THE NILPOTENT CONE FOR CLASSICAL LIE SUPERALGEBRAS

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ABSTRACT. In this paper the authors introduce an analog of the nilpotent cone, \( N \), for a classical Lie superalgebra, \( g \), that generalizes the definition for the nilpotent cone for semisimple Lie algebras. For a classical simple Lie superalgebra, \( g = g_0 \oplus g_1 \) with Lie \( G_0 = g_0 \), it is shown that there are finitely many \( G_0 \)-orbits on \( N \) unless \( g = D(2,1, \alpha) \). Later the authors prove that the Duflo-Serganova commuting variety, \( X \), is contained in \( N \) for any classical simple Lie superalgebra. Consequently, our finiteness result generalizes and extends the work of Duflo-Serganova on the commuting variety. Further applications are given at the end of the paper.

1. INTRODUCTION

1.1. Let \( g \) be a finite-dimensional Lie algebra over the complex numbers and \( N \) be the set of nilpotent elements, often referred to as the nilpotent cone. In the case when \( g \) is semisimple with \( g = \text{Lie}\ G \), it is well-known that \( N \) has finitely many \( G \)-orbits. For the classical families of simple Lie algebras (root systems of types \( A-D \)) a parametrization of orbits is given by partitions under suitable conditions, and for the exceptional Lie algebras one can either use the Bala-Carter labelling or weighted Dynkin diagrams. For a semisimple Lie algebra \( g \), fundamental results in geometric representation theory have involved investigating the geometry of the nilpotent cone \( N \) (also its Springer resolution \( \widetilde{N} \)) and its relationship to the representation theory for \( g \) (cf. [HIT]).

The nilpotent cone \( N \) can be realized as the zero set of the non-constant \( G \)-invariant polynomials on \( g \). The \( G \)-invariant polynomials also have a direct connection with the semisimple elements. Under the Chevalley isomorphism theorem the restriction map induces an isomorphism

\[
\text{res} : S(\mathfrak{g}^*)^G \to S(t^*)^W
\]

where \( t \) is a maximal torus of \( g \) and \( W \) is the Weyl group. The semisimple elements are those elements in \( g \) that are \( G \)-conjugate to an element in \( t \) [Hum, Section 0.1].

1.2. A similar picture arises in the study of classical simple Lie superalgebras, \( g = g_0 \oplus g_1 \). Boe, Kujawa and Nakano [BKN1] used invariant theory for reductive groups to show that there are natural classes of “subalgebras” that detect the cohomology. These subalgebras arise from considering “semisimple” elements of the \( G_0 \) action on \( g_1 \), and fall into two families: \( f \) (when \( g \) is stable) and \( e \) (when \( g \) is polar). If \( g \) is a classical simple Lie superalgebra, then \( g \) admits a stable action and in most cases \( g \) admits a polar action (cf [BKN1, Table 5]).

When stable and polar actions exist, the restriction maps induce isomorphisms:

\[
\begin{align*}
\text{H}^\bullet(\mathfrak{g}, \mathfrak{g}_0, \mathbb{C}) & \longrightarrow \text{H}^\bullet(f, f_0, \mathbb{C})^N \longrightarrow \text{H}^\bullet(e, e_0, \mathbb{C})^W \\
S^\bullet(\mathfrak{g}_1)^G & \longrightarrow S^\bullet(f_1)^N \longrightarrow S^\bullet(e_1)^W 
\end{align*}
\]

where \( N \) is a reductive group and \( W_e \) is a finite pseudoreflection group. The finite generation of these cohomology rings was used in [BKN1] to define support varieties for \( g \)-modules. In [GGNW] it was shown that the support varieties (appropriately) defined over \( g \), \( f \) and \( e \) are isomorphic.

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In order to have a complete picture, it is natural to ask whether there exists an appropriate algebraic variety consisting of “nilpotent elements” for classical Lie superalgebras that fits into this framework. Since \( g \) is classical, \( g_0 = \text{Lie } G_0 \) where \( G_0 \) is a reductive algebraic group. In this paper we study a generalization of the nilpotent cone

\[
\mathcal{N} = \mathcal{N}_0 = Z(S^\bullet (g_1^\bullet)_{G_0^0}^+)\]

where \( Z(S^\bullet (g_1^\bullet)_{G_0^0}^+) \) is the zero-set of non-constant \( G_0^0 \)-invariant polynomials on \( g_1 \). When \( g = q(n) \), one obtains the nilpotent cone for the Lie algebra \( g_1^n(C) \).

The construction in our paper is inspired by work of Kac [K] in 1980. He defined the nilvariety for a \( G_0^0 \)-module, \( V \), as the zero locus of the non-constant \( G_0^0 \)-invariant polynomials on \( V \). Kac’s results are in a more general context than our paper and there is some overlap in our results. We anticipate that the varieties \( \mathcal{N} \) will play an important role in the representation theory for Lie superalgebras. There is evident in Section 5 with the strong connections with the Duflo-Serganova commuting varieties that were introduced in the mid 2000’s. The aim of our paper is to present a self-contained treatment of \( \mathcal{N} \) for classical simple Lie superalgebras that can be easily referenced by those working in super representation theory.

1.3. The paper is organized as follows. In Section 2, the nilpotent cone for classical Lie superalgebras is defined. We also indicate how this definition generalizes the definition of the nilpotent cone for complex semisimple Lie algebras. Our first main result (Theorem 3.1.1) in Section 3 demonstrates that the nilpotent cone for \( \text{gl}(m|n) \) has finitely many \( G_0^0 \)-orbits. Explicit orbit representatives are determined. In Section 4, we prove a theorem that allows us to extend the finiteness result to the nilpotent cone for other classical simple Lie superalgebras. The ideas of the theorem are originally due to Richardson and can be applied in cases when there is a suitable embedding of the classical simple Lie superalgebras into a general linear Lie superalgebra. With these tools, we show that for classical simple Lie superalgebras other than \( D(2,1,\alpha) \), \( \mathcal{N} \) has finitely many \( G_0^0 \)-orbits. Later it is explicitly shown that for \( g = D(2,1,\alpha) \), \( \mathcal{N} \) has infinitely many \( G_0^0 \)-orbits.

Duflo and Serganova [DS] introduced the commuting variety \( \mathcal{X} \) for any finite-dimensional Lie superalgebra. They proved that for basic classical Lie superalgebras, \( \mathcal{X} \) has finitely many \( G_0^0 \)-orbits. In Section 5, we prove that for all classical simple Lie superalgebras, one has an inclusion \( \mathcal{X} \subseteq \mathcal{N} \). In this way, one should consider the nilpotent cone a larger algebraic variety whose geometric properties should encompass that of \( \mathcal{X} \). We show that our results on the finiteness of orbits for \( \mathcal{N} \) allows us to extend the finiteness results in [DS] to a wider class of Lie superalgebras (cf. Corollary 5.3.1).

2. Preliminaries

2.1. Notation: Throughout this paper we will use the conventions in [BKN1, BKN2, BKN3, GGNW]. Let \( g = g_\mathfrak{r} \oplus g_\mathfrak{t} \) be a classical Lie superalgebra over \( k = \mathbb{C} \). This means there exists \( G_\mathfrak{t} \) a corresponding connected reductive algebraic group such that \( \text{Lie } G_\mathfrak{t} = g_\mathfrak{t} \) where \( g_\mathfrak{t} \) is a \( G_\mathfrak{t} \)-module via the adjoint action. The Lie superalgebra \( g \) is a basic classical if it is a classical Lie superalgebra with a nondegenerate invariant supersymmetric even bilinear form.

