SZEGÖ TYPE LIMIT THEOREMS ON THE HEISENBERG GROUP

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Abstract. Let $\mathcal{H} = -\Delta_{\mathbb{H}} + V$ be the Schrödinger operator on the Heisenberg group $\mathbb{H}^n$, where $\Delta_{\mathbb{H}}$ is the full laplacian on $\mathbb{H}^n$ and $V$ is a positive smooth potential, bounded below and grows like $|g|^\kappa$, $\kappa > 0$ for large $|g|$. Let $\mathcal{P}_r$ be the orthogonal projection of $L^2(\mathbb{H}^n)$ onto the space of eigenfunctions of $\mathcal{H}$ with eigenvalue $\leq r$; Let $b$ be a bounded real valued integrable function on $\mathbb{H}^n$ and $M_b$ be the operator of multiplication by $b$ on $L^2(\mathbb{H}^n)$. Then for any $f \in C(\mathbb{R})$ we have

$$\lim_{r \to \infty} \frac{\text{tr}(f(\mathcal{P}_r M_b \mathcal{P}_r))}{\text{tr} \mathcal{P}_r} = \int_{\mathbb{H}^n} f(b(g)) \, dg.$$

Further, if $A$ be a 0-th order self-adjoint pseudo-differential operator on $L^2(\mathbb{H}^n)$ relative to the operator $1 + |\lambda|H + V(g), g \in \mathbb{H}^n, \lambda \in \mathbb{R}$ with symbol $a(g, \lambda)$, where $H$ is the Hermite operator on $L^2(\mathbb{R}^n)$ then

$$\lim_{r \to \infty} \frac{\text{tr}(f(\mathcal{P}_r A \mathcal{P}_r))}{\text{tr} \mathcal{P}_r} = \frac{\int_{G^r} f(a(g, \lambda)(\xi, x)) \, d\xi \, dx \, dg \, d\mu(\lambda)}{\int_{G^r} d\xi \, dx \, dg \, d\mu(\lambda)},$$

(Assuming one limit exists)

where $G^r = \{(g, \lambda, \xi, x) \in \mathbb{H}^n \times \mathbb{R}^* \times \mathbb{R}^n \times \mathbb{R}^n : |\lambda|(1 + |\xi|^2 + |x|^2) + V(g) \leq r\}$, $a(g, \lambda) = Op_W(a_g, \lambda)$, and $\mu(\lambda)$ is the Plancherel measure on the Heisenberg group. Also we show that the above limit on the right hand side remains unaltered under a compact perturbation of the pseudo-differential operator $A$ or a perturbation of the Schrödinger operator $\mathcal{H}$ by bounded self-adjoint operators on $L^2(\mathbb{H}^n)$.

1. Introduction

The observable quantities in the classical system are described by real valued functions on the phase space whereas in quantum systems they are given by self-adjoint operators on a Hilbert space. Therefore it is important to study the correspondence between the classical and quantum statistical mechanics. Pseudo-differential operator theory provides a natural platform to relate the classical and quantum mechanics. For instance in [22], Zelditch considered the Schrödinger operator on $\mathbb{R}^n$ of the form $\tilde{H} = -\frac{i}{2} \Delta + V$, where $V$ is a smooth positive function that grows like $V_0|x|^\kappa$, $\kappa > 0$ at infinity. He took a 0-th order self-adjoint pseudo-differential operator $A$ associated with a symbol $a(x, \xi)$ relative to Beals-Fefferman weights $\varphi_1(x, \xi) = 1, \varphi_2(x, \xi) = (1 + |\xi|^2 + V(x))^{1/2}$ and proved the following Szegö type theorem: For any continuous function $f$,

$$\lim_{\lambda \to \infty} \frac{\text{tr}(f(\mathcal{P}_\lambda A \mathcal{P}_\lambda))}{\text{rank} \mathcal{P}_\lambda} = \lim_{\lambda \to \infty} \frac{\int_{\tilde{H}(x, \xi) \leq \lambda} f(a(x, \xi)) \, dx \, d\xi}{\text{Vol}(\tilde{H}(x, \xi) \leq \lambda)},$$

(1.1)

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where \( \tilde{H}(x, \xi) = \frac{1}{2}|\xi|^2 + V(x) \) and \( P_\lambda \) is the orthogonal projection of \( L^2(\mathbb{R}^n) \) onto the space of the eigenfunctions of \( \tilde{H} \) with eigenvalue less equal to \( \lambda \), assuming one limit exists. Such asymptotic spectral formulae expressing the relation between functions of pseudo-differential operators and their symbols is an important and interesting problem in mathematical analysis. We refer to [6–8, 18, 21] for similar results in the literature.

We consider the Schrödinger operator \( H = -\Delta_H + V \) on the Heisenberg group \( \mathbb{H}^n \), where \( \Delta_H \) is the full laplacian on \( \mathbb{H}^n \) and \( V \) is a positive smooth potential, bounded below and grows like \(|g|^\kappa, \kappa > 0 \) for large

\[
|g| := (|x|^4 + |t|^2)^{\frac{1}{4}}, \quad g = (x, t) \in \mathbb{H}^n,
\]

defining the homogenous norm on \( \mathbb{H}^n \). Such operators are well known to have purely discrete spectrum whose eigenfunctions form a complete set orthonormal basis for \( L^2(\mathbb{H}^n) \) (see Theorem 2 of [17] and the \( L^2 - L^\infty \) boundedness of \( e^{-t\Delta_H} \) can be obtained from (2.2.1) of [10]). Let \( A = Op(Op^W(a_{g, \lambda})) \) be a bounded self-adjoint 0-th order pseudo-differential operator on \( L^2(\mathbb{H}^n) \) relative to the operator \( 1 + |\lambda|H + V(g) \) (defined in Subsection 3.1).

For each \( r > 0 \), \( \mathcal{P}_r A \mathcal{P}_r \) is a finite rank symmetric operator with spectral measure defined as the sum of Dirac delta functions at its eigen values. We show that the sequence of measures \( \frac{\text{tr} f(\mathcal{P}_r A \mathcal{P}_r)}{\text{tr} (\mathcal{P}_r)} \) converges to the weak limit \( \frac{\int_{\mathbb{H}^n} f(a_{g, \lambda}(x)) \, dx \, dg \, d\mu(\lambda)}{\int_{\mathbb{H}^n} \, dx \, dg \, d\mu(\lambda)} \). In particular, if \( b \) is a bounded real valued integrable function on \( \mathbb{H}^n \) then we obtain the following result with respect to the operator of multiplication \( M_b \):

**Theorem 1.1.** Consider the Schrödinger operator of the form \( H = -\Delta_H + V \) on the Heisenberg group \( \mathbb{H}^n \). Let \( \mathcal{P}_r \) be the orthogonal projection of \( L^2(\mathbb{H}^n) \) onto the space of eigenfunctions of \( H \) with eigenvalue \( \leq r \). Let \( b \) be a bounded real valued integrable function on \( \mathbb{H}^n \) and \( M_b \) be the operator of multiplication by \( b \) on \( L^2(\mathbb{H}^n) \). Then for any \( f \in C(\mathbb{R}) \) we have

\[
\lim_{r \to \infty} \frac{\text{tr} f(\mathcal{P}_r M_b \mathcal{P}_r)}{\text{tr} (\mathcal{P}_r)} = \int_{\mathbb{H}^n} f(b(g)) \, dg.
\]

We generalize Theorem 1.1 by taking a 0-th order self-adjoint pseudo-differential operator on \( L^2(\mathbb{H}^n) \) relative to the operator \( 1 + |\lambda|H + V(g) \), where \( H \) is the Hermite operator on \( L^2(\mathbb{R}^n) \) and \( \lambda \in \mathbb{R}^* \), in place of the multiplication operator \( M_b \) and obtain the following Szegö type limit theorem:

**Theorem 1.2.** Consider the Schrödinger operator of the form \( H = -\Delta_H + V \) on the Heisenberg group \( \mathbb{H}^n \). Let \( \mathcal{P}_r \) be the orthogonal projection of \( L^2(\mathbb{H}^n) \) onto the space of eigenfunctions of \( H \) with eigenvalue \( \leq r \); let \( A \) be a 0-th order self-adjoint pseudo-differential operator relative to the operator \( 1 + |\lambda|H + V(g) \) on \( L^2(\mathbb{H}^n) \) with symbol \( a_{g, \lambda} \),
where \( g \in \mathbb{H}^n, \lambda \in \mathbb{R}^+ \) and let \( f \in C(\mathbb{R}) \). Then
\[
\lim_{r \to \infty} \frac{\text{tr} f(P_r A P_r)}{\text{tr} (P_r)} = \lim_{r \to \infty} \frac{\int_{G^r} f(a_{g,\lambda}(\xi, x)) \, d\xi \, dx \, dg \, d\mu(\lambda)}{\int_{G^r} \, d\xi \, dx \, dg \, d\mu(\lambda)},
\]
(Assuming one limit exists)
where \( G^r = \{ (g, \lambda, \xi, x) \in \mathbb{H}^n \times \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n : |\lambda| (1 + |\xi|^2 + |x|^2) + V(g) \leq r \} \), \( a(g, \lambda) = Op^W(a_{g,\lambda}) \), and \( \mu(\lambda) \) is the Plancherel measure on the Heisenberg group.

We also show that the right hand limit in \((1.3)\) remains unaltered under a perturbation of the Schrödinger operator by a bounded self-adjoint operator \( B \) on \( L^2(\mathbb{H}^n) \) such that \( B + \mathcal{H} \) has discrete spectrum and the eigenfunctions of \( B + \mathcal{H} \) form a complete orthogonal basis for \( L^2(\mathbb{H}^n) \). Note that the operator \( e^{-t(B+\mathcal{H})} = e^{-tB} e^{-t\mathcal{H}} \) is a compact operator as \( e^{-tB} \) is a bounded operator for any \( t > 0 \) (see Theorem 2 of [17]).

**Theorem 1.3.** Consider the operator \( \mathcal{H}_1 = B + \mathcal{H} \) on the Heisenberg group \( \mathbb{H}^n \), where \( B \) is a bounded self-adjoint operator on \( \mathbb{H}^n \) such that \( \mathcal{H}_1 \) has purely discrete spectrum and the eigenfunctions of \( \mathcal{H}_1 \) form a complete orthogonal basis for \( L^2(\mathbb{H}^n) \). Let \( P'_r \) be the orthogonal projection of \( L^2(\mathbb{H}^n) \) onto the space of eigenfunctions of \( \mathcal{H}_1 \) with eigenvalue \( \leq r \); let \( A \) be a 0-th order self-adjoint pseudo-differential operator relative to the operator \( 1 + |\lambda|H + V(g) \) on \( L^2(\mathbb{H}^n) \) with symbol \( a(g, \lambda) \), where \( g \in \mathbb{H}^n, \lambda \in \mathbb{R}^+ \) and let \( f \in C(\mathbb{R}) \). Then
\[
\lim_{r \to \infty} \frac{\text{tr} f(P'_r A P'_r)}{\text{tr} (P'_r)} = \lim_{r \to \infty} \frac{\int_{G^r} f(a_{g,\lambda}(\xi, x)) \, d\xi \, dx \, dg \, d\mu(\lambda)}{\int_{G^r} \, d\xi \, dx \, dg \, d\mu(\lambda)},
\]
(Assuming one limit exists)
where \( G^r = \{ (g, \lambda, \xi, x) \in \mathbb{H}^n \times \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n : |\lambda| (1 + |\xi|^2 + |x|^2) + V(g) \leq r \} \), \( a(g, \lambda) = Op^W(a_{g,\lambda}) \), and \( \mu(\lambda) \) is the Plancherel measure on the Heisenberg group.

We show that the above theorems are valid under a compact perturbation of the pseudo-differential operator \( A \) in Corollary 6.3.

To establish our main results we need to consider the ratios of distributions associated to different measures and their asymptotic behaviours. The asymptotic limit of such ratios is computed using Tauberian theorem. For instance, Zelditch in [22] used Karamata’s Tauberian theorem (see [20]), whereas Robert [16] used the Keldysh Tauberian theorem (see [9]). However, we use the recent version of Tauberian theorem of Keldysh by Grishin-Poedintseva [5] and a theorem of Laptev-Safarov [12, 13] for estimate the error term to prove our main results.

Also we provide an alternative proof of the error estimate for \( \kappa \in (0, 1) \) without using pseudo-differential symbolic calculus, but by proving the boundedness of the operators \([A, V]\) and \([A, \mathcal{L}]\) on \( L^2(\mathbb{H}^n) \).
We build up the calculus of symbols the pseudo-differential operators relative to the operator $1 + |\lambda|H + V(g)$ on $L^2(\mathbb{H}^n)$ using similar techniques used in [3,4] and establish the link between these symbols and the scalar valued $(\lambda, V(g))$-Shubin classes. Then we construct pseudo-differential approximations to the operator $(H + u)^{-m}$ on $L^2(\mathbb{H}^n)$ and $(1 + |\lambda|(H + I) + V(g) + u)^{-m}$ on $L^2(\mathbb{H}^n)$ within the calculus of symbols defined related to $1 + |\lambda|H + V(g)$ and $1 + |\lambda|(1 + |\xi|^2 + |x|^2) + V(g)$ respectively. Constructing pseudo-differential approximations is almost classical. We refer to [1] for a detailed study.

We organize the paper as follows. In Section 2, we provide necessary background on the Hermite operator, pseudo-differential operators on $\mathbb{R}^n$, and discuss some basic results on the Heisenberg group. In Section 3, we develop the calculus of symbols relative to the operator $1 + |\lambda|H + V(g)$ on $L^2(\mathbb{H}^n)$ and establish the link between these symbols and the scalar valued $(\lambda, V(g))$-Shubin class symbols. We construct pseudo-differential approximation to the operator $(H + u)^{-m}$ on $L^2(\mathbb{H}^n)$ in Section 4. In Section 5 and 6, we prove our main results Theorem 1.1, 1.2, and 1.3. Finally, we show that our main results are valid under a compact perturbation of the pseudo-differential operator $A$. We conclude with an alternative proof of the error estimate without using pseudo-differential calculus for $\kappa \in (0, 1)$.

2. Notations and Background

The main aim of this section is to define the symbol classes on the Heisenberg group via the left invariant vector fields and their correspondence with certain symbol classes on $\mathbb{R}^n$. We start with the definition of the Hermite operator.

2.1. Hermite Operator. Let $H_k$ denote the Hermite polynomial on $\mathbb{R}$, defined by

$$H_k(x) = (-1)^k \frac{d^k}{d x^k}(e^{-x^2}) e^{x^2}, \quad k = 0, 1, 2, \cdots,$$

and $h_k$ denote the normalized Hermite functions on $\mathbb{R}$ defined by

$$h_k(x) = \left(2^k \sqrt{\pi} k!\right)^{-\frac{1}{2}} H_k(x) e^{-\frac{1}{2} x^2}, \quad k = 0, 1, 2, \cdots,$$

The Hermite functions $\{h_k\}$ are the eigenfunctions of the Hermite operator $H = -\frac{d^2}{d x^2} + x^2$ with eigenvalues $2k + 1, k = 0, 1, 2, \cdots$. These functions form an orthonormal basis for $L^2(\mathbb{R})$. The higher dimensional Hermite functions denoted by $\Phi_\alpha$ are then obtained by taking tensor product of one dimensional Hermite functions. Thus for any multi-index $\alpha \in \mathbb{N}_0^n$ and $x \in \mathbb{R}^n$, we define $\Phi_\alpha(x) = \prod_{j=1}^n h_{\alpha_j}(x_j)$. The family $\{\Phi_\alpha\}$ is then an orthonormal basis for $L^2(\mathbb{R}^n)$. They are eigenfunctions of the Hermite operator $H = -\Delta + |x|^2$ corresponding to eigenvalues $(2|\alpha| + n)$, where $|\alpha| = \sum_{j=1}^n \alpha_j$. 


2.2. Pseudo-Differential Operator on $\mathbb{R}^n$. Given a reasonable function $a$ on $\mathbb{R}^n \times \mathbb{R}^n$, the corresponding operator $T_a$ associated with the function $a$ given by

$$T_a f(x) = a(x, D) f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} a(x, \xi) \hat{f}(\xi) \, d\xi, \quad \forall x \in \mathbb{R}^n$$

for all Schwartz class functions $f$ on $\mathbb{R}^n$, where the Fourier transform of $f$ is defined by

$$\hat{f}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f(x) e^{-ix \cdot \xi} \, dx, \quad \forall \xi \in \mathbb{R}^n.$$

The operator $T_a$ is called pseudo-differential operator corresponding to the symbol $a$. Let $m \in \mathbb{R}$, $0 \leq \delta < \rho \leq 1$. Then the symbol class $S^m_{\rho, \delta}(\mathbb{R}^n)$ consists of those functions $a(x, \xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ satisfying

$$|\partial_x^\alpha \partial_{\xi}^\beta a(x, \xi)| \leq C_{\alpha, \beta} (1 + |\xi|^2)^{-m-\delta|\alpha|+\rho|\beta|}$$

(2.1)

for all multi-indices $\alpha, \beta$. We take $\rho = 1$ and $\delta = 0$ throughout the paper and denote the symbol class by $S^m(\mathbb{R}^n)$.

