Emergent Unitarity from the Amplituhedron

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ABSTRACT: We present a proof of perturbative unitarity for $\mathcal{N} = 4$ SYM, following from the geometry of the amplituhedron. This proof is valid for amplitudes of arbitrary multiplicity $n$, loop order $L$ and MHV degree $k$. 
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1 Introduction

Unitarity is at the heart of the traditional, Feynman diagramatic approach to calculating scattering amplitudes. It is built into the framework of quantum field theory. Modern on-shell methods provide an alternative way to calculate scattering amplitudes. While they eschew lagrangians, gauge symmetries, virtual particles and other redundancies associated with the traditional formalism of QFT, Unitarity remains a central principle that needs to be imposed. It allowed the construction of loop amplitudes from tree amplitudes via generalized unitarity methods [1–5] and loop level BCFW recursion relations [6, 7]. These on-shell methods were particularly fruitful in $\mathcal{N} = 4$ SYM and led to the development of the on-shell diagram approach in [8] and the discovery of the underlying grassmannian structure. Locality and unitarity seemed to be the only guiding principle behind gluing together the on-shell diagrams. The discovery of the amplituhedron [9], [10] revealed the deeper picture behind the process of gluing of on-shell diagrams - positive geometry. Positivity dictated how the on-shell diagrams were to be glued together. The resulting scattering amplitudes were local and unitary!

This discovery of the amplituhedron was inspired by the polytope structure of the NMHV for scattering amplitudes first elucidated in [11] and expanded upon in [12]. These motivated a definition of the amplituhedron analogous to the definition of the interior of a polygon. The tree amplituhedron $\mathcal{A}_{n,k,0}$ is the span of $k$ planes $Y^I$ living in $(k + 4)$ dimensions.

$$Y^I_a = C^{aI}_a Z^I_a \quad (1.1)$$

where $Z^I_a$ are positive external data in $(k + 4)$ dimensions, i.e. $\langle Z_{a_1} \ldots Z_{a_{k+4}} \rangle > 0$ if $a_1 < \cdots < a_{k+4}$ and $C^{aI}_a \in G_+(k,n)$. The extension to loop level is more involved and can be found in [9].

The amplituhedron thus replaced the principles of unitarity and locality by a central tenant of positivity. Tree level locality emerges as a simple consequence of the boundary structure of the amplituhedron, which in turn dictated is by positivity. The emergence of unitarity is more obscure. It is reflected in the factorization of the geometry on approaching certain boundaries. This was proved for $\mathcal{A}_{4,0,L}$ in [10]. The extension of this proof to arbitrary multiplicity is cumbersome using this definition of the amplituhedron. The topological definition of the amplituhedron introduced in [13] is central to extending the proof of unitarity.

The topological definition can be stated entirely in terms of 4 dimensional data (the familiar momentum twistors). The amplituhedron $\mathcal{A}_{n,k,L}$ depends on the $n$ momentum twistors corresponding to the external legs of the amplitude $\{Z_1,\ldots Z_n\}$ and the lines in twistor space corresponding to the loop momenta, $(AB)_1,\ldots (AB)_L$. 

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It satisfies the following constraints.

\begin{align}
\text{Tree level} & \quad \langle ii + 1 jj + 1 \rangle > 0 \quad \text{and the sequence} \\
S_{\text{tree}} & \quad \{ \langle ii + 1 i + 2 i + 3 \rangle, \ldots \langle ii + 1 i + 2 i - 1 \rangle (-1)^{k-1} \} \quad \text{has} \ k \ \text{sign flips}. \\
\text{Loop level} & \quad \langle (AB)_a ii + 1 \rangle > 0 \quad \text{and the sequence} \\
S_{\text{loop}} & \quad \{ \langle (AB)_a 12 \rangle, \langle (AB)_a 13 \rangle, \ldots \langle (AB)_a 1 n \rangle \} \quad \text{has} \ k + 2 \ \text{sign flips}. \\
\text{Mutual Positivity} & \quad \langle (AB)_a (AB)_b \rangle > 0
\end{align}

This topological definition has already been used to investigate the structure of ”deep” cuts to all loop orders in [14, 15]. It is well known that the branch cut structure of amplitudes is intimately tied to perturbative unitarity. This is encapsulated in the optical theorem which related the discontinuity across a double cut to the product of tree amplitudes. However, the location of branch points in loop amplitudes is governed by the boundary structure of the amplituhedron. The discontinuity across a branch cut is calculated by the residue on an appropriate boundary of the amplituhedron. The optical theorem thus translates into a statement about the factorization of the residue on this boundary. We expect this factorization to emerge as a consequence of the positive geometry.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{Figure1.png}
\caption{Structure of a unitarity cut of MHV amplitudes}
\end{figure}

In order to make contact with the amplituhedron, it is helpful to rewrite this using momentum twistors. For now, we will focus on the MHV amplitudes $A_{n,0,L}$. We are interested in the case where one of the loops, call it $AB$, cuts lines $ii + 1$ and $jj + 1$. Let us label the other loops as $(AB)_a$. We can parametrize $AB$ as

\begin{align}
A = Z_i + xZ_{i+1} + w_1 \quad & \quad B = yZ_j + Z_{j+1} + w_2
\end{align}

The term in the $L$-loop integrand which contributes to this cut is

\begin{align}
\mathcal{M}_n^L & = \frac{\langle ABd^2A \rangle \langle ABd^2B \rangle \langle AB(\ast ii + 1 \cap \ast jj + 1) \rangle^2}{\langle AB \ast i \rangle \langle AB \ast i + 1 \rangle \langle AB \ast j \rangle \langle AB \ast j + 1 \rangle \langle AB ii + 1 \rangle \langle AB jj + 1 \rangle} \\
& \quad \times \prod_a \langle (AB)_a d^2A_a \rangle \langle (AB)_a d^2B_a \rangle f(x,y,w_1,w_2,(AB)_a)
\end{align}
The residue on the cut $\langle ABii + 1 \rangle = \langle ABjj + 1 \rangle = 0$ is
\[
Res_{w_1 = w_2 = 0} M_n^L = \frac{dx}{x} \frac{dy}{y} \prod_a \langle (AB)_a d^2 A_a \rangle \langle (AB)_a d^2 B_a \rangle f(x, y, 0, 0, (AB)_a)
\] (1.4)

Unitarity predicts that the function $f(x, y, 0, 0, (AB)_a)$ is related to lower point amplitudes (see figure 1) and is of the form
\[
f(x, y, 0, 0, (AB)_a) = \sum_{L_1 + L_2 = L - 1} M_{L_1} (Z_{j+1}, \ldots, Z_i, A, B) M_{L_2} (B, A, Z_{i+1}, \ldots, Z_j)
\] (1.5)

We will show that this structure follows simply from the geometry of the amplituhedron. We will first present a proof for the four point case. This is just a rewriting of the proof found in [10] in the language of sign flips. This proof will then admits a generalization to amplitudes of higher multiplicity.

