INEQUALITIES FOR THE ANGULAR DERIVATIVES OF CERTAIN CLASSES OF HOLOMORPHIC FUNCTIONS IN THE UNIT DISC

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Abstract. In this paper, a boundary version of the Schwarz lemma is investigated. We take into consideration a function $f(z) = z + c_{p+1}z^{p+1} + \cdots$ holomorphic in the unit disc and $|f(z)| < \lambda f(z) + (1 - \lambda)z - \alpha$ for $|z| < 1$, where $\frac{1}{2} < \alpha \leq \frac{1}{1+\lambda}$, $0 \leq \lambda < 1$. If we know the second and the third coefficient in the expansion of the function $f(z) = z + c_{p+1}z^{p+1} + c_{p+2}z^{p+2} + \cdots$, then we can obtain more general results on the angular derivatives of certain holomorphic function on the unit disc at boundary by taking into account $c_{p+1}$, $c_{p+2}$ and zeros of $f(z) - z$. We obtain a sharp lower bound of $|f'(b)|$ at the point $b$, where $|b| = 1$.

1. Introduction

Let $f$ be a holomorphic function in the unit disc $D = \{z : |z| < 1\}$, $f(0) = 0$ and $|f(z)| < 1$ for $|z| < 1$. In accordance with the classical Schwarz lemma, for any point $z$ in the disc $D$, we have $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$. Equality in these inequalities (in the first one, for $z \neq 0$) occurs only if $f(z) = \gamma z$, $|\gamma| = 1$ ([4], p. 329).

Let $f(z) = z + c_{p+1}z^{p+1} + c_{p+2}z^{p+2} + \cdots$, $p \in \mathbb{N}$ be a holomorphic function on $D$ and let $\frac{f(z)}{\lambda f(z) + (1 - \lambda)z - \alpha} < \alpha$ for $|z| < 1$, where $\frac{1}{2} < \alpha \leq \frac{1}{1+\lambda}$, $0 \leq \lambda < 1$.

Consider the functions

$$\varphi(z) = \frac{\vartheta(z) - \alpha}{\alpha},$$

where

$$\vartheta(z) = \frac{f(z)}{\lambda f(z) + (1 - \lambda)z}$$

Received October 8, 2013; Revised March 5, 2015.

2010 Mathematics Subject Classification. 30C80.

Key words and phrases. Schwarz lemma on the boundary, holomorphic function, angular derivative, Julia-Wolff-Lemma.

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and

\[ \psi(z) = \frac{\varphi'(z) - \varphi(0)}{1 - \varphi(0)\varphi(z)} \]

\(\varphi(z)\) and \(\psi(z)\) are holomorphic functions in the unit disc \(D\), \(|\psi(z)| < 1\) for \(|z| < 1\) and \(\psi(0) = 0\). Therefore, from the Schwarz lemma, we obtain

\[ |f(z)| \leq \frac{\alpha(1 - \lambda)|z|(1 + |z|^p)}{\alpha + (1 - \alpha)|z|^p - \alpha\lambda(1 + |z|^p)} \]

and

\[ |c_{p+1}| \leq \frac{2\alpha - 1}{\alpha(1 - \lambda)} \]

Equality is achieved in (1.1) (for some nonzero \(z \in D\)) or in (1.2) if and only if \(f(z)\) is the function of the form

\[ f(z) = \frac{a(1 - \lambda)z(1 + z^p e^{i\theta})}{\alpha(1 - \alpha)(1 + 2\alpha) e^{i\theta} - \alpha\lambda(1 + 2\alpha e^{i\theta})} \]

where \(\theta\) is a real number.

