Extended form method of antifield-BRST formalism for $BF$ theories

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ABSTRACT

The Batalin-Vilkovisky antifield action for the $BF$ theories is constructed by means of the extended form method. The BRST invariant BV antifield action is directly written down by making use of the extended forms that involve all the required ghosts and antifields.

The $BF$ theory is a kind of the topological quantum field theories (TQFT’s), which can be thought as a generalization of the Chern-Simons theory to arbitrary dimensions [1-4]. Though they are related to the generalized linking numbers of the extended objects, as far as the abelian $BF$ theories are concerned, there are still some obstacles to proceed a quantization of the non-abelian $BF$ theories in arbitrary dimensions ($\geq 4$). The problems mainly come from the on-shell reducibility of the symmetry in these systems. It is well known that the Batalin-Vilkovisky (BV) antifield-antibracket formalism is a useful procedure to construct a BRST invariant gauge fixed action in a covariant manner [5]. Although the BV algorism is a sure method even for reducible systems, one is required to do a tedious task to solve the BV master equation especially for highly reducible theories. There is known, however, an algebraic method to obtain the BRST algebra for the $BF$ theories without solving directly the master equation [4].

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In this paper, we present a modified version of this algebraic method to construct a BRST invariant BV action for BF theories, which has obviously some advantages to the unmodified one. The BV action that is a solution to the master equation is explicitly written down by means of the extended differential forms that involves “all” the required ghosts and antifields. Though the BRST algebra is also derived from a simple condition, the BV action itself can be obtained without referring to a concrete form of the algebra.

Let us begin with a quick recollection of the method of extended differential calculus on the universal bundle [6-8] in the case of topological Yang-Mills (TYM) theory. The basic idea is an extension of the exterior derivative to a sum of the usual exterior derivative \( d \) and the BRST operator \( s \),

\[
\tilde{d} = d + s .
\]

According to this extension, the differential form of type \((p,q)\) should be thought as an extended form of total degree \(p + q\), where we follow the convention that a \(p\)-form with ghost number \(q\) is called \((p,q)\)-form and we will often attach a subscript to the form indicating these degrees as \( \Phi_{(p,q)} \) when necessary.

The BRST algebra of TYM theory is obtained as follows. The universal connection \( \tilde{A} \) is considered as a sum of the Yang-Mills connection \( A \) and the Faddev-Popov ghost \( c \),

\[
\tilde{A} = A_{(1,0)} + c_{(0,1)} .
\]

The curvature 2-form of \( \tilde{A} \) whose elements we will denote as

\[
\tilde{F} = F_{(2,0)} + \psi_{(1,1)} + \phi_{(0,2)} ,
\]

is defined by

\[
\tilde{F} := \tilde{d}\tilde{A} + \frac{1}{2} [\tilde{A}, \tilde{A}] ,
\]
which is required to satisfy the Bianchi identity

\[ \tilde{D} \tilde{F} = 0 . \quad (5) \]

The explicit calculation of (4) and (5) with the components (2) and (3) leads to the BRST algebra

\[
\begin{align*}
    sA &= \psi - Dc \\
    sc &= \phi - \frac{1}{2}[c, c] \\
    s\psi &= -D\phi - [c, \psi] \\
    s\phi &= -[c, \phi].
\end{align*}
\]

An algebraic method to obtain a BRST transformations for the BF theories has been presented in ref. [4], which resembles the method above for the TYM theory. We will present a modified and extended version of this algebraic method.

The classical invariant action for the BF theory in \( D \)-dimensional spacetime has the form of

\[ S = \int_{M_D} \text{Tr}(B \wedge F) . \quad (7) \]

The fundamental fields in the action above are Lie algebra valued 1-form \( A \) and \( n \)-form \( B \), where \( F = dA + \frac{1}{2}[A, A] \) is the curvature 2-form of connection \( A \) and \( n = D - 2, \quad (D \geq 4) \).

