Flavor-mass majorization uncertainty relations and their links to the mixing matrix

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Uncertainties in flavor and mass eigenstates of neutrinos are considered within the majorization approach. Nontrivial bounds reflect the fact that neutrinos cannot be simultaneously in flavor and mass eigenstates. As quantitative measures of uncertainties, both the Rényi and Tsallis entropies are utilized. Within the current amount of experience concerning the mixing matrix, majorization uncertainty relations need to put values of only two parameters, viz. $\theta_{12}$ and $\theta_{13}$. That is, the majorization approach is applicable within the same framework as the Maassen–Uffink relation recently utilized in this context. We also consider the case of detection inefficiencies, since it can naturally be incorporated into the entropic framework. Short comments on applications of entropic uncertainty relations with quantum memory are given.

Keywords: uncertainty relations, Rényi entropy, Tsallis entropy, neutrino oscillations

I. INTRODUCTION

The Heisenberg uncertainty principle is one of cornerstones of modern physics. The traditional position-momentum uncertainty relation was formally derived by Kennard. For any pair of observables, the corresponding formulation was presented by Robertson. This formulation was criticized for several reasons. Entropic uncertainty relations were proposed as an alternative to more traditional approach dealing with the product of standard deviations. For the case of canonically conjugate variables, this approach was initiated by Hirschman and later developed in. In effect, it seems to be more fruitful for discrete observables. For basic advantages of the entropic approach, see. Another reason to use entropic uncertainty relations is inspired by the role of “side information”.

Entropic uncertainty relations of the Maassen–Uffink type are actually Kraus’ conjecture proved on the base of Riesz’s theorem. There are efforts to formulate entropic uncertainty relations beyond this restriction. Sometimes, the corresponding optimization can be carried out explicitly. The authors of examined a constrained optimization problem without the conjugacy restriction on the entropic indices. Majorization relations are a way to characterize uncertainties in terms of probabilities for any. First majorization entropic uncertainty relations were based on tensor products of probability vectors. Stronger bounds obtained in are based on majorization relations applied to direct sums of probability vectors.

New interest to uncertainty relations is stimulated by recent progress in quantum information science. In the context of particle physics, uncertainty relations have found less attention than they deserve. Neutrino physics had come across radical ideas right since its appearing. One of explanations of experimental data in studying β-decay claimed that conservation laws hold only statistically. Pauli’s solution was an alternative to such proposals. Neutrino oscillations predicted by Pontecorvo were for a time the long-standing question. The flavor eigenstates of neutrinos and the mass ones form two different bases related by the Pontecorvo–Maki–Nakagawa–Sakata (PMNS) matrix. Reference studied corollaries of this fact via both the Robertson and Maassen–Uffink formulations. There are physically realizable situations, where the above issues are applicable.

The following issues were discussed as an arena for the use of flavor-mass uncertainty relations. Studies of cosmic neutrinos inevitably involve processing of large amounts of data. Entropic functions are a standard tool in such questions. Being almost free from decoherence effects, neutrinos are candidates to implement some protocols of quantum information. Entangled states of neutrino-lepton pairs produced by decaying gauge bosons can be used to probe experimentally parameters of the mixing matrix. This work is devoted to flavor-mass majorization uncertainty relations. In Section II we briefly recall majorization uncertainty relations and related material, including results of. In Section III majorization approach is applied to the flavor and mass eigenbases. The case of detection inefficiencies is addressed as well. In Section IV entropic uncertainty relations for entangled neutrino-lepton states are shortly discussed. In Section V we conclude the paper.

II. ON MAJORIZATION UNCERTAINTY RELATIONS IN GENERAL

For two integers $m, n \geq 1$ the symbol $\mathbb{M}_{m \times n}(\mathbb{C})$ denotes the space of all $m \times n$ complex matrices. For any $A \in \mathbb{M}_{m \times n}(\mathbb{C})$, the square matrices $A^\dagger A$ and $AA^\dagger$ have the same non-zero eigenvalues. Taking the square root of these
eigenvalues, we obtain non-zero singular values \( \sigma_j(A) \) of \( A \). We further use the spectral norm of \( A \) defined as

\[
\|A\|_\infty = \max_j \sigma_j(A).
\] (2.1)

Let \( X = \{|x_i\rangle\} \) and \( Z = \{|z_j\rangle\} \) with \( i,j = 1,\ldots,d \) be two orthonormal bases. For the pre-measurement state \( \rho \), the probability vectors \( p \) and \( q \) have elements \( p_i = \langle x_i|\rho|x_i \rangle \) and \( q_j = \langle z_j|\rho|z_j \rangle \).

