Abstract. Let $V$ and $F$ be holomorphic bundles over a complex manifold $M$, and $s$ be a holomorphic section of $V$. We study the cohomology associated to the Koszul complex induced by $s$, and prove a generalized Serre duality theorem for them.

1. Introduction

The Serre duality theorem is a fundamental result in complex manifold, which establishes a duality between the cohomology of a complex manifold and the cohomology of with compact supports, provided that the $\overline{\partial}$ operator has closed range in appropriate degrees. In this paper we extend the Serre duality to the cohomology of Landau-Ginzburg models.

Let $V$ be a holomorphic bundle over a complex manifold (usually noncompact) $M$ with rank $V = \dim M = n$, and $s$ be a holomorphic section of $V$ with compact zero loci $Z := (s^{-1}(0))$. Let $V^\vee$ be the dual bundle of $V$, then $s$ induced the following Koszul complex

$$(1.1) \quad 0 \to \wedge^n V^\vee \overset{\iota_s}{\to} \cdots \overset{\iota_s}{\to} \wedge^2 V^\vee \overset{\iota_s}{\to} V^\vee \overset{\iota_s}{\to} \mathbb{C} \to 0,$$

where $\iota_s$ is the contraction operator induced by $s$.

Let $F$ be another holomorphic bundle over $M$, we have the following complex from (1.1)

$$(1.2) \quad 0 \to \wedge^n V^\vee \otimes F \overset{\iota_s \otimes 1}{\to} \cdots \overset{\iota_s \otimes 1}{\to} \wedge^2 V^\vee \otimes F \overset{\iota_s \otimes 1}{\to} V^\vee \otimes F \overset{\iota_s \otimes 1}{\to} F \to 0.$$

Denote by $\mathbb{H}^\bullet(M; V, F)$ the hypercohomology associated to the above complex. We will, somewhat abusively, write $\iota_s \otimes 1$ as $\iota_s$. Because (1.2) is exact outside the compact set $Z$, the cohomology $\mathbb{H}^\bullet(M; V, F)$ is finite dimensional over $\mathbb{C}$. The study of this type cohomology origins to the mathematical interpretation of the Landau-Ginzburg models, which had been widely studied in the following papers $1, 2, 3, 4, 5, 6$.

Let $\psi \in \Gamma(M, \det V \otimes \det \Omega_M)$ be a holomorphic section, where $\Omega_M$ is the holomorphic cotangent bundle of $M$. There is a canonical pairing

$$(-, -)_{\psi} : \mathbb{H}^\bullet(M; V, F) \times \mathbb{H}^\bullet(M; V, F^\vee) \to \mathbb{C},$$

see (3.24). Then we have the following duality theorem.

Theorem 1.1. Let $V, F$ be holomorphic bundles over the complex manifold $M$ with rank $V = \dim M = n$, and $s$ be a holomorphic section of $V$ with compact zero
loci $Z = s^{-1}(0)$. Assume that $\psi \in \Gamma(M, \det V \otimes \det \Omega_M)$ is nowhere vanishing. Then the above pairing $(-,-)_\psi$ is non-degenerate. Thus for $-n \leq k \leq n$,

$$H^k(M; V, F) \cong H^{-k}(M; V, F^\vee).$$

This is a generalization of the non-degenerate theorem of [3, Theorem A] and [10, Theorem 1.2].

Let $V = T_M$ be the holomorphic tangent bundle of the compact complex manifold $M$, and $F$ be a holomorphic bundle over $M$. Let $s$ be the zero section of $T_M$ and $\psi = c \in \Gamma(M, \mathcal{O}_M) \cong \Gamma(M, \det T_M \otimes \det \Omega_M)$ be a nonzero constant, then we recover the classical Serre duality theorem.

**Corollary 1.2.** Let $F$ be a holomorphic bundle over a compact complex manifold $M$, then

$$H^{p,q}(M, F) \cong H^{n-p,n-q}(M, F^\vee).$$

Let $V = \Omega_M$ be the holomorphic cotangent bundle of a small open ball $M = \{z \in \mathbb{C}^n | |z| < \epsilon\}$, and $F = \mathcal{O}_M$. Let $s = df$ be a holomorphic section of $V$, where $f$ is a holomorphic function on $M$. Let $\psi = dz_1 \wedge \cdots \wedge dz_n \otimes e_1 \wedge \cdots \wedge e_n$, where $\{z_i\}$ is the coordinate of $\mathbb{C}^n$ and $\{e_i\}$ is the holomorphic frame of $\Omega_M$. Assume that $s = df = f_1 e_1 + \cdots + f_n e_n$ and $Z = s^{-1}(0) = 0$, then using Theorem [1,1] and the proof of [10, Theorem 1.3] we have

$$H^0(M; V, F) \cong \Gamma(M, \mathcal{O}_M)/(f_1, \cdots, f_n); \quad H^k(M; V, F) = 0, \quad k \neq 0.$$

For $g, h \in \Gamma(M, \mathcal{O}_M)$, let $\psi' = gh\psi$. By (3.31), we have

$$(g, h)_\psi = (-1)^{\lfloor \frac{n+3}{2} \rfloor + \frac{n(n+1)}{2}}(-2\pi i)^n \text{Res} \frac{\psi'}{s},$$

where $\text{Res} \frac{\psi'}{s}$ is the virtual residue associated to $\psi'$ and $s$, the symbol $\lfloor \frac{n+3}{2} \rfloor$ is the greatest integer less than or equal to $\frac{n+3}{2}$. The virtual residue, which had been constructed by Chang and the author in [3], coincides with the Grothendieck residue up to a sign. Therefore $\text{Res} \frac{\psi'}{s}$ equals to the Grothendieck residue $\text{res}_s(g, h) = \int_{|f| = \epsilon} \frac{ghdz_1 \wedge \cdots \wedge dz_n}{f_1 \cdots f_n}$ up to a sign, see formula (4.3). Thus we recover the following local duality theorem, see [3, Page 659].

