Critical behavior of two-dimensional cubic and $MN$ models in the five-loop renormalization-group approximation

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Abstract

The critical thermodynamics of the two-dimensional $N$-vector cubic and $MN$ models is studied within the field-theoretical renormalization-group (RG) approach. The $\beta$ functions and critical exponents are calculated in the five-loop approximation and the RG series obtained are resummed using the Borel-Leroy transformation combined with the generalized Padé approximant and conformal mapping techniques. For the cubic model, the RG flows for various $N$ are investigated. For $N = 2$ it is found that the continuous line of fixed points running from the $XY$ fixed point to the Ising one is well reproduced by the resummed RG series and an account for the five-loop terms makes the lines of zeros of both $\beta$ functions closer to each another. For the cubic model with $N \geq 3$, the five-loop contributions are shown to shift the cubic fixed point, given by the four-loop approximation, towards the Ising fixed point. This confirms the idea that the existence of the cubic fixed point in two dimensions under $N > 2$ is an artifact of the perturbative analysis. For the quenched dilute $O(M)$ models ($MN$ models with $N = 0$) the results are compatible with a stable pure fixed point for $M \geq 1$. For the $MN$ model with $M, N \geq 2$ all the non-perturbative results are reproduced. In addition a new stable fixed point is found for moderate values of $M$ and $N$.

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I. INTRODUCTION

The two-dimensional model with $N$-vector order parameter and cubic anisotropy is known to have a rich phase diagram; it contains, under different values of $N$ and of the anisotropy parameter, the Ising-like and Kosterlitz-Thouless critical points, lines of the first-order phase transitions, and the line of the second-order transitions with continuously varying critical exponents (see, e.g. [1–3] for review). This model is related to many other familiar models in various particular cases, while for $N \to 0$ it describes the critical behavior of two-dimensional weakly disordered Ising systems. Moreover, exact solutions are known for the two-dimensional cubic model in the several limits such as an Ising decoupled limit, the limit of extremely strong anisotropy for $N > 2$ [2,4] and the replica limit $N \to 0$ [2,5,6]. The correspondence, in particular regions of the phase diagram, with the $N$-color Ashkin-Teller models, discrete cubic models, and planar model with fourth order anisotropy give further informations about the critical behavior. All these issues are reviewed in Ref. [3] and we will not repeat them here. These features make the two-dimensional $N$-vector cubic model a convenient and, perhaps, unique testbed for evaluation of the analytical and numerical power of perturbative methods widely used nowadays in the theory of critical phenomena. The field-theoretical renormalization-group (RG) approach in physical dimensions is among of them.

Recently, the critical behavior of the two-dimensional $N$-vector cubic model was explored using the renormalization-group technique in the space of fixed dimensionality [3]. The four-loop expansions for the $\beta$-functions and critical exponents were calculated and analyzed using the Borel transformation combined with the conformal mapping and Padé-approximant techniques as a tool for resummation of the divergent RG series. The most part of predictions obtained within the renormalization group approach turned out to be in accord with known exact results. At the same time, some findings were quite new. In particular, for $N > 2$ the resummed four-loop RG expansions for $\beta$-functions were found to yield a cubic fixed point with (almost) marginal stability; this point does not correspond to any of the critical asymptotics revealed by exact methods ever applied. Although the stability properties of the cubic fixed point look very similar to those of its Ising counterpart, these points were found to lie too far from each other (for moderate $N$) to consider the distance between them as a splitting caused by the limited accuracy of the RG approximation employed.

It is worthy to note that this situation is quite different from what we have in three dimensions. Indeed, for the three-dimensional cubic model the structure of the RG flow diagram is known today with a rather high accuracy. Recent five-loop [7–9] and six-loop [10,11] RG calculations certainly confirmed that for $N > 2$ the cubic fixed point does not merge with any other fixed point and governs the specific anisotropic mode of critical behavior, distinguishable from the Ising and Heisenberg modes (see, e.g. [12]).

It is very desirable, therefore, to clear up to what extent the location of the cubic fixed point in two dimensions is sensitive to the order of the RG calculations and, more generally, whether this point really exists at the flow diagram or its appearance is the approximation artifact caused by the finiteness of the perturbative series and by an ignorance of the confluent singularities significant in two dimensions [13–16].

Of prime interest is also the situation with the line of fixed points that should run, under $N = 2$, from the Ising fixed point to the XY one. Within the four-loop approximation, the
zeros of $\beta$-functions for the $O(N)$-symmetric and anisotropic coupling constants form two
lines that for $N = 2$ are practically parallel to each other and separated by the distance that
is smaller than the error bar appropriate to the working approximation [3]. Will the higher-
order contributions keep these two lines parallel? Will an account for the higher-order terms
further diminish the distance between these lines or their splitting should be attributed, at
least partially, to the influence of the singular terms just mentioned?

To answer the above questions, it is necessary to analyze the critical behavior of the two-
dimensional cubic model in the higher perturbative orders. Recently, the renormalization-
group expansions for the two-dimensional $O(N)$-symmetric model were obtained within the
five-loop approximation [15]. In the course of this study, all the integrals corresponding
to the five-loop four-leg and two-leg Feynman graphs have been evaluated. This makes it
possible to investigate the critical thermodynamics of anisotropic two-dimensional models
with several couplings in the five-loop approximation. In this paper, such an investigation
will be carried out for the two-dimensional $N$-vector model with cubic anisotropy.

For studying the effect of cubic anisotropies one usually considers the $\phi^4$ theory [17,18]:

$$
\mathcal{H} = \int d^d x \left\{ \frac{1}{2} \sum_{i=1}^{N} \left[ (\partial_{\mu} \phi_i)^2 + r \phi_i^2 \right] + \frac{1}{4!} \sum_{i,j=1}^{N} \left( u_0 + v_0 \delta_{ij} \right) \phi_i^2 \phi_j^2 \right\},
$$

(1.1)
in which the added cubic term breaks explicitly the $O(N)$ invariance leaving a residual
discrete cubic symmetry given by the reflections and permutations of the field components.
This term favors the spin orientations towards the faces or the corners of an $N$-dimensional
hypercube for $v_0 < 0$ or $v_0 > 0$ respectively. In two dimensions the effect of anisotropy is
particularly important: systems possessing continuous symmetry do not exhibit conventional
long-range order at finite temperature, while models with discrete symmetry do undergo
phase transitions into conventionally ordered phase.

In general, the model (1.1) has four fixed points: the trivial Gaussian one, the Ising
one in which the $N$ components of the field decouple, the $O(N)$-symmetric and the cubic
fixed points. The Gaussian fixed point is always unstable, and so is the Ising fixed point for
$d > 2$ [17]. Indeed, in the latter case, it is natural to interpret Eq. (1.1) as the Hamiltonian
of $N$ Ising-like systems coupled by the $O(N)$-symmetric term. But this interaction is the
sum of the products of the energy operators of the different Ising systems. Therefore, at the
Ising fixed point, the crossover exponent associated with the $O(N)$-symmetric quartic term
is given by the specific-heat critical exponent $\alpha_I$ of the Ising model, independently of $N$
(for $N = 0$ this argument is equivalent to the Harris criterion [19]). Since $\alpha_I$ is positive for all
d $> 2$ the Ising fixed point is unstable. Obviously, in two dimensions this argument only
tells us that the crossover exponent at this fixed point vanishes. Higher order corrections
to RG equations may lead either to a marginally stable fixed point [20] or to a line of fixed
points. It was argued that for $N \geq 3$ the former possibility is realized, while for $N = 2$ the
latter one holds (see Ref. [1–3] and references therein).

The stability properties of the fixed points depend on $N$. For sufficiently small values of
$N$, $N < N_c$, the $O(N)$-symmetric fixed point is stable and the cubic one is unstable. For
$N > N_c$ and $d > 2$, the opposite is true: the renormalization-group flow is driven towards
the cubic fixed point, which now describes the generic critical behavior of the system. At
$N = N_c$, the two fixed points should coincide for $d > 2$. At $d = 2$, it is expected that
$N_c = 2 \ [21]$ and a line of fixed points connecting the Ising and the $O(2)$-symmetric fixed points exists $[1,3]$. A generalization of the Hamiltonian Eq. (1.1) is obtained by considering $N$ coupled $O(M)$ vector models, instead of $N$ Ising models. The resulting Hamiltonian (defining the so called $MN$ model) is $[17,18]$

$$
H_{MN} = \int d^d x \left\{ \sum_{i,a} \frac{1}{2} \left( \partial_\mu \phi_{a,i} \right)^2 + r \phi_{a,i}^2 \right\} + \sum_{ij,ab} \frac{1}{4!} \left( u_0 + u_0 \delta_{ij} \right) \phi_{a,i}^2 \phi_{b,j}^2 ,
$$

(1.2)

where $a,b = 1, \ldots M$ and $i,j = 1, \ldots N$. The continuous $O(MN)$ symmetry is explicitly broken by the $v_0$ term to $C_N \times O(M)$ where $C_N$ is the discrete group of permutations of $N$ elements. The presence of such discrete symmetry allows for a finite temperature phase transition under general values of $N$ and $M$ (differently from models with $v_0 = 0$). For $M = 1$ it reduces to the Hamiltonian of the cubic model, but it has physical applications for $M \neq 1$ too. For $N \to 0$ and generic $M$, under the condition $u_0 < 0$ and $v_0 > 0$, it describes the critical behavior of quenched dilute $O(M)$ models $[18,22]$. For $M = 2$ and $N = 2$ ($N = 3$) and $v_0 > 0$, it is relevant for the second-order phase transition in planar (isotropic) antiferromagnets with complicated ordering as the three dimensional sinusoidal magnets TbAu$_2$, DyC$_2$, type-II antiferromagnets TbAs, TbP (type-III antiferromagnets K$_2$IrCl$_6$, sinusoidal TbD$_2$) and many others $[23]$. For $M = N = 2$ and $v_0 < 0$ an exact mapping $[24]$ relates the Hamiltonian (1.2) with the $O(2) \times O(2)$ symmetric one, describing the critical behavior of frustrated antiferromagnets with non-collinear order $[25,18]$. For $N \to \infty$ it describes $O(M)$ models with constrains $[26]$ (as $O(M)$ models with annealed disorder).

