Twisted modules for vertex operator algebras

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Abstract

This contribution is mainly based on joint papers with Lepowsky and Milas, and some parts of these papers are reproduced here. These papers further extended works by Lepowsky and by Milas. Following our joint papers, I explain the general principles of twisted modules for vertex operator algebras in their powerful formulation using formal series, and derive general relations satisfied by twisted and untwisted vertex operators. Using these, I prove new “equivalence” and “construction” theorems, identifying a set of sufficient conditions in order to have a twisted module for a vertex operator algebra, and a simple way of constructing the twisted vertex operator map. This essentially combines our general relations for twisted modules with ideas of Li (1996), who had obtained similar construction theorems using different relations. Then, I show how to apply these theorems in order to construct twisted modules for the Heisenberg vertex operator algebra. I obtain in a new way the explicit twisted vertex operator map, and in particular give a new derivation and expression for the formal operator \( \Delta_x \) constructed some time ago by Frenkel, Lepowsky and Meurman. Finally, I reproduce parts of our joint papers. I use the untwisted relations in the Heisenberg vertex operator algebra in order to understand properties of a certain central extension of a Lie algebra of differential operators on the circle: the connection between the structure of the central term in Lie brackets and the Riemann Zeta function at negative integers. I then use the twisted relations in order to construct in a simple way a family of representations for this algebra based on twisted modules for the Heisenberg vertex operator algebra. As a simple consequence of the twisted relations, the construction involves the Bernoulli polynomials at rational values in a fundamental way.

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1 Introduction

This contribution is mainly based on, and partly reproduces, the recent papers by the present author, Lepowsky and Milas [DLMi1], [DLMi2]. These works were a continuation of a series of papers of Lepowsky and Milas [L3], [L4], [M1]–[M3], stimulated by work of Bloch [Bl].

In those papers, we used the general theory of vertex operator algebras to study central extensions of classical Lie algebras and superalgebras of differential operators on the circle in connection with values of $\zeta$-functions at the negative integers, and with the Bernoulli polynomials at rational values. Parts of the present contribution recall the main results of [DLMi1, DLMi2]: Using general principles of the theory of vertex operator algebras and their twisted modules, we obtain a bosonic, twisted construction of a certain central extension of a Lie algebra of differential operators on the circle, for an arbitrary twisting automorphism. The construction involves the Bernoulli polynomials in a fundamental way. This is explained through results in the general theory of vertex operator algebras, including an identity discovered in [DLMi1, DLMi2] which was called “modified weak associativity”, and which is a consequence of the twisted Jacobi identity.

More precisely, we combine and extend methods from [L3], [L4], [M1]–[M3], [FLM1], [FLM2] and [DL2]. In those earlier papers, vertex operator techniques were used to analyze untwisted actions of the Lie algebra $\hat{D}^+$, studied in [Bl], on a module for a Heisenberg Lie algebra of a certain standard type, based on a finite-dimensional vector space equipped with a nondegenerate symmetric bilinear form. Now consider an arbitrary isometry $\nu$ of period say $p$, that is, with $\nu^p = 1$. Then, it was shown in [DLMi1, DLMi2] that the corresponding $\nu$–twisted modules carry an action of the Lie algebra $\hat{D}^+$ in terms of twisted vertex operators, parametrized by certain quadratic vectors in the untwisted module. This extends a result from [FLM1], [FLM2], [DL2] where actions of the Virasoro algebra were constructed using twisted vertex operators.

Still following [DLMi1, DLMi2], we explicitly compute certain “correction” terms for the generators of the “Cartan subalgebra” of $\hat{D}^+$ that naturally appear in any twisted construction. These correction terms are expressed in terms of special values of certain Bernoulli polynomials. They can in principle be generated, in the theory of vertex operator algebras, by the formal operator $e^{\Delta x}$ [FLM1], [FLM2], [DL2] involved in the construction of a twisted action for a certain type of vertex operator algebra, the Heisenberg vertex operator algebra. We generate those correction terms in an easier way, using the modified weak associativity relation.

Then, the present contribution extends the works [DLMi1, DLMi2] described above by providing a detailed analysis of the modified weak asso-
ciativity relation. We state and prove a new theorem (Theorem 5.1) about the equivalence of modified weak associativity and weak commutativity with the twisted Jacobi identity, and a new “construction” theorem (Theorem 5.5), where we identify a set of sufficient conditions in order to have a twisted module for a vertex operator algebra, and a simple way of constructing the twisted vertex operator map. The latter theorem essentially combines modified weak associativity with ideas of Li [Li1, Li2], where similar construction theorems were proven using different general relations of vertex operator algebras and twisted modules – there may be a “direct” path from Li’s construction theorems to ours, but we haven’t investigated this. The use of modified weak associativity seems to have certain advantages in the twisted case. As an illustration, we give a new proof that the \( \nu \)-twisted Heisenberg Lie algebra modules mentioned above are also twisted modules for the Heisenberg vertex operator algebra. Using our theorems, we explicitly construct the twisted vertex operator map (Theorem 6.2). This gives a new and relatively simple derivation and expression for this map, and in particular for the formal operator \( \Delta_x \) mentioned above. A consequence of this is that one minor technical assumption that had to be made in [DLMi1, DLMi2], about the action of the automorphism \( \nu \), can be taken away.

We should mention that in [KR] Kac and Radul established a relationship between the Lie algebra of differential operators on the circle and the Lie algebra \( \hat{\mathfrak{gl}}(\infty) \); for further work in this direction, see [AFMO], [KWY]. Our methods and motivation for studying Lie algebras of differential operators, based on vertex operator algebras, are new and very different, so we do not pursue their direction.

Although we will present many of the main results of [DLMi2] with some of the proofs, we refer the reader to this paper for a more extensive discussion of those results.

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2 Vertex operator algebras, untwisted modules and twisted modules

In this section, we recall the definition of vertex operator algebras, (untwisted) modules and twisted modules. For the basic theory of vertex operator algebras and modules, we will use the viewpoint of [LL].

In the theory of vertex operator algebras, formal calculus plays a fundamental role. Here we recall some basic elements of formal calculus (cf. [LL]). Formal calculus is the calculus of formal doubly–infinite series of formal variables, denoted below by \( x, y \), and by \( x_1, x_2, \ldots, y_1, y_2, \ldots \). The central object
of formal calculus is the formal delta–function
\[ \delta(x) = \sum_{n \in \mathbb{Z}} x^n \]
which has the property
\[ \delta \left( \frac{x_1}{x_2} \right) f(x_1) = \delta \left( \frac{x_1}{x_2} \right) f(x_2) \]
for any formal series \( f(x_1) \). The formal delta–function enjoys many other properties, two of which are:
\[
x_2^{-1} \delta \left( \frac{x_1 - x_0}{x_2} \right) = x_1^{-1} \delta \left( \frac{x_1 + x_0}{x_1} \right)
\]
(2.1)
and
\[
x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) + x_0^{-1} \delta \left( \frac{x_2 - x_1}{x_2} \right) = x_2^{-1} \delta \left( \frac{x_1 - x_0}{x_2} \right).
\]
(2.2)
In these equations, binomial expressions of the type \((x_1 - x_2)^n, n \in \mathbb{Z}\) appear. Their meaning as formal series in \(x_1\) and \(x_2\), as well as the meaning of powers of more complicated formal series, is summarized in the “binomial expansion convention” – the notational device according to which binomial expressions are understood to be expanded in nonnegative integral powers of the second variable. When more elements of formal calculus are needed below, we shall recall them.

2.1 Vertex operator algebras and untwisted modules

We recall from [FLM2] the definition of the notion of vertex operator algebra, a variant of Borcherds’ notion [Bo] of vertex algebra:

**Definition 2.1** A vertex operator algebra \((V, Y, 1, \omega)\), or \(V\) for short, is a \(\mathbb{Z}\)-graded vector space
\[
V = \bigoplus_{n \in \mathbb{Z}} V_{(n)}; \text{ for } v \in V_{(n)}, \text{ wt } v = n,
\]
such that
\[
V_{(n)} = 0 \text{ for } n \text{ sufficiently negative,}
\]
\[
\dim V_{(n)} < \infty \text{ for } n \in \mathbb{Z},
\]
equipped with a linear map \(Y(\cdot, x)\):
\[
Y(\cdot, x) : V \to (\text{End } V)[[x, x^{-1}]]
\]
\[
v \mapsto Y(v, x) = \sum_{n \in \mathbb{Z}} v_n x^{-n-1}, \text{ } v_n \in \text{End } V,
\]
(2.3)
where $Y(v, x)$ is called the vertex operator associated with $v$, and two particular vectors, $1, \omega \in V$, called respectively the vacuum vector and the conformal vector, with the following properties:

truncation condition: For every $v, w \in V$

$$v_n w = 0$$

(2.4)

for $n \in \mathbb{Z}$ sufficiently large;

vacuum property:

$$Y(1, x) = 1_V \quad (1_V \text{ is the identity on } V);$$

(2.5)

creation property:

$$Y(v, x)1 \in V[[x]] \quad \text{and} \quad \lim_{x \to 0} Y(v, x)1 = v;$$

(2.6)

Virasoro algebra conditions: Let

$$L(n) = \omega_{n+1} \text{ for } n \in \mathbb{Z}, \ i.e., \ Y(\omega, x) = \sum_{n \in \mathbb{Z}} L(n)x^{-n-2}.$$  

(2.7)

Then

$$[L(m), L(n)] = (m - n)L(m + n) + cV \frac{m^3 - m}{12} \delta_{n+m,0} 1_V$$

for $m, n \in \mathbb{Z}$, where $cV \in \mathbb{C}$ is the central charge (also called “rank” of $V$),

$$L(0)v = (\text{wt } v)v$$

for every homogeneous element $v$, and we have the $L(-1)$–derivative property:

$$Y(L(-1)u, x) = \frac{d}{dx}Y(u, x);$$

(2.8)

Jacobi identity:

$$x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) Y(u, x_1)Y(v, x_2) - x_0^{-1} \delta \left( \frac{x_2 - x_1}{-x_0} \right) Y(v, x_2)Y(u, x_1)$$

$$= x_2^{-1} \delta \left( \frac{x_1 - x_0}{x_2} \right) Y(Y(u, x_0)v, x_2).$$

(2.9)

An important property of vertex operators is skew–symmetry, which is an easy consequence of the Jacobi identity (cf. [FHL]):

$$Y(u, x)v = e^{xL(-1)}Y(v, -x)u.$$  

(2.10)

Another easy consequence of the Jacobi identity is the $L(-1)$–bracket formula:

$$[L(-1), Y(u, x)] = Y(L(-1)u, x).$$

(2.11)

Fix now a vertex operator algebra $(V, Y, 1, \omega)$, with central charge $cV$. 

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Definition 2.2 A (\(\mathbb{Q}\)-graded) module \(W\) for the vertex operator algebra \(V\) (or \(V\)-module) is a \(\mathbb{Q}\)-graded vector space,\\[ W = \coprod_{n \in \mathbb{Q}} W(n); \quad \text{for } v \in W(n), \ \text{wt} \ v = n, \] such that\\[ W(n) = 0 \quad \text{for } n \text{ sufficiently negative}, \]
\[ \dim W(n) < \infty \quad \text{for } n \in \mathbb{Q}, \]
equipped with a linear map\\[ Y_W(\cdot, x) : V \to (\text{End } W)[[x, x^{-1}]] \]
\[ v \mapsto Y_W(v, x) = \sum_{n \in \mathbb{Z}} v_n^W x^{-n-1}, \quad v_n^W \in \text{End } W, \quad (2.12) \]
where \(Y_W(v, x)\) is still called the vertex operator associated with \(v\), such that the following conditions hold:

- **truncation condition:** For every \(v \in V\) and \(w \in W\)
  \[ v_n^W w = 0 \quad \text{for } n \in \mathbb{Z} \text{ sufficiently large}; \]
- **vacuum property:**
  \[ Y_W(1, x) = 1_W; \quad (2.14) \]

**Virasoro algebra conditions:** Let
\[ L_W(n) = \omega_{n+1}^W \quad \text{for } n \in \mathbb{Z}, \quad \text{i.e.,} \quad Y_W(\omega, x) = \sum_{n \in \mathbb{Z}} L_W(n)x^{-n-2}. \]
We have
\[ [L_W(m), L_W(n)] = (m-n)L_W(m+n) + c_v \frac{m^3-m}{12} \delta_{m+n,0} 1_W, \]
\[ L_W(0)v = \text{wt } v \]
for every homogeneous element \(v \in W\), and
\[ Y_W(L(-1)u, x) = \frac{d}{dx} Y_W(u, x); \quad (2.15) \]

**Jacobi identity:**
\[ x_0^{-1}\delta \left( \frac{x_1 - x_2}{x_0} \right) Y_W(u, x_1)Y_W(v, x_2) - x_0^{-1}\delta \left( \frac{x_2 - x_1}{-x_0} \right) Y_W(v, x_2)Y_W(u, x_1) \]
\[ = x_2^{-1}\delta \left( \frac{x_1 - x_0}{x_2} \right) Y_W(Y(u, x_0)v, x_2). \quad (2.16) \]
From the Jacobi identity (2.16), one can derive the weak commutativity and weak associativity relations, respectively:

\[
(x_1 - x_2)^{k(u,v)} Y_W(u, x_1) Y_W(v, x_2) = (x_1 - x_2)^{k(u,v)} Y_W(v, x_2) Y_W(u, x_1)
\]

\[
(x_0 + x_2)^{l(u,w)} Y_W(u, x_0 + x_2) Y_W(v, x_2) w = (x_0 + x_2)^{l(u,w)} Y_W(u, x_0) Y_W(v, x_2) w,
\]

where \( u, v \in V \) and \( w \in W \), valid for large enough nonnegative integers \( k(u,v) \) and \( l(u,w) \), their minimum value depending respectively on \( u, v \) and on \( u, w \). For definiteness, we will pick the integers \( k(u,v) \) and \( l(u,w) \) to be the smallest nonnegative integers for which the relations above are valid.

### 2.2 Twisted modules for vertex operator algebras

The notion of twisted module for a vertex operator algebra was formalized in \cite{FFR} and \cite{D} (see also the geometric formulation in \cite{GJS}; see also \cite{DLM}), summarizing the basic properties of the actions of twisted vertex operators discovered in \cite{FLM1}, \cite{FLM2} and \cite{L2}; the main nontrivial axiom in this notion is the twisted Jacobi identity of \cite{FLM2} (and \cite{L2}); cf. \cite{FLM1}.

