Two-dimensional motion equations in water flow zone

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Abstract. The paper studies a two-dimensional water flow from a non-pressure rectangular or round pipe placed into a wide horizontal channel. To simplify the problem, the real three-dimensional flow is modeled as a two-dimensional zone by eliminating the liquid particles’ velocities and accelerations in the direction perpendicular to the flow zone. To describe the water flow motion law, L. Euler’s equations for the ideal fluid are used, taking into account the continuity equations and the Bernoulli equation. The two-dimensional flow models in the spreading zone with the adequacy degree sufficient for practice, describe the movement of water flows arising in the lower road drainage systems races, Liman irrigation systems, small bridges, water volley channels, various culverts and water-crossing facilities. The obtained dependences of the velocity distribution, depth and water flow geometry give an accuracy exceeding that known by the previously used methods both by the velocity values and by the boundary current lines geometry.

Introduction

The paper is aimed at reviewing the various types of two-dimensional open water flow motion equations and some practical problems solution with the help of these equations. The basic assumptions and the initial physical prerequisites of two-dimensional in zone of water flows are as follows [1, 2]:

a) vertical (or normal to the selected coordinate plane) components of local averaged velocities and accelerations are small;

b) the liquid particles velocity vectors located on the same vertical lie in the same plane;

c) the velocities’ distribution on any vertical is almost proportional.

We can distinguish a fairly wide class of flows which parameters meet these assumptions.

Such flows are called two-dimensional in zone, reflecting the fact that two geometric coordinates x and y are enough to describe them.

From the technical literature [3-5] it is known that for the formulation and solution of various applied problems in water flows it is necessary to use the motion equations describing the fluid flow process and to know the initial and boundary conditions.

Historically, the two-dimensional theory founders in water flows zone came from the dynamic equations of motion of an ideal two-dimensional open water flow in the form of L. Euler (the ideal fluid motion equations), supplemented by the terms taking into account the fluid resistance forces [1, 2]:

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\[
\begin{align*}
X - \frac{1}{\rho} \frac{\partial p}{\partial x} - T_x &= \frac{du_x}{dt}; \\
Y - \frac{1}{\rho} \frac{\partial p}{\partial y} - T_y &= \frac{du_y}{dt}; \\
Z - \frac{1}{\rho} \frac{\partial p}{\partial z} - T_z &= \frac{du_z}{dt},
\end{align*}
\]

(1)

where \(X, Y, Z\) are the volume forces components; \(T_x, T_y, T_z\) are the resistance forces components related to the liquid unit mass; \(\rho\) is the liquid density; \(p\) is the local pressure; \(u_x, u_y, u_z\) are the local velocity vector components.

**The water flow motion equations**

For steady flow in the vertical direction of the \(z\) axis and the action of a single volume force (gravity) in the liquid, the system (1) takes the form:

\[
\begin{align*}
- \frac{1}{\rho} \frac{\partial p}{\partial x} - T_x &= u_x \frac{du_x}{dx} + u_y \frac{du_y}{dy} + u_z \frac{du_z}{dz}; \\
- \frac{1}{\rho} \frac{\partial p}{\partial y} - T_y &= u_x \frac{du_x}{dx} + u_y \frac{du_y}{dy} + u_z \frac{du_z}{dz}; \\
-g - \frac{1}{\rho} \frac{\partial p}{\partial z} - T_z &= u_x \frac{du_x}{dx} + u_y \frac{du_y}{dy} + u_z \frac{du_z}{dz}.
\end{align*}
\]

(2)

Due to the small vertical velocity and acceleration components’ assumption, all the inertial components containing \(u_z\) and its derivatives can be discarded.

Then the third system equation (2) will take the form:

\[
\frac{\partial p}{\partial z} = -g\rho
\]

(3)

where \(g\) is the gravity acceleration, \(\rho\) – is the water density.

Integrating the equation (3), we obtain:

\[
p = -\gamma z + f(x, y)
\]

(4)

where \(f(x, y)\) is an arbitrary function, \(\gamma = g\rho\).

Taking into account the fact that on the free surface \(z = z_n\) and \(p = p_n = \text{const}\) come to the pressures distribution hydrostatic law on the vertical:

\[
p - p_n = \gamma (z_n - z)
\]

(5)

Denoting by \(z_0\) the water stream bottom coordinates, we get:
\[
\frac{\partial p}{\partial x} = \gamma \frac{\partial}{\partial x} (z_0 + h); \quad \frac{\partial p}{\partial y} = \gamma \frac{\partial}{\partial y} (z_0 + h)
\]  
(6)

Then the flow motion dynamic equations system can be written as:

\[
\begin{align*}
\frac{du_x}{dx} + u_x \frac{du_x}{dy} &= -g \frac{\partial}{\partial x} (z_0 + h) - T_x; \\
\frac{du_y}{dx} + u_y \frac{du_y}{dy} &= -g \frac{\partial}{\partial y} (z_0 + h) - T_y.
\end{align*}
\]  
(7)