The Hamiltonian (1.2) has been largely studied in the framework of $\epsilon$ expansion ($\epsilon = 4-d$) $[27,28]$ and directly in three dimensions $[28-32]$, mainly to understand the critical behavior of the antiferromagnets quoted above. Much less studies were devoted to the two dimensional case for $M \neq 1$, since its main features can be understood by non-perturbative arguments. In fact, three fixed points always exist in the RG flow for any $N$ and $M$: the always unstable Gaussian ($u_0 = v_0 = 0$), the $O(M)$ (with $u_0 = 0$) and the $O(NM)$ (with $v_0 = 0$). Their stability properties are known from exact arguments. As for the Ising fixed point in the cubic model, the stability of the $O(M)$ fixed point is governed by the specific-heat critical exponent $\alpha_M$ of the $O(M)$ model. Being (in two dimensions) $\alpha_M$ negative for all $M > 1$, the $O(M)$ fixed point is stable for physical values of $M$. An exact argument ensures that the $O(NM)$ fixed point is unstable when $NM > N_c = 2 \ [21,33]$, thus it is always unstable for physical values (apart from the replica limit $N \to 0$, but in this case it is not reachable from the physical initial condition with $u_0 > 0$). In the framework of $\epsilon$ expansion another fixed point (called mixed) exists $[17]$. It is the generalization to $M \neq 1$ of the cubic one. Its analytic continuation to $d=2$ is expected to be in the region $u_0 < 0$ and $v_0 > 0$ for $N,M \geq 2$ and in the $u_0,v_0 > 0$ region (unphysical) in the replica limit. It is expected to be unstable in both the cases.

Anyway, these arguments still do not completely characterize the RG flow of the $MN$-model. Indeed for $N,M \geq 2$ and $v_0 < 0$ the situation is more controversial. The $\epsilon$ expansion indicates that no fixed point exists and consequently it was longly believed that no continuous transition can take place in these systems. Nevertheless, for $N = M = 2$ the quoted mapping between the $MN$ and the $O(2) \times O(2)$ models shows the presence of a fixed point (found in
the $O(2) \times O(2)$ model at four [34] and five loops [35,36]) in the region $\nu_0 < 0$. Consequently the transition should be second order in the chiral universality class. The natural question arising is whether the fixed point found at $N = M = 2$ is a peculiar feature or it persists to some larger values of $N$ and $M$ (not too large, since for $N \gg 1$, at fixed $M$, we expect that the $\epsilon$ expansion can be smoothly continued up to $d = 2$). An answer to this question can be found within higher order perturbative expansion.

The paper is organized as follows. In Section II the five-loop contributions to the renormalization-group functions are calculated and the singularities of Borel transforms of the renormalization-group series are discussed for the general $MN$ model. Section III is devoted to the analysis of the critical behavior of the model in the five-loop approximation. The existence of a cubic fixed point for $N \geq 3$ is investigated. The critical behavior of quenched random $O(M)$ models is considered. Attention is paid to the reproducibility of a continuous line of fixed points, predicted earlier for the planar ($N = 2$, $M = 1$) model, in the framework of the field-theoretical RG approach. The critical exponents along this line are evaluated in this section as well. Finally, the critical behavior for $N, M \geq 2$ is addressed. In Section IV we summarize the main results obtained and make concluding remarks.

II. FIXED DIMENSION PERTURBATIVE EXPANSIONS

A. Renormalization of the theory

The fixed-dimension field-theoretical approach represents a powerful procedure in the study of the critical properties of three-dimensional systems belonging to the $O(N)$ and more complicated universality classes (see, e.g., Ref. [18,37,38]). In this approach one performs an expansion in powers of appropriately defined zero-momentum quartic couplings and renormalizes the theory by a set of zero-momentum conditions for the (one-particle irreducible) two-point and four-point correlation functions. For the $MN$ model they read:

$$
\Gamma^{(2)}_{ab,ij}(p) = \delta_{ai,bj} Z_{\phi}^{-1} \left[ m^2 + p^2 + O(p^4) \right],
$$

(2.1)

where $\delta_{ai,bj} = \delta_{ab} \delta_{ij}$, and

$$
\Gamma^{(4)}_{abcd,ijkl}(0) = Z_{\phi}^{-2} m^2 \left[ \frac{u}{3} (\delta_{ai,bj} \delta_{ck,dl} + \delta_{ai,ck} \delta_{bj,dl} + \delta_{ai,dl} \delta_{bj,ck}) + \frac{v}{3} \delta_{ij} \delta_{ik} (\delta_{ab} \delta_{cd} + \delta_{ac} \delta_{bd} + \delta_{ad} \delta_{bc}) \right].
$$

(2.2)

They relate the inverse correlation length (mass) $m$ and the zero-momentum quartic couplings $u$ and $v$ to the corresponding Hamiltonian parameters $r$, $u_0$, and $v_0$:

$$
u_0 = m^2 u Z_u Z_{\phi}^{-2}, \quad v_0 = m^2 v Z_v Z_{\phi}^{-2}.
$$

(2.3)

In addition, one introduces the function $Z_t$ defined by the relation

$$
\Gamma^{(1,2)}_{ai,bj}(0) = \delta_{ai,bj} Z_t^{-1},
$$

(2.4)

where $\Gamma^{(1,2)}$ is the (one-particle irreducible) two-point function with an insertion of $\frac{1}{2} \phi^2$. 

5
From the perturbative expansion of the correlation functions $\Gamma^{(2)}$, $\Gamma^{(4)}$, and $\Gamma^{(1,2)}$ and the above relations, one derives the functions $Z_\phi(u, v)$, $Z_u(u, v)$, $Z_v(u, v)$, and $Z_t(u, v)$ as double expansions in $u$ and $v$.

The fixed points of the theory are given by the common zeros of the $\beta$-functions

$$
\beta_u(u, v) = m \frac{\partial u}{\partial m}\bigg|_{u_0, v_0}, \quad \beta_v(u, v) = m \frac{\partial v}{\partial m}\bigg|_{u_0, v_0}.
$$

(2.5)

The stability properties of the fixed points are controlled by the eigenvalues $\omega_i$ of the matrix

$$
\Omega = \begin{pmatrix}
\frac{\partial \beta_u(u, v)}{\partial u} & \frac{\partial \beta_u(u, v)}{\partial v} \\
\frac{\partial \beta_v(u, v)}{\partial u} & \frac{\partial \beta_v(u, v)}{\partial v}
\end{pmatrix},
$$

(2.6)

computed at the given fixed point: a fixed point is stable if both eigenvalues are positive. The eigenvalues $\omega_i$ are related to the leading scaling corrections, which vanish as $\xi^{-\omega_i} \sim |t|^\Delta_i$ where $\Delta_i = \nu \omega_i$. If $\omega_i$ has a non-vanishing imaginary part, the approaching to the FP is spiral-like and the FP is called focus [35].

One also introduces the functions

$$
\eta_{\phi,t}(u, v) = \frac{\partial \ln Z_{\phi,t}}{\partial \ln m}\bigg|_{u_0, v_0} = \beta_u \frac{\partial \ln Z_{\phi,t}}{\partial u} + \beta_v \frac{\partial \ln Z_{\phi,t}}{\partial v},
$$

(2.7)

in terms of that, the critical exponents are obtained as

$$
\eta = \eta_{\phi}(u^*, v^*),
$$

$$
\nu = [2 - \eta_{\phi}(u^*, v^*) + \eta_t(u^*, v^*)]^{-1},
$$

$$
\gamma = \nu(2 - \eta),
$$

(2.8) \quad (2.9) \quad (2.10)

where $(u^*, v^*)$ is the location of the stable fixed point.

B. The five loop series

We calculate the two-dimensional perturbative RG functions of the $MN$ model in the zero-momentum massive renormalization approach introduced before up to five loops. Corresponding contributions to the functions Eqs. (2.1), (2.2), and (2.4) are given by 162 diagrams for the four-point correlators and by 26 graphs for the two-point one [39]. We handle them with a symbolic manipulation program, which generates the diagrams and computes the symmetry and group factors of each of them. The RG functions are written in terms of the rescaled couplings

$$
u \equiv \frac{8\pi}{3} R_M \bar{\nu},
$$

(2.11)

where $R_K = 9/(8 + K)$.

