Jacobi relations on naturally reductive spaces

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Abstract

Naturally reductive spaces in general can be seen as an adequate generalization of symmetric spaces. Nevertheless there are non-symmetric naturally reductive spaces whose geometric properties come closer to symmetric spaces than others. We consider a distinguished class of non-symmetric naturally reductive spaces intimately related to special geometries in dimension six and seven. For this class we establish the following property: along every geodesic the Jacobi operator satisfies an ordinary differential equation with constant coefficients which can be chosen independent of the given geodesic.

1 Jacobi relations on Riemannian manifolds

Let \((M, g)\) be a Riemannian manifold with Levi-Civita connection \(\nabla\) and curvature tensor \(R\). In the following \(\nabla^k R\) denotes the \(k\)-fold iterated covariant derivative of the curvature tensor. We also denote by \(\text{Sym}^k M\) the bundle of symmetric \(k\)-tensors and let

\[
\text{Sym}^{k,2} M := \text{Sym}^k M \otimes \text{Sym}^2 M. \tag{1}
\]

Then \(\text{Sym}^{k,2} M\) is a tensor bundle over \(M\) whose fibers \(\text{Sym}^{k,2} p M\) should be seen as polynomials on \(T_p M\) via the polarization formula. Furthermore we identify the target space \(\text{Sym}^{k,2} p M\) of these polynomials with the space of symmetric endomorphisms of \(T_p M\) via the metric tensor. The section \(\mathcal{R}^k\) of \(\text{Sym}^{k+2,2} M\) defined by

\[
x \mapsto (u \mapsto \mathcal{R}^k(x)u := \nabla^k_{x,...,x} R(u,x,x)) \tag{2}
\]

is called the symmetrized \(k\)-th covariant derivative of \(R\). For the following definition note that \(\text{Sym}^{*,2} M\) is a graded vector bundle which is pointwise a graded module over \(\text{Sym}^* p M\) with values in \(\text{Sym}^2 p M\). Hence the graded vector space \(\Gamma(\text{Sym}^{*,2} M)\) is a graded module over \(\Gamma(\text{Sym}^* M)\) (where \(\Gamma(\ )\) denotes the space of sections of some vector bundle).

Definition 1. A Jacobi relation of order \(k\) is a \(\Gamma(\text{Sym}^* M)\)-linear relation in \(\Gamma(\text{Sym}^{k+3,2} M)\)

\[
\mathcal{R}^{k+1} = a_1 \mathcal{R}^k + a_2 \mathcal{R}^{k-1} + \cdots + a_{k+1} \mathcal{R}^0 \tag{3}
\]

with coefficients \(a_i \in \Gamma(\text{Sym}^i M)\).

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Remark 1. For every geodesic $\gamma$ let $\mathcal{R}_\gamma$ denote the Jacobi operator and $\mathcal{R}_\gamma^i = \frac{\partial}{\partial t} \mathcal{R}_\gamma$ the $i$-fold iterated covariant derivative. Then (3) holds if and only if

$$\mathcal{R}_\gamma^{k+1} = a_1(\dot{\gamma}) \mathcal{R}_\gamma^k + a_2(\dot{\gamma}) \mathcal{R}_\gamma^{k-1} + \cdots + a_{k+1}(\dot{\gamma}) \mathcal{R}_\gamma$$

for all geodesics $\gamma$, which justifies the name.

Recall that a reductive homogeneous space is a Riemannian manifold $(M, g)$ with an Ambrose-Singer connection $\tilde{\nabla}$ (cf. [6]). The triple $(M, g, \tilde{\nabla})$ is called a naturally reductive (homogeneous) space if the torsion tensor is a 3-form (cf. [44]). Furthermore recall that a symmetric tensor field $a(x)$ is a Killing tensor if $\nabla_x a(x) = 0$ (cf. [8]).

**Theorem 1.** On every naturally reductive space $M^n$ there exists a Jacobi relation (4) of order $k := \frac{n(n-1)}{2} + 1$ whose polynomial coefficients $a_i$ are Killing tensors.

Killing tensors are as well characterized by the property that for every geodesic $\gamma$ the function $a(t) := a(\dot{\gamma}(t))$ is constant along every geodesic. In particular the coefficients $a_i(\dot{\gamma})$ of (4) are constant in the geodesic parameter. We conclude the well known fact that the Jacobi operator $\mathcal{R}_\gamma$ satisfies an ordinary differential equation of order $N := \frac{n(n-1)}{2}$ with constant coefficients along every geodesic $\gamma$ (cf. [19]).

1.1 Linear Jacobi relations

To be more specific we consider now Jacobi relations where the polynomial coefficients are multiples of symmetric powers of the metric tensor. Suppressing the explicit occurrence of the latter via the natural inclusions $\text{Sym}^k M \hookrightarrow \text{Sym}^{k+2} M$ this yields the following concept:

**Definition 2.** A linear Jacobi relation of order $k$ is a linear dependence relation in $\Gamma(\text{Sym}^{k+3,2}M)$

$$\mathcal{R}_\gamma^{k+1} = a_1 \mathcal{R}_\gamma^k + a_2 \mathcal{R}_\gamma^{k-3} + \cdots + \begin{cases} \frac{a_k \mathcal{R}_\gamma^1}{2} & \text{if } k \text{ is even} \\ \frac{a_{k+1} \mathcal{R}_\gamma^0}{2} & \text{if } k \text{ is odd} \end{cases}$$

with real numbers $a_i$. Equivalently we have

$$\mathcal{R}_\gamma^{k+1} = a_1 \|\dot{\gamma}\|^2 \mathcal{R}_\gamma^{k-1} + a_2 \|\dot{\gamma}\|^4 \mathcal{R}_\gamma^{k-3} + \cdots$$

along every geodesic $\gamma$.

Of course every symmetric space has a linear Jacobi relation of order zero. In contrast, experiments with computational programs like Maple or Mathematica show that on non-symmetric naturally reductive spaces linear Jacobi relations occur only very scarcely.

1.2 Outline of the article

We establish the existence of linear Jacobi relations for all naturally reductive spaces satisfying the following condition:

The torsion 3-form $\tau$ of the Ambrose-Singer connection is a generalized vector cross product in the sense of [8].
In three dimensions this condition is not an obstruction. In dimensions six and seven we will see that it yields precisely the well-known classes of (six-dimensional) nearly Kähler 3-symmetric spaces and normal homogeneous nearly parallel $G_2$-spaces. Then we show that in each class there is one and the same linear Jacobi relation:

**Theorem 2.** Let $(M, g, \nabla)$ be a naturally reductive space whose torsion 3-form is a generalized vector cross product. There are the following linear Jacobi relations:

(a) For the three-dimensional naturally reductive space with the Berger metric $g_{\kappa, \tau}$ we have

$$\mathcal{R}^3 = -\tau^2 \mathcal{R}^1.$$  

(7)

(b) For a six-dimensional nearly Kähler 3-symmetric space with scalar curvature $s = 30$ we have

$$\mathcal{R}^5 = -\frac{5}{4} \mathcal{R}^3 - \frac{1}{4} \mathcal{R}^1.$$  

(8)

(c) For a naturally reductive nearly-parallel $G_2$-manifold with $s = \frac{21}{8}$ we have

$$\mathcal{R}^3 = -\frac{1}{36} \mathcal{R}^1.$$  

(9)

Moreover (7)-(9) are minimal unless $M$ has constant sectional curvature.

The proof of Theorem 1 and 2 is given in Section 5. In Section 5.1 we compare our result with the before known examples. In Sections 6 and 7 we prove that (7)-(9) are minimal relations. In Section 8 we remark that $\mathcal{R}^k$ is a generalized twistor in the sense of [12] for the spaces mentioned in (b) and (c) of Theorem 2.

