Gauging of 1d-space translations for nonrelativistic point particles

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Abstract

Gauging of space translations for nonrelativistic point particles in one dimension leads to general coordinate transformations with fixed Newtonian time. The minimal gauge invariant extension of the particle velocity requires the introduction of two gauge fields whose minimal self interaction leads to a Maxwellian term in the Lagrangean. No dilaton field is introduced. We fix the gauge such that the residual symmetry group is the Galilei group. In case of a line the two-particle reduced Lagrangean describes the motion in a Newtonian gravitational potential with strength proportional to the energy. For particles on a circle with certain initial conditions we only have a collective rotation with constant angular velocity.

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1 Introduction

Gauging a symmetry means to introduce space-time dependent functions instead of the constant parameters of a global symmetry group. The original action, invariant with respect to the global symmetry group, will in general not be gauge invariant. In order to cancel the terms which violate the invariance one has to introduce compensating fields as functions of space but also of time. These gauge fields have to obey suitable transformation laws with respect to the now local symmetry.

The most important interactions in current physics are gauge interactions. In elementary particle physics internal symmetry groups are gauged (Quantumelectrodynamics, Quantumchromodynamics, Standard Model). In general relativity, the space-time symmetry group of special relativity (Poincaré group) is gauged. Gauge

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interactions are also of importance in condensed matter physics: the fractional quantum hall effect may be described by an abelian gauge field with a Chern-Simons term minimally coupled to charged matter. Therefore the “gauge principle” may be called the leading principle for the construction of fundamental interactions in physics [1].

What about gauging the Galilei group in nonrelativistic physics? In a recent paper by De Pietri et al. [2] this task has been taken up for point particles in (3+1)-dimensions. The authors of [2] started with the nonrelativistic limit of general relativity and threw away all fields not coupled to matter in this limit. Our work will be distinct from [2] in two respects:

i) We ask for the smallest number of gauge fields doing the job with a minimal coupling Lagrangean,

ii) we don’t start with the nonrelativistic limit of a relativistic theory.

In this letter we treat the simplest example: point particles moving in one dimension. The underlying global transformations are space translations whose gauging leads to general coordinate transformations at fixed Newtonian time. For particles on a line we will show in section 2 that the minimal coupling for this example consists of only two gauge fields with an Maxwellian interaction term. In particular we don’t introduce an additional dilaton field as in (1+1)-gravity [3]. The arbitrary gauge function in the solution for the gauge fields will be chosen such, that the residual symmetry group of the action will be the Galilei group and the boundary term for field variations in the action vanishes (section 3). In section 4 we derive the reduced Lagrangean for one and two particles respectively and discuss the solutions of their equations of motion. In section 5 we extend our results to particles on a circle. With certain initial conditions relative motions of particles are absent. We obtain a collective rotation with constant angular velocity only.

2 Minimal-coupling Lagrangean

Let us start with $N$ nonrelativistic particles in free motion on a line ($\mathbb{R}^1$) described by the Lagrangean\footnote{For reasons of simplicity we give all particles the same mass $m=1$ in appropriate units.}

$$L_0 = \frac{1}{2} \sum_{\alpha=1}^{N} (\dot{x}_\alpha(t))^2$$

(1)

The equations of motion (EOM) following from (1)

$$\ddot{x}_\alpha = 0$$

(2)

are invariant with respect to global Galilei-transformations

$$(x,t) \rightarrow (x', t')$$

(3)
with
\[ x' = x + a + vt \] (4)
and
\[ t' = t + b \] (5)
where the parameters \( a, v \) and \( b \) take values in \( \mathbb{R}^1 \). Now we generalize (4) to a local transformation, given in infinitesimal form by
\[ \delta x = a(x, t) \] (6)
where \( a(x, t) \) is an arbitrary, twice differentiable and bounded function of its arguments. Eq. (6) describes local translations (including local boosts). Time translations are not considered for the moment.

