Stable reconstruction of generalized impedance boundary conditions

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Abstract
We are interested in the identification of a generalized impedance boundary condition from the far fields created by one or several incident plane waves at a fixed frequency. We focus on the particular case where this boundary condition is expressed with the help of a second-order surface operator: the inverse problem then amounts to retrieve the two functions $\lambda$ and $\mu$ that define this boundary operator. We first derive a global Lipschitz stability result for the identification of $\lambda$ or $\mu$ from the far field for bounded piecewise constant impedance coefficients and give a new type of stability estimate when inexact knowledge of the boundary is assumed. We then introduce an optimization method to identify $\lambda$ and $\mu$, using in particular an $H^1$-type regularization of the gradient. Finally, we show some numerical results in two dimensions, including a study of the impact of various parameters, and by assuming either an exact knowledge of the shape of the obstacle or an approximate one.

1. Introduction

In this work, we are interested in the identification of boundary coefficients in the so-called generalized impedance boundary conditions (GIBCs) on a given obstacle from measurements of the scattered field far from the obstacle associated with one or several incident plane waves at a given frequency. More specifically, we shall consider the boundary conditions of the type

$$\frac{\partial u}{\partial \nu} + \text{div}_{\partial D}(\mu \nabla_{\partial D} u) + \lambda u = 0 \quad \text{on} \quad \partial D,$$

where $\mu$ and $\lambda$ are the complex valued functions, $\text{div}_{\partial D}$ and $\nabla_{\partial D}$ are respectively the surface divergence and the surface gradient on $\partial D$ and $\nu$ denotes the outward unit normal on $\partial D$. In the case $\mu = 0$, this condition is the classical impedance boundary condition (also known as the Leontovitch boundary condition) used for instance to model imperfectly conducting
obstacles. The wider class of GIBCs is commonly used to model thin coatings or gratings as well as more accurate models for imperfectly conducting obstacles (see [4, 14, 17, 18]). Addressing this problem is motivated by applications in non-destructive testing, identification problems or modelling related to stealth technology or antennas. For instance, one may think of ultrasonic non-destructive testing for the transverse electric polarization of a medium which contains a perfect conductor coated with a thin layer. In this case, a GIBC as presented above is satisfied with $\mu = \delta$ and $\lambda = 2\delta k^2 n$, where $k$ denotes the wave number, $\delta$ is the width of the layer and $n$ is the mean value of the layer index with respect to the normal coordinate.

The use of GIBCs has at least two advantages for the inverse problem as compared to the use of an exact model. First, the identification problem becomes less unstable. Second, since solving the forward problem with GIBC is less time consuming, using such a model in iterative nonlinear methods is more advantageous.

The classical case $\mu = 0$ has been addressed in the literature by several authors, from the mathematical point of view in [24, 32] and from the numerical point of view in [10, 11]. The problem of recovering both the shape of the obstacle and the impedance coefficient is also considered in [19, 26, 28, 31]. The case of GIBC has only been recently addressed in [7] where uniqueness and local stability results have been reported.

The present work complements these first investigations in two directions. The first one is on the theoretical level. Initially, we derive a global Lipschitz stability estimate for the bounded piecewise constant impedance coefficients $\lambda$ and $\mu$. This result is similar to the one obtained in [33] for classical impedances and the Laplace equation in a bounded domain. Other or complementary stability results for the inverse coefficient problem can be found in [1, 2, 13, 21]. The main particularity and difficulty of our analysis are related to the treatment of the second-order surface operator appearing in the GIBC. Also in contrast with the work in [33], here we make use of Carleman estimates instead of doubling properties as a main tool in deriving the stability estimates. We then prove the stability of the reconstruction of the impedances when only inexact knowledge of the geometry is available. The proof of this result relies on two properties:

- continuity of the measurements with respect to the obstacle, uniformly with respect to the impedance coefficients;
- stability for the inverse coefficient problem for a known obstacle.

Up to our knowledge, this kind of stability result is new. It would be useful for instance when the geometry has itself been reconstructed from the measurements using some qualitative methods (e.g. sampling methods [9, 16]) and therefore is known only approximately. It may also be useful in understanding the convergence of iterative methods to reconstruct both the obstacle and the coefficients where the updates for the geometry and the physical parameters are made alternatively. Let us also mention that the proof of our stability result can straightforwardly be extended to other identification problems that enjoy the two properties indicated above.

In a second part, we investigate a numerical optimization method to identify the boundary coefficients. We propose a reconstruction procedure based on a steepest descent method with the $H^1(\partial D)$ regularization of the gradient. The accuracy and stability of the inversion scheme is tested through various numerical experiments in a 2D setting. Special attention is given to the case of non-regular coefficients and inexact knowledge of the boundary $\partial D$.

The outline of our paper is as follows. In section 2, we introduce and study the forward and inverse problems. Section 3 is dedicated to the derivation of a stability result with inexact geometry. The numerical part is the subject of section 4.
2. The forward and inverse problems

2.1. The forward scattering problem

Let $D$ be an open bounded domain of $\mathbb{R}^d$, $d = 2$ or $3$ with a Lipschitz continuous boundary $\partial D$, $\Omega := \mathbb{R}^d \setminus \overline{D}$ and $(\lambda, \mu) \in (L^\infty(\partial D))^2$ be the impedance coefficients. The scattering problem with GIBCs consists in finding $u = u^i + u^s$ such that

$$\begin{cases} \Delta u + k^2 u = 0 & \text{in } \Omega, \\ \text{div}_D(\mu \nabla_D u) + \frac{\partial u}{\partial v} + \lambda u = 0 & \text{on } \partial D, \end{cases}$$

and $u^s$ satisfies the Sommerfeld radiation condition

$$\lim_{r \to \infty} \int_{|s|=r} \left| \frac{\partial u^s}{\partial r} - ik u^s \right|^2 ds = 0,$$

where $k$ is the wave number, $u^i = e^{ikx}$ is an incident plane wave where $\hat{a}$ belongs to the unit sphere of $\mathbb{R}^d$ denoted $S^{d-1}$ and $u^s \in V : = \{ v \in D'(\Omega), \varphi v \in H^1(\Omega) \forall \varphi \in D(\mathbb{R}^d) \text{ and } \varphi|_{\partial D} \in H^1(\partial D) \}$ is the scattered field. For $v \in H^1(\partial D)$, the surface gradient $\nabla_D v$ lies in $L^2(\partial D) : = \{ V \in (L^2(\partial D))^d, \ V \cdot v = 0 \}$. Moreover, $\text{div}_D(\mu \nabla_D u)$ is defined in $H^{-1}(\partial D)$ for $\mu \in L^\infty(\partial D)$ by

$$\langle \text{div}_D(\mu \nabla_D u), v \rangle_{H^{-1}(\partial D), H^1(\partial D)} := -\int_{\partial D} \mu \nabla_D u \cdot \nabla_D v ds \quad \forall v \in H^1(\partial D).$$

Let us define $\Omega_R := \Omega \cap B_R$ where $B_R$ is the ball of radius $R$ such that $D \subset B_R$ and let $S_R : H^{1/2}(\partial B_R) \to H^{-1/2}(\partial B_R)$ be the Dirichlet-to-Neumann map defined for $g \in H^{1/2}(\partial B_R)$ by $S_R g := \partial u^e/\partial r|_{\partial B_R}$ where $u^e$ is the radiating solution of the Helmholtz equation outside $B_R$ and $u^e = g$ on $\partial B_R$. Then solving (1) is equivalent to find $u$ in $V_R : = \{ v \in H^1(\Omega_R); \ \varphi v\}_{\partial D} \in H^1(\partial D) \}$ such that

$$\mathcal{P}(\lambda, \mu, \partial D) \begin{cases} \Delta u + k^2 u = 0 & \text{inside } \Omega_R, \\ \text{div}_D(\mu \nabla_D u) + \frac{\partial u}{\partial v} + \lambda u = 0 & \text{on } \partial D, \\ \frac{\partial u}{\partial r} - S_R(u) = \frac{\partial u^i}{\partial r} - S_R(u^i) & \text{on } \partial B_R. \end{cases}$$

Remark that the space $V_R$ equipped with the graph norm is a Hilbert space. We define the operator $A$ of $V_R$ and the bilinear form $a$ of $V_R \times V_R$ by

$$(Au, v)_{V_R} := a(u, v) := \int_{\Omega_R} (\nabla u \cdot \nabla v - k^2 u v) dx + \int_{\partial D} (\mu \nabla_D u \cdot \nabla_D v - \lambda u v) \right| = (S_R u, v),$$

for $(u, v) \in V_R \times V_R$ where $(\cdot, \cdot)$ is the duality product between $H^{-1/2}(\partial B_R)$ and $H^{1/2}(\partial B_R)$. Furthermore, we define $l$ a linear form on $V_R$ and $F \in V_R$ by

$$(F, v)_{V_R} := l(v) := \int_{\partial B_R} \left( \frac{\partial u^i}{\partial r} - S_R(u^i) \right) v ds$$

for all $v \in V_R$. Therefore, $u$ is a solution to $\mathcal{P}(\lambda, \mu, \partial D)$ if and only if

$$a(u, v) = l(v) \quad \forall v \in V_R$$

or $Au = F$.

**Hypothesis $\mathcal{H}$**: $(\lambda, \mu) \in (L^\infty(\partial D))^2$ are such that

$$\text{Im}(\lambda) \geq 0, \quad \text{Im}(\mu) \leq 0 \quad \text{a.e. in } \partial D$$
and there exists $c > 0$ such that
\[ \text{Re}(\mu) \geq c \quad \text{a.e. in } \partial D. \]

In the assumption $H_\lambda$, the signs of $\text{Im}(\lambda)$ and $\text{Im}(\mu)$ are governed by physics, since these quantities represent absorption. On the contrary, the assumption on $\text{Re}(\mu)$ is technical and ensures coercivity. However, it is satisfied in the example of a medium with a thin coating which is presented in the introduction. In the following, $K$ will denote a compact set of $(L^\infty(\partial D))^2$ such that there exists a constant $c_K > 0$ for which the assumption $H$ holds with $c = c_K$ for all $(\lambda, \mu) \in K$.

**Proposition 2.1.** If the assumption $H$ is satisfied, then the problem $P(\lambda, \mu; \partial D)$ has a unique solution $u$ in $V_R$. In addition, there exists a constant $C_K > 0$ such that
\[ ||| A^{-1} ||| \leq C_K \quad \text{for all } (\lambda, \mu) \in K, \]
where $||| \cdot |||$ stands for the operator norm.

**Proof.** The proof is quite classical, and we refer to [6] for details. □

Under the sufficient conditions $H$ on the impedance coefficients $\lambda$ and $\mu$ that ensure the existence and uniqueness for the forward problem, we can study the inverse coefficient problem and this is the aim of the next section.

