Exact normalized eigenfunctions for general deformed Hulthén potentials

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The exact solutions of Schrödinger’s equation with the deformed Hulthén potential \( V_q(x) = -\mu e^{-\delta x}/(1 - q e^{-\delta x}) \), \( \delta, \mu, q > 0 \) are given, along with a closed-form formula for the normalization constants of the eigenfunctions for arbitrary \( q > 0 \). The Crum-Darboux transformation is then used to derive the corresponding exact solutions for the extended Hulthén potentials \( V(x) = -\mu e^{-\delta x}/(1 - q e^{-\delta x}) + q j(j + 1) e^{-\delta x}/(1 - q e^{-\delta x})^2, j = 0, 1, 2, \ldots \) A general formula for the new normalization condition is also provided.

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I. INTRODUCTION

The Hulthén potentials [1, 2]

\[ V(x) = -\frac{\mu e^{-\delta x}}{1 - e^{-\delta x}}, \quad x \in (0, \infty), \]

(1)

where \( \mu \) is a constant and \( \delta > 0 \) is the screening parameter that determines the potential range, have important applications in nuclear and particle physics, in atomic physics, and in condensed–matter physics [3–8]. Except for the \( \ell=0 \) case, where the exact energy eigenvalues are known, a variety of numerical methods have been employed in the literature to obtain the eigenvalues and eigenfunctions. The so called centrifugal term approximation [10] has been widely used in these calculations. Interesting energy level ordering involving this potential has been obtained using the elementary comparison theorem of quantum mechanics which serves to provide deeper insights [11, 12] into the various other screened Coulomb potentials.

To our knowledge, no closed-form expression for the normalization constants for arbitrary \( q > 0 \) was given in the literature (including the classical Hulthén potential [1]). The present work provides the complete normalized solutions of Schrödinger’s equation (3). Crum-Darboux transformations are then used to generate the eigenfunctions of the equation (4). The approach given here provides the logical framework [16–19] behind the success of the centrifugal–term approximation [10] for the normalization constants. These ideas are used in section V to generate the exact eigenfunctions of equation (5).

Although such potential variations may be found in the literature [13–15] for a variety of applications, the potential in (2) is essentially the same as the original potential [1] since a simple shift of the independent variable such as \( x \to x + \log(q)/\delta \) transforms \( V_q(x) \) back into \( V(x) \) with the only difference being a new coupling constant \( \mu \). In future developments, in which angular momentum will be considered, it may be advantageous to revert the viewpoint to [1] so that the potential and the centrifugal terms all have singularities at the origin. However, since \( V_q(x) \) has been widely used in the literature, we shall adopt this form in the present work. Therefore, in atomic units, Schrödinger’s equation for our problem becomes

\[ \frac{d^2 \psi(x)}{dx^2} - \frac{\mu e^{-\delta x}}{1 - q e^{-\delta x}} \psi(x) = E \psi(x), \quad \int_{\log(q)/\delta}^{\infty} |\psi(x)|^2 dx = 1, \quad \psi(\log(q)/\delta) = \psi(\infty) = 0. \]

(3)

To our knowledge, no closed-form expression for the normalization constants for arbitrary \( q > 0 \) was given in the literature (including the classical Hulthén potential [1]). The present work provides the complete normalized solutions of Schrödinger’s equation (3). Crum-Darboux transformations are then used to generate the eigenfunctions of the equation

\[ \frac{d^2 \phi(x)}{dx^2} + \left( \frac{\mu e^{-\delta x}}{(1 - q e^{-\delta x})^2} - \frac{\nu e^{-\delta x}}{1 - q e^{-\delta x}} \right) \psi(x) = E \phi(x), \quad \int_{\log(q)/\delta}^{\infty} |\phi(x)|^2 dx = 1, \quad \phi(\log(q)/\delta) = \phi(\infty) = 0. \]

(4)

A general formula is provided for the normalization constants of the eigenfunctions \( \phi(x) \) in terms of the eigenfunctions \( \psi(x) \), for arbitrary \( q > 0 \).

The paper is organized as follows: in section II, we discuss the exact solutions of Schrödinger’s equation (3). In section III, we develop an analytic expression for the normalization constants in terms of the generalized hypergeometric function \( _3F_2 \). In section IV, we give a general review of the Crum-Darboux transformation, and a simplified formula for the normalization constants. These ideas are used in section V to generate the exact eigenfunctions of equation (5). The approach given here provides the logical framework [16–19] behind the success of the centrifugal–term approximation \( 1/x^2 \approx e^{-\delta x}/(1 - e^{-\delta x})^2 \) used to estimate the eigenvalues and eigenfunctions for the Hulthén potential (\( q = 1 \)) for \( \ell \neq 0 \).

II. GENERALIZED HULTHÉN POTENTIAL: EXACT SOLUTIONS

In this section, we show that the exact solutions of Schrödinger’s equation (3) may be expressed in terms of the Gauss hypergeometric functions as

\[ \psi(x) = e^{-\sqrt{-2E}x} (1 - q e^{-\delta x}) _2F_1 \left( 1 + \frac{\sqrt{-2E}}{\delta} - \frac{1}{\delta} \sqrt{\frac{2\mu}{q} - 2E}, 1 + \frac{\sqrt{-2E}}{\delta} + \frac{1}{\delta} \sqrt{\frac{2\mu}{q} - 2E}; 1 + \frac{2\sqrt{-2E}}{\delta}; q e^{-\delta x} \right). \]

(5)

Indeed, the change of variable \( z = e^{-\delta x} \) allows equation (3) to be written as

\[ z^2 \frac{d^2 \psi}{dz^2} + z \frac{d\psi}{dz} + \frac{\nu z}{1 - q z} \psi = -\varepsilon \psi, \quad 0 < z < \frac{1}{q}, \quad \psi(0) = \psi(1/q) = 0. \]

(6)
where $v = 2\mu/\delta^2$ and $\varepsilon = 2E/\delta^2$. The differential equation \ref{eq:10} has two regular singular points $z = 0$ and $z = 1/q$ with the indicial equations $\eta^2 + \varepsilon = 0$ (or $\eta = \sqrt{-2E/\delta}$) and $s(s-1) = 0$ (or $s = 0, 1$), respectively, in addition to an irregular singular point at $z = \infty$. Thus, the general solution of equation \ref{eq:10} assumes the form

$$ \psi(z) = z^n (1 - qz)^s f(z). \quad (7) $$

The factor $(1 - qz)^s$ must have $s$ a positive integer so that $\psi(1/q) = 0$. Thus $s$ cannot be zero. On substituting the ansatz \ref{eq:7} in equation \ref{eq:10}, it is not difficult to show that the function $f(z)$, after implementing the indicial equations, satisfies the differential equation

$$ z(1 - qz)f''(z) + (-q(2\eta + 2s + 1)z + 2\eta + 1)f'(z) + (v - qsz(2\eta + s))f(z) = 0. \quad (8) $$

