Path Integral Method for Proportional Double-Barrier Step Option Pricing

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(Dated: May 13, 2022)

Path integral method in quantum mechanics provides a new thinking for barrier option pricing. For proportional double-barrier step options, option price changing process is analogous to a particle moving in a finite symmetric square potential well. Using the approximate energy level formulas, the analytical expressions of pricing kernel and option price could be derived. Numerical results of option price as a function of underlying price, potential and exercise price are shown, which are consistent with the results given by mathematical method.

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I. INTRODUCTION

In 1973, Black and Scholes derived the analytical expression for fixed-volatility option price by solving stochastic differential equations [1]. Financial mathematics applied in derivative pricing has made great progress from then on [2, 3]. Recently, more complex options have emerged in financial engineering, which collectively called exotic options. A kind of exotic option is the one attached some conditions to an ordinary option, taking barrier options for example: when the underlying price touches this barrier, the option contract will be activated, which is called knock-in option; when the underlying price touches the barrier, the option contract is invalid, which is call knock-out option. Snyder had discussed down-and-out option in 1969 [4]. Baaquie et al discussed this kind of option by path integral method, and derived the corresponding analytical expression [5]. Similar to one-dimensional infinite square potential well in quantum mechanics, the analytical expression for double-knock-out option price was also derived [6], which is in accordance with the result derived by mathematical method [7]. In addition, path integral method has been applied to the research of interest rate derivative pricing [8, 9]. Early works investigating step options appear in [10, 11]. Studies of option pricing by path integral method can be found in [5, 6].

In this paper, we will discuss a kind of barrier option, which is called proportional double-barrier step option: when the underlying price touches and passes the barrier, the option contract is not invalid immediately, but the option is knocked out gradually [10]. We focus on the relation between double-barrier step option pricing and finite symmetric square well: the boundaries of the well are regarded as two barriers. When a particle moving ahead and going through the boundary, the wave function begins to decay exponentially, which is similar to a proportional double-barrier step option knocked out over time.

Our work is organized as follows. In Section 2, we will derive the approximate analytical expression for proportional double-barrier step call option pricing in path integral method. In Section 3, we show the numerical results for option price as a function of underlying price, exercise price and discounting time, respectively. We summarize our main results in Section 4. The pricing formulas for Black-Scholes model and the standard double-barrier option are reviewed in Appendix A and B [6].
II. PROPORTIONAL DOUBLE-BARRIER STEP OPTION PRICING

Making the following variable substitution

\[ S = e^x, \quad x \in (-\infty, +\infty) \]  \hspace{1cm} (1)

where \( S \) is the underlying price. The price changing of a proportional double-barrier step option could be analogous to a particle moving in a symmetric square potential well with the potential

\[ V(x) = \begin{cases} 
0, & a < x < b, \\
V_0, & x < a, \quad x > b.
\end{cases} \hspace{1cm} (2)\]

for \( x < a \) or \( x > b \), the wave function decays with the increasing distances from the well, which is similar to an option touches a barrier and knocks out gradually. The Hamiltonian for a double-barrier step option is [5]

\[ H_{PDBS} = -\frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2} + \left( \frac{1}{2} \sigma^2 - r \right) \frac{\partial}{\partial x} + r + V(x) \hspace{1cm} (3)\]

which is a non-Hermitian Hamiltonian. Considering the following transformation

\[ H_{PDBS} = e^{\alpha x} H_{\text{eff}} e^{-\alpha x} = e^{\alpha x} \left( -\frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2} + \gamma \right) e^{-\alpha x} + V(x) \hspace{1cm} (4)\]

where

\[ \alpha = \frac{1}{\sigma^2} \left( \frac{\sigma^2}{2} - r \right), \quad \gamma = \frac{1}{2\sigma^2} \left( \frac{\sigma^2}{2} + r \right)^2 \hspace{1cm} (5)\]

and \( H_{\text{eff}} \) is a Hermitian Hamiltonian which is considered as the symmetric square potential well Hamiltonian. The stationary state Schrödinger equation for option price is

\[ \begin{cases} 
-\frac{\sigma^2}{2} \frac{d^2C}{dx^2} + \gamma C = EC, & a < x < b, \\
-\frac{\sigma^2}{2} \frac{d^2C}{dx^2} + (\gamma + V_0) C = EC, & x < a, \quad x > b.
\end{cases} \hspace{1cm} (6)\]

where \( C \) is the option price, \( E \) is corresponding to bound state energy levels in the potential well. (6) could be simplified into

\[ \begin{cases} 
\frac{d^2C}{dx^2} + k_1^2 C = 0, & a < x < b, \\
\frac{d^2C}{dx^2} - k_2^2 C = 0, & x < a, \quad x > b.
\end{cases} \hspace{1cm} (7)\]
where
\[ k_1^2 = \frac{2(E - \gamma)}{\sigma^2}, \quad k_2^2 = \frac{2(V_0 + \gamma + E)}{\sigma^2} \]  
(8)

