Confinement of spin-0 and spin-1/2 particles in a mixed vector–scalar coupling with unequal shapes for the potentials

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Abstract

The Klein–Gordon and the Dirac equations with vector and scalar potentials are investigated under a more general condition, \( V_v = V_s + \text{constant} \). These isospectral problems are solved in the case of squared trigonometric potential functions and bound states for either particles or antiparticles are found. The eigenvalues and eigenfunctions are discussed in some detail. It is revealed that a spin-0 particle is better localized than a spin-1/2 particle when they have the same mass and are subjected to the same potentials.

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There has been continuous interest in solving the Klein–Gordon (KG) and Dirac equations in four-dimensional (4D) space-time, as well as in lower dimensions for a variety of potentials. It is well known from the quarkonium phenomenology that the best fit for meson spectroscopy is found for a convenient mixture of vector and scalar potentials put by hand into the equations (see, e.g. [1] and references therein). The same can be said about the treatment of the nuclear phenomena describing the influence of the nuclear medium on the nucleons [2]. The mixed vector–scalar potential has also been analysed in 1+1 dimensions. In this mixed 2D context, all the work has been devoted to the investigation of the solutions of the relativistic equations by assuming that the vector and scalar potential functions are proportional [3]. In the present study, the problem of relativistic particles is considered with a mixing of vector and scalar Lorentz structures with unequal potential functions. The mixing for this enlarged class of problems is chosen in such a way that these relativistic problems are mapped into a Sturm–Liouville problem with the exactly solvable effective symmetric Pöschl–Teller potential [4, 5]. Then, the whole relativistic spectrum is found, whether the particle is massless or not. The process of solving the KG and the Dirac equations for the eigenenergies has been transmuted into the simpler and more efficient process of solving an irrational algebraic equation. Apart from the intrinsic interest as new solutions of fundamental equations in physics, the bound-state solutions of these systems are important in condensed matter mainly because of their potential applications ranging from ferroelectric domain walls in solids and magnetic chains (see e.g. [6]).

In the presence of vector and scalar potentials, the 1+1 dimensional time-independent KG equation for a particle of rest mass \( m \) reads

\[
-\hbar^2 c^2 \phi'' + (mc^2 + V_s)^2 \phi = (E - V_v)^2 \phi, \tag{1}
\]

where the prime denotes differentiation with respect to \( x \), \( E \) is the energy of the particle, \( c \) is the velocity of light and \( \hbar \) is the Planck constant. The subscripts for the terms of potential denote their properties under a Lorentz transformation: ‘\( v \)’ for the time component of the 2-vector potential and ‘\( s \)’ for the
scalar term. In the presence of time-independent vector and scalar potentials the 1+1 dimensional time-independent Dirac equation for a fermion of rest mass \( m \) reads

\[
[c\alpha p + \beta (mc^2 + V_\phi) + V_\psi]\psi = E \psi,
\]

where \( p \) is the momentum operator, \( \alpha \) and \( \beta \) are Hermitian square matrices satisfying the relations \( \alpha^2 = \beta^2 = 1 \), \( [\alpha, \beta] = 0 \). From the last two relations it follows that both \( \alpha \) and \( \beta \) are traceless and have eigenvalues equal to \( \pm 1 \), so that one can conclude that \( \alpha \) and \( \beta \) are even-dimensional matrices. One can choose the \( 2 \times 2 \) Pauli matrices satisfying the same algebra as \( \alpha \) and \( \beta \), resulting in a 2-component spinor \( \psi \). We use \( \alpha = \sigma_1 \) and \( \beta = \sigma_3 \). Provided that the spinor is written in terms of the upper and the lower components, \( \psi_+ \) and \( \psi_- \) respectively, the Dirac equation decomposes into:

\[
\text{i} \hbar c \psi'_\pm = [V_\psi - E \mp (mc^2 + V_\phi)]\psi_\pm.
\]

