Hyperbolic Discounting of the Far-Distant Future

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Abstract. We prove an analogue of Weitzman’s [7] famous result that an exponential discounter who is uncertain of the appropriate exponential discount rate should discount the far-distant future using the lowest (i.e., most patient) of the possible discount rates. Our analogous result applies to a hyperbolic discounter who is uncertain about the appropriate hyperbolic discount rate. In this case, the far-distant future should be discounted using the probability-weighted harmonic mean of the possible hyperbolic discount rates.

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1 Introduction

Consider an individual – or Social Planner – who ranks streams of outcomes over a continuous and unbounded time horizon $T = [0, \infty)$ using a discounted utility criterion with discount function $D: T \to (0, 1]$. We assume throughout that $D$ is differentiable, strictly decreasing and satisfies $D(0) = 1$. For example, $D$ might have the exponential form

$$D(t) = e^{-rt}$$

for some constant discount (or time preference) rate, $r > 0$. Such discounting may be motivated by suitable preference axioms ([4]) or as a survival function with constant hazard rate, $r$ ([6]). For an arbitrary (i.e., not necessarily exponential) discount function, Weitzman ([7]) defines the local (or instantaneous) discount rate, $r(t)$, using the relationship:

$$D(t) = \exp\left(-\int_0^t r(\tau)d\tau\right) \Leftrightarrow r(t) = -\frac{D'(t)}{D(t)}$$

(1)

Note that $r(t)$ is constant if (and only if) $D$ has the exponential form.

Weitzman ([7]) considers a scenario in which the decision-maker is uncertain about the appropriate discount function to use. She may, for example, be uncertain about the true (constant) hazard rate of her survival function, as in [6]. The decision-maker entertains $n$ possible scenarios corresponding to $n$ possible discount functions $D_i$, $i = 1, 2, \ldots, n$, with associated local discount rate functions $r_i$. Suppose that scenario $i$ has probability $p_i > 0$, with $\sum_{i=1}^n p_i = 1$, and that the decision-maker discounts according to the expected (or certainty equivalent) discount function

$$D = \sum_{i=1}^n p_i D_i$$

(2)

(Such a discount function may also arise if the decision-maker is a utilitarian Social Planner for a population of $n$ heterogeneous individuals, as in [5].)

Let $r$ be the local discount rate function associated with certainty equivalent discount function (2). Weitzman ([7]) studies the limit behaviour of $r(t)$ as $t \to \infty$. He proves that if each $r_i(t)$ converges to a limit, then $r(t)$ converges to the lowest of these limits. In other words, if

$$\lim_{t \to \infty} r_i(t) = r_i^*$$

for each $i$, then

$$\lim_{t \to \infty} r(t) = \min\{r_1^*, \ldots, r_n^*\}$$

(3)

Moreover, if each $r_i$ is constant (i.e., each $D_i$ is exponential), then $r(t)$ declines monotonically to this limit ([7]).

Example 1. Suppose each $D_i$ is exponential, so that $r_i(t) = r_i$ is constant. Then the results in [7] imply that $r(t)$ declines monotonically with limit $\min_i r_i$. Figure illustrates for the case $n = 3$, $r_1 = 0.01$, $r_2 = 0.02$, $r_3 = 0.03$ and $p_1 = p_2 = p_3 = 1/3$.

\[1\]In fact, this is true more generally – see [7] footnote 6.
Weitzman’s result may be interpreted as saying that the mixed discount function (2) behaves locally as an exponential discount function with discount rate (3) when discounting outcomes in the far distant future. This result is most salient if the the individual $D_i$ functions are themselves exponential, as in Example 1. However, many individuals do not discount exponentially (2). If the $D_i$ functions all fall within some non-exponential class, it is natural to characterise the local asymptotic behaviour of (2) using the same class of functions. The next section considers the hyperbolic class.

2 The case of proportional hyperbolic discounting

In this section we assume that each $D_i$ has the (proportional) hyperbolic form

$$D_i(t) = \frac{1}{1 + h_i t}$$

where $h_i > 0$ is the hyperbolic discount rate. We further assume that $h_1 > h_2 > \ldots > h_n$. In particular, $D_1$ exhibits the most “patience” and $D_n$ the least – see [1] and Example 2.