H. Unkelbach ([9]) and Robert Osserman ([7]) have given the inequalities which are called the boundary Schwarz lemma. They have first showed that

\[ |f'(b)| \geq \frac{2}{1 + |f'(0)|} \]

and

\[ |f'(b)| \geq 1 \]

under the assumption \(f(0) = 0\) where \(f\) is a holomorphic function mapping the unit disc into itself and \(b\) is a boundary point to which \(f\) extends continuously and \(|f(b)| = 1\). Moreover, equality in (1.3) holds if and only if \(f\) is of the form

\[ f(z) = ze^{i\theta} \frac{z - a}{1 - az} \]

where \(\theta \in \mathbb{R}\) and \(a \in D\) satisfies \(\arg a = \arg b\). Also, the equality in (1.4) holds if and only if \(f(z) = ze^{i\theta}, \theta \in \mathbb{R}\).

One does not need to assume that \(f\) extends continuously to \(b\). For example, if \(f\) has a radial limit \(f(b)\) at \(b\) with \(|f(b)| = 1\), and if \(f\) has a radial derivative at \(b\), then that derivative also satisfies the inequalities (1.3) and (1.4). Accordingly, using the Möbius transformation, they have generalized the inequality on the case of \(f(0) \neq 0\).

If, in addition, the function \(f\) has an angular limit \(f(b)\) at \(b \in \partial D\), \(|f(b)| = 1\), then by the Julia-Wolff lemma the angular derivative \(f'(b)\) exists and \(1 \leq |f'(b)| \leq \infty\) (see [8]).

Inequality (1.4) and its generalizations have important applications in geometric theory of functions (see, e.g., [4], [8]). Therefore, the interest to such type results is not vanished recently (see, e.g., [1], [2], [3], [5], [6] and references therein).
Vladimir N. Dubinin ([2]) has continued this line of research and has made a refinement on the boundary Schwarz lemma under the assumption that \( f(z) = c_p z^p + c_{p+1} z^{p+1} + \cdots \), with a zero set \( \{a_k\} \).

Some other types of the strengthening inequalities are obtained in (see [1], [6]). That is, in ([1]), we gave estimate below \(|f'(b)|\) according to the first nonzero Taylor coefficient of \( f \) about two zeros, namely \( z = 0 \) and \( z_0 \neq 0 \). In ([6]), we obtained such type results for other than above mentioned class. An other interpretation of the results in ([6]) is given in ([5]).

## 2. Main results

In this section, we can obtain more general results on the angular derivatives of certain holomorphic function on the unit disc at boundary by taking into account \( c_{p+1}, c_{p+2} \) and zeros of \( f(z) - z \) if we know the second and the third coefficient in the expansion of the function \( f(z) = z + c_{p+1} z^{p+1} + c_{p+2} z^{p+2} + \cdots \). We obtain a sharp lower bound of \(|f'(b)|\) at the point \( b \), where \(|b| = 1 \).

**Theorem 2.1.** Let \( f(z) = z + c_{p+1} z^{p+1} + c_{p+2} z^{p+2} + \cdots \), \( c_{p+1} \neq 0 \), \( p > 1 \) be a holomorphic function in the unit disc \( D \) and let \( \left| \frac{f(z)}{f(z) + (1-\lambda)z} - \alpha \right| < \alpha \) for \(|z| < 1 \), where \( \frac{1}{2} < \alpha \leq \frac{1}{1+\lambda} \), \( 0 \leq \lambda < 1 \). Further assume that, for some \( b \in \partial D \), \( f \) has an angular limit \( f(b) \) at \( b \), \( f(b) = 0 \). Then

\[
(1.5) \quad |f'(b)| \geq \frac{\alpha (1-\lambda)}{2\alpha - 1} \left[ p + \frac{2 \left( 1 - \frac{\alpha(1-\lambda)}{2\alpha - 1} |c_{p+1}| \right)^2}{1 - \left( \frac{\alpha(1-\lambda)}{2\alpha - 1} |c_{p+1}| \right)^2 + \frac{\alpha(1-\lambda)}{2\alpha - 1} |c_{p+2}|} \right].
\]

Moreover, the equality in (1.5) occurs for the function

\[
f(z) = \frac{\alpha (1-\lambda)}{\alpha - (1-\alpha) z^p - \alpha \lambda (1 - z^p)}.
\]

