STUDIES OF CERTAIN CLASSES OF FUNCTIONS AND ITS CONNECTION WITH $S$-EMBEDDEDNESS

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ABSTRACT. We call a function $f$ in $C(X)$ to be hard-bounded if $f$ is bounded on every hard subset, a special kind of closed subset, of $X$. We call a subset $T$ of $X$ to be $S$-embedded if every hard-bounded continuous function of $T$ can be continuously extended upto $X$. Every $S$-embedded subset is $C^*$-embedded. In this paper we have given a characterization of the converse part. To get the converse, we came across a type of function which are bounded away from zero on every hard subset of a subset. We further studied few properties of this type of functions and also of hard-bounded functions.

1. INTRODUCTION

Throughout this paper, we shall always assume a space to be Tychonoff unless otherwise mentioned. For a space $X$, $C(X)$ and $C^*(X)$ usually denote respectively the rings of real-valued and bounded real-valued continuous functions on $X$. $\beta X$ and $\nu X$ respectively denote the Stone-$\check{C}$ech compactification and Hewitt realcompactification of a space $X$. A subset $H$ of $X$ is called hard in $X$ if $H$ is closed in $X \cup K$ where $K = cl_{\beta X}(\nu X \setminus X)$. The notion of hard set was introduced by M. C. Rayburn in the year 1976 [2]. In our last paper [6], “Construction of Nearly Pseudocompactifications” we introduced the notion of $S$-embeddedness. A continuous function $f$ on $X$ is called hard-bounded if $f$ is bounded on every hard subset of $X$. Let $S(X)$ denote the family of all hard-bounded continuous functions. Then $S(X)$ forms a subring of $C(X)$ containing $C^*(X)$. A subset $T$ of $X$ is called $S$-embedded in $X$ if every hard-bounded continuous function of $T$ has a continuous extension upto $X$. In our last paper, we introduced this notion in order to study some extension properties of nearly pseudocompactification. No further investigations were done in that paper.

In this paper, we are focused in detailing of $S$-embeddedness. We first proved a criteria for $S$-embeddedness. We have shown, a $C^*$-embedded subset $T$ of $X$ is $S$-embedded if and only if $T$ is completely separated from every zero set $Z(g)$ in $X$ where $g$ is bounded away from zero on every hard subset of $T$. Then we have obtained some necessary conditions of a continuous function $f$ which is bounded away from zero on every hard subset of $T$. Indeed we have shown that within the class of pseudocompact spaces a real-valued continuous function is bounded away from zero on every hard subset of $T$ if and only if $Z(f)$ and any hard subset of $T$ are completely separated. Finally we have investigated the following problem. If a sequence of hard-bounded continuous functions converges uniformly to a function $f$ then under what restriction, is $f$ hard-bounded? We have given a necessary and sufficient condition to the above problem.

2. PRELIMINARIES

In this paper, we used the most of preliminary concepts, notations and terminologies from the classic monograph of L. Gillman and M. Jerison, Rings of Continuous Functions [2]. However for ready references, we recall few notations, frequently used over here. For any $f \in C(X)$ or $C^*(X)$, $Z(f) = \{ x \in X : f(x) = 0 \}$, called zero set of $f$. Complement of zero set is called cozero set or cozero part of $f$, denoted as $cozf$. For any $f \in C(X)$ or $C^*(X)$, $cl_X(X \setminus Z(f))$ is called the support of $f$. Two subsets $A$ and $B$ of $X$ are said to be completely separated in $X$...
if $A$ and $B$ are contained in two disjoint zero sets. A subspace $Y$ of $X$ is called $C$-embedded in $X$ if every function in $C(Y)$ can be extended to a function in $C(X)$ and $Y$ is $C^*$-embedded in $X$ if every function in $C^*(Y)$ can be extended to a continuous function in $C(X)$. If $f \in C(X)$ is unbounded on $E$, then $E$ contains a copy of $\mathbb{N}$, $C$-embedded in $X$ along which $f$ tends to $\infty$. A function $f$ in $C(X)$ is called bounded away from zero on $X$ if there exists $m > 0$ such that $|f(x)| \geq m$ for all $x \in X$. A non-zero function $f \in C(X)$ (or $C^*(X)$) is a unit if and only if $Z(f) = \varnothing$ (or $f$ is bounded away from zero). A space is realcompact if and only if every $\omega$-ultrafilter with countable intersection property is fixed. In the year 1976, Rayburn [1] introduced hard set.

