FREE EQUATIONS FOR MASSIVE MATTER FIELDS IN 2+1-DIMENSIONAL ANTI-DE SITTER SPACE From DEFORMED OSCILLATOR ALGEBRA

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Abstract
We reformulate free equations of motion for massive spin 0 and spin 1/2 matter fields in 2+1 dimensional anti-de Sitter space in the form of some covariant constantness conditions. The infinite-dimensional representation of the anti-de Sitter algebra underlying this formulation is shown to admit a natural realization in terms of the algebra of deformed oscillators with a deformation parameter related to the parameter of mass.
1 Introduction

In [1] free equations of spin 0 and spin 1/2 matter fields in 2+1 - dimensional anti-de Sitter (AdS) space were reformulated in a form of certain covariant constantness conditions ("unfolded form"). Being equivalent to the standard one, such a formulation is useful at least in two respects. It leads to a simple construction of a general solution of the free equations and gives important hints how to describe non-linear dynamics exhibiting infinite-dimensional higher-spin symmetries. In [1] it was also observed that the proposed construction admits a natural realization in terms of the Heisenberg-Weyl oscillator algebra for the case of massless fields. Based on this realization, non-linear dynamics of massless matter fields interacting through higher-spin gauge fields was then formulated in [2] in all orders in interactions.

In the present paper we address the question how one can extend the oscillator realization of the massless equations of [1] to the case of an arbitrary mass of matter fields. We show that the relevant algebraic construction is provided by the deformed oscillator algebra suggested in [3] with the deformation parameter related to the parameter of mass. In a future publication of the two of the authors [4] the results of this paper will be used for the analysis of non-linear dynamics of matter fields in 2+1 dimensions, interacting through higher-spin gauge fields. The 2+1 dimensional model considered in this paper can be regarded as a toy model exhibiting some of the general properties of physically more important higher-spin gauge theories in higher dimensions \( d \geq 4 \).

2 Preliminaries

We describe the 2+1 dimensional AdS space in terms of the Lorentz connection one-form \( \omega^{\alpha\beta} = dx^\nu \omega^{\alpha\beta}_\nu(x) \) and dreibein one-form \( h^{\alpha\beta} = dx^\nu h^{\alpha\beta}_\nu(x) \). Here \( x^\nu \) are space-time coordinates (\( \nu = 0,1,2 \)) and \( \alpha,\beta,\ldots = 1,2 \) are spinor indices, which are raised and lowered with the aid of the symplectic form \( \epsilon^{\alpha\beta} = -\epsilon^{\beta\alpha}, A^\alpha = \epsilon^{\alpha\beta} A_\beta, A_\alpha = A^\beta \epsilon_{\beta\alpha}, \epsilon_{12} = \epsilon^{12} = 1 \). The AdS geometry can be described by the equations

\[
d\omega^{\alpha\beta} = \omega^{\alpha\gamma} \wedge \omega^{\beta\gamma} + \lambda^2 h^{\alpha\gamma} \wedge h^{\beta\gamma},
\]

\[
dh = \omega^{\alpha\gamma} \wedge h^{\beta\gamma} + \omega^{\beta\gamma} \wedge h^{\alpha\gamma},
\]

which have a form of zero-curvature conditions for the \( o(2,2) \sim sp(2) \oplus sp(2) \) Yang-Mills field strengths. Here \( \omega^{\alpha\beta} \) and \( h^{\alpha\beta} \) are symmetric in \( \alpha \) and \( \beta \). For the space-time geometric interpretation of these equations one has to assume that the dreibein \( h^{\alpha\beta}_\nu \) is a non-degenerate \( 3 \times 3 \) matrix. Then (2) reduces to the zero-torsion condition which expresses Lorentz connection via dreibein \( h^{\alpha\beta}_\nu \) and (1) implies that the Riemann tensor 2-form \( R^{\alpha\beta} = d\omega^{\alpha\beta} - \omega^{\alpha\gamma} \wedge \omega^{\beta\gamma} \) acquires the AdS form

\[
R^{\alpha\beta} = \lambda^2 h^{\alpha\gamma} \wedge h^{\beta\gamma}
\]

with \( \lambda^{-1} \) identified with the AdS radius.
In [1] it was shown that one can reformulate free field equations for matter fields in 2+1 dimensions in terms of the generating function $C(y|x)$

$$C(y|x) = \sum_{n=0}^{\infty} \frac{1}{n!} C_{\alpha_1...\alpha_n}(x) y^{\alpha_1} \ldots y^{\alpha_n}$$  \hspace{1cm} (4)

