Abstract. Spherical symmetry arguments are used to produce a general device to convert identities and inequalities for the $p$th absolute moments of real-valued random variables into the corresponding identities and inequalities for the $p$th moments of the norms of random vectors in Hilbert spaces. Particular results include the following: (i) an expression of the $p$th moment of the norm of such a random vector $X$ in terms of the characteristic functional of $X$; (ii) an extension of a previously obtained von Bahr–Esseen-type inequality for real-valued random variables with the best possible constant factor to random vectors in Hilbert spaces, still with the best possible constant factor; (iii) an extension of a previously obtained inequality between measures of “contrast between populations” and “spread within populations” to random vectors in Hilbert spaces.

1. Introduction

In this note, we will use averaging with respect to spherically symmetric measures over $\mathbb{R}^d$ to extend certain probability identities and inequalities for real-valued random variables (r.v.’s) to random vectors in Hilbert spaces.

Let $\mu$ be a finite spherically symmetric measure over $\mathbb{R}^d$, so that $\mu(TB) = \mu(B)$ for all orthogonal transformations $T: \mathbb{R}^d \to \mathbb{R}^d$ and Borel sets $B \subseteq \mathbb{R}^d$. Let $g: \mathbb{R} \to \mathbb{R}$ be a Borel-measurable function such that $\int_{\mathbb{R}^d} \mu(dt) |g(t \cdot x)| < \infty$ for all $x \in \mathbb{R}^d$, where $t \cdot x$ denotes the dot product of vectors $t$ and $x$ in $\mathbb{R}^d$. Let $e_1 := (1, 0, \ldots, 0) \in \mathbb{R}^d$. Let $|\cdot|$ denote the Euclidean norm on $\mathbb{R}^d$, for all natural $d$.

Then, for any $x \in \mathbb{R}^d \setminus \{0\}$ and the unit vector $y := x/|x|$, 

$$H_{g, \mu}(x) := \int_{\mathbb{R}^d} \mu(dt) g(t \cdot x) = \int_{\mathbb{R}^d} \mu(dt) g(|x|t \cdot y) = \int_{\mathbb{R}^d} \mu(dt) g(|x|t \cdot e_1) =: h_{g, \mu}(|x|),$$

and also $H_{g, \mu}(0) = g(0) \int_{\mathbb{R}^d} \mu(dt) = h_{g, \mu}(0)$, so that the equality

$$H_{g, \mu}(x) = h_{g, \mu}(|x|)$$

holds for all $x \in \mathbb{R}^d$. It follows that the function $H_{g, \mu}$ is spherically symmetric.
In particular, for any real \( p > 0 \) and the function \( g_p : \mathbb{R} \to \mathbb{R} \) defined by the formula
\[
g_p(u) := |u|^p
\]
for real \( u \), identity (1.1) can be rewritten as
\[
|x|^p = \frac{1}{c_{p,\mu}} \int_{\mathbb{R}^d} \mu(dt)|t \cdot x|^p
\]
for \( x \in \mathbb{R}^d \), assuming that
\[
c_{p,\mu} := \int_{\mathbb{R}^d} \mu(dt)|t \cdot e_1|^p \in (0, \infty);
\]
the latter condition will hold if e.g. \( \mu \) is the uniform distribution on the unit sphere \( S_{d-1} \) in \( \mathbb{R}^d \).

It follows immediately that, for any random vector \( X \) in \( \mathbb{R}^d \),
\[
(1.2) \quad \mathbb{E} |X|^p = \frac{1}{c_{p,\mu}} \int_{\mathbb{R}^d} \mu(dt) \mathbb{E} |t \cdot X|^p.
\]
Thus, the \( p \)th moment of the norm \( |X| \) of the random vector \( X \) is expressed as a mixture of the \( p \)th absolute moments of the real valued r.v.’s \( t \cdot X \).

Another kind of integral representation of \( \mathbb{E} |X|^p \), for any random vector \( X \) in \( \mathbb{R}^d \) with \( \mathbb{E} |X|^p < \infty \), in terms of the distributions of the real-valued r.v.’s \( t \cdot X \) for \( t \in \mathbb{R}^d \) will be established in Theorem 2.1 in Section 2.

