Instability of the Non-Fermi-Liquid Fixed Point
in the Dissipative Gauge Theory of Fermions (I):
Impurity Effects

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Abstract

We study a dissipative gauge theory of nonrelativistic fermions in 2+1 dimensions at zero temperature by the Wilsonian renormalization-group method. In this theory, we incorporate in the fermion propagator a new term of the form \( i\kappa \cdot \text{sgn}(\omega) \) (where \( \kappa \) is a parameter and \( \omega \) is the fermion frequency), which is usually induced by impurity effects. In the previous papers, we studied this system for \( \kappa = 0 \), and showed that there exists a non-Fermi-liquid infrared fixed point. In this paper, we address the question whether this non-Fermi-liquid behavior remains stable or not in the presence of impurity effects, i.e., the \( \kappa \) term. Our results show that the non-Fermi-liquid fixed point is unstable for \( \kappa \neq 0 \) and an effective gauge coupling constant tends to vanish at low energies. However, in intermediate energy scales, the behavior of correlation functions for \( \kappa \neq 0 \) is controlled by the non-Fermi-liquid fixed point at \( \kappa = 0 \), i.e., a crossover phenomenon appears. Physical implications of this phenomenon are discussed.

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1 Introduction

In condensed matter physics, low-dimensional electron systems are of current interest, which include the high-$T_c$ superconductivity, the fractional quantum Hall effect, and the recently discovered new spin ladder systems which exhibit superconductivity. Theory of nonrelativistic fermions interacting with a gauge field plays an important role as an effective reference theory for these interesting systems. A typical example of such gauge theory is found in the work of Halperin, Lee, and Read [1], in which they studied a system of electrons in the half-filled Landau level. This gauge theory is also regarded as an important constituent to compose a low-energy effective field theory for the t-J model [2].

Recently, it has been recognized that the renormalization-group (RG) method can be very efficiently applicable to a system of nonrelativistic fermions for investigating its low-energy behavior [3]. Nayak and Wilczek [4] made a RG analysis of a kind of $\epsilon$-expansion for the above gauge theory in Ref. [1] at zero temperature, and claimed existence of a nontrivial infrared (IR) fixed point in some parameter region. Following their works there appeared some related works [5, 6]. This fixed point is very interesting because it implies a departure from the conventional Fermi liquid theory; At that fixed point, the theory exhibits non-Fermi-liquid-like behavior similar to that of a marginal Fermi liquid for the high-$T_c$ superconductivity [7] or of a Luttinger liquid in one dimension [8].

In this theory, a dissipative term is generated in the gauge-field propagator via quantum fluctuations of fermions, and this term becomes important at low energies. Thus its effect to the above nontrivial fixed point should be clarified. In the Nayak and Wilczek approach, the relevance of this term to the above fixed point is not reflected well, since the dissipative term has no divergence in their $\epsilon$-expansion, hence not being treated as a relevant coupling. In the previous paper [9], we considered a two-dimensional model of nonrelativistic fermions interacting with a gauge field having a dissipative term from the beginning, and carried out the Wilsonian RG (WRG) analysis, treating this dissipative term as a genuine running coupling constant. We found that there still exists a nontrivial fixed point. In this sense, the fixed point found by Nayak and Wilczek is stable against the dissipative term.
Since this fixed point is so important, it is worth studying its stability from a more general point of view. In this paper, we study its stability against “impurities” in detail. Explicitly, we incorporate an additional term in the fermion propagator of the previous dissipative gauge theory of Ref.[9], and study its low-energy effects by the WRG method. The term added is

$$-i\kappa \cdot \text{sgn}(\omega),$$

where \(\kappa\) is a parameter, \(\omega\) is the fermion frequency, and

$$\text{sgn}(x) = \begin{cases} 
1 & (x > 0) \\
-1 & (x < 0) 
\end{cases}.$$  \hspace{1cm} (1.1)

Motivation for adding the above term is twofold. First, this term is just generated via scattering of fermions by impurities. Real physical systems necessarily contain impurities, and they often play an important role; for example, in localization phenomenon in the quantum Hall effect. It is natural to extend the previous analysis in Ref.[9] to a system containing impurities. Usually, one treats effects of random impurities by letting them to interact with fermions through a potential \(v(x)\), so the interaction Hamiltonian is given by

$$H_{\text{int}} = \int dx \psi^\dagger(x) \sum_i v(x - x_i) \psi(x),$$  \hspace{1cm} (1.2)

where \(\psi(x)\) is the fermion field operator and \(x_i\) are coordinates of impurities. Practical calculations at one-loop level show that the self energy \(\Sigma_k(\omega)\) of fermions has the following signature term

$$\Sigma_k(i\omega) = -i\kappa \cdot \text{sgn}(\omega),$$  \hspace{1cm} (1.3)

where \(\kappa(>0)\) is related with the Fourier modes of \(v(x)\). In the classical equation of motion of a fermion, the resistive (friction) force is given by \(\propto -\kappa \vec{v}\), where \(\vec{v}\) is the velocity of fermion. In the random phase approximation (RPA) with respect to impurities, the fermion propagator is simply given as

$$G_0(k, \omega) = \frac{1}{i\omega + i\kappa \text{sgn}(\omega) + \mu - E_k},$$  \hspace{1cm} (1.4)

where \(E_k\) is the energy of fermion with momentum \(k\), and \(\mu\) is the chemical potential.
The constant $\kappa(> 0)$ has another physical significance. This term modifies the momentum distribution of the fermion density which exhibits a step-function behavior at zero temperature $T = 0$ and $\kappa = 0$. Actually, from the propagator (1.4), the density of fermions $N(E_k)$ at $E_k$ is obtained as

$$N(E_k) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi i\omega + i\kappa \text{sgn}(\omega) + \mu - E_k} + \frac{1}{2} = \text{sgn}(\mu - E_k)\left(\frac{1}{2} - \frac{1}{\pi} \arctan \frac{\kappa}{|\mu - E_k|}\right) + \frac{1}{2}. \quad (1.5)$$

The curve $N(E_k)$ is plotted in Fig.1. From (1.3) and Fig.1, we see that the $\kappa$ term removes the singularity at $E_k = 0$.

As is well known, this smooth momentum distribution (1.5) looks similar to that of a system of fermions at finite temperature ($T$). The second motivation is related with this point. When we consider thermal effects at finite $T$ in this model, quantum corrections induce this signature term in the fermion self energy, even if we start with $\kappa = 0$.[10]. Thus we must consider this signature term from the beginning when we investigate the system at $T > 0$ by using the WRG analysis, treating $\kappa$ as a running coupling constant. These motivations let us start with the fermion action with the signature term as given by Eq.(2.2) in Sec.2.

As we explained above, in the case of $\kappa = 0$ there is the nontrivial IR fixed point at which the system exhibits non-Fermi-liquid-like behavior[1, 4]. We want to study the stability of this non-Fermi-liquid fixed point and the RG flow in the presence of the signature term. Throughout the paper we consider the case of zero temperature ($T = 0$). The other important case of finite $T$ will be discussed separately[10].

This paper is organized as follows. In Sec.2, we shall introduce the model. In Sec.3, the RG prescription is explained. We shall employ some specific momentum regularization, which is suitable for the present gauge-fermion system. In Sec.4, we shall calculate one-loop corrections of fermions and gauge bosons by the momentum-shell integration. In Sec.5, renormalization constants and RG equations are explicitly obtained. Qualitative and numerical solutions of the RG equations are given. It is shown that the non-Fermi-liquid fixed point at $\kappa = 0$ is unstable for nonzero $\kappa$. However, at certain intermediate
energy scales, the behavior of the system is controlled by that fixed point. This means that the system exhibits a cross-over phenomenon. In Sec.6, by solving the RG equation for the fermion propagator, we obtain its explicit form at intermediate energy scales and also in the low-energy limit, confirming the above conclusion of the cross-over. Section 7 is devoted to concluding remarks.

2 The Model

In this section we explain our model in detail, which describes a 2-dimensional system of nonrelativistic spinless fermions \( \psi(x, \tau) \) interacting with a dissipative gauge field \( A_i(x, \tau) \) \((i = 1, 2)\), where \(0 < \tau < \infty\) is the imaginary time. Relationships of this model with electrons in the half-filled Landau level and with strongly correlated electron systems for high-\( T_c \) superconductivity are discussed in Refs.\[1, 2\].

To study the low-energy excitations around the Fermi surface and perform the WRG analysis for the gauge-fermion system, it is useful to introduce a momentum cutoff as shown in Figs.2 and 3\[9\]. The region around the Fermi surface is divided into \( N \) segments, and the fermion field in each divided part is labeled by \( a, a = 1, ..., N \), as \( \psi(a; p, \omega) \). The fermion momentum \( p \) is measured from a point on the Fermi surface \( k_{F,a} \), the center of the \( a \)-th part. The momentum cutoff \( \Lambda \) is introduced as \( |p| = |k - k_{F,a}| < \Lambda \), where \( k \) is the original fermion momentum.

In the standard path integral formalism, the partition function of this model is given by

\[
Z = \int [d\bar{\psi}] [d\psi] [dA] \exp(-S_\psi - S_A - S_{int.3}). \tag{2.1}
\]

In (2.1), \( S_\psi \) is the free part of fermions given in the Fourier (momentum and energy) representation by

\[
S_\psi = \int d\omega \int dp \sum_a \bar{\psi}(a; p, \omega)(-i\omega - ik\text{sgn}(\omega) + v_F \epsilon_a \cdot p)\psi(a; p, \omega), \tag{2.2}
\]

where \( \epsilon_a \) is the unit vector in the direction of \( k_{F,a} \) and

\[
\frac{dp}{(2\pi)^2}, \quad \frac{d\omega}{2\pi}, \quad v_F = k_F/m. \tag{2.3}
\]
The free part of gauge field $S_A$ is given by
\[
S_A = \int d\epsilon \int dq A(-q, -\epsilon) \Lambda^{b-1} v_B \left( q^{2-b} + \Lambda^{2-b} \lambda \frac{|\epsilon|}{v_F q} \right) A(q, \epsilon).
\] (2.4)

The second term, with the coefficient $\lambda > 0$, is the so-called dissipative term (Landau dumping factor) that is induced by the vacuum polarizations of fermions at one-loop level. It is shown in the following discussion that the parameter $\lambda$ behaves nontrivially under RG transformation. We have taken the Coulomb gauge $\partial_i A_i = 0$, then the gauge field $A_i$ has only one dynamical degree of freedom, $A(q, \epsilon) \equiv i\epsilon_{ij} q_j A(q, \epsilon)/q$. We have introduced a parameter $b$ which controls fluctuations of the gauge field. This generalization was first introduced in Ref.[1]. The case $b = 1$ corresponds to the Coulomb potential for electrons in the half-filled Landau level, and the case $b = 0$ corresponds to the case of the $t$-$J$ model. The three-point interaction term of the action $S_{int,3}$ is given by
\[
S_{int,3} = igv_F \int d\omega d\epsilon \int dp dq \sum_a \frac{\epsilon_a}{q} A(q, \epsilon) \bar{\psi}(a; p + q, \omega + \epsilon) \psi(a; p, \omega),
\] (2.5)
where $p \times q \equiv \epsilon_{ij} p_i q_j$. In a nonrelativistic fermion-gauge field system, the action also contains the four-point interaction term like $AA\bar{\psi}\psi$. We shall neglect this term, because this term contributes only to the renormalization of the chemical potential of the fermion and the effective mass of the gauge field. Actually, in the field theoretical approach[5], it is shown that the mass of gauge field vanishes because the contribution from the four-point interaction cancels that from the three-point interaction, as guaranteed by the gauge invariance. In the following discussion, we shall not discuss the contribution from these terms to the renormalization of the chemical potential and the gauge field mass.

