When Adaptive Diffusion Algorithm Converges to True Parameter? *

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Abstract

We attempt to answer the question what data brings adaptive diffusion algorithms converging to true parameters. The discussion begins with the diffusion recursive least squares (RLS). When unknown parameters are scalar, the necessary and sufficient condition of the convergence for the diffusion RLS is established, in terms of the strong consistency and mean-square convergence both. However, for the general high dimensional parameter case, our results suggest that the diffusion RLS in a connected network might cause a diverging error, even if local data at every node could guarantee the individual RLS tending to true parameters. Due to the possible failure of the diffusion RLS, we prove that the diffusion Robbins-Monro (RM) algorithm could achieve the strong consistency and mean-square convergence simultaneously, under

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some cooperative information conditions. The convergence rates of the diffusion RM are derived explicitly.

1 Introduction

Perhaps, it is only natural that this paper is intended to prove adaptive diffusion algorithms outperform their individual counterparts in terms of estimation performances. It is an accepted fact for the diffusion least mean squares (LMS) with regard to mean stability and mean-square stability (see [20], [32], [33]). But this time, involving the sophisticated recursive least squares (RLS) in diffusion strategies, situation changes.

An adaptive network is built up from a set of nodes which could communicate with their neighbors through interlinks. Each node observes partial information related to an unknown parameter of common interest and performs local estimation separately. There are two main types of fully decentralized strategies in distributed estimation, namely, consensus strategies [3], [9], [17], [34] and diffusion strategies [1], [14], [18], [26], [27], [31]. In light of local parameter estimation and processed information sharing, the two networks enjoy a certain advantages in robustness and privacy. In particular, compared with individual identification, producing better estimates in collaborative manners is very likely to be an absolute cinch. This guess was first proved false by [32], since it found consensus networks can become unstable when all its nodes exhibit stable behaviors in individual estimation processes. But at the same time, it showed that stability of the individual LMS always infers stability of the diffusion LMS. So, to some extent, diffusion networks are more stable than consensus ones. It was confirmed again in [33] recently by considering the normalized least mean squares (NLMS). Establishing a cooperative information condition, [33] concluded that the diffusion NLMS could track parameters effectively when none of the local data provides sufficient information for individual identification. Almost all
the existing literatures on the diffusion LMS-type algorithms suggest diffusion networks behave superiorly to non-cooperative schemes (see [26], [32], [33]). Interestingly, as regard to the diffusion RLS, we cannot take it for granted.

The diffusion RLS was proposed in [5], which discussed a typical scenario attaining bounded mean-square errors. At each node \( i \), the data is required to be independent and tend towards steady that matrix \( EP_{k,i}^{-1} \) becomes constant for all large time \( k \). These constraints are retained in other relevant studies [2], [4], [5], [19], [22], [28] simply to make the problem tractable. However, for a variety of reasons, connections between data might be inevitable. More importantly,

\[
\lambda_{\text{min}} \left( P_{k,i}^{-1} \right) \to +\infty,
\]

intuitively generates more informative excitation signals than those for steady \( P_{k,i}^{-1} \). So what conclusions will survive, if the data utilized for estimation admits no such constraints? Digging into this case, connections between the diffusion RLS and the non-cooperative RLS are brought to the surface.

Indeed, for scalar unknown parameters, the idea that cooperations among nodes through diffusion networks help to promote estimation performances is verified as expected here, the conclusion for high dimensional parameters turns out to be quite different. Opposite to [33], when parameters are vectors, our results suggest that the convergence of the individual RLS to true parameters at every node cannot even guarantee the stability of the diffusion RLS in a connected network, let alone the identification task.

To be more precise, for a linear regression model with a scalar unknown parameter, we find the necessary and sufficient condition on the regressor data, in a cooperation form, to guarantee the convergence of the diffusion RLS to the true parameter, in the sense of the strong consistency and mean-square convergence. This critical condition degenerates to the necessary and sufficient condition of the above two convergences for the individual RLS,
when the underlying network has only one node. But this critical convergence condition can no longer be extended here in the high dimensional parameter case. Worse still, the cooperation of the nodes in a connected network might cause a diverging error even (1) holds for every node $i$, which means the individual RLS at each node, if is employed, tending to true parameters \cite{11, 24}. As a supplement, we prove that the diffusion Robbins-Monro (RM) algorithm could achieve the strong consistency and mean-square convergence simultaneously, when regressor data fails the diffusion RLS for high dimensional parameters. The two convergence rates of the diffusion RM are explicitly derived.

The rest of the paper is organized as follows. In the next section, we present the main theorems with the proofs given in Sections 3–4. The concluding remarks are included in Section 5.

\section{Main Results}

Consider a network consisting of $n$ nodes that trying to identify an unknown parameter in a collaborative manner. At time $k$, each node $i$ observes a noisy signal $y_{k,i} \in \mathbb{R}$ and a data signal $\phi_{k,i} \in \mathbb{R}^m$. This process is described by a stochastic linear regression model

$$y_{k,i} = \theta^T \phi_{k,i} + \varepsilon_{k,i}, \quad k \geq 0, \ i = 1, \ldots, n,$$

where $(\cdot)^T$ denotes the transpose operator, $\varepsilon_{k,i}$ is a scalar noise sequence and $\theta \in \mathbb{R}^m$ is an unknown deterministic parameter.

Let the network topology be depicted by a directed weighted graph $G = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, where $\mathcal{V} = \{1, 2, \ldots, n\}$ is the set of the nodes and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of the edges that any $(i, j) \in \mathcal{E}$ means $G$ contains a directed path from $j$ to $i$. The structure of the graph $G$ is described by the weighted adjacency matrix $\mathcal{A} = \{a_{ij}\}_{n \times n}$, where $a_{ij} > 0$ for $(i, j) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise.
We employ the adapt-then-combine (ATC) diffusion strategy for the estimation algorithm, which is recursively defined for each node $i$ by

1. Adaption:

$$\beta_{k+1,i} = \theta_{k,i} + L_{k,i}(y_{k,i} - \theta_{k,i}^T \phi_{k,i})$$

with initial estimate $\theta_{0,i} \in \mathbb{R}^m$, where $L_{k,i} \in \mathbb{R}^m$ is to be designed based on data $\phi_{0,i}, \ldots, \phi_{k,i}$.

2. Combination:

$$\theta_{k+1,i} = \sum_{j=1}^{n} a_{ij} \beta_{k+1,i}.$$ 

Denote $\tilde{\theta}_{k,i} \triangleq \theta_{k,i} - \theta$, then

$$\tilde{\Theta}_{k+1} = (A \otimes I_m)(I_{mn} - F_k)\tilde{\Theta}_k + (A \otimes I_m)L_kV_k,$$

where

$$\tilde{\Theta}_k \triangleq \text{col}\{\tilde{\theta}_{k,1}, \ldots, \tilde{\theta}_{k,n}\},$$

$$L_k \triangleq \text{diag}\{L_{k,1}, \ldots, L_{k,n}\},$$

$$\Phi_k \triangleq \text{diag}\{\phi_{k,1}, \ldots, \phi_{k,n}\},$$

$$V_k \triangleq \text{col}\{\varepsilon_{k,1}, \ldots, \varepsilon_{k,n}\},$$

$$F_k \triangleq L_k \Phi_k^T.$$ 

Different $\{L_{k,i}\}$ result in variant types of adaptive algorithms, like the RLS, LMS and Kalman filtering. Since the parameter to be identified is time-invariant, we focus on the RLS and the Robbins-Monro algorithm.
Remark 2.1. Another well studied diffusion scheme is the combine-then-adapt (CTA) rule (see [20], [32]). Since the two strategies are essentially the same for our problem, we only study the ATC diffusion strategy. All the results in this paper still hold for the CTA diffusion strategy.

2.1 Diffusion Recursive Least-Squares Algorithm

In this section, we apply the RLS algorithm to estimate the unknown parameter $\theta$ based on the ATC diffusion strategy. That is, $\{L_{k,i}; k \geq 0, 1 \leq i \leq n\}$ are designed as

$$
\begin{align*}
L_{k,i} &= P_{k+1,i} \phi_{k,i} = \frac{P_{k,i} \phi_{k,i}}{1 + \phi_{k,i}^T P_{k,i} \phi_{k,i}}, \\
P_{k+1,i}^{-1} &= I_m + \sum_{j=0}^{k} \phi_{j,i} \phi_{j,i}^T.
\end{align*}
$$

2.1.1 A Critical Convergence Theorem

We analyze the estimation performance of the diffusion RLS algorithm under

A1 $A$ is an irreducible and aperiodic doubly stochastic matrix with $A^T A$ being irreducible.

A2 The noises $\{(\varepsilon_{k,1}, \ldots, \varepsilon_{k,n})^T\}_{k \geq 0}$ are mutually independent and for each $i = 1, \ldots, n$,

$$
E \varepsilon_{k,i} = 0, \forall k \geq 0 \quad \text{and} \quad \sup_{k \geq 1} E \varepsilon_{k,i}^2 < M,
$$

where $M > 0$ is a constant.

A3 $\phi_{k,i}, i = 1, \ldots, n, k \geq 1$ are non-random constants.

Remark 2.2. If graph $G$ is undirected, connected and containing a self-loop at each node, then it corresponds to a special case of Assumption A1. See the network topology of [33].

Recalling the well-known results [11, Theorem 1] and [24, Theorem 3.1] on the least-squares (LS) estimator, we know that under Assumptions A2–A3, for each single node $i$,
if \( \inf_{k \geq 0} E \varepsilon_{k,i}^2 > 0 \), then
\[
\hat{\theta}_{k,i} \xrightarrow{a.s.} \theta \quad \text{and} \quad E(\hat{\theta}_{k,i} - \theta)^2 \to 0
\]
are both equivalent to
\[
\lambda_{\min} \left( P_{-1,k,i} \right) \to +\infty, \quad (3)
\]
where for any initial \( \hat{\theta}_{0,i} \) and \( k \geq 0 \),
\[
\hat{\theta}_{k+1,i} = \hat{\theta}_{k,i} + L_{k,i} (y_{k,i} - \phi_{k,i}^T \hat{\theta}_{k,i}). \quad (4)
\]

Let \( \| \cdot \| \) denotes the spectral norm of a matrix. The two convergences are now derived at every node in a collaborative manner when the unknown parameter is a scalar.

**Theorem 2.1.** Let \( m = 1 \). Under Assumptions A1–A3,
\[
\| \tilde{\Theta}_k \| \xrightarrow{a.s.} 0 \quad \text{and} \quad E\| \tilde{\Theta}_k \|^2 \to 0 \quad \text{as} \quad k \to +\infty \quad (5)
\]
for any initial \( \Theta_0 \in \mathbb{R}^n \), if and only if
\[
\lim_{k \to +\infty} \sum_{i=1}^n P_{-1,k+1,i} = +\infty. \quad (6)
\]

**Remark 2.3.** (i) Discussions on the necessity of Theorem 2.1:

(a) if (6) fails, as proved in Section 3, any initial values \( \{\theta_{0,i}, 1 \leq i \leq n\} \) except the ones satisfying \( \sum_{i=1}^n \mu_i (\theta_{0,i} - \theta) = 0 \) will lead to
\[
\liminf_{k \to +\infty} E\| \tilde{\Theta}_k \|^2 > 0 \quad \text{and} \quad \| \tilde{\Theta}_k \| \xrightarrow{p} 0,
\]
where \( \mu_1, \ldots, \mu_n > 0 \) are some constants determined by data \( \{\phi_{k,i}\} \) and matrix \( \mathcal{A} \).

(b) when the noises and data satisfy
\[
\begin{cases}
E \varepsilon_{k,i}^2 > 0, & \text{for all} \ k \geq 0, 1 \leq i \leq n \\
E \varepsilon_{k,i} \varepsilon_{k,j} = 0, & \text{for all} \ k \geq 0, 1 \leq i < j \leq n , \\
\sum_{k=0}^{+\infty} \sum_{i=1}^n \phi_{k,i}^2 \neq 0
\end{cases} \quad (7)
\]
then given any initial $\Theta_0 \in \mathbb{R}^n$ (including $\theta_{0,i} = \theta, i \in [1, n]$), (5) is equivalent to (6). See Appendix 5.

(ii) As for the sufficient part of Theorem 2.1, the convergence rate (see (30)) of the estimation error satisfies

$$\sum_{k=0}^{+\infty} \sum_{i=1}^{n} (1 - (P_{k+1,i} P_{k,i}^{-1})^2) \|\tilde{\Theta}_k\|^2 < +\infty, \quad a.s.$$  

with $\sum_{k=0}^{+\infty} \sum_{i=1}^{n} (1 - (P_{k+1,i} P_{k,i}^{-1})^2) = +\infty$.