This is a hypergeometric differential equation with exact solutions given, in terms of the Gauss hypergeometric functions, as

$$ f(z) = C_1 z^\eta (1 + 2\eta; 2; 1 + qz), $$

$$ f(z) = C_2 z^{2\eta} (1 + 2\eta; 2; 1 - qz), \quad (9) $$

However, the boundary condition $\psi(0) = 0$ forces that the vanishing of the constant $C_2 = 0$. Thus,

$$ f(z) = 2F_1 \left( \eta + s - \sqrt{\eta^2 + \frac{v}{q}}, \eta + s + \sqrt{\eta^2 + \frac{v}{q}}; \eta + 1; qz \right) $$

from which the general solution of Eq. \ref{eq:3} takes the form \ref{eq:10}. For polynomial solutions $f_n(z)$, the termination of the hypergeometric series \ref{eq:10} requires $\eta + 1 - \sqrt{\eta^2 + \frac{v}{q}} = -n, \quad n = 0, 1, 2, \ldots$ that yields the following expression for the eigenvalues (using $\eta = \sqrt{-2En/\delta}, \quad v = 2\mu/\delta^2$ and $s = 1$)

$$ E_n = -\frac{1}{2} \left( \frac{\mu}{q \delta (1 + n)} - \frac{\delta (1 + n)}{2} \right)^2 \quad (11) $$

with the exact (unnormalized) wave functions:

$$ \psi_n(x) = N_n \left( 1 - q e^{-\delta x} \right) e^{-\left( \frac{\mu}{q \delta (1 + n)} - \frac{(n + 1)\delta}{2} \right)x} 2F_1 \left( -n, 1 + \frac{2\mu}{q \delta (1 + n)}; \frac{2\mu}{q \delta^2 (1 + n)} - n; q e^{-\delta x} \right) \quad (12) $$

up to normalization constant $N_n$. Note, by using the Pfaff transformation identity

$$ 2F_1(\alpha,\beta;\gamma;z) = (1 - z)^{\gamma - \alpha - \beta} 2F_1(\gamma - \alpha, \gamma - \beta; \gamma;z), \quad (13) $$

we can write the exact solution \ref{eq:12} as

$$ \psi_n(x) = N_n e^{-\left( \frac{\mu}{q \delta (1 + n)} - \frac{(n + 1)\delta}{2} \right)x} 2F_1 \left( -n - 1, \frac{2\mu}{q \delta^2 (1 + n)}; \frac{2\mu}{q \delta^2 (1 + n)} - n; q e^{-\delta x} \right) \quad (14) $$

Thus, the number of the discrete bound-states is bounded above by the inequality

$$ 0 \leq n < -1 + \frac{1}{\delta} \sqrt{\frac{2\mu}{q}}, \quad \text{where} \quad 0 < q < \frac{2\mu}{\delta^2}. \quad (15) $$

### III. GENERALIZED HULTHÉN POTENTIAL: NORMALIZATION CONSTANT

The normalization constant $N_n$ in equation \ref{eq:12} can be evaluated, for $2\mu > q \delta^2 (m + 1)(n + 1)$, using the following definite integral

$$ I_{nm} = \int_{\log(q)/\delta}^{\infty} e^{-\left( \frac{2m+n}{2(1+m)} \right) x} \left( 1 - q e^{-\delta x} \right)^2 2F_1 \left( -n, 1; \frac{2\mu}{\delta^2 (n+1)q}; 1; \frac{2\mu}{\delta^2 (n+1)q} - n; q e^{-\delta x} \right) dx \quad (16) $$

$$ \times 2F_1 \left( -m, 1; \frac{2\mu}{\delta^2 (m+1)q}; 1; \frac{2\mu}{\delta^2 (m+1)q} - m; q e^{-\delta x} \right) dx. \quad (16) $$
To evaluate this definite integral, we use the change of variables $\tau = q e^{-\delta x}$ and note for $x = \log(q) / \delta$, that $\tau = 1/q$, while for $x = \infty$, $\tau = 0$. Further, by the series representation of the Hypergeometric function, it follows that

$$I_{nm} = \frac{1}{\delta} \sum_{i=0}^{n} \sum_{j=0}^{m} \frac{(-n)_i}{i!} \left( 1 + \frac{2\mu}{\delta \sigma^2(1+m)} \right)_i \frac{(-m)_j}{j!} \left( 1 + \frac{2\mu}{\delta \sigma^2(1+m)} \right)_j \int_0^{1/q} (q^\tau - 1)^{2\tau \frac{\mu}{\delta \sigma^2(1+m)+\nu}} \frac{d\tau}{\tau^{m+n+2}}. \quad (17)$$

The integral on the left-hand side can be evaluated in terms of the Gamma function to yield

$$I_{nm} = \frac{2q^{1+n-m} - \frac{\mu(m+n+2)}{\delta \sigma^2(m+1)(n+1)}}{\delta \Gamma \left( 2 - \frac{m+n}{2} - \frac{\mu(m+n+2)}{\delta \sigma^2(m+1)(n+1)} \right)} \times \sum_{i=0}^{n} \sum_{j=0}^{m} \frac{(-n)_i}{i!} \left( 1 + \frac{2\mu}{\delta \sigma^2(1+m)} \right)_i \frac{(-m)_j}{j!} \left( 1 + \frac{2\mu}{\delta \sigma^2(1+m)} \right)_j \Gamma \left( 2 - \frac{m+n}{2} - \frac{\mu(m+n+2)}{\delta \sigma^2(m+1)(n+1)} \right). \quad (18)$$

Using the Pochhammer identity $(\alpha)_i + j = (\alpha + i)_i (\alpha)_i$, equation (18) can be written as

$$I_{nm} = \frac{2q^{1+n-m} - \frac{\mu(m+n+2)}{\delta \sigma^2(m+1)(n+1)}}{\delta \Gamma \left( 2 - \frac{m+n}{2} - \frac{\mu(m+n+2)}{\delta \sigma^2(m+1)(n+1)} \right)} \times \sum_{i=0}^{n} \sum_{j=0}^{m} \frac{(-n)_i}{i!} \left( 1 + \frac{2\mu}{\delta \sigma^2(1+m)} \right)_i \left( 1 + \frac{2\mu}{\delta \sigma^2(1+m)} \right)_j \Gamma \left( 2 - \frac{m+n}{2} - \frac{\mu(m+n+2)}{\delta \sigma^2(m+1)(n+1)} \right). \quad (19)$$

For $m = n$, it follow that

$$I_{nn} = \frac{2q^{1+n-n} - \frac{2\mu}{\delta \sigma^2(1+n)}}{\delta \Gamma \left( 2 - \frac{2\mu}{\delta \sigma^2(1+n)} \right)} \times \sum_{i=0}^{n} \frac{(-n)_i}{i!} \left( 1 + \frac{2\mu}{\delta \sigma^2(1+n)} \right)_i \Gamma \left( 2 - \frac{2\mu}{\delta \sigma^2(1+n)} \right). \quad (20)$$

Breaking the finite sum, the equation can be written as

$$I_{nn} = \frac{2q^{1+n-n} - \frac{2\mu}{\delta \sigma^2(1+n)}}{\delta \Gamma \left( 2 - \frac{2\mu}{\delta \sigma^2(1+n)} \right)} \times \sum_{i=1}^{n} \frac{(-n)_i}{i!} \left( 1 + \frac{2\mu}{\delta \sigma^2(1+n)} \right)_i \Gamma \left( 2 - \frac{2\mu}{\delta \sigma^2(1+n)} \right). \quad (21)$$

