Umbilical routes along geodesics and hypercycles in the hyperbolic space

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ABSTRACT

Given a geodesic line γ in the hyperbolic space \( \mathbb{H}^n \) we formulate a necessary and sufficient condition for a function along this geodesic which measure the mean curvature of totally umbilical leaves of a foliation orthogonal to \( \gamma \). Then we extend the result to \( \gamma \) being a hypercycle i.e. a geodesic on a hypersurface equidistant from the totally geodesic one.

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0. Introduction

In the geometric theory of foliations, a question on foliations with totally umbilical leaves comes just after that on totally geodesic foliations. The last one for compact or finite volume manifolds has definite and negative answer (see [8] for some history). In [13] Langevin and Walczak proved that on a closed manifold of constant non-zero curvature there are no totally umbilical foliations. For open manifolds there are geometrical classifications of totally geodesic foliations in the hyperbolic space by Ferus ([10]) and Browne ([4]).

The question on totally umbilical routes along curves in the real hyperbolic space \( \mathbb{H}^n \) was formulated in [8]. In [1] we announced a solution for geodesics, now extended to transversals which are horocycles or hypercycles. More general result was obtained by the author and Langevin – see Remark 4.3 for a mention.

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The most general result of the paper is Theorem 3.1 giving necessary and sufficient condition for a function along arc-length parametrized hypercycle to generate totally umbilical foliations. Namely, if the hypercycle has (constant) geodesic curvature $\cos \varphi$ then this condition states that the mean curvature of leaves $h$ starts with at least $-\sin \varphi$, ends with at most $\sin \varphi$ and between its modified hyperbolic arcus tangent is a $(\sin \varphi)$-Lipschitz function. The condition is more visible in case of a geodesic transversal (Theorem 2.2): $\mathsf{ath} \circ h$ is 1-Lipschitz. Respective inequalities for differentiable $h$ appear in Theorem 3.5 and Theorem 2.6. The last one formulates as $h' \geq h^2 - 1$ and shows that close to leaves of small mean curvature there is more room for change of $h$.

1. Umbilical hypersurfaces of the hyperbolic space

Umbilicity is a standard notion in Riemannian geometry and one of the easiest which is conformally invariant.

A point on a submanifold of a Riemannian manifold is called umbilical if all eigenvalues of the shape operator at this point are equal. In this case the mean curvature at the points is this common value (or its opposite, depending on orientation). Consequently, a submanifold is totally umbilical if consists only of umbilical points and a totally umbilical foliation on a Riemannian manifold is a foliation with all the leaves totally umbilical.

For the real $n$-dimensional hyperbolic space $\mathbb{H}^n$ consider its half-space model i.e. the set $\Pi^{n,+} = \{ x \in \mathbb{R}^n \mid x_n > 0 \}$ endowed with the Riemannian metric

$$ g(X,Y)_x = \frac{1}{x_n^2} \langle X, Y \rangle $$

where $\langle ., . \rangle$ denotes the standard Euclidean inner product.

The hyperbolic distance in the half-space is given by the formula (cf. [2])

$$ d(x,y) = 2 \mathsf{ath} \sqrt{\frac{||\hat{x} - \hat{y}||^2 + (x_n - y_n)^2}{||\hat{x} - \hat{y}||^2 + (x_n + y_n)^2}} $$

where $\hat{x} = (x_1, \ldots, x_{n-1})$ and analogously for $y$. Here

$$ \mathsf{ath} = (\tanh)^{-1} : t \mapsto \ln \frac{1 + t}{1 - t}. $$

In the particular case $\Pi^{2,+} \subset \mathbb{C}$,

$$ d(z,w) = 2 \mathsf{ath} \left| \frac{z - w}{z - \bar{w}} \right|, $$

especially $d(ai, bi) = \left|\ln \frac{a}{b}\right|$ for $a, b > 0$.

Every isometry of the half-space model is a conformal diffeomorphism $\Pi^{n,+}$ onto itself i.e. a composition of a horizontal translation, a dilation, an inversion in a sphere orthogonal to the ideal boundary or identity, and an orthogonal transformation in the first $n - 1$ variables (cf. [2]).

In particular, for any two geodesic lines there is an isometry sending one to another.

$\mathbb{H}^n$ is an Hadamard manifold so in the purely metric way (cf. [3]) we could define the ideal boundary and horospheres. In the half-space model, the ideal boundary is a topological $(n - 1)$ sphere $\left( \mathbb{R}^{n-1} \times \{0\} \right) \cup \{\infty\}$. A horosphere is a sphere tangent to $\mathbb{R}^{n-1} \times \{0\}$ (without tangency point) or a hyperplane parallel to it.
Totally geodesic hypersurfaces are open hemi-spheres or open half-hyperplanes orthogonal to \( \mathbb{R}^{n-1} \times \{0\} \). A connected component of a set equidistant from a totally geodesic hypersurface is called a hypersphere. In the half-space model, a hypersphere is a part of a sphere or hyperplane transversely intersecting \( \mathbb{R}^{n-1} \times \{0\} \) included in \( \Pi^{n,+} \). (See Fig. 1.)