The general solution for (7) is
\[
C(x) = \begin{cases} 
A_3 e^{k_2(x - \frac{b + a}{2})}, & x \leq a, \\
A_1 \sin(k_1 x + \delta), & a < x \leq b, \\
A_2 e^{-k_2(x - \frac{b + a}{2})}, & x > b.
\end{cases}
\]  
(9)

Now we will use the method provided in [12] to derive the approximate energy level formulas for (9). Considering the continuity for both wave function and its derivative at \(x = a\) and \(x = b\), we have
\[
\delta = \frac{\ell \pi}{2} - k_1 \frac{b + a}{2}, \quad \ell = 0, 1, 2, ...
\]  
(10)

According to different \(\ell\)s in (10), (9) could be split into two parts
\[
C_1(x) = \begin{cases} 
-A_2 e^{k_2(x - \frac{b + a}{2})}, & x \leq a, \\
A_1 \sin k_1 \left(x - \frac{b + a}{2}\right), & a \leq x \leq b, \quad \text{for } \ell = 0, 2, 4, ... \\
A_2 e^{-k_2(x - \frac{b + a}{2})}, & x > b.
\end{cases}
\]  
(11)

and
\[
C_2(x) = \begin{cases} 
A_2 e^{k_2(x - \frac{b + a}{2})}, & x \leq a, \\
A_1 \cos k_1 \left(x - \frac{b + a}{2}\right), & a \leq x \leq b, \quad \text{for } \ell = 1, 3, 5, ... \\
A_2 e^{-k_2(x - \frac{b + a}{2})}, & x > b.
\end{cases}
\]  
(12)

where
\[
A_1 = \sqrt{\frac{2k_2}{k_2(b - a) + 2}}, \quad A_2 = A_1 \sin k_1 \frac{b - a}{2} e^{k_2 \frac{b - a}{2}}
\]  
(13)

here the normalization condition has been used. Considering boundary conditions for (11) and (12) at \(x = b\) respectively, we have
\[
\cot k_1 \frac{b - a}{2} = -\frac{k_2}{k_1}
\]  
(14)
\[
\tan k_1 \frac{b - a}{2} = \frac{k_2}{k_1}
\]  

(15) \hspace{2cm}

Let

\[
\theta = \arcsin \frac{k_1}{\beta} \in \left(0, \frac{\pi}{2}\right), \quad \beta = \sqrt{k_1^2 + k_2^2} = \frac{\sqrt{2V_0}}{\sigma}
\]  

(16) \hspace{2cm}

(14) and (15) could be combined into

\[
k_1 \frac{b - a}{2} = \frac{n\pi}{2} - \theta, \quad n = 1, 2, 3, ...\]

(17) \hspace{2cm}

allowing for \(\theta \in (0, \pi/2)\), the range of \(k_1n\) is

\[
\frac{(n - 1)\pi}{b - a} < k_1n < \frac{n\pi}{b - a}
\]  

(18) \hspace{2cm}

when \(n \to n_{\text{max}}\), the energy \(E_n \approx V_0\), and

\[
k_1n = \frac{2(E_n - \gamma)}{\sigma} \to \sqrt{\beta^2 - \frac{2\gamma}{\sigma^2}} \approx \frac{n_{\text{max}}\pi}{b - a}
\]  

(19) \hspace{2cm}

where \(n_{\text{max}}\) is the maximum number of energy levels, and

\[
n_{\text{max}} = \left[\frac{b - a}{\pi} \sqrt{\beta^2 - \frac{2\gamma}{\sigma^2}}\right]
\]  

(20) \hspace{2cm}

\([x]\) indicates the minimal integer not less than \(x\). In general, there is no analytical solution for energy eigenvalues. For low energy case \((E \ll V_0)\), considering only the first order approximation of (17),

\[
k_1n \frac{b - a}{2} = \frac{n\pi}{2} - \arcsin \frac{k_1n}{\beta} \approx \frac{n\pi}{2} - \frac{k_1n}{\beta}
\]  

(21) \hspace{2cm}

and the low energy level formula is

\[
k_1n \approx \frac{\beta n\pi}{\beta(b - a) + 2}
\]  

