On a Nonlocal Boundary Value Problem of a State-Dependent Differential Equation

Ahmed El-Sayed †, Eman Hamdallah *† and Hanaa Ebead †

Faculty of Science, Alexandria University, Alexandria 21500, Egypt; amasayed@alexu.edu.eg (A.E.-S.); HanaaRezqalla@alexu.edu.eg (H.E.)
* Correspondence: eman.hamdallah@alexu.edu.eg
† These authors contributed equally to this work.

Abstract: In this paper, the existence of absolutely continuous solutions and some properties will be studied for a nonlocal boundary value problem of a state-dependent differential equation. The infinite-point boundary condition and the Riemann–Stieltjes integral condition will also be considered. Some examples will be provided to illustrate our results.

Keywords: state-dependence; solutions; integral boundary condition; infinite-point boundary condition; examples

MSC: 34A12; 39B12; 47H09

1. Introduction

The delay differential equations serve as an important branch of nonlinear analysis that has many applications in most fields. Usually, the deviation of the arguments depends only on the time (see [1–6]); however, when the deviation of the arguments depends upon the state variable x and also the time t is incredibly important theoretically and practically, this type of equations is known as self-reference or state-dependent equations. Equations with state-dependent delays have gained great attention to specialists since they have many application models, like the two-body problem of classical electrodynamics, even have numerous applications within the class of problems that have past memories, as an example, in hereditary phenomena, see [7,8]. Several papers studied this kind of equations, (see [9–24]).

Eder [12], where the author studied the problem

$$x'(t) = x(x(t)), \quad t \in [a, b] \text{ and } x(0) = x_0.$$ 

The existence and the uniqueness of the solution of the problem

$$x'(t) = f(t, x(x(t))), \quad t \in [a, b]$$
$$x(0) = x_0$$

were studied by Buică [11].

In [16], the assumptions of [11], have been relaxed and generalized to the equation

$$x(t) = g(t, \int_0^t f(s, x(x(s))) \, ds), \quad t \in [0, T],$$

where f satisfies Carathéodory condition.
In [14,15], some other results have been obtained for the problem
\[
\frac{dx(t)}{dt} = h_1(t, x(h_2(t, x(t)))), \quad a.e. \ t \in (0, T],
\]
\[
x(0) = x_0.
\]

EL-Sayed and Ebead [17] studied the IVP of state-dependent hybrid functional differential equation
\[
\frac{d}{dt} x(t) - a(t) \frac{d}{dt} h_1(t, x(x(\varphi(t)))) = h_2(t, x(x(\varphi(t)))), \quad t \in [0, T]
\]
with the initial data
\[
x(0) = a(0).
\]

Our aim in this work is to study the m-point boundary value problem (BVP)
\[
\frac{dx(t)}{dt} = f(t, x(x(\varphi(t)))) \quad a.e. \ t \in (0, T]
\]
\[
\sum_{k=1}^{m} a_k x(t_k) = x_0, \quad \tau_k \in [0, T], \quad a_k > 0.
\]

The existence and the uniqueness have been proved for the BVP (1) and (2). Moreover, we show that the solution of our problem depends continuously on \(x_0\) and on the nonlocal data \(a_k\). Furthermore, we study (1) with the nonlocal integral condition
\[
\int_0^T x(s) \, dh(s) = x_0,
\]
where \(h : [0, T] \to [0, T]\) is an increasing function. Finally, we study (1) with the infinite point boundary condition
\[
\sum_{k=1}^{\infty} a_k x(\tau_k) = x_0,
\]
where \(\sum_{k=1}^{\infty} a_k\) is convergent.

2. Main Results

Consider the BVP (1) and (2) under the following hypothesis:

(i) \(f : [0, T] \times [0, T] \to \mathbb{R}^+\) satisfies Carathéodory condition.

(ii) There exist \(m : [0, T] \to \mathbb{R}^+\) bounded measurable function and a constant \(b > 0\) such that
\[
f(t, x) \leq m(t) + b x,
\]
where, \(m(t) \leq M\).

(iii) \(L T \leq A x_0 \leq T (1 - 2 L)\) and \(L = (M + b T) \in [0, \frac{1}{2}]\), where \(A = 1/\sum_{k=1}^{m} a_k\) and \(M\) is a positive constant such that \(|m(t)| \leq M\).

(iv) \(\varphi : [0, T] \to [0, T]\) is continuous and \(\varphi(t) \leq t\).

2.1. Integral Representation

Lemma 1. The BVP (1) and (2) and the integral equation
\[
x(t) = A [x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f(s, x(x(\varphi(s)))) \, ds] + \int_0^t f(s, x(x(\varphi(s)))) \, ds, \quad t \in [0, T]
\]
are equivalent.
Let the hypothesis (i)–(iv) be held, then (1) and (2) has a solution $x$.