A horosphere has natural orientation inside it. For a hypersphere \( S \) we induce its orientation from orientation of the corresponding totally geodesic hypersurface \( S_0 \) from which \( S \) is equidistant in such a way that a hyperball bounded by \( S \) shares the ideal boundary with the hyperbolic half-space bounded by \( S_0 \).

**Definition 1.1.** We use the common name generalized hypersphere for a complete hypersurface in \( \mathbb{H}^n \) which is either horosphere, hypersphere or a totally geodesic hypersurface and attach to such a generalized hypersphere its angle of intersection with the ideal boundary.

Thus a horosphere is 0-or-\( \pi \)-hypersphere (depending on its end) while a totally geodesic hypersurface is a \( \frac{\pi}{2} \)-hypersphere.

**Proposition 1.2.** Let a generalized hypersphere be oriented inside (i.e. “down” in the half-space model) and makes external angle \( \beta \) with the ideal boundary. Then its mean curvature is constant and equals \( -\cos \beta \).

A horosphere of end \( \infty \) is of constant mean curvature 1.

**Proof.** Using Christoffel symbols for a conformal change of Riemannian metric ([9]) one can calculate the second fundamental form for hyperplanes in \( \Pi^{n,+} \).

Assume that the hypersphere is represented by a part of hyperplane \( S \) making angle \( \beta \) with \( \mathbb{R}^{n-1} \times \{0\} \) and \( v \) its unit normal vector i.e. \( v = (v_1, \ldots, v_n) \perp S \) and \( \|v\| = 1 \). Then obviously \( v_n = \pm \cos \beta \).

Lużyńczyk in [14] observed that the shape operator of \( S \cap \Pi^{n,+} \) is simply \( v_n \cdot \text{Id} \) so with the given orientation \( h = -\cos \beta \).

**Proposition 1.3.** A connected complete unbounded hypersurface of \( \mathbb{H}^n \) is totally umbilical iff it is a generalized hypersphere.

**Proof.** It is classical (cf. [16]) that any totally umbilical hypersurface of \( \mathbb{R}^n \) is contained in a sphere or in a hyperplane.

The half-space model of \( \mathbb{H}^n \) is conformally equivalent to \( \mathbb{R}^n \). Conformal diffeomorphisms preserve umbilicity hence all connected complete totally umbilical hypersurfaces in the half-space model are nonempty intersections of \( \Pi^{n,+} \) by a sphere or a hyperplane.

Among them there are metric spheres which are bounded so any unbounded complete umbilical hypersurface is the cross-section of a sphere or hyperplane not disjoint with ideal boundary i.e. a generalized hypersphere. More detailed explanation can be found in [5].
Definition 1.4. A \( \varphi \)-hypercycle is a geodesic line on a \( \varphi \)-hypersphere, \( \varphi \in [0, \pi] \). In the half-space model a \( \varphi \)-hypercycle is a cross-section of a \( \varphi \)-hypersphere with a 2-dimensional plane through its center or simply open ray making angle \( \varphi \) with the ideal boundary.

Example 1.5. A generalized \( \varphi \)-hypercycle has constant geodesic curvature equal \( |\cos \varphi| \) (cf. [6]). In \( \mathbb{H}^{2+} \) hypercycles (at the same time hyperspheres, \( n = 2 \)) are

1. geodesic \( (\varphi = \frac{\pi}{2}) \) of ideal ends 0 and \( \infty \) being positive imaginary half-axis \( i\mathbb{R}_+ \) parametrized by arc-length as \( t \mapsto \text{ie}^{t} \);
2. \( \varphi \)-hypercycle \( E_{\varphi} = e^{it\varphi}i\mathbb{R}_+ \), \( \varphi \in (0, \frac{\pi}{2}) \), of ideal ends 0 and \( \infty \) parametrized by arc-length as \( t \mapsto e^{t \sin \varphi+ie} \);
3. horocycle \( (\varphi = \pi) \) with the ideal end \( \infty \) having arc-length parametrizations \( t \mapsto t + ia \) with \( a > 0 \).

2. Umbilical routes along geodesics

In this section we shall study totally umbilical foliations of \( \mathbb{H}^n \) orthogonal to a given geodesic line. We shall provide a Lipschitz-type condition (Theorem 2.2) and a differentiable condition (Theorem 2.6) for change of the mean curvature of leaves along the geodesic.