**Proof.** Consider the functions

\[
\psi(z) = \frac{\varphi(z) - \varphi(0)}{1 - \varphi(0)\varphi(z)}, \quad \omega(z) = z^p.
\]

\( \psi(z) \) and \( \omega(z) \) are holomorphic functions in \( D \), and \(|\psi(z)| < 1\), \(|\omega(z)| < 1\) for \(|z| < 1\). By the maximum principle for each \( z \in D \), we have the inequality

\[
|\psi(z)| \leq |\omega(z)|.
\]

Therefore, the absolute value of the holomorphic function

\[
h(z) = \frac{\psi(z)}{\omega(z)}
\]

in \( D \) is bounded by 1 in \( D \).

In particular, we have

\[
(1.6) \quad |h(0)| = \frac{\alpha (1-\lambda)}{2\alpha - 1} |c_{p+1}| \leq 1
\]
Moreover, we can show that

$$|h'(0)| = \frac{\alpha (1 - \lambda)}{2\alpha - 1} |c_{p+2}|.$$ 

Moreover, we can show that

$$\frac{b\psi'(b)}{\psi(b)} = |\psi'(b)| \geq |\omega'(b)| = \frac{b\omega'(b)}{\omega(b)}.$$ 

The function

$$T(z) = \frac{h(z) - h(0)}{1 - \overline{h(0)}h(z)}$$

is holomorphic in the unit disc $D$, $|T(z)| < 1$, $T(0) = 0$ and $|T(b)| = 1$ for $b \in \partial D$.

From (1.3), we obtain

$$\frac{2}{1 + |T'(0)|} \leq |T'(b)| = \frac{1 - |h(0)|^2}{|1 - \overline{h(0)}h(b)|} |h'(b)|$$

$$\leq \frac{1 + |h(0)|}{1 - |h(0)|} |h'(b)|$$

$$= \frac{1 + \frac{\alpha (1 - \lambda)}{2\alpha - 1} |c_{p+1}|}{1 - \frac{\alpha (1 - \lambda)}{2\alpha - 1} |c_{p+1}|} \{|\psi'(b)| - |\omega'(b)|\}$$

$$= \frac{1 + \frac{\alpha (1 - \lambda)}{2\alpha - 1} |c_{p+1}|}{1 - \frac{\alpha (1 - \lambda)}{2\alpha - 1} |c_{p+1}|} \left\{ \frac{1 - |\varphi(0)|^2}{|1 - \overline{\varphi(0)}\varphi(b)|^2} |\varphi'(b)| - p \right\}$$

$$= \frac{1 + \frac{\alpha (1 - \lambda)}{2\alpha - 1} |c_{p+1}|}{1 - \frac{\alpha (1 - \lambda)}{2\alpha - 1} |c_{p+1}|} \left\{ \frac{1 - \frac{1 - \alpha}{\alpha}}{(1 + \frac{1 - \alpha}{\alpha})^2} \frac{|\varphi'(b)|}{\alpha} - p \right\}$$

and

$$\frac{2}{1 + \frac{\alpha (1 - \lambda)}{2\alpha - 1} |c_{p+2}|} \leq \frac{1 + \frac{\alpha (1 - \lambda)}{2\alpha - 1} |c_{p+1}|}{1 - \frac{\alpha (1 - \lambda)}{2\alpha - 1} |c_{p+1}|} \left\{ \frac{2\alpha - 1}{\alpha (1 - \lambda)} |f'(b)| - p \right\}.$$ 

Therefore, we take the inequality (1.5).

