A Note on Bulk Quantum Turing Machine

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Abstract

Recently, among experiments for realization of quantum computers, NMR quantum computers have achieved the most impressive succession. There is a model of the NMR quantum computation, namely Atsumi and Nishino’s bulk quantum Turing machine. It assumes, however, an unnatural assumption with quantum mechanics. We, then, define a more natural and quantum mechanically realizable modified bulk quantum Turing machine, and show its computational ability by comparing complexity classes with quantum Turing machine’s counterpart.

1 Introduction

Recently, among experiments for realization of quantum computers, Nuclear Magnetic Resonance (NMR) quantum computers have achieved the most impressive succession. For example, Vandersypen and his colleagues of IBM and Stanford University compose organic molecules realizing 7 quantum bits, and factor 15 with them using Shor’s algorithm [6].

The NMR quantum computation differs from the ordinary quantum computation; since it manipulates many molecules simultaneously, one cannot utilize the projection property of measurement but can only obtain the ensemble average. Note that, however, it is merely unusable the projection in order to select a specific state, but in fact each molecules is projected to the state when it is measured so that the whole ensemble is the ensemble which gives the measurement value of the ensemble average. It is a basic assumption of the quantum mechanics.

Today, there is a model of the NMR quantum computation by Atsumi and Nishino: bulk quantum Turing machine. It has, however, two assumptions contradicting with quantum mechanics: (1) a measurement does not cause projection on each molecules, and (2) the probability to obtain the ensemble average value in a certain interval is 1. We already mentioned about the first point above, and the second point will be shown false later.

We shall define a new model of the NMR quantum computer removing the difficulties of Atsumi and Nishino’s model. Then we shall show the computa-

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tional power is the same with both Atsumi and Nishino’s bulk quantum Turing machine and the original quantum Turing machine for decision problems.

2 Definitions

At first, we recall the definitions of the quantum Turing machine and bulk quantum Turing machine. The first part of the definition is common to both.

**Definition 1 (Tape, Symbols and Unitarity).** A quantum Turing machine (QTM) (or bulk quantum Turing machine (BQTM)) $M$ is defined by a triplet $(\Sigma, Q, \delta)$ where: $\Sigma$ is a finite alphabet with an identified blank symbol $\#$, $Q$ is a finite set of states with an identified initial state $q_0$ and final state $q_f$ which is not equal to $q_0$, and $\delta$ is a function called quantum transition function:

$$\delta : Q \times \Sigma \rightarrow \tilde{C}^{\Sigma \times Q \times D}$$

where $D$ denotes directions $\{L, R\}$ or $\{L, N, R\}$, and $\tilde{C}$ denotes the set of computable numbers. The quantum Turing machine (or bulk quantum Turing machine) has a two-way infinite tape of cells indexed by $Z$ and a single tape head that moves along the tape.

The quantum transition function $\delta$ induces a time evolution operator $U_M$ of the inner-product space of finite complex linear combination of configurations of $M$. The time evolution operator $U_M$ must be unitary.

The observation of Quantum Turing Machine is carried out as usual physics.

**Definition 2 (Observation of Quantum Turing Machine).** When a quantum Turing machine $M$ in a superposition $\Psi = \Sigma_i \alpha_i c_i$ is observed, each configuration $c_i$ is obtained with probability $|\alpha_i|^2$ and the superposition of $M$ is updated to $\Psi' = c_i$. A partial observation is also allowed.

Atsumi and Nishino replaced the observation of quantum Turing machine above with the measurement of bulk quantum Turing machine below in $[1]$.

**Definition 3 (Measurement of Bulk Quantum Turing Machine).** A measurement is an observation such that $|\alpha|^2 - |\beta|^2$ is obtained for a qubit $\alpha|1\rangle + \beta|0\rangle$ with error range less than $\theta$ and with probability 1. The measurement does not disturb the superposition which $M$ is in, and can be repeated several times. No partial observation is allowed.

We make a modification to the measurement.