3 Wilsonian renormalization group program

In this section, we shall explain the WRG program for our gauge-fermion system introduced in Sec.2. In the previous paper[9], we studied the case $\kappa = 0$ by the WRG and showed that there is a nontrivial IR fixed point for $b < 1$. Let us explain the procedures of WRG program briefly once again. It consists of the following three steps (i)-(iii).

**Step (i)** Integrate over the high-momentum modes (modes with the momentum $\Lambda/e^t < p < \Lambda; t(>0)$ is a scaling parameter) of fermion and gauge fields. Then the action of the
low-momentum modes is obtained as follows;

\[
\tilde{S}_\psi = \int d\tilde{\omega} \int^{\Lambda/e^t} dp \times \sum_a \tilde{\psi}(a; p, \omega) R(t) \left(-i\tilde{\omega} - iR_\kappa(t)\kappa\text{sgn}(\omega) + R_{v_F}(t)v_F e_a \cdot \tilde{p} \right) \psi(a; p, \omega),
\]

\[
\tilde{S}_A = \int d\epsilon \int^{\Lambda/e^t} dq A(-q, -\epsilon)A^{b-1}v_B \left( q^{2-b} + A^{2-b}R_\lambda(t)\frac{|\epsilon|}{v_F q} \right) A(q, \epsilon),
\]

\[
\tilde{S}_{\text{int},3} = \int d\omega d\epsilon \int^{\Lambda/e^t} dp dq \times \sum_a iR_g(t)gv_F \frac{e_a \times q}{q} A(q, \epsilon)\tilde{\psi}(a; p + q, \omega + \epsilon)\tilde{\psi}(a; p, \omega),
\] (3.1)

where \(R(t),\) etc. are \(t\)-dependent renormalization constants. We have neglected terms that are higher-orders in momentum or fields as irrelevant terms. This point is discussed somewhat in detail in Ref. [11].

**Step (ii)** The original action \(S\) and the new action \(\tilde{S} = \tilde{S}_\psi + \tilde{S}_A + \tilde{S}_{\text{int},3}\) are defined in the two different kinematical regions, and then we rescale momenta in order to return the new momentum cutoff \(\Lambda/e^t\) to the original value \(\Lambda\). Similarly, we require so that certain quadratic terms in the effective action have the same coefficients with the original ones by rescaling fields. At the same time we scale energy, momentum and the fields. In nonrelativistic theories like the present model, there is an ambiguity in assigning scaling dimensions of energy and fields, because there is no principle for that. This is in sharp contrast with relativistic systems where it is natural to assign same scalings for energy and momentum. Therefore we introduce a general scaling law which contains a free parameter \(\xi > 0\) as follows,

\[
\omega \rightarrow \tilde{\omega} = e^{\xi t} \omega,
\]

\[
p \rightarrow \tilde{p} = e^t p, \ (|p| \leq \Lambda/e^t, \ |\tilde{p}| \leq \Lambda)
\]

\[
\psi(a; p, \omega) \rightarrow \tilde{\psi}(a; \tilde{p}, \tilde{\omega}) = e^{-(\xi+1)t} \sqrt{R(t)} \psi(a; p, \omega),
\]

\[
A(p, \omega) \rightarrow \tilde{A}(\tilde{p}, \tilde{\omega}) = e^{-(2+\frac{b}{\xi})t} A(p, \omega)
\] (3.2)

**Step (iii)** The effective action \(\tilde{S}\) is expressed in terms of the above variables;

\[
\tilde{S}_\psi = \int d\tilde{\omega} \int^{\Lambda} d\tilde{p} \sum_a \tilde{\psi}(a; \tilde{p}, \tilde{\omega})(-i\tilde{\omega} - i\kappa(t)\text{sgn}(\tilde{\omega}) + v_F(t)e_a \cdot \tilde{p})\tilde{\psi}(a; \tilde{p}, \tilde{\omega}),
\]
\[ \tilde{S}_A = \int d\epsilon \int A(-\tilde{q}, -\tilde{\epsilon}) A^{b-1} v_B \left( \tilde{q}^{2-b} + \Lambda^{2-b} \lambda(t) \frac{|\tilde{\epsilon}|}{v_F(t) q} \right) \tilde{A}(\tilde{q}, \tilde{\epsilon}), \]

\[ \tilde{S}_{\text{int,3}} = \int d\tilde{\omega} d\tilde{\epsilon} \int A \sum_a ig(t) v_F(t) \frac{\epsilon_a \times \tilde{q}}{q} \tilde{A}(\tilde{q}, \tilde{\epsilon}) \tilde{\psi}(a; \tilde{p} + \tilde{q}, \tilde{\epsilon} + \tilde{\omega}) \tilde{\psi}(a; \tilde{p}, \tilde{\omega}). \] (3.3)

The effective parameters (coupling constants) which flow as the scaling parameter \( t \) varies have been defined as follows,

\[
\begin{align*}
g(t) &= e^{(1-b+\xi)t} \frac{R_g(t)}{R(t) R_{v_F}(t)} g, \\
v_F(t) &= e^{(\xi-1)t} R_{v_F}(t) v_F, \\
\lambda(t) &= e^{(2-b)t} R_\lambda(t) \lambda, \\
\kappa(t) &= e^{\xi t} R_\kappa(t) \kappa. \tag{3.4}
\end{align*}
\]

The simplest assignment of scaling dimensions is \( \xi = 1 \). One-loop calculation shows that RG recursion equations of the effective parameters depend on the values of \( \xi \). But this \( \xi \)-dependence must be superficial. Actually, as we show in later discussion, physical quantities are independent of \( \xi \).

4 One-loop quantum corrections and renormalization constants

In this section, we shall perform the momentum-shell integration for an infinitesimal RG transformation with \( \exp(dt) \simeq 1 + dt \) at one-loop level and calculate the renormalization constants \( R, R_{v_F}, R_g, R_\lambda \) and \( R_\kappa \). With this result of infinitesimal RG transformation, we derive the RG equations of our running coupling constants, \( g(t), v_F(t), \lambda(t) \) and \( \kappa(t) \). We consider the fermion part in the action in Sec.4.1, the gauge-boson part in Sec.4.2, and the vertex part in Sec.4.3.

4.1 Renormalization of the fermion part

The bare action is renormalized by the one-loop quantum correction which is obtained by integrating over the high-momentum modes of fermion field and gauge field in the momentum region \( \Lambda(1-dt) < p < \Lambda \). The effective action is obtained as follows

\[ \tilde{S}_\psi = \sum_a \int d\omega \int_0^{\Lambda(1-dt)} dp \tilde{\psi}(a; p, \omega) \left[ -i\omega - i \kappa \text{sgn}(\omega) + v_F e_a \cdot p + \Sigma(a; p, \omega) \right] \psi(a; p, \omega), \tag{4.1} \]
where $\Sigma(a; p, \omega)$ is the fermion self-energy coming from the high-momentum modes. It is given by

$$\Sigma(a; p, \omega) = g^2 v_F^2 \int d\epsilon \int_{\Lambda(1-dt)}^\Lambda dq \left( \frac{e_a \times q}{q} \right)^2 D_0(q, \epsilon) G_0(a; p + q, \omega + \epsilon), \quad (4.2)$$

where $D_0(q, \epsilon)$ is propagator of the gauge field obtained from (2.4),

$$D_0(q, \epsilon) = \frac{1}{\gamma(q)(|\epsilon| + F(q))},$$

$$\gamma(q) = \frac{v_B \Lambda}{v q},$$

$$F(q) = \frac{\gamma^{-1}(q) \Lambda^{b-1} v_B q^{2-b}}{\lambda} = \frac{v_B q}{\Lambda} (\frac{q}{\Lambda})^{2-b}. \quad (4.3)$$

Similarly, $G_0(a; p, \omega)$ is the fermion propagator, obtained from $S_\psi$ of (2.2) as

$$G_0(a; p, \omega) = \frac{1}{i \omega + i \kappa \text{sgn}(\omega) - E_p},$$

$$E_p \equiv v_F e_a \cdot p, \quad (4.4)$$

with $\kappa > 0$.

The momentum-shell integration over the radial direction in Eq.(4.2) can be easily performed to obtain the self-energy as

$$\Sigma(a; p, \omega) = -i \alpha F_\Lambda dt \int_{-\lambda}^\lambda dy \sin \phi(y) I(\bar{\epsilon}, \bar{\omega}, \bar{\pi}), \quad (4.5)$$

where

$$I(\bar{\epsilon}, \bar{\omega}, \bar{\pi}) \equiv \int_{-\infty}^{\infty} d\bar{\epsilon} \frac{1}{\bar{\epsilon} + \bar{\omega} + \bar{\pi} \text{sgn}(\bar{\epsilon} + \bar{\omega}) + i\bar{\pi}} \cdot \frac{1}{|\bar{\epsilon}| + 1}. \quad (4.6)$$

In the above we have introduced

$$\alpha \equiv \frac{g^2 v_F}{2\pi^2 v_B},$$

$$F_\Lambda \equiv F(\Lambda) = \frac{v_F \Lambda}{\lambda},$$

$$\bar{E} \equiv y + \frac{v_F e_a \cdot q}{F_\Lambda}, \quad y \equiv \frac{v_F \Lambda \cos \phi}{F_\Lambda},$$

$$\bar{\epsilon} \equiv \frac{\epsilon}{F_\Lambda}, \quad \bar{\omega} \equiv \frac{\omega}{F_\Lambda}, \quad \bar{\pi} \equiv \frac{\kappa}{F_\Lambda}. \quad (4.7)$$

The $\phi$ is the angle between the two vectors $p$ and $e_a$. In the above momentum integration, we changed the integration variable from $\phi$ to $y$. 