Due to lack of further examples we conjecture that for naturally reductive spaces there are no linear Jacobi relations other than those described by the previous theorem. However, in [23] it was shown that the same Jacobi relation (8) also exists on the generalized Heisenberg group $N^6$ known as Kaplan’s example of a g.o.-space which is not naturally reductive (cf. [29]). Moreover the results of [5] suggest that linear Jacobi relations might not exist on strict g.o.-spaces other than Kaplan’s example.

## 2 Generalized vector cross products

Let $(V, \langle \ , \rangle)$ be a Euclidean vector space and $\tau : V \times V \times V \to \mathbb{R}$ be an alternating 3-tensor. For each $x \in V$ we denote by $\tau_x$ the skew-symmetric endomorphism defined by $\langle \tau_x y, z \rangle := \tau(x, y, z)$. Recall that $\tau$ is a vector cross product in the sense of [24] if $|\tau_x(y)|^2 = \|x \wedge y\|^2$ for all $x, y \in V$. Besides the well-known vector cross product $e_{12} + e_{23} + e_{31}$ of $\mathbb{R}^3$ there exists also the following in dimension seven

$$\sigma = e_{127} + e_{347} + e_{567} + e_{135} - e_{146} - e_{236} - e_{245},$$  

(10)

whose stabiliser defines the exceptional holonomy group $G_2$ as a subgroup of SO(7) (cf. [8, 19]). If $\sigma$ is given by (10), then the basis $e_1, \ldots, e_7$ is a Cayley basis up to permutation (cf. [11, Ch. 4]).

For the following definition note that $\tau_x$ is a Hermitian structure on $x^\perp$ for every unit vector $x \in V$ and hence $\tau_{x_1}$ and $\tau_{x_2}$ are conjugate in $O(V)$ for all $x_1, x_2 \in V$ with $\|x_i\| = 1$. 


Definition 3. [8] Definition 2.3] A generalized vector cross product on $V$ is an element $\tau \in \Lambda^3 V$ such that $\tau_{x_1}$ and $\tau_{x_2}$ are conjugate in $O(V)$ for all $x_1, x_2 \in V$ with $\|x_i\| = 1$. The conjugacy class of the endomorphism $\tau_x$ with $x \in S^1(V)$ is called the endomorphism associated with $\tau$.

A different characterization of a generalized vector cross product is the following: Let $\tau$ be a 3-form and $x_0 \in V$ with $\|x_0\| = 1$. Let $\lambda_1 > \lambda_2 > \cdots \geq 0$ denote the distinct (non-negative) eigenvalues of the square $-\tau^2_{x_0}$ on $T_pM$. In general these define smooth functions $\lambda_i(x)$ in a small neighbourhood of $x_0$. Then $\tau$ is a generalized vector cross product if and only if the $\lambda_i$ are constant functions on $S^1(V)$.

An example of a generalized vector cross product which is not a classical vector cross product is the following: Let $(V, g, J)$ be a six-dimensional Hermitian space and $e_1, e_2 := Je_1, e_3, e_4 := Je_3, e_5, e_6 := Je_5$ be an adapted orthonormal basis and $d z_i := e_{2i-1} - iJ e_{2i}$ for $i = 1, 2, 3$ the standard $(1, 0)$ forms. Then $d z_1 \wedge d z_2 \wedge d z_3 = \psi_+ + i \psi_-$ is a complex volume form. The real part

$$\psi_+ = e_{135} - e_{146} - e_{236} - e_{245}$$

(11)
is a generalized vector cross product. For $\|x\| = 1$ the complex eigenvalues of $x_0 \psi_+$ are 0 and $\pm i$ and the kernel is the two-dimensional space $\mathbb{C} x := \mathbb{R} x \oplus \mathbb{R} J x$, whereas the kernel of a classical vector cross product is $\mathbb{R} x$. The stabilizer of $\psi_+$ defines the special unitary group as a subgroup of the orthogonal group of $(V, g)$. Then

$$\sigma = \psi_+ + \omega \wedge e_7$$

(12)
where $\sigma$ and $\psi_+$ are given by (10) and (11) with $\omega = e_{12} + e_{34} + e_{56}$ beeing the standard symplectic form of $\mathbb{R}^6$.

Theorem 3. [8] Theorem 2.6] Let $V$ be a Euclidean vector space of dimension $n$.

(a) If $n = 2m + 1$ is odd, then every generalized vector cross product on $V$ is, up to constant rescaling, a standard vector cross product. Hence $n = 3$ or $n = 7$.

(b) If $n = 2m + 2$ and there exists a generalized vector cross product $\tau$ on $V$, then $n = 6$ and there exists an orthonormal basis of $V$ such that, up to constant rescaling, $\tau$ is given by (11).

(c) In dimension $n = 4m$ there exists no generalized vector cross product.

3 Naturally reductive SU(3)-structures

Following [5] an SU(3)-structure on a six-dimensional Riemannian manifold $(M, g)$ is a reduction of the principal bundle of orthonormal frames to the subgroup $SU(3) \subset O(6)$. Equivalently, an SU(3)-structure is characterized by a triple $(J, \psi_+, \psi_-)$ where $J$ is a complex structure compatible with the metric tensor and $\psi_+ + i \psi_-$ is a complex volume form whose real part $\psi_+$ has constant length. The following lemma shows the role played by the generalized vector cross product given by (11).

Lemma 1. Let $(M, g)$ be a six-dimensional Riemannian manifold. For a 3-form $\psi$ the following are equivalent:

(a) for every point $p$ there exists an orthonormal basis $e_1, \ldots, e_6$ of $T_pM$ such that $\psi$ is given by (11);
(b) the length of $\psi$ is two and there exists an almost complex structure $J$ compatible with the given metric such that $\psi$ is a $(3,0) + (0,3)$-form.

In this case there exists a unique $\text{SU}(3)$-structure $(J,\psi_+,\psi_-)$ with $\psi = \psi_+$. Then (cf. [31, Ch. 2])

$$\begin{align*}
\psi_- &= *\psi_+ = e_{136} + e_{145} + e_{235} - e_{246} \\
\psi_+ &= \psi_- \circ J \times \text{Id} \times \text{Id} \tag{13}
\end{align*}$$

(14)

The most important example of an $\text{SU}(3)$ structure comes from six-dimensional nearly Kähler manifolds:

**Definition 4.** [13, 20, 27, 31] An almost Hermitian manifold $(M^{2n},g,J)$ with Levi-Civita connection $\nabla$ is called nearly Kähler if $\nabla J$ is skew, i.e.

$$\nabla_x Jy = -\nabla_y Jx \tag{15}$$

for all $p \in M$ and $x,y \in T_pM$. A nearly Kähler manifold is called strict if $\nabla J \neq 0$.

For every six-dimensional strict nearly Kähler manifold the three-form $\psi := J \circ \nabla J$ is a $(3,0) + (0,3)$ form of constant length (cf. [10]). In particular the constant type equation for nearly Kähler manifolds holds (cf. [27, Theorem 5.2]):

$$\forall x,y \in T_pM : \langle \nabla_x Jy, \nabla_x Jy \rangle = c^2(\langle x,x \rangle \langle y,y \rangle - \langle x,Jy \rangle^2 - \langle x,y \rangle^2) \tag{16}$$

Furthermore $M$ is a simply connected Einstein space with scalar curvature $s = 30c^2$. Therefore $(J,\psi_+ := J \circ \nabla J, \psi_- := *\psi_-)$ defines an $\text{SU}(3)$-structure in the standard normalization $s = 30$.

For further proceeding recall that for every almost Hermitian manifold $(M,g,J)$ the minimal Hermitian connection is given by

$$\nabla_x y := \nabla xy + \frac{1}{2}\tau_x \tag{17}$$

where $\tau(x,y,z) := \langle J\nabla_x Jy,z \rangle$ is the intrinsic torsion (cf. [14]) . Then one knows the following:

- The Ambrose-Singer connection of a Riemannian 3-symmetric space $(M,g,J)$ coincides with the minimal Hermitian connection. (cf. [13, Proposition 1.7]).