Definition 2.1. Let \( \gamma : \mathbb{R} \to \mathbb{H}^n \) be an arc-length parametrized curve. We say that a real function \( h \) is an umbilical route along \( \gamma \) if the family \((L_t)\) of generalized hyperspheres orthogonal to \( \gamma \) and having mean curvature \( h(t) \) at \( \gamma(t) \) could be extended to a totally umbilical foliation of \( \mathbb{H}^n \).

In codimension 1 the real hyperbolic space is the only carrying non-trivial umbilical routes.

In an open ball there is no foliation tangent to the boundary and totally umbilical complete hypersurfaces in other constant curvature spaces, \( \mathbb{R}^n \) and \( S^n \), are full spheres or hyperplanes. On the other hand, any two nonparallel hyperplanes in \( \mathbb{R}^n \) intersect. Thus in \( S^n \) there is no umbilical route and in \( \mathbb{R}^n \) the only one is constantly equal 0 along a straight line.

In nonconstant curvature even very regular symmetric spaces like the complex hyperbolic space \( \mathbb{C}H^n \) have no totally umbilical hypersurfaces.

We start with a very simple case of umbilical route along a geodesic where the situation is clear and formulae are predictable.

Theorem 2.2. Let \( F \) be a transversely \( C^0 \) codimension 1 totally umbilical foliation of \( \mathbb{H}^n \) orthogonal to an arc-length parametrized geodesic line \( \gamma \). If for any \( t \in \mathbb{R} \) the mean curvature of the leaf \( (\text{taken with orientation opposite to } \gamma) \) through \( \gamma(t) \) is \( h(t) \) then \( |h| \leq 1 \) and there are \( t_-, t_+ \in [-\infty, +\infty] \) such that

\[
\begin{align*}
(i) & \quad h|_{(-\infty, t_-]} \equiv -1, \\
(ii) & \quad \text{the function } (\text{ath } \circ h)|_{(t_-, t_+)} \text{ is } 1-\text{Lipschitz}, \\
(iii) & \quad h|_{[t_+, +\infty)} \equiv 1.
\end{align*}
\]

Conversely, if \( h : \mathbb{R} \to [-1, 1] \) is a continuous function satisfying (1) then \( h \) is an umbilical route along any geodesic line in \( \mathbb{H}^n \).

For the proof we need elementary lemmas.

Lemma 2.3. Let \( s > 0, \beta \in [0, \pi) \) and \( C \) be a circle of center \( C \in \mathbb{C} \) and radius \( R \) orthogonal to the imaginary axis \( i\mathbb{R} \) at point \( i \cdot s \) and making external angle \( \beta \) with the real axis.
Then
\[ R = \frac{s}{1 + \cos \beta}, \quad C = i \frac{s \cos \beta}{1 + \cos \beta} \]
and the point(s) of the intersection \( C \cap \mathbb{R} \) are of the form
\[ a_\pm = \pm s \tan \frac{\beta}{2}. \]

**Proof.** Since \( C \) is orthogonal to \( i \mathbb{R} \) its center \( C \in i \mathbb{R} \) and \( \Im C = s - R \) (Fig. 2). At a point \( a \in C \cap \mathbb{R} \) radius is perpendicular to the tangent. If \( \beta \) is acute (other cases are similar) then \( \angle 0 Ca = \pi - \beta \) and
\[ \frac{s - R}{R} = \sin \left( \frac{\pi}{2} - \beta \right) \]
which implies \( R = \frac{s}{1 + \cos \beta} \).

Thus we have \( C \) and
\[ a = \pm R \sin \beta = \pm s \frac{\sin \beta}{1 + \cos \beta} = \pm s \tan \frac{\beta}{2}. \]

**Lemma 2.4.** Let \( 0 < s_1 < s_2, \ \beta_1, \beta_2 \in [0, \pi) \) and \( C_1, C_2 \subset \mathbb{C} \) be circles orthogonal to the imaginary axis \( i \mathbb{R} \) at points \( is_1, is_2 \) and meeting the real axis \( \mathbb{R} \) at angles \( \beta_1, \beta_2 \), respectively.

Then \( C_1 \) and \( C_2 \) do not intersect in the upper half-plane \( \Pi^{2,+} \) iff
\[ \frac{\cot \beta_2}{\cot \frac{\pi}{2}} \leq \frac{s_2}{s_1} \]
provided that \( \beta_1, \beta_2 \in (0, \pi) \) or \( \beta_1 = \beta_2 = 0 \).