in the following “unfolded” form

$$DC = h^{\alpha\beta} \left[ a(N) \frac{\partial}{\partial y^\alpha} \frac{\partial}{\partial y^\beta} + b(N) y_\alpha \frac{\partial}{\partial y^\beta} + e(N) y_\alpha y_\beta \right] C,$$  \hspace{1cm} (5)

where $D$ is the Lorentz covariant differential

$$D = d - \omega^{\alpha\beta} y_\alpha \frac{\partial}{\partial y^\beta}$$  \hspace{1cm} (6)

and $N$ is the Euler operator

$$N \equiv y^\alpha \frac{\partial}{\partial y^\alpha}.$$  \hspace{1cm} (7)

The integrability conditions of the equations (5) (i.e. the consistency with $d^2 = 0$) require the functions $a, b$ and $e$ to satisfy the following restrictions [1]

$$\alpha(n) = 0 \quad \text{for} \quad n \geq 0, \quad \gamma(n) = 0 \quad \text{for} \quad n \geq 2,$$  \hspace{1cm} (8)

$$\beta(n) = 0 \quad \text{for} \quad n \geq 1,$$

where

$$\alpha(N) = a(N) [(N + 4) b(N + 2) - Nb(N)],$$  \hspace{1cm} (9)

$$\gamma(N) = e(N) [(N + 2) b(N) - (N - 2) b(N - 2)],$$  \hspace{1cm} (10)

$$\beta(N) = (N + 3) a(N) e(N + 2) - (N - 1) e(N) a(N - 2) + b^2(N) + \lambda^2.$$  \hspace{1cm} (11)

It was shown in [1] that, for the condition that $a(n) \neq 0 \forall n \geq 0$ and up to a freedom of field redefinitions $C \rightarrow \tilde{C} = \varphi(N) C$, $\varphi(n) \neq 0 \forall n \in \mathbb{Z}^+$, there exist two one parametric classes of independent solutions of (5),

$$a(n) = 1, \quad b(n) = 0, \quad e(n) = \frac{1}{4} \lambda^2 - \frac{M^2}{2(n+1)(n-1)}, \quad n - \text{even},$$  \hspace{1cm} (12)

$$a(n) = b(n) = e(n) = 0, \quad n - \text{odd},$$

and

$$a(n) = b(n) = e(n) = 0, \quad n - \text{even},$$

$$a(n) = 1, \quad b(n) = \sqrt{2M} \frac{M}{n(n+2)}, \quad e(n) = \frac{1}{4} \lambda^2 - \frac{M^2}{2n^2}, \quad n - \text{odd},$$  \hspace{1cm} (13)

with an arbitrary parameter $M$. As a result, the system (5) reduces to two independent infinite chains of equations for bosons and fermions described by multispinors with even and odd number of indices, respectively. To elucidate the physical content of these equations one has to identify the lowest components of the expansion (5), $C(x)$ and $C_{\alpha}(x)$,
with the physical spin-0 boson and spin 1/2 fermion matter fields, respectively, and to check, first, that the system (5) amounts to the physical massive Klein-Gordon and Dirac equations,

$$\Box C = \left(\frac{3}{2}\lambda^2 - M^2\right)C,$$

$$h^\nu_{\alpha\beta} D_\nu C_\beta = \frac{M}{\sqrt{2}} C_\alpha,$$

and, second, that all other equations in (5) express all highest multispinors via highest derivatives of the matter fields $C$ and $C_\alpha$ imposing no additional constraints on the latter. Note that the D’Alambertian is defined as usual

$$\Box = D^\mu D_\mu,$$

where $D_\mu$ is a full background covariant derivative involving the zero-torsion Christoffel connection defined through the metric postulate $D_\mu h^\alpha_{\nu\beta} = 0$. The inverse dreibein $h^\nu_{\alpha\beta}$ is defined as in [1],

$$h^\nu_{\alpha\beta} h^\nu_{\gamma\delta} = \frac{1}{2} \delta^\nu_{\gamma\delta} + \delta^\nu_{\delta\gamma}.$$

(17)