### 2.2. Formulation of the inverse problem

We recall the following asymptotic behaviour for the scattered field (see [12]):
\[ u^s(x) \sim \frac{e^{ikr}}{r^{(d-1)/2}} \left( u^\infty(\hat{x}) + O \left( \frac{1}{r} \right) \right) \quad r \to +\infty \]
uniformly for all the directions $\hat{x} = x/r \in S^{d-1}$. The far field $u^\infty \in L^2(S^{d-1})$ is given by
\[ u^\infty(\hat{x}) = \int_{\Gamma} \left( u^s(y) \frac{\partial \Phi^\infty(y, \hat{x})}{\partial y} - \frac{\partial u^s(y)}{\partial y} \Phi^\infty(y, \hat{x}) \right) ds(y) \quad \forall \hat{x} \in S^{d-1}, \quad (5) \]
where $\Gamma$ is the boundary of some regular open domain that contains $D$ and $\Phi^\infty$ is the far field associated with the Green function of the Helmholtz equation defined in $\mathbb{R}^2$ by $\Phi^\infty(y, \hat{x}) := \frac{e^{ik|x-y|}}{\sqrt{4\pi|x-y|}}$ and in $\mathbb{R}^3$ by $\Phi^\infty(y, \hat{x}) := \frac{1}{4\pi} e^{-ik|x-y|}$.

**Remark 2.2.** Since $u^\infty$ is an analytical function on $S^{d-1}$ (see [12]), assuming that the far field is known everywhere on $S^{d-1}$ is equivalent to assuming that it is known on a non-empty open set of $S^{d-1}$.

Let us define the far-field map
\[ T: (\lambda, \mu, \partial D) \to u^\infty, \]
where $u^\infty$ is the far field associated with the scattered field $u^s = u - u^i$ and $u$ is the unique solution of the problem $P(\lambda, \mu, \partial D)$. The inverse coefficient problem is as follows: given an obstacle $D$, an incident direction $\hat{d} \in S^{d-1}$ and its associated far-field pattern $u^\infty$ for all $\hat{x} \in S^{d-1}$, reconstruct the corresponding impedance coefficients $\lambda$ and $\mu$. In other words, the inverse problem amounts to invert the map $T$ with respect to the coefficients $\lambda$ and $\mu$ for a given $\partial D$. The first natural question related to this inverse problem is injectivity of $T$ and stability properties of the inverse map. These questions have been addressed in [7] where for instance results on local stability in compact sets have been reported. Our subsequent
analysis on the stability of the reconstruction of $\lambda$ and $\mu$ with respect to perturbed obstacles will depend on the stability for the inverse map of $T$. We shall first give an improvement of the stability results in [7] for the reconstruction of piecewise constant impedance values. Let us note that uniqueness (and therefore stability results) with single incident wave fails in general except if one assumes that parts of $\lambda$ and $\mu$ are known a priori. Moreover, we may need to add some restriction for the incident direction or for the geometry of the obstacle (see [7] for more details).

2.3. A global stability estimate for the generalized impedance functions

In this section, we shall assume that $\partial D$ is a $C^3$ boundary and $K_I$ is a compact set of $(L^\infty(\partial D))^2$ of piecewise constant functions defined as follows: let $I$ be an integer and $(\partial D_i)_{i=1,...,I}$ be $I$ non-overlapping open sets of $\partial D$ such that $\cup_{i=1}^I \partial D_i = \partial D$. Then $(\lambda, \mu) \in K_I$ if there exist $2I$ constants $(\lambda_i)_{i=1,...,I}$ and $(\mu_i)_{i=1,...,I}$ respectively such that for $x \in \partial D$,

$$
\lambda(x) = \sum_{i=1}^I \lambda_i \chi_{\partial D_i}(x) \quad \text{and} \quad \mu(x) = \sum_{i=1}^I \mu_i \chi_{\partial D_i}(x),
$$

and there exist $c_{K_i} > 0$ and $C_{K_i} > 0$ such that

$$
c_{K_i} \leq \Re(\mu_i) \leq C_{K_i}, \quad -C_{K_i} \leq \Im(\mu_i) \leq 0
$$

and

$$
0 \leq \Im(\lambda_i) \leq C_{K_i}, \quad |\Re(\lambda_i)| \leq C_{K_i}
$$

for all $i$. From now on, $C_{K_i}$ and $c_{K_i}$ will be generic constants that can change, but they remain independent of $\lambda$ and $\mu$. Using that $\mu$ is a piecewise constant function, we shall first explicitly give a regularity result for the solution $u$ of the scattering problem.

For convenience, for sufficiently small $\rho > 0$, we denote by $\Xi^\rho_R$ the subset of $\Omega_R$ defined as follows. If $\delta D$ denotes the set of all points of $\partial D$ which are shared by two sets $\partial D_i$ and $\partial D_j$ for $i, j = 1, \ldots, I$ and $i \neq j$, we have

$$
\Xi^\rho_R = \{x \in \Omega_R, d(x, \delta D) > \rho\},
$$

where $d$ denotes the distance function.

**Lemma 2.3.** There exists a constant $C_{K_i}$ (depending on $\rho$ and $R$) such that for all $(\lambda, \mu) \in K_I$, the solution $u$ to $P(\lambda, \mu, \partial D)$ satisfies

$$
\|u\|_{H^1(\Xi^\rho_R)} \leq C_{K_i}.
$$

**Proof.** From the definition of $\mu$, we obtain

$$
\mu \Delta_{\partial \Omega} u + \frac{\partial u}{\partial v} + \lambda u = 0 \quad \text{on} \quad \partial \Xi^\rho_R \cap \partial D,
$$

where $\Delta_{\partial \Omega} u := \text{div}_{\partial \Omega}(\nabla_{\partial \Omega} u)$ is the Laplace–Beltrami operator. Since $\Delta u = -k^2 u$ in $\Omega_R$, we have by a standard trace result in space $\{u \in H^1(\Omega_R), \Delta u \in L^2(\Omega_R)\}$

$$
\left\| \frac{\partial u}{\partial v} \right\|_{H^{-1/2}(\partial D)} \leq C \|u\|_{V_k}.
$$

(6)

Now we consider the regularity of $u$ solving for $f \in H^{-1/2}(\partial \Xi^\rho_R \cap \partial D)$ the equation

$$
\Delta_{\partial \Omega} u + \frac{\lambda}{\mu} u = f \quad \text{on} \quad \partial \Xi^\rho_R \cap \partial D.
$$

(7)
Following section 2.5.6 in [29], by using a local map \( \varphi \) and a local parametrization \( \xi_i \) of \( \partial D \), \( i = 1, \ldots, d - 1 \), the Laplace–Beltrami operator has the local expression

\[
\Delta_{\partial D} u = \frac{1}{\sqrt{\det g}} \sum_{i=1}^{d-1} \frac{\partial}{\partial \xi_i} \sqrt{\det g^{ij}} \frac{\partial u}{\partial \xi_j},
\]

where the vectors \( e_i = \frac{\partial \varphi}{\partial \xi_i} \) form a basis in the tangent plane and the matrix \( g_{ij} = (e_i, e_j) \) forms the metric tensor \( g \), its inverse matrix \( g^{-1} \) being denoted as \( g^{ij} \). The local regularity of the solution to equation (7) on \( \partial D \) hence amounts to a regularity problem for a standard elliptic problem in the divergence form in \( \mathbb{R}^{d-1} \). Hence, applying the local regularity results for elliptic operators of chapter 8 in [15] with \( a_{ij} = \sqrt{\det g} g^{ij} \), we obtain that if the second member of the equation is locally in \( L^2 \) and \( \partial D \) is \( C^2 \) (that is the coefficients \( a_{ij} \) are \( C^1 \)), then \( u \) is locally in \( H^2 \). By using a cut-off function and the interpolation theorem (see [25]) for equation (7), if the second member is locally in \( H^{-1/2} \), then \( u \) is locally in \( H^{3/2} \). Gathering all local estimates on \( \partial/\xi^1 \rho \cap \partial D \) and using (6), we obtain

\[
\|u\|_{H^{3/2}(\partial/\xi^1 \rho \cap \partial D)} \leq C_{KI} \|u\|_{V^\rho}.
\]

Standard regularity results for the Laplace operator with Dirichlet boundary condition lead to

\[
\|u\|_{H^{2}(\partial D)} \leq C_{KI} \|u\|_{V^\rho}
\]

and using once again regularity for the Laplace–Beltrami operator we obtain, since \( \partial D \) is \( C^3 \),

\[
\|u\|_{H^{5/2}(\partial/\xi^1 \rho \cap \partial D)} \leq C_{KI} \|u\|_{V^\rho}.
\]

We finally deduce the desired estimate using the regularity result for the Laplace operator and proposition 2.1. \( \square \)

Uniqueness of the reconstruction of \( \mu \) (with a known \( \lambda \)) is established in the following proposition; then we will derive a uniform stability estimate for \( \mu \).

**Proposition 2.4.** Take \( (\lambda, \mu^1) \) and \( (\lambda, \mu^2) \) in \( K_I \) and assume that their corresponding far fields \( u^{1,\infty} = T(\lambda, \mu^1, \partial D) \) and \( u^{2,\infty} = T(\lambda, \mu^2, \partial D) \) satisfy

\[
u^{1,\infty} = u^{2,\infty}, \quad \forall \hat{x} \in S^{d-1}.
\]

If for all \( \hat{x} \in \partial D_i \), there exist \( \tilde{x}_i \in \partial D_i \) and \( \eta_i > 0 \) such that \( \partial D_i = \partial D_i \cap B(\tilde{x}_i, \eta_i) \) is

- for \( d = 2 \) either a segment or a portion of a circle,
- for \( d = 3 \) either a portion of a plane or a portion of a cylinder or a portion of a sphere,

and the sets \( \{x + \gamma \nu(x), x \in \partial D_i, \gamma > 0\} \) are included in \( \Omega \), then \( \mu^1 = \mu^2 \).

**Proof.** Since the far fields \( u^{1,\infty} \) and \( u^{2,\infty} \) coincide, from Rellich’s lemma and unique continuation principle, the associated total fields \( u^1 \) and \( u^2 \) coincide up to the boundary \( \partial D_i \), which implies by denoting \( u := u^1 = u^2 \) that

\[
\frac{\partial u}{\partial \nu} + \text{div}_D(\mu^j \nabla_{\partial D} u) + \lambda u = 0 \quad \text{on} \quad \partial D
\]

with \( j = 1, 2 \). Since \( \lambda \) and \( \mu^j \) are constant on \( \partial D_i \), \( i = 1, \ldots, I \), we have

\[
\frac{\partial u}{\partial \nu} + \mu^j \Delta_{\partial D} u + \lambda u = 0 \quad \text{on} \quad \partial D_i
\]

and

\[
(\mu^1_j - \mu^2_j) \Delta_{\partial D} u = 0 \quad \text{on} \quad \partial D_i.
\]
In what follows, we focus our attention on some particular portion $\tilde{\partial}D_i$ and then drop the reference to index $i$ for the sake of simplicity. Suppose that $\mu^1 \neq \mu^2$; then we have

$$\Delta_{\tilde{\partial}D} u = 0, \quad \frac{\partial u}{\partial \nu} = -\lambda u \quad \text{on} \quad \tilde{\partial}D.$$ 

We only consider the case $d = 3$ and $\tilde{\partial}D = \partial D \cap B(\tilde{x}, \eta)$ is a portion of the plane of outward normal $\nu$ such that the set $\tilde{Q} = \{x + \gamma \nu, x \in \tilde{\partial}D, \gamma > 0\}$ is included in $\Omega$. We omit the proofs in the other cases because they are very similar (they are addressed in [7] in the case of constant $\lambda$ and $\mu$). There exists a system of coordinates $(x_1, x_2, x_3)$ such that $x(0, 0, 0) = \tilde{x}$ and

$$\tilde{Q} = \{x(x_1, x_2, x_3), \sqrt{x_1^2 + x_2^2} < \eta, x_3 > 0\},$$

$$\tilde{\partial}D = \{x(x_1, x_2, x_3), \sqrt{x_1^2 + x_2^2} < \eta, x_3 = 0\}.$$ 

We now consider the function $\tilde{u}$ defined in $\tilde{Q}$ from $u$ by

$$\tilde{u}(x_1, x_2, x_3) = u(x_1, x_2, 0)c(x_3),$$

where the function $c$ is uniquely defined by

$$\frac{d^2 c}{dx_3^2} + k^2 c = 0, \quad c(0) = 1, \quad \frac{dc}{dx_3}(0) = -\lambda.$$ 

(8)

Proceeding as in [7], we obtain that $u$ and $\tilde{u}$ solve the same Helmholtz equation in $\tilde{Q}$ and satisfy $\tilde{\partial}u = \partial u$ and $\tilde{\partial}_\nu u = \partial_\nu u$ on $\tilde{\partial}D$. Hence, unique continuation implies $\tilde{u} = u$ in $\tilde{Q}$. 