Note that

$$r_i(t) = -\frac{D_i'(t)}{D_i(t)} = \frac{h_i}{1 + h_i t}$$

and hence $r_i^* = 0$ for each $i$. In other words, the limiting local (exponential) discount rate is the same for each discount function, reflecting the fact that hyperbolic functions decline
more slowly than exponentials for large $t$. Weitzman’s result is not very informative for this scenario.

Instead, we should like to have a local hyperbolic approximation to the mixed discount function (2), which may not itself have the proportional hyperbolic form. We therefore follow Weitzman’s example and define the local (or instantaneous) hyperbolic discount rate, $h(t)$, as follows:

$$D(t) = \frac{1}{1 + h(t)t} \Leftrightarrow h(t) = \left( \frac{1}{D(t)} - 1 \right) \frac{1}{t} \quad (4)$$

Note that $h(t)$ is constant if (and only if) $D$ has the proportional hyperbolic form.

How does $h(t)$ behave as $t \to \infty$?

The following two results, which are proved in the Appendix, answer this question. In order to state the second result, we remind the reader that the weighted harmonic mean of non-negative values $x_1, x_2, \ldots, x_n$ with non-negative weights $a_1, a_2, \ldots, a_n$ satisfying $a_1 + \ldots + a_n = 1$ is

$$H(x_1, a_1; \ldots; x_n, a_n) = \left( \sum_{i=1}^{n} \frac{a_i}{x_i} \right)^{-1}.$$

It is well-known that the weighted harmonic mean is smaller than the corresponding weighted arithmetic mean (i.e., expected value).

**Theorem 1.** The local hyperbolic discount rate, $h(t)$, is strictly decreasing.

**Theorem 2.** The local hyperbolic discount rate of the certainty equivalent discount function converges to the probability-weighted harmonic mean of the individual hyperbolic discount rates. That is

$$h(t) \to H(h_1, p_1; \ldots; h_n, p_n)$$

as $t \to \infty$.

The following example illustrates both results.

**Example 2.** Suppose $n = 3$, $h_1 = 0.01$, $h_2 = 0.02$, $h_3 = 0.03$ and $p_1 = p_2 = p_3 = \frac{1}{3}$. Note that $h_2 = 0.02$ corresponds to the arithmetic mean of $h_1$, $h_2$ and $h_3$. Figure 2 displays the monotonic decline of $h(t)$ towards the weighted harmonic mean $H(h_1, p_1; h_2, p_2, h_3, p_3) \approx 0.0164$.

### 3 Discussion

With exponential discounting, uncertainty about the (exponential) discount rate implies that the far-distant future is discounted according to the most “patient” of the possible discount functions. If discounting is hyperbolic, with uncertainty about the (hyperbolic) discount rate, all possible discount functions matter for the discounting of the far-distant future. The asymptotic local hyperbolic discount rate is, however, below the average (i.e., arithmetic mean) of the possible rates.

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\(^2\)See, in particular, the important reformulation of Weitzman’s result by Gollier and Weitzman (3), which resolves the so-called “Weitzman-Gollier puzzle”.

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3
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A Appendix

A.1 Proof of Theorem 1

We prove this statement by induction on $n$. First we need to prove that the statement holds for $n = 2$. In this case:

$$h(t) = \left[ \frac{1}{p_1(1 + h_1t)^{p_2(1 + h_2t)^{-1}} - 1} \right] \frac{1}{t}$$

for each $t > 0$. Rearranging:

$$h(t) = \left[ \frac{(1 + h_1t)(1 + h_2t)}{p_1(1 + h_2t) + p_2(1 + h_1t)} - 1 \right] \frac{1}{t} = \left[ \frac{1 + (h_1 + h_2)t + h_1h_2t^2}{p_1 + p_2 + (p_1h_2 + p_2h_1)t} - 1 \right] \frac{1}{t}.$$ 

Since $p_1 + p_2 = 1$ we obtain:

$$h(t) = \left[ \frac{1 + (h_1 + h_2)t + h_1h_2t^2}{1 + (p_1h_2 + p_2h_1)t} - 1 \right] \frac{1}{t} = \frac{p_1h_1 + p_2h_2 + h_1h_2t}{1 + (p_1h_2 + p_2h_1)t}.$$
By differentiating $h(t)$:

$$h'(t) = \frac{h_1h_2(1 + (p_1h_2 + p_2h_1)t) - (p_1h_1 + p_2h_2 + h_1h_2t)(p_1h_1 + p_2h_1)}{[1 + (p_1h_2 + p_2h_1)t]^2} \tag{5}$$

