Rainbow triangles in edge-colored complete graphs

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Abstract

Let $G$ be a graph of order $n$ with an edge-coloring $c$, and let $\delta^c(G)$ denote the minimum color-degree of $G$. A subgraph $F$ of $G$ is called rainbow if any two edges of $F$ have distinct colors. There have been a lot results in the existing literature on rainbow triangles in edge-colored complete graphs. Fujita and Magnant showed that for an edge-colored complete graph $G$ of order $n$, if $\delta^c(G) \geq \frac{n+1}{2}$, then every vertex of $G$ is contained in a rainbow triangle. In this paper, we show that if $\delta^c(G) \geq \frac{n+k}{2}$, then every vertex of $G$ is contained in at least $k$ rainbow triangles, which can be seen as a generalization of their result. Li showed that for an edge-colored graph $G$ of order $n$, if $\delta^c(G) \geq \frac{n+1}{2}$, then $G$ contains a rainbow triangle. We show that if $G$ is complete and $\delta^c(G) \geq \frac{n}{2}$, then $G$ contains a rainbow triangle and the bound is sharp. Hu et al. showed that for an edge-colored graph $G$ of order $n \geq 20$, if $\delta^c(G) \geq \frac{n+2}{2}$, then $G$ contains two vertex-disjoint rainbow triangles. We show that if $G$ is complete with order $n \geq 8$ and $\delta^c(G) \geq \frac{n+1}{2}$, then $G$ contains two vertex-disjoint rainbow triangles. Moreover, we improve the result of Hu et al. from $n \geq 20$ to $n \geq 7$, the best possible.

Keywords: edge-coloring; edge-colored complete graph; rainbow triangle; color-degree condition

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1 Introduction

In this paper, we consider finite simple undirected graphs. An edge-coloring of a graph $G$ is a mapping $c : E(G) \to \mathbb{N}$, where $\mathbb{N}$ denotes the set of natural numbers. A graph $G$ is called an edge-colored graph if $G$ is assigned an edge-coloring. The color of an edge $e$
of $G$ and the set of colors assigned to $E(G)$ are denoted by $c(e)$ and $C(G)$, respectively. For subset $X$ of vertices of $G$, we use $G[X]$ to denote the subgraph of $G$ induced by $X$. For $V_1, V_2 \subset V(G)$ and $V_1 \cap V_2 = \emptyset$, we set $E(V_1, V_2) = \{xy \in E(G), x \in V_1, y \in V_2\}$, and when $V_1 = \{v\}$, we write $E(u, V_2)$ for $E(\{u\}, V_2)$. The set of colors appearing on the edges between $V_1$ and $V_2$ in $G$ is denoted by $C(V_1, V_2)$. When $V_1 = \{v\}$, use $C(v, V_2)$ instead of $C(\{v\}, V_2)$. The set of colors appearing on the edges of a subgraph $H$ of $G$, is denoted by $C(H)$; moreover if $H = G[V_1]$, we write $C(V_1)$ for $C(G[V_1])$. A subset $F$ of edges of $G$ is called rainbow if no pair of edges in $F$ receive the same color, and a graph is called rainbow if its edge-set is rainbow. In this paper, we only consider rainbow triangles in an edge-colored complete graph.