We come to an analogous conclusion on the strong consistency of Theorem 2.1 when data

$$\Phi_k = \text{diag}\{\phi_{k,1}, \ldots, \phi_{k,n}\}, \quad k \geq 0$$

is a random sequence. Assume

A2’ $\{\varepsilon_{k,1}, \ldots, \varepsilon_{k,n}\}$ are mutually independent and there is a constant $M > 0$ such that for all $i \in [1, n]$,

$$\begin{cases} 
E(\varepsilon_{k,i} | \Phi_j, 0 \leq j \leq k) = 0 \\
\sup_{k \geq 1} E(\varepsilon_{k,i}^2 | \Phi_j, 0 \leq j \leq k) \leq M, \quad a.s. 
\end{cases}$$

Theorem 2.2. Under Assumptions A1 and A2’, for any initial $\Theta_0 \in \mathbb{R}^n$, on set $\{\lim_{k \to +\infty} \sum_{i=1}^{n} P_{k+1,i}^{-1} = +\infty\}$,

$$\|\tilde{\Theta}_k\| \xrightarrow{a.s.} 0, \quad \text{as } k \to +\infty.$$  

The proof of Theorem 2.2 is similar to that of Theorem 2.1 and given in Appendix 5. The above two theorems suggest that when the unknown parameter is a scalar, the informative data of one single node is sufficient to guarantee the strong consistency (mean-square convergence) of the diffusion RLS via the connectivity of the underlying network.
2.1.2 Diffusion Strategy Could Fail the Convergence

When the unknown parameter is of high dimension, a little surprising result emerges, indicating that a diffusion strategy could play a destructive role, if the network topology is strongly connected:

**A1’** $\mathcal{A}$ is irreducible.

**Theorem 2.3.** Let $m > 1$ and Assumptions A1’ and A2 hold. If $\|E\tilde{\Theta}_0\| \neq 0$, then there is a series of data $\{\Phi_k\}_{k=0}^{+\infty}$ satisfying

$$
\lim_{k \rightarrow +\infty} \lambda_{\min}(P^{-1}_{k,i}) = +\infty, \quad i = 1, \ldots, n, \tag{8}
$$

such that $\sup_{k \geq 0} E\|\tilde{\Theta}_k\|^2 = +\infty$.

More divergences of the diffusion RLS occur, if the noises in Assumption A2 are specified by

**A2”** $\{(\varepsilon_{k,1}, \ldots, \varepsilon_{k,n}) \} \{k \geq 0 \}$ is an i.i.d random sequence with a multivariate normal distribution $N(0, \Sigma)$.

**Theorem 2.4.** Let $m > 1$ and Assumptions A1’ and A2” hold. If $\|E\tilde{\Theta}_0\| \neq 0$, then there is a series of data $\{\Phi_k\}_{k=0}^{+\infty}$ satisfying (8) such that

(i) for some set $D_0$ with $P(D_0) > 0$,

$$
\sup_{k \geq 0} \|\tilde{\Theta}_k\| = +\infty, \quad a.s. \ on \ D_0; \tag{9}
$$

(ii) for any $\varepsilon > 0$,

$$
\limsup_{k \rightarrow +\infty} P(\|\tilde{\Theta}_k\| > \varepsilon) > 0.
$$

**Remark 2.4.** Although parameter $\theta$ is modeled as a deterministic vector here, Theorems 2.3 still holds for random parameter $\theta$. Furthermore, if $\theta$ has a normal distribution, then Theorem 2.4 can be derived as well. See Section 4.
Remark 2.5. Let parameter $\theta$, data $\{\phi_{k,i}\}$ and noises $\{\varepsilon_{k,i}\}$ in model (2) all be random. If $\theta$, independent of $\{\phi_{k,i}\}$, is Gaussian distributed and $\{\varepsilon_{k,i}\}$ possesses the standard normal distributions, then in view of [30], for each single node $i$ and any initial value $\hat{\theta}_{0,i}$,

$$\{\lambda_{\min}(P_{k,i}^{-1}) \to +\infty\} \subset \left\{ \lim_{t \to +\infty} \hat{\theta}_{k,i} = \theta \right\},$$

where $\hat{\theta}_{k,i}$ is the individual RLS defined by (4). So, Remark 2.4 means in stochastic framework, the diffusion strategy still possibly do a disservice to estimation. In this sense, we might need a stronger condition to ensure the strong consistency of the diffusion RLS, compared with the individual case.

### 2.2 Diffusion Robbins-Monro Algorithm

Now, we are going to seek an adaptive algorithm competent for distributed estimation, no matter the parameter to be identified is a scalar or a vector. The diffusion RM is a suitable candidate. It achieves the strong consistency and mean-square convergence simultaneously, under the cooperative information condition below:

**A3’** There are two constants $c > 0, \alpha \in [0, \frac{1}{2})$ such that

$$\inf_{k \geq 1} k^\alpha \lambda_{\min} \left( E \left[ \frac{1}{n} \sum_{i=1}^{n} \frac{\phi_{k,i} \phi_{k,i}^T}{1 + \|\phi_{k,i}\|^2} \bigg| \mathcal{F}_{k-1} \right] \right) > c,$$

where $\mathcal{F}_k \triangleq \sigma\{\phi_{j,i}, \varepsilon_{j,i}, 0 \leq j \leq k, 1 \leq i \leq n\}$.

Alternatively, denoting

$$\lambda_k(h) \triangleq \lambda_{\min} \left( E \left[ \frac{1}{nh} \sum_{i=1}^{n} \sum_{j=k}^{k+h-1} \frac{\phi_{j,i} \phi_{j,i}^T}{1 + \|\phi_{j,i}\|^2} \bigg| \mathcal{F}_{k-1} \right] \right),$$

where $h$ is a fixed positive integer, a more useful condition is
A3” Regressr \( \{ \phi_{k,i} \} \) satisfies

(i) for some \( c > 0, \alpha \in [0, \frac{1}{2}) \) and \( h \in \mathbb{N}^+) \)

\[
\inf_{k \geq 1} k^\alpha \lambda_k(h) > c. \tag{10}
\]

(ii) \( \{ \phi_{k,i} \} \) is independent of noises \( \{ \varepsilon_{k,i} \} \).

**Theorem 2.5.** Under Assumptions A1, A2’ and A3’(or A3”), if the diffusion Robbins-Monro algorithm takes

\[
L_{k,i} = \frac{1}{(k + 1)^\beta} \frac{\phi_{k,i}}{1 + \| \phi_{k,i} \|^2}, \quad k \geq 0, \ i = 1, \ldots, n,
\]

where \( \beta \in (\frac{1}{2}, 1 - \alpha) \), then

(i) as \( k \to +\infty \),

\[
E \| \tilde{\Theta}_k \|^2 \to 0 \quad \text{and} \quad \| \tilde{\Theta}_k \| \overset{a.s.}{\longrightarrow} 0;
\]

(ii) the mean-square convergence rate is

\[
\limsup_{k \to +\infty} k^{\beta - \alpha} E \| \tilde{\Theta}_k \|^2 \leq \frac{M}{sC}, \tag{11}
\]

where \( s \) and \( M \) are two constants defined in Lemma 3.3 and Assumption A2’. In addition, if the noises further satisfy

\[
\sup_{k \geq 1} E \left[ (V_k^TV_k)^l | \Phi_j, 0 \leq j \leq k \right] < +\infty, \quad a.s. \tag{12}
\]

for some \( l > \frac{1}{\beta - \alpha} \), then for any \( \varepsilon \in (0, \beta - \alpha - \frac{1}{l+1}) \),

\[
\| \tilde{\Theta}_k \|^2 = o(k^{-\varepsilon}), \quad a.s. \tag{13}
\]

**Remark 2.6.** (i) In Theorem 2.5 let \( m = 1, \theta_{0,i} = 0, \phi_{k,i} = \sqrt{2c}, \varepsilon_{k,i} = \varepsilon_{k,1}, E\varepsilon_{k,1}^2 = M \).
for all $k \geq 0$ and $i = 1, \ldots, n$. Then, $\alpha = 0$ in Assumption A3’ and $\tilde{\Theta}_k = 1 \cdot \tilde{\theta}_{k,1}$ with
\[
E\tilde{\Theta}_{k+1}^2 = E\left[\tilde{\Theta}_k^2(I_{mn} - F_k)(B \otimes I_m)(I_{mn} - F_k)\tilde{\Theta}_k\right] + tr(E[(A \otimes I_m)L_kV_k^2L_k^*(A^T \otimes I_m)])
\]
\[
= \left(1 - \frac{2c}{(1 + 2c)(k + 1)^3}\right)^2 E\tilde{\Theta}_k^2\tilde{\Theta}_k + \frac{2cnM}{(1 + 2c)^2(k + 1)^2}\]
\[
\geq \left(1 - \frac{4c}{(1 + 2c)(k + 1)^3}\right) E\tilde{\Theta}_k^2\tilde{\Theta}_k + \frac{2cnM}{(1 + 2c)^2(k + 1)^2},
\]
which by \[13, Lemma 4.2\] yields
\[
\liminf_{k \to +\infty} k^\beta E\|\tilde{\Theta}_k\|^2 \geq \frac{nM}{2 + 4c}.
\]
So, generally speaking, the order of magnitude of the convergence rate in (11) can not be improved if no further conditions are imposed.

(ii) By (11), constant $s$ is important to the performance of the mean-square convergence for the diffusion RM. Note that if $A$ is symmetric and $\inf_{i \in [1,n]} a_{ii} > 0$, an analogous proof of \[33, Lemma 5.10\] shows that in Lemma 3.3, we can select
\[
s = \frac{\inf_{i \in [1,n]} a_{ii}}{32n(1 + 4h)^2} \lambda(G),
\]
where $\lambda(G)$ is the smallest positive eigenvalue of the Laplacian matrix $I_n - A$ and $h$ is defined in Assumption A3”. See Appendix 5. By Cheeger’s inequality \[10\], $\lambda(G) \geq h_G^2/2$, where $h_G$ is the Cheeger constant that describes the difficulty of breaking the connectivity of $G$. Rewrite (11) as
\[
\limsup_{k \to +\infty} k^{\beta-\alpha} \frac{\sum_{i=1}^n E\|\tilde{\theta}_{k,i}\|^2}{n} \leq \frac{M}{snc},
\]
then
\[
\limsup_{k \to +\infty} k^{\beta-\alpha} \frac{\sum_{i=1}^n E\|\tilde{\theta}_{k,i}\|^2}{n} \leq \frac{64(1 + 4h)^2M}{ch_G^2 \inf_{i \in [1,n]} a_{ii}}.
\]
So, for symmetric $A$ with $\inf_{i \in [1,n]} a_{ii} > 0$, the convergence performance of the diffusion RM could be enhanced by promoting the connectivity of $G$. 

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Remark 2.7. To better understand the problem, we compare the diffusion RM with the diffusion RLS and the diffusion NMLS.

(i) It is easy to verify that data \(\{\phi_{k,i}\}\) constructed in Section 4 satisfies Assumption A3”.

So, for high dimensional parameters, even if \(\{\phi_{k,i}\}\) corresponds to a diverging error of the diffusion RLS, it still stands a chance to generate estimates converging to true parameters, by applying the diffusion RM.

(ii) The cooperative information condition derived in [33] requires \(\{\lambda_{k,k} \geq 0\} \in S^0(\lambda)\), where \(\lambda \in (0,1)\) and

\[
S^0(\lambda) \triangleq \left\{ \{a_k\} : a_k \in [0,1], \sum_{k=i+1}^{\infty} \left(1 - a_j\right) \leq K\lambda^{k-i}, \forall k > i, i \geq 0, \text{ for some } K > 0 \right\}.
\]

Note that this cooperative information condition is necessary and sufficient for the stability of the diffusion NLMS algorithm in [33], whenever \(\{\phi_{k,i}\}\) is \(\phi\)-mixing. However, by [15, Theorem 2.3], \(\{\lambda_{k,k} \geq 0\} \in S^0(\lambda)\) implies (10) with \(\alpha = 0\) for any \(\phi\)-mixing data \(\{\phi_{k,i}\}\).

So, the diffusion RM could deal with some data beyond the capability of the diffusion NLMS, as far as the time-invariant-parameter case is concerned.

3 Proof of Theorem 2.1

We preface the proof with a simple lemma below.

Lemma 3.1. Let \(\{e_k\}\) be a series of nonnegative real numbers.

(i) If for some \(d_k \geq 0\) and \(\sum_{k=0}^{+\infty} d_k < +\infty\),

\[
e_{k+1} \leq e_k + d_k, \quad \forall k \geq 0,
\]

then \(\lim_{k \to +\infty} e_k\) exists.

(ii) If there exist two nonnegative sequences \(\{a_k\}\) and \(\{b_k\}\) with \(\sum_{k=0}^{+\infty} a_k = +\infty\) and
∑_{k=0}^{+∞} b_k < +∞ such that

\[ e_{k+1} \leq (1 - a_k)e_k + b_k, \quad \forall k \geq 0, \]

then \( \lim_{k \to +∞} e_k = 0. \)

Proof. (i) Fix an integer \( k > 0. \) Then, for any \( l \geq k, \)

\[ e_l \leq e_k + \sum_{i=k}^{l-1} d_i \leq e_k + \xi_k, \]

where \( \xi_k = \sum_{i=k}^{+∞} d_i. \) So,

\[ e_k \geq \limsup_{l \to +∞} e_l - \xi_k, \]

which together with \( \lim_{k \to +∞} \xi_k = 0 \) yields

\[ \liminf_{k \to +∞} e_k \geq \limsup_{l \to +∞} e_l - \lim_{k \to +∞} \xi_k = \limsup_{l \to +∞} e_l. \]

Then, \( \lim_{k \to +∞} e_k \) exists.