A critical ingredient in formal calculus needed in the theory of twisted modules is the appearance of fractional powers of formal variables, like \( x^{1/p} \), \( p \in \mathbb{Z}_+ \) (the positive integers). For the purpose of formal calculus, the object \( x^{1/p} \) is to be treated as a new formal variable whose \( p \)-th power is \( x \). The binomial expansion convention is applied as stated at the beginning of Section 2 to binomials of the type \( (x_1 + x_2)^{1/p} \). From a geometrical point of view, these rules correspond to choosing a branch in the “orbifold structure” described (locally) by the twisted vertex operator algebra module.

We now fix a positive integer \( p \) and a primitive \( p \)-th root of unity

\[
\omega_p \in \mathbb{C}.
\]

We record here two important properties of the formal delta–function involving fractional powers of formal variables:

\[
\delta(x) = \frac{1}{p} \sum_{r=0}^{p-1} \delta(\omega_p^r x^{1/p})
\]

and

\[
x_2^{-1} \delta \left( \omega_p^r \left( \frac{x_1 - x_0}{x_2} \right)^{1/p} \right) = x_1^{-1} \delta \left( \omega_p^{-r} \left( \frac{x_2 + x_0}{x_1} \right)^{1/p} \right).
\]

(2.21)
The latter formula can be found (in a slightly different form) in [Li2]. For the sake of completeness, we present here a proof.

**Proof:** The coefficient of $x_0^0$ in equation (2.21) is immediate. Consider some formal series $f(x) = \sum_{n \in \mathbb{C}} f_n x^n$, $f_n \in \mathbb{C}$. From the formula

\[ (-1)^k (\partial/\partial x_1)^k (x_1^s x_2^{-s-1}) = (\partial/\partial x_2)^k (x_1^{s-k} x_2^{-s-1+k}) \]

for any $s \in \mathbb{C}$ and $k$ a nonnegative integer, we find that

\[ (-1)^k \left( \frac{\partial}{\partial x_1} \right)^k (x_2^{-1} f(x_1/x_2)) = \left( \frac{\partial}{\partial x_2} \right)^k (x_1^{-1}(x_1/x_2)^{1-k} f(x_1/x_2)) \tag{2.22} \]

With $f(x) = \delta (\omega_p^r x^{1/p})$, we use the formal delta-function property to get

\[ (x_1/x_2)^{1-k} \delta (\omega_p^r (x_1/x_2)^{1/p}) = \delta (\omega_p^r (x_1/x_2)^{1/p}) \]

and thus

\[ (-1)^k \left( \frac{\partial}{\partial x_1} \right)^k (x_2^{-1} \delta (\omega_p^r (x_1/x_2)^{1/p})) = \left( \frac{\partial}{\partial x_2} \right)^k (x_1^{-1} \delta (\omega_p^{-r} (x_2/x_1)^{1/p})) \tag{2.23} \]

Summing over nonnegative integers $k$ with the coefficients $x_0^k/k!$ on both sides, we obtain (2.21). \[ \square \]

Recall the vertex operator algebra $(V, Y, 1, \omega)$ with central charge $c_V$ of the previous subsection. Fix an automorphism $\nu$ of period $p$ of the vertex operator algebra $V$, that is, a linear automorphism of the vector space $V$ preserving $\omega$ and $1$ such that

\[ \nu Y(v, x) \nu^{-1} = Y(\nu v, x) \text{ for } v \in V, \tag{2.24} \]

and

\[ \nu^p = 1_V. \tag{2.25} \]

**Definition 2.3** A (Q-graded) $\nu$-twisted $V$-module $M$ is a Q-graded vector space,

\[ M = \coprod_{n \in \mathbb{Q}} M_{(n)}; \text{ for } v \in M_{(n)}, \text{ wt } v = n, \]

such that

\[ M_{(n)} = 0 \text{ for } n \text{ sufficiently negative}, \]

\[ \dim M_{(n)} < \infty \text{ for } n \in \mathbb{Q}, \]

equipped with a linear map

\[ Y_M(\cdot, x) : V \rightarrow (\text{End } M)[[x^{1/p}, x^{-1/p}]] \]

\[ v \mapsto Y_M(v, x) = \sum_{n \in \mathbb{Z}} v_n^\nu x^{-n-1}, \quad v_n^\nu \in \text{End } M, \tag{2.26} \]
where $Y_M(v, x)$ is called the twisted vertex operator associated with $v$, such that the following conditions hold:

truncation condition: For every $v \in V$ and $w \in M$

$$v_n^\nu w = 0$$  \hspace{0.5cm} (2.27)

for $n \in \frac{1}{p}Z$ sufficiently large;

vacuum property:

$$Y_M(1, x) = 1_M;$$  \hspace{0.5cm} (2.28)

Virasoro algebra conditions: Let

$$L_M(n) = \omega_n^\nu_{n+1}$$ for $n \in Z$, i.e.,

$$Y_M(\omega, x) = \sum_{n \in Z} L_M(n)x^{-n-2}.$$  \hspace{0.5cm} (2.29)

We have

$$\left[L_M(m), L_M(n)\right] = (m-n)L_M(m+n) + c_V \frac{m^3 - m}{12} \delta_{m+n,0} 1_M,$$

$$L_M(0)v = (\text{wt } v)v$$  \hspace{0.5cm} (2.29)

for every homogeneous element $v$, and

$$Y_M(L(-1)u, x) = \frac{d}{dx}Y_M(u, x);$$  \hspace{0.5cm} (2.30)

Jacobi identity:

$$x_0^{-1}\delta \left(\frac{x_1 - x_2}{x_0}\right) Y_M(u, x_1)Y_M(v, x_2) - x_0^{-1}\delta \left(\frac{x_2 - x_1}{-x_0}\right) Y_M(v, x_2)Y_M(u, x_1)$$

$$= \frac{1}{p}x_2^{-1}\sum_{r=0}^{p-1}\delta \left(\omega_p^r \left(\frac{x_1 - x_0}{x_2}\right)^{1/p}\right) Y_M(Y(\nu^r u, x_0)v, x_2).$$  \hspace{0.5cm} (2.31)

Note that when restricted to the fixed–point subalgebra $\{u \in V | \nu u = u\}$, a twisted module becomes a true module: the twisted Jacobi identity (2.31) reduces to the untwisted one (2.16), by (2.20). This will enable us to construct natural representations of a certain infinite-dimensional algebra $\hat{D}^+$ (see Section 7) on suitable twisted modules.

3 Heisenberg vertex operator algebra and its twisted modules

It is appropriate at this point to make these definitions more substantial by giving a simple but important example of a vertex operator algebra, and of some of its twisted modules.
3.1 Heisenberg vertex operator algebra

Following [FLM2], let $\mathfrak{h}$ be a finite-dimensional abelian Lie algebra (over $\mathbb{C}$) of dimension $d$ on which there is a nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$. Let $\nu$ be an isometry of $\mathfrak{h}$ of period $p > 0$:

$$\langle \nu \alpha, \nu \beta \rangle = \langle \alpha, \beta \rangle, \quad \nu^p \alpha = \alpha$$

for all $\alpha, \beta \in \mathfrak{h}$. Consider the affine Lie algebra $\hat{\mathfrak{h}}$,

$$\hat{\mathfrak{h}} = \bigoplus_{n \in \mathbb{Z}} \mathfrak{h} \otimes t^n \oplus \mathbb{C},$$

with the commutation relations

$$[\alpha \otimes t^m, \beta \otimes t^n] = \langle \alpha, \beta \rangle m \delta_{m+n,0} C \quad (\alpha, \beta \in \mathfrak{h}, \ m, n \in \mathbb{Z})$$

$$[C, \hat{\mathfrak{h}}] = 0.$$

Set

$$\hat{\mathfrak{h}}^+ = \bigoplus_{n > 0} \mathfrak{h} \otimes t^n, \quad \hat{\mathfrak{h}}^- = \bigoplus_{n < 0} \mathfrak{h} \otimes t^n.$$

The subalgebra

$$\hat{\mathfrak{h}}^+ \oplus \hat{\mathfrak{h}}^- \oplus \mathbb{C}C$$

is a Heisenberg Lie algebra. Form the induced (level–one) $\hat{\mathfrak{h}}$–module

$$S = \mathcal{U}(\hat{\mathfrak{h}}) \otimes_{\mathcal{U}(\hat{\mathfrak{h}}^+ \oplus \hat{\mathfrak{h}}^- \oplus \mathbb{C}C)} \mathbb{C} \simeq S(\hat{\mathfrak{h}}^-) \quad (\text{linearly}),$$

where $\hat{\mathfrak{h}}^+ \oplus \mathfrak{h}$ acts trivially on $\mathbb{C}$ and $\mathbb{C}$ acts as 1; $\mathcal{U}(\cdot)$ denotes universal enveloping algebra and $S(\cdot)$ denotes the symmetric algebra. Then $S$ is irreducible under the Heisenberg algebra $\hat{\mathfrak{h}}^+ \oplus \hat{\mathfrak{h}}^- \oplus \mathbb{C}C$. We will use the notation $\alpha(n)$ ($\alpha \in \mathfrak{h}, \ n \in \mathbb{Z}$) for the action of $\alpha \otimes t^n \in \hat{\mathfrak{h}}$ on $S$.

The induced $\mathfrak{h}$–module $S$ carries a natural structure of vertex operator algebra. This structure is constructed as follows (cf. [FLM2]). First, one identifies the vacuum vector as the element 1 in $S$: $1 = 1$. Consider the following formal series acting on $S$:

$$\alpha(x) = \sum_{n \in \mathbb{Z}} \alpha(n) x^{-n-1} \quad (\alpha \in \mathfrak{h}).$$

Then, the vertex operator map $Y(\cdot, x)$ is given by

$$Y(\alpha_1(-n_1) \cdots \alpha_j(-n_j) 1, x)$$

$$= \frac{1}{(n_1 - 1)!} \left( \frac{d}{dx} \right)^{n_1-1} \alpha_1(x) \cdots \frac{1}{(n_j - 1)!} \left( \frac{d}{dx} \right)^{n_j-1} \alpha_j(x) \quad (3.1)$$
for $\alpha_k \in \mathfrak{h}$, $n_k \in \mathbb{Z}_+$, $k = 1, 2, \ldots, j$, for all $j \in \mathbb{N}$, where $:\cdot:\cdot$ is the usual normal ordering, which brings $\alpha(n)$ with $n > 0$ to the right. Choosing an orthonormal basis $\{\tilde{\alpha}_q | q = 1, \ldots, d\}$ of $\mathfrak{h}$, the conformal vector is $\omega = \frac{1}{2} \sum_{q=1}^{d} \tilde{\alpha}_q(-1)\tilde{\alpha}_q(-1)\mathbf{1}$. This implies in particular that the weight of $\alpha(-n)\mathbf{1}$ is $n$:

$$L(0)\alpha(-n)\mathbf{1} = n\alpha(-n)\mathbf{1} \quad (\alpha \in \mathfrak{h}, n \in \mathbb{Z}_+$$

where we used

$$L(0) = \frac{1}{2} \sum_{n \in \frac{1}{p}\mathbb{Z}} \sum_{q=1}^{d} \tilde{\alpha}_q(n)\tilde{\alpha}_q(-n)\,.\,$$

The isometry $\nu$ on $\mathfrak{h}$ lifts naturally to an automorphism of the vertex operator algebra $S$, which we continue to call $\nu$, of period $p$.

Then (cf. [FLM2]), the various properties of a vertex operator algebra are indeed satisfied by the quadruplet $(V, Y, \mathbf{1}, \omega)$ just defined.

### 3.2 Twisted modules

We now proceed as in [L1], [FLM1], [FLM2] and [DL2] to construct a space $S[\nu]$ that carries a natural structure of $\nu$–twisted module for the vertex operator algebra $S$. In these papers, the twisted module structure was constructed assuming the minor hypothesis that $\nu$ preserves a rational lattice in $\mathfrak{h}$. We show in Section [6] that the space $S[\nu]$ is a twisted module, without the need for this minor assumption.

Consider a primitive $p$–th root of unity $\omega_p$. For $r \in \mathbb{Z}$ set

$$\mathfrak{h}(r) = \{ \alpha \in \mathfrak{h} \mid \nu \alpha = \omega_p^r \alpha \} \subset \mathfrak{h}.$$

For $\alpha \in \mathfrak{h}$, denote by $\alpha_{(r)}$, $r \in \mathbb{Z}$, its projection on $\mathfrak{h}(r)$. Define the $\nu$-twisted affine Lie algebra $\hat{\mathfrak{h}}[\nu]$ associated with the abelian Lie algebra $\mathfrak{h}$ by

$$\hat{\mathfrak{h}}[\nu] = \bigoplus_{n \in \frac{1}{p}\mathbb{Z}} \mathfrak{h}(pn) \otimes t^n \oplus \mathbb{C}$$

with

$$[\alpha \otimes t^m, \beta \otimes t^n] = \langle \alpha, \beta \rangle m \delta_{m+n,0} \mathbb{C} \quad (\alpha \in \mathfrak{h}(pn), \beta \in \mathfrak{h}(pm), m, n \in \frac{1}{p}\mathbb{Z})$$

$$[C, \hat{\mathfrak{h}}[\nu]] = 0. \quad (3.3)$$

Set

$$\hat{\mathfrak{h}}[\nu]^+ = \bigoplus_{n > 0} \mathfrak{h}(pn) \otimes t^n, \quad \hat{\mathfrak{h}}[\nu]^− = \bigoplus_{n < 0} \mathfrak{h}(pn) \otimes t^n. \quad (3.4)$$
The subalgebra
\[ \mathfrak{h}[\nu]^+ \oplus \mathfrak{h}[\nu]^- \oplus \mathbb{C} \mathfrak{C} \] (3.5)
is a Heisenberg Lie algebra. Form the induced (level-one) \( \mathfrak{h}[\nu] \)-module
\[ S[\nu] = \mathcal{U}(\mathfrak{h}[\nu]^+ \oplus \mathfrak{h}[\nu]^- \oplus \mathbb{C} \mathfrak{C}) \cong S(\mathfrak{h}[\nu]^-) \text{ (linearly)}, \] (3.6)
where \( \mathfrak{h}[\nu]^+ \oplus \mathfrak{h}[\nu]^0 \) acts trivially on \( \mathfrak{C} \) and \( \mathfrak{C} \) acts as 1; \( \mathcal{U}(\cdot) \) denotes universal enveloping algebra. Then \( S[\nu] \) is irreducible under the Heisenberg algebra \( \mathfrak{h}[\nu]^+ \oplus \mathfrak{h}[\nu]^- \oplus \mathbb{C} \mathfrak{C} \). We will use the notation \( \alpha \nu(\cdot) \) for the action of \( \alpha \otimes t^n \in \mathfrak{h}[\nu] \) on \( S[\nu] \).