For a vertex $v \in V(G)$, the color-degree of $v$ in $G$ is the number of distinct colors assigned to the edges incident to $v$, denoted by $d^c_G(v)$. We use $\delta^c(G) = \min\{d^c_G(v) : v \in V(G)\}$ to denote the minimum color-degree of $G$. The set of neighbors of a vertex $v$ in a graph $G$ is denoted by $N_G(v)$. Let $N_i(v)$ denote the set of vertices with edges of color $i$ adjacent to $v$ for $1 \leq i \leq d^c(v)$, that is $N_i(v) = \{u \in N_G(v), c(uw) = i\}$. Let $\Delta^\text{mon}(v)$ be the maximum number of incident edges of $v$ with the same color, that is $\Delta^\text{mon}(v) = \max\{|N_i(v)|, 1 \leq i \leq d^c(v)\}$. Then the monochromatic-degree of $G$ is the maximum $\Delta^\text{mon}(v)$ over all vertices $v$ in $G$, denoted by $\Delta^\text{mon}(G)$. Let $\psi$ be the incidence function that associates with each edge of $G$ an unordered pair of (not necessarily distinct) vertices of $G$. If $e$ is an edge and $u$ and $v$ are vertices such that $\psi(e) = \{u, v\}$, then $e$ is said to join $u$ and $v$, and the vertices $u$ and $v$ are called the ends of $e$. Let $R$ be a subset of $E(G)$. Then $\psi(R)$ denotes the set of all vertices incident with the edges in $R$, that is $\psi(R) = \cup_{e \in R}\psi(e)$. For other notation and terminology not defined here, we refer to [1].

There have been many results on rainbow triangles in the existing literature. These results can be divided into two parts: local property and global property on rainbow triangles. As for local property, Fujita and Magnant showed the following result.

**Theorem 1** ([2]). Let $G$ be an edge-colored complete graph of order $n$. If $\delta^c(G) \geq \frac{n+1}{2}$, then every vertex of $G$ is contained in a rainbow triangle.

The lower bound on $\delta^c(G)$ in Theorem 1 is sharp. To see this, we obtain the following construction. It will show that $v$ is not contained in any rainbow triangle.

**Construction 2.** Consider a complete graph $G = K_{2n}$. Let $v$ be a vertex of $G$ such that $d^c(v) = n$. Set $|N_1(v)| = 1$ and $|N_i(v)| = 2$ for $2 \leq i \leq n$. Color the edges between $N_1(v)$ and $N_i(v)$ by $i$ for $2 \leq i \leq n$. For any vertex $u \in N_i(v)$, color two edges between $u$ and $N_j(v)$ by $i$ and $j$, respectively, for $2 \leq i \neq j \leq n$. Color the edge in $G[N_i(v)]$ by a
new color different from the colors of edges incident with \( v \). Then we get an edge-colored complete graph with \( \delta^c(G) = n \); see Figure 1.

Using Theorem 1 and repeatedly deleting the vertices of rainbow triangles at \( v \), it is easy to obtain the following sufficient condition for the existence of \( k \) edge-disjoint rainbow triangles at \( v \) and the lower bound is sharp.

**Fact 3.** Let \( G \) be an edge-colored complete graph of order \( n \). If \( \delta^c(G) \geq \frac{n-1}{2} + k \), then every vertex of \( G \) is contained in at least \( k \) edge-disjoint rainbow triangles.

In this paper, we will show the following result, which can be seen as a generalization of Theorem 1.

**Theorem 4.** Let \( G \) be an edge-colored complete graph of order \( n \). If \( \delta^c(G) \geq \frac{n+k}{2} \), then every vertex of \( G \) is contained in at least \( k \) rainbow triangles.

As for global property on rainbow triangle, there are some results in an edge-colored general graph.

**Theorem 5** ([4]). Let \( G \) be an edge-colored graph of order \( n \). If \( \delta^c(G) \geq \frac{n+1}{2} \), then \( G \) contains a rainbow triangle.

**Theorem 6** ([6]). Let \( G \) be an edge-colored graph of order \( n \). If \( \delta^c(G) \geq \frac{n}{2} \) and \( G \) contains no rainbow triangles, then \( n \) is even and \( G \) is the complete bipartite graph \( K_{\frac{n}{2}, \frac{n}{2}} \), unless \( G = K_4 - e \) or \( K_4 \) when \( n = 4 \).

Recently, Hu et al. proved the following result in an edge-colored general graph.

**Theorem 7** ([3]). Let \( G \) be an edge-colored graph of order \( n \geq 20 \). If \( \delta^c(G) \geq \frac{n+2}{2} \), then \( G \) contains two vertex-disjoint rainbow triangles.
In this paper, we are seeking for sufficient condition for the existence of rainbow triangles in an edge-colored complete graph.

