RESEARCH ARTICLE

On the minimal distance between two surfaces

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(Received 00 Month 200x; in final form 00 Month 200x)

This article revisits previous results presented in [9] which were challenged in [19]. We aim to use the points of view presented in [19] to modify the original results and highlight that the consideration of the so-called Gao-Strang total complementary function is indeed quite useful for establishing necessary conditions for solving this problem.

Keywords: canonical duality; triality theory; global optimization

AMS Subject Classification: 49K99; 49N15

1. Introduction and Primal Problem

Minimal distance problems between two surfaces arise naturally from many applications, which have been recently studied by both engineers and scientists (see [13, 15]). In this article, the problem presents a quadratic minimization problem with equality constraints: we let \( x := (y, z) \) and

\[
(P) : \min \left\{ \Pi(x) = \frac{1}{2} \| y - z \|^2 : h(y) = 0, \ g(z) = 0 \right\},
\]

where \( h : \mathbb{R}^n \to \mathbb{R} \) and \( g : \mathbb{R}^n \to \mathbb{R} \) are defined by

\[
h(y) := \frac{1}{2} (y^t A y - r^2),
\]

\[
g(z) := \frac{1}{2} \alpha \left( \frac{1}{2} \| z - c \|^2 - \eta \right)^2 - f^t(z - c),
\]

in which, \( A \in \mathbb{R}^{n \times n} \) is a positive definite matrix, \( \alpha, r \) and \( \eta \) are positive numbers, and \( f, c \in \mathbb{R}^n \) are properly chosen so that these two surfaces

\( Y_c := \{ y \in \mathbb{R}^n : h(y) = 0 \} \)

and

\( Z_c := \{ z \in \mathbb{R}^n : g(z) = 0 \} \)

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ISSN: 0233-1934 print/ISSN 1029-4945 online
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DOI: 10.1080/02331939Yxxxxxxxx
http://www.informaworld.com
are disjoint such that if \( z \in Z_c \) then \( h(z) > 0 \). For example, it can be proved that if \( c = 0, r > 0, \eta > 0.5r^2 \) and \( \|f\| < 0.5(0.5r^2 - \eta)^2/r \) then, \( Y_c \cap Z_c = \emptyset \) and if \( z \in Z_c \) then \( h(z) > 0 \). Notice that the feasible set \( X_c = Y_c \times Z_c \subset \mathbb{R}^n \times \mathbb{R}^n \), defined by

\[
X_c = \{ x \in \mathbb{R}^n \times \mathbb{R}^n : h(y) = 0, g(z) = 0 \},
\]

is, in general, non-convex.

By introducing Lagrange multipliers \( \lambda, \mu \in \mathbb{R} \) to relax the two equality constraints in \( X_c \), the classical Lagrangian associated with the constrained problem \((P)\) is

\[
L(x, \lambda, \mu) = \frac{1}{2} \|y - z\|^2 + \lambda h(y) + \mu g(z).
\]

Due to the non-convexity of the constraint \( g \), the problem may have multiple local minima. The identification of the global minimizer has been a fundamentally difficult task in global optimization. The canonical duality theory is a newly developed, potentially useful methodology, which is composed mainly of (i) a canonical dual transformation, (ii) a complementary-dual principle, and (iii) an associated triality theory. The canonical dual transformation can be used to formulate dual problems without duality gap; the complementary-dual principle shows that the canonical dual problem is equivalent to the primal problem in the sense that they have the same set of KKT points; while the triality theory can be used to identify both global and local extrema. In global optimization, the canonical duality theory has been successfully used for solving many non-convex/non-smooth constrained optimization problems, including polynomial minimization [3, 6], concave minimization with inequality constraints [5], nonlinear dynamical systems [17], non-convex quadratic minimization with spherical [4], box [7], and integer constraints [1].

In the next section, we will show how to correctly use the canonical dual transformation to convert the non-convex constrained problem into a canonical dual problem. The global optimality condition is proposed in Section 2. Applications are illustrated in Section 3. The global minimizer is uniquely identified by the triality theory proposed in [2].