**Definition 1.** A subspace $H$ of $X$ is called hard in $X$ if $H$ is closed in $X \cup K$, $K = \text{cl}_X(\nu X \setminus X)$ where $\beta X$ and $\nu X$ are the Stone-Čech compactification and Hewitt realcompactification of $X$ respectively.

It immediately follows that every hard set is closed in $X$, but the converse is obviously not true. Clearly every compact subset of $X$ is hard, but the converse may not be true. A hard set is compact if and only if $X$ is nearly pseudocompact.

3. MAIN RESULTS

In this section we shall discuss about hard-bounded continuous functions and $S$-embedded subsets of a space $X$.

**Definition 2.** A continuous function is said to be hard-bounded if it is bounded on every hard subsets of $X$.

Let $X$ be a Tychonoff space and $S(X)$ denotes the family of all hard-bounded continuous functions on $X$, that is $S(X) = \{ f \in C(X) : f$ is bounded on every hard set in $X \}$. Then $S(X)$ is an intermediate subring of $C(X)$ containing $C^*(X)$. It is given in [8] that $X$ is nearly pseudocompact if and only if $S(X) = C(X)$. It will be an interesting project to investigate those spaces where $S(X) = C^*(X)$. However our present interest lies on $S$-embeddedness and some related results.

**Definition 3.** A subset $T$ of $X$ is said to be $S$-embedded if any hard bounded continuous function on $T$ can be continuously extended upto $X$.

It follows that every $S$-embedded subset is $C^*$-embedded. We now give a necessary and sufficient condition for a $C^*$-embedded subset to be $S$-embedded.

**Theorem 4.** A $C^*$-embedded subset $T$ of $X$ is $S$-embedded if and only if $T$ is completely separated from every zero set $Z(g)$ in $X$ where $g$ is bounded away from zero on every hard subset of $T$.

**Proof.** Suppose, $T$ be a $C^*$-embedded subset of $X$, which is $S$-embedded in $X$, that is every hard-bounded continuous function on $T$ can be continuously extended upto $X$. Now let $Z(g)$ be a zero set in $X$ such that $g$ is bounded away from zero on every hard subset of $T$. Clearly $Z(g) \cap T = \varnothing$ and $g$ is bounded away from zero on every hard zero subset of $T$, so $\frac{1}{\gamma} |T| = \gamma$ is a hard-bounded continuous function on $T$. Let $H$ be a hard set in $T$, then there exists $m > 0$ such that $|g(x)| \geq m$, for all $x \in H \Rightarrow \frac{1}{\gamma} |x| \leq \frac{1}{m}$, for all $x \in H \Rightarrow \gamma(x) \leq \frac{1}{m}$, for all $x \in H \Rightarrow \gamma$ is bounded on every hard set in $T$. So $\gamma$ is a hard-bounded continuous function on $T$. Since $T$ is $S$-embedded in $X$, so there exists $h \in C(X)$ such that $h|_T = \gamma$. Hence $hg \in C(X)$. Now on $T$, $hg(x) = h(x).g(x) = \gamma(x).g(x) = \frac{1}{g(x)} g(x) = 1$ and $hg(x) = h(x).g(x) = 0$ when $x \in Z(g)$. Therefore $T$ and $Z(g)$ are completely separated.

