Constant factor Approximation Algorithms for Uniform Hard Capacitated Facility Location Problems: Natural LP is not too bad

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Abstract. In this paper, we study the uniform hard capacitated knapsack median problem, a generalization of the k-median problem. Natural LP of the problem is known to have an unbounded integrality gap. We give first constant factor approximation for the problem violating the budget only by an additive $f_{\text{max}}$ where $f_{\text{max}}$ is the maximum cost of a facility opened by the optimal and violating capacities by $(3 + \epsilon)$ factor. To the best of our knowledge, no constant factor approximation is known for the problem even with capacity/budget/both violations. We also give the first deterministic constant factor ($O(1/\epsilon)$) approximation for the capacitated k facility location problem with uniform capacities violating the capacities by $(2 + \epsilon)$ and cardinality by an additional 1. Existing results either use randomized rounding or assume facility opening costs to be uniform.

For capacitated facility location problem with uniform capacities, a constant factor approximation algorithm is presented violating the capacities a little $(1 + \epsilon)$. Though constant factor results are known for the problem without violating the capacities, the result is interesting as it is obtained by rounding the solution to the natural LP, which is known to have an unbounded integrality gap without violating the capacities. Thus, we achieve the best possible from the natural LP for the problem. The result shows that natural LP is not too bad.

Keywords: Capacitated Knapsack Median, Capacitated k-Facility Location

1 Introduction

Facility location problem (FLP) is one of the widely studied problems in computer science and operational research. The problem is well known to be NP-hard. Li [22] gave a 1.488 factor algorithm almost closing the gap between the best known and best possible of 1.463 by Guha et al. [14] for the metric uncapacitated case. Best known approximation ratio of $2.675 + \epsilon$ for (uncapacitated) k-median was given by Byrka et al. [7].
On the other hand, for the capacitated version of the problem (CapFLP), standard LP is known to have an unbounded integrality gap even when the capacities are uniform for the most basic variant of the problem, namely, metric facility location problem.

**Integrality gap example for Capacitated FLP** Consider two facilities such that \( f_1 = 0, \ f_2 = 1 \), each with \( M \) units of capacity and \( M + 1 \) clients, each with unit demand. The LP solution will set \( y_1 = 1, \ y_2 = 1/M \) at a cost of 1. The IP solution will set \( y_1 = y_2 = 1 \) and will incur a cost of 1. Note that the example breaks when the capacities are allowed to be violated a little. In particular, for a fixed \( \epsilon > 0 \) if we allow the capacities to be violated by a factor of \((1 + \epsilon)\), then the gap is bounded by \( O(1/\epsilon) \).

Local search technique has been particularly found useful to deal with capacities. The approach provides 3 factor for uniform capacities \([2]\) and 5 factor for the non-uniform case \([4]\). Few LP-based algorithms known for the problem are due to \([8, 5, 20]\). Levi, Shmoys and Swamy \([20]\) gave a 5-factor algorithm for CapFLP, for a restricted version of the problem in which all facility costs are same. In a recent work, An, Singh and Svensson \([3]\) gave a constant factor approximation by strengthening the natural LP. Recent result on capacitated \( k \)-Facility Location problem by Byrka et al. \([5]\) give a result for CapFLP as a particular case. They give an \( O(1/\epsilon^2) \) factor approximation violating the capacities by a factor of \((2 + \epsilon)\).

Capacitated \( k \)-median is much less understood. Obtaining a constant factor approximation algorithm for the problem is open. Existing solutions giving constant-factor approximation, violate at least one of the two (cardinality and capacity) constraints. Natural LP is known to have an unbounded integrality gap when any one of the two constraints is allowed to be violated by a factor less than 2.

For the case of uniform capacities, several results \([9, 11, 5, 21, 17]\) have been obtained that violate either the capacities or the cardinality by a factor of 2 or more. In case of non-uniform capacities, a \((7 + \epsilon)\) algorithm was given by Aardal et al. \([1]\) violating the cardinality constraint by a factor of 2 as a special case of Capacitated \( k \)-FLP when the facility costs are all zero.

Li \([23]\) broke the barrier of 2 in cardinality and gave an \( \exp(O(1/\epsilon^2)) \) approximation using at most \((1 + \epsilon)k\) centers (facilities) for uniform capacities. Li gave a sophisticated algorithm using a novel linear program called rectangle LP. The result was extended to non-uniform capacities in \([24]\) by the same author using a new LP called configuration LP. The approximation ratio was also improved from \( \exp(O(1/\epsilon^2)) \) to \( O(1/\epsilon^2 \log(1/\epsilon)) \). Though the algorithm violates the cardinality only by \( 1 + \epsilon \), it introduces a softness bounded by a factor of 2. The running time of the algorithm is \( n^{O(1/\epsilon)} \).

Very recently, Byrka et al. \([8]\) broke the barrier of 2 in capacities and gave an \( O(1/\epsilon^2) \) approximation violating capacities by a factor of \((1 + \epsilon)\) factor for uniform capacities. The algorithm uses randomized rounding to round a fractional solution to the configuration LP. For non-uniform capacities also, a similar result has been obtained by Demirci et al. in \([13]\). The paper presents an \( O(1/\epsilon^5) \)
approximation algorithm with capacity violation by a factor of at most \((1 + \epsilon)\). The running time of the algorithm is \(n^{O(1/\epsilon)}\).

In this paper, we study the uniform capacitated variants of some facility location problems. In particular, we study capacitated knapsack median problem and capacitated \(k\)-facility location problem. Knapsack median is a generalization of \(k\)-median whereas \(k\)-facility location problem is a common generalization of \(k\)-median and facility location problems.

For knapsack median, the natural LP is known to have an unbounded integrality gap \([10]\) even for the uncapacitated variant of the problem. Krishnaswamy et al. \([18]\) showed that the integrality gap holds even on adding the covering inequalities to strengthen the LP, and gave a 16 factor approximation that violates the budget constraint by an additive factor of \(f_{\text{max}}\), the maximum opening cost of a facility in optimal. Kumar \([19]\) gave first constant factor approximation without violating the budget constraint. Kumar strengthened the natural LP by obtaining a bound on the maximum distance a client can travel. Charikar and Li \([12]\) reduced the large constant obtained by Kumar to 34 which was further improved to 32 by Swamy in \([25]\). Byrka et al. \([6]\) extended the work of Swamy to obtain a factor of 17.46. To the best of our knowledge, no constant factor algorithm is known for the capacitated variant of the problem even when capacity/budget/both are allowed to be violated by a constant factor. In particular, we present the following result:

**Theorem 1.** There is a polynomial time algorithm that approximates hard uniform capacitated knapsack median problem within a constant factor violating the capacities by a factor of \(3 + \epsilon\) and budget only by an additive factor of \(f_{\text{max}}\) where \(f_{\text{max}}\) is the maximum cost of a facility in the optimal.

For \(k\)-FLP, a few results \([11,16,15,26]\) have been obtained by combining the ideas from \(k\)-median and FLP. CkFLP is NP-hard even when there is only one client and there are no facility costs \([1]\). Aardal et al. \([1]\) extended the FPTAS for knapsack problem to give an FPTAS for single client capacitated \(k\)-FLP. They also extend an \(\alpha\) - approximation algorithm for (uncapacitated) \(k\)-median to give a \((2\alpha + 1)\) - approximation for CkFLP with uniform opening costs using at most 2\(k\) facilities for non-uniform and 2\(k - 1\) facilities for uniform capacities. In a parallel work, Byrka et al. \([5]\) gave an \(O(1/\epsilon^2)\) algorithm for uniform capacities violating capacities by a factor of \(2 + \epsilon\). They use randomized rounding to bound the expected cost.

We give the first deterministic constant factor \((O(1/\epsilon))\) approximation for the capacitated \(k\) facility location problem handling non uniform opening costs, with uniform capacities violating the capacities by \((2 + \epsilon)\) opening at most \(k + 1\) facilities. To the best of our knowledge, no deterministic constant factor approximation is known for the problem even with capacity/cardinality/both violations. We also improve upon the factor of \([5]\) from \(O(1/\epsilon^2)\) to \(O(1/\epsilon)\). In particular, we give the following result:

**Theorem 2.** There is a polynomial time algorithm that approximates hard uniform capacitated \(k\)-facility location problem within a constant factor \((O(1/\epsilon))\)
violating the capacities by a factor of at most \((2 + \epsilon)\) for a fixed \(\epsilon > 0\) and using at most \(k + 1\) facilities.

