RADON NUMBERS
AND THE FRACTIONAL HELLY THEOREM

BY

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ABSTRACT

A basic measure of the combinatorial complexity of a convexity space is its Radon number. In this paper we answer a question of Kalai, by showing a fractional Helly theorem for convexity spaces with bounded Radon number. As a consequence we also get a weak $\varepsilon$-net theorem for convexity spaces with bounded Radon number. This answers a question of Bukh and extends a recent result of Moran and Yehudayoff.

1. Introduction

One of the fundamental statements of combinatorial convexity is Radon’s lemma [34] which states that any set of $d + 2$ points in $\mathbb{R}^d$ can be partitioned into two parts whose convex hulls intersect. This property was extended to partitions into $k$ parts, by the celebrated theorem of Tverberg [39], stating that any set of $(d + 1)(k - 1) + 1$ points in $\mathbb{R}^d$ can be partitioned into $k$ parts whose convex hulls share a common point. There are numerous generalizations, variations, and extensions of these types of results, and we refer the reader to the surveys [9, 10, 14, 16] for more information and further references.

Radon introduced his lemma in order to prove one of the other fundamental theorems of convexity, namely Helly’s theorem [22], which states that if the members of a finite family of convex sets in $\mathbb{R}^d$ have an empty intersection, then

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there are some \( d+1 \) or fewer members of the family whose intersection is empty. A far reaching generalization of Helly’s theorem is the famous \((p,q)\) theorem due to Alon and Kleitman [4], whose proof combined a large number of sophisticated tools and results that had been developed over the years since Helly’s original theorem. For more information on the great number of extensions and generalizations of Helly’s theorem we refer the reader to [5, 14, 16, 18, 24] and the references therein.

Here we will be concerned with one particular (and important) generalization of Helly’s theorem due to Katchalski and Liu [29] known as the fractional Helly theorem. It states the following. Let \( F \) be a finite family of \( n \geq d + 1 \) convex sets in \( \mathbb{R}^d \), and suppose the number of \((d+1)\)-tuples of \( F \) with non-empty intersection is at least \( \alpha \binom{n}{d+1} \), for some constant \( \alpha > 0 \). Then there are at least \( \beta n \) members of \( F \) whose intersection is non-empty, where \( \beta > 0 \) is a constant which depends only on \( \alpha \) and \( d \).

The fractional Helly theorem plays a crucial role in the proof of the \((p,q)\) theorem (some might even say the most crucial role [3]), and various fractional Helly theorems are known [2, 7, 15, 26]. It is also of considerable interest to understand what conditions can be imposed on a set system which guarantees that it admits the “fractional Helly property” (see, e.g., [3, 32]).

A question in this direction, which we learned from Gil Kalai (personal communication; see also [27, Problem 18]), is whether “Radon implies fractional Helly”? Although this may be a (purposefully) vague question, we now describe an axiomatic setting in which it can be made precise.

A **convexity space** is a pair \((X, C)\) where \( X \) is a non-empty set and \( C \) is a family of subsets of \( X \) satisfying the following properties:

1. **(C1)*** \( \emptyset \) and \( X \) are in \( C \).
2. **(C2)*** If \( D \subset C \) is non-empty, then \( \bigcap_{C \in D} C \) is in \( C \).
3. **(C3)*** If \( D \subset C \) is non-empty and totally ordered by inclusion, then \( \bigcup_{C \in D} C \) is in \( C \).

The fundamental example of a convexity space is the standard (Euclidean) convexity \((\mathbb{R}^d, C^d)\), where \( C^d \) is the family of all convex sets in \( \mathbb{R}^d \). Another typical example is the **integer lattice convexity** \((\mathbb{Z}^d, L^d)\) where

\[
L^d = \{ \mathbb{Z}^d \cap C : C \in C^d \}.
\]

For a detailed overview of the theory of convexity spaces we refer the reader to the book by van de Vel [40].
For a convexity space \((X, C)\), the members of \(C\) are called \textbf{convex sets}. Given a subset \(Y \subset X\) we define the \textbf{convex hull} of \(Y\), denoted by \(\text{conv}(Y)\), to be the intersection of all the convex sets containing \(Y\). This is the minimal convex set containing \(Y\). In some situations it will be convenient to allow \(Y\) to be a multiset, in which case \(\text{conv}(Y)\) is simply the convex hull of the underlying set.