**2 Proof for 4 point Amplitudes**

At four points ($n = 4, k = 0$) we focus on the unitarity cut $\langle AB12 \rangle = \langle AB34 \rangle = 0$. On this cut, we can parametrize $(AB)$ as
\[
A = Z_1 + xZ_2 \quad B = yZ_3 + Z_4
\]
with $x > 0, y > 0$. Denoting the uncut loops as $(AB)$, the mutual positivity conditions are
\[
\langle ABA_i B_i \rangle = \langle A_i B_1 13 \rangle y + \langle A_i B_1 14 \rangle + \langle A_i B_2 23 \rangle xy + \langle A_i B_2 24 \rangle x > 0
\] (2.1)

This can be "factorized"
\[
((\langle AB \rangle 2B) \langle \langle AB \rangle A3 \rangle)
= (\langle AB \rangle A3) + \langle A, B, 23 \rangle y (\langle A, B, 13 \rangle + \langle A, B, 23 \rangle x)
= \langle ABA_i B_i \rangle \langle A_i B_2 23 \rangle - \langle A_i B_1 14 \rangle \langle A, B, 23 \rangle + \langle A_i B_2 24 \rangle (\langle A_i B_1 1 \cap A_i B_2 34 \rangle)
= \langle ABA_i B_i \rangle \langle A_i B_2 23 \rangle + \langle A_i B_1 12 \rangle \langle A, B, 34 \rangle > 0
\]

because each term is individually positive. There are two possible solutions are
\[
(\langle (AB)_i 2B \rangle > 0 \quad \langle (AB)_i A3 \rangle > 0
\] (2.2)

and
\[
(\langle (AB)_i 2B \rangle < 0 \quad \langle (AB)_i A3 \rangle < 0
\] (2.3)

If $L_1$ loops, $(AB)_a$ obey (2.2) and $L_2 = L - L_1 - 1$ loops, $(AB)_a$ obey (2.3), it is easy to see that each loop $(AB)_a$ and $(AB)_a$ is in an amplituhedron with lower $n, L$. 
\((AB)_{\alpha}\) belong to a 4 point, \(L_2\) loop amplituhedron with legs \(\{A, Z_2, Z_3, B\}\) as seen from the conditions below.

\[
\begin{align*}
\text{Tree Level} & \quad \langle AB23 \rangle = \langle AB23 \rangle > 0 \\
\text{Loop level} & \quad \langle (AB)_{\alpha}A2 \rangle = \langle (AB)_{\alpha}12 \rangle > 0 \quad \langle (AB)_{\alpha}23 \rangle > 0 \\
& \quad \langle (AB)_{\alpha}3B \rangle = \langle (AB)_{\alpha}34 \rangle > 0 \\
& \quad \text{The sequence } \{(\langle (AB)_{\alpha}A2 \rangle, (\langle AB)_{\alpha}A3 \rangle, (\langle AB)_{\alpha}AB \rangle\} \text{ has 2 sign flips}
\end{align*}
\]

Mutual positivity \(\langle (AB)_{\alpha}(AB)_{\beta} \rangle > 0\),

All these conditions are satisfied because of (2.2) and the fact that \((AB)_{\alpha}\) and \((AB)\) are in the one-loop amplituhedron with legs \(\{Z_1, \ldots, Z_4\}\).

Similarly the \(L_1\) loops \((AB)_{\alpha}\) are in the \(L_1\) loop amplituhedron with legs \(\{Z_1, A, B, Z_4\}\) as seen from the conditions below.

\[
\begin{align*}
\text{Tree Level} & \quad \langle 1AB4 \rangle = \langle AB14 \rangle > 0 \\
\text{Loop level} & \quad \langle (AB)_{\alpha}1A \rangle = x \langle (AB)_{\alpha}12 \rangle > 0 \quad \langle (AB)_{\alpha}AB \rangle > 0 \\
& \quad \langle (AB)_{\alpha}B4 \rangle = y \langle (AB)_{\alpha}34 \rangle > 0 \\
& \quad \text{The sequence } \{(\langle (AB)_{\alpha}A \rangle, (\langle AB)_{\alpha}B)\} \text{ has 2 sign flips}
\end{align*}
\]

Mutual positivity \(\langle (AB)_{\alpha}(AB)_{\beta} \rangle > 0\),

These conditions are again guaranteed because of (2.3) and the fact that the loops \((AB)_{\alpha}\) and \((AB)\) are in the 1 loop amplituhedron with legs \(\{Z_1, \ldots, Z_n\}\). The flip condition follows from the Plücker relation

\[
\langle (AB)_{\alpha}1B \rangle \langle (AB)_{\alpha}23 \rangle - \langle (AB)_{\alpha}2B \rangle \langle (AB)_{\alpha}13 \rangle = \langle (AB)_{\alpha}12 \rangle \langle (AB)_{\alpha}B3 \rangle < 0 \tag{2.6}
\]

which guarantees \(\langle (AB)_{\alpha}1B \rangle < 0\).

To complete the proof of the factorization of the residue into the product of two lower loop amplitudes, we must show that the mutual positivity between the loops \((AB)_{\alpha}\) and \((AB)\) imposes no constraints. To see this, we can expand the loop \((AB)_{\alpha}\) in terms of \(\{Z_1, A, B, Z_4\}\) as

\[
A_{\alpha} = Z_1 + \alpha_1 A + \alpha_2 B \quad B_{\alpha} = -Z_1 + \beta_1 B + \beta_2 Z_4
\]

This gives

\[
\langle (AB)_{\alpha}(AB)_{\alpha} \rangle = y \langle (AB)_{\alpha}1B \rangle \beta_1 + \langle (AB)_{\alpha}14 \rangle \beta_2 + \langle (AB)_{\alpha}1A \rangle (\alpha_1) + \langle (AB)_{\alpha}AB \rangle \alpha_1 \beta_1 + \langle (AB)_{\alpha}A4 \rangle \alpha_1 \beta_2 + \langle (AB)_{\alpha}1B \rangle (\alpha_2) + \langle (AB)_{\alpha}B4 \rangle \alpha_2 \beta_2 \tag{2.7}
\]

Note that all the terms except for \(\langle (AB)_{\alpha}A4 \rangle\) and \(\langle (AB)_{\alpha}B1 \rangle\) are obviously positive.

\[
\langle (AB)_{\alpha}A4 \rangle = \langle (AB)_{\alpha}A(B - y3) \rangle = \langle (AB)_{\alpha}AB \rangle - y \langle (AB)_{\alpha}A3 \rangle > 0 \\
\langle (AB)_{\alpha}B1 \rangle = \langle (AB)_{\alpha}(A - x3) \rangle = \langle (AB)_{\alpha}AB \rangle - x \langle (AB)_{\alpha}2B \rangle > 0
\]

These follow from (2.4) and we can conclude that \(\langle (AB)_{\alpha}(AB)_{\alpha} \rangle > 0\) imposes no further constraints. The residue factorizes into \(\mathcal{M}_L\) and \(\mathcal{M}_R\).
3 Proof for MHV amplitudes of arbitrary multiplicity

We will extend the above results to amplitudes of arbitrary multiplicity. However, the existence of higher $k$ sectors beginning with $n = 5$ complicates the proof. In this section we will focus on a proof of unitarity for MHV amplitudes. This allows us to sketch the essentials of the proof without additional complications. In the next section, we modify the proof to account for higher $k$ sectors.

We are interested in examining the residue of the MHV amplituhedron $A_{n,0,L}$, with external data $\{Z_1, \ldots, Z_n\}$, on the cut $\langle ABii + 1 \rangle = \langle ABjj + 1 \rangle = 0$ where we can parametrize $AB$ as

$$A = Z_i + xZ_{i+1} \quad B = yZ_j + Z_{j+1} \quad x, y > 0.$$  \hspace{1cm} (3.1)

We need to show that the canonical form for every configuration on this cut can written as a product of canonical forms for lower loop, “left” and “right” MHV amplituhedra $A_{f,0,L_1}$ and $A_{n_2,0,L_2}$. The precise definitions and the proof that they exist are in the following section. Their existence is tied to the fact that the external data $\{Z_1, \ldots, Z_n\}$ and the uncut loops $(AB)_a$ are all in the amplituhedron $A_{n,0,L}$ and satisfy the following conditions.

**Tree Level** \(\langle ijkl \rangle > 0\) for \(i < j < k < l\) \hspace{1cm} (3.2)

**Loop level** \(\langle ii + 1 \rangle > 0\)

The sequence \(S_L = \{(i+1,i+2), \ldots, (i+n), -\langle i+11 \rangle, \ldots -\langle i+1i \rangle\}\) has 2 sign flips.