2. Canonical dual problem

In order to use the canonical dual transformation method, the key step is to introduce a so-called geometrical operator \( \xi = \Lambda(z) \) and a canonical function \( V(\xi) \) such that the non-convex function

\[
W(z) = \frac{1}{2} \alpha \left( \frac{1}{2} \|z - c\|^2 - \eta \right)^2
\]

in \( g(z) \) can be written in the so-called canonical form \( W(z) = V(\Lambda(z)) \). By the definition introduced in [2], a differentiable function \( V : \mathcal{V}_a \subset \mathbb{R} \to \mathcal{V}_a^* \subset \mathbb{R} \) is called a canonical function if the duality relation \( \varsigma = DV(\xi) : \mathcal{V}_a \to \mathcal{V}_a^* \) is invertible. Thus, for the non-convex function defined by (5), we let

\[
\xi = \Lambda(z) = \frac{1}{2} \|z - c\|^2,
\]
then the quadratic function $V(\xi) := \frac{1}{2}\alpha(\xi - \eta)^2$ is a canonical function on the domain $\mathcal{V}_a = \{\xi \in \mathbb{R} : \xi \geq 0\}$ since the duality relation

$$\varsigma = DV(\xi) = \alpha(\xi - \eta) : V_a \to V_a^* = \{\varsigma \in \mathbb{R} : \varsigma \geq -\alpha\eta\}$$

is invertible. By the Legendre transformation, the conjugate function of $V(\xi)$ can be uniquely defined by

$$V^*(\varsigma) = \{\xi\varsigma - V(\xi) : \varsigma = DV(\xi)\} = \frac{1}{2\alpha}\varsigma^2 + \eta\varsigma. \quad (6)$$

It is easy to prove that the following canonical relations

$$\xi = DV^*(\varsigma) \Leftrightarrow \varsigma = DV(\xi) \Leftrightarrow V(\xi) + V^*(\varsigma) = \xi\varsigma \quad (7)$$

hold in $\mathcal{V}_a \times \mathcal{V}_a^*$. Thus, replacing $W(z)$ in the non-convex function $g(z)$ by $V(\Lambda(z)) = \Lambda(z)\varsigma - V^*(\varsigma)$, the non-convex Lagrangian $L(x, \lambda, \mu)$ can be written in the Gao-Strang total complementary function form

$$\Xi(x, \lambda, \mu, \varsigma) = \frac{1}{2}\|y - z\|^2 + \lambda h(y) + \mu(\Lambda(z)\varsigma - V^*(\varsigma) - f^t(z - c)). \quad (8)$$

Through this total complementary function, the canonical dual function can be defined by

$$\Pi^d(\lambda, \mu, \varsigma) = \{\Xi(x, \lambda, \mu, \varsigma) : \nabla_x \Xi(x, \lambda, \mu, \varsigma) = 0\}. \quad (9)$$

Let the dual feasible space $\mathcal{S}_a$ be defined by

$$\mathcal{S}_a := \{(\lambda, \mu, \varsigma) \in \mathbb{R}^3 : (1 + \mu\varsigma)(I + \lambda A) - I \text{ is invertible}\}, \quad (10)$$

where $I \in \mathbb{R}^{n \times n}$ is the identity matrix. Then the canonical dual function $\Pi^d$ is well defined by (9). In order to have the explicit form of $\Pi^d$, we need to calculate

$$\nabla_x \Xi(x, \lambda, \mu, \varsigma) = \left[\begin{array}{c}
y - z + \lambda Ay \\
z - y + \mu\varsigma(z - c) - \mu f\end{array}\right].$$

Clearly, if $(\lambda, \mu, \varsigma) \in \mathcal{S}_a$ we have that $\nabla_x \Xi(x, \lambda, \mu, \varsigma) = 0$ if and only if

$$x(\lambda, \mu, \varsigma) = \left[\begin{array}{c}
\mu((1 + \mu\varsigma)(I + \lambda A) - I)^{-1}(f + \varsigma c) \\
\mu(I + \lambda A)((1 + \mu\varsigma)(I + \lambda A) - I)^{-1}(f + \varsigma c)\end{array}\right]. \quad (11)$$

Therefore,

$$\Pi^d(\lambda, \mu, \varsigma) = \Xi(x(\lambda, \mu, \varsigma), \lambda, \mu, \varsigma),$$

where $x(\lambda, \mu, \varsigma)$ is given by (11).

