Non-Markovian dynamics revealed at the bound state in continuum

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We propose a methodical approach to controlling and enhancing deviations from exponential decay in quantum and optical systems by exploiting recent progress surrounding another subtle effect: the bound states in continuum, which have been observed in optical waveguide array experiments within this past decade. Specifically, we show that by populating an initial state orthogonal to that of the bound state in continuum, it is possible to engineer system parameters for which the usual exponential decay process is suppressed in favor of inverse power law dynamics and coherent effects that typically would be extremely difficult to detect in experiment. We demonstrate our method using a model based on an optical waveguide array experiment, and further show that the method is robust even in the face of significant detuning from the precise location of the bound state in continuum.

A bound state in continuum (BIC) represents a localized eigenmode with energy eigenvalue that, counter-intuitively, resides directly within the scattering continuum of a given physical system. Although the existence of such modes were first predicted in 1929 [1], the phenomenon is so delicate that they were not observed until much more recently [2], for example, in optical waveguide array experiments [3,4]. Lasing action has also recently been reported for a cavity supporting BICs [5]. In this work, we propose to apply these recent technical advances in optical control of the BIC to the study of another often elusive phenomenon: long-time non-exponential decay.

In many familiar circumstances, such as atomic relaxation, we tend to think of quantum decay as essentially an exponential process. More precisely, exponential decay tends to manifest when an unstable eigenmode (such as an excited atomic level) is resonant with an energy continuum (environmental reservoir, such as the electromagnetic vacuum) to which it is coupled. However, it can be shown that in fact all quantum systems follow non-exponential dynamics on very short and extremely long timescales. These deviations occur as a direct result of the existence of at least one threshold on the energy continuum in such systems [6–14]. While these effects are ubiquitous in quantum systems, they are unfortunately quite difficult to detect under ordinary circumstances and hence have been measured only in a small handful of experiments [15–17]. The difficulty in observing the long-time deviation, for example, originates in that the effect usually does not appear until after many lifetimes of the exponential decay have passed, by which time the survival probability is so depleted that it is rendered undetectable. A handful of theory papers have suggested special circumstances to enhance the non-exponential effects; these mostly require an initially prepared state near the threshold, usually combined with other conditions [12,18–23]. See also the recent experiment in Ref. [24].

In this paper we take advantage of the simple geometric shape of the BIC to present a qualitatively different and more easily generalized scheme by which the long-time non-exponential effect can be enhanced. While it is clear from the outset that the usual exponential decay associated with the resonance is suppressed when the BIC condition is satisfied, if one were to directly populate the BIC state itself then one would observe a simple stable evolution, as the BIC is of course an eigenstate of the Hamiltonian. However, we show that by populating a state that is orthogonal to the BIC we can take advantage of the suppression of the exponential effect while avoiding the stability associated with the BIC itself. The non-exponential dynamics can then drive the evolution on all timescales. What’s more, we demonstrate in our example below that the exponential effect can still be dramatically suppressed even with significant detuning from the BIC.

We illustrate our method relying on a simple tight-binding model that can be viewed analogously to one of the previously mentioned optical waveguide array experiments. Our Hamiltonian is written

$$H = \epsilon_d |d\rangle\langle d| - J \sum_{n=1}^{\infty} (|n\rangle\langle n+1| + |n+1\rangle\langle n|)$$

$$- g (|d\rangle\langle 2| + |2\rangle\langle d|),$$

(1)

in which the second term represents the semi-infinite array with nearest-neighbor hopping parameter $-J$ and the
chain is side-coupled at site \( |2 \rangle \) to an “impurity” element \( |d \rangle \). After we set the energy units according to \( J = 1 \), the adjustable parameters in the system are the chain–impurity coupling \(-g\) and the impurity energy level \( \epsilon_d \).

