-6   | 4.85723e-17 | 4.07233e+4 |

Table 2. $L^2$, broken $H^1$, nodal errors, and scaled condition numbers with discontinuous jump conditions for quadratic basis functions where $\beta^- = 100$, $\beta^+ = 1$ and $m = 6$.

| $h = 1/8$ | $L_2$ error | $H^1$ error | nodal error | SCN    |
|-----------|--------------|--------------|-------------|--------|
|           | 6.03368e-9   | 2.46764e-16  | 1.20800e+5  |        |
| $h = 1/16$| 7.4355e-10   | 1.44014e-7   | 2.51099e+5  |        |
| $h = 1/32$| 9.81711e-11  | 1.80399e-8   | 8.64010e+4  |        |
| $h = 1/64$| 1.94678e-12  | 2.39218e-15  | 3.45601e+5  |        |
| $h = 1/128$| 2.97116e-12 | 2.39218e-15  | 3.45601e+5  |        |
| $h = 1/256$| 3.97455e-13 | 2.39218e-15  | 3.45601e+5  |        |
| $h = 1/512$| 6.92556e-13 | 2.39218e-15  | 3.45601e+5  |        |
| $h = 1/1024$| 1.17422e-13 | 2.39218e-15  | 3.45601e+5  |        |

Table 3. $L^2$, broken $H^1$, nodal errors, and scaled condition numbers with discontinuous jump conditions for cubic basis functions where $\beta^- = 100$, $\beta^+ = 1$ and $m = 6$.

3.2. Multi-layer Porous Wall Model. In this subsection, we test our method using the multi-layer porous wall model for the drug-eluting stents [25]. In this one-dimensional wall model of layers, a drug is injected or released at an interface and gradually diffuses rightward. The concentration is thus discontinuous across the injection interface and continuous in the other layers. At all interface points, a zero-flux condition is imposed. We run tests using the enriched linear, quadratic, and cubic finite element spaces. For Problem 1, we place only one interface point to model the layer where the drug is delivered. For Problem 2, we place two interfaces to model the layers where the concentration is continuously spread. Finally, for Problem 3 we combine the previous two cases and place three interface points to simulate the full wall model. In each of the three problems, we display run results using linear, quadratic, and cubic enriched finite elements. We confirm that our method is indeed an SGFEM or SSGFEM [3, 11, 33, 34] in the sense that the condition numbers are comparable with those in the continuous case with robustness, and that it has optimal order convergence in the $L_2$ and broken $H^1$-norms.
Problem 1. Discontinuous Solution. Consider the two-point boundary value problem with one interface point $\alpha_0 = 1/9$

$$\frac{\partial}{\partial x} \left( -D \frac{\partial u}{\partial x} + 2\delta u \right) + \eta u = f \quad \text{in } (0, 1)$$

subject to the no-flux Neumann condition at $x = 0$ and the Dirichlet condition at $x = 1$:

$$D_0 u'(0) = 0, \quad u(1) = \frac{1}{3}.$$ 

Here the drug reaction coefficient $\eta = 0$, and the drug diffusivity $D$ and the characteristic convection parameter $\delta$ are piecewise continuous with respect to $[0, 1/9]$ and $[1/9, 1]$:

$$D(x) = \begin{cases} D_0 = 1 & x \in [0, 1/9] \\ D_1 = \frac{18(n-1)}{10n} & x \in [1/9, 1] \end{cases}$$

$$\delta(x) = \begin{cases} \delta_0 = 0 & x \in [0, 1/9] \\ \delta = 0.5(9nD_1 - 8.1(n - 1)) & x \in [1/9, 1] \end{cases}.$$

Furthermore, at the interface point $\alpha_0$, one of the jump conditions is implicit

$$\begin{cases} [u]_{\alpha_0} = \lambda D_0 u'(\alpha_0), \\ -D_0 u'(\alpha_0) = -D_1 u'(\alpha_0^+) + 2\delta_1 u(\alpha_0^+) \end{cases}$$

where $\lambda = 1/81(n - 1)D_0$. The exact solution

$$u(x) = \begin{cases} u_0 = x^{n-1}/30, & x \in [0, 1/9], \\ u_1 = x^n/3, & x \in [1/9, 1] \end{cases}.$$

We test the effectiveness of the method with $n = 4$ and with the enrichment function in (10). The results using linear, quadratic, and cubic elements are displayed in Table 4, Table 5, and Table 6 respectively.

