Computing discrete equivariant harmonic maps

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Abstract

We present effective methods to compute equivariant harmonic maps from the universal cover of a surface into a nonpositively curved space. By discretizing the theory appropriately, we show that the energy functional is strongly convex and derive convergence of the discrete heat flow to the energy minimizer, with explicit convergence rate. We also examine center of mass methods, after showing a generalized mean value property for harmonic maps. We feature a concrete illustration of these methods with Harmony, a computer software that we developed in C++, whose main functionality is to numerically compute and display equivariant harmonic maps.

Key words and phrases: Harmonic maps · Heat flow · Convexity · Gradient descent · Centers of mass · Discrete geometry · Riemannian optimization · Mathematical software

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**Introduction**

The theory of harmonic maps has its roots in the foundations of Riemannian geometry and the essential work of Euler, Gauss, Lagrange, and Jacobi. It includes the study of real-valued harmonic functions, geodesics, minimal surfaces, and holomorphic maps between Kähler manifolds.

The theory of harmonic maps was brought into a modern context for Riemannian manifolds with the seminal work of Eells-Sampson [ES64] (also Hartman [Har67] and Al’ber [Al’68]), who made the following definition: a harmonic map \( f : M \rightarrow N \) is a critical point of the energy functional

\[
E(f) = \frac{1}{2} \int_M \| df \|^2 \, dv_M .
\]

Eells-Sampson studied the heat flow associated to the energy, i.e. the nonlinear parabolic PDE

\[
\frac{d}{dt} f_t = \tau(f_t),
\]

where \( \tau(f) \) is the tension field of \( f \). The tension field can be described as minus the gradient of the energy functional on the infinite-dimensional Riemannian manifold \( C^\infty(M, N) \), so that the heat flow is just the gradient flow for the energy. When \( M \) is compact and \( N \) nonpositively curved, the heat flow is shown to converge to an energy-minimizing map as \( t \to \infty \).

The theory has since been developed and generalized to various settings where the domain or the target are not smooth manifolds [GS92, KS93, Che95, Jos97, EF01, Mes02, DM08]. In particular, Korevaar-Schoen developed an extensive Sobolev theory when the domain is Riemannian but the target is a nonpositively curved metric space [KS93, KS97]. Jost generalized further to a domain that is merely a measure space [Jos84, Jos94, Jos95, Jos96]. Both took a similar approach, constructing the energy functional \( E \) as a limit of approximate energy functionals.

These tools have become powerful and widely used, with celebrated rigidity results [Siu80, GS92, Wan00, DMV11, IN05] and dramatic implications for the study of deformation spaces and Teichmüller theory when the domain \( M \) is a surface (see e.g. [DW07]), especially via the nonabelian Hodge correspondence [Don87, Hit87, Cor88, Wol89, Lab91, Sim91].

This paper and its sequel [GLM18] are concerned with effectiveness of methods for finding harmonic maps. In addition to the mathematical content, we feature Harmony, a freely available computer program that we developed in C++ whose main functionality is to numerically compute equivariant harmonic maps. Our project was motivated by the question: is it possible to study the nonabelian Hodge correspondence experimentally? Though the heat flow is constructive to some extent, it does not provide qualitative information about the convergence in general. We show that one can design entirely effective methods to compute harmonic maps by discretizing appropriately.

Some of the existing literature treats similar questions. For example, Bartels applies finite element methods on submanifolds of \( \mathbb{R}^n \) to nonlinear PDEs such as the Euler-Lagrange equations for minimization of the energy [Bar05, Bar10, Bar15]. There is also an extensive literature on discrete energy functionals of the form we consider in this paper [CdV91, PP93, Leb96, KS01, JT07, Wan00, IN05, EF01, Fug08, HS15].

However, there are a number of features that set our work apart. First of all, we focus on surface domains, and we take sequences of arbitrarily fine discretizations as opposed to one fixed approximation of the surface. Moreover, we follow the maxim of Bobenko-Suris [BS08, p. xiv]:

*Discretize the whole theory, not just the equations.*
Our discrete structures record additional data beyond the commonly employed ‘cotangent edge weights’ of Pinkall-Polthier [PP93]. This data takes the form of a measure on the 0-skeleton of the triangulation, i.e. a system of vertex weights. One can think of this as a discrete record of the area form (i.e. Kähler form) on the domain surface, which, together with the discrete conformal structure recorded by the edge weights, provides a reasonable approximation of the surface’s metric.

This discretization endows the space of discrete maps with a finite-dimensional Riemannian structure, approximating the $L^2$ metric on $C^\infty(M,N)$. We obtain the right setting for a study of the convexity of the discrete energy, and for the definitions of a discrete tension field and, crucially, a discrete heat flow. Among the practical benefits, we find that the discrete energy satisfies stronger convexity properties than those known to hold in the smooth setting.

Another aspect of harmonic maps is revealed via the study of centers of mass. It is well-known in the Euclidean setting that real-valued harmonic functions satisfy the essential mean value property. Strictly speaking, this property does not hold in more general settings, but a fine analysis of the interaction between harmonic maps and centers of mass is still possible. It turns out that averaging a function can be a viable alternative to the heat flow in order to decrease its energy, a viewpoint well adapted to Jost’s theory of generalized harmonic maps [Jos94, Jos95, Jos96, Jos97]. As an alternative to the discrete heat flow, we also pursue a discretization of the theory of Jost by analyzing discrete center of mass methods. First, we explore the smooth setting and obtain novel aspects of the relationship between harmonic maps and Riemannian centers of mass.

For both heat flow and center of mass methods, the present paper focuses on a fixed discretization of the domain, while the forthcoming paper [GLM18] analyzes convergence of the discrete theory to the smooth one as we take finer and finer discretizations approximating a smooth domain.

Now let us describe more precisely some of the main theorems of the paper. After discussing harmonic maps in § 1 and developing a discretized theory in § 2, we study the convexity of the energy in § 3. We show:

**Theorem (Theorem 3.25).** Let $S$ be a discretized hyperbolic surface and let $N$ be a compact manifold with negative sectional curvature. Then the energy functional is strongly convex in the homotopy class of any $\pi_1$-injective map $S \to N$.

We actually show a more general version of this theorem involving equivariant maps and the notion of *biweighted triangulated graph* which we introduce in § 2. See Theorem 3.25 for the precise statement.

When the target $N$ is specialized to a hyperbolic surface, we find explicit bounds for the Hessian of the energy functional (see Theorem 3.21). We achieve this through detailed calculations in the hyperbolic plane, which we then generalize to negatively curved target manifolds using CAT($k$)-type comparisons. Roughly speaking, the key idea is that if the energy of some function has a very small second variation, then each triangle in the domain must be mapped to an almost flat triangle in the target; however this is not possible by the Gauss-Bonnet theorem. § 3 is concerned with the significant work of making this argument precise and quantitative.

In § 4 we study gradient descent methods in Riemannian manifolds and specialize to the convergence of the discrete heat flow. We show:

**Theorem (Theorem 4.4).** Let $S$ be a discretized hyperbolic surface and let $N$ be a compact manifold of negative curvature. Then there exists a unique discrete harmonic map $f^*: S \to N$. Moreover,
for any map \( f_0 : S \to N \) and for any sufficiently small \( t > 0 \), the discrete heat flow with initial value \( f_0 \) and fixed stepsize \( t \) converges to \( f^* \) with convergence rate 
\[
d(f_k, f^*) \leq c q^k,
\]
where \( c > 0 \) and \( q \in [0, 1) \) and \( d(f_k, f^*) \) is the \( L^2 \) distance in the space of discrete maps.

Again, this is a simplified version of the theorem we show; see Theorem 4.4 for the precise statement. When the target \( N \) is a hyperbolic surface, the constants \( c \) and \( q \) can be made explicit.

Next we discuss center of mass methods in § 5. In order to decrease the energy of any given map \( f : M \to N \), an interesting alternative to the heat flow consists in averaging \( f \) on balls (or spheres) of small radius \( r > 0 \), producing a new map \( B_r f : M \to N \). Repeating this process potentially produces energy-minimizing sequences for an approximate version of the energy \( E_r \), a phenomenon that has been explored by Jost [Jos94]. The central theorem we prove in § 5.2 is that in the Riemannian setting, this iterative process is almost the same as a fixed stepsize time-discretization of the heat flow. See Theorem 5.8 for a precise statement.

An immediate but noteworthy consequence of this theorem is the following generalized mean value property for harmonic maps:

**Theorem (Theorem 5.9).** Let \( f : M \to N \) be a smooth map. Then \( f \) is harmonic if and only if 
\[
d(f(x), B_r f(x)) = O(r^4)
\]
as \( r \to 0 \), for all \( x \in M \).

Under suitable conditions, we show that the center of mass method converges to a minimizer of the approximate energy \( E_r \) (see Theorem 5.17), recovering a theorem of Jost [Jos94, §3]. Jost’s result is more general, but our conclusion is slightly stronger.

In the space-discretized setting, the approximate energy coincides with the discrete energy, making the discrete center of mass method an appropriate alternative to the discrete heat flow.

**Theorem (Theorem 5.21).** Let \( S \) be a discretized hyperbolic surface, and let \( N \) be a compact manifold of negative sectional curvature. In a homotopy class of \( \pi_1 \)-injective maps \( S \to N \), the center of mass method from any initial map converges to the unique discrete harmonic map.

As a concrete demonstration of the effectiveness of our algorithms, we present in § 6 our own computer implementation of a harmonic map solver: Harmony is a freely available computer software with a graphical user interface written in C++ code, using the Qt framework. This program takes as input the Fenchel-Nielsen coordinates for a pair of Fuchsian representations \( \rho_L, \rho_R : \pi_1 S \to \text{Isom}(\mathbb{H}^2) \) and computes and visualizes the unique equivariant harmonic map. In future development of Harmony, we plan to compute and visualize harmonic maps for more general target representations that are not necessarily discrete, and for more general target spaces, such as \( \mathbb{H}^3 \) and other nonpositively curved symmetric spaces.

We have implemented both the discrete heat flow method, with fixed and optimal stepizes separately, and the cosh-center of mass method, a clever variant of the center of mass method suggested to us by Nicolas Tholozan (see § 5.4). In practice, the cosh-center of mass method is the most effective, both in number of iterations and execution time (see § 4.3).
Figure 1: Harmony’s main user interface. Both the left and right canvas display the hyperbolic plane $\mathbb{H}^2$ in the Poincaré disk model.

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1 Harmonic maps

In this section we review the theory of harmonic maps between Riemannian manifolds and between more general spaces for the purpose of the present paper.

1.1 Energy functional and harmonic maps

Let $(M, g)$ and $(N, h)$ be two smooth Riemannian manifolds. Assuming $M$ is compact, the energy of a smooth map $f: M \to N$ is:

$$E(f) = \frac{1}{2} \int_M \| df \|^2 \, dv_g$$  \hspace{1cm} (1)
where \( v_g \) is the volume density of the metric \( g \). Note that \( df \) can be seen as a smooth section of the bundle \( T^*M \otimes f^*TN \) over \( M \), which admits a natural metric induced by \( g \) and \( h \), giving sense to \( \| df \| \). This is the so-called Hilbert-Schmidt norm of \( df \), which can also be described as \( \| df \|^2 = \text{tr}_g(f^*h) \).

**Definition 1.1.** A map \( f : M \to N \) is harmonic if it is a critical point of the energy functional (1).

This means concretely that:

\[
\frac{d}{dt}_{|t=0} E(f_t) = 0
\]

for any smooth deformation \( (f_t): (-\delta, \delta) \times M \to N \) of \( f = f_0 \). Note that one should work with compactly supported deformations when \( M \) is not compact, as the energy could be infinite.

A more tangible characterization of harmonicity is given by the Euler-Lagrange equation for \( E \), which takes the form \( \tau(f) = 0 \) where \( \tau(f) \) is the tension field of \( f \): this is an immediate consequence of the first variational formula below (Proposition 1.3). First we define the tension field. Note that the bundle \( T^*M \otimes f^*TN \) admits a natural connection \( \nabla \) induced by the Levi-Civita connection of \( g \) and the pullback connection of the Levi-Civita connection of \( h \). Hence one can take the covariant derivative \( \nabla(df) \in \Gamma(T^*M \otimes T^*M \otimes f^*TN) \) (we use the notation \( \Gamma \) for the space of smooth sections), also denoted \( \nabla^2 f \). It is easily shown that it is symmetric in the first two factors.

**Definition 1.2.** The vector-valued Hessian of \( f \) is

\[
\nabla^2 f := \nabla(df) \in \Gamma(T^*M \otimes T^*M \otimes f^*TN).
\]

The contraction (trace) of \( \nabla^2 f \) on its first two indices using the metric \( g \) is the tension field of \( f \):

\[
\tau(f) := \text{tr}_g(\nabla^2 f) \in \Gamma(f^*TN).
\]

Note that the vector-valued Hessian generalizes both the usual Hessian (when \( N = \mathbb{R} \)) and the (vector-valued) second fundamental form (when \( f \) is an isometric immersion). Accordingly, the tension field generalizes both the Laplace-Betrami operator and the (vector-valued) mean curvature.

**Proposition 1.3** (First variational formula for the energy). Let \( f : (M, g) \to (N, h) \) be a smooth map and let \( (f_t) \) be a smooth deformation of \( f \). Denote by \( V \in \Gamma(f^*TN) \) the associated infinitesimal deformation defined as \( V_x = \frac{d}{dt}_{|t=0} f_t(x) \). Then

\[
\frac{d}{dt}_{|t=0} E(f_t) = - \int_M \langle \tau(f)_x, V_x \rangle \ dv_g(x)
\]

(2)

where \( h = \langle \cdot, \cdot \rangle \) is the Riemannian metric in \( N \).

One can introduce a natural \( L^2 \) inner product of two infinitesimal deformations \( V, W \in \Gamma(f^*TN) \) (also called vector fields along \( f \)):

\[
\langle V, W \rangle = \int_M \langle V_x, W_x \rangle \ dv_g(x).
\]

(3)

There is in fact a natural smooth structure on \( C^\infty(M, N) \), making it an infinite-dimensional manifold, which identifies the tangent space at \( f \) as

\[
T_f C^\infty(M, N) = \Gamma(f^*TN),
\]

(4)
we refer to [KM97, Chapter IX] for details. With respect to this smooth structure, (3) defines a Riemannian metric on $C^\infty(M,N)$, and (2) can simply be put:

$$\text{grad } E(f) = -\tau(f).$$

Next we can compute the second variation of the energy (as the first variation, this is done in the foundational paper [ES64]):

**Proposition 1.4** (Second variational formula for the energy). Let $(f_s)_t: (-\delta, \delta)^2 \times M \to N$ be a smooth deformation of $f = f_{00}$ on two real parameters $s,t$. Denote by $V$ and $W$ the vector fields along $f$ defined by $V = \frac{\partial f}{\partial s}|_{s=0}$ and $W = \frac{\partial f}{\partial t}|_{t=0}$. Then

$$\frac{\partial^2 E(f_{st})}{\partial s \partial t}|_{s=t=0} = \int_M \left( \langle \nabla V, \nabla W \rangle - \text{tr}_g \left( R^N(df, W)df \right) + \left( \nabla^{\nabla} \frac{\partial f}{\partial s}, \tau(f) \right) \right) \text{d}v_g$$

where $R^N$ is the Riemann curvature tensor\(^1\) on $N$.

When $(f_{st})$ is a geodesic variation, i.e. $f_{st}(x) = \exp_{f(x)}(sv_x + tW_x)$, the third term in the integral vanishes. This yields the formula for the Hessian of the energy functional:

$$\text{Hess}(E)_{ij}(V,W) = \int_M \left( \langle \nabla V, \nabla W \rangle - \text{tr}_g \left( R^N(V, df)df \right) \right) \text{d}v_g.$$

This can also be written $\text{Hess}(E)_{ij}(V,W) = \langle J(V), W \rangle$ using the $L^2$ Riemannian metric (3), where $J(V) = -\text{tr}_g(\nabla^2 V + R^N(V, df)df)$ is the *Jacobi operator*.

### 1.2 Energy functional on $L^2(M, N)$ and more general spaces

The energy functional can be extended to maps that are merely in $L^2(M, N)$. First let us define this function space. The $L^2$-distance between two measurable maps $f_1, f_2: M \to N$ is

$$d(f_1, f_2) = \left( \int_M d(f_1(x), f_2(x))^2 \text{d}v_g(x) \right)^{\frac{1}{2}}.$$

If $f_1, f_2$ are both smooth, this is the distance induced by the $L^2$ Riemannian metric (3), provided there exists a geodesic between $f_1$ and $f_2$. A measurable map $f: M \to N$ is declared in $L^2(M,N)$ when it is within finite distance of a constant map. For $r > 0$, one can then define an approximate $r$-energy of $f \in L^2(M,N)$:

$$E_r(f) = \frac{1}{2} \int_M \int_M \eta_r(x,y) \ d(f(x), f(y))^2 \text{d}v_g(y) \text{d}v_g(x)$$

(7)

where $\eta_r(x,y)$ is some *kernel* that may be chosen as $\eta_r(x,y) = \frac{\mathbf{1}_r(x,y)}{V_m(r)}$, where $V_m(r)$ denotes the volume of a ball of radius $r$ in a Euclidean space of dimension $m = \dim M$ and $\mathbf{1}_r(x,y)$ is the characteristic function of the set $\{(x,y) \in M^2 : d(x,y) < r\}$ in $M \times M$ (see [Jos97, §4.1] for a

\(^1\)In this paper, the curvature tensor is defined by $R(X,Y)Z = \nabla^2_{X,Y}Z - \nabla^2_{Y,X}Z$. Some authors’ convention differs from ours by a minus sign (e.g. [GHL04]).
discussion of the choice of kernel). One can show that the functional $E_r$ is continuous on $L^2(M, N)$. Moreover, the limit:

$$E(f) := \lim_{r \to 0} E_r(f)$$  \hfill (8)

exists in $[0, \infty)$ for every $f \in L^2(M, N)$. The resulting energy functional is lower semi-continuous on $L^2(M, N)$ and coincides with (1) on $C^\infty(M, N)$. A measurable map $f : M \to N$ is declared in the Sobolev space $H^1(M, N)$ if it is in $L^2(M, N)$ and has finite energy. The spaces $L^2_{\text{loc}}(M, N)$ and $H^1_{\text{loc}}(M, N)$ are similarly defined by restricting to compact sets. One can then define a (weakly) harmonic map as a critical point of the energy functional in $H^1_{\text{loc}}(M, N)$. Jost shows that $f \in H^1_{\text{loc}}(M, N)$ is weakly harmonic if and only if it is a weak solution of the Euler-Lagrange equation $\tau(f) = 0$: see [Jos11, Lemma 8.1.2] for details. Moreover, any continuous weakly harmonic map is smooth [Jos11, Theorem 8.4.1] (the continuity assumption can be dropped when $M$ and $N$ are compact and $N$ has nonpositive curvature: [Jos11, Corollary 8.6.1]), and even real-analytic when $M$ and $N$ are real-analytic Riemannian manifolds.

In addition to opening the way for tools from functional analysis, this approach has the advantage of generalizing to much more general spaces than smooth Riemannian manifolds. Indeed, assume $M = (M, \mu)$ is a measure space and $N = (N, d)$ is a metric space. The space $L^2(M, N)$ may be defined as before, and given a choice of kernel function $\eta_r$ for $r > 0$, one can define energy functionals $E_r : L^2(M, N) \to \mathbb{R}$ using (7). For a suitable choice of $\eta_r$ and of a sequence $r_n \to 0$, the energy functional is $E = \lim_{r \to +\infty} E_{r_n}$. More precisely, the choices made must ensure that $E$ is the $\Gamma$-limit of the functionals $E_{r_n}$. We refer to [Jos97, Chap. 4] for details and [DM93] for the general theory of $\Gamma$-convergence. Let us just mention that $\Gamma$-convergence is adequate here because it ensures that minimizers of the approximate energies converge to minimizers the energy. This point of view on the theory of harmonic maps was developed by Jost [Jos94, Jos95, Jos96, Jos97].

A similar approach was developed independently by Korevaar-Schoen [KS93, KS97] that applies to a variety of contexts, including the smooth setting of § 1.1 and the discrete setting that we develop and study in § 2.

1.3 The heat flow

Going back to the smooth setting, assume now that $M$ is compact and $N$ is complete and has nonpositive sectional curvature. The formula for the Hessian of the energy (6) shows that it is nonnegative, in other words $E$ is a convex function on $C^\infty(M, N)$ with respect to the $L^2$ Riemannian metric. This makes it somewhat reasonable to expect existence and in certain cases uniqueness of harmonic maps, which are necessarily energy-minimizing in this setting (we discuss this further in § 3.2). A natural approach to minimize a convex function is the gradient flow method, called heat flow in this setting: given $f_0 \in C^\infty(M, N)$, one can study the initial value problem $\frac{d}{dt} f_t = -\text{grad} E(f_t)$, that is in light of (5):

$$\frac{d}{dt} f_t = \tau(f_t).$$

This flow can be shown to exist for all $t \geq 0$. Moreover, if the range of $f_t$ remains in some fixed compact subset of $N$, then $f_t$ converges to a harmonic map as $t \to \infty$, uniformly and in $L^2(M, N)$ (and, in fact, in $C^\infty(M, N)$). Otherwise, there does not exist a harmonic map homotopic to $f$. In particular, when $N$ is compact, any $f_0 \in C^\infty(M, N)$ is homotopic to a smooth energy-minimizing
harmonic map. Moreover, such a harmonic map is unique, unless it is constant or maps into a totally geodesic flat submanifold of $N$ in which case non-uniqueness is realized by translating $f$ in the flat. These foundational results are due to Eells-Sampson [ES64] and Hartman [Har67].

1.4 Equivariant harmonic maps

Instead of working with maps between compact manifolds, it can be useful to study their equivariant lifts to the universal covers. Indeed, up to being careful with basepoints, any continuous map $f : M \to N$ lifts to a unique $\rho$-equivariant map $\tilde{f} : \tilde{M} \to \tilde{N}$, where $\rho : \pi_1 M \to \pi_1 N$ is the group homomorphism induced by $f$. Note that $\rho$ only depends on the homotopy class of $f$, and if $N$ is aspherical (i.e. $\tilde{N}$ is contractible), then conversely any $\rho$-equivariant continuous map $M \to N$ is the lift of some continuous map $\tilde{M} \to \tilde{N}$ homotopic to $\tilde{f}$.