Using the hypergeometric identity [20] formula 7.4.4.1

$$\sum_{a, b, c} \Gamma(d) \Gamma(\frac{e - a - b - c}{d + e - a - b} \Gamma(d - a + b) \Gamma(d - c) \cdot \cdot \cdot \left( 1 \right)_{ Re(d + e - a - b) > 0, Re(d - c) > 0),}$$
it follows

\[ I_{nn} = \frac{2q^{1+n}\frac{2\mu}{\gamma^2(n+1)q} \Gamma \left( \frac{2\mu}{\gamma^2(n+1)q} - n - 1 \right) \left. 3F_2 \left( -n, \frac{2\mu}{\gamma^2(n+1)q} - n, \frac{2\mu}{\gamma^2(n+1)q} + 1, \frac{2\mu}{\gamma^2(n+1)q} - n - 1 \right| 1 \right)}{\delta \Gamma \left( \frac{2\mu}{\gamma^2(n+1)q} - n + 2 \right)} \]

\[ + \frac{2q^{1+n}\frac{2\mu}{\gamma^2(n+1)q} \Gamma \left( \frac{2\mu}{\gamma^2(n+1)q} - n - 1 \right) \sum_{i=1}^{n} \frac{(-1)^i \left( \frac{2\mu}{\gamma^2(n+1)q} + 1 \right)_i \left( \frac{(2n+2)\mu}{\gamma^2(n+1)q} - n - 1 \right)_i \Gamma \left( i - n + \frac{2\mu}{\gamma^2(n+1)q} + 2 \right)}{\delta \Gamma \left( \frac{2\mu}{\gamma^2(n+1)q} - n + 2 \right) \left( \frac{2\mu}{\gamma^2(n+1)q} + 3 \right)} \times 3F_2 \left( 1 - i, \frac{2\mu}{\delta^2(n+1)q} + 1, \frac{2\mu}{\delta^2(n+1)q} + 3, \frac{2\mu}{\delta^2(n+1)} - n - 1 \right) \right] \]

However, because of the reciprocal of the Gamma function \(1/\Gamma(i - n + 1)\), where \(1/\Gamma(-m) = 0, m = 0, 1, 2, \ldots\), the finite sum survives only for \(i = n\) (otherwise each term is zero) whence

\[ I_{nn} = \frac{2q^{1+n}\frac{2\mu}{\gamma^2(n+1)q} \Gamma \left( \frac{2\mu}{\gamma^2(n+1)q} - n - 1 \right) \left. 3F_2 \left( -n, \frac{2\mu}{\gamma^2(n+1)q} - n, \frac{2\mu}{\gamma^2(n+1)q} + 1, \frac{2\mu}{\gamma^2(n+1)q} - n - 1 \right| 1 \right)}{\delta \Gamma \left( \frac{2\mu}{\gamma^2(n+1)q} - n + 2 \right)} \]

\[ + \frac{2q^{1+n}\frac{2\mu}{\gamma^2(n+1)q} \Gamma \left( \frac{2\mu}{\gamma^2(n+1)q} - n - 1 \right) \left( -n \right)_n \frac{2\mu}{\gamma^2(n+1)q} + 1 \frac{2\mu}{\gamma^2(n+1)q} - n - 1 \frac{2\mu}{\gamma^2(n+1)q} + 2 \Gamma \left( \frac{2\mu}{\gamma^2(n+1)q} + 3 \right)}{\delta \Gamma \left( \frac{2\mu}{\gamma^2(n+1)q} - n + 2 \right) \left( \frac{2\mu}{\gamma^2(n+1)q} + 3 \right)} \times 3F_2 \left( 1 - n, \frac{2\mu}{\delta^2(n+1)q} + 1, \frac{2\mu}{\delta^2(n+1)q} + 3, \frac{2\mu}{\delta^2(n+1)} - n - 1 \right) \right] . \quad (23) \]

Upon using the Pochhammer identities

\[ (-n)_n = (-1)^n n!, \quad \left( \frac{2\mu}{\gamma^2(n+1)q} - n - 1 \right)_n = \frac{2\mu}{\gamma^2(n+1)q} - n - 1 \left( \frac{2\mu}{\gamma^2(n+1)q} + 1 \right)_n = \frac{(-1)^n \left( \frac{2\mu}{\gamma^2(n+1)q} + 1 \right)_n}{\left( \frac{2\mu}{\gamma^2(n+1)q} - n + 2 \right)_n} , \]

it follows finally that

\[ \int_{\log(q)/\delta}^{\infty} (1 - q e^{-\delta x})^2 e^{-\left( \frac{2\mu}{\delta^2(n+1)} - (1+n)\delta \right)x} \left[ 2F_1 \left( -n, \frac{2\mu}{\delta^2(n+1)} + 1, \frac{2\mu}{\delta^2(n+1)} - n; q e^{-\delta x} \right) \right]^2 dx \]

\[ = \frac{2q^{n+1}\frac{2\mu}{\gamma^2(n+1)q} \Gamma \left( \frac{2\mu}{\gamma^2(n+1)q} - n - 1 \right) \left[ q \delta^2(n+1) (2\mu - \delta^2(n+1)^2) \left( \frac{2\mu}{\delta^2(n+1)} + 1 \right)_n \right]}{\delta \Gamma \left( \frac{2\mu}{\gamma^2(n+1)q} - n + 2 \right) \left( 2(2\mu - \delta^2(n+1)) (\delta^2(n+1) + \mu) \left( \frac{2\mu}{\delta^2(n+1)} - 1 \right)_n \right)} \times 3F_2 \left( 1 - n, \frac{2\mu}{\delta^2(n+1)q} + 1, \frac{2\mu}{\delta^2(n+1)q} - n \right) \left| 1 \right) + 3F_2 \left( -n, \frac{2\mu}{\delta^2(n+1)q} - n, \frac{2\mu}{\delta^2(n+1)q} - n + 2 \right) \left| 1 \right) . \quad (24) \]

Next, for the case where \(m \neq n\),

\[ I_{nm} = \frac{2q^{1+n-m}\frac{2\mu(m+n+2)}{\gamma^2(m+1)q} \Gamma \left( \frac{2\mu(m+n+2)}{\gamma^2(m+1)q} - m - n - 1 \right) \sum_{i=0}^{n} \left( -n \right)_i \left( \frac{2\mu}{\gamma^2(m+1)q} + 1 \right)_i \left( \frac{m+n+2}{\gamma^2(m+1)q} - m - n - 1 \right)_i \Gamma \left( i - m - n + \frac{2\mu}{\gamma^2(m+1)q} + 2 \right)}{\delta \Gamma \left( \frac{2\mu}{\gamma^2(m+1)q} - m + 2 \right) \left( \frac{2\mu}{\gamma^2(m+1)q} + 2 \right)_i} \times 3F_2 \left( -n, \frac{2\mu}{\gamma^2(m+1)q} + 1, \frac{2\mu}{\gamma^2(m+1)q} + \mu(2+m+n) \right) \left| 1 \right) . \quad (25) \]