**Proof.** For a given \( C_1 \) which intersects \( \mathbb{R} \) transversally (\( \beta_1 > 0 \)) the only situation of \( C_1 \cap C_2 \cap \Pi^{2,+} = \emptyset \) is that \( a_{2-} \leq a_{1-} \) and \( a_{1+} \leq a_{2+} \). Hence we obtain the inequality by Lemma 2.3.

**Corollary 2.5.** Let \( S \) be a hypersphere which is of constant distance \( \delta \) from a totally geodesic hypersurface. If \( \beta \in (0, \frac{\pi}{2}) \) is such that the mean curvature of \( S \) is equal to \( -\cos \beta \) then \( \cos \beta = \tanh \delta \).

**Proof.** After a horizontal translation and a dilation (both are hyperbolic isometries) we may assume that \( S \) is contained in a sphere centered on the imaginary axis and intersects this axis at point \( is, \ s > 1 \), while
the corresponding totally geodesic hypersurface meets $i\mathbb{R}$ at the point $i$. Then by definition of hyperbolic distance in $\Pi^{n,+}$ we have $\delta = \ln s$.

Now we have equality in Lemma 2.4 and the second angle is $\frac{\pi}{2}$ so $e^\delta = \cot \frac{\beta}{2}$. Hence

$$
\delta = \ln \cot \frac{\beta}{2} = \ln \frac{1 + \cos \beta}{\sin \beta} = \ln \frac{1 + \cos \beta}{\sqrt{1 - \cos^2 \beta}} = \text{ath}(\cos \beta).
$$

In [11] we could find this formula in the equivalent form $\cot \beta = \sinh \delta$. □

Now we are prepared for

**Proof of Theorem 2.2.** A geodesic sphere in $\mathbb{H}^n$ cannot serve as a leaf of a codimension 1 foliation because its interior has nonzero Euler characteristic and carries no foliation tangent to the boundary. Thus the only possible leaves of totally umbilical foliations on $\mathbb{H}^n$ are generalized hyperspheres and in fact $|h| \leq 1$.

To prove (i) observe that if some leaf $L_{\gamma(t)}$ is a horosphere “centered” at the begin $\gamma(-\infty)$ then all preceding leaves must be horospheres of the same “centre” – there is no room for other ideal boundary. This argument works in proof of (iii) as well.

We use a conformal transformation of $\Pi^{n,+}$ which is then hyperbolic isometry to put the geodesic $\gamma$ as the $n$-th half-axis

$$
A_{n,+} = \{x_1 = \ldots = x_{n-1} = 0, x_n > 0\}
$$

oriented “up”. Any generalized sphere representing a generalized hypersphere orthogonal to $\gamma$ has its center on the $A_{n,+}$.

Consider section of $\Pi^{n,+}$ by any 2-dimensional plane $P$ containing the $n$-th axis and orthogonal to the ideal boundary. Then $P \cap \Pi^{n,+}$ is isometric to $\Pi^{2,+}$ and $\mathcal{F} \cap P$ is generalized hypercycle foliation orthogonal to $\gamma = P \cap A_{n,+}$.

In $\Pi^{2,+}$ we parametrize the geodesic $\gamma$ by arc-length, namely $\gamma(t) = ie^t$. Fix $t_1$ and for any $t_2 > t_1$ use criterion from Lemma 2.4 to have $L_{\gamma(t_1)} \cap L_{\gamma(t_2)} = \emptyset$. By Proposition 1.2 the mean curvature of $L_{\gamma(t_i)}$ equals $h_{(t_i)} = -\cos \beta(t_i), i = 1, 2$, thus

$$
\frac{e^{t_2}}{e^{t_1}} \geq \frac{\cot \frac{\beta(t_2)}{2}}{\cot \frac{\beta(t_1)}{2}} = \frac{1 - h(t_2)}{\sqrt{1 - (h(t_2))^2}} = \left(\frac{1 + h(t_2)}{1 - h(t_2)}\right)^{-1} = \frac{e^{-\text{ath}(h(t_2))}}{e^{-\text{ath}(h(t_1))}}
$$

which is exactly (ii).

Now assume that $h$ is continuous, bounded by 1 and satisfy (1). Conditions (i) and (iii) imply proper foliation inside the last of initial horospheres and the first of finishing ones. From (ii) we know that generalized hyperspheres of given mean curvature are pairwise disjoint.

Since every leaf divides $\mathbb{H}^n$ into two hyperbolic half-spaces and $h$ is continuous, there is a leaf through every point between initial and finishing horospheres and family of hyperspheres with given $h$ along $\gamma$ covers all the $\mathbb{H}^n$. □

If we assume that a foliation is transversely differentiable then the condition on umbilical route is even simpler.

