Possible direct method to determine THE radius of a star from the spectrum of gravitational wave signals

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We computed the spectrum of gravitational waves from a dust disk star of radius \(R\) inspiraling into a Kerr black hole of mass \(M\) and specific angular momentum \(a\). We found that when \(R\) is much larger than the wave length of the quasinormal mode, the spectrum has several peaks and the separation of peaks \(\Delta \omega\) is proportional to \(R^{-1}\) irrespective of \(M\) and \(a\). This suggests that the radius of the star in coalescing binary black hole - star systems may be determined directly from the observed spectrum of gravitational wave. This also suggests that the spectrum of the radiation may give us important information in gravitational wave astronomy as in optical astronomy.

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A number of worldwide projects on the direct detection of gravitational waves using laser interferometers such as LIGO, VIRGO, GEO600, TAMA300, and LISA as well as several projects using resonant type detectors are in progress.\(^1\) One of the most important sources of gravitational waves for these detectors is coalescing binary black hole (BH) - star systems. In the inspiral phase, the radius of the star has little effect on gravitational waves so that the star can be treated as a point particle. Comparing the waveform of the inspiral phase with a theoretical template using matched filtering techniques, we may determine each mass and spin of the star and BH, respectively\(^2\). While, in the final merging phase, the effect of the radius is important and some information on the radius of the star may be extracted from gravitational wave signals. However, this merger phase is much less well understood than the inspiral one.

In this Letter, as a model of this final merging phase, we computed the spectrum of gravitational waves from a dust disk star of radius \(R\) inspiraling into a Kerr BH of mass \(M\) and specific angular momentum \(a\). We found that when \(R\) is much larger than the wave length of the quasinormal mode (QNM) of Kerr BH, the spectrum has several peaks and the separation of peaks \(\Delta \omega\) is proportional to \(R^{-1}\) irrespective of \(M\) and \(a\). This suggests that the radius of the star in coalescing binary BH - star systems may be determined directly from the observed spectrum of gravitational waves without any assumption to equation of state.\(^1\)

First, let us consider gravitational waves from a particle plunging into a Kerr BH of mass \(M\) and specific angular momentum \(a\). The Fourier component of the radial wave function of gravitational waves \(X_{\ell m \omega}(r^*)\) obeys the Sasaki-Nakamura equation (Eq. (2.28) of [4]) given by

\[
\frac{d^2}{dr^*^2} - F(r^*) \frac{d}{dr^*} - U(r^*) \right) X_{\ell m \omega}(r^*) = S_{\ell m \omega}(r^*),
\]

where \(r^*, S_{\ell m \omega}(r^*), F(r^*)\) and \(U(r^*)\) are the tortoise coordinate of Kerr BH, the source term from the test particle of mass \(\mu\) (Eq. (2.29) of [4]) and two potential functions (Eqs. (2.12a) and (2.12b) of [4]), respectively. To solve \(X_{\ell m \omega}(r^*)\) using a Green function method, we need two independent homogeneous solutions whose boundary conditions are given by

\[
X_{\ell m \omega}^{\text{in}(0)}(r^*) = \begin{cases} e^{-ikr^*} & r^* \to -\infty, \\ A_{\ell m \omega}^{\text{in}} e^{-i\omega r^*} + A_{\ell m \omega}^{\text{out}} e^{i\omega r^*} & r^* \to \infty, \\ \end{cases}
\]

\[
X_{\ell m \omega}^{\text{out}(0)}(r^*) = \begin{cases} B_{\ell m \omega}^{\text{in}} e^{-ikr^*} + B_{\ell m \omega}^{\text{out}} e^{ikr^*} & r^* \to -\infty, \\ e^{i\omega r^*} & r^* \to \infty, \\ \end{cases}
\]

where \(k = \omega - ma/[2(M + \sqrt{M^2 - a^2})]\). Then, the homogeneous solution to Eq. (4) becomes

\[
X_{\ell m \omega}(r^*) = X_{\ell m \omega}^{\text{in}(0)} \int_{r^*}^{\infty} S_{\ell m \omega}(r^*) \frac{dX_{\ell m \omega}^{\text{out}(0)}}{dr^*} \cdot dr^* + X_{\ell m \omega}^{\text{out}(0)} \int_{-\infty}^{r^*} S_{\ell m \omega}(r^*) X_{\ell m \omega}^{\text{in}(0)} \cdot dr^*.
\]

where \(W\) is the Wronskian,

\[
W \equiv X_{\ell m \omega}^{\text{in}(0)} \frac{dX_{\ell m \omega}^{\text{out}(0)}}{dr^*} - X_{\ell m \omega}^{\text{out}(0)} \frac{dX_{\ell m \omega}^{\text{in}(0)}}{dr^*}.
\]