The main invariant of a convexity space that we will be concerned with is its \textbf{Radon number}. This is the smallest integer \(r_2\) (if it exists) such that any subset \(P \subset X\) with \(|P| \geq r_2\) can be partitioned into two parts \(P_1\) and \(P_2\) such that \(\text{conv}(P_1) \cap \text{conv}(P_2) \neq \emptyset\). Such a partition is called a \textbf{Radon partition}. For instance, Radon’s lemma states that every set of \(d+2\) points in \(\mathbb{R}^d\) admits a Radon partition, and it is easy to see that the Radon number of the standard convexity \((\mathbb{R}^d, C^d)\) equals \(d+2\). Let us also point out that the definition of the Radon number trivially extends to multisets. This is because if \(P\) is a multiset containing a point with multiplicity greater than one, then any partition into two parts such that the repeated point appears in both parts is obviously a Radon partition. (Throughout the paper we deal only with convexity spaces in which \(r_2 \geq 3\), thereby excluding degenerate/trivial cases.)

\textbf{RESULTS.} Our main result is a fractional Helly theorem for convexity spaces with bounded Radon number. This answers Kalai’s question.

\textbf{Theorem 1.1:} For every \(r \geq 3\) and \(\alpha \in (0, 1)\) there exists an \(m = m(r)\) and a \(\beta = \beta(\alpha, r) \in (0, 1)\) with the following property: Let \(F\) be a finite family of \(n \geq m\) convex sets in a convexity space with Radon number at most \(r\). If at least \(\alpha \binom{n}{m}\) of the \(m\)-tuples of \(F\) have non-empty intersection, then there are at least \(\beta n\) members of \(F\) whose intersection is non-empty.

\textbf{Remark:} Let us stress that the integer \(m\) depends only on \(r\) and not on \(\alpha\). Our bound on \(m\) is quite large in terms of \(r\) and is expressed by certain Stirling numbers of the second kind. Here is how we plan to prove Theorem 1.1. First we establish a colorful Helly theorem for convexity spaces with bounded Radon number (Theorem 2.2), and this is where the integer \(m = m(r)\) appears as the number of colors needed. Next, we consider the “intersection hypergraph” carrying the information of which subfamilies of \(F\) are intersecting. The colorful Helly theorem may then be interpreted as forbidding certain patterns from the intersection hypergraph (to be made precise in section 3), and we can then apply a recent result by the first author [23] concerning the clique number of dense uniform hypergraphs with forbidden substructures.
Now we turn to an application of Theorem 1.1. Let \( F \) be a finite set system on a ground set \( X \) (which may be infinite). Recall that the \textbf{transversal number} of \( F \), denoted by \( \tau(F) \), is the minimum cardinality of a subset \( T \subset X \) such that \( T \) intersects every member of \( F \). The \textbf{fractional transversal number} of \( F \), denoted by \( \tau^*(F) \), is the infimum of \( \sum_{x \in X} \varphi(x) \) over all functions \( \varphi : X \to [0,1] \) such that \( \sum_{x \in S} \varphi(x) \geq 1 \) for every \( S \in F \). (Note: In the case when \( X \) is infinite we consider only functions \( \varphi : X \to [0,1] \) which are non-zero on a finite number of elements of \( X \).) Trivially, we have \( \tau^*(F) \leq \tau(F) \), while in general there is no universal bound on \( \tau(F) \) in terms of \( \tau^*(F) \). Nevertheless, there are non-trivial classes of set systems for which such bounds do exist, such as finite set systems with bounded VC-dimension (the \( \varepsilon \)-net theorem [21]), finite families of convex sets in \( \mathbb{R}^d \) (weak \( \varepsilon \)-net theorem [1]), and finite families of convex sets in separable convexity spaces with bounded Radon number [33].

Our second result shows that \( \tau(F) \) can be bounded by a function of \( \tau^*(F) \) when \( F \) is a finite family of convex sets in a convexity space with bounded Radon number. (This is a contribution towards a general problem brought up in [3, Problem 7(i)].)