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We divide the integrand \( I \) \((4.6)\) into six terms as there are two signature terms \(\text{sgn}(\tau+\bar{\tau})\) and \(|\tau|\).

\[
I(\omega, E, \kappa) = \Theta(\omega) \sum_{i=1}^{3} I_i(\omega, E, \kappa) - \Theta(-\omega) \sum_{i=1}^{3} I_i(-\omega, -E, \kappa),
\]

where

\[
I_1(\omega, E, \kappa) \equiv \int_{-\infty}^{-\omega} d\tau \frac{1}{\tau + \bar{\omega} + \kappa + iE} \cdot \frac{1}{-\tau + 1},
\]

\[
I_2(\omega, E, \kappa) \equiv \int_{-\omega}^{0} d\tau \frac{1}{\tau + \omega + \kappa + iE} \cdot \frac{1}{-\tau + 1},
\]

\[
I_3(\omega, E, \kappa) \equiv \int_{0}^{\infty} d\tau \frac{1}{\tau + \omega + \kappa + iE} \cdot \frac{1}{\tau + 1},
\]

(4.9)

and \(\Theta(x)\) is the step function,

\[
\Theta(x) = \begin{cases} 
1 & (x > 0) \\
0 & (x < 0) 
\end{cases}.
\]

(4.10)

The integrations in \((4.9)\) can be evaluated as

\[
I_1(\omega, E, \kappa) = -\frac{1}{z_2 + 1} \left[ \frac{1}{2} \ln \frac{(\omega + 1)^2}{\kappa^2 + E^2} + i\text{sgn}(E)(\frac{\pi}{2} - \arctan \frac{\kappa}{|E|}) \right],
\]

\[
I_2(\omega, E, \kappa) = \frac{1}{z_1 + 1} \left[ \frac{1}{2} \ln \frac{(\omega + \kappa)^2 + E^2}{\omega^2 + E^2} + \ln |\omega + 1|^2 \right.
\]

\[
- i\text{sgn}(E)(\arctan \frac{\omega + \kappa}{|E|} - \arctan \frac{\kappa}{|E|}) \left. \right],
\]

\[
I_3(\omega, E, \kappa) = \frac{1}{z_1 - 1} \left[ \frac{1}{2} \ln((\omega + \kappa)^2 + E^2) + i\text{sgn}(E)(\frac{\pi}{2} - \arctan \frac{\omega + \kappa}{|E|}) \right],
\]

(4.11)

where

\[
z_1 \equiv \omega + iE + \kappa, \quad z_2 \equiv \omega + iE - \kappa.
\]

(4.12)

We are interested in the renormalization constants, so we expand the self-energy \(\Sigma(a; p, \omega)\) in powers of \(i\omega\) and \(E_p \equiv v_F e_a \cdot p\) and keep only the linear terms;

\[
\Sigma(a; p, \omega) = \Sigma_0 + \Sigma_\omega \cdot i\omega + \Sigma_p \cdot E_p.
\]

(4.13)

Detailed calculations are given in Appendix A, and the expansion coefficients are obtained as follows;

\[
\Sigma_0 = 0.
\]
\[
\Sigma_\omega = -\alpha dt K_1(\kappa_v, \lambda), \\
\Sigma_p = -\alpha dt \left( K_2(\kappa_v, \lambda) - K_1(\kappa_v, \lambda) \right), \quad (4.14)
\]

where we introduced the parameter \( \kappa_v \), a rescaled \( \kappa \), and two functions \( K_{1,2}(\kappa_v, \lambda) \):

\[
\kappa_v \equiv \kappa / (v_F \Lambda), \\
K_1(\kappa_v, \lambda) \equiv \frac{2}{\pi} \int_0^1 dx \sqrt{1 - x^2} \frac{1}{\lambda (\nu^2 + x^2)} \\
\times \left[ 2\lambda \nu - \frac{\nu^2 - x^2}{\nu^2 + x^2} \ln \lambda^2 (\kappa_v^2 + x^2) - \frac{4\lambda x \nu}{\nu^2 + x^2} \left( \frac{\pi}{2} - \arctan \frac{\kappa_v}{x} \right) \right], \\
K_2(\kappa_v, \lambda) \equiv \frac{2\kappa_v}{\pi} \int_0^1 dx \sqrt{1 - x^2} / \kappa_v^2 + x^2, \\
\nu \equiv \kappa_v - \frac{1}{\lambda}. \quad (4.15)
\]

Below we mainly use \( \kappa_v \) instead of \( \kappa \). We find that no terms proportional to \( \text{sgn}(\omega) \) appear in the above expansion, and so the renormalization constant of \( \kappa \), \( R_\kappa(t) \), equals to unity. Therefore the renormalization of \( \kappa \) comes only from the wave function renormalization \( R(t) \). From (4.14), we obtain the renormalization constants for infinitesimal RG transformation as follows

\[
R_\kappa(dt) = 1, \\
R(dt) = 1 - \Sigma_\omega \\
= 1 + \alpha dt K_1(\kappa_v, \lambda), \\
R_{\nu}(dt) = 1 + \Sigma_\omega + \Sigma_p \\
= 1 - \alpha dt K_2(\kappa_v, \lambda). \quad (4.16)
\]

In Eq.(4.15) the dominant contributions to the integrals for \( K_1 \) and \( K_2 \) come from the region \( x \simeq 0 \), so we can approximate as \( \sqrt{1 - x^2} \simeq 1 \) in the integrand. Then the \( x \) integrations can be performed readily as

\[
K_1(\kappa_v, \lambda) \simeq \frac{2}{\pi} \arctan \frac{1}{\kappa_v} - \frac{1}{\pi \lambda (1 + \nu^2)} \left( \ln \lambda^2 + \ln (\kappa_v^2 + 1) - 2\nu \arctan \frac{1}{\kappa_v} \right), \\
K_2(\kappa_v) \simeq \frac{2}{\pi} \arctan \frac{1}{\kappa_v}. \quad (4.17)
\]
4.2 Renormalization of the gauge-field part

By integrating the high-momentum modes, we obtain the one-loop correction to the action $S_A$, so the renormalized action $\tilde{S}_A$ becomes

$$\tilde{S}_A = \int \frac{d\epsilon}{\epsilon} \int_{\Lambda(1-dt)} A(-q, -\epsilon) \left[ \Lambda^{1-b} v_B (q^{2-b} + \Lambda^{2-b} \frac{|\epsilon|}{v_F q}) + \Pi(q, \epsilon) \right] A(q, \epsilon), \quad (4.18)$$

where the vacuum polarization $\Pi(q, \epsilon)$ reads explicitly as

$$\Pi(q, \epsilon) = g^2 v_F^2 \sum_a \left( \frac{e_a \times q}{q} \right)^2 \int \frac{d\omega}{\Lambda} \int_{\Lambda(1-dt)} dp G_0(a; p + q, \omega + \epsilon) G_0(a; p, \omega). \quad (4.19)$$

After doing the integration in the radial direction of $q$, $\Pi(q, \epsilon)$ is expressed as

$$\Pi(q, \epsilon) = g^2 v_F \frac{\Lambda}{2\pi^2} \sum_a \left( \frac{e_a \times q}{q} \right)^2 \int_{-\Lambda}^{\Lambda} dy \frac{1}{\sin \phi(y)} P(\epsilon, E_q, y, \kappa), \quad (4.20)$$

where

$$P(\epsilon, E_q, y, \kappa) \equiv \Theta(\epsilon) \sum_{i=1}^{3} P_i(\epsilon, E_q, y, \kappa) + \Theta(-\epsilon) \sum_{i=1}^{3} P_i(-\epsilon, -E_q, -y, \kappa), \quad (4.21)$$

with

$$P_1(\epsilon, E_q, y, \kappa) = \frac{i}{2\pi i(\epsilon - E_q)} \left[ \frac{1}{2} \ln \frac{\kappa^2 + E^2}{(\epsilon + \kappa)^2 + E^2} - \text{sgn}(E) \left( \frac{\pi}{2} - \arctan \frac{\kappa}{|E|} \right) \right. \left. + \text{sgn}(y) \left( \frac{\pi}{2} - \arctan \frac{\epsilon + \kappa}{|y|} \right) \right],$$

$$P_2(\epsilon, E_q, y, \kappa) = \frac{i}{2\pi i(\epsilon + 2\kappa - E_q)} \left[ \frac{1}{2} \ln \frac{(\epsilon + \kappa)^2 + E^2}{\kappa^2 + E^2} - \ln \frac{\kappa^2 + E^2}{(\epsilon + \kappa)^2 + y^2} \right. \left. - \text{sgn}(E) \left( \arctan \frac{\epsilon + \kappa}{|E|} - \arctan \frac{\kappa}{|E|} \right) \right. \left. + \text{sgn}(y) \left( \arctan \frac{\kappa}{|y|} - \arctan \frac{\epsilon + \kappa}{|y|} \right) \right],$$

$$P_3(\epsilon, E_q, y, \kappa) = \frac{i}{2\pi i(\epsilon - 2\kappa - E_q)} \left[ \frac{1}{2} \ln \frac{(\epsilon - \kappa)^2 + E^2}{\kappa^2 + E^2} - \ln \frac{\kappa^2 + E^2}{(\epsilon - \kappa)^2 + y^2} \right. \left. - \text{sgn}(E) \left( \arctan \frac{\epsilon - \kappa}{|E|} - \arctan \frac{\kappa}{|E|} \right) \right. \left. + \text{sgn}(y) \left( \arctan \frac{\kappa}{|y|} - \arctan \frac{\epsilon - \kappa}{|y|} \right) \right].$$
\[ P_3(\epsilon, E_q, y, \kappa) = \frac{i}{2\pi(i\epsilon - E_q)} \left[ \frac{1}{2} \ln \frac{(\epsilon + \kappa)^2 + E^2}{\kappa^2 + y^2} \right. \\
\left. - \text{sgn}(E) \left( \frac{\pi}{2} - \arctan \frac{\epsilon + \kappa}{|E|} \right) + \text{sgn}(y) \left( \frac{\pi}{2} - \arctan \frac{\kappa}{|y|} \right) \right], \tag{4.23} \]

where \( E = E_q + y \). We are interested in the low-energy modes, so we expand these results in powers of \( q \) and \( E_q \). Detailed calculations are given in Appendix B. Explicitly, for \( |\epsilon|/(v_F q) << 1 \) \([5, 9]\) we obtain

\[ \Pi(q, \epsilon) = \frac{g^2 v_F \Lambda dt}{2\pi^2} \sum_a K_a E_q \int_0^1 dx \frac{4}{\sqrt{1 - x^2}} \frac{\kappa_v}{\kappa_v^2 + x^2} \approx v_B \Lambda \alpha dt \frac{\pi k_F}{v_F q} \frac{|\epsilon|}{2} \arctan \frac{1}{\kappa_v}, \tag{4.24} \]

where

\[ K_a \equiv \left( \frac{e_a \times q}{q} \right)^2 \tag{4.25} \]

and we set \( \sqrt{1 - x^2} \simeq 1 \) in the \( x \)-integration above as before. This result has a form of the dissipative (\( \lambda \)) term. We neglected a constant term, which is a mass renormalization of gauge field, because this term is canceled by the contribution from the four-point interaction as explained in the introduction.