**Corollary 1.3.** Let $V = \Omega_M$ be the holomorphic cotangent bundle of a small open ball $M = \{z \in \mathbb{C}^n | |z| < \epsilon\}$, and $F = \mathcal{O}_M$. Let $s = df$ be a holomorphic section of $V$, where $f$ is a holomorphic function on $M$. Let $\psi = dz_1 \wedge \cdots \wedge dz_n \otimes e_1 \wedge \cdots \wedge e_n$, where $\{z_i\}$ is the coordinate of $\mathbb{C}^n$ and $\{e_i\}$ is the holomorphic frame of $\Omega_M$. Assume that $s = df = f_1 e_1 + \cdots + f_n e_n$ and $Z = s^{-1}(0) = 0$. Then

$$\text{res}_s : \Gamma(M, \mathcal{O}_M)/(f_1, \cdots, f_n) \times \Gamma(M, \mathcal{O}_M)/(f_1, \cdots, f_n) \to \mathbb{C}$$

is non-degenerate.

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2. Cohomology with compact support

In this section, we study different types of hypercohomology associate to the exact sequence (1.2). It is similar to Section 2 in [10]. As before let \( V, F \) be holomorphic bundles over a complex manifold \( M \) with rank \( V = \dim M = n \), and \( s \) is a holomorphic section of \( V \) with compact zero loci \( Z = s^{-1}(0) \). Let \( V^\vee \) be the dual bundle of \( V \).

Let \( \mathcal{A}^{i,j}(\wedge^l V^\vee \otimes F) \) be the sheaf of smooth \((i,j)\) forms on \( M \) with value in \( \wedge^l V^\vee \otimes F \). Let \( \Omega^{i,j}(\wedge^l V^\vee \otimes F) := \Gamma(M, \mathcal{A}^{i,j}(\wedge^l V^\vee \otimes F)) \) and assign its element \( \alpha \) to have degree \( \sharp \alpha = i + j - l \). Let

\[
\Omega^{(i,j)}(\wedge^l V^\vee \otimes F) := \{ \alpha | \alpha \in \Gamma(M, \mathcal{A}^{i,j}(\wedge^l V^\vee \otimes F)) \text{ with compact support} \}.
\]

Then

\[
\mathcal{B} := \oplus_{i,j,l} \Omega^{(i,j)}(\wedge^l V^\vee \otimes F)
\]

is a graded commutative algebra with the (wedge) product uniquely extending wedge products in \( \Omega^* \), \( \wedge^* V^\vee \) and mutual tensor products. Denote

\[
C_M^k = \bigoplus_{i-j=k} C_M^{i,j} \subset \mathcal{B} \quad \text{with} \quad C_M^{i,j} := \Omega^{(0,0)}(\wedge^l V^\vee \otimes F) = \Gamma(M, \mathcal{A}^{0,0}(\wedge^l V^\vee \otimes F)),
\]

and

\[
C_M^{k,M} = \bigoplus_{i-j=k} C_M^{i,j} \quad \text{with} \quad C_M^{i,j} := \{ \alpha \in C_M^{i,j} | \alpha \text{ has compact support} \}.
\]

Let \( \mathcal{C}_M := \oplus_k \mathcal{C}_M^k \) and \( \mathcal{C}_M^{M} := \oplus_k \mathcal{C}_M^{k,M} \). For \( \alpha \in \mathcal{C}_M \) we denote \( \alpha_{i,j} \) to be its component in \( C_M^{i,j} \). Clearly, \( \mathcal{C}_M \) is a bi-graded \( C^\infty(M) \)-module. Under the operations

\[
\overline{\partial} : C_M^{i,j} \rightarrow C_M^{i+1,j} \quad \text{and} \quad \iota_s : C_M^{i,j} \rightarrow C_M^{i,j-1}
\]

the space \( \mathcal{C}_M^{M} \) becomes a double complex and \( \mathcal{C}_M^{M} \) is a subcomplex. We shall study the cohomology of \( \mathcal{C}_M^{M} \) and \( \mathcal{C}_M^{M} \) with respect to the following coboundary operator

\[
\overline{\partial}_s := \overline{\partial} + \iota_s.
\]

One checks \( \overline{\partial}_s^2 = 0 \) using Leibniz rule of \( \overline{\partial} \) and \( \overline{\partial}s = 0 \). Denote by

\[
\overline{\partial}^k(M; V, F) = H^k(\mathcal{C}_M^{M}),
\]

and

\[
\iota_s^k(M; V, F) = H^k(\mathcal{C}_M^{M}).
\]