The action has a symmetry with \( (n - 1) \)-form \( \varepsilon_{n-1} \) :

\[ \delta_{\varepsilon_{n-1}} A = 0 \quad , \quad \delta_{\varepsilon_{n-1}} B = D\varepsilon_{n-1} , \quad (8) \]

as well as the ordinary Yang-Mills symmetry with 0-form \( \omega \) :

\[ \delta_\omega A = D\omega \quad , \quad \delta_\omega B = [\omega, B] . \quad (9) \]

The on-shell reducibility of the \( \varepsilon \)-symmetry requires us to introduce a sequence of ghosts and ghosts-for-ghosts, when we apply the BV algorism to this system.
We denote the sequence of ghosts descending from $B = B_{(n,0)}$ as

\[ B_{(n,0)} \to B_{(n-1,1)} \to \ldots \to B_{(n-q,q)} \to \ldots \to B_{(0,n)} . \tag{10} \]

There is of course, as required, an ordinary ghost $c$ for the Yang-Mills symmetry.

We propose that the extended forms should be defined so as to include all the required ghosts and BV antifields. Therefore the extended forms $\tilde{A}$ and $\tilde{B}$ are defined by

\[
\tilde{A} = c_{(0,1)} + A_{(1,0)} + \sum_{q=0}^{n} B_{(2+q,-1-q)}^* , \tag{11a}
\]

\[
\tilde{B} = c_{(n+2,-2)}^* + A_{(n+1,-1)}^* + \sum_{q=0}^{n} B_{(n-q,q)}^* , \tag{11b}
\]

where $c_{(n+2,-2)}^*$, $A_{(n+1,-1)}^*$ and $B_{(2+q,-1-q)}^*$ are the BV antifield for $c_{(0,1)}$, $A_{(1,0)}$ and $B_{(n-q,q)}$ respectively.

The conditions that lead to the BRST transformations are

\[
\tilde{F} = 0 , \quad \tilde{D}\tilde{B} = 0 , \tag{12a, b}
\]

which have the same form as the field equations from the classical action (7) except for $\tilde{\cdot}$. Expansion of the conditions (12a,b) in the ghost number leads to the total BRST algebra

\[
\begin{cases}
s c &= -\frac{1}{2} [c, c] \\
s A &= -Dc \\

sB_{(2,-1)}^* &= -F - [c, B_{(2,-1)}^*] \\

sB_{(3,-2)}^* &= -DB_{(2,-1)}^* - [c, B_{(3,-2)}^*] \\

sB_{(2+q,-q-1)}^* &= -DB_{(1+q,-q)}^* - \frac{1}{2} \sum_{q'=0}^{q-2} [B_{(2+q',-1-q')}^*, B_{(q-q',q'+1-q)}^*] \\
&\quad - [c, B_{(2+q,-1-q)}^*] , \quad (2 \leq q \leq n) ,
\end{cases} \tag{13a}
\]
\[
\begin{align*}
sc^* &= -DA^* - \sum_{q'=0}^{n} [B^*_{(2+q',-1-q')}, B_{(n-q',q')} Reserve] - [c, c^*] \\
\text{sA}^* &= -DB_{(n,0)} - \sum_{q'=0}^{n-1} [B^*_{(2+q',-1-q')}, B_{(n-1-q',1+q')} Reserve] - [c, A^*] \\
sB_{(n-q,q)} &= -DB_{(n-q-1,q+1)} - \sum_{q'=0}^{n-q-2} [B^*_{(2+q',-1-q')}, B_{(n-q-2-q',q+2+q')} Reserve] - [c, B_{(n-q,q)} Reserve], \quad (0 \leq q \leq n - 2) \\
\text{sB}_{(1,n-1)} &= -DB_{(0,n)} - [c, B_{(1,n-1)} Reserve] \\
\text{sB}_{(0,n)} &= -[c, B_{(0,n)} Reserve].
\end{align*}
\]

(13b)

It should be remarked that our definition of the extended forms (11a,b) is enlarged from that of ref. [4] so as to include \( A^* \) and \( c^* \) together with \( c \). They have employed a definition of the form

\[
\tilde{A} = A_{(1,0)} + \sum_{q=0}^{n} B^*_{(2+q,-1-q)} Reserve, \quad \tilde{B} = \sum_{q=0}^{n} B_{(n-q,q)} Reserve, \quad (14a,b)
\]

in ref. [4] and have separately treated the BRST algebra for the Yang-Mills symmetry. It is due to the omission of \( A^* \)'s that there is a restriction on ghost number \( (q \geq 1) \) in the expansion of their conditions for BRST algebra,

\[
\tilde{F} = 0 Reserve, \quad \left( \tilde{D} \tilde{B} \right)_{q \geq 1} = 0 Reserve. \quad (15a,b)
\]

Our definition of the extended forms seems to be more natural, because the extension is a maximal one in \( D \)-dimensional spacetime and there appear all the forms of degree \( p = 0 \) to \( D \) (once and for all). The remarkable feature of our extended forms \( \tilde{A} \) and \( \tilde{B} \) is that they look like antifields of each other, that is, \( \tilde{A}^* = \tilde{B} \) and \( \tilde{A} = \tilde{B}^* \).