**Remark 3.1** The special case where \( p = 1 \) (\( \nu = 1 \)) corresponds to the \( \mathfrak{h} \)-module \( S \).

The \( \mathfrak{h}[\nu] \)-module \( S[\nu] \) is naturally a \( \nu \)-twisted module for the vertex operator algebra \( S \). One first constructs the following formal series acting on \( S[\nu] \):
\[ \alpha \nu(x) = \sum_{n \in \mathbb{Z}^+} \alpha \nu(n)x^{-n-1}, \] (3.7)
as well as the formal series \( W(v, x) \) for all \( v \in S \):
\[ W(\alpha_1(-n_1) \cdots \alpha_j(-n_j)1, x) = \frac{1}{(n_1 - 1)!} \left( \frac{d}{dx} \right)^{n_1-1} \alpha \nu_1(x) \cdots \frac{1}{(n_j - 1)!} \left( \frac{d}{dx} \right)^{n_j-1} \alpha \nu_j(x). \] (3.8)
where \( \alpha_k \in \mathfrak{h}, \) \( n_k \in \mathbb{Z}^+ \), \( k = 1, 2, \ldots, j \), for all \( j \in \mathbb{N} \). The twisted vertex operator map \( Y_{S[\nu]}(\cdot, x) \) acting on \( S[\nu] \) is then given by
\[ Y_{S[\nu]}(v, x) = W(e^{\Delta_x}v, x) \quad (v \in S) \] (3.9)
where \( \Delta_x \) is a certain formal operator involving the formal variable \( x \) \cite{FLM1, FLM2, DL2}. This operator is trivial on \( \alpha(-n)1 \in S \) \( (n \in \mathbb{Z}_+) \), so that one has in particular
\[ Y_{S[\nu]}(\alpha(-n)1, x) = \frac{1}{(n - 1)!} \left( \frac{d}{dx} \right)^{n-1} \alpha \nu(x). \] (3.10)

One crucial (among others) role of the formal operator \( \Delta_x \) is to make the fixed–point subalgebra \( \{ u \mid \nu u = u \} \) act according to a true module action. For instance, the conformal vector \( \omega \) is in the fixed–point subalgebra, so that the vertex operator \( Y_{S[\nu]}(\omega, x) \) generates a representation of the Virasoro algebra on the space \( S[\nu] \). This representation of the Virasoro algebra was
explicitly constructed in [DL2]. As one can see in the results of [DL2] and as will become clear below, the resulting representation $\text{Res}_x x Y_{S[\nu]}(\omega, x)$ of the Virasoro generator $L(0)$ is not an (infinite) sum of normal-ordered products the type $\sum_{n \in \mathbb{Z}_+} \alpha^\nu(n) \beta^\nu(-n); \alpha \beta$; rather, there is an extra term proportional to the identity on $S[\nu]$, the so-called correction term, which appears because of the operator $\Delta_x$. The correction term was calculated in [DL2] using the explicit action of $e^{\Delta_x}$ on $\omega$. In the case of the period–2, $\nu = -1$ automorphism, this action is given by [FLM1], [FLM2]:

$$e^{\Delta_x} \omega = \omega + \frac{1}{16} (\dim h)x^{-2},$$

and for general automorphism, the calculation was carried out in [DL2] (see also [FFR] and [FLM2]). These results are relevant, for instance, in the construction of the moonshine module [FLM2].

The calculation of the action of $\Delta_x$ on arbitrary elements of $S$ is, however, a much more complicated task. Below we will derive some identities among twisted vertex operators. One of the important consequences of these identities, for us, will be to give a tool to explicitly construct the twisted vertex operators associated to elements of $S$ from the knowledge of the twisted vertex operators associated to “simpler” elements, without requiring the explicit knowledge of $\Delta_x$. In fact, these identities allow us to construct recursively twisted vertex operators associated to all elements of $S$ and to compute $\Delta_x$, only starting from the knowledge that $\Delta_x$ is trivial on $\alpha(-n)1 \in S \ (n \in \mathbb{Z}_+)$. 

4 Commutativity and associativity properties

This section follows closely similar sections of [DLMi1] and [DLMi2], and reproduces the results and some of the proofs. We first recall the main commutativity and associativity properties of vertex operators in the context of modules ([FLM2], [FHL], [DL1], [Li1]; cf. [LL]), and then we derive other identities somewhat analogous to these. These other identities were stated and proven in [DLMi2], and the most important ones were stated in [DLMi1]. All these identities will be generalized to twisted modules, still following [DLMi1] [DLMi2]. Note that taking the module to be the vertex operator algebra $V$ itself, the relations below specialize to commutativity and associativity properties in vertex operator algebras. We will give the proofs of the simplest identities only, referring the reader to [DLMi2] for all the proofs. Throughout this and the next sections, we fix a vertex operator algebra $V$ and a $V$-automorphism $\nu$ of period $p$, $\nu^p = 1_V$. 

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4.1 Formal commutativity and associativity for untwisted modules

We already stated the weak commutativity relation (2.17) and the weak associativity relation (2.18). They imply the main “formal” commutativity and associativity properties of vertex operators, which, along with the fact that these properties are equivalent to the Jacobi identity, can be formulated as follows (see [LL]):

**Theorem 4.1** Let $W$ be a vector space (not assumed to be graded) equipped with a linear map $Y_W(\cdot, x)$ (2.12) such that the truncation condition (2.13) and the Jacobi identity (2.16) hold. Then for $u, v \in V$ and $w \in W$, there exist $k(u, v) \in \mathbb{N}$ and $l(u, w) \in \mathbb{N}$ and a (nonunique) element $F(u, v, w; x_0, x_1, x_2)$ of $W((x_0, x_1, x_2))$ such that

$$x_0^{k(u,v)}F(u, v, w; x_0, x_1, x_2) \in W[[x_0]]((x_1, x_2)),$$

$$x_1^{l(u,w)}F(u, v, w; x_0, x_1, x_2) \in W[[x_1]]((x_0, x_2))$$

and

$$Y_W(u, x_1)Y_W(v, x_2)w = F(u, v, w; x_1-x_2, x_1, x_2),$$

$$Y_W(v, x_2)Y_W(u, x_1)w = F(u, v, w; -x_2+x_1, x_1, x_2),$$

$$Y_W(Y(u, x_0)v, x_2)w = F(u, v, w; x_0, x_2+x_0, x_2)$$

(4.1)

and

$$Y_W(u, x_1)Y_W(v, x_2)w = F(u, v, w; x_1-x_2, x_1, x_2),$$

$$Y_W(v, x_2)Y_W(u, x_1)w = F(u, v, w; -x_2+x_1, x_1, x_2),$$

$$Y_W(Y(u, x_0)v, x_2)w = F(u, v, w; x_0, x_2+x_0, x_2)$$

(4.2)

(where we are using the binomial expansion convention). Conversely, let $W$ be a vector space equipped with a linear map $Y_W(\cdot, x)$ (2.12) such that the truncation condition (2.13) and the statement above hold, except that $k(u, v) (\in \mathbb{N})$ and $l(u, w) (\in \mathbb{N})$ may depend on all three of $u, v$ and $w$. Then the Jacobi identity (2.16) holds.

It is important to note that since $k(u, v)$ can be (and typically is) greater than 0, the formal series $F(u, v, w; x_1-x_2, x_1, x_2)$ and $F(u, v, w; -x_2+x_1, x_1, x_2)$ are not in general equal. Along with (4.1), the first two equations of (4.2) represent formal commutativity, while the first and last equations of (4.2) represent formal associativity, as formulated in [LL] (see also [PLM2] and [FHL]). The twisted generalization of this theorem, written below, was proven in [DLM2].

4.2 Additional relations in untwisted modules

From the equations in Theorem 4.1, we can derive a number of relations similar to weak commutativity and weak associativity but involving formal limit procedures (the meaning of such formal limit procedures is recalled
below). Even though only one of these will be of use in the following sections, we state here for completeness of the discussion the two relations that are not “easy” consequences of weak commutativity and weak associativity. These relations were proven in [DLMi2]; we report the proofs here.

The first relation can be expressed as follows:

**Theorem 4.2** With $W$ as in Theorem 4.1,

$$
\lim_{x_0 \to -x_2 + x_1} ((x_0 + x_2)^{l(u,w)} Y_W(Y(u, x_0)v, x_2)w) = x_1^{l(u,w)} Y_W(v, x_2)Y(u, x_1)w
$$

(4.3)

for $u, v \in V$.

The meaning of the formal limit

$$
\lim_{x_0 \to -x_2 + x_1} ((x_0 + x_2)^{l(u,w)} Y_W(Y(u, x_0)v, x_2)w)
$$

(4.4)

is that one replaces each power of the formal variable $x_0$ in the formal series $(x_0 + x_2)^{l(u,w)} Y_W(Y(u, x_0)v, x_2)w$ by the corresponding power of the formal series $-x_2 + x_1$ (defined using the binomial expansion convention). Notice again that the order of $-x_2$ and $x_1$ is important in $-x_2 + x_1$, according to the binomial expansion convention.

**Proof of Theorem 4.2** Apply the limit $\lim_{x_0 \to -x_2 + x_1}$ to the expression

$$(x_0 + x_2)^{l(u,w)} Y_W(Y(u, x_0)v, x_2)w$$

written as in the right-hand side of the third equation of (4.2). This limit is well defined; indeed, the only possible problems are the negative powers of $x_2 + x_0$ in $F(u, v, w, x_0, x_2 + x_0, x_2)$, but they are cancelled out by the factor $(x_0 + x_2)^{l(u,w)}$. The resulting expression is read off the second relation of (4.2).

**Remark 4.2** It is instructive to consider the following relation, deceptively similar to (4.3), but that is in fact an immediate consequence of weak associativity (2.18):

$$
\lim_{x_0 \to x_1 - x_2} ((x_0 + x_2)^{l(u,w)} Y_W(Y(u, x_0)v, x_2)w) = x_1^{l(u,w)} Y_W(u, x_1)Y_W(v, x_2)w.
$$

(4.5)

More precisely, it can be obtained by noticing that the replacement of $x_0$ by $x_1 - x_2$ independently in each factor in the expression as written on the left-hand side of (2.18) is well defined. We emphasize that, by contrast, the relation (4.3) cannot be obtained in such a manner. Indeed, although the formal limit procedure $\lim_{x_0 \to -x_2 + x_1}$ is of course well defined on the series on both sides of (2.18), one cannot replace $x_0$ by $-x_2 + x_1$ either in the factor $Y_W(u, x_0 + x_2)Y_W(v, x_2)w$ on the left-hand side or in the factor $Y_W(Y(u, x_0)v, x_2)w$ on the right-hand side of (2.18).
The second nontrivial relation, which we call modified weak associativity, will be important when generalized to twisted modules. It was first written in [DLMi1]. It is stated as:

**Theorem 4.3** With $W$ as in Theorem 4.1,

$$\lim_{x_1 \to x_2 + x_0} \left( (x_1 - x_2)^{k(u,v)}Y_W(u, x_1)Y_W(v, x_2) \right) = x_0^{k(u,v)}Y_W(Y(u, x_0)v, x_2)$$

(4.6)

for $u, v \in V$.

**Proof:** Apply the limit $\lim_{x_1 \to x_2 + x_0}$ to the expression

$$(x_1 - x_2)^{k(u,v)}Y_W(u, x_1)Y_W(v, x_2)$$

written as in the right-hand side of the first equation of (4.2). This limit is well defined, since negative powers of $x_1 - x_2$ in $F(u, v, w; x_1 - x_2, x_1, x_2)$ are cancelled out by the factor $(x_1 - x_2)^{k(u,v)}$. The resulting expression is read off the third relation of (4.2).

**Remark 4.3** Equation (4.6) can be written in the following form:

$$\lim_{x_1 \to x_2 + x_0} \left( \frac{x_1 - x_2}{x_0} \right)^{k(u,v)}Y_W(u, x_1)Y_W(v, x_2) = Y_W(Y(u, x_0)v, x_2).$$

(4.7)

The factor $\left( \frac{x_1 - x_2}{x_0} \right)^{k(u,v)}$ appearing in front of the product of two vertex operators on the left-hand side is crucial in giving a well-defined limit, but when the limit is applied to this factor without the product of vertex operators, the result is simply 1. We will call such a factor a “resolving factor”. Its power is apparent, in particular, in the proof of the main commutator formula (6.1) of [DLMii2]: it allows one to evaluate nontrivial limits of sums of terms with cancelling “singularities” in a straightforward fashion, evaluating the limit of each term independently. Its power will also be clear, in the present paper, when constructing the twisted vertex operator map $Y_{S[\nu]}(\cdot, x)$ and when studying the algebra $\hat{D}^+$ defined in Section 7.

### 4.3 Formal commutativity and associativity for twisted modules

We derive below various commutativity and associativity properties of twisted vertex operators. In order to express some of these properties, we need one more element of formal calculus: a certain projection operator (see [DLMii2]). Consider the operator $P_{[[x_0, x_0^{-1}]]}$ acting on the space $\mathbb{C}\{x_0\}$ of formal series...
with any complex powers of $x_0$, which projects to the formal series with integral powers of $x_0$:

$$P_{[[x_0,x_0^{-1}]]} : \mathbb{C}\{x_0\} \to \mathbb{C}[[x_0,x_0^{-1}]]. \quad (4.8)$$

We will extend the meaning of this notation in the obvious way to projections acting on formal series with coefficients lying in vector spaces other than $\mathbb{C}$, vector spaces which might themselves be spaces of formal series in other formal variables. Notice that when this projection operator acts on a formal series in $x_0$ with powers that are in $\frac{1}{p}\mathbb{Z}$, for instance on $f(x_0) \in \mathbb{C}[[x_0^{1/p}, x_0^{-1/p}]]$, it can be described by an explicit formula:

$$P_{[[x_0,x_0^{-1}]]}f(x_0) = \frac{1}{p} \sum_{r=0}^{p-1} \left( \lim_{x^{1/p} \to \omega^r x_0^{1/p}} f(x) \right).$$

(See Remark 4.4 below for the meaning of formal limit procedures involving fractional powers of formal variables.) We will also extend this projection notation to different kinds of formal series in obvious ways. For instance,

$$P_{x_0^{q/p}[[x_0,x_0^{-1}]]} : \mathbb{C}\{x_0\} \to \mathbb{C} x_0^{q/p}[[x_0,x_0^{-1}]].$$

Again, of course, we will extend the meaning of this notation to formal series with coefficients in vector spaces other than $\mathbb{C}$.