**Proof.** Integrating (1), we obtain

$$x(t) = x(0) + \int_0^t f(s, x(x(\phi(s)))) ds$$

and

$$\sum_{k=1}^m a_k x(\tau_k) = x(0) \sum_{k=1}^m a_k + \sum_{k=1}^m a_k \int_0^{\tau_k} f(s, x(x(\phi(s)))) ds,$$

then we can get

$$x(0) = A[x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} f(s, x(x(\phi(s)))) ds]$$

and

$$x(t) = A[x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} f(s, x(x(\phi(s)))) ds] + \int_0^t f(s, x(x(\phi(s)))) ds.$$

Conversely, differentiating (5), we obtain

$$\frac{dx(t)}{dt} = \frac{d}{dt} \int_0^t f(s, x(x(\phi(s)))) ds = f(t, x(x(\phi(t)))) ds \text{ a.e. } t \in [0, T].$$

Furthermore, from (5), we can show that

$$\sum_{k=1}^m a_k x(\tau_k) = x_0.$$

Hence, the BVP (1) and (2) and the integral Equation (5) are equivalent. \(\square\)

### 2.2. Existence of Solution

Define the set $S_L$ by

$$S_L = \{ x \in C[0, T] : |x(t_2) - x(t_1)| \leq L|t_2 - t_1| \}$$

**Theorem 1.** Let the hypothesis (i)–(iv) be held, then (1) and (2) has a solution $x \in S_L \subset C[0, T]$.

**Proof.** Define the operator $F$ by

$$F_x(t) = A[x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} f(s, x(x(\phi(s)))) ds] + \int_0^t f(s, x(x(\phi(s)))) ds, \ t \in [0, T].$$

Let $x \in S_L$, then we have

$$|F_x(t)| = |A[x_0 - \sum_{k=1}^m a_k \int_0^{\tau_k} f(s, x(x(\phi(s)))) ds] + \int_0^t f(s, x(x(\phi(s)))) ds|$$

$$\leq Ax_0 + A \sum_{k=1}^m a_k \int_0^{\tau_k} |f(s, x(x(\phi(s))))| ds + \int_0^t |f(s, x(x(\phi(s))))| ds$$

$$\leq Ax_0 + A \sum_{k=1}^m a_k \int_0^{\tau_k} \{ m(s) + b|x(x(\phi(s)))| \} ds + \int_0^t \{ m(s) + b|x(x(\phi(s)))| \} ds$$

$$\leq Ax_0 + A \sum_{k=1}^m a_k (M + b T) \int_0^{\tau_k} ds + (M + b T) \int_0^t ds$$

$$\leq Ax_0 + 2LT \leq T.$$

Hence, $\{F_x\}$ is uniformly bounded.
Let \( x \in S_L \) and \( t_1, t_2 \in [0, T] \) with \( t_1 < t_2 \) such that \( |t_2 - t_1| < \delta \), then

\[
|F(x(t_2)) - F(x(t_1))| = |\int_{t_1}^{t_2} f(s, x(x(\phi(s)))) ds - \int_{t_1}^{t_1} f(s, x(x(\phi(s)))) ds| \\
= |\int_{t_1}^{t_2} f(s, x(x(\phi(s)))) ds| \\
\leq |\int_{t_1}^{t_2} |f(s, x(x(\phi(s))))| ds| \\
\leq |\int_{t_1}^{t_2} \{m(s) + b|x(x(\phi(s)))|\} ds| \\
\leq (M + b T)|t_2 - t_1| = L|t_2 - t_1|.
\]

This proves that \( F : S_L \rightarrow S_L \) and \( \{Fx\} \) are equi-continuous.

By Arzela–Ascoli Theorem ([25] p. 54), we find that \( F \) is compact.

Let \( \{x_n\} \subset S_L \) such that \( x_n \rightarrow x \) as \( n \rightarrow \infty \) on \([0, T]\) (i.e., \( |x_n(t) - x(t)| \leq \epsilon_1 \)). This implies that \( |x_n(x(\phi(t))) - x(x(\phi(t)))| \leq \epsilon_2 \) for arbitrary \( \epsilon_1, \epsilon_2 \geq 0 \), then

\[
|x_n(x(\phi(t))) - x(x(\phi(t)))| \\
\leq |x_n(x(\phi(t))) - x_n(x(\phi(t)))| + |x_n(x(\phi(t))) - x(x(\phi(t)))| \\
\leq L|x_n(x(\phi(t))) - x(x(\phi(t)))| + |x_n(x(\phi(t))) - x(x(\phi(t)))| \\
\leq \epsilon_1 + \epsilon_2 = \epsilon
\]

and

\[
x_n(x(\phi(t))) \rightarrow (x(x(\phi(t)))) \text{ in } S_L \text{ as } n \rightarrow \infty.
\]