**Mutual Positivity** \(\langle (AB)_a (AB)_b \rangle > 0\).

Here, \(\langle ij \rangle \equiv \langle (AB)_a ij \rangle\) where $(AB)_a$ are the uncut loops.

### 3.1 Left and right Amplituhedra

#### 3.1.1 The left amplituhedron $A_{n_1,0,L_1}$

The left amplituhedron $A_{n_1,0,L_1}$ is defined by three sets of conditions similar to (3.2). In this case, the external data is the set $\mathcal{L} = \{Z_1, \ldots, Z_i, A, B, Z_{j+1}, , \ldots, Z_n\}$. Letting $a, b, c, d$ denote elements of this set and $(ij) \equiv \langle (AB)_a ij \rangle$, the defining conditions are

**Tree Level** \(\langle abcd \rangle > 0\), \(\langle iAab \rangle > 0\), \(\langle ABab \rangle > 0\)

\(\langle Bj + 1ab \rangle > 0\) \hspace{1cm} a < b < c < d

**Loop Level** \(\langle aa + 1 \rangle > 0\), \(\langle iA \rangle > 0\), \(\langle AB \rangle > 0\), \(\langle Bj + 1 \rangle > 0\)

The sequence \(S_L = \{(iA), \langle iB \rangle, \langle ij + 1 \rangle, \ldots, \langle in \rangle, -\langle i1 \rangle, \ldots -\langle ii - 1 \rangle\}\) has 2 sign flips.

**Mutual Positivity** \(\langle (AB)_a (AB)_b \rangle > 0\).
For consistency, we must also verify that the all sequences

\[ S_1 : \{ \langle AB \rangle, \langle Aj + 1 \rangle, \ldots, -\langle Ai \rangle \} \]
\[ S_2 : \{ \langle Bj + 1 \rangle, \langle Bj + 2 \rangle, \ldots, -\langle BA \rangle \} \]

\vdots
\[ S_{n+2+i-j} : \{ \langle i - 1j \rangle, \langle i - 1A \rangle, \ldots - \langle i - 1i - 2 \rangle \} \]

and \( S_L \) have the same number of sign flips. Note that the positivity conditions on the loop data ensures that all the first and last entries of these sequences are positive. Furthermore any two sequences in the above set are of the form \( \{ \langle ak \rangle \} \) and \( \{ \langle a + 1k \rangle \} \) which satisfy

\[ \langle ak \rangle \langle a + 1k + 1 \rangle - \langle ak + 1 \rangle \langle a + 1k \rangle = \langle aa + 1 \rangle \langle kk + 1 \rangle > 0 \quad (3.3) \]

The equality of sign flips now follows immediately from the analysis in Appendix A. This shows that the left amplituhedron can be consistently defined.

All the tree level, mutual positivity and the loop level positivity conditions are automatically satisfied because of (3.1) and (3.2). The flip condition is the only one that requires a detailed analysis which is presented in Section 3.2.

3.1.2 The right Amplituhedron \( A_{n_2,0,L_2} \)

The external data for the right amplituhedron \( A_{n_2,0,L_2} \) is \( R = \{ A, Z_{i+1}, \ldots, Z_j, B \} \) and the defining inequalities are listed below. \( a, b, c, d \in L \) and \( \langle ij \rangle \equiv (\langle AB \rangle)_a^{ij} \) with \( (AB)_a \) being an uncut loop.

**Tree Level** \( \langle abcd \rangle > 0, \langle Ai + 1ab \rangle > 0, \langle abjB \rangle > 0, \quad (3.4) \)

\( \langle ABab \rangle > 0 \quad \text{with } a < b < c < d \)

**Loop Level** \( \langle Ai + 1 \rangle > 0, \langle jB \rangle > 0, \langle aa + 1 \rangle > 0 \)

The sequence \( S_R = \{ \langle i + 1i + 2 \rangle, \ldots \langle i + 1j \rangle, \langle i + 1B \rangle, -\langle i + 1A \rangle \} \) has 2 sign flips

**Mutual Positivity** \( \langle (AB)_a (AB)_b \rangle > 0 \)

Once again, for consistency we should verify that

\[ S_1 : \{ \langle Ai + 1 \rangle, \langle Ai + 2 \rangle, \ldots, \langle AB \rangle \} \]
\[ S_2 : \{ \langle i + 2i + 3 \rangle, \langle i + 2i + 4 \rangle, \ldots, -\langle i + 2i + 1 \rangle \} \]

\vdots
\[ S_{2+j-i} : \{ -\langle BA \rangle, -\langle Bi + 1 \rangle, \ldots - \langle Bj \rangle \} \]

and \( S_R \) all have the same number of sign flips. The proof is identical to the one for the left amplituhedron. The tree level, mutual positivity and loop level positivity conditions are once again guaranteed by (3.1) and (3.2) and an analysis of the flip condition is in Section 3.2.
3.2 Factorization on the unitarity cut

The external data divides into two sets \( \mathcal{L} = \{Z_1, \ldots, Z_i, A, B, Z_{j+1}, \ldots, Z_n\} \) and \( \mathcal{R} = \{A, Z_{i+1}, \ldots Z_j, B\} \) which are both positive as observed in the previous section. To see the factorization at loop level, consider any uncut loop \( (AB)_a \) and a corresponding sequence \( S \).

\[ S = \{\langle i + 1i + 2 \rangle, \ldots \langle i + 1j \rangle \langle i + 1j + 1 \rangle, \ldots \langle ii + 1 \rangle\} \]

We have divided the sequence in a suggestive way. The left half of \( S \) looks very similar to \( S_R \). It natural to label the different flip patterns of \( S \) as \( S_{ablr} \) where \( a, b = \pm \) are the signs of \( \langle i + 1j \rangle \) and \( \langle i + 1j + 1 \rangle \) and \( l, r \) are the number of flips in the left and right parts of \( S \). In order to compare \( S_L \) to \( S \), we introduce the sequence \( S'_L = \{\langle i + 1A \rangle, \langle i + 1B \rangle, \langle i + 1j + 1 \rangle, \ldots \langle i + 1i - 1 \rangle\} \) with \( k'_L \) flips. \( S_L \) and \( S'_L \) are obviously connected by a Plücker relation and following Appendix[A], the relation between \( k_L \) and \( k'_L \) is determined entirely by the signs of the first and last elements

\[
\begin{pmatrix}
\langle iA \rangle \\
\langle i + 1A \rangle
\end{pmatrix}
\begin{pmatrix}
\langle i - 1i \rangle \\
\langle i - 1i + 1 \rangle
\end{pmatrix}
= \begin{pmatrix}
+ & + \\
- & \langle i - 1i + 1 \rangle
\end{pmatrix}
\]

where \( k_L \) is the number of sign flips in \( S_L \). If \( \langle i - 1i + 1 \rangle > 0 \), then \( k_L = k'_L - 1 \) otherwise \( k_L = k'_L \). \( S \) now looks almost like a juxtaposition of \( S_R \) and \( S'_L \). Each flip pattern determines whether the corresponding loop \( (AB)_a \) belongs to the left or the right amplituhedron.

- \( S_{++20} = \{+, \ldots 2 \text{ flips} \cdots +| + \ldots 0 \text{ flips} \cdots +\} \)

The sequence \( S_R \) clearly has 2 sign flips since

\( -\langle i + 1A \rangle > 0 \text{ and } \langle i + 1j \rangle > 0, \langle i + 1j + 1 \rangle > 0 \implies \langle i + 1B \rangle > 0 \)

\( S'_L \) has one sign flip since

\( \langle i + 1A \rangle > 0, \langle i + 1B \rangle > 0, \langle i - 1i + 1 \rangle > 0. \)

Furthermore \( k_L = k'_L - 1 = 0 \) and the loop \( (AB)_a \) belongs only to the right amplituhedron.