Conversely, let $T$ is completely separated from every zero set $Z(g)$ in $X$ where $g$ is bounded away from zero on every hard subset of $T$. Let $T$ is $C^*$-embedded in $X$. Let $f \in S(T)$, that is $f$ is hard-bounded continuous function on $T$. To show there exists $u \in C(X)$ such that $u|_T = f$. Take $\tan^{-1} f \in C^*(T)$. As $T$ is $C^*$-embedded in $X$, there exists a continuous map $h : X \to \mathbb{R}$ such that $h|_T = \tan^{-1} f$. Now let $Z = \{ x \in X : |h(x)| \geq \frac{\pi}{2} \} = \{ x \in X : (|h(x)| - \frac{\pi}{2}) \geq 0 \} = Z(|h(x)| - \frac{\pi}{2} \wedge 0) = Z(g)$ where $g = (|h| - \frac{\pi}{2} \wedge 0$. Clearly $Z(g) \cap T = \varnothing$. Remaining to show that $g$ is bounded away from zero on every hard set in $T$. Let $H$ be a hard set in $T$. Then
Theorem 8. Let every hard subset of $T$ be uniformly. Then $|Z| \exists \{f \}$ converges. Let $\phi \in Z$. Then $h \in \phi \subseteq X$. So we can completely separate $T$ and $Y \in X$. Therefore $T$ is $S$-embedded in $X$.

**Theorem 5.** $f$ is hard-bounded continuous function on $X$ if and only if $f$ is bounded on every realcompact cozero set in $X$.

**Proof.** Let $f$ be a hard-bounded continuous function on $X$. Let $P$ be a realcompact cozero set in $X$. If $f$ is unbounded on $P$. Then $P$ contains a $C$-embedded copy of $\mathbb{N}$ along which $f$ tends to infinity. Then $\mathbb{N}$ and $X \setminus P$ are completely separated. So $\mathbb{N}$ is hard in $X$ on which $f$ is unbounded, a contradiction.

Conversely, let $H$ be a hard set in $X$. There exists a compact set $K$ which satisfies the property of hardness. There exist a positive real number $\epsilon$ such that $K \subseteq x \in X : |f(x)| < \epsilon = V(say)$. Then $H \setminus V$ is contained in a realcompact cozero set. Thus $f$ is bounded on $H \setminus V$ also. Hence $f$ is bounded on $H$.

**Theorem 6.** If $Y$ is expressed as finite union of hard sets and $Y$ is $C'$-embedded then $Y$ is $S$-embedded.

**Proof.** Let $Y = \bigcup_{i=1}^{m} H_i$. Let $g \in C(X)$ such that $g$ is bounded away from zero on every hard subset of $Y$. For each $i$, there exists $n_i > 0$ such that $Z_{n_i}^{g} = \{x : |g(x)| \geq n_i \} \supset H_i \Rightarrow \bigcup_{i=1}^{m} Z_{n_i}^{g}$ is a zero set and call it $Z$. Then $Z \cap Z(g) = \phi$ and $Y \subset Z$. So $Z(g)$ is completely separated with $Y$. Hence $Y$ is $S$-embedded.

We shall now investigate nature of hard-bounded continuous functions under uniform convergence.

**Theorem 7.** Let $\{f_n\}$ be a sequence of hard-bounded continuous function such that $f_n \to f$ uniformly. Then $f$ is hard-bounded continuous and if only if for every hard set $H$ in $X$, there exists a sequence $\{m_n^H\}$ and $m^H$ such that $m_n^H \leq m^H$ for all $n \geq k$ for some integer $k$ and $|f_n(H)| \leq m_n^H$, $|f(H)| \leq m^H$.