By substituting these results into Eq.(4.18), we get the infinitesimal renormalization constant for the coefficient of the dissipative term,

\[ R_\lambda(dt) \lambda = \lambda + \alpha dt K_3(\kappa_v), \tag{4.26} \]

where

\[ K_3(\kappa_v) \equiv \frac{2k_F}{\Lambda} \arctan \frac{1}{\kappa_v}. \tag{4.27} \]

### 4.3 Vertex correction

The renormalization of the vertex part is similarly obtained by the momentum-shell integration, and the action \( \tilde{S}_{\text{int.3}} \) is expressed as

\[ \tilde{S}_{\text{int.3}} = igv_F \int d\omega d\epsilon \int^{\Lambda(1-dt)} dp dq \sum_a \frac{e_a \times q}{q} \left( 1 + \Gamma_3(a; p, \omega, p + q, \omega + \epsilon) \right) \times A(q, \epsilon) \bar{\psi}(a; p + q, \omega + \epsilon) \psi(a; p, \omega), \tag{4.28} \]
where the vertex correction $\Gamma_3$ is given by

$$
\Gamma_3(a; p, \omega, p', \omega') = \int_{\Lambda(1-dt)}^{\Lambda} \frac{dq'}{\gamma(q')} \left( \frac{e_a \times q'}{q'} \right)^2 \times \frac{U(E_p + E_{p'}, \omega; E_{p'} + E_{p'}, \omega'; F(q'))}{q'} \times U(E_p + E_{p'}, \omega; E_{p'} + E_{p'}, \omega'; F(q')), \quad (4.29)
$$

and

$$
U(E, \omega; E', \omega'; F) \equiv \int d\epsilon G_0(a; E, \omega + \epsilon)G_0(a; E', \omega' + \epsilon') \frac{1}{|\epsilon'| + F} = \int d\epsilon' \left( G_0(a; E, \omega + \epsilon') - G_0(a; E', \omega' + \epsilon') \right) \frac{1}{|\epsilon'| + F} \times \frac{1}{i(\omega' - \omega) + i\kappa(\text{sgn}(\omega' + \epsilon') - \text{sgn}(\omega + \epsilon')) - (E' - E)}. \quad (4.30)
$$

The corrected three-point coupling has a momentum dependence. However, we take the “on-shell” limit since we are not interested in irrelevant higher-order terms of the Taylor-expansion in powers of momenta and energies. The on-shell limit of $\Gamma_3$ should be taken as

$$
\lim_{p,p' \to 0} \lim_{\omega, \omega' \to 0} \Gamma_3(a; p, \omega, p', \omega') = \lim_{p,p' \to 0} \frac{1}{E_p - E_{p'}}(\Sigma(0, E_p) - \Sigma(0, E_{p'}) + O(E_p^2, E_{p'}^2))
$$

$\Sigma_p$ is given by Eq.(4.14). Therefore we obtain the renormalization constant $R_g(dt)$,

$$
R_g(dt) = 1 + \Sigma_p = R(dt)R_{v_F}(dt). \quad (4.32)
$$

This relationship between $R_g = R R_{v_F}$ is nothing but the Ward-Takahashi identity which is discussed in Ref.[5].

5 WRG equations and the RG flows

In the previous section, we obtained infinitesimal forms of renormalization constants, Eqs.(4.16), (4.26) and (4.32). These results and the scaling laws (3.4) give the differential
equations for the running parameters, \( g(t), v_F(t), \lambda(t) \) and \( \kappa_v(t) \) as follows,

\[
\frac{dg(t)}{dt} = \frac{2 - b - \xi}{2} g(t),
\]

\[
\frac{dv_F(t)}{dt} = \left( \xi - 1 - \frac{g(t)^2 v_F(t)}{2\pi^2 v_B} K_2(t) \right) v_F(t),
\]

\[
\frac{d\lambda(t)}{dt} = \left( 2 - b - \frac{g(t)^2 v_F(t)}{2\pi^2 v_B} K_2(t) \right) \lambda(t) + \frac{g(t)^2 v_F(t)}{2\pi^2 v_B} K_3(t),
\]

\[
\frac{d\kappa_v(t)}{dt} = \left( 1 - \frac{g(t)^2 v_F(t)}{2\pi^2 v_B} (K_1(t) - K_2(t)) \right) \kappa_v(t),
\]

(5.1)

where \( K_i(t) \) \((i = 1, 2 \text{ and } 3)\) means \( K_i(\kappa(t), \lambda(t)) \) given by (4.17) and (4.27).

In these equations, in place of the gauge coupling \( g \) itself, one can regard the following combination \( \gamma \) as an effective expansion parameter in the perturbative calculation of the present theory,

\[
\gamma(t) \equiv \frac{g(t)^2 v_F(t)}{2\pi^2 v_B} K_2(t).
\]

(5.2)

In Ref.3 we have introduced a similar combination (we called it \( \alpha = g(t)^2 v_F(t)(2\pi^2 v_B)^{-1} \) there), where \( \kappa = 0 \) and so \( K_2(t) = 1 \). This \( \gamma \) is a generalization of it in the case of \( \kappa > 0 \). The reason why the extra factor \( K_2(t) \) appears is physically obvious from (4.3) and Fig.1. That is, the density of states around the Fermi surface decreases by the existence of the \( \kappa \)-term as \( N(E_k) \sim 1/2 (\arctan \kappa_v^{-1})/\pi \sim (1 \pm K_2)/2 \), and so the interaction between the gauge field and fermions is weakened by the factor \( K_2(t) \).

Then we obtain the following coupled differential equations,

\[
\frac{d\gamma(t)}{dt} = \left[ 1 - b - L_1(t) - (1 - L_1(t)L_2(t))\gamma(t) \right] \gamma(t),
\]

(5.3)

\[
\frac{dv_F(t)}{dt} = \left( \xi - 1 - \gamma(t) \right) v_F(t),
\]

(5.4)

\[
\frac{d\kappa_v(t)}{dt} = \left( 1 - L_2(t)\gamma(t) \right) \kappa_v(t),
\]

(5.5)

\[
\frac{d\lambda(t)}{dt} = \left( 2 - b - \gamma(t) \right) \lambda(t) + \frac{\pi k_F}{\Lambda} \gamma(t),
\]

(5.6)

where \( L_1(t) \) and \( L_2(t) \) mean the following \( L_1(\kappa_v(t), \lambda(t)) \) and \( L_2(\kappa(t), \lambda(t)) \), respectively;

\[
L_1(\kappa_v, \lambda) \equiv \frac{\kappa_v}{(\kappa_v^2 + 1) \arctan \kappa_v^{-1}},
\]

\[
L_2(\kappa_v, \lambda) \equiv \frac{-1}{2\pi \lambda(1 + \nu^2) \arctan \kappa_v^{-1}} \left( \ln \lambda^2 + \ln(1 + \kappa_v^2) - 2\nu \arctan \kappa_v^{-1} \right).
\]

(5.7)
For latter discussions on the IR behavior of the parameters, it is somewhat useful to refer to the system with $\kappa_v = 0$ and $K_2 = 1$. Therefore we rewrite below these differential equations in terms of the combination,

$$\alpha(t) \equiv \frac{\gamma(t)}{K_2(t)} = \frac{g(t)^2 v_F(t)}{2\pi^2 v_B},$$

as follows;

$$\frac{d\alpha(t)}{dt} = \left(1 - b - K_2(t)\alpha(t)\right)\alpha(t), \quad (5.9)$$

$$\frac{d\kappa_v(t)}{dt} = \left(1 - (K_1(t) - K_2(t))\alpha(t)\right)\kappa_v(t), \quad (5.10)$$

$$\frac{d\lambda(t)}{dt} = \left(2 - b - K_2(t)\alpha(t)\right)\lambda(t) + K_3(t)\alpha(t). \quad (5.11)$$

The equations (5.3), (5.5) and (5.6) are a set of closed differential equations with respect to the coupling constants, $\gamma(t), \kappa_v(t)$ and $\lambda(t)$, and they do not depend on the scaling parameter $\xi$. The behavior of $v_F(t)$ is calculated from these quantities. Then, in the following discussions, we shall consider RG flows of only these couplings.

As stated, we want to know whether the present theory including the $\kappa$-term has a nontrivial fixed point or not. In the case $\kappa_v = 0$, this theory has the nontrivial IR fixed point in the parameter region $b < 1$ such as $\gamma(\infty) = \gamma_0^* = 1 - b > 0$. The existence of this fixed point can be seen in Eq.(5.9) by simply setting $K_2(\kappa_v = 0) = 1$, where $k_v = 0$ is certainly a solution. We shall discuss how the existence of the $\kappa$-term influences the stability of this fixed point.