Let \( g \) be a classical Lie superalgebra and \( S^\bullet (g_1^\bullet) \) be the symmetric algebra on the dual of \( g_1 \). We will often regard \( S^\bullet (g_1^\bullet) \) as the polynomial functions on \( g_1 \). Let \( S^\bullet (g_1^\bullet)_+ \) be the non-constant polynomials. The algebraic group \( G_0 \) acts on \( g_1 \), so we can consider the non-constant \( G_0^0 \)-invariant polynomials on \( g_1 \) denoted by \( S^\bullet (g_1^\bullet)_{G_0^0}^+ \). The nilpotent cone for \( g \), \( \mathcal{N} \), is the zero set of these polynomials:

\[
\mathcal{N} = Z(S^\bullet (g_1^\bullet)_{G_0^0}^+).
\]

Observe that \( \mathcal{N} \subseteq g_1 \). The algebraic variety \( \mathcal{N} \) is a \( G_0^0 \)-invariant closed cone in \( g_1 \).
2.2. Simple Classical Lie Superalgebra. The main results of the paper will be stated for classical “simple” Lie superalgebras. We will use the term simple Lie superalgebra to refer to the Lie superalgebra of general interest that are not simple in the true sense, but close enough to being simple (cf. [GGNW]). The Lie superalgebras that will be considered “simple” include:

- $\mathfrak{gl}(m|n)$, $\mathfrak{sl}(m|n)$, $\mathfrak{psl}(n|n)$
- $\mathfrak{osp}(m,n)$
- $D(2,1,\alpha)$
- $G(3)$
- $F(4)$
- $\mathfrak{q}(n)$, $\mathfrak{psq}(n)$
- $\mathfrak{p}(n)$, $\bar{\mathfrak{p}}(n)$.

For the Lie superalgebras of Type Q, $\mathfrak{q}(n)$ is the Lie superalgebra with even and odd parts $\mathfrak{gl}(n)$, while $\mathfrak{psq}(n)$ is the corresponding simple subquotient of $\mathfrak{q}(n)$ (cf. [PS]). The Lie superalgebras that fall into the family of Type P include $\mathfrak{p}(n)$ and its enlargement $\bar{\mathfrak{p}}(n)$.

2.3. Generalization of the ordinary nilpotent cone. We now indicate how our results generalize known results for the nilpotent cone for complex semisimple Lie algebras. Let $\mathfrak{g}$ be a complex semisimple Lie algebra, and set $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with $\mathfrak{g}_0 = \mathfrak{a} = \mathfrak{g}_1$ as vector spaces. We can make $\mathfrak{g}$ into a Lie superalgebra by defining the bracket on $\mathfrak{g}_0$ to be the ordinary Lie bracket on $\mathfrak{a}$. The bracket of an element in $\mathfrak{g}_0$ on $\mathfrak{g}_1$ is given by the adjoint action, and the bracket of any two elements in $\mathfrak{g}_1$ is zero.

Let $G_0$ be the semisimple simply connected group such that Lie $G_0 = G_0$. Then $\mathcal{N}$ is the ordinary nilpotent cone for $\mathfrak{g}_0 = \mathfrak{a}$. One can also set this up for fields of prime characteristic, if one considers Lie algebras that arise as the Lie algebra of a semisimple algebraic group.

3. $G_0$-orbits on $\mathcal{N}$: $\mathfrak{gl}(m|n)$ case

3.1. The adjoint action of $G_0 = GL_m(\mathbb{C}) \times GL_n(\mathbb{C})$ on $\mathfrak{g}_1$ is given by conjugation. Explicitly,\
$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} 0 & X^+ \\ X^- & 0 \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}^{-1} \begin{bmatrix} 0 & X^+ \\ X^- & 0 \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = \begin{bmatrix} 0 & A^{-1}X^+B \\ B^{-1}X-A & 0 \end{bmatrix}.$$\

In this case, [Fuchs, Section 2.6] has determined the generators of the invariants, $S(\mathfrak{g}_1^*)^{G_0}_+$, to be $Tr((X^+X^-)^k), \quad k = 1, \ldots, l$ where $l = \min\{m,n\}$.

For the following theorem we recall the notion of a matrix in column echelon form. A matrix is in (reduced) column echelon form if it satisfies the following conditions:

- all columns that consist entirely of zero entries appear as the right most columns of the matrix
- the first nonzero entry of each column is called the pivot, and the pivot is the only nonzero entry in its row
- if $j > i$, then the pivot of the nonzero column $c_j$ lies in a row strictly below the row of the pivot of column $c_i$.

We also use the convention that each pivot element is 1. By transposing the matrix, the notion of row echelon form can defined in a similar way.

In this section we will show that $\mathfrak{gl}(m|n)$ has finitely many $G_0$-orbits. Furthermore, explicit orbit representatives for this action will be exhibited. The results are summarized in the following theorem and the proof will be given in the next section.

**Theorem 3.1.1.** Let $\mathfrak{g} = \mathfrak{gl}(m|n)$.

(a) The number of $G_0$-orbits of the adjoint action on $\mathcal{N}$ is finite.
(b) The complete set of orbit representatives is given by matrices

\[ y = \begin{bmatrix} 0 & Y^+ \\ Y^- & 0 \end{bmatrix} \]

where

\[ Y^+ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad Y^- = \begin{bmatrix} J_1 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & J_t \end{bmatrix} \]

Here \( I_r \) (resp. \( I_s \)) is a \( r \times r \) (resp. \( s \times s \)) identity matrix, \( J_1, \ldots, J_t \) are Jordan blocks with zero eigenvalues, where the Jordan block \( J_i \) is of size \( k_i \times k_i \), \( i = 1, 2, \ldots, t \) with \( k_1 \geq k_2 \geq \cdots \geq k_t \). Furthermore, the matrix \( C_{r_1} \) (resp. \( R_{r_2} \)) are of size \( t \times n \) (resp. \( m \times t \)) and of the form

\[ C_{r_1} = (e_{i_1} \ e_{i_2} \ \cdots \ e_{i_{r_1}} \ 0 \ \cdots \ 0), \quad R_{r_2} = (e_{j_1} \ e_{j_2} \ \cdots \ e_{j_{r_2}} \ 0 \ \cdots \ 0)^T \]

where \( e_{i_j} \) is the column vector with a single 1 in the \( i_j \)-th row and zeros elsewhere.

3.2. Proof of Theorem 3.1.1. The proof of theorem in the prior section will entail several steps. We start will a general element \( X \in \mathcal{N} \) where \( g = gl(m|n) \). Through a series of conjugations (i.e., applications of elements in \( G_0 \)), the element \( X \) will be transformed into a matrix \( Y \) of the form in Theorem 3.1.1(b). In the process of this transformation, we will use \( Y \) (including \( Y^+ \) and \( Y^- \)) to denote the current matrix under the series of transformations.

3.2.1. Let \( X \in \mathcal{N} \). Using standard results in linear algebra (involving equivalence of matrices), and the action of \( (A, B) \in G_0 \) on \( X \), there exists \( (A, B) \in G_0 \) such that \( A^{-1}X^+B = Y^+ \) where

\[ Y^+ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \]

where \( r \) is the rank of \( X^+ \).