The twisted Jacobi identity (2.31) implies twisted versions of weak commutativity and weak associativity ($u, v \in V, w \in M$):

$$(x_2 - x_1)^k Y_M(v, x_2) Y_M(u, x_1) = (x_2 - x_1)^k Y_M(u, x_1) Y_M(v, x_2)$$

$$P_{[[x_0,x_0^{-1}]]} \left((x_0 + x_2)^l Y_M(u, x_0 + x_2) Y_M(v, x_2)w\right)$$

$$= (x_2 + x_0)^l \frac{1}{p} \sum_{r=0}^{p-1} \omega^{-lr} Y_M(Y(\nu^r u, x_0)v, x_2)w. \quad (4.9)$$

These relations are valid for all large enough $k \in \mathbb{N}$ and $l \in \frac{1}{p}\mathbb{N}$, their minimum value depending respectively on $u$, $v$ and on $u$, $w$. For definiteness, we will denote these minimum values by $k(u, v)$ and $l(u, w)$, respectively (they depend also on the module $M$; in particular, they differ from the integer numbers $k(u, v)$ and $l(u, w)$ used in the previous subsection in connection with the module $W$). As in the untwisted case, these relations imply the main “formal” commutativity and associativity properties of twisted vertex operators [Li2], which, along with with the fact that these properties are equivalent to the Jacobi identity, can be formulated as follows (it was first formulated in this form in [DLMH1]):

\[17\]
Theorem 4.4 Let $M$ be a vector space (not assumed to be graded) equipped with a linear map $Y_M(\cdot, x)$ \[2.26\] such that the truncation condition \[2.27\] and the Jacobi identity \[2.31\] hold. Then for $u, v \in V$ and $w \in M$, there exist $k(u, v) \in \mathbb{N}$ and $l(u, w) \in \frac{1}{p} \mathbb{N}$ and a (nonunique) element $F(u, v, w; x_0, x_1, x_2)$ of $M((x_0, x_1^{1/p}, x_2^{1/p}))$ such that

\[
x_0^{k(u,v)} F(u, v, w; x_0, x_1, x_2) \in M[[x_0]][(x_1^{1/p}, x_2^{1/p})],
\]
\[
x_1^{l(u,w)} F(u, v, w; x_0, x_1, x_2) \in M[[x_1^{1/p}]][(x_0, x_2^{1/p})]
\]

and

\[
Y_M(u, x_1)Y_M(v, x_2)w = F(u, v, w; x_1 - x_2, x_1, x_2),
\]
\[
Y_M(v, x_2)Y_M(u, x_1)w = F(u, v, w; -x_2 + x_1, x_1, x_2),
\]
\[
Y_M(Y(\nu^{-s} u, x_0)v, x_2)w = \lim_{x_1^{1/p} - \omega_p^{s}(x_2 + x_0)^{1/p}} F(u, v, w; x_0, x_1, x_2)
\]

for $s \in \mathbb{Z}$ (where we are using the binomial expansion convention). Conversely, let $M$ be a vector space equipped with a linear map $Y_M(\cdot, x)$ \[2.26\] such that the truncation condition \[2.13\] and the statement above hold, except that $k(u, v) \in \mathbb{N}$ and $l(u, w) \in \frac{1}{p} \mathbb{N}$ may depend on all three of $u, v$ and $w$. Then the Jacobi identity \[2.31\] holds.

This theorem, as well as \[4.9\] and \[4.10\], were proven in [DLMi2].

Remark 4.4 Formal limit procedures involving fractional powers of formal variables like $x_1^{1/p}$ have the same meaning as in \[4.4\], but with $x_1^{1/p}$ being treated as a formal variable by itself. For instance, the formal limit procedure

\[
\lim_{x_1^{1/p} - \omega_p^{s}(x_2 + x_0)^{1/p}} F(u, v, w; x_0, x_1, x_2)
\]

above means that one replaces each integral power of the formal variable $x_1^{1/p}$ in the formal series $F(u, v, w; x_0, x_1, x_2)$ by the corresponding power of the formal series $\omega_p^{s}(x_2 + x_0)^{1/p}$ (defined using the binomial expansion convention).

Remark 4.5 Note that this theorem, and in particular its proof in [DLMi2], illustrates the phenomenon, which arises again and again throughout the theory of vertex operator algebras, that formal calculus inherently involves just as much “analysis” as “algebra”: in many relations there are integers that can be left unspecified, except for their minimum values, and the proof involves taking these integers “large enough”. Recall that essentially the same issues arose for example in the use of formal calculus for the proof of the Jacobi identity for (twisted) vertex operators in [FLM2] (see Chapters 8 and 9). This is certainly not surprising, since we are using the Jacobi identity (for all twisting automorphisms) in order to prove, in a different approach, properties of (twisted) vertex operators.
Along with (4.11), the first two equations of (4.12) represent what we call **formal commutativity** for twisted vertex operators, while the first and last equations of (4.12) represent **formal associativity** for twisted vertex operators. When specialized to the untwisted case $p = 1$ ($\nu = 1_V$), these two relations lead respectively to the usual formal commutativity and formal associativity for vertex operators, as described in (4.2).

### 4.4 Additional relations in twisted modules

As in the case of ordinary vertex operators, one can write other relations involving formal limit procedures. These relations were proven in [DLM2], we report the proofs here. Among them, two cannot be directly obtained from weak commutativity and weak associativity. One of these, the relation generalizing (4.3), is stated as follows:

**Theorem 4.5** With $M$ as in Theorem 4.4,

\[
\lim_{x_0 \to x_1-x_2} \left( (x_2 + x_0)^l \frac{1}{p} \sum_{r=0}^{p-1} \omega_p^{-l r p} Y_M(Y(\nu^r u, x_0) v, x_2) w \right) = P_{[[x_1,x_1^{-1}]]} \left( x_1^l Y_M(u, x_2) Y_M(v, x_1) w \right),
\]

(4.13)

for all $l \in \frac{1}{p} \mathbb{Z}$, $l \geq l(u, w)$.

**Proof:** This is proved along the lines of the proof of Theorem 4.2 with some additions due to the fractional powers. One uses the third equation of (4.12) in order to rewrite the left–hand side of (4.13) as

\[
\lim_{x_0 \to -x_2+x_1} \left( (x_2 + x_0)^l \frac{1}{p} \sum_{r=0}^{p-1} \omega_p^{-l r p} \sum_{r=0}^{x_1^{1/p} \to x_2^{1/p} + x_0^{1/p}} F(u, v, w; x_0, x_3, x_2) \right).
\]

The sum over $r$ keeps only the terms in which $x_2 + x_0$ is raised to a power which has a fractional part equal to the negative of the fractional part of $l$. Multiplying by $(x_2 + x_0)^l$, for any $l \in \frac{1}{p} \mathbb{Z}$, $l \geq l(u, w)$, brings the remaining series to a series with finitely many negative powers of $x_2$ (as well as $x_0$), to which it is possible to apply the limit $\lim_{x_0 \to -x_2+x_1}$. This limit of course brings only integer powers of $x_1$, and the right–hand side of (4.13) can be obtained from the second equation of (4.12).

**Remark 4.6** A relation similar to the last one, but that is a direct consequence of weak associativity (4.10), is

\[
\lim_{x_0 \to x_1-x_2} \left( (x_2 + x_0)^l \frac{1}{p} \sum_{r=0}^{p-1} \omega_p^{-l r p} Y_M(Y(\nu^r u, x_0) v, x_2) w \right) = P_{[[x_1,x_1^{-1}]]} \left( x_1^l Y_M(u, x_1) Y_M(v, x_2) w \right)
\]

(4.14)
for all \( l \in \mathbb{Z} \), \( l \geq l(u, w) \). This generalizes (4.5) (see the comments in Remark 4.2). It can be obtained by applying the formal limit involved in the left-hand side to both sides of (4.10).

The most important relation for our purposes, which was first stated in [DLMTI], generalizing (4.6) and which we call modified weak associativity for twisted vertex operators, is given by the following theorem:

**Theorem 4.6**  With \( M \) as in Theorem 4.4,

\[
\lim_{x_1^{1/p} \to \omega_p^s(x_2 + x_0)^{1/p}} \left( (x_1 - x_2)^{k(u,v)}Y_M(u, x_1)Y_M(v, x_2) \right)
= x_0^{k(u,v)}Y_M(Y(\nu^{-s}u, x_0)v, x_2)
\]

(4.15)

for \( u, v \in V \) and \( s \in \mathbb{Z} \).

**Proof:** The proof is a straightforward generalization of the proof of Theorem 4.3. \( \square \)

**Remark 4.7** The specialization of Theorems 4.4 and 4.6 to the untwisted case \( p = 1 \) and \( M = W \) gives, respectively, Theorems 4.1 and 4.3.

Finally, we derive a simple relation, proved in [Li2], that specifies the structure of the formal series \( Y_M(u, x) \).

**Theorem 4.7**  With \( M \) as in Theorem 4.4,

\[
\lim_{x_1^{1/p} \to \omega_p^s x_1^{1/p}} Y_M(\nu^s u, x_1) = Y_M(u, x)
\]

(4.16)

for \( u \in V \) and \( s \in \mathbb{Z} \).

**Proof:** In the Jacobi identity (2.31), replace \( u \) by \( \nu^s u \) and \( x_1^{1/p} \) by \( \omega_p^s x_1^{1/p} \). The right-hand side becomes

\[
\frac{1}{p}x_2^{-1} \sum_{r=0}^{p-1} \delta \left( \omega_p^{r+s} \left( \frac{x - x_0}{x_2} \right)^{1/p} \right) Y_M(Y(\nu^{r+s}u, x_0)v, x_2),
\]

which is independent of \( s \), as is apparent if we make the shift in the summation variable \( r \mapsto r-s \). Hence the left-hand side is also independent of \( s \). Choosing \( v = 1 \) and using the vacuum property (2.28), this gives

\[
\left( x_0^{-1} \delta \left( \frac{x - x_2}{x_0} \right) - x_0^{-1} \delta \left( \frac{x_2 - x}{-x_0} \right) \right) \lim_{x_1^{1/p} \to \omega_p^s x_1^{1/p}} Y_M(\nu^s u, x_1)
\]

(4.17)

\[
= \left( x_0^{-1} \delta \left( \frac{x - x_2}{x_0} \right) - x_0^{-1} \delta \left( \frac{x_2 - x}{-x_0} \right) \right) Y_M(u, x)
\]

which, upon using (2.2) and taking Res\(_{x_2} \), gives (4.16).

\( \square \)

From Theorem 4.7, we directly have the following corollary:
Corollary 4.8 With $M$ as in Theorem 4.4, 

$$Y_M(u, x) = \sum_{n \in \mathbb{Z} + q/p} u_n x^{-n-1} \text{ for } u \in V, \quad \nu u = \omega^q_p u, \quad q \in \mathbb{Z}.$$ 

5 Equivalence and construction theorems from modified weak associativity

This section presents new results related to modified weak associativity. We will show, loosely speaking, that modified weak associativity (4.15) and weak commutativity (4.9) are equivalent to the Jacobi identity (2.31) ("equivalence theorem"). Then we will show that if the modified weak associativity for twisted vertex operators is valid for all pairs $u, w$ (to be put in (4.15) instead of the ordered pair $u, v$) with $u \in U$ and $w \in V$, where $U$ is a generating subset of a vertex operator algebra $V$, and if weak commutativity for twisted vertex operators is valid for all pairs $u, v$ with $u, v \in U$, then both modified weak associativity and weak commutativity hold for the whole vertex operator algebra $V$ ("construction theorem"). Similar construction theorems were proved by Li in the untwisted and twisted cases [Li1, Li2] (cf. [LL]), using the powerful idea of "local systems of (twisted) vertex operators." Here we start from similar ideas but we make use of modified weak associativity discovered in [DLMi1, DLMi2], in order to illustrate one of its applications. We find it instructive to give a direct proof of our construction theorem, although there may be a shorter route from Li’s construction theorems. In the next section, our two theorems will allow us to construct in a relatively simple way the twisted vertex operator map for $S[\nu]$ – in particular, we will show how the use of modified weak associativity gives a new explicit form for the operator $\Delta_x$ – and to prove the twisted module structure for $S[\nu]$. We recall that throughout this section, $V$ is a vertex operator algebra and $\nu$ is an automorphism of $V$.

5.1 Equivalence theorem

It is well known in the theory of vertex operator algebras that, under natural conditions, the weak commutativity relation (2.17) and the weak associativity relation (2.18) for untwisted modules are equivalent to the Jacobi identity (2.16). It is a simple matter to show that this statement is also true when weak associativity is replaced by modified weak associativity. We state this more generally for twisted modules (and for the twisted Jacobi identity (2.31)) in the following theorem.
Theorem 5.1 Let $M$ be a vector space (not assumed to be graded) equipped with a linear map $Y_M(\cdot, x)$ (2.26) such that the truncation condition (2.27) hold.

If modified weak associativity (4.15) holds, for some $k'(u,v) \in \mathbb{N}$, and weak commutativity (4.9) holds, for a possibly different $k(u,v) \in \mathbb{N}$, then we may change $k'(u,v)$ (in particular, we may lower it) to $k'(u,v) = k(u,v)$, and the twisted Jacobi identity (2.31) holds. Also, the vector space $M$ is a twisted module if, additionally, $M$ is $\mathbb{Q}$–graded and quasi-finite as in the first lines of Definition 2.3, with the $L(0)$–weight property (2.29), and the vacuum property (2.28) holds.