\textbf{Theorem 1.2:} For every \( r \geq 3 \) there exist positive constants \( c_1 = c_1(r) \) and \( c_2 = c_2(r) \) with the following property: For any finite family \( F \) of convex sets in a convexity space with Radon number at most \( r \), we have

\[ \tau(F) \leq c_1 \cdot \tau^*(F)^{c_2}. \]

\textbf{Remark:} This result can be reformulated in terms of what are known as weak \( \varepsilon \)-nets (to be made precise in Section 4). Consequently, Theorem 1.2 implies an affirmative answer to a question of Bukh [11, Question 3] concerning the existence of finite size weak \( \varepsilon \)-nets in convexity spaces with bounded Radon number (Theorem 4.1). A partial answer to Bukh’s question was recently given by Moran and Yehudayoff [33], and Theorem 1.2 also generalizes their result, but the proof methods differ significantly. While their proof relies on VC-dimension arguments, our proof employs the theory developed by Alon, Kalai, Matoušek, and Meshulam [3] which shows that for abstract set systems, a suitable “fractional Helly property” will imply a “weak \( \varepsilon \)-net theorem”. From this point of view, Theorem 1.2 is a straight-forward consequence of Theorem 1.1 and the work done in [3]. The details of this discussion and the proof of Theorem 1.2 will be given in Section 4.
Outline of the paper. In Section 2 we establish a colorful Helly theorem for convexity spaces with bounded Radon number (Theorem 2.2), which we use to prove Theorem 1.1 in Section 3. In Section 4 we discuss weak \( \varepsilon \)-nets and review the main results and concepts from [3] needed to prove Theorem 1.2.

Notation. We use the following standard notation and terminology. For a natural number \( n \), the set \( \{1, \ldots, n\} \) is denoted by \([n]\), and for a finite set \( X \), the set of \( k \)-tuples (\( k \) element subsets) of \( X \) is denoted by \( \binom{X}{k} \). A \( k \)-partition of \( X \) is a partition of the set \( X \) into \( k \) non-empty unlabeled parts. The number of \( k \)-partitions of \( [n] \) is denoted by \( S(n, k) \). (The numbers \( S(n, k) \) are commonly referred to as Stirling numbers of the second kind [38, section 1.9].) By a multiset, we mean a set \( Y \) where each element of \( Y \) has a finite multiplicity. The cardinality of a multiset \( Y \) is the sum of the multiplicities of its elements, and when the cardinality is finite we call it a finite multiset. We allow for \( k \)-partitions of a finite multiset, where the multiplicity of a point determines the maximum number of parts in which the point may appear. Note that the number of \( k \)-partitions of a multiset of cardinality \( n \) is bounded above by \( S(n, k) \).

2. A colorful Helly theorem

The colorful Helly theorem discovered by Lovász, and independently by Bárány [6], states that if \( F_1, \ldots, F_{d+1} \) are non-empty finite families of convex sets in \( \mathbb{R}^d \) such that \( \bigcap_{i=1}^{d+1} S_i \neq \emptyset \) for all \( S_i \in F_i \) and \( i \in [d+1] \), then for some \( 1 \leq i \leq d+1 \) we have \( \bigcap_{S \in F_i} S \neq \emptyset \). Note that this implies Helly’s theorem by setting

\[
F_1 = \cdots = F_{d+1}.
\]

The colorful Helly theorem has many applications in discrete geometry and was originally used by Bárány (in a dual form) to prove the “first selection lemma” [6, Theorem 5.1] (see also [31, chapter 9]). Later Sarkaria [37] showed that the same dual form implies Tverberg’s theorem (see also [8] and [31, chapter 8]). It should also be noted that the colorful Helly theorem has a topological generalization due to Kalai and Meshulam [28], and an algebraic generalization due to Fløystad [19].
The main goal of this section is to establish a colorful Helly theorem for convexity spaces, where the number of colors required will depend on the Radon number. Before stating the general theorem we consider as an example the simplest case when the Radon number equals 3.