The obtained series are
\[ \bar{\beta}_\pi = -\bar{\pi} + \bar{\pi}^2 + \frac{2(M + 2)}{(M + 8)} \bar{\pi} \bar{\pi} + \bar{\pi} \sum_{i+j \geq 2} b_{ij}^{(u)} \bar{\pi}^i \bar{\pi}^j, \]  

(2.12)

\[ \bar{\beta}_\nu = -\bar{\nu} + \bar{\nu}^2 + \frac{12}{8 + NM} \bar{\nu} \bar{\nu} + \bar{\nu} \sum_{i+j \geq 2} b_{ij}^{(v)} \bar{\nu}^i \bar{\nu}^j, \]  

(2.13)

\[ \eta_\phi = \sum_{i+j \geq 2} e_{ij}^{(\phi)} \bar{\pi}^i \bar{\nu}^j, \]  

(2.14)

\[ \eta_t = -\frac{2(2 + NM)}{(8 + NM)} \bar{\pi} - \frac{2 + M}{8 + M} \bar{\nu} + \sum_{i+j \geq 2} e_{ij}^{(t)} \bar{\pi}^i \bar{\nu}^j \]  

(2.15)

where

\[ \bar{\beta}_\pi = \frac{3}{16\pi} R^{-1}_{NM} \beta_u, \quad \bar{\beta}_\nu = \frac{3}{16\pi} R^{-1}_{M} \beta_v. \]  

(2.16)

The coefficients \( b_{ij}^{(u)} \), \( b_{ij}^{(v)} \), \( e_{ij}^{(\phi)} \), and \( e_{ij}^{(t)} \) are reported in the Tables I, II, III, and IV. Note that due to the rescaling (2.16) and (2.11), the matrix element of \( \Omega \) are the double of the derivative of \( \bar{\beta} \) with respect to \( \bar{\pi} \) and \( \bar{\nu} \).

We have verified the exactness of our series by the following relations:

(i) \( \bar{\beta}_\pi(\pi, 0), \eta_\phi(\pi, 0) \) and \( \eta_t(\pi, 0) \) reproduce the corresponding functions of the \( O(NM) \)-symmetric model [15,40].

(ii) \( \bar{\beta}_\nu(0, \nu), \eta_\phi(0, \nu) \) and \( \eta_t(0, \nu) \) reproduce the corresponding functions of the \( O(M) \)-symmetric \( \phi^4 \) theory [15,40].

(iii) The following relations hold close to Heisenberg \( O(NM) \) [41,42]

\[ \frac{\partial \eta_{\phi,t}}{\partial \bar{\pi}} \bigg|_{(\pi,0)} = \frac{MN + 2}{M + 2} \frac{M + 8}{MN + 8} \frac{\partial \eta_{\phi,t}}{\partial \bar{\nu}} \bigg|_{(\pi,0)}, \]  

(2.17)

\[ \frac{\partial \bar{\beta}_\pi}{\partial \bar{\pi}} \bigg|_{(\pi,0)} - \frac{\partial \bar{\beta}_\pi}{\partial \bar{\nu}} \bigg|_{(\pi,0)} = \frac{MN + 2}{M + 2} \frac{M + 8}{MN + 8} \frac{\partial \bar{\beta}_\pi}{\partial \bar{\nu}} \bigg|_{(\pi,0)}, \]  

(2.18)

and to the \( O(M) \) fixed points [42]

\[ \frac{\partial \bar{\beta}_\nu}{\partial \bar{\pi}} \bigg|_{(0,\nu)} - \frac{\partial \bar{\beta}_\nu}{\partial \bar{\nu}} \bigg|_{(0,\nu)} = \frac{MN + 8}{M + 8} \frac{\partial \bar{\beta}_\nu}{\partial \bar{\nu}} \bigg|_{(0,\nu)}, \]  

(2.19)

\[ \frac{\partial \eta_\phi}{\partial \bar{\pi}} \bigg|_{(0,\nu)} = \frac{MN + 8}{M + 8} \frac{\partial \eta_\phi}{\partial \bar{\nu}} \bigg|_{(0,\nu)}. \]  

(2.20)

(iv) The following relations hold for \( N = 1 \) and generic \( M \):

\[ \bar{\beta}_\pi(\pi, x - \pi) + \bar{\beta}_\nu(\pi, x - \pi) = \bar{\beta}_\pi(0, x), \]  

\[ \eta_\phi(\pi, x - \pi) = \eta_\phi(0, x), \]  

\[ \eta_t(\pi, x - \pi) = \eta_t(0, x). \]  

(2.21)
(v) For $N = 2$ and $M = 1$, one easily obtains the identities [43,3]

$$\bar{\beta}_\pi(\bar{u} + \frac{5}{3}v, -v) + \frac{5}{3} \bar{\beta}_\pi(\bar{u} + \frac{5}{3}v, -v) = \bar{\beta}_\pi(\bar{u}, v), \quad (2.22)$$

$$\bar{\beta}_\pi(\bar{u} + \frac{5}{3}v, -v) = -\bar{\beta}_\pi(\bar{u}, v),$$

$$\eta_\phi(\bar{u} + \frac{5}{3}v, -v) = \eta_\phi(\bar{u}, v),$$

$$\eta_t(\bar{u} + \frac{5}{3}v, -v) = \eta_t(\bar{u}, v).$$

These relations are also exactly satisfied by our five-loop series. Note that, since the Ising fixed point is $(0, g_I^*)$, and $g_I^*$ is known with high precision [44]

$$g_I^* = 1.7543637(25), \quad (2.23)$$

the above symmetry gives us the location of the cubic fixed point: $(\frac{5}{3}g_I^*, -g_I^*)$.

(vi) In the large-$N$ limit the critical exponents of the mixed fixed point are related to those of the $O(M)$ model: $\eta = \eta_M$ and $\nu = \nu_M$ [26,45] (note that in general the latter equivalence holds only for $M \geq 1$ and $d = 2$, in the general case Fisher renormalization of exponents [46] has to be taken into account [45]). One can easily see that, for $N \to \infty$, $\eta_\phi(u, v) = \eta_\phi(0, v)$, where $\eta_\phi(0, v)$ is the perturbative series that determines the exponent $\eta_M$ of the $O(M)$ model. Therefore, the first relation is trivially true. On the other hand, the second relation $\nu = \nu_M$ is not identically satisfied by the series, and is verified only at the critical point [10].

(vii) For $M = 0$, the $N$-independent series satisfy [47]

$$\bar{\beta}_\pi(\bar{u}, x - \bar{u}) + \bar{\beta}_\pi(\bar{u}, x - \bar{u}) = \bar{\beta}_\pi(x, 0),$$

$$\eta_\phi(\bar{u}, x - \bar{u}) = \eta_\phi(x, 0),$$

$$\eta_t(\bar{u}, x - \bar{u}) = \eta_t(x, 0). \quad (2.24)$$

(viii) For $M \to \infty$ the series, as expected, reduce to

$$\bar{\beta}_\pi = -\bar{u} + \bar{u}^2 + 2\bar{u}v, \quad \bar{\beta}_6 = -\bar{v} + \bar{v}^2,$$

$$\eta_t = -2\bar{v} - \bar{v}, \quad \eta_\phi = 0, \quad (2.25)$$

with two Gaussian and two spherical (i.e. $O(\infty)$) fixed points (as the $O(N) \times O(M)$ model [48]).

(ix) For $N = M = 2$ they agree, according to the exact mapping [24]:

$$\bar{u} = u_{ch} + \frac{v_{ch}}{2}, \quad \bar{v} = -\frac{5}{6} v_{ch}, \quad (2.26)$$

with the five-loop series of the $O(2) \times O(2)$ model [36], written in terms of $u_{ch}$ and $v_{ch}$.

(x) The series reproduce the previous four-loop results for $M = 1$ [3].

C. Resummations of the series and analysis method

The obtained RG series are asymptotic and some resummation procedure is needed in order to extract accurate numerical values for the physical quantities. Exploiting the
property of Borel summability of $\phi^4$ theories in two and three dimensions [49], we resum the divergent asymptotic series by a Borel transformation combined with an analytic extension of the Borel transform to the real positive axis. This extension can be obtained by a Padé approximant or by a conformal mapping [50] which maps the domain of analyticity of the Borel transform onto a circle (see Refs. [50,37] for details).

The conformal mapping method takes advantage of the knowledge of the large order behavior of the perturbative series $F(\nu, z) = \sum_k f_k(z)\nu^k$ [50,37]

$$f_k(z) \sim k! \left( -a(z) \right)^k k^b \left[ 1 + O(k^{-1}) \right] \quad \text{with} \quad a(z) = -1/\bar{\nu}_b(z), \quad (2.27)$$

where $\bar{\nu}_b(z)$ is the singularity of the Borel transform closest to the origin at fixed $z = \bar{v}/\bar{u}$, given by [10,3,30]

$$\frac{1}{\bar{\nu}_b(z)} = - a \left( R_{MN} + R_M z \right) \quad \text{for} \quad 0 < z \quad \text{and} \quad z < -\frac{2N}{N+1} \frac{R_{MN}}{R_M}, \quad (2.28)$$

$$\frac{1}{\bar{\nu}_b(z)} = - a \left( R_{MN} + \frac{1}{N} R_M z \right) \quad \text{for} \quad 0 > z > -\frac{2N}{N+1} \frac{R_{MN}}{R_M},$$

where $a = 0.238659217 \ldots$ [37]. Note that the condition of Borel summability (that coincides with the mean-field boundness condition) is $z > z_1 = -R_{MN}/R_M$ for $\nu < 0$. So in this region, even if the second of Eq. (2.28) takes into account the singularity of the Borel transform closest to the origin, there is another singularity on the real positive axis that makes the series not Borel summable. The situation is the same as for $O(M) \times O(N)$ under $\nu_{ch} > 0$ [51,34,35]. As in the latter case we resum the series even where they are not Borel summable. Although in this case the sequence of approximation is only asymptotic, it should provide reasonable estimates as long as we are taking into account the leading large-order behavior (i.e. as long as $z > z_2 = -\frac{2N}{N+1} \frac{R_{MN}}{R_M}$).