On the other hand, if the twisted Jacobi identity (2.31) holds, then modified weak associativity (4.15) and weak commutativity (4.9) hold.

Remark 5.2 Note that this theorem can be specialized to $p = 1$, applying then to modules $W$. It can also be used to show that some vector space $V$ is a vertex operator algebra; more precisely, in the definition 2.1, the Jacobi identity can be replaced by modified weak associativity and weak commutativity.

Proof of Theorem 5.1. The last sentence of the theorem was proven already by proving modified weak associativity and weak commutativity above.

Weak commutativity (4.9), when both sides are applied on an element $w$ of $M$, and the truncation condition immediately imply the first equation of (4.11), and the first two equations of (4.12). Then, the limit on the left-hand side of (4.15) with $k(u,v)$ replaced by $k''(u,v)$ certainly exists for all $k''(u,v) \geq k(u,v)$, and the result is the same for any $k''(u,v)$ that makes the limit exist, up to the obvious power of $x_0$ (because if all limits exist, then the product of the limits is the limit of the product). Since we know that this limit gives the right-hand side for some $k'(u,v)$, we may change it (and lower it if necessary) to $k'(u,v) = k(u,v)$. Then, writing the product of twisted vertex operators on the left-hand side of (4.15) as on the right-hand side of the first equation of (4.12), we see that (4.15) implies the third equation of (4.12). Hence, by Theorem 4.4, the twisted Jacobi identity holds. With the additional conditions stated in the theorem, all other parts of the definition (2.3) are satisfied (in particular, the fact that $V$ is a vertex operator algebra implies that the Virasoro commutation relations are also satisfied, and the $L(-1)$–derivative property for vertex operator algebra (2.8) implies, using (1.15) with $v = 1$, the corresponding property for twisted modules (2.30)) and $M$ is a twisted module for $V$.

5.2 Construction theorem

Consider a generating subset $U \subset V$ of the vertex operator algebra $V$, defined as follows.
Definition 5.3 A generating subset $U \subset V$ of a vertex operator algebra is a subset such that all elements of $V$ can be written as linear combinations of elements of the form $u_m v_n \cdots 1$ for $u, v, \ldots \in U$ and $m, n, \ldots \in \mathbb{Z}$.

Suppose we are able to define a twisted vertex operator map $Y_M(\cdot, x)$ as in (2.26) such that the truncation condition (2.27) holds for all $v \in V$, weak commutativity (4.9) holds for all $u, v \in U$, and modified weak associativity (4.15) holds for all $u \in U$ and $v \in V$. Theorem 5.5 below along with Theorem 5.1 tell us that this is enough to have the twisted Jacobi identity, and, with additional mild conditions (see Theorem 5.1) to have a twisted module.

Remark 5.4 Theorem 5.5 of course requires us to already have a twisted vertex operator map $Y_M(\cdot, x)$ on the full vertex operator algebra $V$. On the other hand, once we have the map on $U$ only (with the properties above), it is easy to extend it to a map on the set of symbols of the type $u_m v_n \cdots w$ for $u, v, \ldots, w \in U$ and $m, n, \ldots \in \mathbb{Z}$ by recursive use of modified weak associativity (4.15) with $u \in U$ and $v$ an element in this set of symbols. However, the vertex operator algebra $V$ is the span of this set of symbols with linear relations amongst them; these relations depend on the particular vertex operator algebra at hand. In order to have a twisted module, it is essential to verify that the map on this set of symbols is well-defined on $V$; that is, that it is in agreement with these linear relations. This is extremely nontrivial, and there does not seem to be yet a general theorem as to when that happens. Whatever these relations are, they have to imply those coming from the Jacobi identity (since $V$ is a vertex operator algebra). It is possible to show that all linear relations coming from the Jacobi identity are satisfied by this construction. This is beyond the scope of the present paper, but we hope to clarify some of these issues in a future work.

Note again that the theorem below and the issues discussed in the remark above are in close relation with results of Li [Li2]; cf. [LL].

Theorem 5.5 Let $M$ be a vector space (not assumed to be graded) equipped with a linear map $Y_M(\cdot, x)$ (2.26) such that the truncation condition (2.27) holds. Fix a generating subset $U \subset V$, as defined in Definition 5.3.

If weak commutativity (4.9) is satisfied for all $u, v \in U$ (and fix $k(u, v) \in \mathbb{N}$ to be the lowest integer that can be taken both in (4.9) and in weak commutativity for the vertex operator algebra $V$), and if modified weak associativity (4.15) is satisfied for all $u \in U$ and for all $v \in V$ (for some $k'(u, v) \in \mathbb{N}$ that may be different from $k(u, v)$), then both weak commutativity and modified weak associativity are satisfied for all $u, v \in V$. Further, we may change $k'(u, v)$ (in particular, we may lower it) to $k'(u, v) = k(u, v)$ for $u, v \in V$, and these integers may be taken to satisfy the formula

$$k(u_{n_0+m}v, w) = k(w, u) + k(w, v) + m$$

(5.1)
for all $u, v, w \in V$ where $n_0$ is the highest integer such that $u_{n_0} v \neq 0$. The same integers may also be taken in weak commutativity for the vertex operator algebra $V$.

Proof: We start by showing weak commutativity (4.9), and the equation (5.1). Under the assumptions of the theorem, we have, for $u, v, w \in U$,

\[
(x - x_2 - x_0)^{k(u,u)}(x - x_2)^{k(w,v)}Y_M(w, x) \cdot \\
\cdot \lim_{x_1^{1/p} \to \omega_p^s (x_2 + x_0)^{1/p}} \left( (x_1 - x_2)^{k(u,v)}Y_M(u, x_1)Y_M(v, x_2) \right)
\]

\[
= \lim_{x_1^{1/p} \to \omega_p^s (x_2 + x_0)^{1/p}} \left( (x_1 - x_2)^{k(u,u)}(x - x_2)^{k(w,v)}Y_M(w, x) \cdot \\
\cdot (x_1 - x_2)^{k(u,v)}Y_M(u, x_1)Y_M(v, x_2) \right)
\]

\[
= \lim_{x_1^{1/p} \to \omega_p^s (x_2 + x_0)^{1/p}} \left( (x_1 - x_2)^{k(u,v)}Y_M(u, x_1)Y_M(v, x_2) \right) \\
\cdot (x - x_2 - x_0)^{k(u,u)}(x - x_2)^{k(w,v)}Y_M(w, x)
\]

(5.2)

hence

\[
(x - x_2 - x_0)^{k(u,u)}(x - x_2)^{k(w,v)}Y_M(w, x)Y_M(Y(v^{-s}u, x_0) v, x_2)
\]

\[
= (x - x_2 - x_0)^{k(u,u)}(x - x_2)^{k(w,v)}Y_M(Y(v^{-s}u, x_0) v, x_2)Y_M(w, x).
\]

(5.3)

Both sides have finitely many negative powers of $x_0$. Take the lowest power:

\[
(x - x_2)^{k(u,u) + k(w,v)}Y_M(w, x)Y_M((v^{-s}u)_{n_0} v, x_2)
\]

\[
= (x - x_2)^{k(u,u) + k(w,v)}Y_M((v^{-s}u)_{n_0} v, x_2)Y_M(w, x).
\]

(5.4)

This proves weak commutativity for all pairs $u_{n_0} v, w$ with $u, v, w \in U$, and we may take $k(u_{n_0} v, w) = k(w, u) + k(w, v)$ (although we have not proven that this value is the minimum one for which weak commutativity is valid), where $n_0$ is the highest integer such that $u_{n_0} v \neq 0$.

The next power of $x_0$ in the equation (5.3) contains, on each side, two terms of the same type as those above: one with $(v^{-s}u)_{n_0+1} v$, the other with $(v^{-s}u)_{n_0} v$. The terms with $(v^{-s}u)_{n_0} v$ are multiplied by $(x - x_2)^{k(u,u) + k(w,v) - 1}$, and those with $(v^{-s}u)_{n_0+1} v$, by $(x - x_2)^{k(u,u) + k(w,v)}$. Multiplying the resulting equation through by $x - x_2$, we can use the result (5.3) and we obtain weak commutativity for all pairs $u_{n_0+1} v, w$ with $u, v, w \in U$, where $n_0$ is again highest integer such that $u_{n_0} v \neq 0$. We can take $k(u_{n_0+1} v, w) = k(w, u) + k(w, v) + 1$. Repeating the process, we obtain weak commutativity for all pairs
For $u, v, w \in V$ and $\phi \in M$, there exists a (nonunique) element $F(u, v, w, \phi; x_0, x_4, x_5, x_1, x_2, x_3)$ of $M((x_0, x_4, x_5, x_1, x_2, x_3))$ such that

\begin{align}
&x_0^{k(u,v)} F(u, v, w, \phi; x_0, x_4, x_5, x_1, x_2, x_3) \in M[[x_0]]((x_4, x_5, x_1, x_2, x_3)), \\
x_4^{k(u,w)} F(u, v, w, \phi; x_0, x_4, x_5, x_1, x_2, x_3) \in M[[x_4]]((x_0, x_5, x_1, x_2, x_3)), \\
x_5^{k(v,w)} F(u, v, w, \phi; x_0, x_4, x_4, x_5, x_1, x_2, x_3) \in M[[x_5]]((x_0, x_4, x_1, x_2, x_3))
\end{align}

and

\[ Y_M(u, x_1)Y_M(v, x_2)Y_M(w, x_3)\phi = F(u, v, w, \phi; x_0, x_4, x_5, x_1, x_2, x_3) \]

(5.6)

where $k(u, v), k(u, w), k(v, w) \in \mathbb{N}$ can be taken as those that appear in the weak commutativity relation (4.9).

This is an immediate consequence of weak commutativity (4.9). Indeed, consider

\[(x_1 - x_2)^{k(u,v)}(x_1 - x_3)^{k(u,w)}(x_2 - x_3)^{k(v,w)}Y_M(u, x_1)Y_M(v, x_2)Y_M(w, x_3)\phi . \]

Thanks to the factor $k(u,v)$, both vertex operators may be written in any order. Looking at different orders and using the truncation property, we see the expression has finitely many
negative powers of $x_4$, finitely many negative powers of $x_2$ and finitely many
negative powers of $x_1$. This shows the lemma.

Now, take $u, v, w \in U$ and $w \in V$, and again $\phi \in M$. From modified weak
associativity for vertex operator algebra \textbf{(4.6)} (recall that the module $W$ in
that equation may be replaced by the vertex operator algebra itself $V$), which
we will need only for the pair $u, v$, and from modified weak associativity for
twisted modules, which is valid by assumption when, in \textbf{(4.15)}, the (ordered)
pair $u, v$ is replaced by the pair $v, w$ as well as when it is replaced by the pair
$u, v_n w$, we have

$$Y_M(Y(Y(u, x_0)v, x_5)w, x_3)\phi$$

$$= \lim_{x_4 \to x_5 + x_0} \left( \frac{x_4 - x_5}{x_0} \right)^{k(u,v)} Y_M(Y(u, x_4)Y(v, x_5)w, x_3)$$

$$= \lim_{x_4 \to x_5 + x_0} \left( \frac{x_4 - x_5}{x_0} \right)^{k(u,v)} \sum_{n \in \mathbb{Z}} x_5^{-n-1} Y_M(Y(u, x_4)v_n w, x_3)\phi$$

$$= \lim_{x_4 \to x_5 + x_0} \left( \frac{x_4 - x_5}{x_0} \right)^{k(u,v)} \sum_{n \in \mathbb{Z}} x_5^{-n-1} \lim_{x_1^{\frac{1}{p}} \to (x_2 + x_4)^{\frac{1}{p}}} \left( \frac{x_1 - x_3}{x_4} \right)^{k(u,v,w)} \cdot Y_M(u, x_1)Y_M(v_n w, x_3)\phi$$

where $P_{x_5}^{n-1}$ projects onto the formal series with only the terms having the
factor $x_5^{-n-1}$. Observe that the lowest power of $x_5$ is $-n_0 - 1$ where $n_0$ is the
highest integer such that $v_{n_0} w \neq 0$. Continuing, we find

$$Y_M(Y(Y(u, x_0)v, x_5)w, x_3)\phi$$

$$= \lim_{x_4 \to x_5 + x_0} \left( \frac{x_4 - x_5}{x_0} \right)^{k(u,v)} \sum_{n \in \mathbb{Z}} \lim_{x_1^{\frac{1}{p}} \to (x_3 + x_4)^{\frac{1}{p}}} \left( \frac{x_1 - x_3}{x_4} \right)^{k(u,v,w)} \cdot P_{x_5}^{n-1} \lim_{x_2^{\frac{1}{p}} \to (x_3 + x_5)^{\frac{1}{p}}} \left( \frac{x_2 - x_3}{x_5} \right)^{k(v,w)} Y_M(u, x_1)Y_M(v, x_2)Y_M(w, x_3)\phi$$

$$= \lim_{x_4 \to x_5 + x_0} \left( \frac{x_4 - x_5}{x_0} \right)^{k(u,v)} \sum_{n \in \mathbb{Z}} \lim_{x_1^{\frac{1}{p}} \to (x_3 + x_4)^{\frac{1}{p}}} \left( \frac{x_1 - x_3}{x_4} \right)^{k(u,v,w)} \cdot P_{x_5}^{n-1} \lim_{x_2^{\frac{1}{p}} \to (x_3 + x_5)^{\frac{1}{p}}} \left( \frac{x_2 - x_3}{x_5} \right)^{k(v,w)} \cdot P_{x_5}^{n-1} \lim_{x_2^{\frac{1}{p}} \to (x_3 + x_5)^{\frac{1}{p}}} \left( \frac{x_2 - x_3}{x_5} \right)^{k(v,w)} F(u, v, w, \phi, x_1 - x_2, x_1 - x_3, x_2 - x_3, x_1, x_2, x_3)$$

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\[
\begin{align*}
\lim_{x_4 \to x_5 + x_0} \left( \frac{x_4 - x_5}{x_0} \right)^{k(u,v)} \sum_{n \in \mathbb{Z}} \lim_{x_1 \to (x_3 + x_4)^{1/p}} \left( \frac{x_1 - x_3}{x_4} \right)^{k(u,v,w)} \\
\cdot P_{x_5^{-n-1}} F(u, v, w, \phi; x_1 - x_3, x_5, x_1, x_3 + x_5, x_3) 
\end{align*}
\]

