Multiparametric oscillator Hamiltonians with exact bound states in infinite-dimensional space

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Abstract

Bound states in quantum mechanics must almost always be constructed numerically. One of the best known exceptions concerns the central $D$-dimensional (often called “anharmonic”) Hamiltonian $H = p^2 + a |\vec{r}|^2 + b |\vec{r}|^4 + \ldots + z |\vec{r}|^{4q+2}$ (where $z = 1$) with a complete and elementary solvability at $q = 0$ (central harmonic oscillator, no free parameters) and with an incomplete, $N$-level elementary analytic solvability at $q = 1$ (so called “quasi-exact” sextic oscillator containing one free parameter). In the limit $D \to \infty$, numerical experiments revealed recently a highly unexpected existence of a new broad class of the $q$-parametric quasi-exact solutions at the next integers $q = 2, 3, 4$ and $q = 5$. Here we show how a systematic construction of the latter, “privileged” $D \gg 1$ exact bound states may be extended to much higher $q$s (meaning an enhanced flexibility of the shape of the force) at a cost of narrowing the set of wavefunctions (with $N$ restricted to the first few non-negative integers). At $q = 4K + 3$ we conjecture a closed formula for the $N = 3$ solution at all $K$.

MSC 2000: 81Q05 13P05 14M12
1 Introduction

In quantum mechanics, bound states of a particle confined in a central potential well \( V(|\vec{r}|) \) in \( D \) dimensions are constructed as normalizable solutions of the ordinary differential Schrödinger equation

\[
\left[ -\frac{d^2}{dr^2} + \frac{\ell(\ell + 1)}{r^2} + V(r) \right] \psi(r) = E \psi(r).
\] (1)

usually, this equation (or, more precisely, their infinite set numbered by integer argument \( L = 0, 1, \ldots \) of the “angular momentum” \( \ell(L) = L + (D - 3)/2 \) must be solved by purely numerical means.

A notable exception concerns all the models where the spatial dimension \( D \) proves “sufficiently” large, \( D \gg 1 \). In a way outlined, say, in reviews [1], a semi-analytic, perturbative construction of the solutions may be then based on the so-called large-\( D \) expansion technique. In essence, this approach combines our knowledge of the asymptotic growth of \( V(r) \approx r^\alpha \) at \( r \gg 1 \) with the presence of the strong repulsive core \( \ell(\ell+1)/r^2 \) near the origin. This implies the existence of a (presumably, pronounced) absolute minimum of the combined forces at a point \( r = R(\alpha) \) where we have

\[
\partial_y \left[ \frac{(\ell+1)}{(R+y)^2} + (R+y)^\alpha \right] \bigg|_{y=0} = 0, \quad \Rightarrow \quad R = R(\alpha) = \left[ \frac{2\ell(\ell+1)}{\alpha} \right]^{1/(\alpha+2)} \gg 1
\]

whenever we neglect all the less relevant corrections. The main merit of this approach lies in its simplicity and universality but, unfortunately, its expected rate of convergence (measured by the smallness of \( 1/R \)) will, obviously, increase quite slowly with the growth of \( D \) and decrease quite quickly with an increase of the dominant power of the potential \( \alpha \). As long as we are going to pay attention to the class of potentials

\[
V(q,k)(r) = \frac{1}{r^2} \left[ g_{-2} + g_{-1} r^{2/(k+1)} + g_0 r^{4/(k+1)} + \ldots + g_{2q} r^{(4q+4)/(k+1)} \right]
\] (2)

with a fixed \( k \) (say, for the sake of simplicity, \( k = 0 \)) and growing integers \( q \), an efficient use of the above large-\( D \) approach does not seem too promising.

In an alternative semi-analytic approach to the problem (1) + (2) one could try to employ a power-series method where

\[
\psi^{(PS)}(q,k)(r) \approx r^{\ell+1} \times U(r) \times \exp W(r),
\] (3)

and where approximations \( U(r) = polynomial [r^{2/(k+1)}] \) (of a sufficiently large degree \( m_U \)) and \( W(r) = polynomial [r^{2/(k+1)}] \) (of any degree \( m_W \)) may be re-constructed in a more or less algebraic manner after an insertion of this ansatz in the original differential equation. Unfortunately, virtually all practical implementations of this method prove even less efficient in computations [2].

The third eligible non-numerical approach to the construction represents a modification of the power-series method where also the second polynomial degree \( m_U < \infty \).
is fixed. This method has been proposed by E. Magyari [3] and is based on the evaluation of the solutions (3) via an ad hoc tuning of the potential (2) itself. Basically, the recipe requires the absence of any errors in eq. (3). This idea just extends the well known polynomial solvability of eq. (1) + (2) at $q = k = 0$ (harmonic oscillator) and $q = k - 1 = 0$ (Coulomb field). The details of the Magyari’s recipe have been reviewed in ref. [4] where we emphasized that its practical merits are virtually nonexistent because the underlying and obvious selfconsistency requirements (we called them there the Magyari-Schrödinger equations) may be characterized as a coupled set of determinantal equations which are extremely complicated to solve in general.

The main reason why we decided to pay attention to the Magyari’s approach is that we discovered, many year ago [5], that this method encounters enormous simplifications during a limiting transition to the large dimensions $D \gg 1$. In this setting one has to follow very consequently all the analogies with the harmonic oscillators. Firstly, one assumes that the polynomial $W(r)$ in the exponential represents an exact $r \gg 1$ asymptotic solution at any $q$ and $k$. Both the degree $m_W$ and all the coefficients in $W(r)$ are uniquely determined, in this way, by our choice of the potential (for example, we have $W(r) = -r^2/2$ for harmonic oscillator). On this background, the Magyari’s second key idea parallels the essence of the quasi-exact solvability (i.e., an incomplete solvability, see [6]) and extends, to all the $q \geq 1$ cases, the requirement that our choice of the polynomial $U(r)$ of any integer degree $N = m_U$ leads to some exact solution (3) for a suitably modified, adapted polynomial potential (2).