| $i,j$ | $R_{MN}^i R_M^j b_{ij}^{(u)}$ |
|-------|--------------------------------|
| 2,0   | 0.588581 - 0.127593 $M N$     |
| 1,1   | -0.621503 - 0.310751 $M$     |
| 0,2   | 0.144054 - 0.072026 $M$     |
| 3,0   | 0.193909 + 0.204598 $M N + 0.0068909 M^2 N^2$ |
| 2,1   | 1.19188 + 0.595983 $M + 0.0598763 M N + 0.029488 M^2 N$ |
| 1,2   | 0.66132 + 0.410858 $M + 0.0401129 M^2 + 0.00432068 M N + 0.00216029 M^2 N$ |
| 0,3   | 0.313463 + 0.088654 $M + 0.011459 M^2$ |
| 4,0   | -1.157005 - 0.397981 $M N - 0.0273887 M^2 N^2 - 0.000013541 M^3 N^3$ |
| 3,1   | -2.6265 - 1.31325 M - 0.26635 $M N - 0.133175 M^2 N + 0.00011597 M^2 N^2 + 0.00005798 M^3 N^2$ |
| 2,2   | -2.31478 - 1.56359 $M - 0.203103 M^2 - 0.0493674 M N - 0.0244594 M^2 N + 0.00011262 M^3 N$ |
| 1,3   | -0.970048 - 0.704633 $M - 0.10947 M^2 + 0.00016724 M^3 - 0.00381205 M N - 0.00238253 M^2 N - 0.00023825 M^3 N$ |
| 0,4   | -0.166668 - 0.124257 $M - 0.0206675 M^2 - 0.000103042 M^3$ |
| 5,0   | 2.26883 + 0.901262 $M N + 0.0880695 M^2 N^2 + 0.00136005 M^3 N^3 - 6.908710^{-6} M^4 N^4$ |
| 4,1   | 6.58636 + 3.29318 $M + 0.970818 M N + 0.485409 M^2 N + 0.0167572 M^2 N^2 + 0.00837862 M^3 N^2$ |
| 3,2   | 8.02544 + 5.60644 $M + 0.826059 M^2 + 0.359772 M N + 0.221805 M^2 N + 0.0290597 M^3 N + 0.00113766 M^2 N^2$ |
| 2,3   | 5.23732 + 4.0716 $M + 0.774063 M^2 + 0.0219952 M^3 + 0.058081 M N + 0.039797 M^2 N + 0.00542573 M^3 N + 0.00023594 M^4 N$ |
| 1,4   | 1.84567 + 1.48727 $M + 0.302198 M^2 + 0.012745 M^3 + 0.000017524 M^4 + 0.00401992 M N + 0.003472 M^2 N$ |
| 0,5   | 0.278724 + 0.226356 $M + 0.0478936 M^2 + 0.00219097 M^3 - 3.68575 \times 10^{-6} M^4$ |

**Table I.** The coefficients $b_{ij}^{(u)}$, cf. Eq. (2.12).
TABLE II. The coefficients $b^{(v)}_{ij}$, cf. Eq. (2.13).

| $i,j$ | $R^{-1}_{NM}R^{-1}_{MN}b^{(v)}_{ij}$ |
|-------|-------------------------------------|
| 2,0   | $-1.1424 - 0.0720268 MN$            |
| 1,1   | $-1.62169 - 0.310751 M$             |
| 0,2   | $-0.588581 - 0.127593 M$            |
| 3,0   | $1.68536 + 0.162556 MN - 0.00251242 M^2 N^2$ |
| 2,1   | $3.65454 + 0.776108 MN + 0.036274 MN - 0.0011187 M^2 N$ |
| 1,2   | $2.74577 + 0.729746 M + 0.0159795 M^2$ |
| 0,3   | $0.719309 + 0.204598 M + 0.00685909 M^2$ |
| 4,0   | $-3.15852 + 0.410304 MN - 0.00388438 M^2 N^2 - 0.00012569 M^3 N^3$ |
| 3,1   | $-0.95252 + 0.20553 M - 0.273324 MN - 0.0473612 M^2 N^2 + 0.00103599 M^2 N^2 - 0.000247572 M^3 N^2$ |
| 2,2   | $-10.5999 - 3.25607 M - 0.159554 M^2 - 0.0210851 MN - 0.00248042 M^2 N - 0.0000643977 M^3 N$ |
| 1,3   | $-5.61836 - 1.86565 M - 0.116276 M^2 + 0.0000353371 M^3$ |
| 1,4   | $-1.157 - 0.397981 M - 0.0273887 M^2 - 0.000135411 M^3$ |
| 0,4   | $0.70266 + 1.14357 M N + 0.0316503 M^2 N^2 - 0.000233057 M^3 N^3 - 7.79352 \times 10^{-6} M^4 N^4$ |
| 3,2   | $30.6075 + 3.492 M + 0.87123 M^2 + 0.403601 MN + 0.0984158 M^2 N - 0.00021126 M^3 N - 0.000514422 M^2 N^2$ |
| 2,3   | $32.106 + 12.0427 M + 1.03329 M^2 + 0.00813502 M^2 - 0.00099968 MN - 0.00040252 M^2 N$ |
| 2,2   | $-0.0000485465 M^3 N - 3.956 \times 10^{-6} M^4 N^4$ |
| 1,4   | $13.3343 + 5.18122 M + 0.485923 M^2 + 0.00596934 M^3 + 3.27121 \times 10^{-6} M^4$ |
| 0,5   | $2.26883 + 0.901262 M + 0.0889695 M^2 + 0.00136005 M^3 - 6.90887 \times 10^{-8} M^4$ |

These results do not apply to the case $N = 0$. Indeed, in this case, additional singularities in the Borel transform are expected [52].

For each perturbative series $R(\bar{u}, \bar{v})$, we consider the following approximants [50]

$$E(R)_p(\alpha, b; \bar{\tau}, \bar{\tau}) = \sum_{k=0}^{p} B_k(\alpha, b; \bar{\tau}/\bar{\tau}) \int_{0}^{\infty} dt t^k e^{-t} \frac{[y(\bar{\tau}t; \bar{\tau}/\bar{\tau})]^k}{[1 - y(\bar{\tau}t; \bar{\tau}/\bar{\tau})]^\alpha}, \quad (2.29)$$

where

$$y(x; z) = \frac{\sqrt{1 - x/\bar{\tau}b(z)} - 1}{\sqrt{1 - x/\bar{\tau}b(z)} + 1}. \quad (2.30)$$

The coefficients $B_k$ are determined by the condition that the expansion of $E(R)_p(\alpha, b; \bar{\tau}, \bar{\tau})$ in powers of $\bar{\tau}$ and $\bar{\tau}$ gives $R(\bar{\tau}, \bar{\tau})$ to order $p$. Within this method we end up with several values for each resummed quantity, depending upon the free parameters $\alpha$ and $b$. To have a single estimate we search for the values of $\alpha_{\text{opt}}$ and $b_{\text{opt}}$ minimizing the difference between the highest orders and, as usual [50,10,3], we consider as final estimate (error bar) the average (the variance) of the approximants with $\alpha \in [\alpha_{\text{opt}} - \Delta \alpha, \alpha_{\text{opt}} + \Delta \alpha]$ and $b \in [b_{\text{opt}} - \Delta b, b_{\text{opt}} + \Delta b]$, with $\Delta \alpha = 2$ and $\Delta b = 3$ (see e.g. Ref. [10] for a discussion about the effectiveness of such choice of $\Delta \alpha$ and $\Delta b$).

Within the second resummation procedure, the Borel-Leroy transform is analytically extended by means of a generalized Padé approximant, using the resolvent series trick (see, e.g. [9]). Explicitly, once introduced the resolvent series of the perturbative one $R(\bar{u}, \bar{v})$

$$\tilde{P}(R)(\bar{\tau}, \bar{\tau}, b, \lambda) = \sum_{n} \lambda^n \sum_{k=0}^{n} \frac{\tilde{P}_{k,n}}{\Gamma(n + b + 1)} \bar{\tau}^{n-k} \bar{\tau}^k, \quad (2.31)$$

10
which is a series in powers of $\lambda$ with coefficients being uniform polynomials in $\pi, \tau$. The analytical continuation of the Borel transform is the Padé approximant $[N/M]$ in $\lambda$ at $\lambda = 1$. Obviously, the sum for each perturbative series depends on the chosen Padé approximant and on the parameter $b$. We consider for the final estimates all the non-defective (i.e. having no singularities on the real positive axis) Padé approximants at the value of $b$ minimizing the difference between them.

An important issue in the fixed dimension approach to critical phenomena (and in general of all the field theoretical methods) concerns the analytic properties of the RG functions. As shown in Ref. [16] for the $O(N)$ model, the presence of confluent singularities in the zero of the perturbative $\beta$ function causes a slow convergence of the results given by the resummation of the perturbative series to the correct fixed point value. The $O(N)$ two-dimensional field-theory estimates of physical quantities [50,15] are less accurate than the three-dimensional ones [50] due to the stronger non-analyticities at the fixed point [16,13,14], to say nothing about the stronger growth of the series coefficients themselves. In Ref. [16] it is shown that the non-analytic terms may cause large imprecision in the estimate of the exponent related to the leading correction to the scaling $\omega$. At the same time, the result for the fixed point location turns out to be a rather good approximation compared with those coming from different techniques. Non-analyticities cause slow convergence to the exact critical exponents too.