One advantage of this approach is to enable the following generalization: let $X$ and $Y$ be two Riemannian manifolds, denote by $\text{Isom}(X)$ and $\text{Isom}(Y)$ their respective groups of isometries. Let $\Gamma$ be a discrete group. Given group homomorphisms $\rho_L : \Gamma \to \text{Isom}(X)$ and $\rho_R : \Gamma \to \text{Isom}(Y)$, a map $f : X \to Y$ is called $(\rho_L, \rho_R)$-equivariant if:

$$f \circ \rho_L(\gamma) = \rho_R(\gamma) \circ f$$

for every $\gamma \in \Gamma$. Note that the quotients $X/\rho_L(\Gamma)$ and $Y/\rho_R(\Gamma)$ can be pathological if $\rho_L(\Gamma)$ and $\rho_R(\Gamma)$ do not act freely and properly, but the space of equivariant harmonic maps $X \to Y$ remains ripe for study.

The heat flow approach used by Eells-Sampson [ES64] to show existence of harmonic maps between compact Riemannian manifolds when the target is nonpositively curved has been successfully adapted to the equivariant setting by various authors. It has been found that the adequate condition for guaranteeing existence of equivariant harmonic maps is the reducitivity of the target representation. More precisely, let us quote Labourie’s theorem:

**Theorem 1.5** ([Lab91]). Let $M$ and $N$ be Riemannian manifolds, assume $N$ is Hadamard. Denote by $\rho_L : \pi_1 M \to \text{Isom}(M)$ the action by deck transformations and let $\rho_R : \pi_1 M \to \text{Isom}(N)$ be any group homomorphism. If $\rho_R$ is reducitive, then there exists a $(\rho_L, \rho_R)$-equivariant harmonic map $\tilde{M} \to N$. The converse also holds provided $N$ is without flat half-strips.

Less general versions of this theorem had previously been established by Donaldson [Don87] (for $N = \mathbb{H}^3$) and Corlette [Cor88] (for $N$ a Riemannian symmetric space of noncompact type). The condition without flat half-strips is a mild restriction that is met, for example, when $N$ is a real-analytic Riemannian manifold or when $N$ has negative sectional curvature.

The notion of being reducitive for a group homomorphism $\rho : \pi_1 M \to G$ when $G$ is the isometry group of a Hadamard manifold $N$ (or more generally a Hadamard metric space) can be described algebraically when $N = G/K$ is a Riemannian symmetric space of noncompact type\(^2\). Labourie [Lab91] generalized it to arbitrary Hadamard manifolds. The condition can be expressed simply

---

\(^2\)Specifically, when $G$ is an algebraic group, a subgroup $H \subset G$ is called completely reducible if, for every parabolic subgroup $P \subset H$ containing $H$, there is a Levi subgroup of $P$ containing $H$. Equivalently, the identity component of the algebraic closure of $H$ is a reductive subgroup (with trivial unipotent radical). A $G$-valued group homomorphism $\rho$ is then called reductive (or completely reducible) when its image is a completely reducible subgroup. We refer to [Sik12] for details.
when $N$ has negative curvature: the action $\rho$ on $N$ must either fix no point on the Gromov boundary $\partial_{\infty} N$ or it must preserve a geodesic in $N$.

Wang [Wan00] and Izeki-Nayatani [IN05] then extended the theorem of Labourie to Hadamard metric spaces, using Jost’s extended notion of reductivity [Jos97, Def. 4.2.1]. Less general or different versions of this theorem had previously been established by Gromov-Schoen [GS92], Jost [Jos97, Thm 4.2.1], Korevaar-Schoen [KS97].

1.5 Harmonic maps from surfaces

When $M = S$ is a surface, i.e. $\dim M = 2$, it is easy to check that the energy density $e(f) = \frac{1}{2} \| df \|^2 dv_g$ is invariant under conformal changes of the metric ($g \mapsto e^{2u} g$ with $u \in C^\infty(M, \mathbb{R})$). It follows that the energy functional only depends on the conformal class of $g$, as does the harmonicity of a map $S \to N$. Recall that a conformal structure on an oriented surface $S$ is equivalent to a Riemann surface structure (this follows from the existence of isothermal coordinates for any metric on $S$, a result that goes back to Gauss [Gau25]). Thus it makes sense to talk about the energy functional and harmonic maps $X \to (N, h)$, where $X$ is a Riemann surface (with underlying smooth surface $S$) and $(N, h)$ is a Riemannian manifold. Note however that the $L^2$ Riemannian metric (3) does change under conformal changes of $g$, and therefore the tension field $\tau(f)$ does too. Furthermore, the strong convexity (see § 3.1) of the energy functional only depends on the conformal structure of the domain, though the modulus of strong convexity may depend on the metric in a fixed conformal class.

One can see in fact that the energy density only depends on the complex structure $X$ on $S$ by writing the pullback metric $f^* h$ on $X$. Splitting it into types, one finds that

$$f^* h = \varphi_f + g_f + \bar{\varphi}_f$$

where $\varphi_f = (f^* h)^{(2,0)}$ is a complex quadratic differential on $X$, called the Hopf differential of $f$, and $g_f = (f^* h)^{(1,1)}$ is $e(f)$ (more precisely, $g_f$ is the conformal metric with volume density $e(f)$). The Hopf differential $\varphi_f$ plays a very important role in Teichmüller theory. First note that $f$ is conformal if and only if $\varphi_f = 0$. A key fact is that if $f$ is harmonic, then $\varphi_f$ is a holomorphic quadratic differential on $X$. Wolf [Wol89] proved that the Teichmüller space of $X$ is diffeomorphic to the vector space of holomorphic quadratic differentials on $X$ by taking Hopf differentials of harmonic maps $X \to (S, h)$, where $h$ is a hyperbolic metric on $S$. We refer to [DW07] for more details and a beautiful review of the connections between harmonic maps and Teichmüller theory.

On a closed surface $S$ of negative Euler characteristic, it is convenient to choose the Poincaré metric within a conformal class of metrics: it is the unique metric of constant curvature $-1$ (its existence is precisely the celebrated uniformization theorem). This provides an identification of $\tilde{S}$ with the hyperbolic plane $\mathbb{H}^2$ and an action of $\pi_1(S)$ on $\mathbb{H}^2$ by isometries. Turning this identification around, whenever a Fuchsian (i.e. faithful and discrete) representation $\rho_L : \pi_1(S) \to \text{Isom}^+(\mathbb{H}^2)$ is chosen, we obtain a hyperbolic surface $\mathbb{H}^2/\rho_L(\pi_1(S)) \approx S$. 

11
2 Discretization

We fix some notation: for the remainder of the paper, $S$ is a smooth, closed, oriented surface of negative Euler characteristic (genus $\geq 2$). We denote $\pi_1 S$ the fundamental group of $S$ with respect to some basepoint that can be safely ignored.

In § 2.1 we explain how to approach the energy minimization problem for smooth equivariant maps $\mathbb{H}^2 \to N$ in a way that allows effective computation by introducing meshes and subdivisions, discrete equivariant maps, and discrete energy. Several of these notions will be discussed more thoroughly in the subsequent paper [GLM18]. In the present paper they serve as a preamble to the more formal setting we develop in § 2.2, and they justify the choices made in the software Harmony.

2.1 Meshes and discrete harmonic maps

In this subsection we fix a hyperbolic structure on $S$ given by a Fuchsian representation $\rho_L$, i.e. an injective group homomorphism $\pi_1 S \to \text{Isom}^+(\mathbb{H}^2)$ with discrete image. This setup can be easily generalized to any nonpositively curved metric on $S$, but recall that the notion of harmonicity only depends on the conformal class of the metric. As remarked above, the hyperbolic metric is best suited for explicit computations.

Meshes and subdivisions

Given a group homomorphism $\rho_R : \pi_1 S \to \text{Isom}(N)$ where $N$ is a Riemannian manifold (or possibly a metric space), we would like to discretize $(\rho_L, \rho_R)$-equivariant smooth maps $\mathbb{H}^2 \to N$. To this end, we start by discretizing the domain hyperbolic surface with the notion of invariant mesh:

**Definition 2.1.** A $\rho_L$-invariant mesh of $\mathbb{H}^2$ is an embedded graph $M$ in $\mathbb{H}^2$ such that:

(i) The vertex set $M(0) \subset \mathbb{H}^2$ is invariant under a cofinite action of $\rho_L(\pi_1 S)$.

(ii) Every edge $e \in M(1)$ is an embedded geodesic segment in $\mathbb{H}^2$.

(iii) The complementary components are triangles.

The elements of the vertex set $M(0) \subset \mathbb{H}^2$ are the meshpoints.

For the purpose of approximating smooth maps, we will need to take finer and finer meshes. This will be discussed in detail in [GLM18], but let us describe the strategy that we will adopt there, and have implemented in the software Harmony. A natural way to obtain a finer mesh from a given one is via geodesic subdivision. We indicate below an edge of $M$ with endpoints $x, y \in M(0)$ by $e_{xy}$, and let $m(x, y) \in \mathbb{H}^2$ be the midpoint of $x$ and $y$. It is easy to see that the following is well-defined:

**Definition 2.2.** Let $M$ be a $\rho_L$-invariant mesh. The refinement of $M$ is the $\rho_L$-invariant mesh $M'$ obtained as follows:

(i) The vertices of $M'$ are the vertices of $M$ plus all midpoints of edges of $M$.

(ii) The edges of $M'$ are given by $x \sim m(x, y)$ and $y \sim m(x, y)$ for each edge $e_{xy}$, and $m(x, y) \sim m(x, z)$ for each triple of vertices $x, y, z$ that span a triangle in $M$. 
Evidently, this refinement may be iterated. See Figure 2 for an illustration of a $\rho_L$-invariant mesh and its refinement generated by the software Harmony.

For practical purposes, the $\rho$-invariant geodesic mesh $M$ can be interpreted as a finite data set: we choose a fundamental domain $D \subset M$ for the action of $\rho(\pi_1 S)$ together with side-pairing transformations. Observe that $M$ induces a triangulation of $D$, and the invariance under $\rho(\pi_1 S)$ implies that the set of mesh points of $M$ can be recovered from this triangulation, which consists of a finite set of vertices and edges.

Discrete equivariant maps

**Definition 2.3.** Given a $\rho_L$-invariant mesh $M$ and a group homomorphism $\rho_R : \pi_1 S \to \text{Isom}(N)$, a discrete equivariant map $\mathbb{H}^2 \to N$ along $M$ is a $(\rho_L, \rho_R)$-equivariant map $M^{(0)} \to N$.

Fixing $\rho_L$ and $\rho_R$, let us denote $\text{Map}_M(\mathbb{H}^2, N)$ the set of discrete equivariant maps $\mathbb{H}^2 \to N$ along $M$. Note that $\text{Map}_M(\mathbb{H}^2, N) \approx N^V$ where $V = M^{(0)}/\rho_L(\pi_1 S)$ is the set of equivalence classes of meshpoints (equivalently, $V$ is the set of meshpoints on the hyperbolic surface $\mathbb{H}^2/\rho_L(\pi_1 S)$), which is finite. Therefore we obtain the elementary proposition:

**Proposition 2.4.** If $N$ is a finite-dimensional smooth manifold, then so is $\text{Map}_M(\mathbb{H}^2, N)$.

Denoting $C^0_{\text{eq}}(\mathbb{H}^2, N)$ the space of continuous equivariant maps $\mathbb{H}^2 \to N$, note that we have the forgetful map

$$C^0_{\text{eq}}(\mathbb{H}^2, N) \to \text{Map}_M(\mathbb{H}^2, N)$$

which consists in restricting a continuous function to the set of meshpoints $M^{(0)}$. 

---

Figure 2: An invariant mesh of the Poincaré disk model of $\mathbb{H}^2$ on the left, its refinement of order 1 on the right. The brighter central region is a fundamental domain. The blue circle arcs are the axes of the generators of $\rho_L(\pi_1 S)$. Pictures generated by Harmony.
Definition 2.5. We shall call a left inverse of the forgetful map (9) an interpolation scheme.

While there is one most natural way to interpolate discrete maps between Euclidean spaces (affine interpolation), there is no preferred way to interpolate a discrete map between arbitrary Riemannian manifolds. Even in the easiest case we consider where both the domain and target manifolds are the hyperbolic plane $\mathbb{H}^2$, there are several reasonable interpolations to consider, such as the barycentric interpolation and the harmonic interpolation. However, these are not explicit, and for the purpose of our software Harmony we work with a neat variant, the cosh-center of mass interpolation, that allows direct computations. The cosh-center of mass is discussed in § 5.4; interpolations will be discussed more thoroughly in [GLM18].

Discretization of energy

We turn now to a discretization of the energy functional. For a smooth $(\rho_L, \rho_R)$-equivariant map $f : \mathbb{H}^2 \to N$ where $N$ is a Riemannian manifold, one can define the total energy of $f$ as:

$$E(f) = \int_D \| df \|^2 \mathrm{d}v_g$$

(10)

where $D \subset \mathbb{H}^2$ is any fundamental domain for the action of $\pi_1 S$. If one picks $D$ so that it coincides with a union of triangles defined by the mesh $M$, then the energy integral can be written as a finite sum of energy integrals over each triangle. When $f$ is discretized along the mesh $M$, only the values of $f$ on the meshpoints are recorded. Thus, a natural discretization of $E$ is provided by specifying a way to assign an energy to a map from a triangle whose values are only known at the vertices. A choice of interpolation scheme (cf Definition 2.5) gives a solution to this: just take the energy of the interpolated map.

Definition 2.6. Let $I : \text{Map}_M(\mathbb{H}^2, N) \to C^0_{\text{eq}}(\mathbb{H}^2, N)$ be an interpolation scheme (cf Definition 2.5). The discrete energy of $f \in \text{Map}_M(\mathbb{H}^2, N)$ relative to $I$ is the total energy of $I(f)$.

In order for Definition 2.6 to make sense, we only require that $I(f) : \mathbb{H}^2 \to N$ be piecewise smooth in the sense that it is continuous and smooth in restriction to any triangle relative to $M$. In practice, we will only consider interpolation schemes that send edges of $M$ to geodesic arcs in $N$. The latter assumption is geometrically natural: the energy functional $E_M$ of Definition 2.8 extends to the (usually infinite-dimensional) space of equivariant maps $M \to N$, but any minimizers of $E_M$ will satisfy this assumption about edges (see for example [HS15, Pf. of Prop. 5.5]).

Another approach consists in defining the energy à la Jost/Korevaar-Schoen as in § 1.2. One can take the graph defined by $M$ with metric induced from $\mathbb{H}^2$ as a domain metric space, introduce a measure that approximates the area density of $\mathbb{H}^2$, and choose an appropriate kernel $\eta(x, y)$.

A third approach consists in choosing a discrete energy that is a weighted sum of distances squared as in Definition 2.8 and Definition 2.15. This approach provides a natural extension of the classical notion of real-valued harmonic functions defined on graphs [BH12, Chu97, GR01], that is, functions whose value at any vertex is the average of the values on the neighbors.

It turns out that all three approaches can be made to coincide (Proposition 2.23), or almost coincide for fine meshes (Theorem 2.7), for the appropriate choices involved in the different definitions. This will be discussed in greater detail in the paper [GLM18], but let us quote the following theorem illustrating our claim:
Theorem 2.7 ([GLM18]). Let $T^L \subset \mathbb{H}^2$ and $T^R \subset N$ be geodesic triangles, given with an order on their respective sets of vertices. Denote by $\theta_i$ the interior angles of $T^L$ and denote by $d_i$ the opposite side lengths of $T^R$ ($i \in \{1, 2, 3\}$). Let $\delta$ be any positive real number such that $\delta \leq \min\{\theta_i\}$, and let $D = \max\{\text{diam}(T^L), \text{diam}(T^R)\}$. Then there is a constant $C = C(\delta)$ so that the energy of the barycentric interpolation map from $T^L$ to $T^R$ is approximately equal to

$$\frac{1}{2} \sum_{i=1}^{3} \omega_i \cdot d_i^2$$

with error bounded above by $C(\delta)D^4$, where the weights $\omega_i$ are given by $\omega_i = \cot \theta_i$.

An elementary fact popularized by Pinkall-Polthier [PP93] is that in the Euclidean plane, the barycentric (i.e. affine) map between a pair of triangles has energy given by (11). As the diameter $D \to 0$ the Riemannian metric in the target looks more and more Euclidean; thus it should be no surprise that we recover this expression.

Definition 2.8. Let $M$ be a $\rho_L$-invariant mesh in $\mathbb{H}^2$ such that all the complementary triangles are acute. The discrete energy of a discrete equivariant map $f \in \text{Map}_M(\mathbb{H}^2, N)$ is defined by

$$E_M(f) = \frac{1}{2} \sum_{e \in \mathcal{E}} \omega_e \cdot d(f(x), f(y))^2$$

where:

- $\mathcal{E} \subset M^{(1)}$ is any transversal for the action of $\pi_1 S$ on the set of edges.
- Inside the sum, $x$ and $y$ are the vertices connected by the edge $e$.
- $\omega_e$ is the sum of the cotangents of two angles: one for each of the two triangles sharing the edge $e$, in which we take the angle of the vertex facing the edge $e$.

This definition of discrete energy is a generalization of the energy functional considered by Pinkall-Polthier [PP93], for whom the domain is a triangulated surface with a piecewise Euclidean metric and $N = \mathbb{R}^n$. In their setting, the discrete energy coincides with the energy relative to the linear interpolation (cf. Definition 2.6). In [GLM18] we show that the discrete energy $E_M$ converges to the smooth energy $E$ under iterated refinement of the mesh $M$ in a suitable function space.

Of course, we can now define:

Definition 2.9. A discrete equivariant harmonic map $f \in \text{Map}_M(\mathbb{H}^2, N)$ is a critical point of the discrete energy functional $E_M$.

While several authors have shown the existence and uniqueness of minimizers of the discrete energy in various contexts (e.g. [Wan00, EF01, Mes02]), in this paper we analyze its strong convexity, which makes it better suited for effective minimization. Our approach to strong convexity requires a Riemannian metric on $\text{Map}_M(\mathbb{H}^2, N)$ (cf. § 3.1), which reasonably should approach the $L^2$ Riemannian metric of $C^{\infty}(M, N)$ (cf. § 1.1). We emphasize that such a Riemannian structure is necessary to make sense of both convexity of $E_M$ and a discrete tension field. Though the ‘edge weights’ allow the definition of $E_M$, they are not enough to obtain the needed Riemannian structure. In the next subsection, we develop a more general framework where these ideas apply.
2.2 Equivariant harmonic maps from graphs

The definition of the discrete energy functional $E_M$ (Definition 2.8) is easily generalized to any system of positive weights indexed by the edges of $M$. On the other hand, the Riemannian structure of $\text{Map}_M(\mathbb{H}^2, N)$ requires a measure on the domain: while in the smooth case one has the density of the Riemannian metric, in the discrete case it can be recorded by a system of weights on the vertices. All of this information can be captured using only the graph structure of $M$.

$\tilde{S}$-triangulated graphs

Recall that a triangulation $T$ of a surface is the data of a simplicial complex $K$ and a homeomorphism $h$ from $K$ to the surface. Lifting $T$ to the universal cover, we find a triangulation whose underlying graph (i.e. 1-skeleton) is locally cyclic, meaning that the open neighborhood of any vertex (subgraph induced on the neighbors) is a cycle. This motivates the following definition:

Definition 2.10. Given a topological surface $S$, an $\tilde{S}$-triangulated graph is a locally cyclic graph $G$ with a free, cofinite action of $\pi_1 S$ by graph automorphisms.

Evidently, $\tilde{S}$-triangulated graphs are precisely the graphs that arise as 1-skeleta of triangulations (slight modifications are needed when $S$ has boundary).

When $G$ is an $\tilde{S}$-triangulated graph, we denote the associated group action on the set of vertices by $\rho_L: \pi_1 S \to \text{Aut}(G^{(0)})$. Let $N$ be a metric space and $\rho_R: \pi_1 S \to \text{Isom}(N)$ a group homomorphism.

Definition 2.11. Given $\rho_L$ and $\rho_R$ as above, we call a $(\rho_L, \rho_R)$-equivariant map $G^{(0)} \to N$ an equivariant map from $G$ to $N$. The space of such equivariant maps will be denoted $\text{Map}_{eq}(G, N)$.

As in Proposition 2.4 we have:

Proposition 2.12. If $N$ is a finite-dimensional smooth manifold, so is $\text{Map}_{eq}(G, N)$.

Edge-weighted graphs and the energy functional

Definition 2.13. Let $G$ be an $\tilde{S}$-triangulated graph. We say that $G$ is edge-weighted if it is given a system of edge weights, i.e. a family of positive real numbers $(\omega_e)_{e \in G^{(1)}}$ indexed by the set of edges $G^{(1)}$, that is invariant under the action of $\pi_1 S$.

Clearly, the data of a system of edge weights is equivalent to the data of a function

$$\eta_0: G^{(0)} \times G^{(0)} \to [0, +\infty)$$

that is symmetric, invariant under the diagonal action of $\pi_1 S$, and such that $\eta_0(x, y) > 0$ if and only if $x$ and $y$ are adjacent.

Definition 2.14. A function $\eta_0$ as above is called a pre-kernel on the $\tilde{S}$-triangulated graph $G$.

The motivation for introducing this notion will become clear in Definition 2.22 and Proposition 2.23.

We are now ready to define the energy functional:
Definition 2.15. Let $\mathcal{G}$ be an $\tilde{S}$-triangulated graph with a system of edge weights $(\omega_e)_{e \in \mathcal{G}^{(1)}}$, and let $\rho_R : \pi_1 S \to \text{Isom}(N)$ be a group homomorphism where $N$ is a metric space. The energy functional $E_\mathcal{G} : \text{Map}_\text{eq}(\mathcal{G}, N) \to \mathbb{R}$ is defined by

$$E_\mathcal{G}(f) = \frac{1}{2} \sum_{e \in \mathcal{E}} \omega_e \, d(f(x), f(y))^2$$

(12)

where $\mathcal{E} \subset \mathcal{G}^{(1)}$ is any transversal for the action of $\pi_1 S$, and inside the sum $x$ and $y$ are the endpoints of the edge $e$.

When $N$ is a Hadamard manifold, $E_\mathcal{G}$ is a smooth function on the manifold $\text{Map}_\text{eq}(\mathcal{G}, N)$. Of course we define harmonic maps as:

Definition 2.16. $f \in \text{Map}_\text{eq}(\mathcal{G}, N)$ is called harmonic when it is a critical point of the energy functional $E_\mathcal{G}$.