Our present text is inspired by the series of demonstrations that the latter combined method proves unexpectedly efficient in practice. As we mentioned, our study was initiated by the observation that for one of the most popular, viz., quartic polynomial interaction, the complicated Magyari-Schrödinger equations exhibit a remarkable and fairly surprising simplification. In the next stage of our work on this project [7] we choose the “first nontrivial” $q = 2$ oscillators and complemented the above observations by the formulation of the method of an explicit perturbative construction of $1/D$ corrections. We must emphasize that the improved convergence of our innovated solutions (which were defined by the series in the powers of $1/D^2$) was in a sharp contrast to the steady worsening of the performance of the above-mentioned large–$D$ expansions at the larger $q$. Moreover, in the “minimal” sextic example at $q = 1$, we succeeded in showing that and why our innovated series in the powers of $1/R \approx 1/D^{1/4}$ was absolutely convergent [8].

A real climax of our effort came with the papers [4] and [9] where we performed an explicit construction of the zero-order solutions and discovered that at the next few less trivial exponents $q = 3$, $q = 4$ and $q = 5$ we were still able to construct the $D \to \infty$ solutions in closed form. Unfortunately, we were just able to work by the brute-force methods, based on the direct solution of the $D \to \infty$ limit of the Magyari-Schrödinger coupled algebraic nonlinear equations by the elimination method using the Groebner bases [10]. For this reason, we were never able to find any solution at $q \geq 6$. This was also the main motivation for the study which we are going to describe in what follows.
2 A concise formulation of the problem

2.1 Asymptotic Magyari-Schrödinger equations

In a way discussed thoroughly in [4], the use of our ansatzs in the limit $D \rightarrow \infty$ implies an immediate reduction of the differential radial Schrödinger equation to its algebraic equivalent

\[
\begin{pmatrix}
  s_1 & 1 & & & & & & \\
  s_2 & s_1 & 2 & & & & & \\
  & \ddots & \ddots & \ddots & & & & \\
  s_q & & & & & & & N - 2 \\
  N - 1 & s_q & & & s_1 & & & N - 1 \\
  N - 2 & & & s_q & & \ddots & \ddots & \ddots \\
  & & & & 2 & & s_q & s_{q-1} \\
  & & & & & & 1 & s_q \\
\end{pmatrix}
\begin{pmatrix}
p_0 \\
p_1 \\
\vdots \\
p_{N-2} \\
p_{N-1}
\end{pmatrix} = 0. \tag{4}
\]

Here we are going to study its solvability, referring to [4] (we shall call it paper PI in what follows) for all the explanations of its origin and interpretations. Calling this deeply nonlinear non-square-matrix algebraic problem the Magyari-Schrödinger equation, we may just summarize that the $N$ quantities $p_j$ are in a one-to-one correspondence with the Taylor coefficients in the parts $U(r)$ of wavefunctions (3). Moreover, all $q$ generalized eigenvalues $s_a$ [cf. their definitions $s_a = s_a(g_{a-2})$ by eq. (30) in PI] are just certain re-scaled forms of the energies and/or of coupling constants in our potential (2).

2.2 Upside-down symmetry

As we already explained in paper PI, the practical use of the explicit quasi-exact (QE) solutions requires a purely numerical determination of their QE-compatible eigenvalues $g_{a-2}$ with $a = 1, 2, \ldots, q$. This means that the simplified Magyari-Schrödinger eq. (4) should be interpreted as an implicit definition of a QE solution on a predetermined level of a finite precision. All the sufficiently small $O(1/D)$ variations of the parameters should be ignored, within this convention, as negligible and irrelevant. Alternatively, they may all be treated, if necessary, by perturbation methods in a way exemplified in [11] at $q = 1$ and in [7] at $q = 2$.

In such a setting, one of the main messages delivered by paper PI was that all the perturbation constructions of any type may remain exact and non-numerical, at the first few smallest $q$ at least. This follows from the observation that all the exact QE parameters in PI proved expressible through integers in the zero-order limit $D \rightarrow \infty$. The necessary condition of this simplification lies in the elementary form of our algebraized QE Magyari-Schrödinger equation (4). Still, one has to overcome a few further obstacles. In particular, the brute-force origin of the results in PI made
it impossible to move beyond $q \leq 5$. In the other words, the key weakness of paper PI lies in the too rapid growth of the difficulties with the increasing $q$. One needs all the capacity of the available computers to reveal the structure of solutions at the first few $q$. This means that whenever one needs an improvement of the insight in the structure of the solutions, one has to make a better use of their symmetries.