This model captures the essential features of the waveguide array experiment in Ref. [3] (see Ref. [27] as well as [29]), when we view time evolution in the present context analogously to longitudinal propagation within the waveguides. This model can be partially diagonalized by introducing a half-range Fourier series on the chain according to \( |n \rangle = \sqrt{\frac{2}{\pi}} \int_0^\pi dk \sin nk |k \rangle \), after which we have

\[
H = \epsilon_d |d \rangle \langle d| + \int_0^\pi dk E_k |k \rangle \langle k| + g \int_0^\pi dk V_k (|d \rangle \langle k| + |k \rangle \langle d|) \tag{2}
\]

where \( V_k = -\sqrt{\frac{2}{\pi}} \sin 2k \) and the continuum is given by \( E_k = -2J \cos k \) over \( k \in [0, \pi] \). Note from here we will measure the energy in units in which \( J = 1 \).

The discrete spectrum for this model can be obtained, for example, from the resolvent operator

\[
\frac{1}{z - H} |d \rangle = \frac{1}{z - \epsilon_d - \Sigma(z)} |d \rangle \tag{3}
\]

in which the self-energy function \( \Sigma(z) \) is given by

\[
\Sigma(z) = g^2 \int_0^\pi \frac{|V_k|^2}{z - E_k} dk = \frac{2g^2}{2} \left[ z^2 - 2 - z \sqrt{z^2 - 4} \right]. \tag{4}
\]

Notice that a pole occurs in Eq. (3) at \( z = 0 \) after choosing \( \epsilon_d = 0 \); this is the BIC solution for this model, which resides directly at the center of the continuum \( z \in [-2, 2] \) (defined by the range of \( E_k \)) and which takes the form

\[
|\psi_{\text{BIC}} \rangle = \frac{1}{\sqrt{1 + g^2}} (|d \rangle - g|1 \rangle). \tag{5}
\]

We here emphasize that the BIC state can be understood as a resonance with vanishing decay width \([2, 20, 30]\). In this picture, the ordinary resonance represents a generalized eigenstate with complex energy eigenvalue, for which the imaginary part of the eigenvalue gives the exponential decay half-width. When the BIC condition \( \epsilon_d = 0 \) is fulfilled the imaginary part of this eigenvalue vanishes, yielding a bound state residing directly in the scattering continuum. When \( \epsilon_d \neq 0 \), then the complex eigenvalue is restored and the exponential decay would generally be expected to reappear. Meanwhile note that the BIC solution exists in the present model for any value of the coupling \( g \) as long as \( \epsilon_d = 0 \) is satisfied.

It is easy to show that there exist two further solutions for the \( \epsilon_d = 0 \) case with eigenvalues given by \( z_\pm = \pm g \), in which

\[
z_g = g + \frac{1}{g}. \tag{6}
\]

For \( g > 1 \) these two solutions constitute localized bound states residing on the first Riemann sheet of the complex energy plane, while for \( g < 1 \) they transition to so-called virtual bound states (or anti-bound states), which are delocalized pseudo-states with real eigenvalue resting in the second sheet \([12, 22, 41–45] \), see Fig 1. Note that precisely such a delocalization transition is observed in the optical waveguide array experiment in Fig. 2(b) of Ref. [4]. While the virtual bound states do not appear in the diagonalized Hamiltonian, they nevertheless have a similar influence on the long-time power law decay as do the bound states \([12] \). Specifically, we will show that the timescale characterizing the non-exponential decay is proportional to \( \Delta_g^{-1} \), where

\[
\Delta_g \equiv z_g - 2. \tag{7}
\]

defined as the gap between either of the (virtual) bound state energies and the nearest band edge. Note we will particularly focus on the \( g \leq 1 \) portion of the parameter space as the absence of bound states here means that nothing inhibits the non-exponential decay. (For comparison purposes, we will also briefly discuss the \( g > 1 \) evolution.)

