The quantum harmonic oscillator on the sphere and the hyperbolic plane

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Abstract

A nonlinear model of the quantum harmonic oscillator on two-dimensional spaces of constant curvature is exactly solved. This model depends of a parameter λ that is related with the curvature of the space. Firstly the relation with other approaches is discussed and then the classical system is quantized by analyzing the symmetries of the metric (Killing vectors), obtaining a λ-dependent invariant measure $d\mu_\lambda$ and expressing the Hamiltonian as a function of the Noether momenta. In the second part the quantum superintegrability of the Hamiltonian and the multiple separability of the Schrödinger equation is studied. Two λ-dependent Sturm-Liouville problems, related with two different λ-deformations of the Hermite equation, are obtained. This leads to the study of two λ-dependent families of orthogonal polynomials both related with the Hermite polynomials. Finally the wave functions $\Psi_{m,n}$ and the energies $E_{m,n}$ of the bound states are exactly obtained in both the sphere $S^2$ and the hyperbolic plane $H^2$.

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1 Introduction

This article can be considered as a sequel or continuation of a previous paper [1] which was devoted to the study of a quantum exactly solvable one-dimensional nonlinear oscillator with quasi-harmonic behaviour. Now, our idea is to extend the results and present a similar analysis but for the quantum version of the two-dimensional nonlinear system. We follow the approach of [1], which contains the fundamental ideas and motivation, and we will also make use of some properties discussed, at the classical level, in [2] (other related papers are [3]–[6]).

The following nonlinear differential equation

\[(1 + \lambda x^2) \ddot{x} - \lambda x \dot{x} + \alpha^2 x = 0, \quad (\lambda \text{ a constant}), \tag{1}\]

was studied in Refs. [7, 8] as an example of a nonlinear oscillator. In Lagrangian terms, the equation (1) can be obtained from the following function

\[L(x, v_x; \lambda) = \frac{1}{2} \left( \frac{v_x^2}{1 + \lambda x^2} \right) - \frac{\alpha^2}{2} \left( \frac{x^2}{1 + \lambda x^2} \right), \tag{2}\]

that clearly displays two very interesting characteristics: the potential \(V(\lambda)\) has a nonpolynomial character (this is not a problem of an harmonic oscillator perturbed by higher order terms of the form \(\lambda x^{2m}\) with \(m > 1\)), and the kinetic term depends on a position-dependent mass.

Let us briefly comment these two characteristics.

The form of the potential \(V(\lambda)\) is shown in Figures I and II for several values of \(\lambda\) (\(\lambda < 0\) in Figure I and \(\lambda > 0\) in Figure II). We see that for \(\lambda < 0\) the potential is a well with a boundless wall at \(x^2 = 1/|\lambda|\) and for \(\lambda > 0\) we have that \(V \to (1/2)(\alpha^2/\lambda)\) for \(x \to \pm \infty\). It can be proved that

1. If \(\lambda < 0\) then the general solution of (1) is given by
   \[x = A \sin(\omega t + \phi), \quad \alpha^2 = (1 + \lambda A^2) \omega^2.\]

2. If \(\lambda > 0\) then the general solution is given by
   \[x = A \sin(\omega t + \phi), \quad \alpha^2 = (1 + \lambda A^2) \omega^2,\]
   when the energy \(E\) is smaller than the value \(E_{\alpha,\lambda} = \alpha^2/(2\lambda)\), and by
   \[x = B \sinh(\Omega t + \phi_1), \quad \alpha^2 = (\lambda B^2 - 1) \omega^2,\]
   when the energy \(E\) is greater than \(E_{\alpha,\lambda}\).

The Schrödinger equation involving the potential \(\lambda(x^2/(1 + gx^2))\) has been studied by many authors making use of different approaches [9]-[30]. In some cases the idea was to study the Hamiltonian \(H = -d^2/dx^2 + x^2 + \lambda(x^2/(1 + gx^2))\) by applying perturbative, variational or numerical techniques previously used for the system \(H_m = -d^2/dx^2 + x^2 + \lambda x^{2m}\) with \(m > 1\). It is important to note that in most of these papers the derivative part of the Schrödinger equation was the standard one, that is, the equation arising from a classical Hamiltonian with a quadratic term of the form \((1/2)p^2\) and leading to a derivative term of the form \(-d^2/dx^2\), or to the corresponding two or
three-dimensional versions involving the Laplace operator in $\mathbb{E}^2$ or $\mathbb{E}^3$. Nevertheless, we point out that Mathews and Lakshmanan studied in Ref. [10] the following quantum Hamiltonian

$$H = \frac{1}{2} \left\{ p^2, (1 - gx^2) \right\} + \frac{kx^2}{(1 - gx^2)} ,$$

where the notation $\{A,B\} = AB + BA$ is used.

The second important point is the presence of a position-dependent mass $m = (1 + \lambda x^2)^{-1}$ since, if the mass becomes a spatial function, then the quantum version of the mass no longer commutes with the momentum. Therefore, different forms of presenting the kinetic term in the Hamiltonian $H$, that are equivalent at the classical level, lead to different and nonequivalent Schrödinger equations [31]-[42]. This is an old question that remains as an important open problem in the theory of quantization.

In spite of these two characteristics, the quantum version of this $\lambda$-dependent non-linear oscillator has been proved to be exactly solvable [1]. The question of the order ambiguity in the quantization of the Hamiltonian was solved by introducing a prescription obtained from the analysis of the properties of the classical system (existence of a Killing vector and a $\lambda$-dependent invariant measure) and concerning the problems arising from the nonpolynomial character of the potential, all of them disappear when $V(\lambda)$ is studied with the appropriate quantization of the position-dependent kinetic term.

On the other hand the following two-dimensional Lagrangian

$$L = \frac{1}{2} \left( \frac{1}{1 + \lambda r^2} \right) \left[ v_x^2 + v_y^2 + \lambda (xv_y - yv_x)^2 \right] - \frac{\alpha^2}{2} \left( \frac{r^2}{1 + \lambda r^2} \right), \quad r^2 = x^2 + y^2 ,$$

was proposed in Ref. [2] as the appropriate two-dimensional generalization, at the classical level, of the Lagrangian (2). In fact the general solution of the Euler-Lagrange equations which are given by

$$(1 + \lambda r^2) \ddot{x} - \lambda [ \dot{x}^2 + \dot{y}^2 + \lambda (x\dot{y} - y\dot{x})^2 ] x + \alpha^2 x = 0 ,

(1 + \lambda r^2) \ddot{y} - \lambda [ \dot{x}^2 + \dot{y}^2 + \lambda (x\dot{y} - y\dot{x})^2 ] y + \alpha^2 y = 0 ,$$

is:

1. If $\lambda < 0$ then the general solution is given by

$$x = A \sin(\omega t + \phi_1), \quad y = B \sin(\omega t + \phi_2) ,$$

for all the values of the energy $E$.

2. If $\lambda > 0$ then the general solution is given by

$$x = A \sin(\omega t + \phi_1), \quad y = B \sin(\omega t + \phi_2) ,$$

when the energy $E$ is smaller than a certain value $E_{\alpha,\lambda}$, and by

$$x = A \sinh(\Omega t + \phi_1), \quad y = B \sinh(\Omega t + \phi_2) ,$$

when the energy $E$ is greater than this value, $E > E_{\alpha,\lambda}$.
In both cases the coefficients $A$ and $B$ are related with the coefficient $\alpha$ and the frequency $\omega$ (oscillatory motions) or with $\alpha$ and $\Omega$ (unbounded motions). So we have “quasi-harmonic” nonlinear oscillations in the case of bounded motions and high energy scattering solutions when $\lambda > 0$. Moreover, the analysis of this nonlinear system proved the existence of a relation with the linear harmonic oscillator on the sphere $S^2$ or on the hyperbolic plane $H^2$ with the parameter $\lambda$ playing the role of the (negative of the) curvature $\kappa$.

The main objective of this article is to quantize this two-dimensional nonlinear oscillator as a deformation of the harmonic oscillator in the sense that

1. All the fundamental properties of the linear system continue to hold for $\lambda \neq 0$ but modified in a $\lambda$-dependent way.

2. The limit when $\lambda \to 0$ is well defined and when $\lambda = 0$ all the characteristics of the quantum harmonic oscillator are recovered.

The idea is to prove that it is exactly solvable and to obtain the energies and the corresponding wave functions. Some of the main questions to be discussed in this paper can be summarized in the following four points:

- Relation of this $\lambda$-dependent nonlinear model with the harmonic oscillator on spaces of constant curvature.
  This quasi-harmonic nonlinear oscillator is related, at the classical level, with the harmonic oscillator on the three spaces of constant curvature $(S^2, \mathbb{R}^2, H^2)$. Now this relation is considered for the quantized systems.

- Analysis of the transition from the classical $\lambda$-dependent system to the quantum one.
  The two-dimensional $\lambda$-dependent kinetic term possesses three Noether symmetries. The main idea is to quantize the system by using as Hilbert space the space $L^2(\mathbb{R}, d\mu_\lambda)$ where $d\mu_\lambda$ is a measure invariant under the Killing vectors associated to the Noether symmetries.

- Schrödinger separability and ‘quantum superintegrability’.
  The two-dimensional Schrödinger equation is not separable in $(x, y)$ coordinates because of the $\lambda$-dependent coupling between the two degrees of freedom; nevertheless it is proved that it admits separability in several coordinate systems. The existence of this multiple separability is a property related with ‘quantum superintegrability’.

- Exact resolution of the $\lambda$-dependent Schrödinger equation and families of $\lambda$-dependent orthogonal polynomials.
  Two $\lambda$-dependent Sturm-Liouville problems related with two different $\lambda$-deformations of the Hermite equation are obtained. This leads to the study of two $\lambda$-dependent families of orthogonal polynomials both related with the Hermite polynomials.

In more detail, the plan of the article is as follows: In Sec. 2 we study the relation of this nonlinear model with the harmonic oscillator on spaces of constant curvature. In Sec. 3 we first introduce a $\lambda$-dependent measure, we use it for introducing a quantization rule and we obtain the
\(\lambda\)-dependent Schrödinger equation. Sec. 4 is devoted to the analysis of the \(\lambda\)-dependent Schrödinger separability and to solve two Hermite-related equations and Sec. 5 to obtain the eigenfunctions \(\Psi_{m,n}\) and energies \(E_{m,n}\). In Sec. 6 we briefly analyze the existence of another possible approach and its relation with the presence of the angular momentum. Finally, in Sec. 7 we make some final comments.

2 On the relation of this nonlinear model with the harmonic oscillator on spaces of constant curvature

Although the first studies of the harmonic oscillator on curved spaces are rather old (the last chapter of Ref. [43], that is called ‘Nichteuklidische Mechanik’, is devoted to the dynamics on spaces with curvature; it first studies general properties and then it consider the harmonic oscillator as a particular case; the approach is mainly Newtonian), Ref. [44] is usually considered as the more relevant paper for the modern approach to this system (by modern we mean that it studies subjects such as dynamical symmetries or quantum dynamics). We recall that the harmonic oscillator is a system that is well defined not only in the Euclidean plane \(\mathbb{E}^2\) but also in the other two-dimensional spaces of constant curvature, sphere \(S^2\) and hyperbolic plane \(H^2\).