Finally, we present a constant factor \((O(1/\epsilon))\) approximation for capacitated facility location problem with uniform capacities violating the capacities a little, by \((1 + \epsilon)\). Though constant factor results are known for the problem without violating the capacities \([28]\), the result is interesting as it is obtained by rounding the solution to the natural LP. In particular, we give the following result:

**Theorem 3.** There is a polynomial time algorithm that approximates hard uniform capacitated facility location problem within a constant factor \((O(1/\epsilon))\) violating the capacities by a factor of at most \((1 + \epsilon)\) for a fixed \(0 < \epsilon < 1/2\).

The result shows that natural LP is not too bad for capacitated facility location problem.

**High Level Idea:** We combine the ideas from \([5]\) and \([18]\). The set of facilities and demands are partitioned into ”clusters”, a binary tree structure is defined on the centers of clusters and meta-clusters are formed as in \([5]\). However, instead of using dependent rounding to round the fractional solution, we define a new LP. The iterative algorithm of \([18]\) is modified to obtain a solution with at most two fractionally opened facilities using the properties of extreme point solutions. By opening these fractional facilities suitably, we obtain an integrally open solution violating the budget by an additional \(f_{\text{max}}\). Min-cost flow is then used to obtain integral assignments.

The major challenge lies in defining the LP so that it opens sufficient number of facilities in order to be able to assign the demands violating the capacities in the claimed bound and in obtaining a feasible solution of bounded cost to the new LP.

For CkFLP, meta-clusters are formed in the same way but the approach of opening facilities is slightly different for clusters with large demands. A simple greedy approach is used to obtain a solution having at most one fractionally opened facility in each cluster (this cannot be done for the knapsack problem) with large demand. A covering knapsack problem is then solved within each meta-cluster to collect the fractional openings of these clusters to open few more facilities integrally in the meta-cluster. This gives us some more capacity in a meta-cluster which leads to reduction in capacity violation from \((3 + \epsilon)\) to \((2 + \epsilon)\).

## 2 Capacitated Knapsack Median Problem

Knapsack median problem is a generalization of the \(k\)-median problem where-in facilities have opening costs and there is a budget on the total cost of open facilities. We are given a set of clients \(C\), a set of facilities \(F\) and a real valued distance function \(c(i, j)\) on \(F \cup C\) in metric space. Each client \(j\) has some demand associated with it, each facility \(i\) has an opening cost \(f_i\), and we have a budget \(B\). The goal is to open a set of facilities and assign demands to them so as to minimize the total connection cost subject to the constraint that the
total facility cost of the opened facilities is at most $B$. When $B = k$ and $f_i = 1$, problem reduces to the $k$-median problem. In capacitated knapsack median, each facility $i$ also has an associated capacity $u_i$ which limits the maximum amount of demand it can serve. We deal with the case when $u_i = u \forall i$ and demands are unit and, denote it by unifKnM.

The Integer Program (IP) for instance $(C, F, c, f, u, B)$ of unifKnM is given as follows:

$$
\text{Minimize } \text{CostKM}(x, y) = \sum_{j \in C} \sum_{i \in F} c(i, j)x_{ij}
$$

subject to

$$
\sum_{i \in F} x_{ij} = 1 \quad \forall j \in C \quad (1)
$$

$$
\sum_{j \in C} x_{ij} \leq u_i y_i \quad \forall i \in F \quad (2)
$$

$$
x_{ij} \leq y_i \quad \forall i \in F, j \in C \quad (3)
$$

$$
\sum_{i \in F} f_i y_i \leq B \quad (4)
$$

$$
y_i, x_{ij} \in \{0, 1\} \quad (5)
$$

where $y_i = 1$ if and only if facility $i$ is open and $x_{ij} = 1$ if client $j$ is assigned to facility $i$. First constraint makes sure that every client is served, second constraint ensures that the total assignment on an open facility does not exceed its capacity. Constraint (3) ensures that a client is served only by an open facility. Constraint (4) says that total cost of opened facilities is within the given budget. LP-Relaxation of the problem is obtained by allowing the variables $y_i$ and $x_{ij}$ to be fractional. Call it $LP_1$.

We first define some notations which will be used throughout the paper. For an LP solution $\sigma = <x, y>$, $j \in C$ and a subset $T$ of facilities, let $\text{Size}(y, T) = \sum_{i \in T} y_i$ denote the total extent up to which facilities are opened in $T$, $A_\sigma(j, T) = \sum_{i \in T} x_{ij}$ denote the total assignment of $j$ on facilities in $T$ and $\text{Cost}(\sigma, T)$ denote the total opening cost and the connection cost paid by all clients for getting service from a given subset of facilities $T$ under solution $\sigma$.

To begin with, we guess the facility with maximum opening cost, $f_{max}$, in the optimal solution, and remove all the facilities with facility cost > $f_{max}$ before applying the algorithm. The algorithm runs for all possible choices for $f_{max}$ and the solution with minimum cost is selected. This can be done in polynomial time because there are only $|F|$ choices for $f_{max}$. Our algorithm consists of three main stages. First stage consists of partitioning the set of facilities and demands into clusters. This step is exactly same as that in [5]. In the second stage, we
define a new LP to obtain what we call as pseudo-integral solution. A solution is said to be pseudo-integral if at most two facilities are opened fractionally. This step is the main contribution of our work for the knapsack median problem. The pseudo-integral solution is rounded in the third stage to obtain an integrally open solution. Min-cost flow is finally used to obtain the integral assignments.

2.1 Stage I: Clustering

In this section, we partition the set of facilities and demands into clusters at a loss of constant factor in cost. Let \( \sigma^* = x^*, y^* \) denote the optimal solution of \( LP_1 \). Let \( \hat{C}_j \) denote the average connection cost of a client \( j \) in \( \sigma^* \) i.e. \( \hat{C}_j = \sum_{i \in F} x^*_i c(i, j) \). Let \( l \geq 2 \) be a fixed parameter and \( ball(j) \) be the set of facilities within a distance of \( l\hat{C}_j \) of \( j \) i.e. \( ball(j) = \{ i \in F : c(i, j) \leq l\hat{C}_j \} \). Then, \( \text{Size}(y^*, ball(j)) \geq 1 - \frac{1}{l} \) by the standard averaging argument. Let \( R_j = l\hat{C}_j \) denote the radius of \( ball(j) \). Consider the clients in non-decreasing order of their radii. Let \( S = C, C' = \emptyset \) and \( j \) be a client with smallest radius \( R_j \) in \( S \), breaking ties arbitrarily. Add \( j \) to \( C' \) and delete it from \( S \). \( \forall j' \in S \) with \( c(j, j') \leq 2l\hat{C}_j \), remove \( j' \) from \( S \) and let \( ctr(j') = j \). Repeat until \( S \) is empty. Note that, unlike un-capacitated scenarios, we do not move the demand of \( j' \) to \( j \).

For each \( j \in C' \), define cluster \( N_j \) as the set of facilities to which \( j \) is nearest amongst all the clients in \( C' \); that is \( N_j = \{ i \in F \mid \forall j' \in C' : j \neq j' \Rightarrow c(i, j) < c(i, j') \} \) assuming that the distances are distinct. \( j \) is called the center of the cluster. Thus, \( C' \) is the set of cluster centers. Note that \( ball(j) \subseteq N_j \) and the sets \( N_j \) partition \( F \).

Let \( l_i \) denote the total demand of clients in \( C \) serviced by facility \( i \) in \( \sigma^* \) i.e. \( l_i = \sum_{j \in C} x^*_{ij} \). Let \( d_j = \sum_{i \in N_j} l_i \). Move the demand \( d_j \) to center \( j \) of the cluster. Lemma (1) shows that the cost of moving the demand is bounded by \( 2(l+1)LP_{opt} \). Note that this demand is different from the demand consolidated at the centers in case of un-capacitated variant of the problem. In particular, it is different from the demand consolidated at the centers in [18].

**Lemma 1.** [19] The following holds-

1. (Separation Property) For any \( j \neq j' \in C' \), we have \( c(j, j') > 2l \max\{ \hat{C}_j, \hat{C}_{j'} \} \).
2. For any \( j' \in C \setminus C' \), there is a \( j \in C' \) with \( ctr(j') = j \) and \( c(j, j') \leq 2l\hat{C}_j \).