The Weyl quantization $Op^W$ for a “reasonable” symbol $a$ in $\mathbb{R}^n \times \mathbb{R}^n$ is given by

$$Op^W(a) f(u) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(u-v) \cdot \xi} a \left( \frac{u+v}{2}, \xi \right) f(v) \, dv \, d\xi, \quad \forall u \in \mathbb{R}^n,$$

for all Schwartz class functions $f$ on $\mathbb{R}^n$. The composition of two Weyl quantized operators $Op^W(a)$ and $Op^W(b)$ is given by $Op^W(a)Op^W(b) = Op^W(a \# b)$, where (see [14])

$$a \# b(\xi, u) = (2\pi)^{-2n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-i(\xi-u) \cdot \eta} \frac{a(\xi, u)}{2} b(\eta, v) \, d\eta \, dv \, d\xi,$$

and asymptotically

$$a \# b(x, \xi) \sim \sum_{j=0}^{N} \frac{1}{j!} \left( \frac{i}{2} \right)^j a(x, \xi) \left( \frac{\partial}{\partial \xi} \frac{\partial}{\partial u} - \frac{\partial}{\partial \xi} \frac{\partial}{\partial u} \right)^j b(x, \xi) + S_N(x, \xi)$$

(2.2)

(arrow points towards the factor to be differentiated) with $S_N \in S^{m_1+m_2-N}(\mathbb{R}^n)$.

Further, if $Op^W(a)$ is a trace class operator whose symbol $a(x, \xi) \in L^1(\mathbb{R}^n \times \mathbb{R}^n)$, then

$$tr(Op^W(a)) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} a(x, \xi) \, dx \, d\xi.$$

Moreover, the correspondence $a \rightarrow Op^W(a)$ is an isometry of $L^2(\mathbb{R}^n \times \mathbb{R}^n)$ onto the Hilbert-Schmidt operators on $L^2(\mathbb{R}^n)$. This yields

$$tr(AB^*) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (a \# b)(x, \xi) \, dx \, d\xi = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} a(x, \xi) \overline{b(x, \xi)} \, dx \, d\xi,$$

(2.3)

where $A = Op^W(a)$ and $B = Op^W(b)$.

2.3. Heisenberg Group. One of the simple and natural example of non-abelian, non-compact group is the famous Heisenberg group $\mathbb{H}^n$, which plays an important role in several branches of mathematics. The Heisenberg group $\mathbb{H}^n$ is a Nilpotent Lie group whose underlying manifold is $\mathbb{R}^{2n+1}$ and the group operation is defined by

$$(x, y, t)(x', y', t') = (x + x', y + y', t + t' + \frac{1}{2}(xy' - x'y)),$$
where \((x, y, t)\) and \((x', y', t')\) are in \(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}\). Moreover, \(\mathbb{H}^n\) is a unimodular Lie group on which the Haar measure is the usual Lebesgue measure \(dx\,dy\,dt\). The canonical basis for the Lie algebra \(h_n\) of \(\mathbb{H}^n\) is given by the left-invariant vector fields:

\[
X_j = \frac{\partial}{\partial x_j} - \frac{y_j}{2} \frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} + \frac{x_j}{2} \frac{\partial}{\partial t}, \quad j = 1, 2, \ldots, n, \quad \text{and} \quad T = \frac{\partial}{\partial t},
\]

satisfying the commutator relation \([X_j, Y_j] = T, \quad j = 1, 2, \ldots, n\).

The sublaplacian and the full laplacian on the Heisenberg group are defined as

\[
L = \sum_{j=1}^n (X_j^2 + Y_j^2) = \sum_{j=1}^n \left( \left( \frac{\partial}{\partial x_j} - \frac{y_j}{2} \frac{\partial}{\partial t} \right)^2 + \left( \frac{\partial}{\partial y_j} + \frac{x_j}{2} \frac{\partial}{\partial t} \right)^2 \right)
\]

and

\[
\Delta_H = \sum_{j=1}^n (X_j^2 + Y_j^2 + T_j^2)
\]

respectively. By Stone-von Neumann theorem, the only infinite dimensional unitary irreducible representations (up to unitary equivalence) are given by \(\pi_\lambda, \lambda \in \mathbb{R}^*\), where \(\pi_\lambda\) is defined by

\[
\pi_\lambda(x, y, t) f(u) = e^{i\lambda(t+\frac{1}{2}xy)} e^{i\sqrt{|\lambda|} u} f(u + \sqrt{|\lambda|} x), \quad \forall u \in \mathbb{R}^n, \forall f \in L^2(\mathbb{R}^n) \text{ and } (x, y, t) \in \mathbb{H}^n.
\]

We use the convention

\[
\sqrt{\lambda} := \text{sgn}(\lambda) \sqrt{|\lambda|} = \begin{cases} \sqrt{\lambda}, & \lambda > 0, \\ -\sqrt{|\lambda|}, & \lambda < 0. \end{cases}
\]

For each \(\lambda \in \mathbb{R}^*\), the group Fourier transform of \(f \in L^1(\mathbb{H}^n)\) is a bounded linear operator on \(L^2(\mathbb{R}^n)\) defined by

\[
\hat{f}(\lambda) \equiv \pi_\lambda(f) = \int_{\mathbb{H}^n} f(x, y, t) \pi_\lambda^*(x, y, t) \, dx \, dy \, dt.
\]

We denote \(B(L^2(\mathbb{R}^n))\) to be the set of all bounded operators on \(L^2(\mathbb{R}^n)\). If \(f \in L^2(\mathbb{H}^n)\), then \(\hat{f}(\lambda)\) is a Hilbert-Schmidt operator on \(L^2(\mathbb{R}^n)\) and satisfies the Plancherel formula

\[
\int_{\mathbb{R}^*} \|\hat{f}(\lambda)\|_{S_2}^2 \, d\mu(\lambda) = \|f\|_{L^2(\mathbb{H}^n)},
\]

where \(\|\cdot\|_{S_2}\) stands for the norm in the Hilbert space \(S_2\) of all Hilbert-Schmidt operators on \(L^2(\mathbb{R}^n)\) and \(d\mu(\lambda) = c_n |\lambda|^n \, d\lambda\) where \(c_n\) is a constant.

**Theorem 2.1.** For all Schwartz class functions on \(\mathbb{H}^n\), the following inversion formula holds:

\[
f(g) = \int_{\mathbb{R}^*} \text{tr} \left( \pi_\lambda(g) \hat{f}(\lambda) \right) \, d\mu(\lambda), \quad \forall g \in \mathbb{H}^n.
\]

For a detailed study on the Heisenberg group we refer to Thangavelu [19].
**Definition 2.2.** Let \( \sigma : \mathbb{H}^n \times \mathbb{R}^* \to B(L^2(\mathbb{R}^n)) \) be a operator valued function. The pseudo-differential operator \( T_\sigma \) corresponding to \( \sigma \) is defined by

\[
T_\sigma f(g) = \int_{\mathbb{R}^*} \text{tr} \left( \pi_\lambda(g) \sigma(g, \lambda) \hat{f}(\lambda) \right) d\mu(\lambda), \quad g \in \mathbb{H}^n
\]

for all \( f \in \mathcal{S}(\mathbb{H}^n) \). The operator valued function \( \sigma \) is called the symbol of the pseudo-differential operator \( T_\sigma \). We also often denote the pseudo-differential operator \( T_\sigma \) as \( \text{Op}(\sigma) \).

3. \((\lambda, V(g))\)-Shubin classes \( \Sigma^m_{\rho, \lambda, V}(\mathbb{R}^n) \)

We define the Shubin metric \( g^{(\rho, \lambda, V(g))}_{\xi, u} \) depending on both the parameter \( \lambda \in \mathbb{R}^* \) and \( V(g), g \in \mathbb{H}^n \) on \( \mathbb{R}^{2n} \) as

\[
g^{(\rho, \lambda, V(g))}_{\xi, u}(d\xi, du) := \left( \frac{|\lambda|}{1 + |\lambda| (1 + |\xi|^2 + |u|^2) + V(g)} \right)^\rho (d\xi^2 + du^2).
\]

The associated positive function \( M^{(\lambda, V(g))}(\xi, u) \) is

\[
M^{(\lambda, V(g))}(\xi, u) := (1 + |\lambda| (1 + |\xi|^2 + |u|^2) + V(g))^{\frac{\rho}{2}}.
\]

We consider these \((\lambda, V(g))\)-families of metrics for the case \( \rho = 1 \) as introduced in [2].

**Proposition 3.1.** For each \( \lambda \in \mathbb{R}^* \) and \( g \in \mathbb{H}^n \), the metric \( g^{(\rho, \lambda, V(g))} \) is of Hörmander type i.e., \( g \) is uncertain, slowly varying and temperate (see Definition 6.4.2 page 456 of [4]) where the conjugate of \( g^{(\rho, \lambda, V(g))} \) is \( (g^{(\rho, \lambda, V(g))})^\omega \) given by

\[
(g^{(\rho, \lambda, V(g))})^\omega(d\xi, du) = \left( \frac{1 + |\lambda| (1 + |\xi|^2 + |u|^2) + V(g)}{|\lambda|} \right)^\rho (d\xi^2 + du^2).
\]

Moreover the gain is given by

\[
\Lambda_{g^{(\rho, \lambda, V(g))}} = \left( \frac{1 + |\lambda| (1 + |\xi|^2 + |u|^2) + V(g)}{|\lambda|} \right)^{2\rho}.
\]

**Proof.** The proof of the proposition follows exactly as in Proposition 1.20 of [2] for \( \rho = 1 \). \( \square \)

For each parameter \( \lambda \in \mathbb{R}^* \) and \( V(g), g \in \mathbb{H}^n \) we define the \((\lambda, V(g))\)-Shubin class as

\[
\Sigma^m_{\rho, \lambda, V}(\mathbb{R}^n) := \{ a \in C(\mathbb{R}^n \times \mathbb{R}^n) : \|a\|_{\Sigma^m_{\rho, \lambda, V}(\mathbb{R}^n)} < \infty \text{ for each } N \in \mathbb{N}_0 \},
\]

where

\[
\|a\|_{\Sigma^m_{\rho, \lambda, V}(\mathbb{R}^n), N} := \sup_{(\xi, u) \in \mathbb{R}^n \times \mathbb{R}^n, |\alpha|, |\beta| \leq N} |\lambda|^{-\rho(|\alpha| + |\beta|)} (1 + |\lambda| (1 + |\xi|^2 + |u|^2) + V(g))^{-\frac{m - \rho(|\alpha| + |\beta|)}{2}} \left| \partial_\xi^\alpha \partial_u^\beta a(\xi, u) \right|
\]
is finite. In other words, a symbol \( a = \{a(\xi, u)\} \) is in \( \Sigma^m_{p,\lambda, V(g)}(\mathbb{R}^n) \) if and only if it satisfies
\[
\forall \alpha, \beta \in \mathbb{N}^n, \forall (\xi, u) \in \mathbb{R}^n \times \mathbb{R}^n, \text{ there exists } C = C_{\alpha, \beta} > 0 \text{ such that }
|\partial_\xi^\alpha \partial_u^\beta a(\xi, u)| \leq C|\lambda|^{\frac{m}{2}|\alpha| + \frac{3}{2}|\beta|} (1 + |\lambda| (1 + |\xi|^2 + |u|^2) + V(g))^{-\frac{m-3}{2}}.
\]

3.1. The symbol class \( S^m_{p,\delta, H}(\mathbb{H}^n) \). We define the symbol class \( S^m_{p,\delta, H}(\mathbb{H}^n) \) relative to the operator \( 1 + |\lambda| H + V(g) \) as in Definition 5.2.11 of [4] by the following family of seminorms which are finite:
\[
\|\sigma\|_{S^m_{p,\delta, H}, a,b,c} := \sup_{g \in H^m, \lambda, \in \mathbb{R}} \|\sigma(g, \lambda)\|_{S^m_{p,\delta, H}, a,b,c}, \quad a, b, c \in \mathbb{N}_0
\]
where
\[
\|\sigma(g, \lambda)\|_{S^m_{p,\delta, H}, a,b,c} := \sup_{|\alpha| \leq a, |\beta| \leq b, |\gamma| \leq c} \||\pi_\lambda(I - L) + V(g)\|^{\frac{m - n - |\beta| - |\gamma| - |\alpha|}{2}} \|X^\beta_{\gamma} \Delta^\alpha \sigma(g, \lambda)(\pi_\lambda(I - L) + V(g))^{-\frac{1}{2}}\|_{op}.
\]
with \( \alpha = (\alpha_1, \alpha_2, \alpha_3) = (\alpha_{11}, \alpha_{12}, \ldots, \alpha_{3n}, \alpha_{21}, \alpha_{22}, \ldots, \alpha_{2n}, \alpha_3) \in \mathbb{N}^n \times \mathbb{N}^n \times \mathbb{N}_0, |\alpha| = |\alpha_1| + |\alpha_2| + 2|\alpha_3| \) and \( \| \cdot \|_{op} \) denote the operator norm on \( B(L^2(\mathbb{H}^n)) \). The difference operators are
\[
\Delta^\alpha := \Delta_{x,1}^{\alpha_1} \Delta_{y,2}^{\alpha_2} \Delta_{\xi,3}^{\alpha_3}, \quad \Delta_{x,1}^{\alpha_1} = \Delta_{x,1}^{\alpha_{11}} \Delta_{x,2}^{\alpha_{12}} \ldots \Delta_{x,n}^{\alpha_{1n}}, \quad \Delta_{y,2}^{\alpha_2} = \Delta_{y,1}^{\alpha_{21}} \Delta_{y,2}^{\alpha_{22}} \ldots \Delta_{y,n}^{\alpha_{2n}}
\]
and
\[
X^\alpha = X^{\alpha_{11}} X^{\alpha_{12}} \ldots X^{\alpha_{1n}} , \quad Y^{\alpha_1} = Y^{\alpha_{11}} Y^{\alpha_{12}} \ldots Y^{\alpha_{1n}}.
\]
The symbol in \( S^m_{p,\delta, H}(\mathbb{H}^n) \) relative to the operator \( 1 + |\lambda| H + V(g) \) can be written in terms of scalar-valued \((\lambda, V(g))\)-symbol. More precisely, the symbols \( \sigma = \sigma(g, \lambda) \) in \( S^m_{p,\delta, H}(\mathbb{H}^n) \) are all of the form
\[
\sigma(g, \lambda) = Op^W(a_{g,\lambda}(\xi, u)),
\]
with the \((\lambda, V(g))\)-symbols \( a_{g,\lambda} \) satisfying some properties described below in terms of the family of \((\lambda, V(g))\)-Shubin classes.

**Theorem 3.2.** Let \( m, p, \delta, \in \mathbb{R} \) be real numbers such that \( 1 \geq p \geq \delta \geq 0 \) and \( (p, \delta) \neq (0, 0) \).

If \( \sigma = \sigma(g, \lambda) \) is in \( S^m_{p,\delta, H}(\mathbb{H}^n) \), then there exist a smooth function \( a = a(g, \lambda, \xi, u) = a_{g,\lambda}(\xi, u) \) on \( \mathbb{H}^n \times \mathbb{R}^* \times \mathbb{R}^n \times \mathbb{R}^n \) such that
\[
\sigma(g, \lambda) = Op^W(a_{g,\lambda})
\]
with \( \partial_{\lambda,\xi,u}^\alpha X_{\gamma}^\beta a_{g,\lambda} \in \Sigma^{m - 2|\alpha| + |\beta|}_{p,\lambda, V(g)}(\mathbb{R}^n) \) for each \((g, \lambda) \in \mathbb{H}^n \times \mathbb{R}^* \) satisfying
\[
\sup_{(g, \lambda) \in \mathbb{H}^n \times \mathbb{R}^*} \left\| \partial_{\lambda,\xi,u}^\alpha X_{\gamma}^\beta a_{g,\lambda} \right\|_{\Sigma^{m - 2|\alpha| + |\beta|}_{p,\lambda, V(g)}(\mathbb{R}^n), N} < \infty
\]
for every \( N \in \mathbb{N}_0 \). More precisely, for every \( N \in \mathbb{N}_0 \) there exist \( C > 0 \) and \( a, b, c \) such that
\[
\sup_{(g, \lambda) \in \mathbb{H}^n \times \mathbb{R}^*} \left\| \partial_{\lambda,\xi,u}^\alpha X_{\gamma}^\beta a_{g,\lambda} \right\|_{\Sigma^{m - 2|\alpha| + |\beta|}_{p,\lambda, V(g)}(\mathbb{R}^n), N} \leq C \|\sigma\|_{S^m_{p,\delta, H}, a,b,c}.
\]
Conversely, if \( a = \{ a_{g,\lambda,\xi,u} = a_{g,\lambda}(\xi,u) \} \) is a smooth function on \( \mathbb{H}^n \times \mathbb{R}^* \times \mathbb{R}^n \times \mathbb{R}^n \) satisfying (3.2) for every \( N \in \mathbb{N}_0 \), then there exist a unique symbol \( \sigma \in S_{p,\delta,H}^m(\mathbb{H}^n) \) such that \( \sigma(g,\lambda) = Op^W(a_{g,\lambda}) \). Furthermore, for every \( a,b,c \) there exists \( C > 0 \) and \( N \in \mathbb{N}_0 \) such that

\[
\|\sigma\|_{S_{p,H}^m,\sigma,\beta} \leq C \sup_{(g,\lambda) \in \mathbb{H}^n \times \mathbb{R}^*} \left\| \partial^\beta g a_{g,\lambda} \right\|_{S_{p,H}^{m-2|\alpha|+4|\beta|}(\mathbb{R}^n),N}.
\]

**Proof.** The proof is similar to the proof of Theorem 6.5.1 of [4]. \( \square \)

In other words, Theorem 3.2 yields that \( \sigma \in S_{p,\delta,H}^m(\mathbb{H}^n) \) is equivalent to \( \sigma(g,\lambda) = Op^W(a_{g,\lambda}) \) for each \( (g,\lambda) \in \mathbb{H}^n \times \mathbb{R}^* \) with \( a_{g,\lambda} \in C^\infty(\mathbb{R}^{2n}) \) satisfying: For any \( \alpha \in \mathbb{N}_0^{2n+1} \) there exists a constant \( C > 0 \) such that for every \( (\xi,u) \in \mathbb{R}^n \times \mathbb{R}^n \)

\[
|\partial^\beta \hat{\partial}^\beta \partial^\beta g a_{g,\lambda}(\xi,u)| \leq C_{\alpha,\beta,\alpha,\beta} |\lambda|^{\frac{|\alpha|+2|\beta|}{2}} (1 + |\lambda|^2 + |\xi|^2 + |u|^2) + V(g) \]

We take \( \rho = 1 \) and \( \delta = 0 \) throughout the article and denote the symbol classes \( S_{1,\delta,H}^m(\mathbb{H}^n) \) by \( S_{\delta}^m(\mathbb{H}^n) \).