The stationary points of the function $\Xi$ play a key role in identifying the global minimizer of $(P)$. Because of this, let us put in evidence what conditions the stationary points of $\Xi$ must satisfy:
\[
\n\nabla_x \Xi(x, \lambda, \mu, \varsigma) = \begin{bmatrix} y - z + \lambda Ay \\ z - y + \mu \varsigma(z - c) - \mu f \end{bmatrix} = 0,
\]

\begin{align*}
\frac{\partial \Xi}{\partial \lambda}(x, \lambda, \mu, \varsigma) &= h(y) = 0, \\
\frac{\partial \Xi}{\partial \mu}(x, \lambda, \mu, \varsigma) &= \Lambda(z) \varsigma - V^*(\varsigma) - f^t(z - c), \\
\frac{\partial \Xi}{\partial \varsigma}(x, \lambda, \mu, \varsigma) &= \mu (\Lambda(z) - DV^*(\varsigma)).
\end{align*}

The following result can be found in [19]. Their proof will be presented for completeness.

**Lemma 2.1:** Consider \((x, \lambda, \mu, \varsigma)\) a stationary point of \(\Xi\) then the following are equivalent:

\(a)\) \(\mu = 0\),
\(b)\) \(\lambda = 0\),
\(c)\) \(x \not\in X_c\).

**Proof:**

\(a) \rightarrow b)\) If \(\mu = 0\), then from (12) we have \(y = z\). This implies that \(\lambda Ay = 0\) but \(y \neq 0\) since \(||y|| = r\) by (13) and \(A\) is invertible, therefore \(\lambda = 0\).

\(b) \rightarrow c)\) If \(\lambda = 0\), then from (12), \(y = z\) and so \((y, z) \not\in X_c\) because \(Y_c \cap Z_c = \emptyset\).

\(c) \rightarrow a)\) Consider the counter-positive form of this statement, namely, if \(\mu \neq 0\) then from (15), \(\Lambda(z) = DV^*(\varsigma)\) which combined together with (7) and (14) provides \(z \in Z_c\). Since \(y \in Y_c\), from (13), it has been proven that \(x \in X_c\).

Now we are ready to re-introduce Theorems 1 and 2 of Gao and Yang ([9]).

**Theorem 2.2:** *(Complementary-dual principle)* If \((\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\varsigma})\) is a stationary point of \(\Xi\) such that \((\bar{\lambda}, \bar{\mu}, \bar{\varsigma}) \in S_a\) then \(\bar{x}\) is a critical point of \((P)\) with \(\bar{\lambda}\) and \(\bar{\mu}\) its Lagrange multipliers, \((\bar{\lambda}, \bar{\mu}, \bar{\varsigma})\) is a stationary point of \(\Pi^d\) and

\[
\Pi(\bar{x}) = L(\bar{x}, \bar{\lambda}, \bar{\mu}) = \Xi(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\varsigma}) = \Pi^d(\bar{\lambda}, \bar{\mu}, \bar{\varsigma}).
\]

**Proof:** From Lemma 2.1, we must have that \(\bar{\lambda}\) and \(\bar{\mu}\) are different than zero, otherwise they both will be zero and \((0, 0, \varsigma) \not\in S_a\) for any \(\varsigma \in \mathbb{R}\) which contradicts the assumption that \((\bar{\lambda}, \bar{\mu}, \bar{\varsigma}) \in S_a\). Furthermore \(\bar{x} \in X_c\), clearly \(\bar{x}\) is a critical point of \((P)\) with \(\bar{\lambda}\) and \(\bar{\mu}\) its Lagrange multipliers and

\[
\Pi(\bar{x}) = L(\bar{x}, \bar{\lambda}, \bar{\mu}) = \Xi(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\varsigma}).
\]