Fix a Hermitian metric \( h_V \) on \( V \). For a nonzero \( s \) on \( U := M \setminus Z \), we can form the following smooth section

\[
\tilde{s} := \frac{(s, s)_{h_V}}{(s,s)_{h_V}} \in \Gamma(U, \mathcal{A}^{0,0}(V^\vee \otimes F)).
\]

It associates a map

\[
\tilde{s}^\wedge : \Gamma(U, \mathcal{A}^{0,0}(\wedge^j V^\vee \otimes F)) \rightarrow \Gamma(U, \mathcal{A}^{0,0}(\wedge^{j+1} V^\vee \otimes F)).
\]

To distinguish it in later calculation, we denote \( \mathcal{T}_s := \tilde{s}^\wedge : \mathcal{C}_U^{M} \rightarrow \mathcal{C}_U^{M+1} \), where \( \mathcal{C}_U^{M} := \Gamma(U, \mathcal{A}^{0,0}(\wedge^M V^\vee \otimes F)) \).

The injection \( j : U \rightarrow M \) induces the restriction \( j^* : \mathcal{C}_M^{M} \rightarrow \mathcal{C}_U^{M} \). Let \( \rho \) be a smooth cut-off function on \( M \) such that \( \rho|_{U_1} \equiv 1 \) and \( \rho|_{M \setminus U_2} \equiv 0 \) for some relatively compact open neighborhoods \( U_1 \subset U_2 \subset \overline{U_2} \) of \( Z \) in \( M \).

We define the degree of an operator to be its change on the total degree of elements in \( \mathcal{C}_M \). Then \( \overline{\partial} \) and \( \mathcal{T}_s \) are of degree 1 and \(-1\) respectively, and
\[ [\bar{\partial}, T_s] = \bar{\partial}T_s + T_s\bar{\partial} \] is of degree 0. Consider two operators introduced in [3, (3.1), (3.2)] or [12, page 11]

\begin{equation} \tag{2.1} T_\rho : \mathcal{C}_M \to \mathcal{C}_{c,M} \quad T_\rho(\alpha) := \rho\alpha + (\bar{\partial}\rho)T_s \frac{1}{1 + [\bar{\partial}, T_s]} (j^*\alpha) \end{equation}

and

\begin{equation} \tag{2.2} R_\rho : \mathcal{C}_M \to \mathcal{C}_M \quad R_\rho(\alpha) := (1 - \rho)T_s \frac{1}{1 + [\bar{\partial}, T_s]} (j^*\alpha). \end{equation}

Here as an operator

\[ \frac{1}{1 + [\bar{\partial}, T_s]} := \sum_{k=0}^{\infty} (-1)^k [\bar{\partial}, T_s]^k \]

is well-defined since \([\bar{\partial}, T_s]^k(\alpha) = 0\) whenever \(k > n\). Clearly \(T_\rho\) is of degree zero and \(R_\rho\) is of degree by \(-1\). Also \(R_\rho(\mathcal{C}_{c,M}) \subset \mathcal{C}_{c,M}\) by definition.

**Lemma 2.1.** \([\bar{\partial}_s, R_\rho] = 1 - T_\rho\) as operators on \(\mathcal{C}_M\).

**Proof.** It is direct to check that

\begin{equation} \tag{2.3} \{\iota_s, T_s\} = 1 \quad \text{on } \mathcal{C}_U. \end{equation}

Moreover,

\[ [P, [\bar{\partial}, T_s]] = 0 \]

for \(P\) being \(\iota_s, \bar{\partial}\) or \(T_s\). Therefore, we have

\[ [\bar{\partial}_s, R_\rho] = [\bar{\partial}_s, 1 - \rho]T_s \frac{1}{1 + [\bar{\partial}, T_s]} j^* + (1 - \rho)[\bar{\partial}_s, T_s] \frac{1}{1 + [\bar{\partial}, T_s]} j^* \]

\[ = -(\bar{\partial}\rho)T_s \frac{1}{1 + [\bar{\partial}, T_s]} j^* + (1 - \rho)j^* \]

\[ = -(\bar{\partial}\rho)T_s \frac{1}{1 + [\bar{\partial}, T_s]} j^* + (1 - \rho) = 1 - T_\rho. \]

\[ \square \]

**Proposition 2.2.** The embedding \((\mathcal{C}_{c,M}, \bar{\partial}_s) \to (\mathcal{C}_M, \bar{\partial}_s)\) is a quasi-isomorphism. Thus for \(-n \leq k \leq n, \)

\[ \mathbb{H}^k(M; V, F) \cong \mathbb{H}^k(M; V, F). \]

**Proof.** By Lemma \[2.1\] \[H^*(\mathcal{C}_M/\mathcal{C}_{c,M}, \bar{\partial}_s) \cong 0, \] and thus the proposition follows. \[ \square \]

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1 As a notation convention, we always denote \([,]\) for the graded commutator, that is for operators \(A, B\) of degree \(|A|\) and \(|B|\), the bracket is given by

\[ [A, B] = AB - (-1)^{|A||B|} BA. \]
3. Non-degeneracy

First we recall the definition of operators on wedge products of vector bundles over $M$, see [2] Appendix. Let $\Omega^{i,j}(\wedge^k V \otimes \wedge^l V^\vee) := \Gamma(M,\mathcal{A}^{i,j}(\wedge^k V \otimes \wedge^l V^\vee))$ be the smooth differential forms valued in $\wedge^k V \otimes \wedge^l V^\vee$. Let

$$\mathfrak{B} := \oplus_{i,j,k,l} \Omega^{i,j}(\wedge^k V \otimes \wedge^l V^\vee)$$

be a graded commutative algebra extending the wedge products of $\Omega^*,\wedge^*V$ and $\wedge^*V^\vee$. The degree of $\alpha \in \Omega^{i,j}(\wedge^k V \otimes \wedge^l V^\vee)$ is $\sharp \alpha := i + j + k - l$. We briefly denote $A^0(\wedge^k V \otimes \wedge^l V^\vee) = \Omega^{(0,0)}(\wedge^k V \otimes \wedge^l V^\vee)$.