**Theorem 8.** Let $G$ be an edge-colored complete graph of order $n$. If $\delta^c(G) \geq \frac{n}{2}$, then $G$ contains a rainbow triangle.

With more effort, we can obtain the following stronger theorem.

**Theorem 9.** Let $G$ be an edge-colored complete graph of order $n$. If $\delta^c(G) \geq \frac{n-1}{2}$ and $G$ contains no rainbow triangle, then $V(G)$ can be partitioned into $\frac{n+1}{2}$ parts $\{A_0, A_1, \ldots, A_{\frac{n-1}{2}}\}$ (see Figure 2), such that the following properties hold:

1. $n$ is odd and $d^c(v) = \frac{n-1}{2}$ for all $v \in V(G)$;
2. $|A_0| = 1$ and $|A_i| = 2$ for $1 \leq i \leq \frac{n-1}{2}$;
3. for any vertex $u \in A_i$, $C(u, A_i) = \{i, j\}$ for $1 \leq i \neq j \leq \frac{n-1}{2}$;
4. if $\frac{n-1}{2} \leq 2$, then $C(A_i) \in \{1, 2\}$, $i = 1, 2$; if $\frac{n-1}{2} \geq 3$, then $C(A_i) = \{i\}$, $1 \leq i \leq \frac{n-1}{2}$.

![Figure 2: The structure of $G$ in Theorem 9](image)

Theorem 9 shows that the lower bound on $\delta^c(G)$ in Theorem 8 is tight. Using Theorem 8 and repeatedly deleting the vertices of rainbow triangles it is easy to obtain the following sufficient condition for the existence of $k$ vertex-disjoint rainbow triangles.

**Fact 10.** Let $G$ be an edge-colored complete graph of order $n$. If $\delta^c(G) \geq \frac{n-3+3k}{2}$, then $G$ has at least $k$ vertex-disjoint rainbow triangles.

The lower bound on $\delta^c(G)$ is far from tight. We will investigate the minimum color-degree condition that guarantees the existence of two vertex-disjoint rainbow triangles in an edge-colored complete graph.
Theorem 11. Let $G$ be an edge-colored complete graph of order $n$. If $n \geq 8$ and $\delta^c(G) \geq \frac{n+1}{2}$, then $G$ contains two vertex-disjoint rainbow triangles, and the bound $n \geq 8$ cannot be improved (see Figure 3).

We will also improve the result of Theorem 7 as follows.

Theorem 12. Let $G$ be an edge-colored graph of order $n$. If $n \geq 7$ and $\delta^c(G) \geq \frac{n+2}{2}$, then $G$ contains two vertex-disjoint rainbow triangles, and the bound $n \geq 7$ cannot be improved (see Figure 4).

As far as short cycles are concerned in an edge-colored complete graph, Li et al. showed the following results.

Theorem 13 (5). Let $G$ be an edge-colored complete graph of order $n$. If $\Delta^{mon}(G) \leq n - 2$, then $G$ contains a properly colored cycle of length at most 4.

Theorem 14 (5). Let $G$ be an edge-colored complete graph of order $n$. If $\Delta^{mon}(G) \leq n - 5$, then $G$ contains two disjoint properly colored cycle of length at most 4.

In the following sections, we will give the proofs of our Theorems 4, 8, 9, 11 and 12 separately.
2 Proof of Theorem 4