Note also that the indices $\mu, \nu$ are raised and lowered by the metric tensor

$$g_{\mu\nu} = h^\mu_{\alpha\beta} h^\nu_{\nu\alpha\beta}.$$

As emphasized in [1], the equations (5) provide a particular example of covariant constantness conditions

$$dC_i = A_i^i C_j,$$

(18)

with the gauge fields $A_i^i = A^a (T_a)_i^i$ obeying the zero-curvature conditions

$$dA^a = U^a_{bc} A^b \wedge A^c,$$

(19)

where $U_{bc}^a$ are structure coefficients of the Lie (super)algebra which gives rise to the gauge fields $A^a$ (cf (1), (2)). Then the requirement that the integrability conditions for (18) must be true is equivalent to the requirement that $(T_a)_i^i$ form some matrix representation of the gauge algebra. Thus, the problem consists of finding an appropriate representation of the space-time symmetry group which leads to correct field equations. As a result, after the equations are rewritten in this “unfolded form”, one can write down their general solution in a pure gauge form $A(x) = -g^{-1}(x)dg(x)$, $C(x) = T(g^{-1})(x)C_0$, where $C_0$ is an arbitrary $x$-independent element of the representation space. This general solution has a structure of the covariantized Taylor type expansion [1]. For the problem under consideration the relevant (infinite-dimensional) representation of the AdS algebra is characterized by the coefficients (12) and (13).

### 3 Operator Realization for Arbitrary Mass

Let us now describe an operator algebra that leads automatically to the correct massive field equations of the form (5).
Following to [3] we introduce oscillators obeying the commutation relations

\[ [\hat{y}_\alpha, \hat{y}_\beta] = 2i\epsilon_{\alpha\beta}(1 + \nu k), \]  

(20)

where \( \alpha, \beta = 1, 2 \), \( k \) is the Klein operator anticommuting with \( \hat{y}_\alpha \),

\[ k\hat{y}_\alpha = -\hat{y}_\alpha k, \quad k^2 = 1 \]  

(21)

and \( \nu \) is a free parameter. The main property of these oscillators is that the bilinears

\[ T_{\alpha\beta} = \frac{1}{4i}\{\hat{y}_\alpha, \hat{y}_\beta\} \]  

(22)

fulfill the standard \( sp(2) \) commutation relations

\[ [T_{\alpha\beta}, T_{\gamma\delta}] = \epsilon_{\alpha\gamma}T_{\beta\delta} + \epsilon_{\beta\delta}T_{\alpha\gamma} + \epsilon_{\alpha\delta}T_{\beta\gamma} + \epsilon_{\beta\gamma}T_{\alpha\delta} \]  

(23)

as well as

\[ [T_{\alpha\beta}, \hat{y}_\gamma] = \epsilon_{\alpha\gamma}\hat{y}_\beta + \epsilon_{\beta\gamma}\hat{y}_\alpha \]  

(24)

for any \( \nu \). Note that a specific realization of this kind of oscillators was considered by Wigner [5] who addressed a question whether it is possible to modify the oscillator commutation relations in such a way that the relation \( [H, a_\pm] = \pm a_\pm \) remains valid. This relation is a particular case of (24) with \( H = T_{12} \) and \( a_\pm = \hat{y}_{1,2} \).

The property (23) allows us to realize the \( o(2, 2) \) gravitational fields as

\[ W_{gr}(x) = \omega + \lambda h; \quad \omega \equiv \frac{1}{8i}\omega^{\alpha\beta}\{\hat{y}_\alpha, \hat{y}_\beta\}, \quad h \equiv \frac{1}{8i}h^{\alpha\beta}\{\hat{y}_\alpha, \hat{y}_\beta\}\psi, \]  

(25)

where \( \psi \) is an additional central involutive element,

\[ \psi^2 = 1, \quad [\psi, \hat{y}_\alpha] = 0, \quad [\psi, k] = 0, \]  

(26)

which is introduced to describe the 3d AdS algebra \( o(2, 2) \sim sp(2) \oplus sp(2) \) spanned by the generators

\[ L_{\alpha\beta} = \frac{1}{4i}\{\hat{y}_\alpha, \hat{y}_\beta\}, \quad P_{\alpha\beta} = \frac{1}{4i}\{\hat{y}_\alpha, \hat{y}_\beta\}\psi. \]  

(27)

Now the equations (1) and (2) describing the vacuum anti-de Sitter geometry acquire a form

\[ dW_{gr} = W_{gr} \wedge W_{gr}. \]  

(28)

Let us introduce the operator-valued generating function \( C(\hat{y}, k|x) \)

\[ C(\hat{y}, k|\psi|x) = \sum_{A, B=0,1} \sum_{n=0}^{\infty} \frac{1}{n!} \lambda^{-\frac{|n|^2}{2}} C_{\alpha_1...\alpha_n}^{AB}(x)k^A\psi^B\hat{y}^{\alpha_1}...\hat{y}^{\alpha_n}, \]  