On the other hand, since \((\bar{\lambda}, \bar{\mu}, \bar{\varsigma}) \in S_a\), Equations (11) and (12) are equivalent, therefore it is easily proven that

\[
\frac{\partial \Xi}{\partial t}(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\varsigma}) = \frac{\partial \Pi^d}{\partial t}(\bar{\lambda}, \bar{\mu}, \bar{\varsigma}) = 0,
\]

where \(t\) is either \(\lambda\), \(\mu\) or \(\varsigma\). This implies that \((\bar{\lambda}, \bar{\mu}, \bar{\varsigma})\) is a stationary point of \(\Pi^d\) and

\[
\Xi(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\varsigma}) = \Pi^d(\bar{\lambda}, \bar{\mu}, \bar{\varsigma}).
\]
The proof is complete. □

Following the canonical duality theory, in order to identify the global minimizer of (P), we first need to look at the Hessian of Ξ:

$$\nabla_x^2 \Xi(x, \lambda, \mu, \varsigma) = \begin{bmatrix} I + \lambda A & -I \\ -I & (1 + \mu \varsigma)I \end{bmatrix}. \tag{17}$$

This matrix is positive definite if and only if \(I + \lambda A\) and \((1 + \mu \varsigma)(I + \lambda A) - I\) are positive definite (see Theorem 7.7.6 in [12]). With this, we define \(S_a^+ \subset S_a\) as follows:

$$S_a^+ := \{ (\lambda, \mu, \varsigma) \in S_a : I + \lambda A \succ 0 \text{ and } (1 + \mu \varsigma)(I + \lambda A) - I \succ 0 \}. \tag{18}$$

**Theorem 2.3:** Suppose that \((\bar{\lambda}, \bar{\mu}, \bar{\varsigma}) \in S_a^+\) is a stationary point of \(\Pi^d\). Then \(\bar{x}\) defined by (11) is the only global minimizer of \(\Pi\) on \(X_c\).

**Proof:** Since \((\bar{\lambda}, \bar{\mu}, \bar{\varsigma}) \in S_a^+\), it is clear that \(\bar{x} \in X_c\) and is the only global minimizer of \(\Xi(\cdot, \bar{\lambda}, \bar{\mu}, \bar{\varsigma})\). From (7), notice that \(V\) is a strictly convex function, therefore \(V^*(\varsigma) = \sup \{ \varsigma \xi - V(\xi) : \xi \geq 0 \}\) and

$$\Xi(x, \bar{\lambda}, \bar{\mu}, \bar{\varsigma}) \leq L(x, \bar{\lambda}, \bar{\mu}), \forall x \in \mathbb{R}^{n \times n}, \tag{19}$$

in particular, \(\Xi(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\varsigma}) = L(\bar{x}, \bar{\lambda}, \bar{\mu})\). Suppose now that there exists \(x' \in X_c \setminus \{ \bar{x} \}\) such that

$$\Pi(x') \leq \Pi(\bar{x}),$$

we would have the following:

$$L(x', \bar{\lambda}, \bar{\mu}) = \Pi(x') \leq \Pi(\bar{x}) = L(\bar{x}, \bar{\lambda}, \bar{\mu}),$$

but because of (19) this is equivalent to

$$\Xi(x', \bar{\lambda}, \bar{\mu}, \bar{\varsigma}) \leq L(x', \bar{\lambda}, \bar{\mu}) \leq L(\bar{x}, \bar{\lambda}, \bar{\mu}) = \Xi(\bar{x}, \bar{\lambda}, \bar{\mu}, \bar{\varsigma}).$$

This contradicts the fact that \(\bar{x}\) is the only global minimizer of \(\Xi(\cdot, \bar{\lambda}, \bar{\mu}, \bar{\varsigma})\), therefore, we must have that

$$\Pi(\bar{x}) < \Pi(x), \forall x \in X_c \setminus \{ \bar{x} \}. \tag{20}$$

□

**Remark 1:** Notice that Theorem 2.3 ensures that a stationary point in \(S_a^+\) will give us the only solution of (P). Therefore, the existence and uniqueness of the solution of (P) is necessary in order to find a stationary point of \(\Pi^d\) in \(S_a^+\). From this it should be evident that the examples provided in [19] does not contradict any of the results established under the new conditions of Theorems 2.2 and 2.3.