Set $\kappa : \mathfrak{B} \to \Omega^*$ which sends $\omega(e \otimes e')$ (for $\omega \in \Omega^{i,j}$, $e \in \wedge^k V, e' \in \wedge^l V^\vee$) to $\omega(e, e')$, where $\langle \cdot, \cdot \rangle$ is the dual pairing between $\wedge^k V$ and $\wedge^l V^\vee$, and $\langle e, e' \rangle = 0$ when $k \neq l$. We further extend the pairing by setting $\langle \alpha, \beta \rangle := \kappa(\alpha\beta)$ for $\alpha, \beta \in \mathfrak{B}$. It is direct to verify

$$\overline{\mathfrak{g}}(\alpha, \beta) = \langle \mathfrak{g}(\alpha), \beta \rangle + (-1)^{\sharp \alpha} \langle \alpha, \overline{\mathfrak{g}}(\beta) \rangle. \quad (3.1)$$

We now define different types of contraction maps. Given $u \in \Omega^{i,j}(\wedge^k V)$ and $k \geq l$, we define

$$u_{\beta} : \Omega^{(p,q)}(\wedge^l V^\vee) \longrightarrow \Omega^{(p+i,q+j)}(\wedge^{k-l} V)$$

where for $\theta \in \Omega^{(p,q)}(\wedge^l V^\vee)$, the $u_{\beta}\theta$ is determined by

$$\langle u_{\beta}\theta, \nu^* \rangle = (-1)^{(i+j)(l+p+q)+w+\frac{l(l-1)}{2}} \langle u, \theta \wedge \nu^* \rangle, \quad \forall \nu^* \in A^0(\wedge^{k-l} V^\vee). \quad (3.2)$$

Given $\alpha \in A^0(V)$, we define

$$\iota_{\alpha} : \Omega^{(i,j)}(\wedge^k V^\vee) \longrightarrow \Omega^{(i,j)}(\wedge^{k-1} V^\vee)$$

where for $w \in \Omega^{(i,j)}(\wedge^k V^\vee)$, the $\iota_{\alpha}(w)$ is determined by

$$\langle \nu, \iota_{\alpha}(w) \rangle = \langle \alpha \wedge \nu, w \rangle, \quad \forall \nu \in A^0(\wedge^{k-1} V). \quad (3.3)$$

For above $\alpha, \theta$ and $w$ one has $\iota_{\alpha}(w \wedge \theta) = \iota_{\alpha}(w) \wedge \theta + (-1)^{t\omega} w \wedge \iota_{\alpha}(\theta)$. Given $\gamma \in A^0(V^\vee)$, we define

$$\iota_{\gamma} : \Omega^{(i,j)}(\wedge^k V) \longrightarrow \Omega^{(i,j)}(\wedge^{k-1} V)$$

where for $\nu \in \Omega^{(i,j)}(\wedge^k V)$, the $\iota_{\gamma}(\nu)$ is determined by

$$\langle \iota_{\gamma}(\nu), w \rangle = \langle \nu, \gamma \wedge w \rangle, \quad \forall \nu \in A^0(\wedge^{k-1} V^\vee). \quad (3.4)$$

We have the following identities.

**Lemma 3.1 (3).** Given $u \in \Omega^{i,j}(\wedge^k V)$, and $\theta, \alpha, \gamma$ as above, one has

$$\alpha \wedge (u_{\beta}\theta) = u_{\beta}(\alpha \theta), \quad \iota_{\gamma}(u_{\beta}\theta) = u_{\beta}(\iota_{\gamma}(\theta)). \quad (3.5)$$

**Lemma 3.2 (3).** For $u \in \Omega^{(i,j)}(\wedge^k V)$, $\theta \in \Omega^{(p,q)}(\wedge^l V^\vee)$, $k \geq l$ and a smooth form $\alpha \in \Omega^{(a,b)}(M)$, we have

$$\alpha \wedge (u_{\beta}\theta) = u_{\beta}(\alpha \theta) \quad \text{and} \quad \overline{\mathfrak{g}}(u_{\beta}\theta) = (-1)^{\beta\theta} \overline{\mathfrak{g}}(u_{\beta}\theta) + u_{\beta}(\overline{\mathfrak{g}}(\theta)). \quad (3.6)$$

Denote by $\mathcal{D}^{p,q}(\wedge^k V \otimes F^\vee)$ the space of $\wedge^k V \otimes F^\vee$-valued $(p,q)$-current, which is the dual of the space $\Omega^{n-p,n-q}_c(\wedge^l V^\vee \otimes F)$. There is a naturally pairing

$$\langle \alpha, \beta \rangle : \mathcal{D}^{n-p,n-q}(\wedge^k V \otimes F^\vee) \times \Omega^{n-i}_c(\wedge^l V^\vee \otimes F) \to \mathbb{C}, \quad (3.7)$$

where

$$\langle \alpha, \beta \rangle := \int_M \langle \alpha, \beta \rangle.$$
Denote
\[ D^k_M = \bigoplus_{i+j=n=k} D_M^{ij} \quad \text{with} \quad D_M^{ij} := \mathcal{O}^{(n,i)}(\wedge^j V \otimes F^\vee). \]