It seems also a remarkable advantage of our method that a BRST invariant BV action is obtained quite easily and directly by means of the extended forms as
follows. We point out that the BV antifield action is given by

\[ S_{BV} = \int_{M_D} \text{Tr}(\tilde{B} \wedge \tilde{F} - \tilde{B} \wedge s \tilde{A}) \]  

in terms of the extended forms, which turns out to be equivalent to

\[ S_{BV} = \int_{M_D} \left[ (B_{(n,0)} \wedge F_{(2,0)}) + c^* \wedge \frac{1}{2} [c, c] + A^* \wedge Dc 
+ B_{(n,0)} \wedge [c, B_{*-1}^*] + B_{(n-1,1)} \wedge \left(DB_{*(2,-1)}^* + [c, B_{(3,-2)}^*]\right) 
+ \sum_{q=2}^{n} B_{(n-q,q)} \wedge 
\left(DB_{*(1+q,-q)}^* + \frac{1}{2} \sum_{q'=0}^{q-2} [B_{(2+q',-1-q')}, B_{(q-q',q'+1-q')}^*] + [c, B_{(2+q,-1-q)}^*]\right) \right] \]  

in the component forms. What we call BV action is a minimal solution to the master equation and is considered as a generator for the BRST transformation in the BV formalism. It still contains antifields therefore is to be gauge fixed by introducing a gauge fermion together with antighosts and multipliers.

The BV action (16) seems to have some relations to the straightforward extension of the classical invariant action (7),

\[ \tilde{S} = \int_{M_D} \text{Tr}(\tilde{B} \wedge \tilde{F}) \]  

It is obvious that the conditions (12a,b) leading to the BRST algebra are nothing but formal field equations from this extended action. Though the extended action itself is not our object, our BV action seems to be something like a Legendre transform of it. Here we imply the Legendre transform of \( s \tilde{A} \) to \( \tilde{A}^* = \tilde{B} \), which is an analogy of translation from a Lagrangian to a Hamiltonian provided that \( s \tilde{A} \) pretends to be a time derivative of \( \tilde{A} \), though it is an odd time.
It may be understood as follows. If we define odd “canonical momenta” $\tilde{\pi}_A$ and $\tilde{\pi}_B$ of $\tilde{A}$ and $\tilde{B}$ as

$$\tilde{\pi}_A := \frac{\partial \tilde{\mathcal{L}}}{\partial (s\tilde{A})} = \tilde{B}, \quad \tilde{\pi}_B := \frac{\partial \tilde{\mathcal{L}}}{\partial (s\tilde{B})} = 0,$$

provided $\tilde{S} = \int_{\tilde{M}_D} \tilde{\mathcal{L}}$, then it follows that “Hamiltonian” $\tilde{H}$ is defined by

$$\tilde{H} := \text{Tr} \left( \tilde{\pi}_A \wedge s\tilde{A} + \tilde{\pi}_B \wedge s\tilde{B} \right) - \tilde{\mathcal{L}} = \text{Tr} \left( \tilde{B} \wedge s\tilde{A} - \tilde{B} \wedge \tilde{F} \right).$$

Our BV action (16) is nothing but this “Hamiltonian”, provided that its total sign is changed and it is integrated on the manifold, that is,

$$\mathcal{S}_{\text{BV}} = -\int_{\tilde{M}_D} \tilde{H}.$$

It seems to be not an accident that the BV action can be written in such a simple form as (21) or (16). A general structure may be found in our construction which can be applicable to the other TQFT’s. There seems to be also a connection with the odd time canonical formulation of the BV formalism [9]. We will discuss these features of our extended form method in future publication [10].

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