2. Moments of the norm of a random vector in \( \mathbb{R}^d \) via the characteristic functional

Let \( X \) be a random vector in \( \mathbb{R}^d \), with the characteristic functional (c.f.) \( f_X \), so that
\[
f_X(t) = \mathbb{E} e^{it \cdot X}
\]
for \( t \in \mathbb{R}^d \).

Take any real \( p > 0 \) which is not an even integer, and let
\[
m := \lfloor p/2 \rfloor.
\]
For nonnegative integers \( k \) and real \( z \), let
\[
(2.1) \quad c_k(z) := \cos z - \sum_{j=0}^{k} (-1)^j \frac{2j}{(2j)!}, \quad s_k(z) := \sin z - \sum_{j=0}^{k} (-1)^j \frac{z^{2j+1}}{(2j+1)!}.
\]
Noting that \( c'_k = -s_{k-1} \) and \( s'_k = c_{k-1} \) if \( k \geq 1 \), and \( c_k(0) = 0 = s_k(0) \), it is easy to check by induction that \((-1)^kc_k(z) \leq 0 \) and \((-1)^ks_k(z) \leq 0 \), again for all nonnegative integers \( k \) and real \( z \).

So, for any nonzero vector \( x \in \mathbb{R}^d \) and the unit vector \( y := x/|x| \), by the Tonelli theorem we have
\[
\int_{\mathbb{R}^d} \frac{dt}{|t|^{p+d}} c_m(t \cdot x) = \int_{\mathbb{R}^d} \frac{dt}{|t|^{p+d}} c_m(|x|t \cdot y)
= |x|^p \int_{\mathbb{R}^d} \frac{ds}{|s|^{p+d}} c_m(s \cdot y)
= |x|^p \Omega_d \int_{S_{d-1}} d\omega \int_0^\infty \frac{dr}{r^{p+d}} c_m(r\omega \cdot y),
\]
where

\[(2.2) \quad \Omega_d := \frac{2\pi^{(d+1)/2}}{\Gamma((d+1)/2)} \]

is the surface area of the unit sphere \(S_{d-1}\) in \(\mathbb{R}^d\) and \(\int_{S_{d-1}} d\omega \cdots\) is the integral with respect to the uniform distribution on \(S_{d-1}\). Using now the spherical symmetry of the uniform distribution on \(S_{d-1}\) and recalling the definition \(e_1 := (1, 0, \ldots, 0) \in S_{d-1}\), we see that for any nonzero vector \(x \in \mathbb{R}^d\)

\[
\int_{\mathbb{R}^d} \frac{dt}{|t|^{p+d}} c_m(t \cdot x) = C_{d,p}|x|^p,
\]

where

\[
C_{d,p} := \Omega_d \int_{S_{d-1}} d\omega \int_0^\infty \frac{dr}{r^{p+1}} c_m(r\omega \cdot e_1)
\]

\[(2.3) \quad = \Omega_d \int_{S_{d-1}} d\omega \int_0^\infty \frac{dr}{r^{p+1}} c_m(r|\omega \cdot e_1|)
\]

\[
= \Omega_d K_p \int_{S_{d-1}} d\omega |\omega \cdot e_1|^p
\]

and

\[
(2.4) \quad K_p := \int_0^\infty \frac{dz}{z^{p+1}} c_m(z) = -\frac{\pi}{2\Gamma(p+1)\sin(\pi p/2)};
\]

the equality in (2.3) holds because the function \(c_m\) is even, and the latter equality in (2.4) is the special case of formula (5) in [10] obtained by replacing \(x\) there by 1.