In the non-relativistic approximation (potential energies small compared to \( mc^2 \) and \( E \approx mc^2 \)), equation (1) becomes the Schrödinger equation with binding energy equal to \( E - mc^2 \) and a potential given by \( V_\psi + V_\phi \), so that \( \phi \) obeys the Schrödinger equation without distinguishing the contributions of vector and scalar potentials. In this approximation, equation (3) becomes \( \psi_\pm = p/(2mc)\psi_\pm \), and because of this \( \psi \) obeys the same equations as \( \phi \) while \( \psi_- \) is of order \( v/c \ll 1 \) relative to \( \psi_+ \). It is remarkable that the KG and the Dirac equations with a scalar potential, or a vector potential contaminated with some scalar coupling, is not invariant under the simultaneous changes \( V \to V + \text{constant} \) and \( E \to E + \text{constant} \), this is so because only the vector potential couples to the charge, whereas the scalar potential couples to the mass of the particle. Therefore, if there is any scalar coupling the energy itself has physical significance and not just the energy difference.

It is well known that a confining potential in the non-relativistic approach is not confining in the relativistic approach when it is considered as a Lorentz vector. It is surprising that relativistic confining potentials may result in non-confinement in the non-relativistic approach, simply because there is pair creation and the single-particle picture no longer holds. This last phenomenon is a consequence of the fact that vector and scalar potentials couple differently in the KG and in the Dirac equations, whereas there is no such distinction between them in the Schrödinger equation. This observation permits us to conclude that even a ‘repulsive’ potential can be a confining potential. The case \( V_\psi = -V_\phi \) presents bounded solutions in the relativistic approach, although it reduces to the free-particle problem in the non-relativistic limit. The attractive vector potential for a particle is, of course, repulsive for its corresponding antiparticle, and vice versa. However, the attractive (repulsive) scalar potential for particles is also attractive (repulsive) for antiparticles. For \( V_\psi = V_\phi \) and an attractive vector potential for particles, the scalar potential is counterbalanced by the vector potential for antiparticles as long as the scalar potential is attractive and the vector potential is repulsive. As a consequence there is no bounded solution for antiparticles. For \( V_\psi = 0 \) and a pure scalar attractive potential, one finds energy levels for particles and antiparticles arranged symmetrically about \( E = 0 \). For \( V_\psi = -V_\phi \) and a repulsive vector potential for particles, the scalar and the vector potentials are attractive for antiparticles but their effects are counterbalanced for particles. Thus recurring to this simple standpoint one can anticipate in the mind that there is no bound-state solution for particles in this last case of mixing. Regarding the structure of the wavefunctions under the simultaneous changes \( V_\psi \to -V_\psi \) and \( E \to -E \), from the charge-conjugation operation, one can see that if \( \psi \) is a solution with energy \( E \) for the potential \( V_\psi \), then \( \alpha \psi^* \) is also a solution with energy \( -E \) for the potential \( -V_\psi \). Thus, one has \( (\psi_{\pm})^* = \psi_{\mp}^* \) and that means that the upper and lower components of the Dirac spinor have their roles changed. As for the KG wavefunction, its nodal structure is trivially preserved in such a way that particle and antiparticle can be distinguished only by the eigenenergies.

Supposing that the vector and scalar potentials are constrained by the relation \( V_\psi - V_\phi = 0 \), where \( V_0 \) is a constant, and defining

\[
\varepsilon = E - V_0, \quad F_{\text{eff}} = \frac{\varepsilon^2 - mc^4}{2mc^2}, \quad V_{\text{eff}} = \frac{\varepsilon + mc^2}{mc^2} V_0,
\]

the KG equation can be written as

\[
-\frac{\hbar^2}{2m} \phi'' + V_{\text{eff}} \phi = F_{\text{eff}} \phi.
\]

On the other hand, for \( \varepsilon \neq -mc^2 \) the same Sturm–Liouville equation for \( \phi \) is obeyed by \( \psi_+ \), whereas \( \psi_- = -\text{i}h c \psi_+^* / (\varepsilon + mc^2) \). Otherwise, for \( \varepsilon = -mc^2 \), it might be possible the existence of an isolated solution given by

\[
\psi_+ = \text{constant}, \quad \psi_- = \frac{2\psi_+}{\text{i}h c} \int^x dx (V_\psi + mc^2).
\]

Of course, this solution does not exist if the domain is infinity because \( \psi_+ \) would not be square integrable. Note that apart from the possible isolated solution, \( \psi_+ \), satisfies the KG equation. An equally interesting result in the case of vanishing mass is that the spectrum just changes sign when \( V_0 \) does. As for the eigenfunctions, \( \phi \) and \( \psi_+ \) are invariant under the change of the sign of \( V_0 \) whereas \( \psi_- \) changes sign.