### III. ANALYSIS OF FIVE LOOP SERIES

#### A. Stability of the $O(NM)$ and $O(M)$ fixed points

We start the analysis of five-loop perturbative series in the case when one coordinate of the fixed point (FP) vanishes. Since in this case a lot of exact results are known, these

| $i,j$ | $e^{(6)}_{ij}$ | $R_{NM}^{(6)}R_{MN}^{(6)}$ |
|-------|----------------|----------------------|
| 2,0   | 0.0226441 + 0.011322 $MN$ | |
| 1,1   | 0.0452882 + 0.0226441 $M$ | |
| 0,2   | 0.0226441 + 0.011322 $M$ | |
| 3,0   | $-0.00119855 - 0.000749094 M N - 0.0000461744 M^2 N^2$ | |
| 2,1   | $-0.00359565 - 0.00179783 M N - 0.0000240177 M^2 N^2$ | |
| 1,2   | $-0.00359565 - 0.00224728 M - 0.0000224728 M^2$ | |
| 0,3   | $-0.00119855 - 0.000749094 M - 0.0000461744 M^2 N^2$ | |
| 4,0   | $0.00033062 + 0.00445834 M N + 0.000616201 M^2 N^2 - 0.0000141266 M^3 N^3$ | |
| 3,1   | $0.0253225 + 0.01266121 M N + 0.00517111 M^2 N^2 - 0.000013013 M^2 N^2 - 0.0000565063 M^3 N^2$ | |
| 2,2   | $0.0379337 + 0.0254875 M + 0.00524753 M^2 + 0.0016253 M N + 0.000461744 M^2 N - 0.0000847595 M^3 N$ | |
| 1,3   | $0.0253225 + 0.0178334 M + 0.00247304 M^2 - 0.0000565063 M^3$ | |
| 0,4   | $0.00633062 + 0.00445834 M + 0.000616201 M^2 - 0.0000141266 M^3 N^3$ | |
| 5,0   | $-0.0072953 - 0.00550947 M N - 0.000978196 M^2 N^2 - 0.0000178225 M^3 N^3 - 1.20103 \times 10^{-6} M^4 N^4$ | |
| 4,1   | $-0.0361476 - 0.0180738 M - 0.00947355 M N - 0.00473677 M^2 N - 0.000154204 M^2 N^2 - 0.0000771022 M^3 N^2$ | |
|       | $-0.000120103 M^3 N^3 - 6.0517 \times 10^{-6} M^4 N^4$ | |
| 3,2   | $-0.072953 - 0.0502342 M - 0.0074333 M N - 0.00486051 M^2 N - 0.00273867 M^2 N^2 - 0.000154207 M^3 N + 5.95387 \times 10^{-9} M^3 N^2 - 0.0000240177 M^2 N^3 - 0.0000120103 M^4 N^4$ | |
| 2,3   | $-0.072953 - 0.0544783 M - 0.00942071 M^2 - 0.000127693 M^3 - 0.000161647 M N - 0.000361246 M^2 N$ | |
|       | $-0.0000505316 M^3 N - 0.0000120103 M^4 N^4$ | |
| 1,4   | $-0.0361476 - 0.0275474 M - 0.00489998 M^2 - 0.0000891125 M^3 + 6.0517 \times 10^{-6} M^4 N^4$ | |
| 0,5   | $-0.0072953 - 0.00550947 M - 0.000978196 M^2 - 0.0000178225 M^3 - 1.20103 \times 10^{-6} M^4 N^4$ | |
calculated will be a check of the goodness of our resummation procedure and of the error made considering only five loops in the perturbative expansion.

The location of the FP on the axis $\pi = 0$ (that is equivalent to $\bar{\pi} = 0$, with replacing $NM \rightarrow M$) was already studied in Ref. [15]. Also the critical exponents were considered there. The agreement of these results with the available exact ones was satisfactory (see also footnote [40]). What still remains to be computed is the stability of the FP’s with respect to perpendicular perturbation.

First of all, we analyze the stability properties of the $O(NM)$-symmetric fixed point. Since $\partial^2 \ln \tilde{\omega}^3(\pi,0)/\partial \pi^2 = 0$, the stability with respect to an anisotropic cubic perturbation is given by $\omega_v(\pi) = 2\tilde{\omega}^3(\pi,0)$, whose five-loop expansion is ($n = NM$)

$$
\frac{\omega_v(\pi)}{2} = -1 + \frac{12}{n + 8} \frac{\pi - 92.6834 + 5.83417n - 1.83156n^2 - (n + 8)^3}{(n + 8)^2} + \frac{1228.63 + 118.503n - 1.83156n^2 - 1228.63 \pi^2}{(n + 8)^3} + \frac{20723.1 + 2692.00n + 5.83417n^2 - 8.824655n^3 - 20723.1 \pi^4}{(n + 8)^4} + \frac{414915.7 + 67526.8n + 1868.92n^2 - 13.7618n^3 - 0.4602n^4}{(n + 8)^5}. \quad (3.1)
$$

This series must be calculated at the $O(n)$ FP located at $\pi = \bar{\pi}^*$. In Table V we report the results for $\omega_v$ for several values of $n$, obtained resumming the series (3.1) at the FP’s calculated in Ref. [15]. The $O(n)$ FP results unstable for $n \geq 3$. For $n = 2$ our result is compatible with the expected result $\omega_v = 0$, which is essential in the context of a continuous line of the fixed points.

We can also use these results to discuss the nature of the multicritical point in two-dimensional models with symmetry $O(N_1) \oplus O(N_2)$ [53]. Indeed, they allow to exclude that the multicritical point has enlarged symmetry $O(N_1 + N_2)$ if $N_1 + N_2 = n > 2$ [33]. In RG
TABLE V. Half of the exponent $\omega_v$ at the $O(n)$ fixed point. CM is the value obtained using conformal mapping technique and PB the one using Padé-Borel.

| $n$ | CM 4-loop | PB 4-loop | CM5-loop | PB 5-loop |
|-----|-----------|-----------|----------|-----------|
| 2   | 0.03(3)   | 0.06(4)   | 0.025(40)| 0.00(5)   |
| 3   | -0.08(3)  | -0.07(3)  | -0.10(6) | -0.10(5)  |
| 4   | -0.18(4)  | -0.17(5)  | -0.17(4) | -0.20(4)  |
| 8   | -0.45(5)  | -0.44(6)  | -0.48(4) | -0.50(5)  |

terms, this can generally occur if the $O(N_1+N_2)$ fixed point has only two relevant $O(N_1) \oplus O(N_2)$-symmetric perturbations. But, when $n > 2$, the instability of the $O(n)$ fixed point with respect to the cubic perturbation shows that at least one extra relevant perturbation exists (see for details Ref. [33]).

Then we focus our attention on the stability properties of the $O(M)$ fixed point. Also in this case the stability is given by $\omega_u(\bar{\tau}) = 20m^3[\bar{\tau}](0,\bar{\tau})$, evaluated at the $O(M)$ fixed point $\bar{\tau}^*$. As expected, the series $\omega_u(\bar{\tau})$ is independent of $N$

$$\frac{\omega_u(\bar{\tau})}{2} = -1 + 2\frac{M+2}{M+8}\bar{\tau} - \frac{5.83417(M+2)}{(M+8)^2}\bar{\tau}^2 + \frac{(47.9183 + 8.35207M)(M+2)}{(M+8)^3}\bar{\tau}^3$$

$$-\frac{(546.755 + 134.247M + 0.67606M^2)(M+2)}{(M+8)^4}\bar{\tau}^4$$

$$+\frac{(8229.19 + 2568.45M + 129.81M^2 - 0.2176M^3)(M+2)}{(M+8)^5}\bar{\tau}^5. \quad (3.2)$$

For $M = 1$, the fixed point value of this exponent is $\omega_u^f/2 = -0.09(8)$, using the conformal mapping method, and $-0.08(10)$ with the Padé-Borel. We note that the $[4/1]$ approximant with $b = 1$ leads to $\omega_u^f/2 = -0.031$. These values are compatible with the exact known result $\alpha_I = 0$. For $M \geq 2$, we always find $\omega_u > 0$, in agreement with the exact results predicting the stability of the $O(M)$ fixed point.

An alternative approach to the analysis of the series Eqs (3.1) and (3.2) is the so called pseudo-$\epsilon$ expansion [54]. Within this method, one multiplies the linear term in the $\beta$ functions by a fictitious small parameter $\tau$, finds the common zeros of the $\beta$ functions as series in $\tau$, and finally the exponents (as series in $\tau$) are obtained introducing such expansions in the corresponding RG functions. This approach has twofold advantage. First, in the estimates of the critical exponents the cumulation of errors due to not exact knowledge of the FP and that of the proper uncertainty of the RG function is avoided [50]. Second, it allows for very precise estimates of the marginal number of components since these are series in $\tau$ that must be evaluated at $\tau = 1$ [32,31,55,56]. We use this method only for the second purpose.