In our present continuation and completion of paper PI we are going to exploit the most obvious symmetry of eq. (4) with respect to its upside-down transposition. For this purpose we may modify our notation slightly, replacing eq. (4) by its reincarnation

\[
\begin{pmatrix}
\alpha_1 & 1 \\
\alpha_2 & \alpha_1 & 2 \\
\vdots & \ddots & \ddots \\
\tilde{\alpha}_1 & \vdots & \alpha_1 & N - 2 \\
N - 1 & \tilde{\alpha}_1 & \alpha_1 & N - 1 \\
N - 2 & \tilde{\alpha}_1 & \vdots & \alpha_1 \\
\vdots & \vdots & \ddots & \ddots \\
2 & \tilde{\alpha}_1 & \tilde{\alpha}_2 & 1 \\
1 & \tilde{\alpha}_1 & & \\
\end{pmatrix}
\begin{pmatrix}
p_0 \\
p_1 \\
\vdots \\
\tilde{p}_1 \\
\tilde{p}_0 \\
\end{pmatrix} = 0. 
\] (5)

However trivial, such a change of notation implies that all the separate lines have a tilded partner in this set. For example, the definition $p_0 = -p_1/\alpha$ (where $\alpha = \alpha_1$ is tacitly assumed non-zero) is always accompanied by its tilded counterpart $\tilde{p}_0 = -\tilde{p}_1/\tilde{\alpha}$ whenever $\tilde{\alpha} = \tilde{\alpha}_1 \neq 0$, etc.

### 3 Solutions of the symmetrized recurrences

#### 3.1 Trivial case at $N = 2$

In our tilded notation, the first nontrivial version of our problem (5) with $N = 2$ is particularly transparent and instructive,

\[
\begin{pmatrix}
\alpha_1 & 1 \\
\alpha_2 & \alpha_1 \\
\alpha_3 & \alpha_2 \\
\vdots & \vdots \\
\tilde{\alpha}_1 & \tilde{\alpha}_2 \\
1 & \tilde{\alpha}_1 \\
\end{pmatrix}
\begin{pmatrix}
p \\
\tilde{p} \\
\end{pmatrix} = 0. 
\] (6)

Its upside-down or “tilding” symmetry separates its first and last line as giving a constraint upon the doublet of unknowns $\alpha_1 = s_1 = \alpha$ and $\tilde{\alpha}_1 = s_q = \tilde{\alpha}$ in a tilding-symmetric manner,

\[
\det \begin{pmatrix}
\alpha & 1 \\
1 & \tilde{\alpha} \\
\end{pmatrix} = 0, \quad \alpha \tilde{\alpha} = 1. 
\] (7)
The fully tilding-symmetric way of dealing with the rest of eq. (6) consists now in the equivalence of its recurrent downward or upward treatment.

After we normalize \( p_1 = \tilde{p} = 1 \), the first line defines \( p_0 = p = -1/\alpha \) and may be omitted. Step-by-step, the \( k \)-th line of eq. (6) may be multiplied by the factor \( \tilde{\alpha}/\alpha^{k-1} \). This converts the old components into the known constants and we get the following sequence of definitions

\[
\tilde{\alpha}_1 \alpha = 1, \quad \tilde{\alpha}_2/\alpha = 1, \quad \tilde{\alpha}_3/\alpha^2 = 1, \ldots ,
\]

with elementary consequence: \( \alpha_k = \alpha^k \). We may imagine that the last definition prescribes that \( \alpha_{q+1} = \alpha^{q+1} = 1 \). This equation may be read as a boundary condition for our recurrences, fixing the physical value of our single free parameter \( \alpha \). It has many unphysical complex roots and just a single real one, viz., the physical root \( \alpha = 1 \) at any even \( q \). Similarly, two different real roots \( \alpha = \pm 1 \) become available at all the odd \( q \)'s. This makes the final reconstruction of all the original QE-compatibility “eigenvalues” \( s_1, \ldots , s_q \) trivial.

We may add a comment. Knowing that the last line of recurrences (6) defines the function of \( \alpha (\alpha_{q+1} = \alpha^{q+1}) \) with a prescribed value \( (\alpha^{q+1} = 1) \), the latter constraint may be interpreted as an algebraic equation which fixes the eligible values of \( \alpha \). Such a type of the boundary condition is not unique. The same role may be played by any other line of eq. (6), once we re-direct these recurrences and demand that

\[
\tilde{\alpha}_{1+j} = \alpha_{q-j}
\]

at any shift \( j \leq q - 1 \). At \( N = 2 \) all this is trivial since after being multiplied by \( \alpha^{j+1} \), all relations (9) degenerate to the same rule \( \alpha^{q+1} = 1 \).

### 3.2 A transition to the single variable at \( N = 3 \)

A separate treatment of the first nontrivial \( N = 3 \) version of eq. (4) is necessary in the degenerate case with \( p_1 = 0 \). We may infer that \( s_1 = s_3 = s_5 = \ldots = s_q = 0 \). This means that \( q = 2Q + 1 \) must be odd and that we in effect return to the \( N = 2 \) structure. We only have to replace the old unknowns \( s_k \) by the new ones, re-scaled to \( s_{2k}/2 \). Otherwise, the construction of the solutions remains strictly the same, giving the nontrivial roots \( s_{2k} = 2\tilde{g}^k \) where \( \tilde{g}^{Q+1} = 1 \).