(29)

where \( C_{\alpha_1...\alpha_n}^{AB} \) are totally symmetric tensors (which implies the Weyl ordering with respect to \( \hat{y}_\alpha \)). It is easy to see that the following two types of equations

\[ DC = \lambda[h, C], \]  

(30)
and
\[ DC = \lambda \{ h, C \}, \]  
where
\[ DC \equiv dC - [\omega, C] \]  
are consistent (i.e. the integrability conditions are satisfied as a consequence of the vacuum conditions (28)). Indeed, (30) corresponds to the adjoint action of the space-time algebra (27) on the algebra of modified oscillators. The equations (31) correspond to another representation of the space-time symmetry which we call twisted representation. The fact that one can replace the commutator by the anticommutator in the term proportional to dreibein is a simple consequence of the property that AdS algebra possesses an involutive automorphism changing a sign of the AdS translations. In the particular realization used here it is induced by the automorphism \( \psi \rightarrow -\psi \).

There is an important difference between these two representations. The first one involving the commutator decomposes into an infinite direct sum of finite-dimensional representations of the space-time symmetry algebra. Moreover, because of the property (24) this representation is \( \nu \)-independent and therefore is equivalent to the representation with \( \nu = 0 \) which was shown in [1] to describe an infinite set of auxiliary (topological) fields. The twisted representation on the other hand is just the infinite-dimensional representation needed for the description of matter fields (in what follows we will use the symbol \( C \) only for the twisted representation).

To see this one has to carry out a component analysis of the equations (31) which consists of some operator reorderings bringing all terms into the Weyl ordered form with respect to \( \hat{y}_\alpha \). As a result one finds that (31) takes the form of the equation (5) with the following values of the coefficients \( a(n) \), \( b(n) \) and \( e(n) \):

\[
a(n) = \frac{i\lambda}{2} \left[ 1 + \nu k \frac{1 + (-1)^n}{(n + 2)^2 - 1} \nu^2 \right] - \frac{(n + 2)^2((n + 2)^2 - 1)}{(n + 2)^2} \left( (n + 2)^2 - \frac{1}{2} \right) \right], \tag{33}
\]
\[
b(n) = -\nu k \lambda \frac{1 - (-1)^n}{2n(n + 2)}, \tag{34}
\]
\[
e(n) = -\frac{i\lambda}{2}. \tag{35}
\]

As expected, these expressions satisfy the conditions (8).

Now let us remind ourselves that due to the presence of the Klein operator \( k \) we have a doubled number of fields compared to the analysis in the beginning of this section. One can project out the irreducible subsets with the aid of the two projectors \( P_\pm \),

\[ C_\pm \equiv P_\pm C, \quad P_\pm \equiv \frac{1 \pm k}{2}. \tag{36}\]

As a result we get the following component form of eq. (5) with the coefficients (33)-(35),

\[
DC_{\alpha(n)}^\pm = \frac{i}{2} \left[ \left( 1 - \frac{\nu(\nu + 2)}{(n + 1)(n + 3)} \right) h^{\beta\gamma} C_{\beta\gamma\alpha(n)}^\pm - \lambda^2 n(n - 1) h_{\alpha\alpha} C_{\alpha(n-2)}^\pm \right] \tag{37}
\]
for even $n$, and
\[
DC_{a(n)}^{\pm} = \frac{i}{2} \left(1 - \frac{\nu^2}{(n+2)^2}\right) h^{\beta\gamma} C_{\beta\gamma a(n)}^{\pm} \pm \frac{\nu \lambda}{n+2} h^\alpha \beta C_{\beta a(n-1)}^{\pm}
\]
\[-\frac{i}{2} \lambda^2 n(n-1) h_{\alpha\alpha} C_{a(n-2)}^{\pm}
\]
for odd $n$. Here we use the notation $C_{a(n)} = C_{a_1,...,a_n}$ and assume the full symmetrization of the indices denoted by $\alpha$.