Since $u'$ satisfies the radiation condition and

$$u(x_1, x_2, x_3) = u'(x_1, x_2, x_3) + e^{ik(d_1x_1 + d_2x_2 + d_3x_3)},$$

we obtain

$$u(x_1, x_2, x_3) \sim e^{ik(d_1x_1 + d_2x_2 + d_3x_3)}, \quad x_3 \to +\infty,$$

and in particular when $x_1 = x_2 = 0$,

$$u(0, 0, x_3) \sim e^{ikd_3x_3}, \quad x_3 \to +\infty,$$

that is,

$$c(x_3) \sim C e^{ikd_3x_3}, \quad x_3 \to +\infty.$$ 

(9)

To see that the asymptotic behaviour (9) is impossible, we solve explicitly equation (8) and obtain

$$c(x_3) = \frac{1}{2} \left(1 + \frac{\lambda}{k}\right) e^{ikx_3} + \frac{1}{2} \left(1 - \frac{\lambda}{k}\right) e^{-ikx_3}.$$ 

But from (9), we have

$$d_3 = 1 \quad \text{and} \quad 1 - \frac{\lambda}{k} = 0;$$

hence, $\text{Im}(\lambda) = -ik < 0$ which is forbidden from the assumption $\gamma'$. \hfill \Box

**Theorem 2.5.** Under the same assumptions as in proposition 2.4, there exists a constant $C_K$, such that for all $(\lambda, \mu^1)$ and $(\lambda, \mu^2)$ in $K$, 

$$\|\mu^1 - \mu^2\|_{L^\infty(\tilde{\partial}D)} \leq C_K \|T(\lambda, \mu^1, \tilde{\partial}D) - T(\lambda, \mu^2, \tilde{\partial}D)\|_{L^2(S^{d-1})},$$

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In order to prove theorem 2.5, we will need the following results on the quantification of unique continuation, the proof of which can be found in [5, proposition 2.4] for the first one and in [30, lemma 3.1] for the second one.

**Proposition 2.6.** For all \( x_i \in \partial D_i \cap \partial \Sigma_R^0 \), there exist \( r_i > 0 \) and an open domain \( \omega_i \in \Sigma_R^0 \) such that for all \( \kappa \in (0, 1) \), there exist \( c, \epsilon_0 > 0 \) such that for all \( \epsilon \in (0, \epsilon_0) \) and for all \( u \in H^1(\Sigma_R^0) \) which satisfies \( \Delta u + k^2 u = 0 \) in \( \Sigma_R^0 \), we have

\[
\| u \|_{H^1(\Sigma_R^0 \cap \partial \omega_i, r_i)} \leq C \epsilon \| u \|_{H^1(\omega_i)} + \epsilon \| u \|_{H^1(\Sigma_R^0)}.
\]

**Proposition 2.7.** Let \( \omega_0, \omega_1 \) be two open domains such that \( \omega_0, \omega_1 \subset \Sigma_R^0 \). There exist \( s, c, \epsilon_0 > 0 \) such that for all \( \epsilon \in (0, \epsilon_0) \) and for all \( u \in H^1(\Sigma_R^0) \) which satisfies \( \Delta u + k^2 u = 0 \) in \( \Sigma_R^0 \),

\[
\| u \|_{H^1(\omega_0)} \leq C \epsilon \| u \|_{H^1(\omega_1)} + \epsilon \| u \|_{H^1(\Sigma_R^0)}.
\]

**Proof of theorem 2.5.** Let \( u^1 \) and \( u^2 \) be the solutions to the problems \( \mathcal{P}(\lambda, \mu^1, \partial D) \) and \( \mathcal{P}(\lambda, \mu^2, \partial D) \) respectively with \( u^1(x) = e^{i k d x} \) as an incident plane wave. Following [33], we introduce an auxiliary function

\[
v := \frac{u^2 - u^1}{\| \mu^2 - \mu^1 \|_{L^\infty(\partial D)}},
\]

which is a solution of the scattering problem

\[
\begin{cases}
\Delta v + k^2 v = 0 & \text{inside } \Omega_R, \\
\text{div}_D(\mu^1 \nabla_D v) + \frac{\partial v}{\partial r} + \lambda v = \frac{1}{\| \mu^2 - \mu^1 \|_{L^\infty(\partial D)}} \text{div}_D[(\mu^1 - \mu^2) \nabla_D u^2] & \text{on } \partial D, \\
\frac{\partial v}{\partial r} - S_R(v) = 0 & \text{on } \partial B_R.
\end{cases}
\]

Using proposition 2.1, there exists a constant \( C_{K_1} \), independent of \( \lambda, \mu^1 \) and \( \mu^2 \) such that

\[
\| v \|_{V_k} \leq C_{K_1} \sup_{u \in H^1(\partial D)} \int_{\partial D} \frac{\nabla_M u^2 \cdot \nabla_M w}{\| u \|_{H^1(\partial D)}} \, ds \leq C_{K_1} \| u^2 \|_{V_k},
\]

and then using proposition 2.1 once more for \( u^2 \), we obtain

\[
\| v \|_{V_k} \leq C_{K_1}.
\]

Using a similar argument as in the proof of lemma 2.3, one obtains that

\[
\| v \|_{H^1(\Sigma_R^0)} \leq C_{K_1}
\]

for another constant \( C_{K_1} \).

Let us focus now on the boundary condition for \( v \) on \( \partial D_i \) for a fixed \( i = 1, \ldots, I \),

\[
\mu^1_i \Delta_{\partial D_i} v + \frac{\partial v}{\partial r} + \lambda v = \frac{\mu^1_i - \mu^2_i}{\| \mu^2 - \mu^1 \|_{L^\infty(\partial D)}} \Delta_{\partial D_i} u^2 \quad \text{on } \partial D_i.
\]

Using the notation of proposition 2.4, for a part \( \gamma_i \) of \( \partial D_i \) which is strictly included into \( \partial \Sigma_R^0 \cap B(\bar{\gamma}_i, r_i) \), where \( r_i \) comes from proposition 2.6, we have

\[
\left\| \mu^1_i \Delta_{\partial D_i} v + \frac{\partial v}{\partial r} + \lambda v \right\|_{L^2(\gamma_i)} \leq C_{K_1} \left\| \mu^1_i - \mu^1 \right\|_{L^\infty(\partial D)} \left\| \Delta_{\partial D_i} u^2 \right\|_{L^2(\gamma_i)}.
\]
The strategy now consists in bounding the left-hand side from above and the right-hand side from below.

(i) Upper bound for the left-hand side of (11). Let $\phi_i \in C^\infty_0(B(\overline{\xi}_i, r_i))$ with $\phi_i = 0$ on $\overline{\Omega}_R \setminus \overline{\Xi}_R$ and $\phi_i = 1$ on $\gamma_i$; then

$$
\|\mu_1^1 A_D v + \frac{\partial v}{\partial v} + \lambda v\|_{L^2(\gamma_i)} \leq C_{K_i} \left[ \|\Delta_D v\|_{L^2(\gamma_i)} + \|\frac{\partial v}{\partial v}\|_{L^2(\gamma_i)} + \|v\|_{L^2(\gamma_i)} \right] \leq C_{K_i} \left[ \|\Delta_D (\phi_i v)\|_{L^2(\partial D)} + \|\frac{\partial (\phi_i v)}{\partial v}\|_{L^2(\partial D)} + \|\phi_i v\|_{L^2(\partial D)} \right].
$$

By interpolation, we obtain

$$
\|\Delta_D (\phi_i v)\|_{L^2(\partial D)} + \|\frac{\partial (\phi_i v)}{\partial v}\|_{L^2(\partial D)} + \|\phi_i v\|_{L^2(\partial D)} \leq C \left( \|\phi_i v\|_{H^1(\partial D)} + \|\frac{\partial (\phi_i v)}{\partial v}\|_{L^2(\partial D)} \right) \leq C \left( \|\phi_i v\|_{H^{1/2}(\partial D)} + \|\frac{\partial (\phi_i v)}{\partial v}\|_{H^{1/2}(\partial D)} \right) \leq C_{K_i} \|\phi_i v\|_{H^{1/2}(\partial D)}\]

where the last inequality comes from the trace theorems, the fact that $\Delta u = -k^2 u$ in $\Omega_R$ and the bound (10). Consequently,

$$
\left\|\mu_1^1 A_D v + \frac{\partial v}{\partial v} + \lambda v\right\|_{L^2(\gamma_i)} \leq C_{K_i} \|v\|_{H^{1/2}(\Xi_R \cap B(\overline{\xi}_i, r_i))}.
$$

(12)

By applying proposition 2.6 to $v$, there exists an open set $\omega_i \subset \Xi_R$ such that for some fixed $\kappa \in (0, 1)$, there exists $c > 0$ such that for small $\epsilon$, $\|v\|_{H^1(\Xi_R \cap B(\overline{\xi}_i, r_i))} \leq C_{K_i} \|v\|_{H^1(\omega_i)} + \epsilon^c \|v\|_{H^1(\Xi_R)}$.

Take $R$ sufficiently large such that there exists $\tilde{R}$ such that $3\tilde{R} + 1 < R$ and $D \subset B_{\tilde{R}}$. Then, applying proposition 2.7 to $v$ with $\omega_0 = B(x_0, 1/4)$ for $x_0 \in \partial B_{3R+1/2}$ and $\omega_1 = \omega_i$, there exist constants $c, s > 0$ such that for small $\epsilon$, $\|v\|_{H^1(\omega_0)} \leq C_{K_i} \|v\|_{H^1(\Xi_R)} + \epsilon^c \|v\|_{H^1(\Xi_R)}$.