(22) \hspace{2cm}

the error of \(k_1n\) is

\[
\Delta k_1n \approx \frac{1}{6} \left(\frac{k_1n}{\beta}\right)^3 = \frac{n^3\pi^3}{6[\beta(b - a) + 2]^3}
\]  

(23) \hspace{2cm}

where \(O(k_1n^5)\) and higher orders have been ignored. The relative error for \(k_1n\) is

\[
\delta k_1n = \frac{\Delta k_1n}{k_1n} \approx \frac{n^2\pi^2}{6[\beta(b - a) + 2]^2}
\]  

(24) \hspace{2cm}
For high energy case \((E_n \approx V_0)\), the approximation of (17) is
\[
k_{1n}b - a \approx \frac{n\pi}{2} - \left[\frac{\pi}{2} - \sqrt{2 \left(1 - \frac{k_{1n}}{\beta}\right)}\right] = \frac{(n - 1)\pi}{2} + \sqrt{2 - \frac{2(n - 1)\pi}{\beta(b - a)}} \tag{25}
\]
where the Taylor expansion
\[
\arcsin(1 - x) \approx \frac{\pi}{2} - \sqrt{2x} - \frac{(\sqrt{2x})^3}{24} - ...
\tag{26}
\]
has been used. The error and relative error are
\[
\Delta k_{1n} = \frac{1}{12(b - a)} \left[2 - \frac{2(n - 1)\pi}{\beta(b - a)}\right]^{3/2}
\]
\[
\delta k_{1n} = \frac{1}{12(n - 1)\pi} \left[2 - \frac{2(n - 1)\pi}{\beta(b - a)}\right]^{3/2} \tag{27}
\]

Now we calculate the pricing kernel of proportional double-barrier step option. \(a\) and \(b\) in (2) could be considered as the lower and the upper barriers of the option. Let \(\tau_1\) indicates the occupation time between the lower barrier \(a\) and the upper barrier \(b\), and \(\tau_2\) is the occupation time below the lower barrier \(a\) and above the upper barrier \(b\). The pricing kernel is
\[
p_{\text{PDBS}}(x, x'; \tau) = \langle x | e^{-\tau_1 H_1 - \tau_2 H_2} | x' \rangle = \int_{-\infty}^{+\infty} dx'' \langle x | e^{-\tau_1 H_1} | x'' \rangle \langle x'' | e^{-\tau_2 H_2} | x' \rangle = e^{-\gamma} \int_{-\infty}^{+\infty} dx'' e^{\alpha(x-x'')} \sum_n e^{-\tau_1 E_1n - \tau_2 E_2n} \phi_n(x)\phi_n(x'')\phi_n(x') \tag{28}
\]
and the proportional double-barrier call price is
\[
C_{\text{PDBS}}(x; \tau) = \int_{\ln K}^{+\infty} dx' p_{\text{PDBS}}(x, x'; \tau)(e^{x'} - K) \tag{29}
\]
where
\[
H_1 = e^{\alpha x} \left(-\frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2} + \gamma\right) e^{-\alpha x}
\]
\[
H_2 = e^{\alpha x} \left(-\frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2} + \gamma\right) e^{-\alpha x} + V_0 \tag{30}
\]
$\phi_n(x)$ is the energy eigenstate in coordinate representation, $K$ is the exercise price, and $\tau = \tau_1 + \tau_2$ is the expiration time.

Considering different wave functions (11) and (12), and different energy level formulas (22) and (25), $k_{1n}$ would be divided into four cases

$$k_{111} = \frac{2m\beta\pi}{\beta(b-a) + 2}, \quad \text{for } E \ll V_0, \ C_1(x)$$

$$k_{112} = \frac{(2m-1)\beta\pi}{\beta(b-a) + 2}, \quad \text{for } E \ll V_0, \ C_2(x)$$

$$k_{121} = \frac{2}{b-a} \left[ \frac{(2m-1)\pi}{2} + \sqrt{2 - \frac{2(2m-1)\pi}{\beta(b-a)}} \right], \quad \text{for } E \approx V_0, \ C_1(x)$$

$$k_{122} = \frac{2}{b-a} \left[ (m-1)\pi + \sqrt{2 - \frac{4(m-1)\pi}{\beta(b-a)}} \right], \quad \text{for } E \approx V_0, \ C_2(x)$$

where $m = 1, 2, 3, \ldots$. The barriers $a$ and $b$ divide the integral interval into three parts: $(-\infty, a), (a, b), (b, +\infty)$.