**Proof.** Suppose $f$ is hard-bounded continuous function on $X$. Let $H$ be a hard set in $X$. Let $m_n^H = \sup_{x \in H} |f_n(x)|$. Now for all $x \in H$, $|f_n(x)| \leq |f_n(x) - f(x)| + |f(x)|$. As $f_n$ converges to $f$ uniformly, for any $\epsilon > 0$, there exists $n > 0$ such that for all $m \geq n$, $|f_m(x) - f(x)| < \epsilon$. So $|f_m(x)| \leq |f_m(x) - f(x)| + |f(x)| < \epsilon + |f(x)|$ for all $x \in H$. Since $f$ is hard-bounded, there exists $m^H$ such that $|f(x)| \leq m^H$ for all $x \in H \Rightarrow |f_m(x)| < \epsilon + m^H$ for all $x \in H \Rightarrow \sup_{x \in H} |f_n(x)| \leq \epsilon + m^H \Rightarrow m_n^H \leq \epsilon + m^H$. Letting $\epsilon \to 0$, we get $m_n^H \leq m^H$ for all $m \geq n$.

Conversely, let for every hard set $H$ in $X$, there exists a sequence $\{m_n^H\}$ and $m^H$ such that $m_n^H \leq m^H$ for all $n \geq k$ for some integer $k$ and $|f_n(H)| \leq m_n^H$, $|f(H)| \leq m^H$. We have to show that $f$ is hard-bounded. Let, $H, B$ be a hard set in $X$. Therefore $|f_n(H)| \leq m_n^H$ and $m_n^H \leq m^H$ for all $n \geq k \Rightarrow |f_n(H)| \leq m^H \Rightarrow |f(H)| \leq m^H$. Hence $f$ is hard-bounded continuous map.

In next few theorems, we discussed about those function which are bounded away from zero on every hard subset of a subset of $X$:

**Theorem 8.** Let $X$ be a pseudocompact space. An $f \in C(X)$ is bounded away from zero on every hard subset of $T$ if and only if $Z(f) \cap T = \phi$. 
Proof. Suppose \( f \in C(X) \) is bounded away from zero on every hard subset of \( T \). Then, \( \{x\} \) being hard subset of \( T \), \( f(x) \neq 0 \) for all \( x \in T \). Therefore \( Z(f) \cap T = \emptyset \).

Conversely, suppose \( Z(f) \cap T = \emptyset \). We have to show that, \( f \) is bounded away from zero on every hard subset of \( T \). Let us assume that \( f \) is not bounded away from zero on every hard subset of \( T \). Then there exists a hard subset \( H \) of \( T \) in which \( f \) is not bounded away from zero. So for every \( m > 0 \), \( H \not\subseteq \{x \in X : |f(x)| \geq m\} \Rightarrow \forall m \) there exists \( x_m \in H \) such that \( |f(x_m)| < m \). Again, as \( Z(f) \cap H = \emptyset \), so \( |f(x)| > 0 \), \( \forall m \). Then we have a sequence \( \{x_n\} \) in \( H \) satisfying the following properties: \( |f(x_1)| < 1 \) and \( |f(x_n)| < \min\{|f(x_{n-1})|, \frac{1}{m}\} \). So \( |f(x)| < |f(x_{n-1})| \) and \( |f(x)| < \frac{1}{m} \), for all \( n \in \mathbb{N} \). Now we choose a sequence of closed intervals \( \{I_n\} \) such that \( |f(x)| \in \text{int}I_n \), for all \( n \in \mathbb{N} \) and \( I_n \cap I_m = \emptyset \), for all \( n \neq m \). Accordingly we have a sequence of closed sets \( \{V_n\} \) such that \( x_n \in \text{int}V_n \subset V_n \) and for all \( x \in V_n \), \( |f(x)| \in I_n \). Clearly \( V_n \cap V_m = \emptyset \) for \( m \neq n \) and \( V_n \neq \emptyset \) for all \( n \). We shall now show that for \( m \in \mathbb{N} \), \( \bigcup_{n \neq m} V_n \) is closed in \( X \). Let \( x \in X \setminus \bigcup_{n \neq m} V_n \), then \( |f(x)| > 0 \) and there exists a \( k \) such that \( |f(x_{k+1})| \leq |f(x)| < |f(x_k)| \) for \( k \leq 1 \) or \( |f(x)| \geq |f(x_1)| \). In any case there exists a neighbourhood of \( x \) which intersects at most two \( V_k \)'s. Thus \( \{V_n : n \neq m\} \) is locally finite and hence \( \bigcup_{n \neq m} V_n \) is closed in \( X \). By \cite{2} (Exercise 3L.), \( N = \{x_n : n \in \mathbb{N}\} \) is a copy of \( \mathbb{N} \), \( C \)-embedded in \( X \). Therefore \( X \) is not pseudocompact, a contradiction. \( \square \)