To show that the inequality (1.5) is sharp, take the holomorphic function

$$f(z) = \frac{\alpha (1 - \lambda)}{\alpha - (1 - \alpha)} \frac{z (1 - z^p)}{z^p - \alpha \lambda (1 - z^p)}.$$ 

Then

$$f'(z) = \alpha (1 - \lambda) \frac{(1 - (p + 1)z^p) (\alpha (1 - \alpha) z^p - \alpha \lambda (1 - z^p))}{(\alpha - (1 - \alpha) z^p - \alpha \lambda (1 - z^p))^2}$$

$$- \alpha (1 - \lambda) \frac{(-p(1 - \alpha)z^{p-1} + \alpha \lambda z^{p-1}) (z - z^{p+1})}{(\alpha - (1 - \alpha) z^p - \alpha \lambda (1 - z^p))^2}.$$
and
\[ f'(1) = -p \frac{a (1 - \lambda)}{2a - 1}. \]
Since \(|c_{p+1}| = \frac{2a - 1}{a (1 - \lambda)}\), (1.5) is satisfied with equality. \(\square\)

If \(f(z) - z\) has no zeros different from \(z = 0\) in Theorem 2.1, the inequality (1.5) can be further strengthened. This is given by the following theorem.

**Theorem 2.2.** Let \(f(z) = z + c_{p+1}z^{p+1} + c_{p+2}z^{p+2} + \cdots\), \(c_{p+1} \neq 0\), \(p > 1\) be a holomorphic function in the unit disc \(D\) and \(f(z) - z\) has no zeros in \(D\) except \(z = 0\), and let \(\left| f(z) \right| - \alpha \left| z \right| < \alpha\) for \(|z| < 1\), where \(\frac{1}{2} < \alpha \leq \frac{1}{4\lambda}\), \(0 \leq \lambda < 1\). Further assume that, for some \(b \in \partial D\), \(f(z)\) has an angular limit \(f(b)\) at \(b\), \(f(b) = 0\). Then

\[ |f'(b)| \geq \frac{\alpha (1 - \lambda)}{2\alpha - 1} \left[ p - \frac{2|c_{p+1}|}{2|c_{p+1}|} \left( \ln \frac{\alpha (1 - \lambda)}{2\alpha - 1} |c_{p+1}| \right)^2 \right] \]

and

\[ |c_{p+2}| \leq 2 \left| c_{p+1} \ln \left( \frac{\alpha (1 - \lambda)}{2\alpha - 1} |c_{p+1}| \right) \right|. \]

In addition, the equality in (1.7) occurs for the function

\[ f(z) = \frac{\alpha (1 - \lambda) z(1 - z^p)}{\alpha - (1 - \alpha) z^p - \alpha \lambda (1 - z^p)} \]

and the equality in (1.8) occurs for the function

\[ f(z) = z + z^{p+1} \frac{(2\alpha - 1) e^Q}{\alpha + (1 - \alpha) z^p e^Q - \alpha \lambda (1 + z^p e^Q)}, \]

where \(0 < c_{p+1} < 1\), \(\ln \left( \frac{\alpha (1 - \lambda)}{2\alpha - 1} c_{p+1} \right) < 0\) and \(Q = \frac{1+z}{1-z} \ln \left( \frac{\alpha (1 - \lambda)}{2\alpha - 1} c_{p+1} \right)\).

**Proof.** We can assume that \(c_{p+1} > 0\). Let \(\psi(z), h(z)\) and \(\omega(z)\) be as in the proof of Theorem 2.1. Bearing in the mind inequality (1.6), we denote by \(\ln h(z)\) the holomorphic branch of the logarithm normed by the condition

\[ \ln h(0) = \ln \left( \frac{\alpha (1 - \lambda)}{2\alpha - 1} c_{p+1} \right) < 0. \]

The composite function

\[ \Theta(z) = \frac{\ln h(z) - \ln h(0)}{\ln h(z) + \ln h(0)} \]