In differential geometric terms, the three spaces with constant curvature, sphere \(S^2_\kappa\) \((\kappa > 0)\), Euclidean plane \(\mathbb{E}^2\), and hyperbolic plane \(H^2_\kappa\) \((\kappa < 0)\), can be considered as three different situations inside a family of Riemannian manifolds \(M^2_\kappa = (S^2_\kappa, \mathbb{E}^2, H^2_\kappa)\) with the curvature \(\kappa\) as a parameter \(\kappa \in \mathbb{R}\). In order to obtain mathematical expressions valid for all the values of \(\kappa\), it is convenient to make use of the following \(\kappa\)-trigonometric functions

\[
C_\kappa(x) = \begin{cases} 
\cos \sqrt{\kappa} x & \text{if } \kappa > 0, \\
1 & \text{if } \kappa = 0, \\
\cosh \sqrt{-\kappa} x & \text{if } \kappa < 0,
\end{cases}
\]

\[
S_\kappa(x) = \begin{cases} 
\frac{1}{\sqrt{\kappa}} \sin \sqrt{\kappa} x & \text{if } \kappa > 0, \\
x & \text{if } \kappa = 0, \\
\frac{1}{\sqrt{-\kappa}} \sinh \sqrt{-\kappa} x & \text{if } \kappa < 0,
\end{cases}
\]

then the expression of the differential element of distance in geodesic polar coordinates \((R, \Phi)\) on the family \(M^2_\kappa = (S^2_\kappa, \mathbb{E}^2, H^2_\kappa)\), can be written as follows

\[
ds^2_\kappa = dR^2 + S^2_\kappa(R) d\Phi^2,
\]

so it reduces to

\[
ds^2_1 = dR^2 + (\sin^2 R) d\Phi^2, \quad ds^2_0 = dR^2 + R^2 d\Phi^2, \quad ds^2_{-1} = dR^2 + (\sinh^2 R) d\Phi^2,
\]

in the three particular cases of the unit sphere, the Euclidean plane, and the ‘unit’ Lobachewski plane. Note that \(R\) denotes the distance along a geodesic on the manifold \(M^2_\kappa\); for example, in the spherical \(\kappa > 0\) case, \(R\) is the distance of the point to the origin (e.g., the North pole) along a maximum circle.

If we make use of this formalism then the Lagrangian of the harmonic oscillator on \(M^2_\kappa\) is given by [45, 46]

\[
\mathcal{L}(\kappa) = \left(\frac{1}{2}\right) \left(v_R^2 + S^2_\kappa(R)v_\Phi^2\right) - \left(\frac{1}{2}\right) \alpha^2 \left(T^2_\kappa(R)\right).
\]
where the $\kappa$-dependent tangent is defined in the natural way $T_\kappa(R) = S_\kappa(R)/C_\kappa(R)$. In this way, the potential of the harmonic oscillator on the unit sphere, on the Euclidean plane, or on the unit Lobachewski plane, arise as the following three particular cases

$$U_1(R) = \left(\frac{1}{2}\right)\alpha^2 \tan^2 R, \quad U_0(R) = \left(\frac{1}{2}\right)\alpha^2 R^2, \quad U_{-1}(R) = \left(\frac{1}{2}\right)\alpha^2 \tanh^2 R.$$ 

The Euclidean oscillator $U_0(R)$ appears in this formalism as a parabolic curve making a separation between two different situations (see Figure III). Note also that in spherical $\kappa > 0$ case, this Lagrangian describes in fact two oscillators with centers in the north ($R = 0$) and south ($\sqrt{\kappa} R = \pi$) poles and with a boundary barrier in the equatorial circle.

Next we study the behaviour of $\mathbb{L}(\kappa)$ under two different changes of variables.

1. Let us consider the $\kappa$-dependent change $(R, \Phi) \rightarrow (r, \phi)$ given by

$$r = S_\kappa(R), \quad \phi = \Phi, \quad \lambda = -\kappa.$$ 

Then the Lagrangian $\mathbb{L}(\kappa)$ becomes

$$L(\lambda) = \frac{1}{2} \left( \frac{v_r^2}{1 + \lambda r^2} + r^2 v_\phi^2 \right) - \frac{\alpha^2}{2} \left( \frac{r^2}{1 + \lambda r^2} \right).$$

Therefore, if we change to Cartesian coordinates $(x, y)$ we arrive to

$$L(\lambda) = \frac{1}{2} \left( \frac{1}{1 + \lambda r^2} \right) \left[ v_x^2 + v_y^2 + \lambda (x v_y - y v_x)^2 \right] - \frac{\alpha^2}{2} \left( \frac{r^2}{1 + \lambda r^2} \right), \quad r^2 = x^2 + y^2,$$

This function is just the Lagrangian obtained in Ref. [2] as the natural generalization of the one-dimensional Lagrangian $L(x, v_x; \lambda)$ for the nonlinear equation (1) of Mathews and Lakshmanan.

2. Let us now consider the $\kappa$-dependent change $(R, \Phi) \rightarrow (r', \phi)$ given by

$$r' = T_\kappa(R), \quad \phi = \Phi.$$ 

Then the Lagrangian $\mathbb{L}(\kappa)$ becomes

$$L_H(\kappa) = \frac{1}{2} \left( \frac{v_r'^2}{(1 + \kappa r'^2)^2} + \frac{r'^2 v_\phi^2}{(1 + \kappa r'^2)^2} \right) - \frac{1}{2} \alpha^2 r'^2,$$

Therefore, if we change to Cartesian coordinates $(x, y)$ we arrive to

$$L_H(\kappa) = \frac{1}{2} \frac{1}{(1 + \kappa r'^2)^2} \left[ v_x^2 + v_y^2 + \kappa (x v_y - y v_x)^2 \right] - \frac{1}{2} \alpha^2 r'^2, \quad r'^2 = x^2 + y^2,$$

This function is the Lagrangian studied by Higgs in Ref. [44] (the study of Higgs was originally limited to a spherical geometry but the idea can be easily extended to the hyperbolic space).
We note that these two changes are correct and both radial variables, \( r \) and \( r' \), are well defined. In the hyperbolic \( \kappa < 0 \) case the two functions \( S_\kappa(R) \) and \( T_\kappa(R) \) are positive for \( R > 0 \) and concerning the spherical \( \kappa > 0 \) case this property is also true because then \( R \) is restricted to a bounded interval.

The situation can be summarized as follows. We have obtained three alternative ways of describing the harmonic oscillator on spaces of constant curvature: the original system \( I_L(\kappa) \) and the two other approaches, \( L(\lambda) \) and \( L_H(\kappa) \), obtained from it. Of course, everyone of these three different approaches, \( I_L(\kappa), L(\lambda) \) and \( L_H(\kappa) \), has its own characteristics and advantages.

A two–dimensional manifold \( M \) can be described by using different coordinate systems. If we consider it as an imbedded submanifold of \( \mathbb{R}^3 \), then the points of \( M \) can be characterized by the three external coordinates, as e.g. \( (x, y, z) \), plus an additional constraint. Nevertheless, in differential geometric terms, a more appropriate approach is to develop the study by using two–dimensional systems of coordinates intrinsically defined in \( M \) (and without make reference to the external space). The Lagrangian \( I_L(\kappa) \) is directly defined on the manifold \( M^2_\kappa = (S^2_\kappa, \mathbb{E}^2, H^2_\kappa) \) and it uses the expression of the differential element of distance \( ds^2_\kappa \) in geodesic polar coordinates \( (R, \Phi) \) (see the Appendix). Therefore, in differential geometric terms, this approach can be considered as more formally correct than the other two.

The Higgs approach [44, 47] consider the motion on \( S^n \), embedded in the Euclidean space \( \mathbb{E}^{n+1} \), by means of a central (also known as gnomonic) projection on a plane \( \Pi^n \) tangent to \( S^n \) at a chosen point. This particular formalism leads to a dynamics that is described, when \( n = 2 \), by the Lagrangian \( L_H(\kappa) \). This approach is very interesting because it states a direct relation between the motion on a curved space, the sphere \( S^n \), and the motion on a plane. In fact, it has been studied by many authors (see e.g. [48]–[59] and references therein) mainly in relation of the theory of dynamical symmetries.

The \( \lambda \)-dependent Lagrangian \( L(\lambda) \) has a certain similarity with the Lagrangian of Higgs but nevertheless it does not coincide with it: in the model of Higgs \( \kappa \) (or \( \lambda \)) is present in the kinetic term \( T \) in a different way and the potential \( V \) appears as \( \kappa \)-independent; we will see that this affects, via the Legendre transformation, to the Hamiltonian formalism. As we have seen in Sec. I (Introduction) one of the advantages of this approach is that the Euler-Lagrange equations can be directly solved and the general solution has a rather simple form that can be interpreted as “quasi-harmonic” nonlinear oscillations. It is clear that this \( \lambda \)-dependent formalism seems very appropriate for solving equations or for other related calculus.

In what follows we will focus our attention on the Hamiltonian dynamics determined by the \( \lambda \)-dependent Lagrangian \( L(\lambda) \).

### 3 Quantization and \( \lambda \)-dependent Schrödinger equation

Let us start our study considering the following Lagrangian

\[
L(\lambda) = \frac{1}{2} \left( \frac{1}{1 + \lambda r^2} \right) \left[ v_x^2 + v_y^2 + \lambda (x v_y - y v_x)^2 \right] - \frac{\alpha^2}{2} \left( \frac{r^2}{1 + \lambda r^2} \right), \quad r^2 = x^2 + y^2, \tag{4}
\]

where the parameter \( \lambda \) can take both positive and negative values; of course it is clear that for \( \lambda < 0, \lambda = -|\lambda| \), the function (and the associated dynamics) will have a singularity at \( 1 - |\lambda| r^2 = 0 \).
and we shall restrict the study of the dynamics to the interior of the interval \( r^2 < 1/|\lambda| \) where the kinetic energy function is positive definite.

The Legendre transformation is given by

\[
p_x = \frac{v_x - \lambda Jy}{1 + \lambda r^2}, \quad p_y = \frac{v_y + \lambda Jx}{1 + \lambda r^2},
\]

and the expression of the \( \lambda \)-dependent Hamiltonian turns out to be

\[
H(\lambda) = \frac{1}{2} \left[ p_x^2 + p_y^2 + \lambda (xp_x + yp_y)^2 \right] + \frac{\alpha^2}{2} \left( \frac{r^2}{1 + \lambda r^2} \right).
\]

(5)

The transition from the classical system to the quantum one is a difficult problem because of the ambiguities in the order of positions and momenta. In Refs. [1, 3] the one-dimensional nonlinear oscillator was quantized by using a prescription obtained from the existence of a one-dimensional Killing vector \( X(\lambda) \) and a \( \lambda \)-dependent measure \( d\mu_\lambda \) in \( \mathbb{R} \) preserved by \( X(\lambda) \). Next we prove that this approach, that was successful for the one-dimensional system, admits a direct generalization to this more difficult two-dimensional case. We must begin with an analysis of the symmetries of the kinetic energy term.