Proof of Theorem 4: Let \( G \) be a graph satisfying the assumptions of Theorem 4 and \( v \) be a vertex of \( G \), and let \( t = d^c(v) \). Suppose \( |N_1(v)| = \cdots = |N_s(v)| = 1 \) and \( 2 \leq |N_{s+1}(v)| \leq \cdots \leq |N_t(v)| \). Clearly \( t - s \leq \frac{n+k-1}{2} \). Let \( N_1 = \bigcup_{1 \leq i \leq s} N_i(v) \) and \( N_2 = \bigcup_{s+1 \leq i \leq t} N_i(v) \). Let \( R(v) \) be a maximum subset of \( E(G) \) such that for any edge \( xy \in R(v) \), \( vxyv \) is rainbow. Then the number of rainbow triangles containing \( v \) is equal to \( |R(v)| \). Now we give an orientation to \( G[N_1] \) in such a way: for an edge \( xy \), if \( c(xy) = c(vx) \), then the orientation of the edge is from \( y \) to \( x \), (if \( c(xy) = c(vx) = c(vy) \), then we give the orientation arbitrarily); if \( xy \in R(v) \), then we give the orientation arbitrarily. The oriented graph is denoted by \( D \). Now we proceed by proving the following claims.

Apparently, all out-arcs from a vertex \( u \in N_1 \) are assigned colors different from \( c(uv) \). Then we can get the following claim.

Claim 2.1. For all \( u \in N_1 \), \( d^c_{N_1 \cup \{v\}}(u) \leq d^+_D(u) + 1 \).

Claim 2.2. \( C(N_i(v), N_j(v)) \setminus R(v) \subseteq \{i,j\} \) for \( 1 \leq i \neq j \leq t \).

Claim 2.3. \( \psi(R(v)) \cap N_1 \neq \emptyset \).

Proof. Suppose not, then all in-arcs to \( u \in N_1 \) are assigned color \( c(uv) \) in \( D \). Then from Claim 2.1 there is a vertex \( w \) in \( N_1 \), such that \( d^c_{N_1 \cup \{v\}}(w) \leq \frac{n+k}{2} \). Since there is no edge in \( R(v) \) incident with \( w \), from Claim 2.2 we have \( C(w, N_j(v)) \subseteq \{c(uv), j\} \). Hence, \( d^c(w) \leq \frac{n+k}{2} + t - s \leq \frac{n}{2} \), a contradiction.

Now we proceed the proof of Theorem 4 by induction on \( k \). The case \( k = 1 \) follows from Theorem 4. Let \( k \geq 2 \) and suppose Theorem 4 holds for \( k - 1 \). Suppose to the contrary, that there is a vertex \( v \) such that \( |R(v)| < k \). Since \( \delta^c(G) \geq \frac{n+k}{2} \geq \frac{n+k-1}{2} \), we have \( |R(v)| = k - 1 \). Certainly, \( s \geq k + 1 \) since \( \delta^c(G) \geq \frac{n+k}{2} \). For \( u \in N_1 \), let \( R_u(v) \) be the subset of \( R(v) \) in which each edge is an in-arc to \( u \) and \( \psi(R_u(v)) \subseteq N_1 \) and \( R'_u(v) \) be the subset of \( R(v) \) in which each edge is incident with \( u \) and \( \psi(R'_u(v)) \setminus \{u\} \subseteq N_2 \), that is, \( R_u(v) = \{uw \in R(v) : \hat{u}w \in D\} \) and \( R_u(v) = \{uw \in R(v) : w \in N_2\} \). Hence, from Claims 2.1 and 2.2 we have \( d^c_{N_1}(u) \leq d^+_D(u) + 1 + |R_u(v)| \) and \( |C(u, N_2) \setminus \{c(uv)\}| \leq t - s + |R'_u(v)| \). Then,

\[
d^c(u) \leq d^+_D(u) + 1 + |R_u(v)| + |R'_u(v)| + t - s.
\]