In what follows, similar detailed qualification will be omitted and, with the degenerate solutions ignored, we shall normalize \( p_1 = 1 \) at \( N = 3 \), etc. From the two outer lines of the \( N = 3 \) version of eq. (4) we deduce that \( p_0 = -1/\alpha \) while \( p_2 = -1/\tilde{\alpha} \). The rest of equation (4) acquires the tilding-symmetric matrix form

\[
\begin{pmatrix}
\beta & \alpha & 2 \\
\gamma & \beta & \alpha \\
\delta & \gamma & \beta \\
\vdots & \vdots & \vdots \\
\tilde{\alpha} & \tilde{\beta} & \tilde{\gamma} \\
2 & \tilde{\alpha} & \tilde{\beta}
\end{pmatrix}
\begin{pmatrix}
-1/\alpha \\
1 \\
-1/\tilde{\alpha}
\end{pmatrix} = 0 .
\]

(10)
It gets facilitated when pre-multiplied by an auxiliary row vector. This observation results from the step-by-step analysis of this system of equations re-written in the form

$$\tilde{\alpha} \beta / \alpha = \tilde{\alpha} \alpha - 2, \quad \tilde{\alpha} \gamma / \alpha = \tilde{\alpha} \beta - \alpha, \quad \tilde{\alpha} \delta / \alpha = \tilde{\alpha} \gamma - \beta, \quad \ldots. \quad (11)$$

In the first item the right-hand-side part $\tilde{\alpha} \alpha - 2 = \alpha \tilde{\alpha} - 2 \equiv \xi - 2 = \tilde{\xi} - 2$ is tilding symmetric. This means that the same tilding-invariance must hold for the left-hand-side expression as well. The second item is not tilding-invariant but the invariance is restored after we divide all this equation by $\alpha$. This gives a consistent picture because one can deduce that also in all the subsequent rows the full tilding-invariance is achieved when we replace $\alpha$, $\beta$, $\gamma$, $\ldots$ by their renormalized and tilding-invariant forms $\tilde{\alpha} / \alpha^0$, $\tilde{\beta} / \alpha$, $\tilde{\gamma} / \alpha^2$, $\ldots$, respectively. In the other words, the system (11) must be pre-multiplied by the row of the factors $1, 1 / \alpha, \tilde{\alpha} / \alpha, \tilde{\alpha}^2 / \alpha^2, \ldots$ obtained, in recurrent manner, by the multiplication by the quotient which depends on the parity, i.e., equals to $1 / \alpha$ and to $\tilde{\alpha}$ in subsequent steps. This means that the even and odd items in eq. (11) have a different structure.

This difference may be reflected by the change of the notation. Once we put $\alpha = s_1 = a$, $\beta = s_2 = A$, $\gamma = s_3 = b$, $\delta = s_4 = B$, $\epsilon = s_5 = c$ (while denoting also $\tilde{\alpha} = s_q = \tilde{a}$ etc) etc, equation (4) acquires another formally tilding-symmetric matrix form

$$\begin{pmatrix}
a & 1 & 0 \\
A & a & 2 \\
b & A & a \\
B & b & A \\
\vdots & \vdots & \vdots
\end{pmatrix}
\begin{pmatrix}
\tilde{\alpha} / a \\
\tilde{\alpha}^2 / a \\
\tilde{\alpha} / \rho \\
\tilde{\alpha}^2 / \rho
\end{pmatrix} = 0. \quad (12)$$

The pair of the old variables $a$ and $\tilde{\alpha}$ must be replaced by their tilding-invariant product $\xi = a \tilde{\alpha} = \tilde{\xi}$ and its tilding-covariant complement $\rho = a / \tilde{\alpha} = 1 / \tilde{\rho}$. In an opposite direction, whenever needed, we may re-construct $a$ and $\tilde{\alpha}$ from the two quadratic relations $a^2 = \rho \xi$ and $\tilde{\alpha}^2 = \xi / \rho$, i.e., up to an inessential indeterminacy in sign. After we abbreviate

$$\Sigma_1 = \frac{A}{\rho}, \quad \Sigma_2 = \frac{B}{\rho^2}, \quad \Sigma_3 = \frac{C}{\rho^3}, \quad \ldots, \quad \sigma_1 = \frac{a}{\tilde{\alpha} \rho^0}, \quad \sigma_2 = \frac{b}{\tilde{\alpha} \rho^1}, \quad \sigma_3 = \frac{c}{\tilde{\alpha} \rho^2}, \quad \ldots$$

and postulate that $\Sigma_0 = 2$ and $\sigma_1 = 1$, this procedure results in the conclusion that our recurrences may be re-written as the following sequence of the coupled pairs of the recurrent relations,

$$\Sigma_k = \xi \sigma_k - \Sigma_{k-1}, \quad \sigma_{k+1} = \Sigma_k - \sigma_k, \quad k = 1, 2, \ldots. \quad (13)$$

One re-interprets eqs. (13) as the mere recurrent definition of the auxiliary sequence of functions of our auxiliary real variable $\xi$,

$$\Sigma_1 = \xi - 2, \quad \sigma_2 = \xi - 3, \quad \Sigma_2 = \xi^2 - 4 \xi + 2, \quad \ldots.$$  

We see that the functions $\Sigma_k(\xi)$ and $\sigma_{k+1}(\xi)$ are, by their construction, both polynomials of the same degree $k$. 

6
Our final change in the notation will prescribe $\xi$ replaced by $\xi = 4x^2$, with $\Sigma_k$ represented as $\Sigma_k = 2T_k(x)$ and with $\sigma_k$ re-scaled into $\sigma_k = T_{2k-1}(x)/x$. We notice that our recurrences become simpler in the new notation but what is more important is that after such a transformation, our new polynomials $T_n(x)$ coincide precisely with the classical orthogonal Chebyshev polynomials of the first kind [12].

$$T_0(x) = 1, \ T_1(x) = x, \ T_2(x) = 2x^2 - 1, \ \ldots.$$  \hspace{1cm} (14)

In this sense, our $N = 3$ recurrences are solved exactly in closed form.