- A Riemannian 3-symmetric space $(M,g,J)$ is naturally reductive if and only if it is nearly Kähler (cf. [24, Proposition 5.6]).

Hence a nearly Kähler 3-symmetric space is the same as a naturally reductive 3-symmetric space. In dimension six there are the following examples:

- the round sphere $S^6 = G_2/\text{SU}(3)$ realized as the purely imaginary octonions of unit length and with the nearly Kähler structure coming from the octonionic multiplication,

- the complex flag manifold $\text{SU}(3)/\text{U}(1) \times \text{U}(1)$ seen as the twistor space of $\mathbb{C}P^2$,

- the generalized flag manifold $\text{SO}(5)/\text{U}(2)$ seen as the twistor space of $S^4$,

- the 3-symmetric metric on $S^3 \times S^3 = \text{SU}(2) \times \text{SU}(2) \times \text{SU}(2)/\text{SU}(2)$ constructed by Ledger-Obata.

We recall that in [13] it is shown that there are no other six-dimensional homogeneous nearly Kähler manifolds.
Proposition 1. Let $M$ be a six-dimensional naturally reductive space with Ambrose-Singer connection $\nabla$ and torsion 3-form $\tau$. The following are equivalent:

(a) $M$ is a nearly Kähler 3-symmetric space $(M,g,J)$;
(b) a multiple of $\tau$ defines an SU(3)-structure.

In the standard normalization $s = 30$ the correspondence is moreover such that $\tau = J \circ \nabla J$.

Proof. For (a) $\Rightarrow$ (b): Since $M$ is 3-symmetric, the Ambrose-Singer connection coincides with the minimal Hermitian connection $\langle J \rangle$. Thus we have $\tau = J \circ \nabla J$. Because $M$ is nearly Kähler, the latter is a $(3,0) + (0,3)$-form of constant length. Therefore a multiple of $\tau$ defines an SU(3)-structure.

For (b) $\Rightarrow$ (a): Suppose there exists a non-zero constant $c$ and pointwise an orthonormal basis of $e_1, \cdots, e_6$ of $T_pM$ such that $c\tau$ is given by $\langle \tau \rangle$. By scaling the metric with a constant factor we can assume that $c = 1$.

Hence $M$ carries a unique SU(3)-structure $(J,\psi_+)$ such that $\tau = \psi_+$ according to Lemma 1. Moreover $\nabla \tau = 0$ implies that $\nabla$ is an SU(3)-connection. In particular, since $\nabla J = 0$, we have $\nabla_x J = -\frac{1}{2} \tau_x \cdot \omega$ where $A \cdot \omega$ denotes the usual left action of a skew-symmetric endomorphism $A$ on a 2-form $\omega$. Furthermore we claim that $\tau_x \cdot \omega$ is given by $-2 x \cdot \tau$. Since the equation in question is SU(3)-invariant and SU(3) acts transitively on the unit sphere, it suffices to consider $x := e_1$. From

$$x_j \tau = e_{1j} \psi_+ e_{35} - e_{46}$$

$$x_j \tau = e_{1j} \psi_+ e_{36} + e_{45}.$$

we obtain by means of the formula $A \cdot \omega = \sum_{i=1}^6 A e_i \wedge e_i \omega$

$$x_j \tau \cdot \omega = \sum_{i=1}^6 e_{i} \omega e_{1j} \psi_+ \wedge e_i \omega = -2(e_{36} + e_{45}).$$

Therefore $\nabla_x J y, z \rangle = -\frac{1}{2} (\tau_x \cdot J y, z \rangle = \ast \tau(x, y, z)$. We conclude that $\nabla \omega = \ast \tau$. Hence $(M,g,J)$ is nearly Kähler. Also we have $\ast \tau(x, J y, z) \equiv \tau(x, y, z)$. It follows that $x_j \tau = \nabla_x J \circ J = -J \circ \nabla_x J$. Thus the Ambrose-Singer connection is equal to the canonical Hermitian connection $\nabla - \frac{1}{2} J \circ \nabla J$. Therefore $M$ is a nearly Kähler 3-symmetric space with $\tau = J \circ \nabla J$ (cf. [13] Ch. 1]).

4 Naturally reductive $G_2$-manifolds

Following [2] [19] a $G_2$-structure on a Riemannian manifold $(M^7, g)$ is a reduction of the O(7)-principal bundle of orthonormal frames to the subgroup $G_2 \subset O(7)$. Equivalently there exists a 3-form $\sigma$ which has the specific form $\langle \sigma \rangle$ with respect to a suitable orthonormal basis $e_1, \cdots, e_7$ of $T_pM$ at each point $p$. There are several equivalent ways to define a nearly parallel $G_2$-structure (cf. [2] Proposition 2.3]). We choose the following one:

Definition 5. A $G_2$-structure $\sigma$ is called nearly parallel if there exists a constant $\tau_0 \neq 0$ such that

$$\nabla_x \sigma = \frac{\tau_0}{4} x_j \ast \sigma$$

holds for all $x \in T_pM$. 

6
A nearly parallel $G_2$-manifold can be seen as the seven-dimensional analogue of a six-dimensional strict nearly Kähler manifold. It is known that every nearly parallel $G_2$-manifold is an irreducible Einstein manifold with scalar curvature $s = \frac{21}{8} \tau_0^2$. Moreover, the existence of such a structure is equivalent to the existence of a spin structure with a Killing spinor (cf. [2, 19]). In this case the intrinsic torsion of the $G_2$-connection is given by $-\frac{1}{6} \sigma$. The minimal connection $\bar{\nabla}_x := \nabla_x - \frac{1}{12} x \sigma$ satisfies $\bar{\nabla} \sigma = 0$.

In order to define the nearly parallel analogue of a six-dimensional nearly Kähler 3-symmetric space the following definition turns out to be suitable.

**Definition 6.** A nearly parallel $G_2$-space $(M,g,\sigma)$ is called naturally reductive if there exists an Ambrose-Singer connection $\bar{\nabla}$ with skew-symmetric torsion such that $\bar{\nabla} \sigma = 0$.

**Remark 2.** The above definition demands not only that the given metric is nearly parallel and naturally reductive at the same time but also that these two structures are compatible with each other. Furthermore we have $\bar{\nabla} \sigma = 0$ if and only if the subgroup $G$ of the isometry group $I(M)$ generated by the transvections of $\bar{\nabla}$ is contained in the automorphism group of $\sigma$. Hence statements in this article on whether a given nearly parallel $G_2$-manifold is naturally reductive or not will always be subjected to the restriction $G \subseteq \text{Aut}(\sigma)$.

**Theorem 4.** ([17, Corollary 4.3]) For a fixed $G_2$-structure $\sigma$ there exists at most one affine connection $\bar{\nabla}$ with skew-symmetric torsion such that $\bar{\nabla} \sigma = 0$.

**Corollary 1.** Let $(M,g,\bar{\nabla})$ be a seven-dimensional naturally reductive space with Ambrose-Singer connection $\bar{\nabla}$ and torsion tensor $\tau$. The following are equivalent:

(a) There exists a nearly parallel structure $\sigma$ such that $\bar{\nabla} \sigma = 0$;

(b) A multiple of $\tau$ defines a $G_2$-structure.

In the standard normalization $s = \frac{21}{8}$ the correspondence is such that $\tau = -\frac{1}{12} \sigma$. In particular the nearly parallel structure is uniquely determined by the naturally reductive structure.

**Proof.** Every $G$-invariant tensor on a naturally reductive space is parallel with respect to the Ambrose Singer connection. Hence $\bar{\nabla} \sigma = 0$. Thus $\bar{\nabla}$ is the minimal connection of the nearly-parallel $G$-structure $\sigma$ according to Theorem 4. Therefore $\bar{\nabla} = \nabla - \frac{\tau}{12} = \nabla + \frac{s}{2}$. The result follows. □

The following proposition is the analogue of Proposition 1.