**Theorem 2.6.** For function $h$ of mean curvature of leaves of a transversely $C^1$ totally umbilical codimension 1 foliation of $\mathbb{H}^n$ along arc-length parametrized geodesic (transversal orientation opposite to the geodesic) there are $t_-, t_+ \in [-\infty, +\infty]$ such that
Fig. 3. “Rising sun” foliation in $\mathbb{H}^2$ of common ends 0 and $\infty$.

\begin{align*}
(\text{i}) \quad h|_{(\infty,-t_-]} &\equiv -1, \\
(\text{ii}) \quad h' &\geq h^2 - 1, \\
(\text{iii}) \quad h|_{(t_+,+\infty]} &\equiv 1.
\end{align*}

Conversely, if $h : \mathbb{R} \to [-1, 1]$ is a $C^1$-function satisfying (2) then $h$ is an umbilical route along any geodesic line in $\mathbb{H}^n$.

**Proof.** For any $\varepsilon > 0$ (1)(ii) gives

$$-\frac{\text{ath}(h(t + \varepsilon)) - \text{ath}(h(t))}{\varepsilon} \leq 1$$

As limit in $\varepsilon \to 0$ we obtain

$$\frac{h'}{1 - h^2} \geq -1. \quad \square$$

**Remark 2.7.** The condition (2)(ii) on the derivative of $h$ could be reformulated in terms of intersection angle as

$$\beta' \geq -\sin \beta.$$

**Example 2.8.**

1. Totally geodesic foliation with $h \equiv 0$ is represented by concentric spheres.
2. Horospherical foliation with $h \equiv -1$ is represented by spheres tangent at one point.
3. A function $h = -\tanh$ is extremal for estimation (2)(ii). A foliation defined by $h$ is represented by spheres having an $(n - 2)$-dimensional sphere $\subseteq \mathbb{R}^{n-1} \times \{0\} \cup \{\infty\}$ in common. This is so-called *pencil of spheres* defined by linear combinations of equations of two intersecting spheres.

In dimension 2 such a foliation looks like “rising sun” (Fig. 3).

**Proposition 2.9.** For a given (even non-discrete) family of generalized hyperspheres pairwise disjoint and orthogonal to a given geodesic there is a totally umbilical foliation of whole $\mathbb{H}^n$ containing these hypercycles as leaves.

**Proof.** Let $\gamma$ be an arc-length parametrized geodesic line in $\mathbb{H}^n$ and $(L_{t_\alpha})_{\alpha \in A}$ be a family of generalized hyperspheres which are pairwise disjoint. Assume that $L_{t_\alpha}$ contains $\gamma(t_\alpha)$ and has mean curvature equal to $h(t_\alpha)$. 

We shall construct an umbilical route along $\gamma$. First add to the set $X = \{t_\alpha \mid \alpha \in A\}$ set $X'$ of its accumulation points. In $X'$ define values of $h$ as limits. Observe that function $h$ on $X \cup X'$ is such that $\text{ath} \circ h$ is 1-Lipschitz. This comes from disjointness of the family and Lemma 2.3.

The set $\mathbb{R} \setminus (X \cup X')$ is open so it is a union of disjoint open intervals. On every of its closure define $h$ as linear function. If the interval is infinite then $h$ is constant. Hence we obtain a continuous function $h: \mathbb{R} \to [-1,1]$ such that the function $\text{ath} \circ h$ is 1-Lipschitz.

In fact, if interval $[a,b]$ such that $(a, b) \subset \mathbb{R} \setminus (X \cup X')$ then we use convexity of $\text{ath}$ if $h(a), h(b) \geq 0$ (respectively, convexity of $-\text{ath}$ if $h(a), h(b) \leq 0$). Otherwise, there is $c \in [a, b]$ such that $h(c) = 0$ and the argument above works for intervals $[a, c]$ and $[c, b]$ by adding totally geodesic hypersurface as $L_c$ to the family.

By the Theorem 2.2 function $h$ is then umbilical route along $\gamma$. □

Remark 2.10. The above construction could be modified to obtain a smooth umbilical route if the family of hyperspheres is discrete.

On the other hand, omitting assumption on orthogonality to some geodesic line makes Proposition 2.9 false. It is enough to take two disjoint totally geodesic hypersurfaces and a third which is disjoint but not separating them.

3. Umbilical routes along hypercycles

Now we shall consider totally umbilical foliations orthogonal to a hypercycle. Similarly to the case of a geodesic we shall provide a Lipschitz-type and differential condition. They are something technical but in terms of intersection angle (Remark 3.6) inequality is easier to apply.