To prove (ii), note that \( e_{k+1} \leq e_k + b_k, \) where \( \sum_{k=0}^{+∞} b_k < +∞. \) Therefore, \( \lim_{k \to +∞} e_k \) exists by (i). Suppose \( e \triangleq \lim_{k \to +∞} e_k > 0, \) so there is a \( N > 0 \) such that \( e_k > \frac{e}{2} \) for all \( k > N. \) Consequently,

\[ e_{N+i} - e_{N+1} = \sum_{k=N+1}^{N+i-1} (e_{k+1} - e_k) \leq - \sum_{k=N+1}^{N+i-1} a_k e_k + \sum_{k=N+1}^{N+i-1} b_k \leq - \frac{a}{2} \sum_{k=N+1}^{N+i-1} a_k + \sum_{k=N+1}^{+∞} b_k, \]

which shows \( e_{N+i} \to -∞ \) by letting \( i \to +∞. \) This leads to a contradiction and hence \( e = 0. \)

\[ \square \]

Lemma 3.2. Let \( \{e_k, k \geq 0\} \) and \( \{d_k, k \geq 0\} \) be two non-negative processes adapted to a filtration \( \{\mathcal{G}_k, k \geq 0\}. \) If

\[ E[e_{k+1}|\mathcal{G}_k] \leq e_k + b_k - d_k, \quad k \geq 0 \]
for some $b_k \geq 0$ with $\sum_{k=0}^{+\infty} b_k < +\infty$, then
\[
\sum_{k=0}^{+\infty} d_k < +\infty \quad a.s.
\] (14)

In addition, if $\lim_{k \to +\infty} E e_k = 0$, then
\[
\lim_{k \to +\infty} e_k = 0, \quad a.s.
\]

Proof. As a matter of fact, (14) is a direct result of [7, Lemma 1.2.2] and this lemma further shows that there exists a random variable $e_\infty$ such that $E|e_\infty| < +\infty$ and
\[
\lim_{k \to +\infty} e_k = e_\infty, \quad a.s.
\]

Since $e_k \geq 0$, by Fatou’s lemma,
\[
0 = \liminf_{k \to +\infty} E e_k \geq E e_\infty,
\]
which indicates $e_\infty = 0$ almost surely. \qed

Fix an integer $h \geq 1$. Let $\{A_{k,i}; k = 1, \ldots, h, i = 1, \ldots, n\}$ be a sequence of $m \times m$ symmetric random matrices satisfying $0 \leq A_{k,i} \leq I_m$. Denote $I_k(A) \triangleq \text{diag}\{A_{k,1}, \ldots, A_{k,n}\}$ and
\[
\begin{align*}
\psi_0 &\triangleq I_{mn} \\
\psi_k &\triangleq \prod_{j=k}^{1}(A \otimes I_m)I_j(A), \quad k = 1, \ldots, h.
\end{align*}
\] (15)

The following lemma shows

**Lemma 3.3.** Under Assumption A1, for any $\sigma$-algebra $\mathcal{F}$, there is a constant $s \in (0, 1)$ determined by $h$ and $A$ such that
\[
\lambda_{\min}(E [I_{mn} - \psi_h^T\psi_h | \mathcal{F}]) \geq s\lambda_{\min}\left(E \left[ \sum_{k=1}^{h} \sum_{i=1}^{n} (I_m - A^2_{k,i}) | \mathcal{F} \right] \right).
\]

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Proof. Denote \( \mathcal{B} \triangleq \mathcal{A}^\top \mathcal{A} \). Since \( \mathcal{B} \) is irreducible, for any \( i \in [1,n-1] \), there is an integer \( d_i \geq 2 \) and some distinct \( c_1^i, \ldots, c_{d_i}^i \in [1,n] \) such that

\[
\begin{cases}
  c_1^i = i, & c_{d_i}^i = i + 1 \\
  \mathcal{B}[c_j^i, c_{j+1}^i] > 0, & j \in [1,d_i - 1],
\end{cases}
\]

where \( M[i,j] \) refers to the \((i,j)\)th entry of a matrix \( M \). Let \( q \triangleq \sum_{i=1}^{n-1} d_i - (n - 2) \) and define a sequence of \( b_j, j = 1, \ldots, q \) with \( b_1 = c_1^1 \) and \( b_j = c_{l+1}^j - \sum_{i=1}^{l} (d_i - 1) \), where \( l \in [0,n-2] \) and

\[
1 + \sum_{i=1}^{l} (d_i - 1) < j \leq 1 + \sum_{i=1}^{l+1} (d_i - 1).
\]

Hence \( \mathcal{B}[b_j, b_{j+1}] > 0 \) for all \( j \in [1,q - 1] \).

Select

\[
0 < s < \frac{\min_{j \in [1,q-1]} \mathcal{B}[b_j, b_{j+1}]}{512 h^3 n^4 q(1 + n^2)}
\]

and denote

\[
\rho \triangleq \lambda_{\min} \left( \mathbb{E} \left[ \sum_{k=1}^{h} \sum_{i=1}^{n} (I_m - A^2_{k,i}) \right] \mathcal{F} \right).
\]

Now, suppose for a constant vector \( x \in \mathbb{R}^{mn} \) with \( \|x\| = 1 \),

\[
x^\top \mathbb{E} \left[ I_{mn} - \psi_h^\top \psi_h | \mathcal{F} \right] x < s \rho
\]

on some trajectory. We prove that on this trajectory, for any \( k \in [1,h] \),

\[
E[\|\psi_k x - x\|^2 | \mathcal{F}] < \frac{\rho}{64hn}.
\]

To this end, write \( \psi_k x = \text{col}\{z_{k,1}, \ldots, z_{k,n}\} \in \mathbb{R}^{mn}, k \in [0,h] \). Observe that

\[
x^\top (I_{mn} - \psi_{k+1}^\top \psi_{k+1}) x = (\psi_k x)^\top (I_{mn} - I_{k+1}(A)(\mathcal{B} \otimes I_m)I_{k+1}(A)) (\psi_k x) + x^\top (I_{mn} - \psi_k^\top \psi_k) x,
\]
a direct calculation yields

\[ x^\tau E \left[ I_{mn} - \psi_k^\tau \psi_k \right] x = \sum_{1 \leq i < j \leq n} B[i, j] \sum_{k=0}^{h-1} E[\|A_{k+1,i} z_{k,i} - A_{k+1,j} z_{k,j}\|^2|\mathcal{F}] + \sum_{k=0}^{h-1} x^\tau E[\psi_k^\tau (I_{mn} - I_{k+1}^2(A)) \psi_k |\mathcal{F}] x, \] (18)

which, together with (16), implies that for any \( i \in [1, n-1], \)

\[ \sum_{j=1}^{d_i-1} E \left[ \left\| A_{k+1,c_j^i z_{k,c_j^i} - A_{k+1,c_{j+1}^i z_{k,c_{j+1}^i}} \right\|^2 \right] |\mathcal{F} < s\rho \min_{j \in [1, d_i-1]} B[c_j^i, c_{j+1}^i], \]

and hence

\[ \sum_{j=1}^{q-1} E \left[ \left\| A_{k+1,b_j z_{k,b_j} - A_{k+1,b_{j+1} z_{k,b_{j+1}}} \right\|^2 \right] |\mathcal{F} < n\rho \min_{j \in [1, q-1]} B[b_j, b_{j+1}]. \] (19)

By (19) and Cauchy-Schwarz inequality,

\[ E \left[ \left\| A_{k+1,i} z_{k,i} - A_{k+1,j} z_{k,j} \right\|^2 \right] |\mathcal{F} < \frac{qns\rho}{\min_{j \in [1, q-1]} B[b_j, b_{j+1}]}, \quad \forall i \neq j. \] (20)

Furthermore, since

\[ z_{k,i}^\tau (I_m - A_{k+1,i}^2) z_{k,i} + z_{k,j}^\tau (I_m - A_{k+1,j}^2) z_{k,j} \geq \frac{1}{2} \| (I_m - A_{k+1,i}) z_{k,i} - (I_m - A_{k+1,j}) z_{k,j} \|^2, \]

(16) and (18) imply

\[ \frac{1}{2} \max_{i,j} E[\| (I_m - A_{k+1,i}) z_{k,i} - (I_m - A_{k+1,j}) z_{k,j}\|^2|\mathcal{F}] \leq \sum_{i=1}^{n} E[z_{k,i}^\tau (I_m - A_{k+1,i}^2) z_{k,i}|\mathcal{F}] < s\rho \]

and

\[ E[\| I_{k+1}(A) \psi_k x - \psi_k x \|^2|\mathcal{F}] = \sum_{i=1}^{n} E[z_{k,i}^\tau (I_m - A_{k+1,i}^2) z_{k,i}|\mathcal{F}] \leq \sum_{i=1}^{n} E[z_{k,i}^\tau (I_m - A_{k+1,i}^2) z_{k,i}|\mathcal{F}] < s\rho. \]
So,

\[
E[\|\psi_{k+1}x - (\mathcal{A} \otimes I_m)\psi_kx\|^2|\mathcal{F}] = E[\|(\mathcal{A} \otimes I_m)(I_{k+1}(\mathcal{A})\psi_kx - \psi_kx)\|^2|\mathcal{F}]
\]
\[
\leq E[\|I_{k+1}(\mathcal{A})\psi_kx - \psi_kx\|^2|\mathcal{F}] < s\rho \quad (21)
\]

and

\[
\sum_{j=1}^{q-1} B[b_j, b_{j+1}] E[\|z_{k,b_j} - z_{k,b_{j+1}}\|^2|\mathcal{F}] \leq \sum_{1 \leq i < j \leq n} B[i, j] E[\|z_{k,i} - z_{k,j}\|^2|\mathcal{F}]
\]
\[
\leq 2 \sum_{1 \leq i < j \leq n} B[i, j] (E[\|A_{k+1,i}z_{k,i} - A_{k+1,j}z_{k,j}\|^2 + \|(I_m - A_{k+1,i})z_{k,i} - (I_m - A_{k+1,j})z_{k,j}\|^2|\mathcal{F}])
\]
\[
\leq 2s\rho + 2n^2s\rho. \quad (22)
\]

Similar to (20), by Cauchy-Schwarz inequality and (22),

\[
E[\|z_{k,i} - z_{k,j}\|^2|\mathcal{F}] \leq q \sum_{j=1}^{q-1} E[\|z_{k,b_j} - z_{k,b_{j+1}}\|^2|\mathcal{F}] < \frac{2qs\rho(1 + n^2)}{\min_{j \in [1, q-1]} B[b_j, b_{j+1}]},
\]
\[
< \frac{\rho}{256h^3n^3}, \quad i < j. \quad (23)
\]

Since \(\mathcal{A}\) is a stochastic matrix,

\[
E[\|\psi_kx - (\mathcal{A} \otimes I_m)\psi_kx\|^2|\mathcal{F}] = \sum_{i=1}^{n} E\left[\left\| \sum_{j=1}^{n} \mathcal{A}[i, j](z_{k,i} - z_{k,j}) \right\|^2|\mathcal{F}\right] < \frac{\rho}{256h^3n},
\]

which together with (21) leads to

\[
E[\|\psi_{k+1}x - \psi_kx\|^2|\mathcal{F}] \leq 2E[\|\psi_{k+1}x - (\mathcal{A} \otimes I_m)\psi_kx\|^2|\mathcal{F}] + 2E[\|\psi_kx - (\mathcal{A} \otimes I_m)\psi_kx\|^2|\mathcal{F}]
\]
\[
< \frac{\rho}{64h^3n}. \quad (24)
\]

Note that (24) holds for \(k = 0, \ldots, h-1\), by Cauchy-Schwarz inequality again, for \(k \in [1, h]\),

\[
E[\|\psi_kx - x\|^2|\mathcal{F}] \leq k \sum_{j=1}^{k} E[\|\psi_jx - \psi_{j-1}x\|^2|\mathcal{F}] < h^2 \left(\frac{\rho}{64h^3n}\right) = \frac{\rho}{64hn},
\]

which is exactly (17).
Now, let $k = 0$ in (23), it yields that $\|x_1 - x\|^2 < \frac{\rho}{16hn^2}$ for all $i > 1$. Since $\sum_{i=1}^{n} \|x_i\|^2 = 1$,
\[
\|x_1\|^2 \geq \frac{1}{2n - 1} - \frac{1}{16hn^2}\rho > \frac{1}{4n}.
\]
Moreover, by applying Cauchy-Schwarz inequality,
\[
\begin{align*}
\frac{1}{2}x^\tau (I_{mn} - I_{k+1}^2(A))x & \leq x^\tau \psi^\tau_k(I_{mn} - I_{k+1}^2(A))\psi_kx + (\psi_{k+1}x - x)^\tau (I_{mn} - I_{k+1}^2(A))(\psi_{k+1}x - x) \\
& \leq x^\tau \psi^\tau_k(I_{mn} - I_{k+1}^2(A))\psi_kx + \|\psi_{k+1}x - x\|^2,
\end{align*}
\]
therefore,
\[
\begin{align*}
x^\tau E \left[ I_{mn} - \psi^\tau_k \psi_h | \mathcal{F} \right] x & \geq \sum_{k=0}^{h-1} x^\tau E[\psi^\tau_k(I_{mn} - I_{k+1}^2(A))\psi_k | \mathcal{F}]x \\
& \geq \frac{1}{2} \sum_{k=0}^{h-1} x^\tau E[(I_{mn} - I_{k+1}^2(A)) | \mathcal{F}]x - \sum_{k=0}^{h-1} E[\|\psi_{k+1}x - x\|^2 | \mathcal{F}] \\
& = \frac{1}{2} \sum_{i=1}^{n} x_i^\tau E \left[ \sum_{k=0}^{h-1} (I_m - A_{k+1,i}^2) | \mathcal{F} \right] x_i - \sum_{k=0}^{h-1} E[\|\psi_{k+1}x - x\|^2 | \mathcal{F}] \\
& \geq \frac{1}{4} \sum_{k=0}^{h-1} \sum_{i=1}^{n} (I_m - A_{k+1,i}^2) | \mathcal{F} \right] x_1 - \frac{h}{2} \sum_{i=2}^{n} \|x_1 - x_i\|^2 - \frac{\rho}{64n} \\
& \geq \frac{\rho}{16n} - \frac{\rho}{32n} - \frac{\rho}{64n} > s\rho,
\end{align*}
\]
which contradicts to (16). So, on every trajectory,
\[
x^\tau E \left[ I_{mn} - \psi^\tau_k \psi_h | \mathcal{F} \right] x \geq s\rho
\]
holds for all unit vector $x \in \mathbb{R}^m$ and Lemma 3.3 follows. \(\square\)