**Proof.** 1. Wlog, assume that \( j \) was added to \( C' \) before \( j' \), then \( \hat{C}_j \leq \hat{C}_{j'} \). Hence, \( c(j, j') > 2l\hat{C}_{j'} \). For otherwise, \( j' \) would have been removed from \( S \) when \( j \) was considered.
2. If \( j' \notin C' \), then there must have been a \( j \in C' \) such that \( j' \) was removed when \( j \) was added to \( C' \). Thus, \( ctr(j') = j \) and \( c(j, j') \leq 2l\hat{C}_j \).

**Lemma 2.** Let \( j' \in C \setminus C' \) and \( j \in C' \) such that \( c(j, j') \leq R_j \), then \( R_j \geq 2R_{j'} \).
Proof. Suppose, if possible, \( R_j > 2R_{j'} \). Let \( ctr(j') = k \) \((k \text{ could be } j \text{ itself})\). Then, \( c(j', k) \leq 2R_{j'} \). And, \( c(k, j) \leq c(k, j') + c(j', j) \leq 2R_{j'} + R_j < 2R_j = 2l\hat{C}_j \), which is a contradiction to claim (1) of Lemma (1). \( \square \)

**Lemma 3.** \([\underline{5}]\) Let \( j \in C' \) and \( i \in N_j \) then,

1. For \( j' \in C' \), \( c(j, j') \leq 2c(i, j') \).
2. For \( j' \in C \setminus C' \), \( c(j, j') \leq 2c(i, j') + 2R_{j'} \).

**Proof.** If \( j' \in C' \) then using triangle inequality and using the fact that \( i \) belongs to \( N_j \) and not \( N_{j'} \), we get \( c(j, j') \leq 2c(i, j') \).

If \( j' \notin C' \), there exists \( k \in C' \) such that \( ctr(j') = k \) and \( c(j', k) \leq 2l\hat{C}_j \).

Also, \( c(i, j) \leq c(i, k) \) because \( i \in N_j \) and not \( N_k \). By triangle inequality, \( c(i, k) \leq c(i, j') + c(j', k) \leq c(i, j') + 2l\hat{C}_j = c(i, j') + 2R_{j'} \). Therefore, \( c(j, j') \leq 2c(i, j') \). \( \square \)

**Lemma 4.** \( \sum_{j' \in C'} \sum_{j \in C'} c(j, j')A_{\sigma^*}(j', N_j) \leq 2(l + 1)LP_{opt} \).

**Proof.** From Lemma \( \underline{3} \), \( c(j, j') \leq 2c(i, j') + 2R_{j'} \) for all \( j \in C', i \in N_j \). Multiplying both sides by \( A_{\sigma^*}(j', N_j) \) and summing over all \( j \in C' \) we get

\[
\sum_{j \in C'} \sum_{j' \in C'} c(j, j')A_{\sigma^*}(j', N_j) \leq \sum_{j \in C'} \sum_{i \in N_j} 2c(i, j')x_{ij'} + \sum_{j \in C'} 2R_{j'}A_{\sigma^*}(j', N_j) \leq 2\hat{C}_j + 2l\hat{C}_j = 2(l + 1)\hat{C}_j .
\]

Summing over all \( j' \in C \) we have

\[
\sum_{j' \in C} \sum_{j \in C} c(j, j')A_{\sigma^*}(j', N_j) \leq 2(l + 1)LP_{opt}
\]

\( \square \)

Let \( C_S \) be the set of cluster centers for which \( d_j < u \) and \( C_D \) be the set of remaining centers in \( C' \). We call the clusters centered at \( j \in C_S \) as sparse and those centered at \( j \in C_D \) as dense. Define \( N_{C_D} = \bigcup_{j \in C_D} N_j \) and \( N_{C_S} = \bigcup_{j \in C_S} N_j \).

Sparse clusters have the nice property that they need to take care of small demand (less than \( u \) each) and dense clusters have the nice property that the total opening within each cluster is at least 1. These properties are exploited to obtain the claimed bounds.

### 2.2 Stage II: Obtaining a pseudo-integral solution

In this section, we define a new LP which provides us with a solution having at most two fractionally opened facilities. We define a set of meta-clusters(MCs) of appropriate size so that LP opens sufficient number of facilities in each MC. To be able to define our MCs, a tree structure is defined on the cluster centers. But, before that we present some results which will motivate us to define our trees. Next two lemmas show that if no facility is opened in a sparse cluster then the cost of moving its demand to the nearest other cluster center is bounded.
Lemma (5) bounds the cost of distributing the demand $d_j$ of a cluster centered at $j$ to the facilities that served $j$ in $\sigma^*$, proportionately. This, together with Lemma (6) bounds the cost of transferring a part of $d_j$ (that is not served from inside) to the nearest cluster center in $C'$. Though results in Lemmas (5) and (6) look like the results in [18], they are different as the demand consolidated at the centers are different.

**Lemma 5.**\[
\sum_{j \in C'} d_j \sum_{i \in F} c(i, j)x^*_{ij} \leq 3 \sum_{j \in C} \sum_{i \in F} c(i, j)x^*_{ij}.
\]

**Proof.**\[
\sum_{j \in C'} d_j \sum_{i \in F} c(i, j)x^*_{ij} = \sum_{j \in C'} d_j \hat{C}_j = \sum_{j \in C'} \sum_{j' \in C} A_{\sigma^*}(j', N_j) \hat{C}_j = \sum_{j \in C'} \left( \sum_{j' : (j, j') \leq R_j} A_{\sigma^*}(j', N_j) \hat{C}_j + \sum_{j' : (j, j') > R_j} A_{\sigma^*}(j', N_j) \hat{C}_j \right)
\]

Let $j \in C'$ and $j' \in C$. If $c(j, j') \leq R_j$ then we know $j' \notin C'$. Thus, using Lemma (2), we have $\hat{C}_j \leq 2\hat{C}_{j'}$, and,

\[
\sum_{j \in C'} \sum_{j' : (j, j') \leq R_j} A_{\sigma^*}(j', N_j) \hat{C}_j = 2 \sum_{j \in C'} \sum_{j' : (j, j') \leq R_j} A_{\sigma^*}(j', N_j) \hat{C}_{j'} \quad (1)
\]

Otherwise, (i.e. if $c(j, j') > R_j$) using Lemma (3) we have $c(j, j') \leq 2c(i, j') + 2R_{j'}$, and,

\[
\sum_{j \in C'} \sum_{j' : (j, j') > R_j} A_{\sigma^*}(j', N_j) \hat{C}_j < \frac{1}{2} \sum_{j \in C'} \sum_{j' : (j, j') > R_j} A_{\sigma^*}(j', N_j) c(j, j')
\]

\[
\leq \frac{1}{2} \sum_{j' \in C'} \sum_{j : (j, j') > R_j} \sum_{i \in N_j} x^*_{ij} \left(2c(i, j') + 2R_{j'} \right)
\]

\[
\leq \sum_{j' \in C'} \sum_{j : (j, j') > R_j} \sum_{i \in N_j} x^*_{ij} c(i, j') + 2 \sum_{j' \in C'} \sum_{j : (j, j') > R_j} \left( \sum_{i \in C} A_{\sigma^*}(j', N_j) \hat{C}_{j'} \right) \quad (2)
\]

Adding (1) and (2), we have

\[
\sum_{j \in C'} \left( \sum_{j' : (j, j') \leq R_j} A_{\sigma^*}(j', N_j) \hat{C}_j + \sum_{j' : (j, j') > R_j} A_{\sigma^*}(j', N_j) \hat{C}_j \right)
\]

\[
\leq 2 \sum_{j \in C' \setminus (j, j') \leq R_j} A_{\sigma^*}(j', N_j) \hat{C}_j + \sum_{j' \in C'} \sum_{j \in N_j} x^*_{ij} c(i, j') + 2 \sum_{j' \in C'} \sum_{j : (j, j') > R_j} A_{\sigma^*}(j', N_j) \hat{C}_{j'}
\]

\[
= \sum_{j \in C'} \sum_{i \in F} c(i, j') x^*_{ij'} + 2 \sum_{j' \in C} A_{\sigma^*}(j', N_j) \hat{C}_{j'}
\]

\[
= \sum_{j \in C'} \sum_{i \in F} c(i, j') x^*_{ij'} + 2 \sum_{j' \in C} \hat{C}_{j'} = 3 \sum_{j \in C} \sum_{i \in F} c(i, j') x^*_{ij'}
\]
Next, for each \( j \in C' \), define \( \eta(j) \) as follows: \( \eta(j) = j' \neq j \) for some \( j' \in C' \), such that \( c(j, j') < c(j, j'') \) \( \forall j'' \in C' \), assuming distances are distinct.