### 3.3 The two tilding-covariant variables at $N = 4$

Using the abbreviations $\alpha = s_1$ and $\tilde{\alpha} = s_q$ etc., let us interpret the $N = 4$ QE recurrences (4) as a tilding-symmetric problem

$$
\begin{pmatrix}
\alpha & 1 \\
\beta & \alpha & 2 \\
\gamma & \beta & \alpha & 3 \\
\delta & \gamma & \beta & \alpha \\
\epsilon & \delta & \gamma & \beta \\
\vdots & \vdots & \vdots & \vdots \\
3 & \tilde{\alpha} & \tilde{\beta} & \tilde{\gamma} \\
2 & \tilde{\alpha} & \tilde{\beta} & \tilde{\gamma} \\
1 & \tilde{\alpha} \\
\end{pmatrix}
\begin{pmatrix}
p \\
t \\
\tilde{t} \\
\tilde{p} \\
\end{pmatrix} = 0.
$$  \hspace{1cm} (15)

The first and the last line may be dropped as defining merely $p = -t/\alpha$ and $\tilde{p} = -\tilde{t}/\tilde{\alpha}$. From the next two outer lines we may express $t$ in terms of $\tilde{t}$ or vice versa. Once we normalize $\tilde{t} = 1$ and re-parametrize $s_2 = \beta = \beta(b) = \alpha^2 + 2b\alpha$ and, in parallel, $s_{q-1} = \tilde{\beta} = \tilde{\beta}(\tilde{b}) = \tilde{\alpha}^2 + 2\tilde{b}\tilde{\alpha}$, the vanishing of the related secular two-by-two determinant may be re-read as the constraint $\tilde{b}\tilde{b} = 1$. The rest of eq. (15) reads

$$
\begin{pmatrix}
\gamma & \beta(b) & \alpha & 3 \\
\delta & \gamma & \beta(b) & \alpha \\
\epsilon & \delta & \gamma & \beta(b) \\
\vdots & \vdots & \vdots & \vdots \\
3 & \tilde{\alpha} & \tilde{\beta}(\tilde{b}) & \tilde{\gamma} \\
\end{pmatrix}
\begin{pmatrix}
-\tilde{b}\tilde{\alpha}/\alpha \\
\tilde{b}\tilde{\alpha} \\
\tilde{\alpha} \\
-1 \\
\end{pmatrix} = 0.
$$  \hspace{1cm} (16)

Its first row expresses $\gamma$ as a function of $\alpha$ and $b$ and $\tilde{\alpha}$ and $\tilde{b}$,

$$\gamma\tilde{b}\tilde{\alpha}/\alpha = Q = \beta(b)\tilde{b}\tilde{\alpha} + \alpha\tilde{\alpha} - 3.$$  \hspace{1cm} (17)

Fortunately, the quantity $Q$ may be read as a polynomial in the mere two new, auxiliary variables $Z = \alpha\tilde{\alpha} \equiv \tilde{Z}$ and $Y = \alpha\tilde{b} = Z/\tilde{Y}$. In the light of the preceding subsection, this function $Q = Q(Z,Y) = ZY + 3Z - 3$ of two variables will obviously
play the role of a generalized Hermite polynomial. We may only regret that it is not a tilding-invariant function anymore.

The second row of eq. (16) must be multiplied by $Y\tilde{\alpha}$ to give the next-order polynomial in the same two variables,

$$R = \delta\tilde{b}\tilde{\alpha}^2 Y/\alpha = Z Y Q(Z,Y) + Z^2 Y + 2 Z^2 - Z Y = R(Z,Y).$$  \hspace{1cm} (18)

This defines the new quantity (= a rescaled $\delta$) and we may proceed to the third row multiplied by $Y\tilde{\alpha}^3$,

$$S = \epsilon\tilde{b}\tilde{\alpha}^3 Y/\alpha = Z R(Z,Y) + Z^2 Q(Z,Y) - Z^2 Y - 2 Z^2 = S(Z,Y).$$  \hspace{1cm} (19)

After the multiplication by $Y^2\tilde{\alpha}^3$ the fourth row reads

$$T = \zeta\tilde{b}\tilde{\alpha}^4 Y^2/\alpha = Z S(Z,Y) + Z^2 R(Z,Y) - Y Z^2 Q(Z,Y)$$

with the next factor $Y^2\tilde{\alpha}^4$ giving the next, fifth row

$$U = \eta\tilde{b}\tilde{\alpha}^5 Y^2/\alpha = Z T(Z,Y) + Z^2 S(Z,Y) - Z^2 R(Z,Y)$$

etc. Step by step we construct, in this manner, the two sequences of functions denoted as $P_n(Y,Z)$ and $Q_n(Y,Z)$ and defined by the pair of the coupled recurrences,

$$
P_{n+1} = Y Z Q_n + Z^2 P_n - Y Z^2 Q_{n-1},
Q_{n+1} = Z P_{n+1} + Z^2 Q_n - Z^2 P_n,
$$

$$n = 0, 1, \ldots$$  \hspace{1cm} (22)

from the initial values $Q_{-1} = 1/Z$, $P_0 = Y + 2$ and $Q_0 = Q = Y Z + 3 Z - 3$ generating $R = P_1$ etc.