Obviously the EOM (2) are not invariant with respect to the transformation (6). In order to repair that, we introduce two gauge fields \( h(x, t) \) and \( e(x, t) \) and replace \( \dot{x} \) in (1) by the function\footnote{Our procedure differs from the corresponding one in [2] applied to one space dimension. In [2] \( L_0 \) would be replaced by a polynomial of second order in \( \dot{x} \) requiring three gauge fields instead of two.}
\[ \xi = h(x, t)\dot{x} + e(x, t). \] (7)
Invariance of \( \xi \) with respect to (6) requires the following transformation rules for the gauge fields
\[ \delta h = -h\partial_x a, \quad \delta e = -h\partial_t a \] (8)
where we defined
\[ \delta f(x, t) := f'(x + \delta x, t) - f(x, t). \] (9)
Therefore, (7) supplies the minimal gauge invariant extension of \( \dot{x} \). Now \( L_0 \) in (1) has to be replaced by
\[ L_{\text{matter}} = \frac{1}{2} \sum_{\alpha=1}^{N} (\xi_\alpha(t))^2 \] (10)
with
\[ \xi_\alpha(t) := h(x_\alpha(t), t)\dot{x}_\alpha(t) + e(x_\alpha(t), t) \] (11)
We must supplement (10) by an invariant (or quasi-invariant) Lagrangean \( L_{\text{field}} \) describing the self-interaction of the gauge fields. Let us define a field strength \( F \)
\[ F := \frac{1}{\hbar}(\partial_t h - \partial_x e) \] (12)
From the definition (9) we obtain easily the commutator between \( \partial \) and any partial differentiation \( \partial \in (\partial_t, \partial_x) \)
\[ \delta \partial f = \partial \delta f - (\partial a) \partial_x f \] (13)
We infer from (13) and (8) that our field strength $F$ is gauge invariant
\[ \delta F = 0 \tag{14} \]

Therefore, any integral of the form
\[ \int_{\mathbb{R}^1} d\mu(x)K(F(x, t)) \tag{15} \]
with the invariant measure
\[ d\mu(x) := h(x, t)dx \tag{16} \]
is a candidate for $L_{\text{field}}$. The simplest, nontrivial example for the arbitrary function $K$ is given by a quadratic
\[ K(Z) = Z^2 \]

With this Maxwellian choice for $L_{\text{field}}$ our action takes the form
\[ S = \int dt(L_{\text{matter}} + L_{\text{field}}) \tag{17} \]
with
\[ L_{\text{field}} = \frac{1}{2g} \int dx h(x, t) F^2(x, t) \tag{18} \]
where $g$ is a coupling strength. By varying $S$ with respect to $x, h$ and $e$ we get the EOM
\[ \dot{\xi}_\alpha + \xi_\alpha F_\alpha = 0 \tag{19} \]
\[ \partial_t F + \frac{1}{2} F^2 = g \sum_\alpha \xi_\alpha \dot{x}_\alpha \delta(x - x_\alpha) \tag{20} \]
\[ \partial_x F = -g \sum_\alpha \xi_\alpha \delta(x - x_\alpha) \tag{21} \]

In order that the boundary term arising in the derivation of (21) vanishes and the integral (18) for $L_{\text{field}}$ exists, our gauge fields have to satisfy the following boundary conditions (A) at spatial infinity:

(A1) $e(x, t)$ and $h(x, t)$ are finite at $x = \pm \infty$,

(A2) $F(x, t)$ vanishes at $x = \pm \infty$. 

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3 Solutions of the field equations and gauge fixing

The solution of (21) satisfying (A2) is given by (ε(x) := x/|x|)

\[ F(x, t) = -\frac{g}{2} \sum_\alpha \xi_\alpha(t) \epsilon(x - x_\alpha(t)) \]  (22)

with the constraint

\[ \sum_\alpha \xi_\alpha(t) = 0 \]  (23)

With (19), (22) and (23) the EOM (20) is satisfied identical.

From the expression (22) for the field strength \( F \) and its relation to the gauge fields (12) we obtain \( h \) and \( e \) only modulo an arbitrary gauge function \( \Lambda(x, t) \)

\[ h(x, t) = \partial_x \Lambda(x, t) \]  (24)

\[ e(x, t) = \partial_t \Lambda(x, t) + \frac{g}{2} \sum_\alpha \xi_\alpha(t) \epsilon(x - x_\alpha(t))(\Lambda(x, t) - \Lambda_\alpha(t)) \]  (25)

Gauge fixing means to choose an appropriate function (class of functions) for \( \Lambda \).

Thereby we have to keep in mind, that the relation between \( \xi_\beta \) and the particle variables \( \{x_\alpha, \dot{x}_\alpha\} \) is gauge dependent\(^3\). A physical choice for \( \Lambda \) should fulfill two conditions:

i) The \( x_\alpha(t) \) describe particle motion in an inertial frame. Therefore, the symmetry remaining after gauge fixing (residual symmetry) is the Galilean symmetry in (1+1)-dimension.

ii) The boundary condition (A1) is satisfied.