Using the particular value of \(k(u, v, w)\) given by (5.7), we can continue evaluating the limits:

\[
Y_M(Y(u, x_0)v, x_5)\phi
= \lim_{x_4 \to x_5 + x_0} \left( \frac{x_4 - x_5}{x_0} \right)^{k(u,v)} F(u, v, w, \phi; x_4 - x_5, x_4, x_5, x_3 + x_4, x_3 + x_5, x_3)
= F(u, v, w, \phi; x_0, x_5 + x_0, x_5 + x_5 + x_0, x_3 + x_5, x_3). \tag{5.7}
\]

Now, consider the following formal series, and use similar arguments as those above in order to evaluate it:

\[
\sum_{n \in \mathbb{Z}} x_0^{n-1} \lim_{x_2 \to (x_3 + x_0)^{1/p}} \left( \frac{x_2 - x_3}{x_5} \right)^{k(u,v,w)} Y_M(u_n v, x_2) Y_M(w, x_3) \phi
= \sum_{n \in \mathbb{Z}} \lim_{x_2 \to (x_3 + x_0)^{1/p}} \left( \frac{x_2 - x_3}{x_5} \right)^{k(u,v,w)} P_{x_0^{-n-1}} Y_M(Y(u, x_0)v, x_2) Y_M(w, x_3) \phi
= \sum_{n \in \mathbb{Z}} \lim_{x_2 \to (x_3 + x_0)^{1/p}} \left( \frac{x_2 - x_3}{x_5} \right)^{k(u,v,w)} .
\]

\[
\cdot P_{x_0^{-n-1}} \lim_{x_1 \to (x_2 + x_0)^{1/p}} \left( \frac{x_1 - x_2}{x_0} \right)^{k(u,v)} Y_M(u, x_1) Y_M(v, x_2) Y_M(w, x_3) \phi
= \sum_{n \in \mathbb{Z}} \lim_{x_2 \to (x_3 + x_0)^{1/p}} \left( \frac{x_2 - x_3}{x_5} \right)^{k(u,v,w)} .
\]

\[
\cdot P_{x_0^{-n-1}} \lim_{x_1 \to (x_2 + x_0)^{1/p}} \left( \frac{x_1 - x_2}{x_0} \right)^{k(u,v)} F(u, v, w, \phi; x_1 - x_2, x_3 + x_2 - x_3, x_1, x_3 + x_2, x_3)
= \sum_{n \in \mathbb{Z}} \lim_{x_2 \to (x_3 + x_0)^{1/p}} \left( \frac{x_2 - x_3}{x_5} \right)^{k(u,v,w)} .
\]

\[
\cdot P_{x_0^{-n-1}} F(u, v, w, \phi; x_0, x_2 + x_0 - x_3, x_2 - x_3, x_2 + x_0, x_2, x_3)
= F(u, v, w, \phi; x_0, x_5 + x_0, x_3 + x_5 + x_0, x_3 + x_5, x_3). \tag{5.8}
\]

Comparing with (5.7), and taking any fixed power of \(x_0\), we have

\[
Y_M(Y(u_n v, x_5)w, x_3) = \lim_{x_2 \to (x_3 + x_0)^{1/p}} \left( \frac{x_2 - x_3}{x_5} \right)^{k(u,v,w)} Y_M(u_n v, x_2) Y_M(w, x_3) \phi . \tag{5.9}
\]

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This is modified weak associativity (4.15) for $s = 0$ and for the pairs $u_n v, w$, for all $n \in \mathbb{Z}$. It is a simple matter to compare, instead, the expression

$$Y_M(Y(\nu^{-s} Y(u, x_0)v, x_5)w, x_3)\phi$$

with the expression

$$\sum_{n \in \mathbb{Z}} x_0^{-n-1} \lim_{x_2^{1/p} \to \omega_0^p(x_3+x_5)^{1/p}} \left( x_2 - x_3 \over x_5 \right)^{k(u_n v, w)} Y_M(u_n v, x_2) Y_M(w, x_3) \phi$$

using similar steps, and using the fact that $\nu$ is an automorphism of $V$. We obtain modified weak associativity (4.15) for arbitrary $s \in \mathbb{Z}$ and for the pairs $u_n v, w$, for all $n \in \mathbb{Z}$.

Repeating the argument with $v$ replaced by $\tilde{u}_n v$ for all $\tilde{u} \in U$ and for all $n \in \mathbb{Z}$, and so on, we find modified weak associativity for all pairs $v, w$ with $v \in V$ and $w \in V$. This proves the last part of the theorem.

Remark 5.8 As usual, it is possible to specialize the theorems above to the case $p = 1$ in order to obtain theorems applying to untwisted modules.

Remark 5.9 In the theorem above (in close relation with theorems of [Li2]), concerned with twisted modules for vertex operators algebras, we assumed the existence of a vertex operator algebra $V$. With slight adjustments, the theorem can be made into a construction theorem for vertex operator algebras themselves, in relation with constructions of [Li1] (a more complete construction theory, the representation theory of vertex operator algebras of [Li1], is explained at length in [LL], cf. Theorems 5.7.6, 5.7.11 for instance).

6 Proof of the twisted module structure for $S[\nu]$ and construction of the twisted vertex operator map

The twisted module structure of $S[\nu]$, for the vertex operator $S$, was established in [Li1], [FLM1], [FLM2] and [DL2] (assuming that $\nu$ preserves a rational lattice in $\mathfrak{h}$). In this section, we present a new proof of the twisted module structure of $S[\nu]$ (which does not require this minor assumption), and we construct explicitly the twisted vertex operators using modified weak associativity (and in particular, we calculate the explicit form of $\Delta_x$ in a simple way).

Starting from (3.10), we will construct the twisted vertex operator $Y_{S[\nu]}(u, x)$ for all $u \in S$. As stated above, it is usual in studying vertex operator algebras
to construct vertex operators associated to elements of the vertex operator algebra from vertex operator associated to “simpler” elements, by using associativity. However, for twisted vertex operators, the natural weak associativity (4.10), that is immediately obtained from the Jacobi identity, is somewhat complicated by the projection operator and hides the simple structure of the construction. On the other hand, the modified weak associativity (4.15) is simpler, especially when written in the form (specialized for convenience to $s = 0$)

$$Y_M(Y(u, x_0)v, x_2) = \lim_{x_1^{1/p} \rightarrow (x_2 + x_0)^{1/p}} \left( \frac{x_1 - x_2}{x_0} \right)^{k(u,v)} Y_M(u, x_1)Y_M(v, x_2).$$

(6.1)

In order to better understand this formula, note that, as remarked in Remark 4.3, the pre-factor $\left( \frac{x_1 - x_2}{x_0} \right)^{k(u,v)}$ gives exactly 1 when the limit procedure is applied on it alone. The formula above cannot be simplified by replacing this pre-factor by 1, however, because the limit is not well defined on the other factor $Y_M(u, x_1)Y_M(v, x_2)$ alone. The construction principle will be to replace this product of vertex operators by a normal-ordered product plus extra terms. On the normal-ordered product, the limit is well defined, so that the pre-factor multiplying the normal-ordered product can be set to 1.

It is immediate to see that the set $U = \{1, \alpha(-1)1 | \alpha \in h\}$ is a generating subset for the vertex operator algebra $S$. Hence, in the proof of the twisted module structure of $S[[\nu]]$ using Theorems 5.5 and 5.1 we need first to prove weak commutativity (4.9) (and it turns out that it holds with $k(u, v) = 2$ – the integer $k(u, v) = 2$ is the one involved in the corresponding weak commutativity for the vertex operator algebra $S$) for the operators defined by (3.10) with $n = 1$. Then, we need to construct explicitly the map $Y_{S[[\nu]]}(\cdot, x)$ for all elements of $S$ recursively by using modified weak associativity. Finally, we need to check modified weak associativity (4.15) for $u \in U$ and $v \in V$. All other requirements of Theorem 5.1 are immediate to see from the construction of Section 3.

**Theorem 6.1** The operators (3.10) with $n = 1$ satisfy weak commutativity (4.9) with $k(u, v) = 2$.

**Proof:** We have

$$\alpha^\nu(x_1)\beta^\nu(x_2) = :\alpha^\nu(x_1)\beta^\nu(x_2): + h(\alpha, \beta, x_1, x_2)$$

(6.2)

with

$$h(\alpha, \beta, x_1, x_2) = \sum_{m \in \frac{1}{p} \mathbb{Z}, m > 0} x_1^{-m-1}x_2^{-m-1}m\langle \alpha_{(mp)}, \beta \rangle$$
It is a simple matter to see that

\[(x_1 - x_2)^2 h(\alpha, \beta; x_1, x_2) = \sum_{r=1}^{p} \left( \frac{x_2}{x_1} \right) \frac{r}{p} \left( 1 - \frac{x}{p} + \frac{r x_2}{p x_2} \right) (x_1 - x_2)^2 (\alpha_r, \beta) . \tag{6.3} \]

which proves the theorem.

**Theorem 6.2** The space \( S[\nu] \) has the structure of a twisted module for the vertex operator algebra \( S \). The general form of the twisted vertex operator \( Y_{S[\nu]}(u, x) \), for any element \( u \) of the vertex operator algebra \( S \), is:

\[
Y_{S[\nu]}(\alpha_j(-n_j) \cdots \alpha_1(-n_1) \mathbf{1}, x) = \sum_{J \subset \{1, \ldots, j\}} f_{\{1, \ldots, j\} \setminus J}(x) \prod_{l \in J} \left( \frac{1}{n_l - 1} \right) \left( \frac{d}{dx} \right)^{n_l - 1} \alpha_l^\nu(x);
\]

with \( n_1, \ldots, n_j \in \mathbb{Z}_+ \), for some factors \( f_l(x) \), where we just write the dependence on the index set \( I \), but that really depend on the elements \( \alpha_i \) and the integer numbers \( n_i \) for all \( i \in I \). The set \( J \) on which we sum takes the values \( \emptyset \) (the empty set), \( \{1, \ldots, j\} \) (if it is different from \( \emptyset \)), and all other proper subsets of \( \{1, \ldots, j\} \) (if any). The factors \( f_l(x) \) are given by

\[
f_l(x) = \begin{cases} 
0 & |I| \text{ odd} \\
\sum_{s \in \text{Pairings}(I)} \prod_{l=1}^{\lfloor |I|/2 \rfloor} g_{s_l}(x) & |I| \text{ even}
\end{cases} \tag{6.6}
\]

where \( |I| \) is the cardinal of \( I \), where \( \text{Pairings}(I) \) is the set of all distinct sets \( s = \{s_1, \ldots, s_{|I|/2}\} \) of distinct (without any element in common) pairs \( s_l = (i_l, i_l') \) (where the order of elements is not important) of elements \( i_l \neq i_l' \) of \( I \) such that \( \{i_1, \ldots, i_{|I|/2}, i_1', \ldots, i_{|I|/2}'\} = I \), and where

\[
g_{(i, i')}(x) = g(\alpha_i, n_i, \alpha_{i'}, n_{i'}, x) \tag{6.7}
\]

with

\[
g(\alpha, m, \beta, n, x) = \text{Res}_{x_0} \text{Res}_{x_2} x_0^{-m} x_2^{-n} \sum_{r=1}^{p} \left( \frac{x + x_2}{x + x_0} \right) \frac{r}{p} \left( 1 - \frac{x}{p} + \frac{r x_0}{p x_0} \right) (x_0 - x_2)^2 (\alpha_r, \beta) . \tag{6.8}
\]
Proof: Clearly, for \( j = 0 \) and \( j = 1 \) the form (6.5) is consistent with modified weak associativity and well defined on \( S \), and we must have \( f_I(x) = 1 \) and \( f_I(x) = 0 \) if \( |I| = 1 \). Assume (6.5) to be valid for \( j \) replaced with \( j - 1 \). With \( k \) a nonnegative integer large enough and \( n_1, \ldots, n_j \in \mathbb{Z}_+ \), we have

\[
Y_{S[\nu]}(\alpha_j(-n_j) \cdots \alpha_1(-n_1)1, x) = \text{Res}_{x_0} x_0^{-n_j} Y_{S[\nu]}(Y(\alpha_j(-1)1, x_0)\alpha_{j-1}(-n_{j-1}) \cdots \alpha_1(-n_1)1, x) = \text{Res}_{x_0} x_0^{-n_j} \lim_{x_1^{1/p} \to (x+x_0)^{1/p}} \left( \frac{x_1 - x}{x_0} \right)^k Y_{S[\nu]}(\alpha_j(-1)1, x_1) Y_{S[\nu]}(\alpha_{j-1}(-n_{j-1}) \cdots \alpha_1(-n_1)1, x). \]

Now, using the commutation relations (3.3), it is a simple matter to obtain

\[
\alpha_j^\nu(x_1) : \prod_{i \in J} \left( \frac{d}{dx} \right)^{n_i-1} \alpha_i^\nu(x) : = : \alpha_j^\nu(x_1) \prod_{i \in J} \left( \frac{d}{dx} \right)^{n_i-1} \alpha_i^\nu(x) : 
\]

\[
+ \sum_{i \in J} \left( \frac{\partial}{\partial x} \right)^{n_i-1} h(\alpha_j, \alpha_i, x_1, x) : \prod_{i \in J \setminus \{i\}} \left( \frac{d}{dx} \right)^{n_i-1} \alpha_i^\nu(x) : 
\]

with \( h(\alpha, \beta, x_1, x_2) \) defined in (6.3). Note that, comparing with (6.8), we have

\[
g(\alpha, m, \beta, n, x) = \text{Res}_{x_0} x_0^{m} \lim_{x_1^{1/p} \to (x+x_0)^{1/p}} \left( \frac{x_1 - x}{x_0} \right)^k \text{Res}_{x_2} x_2^{-n} h(\alpha, \beta, x_1, x + x_2) \right). \]

Using the relation

\[
\frac{1}{(n-1)!} \left( \frac{d}{dx} \right)^{n-1} f(x) = \text{Res}_{x_2} x_2^{-n} f(x + x_2) \]

for formal series \( f(x) \) with finitely many negative powers of \( x \) and for \( n \in \mathbb{Z}_+ \), we can now evaluate the limit and the residue on the right-hand side of the last equality of (6.9):

\[
Y_{S[\nu]}(\alpha_j(-n_j) \cdots \alpha_1(-n_1)1, x) \quad (6.12)
\]
\[ = \sum_{J \subset \{1, \ldots, j-1\}} f_{\{1, \ldots, j-1\} \setminus J}(x) \cdot \left( \begin{array}{c} 1 \\ (n_j - 1)! \end{array} \right) n_j^{-1} \frac{d^r}{dx^r} \prod_{l \in J} \left( \begin{array}{c} 1 \\ (n_l - 1)! \end{array} \right) n_l^{-1} \alpha'_l(x) \cdot \right. \\
+ \sum_{i \in J} g(\alpha_j, n_j, \alpha_i, n_i, x) \cdot \prod_{l \in J \setminus \{i\}} \left( \begin{array}{c} 1 \\ (n_l - 1)! \end{array} \right) n_l^{-1} \alpha'_l(x) \cdot \right) .