It was shown in [2] that function \( T(\lambda) \) representing the kinetic energy

\[
T(\lambda) = \frac{1}{2} \left( \frac{1}{1 + \lambda r^2} \right) \left[ v_x^2 + v_y^2 + \lambda (xv_y - yv_x)^2 \right]
\]

is invariant under the action of the vector fields \( X_1(\lambda), X_2(\lambda), \) and \( X_J \), given by

\[
X_1(\lambda) = \sqrt{1 + \lambda r^2} \frac{\partial}{\partial x}, \quad X_2(\lambda) = \sqrt{1 + \lambda r^2} \frac{\partial}{\partial y}, \quad X_J = x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x},
\]

in the sense that, if we denote by \( X^r_\tau \), \( r = 1, 2, J \), the natural lift to the tangent bundle (phase space \( \mathbb{R}^2 \times \mathbb{R}^2 \)) of the vector field \( X_\tau \),

\[
X^1_\tau(\lambda) = \sqrt{1 + \lambda r^2} \frac{\partial}{\partial x} + \lambda \left( \frac{xxv_y + yv_x}{\sqrt{1 + \lambda r^2}} \right) \frac{\partial}{\partial v_x}, \\
X^2_\tau(\lambda) = \sqrt{1 + \lambda r^2} \frac{\partial}{\partial y} + \lambda \left( \frac{xxv_y + yv_x}{\sqrt{1 + \lambda r^2}} \right) \frac{\partial}{\partial v_y}, \\
X^J_\tau = x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} + v_x \frac{\partial}{\partial v_y} - v_y \frac{\partial}{\partial v_x},
\]

then the Lie derivatives of \( T(\lambda) \) with respect to \( X^r_\tau(\lambda) \) vanish, that is

\[
X^r_\tau(\lambda) \left( T(\lambda) \right) = 0, \quad r = 1, 2, J.
\]

It is known that a symmetric bilinear form in the velocities \( (v_x, v_y) \) can be considered as associated to a two-dimensional metric \( ds^2 \) in \( \mathbb{R}^2 \). In this particular case, the function \( T(\lambda) \) considered as a bilinear form determines the following \( \lambda \)-dependent metric

\[
ds^2(\lambda) = \left( \frac{1}{1 + \lambda r^2} \right) \left[ (1 + \lambda y^2) dx^2 + (1 + \lambda x^2) dy^2 - 2\lambda xy dx dy \right].
\]

(6)
Thus, in differential geometric terms, the three vector fields $X_1(\lambda)$, $X_2(\lambda)$, and $X_J$, must be considered as three Killing vector fields (infinitesimal generators of isometries) of $ds^2(\lambda)$. These three symmetries of the kinetic term determine three associate Noether momenta (a Noether momentum is a constant of motion for the geodesic motion) given by

\[ P_1(\lambda) = \frac{v_x - \lambda J y}{\sqrt{1 + \lambda r^2}}, \quad P_2(\lambda) = \frac{v_y + \lambda J x}{\sqrt{1 + \lambda r^2}}, \quad J = xv_y - yv_x, \]

that become

\[ P_1(\lambda) = \sqrt{1 + \lambda r^2} p_x, \quad P_2(\lambda) = \sqrt{1 + \lambda r^2} p_y, \quad J = xp_y - yp_x, \]

in the Hamiltonian formalism. We note that the form of the angular momentum $J$ is preserved by the Legendre map, in the sense that we have $xp_y - yp_x = xv_y - yv_x$ (this is another one of the differences with the Higgs model).

**Proposition 1** The only measure on the space $\mathbb{R}^2$ that is invariant under the action of the three vector fields $X_1(\lambda)$, $X_2(\lambda)$, and $X_J$, is given by

\[ d\mu_\lambda = \left( \frac{1}{\sqrt{1 + \lambda r^2}} \right) dx \, dy, \]

up to a constant factor.

**Proof:** The most general expression for a volume two-form on the space $\mathbb{R}^2$ is given by

\[ \omega = \rho(x, y) \, dx \wedge dy \]

Then the Lie derivatives of $\omega$ under $X_1(\lambda)$, $X_2(\lambda)$, and $X_J$, are given by

\[
\begin{align*}
\mathcal{L}_{X_1} \omega &= \left( \sqrt{1 + \lambda r^2} \frac{\partial \rho}{\partial x} \right) dx \wedge dy + \rho \left( d \sqrt{1 + \lambda r^2} \right) \wedge dy \\
\mathcal{L}_{X_2} \omega &= \left( \sqrt{1 + \lambda r^2} \frac{\partial \rho}{\partial y} \right) dx \wedge dy + dx \wedge \rho \left( d \sqrt{1 + \lambda r^2} \right) \\
\mathcal{L}_{X_J} \omega &= \left( x \frac{\partial \rho}{\partial y} - y \frac{\partial \rho}{\partial x} \right) dx \wedge dy
\end{align*}
\]

The condition $\mathcal{L}_{X_J} d\mu_\lambda = 0$ implies that $\rho(x, y)$ must be a function $f(r)$ of $r$. Then the two other conditions, $\mathcal{L}_{X_1} d\mu_\lambda = 0$ and $\mathcal{L}_{X_2} d\mu_\lambda = 0$, lead to (for $r \neq 0$):

\[ \sqrt{1 + \lambda r^2} \frac{1}{r} \frac{df}{dr} + \frac{\lambda f}{\sqrt{1 + \lambda r^2}} = 0, \]

with general solution given by

\[ f = \frac{k}{\sqrt{1 + \lambda r^2}} \]

where $k$ is an arbitrary numerical constant.

We will consider this proposition as the fundamental point for the study of transition from the classical system to the quantum one. In fact, this property suggests us to work with functions and
linear operators defined on the space obtained by considering the two-dimensional real plane \( \mathbb{R}^2 \) endowed with the measure \( d\mu_\lambda \) given by

\[
d\mu_\lambda = \left( \frac{1}{\sqrt{1 + \lambda r^2}} \right) dx dy.
\]  

(7)

This means, in the first place, that the operators \( \hat{P}_x \) an \( \hat{P}_y \) representing the quantum version of the Noether momenta must be self-adjoint not in the standard space \( L^2(\mathbb{R}) \) but in the space \( L^2(\mathbb{R}, d\mu_\lambda) \). If we assume

\[
\hat{P}_x = -i\hbar \sqrt{1 + \lambda r^2} \frac{\partial}{\partial x},
\]

\[
\hat{P}_y = -i\hbar \sqrt{1 + \lambda r^2} \frac{\partial}{\partial y},
\]

then we arrive to the following correspondence

\[
(1 + \lambda r^2) p^2_x \to -\hbar^2 \left( \sqrt{1 + \lambda r^2} \frac{\partial}{\partial x} \right) \left( \sqrt{1 + \lambda r^2} \frac{\partial}{\partial x} \right),
\]

\[
(1 + \lambda r^2) p^2_y \to -\hbar^2 \left( \sqrt{1 + \lambda r^2} \frac{\partial}{\partial y} \right) \left( \sqrt{1 + \lambda r^2} \frac{\partial}{\partial y} \right),
\]

as well as

\[
J^2 \to -\hbar^2 \left( x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) \left( x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right),
\]

in such a way that the quantum version of the classical Hamiltonian

\[
H = \left( \frac{1}{2m} \right) \left[ p^2_x + p^2_y + \lambda (xp_x + yp_y)^2 \right] + \left( \frac{1}{2} \right) g \left( \frac{r^2}{1 + \lambda r^2} \right), \quad g = ma^2.
\]

(8)

that can be rewritten as follows

\[
H = \left( \frac{1}{2m} \right) \left[ P^2_1 + P^2_2 - \lambda J^2 \right] + \left( \frac{1}{2} \right) g \left( \frac{r^2}{1 + \lambda r^2} \right),
\]

is

\[
\hat{H} = -\frac{\hbar^2}{2m} \left[ (1 + \lambda r^2) \frac{\partial^2}{\partial x^2} + \lambda x \frac{\partial}{\partial x} \right] - \frac{\hbar^2}{2m} \left[ (1 + \lambda r^2) \frac{\partial^2}{\partial y^2} + \lambda y \frac{\partial}{\partial y} \right]
\]

\[
+ \lambda \frac{\hbar^2}{2m} \left[ x^2 \frac{\partial^2}{\partial y^2} + y^2 \frac{\partial^2}{\partial x^2} - 2xy \frac{\partial^2}{\partial x \partial y} - x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} \right] + \left( \frac{1}{2} \right) g \left( \frac{r^2}{1 + \lambda r^2} \right).
\]

(9)

The first important property of this Hamiltonian is that it admits the following decomposition

\[
\hat{H} = \hat{H}_1 + \hat{H}_2 - \lambda J^2
\]

where the three partial operators \( \hat{H}_1, \hat{H}_2 \) y \( J^2 \) are given by

\[
\hat{H}_1 = -\frac{\hbar^2}{2m} \left[ (1 + \lambda r^2) \frac{\partial^2}{\partial x^2} + \lambda x \frac{\partial}{\partial x} \right] + \left( \frac{1}{2} \right) g \left( \frac{x^2}{1 + \lambda r^2} \right)
\]

\[
\hat{H}_2 = -\frac{\hbar^2}{2m} \left[ (1 + \lambda r^2) \frac{\partial^2}{\partial y^2} + \lambda y \frac{\partial}{\partial y} \right] + \left( \frac{1}{2} \right) g \left( \frac{y^2}{1 + \lambda r^2} \right)
\]
\[ J^2 = -\frac{\hbar^2}{2m} \left[ x^2 \frac{\partial^2}{\partial y^2} + y^2 \frac{\partial^2}{\partial x^2} - 2xy \frac{\partial^2}{\partial x \partial y} - x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} \right] \]

in such a way that the total Hamiltonian \( \hat{H} \) commutes, for any value of the parameter \( \lambda \), with each one of the three partial terms

\[ [\hat{H}, \hat{H}_1] = 0, \quad [\hat{H}, \hat{H}_2] = 0, \quad [\hat{H}, \hat{J}^2] = 0. \]

The vanishing of these three commutators means that the \( \lambda \)-dependent Hamiltonian (9) describes a quantum superintegrable system [60]-[71]. This property was analyzed at the classical level in Ref. [2]; now we see that the quantization rule we have applied preserves the superintegrability.

Now, if we consider the Schrödinger equation

\[ \hat{H} \Psi = E \Psi, \]

as we have the following property

\[ [\hat{H}_1, \hat{H}_2 - \lambda \hat{J}^2] = 0, \quad [\hat{H}_1 - \lambda \hat{J}^2, \hat{H}_2] = 0, \quad [\hat{H}_1 + \hat{H}_2, \hat{J}^2] = 0, \]

then, we have three different sets of compatible observables and therefore three different ways of obtaining a Hilbert basis of common eigenstates.

1. The two operators \( \hat{H}_1 \) and \( \hat{H}_2 - \lambda \hat{J}^2 \) are a (complete) set of commuting observables; therefore they represent two quantities that can be simultaneously measured. Thus, the first way of looking for \( \Psi \) is as a solution of the following two equations

\[ \hat{H}_1 \Psi = E_1 \Psi, \quad (\hat{H}_2 - \lambda \hat{J}^2) \Psi = E_{2j} \Psi. \]

In this case the total energy is given by \( E = E_1 + E_{2j} \) and the associated wave function can be denoted by \( \Psi(E_1, E_{2j}) \).