Theorem 3.1. Let $0 < \varphi < \frac{\pi}{2}$. Assume that $\mathcal{F}$ is a transversely $C^0$ codimension 1 totally umbilical foliation of $\mathbb{H}^n$ orthogonal to an arc-length parametrized $\varphi$-hypercycle $\gamma$. If for any $t \in \mathbb{R}$ the mean curvature of the leaf (taken with orientation opposite to $\gamma$) through $\gamma(t)$ is $h(t)$ then $|h| \leq \sin \varphi$ and there are $t_-, t_+ \in [-\infty, +\infty]$ such that

\[
\begin{align*}
(i) & \quad h|_{(-\infty, t_-]} \equiv -\sin \varphi, \\
(ii) & \quad t \mapsto \ln \frac{\sin \varphi - h(t)}{h(t) \cos \varphi + \sqrt{1 - (h(t))^2}} \text{ is a } (\sin \varphi)\text{-Lipschitz function on } (t_-, t_+), \\
(iii) & \quad h|_{[t_+, +\infty)} \equiv \sin \varphi.
\end{align*}
\] (3)

Conversely, if $h: \mathbb{R} \to [-\sin \varphi, \sin \varphi]$ is a continuous function satisfying (3) then $h$ is an umbilical route along any $\varphi$-hypercycle in $\mathbb{H}^n$.

We shall modify Lemmas 2.3 and 2.4. Denote by $E_\varphi$ the open ray $e^{i\varphi} \mathbb{R}^+ \subset \mathbb{C}$.

Lemma 3.2. Let $\varphi \in (0, \frac{\pi}{2})$, $s > 0$, $\beta \in [0, \pi)$, and $\beta \neq \frac{\pi}{2} + \varphi$. Assume that $\mathcal{C} \subset \mathbb{C}$ is a circle of center $\mathcal{C}$ and radius $R$ meeting orthogonally $E_\varphi$ at only one point $se^{i\varphi}$ and meeting the real axis $\mathbb{R}$ at angle $\beta$.

Then $\beta \in \left[\frac{\pi}{2} - \varphi, \frac{\pi}{2} + \varphi\right)$,

\[
R = \frac{s \sin \varphi}{\sin \varphi + \cos \beta}, \quad C = \frac{s \cos \varphi \cos \beta}{\sin \varphi + \cos \beta} + i \frac{s \sin \varphi \cos \beta}{\sin \varphi + \cos \beta}
\]

and the point(s) of the intersection $\mathcal{C} \cap \mathbb{R}$ are of the form

\[
a = \frac{s \cos(\varphi \pm \beta)}{\sin \varphi + \cos \beta}.
\]
Proof. Since $C \perp E_\varphi$, $C = ce^{i\varphi}$ for some $c \in \mathbb{R}$ (Fig. 4). Thus

$$R = |ce^{i\varphi} - se^{i\varphi}| = |s - c| = s - c$$

because if $c = s + R$ then $(s + 2R)e^{i\varphi}$ would be the second point of intersection $E_\varphi \cap C$. The top point of $C$ is

$$C + iR = (s - R)e^{i\varphi} + iR = (s - R)\cos \varphi + i(s \sin \varphi + (1 - \sin \varphi)R).$$

Shifting horizontally by the real part of $C$ to the left we reduce the situation to Lemma 2.3. Now

$$R = \frac{s \sin \varphi + (1 - \sin \varphi)R}{1 + \cos \beta}$$

hence

$$R = \frac{s \sin \varphi}{\sin \varphi + \cos \beta} \text{ and } C = (s - R)e^{i\varphi}$$

hence the real and imaginary part of $C$ are

$$\Re C = \frac{s \cos \varphi \cos \beta}{\sin \varphi + \cos \beta}, \quad \Im C = \frac{s \sin \varphi \cos \beta}{\sin \varphi + \cos \beta}$$

Moreover, $C$ intersects the real axis at points

$$\Re C \pm (\Im C + R) \tan \frac{\beta}{2} = \frac{s \cos \varphi \cos \beta}{\sin \varphi + \cos \beta}$$

$$\pm \left( \frac{s \sin \varphi \cos \beta}{\sin \varphi + \cos \beta} + \frac{s \sin \varphi}{\sin \varphi + \cos \beta} \right) \frac{\sin \beta}{1 + \cos \beta}$$

$$= \frac{s \cos(\varphi \pm \beta)}{\sin \varphi + \cos \beta}.$$

Observe that $C$ cuts $E_\varphi$ at only one point iff $|C| \leq R$ i.e. $|\cos \beta| \leq \sin \varphi$ so $\frac{\pi}{2} - \varphi \leq \beta < \frac{\pi}{2} + \varphi$. But this means that the left and right hand points of $C \cap \mathbb{R}$ are respectively
\[
a_+ = \frac{s \cos(\varphi - \beta)}{\sin \varphi + \cos \beta} > 0, \quad a_- = \frac{s \cos(\varphi + \beta)}{\sin \varphi + \cos \beta} \leq 0. \quad \square
\]