Imposing the vanishing of $\omega_v$ at the $O(NM)$ fixed point, the series for $N_c$ determining the relevance of cubic anisotropy (see the Introduction) is:

$$N_c = 4 - 2.2504\tau + 0.6230\tau^2 - 0.7509\tau^3 + 1.1761\tau^4. \quad (3.3)$$

Analogously, imposing $\omega_u = 0$ at the $O(M)$ FP leads to

$$M_c = 4 - 4.5008\tau + 2.1693\tau^2 - 1.2165\tau^3 + 1.3055\tau^4, \quad (3.4)$$
determining the relevance of an energy-like interaction, as e.g. the relevance of quenched randomness. Such series do not behave asymptotically with a factorial growth of the coefficients (at least up to the considered order), thus simple Padé approximants without Borel resummation are expected to give reliable estimates. This in fact turned out to be the case for their three dimensional analogs \([32,31]\) and also for more complicated series as those for the marginal number of components of \(O(N) \times O(M)\) \([55]\) and \(U(N) \times U(M)\) \([56]\) models.

The Padé table for the series Eq. (3.3) at \(\tau = 1\) is

\[
\begin{bmatrix}
4 & 1.75 & 2.37 & 1.62 & 2.53 \\
2.56 & 2.24 & 2.03 & 2.08 \\
2.32 & 5.76_{[0.9]} & 2.08 \\
2.08 & 2.17 \\
2.21_{[3.1]}
\end{bmatrix},
\]

(3.5)

where we indicated in small brackets the closest pole to the origin on the real axis (when exists). Whenever this pole is close to \(\tau = 1\), the approximant has not to be considered in the average procedure. This Padé table leads to the estimates \(N_c = 2.10(7)\) that includes all the four- and five-loop estimates without poles (excluding the first line and column that are unreliable). This is in quite good agreement with the known exact result \(N_c = 2\) \([21]\), signaling about the high effectiveness of the perturbative expansion technique at the five-loop level.

Similarly, the Padé table for the series Eq. (3.4) is

\[
\begin{bmatrix}
4 & -0.5 & 1.66 & 0.45 & 1.75 \\
1.88 & 0.96 & 0.89 & 1.08 \\
1.40 & 0.88_{[0.2]} & 0.95 \\
1.19 & 1.09 \\
1.13
\end{bmatrix}.
\]

(3.6)

The same averaging procedure as before leads to \(M_c = 0.99(10)\), that perfectly agrees with the exact value \(M_c = 1\) at which the specific-heat exponent is vanishing. This demonstrates under \(\bar{\mu} = 0\) good approximating properties of the five-loop series.

We analyzed the series (3.3) and (3.4) with the Padé-Borel resummation as well, having obtained equivalent results. So, indeed at the considered order these series do not behave as asymptotic.

**B. The cubic model for \(N \geq 3\)**

In this Section we consider the existence of the cubic fixed point for \(N \geq 3\), previously found in the four-loop approximation \([3]\). It was claimed in Ref. \([3]\) that the quite peculiar features of this fixed point (like the marginal instability) make its existence quite doubtful and that it might be an artifact of the relatively short series available at that time. Now we are in a position to analyze longer series and to further confirm or to reject this statement.

The results obtained with the conformal mapping methods are reported in Table VI together with the previous four loop results \([3]\), in order to make the comparison visible at
TABLE VI. Location of the apparent cubic fixed point for some $N \geq 3$.

| $N$ | CM 4-loop               | CM 5-loop               | Padé [4/1] $b = 1$ |
|-----|-------------------------|-------------------------|-------------------|
| 3   | [0.83(12),1.12(9)]     | [0.54(6),1.35(4)]     | [0.050,1.757]     |
| 4   | [0.54(10),1.43(8)]     | [0.32(5),1.58(4)]     | [0.031,1.774]     |
| 8   | [0.24(8),1.72(10)]     | [0.14(4),1.74(4)]     | [0.015,1.788]     |

first sight. This Table shows that the cubic fixed point drastically moves towards the Ising fixed point with increasing the order of perturbation theory from four to five loops. It may be considered as an argument in favor of the statement that, within the exact theory, the cubic and the Ising fixed points coincide. On the other hand, both at four and at five loops, the quoted errors are less than the difference between the two estimates. This leads to the conclusion that the reported uncertainty is actually an underestimate of the correct one. We remind to the reader that this error come from the so called stability criterion, i.e. it is obtained looking at those approximants that minimize the difference between the estimates at the highest available orders (see Sec. II C). So, the considerable discrepancies between the four- and five-loop results lead to serious doubts in the existence of the cubic fixed point in two dimensions.

In order to understand better these somewhat strange results, we report the values of the coordinate $\mathbf{v}^*$ obtained for the cubic fixed point using several Padé approximants for $N = 3, 4, 8$; the estimates are presented in Fig. 1 as functions of $b$. Let us consider first the case $N = 3$ as a typical example. If one limits himself with only three lower-order approximants [2/1], [3/1], and [2/2], he easily finds that they minimize their differences under $b \sim 0$, leading in such a way to the estimate $\mathbf{v}^* \sim 0.7$. Taking into account two non-defective Padé approximants [4/1] and [3/2] (note the oscillating behavior of the [3/2] approximant for $b < 1$, signaling the presence of close singularities) existing at the five-loop level shifts the zone of stability to $b \sim 2$, thus leading to the estimate $\mathbf{v}^* \sim 0.5$. Moreover, the approximant [4/1] with $b = 1$, that is usually considered as one of the best approximants, results in the estimate $\mathbf{v}^* = 0.050$ very close to zero. Because of the alternative character of RG expansions, it looks very likely that the unknown six-loop contribution (and the higher-order ones) will locate the stability region somewhere near $b \sim 1$, leading finally to the coalescence of the Ising and cubic fixed point. According to this scenario, the cubic fixed point, found at finite order in perturbation theory, is probably only an artifact due to the finiteness of the perturbative series.

The same scenario is possible also for other values of $N$. From Fig. 1 we see that the region of maximum stability always shifts from $b \sim 0$ to $b \sim 2$ with increasing the order of approximation from four to five loops, moving the coordinate of the fixed point from $\mathbf{v}^* \sim 0.5$ for $N = 4$ ($\mathbf{v}^* \sim 0.2$ for $N = 8$) to $\mathbf{v}^* \sim 0.3$ ($\mathbf{v}^* \sim 0.1$). Let us stress again that the value given by the approximant [4/1] with $b = 1$ is always very close to zero, as is seen from Table VI. Note that, with increasing $N$, the distance between the cubic fixed point and the Ising one reduces rapidly.

In the limit $N \rightarrow \infty$ the series simplify as at the four-loop level (see Ref. [3]). We only mention that with increasing the length of the RG series the coordinate $\mathbf{v}^*$ of the cubic fixed point shifts from $\mathbf{v}^* \sim 0.08$ to $\mathbf{v}^* \sim 0.03$ that again is much more close to zero.
C. The random $O(M)$ model

The $MN$ model in the limit $N \to 0$ describes the critical behavior of quenched dilute $O(M)$ models. Being $u_0$ proportional to minus the variance of the disorder [18,22] only the region with $\overline{u} < 0$ is of physical interest. We have already shown that for $M \geq 2$ the $O(M)$ fixed point is stable. To analyze the RG flow in the whole physical region we do not use here advanced resummation procedures developed [30] to avoid Borel non-summability at fixed $u/v$ [52], but limit ourselves by a simple Padé analysis, since it is sufficient for our aims. Within this method we check for several $M \geq 2$ that no other FP with $\overline{u} < 0$ exists when the resummation is effective. Thus, whenever the transition of dilute models is second-order (i.e. under the percolation threshold), the critical behavior is of $O(M)$ type. An unstable FP is found in the unphysical region $\overline{u}, \overline{v} > 0$, which delimits the domain of attraction of the $O(M)$ and $O(0)$ fixed points, in agreement with the extrapolation of $\epsilon$-expansion at $\epsilon = 2$ [17].

Finally, we discuss the fate of the random fixed point governing the critical behavior of the weakly-disordered Ising model ($M = 1$ and $N = 0$) that is found in three dimensions
and close to four dimensions by means of the $\sqrt{\epsilon}$ expansion [57,8,22]. In this case the Ising FP is marginally stable for $\overline{\tau} > 0$ [2], so the RG flow is driven to $\overline{\tau^*} = 0$ from the physical region (in the unphysical region the Ising FP is marginally unstable and the RG flows run into the $O(0)$ FP). We find, for the majority of the considered approximants, a fixed point with negative $\overline{\tau}^*$, as reported in Fig. 1. A possible estimate, according to stability criteria is $\overline{\tau}^* = -0.1(1)$, but if we concentrate on some certain approximants we obtain $\overline{\tau}^* = -0.090$ for the $[3/1]$ and $\overline{\tau}^* = -0.030$ for the $[4/1]$ (both with $b = 1$). In particular, the last value is very close to zero, i.e. to the value predicted by the asymptotically exact solution that has been obtained in the framework of the fermionic representation [5,6,2]. It is worthy to note that the five-loop results for $N = 0$ seem to be less scattered than analogous four-loop estimates obtained by means of Chisholm-Borel resummation technique [58], and they look more precise than their five-loop counterparts for finite $N$.

D. The cubic model for $N = 2$

The four-loop analysis of Ref. [3] for $N = 2$ turned out to be compatible with the presence of a line of the fixed points joining the $O(2)$-symmetric and the decoupled Ising fixed points. The lines of zeros of the two $\beta$ functions were found to be practically parallel and the quoted error was bigger than distance between them. This line of fixed points with continuously varying critical exponents is in agreement with what is expected from the correspondence, at the critical point, between the cubic model and the Ashkin-Teller and the planar model with fourth-order anisotropy [1,3]. We are now in a position to verify this statement at the five-loop level.