**Proposition 2.** Let $(M,g,\bar{\nabla})$ be a seven-dimensional naturally reductive space with Ambrose-Singer connection $\bar{\nabla}$ and torsion tensor $\tau$. The following are equivalent:

(a) There exists a nearly parallel structure $\sigma$ such that $\bar{\nabla} \sigma = 0$;

(b) A multiple of $\tau$ defines a $G_2$-structure.

In the standard normalization $s = \frac{21}{8}$ the correspondence is such that $\tau = -\frac{1}{12} \sigma$.

**Proof.** (a) $\Rightarrow$ (b) follows from Corollary 1 (with $c = -\frac{\tau}{12}$). For the other direction cf. [2, Lemma 7.1]. □

In the following we call a naturally reductive $G_2$-manifold normal if the naturally reductive structure is normal. It remains to show that every naturally reductive $G_2$-manifold is normal by comparing the list of homogeneous nearly parallel $G_2$-manifolds from [19, Ch. 4] with the classification of seven-dimensional naturally reductive structures given in [16].
Lemma 2. ([44],[22],[46]) Let $(M,g,\bar{\nabla})$ be a compact simply connected seven-dimensional naturally reductive space with Ambrose-Singer connection $\bar{\nabla}$. Suppose that the Lie algebra $\mathfrak{g}$ of the connected subgroup $G \subset \mathfrak{I}_0(M)$ generated by the transvections is not semisimple. Then there exists a six-dimensional naturally reductive space $(\tilde{M},\tilde{g},\tilde{\nabla})$ whose transvection algebra $\tilde{\mathfrak{g}}$ is semisimple together with the following:

(a) a principal $\mathbb{S}^1$-fiber bundle $\pi : M \to \tilde{M}$ such that the bundle map respects the naturally reductive structures,

(b) an orthogonal splitting $T_p M = T_{\pi(p)} \tilde{M} \oplus \mathbb{R} e_7$ where $e_7$ is a unit vector of the tangent space of the fiber of $\pi$,

(c) some element $Y$ of the center of $\tilde{\mathfrak{h}}$ such that the torsion 3-forms $\tau$ and $\tilde{\tau}$ of $M$ and $\tilde{M}$ are related by

$$\tau = \tilde{\tau} + \phi(Y) \wedge e_7$$

where $\phi : \tilde{\mathfrak{h}} \to \mathfrak{so}(T_{\pi(p)} \tilde{M})$ denotes the linearized isotropy representation.

Proof. Part (a) follows from the classification given in [46]. Part (b) is obvious. For (19) cf. [44, (3.2)] and the discussion in [44, Ch. 4.1].

Corollary 2. The transvection algebra $\mathfrak{g}$ of a simply connected naturally reductive nearly parallel space $(M,g,\sigma)$ is semisimple.

Proof. Let $\bar{\nabla}$ be an Ambrose-Singer connection with skew-symmetric torsion $\tau$ such that $\bar{\nabla}\sigma = 0$. By means of Proposition 2 we can assume that $\tau = \sigma$. If we assume for a contradiction that $\mathfrak{g}$ is not semisimple, then there exists a six-dimensional naturally reductive space $(\tilde{M},\tilde{g},\tilde{\nabla})$ satisfying (a) - (c) from Lemma 2. By comparing (10) with (19), we see that

$$\tilde{\tau} = e_{135} - e_{146} - e_{236} - e_{245},$$

$$\phi(Y) = e_{12} + e_{34} + e_{56}$$

Hence $\psi_+ := \tilde{\tau}$, $\psi_- := *\tilde{\tau}$ and $J := \phi(Y)$ defines an SU(3)-structure $(\psi_+,\psi_-,J)$ on $\tilde{M}$. Thus $\tilde{M}$ is nearly Kähler 3-symmetric according to Proposition 1. Moreover the nearly Kähler structure $J$ lies in the image of the linearized isotropy representation $\phi : \tilde{\mathfrak{h}} \to \mathfrak{so}(T_{\pi(p)} \tilde{M})$. Since the torsion tensor of a naturally reductive space is invariant under the isotropy action, this implies $J \cdot \tilde{\tau} = 0$. On the other hand the torsion tensor of a nearly Kähler 3-symmetric space is a $(3,0) + (0,3)$ form, hence $J \cdot \tilde{\tau}(x,y,z) = -3\tilde{\tau}(Jx,y,z)$. Therefore $J \cdot \tilde{\tau}$ does not vanish.

Theorem 5. Every simply connected naturally reductive nearly parallel $G_2$-space is normal. Therefore it is a standard normal space from the following list

- The round sphere $S^7 = \text{Spin}(7)/G_2$,
- Berger’s manifold $V_1 = \text{SO}(5)/\text{SO}(3)$,
- Wilking’s manifold $V_3 = (\text{SO}(3) \times \text{SU}(3))/\text{U}^\star(2)$,
- the squashed sphere $\text{Sp}(2) \times \text{Sp}(1)/\text{Sp}(1) \times \text{Sp}(1)$.
Proof. We compare the list of seven-dimensional naturally reductive spaces from [16] with the list homogeneous nearly parallel $G_2$-manifolds from [19, Ch. 4]. For this we can restrict us to the naturally reductive structures of semisimple type from [16, Ch. 3.1]. Then we observe that all possible naturally reductive metrics are normal. Furthermore a simply connected normal nearly parallel $G_2$-manifold is a standard normal space from the above list (cf. [2, Ch. 8]). Now the result follows. \hfill \square

We thus see that the analogue of the main result from [13] is not true for nearly parallel $G_2$-manifolds: there exist homogeneous nearly parallel $G_2$-manifolds which are not naturally reductive.

5 Jacobi relations on naturally reductive spaces

Before we come to the proof of Theorem [11] we will first show how to reduce the occurrence of Jacobi relations on naturally reductive spaces to a question of linear algebra. Let $(M, g, \nabla)$ be a naturally reductive space. By the first Bianchi identity $\mathcal{R}^k(x)$ actually belongs to $\text{Sym}^2(x^\perp)$ for each $x \in T_pM$ (where $\perp$ denotes the orthogonal complement). In other word $\mathcal{R}^k$ is a section of the smaller subbundle $\mathcal{R}^k_M \subset \text{Sym}^{k+2,2}M$ defined by

$$\mathcal{R}^k_M := \{ R \in \text{Sym}^{k+2,2}M \mid \forall x \in T_pM : x.R(x) = 0 \}.$$ 

Next we consider for each $x \in T_pM$ the linear operator

$$T_x : \text{Sym}^2(x^\perp) \rightarrow \text{Sym}^2(x^\perp), T_x(\beta(u,v)) := \frac{1}{2} \tau_x \cdot \beta(u,v) = \frac{1}{2} (\beta(\tau_x u,v) + \beta(u,\tau_x v)).$$

By definition $-T_x$ is the usual action of the skew-symmetric endomorphism $\tau_x$ acting on $\text{Sym}^2(x^\perp)$ via the canonical infinitesimal left action of $\mathfrak{so}(T_pM)$.

The endomorphism valued 1-form $x \mapsto T_x$ induces the partial algebraic derivation

$$\mathcal{T} : \mathcal{R}^k_M \rightarrow \mathcal{R}^{k+1}_M,$$

$$(\mathcal{T}R)(x) := T_x(R(x)).$$

Lemma 3. Let $(M, g)$ be a naturally reductive space. Then the symmetrized $k$-th covariant derivative of the curvature tensor is given by

$$\mathcal{R}^k = \mathcal{T}^k \mathcal{R}.$$  

Proof. Let $\nabla$ be an Ambrose-Singer connection and $G$ denote the group generated by the transvections of $\nabla$. The subspace of $G$-invariant sections of $\mathcal{R}^k M$ will be denoted by $[\Gamma(\mathcal{R}^k M)]_G$. We have $\mathcal{R} \in [\Gamma(\mathcal{R}^2 M)]_G$. By induction it suffices to show that $T\mathcal{R} \in [\Gamma(\mathcal{R}^{k+1} M)]_G$ and $\nabla_x \mathcal{R} = T_x(\mathcal{R}(x))$ for each $R \in [\Gamma(\mathcal{R}^k M)]_G$.