Using the identity \[20\] formula 7.4.4.90

\[ 3F_2 \left( -m, a, a - \ell, b - s, b, 1 \right) = 0, \quad \text{if} \quad (\ell + s = 1, 2, 3, \ldots, m - 1) , \quad (26) \]
we see that it is enough to consider the case \( n = 0 \) and \( m \neq 0 \), for every other (fixed) value of \( n \), the parameter \( a \) is varied by a constant factor. Thus,

\[
I_{0m} = 2q^{1 + \frac{n}{2} - \frac{\mu(2m + 1)}{q^{\delta(n+1)}}} \Gamma \left( \frac{2m}{q^{\delta(n+1)}} - n + 1 \right) \frac{2^m}{q^{\delta(n+1)}} - n + 1 \right) \Gamma \left( \frac{2m}{q^{\delta(n+1)}} - n - 1 \right) \right)
\]

\[
\int_{\log(q)/\delta}^{\infty} (1 - q^{-\delta x})^2 e^{-\left(\frac{2m}{q^{\delta(n+1)}} - (1+n)\delta x\right)} 2F_1 \left( -n, \frac{2m}{q^{\delta(n+1)}} - n + 1; \frac{2m}{q^{\delta(n+1)}} - n + 1 \right) \right) dx = I_{nm} \delta_{nm}, \quad (28)
\]

where

\[
I_{nm} = \frac{2q^{n+1} - \frac{2m}{q^{\delta(n+1)}}}{\delta \Gamma \left( \frac{2m}{q^{\delta(n+1)}} - n + 2 \right)} \left[ \frac{q \delta^2(n+1) (2m - q \delta^2(n+1)^2)}{2 (2m - q \delta^2(n+1)) (q \delta^2(n+1) + \mu) \left( -\frac{2m}{q^{\delta(n+1)}} - n \right)} \right] + 3 \left. 2F_2 \left( \frac{2m}{q^{\delta(n+1)}} - n, \frac{2m}{q^{\delta(n+1)}} - n + 1, \frac{2m}{q^{\delta(n+1)}} - n + 1 \right) \right] \right) \right.
\]

\[
IV. CRUM-DARBOUX TRANSFORMATION: SEQUENTIAL TRANSFORMATIONS
\]

An important technique for generating classes of exactly-solvable quantum potentials is build on the concept of intertwining operators \([21]\). Two Hamiltonian operators \( H_0 \) and \( H_1 \) are said to be intertwined if there exist an operator \( \mathcal{L} \) so that

\[
H_1 \mathcal{L} = \mathcal{L} H_0.
\]

In this case, if \( \phi_{0,n}(x) \) and \( \psi_{1,n}(x) \) are eigenfunctions of the intertwined operators \( H_0 \) and \( H_1 \) respectively, then the two sets of eigenfunctions are related \([22], p. 63\) by the operator \( \mathcal{L} \) through the relations

\[
\psi_{1,n}(x) = \mathcal{L} \phi_{0,n}(x), \quad \phi_{0,n}(x) = \mathcal{L}^\dagger \psi_{0,n}(x),
\]

and the Hamiltonians \( H_0 \) and \( H_1 \) are said to be isospectral, i.e. share the same spectrum, except for those states that are annihilated by \( \mathcal{L} \) or \( \mathcal{L}^\dagger \). In the context of the one-dimensional quantum mechanics, \( \mathcal{L} \) is taken to be a first-order linear differential operator \( \mathcal{L} = \partial_x + f(x) \) intertwines two one-dimensional Schrödinger Hamiltonians \( H_0 = -\partial_{xx} + V(x) \) and \( H_1 = -\partial_{xx} + V(x) \) where \( x \in (a, b) \) that can be finite or infinite interval. Here, \( \partial_x \) refers to the first-derivative with respect to the variable \( x \). Direct computation, using \([30]\), yields

\[
(-2\partial_x f(x) + V(x) - V(x)) \partial_x \phi_n(x) = \left( \partial_x V(x) + [V(x) - V(x)] f(x) + \partial_x^2 f(x) \right) \phi_n(x).
\]

The consistency condition of \([32]\) then requires

\[
-2\partial_x f(x) + V(x) - V(x) = 0, \quad \text{and} \quad \partial_x V(x) + [V(x) - V(x)] f(x) + \partial_x^2 f(x) = 0.
\]

Substituting the first condition into the second one gives

\[
\partial_x \left( V(x) - f^2(x) + \partial_x f(x) \right) = 0 \quad \text{or} \quad V(x) - f^2(x) + \partial_x f(x) = \lambda
\]

for some constant \( \lambda \). Clearly, the function \( f(x) = -\partial_x \phi(x)/\phi(x) = -\partial_x \log \phi(x) \) is a particular solution of the Riccati equation \([34]\) provided that

\[
-\partial_{xx} \phi(x) + V(x) \phi(x) = \lambda \phi(x),
\]
that is to say, provided that $\phi$ is an eigenfunction of the Hamiltonian $H_0$ with eigenvalue $\lambda$. An important conclusion from this approach is that every no-node eigenfunction, called a seed function, $\phi = \phi_{0,0}$ of $H_0$ (regardless of the normalizability) generates a new solvable Hamiltonian $H_1$ with the potential $V$ expressed in terms of $V$ and the seed function as

$$V(x) = V(x) + 2 \frac{d^2}{dx^2} \log \phi_{0,0}(x).$$  \hfill (36)

The first-order differential intertwining operator

$$\mathcal{L} = \frac{d}{dx} - \frac{1}{\phi_{0,0}} \frac{d\phi_{0,0}}{dx}$$  \hfill (37)

is known as a Darboux transformation \[23\]. The following theorem summarize these results:

**Theorem IV.1.** The eigenfunctions $\psi_{1,n}(x), n = 1, 2, \ldots$ of the Schrödinger equation

$$\left(-\frac{d^2}{dx^2} + V(x)\right) \psi_{1,n}(x) = E_n \psi_{1,n}(x) \text{ where } V(x) = V(x) - 2 \frac{d^2}{dx^2} \log \phi_{0,0}(x), \quad x \in (a, b)$$  \hfill (38)

are generated using

$$\psi_{1,n}(x) = \left(\frac{d}{dx} - \frac{\phi'_{0,0}(x)}{\phi_{0,0}(x)}\right) \phi_{0,n}(x) = \frac{W(\phi_{0,0}(x), \phi_{0,n}(x))}{W(\phi_{0,0}(x))}, \quad n = 1, 2, \ldots,$$  \hfill (39)

where $W(\psi_{0,0}(x)) \equiv \phi_{0,0}(x)$ and $W(\phi_{0,0}(x), \phi_{0,n}(x)) = \psi_{0,0}(x)\psi'_{0,n}(x) - \psi_{0,n}(x)\psi'_{0,0}(x)$ is the classical Wronskian. Here, $\phi_{0,n}(x), n = 0, 1, 2, \ldots,$ are solutions of the Schrödinger equation

$$-\phi''_{0,n}(x) + V(x)\phi_{0,n}(x) = E_n\phi_{0,n}(x), \quad n = 0, 1, 2, \ldots, \quad \phi_{0,n}(a) = \phi_{0,n}(b) = 0, \quad \phi_{0,0}(x) \neq 0 \forall x \in (a, b).$$  \hfill (40)