As it was shown in [1], the D’Alambertian corresponding to eq. (5) has the following form
\[
\Box C = \left( (N + 3)(N + 2) a(N) e(N + 2) + N(N - 1) e(N) a(N - 2) - \frac{1}{2} N(N + 2) b^2(N) \right) C .
\]
Insertion of (33)-(35) into (39) yields
\[
\Box C_{\pm} = \left[ \lambda^2 \frac{N(N + 2)}{2} + \lambda^2 \frac{3}{2} - M_{\pm}^2 \right] C_{\pm},
\]
with
\[
M_{\pm}^2 = \lambda^2 \frac{\nu(\nu + 2)}{2}, \quad n -\text{even},
\]
\[
M_{\pm}^2 = \lambda^2 \frac{\nu^2}{2}, \quad n -\text{odd}.
\]

Thus, it is shown that the modification (20) allows one to describe matter fields with an arbitrary mass parameter related to $\nu$. This construction generalizes in a natural way the realization of equations for massless matter fields in terms of the ordinary ($\nu = 0$) oscillators proposed in [1]. An important comment however is that this construction not necessarily leads to non-vanishing coefficients $a(n)$. Consider, for example, expression (33) for the bosonic part of $C_{\pm}$, i.e., set $k = 1$, $n = 2m$, $m$ is some integer,

\[
a(2m) = \frac{i \lambda}{2} \left[ 1 - \frac{\nu(\nu - 2)}{(2m + 1)(2m + 3)} \right].
\]

We observe that $a(2l) = 0$ at $\nu = \pm 2(l + 1) + 1$. It is not difficult to see that some of the coefficients $a(n)$ vanish if and only if $\nu = 2k + 1$ for some integer $k$. This conclusion is in agreement with the results of [4] where it was shown that for these values of $\nu$ the enveloping algebra of the relations (20), $Aq(2; \nu|C)$, possesses ideals. Thus, strictly speaking for $\nu = 2k + 1$ the system of equation derived from the operator realization (31) is different from that considered in [1]. The specificities of the degenerated systems with $\nu = 2k + 1$ will be discussed in the section 5.

\footnote{Let us remind the reader that the physical matter field components are singled out by the conditions $NC_{\pm} = 0$ in the bosonic sector and $NC_{\pm} = C_{\pm}$ in the fermionic sector.}
In [6] it was shown that the algebra $Aq(2, \nu)$ is isomorphic to the factor algebra $U(osp(1, 2))/I(C_2 - \nu^2)$, where $U(osp(1, 2))$ is the enveloping algebra of $osp(1, 2)$, while $I(C_2 - \nu^2)$ is the ideal spanned by all elements of the form

$$(C_2 - \nu^2) x, \quad \forall x \in U(osp(1, 2)),$$

where $C_2$ is the quadratic Casimir operator of $osp(1, 2)$. From this observation it follows in particular that the oscillator realization described above is explicitly supersymmetric. In fact it is $N=2$ supersymmetric [6] with the generators of $osp(2, 2)$ of the form

$$T_{\alpha\beta} = \frac{1}{4i} \{\hat{y}_\alpha, \hat{y}_\beta\}, \quad Q_\alpha = \hat{y}_\alpha, \quad S_\alpha = \hat{y}_{\alpha k}, \quad J = k + \nu.$$

This observation guarantees that the system of equations under consideration possesses $N=2$ global supersymmetry. It is this $N = 2$ supersymmetry which leads to a doubled number of boson and fermion fields in the model.

4 **Bosonic Case and U(o(2,1))**

In the purely bosonic case one can proceed in terms of bosonic operators, avoiding the doubling of fields caused by supersymmetry. To this end, let us use the orthogonal realization of the AdS algebra $o(2, 2) \sim o(2, 1) \oplus o(2, 1)$. Let $T_a$ be the generators of $o(2, 1)$,

$$[T_a, T_b] = \epsilon_{abc} T_c,$$  

where $\epsilon_{abc}$ is a totally antisymmetric 3d tensor, $\epsilon_{012} = 1$, and Latin indices are raised and lowered by the Killing metrics of $o(2, 1)$,

$$A^a = \eta^{ab} A_b, \quad \eta = diag(1, -1, -1).$$

Let the background gravitational field have a form

$$W^\mu = \omega_\mu^a T_a + \tilde{\lambda} \psi h^a T_a,$$  

where $\psi$ is a central involutive element,

$$\psi^2 = 1, \quad [\psi, T_a] = 0,$$

and let $W$ obey the zero-curvature conditions (28). Note, that the inverse dreibein $h^{-a}_\mu$ is normalized so that

$$h^{-a}_\mu h^a = \eta^{ab}.$$  

Let $T_a$ be restricted by the following additional condition on the quadratic Casimir operator

$$C_2 \equiv T_a T^a = \frac{1}{8} \left( \frac{3}{2} - \frac{M^2}{\chi^2} \right).$$