Since \( |R_u(v)| + |R'_u(v)| \leq k - 1 \), if \( d^+_D(u) \leq \frac{s-k}{2} \), we have \( d^c(u) \leq \frac{n+k-1}{2} \), a contradiction. Therefore, \( d^+_D(u) \geq \frac{s-k+1}{2} \) for \( u \in N_1 \). Let \( w \) be a vertex with minimum out-degree in \( D \). Then \( \frac{s-k+1}{2} \leq d^+_D(w) \leq \frac{s-1}{2} \). Assume that \( d^+_D(w) = \frac{s-k+a}{2} \), \( 1 \leq a \leq k - 1 \). Then
\[ |R_w(v)| + |R'_w(v)| \geq k - \frac{a+1}{2}, \text{ otherwise } d^c(w) < \frac{n+k}{2}. \] Since the edges in \( R_w(v) \) are in-arcs to \( w \), they are out-arcs from the vertices in \( \psi(R_w(v)) \setminus \{w\} \). Then for \( u \in N_1 \setminus \{w\} \), we have \( |R_u(v)| + |R'_u(v)| \leq \frac{a-1}{2} \). So, \( d^+_D(u) \geq \frac{s+k-a}{2} \) for all \( u \in N_1 \setminus \{w\} \). Hence, \[ \sum_{u \in N_1} d^+_D(u) \geq (s-1)\frac{s+k-a}{2} + \frac{s-k+a}{2}. \] Since \( s \geq k+1 \) and \( 1 \leq a \leq k-1 \), we have \[ \sum_{u \in N_1} d^+_D(u) \geq \frac{s(s-1)}{2}, \] a contradiction. \( \square \)

### 3 Proofs of Theorems 8 and 9

**Proof of Theorem 8:** Let \( G \) be a graph satisfying the assumptions of Theorem 8 and \( v \) be a vertex of \( G \). Suppose, to the contrary, that \( G \) has no rainbow triangle. Assume that \( |N_1(v)| = \cdots = |N_s(v)| = 1 \) and \( 2 \leq |N_{s+1}(v)| \leq \cdots \leq |N_t(v)| \) (\( t = d^c(v) \)). Clearly, \( t-s \leq \frac{n-1-s}{2} \). Let \( N_1 = \cup_{1 \leq i \leq s} N_i(v) \) and \( N_2 = \cup_{s+1 \leq i \leq t} N_i(v) \). Now we proceed by proving the following claims.

**Claim 3.1.** \( C(N_i(v), N_j(v)) \subseteq \{i, j\}, \) for \( 1 \leq i \neq j \leq t \).

**Claim 3.2.** If \( |N_1| \geq 2 \), then there is a vertex \( u \in N_1 \) such that \( C(u, N_1 \setminus \{u\}) = \{c(vu)\} \).

*Proof.* Suppose not, let \( u \) be a vertex in \( N_1 \) with minimum color-degree in \( G[N_1] \). Set \( W_1 = \{w \in N_1 \mid c(uw) = c(vu)\} \) and \( W_2 = \{w \in N_1 \mid c(uw) = c(vw)\} \). From Claim 3.1 we have \( N_1 \setminus \{u\} = W_1 \cup W_2 \). For any vertex \( w_1 \in W_1 \) and \( w_2 \in W_2 \), from Claim 3.1 we have \( c(w_1w_2) \in \{c(vw_1), c(vw_2)\} \). According to the definitions of \( N_1 \), \( W_1 \) and \( W_2 \), we have \( c(w_1w_2) \neq c(uw_2) \). Then, \( c(uw_1w_2) = c(uw_2) \), otherwise \( uw_1w_2u \) is a rainbow triangle, a contradiction. Hence, for any vertex \( w \in W_2 \), \( C(w, W_1) = \{c(vu)\} \). Then, there is a vertex \( w \in W_2 \) such that \( d^c_{N_1}(w) < d^c_{N_1}(u) \). Therefore, we can find a vertex \( u \in N_1 \) such that \( C(u, N_1 \setminus \{u\}) = \{c(vu)\} \). \( \square \)

Next we distinguish two cases.

**Case 1.** \( |N_1| \geq 2 \).

Let \( u \) be a vertex in \( N_1 \) such that \( C(u, N_1 \setminus \{u\}) = \{c(vu)\} \). From Claim 3.1 we have \( C(u, N_j(v)) \subseteq \{c(vu), j\} \) for \( s+1 \leq j \leq t \). Hence, \( d^c(u) \leq t-s+1 \leq \frac{n-1-s}{2} + 1 < \frac{n-1}{2} \), a contradiction.