We need to show that $h'(t) < 0$. Since the denominator of (5) is positive, the sign of $h'(t)$ depends on the sign of the numerator. Therefore, we denote the numerator of (5) by $Q$ and analyse it separately:

$$Q(t) = h_1h_2[1 + (p_1h_2 + p_2h_1)t] - (p_1h_1 + p_2h_2 + h_1h_2t)(p_1h_1 + p_2h_1)$$

$$= h_1h_2 - (p_1h_1 + p_2h_2)(p_1h_2 + p_2h_1).$$

By expanding the brackets and using the fact that $p_1 + p_2 = 1$ implies $1 - p_1^2 - p_2^2 = 2p_1p_2$ expression $Q$ can be simplified further:

$$Q(t) = h_1h_2 - p_1^2h_1h_2 - p_1p_2h_1^2 - p_1p_2h_2^2 - p_2^2h_1h_2$$

$$= h_1h_2(1 - p_1^2 - p_2^2) - p_1p_2(h_1^2 + h_2^2)$$

$$= -p_1p_2(h_1 - h_2)^2.$$ 

Therefore, since $h_1 \neq h_2$ we have $Q < 0$. Hence it follows that $h'(t) < 0$ and $h(t)$ is strictly decreasing.

Suppose that the proposition holds for $n = k$. We need to show that it also holds for $n = k + 1$. When $n = k + 1$ we have:

$$D = \sum_{i=1}^{k+1} p_i D_i = (1 - p_{k+1}) \left( \sum_{i=1}^{k} \frac{p_i}{1 - p_{k+1}} D_i \right) + p_{k+1} D_{k+1}.$$ 

Since

$$\sum_{i=1}^{k} \frac{p_i}{1 - p_{k+1}} = 1,$$

we may write

$$D = (1 - p_{k+1}) D^{(k)} + p_{k+1} D_{k+1},$$

where

$$D^{(k)} = \sum_{i=1}^{k} \frac{p_i}{1 - p_{k+1}} D_i.$$ 

By the induction hypothesis it follows that

$$D^{(k)} = \frac{1}{1 + h^{(k)}(t)t},$$

where $h^{(k)}$ is strictly decreasing. Therefore,

$$h(t) = \left[ \frac{1}{(1 - p_{k+1})D^{(k)} + p_{k+1}D_{k+1}} - 1 \right] \frac{1}{t}$$

$$= \left[ \frac{1}{(1 - p_{k+1}) (1 + h^{(k)}(t)t)^{-1} + p_{k+1} (1 + h_{k+1}t)^{-1}} - 1 \right] \frac{1}{t},$$
Let $\hat{p}_1 = 1 - p_{k+1}$, $\hat{p}_2 = p_{k+1}$, $\hat{h}_1(t) = h^{(k)}(t)$ and $\hat{h}_2 = h_{k+1}$. Then we have

$$h(t) = \left[ \frac{1}{\hat{p}_1(1 + \hat{h}_1(t))^{-1} + \hat{p}_2(1 + \hat{h}_2(t))^{-1}} - 1 \right] \frac{1}{t}.$$ \hfill (1)

Analogously to the case $n = 2$, this expression can be rearranged to give:

$$h(t) = \frac{\hat{p}_1\hat{h}_1(t) + \hat{p}_2\hat{h}_2 + \hat{h}_1(t)\hat{h}_2 t}{1 + \hat{p}_1\hat{h}_2 t + \hat{p}_2\hat{h}_1(t) t}.$$ \hfill (5)