We introduce the dynamical 0-form $C$ as a function of $T_a$ and $\psi$

$$C = \sum_{n=0}^{\infty} \sum_{A=0,1} \frac{1}{n!} \psi^A C_A^{a_1...a_n}(x) T_{a_1} \cdots T_{a_n},$$
where $C_{A^1 \cdots A^n}$ are totally symmetric traceless tensors. Equivalently one can say that $C$ takes values in the algebra $A_M \oplus A_M$ where $A_M = U(o(2,1))/I_{(C_2 - 1/8(3^2 - M^2))}$. Here $U(o(2,1))$ is the enveloping algebra for the relations (44) and $I_{(C_2 - 1/8(3^2 - M^2))}$ is the ideal spanned by all elements of the form

$$[C_2 - \frac{1}{8} \left(\frac{3}{2} - \frac{M^2}{\lambda^2}\right)] x, \quad \forall x \in U(o(2,1)).$$

We can then write down the equation analogous to (31) in the form

$$D_\mu C = \tilde{\lambda} \psi h_\mu^a \{T_a, C\}, \quad (50)$$

where

$$D_\mu C = \partial_\mu C - \omega_\mu^a [T_a, C]. \quad (51)$$

Acting on the both sides of eq. (50) by the full covariant derivative $D^\mu$, defined through the metric postulate $D^\mu(h_\nu^a T_a) = 0$ under the condition that the Christoffel connection is symmetric, one can derive

$$\Box C_n = \frac{1}{2} \tilde{\lambda}^2 \left[2n(n+1) + \frac{3}{2} - \frac{M^2}{\lambda^2}\right] C_n, \quad (52)$$

where $C_n$ denotes a $n$-th power monomial in (49). We see that this result coincides with (40) at $N = 2n$ and

$$\lambda^2 = \frac{1}{2} \lambda^2. \quad (53)$$

Also one can check that the zero-curvature conditions for the gauge fields (25) and (45) are equivalent to each other provided that (53) is true. The explicit relationships are

$$\omega^a_\mu \sigma_\alpha^\beta = -\frac{1}{2} \omega^a_\mu \sigma_\alpha^\beta, \quad h_\mu^a \sigma_\alpha^\beta = -\frac{1}{\sqrt{2}} h_\mu^a \sigma_\alpha^\beta, \quad T_a = \frac{1}{16i} \sigma_\alpha^\beta (\hat{y}_\alpha, \hat{y}_\beta),$$

where $\sigma_\alpha^\beta = (I, \sigma_1, \sigma_3)$, $\sigma_1, \sigma_3$ are symmetric Pauli matrices.

One can also check that, as expected, eq. (50) possesses the same degenerate points in $M$ as eq. (5) does according to (43).

5 Degenerate Points

In this section we discuss briefly the specificities of the equation (31) at singular points in $\nu$. Let us substitute the expansion (29) into (5) with the coefficients defined by (33)-(35) and project (5) to the subspace of bosons $C_+$ by setting $k = 1$ and $n$ to be even. Then we get in the component form

$$DC_{\alpha(n)} = i \left[\left(1 - \frac{\nu(\nu - 2)}{(n+1)(n+3)}\right) h^{\beta\gamma} C_{\beta\gamma\alpha(n)} - \lambda^2 n(n-1) h_{\alpha\alpha} C_{\alpha(n-2)}\right]. \quad (54)$$

In the general case (i.e., $\nu \neq 2l + 1$, $l$-integer) this chain of equations starts from the scalar component and is equivalent to the dynamical equation (14) with $M^2 = \lambda^2 \frac{\nu(\nu-2)}{2}$.
supplemented either by relations expressing highest multispinors via highest derivatives of $C$ or identities which express the fact that higher derivatives are symmetric.

At $\nu = 2l + 1$ the first term on the r.h.s. of (54) vanishes for $n = 2(\pm l - 1)$. Since $n$ is non-negative let us choose for definiteness a solution with $n = 2(l - 1)$, $l > 0$. One observes that the rank-2$l$ component is not any longer expressed by (54) via derivatives of the scalar $C$, thus becoming an independent dynamical variable. Instead, the equation (54) tells us that (appropriately AdS covariantized) $l$-th derivative of the scalar field $C$ vanishes. As a result, at degenerate points the system of equations (54) acquires a non-decomposable triangle-type form with a finite subsystem of equations for the set of multispinors $C_{\alpha(2n)}$, $n < l$ and an infinite system of equations for the dynamical field $C_{\alpha(2l)}$ and higher multispinors, which contains (derivatives of) the original field $C$ as a sort of sources on the right hand side.