For the given probability distribution, the Shannon entropy is written as

\[
H_1(p) = -\sum_i p_i \ln p_i.
\] (2.2)

Let \( \eta_1 \) be the largest among \( |\langle x_i|z_j \rangle| \). Maassen and Uffink proved that \[3\]

\[
H_1(X;\rho) + H_1(Z;\rho) \geq -2 \ln \eta_1,
\] (2.3)

where \( H_1(X;\rho) \) and \( H_1(Z;\rho) \) are obtained by substituting the above probabilities. Let \( \eta_2 \) denote the second largest value among \( |\langle x_i|z_j \rangle| \); then \[30\]

\[
H_1(X;\rho) + H_1(Z;\rho) \geq -2 \ln \eta_1 + (1-\eta_1) \ln \left( \frac{\eta_1}{\eta_2} \right).
\] (2.4)

The latter improves (2.3), whenever \( \eta_2 < \eta_1 \). The authors of \[30\] also derived entropic uncertainty relations with quantum side information.

To the unitary matrix \( W = [[[|x_i|z_j \rangle]\rangle] \), we assign the set of all submatrices of class \( k \), viz.

\[
\text{SUB}(W,k) := \{ M \in M_{r \times r'}(\mathbb{C}) : r + r' = k + 1, M \text{ is a submatrix of } W \}.
\] (2.5)

The majorization relations are formulated in terms of positive quantities

\[
\zeta_k := \max \{ \|M\|_\infty : M \in \text{SUB}(W,k) \}.
\] (2.6)

Majorization relations of the tensor-product type are formally posed as \[20\]

\[
p \otimes q \prec \omega',
\] (2.7)

where

\[
\omega' = (\xi_1, \xi_2 - \xi_1, \ldots, \xi_d - \xi_{d-1}), \quad \xi_k = \frac{(1 + \zeta_k)^2}{4}.
\] (2.8)

Majorization relations of the direct-sum type follow from the formula \[22\]

\[
p \oplus q \prec \{1\} \oplus \omega,
\] (2.9)

in which \( \omega = (\zeta_1, \zeta_2 - \zeta_1, \ldots, \zeta_d - \zeta_{d-1}) \) and \( \zeta_d = 1 \) by the unitarity. The relations \[24\] and \[25\] are both based on lemma 1 of \[20\]. As the right-hand side of its formula gives a correct upper bound for any state, the majorization relations hold for all states.

It is very helpful to convert (2.7) and (2.9) into inequalities between entropic functions. For \( 0 < \alpha \neq 1 \), the Rényi \( \alpha \)-entropy is defined as \[32\]

\[
R_\alpha(p) := \frac{1}{1-\alpha} \ln \left( \sum_i p_i^\alpha \right).
\] (2.10)

It is not greater than the logarithm of the number of non-zero probabilities. The Tsallis \( \alpha \)-entropy of degree \( 0 < \alpha \neq 1 \) reads as \[33\]

\[
H_\alpha(p) := \frac{1}{1-\alpha} \left( \sum_i p_i^\alpha - 1 \right).
\] (2.11)

In the limit \( \alpha \to 1 \), both the entropies (2.10) and (2.11) reduce to (2.2). For \( \alpha \to +0 \), the entropy (2.10) tends to the logarithm of the number of non-zero probabilities. Basic properties of the above entropies with applications are discussed in \[34\].
Since the Rényi entropy is Schur concave, the majorization relation \[2.7\] implies that, for \(\alpha > 0\),
\[
R_\alpha(X; \rho) + R_\alpha(Z; \rho) \geq R_\alpha(\omega').
\]
(2.12)
The result \[40\] allows us to improve entropic bounds \[22\]. For \(0 < \alpha \leq 1\), one obtains
\[
R_\alpha(X; \rho) + R_\alpha(Z; \rho) \geq R_\alpha(\omega).
\]
(2.13)
It is stronger, since \(\omega < \omega'\) and, therefore, \(R_\alpha(\omega) \geq R_\alpha(\omega')\) \[22\]. For \(\alpha > 1\), relation \[2.13\] is not valid in general.
The authors of \[22\] proved another inequality
\[
R_\alpha(X; \rho) + R_\alpha(Z; \rho) \geq \frac{2}{1 - \alpha} \ln \left(\frac{1}{2} + \frac{1}{2} \sum_i \omega_i^\alpha\right),
\]
(2.14)
which holds for \(\alpha > 1\). The sum of two Tsallis \(\alpha\)-entropies is bounded from below similarly to \[2.13\]. For all \(\alpha > 0\), we have
\[
H_\alpha(X; \rho) + H_\alpha(Z; \rho) \geq H_\alpha(\omega).
\]
(2.15)
Several generalizations of the above majorization relations were also considered in the literature \[21, 22, 35, 36\].