Now the function \( f \) is continuous in the second argument, then

\[
f(t, x_n(x(\phi(t)))) \rightarrow f(t, x(x(\phi(t))))
\]

Using assumption \((ii)\) and Lebesgue dominated convergence theorem ([26] p. 151), we get

\[
\lim_{n \rightarrow \infty} \int_0^t f(s, x_n(x(\phi(s)))) ds = \int_0^t f(s, x(x(\phi(s)))) ds.
\]

Similarly,

\[
\lim_{n \rightarrow \infty} \int_0^{t_k} f(s, x_n(x(\phi(s)))) ds = \int_0^{t_k} f(s, x(x(\phi(s)))) ds.
\]

Now we have

\[
\lim_{n \rightarrow \infty} (Fx_n)(t) = A[x_0 - \sum_{k=1}^m a_k \int_0^{t_k} f(s, x_n(x(\phi(s)))) ds] + \int_0^t f(s, x_n(x(\phi(s)))) ds \\
= A[x_0 - \sum_{k=1}^m a_k \int_0^{t_k} f(s, x(x(\phi(s)))) ds] + \int_0^t f(s, x(x(\phi(s)))) ds \\
= (Fx)(t).
\]

This proves that \( F \) is continuous.

Applying Schauder Theorem [27], there is a solution \( x \in C[0, T] \) of (5). This means that the BVP (1) and (2) has at least one solution \( x \in AC[0, T] \).

2.3. Riemann-Stieltjes Integral

Let \( x \in AC[0, T] \) be a solution of BVP (1) and (2), then we can formulate the next theorem.
Theorem 2. Let the hypothesis (i)–(iv) hold. Let \( h : [0, T] \to [0, T] \) be an increasing function, then there is a solution \( x \in AC[0, T] \) of (1) with the Riemann–Stieltjes integral condition (3) and this solution given by

\[
x(t) = \frac{1}{h(T) - h(0)} \left[ x_0 - \int_0^T \int_0^t f(s, x(x(\phi(s)))) \, ds \, dh(t) \right] + \int_0^t f(s, x(x(\phi(s)))) \, ds,
\]

\( t \in [0, T] \).

Proof. Let \( a_k = h(t_k) - h(t_{k-1}) \), \( 0 = t_0 < t_1 < t_2 < \ldots < t_m = T \) and \( \tau_k \in (t_{k-1}, t_k) \), then the multi-point nonlocal condition (2) will be

\[
\sum_{k=1}^{m} x(\tau_k) \left( h(t_k) - h(t_{k-1}) \right) = x_0 \quad \text{and} \quad \sum_{k=1}^{m} a_k = h(T) - h(0).
\]

Hence,

\[
\lim_{m \to \infty} \sum_{k=1}^{m} a_k x(\tau_k) = \lim_{m \to \infty} \sum_{k=1}^{m} x(\tau_k) \left( h(t_k) - h(t_{k-1}) \right) = \int_0^T x(t) \, dh(t) = x_0
\]

and

\[
x(t) = \frac{1}{h(T) - h(0)} \left[ x_0 - \lim_{m \to \infty} \sum_{k=1}^{m} \left( h(t_k) - h(t_{k-1}) \right) \int_0^{\tau_k} f(s, x(x(\phi(s)))) \, ds \right]
\]

\[
+ \int_0^t f(s, x(x(\phi(s)))) \, ds
\]

\[
= \frac{1}{h(T) - h(0)} \left[ x_0 - \int_0^T \int_0^t f(s, x(x(\phi(s)))) \, ds \, dh(t) \right] + \int_0^t f(s, x(x(\phi(s)))) \, ds
\]

which is the solution of (1) with the Riemann–Stieltjes integral condition (3). This completes the proof. \( \square \)

2.4. Infinite-Point Boundary Condition

Let \( x \in AC[0, T] \) be a solution of the BVP (1) and (2). Then, we can formulate the next theorem.

Theorem 3. Let the assumptions (i)–(iv) be satisfied. Assume that the series:

\[
\sum_{k=1}^{\infty} a_k = \frac{1}{B}
\]

is convergent. Then, there is a solution \( x \in AC[0, T] \) of (1) and (4), and this solution is given by

\[
x(t) = B \left[ x_0 - \sum_{k=1}^{\infty} a_k \int_0^{\tau_k} f(s, x(x(\phi(s)))) \, ds \right] + \int_0^t f(s, x(x(\phi(s)))) \, ds.
\]