To evaluate the integral \(\int_{S_{d-1}} d\omega |\omega \cdot e_1|^p\), note that the distribution of \(\omega \cdot e_1\) coincides with that of \(G_1^2 + \cdots + G_d^2)^{1/2}\), where \(G_1, \ldots, G_d\) are independent standard normal random variables. Hence, \((\omega \cdot e_1)^2\) has the beta distribution with parameters \(1/2, (d-1)/2\). So,

\[
\int_{S_{d-1}} d\omega |\omega \cdot e_1|^p = \frac{\Gamma(d/2)}{\Gamma(1/2)\Gamma((d-1)/2)} \int_0^1 dz z^{p/2-1/2} (1 - z)^{(d-1)/2-1}
\]

\[
= \frac{\Gamma(d/2)\Gamma((p+1)/2)}{\sqrt{\pi}\Gamma((p+d)/2)}.
\]

It follows that

\[
(2.5) \quad C_{d,p} = -\frac{\pi^{d/2}}{\sin(\pi p/2)} \frac{\Gamma(d/2)\Gamma((p+1)/2)}{\Gamma((d+1)/2)\Gamma((p+1)\Gamma((p+d)/2))}
\]

and

\[
(2.6) \quad |x|^p = \frac{1}{C_{d,p}} \int_{\mathbb{R}^d} \frac{dt}{|t|^{p+d}} c_m(t \cdot x)
\]

for all nonzero vectors \(x \in \mathbb{R}^d\). Identity (2.6) also trivially holds if \(x\) is the zero vector, because \(c_m(0) = 0\).

Thus, using the Tonelli theorem again, we immediately obtain the following expression of the \(p\)th moment of the norm \(|X|\) of a random vector \(X\) in terms of the c.f. \(f_X\) of \(X\):
Theorem 2.1. For any real $p > 0$ which is not an even integer and any random vector $X$ in $\mathbb{R}^d$ with $E|X|^p < \infty$, 

$$E|X|^p = \frac{1}{C_{d,p}} \int_{\mathbb{R}^d} \frac{dt}{|t|^{p+d}} \left( \Re f_X(t) - \sum_{j=0}^{m} \frac{(-1)^j}{(2j)!} E(t \cdot X)^{2j} \right)$$

$$= \frac{1}{C_{d,p}} \int_{\mathbb{R}^d} \frac{dt}{|t|^{p+d}} \left( \Re f_X(t) - \sum_{j=0}^{m} \frac{(D_i^{(2j)} f_X)(0)}{(2j)!} \right),$$

where $C_{d,p}$ is as in [2,5] and $(D_i^{(2j)} f_X)(0) := \left. \frac{d^{2j} f_X(z)}{dz^{2j}} \right|_{z=0}$, the $(2j)$th “directional” derivative of $f_X$ at 0 along the vector $t$.

Formula (6) in [10] is the special, “one-dimensional” case of Theorem 2.1 corresponding to $d = 1$. (For other expressions of the absolute and positive-part moments of r.v.’s in terms of the corresponding characteristic functions, see [5, 8] and references there.)

Another special case of Theorem 2.1 is given by

Corollary 2.2. For any $p \in (0,2)$ and any random vector $X$ in $\mathbb{R}^d$, 

$$E|X|^p = \frac{1}{C_{d,p}} \int_{\mathbb{R}^d} \frac{dt}{|t|^{p+d}} \left( \Re f_X(t) - 1 \right).$$

(If $E|X|^p = \infty$, then the right-hand side of (2.7) is $\infty$ as well.)

3. An exact von Bahr–Esseen-type inequality in Hilbert spaces

Given any sequence $(S_j)_{j=1}^n$ of random vectors in a separable Hilbert space $H$ with the corresponding norm $\|\cdot\|$ on $H$, let $X_j := S_j - S_{j-1}$ denote the corresponding differences, for $j \in \mathbb{N}$, with the convention $S_0 := 0$, so that $X_1 = S_1$; here and in what follows, for any $m$ and $n$ in the set \{0, 1, \ldots, \infty\} we let $\overline{m, n}$ stand for the set of all integers $i$ such that $m \leq i \leq n$.

If $E\|X_j\| < \infty$ and $E(X_j|S_{j-1}) = 0$, let us say that the sequence $(S_j)_{j=1}^n$ is a $v$-martingale (where “v” stands for “virtual”); in such a case, let us also say that $(X_1, \ldots, X_n)$ is a $v$-martingale difference sequence, or simply that the $X_j$’s are $v$-martingale differences. Note that, for a general $v$-martingale difference sequence $(X_1, \ldots, X_n)$, $X_1$ may be any random vector in $H$; in particular, the expectation $E X_1$ of $X_1$ (if it exists) may or may not be 0. It is clear that any martingale $(S_j)_{j=1}^n$ is a $v$-martingale.