A formal solution to the differential equations (5.3), (5.5) and (5.6) is obtained as

$$\gamma(t) = e^{(1-b)t}\frac{e^{(1-b)\int_0^t ds e^{(1-b)s} K_2(s)} + \alpha_0^{-1}}{\int_0^t ds e^{(1-b)s} K_2(s)},$$

$$\kappa_v(t) = \kappa_{v0} \exp(t - \int_0^t ds L_2(s)\gamma(s)), $$

$$\lambda(t) = \lambda_0 e^{(2-b)t - \int_0^t ds \gamma(s)} + \frac{\pi k_F}{\Lambda}. \quad (5.12)$$

One can determine the qualitative IR behavior of the parameters by the following arguments;

(i) First, let us assume $\kappa_v$ has a finite constant value in the IR limit, i.e., $\lim_{t \to \infty} \kappa_v(t) =$
\( \kappa^*_v > 0 \). Then, from Eq. (5.9), \( \alpha(t) \) has the following fixed points \( \alpha^* \),

\[
\alpha^* = \begin{cases} 
\frac{1-b}{K_2(\kappa)} & (b < 1) \\
0 & (1 \leq b \leq 2)
\end{cases}.
\] (5.13)

Then \( \lambda(t) \) becomes large according to (5.11) as

\[
\dot{\lambda} \simeq (2 - b - K_2 \alpha^*)\lambda + K_3 \alpha^* \\
= \begin{cases} 
\lambda + K_3 \alpha^* & > 0 \ (b < 1) \\
(2 - b)\lambda + K_3 \alpha^* & > 0 \ (1 \leq b \leq 2),
\end{cases}
\] (5.14)

where \( \dot{\lambda} \equiv d\lambda(t)/dt \). Since \( \lambda \) becomes large, the factor \( K_1 - K_2 \) behaves like,

\[
K_1 - K_2 \simeq -\frac{\ln \lambda^2}{\pi \lambda} \to 0.
\] (5.15)

This fact and Eq. (5.10) indicate that \( \kappa_v \) increases exponentially. This is an obvious contradiction to the starting assumption. This implies that \( \kappa_v \) cannot have a finite value in the IR limit.

(ii) From (i) we conclude that the parameter \( \kappa_v \) tends to large in the IR limit. Thus \( L_1 \) goes to unity and \( L_2 \) is given as follows,

\[
L_2 \simeq -\frac{1}{\kappa_v^2 x^2 + (x - 1)^2} (x \ln x + 1 - x),
\] (5.16)

where \( x \equiv \kappa_v \lambda \). We note that the value of \( L_2 \) is restricted to \(-1 \leq L_2 \leq 0\). The coupled equations are approximated as follows,

\[
\dot{\gamma} \simeq \left( -b + (L_2 - 1)\gamma \right)\gamma, \\
\dot{\kappa}_v \simeq (1 - L_2\gamma)\kappa_v, \\
\dot{\lambda} \simeq (2 - b - \gamma)\lambda + \frac{\pi k_F}{\Lambda} \gamma,
\] (5.17)

where we have omitted the argument \( t \) in \( \gamma(t) \) etc. In (ii-1,2,3) below we argue that \( \gamma(t) \to 0 \) in the IR limit.

(ii-1) Let us assume that \( \gamma(t) \) has a nontrivial fixed point \( \gamma^* = \frac{b}{L_2 - 1} > 0 \). Since \( \gamma(t) \) should be positive, \( L_2 \) should become larger than unity. However we found that \( L_2 < 0 \) in (ii) above, and so this cannot happen.
(ii-2) Let us assume that $\gamma(t)$ tends to infinity. Then, $\lambda(t)$ satisfies the equation

$$\dot{\lambda} \simeq -\gamma \lambda + (\pi k_F/\Lambda) \gamma,$$

and we obtain $\lambda(t)$ as

$$\lambda(t) = \exp(-\int_0^\infty ds \gamma(s)) + \frac{\pi k_F}{\Lambda}.$$  \hspace{1cm} (5.18)

From the behavior of $\kappa_v(t)$ and $\lambda(t)$ at large $t$, $\kappa_v \to \infty$ and $\lambda \to \pi k_F/\Lambda$ (constant), we find that $L_2 \sim -\ln x/x \to 0$. This gives $\dot{\gamma} \simeq (-b - \gamma) \gamma$ and $\gamma(t)$ ($\gamma(t)$ should be positive) tends to vanishingly small. This contradicts the starting assumption. Therefore, $\gamma(t)$ cannot tend to large in the IR limit.

(ii-3) After all, the only possible situation is that $\gamma^* = 0$, i.e., IR limit is the trivial fixed point. In this case, $\kappa_v$ and $\lambda$ become large (for $0 \leq b \leq 2$) according to the original canonical scaling such as

$$\kappa_v \simeq \kappa_v,$$

$$\dot{\lambda} \simeq (2 - b) \lambda.$$ \hspace{1cm} (5.19)

Since we have obtained the qualitative behavior of the parameters in the IR limit, let us turn to the numerical calculations of the RG flows. The result shows not only that the above qualitative argument is correct but also that some interesting crossover phenomenon appears. We discuss the two cases (1) $0 \leq b < 1$ and (2) $1 \leq b \leq 2$ separately.

(1) The case $0 \leq b < 1$.

When $\kappa_v = 0$, as mentioned, $\gamma(t)$ has the nontrivial IR fixed point $\gamma^*_0 \equiv 1 - b > 0$, and at this fixed point the system behaves as a non-Fermi liquid. On the line of $\kappa_v = 0$, the RG flows converges to this point. The RG flow whose initial value of $\kappa$ is very small, $\kappa_v \simeq 0$, first approaches the point $P_0(\kappa_v = 0, \gamma = \gamma^*_0 > 0)$. Then $\kappa_v$ increases and the flow goes away from the point $P_0$, finally approaching the trivial fixed point $P_1(\kappa_v = \infty, \gamma = 0)$. Therefore, the point $P_0$ is an unstable fixed point of this theory and the IR limit of the system is controlled by the point $P_1$ which describes a Fermi liquid with impurity effects. Such behavior is shown in Fig.4. and Fig.5. In Fig.4 we plot the flows close to $\gamma(t) = 1 - b$. In Fig.5 we show the global behavior of RG flows in a wider region of $\kappa_v$. In Fig.6 we show the wave-function renormalization constant $R(t)$, together with the coupling constant $\gamma(t)$. As long as $\gamma(t)$ is staying in the vicinity of the unstable fixed
point \( P_0 \), \( R(t) \) increases exponentially \( \sim \exp(2\eta t) \), where \( \eta \) is the anormalous dimension of fermion field (for \( \kappa_v = 0 \)). As discussed in Ref.[4], this behavior of \( R(t) \) implies that the system in these energy scales is controlled by the unstable fixed point \( P_0 \) and it has non-Fermi-liquid-like behavior. This result will be confirmed by the discussion on the fermion propagator which is given in Sec.6. As RG flow is going away from this unstable fixed point, approaching the trivial fixed point \( P_1 \), \( \gamma \) decreases quickly and \( R(t) \) becomes constant, as shown in Fig.6.

(2) \( 1 \leq b \leq 2 \).

The RG flows of this situation are shown in Fig.7. They reach the point \( P_1 \) in the IR limit. In both cases, \( \kappa_v \) and \( \lambda \) increase exponentially.

6 Low-energy behavior of fermion propagator

In this section, let us consider the behavior of full fermion propagator under the RG transformation with the generalized scaling assignment (parameterized by \( \xi \)):

\[
\tilde{\omega} = e^{\xi t} \omega, \quad \tilde{p} = e^{\xi t} p,
\]

\[
\tilde{\psi}(a; \tilde{p}, \tilde{\omega}) = e^{-(\xi+1)t} \{ R(t) \} \frac{1}{2} \psi(a; p, \omega).
\] (6.1)

We begin with the definition,

\[
\langle \psi(a; p, \omega) \tilde{\psi}(a'; p', \omega') \rangle \equiv -\delta_{a,a'} \delta(p - p') \delta(\omega - \omega') \times G(p, \omega; v_F, \gamma, \lambda, \kappa; \Lambda). \] (6.2)

By substituting the last line of Eq.(6.1) into the LHS of Eq.(6.2) and then expressing it by the propagator at \( \tilde{p} \), we obtain

\[
\langle \psi(a; p, \omega) \tilde{\psi}(a'; p', \omega') \rangle = e^{2(\xi+1)t} \{ R(t) \}^{-1} \langle \tilde{\psi}(a; \tilde{p}, \tilde{\omega}) \tilde{\psi}(a'; \tilde{p}', \tilde{\omega}') \rangle
\]

\[
= -e^{2(\xi+1)t} \{ R(t) \}^{-1} \delta_{a,a'} \tilde{\delta}(p - p') \tilde{\delta}(\omega - \omega') \times G(\tilde{p}, \tilde{\omega}; v_F(t), \gamma(t), \lambda(t), \kappa(t); \Lambda). \] (6.3)

From (6.2) and (6.3) we have

\[
G(p, \omega; \rho_i; \Lambda) = e^{\xi t} \{ R(t) \}^{-1} G(\tilde{p}, \tilde{\omega}; \rho_i(t); \Lambda) \] (6.4)
where $\rho_i(t)$'s denote the running parameters $v_F(t), \gamma(t), \lambda(t)$ and $\kappa(t)$. This equation can be used to evaluate the full fermion propagator simply by substituting the solutions of the RG equations to the RHS of Eq. (6.4).

Explicitly, the RHS of (6.4) is written as follows;

$$e^{-\xi t}R(t)G^{-1}(\bar{\rho}, \bar{\omega}; \rho_i(t); \Lambda)$$

$$\simeq e^{-\xi t}R(t)[i\bar{\omega} + i\kappa(t)\text{sgn}(\bar{\omega}) - v_F(t)e_a \cdot \bar{p} - \Sigma_f(0, \bar{\omega}; \rho_i(t); \Lambda)]$$

$$\simeq iR(t)\omega + iR(t)R_{\kappa}(t)\kappa \text{sgn}(\omega) - R(t)R_{v_F}(t)v_F e_a \cdot \bar{p}$$

$$- e^{-\xi t}R(t)\Sigma_f(0, \bar{\omega}; \rho_i(t); \Lambda), \quad (6.5)$$

where $\Sigma_f(\bar{p} = 0, \bar{\omega}; \rho_i(t); \Lambda)$ is the fermion self-energy which is obtained by integrating out all the modes of fields (i.e., with momenta smaller than the cut off $\Lambda$). We have neglected its momentum-dependent part which is less dominant than $v_F e_a \cdot \bar{p}$ at low energies. We evaluate the self-energy $\Sigma_f(0, \bar{\omega}; \rho_i; \Lambda)$ at one-loop level as follows;

$$\Sigma_f(0, \bar{\omega}; \rho_i(t); \Lambda)$$

$$= \frac{g^2(t) v_F^2(t)}{v_B \Lambda} \int d\bar{p} \int_0^\Lambda d\bar{q} \left( \frac{e_a \times \bar{q}}{\bar{q}} \right)^2 \frac{1}{i(\bar{\omega} + \bar{\epsilon}) + i\kappa \text{sgn}(\bar{\omega} + \bar{\epsilon}) - v_F(t)e_a \cdot \bar{q}}$$