3.2.2. The next step is to identify and later work with \( (A, B) \in G_0 \) that centralize \( Y^+ \), which is equivalent to the condition: \( A^{-1}Y^+B = Y^+ \). In order elaborate further, it will be useful to consider \( (A, B) \) in block matrix form

\[ A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad B = \begin{bmatrix} B_{11} & B_{21} \\ B_{21} & B_{22} \end{bmatrix}. \]

The centralizing condition is equivalent to

\[ \begin{bmatrix} B_{11} & B_{12} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A_{11} & 0 \\ A_{21} & 0 \end{bmatrix}. \]

Therefore, one has

\[ A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \quad B = \begin{bmatrix} A_{11} & 0 \\ B_{21} & B_{22} \end{bmatrix}. \]

Note we have a formula for \( B^{-1} \) in terms of blocks:

\[ B^{-1} = \begin{bmatrix} A_{11}^{-1} & 0 \\ -B_{22}^{-1}B_{21}A_{11}^{-1} & B_{22}^{-1} \end{bmatrix}. \]

We can now provide a formula for the action of \( (A, B) \) in the centralizer of \( Y^+ \) on \( Y^- \) in block form:
\[ B^{-1}Y^{-A} = \begin{bmatrix} A_{11}^{-1} & 0 \\ -B_{22}^{-1}B_{21}A_{11}^{-1} & B_{22}^{-1} \end{bmatrix} \begin{bmatrix} Y_{11}^{-} \ Y_{12}^{-} \\ Y_{21}^{-} \ Y_{22}^{-} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \]

\[ = \begin{bmatrix} A_{11}^{-1}Y_{11}^{-}A_{11} \\ B_{22}^{-1}(Y_{21}^{-}A_{11} - B_{21}^{-}A_{11}^{-1}Y_{11}^{-}A_{11}) + B_{22}^{-1}(Y_{21}^{-}A_{12} + Y_{22}^{-}A_{22}) - B_{22}^{-1}B_{21}A_{11}^{-1}(Y_{11}^{-}A_{12} + Y_{12}^{-}A_{22}) \end{bmatrix}. \]

As long as we work with \((A, B) \in G_0\) that centralize \(Y^+\) (i.e., \((A, B) \in c_{G_0}(Y^+)\)), we can focus on transforming \(Y^-\) into the desired form.

3.2.3. Observe that \(Y_{11}^{-}\) can be put into its Jordan form \(J\) (upper triangular) via \(A_{11}\). If we then choose \(A_{11}\) to be in the centralizer of \(J\), we can replace both \(Y_{11}^{-}\) and \(A_{11}^{-1}Y_{11}A_{11}\) with \(J\) in the expression above. By operating with elements \((A, B) \in c_{G_0}(Y^+)\) with \(A_{11}\) that centralizes \(J\), our new expression for the action on \(Y^-\) is

\[ B^{-1}Y^-A = \begin{bmatrix} J \\ B_{22}^{-1}(Y_{21}^{-}A_{11} - B_{21}^{-}J) \\ B_{22}^{-1}(Y_{21}^{-}A_{12} + Y_{22}^{-}A_{22}) - B_{22}^{-1}B_{21}A_{11}^{-1}(J A_{12} + Y_{12}^{-}A_{22}) \end{bmatrix}. \]

3.2.4. We also observe that all the eigenvalues of \(J\) are zero since

\[ Y^+Y^- = \begin{bmatrix} J \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} J \\ Y_{21}^{-} \\ Y_{22}^{-} \end{bmatrix} = \begin{bmatrix} J \\ 0 \\ 0 \end{bmatrix}. \]

The condition that \(Tr((Y^+Y^-)^k) = 0\) for \(k = 1, \ldots, m\) where \(m = \min\{m, n\}\) implies that if \(\lambda_1, \lambda_2, \ldots, \lambda_r\) are the eigenvalues of \(J\) then \(\lambda_1^k + \lambda_2^k + \cdots + \lambda_r^k = 0\) for \(k = 1, 2, \ldots, m\). Since \(r \leq m\) this implies that \(\lambda_j = 0\) for all \(j\).

3.2.5. Let \(J\) be Jordan canonical form with Jordan blocks \(J_1, J_2, \ldots, J_t\) of sizes \(k_1 \geq k_2 \geq \cdots \geq k_t\) with all Jordan blocks having eigenvalue zero.

\[ J = \begin{pmatrix} J_1 & 0 & 0 & \cdots & 0 \\ 0 & J_2 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & J_t \end{pmatrix}. \]

Consider the action of \((A, B) \in c_{G_0}(Y^+)\) with \(A_{11}, A_{22},\) and \(B_{22}\) to be (appropriately sized) identity matrices. Then

\[ B^{-1}Y^-A = \begin{bmatrix} J \\ Y_{21}^{-} - B_{21}^{-}J \end{bmatrix} \begin{bmatrix} J A_{12} + Y_{12}^- \end{bmatrix}. \]

The matrix \(J A_{12}\) (resp. \(B_{21}J\)) consists of matrix entries with zeros in rows (resp. columns) \(k_1, k_1 + k_2, \ldots, k_1 + k_2 + \cdots + k_t\) (resp. \(1, 1 + k_1, 1 + k_1 + k_2, \ldots, 1 + k_1 + k_2 + \cdots + k_t\)) and arbitrary entries in the other rows (resp. columns). Therefore, one can choose entries in \(A_{12}\) (resp. \(B_{21}\)) to make \(J A_{12} + Y_{12}^-\) (resp. \(Y_{21}^- - B_{21}J\)) into a matrix with possibly non-zero entries in rows (resp. columns) \(k_1, k_1 + k_2, \ldots, k_1 + k_2 + \cdots + k_t\) (resp. \(1, 1 + k_1, 1 + k_1 + k_2, \ldots, 1 + k_1 + k_2 + \cdots + k_t\)) and zeros in the other rows (resp. columns).
3.2.6. Now consider the action of \((A, B) \in c_{G_0}(Y^+)\) with \(A_{11}\) being the identity matrix, \(A_{12} = 0\) and \(B_{21} = 0\). Then

\[
B^{-1}Y^{-1}A = \begin{bmatrix}
J \\
B^{-1}Y_{21}
\end{bmatrix}
\begin{bmatrix}
Y_{12}^{-1}A_{22} \\
* 
\end{bmatrix}.
\]

Now one can make \(A_{22}\) (resp. \(B_{22}^{-1}\)) into a product of a permutation matrix and elementary matrices to transform \(Y_{12}^{-1}A_{22}\) (resp. \(B_{22}^{-1}Y_{21}\)) into a column (resp. row) echelon form with possibly non-zero entries in rows (resp. columns) \(k_1, k_1 + k_2, \ldots, k_1 + k_2 + \cdots + k_t\) (resp. \(1, 1 + k_1, 1 + k_1 + k_2, \ldots, 1 + k_1 + k_2 + \cdots + k_t\)) and zeros in the other rows (resp. columns).

3.2.7. The next step is to transform \(Y_{12}^{-1}\) into a matrix \(C_{r_1}\) of the form stated in the theorem. Let \((A, B) \in c_{G_0}(Y^+)\) with \(A_{12} = 0\), \(B_{21} = 0\) and \(A_{22}, B_{22}\) being identity matrices. Then

\[
B^{-1}Y^{-1}A = \begin{bmatrix}
A_{11}^{-1}JA_{11} \\
Y_{21}^{-1}A_{11}
\end{bmatrix}
\begin{bmatrix}
A_{11}^{-1}Y_{12}^{-1} \\
*
\end{bmatrix}.
\]

By using the action of \(A_{11} \in c(J)\) (centralizer of \(J\)) without loss of generality we can assume that the pivots in \(Y_{12}^{-1}\) are all 1’s. Normally we can transform \(Y_{12}^{-1}\) into a elementary row echelon form by making \(A_{11}^{-1}\) into a product of elementary matrices that performs the row operations used in Gaussian elimination, and since \(Y_{12}^{-1}\) is already in column echelon form from the previous step, this is equivalent to the form of \(C_{r_1}\). The issue is that this product of elementary matrices need not centralize \(J\). This problem can be remedied as follows.

The matrix \(Y_{12}^{-1}\) is in column echelon form and has non-zero entries in rows \(k_1, k_1 + k_2, \ldots, k_1 + k_2 + \cdots + k_t\) with zeros in the other rows. Set \(f(j) = k_1 + k_2 + \cdots + k_j, j = 1, 2, \ldots, t\). Furthermore, the non-zero pivots can lie in matrix positions \((f(j), j)\) for \(j = 1, 2, \ldots, t\). We want to use \(A_{11}^{-1}\) to clear the matrix entries in the columns directly below the pivots. If there was no restriction on \(A_{11}^{-1}\) one can accomplish this with a product of elementary matrices. For example, if one wants to eliminate a non-zero entry \(\alpha\) in position \((f(i), j)\), then one can apply the elementary matrix \(E_{f(j), f(i)}(-\alpha)\) on the left which replaces Row \(f(j)\) with \(-\alpha\times \text{Row } f(j) + \text{Row } f(i)\).