In the rest of this section, it will be always assumed that $(S_j)_{j=1}^n$ is a $v$-martingale in $H$.

Take any $p \in (1,2]$.

In the case when $H = \mathbb{R}$, the best possible constant $C_p$ in the von Bahr–Esseen-type inequality

$$E|S_n|^p \leq E|X_1|^p + C_p \sum_{j=2}^n E|X_j|^p$$

was obtained in [6] – see formula (1.11) there. Specifically, it was shown in [6, Proposition 1.8] that

$$C_p = \max_{x \in (0,1)} \ell(p, x),$$
where
\[ \ell(p, x) := (1 - x)^p - x^p + px^{p-1}. \]

It was also shown in [6] that \( C_p \) is continuously and strictly decreasing in \( p \in (1, 2] \) from \( C_{1+} = 2 \) to \( C_2 = 1 \).

Inequality (3.1) with 2 in place of \( C_p \) was obtained by von Bahr and Esseen [11].

In this section we will show that inequality (3.1), previously obtained in the case \( H = \mathbb{R} \), extends to arbitrary separable Hilbert spaces \( H \), with the same best possible constant \( C_p \):

**Theorem 3.1.**

(3.2) \[ E \| S_n \|^p \leq E \| X_1 \|^p + C_p \sum_{j=2}^{n} E \| X_j \|^p. \]

**Proof.** Consider first the special case when \( H = \mathbb{R}^d \), for some natural \( d \). Take any \( t \in \mathbb{R}^d \). Then the sequence \( (t \cdot S_j)_{j=1}^{n} \) is a v-martingale in \( \mathbb{R} \), because

(3.3) \[ E(t \cdot X_j | t \cdot S_{j-1}) = t \cdot E(X_j | t \cdot S_{j-1}) = t \cdot E((X_j | S_{j-1})|t \cdot S_{j-1}) = t \cdot E(0|t \cdot S_{j-1}) = 0 \]

for \( j \in \mathbb{Z}_{\geq 1} \). So, by (3.1),

\[ E |t \cdot S_n|^p \leq E |t \cdot X_1|^p + C_p \sum_{j=2}^{n} E |t \cdot X_j|^p. \]
Using now the Tonelli theorem and (1.2) (with $\mu$ being, say, the uniform distribution on the unit sphere $S_{d-1}$ in $\mathbb{R}^d$), we have

\begin{align*}
&c_{p,\mu} \mathbb{E} \|S_n\|^p = \mathbb{E} \int_{\mathbb{R}^d} \mu(dt) |t \cdot S_n|^p \\
&= \int_{\mathbb{R}^d} \mu(dt) \mathbb{E} |t \cdot S_n|^p \\
&\leq \int_{\mathbb{R}^d} \mu(dt) \left( \mathbb{E} |t \cdot X_1|^p + C_p \sum_{j=2}^{n} \mathbb{E} |t \cdot X_j|^p \right) \\
&= \int_{\mathbb{R}^d} \mu(dt) \mathbb{E} |t \cdot X_1|^p + C_p \sum_{j=2}^{n} \int_{\mathbb{R}^d} \mu(dt) \mathbb{E} |t \cdot X_j|^p \\
&= c_{p,\mu} \left( \mathbb{E} \|X_1\|^p + C_p \sum_{j=2}^{n} \mathbb{E} \|X_j\|^p \right).
\end{align*}

This proves (3.2) for $H = \mathbb{R}^d$ and therefore for any finite-dimensional Euclidean space.

It remains to consider the case when $H$ is infinite dimensional. Let $(e_1, e_2, \ldots)$ be any orthonormal basis of the separable Hilbert space $H$. For any natural $d$, let $H_d$ be the span of the set $\{e_1, \ldots, e_d\}$, so that $\dim H_d = d < \infty$. Let $P_d$ denote the orthogonal projector of $H$ onto $H_d$. Note that the sequence $(P_dS_j)_{j=1}^{n}$ is a $v$-martingale in $H_d$—cf. (3.3). Hence, by what has been proved for finite-dimensional Euclidean spaces,

\begin{equation}
\mathbb{E} \|P_dS_n\|^p \leq \mathbb{E} \|P_dX_1\|^p + C_p \sum_{j=2}^{n} \mathbb{E} \|P_dX_j\|^p.
\end{equation}