As previously discussed, if we were to consider the evolution of the BIC state itself, the initial state would simply remain occupied for all time as \( |\psi_{\text{BIC}} \rangle \) is an eigenstate of \( H \) with energy eigenvalue \( z = 0 \). However, by instead choosing the (simplest) BIC orthogonal state

\[
|\psi_\perp \rangle = \frac{1}{\sqrt{1 + g^2}} (g|d \rangle + |1 \rangle) \tag{8}
\]

as our initial state, we obtain pure non-exponential decay for any value \( g \leq 1 \), as shown below. To analyze the evolution of \( |\psi_\perp \rangle \), we evaluate the survival probability \( P_\perp (t) = |A_\perp (t)|^2 \), in which the survival amplitude is
Figure 2. (color online) Numerical simulations for the survival probability of $|\psi_\perp\rangle$ at time $t$ for (a) $g = 1.1$ (linear plot, inset: log-log plot), (b) $g = 1.0$ (log-log plot), and (c, d, e) $g = 0.98$ (c: log-log plot, d: early near zone close-up, and e: far zone close-up). The green dashed (orange dotted) lines indicate the $1/t$ ($1/t^2$) dynamics.

The survival amplitude in the contour for Eq. (9). The survival amplitude in terms of the eigenstates of the generalized discrete spectrum of the model as in Ref. [15]. By either method we obtain the following results.

For the case $g > 1$ there are two bound states included in the contour for Eq. (9). The survival amplitude in this case evaluates as

$$A_\perp(t) = \langle \psi_\perp|e^{-iHt}|\psi_\perp\rangle = \frac{1}{2\pi i} \int_{C_E} e^{-izt}(\psi_\perp| \frac{1}{z-H}|\psi_\perp\rangle) dz,$$

(9)

where $C_E$ is an integration contour surrounding the entire real axis in the first Riemann sheet of the complex energy plane, which includes the branch cut along $z \in [-2, 2]$ as well as any bound states. We can apply various methods to evaluate this integral, for example by directly computing the relevant matrix elements of the resolvent operator and integrating over these or by applying an expansion in terms of the eigenstates of the generalized discrete spectrum of the model as in Ref. [15]. By either method we obtain the following results.

Meanwhile for the case $g \leq 1$, the bound states have become virtual bound states and the evolution is now determined entirely by the non-Markovian branch cut contribution $A_\perp(t) = A_{br}(t)$. We find that this expression yields two distinct time regions, in which the integral is most easily estimated by somewhat different methods. First there is a short/intermediate time region, in which we first apply a fraction decomposition to the denominator of Eq. (11); this yields two simpler integrals, one associated with the upper virtual bound state and the other associated with the lower. As outlined in [23], these two integrals can be evaluated in terms of Bessel functions by methods similar to those used in Ref. [23], which yields

$$A_{br}(t) = \frac{1 + g^2}{4\pi i g^2} \int_{C_{br}} dz e^{-izt} \sqrt{z^2 - \frac{4}{z^2 - z_g^2}}.$$

(11)

The presence of the bound states in this case results in partial decay, which tends to de-emphasize the non-exponential evolution associated with the branch cut. This can be seen for the case $g = 0.98$ in Fig. 2(a).
Then, in the intermediate time region $T_z \ll t \ll T_\Delta$, we can approximate the Bessel function in the first term of Eq. (12) to write
\[ A_{NZ}(t) \approx \frac{\cos(2t - \pi/4)}{\sqrt{\pi t}} - \frac{1 - g}{g} \cos z_g t. \quad (13) \]

We refer to this time region including characteristic $1/t$ decay as the non-exponential near zone (NZ) [12], which we can roughly think of as having replaced the usual exponential decay regime. For values $g \lesssim 1$ fairly close to the $g = 1$ localization transition, the first term in Eq. (13) tends to dominate the evolution early in the near zone, while the second term provides only a small correction. Estimating the evolution in this case yields
\[ P_{NZ, early}(t) \approx \frac{\cos^2(2t - \pi/4)}{\pi t}, \quad (14) \]
which can be seen for the case $g = 0.98$ in Fig. 2(c,d).