The coboundary map \( \delta_s \) of \( D^*_M \) is defined as follows
\begin{align*}
\delta_s \alpha &= \overline{D}\alpha + (-1)^{i+j}s \wedge \alpha, \quad \text{for} \quad \alpha \in \mathcal{O}^{p,q}(\wedge^i V \otimes F^\vee).
\end{align*}

By (3.8) and (3.9),
\begin{align*}
(\alpha, \overline{D}s\beta)_N + (-1)^{s}\delta_s(\alpha, \beta)_N &= 0.
\end{align*}

Thus \( (D^*_M, (-1)^{s+1}\delta_s) \) is the dual complex of \( (C^*_M, \overline{\partial}) \). Let
\begin{align*}
\mathcal{H}^k(M; V, F^\vee) := \frac{\text{Ker}(\delta_s : D^k_M \to D^{k+1}_M)}{\text{Im}(\delta_s : D^{k-1}_M \to D^k_M)}.
\end{align*}

Because the complex \( (D^*_M, (-1)^{s+1}\delta_s) \) is quasi-isomorphic to the complex \( (D^*_M, \delta_s) \), the pairing (3.7) induces a pairing on the hypercohomologies which we denote by
\begin{align*}
(\alpha, \beta)_N : \mathcal{H}^*(M; V, F^\vee) \times \mathbb{H}^*(M; V, F) \to \mathbb{C}.
\end{align*}

**Theorem 3.3.** Let \( V, F \) be holomorphic bundles over the complex manifold \( M \) with rank \( V = \dim M = n \) and \( s \) be a holomorphic section of \( V \) with compact zero loci \( Z = s^{-1}(0) \). Then the above pairing (3.11) is non-degenerate. Thus for \( -n \leq k \leq n \)
\[ \mathbb{H}^k_c(M; V, F) \cong \mathcal{H}^{-k}(M; V, F^\vee)^\vee. \]

**Proof.** Because \( \mathbb{H}^k_c(M; V, F) \) is finite dimensional and the complex \( (D^*_M, (-1)^{s+1}\delta_s) \) is the dual complex of \( (C^*_M, \overline{\partial}) \), by applying [11, Theorem 1.6] and [11, Corollary 1.7], we obtain the theorem. \( \Box \)

**Remark 3.4.** By the Dolbeault-Grothendieck Lemma [4, 3.29] for current, \( \mathcal{H}^*(M; V, F^\vee) \)
\begin{align*}
\text{(up to a shift on degree) is the hypercohomology of the following complex,}
\end{align*}
\begin{align*}
0 \to \det \Omega_M \otimes F^\vee \to \det \Omega^*_M \otimes V \otimes F^\vee \to (-1)^{s} \cdots \to (-1)^{n} \wedge^V \to 0,
\end{align*}

which is quasi-isomorphic to the complex
\begin{align*}
0 \to \det \Omega_M \otimes V \otimes F^\vee \to \det \Omega^*_M \otimes V \otimes F^\vee \to \cdots \to \det \Omega^*_M \otimes \wedge^n V \otimes F^\vee \to 0.
\end{align*}

Let \( (\overline{D^*_M}, \overline{\partial}) \) be the complex with
\[ \overline{D}^k_M = \bigoplus_{i+j=k} \overline{D}_M^{ij} \quad \text{where} \quad \overline{D}_M^{ij} := \mathcal{O}^{(n,i)}(\wedge^j V \otimes F^\vee), \]

and the coboundary map \( \overline{\partial} \) is defined as follows
\begin{align*}
\overline{\partial}\alpha &= D\alpha + (-1)^{n-i+1}s\wedge \alpha, \quad \text{for} \quad \alpha \in \mathcal{O}^{p,q}(\wedge^i V \otimes F^\vee).
\end{align*}

By the Dolbeault-Grothendieck Lemma [4, 3.29] for current, the complex \( (\overline{D}^*_M, \overline{\partial}) \) is the Dolbeault resolution of the following complex
\begin{align*}
0 \to \wedge^n V^\vee \otimes F^\vee \to \cdots \to (-1)^{n} V^\vee \otimes F^\vee \to F^\vee \to 0.
\end{align*}

Denote by
\begin{align*}
\mathcal{H}^k(\overline{D}^*_M) := \frac{\text{Ker}(\overline{\partial}_s : \overline{D}^k_M \to \overline{D}^{k+1}_M)}{\text{Im}(\overline{\partial}_s : \overline{D}^{k-1}_M \to \overline{D}^k_M)}
\end{align*}
the cohomology of the complex \( D_M, \delta_s \).

Let \( \psi \in \Gamma(M, \det V \otimes \det \Omega_M) \) be a holomorphic section which is nowhere vanishing. Then the pairing such that \( (\alpha, \beta) \)

\[ H^k(\nu, \Omega) \]

is isomorphic to \( H^k(M, V, F^\vee) \).