This is still of the form (6.5). Moreover, it is simple to understand, from comparing (6.12) with (6.9), that the solution (6.6) is correct.

This construction certainly gives us a map \( Y_{S[\nu]}(\cdot, x) \) on the vector space \( S \simeq S(\hat{\mathfrak{h}}^-) \). We need to verify that this map satisfies modified weak associativity (4.15). This requires three steps.

First, we need to check that operators on the right-hand side of (6.5) are independent of the order of the pairs \((\alpha_1, n_1), \ldots, (\alpha_j, n_j)\) for any positive integers \(n_1, \ldots, n_j\) and any elements \(\alpha_1, \ldots, \alpha_j\) of \(\mathfrak{h}\). The sum over pairings has this symmetry, and we need to check that

\[ g(\alpha, m, \beta, n, x) = g(\beta, n, \alpha, m, x) . \]  

This is not immediately obvious, because, by the binomial expansion convention, \((x_0 - x_2)^{-2} \neq (x_2 - x_0)^{-2}\). This symmetry can indeed be checked:

\[ g(\alpha, m, \beta, n, x) - g(\beta, n, \alpha, m, x) \]

\[ = \text{Res}_{x_0} \text{Res}_{x_2} x_0^{-m} x_2^{-n} \sum_{r=1}^p \left( \frac{x + x_2}{x + x_0} \right)^{-r} \frac{\partial}{\partial x_2} \delta \left( \frac{x_2 - x_0}{x_2 - x_0^2} \right) \langle \alpha_{(r)}, \beta \rangle \]

\[ = 0 \]  

where in the last step, we moved the derivative \(\frac{\partial}{\partial x_2}\) towards the left using Leibniz’s rule, and we used the formal delta-function property and the fact that \(m, n \in \mathbb{Z}_+\).

Second, we need to check that modified weak associativity (4.15) with \( Y_{S[\nu]}(Y(\alpha_j(-1)1, x_0)\alpha_{j-1}(-n_{j-1}) \cdots \alpha_1(-n_1)1, x) \) for its left-hand side is in agreement, at negative powers of \(x_0\), with

\[ Y_{S[\nu]}(\alpha_j(n_j)\alpha_{j-1}(-n_{j-1}) \cdots \alpha_1(-n_1)1, x) = \]  

\[ 32 \]
\[ \sum_{i=1}^{j-1} n_j \delta_{n_j,n_i} \langle \alpha_j, \alpha_i \rangle Y S[\nu] (\alpha_{j-1} (-n_{j-1}) \cdots \alpha_i (-n_i) \cdots \alpha_1 (-n_1) 1, x) \]

for \( n_j \in \mathbb{N} \) (the nonnegative integers) where \( \alpha_i (-n_i) \) means that the operator \( \alpha_i \) is omitted. Repeating the derivation (6.9), (6.12), we see that this is equivalent to requiring (with the definition (6.8))

\[ g(\alpha, -m, \beta, n) = m \delta_{m,n} \langle \alpha, \beta \rangle \] (6.17)

for \( m \in \mathbb{N}, n \in \mathbb{Z}_+ \). This is a consequence of the fact that

\[ \frac{(x + x_2)^s}{(x + x_0)^2} \left( 1 - s + \frac{s x + x_0}{x + x_2} \right) - \frac{1}{(x_0 - x_2)^2} + \mathbb{C}[x_0, x_2, x^{-1}] \] (6.18)

for any \( s \in \mathbb{C} \). The quantity

\[ \left( \frac{x + x_2}{x + x_0} \right)^s \frac{1 - s + \frac{x + x_0}{x + x_2}}{x_0 - x_2} = \frac{1}{(x_0 - x_2)^2} \]

obviously has only nonnegative powers of \( x_2 \) and nonpositive powers of \( x \). On the other hand, it is equal to

\[
\left( \frac{x + x_2}{x + x_0} \right)^s \left( 1 - s + \frac{x + x_0}{x + x_2} \right) - 1 \right) x_0^{-1} \frac{\partial}{\partial x_2} \delta \left( x_2 - x_0 \right).
\]

The first two terms obviously have only nonnegative powers of \( x_0 \). The third term can be evaluated using Leibniz's rule and gives zero. This completes the proof of (6.18).

Third, the structure of the construction of Section 3 shows that (4.15) is valid as well for \( s \neq 0 \) (equation (4.16) is satisfied).

The other requirements of Theorems 5.1 can be checked from the construction of Section 3, and with Theorems 5.5 and 6.1, this completes the proof.

Finally, let us mention that formula (6.5) with (6.6) immediately leads to the following formula for the operator \( \Delta_x \) introduced in (3.9):

\[ \Delta_x = \sum_{q_1,q_2=1}^{d} \sum_{m,n \in \mathbb{Z}_+} \bar{\alpha}_{q_1}(m) \bar{\alpha}_{q_2}(n) \frac{g(\bar{\alpha}_{q_1}, m, \bar{\alpha}_{q_2}, n, x)}{mn} \] (6.19)

where we recall that \( \bar{\alpha}_q, q = 1, \ldots, d \) form an orthonormal basis of \( \mathfrak{h} \). This was first constructed, in a different form, in [FLM1] and [FLM2].
7 The Lie algebra $\hat{D}^+$

We will now apply modified weak associativity and the results of the previous section, that $S[\nu]$, constructed in Section 3 is a twisted module for the vertex operator algebra $S$, in order to study a certain infinite-dimensional Lie algebra $\hat{D}^+$ and its representations. This follows closely the results of [DLMi1] and [DLMi2], and does not give new results with respect to these works.

Let $D$ be the Lie algebra of formal differential operators on $\mathbb{C}^\times$ spanned by $t^n D^r$, where $D = t \frac{d}{dt}$ and $n \in \mathbb{Z}$, $r \in \mathbb{N}$ (the nonnegative integers). This Lie algebra has an essentially unique one-dimensional central extension $\hat{D} = \mathbb{C} c \oplus D$ (denoted in the physics literature by $\mathcal{W}_{1+\infty}$).

The representation theory of the highest weight modules of $\hat{D}$ was initiated in [KR], where, among other things, the complete classification problem of the so-called quasi-finite representations [34] was settled. The detailed study of the representation theory of certain subalgebras of $\hat{D}$ having properties related to those of certain infinite–rank “classical” Lie algebras was initiated in [KWY] along the lines of [KR]. In [Bl] and [M2], related Lie algebras (and superalgebras) are considered from different viewpoints. As in [DLMi1] [DLMi2], we will follow these lines and concentrate on the Lie subalgebra $\hat{D}^+$ described in [Bl] and recalled below.

View the elements $t^n D^r$ ($n \in \mathbb{Z}$, $r \in \mathbb{N}$) as generators of the central extension $\hat{D}$. They can be taken to satisfy the following commutation relations (cf. [KR]):

$$[t^m f(D), t^n g(D)] =$$
$$t^{m+n}(f(D + n)g(D) - g(D + m)f(D)) + \Psi(t^m f(D), t^n g(D))c,$$

where $f$ and $g$ are polynomials and $\Psi$ is the 2–cocycle (cf. [KR]) determined by

$$\Psi(t^m f(D), t^n g(D)) = -\Psi(t^n g(D), t^m f(D)) = \delta_{m+n,0} \sum_{i=1}^{m} f(-i)g(m-i), \ m > 0.$$

We consider the Lie subalgebra $D^+$ of $D$ generated by the formal differential operators

$$L_n^{(r)} = (-1)^{r+1} D^r(t^n D)D^r,$$

where $n \in \mathbb{Z}$, $r \in \mathbb{N}$ [Bl]. The subalgebra $D^+$ has an essentially unique central extension (cf. [N]) and this extension may be obtained by restriction of the 2–cocycle $\Psi$ to $D^+$. Let $\hat{D}^+ = \mathbb{C} c \oplus D^+$ be the nontrivial central extension defined via the slightly normalized 2–cocycle $-\frac{1}{2} \Psi$, and view the elements

1These are representations with finite-dimensional homogeneous subspaces.
\( L_n^{(r)} \) as elements of \( \hat{\mathcal{D}}^+ \). This normalization gives, in particular, the usual Virasoro algebra bracket relations

\[
[L_m^{(0)}, L_n^{(0)}] = (m - n)L_{m+n}^{(0)} + \frac{m^3 - m}{12}\delta_{m+n,0} c. \tag{7.2}
\]

In \([B]\) Bloch discovered that the Lie algebra \( \hat{\mathcal{D}}^+ \) can be defined in terms of generators that lead to a simplification of the central term in the Lie bracket relations. Oddly enough, if we let

\[
\bar{L}_n^{(r)} = L_n^{(r)} + \frac{(-1)^r}{2}\zeta(-1 - 2r)\delta_{n,0} c, \tag{7.3}
\]

then the central term in the commutator

\[
[\bar{L}_m^{(r)}, \bar{L}_n^{(s)}] = \sum_{i = \min(r,s)}^{r+s} a^{(r,s)}_i(m, n)\bar{L}_{m+n}^{(i)} + \frac{(r + s + 1)!}{2(2r + 3)!}m^{2(r+s)+3}\delta_{m+n,0} c \tag{7.4}
\]

is a pure monomial (here \( a^{(r,s)}_i(m, n) \) are structure constants), in contrast to the central term in (7.2) and in other bracket relations that can be found from (7.1). As was announced in \([L3], [L4]\) and shown in \([DLM1, 2]\), in order to conceptualize this simplification (especially the appearance of zeta-values) one can construct certain infinite-dimensional projective representations of \( \mathcal{D}^+ \) using vertex operators.

Let us explain Bloch’s construction \([B]\). Consider the Lie algebra \( \hat{\mathfrak{h}} \) introduced in Section \( 3 \) and its induced (level-one) module \( \mathcal{S} \). Then the correspondence

\[
L_n^{(r)} \leftrightarrow \frac{1}{2}\sum_{q=1}^{d} \sum_{j \in \mathbb{Z}} j^r(n - j)^r \cdot \tilde{\alpha}_q(j)\tilde{\alpha}_q(n - j) \cdot \quad (n \in \mathbb{Z}), \quad c \mapsto d, \tag{7.5}
\]

where we recall that \( \{\tilde{\alpha}_q\} \) is an orthonormal basis of \( \mathfrak{h} \) and \( \cdot \cdot \cdot \) is the usual normal ordering, gives a representation of \( \hat{\mathcal{D}}^+ \). Let us denote the operator on the right–hand side of (7.5) by \( \bar{L}^{(r)}(n) \). In particular, the operators \( L_m^{(0)}(m) \ (m \in \mathbb{Z}) \) give a well-known representation of the Virasoro algebra with central charge \( c \mapsto d \),

\[
[L_m^{(0)}, L_n^{(0)}] = (m - n)L_{m+n}^{(0)} + \frac{m^3 - m}{12}\delta_{m+n,0},
\]

and the construction (7.5) for those operators is the standard realization of the Virasoro algebra on a module for a Heisenberg Lie algebra (cf. \([FLM2]\)).

Without going into any detail, let us mention that Bloch \([B]\) also studied certain natural graded traces using this representation of \( \bar{L}_n^{(r)} \), and found that,
as in the well-known case of the Virasoro algebra $r = 0$, they possess nice modular properties.

The appearance of zeta–values in (7.3) can be conceptualized by the following heuristic argument [Bl]: Suppose that we remove the normal ordering in (7.5) and use the relation $[\bar{\alpha}_q(m), \bar{\alpha}_q(-m)] = m$ to rewrite $\bar{\alpha}_q(m)\bar{\alpha}_q(-m)$, with $m \geq 0$, as $\bar{\alpha}_q(-m)\bar{\alpha}_q(m) + m$. It is easy to see that the resulting expression contains an infinite formal divergent series of the form

$$1^{2r+1} + 2^{2r+1} + 3^{2r+1} + \ldots.$$ 

A heuristic argument of Euler’s suggests replacing this formal expression by $\zeta(-1 - 2r)$, where $\zeta$ is the (analytically continued) Riemann $\zeta$–function. The resulting (zeta–regularized) operator is well defined and gives the action of $\bar{L}^{(r)}_n$; such operators satisfy the bracket relations (7.4).

## 7.1 Realization in $S$: zeta function at negative integers

In order to understand the appearance of the zeta function at negative integers using the vertex operator algebra $S$, following [L3, L4, DLMi1, DLMi2], we need to introduce slightly different vertex operators. Consider a vertex operator algebra $V$. The homogeneous vertex operators are defined by

$$X(u, x) = Y(x^{L(0)}u, x) \quad (u \in V). \quad (7.6)$$

The most important property of these operators, for us, is the homogenous version of modified weak associativity:

$$\lim_{x_1 \to e^y x_2} \left( \frac{x_1 - 1}{x_2} \right)^{k(u,v)} X(u, x_1)X(v, x_2) = (e^y - 1)^{k(u,v)} X(Y[u, y]v, x_2) \quad (7.7)$$

for $u, v \in V$, and $k(u,v)$ as in Theorem [4.3]. Interestingly, in this relation, yet a new type of vertex operator appears:

$$Y[u, y] = Y(e^y L(0)u, e^y - 1) \quad (u \in V). \quad (7.8)$$

This vertex operator map generates a vertex operator algebra that is isomorphic to $V$, and geometrically corresponding to a change to cylindrical coordinates. Those properties were proven in [Z1, Z2].