Taking $m = 1$ and $h = 1$ in Lemma 3.3 gives

**Corollary 3.1.** Let $c = (c_1, \ldots, c_n)^\tau \in \mathbb{R}^n$ be a sequence of random variables satisfying $c_i \in [0,1]$ for all $i \in [1, n]$. Denote $I(c) \triangleq \text{diag}\{c_1, \ldots, c_n\}$, then there is a constant
\( s \in (0, 1) \) depending on \( A \) such that

\[
\lambda_{\max}(I(c)A^TAI(c)) \leq 1 - s \sum_{i=1}^{n}(1 - c_i^2).
\]

**Proof of Theorem 2.1.** First, we show the sufficiency. Without loss of generality, assume

\[
\lim_{k \to +\infty} P_{k+1,1}^{-1} = +\infty.
\]

Since \( m = 1 \),

\[
\tilde{\Theta}_{k+1} = A(I - F_k)\tilde{\Theta}_k + A\Lambda_k V_k.
\]  

(25)

Denoting \( \Lambda_k = E[\tilde{\Theta}_k\tilde{\Theta}_k^T] \), Assumption A3 shows

\[
\Lambda_{k+1} = A(I - F_k)\Lambda_k(I - F_k)A^T + A\Lambda_k E[V_k V_k^T]L_k A^T.
\]  

(26)

In view of Assumption A2, applying Neumann inequality and Corollary 3.1 leads to

\[
\text{tr}(\Lambda_{k+1}) \leq \left(1 - s \sum_{i=1}^{n}(1 - (P_{k+1,i}P_{k+1,i}^{-1})^2)\right)\text{tr}(\Lambda_k) + nM \sum_{i=1}^{n} P_{k+1,i}^2\phi_{k,i}^2.
\]  

(27)

Because \( \lim_{k \to +\infty} P_{k+1,1}^{-1} = +\infty \),

\[
\prod_{k=0}^{+\infty}(1 - (1 - (P_{k+1,i}P_{k+1,i}^{-1})^2)) = \prod_{k=0}^{+\infty}(P_{k+1,i}P_{k+1,i}^{-1})^2 = 0,
\]

which infers

\[
\sum_{k=0}^{+\infty}(1 - (P_{k+1,i}P_{k+1,i}^{-1})^2) = +\infty.
\]  

(28)

Furthermore,

\[
\sum_{i=1}^{n} \sum_{k=0}^{+\infty} P_{k+1,i}^2\phi_{k,i}^2 = \sum_{i=1}^{n} \sum_{k=0}^{+\infty} (1 - P_{k+1,i}P_{k+1,i}^{-1})P_{k+1,i} < \sum_{i=1}^{n} \sum_{k=0}^{+\infty} (P_{k+1,i}P_{k,i} - 1)P_{k+1,i} \leq \sum_{i=1}^{n} P_{0,i} < +\infty,
\]

(29)

we thus conclude \( \lim_{k \to +\infty} \text{tr}(\Lambda_k) = 0 \) from Lemma 3.1(ii).
To prove the strong consistency, let $G_k = \sigma\{V_l, 0 \leq l \leq k-1\}$. Then, (29) and Corollary 3.1 yield

$$E[\tilde{\Theta}_{k+1}^r|G_k] \leq \tilde{\Theta}_k^r - s \sum_{j=1}^{n} (1 - (P_{k+1,j}P_{k,j}^{-1})^2)\|\tilde{\Theta}_k\|^2 + nM \sum_{i=1}^{n} P_{k+1,i}^2\phi_{k,i}^2.$$  

Since $\tilde{\Theta}_k^r \tilde{\Theta}_k \in G_k$, by (29) and Lemma 3.2, $\tilde{\Theta}_{k+1}^r \tilde{\Theta}_{k+1} \to 0$ as $k \to +\infty$ almost surely with the convergence rate

$$\sum_{k=0}^{+\infty} \sum_{i=1}^{n} (1 - (P_{k+1,i}P_{k,i}^{-1})^2)\|\tilde{\Theta}_k\|^2 < +\infty, \text{ a.s.} \quad (30)$$

Now, we prove the necessity under

$$\lim_{k \to +\infty} \sum_{i=1}^{n} P_{k+1,i}^{-1} < +\infty.$$  

In this case,

$$\sum_{k=0}^{+\infty} \sum_{i=1}^{n} P_{k+1,i}\phi_{k,i}^2 < +\infty. \quad (31)$$

Denote $\Pi_k \triangleq \prod_{i=k}^{0} A(I_n - F_i)$, we first prove $\lim_{k \to +\infty} \Pi_k$ exists. In fact, since $A$ is an irreducible and aperiodic doubly stochastic matrix, we have $\lim_{k \to} A^k = \frac{1}{n} \cdot 11^r$. Then, by (31), given any $\varepsilon > 0$, there is a $k_1 > 0$ such that

$$\begin{cases}
\sum_{k=k_1}^{+\infty} \sum_{i=1}^{n} P_{k+1,i}\phi_{k,i}^2 < \frac{\varepsilon}{3} \\
\|A^k - A^l\|_1 < \frac{\varepsilon}{3}, \quad \forall k, l > k_1
\end{cases} \quad (32)$$

here $\|X\|_1 \triangleq \max_{1 \leq j \leq r} \sum_{i=1}^{p} X[i,j]$ for any $X \in \mathbb{R}^{p \times r}, p, r \geq 1$.

Therefore, for every $k > 2k_1$,

$$\|\Pi_k - A^{k-k_1}\Pi_{k_1}\|_1 = \left\| \sum_{j=k_1}^{k-1} A^{k-j-1}(I_{j+1} - A\Pi_j) \right\|_1 = \left\| \sum_{j=k_1}^{k-1} A^{k-j}F_{j+1}\Pi_{j} \right\|_1 \leq \sum_{j=k_1}^{k-1} \|A^{k-j}F_{j+1}\Pi_{j}\|_1 \leq \sum_{j=k_1}^{k-1} \sum_{i=1}^{n} P_{j+2,i}\phi_{j+1,i}^2 < \frac{\varepsilon}{3}. \quad (33)$$
Combining (32) and (33) infers that for all $k, l > 2k_1$,

$$\|\Pi_k - \Pi_l\|_1 \leq \|\Pi_k - A^{k-k_1}\Pi_{k_1}\|_1 + \|\Pi_l - A^{l-k_1}\Pi_{k_1}\|_1 + \|(A^{l-k_1} - A^{k-k_1})\Pi_{k_1}\|_1$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,$$

which means $\lim_{k \to +\infty} \Pi_k$ exists.

Now, denote $\Pi \triangleq \lim_{k \to +\infty} \Pi_k$. Observe that

$$\begin{align*}
\Pi_{k+1} &= A(I_n - F_{k+1})\Pi_k, \\
\lim_{k \to +\infty} A(I_n - F_k) &= A,
\end{align*}$$

then $\Pi = A\Pi$. Consequently, $\Pi = 1 \cdot (\mu_1, \ldots, \mu_n)$ for some $\mu_i \geq 0, i = 1, \ldots, n$. We now prove $\mu_i > 0$ for all $i = 1, \ldots, n$. First, (31) infers that there is a $k_2 > k_1$ such that

$$\sum_{k=k_2}^{+\infty} \sum_{i=1}^{n} P_{k+1, i} \phi_{k,i}^2 < 1.$$ 

Furthermore, $1 - P_{k+1, i} \phi_{k,i}^2 > 0$ for all $k \geq 0, i = 1, \ldots, n$ and $\lim_{k \to +\infty} A^k = \frac{1}{n} \cdot 11^T$, we then conclude that as long as $k_2$ is sufficiently large,

$$\min_{i,j} \Pi_{k_2}[i,j] > 0.$$ (35)

Further, since $A$ is a doubly stochastic matrix, for all $k \geq 0$,

$$\min_{i,j} \Pi_{k+1}[i,j] \geq \left( 1 - \max_j P_{k+1, j} \phi_{k,j}^2 \right) \min_{i,j} \Pi_k[i,j].$$

As a result, by (32) and (35),

$$\min \{\mu_1, \ldots, \mu_n\} = \liminf_{k \to +\infty} \min_{i,j} \Pi_{k+1}[i,j] \geq \min_{i,j} \Pi_{k_2}[i,j] \prod_{k=k_2}^{+\infty} \left( 1 - \sum_{i=1}^{n} P_{k+1, i} \phi_{k,i}^2 \right) > 0.$$ 

So, in view of (26),

$$\liminf_{k \to +\infty} \text{tr}(\Lambda_{k+1}) \geq \liminf_{k \to +\infty} \text{tr}(\Pi_k \widetilde{\Theta}_0 \widetilde{\Theta}_0^T \Pi_k) = n \left( \sum_{j=1}^{n} \mu_j(\theta_{0,j} - \theta) \right)^2,$$ (36)

which infers $\liminf_{k \to +\infty} E \widetilde{\Theta}_k^T \widetilde{\Theta}_k > 0$ if $\sum_{j=1}^{n} \mu_j(\theta_{0,j} - \theta) \neq 0$. 

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The last part is addressed to proving $\tilde{\Theta}_k \xrightarrow{p} 0$. By (27) and (29), Lemma 3.1(i) shows that $\lim_{k \to +\infty} E\|\tilde{\Theta}_k\|^2$ exists. Denote

$$Q \triangleq \sup_{k \geq 0} E\|\tilde{\Theta}_k\|^2$$

$$\Pi(k, i) \triangleq \Pi^{k-i+1} A(I_n - F_k)$$

By (29), for any fixed $\varepsilon > 0$, there is a $k_3 > 0$ such that

$$\sum_{j=k_3}^{+\infty} \sum_{i=1}^{n} P^2_{j+1, i} \phi^2_{j, i} < \frac{\varepsilon}{4M}.$$ 

In addition, similar to (34), there is a $k_4 > k_3$ such that for any $k, l > k_4$,

$$\|\Pi(k, k - k_3 - 1) - \Pi(l, l - k_3 - 1)\|_2 < \frac{\varepsilon}{2Q}.$$

So, as long as $k, l > k_4$,

$$\tilde{\Theta}_{k+1} - \tilde{\Theta}_{l+1} = \Pi(k, k - k_3 + 1)\tilde{\Theta}_{k_3} + \sum_{j=k_3}^{k} \Pi(k, k - j)AL_jV_j$$

$$- \Pi(l, l - k_3 + 1)\tilde{\Theta}_{k_3} + \sum_{j=k_3}^{l} \Pi(l, l - j)AL_jV_j,$$

which infers

$$E\|\tilde{\Theta}_{k+1} - \tilde{\Theta}_{l+1}\|^2$$

$$\leq E\|(\Pi(k, k - k_3 + 1) - \Pi(l, l - k_3 + 1))\tilde{\Theta}_{k_3}\|^2 + 2 \sum_{j=k_3}^{\max\{k, l\}} E\|L_jV_j\|^2$$

$$< \frac{\varepsilon}{2Q} \cdot Q + 2M \sum_{j=k_3}^{+\infty} \sum_{i=1}^{n} P^2_{j+1, i} \phi^2_{j, i} < \varepsilon.$$ 

This means $\{\tilde{\Theta}_k\}_{k \geq 0}$ is a Cauchy sequence in $L^2(dP)$, and hence there exists a random vector $Z \in L^2(dP)$ such that $\lim_{k \to +\infty} E\|\tilde{\Theta}_k - Z\|^2 = 0$. So, $\tilde{\Theta}_k \xrightarrow{p} Z$. Note that $Z \neq 0$ due to $\lim_{k \to +\infty} E\|\tilde{\Theta}_k\|^2 \neq 0$. □
4 Proofs of Theorems 2.3–2.4

Since a deterministic parameter can be viewed as a random variable having a degenerate Gaussian distribution with zero variance, it suffices to prove Remark 2.4 by assuming that \( \theta \) in Theorems 2.3–2.4 is random. In addition, let \( \theta \) in Theorem 2.4 be Gaussian distributed.

We first prove a technical lemma. Fix a \( j^* \in \{1, \ldots, n\} \) and let \( d \) be the smallest integer that \( A_{d+1}[j^*, j^*] > 0 \). Define a sequence of vectors in \( \mathbb{R}^{mn} \):

\[
\begin{align*}
\mathcal{P}_0 & \triangleq \{ C : C[m(j^* - 1) + 1, 1] = 0 \} \\
\mathcal{P}_l & \triangleq \{ C : \sum_{k=1}^{n} b_{l,k} C[m(k - 1) + 1, 1] = 0 \}, \quad l \in [1, d],
\end{align*}
\]

where for \( l \geq 1 \) and \( k = 1, \ldots, n \),

\[
b_{l,k} = \sum_{i \neq j^*, i_1, \ldots, i_{l-1}} a_{j^*i_1i_2 \ldots i_{l-1}k}.
\]

**Lemma 4.1.** Given \( k \geq 1 \), let \( f = (f_1, \ldots, f_{mn})^T : \mathbb{R}^k \to \mathbb{R}^{mn} \) be a map that each \( f_i(z) \) is a polynomial of \( z \in \mathbb{R}^k, 1 \leq i \leq mn \). If \( f(\mathbb{R}^k) \not\subset \mathcal{P}_l \) for some \( l \in [0, d] \), then for any nonempty open set \( U \subset \mathbb{R}^k \), there is a \( z \in U \) such that \( f(z) \not\in \mathcal{P}_l \).