The entropic framework allows one to take into account detection efficiencies, when the “no-click” event appears.

To the given detector efficiency \(\kappa \geq 1/2\) and probability distribution \(\{p_i\}\), we assign a “distorted” distribution \(\tilde{p}(\kappa)\) such that
\[
\tilde{p}_i(\kappa) = \kappa p_i, \quad \tilde{p}_0(\kappa) = 1 - \kappa.
\]
(2.16)
The probability \(p(\kappa)_0\) corresponds to the no-click event. The above formulation was proposed in studying cycle scenarios of the Bell type \[37\]. It follows that \[38, 39\]
\[
H_\alpha(p(\kappa)) = \kappa^\alpha H_\alpha(p) + h_\alpha(\kappa),
\]
(2.17)
where the binary entropy \(h_\alpha(\kappa) = (1 - \alpha)^{-1}(\kappa^\alpha + (1 - \kappa)^\alpha - 1)\).

### III. Uncertainty Relations for Flavor and Mass

Let us proceed to uncertainty relations for measurements defined via kets from the flavor basis \(\mathcal{F} = \{|\nu_e\rangle, |\nu_\mu\rangle, |\nu_\tau\rangle\}\) and the mass basis \(\mathcal{M} = \{|\nu_1\rangle, |\nu_2\rangle, |\nu_3\rangle\}\). The flavor basis states are linked to the mass basis ones as
\[
|\nu_\beta\rangle = \sum_i u_{\beta i} |\nu_i\rangle, \quad |\nu_i\rangle = \sum_\beta u_{\beta i} |\nu_\beta\rangle.
\]
(3.1)
The \(3 \times 3\)-matrix \(U = [u_{\beta i}]\) is now referred to as the Pontecorvo–Maki–Nakagawa–Sakata (PMNS) one \[27, 28\]. The PMNS matrix is further assumed to be unitary, though this does not holds in some models. Questions of testing such models are discussed in \[40\]. In the case considered, the bases are orthonormal:
\[
\langle \nu_\beta | \nu_\gamma \rangle = \delta_{\beta \gamma}, \quad \langle \nu_i | \nu_\gamma \rangle = \delta_{ij}.
\]
(3.2)
The PMNS matrix is parametrized by mixing angles \(\theta_{12}, \theta_{23}, \theta_{13}\), and the CP-violating phase \(\delta\), viz.

\[
U = \begin{pmatrix}
1 & 0 & 0 \\
0 & c_{13} & s_{13}e^{-i\delta} \\
0 & -s_{13}e^{-i\delta} & c_{13}
\end{pmatrix}\begin{pmatrix}
c_{12} & s_{12} & 0 \\
-s_{12} & c_{12} & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

\[
= \begin{pmatrix}
c_{12}c_{13} & s_{12}s_{13}e^{i\delta} & 0 \\
-s_{12}c_{13} & c_{12}s_{13}e^{-i\delta} & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

(3.3)
where \(c_{ij} \equiv \cos \theta_{ij}\) and \(s_{ij} \equiv \sin \theta_{ij}\). Since Majorana-like phases do not affect majorization uncertainty relations, they are not recalled.