$$\times \frac{1}{\left( \frac{\bar{q}}{\Lambda} \right)^{2-b} + \frac{\lambda(t) \bar{\epsilon}}{v_F(t)\bar{q}}}$$

$$= -\frac{i g^2(t) v_F(t)}{(2\pi)^2 v_B \Lambda} \int d\bar{p} \int_0^\Lambda d\bar{q} \frac{\text{sgn}(\bar{\omega} + \bar{\epsilon})}{\left( \frac{\bar{q}}{\Lambda} \right)^{2-b} + \frac{\lambda(t) \bar{\epsilon}}{v_F(t)\bar{q}}}$$

$$\times \left[ 1 + \left\{ \frac{\left| \bar{\omega} + \bar{\epsilon} + \kappa \text{sgn}(\bar{\omega} + \bar{\epsilon}) \right|}{v_F(t)\bar{q}} \right\}^2 - \frac{|(\bar{\omega} + \bar{\epsilon}) + \kappa \text{sgn}(\bar{\omega} + \bar{\epsilon})|}{v_F(t)\bar{q}} \right], \quad (6.6)$$

where we have performed the angle integration, and used the fact $\text{sgn}(\bar{\omega} + \bar{\epsilon} + \kappa \text{sgn}(\bar{\omega} + \bar{\epsilon})) = \text{sgn}(\bar{\omega} + \bar{\epsilon})$. In the $\bar{\epsilon}$-integration above, the region of $\bar{\epsilon} \simeq 0$ gives dominant contribution, so we treat $\bar{\epsilon}$ as small and restrict the integration region to $0 < \bar{\epsilon} < v_F(t)\bar{q}$. We also treat $\bar{\omega}$ small because we are interested in the low-energy behavior of the propagator. Below, we discuss the three cases, (I) $0 < b < 1$, (II) $b = 1$, (III) $1 < b < 2$ separately.

(I) $0 < b < 1$

In the previous section, we showed the RG flows of the parameters $\gamma(t), \kappa_v(t)$ and $\lambda(t)$.
In the small κ_v(t) region, the expansion parameter γ(t) approaches the saddle point P_0 with γ_0 = 1 - b. As κ_v(t) tends to large, the parameter γ(t) leaves away from P_0 and approaches the trivial fixed point P_1. We study the low-energy behavior of the propagator in two regions; (i) region near the saddle point P_0 where κ_v(t) is small, and (ii) region near the trivial fixed points P_1 where κ_v(t) is large.

(i) Small κ_v(t)

For small κ_v(t), the self energy is evaluated as

\[
\Sigma_f(0, \tilde{\omega}; \rho_i(t); \Lambda) = -i \frac{\gamma(t)}{2\Lambda K_2(t)} \int d\tilde{\omega} \int_0^\Lambda dq \frac{q}{(\frac{4}{\Lambda})^{2-b} + \frac{\lambda(t)|\tilde{\omega}|}{v_F(t)q}} \text{sgn}(\tilde{\omega} + \tilde{\epsilon}) \left( 1 - \frac{\tilde{\omega} + \tilde{\epsilon} + \kappa(t)}{v_F(t)\tilde{q}} \right) + O(\tilde{\omega}^2, \kappa_v^2(t)) \approx \Sigma_1 + \Sigma_2 + \Sigma_3
\]

where

\[
\Sigma_1 = -i \frac{\gamma(t)}{2K_2(t)} \int d\tilde{\omega} \text{sgn}(\tilde{\omega} + \tilde{\epsilon}) \int_0^1 dq q^{3-b + \lambda_v(t)|\tilde{\omega}|} \\
\Sigma_2 = i \frac{\gamma(t)}{K_2(t)} \kappa_v(t) \tilde{\omega}(\lambda_v(t)|\tilde{\omega}|)^{-\frac{\gamma}{3}} H_{1b}(\lambda_v(t)|\tilde{\omega}|) \\
\Sigma_3 = i \frac{\gamma(t)}{K_2(t)} \frac{\tilde{\omega}}{v_F(t)\Lambda} \int_{v_F(t)\Lambda}^{v_F(t)\Lambda} d\tilde{\omega} \int_0^1 dq q^{3-b + \lambda_v(t)|\tilde{\omega}|} \\
\lambda_v(t) = \frac{\lambda(t)}{v_F(t)\Lambda} \\
H_{1b}(c) = \frac{1}{3 - b} \int_c^\infty dy y^{\frac{4-b}{3-b}} \ln(1 + y) \\
H_{2b}(c) = \frac{1}{3 - b} \int_c^\infty dy y^{\frac{4-b}{3-b}} \ln(1 + y).
\]
The upper limit of integral region with respect of \(\tilde{\epsilon}\) in \(\Sigma_3\) is \(v_F(t)\Lambda\) then the argument contained in \(\Sigma_3\) is \(\lambda(t)\) and is not \(\lambda_v(t)\). To obtain an explicit form of the propagator, we need to evaluate these \(H\)'s.

Now we consider the case where \(t\) increases so that \(\lambda(t)\) becomes large, while \(\gamma(t)\) is still staying near the saddle point and the argument \(c = \lambda_v(t)\tilde{\omega}\) of \(\Sigma_1, \Sigma_2\) is still small. Then, we evaluate the above integrals for small \(c\)

\[
H_{1b}(c) \simeq \frac{1}{1 - b} (1 - c \frac{1 + b}{b})
\]

\[
H_{2b}(c) \simeq \frac{1}{2 - b} (1 - c \frac{2 + b}{b}). \tag{6.10}
\]

On the other hand, the argument \(\lambda(t)\) of \(H_{2b}\) in \(\Sigma_3\) is large, and then for large \(c\),

\[
H_{2b}(c) \simeq c \frac{1}{3 - b} (\ln c + 3 - b). \tag{6.11}
\]

We have used the assumption of small \(\lambda_v(t)|\tilde{\omega}|\) above just to evaluate the integrals \(H_{1b}\) and \(H_{2b}\). The discussion later on shows that the Green function obtained below by using (6.10) is not only a solution to the RG equation at low energies for small \(\lambda_v(t)|\tilde{\omega}|\) but also remains to be an approximate solution even for large \(\lambda_v(t)|\tilde{\omega}|\).

By using (6.10) and (6.11), we obtain the Green function as follows;

\[
e^{-\xi t}G^{-1}(\tilde{p}, \tilde{\omega}; \rho_i(t); \Lambda)
= iR(t)\omega + ie^{-\xi t}R(t)\kappa(t)\text{sgn}(\omega) - R(t)R_{vF}(t)v_F a \cdot p
- ie^{-\xi t}R(t)\frac{\gamma(t)}{K_2(t)}\left(\text{sgn}(\omega) \left(\frac{\kappa_v(t)}{2 - b} |\tilde{\omega}|^{-\frac{1}{2 - b}} \lambda_v(t)\frac{1}{2 - b} - \frac{1}{1 - b} |\tilde{\omega}|^{-\frac{1}{2 - b}} \lambda_v(t)\frac{1 + b}{2 - b}\right)
+ \tilde{\omega} \left(\frac{\kappa_v(t)}{2 - b} - \frac{1}{1 - b} + \frac{1}{\lambda(t)}(\ln \lambda(t) + 3 - b)\right)\right). \tag{6.12}
\]

In the situation where \(\lambda(t)\) becomes large, while \(\gamma(t)\) is still staying near the saddle point and \(\kappa_v\) is small value, we obtain the following equation from (4.17) such as

\[
K_1 \simeq K_2 = 1 - \frac{2\kappa_v}{\pi} + O(\kappa_v^2). \tag{6.13}
\]

From this relation and \(R(t), R_{vF}(t)\) in Eq.(4.16), we obtain

\[
R(t) \simeq e^{\int_0^t d\gamma(s)}
\]

\[
R(t)R_{vF}(t) \simeq 1. \tag{6.14}
\]
The formal solution of $\gamma(t)$ which is given by (5.12) and the above equation give the following relation,

$$R(t)\gamma(t) = \alpha_0 e^{(1-b)t} K_2(t),$$

(6.15)

where $\alpha_0$ is the initial value of $\alpha(t)$ of (5.8). From (5.1), we also find

$$v_F(t) \simeq e^{(\xi-1)t} v_{F0},$$

$$\kappa_v(t) \simeq e^t \kappa_{v0} \ll 1,$$

$$\lambda_v(t) \simeq \lambda_{v0} e^{(3-b-\xi)t}.$$  

(6.16)

By using these equations, $R(t)$ is obtained as follows;

$$R(t) = 1 + \alpha_0 \int_0^t ds e^{(1-b)s} K_2(s) = 1 + \frac{\alpha_0}{1-b} \left(e^{(1-b)t} - 1\right) - \frac{2\alpha_0 \kappa_{v0}}{\pi(2-b)} \left(e^{(2-b)t} - 1\right).$$

(6.17)

We substitute these relations, which depend on $t$, to Eq.(6.12), and obtain the Green function as

$$G^{-1}(p, \omega; \rho_i; \Lambda) = e^{-\xi t} R(t) G^{-1}(\tilde{p}, \tilde{\omega}; \rho_i(t); \Lambda)$$

$$\simeq i\alpha \sgn(\omega) \left(\frac{\lambda_v}{1-b} \frac{1}{\sqrt{2\pi}} - \frac{\kappa_v \lambda_v}{1-b} \frac{2}{\pi} \right)$$

$$+ i\omega \left(1 - \frac{\alpha}{1-b} + \frac{2\alpha \kappa_v}{\pi(2-b)} - \frac{\kappa_v}{\pi} \right) + i k \sgn(\omega) - v_F e_a \cdot p + \Delta G^{-1}$$

$$= \left(G_{\text{leading}}\right)^{-1} + O(\omega),$$

(6.18)

where

$$\Delta G^{-1} = i\omega \alpha \left(e^{(2-b)t} \frac{2\kappa_v}{\pi(2-b)} \frac{\pi}{2} - 1 - \frac{e^{-bt}}{\lambda} \right),$$

(6.19)

and we have omitted the suffix 0 for initial values of $\alpha, \lambda, \lambda_v, \kappa_v$ and $v_F$.