When $N$ is not a Hadamard manifold but merely a metric space, one can still define (locally) energy-minimizing harmonic maps as points where $E_\mathcal{G}$ has (local) minima.

Note that, taking $\omega_e = 1$ for all $e \in \mathcal{G}^{(1)}$ and $N = \mathbb{R}$, a harmonic map from $\mathcal{G}$ to $\mathbb{R}$ in the sense above coincides with the classical notion of harmonicity for real-valued functions on graphs. The well-known mean value property of harmonic functions generalizes (see e.g. [HS15, Lem. 6.1] or [IN05, Prop. 2.5] for similar observations).

Proposition 2.17. Let $\mathcal{G}$ be an edge-weighted triangulated graph and let $N$ be a metric space. If $f \in \text{Map}_\text{eq}(\mathcal{G}, N)$ is an energy-minimizing harmonic map then $f(x)$ is a center of mass of the weighted system of points $\{(f(y), \omega_{xy})\}_{y \sim x}$ in $N$ for every $x \in \mathcal{G}^{(0)}$.

Proof. If $f(x)$ was not the center of mass of its neighbors, then the summands of (12) that involve $x$ could be decreased, while leaving the others constant. □

Vertex weighted-graphs and the Riemannian structure

Definition 2.18. Let $\mathcal{G}$ be an $\tilde{S}$-triangulated graph. We say that $\mathcal{G}$ is vertex-weighted if it is given a system of vertex weights, i.e. a family of positive real numbers $(\mu_v)_{v \in \mathcal{G}^{(0)}}$ indexed by the set of vertices $\mathcal{G}^{(0)}$, that is invariant under the action of $\pi_1 S$.

We will see a system of vertex weights as a $\pi_1 S$-invariant Radon measure $\mu$ on $\mathcal{G}^{(0)}$ with full support. Of course, $\mathcal{G}^{(0)}$ being discrete and cofinite, this is simply a $\pi_1 S$-invariant function $\mu : \mathcal{G}^{(0)} \to (0, +\infty)$, but we emphasize that our viewpoint for discretization is to approximate the smooth theory; the measure $\mu$ is an approximation of the volume density of a Riemannian manifold, i.e. a Radon measure.

Assume now that $N$ is a finite-dimensional Riemannian manifold and let $\rho_R : \pi_1 S \to \text{Isom}(N)$ be a group homomorphism. We saw (Proposition 2.12) that $\text{Map}_\text{eq}(\mathcal{G}, N)$ is a smooth manifold. Moreover, it is easy to describe its tangent space.

\footnote{A Hadamard manifold is a complete, simply connected Riemannian manifold of nonpositive curvature. On a Hadamard manifold the distance squared function to a fixed point is smooth, while in general it may not be differentiable on the cut locus.}
Proposition 2.19. The tangent space at \( f \in \text{Map}_{\text{eq}}(G, N) \) is:

\[
T_f \text{Map}_{\text{eq}}(G, N) = \Gamma_{\text{eq}}(f^*TN)
\]

where \( f^*TN \) is the pullback of the tangent bundle \( TN \) to \( G^{(0)} \) and \( \Gamma_{\text{eq}}(f^*TN) \) is its space of \( \pi_1S \)-equivariant smooth sections. Equivalently, if \( V \subseteq G(0) \) is any transversal for the action of \( \pi_1S \), we have:

\[
T_f \text{Map}_{\text{eq}}(G, N) = \bigoplus_{x \in V} T_{f(x)}N.
\]

Notice of course the similarity of (13) with (4). Using the measure \( \mu \), one can define a natural \( L^2 \) Riemannian metric on \( \text{Map}_{\text{eq}}(G, N) \) analogous to (3):

Definition 2.20. Let \( G \) be an \( \tilde{S} \)-triangulated graph with a system of vertex weights \( \mu \), and let \( \rho_R : \pi_1S \to \text{Isom}(N) \) where \( N \) is a Riemannian manifold. The \( L^2 \) Riemannian metric on \( \text{Map}_{\text{eq}}(G, N) \) is given by:

\[
\langle V, W \rangle = \int_V \langle V_x, W_x \rangle \, d\mu(x)
\]

where \( V, W \in \Gamma_{\text{eq}}(f^*TN) \) and \( V \subseteq G(0) \) is any transversal for the action of \( \pi_1S \).

Of course, one can write more concretely:

\[
\langle V, W \rangle = \sum_{x \in V} \mu(x) \langle V_x, W_x \rangle.
\]

One can easily derive that the unit speed geodesics in \( \text{Map}_{\text{eq}}(G, N) \) are the one-parameter families of functions \( (f_t(x))_{x \in G^{(0)}} \) given by \( f_t(x) = \exp(tV_x) \), where \( V \in \Gamma_{\text{eq}}(f^*TN) \) is a unit vector, and, provided \( N \) is connected, the Riemannian distance in \( \text{Map}_{\text{eq}}(G, N) \) is simply given by

\[
d(f, g)^2 = \sum_{x \in V} \mu(x) d(f(x), g(x))^2,
\]

where on the right hand-side \( d \) is the Riemannian distance in \( N \).

Biweighted graphs

Definition 2.21. Let \( G \) be an \( \tilde{S} \)-triangulated graph. We say that \( G \) is biweighted if it is both edge-weighted (Definition 2.13) and vertex-weighted (Definition 2.18).

From the discussion of the previous paragraph, when \( G \) is an \( \tilde{S} \)-triangulated biweighted graph and \( N \) is a Riemannian manifold with a group homomorphism \( \rho_R : \pi_1S \to N \), the space of equivariant maps \( \text{Map}_{\text{eq}}(G, N) \) is a Riemannian manifold, and the energy functional is a continuous function \( E_G : \text{Map}_{\text{eq}}(G, N) \to \mathbb{R} \). Moreover \( E_G \) is smooth when \( N \) is Hadamard. In § 3 we show that \( E_G \) is strongly convex under suitable restrictions on \( \rho_R \), with an explicit bound on the modulus of strong convexity (Theorem 3.25). This implies that there exists a unique equivariant harmonic map \( G \to N \) that can be computed effectively through gradient descent (§ 4).

We pause to point out that our definition of the energy functional and harmonic maps in this setting coincides with Jost’s theory briefly described in § 1.2 (we refer to [Jos96, Jos97] for details). First we introduce the kernel function associated to a biweighted graph:
**Definition 2.22.** The kernel function associated to a biweighted graph $G$ is the function

$$\eta: G(0) \times G(0) \to \mathbb{R}$$

$$(x, y) \mapsto \frac{\eta_0(x, y)}{2\mu(x)\mu(y)}$$

where $\eta_0$ is the pre-kernel associated to the underlying edge-weighted graph (cf. Definition 2.14) and $\mu$ is the measure on $G(0)$ giving the vertex weights.

The next proposition is trivial but conceptually significant:

**Proposition 2.23.** The energy functional on $\text{Map}_{eq}(G, N)$ is given by

$$E_G(f) = \frac{1}{2} \int_{\mathcal{V}} \int_{\mathcal{V}} \eta(x, y) d(f(x), f(y))^2 \, d\mu(y) \, d\mu(x)$$

where $\mathcal{V} \subseteq G(0) \times G(0)$ is a transversal for the diagonal action of $\pi_1S$.

**Proposition 2.23** implies that, choosing $\eta_r = \eta$ for all $r > 0$, the Jost energy functional $E = \lim_{r \to 0} E_r$ (compare with (8)) coincides with the energy functional $E_G$. In particular, our notion of harmonic maps from graphs is a specialization of Jost’s generalized harmonic maps.

Next we observe that the Riemannian structure of $\text{Map}_{eq}(G, N)$ allows us to define the discrete tension field as:

**Definition 2.24.** The tension field of $f \in \text{Map}_{eq}(G, N)$ is the vector field along $f$ denoted $\tau_G(f) \in \Gamma_{eq}(f^*TN)$ given by:

$$\tau_G(f)|_x = \frac{1}{\mu(x)} \sum_{y \sim x} \omega_{xy} \exp^{-1}_{f(x)}(f(y))$$

where we have denoted $\omega_{xy}$ the weight of the edge connecting $x$ and $y$.

We have the discrete version of the first variational formula for the energy (Proposition 1.3):

**Proposition 2.25.** The tension field is minus the gradient of the energy functional:

$$\tau_G(f) = -\nabla E_G(f)$$

for any $f \in \text{Map}_{eq}(G, N)$.

**Proof.** In a Riemannian manifold $N$, when $x_0 \in N$ is chosen such that $\exp_{x_0}$ is a diffeomorphism (any $x_0$ works when $N$ is Hadamard), the function $g : x \mapsto \frac{1}{2}d(x_0, x)^2$ is smooth and its gradient is given by $\nabla g(x) = -\exp^{-1}_{x_0}(x_0)$. □

It follows, of course, that an equivariant map $G \to N$ is harmonic if and only if its tension field is zero, and we obtain a characterization of discrete harmonic maps:

**Proposition 2.26.** Let $G$ be an edge-weighted triangulated graph and let $N$ be a Hadamard manifold. Then $f \in \text{Map}_{eq}(G, N)$ is a harmonic map if and only if $f(x)$ is a center of mass of the weighted system of points $\{(f(y), \omega_{xy})\}_{y \sim x}$ in $N$ for every $x \in G(0)$.
We conclude this section by looping back to § 2.1 and the approximation problem. The point is that when $S$ is equipped with a hyperbolic structure (or more generally any nonpositively curved metric), a mesh in the sense of Definition 2.1 induces a biweighted graph structure:

**Definition 2.27.** Let $\rho_L : \pi_1 S \to \text{Isom}^+(\mathbb{H}^2)$ be a Fuchsian representation and let $M$ be an invariant mesh (cf. Definition 2.1). The biweighted graph underlying $M$ is the biweighted graph $G$ such that:

- $G$ is the abstract graph underlying $M$ (which is evidently $\tilde{S}$-triangulated).
- The edge weights are the $\omega_e$ as in Definition 2.8.
- The vertex weights are given by, for every vertex $x$:

$$\mu(x) = \frac{1}{3} \sum_T \text{Area}(T)$$

where the sum is taken over all triangles incident to the vertex $x$.

Clearly, any discrete equivariant map along $M$ from $\mathbb{H}^2$ to a Riemannian manifold $N$ induces an equivariant map $\tilde{G} \to N$, and the energy $E_M$ agrees with the energy $E_G$. Of course, the systems of weights are chosen so that the discrete energy functional $E_G$ approximates the smooth energy functional (10), and the Riemannian structure of $\text{Map}_{\text{eq}}(G, N)$ approximates the $L^2$ Riemannian metric on $C^\infty(M, N)$ (or $L^2(M, N)$), with finer approximation when one takes finer meshes. The analysis of this phenomenon is treated in [GLM18].

**Remark 2.28.** With this construction in mind, biweighted triangulated graphs can be roughly thought of as follows: the edge weights are a discrete record of the conformal structure of $S$, and the vertex weights, the metric structure (or alternatively, the symplectic structure). Note that this is reminiscent of [BPS15]– though distinct– in which two graphs with edge weights are considered conformally equivalent if there is a function of the vertices that scales one set of weights to another.

**Remark 2.29.** Other authors have considered similar discretizations of the theory of harmonic maps, in which the domain is a simplicial complex endowed with admissible weights [Wan00, IN05]. In our language, the latter records only the discrete conformal structure, and not the discrete metric structure. This difference is essential in the effective program we have in mind (i.e. strong convexity of $E_G$), as the latter would be inaccessible without a Riemannian structure on $\text{Map}_{\text{eq}}(G, N)$.

This extra structure is captured for example by Eells-Fuglede, who pursued harmonic maps from Riemannian polyhedra [EF01] (i.e. spaces glued from simplices that are endowed with Riemannian metrics satisfying some regularity conditions), but we remark that asking for the discretization to remember such strong information is too restrictive. For instance, it would make any computer program significantly more computationally expensive, and it is hard to imagine an implementation of this viewpoint that would be practical.

### 3 Strong convexity of the energy

In this section we study the convexity of the discrete energy functional $E_G : \text{Map}_{\text{eq}}(G, \mathbb{H}^2) \to \mathbb{R}$ introduced in the previous section (Definition 2.15). In § 3.1 we recall basics about convexity and strong convexity in Riemannian manifolds. In § 3.2 we review the convexity of the energy
functional for nonpositively curved target spaces. Next we turn to proving the strong convexity of the discrete energy when the target space is $\mathbb{H}^2$ with a Fuchsian representation: we first perform some preliminary computations in the hyperbolic plane in § 3.3, and then prove the main theorem in § 3.4. Finally in § 3.5 we extend this result to more general target spaces, namely Hadamard spaces with pinched negative curvature.

3.1 Convexity in Riemannian manifolds

The classical notion of convexity in Euclidean vector spaces naturally extends to the Riemannian setting—as Udrişte puts it [Udr94, Chapter 1],

Riemannian geometry is the natural frame for convexity.

We shall first give a definition for metric spaces. Recall that geodesics in a metric space $(M,d)$ are harmonic maps from intervals of the real line; more concretely, a curve $\gamma: I \subseteq \mathbb{R} \to M$ in a metric space $(M,d)$ is called a geodesic if $d(\gamma(t_1),\gamma(t_2)) = v|t_2 - t_1|$ for any sufficiently close $t_1, t_2 \in I$, where $v$ is a positive constant. A function on $M$ is then called (geodesically) convex when it is convex along geodesics. More precisely:

**Definition 3.1.** Let $(M,d)$ be a metric space. A function $f: M \to \mathbb{R}$ is convex if, for every geodesic $\gamma: [a,b] \to M$ and for all $t \in [0,1]$:

$$f(\gamma((1-t)a + tb)) \leq (1-t)f(\gamma(a)) + tf(\gamma(b)).$$

When the inequality is strict for all $t \in (a,b)$, $f$ is called strictly convex. Furthermore $f$ is called $\alpha$-strongly convex, where $\alpha > 0$, if:

$$f(\gamma((1-t)a + tb)) \leq (1-t)f(\gamma(a)) + tf(\gamma(b)) - \alpha \frac{(1-t)}{2} l(\gamma)^2$$

where $l(\gamma)$ is the length of $\gamma$ (when $\gamma$ is a minimizing geodesic, this is just $d(\gamma(a), \gamma(b))$). The largest such $\alpha$ is called the modulus of strong convexity of $f$.

When $M = (M,g)$ is a Riemannian manifold and $f$ is $C^2$, one can quickly characterize convex functions in terms of the positivity of their Hessian as a quadratic form. Recall that the Hessian of a $C^2$ function $f: M \to \mathbb{R}$ is the symmetric 2-covariant tensor field on $M$ defined by $\text{Hess}(f) = \nabla(df)$.

**Proposition 3.2.** Let $f: M \to \mathbb{R}$ be a $C^2$ function on a Riemannian manifold $(M,g)$. Then:

- $f$ is convex if and only if it has positive semidefinite Hessian everywhere.
- $f$ is strictly convex if and only if it has positive definite Hessian everywhere.
- $f$ is $\alpha$-strongly convex if and only if it has $\alpha$-coercive Hessian everywhere:

$$\forall v \in TM \quad \text{Hess}(f)(v,v) \geq \alpha \|v\|^2$$

Convex functions enjoy several attractive properties. Among them, we highlight the straightforward fact that any sub-level set of a convex function is totally convex (i.e. it contains any geodesic...
whose endpoints belong to it). Definition 3.1 and Proposition 3.2 work when $M$ is an infinite-dimensional Riemannian manifold (e.g. $C^\infty(M, N)$ as in §1.1), however note that a convex function is not necessarily continuous in that case, whereas it is always locally Lipschitz in finite dimension. We refer to [Udr94, Chap. 3] for convex functions on finite-dimensional Riemannian manifolds. 4

### 3.2 Convexity of the energy functional

We review the convexity of the energy functional when the target is nonpositively curved, whether in the Riemannian sense or in the sense of Alexandrov, and we also address the possibility of strict or strong convexity in these settings.

**Remark 3.3.** While strict convexity of the energy is a clear-cut way to prove uniqueness of harmonic maps and strong convexity their existence, neither are necessary, and the existence and uniqueness of harmonic maps has been properly characterized both in the smooth case and in more general spaces: see § 1.3 and § 1.4.

**Convexity of the energy in the smooth setting**

The second variation of the energy functional in the smooth context was first calculated by Eells-Sampson [ES64] (cf. Proposition 1.4). Indeed, the next proposition follows immediately from (6):

**Proposition 3.4.** Let $M$ be and $N$ be smooth Riemannian manifolds. If $N$ has nonpositive sectional curvature, then the Hessian of the energy functional satisfies:

$$
\forall V \in \Gamma(f^*TN) \quad \text{Hess}(E)_{ff}(V, V) \geq \int_M \|\nabla V\|^2 \, dv_g .
$$

In particular, it is clear that the energy functional is convex. It is tempting to try and get more out of (15): is $E$ strictly convex? Is it strongly convex? Evidently, neither can be true without some obvious restrictions: if $f$ maps into a flat (a totally geodesic submanifold of zero sectional curvature), then the energy is constant along the path that consists in translating $f$ along some constant vector field on the flat. Even when $N$ has negative sectional curvature, whence it has no flats of dimension $> 1$, this issue remains for constant maps and maps into a curve.

However, one can restrict to a connected component of $C^\infty(M, N)$ that does not contain such maps, and there the question becomes interesting. An example of particular interest is when $M$ is compact and $\dim N = \dim M > 1$, in which case the degree of maps is an invariant on the components of $C^\infty(M, N)$, and any component of nonzero degree contains only surjective maps (cf. Lemma 3.17).

When the target is negatively curved, (6) does guarantee strict convexity:

**Proposition 3.5.** Let $M$ be a Riemannian manifold, let $N$ be a Riemannian manifold of negative sectional curvature. Then the energy functional is strictly convex on any connected component of $C^\infty(M, N)$ that does not contain any map of rank everywhere $\leq 1$. 22
Proof. Let \((E_i)\) be a local orthonormal frame in \(M\). The integrand for the Hessian of the energy functional (cf. (6)) is:
\[
\|\nabla V\|^2 - \sum_{i=1}^{n} \langle R^N(V, d f(E_i)) d f(E_i), V \rangle
\]
Note that each term \(\langle R^N(V, d f(E_i)) d f(E_i), V \rangle\) in the sum is nonpositive, and it is nonzero unless \(V\) and \(d f(E_i)\) are collinear. Indeed, when \(V\) and \(d f(E_i)\) are not collinear:
\[
\langle R^N(V, d f(E_i)) d f(E_i), V \rangle = K^N(V, d f(E_i)) \left(\|\|V\|\|^2 \| d f(E_i)\|^2 - \langle V, d f(E_i)\rangle^2\right) < 0
\]
where \(K^N(V, d f(E_i))\) denotes the sectional curvature of the plane spanned by \(V\) and \(d f(E_i)\). If \(\text{Hess}(E)_f(V, V)\) vanishes, then the integrand must vanish everywhere, so that (1) \(\nabla V = 0\) everywhere, and (2) \(d f(E_i)\) and \(V\) must be collinear for every \(i\). From (1) it follows that \(V\) has constant length, and from (2) and the fact that \(V_x \neq 0\) it follows that \(d_x f\) maps into \(\text{span}(V_x)\) for every \(x \in M\). In particular, \(f\) has rank \(\leq 1\) everywhere. \(\square\)

As far as the authors are aware, no sufficient conditions for strong convexity of the energy functional are known in the smooth setting. We conjecture that strong convexity holds when \(N\) has pinched negative curvature. More precisely:

**Conjecture 3.6.** Let \(M\) and \(N\) be smooth Riemannian manifolds with \(M\) compact and \(N\) of sectional curvature \(\leq k < 0\). Then the energy functional is strongly convex on any connected component of \(C^\infty(M, N)\) that does not contain any map of rank everywhere \(\leq 1\).

**Convexity of the energy for more general spaces**

Defining the energy functional with the general approach of Jost sketched in § 1.2, it is straightforward to show that the energy functional is convex when the target space is negatively curved in a suitable sense. Indeed, let \((M, \mu)\) be a measure space and let \((N, d)\) be a Hadamard metric space, i.e. a complete CAT(0) metric space. Recall that a CAT(0) space is a geodesic metric space where any geodesic triangle \(T\) is ‘thinner’ than the triangle \(T’\) with same side lengths in the Euclidean plane—more precisely, the comparison map \(T \rightarrow T’\) is distance nonincreasing. A Hadamard space has the property that the distance squared function
\[
d^2: N \times N \rightarrow \mathbb{R}
\]
is convex (see [BH99] for details). It follows easily that for any choice of nonnegative symmetric kernel \(\eta\) (cf § 1.2), the energy functional \(E_r\) is convex on \(L^2(M, N)\). Furthermore if the energy functional \(E\) on \(L^2(M, N)\) is obtained as a \(\Gamma\)-limit of \(E_r\), then it must also be convex [DM93, Thm 11.1]. In particular, this applies to our energy functional \(E_G\) by way of Proposition 2.23:

**Proposition 3.7.** Let \(G\) be any \(\hat{S}\)-triangulated biweighted graph (Definition 2.21) and let \(N\) be a Hadamard metric space. Then the energy functional \(E_G: \text{Map}_{eq}(G, N) \rightarrow \mathbb{R}\) (Definition 2.15) is convex.
We point out that the convexity is relative to a metric structure on $\text{Map}_{\text{eq}}(G, N)$ which depends on a system of vertex weights (see Definition 2.18), but the fact that the energy is convex (respectively strictly, or even strongly convex) does not depend on the choice of such vertex weights. Evidently, the modulus of strong convexity does depend on such a choice.

We will examine conditions that ensure $E_G$ is strongly convex, first for $N = \mathbb{H}^2$ (Theorem 3.21), then in Hadamard manifolds with pinched negative curvature (Theorem 3.25).

We highlight some important context: Korevaar-Schoen obtained yet another form of convexity of the energy when the domain $M$ is Riemannian. Their energy functional $E$, which coincides with Jost’s for suitable choices [Chi07], satisfies the convexity inequality

$$E(f_t) \leq (1 - t)E(f_0) + tE(f_1) - t(1 - t)\|\nabla d(f_0, f_1)\|^2,$$

(16)

where $(f_t) \in L^2(M, N)$ is a geodesic, i.e. $f_t(x)$ is a geodesic in $N$ for all $x \in M$. This is a weaker analog of Proposition 3.4. It is again tempting to investigate strong convexity when $N$ has pinched negative curvature and $f$ does not have rank everywhere $\leq 1$, but work remains to be done.