Within the considered framework, flavor and mass measurements are formally treated in terms of the projectors \(|\nu_\beta\rangle \langle \nu_\beta|\) and \(|\nu_i\rangle \langle \nu_i|\), respectively. As flavor fields contribute to the electroweak charged current, they can be produced
from the decay of a charged gauge boson \[^{29}\]. For instance, neutrino–gauge boson interaction produces a lepton further identified as electron, muon, or tau. In other words, flavor can physically be measured through electroweak interactions. Instead, the gravitation must inevitably participate in a mass measurement. In practice, mass eigenstates can rather be observed by absence of neutrino oscillations. Decay processes of charged gauge bosons give a tool to entanglement-assisted determination of the PMNS matrix \[^{29}\]. Of course, in neutrino physics any question of experimental character is certainly not easy. Even so, the measurements of interest are within capabilities of modern physics. As will be discussed later, entropic formulation also allows us to take into account detection inefficiencies.

To concretize majorization uncertainty relations, we need to have evaluate numbers \(\zeta_k\) for the matrix \(^{33}\). Since complete rows and columns of a unitary matrix are unit vectors, we have \(\zeta_3 = 1\). Roughly, the experimental findings are consistent with a tri-bimaximal mixing matrix \(^{41}\), which for \(\delta = 0\) reads as

\[
\begin{pmatrix}
\sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} & 0 \\
-\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} \\
\sqrt{\frac{1}{6}} & -\sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}}
\end{pmatrix}.
\]

Due to \(^{33}\), one has \(\eta_1 = \sqrt{2/3}, \eta_2 = \sqrt{1/2}\), and \((\zeta_1, \zeta_2, \zeta_3) = (\sqrt{2/3}, 1, 1)\). These values should be treated only as a first attempt. Let us quote some results of NuFIT \[^{12, 42}\] with an updated global analysis of neutrino oscillation measurements. We will use the best-fit values based on data available in July 2020. The best-fit parameters (bfp-values) from NuFIT \[^{12}\] are reproduced in Table I. The second of two imprints is obtained with the inclusion of data on atmospheric neutrinos provided by Super-Kamiokande \[^{14}\]. The 3\(\sigma\) ranges of two imprints are shown as well.

Irrespectively to the inclusion of data on atmospheric neutrinos, the term \(\eta_1 = \zeta_1\) is equal to \(|u_{e1}| = c_{12}c_{13}\) within the 3\(\sigma\) ranges provided by NuFIT \[^{12}\]. Under the same circumstances, we have \(\eta_2 = c_{13}\max\{s_{23}, s_{33}\}\). Appositely, this choice of \(\eta_1\) and \(\eta_2\) remains valid with elements of the matrix \(^{33}\). With the bfp-values, the right-hand side of \(^{23}\) exceeds the Maassen–Uffink bound \(−2\ln \eta_1\) by 4.5\%. To write explicitly majorization uncertainty relations, more narrow ranges should be taken. It turned out that \(\zeta_2 = c_{13}\) within the 1\(\sigma\) ranges given by NuFIT \[^{12}\]. This conclusion is also independent of the inclusion of data on atmospheric neutrinos. With the bfp-values and \(\alpha = 1\), the right-hand side of \(^{213}\) exceeds \(−2\ln \eta_1\) by around 31.3\%. The structure of \(^{33}\) is such that

\[
\sqrt{|u_{e1}|^2 + |u_{e2}|^2} = c_{13} = \sqrt{|u_{\mu3}|^2 + |u_{\tau3}|^2}.
\]

It is natural that the answer \(\zeta_2 = c_{13}\) also follows from \(^{33}\). From the unitarity condition, we inevitably have \(\zeta_3 = 1\). In contrast to the NuFIT results, the matrix \(^{33}\) leads to \(\zeta_2 = 1\). In each of the relations \(^{212}, 213\) and \(^{214}\), the right-hand side then becomes the binary entropy. It is instructive to visualize this distinction.

The product-type inequality \(^{212}\) holds for all \(\alpha > 0\). Another entropic inequality is expressed as \(^{213}\) for \(0 < \alpha \leq 1\) and as \(^{214}\) for \(\alpha > 1\). For \(0 < \alpha \leq 2\), the right-hand sides of these inequalities are shown in Fig. 1 with the use of the NuFIT best-fit values and the elements of \(^{33}\). For concreteness, the corresponding two curves use the best-fit values with the inclusion of data on atmospheric neutrinos \[^{12}\]. A contrast with the best-fit values without such data is so small that does not have actual bearing. In opposite, substituting the elements of \(^{33}\) leads to principally different curves. It is natural since the use of \(^{33}\) gives \(s_{13} = 0\) and, further, \(\omega_3 = \omega_3' = 0\). So, the entropies \(R_\alpha(\omega)\) and \(R_\alpha(\omega')\) become binary. For \(\alpha \rightarrow +0\), the corresponding two curves approach in 2 instead of In3. Probing the number of non-zero probabilities, Rényi entropies with small \(\alpha\) mirror the number of neutrino species.