- \( S_{+02} = \{+, \ldots 0 \text{ flips} \cdots +| + \ldots 2 \text{ flips} \cdots +\} \)

\( S_R \) obviously has 0 sign flips. If \( \langle i - 1i + 1 \rangle > 0 \), \( k_L = k'_L + 1 = 2 \) and if \( \langle i - 1i + 1 \rangle < 0 \), \( k'_L = k_L = 2 \). In both cases, the loop belongs to the left amplituhedron and not the right.

- \( S_{+-01} = \{+, \ldots 0 \text{ flips} \cdots +| - \ldots 1 \text{ flip} \cdots +\} \)
If $\langle i + 1B \rangle > 0$, then the sequence $S_R$ has 0 flips and the loop doesn’t belong to the right amplituhedron. If $\langle i - 1i + 1 \rangle > 0$, $k'_L = 3$ and $k_L = k'_L - 1 = 2$. Otherwise, $k'_L = 2$ and $k_L = k'_L = 2$. Thus irrespective of the sign of $\langle i - 1i + 1 \rangle$, the loop $(AB)_a$ belongs to the left amplituhedron.

If $\langle i + 1B \rangle < 0$, then $S_R$ has 2 sign flips and it can be shown that $k_L = 0$ by analysis similar to the cases above. This $(AB)_a$ belongs to the right amplituhedron.

- $S_{-10} = \{+ \ldots 1 \text{ flip } \ldots -| \ldots 2 \text{ flips } \ldots +\}$

In this case, $S_R$ has two flips and $S'_L$ has one flip irrespective of the sign of $\langle i + 1B \rangle$. Since $k_L = k'_L - 1$, we have $k_L = 0$ and the loop belongs to the right amplituhedron.

- $S_{-11} = \{+ \ldots 1 \text{ flip } \ldots -| \ldots 1 \text{ flip } \ldots +\}$

Once again, it is simple to show that $S_R$ has two sign flips and $S_L$ has 0 sign flips in this configuration.

### 3.2.1 Trivialized mutual positivity

It remains to be shown that the mutual positivity between a loop $(AB)_L$ in the left amplituhedron and a loop $(AB)_R$ in the right amplituhedron is trivially true. It is easiest to see this if we expand each loop $(AB)_L$ in the left amplituhedron and $(AB)_R$ right amplituhedron in a Kermit expansion by using the following parametrizations.

$$
A_R = A + \alpha_1 Z_{r_1} + \alpha_2 Z_{r_1+1} \quad B_R = -A + \beta_1 Z_{r_2} + \beta_2 Z_{r_2+1}
$$

$r_1 < r_2 \in \{A, Z_{i+1}, \ldots, Z_j, B\}$. $(AB)_L$ is in one of the following cells of the left one-loop amplituhedron

$$
A_L = A + \alpha_3 Z_{l_1} + \alpha_4 Z_{l_1+1} \quad B_L = -A + \beta_3 Z_{l_2} + \beta_4 Z_{l_2+1}
$$

with $l_1 < l_2 \in \{Z_1, \ldots, Z_i, A, B, Z_{j+1}, \ldots, Z_n\}$. On expanding $\langle (AB)_L(AB)_R \rangle$, every term is of the form $\langle l_1 l_2 r_1 r_2 \rangle$. Since the external data are positive, i.e. $\langle ijkl \rangle > 0$ for $i < j < k < l$.

This completes the proof of factorization on the Unitarity cut for MHV amplituhedra. In the next section, we will demonstrate that this proof can be extended to higher $k$ sectors.
4 Proof for higher \( k \) sectors

The proof of unitarity for higher \( k \) is similar in spirit to that for the MHV sector. However, there are a lot additional details that we must take into account. Firstly, we must modify (1.5) to include products with different \( k \). Suppose the left amplitude has \( n_L \) negative helicity gluons and the right amplitude has \( n_R \) negative helicity gluons, then we have \( n_L + n_R = n + 2 \). With the MHV degrees are defined as \( k_L = n_L - 2, k_R = n_R - 2, k = n - 2 \), this equation reads \( k_L + k_R = k \). Thus we expect

\[
f(x, y, 0, 0, (AB)_a) = \sum_{k_L + k_R = k} \sum_{L_1 + L_2 = L} \mathcal{M}^{k_L, L_1}_{L} \mathcal{M}^{k_R, L_2}_{R}
\]

![Figure 2. Unitarity cut for an N\(^k\)MHV amplitude](image)

We expect that unitarity emerges from a factorization property of the geometry in a manner similar to the MHV case. In order to make this statement more precise, we will have to define analogues of the left and right MHV amplituhedra. We start with \( A_{n,k,L} \), the \( N^k \)MHV amplituhedron which is defined by the conditions

- **Tree level**: \( \langle ii + 1 jj + 1 \rangle > 0 \) and the sequence
  \( S^{\text{tree}}: \{\langle ii + 1 i + 2 i + 3 \rangle, \ldots, \langle ii + 1 i + 2 i - 1 \rangle (-1)^{k-1} \} \) has \( k \) sign flips.
- **Loop level**: \( \langle (AB)_a ii + 1 \rangle > 0 \) and the sequence
  \( S^{\text{loop}}: \{\langle (AB)_a 12 \rangle, \langle (AB)_a 13 \rangle, \ldots, \langle (AB)_a 1n \rangle \} \) has \( k + 2 \) sign flips.
- **Mutual Positivity**: \( \langle (AB)_a (AB)_b \rangle > 0 \)

Note that this definition is invariant under a twisted cyclic symmetry \( Z_{n+1} \rightarrow (-1)^{k-1} Z_1 \). On the unitarity cut (\( \langle ABii + 1 \rangle = \langle ABjj + 1 \rangle = 0 \)), there is a natural division of the external data into "left" and "right" sets, \( \{Z_1, \ldots, Z_i, A, B, Z_{i+1}, \ldots, Z_n\} \) and \( \{A, Z_{i+1}, \ldots, Z_j, B\} \). However, the data obeys a twisted cyclic symmetry with a fixed \( k \). In order to generate data with \( k_L \) which has a different even/odd parity, we will have to allow for arbitrary signs on the \( Zs \) and define two sets of external data.

\[
\mathcal{L} = \{\sigma(1)Z_1, \ldots, \sigma_L(i)Z_i, \sigma_L(A)A, \sigma_L(B)B, \sigma_L(j + 1)Z_{j+1}, \ldots, \sigma_L(n)Z_n\}
\]

\[
\mathcal{R} = \{\sigma_R(A)A, \sigma_R(i + 1)i + 1, \ldots, \sigma_R(j)j, \sigma_R(B)B\}
\]

(4.2)
where \( \sigma(k) = \pm 1 \). These will be determined by conditions like (4.1) which define the left and right amplituhedra along with the appropriate twisted cyclic symmetry. We will then show that the canonical form for every configuration in \( \mathcal{A}_{n,k,L} \) can be mapped into a product of canonical forms on suitably defined left and right amplituhedra \( \mathcal{A}_{n1,kL,L1}^L \) and \( \mathcal{A}_{n2,kR,L2}^R \).