By Lemma 3.2 and Theorem 3.3, we obtain the following statement.

Combining (3.11) and (3.21), there is a pairing

\[ (\alpha, \beta)_\psi : H^\bullet(M; V, F^\vee) \times H^\bullet_c(M; V, F) \to \mathbb{C}, \]

where \( (\alpha, \beta)_\psi := (\psi_\bullet(H^\bullet(\nu))(\alpha)), \beta)_N. \)

From (3.21) and Theorem 3.3, we obtain the following statement.

**Corollary 3.5.** Let \( V, F \) be holomorphic bundles over the complex manifold \( M \) with rank \( V = \dim M = n \) and \( s \) be a holomorphic section of \( V \) with compact zero loci \( Z = s^{-1}(0) \). Assume that \( \psi \in \Gamma(M, \det V \otimes \det \Omega_M) \) is a holomorphic section which is nowhere vanishing. Then the pairing \( (\beta, \alpha)_\psi \) is non-degenerate. Thus for \(-n \leq k \leq n\)

\[ H^k(M; V, F) \cong H^{-k}(M; V, F^\vee)^\vee. \]

We define the following pairing

\[ (\beta, \alpha)_\psi : H^\bullet(M; V, F^\vee) \times H^\bullet(M; V, F) \to \mathbb{C}, \]

such that \( (\alpha, \beta)_\psi := ([\alpha], [\beta])_\psi \), where \([\alpha]\) and \([\beta]\) are the image of \( \alpha, \beta \) under the isomorphic \( H^\bullet(M; V, F) \cong H^\bullet_c(M; V, F) \) and \( H^\bullet(M; V, F^\vee) \cong H^\bullet_c(M; V, F^\vee) \). The pairing (3.34) is well defined because it does not depend on the compact support representation. Therefore

**Corollary 3.6.** Under the conditions as in Corollary 3.5, the pairing \( (\beta, \alpha)_\psi \) is non-degenerate. Thus for \(-n \leq k \leq n\)

\[ H^k(M; V, F) \cong H^{-k}(M; V, F^\vee)^\vee. \]
Let $V = \Omega_M$ be the holomorphic cotangent bundle of a small open ball $M = \{ z \in \mathbb{C}^n \mid |z| < \epsilon \}$, and $F = \mathcal{O}_M$. Let $s = df$ be a holomorphic section of $V$, where $f$ is a holomorphic function on $M$. Let $\psi = dz_1 \wedge \cdots \wedge dz_n \otimes e_1 \wedge \cdots \wedge e_n$, where $\{ z_i \}$ is the coordinate of $\mathbb{C}^n$ and $\{ e_i \}$ is the holomorphic frame of $\Omega_M$. Assume that $s = df = f_1 e_1 + \cdots + f_n e_n$ and $Z = s^{-1}(0)$, then using Corollary 3.6 and the proof of [10, Theorem 1.3]
\[
\mathbb{H}^0(M; V_F) \cong \Gamma(M, \mathcal{O}_M)/(f_1, \cdots, f_n); \quad \mathbb{H}^k(M; V_F) = 0, \quad k \neq 0.
\]
On the other hand $s$ induced the following complex
\[
(3.26) \quad 0 \to \det \Omega_M \to \det \Omega_M \otimes V^{(-1)^s} \otimes \cdots \otimes (-1)^n \otimes \det \Omega_M \otimes \wedge^n V \to 0.
\]
As in the definition of (3.10), let $\mathcal{H}^*(M; V, \mathcal{O}_M)$ be the hypercohomology of (3.26), and $\mathcal{H}^*_c(M; V, \mathcal{O}_M)$ be its hypercohomology with compact support. By [3, Proposition 3.2]
\[
(3.27) \quad \mathcal{H}^k(M; V, \mathcal{O}_M) \cong \mathcal{H}^k_c(M; V, \mathcal{O}_M),
\]
for $-n \leq k \leq n$.
Because $\psi = dz_1 \wedge \cdots \wedge dz_n \otimes e_1 \wedge \cdots \wedge e_n$ is nowhere vanishing. It induces a bundle isomorphism
\[
\psi \mapsto \wedge^i V^\vee \to \det \Omega_M \otimes \wedge^{n-i} V.
\]
So is the Dolbeault resolution of the two complex (1.1) and (3.26).

For $g, h \in \Gamma(M, \mathcal{O}_M)$, let $[g], [h]$ be the images of $g, h$ under the isomorphism $\mathbb{H}^0(M; V, \mathcal{O}_M) \cong \mathbb{H}^0_c(M; V, \mathcal{O}_M)$. By (3.27), (3.22) and (3.24) the pairing
\[
(g, h) = \int_M \langle (-1)^{n+1} g \psi, [h] \rangle
\]
\[
= (-1)^{n+1} \int_M \langle g h \psi, [1] \rangle
\]
\[
= (-1)^{n+1} \int_M (g h \psi)_1[1].
\]
Let $\rho$ be a smooth cut-off function on $M$ such that $\rho|_{U_1} \equiv 1$ and $\rho|_{M \setminus U_2} \equiv 0$ for some relatively compact open neighborhoods $U_1 \subset U_2 \subset U_0$ of $0$ in $M$. By Lemma 2.1 we have
\[
[\mathcal{G}_s, R_\rho] = 1 - T_\rho.
\]
Therefore
\[
(3.28) \quad \int_M (g h \psi)_1[1] = \int_M (g h \psi)_1[T_\rho 1].
\]
For the nonzero $s$ on $U := M \setminus Z$, the smooth section $\tilde{s} := \frac{(s, s)_{\lambda V}}{(s, s)_{V V}} \in \Gamma(U, \mathcal{A}^{0,0}(V^\vee))$ induces a contraction
\[
\iota_\tilde{s} : \Gamma(U, \mathcal{A}^{0,0}(\lambda^j V)) \to \Gamma(U, \mathcal{A}^{0,0}(\lambda^{j-1} V)).
\]