**Case 2.** \( |N_1| = 1 \).

Let \( N_1 = \{u\} \). Then \( |N_j(v)| = 2 \) and \( j \in C(u, N_j(v)) \) for \( 2 \leq j \leq t \), otherwise \( d^c(u) < \frac{n}{2} \). Assume there is a set \( N_k(v) = \{x, y\} \). W.l.o.g., suppose \( c(ux) = c(vx) = k \). Since \( n \geq 8 \), there exists a vertex \( z \in N_2 \setminus N_k(v) \) such that \( c(xz) = c(vz) \). Hence, \( c(uz) = c(vz) \), otherwise from Claim 3.1 \( uzzu \) is a rainbow triangle. Therefore, \( c(zy) = c(vy) = k \), otherwise \( d^c(z) \leq t-s-2+2 < \frac{n}{2} - 1 \), a contradiction. Hence, \( c(zx) \neq c(zy) \).
Then, \(c(xy) \in \{c(zx), c(zy)\} = \{c(vx), c(vz)\}\), otherwise \(xyzx\) is a rainbow triangle. So, 
\[d^c(x) \leq t - s - 1 + 1 \leq \frac{n}{2} - 1,\] a contradiction. \(\Box\)

**Proof of Theorem 3.4.** Let \(G\) be a graph satisfying the assumptions of Theorem 3.4. Since \(G\) has no rainbow triangle and \(\delta^c(G) \geq \frac{n-1}{2}\), there exist a vertex \(v\) such that \(d^c(v) = \frac{n-1}{2} = t\) in \(G\). Assume that \(|N_1(v)| = \cdots = |N_7(v)| = 1\) and \(2 \leq |N_{s+1}(v)| \leq \cdots \leq |N_t(v)|\). Let \(N_1 = \bigcup_{1 \leq i \leq s} N_i(v)\) and \(N_2 = \bigcup_{s+1 \leq i \leq t} N_i(v)\). Now we proceed by proving the following claims.

**Claim 3.3.** \(C(N_i(v), N_j(v)) \subseteq \{i, j\}\), for \(1 \leq i \neq j \leq t\).

**Claim 3.4.** \(N_1 = \emptyset\).

**Proof.** Suppose not, since \(d^c(v) = \frac{n-1}{2}\), there is a set \(N_k(v), s + 1 \leq k \leq t\), such that \(|N_k(v)| \geq 3\). Thus, \(s - t \leq \frac{n-2-t}{2}\). From Claim 3.3 \(C(u, N_j(v)) \subseteq \{c(vu), j\}\) for \(s + 1 \leq j \leq t\).

If \(|N_1| \geq 2\), as in the proof of Theorem 3.3 there is a vertex \(u \in N_1\) such that \(C(u, N_1 \setminus \{u\}) = \{c(vu)\}\). Then, \(d^c(u) \leq t - s - 1 \leq \frac{n-2-s}{2} - 1 \leq \frac{n}{2} - 1\), a contradiction. If \(|N_1| = 1\), let \(N_1 = \{u\}\), and assume that \(|N_2(v)| = 3\) and \(|N_3(v)| = 2\) for \(3 \leq j \leq t\). Then there must exist a vertex \(x \in N_i(v)\) such that \(c(wx) = c(vx), 3 \leq l \leq t\). Let \(N_l(v) = \{x, y\}\). Then there is a vertex \(z \in N_k(v)\) such that \(c(xz) = c(vz)\), otherwise \(d^c(x) < \frac{n-1}{2}\).