In fact, Mese [Mes02] claims both an improvement of the convexity statement (16), as well as the strict convexity of the energy functional at maps of rank $\leq 1$ (i.e. Proposition 3.5). However, neither of these claims is explained as far as we can tell, and there is a mistake in the curvature term of [Mes02, eq. (1)]. (In fairness, this lack of explanation is probably due to Mese’s focus on the significant task of extending the uniqueness of Korevaar-Schoen to the setting where $\partial M = \emptyset$.)

Although we prove strong convexity for biweighted graph domains under appropriate restrictions, we suspect that a more general version of this theorem is true, namely an analog of Conjecture 3.6 for singular spaces. In fact, one can further explore extensions to the equivariant setting, with a suitable condition on the target representation (strengthening the reductivity of Labourie).

### 3.3 Convexity estimates in the hyperbolic plane

In order to study the second variation of the discrete energy for $\mathbb{H}^2$-valued equivariant maps, we first need some convexity estimates for the energy of two and three points in the hyperbolic plane.

We start with a formula for quadrilaterals in $\mathbb{H}^2$. 

![Figure 3](image1.png)  
![Figure 4](image2.png)
**Proposition 3.8.** Let \( A, B, C, D \) be four points in the hyperbolic plane. Let \( \alpha \) and \( \beta \) denote the oriented angles as shown in Figure 3. Then:

\[
\cosh(DC) = \cosh(AB) \left[ \cosh(DA) \cosh(BC) + \sinh(DA) \sinh(BC) \cos \alpha \cos \beta \right] \\
- \sinh(AB) \left[ \cosh(DA) \sinh(BC) \cos \beta + \sinh(DA) \cosh(BC) \cos \alpha \right] \\
- \sinh(DA) \sinh(BC) \sin \alpha \sin \beta .
\]

**Remark 3.9.** This equation holds without restriction on \( \alpha \) and \( \beta \); the angles may be negative or obtuse.

**Proof.** Referring to Figure 4, the hyperbolic law of cosines implies:

\[
\cosh(DC) = \cosh(DA) \cosh(AC) - \sinh(DA) \sinh(AC) \cos(\alpha_2) .
\] (17)

The hyperbolic laws of sines and cosines in the triangle \( ABC \) give

\[
\cosh(AC) = \cosh(AB) \cosh(BC) - \sinh(AB) \sinh(BC) \cos(\beta) , \quad \text{and}
\]

\[
\sinh(AC) \cos(\alpha_2) = \sinh(AC) \cos(\alpha_1 - \alpha)
\]

\[
= \sinh(AC) \cos(\alpha_1) \cos(\alpha) + \sinh(AC) \sin(\alpha_1) \sin(\alpha)
\]

\[
= \sinh(AC) \cos(\alpha) + \sinh(BC) \sin(\beta) \sin(\alpha) .
\] (19)

Moreover, it is a consequence of the two forms of the hyperbolic law of cosines (see e.g. [Rat06, p. 82]) in the triangle \( ABC \) that we have

\[
\sinh(AC) \cos(\alpha_1) = \sinh(AB) \cosh(BC) - \sinh(BC) \cosh(AB) \cos(\beta) .
\] (20)

Equation (20) allows us to rewrite equation (19) as:

\[
\sinh(AC) \cos(\alpha_2) = \left( \sinh(AB) \cosh(BC) - \sinh(BC) \cosh(AB) \cos(\beta) \right) \cos(\alpha)
\]

\[
+ \sinh(BC) \sin(\beta) \sin(\alpha) .
\] (21)

Together (18) and (21) and (17) imply the desired equation. \( \square \)

Next we study the convexity of the energy for two points, which amounts to analyzing the second variation of the half-distance squared function \( \frac{d^2}{dt^2} : \mathbb{H}^2 \times \mathbb{H}^2 \to \mathbb{R} \). We perform this computation in two stages: first we study instead the function \( (\cosh d) - 1 : \mathbb{H}^2 \times \mathbb{H}^2 \to \mathbb{R} \), as it is better suited to computations, and then we relate the second variation of the two functions.

**Proposition 3.10.** Let \( A \) and \( B \) be two points in the hyperbolic plane at distance \( D \). Let \( \vec{u} \) and \( \vec{v} \) be tangent vectors at \( A \) and \( B \) respectively. Let \( A_t = \exp_A(t\vec{u}) \) and \( B_t = \exp_B(t\vec{v}) \) for \( t \in \mathbb{R} \), and consider the function \( F_{AB}(t) = \cosh(d(A_t, B_t)) - 1 \). Then:

\[
\frac{d}{dt}_{t=0} F_{AB}(t) = -\sinh(D) (\|\vec{u}\| \cos \alpha - \|\vec{u}\| \cos \beta)
\]

\[
\frac{d^2}{dt^2}_{t=0} F_{AB}(t) = \cosh(D) (\|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\|\vec{v}\| \cos \alpha \cos \beta) - 2\|\vec{u}\|\|\vec{v}\| \sin \alpha \sin \beta .
\]

where \( \alpha \) (resp. \( \beta \)) is the oriented angle between the oriented geodesic \( AB \) and the vector \( \vec{u} \) (resp. the vector \( \vec{v} \)).
Proposition 3.11. The result follows immediately by taking the first and second derivatives at the sine function is concave, we have

\[ 1 + F_{AB}(t) = \cosh(D) \left[ \cosh(t\|\tilde{u}\|) \cosh(t\|\tilde{v}\|) - \sinh(t\|\tilde{u}\|) \sinh(t\|\tilde{v}\|) \cos \alpha \cos \beta \right] - \sinh(D) \left[ - \cosh(t\|\tilde{u}\|) \sinh(t\|\tilde{v}\|) \cos \beta + \sinh(t\|\tilde{u}\|) \cosh(t\|\tilde{v}\|) \cos \alpha \right] - \sinh(t\|\tilde{u}\|) \sinh(t\|\tilde{v}\|) \sin \alpha \sin \beta. \]

The result follows immediately by taking the first and second derivatives at \( t = 0 \). \( \square \)

**Proposition 3.11.** We keep the same setup as Proposition 3.10, and let \( E_{AB}(t) = \frac{1}{2}d(A_t, B_t)^2 \). Then:

\[ \frac{d^2}{dt^2} E_{AB}(t) = a + b \, D \tanh(D/2) + c \left( D \coth D - D - D \tanh(D/2) \right), \]

where \( a, b, \text{ and } c \geq 0 \) are given by

\[ a = (\|\tilde{u}\| \cos \alpha - \|\tilde{v}\| \cos \beta)^2, \]
\[ b = \|\tilde{u}\|^2 \sin^2 \alpha + \|\tilde{v}\|^2 \sin^2 \beta, \text{ and } \]
\[ c = (\|\tilde{u}\| \sin \alpha - \|\tilde{v}\| \sin \beta)^2. \]

**Proof.** For ease in notation, we leave the subscripts \( AB \) from \( E_{AB} \) and \( F_{AB} \) in what follows. We have \( E(t) = \phi \circ F(t), \) where \( \phi(x) = \frac{1}{2} \left( \arccosh(1 + x) \right)^2. \) It is straightforward to check that

\[ \phi'(\cosh x - 1) = \frac{x}{\sinh x}, \quad \text{and} \quad \phi''(\cosh x - 1) = \frac{\sinh x - x \cosh x}{\sinh^3 x}. \]

Since we have \( E''(0) = \phi''(0)(F'(0))^2 + \phi'(0)F''(0) \), using Proposition 3.10 we find that

\[ E''(0) = \frac{\sinh D - D \cosh D}{\sinh^3 D} \cdot \sinh^2 D \left( \|\tilde{u}\| \cos \alpha - \|\tilde{v}\| \cos \beta \right)^2 \]
\[ + \frac{D}{\sinh D} \left( \cosh D \left( \|\tilde{u}\|^2 + \|\tilde{v}\|^2 - 2 \|\tilde{u}\| \|\tilde{v}\| \cos \alpha \cos \beta \right) - 2 \|\tilde{u}\| \|\tilde{v}\| \sin \alpha \sin \beta \right) \]
\[ = \left( \|\tilde{u}\| \cos \alpha - \|\tilde{v}\| \cos \beta \right)^2 + D \coth D \left( \|\tilde{u}\| \cos \alpha - \|\tilde{v}\| \cos \beta \right)^2 + \]
\[ \left( \|\tilde{u}\|^2 + \|\tilde{v}\|^2 - 2 \|\tilde{u}\| \|\tilde{v}\| \cos \alpha \cos \beta \right) - D \operatorname{csch} D \left( 2 \|\tilde{u}\| \|\tilde{v}\| \sin \alpha \sin \beta \right) \]
\[ = a + b \, D \coth D + (c - b) \, D \operatorname{csch} D. \]

To finish, note that \( \coth D - \operatorname{csch} D = \tanh(D/2) \). \( \square \)

With the goal in mind of determining when the second derivative of \( E_{AB}(t) \) may become small, we will need an elementary estimate:

**Lemma 3.12.** Let \( \varepsilon > 0 \) and \( x \in [-\pi, \pi] \) such that \( 1 - \cos(x) < \varepsilon \). Then \( |x| < \pi \sqrt{\frac{\varepsilon}{2}} \).

**Proof.** Recall that \( 1 - \cos(x) = 2 \sin^2 \left( \frac{x}{2} \right) = 2 \sin^2 \left( \frac{|x|}{2} \right) \). Since \( \frac{|x|}{2} \) is in the interval \([0, \frac{\pi}{2}]\) where the sine function is concave, we have \( \sin^2 \left( \frac{|x|}{2} \right) \leq \frac{2}{\pi} \frac{|x|}{2} \), and the conclusion follows. \( \square \)
The following quantitative control is at the core of strong convexity for $E_G$:

**Proposition 3.13.** Let $E_{AB}(t)$ be the function as in Proposition 3.11. Then $\frac{d^2}{dt^2} E_{AB}(t) \geq 0$. Furthermore, if $\varepsilon > 0$ is such that $\frac{d^2}{dt^2} |_{t=0} E_{AB}(t) < \varepsilon$ then:

(i) $\|\mathbf{u}\| - \|\mathbf{v}\| < \sqrt{\varepsilon}$.

(ii) $\text{pv}(\alpha - \beta) < \frac{\pi}{2} \sqrt{\varepsilon \|\mathbf{u}\| \|\mathbf{v}\|}$.

Note that $\text{pv}(\theta)$ refers to the principal value of the angle $\theta$, an element of $(-\pi, \pi]$.

**Proof.** By rewriting $E''_{AB}(0)$ using Proposition 3.11, and noting that $2b \geq c$ by the Cauchy-Schwarz inequality, we find

$$E''_{AB}(0) = a + b D \tanh \frac{D}{2} + c \left( D \coth D - D \tanh \frac{D}{2} \right) \geq a + c \left( D \coth D - \frac{D}{2} \tanh \frac{D}{2} \right) = a + c \cdot D \coth \frac{D}{2}.$$

Because $x \coth x > 1$, we find that $E''_{AB}(0) \geq a + c$. In particular, $E''_{AB}(0) \geq 0$.

Observe that $a + c$ may be rewritten as

$$a + c = \left( \|\mathbf{u}\| - \|\mathbf{v}\| \right)^2 + 2\|\mathbf{u}\| \|\mathbf{v}\| (1 - \cos(\alpha - \beta)).$$

(22)

Now the inequality $a + c < \varepsilon$ implies that each term is $< \varepsilon$:

$$(\|\mathbf{u}\| - \|\mathbf{v}\|)^2 < \varepsilon$$

(23)

$$2\|\mathbf{u}\| \|\mathbf{v}\| (1 - \cos(\alpha - \beta)) < \varepsilon.$$  

(24)

Inequality (23) clearly gives us (i), while inequality (24) gives us (ii) as a direct application of Lemma 3.12. □

**Remark 3.14.** More geometrically, the quantity $a + c$ of (22) is $\|\mathbf{u} - P_{[BA]} \mathbf{v}\|^2$, where $P_{[BA]} \mathbf{v}$ denotes the parallel transport of $\mathbf{v}$ along the geodesic segment $BA$. Thus the proof above shows that $E''_{AB}(0) > \|\mathbf{u} - P_{[BA]} \mathbf{v}\|^2$.

Now we upgrade the estimates of Proposition 3.13 from edges to triangles. Let $ABC$ be a hyperbolic triangle, and suppose that $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are tangent vectors at $A$, $B$, and $C$ respectively as in Figure 5. Suppose that $A_t = \exp_A(t \mathbf{u})$, $B_t = \exp_B(t \mathbf{v})$, and $C_t = \exp_C(t \mathbf{w})$, and let

$$E_{ABC}(t) = \omega_A E_{BC}(t) + \omega_B E_{AC}(t) + \omega_C E_{AB}(t)$$

for some weights $\omega_A, \omega_B, \omega_C > 0$. 

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Figure 5: Oriented angles for the tangent vectors to triangle $ABC$.

**Proposition 3.15.** We have $\frac{d^2}{dt^2}|_{t=0} E_{ABC}(t) \geq 0$. Furthermore, if $\frac{d^2}{dt^2}|_{t=0} E_{ABC}(t) < \varepsilon$ for $\varepsilon > 0$, then

$$\text{Area}(ABC) < \frac{\pi \sqrt{\varepsilon}}{2} \left( \frac{1}{\sqrt{\omega_A ||\vec{v}|| ||\vec{w}||}} + \frac{1}{\sqrt{\omega_B ||\vec{u}|| ||\vec{w}||}} + \frac{1}{\sqrt{\omega_C ||\vec{u}|| ||\vec{v}||}} \right).$$

The reader should interpret the above statement as vacuous if any of the vectors $\vec{u}$, $\vec{v}$, or $\vec{w}$ vanish.

**Proof.** Nonnegativity is immediate. For the requisite estimates, let $\alpha_A$ and $\beta_A$ indicate the oriented angles from $BC$ to $\vec{v}$ and $\vec{w}$, respectively, and similarly for $\alpha_B$, $\beta_B$, $\alpha_C$, and $\beta_C$ (see Figure 5), and let $\hat{A}$, $\hat{B}$, and $\hat{C}$ indicate the oriented interior angles of $ABC$.

Observe that at each vertex of $ABC$, the labelled angles imply the equalities

$$\beta_B - \alpha_C + \hat{A} = \pi,$$
$$\beta_C - \alpha_A + \hat{B} = \pi,$$
$$\beta_A - \alpha_B + \hat{C} = \pi.$$

Summing these modulo $2\pi$ and rearranging we obtain

$$\text{pv}((\beta_A - \alpha_A) + (\beta_B - \alpha_B) + (\beta_C - \alpha_C)) = \text{Area}(ABC). \quad (25)$$

Of course, if $\frac{d^2}{dt^2}|_{t=0} E_{ABC}(t) < \varepsilon$ then each of $\frac{d^2}{dt^2}|_{t=0} E_{AB}(t)$, $\frac{d^2}{dt^2}|_{t=0} E_{AC}(t)$, and $\frac{d^2}{dt^2}|_{t=0} E_{BC}(t)$ are bounded by $\varepsilon/\omega_A$, $\varepsilon/\omega_B$, and $\varepsilon/\omega_C$, respectively, as they are all nonnegative. **Proposition 3.13** then implies that each of the summands on the left of (25) is controlled. We find that

$$| \text{pv}(\beta_A - \alpha_A) + \text{pv}(\beta_B - \alpha_B) + \text{pv}(\beta_C - \alpha_C) | < \frac{\pi \sqrt{\varepsilon}}{2} \left( \frac{1}{\sqrt{\omega_A ||\vec{v}|| ||\vec{w}||}} + \frac{1}{\sqrt{\omega_B ||\vec{u}|| ||\vec{w}||}} + \frac{1}{\sqrt{\omega_C ||\vec{u}|| ||\vec{v}||}} \right). \quad (26)$$

As the quantity $\text{Area}(ABC)$ is in $[0, \pi]$, (25) and (26) together imply the desired result.
3.4 Strong convexity of the discrete energy in $\mathbb{H}^2$

We are ready to prove strong convexity for $E_G$. For the remainder of the section, we fix an equivariant map $f \in \text{Map}_\text{eq}(G, \mathbb{H}^2)$, we choose a transversal to the action of $\pi_1S$ on $G^{(0)}$ indicated by the vertices $p_1, \ldots, p_n$, and we record the data of $f$ on this transversal via the tuple $(x_1, \ldots, x_n)$ (with the remaining images determined by equivariance). Moreover, indicate the finitely many $\pi_1S$-orbits of edges of $G$ by $\mathcal{E}$, and the finitely many orbits of triangles of $G$ by $\mathcal{T}$.

Recall that $G$ is equipped with vertex- and edge-weights; the former via the measure $\mu$ (Definition 2.18), and the latter via $\omega : \mathcal{E} \to \mathbb{R}_{>0}$ (Definition 2.13). For brevity we introduce the notation $\mu_i = \mu(x_i)$ and $\omega_{ij} = \omega(e)$, where $e \in \mathcal{E}$ is an edge between $x_i$ and $x_j$.

Finally, we fix a unit tangent vector $\bar{v} \in T_f \text{Map}_\text{eq}(G, \mathbb{H}^2)$. With (14) and Definition 2.20 in mind, we record $\bar{v}$ as the tuple $(\bar{v}_1, \ldots, \bar{v}_n)$, where $v_i \in T_{x_i} \mathbb{H}^2$ and

$$1 = \sum_i \mu_i \|\bar{v}_i\|^2. \quad (27)$$

We are after an understanding of the second derivative at $t = 0$ of $E_G \circ \exp_f(t\bar{v})$.

We introduce more notation for clarity: with $f$ and $\bar{v}$ understood, we indicate $E_G \circ \exp_f(t\bar{v})$ by $E_G(t)$, and for each edge $e_{ij} \in E_G$ between points $x_i$ and $x_j$ we fix the notation

$$E_{ij}(t) = \frac{1}{2} d(\exp_{x_i}(t\bar{v}_i),\exp_{x_j}(t\bar{v}_j))^2.$$ 

Let $T_{ijk} \in \mathcal{T}$ denote a triangle whose vertices are the points $p_i$, $p_j$, and $p_k$. We let $E_{ijk}(t) = \frac{1}{2} (\omega_{ij} E_{ij}(t) + \omega_{jk} E_{jk}(t) + \omega_{ki} E_{ki}(t))$ for each triangle $T_{ijk} \in \mathcal{T}$. Evidently, the energy $E_G(t)$ may be rewritten either as a sum over edges of $G$ or over triangles:

$$E_G(t) = \sum_{T_{ijk} \in \mathcal{T}} E_{ijk}(t) = \sum_{e_{ij} \in \mathcal{E}} \omega_{ij} E_{ij}(t). \quad (28)$$

First, we note that if the second variation of $E_G$ is small enough, then Proposition 3.13 implies that the tangent vectors $\bar{v}_i$ all have approximately the same length. Precisely,

**Lemma 3.16.** Given $\varepsilon > 0$, there is a $\delta > 0$ so that if $\frac{d^2}{dt^2} |_{t=0} E_G(t) \leq \delta$ then

$$\left| \frac{\|\bar{v}_i\| - 1}{\sqrt{A}} \right| < \varepsilon$$

for all $i$, where $A = \sum_i \mu_i$.

**Proof.** We temporarily postpone the choice of $\delta$. Let $e_{ij} = \frac{d^2}{dt^2} |_{t=0} E_{ij}(t)$ for each edge $e_{ij} \in \mathcal{E}$, so that $\frac{d^2}{dt^2} |_{t=0} E_G(t) = \sum_{e_{ij} \in \mathcal{E}} e_{ij} e_{ij}$.

Observe that for each edge $e_{ij} \in \mathcal{E}$, by Proposition 3.13 we find that $\|\bar{v}_i\| - \|\bar{v}_j\| \leq \sqrt{\varepsilon_{ij}}$.

Let $U = \min \{ \mu_i \}$, and observe that (27) implies $\|\bar{v}_i\| \leq 1/U$ for all $i$. Therefore we see that

$$\left| \frac{\|\bar{v}_i\|^2 - \|\bar{v}_j\|^2}{\|\bar{v}_i\|^2} \right| = \frac{\|\bar{v}_i\| - \|\bar{v}_j\|}{\|\bar{v}_i\|} \cdot \frac{\|\bar{v}_i\| + \|\bar{v}_j\|}{\|\bar{v}_i\|} \leq \frac{2}{U} \sqrt{\varepsilon_{ij}}$$

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for each edge $e_{ij} \in \mathcal{E}$. As this estimate holds for every edge, for any pair of points $p_i$ and $p_j$ we choose a (non-backtracking) path $p_i = p_{i_0}, p_{i_1}, \ldots, p_{i_r} = p_j$ and observe that by the triangle inequality and the Cauchy-Schwarz inequality we have

$$\|\mathbb{v}_i\|^2 - \|\mathbb{v}_j\|^2 < \frac{2}{U} \sum_{k=1}^r \sqrt{E_{i_k-i_{k+1}}} \cdot \sqrt{1} \leq \frac{2}{U} \left( \sum_{k=1}^r E_{i_k-i_{k+1}} \right)^{1/2} \left( \sum_{k=1}^r 1 \right)^{1/2} \leq \frac{2\sqrt{r\delta}}{U}. $$

Because $\pi_1 \mathcal{S} \acts \mathcal{G}$ acts cofinitely, the quotient has bounded diameter, which we indicate by $D = \text{diam}(\mathcal{G})$. Now the path may be taken with length $r \leq D$, and the above inequality now reads

$$\|\mathbb{v}_i\|^2 - \|\mathbb{v}_j\|^2 < \frac{2\sqrt{D\delta}}{U}$$

for all $i, j$.