Hence, for some fixed $\kappa \in (0, 1)$, there exists a constant $c > 0$ such that for small $\epsilon$, $\|v\|_{H^1(\Xi_R \cap B(\overline{\xi}_i, r_i))} \leq C_{K_i} \|v\|_{H^1(\omega_0)} + \epsilon^c \|v\|_{H^1(\Xi_R)}$.

Estimate (10) yields the existence of the constants $C_{K_i}$, $c$ such that for small $\epsilon$, $\|v\|_{H^1(\Xi_R \cap B(\overline{\xi}_i, r_i))} \leq C_{K_i} \|v\|_{H^1(\omega_0)} + \epsilon^c \|v\|_{H^1(\Xi_R)}$.

(13)

From the definition of $x_0$, we obtain $B(x_0, 3/8) \subset B_{3\tilde{R}+1} \setminus \overline{B_{\tilde{R}}}$ and the following inequalities

$$
\|v\|_{H^1(B(x_0, 1/4))} \leq C \|v\|_{L^2(B(x_0, 3/8))} \leq C \|v\|_{L^2(B_{3\tilde{R}+1} \setminus \overline{B_{\tilde{R}}})},
$$

(14)

where the first inequality is obtained by taking $v = \chi^2 u$ in the variational formulation (3), where $\chi \in C^\infty_0(B(x_0, 3/8))$ is a positive function and $\chi = 1$ in $B(x_0, 1/4)$. In addition, from the continuous embedding of $H^2(\Xi_R)$ into the space of continuous functions for $d \leq 3$ and the uniform bound (10), we have the following $L^\infty$ estimate:

$$
\|v\|_{L^\infty(\Xi_R)} \leq C_{K_i}.
$$
also \((10)\), and there exists \(0 < \beta_0 < 1\) such that if
\[
\beta := \|v\|_{L^2(S^{d-1})} \leq \beta_0,
\]
then
\[
\|v\|_{L^2(B_{\delta_0})} \leq C_{K_I} \beta^\eta(\beta)
\]
with
\[
\eta(\beta) := \frac{1}{1 + \ln(-\ln(\beta) + \epsilon)}.
\]
Plugging the last inequality into (14) and then using (13), one obtains, for small \(\epsilon\),
\[
\|v\|_{H^1(B_{\delta_0})(B(\xi_I, r_I))} \leq C_{K_I}(e^{c/\epsilon} \beta^{\eta(\beta)} + \epsilon^c)
\]
and the constants \(c, C_{K_I}, \delta_0\) do not depend on \(\lambda\) and \(\mu\). Now by using the optimization technique of [5, corollary 2.1], which is applicable since \(\|v\|_{H^1(B_{\delta_0}(B(\xi_I, r_I)))} \leq C_{K_I}\), it follows from (15) that there exist \(C_{K_I}, \delta_0 > 0\) such that for \(\beta^{\eta(\beta)} \leq \delta_0\),
\[
\|v\|_{H^1(B_{\delta_0}(B(\xi_I, r_I)))} \leq \frac{C_{K_I}}{(\ln(C_{K_I}/(\beta^{\eta(\beta)})))^{\kappa}}.
\]
But \(\beta \mapsto \beta^{\eta(\beta)}\) is an increasing function on \((0, 1)\); hence, there exists \(\beta_1 \leq \beta_0\) such that inequality (16) is satisfied for all \(\beta \leq \beta_1\). By using (12), there exist \(\kappa, C_{K_I}, \beta_1 > 0\) such that for \(\beta \leq \beta_1\),
\[
\left\|\mu_I^1 \Delta_\partial u + \frac{\partial v}{\partial \nu} + \lambda v\right\|_{L^2(\gamma_I)} \leq \frac{C_{K_I}}{(\ln(C_{K_I}/(\beta^{\eta(\beta)})))^{\kappa/8}} := f(\beta).
\]
(ii) Lower bound for the right-hand side of (11). Let us denote \(c_{K_I}^i = \inf_{(\lambda, \mu) \in \Delta} \|\Delta_\partial u\|_{L^2(\gamma_I)}\), where \(u\) is solution to \(\mathcal{P}(\lambda, \mu, \partial D)\). Assume that \(c_{K_I}^i = 0\). The mapping \((\lambda, \mu) \in K_I \mapsto \|\Delta_\partial u\|_{L^2(\gamma_I)}\) is continuous from proposition 2.1 and \(K_I\) is a compact subset of \((L^\infty(\partial D))^2\). Then \(c_{K_I}^i\) is reached for some \((\lambda^0, \mu^0) \in K_I\). The corresponding solution \(u^0\) satisfies \(\Delta_\partial u^0 = 0\) on \(\gamma_I\) which is impossible using the same argument as in the uniqueness proof (proposition 2.4). Therefore \(c_{K_I} > 0\), and from (11) and (17), we obtain for \(c_{K_I} := \min_{i=1, \ldots, I} c_{K_I}^i > 0\)
\[
\frac{\|\mu^1 - \mu^2\|_{L^\infty(\partial D)}}{\|\mu^1 - \mu^2\|_{L^\infty(\partial D)}} c_{K_I} \leq f(\beta)
\]
for \(\beta \leq \beta_1\). By taking the max over \(i\), one obtains
\[
c_{K_I} \leq f(\beta),
\]
and since \(f\) is an increasing function, there exists \(\beta_2 > 0\) which is independent of \(\lambda\) and \(\mu\) such that if \(\beta \leq \beta_1\), then \(\beta \geq \beta_2\). Finally, we have
\[
\min(\beta_1, \beta_2) \leq \beta \leq \frac{\|\mu^1 - \mu^2\|_{L^\infty(\partial D)}}{\|\mu^1 - \mu^2\|_{L^\infty(\partial D)}} = \frac{\|u^1\|_{L^{\infty}(\partial D)} - \|u^2\|_{L^{\infty}(\partial D)}}{\|\mu^1 - \mu^2\|_{L^\infty(\partial D)}}
\]
which is the desired result since \(\beta_1\) and \(\beta_2\) are independent of \(\lambda\) and \(\mu\). \(\square\)

Conversely, if one assumes that \(\mu\) is known, a slight adaptation of this last proof gives the following stability estimate for \(\lambda\).

Theorem 2.8. There exists a constant \(C_{K_I}\) such that for all \((\lambda^1, \mu)\) and \((\lambda^2, \mu)\) in \(K_I\),
\[
\|\lambda^1 - \lambda^2\|_{L^\infty(\partial D)} \leq C_{K_I}\|T(\lambda^1, \mu, \partial D) - T(\lambda^2, \mu, \partial D)\|_{L^{2}(S^{d-1})},
\]
Proof. The proof of this result follows the same lines as the proof of theorem 2.5. The details are left to the readers.

We expect that when $I \to +\infty$, the constant $C_k$ in theorems 2.5 and 2.8 grows exponentially with $I$, as proved for similar Lipschitz stability estimates in [33]. The proof would be based on results established in [13, 27]. Let us note that up to our knowledge, the problem of global stability when both coefficients $\lambda$ and $\mu$ are unknowns is still open. For instance, the technique used in the proof of theorem 2.5 cannot be applied in this case.

3. A stability estimate in the case of inexact knowledge of the obstacle

3.1. The forward and inverse problems for a perturbed obstacle

Here, we are interested in the stability of the reconstruction of $\lambda$ and $\mu$ when exact knowledge of the geometry $\partial D$ is not available. We assume that $D$ is of class $C^1$ and consider problem (2) with a 'perturbed' geometry $D_\varepsilon$ of $D$ such that one can find a function $\varepsilon \in (C^1(\mathbb{R}^d))^d$ which is compactly supported in $B_R$ and

$$\partial D_\varepsilon = \{x + \varepsilon(x) : x \in \partial D\}.$$ 

Whenever $\|\varepsilon\| < 1$, where $\|\cdot\|$ stands for the $(W^{1,\infty}(\mathbb{R}^d))^d$ norm

$$\|\cdot\| = \|\cdot\|_{L^\infty(\mathbb{R}^d)^d} + \|\nabla \cdot \|_{L^\infty(\mathbb{R}^d)^{d^2}}$$

the map $f_\varepsilon := I_d + \varepsilon$, where $I_d$ stands for the identity of $\mathbb{R}^d$, is a $C^1$-diffeomorphism of $\mathbb{R}^d$ (see chapter 5 of [20]).

The inverse problem. Assume that $\partial D_\varepsilon$ is known and that an approximation $\lambda_{\varepsilon}$ and $\mu_{\varepsilon}$ of the exact impedance coefficients is available, i.e. for some $\delta \geq 0,$

$$\|T(\lambda_{\varepsilon}, \mu_{\varepsilon}, \partial D_\varepsilon) - T(\lambda, \mu, \partial D)\|_{L^\infty(\mathbb{R}^d)} \leq \delta.$$ 

From this information and from the distance between $\partial D$ and $\partial D_\varepsilon$, can we have an estimate on the boundary coefficients? In other words, do we have

$$\|\lambda_{\varepsilon} \circ f_\varepsilon - \lambda\|_{L^\infty(\partial D)} + \|\mu_{\varepsilon} \circ f_\varepsilon - \mu\|_{L^\infty(\partial D)} \leq g(\delta, \varepsilon)$$

(18)
for some function $g$ such that $g(\delta, \varepsilon) \to 0$ as $\delta \to 0$ and $\varepsilon \to 0$? In order to prove such a result, we first need a continuity property of $T$ with respect to $\partial D$.

### 3.2. Continuity of the far field with respect to the obstacle

In the following, we assume that $\|\varepsilon\| < 1$. To prove the continuity of the far field with respect to the obstacle, we first establish this result for the scattered field. Define $\tilde{\lambda}_\varepsilon := \lambda \circ f_\varepsilon^{-1}$ and $\tilde{\mu}_\varepsilon := \mu \circ f_\varepsilon^{-1}$ as two elements of $L^\infty(\partial D_\varepsilon)$. To evaluate the distance between the solution $u$ of $P(\lambda, \mu, \partial D)$ and the solution $u_{\varepsilon}$ of $P(\tilde{\lambda}_\varepsilon, \tilde{\mu}_\varepsilon, \partial D_\varepsilon)$, we first transport $u_{\varepsilon}$ on the fixed domain $\Omega_R$ by using the $C^1$-diffeomorphism $f_\varepsilon$ between $\Omega_1 \setminus \overline{D_\varepsilon}$ and $\Omega_1 \setminus \overline{D}$. Define $\tilde{u}_{\varepsilon} := u_{\varepsilon} \circ f_\varepsilon$ where $u_{\varepsilon}$ is the solution of $P(\tilde{\lambda}_\varepsilon, \tilde{\mu}_\varepsilon, \partial D_\varepsilon)$; then the following estimate holds.

**Theorem 3.1.** Let $(\lambda, \mu)$ be in $K$. There exist two constants $\varepsilon_0 > 0$ and $C_K$ which depend only on $K$ such that for all $\|\varepsilon\| \leq \varepsilon_0$, we have

$$\|\tilde{u}_{\varepsilon} - u\|_{VR} \leq C_K \|\varepsilon\|.$$

We denote by $O(y)$ a $C^\infty(\mathbb{R})$ function such that $|O(y)| \leq C|y|$ for all $y \in \mathbb{R}$.