is holomorphic in the unit disc \(D\), \(|\Theta(z)| < 1\), \(|\Theta(0)| = 0\) and \(|\Theta(b)| = 1\) for \(b \in \partial D\).
From (1.3), we obtain
\[
\frac{2}{1 + |\Theta'(0)|} \leq |\Theta'(b)| = \frac{|2 \ln h(0)|}{|\ln h(b) + \ln h(0)|} \left| \frac{h'(b)}{h(b)} \right|
\]
\[
\frac{2}{1 - \frac{c_{p+1}}{2(c_{p+1} \ln(\frac{\alpha(1-\lambda)}{2\alpha-1} c_{p+1}))}} \leq \frac{-2 \ln h(0)}{\arg^2 h(b) + \ln^2 h(0)} \left| h'(b) \right|
\]
\[
= \frac{-2 \ln h(0)}{\arg^2 h(b) + \ln^2 h(0)} \left\{ \left| \psi'(b) \right| - \left| \omega'(b) \right| \right\}
\]
\[
= \frac{-2 \ln h(0)}{\arg^2 h(b) + \ln^2 h(0)} \left\{ \frac{1 - |\varphi(0)|^2}{1 - \varphi(0)\varphi(b)} \left| \varphi'(b) \right| - p \right\}
\]
\[
= \frac{-2 \ln h(0)}{\arg^2 h(b) + \ln^2 h(0)} \left\{ \frac{1 - (1 - \frac{\alpha}{\alpha})^2}{(1 + \frac{\lambda}{\alpha})^2} \left| \varphi'(b) \right| - p \right\}
\]
\[
= \frac{-2 \ln h(0)}{\arg^2 h(b) + \ln^2 h(0)} \left\{ \frac{2\alpha - 1}{\alpha(1 - \lambda)} \left| f'(b) \right| - p \right\}
\]
and replacing \( \arg^2 h(b) \) by zero, we obtain (1.7) with an obvious equality case.

Similarly, \( \Theta(z) \) satisfies the assumptions of the Schwarz lemma, we obtain
\[
1 \geq |\Theta'(0)| = \frac{|2 \ln h(0)|}{|\ln h(0) + \ln h(0)|^2} \left| \frac{h'(0)}{h(0)} \right| = \frac{-1}{2 \ln \left( \frac{\alpha(1 - \lambda)}{2\alpha - 1} c_{p+1} \right)} |c_{p+1}|
\]
Therefore, we have the inequality (1.8).

Now, we shall show that the inequality (1.8) is sharp. Let
\[
f(z) = z + z^{p+1} g(z),
\]
where
\[
g(z) = \frac{(2\alpha - 1) e^{\frac{1 + \lambda}{\alpha + (1 - \alpha) z^p e^{\frac{1 + \lambda}{\alpha + (1 - \alpha) z^p e^{\frac{1 + \lambda}{\alpha + (1 - \alpha) z^p e^{\frac{1 + \lambda}{\alpha + (1 - \alpha) z^p e^{\frac{1 + \lambda}{\alpha + (1 - \alpha) z^p e^{\frac{1 + \lambda}{\alpha + (1 - \alpha) z^p e^{\frac{1 + \lambda}{\alpha + (1 - \alpha) z^p e^{\frac{1 + \lambda}{\alpha + (1 - \alpha) z^p e^{\frac{1 + \lambda}{\alpha + (1 - \alpha) z^p e^{1 + \lambda}}}}}}}}}}}}}}}}}{\alpha + (1 - \alpha) z^p e^{\frac{1 + \lambda}{\alpha + (1 - \alpha) z^p e^{\frac{1 + \lambda}{\alpha + (1 - \alpha) z^p e^{\frac{1 + \lambda}{\alpha + (1 - \alpha) z^p e^{\frac{1 + \lambda}{\alpha + (1 - \alpha) z^p e^{\frac{1 + \lambda}{\alpha + (1 - \alpha) z^p e^{\frac{1 + \lambda}{\alpha + (1 - \alpha) z^p e^{\frac{1 + \lambda}{\alpha + (1 - \alpha) z^p e^{1 + \lambda}}}}}}}}}}}}}}}
\]
Then
\[
g'(0) = c_{p+2}.
\]
Under the simple calculations, we take
\[
c_{p+2} = 2c_{p+1} \ln \left( \frac{\alpha(1 - \lambda)}{2\alpha - 1} c_{p+1} \right)
\]
Therefore, we obtain
\[
|c_{p+2}| = 2 \left| c_{p+1} \ln \left( \frac{\alpha(1 - \lambda)}{2\alpha - 1} |c_{p+1}| \right) \right|.
\]
\[ \square \]
Relation (1.8) shows that inequality (1.7) is more stronger than inequality (1.5).