Problem 2. Continuous Solution. Consider the two-point boundary value problem with two interface points $\alpha_1 = 1/3, \alpha_2 = 2/3$

$$\frac{\partial}{\partial x} \left( -D \frac{\partial u}{\partial x} + 2\delta u \right) + \eta u = f, \quad x \in (0, 1)$$

| Problem 1 | $L_2$ error | $H^1$ error | nodal error | SCN |
|-----------|-------------|-------------|-------------|-----|
| $h = 1/8$ | 2.99833e-3  | 1.26987e-1  | 1.26987e-2  | 9.94155e+3 |
| $h = 1/16$| 6.9536e-4   | 2.62730e-3  | 4.70183e+8  |
| $h = 1/32$| 1.61783e-4  | 3.8533e+6   |
| $h = 1/64$| 4.00845e-5  | 1.03453e+7  |
| $h = 1/128$| 9.99837e-6 | 6.12808e+6  |
| $h = 1/256$| 2.49618e-6 | 1.85688e+7  |

Table 4. $L_2$, broken $H^1$ nodal errors, and scaled condition numbers with discontinuous jump conditions for linear basis functions.
Problem 1

|                  | $L_2$ error | $H^1$ error | nodal error | SCN       |
|------------------|-------------|-------------|-------------|-----------|
| $h = 1/8$        | 1.72319e-4 | 9.42142e-3  | 2.80566e-4  | 7.56163e+8|
| $h = 1/16$       | 2.13339e-5 | 2.24515e-3  | 1.70134e-5  | 1.21704e+7|
| $h = 1/32$       | 2.65693e-6 | 5.53095e-4  | 1.08616e-6  | 4.05717e+6|
| $h = 1/64$       | 3.31770e-7 | 1.37738e-4  | 6.75489e-8  | 3.58170e+6|
| $h = 1/128$      | 4.14600e-8 | 3.44007e-5  | 4.21720e-9  | 1.29114e+6|
| $h = 1/256$      | 5.18215e-9 | 8.59806e-6  | 2.63633e-10 | 2.14262e+6|
| $h = 1/512$      | 6.47758e-10| 3.44007e-5  | 1.08616e-6  | 4.05717e+6|
| $h = 1/1024$     | 8.09702e-11| 1.76583e+7  | 2.27815e-12 | 5.18931e+6|

order ≈ 3  ≈ 2  ≈ 4

Table 5. $L_2$, broken $H^1$, nodal errors, and scaled condition numbers with discontinuous jump conditions for quadratic basis functions

Problem 1

|                  | $L_2$ error | $H^1$ error | nodal error | SCN       |
|------------------|-------------|-------------|-------------|-----------|
| $h = 1/8$        | 4.20870e-6 | 3.39467e-4  | 2.36926e-6  | 5.55914e+8|
| $h = 1/16$       | 2.69392e-7 | 4.16077e-5  | 4.00497e-8  | 1.35352e+7|
| $h = 1/32$       | 1.69519e-8 | 5.16948e-6  | 6.46344e-10 | 3.32974e+6|
| $h = 1/64$       | 1.06136e-9 | 6.45145e-7  | 1.01912e-11 | 5.18931e+6|
| $h = 1/128$      | 6.63646e-11| 8.06104e-8  | 6.94479e-14 | 1.55749e+7|
| $h = 1/256$      | 4.15956e-12| 1.00753e-8  | 8.35193e-13 | 5.14193e+7|
| $h = 1/512$      | 9.38187e-13| 1.25944e-9  | 2.44074e-12 | 5.14193e+7|
| $h = 1/1024$     | 4.79586e-12| 1.70856e-10 | 1.30979e-11 | 1.90080e+8|

order ≈ 4  ≈ 3  ≈ 6

Table 6. $L_2$, broken $H^1$, nodal errors, and scaled condition numbers with discontinuous jump conditions for cubic basis functions

with the boundary conditions

$$D_0 u'(0) = 0 \quad u(1) = 0.$$  

Here with $n = 4$  

$$D(x) = \begin{cases}  
D_1 = \frac{18(n-1)}{10n} & x \in [0, 1/3] \\
D_2 = \frac{6nD_1 - 3\delta}{3(n+1)} & x \in [1/3, 2/3]  \\
D_3 = \frac{8\delta_2 - 3(n+1)D_2}{3(n+5)} & x \in [2/3, 1]; 
\end{cases}$$  