Note also that, for each $x \in H$, $\|P_dx\|$ is nondecreasing in $d$, and $\|P_dx\| \to \|x\|$ as $d \to \infty$. Thus, inequality (3.2) follows from (3.4) by the monotone convergence theorem. □

4. Extensions and applications

4.1. A general device to convert inequalities for the $p$th absolute moments of real-valued random variables into the corresponding inequalities for the $p$th moments of the norms of random vectors in Hilbert spaces. Let $\mathcal{P}(X_1, \ldots, X_n)$ denote some property of random vectors $X_1, \ldots, X_n$ in a separable Hilbert space stated in terms applicable to all separable Hilbert spaces. Let us say that $\mathcal{P}(X_1, \ldots, X_n)$ is linearly invariant if for any separable Hilbert space $G$ and any bounded linear operator $A: H \to G$ we have the implication

\[ \mathcal{P}(X_1, \ldots, X_n) \implies \mathcal{P}(AX_1, \ldots, AX_n). \]

For instance, any one of the following three properties is linearly invariant:

(i) $(X_1, \ldots, X_n)$ is a $v$-martingale difference sequence;
(ii) $X_1, \ldots, X_n$ are independent;
(iii) $X_1, \ldots, X_n$ are symmetrically distributed.

Clearly, the reasoning in the proof of Theorem 3.1 can be extended to yield the following general result:
Theorem 4.1. Let \( P(X_1, \ldots, X_n) \) be a linearly invariant property of random vectors \( X_1, \ldots, X_n \) in a separable Hilbert space. Suppose that the implication

\[
P(Y_1, \ldots, Y_n) \implies \sum_{k=1}^{K} a_k E \left\| \sum_{j=1}^{n} b_{k,j} Y_j \right\|^p \geq 0
\]

holds for some real \( p > 0 \), some natural \( K \), some vector \((a_k: k \in \mathbb{N}, K) \in \mathbb{R}^K \), some matrix \((b_{k,j}: k \in \mathbb{N}, K, j \in \mathbb{N}, n) \in \mathbb{R}^{K \times n} \), and all real-valued r.v.’s \( Y_1, \ldots, Y_n \). Then

\[
P(X_1, \ldots, X_n) \implies \sum_{k=1}^{K} a_k E \left\| \sum_{j=1}^{n} b_{k,j} X_j \right\|^p \geq 0
\]

for all random vectors \( X_1, \ldots, X_n \) in any separable Hilbert space.

(For convenience, we assume here that the inequality in (4.1) always holds if at least one of the \( K \) summands in the sum there equals \( -\infty \) – even if another summand in that sum equals \( -\infty \). The similar convention is assumed for (4.2). That is, we assume that the inequalities in (4.1) and (4.2) are/can be rewritten in the more general way so that the summands with \( a_k > 0 \) be retained on the left and the summands with \( a_k < 0 \) be moved to the right. This allows us to avoid requiring the finiteness of the \( p \)th moments.)

Thus, Theorem 4.1 provides a general device to convert inequalities for the \( p \)th moments of real-valued random variables into corresponding inequalities for the \( p \)th moments of the norms of random vectors in Hilbert spaces.

Now Theorem 4.1 can be viewed as a particular case of Theorem 4.1 – corresponding to the following settings: \( p \in (1, 2] \); \( P(X_1, \ldots, X_n) \) meaning that \((X_1, \ldots, X_n)\) is a v-martingale difference sequence; \( K = n + 1 \); \( a_1 = 1 \), \( a_2 = \cdots = a_n = C_p \), \( a_{n+1} = -1 \); \( b_{k,j} = I\{k = j\} \) for \((k, j)\in \mathbb{N} \times \mathbb{N} \) and \( b_{n+1,j} = 1 \) for \( j \in \mathbb{N} \), where \( I\{\cdot\} \) denotes the indicator.