Set $x \in (a, b)$, $\ln K \in (a, b)$, and the option price expression is calculated as

$$C_1(x; \tau) = e^{-\tau\gamma} \int_{\ln K}^{b} dx' e^{\alpha(x-x')} \int_{-\infty}^{a} dx'' \left[ \sum_{m=1}^{m_{111}} e^{-\frac{1}{2}\tau \sigma^2 k_{111}^2} \phi_{111}(x) \phi_{211}^2(x'') \right]_{x''<a} \phi_{111}(x') \left| \phi_{111}(x') \right|_{\ln K<x'<b}$$

$$+ \sum_{m=m_{111}+1}^{m_{121}} e^{-\frac{1}{2}\tau \sigma^2 k_{121}^2} \phi_{121}(x) \phi_{221}^2(x'') \left| \phi_{121}(x') \right|_{x''<a} \phi_{121}(x') \left| \phi_{121}(x') \right|_{\ln K<x'<b} +$$

$$+ \sum_{m=1}^{m_{121}} e^{-\frac{1}{2}\tau \sigma^2 k_{122}^2} \phi_{122}(x) \phi_{222}^2(x'') \left| \phi_{122}(x') \right|_{x''<a} \phi_{122}(x') \left| \phi_{122}(x') \right|_{\ln K<x'<b} +$$

$$+ \sum_{m=m_{211}+1}^{m_{222}} e^{-\frac{1}{2}\tau \sigma^2 k_{222}^2} \phi_{121}(x) \phi_{221}^2(x'') \left| \phi_{122}(x') \right|_{x''<a} \phi_{122}(x') \left| \phi_{122}(x') \right|_{\ln K<x'<b} (e^{x'} - K)$$
\[ C_2(x; \tau) = e^{\gamma} \int_{b}^{+\infty} dx' e^{a(x-x')} \int_{-\infty}^{a} dx'' \left[ \sum_{m=1}^{m_{\text{max}1}} e^{-\frac{1}{2}\sigma^2k_1^2} \phi_{111}(x) \phi_{211}(x') \right] \phi_{211}(x') \bigg|_{x'' < a} \bigg|_{x' > b} + \sum_{m=m_{\text{max}1}+1}^{m_{\text{max}2}} e^{-\frac{1}{2}\sigma^2k_1^2} \phi_{121}(x) \phi_{221}(x') \bigg|_{x'' < a} \bigg|_{x' > b} + \sum_{m=1}^{m_{\text{max}2}} e^{-\frac{1}{2}\sigma^2k_1^2} \phi_{112}(x) \phi_{212}(x') \bigg|_{x'' < a} \bigg|_{x' > b} + \sum_{m=m_{\text{max}2}+1}^{m_{\text{max}2}} e^{-\frac{1}{2}\sigma^2k_1^2} \phi_{122}(x) \phi_{222}(x') \bigg|_{x'' < a} \bigg|_{x' > b} \right] (e^{x'} - K) \]

\[ C_3(x; \tau) = e^{\gamma} \int_{\ln K}^{b} dx' e^{a(x-x')} \int_{a}^{\ln K} dx'' \left[ \sum_{m=1}^{m_{\text{max}1}} e^{-\frac{1}{2}\sigma^2k_1^2} \phi_{111}(x) \phi_{211}(x') \right] \phi_{111}(x') \bigg|_{a < x'' < b} \bigg|_{\ln K < x' < b} + \sum_{m=m_{\text{max}1}+1}^{m_{\text{max}2}} e^{-\frac{1}{2}\sigma^2k_1^2} \phi_{121}(x) \phi_{221}(x') \bigg|_{a < x'' < b} \bigg|_{\ln K < x' < b} + \sum_{m=1}^{m_{\text{max}2}} e^{-\frac{1}{2}\sigma^2k_1^2} \phi_{112}(x) \phi_{212}(x') \bigg|_{a < x'' < b} \bigg|_{\ln K < x' < b} + \sum_{m=m_{\text{max}2}+1}^{m_{\text{max}2}} e^{-\frac{1}{2}\sigma^2k_1^2} \phi_{122}(x) \phi_{222}(x') \bigg|_{a < x'' < b} \bigg|_{\ln K < x' < b} \right] (e^{x'} - K) \]