$$\delta(x) = \begin{cases}  
\delta = 0.5(9nD_1 - 8.1(n-1)) & x \in [0, 1/3] \\
\delta_2 = 0.5(3(n+1)D_2 - 3nD_1 + 2\delta) & x \in [1/3, 2/3]  \\
\delta_3 = 0.25(3(n-1)D_3 - 3(n+1)D_2 + 4\delta_2) & x \in [2/3, 1], 
\end{cases}$$  

and $\eta = 10, 1, 0.1$ in respective subintervals. At the interface points $\alpha_i$ for $i = 1, 2$, the solution $u$ is continuous and

$$[u]_{\alpha_i} = 0,$$

$$-D_i u'(\alpha_i^-) + 2\delta_i u(\alpha_i^-) = -D_{i+1} u'(\alpha_i^+) + 2\delta_{i+1} u(\alpha_i^+).$$  

(50)
The exact solution is

\[
 u(x) = \begin{cases} 
 x^n / 3 & x \in [0, 1/3] \\
 x^{n+1} & x \in [1/3, 2/3] \\
 3(1 - x)x^{n+1} & x \in [2/3, 1]. 
\end{cases}
\]

The enrichment function \( \psi \) is well-known \[3, 10, 2\]:

\[
 \psi(x) = \begin{cases} 
 0 & x \in [0, x_k] \\
 (x_{k+1} - \alpha)(x_{k+1} - x) / (x_{k+1} - x_k) & x \in [x_k, \alpha] \\
 (\alpha - x_k)(x_{k+1} - x) / (x_{k+1} - x_k) & x \in [\alpha, x_{k+1}] \\
 0 & x \in [x_{k+1}, 1]. 
\end{cases}
\]

The test results using linear, quadratic, and cubic elements are displayed in Table 7, Table 8, and Table 9, respectively.
Problem 2

\[ L_2 \text{ error} \quad H^1 \text{ error} \quad \text{nodal error} \quad \text{SCN} \]

\begin{array}{|c|c|c|c|c|}
\hline
h = 1/8 & 4.63161e-5 & 4.44106e-3 & 1.16476e-5 & 9.80234e+6 \\
\hline
h = 1/16 & 2.99600e-6 & 5.69343e-4 & 1.77027e-7 & 7.06792e+6 \\
\hline
h = 1/32 & 1.88766e-7 & 7.15455e-5 & 2.84738e-9 & 1.78519e+7 \\
\hline
h = 1/64 & 1.18315e-8 & 8.96179e-6 & 4.48673e-11 & 9.80234e+6 \\
\hline
h = 1/128 & 7.40092e-10 & 1.12095e-6 & 9.76080e-13 & 7.06792e+6 \\
\hline
h = 1/256 & 4.62726e-11 & 1.40160e-7 & 6.82204e-13 & 1.78519e+7 \\
\hline
h = 1/512 & 3.66783e-12 & 1.75222e-8 & 5.15474e-12 & 9.80234e+6 \\
\hline
h = 1/1024 & 8.16340e-12 & 2.19198e-9 & 1.84626e-11 & 7.06792e+6 \\
\hline
\end{array}

Table 9. \( L_2 \), broken \( H^1 \) nodal errors, and scaled condition numbers with discontinuous jump conditions for cubic basis functions

Problem 3. Implicit and Explicit Conditions Both Present. In this problem, we combine the interfaces of the last two problems. The interface points are \( \alpha_0 = 1/9 \), \( \alpha_1 = 1/3 \) and \( \alpha_2 = 2/3 \). The two-point boundary value problem is

\begin{equation}
\frac{\partial}{\partial x} \left( -D \frac{\partial u}{\partial x} + 2\delta u \right) + \eta u = f \quad x \in (0, 1)
\end{equation}

subject to the boundary conditions

\[ D_0 u'(0) = 0, \quad u(1) = 0. \]

The coefficients are defined as follows:

\[ D(x) = \begin{cases}
D_0 = 1 & x \in [0, 1/9] \\
D_1 = \frac{18(n-1)}{10n} & x \in [1/9, 1/3] \\
D_2 = \frac{6nD_1 - 2\delta_2}{3(n+1)} & x \in [1/3, 2/3] \\
D_3 = \frac{8\delta_2 - 3(n+1)D_2}{3(n+2)} & x \in [2/3, 1]; \\
\end{cases} \]

\[ \delta(x) = \begin{cases}
\delta_0 = 0 & x \in [0, 1/9] \\
\delta = 0.5(9nD_1 - 8.1(n-1)) & x \in [1/9, 1/3] \\
\delta_2 = 0.5(3(n+1)D_2 - 3nD_1 + 2\delta) & x \in [1/3, 2/3] \\
\delta_3 = 0.25(3(n-1)D_3 - 3(n+1)D_2 + 4\delta_2) & x \in [2/3, 1]; \\
\end{cases} \]