If \( f(z) - z \) have zeros different from \( z = 0 \), taking into account these zeros, the inequality (1.5) can be strengthened in another way. This is given by the following theorem.

**Theorem 2.3.** Let \( f(z) = z + c_{p+1}z^{p+1} + c_{p+2}z^{p+2} + \cdots \), \( c_{p+1} \neq 0 \), \( p > 1 \) be a holomorphic function in the unit disc \( D \) and let \( \left| \frac{f(z)}{f(z)+(1-\lambda)z} - \alpha \right| < \alpha \) for \( |z| < 1 \), where \( \frac{1}{2} < \alpha \leq \frac{1}{1+\lambda} \). Further assume that, for some \( b \in \partial D \), \( f \) has an angular limit \( f(b) \) at \( b \), \( f(b) = 0 \). Let \( a_1, a_2, \ldots, a_n \) be zeros of the function \( f(z) - z \) in \( D \) that are different from zero. Then we have the inequality

\[
|f'(b)| \geq \frac{\alpha (1 - \lambda)}{2\alpha - 1} \left( p + \sum_{k=1}^{n} \frac{1 - |a_k|^2}{|b - a_k|^2} + \frac{2 \left( 1 - \frac{\alpha(1-\lambda)}{2\alpha - 1} \prod_{k=1}^{n} |a_k| \right)^2}{1 - \left( \frac{\alpha(1-\lambda)}{2\alpha - 1} \prod_{k=1}^{n} |a_k| \right)^2} \right).
\]

In addition, the equality in (1.9) occurs for the function

\[
f(z) = \frac{\alpha (1 - \lambda) z \left( 1 - z^p \prod_{k=1}^{n} \frac{z - a_k}{1 - \overline{a_k} z} \right)}{\alpha - (1 - \alpha) z^p \prod_{k=1}^{n} \frac{z - a_k}{1 - \overline{a_k} z} - \alpha \lambda \left( 1 - z^p \prod_{k=1}^{n} \frac{z - a_k}{1 - \overline{a_k} z} \right)},
\]

where \( a_1, a_2, \ldots, a_n \) are positive real numbers.

**Proof.** Let \( \psi(z) \) be as in the proof of Theorem 2.1 and \( a_1, a_2, \ldots, a_n \) be zeros of the function \( f(z) - z \) in \( D \) that are different from zero.

\[
B(z) = z^p \prod_{k=1}^{n} \frac{z - a_k}{1 - \overline{a_k} z}
\]

is a holomorphic function in \( D \) and \( |B(z)| < 1 \) for \( |z| < 1 \). By the maximum principle for each \( z \in D \), we have

\[
|\psi(z)| \leq |B(z)|.
\]

The function

\[
s(z) = \frac{\psi(z)}{B(z)}
\]
is a holomorphic function in $D$ and $|s(z)| < 1$ for $|z| < 1$. In particular, we have
\[ |s(0)| = \frac{\alpha (1 - \lambda)}{2\alpha - 1} \frac{|c_{p+1}|}{\prod_{k=1}^{n} |a_k|} \leq 1 \]
and
\[ |s'(0)| = \frac{\alpha (1 - \lambda)}{2\alpha - 1} \frac{|c_{p+2}|}{\prod_{k=1}^{n} |a_k|} \]

Moreover, it can be seen that
\[ \frac{b\psi'(b)}{\psi(b)} = |\psi'(b)| \geq |B'(b)| = \frac{bB'(b)}{B(b)} \]

Besides, with the simple calculations, we take
\[ |B'(b)| = \frac{bB'(b)}{B(b)} = p + \sum_{k=1}^{n} \frac{1 - |a_k|^2}{|b - a_k|^2} \]

The auxiliary function
\[ d(z) = \frac{s(z) - s(0)}{1 - s(0)s(z)} \]
is holomorphic in the unit disc $D$, $|d(z)| < 1$, $d(0) = 1$ and $|d(b)| = 1$ for $b \in \partial D$.