Here we comment on the last term $\Delta G^{-1}$ which depends on the RG scale $t$. The LHS of Eq.(6.18) is independent of the scale parameter $t$, so the representation on the RHS should not have terms depending on $t$ if we calculated the fermion self-energy exactly. However one cannot do the exact calculation, and we evaluated the self-energy perturbatively at
one loop level assuming the RG parameters lie in the region where \( \gamma(t) \ll 1 \). If this region corresponds to the infrared limit (like the case \( \kappa_v \gg 1 \) and \( 1 \leq b < 2 \) which we will discuss later), the \( t \)-dependent terms should disappear in the infrared limit \( (t \to \infty) \) with exponentially dumping factors, for example see Eq.(6.25) and Eq.(6.30). In that case, the one-loop calculation is almost exact. Right now, we are considering the situation with some finite \( t \), \( t \sim t^* \), in which \( \gamma(t) \) is staying near the saddle point \( \gamma_0^* = 1 - b \). We assume \( 1 - b \ll 1 \) so that the perturbative calculation assuming small \( \gamma \) is meaningful.

However, the above \( \Delta G^{-1} \) does not disappear even in the limit \( t \to t^* \) with the \( t \)-dependent factor such as \( e^{(2-b)t} \). Of course, we find that the leading terms of Eq.(6.18), \( G_{\text{leading}} \), are independent of \( t \) and are in fact a solution to Eq.(6.4) up to the first order of \( \kappa_v(t) \) and at low energies. This fact suggests that the \( \Delta G^{-1} \) term shall be canceled out with higher-loop corrections to the fermion self-energy. Actually, the two-loop corrections are known to contain terms of the order \((\gamma_0^*)^2/(1 - b) = \gamma_0^*\), which are of the same order in \( \gamma_0^* = 1 - b \ll 1 \) as the one-loop correction.

From (6.18), we find that the dominant terms of the Green function \((G_{\text{leading}})^{-1}\) are \( O(\omega^{\frac{b}{3-b}}) \) and \( O(\kappa_v\omega^{\frac{1}{3-b}}) \) in the present parameter region \( 0 < b < 1 \). Therefore, at intermediate energy scales, the Green function has the behavior of a non-Fermi liquid; a branch cut rather than a pole in \( \omega \) appears as in the Luttinger liquid in one dimension\[^9\].

**ii) Large \( \kappa_v(0) \)**

In this case, the self-energy of (6.6) is evaluated as

\[
\Sigma_f(0, \bar{\omega}; \rho_i(t); \Lambda) \\
\approx -i \frac{\gamma(t)}{2\Lambda K_2(t)} \text{sgn}(\bar{\omega} + \bar{\epsilon}) \int d\bar{\epsilon} \int_0^\Lambda d\tilde{q} \frac{1}{(\tilde{q}/\lambda)^{2-b} + \lambda(t)|\bar{\epsilon}|} \frac{v_F(t)\tilde{q}}{2\kappa(t)}
\]

\[
= -i \frac{\gamma(t)}{2\kappa_v(t)\lambda_v(t)K_2(t)} \text{sgn}(\bar{\omega})(\lambda_v(t)|\bar{\omega}|)^{\frac{b}{3-b}} H_{3b}(\lambda_v(t)|\bar{\omega}|),
\]

where

\[
H_{3b}(c) \equiv \frac{1}{3 - b} \int_c^\infty dyy^{\frac{b}{3-b}} \ln(1+y).
\]

For small \( c \), \( H_{3b} \) is evaluated as

\[
H_{3b}(c) \simeq \frac{1}{b} \left(1 - c^{\frac{b}{3-b}}\right).
\]
Then we obtain $G^{-1}(p, \omega; \rho_i; \Lambda)$ as

$$e^{-\xi t} R(t) G^{-1}(\tilde{p}, \tilde{\omega}; \rho_i(t); \Lambda)$$

$$\simeq i R(t) \omega + i k \text{sgn}(\omega) - v_F e_a \cdot p + i e^{-\xi t} \frac{\pi R(t) \gamma(t)}{4b} \tilde{\omega} \left( (\lambda_v |\tilde{\omega}|)^{\frac{1}{4b}} - 1 \right), \quad (6.23)$$

where we have used $K_2(t) \simeq 2/(\pi \kappa_v(t))$. Now $\kappa$ is large enough, so $R(t)$ behaves as

$$R(t) \simeq 1 + \frac{2 \alpha_0}{\pi} \int_0^t ds \frac{1}{\kappa_v(s)}$$

$$\simeq 1 + \gamma_0 (1 - e^{-t}), \quad (6.24)$$

where $\gamma_0 \equiv 2 \alpha_0 / (\pi \kappa_v(t))$. Thus, in this case, the Green function is obtained as

$$e^{-\xi t} R(t) G^{-1}(\tilde{p}, \tilde{\omega}; \rho_i(t); \Lambda)$$

$$\simeq i \kappa \text{sgn}(\omega) - v_F e_a \cdot p + i (1 + 2 \alpha \kappa v \pi) \omega + i \alpha \frac{1}{2} \omega \ln \left( \lambda_v |\omega| \right)^{\frac{1}{4b}} + O(e^{-t}) \quad (6.25)$$

where we have omitted the suffix 0 as before.

In the low-energy limit, we find that the dominant term is linear in $\omega$, so the Green function has the Fermi-liquid behavior. This result of course agrees with the conclusion in Sec.5 obtained based on the RG flow.

(II) $b = 1$

In this case, $\gamma(t)$ goes to the trivial fixed point $P_1$.

(i) Small $\kappa_v(t)$

For small $\kappa_v(t)$, we obtain the self energy by setting $b = 1$ in Eq.(6.8). The functions $H_{11}$ and $H_{21}$ are easily obtained as

$$H_{11}(c) = \frac{1}{2} \left( \left( \frac{1}{c} + 1 \right) \ln(1 + c) - \ln c \right)$$

$$H_{21}(c) = \frac{1}{\sqrt{c}} \ln(1 + c) + 2 \arctan \frac{1}{\sqrt{c}}. \quad (6.26)$$

From these equations and the $t$-dependences of parameters in (6.15), (6.16) and (6.17) with $b = 1$, we obtain the Green function as follows;

$$e^{-\xi t} R(t) G^{-1}(\tilde{p}, \tilde{\omega}; \rho_i(t); \Lambda)$$

$$\simeq i k \text{sgn}(\omega) - v_F e_a \cdot p + i \left( 1 + \frac{2 \alpha \kappa_v}{\pi} \right) \omega + i \alpha \frac{1}{2} \omega \ln \left( \lambda_v |\omega| \right) + O(t) \quad (6.27)$$
where the suffix 0 for initial values are omitted as before. The last term $O(t)$ is the $t$-dependent part which should be canceled out with higher-order corrections as explained above.

This expression contains $\omega\ln(\omega)$ term, leading to the weight of quasiparticles that vanishes logarithmically at low energies. This behavior is like that of the marginal-Fermi liquid [9].

(ii) Large $\kappa_v(t)$

As $t$ becomes large, $\kappa_v(t)$ becomes large too. Then we evaluate the self energy as

$$\Sigma(0, \tilde{\omega}; \rho_i(t); \Lambda) \simeq -i\text{sgn}(\tilde{\omega}) \frac{\gamma(t)}{2\kappa_v(t)\lambda_v(t)K_2(t)}(\lambda_v(t)|\tilde{\omega}|)\frac{3}{2}H_{31}(\lambda_v(t)|\tilde{\omega}|).$$

(6.28)

Here $H_{31}$ is evaluated as follows

$$c^3 H_{31}(c) = \frac{1}{3} \left( \ln(1 + c) + 2c(1 - \sqrt{c}\arctan \frac{1}{\sqrt{c}}) \right).$$

(6.29)

By using (5.16) and (5.24) with $b = 1$, we obtain the Green function as

$$e^{-\xi t} R(t) G^{-1}(\tilde{p}, \tilde{\omega}; \rho_i(t); \Lambda) \simeq i\text{sgn}(\omega) - v_F e_a \cdot \rho + i(1 + \gamma)\omega + O(e^{-t}).$$

(6.30)

The logarithmic term has disappeared but only the term linear in $\omega$ survives (together with the signature term). This behavior of the Green function is that of the ordinary Fermi liquid.

(III) $1 < b < 2$

In this parameter region, the Green function has the same form with (6.18) for $\kappa_v(t) \ll 1$ and (6.25) for $\kappa_v(t) \gg 1$. But the parameter $b > 1$ here, and so the dominant term of the Green function is linear in $\omega$ at low energies. Therefore, we find that the Green function has the Fermi-liquid behavior in both cases of $\kappa_v(t) \ll 1$ and $\kappa_v(t) \gg 1$.

7 Concluding Remarks

In this paper, we have given detailed studies on low-energy behavior of the system of nonrelativistic fermions interacting with a dissipative gauge field by using the WRG. Especially, we investigated the stability problem of the nontrivial IR fixed point in the
clean system against impurities. The $\kappa$-term in the fermion propagator, which is generated by the interaction between fermions and impurities, is introduced for this purpose. We showed that this term makes the non-Fermi-liquid fixed point unstable, and in the IR limit the effective gauge-coupling constant tends to vanish. However in intermediate energy scales, the system is controlled by the unstable fixed point as a saddle point, exhibiting non-Fermi-liquid behavior for some parameter regions of $b$ and $\kappa$. This result is confirmed by obtaining the fermion propagator by solving the RG equation and also by calculating the wave function renormalization constant.

It is also quite interesting to investigate finite-temperature effects in the present gauge-fermion system by the RG analysis; in particular, the same stability problem. This subject is under study and we plan to publish the result as a paper subsequent to the present one[10].