From the very definition of an Ambrose-Singer connection it follows that every $G$-invariant section of a tensor bundle over $M$ is $\nabla$-parallel. Therefore $\nabla_x$ acts on $[\Gamma(\mathcal{R}^k M)]_G$ by the total algebraic derivation $-(\tau_x \cdot \mathcal{R})(x) := T_x(\mathcal{R}(x)) + k \mathcal{R}(\tau_x x)$. Since $\tau$ is a 3-form, we have $\tau_x x = 0$ and thus the second summand vanishes. We conclude that $(\nabla_x \mathcal{R})(x) = T_x(\mathcal{R}(x))$ for each $R \in [\Gamma(\mathcal{R}^k M)]_G$. Since further $\nabla$ and $\tilde{\nabla}$ both are $G$-invariant, so is their difference $\tilde{\nabla} = \nabla - \nabla$. Hence also $T$ is $G$-invariant. We conclude that

$$\mathcal{T}([\Gamma(\mathcal{R}^k M)]_G) \subset [\Gamma(\mathcal{R}^{k+1} M)]_G.$$ 

By induction (24) now follows. \hfill \square
Proposition 3. Let \((M,g,\nabla)\) be a naturally reductive space with transvection group \(G\). Then every \(G\)-invariant symmetric tensor is a Killing tensor.

Proof. For an Ambrose-Singer connection every \(G\)-invariant tensor is \(\nabla\)-parallel. Because \(\tau_x x = 0\), we also have
\[
\tau_x \cdot a(x) = -ka(\tau_x x, x, \cdots, x) = 0
\]
for every symmetric \(k\)-tensor \(a\). Therefore
\[
\nabla_x a(x) = \nabla_x a(x) - \tau_x \cdot a(x) = 0.
\]

Corollary 3. On every naturally reductive space \(M^n\) with Ambrose-Singer connection \(\nabla\) and corresponding torsion 3-form \(\tau\) the coefficients \(a_i = a_i(x)\) of the characteristic polynomial \(p_\tau(\lambda) = \lambda^N - \sum_{i=1}^N a_i(x)\lambda^{N-i}\) of \(T_x : \text{Sym}^2(x^+) \to \text{Sym}^2(x^+)\) are Killing tensors of degree \(i\).

Proof. The torsion 3-form \(\tau\) is \(G\)-invariant \(\tau_g x = g\tau_x g^{-1}\). Because the characteristic polynomials of conjugate endomorphisms are identical, we see that the coefficients of the characteristic polynomial \(q = q_x\) of \(\tau\) are \(G\)-invariant polynomials in \(x\) and hence Killing tensors according to Proposition 3. Thus the \(a_i\) are Killing tensor, too, since these are polynomial in the coefficients of \(q\).

Proof of Theorems 1 and 2. Let \((M,g,\nabla)\) be a naturally reductive space with torsion 3-form \(\tau\). For each \(x \in T_x M\) let \(T_x\) denote the action on \(\text{Sym}^2(x^+)\) induced by \(-\frac{1}{2}\tau_x\), see (22). For Theorem 1 we consider the characteristic polynomial \(p_{\text{char}}(\lambda) = \lambda^N - \sum_{i=1}^N a_i(x)\lambda^{N-i}\) of \(T_x\). Here the coefficients \(a_i = a_i(x)\) are symmetric Killing tensors according to Corollary 3. Lemma 3 implies that \(p_k(R_x) = R^k(x)\) for the monomial \(p_k(\lambda) := \lambda^k\). Hence the theorem of Cayley-Hamilton \(p_{\text{char}}(R_x) = 0\) yields the Jacobi relation
\[
R^N(x) = \sum_{i=1}^N a_i(x) R^{N-i}(x).
\]
This finishes the proof of Theorem 1. For Theorem 2 we assume that \(\tau\) is a generalized vector cross product and consider the minimal polynomial \(p_{\text{min}}(\lambda)\) of \(T_x\). Since the coefficients of \(p_{\text{min}}\) do not depend on \(x \in S^1(T_x M)\), the theorem of Cayley-Hamilton \(p_{\text{min}}(R_x) = 0\) yields a linear Jacobi relation. In fact by Theorem 1 it suffices to study two cases anyway. If the torsion 3-form \(\tau\) is a multiple \(cP\) of a classical vector cross product \(P \in \Omega^2 M\), then the eigenvalues of \(\frac{\tau}{P}\) on \(T_x M\) are \(\{\pm \frac{\sqrt{3}}{2} i \|x\|\}\). Thus the eigenvalues of \(T_x\) on symmetric 2-tensors of \(x^+\) are \(\{0, \pm \sqrt{3} \|x\|\}\) and the minimal polynomial is given by
\[
p(\lambda) := \lambda(\lambda^2 + c^2 \|x\|^2).
\]
For \(c^2 = \tau^2\) or \(c^2 = \frac{1}{3\tau}\) this yields (7) and (9). Similarly, if \(\dim(M) = 6\) and \(\tau\) is the real part \(\psi_+\) of a complex volume form, then the eigenvalues of \(\frac{\tau}{\psi_+}\) on \(x^+\) are \(\{0, \pm \frac{\sqrt{3}}{2} \|x\|\}\) according to (11). Thus the eigenvalues of \(T_x\) on symmetric 2-tensors of \(x^+\) are \(\{0, \pm \frac{\sqrt{3}}{2} \|x\|, \pm \sqrt{3} \|x\|\}\) and the minimal polynomial is
\[
p(\lambda) := \lambda(\lambda^2 + \frac{1}{4} \|x\|^2)(\lambda^2 + 2 \|x\|^2) = \lambda^5 + \frac{5}{4} \|x\|^2 \lambda^3 + \frac{1}{4} \|x\|^4 \lambda.
\]
which establishes (8). It remains to show that (7)-9 are minimal. For this we have to refer to Sections 6 and 7.

\[\square\]
5.1 Comparison with known examples of linear Jacobi relations

We now compare the results with the examples of linear Jacobi relations found by author others. In the following we let $\mathfrak{B}_g$ denote the Killing form of some semisimple Lie algebra $\mathfrak{g}$.

- Simply connected three-dimensional naturally reductive homogeneous spaces are parametrised by pairs $(\kappa, \tau)$ of real numbers. The corresponding naturally reductive metric is called the Berger metric. According to [22, Theorem 4.3] there exists a Jacobi relation
  \begin{equation}
  \mathcal{R}^3 = -\tau^2 \mathcal{R}^1.
  \end{equation}
  (28)

- On the Berger manifold $V_1 := \text{Sp}(2)/\text{SU}(2) = \text{SO}(5)/\text{SO}(3)$ (cf. [11]) and the Wilking manifold $V_3 := \text{SO}(3) \times \text{SU}(3)/U^*(2)$ (cf. [50]) there exist minimal linear Jacobi relations of order two. For the normal metric corresponding to $-\frac{1}{12} B_{\text{so}(5)}$ on $V_1$ it was shown in [35] that
  \begin{equation}
  \mathcal{R}^3 = -\mathcal{R}^1.
  \end{equation}
  (29)

For the normal metric induced by $-\frac{1}{12} B_{\text{so}(3) \oplus \text{su}(3)}$ on $V_3$ it was shown in [30] that
  \begin{equation}
  \mathcal{R}^3 = -\frac{2}{5} \mathcal{R}^1.
  \end{equation}
  (30)

- On the complex flag manifold $F(1, 2) := \text{SU}(3)/T^2$ and the generalized flag manifold $F^6 := \text{SO}(5)/U(2)$ (cf. [4]) there exist minimal linear Jacobi relations of order four. For the normal metric induced by $-\frac{1}{6} B_{\text{su}(3)}$ on $F(1, 2)$ it was shown in [3] that
  \begin{equation}
  \mathcal{R}^5 = -\frac{5}{8} \mathcal{R}^3 - \frac{1}{16} \mathcal{R}^1.
  \end{equation}
  (31)

For the normal metric induced by $-\frac{1}{12} B_{\text{su}(3)}$ on $F^6$ we have according to [5]
  \begin{equation}
  \mathcal{R}^5 = -\frac{5}{4} \mathcal{R}^3 - \frac{1}{4} \mathcal{R}^1.
  \end{equation}
  (32)

In order to compare these results with Theorem 2, note that a scaling $\bar{g} = cg$ of the metric tensor by a constant factor obviously transforms the coefficients $a_i$ of a linear Jacobi relation (5) according to $\bar{a}_i = c^{-i} a_i$. Hence (29)-(32) are a consequence of the following lemma:

**Lemma 4.** Let $G/K$ be a normal homogeneous space whose metric is induced by $-cB$, where $B$ is the Killing form of the Lie algebra $\mathfrak{g}$ of $G$.