**Definition 4 ((\(\epsilon, \theta\))-measurement).** An $(\epsilon, \theta)$-measurement is an observation such that $|\alpha|^2 - |\beta|^2$ is obtained for a qubit $\alpha|1\rangle + \beta|0\rangle$ with error range less than $\theta$ and error probability less than $\epsilon$. If either $|\alpha|^2$ or $|\beta|^2$ is zero, the error probability is exceptionally zero. The measurement does disturb the superposition which $M$ is in, and can not be repeated multiple times. No partial observation is allowed.
We call a quantum computer defined by definition 1 and 4 a modified bulk quantum Turing machine (MBQTM).

Since a set of parallel statistically independent quantum Turing machines gives a model of modified bulk quantum Turing machine; the bigger the number of quantum Turing machines gets, the smaller the parameters $\theta$ and $\epsilon$ of modified bulk quantum Turing machine get. A quantitative relationship among parameters is given in the following subsection, but beforehand we give a qualitative statement.

**Lemma.** If a quantum Turing machine without partial measurements and resulting 0 or 1 exists, corresponding modified bulk quantum Turing machine exists for $\frac{1}{2} > \forall \theta, \epsilon > 0$.

**Proof.** Consider $n$ parallel independent quantum Turing machines without partial measurements and resulting 0 or 1. Since partial measurements are not used, the computation is also carried out by modified bulk quantum Turing machine having the same quantum transition function. Therefore, the only difference is the final observation or measurement. If the final superposition of the cell is $\alpha|1\rangle + \beta|0\rangle$ and one assigns $-1$ to $|0\rangle$, the ensemble average is $|\alpha|^2 - |\beta|^2$. The law of large numbers assures, the bigger $n$ gets, the smaller the error probability $\epsilon$ becomes with given value range of $\theta$.

Note that the measurement of bulk quantum Turing machine is understood as the $(0, \theta)$-measurement, if we forget about the disturbance on the superposition. However, the parameters of positive $\theta$ with $\epsilon = 0$ is not realized by the parallel independent quantum Turing machines until the superposition is in an eigen state ($|0\rangle$ or $|1\rangle$).

### 2.1 Relationship among $\epsilon$, $\theta$ and the Number of Quantum Turing Machine’s

As stated above, modified bulk quantum Turing machine is realizable by parallel independent quantum Turing machines and the parameters $\theta$ and $\epsilon$ of modified bulk quantum Turing machine and the number $n$ of quantum Turing machines are dependent. The relationship among them are known by de Moivre - Laplace’s theorem asymptotically:

$$\frac{1}{\sqrt{2\pi}} \int_{-t}^{t} e^{-\frac{x^2}{2}} dx \sim 1 - \epsilon$$

where $t = 2\theta \sqrt{n}$.

By this formula, the table is obtained. The number $n$ is considered as the number of molecules in NMR to be observed.

### 3 Complexity Classes

There are some complexity classes for quantum Turing machine and corresponding classes for bulk quantum Turing machine. We shall define the corresponding
Table 1: value of $n$ for $\theta$, $\epsilon$

| value of $n$ | $\epsilon$ |
|-------------|-------------|
| $0.04550$   | $0.02000$   | $0.01000$   |
| $2^{-5}$    | $1024$      | $1699$      | $2018$      |
| $2^{-6}$    | $4096$      | $6795$      | $8069$      |
| $2^{-7}$    | $16384$     | $27177$     | $32275$     |

classes for modified bulk quantum Turing machine, then to show their equalences. Since we concern classes of decision problems, we assume that the alphabet $\Sigma$ includes \{0, 1\}. Moreover, a tape cell called acceptance cell be in the superposition $\alpha|1\rangle + \beta|0\rangle$ when it is observed, i.e. there is no possibility to have blank or any other symbols.

We shall discuss about three kinds of classes. The classes for quantum Turing machine was defined by Bernstein and Vazirani [2], and for bulk quantum Turing machine by Nishino et al. [4][5].

3.1 Exact Quantum Polynomial Time

First of all, we see the classes of exact quantum polynomial time languages.