**Lemma 3.3.** Let \( \varphi \in (0, \frac{\pi}{2}) \), \( 0 < s_1 < s_2 \), and \( \beta_1, \beta_2 \in \left[ \frac{\pi}{2} - \varphi, \frac{\pi}{2} + \varphi \right) \). Assume that \( C_1, C_2 \subset \mathbb{C} \) are circles orthogonal to \( E_\varphi \) at points \( s_1 e^{i \varphi}, s_2 e^{i \varphi} \) and meeting the real axis \( \mathbb{R} \) at angles \( \beta_1, \beta_2 \), respectively.

Then \( C_1 \) and \( C_2 \) do not intersect in the upper half-plane \( \Pi^{2,+} \) iff

\[
\frac{s_1}{s_2} \leq \frac{\sin \varphi + \cos \beta_2}{\sin \varphi + \cos \beta_1} \leq \frac{\sin \varphi + \cos \beta_1}{\sin \varphi + \cos \beta_2}.
\]

**Proof.** Under these assumptions circles \( C_1 \) and \( C_2 \) do not intersect in \( \Pi^{2,+} \) iff \( C_1 \) is inside \( C_2 \) (including internal tangency) or they intersect on the side of \(-s_1 e^{i \varphi}\) not “too high” i.e. upper intersection point is still under or on \( \mathbb{R} \).

Since centers of \( C_1 \) and \( C_2 \) lie on the line spanned by \( E_\varphi \), points \( C_1 \cap C_2 \) are symmetric in this line. Thus if the left one is under the real axis then the same is true for the right one.

This implies that the second condition is equivalent to the request that the left hand point of \( C_1 \cap \mathbb{R} \) is to the right of the left hand point of \( C_2 \cap \mathbb{R} \). Now it is enough to use Lemma 3.2. \( \square \)

**Remark 3.4.** If \( \beta = \frac{\pi}{2} + \varphi \) the role of circle meeting \( \mathbb{R} \) at this angle plays a straight line and their unique common point is \( \frac{s}{\cos \beta} \).

If \( C_1 \) is such a line then \( C_2 \) has no room to bend and must a line parallel to \( C_1 \). Similarly, if \( C_2 \) is a line meeting \( \mathbb{R} \) at angle \( \frac{\pi}{2} - \varphi \) then \( C_1 \parallel C_2 \).

**Proof of Theorem 3.1.** As in the proof of Theorem 2.2 we conclude that the only possible leaves are generalized hypercycles. Every of them is diffeomorphic to \( \mathbb{R}^{n-1} \) and divides \( \mathbb{H}^n \) into two parts diffeomorphic to \( \mathbb{R}^n \). From the topological point of view the foliation is a product \( \mathbb{R}^{n-1} \times \mathbb{R} \) so any transversal meets any leaf at most once.

Like in Theorem 2.2 we reduce the situation to the dimension 2 with \( A_{n,+} \) being the geodesic from which the \( \varphi \)-hypercycle is equidistant. Lemma 3.2 implies \( |h| \leq \sin \varphi \). (i) and (iii) are explained in Remark 3.4.

To prove (ii) recall that \( E_\varphi \) has arc-length parametrization

\[
\gamma(t) = e^{t \sin \varphi + i \varphi}, \quad t \in \mathbb{R}
\]

and use Lemma 3.3. In fact, for given \( t_1 < t_2 \) we have

\[
(e^{t_2 - t_1}) \sin \varphi = e^{t_2} \sin \varphi \quad e^{t_1} \sin \varphi \geq \frac{\sin \varphi - h(t_2)}{-h(t_2) \cos \varphi - \sqrt{1 - (h(t_2))^2}} \sin \varphi
\]

\[
= \frac{\sin \varphi - h(t_1)}{-h(t_1) \cos \varphi - \sqrt{1 - (h(t_1))^2}} \sin \varphi
\]

Now applying logarithm to both sides we see that the function

\[
t \mapsto \ln \frac{\sin \varphi - h(t)}{h(t) \cos \varphi + \sqrt{1 - (h(t))^2}} \sin \varphi
\]

is \((\sin \varphi)\)-Lipschitz.