The subsystem for lower multispinors describes a system analogous to that of topological fields (30) which can contain at most a finite number of degrees of freedom. In fact this system should be dynamically trivial by the unitarity requirements (there are no finite-dimensional unitary representations of the space-time symmetry groups)\footnote{The only exception is when the degeneracy takes place on the lowest level and the representation turns out to be trivial (constant).}. Physically, this is equivalent to imposing appropriate boundary conditions at infinity which must kill these degrees of freedom because, having only a finite number of non-vanishing derivatives, these fields have a polynomial growth at the space-time infinity (except for a case of a constant field $C$).

Thus one can factor out the decoupling lowest components arriving at the system of equations which starts from the field $C_{\alpha(2l)}$. These systems are dynamically non-trivial and correspond to certain gauge systems. For example, one can show that the first degenerate point $\nu = 3$ just corresponds to 3d electrodynamics. To see this one can introduce a two-form

$$F = h^\alpha_\gamma \wedge h^{\gamma\beta} C_{\alpha\beta}$$

and verify that the infinite part of the system (54) with $n \geq 2$ (i.e. with the scalar field factored out) is equivalent to the Maxwell equations

$$dF = 0, \quad d^* F = 0$$

supplemented with an infinite chain of higher Bianchi identities (here $^* F$ denotes a form dual to $F$). Note that, for our normalization of a mass, electrodynamics turns out to be massive with the mass $M^2 = \frac{3}{2} \lambda^2$ which vanishes in the flat limit $\lambda \to 0$. A more detailed analysis of this formulation of electrodynamics and its counterparts corresponding to higher degenerate points will be given in [4].

Now let us note that there exists an alternative formulation of the dynamics of matter fields which is equivalent to the original one of [1] for all $\nu$ and is based on the co-twisted representation $\bar{C}$. Namely, let us introduce a non-degenerate invariant form

$$\langle C, \bar{C} \rangle = \int d^4 x \sum_{n=0}^{\infty} \frac{1}{(2n)!} C_{\alpha(2n)} \bar{C}^{\alpha(2n)}$$

confining ourselves for simplicity to the purely bosonic case in the sector $C_+$.\footnote{The only exception is when the degeneracy takes place on the lowest level and the representation turns out to be trivial (constant).}
The covariant differential corresponding to the twisted representation $C$ of $o(2, 2)$ has the form

$$DC = dC - \{\omega, C\} - \lambda \{h, C\}, \quad (58)$$

so that eq. (51) acquires a form $DC = 0$. The covariant derivative in the co-twisted representation can be obtained from the invariance condition

$$\langle C, DC \rangle = -\langle DC, \bar{C} \rangle. \quad (59)$$

It has the following explicit form

$$D\bar{C}^{(n)} = d\bar{C}^{(n)} - n\omega_{\beta} \bar{C}^{(n-1)} - \frac{i}{2} \left[h_{\beta\gamma} \bar{C}^{(n)} - \lambda^2 n(n-1) \left(1 - \frac{\nu(n-2)}{(n-1)(n+1)}\right) h^{\alpha\alpha} \bar{C}^{(n-2)}\right]. \quad (60)$$

As a result the equation for $\bar{C}$ analogous to (54) reads

$$D\bar{C}^{(n)} = \frac{i}{2} \left[h_{\beta\gamma} \bar{C}^{(n)} - \lambda^2 n(n-1) \left(1 - \frac{\nu(n-2)}{(n-1)(n+1)}\right) h^{\alpha\alpha} \bar{C}^{(n-2)}\right]. \quad (61)$$

We see that now the term containing a higher multispinor appears with a unite coefficient while the coefficients in front of the lower multispinor sometimes vanish. The equations (61) identically coincide with the equations derived in [1] which are reproduced in the section 2 of this paper. Let us note that the twisted and co-twisted representations are equivalent for all $\nu \neq 2l + 1$ because the algebra of deformed oscillators possesses an invariant quadratic form which is non-degenerate for all $\nu \neq 2l + 1$ [3]. For $\nu = 2l + 1$ this is not the case any longer since the invariant quadratic form degenerates and therefore twisted and co-twisted representations turn out to be formally inequivalent.