\[ C_4(x; \tau) = e^{\gamma} \int_{b}^{+\infty} dx' e^{a(x-x')} \int_{a}^{b} dx'' \left[ \sum_{m=1}^{m_{\text{max}1}} e^{-\frac{1}{2}\sigma^2k_1^2} \phi_{111}(x) \phi_{211}(x') \right] \phi_{221}(x') \bigg|_{a < x'' < b} \bigg|_{x' > b} + \sum_{m=m_{\text{max}1}+1}^{m_{\text{max}2}} e^{-\frac{1}{2}\sigma^2k_1^2} \phi_{121}(x) \phi_{221}(x') \bigg|_{a < x'' < b} \bigg|_{x' > b} + \sum_{m=1}^{m_{\text{max}2}} e^{-\frac{1}{2}\sigma^2k_1^2} \phi_{112}(x) \phi_{212}(x') \bigg|_{a < x'' < b} \bigg|_{x' > b} + \sum_{m=m_{\text{max}2}+1}^{m_{\text{max}2}} e^{-\frac{1}{2}\sigma^2k_1^2} \phi_{122}(x) \phi_{222}(x') \bigg|_{a < x'' < b} \bigg|_{x' > b} \right] (e^{x'} - K) \]
\[ C_5(x; \tau) = e^{\gamma \int_b^{+\infty} dx' e^{\alpha(x-x')} \int_b^{+\infty} dx'' \left[ \sum_{m=1}^{m_{\text{max}1}} e^{-\frac{1}{2} \tau \sigma^2 k_{111}^2 \phi_{111}(x) \phi_{211}(x'')} \phi_{111}(x') \bigg|_{x'' > b} \right]_{\ln K < x' < b} + \sum_{m=m_{\text{max}1}+1}^{m_{\text{max}2}} e^{-\frac{1}{2} \tau \sigma^2 k_{121}^2 \phi_{121}(x) \phi_{221}(x'')} \phi_{121}(x') \bigg|_{x'' > b} \right]_{\ln K < x' < b} + \sum_{m=1}^{m_{\text{max}2}} e^{-\frac{1}{2} \tau \sigma^2 k_{112}^2 \phi_{112}(x) \phi_{212}(x'')} \phi_{112}(x') \bigg|_{x'' > b} \right]_{\ln K < x' < b} + \sum_{m=m_{\text{max}2}+1} \left[ e^{-\frac{1}{2} \tau \sigma^2 k_{122}^2 \phi_{122}(x) \phi_{222}(x'')} \phi_{122}(x') \bigg|_{x'' > b} \right]_{\ln K < x' < b} (e^{x'} - K) \]

\[ C_6(x; \tau) = e^{\gamma \int_b^{+\infty} dx' e^{\alpha(x-x')} \int_b^{+\infty} dx'' \left[ \sum_{m=1}^{m_{\text{max}1}} e^{-\frac{1}{2} \tau \sigma^2 k_{111}^2 \phi_{111}(x) \phi_{211}(x'')} \phi_{211}(x') \bigg|_{x'' > b} \right]_{x' > b} + \sum_{m=m_{\text{max}1}+1}^{m_{\text{max}2}} e^{-\frac{1}{2} \tau \sigma^2 k_{121}^2 \phi_{121}(x) \phi_{221}(x'')} \phi_{221}(x') \bigg|_{x'' > b} \right]_{x' > b} + \sum_{m=1}^{m_{\text{max}2}} e^{-\frac{1}{2} \tau \sigma^2 k_{112}^2 \phi_{112}(x) \phi_{212}(x'')} \phi_{212}(x') \bigg|_{x'' > b} \right]_{x' > b} + \sum_{m=m_{\text{max}2}+1} \left[ e^{-\frac{1}{2} \tau \sigma^2 k_{122}^2 \phi_{122}(x) \phi_{222}(x'')} \phi_{222}(x') \bigg|_{x'' > b} \right]_{x' > b} (e^{x'} - K) \]

the option price is

\[ C_{\text{PBS}}(x; \tau) = C_1(x; \tau) + C_2(x; \tau) + C_3(x; \tau) + C_4(x; \tau) + C_5(x; \tau) + C_6(x; \tau) \]

where

\[
\begin{align*}
\phi_{111}(x) &= A_{111} \sin \left[ k_{111} \left( x - \frac{b + a}{2} \right) \right] \\
\phi_{121}(x) &= A_{121} \sin \left[ k_{121} \left( x - \frac{b + a}{2} \right) \right] \\
\phi_{112}(x) &= A_{112} \cos \left[ k_{112} \left( x - \frac{b + a}{2} \right) \right] \\
\phi_{122}(x) &= A_{122} \cos \left[ k_{122} \left( x - \frac{b + a}{2} \right) \right]
\end{align*}
\]
respectively. We will use different energy formulas for different $m$