Consider the vertex operator algebra $S$. Recall that the Virasoro generators $L(n)$ acting on $S$ are given by the operators on the right-hand side of (7.3) with $r = 0$. It is a simple matter to verify that $X(\alpha(-1)1, x) = \alpha(x)$, with

$$\alpha(x) = \sum_{n \in \mathbb{Z}} \alpha(n)x^{-n}. \quad (7.9)$$
Consider now the following formal series, acting on $S$:

$$L^{y_1, y_2}(x) = X \left( \frac{1}{2} \sum_{q=1}^{d} Y[\bar{\alpha}_q(-1)\mathbf{1}, y_1 - y_2] \bar{\alpha}_q(-1)\mathbf{1}, e^{y_2}x]\right). \quad (7.10)$$

By (7.7), we have

$$\bar{L}^{y_1, y_2}(x_2) = \frac{1}{2} \lim_{x_1 \to x_2} \sum_{q=1}^{d} \left( \left( \frac{\frac{z_1}{x_2} e^{y_1 - y_2} - 1}{e^{y_1 - y_2} - 1} \right)^k \bar{\alpha}_q(e^{y_1} x_1) \bar{\alpha}_q(e^{y_2} x_2) \right) \quad (7.11)$$

for any fixed $k \in \mathbb{N}$, $k \geq 2$. Using

$$\sum_{q=1}^{d} \bar{\alpha}_q(e^{y_1} x_1) \bar{\alpha}_q(e^{y_2} x_2) = \sum_{q=1}^{d} \bar{\alpha}_q(e^{y_1} x_1) \bar{\alpha}_q(e^{y_2} x_2) : - \frac{\partial}{\partial y_1} \left( \frac{1}{1 - \frac{z_1}{x_2} e^{-y_1 + y_2}} \right),$$

we immediately find that

$$\bar{L}^{y_1, y_2}(x) = \frac{1}{2} \sum_{q=1}^{d} \bar{\alpha}_q(e^{y_1} x) \bar{\alpha}_q(e^{y_2} x) : - \frac{1}{2} \frac{\partial}{\partial y_1} \left( \frac{1}{1 - e^{-y_1 + y_2}} \right). \quad (7.12)$$

Defining the operators $\bar{L}^{r_1, r_2}(n)$, $r_1, r_2 \in \mathbb{N}$, $n \in \mathbb{Z}$ via

$$\bar{L}^{y_1, y_2}(x) = \frac{1}{2 \left(y_1 - y_2\right)^2} + \sum_{n \in \mathbb{Z}, r_1, r_2 \in \mathbb{N}} \bar{L}^{r_1, r_2}(n)x^{-n} \frac{y_1^{r_1} y_2^{r_2}}{r_1! r_2!}, \quad (7.13)$$

it is simple to see, using (7.3), that the correspondence

$$\bar{L}^{(r)}_n \mapsto \bar{L}^{r}(n), \quad c \mapsto d \quad (7.14)$$

for $n \in \mathbb{Z}$, $r \in \mathbb{N}$ gives a representation of the generators (7.3) of the algebra $\mathcal{D}_+$. Recall that these generators were introduced by Bloch in order to simplify the central term in the commutation relations.

As was shown in [DLMi2], from the expression (7.11) of the formal series $L^{y_1, y_2}(x)$, involving formal limits, it is a simple matter to compute the following commutators, first written in [L3]:

$$[\bar{L}^{y_1, y_2}(x_1), \bar{L}^{y_3, y_4}(x_2)] = - \frac{1}{2} \frac{\partial}{\partial y_1} \left( \bar{L}^{-y_1 + y_2 + y_3, y_4}(x_2) \delta \left( \frac{e^{y_1} x_1}{e^{y_3} x_2} \right) + \bar{L}^{-y_1 + y_2 + y_4, y_3}(x_2) \delta \left( \frac{e^{y_1} x_1}{e^{y_4} x_2} \right) \right)$$

$$- \frac{1}{2} \frac{\partial}{\partial y_2} \left( \bar{L}^{-y_1 + y_3, y_4}(x_2) \delta \left( \frac{e^{y_2} x_1}{e^{y_3} x_2} \right) + \bar{L}^{-y_1 + y_3, y_4}(x_2) \delta \left( \frac{e^{y_2} x_1}{e^{y_3} x_2} \right) \right). \quad (7.15)$$

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As was announced in [L3, L4] and explained in [DLMi2], a simple analysis of this commutator shows that the central term in the commutators of the generators (7.14) is a pure monomial, as in (7.4). This gives a simple explanation of Bloch’s phenomenon using the vertex operator algebras $S$. Moreover, the definition (7.10) says that the operators $L^{y_1,y_2}(x)$ represent on $V$ the image of some fundamental algebra elements of $V$ under the transformation to the cylinder. These fundamental elements, being closely related to the Virasoro element $\omega$, can be expected, when transformed to the cylinder, to lead to graded traces with simple modular properties, in agreement with the observations of Bloch [B].

7.2 Representations on $S[\nu]$: Bernoulli polynomials at rational values

Following [DLMi1, DLMi2], we will now construct a representation of $\hat{D}^+$ on $S[\nu]$. The property that a twisted module is a true module on the fixed-point subalgebra will be essential below in this construction. This property is guaranteed by the operator $\Delta_x$ that we calculated above (6.19). In order to have the correction terms for the representation of the algebra $\hat{D}^+$ on the twisted space $S[\nu]$, one could apply $e^{\Delta_x}$ on the vectors generating the representation of the whole algebra $\hat{D}^+$. This can be a complicated problem, mainly because generators of $\hat{D}^+$ have arbitrarily large weights. In line with [DLMi1, DLMi2], below we will calculate the correction terms directly using the modified weak associativity relation for twisted operators, as well as the simple result (3.10). Hence in this argument, the explicit action of $\Delta_x$ on vectors generating the representation of the algebra $\hat{D}^+$ is not of importance; all we need to know is that there exists such an operator $\Delta_x$ giving to the space $S[\nu]$ the properties of a twisted module for the vertex operator algebra $S$.

In parallel to the previous sub-section, we need to introduce homogeneous twisted vertex operators. Being given a vertex operator algebra $V$ and a $\nu$-twisted $V$-module $M$, they are defined by

$$X_M(u, x) = Y_M(x^{L(0)}u, x) \quad (u \in V).$$

(7.16)

Again, the most important property of these operators, for us, is the homogeneous version of modified weak associativity for twisted vertex operators:

$$\lim_{x_1^{1/p} \rightarrow \omega_p^\nu(e^y x_2)^{1/p}} \left( \left( \frac{x_1}{x_2} - 1 \right)^{k(u,v)} X_M(u, x_1)X_M(v, x_2) \right) = (e^y - 1)^{k(u,v)} X_M(Y[\nu^{-s}u, y]v, x_2).$$

(7.17)
for $u, v \in V$, $s \in \mathbb{Z}$, and $k(u, v)$ as in Theorem 4.6. Recall the definition of $Y[u, y]$ in (7.8).

Consider the vertex operator algebra $S$ and its twisted module $S[\nu]$. It is a simple matter to verify that $X_{S[\nu]}(\alpha(-1)1, x) = \alpha^\nu(x)$, with

$$\alpha^\nu(x) = \sum_{n \in \mathbb{Z}} \alpha^\nu(n)x^{-n}.$$  \hspace{1cm} (7.18)

Consider now the following formal series, acting on $S[\nu]$:

$$L^{\nu} y_1 y_2 \langle x \rangle = X_{S[\nu]} \left( \frac{1}{2} \sum_{q=1}^{d} Y[\bar{\alpha}_q(-1)1, y_1 - y_2] \bar{\alpha}_q(-1)1, e^{y_2}x \right).$$ \hspace{1cm} (7.19)

Since the operator $\frac{1}{2} \sum_{q=1}^{d} Y[\bar{\alpha}_q(-1)1, y_1 - y_2] \bar{\alpha}_q(-1)1$ is in the fixed point subalgebra of $S$, it is immediate that these operators satisfy the same commutation relations as (7.15):

$$[\bar{L}^{\nu} y_1 y_2 \langle x_1 \rangle, \bar{L}^{\nu} y_3 y_4 \langle x_2 \rangle]$$ \hspace{1cm} (7.20)

$$= -\frac{1}{2} \frac{\partial}{\partial y_1} \left( \bar{L}^{\nu} y_1 y_2 y_3 y_4 \langle x_1 \rangle \delta \left( \frac{e^{y_1}x_1}{e^{y_3}x_2} \right) + \bar{L}^{\nu} y_1 y_2 y_3 y_4 \langle x_2 \rangle \delta \left( \frac{e^{y_1}x_1}{e^{y_3}x_2} \right) \right)$$

$$-\frac{1}{2} \frac{\partial}{\partial y_2} \left( \bar{L}^{\nu} y_1 y_2 y_3 y_4 \langle x_1 \rangle \delta \left( \frac{e^{y_2}x_1}{e^{y_3}x_2} \right) + \bar{L}^{\nu} y_1 y_2 y_3 y_4 \langle x_2 \rangle \delta \left( \frac{e^{y_2}x_1}{e^{y_3}x_2} \right) \right).$$

Hence, defining the operators $\bar{L}^{\nu r_1 r_2}(n), r_1, r_2 \in \mathbb{N}, n \in \mathbb{Z}$ via

$$\bar{L}^{\nu} y_1 y_2 \langle x \rangle = \frac{1}{2} \frac{d}{(y_1 - y_2)^2} + \sum_{n \in \mathbb{Z}, r_1, r_2 \in \mathbb{N}} \bar{L}^{\nu r_1 r_2}(n) x^{-n} \frac{y_1^{r_1} y_2^{r_2}}{r_1! r_2!}$$ \hspace{1cm} (7.21)

(in particular, only integer powers of $x$ appear in this expansion), we conclude that they satisfy the same commutation relations as the operators $\bar{L}^{r_1 r_2}(n)$ introduced in (7.13). Then, as in (7.14), we can expect that the correspondence

$$\bar{L}^{(r)}_n \mapsto \bar{L}^{\nu r}(n), \quad c \mapsto d$$ \hspace{1cm} (7.22)

for $n \in \mathbb{Z}, r \in \mathbb{N}$ gives a representation of the generators (7.3) of the algebra $\bar{D}^+$ on $S[\nu]$. Bringing this expectation to a proof needs a little more analysis (in particular, one needs to show that the $L^{\nu r_1 r_2}(n)$ are related to the $L^{\nu r}(n)$ in the same way as the $L^{r_1 r_2}(n)$ are related to the $L^r(n)$), which is done in detail in [DLM12].

Now, by (7.17) we have

$$\bar{L}^{\nu} y_1 y_2 \langle x_2 \rangle = \frac{1}{2} \lim_{x_1 \to x_2} \sum_{q=1}^{d} \left( \frac{x_1 e^{y_1} - y_2}{x_2 e^{y_1} - y_2} \right) \bar{\alpha}_q \left( e^{y_1} x_1 \right) \bar{\alpha}_q \left( e^{y_2} x_2 \right)$$ \hspace{1cm} (7.23)
for any fixed $k \in \mathbb{N}$, $k \geq 2$, which gives

$$\bar{L}^\nu_{y_1,y_2}(x) = \frac{1}{2} \sum_{q=1}^d \bar{\alpha}_q^\nu(e^{y_1}x)\bar{\alpha}_q^\nu(e^{y_2}x) - \frac{1}{2} \frac{\partial}{\partial y_1} \left( \sum_{k=0}^{p-1} \frac{e^{\frac{k(-y_1+y_2)}{p}} \dim h(k)}{1 - e^{-y_1+y_2}} \right)$$

using

$$\sum_{q=1}^d \bar{\alpha}_q^\nu(e^{y_1}x_1)\bar{\alpha}_q^\nu(e^{y_2}x_2) = \sum_{q=1}^d \bar{\alpha}_q^\nu(e^{y_1}x_1)\bar{\alpha}_q^\nu(e^{y_2}x_2) - \frac{\partial}{\partial y_1} \left( \sum_{k=0}^{p-1} \frac{e^{\frac{k(-y_1+y_2)}{p}} \dim h(k)}{1 - \frac{x_1}{x_1} e^{-y_1+y_2}} \right).$$

Evaluating the operators $\bar{L}^\nu_{r,r}(n)$ from (7.21), we conclude that the operators

$$\bar{L}^\nu_{r,r}(n) = \frac{1}{2} \sum_{q=1}^d \sum_{j \in \frac{1}{p} \mathbb{Z}} j^r(n-j)^r :\bar{\alpha}_q^\nu(j)\bar{\alpha}_q^\nu(n-j):$$

$$-\delta_{n,0} \frac{(-1)^r}{4(r+1)} \sum_{k=0}^{p-1} \dim h(k)B_{2(r+1)}(k/p)$$

form a representation, on $S[\nu]$, of the generators (7.3) for the Lie algebra $\hat{D}^+$. Notice the appearance of the Bernoulli polynomials. From our construction, this is seen to be directly related to general properties of homogeneous twisted vertex operators.

The next result is a simple consequence of the discussion above. It was shown in [DLMi2]. It describes the action of the “Cartan subalgebra” of $\hat{D}^+$ on a highest weight vector of a canonical quasi-finite $\hat{D}^+$–module; here we are using the terminology of [KR]. This corollary gives the “correction” terms referred to in the introduction.

**Corollary 7.1** Given a highest weight $\hat{D}^+$–module $W$, let $\delta$ be the linear functional on the “Cartan subalgebra” of $\hat{D}^+$ (spanned by $L_0^{(k)}$ for $k \in \mathbb{N}$) defined by

$$L_0^{(k)} \cdot w = (-1)^k \delta \left( L_0^{(k)} \right) w,$$

where $w$ is a generating highest weight vector of $W$, and let $\Delta(x)$ be the generating function

$$\Delta(x) = \sum_{k \geq 1} \frac{\delta(L_0^{(k)})x^{2k}}{(2k)!}$$

(cf. [KR]). Then for every automorphism $\nu$ of period $p$ as above,

$$\mathcal{U}(\hat{D}^+) \cdot 1 \subset S[\nu]$$
is a quasi–finite highest weight $\hat{D}^+-$module satisfying

$$\Delta(x) = \frac{1}{2} \frac{d}{dx} \sum_{k=0}^{p-1} \frac{(e^{\frac{ikx}{p}} - 1)\dim h(k)}{1 - e^x}.$$  

(7.26)

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