Further, if $\phi_{0,n}(x)$ are normalized according to the condition

$$\int_a^b \phi_{0,n}(x)\phi_{0,m}(x)dx = \eta_n \delta_{mn},$$  \hfill (41)

where $\delta_{mn}$ is the known Kronecker delta, then

$$\int_a^b \phi_{1,n}(x)\phi_{1,m}(x)dx = (E_n - E_0) \eta_n \delta_{mn}, \quad n = 1, 2, \ldots.$$  \hfill (42)

The proof of the normalization relation is given in the Appendix. It is evident that such uses of Darboux’s transformation can be applied to \[19\] using $\mathcal{L}_2 = \frac{d}{dx} - \frac{\phi'_{0,1}(x)}{\phi_{0,1}(x)}$, again to reproduce a new solvable eigenvalue problem $H_2$ and such a procedure may be repeated an arbitrary number of times \[24\] as long as the consecutive (generated) Hamiltonians support the existence of (discrete) states, as illustrated by the following diagram:

$$H_0 \xrightarrow{\mathcal{L}_0} H_1 \xrightarrow{\mathcal{L}_1} H_2 \xrightarrow{\mathcal{L}_3} \cdots \xrightarrow{\mathcal{L}_{j-1}} H_j$$

**Sequential Darboux transformations**

M. M. Crum \[25\] in his remarkable paper 1955 introduced an elegant approach to evaluate the eigenfunctions of the Hamiltonian $H_j$, expressed entirely in terms of the eigenfunctions of the initial Hamiltonian $H_0$, without any reference to the intermediate Hamiltonians:
This approach can be illustrated as follows: consider the eigenfunctions $\phi_{1,n}(x)$, $n = 1, 2, \ldots$ of the Hamiltonian $\mathcal{H}_1$ generated using $\phi_{0,n}(x)$, $n = 0, 1, 2, \ldots$; the solutions of an initial Hamiltonian $\mathcal{H}_0 = -\partial_{xx} + V(x)$; and employed in a second Darboux transformation to obtain

$$\phi_{2,n}(x) = \left(\frac{d}{dx} - \frac{d}{dx} \log(\phi_{1,1}(x))\right) \phi_{1,n}(x) = \frac{W(\phi_{0,0}(x), \phi_{0,1}(x), \phi_{0,n}(x))}{W(\phi_{0,0}(x), \phi_{0,1}(x))}, \quad n = 2, 3, \ldots \quad (43)$$

where we perform the computations using (39) and the definition of the Wronskian. This approach, in turn, generates a new class of solvable Schrödinger equations

$$-\frac{d^2}{dx^2} \phi_{2,n}(x) + \left(V(x) - 2 \frac{d^2}{dx^2} \left(\log \left[W(\phi_{0,0}(x), \phi_{0,1}(x))\right]\right)\right) \phi_{2,n}(x) = E_n \phi_{2,n}(x), \quad n = 2, 3, \ldots \quad (44)$$

expressed entirely in terms of the eigenfunctions of the Hamiltonian $\mathcal{H}_0$ with no reference to the intermediate Hamiltonian $\mathcal{H}_1$. For the eigenvalue problem (44), we may now employ the third transformation

$$\phi_{3,n}(x) = \left(\frac{d}{dx} - \frac{d}{dx} \log \phi_{2,2}(x)\right) \phi_{2,n}(x) = \frac{W(\phi_{0,0}(x), \phi_{0,1}(x), \phi_{0,2}(x), \phi_{0,n}(x))}{W(\phi_{0,0}(x), \phi_{0,1}(x), \phi_{0,2}(x))}, \quad n = 3, 4, \ldots \quad (45)$$

to obtain the third solvable class of Schrödinger equations

$$-\frac{d^2}{dx^2} \phi_{3,n}(x) + \left(V(x) - 2 \frac{d^2}{dx^2} \left(\log \left[W(\phi_{0,0}(x), \phi_{0,1}(x), \phi_{0,2}(x))\right]\right)\right) \phi_{3,n}(x) = E_n \phi_{3,n}(x), \quad n = 3, 4, \ldots \quad (46)$$

without reference to the intermediate Hamiltonians $\mathcal{H}_1$ and $\mathcal{H}_2$. This process may be generalized to include the case of $j$-times repeated Darboux transformation, expressed entirely in terms of the eigenfunctions for the initial Hamiltonian $\mathcal{H}_0$ to give: The transformed functions

$$\psi_{j,n}(x) = \left(\frac{d}{dx} - \frac{d}{dx} \log \psi_{j-1,1}(x)\right) \psi_{j-1,n}(x) = \frac{W(\phi_{0,0}(x), \phi_{0,1}(x), \phi_{0,2}(x), \ldots, \phi_{0,k-1}(x), \phi_{0,n}(x))}{W(\phi_{0,0}(x), \phi_{0,1}(x), \phi_{0,2}(x), \ldots, \phi_{0,k-1}(x))}, \quad n = k, k+1, k+2, \ldots \quad (47)$$

satisfy the Schrödinger equation

$$-\frac{d^2}{dx^2} \phi_{j,n}(x) + \left(V(x) - 2 \frac{d^2}{dx^2} \left(\log \left[W(\phi_{0,0}(x), \phi_{0,1}(x), \phi_{0,2}(x), \ldots, \phi_{0,j-1}(x))\right]\right)\right) \phi_{j,n}(x) = E_n \phi_{j,n}(x), \quad (48)$$

where $n = j, j+1, j+2, \ldots$. The proof of Curm’s formulation [25] is based on the following Wronski identity:

$$W(\phi_{0,0}, \phi_{0,1}, \ldots, \phi_{0,j-1})W(\phi_{0,0}, \phi_{0,1}, \ldots, \phi_{0,j}, \phi_{0,n}) = W(\phi_{0,0}, \phi_{0,1}, \ldots, \phi_{0,j}), W(\phi_{0,0}, \phi_{0,1}, \ldots, \psi_{j-1,1}, \phi_{0,n}). \quad (49)$$

The general expression of the normalization constants is given in terms of the normalization of $\mathcal{H}_0$ as:

$$\int_a^b \phi_{j,n}(x) \phi_{j,m}(x) dx = \mu_n \delta_{nm} \prod_{i=0}^{j-1} (E_n - E_i), \quad \text{where} \quad \mu_n = \int_a^b [\phi_{0,n}(x)]^2 dx. \quad (50)$$
V. THE CRUM-DARBOUX TRANSFORMATION AND THE GENERALIZED HULTHÉN POTENTIAL