4.2. Applications to von Bahr–Esseen-type inequalities. Here is another special case of Theorem 4.1.

Corollary 4.2. Take any \( p \in [1, 2] \). Suppose that \((X_1, \ldots, X_n)\) is a v-martingale difference sequence in a separable Hilbert space \( H \) such that the conditional distribution of \( X_j \) given \( S_{j-1} \) is symmetric for each \( j \in \mathbb{N} \). Then (cf. (3.2))

\[
E \|S_n\|^p \leq \sum_{j=1}^{n} E \|X_j\|^p.
\]

The particular case of Corollary 4.2 with \( H = \mathbb{R} \) is Theorem 1 in [11]. So, Corollary 4.2 follows by Theorem 4.1 since the property of \((X_1, \ldots, X_n)\) assumed in the second sentence of Corollary 4.2 is linearly invariant.

Theorem 1 in [11] was obtained as an almost immediate corollary of one of the celebrated Clarkson inequalities – see the first inequality in [3, formula (3)], reversed for \( p \in (1, 2] \). That inequality by Clarkson can be stated as

\[
E |x + \varepsilon y|^p \leq |x|^p + |y|^p,
\]

where \( x \) and \( y \) are real numbers and \( \varepsilon \) is a Rademacher r.v., so that \( P(\varepsilon = 1) = P(\varepsilon = -1) = 1/2 \).
Corollary 4.2 can also be obtained directly, without using [11, Theorem 1] or Theorem 4.1 in the present paper. To do that, instead of the Clarkson inequality (4.3), use the inequality

\[ E \|x + \varepsilon y\|^p \leq \|x\|^p + \|y\|^p \]

for \( p \in (1, 2] \) and \( x, y \) in \( H \). To prove (4.4), just note that

\[ E \|x + \varepsilon y\|^p \leq (E \|x + \varepsilon y\|^2)^{p/2} = (\|x\|^2 + \|y\|^2)^{p/2} \leq \|x\|^p + \|y\|^p. \]

**Corollary 4.3.** Take any \( p \in [1, 2] \). Let \( X \) and \( \tilde{X} \) be independent identically distributed random vectors in \( H \). Then

\[ E \|X - \tilde{X}\|^p \leq 2 E \|X\|^p. \]

The weaker version of (4.5) with \( 2^{p-1} \) in place of 2 follows immediately from the norm inequality \( \|X - \tilde{X}\| \leq \|X\| + \|\tilde{X}\| \) and the inequality \( (a + b)^p \leq 2^{p-1}(a^p + b^p) \) for nonnegative \( a \) and \( b \).

The special case of (4.5) for \( H = \mathbb{R} \) is the first inequality in formula (10) in Lemma 4 in [11]. Now Corollary 4.3 follows by Theorem 4.1.

Alternatively, Corollary 4.3 follows from Corollary 2.2 the same way [11, Lemma 4] (which is a special case of Corollary 4.3) follows from [11] Lemma 2 (which is a special case of Corollary 2.2).

Comparatively recently, the von Bahr–Esseen inequality was extended to pairwise independent r.v.’s [12]. In this regard, let us present

**Theorem 4.4.** Let \( X_1, \ldots, X_n \) be pairwise independent zero-mean random vectors in a separable Hilbert space \( H \). Then

\[ E \|S_n\|^p \leq \frac{4}{2 - p} \sum_{j=1}^n E \|X_j\|^p \]

for \( p \in [1, 2) \).

If the \( X_j \)’s are symmetric, then the factor \( \frac{4}{2 - p} \) in (4.6) can be replaced by \( \frac{2}{2 - p} \).

**Proof.** Consider first the case when \( X_1, \ldots, X_n \) are symmetric real-valued r.v.’s. Then the r.v.’s \( X_j I\{|X_j| < cx\} \) are pairwise independent and zero-mean for all positive \( c \) and \( x \). Hence, by the standard truncation argument and Chebyshev’s inequality,

\[ P(\|S_n\| \geq x) \leq \sum_{j=1}^n P(|X_j| \geq cx) + \frac{1}{x^2} \sum_{j=1}^n E X_j^2 I\{|X_j| < cx\}. \]
So,

\[
E|S_n|^p = \int_0^\infty px^{p-1} P(|S_n| \geq x) \, dx
\]