Hence, \(c(zy) = c(vy)\), otherwise \(d^c(z) < \frac{n-1}{2}\). Then \(c(xz) \neq c(zy)\). Therefore, \(c(xy) \in \{c(vz), c(vx)\}\), otherwise \(xyzx\) is a rainbow triangle. Then, \(d^c(x) \leq \frac{n-2-1}{2} < \frac{n-1}{2}\), a contradiction. \(\Box\)

By Claim 3.3 \(|N_i(v)| = 2\) for all \(1 \leq i \leq t\). Thus, \(n\) is odd. Since \(\delta^c(G) \geq \frac{n-1}{2}\), by Claim 3.3 we have that for any vertex \(u \in N_i(v), j \in C(u, N_j(v))\) for \(1 \leq j \neq i \leq t\), that is \(C(u, N_j(v)) = \{c(vu), j\}\). If \(n \leq 5\), it is easy to verify that \(C(N_i(v)) \subseteq \{1, 2\}\). If \(n \geq 7\), then \(C(N_j(v)) = \{j\}\). Otherwise, suppose that there is a set \(N_k(v) = \{x, y\}\) such that \(c(xy) \neq k\). Then there is a vertex \(z \in N_2 \setminus N_k(v)\) such that \(xyzx\) is a rainbow triangle, a contradiction. Therefore, let \(A_0 = \{v\}\) and \(A_i = N_i(v)\) for \(1 \leq i \leq \frac{n-1}{2}\). This completes the proof. \(\Box\)

**4 Proofs of Theorems 11 and 12**

At first we need the following lemmas.
Lemma 15. Let $G$ be an edge-colored complete graph of order $n \geq 8$. If $\delta^c(G) \geq \frac{n+1}{2}$ and there are two vertices $y, z$ such that $G' = G - \{y, z\}$ has no rainbow triangles, then $G$ has two vertex-disjoint rainbow triangles containing $y$ and $z$, respectively.

Proof. Since $\delta^c(G) \geq \frac{n+1}{2}$, we have $\delta^c(G') \geq \delta^c(G) - 2 \geq \frac{|G|-1}{2}$. From Theorem 10, $d^c_{G'}(v) = \frac{|G'|-1}{2}$ for $v \in V(G')$ and $G'$ has a partition $\{A_i, 0 \leq i \leq \frac{n-1}{2}\}$. Assume that $A_0 = \{v\}$ and $A_i = N_i(v)$ for $1 \leq i \leq \frac{n-1}{2}$. Since $\delta^c(G) \geq \frac{n+1}{2}$, the edges from every vertex in $G'$ to $z$ and $y$ are assigned two new colors. Let $N_i(v) = \{a_i, b_i\}$. If there is a set $N_k(v)$ such that $c(za_k) \neq c(zb_k)$, then $C(y, G' \setminus (\{v\} \cup N_k(v))) = \{c(vy)\}$. If not, there is a vertex $u \in G' \setminus (\{v\} \cup N_k(v))$ such that $c(uv) \neq c(vy)$, and then $uvy$ is a disjoint rainbow triangle from $xakbyx$. So, $d^c(y) \leq 4$, a contradiction. Hence, for any set $N_i(v)$, $c(za_i) = c(zb_i)$ and $c(ya_i) = c(yb_i)$. Since $d^c(z) \geq \frac{n+1}{2}$, there are two sets $N_i(v)$ and $N_j(v)$, $i \neq j$, such that $c(za_i) \neq c(za_j)$. Then $za_ia_jz$ is a rainbow triangle. Similarly, we can find another rainbow triangle $yb/by$ a contradiction. \qed

Lemma 16. Let $G$ be an edge-colored graph of order $n \geq 7$. If $\delta^c(G) \geq \frac{n+2}{2}$ and there are two vertices $y, z$ such that $G' = G - \{y, z\}$ has no rainbow triangles, then $G$ has two vertex-disjoint rainbow triangles containing $y$ and $z$, respectively.