\( n = 4 \) and \( \eta = 0, 10, 1, 0.1 \) in respective subintervals. The exact solution is

\[ u(x) = \begin{cases}
u_0 = x^{n-1}/30 & [0, 1/9] \\
u_1 = x^n/3 & [1/9, 1/3] \\
u_2 = x^{n+1} & [1/3, 2/3] \\
u_3 = 3(1-x)x^{n+1} & [2/3, 1] \\
\end{cases} \]

and it satisfies the jump condition (48) at 1/9 and (50) at the interface points 1/3 and 2/3. For the discontinuous interface point 1/9 we use the enrichment function defined in (10) and for the continuous interface points 1/3 and 2/3 we use the enrichment function in (51). The test results using linear, quadratic, and cubic elements are displayed in Table 10, Table 11 and Table 12 respectively.
### Table 10. $L^2$, broken $H^1$, nodal errors, and scaled condition numbers with discontinuous jump conditions for linear basis functions

| Problem 3 | $L^2$ error | $H^1$ error | nodal error | SCN         |
|-----------|-------------|-------------|-------------|-------------|
| $h = 1/8$ | 9.83346e-3 | 2.9708e-1   | 2.91989e-2  | 2.34756e+4  |
| $h = 1/16$| 2.45247e-3 | 6.42991e-3  | 2.9346e+4   |
| $h = 1/32$| 6.12229e-4 | 1.5276e-2   | 2.3464e+7   |
| $h = 1/64$| 1.52940e-4 | 3.81729e-4  | 2.1264e+7   |
| $h = 1/128$| 3.82602e-5 | 1.05477e-2  | 1.7103e+7   |
| $h = 1/256$| 9.56518e-6 | 2.63761e-3  | 1.2215e+8   |
| $h = 1/512$| 2.39193e-6 | 5.27506e-3  | 3.4957e+8   |
| $h = 1/1024$| 5.97993e-7 | 1.05477e-2  | 3.4957e+8   |
| order     | $\approx 2$ | $\approx 2$ | $\approx 2$ |

### Table 11. $L^2$, broken $H^1$, nodal errors, and scaled condition numbers with discontinuous jump conditions for quadratic basis functions

| Problem 3 | $L^2$ error | $H^1$ error | nodal error | SCN         |
|-----------|-------------|-------------|-------------|-------------|
| $h = 1/8$ | 1.02606e-3 | 5.53014e-2  | 1.01268e-3  | 4.62348e+8  |
| $h = 1/16$| 1.34864e-4 | 1.41241e-2  | 5.81152e-5  | 3.09158e+7  |
| $h = 1/32$| 1.70899e-5 | 3.55328e-3  | 3.71636e-6  | 1.04108e+7  |
| $h = 1/64$| 2.14269e-6 | 8.89303e-4  | 2.30931e-7  | 1.22150e+8  |
| $h = 1/128$| 2.68202e-7 | 2.22519e-4  | 1.44122e-8  | 1.04108e+7  |
| $h = 1/256$| 3.35320e-8 | 5.6344e-5   | 9.0412e-10  | 1.04108e+7  |
| $h = 1/512$| 4.19240e-9 | 1.39111e-5  | 1.20355e+7  |
| $h = 1/1024$| 5.24060e-10| 3.47781e-6  | 6.0974e+7   |
| order     | $\approx 3$ | $\approx 2$ | $\approx 4$ |

### Table 12. $L^2$, broken $H^1$, nodal errors, and scaled condition numbers with discontinuous jump conditions for cubic basis functions

| Problem 3 | $L^2$ error | $H^1$ error | nodal error | SCN         |
|-----------|-------------|-------------|-------------|-------------|
| $h = 1/8$ | 4.63166e-5 | 1.44106e-3  | 1.6423e-5   | 6.98538e+7  |
| $h = 1/16$| 2.99901e-6 | 5.69343e-4  | 1.76945e-7  | 1.83429e+7  |
| $h = 1/32$| 1.88766e-7 | 7.15455e-5  | 2.84985e-9  | 1.09970e+7  |
| $h = 1/64$| 1.18315e-8 | 8.96179e-6  | 4.48963e-11 | 1.58205e+7  |
| $h = 1/128$| 7.40092e-10| 1.12095e-6  | 9.61980e-13 | 8.98704e+7  |
| $h = 1/256$| 4.62732e-11| 1.40160e-7  | 8.69610e-13 | 1.71391e+8  |
| $h = 1/512$| 3.51944e-12| 1.75222e-8  | 5.27884e-12 | 3.0313e+8   |
| $h = 1/1024$| 6.20008e-12| 2.19319e-9  | 3.20313e+8  |
| order     | $\approx 4$ | $\approx 3$ | $\approx 6$ |
4. Concluding Remarks.