Proof. Assume that \( x \in AC[0, T] \) be a solution of the BVP (1) and (2), thus we have

\[
|a_k x(\tau_k)| \leq a_k \|x\|
\]

and

\[
|a_k \int_0^{\tau_k} f(s, x(x(\phi(s)))) \, ds| \leq a_k \int_0^{\tau_k} |f(s, x(x(\phi(s))))| \, ds
\]

\[
\leq a_k \int_0^{\tau_k} (M + b T) \, ds \leq a_k LT.
\]
Using the comparison test, we deduce that the series
\[
\sum_{k=1}^{\infty} a_k x(\tau_k) \quad \text{and} \quad \sum_{k=1}^{\infty} a_k \int_0^{\tau_k} f(s, x(\phi(s))) ds
\]
are convergent. Then as \( m \to \infty \) in (5), we get
\[
x(t) = [x_0 - \sum_{k=1}^{\infty} a_k \int_0^{\tau_k} f(s, x(\phi(s))) ds] B + \int_0^t f(s, x(\phi(s))) ds.
\]
Furthermore, we have
\[
\sum_{k=1}^{\infty} a_k x(\tau_k) = B^{-1} B x_0 - B^{-1} B \sum_{k=1}^{\infty} a_k \int_0^{\tau_k} f(s, x(\phi(s))) ds
\]
\[
+ \sum_{k=1}^{\infty} a_k \int_0^{\tau_k} f(s, x(\phi(s))) ds = x_0.
\]
This proves that the solution of (8) satisfies (1) under infinite-point boundary condition (4). This completes the proof. \( \square \)

3. Uniqueness of the Solution

Here we prove the uniqueness of the solution of the BVP (1) and (2). Assume that
\[
(1') \quad |f(t, u) - f(t, v)| \leq b |u - v|.
(2') \quad |f(t, 0)| \leq M.
\]

**Theorem 4.** Let the hypothesis (1'), (2'), (iii), and (iv) be held. If \( 2b (L + 1) T < 1 \), then the solution \( x \in AC[0, T] \) of the BVP (1) and (2) is unique.

**Proof.** By putting \( y = 0 \) in (1') and using (2'), we obtain
\[
|f(t, x)| \leq b |x| + |f(t, 0)| \leq b |x| + M,
\]
thus we deduce that all assumptions of Theorem 1 are satisfied. Then the BVP (1) and (2) has a solution.

Now, let \( x, y \) be two solutions of (1) and (2), then
\[
|x(t) - y(t)| = |A[x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f(s, x(\phi(s))) ds] + \int_0^t f(s, x(\phi(s))) ds
\]
\[
- A[x_0 - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f(s, y(\phi(s))) ds] - \int_0^t f(s, y(\phi(s))) ds|
\]
\[
\leq A[\sum_{k=1}^{m} a_k \int_0^{\tau_k} f(s, x(\phi(s))) ds - \sum_{k=1}^{m} a_k \int_0^{\tau_k} f(s, y(\phi(s))) ds]
\]
\[
+ \int_0^t |f(s, x(\phi(s))) - f(s, y(\phi(s)))| ds
\]
Let the hypothesis of Theorem 4 be held. Then the solution of BVP (1) and (2) depends continuously on $x_0$ and $\tau_k$ such that Definition 1. The solution of (1) and (2) depends continuously on $x_0$ if $\forall \epsilon_1 > 0$, $\exists \delta_1(\epsilon_1) > 0$ such that $|x_0 - x^*_0| \leq \delta_1 \Rightarrow \|x - x^*\| \leq \epsilon_1$, $x^*$ is the unique solution of the BVP

\[
\frac{dx^*_t(t)}{dt} = f(t, x^*(\phi(t))) \quad \text{a.e. } t \in (0, T)
\]  

(10)

and

\[
\sum_{k=1}^{m} a_k x^*(\tau_k) = x^*_0, \quad \tau_k \in (0, T], \quad a_k > 0.
\]  

(11)

Theorem 5. Let the hypothesis of Theorem 4 be held. Then the solution of BVP (1) and (2) depends continuously on $x_0$. 

\[
\leq A \sum_{k=1}^{m} a_k \int_{0}^{\tau_k} |f(s, x(x(\phi(s)))) - f(s, y(\phi(s))))| ds
\]

\[
+ \int_{0}^{1} |f(s, x(x(\phi(s)))) - f(s, y(\phi(s))))| ds
\]

\[
\leq A \sum_{k=1}^{m} a_k b \int_{0}^{\tau_k} |x(x(\phi(s))) - y(\phi(s)))| ds
\]

\[
+ b \int_{0}^{1} |x(x(\phi(s))) - y(\phi(s)))| ds.
\]

However,

\[
|x(x(\phi(s))) - y(\phi(s)))| = |x(x(\phi(s))) - y(\phi(s))) + x(y(\phi(s))) - x(\phi(s)))|
\]

\[
\leq L|x(\phi(s))) - y(\phi(s)))| + |x(y(\phi(s))) - y(\phi(s)))|
\]

\[
\leq L\|x - y\| + \|x - y\| = (L + 1)\|x - y\|.
\]  

(9)

Using (9), we get

\[
|x(t) - y(t)| \leq A \sum_{k=1}^{m} a_k b (L + 1)\|x - y\| \int_{0}^{\tau_k} ds + b(L + 1)\|x - y\| \int_{0}^{1} ds
\]

\[
\leq b (L + 1)\|x - y\| A \sum_{k=1}^{m} a_k \tau_k + b(L + 1)\|x - y\| t
\]

\[
\leq b (L + 1) T \left( A \sum_{k=1}^{m} a_k + 1 \right) \|x - y\|
\]

thus we have

\[
\|x - y\| [1 - 2b (L + 1) T] \leq 0.
\]

Since $2b (L + 1) T < 1$, then we get $x = y$ and the solution of the BVP (1) and (2) is unique.  