Consider the matrix \(L_{f(j), f(i)}(-\alpha)\) which is in the centralizer of \(J\) where

\[
L_{f(j), f(i)}(-\alpha) = \begin{bmatrix}
I_{k_1} & 0 & 0 & \cdots & 0 \\
Z_{2,1} & I_{k_2} & 0 & \cdots & 0 \\
Z_{3,1} & Z_{3,2} & I_{k_3} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & 0 \\
Z_{t,1} & Z_{t,2} & Z_{t,3} & \cdots & I_{k_t}
\end{bmatrix}.
\]

Here \(I_{k_j}\) are \(k_j \times k_j\) identity matrices and \(Z_{j,i}\) are \(k_j \times k_i\) matrices where \(i = 1, 2, \ldots, t - 1\) and \(j = 2, 3, \ldots, t\). Moreover, set \(Z_{j',i'} = 0\) for \((j, i) \neq (j', i')\), and the block matrix

\[
Z_{j,i} = \begin{bmatrix} 0 & -\alpha I_{k_i} \end{bmatrix}.
\]

It can be directly verified that \(L_{f(j), f(i)}(-\alpha)\) will have the same effect as \(E_{f(j), f(i)}(-\alpha)\) on the Gaussian elimination process on \(Y_{12}^{-1}\). Therefore, if the rank of \(Y_{12}^{-1}\) is \(r_1\) with pivots in rows \(i_1, \ldots, i_r\), then it is transformed to

\[
C_{r_1} = \begin{bmatrix} e_{i_1} & e_{i_2} & \cdots & e_{i_r} & 0 & \cdots & 0 \end{bmatrix}
\]

where \(e_{i_j}\) is the column vector with a 1 in the \(i_j\)-th row and zeroes elsewhere.

3.2.8. We now perform that same procedure as in the last step to to transform \(Y_{21}^{-1}\) into a matrix \(R_t\) with form analogous to the transpose of \(C_t\). Let \((A, B) \in c_{G_0}(Y^+)\) with \(A_{12} = 0\), \(B_{21} = 0\) and \(A_{22}, B_{22}\) being identity matrices. This time one makes \(A_{11}\) into a product of matrices that are upper triangular, are in the centralizer of \(J\), and perform column operations to clear out the entries in the pivot rows of all non-zero entries (except for the pivot).
Note that in the process we have changed $Y_{12}^-$ out of the form of $C_1$. Consider $A_{11}$. Then $A_{11}^{-1}$ will still be upper triangular and $A_{11}^{-1}Y_{12}^-$ will be a matrix with 1’s in the same pivot entries as $Y_{12}^-$ with possible non-zero entries above and to the right of the position of the pivot entries in $Y_{12}^-$. The next two steps will correct this issue.

3.2.9. Now let $(A, B) \in cG_0(Y^+)$ with $A_{11}$, $A_{22}$, and $B_{22}$ be identity matrices, and $B_{21} = 0$. Then

$$B^{-1}Y^-A = \begin{bmatrix} J & JA_{12} + Y_{12}^- \\ Y_{21}^- & * \end{bmatrix}.$$  

We can now transform $Y_{12}^-$ into a matrix in row echelon form by choosing $A_{12}$ to kill the non-zero entries in the "non-pivot" rows. Note that $Y_{21}^-$ is unchanged.

3.2.10. Let $(A, B) \in cG_0(Y^+)$ with $A_{11}$, $B_{22}$ being identity matrices, and $A_{12} = 0$, $B_{21} = 0$. Then

$$B^{-1}Y^-A = \begin{bmatrix} J & Y_{12}^-A_{22}^- \\ Y_{21}^- & * \end{bmatrix}.$$  

The matrix $A_{22}$ can be chosen to clear out the non-zero entries that are not pivots to make $Y_{12}^-$ in column echelon form. Again $Y_{21}^-$ is unchanged in the process.

3.2.11. The final step is to transform $Y_{22}^-$ into a matrix of the desired form. We will work with $(A, B)$ in the centralizer of $Y^+$ with $A_{11}$ centralizing $J$. Let $A_{11} = I$ and $A_{12}$, $B_{21} = 0$. Then $Y_{22}^-$ is transformed to $B_{22}^{-1}Y_{22}^-A_{22}$. Let $\text{rank}(C_{r_1}) = r_1$ and $\text{rank}(R_{r_2}) = r_2$.

Write

$$Y_{22}^- = \begin{pmatrix} \xi_{11} & \xi_{12} \\ \xi_{21} & \xi_{22} \end{pmatrix}$$

where $\xi_{11}$ is a $r_2 \times r_1$ matrix. Moreover, $\xi_{12}$ is $r_2 \times (n - k - r_1)$, $\xi_{21}$ is $(n - k - r_2) \times r_1$, and $\xi_{22}$ is $(n - k - r_2) \times (n - k - r_1)$.

Now to centralize $C_{r_1}$ and $R_{r_2}$ we require the first $r_1$ rows of $A_{22}$ to be the first $r_1$ rows of the identity matrix and similarly the first $r_2$ columns of $B_{22}^{-1}$ to be the first $r_2$ columns of the identity matrix. Therefore we can write $A_{22}$ and $B_{22}^{-1}$ in block form (in a similar way with $Y_{22}^-$) as

$$A_{22} = \begin{pmatrix} I_{r_1} \\ 0_{r_1 \times (n - k - r_1)} \end{pmatrix} \quad \text{and} \quad B_{22}^{-1} = \begin{pmatrix} I_{r_2} \\ 0_{r_2 \times (n - k - r_2)} \end{pmatrix}.$$  

Then

$$B_{22}^{-1}Y_{22}^-A_{22} = \begin{pmatrix} \xi_{11} + \beta_{12}\xi_{21} + \xi_{12}\alpha_{21} + \beta_{12}\xi_{22}\alpha_{21} \\ \beta_{22}\xi_{21} + \beta_{22}\xi_{22}\alpha_{21} \end{pmatrix}.$$  

This shows that we can send $\xi_{22}$ to

$$\begin{pmatrix} I_s & 0 \\ 0 & 0 \end{pmatrix}$$

where $s = \text{rank}(\xi_{22})$.

3.2.12. Now choose $(A, B) \in cG_0(Y^+)$ with $A_{11}$, $A_{22}$ and $B_{22}^{-1}$ to be the identity, while allowing $A_{12}$ and $B_{21}$ to be free. This action will fix $\xi_{22}$ and image of $Y_{22}^-$ is

$$\xi_{11} + \beta_{12}\xi_{21} + \xi_{12}\alpha_{21} + \beta_{12}\xi_{22}\alpha_{21} \quad \text{and} \quad \beta_{22}\xi_{21} + \beta_{22}\xi_{22}\alpha_{21}$$

Set $B_{21}$ to be zero. Let the pivots of $Y_{21}^-$ be in columns $i_1, \ldots, i_{r_2}$. Then set the $i_1$-th row of $A_{12}$ to be the negative of the 1st row of $Y_{22}^-$, the $i_2$-th row of $A_{12}$ to be the negative of the 2nd row of $Y_{22}^-$ and so forth. Then by (3.2.1), $(A, B)$ sends $\xi_{11}$ and $\xi_{12}$ to 0.
3.2.13. Finally, we need to choose \((A, B) \in c_{G_0}(Y^+)\) that stabilizes the current \(Y_{11}, Y_{12}, Y_{21},\) \(\xi_1, \xi_2, \xi_3\) and sends \(\xi_{21}\) to 0. This can be accomplished by setting \(A_{11}, A_{22}\) and \(B_{22}^{-1}\) to be the identity, and \(A_{12}\) to be zero. If the pivots of \(Y_{12}\) are in rows \(j_1, \ldots, j_n\). Then set the \(j_1\)-th column of \(B_{21}\) to be the 1st column of \(Y_{22}\), the \(j_2\)-th column of \(B_{21}\) to be the 2nd column of \(Y_{22}\) and so forth.