(a) (cf. [22, Lemma 5.4]) Let $G/K$ be a six-dimensional homogeneous strict nearly Kähler manifold. Then the scalar curvature is normalized to $s = 30$ for $c = \frac{1}{12}$.

(b) (cf. [2] Lemma 7.1) Let $G/K$ be a normal homogeneous nearly parallel $G_2$-space. Then the scalar curvature is normalized to $\frac{1}{8}$ for $c = \frac{6}{5}$.  

11
6 Holomorphic sectional curvature of six-dimensional nearly Kähler manifolds

We want to show that the Jacobi relation (8) on six-dimensional nearly Kähler 3-symmetric spaces is of minimal order. Let \((M,g,J)\) be a nearly Kähler manifold. The sectional curvature \(K(x,y)\) and the holomorphic sectional curvature \(H(x)\) are defined by (cf. [25, Ch. 2])

\[
\|x\|^2 \|y\|^2 K(x,y) := R(x,y,y,x), \tag{33}
\]

\[
\|x\|^4 H(x) := R(x,Jx,Jx,x). \tag{34}
\]

for all \(p \in M\) and \(x,y \in T_pM\). In the following we consider the complex line \(C_x := \{x,Jx\} \subseteq \mathbb{R}\) and its orthogonal complement \(C_{x}^\perp \subset T_pM\).

**Proposition 4.** For a six-dimensional strict nearly Kähler manifold \(M\) the following are equivalent:

(a) The sectional curvature \(K(x,y)\) is constant;

(b) The holomorphic sectional curvature \(H(x)\) is constant;

(c) We have \(\mathcal{R}_xJx \in C_x\) for all \(x \in T_pM\) and \(p \in M\);

(d) We have \(\mathcal{R}_x(\tau_xu,\tau_xu) = (x,x)\mathcal{R}_x(u,u)\) for each \(x \in T_pM\), every \(u \in \mathbb{C}x^\perp\) and each \(p \in M\).

**Proof.** The equivalence of \((a)\) and \((b)\) is seen as follows: \((a) \Rightarrow (b)\) is obvious. For the other direction, suppose that the holomorphic sectional curvature \(H\) is constant. Then according to [25, Proposition 3.4] we have the following relation

\[
4\langle x,x\rangle\langle y,y\rangle K(x,y) = H(\langle x,x\rangle\langle y,y\rangle + 3\langle Jx,y\rangle^2) + 3\|\nabla_x Jy\|^2
\]

for all \(y\) with \(\langle x,y\rangle = 0\). Via the constant type equation \([10]\) this becomes

\[
4\langle x,x\rangle\langle y,y\rangle K(x,y) = (H + 3)\langle x,x\rangle\langle y,y\rangle + 3(H - 1)\langle Jx,y\rangle^2 \tag{35}
\]

Hence the Ricci tensor satisfies

\[
4 \text{ric}(x,x) = (8H + 12)\langle x,x\rangle.
\]

On the other hand using that \(\text{ric}(x,x) = 5\langle x,x\rangle\) we conclude that \(H \equiv 1\). Resubstituting this into \((35)\), we see that the sectional curvature is constant.

The equivalence of \((b)\) and \((c)\) was shown by Tanno (cf. [47, Theorem 3.4]).

For \((a) \Rightarrow (d)\), we have \(\langle \tau_xu,\tau_xu\rangle = \langle x,x\rangle\langle u,u\rangle\) for all \(u \in \mathbb{C}x^\perp\) according to \([10]\). Hence \(I_x := \frac{1}{\langle x,x\rangle}\tau_x|_{\mathbb{C}x^\perp}\) defines a second Hermitian structure on \(\mathbb{C}x^\perp\). Furthermore if the sectional curvature \(K\) is constant, then \(\mathcal{R}_x = K(x,x)\text{Id}\). This yields \((d)\).

It remains to show that \((d)\) implies \((b)\): Recall that the 3-form \(\tau(x,y,z) = \langle \tau_xy,z\rangle\) is of type \((3,0) + (0,3)\) and hence

\[
\langle \tau_xy,z\rangle = -\langle \tau_Jxy,Jz\rangle = \langle J\tau_Jxy,z\rangle, \\
\langle \tau_xJy,z\rangle = \langle \tau_xy,Jz\rangle = -\langle J\tau_xy,z\rangle
\]
We conclude that $I_x = J \circ I_{Jx}$ and $J \circ I_x = -I_x \circ J$. The latter implies that $\mathbb{C}x^\perp$ is the four dimensional standard representation of the quaternionic numbers $\mathbb{H}$. Let us now assume that $R_x$ commutes with $I_x$ for all $x$. We have

$$R_x(Ju, Ju) = R(x, Ju, Ju, x)$$

$$= R(Jx, u, u, Jx)$$

$$= R(Jx, I_{Jx}u, I_{Jx}u, Jx)$$

$$= R(x, I_{Jx}u, I_{Jx}u, x)$$

$$= R_x(I_{Jx}u, I_{Jx}u)$$

$$= R_x(u, u)$$

Here we used in the second and the fifth line a known curvature identity for nearly Kähler manifolds (cf. Corollary 2.2]). Hence $R_x$ commutes with $J$, too. Therefore $R_x$ is $\mathbb{H}$-homomorphism and hence a multiple of the identity on $\mathbb{C}x^\perp$ according to Schur’s Lemma. Thus there exists a real constant $\mu = \mu(x)$ such that $R(u, x, x, u) = \mu(x) \langle x, x \rangle \langle u, u \rangle$ for all $u \in \mathbb{C}x^\perp$. We claim that $\mu$ does not depend on $x$: Because $J$ is a unitary structure, we have $u \in \mathbb{C}x^\perp$ if and only if the converse relation $x \in \mathbb{C}u^\perp$ is satisfied. Hence

$$\mu(x) \langle x, x \rangle \langle u, u \rangle = \mu(u) \langle x, x \rangle \langle u, u \rangle$$

whenever $u \in \mathbb{C}x^\perp$. Furthermore given $x$ and $y$ both different from zero, the dimension of the linear span $\{x, y, Jx, Jy\}$ is at most four. Thus there exists $u$ different from zero with $\{x, y\} \subset \mathbb{C}u^\perp$. Therefore,

$$\mu(x) \langle x, x \rangle \langle u, u \rangle = \mu(u) \langle x, x \rangle \langle u, u \rangle$$

$$\mu(y) \langle y, y \rangle \langle u, u \rangle = \mu(u) \langle y, y \rangle \langle u, u \rangle.$$

We conclude that $\mu(x) = \mu(y)$ and $\mu$ does actually not depend on $x$. Calculating the Ricci tensor, we see that

$$\text{ric}(x, x) = \langle x, x \rangle (4\mu + H(x)).$$

Using the Einstein equation $\text{ric}(x, x) = 5 \langle x, x \rangle$ we conclude $H(x) \equiv 5 - 4\mu$. Thus $H \equiv \mu \equiv 1$. 