**Lemma 6.** \( \sum_{j \in C_s} d_j \left( \sum_{i \in N_j} c(i, \, j)x^{*}_{ij} + c(j, \, \eta(j)) \left(1 - \sum_{i \in N_j} x^*_{ij}\right) \right) \leq 2 \sum_{j \in C'} d_j \left( \sum_{i \in F} c(i, \, j)x^*_{ij} \right) \)

**Proof.** Let us first consider the second term of LHS.

\[
\sum_{j \in C_s} d_j \left( c(j, \, \eta(j)) \left(1 - \sum_{i \in N_j} x^*_{ij}\right) \right) = \sum_{j \in C_s} d_j \left( \sum_{i \notin N_j} c(i, \, j)x^*_{ij} \right) \\
\leq \sum_{j \in C_s} d_j \left( \sum_{j' \in C': j' \neq j} \sum_{i \in N_{j'}} c(j, \, j')x^*_{ij} \right) \quad (\text{by the definition of } \eta(j)) \\
\leq \sum_{j \in C_s} d_j \left( \sum_{j' \in C': j' \neq j} \sum_{i \in N_{j'}} 2c(i, \, j)x^*_{ij} \right) \quad (\text{by Lemma } \ref{lem:triangle} (1)) \\
= 2 \sum_{j \in C_s} d_j \left( \sum_{i \notin N_j} c(i, \, j)x^*_{ij} \right)
\]

Thus,

\[
\sum_{j \in C_s} d_j \left( \sum_{i \in N_j} c(i, \, j)x^*_{ij} \right) + \sum_{j \in C_s} d_j \left( c(j, \, \eta(j)) \left(1 - \sum_{i \in N_j} x^*_{ij}\right) \right) \\
\leq \sum_{j \in C_s} d_j \left( \sum_{i \in N_j} c(i, \, j)x^*_{ij} \right) + 2 \sum_{j \in C_s} d_j \left( \sum_{i \notin N_j} c(i, \, j)x^*_{ij} \right) \\
\leq 2 \sum_{j \in C'} d_j \left( \sum_{i \in F} c(i, \, j)x^*_{ij} \right)
\]

Next, we define our tree structure on the cluster centers. We construct a forest \( F \). The forest consists of a collection of rooted trees with client centers as nodes and \((j, \, \eta(j))\) as directed edges. In case, we have a 2 length cycle at the root, we arbitrarily remove one of the cycle edges. Note that, the in-degree of a node in a tree may be unbounded. We modify the trees to bound the in-degree of nodes. Let \( T \) be a tree in \( F \). For every node \( j \in T \), sort all the children of \( j \) from left to right by non-decreasing distance from \( j \). Remove all incoming edges of \( j \) from \( T \) except the shortest one. Add a directed edge from each child of \( j \) to its left sibling (if it exists). Let \( T' \) denote the new tree obtained, \( F' \) be the resulting forest and let \( \sigma(j) \) be the parent of \( j \) in \( F' \).

**Lemma 7.** \( \sum_{j \in C_s} d_j \left( \sum_{i \in N_j} c(i, \, j)x^{*}_{ij} + c(j, \, \sigma(j)) \left(1 - \sum_{i \in N_j} x^*_{ij}\right) \right) \leq 12 \sum_{j \in C} \sum_{i \in F} c(i, \, j)x^*_{ij} \)

**Proof.** We will show that for a node \( j \) in \( F' \), \( c(j, \, \sigma(j)) \leq 2c(j, \, \eta(j)) \). The claim then follows using Lemmas \ref{lem:triangle} and \ref{lem:binarytrees}. While converting \( F \) into \( F' \), some edges were changed. \( \sigma(j) = \eta(j) \) for a node \( j \) whose edge did not change and hence, the result holds trivially. Consider a \( j \) such that \( \sigma(j) \) was the left brother of \( j \) in \( T \). Using triangle inequality we have \( c(j, \, \sigma(j)) \leq c(\sigma(j), \, \eta(j)) + c(j, \, \eta(j)) \) and by construction of binary trees \( c(\sigma(j), \, \eta(j)) \leq c(j, \, \eta(j)) \), so we can write \( c(j, \, \sigma(j)) \leq 2c(j, \, \eta(j)) \).

\( \square \)
Next, we define our meta-clusters. Let $T$ be a tree in $F'$. Nodes in $T$ are grouped in meta-clusters by a top-down greedy procedure starting from the root node $r$. To form a new meta-cluster, let $j$ be a topmost node in $T$ not yet grouped. Include $j$ in the new meta-cluster, denoted by $G_j$. Include $g$ in $G_j$ such that $g$ has the cheapest edge connecting it to some node in $G_j$, i.e., $g = \text{argmin}_u\{c(u, v) \in T : v \in G_j\}$. We make meta-clusters of size $l$ (if possible) and remove these nodes from further consideration in the formation of meta-clusters. There may be some meta-clusters with fewer nodes than $l$, towards the leaves of the tree. With a slight abuse of notation, we will use $G_j$ to denote the collection of centers of the clusters in it as well as the set of clusters themselves.

Let $H(G_j)$ denote the subgraph of $T$ with nodes in $G_j$.

Now, we define our new LP so that sufficient number of facilities are opened in each meta-cluster, opened facilities are well spread out amongst the clusters (we make sure that at most 1 (sparse) cluster has no facility opened in it) and demand of a dense cluster is satisfied within the cluster itself. Define volume of a cluster as follows: $\text{vol}(g) = 1 - \frac{1}{l}$ if $g \in C_S$ and $\text{vol}(g) = \lfloor \frac{d_j}{u} \rfloor$ if $g \in C_D$. Then, volume of a meta-cluster $G_j$, rooted at $j$ is defined naturally as,

$$\text{vol}(G_j) = \sum_{g \in G_j} \text{vol}(g).$$

LP is defined so as to open at least $\alpha_j = \lfloor \text{vol}(G_j) \rfloor$ facilities in $G_j$. Let $\tau(g) = \{i \in N_g : c(i, g) \leq c(g, \sigma(g))\}$ if $g \in C_S$ and $\tau(g) = N_g$ if $g \in C_D$. Also, let $S_j = G_j, s_j = \alpha_j$ and $B' = B$. We now write a new LP, called $LP_2$, with the objective function

$$\text{CostKM}(w) = \sum_{j \in C_S} d_j \left[ \sum_{i \in N_j} c(i, j)w_i + c(j, \sigma(j)) \left( 1 - \sum_{i \in N_j} w_i \right) \right] + u \sum_{j \in C_D} \sum_{i \in N_j} c(i, j)w_i$$

$$LP_2 :$$

Minimize $\text{CostKM}(w)$

subject to

$$\sum_{i \in \tau(j)} w_i \leq 1 \quad \forall \ j \in C_S$$

$$\sum_{i \in \tau(j)} w_i = \lfloor \frac{d_j}{u} \rfloor \quad \forall \ j \in C_D$$

$$\sum_{j' \in S_j} \sum_{i \in \tau(j')} w_i \geq s_j \quad \forall \ j : G_j \text{ is a MC}$$

$$\sum_{i \in F} f_iw_i \leq B'$$

$$0 \leq w_i \leq 1$$

Constraint (8) ensures that sufficient number of facilities are opened in a meta-cluster. We will show later that this much opening in an MC is sufficient. Constraint (6) and (7) ensure that the opened facilities are well spread out amongst the clusters as no more than 1 facility is opened in a sparse cluster and no more than $\lfloor \frac{d_j}{u} \rfloor$ facilities are opened in a dense cluster. Constraint (7) ensures
that at least \( \frac{d_j}{u} \) facilities are opened in a dense cluster. This requirement is essential to make sure that the demand of a dense cluster is served within the cluster only. Hence, equality in constraint (7) is important.

**Lemma 8.** A feasible solution \( w' \) to \( LP_2 \) can be obtained such that \( \text{Cost}_{KM}(w') \leq (2l + 13)LP_{opt} \).

**Proof.** Define a feasible solution to the above LP as follows: let \( j \in C_D, i \in \tau(j) \), set \( w_i' = \frac{l_i}{u} \lfloor d_j/u \rfloor = \frac{l_i}{u} d_j/u \leq \frac{l_i}{u} \leq y_i^* \). For \( j \in C_S \), we set \( w_i' = \min \{ x_{ij}^*, y_i^* \} = x_{ij}^* \) for \( i \in \tau(j) \) and \( w_i' = 0 \) for \( i \in N_j \setminus \tau(j) \). We will next show that the solution is feasible.