In a way paralleling the previous $N = 3$ case, we might slightly modify the functions and define $P_{n+1} = \sqrt{Y} W_{2n+1}$ while $Q_{n+1} = W_{2n+2}$. It is easy to verify that we can now use just the single common recurrence

$$W_{n+1} = \sqrt{Y} Z W_n + Z^2 W_{n-1} - \sqrt{Y} Z^2 W_{n-2},
$$

$$n = 0, 1, \ldots$$  \hspace{1cm} (23)

with the merely slightly modified initialization by $W_{-2} = Q_{-1} = 1/Z$, $W_{-1} = P_0/\sqrt{Y} = \sqrt{Y} + 2/\sqrt{Y}$ and $W_0 = Q_0 = Q = Y Z + 3 Z - 3$. We may see the clear parallels with the previous $N = 3$ case, noticing that the polynomials $W_{3n}$ and $W_{3n-1}$ are both divisible by $Z^{2n}$ while $W_{3n-2}$ is only divisible by $Z^{2n-1}$. We shall skip the further technical analyses of this sort here.
4 Matching and secular polynomials at $N = 3$

For the sake of simplicity, let us only pay attention to the choice of $N = 3$. Then, the knowledge of the closed form of the polynomials $\Sigma_k(\xi)$ and $\sigma_{k+1}(\xi)$ enables us to define the explicit values of all our coupling constants as functions of the mere two parameters $a$ and $\tilde{a}$ entering $\xi = a \tilde{a}$ and $\rho = a / \tilde{a}$,

$$a_1 = a = a \rho^0 \sigma_1(\xi), \quad A_1 = A = \rho \Sigma_1(\xi), \quad a_2 = b = a \rho \sigma_2(\xi), \quad A_2 = B = \rho^2 \Sigma_2(\xi), \quad a_3 = c = a \rho^2 \sigma_3(\xi), \quad \ldots .$$

One could also have constructed this general solution of our recurrences (10) in an opposite, upward direction. For this purpose, it suffices when all the above formulae are modified by a consequent application of the tilding operation.

We have seen that the $N = 3$ case operates with two unknowns. At the same time, the set of recurrences (10) contains precisely two redundant items. In one extreme example we may read whole this set as a sequence of definitions of $\beta = s_2 = s_2(s_1, s_q)$, $\gamma = s_3 = s_3(s_1, s_q)$, $\ldots$, $s_{q+1} = s_{q+1}(s_1, s_q)$ where the last two lines are redundant since we already knew the outcome, viz., $s_{q+1} = 2$ and $s_q(s_1, s_q) = s_q$. This may be understood as a source of our final pair of boundary conditions determining the QE-compatible values of the pair of the unknown parameters.

In a way paralleling the previous $N = 2$ example, any other two lines of eq. (10) might be selected as boundary conditions. In contrast to the $N = 2$ example, almost all of the non-extreme choices of matching conditions would be preferable in practice, lowering the degree of the resulting secular polynomials.

This observation deserves to be explained in more detail. Indeed, it makes sense to distinguish between the four possible selections of the optimal matching conditions.

4.1 $q = 4K$

Whenever $q = 4K$ where $K = 1, 2, \ldots$, the above-mentioned recurrent construction may be started, simultaneously, at both the upper and lower end of eq. (10). Without any difficulties and using eq. (24), the recipe defines all the unknown quantities, i.e., the doublets of pairs of the values

$$(a_j, A_j), \quad (\tilde{a}_j, \tilde{A}_j), \quad j = 1, 2, \ldots, K .$$

The two middle lines of eq. (10) define the other two redundant functions (or “non-existent couplings”) $a_{K+1}$ and $\tilde{a}_{K+1}$. This induces no real difficulty since the two parameters $a_1 = a$ and $\tilde{a}_1 = \tilde{a}$ are not yet specified. The latter two definitions are not redundant, therefore, as they have to fix these initial values.

The inspection of eq. (10) reveals that our symbol $a_{K+1}$ is an alternative name for another and well defined coupling $\tilde{A}_K$. Similarly, the quantity $\tilde{a}_{K+1}$ is an “alias” for $A_K$. We determine the missing QE roots $a$ and $\tilde{a}$ via the two redundant equations $a_{K+1} = \tilde{A}_K$ and $\tilde{a}_{K+1} = A_K$ or, in the notation of eq. (24),

$$a \sigma_{K+1} \rho^K = \Sigma_K \tilde{\rho}^K, \quad \tilde{a} \sigma_{K+1} \tilde{\rho}^K = \Sigma_K \rho^K .$$
Their ratio reads $\rho^{4K+1} = 1$ and gives the unique real root $\rho = 1$. Our first conclusion is that we must have $a = \tilde{a}$. The above two equations coincide and any of them represents our ultimate matching condition or constraint imposed upon $\xi = a^2$,

$$a \sigma_{K+1}(a^2) = \Sigma_K(a^2), \quad q = 4K.$$  \hspace{1cm} (26)

A sample of its roots may be found in Table 1 at $q = 4$ and $q = 8$. The inspection of the subsequent Table 2 reveals that with the further growth of $K$, the determination of these roots becomes purely numerical very quickly.

4.2 $q = 4K + 2$

After a move to $q = 4K + 2$ with $K = (0,) 1, 2, \ldots$, the previous recipe does not change too much. This time we define all the unknowns in a reversed order,

$$(A_j, a_{j+1}), \quad (\tilde{A}_j, \tilde{a}_{j+1}), \quad j = 1, 2, \ldots, K.$$  \hspace{1cm} (27)

*Mutatis mutandis* we find that the central part of eq. (10) defines the other “non-existent” couplings $A_{K+1}$ and $\tilde{A}_{K+1}$ so that the doublet of equations $A_{K+1} = \tilde{a}_{K+1}$ and $\tilde{A}_{K+1} = a_{K+1}$ leads to another set of the selfconsistency conditions,

$$a \sigma_{K+1} \rho^K = \Sigma_{K+1} \rho^{K+1}, \quad \tilde{a} \sigma_{K+1} \rho^K = \Sigma_{K+1} \rho^{K+1}.$$  \hspace{1cm} (28)

Their ratio degenerates to the modified constraint $\rho^{4K+3} = 1$ with the same unique real root as above, $a/\tilde{a} = \rho = 1$. Both our innovated identities coincide,

$$a \sigma_{K+1}(a^2) = \Sigma_{K+1}(a^2), \quad q = 4K + 2$$  \hspace{1cm} (29)

and guarantee the desired matching. Their numerical aspects are sampled again in Table 1 (easily solvable cases at $K = 0, 1$ and 2). The adjacent Table 2 complements this list and facilitates the determination of the explicit form of the secular equation (28) at all the integers $K$.