\[
\phi_{211}(x) = A_{211} e^{-k_{211}(x - \frac{b+a}{2})}, \quad \phi_{211}(x) = -A_{211} e^{k_{211}(x - \frac{b+a}{2})}
\]

\[
\phi_{221}(x) = A_{221} e^{-k_{221}(x - \frac{b+a}{2})}, \quad \phi_{221}(x) = -A_{221} e^{k_{221}(x - \frac{b+a}{2})}
\]

\[
\phi_{212}(x) = A_{212} e^{-k_{212}(x - \frac{b+a}{2})}, \quad \phi_{212}(x) = A_{212} e^{k_{212}(x - \frac{b+a}{2})}
\]

\[
\phi_{222}(x) = A_{222} e^{-k_{222}(x - \frac{b+a}{2})}, \quad \phi_{222}(x) = A_{222} e^{k_{222}(x - \frac{b+a}{2})}
\]

\[
k_{211} = \sqrt{\frac{2V_0}{\sigma^2} - k_{111}^2}, \quad k_{221} = \sqrt{\frac{2V_0}{\sigma^2} - k_{121}^2}
\]

\[
k_{212} = \sqrt{\frac{2V_0}{\sigma^2} - k_{112}^2}, \quad k_{222} = \sqrt{\frac{2V_0}{\sigma^2} - k_{122}^2}
\]

\[
A_{111} = \sqrt{\frac{2k_{211}}{k_{211}(b - a) + 2}}, \quad A_{112} = \sqrt{\frac{2k_{212}}{k_{212}(b - a) + 2}}
\]

\[
A_{121} = \sqrt{\frac{2k_{221}}{k_{221}(b - a) + 2}}, \quad A_{122} = \sqrt{\frac{2k_{222}}{k_{222}(b - a) + 2}}
\]

\[
A_{211} = A_{111} \sin \left( k_{111} \frac{b - a}{2} \right) e^{k_{211}\frac{b-a}{2}}, \quad A_{212} = A_{112} \cos \left( k_{112} \frac{b - a}{2} \right) e^{k_{212}\frac{b-a}{2}}
\]

\[
A_{221} = A_{121} \sin \left( k_{121} \frac{b - a}{2} \right) e^{k_{221}\frac{b-a}{2}}, \quad A_{222} = A_{122} \cos \left( k_{122} \frac{b - a}{2} \right) e^{k_{222}\frac{b-a}{2}}
\]

(41), (42) and (43) are derived from (13) and (16). $m_{max1}$ and $m_{max2}$ are corresponding to the maximum value of $m$ in high energy ($E \approx V_0$) formulas (32), $m_1$ and $m_2$ are corresponding to the maximum value of $m$ in low energy ($E \ll V_0$) formulas (31), respectively. The option price for $x < a$ and $x > b$ could be obtained similarly.

In Table. I, the relative errors for (31) and (32) are shown. For $a = \ln 90 = 4.5$, $b = \ln 130 = 4.867, V_0 = 55$, $\sigma = 0.3$ and $r = 0.05$, we have $\beta = 34.96$ and $n_{max} = 4$. It is shown that, with the increasing of $n$, the error in the second column increases while the error in the third column decreases. For low energy levels ($n \leq 3$), the error of (31) is smaller, and for high energy level ($n = 4$), the error of (32) is smaller. In Table. II, $m_{max1}$, $m_{max2}$ and $m_1$, $m_2$ for $V_0 = 55, 26, 13$ (or daily knock-out factors $d = 0.8, 0.9, 0.95$ [10]) are shown, respectively. We will use different energy formulas for different $m$ to give a more accurate pricing kernel.
| energy level $n$ | relative error for low energy formula(31) | relative error for high energy formula(32) |
|-----------------|------------------------------------------|------------------------------------------|
| $n = 1$         | $2.14 \times 10^{-4}$                    | $0.0833$                                 |
| $n = 2$         | $8.55 \times 10^{-4}$                    | $0.0276$                                 |
| $n = 3$         | $0.002$                                  | $0.01$                                   |
| $n = 4$         | $0.0034$                                 | $0.00296$                                |

TABLE I. relative errors for different energy levels at $V_0 = 55$. Parameters: $a = \ln90 = 4.5$, $b = \ln130 = 4.867$, $V_0 = 55$, $\sigma = 0.3$, $r = 0.05$.

| $V_0$ | $\beta$ | $n_{max}$ | $m_{max1}$ | $m_{max2}$ | $m_1$ | $m_2$ |
|-------|----------|-----------|------------|------------|-------|-------|
| 55    | 35       | 4         | 2          | -          | 1     | 2     |
| 26    | 24       | 2         | 1          | -          | -     | 1     |
| 13    | 17       | 1         | -          | -          | -     | 1     |

TABLE II. $m_{max1}$, $m_{max2}$ and $m_1$, $m_2$ for $V_0 = 55, 26, 12$, respectively. Parameters: $a = \ln90 = 4.5$, $b = \ln130 = 4.867$.