In this paper, we extend the lower-order SGFEM developed recently for interface problems with discontinuous solutions to arbitrarily high-order elements in the spatial dimension. The main challenge has been the fact that the enrichment function constructed for linear SGFEM is not sufficient to capture the discontinuous feature of the solution to the interface problem. We herein introduce two enrichment functions locally on the interval that contains an interface. These two enrichment functions, however, introduce linear dependence among the basis functions. We overcome this issue by removing, from the standard FEM space, the bubble functions that are associated with the interval containing the interface. We establish optimal error estimates for arbitrary-order elements and demonstrated the performance of the method with various numerical examples.

This work serves as preliminary development of SGFEM for a larger class of interface problems as it is focused on 1D. A natural direction of future work is the extension to multiple dimensions. The challenge lies in the construction of enrichment functions such that the enriched space has the optimal approximability while keeping the condition number from fast growth with respect to mesh size. Another future work direction is the study of the related time-dependent interface problems. Many real application interface problems are time-dependent, for example, the precipitating quasigeostrophic equations for climate modeling. Herein, the SGFEM provides an alternative numerical solver that is of high-order accuracy, stable, and robust.

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In this appendix we examine the order of magnitude of the diagonal entries of the stiffness matrix \( A \) whose diagonal entries take the form of

\[
\tilde{a}_{ii} = a(\xi_i, \xi_i) = \int_{\Omega} \beta \xi_i^2 \xi_i' dx + \frac{[\xi_i]^2}{\lambda} := a_{ii} + r_{ii}
\]

with

\[
\{\xi_i\} := \{\phi_1, \phi_2, \ldots, \phi_{n-1}, \phi_k \psi_0, \phi_{k+1} \psi_0, \phi_k \psi_1, \phi_{k+1} \psi_1\}.
\]

Note that the jump-related terms \( r_{ij} = 0 \) except for \( r_{jj}, j = n, n + 1, n + 2, n + 3 \).

Assume the diffusion coefficient has only two values \( \beta^-, \beta^+ \) for the time being (to get the general result we can use \( \beta_{\min} = \min_{x \in I} \{\beta^-(x), \beta^+(x)\}, \beta_{\max} = \max_{x \in I} \{\beta^-(x), \beta^+(x)\} \}). Then

\[
a_{ii} = \int_{0}^{1} \beta \phi_i \phi_i' dx = \begin{cases} \frac{2\beta^-}{h} + \frac{\beta^-}{h} (\alpha - x_k) + \frac{\beta^+}{h} (x_{k+1} - \alpha) & i = 1, \ldots, k - 1 \\ \frac{\beta^+}{h} + \frac{\beta^-}{h} (\alpha - x_k) + \frac{\beta^+}{h} (x_{k+1} - \alpha) & i = k + 1 \\ \frac{2\beta^-}{h} & i = k + 2, \ldots, n - 1 \end{cases}
\]

and thus

\[
\frac{2\beta_{\min}}{h} \leq a_{ii} \leq \frac{2\beta_{\max}}{h}, \quad 1 \leq i \leq n - 1.
\]

Thus, there holds

\[
a_{nn} = \int_{x_k}^{x_{k+1}} \beta (\phi_k \psi_0)^2 dx = \beta^- m_1^2 \int_{x_k}^{x_{k+1}} (\phi_k' (x - x_k) + \phi_k)^2 dx
\]

\[
a_{n+1,n+1} = \int_{x_k}^{x_{k+1}} \beta (\phi_{k+1} \psi_0)^2 dx = \beta^- m_1^2 \int_{x_k}^{x_{k+1}} (\phi_{k+1}' (x - x_k) + \phi_{k+1})^2 dx
\]

\[
a_{n+2,n+2} = \int_{x_k}^{x_{k+1}} \beta (\phi_k \psi_1)^2 dx = \beta^+ m_2 \int_{x_k}^{x_{k+1}} (\phi_k' (x - x_{k+1}) + \phi_k)^2 dx
\]

\[
a_{n+3,n+3} = \int_{x_k}^{x_{k+1}} \beta (\phi_{k+1} \psi_1)^2 dx = \beta^+ m_2 \int_{x_k}^{x_{k+1}} (\phi_{k+1}' (x - x_{k+1}) + \phi_{k+1})^2 dx.
\]