It is a conjecture proposed in [7] that in nonconvex optimization with box/integer constraints, if the canonical dual problem does not have a critical point in \(S_a^+\), the primal problem could be NP-hard.
3. Numerical Results

The graphs in this section were obtained using WINPLOT [14].

3.1. Distance between a sphere and a non-convex polynomial

Let \( n = 3, \eta = 2, \alpha = 1, f = (2, 1, 1), c = (4, 5, 0), r = 2\sqrt{2} \) and \( A = I \). In this case, the sets \( S_a \) and \( S^+_a \) are given by:

\[
S_a = \{(\lambda, \mu, \varsigma) \in \mathbb{R}^3 : (1 + \mu \varsigma)(1 + \lambda) \neq 1\},
\]

\[
S^+_a = \{(\lambda, \mu, \varsigma) \in \mathbb{R}^3 : 1 + \lambda > 0, (1 + \mu \varsigma)(1 + \lambda) > 1\}.
\]

Using Maxima [16], we can find the following stationary point of \( \Pi^d \) in \( S^+_a \):

\[
(\bar{\lambda}, \bar{\mu}, \bar{\varsigma}) = (0.9502828628898, 1.06207786194864, 0.30646555192966).
\]

Then the global minimizer of \( (P) \) is given by Equation (11):

\[
\bar{y} = \begin{pmatrix}
2.161477484004744 \\
1.696777196962463 \\
0.67004643869564
\end{pmatrix}, \quad \bar{z} = \begin{pmatrix}
4.215492495576614 \\
3.309195489378083 \\
1.306780086728456
\end{pmatrix}.
\]

Figure 1.: Distance between a sphere and a non-convex polynomial
3.2. **Distance between an ellipsoid and a non-convex polynomial**

Let \( n = 3, \eta = 2, \alpha = 1, f = (-2, -2, 1), c = (-4, -5, 0), r = 2\sqrt{2} \) and

\[
A = \begin{bmatrix}
3 & 1 & 1 \\
1 & 4 & 1 \\
1 & 1 & 5
\end{bmatrix}.
\]

Using Maxima [16], we can find the following stationary point of \( \Pi_d \) in \( S^+_a \):

\[
(\bar{\lambda}, \bar{\mu}, \bar{\varsigma}) = (0.84101802234162, 1.493808342458642, 0.12912817444352).
\]

To put in evidence that this stationary point is in fact in \( S^+_a \), notice that the eigenvalues of \( A \) are given by:

\[
\beta_1 = \frac{4}{\sqrt{3}} \cos \left( \frac{4\pi}{3} + \theta \right) + 4 \approx 3.46081127
\]
\[
\beta_2 = \frac{4}{\sqrt{3}} \cos \left( \frac{2\pi}{3} + \theta \right) + 4 \approx 2.324869129
\]
\[
\beta_3 = \frac{4}{\sqrt{3}} \cos \left( \frac{\pi}{3} \right) + 4 \approx 6.214319743,
\]

with \( \theta = \cos^{-1} \left( \frac{3\sqrt{3}}{8} \right) \). Then, the matrices \( I + \bar{\lambda}A \) and \((1 + \bar{\mu} \bar{\varsigma})(I + \bar{\lambda}A) - I \) are similar to

\[
\begin{bmatrix}
3.910604529727413 & 0 & 0 \\
0 & 2.955256837074665 & 0 \\
0 & 0 & 6.226354900456345
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
3.664931769065526 & 0 & 0 \\
0 & 2.525304438283014 & 0 \\
0 & 0 & 6.42737358375643
\end{bmatrix}
\]

respectively. Finally, the global minimizer of \( (P) \) is given by Equation (11):

\[
\tilde{y} = \begin{pmatrix}
-1.121270493506938 \\
-0.830254436735376 \\
0.66262025515374
\end{pmatrix}, \quad \tilde{z} = \begin{pmatrix}
-4.091279940255224 \\
-4.009023330835817 \\
1.807730500535487
\end{pmatrix}.
\]