As we move later into the near zone, the first term decays sufficiently so that the second term becomes non-negligible; we can estimate this as when the second term is about 10% of the first, which gives $t \approx T_{VR} \sim g^2/100\pi(1 - g)^2 = g/100\pi \cdot T_\Delta$. This implies we should be fairly close to the transition point $g = 1$ to observe the 'pure' $1/t$ dynamics. For example, in the case $g = 0.98$ shown in Fig. 2(d), we can already see a small influence from the second term of Eq. (13) around $t \approx T_{VR} \approx 8.0$ as the last three visible oscillation cycles show a slight deviation from the Eq. (14) prediction. We will return to discuss the physical interpretation of this second term in greater detail momentarily.

Next appears the asymptotic time region $T_\Delta \ll t$ during which the dynamics are instead described by a $1/t^3$ power law decay. To show this, we return to the (exact) integral expression for the survival amplitude appearing in Eq. (14) and instead proceed by deforming the contour $C_{\psi}$ surrounding the branch cut by dragging it out to infinity in the lower half of the complex energy plane, as described in [16]. Following this procedure, we finally obtain
\[ P_{FZ}(t) \approx \frac{(1 + g^2)\cos(2t - 3\pi/4)}{\pi g^4 \Delta_g^2 (2 + z_g)^2 t^3}, \quad (15) \]

with the characteristic $1/t^3$ decay that is typical of odd dimensional systems on long timescales [12, 17, 51]. We refer to this as the non-exponential far zone (FZ). The far zone dynamics can be seen for the $g = 0.98$ case in Fig. 2(c,e).

We emphasize three further points about these results as follows. First, we draw attention more carefully to the occurrence of oscillations in both time zones, which are due to interference between the contributions from the two band edges. These contributions are equally weighted because the BIC occurs at the center of the continuum band in the present case. Notice further that a $\pi/2$ phase shift occurs between the early near zone result Eq. (14) and the far zone Eq. (15). These oscillations and the resulting phase shift are highlighted in Fig. 2(d,e). While similar oscillations have been previously predicted in the far zone [21, 23, 51], we believe the near zone oscillations as well as the resulting phase shift are new — indeed, outside of our choice for the initial state, these would almost certainly be obscured by the exponential decay. Second, we return our attention to the second term of Eq. (13), which becomes relatively more pronounced in the late near zone; however, counterintuitively perhaps, it vanishes in the far zone. Notice this term takes precisely the form of a Rabi oscillation between bound states [compare with the first term of Eq. (10)]; however, in this case it is an oscillation between the virtual bound states. Hence, we refer to this effect as a virtual Rabi oscillation, which is intended to reflect its origin as well as its transient nature. A further interesting point is that the virtual Rabi oscillation plays a role in facilitating the phase shift from the early near zone into the far zone [16].

Third, notice that when we are directly at the localization transition at $g = 1$, the second term in Eq. (13) vanishes. Further, since the key timescale $T_\Delta$ is inversely proportional to $\Delta_\psi$, as we approach $g = 1$ from below the energy gap $\Delta_\psi$ closes and $T_\Delta$ diverges. Hence, in this case, Eq. (14) describes the dynamics accurately for all $T_z \ll t$, which is shown in Fig. 2(b) (see also Ref. [12] for discussion relevant to this point as well as the influence of a virtual bound state on the power law decay).

While the preceding analysis gives a clear picture of the types of evolution we can expect for the state $|\psi_\perp\rangle$, it is still a bit idealized in comparison to experiment in two ways that we will account for below. First, in a real experiment it would be difficult to tune exactly to the BIC at $\epsilon_d = 0$; since the BIC is just the special case of a resonance with zero decay width, as we introduce detuning $\epsilon_d \neq 0$ the resonance must reappear, which we could expect might perturb the non-exponential evolution of $P_\perp(t)$. The complex eigenvalue of the resonance state can be expanded in the vicinity of the BIC up to second order in $\epsilon_d$ as $z_{res} \approx \epsilon_d/(1 + g^2) - i\Gamma/2$ with $\Gamma = 2g^2\epsilon_d^2/(1 + g^2)^3$, which of course reduces to $z_{BIC} = 0$ in the limit $\epsilon_d = 0$. However, when we examine $P_\perp(t)$ (red curve in Fig. 3 for $g = 0.9$, as an example), we find that the resonance has virtually no influence on the survival probability, even for moderately large detuning values $\epsilon_d \neq 0$. We can obtain an understanding for this by calculating the resonance pole contribution to $|\psi_\perp\rangle$.