**Proof.** Since \( \sum_{k=1}^{n} b_{l,k} f_{m(k-1)+1}(z) \) is a polynomial, if for some nonempty open set \( U \subset \mathbb{R}^k \),

\[
\sum_{k=1}^{n} b_{l,k} f_{m(k-1)+1}(z) \equiv 0 \quad \text{for all} \quad z \in U
\]

then the polynomial must be identically zero on \( \mathbb{R}^k \). This contradicts to \( f(\mathbb{R}^k) \not\subset \mathcal{P}_l, l \in [0, d] \). \( \square \)

**Lemma 4.2.** Let \( C \in \mathbb{R}^{mn} \) be a vector and \( B_i \in \mathbb{R}^{m \times m}, i = 1, \ldots, n \) be a sequence of positive definite matrices. Define a map \( Q_0 : \mathbb{R}^{mn} \to \mathbb{R}^{mn \times mn} \) by

\[
Q_0(z) \triangleq \text{diag}\{(B_i^{-1} + v_i v_i^T)^{-1}B_i^{-1}, i = 1, \ldots, n\},
\]

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where $z = \text{col}\{v_1, \ldots, v_n\}$ and $v_i \in \mathbb{R}^m$, $1 \leq i \leq n$. For each $l \in [0, d]$,

(i) if $C \not\in \mathcal{P}_{l+1}$, then for any nonempty open set $U \in \mathbb{R}^{mn}$, there is a $z \in U$ such that

$$(\mathcal{A} \otimes I_m)(Q_0(z)C) \not\in \mathcal{P}_l;$$

(ii) if $C \not\in \mathcal{P}_0$, then for any nonempty open set $U \in \mathbb{R}^{mn}$, there is a $z \in U$ such that

$$(\mathcal{A} \otimes I_m)(Q_0(z)C) \not\in \mathcal{P}_d.$$ 

Proof. (i) Let $D(z) \triangleq (\mathcal{A} \otimes I_m)Q_0(z)C$, then

$$D[m(i-1) + 1, 1](z) = -\sum_{k=1}^{n} a_{ik} \frac{v_k^T C^{(k)}_1 B_k v_k}{1 + v_k^T B_k v_k} + \sum_{k=1}^{n} a_{ik} C[m(k-1) + 1, 1],$$

where $C^{(k)}_1 \in \mathbb{R}^{m \times m}$ satisfies

$$C^{(k)}_1 = \begin{bmatrix}
C[m(k-1) + 1, 1] & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
C[mk, 1] & 0 & \cdots & 0
\end{bmatrix}, \quad k = 1, \ldots, n.$$ 

Because each component of $D(z) \prod_{k=1}^{n} (1 + v_k^T B_k v_k)$ is a polynomial and $\prod_{k=1}^{n} (1 + v_k^T B_k v_k) > 0$, in view of Lemma 4.1, it is sufficient to prove

$$D(\mathbb{R}^{mn}) \prod_{k=1}^{n} (1 + v_k^T B_k v_k) \not\subset \mathcal{P}_l.$$ 

Suppose $(\mathcal{A} \otimes I_m)(Q_0(\mathbb{R}^{mn})C) \subset \mathcal{P}_l$ and let $v_j = (x_{1j}, \ldots, x_{mj})^\tau$, $j \in [1, n]$. If $l \geq 1$, the constant term of

$$\prod_{k=1}^{n} (1 + v_k^T B_k v_k) \sum_{k=1}^{n} b_{l,k} D[m(k-1) + 1, 1]$$

is

$$\sum_{k=1}^{n} \sum_{i=1}^{n} b_{l,i} a_{i,k} C[m(k-1) + 1, 1] = \sum_{k=1}^{n} b_{l+1,k} C[m(k-1) + 1, 1] = 0,$$
which implies $C \in \mathcal{P}_{l+1}$. It contradicts to $C \notin \mathcal{P}_{l+1}$. If $l = 0$, the coefficient of $x_{j^*}^2$ of

$$\prod_{k=1}^{n}(1 + v_k^* B_k v_k) D[m(j^* - 1) + 1, 1]$$

is

$$B_{j^*}[1, 1] \sum_{k \neq j^*} a_{j^* k} C[m(k - 1) + 1, 1] = 0,$$

which implies $C \in \mathcal{P}_1$ since $B_j$ is positive definite. Hence, it leads to a contradiction again.

(ii) If $(\mathcal{A} \otimes I_m)(Q_0(\mathbb{R}^{mn})C) \subset \mathcal{P}_d$, then the coefficient of $x_{j^*}^2$ and the constant term of

$$\prod_{k=1}^{n}(1 + v_k^* B_k v_k) \sum_{k=1}^{n} b_{d,k} D[m(k - 1) + 1, 1]$$

are

$$B_{j^*}[1, 1] \sum_{k \neq j^*} \sum_{i=1}^{n} b_{d,i} a_{i,k} C[m(k - 1) + 1, 1] = 0,$$

and

$$\sum_{k=1}^{n} \sum_{i=1}^{n} b_{d,i} a_{i,k} C[m(k - 1) + 1, 1] = 0,$$

respectively. As a result,

$$\sum_{i=1}^{n} b_{d,i} a_{i,j^*} C[m(j^* - 1) + 1, 1] = \sum_{i \neq j^*} \sum_{i_1, \ldots, i_d} a_{j^* i} a_{i_1 i} \ldots a_{i_d j^*} C[m(j^* - 1) + 1, 1] = 0.$$

So, by $C \notin \mathcal{P}_0$,

$$\sum_{i \neq j^*} \sum_{i_1, \ldots, i_d} a_{j^* i} a_{i_1 i} \ldots a_{i_d j^*} = 0.$$

Hence

$$\mathcal{A}^{d+1}[j^*, j^*] = \sum_{i=1}^{n} \sum_{i_1, \ldots, i_d} a_{j^* i} a_{i_1 i} \ldots a_{i_d j^*} = a_{j^* j^*} \left( \sum_{i_1, \ldots, i_d} a_{j^* i_1} \ldots a_{i_d j^*} \right),$$

which together with $\mathcal{A}^{d+1}[j, j] > 0$ implies

$$\mathcal{A}^d[j^*, j^*] = \sum_{i_1, \ldots, i_d} a_{j^* i_1} \ldots a_{i_d j^*} > 0.$$

This contradicts to the definition of $d$. □
Now, letting
\[ z = \text{col}\{0, \ldots, 0, v, 0, \ldots, 0\}, \]
in Lemma 4.2 shows

**Corollary 4.1.** Let \( C \in \mathbb{R}^{mn} \) and \( B \in \mathbb{R}^{m \times m} \) be a vector and a positive definite matrix. Denote \( Q_0^* : \mathbb{R}^m \to \mathbb{R}^{mn \times mn} \) by
\[
Q_0^*(v) \triangleq \text{diag}\{I_{m}, \ldots, I_{m}, (B^{-1} + vv^\tau)^{-1}B^{-1}, I_{m}, \ldots, I_{m}\},
\]

(i) If \( C \notin \mathcal{P}_{l+1} \), then for any nonempty open set \( U \in \mathbb{R}^m \), there is \( z \in U \) such that
\[
(A \otimes I_m)(Q_0^*(z)C) \notin \mathcal{P}_l.
\]

(ii) If \( C \notin \mathcal{P}_0 \), then for any nonempty open set \( U \in \mathbb{R}^m \), there is \( z \in U \) such that
\[
(A \otimes I_m)(Q_0^*(z)C) \notin \mathcal{P}_d.
\]

The next lemma with the proof given in Appendix 5 is the main reason for the failure of the diffusion RLS in Remark 2.4. We introduce some necessary notations. For \( C \in \mathbb{R}^{mn} \) and \( B \in \mathbb{R}^{m \times m} \) defined in Corollary 4.1, denote maps \( Q_1 : \mathbb{R}^m \to \mathbb{R}^{mn \times mn} \), \( Q_2 : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^{mn \times mn} \) and \( Q_3 : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^{mn \times mn} \) by
\[
\begin{aligned}
Q_1(v_1) &\triangleq \text{diag}\{I_{m}, \ldots, I_{m}, B_1B^{-1}, I_{m}, \ldots, I_{m}\}, \\
Q_2(v_1, v_2) &\triangleq \text{diag}\{I_{m}, \ldots, I_{m}, B_2B_1^{-1}, I_{m}, \ldots, I_{m}\}, \\
Q_3(v_1, v_2) &\triangleq (A \otimes I_m)Q_2(v_1, v_2)(A \otimes I_m)Q_1(v_1)C
\end{aligned}
\]
where \( B_1 \triangleq (B^{-1} + v_1v_1^\tau)^{-1} \), \( B_2 \triangleq (B^{-1} + v_1v_1^\tau + v_2v_2^\tau)^{-1} \) and \( v_1, v_2 \in \mathbb{R}^m \).
Lemma 4.3. Let $a_{j^*} > 0$ for some $l \in [1, n]$, where $j^*$ is the fixed index defined before. If $C \notin \mathcal{P}_1$, then for any $L > 0$, there exist some $v_1, v_2 \in \mathbb{R}^m$ such that

$$
\begin{cases}
(A \otimes I_m)(Q_1(v_1))C \notin \mathcal{P}_0 \\
Q_3(v_1, v_2) \notin \mathcal{P}_d \\
|Q_3(v_1, v_2)[m(l - 1) + 1, 1]| > L.
\end{cases}
$$

Lemma 4.4. Let $C \in \mathbb{R}^{mn}$ and $\{B_i \in \mathbb{R}^{m \times m}\}$ be defined in Lemma 4.2. For any $K > 0$, if $C \notin \mathcal{P}_d$, then there exists some $z_j = \text{col}\{v_{j,1}, \ldots, v_{j,n}\} \in \mathbb{R}^{mn}, j \in [1, m]$, such that

$$
\begin{cases}
\inf_{i \in [1, n]} \lambda_{\min} \left(B_i^{-1} + \sum_{j=1}^m v_{j,i} v_{j,i}^\top\right) > K \\
\prod_{k=j}^{1}(A \otimes I_m)G_k(z_1, \ldots, z_k)C \notin \mathcal{P}_{d-i}, \quad j \in [1, m],
\end{cases}
$$

where $\mathcal{P}_{-l} \triangleq \mathcal{P}_{d-l+1}, \ l \geq 1$ and

$$
G_j(z_1, \ldots, z_j) \triangleq \text{diag}\{B_{1,j} B_{1,j-1}^{-1}, \ldots, B_{n,j} B_{n,j-1}^{-1}\}
$$

for $1 \leq j \leq m$ and $B_{i,0} \triangleq B_i, \ 1 \leq i \leq n.$

Proof. Let $e_j$ denote the $j$th column of the identity matrix $I_m, j \in [1, m]$ and

$$z_j^* = \text{col}\{v_{j,1}^*, \ldots, v_{j,n}^*\} = \text{col}\{\sqrt{nK} \cdot e_j, \ldots, \sqrt{nK} \cdot e_j\}.$$

Then, for $i \in [1, n],

$$\lambda_{\min} \left(B_i^{-1} + \sum_{j=1}^m v_{j,i}^* (v_{j,i}^*)^\top\right) \geq \lambda_{\min} \left(\sum_{j=1}^m v_{j,i}^* (v_{j,i}^*)^\top\right) = nK > K.$$

Since

$$\lambda(z_1, z_2, \ldots, z_m) \triangleq \inf_{i \in [1, n]} \lambda_{\min} \left(B_i^{-1} + \sum_{j=1}^m v_{j,i} v_{j,i}^\top\right)$$

is continuous in $z_1, \ldots, z_m$, there exists a neighbourhood $U_1$ of $z_1^*$ such that $\lambda(s, z_2^*, \ldots, z_m^*) > K$ for all $s \in U_1$. By Lemma 4.2, there is a $z_1 \in U_1$ such that $(A \otimes I_m)G_1(z_1)C \notin \mathcal{P}_{d-1}$. 28
An analogous argument shows that we can select a series of \( z_1, \ldots, z_m \) satisfying

\[
\begin{cases}
\inf_{i \in [1, n]} \lambda_{\min} \left( B_i^{-1} + \sum_{j=1}^m v_{j,i}v_{j,i}^T \right) > K, \\
\prod_{k=j}^m (A \otimes I_m)G_k(z_1, \ldots, z_k)C \not\in \mathcal{P}_{d-j}
\end{cases}
\]

which is exactly the result as desired. \( \square \)

**Lemma 4.5.** Let \( E\tilde{\Theta}_0[m(j^* - 1) + 1, 1] \neq 0 \) and \( a_{i,j^*} > 0 \) for some \( l \in [1, n] \). Then, under Assumption A1’, there is a sequence of deterministic matrices \( \{\Phi_i\}_{i=0}^{+\infty} \) such that

\[
\lim_{t \to +\infty} \lambda_{\min} \left( \sum_{i=0}^t \Phi_i\Phi_i^T \right) = +\infty
\]

and for \( R_t = \prod_{i=0}^t (A \otimes I_m)(I_{mn} - F_i)E\tilde{\Theta}_0 \),

\[
\limsup_{t \to +\infty} \frac{|R_t|m(l-1) + 1, 1|}{16(t+1)^4} > 1.
\]