\[
\leq \int_0^\infty px^{p-1} \left( \sum_{j=1}^n P(|X_j| \geq cx) + \frac{1}{x^2} \sum_{j=1}^n E X_j^2 \mathbb{1}_{|X_j| < cx} \right) \, dx
\]

\[
= \frac{1}{c^p} \sum_{j=1}^n E |X_j|^p + \sum_{j=1}^n E \int_0^\infty px^{p-3} X_j^2 \mathbb{1}_{|X_j| < cx} \, dx
\]

\[
= \frac{1}{c^p} \sum_{j=1}^n E |X_j|^p + \sum_{j=1}^n E X_j^2 \int_{|X_j|/c}^\infty px^{p-3} \, dx
\]

\[
= \left( \frac{1}{c^p} + \frac{p}{2 - p} c^{2-p} \right) \sum_{j=1}^n E |X_j|^p.
\]

The minimum of the latter expression is attained at \( c = 1 \). So, inequality (4.6) with the twice better constant factor \( \frac{2}{2-p} \) (in place of \( \frac{4}{2-p} \)) holds when \( X_1, \ldots, X_n \) are symmetric pairwise independent real-valued r.v.'s. So, by Theorem 4.1 (4.6) with constant factor \( \frac{2}{2-p} \) holds when \( X_1, \ldots, X_n \) are symmetric pairwise independent random vectors in \( H \), since the property that \( X_1, \ldots, X_n \) are symmetric pairwise independent random vectors is linearly invariant.

Finally, by a standard symmetrization argument and in view of Corollary 4.3 (4.6) holds for any pairwise independent zero-mean random vectors \( X_1, \ldots, X_n \) in \( H \).

In the case when \( H = \mathbb{R} \), inequality (4.6) was obtained in [1] but with the constant factor

\[
K_1(p) := \min_{\varepsilon > 0} \left( \frac{4(\varepsilon + 1)^2}{\varepsilon^2(2-p)^2} + \varepsilon + 2 \right)
\]

in place of \( \frac{4}{2-p} \). Note that \( K_1(p) \) is a root of a certain polynomial whose coefficients are polynomials in \( p \). Clearly, \( K_1(p) \geq \frac{4}{(2-p)^2} + 2 \), so that \( K_1(p) \) explodes to \( \infty \) inversely-quadratically fast as \( p \uparrow 2 \). The constant factor \( K_1(p) \) was partially improved [2, Corollary 2.1] to

\[
K_2(p) := \frac{4}{p-1} + \frac{8}{2-p}.
\]

Indeed, \( K_2(p) \) explodes to \( \infty \) only inversely-linearly fast as \( p \uparrow 2 \). However, in contrast with \( K_1(p) \), \( K_2(p) \) explodes to \( \infty \) also when \( p \downarrow 1 \).

It is not hard to see that, for all \( p \in [1, 2) \), the constant factor \( \frac{1}{2-p} \) in (4.6) is at least as twice as good (that, at least as twice as small) as the combined factor \( K_{12}(p) := \min(K_1(p), K_2(p)) \). This remark is illustrated in Figure 1.

One may also recall here that the just mentioned results in [1, 2] were obtained only for \( H = \mathbb{R} \).

It remains an open question whether it is possible to get rid of an explosion to \( \infty \) for \( p \uparrow 2 \) of the constant factor in the von Bahr–Esseen-type inequality for pairwise independent zero-mean r.v.'s (or, equivalently, for pairwise independent zero-mean random vectors in any separable Hilbert space). Ideas from [9] – where it was shown that the condition of pairwise independence may be starkly different from...
the complete independence condition – may turn out to be of use in this regard. However, whatever constant factor one would be able to obtain in a von Bahr–Esseen-type inequality for pairwise independent random vectors $X_j$ in $H = \mathbb{R}$, the same constant factor would be available for any separable Hilbert space $H$.

Pairwise independence plays a notable role in theoretical computer science; see e.g. [4].