Note that

\[Q. \text{ Zhang and I. Babuška, A stable generalized finite element method (sgfem) of degree two for interface problems, Computer Methods in Applied Mechanics and Engineering, 363} (2020), \text{p. 112889.}\]

\[Q. \text{ Zhang, U. Banerjee, and I. Babuška, Strongly stable generalized finite element method (ssgfem) for a non-smooth interface problem, Computer Methods in Applied Mechanics and Engineering, 344} (2019), \text{pp. 538–568.}\]

\[Q. \text{ Zhang and I. Babuška, Strongly stable generalized finite element method (ssgfem) for a non-smooth interface problem ii: A simplified algorithm, Computer Methods in Applied Mechanics and Engineering, 363} (2020), \text{p. 112926.}\]
\[
\int_{x_k}^{x} (\phi'_k(x-x_k) + \phi_k)^2 dx = h_k \int_0^{(\alpha-x_k)/h_k} (2\hat{x} - 1)^2 d\hat{x} \\
= \frac{1}{6h_k^2} ((2(\alpha - x_k) - h_k)^3 + h_k^3) \\
= \frac{1}{6h_k^2} 2(\alpha - x_k) (2(\alpha - x_k - x_{k+1})^2 - (2\alpha - x_k - x_{k+1})h_k + h_k^2) \\
\geq \frac{1}{6h_k^2} 2(\alpha - x_k) \frac{1}{2} ((2\alpha - x_k - x_{k+1}) - h_k)^2 \quad \text{by} \quad a^2 - ab + b^2 \geq \frac{1}{2}(a-b)^2 \\
= \frac{2}{3h_k^2} (\alpha - x_k)(\alpha - x_{k+1})^2.
\]

Also, a simple calculation leads to
\[
\int_{x_k}^{x_{k+1}} (\phi'_k(x-x_{k+1}) + \phi_k)^2 dx = \frac{4}{3h_k^2} (\alpha - x_k)^3.
\]

\[
\int_{\alpha}^{x_{k+1}} (\phi'_k(x-x_{k+1}) + \phi_k)^2 dx = h_k \int_{(\alpha-x_k)/h_k}^{1} (2\hat{x} - 1)^2 d\hat{x} \\
= \frac{1}{6h_k^2} ((-2(\alpha - x_k) + h_k)^3 + h_k^3) \\
= \frac{1}{6h_k^2} 2(x_{k+1} - \alpha) (2(\alpha - x_k - x_{k+1})^2 + (2\alpha - x_k - x_{k+1})h_k + h_k^2) \\
\geq \frac{1}{6h_k^2} 2(x_{k+1} - \alpha) \frac{1}{2} ((2\alpha - x_k - x_{k+1}) + h_k)^2 \quad \text{by} \quad a^2 + ab + b^2 \geq \frac{1}{2}(a+b)^2 \\
= \frac{2}{3h_k^2} (x_{k+1} - \alpha)(\alpha - x_k)^2.
\]

Recall the jump-related terms \(r_{ij} = 0\) except for \(r_{jj}, j = n, n+1, n+2, n+3\) and are of the lower order compared with \(a_{ij}\) in magnitude due to the order of derivative:
\[
\frac{1}{\lambda} \begin{pmatrix}
[\phi_k \psi_0]_{\alpha} = m_1^2 (\alpha - x_k)^2 (\alpha - x_{k+1})^2 h_k^{-2} \\
[\phi_k+1 \psi_0]_{\alpha} = m_2^2 (\alpha - x_k)^2 (\alpha - x_{k+1})^2 h_k^{-2}
\end{pmatrix}
\]

and
\[
\frac{1}{\lambda} \begin{pmatrix}
[\phi_k \psi_1]_{\alpha} = m_1^2 (\alpha - x_k)^2 (\alpha - x_{k+1})^2 h_k^{-2} \\
[\phi_k+1 \psi_1]_{\alpha} = m_2^2 (\alpha - x_k)^2 (\alpha - x_{k+1})^2 h_k^{-2}
\end{pmatrix}
\]

From the above estimates, it is clear that \(m_1 = (\alpha - x_{k+1})/h_k, m_2 = (\alpha - x_k)/h_k\) generate a system of condition number with a scale comparable to the system from the standard FEM.