**Corollary 4.** The Jacobi rank of a six-dimensional nearly Kähler 3-symmetric space different from the round sphere $S^6$ is equal to four.

**Proof.** Let $M$ be a six-dimensional nearly Kähler 3-symmetric space other than $S^6$. We will show that $[\mathbb{S}]$ is a minimal relation. In other words, we claim that $p(\lambda) := \lambda^5 + \frac{2}{3} \lambda^3 + \frac{1}{3} \lambda$ is the smallest polynomial which satisfies $p(R_x) = 0$ for all $x \in T_p M$. For this it suffices to show that $R_x$ has a non-vanishing component in $-\mu^2$-eigenspace of $T_x^2$ for every eigenvalue $-\mu^2 \in \{0, \frac{1}{3} \langle x, x \rangle, \langle x, x \rangle\}$ for at least one $x$. As usual we can assume that the scalar curvature is normalized to thirty. For $x \neq 0$ we have a direct sum decomposition $T_p M = \mathbb{C}x \oplus \mathbb{C}x^\perp$ such that $r_x^2$ acts as the diagonal matrix $\text{diag}(0, -\langle x, x \rangle)$. Hence $T_x^2$ respects the induced decomposition

$$\text{Sym}^2(T_p M) = \text{Sym}^2(\mathbb{C}x) \oplus \text{Hom}(\mathbb{C}x, \mathbb{C}x^\perp) \oplus \text{Sym}^2(\mathbb{C}x^\perp).$$

On the first two components, $T_x^2$ is the diagonal matrix $\text{diag}(0, -\frac{1}{3} \langle x, x \rangle)$. On the last one it has two eigenvalues 0 and $-\langle x, x \rangle$. It thus suffices to show that the first two components of $R_x$ do not vanish and that the last one does not belong to the kernel of $T_x$. 

13
• Suppose for a contradiction that the Sym²(ℂx)-component of \( R_x \) vanishes. Thus \( R_x(u, u) = 0 \) for all \( x \in T_pM \) and \( u \in ℂx \). This means that the holomorphic sectional curvature of \( M \) vanishes identically. According to (b) of Proposition 4 this implies that \( M \) is flat, which is not possible for a strict nearly Kähler manifold.

• Suppose that the Hom(ℂx, ℂx⁺)-component of \( R_x \) vanishes. By means of (c) of Proposition 4 this implies that the sectional curvature of \( M \) is constant.

• Suppose that the Sym²(ℂx⁺)-component of \( R_x \) is annihilated by \( T_x \). Thus \( R_x(τ_x u, v) + R_x(u, τ_x v) = 0 \) for all \( u, v \in ℂx⁺ \). Since moreover \( τ_x^2 = \langle x, x \rangle \text{Id} \) on \( ℂx⁺ \), we obtain that \( R_x(τ_x u, τ_x v) = \langle x, x \rangle R_x(u, v) \) for all \( u, v \in ℂx⁺ \). Hence (d) of Proposition 4 implies that the sectional curvature of \( M \) is constant.

\[ \square \]

7 The structure of linear Jacobi relations

It remains to show that the Jacobi relations (7) and (9) on three-dimensional naturally reductive homogeneous spaces and naturally reductive nearly parallel \( G_2 \)-manifolds both are of minimal order. This will follow from a general structure result for linear Jacobi relations on compact or naturally reductive spaces. For this note that (6) becomes an ordinary differential equation with constant coefficients \( p(\overline{s})r_{ij} = 0 \) for the matrix coefficients \( r_{ij} \) of \( R_γ \) with respect to a \( ∇ \)-parallel frame along \( γ \). Here \( p(λ) \) denotes the polynomial corresponding to (6),

\[ p(λ) := λ^{k+1} - a_1 λ^{k-1} - a_2 λ^{k-3} - \cdots \left\{ \begin{array}{ll} a_{\frac{k}{2}} λ^\frac{k}{2} & \text{if } k \text{ is even} \\ a_{\frac{k+1}{2}} & \text{if } k \text{ is odd} \end{array} \right. \]

(36)

Lemma 5. Let \( M \) be a compact manifold or a naturally reductive space. Suppose there exists a linear Jacobi relation (5) where we assume that the order \( k \) is minimal.

(a) Each root of \( p(λ) \) defined by (36) is purely imaginary and simple. Therefore there exist strictly positive numbers \( 0 < λ_1 < \cdots < λ_\frac{k}{2} \) such that \( p(λ) = q(λ) \) or \( p(λ) = λ q(λ) \) where \( q(λ) = \prod_{i=1}^{\frac{k}{2}} (λ^2 + λ_i^2) \). In particular the coefficients \( a_j \) are strictly negative.

(b) If the Ricci tensor is a non-vanishing Killing tensor, then \( p(λ) = λ q(λ) \). In particular \( k \) is even.

Note that the Ricci tensor of a naturally reductive homogeneous space is automatically a Killing tensor.

Proof. On a compact Riemannian manifold the norm of the Riemannian curvature tensor is bounded. Therefore the matrix elements \( r_{ij} \) of \( R_γ \) with respect to a \( ∇ \)-parallel orthonormal frame along \( γ \) are bounded functions in the geodesic parameter. The same is true for naturally reductive spaces. In fact \( R \) is parallel with respect to the Ambrose-Singer connection \( ∇ \) and geodesics of \( ∇ \) are geodesics of \( ∇ \). Hence the matrix coefficients of \( R_γ \) are constant with respect to a \( ∇ \)-orthonormal frame along \( γ \). Thus the \( r_{ij} \) are bounded functions, too, since the rows of orthogonal matrices are unit vectors.

Furthermore the space of solutions for the ODE \( p(\overline{s}) f = 0 \) is generated by functions of the form \( p_μ(t)e^{μt} \) where \( μ \) is a root of \( p \) and \( p_μ(t) \) is a polynomial in \( t \) whose degree is strictly smaller than the multiplicity of
The space of bounded solutions for $p(\frac{d}{dt})f = 0$ is thus generated by functions of the form $c_\mu e^{\mu t}$ where $\mu$ is a purely imaginary root of $p$ and $c_\mu$ is a constant. Since $p$ is a real polynomial, there exist real numbers $0 < \lambda_1 < \cdots < \lambda_{\frac{n}{2}}$ such that the non-zero roots $\mu_i$ of $p$ are given by $\pm i\lambda_i$. It follows that bounded solutions of $p(\frac{d}{dt})f = 0$ also satisfy $q(\frac{d}{dt})f' = 0$. By minimality of $p$ we have $p(\lambda) = q(\lambda)$ or $p(\lambda) = \lambda q(\lambda)$ depending on whether zero is an eigenvalue of $p$ or not.