2. The two operators \( \hat{H}_1 - \lambda \hat{J}^2 \) and \( \hat{H}_2 \) are a (complete) set of commuting observables. Thus, the second way of looking for \( \Psi \) is as a solution of the following two equations

\[ (\hat{H}_1 - \lambda \hat{J}^2) \Psi = E_{1j} \Psi, \quad \hat{H}_2 \Psi = E_2 \Psi. \]

In this case we have \( E = E_{1j} + E_2 \) and \( \Psi \) can be denoted by \( \Psi(E_{1j}, E_2) \).

3. The third (complete) set of commuting observables is provided by \( \hat{H}_1 + \hat{H}_2 \) and \( \hat{J}^2 \). So in this case we have

\[ (\hat{H}_1 + \hat{H}_2) \Psi = E_{12} \Psi, \quad \hat{J} \Psi = j \Psi. \]

Thus, the two physically measurable quantities are \( E_{12} \) and the angular momentum \( j \), the total energy is given by \( E = E_{12} - \lambda j^2 \) and the wave function so defined can be denoted by \( \Psi(E_{12}, j) \).

The existence of these three alternative descriptions arises from the presence of the term \( \lambda \hat{J}^2 \) inside the kinetic part of the Hamiltonian. Notice that the second approach is just the symmetric of the first one but, although they are closely related, they lead however to different solutions with
different properties; that is, \( \Psi(E_1, E_2) \neq \Psi(E_1, E_2) \). This fact is a consequence of the nonlinear character of the model since in the linear limit, when \( \lambda \to 0 \), then both descriptions coincide.

Let us consider the following quantum Hamiltonian

\[
\hat{H} = -\hbar^2 \left( \frac{1 + \Lambda r^2}{2m} \frac{\partial^2}{\partial x^2} + \lambda x \frac{\partial}{\partial x} \right) - \hbar^2 \left( \frac{1 + \Lambda r^2}{2m} \frac{\partial^2}{\partial y^2} + \lambda y \frac{\partial}{\partial y} \right) + \lambda \hbar^2 \left[ x^2 \frac{\partial^2}{\partial y^2} + y^2 \frac{\partial^2}{\partial x^2} - 2xy \frac{\partial^2}{\partial x \partial y} - x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} \right] + \left( \frac{1}{2} \right) m \alpha \left( \alpha + \frac{\hbar}{m} \lambda \right) \left( r^2 + 1 + \Lambda (x^2 + y^2) \right) \tag{10}
\]

where we have slightly modified the value of the parameter \( g \) that now is given by \( g = m \alpha + \hbar \alpha \) (this is done to coincide with the notation of the one-dimensional system in [1]). It is also convenient to simplify this function \( \hat{H} \) by introducing adimensional variables \((\tilde{x}, \tilde{y}, \Lambda, E)\) defined by

\[
x = \left( \sqrt{\frac{\hbar}{ma}} \right) \tilde{x}, \quad y = \left( \sqrt{\frac{\hbar}{ma}} \right) \tilde{y}, \quad \lambda = \left( \frac{ma}{\hbar} \right) \Lambda, \quad E = (\hbar \alpha) e,
\]

in such a way that the following relation holds

\[1 + \lambda r^2 = 1 + \Lambda \tilde{r}^2, \quad \tilde{r}^2 = \tilde{x}^2 + \tilde{y}^2.\]

The Schrödinger equation takes then the following form

\[
-\frac{1}{2} \left[ \left( 1 + \Lambda \tilde{r}^2 \right) \frac{\partial^2}{\partial \tilde{x}^2} + \Lambda \tilde{x} \frac{\partial}{\partial \tilde{x}} \right] \Psi - \frac{1}{2} \left[ \left( 1 + \Lambda \tilde{r}^2 \right) \frac{\partial^2}{\partial \tilde{y}^2} + \Lambda \tilde{y} \frac{\partial}{\partial \tilde{y}} \right] \Psi + \frac{\Lambda}{2} \left[ \tilde{x}^2 \frac{\partial^2}{\partial \tilde{y}^2} + \tilde{y}^2 \frac{\partial^2}{\partial \tilde{x}^2} - 2\tilde{x} \tilde{y} \frac{\partial^2}{\partial \tilde{x} \partial \tilde{y}} - \tilde{x} \frac{\partial}{\partial \tilde{x}} - \tilde{y} \frac{\partial}{\partial \tilde{y}} \right] \Psi + \left( \frac{1}{2} \right) (1 + \Lambda) \left( \frac{\tilde{r}^2}{1 + \Lambda \tilde{r}^2} \right) \Psi = E \Psi \tag{11}
\]

4 Resolution of the Schrödinger equation I

In the following all the variables, parameters and equations are adimensional. Nevertheless, and for ease of notation, we will drop the use of the tilde and write the variables just as \( x, y, r \) and so on.

4.1 Separability

The Schrödinger equation (11) is not separable in Cartesian \((x, y)\) coordinates because of the \( \Lambda \)-dependent term. Nevertheless, at this point we recall that it was proved in [2] that, at the classical level, the Hamilton-Jacobi equation

\[
\left( \frac{\partial S}{\partial x} \right)^2 + \left( \frac{\partial S}{\partial y} \right)^2 + \lambda \left( x \frac{\partial S}{\partial x} + y \frac{\partial S}{\partial y} \right)^2 + \alpha^2 \left( \frac{x^2 + y^2}{1 + \Lambda (x^2 + y^2)} \right) = 2E.
\]

admits separability in the following three different orthogonal coordinate systems:

1. \( \Lambda \)-dependent coordinates \((z_x, y)\) with \( z_x \) defined by \( z_x = x / \sqrt{1 + \Lambda y^2} \).
2. A-dependent coordinates \((x, z_y)\) with \(z_y = y/\sqrt{1 + \Lambda x^2}\).

3. Polar coordinates \((r, \phi)\).

The expression of the potential \(V(\lambda)\) in these three systems is

\[
V(\lambda) = \frac{1}{2} \left( \frac{1}{1 + \Lambda y^2} \right) \left[ \frac{z_x^2}{1 + \Lambda z_x^2} + y^2 \right]
\]

\[
= \frac{1}{2} \left( \frac{1}{1 + \Lambda x^2} \right) \left[ x^2 + \frac{z_y^2}{1 + \Lambda z_y^2} \right]
\]

\[
= \frac{1}{2} \left( \frac{r^2}{1 + \Lambda r^2} \right).
\]

Since it is known the existence of a close relation between (additive) classical Hamilton-Jacobi separability and (multiplicative) quantum Schrödinger separability [72, 73] it seems natural to make use of these three coordinate systems for the study of this quantum problem.

Next we start our study with the first coordinate system.

Using \((z_x, y)\) coordinates, the Schrödinger equation becomes

\[
-\frac{1}{2} \left[ \left( \frac{1 + \Lambda z_x^2}{1 + \Lambda y^2} \right) \frac{\partial^2}{\partial z_x^2} + \left( \frac{\Lambda z_x}{1 + \Lambda y^2} \right) \frac{\partial}{\partial z_x} \right] \Psi - \frac{1}{2} \left[ (1 + \Lambda y^2) \frac{\partial^2}{\partial y^2} + 2\Lambda y \frac{\partial}{\partial y} \right] \Psi
\]

\[
+ \left( \frac{1}{2} \right) (1 + \Lambda \left( \frac{1}{1 + \Lambda z_x^2} \right) \left( \frac{z_x^2}{1 + \Lambda z_x^2} + y^2 \right) \Psi = e \Psi.
\] (12)

We assume a solution in the form

\[
\Psi(z, y) = Z(z_x) Y(y),
\]

where \(Z\) and \(Y\) are, respectively, functions of \(z_x\) and \(y\) alone. Substituting this expression in the Eq. (12) we arrive to

\[
-\frac{1}{2} \frac{1}{Z} \left[ (1 + \Lambda z_x^2) Z'' + (\Lambda z_x) Z' \right] + \frac{1}{2} (1 + \Lambda) \left( \frac{z_x^2}{1 + \Lambda z_x^2} \right)
\]

\[
= \frac{1}{2} \frac{1}{Y} \left[ (1 + \Lambda y^2)^2 Y'' + 2\Lambda y (1 + \Lambda y^2) Y' \right] - \left( \frac{1}{2} \right) (1 + \Lambda) y^2 + (1 + \Lambda y^2) e
\]

that leads to the two following ordinary equations

\[
-\frac{1}{2} \frac{1}{Z} \left[ (1 + \Lambda z_x^2) Z'' + (\Lambda z_x) Z' \right] + \left( \frac{1}{2} \right) (1 + \Lambda) \left( \frac{z_x^2}{1 + \Lambda z_x^2} \right) = \mu,
\]

\[
-\frac{1}{2} \frac{1}{Y} \left[ (1 + \Lambda y^2)^2 Y'' + 2\Lambda y (1 + \Lambda y^2) Y' \right] + \left( \frac{1}{2} \right) (1 + \Lambda) y^2 - (1 + \Lambda y^2) e = -\mu,
\]

where \(\mu\) denotes the separation constant.

Consequently the A-dependent Schrödinger equation is in fact separable in the \((z_x, y)\) coordinates. Thus the two-dimensional problem has been decoupled in two one-dimensional equations.
4.2 Power series resolution of the $Z$-equation

The first equation to be solved is

$$(1 + \Lambda z^2_x) Z'' + (\Lambda z_x) Z' - (1 + \Lambda) \left( \frac{z^2_x}{1 + \Lambda z^2_x} \right) Z = -2\mu Z$$

(13)

This equation coincides (up to the appropriate changes of notation) with the equation of the one-dimensional nonlinear oscillator studied in Ref. [1]. Therefore, we directly explain the characteristics of the solution.

Firstly, using the following factorization for the function $Z$

$$Z(z_x, \Lambda) = p(z_x, \Lambda) (1 + \Lambda z^2_x)^{-1/(2\Lambda)},$$

(14)

the function $p = p(z_x, \Lambda)$ must satisfy the differential equation

$$(1 + \Lambda z^2_x)p'' + (\Lambda - 2) z_x p' + (2\mu - 1) p = 0,$$

(15)

that represents a $\Lambda$-deformation of the Hermite equation. Secondly, this new equation can be solved by the use of power series expansions. Assuming

$$p(z_x, \Lambda) = \sum_{n=0}^{\infty} c_n(\Lambda) z^n_x = c_0(\Lambda) + c_1(\Lambda) z_x + c_2(\Lambda) z^2_x + \ldots$$

the following $\Lambda$-dependent recursion relation is obtained

$$c_{n+2} = (-1)^r \frac{c_n}{(n+2)(n+1)} \left[ n (\Lambda n - 2) + (2\mu - 1) \right], \quad n = 0, 1, 2, \ldots$$

The general solution $p(z_x)$, defined in the interval $z^2_x < 1/\Lambda$, is given by the linear combination $p = c_0 p_1 + c_1 p_2$ where $p_1(z_x)$ and $p_2(z_x)$ are the two solutions determined by $p_1(0) = 1$, $p'_1(0) = 0$ and $p_2(0) = 0$, $p'_2(0) = 1$, respectively. If there exists a certain integer $m$ such that the coefficient $\mu = \mu_m$ is given by

$$2\mu_m = 2m + 1 - \Lambda m^2,$$

then we have $c_m \neq 0$, $c_{m+2} = 0$, and one of the two solutions is a polynomial of order $m$.