The curves of Fig. 1A witness a sensitivity of the majorization bounds to \(\theta_{13}\), especially with varying \(\alpha\). This gives an example of utility of parametrized uncertainty relations \[^{3}\]. Of course, these curves also reflect some features of deriving the used uncertainty bounds. Nonetheless, relations with parametrized entropies quite deserve to be used, whenever we become delving into the flavor-mass complementarity.
Let obtained data with respect to the bases $\mathcal{F}$ and $\mathcal{M}$ be characterized by the efficiencies $\kappa_F$ and $\kappa_M$, respectively. Combining (2.15) with (2.17) gives

$$H_{\alpha}(\kappa_F) + H_{\alpha}(\kappa_M) \geq \kappa \alpha H_\alpha(\omega) + h_\alpha(\kappa_F) + h_\alpha(\kappa_M).$$

(3.6)

Replacing $H_1(\omega)$ with $-2 \ln \eta_1$ here, we get the corresponding corollary of the the Maassen–Uffink relation [5]. Thus, detection inefficiencies produce additional uncertainties in the entropies of actually measured data. The above discussion completes the results of [29] in this regard as well. As follows from the performed inspection, only $\theta_{12}$ and $\theta_{13}$ are actually needed to apply (2.7) and (2.9) in the case of interest. Hence, the inequality (3.7) is valid within the same framework as corollaries of the Maassen–Uffink uncertainty relation.

IV. ON UNCERTAINTY RELATIONS WITH QUANTUM SIDE INFORMATION

The authors of [29] formulated a protocol to probe parameters of the PMNS matrix from quantum manipulations and measurements on entangled neutrino-lepton pairs. For such experiments, the “quantum-memory” uncertainty relation of [13] was proposed to be used. As was already mentioned, one very likely deals with the case, when $\eta_2$ is strictly less than $\eta_1$. Hence, the stronger results of [30] should also be kept in mind. Entangled neutrino-lepton pairs could result from decay processes of the type $W^+ \rightarrow \nu + \tilde{\ell}^+$, namely

$$\frac{1}{\sqrt{3}} \left( |\nu_e\rangle \otimes |e^+\rangle + |\nu_\mu\rangle \otimes |\mu^+\rangle + |\nu_\tau\rangle \otimes |\tau^+\rangle \right) = \frac{1}{\sqrt{3}} \sum_j |\nu_j\rangle \otimes |\tilde{\ell}^+_j\rangle.$$

(4.1)

As direct manipulations in flavor space are challenging, projective measurements in the basis of leptonic states $|\tilde{\ell}^+_j\rangle$ can be used instead [29]. In each summand of the right-hand side of (4.1), the corresponding neutrino is in a definite mass eigenstate and does not show oscillations. To implement proper manipulations on the lepton side, the exact matrix elements are required. In reality, however, actual values of the four angles should be treated only as a guess. This motivates the use of uncertainty relations with quantum memory.

Uncertainty relations in the presence of quantum memory are posed as follows [13, 30]. Let $\rho_{AB}$ be density matrix of a system of two subsystems $A$ and $B$. The reduced densities are obtained by partial tracing, viz.

$$\rho_A = \text{Tr}_B(\rho_{AB}), \quad \rho_B = \text{Tr}_A(\rho_{AB}).$$

(4.2)

To the given orthonormal basis $\mathcal{X} = \{|x_i\rangle\}$ in $\mathcal{H}_A$, we assign the linear map

$$\rho_A \mapsto \Phi_{\mathcal{X}}(\rho_A) := \sum_i |x_i\rangle\langle x_i| \rho_A |x_i\rangle\langle x_i|.$$