\footnote{The reason for this is described in pp. 21-22 of Ref. [45].}
which to lowest order is given by

\[ P_{\perp, \text{res}}(t) \approx \frac{g^4 \epsilon_d^4}{(1 + g^2)^4} e^{-\Gamma t}. \]  

The pre-factor in this expression, which is fourth order in \( \epsilon_d \), assures that the exponential effect will be quite small for almost any \( \epsilon_d \approx 0 \) regardless of the value of \( g \). For example, even for modest detuning \( \epsilon_d = 0.2 \) and \( g = 0.9 \) in Fig. 3 (b) [red curve], we have \( g^4 \epsilon_d^4 / (1 + g^2)^4 \approx 10^{-5} \).

Second, while preparation of the initial state \( |\psi_\perp\rangle \) seems feasible, measuring the precise output state \( |\psi_\perp\rangle \) might prove more challenging. Instead, it may be more realistic to consider the quantity

\[ P_{1d}(t) = |\langle 1 | e^{-iHt} | \psi_\perp \rangle|^2 + |\langle d | e^{-iHt} | \psi_\perp \rangle|^2, \]  

which is equivalent to the non-escape probability that has appeared in the literature previously \[11, 48–50, 52\]. It can easily be shown that \( P_{1d}(t) = P_1(t) \) for the case \( \epsilon_d = 0 \), and hence all of our preceding detailed analytical results still apply directly at the BIC. As shown in Fig. 3 the difference between \( P_{1d}(t) \) [blue curve] and \( P_\perp(t) \) [red curve] appears first well in the long time region for small \( \epsilon_d \neq 0 \) and moves gradually to earlier times as we increase the detuning. The origin of the difference between the two quantities is easy to understand, as it seems to be entirely attributable to the fact that the lowest-order resonance pole contribution for \( P_{1d}(t) \) is given by

\[ P_{1d, \text{res}}(t) \approx \frac{g^2 \epsilon_d^2}{(1 + g^2)^4} e^{-\Gamma t}, \]  

which is still small, but has some noticeable influence on the spectrum in some cases. For example, in Fig. 3 (b) for \( \epsilon_d = 0.2 \) we see the resonance pole with magnitude \( g^2 \epsilon_d^2 / (1 + g^2)^4 \approx 0.003 \) introduces exponential dynamics into \( P_{1d}(t) \) around \( t \gtrsim 10 \), although this only lasts for a few lifetimes \( \tau = 2/\Gamma \sim 360 \), which leaves the non-escape probability relatively intact when this quantity rejoins with \( P_\perp(t) \) as the \( 1/t^3 \) far zone dynamics kick in. We note that \( P_{1d}(t) \) also exhibits the interesting feature of pre-exponential decay that extends beyond the usual parabolic dynamics in the region \( 1 \lesssim t \lesssim 10 \). As we further increase \( \epsilon_d \) as in Fig. 3 (c), we find the exponential decay region lasts even fewer lifetimes as the difference between \( P_{1d}(t) \) and \( P_\perp(t) \) again becomes diminished.

In this work we have shown that by populating a state that lies orthogonal to a bound state in continuum one can observe non-exponential dynamics that are usually overwhelmingly suppressed when the resonance condition is satisfied. Note that for the present model we could consider the evolution of more general BIC orthogonal states such as \( |d⟩ + |1⟩ + \sum_{n=2}^∞ w_n |n⟩ \) that include elements of the chain beyond the BIC sector. We briefly comment on a representative example of this more general configuration in Ref. [49], where we show that including a single site from the chain can result in an effective decoherence.