The asymptotic behavior of the radial wave function \(X_{\ell m \omega}(r^*)\) is given by

\[
X_{\ell m \omega}(r^*) = A_{\ell m \omega} e^{i\omega r^*},
\]

\[
A_{\ell m \omega} = \int_{-\infty}^{\infty} \frac{S_{\ell m \omega}(r^*)}{W} X_{\ell m \omega}^{\text{in}(0)} dr^*.
\]
Then, the energy spectrum and waveform of gravitational waves (Eq. (3.6) and (3.10) of [3]) are given by

\[
\frac{dE}{d\omega}_{lm\omega} = 8\omega^2 \left[ \frac{A_{lm\omega}}{c_0} \right]^2 \left( -\infty < \omega < \infty \right),
\]

\[
h_+ - ih_\times = \frac{8}{r} \int_{-\infty}^{\infty} d\omega e^{i\omega(r-t)} \times \sum_{l,m} \left[ \left( \frac{A_{lm\omega}}{c_0} \right) -2S_{lm\omega}(\theta) \frac{e^{ilm\varphi}}{\sqrt{2\pi}} \right],
\]

where \( r \) is the coordinate radius from the center of BH, \(-2S_{lm}\) is the spin -2 weighted spherical harmonics, \(c_0\) is a constant given in [3].

Next, we consider gravitational waves from a dust disk [5]. We note that a geodesic in the equatorial plane in Kerr BH is characterized by the specific energy \( \langle E \rangle \) and angular momentum \( \langle L_z \rangle \). Let \( t = T(r) \) and \( \phi = \Phi(r) \) express an orbit of the geodesic for given \( E \) and \( L_z \). Then another geodesic for the same \( E \) and \( L_z \) is expressed by \( t = T(r) + c_t \) and \( \phi = \Phi(r) + c_\phi \) where \( c_t \) and \( c_\phi \) are constants. Specifically the geodesic, whose location at \( t = T(r_0) \) is \( r = r_1 \) and \( \phi = \phi_1 \), is expressed by \( t = T(r) + T(r_0) - T(r_1) \) and \( \phi = \Phi(r) + \phi_1 - \Phi(r_1) \).

It is possible to set a number of particles to form a disk of radius \( R \) whose center is \( r = r_0 \) and \( \phi = \Phi(r_0) \) at \( t = T(r_0) \).

Each particle in the disk emits the same gravitational waves as the particle in the center of the disk except that the phase in \( t \) and \( \phi \) is different from the central particle \( \Phi \). Then the amplitude of the gravitational waves from the dust disk star is computed as

\[
A_{lm\omega}^{(\text{disk})} = f_{m\omega} A_{lm\omega}^{(\text{particle})},
\]

\[
f_{m\omega} = 2\mu S \int_{r_0 - R}^{r_0 + R} dr \sin(m\phi_0(r)) \frac{\sin(m\phi_0(r))}{m} \times e^{i[\omega(T(r) - T(r_0)) - m(\Phi(r) - \Phi(r_0))]},
\]

\[
\phi_0(r) = \cos^{-1} \left[ \frac{r^2 + r_0^2 - R^2}{2rr_0} \right],
\]

where \( f_{m\omega} \) and \( A_{lm\omega}^{(\text{particle})} \) are a form factor and the amplitude of gravitational waves at infinity for a single particle case, respectively. \( S \) is the normalization factor given by

\[
S = 2 \int_{r_0 - R}^{r_0 + R} dr r \phi_0(r).
\]

In this Letter, we only show the result for \( r_0 = 10M \), \( a/M = 0.9 \), \( E = 1 \) and \( L_z = 2M \). In this case, the orbit rotates about \( \pi \) angle around the BH within the radius of \( r = r_0 \), and the velocity at \( r = 3M \), the location where the particle might radiate the characteristic QNM frequency, is \( v \sim 0.48c \), where \( c \) is the speed of light. As for the choice of the parameter \( r_0 = 10M \), we can refer the result of the relativistic Roche limit (Eq. (5.1) of [3]) as

\[
M < 4.64M_\odot \left( \frac{1.4M_\odot}{M_s} \right)^{1/2} \left( \frac{R}{10\text{km}} \right)^{3/2} \left( \frac{6M_\odot}{r_c} \right)^{3/2},
\]

where \( M_s \) and \( r_c \) are the mass of the star and the Roche radius, respectively. For appropriate choice of parameters \( r_0 = 10M \) corresponds to the Roche limit.