Two questions are now in order. First, what is a physical difference between the equations corresponding to twisted and co-twisted representations at the degenerate points, and second which of these two representations can be used in an interacting theory. These issues will be considered in more detail in [4]. Here we just mention that at the free field level the two formulations are still physically equivalent and in fact turn out to be dual to each other. For example for the case of electrodynamics the scalar field component $C$ in the co-twisted representation can be interpreted as a magnetic potential such that $* F = dC$. A non-trivial question then is whether such a formulation can be extended to any consistent local interacting theory. Naively one can expect that the formulation in terms of the twisted representation has better chances to be extended beyond the linear problem. It will be shown in [4] that this is indeed the case.

6 Conclusion

In this paper we suggested a simple algebraic method of formulating free field equations for massive spin-0 and spin 1/2 matter fields in 2+1 dimensional AdS space in the form of covariant constantness conditions for certain infinite-dimensional representations of the space-time symmetry group. An important advantage of this formulation is that it allows
one to describe in a simple way a structure of the global higher-spin symmetries. These symmetries are described by the parameters which take values in the infinite-dimensional algebra of functions of all generating elements $y_\alpha, k$ and $\psi$, i.e. $\varepsilon = \varepsilon(y_\alpha, k, \psi|x)$. The full transformation law has a form

$$\delta C = \varepsilon C - C\varepsilon,$$

where

$$\varepsilon(y_\alpha, k, \psi|x) = \varepsilon(y_\alpha, k, -\psi|x)$$

and the dependence of $\varepsilon$ on $x$ is fixed by the equation

$$d\varepsilon = W_{gr}\varepsilon - \varepsilon W_{gr},$$

which is integrable as a consequence of the zero-curvature conditions (63) and therefore admits a unique solution in terms of an arbitrary function $\varepsilon_0(y_\alpha, k, \psi|x_0) = \varepsilon(y_\alpha, k, \psi|x_0)$ for an arbitrary point of space-time $x_0$. It is obvious that the equations (63) are indeed invariant with respect to the transformations (62).

Explicit knowledge of the structure of the global higher-spin symmetry is one of the main results obtained in this paper. In [4] it will serve as a starting point for the analysis of higher-spin interactions of matter fields in 2+1 dimension. An interesting feature of higher-spin symmetries demonstrated in this paper is that their form depends on a particular dynamical system under consideration. Indeed, the higher-spin algebras with different $M^2(\nu)$ are pairwise non-isomorphic. This is obvious from the identification of the higher-spin symmetries with certain factor-algebras of the enveloping algebras of space-time symmetry algebras along the lines of the Section 4. Ordinary space-time symmetries on their turn can be identified with (maximal) finite-dimensional subalgebras of the higher-spin algebras which do not depend on the dynamical parameters like $\nu$ (cf (21)).

The infinite-dimensional algebras isomorphic to those considered in the section 4 have been originally introduced in [4, 8] as candidates for 3d bosonic higher-spin algebras, while the superalgebras of deformed oscillators described in the section 3 were suggested in [3] as candidates for 3d higher-spin superalgebras. Using all these algebras and the definition of supertrace given in [3] it was possible to write a Chern-Simons action for the 3d higher-spin gauge fields which are all dynamically trivial in the absence of matter fields (in a topologically trivial situation). Originally this was done by Blencowe [1] for the case of the Heisenberg algebra (i.e. $\nu = 0$). It was not clear, however, what is a physical meaning of the ambiguity in the continuous parameter like $\nu$ parametrizing pairwise non-isomorphic 3d higher-spin algebras. In this paper we have shown that different symmetries are realized on different matter multiplets, thus concluding that higher-spin symmetries turn out to be dependent on a particular physical model under consideration.

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References

[1] M.A. Vasiliev, Class. Quant. Grav. 11 (1994) 649.
[2] M.A. Vasiliev, Mod. Phys. Lett. A7 (1992) 3689.
[3] M.A. Vasiliev, JETP Lett. 50 (1989) N8, 374; Int. J. Mod. Phys. A6 (1991) 1115.
[4] S.F. Prokushkin and M.A. Vasiliev, in preparation.
[5] E.P. Wigner, Phys. Rev. 77 (1950) 711.
[6] E. Bergshoeff, B. de Wit and M. A. Vasiliev, Nucl. Phys. B366 (1991) 315.
[7] E. Bergshoeff, M.P. Blencowe and K.S. Stelle, Commun. Math. Phys. 128 (1990) 213.
[8] M. Bordemann, J. Hoppe and P. Schaller, Phys. Lett. 232 (1989) 199.
[9] M.P. Blencowe, Class. Quantum Grav. 6 (1989) 443.