4. Continuous Dependence

Definition 1. The solution of (1) and (2) depends continuously on $x_0$ if $\forall \epsilon_1 > 0$, $\exists \delta_1(\epsilon_1) > 0$ such that $|x_0 - x^*_0| \leq \delta_1 \Rightarrow \|x - x^*\| \leq \epsilon_1$, $x^*$ is the unique solution of the BVP
Theorem 6. Let the hypothesis of Theorem 4 be hold, then the solution of BVP (1) and (2) depends continuously on the nonlocal data $a_k$,

Proof. Consider the two solutions $x$, $x^\ast$ of (1) and (2) and (10) and (11), respectively, thus we have

$$
|x(t) - x^\ast(t)| = |A[x_o - \sum_{k=1}^{m} a_k \int_0^{T_k} f(s, x(x(\phi(s)))) ds] + \int_0^t f(s, x(x(\phi(s)))) ds |
$$

$$
- A[x_o^\ast - \sum_{k=1}^{m} a_k \int_0^{T_k} f(s, x^\ast(x^\ast(\phi(s)))) ds] - \int_0^t f(s, x^\ast(x^\ast(\phi(s)))) ds |
$$

$$
= |A(x_o - x_o^\ast) + A \sum_{k=1}^{m} a_k \int_0^{T_k} (f(s, x^\ast(x^\ast(\phi(s)))) - f(s, x(x(\phi(s)))) ds |
$$

$$
+ \int_0^t |f(s, x(x(\phi(s)))) - f(s, x^\ast(x^\ast(\phi(s))))| ds |
$$

$$
\leq A|x_o - x_o^\ast| + A \sum_{k=1}^{m} a_k \int_0^{T_k} |f(s, x(x(\phi(s)))) - f(s, x^\ast(x^\ast(\phi(s))))| ds |
$$

$$
+ \int_0^t |f(s, x(x(\phi(s)))) - f(s, x^\ast(x^\ast(\phi(s))))| ds |
$$

$$
\leq A \delta_1 + b A \sum_{k=1}^{m} a_k \int_0^{T_k} |x(x(\phi(s))) - x^\ast(x^\ast(\phi(s))))| ds |
$$

$$
+ b \int_0^t |x(x(\phi(s))) - x^\ast(x^\ast(\phi(s))))| ds |
$$

Using (9), we get

$$
|x(t) - x^\ast(t)| \leq A \delta_1 + b A \sum_{k=1}^{m} a_k (L + 1)||x - x^\ast|| \int_0^{T_k} ds + b (L + 1)||x - x^\ast|| \int_0^t ds |
$$

$$
\leq A \delta_1 + 2b T (L + 1)||x - x^\ast|| |
$$

thus we have

$$
||x - x^\ast||(1 - 2b T (L + 1)) \leq A \delta_1 |
$$

$$
||x - x^\ast|| \leq \frac{A \delta_1}{1 - 2b T (L + 1)} = \epsilon_1 |
$$

Since $2b T (L + 1) < 1$, then the result follows. \qed

Definition 2. The solution of the BVP (1) and (2) depends continuously on the nonlocal data $a_k$ if \forall $\epsilon_2 > 0$, \exists $\delta_2(\epsilon_2) > 0$ such that

$$
|a_k - a_k^\ast| \leq \delta_2 \Rightarrow ||x - x^\ast|| \leq \epsilon_2 |
$$

$x^\ast$ is the unique solution of the BVP

$$
\frac{dx^\ast(t)}{dt} = f(t, x^\ast(x^\ast(\phi(t)))) \text{ a.e. } t \in (0, T) |
$$

and

$$
\sum_{k=1}^{m} a_k^\ast x^\ast(\tau_k) = x_o, \text{ } \tau_k \in (0, T], \text{ } a_k^\ast > 0 |
$$