### 4.1 The left and right amplituhedra

#### 4.1.1 The left amplituhedron \( \mathcal{A}_{n1,kL,L1}^L \)

We must demand that the set \( \mathcal{L} \) satisfies all the conditions in (4.1). In addition, this must also be compatible with the fact that the \( Z_i \) are the external data for \( \mathcal{A}_{n,k,L} \).

\[
\langle aa + 1bb + 1 \rangle \sigma_L(a)\sigma_L(a + 1)\sigma_L(b)\sigma_L(b + 1) > 0 \\
\langle ABaa + 1 \rangle \sigma_L(A)\sigma_L(B)\sigma_L(a)\sigma_L(a + 1) > 0 \quad (4.3)
\]

\( \forall a, b \in \{1, \ldots, j + 1, n - 1\} \)

\( AB \) and the \( Z_i \) automatically satisfy \( \langle aa + 1bb + 1 \rangle > 0 \) and \( \langle ABaa + 1 \rangle > 0 \). Thus we have, \( \sigma_L(a)\sigma_L(a + 1)\sigma_L(b)\sigma_L(b + 1) > 0 \) and \( \sigma_L(A)\sigma_L(B)\sigma_L(a)\sigma_L(a + 1) > 0 \). Furthermore, we have new constraints on \( A \) and \( B \) coming from

\[
\langle iABj + 1 \rangle \sigma_L(i)\sigma_L(A)\sigma_L(B)\sigma_L(j + 1) > 0 \\
\langle iAkk + 1 \rangle \sigma_L(i)\sigma_L(A)\sigma_L(k)\sigma_L(k + 1) > 0 \\
\langle Bj + 1kk + 1 \rangle \sigma_L(B)\sigma_L(j + 1)\sigma_L(k)\sigma_L(k + 1) > 0 \quad (4.4)
\]

Finally, since the set \( \mathcal{L} \) is the external data for \( \mathcal{A}_{n1,kL,L1}^L \), it must satisfy a twisted cyclic symmetry

\[
\langle aa + 1n1 \rangle \sigma_L(a)\sigma_L(a + 1)\sigma_L(n)\sigma_L(1)(-1)^{kL-1} > 0 \quad (4.5)
\]

Since \( \langle aa + 1n1 \rangle (-1)^{k-1} > 0 \), consistency requires \( (-1)^{k+kL}\sigma_L(a)\sigma_L(a+1)\sigma_L(n)\sigma_L(1) > 0 \). This divides into two cases

- \((-1)^{k+kL} < 0 \)

An allowed set \( \{\sigma_L(k)\} \) satisfying (4.3) and (4.5) is

\[
\{\sigma_L(1), \ldots, \sigma_L(i), \sigma_L(A), \sigma_L(B), \sigma_L(j + 1), \ldots, \sigma_L(n)\} \\
\{+, \ldots, +, \ldots, ? , \ldots, - , \ldots, - \}
\]

with \( \sigma_L(A) \) and \( \sigma_L(B) \) undetermined. (4.4) now requires \( \sigma_L(A)\sigma_L(B) > 0 \) and the constraints in (4.4) now read

\[
\langle iABj + 1 \rangle < 0 \quad \langle iAkk + 1 \rangle\sigma_L(A) > 0 \quad \langle Bj + 1kk + 1 \rangle\sigma_L(B) < 0
\]

Thus the solutions are

\( \mathcal{L}_1 : \sigma_L(A) > 0, \sigma_L(B) > 0 \) with \( \langle iAkk + 1 \rangle > 0, \langle Bj + 1kk + 1 \rangle < 0, \langle iABj + 1 \rangle < 0 \)

\( \mathcal{L}_2 : \sigma_L(A) < 0, \sigma_L(B) < 0 \) with \( \langle iAkk + 1 \rangle < 0, \langle Bj + 1kk + 1 \rangle > 0, \langle iABj + 1 \rangle < 0 \)
\[ (-1)^{k+k_L} > 0 \]

In this case \( \{ \sigma_L(k) \} \) satisfying (4.3) and (4.5) is

\[ \{ \sigma_L(1), \ldots, \sigma_L(i), \sigma_L(A), \sigma_L(B), \sigma_L(j+1), \ldots, \sigma_L(n) \} \]

\[ \{+, \ldots, +, ?, ?, +, \ldots, + \} \]

which again requires \( \sigma_L(A)\sigma_L(B) > 0 \) and turns (4.4) into

\[ \langle iABj + 1 \rangle > 0 \quad \langle iAkk + 1 \rangle \sigma_L(A) > 0 \quad \langle Bj + 1kk + 1 \rangle \sigma_L(B) > 0 \]

which has the following solutions

\( \mathcal{L}_3: \sigma_L(A) > 0, \sigma_L(B) > 0 \) with \( \langle iAkk + 1 \rangle > 0, \langle Bj + 1kk + 1 \rangle > 0, \langle iABj + 1 \rangle > 0 \)

\( \mathcal{L}_4: \sigma_L(A) < 0, \sigma_L(B) < 0 \) with \( \langle iAkk + 1 \rangle < 0, \langle Bj + 1kk + 1 \rangle < 0, \langle iABj + 1 \rangle > 0 \)

Each of these regions is characterized by a particular sign of \( \langle iAkk + 1 \rangle \) and \( \langle Bj + 1kk + 1 \rangle \) along with a pattern of sign flips for the sequence

\[ S^\text{tree}_L : \{ (i - 1)AB) \sigma_L(B), (i - 1)Ajj + 1) \sigma_L(j + 1), \ldots, (i - 1)Ai - 2)(-1)^{k_L - 1} \sigma_L(i - 2) \} \]

Each region allows parametrization of the line \( (AB) \) as \( A = \pm Z_i \pm xZ_{i+1} \) and \( B = \pm yZ_j \pm Z_{j+1} \) with \( x > 0, y > 0 \). In the table below, we list the different possibilities.

| Region | \( A \) | \( B \) | \( S^\text{tree}_L \) |
|--------|--------|--------|----------------|
| \( \mathcal{L}_1 \) | \( \pm Z_i + xZ_{i+1} \) | \( -yZ_j \pm Z_{j+1} \) | \( \{+, \ldots, (-1)^{k_L} \} \) |
| \( \mathcal{L}_2 \) | \( \pm Z_i - xZ_{i+1} \) | \( yZ_j \pm Z_{j+1} \) | \( \{-, \ldots, (-1)^{k_L - 1} \} \) |
| \( \mathcal{L}_3 \) | \( \pm Z_i + xZ_{i+1} \) | \( yZ_j \pm Z_{j+1} \) | \( \{+, \ldots, (-1)^{k_L} \} \) |
| \( \mathcal{L}_4 \) | \( \pm Z_i - xZ_{i+1} \) | \( -yZ_j \pm Z_{j+1} \) | \( \{-, \ldots, (-1)^{k_L - 1} \} \) |

Table 1. Parametrization of \((AB)\) in the four regions

4.1.2 The right amplituhedron \( \mathcal{A}_{R_2,k_L}^{\mathcal{R}} \)

A similar analysis of the effects of (4.1) on the set \( \mathcal{R} \) yields the following constraints on \( \{ \sigma_R \} \).

\[ \sigma_R(a)\sigma_R(a + 1)\sigma_R(b)\sigma_R(b + 1) > 0 \quad (4.6) \]

\[ \sigma_R(A)\sigma_R(i + 1)\sigma_R(k)\sigma_R(k + 1)\langle Ai + 1kk + 1 \rangle > 0 \]

\[ \sigma_R(j)\sigma_R(B)\sigma_R(k)\sigma_R(k + 1)\langle jBkk + 1 \rangle > 0 \]

\[ \sigma_R(B)\sigma_R(A)\sigma_R(k)\sigma_R(k + 1)\langle Bakk + 1 \rangle(-1)^{k_L - 1} > 0 \]

\[ \sigma_R(A)\sigma_R(B)\sigma(i + 1)\sigma_R(j)\langle ABi + 1j \rangle > 0 \]