### III. NUMERICAL RESULTS

In Fig. 1, we show the double-barrier step call price as a function of underlying price. The black line is corresponding to the standard double-barrier call for comparison. It is shown that the option price decreases with the increasing of potential $V_0$. In the limit $V_0 \to \infty$, the option payoff tends to be the payoff of an standard double-barrier option. The results agree with the results given in [10].

In Fig. 2, we show the double-barrier step call price as a function of exercise price. In the limit $V_0 \to \infty$, the option payoff tends to be the payoff of an standard double-barrier option. For a fixed $V_0$, the option price decreases with the increasing of exercise price $K$.

In Fig. 3, we show the proportional double step call price as a function of potential $V_0$. The red line is corresponding to the result in [10], where the independent variable is the daily knock-out factor $d = e^{-V_0/250}$. 
FIG. 1. Double-barrier step call price against underlying price for different potentials. Parameters: $a = 4.5$, $b = 4.867$, $K = 100$, $r = 0.05$, $\sigma = 0.3$, $\tau = 1$.

The black line is corresponding to the case of $S = 110$, which gives the maximum option price for the same $V_0$.

In Fig. 4, we show the proportional double step call delta as a function of underlying price. The definition of delta is

$$\Delta = \frac{\partial C}{\partial S} = e^{-x} \frac{\partial C}{\partial x}$$  \hspace{1cm} (45)$$

and our result is consistent with the result in [10]. The black line is corresponding to the standard double-barrier call delta for comparison.
FIG. 2. Double-barrier step call price against exercise price for different potentials. Parameters: $a = 4.5$, $b = 4.867$, $x = 4.605$, $r = 0.05$, $\sigma = 0.3$, $\tau = 1$.

IV. CONCLUSION

Path-integral is an effective method linking option price changing to a particle moving in potential wells. Here we have studied pricing of the proportional double-barrier step call option, which could be analogous to a particle moving in a symmetric square potential well. We have presented option prices changing with the initial underlying prices, potential, and exercise prices, respectively. The numerical results are in accordance with the results using mathematical method in [10]. The pricing of other barrier options could be studied by defining appropriate potentials $V$. 
FIG. 3. Double-barrier step call price against potential for different initial underlying prices. Parameters: $a = 4.5$, $b = 4.867$, $K = 100$, $r = 0.05$, $\sigma = 0.3$, $\tau = 1$.

Appendix A: Path Integral Method for Black-Scholes Model Pricing

According to Ref [6], starting from Black-Scholes pricing formula, the price of European option can be derived by path integral method. The Black-Scholes formula is

$$\frac{\partial C}{\partial t} + rS \frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} = 0 \quad (A1)$$

where $C$ is European option price, $S$ is the underlying asset price, $\sigma$ is the fixed volatility, and $r$ is the interest rate. Let

$$S = e^x, \quad (-\infty < x < +\infty) \quad (A2)$$
FIG. 4. Proportional double-barrier step call delta against underlying price for different potentials. Parameters: $a = 4.5$, $b = 4.867$, $K = 100$, $x = 4.605$, $r = 0.05$, $\sigma = 0.3$, $\tau = 1$.

and (A1) can be denoted as

$$\frac{\partial C}{\partial t} = \left[-\frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2} + \left(\frac{1}{2} \sigma^2 - r\right) + r\right] C$$

(A3)

let

$$H_{BS} = -\frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2} + \left(\frac{1}{2} \sigma^2 - r\right) \frac{\partial}{\partial x} + r$$