**Example 2.1:** Let $F_1, F_2, F_3$ be non-empty finite families of convex sets in a convexity space $(X, C)$ whose Radon number equals 3, and suppose that $\bigcap_{i=1}^{3} S_i \neq \emptyset$ for all $S_i \in F_i$ and $i \in [3]$. We claim that one of the $F_i$ has a non-empty intersection, that is, $\bigcap_{S \in F_i} S \neq \emptyset$ for some $1 \leq i \leq 3$. The argument goes as follows. For each $i$ choose a pair of sets $\{A_i, B_i\} \subset F_i$. (If $|F_i| = 1$ for some $i$, then the claim is trivially true.) Next, choose points $x_1, x_2, x_3 \in X$ such that

- $x_1 \in A_1 \cap B_2 \cap B_3$,
- $x_2 \in B_1 \cap A_2 \cap B_3$,
- $x_3 \in B_1 \cap B_2 \cap A_3$.

(Note how each column of $A_i$’s and $B_i$’s “encodes” one of the three possible 2-partitions of [3].) By assumption, the set $\{x_1, x_2, x_3\}$ admits a Radon partition, say for instance, $\text{conv}\{x_2\} \cap \text{conv}\{x_1, x_3\} \neq \emptyset$. Referring to the second column we see that $x_2 \in A_2$ and $\{x_1, x_3\} \subset B_2$, which implies that $A_2 \cap B_2 \neq \emptyset$. Any other Radon partition would work similarly, since the array above covers all possible 2-partitions of [3]. This argument shows that it is impossible that each of the $F_i$ contains a pair of disjoint sets. Therefore one of the $F_i$ must have pairwise intersecting members, and since the Radon number equals 3, this $F_i$ has a non-empty intersection, by Lemma 2.4, below.

Here is the general version of the colorful Helly theorem for convexity spaces with bounded Radon number.

**Theorem 2.2:** For every integer $r \geq 3$ there exists an integer $m = m(r)$ with the following property: Let $F_1, \ldots, F_m$ be non-empty finite families of convex sets in a convexity space with Radon number at most $r$. If $\bigcap_{i=1}^{m} S_i \neq \emptyset$ for all $S_i \in F_i$ and $i \in [m]$, then for some $1 \leq i \leq m$ we have

$$\bigcap_{S \in F_i} S \neq \emptyset.$$ 

Our proof is a direct generalization of the argument given in Example 2.1, but before getting to it we need to introduce a few more notions from the theory of convexity spaces.
For an integer \( k \geq 2 \), the \( k \)th **partition number** of a convexity space \((X, \mathcal{C})\), denoted by \( r_k \), is the smallest integer (if it exists) such that for any multiset \( Y \subset X \) with cardinality \( r_k \) (counting multiplicities), there exists a \( k \)-partition of \( Y \) into parts \( Y_1, \ldots, Y_k \) such that \( \text{conv}(Y_1) \cap \cdots \cap \text{conv}(Y_k) \neq \emptyset \). Observe that for \( k = 2 \) this coincides with our definition of the Radon number.

In the case when the ground set \( X \) is finite and \( k > |X| \) we adopt the convention that \( r_k = |X| + 1 \).

**Remark:** In the literature the \( k \)th partition number is sometimes referred to as the \( k \)th Radon number or the \( k \)th Tverberg number, but we will only use the term Radon number when referring to \( r_2 \). Let us also remark that the partition numbers dealing with multisets are sometimes referred to as **unrestricted**, while the \( k \)th **restricted** partition number, denoted by \( \bar{r}_k \), deals only with sets of distinct points. For any convexity space with bounded Radon number we have \( \bar{r}_2 = r_2 \), while for \( k \geq 3 \) we always have \( \bar{r}_k \leq r_k \), and there are examples showing that this inequality may be strict. [17, 25].

In general, we have the following bound on the \( k \)th partition number of a convexity space.