Now we fix $i$, and observe that because $\mathbb{v}$ is a unit tangent vector we have

$$1 = \sum_j \mu_j \|\mathbb{v}_j\|^2 = \sum_j \mu_j \|\mathbb{v}_i\|^2 + \sum_j \mu_j \left( \|\mathbb{v}_j\|^2 - \|\mathbb{v}_j\|^2 \right).$$

Rearranging and using (29) we find that

$$A\|\mathbb{v}_i\|^2 - 1 < \frac{2A\sqrt{D\delta}}{U},$$

and hence

$$\left| \|\mathbb{v}_i\| - \frac{1}{\sqrt{A}} \right| < \sqrt{\frac{2\sqrt{D\delta}}{U}}.$$

Taking $\delta = \frac{e^A U^2}{4D}$ proves the claim.

\begin{lemma}
For any $f \in \text{Map}_{\text{eq}}(\mathcal{G}, \mathbb{H}^2)$, we have

$$\sum_{T_{ijk} \in \mathcal{T}} \text{Area}(T_{ijk}) \geq \text{Area}(\mathcal{Y}) = 2\pi|\chi(\mathcal{S})|.$$

\end{lemma}

\begin{proof}
By pasting in triangles to the graph $\mathcal{G}$, the point $f \in \text{Map}_{\text{eq}}(\mathcal{G}, \mathbb{H}^2)$ induces a $\rho$-equivariant map $\mathcal{D} \rightarrow \mathbb{H}^2$ (see §2.1 and §2.2). This map descends to a continuous map $\mathcal{S} \rightarrow \mathcal{S}$ that is homotopic to the identity, so that it is in particular of degree one. Endowing the domain with the smooth structure given by pulling back that of the target, the map we find is a smooth map of degree one, so it follows that the image is full measure. The result is now immediate: the triangles in $\mathcal{T}$ have full measure in $\mathcal{Y}$.

This is enough to produce a lower bound for the second variation:

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Proposition 3.18. There exists an \( \alpha = \alpha(G) > 0 \) so that we have

\[
\frac{d^2}{dt^2} |_{t=0} E_G(t) \geq \alpha.
\]

Proof. Let \( R_{ijk} \) indicate the geodesic triangle with vertices \( x_i, x_j, \) and \( x_k \). By Proposition 3.15, for each triangle \( T_{ijk} \in \mathcal{T} \) we have

\[
\text{Area}(R_{ijk}) < \frac{\pi}{2} \sqrt{\frac{d^2}{dt^2} |_{t=0} E_{ijk}(t)} \cdot K_{ijk} (\vec{v})
\]

with

\[
K_{ijk} (\vec{v}) = \frac{1}{\sqrt{\omega_{ij} \|v_i\| \|v_j\|}} + \frac{1}{\sqrt{\omega_{ik} \|v_i\| \|v_k\|}} + \frac{1}{\sqrt{\omega_{jk} \|v_j\| \|v_k\|}}.
\]

Thus we have

\[
\text{Area}(S) \leq \sum_{T_{ijk} \in \mathcal{T}} \text{Area}(R_{ijk}) \quad \text{by Lemma 3.17}
\]

\[
\leq \frac{\pi}{2} \sqrt{\sum_{T_{ijk} \in \mathcal{T}} \frac{d^2}{dt^2} |_{t=0} E_{ijk}(t)} \cdot \sqrt{\sum_{T_{ijk} \in \mathcal{T}} K_{ijk} (\vec{v})^2}
\]

where we have used the Cauchy-Schwarz inequality in the last line. We indicate

\[
K(\vec{v}) = \sum_{T_{ijk} \in \mathcal{T}} K_{ijk} (\vec{v})^2
\]

and note that the inequality above may be rearranged to

\[
\frac{d^2}{dt^2} |_{t=0} E_G(t) \geq \frac{8|\chi(S)|^2}{K(\vec{v})}.
\]

Thus in order to complete the proof of the claim, it remains to bound \( K(\vec{v}) \). Unfortunately, though we have made the assumption that \( \vec{v} \) is a unit tangent vector to \( \text{Map}_G(\mathcal{G}, \mathbb{H}^2) \), it remains possible that some components of \( \vec{v} \) may vanish, and therefore \( K(\vec{v}) \) may be infinite.

Lemma 3.16 may be brought to bear: There is a \( \delta_0 \) small enough so that if \( \frac{d^2}{dt^2} |_{t=0} E_{ij}(t) \leq \delta_0 \) for all edges \( e_{ij} \in \mathcal{E} \), then

\[
\|v_i\| = \frac{1}{\sqrt{A}} < \frac{1}{2 \sqrt{A}}
\]

where \( A = \sum \mu_i \). Let \( \Omega = \min \{ \omega_{ij} \} \) and \( \varepsilon = \delta_0 \Omega \). By (28) if \( \frac{d^2}{dt^2} |_{t=0} E_G(t) \leq \varepsilon \) we find that

\[
\frac{d^2}{dt^2} |_{t=0} E_{ij}(t) \leq \varepsilon / \Omega = \frac{\delta_0}{\Omega}.
\]
for all edges $x_i \sim x_j$, and hence $\|\vec{v}_i\| \geq \frac{1}{2\sqrt{A}}$ for each $i$.

Now we find that

$$K(\vec{v}) \leq \sum_{T_{ijk} \in T} \frac{36A}{\Omega} = \frac{36A|T|}{\Omega}.$$  

Therefore, we conclude that either $\frac{d^2}{dt^2}|_{t=0} E_{\vec{v}}(t) \geq \varepsilon$ or $\frac{d^2}{dt^2}|_{t=0} E_{\vec{v}}(t) \geq 2\frac{|\chi(S)|^2\Omega}{9AT}$.

**Remark 3.19.** Tracing both the proofs of Lemma 3.16 and Proposition 3.18 above and collecting the relevant constants, we find that

$$\frac{d^2}{dt^2}|_{t=0} E_{\vec{v}}(t) \geq \frac{\Omega}{64} \min \left\{ \frac{U^2}{A^2D} \left| \frac{|\chi(S)|^2}{AT} \right| \right\} =: \alpha \quad (31)$$

gives a lower bound for the modulus of convexity, where $U = \min\{\mu_i\}$, $A = \sum \mu_i$, $\Omega = \min\{\omega_{ij}\}$, $D = \text{diam}(G)$, and $T = |T|$ (number of triangles). Note: if the step (30) is done more carefully using the Cauchy-Schwarz inequality, the denominator of the first term in the minimum would lose one factor of $A$.

Finally, the discrete heat flow will require an upper bound for the Hessian of $E_{\vec{v}}$ (see §4). With the setup as above, such an estimate is apparent.

**Proposition 3.20.** Suppose that $E_{\vec{v}}(0) \leq E_0$. Then we have

$$\frac{d^2}{dt^2}|_{t=0} E_{\vec{v}}(t) \leq \frac{2V}{U} \left( 1 + \sqrt{\frac{E_0}{\Omega}} \coth \sqrt{\frac{E_0}{\Omega}} \right) =: \beta , \quad (32)$$

where $V$ is the maximum valence of vertices of $G$, $U = \min\{\mu_i\}$ is the minimum vertex weight, and $\Omega = \min\{\omega_{ij}\}$ is the minimum edge weight.

**Proof.** It is not hard to see from Proposition 3.11 that we have

$$\frac{d^2}{dt^2}|_{t=0} E_{ij}(t) \leq 2 \left( \|\vec{v}_i\|^2 + \|\vec{v}_j\|^2 \right) \left( 1 + d(x_i, x_j) \coth d(x_i, x_j) \right).$$

Letting $L = \max\{d(x_i, x_j)\}$ we find

$$\frac{d^2}{dt^2}|_{t=0} E_{\vec{v}}(t) = \sum_{e_{ij} \in E} \omega_{ij} \frac{d^2}{dt^2}|_{t=0} E_{ij}(t) \leq \frac{2\Omega}{U} (1 + L \coth L) \sum_{e_{ij} \in E} \left( \mu_i \|\vec{v}_i\|^2 + \mu_j \|\vec{v}_j\|^2 \right) \leq \frac{2\Omega V}{U} (1 + L \coth L) \sum_i \mu_i \|\vec{v}_i\|^2 \leq \frac{2\Omega \Omega V}{U} (1 + L \coth L).$$

The assumption $E_{\vec{v}}(0) \leq E_0$ implies that $\Omega L^2 \leq E_0$, so we are done. $\square$

Together, Proposition 3.20 and Proposition 3.18 imply:
Theorem 3.21. Suppose that \( G \) is a biweighted triangulated graph, \( N = \mathbb{H}^2 \) is the hyperbolic plane and \( \rho : \pi_1 S \rightarrow \text{Isom}^+(\mathbb{H}^2) \) is Fuchsian. The energy functional \( E_G : \text{Map}_{eq}(G, N) \rightarrow \mathbb{R} \) is strongly convex. More precisely,

\[
\forall \bar{v} \quad \alpha ||\bar{v}||^2 \leq \text{Hess}(E_G)(\bar{v}, \bar{v})
\]

where \( \alpha \) is given explicitly by (31). Moreover, on the compact convex set \( \{E \leq E_0\} \subseteq \text{Map}_{eq}(G, N) \),

\[
\forall \bar{v} \quad \text{Hess}(E_G)(\bar{v}, \bar{v}) \leq \beta ||\bar{v}||^2
\]

where \( \beta \) is given explicitly by (32).

3.5 More general target spaces

Many of the steps in the proof of Theorem 3.21 actually hold in much greater generality. For instance, Proposition 3.13 holds verbatim if we replace \( \mathbb{H}^2 \) with any Hadamard manifold. Keeping the same setup as in § 3.3, let \( A, B \in N \), let \( \bar{u} \) and \( \bar{v} \) be vectors in \( T_A N \) and \( T_B N \), respectively, let \( A_t = \exp_A(t\bar{u}) \) and \( B_t = \exp_B(t\bar{v}) \), and let \( E_{AB}(t) = \frac{1}{2} d_N(A_t, B_t)^2 \).

Proposition 3.22. Suppose that \( N \) is a Hadamard manifold. Then \( \frac{d^2}{dt^2} E_{AB}(t) \geq 0 \). Furthermore, if \( \epsilon > 0 \) is such that \( \frac{d^2}{dt^2} |_{t=0} E_{AB}(t) < \epsilon \), then

(i) \[ \left| ||\bar{u}|| - ||\bar{v}|| \right| < \sqrt{\epsilon} \]

(ii) \[ \left| \text{pv} (\alpha - \beta) \right| < \frac{\pi}{2} \sqrt{\frac{\epsilon}{||\bar{u}|| ||\bar{v}||}} \]

where \( \alpha \) (resp. \( \beta \)) are the oriented angles between the geodesic \( AB \) and \( \bar{u} \) (resp. \( \bar{v} \)).

The proof below builds on § 3.3.

Proof. We start by noting that the computations for \( E_{AB}''(0) \) (Proposition 3.10 and Proposition 3.11) simplify dramatically when \( N = \mathbb{R}^2 \). It is not hard to use trigonometry of quadrilaterals (this is the simpler Euclidean version of Proposition 3.8) to see that in this case we have

\[
\frac{d^2}{dt^2} E_{AB}(t) \equiv \frac{1}{2} ||\bar{u} - \bar{v}||^2.
\]

In particular, if \( \bar{u} \neq \bar{v} \) then \( E_{AB}(t) \) is \( \frac{1}{2} ||\bar{u} - \bar{v}||^2 \)-strongly convex. (Note that we are implicitly using the flatness of \( \mathbb{R}^2 \) to speak of \( ||\bar{u} - \bar{v}|| \); a more general analysis would require more care using parallel transport to compare \( \bar{u} \) and \( \bar{v} \).)

Now we turn to the general case. Let \( r, s \in \mathbb{R} \), and consider the quadrilateral through \( A_r, B_r, B_s, \) and \( A_s \). Because Hadamard manifolds are CAT(0), this quadrilateral has a comparison quadrilateral
in \( \mathbb{R}^2 \) with vertices \( A', B', B', A' \). For any \( t \in \mathbb{R} \), as a consequence of (33) we have
\[
d(A_{(1-t)r+ts}, B_{(1-t)r+ts})^2 \leq d(A'_{(1-t)r+ts}, B'_{(1-t)r+ts})^2
\leq (1-t)d(A', B')^2 + td(A', B')^2 - \| \bar{u}' - \bar{v}' \|^2 \frac{2t(1-t)}{4} |r-s|^2
\leq (1-t)d(A, B)^2 + td(A, B)^2 - \| \bar{u}' - \bar{v}' \|^2 \frac{2t(1-t)}{4} |r-s|^2,
\]
where \( \bar{u}' = A' - A' \) and \( \bar{v}' = B' - B' \) in \( \mathbb{R}^2 \), respectively. In terms of \( E_{AB} \) we have
\[
E_{AB}((1-t)r+ts) \leq (1-t)E_{AB}(r) + tE_{AB}(s) - \| \bar{u}' - \bar{v}' \|^2 \frac{2t(1-t)}{4} |r-s|^2.
\]

This is almost an equivalent formulation of the strong convexity of \( E_{AB} \) (see Definition 3.1), however the term \( \| \bar{u}' - \bar{v}' \|^2 \) depends on both \( r \) and \( s \). In fact, this detail is essential, as \( E_{AB} \) may fail to be strongly convex (see Remark 3.23). Moreover, this quantity is naturally associated to the quadrilateral in \( \mathbb{R}^2 \), as opposed to the geometric context in \( N \).

For \( \delta > 0 \) we take \( r = s = \delta \) and \( t = \frac{1}{2} \), and we find
\[
E_{AB}(0) = E_{AB} \left( \frac{1}{2} \delta + \frac{1}{2} (-\delta) \right) \leq \frac{1}{2} E_{AB}(\delta) + \frac{1}{2} E_{AB}(-\delta) - \| \bar{u}' - \bar{v}' \|^2 \frac{\delta^2}{2}.
\]
(We stress the dependence of \( \| \bar{u}' - \bar{v}' \|^2 \) on \( \delta \), though we suppress the excess notation.) Rearranging we find that
\[
\| \bar{u}' - \bar{v}' \|^2 \leq \frac{E_{AB}(\delta) - 2E_{AB}(0) + E_{AB}(-\delta)}{\delta^2}.
\]
(Equation 34)

Evidently, as \( \delta \to 0 \), the righthand side approaches \( \frac{d^2}{dt^2} E_{AB}(t) \).

As for the lefthand side, we may rewrite
\[
\| \bar{u}' - \bar{v}' \|^2 = \| \bar{u}' \|^2 + \| \bar{v}' \|^2 - 2\| \bar{u}' \||\| \bar{v}' \|| \cos(\alpha' - \beta')
= \| \bar{u}_\delta \|^2 + \| \bar{v}_\delta \|^2 - 2\| \bar{u}_\delta \||\| \bar{v}_\delta \|| \cos(\alpha' - \beta'),
\]
where \( \bar{u}_\delta = \exp_{A_{\delta}}^{-1}(A_{\delta}) \) and \( \bar{v}_\delta = \exp_{B_{-\delta}}^{-1}(B_{-\delta}) \) (so that \( \| \bar{u}_\delta \| = \| \bar{u}' \| \) and \( \| \bar{v}_\delta \| = \| \bar{v}' \| \)), and \( \alpha' \) (resp. \( \beta' \)) are the oriented angles (measured in \( \mathbb{R}^2 \)) between the geodesic \( A_{\delta}B_{-\delta} \) and \( \bar{u}' \) (resp. \( \bar{v}' \)). By a well-known theorem of Alexandrov (see [Ale51] or [BH99]), the interior angles of a quadrilateral of \( N \) are smaller than those of the model, and we conclude that \( \alpha_\delta \leq \alpha' \) and \( \pi - \beta_\delta \leq \pi - \beta' \), where \( \alpha_\delta \) and \( \beta_\delta \) are the interior angles at \( A_{-\delta} \) and \( B_{-\delta} \), respectively, of the quadrilateral with vertices \( A_{-\delta}, B_{-\delta}, B_{\delta} \), and \( A_{\delta} \). Thus \( \alpha_\delta - \beta_\delta \leq \alpha' - \beta' \), and we may conclude that
\[
\| \bar{u}' - \bar{v}' \|^2 \geq \| \bar{u}_\delta \|^2 + \| \bar{v}_\delta \|^2 - 2\| \bar{u}_\delta \||\| \bar{v}_\delta \|| \cos(\alpha_\delta - \beta_\delta).
\]

Using the latter for the lefthand side of (34) and taking the limit as \( \delta \to 0 \), we find that
\[
\| \bar{u}' \|^2 + \| \bar{v}' \|^2 - 2\| \bar{u}' \||\| \bar{v}' \|| \cos(\alpha - \beta) \leq \frac{d^2}{dt^2} E_{AB}(t).
\]

Now the hypothesis \( \frac{d^2}{dt^2} E_{AB}(t) < \varepsilon \) implies that
\[
\| \bar{u}' \|^2 + \| \bar{v}' \|^2 - 2\| \bar{u}' \||\| \bar{v}' \|| \cos(\alpha - \beta) < \varepsilon.
\]

Note that this is precisely the quantity \( a+c \) from (22), and the remainder of the proof is as before.
Remark 3.23. Remarkably, though we exploit convexity properties of the distance squared function, it is in general false that $E_{AB}$ is strongly convex: one could have $\bar{u}$ and $\bar{v}$ of equal length and parallel to the geodesic $AB$. (Of course the latter occurs as well for $N = \mathbb{R}^2$, since in this case $\bar{u} - \bar{v} = 0$.) Slightly more surprisingly, strong convexity may fail differently in negative curvature. For instance, when $N = \mathbb{H}^2$ one could have $\bar{u}$ and $\bar{v}$ of equal length and both pointing towards a fixed point on $\partial_{\infty} \mathbb{H}^2$. It is an exercise in hyperbolic geometry to show that these are the only situations in which $E_{AB}$ is not strongly convex.

The other crucial piece of strong convexity for $E_{G}$ is Proposition 3.15. The latter step also admits a straightforward generalization using the Gauss-Bonnet theorem:

**Proposition 3.24.** Suppose that $N$ is a Hadamard manifold of curvature bounded above by $\kappa < 0$. We have $\frac{d^2}{dt^2} \left. E_{ABC}(t) \right|_{t=0} \geq 0$. Furthermore, if $\frac{d^2}{dt^2} \left. E_{ABC}(t) \right|_{t=0} < \varepsilon$ for $\varepsilon > 0$, then

$$\text{Area}(ABC) < \frac{\pi \sqrt{\varepsilon}}{2\kappa} \left( \frac{1}{\sqrt{\omega_A \|\bar{v}\| \|\bar{w}\|}} + \frac{1}{\sqrt{\omega_B \|\bar{u}\| \|\bar{w}\|}} + \frac{1}{\sqrt{\omega_C \|\bar{u}\| \|\bar{v}\|}} \right)$$

This in turn makes a generalization of Theorem 3.21 to other negatively curved target manifolds $N$ possible, provided one has a target representation $\rho : \pi_1 S \to \text{Isom}^+(N)$ that allows a lower bound for the sum of areas in $N$ of the triangles of $G$. For instance, when $\rho$ factors through a $\pi_1$-injection induced by $S \to Y$, where $Y$ is a compact manifold whose universal cover is isometric to $N$, any sequence of maps with second variation approaching zero would have a uniformly convergent subsequence whose limit has zero second variation, via the Arzelà-Ascoli theorem. Because the fundamental group of the limit’s image is a surface group (and, in particular, not free) one finds that the image of the limiting map must have positive area, contradicting Proposition 3.24. We obtain:

**Theorem 3.25.** Let $G$ be be a biweighted $\tilde{S}$-triangulated graph, let $N$ be a Hadamard manifold of pinched negative curvature and suppose that $\rho : \pi_1 S \to \text{Isom}(N)$ is a faithful representation whose image is contained in a discrete subgroup of $\text{Isom}(N)$ acting freely, properly, and cocompactly. Then the energy functional $E_{G} : \text{Map}_{eq}(G, N) \to \mathbb{R}$ is strongly convex.

4 Discrete heat flow

4.1 Gradient descent in Riemannian manifolds

The area of mathematics concerned with methods for finding the minima of a convex function $F : \Omega \to \mathbb{R}$, called convex optimization, has been intensely developed in the last few decades and finds countless applications, including in machine learning. The majority of the existing literature deals with the classical case where $\Omega$ is a (convex) subset of a Euclidean space; the more general case where $\Omega$ is a subset of a Riemannian manifold has been far less explored although it is a very natural and useful extension. Udrişte’s book [Udr94] is a good standard reference for Riemannian convex optimization (see e.g. [AMS08], [ZS16] for more recent developments). Our goal here is to present a simple but effective method that can be implemented to find the minimum of the discrete energy functional, with a rigorous proof of convergence and explicit control of the convergence
rate. Of course, there are more sophisticated and faster algorithms for Riemannian optimization in practice. For example, the C++ library ROPTLIB was developed specifically for this purpose (see [HAGH16] for details).

A gradient descent method is an iterative algorithm for minimizing a function \( F: \Omega \subseteq \mathbb{R}^N \to \mathbb{R} \) which produces a sequence \( (x_k)_{k\geq0} \) of points in \( \Omega \), defined inductively by the equation:

\[
x_{k+1} = x_k - t_k \text{ grad } F(x_k).
\]  

(35)

In this relation, \( t_k \in \mathbb{R} \) is a chosen stepsize. If \( F \) has good convexity properties (such as being strongly convex), then a small enough fixed stepsize \( t_k = t \) guarantees convergence of the sequence \( (x_k) \) to a minimum of \( F \), with explicit control of the convergence rate.

