The symplectic form for the fiber bundles over a
Riemann surface II

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We obtain an explicit formula for the symplectic form over the double quotient with help of the Green function of a Riemann surface.

1 The symplectic form over the moduli space of fiber bundles

Let a compact Riemann surface $X$ be. We consider the holomorphic topologically trivial fiber bundles as connections over the trivial bundle; the symplectic form is defined over the connections spaces:

$$\omega(\alpha, \beta) = \int_X tr(\alpha \wedge \beta).$$

Where $\alpha$ and $\beta$ are some 1-forms with values in the trivial fiber bundle as elements of the tangent space of the connections. We can consider the moduli space as a symplectic quotient. The tangent space iss then identified with the harmonic 1-forms with values in the anti-symmetric endomorphisms.

$$H^1(X, A(C^n)).$$

2 The double quotient of the moduli space

A holomorphic fiber bundle and topologically trivial over a Riemann surface can be trivial outside a small disc around a point. The result is that the space of holomorphic fiber bundles can be considered as 1-cocycles of Čech with values in the complex group $Sl_n(C)$. With the equivalence relation over the cocycles, the moduli space is:

$$Sl_n(K)\backslash Sl_n((z))/Sl_n[[z]].$$

Where there has been a choice of local coordinates and $K$ is the set of Laurent series which are the meromorphic functions over the surface with pole in the point $p$. 

1
2.1 Decomposition over the double quotient

It is possible to form a decomposition of the double quotient:

\[ SL_n[z^{-1}].SL_n[[z]]/SL_n \hookrightarrow SL_n((z)), \]

and so an injection, with left quotient, we deduce the loop group:

\[ SL_n(K)ackslash SL_n[z^{-1}] \hookrightarrow SL_n(K)ackslash SL_n((z))/SL_n[[z]], \]

it is so possible to show that the symplectic form can be reduced and calculated over the double quotient, with translation in the identity.

3 The symplectic form of the moduli space \( \mathcal{M}_n(\Sigma) \)

3.1 The symplectic form of the space of moduli, \( \omega \)

When the holomorphic fiber bundles are considered (topologically trivial) as being connections over the trivial fiber bundle, the symplectic form can be considered as being given by the natural symplectic form of the space of unitary connections:

\[ \omega(\alpha_1, \alpha_2) = \int_{\Sigma} tr(\alpha_1 \wedge \alpha_2). \]

Where \( \alpha_1 \) and \( \alpha_2 \) are some 1-forms with values in the anti-symmetric endomorphisms of the trivial fiber bundle considered as elements of the tangent space of the space of connections. The space of moduli of the fiber bundles \( \mathcal{M}_n(\Sigma) \) can be considered as a symplectic quotient of the space of connections and the tangent space is then identified with the harmonic 1-forms with values in the antisymmetric endomorphisms of the fiber bundle:

\[ H^1(\Sigma, A(\mathbb{C}^n)). \]

3.2 The reduction of a holomorphic 1-cocycle in a 1-form over the surface \( \Sigma \)

There is no actual canonical and explicit mean to associate a connection with a holomorphic fiber bundle (stable). The problem is equivalent to find a metric of the stable holomorphic fiber bundle which allows to show him as an unitary representation of the Poincaré group. But, at the level of the tangent spaces in the trivial fiber bundle, the equivalent of the identification is simple, it is the isomorphism between the cohomology of Čech and the cohomology of de Rham obtained via the Hodge theory [9]. The reduction in a harmonic 1-form is then: let a holomorphic 1-cocycle \( f \) be at the point \( p \) and a choice of a partition of the unity adapted with the open sets of the surface which are the disc \( D \) and the surface \( \Sigma \) minus the point \( p \). There is then a function \( \varrho \) with value 1 near the
point and zero outside. Then the form of type $(0,1)$ is considered: $f \overline{\partial} \theta$ reduced in its harmonic counterpart in the Dolbeault-Grothendieck cohomology $[1]$: 

$$\phi(Q) = \int_{P \in \Sigma} f(P) \overline{\partial}_P \theta(P) \wedge \partial_P \overline{\partial}_Q h_A(P,Q).$$

Where $h_A$ is the Green function of the Riemann surface $\Sigma$. The harmonic form of the de Rham cohomology is then: $\alpha = \text{Re}(\phi)$.