**Lemma 2.3** (Jamison [25]): For any integer \( k > 2 \) and convexity space with bounded Radon number, we have

\[
\frac{r_k}{r_2^{\left \lfloor \log_2 k \right \rfloor}}.
\]

For certain convexity spaces better bounds are known. For instance, for the standard convexity \((\mathbb{R}^d, \mathcal{C}^d)\) Tverberg’s theorem states that the \( k \)th partition number equals \((d+1)(k-1)+1\). One of the long-standing conjectures concerning the partition numbers of convexity spaces asserted that \( r_k \leq (k-1)(r_2 - 1) + 1 \), which would imply a purely combinatorial proof of Tverberg’s theorem (see, e.g., Eckhoff’s survey [17]). However, this conjecture was refuted by Bukh [11] who constructed convexity spaces with \( r_2 = 4 \) and \( r_k \geq 3k - 1 \), for all \( k \geq 3 \). It should also be noted that Bukh [11, Theorem 1] gave an improvement on the restricted partition numbers by showing \( \bar{r}_k \leq c(r_2)k^2 \log^2 k \), but since it is convenient for us to allow for multisets we will refer to the general bound by Jamison.

The **Helly number** of a convexity space \((X, \mathcal{C})\) is the smallest integer \( h_C \) (if it exists) such that in any finite family of convex sets whose intersection is empty we can find a subfamily of at most \( h_C \) sets whose intersection is empty.
Helly’s theorem [22] states that for the standard convexity \((\mathbb{R}^d, C^d)\), the Helly number equals \(d + 1\). In general, we have the following bound on the Helly number of a convexity space.

**Lemma 2.4** (Levi [30]): For any convexity space with bounded Radon number, we have \(h_C < r_2\).

We are now in position to prove Theorem 2.2.

**Proof of Theorem 2.2.** Let \(k = r - 1\) and \(n = r^{\lceil \log_2 k \rceil}\). We will prove the theorem for \(m = S(n, k)\). (Recall that \(S(n, k)\) is the Stirling number denoting the number of \(k\)-partitions of \([n]\).) For contradiction, suppose the families \(F_1, \ldots, F_m\) satisfy \(\bigcap_{S \in F_i} S = \emptyset\) for every \(i \in [m]\). From each \(F_i\) choose sets \(S_i^{(1)}, \ldots, S_i^{(k)}\) (with repetitions if necessary) such that \(\bigcap_{j=1}^k S_i^{(j)} = \emptyset\), and set

\[
G_i = \{S_i^{(1)}, \ldots, S_i^{(k)}\}.
\]

This is possible by definition of the Helly number and Lemma 2.4. Let \(P_1, \ldots, P_m\) be the distinct \(k\)-partitions of \([n]\), which we denote by

\[
P_i = \{P_i^{(1)}, \ldots, P_i^{(k)}\},
\]

where

\[
P_i^{(1)} \cup \cdots \cup P_i^{(k)} = [n].
\]

For every \(t \in [n]\) we define the subfamily \(X_t \subset \bigcup_{i=1}^m G_i\) according to the rule

\[
S_i^{(j)} \in X_t \iff t \in P_i^{(j)},
\]

which implies that \(|X_t \cap G_i| = 1\) for every \(t \in [n]\) and \(i \in [m]\). By the hypothesis, we can find a point

\[
x_t \in \bigcap_{S \in X_t} S
\]

for every \(t \in [n]\).

By Lemma 2.3, we have \(n \geq r_k\), and therefore there exists a partition \(P_i\) and a point \(x \in X\) such that

\[
x \in \text{conv}\{x_t\}_{t \in P_i^{(j)}}
\]

for every \(j \in [k]\). But this implies that \(x \in S_i^{(j)}\) for every \(j \in [k]\), which contradicts our initial assumption that \(\bigcap_{j=1}^k S_i^{(j)} = \emptyset\). \(\blacksquare\)
Remark: Using elementary bounds on the $S(n, k)$ [35] our proof gives a bound on $m(r)$ which is roughly $r^{r \lceil \log_2 r \rceil}$. We have little reason to believe that this bound is optimal, and certainly for specific convexity spaces, such as the standard convexity $(\mathbb{R}^d, C^d)$, it is very far from the truth.

3. A fractional Helly theorem

Let $H = (V, E)$ be a $k$-uniform hypergraph with finite vertex set $V$ and edge set $E$. A clique in $H$ is a subset $S \subset V$ such that $\binom{S}{k} \subset E$, and we let $\omega(H)$ denote the maximum number of vertices of a clique in $H$. For an integer $m \geq k$, let $c_m(H)$ denote the number of cliques in $H$ on $m$ vertices.