These conditions are satisfied by

\[ \{ \sigma_R(1), \ldots, \sigma_R(i), \sigma_R(A), \sigma_R(B), \sigma_R(j + 1), \ldots, \sigma_R(n) \} \]

\[ \{+, \ldots, +, ?, ?, +, \ldots, + \} \]

with \( \sigma_R(A) \) and \( \sigma_R(B) \) having solutions depending on \( k_R \).
\( (-1)^{k_R} > 0 \)

\( \mathcal{R}_1 : \sigma_R(A) > 0, \sigma_R(B) > 0 \) with \( \langle Ai + 1k + 1 \rangle > 0, \langle jBkk + 1 \rangle > 0, \langle ABi + 1j \rangle > 0 \)

\( \mathcal{R}_2 : \sigma_R(A) < 0, \sigma_R(B) < 0 \) with \( \langle Ai + 1k + 1 \rangle < 0, \langle jBkk + 1 \rangle < 0, \langle ABi + 1j \rangle > 0 \)

\( \mathcal{R}_3 : \sigma_R(A) > 0, \sigma_R(B) < 0 \) with \( \langle Ai + 1k + 1 \rangle > 0, \langle jBkk + 1 \rangle < 0, \langle ABi + 1j \rangle < 0 \)

\( \mathcal{R}_4 : \sigma_R(A) < 0, \sigma_R(B) > 0 \) with \( \langle Ai + 1k + 1 \rangle < 0, \langle jBkk + 1 \rangle > 0, \langle ABi + 1j \rangle < 0 \)

Once again, each region is characterized by different patterns of sign flips of the sequence

\( S_R^{\text{tree}} : \{ \langle Ai + 1i + 2i + 3 \rangle \sigma_R(i + 3), \ldots, \langle Ai + 1i + 2B \rangle \sigma_R(B) \} \)

where we have ignored an overall factor of \( \sigma_R(A) \sigma_R(i + 1) \sigma_R(i + 2) \). We list the various parametrizations and sign patterns of \( S_R^{\text{tree}} \) below

\[
\begin{array}{|c|c|c|}
\hline
\text{Region} & A & B \\
\hline
\mathcal{R}_1 & Z_i + xZ_{i+1} & \pm yZ_j + Z_{j+1} \\
\mathcal{R}_2 & -Z_i + xZ_{i+1} & \pm yZ_j - Z_{j+1} \\
\mathcal{R}_3 & Z_i + xZ_{i+1} & \pm yZ_j - Z_{j+1} \\
\mathcal{R}_4 & -Z_i + xZ_{i+1} & \pm yZ_j + Z_{j+1} \\
\hline
\end{array}
\]

Table 2. Parametrization of \((AB)\) in the four regions

4.2 Factorization of the external data

We will show that on the unitarity cut, every allowed flip pattern for the sequence \( S_{\text{tree}} \), we can find regions \( L_i, \mathcal{R}_i \) such that \( S_L^{\text{tree}} \) and \( S_R^{\text{tree}} \) have the flip patterns necessary for \( A_{n_1,k_L,L_1} \) and \( A_{n_2,k_R,L_2}^{\mathcal{R}} \). In order to compare \( S_L^{\text{tree}} \) with \( S_{\text{tree}} \), it is useful to introduce another sequence \( S_L^{\text{tree}} \).

\( S_L^{\text{tree}} : \{ (i + 2iAB) \sigma_L(B), (i + 2iAji + 1) \sigma_L(j + 1), \ldots, (i + 2iAi - 2) \sigma_L(i - 2)(-1)^{k_L-1} \} \)

Let \( k, k_L, k'_L, k_R \) be the number of flips in \( S_L^{\text{tree}}, S_L^{\text{tree}}, S_R^{\text{tree}}, S_{\text{tree}} \) respectively. \( k_L \) and \( k'_L \) are related to each other due to the existence of the following Plücker relations which hold in all regions \( (L_i, \mathcal{R}_i) \).

\[
\begin{align*}
\sigma_L(B)\sigma_L(j + 1) & (\langle i - 1iAB \rangle\langle i + 2iAji + 1 \rangle - \langle i - 1iAji + 1 \rangle\langle i + 2iAB \rangle) \\
& = \sigma_L(B)\sigma_L(j + 1)\langle (i - 1iA \cap i + 2iA) Bj + 1 \rangle \\
& = \sigma_L(B)\sigma_L(j + 1)\langle i - 1iA + 2 \rangle\langle iABj + 1 \rangle > 0 \\
& = \langle i - 1iA \cap i + 2iA \rangle kk + 1 \rangle = \langle i - 1iA + 2 \rangle\langle iAk + 1 \rangle + 1 \rangle > 0
\end{align*}
\]
As shown in Appendix [A], we can conclude that the relation between \( k_L \) and \( k'_L \) depends only on the signs of first and last terms which are encoded in the matrix below.

\[
M = \begin{pmatrix}
\text{sign}(\langle i - 1iAB \rangle) \text{ sign}(\langle i - 2i - 1iA \rangle)(-1)^{k_L} \\
\text{sign}(\langle i + 2iAB \rangle) \text{ sign}(\langle i - 2iAi + 2 \rangle)(-1)^{k_L}
\end{pmatrix}
\equiv \begin{pmatrix}
+ \\
\text{sign}(\langle iAaa + 1 \rangle) \text{ sign}(\langle Ai + 1aa + 1 \rangle) \text{ sign}(\langle i - 2ii + 1i + 2 \rangle)
\end{pmatrix}
\]

The relation between \( k_L \) and \( k'_L \) is shown below

| \text{sign}(\langle iAaa + 1 \rangle) | \text{sign}(\langle Ai + 1aa + 1 \rangle) | \text{sign}(\langle i - 2ii + 1i + 2 \rangle) | k_L - k'_L |
|-----------------|-----------------|-----------------|-----------------|
| + | + | + | 0 |
| + | + | - | 1 |
| + | - | + | -1 |
| + | - | - | 0 |
| - | + | + | -1 |
| - | + | - | 0 |
| - | - | + | 0 |
| - | - | - | 1 |

Table 3. Relation between \( k_L \) and \( k'_L \) determined according to Appendix [A]

We will keep track of the flip patterns of \( S^{\text{tree}} \) by labeling them as \( S^{\text{tree}}_{ab} \) where \( a \) and \( b \) are the signs of \( \langle ii + 1i + 2j \rangle \) and \( \langle ii + 1i + 2j + 1 \rangle \) respectively. All the possibilities are listed below.

\[
S^{\text{tree}}: \{\langle ii + 1i + 2i + 3 \rangle, \ldots \langle ii + 1i + 2j \rangle \langle ii + 1i + 2j + 1 \rangle, \ldots, \langle ii + 1i + 2i - 1 \rangle(-1)^{k-1}\}
\]

\[
S^{\text{tree}}_{++}: \{+ \} k_1 \quad + | + \quad k_2 \quad (-1)^{k}
\]

\[
S^{\text{tree}}_{+-}: \{+ \} k_1 \quad + | - \quad k_2 \quad (-1)^{k}
\]

\[
S^{\text{tree}}_{-+}: \{+ \} k_1 \quad - | + \quad k_2 \quad (-1)^{k}
\]

\[
S^{\text{tree}}_{--}: \{+ \} k_1 \quad - | - \quad k_2 \quad (-1)^{k}
\]

For each \( S^{\text{tree}}_{ab} \), we determine \((k_L, k_R)\) for all regions \((\mathcal{L}_i, \mathcal{R}_i)\).
must also ensure that the sequence $L^\nu$ shown below has $k + 1$ sign flips.