**Example 3.3.** For any \( \beta \in \mathbb{R} \), \( \pi(1-L), V(g)^{\beta} \) and \( (1 + \lambda^2)^\beta \) are symbols with order 2, 2, 2, and 4 respectively.

**Remark 3.4.** Let \( \sigma \in S_{\delta}^m(\mathbb{H}^n) \). Then we have the following properties.

1. If \( \alpha \in \mathbb{N}_0^n \) then the symbol \( \{ X^\beta_{g,\lambda} \sigma(g,\lambda), (g,\lambda) \in \mathbb{H}^n \times \mathbb{R}^* \} \) is in \( S_{\delta}^m(\mathbb{H}^n) \) and

\[
\left\| X^\beta_{g,\lambda} \sigma(g,\lambda) \right\|_{S_{\delta,H}^m[a,b,c]} \leq C_{b,\beta} \| \sigma(g,\lambda) \|_{S_{\delta,H}^m[a,b+|\beta|,c]}.
\]

2. If \( \alpha \in \mathbb{N}_0^n \) then the symbol \( \{ \Delta^\alpha \sigma(g,\lambda), (g,\lambda) \in \mathbb{H}^n \times \mathbb{R}^* \} \) is in \( S_{\delta,H}^{m-|\alpha|}(\mathbb{H}^n) \) and

\[
\left\| \Delta^\alpha \sigma(g,\lambda) \right\|_{S_{\delta,H}^{m-|\alpha|}[a,b,c]} \leq C_{a,\alpha} \| \sigma(g,\lambda) \|_{S_{\delta,H}^m[a,b,c]}.
\]

3. If \( \sigma_1 \in S_{H}^m(\mathbb{H}^n) \) and \( \sigma_2 \in S_{H}^m(\mathbb{H}^n) \) then \( \sigma(g,\lambda) = \sigma_1(g,\lambda) + \sigma_2(g,\lambda) \) is in \( S_{H}^{m+n}(\mathbb{H}^n) \) and

\[
\left\| \sigma(g,\lambda) \right\|_{S_{H}^{m+n}[a,b,c]} \leq \left\| \sigma_1(g,\lambda) \right\|_{S_{H}^{m+n}[a,b,c]} + \left\| \sigma_2(g,\lambda) \right\|_{S_{H}^{m+n}[a,b,c]}.
\]

4. If \( \sigma_1 \in S_{H}^m(\mathbb{H}^n) \) and \( \sigma_2 \in S_{H}^m(\mathbb{H}^n) \) then \( \Delta^\alpha \sigma_1 X^\beta_{g,\lambda} \sigma_2 \in S_{H}^{m+n}[a]\).

**Lemma 3.5.** If \( A \) is a trace class pseudo-differential operator on \( L^2(\mathbb{H}^n) \) with symbol \( a(\cdot,\cdot) \in L^1(\mathbb{H}^n \times \mathbb{R}^*,S_1,d\mu(\lambda)) \), then

\[
tr(A) = \int_{\mathbb{H}^n} \int_{\mathbb{R}^*} tr(a(g,\lambda)) \ dg d\mu(\lambda).
\]

**Proof.** For all \( f \in L^2(\mathbb{H}^n) \), we have

\[
(Af)(g) = \int_{\mathbb{R}^*} tr(\pi^*_\lambda(g)a(g,\lambda)\hat{f}(\lambda)) \ d\mu(\lambda)
= \int_{\mathbb{H}^n} \int_{\mathbb{R}^*} tr(\pi^*_\lambda(g)a(g,\lambda)\pi_\lambda(g_1)) \ d\mu(\lambda) f(g_1) \ dg_1
\]
\[ = \int_{\mathbb{H}^n} K(g, g_1) f(g_1) \, dg_1, \]
with
\[ K(g, g_1) = \int_{\mathbb{R}^n} tr(\pi^*_\lambda(g) a(g, \lambda) \pi_\lambda(g_1)) \, d\mu(\lambda). \]
Therefore
\[ tr(A) = \int_{\mathbb{H}^n} K(g, g) \, dg = \int_{\mathbb{H}^n} \int_{\mathbb{R}^n} tr(a(g, \lambda)) \, dg \, d\mu(\lambda). \]

The correspondence \( a \to Op(a) \) is an isometry from \( L^2(\mathbb{H}^n \times \mathbb{R}^*, S_2, d\mu(\lambda)) \) onto the set of Hilbert-Schmidt operators on \( L^2(\mathbb{H}^n) \) via square integrable kernels [15]. This allows us to write
\[ tr(Op(a) \circ Op(b)^*) = \int_{\mathbb{H}^n} \int_{\mathbb{R}^n} tr(a \#_{\mathbb{H}^n} b^{(x)})(g, \lambda) \, dg \, d\mu(\lambda) \]
\[ = \int_{\mathbb{H}^n} \int_{\mathbb{R}^n} tr(a(g, \lambda)b^{(x)}(g, \lambda)) \, dg \, d\mu(\lambda), \quad (3.5) \]
where \( a \#_{\mathbb{H}^n} b \) is the symbol of the composition \( Op(a) \circ Op(b) \) (defined in Theorem 3.8) and \( b^{(x)} \) is the symbol of \( Op(b)^* \), the adjoint of \( Op(b) \) (see page 365 of [4]).

Now as in the proof of Calderón-Vaillancourt theorem (Theorem 5.7.1 of [4]), we get the following Calderón-Vaillancourt theorem for the symbol class \( S^0_\mathcal{H}(\mathbb{H}^n) \).

**Theorem 3.6** (The Calderón-Vaillancourt theorem). Let \( \sigma \in S^0_\mathcal{H}(\mathbb{H}^n) \). Then \( Op(\sigma) \) extends a bounded operator on \( L^2(\mathbb{H}^n) \). Moreover, there exist a constant \( C > 0 \) and a seminorm \( \| \cdot \|_{S^0_{\mathcal{H},a,b,c}} \) with computable integers \( a, b, c \in \mathbb{N}_0 \) independent of \( Op(\sigma) \) such that
\[ \| Op(\sigma)\phi \|_{L^2(\mathbb{H}^n)} \leq C\|\sigma\|_{S^0_{\mathcal{H},a,b,c}}\|\phi\|_{L^2(\mathbb{H}^n)}, \quad \phi \in S(\mathbb{H}^n). \]

**3.2. Composition of symbols.** Let \( a \in S^{m_1}_\mathcal{H}(\mathbb{H}^n) \) and \( b \in S^{m_2}_\mathcal{H}(\mathbb{H}^n) \). Then the composition of pseudo-differential operators corresponding to the symbols \( a \) and \( b \) defines a pseudo-differential operator and the symbol \( \sigma \) of the composition is given by the following asymptotic expansion (3.6). We add a constraint on \( V \) (see [11] and [22]) which guarantees the asymptotic expansion (3.6).

**Definition 3.7.** The potential \( V \) is said to be temperate potential if there exists \( C > 0 \) such that
\[ \| (\pi_\lambda(I - \mathcal{L}) + V(x))^{-1}(\pi_\lambda(I - \mathcal{L}) + V(xx_1)) \|_{op} \leq C|x_1|^k \]
for all \( x, x_1 \in \mathbb{H}^n \) and for some constant \( k > 0 \).

**Theorem 3.8** (Composition formula). Let \( a \in S^{m_1}_\mathcal{H}(\mathbb{H}^n) \) and \( b \in S^{m_2}_\mathcal{H}(\mathbb{H}^n) \). Then the composition \( Op(a) \circ Op(b) \) is a pseudo-differential operator with symbol \( \sigma \in S^{m_1+m_2}_\mathcal{H}(\mathbb{H}^n) \).
having asymptotic expansion

\[ \sigma(x, \lambda) \sim \sum_{\alpha} \Delta^{\alpha} a(x, \lambda) X^{\alpha}_x b(x, \lambda), \]  

(3.6)

where the asymptotic expansion means that for every \( M \in \mathbb{N} \), we have

\[ \sigma - \sum_{|\alpha| \leq M} \Delta^{\alpha} a X^{\alpha}_g b \in S_{\mathcal{H}}^{m_1 + m_2 - M}(\mathbb{H}^n). \]

In order to estimate the remainder term in composition formula, we need the following lemma.

**Lemma 3.9.** Let \( m_1, m_2 \in \mathbb{R}, \beta_0 \in \mathbb{N}_0^n \), and \( M, M_1 \in \mathbb{N}_0 \). Suppose that

\[
\begin{align*}
    m_2 & \leq 2M_1 < M - m_1 + v_1 \\
    m_2 & \leq 2M_1 < -m_1 - M.
\end{align*}
\]

(3.7)

If \( M \geq 2M_1 \), then only the second condition may be assumed. Then there exist a constant \( C > 0 \) and two pseudo-norms \( \| \cdot \|_{S_{\mathcal{H}}^{m_1, r, a_1, b_1}} \), \( \| \cdot \|_{S_{\mathcal{H}}^{m_2, 0, b_2, 0}} \) such that for any two symbol \( a, b \) and for any \((x, \pi) \in \mathbb{H}^n \times \mathbb{R}^* \), we have

\[
\left\| X^\beta_x \left( a \circ b (x, \pi) - \sum_{|\alpha| \leq M} \Delta^{\alpha} a(x, \pi) X^{\alpha}_x b(x, \pi) \right) \right\| \leq C \|a\|_{S_{\mathcal{H}}^{m_1, r, a_1, b_1}} \|b\|_{S_{\mathcal{H}}^{m_2, 0, b_2, 0}}.
\]

**Proof.** Let \( k_1 \) and \( k_2 \) are the kernels of \( Op(a) \) and \( Op(b) \) respectively. Then we have \( Op(a) \circ Op(b) = Op(\sigma) \), where the symbol of the composition is given by

\[ \sigma(x, \lambda) = \int_G k_1(x, z) \pi(z) \ast b(xz^{-1}, \lambda) \, dz. \]

First we consider the case when \( \beta_0 = 0 \). Thus

\[
\begin{align*}
    \sigma(x, \lambda) - \sum_{|\alpha| \leq M} \Delta^{\alpha} a(x, \lambda) X^{\alpha}_x b(x, \lambda) & = \int_{\mathbb{H}^n} k_1(x, z) \pi(\lambda) \ast (\pi(\lambda)(I - \mathcal{L}) + V(x))^{M_1} (\pi(\lambda)(I - \mathcal{L}) + V(x))^{-M_1} \\
    & \times \left( b(xz^{-1}, \lambda) - \sum_{|\alpha| \leq M} q_{\alpha} (z^{-1}) X^{\alpha}_x b(x, \lambda) \right) \, dz \\
    & = \sum_{|\beta| = 1}^{M_1} \int_{\mathbb{H}^n} k_1(x, z) \pi(\lambda) \ast (\pi(\lambda)(I - \mathcal{L}))^{\beta} V(x)^{M_1 - \beta} (\pi(\lambda)(I - \mathcal{L}) + V(x))^{-M_1} \\
    & \times \left( b(xz^{-1}, \lambda) - \sum_{|\alpha| \leq M} q_{\alpha} (z^{-1}) X^{\alpha}_x b(x, \lambda) \right) \, dz \\
    & = \sum_{|\beta_1| + |\beta_2| \leq 2M_1} \sum_{|\beta| = 1}^{M_1} \int_{\mathbb{H}^n} \hat{X}^{\beta_1}_z k_1(x, z) V(x)^{M_1 - \beta} \pi(\lambda)^* \\
    & \times \hat{X}^{\beta_2}_z (\pi(\lambda)(I - \mathcal{L}) + V(x))^{-M_1} \mathcal{R}_{x,M}^{b(\cdot, \lambda)} (z^{-1}) \, dz,
\end{align*}
\]
where \( R_{x,M}^{b(-\lambda)}(z) = b(x,z,\lambda) - \sum_{|\alpha| \leq M} q_{\alpha}(z) X_x^\alpha b(x, \lambda) \). Taking the operator norm on \( B(L^2(\mathbb{R}^n)) \), we have

\[
\|\sigma(x, \lambda) - \sum_{|\alpha| \leq M} \Delta^\alpha a(x, \lambda) X_x^\alpha b(x, \lambda)\|_{op} \leq \sum_{|\beta_1| + |\beta_2| \leq 2M_1} \sum_{|\beta| = 1} M_1 C_M \int_{\mathbb{H}^n} |\hat{\Delta}^{\beta_1} k_1(x, z)V(x)^{M_1 - \beta}|
\times \left\| \hat{\Delta}^{\beta_2} (\pi_\lambda (I - \mathcal{L}) + V(x))^{-M_1} R_{x,M}^{b(-\lambda)}(z^{-1}) \right\|_{op}.
\]

Using Taylor’s estimate for vector-valued functions given in Proposition 3.1.40 and by Theorem 3.1.51 of [4], there is a constant \( c_1 \) (depending on \( M \)) such that

\[
\left\| \hat{\Delta}^{\beta_2} (\pi_\lambda (I - \mathcal{L}) + V(x))^{-M_1} R_{x,M}^{b(-\lambda)}(z^{-1}) \right\|_{op} = \left\| (\pi_\lambda (I - \mathcal{L}) + V(x))^{-M_1} \right\|_{op}
\times \left\| \hat{\Delta}^{\beta_2} b(x, \lambda) \right\|_{op}
\leq C_M \sum_{|\gamma| \leq (M - |\beta_2|) + 1} |z|^{|\gamma|} \sup_{|x_1| \leq c_1 |z|} \left\| (\pi_\lambda (I - \mathcal{L}) + V(x))^{-M_1} \right\|_{op}
\times \left\| (\pi_\lambda (I - \mathcal{L}) + V(x))^{-M_1} \right\|_{op}
\times \left\| \hat{\Delta}^{\beta_2} b(x, \lambda) \right\|_{op}.
\]

Using the fact that \( V \) is a temperate potential, we have

\[
\left\| \hat{\Delta}^{\beta_2} (\pi_\lambda (I - \mathcal{L}) + V(x))^{-M_1} R_{x,M}^{b(-\lambda)}(z^{-1}) \right\|_{op} \leq C_M c_1^{M_1} \sum_{|\gamma| \leq (M - |\beta_2|) + 1} |z|^{|\gamma| + M_1} \sup_{|x_1| \leq c_1 |z|} \left\| (\pi_\lambda (I - \mathcal{L}) + V(x))^{-M_1} \right\|_{op}
\times \left\| (\pi_\lambda (I - \mathcal{L}) + V(x))^{-M_1} \right\|_{op}
\times \left\| \hat{\Delta}^{\beta_2} b(x, \lambda) \right\|_{op}.
\]

Let \( \sigma_1(x, \lambda) = V(x)^{M_1 - \beta} a(x, \lambda) \). Then \( \sigma_1 \in S^{m_1 + 2(M_1 - \beta)}_{\mathcal{H}} \) with kernel \( \tilde{k}_x = V(x)K_1(x, \cdot) \). So \( \sigma_1 = \pi(\tilde{k}_x) \). Choosing \( M, M_1 \) such that it satisfies (3.7) and the conditions of Lemma 5.5.6 in [4]. Thus

\[
\|\sigma(x, \lambda) - \sum_{|\alpha| \leq M} \Delta^\alpha a(x, \lambda) X_x^\alpha b(x, \lambda)\|_{op} \leq \sum_{|\beta_1| + |\beta_2| \leq 2M_1} \sum_{|\beta| = 1} M_1 C_M c_1^{M_1} \int_{\mathbb{H}^n} |z|^{|\gamma| + M_1} \left| \hat{\Delta}^{\beta_1} k_1(x, z)V(x)^{M_1 - \beta} \right| dz
\times \|b\|_{S^{m_2}_{\mathcal{H}}, 0, b_2, 0} \leq C_2 \|\sigma_1\|_{S^{m_1 + 2(M_1 - \beta), R_2, 0, b_2, 0}} \times \|b\|_{S^{m_2}_{\mathcal{H}}, 0, b_2, 0} \leq C \|a\|_{S^{m_1, R, a_1, b_1}} \times \|b\|_{S^{m_2}_{\mathcal{H}}, 0, b_2, 0}.
\]
where \( C = C_1 \| V \|_{\mathcal{S}_R^{2(M_2 - \beta), R, a_1, b_1}} \). The general case \( \beta_0 \neq 0 \) follows by adopting the proof of Lemma 5.5.5 in [4]. \( \square \)