### 3.3 The reduction of the symplectic form of the double quotient

Let two holomorphic fiber bundles be $f_1$ and $f_2$ considered as meromorphic functions over the disc $D$ and a pole in zero.

**Definition**: $h_A(P,Q)$ is the renormalisation of the Green function de Green:

$$h_A(P,Q) := h_A(P,Q) - \ln(|P - Q|),$$

where $P, Q \in \Sigma$ and $|P - Q|$ is the geodesic distance over the Riemann surface $\Sigma$.

**Lemme**: The Green function being biharmonic, it can be written $\partial_z \overline{\partial}_t : h_A(z,t)$ as double series holomorphic, anti-holomorphic.

**Proof**: The Green function de Green being biharmonic is the solution of the Laplacian $\Delta = \partial \overline{\partial} = \overline{\partial} \partial$. The partial derivations can then be written over the disc as holomorphic function in one of the variables and anti-holomorphic with the other; it shows, by the Cauchy formula, that the expression can be developed near zero in a double series.

$$\partial_z \overline{\partial}_t : h_A(z,t) := \sum_{n=0}^{+\infty} \sum_{m=0}^{+\infty} a_{n,m} z^n \overline{t}^m.$$ 

The symplectic form is then given by:

**Theorem**: The symplectic form corresponding with the tangent space of the double quotient at level of the trivial fiber bundle is:

$$\omega_t(f_1, f_2) = (2\pi)^2 \text{Re}(\sum_{n} \sum_{m} a_{n,m} \text{tr}([f_{n-1}^1] f_{m-1}^2))$$

pour $f_1(z) = \sum_{r=q}^{+\infty} f_r^1 z^r$ et $f_2(t) = \sum_{l=s}^{+\infty} f_l^2 t^l$.

**Proof**: The symplectic form is the real part of:

$$\int_{Q \in \Sigma} \text{tr}((\int_{P \in \Sigma} f_1^*(P) \overline{\partial}_P \theta(P) \wedge [\overline{\partial}_P \overline{\partial}_Q h_A(P,Q)]) \wedge$$

$$((\int_{P \in \Sigma} f_2(P) \overline{\partial}_P \theta(P) \wedge [\partial_P \overline{\partial}_Q h_A(P,Q)]).$$
There is a function \( \varrho \) with 1 value near the point and zero outside of the disc, the function goes towards the characteristic function of the disc. Stokes is applied two times and the property of the function of Green is used [10] :

\[
\int_{Q \in \Sigma} \left[ \partial_P \overline{\partial}_Q h_A(P, Q) \right] \wedge \left[ \partial_Q \overline{\partial}_Q' h_A(Q, Q') \right] = \partial_P \overline{\partial}_Q' h_A(P, Q').
\]

Instead an integration over all the surface \( \Sigma \), minus where \( \varrho \) is 1, as \( \varrho \) goes towards the characteristic function of the disc, Stokes is applied to \([\Sigma - D] \). The following expression of the symplectic form is obtained :

\[
\int_{P \in S^1} \int_{Q \in S^1} tr(f_1^*(P) f_2(Q)) [\partial_P \overline{\partial}_Q : h_A(P, Q:)].
\]

\( \partial_P \overline{\partial}_Q : h_A(P, Q) : \) is supposed being developed in a double series over the disc or a smaller; this double integral is explained around zero :

\[
\int \int_{z, t \in S^1} tr(f_1^*(z) f_2(t)) \sum_{n=k}^{+\infty} \sum_{m=p}^{+\infty} a_{n,m} z^n t^m d\bar{z} d\bar{t}.
\]

And so,

\[
\sum_{n=k}^{+\infty} \sum_{m=p}^{+\infty} \sum_{r=q}^{+\infty} \sum_{l=s}^{+\infty} a_{n,m} tr([f_r^1] [f_l^2]) \int \int_{z, t \in S^1} z^n t^m \delta_{n-1,r} \delta_{m-1,l} d\bar{z} d\bar{t}.
\]

developing in series the functions \( f_1 \) and \( f_2 \). So :

\[
(2\pi)^2 \sum_{n=k}^{+\infty} \sum_{m=p}^{+\infty} \sum_{r=q}^{+\infty} \sum_{l=s}^{+\infty} a_{n,m} tr([f_r^1] [f_l^2]) \delta_{n-1,r} \delta_{m-1,l},
\]

with \( \delta \), the symbol of Kronecker.
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