4. \(G_0\)-orbits on \(\mathcal{N}\): general case

4.1. In the ordinary Lie algebra case, the adjoint action of the algebraic group \(G\) on \(\mathfrak{g}\) is known to have finitely many nilpotent orbits via Richardson’s Theorem (see [Hum] Theorem 3.8). In this section, we prove an appropriate generalization for Lie superalgebras. We begin by stating the following lemma whose proof can be found in [Jan2] Section 2.4.

**Lemma 4.1.1.** Let \(G\) be an algebraic group and let \(H\) be a closed subgroup of \(G\). Let \(X\) be a \(G\)-variety and let \(Y\) be a closed and \(H\)-invariant subvariety of \(X\). Suppose that for all \(y \in Y\),

\[(4.1.1) \quad T_y(G \cdot y) \cap T_y(Y) \subseteq (d\pi_y)_{id}(\text{Lie } H),\]

where \(\pi_y : G \to G \cdot y\) sends \(g\) to \(gy\). Then the intersection with \(Y\) of each \(G\)-orbit in \(X\) is a finite union of \(H\)-orbits.

4.2. Generalization of Richardson’s Theorem. We can now state a generalization of Richardson’s theorem in the context of Lie superalgebras.

**Theorem 4.2.1.** Let \(G_0\) be a closed subgroup of some \(GL_m(\mathbb{C}) \times GL_n(\mathbb{C})\). Let \(\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1\) be a Lie superalgebra with \(\text{Lie } G_0 = \mathfrak{g}_0\). Suppose there exists a supersubspace \(M \subseteq \mathfrak{gl}(m|n)\) such that \(\mathfrak{gl}(m|n) = M \oplus \mathfrak{g}\) and \([\mathfrak{g}, M] \subseteq M\). Then the intersection with \(\mathfrak{g}\) of each \(GL_m(\mathbb{C}) \times GL_n(\mathbb{C})\)-orbit in \(\mathfrak{gl}(m|n)\) is a union of finitely many \(G_0\)-orbits.

**Proof.** Assume such a complement \(M\) exists. We show that condition \((4.1.1)\) of Lemma 4.1.1 is satisfied where \(X = \mathfrak{gl}(m|n), Y = \mathfrak{g}, G = GL_m(\mathbb{C}) \times GL_n(\mathbb{C}),\) and \(H = G_0\). This means one must show for every \(y \in \mathfrak{g}\)

\[T_y(G \cdot y) \cap T_y(\mathfrak{g}) \subseteq (d\pi_y)_{id}(\mathfrak{g}_0).\]

We observe that \(T_y(\mathfrak{g}) = \mathfrak{g}\) and \((d\pi_y)_{id}(\mathfrak{g}_0) = [\mathfrak{g}_0, y]\).

Now showing \((4.1.1)\) is equivalent in our case to showing

\[(4.2.1) \quad T_y(G \cdot y) \cap \mathfrak{g} \subseteq [\mathfrak{g}_0, y],\]

for every \(y \in \mathfrak{g}\).

Note that \(\text{Lie } G_x = (\mathfrak{gl}(m) \times \mathfrak{gl}(n))_x\), and by a standard fact in [Bor] Prop. 9.1] this is equivalent to \(T_y(G \cdot y) = [\mathfrak{gl}(m) \times \mathfrak{gl}(n), y]\).

Now apply the complement condition to obtain that

\[[\mathfrak{gl}(m) \times \mathfrak{gl}(n), y] = [M_0 \oplus \mathfrak{g}_0, y] = [M_0, y] + [\mathfrak{g}_0, y].\]

By assumption \([M_0, y] \subseteq M\), so \((4.2.1)\) becomes

\[T_y(G \cdot y) \cap \mathfrak{g} \subseteq (M + [\mathfrak{g}_0, y]) \cap \mathfrak{g}\]

and since \(M \cap \mathfrak{g} = \{0\}\) and \([\mathfrak{g}_0, y] \subseteq \mathfrak{g}\), this reduces to

\[T_y(G \cdot y) \cap \mathfrak{g} \subseteq [\mathfrak{g}_0, y].\]

So \((4.2.1)\) is satisfied and the result follows by Lemma 4.1.1. \(\square\)
4.3. In the case when Richardson’s Theorem is applied to show the finiteness of $G$-orbits for the nilpotent cone for complex semisimple Lie algebras, the existence of such an $M$ is guaranteed via complete reducibility. More specifically, this follows from regarding $\mathfrak{gl}(n)$ as a $G$-module under the adjoint action. Then $\mathfrak{g} = \text{Lie } G$ is a submodule and therefore has a vector space complement $M$ in $\mathfrak{gl}(n)$ that is invariant under the adjoint action of $\mathfrak{g}$.

In the situation for Lie superalgebras, we can apply this same reasoning to $G_0$ acting on $\mathfrak{g}_0$ to produce an $M_0$ satisfying $\mathfrak{gl}(m|n)_0 = \mathfrak{g}_0 \oplus M_0$ and $G_0 \cdot M_0 \subseteq M_0$. Then $M_0$ will also be invariant under the derived $\mathfrak{g}_0$ action, so that $[\mathfrak{g}_0, M_0] \subseteq M_0$. However, an issue arises when considering $\mathfrak{g}_1$. We can still regard $\mathfrak{g}_1$ as a $G_0$-module and produce a complement $M_1$, but since the derived action involves only $\mathfrak{g}_0$, we know nothing about $[\mathfrak{g}_1, M_1]$.

In order to prove finiteness of orbits for $\mathcal{N}$ in the superalgebra case we construct $M_1$ in a case-by-case manner and show directly that $[\mathfrak{g}_1, M_1] \subseteq M_0$. Details can be found in Section 4.4. We can now verify the finiteness of $\mathcal{N}$ though all of the Lie superalgebras considered here are over $\mathbb{C}$, we still need to use ideas from the characteristic $p$ case in order to produce compatible complements.

4.4. We can now verify the finiteness of $G_0$-orbits on $\mathcal{N}$ when $\mathfrak{g}$ is not isomorphic to $D(2,1,\alpha)$.

**Theorem 4.4.1.** Let $\mathfrak{g}$ be a classical simple Lie superalgebra over $\mathbb{C}$ with $\mathfrak{g} \neq D(2,1,\alpha)$. Then $\mathcal{N}$ has finitely many $G_0$-orbits.

**Proof.** For classical simple Lie superalgebras $\mathfrak{g}$ other than $D(2,1,\alpha)$, $G(3)$ and $F(4)$, there exists an embedding $\mathfrak{g} \hookrightarrow \mathfrak{g}' \cong \mathfrak{gl}(m|n)$ and a supersubspace $M \subseteq \mathfrak{g}'$ such that $\mathfrak{g}' = M \oplus \mathfrak{g}$ and $[\mathfrak{g}, M] \subseteq M$. The embeddings and complements are described in Section 4.3 and 6.