Assume now that the Ricci tensor is a Killing tensor and that $p(\lambda) = q(\lambda)$. We claim that $\text{ric} = 0$: Taking the trace in \((\ref{eq:trace})\), we obtain

$$\frac{d^2}{dt^2} + \lambda_1^2) \cdots \cdot \frac{d^2}{dt^2} + \lambda_{\frac{n}{2}}^2) \text{ric}(\dot{\gamma}, \dot{\gamma}) = 0.$$ 

Since $\text{ric}(\dot{\gamma}, \dot{\gamma})$ is constant by the Killing property, this reduces to

$$\lambda_1^2 \cdots \cdot \lambda_{\frac{n}{2}}^2 \text{ric}(\dot{\gamma}, \dot{\gamma}) = 0.$$ 

Because the $\lambda_i^2$ are strictly positive, we obtain that $\text{ric} = 0$. \hfill \Box

**Corollary 5.** The Jacobi relations \((\ref{eq:jacobi})\) and \((\ref{eq:jacobi2})\) are minimal.

**Proof.** According to Lemma \([5]\) a minimal Jacobi relation of order strictly smaller than two has order zero. Thus $\nabla_\dot{\gamma} R(\dot{\gamma}, \dot{\gamma}, \dot{\gamma}) = 0$ holds for all geodesics. It is well known that this is equivalent to $\nabla R = 0$, i.e. $M$ is a symmetric space. Furthermore a simply connected three-dimensional naturally reductive homogeneous space is symmetric if and only if $M$ is the round three-sphere; a simply connected nearly parallel $G_2$-space is a symmetric space if and only if $M$ is the round seven-sphere. \hfill \Box

### 8 The generalized twistor equation for the higher covariant derivatives of the curvature tensor

Let $n \geq 5$ and $\mathfrak{so}(n)$ denote the Lie algebra of the orthogonal group $SO(n)$. A standard choice of a maximal torus is given by matrices $D = \text{diag}(D_1, \ldots, D_{\frac{n}{2}})$ which are diagonal in the $2 \times 2$ matrices $D_i := \left(\begin{smallmatrix} 0 & -\epsilon_i \\ \epsilon_i & 0 \end{smallmatrix}\right)$ with arbitrary complex numbers $\epsilon_i$ for $i = 1, \ldots, \frac{n}{2}$. The weights of the standard representation of $\mathfrak{so}(n)$ on $\mathbb{C}^n$ are the linear functions $\{\pm \epsilon_i\}$ defined by $\epsilon_i(D) := \epsilon_i$. We choose the standard linear ordering on the dual space of the torus such that $\epsilon_1 > \cdots > \epsilon_{\frac{n}{2}}$. Then irreducible complex representations $V_\lambda$ of $\mathfrak{so}(n)$ are labelled by their highest weights: if $n$ is uneven, these are precisely of the form $\lambda = \sum_{i=1}^{\frac{n}{2}} \lambda_i \epsilon_i$ where the $\lambda_i$ are either all integers or all half-integers satisfying $\lambda_1 \geq \cdots \geq \lambda_{\frac{n}{2}} \geq 0$ (if $n$ is even, there are additionally those weights of the form $\lambda = \sum_{i=1}^{\frac{n}{2}-1} \lambda_i \epsilon_i - \lambda_{\frac{n}{2}} \epsilon_{\frac{n}{2}}(\frac{n}{2})$). This yields a representation of $SO(n)$ precisely if all $\lambda_i$ are integers thus encompassing “half” of all possibilities. Weyl showed how to realize these representations as induced representations on trace-free tensors of a distinguished symmetry type (cf. \([21]\) Ch. 9.5]). The other half involves the spin representations.

The corresponding complex vector bundle associated to the $SO(n)$-principal fiber bundle of orthogonal frames over a Riemannian manifold of dimension $n$ will be denoted by $V_\Lambda M$. The tensor product $V_\Lambda \mathbb{C}^n$ decomposes into a sum of isotypic components $\bigoplus_{\mu} V_\mu$ under $\mathfrak{so}(n, \mathbb{C})$ corresponding to a splitting $V_\Lambda M \otimes TM = \bigoplus_{\mu} V_\mu M$ into $\nabla$-parallel subbundles. For every $\mu_0$ such that $V_{\mu_0} \neq 0$ we consider the first order differential operator

$$P_{\mu_0} : \Gamma(V_\Lambda M) \xrightarrow{\nabla} \Gamma(V_\Lambda M \otimes TM) = \bigoplus_{\mu} \Gamma(V_\mu M) \rightarrow \Gamma(V_{\mu_0} M)$$
given by the covariant derivative followed by the projection on $V_{\mu \nu}M$ is called a generalized gradient. The highest weight $\lambda + \epsilon_1$ occurs with multiplicity one (see [12, p. 6]). The corresponding gradient $T_\lambda := P_{\lambda + \epsilon_1}$ is usually called a generalized twistor operator. The generalized twistor equation $T_\lambda s = 0$ for a section $s$ of $\Lambda_3 M$ is a partial differential equation of finite type. Well-known examples are the conformal Killing equation for alternating $p$-forms (for $\lambda := \sum_{i=1}^p \epsilon_i$) (cf. [11]), the conformal Killing equation for symmetric Killing tensors (cf. [28]) and the Penrose twistor equation for positive and negative half-spinors (with $\lambda = \frac{1}{2} (\sum_{i=1}^{n-1} \epsilon_i \pm \epsilon_n)$).

There is a unique decomposition

$$\mathcal{R}^k = \mathcal{R}_0^k + \mathcal{R}_1^k$$

(37)

were $\mathcal{R}_1^k$ is an expression involving the metric tensor at least linearly and all traces of $\mathcal{R}_0^k$ vanish. Therefore $\mathcal{R}_0^k$ is called the completely trace-free part of $\mathcal{R}^k$.

Furthermore by the symmetries of the Riemannian curvature tensor the symmetrized higher covariant derivatives $\mathcal{R}^k$ defined in (2) satisfy the additional property $\mathcal{R}^k(x)x = \nabla_{x,x} R(x,x,x) = 0$. Hence $\mathcal{R}^k$ is a tensor whose index symmetries are described by a Young diagram with two rows of lengths $k + 2$ and 2, respectively (cf. [21, Ch. 6]). Therefore $\mathcal{R}_0^k$ is a section of $V_{(k+2)\epsilon_1 + 2\epsilon_2} M$ according to Weyl’s construction of reducible representations of the orthogonal group (cf. [21, Ch. 19.5]). The parallelity of the metric tensor also implies that

$$T_{(k+2)\epsilon_1 + 2\epsilon_2} \mathcal{R}_0^k = \mathcal{R}_0^{k+1}.$$  

(38)

Thus $\mathcal{R}_0^k$ is a generalized twistor if and only if

$$\mathcal{R}_0^{k+1} = 0.$$  

(39)

**Corollary 6.** (a) Let $M$ be a six-dimensional nearly Kähler 3-symmetric space. Then $\mathcal{R}_0^3$ is a generalized twistor.

(b) Let $M$ be a naturally reductive nearly parallel $G_2$-manifold. Then $\mathcal{R}_0^2$ is a generalized twistor.

**Proof.** Let us assume that a linear Jacobi relation (41) holds. For persuasiveness we make the scalar product $\langle , \rangle$ re-appear in our notation. Then

$$T_{(k+2)\epsilon_1 + 2\epsilon_2} \mathcal{R}_0^k \equiv \mathcal{R}_0^{k+1} \equiv \left( \langle , \rangle (a_1 \mathcal{R}^{k-1} + a_2 \langle , \rangle \mathcal{R}^{k-3} + \cdots) \right)_{0} \equiv 0,$$

where the last equality holds because the traceless part of a term which involves the metric tensor at least linearly vanishes by definition. Hence (43) implies (39) for the same $k$. Applying this to (43) and (2) yields the content of the corollary. □

The Jet Isomorphism Theorem of Riemannian geometry implies that the Taylor expansion of the metric tensor is already encoded in the sequence $(\mathcal{R}^0, \mathcal{R}^1, \mathcal{R}^3, \ldots)$. For an Einstein manifold it even suffices to consider the sequence of the completely traceless parts $(\mathcal{R}_0^0, \mathcal{R}_0^1, \mathcal{R}_0^2, \ldots)$. The latter assertion can be reasoned as follows:

It is straightforward to show that the curvature tensor $R$ of an Einstein manifold and its covariant derivative $\nabla R$ both are completely trace-free. Further the symmetries of the Riemannian curvature tensor can be used to show that all traces of $\mathcal{R}^{k+2}$ are determined by $(\mathcal{R}^0, \mathcal{R}^1, \ldots, \mathcal{R}^k)$. Our assertion follows by induction.

Therefore the generalized twistor equation (39) seems to be in particular interesting for Einstein manifolds, e.g. in the situation of Corollary 6 (a) and (b). This becomes for example evident in the case $k = 0$, since for an Einstein manifold (39) holds with $k = 0$ if and only if $\nabla R = 0$. 

16
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