Theorem 6. Let the hypothesis of Theorem 4 be hold, then the solution of BVP (1) and (2) depends continuously on $a_k$. 
Proof. Consider the two solutions $x$, $x^*$ of BVP (1) and (2) and (12) and (13), respectively, thus we have
\[
|x(t) - x^*(t)| = |A[x_0 - \sum_{k=1}^{m} a_k \int_{0}^{t} f(s, x(s \phi(s))) ds + \int_{0}^{t} f(s, x(s \phi(s))) ds - A^*[x_0 - \sum_{k=1}^{m} a_k^* \int_{0}^{t} f(s, x^*(s \phi(s))) ds - \int_{0}^{t} f(s, x^*(s \phi(s))) ds]| \\
\leq x_0 |A - A^*| + \int_{0}^{t} |f(s, x(s \phi(s))) - f(s, x^*(s \phi(s)))| ds
\]
\[
+ \sum_{k=1}^{m} a_k |f(s, x(s \phi(s))) ds - A^* \sum_{k=1}^{m} a_k^* \int_{0}^{t} f(s, x^*(s \phi(s))) ds| \\
+ \sum_{k=1}^{m} a_k^* \int_{0}^{t} f(s, x^*(s \phi(s))) ds - A^* \sum_{k=1}^{m} a_k^* \int_{0}^{t} f(s, x^*(s \phi(s))) ds| \\
\leq x_0 A^* \delta m + b(L + 1) T \|x - x^*\| \\
+ |A - A^*| | \sum_{k=1}^{m} a_k^* \int_{0}^{t} f(s, x^*(s \phi(s))) ds| \\
+ \sum_{k=1}^{m} a_k |f(s, x(s \phi(s))) ds - \sum_{k=1}^{m} a_k^* \int_{0}^{t} f(s, x^*(s \phi(s))) ds| \\
+ \sum_{k=1}^{m} a_k^* \int_{0}^{t} f(s, x^*(s \phi(s))) ds - \sum_{k=1}^{m} a_k^* \int_{0}^{t} f(s, x^*(s \phi(s))) ds| \\
\leq x_0 A^* \delta m + b(L + 1) T \|x - x^*\| \\
+ A A^* \delta m \sum_{k=1}^{m} a_k^* \int_{0}^{t} |f(s, x^*(s \phi(s)))| ds \\
+ \sum_{k=1}^{m} a_k^* \int_{0}^{t} |f(s, x^*(s \phi(s)))| ds - \sum_{k=1}^{m} a_k^* \int_{0}^{t} f(s, x^*(s \phi(s))) ds| \\
\leq x_0 A^* \delta m + b(L + 1) T \|x - x^*\| \\
+ A A^* \delta m \sum_{k=1}^{m} a_k^* \int_{0}^{t} (M + b|x^*(s \phi(s)))| ds \\
+ A b \sum_{k=1}^{m} a_k^* \int_{0}^{t} |x(s \phi(s)) - x^*(s \phi(s))| ds \\
\leq x_0 A^* \delta m + 2b(L + 1) T \|x - x^*\| + 2A \delta m L T.
\]

Then, we obtain
\[
\|x - x^*\| (1 - 2b T (L + 1)) \leq x_0 A^* \delta m + 2A \delta m L T
\]
and
\[
\|x - x^*\| \leq \frac{x_0 A^* m + 2A m L T}{1 - 2b T (L + 1)} \delta = \varepsilon_2.
\]

Since $2b T (L + 1) < 1$, then the solution of the BVP (1) and (2) depends continuously on $a_k$. \qed
5. Examples

**Example 1.** Consider the nonlinear self-reference differential equation

\[
\frac{dx(t)}{dt} = \frac{1}{14} e^{-t^2} \cos^2 2t + \frac{\ln(1 + |x(x(\beta \, t))|)}{15 - t} \text{ a.e. } t \in (0, 3)
\]  

with the infinite point nonlocal boundary condition

\[
\sum_{k=1}^{\infty} \frac{1}{k^2} x\left(\frac{3k - 1}{k^2 + 1}\right) = 1.7,
\]

where \( \beta \in (0, 1) \), here we have \( \phi(t) = \beta \, t \), \( A = \frac{1}{\sum_{k=1}^{\infty} \frac{1}{k^2} x_0} = \frac{6}{\pi^2} \) and \( x_0 = 1.7 \).

It is clear that series \( \sum_{k=1}^{\infty} \frac{1}{k^2} \) is convergent. Now set

\[
f(t, x(x(\phi(t)))) = \frac{1}{14} e^{-t^2} \cos^2 2t + \frac{\ln(1 + |x(x(\beta \, t))|)}{15 - t}.
\]

Then

\[
f(t, x(x(\phi(t)))) \leq \frac{1}{14} e^{-t^2} \cos^2 2t + \frac{1}{12} x(x(\beta \, t)),
\]

thus we have

\[
m(t) = \frac{1}{14} e^{-t^2} \cos 2t,
\]

hence \( M = \frac{1}{14}, \quad b = \frac{1}{12}, \quad L = \frac{9}{28} \in [0, 1/3]. \)

Furthermore, we have \( \frac{27}{28} = L T < A \, x_0 \approx 1.0335 < T (1 - 2L) = \frac{15}{14} \).