First, we analyze the series with the conformal mapping method. Again we find that zeros of two $\beta$ functions form two parallel lines, while the apparent uncertainties seem to be smaller than their separation. Of course, this fact may simply indicate that the model has no fixed point at all. Let us, however, look more carefully to this result and, in particular, to the accuracy of the quoted error. In fact, as we have already seen for $N > 2$, the error coming from stability criteria is likely an underestimate of the correct one. To understand better the situation, we use the Padé-Borel method. In Fig. 2 we report the curves of zeros of the two $\beta$ functions given by several Padé approximants under $b = 1$, the value that for $N > 2$ always leads to good results and that is the best for the fixed point values of $O(N)$ and Ising model [15]. The four approximants for $\overline{\beta}_\tau$ are always well-defined. They are hardly distinguishable close to the $O(2)$ fixed point and their separation slowly increases moving toward the $\overline{\tau}$ axis. The coordinate of the Ising fixed point $\overline{\tau} \sim 1.8$ is obtained using the $[4/1]$ approximant, since approximants of [L-1/1] type proved to give rather precise estimates for the fixed point location both in two and three [59,60] dimensions. The situation is a bit worse for the function $\overline{\beta}_\tau$. In fact, the working approximants are well defined close to the Ising fixed point, but approaching the $\overline{\tau}$ axis they becomes defective. The approximant $[3/2]$ starts oscillating around $\overline{\tau} \sim 0.8$, while $[4/1]$ is bad in the range $\overline{\tau} \in [1, 1.5]$ and $[3/1]$ for $\overline{\tau} > 1.3$. Also the values of zeros of the approximant $[4/1]$ for $\overline{\tau} > 1.5$ are not reliable enough, since they may suffer of the effect of close singularities.

Despite of these shortcomings, we can obtain a rich information from Fig. 2. Indeed, the line of zeros of $\overline{\beta}_\tau$ given by the approximant $[4/1]$ practically coincide with those of the $\overline{\beta}_\tau$
FIG. 2. Zeros of $\bar{\beta}_u$ (continuous lines) and $\bar{\beta}_v$ (points) for several Padé approximant (all with $b = 1$).

From the Ising fixed point up to $\tau \sim 0.8$. For greater $\tau$ various approximants for $\bar{\beta}_\tau$ results in the lines of zeros that diverge leaving, however, the line [4/1] of $\bar{\beta}_\tau$ zeros between them. Keeping in mind a finite length of the RG series and the influence of non-analytic terms missed by the perturbation theory, we retain this fact as a strong evidence in favor of the continuous line of fixed points. The best estimate for this line is believed to be that given by the approximant [4/1] for $\bar{\beta}_\tau$. Thus it will be used in what follows to calculate continuous varying critical exponents.

We evaluate the smallest eigenvalues of the $\Omega$ matrix along the line of fixed points both with conformal mapping and Padé-Borel method. We find that it is always compatible with zero (with the uncertainty of the resummation), that is a necessary condition for having a stable line of fixed points.

When evaluating the critical exponents, one should keep in mind that the limit $z \to 0$ is not simply accessible perturbatively since it corresponds to the two-dimensional $XY$ model which is known to behave in a quite specific manner. In particular, this model does not undergo an ordinary transition into the ordered phase at any finite temperature and its critical behavior is essentially controlled by the vortex excitations [61]. Such excitations lead to an exponentially diverging correlation length at finite temperature that can not be accounted for within the $\lambda \phi^4$ model Eq. (1.1) dealt with in this paper. Since arriving to the point $z = 0$ a new physics emerges, it is natural to assume that the observables as functions of the fixed point location may be non-analytic near this point. Hence, what we
FIG. 3. The exponent $y$ as function of $x$ from free and constrained resummation. The line "first approximation" is Eq. (3.7) and the line "second approximation" is Eq. (3.8).

can really explore trusting upon our five-loop expansions is a domain corresponding to finite (and not too small) values of $z$. Oppositely, the limit $z \to \infty$ (or $1/z \to 0$) looks quite "undangerous" in the above sense since it corresponds to the critical behavior close to that of the Ising model, which was shown not to be influenced considerably by the non-analytic terms even in two dimensions [15]. Note that just presented results concerning the line of the fixed points confirm this idea. Indeed, as seen from Fig. 2, at the "Ising side", i.e. for $0 < \pi \sim 0.8$, the zeros of both $\beta$ functions form smooth curves running very close to each other. The closer the "XY side", however, the stronger the estimates for $\beta$ function zeros are scattered, indicating, likely, the increasing impact of non-analytic contributions.

The expected value of $\eta$ is $1/4$ independently on the location of a fixed point within the line. The best way to check the constantness of $\eta$ along the line of fixed points is probably to write the RG function in terms of $\pi$ and $s = \pi + \bar{\pi}$, i.e. $\eta_s(s, \pi) = \eta(\pi, s - \pi)$. Then one resum the difference $\Delta(s, \pi) = \eta_s(s, \pi) - \eta_s(s, 0)$. Along the line for all the five-loop approximants we always find $|\Delta(s, \pi)| < 8 \times 10^{-3}$. This leads us to conclude that the two dimensional LGW approach is able to keep the constantness of $\eta$ within an error of about $3\%$. Note that the previous quoted problems concerning non-analyticities close to the $O(2)$ side do not significantly affect the estimates of $\eta$.

For the exponent $y = \eta - \eta_t = 1/(2 - \nu)$ it was conjectured in Ref. [3] that it should behaves like

$$y = \frac{2}{1 + x}, \quad \text{where,} \quad x = \frac{2}{\pi} \arctan \frac{\pi^s}{\pi^t}. \quad (3.7)$$
A direct reliable quantitative estimate of this exponents is impossible because of the effect of non-analyticities, in particular close to the $O(2)$ fixed point for the reasons explained above. In fact, we know from Ref. [15,40] that the resummation of the series for $y$ at the Ising fixed point provides $y = 1.02$, very close to the exact value 1. Instead, at the $O(2)$ fixed point one has $y \sim 1.25$, that is quite far from a diverging $\nu$, i.e. $y = 2$. A direct resummation of this exponent is reported in Fig. 3, and as expected it seems to reproduces the correct critical behavior only close to the Ising fixed point.

In Ref. [3] it was proposed to constrain the exponents to assume the exactly known value along the axes to have better quantitative results. We apply here a different method of constrained analysis with respect to Ref. [3]. We prefer to constrain the series of $y$ expressed in terms of $\varpi$ and $s = \varpi + \varphi$ (as previously for the exponent $\eta$), since the results obtained with this constrain appear more stable. The constrained Padé-Borel approximants are reported in Fig. 3. At the Ising side, they are practically indistinguishable from the unconstrained ones up to $x \sim 0.6$, but then they start oscillating, signaling the presence of singularities leading to bad quantitative estimates. The conformal mapping results are practically equivalent. The numerical data thus obtained are not in agreement with Eq. (3.7), even if this reproduced the right qualitative behavior. A quadratic form in the denominator of $y$ as e.g.

$$y = \frac{4}{4 - (1 - x)(2 - x)}, \quad (3.8)$$

fits the most of the data much better, as shown in Fig. 3. Unfortunately, we are not able to estimate the goodness of our resummation and, as a result, to verify Eq. (3.8). Perhaps, the exact behavior of the exponent $\nu$ along the line of fixed points requires new method of analysis of the perturbative series and, in any case, it deserves for different studies on the subject like Monte Carlo simulation or high temperature expansion.

**E. The $MN$ model for $N, M \geq 2$**

The critical behavior of the $MN$ model for $M, N \geq 2$ and $v_0 > 0$ can be understood by means of non-perturbative arguments. In fact, as we explained in the Introduction, it turns out that the $O(M)$ fixed point is always stable and oppositely the $O(MN)$ one is always unstable. Thus the RG flow is driven to the $O(M)$ fixed point from both positive and negative $\varpi$. The only extra fixed point is provided by the $\epsilon$ expansion with $\varphi^* < 0$ which limits the attraction domain of the $O(M)$ one. These features are reproduced by the resummation of the five-loop series. In particular, we found only one fixed point with non-vanishing coordinates that is the mixed one of the $\epsilon$ expansion, that in fact turns out to be unstable.

More interesting is the RG flow for $v_0 < 0$. In this case the $\epsilon$ expansion does not provide any fixed points, thus the transition (if any) cannot be continuous. Nevertheless for $M = N = 2$ we can relate the $MN$ model to the $O(2) \times O(2)$ one, arguing that a fixed point with negative $\varphi^*$ (located at $\varphi^* \simeq 4.6$ and $\varphi^* \simeq -4.0$ [36]) should exist, describing a finite temperature phase transition in the chiral universality class. Note that, as stressed in Refs. [34–36] (see also [51] for the three-dimensional analogous) such fixed point is in the region of non-Borel summability: in fact, $z^* = \varphi^*/\varpi^* \simeq -0.86 < z_1 = -0.833$, where
\[ z_1 = \frac{R_{MN}}{R_N} \] delimits the region where the series are Borel summable (see Sec. II C).
Anyway, in the course of the resummation with the conformal mapping, the singularity of the Borel transform closest to the origin has been taken into account. The resulting sequence of approximations is thus only asymptotic (as for the random models), but it is expected to provide a reasonable estimate as long as the resummation point is far from the region
\[ z_2 = z_1 2N/(N+1), \] where the singularity on the real positive axis becomes the closest to the origin.