Denote
\[
\tilde{\mathcal{C}}_M^k = \bigoplus_{i+j=k} \tilde{C}_{i,j}^k \quad \text{with} \quad \tilde{C}_{i,j}^k := \Omega^{0,0}(\lambda^j V) = \Gamma(M, \mathcal{A}^{0,0}(\lambda^j V)),
\]
and
\[
\tilde{\mathcal{C}}_{c,M}^k = \bigoplus_{i+j=k} \tilde{C}_{i,j}^k \quad \text{with} \quad \tilde{C}_{i,j}^k := \{ \alpha \in \tilde{C}_{M}^k \mid \alpha \text{ has compact support} \}. 
\]
Let $\tilde{C}_M := \oplus l \tilde{C}_l^{\tilde{C}_M}$ and $\tilde{C}_{c,M} := \oplus l \tilde{C}_l^{\tilde{C}_{c,M}}$. Let $j : U \to M$ be the injection. We can form the following operator by using the contraction $\iota$

\begin{equation}
\tilde{T}_p : \tilde{C}_M \to \tilde{C}_{c,M} \quad \tilde{T}_p(\alpha) := \rho \alpha + (\overline{\partial} \rho) \iota \frac{1}{1 + \overline{\partial} \iota} (j^* \alpha).
\end{equation}

Because $\psi$ is holomorphic, by Lemma 3.1

\begin{equation}
\int_M (gh\psi)_\iota[T_p 1] = \int_M \tilde{T}_p(gh\psi).
\end{equation}

Denote $\psi' = gh\psi$, and applying [3, Proposition 3.3] to $\psi'$ and $s = df$, we have

\begin{equation}
(g, h)_{\psi} = \left( \frac{\sum_{i=1}^{n+1} f_{i} e_{i}}{s} \right) \int_M (gh\psi)_\iota[T_p 1] = \left( \sum_{i=1}^{n+1} f_{i} e_{i} \right) \int_M \tilde{T}_p(gh\psi) = \left( \frac{\sum_{i=1}^{n+1} f_{i} e_{i}}{s} \right) (-2\pi i)^n \text{Res} \frac{s'}{s},
\end{equation}

where $\text{Res} \frac{s'}{s}$ is the virtual residue associated to $\psi'$ and $s$, it coincides with the Grothendieck residue $\text{Res}_s(g, h) = \int_{\{f \leq \epsilon\}} \frac{g h d\zeta \wedge \cdots d\zeta}{f \cdots f}$. Thus we recover the local duality theorem, see [3, Page 659].

**Corollary 3.7.** Let $V = \Omega_M$ be the holomorphic cotangent bundle of a small open ball $M = \{ z \in \mathbb{C}^n | |z| < \epsilon \}$, and $F = \mathcal{O}_M$. Let $s = df$ be a holomorphic section of $V$, where $f$ is a holomorphic function on $M$. Let $\psi = d\zeta_1 \wedge \cdots \wedge d\zeta_n \otimes e_1 \wedge \cdots \wedge e_n$, where $\{z_i\}$ is the coordinate of $\mathbb{C}^n$ and $\{e_i\}$ is the holomorphic frame of $\Omega_M$. Assume that $s = df = f_1 e_1 + \cdots + f_n e_n$ and $Z = s^{-1}(0) = 0$. Then

\[ \text{res}_s : \Gamma(M, \mathcal{O}_M)/(f_1, \cdots, f_n) \times \Gamma(M, \mathcal{O}_M)/(f_1, \cdots, f_n) \to \mathbb{C} \]

is non-degenerate.

4. **Appendix**

In this appendix we recall the construction of the virtual residue given by Chang and the author in [3], and prove the relation between the virtual residue and the Grothendieck residue when the zero loci is zero-dimensional.

Let $V$ be a holomorphic bundle over a compact complex manifold $M$ with rank $V = \dim M = n$. Let $s$ be a holomorphic section of $V$, and $Z = s^{-1}(0)$ be the compact zero loci.