**Proof.** It suffices to construct a series of deterministic \( \{\Phi_i\}_{i=0}^{+\infty} \) such that for any \( k \geq 0, s \in [0, d] \) and \( t_k = k(m+3)(d+1), \)

\[
\begin{cases}
R_{t_k+j} \not\in \mathcal{P}_{d-j} \\
\lambda_{\min} \left( \sum_{i=0}^{t_k+m} \Phi_i\Phi_i^T \right) > t_k + m \\
|R_{t_k+m}[m(l-1) + 1, 1]| > 20(t_k+1)^4
\end{cases}
\]

(37)

First, since \( E\tilde{\Theta}_0 \not\in \mathcal{P}_0 \), by Lemma 4.2, there is a \( \Phi_0 \) such that \( R_0 \not\in \mathcal{P}_d \). Let \( k = 0 \). In view of Lemma 4.3, we can find some \( \Phi_{t_k+j}, j = 1, \ldots, m \) such that for all \( j \in [1, m] \),

\[
\begin{cases}
\lambda_{\min} \left( \sum_{i=0}^{t_k+m} \Phi_i\Phi_i^T \right) > t_k + m \\
R_{t_k+j} = \prod_{i=j}^1 (A \otimes I_m)(I_{mn} - F_{t_k+i})R_{t_k} \not\in \mathcal{P}_{d-j}
\end{cases}
\]

(38)

Moreover, by Lemma 4.2 there are some \( \Phi_{t_k+j}, j = m+1, \ldots, (m+3)(d+1) - 2 \) such that for all \( j \in [m+1, (m+3)(d+1) - 2] \),

\[
R_{t_k+j} = \prod_{i=j}^{m+1} (A \otimes I_m)(I_{mn} - F_{t_k+i})R_{t_k+m} \not\in \mathcal{P}_{d-j}.
\]

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Finally, by noting that \( R_{t_k+(m+3)(d+1)-2} \notin \mathcal{P}_1 \), Lemma 4.3 indicates that for some

\[
\Phi_{t_k+1-i} = \text{diag}(0,\ldots,0,v_{i,j^*},0,\ldots,0), \quad i = 0,1,
\]

one has

\[
\begin{cases}
R_{t_k+1} \notin \mathcal{P}_0, & R_{t_k+1} \notin \mathcal{P}_d \\
|R_{t_k+1}[m(l-1)+1,1]| > 20(t_{k+1}+1)^4.
\end{cases}
\]

So, we obtain a series of \( \{\Phi_j, j = 0,\ldots,t_1\} \) fulfilling (37). By repeating (38) to (39) for all \( k \geq 1 \), (37) is proved immediately based on the mathematical induction. \( \square \)

**Proof of Theorem 2.3** Considering \( \|E\tilde{\Theta}_0\| \neq 0 \) and Assumption A1’, we suppose, without loss of generality, there are some \( j^*, l \in \{1,\ldots,n\} \) such that \( E\tilde{\Theta}_0[m(j^*-1)+1,1] \neq 0 \) and \( a_{lj^*} > 0 \). Let \( \{\Phi_k\}_{k=0}^{+\infty} \) be the deterministic sequence constructed in Lemma 4.5. Then, by virtue of Assumption A2 and (25),

\[
E(\tilde{\Theta}_k[m(l-1)+1,1]) = R_k[m(l-1)+1,1],
\]

and hence

\[
\sup_{k \geq 0} E\|\tilde{\Theta}_k\|^2 \geq \sup_{k \geq 0} \|E\tilde{\Theta}_k\|^2 \geq \sup_{k \geq 0} (E(\tilde{\Theta}_k[m(l-1)+1,1]))^2 \\
= \sup_{k \geq 0} (R_k[m(l-1)+1,1])^2 = +\infty,
\]

where \( R_k \) is define in Lemma 4.5. \( \square \)

**Proof of Theorem 2.4** Let \( \{\Phi_k\}_{k=0}^{+\infty} \) be defined in the proof of Theorem 2.3. Since \( \theta \) is Gaussian distributed, \( \tilde{\Theta}_k[m(l-1)+1,1] \) possesses a normal distribution by Assumption
A2”. Note that for any random variable \( \xi \sim N(E\xi, \sigma^2) \) and \( k \geq 1 \),

\[
P(16(k+1)^3|\xi| < |E\xi|) = I_{\{\sigma \neq 0\}} \cdot \frac{1}{\sqrt{2\pi}} \int_{\frac{|E\xi|}{|\sigma|}}^{\frac{E\xi}{\sigma}} e^{-\frac{x^2}{2}} dx
\]

\[
\leq I_{\{|E\xi| < 8k\}} \cdot \frac{1}{\sqrt{2\pi}} \frac{1}{(k+1)^2} + I_{\{|E\xi| \geq 8k\}} \cdot \frac{1}{\sqrt{2\pi}} \int_{4k}^{+\infty} e^{-\frac{x^2}{2}} dx
\]

\[
\leq I_{\{|E\xi| < 8k\}} \cdot \frac{1}{\sqrt{2\pi}} \frac{1}{(k+1)^2} + I_{\{|E\xi| \geq 8k\}} \cdot \frac{1}{\sqrt{2\pi}} \int_{4k}^{+\infty} \frac{2}{x^3} dx
\]

and

\[
P(|\xi| \leq \varepsilon) = I_{\{\sigma \neq 0\}} \cdot \frac{1}{\sqrt{2\pi}} \int_{\frac{|E\xi|}{|\sigma|}}^{\frac{E\xi}{\sigma}} e^{-\frac{x^2}{2}} dx + I_{\{\sigma = 0, |E\xi| \leq \varepsilon\}}
\]

\[
\leq 2 \cdot I_{\{|E\xi| \leq \varepsilon\}} + I_{\{\sigma > \varepsilon\}} \cdot \frac{2}{\sqrt{2\pi}} + I_{\{\sigma \leq \varepsilon, |E\xi| > \varepsilon\}} \cdot \frac{1}{\sqrt{2\pi}} \int_{\frac{|E\xi| - \varepsilon}{\varepsilon}}^{+\infty} e^{-\frac{x^2}{2}} dx.
\]

Define

\[
D_0 \triangleq \bigcap_{k=1}^{+\infty} \left\{ |\tilde{\Theta}_k[m(l-1) + 1, 1]| \geq \frac{|R_k[m(l-1) + 1, 1]|}{16(k+1)^3} \right\},
\]

then (41) infers

\[
P(D_0) \geq 1 - \sum_{k=1}^{+\infty} P\left(\{16(k+1)^3|\tilde{\Theta}_k[m(l-1) + 1, 1]| < |R_k[m(l-1) + 1, 1]|\right)
\]

\[
\geq 1 - \frac{1}{\sqrt{2\pi}} \sum_{k=1}^{+\infty} \frac{1}{k^2} > 0.
\]

According to Lemma 4.5 and (40),

\[
||\tilde{\Theta}_k|| \geq |\tilde{\Theta}_k[m(l-1) + 1, 1]| \geq \frac{1}{16(k+1)^3} R_k[m(l-1) + 1, 1]
\]

\[
> k + 1, \text{ i.o on } D_0.
\]

So, (9) holds. Moreover, by Lemma 4.5 and (40) again,

\[
\limsup_{k \to +\infty} |E\tilde{\Theta}_k[m(l-1) + 1, 1]| = +\infty.
\]
which together with (42) yields
\[
\liminf_{k \to +\infty} P(|\tilde{\Theta}_k[m(l-1) + 1, 1]| \leq \varepsilon) \leq \frac{2}{\sqrt{2\pi}},
\]
and hence
\[
\limsup_{k \to +\infty} P(\|\tilde{\Theta}_k\| > \varepsilon) \geq \limsup_{k \to +\infty} P(|\tilde{\Theta}_k[m(l-1) + 1, 1]| > \varepsilon) \geq 1 - \frac{2}{\sqrt{2\pi}}.
\]

The proof is completed. □

5 Concluding Remarks

We have established the necessary and sufficient condition that ensures the diffusion RLS converging to true scalar parameters. This condition shows that cooperations among nodes through diffusion networks indeed could help estimation, as long as the parameters to be identified are scalar. But for the general case where parameters are high dimensional, our results reveal that the diffusion RLS do not necessarily outperform the individual RLS.

On the other hand, the convergence theorem on the diffusion RM in this paper and the relevant studies on the diffusion LMS reflect that the ATC and CTA diffusion strategies might be very suitable for the adaptive algorithms in the form of the LMS-type.

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Appendices

Appendix A

Proof of Remark 2.3(i)(b). The argument is based on the proof of Theorem 2.1 from (26)–(36). Considering (1), let $l$ be the smallest integer such that $\sum_{i=1}^{n} \phi_{l,i}^2 \neq 0$. An analogous
proof of Theorem 2.1 shows that for some $\mu'_i > 0, i = 1, \ldots, n$,

$$\lim_{k \to +\infty} \Pi(k, k - l) = 1 \cdot (\mu'_1, \ldots, \mu'_n).$$

As a result,

$$\liminf_{k \to +\infty} \text{tr}(\Lambda_{k+1}^r) \geq \liminf_{k \to +\infty} \text{tr}(\Pi(k, k - l) AL_d E[V_i V_i^T] L_d A^T \Pi(k, k - l)^T)$$

$$= n \sum_{i=1}^{n} \left( \sum_{j=1}^{n} a_{ij} \mu'_i \right)^2 P_{l+1,i}^2 \phi_{l,i}^2 E \varepsilon_{l,i}^2 > 0,$$

and $\widetilde{\Theta}^r \not\to 0$ follows as proved in Theorem 2.1. □

**Proof of Theorem 2.2.** Let

$$G_k \triangleq \sigma\{\Phi_i, V_i, 0 \leq i \leq k, 0 \leq l \leq k - 1\},$$

then by (25) and Corollary 3.1,

$$E[\widetilde{\Theta}^r_{k+1} \widetilde{\Theta}_{k+1} | G_k] \leq \widetilde{\Theta}^r_k \widetilde{\Theta}_k - s \sum_{i=1}^{n} (1 - (P_{k+1,i} P_{k,i}^{-1})^2) \|\widetilde{\Theta}_k\|^2 + nM \sum_{i=1}^{n} P_{k+1,i}^2 \phi_{k,i}^2.$$

Since $\widetilde{\Theta}^r_k \widetilde{\Theta}_k \in G_k$, according to (29) and [7, Lemma 1.2.2], $\lim_{k \to +\infty} \widetilde{\Theta}^r_{k+1} \widetilde{\Theta}_{k+1}$ exists almost surely and

$$\sum_{k=0}^{+\infty} \sum_{i=1}^{n} (1 - (P_{k+1,i} P_{k,i}^{-1})^2) \|\widetilde{\Theta}_k\|^2 < +\infty, \text{ a.s.} \quad (42)$$

Denote $\Theta_\infty \triangleq \lim_{k \to +\infty} \widetilde{\Theta}^r_k \widetilde{\Theta}_k$ and

$$S \triangleq \{\Theta_\infty \neq 0\} \cap \left\{ \lim_{k \to +\infty} \sum_{i=1}^{n} P_{k+1,i}^{-1} = +\infty \right\}$$

$$S' \triangleq \{\Theta_\infty \neq 0\} \cap \left\{ \sum_{k=0}^{+\infty} \sum_{i=1}^{n} (1 - (P_{k+1,i} P_{k,i}^{-1})^2) = +\infty \right\}.$$ 

Note that by (28),

$$\left\{ \lim_{k \to +\infty} \sum_{i=1}^{n} P_{k+1,i}^{-1} = +\infty \right\} \subset \left\{ \sum_{k=0}^{+\infty} \sum_{i=1}^{n} (1 - (P_{k+1,i} P_{k,i}^{-1})^2) = +\infty \right\},$$

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then $S ⊂ S'$. Moreover,
\[ \sum_{k=0}^{+\infty} \sum_{i=1}^{n} (1 - (P_{k+1,i}P_{k,i}^{-1})^2) \| \tilde{\Theta}_k \|^2 = +\infty \quad \text{on } S', \]

which implies $P(S) \leq P(S') = 0$ by (42).

\[ \square \]

## Appendix B

**Proof of Lemma 4.3.** The first step is to seek a pair $(v_1, v_2)$ that
\[ |Q_3(v_1, v_2)[m(l - 1) + 1, 1]| > L. \quad (43) \]

To this end, denote $D(v_1) \triangleq (A \otimes I_m)Q_1(v_1)C$. In the later discussion, we suppress $v_1$ in $D(v_1)$ for brevity. Calculate

\[ Q_3(v_1, v_2)[m(l - 1) + 1, 1] \]
\[ = ((A \otimes I_m)(Q_2(v_1, v_2))D)[m(l - 1) + 1, 1] \]
\[ = a_{ij^*}(1, 0, \ldots, 0)B_2B_1^{-1} \cdot (D[m(j^* - 1) + 1, 1], \ldots, D[mj^*, 1])^\tau + \sum_{i \neq j^*} a_{ii}D[m(i - 1) + 1, 1] \]
\[ = a_{ij^*}(1, 0, \ldots, 0) \left( I_m - \frac{B_1v_2v_2^\tau}{1 + v_2^\tau B_1v_2} \right) \cdot (D[m(j^* - 1) + 1, 1], \ldots, D[mj^*, 1])^\tau \]
\[ + \sum_{i \neq j^*} a_{ii}D[m(i - 1), 1] \]
\[ = -a_{ij^*} \frac{v_2^\tau D_1B_1v_2}{1 + v_2^\tau B_1v_2} + \sum_{i=1}^{n} a_{ii}D[m(i - 1) + 1, 1], \]

where $D_1 \in \mathbb{R}^{m \times m}$ is defined by

\[ D_1 \triangleq \begin{bmatrix} D[m(j^* - 1) + 1, 1] & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ D[mj^*, 1] & 0 & \ldots & 0 \end{bmatrix} \]
Similarly, for all $i = 1, \ldots, n,$

$$D[m(i - 1) + 1, 1] = -a_{ij} \frac{v_i^* C_1 B v_1}{1 + v_i^* B v_1} + \sum_{k=1}^n a_{ik} C[m(k - 1) + 1, 1],$$  \hspace{1cm} (44)$$

where $C_1 \in \mathbb{R}^{m \times m}$ is defined by

$$C_1 \triangleq \begin{bmatrix} C[m(j^* - 1) + 1, 1] & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ C[mj^*, 1] & 0 & \ldots & 0 \end{bmatrix}.$$ 