Therefore, from Theorem 3, the BVP (14) and (15) has at least one solution \( x \in [0, 3] \).

**Example 2.** Consider the nonlinear self-reference differential equation

\[
\frac{dx(t)}{dt} = \frac{1}{16} t^3 e^{-t^2} \cos^2 (3(t + 1))
\]

\[
+ \frac{1}{9} \left( \frac{(x(x(t^4)))^2}{1 + x(x(t^4))} + e^{-t} \frac{1}{3} x(x(3(t + 1))) \right) \text{ a.e. } t \in (0, 1)
\]

with the infinite point nonlocal boundary condition

\[
\sum_{k=1}^{\infty} \left( \frac{2}{3} \right)^k x\left(\frac{2k - 1}{k^2}\right) = 1.
\]

Here, we have \( \phi(t) = t^4 \), \( A = \frac{1}{\sum_{k=1}^{\infty} \left( \frac{2}{3} \right)^k x_0} = \frac{1}{2} \) and \( x_0 = 1.7 \).

The series \( \sum_{k=1}^{\infty} \left( \frac{2}{3} \right)^k \) is convergent. Now set

\[
f(t, x(x(\phi(t)))) = \frac{1}{16} t^3 e^{-t^2} \cos^2 (3(t + 1))
\]

\[
+ \frac{1}{9} \left( \frac{(x(x(t^4)))^2}{1 + x(x(t^4))} + e^{-t} \frac{1}{3} x(x(3(t + 1))) \right).
\]

Then

\[
|f(t, x(x(\phi(t))))| \leq \frac{1}{16} t^3 e^{-t^2} \cos^2 (3(t + 1)) + \frac{4}{27} x(t^4),
\]

thus we have,

\[
m(t) = \frac{1}{16} t^3 e^{-t^2} \cos^2 (3(t + 1)),
\]

so we get \( M = \frac{1}{16}, \quad b = \frac{4}{27}, \quad L = \frac{91}{132} \in [0, 1/3]. \)
Hence, \( \frac{91}{342} = L \ T < A \ x_0 = 0.5 < T \ (1 - 2 \ L) = \frac{125}{216} \).

Therefore, from Theorem 3, the BVP (16) and (17) has at least one solution \( x \in [0, 1] \).

Example 3. Consider the nonlinear self-reference differential equation

\[
\frac{dx(t)}{dt} = \frac{1}{9} \left( e^{-2t} x(x(t^2)) \right) + \frac{1}{17} \left[ e^{-2t} x(x(t^2)) \cos^2 (x(x(t^2))) + t^4 x(x(t^2)) \right] \quad \text{a.e. } t \in (0, 1) \tag{18}
\]

with the nonlocal integral condition

\[
\int_0^1 x(s)d(t) = \frac{1}{3}. \tag{19}
\]

Here, we have \( \phi(t) = t^2 \), the function \( h : [0, 1] \to [0, 1] \) such that \( h(t) = t \) is an increasing function, furthermore, we have \( A = 1 \) and \( x_o = \frac{1}{3} \).

Now set

\[
f(t, x(x(\phi(t)))) = \frac{1}{9} \left( e^{-2t} x(x(t^2)) \right) + \frac{1}{17} \left[ e^{-2t} x(x(t^2)) \cos^2 (x(x(t^2))) + t^4 x(x(t^2)) \right].
\]

Then

\[
f(t, x(x(\phi(t)))) \leq \frac{1}{9} \left( e^{-2t} \right) + \frac{2}{17} x(x(t^2)),
\]

thus we have

\[
m(t) = \frac{1}{9} \left( e^{-2t} \right)
\]

so we get \( M = \frac{1}{9}, b = \frac{2}{17}, L = \frac{35}{125} \in [0, 1/3]. \)

Furthermore, \( \frac{35}{125} = L \ T < A \ x_0 = \frac{1}{3} < T \ (1 - 2 \ L) = \frac{83}{125} \).

Therefore, from Theorem 2, the BVP (18) and (19) has at least one solution \( x \in [0, 1] \).

6. Conclusions

In this paper, we introduce a nonlocal boundary value problem with deviating argument depending on both the state variable \( x \) and the time \( t \); this case is of importance in theory and practice and also has many application models. Here we have proved, the existence of absolutely continuous solutions for the nonlocal problem (1)–(2). The sufficient conditions for the uniqueness have been given and the continuous dependence has been proved. Generalization for the boundary condition (2) to (3) and (4) has been proved. Some examples; to illustrate the obtained results; have been given. Moreover, we have generalized the results in [11,12,18].

Author Contributions: These authors contributed equally to this work. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The authors would like to thank the editor and the referees for their positive comments and useful suggestions which have improved this manuscript.