Let $U := M \setminus Z$, and let $V_U$ be the restriction of $V$ over $U$. Since $s$ is nowhere zero over $U$, the following Koszul sequence is exact over $U$

\[ 0 \to K_U \to K_U \otimes V_U \to \cdots \to K_U \otimes \wedge^{n-1} V_U \to K_U \otimes \wedge^n V_U \to 0. \]

The exact Koszul sequence induces a homomorphism

\begin{equation}
\text{H}^0(U, K_U \otimes \wedge^n V_U) \to \text{H}^{n-1}(U, K_U).
\end{equation}

One also has a canonical Dolbeault isomorphism

\begin{equation}
\text{H}^{n-1}(U, K_U) \cong \text{H}^n_{\partial}(U).
\end{equation}
Applying (4.1) and (4.2) to the holomorphic section \( \psi \in \Gamma(M, K_M \otimes \text{det} V) \), and using that every \((n, n-1)\) form is \( \partial \)-closed, one obtains a (unique) De-Rham cohomology class

\[
\eta_\psi \in H^{2n-1}(U, \mathbb{C}).
\]

Then the virtual residue is defined as

\[
\text{Res}_{\psi} Z \frac{\eta}{s} := \left( \frac{1}{2\pi i} \right)^n \int_N \eta_\psi \in \mathbb{C},
\]

where \( N \) is a real \((2n-1)\)-dimensional piecewise smooth compact subset of \( M \) that surrounds \( Z \), in the sense that \( N = \partial T \) for some compact domain \( T \subset M \), which contains \( Z \) and is homotopically equivalent to \( Z \).

When \( M = \{ z \in \mathbb{C}^n | ||z|| < \varepsilon \} \) is a small open ball, and \( V = \Omega_M \) with the standard Hermitian metric \( h_V \). Let \( F = O_M \) and \( s = df \), where \( f \) is a holomorphic function on \( M \). Let \( \{ z_i \} \) be the coordinate of \( \mathbb{C}^n \), and \( \{ e_i \} \) be the holomorphic frame of \( \Omega_M \). Assume that \( s = df = f_1 e_1 + \cdots + f_n e_n \) and \( Z = s^{-1}(0) = 0 \). Let \( \bar{s} = (s, s)^{-1} \sum_{i=1}^n \bar{f}_i e_i^* \), where \( e_i^* \) is the dual basis of \( V^\perp \). Then we have the following equalities on \( U \)

\[
\bar{\partial}\bar{s} = \sum \left( \frac{\bar{\partial}\bar{f}_i}{\bar{s}, \bar{s}}_{h_V} - \bar{f}_i \bar{\partial}(s, s)_{h_V} \right) e_i^*,
\]

and

\[
(\langle s, s \rangle_{h_V}^{-1} \sum_{i=1}^n \bar{f}_i e_i^*) \left( \sum_{i=1}^n \bar{f}_i \bar{\partial}(s, s)_{h_V} \right) e_i^* = -\left( \bar{\partial}(s, s)_{h_V} \langle s, s \rangle_{h_V}^{-1} \sum_{i=1}^n \bar{f}_i e_i^* \right)^2 = 0.
\]

Let \( g, h \) be holomorphic functions on \( M \). Then \( \psi = gh dz_1 \wedge \cdots \wedge dz_n \otimes e_1 \wedge \cdots \wedge e_n \) is a holomorphic section of \( \Gamma(M, K_M \otimes \text{det} V) \). Therefore

\[
\eta_\psi = \langle s, s \rangle_{h_V}^{-1} \left( \sum_{i=1}^n \bar{f}_i e_i^* \right) \left( \bar{\partial}\bar{s} \right)^{n-1} \psi
\]

\[
= \langle s, s \rangle_{h_V}^{-1} \left( \sum_{i=1}^n \bar{f}_i e_i^* \right) \left( \sum_{i=1}^n \bar{\partial}\bar{f}_i \right) \langle s, s \rangle_{h_V}^{-1} \left( \sum_{i=1}^n \bar{f}_i \bar{\partial}(s, s)_{h_V} \right) e_i^* \psi
\]

\[
= (-1)^{(n+1)/2} \langle s, s \rangle_{h_V}^{-1} \left( \sum_{i=1}^n \bar{f}_i \bar{\partial}(s, s)_{h_V} \right) \langle s, s \rangle_{h_V}^{-1} \left( \sum_{i=1}^n \bar{f}_i \bar{\partial}(s, s)_{h_V} \right) e_i^* \psi
\]

\[
= (-1)^{(n+1)/2} \langle s, s \rangle_{h_V}^{-1} \left( \sum_{i=1}^n \bar{f}_i \bar{\partial}(s, s)_{h_V} \right) \langle s, s \rangle_{h_V}^{-1} \left( \sum_{i=1}^n \bar{f}_i \bar{\partial}(s, s)_{h_V} \right) e_i^* \psi
\]

\[
= \bar{f}_i \langle s, s \rangle_{h_V}^{n} \bar{\partial}\bar{f}_1 \wedge \cdots \wedge \bar{\partial}\bar{f}_n \wedge d z_1 \wedge \cdots \wedge d z_n.
\]

Let \( N \) be a small sphere around 0, the virtual residue
\( (4.5) \ \text{Res} \frac{\psi}{s} = \left( \frac{1}{2\pi \sqrt{-1}} \right)^n \int_N \eta \psi \\
= (-1)^{\frac{n(n+1)}{2}} \cdot \frac{n(n-1)!}{2} \left( \frac{1}{2\pi \sqrt{-1}} \right)^n \int_N gh \sum_{i=1}^n (-1)^{i-1} \overline{\partial f_i} \wedge \cdots \wedge \overline{\partial f_i} \wedge \cdots \wedge \overline{\partial f_i} \wedge dz_1 \wedge \cdots \wedge dz_n. \)

By Lemma in [6, Page 651] and the definition of \( \text{res}_s(g,h) \) in [6, Page 659], we have
\( (4.6) \ \text{Res} \frac{\psi}{s} = (-1)^{\frac{n(n+1)}{2}} \text{res}_s(g,h). \)

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