Now, write $v_i = r_i z_i$, where $r_i > 0$ and $|z_i| = 1$, $i = 1, 2$. Since for any $r_1 > 0$,

$$a_{ij} \frac{|v_i^* C_1 B v_1|}{1 + v_i^* B v_1} = a_{ij} \frac{|z_i^* C_1 B z_1|}{r_1^2 + z_i^* B z_1} < a_{ij} \frac{|z_i^* C_1 B z_1|}{z_i^* B z_1},$$

it is trivial that

$$a_{ij} \frac{|z_i^* C_1 B z_1|}{z_i^* B z_1} \leq a_{ij} \frac{\lambda_{\text{max}}(B^{-1})\|B\|\|C\|_1}{r_1^2 + z_i^* B z_1}.$$ 

Then, by (44), for all $i = 1, \ldots, n,$

$$|D[m(i - 1) + 1, 1]| \leq (1 + \lambda_{\text{max}}(B^{-1})\|B\|\|C\|_1),$$

which infers

$$Q_3(v_1, v_2)[m(l - 1) + 1, 1] < -a_{ij} \frac{v_i^* D_1 B v_2}{1 + v_i^* B v_2} + (1 + \lambda_{\text{max}}(B^{-1})\|B\|\|C\|_1).$$

Next, for any $L > 0$, denote

$$c \triangleq L \cdot a_{ij}^{-1} + a_{ij}^{-1}(1 + \lambda_{\text{max}}(B^{-1})\|B\|\|C\|_1).$$

If we could find a $v_1$ such that $D \notin P_0$ and

$$K \triangleq 2cB_1 - (D_1 B_1 + B_1 D_1^*)$$

is not semi-positive definite, then there is a $v_2'$ such that for any $v_2$ in some sufficiently small neighbourhood of $v_2'$,

$$z_2^*(D_1 - cI_m) B_1 z_2 > \frac{c}{r_2^2}.$$
which can deduce \(13\). So, according to Corollary \(4.1\) there exists a \(v_2\) in this neighbourhood fulfilling both \(13\) and

\[ Q_3(v_1, v_2) = (\mathcal{A} \otimes I_m)Q_2(v_1, v_2)D \notin \mathcal{P}_d. \]

To construct the desired \(v_1\), compute the leading principal minor of order 2 of \(K\) by

\[
K[1, 1]K[2, 2] - K^2[1, 2] = 4(cB_1[1, 1] - D_1[1, 1]B_1[1, 1])(cB_1[2, 2] - D_1[2, 1]B_1[1, 2])
- (2cB[1, 2] - D_1[2, 1]B_1[1, 1] - D_1[1, 1]B_1[1, 2])^2
= 4c(c - D_1[1, 1])(B_1[1, 1]B_1[2, 2] - B_1^2[1, 2]) - (D_1[1, 1]B_1[1, 2] - D_1[2, 1]B_1[1, 1])^2.
\]

Let \(z_1 = (q_1, q_2, 0, \ldots, 0)^\tau\), where \(q_1, q_2\) are two real numbers satisfying \(q_1^2 + q_2^2 = 1\) and \(q_2 \neq 0\). Then,

\[ K[1, 1]K[2, 2] - K^2[1, 2] < 0 \quad (45) \]

is equivalent to

\[
4c(c - D_1[1, 1]) \left( (B_1^{-1})^*[2, 2] - \frac{((B_1^{-1})^*[1, 2])^2}{(B_1^{-1})^*[1, 1]} \right) < \frac{D_1[1, 1](B_1^{-1})^*[1, 2] - D_1[2, 1](B_1^{-1})^*[1, 1])^2}{(B_1^{-1})^*[1, 1]}.
\] (46)

Calculating the adjoint matrix of \(B_1^{-1}\) shows that there exist two constants \(M_1, M_2 > 0\) depending on \(B\) such that \(|l_i| < M_2\) for \(i = 1, 2, 3\), where

\[
\begin{align*}
l_1 &\triangleq (B_1^{-1})^*[1, 1] - r_1^2q_2^2M_1, \\
l_2 &\triangleq (B_1^{-1})^*[1, 2] + r_1^2q_1q_2M_1, \\
l_3 &\triangleq (B_1^{-1})^*[2, 2] - r_1^2q_1^2M_1.
\end{align*}
\]

Therefore,

\[
\left| (B_1^{-1})^*[2, 2] - \frac{((B_1^{-1})^*[1, 2])^2}{(B_1^{-1})^*[1, 1]} \right| = \left| \frac{r_1^2(l_3q_2^2M_1 + l_1q_1^2M_1 + 2q_1q_2M_1l_2) + l_1l_3 - l_2^2}{r_1^2q_2^2M_1 + l_1} \right| \\
\leq \frac{2r_1^2M_1M_2(q_1^2 + q_2^2) + 2M_2^2}{r_1^2q_2^2M_1 + l_1}.
\]
which yields

$$\limsup_{r_1 \to +\infty} \left| (B_1^{-1})^*[2, 2] - \frac{((B_1^{-1})^*[1, 2])^2}{(B_1^{-1})^*[1, 1]} \right| \leq 1 + \frac{2M_2}{q_2^2}. \quad (47)$$

In order to estimate the right hand side of (46), we define two functions $H_1(\cdot)$ and $H_2(\cdot)$ by

$$H_1 \left( \frac{q_1}{q_2} \right) \triangleq \lim_{r_1 \to +\infty} D_1[1, 1] = \sum_{k=1}^{n} a_{j^*k} C[m(k - 1) + 1, 1] - a_{j^*j} \frac{z_1^\top C_1 B z_1}{z_1^\top B z_1}$$

and

$$H_2 \left( \frac{q_1}{q_2} \right) \triangleq \lim_{r_1 \to +\infty} D_1[2, 1] = \sum_{k=1}^{n} a_{j^*k} C[m(k - 1) + 2, 1] - a_{j^*j} \frac{z_1^\top C_2 B z_1}{z_1^\top B z_1},$$

where $C_2 \in \mathbb{R}^{m \times m}$ satisfies

$$C_2 = \begin{bmatrix} 0 & C[m(j^* - 1) + 1, 1] & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & C[mj^*, 1] & 0 & \ldots & 0 \end{bmatrix}.$$  

As a result,

$$\frac{1}{M_1 r_1^2} \frac{(D_1[1, 1](B_1^{-1})^*[1, 2] - D_1[2, 1](B_1^{-1})^*[1, 1])^2}{(B_1^{-1})^*[1, 1]}$$

$$= \frac{(D_1[1, 1](l_2 - r_1^2 q_1 q_2 M_1) - D_1[2, 1](r_1^2 q_2^2 M_1 + l_1))^2}{M_1 r_1^2 (r_1^2 q_2^2 M_1 + l_1)}$$

$$\rightarrow q_2^2 \left( H_1 \left( \frac{q_1}{q_2} \right) \frac{q_1}{q_2} + H_2 \left( \frac{q_1}{q_2} \right) \right)^2 \quad (48)$$

as $r_1 \to +\infty$. Therefore, if

$$H_1 \left( \frac{q_1}{q_2} \right) \frac{q_1}{q_2} + H_2 \left( \frac{q_1}{q_2} \right) \neq 0, \quad (49)$$

then (45) will follow directly from (47) and (48) by letting $r_1 > N(q_1, q_2)$ for some sufficiently large number $N(z_1)$.
So, the remainder is to show that there is a \( x \in \mathbb{R} \) such that

\[
H_1(x)x + H_2(x) \neq 0,
\]

which is equivalent to

\[
- a_{j^*j^*} (x, 1, 0, \ldots, 0)(x C_1 + C_2)B(x, 1, 0, \ldots, 0)^\tau + x \sum_{k=1}^n a_{j^*k}C[m(k - 1) + 1, 1]
\]

\[
+ \sum_{k=1}^n a_{j^*k}C[m(k - 1) + 2, 1] \neq 0.
\]

If (50) fails, then the coefficient of \( x^3 \) of

\[
(x, 1, 0, \ldots, 0)B(x, 1, 0, \ldots, 0)^\tau (H_1(x)x + H_2(x))
\]

is

\[
B[1, 1] \cdot \sum_{k \neq j^*} a_{j^*k}C[m(k - 1) + 1, 1] = 0,
\]

which contradicts to \( C \notin \mathcal{P}_1 \). So, (49) holds if \( \frac{q_1}{q_2} = x \).

We now can conclude that all \( v_1 = r_1(q_1, q_2, 0, \ldots, 0)^\tau \) with \( q_1^2 + q_2^2 = 1, q_2 \neq 0, \frac{q_1}{q_2} = x \) and \( r_1 > N(q_1, q_2) \) will result in (45). Note that \( C \notin \mathcal{P}_1 \), by Corollary 4.1 again, there always exists some \( v_1 \) fulfilling both \( D \notin \mathcal{P}_0 \) and (45), which means \( K \) cannot be a semi-positive definite matrix.

\[ \square \]

### Appendix C

In this appendix, we prove Theorem 2.5 and Remark 2.6(ii).

**Proof of Theorem 2.5** (i) We verify

\[
\|\tilde{\Theta}_k\| \xrightarrow{a.s.} 0 \text{ and } \|\tilde{\Theta}_k\|_{L^2} \xrightarrow{k \to +\infty} 0
\]
separately under Assumptions A3’ and A3”.

Case 1: Consider the case where Assumption A3’ holds. Since

\[ \tilde{\Theta}_{k+1} = (A \otimes I_m)(I_{mn} - F_k)\tilde{\Theta}_k + (A \otimes I_m)L_kV_k, \]  

(51)

by denoting \( B = A^T A \), Assumption A2’ shows

\[
E \tilde{\Theta}_{k+1}^\tau\tilde{\Theta}_{k+1} \\
= E \tilde{\Theta}_k^\tau(I_{mn} - F_k)(B \otimes I_m)(I_{mn} - F_k)\tilde{\Theta}_k \\
+ E[V_k^\tau L_k^\tau(B \otimes I_m)L_kV_k] \\
\leq E \tilde{\Theta}_k^\tau(I_{mn} - F_k)(B \otimes I_m)(I_{mn} - F_k)\tilde{\Theta}_k \\
+ \frac{nM}{(k + 1)^{2\beta}}. \]  

(52)

Note that \( \tilde{\Theta}_k \in \mathcal{F}_k \), by Lemma 3.3 with \( h = 1, \)

\[
E \left[ \tilde{\Theta}_k^\tau(I_{mn} - F_k)(B \otimes I_m)(I_{mn} - F_k)\tilde{\Theta}_k \right] \\
= E \left[ \tilde{\Theta}_k^\tau(I_{mn} - F_k)(B \otimes I_m)(I_{mn} - F_k)\tilde{\Theta}_k | \mathcal{F}_{k-1} \right] \\
= E \left[ \tilde{\Theta}_k^\tau \tilde{\Theta}_k \right] E \left[ (I_{mn} - F_k)(B \otimes I_m)(I_{mn} - F_k) | \mathcal{F}_{k-1} \right] \\
\leq E \left[ \tilde{\Theta}_k^\tau \tilde{\Theta}_k \right] \left( 1 - \frac{s}{(k + 1)^{\beta}} \right) \lambda_{\min} \left( \sum_{i=1}^{n} E \left[ \frac{2\phi_{k,i}\phi_{k,i}^\tau}{1 + \|\phi_{k,i}\|^2} - \frac{1}{(k + 1)^{2\beta}} \left( \frac{\phi_{k,i}\phi_{k,i}^\tau}{1 + \|\phi_{k,i}\|^2} \right)^2 \right] \right) \\
\leq E \left[ \tilde{\Theta}_k^\tau \tilde{\Theta}_k \right] \left( 1 - \frac{s}{(k + 1)^{\beta}} \right) \lambda_{\min} \left( \sum_{i=1}^{n} E \left[ \frac{\phi_{k,i}\phi_{k,i}^\tau}{1 + \|\phi_{k,i}\|^2} \right] \right). \]  

(53)

For \( \alpha \) and \( c \) defined in Assumption A3’, (52)–(53) yield

\[
E \tilde{\Theta}_{k+1}^\tau\tilde{\Theta}_{k+1} \leq \left( 1 - \frac{scn}{(k + 1)^{\alpha+\beta}} \right) E \tilde{\Theta}_k^\tau\tilde{\Theta}_k + \frac{nM}{(k + 1)^{2\beta}}. \]

So, [13, Lemma 4.2] implies

\[
\limsup_{k \to +\infty} k^{\beta-\alpha} E \tilde{\Theta}_k^\tau\tilde{\Theta}_k \leq \frac{M}{sc}. \]  

(54)