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A Taylor series of the self energy

In this Appendix, we make the Taylor expansion of the fermion self energy () to obtain the result (4.14). It is assumed that \( z_0 = \omega + E_p \) takes a small value and the self energy is expanded around \( y = E_q \). After the \( y \) integration, only even functions of \( y \) in \( I_i \) remain, and then we neglect odd functions of \( y \) in the expansion of \( I_i \).

\[
I_1(\omega, E, \kappa) = - \frac{1}{z_2 + 1} \left( \frac{1}{2} \ln \left( \frac{\omega + 1}{\kappa^2 + E^2} \right) + i \text{sgn}(E) \left( \frac{\pi}{2} - \arctan \frac{\kappa}{|E|} \right) \right)
= c_0(\kappa, y) + c_1(\kappa, y) z_0 + c_2(\kappa, y) i E_p,
\]

\[
I_2(\omega, E, \kappa) = \frac{1}{z_1 + 1} \left( \frac{1}{2} \ln \left( \frac{\omega + \kappa}{\kappa^2 + E^2} \right) + \ln |\omega + 1| \right)
- i \text{sgn}(E) \left( \frac{\pi}{2} - \arctan \frac{\omega + \kappa}{|E|} \right)
= d_1(\kappa, y) z_0 + d_2(\kappa, y) i E_p,
\]

\[
I_3(\omega, E, \kappa) = \frac{1}{z_1 - 1} \left( \frac{1}{2} \ln \left( \frac{\omega + \kappa}{\kappa^2 + E^2} \right) + i \text{sgn}(E) \left( \frac{\pi}{2} - \arctan \frac{\omega + \kappa}{|E|} \right) \right)
= g_0(\kappa, y) + g_1(\kappa, y) z_0 + g_2(\kappa, y) i E_p.
\]

where

\[
c_0(\kappa, y) = \frac{1}{2a_-} \left( (1 - \kappa) \ln(\kappa^2 + y^2) - 2|y|(\frac{\pi}{2} - \arctan \frac{\kappa}{|y|}) \right),
\]

\[
c_1(\kappa, y) = \frac{1}{a_-} \left( \kappa - 1 + D(\kappa, y) \right),
\]

\[
c_2(\kappa, y) = - \frac{1}{a_-} \left( (1 - \kappa) \left( \frac{\kappa}{\kappa^2 + y^2} - 1 \right) + \frac{y^2}{\kappa^2 + y^2} \right) + E(\kappa, y),
\]

\[
d_1(\kappa, y) = - d_2(\kappa, y) = \frac{\kappa}{\kappa^2 + y^2},
\]

\[
g_0(\kappa, y) = - c_0(\kappa, y),
\]

\[
g_1(\kappa, y) = \frac{1}{a_-} \left( \frac{(\kappa - 1) \kappa - y^2}{\kappa^2 + y^2} + D(\kappa, y) \right),
\]

\[
g_2(\kappa, y) = E(\kappa, y),
\]

and

\[
D(\kappa, y) \equiv \frac{1}{2a_-}(\kappa, y) \left( - c_- (\kappa, y) - \ln(\kappa^2 + y^2) + 4(1 - \kappa) |y|(\frac{\pi}{2} - \arctan \frac{\kappa}{|y|}) \right),
\]

\[
E(\kappa, y) \equiv - \frac{2(1 - \kappa)}{a_-}(\kappa, y) \left( \frac{\pi}{2} - \arctan \frac{\kappa}{|y|} \right),
\]

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\[ a_-(\kappa, y) \equiv (1 - \kappa)^2 + y^2, \]
\[ c_-(\kappa, y) \equiv (1 - \kappa)^2 - y^2. \]  
(A.3)

The self energy is expanden as follows;
\[ \Sigma(a; p, \omega) = \Sigma_0 + \Sigma_\omega \omega + \Sigma_p E_p. \]  
(A.4)

where
\[ \Sigma_0 \equiv -i\alpha dt F_\Lambda \int_{-\lambda}^{\lambda} dy |\sin \phi(y)| \text{sgn}(\omega)(c_0(\kappa, y) + g_0(\kappa, y)) \]
\[ = 0, \]
\[ \Sigma_\omega \equiv -\alpha dt F_\Lambda \int_{-\lambda}^{\lambda} dy |\sin \phi(y)| \left( c_1(\kappa, y) + d_1(\kappa, y) + g_1(\kappa, y) \right) \]
\[ = \frac{1}{a_-(\kappa, y)} \left( 2(\kappa - 1) - \frac{c_-(\kappa, y)}{a_-(\kappa, y)} \ln(\kappa^2 + y^2) + \frac{4|y|(1 - \kappa)}{a_-(\kappa, y)} \left( \frac{\pi}{2} - \arctan \frac{\kappa}{|y|} \right) \right), \]
\[ \Sigma_p = I_p - \Sigma_\omega. \]  
(A.5)

and
\[ I_p \equiv \alpha dt F_\Lambda \int_{-\lambda}^{\lambda} dy |\sin \phi(y)| \left( c_2(\kappa, y) + d_2(\kappa, y) + g_2(\kappa, y) \right) \]
\[ = \alpha dt F_\Lambda \int_{-\lambda}^{\lambda} dy |\sin \phi(y)| \frac{-2\pi}{\kappa^2 + y^2}, \]  
(A.6)

\( y \) and \( \kappa \) are defined in Eq.(4.7) such as \( y = v_F \Lambda \cos \phi / F_\Lambda \) and \( \kappa = \kappa / F_\Lambda \).

\section{B Vacuum porlarization}

In this Appendix, the vacuum porlarization (1.20) is calculated to give (4.24). In the \( y \) integration, only even functions of \( y \) in \( P_i \) survive, and odd functions of \( y \) are omitted. Then \( P_i \)'s are obtained as follows,
\[ P_1(\epsilon, E_q, y, \kappa) = \left( \frac{1}{2} \ln \frac{L}{L_e} - \frac{\kappa}{L} i E_q \right), \]
\[ P_2(\epsilon, E_q, y, \kappa) = \left( -\ln \frac{L}{L_e} - \left( \frac{\kappa}{L} - \frac{\epsilon + \kappa}{L_e} \right) i E_q \right), \]
\[ P_3(\epsilon, E_q, y, \kappa) = \left( \frac{1}{2} \ln \frac{L}{L_e} - \frac{\epsilon + \kappa}{L_e} i E_q \right), \]  
(B.1)
where
\[ L \equiv \kappa^2 + y^2, \quad L_e \equiv (\epsilon + \kappa)^2 + y^2. \] (B.2)

We assume that the coefficient of the impurity term \( \kappa \) is larger than the momentum and energy, \( q \) and \( \epsilon \), as \( q/\kappa \ll 1 \) and \( \epsilon/\kappa \ll 1 \). In this situation, we find that \( P_1 \) and \( P_3 \) are order \( O(1/\kappa) \) and \( P_2 \) is order \( O(1/\kappa^2) \), and then we can neglect the \( P_2 \) term. We also set \( \ln(L/L_e) \sim 0 \) because \( L \sim L_e \). After all we obtain the approximate form of \( P_i \) as
\[
\sum_{i=1}^{3} P_i(\epsilon, E_q, y, \kappa) \simeq \frac{E_q}{2\pi(i\epsilon - E_q)} \left( \frac{\kappa}{L} + \frac{\epsilon + \kappa}{L_e} \right),
\]
\[
\simeq \frac{E_q}{\pi(i\epsilon - E_q)} \frac{\kappa}{L}.
\] (B.3)

The vacuum polarization part is given as follows
\[
\Pi(q, \epsilon) = \frac{g^2 v_F \Lambda dt}{2\pi^2} \sum_a \frac{K_a E_q}{2\pi(i\epsilon - E_q)} \int_0^1 dx \frac{4}{\sqrt{1 - x^2} \kappa_v + x^2}
\]
\[
\simeq v_B \Lambda \alpha dt \frac{k_F}{2\Lambda} \int_{-\pi}^{\pi} d\phi \frac{\cos \phi \sin^2 \phi}{i\epsilon - v_F q \cos \phi} \int_0^1 dx \frac{4}{\sqrt{1 - x^2} \kappa_v^2 + x^2}
\]
\[
\simeq \text{const} + v_B \Lambda \alpha dt \frac{k_F}{2\Lambda} \frac{|\epsilon|}{v_F q \pi} \frac{2}{\arctan \frac{1}{\kappa_v}},
\] (B.4)

where
\[ K_a \equiv \left( \frac{e_a \times q}{k_F} \right)^2, \] (B.5)
and use the relation \( \sum e_a (\frac{2\Lambda}{k_F}) \simeq \int_{-\pi}^{\pi} d\phi \). In the \( x \) integration, we set \( \sqrt{1 - x^2} \simeq 1 \).
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**Figure Captions**

Fig.1
Momentum distribution of the fermion density. The existence of the $\kappa$-term smears the step-function-like behavior at $\kappa = 0$. The energy scale corresponding to the momentum cut off $\Lambda$ is also shown.

Fig.2
Partition of the two-dimensional space of fermion momentum $\vec{K}$. The big arc $C-C'$ shows a part of Fermi circle of radius $K_F$ around the origin $O$. There are $N = 2\pi K_F/(2\Lambda)$ Fermi wave vectors, $\vec{K}_{F,a}(a = 1, ..., N)$, being separated each other by equal distances $2\Lambda$ along the circle. Here $\Lambda$ is the cutoff momentum. Under the renormalization-group transformations, we integrate gradually the modes with the momenta that sit “near” the Fermi circle within the distance $\simeq \Lambda$. To be explicit, we integrate the momenta which locate within one of the $N$ small circles of radius $\Lambda$, each circle being around the end point of $\vec{K}_{F,a}$ as illustrated. One may enlarge the region of relevant momenta, e.g., to those that locate within the union of small circles of radius $2\Lambda$, to avoid the momenta which sit very near to the Fermi circle but never be integrated. However, the main conclusions of the text do not depend on such detail.

Fig.3
The two-dimensional space of momentum $\vec{k} = (k_x, k_y)$. $\vec{k} = \vec{K} - \vec{K}_{F,a}$ is a deviation of fermion momentum $\vec{K}$ near the Fermi circle measured from the nearest Fermi momentum $\vec{K}_{F,a}$. In the step of renormalization-group transformation from $t$ to $t + dt$, the modes with such momenta that sit in the shaded area are to be integrated. This area is nothing but a shell surrounded by two circles of radii $\Lambda e^{-t}$ and $\Lambda e^{-(t+dt)}$.

Fig.4
Structure of RG flows of $\kappa_v(t)$ and $\gamma(t)$ near the unstable fixed point $P_0 (\gamma^*, \kappa_v) = (1-b, 0)$. Values of the parameters are $b = 0.5$, $k_F = 1$, $\Lambda = 1$ and $\lambda = 0.1$. For sufficiently small
initial value of $\kappa_v^0$, the RG flows approach to the fixed point $P_0$.

Fig. 5
Global structure of the RG flows. Same as Fig.4, but the scale of $\kappa_v$ is enlarged. The effective coupling constant $\gamma(t)$ tends to vanish in the IR limit.

Fig. 6
The solid line is the effective coupling constant $\gamma(t)$, and the dashed line is the renormalization constant $\ln R(t)$. As long as the effective coupling constant stays near the unstable fixed point $P_0$, the renormalization constant grows up exponentially.

Fig. 7
RG flows for $1 \leq b \leq 2$. $\kappa_v(t)$ increase exponentially and $\gamma(t)$ tends to vanish in the IR limit.