The gradient descent method naturally extends to the setting of Riemannian geometry, i.e. when \( F: \Omega \subseteq M \to \mathbb{R} \) is defined on a subset \( \Omega \) of a Riemannian manifold \( M \), in which case the inductive relation (35) should be understood as

\[
x_{k+1} = \exp_{x_k}(-t_k \text{ grad } F(x_k)),
\]  

(36)

where \( \exp \) denotes the Riemannian exponential map (recall that by definition, \( t \mapsto \exp_x(tv) \) is the geodesic through \( x \) with initial tangent vector \( v \)). As in the Euclidean setting, \( -\text{ grad } F(x_k) \) is the direction of steepest descent (maximum decrease rate) for \( F \) at \( x_k \), so it is natural to look for \( x_{k+1} \) in the geodesic ray based at \( x_k \) given by this direction. Note that, as in the Euclidean case, the gradient descent method can simply be described as Euler’s method for the gradient flow ODE

\[
x'(t) = -\text{ grad } F(x(t)).
\]

Gradient descent method with fixed stepsize for strongly convex functions

The gradient descent method with fixed stepsize remains valid for \( C^2 \) strongly convex functions on Riemannian manifolds:

**Theorem 4.1 ([Udr94, Chap. 7, Theorem 4.2]).** Let \( (M, g) \) be a complete Riemannian manifold and let \( F: M \to \mathbb{R} \) be a function of class \( C^2 \). Assume that there exists \( \alpha, \beta > 0 \) such that the Riemannian Hessian of \( F \) satisfies:

\[
\forall v \in TM \quad \alpha \|v\|^2 \leq (\text{Hess } F)(v, v) \leq \beta \|v\|^2
\]

Then \( F \) has a unique minimum \( x^* \). Furthermore, for \( t \in (0, \frac{1}{\beta}] \), the gradient descent method with fixed stepsize \( t_k = t \) converges to \( x^* \) with a linear convergence rate:

\[
d(x_k, x^*) \leq c q^k
\]

for all \( k \geq 0 \), where \( d \) is the Riemannian distance. The constants \( c \geq 0 \) and \( q \in [0, 1) \) are given by:

\[
c = \sqrt{\frac{2}{\alpha} (F(x_0) - F(x^*))} \quad q = \sqrt{1 - \frac{t}{2} \beta \left(1 + \frac{\alpha}{\beta}\right)}.
\]  

(37)
One of the key steps in the proof of Theorem 4.1 is to show that the sequence \((F(x_k) - F(x^*))_{k \geq 0}\) is nonincreasing and limits to 0 with a linear convergence rate itself. In particular, the sequence \((F(x_k))_{k \geq 0}\) is nonincreasing, which implies that any sublevel set \(\Omega = \{ x \in M : F(x) \leq F(x_0) \}\) is stable under the gradient descent method. Moreover, such a set \(\Omega\) is convex and compact by strong convexity of \(F\). In fact, the gradient descent method is valid for a strongly convex \(C^2\) function \(F\) on \(M\) even if the Hessian of \(F\) is not bounded above: one can always restrict to a sub-level set \(\Omega\) where the Hessian of \(F\) is bounded. Consequently, the sequence \((F(x_k))_{k \geq 0}\) remains in \(\Omega\), and the proof of Theorem 4.1 applies verbatim.

Gradient descent method with optimal stepsize for strongly convex functions

There are many variants of the gradient descent method, including in the Riemannian setting, that can be more or less useful depending on the context (see e.g. [ZS16], [FLP18]).

One of them is the optimal stepsize gradient descent, an instance of the gradient descent method (36) where one performs a line search in order to determine the optimal choice of the stepsize \(t_k\), i.e. the one that minimizes \(F(x_{k+1})\). It is clear that when \(F\) is strongly convex, such a \(t_k\) exists, is unique, and is \(> 0\) (unless \(x_k = x^*\)). This algorithm can be significantly faster than the fixed step gradient descent provided one can efficiently find the optimal step \(t_k\) at every iteration. When the Hessian of \(F\) is known analytically, Newton’s method offers an extremely fast line search (with quadratic convergence).

The optimal stepsize gradient descent may in fact converge slower than the fixed step gradient descent in some cases, but nonetheless its convergence rate is at least as fast as that guaranteed by Theorem 4.1.

**Theorem 4.2.** Let \((M, g)\) be a complete Riemannian manifold and let \(F : M \to \mathbb{R}\) be a \(C^2\) strongly convex function as in Theorem 4.1. The optimal stepsize gradient descent has a linear convergence rate at least as fast as the fixed stepsize gradient descent convergence rate specified in Theorem 4.1, whatever the choice of the fixed stepsize.

**Proof.** Theorem 4.2 is easily derived from a careful analysis of the proof of Theorem 4.1 which can be found in [Udr94, Chater 7, Theorem 4.2]. In summary, the proof is a combination of three elementary observations:

(i) For any \(x \in M\):

\[
\frac{\alpha}{2} d(x, x^*)^2 \leq F(x) - F(x^*) \leq \frac{\beta}{2} d(x, x^*)^2 .
\]  

This follows from a Taylor expansion of \(F\) at \(x^*\) along the geodesic \([x^*, x]\).

(ii) For any \(x \in M\):

\[
\| \text{grad} F(x) \|^2 \geq \alpha \left( 1 + \frac{\alpha}{\beta} \right) (F(x) - F(x^*)) .
\]

This follows from a Taylor expansion of \(F\) at \(x\) along the geodesic \([x, x^*]\) and from (38).

(iii) For any \(x \in M\) and for any \(t \in [0, \frac{1}{\beta}]\):

\[
F(x) - F(x^*(t)) \geq \frac{t}{2} \| \text{grad} F(x) \|^2
\]

where we have denoted \(x^*(t) = \exp_x(-t \text{grad} F(x))\). Again this can be proven using a Taylor formula for \(F\) along the segment \([x, x^*(t)]\).
It follows immediately from these three observations that for any \( x \in M \) and for any \( t \in [0, \frac{1}{\beta}] \):

\[
F(x^+(t)) - F(x^+) \leq Q(t) (F(x) - F(x^+))
\]

(39)

where \( Q(t) = 1 - \frac{t}{2} q (1 + \frac{q}{\beta}) = q^2 \).

When one performs the gradient descent method with fixed stepsize \( t \), by assumption \( x_{k+1} = x_k^+(t) \), Theorem 4.2 is then easily concluded by finding \( F(x_k^+(t)) \leq F(x_k) \) from (39) (with an obvious induction) and making one last use of (38).

If instead one performs an optimal stepsize gradient descent, then \( x_{k+1} = x_k^+(t_k) \), where \( t_k \) is the optimal step. Fix \( t \in [0, \frac{1}{\beta}] \). By definition of the optimal step, \( F(x_k^+(t_k)) \leq F(x_k^+(t)) \), so \( F(x_{k+1}) - F(x_k) \leq F(x_k^+(t)) - F(x_k) \). Therefore we can derive from (39) that

\[
F(x_{k+1}) - F(x^+) \leq Q(t) (F(x_k) - F(x^+))
\]

and the conclusion follows like before. \( \square \)

4.2 Convergence of the discrete heat flow

The discrete heat flow can be described as a discretization both in time and space of the heat flow on \( C^\infty(M, N) \). Recall that the smooth heat flow is the gradient flow of the smooth energy functional:

\[
\frac{d}{dt} f_t = \tau(f_t)
\]

(40)

where \( \tau(f_t) = -\text{grad} E(f_t) \) is the tension field of \( f_t \) (cf § 1.1 and § 1.3). A gradient descent for minimizing the energy would consist in doing an Euler method to solve the infinite-dimensional ODE (40), i.e. a time discretization of the heat flow.

We recall the setup of our discretization: Let \( \mathcal{G} \) be a biweighted \( \tilde{S} \)-triangulated graph (see Definition 2.21), let \( Y \) be a Riemannian manifold, and let \( \rho: \pi_1 S \to \text{Isom}(Y) \) be a group homomorphism. Recall that the discrete energy is a function

\[
E_G: \text{Map}_{eq}(\mathcal{G}, Y) \to \mathbb{R},
\]

where \( \text{Map}_{eq}(\mathcal{G}, Y) \) is the space of \( \rho \)-equivariant maps \( \mathcal{G} \to Y \). The latter space has a natural Riemannian structure with respect to which the gradient of the energy is minus the discrete tension field \( \tau_\mathcal{G} \) (see Definition 2.20 and Proposition 2.25). Thus we may define the discrete heat flow:

**Definition 4.3.** The discrete heat flow is the iterative algorithm which, given \( f_0 \in \text{Map}_{eq}(\mathcal{G}, Y) \), produces the sequence \( (f_k)_{k \in \mathbb{N}} \) in \( \text{Map}_{eq}(\mathcal{G}, Y) \) defined inductively by the relation

\[
f_{k+1}(x) = \exp_{f_k(x)} (-t_k (\tau_\mathcal{G} f_k)_x),
\]

where \( t_k \in \mathbb{R} \) is a chosen stepsize, and \( \exp \) is the Riemannian exponential on \( \text{Map}_{eq}(\mathcal{G}, Y) \).

The main theorem of this section is an immediate application of Theorem 3.25 and Theorem 4.1:
**Theorem 4.4.** Let $G$ be a biweighted $\check{S}$-triangulated graph. Assume that $N$ is a Hadamard manifold of pinched negative curvature and $\rho: \pi_1 S \to \text{Isom}(N)$ is a faithful representation whose image is contained in a discrete subgroup acting freely, properly, and cocompactly on $N$. Then there exists a unique $\rho$-equivariant harmonic map $f^*: G \to N$. Moreover, for any $f_0 \in \text{Map}_{eq}(G, N)$ and for any sufficiently small $t > 0$, the discrete heat flow with initial value $f_0$ and fixed stepsize $t$ converges to $f^*$ with a linear convergence rate:

$$d(f_k, f^*) \leq c q^k \quad (41)$$

where $c > 0$ and $q \in (0, 1)$ are constants, and $d(f_k, f^*)$ is the $L^2$ distance in $\text{Map}_{eq}(G, N)$.

Of course, it also follows from Theorem 4.2 that the discrete heat flow with optimal stepsize converges to $f^*$ as well, with a linear convergence rate at least as fast as (41).

We emphasize that in our favorite setting where $N = \mathbb{H}^2$ and $\rho: \pi_1 S \to \text{Isom}^+(\mathbb{H}^2)$ is Fuchsian, Theorem 3.21 enables explicit estimates on the constants $c$ and $q$ in (41): the expressions of $c$ and $q$ are given by (37), in which $\alpha$ is given by (31) and $\beta$ is given by (32) with $E_0 = E(f_0)$.

### 4.3 Experimental comparison of convergence rates

In Figure 6 and Figure 7 we present some numerical experiments performed with Harmony (see § 6): in the first we compare the convergence rate of the discrete heat flow for different stepsizes, and in the second we compare our three methods (discrete heat flow with fixed stepsize, discrete heat flow with optimal stepsize, and center of mass method).

**Comparison of different fixed stepsizes**

We observe the number of iterations required for the discrete heat flow with fixed stepsize to converge (more precisely, to approximate the energy minimizer within some given tolerance) as a function of the stepsize.

Let $S$ be a closed oriented surface of genus 2. We choose a fixed Fuchsian representation $\rho_L: \pi_1 S \to \text{Isom}^+(\mathbb{H}^2)$ for the domain: the representation pictured on the left in Figure 9, and let Harmony construct an invariant mesh (depth 4, 1921 vertices). We let the target Fuchsian representation $\rho_R: \pi_1 S \to \text{Isom}^+(\mathbb{H}^2)$ vary: we choose Fenchel-Nielsen coordinates of lengths $(2, 2, \ell)$ and twists $(-1.5, 2, 0.5)$, where $\ell$ is variable taking the values: 2.5 (red), 1.5 (green), 0.5 (blue), and 0.2 (gold). For each, the number of iterations is recorded as the stepsize varies between 0.01 and 0.054. The dotted line $t = .55$ is the value of a stepsize that makes the fixed step gradient method unstable for all four target surfaces.

We observe that the plotted points resemble in profile functions of the form $-C_1 (\log(1 - C_2 t))^{-1}$, which is exactly the type of function predicted by Theorem 4.4.

**Comparison of our three methods**

For the second experiment (Figure 7) we compare the convergence rate, in terms of number of iterations, of our three methods:
Figure 6: Number of iterations against stepsize in the discrete heat flow with fixed stepsize performed by Harmony.

Figure 7: Comparison of the three methods performed by Harmony.
• Discrete heat flow with fixed stepsize (see § 4.2),
• Discrete heat flow with optimal stepsize (see § 4.2),
• Cosh-center of mass method (see § 5.3).

We keep the same setting as before, letting \( \ell \) this time vary between 0.2 and 4.4. As the figure shows, the cosh-center of mass method is significantly more effective than either gradient descent methods. In fact, its superiority is even more striking in terms of time of execution, even though that is a less objective measure. The optimal stepsize discrete heat flow, on the other hand, turns out to be the slowest in practice. That being said, we have yet to make a serious effort towards optimization of Harmony, and it is likely that these results will change (especially time of execution).

5 Center of mass methods

In this section we investigate a center of mass algorithm towards the minimization of the discrete energy. In some sense, the latter is another variant of the fixed stepsize discrete heat flow. As it turns out, in the current state of our program Harmony the most effective method for convergence to a harmonic map uses the center of mass viewpoint (see § 4.3).

First we recall some facts about centers of mass in Riemannian manifolds and explain how they relate to harmonic maps. Before describing another algorithm to minimize \( E_G \) (Theorem 5.21), we first prove as motivation a generalized mean value property for harmonic maps between Riemannian manifolds (Theorem 5.9). This statement is interesting in its own right, as it adds context to the general principle that harmonic maps are ‘well-balanced’.

5.1 Centers of mass in metric spaces and Riemannian manifolds

Let \((\Omega, \mathcal{F}, \mu)\) be a probability space, \((X, d)\) be a metric space, and \(h: \Omega \to X\) a measurable map. In a typical situation, \(\Omega\) is a finite subset of \(X\), \(\mu\) is a collection of weights indexed by \(A\), and \(h\) is the inclusion map.

Definition 5.1. A center of mass (or barycenter) of \(h\) is a minimizer of the function

\[
P_h: X \to \mathbb{R},
\quad x \mapsto \frac{1}{2} \int_{\Omega} d(x, h(y))^2 \, d\mu(y).
\]  

In general, neither existence nor uniqueness of centers of mass hold. If \(X\) is a Hadamard space and \(h \in L^2(\Omega, X)\) then existence and uniqueness do hold [KS93, Lemma 2.5.1]. For Riemannian manifolds we have:

Theorem 5.2 (Karcher [Kar77]). Assume that \(X\) is a complete Riemannian manifold and \(h\) takes values in a ball \(B = B(x_0, r) \subset X\) such that:

• \(B\) is strongly convex: any two points of \(B\) are joined by a unique minimal geodesic \(\gamma: [0, 1] \to X\), and each such geodesic maps entirely into \(B\).
\( B \) has nonpositive sectional curvature, or \( r < \frac{\pi}{4\sqrt{K}} \) where \( K > 0 \) is an upper bound for the sectional curvature in \( B \).

Under these conditions, the function \( P_h \) of (42) only has interior minimizers on \( \tilde{B} \) and is strongly convex inside \( B \). Consequently, existence and uniqueness of the center of mass hold.

Note that if \( X \) is a complete Riemannian manifold, any sufficiently small \( r > 0 \) meets the requirements of Theorem 5.2. The center of mass of \( h \) is characterized as the unique point \( G \in X \) such that

\[
\int_{\Omega} \exp_{G}^{-1}(h(y)) \, d\mu(y) = 0.
\]  

This simply expresses the vanishing at \( G \) of the gradient of the function \( P_h : X \to \mathbb{R} \) of (42).

Of course, Definition 5.1 generalizes the usual notion of center of mass in \( \mathbb{R}^n \) (or more generally in an affine space): when \( h \in L^2(\Omega, \mathbb{R}^n) \), the center of mass given by the formula

\[
G = \int_{\Omega} h(y) \, d\mu(y).
\]

When \( X \) is not Euclidean, the center of mass is only defined implicitly (by equation (43)), but one can estimate proximity to the center of mass as follows:

**Lemma 5.3.** Let \( h : \Omega \to X \) be a measurable map where \((\Omega, \mathcal{F}, \mu)\) is a probability space. Assume \( X \) is a Hadamard manifold and \( h \) is bounded, or more generally that the conditions of Theorem 5.2 are satisfied. In particular the center of mass \( G \) of \( h \) is well-defined. If \( G' \) is a point in \( X \) such that:

\[
\left\| \int_{\Omega} \exp_{G}^{-1}(h(y)) \, d\mu(y) \right\| < \delta
\]

then

\[
d(G, G') \leq C \delta.
\]

where \( d \) denotes here the Riemannian distance in \( X \), and \( C > 0 \) is a constant that can be taken as \( C = 1 \) when \( X \) has nonpositive curvature, or \( C = C(K, r) \) when \( X \) has sectional curvature bounded above by \( K > 0 \) in a strongly convex ball of radius \( r \) containing the image of \( h \).

**Proof.** This is an immediate consequence of the fact that the function \( P_h \) of (42) is \( C \)-strongly convex under the assumptions of the lemma. Also see [Kar77, Thm 1.5]. \( \square \)

### 5.2 Generalized mean value property

In this section we show a generalized mean value property for smooth harmonic maps between Riemannian manifolds. Let \( f : (M, g) \to (N, h) \) be a smooth map between Riemannian manifolds.

Fix \( x \in M \). Denote by:

- \( S_r \) (resp. \( B_r \)) the sphere (resp. the closed ball) centered at the origin of radius \( r \) in the Euclidean vector space \((T_x M, g)\).
- \( \hat{S}_r \) (resp. \( \hat{B}_r \)) the sphere (resp. the closed ball) centered at \( x \) of radius \( r \) in \( M \), with the induced metric from \( g \). Note that \( \hat{S}_r = \exp_x(S_r) \) and \( \hat{B}_r = \exp_x(B_r) \).
The topological space $S_r$ (resp. $B_r$) can be equipped with a natural Borel probability measure by taking the measure induced from the Euclidean metric $g$ in $T_xM$, renormalized so that it has total mass 1. Similarly, $\mathcal{S}_r$ (resp. $\mathcal{B}_r$) can be equipped with a natural Borel probability measure by taking the measure induced from the Riemannian metric $g$.

**Definition 5.4.** We define four functions $S_r f, B_r f, \mathcal{S}_r f, \mathcal{B}_r f : M \to N$ as follows. Using the notations above, given $x \in M$:

- $S_r f(x)$ is the center of mass of the function $f \circ \exp_x : S_r \to N$.
- $B_r f(x)$ is the center of mass of the function $f \circ \exp_x : B_r \to N$.
- $\mathcal{S}_r f(x)$ is the center of mass of the function $f|_{\mathcal{S}_r} : \mathcal{S}_r \to N$.
- $\mathcal{B}_r f(x)$ is the center of mass of the function $f|_{\mathcal{B}_r} : \mathcal{B}_r \to N$.

**Remark 5.5.** The four functions of Definition 5.4 are well-defined as long as $(N, h)$ is a Hadamard manifold, or as long as $r$ is small enough and $(N, h)$ has sectional curvature bounded above and injectivity radius bounded below by a positive number (e.g. $N$ is compact).

Note that $S_r f(x)$ and $\mathcal{S}_r f(x)$ are different in general, as are $B_r f(x)$ and $\mathcal{B}_r f(x)$. However the following proposition shows that they are very close:

**Proposition 5.6.** Let $f : M \to N$ be a smooth map. Then for all $x \in M$:

\[
\begin{align*}
    d(S_r f(x), \mathcal{S}_r f(x)) &= O(r^4) \\
    d(B_r f(x), \mathcal{B}_r f(x)) &= O(r^4)
\end{align*}
\]

**Proof.** The proof is technical but not very difficult. It is basically derived from a Taylor expansion of the metric in normal coordinates at $x$ and one use of Lemma 5.3. We will do several similar proofs in what follows, so we skip the details for brevity. \qed

Of course, in the case where $M = \mathbb{R}^m$ and $N = \mathbb{R}$ (or $N = \mathbb{R}^n$), $S_r f$ and $\mathcal{S}_r f$ (resp. $B_r f$ and $\mathcal{B}_r f$) coincide. We recall the celebrated mean property for harmonic functions in this setting:

**Theorem 5.7.** $f : \mathbb{R}^m \to \mathbb{R}$ is harmonic if and only if $S_r f = B_r f = f$ for all $r > 0$.

More generally, if $M$ is any Riemannian manifold and $N = \mathbb{R}$, Willmore [Wil50] proved that $\mathcal{S}_r f = f$ characterizes harmonic maps if and only if $M$ is a harmonic manifold.

The central theorem of this subsection is the following:

**Theorem 5.8.** Let $f : M \to N$ be a smooth map. For all $x \in M$, the following estimates holds as $r \to 0$:

\[
\begin{align*}
    d\left(S_r f(x), \exp_{f(x)} \left( \frac{r^2}{2m} \tau(f) x \right) \right) &= O(r^4) \\
    d\left(B_r f(x), \exp_{f(x)} \left( \frac{r^2}{10} \tau(f) x \right) \right) &= O(r^4)
\end{align*}
\]

where $d$ denotes the Riemannian distance in $N$, $m = \dim M$, and $\tau(f)$ is the tension field of $f$.

The following “generalized mean property for harmonic functions between Riemannian manifolds” is an immediate corollary of Theorem 5.8:
Theorem 5.9. Let \( f: M \to N \) be a smooth map. The following are equivalent:

(i) \( f \) is harmonic.
(ii) \( d(f(x), S_r f(x)) = O(r^4) \) for all \( x \in M \).
(iii) \( d(f(x), B_r f(x)) = O(r^3) \) for all \( x \in M \).

Here \( d \) denotes the Riemannian distance in \( N \).

Remark 5.10. It is an immediate consequence of Proposition 5.6 that Theorem 5.8 and Theorem 5.9 also hold for \( \hat{S}_r f(x) \) instead of \( S_r f(x) \), and \( \hat{B}_r f(x) \) instead of \( B_r f(x) \).

In the remainder of this subsection we show Theorem 5.8. We shall only prove (44); the proof of (45) follows exactly the same lines. In what follows, we consider a smooth map \( f: (M, g) \to (N, h) \) and fix \( x \in M \).

Lemma 5.11. Let \( r > 0 \). Denote by \( S_r \) the Euclidean sphere of radius \( r > 0 \) in \( T_x M \) and \( \sigma_r \) its area density. Then, as \( r \to 0 \), the following estimate holds:

\[
\frac{1}{\text{Area}(S_r)} \int_{S_r} \exp^{-1}_{f(x)} \circ f \circ \exp_x(u) \, d\sigma_r(u) = \frac{r^2}{2m} \tau(f)_x + O(r^4)
\]

where \( m = \dim M \).