Definition 5 (EQP, EBQP, EBQP$^*$). The quantum complexity classes of exact quantum polynomial time languages are defined according to the models of the quantum computers:

1. A language $L$ is in the class EQP if and only if there exists a quantum Turing machine and polynomial $p$ such that for any input $x$ an observation of a certain tape cell after calculation of $p(|x|)$ steps gives 1 with probability 1 if $x$ belongs to $L$, 0 with probability 1 otherwise.

2. A language $L$ is in the class EBQP if and only if there exists a bulk quantum Turing machine and polynomial $p$ such that for any input $x$ a measurement of a certain tape cell after calculation of $p(|x|)$ steps gives more than $1 - \theta$ with probability 1 if $x$ belongs to $L$, less than $-1 + \theta$ with probability 1 otherwise.

3. A language $L$ is in the class EBQP$^*$ if and only if there exists a modified bulk quantum Turing machine and polynomial $p$ such that for any input $x$ an $(\epsilon, \theta)$-measurement of a certain tape cell after calculation of $p(|x|)$ steps gives more than $1 - \theta$ with probability 1 if $x$ belongs to $L$, less than $-1 + \theta$ with probability 1 otherwise.

Theorem. EQP and EBQP$^*$ are equivalent.

Proof. It is possible to observe a value from a tape cell of quantum Turing machine with probability 1 only when the cell is equals to one of the eigen states;
Thus, if a language \( L \) is in the class \( \text{EQP} \), there exists a quantum Turing machine \( M \) with a certain cell in the eigen state to be observed at the last step. Suppose modified bulk quantum Turing machine \( M^* \) which has the same quantum transition function \( \delta \) with \( M \). The calculation steps are identical and the tape cell to be read is in the eigen state. Then, by the definition \[1\] it is possible to obtain the value with probability 1. Thus, the language \( L \) is in the class \( \text{EBQP}^* \).

Conversely, if a language \( L \) is in the class \( \text{EBQP}^* \), there exists a modified bulk quantum Turing machine \( M^* \). Since the probability to observe either one of the values 1 or \(-1\) is 1, the tape cell is in either one of the eigen state when it is observed. Suppose quantum Turing machine \( M \) which has the same quantum transition function \( \delta^* \) with \( M^* \). Then, the calculation steps are identical and the final result is obtained with probability 1, i.e. the language \( L \) is in the class \( \text{EQP} \).

Corollary. \( \text{EBQP} \) and \( \text{EBQP}^* \) are equivalent.

Proof. It is a direct consequence of \( \text{EQP} = \text{EBQP} \) \[1\].

3.2 Bounded Error Quantum Polynomial Time

Next, we see the classes of bounded error quantum polynomial time languages. These are the most important classes.

Definition 6 (BQP, BBQP, BBQP\(^*\)). The quantum complexity classes of bounded error quantum polynomial time languages are defined according to the models of the quantum computers:

1. A language \( L \) is in the class \( \text{BQP} \) if and only if there exists a quantum Turing machine such that for any input \( x \) an observation of a certain tape cell after calculation of polynomial time of its size gives 1 with probability more than \( 2/3 \) if \( x \) belongs to \( L \), 0 with probability more than \( 2/3 \) otherwise.

2. A language \( L \) is in the class \( \text{BBQP} \) if and only if there exists a bulk quantum Turing machine such that for any input \( x \) a measurement of a certain tape cell after calculation of polynomial time of its size gives more than \( 1/3 - \theta \) if \( x \) belongs to \( L \), less than \( -1/3 + \theta \) otherwise.

3. A language \( L \) is in the class \( \text{BBQP}^* \) if and only if there exists a modified bulk quantum Turing machine such that for any input \( x \) an \((\epsilon, \theta)\)-measurement of a certain tape cell after calculation of polynomial time of its size gives more than \( 1/3 \) if \( x \) belongs to \( L \), less than \(-1/3 \) otherwise.

Theorem. \( \text{BQP} \) and \( \text{BBQP}^* \) are equivalent.