We try to satisfy the first condition with the ansatz

\[ \Lambda(x, t) = x - (a + vt) \]  (26)

because due to (8) and (13) we have \( \delta \Lambda = 0 \) and therefore with (26)

\[ \delta x = \delta a + t \delta v \]  (27)

i.e. the residual symmetry is given by the Galilei group\(^4\). In section 4 we will show, that the resulting particle EOM describe particle motion in an inertial frame at least for \( N = 1 \) and \( N = 2 \). Now the second condition (A1) is satisfied too. From (23) - (26) we infer

\[ h(x, t) = 1 \]  (28)

and

\[ e(x, t) \xrightarrow{|x| \to \infty} -v - \frac{g}{2} \epsilon(x) \sum_\alpha \xi_\alpha x_\alpha \]  (29)

\(^3\)This situation reminds us of general relativity.

\(^4\)Time translations will be considered in section 4.
4 Reduced particle Lagrangean

With the results of section 3 the following equations remain for the determination of the particle trajectories $x_\alpha(t)$:

i) 
$$\xi_\alpha = \dot{x}_\alpha - v + \frac{g}{2} \sum_\beta \xi_\beta |x_\alpha - x_\beta|$$

(30) determines $\xi_\alpha$ in terms of the $\{x_\beta\}$. It is obtained from (7) and (24) - (26).

ii) 
$$\dot{\xi}_\alpha = \frac{g}{2} \sum_\beta \xi_\alpha \xi_\beta \epsilon(x_\alpha - x_\beta)$$

(31) obtained from (19) and (22),

iii) 
$$\sum_\alpha \xi_\alpha(t) = 0 .$$

(32)

In our gauge the total canonical particle momentum $P$ is given by

$$P = \sum_\alpha \xi_\alpha$$

(33) which is conserved due to (31)$^\text{5}$. It even has to vanish according to the constraint (32). Let us now discuss (30) - (32) for different particle numbers $N$.

$N = 1$

We have $\xi = \dot{x} - v$ and $\xi = 0$ i.e.

$$\ddot{x} = 0$$

(34) as it should be.

$N = 2$

With $\xi := \xi_1 - \xi_2$ and $x := x_1 - x_2$ we obtain from (30)

$$\xi_1 + \xi_2 = \frac{\dot{x}_1 + \dot{x}_2 - 2v}{1 - \frac{g}{2} |x|}$$

(35) and

$$\xi = \frac{\dot{x}}{1 + \frac{g}{2} |x|}$$

(36) According to (35) the constraint (32) leads to an uniform motion of the center of mass

$$\ddot{x}_1 + \ddot{x}_2 = 0$$

(37)

$^5$Our gauge fields carry no dynamical degrees of freedom. Therefore, the field contribution to the momentum vanishes.

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This result confirms the expectation that our gauge fixing (26) together with the constraint (32) describes particle motion in an inertial frame for \( N = 2 \).

Insertion of (36) and (32) into (31) leads to the EOM for \( x(t) \)

\[
\ddot{x} - \frac{g}{4} \frac{\dot{x}^2 \epsilon(x)}{1 + \frac{g^2}{2}|x|} = 0 \tag{38}
\]

This EOM may be derived from the reduced particle Lagrangean

\[
L_{\text{red}} = \frac{1}{4} \frac{\dot{x}^2}{1 + \frac{g^2}{2}|x|} \tag{39}
\]

The EOM (37) and \( L_{\text{red}} \) (39) are invariant with respect to time translations. Therefore our residual symmetry is the full Galilei group.

It is straightforward to show that \( L_{\text{red}} \) follows from our total Lagrangean (10) and (18) by inserting the solutions for the gauge fields. The EOM (38) may be solved explicitly:

With the conserved energy \( E \) corresponding to \( L_{\text{red}} \)

\[
E = \frac{1}{4} \frac{\dot{x}^2}{1 + \frac{g^2}{2}|x|} \tag{40}
\]

(38) takes the form

\[
\ddot{x} - gE \epsilon(x) = 0 \tag{41}
\]

This is Newton’s EOM for a gravitational potential whose strength is proportional to \( E \).

From (40) we conclude, that the relative particle motion described by \( x(t) \) is bounded for \( E = 0 \) and for \( E > 0 \) with \( g < 0 \), but unbounded in all other cases. From (41) we infer that \( x(t) \) is given by a second order polynomial.