Conflicts of Interest: The authors declare no conflict of interest.
References

1. Baker, C.T.H.; Paul, C.A.H. Pitfalls in parameter estimation for delay differential equations. *SIAM J. Sci. Comp.* 1997, 18, 305–314. [CrossRef]
2. Driver, R.D. Existence Theory for a Delay Differential System. *Contrib. Diff. Eq.* 1963, 1, 317–336.
3. Driver, R.D. *Ordinary and Delay Differential Equations*; Applied Mathematical Sciences 20; Springer: New York, NY, USA, 1977.
4. Banasić, J.; Lecko, M.; EL-Sayed, W. Existence Theorems for Some Quadratic Integral Equations. *J. Math. Anal. Appl.* 1998, 222, 276–285. [CrossRef]
5. Tunc, C. Stability and boundedness of solutions of nonlinear differential equations of third-order with delay. *Differ. Equ. Control. Process.* 2007, 3, 1–13.
6. Tunc, C. On Asymptotic Stability of Solutions to Third Order Nonlinear Differential Equations with Retarded Argument. *Commun. Appl. Anal.* 2007, 11, 515–528.
7. Tuan, N.; Nguyen, L. On solutions of a system of hereditary and self-referred partial-differential equations. *Numer. Algorithms* 2010, 55, 101–113. [CrossRef]
8. Van Le, U.; Nguyen, L. Existence of solutions for systems of self-referred and hereditary differential equations. *Electron. J. Differ. Equ.* 2008, 51, 1–7.
9. Anh, P.; Lan, N.; Tuan, N. Solutions to systems of partial differential equations with weighted self-reference and heredity. *Electron. J. Differ. Equ.* 2012, 2012, 1–14.
10. Berinde, V. Existence and approximation of solutions of some first order iterative differential equations. *Miskolc Math. Notes* 2010, 11, 13–26. [CrossRef]
11. Buică, A. Existence and continuous dependence of solutions of some functional differential equations. *Semin. Fixed Point Theory* 1995, 3, 1–14.
12. Eder, E. The functional differential equation $x'(t) = x(x(t))$. *J. Differ. Equ.* 1984, 54, 390–400. [CrossRef]
13. El-Sayed, A.; Ahamed, R. Solvability of a boundary value problem of self-reference functional differential equation with infinite point and integral conditions. *J. Math. Computer Sci.* 2020, 55, 296–308. [CrossRef]
14. El-Sayed, A.; Ebead, H. On an initial value problem of delay-refereed differential Equation. *Int. J. Math. Trends Technol.* 2020, 66, 32–37.
15. El-Sayed, A.; Ebead, H. Positive solutions of an initial value problem of a delay-self-reference nonlinear differential equation. *Malaya J. Mat.* 2020, 8, 1001–1006.
16. El-Sayed, A.; Ebead, H. On the solvability of a self-reference functional and quadratic functional integral equations. *Filomat* 2020, 34, 129–141. [CrossRef]
17. El-Sayed, A.; Ebead, H. Existence of positive solutions for a state-dependent hybrid functional differential equation. *IAENGTran* 2020, 50, 883–889.
18. Féckan, M. On a certain type of functional differential equations. *Math. Slovaca* 1993, 43, 39–43.
19. Feldstein, A.; Neves, K.W. High order methods for state dependent delay differential equations with nonsmooth solutions. *SIAM J. Numer. Anal.* 1984, 21, 844–863. [CrossRef]
20. Gal, C. Nonlinear abstract differential equations with deviated argument. *J. Math. Anal. Appl.* 2007, 333, 971–983. [CrossRef]
21. Haloi, R.; Kumar, P.; Pandey, D. Sufficient conditions for the existence and uniqueness of solutions to impulsive fractional integro-differential equations with deviating arguments. *J. Fract. Calc. Appl.* 2014, 5, 73–84.
22. Lan, N.; Pascali, E. A two-point boundary value problem for a differential equation with self-reference. *Electron. J. Math. Anal. Appl.* 2018, 6, 25–30.
23. Letelier, J.; Kuboyama, T.; Yasuda, H.; Cárdenas, M.; Cornish-Bowden, A. A self-referential equation, $f(f) = f$, obtained by using the theory of (m; r) systems: Overview and applications. *Algerbr. Biol.* 2005, 2015, 115–126.
24. Yang, D.; Zhang, W. Solutions of equivariance for iterative differential equations. *Appl. Math. Lett.* 2004, 17, 759–765. [CrossRef]
25. Kolmogorov, A.; Fomin, S. *Elements of the Theory of Functions and Functional Analysis*; Courier Corporation: North Chelmsford, MA, USA, 1957; pp. 50–60.
26. Dunford, N.; Schwartz, J. *Linear Operators, (Part 1), General Theory*; New York Interscience: New York, NY, USA, 1957; pp. 151–157.
27. Kantorovich, L.; Akilov, G. *Functional Analysis*; Silcock, H.L., Translator; Pergamon Press: New York, NY, USA, 1982.