We refer to the set $M = \binom{V}{k} \setminus E$ as the set of missing edges, and we say that a family $\{\tau_1, \ldots, \tau_m\} \subset M$ is a complete $m$-tuple of missing edges if

1. $\tau_i \cap \tau_j = \emptyset$ for all $i \neq j$, and
2. $\{t_1, \ldots, t_m\}$ is a clique in $H$ for all $t_i \in \tau_i$ and $i \in [m]$.

We need the following result [23, Theorem 1.2] for the proof of Theorem 1.1. It is a generalization of a theorem due to Gyárfás, Hubenko, and Solymosi [20] which deals with the special case $k = m = 2$.

**Lemma 3.1:** For any $m \geq k > 1$ and $\alpha \in (0, 1)$, there exists a constant $\beta = \beta(\alpha, k, m) \in (0, 1)$ with the following property: Let $H$ be a $k$-uniform hypergraph on $n \geq m$ vertices and with $c_m(H) \geq \alpha \binom{n}{m}$. If $H$ does not contain a complete $m$-tuple of missing edges, then

$$\omega(H) \geq \beta n.$$ 

Remark: For fixed $k$ and $m$ the proof in [23] gives a lower bound on $\beta = \beta(\alpha, k, m)$ which is in $\Omega(\alpha^{k(m-1)})$.

**Proof of Theorem 1.1.** Let $m = m(r)$ be the constant from Theorem 2.2 and set $k = r - 1$. For given $\alpha \in (0, 1)$ we prove the theorem with $\beta = \beta(\alpha, k, m) > 0$, which is the constant from Lemma 3.1.

Define a $k$-uniform hypergraph $H(F, E)$ where $E$ is the set of intersecting $k$-tuples of $F$, that is,

$$E = \left\{ \sigma \subset \binom{F}{k} : \bigcap_{S \in \sigma} S \neq \emptyset \right\}.$$
Note that an intersecting \( m \)-tuple in \( F \) corresponds to a clique on \( m \) vertices in \( H \). A complete \( m \)-tuple of missing edges in \( H \) corresponds to pairwise disjoint subfamilies \( F_1, \ldots, F_m \), with \( |F_i| = k \), such that

\[
\bigcap_{S \in F_i} S = \emptyset \quad \text{and} \quad \bigcap_{i=1}^m S_i \neq \emptyset
\]

for all \( S_i \in F_i \) and \( i \in [m] \). By Theorem 2.2 this can not exist, and therefore \( H \) does not contain a complete \( m \)-tuple of missing edges. By the fractional Helly hypothesis we have \( c_m(H) \geq \alpha \binom{n}{m} \), and so by Lemma 3.1 we have \( \omega(H) \geq \beta n \). This means there exists a subfamily \( G \subseteq F \) with \( |G| \geq \beta n \) such that every \( k \)-tuple of \( G \) is intersecting. By Lemma 2.4 it follows that \( \bigcap_{S \in G} S \neq \emptyset \). □

4. Transversal numbers

Let \( X \) be a non-empty set and let \( \mathcal{F} \) be a family of subsets of \( X \). (Here both \( X \) and \( \mathcal{F} \) may be infinite.) Given an \( \varepsilon \in (0, 1) \) and a finite multiset \( Y \subset X \), a \textbf{weak \( \varepsilon \)-net} for \( Y \) (with respect to \( \mathcal{F} \)) is a subset \( N \subset X \) such that \( N \cap S \neq \emptyset \) for any \( S \in \mathcal{F} \) with \( |S \cap Y| \geq \varepsilon |Y| \)

(where elements of \( Y \) are counted with multiplicity).

If we let \( X = \mathbb{R}^d \) and \( \mathcal{F} = C^d \) (that is, \( \mathcal{F} \) is the family of convex sets in \( \mathbb{R}^d \)), the weak \( \varepsilon \)-net theorem [1] states that any finite multiset \( Y \subset \mathbb{R}^d \) admits a weak \( \varepsilon \)-net \( N \subset X \) (with respect to \( C^d \)), where the size of \( N \) is bounded above by a function \( f(\varepsilon, d) \) which is independent of the choice of \( Y \).