From section III, we may write the exact solutions of the Schrödinger equation

\[
-\frac{d^2 \psi_{0,n}(r)}{dr^2} - \frac{v e^{-r}}{1 - q e^{-r}} \psi_{0,n}(r) = \mathcal{E}_n \psi_{0,n}(r), \quad r \in [\log(q), \infty), \quad q > 0, \quad \psi_{0,n}(\log(q)) = \psi_{0,n}(\infty) = 0,
\]

are simply, for \(n = 0, 1, 2, \ldots\),

\[
\mathcal{E}_n = -\left(\frac{v}{2 q (n + 1)} - \frac{n + 1}{2}\right)^2,
\]

\[
\psi_{0,n}(r) = N_n (1 - q e^{-r}) e^{-\left(\frac{v}{nq + q} - \frac{n+1}{2}\right) r} 2F_1 \left(-n, \frac{v}{(n+1)q} + 1; \frac{v}{(n+1)q} - n; q e^{-r}\right),
\]

up to the normalization constant evaluated using the following definite integral (see equation (28) and (29)):

\[
N_n^2 \int_{\log(q)}^{\infty} (1 - q e^{-r})^2 e^{-\left(\frac{v}{nq + q} - n\right) r} \left[2F_1 \left(-n, \frac{v}{q(1+n)} + 1; \frac{v}{q(1+n)} - n; q e^{-r}\right)\right]^2 dr = 1
\]

as

\[
N_n = \left\{\frac{2 q^{n+1} - \frac{v}{nq + q}}{\Gamma\left(\frac{v}{nq + q} - n + 2\right)} \frac{q (n + 1) (v - q (n + 1)^2)}{(v - q (n + 1)) (2 q (n + 1) + v)} \frac{\Gamma\left(\frac{v}{nq + q} - n\right)}{\Gamma\left(\frac{v}{nq + q} - 1\right)} \right\}^{-1/2},
\]

and subject to the parameter constraints

\[
v > 0, \quad 0 < q < v, \quad 0 \leq n < \sqrt{v} - 1,
\]

where \(n\) is an integer.

A. First Transformation

Consider the Darboux transformation

\[
\psi_1,n(r) = \left[\frac{d}{dr} - \frac{1}{\psi_{0,0}(r)} \frac{d\psi_{0,0}(r)}{dr}\right] \psi_{0,n}(r) = W(\psi_{0,0}(r), \psi_{0,n}(r)) / \psi_{0,0}(r), \quad n = 1, 2, \ldots,
\]

where the seed function \(\psi_{0,0}(r)\) given as the ground-state wave-function \([52]\), from which a new solvable potential is obtained:

\[
V_1(r) = V(r) - 2 \frac{d^2}{dr^2} \log \psi_{0,0}(r) = -\frac{v e^{-r}}{1 - q e^{-r}} + \frac{2 q e^{-r}}{(1 - q e^{-r})^2}
\]

Thus, the Schrödinger equation with the potential \(V_1\) is exactly solvable

\[
-\frac{d^2 \psi_{1,n}(r)}{dr^2} + \left(-\frac{v e^{-r}}{1 - q e^{-r}} + \frac{2 q e^{-r}}{(1 - q e^{-r})^2}\right) \psi_{1,n}(r) = -\mathcal{E}_n \psi_{1,n}(r), \quad n = 1, 2, \ldots,
\]

with (up to normalization constant) exact wave function solutions, for \(n = 1, 2, \ldots\),

\[
\psi_{1,n}(r) = \frac{n (e^r - q)}{2 q (n + 1)} e^{-\frac{v}{2 q (n + 1) \log(q - 1)}} \left[2 q (n + 1) \frac{v}{nq + q} \right] 2F_1 \left(-n, \frac{v}{nq + q} + 1; \frac{v}{nq + q} - n; q e^{-r}\right)
\]

\[
+(v - (n + 1)q) 2F_1 \left(-n, \frac{v}{nq + q} + 1; \frac{v}{nq + q} - n; q e^{-r}\right),
\]
subject to \( v > 0, q > 0, 1 \leq n < \sqrt{(q+v)/q} \). Using the contiguous relation
\[
(\gamma - \alpha - \beta)_{2F_1}(\alpha, \beta; \gamma; z) + \alpha(1-z)_{2F_1}(\alpha+1, \beta; \gamma; z) - (\gamma - \beta)_{2F_1}(\alpha, \beta - 1; \gamma; z) = 0, \tag{59}
\]
equation (58) can be written in more compact form as:
\[
\psi_{1,n}(r) = \frac{n(qn + q + v)}{2(n+1)q} e^{-\frac{v}{q}} \left(1 - q e^{-r}\right)^{2\frac{2}{nq + q} - n} 2F_1 \left(1 - n, \frac{q}{nq + q} + 2; \frac{v}{nq + q} + n + e^{-r}q\right), \quad (n = 1, 2, \ldots, n < \sqrt{(q+v)/q}). \tag{60}
\]

\[ \text{B. Second transformation} \]

Using these exact solutions (60), it is possible via Crum’s approach to generate sequential transformation of the Hulthén potential with
\[
\psi_{2,n}(r) = \left[ \frac{d}{dr} - \psi'_{1,1}(r) \right] \psi_{1,n}(r) \equiv \frac{W(\psi_{0,0}(r), \psi_{0,1}(r), \psi_{0,n}(r))}{W(\psi_{0,0}(r), \psi_{0,1}(r))}, \quad n = 2, 3, \ldots, \tag{61}
\]
where the seed function \( \psi_{1,1}(r) \) given by the ground-state wave function (60) for \( n = 1 \). Thus, a new solvable potential is obtained:
\[
V_2(r) = V_1(r) - 2 \frac{d^2}{dr^2} \log \left[ \psi_{1,1}(r) \right] = V(r) - 2 \frac{d^2}{dr^2} \log W(\psi_{0,0}(r), \psi_{0,1}(r)) = - \frac{v e^{-r}}{1 - q e^{-r}} + \frac{6 q e^{-r}}{(1 - q e^{-r})^2}. \tag{62}
\]
This potential has exact analytic solutions of the Schrödinger equation
\[
- \frac{d^2}{dr^2} \psi_{2,n}(r) + \left( - \frac{v e^{-r}}{1 - q e^{-r}} + \frac{6 q e^{-r}}{(1 - q e^{-r})^2} \right) \psi_{2,n}(r) = - \delta_n \psi_{2,n}(r), \quad n = 2, 3, \ldots, \tag{63}
\]
with exact solutions (up to normalization constant) given for \( n = 2, 3, \ldots, \)
\[
\psi_{2,n}(r) = \frac{n(n-1)(v+q(n+1)(v+2q(n+1)))}{8(1+n)^2 q^2} e^{-\frac{v}{q}} \left(1 - q e^{-r}\right)^{2\frac{2}{nq + q} - n} 2F_1 \left(1 - n, \frac{q}{nq + q} + 2; \frac{v}{nq + q} + n + e^{-r}q\right)
\]
\[
\times \left(2 F_1 \left(1 - n, \frac{v}{nq + q} + 2; \frac{v}{nq + q} + n + e^{-r}q\right) - \frac{4 q^2 (n+1)}{(nq + n^2 q - v)} e^{-r} 2 F_1 \left(2 - n, \frac{v}{nq + q} + 3; \frac{v}{nq + q} + n + 1 + e^{-r}q\right) \right). \tag{64}
\]
That can be written as
\[
\psi_{n,2}(r) = \frac{n(n-1)(v+q(n+1)+v+2q(n+1))}{8q^2(n+1)^2} e^{-\frac{v}{q}} \left(1 - q e^{-r}\right)^{3\frac{2}{nq + q} - n} 2F_1 \left(2 - n, \frac{v}{nq + q} + 3; \frac{v}{nq + q} + n + 1 + e^{-r}q\right), \tag{65}
\]
subject to \( v > 0, 0 < q < v/3, 2 \leq n < \sqrt{(q+v)/q} \).