We briefly note we have focused here on bound states in continuum that appear purely due to interference effects as originally proposed by von Neumann and Wigner in 1929 [1]. We have not directly addressed “accidental” BICs [39] that exhibit interesting topological properties [2, 5, 15, 55], although the study of BIC-orthogonal states in this context might prove fruitful as well.

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SUPPLEMENTARY MATERIAL

DETAILS OF APPROXIMATE DYNAMICS

We start with the exact expression for the dynamics associated with the branch cut taken from the main text

\[ A_{br}(t) = \frac{1 + g^2}{4\pi i g^2} \int_{C_{br}} dz e^{-izt} \frac{\sqrt{z^2 - 4}}{z^2 - z_g^2}. \]  

(19)

In Sec. we outline the approximations for the short/intermediate time region and make a brief comment on the influence of the virtual Rabi oscillation on the phase in the near zone, while in Sec. we describe corresponding approximations for the far zone. In Sec. we present some plots for additional \( g \) values compared to the main text and briefly discuss these. Note that \( \epsilon_d = 0 \) throughout this document.

Short/intermediate time region

We begin the derivation by performing a fraction decomposition on the integrand of Eq. (19) in order to rewrite this as

\[ A_{br}(t) = -\frac{1}{2g} (I(z_+) - I(z_-)) \]  

(20)

in which

\[ I(z_n) = -\frac{1}{2\pi} \int_{C_{br}} dz e^{-izt} \frac{\sqrt{1 - z^2 / 4}}{z - z_n}. \]  

(21)

From this point, we can apply methods similar to those appearing in Apps. C and D of Ref. [23] to evaluate this integral; we eventually obtain

\[ I(\pm z_g) = e^{\mp iz_g t} \left[ \mp g - i \int_0^t d\tau e^{iz_g \tau} \frac{J_1(2\tau)}{\tau} \right], \]  

(22)

in which the first term is a pole contribution associated with the virtual bound states. Since we are primarily interested in the vicinity of \( g = 1 \), we can approximate \( z_g \approx 2 \) under the remaining integration, which allows us to write

\[ I(\pm z_g) = \pm e^{\mp iz_g t} \left[ 1 - g - e^{\pm 2i t} (J_0(2t) \mp i J_1(2t)) \right], \]  

(23)

Plugging this result into Eq. (20) gives

\[ A_{br}(t) \approx \frac{1}{g} \left[ (g - 1) \cos z_g t + \cos(\Delta_g t) J_0(2t) - \sin(\Delta_g t) J_1(2t) \right]. \]  

(24)

After applying the approximation \( \Delta_g t \ll 1 \) \( (t \ll T_\Delta) \) we obtain the result in the text Eq. (12).

As mentioned in the main text, the virtual Rabi oscillation plays a role in facilitating the phase shift from the early near zone (with phase \( \pi/4 \)) to the far zone (with phase \( 3\pi/4 \)). For example, at time \( t = g/4\pi \ast T_\Delta \sim T_\Delta/10 \) in the near zone evolution the coefficient of the second term in Eq. (13) from the main text has exactly half the magnitude of that of the first term; in this moment, the effective phase of the two combined terms can be shown to be \( \phi_{1/2} = \arctan(\sqrt{2}/(\sqrt{2} - 1)) \approx 0.4093\pi \), which indeed satisfies \( \pi/4 < \phi_{1/2} < 3\pi/4 \).