It is a central problem in discrete geometry to understand the correct growth rate of the function \( f(\varepsilon, d) \) for fixed \( d \) and \( \varepsilon \rightarrow 0 \). It is known that there are sets \( Y \subset \mathbb{R}^d \) which require weak \( \varepsilon \)-nets of size \( \Omega(\varepsilon^{-1}(\log\varepsilon^{-1})^{d-1}) \) [12], while the best known upper bound is roughly \( \varepsilon^{-d} \) [13]. A recent breakthrough is due to Rubin [36] who showed \( f(\varepsilon, 2) \leq \varepsilon^{-(\frac{d}{2}+\delta)} \) for arbitrary small \( \delta > 0 \).

In [11, Question 3], Bukh asked whether the weak \( \varepsilon \)-net theorem can be extended to arbitrary convexity spaces with bounded Radon number. More precisely, does there exist a function \( \tilde{f}(\varepsilon, r) \) with the following property: Given any convexity space \( (X, C) \) with Radon number at most \( r \) and any finite multiset \( Y \subset X \), does \( Y \) admit a weak \( \varepsilon \)-net \( N \subset X \) (with respect to \( C \)), where \( |N| \leq \tilde{f}(\varepsilon, r) \)?
It is a well-known consequence of Helly’s theorem (see Lemma 2.4, above) that $\tilde{f}(\varepsilon, r)$ exists whenever $\varepsilon > 1 - \frac{1}{r-1}$, in which case we have $\tilde{f}(\varepsilon, r) = 1$, and Bukh showed that $\tilde{f}(\varepsilon, 3) \leq O(\varepsilon^{-2})$ (see the discussion following [11, Proposition 4]). Recently, Moran and Yehudayoff [33] considered Bukh’s question in the setting of separable convexity spaces. In this case they showed that if the Radon number is at most $r$, then there exists weak $\varepsilon$-nets of size at most $(120r^2\varepsilon^{-1})^{4r^2\ln\varepsilon^{-1}}$.

The following generalizes the result of Moran and Yehudayoff and provides an affirmative answer to Bukh’s question.

**Theorem 4.1:** For every $r \geq 3$ there exist positive constants $c_1 = c_1(r)$ and $c_2 = c_2(r)$ with the following property: Let $(X, C)$ be a convexity space with Radon number at most $r$. Then any finite multiset $Y \subset X$ admits a weak $\varepsilon$-net (with respect to $C$) of size at most $c_1 \cdot \varepsilon^{-c_2}$.

**Proof.** Let $c_1$ and $c_2$ be the constants from Theorem 1.2. Given a finite multiset $Y \subset X$ we define the family

$$F = \{\text{conv}(S) : S \subset Y, |S| \geq \varepsilon|Y|\}.\$$

(Here $S$ is a multiset, and as usual elements of $S$ are counted with multiplicity.) Note that any subset of $X$ that intersects every member of $F$ is a weak $\varepsilon$-net for $Y$ (with respect to $C$), so we need to show that $\tau(F) \leq c_1 \cdot \varepsilon^{-c_2}$. Observe that $\tau^*(F) \leq \varepsilon^{-1}$ because there is a fractional transversal $\varphi : X \to [0, 1]$, defined by

$$\varphi(x) = \frac{m_x}{\varepsilon|Y|},$$

where $m_x$ denotes the multiplicity of the point $x \in Y$ and $m_x = 0$ for $x \notin Y$. Since the multiset $Y$ is finite, it follows that $F$ is finite as well, and so we may apply Theorem 1.2 to get $\tau(F) \leq c_1 \cdot \tau^*(F)^{c_2} \leq c_1 \cdot \varepsilon^{-c_2}$. \qed

**Remark:** The proof above is standard in the literature and can also be reversed to show that Theorem 4.1 implies Theorem 1.2 (see for instance [31, Corollary 10.4.4]).