Next we need to show that $\mathcal{N}_0 \subseteq \mathcal{N}'$. We will prove a stronger statement that $\mathcal{N}_0 \cap \mathfrak{g}_1 = \mathcal{N}_0$. First we prove that $\mathcal{N}_0 \subseteq \mathcal{N}' \cap \mathfrak{g}_1$. Let $z \in \mathcal{N}_0$ then $f(z) = 0$ for every $f(x) \in S^\bullet(\mathfrak{g}_1^\ast)^{G_0}$. We have the identifications:

$$S^\bullet((\mathfrak{g}_1^\ast)^{G_0}) \subseteq S^\bullet((\mathfrak{g}_1^\ast)^{G_0}) = [S^\bullet(\mathfrak{g}_1^\ast) \otimes S^\bullet(M_1^\ast)]^{G_0}.$$  

Under this identification, one can regard $g(x) = g(p, q) \in S^\bullet((\mathfrak{g}_1^\ast)^{G_0})$. If $z \in \mathcal{N}_0 \subseteq \mathfrak{g}_1$ then $g(z) = g(z, 0) = 0$, thus $z \in \mathcal{N}'$.

The other inclusion, $\mathcal{N}_0 \cap \mathfrak{g}_1 \subseteq \mathcal{N}_0$, uses property (b). One has $S^\bullet(\mathfrak{g}_1^\ast)^{G_0} = S^\bullet(\mathfrak{g}_1^\ast)^{\mathfrak{g}_0}$, and it will be more convenient to use Lie algebra invariants. We have

$$S^\bullet((\mathfrak{g}_1^\ast)^{\mathfrak{g}_0}) \subseteq [S^\bullet(\mathfrak{g}_1^\ast) \otimes S^\bullet(M_1^\ast)]^{\mathfrak{g}_0} \subseteq [S^\bullet(\mathfrak{g}_1^\ast) \otimes S^\bullet(M_1^\ast)]^{\mathfrak{g}_0},$$

and

$$S^\bullet(\mathfrak{g}_1^\ast)^{\mathfrak{g}_0} \subseteq [S^\bullet(\mathfrak{g}_1^\ast) \otimes S^\bullet(M_1^\ast)]^{\mathfrak{g}_0}.$$  

Let $h(x) \in S^\bullet(\mathfrak{g}_1^\ast)^{G_0}$. Let $p = g_0 + m_0 \in \mathfrak{g}_0^\ast$ where $g_0 \in \mathfrak{g}_0$ and $m_0 \in M_0$. Using the inclusion in (4.4.3) and the fact that $[M_0, \mathfrak{g}_1] \subseteq M_1$, it follows that

$$p.h(x) = -h([g_0, x] + [m_0, x]) = -h([g_0, x]) = g_0.h(x) = 0.$$  

This shows that if $z \in \mathcal{N}_0 \cap \mathfrak{g}_1$ then $z \in \mathcal{N}_0$.

We can now prove the finiteness of $G_0$-orbits on $\mathcal{N} := \mathcal{N}_0$. Let $G_0 \cdot y \in \mathcal{N}$. Set $G_0' = GL_m(\mathbb{C}) \times GL_n(\mathbb{C})$. Then $G_0' \cdot y$ contains $G_0 \cdot y$, and $y \in \mathcal{N}_0$. Now the finiteness of $G_0$-orbits on $\mathcal{N}$ follows from the finiteness of orbits for the nilpotent cone of $\mathfrak{gl}(m|n)$ and the fact that the intersection of any orbit in $\mathcal{N}_0$ with $\mathfrak{g}$ contains only finitely many $G_0$-orbits (see Theorem 4.2.1).

For $\mathfrak{g} = G(3)$ or $F(4)$, one can argue the finiteness as follows. Let $\mathfrak{g} = G(3)$ . In this case $\mathfrak{g}_1 = V \otimes \mathbb{Z}$ where $V$ is the 2-dimensional natural representation for $SL_2 := SL_2(\mathbb{C})$ and $Z$ is the 7-dimensional irreducible representation for $G_2$. Let $v_H = (1, 0)^T$ and $v_L = (0, 1)^T$ and let $z_H$ be a highest weight vector for $Z$. If $x \in \mathcal{N}$ then $x = v_H \otimes p_1 + v_L \otimes p_2$. If $p_1 = 0$ or $p_2 = 0$ then we can
use the fact that and $SL_2$ acts transitively on $V$ and $G_2$ acts transitively on $Z$ to show that $x$ is $G_0$-conjugate to $v_H \otimes z_H$.

Now suppose that $p_1 \neq 0$ and $p_2 \neq 0$. First, we can conjugate $x$ to $x_1 = v_H \otimes z_H + v_L \otimes p_2$. Using the Bruhat decomposition for $G_2$ one can show that if $B$ is a Borel subgroup (corresponding to the positive roots) for $G_2$ then there are finitely many $B$-orbits on $Z$ with orbit representatives given by weight vectors of the form $z_\gamma$ where $\gamma$ is a short root for $G_2$. The group $B = T \times U$ where $U$ acts trivially on $z_H$ and $T$ acts by scaling $z_H$. Thus, $x_1$ is $G_0$-conjugate to $x_2 = \alpha(v_H \otimes z_H) + v_L \otimes z_\gamma$ for $\alpha \neq 0$. Moreover, since $x_2 \in \mathcal{N}$ and satisfies a 4th degree $T_0$-invariant polynomial ($T_0$ a maximal torus for $G_0$), it follows that $z_\gamma$ is not a multiple of a lowest weight vector $z_L$. Now one can use $T_0$ to conjugate $x_2$ to $x_3 = v_H \otimes z_H + v_L \otimes z_\gamma$. Consequently, there are only finitely many $G_0$-orbits on $\mathcal{N}$.

A similar argument can be used to prove the finiteness of $G_0$-orbits for $g = F(4)$. Our conclusions on the finiteness for $G(3)$ and $F(4)$ can also be found in [K Table IV].

**4.5. The case for $g = D(2,1,\alpha)$.** Let $g = D(2,1,\alpha)$. Then $G_0 \cong SL_2 \times SL_2 \times SL_2$ with $g_1 = V \otimes V \otimes V$ where $V$ is the two-dimensional natural representation. Let $(g_1, g_2, g_3) \in G_0$ with

$$g_j = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix}$$

with $a_j d_j - b_j c_j$ for $j = 1, 2, 3$. Let $v_h = (1, 0)^T$, $v_l = (0, 1)^T$ and set $v_{hhh} = v_h \otimes v_h \otimes v_h \in V \otimes V \otimes V$, and $x_{hhh}$ to be the corresponding dual basis element. Similarly, one can define $v_{hhl}$, $v_{hlt}$, etc. (resp. $x_{hhl}$, $x_{hlt}$, etc.). Then a basis for $g_1$ is given by

$$V \otimes V \otimes V = \langle v_{hhh}, v_{hhl}, v_{hlt}, v_{hlt}, v_{hlt}, v_{hlt}, v_{hlt}, v_{hlt} \rangle.$$

Let $T_0$ be the maximal torus in $G_0$ consisting of diagonal matrices in each of the coordinates. Then $S^*\langle g_1^* \rangle_{T_0} = \mathbb{C}[w_1, w_2, w_3, w_4]$ where

$$w_1 = x_{hhh} x_{hlt}, \quad w_2 = x_{hhl} x_{hlt}, \quad w_3 = x_{hlt} x_{hlt}, \quad w_4 = x_{hhh} x_{hhl}$$

Next note that $S^*\langle g_1^* \rangle_{G_0} \subseteq S^*\langle g_1^* \rangle_{T_0}$ and $S^*\langle g_1^* \rangle_{G_0} = \mathbb{C}[f(x)]$ where $\deg f(x) = 4$ (cf. [BKN1, Table 1]). One can now use the facts along with the invariance of $f(x)$ under the Weyl group, $S_2 \times S_2 \times S_2$, of $G_0$ to show that (up to scaling) for some $a, b, c \in \mathbb{C}$:

$$f(x) = w_1^2 + w_2^2 + w_3^2 + w_4^2 + a(w_1 w_2 + w_3 w_4) + b(w_2 w_3 + w_1 w_4) + c(w_1 w_3 + w_2 w_4).$$

Observe that $f(x_{hhh}) = 0$ so $G_0 \cdot x_{hhh} \subseteq \mathcal{N}_1$. Furthermore, $\dim \mathcal{N} = 7$.