For \( j \in C_S \), \( \sum_{i \in \tau(j)} w_i' = \sum_{i \in \tau(j)} \sum_{i \in N_j} w_i' \leq \sum_{i \in N_j} x_{ij}^* \leq 1 \).

Next, let \( j \in C_D \), then \( \sum_{i \in \tau(j)} w_i' = \sum_{i \in N_j} \frac{l_i}{u} \lfloor d_j/u \rfloor = \lfloor d_j/u \rfloor \) as \( \sum_{i \in N_j} l_i = d_j \).

Note that \( \sum_{i \in \tau(j)} w_i' > 1 \) as \( d_j > u \).

For a meta-cluster \( G_j \), we have \( \sum_{j' \in G_j} \sum_{i \in \tau(j')} w_i' = \sum_{j' \in G_j \cap C_S} \sum_{i \in \tau(j')} x_{ij'}^* + \sum_{j' \in G_j \cap C_D} \sum_{i \in N_{j'}} \frac{l_i}{u} \lfloor d_j/u \rfloor \geq \sum_{j' \in G_j \cap C_S} (1 - 1/l) \sum_{j' \in G_j \cap C_D} \lfloor d_j/u \rfloor = \text{vol}(G_j) \geq \alpha_j \).

Since for each \( i \in N_{C_j} \cup N_{C_S} \) we have \( w_i' \leq y_i^* \Rightarrow \sum_{i \in F} f_i w_i' \leq \sum_{i \in F} f_i y_i^* = B \).

Next, consider the objective function. For \( j \in C_D \), we have \( \sum_{i \in \tau(j)} u c(i, j) w_i' = \sum_{i \in N_j} c(i, j) x_{ij}^* = \sum_{i \in N_j} \sum_{j' \in C} c(i, j') x_{ij'}^* \leq \sum_{i \in N_j} \sum_{j' \in C} \left( c(i, j') + 2lC_{j'} \right) x_{ij'}^* \).

Summing over all \( j \in C_D \) we get, \( \sum_{j \in C_D} \sum_{i \in N_j} \sum_{j' \in C} x_{ij'}^* \left[ c(i, j') + 2lC_{j'} \right] \leq (2l + 1)LP_{opt} \).

Also, \( \sum_{j \in C_S} \left( \sum_{i \in N_j} c(i, j) w_i' + c(j, \sigma(j))(1 - \sum_{i \in \tau(j)} w_i') \right) = \sum_{j \in C_S} d_j \left( \sum_{i \in \tau(j)} c(i, j) w_i' + \sum_{i \in N_j \setminus \tau(j)} c(i, j) w_i' + c(j, \sigma(j))(1 - \sum_{i \in \tau(j)} w_i' - \sum_{i \in N_j \setminus \tau(j)} w_i') \right) = \sum_{j \in C_S} d_j \left( \sum_{i \in \tau(j)} c(i, j) x_{ij}^* + \sum_{i \in N_j \setminus \tau(j)} c(i, j) x_{ij}^* + c(j, \sigma(j))(1 - \sum_{i \in \tau(j)} x_{ij}^*) \right) < \sum_{j \in C_S} d_j \left( \sum_{i \in \tau(j)} c(i, j) x_{ij}^* + c(j, \sigma(j))(1 - \sum_{i \in \tau(j)} x_{ij}^*) \right) + \sum_{j \in C_S} d_j \left( \sum_{i \in N_j \setminus \tau(j)} c(i, j) x_{ij}^* \right) \right) as \( c(i, j) > c(j, \sigma(j)) \forall i \in N_j \setminus \tau(j) \)
\[
= \sum_{j \in \mathcal{C}_S} d_j \left( \sum_{i \in \mathcal{N}_j} c(i, j)x_{ij}^* + c(j, \sigma(j))(1 - \sum_{i \in \mathcal{N}_j} x_{ij}^*) \right)
\leq 12LP_{opt} \text{ by Lemma (7).}
\]

Thus, the solution \( w' \) is feasible and \( \text{CostKM}(w') \),
\[
\sum_{j \in \mathcal{C}_S} d_j \left[ \sum_{i \in \mathcal{N}_j} c(i, j)w_i^* + c(j, \sigma(j)) \left( 1 - \sum_{i \in \mathcal{N}_j} w_i^* \right) \right] + u \sum_{j \in \mathcal{C}_D} \sum_{i \in \mathcal{N}_j} c(i, j)w_i^* \leq (2l + 13)LP_{opt}.
\]

\[\square\]

We now present an iterative algorithm that outputs a set \( \mathcal{A} \) of facilities having at most two fractionally opened facilities.

**Iterative Algorithm: Obtaining a pseudo-integral solution**

1. Initialize \( \mathcal{A} = \emptyset \), \( \tilde{\mathcal{F}} = \mathcal{F} \).

2. While \( \tilde{\mathcal{F}} \neq \emptyset \) do:

   (a) Compute an extreme point solution \( \tilde{w} \) to \( LP_2 \).

   (b) Let \( \tilde{\mathcal{F}}_0 = \{ i \in \tilde{\mathcal{F}} : \tilde{w}_i = 0 \} \). Remove all facilities in \( \tilde{\mathcal{F}}_0 \) from \( \tilde{\mathcal{F}} \) i.e. set \( \tilde{\mathcal{F}} = \tilde{\mathcal{F}} \setminus \tilde{\mathcal{F}}_0 \).

   (c) Let \( \tilde{\mathcal{F}}_1 = \{ i \in \tilde{\mathcal{F}} : \tilde{w}_i = 1 \} \). Remove all facilities in \( \tilde{\mathcal{F}}_1 \) from \( \tilde{\mathcal{F}} \) and add them to \( \mathcal{A} \) i.e. set \( \mathcal{F} = \tilde{\mathcal{F}} \setminus \tilde{\mathcal{F}}_1 \), \( \mathcal{A} = \mathcal{A} \cup \tilde{\mathcal{F}}_1 \) and \( B' = B' - \sum_{i \in \tilde{\mathcal{F}}_1} f_i \tilde{w}_i \).

   (d) While \( \exists j' \in \mathcal{S}_j \) for some \( \mathcal{S}_j \) such that constraint \( (10) \) or \( (11) \) is tight:

      i. If \( j' \in \mathcal{C}_S \), set \( \mathcal{S}_j = \mathcal{S}_j \setminus \{ j' \} \), \( s_j = s_j - 1 \).

      ii. If \( j' \in \mathcal{C}_D \), set \( \mathcal{S}_j = \mathcal{S}_j \setminus \{ j' \} \), \( s_j = s_j - \lfloor d_j/w \rfloor \).

   (e) If there does not exist any \( \tilde{w}_i \) that is 0 or 1 then break.

3. Return \( \mathcal{A} \).

**Lemma 9.** The solution \( \tilde{w} \) given by Iterative Algorithm satisfies the following:

1. \( \tilde{w} \) has at most two fractional facilities.

2. \( \sum_{i \in \mathcal{F}} f_i \tilde{w}_i \leq B \).

3. \( \text{CostKM}(\tilde{w}) \leq \text{CostKM}(w') \leq (2l + 13)LP_{opt} \).

**Proof.** Note that no facility is opened in \( \mathcal{N}_j \setminus \tau(j) \): \( j \in \mathcal{C}_S \) for if \( i \in \mathcal{N}_j \setminus \tau(j) : j \in \mathcal{C}_S \) is opened, then it can be shut down and the demand \( d_j \tilde{w}_i \), can be shipped to \( \sigma(j) \), decreasing the cost as \( c(j, \sigma(j)) < c(i, j) \).

Consider the iteration when the algorithm reaches step (2e). Let the linearly independent tight constraints corresponding to \( (10) \) and \( (11) \) be denoted as \( \mathcal{X} \) and the ones corresponding to \( (12) \), i.e. all meta-clusters, be denoted as \( \mathcal{Y} \). Because of the laminar nature of constraints, all sets in \( \mathcal{X} \cup \mathcal{Y} \) are disjoint. Also, each set in \( \mathcal{X} \cup \mathcal{Y} \) has at least two fractional variables because \( \tilde{w}_i = (0, 1) \forall i \) and constraints \( (10), (11) \) and \( (12) \) are tight. This implies that the number of variables is at least \( |2\mathcal{X}| + |2\mathcal{Y}| \), whereas the number of linearly independent tight constraints is at
Lemma 11. Consider a meta-cluster coming on \( G \) opened in \( G \) to itself with cost \((r, \sigma)\) called the connecting edge \( G \) parent meta-cluster of \( r \). Thus, the budget loss is no more than \( f \) there is no loss in connection cost.