Proof. Let us write the Taylor expansion of the function \( \hat{f} = \exp^{-1}_{f(x)} \circ f \circ \exp_x : T_x M \to T_{f(x)} N \):

\[
\hat{f}(u) = \hat{f}(0) + (D \hat{f})_0(u) + \frac{1}{2} \langle D^2 \hat{f} \rangle_0(u, u) + \frac{1}{6} \langle D^3 \hat{f} \rangle_0(u, u, u) + O(||u||^4)
\]

We are now going to integrate this identity over \( S_r \). We have:

\begin{itemize}
  \item \( \hat{f}(0) = 0 \), so \( \frac{1}{\text{Area}(S_r)} \int_{S_r} \hat{f}(0) \, d\sigma_r(u) = 0 \).
  \item \( (D \hat{f})_0(u) = (df)_x(u) \) is an odd function of \( u \), so \( \frac{1}{\text{Area}(S_r)} \int_{S_r} (D \hat{f})_0(u) \, d\sigma_r(u) = 0 \).
  \item It is straightforward to check that \( (D^2 \hat{f})_0(u, u) = (\text{Hess } f)_x(u, u) \) by definition of Hess \( f \).
\end{itemize}

Moreover, since this is a quadratic function of \( u \), we can apply Lemma 5.12:

\[
\frac{1}{\text{Area}(S_r)} \int_{S_r} (D^2 \hat{f})_0(u, u) \, d\sigma_r(u) = \frac{r^2}{m} \tau((\text{Hess } f)_x) = \frac{r^2}{m} \tau(f)_x.
\]

\begin{itemize}
  \item \( (D^3 \hat{f})_0(u, u, u) \) is an odd function of \( u \), so \( \frac{1}{\text{Area}(S_r)} \int_{S_r} (D^3 \hat{f})_0(u, u, u) \, d\sigma_r(u) = 0 \).
\end{itemize}

Putting all this together, we get (46). \( \square \)

The following lemma is required to complete the proof of Lemma 5.11:

Lemma 5.12. Let \( V, g = \langle \cdot, \cdot \rangle \) be a Euclidean vector space and let \( B: V \times V \to \mathbb{R} \) be a symmetric bilinear form. Denote by \( S_r = S(0, r) \) the sphere centered at the origin in \( V \) with radius \( r > 0 \), \( d\sigma_r \) the area density on \( S_r \) induced from the metric \( g \) and \( \text{Area}(S_r) = \int_{S_r} d\sigma_r \) its area. Then:

\[
\frac{1}{\text{Area}(S_r)} \int_{S_r} B(x, x) \, d\sigma_r = \frac{r^2}{\dim V} \tau_B.
\]

Here we have denoted by \( \tau_B \) the \( g \)-trace of \( B \), i.e. the trace of the \( g \)-self adjoint endomorphism of \( V \) associated to \( B \), or, equivalently, the trace of a matrix representing \( B \) in a \( g \)-orthonormal basis.

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Proof. Let \((e_1, \ldots, e_n)\) be basis of \(V\) which is \(g\)-orthonormal and \(B\)-orthogonal (the existence of such a basis is precisely the spectral theorem). Let \(A_k = B(e_k, e_k)\) for \(k \in \{1, \ldots, n\}\). For any vector \(x = \sum_{k=1}^{n} x_k e_k\), the quadratic form is given by \(B(x, x) = \sum_{k=1}^{n} A_k x_k^2\), hence:

\[
\int_{S_r} B(x, x) \, d\sigma_r = \sum_{k=1}^{n} A_k \int_{S_r} x_k^2 \, d\sigma_r .
\]

The integrals \(I_k = \int_{S_r} x_k^2 \, d\sigma_r\) can be swiftly computed using a little trick, starting with the observation that any two of these integrals are equal. Indeed, for any \(k \neq l\), one can easily find a linear isometry \(\varphi \in O(V)\) such that \(x_k^2 \circ \varphi = x_l^2\) (namely, the orthogonal reflection through the line spanned by \(e_k + e_l\)). Since \(\varphi\) preserves \(\sigma_r\), the change of variables theorem ensures that \(I_k = I_1\). Since all the integrals \(I_k\) are equal, one can write \(I_k = \frac{1}{n} \sum_{l=1}^{n} I_l\) for any \(k\). That is \(I_k = \frac{1}{n} \int_{S_r} (\sum_{k=1}^{n} x_k^2) \, d\sigma_r\). However \(\sum_{k=1}^{n} x_k^2 = g(x, x) = r^2\) for any \(x \in S_r\). This yields \(I_k = \frac{1}{n} \int_{S_r} r^2 \, d\sigma_r = \frac{\pi}{n^2} \text{Area}(S_r)\). The desired result follows. \(\square\)

It is easy to see that Theorem 5.8 follows immediately from Lemma 5.11 when \(N = \mathbb{R}^n\). When \(N\) is not Euclidean, centers of mass in \(N\) are only defined implicitly (by equation (43)), so we have to work harder to prove Theorem 5.8. The trick is to use Lemma 5.3.

First we need a Riemannian geometry estimate in the following general setting. Let \(A, B, C\) be three points in a Riemannian manifold \((M, g)\). We assume that \(B\) and \(C\) are contained in a sufficiently small ball centered at \(A\) for what follows to make sense. Denote by \(\bar{u}_A = (\exp_A)^{-1}(B)\), \(\bar{u}_B = (\exp_B)^{-1}(C)\), and \(\bar{u}_C = (\exp_C)^{-1}(A)\). If we were in a Euclidean vector space, we could write:

\[
\bar{u}_A + \bar{u}_B + \bar{u}_C = 0 .
\]

We would like to find an approximate version of this identity when the metric is not necessarily flat. Note that the sum \(\bar{u}_A + \bar{u}_B + \bar{u}_C\) does not even make sense, because these three vectors are based at different points. Let us denote \(\bar{v}\) the parallel transport of \(\bar{u}_C\) along the geodesic segment \([C, A]\) and \(\bar{w}\) the parallel transport of \(\bar{u}_B\) along the geodesic segment \([B, A]\). Note that \(\bar{v}\) is simply \((\exp_A)^{-1}(C)\), but \(\bar{w}\) is more “mysterious”. Let us also write \(\bar{u} = \bar{u}_A\) for aesthetics. Now the vectors \(\bar{u}, \bar{v}, \bar{w}\) are all based at \(A\), and one expects that \(\bar{w} = \bar{v} - \bar{u}\) up to some error term.

**Lemma 5.13.** Using the setting and notations above, the following estimate holds

\[
\bar{w} = \bar{v} - \bar{u} + O(||\bar{u}||^2 ||\bar{v}|| + ||\bar{u}|| ||\bar{v}||^2)
\]

as \(||\bar{u}|| \to 0\) and \(||\bar{v}|| \to 0\).

In fact, let us show the following more precise lemma:

**Lemma 5.14.** Let \((M, g)\) be a Riemannian manifold, fix \(A \in M\). Let \(\bar{U}\) and \(\bar{V}\) be two tangent vectors at \(A\), denote \(B(t) = \exp_A(t\bar{U})\) and \(C(s) = \exp_A(s\bar{V})\). Let \(\bar{w}(t, s)\) be the parallel transport of \(\bar{u}_B := \exp_{B(t)}^{-1}(C(s))\) along the geodesic segment from \(B(t)\) to \(A\). Then:

\[
\bar{w}(t, s) = s\bar{V} - t\bar{U} - \frac{t^2 s}{2} R(\bar{V}, \bar{U})\bar{U} - \frac{t s^2}{3} R(\bar{U}, \bar{V})\bar{V} + O(t^4 + t^3 s + t^2 s^2) .
\]

where \(R\) is the Riemann curvature tensor of \((M, g)\).
Proof. First let us quickly discuss some general Riemannian geometry estimates in normal coordinates. We refer to [Bre96, Bre09] for more details on the computations that follow.

In normal coordinates at a point $A$, the Riemannian metric $g$ has the Taylor expansion

$$g_{ij} = \delta_{ij} - \frac{1}{3} R_{ijkl} x^k x^l + O(|x|^3)$$

where $R_{ijkl}$ is the Riemann curvature tensor at $A$, or rather its purely covariant version. One can derive from the expression for the Christoffel symbols $\Gamma^{k}_{ij} = \frac{1}{2} g^{kl} (g_{li,j} + g_{lj,i} - g_{ij,l})$ that

$$\Gamma^{k}_{ij} = -\frac{1}{3} (R^{k}_{ijkl} - R^{k}_{jili}) x^l + O(|x|^3).$$

One can then find the Taylor expansion of any geodesic $x(s)$, say with initial endpoint $x = x(0)$ and initial velocity $v$, by solving the geodesic equation $rac{d^2 x^k}{ds^2} + \Gamma^{k}_{ij} \frac{dx^i}{ds} \frac{dx^j}{ds} = 0$. One finds:

$$x^k(s) = x^k + sv^k - \frac{s^2}{3} R^{k}_{ijil} v^i v^j x^l + O(|s|^3). \quad (47)$$

We can rewrite (47) as a coordinate-free expression (but still in the chart given by $exp_A$) as

$$x(s) = x + sv - \frac{s^2}{3} R(x, v) v + O(|s|^3). \quad (48)$$

One can also compute the parallel transport of a vector $v$ along a radial geodesic $x(t) = tx$ by solving the parallel transport equation $\frac{dv^k}{dr} + \Gamma^{k}_{ij} (x(t)) v^i \frac{dx^j}{dr} = 0$. One finds that

$$v^k(t) = v^k + \frac{1}{6} R^{k}_{jili} v^i x^j x^l(t) + O(t|x|^3),$$

which we can rewrite as

$$v(t) = v + \frac{1}{6} R(v, x) x + O(t|x|^3). \quad (49)$$

Let us now come back to the setting of Lemma 5.14. We shall work (implicitly) in the chart given by $exp_A$. Note that we can write $B = t\bar{U}$ and $C = s\bar{V}$ in this chart. Let us denote by $x(\cdot)$ the unit speed geodesic from $B$ to $C$, so that $x(0) = B, x(r) = C$ where $r = d(B, C)$, and $x'(0) = \bar{U}_B$ is the unit vector such that $exp_B (r\bar{U}_B) = C$. By (48) we can write

$$x(r) = x(0) + r\bar{U}_B - \frac{r^2}{3} R(x(0), \bar{U}_B) \bar{U}_B + O(r|x(0)|^3).$$

In other words, recalling that $x(0) = B = t\bar{U}$ and $x(r) = C = s\bar{V}$, we have

$$s\bar{V} - t\bar{U} = r\bar{U}_B - \frac{r^2}{3} R(\bar{U}, \bar{U}_B) \bar{U}_B + O(r^3). \quad (50)$$

On the other hand, the parallel transport of $\bar{U}_B = r\bar{U}_B$ back to the origin along the radial geodesic $[A, B]$ is given by, according to (49):

$$\bar{w} = r\bar{U}_B - \frac{r^2}{6} R(\bar{U}_B, \bar{U}_B) \bar{U}_B + O(t^3 r). \quad (51)$$

---

8This well-known fact of Riemannian geometry goes back to Riemann’s original 1854 habilitation work [Ric13].
Comparing (50) and (51), we see that
\[
\tilde{\omega} = s \tilde{V} - t \tilde{U} - \frac{t^2 r}{6} R(\tilde{U}_B, \tilde{U}) \tilde{U} - \frac{t r^2}{3} R(\tilde{U}, \tilde{U}_B) \tilde{U} + O(r^3) .
\] (52)

Finally, let’s work to have \( s \)’s and \( \tilde{V} \)’s appear in this equation instead of \( r \)’s and \( \tilde{U}_B \)’s. First note that
\[
r \tilde{R}(\tilde{U}_B, \tilde{U}) \tilde{U} = s \tilde{V} - t \tilde{U} + O(tr^2 + rt^3)
\]
and using the fact that \( R(\tilde{U}, \tilde{U}) = 0 \), one can write:
\[
\begin{align*}
R(\tilde{U}_B, \tilde{U}) \tilde{U} &= sR(\tilde{V}, \tilde{U}) \tilde{U} + O(tr^2 + rt^3) \\
R(\tilde{U}, \tilde{U}_B) \tilde{U} &= s^2 R(\tilde{U}, \tilde{V}) \tilde{U} + tsR(\tilde{U}, \tilde{V}) \tilde{U} + O(ts^2r^2 + t^2sr + r^3 + r^3 + r^4r) .
\end{align*}
\]
We thus get in lieu of (52):
\[
\tilde{\omega} = s \tilde{V} - t \tilde{U} - \frac{t^2 s}{2} R(\tilde{V}, \tilde{U}) \tilde{U} - \frac{t s^2}{3} R(\tilde{U}, \tilde{V}) \tilde{V} + O(t^3 r + r^2 sr^3) .
\]
The conclusion follows, noting that \( r = O(t + s) \) by the triangle inequality. \( \square \)

**Remark 5.15.** A direct consequence of Lemma 5.14 is a formula for the expansion of the distance squared function:
\[
d^2(\exp_A(t \tilde{U}), \exp_A(s \tilde{V})) = \|s \tilde{V} - t \tilde{U}\|^2 - \frac{1}{3} R(U, V, V, U)s^2t^2 + O((t^2 + s^2)^\frac{5}{2}) .
\]
The same formula has been observed by other authors, see e.g. [Raz15].

We are now ready to wrap up the proof of Theorem 5.8:

**Proof of Theorem 5.8.** We prove that \( d(S_r f(x), T_x^S f(x)) = O(r^4) \). It is straightforward to adapt the proof for \( B_r f \) instead of \( S_r f \).

Fix \( x \in M \). Let \( u_0 \in S_r \subset T_x M \), and consider the triangle in \( N \) with vertices \( A = f(x), B = T_r f(x), \) and \( C(u_0) = f(\exp_A(u_0)) \) in \( N \). With the notations introduced above Lemma 5.13, note that we have \( \tilde{u} = \tilde{u}_A = \frac{t^2}{2} r(f)_x, \tilde{v}(u_0) = f(u_0) = \exp_{f(x)}^1(f(\exp_A(u_0))), \) and \( \tilde{w}(u_0) = P(\tilde{u}_B(u_0)) \) where \( \tilde{u}_B(u_0) = \exp_{f(x)}^{-1}(f(\exp_A(u_0))) \) and \( P: T_A N \rightarrow T_B N \) is the parallel transport along the geodesic segment \([A, B] \). By Lemma 5.13, we have
\[
P(\tilde{u}_B(u_0)) = \tilde{v}(u_0) - \tilde{u} + O(\|\tilde{u}\|^2 \|\tilde{v}(u_0)\| + \|\tilde{u}\| \|\tilde{v}(u_0)\|^2) .
\] (53)

Because we have \( \|\tilde{u}\| = O(r^2) \) and \( \|\tilde{v}(u_0)\| = O(r) \), (53) may be rewritten:
\[
P(\tilde{u}_B(u_0)) = \tilde{v}(u_0) - \tilde{u} + O(r^4) .
\] (54)

We now integrate (54) over \( u_0 \in S_r \):
\[
\frac{1}{\text{Area}(S_r)} \int_{S_r} P(\tilde{u}_B(u_0)) \, d\sigma_r(u_0) = \frac{1}{\text{Area}(S_r)} \int_{S_r} \left( \tilde{v}(u_0) \, d\sigma_r(u_0) - \tilde{u} + O(r^4) \right) ,
\]
which we can rewrite as
\[
P \left( \frac{1}{\text{Area}(S_r)} \int_{S_r} \tilde{u}_B(u_0) \, d\sigma_r(u_0) \right) = \left( \frac{1}{\text{Area}(S_r)} \int_{S_r} \tilde{v}(u_0) \, d\sigma_r(u_0) \right) - \tilde{u} + O(r^4) .
\]
Now, Lemma 5.11 says precisely that \( \frac{1}{\text{Area}(S)} \int_{S_r} v(u_0) \, d\sigma_r(u) = \tilde{u} + O(r^4) \). We thus get

\[
\int_{S_r} \exp_{T_{u_0}}^{-1}(f(\exp_{u_0}(u_0))) \, d\sigma_r(u_0) = O(r^4).
\]

Recalling that \( S_r f(x) \) is by definition the center of mass of the function \( u_0 \in S_r \mapsto f(\exp_{u_0}(u_0)) \), we can apply Lemma 5.3 to conclude that

\[
d(S_r f(x), T_r f(x)) = O(r^4)
\]

where \( d \) is the Riemannian distance in \( N \).

\[\square\]

### 5.3 Center of mass methods

Here we discuss center of mass methods as an alternative to the heat flow in order to minimize the energy functional. The basic idea is to iterate the process of replacing a function \( f : M \to N \) by its average on balls (or spheres) of radius \( r > 0 \), hopefully converging to a map \( f^* \) that is almost harmonic when \( r \) is small. Observe that Theorem 5.8 shows that this iterative averaging process is very close to a constant step gradient flow for the energy functional (for the right choice of stepsize), i.e. an Euler method with fixed stepsize.

The next proposition is claimed in [Jos11, Lemma 4.1.1].

**Proposition 5.16.** Let \((M, \mu)\) be a measure space, let \((N, d)\) be a Hadamard metric space, let \( \eta : M \times M \to [0, +\infty) \) be a measurable symmetric function, and suppose that \( \eta(x, \cdot) \mu \) has finite total mass for all \( x \). Define the Jost energy functional by

\[
E(f) = \frac{1}{2} \int_M \int_M \eta(x, y) d\mu(x) \, d\mu(y) \, d\mu(x).
\]

(55)

For a measurable map \( f : M \to N \), let \( \varphi(f) : M \to N \) be the map such that for all \( x \in M \), \( \varphi(f)(x) \) is the center of mass of \( f \) for the measure \( \eta(x, \cdot) \mu \).

Then for every measurable \( f : M \to N \) with finite energy we have:

\[
E(\varphi(f)) \leq E(f) .
\]

(56)

Moreover, the following are equivalent:

(i) Equality holds in (56).

(ii) \( \varphi(f) = f \) almost everywhere.

(iii) \( f \) is a minimizer of \( E \).

**Center of mass method in the smooth setting**

Now assume that \( M \) and \( N \) are both Riemannian manifolds. For \( r > 0 \) we take the kernel \( \eta_r(x, y) = \frac{1}{r^m V_m(r)} \) as described in § 1.2, so that \( E_r \) is the \( r \)-approximate energy functional. Using the notation of Proposition 5.16, the map \( \varphi(f) \) is the same as the map \( \hat{B}_r f \) that we introduced in
**Definition 5.4.** It is tempting to iterate the process of averaging \( f \) (replacing \( f \) by \( \hat{B}_r f \)) in order to try and minimize \( E_r \). The following theorem guarantees the success of this method under suitable conditions:

**Theorem 5.17.** Let \( M \) and \( N \) be Riemannian manifolds, assume \( N \) is compact and with nonpositive sectional curvature. For any continuous \( f: M \to N \), recall that \( \hat{B}_r f \) is the map obtained by averaging \( f \) on Riemannian balls of radius \( r \) (see Definition 5.4). Let \( f_0: M \to N \) be a continuous map such that the \( r \)-approximate energy \( E_r \) admits a unique minimizer \( f^* \) in the homotopy class of \( f_0 \). Then the sequence \( (f_k)_{k \in \mathbb{N}} \) defined by \( f_{k+1} = \hat{B}_r f_k \) converges locally uniformly to \( f^* \).

**Proof.** We reduce the proof to a combination of Lemma 5.18 and Lemma 5.19 below. Let us denote by \( X \) the connected component of \( f_0 \) in \( C(M, N) \), let \( E: X \to \mathbb{R} \) denote the restriction of \( E_r \), and let \( \varphi: X \to X \) be the map \( f \mapsto \hat{B}_r f \). Note that \( f_k = \varphi^k(f_0) \). Lemma 5.18 guarantees immediately that the sequence \( (f_k)_{k \in \mathbb{N}} \) is equicontinuous. Since \( N \) is compact, it follows that from the Arzelà-Ascoli theorem that the sequence \( (f_k)_{k \in \mathbb{N}} \) is relatively compact in \( X \) for the compact-open topology. Note that by Proposition 5.16 and the assumption that \( f^* \) is unique, we have \( E(\varphi(f)) \leq E(f) \) for all \( f \in X \), with equality only if \( f = f^* \). We conclude by application of Lemma 5.19. \(\square\)

**Lemma 5.18.** Let \( f: M \to N \) where \( M \) and \( N \) are Riemannian manifolds, and assume \( N \) is complete and has nonpositive sectional curvature. If \( f \) is locally Lipschitz continuous, then so is \( \hat{B}_r f \). Moreover, the Lipschitz constant of \( \hat{B}_r f \) is bounded above by the Lipschitz constant of \( f \) on any compact \( K \subseteq M \).

**Proof.** For simplicity, we assume that \( f \) is globally \( L \)-Lipschitz, and argue that \( \hat{B}_r f \) is also \( L \)-Lipschitz; the proof can easily be extended to the general case by restricting to compact sets. First we assume that \( M \) is Euclidean, in fact let us put \( M = \mathbb{R}^m \). Let \( x, y \in M \), write \( y = x + h \) so that \( d(x, y) = ||h|| \). By definition, \( \hat{B}_r f(y) \) is the point of \( N \) such that

\[
\frac{1}{\text{Vol}(B(y, r))} \int_{B(y, r)} \exp^{-1}_{B_r f(y)}(f(v)) \, dv_g(v) = 0.
\]  

(57)

Note that the map \( u \mapsto u + h \) defines an isometry from \( B(x, r) \) to \( B(y, r) \). Making the change of variables \( v = u + h \), we derive from (57):

\[
\int_{B(x, r)} \exp^{-1}_{B_r f(y)}(f(u + h)) \, dv_g(u) = 0.
\]

It follows that

\[
\int_{B(x, r)} \exp^{-1}_{B_r f(y)}(f(u)) \, dv_g(u) = \int_{B(x, r)} \left[ \exp^{-1}_{B_r f(y)}(f(u)) - \exp^{-1}_{B_r f(y)}(f(u + h)) \right] \, dv_g(u).
\]  

(58)

Assume without loss of generality that \( N \) is simply connected (otherwise one can lift to the universal cover). Then \( N \) is a Hadamard manifold and in particular a CAT(0) metric space, which implies that for any \( p \in N \), the map \( \exp^{-1}_p: N \to T_p N \) is distance nonincreasing (in fact, the converse is also true). We can therefore derive from (58):

\[
\left\| \int_{B(x, r)} \exp^{-1}_{B_r f(y)}(f(u)) \, dv_g(u) \right\| \leq \int_{B(x, r)} d(f(u), f(u + h)) \, dv_g(u).
\]  

(59)

49
It follows from (59) and the fact that $f$ is $L$-Lipschitz that
\[
\left\| \frac{1}{\text{Vol}(B(x, r))} \int_{B(x, r)} \exp^{-1} B_{f(y)}(f(u)) \, dv_\gamma(u) \right\| \leq L\|h\|. 
\] (60)

Lemma 5.3 now applies directly to (60) to conclude that
\[
d(\hat{B}_r f(x), \hat{B}_r f(y)) \leq L\|h\|. 
\]
Since $\|h\| = d(x, y)$, we have shown that $\hat{B}_r f$ is $L$-Lipschitz, as desired.