4.3. **Contrast between populations versus spread within populations for vector measurements.** Yet another corollary of Theorem 4.1 is as follows:

**Corollary 4.5.** Take any real $p \in (0, 4]$. Let $X$ and $Y$ be independent random vectors in a separable Hilbert space $H$. Let $X_1, X_2$ be independent copies of $X$, and let $Y_1, Y_2$ be independent copies of $Y$. Assume that $\mathbb{E}\|X\|^p + \mathbb{E}\|Y\|^p < \infty$. For $p > 2$, assume also that $\mathbb{E}X = \mathbb{E}Y$.

Let

$$D_p := \mathbb{E}\|X - Y\|^p - \frac{1}{2}(\mathbb{E}\|X_1 - X_2\|^p + \mathbb{E}\|Y_1 - Y_2\|^p).$$

Then

$$D_p \geq 0 \text{ for } p \in (0, 2], \quad D_p \leq 0 \text{ for } p \in [2, 4].$$

Here $\|X_1 - X_2\|^p$ measures the spread of a vector measurement within an $X$ population, and similarly $\|Y_1 - Y_2\|^p$ measures the spread of a vector measurement within a $Y$ population, so that $\frac{1}{2}(\mathbb{E}\|X_1 - X_2\|^p + \mathbb{E}\|Y_1 - Y_2\|^p)$ is the average spread within the two populations. On the other hand, $\|X - Y\|^p$ measures the contrast between vector measurements in the $X$ population and in the $Y$ population. We see that the comparison between the “average spread within” and the “contrast between” depends on whether $p$ is less or greater than 2.

The particular case of Corollary 4.5 with $H = \mathbb{R}$ is Theorem 1 in [7]. So, Corollary 4.5 follows by Theorem 4.1 since the properties of $(X, Y, X_1, X_2, Y_1, Y_2)$ assumed in the first paragraph of Corollary 4.5 are linearly invariant.

Corollary 4.5 can also be deduced from Theorem 2.1. Indeed, repeat the proof of Theorem 1 in [7] almost literally – except for using (i) Theorem 2.1 of the present paper instead of formula (4) in [5] and (ii) the characteristic functionals instead of the corresponding characteristic functions. Thus, we will prove Corollary 4.5...
for $H = \mathbb{R}^d$ and therefore for any finite-dimensional Euclidean space. Finally, to complete the proof of Corollary 4.5, reason as in the last paragraph of Section 3.

The special case of Corollary 4.5 for $p = 1$ and $H = \mathbb{R}^d$ is part of Corollary 2 in [7].

References

1. Pingyan Chen, Peng Bai, and Soo Hak Sung, *The von Bahr–Esseen moment inequality for pairwise independent random variables and applications*, J. Math. Anal. Appl. 419 (2014), no. 2, 1290–1302. MR 3225435
2. Pingyan Chen and Soo Hak Sung, *Generalized Marcinkiewicz-Zygmund type inequalities for random variables and applications*, J. Math. Inequal. 10 (2016), no. 3, 837–848. MR 3565157
3. James A. Clarkson, *Uniformly convex spaces*, Trans. Amer. Math. Soc. 40 (1936), no. 3, 396–414. MR 1501880
4. Michael Luby and Avi Wigderson, *Pairwise independence and derandomization*, Found. Trends Theor. Comput. Sci. 1 (2005), no. 4, 237–301. MR 2379508
5. Iosif Pinelis, *Positive-part moments via the Fourier–Laplace transform*, J. Theor. Probab. 24 (2011), 409–421.
6. Iosif Pinelis, *Best possible bounds of the von Bahr–Esseen type*, Ann. Funct. Anal. 6 (2015), no. 4, 1–29. MR 3365979
7. Iosif Pinelis, *Contrast between populations versus spread within populations*, Statist. Probab. Lett. 121 (2017), 99–100.
8. Iosif Pinelis, *Positive-part moments via characteristic functions, and more general expressions*, J. Theoret. Probab. 31 (2018), no. 1, 527–555. MR 3769823
9. Iosif Pinelis, *Exact lower bound on an ‘exactly one’ probability*, Bulletin of the Australian Mathematical Society 104 (2021), no. 2, 330–336.
10. Bengt von Bahr and Carl-Gustav Esseen, *Inequalities for the rth absolute moment of a sum of random variables, 1 ≤ r ≤ 2*, Ann. Math. Statist. 36 (1965), 299–303. MR MR0170407 (30 #645)

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