The polynomial solutions are given by

- Even index (even power polynomials)

  $$\mathcal{P}_{2p} = \sum_{r=0}^{p} c_{2r} z_x^{2r}$$

  $$c_{2r} = (-1)^r \frac{a_0}{2r!} (p'(p' - 2)(p' - 4)\ldots(p' - 2(r - 1)))$$

  $$\left[ 2 - \Lambda p' \right] \left[ 2 - \Lambda (p' + 2) \right] \left[ 2 - \Lambda (p' + 4) \right] \ldots \left[ 2 - \Lambda (p' + 2(r - 1)) \right],$$

where we have introduced the notation $p' = 2p$. More specifically, the expressions of the first solution $p_1(z_x)$, in the particular cases of $p' = 0, 2, 4$, are given by:

- $\mathcal{P}_0 = 1$,
- $\mathcal{P}_2 = 1 - 2(1 - \Lambda) z^2_x$,
- $\mathcal{P}_4 = 1 - 4(1 - 2\Lambda) z^2_x + \left( \frac{4}{3} \right)(1 - 2\Lambda)(1 - 3\Lambda) z^4_x - 4\Lambda z^4_x$
• Odd index (odd power polynomials)

\[ P_{2p+1} = \sum_{r=0}^{r=p} a_{2r+1} z_x^{2r+1} \]

where we have introduced the notation \( p' = 2p + 1 \). More specifically, the expressions of the second solution \( P_2(z_x) \) for \( p' = 1, 3, 5 \), are given by:

\[ P_1 = z_x, \]
\[ P_3 = z_x - \left( \frac{2}{3} \right)(1 - 2\Lambda)z_x^3, \]
\[ P_5 = z_x - \left( \frac{4}{3} \right)(1 - 3\Lambda)z_x^3 + \left( \frac{4}{15} \right)(1 - 3\Lambda)(1 - 4\Lambda)z_x^5. \]

4.3 Power series resolution of the Y-equation

The second equation to be solved is

\[
(1 + \Lambda y^2) Y'' + (2\Lambda y) Y' - (1 + \Lambda) \left( \frac{y^2}{1 + \Lambda y^2} \right) Y + 2eY = 2 \left( \frac{\mu}{1 + \Lambda y^2} \right) Y
\]

that, although it has certain similarity with the Eq. (13), it does not coincide with it (two differences: the coefficient \( 2\Lambda \) and the \( \mu \)-dependent right-hand term). The main reason for this asymmetry is that, when introducing separability in the Schrödinger equation, the angular momentum term \( \hat{J}^2 \) was displaced into this second equation.

We start our study with the following two steps.

Step 1. Introduction of a new quantum number

It is convenient to decompose the energy \( e \) as the following sum

\[ e = \mu + \nu \]

where \( \nu \) is a new parameter. Then the equation (16) transforms into

\[
(1 + \Lambda y^2) Y'' + (2\Lambda y) Y' - (1 + \Lambda - 2\Lambda \mu) \left( \frac{y^2}{1 + \Lambda y^2} \right) Y + 2\nu Y = 0, \]

which looks more similar to previous first equation for the function \( Z \).

Step 2. Factorization of the function \( Y \)

Let us rewrite the previous equation as follows

\[
(1 + \Lambda y^2) Y'' + (2\Lambda y) Y' - G_\mu^2 \left( \frac{y^2}{1 + \Lambda y^2} \right) Y + 2\nu Y = 0, \quad G_\mu^2 = 1 + (1 - 2\mu)\Lambda.
\]

Firstly, it can be verified that the function \( \Psi_\infty \) defined by

\[ \Psi_\infty = (1 + \Lambda y^2)^{-G_\mu/(2\Lambda)} \]
satisfies the following property
\[
\left[ (1 + \Lambda y^2) \frac{d^2}{dy^2} + 2\Lambda y \frac{d}{dy} - G_\mu^2 \left( \frac{y^2}{1 + \Lambda y^2} \right) \right] \Psi_\infty = -G_\mu \Psi_\infty.
\]

Thus, \( \Psi_\infty \) is the exact solution of the Eq. (16) in the very particular case of \( \nu = (1/2)G_\mu \) and can be considered as representing, in the general case \( 2\nu \neq G_\mu \), the asymptotic behaviour of the solution. Consequently, this property suggests the following factorization
\[
Y(y, \Lambda) = q(y, \Lambda) (1 + \Lambda y^2) - G_\mu / (2\Lambda),
\]
and then the new function \( q(y, \Lambda) \) must satisfy the differential
\[
(1 + \Lambda y^2)q'' + 2(\Lambda - G_\mu) yq' + (2\nu - G_\mu) q = 0,
\]
that turns out to be a new \( \Lambda \)-deformation of the Hermite equation.

Assuming a power expansion for the solution
\[
q(y, \Lambda) = \sum_{n=0}^{\infty} c_n(\Lambda) y^n = c_0(\Lambda) + c_1(\Lambda) y + c_2(\Lambda) y^2 + \ldots
\]
the equation leads to
\[
\sum_{n=0}^{\infty} [(n + 2)(n + 1) c_{n+2} + \Lambda n(n - 1)c_n y^n + 2(\Lambda - G_\mu) n c_n + (2\nu - G_\mu) c_n] y^n = 0,
\]
and we obtain the following \( \Lambda \)-dependent recursion relation
\[
c_{n+2} = (-1) \frac{c_n}{(n + 2)(n + 1)} \left[ \Lambda n(n - 1) - G_\mu(2n + 1) + 2\nu \right]
\]
Note that this relation shows that, as in the \( \Lambda = 0 \) case, even power coefficients are related among themselves and the same is true for odd power coefficients. In both cases, having in mind that
\[
\lim_{n \to \infty} \frac{c_{n+2} x^{n+2}}{c_n x^n} = \lim_{n \to \infty} \left| \frac{\Lambda n(n + 1) - G_\mu(2n + 1) + 2\nu}{(n + 2)(n + 1)} \right| | x^2 | = | \Lambda | | x^2 |,
\]
the radius of convergence \( R \) is given by
\[
R = \frac{1}{\sqrt{|\Lambda|}}.
\]
Hence, when we consider the limit \( \Lambda \to 0 \), we recover the radius \( R = \infty \) of the Hermite’s equation.

The general solution is given by the linear combination \( q = c_0q_1 + c_1q_2 \) where \( q_1(y) \) and \( q_2(y) \) are the solutions determined by \( q_1(0) = 1, q_1'(0) = 0 \) and \( q_2(0) = 0, q_2'(0) = 1 \), respectively. In the very particular case of the coefficient \( \nu \) be given by \( \nu = \nu_n \) with
\[
2\nu_n = (2n + 1)G_\mu - n(n + 1) \Lambda,
\]
then we have \( c_n \neq 0, c_{n+2} = 0 \), and one of the two solutions becomes a polynomial of order \( n \).

The polynomial solutions are given by
• Even index (even power polynomials)

\[ P_{2p} = \sum_{r=0}^{p} c_{2r} y^{2r} \]

\[ c_{2r} = (-1)^r \frac{a_0}{2r+1} \frac{p' - 2}{(p' - 4) \ldots (p' - 2(r - 1))} \]

\[ [2G - \Lambda(p' + 1)][2G_\mu - \Lambda(p' + 3)][2G_\mu - \Lambda(p' + 5)] \ldots [2G_\mu - \Lambda(p' + 2r - 1)] \]

where we have introduced the notation \( p' = 2p \). More specifically, the expressions of the first solution \( q_1(y) \), in the particular cases of \( p' = 0, 2, 4, 6 \), are given by:

\[ P_0 = 1, \]
\[ P_2 = 1 - (2G_\mu - 3\Lambda)y^2, \]
\[ P_4 = 1 - 2(2G_\mu - 5\Lambda)y^2 + \left( \frac{1}{3} \right)(2G_\mu - 5\Lambda)(2G_\mu - 7\Lambda)y^4. \]

• Odd index (odd power polynomials)

\[ P_{2p+1} = \sum_{r=0}^{p} a_{2r+1} y^{2r+1} \]

\[ a_{2r+1} = (-1)^{r+1} \frac{a_1}{2r+1} \frac{p' - 1}{(p' - 3) \ldots (p' - 2r - 1)} \]

\[ [2G_\mu - \Lambda(p' + 2)][2G_\mu - \Lambda(p' + 4)] \ldots [2G_\mu - \Lambda(p' + 2r)] \]

where we have introduced the notation \( p' = 2p + 1 \). More specifically, the expressions of the second solution \( q_2(y) \) for \( p' = 1, 3, 5 \), are given by:

\[ P_1 = y, \]
\[ P_3 = y - \left( \frac{1}{3} \right)(2G_\mu - 5\Lambda)y^3, \]
\[ P_5 = y - \left( \frac{1}{3} \right)(2G_\mu - 7\Lambda)y^3 + \left( \frac{1}{15} \right)(2G_\mu - 7\Lambda)(2G_\mu - 9\Lambda)y^5. \]

5 Eigenfunctions \( \Psi_{m,n} \) and energies \( E_{m,n} \)

5.1 Sturm-Liouville problems and \( \Lambda \)-dependent Hermite polynomials

We have obtained two different \( \Lambda \)-dependent deformations of the Hermite equation as well as the general solution and the particular polynomial solutions. Nevertheless in quantum mechanics the important point is not the equation by itself but the associate Sturm-Liouville problem. In this case we have two problems, one for the \( Z \)-equation and other for the \( Y \)-equation. Moreover each one of them splits into two: one for \( \Lambda < 0 \) (spherical case) and other for \( \Lambda > 0 \) (hyperbolic case).

Spherical \( \Lambda < 0 \) case The first \( \Lambda \)-dependent differential equation

\[ (1 + \Lambda z_x^2) p'' + (\Lambda - 2) z_x p' + (2\mu - 1) p = 0 \]  \( (19) \)

can be reduced to self-adjoint form by making use of the following integrating factor

\[ \mu(z_x) = (1 + \Lambda z_x^2)^{-(\Lambda+2)/(2\Lambda)}, \]
in such a way that we arrive to the following expression

\[ \frac{d}{dz_x} \left[ A(z_x, \Lambda) \frac{dp}{dz_x} \right] + (2\mu - 1) r(z_x, \Lambda) p = 0, \tag{20} \]

where the two functions \( A = A(z_x, \Lambda) \) and \( r = r(z_x, \Lambda) \) are given by

\[ A = \frac{\sqrt{1 + \Lambda z_x^2}}{(1 + \Lambda z_x^2)^{1/\Lambda}}, \quad r = \frac{1}{\sqrt{1 + \Lambda z_x^2 (1 + \Lambda z_x^2)^{1/\Lambda}}}, \]

that, together with appropriate boundary conditions, constitute a Sturm-Liouville problem.

If \( \Lambda \) is negative the problem is defined in the bounded interval \([-a_\Lambda, a_\Lambda] \) with \( a_\Lambda = 1/\sqrt{\abs{\Lambda}} \). The function \( A(z_x, \Lambda) \) vanishes in the two end points \( z_1 = -a_\Lambda \) and \( z_2 = a_\Lambda \) and the problem is singular. The boundary conditions prescribe that the solutions must be bounded functions at the two end points of the interval. The eigenvalues are the quantized values of the parameter \( \mu \), i.e. \( \mu_m, m = 0, 1, 2, \ldots \), and the eigenfunctions the associated polynomial solutions.