$S^{AB}$: \[ \{ \langle ABii + 1 \rangle, \langle ABii + 2 \rangle, \ldots, \langle ABij \rangle \mid \langle ABij + 1 \rangle, \ldots, \langle ABi - 1i \rangle \langle -1 \rangle^k \} \]
\[ = \{ 0, \langle Bii + 1i + 2 \rangle, \ldots, \langle Bii + 1j \rangle \mid \langle Bii + 1j + 1 \rangle, \ldots, \langle Bi + 1i - 1i + 1 \rangle \langle -1 \rangle^k \} \]

The number of flips of this sequence is almost determined by $k_L$ and $k_R$. However, there is the possibility of an additional flip at the boundary in regions $(L_i, R_i)$ such that $\langle Bii + 1j \rangle$ and $\langle Bii + 1j + 1 \rangle$ have opposite signs.

| $S_{++}^{\text{tree}}$ | $\mathcal{R}_1$ | $\mathcal{R}_2$ | $\mathcal{R}_3$ | $\mathcal{R}_4$ |
|-----------------------|--------------|--------------|--------------|--------------|
| $\mathcal{L}_1$      | $(k_2 + 1, k_1)$ | $(k_2 - 1, k_1)$ | $(k_2 + 1, k_1 + 1)$ | $(k_2 - 1, k_1 + 1)$ |
| $\mathcal{L}_2$      | $(k_2 - 1, k_1)$ | $(k_2 + 1, k_1)$ | $(k_2 - 1, k_1 + 1)$ | $(k_2 + 1, k_1 + 1)$ |
| $\mathcal{L}_3$      | $(k_2, k_1)$    | $(k_2, k_1)$    | $(k_2, k_1 + 1)$    | $(k_2, k_1 + 1)$    |
| $\mathcal{L}_4$      | $(k_2, k_1)$    | $(k_2, k_1)$    | $(k_2, k_1 + 1)$    | $(k_2, k_1 + 1)$    |

| $S_{+-}^{\text{tree}}$ | $\mathcal{R}_1$ | $\mathcal{R}_2$ | $\mathcal{R}_3$ | $\mathcal{R}_4$ |
|-----------------------|--------------|--------------|--------------|--------------|
| $\mathcal{L}_1$      | $(k_2 + 1, k_1 + 1)$ | $(k_2 - 1, k_1 + 1)$ | $(k_2 + 1, k_1)$ | $(k_2 - 1, k_1)$ |
| $\mathcal{L}_2$      | $(k_2 - 1, k_1 + 1)$ | $(k_2 + 1, k_1 + 1)$ | $(k_2 - 1, k_1)$ | $(k_2 + 1, k_1)$ |
| $\mathcal{L}_3$      | $(k_2, k_1 + 1)$    | $(k_2, k_1 + 1)$    | $(k_2, k_1)$    | $(k_2, k_1)$    |
| $\mathcal{L}_4$      | $(k_2, k_1 + 1)$    | $(k_2, k_1 + 1)$    | $(k_2, k_1)$    | $(k_2, k_1)$    |

| $S_{-+}^{\text{tree}}$ | $\mathcal{R}_1$ | $\mathcal{R}_2$ | $\mathcal{R}_3$ | $\mathcal{R}_4$ |
|-----------------------|--------------|--------------|--------------|--------------|
| $\mathcal{L}_1$      | $(k_2, k_1 + 1)$    | $(k_2, k_1 + 1)$    | $(k_2, k_1)$    | $(k_2, k_1)$    |
| $\mathcal{L}_2$      | $(k_2, k_1 + 1)$    | $(k_2, k_1 + 1)$    | $(k_2, k_1)$    | $(k_2, k_1)$    |
| $\mathcal{L}_3$      | $(k_2, k_1 + 1)$    | $(k_2, k_1 + 1)$    | $(k_2, k_1)$    | $(k_2, k_1)$    |
| $\mathcal{L}_4$      | $(k_2, k_1 + 1)$    | $(k_2, k_1 + 1)$    | $(k_2, k_1)$    | $(k_2, k_1)$    |

| $S_{--}^{\text{tree}}$ | $\mathcal{R}_1$ | $\mathcal{R}_2$ | $\mathcal{R}_3$ | $\mathcal{R}_4$ |
|-----------------------|--------------|--------------|--------------|--------------|
| $\mathcal{L}_1$      | $(k_2, k_1 + 1)$    | $(k_2, k_1 + 1)$    | $(k_2, k_1)$    | $(k_2, k_1)$    |
| $\mathcal{L}_2$      | $(k_2, k_1 + 1)$    | $(k_2, k_1 + 1)$    | $(k_2, k_1)$    | $(k_2, k_1)$    |
| $\mathcal{L}_3$      | $(k_2, k_1 + 1)$    | $(k_2, k_1 + 1)$    | $(k_2, k_1)$    | $(k_2, k_1)$    |
| $\mathcal{L}_4$      | $(k_2, k_1 + 1)$    | $(k_2, k_1 + 1)$    | $(k_2, k_1)$    | $(k_2, k_1)$    |

We see that each flip pattern can be covered by many charts $(\mathcal{L}_i, \mathcal{R}_i)$. However we must also ensure that the sequence $S^{AB}$ shown below has $k + 1$ sign flips.

Table 4. $(k_L, k_R)$ in all regions for the configuration $S_{++}$

Table 5. $(k_L, k_R)$ in all regions for the configuration $S_{+-}$

Table 6. $(k_L, k_R)$ in all regions for the configuration $S_{-+}$

Table 7. $(k_L, k_R)$ in all regions for the configuration $S_{--}$
Let the number of flips in $S_{AB}$ at loop level, we need to show that each loop (4.3 Factorization of loop level data)

At loop level, we need to show that each loop $(AB)_a$ belongs either to the left or the right amplituhedron. The relevant sequences are (denoting $(⟨AB⟩_a ij)$ as $(ij)$)

$$S_R^{\text{loop}} = \{ (i+1i+2)σ_R(i+2), \ldots, (i+1j)σ_R(j), (i+1B)σ_R(B), (-1)^{k_r-1}(i+1A)σ_R(A) \}$$

$$S_L^{\text{loop}} = \{ (iA)σ_L(A), (iB)σ_L(B), (ij+1)σ_L(j+1), \ldots, (ii-1)σ_L(i-1)(-1)^{k_l-1} \}$$

Similar to before, it will be convenient to introduce the sequence $S_L^{\text{loop}}$

$$S_L^{\text{loop}} = \{ (i+1A)σ_L(A), (i+1B)σ_L(B), (i+j+1)σ_L(j+1), \ldots, (i+i-1)σ_L(i-1)(-1)^{k_l-1} \}$$

Let the number of flips in $S_R^{\text{loop}}$ and $S_L^{\text{loop}}$ be $k_r$ and $k_l$ respectively. These are not $k_R$ and $k_L$, which are the number of flips in $S_R^{\text{tree}}$ and $S_L^{\text{tree}}$ respectively. The flip patterns of the sequence $(i+1k)$ can be organized as follows.

$$S_R^{\text{loop}} : \{ (i+1i+2), (i+1i+3), \ldots, (i+1j)(i+j+1), \ldots, (i+i)(-1)^{k_l-1} \}$$

$$S_L^{\text{loop}} : \{ + k_1 + | + k_2 (-1)^k \}$$

On the cut, the external data factorizes such that $k_L+k_R = k$ with $k_L,k_R \in \{0,\ldots,k\}$. It is trivially true that each loop belongs to the left or the right amplituhedron. We must show that if a loop $(AB)_a$ belongs to the left amplituhedron, then it cannot belong to the right amplituhedron. First, note that in each configuration, we will have $k_l = k_2 + l$ and $k_r = k_1 + r$ with $r,l = 1$ or 2. Now suppose that $(AB)_a$ belongs to both the left and right amplituhedra. Then we must have $k_l = k_1 + 2$ and

| $\mathcal{L}_i$ | $\mathcal{R}_1$ | $\mathcal{R}_2$ | $\mathcal{R}_3$ | $\mathcal{R}_4$ |
|----------------|----------------|----------------|----------------|----------------|
| $\mathcal{L}_1$ | $k_L + k_R$    | $k_L + k_R + 1$| $k_L + k_R + 1$| $k_L + k_R$    |
| $\mathcal{L}_2$ | $k_L + k_R + 1$| $k_L + k_R$    | $k_L + k_R + 1$| $k_L + k_R$    |
| $\mathcal{L}_3$ | $k_L + k_R + 1$| $k_L + k_R$    | $k_L + k_R$    | $k_L + k_R + 1$|
| $\mathcal{L}_4$ | $k_L + k_R$    | $k_L + k_R + 1$| $k_L + k_R + 1$| $k_L + k_R$    |