Let us consider the specific case of the two-parameter potential functions \( V_\psi = V_0 \sec^2 ax \) and \( V_\phi = V_0 \tan^2 ax \). In this case, the isolated solution of the Dirac equation for \( \psi_+ \) is not normalizable and the effective potential of the Sturm–Liouville problem for both \( \phi \) and \( \psi_+ \) can be expressed as

\[
V_{\text{eff}} = U_0 \tan^2 ax, \quad U_0 = \frac{\varepsilon + mc^2}{mc^2} V_0.
\]

Notice that \( V_{\text{eff}} \) is invariant under the change \( \alpha \to -\alpha \) so that the results can depend only on \( |\alpha| \). Furthermore, the effective potential is an even function under \( x \to -x \) in such a way that \( \phi \) and \( \psi_+ \) can be taken to be even or odd. When, \( \varepsilon < -mc^2 \) for \( V_0 > 0 \) and \( \varepsilon > -mc^2 \) for \( V_0 < 0 \) one has \( U_0 < 0 \). In this case, the effective potential consists of periodic wells.
and barriers. On the other hand, when \( \varepsilon > -mc^2 \) for \( V_0 > 0 \) and \( \varepsilon < -mc^2 \) for \( V_0 < 0 \) one has \( U_0 > 0 \) and the effective potential is identified as the exactly solvable symmetric Pöschl–Teller potential [4, 5]. In this last circumstance, due to the infinities at \( |x| = \pi/(2|\alpha|) \), attention can be restricted to \( |x| < \pi/(2|\alpha|) \). In fact, the effective potential is a well potential limited by infinite barriers at \( x = \pm \pi/(2|\alpha|) \), so that the capacity of the effective potential to hold bound-state solutions with \( \varepsilon > mc^2 \) for \( V_0 > 0 \) and \( \varepsilon < -mc^2 \) for \( V_0 < 0 \) is infinite (with a spectral gap in the interval \( |\varepsilon| < mc^2 \) for \( V_0 > 0 \)).

For the bound-state solutions, one can see that the normalizable eigenfunctions are subject to the boundary conditions \( \phi = \psi_\ast = 0 \) as \( |x| = \pi/(2|\alpha|) \) (where the potential becomes infinitely steep) in such a manner that the solution of our relativistic problem can be developed by taking advantage from the knowledge of the exact solution for the symmetric Pöschl–Teller potential. The corresponding effective eigenenergy is given by [4, 5]

\[
\frac{\varepsilon^2 - m^2c^4}{2mc^2} = \frac{h^2\lambda^2}{2m}(n^2 + 2n\lambda + \lambda), \quad n = 0, 1, 2, \ldots, \tag{8}
\]

where

\[
\lambda = \frac{1}{2} \left( 1 + \sqrt{1 + \frac{8mU_0}{h^2c^2\alpha^2}} \right). \tag{9}
\]

Now, (8) and (9) lead to the quantization condition

\[
2\sqrt{\varepsilon^2 - m^2c^4 + 2(\varepsilon + mc^2)V_0} - \sqrt{h^2c^2\alpha^2 + 8V_0(\varepsilon + mc^2)} = \hbar|\alpha|(2n + 1). \tag{10}
\]

The solutions of (10) determine the eigenvalues of the relativistic problem. This equation can be solved easily with a symbolic algebra program by searching eigenenergies in the range \( \varepsilon > mc^2 \) for \( V_0 > 0 \) and \( \varepsilon < -mc^2 \) for \( V_0 < 0 \), as foreseen by the preceding qualitative arguments. Of course, for \( V_0 > 0 \) one obtains \( \varepsilon \approx mc^2 \) for the lowest quantum numbers when \( V_0 \ll mc^2 \). On the other hand, for \( V_0 < 0 \) one finds \( \varepsilon \approx -mc^2 \) for the lowest quantum numbers when \( |V_0| \ll mc^2 \). It happens that there is at most one solution of (10) for a given quantum number. Figures 1 and 2 show the behaviour of the energies as a function of \( V_0 \) and \( \alpha \), respectively. It is noticeable from both of these figures, for \( V_0 > 0 \), that for a given set of potential parameters one finds that the lowest quantum numbers correspond to the lowest eigenenergies, as it should be for particle energy levels. For \( V_0 < 0 \) the spectrum presents a similar behaviour but the highest energy levels are labelled by the lowest quantum numbers and are to be identified with antiparticle levels. If we had plotted the spectra for a massless particle, we would encounter, up to the sign of \( \varepsilon \), identical spectra for both signs of \( V_0 \). At any circumstance, the spectrum contains either particle-energy levels or antiparticle-energy levels.