If there is no fractionally opened facility, then set \( \hat{w} = \hat{w}_i \forall i \in \mathcal{F} \). Else, let \( \{i_1, i_2\} \) denote the fractionally opened facilities in solution \( \tilde{w} \). Note that we must have \( \hat{w}_i = \hat{w}_i + \hat{w}_{i_2} = 1 \). Set \( \hat{w}_i = \hat{w}_i \forall i \in \mathcal{F} \setminus \{i_1, i_2\} \) and set \( \hat{w}_i = 1, \hat{w}_{i_2} = 1, \hat{A} = \hat{A} \cup \{i_1 \cup i_2\} \).

\[
\begin{align*}
&1. \sum_{i \in \mathcal{F}} f_i \hat{w}_i \leq B + f_{\max}. \\
&2. \text{Cost}KM(\tilde{w}) \leq \text{Cost}KM(\hat{w}) \leq (2l + 13)LP_{\text{opt}}.
\end{align*}
\]

Proof. If there is no fractionally opened facility, then set \( \hat{w}_i = \hat{w}_i \forall i \in \mathcal{F} \). Else, let \( \{i_1, i_2\} \) denote the fractionally opened facilities in solution \( \tilde{w} \). Note that we must have \( \hat{w}_i = \hat{w}_i + \hat{w}_{i_2} = 1 \). Set \( \hat{w}_i = \hat{w}_i \forall i \in \mathcal{F} \setminus \{i_1, i_2\} \) and set \( \hat{w}_i = 1, \hat{w}_{i_2} = 1, \hat{A} = \hat{A} \cup \{i_1 \cup i_2\} \).

\[
\begin{align*}
&f_i \hat{w}_i + f_{i_2} \hat{w}_{i_2} = f_i \hat{w}_i + f_{i_2} \hat{w}_{i_2} + f_{i_2} (1 - \hat{w}_{i_2}) + f_{i_2} \hat{w}_{i_2} = f_i \hat{w}_i + f_{i_2} \hat{w}_{i_2} + f_{\max} (1 - \hat{w}_i, 1 - \hat{w}_{i_2}) = f_i \hat{w}_i + f_{i_2} \hat{w}_{i_2} + f_{\max} (2 - (\hat{w}_i + \hat{w}_{i_2})) = f_i \hat{w}_i + f_{i_2} \hat{w}_{i_2} + f_{\max}, \text{ where the last equality follows because } \hat{w}_i + \hat{w}_{i_2} = 1.
\end{align*}
\]

Thus, the budget loss is no more than \( f_{\max} \) and hence claim (1) holds. Clearly, there is no loss in connection cost.

2.3 Obtaining an integrally open solution

Now, we round the fractionally opened facilities obtained in the previous section violating the budget by an additional cost of \( f_{\max} \).

Lemma 10. Given an optimal pseudo-integral solution \( \tilde{w} \) for \( LP_2 \), an integrally open solution \( \hat{w} \) can be obtained such that

\[
\begin{align*}
&1. \sum_{i \in \mathcal{F}} f_i \hat{w}_i \leq B + f_{\max}. \\
&2. \text{Cost}KM(\tilde{w}) \leq \text{Cost}KM(\hat{w}) \leq (2l + 13)LP_{\text{opt}}.
\end{align*}
\]

Proof. If there is no fractionally opened facility, then set \( \hat{w}_i = \hat{w}_i \forall i \in \mathcal{F} \). Else, let \( \{i_1, i_2\} \) denote the fractionally opened facilities in solution \( \tilde{w} \). Note that we must have \( \hat{w}_i = \hat{w}_i + \hat{w}_{i_2} = 1 \). Set \( \hat{w}_i = \hat{w}_i \forall i \in \mathcal{F} \setminus \{i_1, i_2\} \) and set \( \hat{w}_i = 1, \hat{w}_{i_2} = 1, \hat{A} = \hat{A} \cup \{i_1 \cup i_2\} \).

\[
\begin{align*}
&f_i \hat{w}_i + f_{i_2} \hat{w}_{i_2} = f_i \hat{w}_i + f_{i_2} \hat{w}_{i_2} + f_{i_2} (1 - \hat{w}_{i_2}) + f_{i_2} \hat{w}_{i_2} = f_i \hat{w}_i + f_{i_2} \hat{w}_{i_2} + f_{\max} (1 - \hat{w}_i, 1 - \hat{w}_{i_2}) = f_i \hat{w}_i + f_{i_2} \hat{w}_{i_2} + f_{\max} (2 - (\hat{w}_i + \hat{w}_{i_2})) = f_i \hat{w}_i + f_{i_2} \hat{w}_{i_2} + f_{\max}, \text{ where the last equality follows because } \hat{w}_i + \hat{w}_{i_2} = 1.
\end{align*}
\]

Thus, the budget loss is no more than \( f_{\max} \) and hence claim (1) holds. Clearly, there is no loss in connection cost.

Next, we prove that these many facilities are sufficient to serve the demands coming onto \( G_j \). To be able to do so, we define a tree structure on the meta-clusters defined in previous section. A tree consists of meta-clusters as nodes and there is an edge from a meta-cluster \( G_r \) to a meta-cluster \( G_s \), if there is a directed edge from root \( r \) of \( G_r \) to some node \( s' \in G_s \). \( G_s \) is then called the parent meta-cluster of \( G_r \), \( G_r \) a child meta-cluster of \( G_s \) and the edge \((r, s')\) is called the connecting edge of the child MC \( G_r \). If \( G_r \) is a root MC, add an edge to itself with cost \( (r, \sigma(r)) \). This edge is then called the connecting edge of \( G_r \).

Lemma 11. Consider a meta-cluster \( G_j \). The demand of \( G_j \) and the demand coming on \( G_j \) from the children meta-clusters can be assigned to the facilities opened in \( G_j \) such that

\[
\begin{align*}
&1. \text{Capacity is violated at most by a factor of } 3 + 5/(l - 1) \text{ for } l \geq 2. \\
&2. \text{Each dense cluster is self-sufficient i.e. its demand can be completely assigned within the cluster itself.}
\end{align*}
\]
3. Demands are assigned only to facilities above them in the meta-cluster except for the root cluster of a root meta-cluster.
4. At most \(u\) units of demand in \(G_j\) remains un-assigned.
5. Demand coming from all the children meta-clusters are assigned to facilities within \(G_j\).
6. Total distance traveled by demand \(d_g\) of \(g \in C'\) to reach the centers of the clusters in which they are served is bounded by \(ld_g(c(g, \sigma(g)))\).

Proof. We first prove claim (1). For a meta-cluster \(G_j\), let \(t_j\) be the number of clusters in \(G_j\), \(p_j\) be the number of dense clusters in \(G_j\) and \(q_j\) be the number of sparse clusters respectively in \(G_j\). That is \(t_j = |G_j|\), \(p_j = |G_j \cap C_D|\) and \(q_j = |G_j \cap C_S|\). Then, \(t_j = p_j + q_j\). The total demand of a non-leaf meta-cluster \(G_j\) is at most,

\[
\sum_{g \in G_j} d_g + (t_j + 1)u
\]

\[
= \sum_{g \in C_D \cap G_j} [d_g - u(1-1/l)] + \sum_{g \in C_D \cap G_j} u(1-1/l) + \sum_{g \in C_D \cap G_j} u[d_g/u] + \sum_{g \in C_D \cap G_j} [d_g - u(1-1/l)] + \sum_{g \in C_D \cap G_j} u[d_g/u - [d_g/u]] + (t_j + 1)u
\]

\[
\leq u(\alpha_j + 1) + \mu_j + (t_j + 1)u, \quad \text{where} \quad \mu_j = \sum_{g \in C_D \cap G_j} [d_g - u(1-1/l)] + \sum_{g \in C_D \cap G_j} u[d_g/u - [d_g/u]]
\]

The total available capacity according to Lemma \(\text{(11)}\) is at least \(u \alpha_j\). We will show shortly that \(\mu_j \leq u(\alpha_j + 1)\) and \((t_j + 1) \leq (\alpha_j + 3)\). Thus, capacity violation is at most \(\frac{u(\alpha_j + 1) + \mu_j + (t_j + 1)u}{u \alpha_j} \leq 3 + 5/\alpha_j\). Note that for a non-leaf meta-cluster \(\alpha_j \geq l - 1\).