### Proof of theorem 3.1.

We recall the weak formulation of $P(\lambda, \mu, \partial D)$: find $u \in VR$ such that

$$a(u, v) = l(v) \quad \forall v \in VR.$$

Similarly, the weak formulation of $P(\tilde{\lambda}_\varepsilon, \tilde{\mu}_\varepsilon, \partial D_\varepsilon)$ is as follows: find $u_{\varepsilon} \in V_{\varepsilon} R := \{ v \in H^1(\Omega_1 \setminus \overline{D_\varepsilon}); v|_{\partial D_\varepsilon} \in H^1(\partial D_\varepsilon) \}$ such that

$$a_{\varepsilon}(u_{\varepsilon}, v_{\varepsilon}) = l(v_{\varepsilon}) \quad \forall v_{\varepsilon} \in V^{'\varepsilon},$$

where

$$a_{\varepsilon}(u_{\varepsilon}, v_{\varepsilon}) := \int_{\Omega_1 \setminus \overline{D_\varepsilon}} \nabla u_{\varepsilon} \cdot \nabla v_{\varepsilon} - k^2 u_{\varepsilon} v_{\varepsilon} \, dx + \int_{\partial D_\varepsilon} (\tilde{\mu}_\varepsilon \nabla u_{\varepsilon}, v_{\varepsilon}) \, ds_{\varepsilon} = (S_{\varepsilon} u_{\varepsilon}, v_{\varepsilon}).$$

We define a new bilinear form on $V_{\varepsilon} R$:

$$\tilde{a}_{\varepsilon}(u, v) := a_{\varepsilon}(u \circ f_\varepsilon^{-1}, v \circ f_\varepsilon^{-1}) \quad \forall u, v \in VR.$$

Since $f_\varepsilon(\partial B_R) = \partial B_R$, we have $l(v) = l(v \circ f_\varepsilon^{-1})$ and $\tilde{u}_{\varepsilon}$ is a solution of $P(\tilde{\lambda}_\varepsilon, \tilde{\mu}_\varepsilon, \partial D_\varepsilon)$; then the following estimate holds.

$$\|\tilde{u}_{\varepsilon} - u\|_{VR} \leq C_K \|\varepsilon\|.$$
In addition, for all \( v \in V_R, S_R v = S_R (v \circ f^{-1}_e) \), and as a consequence for all \( u, v \in V_R \), we have \( (S_R u, v) = (S_R (u \circ f^{-1}_e), v \circ f^{-1}_e) \). Using the change of variables formula for integrals (see chapter 5 of [20]), we have

\[
\begin{align*}
\tilde{a}_e(u, v) &= \int_\Omega \left( \nabla u \cdot P_e \cdot \nabla v - k^2 u \nabla v \right) \, dx + \int_{\partial D_e} \tilde{\mu}_e \left[ \nabla_{\partial D_e} (u \circ f^{-1}_e) \cdot \nabla_{\partial D_e} (v \circ f^{-1}_e) \right] \, ds_e \\
&\quad - \int_{\partial D} \lambda u \nabla J_e^0 \, ds - (S_R u, v),
\end{align*}
\]

where \( J_e := |\det(\nabla f_e)|, P_e := (\nabla f_e)^{-1} (\nabla f_e)^{-T} \) and \( J_e^0 := J_e \left| (\nabla f_e)^{-T} v \right| \) (for a matrix \( B, B^{-T} \) denotes the transpose of the inverse of \( B \)). Thanks to the Neumann series of [23]. Actually, as

\[
\begin{align*}
|(\nabla f_e)^{-1}(x)| &= \sum_{k=0}^{\infty} (-1)^k (\nabla \varepsilon(x))^k = (1 + O(\|\varepsilon\|)) I_d.
\end{align*}
\]

As a consequence, \( P_e(x) \) expands uniformly for \( x \in \partial D \) as

\[
P_e(x) = I_d (1 + O(\|\varepsilon\|))
\]

and we also have

\[
J_e(x) = 1 + O(\|\varepsilon\|) \quad \text{and} \quad J_e^0(x) = 1 + O(\|\varepsilon\|)
\]

since \( \det(I_d + \nabla \varepsilon(x)) = 1 + \text{div}(\varepsilon) + O(\|\varepsilon\|^2) \). Using all of these results and lemma 3.2, we infer for \( u \) and \( v \) two functions of \( V_R \):

\[
\begin{align*}
\int_\Omega \left( \nabla u \cdot P_e \cdot \nabla v - k^2 u \nabla v \right) \, dx &\leq C \|u\|_{V_R} \|v\|_{V_R} \|\varepsilon\|, \\
\int_{\partial D} \left[ \lambda u \nabla J_e^0 - \lambda u \nabla v \right] \, ds &\leq C \|\lambda\|_{L^\infty(\partial D)} \|u\|_{V_R} \|v\|_{V_R} \|\varepsilon\|, \\
\int_{\partial D} \left( \mu v \partial D_e (u \circ f^{-1}_e) \cdot \nabla_{\partial D} (v \circ f^{-1}_e) \right) \, ds_e &\quad - \int_{\partial D} \mu \nabla_{\partial D} u \cdot \nabla_{\partial D} v \, ds_e \\
&\leq C \|u\|_{V_R} \|v\|_{V_R} \|\mu\|_{L^\infty(\partial D)} \|\varepsilon\|,
\end{align*}
\]

where the constant \( C \) does not depend on \( \varepsilon \). One obtains for the bilinear forms \( \tilde{a}_e \) and \( a \)

\[
\begin{align*}
\tilde{a}_e(u, v) - a(u, v) &\leq C_K \|u\|_{V_R} \|v\|_{V_R} \|\varepsilon\|,
\end{align*}
\]

where \( C_K \) does not depend on \( \lambda, \mu \) and \( \varepsilon \). Thanks to the Riesz representation theorem, we uniquely define \( A_e \) from \( V_R \) into itself by

\[
(A_e, v, w)_{V_R} = \tilde{a}_e(v, w) \quad \forall \, v, w \in V_R.
\]

We recall the definition of the operator \( A \) of \( V_R \):

\[
(A, v, w)_{V_R} = a(v, w) \quad \forall \, v, w \in V_R,
\]

and \( F \) in \( V_R \) is defined by

\[
(F, w)_{V_R} = l(w) \quad \forall \, w \in V_R.
\]

Thanks to inequality (19), we have

\[
||| A_e - A ||| \leq C_K \| \varepsilon \|.
\]

To have information on the scattered field, we should have information on the inverse of the operators, and we will use once again the results on the Neumann series of [23]. Actually, as
soon as $|||A^{-1}(A_\varepsilon - A)||| \leq 1/2$ which is true when $\varepsilon_0 \leq 1/(2C_K^2)$ (see proposition 2.1), the inverse operator of $A_\varepsilon$ satisfies

$$|||A_\varepsilon^{-1}||| \leq \frac{|||A^{-1}|||}{1 - |||A^{-1}(A_\varepsilon - A)|||} \leq 2C_K.$$  

From the identity $\tilde{u}_\varepsilon - u = A_\varepsilon^{-1}(A - A_\varepsilon)u$, we deduce

$$||\tilde{u}_\varepsilon - u||_{v_\varepsilon} = ||A_\varepsilon^{-1}((A - A_\varepsilon)u)||_{v_\varepsilon} \leq |||A_\varepsilon^{-1}|||((A_\varepsilon - A)u)||_{v_\varepsilon} \leq 2C_K^2 \varepsilon ||u||_{v_\varepsilon} \leq 2C_K^2 \varepsilon ||u||_{F_\varepsilon},$$

where we again used proposition 2.1 for the last inequality. This provides the desired estimate. □

**Proof of lemma 3.2.** Let us consider three functions $u \in H^1(\partial D)$, $v \in H^1(\partial D)$ and $\mu \in L^\infty(\partial D)$ and let $x_0 \in \partial D$. There exists a function $\varphi$ of class $C^1$ and an open set $U \subset \mathbb{R}^{d-1}$ such that

$$\partial D \cap V = \{\varphi(\xi) : \xi \in U\},$$

where $V$ is a neighbourhood of $x_0$ and $\varphi(0) = x_0$ and such that

$$e_i := \frac{\partial \varphi}{\partial \xi_i}(0), \quad \text{for} \quad i = 1, \ldots, d - 1,$$

form a basis of the tangential plane to $\partial D$ at $x_0$. We can also use this parametrization to describe a surfacic neighbourhood of $x_0, \varepsilon = f_\varepsilon(x_0)$; similarly there exists a neighbourhood $V_\varepsilon$ of $x_0$ such that

$$\partial D_\varepsilon \cap V_\varepsilon = \{\varphi_\varepsilon(\xi) : \xi \in U\},$$

where $\varphi_\varepsilon := f_\varepsilon \circ \varphi$. We define the tangential vectors of $\partial D_\varepsilon$ at a point $x_{0,\varepsilon} = \varphi_\varepsilon(0)$ by

$$e_{\varepsilon,i} := \frac{\partial \varphi_\varepsilon}{\partial \xi_i}(0) \quad \text{for} \quad i = 1, \ldots, d - 1$$

and thanks to the chain rule

$$e_{\varepsilon,i} = (\nabla f_\varepsilon(x_0))e_i, \quad (20)$$

Remark that as $\nabla f_\varepsilon(x_0)$ is an invertible matrix ($|||\varepsilon||| < 1$), the family $e_{\varepsilon,i}$ is a basis of the tangent plane to $\partial D$ at $x_{0,\varepsilon}$. Finally, we define the covariant basis of the cotangent planes of $\partial D$ at a point $x_0$ and of $\partial D_\varepsilon$ at a point $x_{0,\varepsilon}$ by

$$e^i \cdot e_j = \delta^i_j \quad \text{and} \quad e^i_{\varepsilon} \cdot e^j_{\varepsilon} = \delta^i_j \quad \text{for} \quad i, j = 1, \ldots, d - 1.$$

Using this definition and (20), we have the relation

$$e^i_{\varepsilon} = (\nabla f_\varepsilon)^{-T}e^i, \quad i = 1, \ldots, d - 1.$$  