\[ \text{C. The } j^{\text{th}} \text{ Transformation} \]

The class of generalized Hulthén potentials given by an arbitrary \( j^{\text{th}} \) transformation, \( j = 1, 2, \ldots \) can be established using Crum’s approach, see Section III to give
\[
V_j(r) \equiv - \frac{v e^{-r}}{1 - q e^{-r}} - 2 \frac{d^2}{dr^2} \log W(\psi_{0,0}(r), \psi_{1,0}(r), \ldots, \psi_{j-1,0}(r)), \quad j = 1, 2, \ldots, n. \tag{66}
\]
We shall show, by induction on \( j \), that

\[
-2 \frac{d^2}{dr^2} \log W(\psi_{0,0}(r), \psi_{1,0}(r), \ldots, \psi_{j-1,0}(r)) = \frac{j(j+1) q e^r}{(e^r - q)^2} \tag{67}
\]

whence

\[
V_j(r) = -\frac{ve^{-r}}{1-qe^{-r}} + \frac{j(j+1) q e^r}{(e^r - q)^2}, \quad j = 1, 2, \ldots \tag{68}
\]

For \( j = 1 \), using \( W(\psi_{0,0}(r)) = \psi_{0,0}(r) \) we note for

\[
\psi_{0,0}(r) = e^{-r(\frac{v}{2} - \frac{1}{2})}(1-qe^{-r}), \quad v > q > 0,
\]

\[
\frac{d}{dr} \log \psi_{0,0}(r) = \frac{1}{2} + \frac{q}{e^r - q} - \frac{v}{2q}; \quad \frac{d^2}{dr^2} \log \psi_{0,0}(r) = -\frac{q e^r}{(e^r - q)^2}. \tag{69}
\]

and equation (67) is true for \( j = 1 \). Assume, equation (67) is true for \( j' \), that is to say

\[
-2 \frac{d^2}{dr^2} \log W(\psi_{0,0}(r), \psi_{1,0}(r), \ldots, \psi_{j'-1,0}(r)) = \frac{j'(j'+1) q e^r}{(e^r - q)^2} \tag{70}
\]

then for \( j = j'+1 \), since

\[
-2 \frac{d^2}{dr^2} \log W(\psi_{0,0}(r), \psi_{1,0}(r), \ldots, \psi_{j'-1,0}(r), \psi_{j,0}(r))
\]

\[
= -2 \frac{d^2}{dr^2} \log W(\psi_{0,0}(r), \psi_{1,0}(r), \ldots, \psi_{j'-1,0}(r)) - 2 \frac{d^2}{dr^2} \log \frac{W(\psi_{0,0}(r), \psi_{1,0}(r), \ldots, \psi_{j'-1,0}(r), \psi_{j,0}(r))}{W(\psi_{0,0}(r), \psi_{1,0}(r), \ldots, \psi_{j'-1,0}(r))}
\]

\[
= \frac{j'(j'+1) q e^r}{(e^r - q)^2} - 2 \frac{d^2}{dr^2} \log \psi_{j'-1,0}(x) = \frac{j'(j'+1) q e^r}{(e^r - q)^2} + V_{j'+1}(r) - V_j(r)
\]

\[
= \frac{ve^{-r}}{1-qe^{-r}} + V_{j'+1}(r), \tag{71}
\]

which ensures the truth of the identity (67). This potential has exact analytic solutions

\[
\psi_{j,n}(r) = \psi_{j-1,n}(r) - \frac{\psi'_{j-1,1}(r)}{\psi_{j-1,1}(r)} \psi_{j-1,n}(r) = \frac{W(\psi_{0,0}(r), \psi_{1,0}(r), \ldots, \psi_{j-1,0}(r), \psi_{n,0}(r))}{W(\psi_{0,0}(r), \psi_{1,0}(r), \ldots, \psi_{j-1,0}(r))}, \quad n = j, j+1, \ldots, \tag{72}
\]

where \( n \) indicate the finiteness of the discrete bound states. The eigenfunctions (69) are the solutions of the Schrödinger equation

\[
-\frac{d^2}{dr^2} \psi_{n,j}(r) + \left( \frac{j(j+1) q e^{-r}}{v (1-qe^{-r})^2} - \frac{ve^{-r}}{1-qe^{-r}} \right) \psi_{n,j}(r) = -\mathcal{E}_n \psi_{n,j}(r), \quad r \in [\log(q), \infty) \tag{73}
\]

with

\[
\mathcal{E}_n = -\left( \frac{v}{2q} - \frac{n+1}{2} \right)^2, \quad n = j, j+1, \ldots \tag{74}
\]

subject to \( v > 0, q > 0, j \leq n < \sqrt{(q+v)/q} \). Obviously, the evaluation of the general expression for \( \psi_{j,n}(r) \) using the relation (69) is not straightforward. However, we can find a general expression by analyzing the possible solutions of the Schrödinger equation

\[
-\frac{d^2}{dr^2} \psi(r) + \left( \frac{\mu e^{-r}}{v (1-qe^{-r})^2} - \frac{ve^{-r}}{1-qe^{-r}} \right) \psi(r) = \mathcal{E}_n \psi(r), \quad r \in [\log(q), \infty) \tag{75}
\]

where \( \mu \) is an arbitrary constant that supports the existence of discrete bound-states.
VI. EXTENDED HULTHÉN’S POTENTIAL: EXACT SOLUTIONS

In this section, we analyze the exact solutions of equation (75) which will allow us to obtain a compact formula for the \( j \)-transformed \( \psi_{j,n}(r) \) as given by (69). Using a similar approach to that discussed in Section II, it is not difficult to show that the change of variable \( z = e^{-r} \) along with the analysis of the regular singular points implies by means of the ansatz solution

\[ \psi(z) = z^\eta \left(1 - q z\right)^s f(z), \quad \text{where} \quad \eta^2 + \mathcal{E}_n = 0, \quad qs^2 - qs - \mu = 0. \]  

(76)

The following second-order differential equation for \( f(z) \)

\[ z \left(1 - q z\right) f''(z) + \left[1 + 2 \eta - q \left(1 + 2 \eta + 2 s\right)z\right] f'(z) + \left[v - q \left(2 \eta s + s^2\right)\right] f(z) = 0 \]

(77)

The exact solutions of this equation are given in terms of the Gauss hypergeometric functions as

\[ f(z) = _2F_1 \left(\eta + s - \sqrt{\eta^2 + \frac{v}{q}}, \eta + s + \sqrt{\eta^2 + \frac{v}{q}}; 1 + 2 \eta; q z\right), \]