4.3 $q = 4K + 1$

The subset of odd $q = 4K + 1$ with $K = (0,) 1, 2, \ldots$ requires a more careful analysis. Although we have the same complete list (27) of the definitions of the QE-fixed couplings as above, its last two items are defined twice, in two different ways. Their necessary compatibility represented by the relation $a_{K+1} = \tilde{a}_{K+1}$ or rather

$$a \sigma_{K+1} \rho^K = \tilde{a} \sigma_{K+1} \rho^K$$

implies that $\rho^{2K+1} = 1$ so that we must put $a/\tilde{a} = \rho = 1$. In the light of this conclusion, the other two consequences $A_{K+1} = \tilde{A}_{K}$ and $\tilde{A}_{K+1} = A_{K}$ of the two other next-to-central rows of eq. (10) coincide and give the same ultimate matching rule

$$\Sigma_{K+1}(\xi) = \Sigma_K(\xi), \quad q = 4K + 1.$$  \hspace{1cm} (29)
Its numerical performance appears illustrated by the corresponding subset of roots in Table 3.

Marginally, let us note that for the specific exponents \( q = 4K + 1 \), the secular polynomial may be re-written in the compact form

\[
R^{(K,-)}(\xi) = (\xi - 4) \left[ \left( \frac{2K}{0} \right) \xi^K - \left( \frac{2K - 1}{1} \right) \xi^{K-1} + \left( \frac{2K - 2}{2} \right) \xi^{K-2} + \ldots \right]
\]

\[
\ldots (-1)^{K+2} \left( \frac{K + 2}{K - 2} \right) \xi^2 + (-1)^{K+1} \left( \frac{K + 1}{K - 1} \right) \xi + (-1)^K \left( \frac{K}{K} \right) \right]. \tag{30}
\]

The secular polynomials \( \sum_{m=0}^{K} \xi^m d_m^{[K]} \) contain the \( (K + 1) \)-plets of coefficients \( K^{(K)} = (d_K^{[K]}, d_{K-1}^{[K]}, \ldots, d_0^{[K]}) \) such that \( K^{(0)} = (1), K^{(1)} = (1, -1), K^{(2)} = (1, -3, 1), K^{(3)} = (1, -5, 6, 1) \), etc. This rule parallels the even \(-q\) recipe of Table 2.

### 4.4 \( q = 4K + 3 \)

The last possible choice of the odd exponents \( q = 4K + 3 \) (with \( K = 0, 1, 2, \ldots \)) in the potentials \( V_{(q,n)}(r) \) of eq. 2 leads to a routine completion of all the above analysis. A marginal modification of the list (25) is needed to specify all the necessary QE couplings, recurrently determined as functions of \( a \) and \( \tilde{a} \) only,

\[
(a_j, A_j), \quad (\tilde{a}_j, \tilde{A}_j), \quad j = 1, 2, \ldots, K + 1. \tag{31}
\]

Nevertheless, the results of the matching become slightly different this time. Although the first, central-line rule \( A_{K+1} = \tilde{A}_{K+1} \) prescribes merely

\[
\Sigma_{K+1} \rho^{K+1} = \Sigma_{K+1} \tilde{\rho}^{K+1}
\]

its consequence \( \rho^{2K+2} = 1 \) admits the two alternative signs in the resulting \( a/\tilde{a} = \rho = \pm1 \). Under this condition, the other two equations (in detail, \( a_{K+2} = \tilde{a}_{K+1} \) and its tilding-conjugate \( \tilde{a}_{K+2} = a_{K+1} \)) coincide as well, giving the same final condition

\[
\sigma_{K+2}(\xi) = \sigma_{K+1}(\xi), \quad q = 4K + 3. \tag{32}
\]

Curiously enough, this equation is the most easily solvable implicit definition of the QE roots \( \xi = a^2 = \tilde{a}^2 \) (cf. Table 4).

Even the shortest glimpse at the results of the factorization of the effective secular polynomial \( R^{(K,+)}(\xi) \) reveals that the sequence of the exponents \( q = 4K + 3 \) might be viewed as the most privileged one. The search for its QE roots becomes by far the easiest. After we omit the roots \( \xi = 0 \) and \( \xi = 4 \) as trivial, we encounter another unexpected and purely empirically observed symmetry. Indeed, the secular roots \( a_1 = \sqrt{\xi_{\pm n}} = \sqrt{2 \pm \Xi_n^{[K]}} \) listed in Table 4 at the indices \( q = 4K + 3 \) appear to be of the very similar form. It is really instructive to list a few sample distances \( \Xi_n^{[K]} \) of \( a_1^2 \) from their median = 2. We have \( \Xi_n^{[1]} = 0, \Xi_n^{[2]} = 1, \Xi_n^{[3]} = 0 \) or \( \sqrt{2} \), two values of \( \Xi_n^{[4]} = (\sqrt{5} \pm 1)/2 \), the three values of \( \Xi_n^{[5]} = 0, 1 \) and \( \sqrt{3} \). One
may see that the full set of the secular roots \( a_1 = \sqrt{\xi} \) exhibits a weird regularity manifested by a reflection symmetry with respect to the center at \( \xi_c = 2 \). All roots become tractable as certain quasi-conjugate pairs \( \xi = \xi_{\pm n} = 2 \pm \Xi^{[K]} \). In this sense, the results listed in Table 4 may be tentatively extrapolated to all the values of \( q \). Indeed, once we omit the permanent pair of the minimal and maximal QE-compatibility roots \( s = s_1 = \pm 2 \) as a trivial, we may use the auxiliary variable \( \Xi^{[K]} \) as defined by the relation \( s_1 = \sqrt{2} \pm \Xi^{[K]} \) at all the integers \( K \). The inspection of Table 4 then reveals that all the complete sets of all the QE-compatibility roots at all the listed integers \( q = 4K + 3 \) (i.e., at the set of \( K = 1, 2, 3, 4 \) and 5) coincide with the complete sets of roots of the Chebyshev polynomials of the second kind,