Now, we prove the strong consistency. Since \( \tilde{\Theta}_k | \mathcal{F}_k \) \( \tilde{\Theta}_k \in \mathcal{F}_k \mapsto \sigma\{\Phi_j, V_l, 0 \leq j \leq k, 0 \leq l \leq k - 1\} \), similar to (52)–(53), (51) infers

\[
E \left[ \tilde{\Theta}_{k+1}^\tau\tilde{\Theta}_{k+1} | \mathcal{F}_k \right] \leq \tilde{\Theta}_k^\tau\tilde{\Theta}_k + \frac{nM}{(k + 1)^{2\beta}}. \]  

(55)
By using Lemma 3.2 and (51) immediately yields
\[
\lim_{{k \to +\infty}} \tilde{\Theta}_{{k+1}} = 0, \text{ a.s.}
\]

**Case 2:** Let Assumption A3” hold. Denote
\[
\Gamma_k \triangleq (A \otimes I_m)L_k V_k V_k^T L_k^T (A^T \otimes I_m)
\]
\[
\Pi(k, i) \triangleq \Pi_{j=k}^{k-i+1} (A \otimes I_m)(I_{mn} - F_k)
\]
Then, (51) together with Assumption A3”(ii) deduces
\[
E[\tilde{\Theta}_{k+h}] = E[\Pi(k + h - 1, h)\tilde{\Theta}_k \Pi(k + h - 1, h)^T] + \sum_{{i=0}}^{h-1} E[\Pi(k + h - 1, i)\Gamma_{{k+i-1}} \Pi(k + h - 1, i)^T],
\]
and hence
\[
E[\tilde{\Theta}_{k+h} \tilde{\Theta}_{k+h}] = tr(E[\tilde{\Theta}_{k+h} \tilde{\Theta}_{k+h}])
\]
\[
= E[\tilde{\Theta}_k \Pi(k + h - 1, h)^T \Pi(k + h - 1, h)\tilde{\Theta}_k] + \sum_{{i=0}}^{h-1} tr(E[\Pi(k + h - 1, i)\Gamma_{{k+i-1}} \Pi(k + h - 1, i)^T])
\]
\[
\leq E[\tilde{\Theta}_k \Pi(k + h - 1, h)^T \Pi(k + h - 1, h)\tilde{\Theta}_k] + \sum_{{i=0}}^{h-1} E\left[ \prod_{{j=k+i-1}}^{k+h-1} \lambda_{\max}\left((I_{mn} - F_j)(B \otimes I_m)(I_{mn} - F_j)\right) \cdot \lambda_{\max}(B \otimes I_m) \lambda_{\max}(L_k^T V_{k+h-i-1} L_{k+h-i-1}) \cdot V_{k+h-i-1}^T V_{k+h-i-1} \right]
\]
\[
\leq E[\tilde{\Theta}_k \Pi(k + h - 1, h)^T \Pi(k + h - 1, h)\tilde{\Theta}_k] + \frac{hnM}{(k+1)^{2\beta}}.
\]

Similar to (53)–(55), by applying Lemma 3.3 and Assumption A3”(i), one has
\[
E[\tilde{\Theta}_k \Pi(k + h - 1, h)^T \Pi(k + h - 1, h)\tilde{\Theta}_k] \leq \left(1 - \frac{shnc}{(k+1)^{\alpha+\beta}}\right) \frac{hnM}{(k+1)^{2\beta}}.
\]
and finally can obtain
\[
\left\{
\begin{align*}
E[\tilde{\Theta}_{k+h} \tilde{\Theta}_{k+h}] &\leq \frac{1 - \frac{shnc}{(k+1)^{\alpha+\beta}}}{E[\tilde{\Theta}_k \tilde{\Theta}_k] + \frac{hnM}{(k+1)^{2\beta}}}, \\
E[\tilde{\Theta}_{k+h} \tilde{\Theta}_{k+h} | G_k] &\leq \frac{shnc}{(k+1)^{\alpha+\beta}} + \frac{hnM}{(k+1)^{2\beta}}.
\end{align*}
\right.
\]

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So, as the arguments for statement (i), given any \( k \geq 0 \),
\[
\begin{align*}
\limsup_{j \to +\infty} (k + hj)^{\beta - \alpha} E \tilde{\Theta}_{k+hj}^r \tilde{\Theta}_{k+hj} \leq \frac{M}{\pi e}.
\end{align*}
\tag{56}
\]

The result is thus proved by taking \( k = 0, \ldots, h - 1 \).

(ii) The mean-square convergence rate has already been derived by (54) and (56). The rest part is devoted to computing the convergence rate of the strong consistency under Assumption \( \text{A3}' \). A similar analysis will lead to the same conclusion under Assumption \( \text{A3}'' \).

For every \( \varepsilon \in (0, \beta - \alpha) \), we first use an induction method to prove that for all \( j \in [1, l+1] \),
\[
\lim_{k \to +\infty} k^j (\beta - \alpha - \varepsilon) E(\tilde{\Theta}_k^r \tilde{\Theta}_k^r)^j = 0.
\tag{57}
\]

Since (57) is obviously true for \( j = 1 \) by (54), we assume that (57) holds for all \( j \leq k_0 \) with some \( k_0 \in [1, l] \). Now, check (57) for \( j = k_0 + 1 \). Calculate

\[
\tilde{\Theta}_{k+1}^r \tilde{\Theta}_{k+1}^r = \tilde{\Theta}_k^r (I_{mn} - F_k) (B \otimes I_m) (I_{mn} - F_k) \tilde{\Theta}_k^r + V_k L_k (B \otimes I_m) L_k V_k + 2 \tilde{\Theta}_k^r (I_{mn} - F_k) (B \otimes I_m) L_k V_k \triangleq H_{k,1} + H_{k,2} + 2H_{k,3},
\]

therefore,
\[
(\tilde{\Theta}_{k+1}^r \tilde{\Theta}_{k+1}^r)^{k_0+1} = \sum_{i_1 + i_2 + i_3 = k_0+1} 2^{i_3} C_{k_0+1}^{i_1} C_{k_0+1}^{i_2} C_{k_0+1}^{i_3} H_{k,1}^{i_1} H_{k,2}^{i_2} H_{k,3}^{i_3}.
\]

We estimate \( E(\tilde{\Theta}_{k+1}^r \tilde{\Theta}_{k+1}^r)^{k_0+1} \) by considering the following three cases.

\textit{Case 1:} \( i_2 + \frac{i_3}{2} \geq 1 \). Then, \( i_1 + \frac{i_3}{2} \leq k_0 \leq l \), and by the induction hypothesis,
\[
\limsup_{k \to +\infty} k^{(i_1 + \frac{i_3}{2}) (\beta - \alpha - \varepsilon)} (E(\tilde{\Theta}_k^r \tilde{\Theta}_k^r)^{i_1 + \frac{i_3}{2}} \leq \limsup_{k \to +\infty} k^{(i_1 + \frac{i_3}{2}) (\beta - \alpha - \varepsilon)} (E(\tilde{\Theta}_k^r \tilde{\Theta}_k^r)^{i_1 + \frac{i_3}{2}} = 0.
\]
Therefore, (12) shows

\[
|E\left[H_{k,1}^{i_1}H_{k,2}^{i_2}H_{k,3}^{i_3}\right]| \leq \frac{E\left[\tilde{\Theta}_k^\tau \tilde{\Theta}_k^{i_1+i_2} \cdot E\left[(V_k^\tau V_k)^{i_2+\frac{i_3}{2}}|\Phi_j, 0 \leq j \leq k\right]\right]}{(k+1)^{2i_2+i_3}}
\]

\[= o \left((k+1)^{-\left(i_1+i_2+i_3\right)}\right)
\]

\[= o \left((k+1)^{-(k_0+1)(\beta-\alpha-\varepsilon)-\beta-\varepsilon}\right).
\]

**Case 2:** \(i_2 = 0, i_3 = 1\). Since

\[E\left[V_k|\Phi_j, V_i, 0 \leq j \leq k, 0 \leq i \leq k-1\right] = 0,
\]

it immediately follows that

\[E\left[H_{k,1}^{i_1}H_{k,2}^{i_2}H_{k,3}^{i_3}\right] = E\left[H_{k,1}^{k_0}\right]
\]

\[= E\left[E\left[H_{k,1}^{k_0}H_{k,3}|\Phi_j, V_i, 0 \leq j \leq k, 0 \leq i \leq k-1\right]\right] = 0.
\]

**Case 3:** \(i_2 = i_3 = 0\). Similar to (13),

\[E\left[H_{k,1}^{i_1}H_{k,2}^{i_2}H_{k,3}^{i_3}\right] = E\left[H_{k,1}^{k_0+1}\right] \leq E\left[\left(\tilde{\Theta}_k^\tau \tilde{\Theta}_k\right)^{k_0}H_{k,1}\right]
\]

\[= E\left[\left(\tilde{\Theta}_k^\tau \tilde{\Theta}_k\right)^{k_0}E\left[H_{k,1}|\mathcal{F}_{k-1}\right]\right] \leq \left(1 - \frac{scn}{(k+1)^{\alpha+\beta}}\right)E\left(\tilde{\Theta}_k^\tau \tilde{\Theta}_k\right)^{k_0+1}.
\]

So, combining Cases 1–3, we deduce that as \(k \to +\infty\),

\[E\left(\tilde{\Theta}_k^{\tau \arctan} \tilde{\Theta}_k\right)^{k_0+1} \leq \left(1 - \frac{scn}{(k+1)^{\alpha+\beta}}\right)E\left(\tilde{\Theta}_k^\tau \tilde{\Theta}_k\right)^{k_0+1} + o\left((k+1)^{-(k_0+1)(\beta-\alpha-\varepsilon)-\beta-\varepsilon}\right).
\]

By [13] Lemma 4.2 again,

\[
\lim_{k \to +\infty} k^{(k_0+1)(\beta-\alpha-\varepsilon)}E\left(\tilde{\Theta}_k^\tau \tilde{\Theta}_k\right)^{k_0+1} = 0,
\]

which means assertion (17) is true for all \(j \in [1, l+1]\).

Next, for any \(\varepsilon \in (0, \beta - \alpha - \frac{1}{l+1})\), select some \(\varepsilon_0 \in (0, \beta - \alpha - \frac{1}{l+1} - \varepsilon)\). So, \(l+1 > (\beta - \alpha - \varepsilon - \varepsilon_0)^{-1}\). By Markov’s inequality, for any \(\delta > 0\),

\[P\left((k+1)^{\varepsilon \arctan} \tilde{\Theta}_k^\tau \tilde{\Theta}_k > \delta\right) \leq \frac{(k+1)^{\varepsilon(l+1)E\left(\tilde{\Theta}_k^\tau \tilde{\Theta}_k\right)^{l+1}}{\delta^{l+1}} = o((k+1)^{-(l+1)(\beta-\alpha-\varepsilon-\varepsilon_0)}).
\]
which implies
\[ \sum_{k=1}^{+\infty} P \left( (k+1)^{\varepsilon} \tilde{\Theta}_k \Theta_k > \delta \right) < +\infty. \]
This together with the Borel-Cantelli lemma yields
\[ P \left( (k+1)^{\varepsilon} \tilde{\Theta}_k \Theta_k > \delta, \text{ i.o.} \right) = 0. \]
So, (13) is true by noting that \( \delta \) can take arbitrary values. \( \square \)

**Proof of Remark 2.6(ii).** At first, it is easy to verify
\[ \lambda_{\max}(I_n - A) \leq 2 - 2 \inf_{i \in [1,n]} a_{ii}. \]
Taking \( \varepsilon = \frac{\inf_{i \in [1,n]} a_{ii}}{2} \) in [33, Lemmas 5.5, 5.10] shows that for any \( k \geq 0 \), if \( l \) is sufficiently large,
\[ B_j^r B_j \leq (1 - \varepsilon)(B_j^r + B_j), \quad j \geq 1, \]  (58)
where
\[ B_j \triangleq F_{k+lh+j-1} + (I_{mn} - A \otimes I_m)(I_{mn} - F_{k+lh+j-1}). \]

Fix \( k, l \geq 0 \) and define \( I_j(A) \triangleq I_{mn} - F_{k+lh+j-1} \) in [15]. Then, (58) and [33] Lemmas 5.5, 5.10 yield
\[
\lambda_{\min}(E \left[ I_{mn} - \psi_h^\tau \psi_h | F_{k-1} \right]) \\
\geq \varepsilon \frac{0.5}{(1 + 4(1 - \varepsilon)h)^2} \cdot \frac{(k + (l + 1)h)^{\beta}}{2 + \lambda(G)} \cdot \lambda(G) h \\
\cdot \lambda_{\min}\left( E \left[ \frac{1}{nh} \sum_{i=1}^{n} \sum_{j=k+lh}^{k+lh+h-1} \phi_{j,i} \phi_{j,i}^\tau \left| F_{k-1} \right. \right] \right) \\
\geq \varepsilon \frac{(k + lh)^{\beta}}{2(1 + 4h)^2} \cdot \frac{\lambda(G)}{4n} \cdot \lambda_{\min} \left( E \left[ \sum_{j=1}^{h} \sum_{i=1}^{n} (I_m - A_{j,i}) | F_{k-1} \right] \right) \\
\geq \frac{\inf_{i \in [1,n]} a_{ii}}{32n(1 + 4h)^2} \lambda(G) \cdot \lambda_{\min} \left( E \left[ \sum_{j=1}^{h} \sum_{i=1}^{n} (I_m - A_{j,i}) | F_{k-1} \right] \right),
\]
where \( l \) is sufficiently large. This remark is thus proved by taking 
\[
    s = \frac{\inf_{i \in [1,n]} a_i}{32n(1+4h)^2} \lambda(G).
\]
\( \square \)