From (1.3), we obtain
\[ \frac{2}{1 + |d'(0)|} \leq |d'(b)| = \frac{1 - |s(0)|^2}{1 - s(0)s(b)} |s'(b)| \]
\[ \leq 1 + \frac{|s(0)|}{1 - |s(0)|} \left\{ \{|\psi'(b)| - |B'(b)|\} \right\} \]
\[ = 1 - \frac{\alpha (1 - \lambda)}{2\alpha - 1} \frac{|c_{p+1}|}{\prod_{k=1}^{n} |a_k|} \left\{ \frac{1 - |\varphi(0)|^2}{1 - |\varphi(0)||\varphi(b)|} |\varphi'(b)| - |B'(b)| \right\} \]

and
\[ \frac{2}{1 + \frac{\alpha (1 - \lambda)}{2\alpha - 1} \frac{|c_{p+1}|}{\prod_{k=1}^{n} |a_k|}} \leq 1 - \frac{\alpha (1 - \lambda)}{2\alpha - 1} \frac{|c_{p+2}|}{\prod_{k=1}^{n} |a_k|} \left\{ \frac{2\alpha - 1}{\alpha (1 - \lambda)} |f'(b)| - |B'(b)| \right\}. \]

Therefore, we take the inequality (1.9) with an obvious equality case. \( \square \)
We note that the inequality (1.3) has been used in the proofs of Theorem 2.1, Theorem 2.2 and Theorem 2.3. Therefore, there are both \(c_{p+1}\) and \(c_{p+2}\) in the right side of the inequalities. But, if we use (1.4) instead of (1.3), we obtain weaker but more simpler inequality (not including \(c_{p+2}\)). It is formulated in the following theorem.

**Theorem 2.4.** Under the hypotheses of Theorem 2.2, we have the inequality

\[
|f'(b)| \geq \frac{\alpha (1 - \lambda)}{2\alpha - 1} \left[ p - \frac{1}{2} \ln \left( \frac{\alpha (1 - \lambda)}{2\alpha - 1} \right) \right].
\]

The equality in (1.10) holds if and only if

\[
f(z) = \frac{\alpha (1 - \lambda) z}{\alpha + (1 - \alpha) z^p \lambda} - \alpha \left(1 + z^p e^Q\right) \ln \left(\frac{\alpha (1 - \lambda)}{2\alpha - 1} \right) e^{i\theta} \text{ and } \theta \text{ is a real number.}
\]

**Proof.** From the proof of Theorem 2.2, using the inequality (1.4) for the function \(\Theta(z)\), we obtain

\[
1 \leq |\Theta'(b)| = \frac{|2 \ln h(0)|}{|\ln h(b) + \ln h(0)|^2} \left| \frac{h'(b)}{h(b)} \right|
\]

and replacing \(\arg^2 h(b)\) by zero

\[
1 \leq -2 \ln h(0) \left\{ \frac{1 - |\varphi(0)|^2}{\left| 1 - \varphi(0) \varphi'(b) \right|^2} |\varphi'(b)| - p \right\}
\]

and

\[
\left\{ \frac{2\alpha - 1}{\alpha (1 - \lambda)} |f'(b)| - p \right\}.
\]

Therefore, we have the inequality (1.10).

If \(|f'(b)| = \frac{\alpha (1 - \lambda)}{2\alpha - 1} \left[ p - \frac{1}{2} \ln \left( \frac{\alpha (1 - \lambda)}{2\alpha - 1} \right) \right]\) from (1.11) and \(|\Theta'(b)| = 1\), we obtain

\[
f(z) = \frac{\alpha (1 - \lambda) z}{\alpha + (1 - \alpha) z^p \lambda} - \alpha \left(1 + z^p e^Q\right) \ln \left(\frac{\alpha (1 - \lambda)}{2\alpha - 1} \right) e^{i\theta} \text{ and } \theta \text{ is a real number.} \]

**Acknowledgement.** The author would like to thank the referee for his/her constructive comments and suggestions on the earlier version of this paper.
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