If \( \Lambda \) is positive the problem is singular since is defined in the whole real line \( \mathbb{R} \). The solutions must decrease when \( z_x \to \pm \infty \) in such a way that their norms, determined with respect to the weight function \( r(z_x) \), be finite. Therefore the eigenfunctions are again the \( \Lambda \)-dependent polynomials \( P_m(z_x, \Lambda) \), \( m = 0, 1, 2, \ldots \).

**Proposition 2** The eigenfunctions \( P_m(z_x, \Lambda), m = 0, 1, 2, \ldots \) of the de Sturm-Liouville problem of the Eq. (20) are orthogonal with respect to the function \( r = (1 + \Lambda z_x^2)^{-(1/2 + 1/\Lambda)} \).

**Proof:** This statement is just a consequence of the properties of the Sturm-Liouville problems. Because of this the polynomial solutions \( P_m(z_x, \Lambda), m = 0, 1, 2, \ldots \) of the equation (20), satisfy

\[ \int_{-a_\Lambda}^{a_\Lambda} \frac{P_m(z_x, \Lambda) P_n(z_x, \Lambda)}{(1 + \Lambda z_x^2)^{1/\Lambda} \sqrt{1 + \Lambda z_x^2}} dz_x = 0, \quad m \neq n, \quad \Lambda < 0, \]

and

\[ \int_{-\infty}^{\infty} \frac{P_m(z_x, \Lambda) P_n(z_x, \Lambda)}{(1 + \Lambda z_x^2)^{1/\Lambda} \sqrt{1 + \Lambda z_x^2}} dz_x = 0, \quad m \neq n, \quad \Lambda > 0. \]

In the \( \Lambda > 0 \) case, as the integral is defined on a infinite interval, the following property must be satisfied

\[ \lim_{|z_x| \to \infty} z_x [P_m(z_x, \Lambda)]^2 (1 + \Lambda z_x^2)^{-(1/\Lambda+1/2)} = 0. \]

The consequence is that if \( \Lambda > 0 \) then the quantum number \( m \) is limited by the condition \( m < 1/\Lambda \), and there is only \( M_\Lambda \) eigenvalues and eigenfunctions where \( M_\Lambda \) denotes the greatest integer lower than \( 1/\Lambda \).

The following “Rodrigues formula”

\[ H_m(z_x, \Lambda) = (-1)^m W_z^{1/\Lambda+1/2} \frac{d^m}{dz_x^m} \left[ W_z W_z^{-1/(\Lambda+1/2)} \right], \quad W_z = 1 + \Lambda z_x^2, \tag{21} \]

leads to a family of \( \Lambda \)-dependent Hermite polynomials \( H_m \) which are proportional to \( P_m \)

\[ H_m = k_m P_m(z_x, \Lambda), \quad m = 0, 1, 2, \ldots \]
where $k_m$ are constants that in the first cases are given by

\[
\begin{align*}
  k_0 &= 1, & k_1 &= (2 - \Lambda), \\
  k_2 &= -(2 - 3\Lambda), & k_3 &= -3(2 - 3\Lambda)(2 - 5\Lambda), \\
  k_4 &= 3(2 - 5\Lambda)(2 - 7\Lambda), & k_5 &= 15(2 - 5\Lambda)(2 - 7\Lambda)(2 - 9\Lambda).
\end{align*}
\]

Alternatively we can obtain these polynomials by using the following function

\[
F(z_x, t, \Lambda) = \left(1 + \Lambda (2tz_x - t^2)\right)^{1/\Lambda}
\]

as a generating function

\[
\left(1 + \Lambda (2tz_x - t^2)\right)^{1/\Lambda} = \sum_{m=0}^{\infty} \left(\frac{1}{m!}\right) \tilde{H}_m(z_x, \Lambda) t^m,
\]

where we have used the notation $\tilde{H}_m$ for the coefficients of the Taylor series. We obtain

\[
\tilde{H}_m = g_m \mathcal{P}_m(z_x, \Lambda), \quad m = 0, 1, 2, \ldots
\]

where $g_m$ are constants that in the first cases are given by

\[
\begin{align*}
  g_0 &= 1, & g_1 &= 2, \\
  g_2 &= -2, & g_3 &= -12(1 - \Lambda), \\
  g_4 &= 12(1 - \Lambda), & g_5 &= 120(1 - \Lambda)(1 - 2\Lambda).
\end{align*}
\]

We can define the $\Lambda$-dependent Hermite functions $Z_m$ by

\[
Z_m(z_x, \Lambda) = \mathcal{H}_m(z_x, \Lambda) (1 + \Lambda z_x^2)^{-1/(2\Lambda)}, \quad m = 0, 1, 2, \ldots
\]

then the above statement admits the following alternative form: The $\Lambda$-dependent Hermite functions $Z_m(z_x, \Lambda)$ are orthogonal with respect to the weight function $\bar{\tau} = 1/\sqrt{1 + \Lambda z_x^2}$:

\[
\int_{-a_\Lambda}^{a_\Lambda} Z_m(z_x, \Lambda) Z_n(z_x, \Lambda) \bar{\tau}(z_x, \Lambda) dz_x = \int_{-a_\Lambda}^{a_\Lambda} Z_m(z_x, \Lambda) Z_n(z_x, \Lambda) \frac{dz_x}{\sqrt{1 + \Lambda z_x^2}} = 0, \quad m \neq n, \quad \Lambda < 0,
\]

and

\[
\int_{-\infty}^{\infty} Z_m(z_x, \Lambda) Z_n(z_x, \Lambda) \bar{\tau}(z_x, \Lambda) dz_x = \int_{-\infty}^{\infty} Z_m(z_x, \Lambda) Z_n(z_x, \Lambda) \frac{dz_x}{\sqrt{1 + \Lambda z_x^2}} = 0, \quad m \neq n, \quad \Lambda > 0.
\]

Figures IV and V show the form of the function $Z_2(z_x, \Lambda)$ for several values of $\Lambda$ ($\Lambda < 0$ in Figure IV and $\Lambda > 0$ in Figure V).

Summarizing, the final solution of the Sturm-Liouville problem for the function $Z(z_x)$ is:

- Spherical $\Lambda < 0$ case:

\[
\begin{align*}
  Z_m(z_x, \Lambda) &= \mathcal{H}_m(z_x, \Lambda) (1 - |\Lambda| z_x^2)^{1/(2|\Lambda|)} \\
  \mu_m &= (m + 1/2) + \frac{1}{2} m^2 |\Lambda|, \quad m = 0, 1, 2, \ldots
\end{align*}
\]
• Hyperbolic $\Lambda > 0$ case:

$$Z_m(z, \Lambda) = \mathcal{H}_m(z, \Lambda) (1 + \Lambda z^2)^{-1/(2\Lambda)},$$

$$\mu_m = (m + \frac{1}{2}) - \frac{1}{2} m^2 \Lambda, \quad m = 0, 1, 2, \ldots, M_\Lambda.$$  

The second $\Lambda$-dependent differential equation

$$a_0 q'' + a_1 q' + a_2 q = 0, \quad G_m = 1 - m \Lambda,$$

$$a_0 = 1 + \Lambda y^2, \quad a_1 = 2(\Lambda - G_m) y, \quad a_2 = 2\nu - G_m, \quad (24)$$

is not self-adjoint since $a_0' \neq a_1$ but it can be reduced to self-adjoint form by making use of the following integrating factor

$$\mu(y) = \frac{1}{a_0} e^\int \frac{a_1}{a_0} dy = (1 + \Lambda y^2)^{-G_m/\Lambda},$$

in such a way that we arrive to the following expression

$$\frac{d}{dy} \left[ B(y, m, \Lambda) \frac{dq}{dy} \right] + (2\nu - G_m) r(y, m, \Lambda) q = 0, \quad (25)$$

where the two functions $B = B(y, m, \Lambda)$ and $r = r(y, m, \Lambda)$ are given by

$$B = (1 + \Lambda y^2)^{1-G_m/\Lambda}, \quad r = (1 + \Lambda y^2)^{-G_m/\Lambda}.$$

that, together with appropriate conditions for the behaviour of the solutions at the end points, constitute a Sturm-Liouville problem. It is to be pointed out that the boundary conditions are in fact different according to the sign of $\Lambda$; therefore we arrive to, no just one, but two different Sturm-Liouville problems:

If $\Lambda$ is negative the range of the variable $y$ is limited by the restriction $y^2 < 1/|\Lambda|$. In this case the problem, defined in the bounded interval $[-a_\Lambda, a_\Lambda]$ with $a_\Lambda = 1/\sqrt{|\Lambda|}$, is singular because the function $q(y, \Lambda)$ vanishes in the two end points $y_1 = -a_\Lambda$ and $y_2 = a_\Lambda$. The conditions to be imposed in this case lead to prescribe that the solutions $q(y, \Lambda)$ of the problem must be bounded functions at the two end points, $y_1 = -a_\Lambda$ and $y_2 = a_\Lambda$, of the interval. It is clear that this leads to the above mentioned polynomial solutions.

If $\Lambda$ is positive the variable $y$ is defined in the whole real line $\mathbb{R}$ and, therefore, the Sturm-Liouville problem is singular. The solutions $q(y, \Lambda)$ must be well defined in all $\mathbb{R}$, and the boundary conditions prescribe that the behaviour of these functions when $y \to \pm \infty$ must be such that their norms, determined with respect to the weight function $r(y)$, be finite. It is clear that in this case the solutions of the problem are again the $\Lambda$-dependent polynomials $\mathcal{H}_m$, $m = 0, 1, 2, \ldots$

**Proposition 3** The eigenfunctions $\mathcal{P}_n(y, m, \Lambda)$, $n = 0, 1, 2, \ldots$ of the de Sturm-Liouville problem of the Eq. (25) are orthogonal with respect to the function $r = (1 + \Lambda y^2)^{-G_m/\Lambda}$, $G_m = 1 - m \Lambda$.

**Proof:** This statement is just a consequence of the properties of the Sturm-Liouville problems. Because of this the polynomial solutions $\mathcal{P}_n = \mathcal{P}_n(y, m, \Lambda)$, $n = 0, 1, 2, \ldots$, of the equation (25), satisfy

$$\int_{-a_\Lambda}^{a_\Lambda} \frac{\mathcal{P}_r(y, m, \Lambda) \mathcal{P}_s(y, m, \Lambda)}{(1 + \Lambda y^2)^G_m/\Lambda} dy = 0, \quad r \neq s, \quad \Lambda < 0,$$

20
\[
\int_{-\infty}^{\infty} \frac{\mathcal{P}_r(y, m, \Lambda) \mathcal{P}_s(y, m, \Lambda)}{(1 + \Lambda y^2)^{G_m/\Lambda}} \, dy = 0, \quad r \neq s, \quad \Lambda > 0.
\]

In the \(\Lambda > 0\) case, as the integral is defined on a infinite interval, the following property must be satisfied

\[
\lim_{|y| \to \infty} y [\mathcal{P}_n(y, m, \Lambda)]^2 (1 + \Lambda y^2)^{m-1/\Lambda} = 0,
\]

and as a consequence, the quantum number \(n\) is limited by the condition

\[
n < \frac{1}{\Lambda} - m - \frac{1}{2}.
\]

That is, for every value of \(m\) there is only \(N_{\Lambda}\) eigenvalues and eigenfunctions where \(N_{\Lambda}\) denotes the greatest integer number lower than \(1/\Lambda - m - 1/2\).