**Proof of Theorem 3.8:** Let \( T = Op(a) \circ Op(b) \). Then
\[
Tf(x) = \int_{\mathbb{H}^n} \int_{\mathbb{R}^n} f(z) k_2(y, z^{-1}y) k_1(x, z) dy dz,
\]
where \( k_1 \) and \( k_2 \) are the kernels of \( Op(a) \) and \( Op(b) \) respectively. Furthermore, we have \( Op(a) \circ Op(b) = Op(\sigma) \), where
\[
\sigma(x, \pi) = \int_G k_1(x, z) \pi(z) * b(xz^{-1}, \pi) \, dz.
\]
By the Taylor series expansion (see [4]) of \( b \) in the first variable we get
\[
\sigma(x, \lambda) \sim \sum_\alpha \Delta^\alpha a(x, \lambda) X^\alpha_x b(x, \lambda).
\]
The remainder term is estimated similar to Theorem 5.5.3 of [4] with few modifications. We will only indicate the main steps with modifications in our setting. Let \( m = m_1 + m_2, \beta_0 \in \mathbb{N}_0 \) and \( M_0 \in \mathbb{N} \). By Theorem 3.6, we have
\[
\left\| X^{\beta_0}_{x} \tau_M(x, \pi)(\pi\lambda(I - \mathcal{L}) + V(x))^{-\frac{m-M_0}{2}} \right\|_{op} \\
= \left\| X^{\beta_0}_{x} \tau_M(x, \pi)(\pi\lambda(I - \mathcal{L}))^{-\frac{m-M_0}{2}} \left[ (\pi\lambda(I - \mathcal{L}))^{-\frac{m-M_0}{2}} (\pi\lambda(I - \mathcal{L}) + V(x))^{-\frac{m-M_0}{2}} \right] \right\|_{op} \\
\leq \left\| X^{\beta_0}_{x} \tau_M(x, \pi) \left( \pi\lambda(I - \mathcal{L}) \right)^{-\frac{m-M_0}{2}} \left[ (\pi\lambda(I - \mathcal{L}))^{-\frac{m-M_0}{2}} (\pi\lambda(I - \mathcal{L}) + V(x))^{-\frac{m-M_0}{2}} \right] \right\|_{op} \\
\leq C \left\| X^{\beta_0}_{x} \tau_M(x, \pi) \left( \pi\lambda(I - \mathcal{L}) \right)^{-\frac{m-M_0}{2}} \right\|_{op}, \tag{3.8}
\]
where \( \tau_M = a \circ b - \sum_{|a| \leq M} \Delta^\alpha a X^\alpha_x b \). We fix \( m_2' := -m_1 + M_0 \). Then we can find \( M \geq \max(M_0, v_1) \) such that \( -m_1 + M - m_2' \geq 2 \). This shows that we can find \( M_1 \) satisfying the second condition of (3.7) for \( m_1, m_2' \) and therefore also the first. Hence we can apply Lemma 3.9 to \( M, M_1 \) and the symbols \( a \) and \( b(\pi\lambda(I - \mathcal{L}))^{-\frac{m-M_0}{2}} \), with orders \( m_1 \) and \( m_2' \). Thus by (3.8) and Theorem 3.6, we have
\[
\left\| X^{\beta_0}_{x} \tau_M(x, \pi) \left( \pi\lambda(I - \mathcal{L}) \right)^{-\frac{m-M_0}{2}} \right\|_{op} \\
\leq \left\| a \right\|_{S^{m_1, R, a_1, b_1}_M} \left\| b \right\|_{S^{m_2', 0, b_2, 0}_M} \\
= \left\| a \right\|_{S^{m_1, R, a_1, b_1}_M} \left\| (\pi\lambda(I - \mathcal{L} + V(g)))^{-\frac{m_0}{2}} b(\pi\lambda(I - \mathcal{L}))^{-\frac{m-M_0}{2}} \right\|_{op} \\
= \left\| a \right\|_{S^{m_1, R, a_1, b_1}_M} \left\| (\pi\lambda(I - \mathcal{L} + V(g)))^{-\frac{m_0}{2}} b(\pi\lambda(I - \mathcal{L} + V(g)))^{-\frac{m-M_0}{2}} \right\|_{op} \\
\times \left( \pi\lambda(I - \mathcal{L} + V(g)) \right)^{-\frac{m-M_0}{2}} \left( \pi\lambda(I - \mathcal{L}) \right)^{-\frac{m-M_0}{2}} \right\|_{op} \\
\leq \left\| a \right\|_{S^{m_1, R, a_1, b_1}_M} \left\| (\pi\lambda(I - \mathcal{L} + V(g)))^{-\frac{m_0}{2}} b(\pi\lambda(I - \mathcal{L} + V(g)))^{-\frac{m-M_0}{2}} \right\|_{op}.
\[
\times \left\| (\pi_\lambda (I - \mathcal{L} + V(g)))^{-\frac{m-M_0}{2}} (\pi_\lambda (I - \mathcal{L}))^{-\frac{m-M_0}{2}} \right\|_{op}
\leq C \|a\|_{S^m_{H^1,r,a_1,b_1}} \left\| b \left( \pi_\lambda (I - \mathcal{L} + V(g)) \right)^{-\frac{m-M_0}{2}} \right\|_{S^m_{H^1,0,b_2,0}^2}.
\]

The rest of proof follows along the similar lines of Theorem 5.5.3 in [4].

4. Symbolic calculus relative to \((1 + |\lambda|H + V(g) + w)\) on the Heisenberg group

Let \(\Gamma \subset \mathbb{C}\) be a curve enclosing \(\mathbb{R}^+\) and \(w\) vary over \(\Gamma\). In particular, let us consider the curve \(\Gamma\) be made up of two half-lines hinged at \(-1\) and makes an angles of \(\pm \frac{\pi}{4}\) with respect to the real axis. In order to construct the pseudo-differential approximation to the operator \((\mathcal{H} + u)^{-m}\), we need to define the following symbol class.

4.1. The symbol class \(S^m_{\rho,\delta,H,w}(\mathbb{H}^n)\). We define the symbol class \(S^m_{\rho,\delta,H,w}(\mathbb{H}^n)\) relative to the operator \(1 + |\lambda|H + V(g) + |w|\) defined as in Subsection 3.1 and \((\lambda, V(g))\)-Shubin class defined as in Section 3 relative to the weight \(1 + |\lambda|(|\xi|^2 + |x|^2 + 1) + V(g) + |w|\). Also we get the similar result for the symbol class \(S^m_{\rho,\delta,H,w}(\mathbb{H}^n)\) as in Theorem 3.2. When \(\rho = 1\) and \(\delta = 0\), we denote the symbol classes \(S^m_{1,0,H,w}(\mathbb{H}^n)\) by \(S^m_{H,w}(\mathbb{H}^n)\).

**Proposition 4.1.** Let \(a_{g,\lambda,w}(\xi, u) = (|\lambda|(1 + |\xi|^2 + |u|^2) + V(g) - w)^s\), \(s \in \mathbb{R}\) and \(\sigma(g, \lambda, w) = Op^W(a_{g,\lambda,w})\). Then \(\sigma \in S^2_{H,w}(\mathbb{H}^n)\).

**Proof.** By Theorem 3.2, \(Op^W(|\lambda|(1 + |\xi|^2 + |u|^2) + V(g) - w) \in S^2_{H,w}(\mathbb{H}^n)\). Now

\[
\partial_\xi^\alpha \partial_u^\beta a_{g,\lambda,w}(\xi, u) = \sum_{1 \leq \theta \leq |\alpha| + |\beta| + |\sigma| - |\beta|} \left( |\lambda|(1 + |\xi|^2 + |u|^2) + V(g) - w \right)^{s - \theta}
\]

\[
\times \bigotimes_{j=1}^\theta \partial_\xi^{\nu_j} \partial_u^{\mu_j} X_{g}^{\beta_j} \left( |\lambda|(1 + |\xi|^2 + |u|^2) + V(g) - w \right).
\]

Since each term is bounded by a constant times

\[
(|\lambda|(1 + |\xi|^2 + |u|^2) + V(g) - w)^{s - \theta} \prod_{j=1}^\theta |\lambda| \frac{|\nu_j| + |\nu_j|}{2} \times (1 + |\lambda|(1 + |\xi|^2 + |u|^2) + V(g) + |w|)^{2 - 2|\nu_j| - 2(|\nu_j| + |\nu_j|)} \leq |\lambda| \frac{|\alpha| + |\beta|}{2} (1 + |\lambda|(1 + |\xi|^2 + |u|^2) + V(g) + |w|)^{2 - 2|\alpha| - 2(|\alpha| + |\beta|)}.
\]
thus for any $w \in \Gamma$, we have

$$
\left| \partial^\alpha_c \partial^\beta_u \tilde{\partial}^\gamma_{\lambda, \xi, w} X^\delta_g a_{g, \lambda, w}(\xi, u) \right| \leq C |\lambda| \left( 1 + |\lambda| (1 + |\xi|^2 + |u|^2) + V(g) + |w| \right)^{-2\left(2|\alpha| + |\beta| + |\gamma| \right) \left( |\alpha| + |\beta| + |\gamma| \right) + 2}.
$$

Now

$$
\left\| \tilde{\partial}^\gamma_{\lambda, \xi, u} X^\delta_g a_{g, \lambda, w} \right\|_{2^{-2\alpha}(\mathbb{R}^n), \mathcal{N}} = \sup_{(\xi, u) \in \mathbb{R}^n \times \mathbb{R}^n} |\lambda| \left( 1 + |\lambda| (1 + |\xi|^2 + |u|^2) + V(g) + w \right)^{-2\left(2|\alpha| + |\beta| + |\gamma| \right) \left( |\alpha| + |\beta| + |\gamma| \right) + 2} \times \left| \partial^\alpha_c \partial^\beta_u \tilde{\partial}^\gamma_{\lambda, \xi, u} X^\delta_g a_{g, \lambda, w}(\xi, u) \right| \leq C_{\alpha, \beta, \mathcal{N}}.
$$

Thus $\sigma = \sigma(g, \lambda, w) = Op^W(a_{g, \lambda, w}) \in S^{2\alpha}_{\mathcal{H}, w}(\mathbb{H}^n)$ by (3.3).

Construct a symbol $R_N(g, \lambda, w)$ such that $(\mathcal{H} - w) \circ Op(R_N(g, \lambda, w)) = I_{L^2(\mathbb{H}^n)} + Op(S_N(g, \lambda, w))$, where $S_N \in S_{\mathcal{H}, w}(\mathbb{H}^n)$ or equivalently $|\lambda| \left( |H + I| + V(g) - w \right) \#_{\mathbb{H}^n} R_N(g, \lambda, w) = I_{L^2(\mathbb{H}^n)} + S_N(g, \lambda, w)$. By substituting the expansion $R_N = R_{-2} + R_{-3} + \cdots + R_{-N}$ with the property that $R_{-2-\ell} \in S_{\mathcal{H}, w}^{2\alpha}(\mathbb{H}^n)$ into the asymptotic expansion (3.6), we get

$$
(\lambda|H + I| + V(g) - w) \#_{\mathbb{H}^n} R_N(g, \lambda, w)
$$

$$
= \sum_{|\alpha| \leq N} \Delta^\alpha (\lambda|H + I| + V(g) - w) X^\alpha_g R_N(g, \lambda, w) + S_N(g, \lambda, w)
$$

$$
= I_{L^2(\mathbb{H}^n)} + S_N(g, \lambda, w).
$$

Now solving for $R_{-2-\ell}$ recursively by comparing the order by order of the symbols so that the sum equals to 1, we get

$$
R_{-2}(g, \lambda, w) = (\lambda|H + I| + V(g) - w)^{-1}
$$

and

$$
R_{-2-\ell}(g, \lambda, w) = (\lambda|H + I| + V(g) - w)^{-1} \sum_{|\alpha| \leq N} \Delta^\alpha (\lambda|H + I| + V(g) - w) X^\alpha_g R_{-2-|\alpha|}(g, \lambda, w)
$$

(4.2)

for $w \in \Gamma$. To understand the dependence on $r$, we express the symbol $R_{-2-\ell}$ differently in the following proposition.

**Proposition 4.2.** Let $w \in \Gamma$. Then

$$
R_{-2-\ell}(g, \lambda, w) = (\lambda|H + I| + V(g) - w)^{-1} \sum_{|\alpha| \leq N} \sum_{|\beta| \leq M \leq \ell} R_{\ell, M}(g, \lambda)(\lambda|H + I| + V(g) - w)^{-M},
$$

where $[\ell]$ denotes the least integer greater than $\ell$ and $R_{\ell, M}(g, \lambda) \in S_{\mathcal{H}, w}^{2\ell-M}(\mathbb{H}^n)$ is a polynomial in $\pi(X)$ and $X^\alpha_g V, |\alpha| \leq \ell$. 


Proof. We prove the proposition by induction on \(\ell\). When \(\ell = 0\), the expression is trivial from (4.1). Assume that the expression (4.3) holds for \(k \leq \ell - 1\). From (4.2), the difference operator \(\Delta\) contributes only some possible factors of \(\pi(X)\) but no \(w\). However, the differential operator \(X_g\) either acts on \((|\lambda|(H + I) + V(g) - w)^{-M}\) or \(R_{\ell,M}\) (after substituting (4.3) for \(k \leq \ell - 1\) in (4.2)) resulting the expressions as in (4.3). It is easy to check that each term in the sum for \(R_{-2-\ell}\) lies in \(S_{H,w}^{2-\ell}(\mathbb{H}^n)\) after expanding by Leibniz rule. Since \((|\lambda|(H + I) + V(g) - w)^{-1}R_{\ell,M}(g,\lambda)(|\lambda|(H + I) + V(g) - w)^{-M} \in S_{H,w}^{2-\ell}(\mathbb{H}^n)\), \(R_{\ell,M} \in S_{H,w}^{2M-\ell}(\mathbb{H}^n)\). So \(R_{\ell,M}\) is a polynomial in \(\pi(X)\) and \(X_g^aV\). The highest power of \((|\lambda|(H + I) + V(g) - w)^{-1}\) in the right comes out when we throw all derivatives on factors of \((|\lambda|(H + I) + V(g) - w)^{-1}\) and count this number which is essentially \(\ell\).

\[
\]
\((I + \mathcal{H})^{-\frac{1}{2}}\) is Hilbert-Schmidt operator. That means, if \(\text{Op}(1 + |\lambda|H + V(g))^{-\frac{1}{2}}\) and \(\text{Op}(F\mathcal{Z}(g, \lambda))\) are Hilbert-Schmidt operators or equivalently \((1 + |\lambda|H + V(g))^{-\frac{1}{2}}, F\mathcal{Z}(g, \lambda) \in L^2(\mathbb{H}^n \times \mathbb{R}, S_2, d\mu(\lambda)), (I + \mathcal{H})^s\) is a trace class operator. By generalized Minkowski’s inequality, we have

\[
\|\left(1 + |\lambda|H + V(g)\right)^{-\frac{1}{2}}\|_{L^2(\mathbb{H}^n \times \mathbb{R}, S_2, d\mu(\lambda))} = \left(\int_{\mathbb{H}^n} \int_{\mathbb{R}^s} |\text{tr}(1 + |\lambda|H + V(g))^{-\frac{1}{2}}|^2 \, dg \, d\mu(\lambda)\right)^{\frac{1}{2}}
\]

\[
\leq C \sum_{\alpha} \left(\int_{\mathbb{H}^n} \int_{\mathbb{R}^s} \frac{\lambda^n}{(1 + |2|\alpha| + n) + V(g)} \, dg \, d\lambda\right)^{\frac{1}{2}}
\]

\[
= C \sum_{\alpha} \frac{1}{(2|\alpha| + n)^{\frac{n}{2}}} \left(\int_{\mathbb{H}^n} \int_{1 + V(g)} \frac{1}{u - 1 - V(g)^n} \, du \, dg\right)^{\frac{1}{2}}
\]

\[
= C \sum_{\alpha} \frac{1}{(2|\alpha| + n)^{\frac{n}{2}}} \left(\int_{\mathbb{H}^n} \frac{1}{(1 + V(g))^{s-n-1}} \, dg\right)^{\frac{1}{2}}.
\]  

(4.5)

Under the assumption \(V(g) \sim V_0 |g|^k\) as \(|g| \to \infty\), the function \(\frac{1}{(1 + V(g))^{s-n-1}}\) is integrable if we choose \((s - n - 1)k > 1\). A similar argument gives \(F\mathcal{Z}\) is also a Hilbert-Schmidt operator for large \(N\). Indeed \((\mathcal{H} - w)^{-1} = \text{Op}(R_N) + (\mathcal{H} - w)^{-1}\text{Op}(S_N(g, \lambda, w))\) is compact and hence has discrete spectrum.