Leaf meta-clusters may have length less than \(l\) but they don’t have any demand coming onto them from the children meta-cluster thus capacity violation is bounded. Consider a leaf meta-cluster with only one cluster in it i.e. \(t_j = 1\). If no facility is opened in this cluster, its demand is served by the parent meta-cluster (note that it cannot be a root MC as a root MC must have at least 2 clusters in it). Next, consider a leaf meta-cluster with at least two clusters in it i.e. \(t_j \geq 2\) and hence \(\alpha_j \geq 1\). A special case when \(\alpha_j = 1\) occurs only when the MC has one dense and one sparse cluster or two sparse clusters. In either case, the total demand is at most \(3u\) and hence violation is no more than 3. For \(\alpha_j \geq 2\), the available capacity in the meta-cluster is at least \(\alpha_j u\) where as the demand is at most \(u(\alpha_j + 1) + \mu_j \leq 2u(\alpha_j + 1)\). Thus, capacity violation is at most \(\frac{u(\alpha_j + 1) + \mu_j}{u \alpha_j} = 2 + 2/\alpha_j \leq 3\) for \(\alpha_j \geq 2\).

Next, we prove that \(\mu_j \leq u(\alpha_j + 1)\).
\[
\mu_j = \sum_{g \in \mathcal{C}_S \cap G_j} [d_g - u(1 - 1/l)] + \sum_{g \in \mathcal{C}_D \cap G_j} [d_g - u[d_g/u]]
\]
\[
< \sum_{g \in \mathcal{C}_S \cap G_j} u/l + \sum_{g \in \mathcal{C}_D \cap G_j} u \text{ as } d_g < u \forall g \in \mathcal{C}_S \text{ and } [d_g - u[d_g/u]] < u \forall g \in \mathcal{C}_D
\]
\[
\leq u \sum_{g \in \mathcal{C}_S \cap G_j} (1 - 1/l) + \sum_{g \in \mathcal{C}_D \cap G_j} u \text{ as } 1/l \leq (1 - 1/l) \forall l \geq 2
\]
\[
\leq u \left( \sum_{g \in \mathcal{C}_S \cap G_j} (1 - 1/l) + \sum_{g \in \mathcal{C}_D \cap G_j} [d_g/u] \right) \text{ as for dense clusters } [d_g/u] \geq 1
\]
\[
= u \text{ vol}(G_j) \leq u(\alpha_j + 1)
\]

Now, we prove, \( t_j \leq (\alpha_j + 2) \), \( \text{vol}(G_j) = \sum_{g \in \mathcal{C}_S \cap G_j} (1 - 1/l) + \sum_{g \in \mathcal{C}_D \cap G_j} [d_g/u] \)
\[
\geq (q_j - 1) + p_j \text{ as } \sum_{g \in \mathcal{C}_S \cap G_j} (1 - 1/l) = q_j(1 - 1/l) = q_j - q_j/l \geq q_j - 1
\]
\[
= t_j - 1 \text{ which implies } t_j \leq \alpha_j + 2
\]

Next, we prove claims (2), (3), (4) (5) and (6). First of all note that there is at most one cluster with no facility opened in it and that it must be a sparse cluster. Constraint (17) ensures that \( \sum_{g \in \mathcal{C}_D \cap G_j} [d_g/u] \) facilities are open in the dense clusters of \( G_j \). Thus the remaining \( \geq \sum_{g \in \mathcal{C}_S \cap G_j} (1 - 1/l) \geq q_j - 1 \) facilities open in sparse clusters. Constraint (18) then ensures that no more than one facility opens in a sparse cluster. Hence there is at most one sparse cluster with no facility opened in it.

Let \( g \in G_j \). If at least one facility is opened in \( \mathcal{N}_g \), then the demand \( d_g \) can be served within the cluster with maximum violation in capacity as stated in (1). This is true irrespective of whether \( \mathcal{N}_g \) is a sparse cluster or a dense cluster. Since at least one facility is opened in each dense cluster, they are self sufficient. For a dense cluster, assign \( u \) units of demand to the integrally opened facilities and distribute the residual demand also uniformly on them.

Let \( g \in \mathcal{C}_S \) such that no facility is opened in \( \tau(j) \). If \( g \) is not the root cluster, then there must be a facility opened in \( \tau(\sigma(g)) \) within \( G_j \). Assign the demand of \( g \) to a facility in \( \sigma(g) \), it clearly does not violate the capacity by more than a factor of 2 if \( \sigma(g) \in \mathcal{C}_S \) and 3 if \( \sigma(g) \in \mathcal{C}_D \). In case if \( g \) is a root cluster and \( G_j \) is not a root meta-cluster, its demand is served in the parent meta-cluster. Otherwise if \( G_j \) is a root meta-cluster, at least one facility is guaranteed to be open in \( \tau(g) \cup \tau(\sigma(g)) \), demand of \( g \) is served by this facility.

Demands of the children meta-cluster are distributed arbitrarily amongst the open facilities in \( G_j \). Claim (1) shows that this can be done with capacity violation within the bounds. Also, since the clusters of children MCs were not included while the meta-cluster \( G_j \) was formed, edges in \( \mathcal{H}(G_j) \) are cheaper than the connecting edges of the children MCs and hence cheaper than the edges in...
subgraphs corresponding to children MCs. Thus, the total distance traveled by demand \(d_g\) to reach the centers of the clusters in which they are served is bounded by \(ld_gc(g, \sigma(g))\).

\[\]

**Lemma 12.** An integrally open solution \(\tilde{\sigma} = \langle \tilde{x}, \tilde{y} \rangle\) to knapsack-median problem instance \((\mathcal{C}, \mathcal{F}, c, f, u, \mathcal{B})\) can be obtained with a capacity violation at most \(3 + \epsilon\) such that \(\text{CostKM}(\tilde{x}, \tilde{y}) = O(1/\epsilon^2)LP_{opt}\) for a fixed \(\epsilon > 0\).

**Proof.** Choose \(l\) such that \(5/(l-1) < \epsilon\). The capacity violation then follows from Lemma (11.1). Let \(j \in \mathcal{C}'\). Let \(\lambda(j)\) be the set of centers such that facilities in \(\lambda(j)\) serve the demand of \(j\). Note that if some facility is opened in \(\tau(j)\), then \(\lambda(j)\) is \(\{j\}\) itself and if no facility is opened in \(\tau(j)\), then \(\lambda(j) = \{j'' : \exists i \in \tau(j'')\}\) such that demand of \(j\) is served by \(i\). Also let \(\psi(j)\) be the set of facilities in \(\lambda(j)\) that serve the demand of \(j\) and \(\theta(j, j'')\) be the extent to which \(d_j\) is served by the facilities in \(N_{j''}, j'' \in \lambda(j)\). Further, let \(\hat{y}_i\) be the total demand served by any facility \(i\). Set \(\tilde{y}_i = \hat{y}_i \forall i \in \mathcal{F}\). Let \(j' \in \mathcal{C}\) be such that \(A_{\sigma^*}(j', N_j) > 0\). For \(j'' \in \lambda(j), i \in \tau(j'')\), set \(\tilde{x}_{ij} = \frac{\hat{y}_i}{\sum_{i' \in N_{j''}} \hat{y}_{i'}} \theta(j, j'')A_{\sigma^*}(j', N_j)\).

\[
\sum_{j' \in \mathcal{C}} \tilde{x}_{ij'} = \sum_{j' \in \mathcal{C}} \left( \frac{\hat{y}_i}{\sum_{i' \in N_{j''}} \hat{y}_{i'}} \theta(j, j'')A_{\sigma^*}(j', N_j) \right) \]

\[
= \left( \frac{\hat{y}_i}{\sum_{i' \in N_{j''}} \hat{y}_{i'}} \right) \theta(j, j'') \sum_{j' \in \mathcal{C}} A_{\sigma^*}(j', N_j) = \left( \frac{\hat{y}_i}{\sum_{i' \in N_{j''}} \hat{y}_{i'}} \right) \theta(j, j'')d_j
\]

Summing over \(j \in \mathcal{C}'\),

\[
\sum_{j \in \mathcal{C}'} \sum_{j' \in \mathcal{C}} \tilde{x}_{ij'} = \left( \frac{\hat{y}_i}{\sum_{i' \in N_{j''}} \hat{y}_{i'}} \right) \sum_{j \in \mathcal{C}'} \theta(j, j'')d_j = \hat{y}_i - (1)
\]

as \(\sum_{j \in \mathcal{C}'} \theta(j, j'')d_j = \sum_{i' \in N_{j''}} \hat{y}_{i'}\).