Proof. Since $\delta^c(G) \geq \frac{n+1}{2}$, we have $\delta^c(G') \geq \delta^c(G) - 2 \geq \frac{|G|-1}{2}$. According to Theorem 10, we know that $G'$ is a properly colored balanced complete bipartite graph. Since $\delta^c(G) \geq \frac{n+2}{2}$, we have that $vz$ and $zy$ are in $E(G)$ and $d^c(v) = \frac{n+2}{2}$ for every vertex $v \in V(G)$. Thus, we can easily find two vertex-disjoint rainbow triangles containing $z$ and $y$, respectively, a contradiction. \qed

Proof of Theorem 11: Since $\delta^c(G) \geq \frac{n}{2}$, according to Theorem 8 $G$ has a rainbow triangle $xyz$. Let $T(G)$ be the set of all rainbow triangles in $G$. Suppose, to the contrary, that $G$ has no vertex-disjoint rainbow triangles. Then each rainbow triangle in $T(G) \setminus \{xyz\}$ meets at least one of $\{x, y, z\}$. Let $W_1(W_2, W_3)$ denote the subset of vertices in $V(G) \setminus \{x, y, z\}$, in which every vertex is contained in a rainbow triangle together with $x(y, z)$. Now we proceed by proving the following claims.

Claim 4.1. $W_i \neq \emptyset$, for $i = 1, 2, 3$.

Proof. W.l.o.g., suppose $W_1 = \emptyset$. Then each rainbow triangle in $T(G) \setminus \{xyz\}$ meets $y$ or $z$. From Lemma 15, we can find two vertex-disjoint rainbow triangles in $G$, a contradiction. \qed

Claim 4.2. For any set $W_i$, there is a vertex in $W_i$ but not in $W_j$, $1 \leq i \neq j \leq 3$.
Proof. W.l.o.g., suppose, to the contrary, that $W_1 \subseteq W_2 \cup W_3$. Then there is no rainbow triangle in $G' = G - \{z, y\}$. Therefore, from Lemma 15 we can find two vertex-disjoint rainbow triangles in $G$, a contradiction.

Claim 4.3. There is a vertex $a_0$ such that all rainbow triangles in $T(G) \setminus \{xyzx\}$ meet at $a_0$.

Proof. By Claim 4.2 let $a_i \in W_i \setminus (W_j \cup W_k)$, $1 \leq i \neq j \neq k \leq 3$. Then there is a vertex $a_0$ such that $xa_1a_0x, ya_2a_0y$ and $za_3a_0z$ are rainbow triangles, otherwise we can easily find two disjoint rainbow triangles. Suppose that there is a rainbow triangle $uvwu$ in $T(G) \setminus \{xyzx\}$ such that $a_0 \notin \{u, v, w\}$. Then we can easily find two vertex-disjoint rainbow triangles, a contradiction.

Let $G'' = G - \{x, a_0\}$. Then $\delta_c(G'') \geq \frac{|G''| - 1}{2}$. From Claim 4.3 there is no rainbow triangle in $G''$. Hence, from Lemma 15 we can find two vertex-disjoint rainbow triangles in $G$ containing $x$ and $a_0$, respectively, a contradiction. This completes the proof of Theorem 11.

Proof of Theorem 12: Using Theorems 5 and 6 in [6], and Lemma 16 as well as by an analogue of the proof of Theorem 11 we can get the result of Theorem 12.

References

[1] J.A. Bondy, U.S.R. Murty, Graph Theory, GTM 244, Springer (2008).
[2] S. Fujita, C. Magnant, Properly colored paths and cycles, Discrete Appl. Math. 159 (2011), 1391–1397.
[3] J. Hu, H. Li, D. Yang, Vertex-disjoint rainbow triangles in edge-colored graphs, Discrete Math. 343 (2020), 112–117.
[4] H. Li, Rainbow $C_3$’s and $C_4$’s in edge-colored graphs, Discrete Math. 313 (2013) 1893–1896.
[5] R. Li, H. Broersma, S. Zhang, Vertex-disjoint properly edge-colored cycles in edge-colored complete graphs, J. Graph Theory 94 (2020), 476–493.
[6] B. Li, B. Ning, C. Xu, S. Zhang, Rainbow triangles in edge-colored graphs, European J. Combin. 36 (2014), 453–459.