With this in mind, we resum the \( \beta \) functions for several values of \( M \) and \( N \) and search for new fixed points with \( v^* < 0 \). We consider for each \( \beta \) functions the 18 conformal mapping approximants with \( \alpha = 0, 1, 2 \) and \( b = 5, 7, 9, 11, 13, 15 \) that (as for the \( O(M) \times O(N) \) model [36]) turned out to be the more stable with changing the number of terms considered in perturbation theory. Up to three loops we do not find a fixed point for any value of \( N \) and \( M \), but at four and five loops a fixed point with \( z^* \) very close to \( z_1 \) appears for the majority of the 324 couples of approximants of the \( \beta \) functions. In Table VII we report the five-loop results for the location of the fixed point \( \vec{w}, \vec{v}^* \) and the percentage of the 324 approximant having it. In the Table are reported also the values of \( z^* \) (obtained independently in the course of the averaging procedure) and the values of \( z_1 \) and \( z_2 \) to make the comparison between them clear at first sight. The reported estimates are the averages over those approximants having a fixed point and the error bars are the variances.

For \( M = N = 2 \) we reproduce the result of the chiral model [36]. Several features for other value of \( M \) and \( N \) can be extracted from Table VII. For each \( M \) there is a maximum \( N \), we call \( N^*(M) \), for which the FP exists. This is true up to a maximum value of \( M \), we call \( M^* \), after that the results of the \( \epsilon \) expansion and large \( M \) are recovered. A reliable determination of \( M^* \) and \( N^*(M) \) is difficult. In fact, one has arbitrarily to fix a confidence level of the percentage for which the existence of the fixed point can be considered firm. For example, we can decide that the fixed point is credible when the percentage of the approximants displaying it is greater than the 70% and its existence is improbable when it is less than 40%. Within this method we have \( 2 < N^*(2) < 4, 3 < N^*(3) < 5 \), etc. Regarding \( M^* \) the situation is even more difficult. With increasing \( M \), the FP moves closer and closer to \( z_2 \), and, in fact, its existence is not really reliable. Anyway, we retain quite safe to state \( M^* < 22 \), but probably its value is much lower.

We check the stability of the results at four and five loops. For small values of \( M \) the location of the FP is rather stable, but it gets worse with increasing \( M \), signaling that the estimate in this case is not so robust.

We evaluated the eigenvalues of the \( \Omega \) matrix for each couple of approximants at their common zero. They always have a quite large positive real part, so the FP is stable. However, it is very hard to obtain reliable numerical estimates since the values strongly oscillate. At the same time, the focus behavior seems to be a peculiarity of the chiral model \( M = N = 2 \); for larger \( M, N \) only few approximants lead to a non-vanishing imaginary part. Even the estimates of the critical exponents are practically impossible probably because of non-analyticities and the large values of the coupling constants involved in the calculation. Anyway, the exponent \( \eta \) for moderate \( M \) seems to be compatible with 1/4 that is the value of the Ising and \( XY \) models.

Let us comment that a parallel analysis in \( d = 3 \) [28] seems to indicate a similar structure of fixed points, but the results are more stable (as usual) both because of the knowledge of
TABLE VII. Location \((u^*, v^*)\) of the stable fixed point with \(v^* < 0\) at five loops for several \(M\) and \(N\). The estimate of \(z^* = v^*/u^*\) and the value of \(z_1\) and \(z_2\) (see the text) are also reported.

| \(M\) | \(N\) | % | \(u^*\) | \(v^*\) | \(z^*\) | \(z_1\) | \(z_2\) |
|---|---|---|---|---|---|---|---|
| 2 | 2 | 82 | 4.65(18) | -3.93(33) | -0.85(5) | -0.83 | -1.11 |
| 2 | 3 | 57 | 4.66(41) | -3.73(46) | -0.80(5) | -0.71 | -1.07 |
| 2 | 4 | 42 | 4.88(49) | -3.72(51) | -0.76(5) | -0.62 | -1 |
| 2 | 5 | 31 | 4.99(51) | -3.64(44) | -0.73(5) | -0.56 | -0.93 |
| 3 | 2 | 86 | 5.37(25) | -4.34(29) | -0.81(4) | -0.79 | -1.05 |
| 3 | 3 | 65 | 5.20(60) | -3.79(53) | -0.71(5) | -0.65 | -0.97 |
| 3 | 4 | 44 | 5.34(62) | -3.79(54) | -0.71(5) | -0.55 | -0.88 |
| 3 | 5 | 36 | 5.81(51) | -3.89(53) | -0.75(5) | -0.68(5) | -0.80 |
| 4 | 2 | 89 | 6.04(39) | -4.71(26) | -0.78(4) | -0.75 | -1 |
| 4 | 3 | 67 | 5.64(79) | -4.01(61) | -0.71(5) | -0.6 | -0.9 |
| 4 | 4 | 52 | 6.03(97) | -4.05(76) | -0.67(5) | -0.5 | -0.8 |
| 5 | 2 | 91 | 6.63(57) | -5.05(31) | -0.76(4) | -0.72 | -0.96 |
| 5 | 3 | 67 | 5.94(89) | -4.07(58) | -0.69(5) | -0.56 | -0.85 |
| 5 | 4 | 50 | 6.5(1.2) | -4.15(77) | -0.64(4) | -0.46 | -0.74 |
| 6 | 2 | 90 | 7.10(75) | -5.32(40) | -0.75(4) | -0.7 | -0.93 |
| 6 | 3 | 67 | 6.2(1.0) | -4.13(61) | -0.67(5) | -0.53 | -0.81 |
| 7 | 2 | 93 | 7.5(1.0) | -5.55(51) | -0.74(4) | -0.68 | -0.91 |
| 8 | 2 | 88 | 7.7(1.1) | -5.68(66) | -0.74(3) | -0.67 | -0.89 |
| 8 | 3 | 64 | 6.5(1.2) | -4.2(6) | -0.58(2) | -0.5 | -0.75 |
| 9 | 2 | 84 | 7.9(1.2) | -5.83(70) | -0.74(3) | -0.65 | -0.87 |
| 10 | 2 | 83 | 8.0(1.2) | -5.87(70) | -0.74(3) | -0.64 | -0.86 |
| 15 | 2 | 75 | 8.1(1.0) | -6.01(55) | -0.74(3) | -0.61 | -0.81 |
| 15 | 4 | 15 | 10(2) | -5.2(9) | -0.52(1) | -0.34 | -0.54 |
| 20 | 2 | 55 | 8.9(1.0) | -6.57(54) | -0.74(2) | -0.58 | -0.78 |
| 20 | 3 | 55 | 8(1) | -4.6(5) | -0.58(2) | -0.41 | -0.62 |
| 22 | 2 | 23 | 9.2(5) | -6.7(3) | -0.73(2) | -0.58 | -0.77 |
| 25 | 2 | 21 | 10(1) | -7.5(5) | -0.73(2) | -0.57 | -0.76 |

six-loop terms [30] and because of the weaker effect of non-analyticities (in particular, for the estimates of the exponents).

To conclude this section, we want to stress that Table VII should be read with care. The point is that all the found FP have \(z^* < z_1\) being located in the region where the series are not Borel summable and all the FP with \(z^*\) close to \(z_2\) are not credible because of the bad behavior of the resummed approximants close to this region.

IV. CONCLUSION

To summarize, the critical behaviors of the two-dimensional \(N\)-vector cubic and \(MN\) models have been studied within the renormalization-group approach. The five-loop contributions to the \(\beta\) functions and critical exponents have been calculated and the five-loop
RG series have been resummed by means of Padé-Borel-Leroy procedure and the conformal mapping technique.

For the cubic planar model ($N = 2$) we have found that the continuous line of fixed points connecting the Heisenberg and the Ising ones is well reproduced by the resummed five-loop RG series. Moreover, the five-loop terms being taken into account make the lines of zeros of $\beta$ functions for $\pi$ and $\overline{\pi}$ closer to each another thus improving the results of the lower-order approximation. For the cubic model with $N > 2$, the five-loop contributions have been shown to shift the cubic fixed point, given by the four-loop approximation, towards the Ising one. This may be considered as an argument in favor of the idea that the existence of cubic fixed point in two dimensions for $N \geq 3$ is an artifact of the perturbative analysis.

The models with $N = 0$ describing the critical thermodynamics of two-dimensional weakly-disordered $O(M)$ systems has been also studied. The results obtained have been found to be compatible with the conclusion that in two dimensions the impure critical behavior is governed by the $O(M)$ fixed point, even in the Ising case, where $\alpha_I = 0$ and the Harris criterion is inconclusive.

The five-loop RG analysis of the two-dimensional $M_N$-model with $M, N \geq 2$ has been also performed. For $v_0 > 0$ we reproduced all the non-perturbative results. It was shown, in particular, that the transition is driven to the $O(M)$ fixed point, that only for $M \leq 2$ describes a finite temperature phase transition. We also found a stable fixed point in the region with $v_0 < 0$ that has no counterpart in $\epsilon$ and large $M$ expansion, but its location is in the region of non-Borel summability of the series and its existence is still doubtful. At fixed $M$, this new fixed point is found for $N < N^*(M)$ up to a maximum value $M^*$, after which it disappears. Whether this fixed point describes a finite temperature phase transition (allowed because of the discrete symmetry $C_N$) or a zero temperature one cannot be discerned by our analysis; only lattice techniques such as Monte Carlo simulation and high temperature expansion or real experiments can completely clarify this point.

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are presented (Table IV, line [5,0]). This correction improves a bit the estimate of the 
exponent $\nu$ for the two-dimensional Ising model. Indeed, applying the same method 
of Ref. [15] (i.e. the Padé-Borel [4/1] with $b = 1$ to the series $1/\nu(g)$), we obtain at 
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