The “Rodrigues formula” for these new family of \(\Lambda\)-dependent Hermite polynomials is given by

\[
\mathcal{H}_n(y, m, \Lambda) = (-1)^n W_y^{G_m/\Lambda} \frac{d^n}{dy^n} \left[ W_y^n W_y^{-G_m/\Lambda} \right], \quad W_y = 1 + \Lambda y^2,
\]

in such a way that we obtain

\[
\mathcal{H}_n = k_n \mathcal{P}_n(y, m, \Lambda), \quad n = 0, 1, 2, \ldots
\]

with the following values for the first constants

\[
\begin{align*}
k_0 &= 1, & k_1 &= 2(G_m - \Lambda), \\
k_2 &= -2(G_m - 2\Lambda), & k_3 &= -12(G_m - 2\Lambda)(G_m - 3\Lambda), \\
k_4 &= 12(G_m - 2\Lambda)(G_m - 3\Lambda), & k_5 &= 120(G_m - 3\Lambda)(G_m - 4\Lambda)(G_m - 5\Lambda).
\end{align*}
\]

The \((\Lambda, m)\)-dependent function defined by

\[
\mathcal{F}(y, t, \Lambda) = \left(1 + \Lambda (2ty - t^2)\right)^{(G_m/\Lambda - 1/2)}
\]

is a generating function with the following power expansion

\[
\left(1 + \Lambda (2ty - t^2)\right)^{(G_m/\Lambda - 1/2)} = \sum_{n=0}^{\infty} \frac{1}{n!} \tilde{\mathcal{H}}_n(y, m, \Lambda) t^n
\]

where we have used the notation \(\tilde{\mathcal{H}}_n\) for the coefficients of the Taylor series. The first \((\Lambda, m)\)-dependent Hermite polynomials obtained in such a way have the following expressions

\[
\tilde{\mathcal{H}}_n = g_n \mathcal{P}_n(y, m, \Lambda), \quad n = 0, 1, 2, \ldots
\]

where the constants \(g_i, i = 0, 1, 2, \ldots, 5\), take the values

\[
\begin{align*}
g_0 &= 1, & g_1 &= (2G_m - \Lambda), \\
g_2 &= -(2G_m - \Lambda), & g_3 &= -3(2G_m - \Lambda)(2G_m - 3\Lambda), \\
g_4 &= 3(2G_m - \Lambda)(2G_m - 3\Lambda), & g_5 &= 15(2G_m - \Lambda)(2G_m - 3\Lambda)(2G_m - 5\Lambda).
\end{align*}
\]
The \( \Lambda \)-dependent Hermite functions \( Y_n \) are defined by
\[
Y_n(y, m, \Lambda) = \mathcal{H}_n(y, m, \Lambda) \left(1 + \Lambda y^2\right)^{-G_m/(2\Lambda)}, \quad n = 0, 1, 2, \ldots
\]
and the above statement admits the following alternative form: The \( \Lambda \)-dependent Hermite functions \( Y_n(y, m, \Lambda) \) are orthogonal with respect to the weight function \( \tilde{r} = 1 \):
\[
\int_{-\Lambda}^{\Lambda} Y_r(y, m, \Lambda) \tilde{r}(y, \Lambda) dy = \int_{-\Lambda}^{\Lambda} Y_s(y, m, \Lambda) \tilde{r}(y, \Lambda) dy = 0, \quad r \neq s, \quad \Lambda < 0,
\]
and
\[
\int_{-\infty}^{\infty} Y_r(y, m, \Lambda) \tilde{r}(y, \Lambda) dy = \int_{-\infty}^{\infty} Y_s(y, m, \Lambda) \tilde{r}(y, \Lambda) dy = 0, \quad r \neq s, \quad \Lambda > 0.
\]
Summarizing, the final solution of the Sturm-Liouville problem for the function \( Y(y) \) is:

- Spherical \( \Lambda < 0 \) case:
  \[
  Y_n(y, m, \Lambda) = \mathcal{H}_n(y, m, \Lambda) \left(1 - |\Lambda| y^2\right)^{G_m/(2\Lambda)}, \quad G_m = 1 + m |\Lambda|,
  \nu_n = \left(n + \frac{1}{2}\right) G_m + \frac{1}{2} n (n + 1) |\Lambda|, \quad n = 0, 1, 2, \ldots, m, \ldots
  \]

- Hyperbolic \( \Lambda > 0 \) case:
  \[
  Y_n(y, m, \Lambda) = \mathcal{H}_n(y, m, \Lambda) \left(1 + \Lambda y^2\right)^{-G_m/(2\Lambda)}, \quad G_m = 1 - m\Lambda,
  \nu_n = \left(n + \frac{1}{2}\right) G_m - \frac{1}{2} n (n + 1) \Lambda, \quad n = 0, 1, 2, \ldots, N_{\Lambda}.
  \]

### 5.2 Final solution

The wave functions of the \( \Lambda \)-dependent nonlinear oscillator are
\[
\Psi_{m,n}(z_x, y) = Z_m(z_x) Y_n(y), \quad z_x = \frac{x}{\sqrt{1 + \Lambda y^2}},
\]
with energies given by
\[
e_{m,n} = \mu_m + \nu_n = \left((m + \frac{1}{2}) + (n + \frac{1}{2})\right) \left[1 - \frac{1}{2}(m + n)\Lambda\right].
\]
So the total energy \( E_{m,n} = (\hbar \alpha)e_{m,n} \) is a linear function of \( \Lambda \) and depends, as in the \( \Lambda = 0 \) case, of the sum \( m + n \) of the two quantum numbers. It is clear that \( \Psi_{m,n} \) is well defined for any value of \( \Lambda \) and that the following limit is satisfied
\[
\lim_{\Lambda \to 0} \Psi_{m,n}(z_x, y) = H_m(x) H_n(y) e^{-\frac{1}{(2)}(x^2+y^2)}, \quad m, n = 0, 1, 2, \ldots
\]
We recall that this approach has considered the space \( L^2(\mathbb{R}^2, d\mu_{\Lambda}) \) as the appropriate Hilbert space. Thus, if \( \Psi_{m,n}(x, y) \) and \( \Psi_{r,s}(x, y) \) are to wave functions representing states of the nonlinear oscillators with quantum number \( (m, n) \) and \( (r, s) \) respectively, then the scalar product is given by
\[
\langle \Psi_{m,n}, \Psi_{r,s} \rangle_{\Lambda} = \int \Psi_{m,n}(x, y) \Psi_{r,s}(x, y) d\mu_{\Lambda}.
\]
The point is that making use of the equality
\[ 1 + \Lambda r^2 = (1 + \Lambda z_x^2) (1 + \Lambda y^2) \]
the measure \( d\mu_\Lambda \) becomes as follows
\[ d\mu_\Lambda = \left( \frac{1}{\sqrt{1 + \Lambda r^2}} \right) dx \, dy = \left( \frac{1}{\sqrt{1 + \Lambda z_x^2}} \right) dz_x \, dy \]
in coordinates \((z_x, y)\). Hence, making use of the factorization of the measure \( d\mu_\Lambda \) in coordinates \((z_x, y)\), we can factorize the scalar product in \( \mathbb{R}^2 \) as a product of two one-dimensional scalar products and arrive to the following important property
\[
\langle \Psi_{m,n}, \Psi_{r,s} \rangle_\Lambda = \int Z_m(z_x) Z_r(z_x) Y_n(y) Y_s(y) \, d\mu_\Lambda \\
= \left( \int Z_m(z_x, \Lambda) Z_r(z_x, \Lambda) \frac{dz_x}{\sqrt{1 + \Lambda z_x^2}} \right) \left( \int Y_n(y, \Lambda) Y_s(y, \Lambda) \, dy \right) \\
= \delta_{m,r} \delta_{n,s}.
\]

The following two points summarize the main characteristics of the energies of the bound states:

1. Spherical \( \Lambda < 0 \) case:
   The Hamiltonian \( \hat{H}(\Lambda) \) describes a quantum oscillator on the sphere \( S^2_\kappa \) (\( \kappa > 0 \)). The oscillator possesses a countable infinite set of bound states \( \Psi_{n,m} \), with \( n, m = 0, 1, 2, \ldots \) and the energy spectrum is unbounded, not equidistant and with a difference between the levels that increases with \( N \)
   \[ e_0 < e_1 < e_2 < e_3 < \ldots < e_N < e_{N+1} < \ldots \]
   \[ e_{N+1} - e_N = 1 + (N + 1) |\Lambda|, \quad N = m + n. \]
The oscillations of the wave functions are reinforced and the values of the energies \( E_{n,m} \) are higher than in the Euclidean \( \Lambda = 0 \) case; that is, \( E_{n,m}(\Lambda) > E_{n,m}(0) \).

2. Hyperbolic \( \Lambda > 0 \) case:
   The Hamiltonian \( \hat{H}(\Lambda) \) describes a quantum oscillator on the hyperbolic plane \( H^2_\kappa \) (\( \kappa < 0 \)). The oscillator possesses only a finite number of bound states \( \Psi_{n,m} \), with \( n + m = 0, 1, 2, \ldots, N_A \), \( N_A < 1/\Lambda - 1/2 \), and the energy spectrum is bounded, not equidistant and with a difference between the levels that decreases with \( N \)
   \[ e_0 < e_1 < e_2 < e_3 < \ldots < e_{N_A} \]
   \[ e_{N+1} - e_N = 1 - (N + 1) \Lambda, \quad N = m + n. \]
The oscillations of the wave functions are smoothed down and the values of the energies \( E_{n,m} \) is lower than in the Euclidean \( \Lambda = 0 \) case; that is \( E_{n,m}(\Lambda) < E_{n,m}(0) \).
The degeneracy of the energy levels is the same that in the Euclidean case.

Figures VI and VII show the values of the energy \( e_{m,n} \) as a function of \( N, N = m + n \), for several values of \( \Lambda \). Figure VI shows as \( E_{n,m}(\Lambda) < E_{n,m}(0) \) when \( \Lambda > 0 \) (hyperbolic \( \Lambda > 0 \) case) and \( E_{n,m}(\Lambda) > E_{n,m}(0) \) when \( \Lambda < 0 \) (spherical \( \Lambda < 0 \) case). Figure VII shows the plot of the energy \( E_{m,n} \) for three different values of \( \Lambda > 0 \); it is clear that when \( \Lambda \) decreases the maximum of the curve moves into the upper right and the number of bound states increases.

6 Resolution of the Schrödinger equation II

The second alternative way of solving the Schrödinger equation (11) is using the property of separability in coordinates \((x, z_y)\) with \( z_y \) defined by \( z_y = y/\sqrt{1 + \Lambda x^2} \). This second approach is symmetric to the first one so the solution can be directly given as

\[
\Phi_{n,m}(x, z_y) = X_n(x) Z_m(z_y) ,
\]

with

\[
X_n(x, m, \Lambda) = \mathcal{H}_n(x, m, \Lambda) (1 + \Lambda x^2)^{(1-m\Lambda)/(2\Lambda)} ,
\]

\[
Z_m(z_y, \Lambda) = \mathcal{H}_m(z_y, \Lambda) (1 + \Lambda z_y^2)^{-1/(2\Lambda)} .
\]

The interesting property of this solution is that it satisfies

\[
(\hat{H}_1 - \lambda \hat{J}^2) \Phi_{n,m}(x, z_y) = \nu_n \Phi_{n,m}(x, z_y) ,
\]

\[
\hat{H}_2 \Phi_{n,m}(x, z_y) = \mu_m \Phi_{n,m}(x, z_y) .
\]

It is interesting to relate the existence of this second alternative approach with the above mentioned property of superintegrability.