Asymptotic time region (far zone)

In the case of the asymptotic time zone \( t \gg T_\Delta \), we find it most convenient to evaluate the integral Eq. (19) using methods similar to those used in Ref. [12]. We begin by dragging the contour \( C_{br} \) surrounding the branch cut out to infinity in the lower half of the complex energy plane. After this, the only non-vanishing portions of the integration are the contours \( A_\mp(t) \) running from the two branch points out to infinity in the lower half plane. These portions are written as

\[ A_\pm(t) = \frac{1 + g^2}{2\pi i g^2} \int_{\mp 2}^{\mp 2 - i\infty} dz e^{-izt} \frac{\sqrt{z^2 - 4}}{(z - z_-)(z - z_+)}. \]  

(25)

Applying an integration variable transform \( s \equiv it(z \mp 2) \) yields

\[ A_\pm(t) = \frac{i (1 + g^2) e^{\pm 2it}}{2\pi g^2 t^2} \int_0^\infty ds e^{-s} \frac{\sqrt{s^2 - 4is t}}{\pm \Delta_g (2 + z_g) + 4i \frac{s^2 - 4is t}{\pm 2z_g}}. \]  

(26)

For very large \( t \) the first term in the denominator is much larger than the other two terms, which can be safely neglected. Performing the remaining simplified integration and combining \( A_\pm \) we obtain the result reported for the far zone in Eq. (15) of the main text.

Near zone/far zone transition: plots for additional cases

In Fig. 4(a-c) we plot the survival probability \( P_\perp(t) \) for \( g = 0.9 \), similar to the case \( g = 0.98 \) that was presented in Fig. 2(c-e) of the main text; only here we are a bit further away from the localization transition that occurs at \( g = 1 \). We see in Fig. 4(b) for this case that the early near zone \( 1/t \) prediction gives only a rough description, as the virtual Rabi oscillation plays a significant role from
early on; however, the early near zone phase prediction
\( \cos^2(2t - \pi/4) \) is still accurate.

We plot the same in Fig. 4(d-f) for \( g = 0.7 \), signifi-
cantly further from the \( g = 1 \) localization transition. In
this case, we see in the near zone close up Fig. 4(e) that
we need to take into account the influence of the virtual
Rabi oscillation on both the maxima and the phase al-
most from the beginning of the evolution. However, the
dynamics remain non-exponential on all timescales.

If we keep decreasing the value of \( g \), eventually around
\( g \approx 0.38 \) we obtain \( T_\Delta \sim 1 \). For this and any smaller
values of \( g \), the near zone is entirely squeezed out and
the system will instead transition from the early time
parabolic (Zeno) dynamics directly into the long time
\( 1/t^3 \) behavior in the far zone.

All numerical results in this work were obtained by
evolving a chain in the site representation (with up to
16000 elements) according to the Schrödinger equation
using a variable-order variable-step Adams method.

**CHAIN-INDUCED EFFECTIVE DECOHERENCE**

Here we briefly consider the evolution of a slightly more
general BIC orthogonal state, written as

\[
|\psi_w\rangle = N_w (g|d\rangle + |1\rangle + w|2\rangle)
\]  

(27)
in which \( N_w^2 = (1 + g^2 + w^2)^{-1} \). Here we have included
a single site \( |2\rangle \) from the chain outside of the subspace
spanned by the BIC itself, with arbitrary amplitude \( w \).

In Fig. 5 we show how the inclusion of this site modifies
the evolution for \( g = 0.9 \). In Fig. 5 (a-c), we see that
increasing the value of \( w \) in the range \( w \leq 1 \) results in
the oscillations we observed in the main text becoming
gradually damped out, with near total damping occurring
for \( w = 1 \). By contrast, the oscillations return for \( w \) values
much larger than 1 as shown in Fig. 5(d).

Extending from this observation, in Fig. 6 we plot
numerical simulations for the survival probability of \( |\psi_w\rangle \)
for \( w = 1 \) over a variety of \( g \) values. We observe that the
near-total suppression of the oscillations occurs for a wide
range of \( g \) values in the vicinity of \( g = 1 \).

We can gain insight into the mechanism of this sup-
pression through the following analytic approximations.
We begin by writing the survival probability for this state

\begin{align*}
\langle \psi_w | \psi_w \rangle &= N_w^2 (g^2 + 3g^2 + (1 + g^2 + w^2))
\end{align*}

Figure 4. Survival probability of \( |\psi_\perp\rangle \) at time \( t \) for (a-c) \( g = 0.9 \) and (d-f) \( g = 0.7 \). (Compare with Fig. 2 from the main text.)