Adding this equations system to the flow continuity equation, we get:

\[
\frac{\partial}{\partial x} (hu_x) + \frac{\partial}{\partial y} (hu_y) = 0
\]  
(8)

in the case of a horizontal channel, we obtain the following flow system in the form of:

\[
\begin{align*}
\frac{du_x}{dx} + u_x \frac{du_x}{dy} + g \frac{\partial h}{\partial x} &= -T_x; \\
\frac{du_y}{dx} + u_y \frac{du_y}{dy} + g \frac{\partial h}{\partial y} &= -T_y; \\
\frac{\partial}{\partial x} (hu_x) + \frac{\partial}{\partial y} (hu_y) &= 0.
\end{align*}
\]  
(9)

The system of equations (9) is reduced to one equation of the second order in partial derivative with respect to the potential function \( \varphi = \varphi(x, y) \):

\[
\frac{\partial^2 \varphi}{\partial x^2} \left[ C^2 - \left( \frac{\partial \varphi}{\partial x} \right)^2 \right] - 2 \frac{\partial \varphi}{\partial x} \frac{\partial \varphi}{\partial y} \frac{\partial^2 \varphi}{\partial x \partial y} + \frac{\partial^2 \varphi}{\partial y^2} \left[ C^2 - \left( \frac{\partial \varphi}{\partial y} \right)^2 \right] = 0
\]  
(10)

where \( C = \sqrt{gh}; \quad h = H_0 - \frac{\left( \frac{\partial \varphi}{\partial x} \right)^2 + \left( \frac{\partial \varphi}{\partial y} \right)^2}{2g} \).

Unfortunately, the flows calculation method using these characteristics is a numerical-graph-analytical method and does not always give sufficient adequacy for the model results’ practical use. Much more accurate results are given by the analytical models presented in monographs [9-11].

It should be noted that the two-dimensional model in the potential turbulent flow zone, despite a significant idealization degree, is of important theoretical and practical importance.

Analytical methods for solving various problems in two-dimensional flow in the potential flows zone are used:

a) in the case of simplifications for the potential model:
- the simple centered wave motion equations;
- radial flow (non-pressure potential source) [1, 2].

b) when using the speed hodograph auxiliary [9, 10].

According to the equations (9) the two-dimensional motions system in the potential water flows zone can be written as:
\[
\begin{aligned}
\frac{\partial \varphi}{\partial \tau} &= \frac{h}{h_0} \frac{3\tau - 1}{2H_0(1-\tau)^2} \frac{\partial \psi}{\partial \theta}; \\
\frac{\partial \varphi}{\partial \theta} &= 2 \frac{h_0}{H_0} \frac{\tau}{1-\tau} \frac{\partial \psi}{\partial \tau}.
\end{aligned}
\]  

The system of equations (12) is already a linear system of partial differential equations in contrast to the system (9). This system is reduced to solving the following equation of mathematical physics:

\[
\frac{\partial}{\partial \tau} \left( \frac{2\tau}{1-\tau} \frac{\partial \psi}{\partial \tau} \right) + \frac{1-3\tau}{2\tau(1-\tau)^2} \frac{\partial^2 \psi}{\partial \theta^2} = 0
\]

This equation is reduced to a number of substitutions hypergeometric equation with real coefficients, which solutions are known.

In papers [1, 2] the equations of characteristics in the speed hodograph plane are given; this result and the equations appearance analogy of perfect gas and the potential two-dimensional in zone of water flow allowed the authors of papers [7-9] to use the S. A. Chaplygin transformation [6] to obtain the system (12) and to divide the boundary value problem solution into two stages:

1- the problem solution in the speed hodograph plane;  
2-obtaining the algorithm for determining the flow parameters in the physical plane of the flow (in the flow zone) by integrating the connection with the functions \( \varphi(\tau, \theta) \) and \( \psi(\tau, \theta) \) found at the 1st stage.

Further we present one of the methods for determining the flow resistance forces when using the system (9).

For flows in which it is impossible to neglect the resistance forces to the flow from the water conduit, the resistance forces account can be carried out at the suggestion of the authors in papers [1, 2, 7] by reducing the surface forces to some equivalent volume.

If \( \tau \) is the tangential stress at the elementary cylindrical volume base, the friction force at the bottom is equal to \( \tau ds \). The force vector direction is opposite to the velocity vector \( \vec{V} \) direction.
Then the absolute values of this force, referred to the selected volume unit mass, expressed by formulas:

\[ T_x = \frac{\tau}{\rho h} \frac{u_x}{V} ; \quad T_y = \frac{\tau}{\rho h} \frac{u_y}{V} \]  

(15)

Or, by introducing the hydraulic friction coefficient:

\[ \lambda = \frac{2\tau}{\rho V^2} \]  

(16)

the resistance forces components are rewritten as:

\[ T_x = \frac{\lambda u_x V}{2h} ; \quad T_y = \frac{\lambda u_y V}{2h} \]  

(17)

As it is known from [10], the Chezy’s velocity factor \( C \) is related to the hydraulic friction coefficient by the ratio [13]: \( \lambda = \frac{2g}{C^2} \),

moreover \( C = \frac{h^m}{n} \); the indicator \( m \) is determined by a specific resistance law, given in the known literature on hydraulics; \( n \) is the conduit walls roughness coefficient.