Fig. 2 (a) and Fig. 3 (a) show the characteristics of gravitational waves from a dust disk star moving on an equatorial plane in Kerr spacetime whose radius is set up at \( r_0 = 10M \) for the case of \( a/M = 0.9 \), \( L_z = 2M \). (a) \( R/M = 0 \) (test particle), (b) \( R/M = 1.56 \), (c) \( R/M = 5.88 \). We only show \( l = 2 \) mode. Solid, dashed, dash-dotted, dotted, and dash-three dotted lines denote the case of \( m = -2, -1, 0, 1, 2 \), respectively.

\[
\text{FIG. 1: Energy spectrum of gravitational waves from a dust disk star moving on an equatorial plane in Kerr spacetime whose radius is set up at } r_0 = 10M \text{ for the case of } a/M = 0.9, L_z = 2M \text{. (a) } R/M = 0 \text{ (test particle), (b) } R/M = 1.56, \text{ (c) } R/M = 5.88 \text{. We only show } l = 2 \text{ mode. Solid, dashed, dash-dotted, dotted, and dash-three dotted lines denote the case of } m = -2, -1, 0, 1, 2 \text{, respectively.}
\]

\[
\text{FIG. 2: Waveform } h_+ \text{ observed from the equatorial plane at infinity. From Fig. 2 (a), we see that most of gravitational waves are radiated near the frequency of the QNM since the QNM frequency for } a/M = 0.9 \text{ is } \text{Re} (M\omega_{\text{QNM}}) = 0.7 \text{ for } l = m = 2, \text{ while}
\]

\[
\text{FIG. 3: Energy spectrum of gravitational waves from a dust disk star moving on an equatorial plane in Kerr spacetime whose radius is set up at } r_0 = 10M \text{ for the case of } a/M = 0.9, L_z = 2M \text{. (a) } R/M = 0 \text{ (test particle), (b) } R/M = 1.56, \text{ (c) } R/M = 5.88 \text{. We only show } l = 2 \text{ mode. Solid, dashed, dash-dotted, dotted, and dash-three dotted lines denote the case of } m = -2, -1, 0, 1, 2 \text{, respectively.}
\]
the imaginary part of QNM (Im (MωQNM)) in this case is 0.07 (Fig. 1a of [3] and Fig. 3 (c) of [9]). Fig. 2 (a) shows clearly that there exists a typical ringing tail at late time (Fig. 4 of [10] and Fig. 4 of [11]). Fig. 1 (b), (c) and Fig. 2 (b), (c) show the spectra and the waveform from the circular dust disk star inspiraling into a Kerr BH with a/M = 0.9. The radius of the disk is set up as R/M = 1.56 for Fig. 1 (b) and Fig. 2 (b) while R/M = 5.88 for Fig. 1 (c) and Fig. 2 (c). For the case of Fig. 2 (b), the peak of the spectrum is located at Mω ∼ 0.5 which is quite different from the QNM frequency Re (MωQNM) ∼ 0.7 although the ringing tail can be seen in Fig. 2 (b). When we turn to look at Fig. 1 (c), there are several peaks in the spectrum and the ringing tail is very low or absent in the waveform of Fig. 2 (c). This is a new finding. The reason for this behavior is as follows. The energy spectrum (dE/dω)\text{(disk)} of gravitational waves from the disk is expressed as
\[
\frac{dE}{d\omega}_{\text{disk}} \propto |f_{m\omega}|^2 \frac{dE}{d\omega}_{\text{particle}},
\]
where (dE/dω)\text{(particle)} is the spectrum from the single test particle. The spectrum from the test particle has only one peak at the frequency ωQNM (Fig. 1 (a)) so that the square of the form factor |f_{m\omega}|^2 is responsible for this behavior. To confirm this, we show |f_{m\omega}|^2 for R/M = 1.56 (Fig. 2 (a)) and R/M = 5.88 (Fig. 2 (b)) and find that |f_{m\omega}|^2 is responsible for the behavior of the spectrum in Fig. 2 (b) and (c). The existence of several peaks in Fig. 2 (a) and (b) can be understood by the approximate estimation of f_{m\omega} as
\[
f_{m\omega} \propto \sin(\omega T'/r_0 R) / \omega,
\]
where T' = dT(r)/dr. The frequency where f_{m\omega} takes zero is
\[
\omega_n \sim \frac{(n + 1)\pi}{T'_{r=r_0} R} \quad n = 0, 1, 2, \ldots,
\]
which agrees quite well with the numerical results of Fig. 2 (a) and (b). Equation (13) suggests that the separation of peaks of the spectrum ∆ω may be in proportion to R^{-1}. In Table 1, we show ∆ω for various R/M and we found that
\[
R = C \frac{1}{\Delta \omega} \text{ for } R/M \gg 1.6,
\]
where a constant C is close to 1 in the present case. In the physical unit, ∆ν ≡ ∆ω/(2π) is given by
\[
\Delta \nu = 5 \text{kHz} \left( \frac{R}{10 \text{km}} \right)^{-1} = 10 \text{Hz} \left( \frac{R}{5000 \text{km}} \right)^{-1},
\]
assuming that there are 4 peaks in the region of ω ∈ [0, ωQNM] in the spectrum. Therefore for neutron stars and white dwarfs, the frequency band is within some laser interferometers and resonant type detectors.