Proof. Assume that \( L \) is in the class \( \text{BQP} \). By the definition there is a quantum Turing machine \( M \) which accepts \( L \). We can assume that the tape cell to be observed is in superposition \( \alpha|1\rangle + \beta|0\rangle \) and \( p = |\alpha|^2 > 2/3 \) if an input belongs
to $L$. By the lemma of section refsecdef, there is a modified bulk quantum Turing machine $M^*$ corresponding to $M$ with $\theta < p - 2/3$. Then, the $(\epsilon, \theta)$-measurement of $M^*$ gives a value in range $(|\alpha|^2 - |\beta|^2 - \theta, |\alpha|^2 - |\beta|^2 + \theta)$.

$$|\alpha|^2 - |\beta|^2 - \theta = p - (1 - p) - \theta$$
$$> 2p - 1 - p + 2/3$$
$$> 1/3. \tag{5}$$

The other case is shown similarly. Thus, we can conclude $L$ is in the class BBQP$^*$. Conversely, Assume that $L$ belongs to BBQP$^*$. By the definition there is a modified bulk quantum Turing machine $M^*$ which accepts $L$. If an input belongs to $L$, the $(\epsilon, \theta)$-measurement of the acceptance cell gives a value more than 1/3. With an identity $|\alpha|^2 + |\beta|^2 = 1$, it gives $|\alpha|^2 > 2/3$. In the case an input does not belong to $L$, it is shown in the same way that $|\beta|^2 > 2/3$. Thus, considering a quantum Turing machine $M$ which has the same quantum transition function with $M^*$ leads to the conclusion that $L$ is in the class BQP

**Corollary.** BBQP and BBQP$^*$ are equivalent.

**Proof.** It is a direct consequence of BQP = BBQP [4].

**3.3 Zero Error Quantum Polynomial Time**

Finally, we see the classes of zero error quantum polynomial time languages.

**Definition 7 (ZQP, ZBQP and ZBQP$^*$).** The quantum complexity classes of zero error quantum polynomial time languages are defined according to the models of the quantum computers:

1. A language $L$ is in the class ZQP if and only if there exists a quantum Turing machine such that for any input $x$ an observation of a certain tape cell (halt cell) after calculation of polynomial time of its size gives 1 with probability more than 1/2 and then an observation of another cell (decision cell) gives 1 with probability 1 if $x$ belongs to $L$, 0 with probability 1 otherwise.

2. A language $L$ is in the class ZBQP if and only if there exists a bulk quantum Turing machine such that for any input $x$ an observation of a certain tape cell (halt cell) after calculation of polynomial time of its size gives more than 0 with probability 1 and then either one of the following cases holds:

- measurement of another cell (accept cell) gives 1 with probability 1 if $x$ belongs to $L$,
- measurement of one another cell (reject cell) gives $-1$ with probability 1 if $x$ does not belong to $L$.  


3. A language $L$ is in the class $\text{ZBQP}^*$ if and only if there exists a modified bulk quantum Turing machine such that for any input $x$ an observation of a certain tape cell (halt cell) after calculation of polynomial time of its size gives more than $0$ with probability more than $1 - \epsilon$ and then either one of the following cases holds:

- $(\epsilon, \theta)$-measurement of another cell (accept cell) gives $1$ with probability $1$ if $x$ belongs to $L$,
- $(\epsilon, \theta)$-measurement of one another cell (reject cell) gives $-1$ with probability $1$ if $x$ does not belong to $L$.

**Theorem.** $\text{ZQP}$ and $\text{ZBQP}^*$ are equivalent.

**Proof.** If a language $L$ is in the class $\text{ZQP}$, there exists a quantum Turing machine $M$ and a polynomial $p$, which is a time estimation polynomial. We construct a modified bulk quantum Turing machine $M^*$ from $M$ in the following way.

At first, we replace the initialization steps of the decision cell of $M$ with the steps to initialize the accept cell to $|1\rangle$ and the reject cell to $|0\rangle$. If there is no initialization steps in $M$, we insert the steps above.

The steps of $M^*$ after initialization are identical to $M$ until it reaches to the step to write the result in decision cell.