For discontinuous flows by flow parameters, two-dimensional planned flow equations (Saint-Venant) are used, written in the form [12]:

\[ \int \int \varphi \, dx \, dy + \Pi \, dy \, dt + \Phi \, dx \, dt = \int \int \Psi \, dx \, dy \, dt \]  

(18)

We present a method for solving an important problem for the conjugate parameters determining practice of a two-dimensional jet in zone of a turbulent flow at its impact on the side wall (Figure 1) based on the motion equations use in the integral form.

![Figure 1. Plan of the impact of the extreme current line on the side wall of the channel](image)

Let at the point \( M \) before the extreme flow line impact on the channel side wall it has parameters \( V_1, h_1, \theta_1 \).

It is necessary to determine the flow parameters at the point \( M \) after impact \( V_2, h_2 \) and the angle \( \delta \) of the exit line deviation from the channel side wall.
To estimate \( V_2, h_2, \delta \) we use the stationary flow equation in integral form excluding the resistance forces to the flow:

\[
\oint \left( \Pi \, dy + \Phi \, dx \right) = 0,
\]

(19)

where \( \Pi = \begin{bmatrix} V h \cos \theta \\ V^2 h \cos^2 \theta + \frac{gh^2}{2} \\ V^2 h \cos \theta \sin \theta \end{bmatrix} \) and \( \Phi = \begin{bmatrix} V h \sin \theta \\ V^2 h \cos^2 \theta \sin \theta \\ V^2 h \sin^2 \theta + \frac{gh^2}{2} \end{bmatrix} \).

Composing the algorithmic equation form with respect to the four-point template \( IM23 \) (see Figure 1), two points of which are on the vanishing line, and solving it with respect to the flow parameters at the point \( M \), we obtain the algebraic system:

\[
\begin{align*}
V h \cos \theta + KV h \sin \theta &= r_1; \\
V^2 h \cos^2 \theta + \frac{gh^2}{2} + KV^2 h \cos \theta \sin \theta &= r_2; \\
V^2 h \cos \theta \sin \theta + K \left( V^2 h \sin^2 \theta + \frac{gh^2}{2} \right) &= r_3.
\end{align*}
\]

(20)

Since before the extreme current line impact on the side wall, the parameters \( V_1, h_1, \theta_1 \) must satisfy the system (20), then

\[
\begin{align*}
r_1 &= a_1 + KC_1; \\
r_2 &= a_2 + KC_2; \\
r_3 &= a_3 + KC_3,
\end{align*}
\]

(21)

where \( a_1 = V_1 h_1 \cos \theta_1; \quad C_1 = V_1 h_1 \sin \theta_1; \quad a_2 = V_1^2 h_1 \cos^2 \theta_1 + \frac{gh_1^2}{2}; \quad C_2 = V_1^2 h_1 \cos \theta_1 \sin \theta_1; \quad a_3 = V_1^3 h_1 \cos \theta_1 \sin \theta_1; \quad C_3 = V_1^3 h_1 \sin^2 \theta_1 + \frac{gh_1^2}{2}.\)

After the jump at point \( M \) the angle is equal to \( \theta_2 = 0 \) and therefore from the system (20) we get:

\[
\begin{align*}
V_2 h_2 &= r_1; \\
V_2^2 h_2 + \frac{gh_2^2}{2} &= r_2; \\
K \frac{gh_2^2}{2} &= r_3.
\end{align*}
\]

(22)

The system (22) has three unknown parameters \( V_2, h_2, K \) and the condition must satisfy its compatibility:

\[
r_1^4 K^3 g - 2r_3 (r_2 K - r_3)^2 = 0.
\]

(23)
Having determined the unknown $K$ from the equation (23), we define the conjugate parameters of the flow $V_2$, $h_2$, $\delta$ after the jump [13, 14]:

$$ h_2 = \frac{2r_3}{Kg}; \quad V_2 = \frac{r_1}{h_2}; \quad \delta = \arctg \frac{1}{K} \quad (24) $$

**Summary**

Earlier in the literature on hydraulic calculations to solve the problem of determining the parameters $V_2$, $h_2$, $\delta$ it was necessary to use bulky nomograms with restrictions $\Theta_1 < 67^\circ$, which significantly complicated obtaining a reliable result at the values of the flow angle on the wall, larger than $67^\circ$.

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