Now we argue that the argument extends to the case where $M$ is an arbitrary Riemannian manifold. It is surprisingly easy to do so using a local to global trick. First note that a function is globally $L$-Lipschitz if and only if it is locally $L$-Lipschitz. Here we mean by locally $L$-Lipschitz the seemingly weak property that for any $x \in M$, there exists $\delta > 0$ such that $d(y, x) < \delta$ implies $d(f(y), f(x)) \leq Ld(y, x)$. We leave it to the reader to show that in any path metric space, locally $L$-Lipschitz in this sense implies globally $L$-Lipschitz (the converse is obvious).

With this observation in mind, let us finish the proof. The key argument that worked above when $M$ is Euclidean and fails in general is that there exists an isometry from $B(x, r)$ to $B(y, r)$ that displaces every point of at most $d(x, y)$. This is no longer true when $M$ is an arbitrary Riemannian manifold, however note that it is almost true when $x$ and $y$ are very close. Quantifying this properly, clearly one can show that for every $x \in M$ and for every $L' > L$, there exists $\delta > 0$ such that $d(y, x) < \delta$ implies $d(\hat{B}_r f(y), \hat{B}_r f(x)) \leq L'd(y, x)$. Thus we have shown that $\hat{B}_r f$ is locally $L'$-Lipschitz, and therefore globally $L'$-Lipschitz. Since this is true for all $L' > L$, $\hat{B}_r f$ is actually $L$-Lipschitz. \(\square\)

**Lemma 5.19.** Let $X$ be a first-countable topological space and let $E : X \to \mathbb{R}$ a continuous function that admits a unique minimizer $x^\ast$. Assume that $\varphi : X \to X$ is a continuous map such that:

(i) $E(\varphi(x)) \leq E(x)$ for all $x \in X$, with equality only if $x = x^\ast$.

(ii) For all $x_0 \in X$, the set $\{ \varphi^k(x_0), k \in \mathbb{N} \}$ is relatively compact in $X$.

Then for any $x_0 \in X$, the sequence $(\varphi^k(x_0))_{k \in \mathbb{N}}$ converges to $x^\ast$.

**Proof.** In any topological space, in order to show that a sequence $(x_k)_{k \in \mathbb{N}}$ converges to a point $x^\ast$, it is enough to show that:

(a) The sequence $(x_k)$ has no cluster points except possibly $x^\ast$.

(b) Any subsequence of $(x_k)$ admits a cluster point.

Indeed, assume that $(x_k)$ does not converge to $x^\ast$, then there exists a subsequence of $(x_k)$ that avoids a neighborhood of $x^\ast$. This subsequence must have a cluster point by (b), which cannot be $x^\ast$. However this point is also a cluster point of the sequence $(x_k)$, contradicting (a).

Coming back to **Lemma 5.19**, let $x_0 \in X$ and denote $x_k = \varphi^k(x_0)$. The sequence $(x_k)$ satisfies (b) because of the assumption (ii). So we need to show that $(x_k)$ satisfies (a) and we are done. Let $y$ be a cluster point of $(x_k)$, we need to show that $y = x^\ast$. Since $X$ is first-countable, there exists a subsequence $(x_{k_n})_{n \in \mathbb{N}}$ converging to $y$. Observe that by assumption (i), since $k_n \leq k_n + 1 \leq k_{n+1}$ we have:
\[
E(x_{k_n}) \leq E(x_{k_{n+1}}) \leq E(x_{k_{n+1}}). 
\] (61)

By continuity of $E$, we have $\lim E(x_{k_n}) = \lim E(x_{k_{n+1}}) = E(y)$, so (61) implies that $\lim E(x_{k_{n+1}}) = E(y)$. On the other hand, since $x_{k_{n+1}} = \varphi(x_{k_n})$ and $\varphi$ is continuous, we have $\lim x_{k_{n+1}} = \varphi(y)$, so $\lim E(x_{k_{n+1}}) = E(\varphi(y))$. Thus $E(\varphi(y)) = E(y)$, and we conclude that $y = x^\ast$ by (i). \(\square\)
Discrete center of mass method

We now prove that Theorem 5.17 also holds in the discrete setting developed in § 2, providing an alternative method to the discrete heat flow (see § 4.2) to compute discrete equivariant harmonic maps effectively. Let $G$ be a biweighted $\tilde{S}$-triangulated graph (see § 2.2), let $N$ be a Hadamard manifold and let $\rho: \pi_1 S \to \text{Isom}(N)$ be a group homomorphism. We recall that the discrete energy functional $E_G: \text{Map}_{\text{eq}}(G, N) \to \mathbb{R}$ coincides with Jost’s energy functional (55) for the appropriate choice of kernel $\eta$ (see Definition 2.22 and Proposition 2.23).

Motivated by Proposition 5.16, we note that in this discrete setting the measure $\eta(x, \cdot)\mu$ is given by the weighted atomic measure

$$\sum_{y-x} \frac{\omega_{xy}}{\mu(x)} \delta_y,$$

where $\delta_y$ is the Dirac measure at $y$.

Now the averaging map $f \mapsto \hat{B}_r f$ takes the following form:

Definition 5.20. The discrete center of mass method on $\text{Map}_{\text{eq}}(G, N)$ is given by $f \mapsto \varphi(f)$, where $\varphi(f)$ is the center of mass of the atomic measure

$$\sum_{y-x} \frac{\omega_{xy}}{\mu(x)} \delta_y.$$

Note that Proposition 5.16 applies in this setting (cf. Proposition 2.26). Under certain assumptions on $N$ and $\rho$, we obtained strong convexity of $E_G$ in Theorem 3.25, so that, in particular, $E_G$ has a unique minimum. The same assumptions have similarly useful consequences here:

Theorem 5.21. Let $N$ be a manifold of pinched negative curvature, and let $\rho$ be a faithful representation whose image is contained in a discrete subgroup of $\text{Isom}(N)$ acting freely, properly, and cocompactly on $N$. From any starting point in $\text{Map}_{\text{eq}}(G, N)$, the discrete center of mass method converges to the unique discrete harmonic map.

Proof. The proof is a similar (but easier) version of the proof of Theorem 5.17.

Let $f_0 \in \text{Map}_{\text{eq}}(G, N)$, and define the sequence $(f_k)_{k \in \mathbb{N}}$ by $f_{k+1} = \varphi(f_k)$. The assumption on $\rho$ means that we can work in a compact quotient of $N$, making the sequence $(f_k)$ pointwise relatively compact. Since the action of $\pi_1 S$ on $G$ is cofinite, it is also easy to see that the condition that the family $\{f_k\}$ is equicontinuous is vacuous. Hence the family $\{f_k\}$ is relatively compact in $\text{Map}_{\text{eq}}(G, N)$ for the compact-open topology, which is just the topology of pointwise convergence in this case. Since Proposition 5.16 holds in this setting, all the requirements are met to conclude with Lemma 5.19.

Remark 5.22. In [JT07], Jost-Todjihonde describe an iterative process to obtain a discrete harmonic map from an edge-weighted triangulated graph $G$ to a target space that admits centers of mass (e.g. Hadamard spaces). Theorem 5.21 can be viewed as a strengthened version of their result in two respects: For one, we avoid Jost-Todjihonde’s passage to a subsequence of $(f_k)$. Moreover, Jost-Todjihonde start by subdividing $G$ and pursuing centers of mass in two phases, separately for vertices and midpoints of edges. Our discrete center of mass method requires no such subdivision.
5.4 cosh-center of mass

Theorem 5.21 provides an effective method to compute discrete equivariant harmonic maps, alternative to the discrete heat flow (see § 4.2). The one obstacle to making this method truly effective is being able to compute centers of mass.

Unfortunately, non-Euclidean centers of mass are often not easily accessible. Even finding the barycenter of three points in the hyperbolic plane is a nontrivial task. While it is possible to use gradient descent method (see [ATV13]), it is computationally expensive and, in any case, in stark contrast to Euclidean centers of mass not possible to do precisely in finite time. We present a clever variant to barycenters, well-suited to hyperbolic space $\mathbb{H}^n$, that avoids this issue. We thank Nicolas Tholozan for bringing this idea to our attention.

Definition 5.23. Let $(\Omega, \mathcal{F}, \mu)$ be a probability space, $(X, d)$ be a metric space, and $h : \Omega \to X$ a measurable map. A cosh-center of mass (or cosh-barycenter) of $h$ is a minimizer of the function

$$P_h : X \to \mathbb{R}$$

$$x \mapsto \int_{\Omega} (\cosh d(x, h(y))) - 1 \, d\mu(y).$$

Note that like the center of mass, the cosh-center of mass is geometrically natural: the image under an isometry $A$ of a cosh-center of mass of $h$ is a cosh-center of mass of $A \circ h$.

When $X$ is a Riemannian manifold, a cosh-center of mass $G$ is characterized by

$$\int_{\Omega} \sinh d(G, h(y)) \exp^{-1}_{G}(h(y)) \, d\mu(y) = 0,$$

(62)

where $\sinhc(x) = \sinh(x)/x$ is the cardinal hyperbolic sine function.

Equation (62) implies that if $\text{supp}(h, \mu)$ is contained in a strongly convex region $U$ (e.g. a ball of small enough radius), then any cosh-center of mass is contained in $U$ as well: if $x$ is outside $U$, then each vector $\exp^{-1}_x(h(y))$, for $y \in \text{supp}(h, \mu)$, is contained in an open half-space in $T_x X$ containing $\exp^{-1}_x(U)$, and (62) cannot be satisfied.

Let us now specialize to the case where $X = \mathbb{H}^n$ is the hyperbolic $n$-space. In this setting the function $F(x) = \cosh(d(x_0, x)) - 1$ is especially amenable to computations, with its gradient and Hessian given by:

$$\text{grad} F(x) = \sinhc(d(x_0, x)) \exp^{-1}_x(x_0)$$

$$\text{Hess}(F)(x, v) = F(x)\|v\|^2.$$

In particular, $F$ is a strongly convex function on $\mathbb{H}^n$ with modulus of strong convexity $\alpha = 1$. Existence and uniqueness of the center of mass of any function $h \in L^2(\Omega, \mathbb{H}^n)$ quickly follows.

The main advantage of the cosh-center of mass is that it admits an explicit description, much like the Euclidean barycenter. For this we work in the hyperboloid model for $\mathbb{H}^n$, i.e.

$$\mathcal{H} = \{ x \in \mathbb{R}^{n,1} : \langle x, x \rangle = -1, x_{n+1} > 0 \},$$

where Minkowski space $\mathbb{R}^{n,1}$ is defined as $\mathbb{R}^{n+1}$ equipped with the indefinite inner product

$$\langle x, y \rangle = x_1 y_1 + \cdots + x_n y_n - x_{n+1} y_{n+1}.$$

This inner product induces a Riemannian metric on $\mathcal{H}$ of constant curvature $-1$.

We state Proposition 5.24 and Proposition 5.25 below for finite collections of weighted points, but the generalization to any probability measure supported in $\mathbb{H}^n$ is standard.
**Proposition 5.24.** The cosh-center of mass of a finite collection of weighted points in $\mathbb{H}^n \approx \mathcal{H}$ is given by the orthogonal projection of their Euclidean barycenter in Minkowski space $\mathbb{R}^{n,1}$ to the hyperboloid $\mathcal{H} \subset \mathbb{R}^{n,1}$.

**Proof.** Consider points $p_1, \ldots, p_n \in \mathcal{H}$ with weights $w_1, \ldots, w_n$ satisfying $\sum_i w_i = 1$, let $p$ be their Euclidean barycenter in $\mathbb{R}^{n,1}$, and let $q$ indicate the orthogonal (i.e. radial) projection of $p$ to $\mathcal{H}$. By (62), it suffices to check that

$$\sum_i w_i \sinh d(q, p_i) \exp^{-1}(p_i) = 0. \tag{63}$$

Let $P$ be the tangent plane to $\mathcal{H}$ at $q$, i.e. the affine plane in $\mathbb{R}^{n,1}$ which is orthogonal to the line $\mathbb{R}q$ at $q$. The orthogonal projection $\pi : \mathbb{R}^{n,1} \to P$ is an affine map, therefore the identity $\sum_i w_i(p_i - p) = 0$ in $\mathbb{R}^{n,1}$ translates to $\sum_i w_i(q_i - q) = 0$ on $P$, where $q_i = \pi(p_i)$. It is straightforward to compute $q$ and $q_i$ explicitly:

$$q = p \frac{1}{\sqrt{-\langle p, p \rangle}}$$

$$q_i = p_i + q + \frac{\langle p_i, p \rangle}{-(p, p)} p$$

Geodesics in the hyperboloid are intersections of 2-dimensional subspaces of $\mathbb{R}^{n,1}$ with $\mathcal{H}$, so $q_i - q$ is a vector in $T_q \mathcal{H}$ pointing towards $p_i$, and we can compute its length:

$$\|q_i - q\|^2 = \langle p_i, q \rangle^2 - 1 = \sinh^2 d_{\mathcal{H}}(q, p_i)$$

Thus we proved that

$$q_i - q = \frac{\sinh d_{\mathcal{H}}(q, p_i)}{d_{\mathcal{H}}(q, p_i)} \exp^{-1}(q_i)$$

and we get (63) as desired. $\square$

Another useful feature of the cosh-center of mass is that it is a good approximation of the center of mass for small distances. This should not come as a surprise, since $\cosh(d(x, y)) - 1$ is close to $\frac{1}{2} d^2(x, y)$ when $d(x, y)$ is small. This will be important in [GLM18].

**Proposition 5.25.** If a collection of weighted points $p_1, \ldots, p_n$ is contained in a ball of radius $D$, then its center of mass $p$ and its cosh-center of mass $q$ and are within $O(D^3)$ of each other.

**Proof.** By (62) we can write:

$$\sum_i w_i \exp^{-1}(p_i) = \sum_i w_i (1 - \sinh d(p_i, q)) \exp^{-1}(p_i). \tag{64}$$

Because $q$ must be contained in the same ball of radius $D$ as $\{p_i\}$, we find that $d(p_i, q) < 2D$ for each $i$. Given that sinh is a nondecreasing function we derive from (64)

$$\left\| \sum_i w_i \exp^{-1}(p_i) \right\| \leq \sum_i w_i (\sinh d(p_i, q) - 1) \|\exp^{-1}(p_i)\|$$

$$\leq \sum_i w_i (\sinh(2D) - 1) \cdot (2D) = 2D (\sinh(2D) - 1).$$
Lemma 5.3 now implies that
\[ d(p, q) \leq 2D \left( \sinhc(2D) - 1 \right). \]
The conclusion follows, since \( 2D \left( \sinhc(2D) - 1 \right) = \frac{4D^3}{3} + O(D^5) \). \(\square\)

Note that we did not use in the proof of Proposition 5.25 that we are working in \( \mathbb{H}^n \): this proposition holds in any Riemannian manifold.

6 Computer implementation: Harmony

6.1 Computer description and availability

Harmony is a computer program with a graphical user interface written in C++ code using the Qt framework. In its current state, it totals about 14,000 lines of code.

Harmony is a free and open source software under the GNU General Public License. It is available on GitHub at https://github.com/seub/Harmony.

Harmony is currently in beta version. Unfortunately we have not yet deployed the software: one needs to compile it from the sources in order to run it. Doing so is fairly straightforward on GNU/Linux operating systems; we are happy to provide assistance to anyone wanting to try it. On Windows and Mac systems, it requires installing the Qt framework beforehand, which is not necessarily a trivial task.

6.2 Algorithms

We now provide a brief overview of Harmony’s algorithms allowing effective computation of discrete equivariant harmonic maps \( \mathbb{H}^2 \to \mathbb{H}^2 \) with respect to a pair of Fuchsian representations. A flowchart showing how the main algorithms fit into the program is pictured in Figure 12.

We fix an identification of the closed oriented topological surface \( S \) of genus \( g \) as \( P_0/\sim \), where \( P_0 \) is a topological \( 4g \)-gon with oriented sides \( s_1, \ldots, s_{4g} \), and \( \sim \) identifies the sides via
\[ s_{4i+1} \sim s_{4i+3} \quad \text{and} \quad s_{4i+2} \sim s_{4i+4} \quad \text{for} \quad i = 0, \ldots, g - 1 \]
where the notation \( \overline{s} \) stands for the reverse of an oriented arc \( s \).

Remark 6.1. The calculations performed by Harmony become numerically unstable outside of some range of inputs that depends on the chosen genus \( g \). All figures contained here were generated with \( g = 2 \); though the algorithm would in theory terminate for any \( g \), in practice Harmony is most reliable with this restriction.

We parametrize hyperbolic structures on \( S \) using the famous Fenchel-Nielsen coordinates. This requires first choosing a pants decomposition of \( S \), i.e. a maximal system of disjoint simple closed curves. Harmony is equipped to make such choices for arbitrary \( g \), in a way that minimizes future error propagation. In light of Remark 6.1 we point out only the choice made for \( g = 2 \): the unique pants decomposition containing a separating curve.
We now describe the algorithms Harmony performs in sequential order. The input at the start is a pair of Fenchel-Nielsen coordinates for hyperbolic structures $X$ and $Y$ on $S$, the domain and target hyperbolic surfaces respectively. These can be entered by the user in a ‘Fenchel-Nielsen selector’ window: see Figure 8. For the user’s convenience, we also offer the ability to let Harmony choose random Fenchel-Nielsen coordinates, and in genus 2 we offer the user a fixed nice representation (the Token nice representation pictured on the left of Figure 9).

**Step 1: Construct the fundamental group and pants decomposition**

After getting the genus $g$ as input, Harmony constructs the fundamental group of the surface as an abstract structure. It then chooses a pants decomposition of the surface, yielding a decomposition of the fundamental group as a combination of amalgamated products and HNN extensions of fundamental groups of pairs of pants. This is done recursively on the genus, using a tree structure introduced for this purpose.
Step 2: Construct representations \( \rho_X \) and \( \rho_Y \)

This step performs the translation of Fenchel-Nielsen coordinates to Fuchsian representations. Harmony starts by computing the representation of the fundamental group of each pair of pants using formulas that can be found in e.g. [Kou94, Prop. 2.3] or [Mas99, Mas01]. It then computes the representation of the whole fundamental group using its decomposition discussed in Step 1.

Step 3: Construct fundamental domains \( P_X \) and \( P_Y \)

This step computes polygonal fundamental domains \( P_X \) and \( P_Y \) in \( \mathbb{H}^2 \) for the Fuchsian groups in the images of \( \rho_X \) and \( \rho_Y \). These polygons should be ‘as convex as possible’ in order to ensure good behavior of the discrete heat flow.

Both \( P_X \) and \( P_Y \) come with \( \pi_1 S \)-equivariant identifications to the topological 4n-gon \( P_0 \) that record side-pairings. Because the vertices of \( P_0 \) are all in a single \( \pi_1 S \)-orbit, the vertices of \( P_X \) are determined by a single point in \( \mathbb{H}^2 \). With this combinatorial setup, the choices for \( P_X \) are parameterized by \( \mathbb{H}^2 \), and a best choice will be determined by minimizing a cost function \( F : \mathbb{H}^2 \to \mathbb{R}^+ \). While various choices for \( F \) are possible, we let \( F(x) \) equal the diameter of the generated polygon. A candidate minimum \( F \) is located by a straightforward Newton method.

Remark 6.2. While this algorithm seems to work well in practice, it is unclear whether it will always produce a convex fundamental domain.

Step 4: Construct a triangulation of \( P_X \)

This step computes a triangulation of the fundamental domain \( P_X \). Finer and finer meshes can then be obtained by subdivision (see Definition 2.2). As we explain in [GLM18], it is important to keep the smallest angle of the triangulation as large as possible.

Unfortunately, \( P_X \) itself already typically has very small angles. In order to avoid subdividing these angles further in the process of triangulating, we first introduce new Steiner vertices evenly spaced along the sides of \( P_X \). Precisely, if \( \ell \) is the shortest side length of \( P_X \), we add \( \lfloor \ell_{\mathbb{H}^2}(s)/\ell \rfloor \) vertices to the side \( s \). We triangulate the resulting polygon with a greedy recursive algorithm maximizing the smallest angle. See Figure 8 for a sample output.

Remark 6.3. The algorithm above seems to always produce acute triangulations, a necessary condition for the definition of the edge weights and the discrete energy functional (see § 2.1), but we make no claims about whether this will always hold. Acute triangulations of surfaces and higher dimensional analogues are part of a fascinating area of current research [CdVM90, Zam13, BG18].

Step 5: Construct the \( \rho_X \)-invariant mesh \( M \)

From Harmony’s viewpoint, the mesh \( M \) consists of a list of meshpoints \( M^{(0)} \subset \mathbb{H}^2 \), each of which is equipped with a list of references to its neighboring meshpoints, and possibly side-pairing information. Initially, this data is recorded by the triangulated polygon \( P_X \). Then, given a user chosen mesh depth \( k \geq 1 \), \( M \) is replaced with the \( k \)th iterated midpoint refinement of \( M \) (see Definition 2.2).
Remark 6.4. Constructing the adequate data structure to efficiently store all the information is quite a difficult challenge from a programming standpoint: the corresponding C++ classes that we defined are by far the most sophisticated in the code of Harmony.

**Step 6: Initialize an equivariant map**

The triangulation of $P_X$ may be transported to one for $P_Y$ via the $\pi_1 S$-equivariant maps

$$P_X \cong P_0 \cong P_Y.$$

Because $M$ is built from midpoint refinements of $P_X$, this identification of a triangulation of $P_Y$ provides the data of an initial discrete equivariant map $f_0: \mathbb{H}^2 \to \mathbb{H}^2$. A sample initial map is showed in Figure 9.

**Step 7: Run the discrete flow**

Finally, Harmony is ready to apply the discrete flow: either the discrete heat flow with fixed or optimal stepsize (see § 4.2) or the cosh-center of mass method (§ 5.3, § 5.4). Harmony computes the heat flow in a separate thread so that it is able to display the flow simultaneously: see Figure 10 for a screenshot of Harmony mid-flow.

The flow is iterated until the error reaches a preset tolerance, or when it is stopped by the user. Both the discrete heat flow and the center of mass method typically converge very well. Refer to § 4.3 for a comparison of the methods. Figure 11 shows a sample output equivariant harmonic map.

![Figure 9: An initial discrete equivariant map $f_0$. The highlighted blue triangles differ by $f_0$.](image-url)
Figure 10: Harmony mid-flow.

Figure 11: Sample output equivariant harmonic map. The brighter central regions are fundamental domains. The darker areas in the target are the most contracted by the harmonic map.
Figure 12: Flowchart representing Harmony’s main algorithms.
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