\[
U_K \left( \frac{2 - s_1^2}{2} \right) = 0, \quad K = 1, 2, \ldots, K_0
\]

with the confirmed \( K_0 = 5 \) at present. This means that in a way complementing the \( N > N_0 \) extrapolations performed in our paper PI we may now tentatively extrapolate also the result (33) and conjecture that this equation determines all the closed and exact solutions of our \( N = 3 \) QE Magyari-Schrödinger eq. (4) also at all the larger integers \( q = 4K + 3 > 4K_0 + 3 = 23 \).

**Acknowledgement**

Work supported by the grant Nr. A 1048302 of GA AS CR.
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- Table 1. Real QE roots $s = s_1$ at the first few even $q$ for $N = 3$.

- Table 2. Double Pascal triangle for coefficients in the reduced secular equations $\sum_{k=0}^{q/2} s_1^k c_k^{(q)} = 0$. This extrapolates Table 1 to all the even $q$’s.

- Table 3. Real QE roots $s = s_1$ at the first few odd $q$ for $N = 3$.

- Table 4. Real QE roots $s = s_1$ at the first few $q \equiv 3 \ (mod \ 4)$ for $N = 3$.

Tables

Table 1. Real QE roots $s = s_1$ at the first few even $q$ for $N = 3$.

| $q$  | roots $s = s_1$                                                                 |
|------|--------------------------------------------------------------------------------|
| 2    | $2 \quad -1$                                                                 |
| 4    | $2 \quad (\sqrt{5} - 1)/2 \quad -(\sqrt{5} + 1)/2$                            |
| 6    | $2 \quad \text{(plus all three roots of } a^3 + a^2 - 2a - 1)$                |
| 8    | $2 \quad -1 \quad \text{(plus all three roots of } a^3 - 3a + 1)$             |
| 10   | $2 \quad \text{(plus all five roots of } a^5 + a^4 - 4a^3 - 3a^2 + 3a + 1)$  |
Table 2. Double Pascal triangle for coefficients in the reduced secular equations \( \sum_{k=0}^{q/2} s_k c_k^{(q)} = 0 \). This extrapolates Table 1 to all the even \( q \)'s.

| \( k \) | coefficients \( c_k^{(q)} \) |
|-------|-------------------|
| \( q \) | 0 | 1 | 2 | 3 | 4 | 5 | ... |
| 2 | 1 | 1 |   |   |   |   |   |
| 4 | -1 | 1 | 1 |   |   |   |   |
| 6 | -1 | -2 | 1 | 1 |   |   |   |
| 8 | 1 | -2 | -3 | 1 | 1 |   |   |
| 10 | 1 | 3 | -3 | -4 | 1 | 1 |   |
| 12 | -1 | 3 | 6 | -4 | -5 | 1 |   |
| 14 | -1 | -4 | 6 | 10 | -5 | -6 |   |
| 16 | 1 | -4 | -10 | 10 | 15 | -6 |   |

Table 3. Real QE roots \( s = s_1 \) at the first few odd \( q \) for \( N = 3 \).

| \( q \) | roots \( s = s_1 \) |
|-------|-------------------|
| 1 | 2 | -2 |
| 3 | 2 | -2 |
| 5 | 2 | 1 | -1 | -2 |
| 7 | 2 | \( \sqrt{2} \) | -\( \sqrt{2} \) | -2 |
| 9 | 2 | \( \sqrt{3}\pm\sqrt{2}/2 \) | \( \sqrt{3}\pm\sqrt{2}/2 \) | -\( \sqrt{3}\pm\sqrt{2}/2 \) | -2 |
| 11 | 2 | \( \sqrt{3} \) | 1 | -1 | -\( \sqrt{3} \) | -2 |

Table 4. Real QE roots \( s = s_1 \) at the first few \( q \equiv 3 \ (mod \ 4) \) for \( N = 3 \).

| \( q \) | roots \( s = s_1 \) |
|-------|-------------------|
| 3 | \( \pm 2 \) |
| 7 | \( \pm 2 \) | \( \pm \sqrt{2} \) |
| 11 | \( \pm 2 \) | \( \pm \sqrt{2}\pm 1 \) |
| 15 | \( \pm 2 \) | \( \pm \sqrt{2} \) | \( \pm \sqrt{2}\pm \sqrt{2} \) |
| 19 | \( \pm 2 \) | \( \pm \sqrt{2}\pm (\sqrt{5}\pm 1)/2 \) |
| 23 | \( \pm 2 \) | \( \pm \sqrt{2} \) | \( \pm \sqrt{2}\pm \sqrt{2}\pm 1 \) |