Finally in the covariant basis, the tangential gradient $\nabla_{\partial D}$ for $w \in H^1(\partial D)$ at a point $x_0$ is

$$\nabla_{\partial D} w(x_0) = \sum_{i=1}^{d-1} \frac{\partial \tilde{w}}{\partial \xi_i}(0) e^i,$$

where $\tilde{w} := w \circ \varphi$. Similarly, at a point $x_{0,\varepsilon}$ we have for $w_\varepsilon \in H^1(\partial D_\varepsilon)$

$$\nabla_{\partial D_\varepsilon} w_\varepsilon(x_{0,\varepsilon}) = \sum_{i=1}^{d-1} \frac{\partial \tilde{w}_\varepsilon}{\partial \xi_i}(0) e^i_{\varepsilon},$$
where \( \tilde{w}_\varepsilon := w_\varepsilon \circ \varphi_\varepsilon \). As a consequence, for \( w \in H^1(\partial D) \),
\[
\nabla_{\partial D}(w \circ f^{-1}_\varepsilon(x_0, \cdot)) = \sum_{i=1}^{d-1} \frac{\partial \tilde{w}}{\partial \xi_i}(0) e_i\varepsilon
= \sum_{i=1}^{d-1} \frac{\partial \tilde{w}}{\partial \xi_i}(0)(\nabla f_\varepsilon(x_0))^{-T} e'_i
= (\nabla f_\varepsilon(x_0))^{-T} \nabla_{\partial D} w(x_0)
\]
because \( w \circ f^{-1}_\varepsilon \circ \varphi_\varepsilon = \tilde{w} \). By this formula, we just proved that for all \( x_\varepsilon = f_\varepsilon(x) \), \( x \in \partial D \), we have
\[
\nabla_{\partial D}(w \circ f^{-1}_\varepsilon(x_\varepsilon)) = (\nabla f_\varepsilon(x))^{-T} \nabla_{\partial D} w(x)
\]  
for all \( w \in H^1(\partial D) \). For \( u, v \) and \( \mu \), change of variables in the boundary integral \( (x = f^{-1}_\varepsilon(x_\varepsilon)) \) gives
\[
\int_{\partial D} (\mu \circ f^{-1}_\varepsilon) \nabla_{\partial D}(u \circ f^{-1}_\varepsilon) \cdot \nabla_{\partial D}(v \circ f^{-1}_\varepsilon) \, dx_\varepsilon
= \int_{\partial D} \mu(x) [\nabla_{\partial D}(u \circ f^{-1}_\varepsilon)(f_\varepsilon(x))] \cdot [\nabla_{\partial D}(v \circ f^{-1}_\varepsilon)(f_\varepsilon(x))] J_{\varepsilon}^y \, dx,
\]
and thanks to relation (21),
\[
\int_{\partial D} (\mu \circ f^{-1}_\varepsilon) \nabla_{\partial D}(u \circ f^{-1}_\varepsilon) \cdot \nabla_{\partial D}(v \circ f^{-1}_\varepsilon) \, dx_\varepsilon
= \int_{\partial D} \mu(x) [\nabla_{\partial D}(u)(x)] [\nabla_{\partial D}(v)(x)]^{-1} [\nabla_{\partial D}(f_\varepsilon(x))] J_{\varepsilon}^y \, dx.
\]
Finally recalling that
\[(\nabla f_\varepsilon(x))^{-1} (\nabla f_\varepsilon(x))^{-T} = P_\varepsilon(x) = (1 + O(||\varepsilon||)) I_d \quad \text{and} \quad J_{\varepsilon}^y = 1 + O(||\varepsilon||),\]
we may write
\[
\int_{\partial D} (\mu \circ f^{-1}_\varepsilon) \nabla_{\partial D}(u \circ f^{-1}_\varepsilon) \cdot \nabla_{\partial D}(v \circ f^{-1}_\varepsilon) \, dx_\varepsilon = (1 + O(||\varepsilon||)) \int_{\partial D} \mu \nabla_{\partial D} u : \nabla_{\partial D} v \, dx,
\]
which is the desired result. \( \Box \)

**Corollary 3.3.** There exist two constants \( \varepsilon_0 > 0 \) and \( C_K \) such that
\[
\|T(\lambda \circ f^{-1}_\varepsilon, \mu \circ f^{-1}_\varepsilon, \partial D_\varepsilon) - T(\lambda, \mu, \partial D)\|_{L^2(S^{d-1})} \leq C_K ||\varepsilon||,
\]
for all \( (\lambda, \mu) \in K \) and \( ||\varepsilon|| \leq \varepsilon_0 \).

**Proof.** Let \( u_\varepsilon^\infty \) be the far field that corresponds to the obstacle \( D \) and \( u_\varepsilon^\infty \) be the one that corresponds to the obstacle \( D_\varepsilon \). We use the integral representation formula for the far field on \( \partial B_{R_\varepsilon} \), and as \( \tilde{u}_\varepsilon|_{\partial B_{R_\varepsilon}} = u_\varepsilon|_{\partial B_{R_\varepsilon}} \) we obtain
\[
\tilde{u}_\varepsilon^\infty(\hat{x}) = \int_{\partial B_{R_\varepsilon}} \tilde{u}_\varepsilon(y) \frac{\partial \Phi^\infty(y, \hat{x})}{\partial v(y)} - \frac{\partial \tilde{u}_\varepsilon(y)}{\partial v} \Phi^\infty(y, \hat{x}) \, ds(y).
\]
The exterior DtN map \( S_\varepsilon \) defined in section 2 is continuous from \( H^{1/2}(\partial B_{R_\varepsilon}) \) to \( H^{-1/2}(\partial B_{R_\varepsilon}) \), and as a consequence,
\[
\left\| \frac{\partial \tilde{u}_\varepsilon}{\partial v} \right\|_{H^{-1/2}(\partial B_{R_\varepsilon})} \leq C \left\| \tilde{u}_\varepsilon \right\|_{H^{1/2}(\partial B_{R_\varepsilon})},
\]
finally
\[ \|u^{\infty}(\lambda) - u^{\infty}(\tilde{\lambda})\|_{L^2(S^{e-1})} \leq C \|\delta\|_{H^{1/2}(\partial BR)}. \]
The trace is continuous from \( H^1(\Omega R) \) into \( H^{1/2}(\partial BR) \), so combining this last inequality with (3.1), one obtains the continuity result
\[ \|T(\lambda \circ f^{-1}_e, \mu \circ f^{-1}_e, \partial D_e) - T(\lambda, \mu, \partial D)\|_{L^2(S^{e-1})} \leq C_K \|\delta\|. \]

3.3 A stability estimate of type (18)

In order to prove a stability estimate of type (18), we first need to formulate a stability result for the case of an exact geometry. Following the stability results derived in section 2.3, we assume that there exists a compact \( K \subset (L^{\infty}(\partial D))^{2} \) such that for \((\tilde{\lambda}, \tilde{\mu}) \in K \), there exists a constant \( C(\lambda, \mu, K) \) which depends on \( \lambda, \mu \) and \( K \) such that for all \((\lambda, \mu) \in K \), we have
\[ \|\lambda - \tilde{\lambda}\|_{L^{\infty}(\partial D)} + \|\mu - \tilde{\mu}\|_{L^{\infty}(\partial D)} \leq C(\lambda, \mu, K)\|T(\lambda, \mu, \partial D) - T(\tilde{\lambda}, \tilde{\mu}, \partial D)\|_{L^2(S^{e-1})}. \] (22)
We also refer to [7, section 4] for examples of such compact \( K \).

**Theorem 3.4.** There exists a constant \( \varepsilon_0 \) which depends only on \( K \) such that for all \((\lambda, \mu) \in K \) there exists a constant \( C(\lambda, \mu, K) \) such that for all \( \|\varepsilon\| \leq \varepsilon_0 \) and for all \((\lambda \circ f_e, \mu \circ f_e) \in K \) that satisfy
\[ \|T(\lambda_e, \mu_e, \partial D_e) - T(\lambda, \mu, \partial D)\|_{L^2(S^{e-1})} \leq \delta \quad \text{for} \quad \delta > 0 \]
we have
\[ \|\lambda_e - \lambda\|_{L^{\infty}(\partial D)} + \|\mu_e - \mu\|_{L^{\infty}(\partial D)} \leq C(\lambda, \mu, K)(\delta + \|\varepsilon\|). \]

**Proof.** The idea of the proof is first to split the uniform continuity with respect to the obstacle and the stability with respect to the coefficients and secondly to use the stability estimate (22). We have
\[ \|T(\lambda_e \circ f_e, \mu_e \circ f_e, \partial D) - T(\lambda, \mu, \partial D)\|_{L^2(S^{e-1})} \leq \|T(\lambda_e, \mu_e, \partial D_e) - T(\lambda_e, \mu_e, \partial D)\|_{L^2(S^{e-1})} \]
but the hypothesis of the theorem tells us that
\[ \|T(\lambda_e, \mu_e, \partial D_e) - T(\lambda, \mu, \partial D)\|_{L^2(S^{e-1})} \leq \delta \]
and thanks to the continuity property of the far field with respect to the obstacle (see corollary 3.3), we have
\[ \|T(\lambda_e, \mu_e, \partial D_e) - T(\lambda_e \circ f, \mu_e \circ f, \partial D)\|_{L^2(S^{e-1})} \leq C_K \|\varepsilon\| \]
because \( \lambda_e \circ f_e \) and \( \mu_e \circ f_e \) belong to the compact set \( K \). Finally
\[ \|T(\lambda_e \circ f_e, \mu_e \circ f_e, \partial D) - T(\lambda, \mu, \partial D)\|_{L^2(S^{e-1})} \leq C(\delta + \|\varepsilon\|), \]
and the stability assumption (22) implies
\[ \|\lambda_e \circ f_e - \lambda\|_{L^{\infty}(\partial D)} + \|\mu_e \circ f_e - \mu\|_{L^{\infty}(\partial D)} \leq C(\lambda, \mu, K)\|T(\lambda_e \circ f_e, \mu_e \circ f_e, \partial D) - T(\lambda, \mu, \partial D)\|_{L^2(S^{e-1})} \]
which concludes the proof. \( \square \)

The local nature of this estimate depends only on the local stability result for the impedances. In [7] the reader can find examples of compact sets \( K \) for which the stability estimate (22) holds locally. In the case of a classic impedance boundary condition \((\mu = 0)\), global stability results of Sincich in [32] or of Labreuche in [24] can be used to obtain a constant \( C(\lambda, \mu, K) \) independent of \( \lambda \) and \( \mu \). Furthermore, theorems 2.5 and 2.8 also provide global stability results in the case where \( \mu \neq 0 \) and \( \lambda \) are piecewise constant functions.
4. A numerical inversion algorithm and experiments

This section is dedicated to the effective reconstruction of impedance functional coefficients \( \lambda_0 \) and \( \mu_0 \) from the observed far field \( u_{\infty obs} := T(\lambda_0, \mu_0, \partial D) \in L^2(S^{d-1}) \) associated with one or several given incident directions and a given obstacle (which is either exactly known or perturbed). In the simplest case of a single incident wave and an exact knowledge of the obstacle, we shall minimize the cost function

\[
F(\lambda, \mu) = \frac{1}{2} \| T(\lambda, \mu, \partial D) - u_{\infty obs} \|_{L^2(S^{d-1})}^2
\]

with respect to \( \lambda \) and \( \mu \) using a steepest descent method. To do so, we first compute the Fréchet derivative of \( F \).

**Theorem 4.1.** The function \( F \) is differentiable for \((\lambda, \mu) \in (L^\infty(\partial D))^2\) that satisfy the assumption \( H \), and its Fréchet derivative is given by

\[
d_F(\lambda, \mu)(h, l) = \text{Re} \langle G, \text{div}_{\partial D}(\lambda \nabla_{\partial D} u) + h u \rangle_{H^1(\partial D), H^{-1}(\partial D)},
\]

where

\[
\begin{align*}
\bullet & \quad u \text{ is the solution of the problem } \mathcal{P}(\lambda, \mu, \partial D), \\
\bullet & \quad G = G^1 + G^2 \text{ is the solution of } \mathcal{P}(\lambda, \mu, \partial D) \text{ with } u \text{ replaced by } \\
& \quad G^1(y) := \int_{S^{d-1}} \Phi_\infty(\hat{x}, \hat{y})(T(\lambda, \mu, \partial D) - u_{\infty obs}) \, d\hat{x}.
\end{align*}
\]

To derive such a theorem, we have to compute the Fréchet derivative of the far-field map \( T \) and hence to prove the following lemma.