Figure 5. Numerical simulations for the survival probability
for the \( |\psi_w\rangle \) state with \( g = 0.9 \), \( \epsilon_d = 0 \) and (a) \( w = 0.1 \), (b) \( w = 0.5 \), (c) \( w = 1.0 \), and (d) \( w = 2.0 \).
be shown that at the impurity site, Eq. (3) from the main text. Here we have $A|Q|g$ and $P$ respectively, in the main text.)

Focusing on the case $w = 1$ as a quick example, it can be shown that $Q(z)$ simplifies a bit as

$$Q(z) = (1 + g^2) z (z - 2) [\sigma_1(z)]^2. \quad (31)$$

Note the useful relation $\sigma_1(z) + 1/\sigma_1(z) = z$ has been applied here. Eq. (31) can in turn be used to simplify the integrand of Eq. (28) such that we obtain

$$A_w(t) = \frac{N_w^2}{4\pi i} \int d\zeta e^{-izt} \frac{(z - g_{\Sigma})^2 \sqrt{\zeta^2 - 4}}{(z - \zeta)(z + \zeta)}. \quad (32)$$

Note the presence of the $(z-g_{\Sigma})^2$ factor in the numerator of the integrand. Since the dominant contribution to the integration comes from around the branch points $z = \pm 2$ and since $g_{\Sigma} = 1 + g^2 \approx 2$ in the vicinity of $g \sim 1$, this factor is very small for the contribution coming from the upper branch cut (if we had chosen $w = -1$, it would instead be the lower branch cut contribution that would be very small). Since there is one overwhelmingly dominant contribution, the oscillations between the two band edges are greatly diminished, which explains the effective decoherence observed in Figs. 5 and 6.

To see this more explicitly, we carry out the integration for the far zone in the $g \neq 1$ case by dragging the integration contour out to infinity in the lower half plane similar to Sec. I. Doing so we find there are again the two contributions $A_w(t) = A_{\pm}(t) + A_{\mp}(t)$ in which

$$A_{\pm,w}(t) = \mp \frac{N_w^2}{2\pi} \int d\zeta e^{izt} (\sigma_1(z) + Q(z)g_{dd}(z)). \quad (28)$$

Here we have

$$\sigma_1(z) = \frac{z - \sqrt{z^2 - 1}}{2}, \quad (29)$$

$$Q(z) = g^2 + [\sigma_1(z)]^2 \left(2g^2 - 2g^2wz + g^2[\sigma_1(z)]^2 - 2wz + wz^2\right), \quad (30)$$

and $G_{dd}(z) = (z - e_d - \Sigma(z))^{-1}$ is the resolvent operator at the impurity site, Eq. (3) from the main text.

Focusing on the case $w = 1$ as a quick example, it can be shown that $Q(z)$ simplifies a bit as

$$Q(z) = (1 + g^2) z (z - 2) [\sigma_1(z)]^2. \quad (31)$$

Hence we immediately see the $A_{+}(t)$ contribution will indeed be quite small for $g \sim 1$. The far zone evolution is then approximately described by

$$P_{\pm,w}(t) \approx \frac{(2 + g_{\Sigma})^4}{4\pi g^4 (2 + g^2)^2 (2 + g_{\Sigma})^2 \Delta g^2/3}, \quad (34)$$

which predicts the slope in Fig. 6 quite well (red dotted lines), even in the case $g = 0.7$ not so near the localization transition at $g = 1$.

In the special case $g = 1$ notice that $z - g_{\Sigma} = z - 2$ exactly, which results in the upper band edge contribution vanishing entirely. The timescale $T_\Delta$ also diverges as the gap $\Delta_g$ closes (just as in the main text), which leads to an asymptotic near zone (1/t) description

$$P_w(t) \approx 16/9\pi t. \quad (35)$$

This again agrees very well (green dashed line) with the numerical simulation in Fig. 6.