(A4)

the Black-Scholes equation is written as

$$\frac{\partial C}{\partial t} = H_{BS} C$$

(A5)
Comparing (A5) to Schrödinger equation, we have

$$\sigma^2 \sim \frac{1}{m^2}, \quad C \sim \psi(x)$$  \hfill (A6)

where \( m \) is the particle mass, and \( \psi(x) \) is the wave function. The Black-Scholes Hamiltonian (A4) in momentum representation can be denoted as

$$H_{BS} = \frac{1}{2} \sigma^2 p^2 + i \left( \frac{1}{2} \sigma^2 - r \right) p + r$$  \hfill (A7)

where \( p = -i \frac{\partial}{\partial x} \). The pricing kernel is

$$\langle x | e^{-\tau H_{BS}} | x' \rangle = \int_{-\infty}^{+\infty} \frac{dp}{2\pi} \langle x | e^{-\tau H_{BS}} | p \rangle \langle p | x' \rangle$$

$$= e^{-\tau r} \int_{-\infty}^{+\infty} \frac{dp}{2\pi} e^{-\frac{1}{2} \tau \sigma^2 \left( p - \frac{x' - x_0}{\tau \sigma} \right)^2 - \frac{(x' - x_0)^2}{2 \tau \sigma^2}} $$

$$= \frac{1}{\sqrt{2\pi \tau \sigma^2}} e^{-\frac{1}{2} \tau \sigma^2 (x' - x_0)^2}$$  \hfill (A8)

where the completeness relation has been used, and

$$x_0 = x + \tau \left( r - \frac{\sigma^2}{2} \right)$$  \hfill (A9)

The European call option price can be denoted as

$$C(x, \tau) = e^{-\tau r} \int_{-\infty}^{+\infty} \frac{dx'}{\sqrt{2\pi \tau \sigma^2}} \left( e^{x'} - K \right)_{+} e^{-\frac{1}{2} \tau \sigma^2 (x' - x_0)^2}$$

$$= e^{-\tau r} \int_{\ln K - x_0}^{+\infty} \frac{dx'}{\sqrt{2\pi \tau \sigma^2}} \left( e^{x'} - K \right) e^{-\frac{1}{2} \tau \sigma^2 x'^2}$$  \hfill (A10)

$$= SN(d_+) - e^{-\tau r} KN(d_-)$$

where

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2} z^2} dz, \quad d_\pm = \frac{\ln \frac{S}{K} + \left( r \pm \frac{\sigma^2}{2} \right) \tau}{\sigma \sqrt{\tau}}$$  \hfill (A11)
Appendix B: Path Integral Method for the Standard Double-Barrier Option Pricing

The standard double-barrier option Hamiltonian is [6]

\[ H_{DB} = H_{BS} + V(x) \]
\[ = -\frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2} + \left( \frac{\sigma^2}{2} - r \right) \frac{\partial}{\partial x} + r + V(x) \]
\[ = e^{ax} \left( -\frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2} + \gamma \right) e^{-ax} + V(x) \]  

(B1)

where
\[ \alpha = \frac{1}{\sigma^2} \left( \frac{\sigma^2}{2} - r \right), \quad \gamma = \frac{1}{2\sigma^2} \left( \frac{\sigma^2}{2} + r \right)^2 \]  

(B2)

and the potential \( V(x) \) is
\[ V(x) = \begin{cases} \infty, & x \leq a, \\ 0, & a < x < b, \\ \infty, & x \geq b. \end{cases} \]  

(B3)

the corresponding eigenstate is
\[ \phi_n(x) = \begin{cases} \sqrt{\frac{n\pi}{b-a}} \sin[p_n(x-a)], & a < x < b, \\ 0, & x < a, x > b. \end{cases} \]  

(B4)

where
\[ p_n = \frac{n\pi}{b-a}, \quad E_n = \frac{1}{2}\sigma^2 p_n^2, \quad n = 1, 2, 3, \ldots \]  

(B5)

The pricing kernel is
\[ p_{DB}(x, x'; \tau) = \langle x|e^{-\tau H_{DB}}|x'\rangle \]
\[ = e^{a(x-x')} \langle x|e^{-\tau \left( -\frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2} + \gamma + V \right)}|x'\rangle \]
\[ = e^{-\tau \gamma} e^{a(x-x')} \sum_{n=1}^{+\infty} e^{-\frac{1}{2}\tau \sigma^2 p_n^2} \phi_n(x) \phi_n(x') \]  

(B6)
and the option price

\[ C_{DB}(x; \tau) = \int_{\ln K}^{b} dx' p_{DB}(x, x'; \tau)(e^{x'} - K) \]

\[ = \frac{2}{b - a} e^{-\gamma} \int_{\ln K}^{b} dx' e^{\alpha(x-x')} \sum_{n=1}^{\infty} e^{-\frac{1}{2} \sigma^2 \frac{n^2 \pi^2}{(b-a)^2}} \sin\left[ \frac{n \pi}{b - a} (x - a) \right] \sin\left[ \frac{n \pi}{b - a} (x' - a) \right] (e^{x'} - K) \]

(B7)

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