There is not presently a satisfactory definition of quantum integrability. The most direct way of considering this question is by direct translation of the classical notions, so that, according to this approach, a quantum Hamiltonian \( \hat{H} \) would be integrable if there exists a set \( \{ A_i \} \) of \( n \) independent observables (including the Hamiltonian itself) that pairwise commute. If there exists an additional set of independent observables \( \{ B_r \} \) commuting with \( \hat{H} \) then the system would be superintegrable [60]-[71]; of course the \( \{ B_r \} \) do not necessary commute between them and every \( B_r \) only commute with some of the \( \{ A_i \} \). The main problem is the definition of independence for quantum operators since the commutation relation \([A_j, A_k] = 0\) can be considered as determining a functional dependence between \( A_j \) and \( A_k \). Several criteria have been analyzed; the most simple is to consider the operators \( \{ A_i \} \) as independent if they are obtained by quantizing classical functions which are functionally independent.

On the other hand most of classical superintegrable systems are actually superseparable (they admit separations of variables in at least two coordinate systems) and the classical
integrals of motion are at most quadratic in the momenta; in this case all the operators \{A_i, B_r\} are first or second order differential operators and this restriction makes easier the study of independence (when third order operators are considered then quantum integrability can lead to properties rather different to those of the classical system [62, 66]). Let us mention that although quantum superintegrability and exact solvability are defined in different ways, it has been conjectured [66] that all maximally superintegrable systems are exactly solvable.

A consequence of the quantum superintegrability is that, as we have a total of \(2n - 1\) operators commuting with \(\hat{H}\), we can construct different complete sets of \(n\) commuting observables and, therefore, different ways of characterizing the wave function \(\Psi\). Concerning this quantum \(\lambda\)-dependent oscillator, it is endowed with the following three sets of commuting observables

\[
\{\hat{H}_1, \hat{H}_2 - \lambda \hat{J}_2^2\}, \quad \{\hat{H}_1 - \lambda \hat{J}_2^2, \hat{H}_2\}, \quad \{\hat{H}_1 + \hat{H}_2, \hat{J}_2^2\},
\]

that correspond to three different ways of decomposing \(\hat{H}\) as a sum of two commuting observables and also to three coordinate systems separating the Schrödinger equation

\[
\begin{align*}
(z_x, y), & \quad (x, z_y), & \quad (r, \phi),
\end{align*}
\]

and to three alternative ways of representing the wave function.

7 Final comments and outlook

The harmonic oscillator is not a specific or special characteristic of the Euclidean space but it is well defined in all the three spaces of constant curvature. In fact if we use the curvature \(\kappa\) (or \(\lambda\)) as a parameter then we can say that there are not three different harmonic oscillators but only one that is defined, at the same time, in the three manifolds. This property, that was known at the classical level, is also true for the quantum system.

If we consider the spherical and hyperbolic systems as a deformation of the well known Euclidean system (in the sense discussed in the Introduction) then this deformation appears as clearly asymmetric. This is a natural result since the sphere \(S^2\) and the hyperbolic plane \(H^2\) are geometrically different and, because of this, some dynamical properties, as the characteristics of the wave functions \(\Psi_{m,n}\) and the energies \(E_{m,n}\), also show differences depending of the sign of \(\lambda\).

We finalize with some open questions.

Firstly, we have focussed our study on the systems \((x, z_y)\) and \((z_x, y)\) because of their relations with the Cartesian coordinates and the Hermite polynomials in the Euclidean \(\lambda = 0\) case. Nevertheless the resolution in polar coordinate must also be studied.

Secondly, we have quantized the system by analyzing the symmetries of the metric, obtaining an invariant measure and expressing the Hamiltonian as a function of the Noether momenta. The use of this quantization procedure for other systems with a position-dependent mass is a matter to be studied.
Thirdly, one of the byproducts of this study is the existence of $\lambda$-dependent deformations (or generalizations) of the Hermite polynomials endowed with the appropriate properties of orthogonality, Rodrigues formula and generating functions. The first family was already obtained in Ref. [1] but the second family is new; it is clear that the three-dimensional oscillator will lead to a new third family. These polynomials are interesting and deserve a more detailed study not only for the quantum problem but also from a mathematical viewpoint. At this point we also recall the existence of another family of “relativistic Hermite polynomials” obtained by Aldaya et al in Ref. [74, 75, 76] in the study of the relativistic quantum harmonic oscillator. We also note that the equations (15) and (18) are of hypergeometric type and the relation of the solutions with the hypergeometric series is an interesting problem.

Four, it was proved that the one-dimensional $\lambda$-oscillator is solvable [1] by the use of the Schrödinger factorization formalism in terms of first order differential operators $A$ and $A^+$ (these operators are known as intertwining operators). The factorization of two-dimensional systems still remains as a very difficult problem but, in any case, the existence of appropriate operators $A$ and $A^+$ for this two-dimensional oscillator must be studied.

Finally, the technique of introducing the curvature $\kappa$ as a parameter for the joint analysis of the dynamics in the three manifolds $(S^2_{\kappa}, E^2, H^2_{\kappa})$ has been generalized to the Cayley-Klein geometries (see Refs. [77]–[80]). In this more general case this technique is used with two parameters, $\kappa_1$ and $\kappa_2$, which correspond to a space $M_{\kappa_1, \kappa_2}$ with constant curvature $\kappa_1$ and signature $(+1, \kappa_2)$. This formalism is more general and includes the $\kappa$-dependent formalism as the particular case $\kappa_1 = \kappa$ and $\kappa_2 = 1$. We think that the study presented in this article for the quantum harmonic oscillator on the three classical spaces of constant curvature can be extended to the de-Sitter, Minkowski and anti-de-Sitter spaces by using this two parameters formalism.

Appendix: Geodesic polar coordinates

A two–dimensional manifold $M$ can be described by using different coordinate systems. If we consider it as an imbedded submanifold of $\mathbb{R}^3$, then the points of $M$ can be characterized by the three external coordinates, as $(x, y, z)$ or $(r, \phi, \theta)$, plus an additional constraint relation. Nevertheless, in differential geometric terms, a more appropriate approach is to develop the study by using two–dimensional systems of coordinates adapted to $M$.

On any general two–dimensional Riemannian space, not necessarily of constant curvature, there are two distinguished types of local coordinate systems: “geodesic parallel” and “geodesic polar” coordinates. They reduce to the familiar Cartesian and polar coordinates on the Euclidean plane and both are based on a origin point $O$ and an oriented geodesic $g_1$ through $O$.

For any point $P$ in some suitable neighbourhood of $O$, there is a unique geodesic $g$ joining $P$ with $O$. The (geodesic) polar coordinates $(R, \Phi)$ of $P$, relative to the origin $O$ and the
positive geodesic ray of $g_1$, are the (positive) distance $R$ between $P$ and $O$ measured along $g$, and the angle $\Phi$ between $g$ and the positive ray $g_1$, measured around $O$. These coordinates are defined in a neighbourhood of $O$ not extending beyond the cut locus of $O$; polar coordinates are singular at $O$, and $\Phi$ is discontinuous on the positive ray of $g_1$.

In the case of $M$ being a space of constant curvature $\kappa$, the expression for the differential element of distance $ds^2$ is given by

$$ds_\kappa^2 = dR^2 + S_\kappa^2(R) d\Phi^2,$$

so that we get $ds^2 = dr^2 + r^2 d\phi^2$ for the particular $\kappa = 0$ Euclidean case.

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Figure Captions

- **Figure I.** Plot of the one-dimensional potential \( V(\lambda) = (1/2)(\alpha^2 x^2)/(1 + \lambda x^2) \), as a function of \( x \), for \( \alpha = 1 \) and \( \lambda < 0 \).

- **Figure II.** Plot of the one-dimensional potential \( V(\lambda) = (1/2)(\alpha^2 x^2)/(1 + \lambda x^2) \), as a function of \( x \), for \( \alpha = 1 \) and \( \lambda > 0 \).

- **Figure III.** Plot of the potential \( U_\kappa(r) \), \( \alpha = 1 \), as a function of \( r \), for \( \kappa = -1 \) (lower curve), \( \kappa = 0 \) (dash line), and \( \kappa = 1 \) (upper curve).

- **Figure IV.** Plot of the \( \Lambda \)-dependent Hermite function \( Z_2(z_x, \Lambda) \) as a function of \( z_x \) for \( \Lambda = 0 \) (dashed curve) and \( \Lambda = -0.15 \) and \( \Lambda = -0.30 \). For very small values of \( |\Lambda| \) the figure is very similar to the standard Hermite curve and when the value of \( |\Lambda| \) increases the oscillations become stronger.

- **Figure V.** Plot of the \( \Lambda \)-dependent Hermite function \( Z_2(z_x, \Lambda) \) as a function of \( z_x \) for \( \Lambda = 0 \) (dashed curve) and \( \Lambda = 0.15 \) and \( \Lambda = 0.30 \). In this case when the value of \( \Lambda \) increases the oscillations become softer and for \( \Lambda \geq 0.5 \) the Hermite function becomes not normalizable.

- **Figure VI.** Plot of the energy \( e_N \) as a function of \( N \), \( N = m + n \), for \( \Lambda = 0.30 \) (lower curve) and \( \Lambda = -0.30 \) (upper curve). The thick points \((N, e_N)\), corresponding to the values \( N = 0, 1, 2 \), represent the three bound states existing for \( \Lambda = 0.30 \) and the first three bound states for \( \Lambda = -0.30 \). The straight line (dashed line) placed in the middle corresponds to the linear harmonic oscillator.

- **Figure VII.** Plot of the energy \( e_{m,n} \) as a function of \( N \), \( N = m + n \), for \( \Lambda = 0.45 \) (lower curve), \( \Lambda = 0.30 \) (middle curve) and \( \Lambda = 0.15 \) (upper curve). The curves also show the plot of the points \((N, e_N)\) for the values \( N = 0, 1 \), \( N = 0, 1, 2 \), and \( N = 0, 1, \ldots, 6 \), respectively. Every thick point represents a certain number of bound states with the same energy \( E_{m,n} \) and characterized by quantum numbers \( m \) and \( n \) such that \( m + n = N \). When \( \Lambda \) decreases the maximum of the curve moves into the upper right, the number of bound states goes up and in the limit \( \Lambda \to 0 \) the curve converges into a straight line parallel to the diagonal (dashed line).
Figure I. Plot of the one-dimensional potential $V(\lambda) = (1/2)(\alpha^2 x^2)/(1 + \lambda x^2)$, as a function of $x$, for $\alpha = 1$ and $\lambda < 0$.

Figure II. Plot of the one-dimensional potential $V(\lambda) = (1/2)(\alpha^2 x^2)/(1 + \lambda x^2)$, as a function of $x$, for $\alpha = 1$ and $\lambda > 0$. 
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