However, in contrast to the case $n = 2$, $\hat{h}_1$ is now a function of $t$. Thus:

$$h'(t) = \frac{\frac{\partial N(t)}{\partial t}}{\left[ 1 + \hat{p}_1\hat{h}_2 t + \hat{p}_2\hat{h}_1(t) t \right]^2} \hfill (6)$$

where

$$N(t) = \left( \hat{p}_1\hat{h}_1(t) + \hat{h}_1(t)\hat{h}_2 + \hat{h}_1'(t)\hat{h}_2 t \right) \left( 1 + \hat{p}_1\hat{h}_2 t + \hat{p}_2\hat{h}_1(t) t \right) - \left( \hat{p}_1\hat{h}_1(t) + \hat{p}_2\hat{h}_2 + \hat{h}_1(t)\hat{h}_2 t \right) \left( \hat{p}_1\hat{h}_2 + \hat{p}_2\hat{h}_1(t) + \hat{p}_2\hat{h}_1'(t) t \right).$$

The denominator of (6) is strictly positive, so the sign of the derivative is the same as that of $N(t)$. Note that

$$N(t) = \hat{Q}(t)\hat{h}_1'(t) \left[ \left( \hat{p}_1 + \hat{h}_2 t \right) \left( 1 + \hat{p}_1\hat{h}_2 t + \hat{p}_2\hat{h}_1(t) t \right) - \hat{p}_2 t \left( \hat{p}_1\hat{h}_1(t) + \hat{p}_2\hat{h}_2 + \hat{h}_1(t)\hat{h}_2 t \right) \right]$$

where $\hat{Q}(t)$ is defined as above, but with $h_1 = \hat{h}_1(t)$ and $h_2 = \hat{h}_2$. Since $\hat{Q}(t) \leq 0$ (with equality if and only if $\hat{h}_1(t) = h_2$) and $h'_1 < 0$, it suffices to show

$$\left( \hat{p}_1 + \hat{h}_2 t \right) \left( 1 + \hat{p}_1\hat{h}_2 t + \hat{p}_2\hat{h}_1(t) t \right) - \hat{p}_2 t \left( \hat{p}_1\hat{h}_1(t) + \hat{p}_2\hat{h}_2 + \hat{h}_1(t)\hat{h}_2 t \right) > 0 \hfill (7)$$

Cancelling terms on the left-hand side of (7) leaves us with:

$$\hat{p}_1 \left( 1 + \hat{p}_1\hat{h}_2 t \right) + \hat{h}_2 t \left( 1 + \hat{p}_1\hat{h}_2 t \right) - \left( \hat{p}_2 \right)^2 \hat{h}_2 t.$$ \hfill (8)

We now use the fact that $(\hat{p}_2)^2 = (1 - \hat{p}_1)^2 = 1 - 2\hat{p}_1 + (\hat{p}_1)^2$ to get

$$\hat{p}_1 \left( 1 + \hat{p}_1\hat{h}_2 t \right) + \hat{h}_2 t \left( 1 + \hat{p}_1\hat{h}_2 t \right) - \left[ 1 - 2\hat{p}_1 + (\hat{p}_1)^2 \right] \hat{h}_2 t = \hat{p}_1 + \left( \hat{h}_2 \right)^2 \hat{p}_1 + 2\hat{p}_1\hat{h}_2 t,$$

which is strictly positive. This establishes the required inequality (7) and completes the proof.
A.2 Proof of Theorem 2

We note that
\[ \frac{p_i}{1 + h_i t} = \frac{p_i}{h_i t} + \epsilon_i(t), \]
where \( \epsilon_i(t)/t^2 \to 0 \) when \( t \to \infty \). Let \( \epsilon(t) = \epsilon_1(t) + \ldots + \epsilon_n(t) \). Hence it follows that:

\[
\frac{1}{1 + h(t)t} = \sum_{i=1}^{n} p_i D_i(t) = \frac{p_1}{1 + h_1 t} + \ldots + \frac{p_n}{1 + h_n t} \\
= \frac{p_1}{h_1 t} + \ldots + \frac{p_n}{h_n t} + \epsilon(t) \\
= \left( \frac{p_1}{h_1} + \ldots + \frac{p_n}{h_n} \right) \frac{1}{t} + \epsilon(t) \\
= \frac{1}{H(h_1, p_1; \ldots; h_n, p_n)t} + \hat{\epsilon}(t),
\]
where \( \hat{\epsilon}(t)/t^2 \to 0 \) as \( t \to \infty \). This implies that \( h(t) \to H(h_1, p_1; \ldots; h_n, p_n) \) as \( t \to \infty \).

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