Finally, the writing step is replaced with those which write the same result in both the accept cell and the reject cell: if $x$ belongs to $L$ then the result is $|1\rangle$, otherwise $|0\rangle$.

The changes increase at most a constant $k$ steps. Thus, after $p(|x|) + k$ steps $(\epsilon, \theta)$-measurement of halt cell can give, if one choose an appropriate $\theta$, more than $0$ with probability more than $1 - \epsilon$, since the corresponding observation of $M$ gives $1$ with probability more than $1/2$. At the moment, if $x$ belongs to $L$, the accept cell has not been changed since the initialization and reading $|1\rangle$ gives $1$ with probability $1$. On the other hand, reject cell is not in the eigen state and it is impossible to obtain $-1$ with probability $1$ by $(\epsilon, \theta)$-measurement. In the case when $x$ does not belong to $L$, the behaviors of the accept and reject cell are switched and an $(\epsilon, \theta)$-measurement of reject cell gives $-1$ with probability $1$. Thus, $L$ is in $\text{ZBQP}^*$.

Conversely, we assume $L$ is in the class $\text{ZBQP}^*$. Then, there exists a modified bulk quantum Turing machine $M^*$ to accept $L$. Consider a quantum Turing machine $M$ which has the same quantum transition function with $M^*$, and identify the accept cell or the reject cell as the decision cell. Obviously, if an observation of the halt cell gives $1$, the result is obtained correctly with probability $1$. Moreover, the probability that the observation of halt cell gives $1$ is more than $1/2$, because $M^*$ gives more than $0$ with probability more than $1 - \epsilon$. Therefore, $L$ is in the class $\text{ZQP}$. \[\square\]

**Corollary.** $\text{ZBQP}$ and $\text{ZBQP}^*$ are equivalent.

**Proof.** It is a direct consequence of $\text{ZQP} = \text{ZBQP}$. \[\square\]
4 Conclusion

We construct a model of NMR quantum computation named modified bulk quantum Turing machine. It can be realized as a set of statistically independent quantum Turing machines and more consistent with quantum physics than bulk quantum Turing machine, but still the computational power is equivalent to that of quantum Turing machine and bulk quantum Turing machine. Since, the main difference between bulk quantum Turing machine and modified bulk quantum Turing machine is the consistency with quantum physics, it is better to replace bulk quantum Turing machine with modified bulk quantum Turing machine.

References

[1] K. Atsumi and T. Nishino, “Solving NP-complete problems and factoring problems by using NMR quantum computation”, Transaction of Information Processing Society Japan Vol.43 No. SIG 7(TOM 6) pp.10–18, September 2002.

[2] E. Bernstein and U. Vazirani, “Quantum complexity theory”, SIAM J. Comput. Vol.26 No.5, pp.1411–1473, 1997.

[3] D. Deutsch, “Quantum theory, the Church-Turing principle and the universal quantum computer”, Proc. R. Soc. Lond. A 400, pp.97–117, 1985.

[4] T. Nishino, “How to design efficient quantum algorithms”, Transaction of Information Processing Society Japan Vol.43 No. SIG 7(TOM 6) pp.1–9, September 2002.

[5] T. Nishino, H. Shibata, K. Atsumi, T. Shima, “Solving function problems and NP-Complete Problems by NMR Quantum Computation”, Technical Report of IEICE COMP 98-71 pp.65–72, December 1998.

[6] L. M. K. Vandersypen, M. Steffen, G. Breyta, C. S. Yannoni, M. H. Sherwood, I. L. Chuang, Experimental Realization of Shor’s Quantum Factoring Algorithm Using Nuclear Magnetic Resonance, Nature Vol.414 20/27 December pp.883–887, December 2001.

[7] A. N. Kolmogorov, I. G. Zhurbenko, A. V. Prokhorov, Vvedenie v Teoriyu Veroiatnostei 2nd ed., Nauka, Moscow, 1995; T. Maruyama, Y. Baba (Japanese translation), Korumogorohu no kakuritsuronnyuunon, Morikita Shuppan, Tokyo, 2003.