Next we want to show that there are infinitely many $G_0$-orbits in $\mathcal{N}$. Let $\sigma = (\tau, \beta, \gamma)$ where $\tau, \beta, \gamma \in \mathbb{C}^*$, and set

$$v_\sigma = v_{hhh} + \tau v_{hlt} + \beta v_{lth} + \gamma v_{lhl} \in \mathcal{N}.$$ 

We want to show that there are infinitely many $\sigma$ with $v_\sigma \in \mathcal{N}$ that yield distinct $G_0$-orbits.

Let $Z = \{ \sigma = (\tau, \beta, \gamma) : \tau, \beta, \gamma \in \mathbb{C}^*, \quad v_\sigma \in \mathcal{N} \}$. The condition that $v_\sigma \in \mathcal{N}$ means that $\tau^2 + \beta^2 + \gamma^2 + a(\tau \beta) + b(\beta \gamma) + c(\tau \gamma) = 0$. It follows that $Z$ is two-dimensional.

If there exists $g \in G_0$ where $g \cdot x_{\sigma_i} = x_{\sigma_2}$, and $\sigma_i = (\tau_i, \beta_i, \gamma_i)$ for $i = 1, 2$ then the following eight equations hold:

---

1One can use a similar argument to prove the vector of highest weight in $g_1$ in contained in $\mathcal{N}$ also holds for $F(4)$ and $G(3)$.
5.2. The theorem below shows that under suitable conditions on \(g\)-govern the representation theory of \(g\) self-commuting variety nilpotent cone of (4.5.6)

Then \(\dim B = 6 + 2 = 8\) and from (i) and (ii) above the inverse image of a point in the image has dimension zero. Therefore, \(\dim B \cdot V = 8\) which is a contradiction, since \(\dim G \cdot x = 7\).

5. Connections with the Duflo-Serganova Self-Commuting Variety

5.1. Let \(g = g_0 \oplus g_1\) be a finite-dimensional complex Lie superalgebra with Lie \(G_0 = g_0\). Duflo and Serganova defined the self-commuting variety as

\[ \mathcal{X} = \{ x \in g_1 : [x, x] = 0 \}. \]

The variety \(\mathcal{X}\) is a \(G_0\)-invariant conical variety of \(g_1\). In [DS], it was shown for a finite-dimensional \(g\)-module, \(M\), one can defined a subvariety \(\mathcal{X}_M\) of \(\mathcal{X}\). The collection of these associated varieties govern the representation theory of \(g\).

5.2. The theorem below shows that under suitable conditions on \(g\), the self-commuting variety is contained in the nilpotent cone of \(g\).

**Theorem 5.2.1.** Let \(g\) be a classical Lie superalgebra such

(a) there exists an embedding \(g \hookrightarrow g' \cong \mathfrak{gl}(m|n)\),

(b) there exists a supersubspace \(M \subseteq g'\) such that \(g' = M \oplus g\) and \([g, M] \subseteq M\).

Then \(\mathcal{X} \subseteq \mathcal{N}\).

**Proof.** Let \(\mathcal{X}_g\) (resp. \(\mathcal{X}_{g'}\)) be the self-commuting variety of \(g\) (resp. \(g'\)). Similarly, denote the nilpotent cone of \(g\) (resp. \(g'\)) by \(\mathcal{N}_g\) (resp. \(\mathcal{N}_{g'}\)).

By using the fact that \(\mathcal{N}_{g'}\) is defined as the zero set of \(\text{Tr}((X^+X^-)^k) k = 1, \ldots, r\) where \(r = \min\{m, n\}\), one has

\[ \mathcal{X}_{g'} \subseteq \mathcal{N}_{g'} \]

Moreover, using the definition of the self-commuting variety, one has

\[ \mathcal{X}_g \subseteq \mathcal{X}_{g'} \]

Now from the proof of Theorem 4.4.1 \(\mathcal{N}_{g'} \cap g_1 \subseteq \mathcal{N}_g\). Consequently, \(\mathcal{X}_g \subseteq \mathcal{N}_g\).
5.3. We can now state and prove generalizations of the finiteness of $G_0$-orbits on $\mathcal{X}$ due to Duflo and Serganova (cf. [DS, Theorem 4.2]). Note that their work is stated under the assumption that $\mathfrak{g}$ is a contragredient Lie superalgebra with indecomposable Cartan matrix.

**Corollary 5.3.1.** Let $\mathfrak{g}$ be a classical simple Lie superalgebra over $\mathbb{C}$. Then

(a) $\mathcal{X} \subseteq \mathcal{N}$,

(b) $\mathcal{X}$ has finitely many $G_0$-orbits.

*Proof.* We handle the case first when $\mathfrak{g}$ is not isomorphic to $D(2,1,\alpha)$, $F(4)$ or $G(3)$. In this situation, Theorem 5.2.1 applies. Therefore, $\mathcal{X} \subseteq \mathcal{N}$ and $\mathcal{X}$ has finitely many $G_0$-orbits.

Now in the case when $\mathfrak{g} = D(2,1,\alpha)$, $F(4)$ or $G(3)$. One can obtain the inclusion $\mathcal{X} \subseteq \mathcal{N}$ because $\mathcal{X}$ is the closure of $G_0 \cdot v_H$ where $v_H$ is the highest weight vector (cf. [DS], pf. of Theorem 4.2]). Since $v_H$ satisfies the defining equation for $\mathcal{N}$, one obtains the inclusion. The finiteness result for $\mathcal{X}$ follows from the finiteness results for $F(4)$ and $G(3)$ in Theorem 4.4.1. Finally, for $D(2,1,\alpha)$, one can directly verify the finiteness result for $\mathcal{X}$.

$\square$

5.4. For $\mathfrak{gl}(m|n)$, we can use the parametrization of $G_0$-orbit representatives for $\mathcal{N}$ to recover the Duflo-Seganova parametrization of $G_0$-orbit representatives for $\mathcal{X}$ (cf. [DS, Theorem 4.2]).

Let $Y$ be an orbit representative as described in Theorem 3.1.1(b). Then $Y \in \mathcal{X}$ if and only if $[Y,Y] = 2Y^2 = 0$. A direct calculation shows that $Y^2 = 0$ if and only if $Y^{-}Y^{+} = 0$ and $Y^{+}Y^{-} = 0$. This is equivalent to the Jordan blocks $J_i = 0$ for $i = 1, 2, \ldots, t$, $C_{r_1} = 0$, and $R_{r_2} = 0$. Hence, $Y \in \mathcal{X}$ if and only if

$$Y^{+} = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad Y^{-} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I_s & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This corresponds to taking a representative of a subset of linearly independent set of mutually orthogonal isotropic odd roots under the action of the Weyl group for $G_0$ (see the paragraph after [DS, Theorem 4.2]).

### 6. Appendix: Construction of Complements $M$

6.1. For each classical Lie superalgebra $\mathfrak{g}$, an explicit matrix realization of $\mathfrak{g}$ is well known (for example, see [K]). We construct a matrix realization for the complement $M$ in Table 6.1.1 below.

6.2. We now check that each of the classical Lie superalgebras $\mathfrak{g}$ except $\mathfrak{gl}(m|n)$ satisfy the hypotheses of Theorem 4.2.1 case-by-case. From the construction of $M_1$ in each case below it follows that $\mathfrak{gl}(m|n) = \mathfrak{g}_1 \oplus M_1$, since each generator $E_{ij}$ of $\mathfrak{gl}(m|n)$ can be written as a sum of an element of $\mathfrak{g}_1$ and an element of $M_1$ in an obvious way. Direct calculation shows that $[\mathfrak{g}_1, M_j] \subseteq M_{i+j}$ in each case. Sample calculations are given below for each classical Lie superalgebra when $i = j = 1$.