**Lemma 4.2.** The far-field map \( T \) is Fréchet differentiable for \((\lambda, \mu) \in (L^\infty(\partial D))^2\) that satisfy the assumption \( H \) and its Fréchet derivative \( dT(\lambda, \mu) : (L^\infty(\partial D))^2 \to L^2(S^{d-1}) \) maps \((h, l)\) to \( v_{\infty h,l} \) such that

\[
v_{\infty h,l}(\hat{x}) := \langle p(\cdot, \hat{x}), \text{div}_{\partial D}(\mu \nabla_{\partial D} u) + h u \rangle_{H^1(\partial D), H^{-1}(\partial D)} \quad \forall \hat{x} \in S^{d-1},
\]

where \( u \) is the solution of problem (1) and \( p(\cdot, \hat{x}) \) is the solution of (1) in which \( u \) is replaced by \( \Phi_\infty(\cdot, \hat{x}) \).

**Proof.** Following the proof of proposition 6 in [7], we obtain that \( T \) is differentiable and that \( dT_{\lambda, \mu}(h, l) = v_{\infty h,l} \) where \( v_{\infty h,l} \) is the far field associated with the \( v_{h,l} \) solution of

\[
\begin{align*}
\Delta v_{h,l} + k^2 v_{h,l} &= 0 \quad \text{in } \Omega \\
\text{div}_{\partial D}(\mu \nabla_{\partial D} v_{h,l}) + \frac{\partial v_{h,l}}{\partial \nu} + \lambda v_{h,l} &= -\text{div}_{\partial D}(\mu \nabla_{\partial D} u) - h u \quad \text{on } \partial D \\
\lim_{k \to \infty} \int_{\partial \Omega} |\partial v_{h,l}/\partial r - i k v_{h,l}|^2 \, ds &= 0.
\end{align*}
\]

From (5), we have for all \( \hat{x} \in S^{d-1} \)

\[
v_{\infty h,l}(\hat{x}) = \int_{\partial D} \left( v_{h,l} \frac{\partial \Phi_\infty(\cdot, \hat{x})}{\partial v} - v_{h,l} \frac{\partial \Phi_\infty(\cdot, \hat{x})}{\partial v} \right) ds.
\]

Since on \( \partial D \)

\[
\frac{\partial v_{h,l}}{\partial v} = -\text{div}_{\partial D}(\mu \nabla_{\partial D} v_{h,l}) - \lambda v_{h,l} - \text{div}_{\partial D}(\mu \nabla_{\partial D} u) - h u,
\]
we obtain with integration by parts
\[
v_{h,l}^\infty(\hat{\xi}) = \left( v_{h,l}^s, \text{div}_D(\mu \nabla_D \Phi^\infty(., \hat{\xi})) + \frac{\partial \Phi^\infty(., \hat{\xi})}{\partial v} + \lambda \Phi^\infty(., \hat{\xi}) \right)_{H^1(\partial D), H^{-1}(\partial D)}
+ \langle \Phi^\infty(., \hat{\xi}), \text{div}_D(\nabla_D u) + hu \rangle_{H^1(\partial D), H^{-1}(\partial D)}.
\] (26)

We introduce the solution \( p(., \hat{\xi}) \) of (1) with \( u' = \Phi^\infty(., \hat{\xi}) \). The associated scattered field \( p^s(., \hat{\xi}) := p(., \hat{\xi}) - \Phi^\infty(., \hat{\xi}) \) satisfies on \( \partial D \)

\[
\text{div}_D(\mu \nabla_D p^s) + \frac{\partial p^s}{\partial v} + \lambda p^s = -\text{div}_D(\mu \nabla_D \Phi^\infty) - \frac{\partial \Phi^\infty}{\partial v} - \lambda \Phi^\infty.
\]

Since \( v_{h,l}^s \) and \( p^s \) are the radiating solutions of a scattering problem, the following identity holds:

\[
\int_{\partial D} \left( \frac{\partial p^s}{\partial v} v_{h,l}^s - p^s \frac{\partial v_{h,l}^s}{\partial v} \right) \, ds = 0.
\]

Using the boundary condition for \( p^s \) and \( v_{h,l}^s \), equation (26) becomes

\[
v_{h,l}^\infty(\hat{\xi}) = - \left( p^s(., \hat{\xi}), \text{div}_D(\mu \nabla_D v_{h,l}^s) + \frac{\partial v_{h,l}^s}{\partial v} + \lambda v_{h,l}^s \right)_{H^1(\partial D), H^{-1}(\partial D)}
+ \langle \Phi^\infty(., \hat{\xi}), \text{div}_D(\nabla_D u) + hu \rangle_{H^1(\partial D), H^{-1}(\partial D)}
= \langle p(., \hat{\xi}), \text{div}_D(\nabla_D u) + hu \rangle_{H^1(\partial D), H^{-1}(\partial D)},
\]

which completes the proof.

We are now in a position to prove theorem 4.1.

**Proof of theorem 4.1.** By composition of derivatives and using the Fubini theorem, we have for \((h, l) \in (L^\infty(\partial D))^2\)

\[
dF(\lambda, \mu) \cdot (h, l) = \text{Re} \left\{ \left( T(\lambda, \mu, \partial D) - u_{\text{obs}}^\infty, dT(\lambda, \mu) \cdot (h, l) \right)_{L^2(\Sigma^\infty)} \right\}
= \text{Re} \int_{\Sigma^\infty} \left\{ (T(\lambda, \mu, \partial D) - u_{\text{obs}}^\infty)(\xi) \langle p(y, \xi), A(u)(y) \rangle_{H^1(\partial D), H^{-1}(\partial D)} \right\} \, d\xi
= \text{Re} \left\langle G, A(u) \right\rangle_{H^1(\partial D), H^{-1}(\partial D)}
\]

with

\[
A(u)(y) = \text{div}_D(f(y) \nabla_D u(y)) + h(y)u(y).
\]

\[\square\]

### 4.1. Numerical algorithm

To minimize the cost function (23), we use a steepest descent method and compute the gradient of \( F \) with the help of theorem 4.1. We solve the direct problems using a finite element method (implemented with FreeFem++ [34]) applied to (2). We look for the imaginary part of a function \( \lambda \) with \( \text{Im}(\lambda) \geq 0 \) and the real part of a function \( \mu \) with \( \text{Re}(\mu)(x) \geq c > 0 \) for almost every \( x \in \partial D \) assuming that \( \text{Re}(\lambda) \) and \( \text{Im}(\mu) \) are known in order to satisfy the hypothesis presented in [7] for which uniqueness and local stability hold. Moreover, for the sake of simplicity, we choose these known parts of the impedances equal to zero. Let us give initial values \( \lambda_{\text{init}} \) and \( \mu_{\text{init}} \) in the same finite element space as the one used to solve the forward problem. We update these values at each time step \( n \) as follows:

\[
\lambda_{n+1} = \lambda_n - i \delta \lambda_n,
\]

where the descent direction \( \delta \lambda_n \) is taken proportional to \( d F(\lambda_n, \mu_n) \). Since the number of parameters for \( \lambda_n \) is in general high, a regularization of \( d F(\lambda_n, \mu_n) \) is needed. We choose
to use a $H^1(\partial D)$-regularization (see [3] for a similar regularization procedure) by taking the \(\delta\lambda_n \in H^1(\partial D)\) solution to
\[
\eta_1(\nabla_{\partial D}(\delta\lambda_n), \nabla_{\partial D}\varphi)_{L^2(\partial D)} + (\delta\lambda_n, \varphi)_{L^2(\partial D)} = \alpha_1 \, dF(\lambda_n, \mu_n) \cdot (i\varphi, 0)
\] for each \(\varphi\) in the finite element space and with \(\eta_1\) the regularization parameter and \(\alpha_1\) the descent coefficient for \(\lambda\). We apply a similar procedure for \(\mu\),
\[
\mu_{n+1} = \mu_n - \delta\mu_n,
\]
where \(\delta\mu_n\) solves
\[
\eta_2(\nabla_{\partial D}(\delta\mu_n), \nabla_{\partial D}\varphi)_{L^2(\partial D)} + (\delta\mu_n, \varphi)_{L^2(\partial D)} = \alpha_2 \, dF(\lambda_n, \mu_n) \cdot (0, \varphi).
\]
We take two different regularization parameters for \(\lambda\) and \(\mu\) since we observed that the algorithm has different sensitivities with respect to each coefficient. From the practical point of view, we choose large \(\eta_i\) at the first steps to quickly approximate the searched \(\lambda\) and \(\mu\), and then we decrease them during the algorithm in order to increase the precision of the reconstruction. In all computations (except for constant \(\lambda\) and \(\mu\)), the parameters \(\alpha_1\) and \(\alpha_2\) are chosen in such a way that the cost function decreases at each step. Finally, we update alternatively \(\lambda\) and \(\mu\) because the cost function is much more sensitive to \(\lambda\) than to \(\mu\) and, as a consequence, if we update both at each time step, we would have a poor reconstruction of \(\mu\). Concerning the stopping criterion, we stop the algorithm if the descent coefficients \(\alpha_1\) and \(\alpha_2\) are too small or if the number of iterations is larger than 100 (in every case, there was no significant improvement of the reconstruction after 80 iterations).

4.2. Numerical experiments

In this section, we will show some numerical reconstructions using synthetic data generated with the code FreeFem++ in two dimensions to illustrate the behaviour of our numerical method. First of all, we will see that the use of a single incident wave is not satisfactory, and we will quickly turn to the use of several incident waves. Then all the simulations will be performed with several incident waves and with limited aperture data. Remark that all the theoretical results still hold in this particular case (see remark 2.2). In all the simulations, the obstacle is an ellipse of semi-axes 0.4 and 0.3; its diameter is hence more or less equal to the wavelength \(2\pi/k\) when \(k = 9\).

Moreover, since we consider a modelling of physical properties, we rescale the equation on the boundary of the obstacle \(\partial D\) in order to deal with dimensionless coefficients \(\lambda\) and \(\mu\). The equation on \(\partial D\) becomes
\[
\text{div}_{\partial D}\left(\frac{\mu}{k} \nabla_{\partial D}u\right) + \frac{\partial u}{\partial v} + k\lambda u = 0.
\]
In all cases (except when we specify it), we simply reconstruct \(\mu\) taking \(\lambda = 0\) because the reconstruction of \(\lambda\) has been investigated for a long time (see [10] or more recently [11]). Finally, as we consider star-shaped obstacles, we can define the impedance functions as functions of the angle \(\theta\). In the following, we will represent \(\lambda\) and \(\mu\) with the help of such a parametrization. Other experiments have been carried out and can be found in [6].