**Proposition 4.3.** Let \(u > 0\) and \(m \in \mathbb{N}\) be sufficiently large such that \((\mathcal{H} + u)^{-m}\) is a trace class operator on \(L^2(\mathbb{H}^n)\). Then for such \(m\), \((\mathcal{H} + u)^{-m} = \text{Op}((|\lambda|(H + I) + V(g) + u)^{-m}) + \text{Op}(E(g, \lambda, u))\) such that

\[
|\text{tr}(\mathcal{H} + u)^{-m} - \text{tr}(\text{Op}((|\lambda|(H + I) + V(g) + u)^{-m}))| = |\text{tr}(\text{Op}(E(g, \lambda, u)))|
\]

\[
\leq \psi_1(u) |\text{tr}(\text{Op}((|\lambda|(H + I) + V(g) + u)^{-m}))|
\]

with \(\psi_1(u) \to 0\) as \(u \to \infty\).

**Proof.** From the discussions in the previous subsections, we write

\[
(\mathcal{H} + u)^{-m} = \text{Op}((|\lambda|(H + I) + V(g) + u)^{-m}) + \text{Op}(E(g, \lambda, u)),
\]

where

\[
E(g, \lambda, u) = \sum_{\ell=1}^{N} \sum_{\frac{1}{2} \leq |M| \leq \ell} \frac{1}{2\pi i} \int (w + u)^{-m}(|\lambda|(H + I) + V(g) - w)^{-1} R_{\ell, M}(g, \lambda)
\]
\( \times (|\lambda|(H + I) + V(g) - w)^{-M} dw + \frac{1}{2\pi i} \int_{\Gamma} (w + u)^{-m}(H - w)^{-1}Op(S_N(g, \lambda, w)) \, dw. \)

For large \( N \), choose \( 0 < s < N \) such that \((I + H)^{-\frac{s}{2}}\) is a trace class operator. Then \( Op(S_N(g, \lambda, w)) \) is a trace class operator with
\[
|\text{tr}(Op(S_N(g, \lambda, w)))| \leq |\text{tr}((I + H)^{-\frac{s}{2}}(I + H)^{\frac{\sigma}{2}}Op(S_N(g, \lambda, w)))| \\
\leq |\text{tr}((I + H)^{-\frac{s}{2}})| \text{tr}((I + H)^{\frac{s}{2}}Op(S_N(g, \lambda, w))) .
\]

But from Theorem 3.6, we have
\[
\| (I + H)^{\frac{s}{2}}Op(S_N(g, \lambda, w)) \| \\
= \| (I + H)^{\frac{s}{2}}(H + |w|)^{-\frac{s}{2}}(H + |w|)^{\frac{s}{2}}Op(S_N(g, \lambda, w)) \| \\
= \| (I + H)^{\frac{s}{2}}(H + |w|)^{-\frac{s}{2}}(H + |w|)^{\frac{s}{2}}Op(S_N(g, \lambda, w)) \| \\
\leq C \| (I + H)^{\frac{s}{2}}(H + |w|)^{-\frac{s}{2}}(H + |w|)^{\frac{s}{2}} \| \| (H + |w|)^{\frac{(s-N)}{2}} \| \\
= O \left( |w|^{\frac{(s-N)}{2}} \right) .
\]

Therefore, from (4.7) and (4.8), we obtain
\[
|\text{tr} \left( \int_{\Gamma} (w + u)^{-m}(H - w)^{-1}Op(S_N(g, \lambda, w)) \, dw \right) | \leq C u^{1-m} \to 0 \text{ as } u \to \infty.
\]

Consequently this part of the error is negligible and the pseudo-differential part of \( E(g, \lambda, u) \) is a trace class operator because it has smooth rapidly decaying symbol. By Lemma 3.5, each term of \( \text{tr}(Op(E(g, \lambda, u))) \) is of the form
\[
\text{tr} \left[ Op \left( \frac{1}{2\pi i} \int_{\Gamma} (w + u)^{-m}(H + I) + V(g) - w)^{-1}R_{\ell,M}(g, \lambda) \right) \\
\times (|\lambda|(H + I) + V(g) - w)^{-M} dw \right] \\
= \int_{\mathbb{H}^n} \int_{\mathbb{R}^*} \text{tr} \left[ \left( \frac{1}{2\pi i} \int_{\Gamma} (w + u)^{-m}(H + I) + V(g) - w)^{-1}M \, dw \right) R_{\ell,M}(g, \lambda) \right] \, dg \, d\mu(\lambda) \\
= C_{m,M} \int_{\mathbb{H}^n} \int_{\mathbb{R}^*} \text{tr} \left( (|\lambda|(H + I) + V(g) + u)^{-m} \, M \right) \, d\mu(\lambda) \\
= C_{m,M} u^{-m-M} \int_{\mathbb{H}^n} \int_{\mathbb{R}^*} \text{tr} \left( (u^{-1}|\lambda|(H + I) + u^{-1}V(g) + 1)^{-m} \, M \right) \, d\mu(\lambda) \\
\sim C_{m,M} u^{-m-M+n+1+\frac{2\alpha}{\pi}+\frac{2}{\pi}} \int_{\mathbb{H}^n} \int_{\mathbb{R}^*} \text{tr} \left( (|\lambda|(H + I) + |g|^k + 1)^{-m} \, M \right) \, |\lambda|^n \, d\lambda,
\]
where \( \tilde{g} = (u^{\frac{1}{2}}x_1, u^{\frac{1}{2}}x_2, \ldots, u^{\frac{1}{2}}x_{2n}, u^{\frac{1}{2}}t) \). Since \( R_{\ell,M} \in S_{\ell,w}^{2M-\ell}(\mathbb{H}^n) \),
\[
\| R_{\ell,M}(\tilde{g}, u\lambda)(u|\lambda|(H + I) + u|\tilde{g}|^k + 1)^{-\frac{2M+\ell}{2}} \|_{op}
\]
is uniformly bounded and so
\[
|tr\left( Op\left( (|\lambda|(H + I) + V(g) + u)^{-m}R_{\ell,M}(g, \lambda)\right)\right)| \\
\sim C u^{n+1+\frac{2n}{n+M}} \int_{\mathbb{H}^n} \int_{\mathbb{R}^*} |tr\left( (u|\lambda|(H + I) + u|g|^\kappa + 1)^{\frac{2M-\ell}{2}}\right) \times (|\lambda|(H + I) + |g|^\kappa + 1)^{-m-M} \right) |\lambda|^n \, dg \, d\lambda
\]
\[
\leq C u^{-m-M+n+1+\frac{2n}{n+M}+\frac{2M-\ell}{2}} \sum_k \int_{\mathbb{H}^n} \int_{\mathbb{R}^*} \frac{|\lambda|^n}{(|\lambda|(2k|n+1) + |g|^\kappa + 1)^{m+M}} \, dg \, d\lambda
\]
\[
\leq C u^{-m-n+1+\frac{2n}{n+M}+\frac{2M-\ell}{2}} \sum_k \int_{\mathbb{R}^*} \frac{|\lambda|^n \, d\lambda}{(|\lambda|(2k|n+1) + |g|^\kappa + 1)^{m+M}} \sum_k \int_{\mathbb{H}^n} \frac{1}{(|g|^\kappa + 1)^{\frac{2n}{n+M}+\frac{2M-\ell}{2}}}
\]
\[
\approx u^{-m-n+1+\frac{2n}{n+M}+\frac{2M-\ell}{2}}. \tag{4.10}
\]

Similarly, we have
\[
|tr\left( Op\left( (|\lambda|(H + I) + V(g) + u)^{-m})\right)\right)| \approx u^{-m-n+1+\frac{2n}{n+M}}. \tag{4.11}
\]

Thus applying trace in (4.6) and using (4.9), (4.11), we get
\[
\left| \frac{tr(Op(E,g,\lambda, u))}{tr\left( Op\left( ((|\lambda|(H + I) + V(g) + u)^{-m})\right)\right)} - 1 \right| \leq C \psi_1(u) \to 0 \text{ as } u \to \infty, \tag{4.12}
\]
where \( \psi_1(u) = \frac{1}{m-1} + \sum_{\ell=1}^N u^{-\frac{2}{\ell}} \). Note that when \( \ell = 0, M = 0 \) and \( R_{\ell,M} = 1, (4.9), (4.11) \) has same decay. If \( \ell \geq 1 \) then (4.12) also holds.

Let \( w \) be the complex number varying over the curve \( \Gamma \) (defined in Section 4). For fixed \( (g, \lambda) \in \mathbb{H}^n \times \mathbb{R}^* \), the class \( S^m_w(\mathbb{R}^n) \) defined as
\[
S^m_w(\mathbb{R}^n) = \left\{ a \in C^\infty(\mathbb{R}^{2n} \times \Gamma) : |\partial^\alpha_x \partial^\beta_x a(x, \xi)| \leq C_{\alpha,\beta}(1 + |\lambda|(1 + |\xi|^2 + |x|^2) + V(g) + |w|)^{m-|\alpha|/2} \right\}. \tag{4.13}
\]

We obtain the following result as in Proposition 4.3.

**Proposition 4.4.** Let \( m > 0 \) be a sufficiently large such that \( (|\lambda|(H + I) + V(g) + u)^{-m} \) is in trace class. Then for a fixed \( (g, \lambda) \in \mathbb{H}^n \times \mathbb{R}^* \), we have
\[
(|\lambda|(H + I) + V(g) + u)^{-m} = Op^W \left( (|\lambda|(1 + |\xi|^2 + |x|^2) + V(g) + u)^{-m} \right) + Op^W(E_{g,\lambda}(u)),
\]
where \( \mathcal{W} = \mathcal{W}(\mathbb{R}^n) \).
where
\[
\begin{align*}
tr(|\lambda|(H + I) + V(g) + u)^{-m} - tr\left(Op^W\left(|\lambda|(1 + |\xi|^2 + |x|^2) + V(g) + u\right)^{-m}\right) \\
= tr(Op^W(E_{g,\lambda}(u))) \\
\leq \psi_2(u) \left| tr\left(Op^W\left(|\lambda|(1 + |\xi|^2 + |x|^2) + V(g) + u\right)^{-m}\right) \right|
\end{align*}
\]
with \( \psi_2(u) \rightarrow 0 \) as \( u \rightarrow \infty \).

**Proof.** The proof is based on the same idea as in Proposition 4.3. For fixed \((g, \lambda) \in \mathbb{H}^n \times \mathbb{R}^s\), there exists \( m \in \mathbb{N} \) such that \(|\lambda|(H + I) + V(g) + u)^{-m}\) is a trace class operator on \(L^2(\mathbb{R}^n)\). We refer to [22] for similar pseudo-differential approximation to \(|\lambda|(H + I) + V(g) + u)^{-m}\) on \(L^2(\mathbb{R}^n)\). However, we will only indicate some intermediate steps. Now
\[
(|\lambda|(H + I) + V(g) + u)^{-m} = Op^W\left(|\lambda|(1 + |\xi|^2 + |x|^2) + V(g) + u\right)^{-m} + Op^W(E_{g,\lambda}(u)),
\]
(4.14)
where
\[
E_{g,\lambda}(u)
\]
\[
= \sum_{\ell=1}^{N} \sum_{|\xi| \leq M \leq \ell} \frac{\Gamma(s + 1)}{\Gamma(s - M)\Gamma(M + 1)} R_{\ell,\xi}^{(g,\lambda)}(\xi, x)(|\lambda|(1 + |\xi|^2 + |x|^2) + V(g) + u)^{-m} M
\]
\[
+ \frac{1}{2\pi i} \int_{\Gamma} (w + u)^{-m} S^{(g,\lambda)}_N(w)(|\lambda|(H + I) + V(g) - w)^{-1} dw
\]
with \( S^{(g,\lambda)}_N \in S_{w}^{-N}(\mathbb{R}^n) \). Let \( 0 < s < N \) and \((I + |\lambda|(H + I) + V(g))^{-\frac{s}{2}}\) is a trace class operator on \(L^2(\mathbb{R}^n)\). Then imitating the similar calculations in [22], we have
\[
\left| tr\left(\int_{\Gamma} (w + u)^{-m} Op^W(S^{(g,\lambda)}_N(w))(|\lambda|(H + I) + V(g) - w)^{-1} dw \right) \right| \rightarrow 0 \text{ as } u \rightarrow \infty,
\]
and
\[
\begin{align*}
\left| tr\left(Op^W\left(R_{\ell,\xi}^{(g,\lambda)}(\xi, x)(|\lambda|(1 + |\xi|^2 + |x|^2) + V(g) + u)^{-m}\right) \right) \right|
\end{align*}
\]
\[
= \left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} R_{\ell,\xi}^{(g,\lambda)}(\xi, x)(|\lambda|(1 + |\xi|^2 + |x|^2) + V(g) + u)^{-m} dx \, d\xi \right|
\]
\[
\leq u^{-m} \left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} R_{\ell,\xi}^{(g,\lambda)}(\xi, x)(u^{-1}|\lambda|(1 + |\xi|^2 + |x|^2) + u^{-1}V(g) + 1)^{-m} dx \, d\xi \right|
\]
\[
\leq u^{-m} \left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} R_{\ell,\xi}^{(g,\lambda)}(\xi, x) \left( u^{-1}|\lambda|(1 + |\xi|^2 + |x|^2) + 1 \right)^{-m} dx \, d\xi \right|
\]
\[
= u^{-m} \left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} R_{\ell,\xi}^{(g,\lambda)}(u^{\frac{1}{2}}\xi, u^{\frac{1}{2}}x) \left( |\lambda|(|\xi|^2 + |x|^2) + 1 \right)^{-m} dx \, d\xi \right|
\]
\[
\times \left| \int_{\mathbb{R}^n} \left(1 + |\xi|(1 + u|\xi|^2 + u|x|^2) + V(g) + |w|^2 \right)^{2M/2} dx \right|
\]
\[
\times \left| \int_{\mathbb{R}^n} \left(1 + |\lambda|(|\xi|^2 + |x|^2) + 1 \right)^{-m} dx \right|
\]
\[ \leq C u^{-m+n-\frac{\lambda}{2}}. \]

Note that if \( \ell = 1, M = 1 \) and \( R^{(g,\lambda)}_{1,1}(x, \xi) = 1 \). So for \( \ell \geq 2 \),
\[
\left| \text{tr} \left( Op^W \left( R^{(g,\lambda)}_{\ell,M}(\xi, x) \left( |\lambda|(1 + |\xi|^2 + |x|) + V(g) + u \right)^{-m-M} \right) \right) \right| \approx u^{-m+n-1}.
\]

Similarly, we have
\[
\left| \text{tr} \left( Op^W \left( (|\lambda|(1 + |\xi|^2 + |x|) + V(g) + u)^{-m-M} \right) \right) \right| \approx u^{-m+n}.
\]

Thus
\[
\frac{\text{tr}(Op^W(E_{g,\lambda}(u)))}{\text{tr} \left( Op^W \left( (|\lambda|(1 + |\xi|^2 + |x|) + V(g) + u)^{-m} \right) \right)} = \frac{\text{tr} \left( Op \left( (|\lambda|(H + I) + V(g) + u)^{-m} \right) \right)}{\text{tr} \left( Op^W \left( (|\lambda|(1 + |\xi|^2 + |x|) + V(g) + u)^{-m-M} \right) \right)} - 1 \leq \psi_2(u) \to 0 \text{ as } u \to \infty,
\]

where \( \psi_2(u) = \frac{1}{u^{m-n}} + \sum_{\ell=1}^N u^{-1} \).

**Remark 4.5.** Note that for sufficiently large \( m \in \mathbb{N} \), the operator
\[
Op \left( Op^W \left( R^{(g,\lambda)}_{\ell,M}(\xi, x) \left( |\lambda|(1 + |\xi|^2 + |x|) + V(g) + u \right)^{-m-M} \right) \right)
\]
is a trace class operator on \( L^2(\mathbb{H}^n) \), since from Proposition 4.4, we have
\[
\left| \text{tr} \left( Op^W \left( R^{(g,\lambda)}_{\ell,M}(\xi, x) \left( |\lambda|(1 + |\xi|^2 + |x|) + V(g) + u \right)^{-m-M} \right) \right) \right| \leq \int_{\mathbb{H}^n} \int_{\mathbb{R}^n} \left| \text{tr} \left( Op^W \left( R^{(g,\lambda)}_{\ell,M}(\xi, x) \left( |\lambda|(1 + |\xi|^2 + |x|) + V(g) + 1 \right)^{-m-M} \right) \right) \right| d\mu(\lambda)
\]
\[
\leq \int_{\mathbb{H}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left| (|\lambda|(1 + |\xi|^2 + |x|) + V(g) + 1)^{-m-M} \right| dx d\xi d\mu(\lambda) < \infty.
\]

Similarly it can be shown that \( Op \left( Op^W \left( (|\lambda|(1 + |\xi|^2 + |x|) + V(g) + u)^{-m} \right) \right) \) is a trace class operator on \( L^2(\mathbb{H}^n) \) for sufficiently large \( m \in \mathbb{N} \).

We take the positive integer \( m \) such that the requirement of \( m \)-th power of the operators discussed earlier to be a trace class operator.
5. Szegö Limit Theorem for $\mathcal{H}$

Now we are in a position to prove our main results. We start this section with the following lemmas.