- $\mathfrak{s}(m|n) : Y = 0$, so $[X,Y] = 0$.
- $\mathfrak{q}(n) : X = \begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix}, Y = \begin{bmatrix} 0 & d \\ -d & 0 \end{bmatrix}$. Then

$$[X,Y] = \begin{bmatrix} db - bd & 0 \\ 0 & -(db - bd) \end{bmatrix}.$$

- $\mathfrak{p}(n) : X = \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}, Y = \begin{bmatrix} 0 & a \\ d & 0 \end{bmatrix}$ with $b, d$ symmetric and $a, c$ skew-symmetric. Then

$$[X,Y] = \begin{bmatrix} bd + ac & 0 \\ 0 & ca + db \end{bmatrix} = \begin{bmatrix} bd + ac & 0 \\ 0 & (bd + ac)^t \end{bmatrix}.$$
Table 6.1.1. Block matrix realization of $M$ for classical Lie superalgebras

| $\mathfrak{g}$ | $M$ |
|----------------|-----|
| $\mathfrak{sl}(m|n)$ | $\begin{bmatrix} kI_m & 0 \\ 0 & -kI_n \end{bmatrix}$, $I_m, I_n$ identity matrices, $k \in \mathbb{C}$ |
| $\mathfrak{osp}(2m+1|2n)$ | $\begin{bmatrix} \delta & u^t & v^t & x & x_1 \\ v & a & b & y & y_1 \\ u & c & a^t & z & z_1 \end{bmatrix}$, $b, c$ symmetric, $e, f$ skew-symmetric |
| $\mathfrak{osp}(2m|2n)$ | $\begin{bmatrix} x^t & z^t_1 & y^t_1 & d & e \\ -x^t & -z^t & -y^t & f & d^t \end{bmatrix}$, $b, c$ symmetric, $e, f$ skew-symmetric |
| $\mathfrak{q}(n)$ | $\begin{bmatrix} a & b \\ -b & -a \end{bmatrix}$, $b$ skew-symmetric, $c$ symmetric |
| $\mathfrak{p}(n)$ | $\begin{bmatrix} a & b \\ c & a^t \end{bmatrix}$ |

- $\mathfrak{osp}(2m+1|2n) : X = \begin{bmatrix} x & x_1 \\ y & y_1 \\ z & z_1 \end{bmatrix}$, $Y = \begin{bmatrix} a & a_1 \\ b & b_1 \\ c & c_1 \end{bmatrix}$. Then $[X, Y] = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ where the matrices $A, B$ have block forms

$A = \begin{bmatrix} xa^t_i - x_1a^t + ax^t_1 + a_1x^t & x^t_1 - x_1c^t - az^t_1 + a_1z^t & \frac{1}{2} (a^t b - a b^t + ay^t_1 + a_1y^t) \\ ya^t_i - y_1a^t - bx^t_1 + b_1x^t & y^t_1 - y_1c^t + b_1z^t + b_1z^t & bx^t_1 - x_1b^t - ay^t_1 + a_1y^t \\ za^t_i - z_1a^t - cx^t_1 + c_1x^t & ze^t_1 - z_1c^t - cz^t_1 + c_1z^t & z^t b - z_1b^t - cy^t_1 + c_1y^t \end{bmatrix}$

$B = \begin{bmatrix} -x^t a - z^t b - y^t c + a^t x + c^t y + b^t z & -x^t a_1 - z^t b_1 - y^t c_1 + a^t x_1 + c^t y_1 + b^t z_1 \\ x^t a + z^t b + y^t c - a^t x - c^t y - b^t z & x^t a_1 + z^t b_1 + y^t c_1 - a^t x_1 - c^t y_1 - b^t z_1 \end{bmatrix}$

with the blocks satisfying the relations

$A_{12} = A_{31}, A_{13} = A_{21}, A_{33} = A_{22}, B_{22} = B_{11}^t$

and with $A_{12}, A_{33}$ symmetric and $B_{12}, B_{21}$ skew-symmetric.

- $\mathfrak{osp}(2m|2n)$: Since this superalgebra is obtained by deleting the first row and first column of $\mathfrak{osp}(2m+1|2n)$, the calculations in this case are obtained in a similar manner.

References

[BKN1] B.D. Boe, J.R. Kujawa, D.K. Nakano, Cohomology and support varieties for Lie superalgebras, Transactions of the AMS, 362 (2010), 6551-6590.

[BKN2] B.D. Boe, J.R. Kujawa, D.K. Nakano, Cohomology and support varieties for Lie superalgebras II, Proc. London Math. Soc., 98 (2009), no. 1, 19-44.

[BKN3] Complexity and module varieties for classical Lie superalgebras, International Math. Research Notices, doi:10.1093/imrn/rnq090, (2010).

[BKN4] Tensor triangular geometry for Lie superalgebras, Advances in Math., 314 (2017), 228-277.

[Bor] A. Borel, Linear Algebraic Groups, 2nd ed., Graduate Texts in Mathematics, 126, Springer, 1991.
[BruKl]  J. Brundan, A. Kleshchev, Modular representations of the supergroup $Q(n)$ I, J. Algebra, 260 (2003), 64-98.

[Bru]  J. Brundan, Modular representations of the supergroup $Q(n)$ II, Pacific J. Math., 224 (2006), 65–90.

[DK]  J. Dadok and V. G. Kac, Polar representations, J. Algebra, 92 (1985), no. 2, 504–524.

[DS]  M. Duflo, V. Serganova, On associated variety for Lie superalgebras, ArXiv: 0507198.

[Fuks]  D.B. Fuks, Cohomology of Infinite Dimensional Lie Algebras, Springer, 1986.

[GGNW]  D. Grantcharov, N. Grantcharov, D.K. Nakano, J. Wu, On BBW parabolics for Lie superalgebras, ArXiv:1810.06980.

[HTT]  R. Hotta, K. Takeuchi, T. Tanisaki, D-modules, Perverse Sheaves, and Representation Theory, Progress in Mathematics, 236, Birkhäuser, 2008.

[Hum]  J.E. Humphreys, Conjugacy Classes in Semisimple Algebraic Groups, Mathematical Surveys and Monographs, Vol. 43, American Mathematical Society, 1995.

[Jan1]  J. Jantzen, Representations of Algebraic Groups, Second Edition, Mathematical Surveys and Monographs, Vol. 107, American Mathematical Society, Providence RI, 2003.

[Jan2]  J. Jantzen, Nilpotent orbits in representation theory, Progress in Mathematics, 228, Birkhauser, 2004, 1-211.

[K]  V. Kac, Some remarks on nilpotent orbits, J. Algebra, 64 (1980), 190-213.

[Kum]  S. Kumar, Kac-Moody Groups, their Flag Varieties and Representation Theory, Progress in Mathematics, Vol. 204, Birkhauser, Boston, MA, 2002.

[LNZ]  G.I. Lehrer, D.K. Nakano, R. Zhang, Detecting cohomology for Lie superalgebras, Advances in Math., 228 (2011), 2008–2115.

[LR]  D. Luna and R. W. Richardson, A generalization of the Chevalley restriction theorem, Duke Math. J., 46 (1979), no. 3, 487–496.

[P]  I. Penkov, Borel-Weil-Bott theory for classical Lie supergroups, (Russian) Translated in J. Soviet Math. 51 (1990), 2108–2140.

[PS]  I. Penkov, V. Serganova, Characters of irreducible $G$-modules and cohomology of $G/P$ for the Lie supergroup $G = Q(N)$, J. Math. Sci. (New York) 84 (1997), 1382–1412.