\[\text{FIG. 2: Waveform of gravitational waves from a dust disk star moving on an equatorial plane in Kerr spacetime whose radius is set up at } r_0 = 10 \text{M for the case of } a/M = 0.9, L_z/M = 2. (a) } R/M = 0 \text{ (test particle), (b) } R/M = 1.56, \text{ (c) } R/M = 5.88. \text{ We set the angle of the observer at infinity as } \theta = \pi/2, \phi = 0. \text{ We only show } l = m = 2 \text{ mode to exclude } m \text{ mode coupling effect.}\]

Equation (14) strongly suggests that from the observed value of ∆ω in the spectrum of the gravitational wave signals, we may determine the radius of the star if R/M \gg 1.6. From the physical point of view, the form factor f_{m\omega} expresses the phase cancellation effect of gravitational waves [3], so that if R < 1/ωQNM gravitational waves from the dust disk star is essentially regarded as that from the test particle. This suggests that our

\[\text{1 We also confirmed that this phase cancellation can also be seen in the 3D dust star assuming that each test particle moves the constant polar angle with no orbital angular momentum.}\]
The proposal is valid only if the inspiraling star is tidally disrupted by the BH finally. It is instructive to note here that in reality the form factor may bring us the information about the form of the source, radius.

Here, we only showed the results for a special set of parameter such as \( r_0 = 10M, a/M = 0.9, \dot{E} = 1, \dot{L}_z = 2M \). To check the robustness of our proposal, we also calculated the spectra from the dust disk star inspiraling into Kerr BH for the wide range of parameters \((\dot{L}_z/M): \) spiraling case, \( 0 \leq a/M \leq 0.9 \) and found that there exist some peaks in the spectrum for \( R \gg 1/\omega_{QNM} \). In these cases we confirmed the relation \( R \sim 1/\Delta \omega \). In this Letter, we did not take into account of the pressure and the self-gravity of the star so that it is urgent to confirm our proposal by full 3D numerical simulations including the determination of the value of the constant \( C \) in Eq. (14).

If the relation of Eq. (14) is confirmed irrespective of the equation of state, Eq. (14) can be adopted as one of the direct methods to determine the radius of the star from gravitational waves. In any case, it is quite possible that the spectrum of the gravitational waves may give us important information in gravitational wave astronomy as in optical astronomy.

\[ \text{FIG. 3: Form factor of the dust disk star moving on an equatorial plane in Kerr spacetime whose radius is set up at } r_0 = 10M \text{ for the case of } a/M = 0.9, \dot{L}_z/M = 2 \text{. (a) } R/M = 1.56, \text{ (b) } R/M = 5.88. \]  

We only show \( l = m = 2 \) mode, because \( m = 2 \) is a dominant mode for above parameters in \( l = 2 \). It is clear that the form factor is responsible for the spectra in Fig. 1.

**TABLE I.** Comparison with the characteristic length from the energy spectrum of gravitational waves to the radius of the disk. \( R \) denotes the coordinate radius.

| \( R/M \) | \( \Delta \omega \) | \( 1/\Delta \omega \) |
|--------|--------|--------|
| 3.09   | 0.335  | 2.99   |
| 3.83   | 0.270  | 3.70   |
| 4.54   | 0.233  | 4.29   |
| 5.22   | 0.200  | 5.00   |
| 5.88   | 0.175  | 5.71   |

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