**Lemma 5.1.** Let $M_b$ be the multiplication operator defined in Theorem 1.1, then for any $f \in C(\mathbb{R})$, $tr(f(P_r M_b P_r)) = tr(P_r f(M_b) P_r)$

*Proof.* Notice that $\|(I - P_r)M_b P_r\|_{B_2}^2 = tr(P_r M_b P_r) = tr(P_r M_b^2 P_r) - tr(P_r M_b P_r)^2$. Also $P_r M_b^2 P_r$ is an operators on $L^2(\mathbb{H}^n)$ with kernel $K_1(g, g_1) = \sum_{k_1, k_2 \leq r} (b^2 e_{k_1} e_{k_2}) e_{k_2}(g) e_{k_1}(g_1)$, for any orthonormal basis $\{e_k\}$ of $L^2(\mathbb{H}^n)$. Therefore $tr(P_r M_b^2 P_r) = \int_{\mathbb{H}^n} K_1(g, g) dg = \sum_{k \leq r} (b^2 e_k, e_k)$. Further, $tr(P_r M_b^2 P_r)^2 = \int_{\mathbb{H}^n} K_2(g, g) dg = \sum_{k \leq r} (b^2 e_k, e_k)$, where the operator $P_r M_b P_r M_b P_r$ is an integral operator with kernel $K_2(g, g_1) = \sum_{k_1, k_2, k_3 \leq r} (b e_{k_1}, e_{k_2}) (b e_{k_2}, e_{k_3}) e_{k_3}(g) e_{k_1}(g_1)$.

So $tr(P_r M_b^2 P_r) = tr(P_r M_b P_r)^2$. So $\|(I - P_r)M_b P_r\|_{B_2}^2 = 0$. Observe that for $n \in \mathbb{N}$, $P_r M_b^2 P_r = P_r M_b (P_r + (I - P_r)) M_b \cdots M_b = (P_r M_b P_r)^n$ + terms with a factor of $(I - P_r)P_r$ is dominated by some constant (depending on $b$) times $\|(I - P_r)M_b P_r\|_{B_2}$. Therefore $|tr(P_r M_b^n P_r) - tr(P_r M_b P_r)^n| = 0$. Thus $tr(f(P_r M_b P_r)) = tr(P_r f(M_b) P_r)$ for $f(x) = x^n, \forall n \in \mathbb{N}$ and this result can be extended to continuous functions as an application of the Weierstrass approximation theorem and spectral theorem.

**Lemma 5.2.** For $r > 0$ define $I_r : L^2(\mathbb{H}^n) \rightarrow L^2(\mathbb{H}^n)$ by

$$I_r(\phi)(g) = \int_{-r}^r tr(\pi_3(g) \phi(\lambda)) d\mu(\lambda).$$

Then

$$\lim_{r \to \infty} \frac{tr(P_r f(M_b) P_r)}{tr(P_r)} = \lim_{r \to \infty} \frac{tr(P_r I_r f(M_b) P_r I_r)}{tr(P_r I_r)}.$$

*Proof.* We know that if $X$ is a positive trace class operator and $Y$ is a bounded operator on $L^2(\mathbb{H}^n)$ then $|tr(XYX)| \leq \|Y\| tr(X^2)$. Using this inequality we get $|tr(P_r) - tr(P_r I_r)| = |tr(P_r (I - I_r))| \leq \|I - I_r\| tr(P_r)$. But for $\psi \in L^2(\mathbb{H}^n)$, an application of Plancherel formula gives $\|(I - I_r) \psi\|_2^2 = \int_{|\lambda| > r} \|\hat{\psi}(\lambda)\|_{B_2}^2 d\mu(\lambda) \to 0$ as $r \to \infty$. Therefore,

$$\left| \frac{tr(P_r I_r)}{tr(P_r)} - 1 \right| \leq \|I - I_r\| \to 0 \text{ as } r \to \infty.$$
Let Proposition 5.3.

Therefore,

\[ \lim_{r \to \infty} \frac{\text{tr}(P_r f(M_b)P_r)}{\text{tr}(P_r f(M_b))} - 1 \leq \left( 1 + \frac{\text{tr}(P_r f(M_b)P_r)}{\text{tr}(P_r f(M_b))} \right) \|I - I_r\| \to 0, \] (5.3)

as \( r \to \infty \). Combining (5.2) and (5.3) we get (5.1). \hfill \Box

Proof of theorem 1.1:

The operator \( P_r I_r f(M_b)P_r I_r \) is an integral operator with kernel

\[ K_r(g, g_1) = \int_{-r}^{r} \text{tr}(\pi^*_\lambda(g) I_r \times f(b(g_1)) \pi_\lambda(g_1)) d\mu(\lambda). \]

Then volume of \( G \) is given by

\[ V(G) = \int_{\mathbb{R}^n} f(b(g)) \, dg. \]

Proof of theorem 1.2:

Now we prove Szegö limit theorem for \( \mathcal{H} \) under certain assumptions on the symbol \( a(g, \lambda) \) to ensure the existence of the RHS limit in Theorem 1.2 (see [22]). We assume

\[ \lim_{E \to \infty} \bar{a}(E) = a, \] (5.4)

where

\[ \bar{a}(E) = \frac{1}{S(E)} \int_{G_E} a_{g, \lambda}(\xi, x) \, d\xi \, dx \, dg \, d\mu(\lambda) \]

and

\[ S(E) = \int_{G_E} d\xi \, dx \, dg \, d\mu(\lambda), \]

with

\[ G_E = \{ (g, \lambda, \xi, x) \in \mathbb{H}^n \times \mathbb{R}^* \times \mathbb{R}^n \times \mathbb{R}^n : |\lambda|(1 + |\xi|^2 + |x|^2) + V(g) = E \} \]

for real \( \kappa > 0 \) in the sense that

\[ V(g) \sim V_0|g|^\kappa \quad \text{as} \quad |g| \to \infty \] (5.5)

for real \( \kappa > 0 \) in the sense that

\[ V(g) \sim V_0|g|^\kappa \quad \text{as} \quad |g| \to \infty \] (5.5)
Proof. : Using the homogeneous norm on $\mathbb{H}^n$ we have

\[ v(E) = \int_{G_E} d\xi \, dx \, dg \, d\mu(\lambda) \]

\[ = C_n \int_{\mathbb{H}^n} \int_{\mathbb{R}^{2n}} \left( \int |\lambda|^n \, d\lambda \right) d\xi \, dx \, dg \]

\[ = 2C_n \int_{\mathbb{H}^n} (E - V(g))^{n+1} \, dg \int_{\mathbb{R}^{2n}} \left( \frac{1}{1 + |x|^2 + |\xi|^2} \right)^{n+1} d\xi \, dx \]

\[ = C'_n E^{n+1} \int_{V \leq E} (1 - E^{-1}V(g))^{n+1} \, dg \]

\[ \sim C'_n E^{n+1} \int_{V \leq E} (1 - E^{-1}(V_0|g|^n + W(g)))^{n+1} \, dg \]

\[ = C'_n E^{n+1} \int_{V \leq E} (1 - V_0|g|^k - E^{-1}W(\tilde{g}))^{n+1} \, dg, \]

where $\tilde{g} = (E^{1/2}x_1, E^{1/2}x_2, \ldots, E^{1/2}x_{2n}, E^{1/2}t)$ for $g = (x_1, x_2 \ldots, x_{2n}, t) \in \mathbb{H}^n$. Since $\lim_{E \to \infty} E^{-1}W(\tilde{g}) = 0$, the right hand side of the above integral converges to $\int_{V \leq E} (1 - V_0|g|^n)^{n+1} \, dg$ by dominated convergence theorem.

Lemma 5.4. Let $\phi(r) = \text{tr}(\mathcal{P}_r)$ and $\psi(r) = \text{tr}(\mathcal{P}_r \mathcal{A} \mathcal{P}_r)$. Then under the assumption (5.4) and (5.5), we have

\[ \Phi(u) = \int_0^\infty \frac{\phi(r)}{(r + u)^{m+1}} \, dr = \int_0^\infty \int_{G_E} \frac{d\xi \, dx \, dg}{(E + u)^{m+1}} \, dE + E_1(u) \]

and

\[ \Psi(u) = \int_0^\infty \frac{\psi(r)}{(r + u)^{m+1}} \, dr = \int_0^\infty \int_{G_E} \frac{a_{g,\lambda}(\xi, x)}{(E + u)^{m+1}} \, d\xi \, dx \, d\mu(\lambda) \, dE + E_2(u), \]

with $|E_i(u)| \to 0$ as $u \to \infty$, $i = 1, 2$.

Proof. The operator $\mathcal{H}$ has discrete spectrum of eigenvalues $0 \leq c_1 \leq c_2 \cdots \infty$. Let $\{\psi_j\}_{j=1}^\infty$ be the complete set of eigenfunctions corresponding to the eigenvalues $\{c_j\}$ on $L^2(\mathbb{H}^n)$. Then $\psi(r) = \text{tr}(\mathcal{P}_r \mathcal{A} \mathcal{P}_r) = \sum_{c_j \leq r} \langle A\psi_j, \psi_j \rangle$ and $\psi'(r) = \sum_{j=1}^\infty \langle A\psi_j, \psi_j \rangle \delta(r - c_j)$. Now

\[ \Psi(u) = \int_0^\infty \frac{\psi(r)}{(r + u)^{m+1}} \, dr = m \sum_{j=1}^\infty \langle A\psi_j, \psi_j \rangle \int_0^\infty \frac{\delta(r - c_j)}{(r + u)^m} \, dr \]

\[ = m \sum_{j=1}^\infty \langle A\psi_j, \psi_j \rangle \frac{1}{(c_j + u)^m} = m \text{ tr } (A(\mathcal{H} + u)^{-m}). \]

By Proposition 4.3, we obtain

\[ \Psi(u) = m \text{ tr } (A(\mathcal{H} + u)^{-m}) \]
respectively. Let

\begin{equation}
\Psi(u) = m \int \left( a(g, \lambda)(|\lambda|(H + I) + V(g) + u)^{-m} \right) dg d\mu(\lambda)
\end{equation}

with \(|tr(A Op(E(g, \lambda, u)))| \leq \|A\| |tr(Op(E(g, \lambda, u)))| \to 0\) as \(u \to \infty\). Thus for large \(u\), using (3.5) and (2.3), we have

\begin{align*}
\Psi(u) &= m \int \left( a(g, \lambda)(|\lambda|(H + I) + V(g) + u)^{-m} \right) dg d\mu(\lambda) \\
&= m \int \left( a(g, \lambda)Op^W \left( |\lambda|(1 + |\xi|^2 + |x|^2) + V(g) + u)^{-m} \right) + a(g, \lambda)Op^W(E_{g,\lambda}(u)) \right) dg d\mu(\lambda) \\
&= m \int_{\mathbb{H}^n \times \mathbb{R}^*} \left( a_{g,\lambda}(\xi, x)(|\lambda|(1 + |\xi|^2 + |x|^2) + V(g) + u)^{-m} \right) d\xi dx dg d\mu(\lambda) + E_1(u) \\
&= \int_0^\infty \int_{\mathbb{H}^n \times \mathbb{R}^*} \left( a_{g,\lambda}(\xi, x) \frac{d\xi dx dg d\mu(\lambda)}{(E + u)^{m+1}} \right) dE + E_1(u),
\end{align*}

where \(E_1(u) = m \int_{\mathbb{H}^n \times \mathbb{R}^*} \left( a(g, \lambda)Op^W(E_{g,\lambda}(u)) \right) dg d\mu(\lambda)\). From Remark 4.5, we conclude that \(|E_1(u)| \to 0\) as \(u \to \infty\) by dominated convergence theorem. Similarly taking \(A = I\), we get \(\phi(r) = tr(P_r)\), and in this case, for large \(u\), we have

\begin{equation}
\Phi(u) = \int_0^\infty \int_{\mathbb{H}^n \times \mathbb{R}^*} \left( a_{g,\lambda}(\xi, x) \frac{d\xi dx dg d\mu(\lambda)}{(E + u)^{m+1}} \right) dE + E_2(u)
\end{equation}

with \(|E_2(u)| \to 0\) as \(u \to \infty\).

In order to prove the Szegő limit theorem for the Schrödinger operator \(\mathcal{H}\), we need to estimate the asymptotic growth of the measures \(tr(P_r AP_r)\) and \(tr(P_r)\). Therefore we apply Keldysh Tauberian Theorem (see Theorem 5.4 in Appendix) to compare the measures.

**Corollary 5.5.** Consider the self-adjoint operator \(P_r\) and \(\psi(r)\) as given in Theorem 1.2 and Proposition 5.3 respectively. Let \(\phi(r) = tr(P_r), \psi(r) = tr(P_r AP_r)\) then we have the following asymptotics:

1. \(v(r) \approx C r^{n+1+\frac{2(n+1)}{\kappa}}\) as \(r \to \infty\).
2. \(\psi\) is multiplicatively continuous.
3. \(tr(P_r) \approx C r^{n+1+\frac{2(n+1)}{\kappa}}\) as \(r \to \infty\).
4. \(\sup_{\mu \leq r}[tr(P_{\mu+r}) - tr(P_\mu)] \leq tr(P_r) \left[ \left( n + 1 + \frac{2(n+1)}{\kappa} \right) \frac{r_1}{r} + O \left( \frac{1}{r} \right) \right]^2\), as \(r \to \infty\).
5. \(\psi\) is multiplicatively continuous.
Proof. Clearly (1) directly follows from Proposition 5.3. Now
\[
\lim_{r \to \infty} \frac{v(\tau r)}{v(r)} = \lim_{r \to \infty} \frac{(\tau r)^{n+1 + \frac{2(n+1)}{\kappa}}}{r^{n+1 + \frac{2(n+1)}{\kappa}}} = \lim_{r \to \infty} \tau^{n+1 + \frac{2(n+1)}{\kappa}} = 1.
\]
Therefore \( v \) is multiplicatively continuous function. We choose sufficiently large \( m \) such that the operator \( (\mathcal{H} + uI)^{-m} \) is a trace class operator. Therefore by Lemma 5.4 and Theorem 8 of Grishin-Poedintseva [5], we get \( \phi(r)/v(r) \to 1 \) as \( r \to \infty \). This proves (3).

Using the asymptotic in (3), it is easy to check that
\[
\sup_{\mu \leq r} |tr(P_{\mu+r}) - tr(P_{\mu})| \leq tr(P_r) \left[ \left( n + 1 + \frac{2(n+1)}{\kappa} \right) \frac{r_1}{r} + O \left( \frac{1}{r} \right)^2 \right].
\]
To prove (5), notice that if \( \varphi \) and \( \chi \) are two distribution functions satisfying \( \lim_{r \to \infty} \frac{\varphi(r)}{\chi(r)} = 1 \) then \( \varphi \) is multiplicatively continuous whenever \( \chi \) is. Therefore \( \psi \) is also a multiplicatively continuous function. \( \Box \)

**Theorem 5.6.** Under the assumption (5.4) and (5.5) we have
\[
\lim_{r \to \infty} \frac{tr(P_r A P_r)}{tr(P_r)} = \lim_{r \to \infty} \frac{\int_{G^*} a_{g,\lambda}(\xi, x) d\xi dx dg d\mu(\lambda)}{\int_{G^*} d\xi dx dg d\mu(\lambda)}.
\]

Proof. The proof follows directly by Lemma 5.4, as all the requirements (by our assumption (5.4) on the symbol \( a(g, \lambda) \)) of Theorem 8 of Grishin-Poedintseva [5] are satisfied. \( \Box \)

**Corollary 5.7.** Let \( P(r) \) be a polynomial in \( \mathbb{R} \). Then
\[
\lim_{r \to \infty} \frac{tr(P_r P(A) P_r)}{tr(P_r)} = \lim_{r \to \infty} \frac{\int_{G^*} P(a_{g,\lambda}(\xi, x)) d\xi dx dg d\mu(\lambda)}{\int_{G^*} d\xi dx dg d\mu(\lambda)}.
\]

Proof. From the asymptotic expression of Theorem 3.8 along with Remark 3.4, we see that the operator \( P(A) \) has symbol \( P(a(g, \lambda)) + E(g, \lambda) + E_{-1}(g, \lambda) \), where \( E(g, \lambda) \), \( E_{-1}(g, \lambda) \in S_{H}^{-1}(\mathbb{R}^n) \) (the term associated with \( [\alpha] = 1 \) is \( E \) and \( E_{-1} \) is the remaining terms with \( [\alpha] > 1 \) in the asymptotic expansion). The proof will be complete if we show
\[
\lim_{r \to \infty} \frac{\int_{G^*} \hat{E}_{g,\lambda}(\xi, x) d\xi dx dg d\mu(\lambda)}{\int_{G^*} d\xi dx dg d\mu(\lambda)} = 0,
\]
where \( E(g, \lambda) + E_{-1}(g, \lambda) = Op^W(\hat{E}_{g,\lambda}(\xi, x)) \). Now proceeding as in proposition 5.3, we get
\[
\int_{G^*} |\hat{E}_{g,\lambda}(\xi, x)| d\xi dx dg d\mu(\lambda)
\]
\[
\leq C \int_{H^n} \int_{\mathbb{R}^{2n}} \left( \int_{|\lambda| \leq \frac{(r - V(g))_{+}}{1 + |x|^2 + |\xi|^2}} (1 + |\lambda| |1 + |\xi|^2 + |x|^2| + V(g))^{-\frac{1}{2}} |\lambda|^n \langle d\lambda \rangle \right) d\xi dx dg
\]
\[
\leq C \int_{H^n} \int_{\mathbb{R}^{2n}} \left( \int_{|\lambda| \leq \frac{(r - V(g))_{+}}{1 + |x|^2 + |\xi|^2}} |\lambda|^{n-\frac{4}{\kappa}} d\lambda \right) d\xi dx dg
\]
Let there exist two symbols \( S \) where \( H \).