3.3. **Example given in [19]**

Let \( n = 2, \alpha = \eta = 1, c = (1, 0), f = \left( \frac{\sqrt{6}}{90}, 0 \right), r = 1 \) and \( A = I \). As it was pointed out in [19], there are no stationary points in \( S^+_a \). Under the new conditions of Theorem 2.3, this is expected since the problem has more than one solution (see figure 3). The following was found ([19]) to be one of the global minimizers of \( (P) \):

\[
\tilde{y} = \begin{pmatrix}
0.5872184947 \\
0.8094284647
\end{pmatrix}, \quad \tilde{z} = \begin{pmatrix}
1.012757759 \\
1.395996491
\end{pmatrix}.
\]
Notice that $\mathcal{S}_a$ and $\mathcal{S}_a^+$ are defined as in Equations (20) and (21).

In order to solve this problem, we will introduce a perturbation. Instead of the given $f$, we will consider $f_n = \left(\frac{\sqrt[6]{696}}{96}, \frac{1}{n}\right)$ for $n > 100$.

The following table summarizes the results for different values of $n$.

| $n$   | $\left(\lambda_n, \mu_n, \zeta_n\right) \in \mathcal{S}_a^+$ | $\mathbf{x}_n = (\mathbf{y}_n, \mathbf{z}_n)$ |
|------|--------------------------------------------------|----------------------------------|
| 64   | $(0.2284381, 5.319007, -0.0219068)$               | $\mathbf{y} = \begin{pmatrix} 0.2250312 \\ 0.9743515 \end{pmatrix}$, $\mathbf{z} = \begin{pmatrix} 0.2764370 \\ 1.1969306 \end{pmatrix}$ |
| 1000 | $(0.6926569, 16.01863, -0.0248297)$               | $\mathbf{y} = \begin{pmatrix} 0.5656039 \\ 0.8246770 \end{pmatrix}$, $\mathbf{z} = \begin{pmatrix} 0.9573734 \\ 1.3958953 \end{pmatrix}$ |
| 10000| $(0.7214940, 16.42599, -0.0254434)$               | $\mathbf{y} = \begin{pmatrix} 0.5850814 \\ 0.8109745 \end{pmatrix}$, $\mathbf{z} = \begin{pmatrix} 1.0072142 \\ 1.3960878 \end{pmatrix}$ |
| 100000| $(0.7243521, 16.46345, -0.0255083)$              | $\mathbf{y} = \begin{pmatrix} 0.5870050 \\ 0.8095833 \end{pmatrix}$, $\mathbf{z} = \begin{pmatrix} 1.0122034 \\ 1.3960066 \end{pmatrix}$ |

**Remark 1:** The combination of the linear perturbation method and canonical duality theory for solving nonconvex optimization problems was first proposed in [18] with successful applications in solving some NP-complete problems [20]. High-order perturbation methods for solving integer programming problems were discussed in [8].
4. Concluding remarks and future research

- The total complementary function (Equation (8)) is indeed useful for finding necessary conditions for solving $(\mathcal{P})$ by means of the Canonical Duality theory.
- The examples presented in [19] do not contradict the new conditions and results presented here.
- As stated by Theorem 2.3, in order to use the canonical dual transformation a necessary condition is that $(\mathcal{P})$ has a unique solution. The question if this condition is sufficient remains open.
- The combination of the perturbation and the canonical duality theory is an important method for solving nonconvex optimization problems which have more than one global optimal solution.
- Finding a stationary point of $\Pi^d$ in $\mathcal{S}_n^+$ is not a simple task. It is worth to continue studying this problem in order to develop an efficient algorithm for solving challenging problems in global optimization.

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