5  Particles on a circle

Let us apply the foregoing results to a compact manifold, a circle. In order to do that we have to substitute everywhere

\[
x \to \varphi, \quad -\pi < \varphi < \pi \tag{42}
\]

and to require 2\( \pi \)-periodicity in \( \varphi \) for our gauge fields. In doing so the only problem arises from the step function contained in the expressions for \( F(\varphi, t) \) and \( e(\varphi, t) \). It decomposes into a periodic and a non-periodic part

\[
\epsilon(\varphi) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sin n \varphi}{n} + \frac{\varphi}{\pi} \tag{43}
\]
Therefore periodicity of $F(\phi, t)$, which now, as the general solution of (21), is given by

$$F(\phi, t) = -\frac{g}{2} \sum_{\alpha} \xi_{\alpha}(t) \epsilon(\phi - \phi_{\alpha}(t)) + f_0(t) \quad (44)$$

forces a vanishing coefficient of $\phi$

$$\sum_{\alpha} \xi_{\alpha} = 0 \quad (45)$$
in accordance with (23).

In order to reproduce the gauge (26) with a periodic gauge function $\Lambda(\phi, t)$ we have to put

$$\Lambda(\phi, t) = \Lambda_0(\phi) - (a + vt) \quad (46)$$

with

$$\Lambda_0(\phi) := 2 \arctan(tg \frac{\phi}{2}) \quad (47)$$

leading to

$$\Lambda_0(\phi) = \phi \quad \text{for } -\pi < \phi < \pi. \quad (48)$$

With this choice $e(\phi, t)$ is given by

$$e(\phi, t) = - v + (\Lambda_0(\phi) - (a + vt)) f_0(t) +$$

$$+ \frac{g}{2} \sum_{\alpha} \xi_{\alpha}(t) \epsilon(\phi - \phi_{\alpha}(t)) (\Lambda_0(\phi) - \phi_{\alpha}(t)) \quad (49)$$

Periodicity of $e(\phi, t)$ enforces the additional constraint

$$\sum_{\alpha} \xi_{\alpha} \phi_{\alpha} = 0 \quad (50)$$

Thus we obtain instead of (30) the relation

$$\xi_{\alpha} = \dot{\phi}_{\alpha} - v - (\phi_{\alpha} - (a + vt)) f_0 +$$

$$+ \frac{g}{2} \sum_{\beta} \xi_{\beta} |\phi_{\alpha\beta}| \quad (51)$$

with $-\pi < \phi_{\beta} < \pi \quad \forall \beta$ and $\phi_{\alpha\beta} := \phi_{\alpha} - \phi_{\beta}$.

In order to have time-translational invariance for our particle dynamics, the $t$-dependence of $\xi_{\alpha}(t)$ must arise solely from the particle trajectories $\phi_{\alpha}(t)$. Eq. (51) tells us, that this requires

$$f_0 = 0 \quad (52)$$

so that we have for the determination of our trajectories the same set of equations as in section 4 supplemented by (50).

Let us consider an initial condition where
i) all particles are at different positions at $t = 0$

\[ \varphi_\alpha(0) \neq \varphi_\beta(0) \quad \forall \quad \alpha \neq \beta \quad (53) \]

ii) the angular velocities at $t = 0$ are equal to $v$ for $N - 2$ particles, i.e.

\[ \dot{\varphi}_\alpha(0) = v \quad \text{for} \quad 1 \leq \alpha \leq N - 2 . \quad (54) \]

Then we obtain from (30) - (32) and (50)

\[ \dot{\varphi}_\alpha(t) = v \quad \forall \quad \alpha, \quad \forall t \in \mathbb{R}^1 \quad (55) \]

i.e. we have no relative particle motion but only a collective rotation with constant angular velocity\footnote{It is quite interesting to note that for $N = 2$ we can distinguish locally whether the particles are on a line or on a large circle.}.

The proof of (55) runs as follows:

Due to (30) and (50) we may represent $\xi_{1,2}$ as linear combinations of the others

\[ \xi_2 = \pm \frac{1}{\varphi_{12}} \sum_{\alpha = 3}^{N} \xi_\alpha \varphi_{12} \alpha \quad (56) \]

At $t = 0$ we obtain from (54), (56) and (30)$_{1 \leq \alpha \leq N - 2}$

\[ \xi_\alpha = 0 \quad \forall \alpha \quad (57) \]

which holds, due to (31) $\forall t \in \mathbb{R}^1$. Inserting (57) for arbitrary $t$ into (30) leads to the desired result (55).

\section{Conclusions}

We have shown that the application of the gauge principle to point particles in one dimension leads to a nontrivial interaction between two particles on a line. On a circle we observe for appropriate initial conditions a collective rotation only. In this case a sensitivity analysis for perturbations of the initial conditions would be useful. Work on the gauging of space translations for nonrelativistic matter fields is in progress. Results will be reported elsewhere.

Next we intend to treat the 2d-case with a Chern-Simons interaction for the gauge fields. Thereby we will include the second central charge of the Galilei group in $(2 + 1)$-dimensions in the free-particle Lagrangean (cp. \footnote{[4]}).

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