Lemma 5.8. Let \( \mathcal{H}, A \) be the operators defined in Theorem 1.2. Then

(a) \( \sqrt{\mathcal{H}} = \mathcal{H}^{\frac{1}{2}} + C \), where \( \mathcal{H}^{\frac{1}{2}} = Op(H^{\frac{1}{2}}(g, \lambda)) \) and \( C \) is a bounded operator on \( L^2(\mathbb{H}^n) \).

(b) the operator \( [\sqrt{\mathcal{H}}, A] \) is bounded on \( L^2(\mathbb{H}^n) \).

(c) under the assumptions of Theorem 1.2, we have

\[
\frac{\left| \text{tr}(f(\mathcal{P}_r A \mathcal{P}_r) - \text{tr}(\mathcal{P}_r f(A) \mathcal{P}_r) \right|}{\text{tr}(\mathcal{P}_r)} \to 0 \text{ as } r \to \infty.
\]

Proof. (a) Let \( f(w) = w^{\frac{1}{2}} \). Proceeding as in Subsection 4.2, we get \( \sqrt{\mathcal{H}} = \mathcal{H}^{\frac{1}{2}} + F_{\frac{1}{2}} \), where \( \mathcal{H}^{\frac{1}{2}} = Op(H^{\frac{1}{2}}(g, \lambda)) \) with \( H^{\frac{1}{2}}(g, \lambda) \in S^1_\mathcal{H}(\mathbb{H}^n) \) and \( F_{\frac{1}{2}} \) is defined in (4.4) with \( S_N \in S_{\mathcal{H}, w}(\mathbb{H}^n) \). We choose \( N > 0 \) such that the integral (4.4) converges in the norm on \( B(L^2(\mathbb{R}^n)) \). Denoting \( C = F_{\frac{1}{2}} \), we have \( \sqrt{\mathcal{H}} = \mathcal{H}^{\frac{1}{2}} + C \) as desired.

(b) From part (a) we have \( \sqrt{\mathcal{H}} = \mathcal{H}^{\frac{1}{2}} + C \). Since \( C \) is bounded, \( [\sqrt{\mathcal{H}}, A] \) is bounded if \( [\mathcal{H}^{\frac{1}{2}}, A] \) is bounded on \( L^2(\mathbb{H}^n) \). Now using the composition formula (3.6) of Theorem 3.8, there exist two symbols \( R_1(g, \lambda), R_2(g, \lambda) \in S^0_{\mathcal{H}}(\mathbb{H}^n) \) such that

\[
[\mathcal{H}^{\frac{1}{2}}, A] = \mathcal{H}^{\frac{1}{2}} A - A \mathcal{H}^{\frac{1}{2}}
\]

(b) From part (a) we have \( \sqrt{\mathcal{H}} = \mathcal{H}^{\frac{1}{2}} + C \). Since \( C \) is bounded, \( [\sqrt{\mathcal{H}}, A] \) is bounded if \( [\mathcal{H}^{\frac{1}{2}}, A] \) is bounded on \( L^2(\mathbb{H}^n) \). Now using the composition formula (3.6) of Theorem 3.8, there exist two symbols \( R_1(g, \lambda), R_2(g, \lambda) \in S^0_{\mathcal{H}}(\mathbb{H}^n) \) such that

\[
[\mathcal{H}^{\frac{1}{2}}, A] = \mathcal{H}^{\frac{1}{2}} A - A \mathcal{H}^{\frac{1}{2}}
\]

\[
= Op\left(H^{\frac{1}{2}}(g, \lambda) #_{\mathbb{H}^n} a(g, \lambda)\right) - Op\left(a(g, \lambda) #_{\mathbb{H}^n} H^{\frac{1}{2}}(g, \lambda)\right)
\]

\[
= Op\left(H_{\frac{1}{2}}(g, \lambda) a(g, \lambda) + \Delta H_{\frac{1}{2}}(g, \lambda) X_g a(g, \lambda) + R_1(g, \lambda)\right)
\]

\[
- Op\left(a(g, \lambda) H_{\frac{1}{2}}(g, \lambda) + \Delta a(g, \lambda) X_g H_{\frac{1}{2}}(g, \lambda) + R_2(g, \lambda)\right)
\]

\[
= Op\left(Op^W(a_{g, \lambda}(\xi, u)) #_{\mathbb{H}^n} H^{\frac{1}{2}}_{g, \lambda}(\xi, u) - H_{g, \lambda}(\xi, u) #_{\mathbb{H}^n} a_{g, \lambda}(\xi, u)\right)
\]

\[
+ Op\left(\Delta H_{\frac{1}{2}}(g, \lambda) X_g a(g, \lambda) - \Delta a(g, \lambda) X_g H_{\frac{1}{2}}(g, \lambda)\right) + Op\left(R_1(g, \lambda) - R_2(g, \lambda)\right)
\]

\[
= Op\left(Op^W(F^1_{g, \lambda}(\xi, u) + F^2_{g, \lambda}(\xi, u))\right) + Op\left(R_1(g, \lambda) - R_2(g, \lambda)\right)
\]

\[
+ Op\left(\Delta H_{\frac{1}{2}}(g, \lambda) X_g a(g, \lambda) - \Delta a(g, \lambda) X_g H_{\frac{1}{2}}(g, \lambda)\right),
\]

(5.7)

where \( F^1_{g, \lambda}(\xi, u), F^2_{g, \lambda}(\xi, u) \in S^0(\mathbb{R}^n) \) (the term associated with \( j = 1 \) is \( F^1_{g, \lambda} \) and \( F^2_{g, \lambda} \) is the remaining terms with \( j > 1 \) in the asymptotic expansion (2.2)). Therefore \( Op^W(F^1_{g, \lambda}(\xi, u) + F^2_{g, \lambda}(\xi, u)) \in S^0_{\mathcal{H}}(\mathbb{H}^n) \). Since each symbol in the last equality of the expression (5.7) belongs to the \( S^0_{\mathcal{H}}(\mathbb{H}^n) \) class, by Theorem 3.6, the operator \( [\mathcal{H}^{\frac{1}{2}}, A] \) is bounded on \( L^2(\mathbb{H}^n) \).
Since $A$ is bounded self-adjoint, the spectrum of $A$, $\sigma(A)$ is a compact subset of $\mathbb{R}$. Since any continuous function can be approximated in the supremum norm by smooth functions, it is enough to assume that $f \in C^2(\sigma(A))$. By Theorem 1.6 of Laptev-Safarov [13], by setting $A = \sqrt{\mathcal{H}}$, $B = A$, $\chi = 0$, $\psi = f$, $P_\lambda = P_{\tau^2}$, we get
\[
|\text{tr}(P_{\tau^2}f(A)P_{\tau^2} - P_{\tau^2}f(P_{\tau^2}A P_{\tau^2})P_{\tau^2})| \leq \frac{1}{2}||f''||_{\infty}N_{r_1}(r^2)\left(\|P_{\tau^2-r_1,\tau^2}A\|^2 + \frac{\pi^2}{6r_1^2}\|P_{\tau^2-r_1}[A, \sqrt{\mathcal{H}}]\|^2\right).
\]
Dividing both sides by $\text{tr}(P_{\tau^2})$ and setting $r_1 = r^{2\alpha}$, $\alpha \in (0, 1)$
\[
\frac{|\text{tr}(P_{\tau^2}f(A)P_{\tau^2} - P_{\tau^2}f(P_{\tau^2}A P_{\tau^2})P_{\tau^2})|}{\text{tr}(P_{\tau^2})} \leq C\frac{N_{r_1}(r^2)}{\text{tr}(P_{\tau^2})} \approx r^{2\alpha - 2}.
\]
So
\[
\frac{|\text{tr}(P_{\tau}f(A)P_{\tau} - P_{\tau}f(P_{\tau}A P_{\tau})P_{\tau})|}{\text{tr}(P_{\tau})} \leq C\frac{N_{r_1}(r^2)}{\text{tr}(P_{\tau})} \approx r^{2\alpha - 2} \rightarrow 0 \text{ as } r \rightarrow \infty
\]
by part (3) and part (4) of Corollary 5.5, where $N_{r_1}(r) = \sup_{\mu \leq r}(\text{tr}(P_{\mu + r_1} - P_{\mu}))$. \qed

The proof of Theorem 1.2 follows from Corollary 5.7 and part (c) of Lemma 5.8.

6. Szegö limit theorem for $\mathcal{H}_1$

Consider the operators $\mathcal{H}_1$ and $\mathcal{H}$ as defined in Theorem 1.3. Since the operators $e^{-t\mathcal{H}_1}$ and $e^{-t\mathcal{H}}$ are compact for $t > 0$, we choose a suitable $m \in \mathbb{N}$ such that $(\mathcal{H}_1 + rI)^{-m}$ and $(\mathcal{H} + rI)^{-m}$ are trace class operators on $L^2(\mathbb{H}^n)$ for $r > 0$. We observe the following facts before proving Theorem 1.3.

**Lemma 6.1.** Consider the self-adjoint operators $\mathcal{H}$ and $\mathcal{H}_1$ as defined in Theorem 1.3. Then
\[(a)\]
\[
\left|\frac{\text{tr}((\mathcal{H}_1 + rI)^{-m})}{\text{tr}((\mathcal{H} + rI)^{-m})} - 1\right| \rightarrow 0 \text{ as } r \rightarrow \infty.
\]
\[(b)\] If $B$ is any bounded operator on $L^2(\mathbb{H}^n)$, then
\[
\left|\frac{\text{tr}(B(\mathcal{H}_1 + rI)^{-m})}{\text{tr}(B(\mathcal{H} + rI)^{-m})} - 1\right| \rightarrow 0 \text{ as } r \rightarrow \infty.
\]

**Proof.** Without loss of generality we prove the result for the positive operator $B$ by adding a suitable constant $c > 0$ which makes the operator $B + cI$ positive.

(a) Since $B$ and $(\mathcal{H} + rI)^{-1}$ are bounded and positive operators, we have
\[
(\mathcal{H}_1 + rI) = (\mathcal{H} + rI)^{\frac{1}{2}}((\mathcal{H} + rI)^{-\frac{1}{2}}(B)(\mathcal{H} + rI)^{-\frac{1}{2}} + 1)(\mathcal{H} + rI)^{\frac{1}{2}}.
\]
Therefore
\[
(\mathcal{H}_1 + rI)^{-m} = (\mathcal{H} + rI)^{-m} + (\mathcal{H} + rI)^{-\frac{m}{2}}((1 + K_r)^{-m} - 1)(\mathcal{H} + rI)^{-\frac{m}{2}},
\]
where
\[
(\mathcal{H}_1 + rI)^{-m} = (\mathcal{H} + rI)^{-m} + (\mathcal{H} + rI)^{-\frac{m}{2}}((1 + K_r)^{-m} - 1)(\mathcal{H} + rI)^{-\frac{m}{2}}.
\]
where \(K_r = (\mathcal{H} + rI)^{-\frac{1}{2}}B(\mathcal{H} + rI)^{-\frac{1}{2}}\). Here \(K_r\) is a positive operator and \(\| (I + K_r)^{-1} \| \leq 1\), for any \(r > 0\). Thus

\[
\left| \text{tr} \left( (\mathcal{H}_1 + rI)^{-m} \right) - \text{tr} \left( (\mathcal{H} + rI)^{-m} \right) \right| = \left| \text{tr} \left( (\mathcal{H} + rI)^{-\frac{m}{2}}((1 + K_r)^{-m} - 1)(\mathcal{H} + rI)^{-\frac{m}{2}} \right) \right|
\leq \text{tr} \left( (\mathcal{H} + rI)^{-m} \right) \|((1 + K_r)^{-m} - 1)\|
\leq m \|K_r\| \text{tr} \left( (\mathcal{H} + rI)^{-m} \right)
\leq m \|B\| \| (\mathcal{H} + rI)^{-1} \| \text{tr} \left( (\mathcal{H} + rI)^{-m} \right).
\]

Therefore,

\[
\left| \frac{\text{tr} \left( (\mathcal{H}_1 + rI)^{-m} \right)}{\text{tr} \left( (\mathcal{H} + rI)^{-m} \right)} - 1 \right| \leq m \|B\| \| (\mathcal{H} + rI)^{-1} \| \rightarrow 0 \text{ as } r \rightarrow \infty.
\]

(b) Using (6.2) we have

\[
\left| \text{tr} \left( B(\mathcal{H}_1 + rI)^{-m} \right) - \text{tr} \left( B(\mathcal{H} + rI)^{-m} \right) \right|
= \left| \text{tr} \left( B(\mathcal{H} + rI)^{-\frac{m}{2}}((1 + K_r)^{-m} - 1)(\mathcal{H} + rI)^{-\frac{m}{2}} \right) \right|
= \left| \text{tr} \left( (\mathcal{H} + rI)^{-\frac{m}{2}}B(\mathcal{H} + rI)^{-\frac{m}{2}}((1 + K_r)^{-m} - 1) \right) \right|
= \left| \text{tr} \left( W_r((1 + K_r)^{-m} - 1) \right) \right|
\leq m \|B\| \| (\mathcal{H} + rI)^{-1} \| \text{tr}(B(\mathcal{H} + rI)^{-m}),
\]

where \(W_r = (\mathcal{H} + rI)^{-\frac{m}{2}}B(\mathcal{H} + rI)^{-\frac{m}{2}}\) is a positive and trace class operator on \(L^2(\mathbb{H}^n)\).

Therefore,

\[
\left| \frac{\text{tr} \left( B(\mathcal{H}_1 + rI)^{-m} \right)}{\text{tr} \left( B(\mathcal{H} + rI)^{-m} \right)} - 1 \right| \leq m \|B\| \| (\mathcal{H} + rI)^{-1} \| \rightarrow 0 \text{ as } r \rightarrow \infty.
\]

\[
\square
\]

**Lemma 6.2.** Let \(\mathcal{H}\) and \(\mathcal{H}_1\) defined as in Theorem 1.3, then

\[
\lim_{r \rightarrow \infty} \frac{\text{tr} \left( B(\mathcal{H}_1 + rI)^{-m} \right)}{\text{tr} \left( (\mathcal{H}_1 + rI)^{-m} \right)} = \lim_{r \rightarrow \infty} \frac{\text{tr} \left( B(\mathcal{H} + rI)^{-m} \right)}{\text{tr} \left( (\mathcal{H} + rI)^{-m} \right)}.
\]

The above equality valid in the sense that if one of limits exist then the other also does and the limits are the same.

**Proof.** For each \(r > 0\), we have

\[
\left( \frac{\text{tr} \left( B(\mathcal{H}_1 + rI)^{-m} \right)}{\text{tr} \left( (\mathcal{H}_1 + rI)^{-m} \right)} \right) = \left( \frac{\text{tr} \left( B(\mathcal{H} + rI)^{-m} \right)}{\text{tr} \left( (\mathcal{H} + rI)^{-m} \right)} \right).
\]

(6.3)

Since the left hand side has limit 1 (by part (b) of Lemma 6.1), the right hand side limit in (6.3) exists and equal to 1. Therefore if the numerator or the denominator in the fraction
in the right hand side has a limit in (6.3), then the other also has a limit and they both agree. Therefore, \( \lim_{r \to \infty} \frac{\text{tr} (B(\mathcal{H}_1 + rI)^{-m})}{\text{tr} ((\mathcal{H}_1 + rI)^{-m})} = \lim_{r \to \infty} \frac{\text{tr} (B(\mathcal{H} + rI)^{-m})}{\text{tr} ((\mathcal{H} + rI)^{-m})}. \)

**Proof of theorem 1.3:** Without loss of generality add a suitable constant to make the function \( f \) positive. Then \( f(A) \) is a positive operator. Setting \( \phi_H(r) = \text{tr}(P_r) \), \( \phi_{H,f}(r) = \text{tr}(P_r f(A)P_r) \) and \( \phi_{H_1,f}(r) = \text{tr}(P'_r f(A)P'_r) \) we have

\[
\lim_{r \to \infty} \frac{\text{tr}(P'_r f(A)P'_r)}{\text{tr}(P'_r)} = \lim_{r \to \infty} \frac{\int_0^\infty \frac{\phi_{H,f}(u)}{(1+u)^{m+1}} \, du}{\int_0^\infty \frac{\phi_{H_1}(u)}{(1+u)^{m+1}} \, du} = \lim_{r \to \infty} \frac{\int_0^\infty \frac{\phi_{H,f}(u)}{(1+u)^{m+1}} \, du}{\int_0^\infty \frac{\phi_{H_1}(u)}{(1+u)^{m+1}} \, du} = \lim_{r \to \infty} \frac{\text{tr}(P_r f(A)P_r)}{\text{tr}(P_r)} = \lim_{r \to \infty} \frac{\int_{G^r} f(a,\lambda,\xi,x) \, d\xi \, dx \, dg \, d\mu(\lambda)}{\int_{G^r} d\xi \, dx \, dg \, d\mu(\lambda)},
\]

(Assuming one limit exists)

where \( G^r = \{(g,\lambda,\xi,x) \in \mathbb{H}^n \times \mathbb{R}^* \times \mathbb{R}^n \times \mathbb{R}^n : |\lambda|(1 + |\xi|^2 + |x|^2) + V(g) \leq r \} \) and \( a(g,\lambda) = Op^W(a_g,\lambda) \). We use Lemma 6.2 for the middle equality and Theorem 7.4 (see Appendix) for the extreme left equalities. The extreme right equality follows from Lemma 5.8.