4.2.1. A single incident wave with full aperture. In this section, we consider the exact framework of the theory developed at the beginning, namely we enlighten the obstacle with a single incident plane wave and measure the far field in all directions. As we expected, the reconstruction is quite good in the enlightened area but rather poor far away from such an area (see figure 2); that is why we will consider a framework with several incident waves.
4.2.2. Several incident waves and limited aperture. From now on, we suppose that we measure several far fields corresponding to several incident directions. Hence, we reproduce an experimental device that would rotate around the obstacle $D$. To be more specific, we denote $u^i(\cdot, d)$ the scattered field associated with the incident plane wave of direction $d$. Considering that we have $N$ incident directions $d_j$ and $N$ areas of observation, $S_j \subset S^1$ such that the angle associated with $S_j$ is $2\pi/N$, we construct a new cost function

$$F(\lambda, \mu) = \frac{1}{2} \sum_{j=1}^{N} \|T(\lambda, \mu, \partial D, d_j) - u_{\text{obs}}^\infty(\cdot, d_j)\|_{L^2(S_j)}^2.$$ 

To minimize $F$, we use the same technique as before but now the Herglotz incident wave $G^j$ is associated with the incident direction $d_j$ and is given by

$$G^j(y, d_j) = \int_{S_j} \Phi^\infty(y, \hat{x}) (T(\lambda, \mu, \partial D, d_j) - u_{\text{obs}}^\infty(\cdot, d_j)) \, d\hat{x}.$$ 

More precisely, for the next experiments, we send $N = 10$ incident waves uniformly distributed on the unit circle, and hence the $S_j$ are the portions of the unit circle of aperture $\pi/5$. In order to evaluate the convergence of the algorithm, we introduce the following relative cost function:

$$\text{Error} := \sqrt{\sum_{j=1}^{N} \|T(\lambda, \mu, d_j) - u_{\text{obs}}^\infty(\cdot, d_j)\|_{L^2(S_j)}^2 / \sum_{j=1}^{N} \|u_{\text{obs}}^\infty(\cdot, d_j)\|_{L^2(S_j)}^2}.$$ 

Moreover, we add some noise on the data to avoid ‘inverse crime’. Precisely we handle some noisy data $u_{\text{nosy}}^\infty(\cdot, d_j)$ such that

$$\frac{\|u_{\text{nosy}}^\infty(\cdot, d_j) - u_{\text{obs}}^\infty(\cdot, d_j)\|_{L^2(S_j)}}{\|u_{\text{obs}}^\infty(\cdot, d_j)\|_{L^2(S_j)}} = \sigma.$$ 

In the next experiments, we study the impact of the level of noise (1% and 5%) on the quality of the reconstruction. Error ($\sigma$) will denote the final error with amplitude of noise $\sigma$.

Influence of the wavelength and the regularization parameter. First of all we are interested in the influence of the wavelength on the accuracy of the results; the first two graphics on
Figure 3. Reconstruction of $\text{Re}(\mu_0) = 0.5(1 + \cos^2(\theta))$, $\mu_{\text{init}} = 0.7$, $\lambda = 0$, with no regularization procedure, the wave number $k = 2$ (top left) and $k = 24$ (top right); with a regularization procedure and $k = 24$ (bottom). (a) Error (1%) = 2%, error (5%) = 11%; (b) error (1%) = 2.8%, error (5%) = 11.5%; (c) error (1%) = 2.1%, error (5%) = 10.2%.

Figure 3 show how the algorithm behaves with respect to the wavelength. We can see that if we decrease the wavelength (figure 3(b)), the reconstructed impedance is very irregular, that is why in figure 3(c) we add some regularization to flatten the solution and then improve the reconstruction compared to figure 3(b).

The case of non-smooth functional coefficients. We are able to handle a non-smooth coefficient $\mu$ since $\mu$ is expressed as a linear combination of functions of the finite element space. We present our results in figure 4 for piecewise constant functions $\mu$. To have a good reconstruction of a piecewise constant function, we need a small wavelength. However, we have just seen before that too small wavelength generates instability due to the noise that contaminates data. That is why we use a two-step procedure. First we use a large wavelength equal to 0.7 ($k = 9$, on the left) to quickly find a good approximation of the coefficient. Secondly, to improve the result, we use a three times smaller wavelength ($k = 24$ on the right).
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Figure 4. Reconstruction of $\text{Re}(\mu_0) = 0.5 + 0.5X[-\pi/2,\pi/2]$; $\mu_{\text{init}} = 0.7$, $\lambda = 0$; the wave number is $k = 9$ on the left and $k = 24$ on the right. (a) Error (1%) = 2.9%, error (5%) = 11.4%; (b) error (1%) = 2.3%, error (5%) = 11%.

Table 1. Simultaneous reconstruction of the constants $\lambda$ and $\mu$ with the wave number $k = 9$; $\sigma = 1\%$ of noise.

| $\lambda_0$ | $\mu_0$ | $\lambda_{\text{init}}$ | $\mu_{\text{init}}$ | Reconstructed $\lambda$ | Reconstructed $\mu$ | Error on the far field |
|-------------|----------|--------------------------|----------------------|------------------------|---------------------|------------------------|
| i           | 1        | 0.5i                     | 0.5                  | i                      | 0.97                | 0.05%                  |
| i           | 0.2      | 0.5i                     | 0.1                  | 0.99i                  | 0.21                | 0.08%                  |
| i           | 5        | 0.5i                     | 2.5                  | 0.97i                  | 5.12                | 0.07%                  |

Hence, we combine the advantage of a large wavelength (low numerical cost) and a small wavelength (good precision on the reconstruction of the discontinuity).

Simultaneous search for $\lambda$ and $\mu$. We now study the simultaneous reconstruction of $\lambda$ and $\mu$. Table 1 represents the simultaneous reconstruction of the constants $\lambda$ and $\mu$, and we observe a good reconstruction in the constant case.

In figure 5, we show the reconstruction of functional impedance coefficients $\lambda$ and $\mu$. The reconstruction is quite good for 1% of noise and remains acceptable for 5% of noise.

Stability with respect to a perturbed geometry. To illustrate the stability result with respect to the obstacle stated in theorem 3.4, we construct numerically $u_{\text{obs}}^\infty(\cdot, d_j) = T(\lambda_0, \mu_0, \partial D, d_j)$ for a given obstacle $D$ and minimize the ‘perturbed’ cost function

$$F_\epsilon(\lambda, \mu) = \frac{1}{2} \sum_{j=0}^{N} \left\| T(\lambda, \mu, \partial D_\epsilon, d_j) - u_{\text{obs}}^\infty(\cdot, d_j) \right\|_{L^2(S_j)}^2$$

for a perturbed obstacle $D_\epsilon$ in order to retrieve $(\lambda_0, \mu_0)$ on $\partial D$. We first consider a perturbed obstacle $D_\epsilon$ which is our previous ellipse of the semi-axes 0.4 and 0.3, the exact obstacle $D$ being such an ellipse once perturbed with an oscillation of amplitude $\gamma$. More precisely, the obstacle $D$ is parametrized by

$$x(t) = 0.4(\cos t + \gamma \cos(20t)), \quad y(t) = 0.3(\sin t + \gamma \sin(20t)), \quad t \in [0, 2\pi].$$
In the following experiments, we evaluate the impact of $\gamma$ on the reconstruction of the coefficients. The corresponding results are represented in figure 6 for $\lambda = 0$ and piecewise constant $\mu$, for two amplitudes $\gamma$ of perturbation.

We now consider a second kind of perturbed obstacle, as indicated in figure 7. The perturbation of the obstacle is again denoted $\gamma$ and defined by

$$\gamma := \frac{\varepsilon_0}{\text{diam}(D)}.$$ 

Note that the perturbed obstacle is the convex hull of the non-convex obstacle $D$. To satisfy the assumptions of theorem 3.4, we have to check that we can find some $\lambda_\varepsilon$ and $\mu_\varepsilon$ such that

$$F_\varepsilon(\lambda_\varepsilon, \mu_\varepsilon) \leq \delta.$$ 

Figure 5. Wave number $k = 9$; limited aperture data with ten incident waves. Error (1%) = 2.7%, error (5%) = 9.8% reconstruction of $\text{Im}(\lambda_0) = 0.5(1 + \sin^2(\theta))$, $\lambda_\text{init} = 0.7i$ (on the left) and $\text{Re}(\mu_0) = 0.5(1 + \cos^2(\theta))$, $\mu_\text{init} = 0.7$ (on the right).

Figure 6. Perturbed ellipse, wave number $k = 9$, $\mu_\text{init} = 0.7$, $\lambda = 0$, and $\gamma = 1\%$ on the left and 3\% on the right. (a) Error (1%) = 3.2\%, error (5%) = 10.8\%; (b) error (1%) = 11\%, error (5%) = 15\%.
Figure 7. Exact (on the left) and perturbed (on the right) geometries. (This figure is in colour only in the electronic version)

Figure 8. Perturbed obstacle (see figure 7) and smooth function \( \mu \), wave number \( k = 9 \), \( \mu_{\text{init}} = 0.7 \), \( \lambda = 0 \), and \( \gamma = 1\% \) on the left and \( 3\% \) on the right. (a) Error (1\%) = 2.8\%, error (5\%) = 11\%; (b) error (1\%) = 8\%, error (5\%) = 10.8\%.

for a small \( \delta \). If we take the same uniformly distributed incident directions with \( N = 10 \) and \( k = 9 \) as before (the wavelength is more or less equal to the diameter of \( D \)), we have

\[
\frac{\sqrt{\sum_{j=0}^{N} \| u_{\text{obs}}(\cdot,d_j) \|_{L^2(S_j)}^2}}{F_\varepsilon(\lambda_0,\mu_0)} = \begin{cases} 
8\% & \text{if} \quad \gamma = 1\% \\
27\% & \text{if} \quad \gamma = 3\%. 
\end{cases}
\]

These levels of perturbation on the cost function are too high to hope a good reconstruction of the coefficients. It is reasonable to consider that we do not enlighten the obstacle in the direction of the non-convexity (since we have poor knowledge of such an area). Let us suppose for example that we still have ten incident waves but now the incident directions belong to \([-\pi/2, \pi/2]\]. We have the following relative errors with the actual impedances:

\[
\frac{\sqrt{\sum_{j=0}^{N} \| u_{\text{obs}}(\cdot,d_j) \|_{L^2(S_j)}^2}}{F_\varepsilon(\lambda_0,\mu_0)} = \begin{cases} 
3\% & \text{if} \quad \gamma = 1\% \\
9\% & \text{if} \quad \gamma = 3\%. 
\end{cases}
\]

In this case, we hope a good reconstruction of the impedance coefficients at least in the directions of incidence. The corresponding results are represented for \( \lambda = 0 \) and a smooth
function $\mu$ in figure 8, and for $\lambda = 0$ and piecewise constant $\mu$ in figure 9, for two amplitudes $\gamma$ of perturbation. We can see that reconstruction is good for $\gamma = 1\%$ even if we put $\sigma = 5\%$ of noise on the measurements. For $\gamma = 3\%$, the reconstruction remains quite good in the non-perturbed area and acceptable in the perturbed area.

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