(4.3)
Taking the two bases $X = \{ |x_i\rangle \}$ and $Z = \{ |z_j\rangle \}$ in $\mathcal{H}_A$, we further define the density matrices

$$\rho_{XB} = (\Phi_X \otimes \text{id})(\rho_{AB}), \quad \rho_{ZB} = (\Phi_Z \otimes \text{id})(\rho_{AB}),$$

where $\text{id} : \rho_B \mapsto \rho_B$ is the identity map. The quantum conditional entropy $S_1(A|B)$ is defined as

$$S_1(A|B) := S_1(\rho_{AB}) - S_1(\rho_B),$$

where $S_1(\rho) = -\text{Tr}(\rho \ln \rho)$ denotes the von Neumann entropy. Coles and Piani showed that

$$S_1(X|B) + S_1(Z|B) \geq -2 \ln \eta_1 + (1 - \eta_1) \ln \left( \frac{\eta_1}{\eta_2} \right) + S_1(A|B).$$

(4.6)

The two terms $S_1(X|B)$ and $S_1(Z|B)$ are nonnegative as entropies of classical probability distributions. On the other hand, the conditional entropy $S_1(A|B)$ can be negative if $\rho_{AB}$ is entangled. For pure states of the form (4.1), we have $S_1(\rho_{AB}) = 0$ and $S_1(\rho_A) = \ln 3$. The second term in the right-hand side of (4.6) is nonnegative. Replacing it with zero, the uncertainty relation of (13) follows,

$$S_1(X|B) + S_1(Z|B) \geq -2 \ln \eta_1 + S_1(A|B).$$

(4.7)

The latter is mentioned as a tool for entanglement-assisted determination of the PMNS matrix. The result is also suitable to use in this context. In the case of product states, we have the additivity property

$$S_1(\rho_A \otimes \rho_B) = S_1(\rho_A) + S_1(\rho_B).$$

(4.8)

If the measured system is not coupled with others, the inequality reduces to

$$H_1(X; \rho_A) + H_1(Z; \rho_A) \geq -2 \ln \eta_1 + (1 - \eta_1) \ln \left( \frac{\eta_1}{\eta_2} \right) + S_1(\rho_A).$$

(4.9)

Without $S_1(\rho_A) \geq 0$, the latter gives. Vanishing the second term in the right-hand side of (4.9), we get the Maassen–Uffink bound added by the von Neumann entropy of the measured state, namely

$$H_1(X; \rho_A) + H_1(Z; \rho_A) \geq -2 \ln \eta_1 + S_1(\rho_A).$$

(4.10)

As was shown, this result also follows from the monotonicity of the quantum relative entropy. Thus, the relation should be used together with (4.10). But the former needs concrete values of all three mixing angles.

V. CONCLUSIONS

We examined applications of majorization uncertainty relations to neutrino states in the three-dimensional flavor-mass space. The recent literature has brought an attention to situations in which flavor-mass uncertainty relations could be used. Since neutrinos are almost free from decoherence effects, they are candidates to carry quantum information. In practice, of course, their applicability in quantum technology is an open question. Applications of information-theoretic concepts deserve to be involved in experiments to analyze the cosmic neutrino background. The authors of proposed an approach with entangled states to probe experimentally parameters on the mixing matrix. The amount of accumulated experimental data allows us to make the following observations.

Applying the entropic approach to flavor-mass uncertainties is coupled with incorporating concrete values of the mixing angles. In effect, the Maassen–Uffink relation as well as the majorization approach use only two concrete parameters, namely $\theta_{12}$ and $\theta_{13}$. At first glance, majorization uncertainty relations may seem to require all the matrix elements to be known. One of unexpected results of our research is that only $\theta_{12}$ and $\theta_{13}$ are actually needed, at least within the current level of experience concerning the PMNS matrix. Hence, the majorization approach is valid within the same framework as the Maassen–Uffink relation discussed for the flavor-mass pair in. In this way, lower entropic bounds are improved by almost one third.

So, the use of two mixing angles characterizes the two of three considered schemes to pose desired entropic uncertainty relations. Further, all the three mixing angles should be put to use the uncertainty relation of Coles and Piani. The same conclusion holds, when the results of are assumed to be used in entanglement-assisted tests of the mixing matrix. The case of detection inefficiencies is naturally incorporated into the entropic framework. Such
inefficiencies will produce some additional level of uncertainty. Thus, majorization uncertainty relations deserve to be used in this context among other formulations. The presented results may be applied in future studies of neutrinos.

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