**Corollary 6.3.** The Theorems 1.1, 1.2 and 1.3 also hold under the compact perturbation of the pseudo-differential operator \( A \).

**Proof.** To prove the above result, enough to show \( \lim_{r \to \infty} \frac{\text{tr}(P_r A^n P_r)}{\text{tr}(P_r)} = \lim_{r \to \infty} \frac{\text{tr}(P_r (A + K)^n P_r)}{\text{tr}(P_r)} \) for any compact operator \( K \) on \( L^2(\mathbb{H}^n) \). Notice that \( (A + K)^n = A^n + \text{terms with factor } A^{p}K^{n-p} \) or \( K^p A^{n-p} \) where \( p \in \{1,2,\ldots,n\} \). Since the class of compact operators form a two sided ideal of the class of bounded operators, \( (A + K)^n = A^n + \text{a compact operator} \). We are done if we can prove that for a compact operator \( T \), \( \lim_{r \to \infty} \frac{\text{tr}(P_r T P_r)}{\text{tr}(P_r)} = 0 \). Since \( T \) is a compact operator, for given \( \epsilon > 0 \) there exist a finite rank operator \( T_k \) such that \( \|T_k - T\| \to 0 \) as \( k \to \infty \). Then \( \left| \frac{\text{tr}(P_r T P_r) - \text{tr}(P_r T_k P_r)}{\text{tr}(P_r)} \right| \leq \|T - T_k\| \to 0 \) as \( k \to \infty \). Therefore for given \( \epsilon > 0 \) there exist \( N_0 \in \mathbb{N} \) such that \( \left| \frac{\text{tr}(P_r T N_0 P_r) - \text{tr}(P_r T_k P_r)}{\text{tr}(P_r)} \right| < \frac{\epsilon}{2} \) for \( k \geq N_0 \). Further, \( \frac{\text{tr}(P_r T N_0 P_r)}{\text{tr}(P_r)} \to 0 \) as \( r \to \infty \) i.e. for given \( \epsilon > 0 \), \( \exists \, N_1 \in \mathbb{N} \) such that \( \left| \frac{\text{tr}(P_r T N_0 P_r)}{\text{tr}(P_r)} \right| < \frac{\epsilon}{2} \) for \( r > N_1 \). Thus \( \left| \frac{\text{tr}(P_r T P_r)}{\text{tr}(P_r)} \right| \leq \frac{\text{tr}(P_r T N_0 P_r)}{\text{tr}(P_r)} + \|T - T_{N_0}\| < \epsilon \) for \( r \geq N_1 \).
Remark 6.4. The proof of part (c) of Lemma 5.8 can also be achieved for \( \kappa \in (0, 1) \) proving the boundedness of the operators \([A, V]\) and \([A, \mathcal{L}]\) on \( L^2(\mathbb{H}^n)\). Now for any \( h \in L^2(\mathbb{H}^n)\), we have

\[
[A, V]h(g) = (AV - VA)h(g) = \int_{\mathbb{H}^n} K_3(g, g_1)h(g_1) \, dg_1,
\]

where

\[
K_3(g, g_1) = (V(g_1) - V(g))\int_{\mathbb{R}^n} \text{tr} (\pi_\lambda(g) \, a(g, \lambda)\pi_\lambda^*(g_1)) \, d\mu(\lambda).
\]

(6.4)

We note that

\[
a(g, \lambda) = \hat{k}_g(\lambda)
\]

\[
= \int_{\mathbb{H}^n} k_g(g_1)\pi_\lambda^*(g_1) \, dg_1
\]

\[
= \int_{\mathbb{H}^n} k_g(g_1)(I - T^2)^N (1 + \lambda^2)^{-N}\pi_\lambda^*(g_1) \, dg_1
\]

\[
= (-1)^N (1 + \lambda^2)^{-N} \int_{\mathbb{H}^n} (I - T^2)^N k_g(g_1)\pi_\lambda^*(g_1) \, dg_1
\]

\[
= (-1)^N (1 + \lambda^2)^{-N} ((I - T^2)^{4N}k_g)(\lambda)
\]

\[
= (-1)^N (1 + \lambda^2)^{-N} \sum_{\beta \leq 2N} \pi(T)^\beta a(g, \lambda).
\]

(6.5)

Then, using the identity (6.5), we have

\[
|\text{tr} (\pi_\lambda(g) \, a(g, \lambda)\pi_\lambda^*(g_1))| = |\text{tr} (\pi_\lambda(g_1^{-1}g) \, a(g, \lambda))|
\]

\[
= |\text{tr} (\pi_\lambda(g_1^{-1}g) (\pi_\lambda(I - \mathcal{L}) + V(g_1^{-1}g))^{-4N} (\pi_\lambda(I - \mathcal{L}) + V(g_1^{-1}g))^{4N} |
\]

\[
\times (1 + \lambda^2)^{-4N} \sum_{\beta \leq 4N} \pi(T)^\beta a(g, \lambda) |
\]

\[
\leq \left| \text{tr} (\pi_\lambda(I - \mathcal{L}) + V(g_1^{-1}g))^{-4N} \right| \left\| (\pi_\lambda(I - \mathcal{L}) + V(g_1^{-1}g))^{4N} (1 + \lambda^2)^{-4N} \right\|_{op}
\]

\[
\times \left\| \sum_{\beta \leq 4N} \pi(T)^\beta a(g, \lambda) \right\|_{op}
\]

\[
\leq C \left| \text{tr} (\pi_\lambda(I - \mathcal{L}) + V(g_1^{-1}g))^{-4N} \right|,
\]

(6.6)

where the second term is bounded by Theorem 3.6 and the last term is bounded by \( ||a|| \).

Now we have

\[
\text{tr} (1 + |\lambda|H + V(g_1^{-1}g))^{-4N} = \sum_{\alpha} \langle (1 + |\lambda|H + V(g_1^{-1}g))^{-4N} \Phi_\alpha, \Phi_\alpha \rangle
\]

\[
= \sum_{\alpha} \frac{1}{(1 + |\lambda|(2|\alpha| + n) + V(g_1^{-1}g))^{4N}}
\]
and consequently
\[
\int_{\mathbb{R}^+} tr\left(\pi_{\lambda}(g) a(g, \lambda) \pi_{\lambda}^*(g_1)\right) d\mu(\lambda) \\
\leq C \sum_\alpha \int_0^\infty \frac{\lambda^n}{(1 + \lambda|2\alpha| + n + V(g_1^{-1} g))^4N} d\lambda \\
= C \sum_\alpha \frac{1}{(2\alpha + n)^{n+1}} \int_{1+V(g_1^{-1} g)}^\infty \frac{(u - 1 - V(g_1^{-1} g))^n}{u^{4N}} du \\
= C_2 \frac{1}{(1 + V(g_1^{-1} g))^{4N-n-1}}.
\]
(6.7)

Thus from (6.4), (6.6), and (6.7), we get
\[
|K_3(g, g_1)| \leq \frac{c|V(g_1) - V(g)|}{(1 + V(g_1^{-1} g))^{4N-n-1}}.
\]

For large $|g|, |g_1|$, using the triangle inequality for the homogeneous norm and the fact that $\kappa \in (0,1)$, we have
\[
\| [A, V]h \|_2^2 \leq \int_{\mathbb{H}^n} \left( \int_{\mathbb{H}^n} K_3(g, g_1) h(g_1) dg_1 \right)^2 dg \\
\leq c_n \int_{\mathbb{H}^n} \left( \int_{\mathbb{H}^n} \frac{|g_1|^k - |g|^k}{(1 + |g_1^{-1} g|^{\kappa})^{4N-n-1}} h(g_1) \right)^2 dg_1 \\
\leq c_n \int_{\mathbb{H}^n} \left( \int_{\mathbb{H}^n} \frac{|g_1|^k - |g|^k}{(1 + |g_1^{-1} g|^{\kappa})^{4N-n-1}} h(g_1) \right)^2 dg_1 \\
\leq c_n \int_{\mathbb{H}^n} \left( \int_{\mathbb{H}^n} \frac{|g_1^{-1} g| h(g_1)}{(1 + |g_1^{-1} g|^{\kappa})^{4N-n-2}} dg_1 \right)^2 dg_1 \\
\leq c_n \int_{\mathbb{H}^n} \left( \int_{\mathbb{H}^n} \frac{h(g_1)}{(1 + |g_1^{-1} g|^{\kappa})^{4N-n-2}} \right)^2 dg_1 \\
= \| [h, K] \|_2^2,
\]
where $K(g) = \frac{1}{(1 + |g|^\kappa)^{4N-n-2}}$. Since for a sufficiently large $N \in \mathbb{N}$, $K \in L^1(\mathbb{H}^n)$, an application of Minkowski’s inequality gives $\|[A, V]h\|_2 \leq C\|K\|_1\|h\|_2$.

If $|g|$ and $|g_1|$ are lying in some compact set $K \subset \mathbb{R}$ then $\int_K \left| \int_K K_3(g, g_1) h(g_1) \right|^2 dg_1 \leq C_K\|h\|_2$. If $|g|$ (or $|g_1|$) lies in $K$ and $|g_1|$ (or $|g|$) is large then an application of Cauchy-Schwarz inequality gives $\|[A, V]h\|_2 \leq \|h\|_2 \int_K \int_{|g|} |K_3(g_1) h(g_1)|^2 dg_1 \ d\mu(\lambda) \leq C_K\|h\|_2$.

For $\kappa \in (0,1)$ the operator $[V, A]$ is bounded. The boundedness of the operator $[\mathcal{L}, A]$ will imply boundedness of the operator $[\mathcal{H}, A]$ on $L^2(\mathbb{H}^n)$ as $[T^2, A] = 0$. Using the identity (6.5), we get
\[
A\mathcal{L}h(g) = \int_{\mathbb{R}^+} tr\left( \pi_{\lambda}(g) a(g, \lambda) \mathcal{L}h(\lambda) \right) d\mu(\lambda)
\]
\[
\int_{\mathbb{R}^n} |A\mathcal{L}h(g)| \leq \int_{\mathbb{R}^n} \left( (\pi_\lambda(I - \mathcal{L}) + V(g))^{-4N+1} \right) \|\hat{h}(\lambda)\|_{op} d\mu(\lambda)
\]

\[
\leq \int_{\mathbb{R}^n} \left( (\pi_\lambda(I - \mathcal{L}) + V(g))^{-4N+1} \right) \|\hat{h}(\lambda)\|_{S_2} d\mu(\lambda)
\]

\[
\leq \left( \int_{\mathbb{R}^n} \left( (\pi_\lambda(I - \mathcal{L}) + V(g))^{-4N+1} \right)^2 d\mu(\lambda) \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^n} \|\hat{h}(\lambda)\|^2_{S_2} d\mu(\lambda) \right)^{\frac{1}{2}}
\]

\[
\leq C\|h\|_2(1 + V(g))^{-\frac{8N+n+3}{2}}.
\]

For sufficiently large \( N \), an application of Cauchy-Schwarz inequality gives \( \|A\mathcal{L}h\|_2 \leq M_1\|h\|_2 \). Further,

\[
\mathcal{L}Ah(g) = \int_{\mathbb{R}^n} tr \left( \pi_\lambda^*(g)\mathcal{L}Ah(\lambda) \right) d\mu(\lambda)
\]

\[
\int_{\mathbb{R}^n} tr \left( \pi_\lambda^*(g)\mathcal{L}Ah(\lambda)\right) d\mu(\lambda)
\]

\[
\int_{\mathbb{R}^n} \int_{\mathbb{H}^n} \mathcal{A}h(g_1) tr \left( \pi_\lambda^*(g)\pi_\lambda(g_1)\right) d\mu(\lambda) d\lambda
\]

\[
\int_{\mathbb{R}^n} \int_{\mathbb{H}^n} \left( \int_{\mathbb{R}^n} tr \left( \pi_\lambda^*(g_1)\pi_\lambda^*(g_1)\right) d\mu(\lambda) \right) tr \left( \pi_\lambda^*(g)\pi_\lambda(g_1)\right) d\mu(\lambda) d\lambda.
\]

Proceeding as in (6.7), we get

\[
\int_{\mathbb{R}^n} \left| tr \left( \pi_\lambda^*(g_1)a(g_1, \lambda_1)\hat{h}(\lambda_1) \right) \right| d\mu(\lambda_1) \leq C\|h\|_2(1 + V(g_1))^{-\frac{8N+n+1}{2}}.
\]

Moreover, a similar way as in (6.7), we obtain

\[
\int_{\mathbb{R}^n} \int_{\mathbb{H}^n} |tr (\pi_\lambda^*(g_1)\pi_\lambda(g_1)|\lambda|H)| (1 + V(g_1))^{-\frac{8N+n+1}{2}} d\mu(\lambda) d\lambda \leq \int_{\mathbb{H}^n} (1 + V(g_1))^{-\frac{8N+n+1}{2}} d\lambda \int_{\mathbb{R}^n} |tr (|\lambda|H^\pi_\lambda(g_1))| d\mu(\lambda)
\]

\[
\leq \int_{\mathbb{R}^n} tr \left( (1 + \lambda^2)^{-4N} \sum_{\beta \leq 8N} \pi(\lambda)^{\beta}|\lambda|H \right) d\mu(\lambda)
\]

\[
\leq C(1 + V(g))^{-8N+n}.
\]

Therefore \( \|[A, \mathcal{L}]h\|_2 \leq M\|h\|_2 \) and so the operator \([A, \mathcal{H}]\) is bounded on \( L^2(\mathbb{H}^n) \).
Now setting $A = H, B = A, \chi = 0, \psi = f, P_\lambda = \pi_r$ in Theorem 1.6 of Laptev-Safarov [13], we get

$$|tr(P_r f(A)P_r - P_r f(P_r A P_r)P_r)| \leq \frac{1}{2} ||f''||_{\infty} N_r(r) \left( ||\pi_r A||^2 + \frac{\pi^2}{6r^2} ||\pi_{r-r}[A, H]||^2 \right).$$

Dividing both sides by $tr(P_r)$ and setting $r = r^\alpha, \alpha \in (0, 1)$ and using the boundness of $A, [A, H]$ we have

$$\frac{|tr(P_r f(A)P_r - P_r f(P_r A P_r)P_r)|}{tr(P_r)} \leq C N_r(r) \to 0 \text{ as } r \to \infty$$

by (4) of Corollary 5.5, where $N_r(r) = \sup_{\mu \leq r} (tr(\pi_\mu - \pi_{\mu-r})).$

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7. Appendix

We collect few definitions and theorems of Grishin-Poedintseva [5], that we use in our paper for the reader’s convenience.

**Definition 7.1.** Let $\phi$ be a positive function on the half line $[0, \infty)$. Let

$$S = \{\alpha : \exists M, R \text{ with } \phi(tr) \leq Mt^\alpha \phi(r), \text{ for all } t \geq 1, r \geq R\}$$

and

$$G = \{\beta : \exists M, R \text{ with } \phi(tr) \geq Mt^\beta \phi(r), \text{ for all } t \geq 1, r \geq R\}$$

Then the numbers $\alpha(\phi) := \inf S$ and $\beta(\phi) := \sup G$ are called the upper and lower Matushevskaya index of $\phi$ respectively.

**Theorem 7.2.** ([5], Theorem 2)

Let $m > -1$. Assume that $\varphi$ is positive measurable function on $[0, \infty)$ that does not vanish identically in any neighborhood of infinity. Let $\Phi(r) = \int_0^\infty \frac{\varphi(rt)}{(1 + t)^{m+1}} dt$ be finite. Then the functions $\varphi$ and $\Phi$ have same growth at infinity if and only if $\beta(\varphi) > -1$ and $\alpha(\varphi) < m$.

**Definition 7.3.** A function $\varphi$ is said to be multiplicatively continuous at infinity if it satisfies

$$\lim_{r \to \infty} \frac{\varphi(\tau r)}{\varphi(r)} = 1.$$  

**Theorem 7.4.** ([5], Theorem 8) Let $\varphi$ and $\psi$ be positive functions on $[0, \infty)$ satisfying the following conditions:
(1) the functions \( \varphi \) and \( \psi \) do not vanish identically in any neighborhood of infinity;
(2) the function \( \varphi \) is multiplicatively continuous at infinity and \( \beta(\varphi) > -1 \);
(3) the function \( \psi \) is increasing;
(4) at least one of the inequalities \( \alpha(\varphi) < m \) and \( \alpha(\psi) < m \) holds, where \( m > -1 \);
(5) the functions
\[
\Phi(r) = \int_0^\infty \frac{\varphi(ru)}{(1 + u)^{m+1}} du \quad \text{and} \quad \Psi(r) = \int_0^\infty \frac{\psi(ru)}{(1 + u)^{m+1}} du
\]
are finite and \( \lim_{r \to \infty} \frac{\Psi(r)}{\Phi(r)} = 1 \) then \( \lim_{r \to \infty} \frac{\psi(r)}{\varphi(r)} = 1 \).

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