(78)

up to the normalization constant where \( \eta = \sqrt{-\mathcal{E}_n} \) and \( s_+ = \frac{1}{2} + \sqrt{\frac{\mu}{q} + \frac{1}{4}} \). Imposing the termination condition on the hypergeometric function, to obtain polynomial solutions, implies the exact solutions of Schrödinger’s equation (75) as

\[ \psi(r) = e^{-\left(\frac{\eta(qn + q) - \frac{n + s}{2}}{2}\right)r} \left(1 - q e^{-r}\right)^{s_+} _2F_1 \left(-n, s_+ + \frac{v}{q(n + s_+)}; 1 - n - s_+ + \frac{v}{q(n + s_+)}; q e^{-r}\right), \]

(79)

where

\[ \eta = \sqrt{-\mathcal{E}_n}, \quad s_+ = \frac{1}{2} + \sqrt{\frac{\mu}{q} + \frac{1}{4}}, \quad \mathcal{E}_n = -\left(\frac{v}{2q(n + s_+)} - \frac{n + s_+}{2}\right)^2, \]

(80)

up to the normalization constant that, as before, can be evaluated exactly. With \( \mu = q j (j + 1) \), we obtain \( s_+ = j + 1, j = 0, 1, 2, \ldots \), which proves the consistency between \( \mathcal{E}_n \) and \( \mathcal{E}_n \) as given by (74). Finally, we can now write, for \( q < (2v)/(1 + 2j + j^2) \), \( 0 \leq n < -1 - j + \sqrt{2v/q} \),

\[ \psi_{j,n}(r) = \frac{W(\psi_{0,0}(r), \psi_{1,0}(r), \ldots, \psi_{j-1,0}(r), \psi_{n,0}(r))}{W(\psi_{0,0}(r), \psi_{1,0}(r), \ldots, \psi_{j-1,0}(r))} \]

\[ = e^{-\left(\frac{\eta(qn + q) - \frac{n + s}{2}}{2}\right)r} \left(1 - q e^{-r}\right)^{j+1} _2F_1 \left(-n, j + 1 + \frac{v}{q(n + j + 1)}; \frac{v}{q(n + j + 1)}; q e^{-r}\right), \]

\[ = e^{-\left(\frac{\eta(qn + q) - \frac{n + s}{2}}{2}\right)r} \left(1 - q e^{-r}\right)^{j+1} _2F_1 \left(j - n, j + 1 + \frac{v}{q(n + 1)}; \frac{v}{q(n + 1)}; q e^{-r}\right), \quad n = j, j + 1, \ldots \]

(81)

which is total agreement with the normalization constant obtained through the identity

\[ \int_{\log q}^{\infty} \psi_{j,n}(r) \psi_{j,m}(r) dx = \frac{\left(-j\right)\left(j + 2n + 2\right)\left(n - \frac{v}{(j + n + 1)q} + 1\right)}{\left(2\left(j + n + 1\right)\right)^2} \times N_n \delta_{nm} \]

(82)

and \( N_n \) is given by equation (54).

VII. CONCLUSION

General expressions for the energy eigenvalues and wave function solutions are obtained for Schrödinger’s equation with the generalized Hulthén potential. The simplified closed-form expressions for the normalization constants for arbitrary \( q > 0 \) in terms of the generalized Hypergeometric functions \( _3F_2 \) with the unit argument are new results. These include, as a particular case, the closed-form expression for the normalization constants of the classical Hulthén potential \( q = 1 \). It is also of interest to note that the double sum in equation (17) can be evaluated in terms of the
terminating generalized Kampé de Fériet function with unit arguments to yield

\[
I_{nm} = \int_{\log(q)/\delta}^{\infty} e^{\left(\frac{2m+2n}{2(n+m)(m+n)q}\right)x} \left(1 - q e^{-\delta x}\right)^2 \frac{2\mu}{\delta^2(n+1)q + 1} - \frac{2\mu}{\delta^2(n+1)q - n} q e^{-\delta x} \right) dx
\times 2F_1 \left( -m, \frac{2\mu}{\delta^2(m+1)q + 1}; \frac{2\mu}{\delta^2(m+1)q - m} q e^{-\delta x} \right) \times 2 \int_{\frac{\mu(n+m+2)}{\delta^2(m+1)(n+1)q} - \frac{m+n-1}{2}}^{1} \left( \frac{\mu(n+m+2)}{\delta^2(m+1)(n+1)q} - \frac{m+n-1}{2} \right)^{\frac{1}{2}}
\]

Thus we obtain as a byproduct a simplified expression for the terminating generalized Kampé de Fériet function.

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APPENDIX I: NORMALIZATION RELATION

To prove the normalization \([32]\), we note first

\[
\int_{a}^{b} \psi_n(x, E_n) \psi_m(x, E_m) dx = \int_{a}^{b} \phi_n(x, E_n) \phi_m(x, E_m) dx - \int_{a}^{b} \phi'_n(x, E_n) \phi_m(x, E_m) \frac{d}{dx} \log \phi_0(x, \lambda_0) dx
\]

Using integration by parts and the boundary conditions, it is not difficult to show that

\[
\int_{a}^{b} \phi'_n(x, E_n) \phi'_m(x, E_m) dx = E_n \mu_n \delta_{nm} - \int_{a}^{b} \phi_n(x, E_m) V(x) \phi_n(x, E_n) dx
\]

\[
\int_{a}^{b} \phi_n(x, E_m) \phi(x, E_n) \frac{d}{dx} \log \phi_0(x, E_0) dx = \phi_n(x, E_n) \phi_m(x, E_m) \frac{d}{dx} \log \phi_0(x, E_0) \bigg|_{a}^{b}
\]

\[
- \int_{a}^{b} \phi_n(x, E_n) \phi'_m(x, E_m) \frac{d}{dx} \log \phi_0(x, E_0) dx - \int_{a}^{b} \phi_n(x, E_n) \phi_m(x, E_m) \frac{d^2}{dx^2} \log \phi_0(x, E_0) dx
\]

\[
\int_{a}^{b} \phi'_n(x, E_n) \phi_m(x, \lambda_n) \frac{d}{dx} \log \phi_0(x, \lambda_0) dx = \phi_n(x, E_n) \phi_m(x, E_m) \frac{d}{dx} \log \phi_0(x, E_0) \bigg|_{a}^{b}
\]

\[
- \int_{a}^{b} \phi_n(x, E_n) \phi'_m(x, E_m) \frac{d}{dx} \log \phi_0(x, E_0) dx - \int_{a}^{b} \phi_n(x, E_n) \phi_m(x, E_m) \frac{d^2}{dx^2} \log \phi_0(x, E_0) dx
\]

\[
\int_{a}^{b} \left( \frac{\phi'_n(x, \lambda_0)}{\phi_0(x, \lambda_0)} \right)^2 \phi_n(x, E_n) \phi_m(x, E_m) dx = \int_{a}^{b} \left( V(x) - E_n - \frac{d^2}{dx^2} \log \phi_0(x, \lambda_0) \right) \phi_n(x, \lambda_n) \phi_m(x, \lambda_m) dx,
\]

from which the assertion \([42]\) is proved.

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