The KG eigenfunction as well as the upper component of the Dirac spinor can be given by [5]

\[
\psi = \psi_\ast = N 2^{-\lambda/2} \frac{\Gamma(n + 1)}{\Gamma(n + 2\lambda)} \frac{\Gamma(2\lambda)}{\Gamma(n + 2\lambda) \Gamma(\lambda + 1/2)} \times (1 - z^2)^{\lambda/2} C_{n+\lambda}^{(2\lambda)}(z), \tag{11}
\]

where \( z = \sin \alpha \varepsilon \) and \( C_{n+\lambda}^{(2\lambda)}(z) \) is the Gegenbauer (ultraspherical) polynomial of degree \( n \). Since \( C_n^{(2\lambda)}(\pm z) = (-)^n C_n^{(2\lambda)}(z) \) and \( C_n^{(2\lambda)}(z) \) has \( n \) distinct zeros (see, e.g. [7]), it becomes clear that \( \psi_\ast \) and \( \psi_\ast \) have definite and opposite parities. The constant \( N \) is the unit in the KG problem and it is chosen such that \( \int_{-\infty}^{\infty} dx (|\psi_\ast|^2 + |\psi|^2) = 1 \) in the Dirac problem. Figure 3 illustrates the behaviour of the upper and lower components of the Dirac spinor \( |\psi_\ast|^2 \) and \( |\psi|^2 \), and the position probability densities \( |\psi|^2 = |\psi_\ast|^2 + |\psi|^2 \) and \( |\phi|^2 \) for \( n = 0 \). The relative normalization constant was calculated numerically. Comparison of \( |\psi_\ast|^2 \) and \( |\psi|^2 \) shows that \( \psi_\ast \) is suppressed relative to \( \psi_\ast \). This result is expected since we have here a particle eigenstate. Surprisingly, the same behaviour is displayed for the antiparticle eigenstates (for \( V_0 < 0 \)). In addition, comparison of \( |\phi|^2 \) and \( |\psi|^2 \) shows that a KG particle tends to be better localized than a Dirac particle.

In summary, the methodology for finding solutions of the KG and the Dirac equations for the enlarged class of mixed vector–scalar potentials satisfying the constraint \( V_\ast = V_0 + V_\ast \) have been put forward. With the two-parameter potential functions \( V_\ast = V_0 \sec^2 \alpha \varepsilon \) and \( V_\ast = V_0 \tan^2 \alpha \varepsilon \), the
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Figure 3. $|\psi_+|^2$ (heavy dashed line), $|\psi_-|^2$ (light dashed line), $|\psi|^2 = |\psi_+|^2 + |\psi_-|^2$ (thick line) and $|\phi|^2$ (thin line) for $n = 0$ ($m = \hbar = c = 1$, $V_0 = 3$ and $\alpha = 5$).

KG equation and the Dirac equation for $\psi_+$ have been mapped into a Schrödinger-like equation with the symmetric Pöschl–Teller potential. The spectrum of these relativistic problems consists of infinitely many discrete eigenenergies related to either particle or antiparticle levels in such a way that Klein’s paradox is absent from the scenario. As has been commented above, changing the sign of $V_0$ allows us to migrate from the particle sector to the antiparticle sector and vice versa just by changing the sign of the eigenenergies as far as the spectra is concerned. These changes imply that $|\phi|$ maintains its nodal structure whereas $|\psi_+|$ and $|\psi_-|$ exchange theirs in such a way that the nodal structure of the position probability density is preserved. Although the KG and Dirac equations exhibit the very same spectrum their eigenfunctions make all the difference. In fact, we have shown that a KG particle tends to be better localized than a Dirac particle.

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