For the converse, argument from Theorem 2.2 works similarly but the foliation orthogonal to one hypercycle does not fill all the \( \mathbb{H}^n \). Anyway leaves of such a foliation have a limit (on both sides) which is an umbilical hypersurface. A domain bounded by a hypersphere can be easily foliated adding leaves of the same mean curvature (Fig. 5). \( \square \)
Differentiation of (3)(ii) leads to

**Theorem 3.5.** The function $h$ of mean curvature of leaves of totally umbilical transversally $C^1$ codimension 1 foliation of $\mathbb{H}^n$ along arc-length parametrized $\varphi$-hypercycle satisfies $|h| \leq \sin \varphi$ and there are $t_-, t_+ \in [-\infty, +\infty]$ such that

\begin{align*}
\text{(i)} & \quad h|_{(-\infty,t_-]} \equiv -\sin \varphi, \\
\text{(ii)} & \quad h'(t) \geq \frac{(h(t) - \sin \varphi)(\frac{h(t) \cos \varphi + \sqrt{1-(h(t))^2} \sin \varphi}{\sqrt{1-(h(t))^2} \cos \varphi})}{1-h \sin \varphi + \sqrt{1-(h(t))^2} \cos \varphi} \quad \text{on } (t_-, t_+), \\
\text{(iii)} & \quad h|_{[t_+ ,+\infty)} \equiv \sin \varphi. \quad (4)
\end{align*}

Conversely, if $h : \mathbb{R} \to [-\sin \varphi, \sin \varphi]$ is a $C^1$-function satisfying (4) then $h$ is an umbilical route along any $\varphi$-hypercycle in $\mathbb{H}^n$.

**Remark 3.6.** The condition from Theorem 3.5 looks shorter in terms of intersection angle

$$
\beta' \geq \frac{(\sin \varphi + \cos \beta) \cos(\varphi + \beta)}{1 + \sin(\varphi + \beta)}.
$$

Indeed, it is enough to reformulate inequality

$$
\left(\frac{\sin \varphi + \cos \beta}{-\cos(\varphi + \beta)}\right)' \leq \sin \varphi.
$$

Any horocycle could be transformed into a line parallel to $\mathbb{R}^{n-1} \times \{0\}$ by inversion centered at the end of the horocycle. Thus any sphere orthogonal to it intersects the horocycle in two points. The only generalized hyperspheres orthogonal to the horocycle and meeting it once are 0-hyperspheres represented by vertical hyperplanes. This motivates the following

**Corollary 3.7.** The only umbilical route along a horocycle is $h \equiv 0$.

**Example 3.8.**

1. A family of disjoint totally geodesic hypersurfaces orthogonal to a hypercycle at any of its point foliates whole the $\mathbb{H}^n$. 
2. Constant curvature foliation with \( h \equiv \sin \varphi \) is represented by parallel hyperplanes which are orthogonal to \((\cos \varphi)\)-hypercycles.

3. Mean curvature of leaves of a foliation by concentric spheres (with the center outside of \(\Pi^{n,+} \)) orthogonal to some \( \varphi \)-hypercycle varies over \((- \sin \varphi, \sin \varphi)\) along the hypercycle but on remaining domain of \(\mathbb{H}^n\) could include even horospheres.

4. Final remarks

4.1. In [10] Fernus classified totally geodesic codimension 1 foliations of \(\mathbb{H}^n\) as those with orthogonal transversal of geodesic curvature \(\leq 1\).

At any point of such a curve in \(\mathbb{H}^n\) one can find a generalized hypercycle in contact of order 2. If curvature of orthogonal transversal exceeded 1 then even totally geodesic leaves would intersect. In such sense, it suffices to consider only generalized hypercycles for local study of totally umbilical foliations.

4.2. If \(k_g(p)\) and \(k_n(p)\) denote respectively the norm of the second fundamental form of the leaf through \(p\) and geodesic curvature of an orthogonal transversal then \(k_g^2 + k_n^2 \leq 1\) in case of totally umbilical foliations along hypercycles.

This is a very special case of Hadamard foliations (cf. [7]) for which this estimation is probably also true.

4.3. Totally umbilical foliations of \(\mathbb{H}^n\) could be described in a purely conformal way. In fact, the ideal boundary and totally umbilical leaves are represented by generalized spheres and the mean curvature depends only on angle of intersection.

This motivates description of such objects in the space of spheres – de Sitter space which is quadric in the Lorentz space (a comprehensive study of this theory can be found in [12] and [15]). The author and Langevin gave a local classification based on boosted time cones in the de Sitter space and deduce some global facts on curvature of orthogonal transversals.

4.4. In the paper we restricted to hypercyclic orthogonal transversals as the most similar to totally umbilical higher-dimensional submanifolds.

We could define a bi-umbilical foliation as totally umbilical foliation with a/all orthogonal transversals being totally umbilical. The classification of bi-umbilical foliations on \(\mathbb{H}^n\) may be of some interest even for codimension 2 in \(\mathbb{H}^4\).

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