The following counter example asserts that we can not drop the condition of pseudocompactness of \( X \) in the above theorem.

Example 9. Take \( X = \mathbb{N} \) with discrete topology. Take \( T = 2\mathbb{N} \). As \( T \) is realcompact, hard sets of \( T \) are precisely the closed subsets of \( T \). Also note that, any subset of a discrete space is a zero set. We can construct a continuous function \( f : X \to \mathbb{R} \) satisfying \( Z(f) = X \setminus T \) and \( f(x) = \frac{1}{x} \), whenever \( x \in T \). It is easy to observe that \( Z(f) \cap T = \emptyset \), but \( f \) is not bounded away from zero on every hard subset of \( T \).

Theorem 10. Let \( T \) be a nearly pseudocompact subset of \( X \). An \( f \in C(X) \) is bounded away from zero on every hard subset of \( T \) if and only if \( Z(f) \cap T = \emptyset \).

Proof. Following the proof of Theorem 3 above, the set \( N \) is hard in \( T \). As \( T \) is nearly pseudocompact, \( N \) is compact which is absurd. Rest follows from the above proof of Theorem[3]. \( \square \)

Theorem 11. \( f \in S(X) \) is a unit of \( S(X) \) if and only if \( f \) is bounded away from zero on every hard subset of \( X \).

Proof. Let \( f \in S(X) \) be a unit of \( S(X) \). So there exists \( g \in S(X) \) such that \( fg = 1 \). Let \( H \) be a hard subset of \( X \). Since \( g \in S(X) \), \( |g(x)| \leq m \) for all \( x \in H \), for some \( m > 0 \) \( \Rightarrow |f(x)| \geq \frac{1}{m} \) for all \( x \in H \) \( \Rightarrow f \) is bounded away from zero on \( H \). Hence \( H \) being arbitrary, \( f \) is bounded away from zero on every hard subset of \( X \).

Conversely, suppose that \( f \in S(X) \) is bounded away from zero on every hard subset of \( X \). Then \( f(x) \neq 0 \) for all \( x \in X \), since \( \{x\} \) being compact, it is hard and hence \( f \) can not be zero at \( x \). So \( \frac{1}{f} \) exists and it is continuous on \( X \). Now we have to show \( \frac{1}{f} \in S(X) \). Let \( H \) be a hard subset of \( X \). By our assumption \( |f(x)| \geq m_H \) for all \( x \in H \), for some \( m_H > 0 \) \( \Rightarrow |\frac{1}{f(x)}| \leq \frac{1}{m_H} \) for all \( x \in H \) \( \Rightarrow \frac{1}{f} \) is bounded on \( H \). Hence \( H \) being arbitrary, \( \frac{1}{f} \) is bounded on every hard subset of \( X \) i.e. \( \frac{1}{f} \in S(X) \). Therefore \( f \) is a unit of \( S(X) \). \( \square \)

Corollary 3.1. A \( C^* \)-embedded subset \( T \) of \( X \) is \( S \)-embedded if and only if for all \( f \in C(X) \), \( f|_T \) is a unit of \( S(T) \) and \( Z(f), T \) are completely separated.

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