Table 8. Number of sign flips of $S^{AB}$ for all regions ($\mathcal{L}_i, \mathcal{R}_i$)
We get impossible. Note that this is possible only if $r = 1$ or 2. We just need to show that $l + r = 4$ is impossible. Note that this is possible only if $l = r = 2$. In this case we will must have

$$((i + 1)j)\sigma_R(j), (i + 1B)\sigma_R(B), (-1)^{k_r} \sim (-1)^{k_l}(+, - , +)$$

and

$$((i + 1A)\sigma_L(A), (i + 1B)\sigma_L(B), (i + 1j + 1)\sigma_L(j + 1))$$

$$= (-\sigma_L(A)\sigma_R(A), (-1)^{k_l}\sigma_R(B)\sigma_L(B), (i + 1j + 1)\sigma_L(j + 1))$$

$$= (+, - , +) \text{ or } (-, +, -)$$

In all these cases, we must have $\sigma_R(A)\sigma_R(B)\sigma_L(A)\sigma_L(B)(-1)^{kn} < 0$. It is easy to verify from Section [4.1] that this is always false. Thus each loop belongs either to the left or the right amplituhedron.

### 4.4 Mutual positivity

To complete the proof of factorization, we need to show that the mutual positivity between a loop in $A^L_{\mu_1 k_L L_1}$ and one in $A^R_{\nu_2 k_R L_2}$ is automatically satisfied. This is easier to see while working with $(k + 2)$ dimensional data. A loop in the left amplituhedron can be parametrized as a $k_L + 2$ plane $Y^L_1 \ldots Y^L_{k_l} A \alpha B$.

$$Y^L_\nu = (-1)^{\nu-1} \sigma_L(A) A + \alpha_\nu \sigma_L(i_{\nu}) Z_{i_{\nu}} + \beta_\nu \sigma_L(\nu + 1) Z_{i_{\nu + 1}}$$

$$A_\alpha = (-1)^{k_L + 1} \sigma_L(A) A + \alpha_{k_L + 1} \sigma_L(i_{k_L + 1}) Z_{i_{k_L + 1}} + \beta_{k_L + 1} \sigma_L(i_{k_L + 1} + 1) Z_{i_{k_L + 1} + 1}$$

$$B_\beta = (-1)^{k_L + 2} \sigma_L(A) A + \alpha_{k_L + 2} \sigma_L(i_{k_L + 2}) Z_{i_{k_L + 2}} + \beta_{k_L + 2} \sigma_L(i_{k_L + 2} + 1) Z_{i_{k_L + 2} + 1}$$

with $\nu = \{1, \ldots, k_L\}$, $Z_{i_{\nu}} \in \{Z_1, \ldots, Z_i, A, B, Z_{i+1}, Z_n\}$ and $i_1 < i_2 < \ldots < i_{k_L} + 2$.

Similarly, a loop in the right amplituhedron can be thought of as a $k_R + 2$ plane $Y^R_1 \ldots Y^R_{k_R} A \alpha B$ and parametrized as

$$Y^R_\mu = (-1)^{\mu-1} \sigma_R(A) A + \alpha_\mu \sigma_R(i_{\mu}) Z_{i_{\mu}} + \beta_\mu \sigma_R(\mu + 1) Z_{i_{\mu + 1}}$$

$$A_\alpha = (-1)^{k_R + 1} \sigma_R(A) A + \alpha_{k_R + 1} \sigma_R(i_{k_R + 1}) Z_{i_{k_R + 1}} + \beta_{k_R + 1} \sigma_R(i_{k_R + 1} + 1) Z_{i_{k_R + 1} + 1}$$

$$B_\beta = (-1)^{k_R + 2} \sigma_R(A) A + \alpha_{k_R + 2} \sigma_R(i_{k_R + 2}) Z_{i_{k_R + 2}} + \beta_{k_R + 2} \sigma_R(i_{k_R + 2} + 1) Z_{i_{k_R + 2} + 1}$$

with $\mu = \{1, \ldots, k_R\}$, $Z_{i_{\mu}} \in \{A, Z_{i+1}, \ldots, Z_J, B\}$ and with $j_1 < j_2 < \ldots < j_{k_R + 2}$.

This reduces the mutual positivity condition $(Y^L(AB)_a Y^R(AB)_b) > 0$ to a condition involving $k + 4$ brackets of the form $(i k l m)$. It is easy to see that with positive $k + 4$ dimensional data $(i_1 \ldots i_{k+4})$ when $i_1 < i_2 < \ldots < i_{k+4}$, mutual positivity is guaranteed. The signs $\sigma_L(k)$ and $\sigma_R(k)$ are crucial in making this work.
5 Conclusions

We have shown that Unitarity can be an emergent feature. The positivity of the geometry inevitably leads to amplitudes identical to those derived from a unitary quantum field theory. This lends further support for the conjecture that the amplituhedron computes all the amplitudes of $\mathcal{N} = 4$ SYM. It also suggests that the notion of positivity is more fundamental than those of unitarity and locality which are the cornerstones of the traditional framework of quantum field theory.

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A Restricting flip patterns

Consider a pair of sequences $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$ which have an equal number of terms. Further suppose that they are connected by the Schöoten identity and satisfy a positivity condition, i.e. there exists a relation $a_i b_{i+1} - a_{i+1} b_i = ab > 0$. We will show that the number of sign flips in these sequences, $k_1$ and $k_2$ respectively, are related and that the relation depends only on the signs of $a_1, a_n, b_1$ and $b_n$.

Firstly, we note that the positivity forces each block in the pair of sequences $(a_i a_{i+1}) (b_i b_{i+1})$ to take one of the following forms.

Type 1: $(++), (++), (--), (++)$

Type 2: $(-+), (+-), (++)$, $(-+), (--), (+-)$

Type 3: $(++), (--)$

Type 4: $(+-), (-+), (++)$.

Blocks of type 1 and 2 leave $k_1 - k_2$ fixed. A block of type 3 changes $k_1 - k_2$ by $-1$ and a block of type 4 changes it by $1$. Two consecutive blocks of type 3 or 4 are prohibited and a block of type 4 must follow a block of type 3 before the sign of the bottom sequence can be flipped without flipping the sign of the top. Thus, if we know the signs of $a_1, a_n, b_1$ and $b_n$, we can determine $k_1 - k_2$. We can list the possibilities by the matrices $\begin{pmatrix} s(a_1) & s(a_n) \\ s(b_1) & s(b_n) \end{pmatrix}$ where $s(x)$ is the sign of $x$.

- $k_1 = k_2$
  
  $\begin{pmatrix} ++ & +- \\ ++ & + - \\ -- & - + \\ -- & + - \\ ++ & + + \\ ++ & + - \\ -- & + + \\ -- & - - \\ -- & - + \\ -- & + - \end{pmatrix}$
\[ k_1 = k_2 + 1 \]
\[ (++] (++) (++) (++)(--) (--) \]
\[ k_1 = k_2 - 1 \]
\[ (--) (++) (++) (++) (--) (--) \]

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