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1 Separable convexity spaces $(X, C)$ are equipped with the additional structure of half-spaces, i.e., convex sets $H \in C$ such that $(X \setminus H) \in C$, and a separation axiom which requires that for every convex set $S \in C$ and $x \in (X \setminus S)$ there exists a half-space $H$ such that $S \subset H$ and $x \notin H$. 
We now review the work of Alon, Kalai, Matoušek and Meshulam [3], in which they investigated the relationship between transversal numbers of set systems and the fractional Helly property. Borrowing their notation, we say that a family of sets \( F \) (which may be infinite) has property FH\((k, \alpha, \beta)\) if for any finite subfamily \( G \subset F \), with \(|G| = n\), in which at least \( \alpha \binom{n}{k} \) of the \( k \)-tuples of \( G \) have non-empty intersection, there exists at least \( \beta n \) members of \( G \) whose intersection is non-empty.

For a finite family of sets \( F \), let \( F^\cap \) denote the family of all intersections of the sets in \( F \), that is,

\[
F^\cap = \left\{ \bigcap_{S \in H} S : H \subseteq F \right\}.
\]

We need the following weak \( \varepsilon \)-net theorem for abstract set systems due to Alon et al.

**Theorem 4.2** ([3], Theorem 9): For every \( d \geq 1 \) there exists an \( \alpha > 0 \) such that the following holds. Let \( F \) be a finite family of sets and suppose \( F^\cap \) satisfies FH\((d + 1, \alpha, \beta)\) with some \( \beta > 0 \). Then we have

\[
\tau(F) \leq c_1 \cdot \tau^*(F)^{c_2},
\]

where \( c_1 \) and \( c_2 \) depend only on \( d \) and \( \beta \).

**Proof of Theorem 1.2.** Note that for any convexity space \((X, C)\) and any finite subfamily \( F \subset C \) we have \( F^\cap \subset C \). If the Radon number of \((X, C)\) is at most \( r \), then Theorem 1.1 implies that there exists an \( m = m(r) \) such that for any \( \alpha > 0 \) there is a \( \beta > 0 \) such that property FH\((m, \alpha, \beta)\) holds for any finite subfamily of \( C \), in particular for a subfamily of \( F^\cap \). Theorem 1.2 therefore follows from Theorem 4.2.

**Remark:** It is pointed out in [3] that their proof yields a bound on \( c_2 \) which is exponential in \( d \). Since we are applying their theorem with \( d + 1 = m(r) \) from Theorem 1.1 we get a bound on \( c_2(r) \) which is exponential in \( r^{\log_2 r} \). (See the remark following the proof of Theorem 2.2.)

It would be interesting to find further properties of set systems which guarantee a weak \( \varepsilon \)-net theorem, and some directions are suggested by Moran and Yehudayoff [33, section 6]. Finally, let us point out that our results also imply a \((p, q)\) theorem in convexity spaces with bounded Radon number. This follows immediately from Theorems 1.1, 1.2, and the results in [3].
Theorem 4.3: Let \( r \geq 3 \) and let \( m = m(r) \) be the value from Theorem 1.1. For any \( p \geq m \) there exists a constant \( c = c(p, r) \) with the following property: Let \( F \) be a finite family of convex sets in a convexity space with Radon number at most \( r \), and suppose among any \( p \) members of \( F \) there are some \( m \) of them with non-empty intersection. Then \( \tau(F) \leq c \).

Sketch of proof. We first apply Theorem 8(i) from [3] which states that for given \( p \geq m \) there exists an \( \alpha = \alpha(p, m) > 0 \) such that if \( F \) satisfies \( FH(m, \alpha, \beta) \) for some \( \beta > 0 \) and among any \( p \) members of \( F \) some \( m \) of them intersect, then \( \tau^*(F) \leq T \), where \( T \) depends only on \( p, m \) and \( \beta \). (This can be proved exactly as in Alon and Kleitman’s original proof of the \( (p, q) \) theorem [4], which is also what is done in [3]. Another excellent exposition can be found in [31, section 10.5].) The existence of a \( \beta > 0 \) such that \( F \) satisfies \( FH(m, \alpha, \beta) \) follows from Theorem 1.1, where \( \beta = \beta(\alpha, r) \). So in our situation, as \( m = m(r) \), it follows that \( T \) depends only on \( p \) and \( r \). Having this absolute bound on \( \tau^*(F) \), we then apply Theorem 1.2 to obtain an absolute bound on \( \tau(F) \) which depends only on \( p \) and \( r \).

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