From Lemma (11), the cost of consolidating the demands at the centers of the clusters is bounded as follows:

\[
\sum_{j \in \mathcal{C}} \sum_{j' \in \mathcal{C}'} c(j, j')A_{\sigma^*}(j', N_j) \leq 2(l + 1)LP_{opt} - (2)
\]

Let \(J_2\) be the set of sparse clusters in which no facility is opened and \(J_1\) be the remaining ones i.e. \(J_1 = C_{S} \setminus J_2\). Then, \(\lambda(j) = \{j\} \forall j \in J_1\). Let \(j \in J_1\) and \(i^*(j)\) be the facility opened in \(\tau(j)\).
\[
\sum_{j \in J_1} \sum_{j'' \in \mathcal{C}_S \cap \lambda(j)} \sum_{i \in N_{j''}} \bar{e}_{ij''} c(i, j'') = \sum_{j \in J_1} \bar{e}_{ij} c(i^*(j), j)
\]

Summing over \(j' \in \mathcal{C}\), we get
\[
\sum_{j \in J_1} \sum_{j'' \in \mathcal{C}_S \cap \lambda(j)} \sum_{i \in N_{j''}} \bar{e}_{ij''} c(i, j'') = \sum_{j \in J_1} \sum_{j' \in \mathcal{C}} \bar{e}_{ij(j')'} c(i^*(j), j) = \sum_{j \in J_1} d_j c(i^*(j), j)
\]

\[
= \sum_{j \in J_1} \sum_{i \in N_j} d_j \hat{w}_{ij}(i, j) — (3)
\]

For \(j \in J_2\), we have \(d_j = \sum_{j'' \in \lambda(j)} \theta(j, j'')d_j\). Using Lemma (116), the cost of moving the demand \(d_j\) to the centers in \(\lambda(j)\) is bounded as follows:

\[
\sum_{j'' \in \lambda(j)} \theta(j, j''d_j c(j, j'') \leq ld_j c(j, \sigma(j))
\]

Summing over \(j \in J_2\) and applying Lemma [10], we get
\[
\sum_{j \in J_2} \sum_{j'' \in \lambda(j)} \theta(j, j'')d_j c(j, j'') \leq l \sum_{j \in J_2} d_j c(j, \sigma(j))(1 - \sum_{i \in N_j} \hat{w}_{ij}) \leq l(2l+13)L P_{\text{opt}} — (4)
\]

Next, we bound the cost of assigning the demands collected at the centers in \(\lambda(j)\)s to the facilities opened in their respective clusters. The cost of assigning a part of the demand \(d_j\) in a facility opened in \(\lambda(j) \cap \mathcal{C}_S\) is bounded differently from the part assigned to facilities in \(\lambda(j) \cap \mathcal{C}_D\).

Let \(j'' \in \mathcal{C}_S \cap \lambda(j)\), \(i \in \tau(j'')\). Then, \(c(j'', i) \leq c(j'', \sigma(j'')) \leq c(j, \sigma(j))\). Last inequality follows as: either \(j''\) is above \(j\) in the same MC (by Lemma [113]) or \(j''\) is in the parent MC (say \(G_s\)) of the MC (say \(G_r\)) to which \(j\) belongs. In the first case, the inequality follows as edge costs are non-increasing as we go up the tree. In the latter case, edges \((j'', \sigma(j''))\) is either in \(G_s\) or it is the connecting edge of \(G_s\); in either case, \(c(j'', \sigma(j'')) \leq c(j, \sigma(j))\) as the edges in \(G_s\) are no costlier than the edges in \(G_r\).

\[
\sum_{j \in J_1} \sum_{j'' \in \mathcal{C}_S \cap \lambda(j)} \sum_{i \in N_{j''}} \sum_{j' \in \mathcal{C}} \bar{e}_{ij''} c(i, j'')
\]

\[
= \sum_{j \in J_1} \sum_{j'' \in \mathcal{C}_S \cap \lambda(j)} \sum_{i \in N_{j''}} \left(\frac{\hat{g}_i}{\sum_{j' \in L_{j''}} \hat{g}_{j'}}\right) \theta(j, j'') A_{\sigma_s(j', N_j)} c(i, j'')
\]

\[
= \sum_{j \in J_1} \sum_{j'' \in \mathcal{C}_S \cap \lambda(j)} \sum_{i \in N_{j''}} \left(\frac{\hat{g}_i}{\sum_{j' \in L_{j''}} \hat{g}_{j'}}\right) \theta(j, j'') d_j c(i, j'')
\]

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\[ \sum_{j \in J_2} \sum_{j'' \in C} \sum_{i \in N_{j''}} \left( \frac{\hat{g}_i}{\sum_{i' \in N_{j''}} \hat{g}_{i'}} \right) \theta(j, j'') d_j c(j, \sigma(j)) \leq \sum_{j \in J_2} d_j c(j, \sigma(j)) = \sum_{j \in J_2} d_j \left(1 - \sum_{i \in N_j} \hat{w}_i \right) c(j, \sigma(j)) - (5) \]

Next, let \( j'' \in C \cap \lambda(j), i \in N_j \). Multiplying both sides of (1) by \( c(i, j'') \), we get

\[ \sum_{j \in J_2} \sum_{j'' \in C} \sum_{i \in N_j} \sum_{j' \in C} \bar{x}_{ij'} c(i, j'') \leq \sum_{i \in C} \sum_{j \in J_2} \sum_{j'' \in C} \bar{x}_{ij''} c(i, j') \leq (3 + 5/(l - 1)) u \hat{w}_i c(i, j'') - (6) \]

The above equation bounds the cost of leaving the total demand accumulated at \( j'' \) (from \( J_2 \)) and including the demand of \( j'' \) itself. Also, recall that for \( j \in C \), \( \lambda(j) = \{j\} \).

Adding (3), (5) and (6) and applying Lemma (11.1), we get

\[ \sum_{j'' \in C} \sum_{i \in N_j} \sum_{j' \in C} \bar{x}_{ij'} c(i, j'') \leq (3 + 5/(l - 1))(2l + 13)LP_{opt} - (7) \]

Adding (2), (4) and (7), we get that the total connection cost is bounded by

\[ l(2l + 13)LP_{opt} + (3 + 5/(l - 1))(2l + 13)LP_{opt} + 2(l + 1)LP_{opt} = O(l^2)LP_{opt} = O(1/\epsilon^2)LP_{opt}. \]

3 Capacitated k Facility Location Problem

Capacitated k-Facility Location Problem is a common generalization of facility location problem and k-median problem. Instead of budget \( B \) we have a bound \( k \) on the maximum number of facilities that can be opened. The goal is to open facilities within the bound and assign clients to them so as to minimize the total facility opening cost and connection cost. When \( f_i = 0 \), the problem reduces to the k-median problem and when \( k = |F| \) it reduces to facility location problem. Here also, we deal with the case when \( u_i = u \forall i \). The IP-formulation for instance \((C, F, c, f, u, k)\) of unif-k-FLP is given as follows:
Minimize \[ \text{Cost}_{kFLP}(x, y) = \sum_{i \in F} f_i y_i + \sum_{i \in F} \sum_{j \in C} c(i, j) x_{ij} \]

subject to

\[ \sum_{i \in F} x_{ij} = 1 \quad \forall j \in C \] (11)

\[ \sum_{j \in C} x_{ij} \leq u_i y_i \quad \forall i \in F \] (12)

\[ x_{ij} \leq y_i \quad \forall i \in F, j \in C \] (13)

\[ \sum_{i \in F} y_i \leq k \] (14)

\[ x_{ij}, y_i \in \{0, 1\} \] (15)

where constraints (11), (12), (13) and (15) are similar to constraints (1), (2), (4) and (5) respectively of the knapsack median problem. Instead of constraint (4) (the budget constraint) we have the cardinality constraint (14) which ensures that no more than \(k\) facilities are opened and facility costs are part of the objective function. LP relaxation of the problem is obtained by allowing the variables to be fractional. Call it \(LP_3\).

We define some more notations which will be used in the following sections. For an LP solution \(\sigma = < x, y >, j \in C\) and a subset \(T\) of facilities, let \(Cost_s(j, \sigma, T) = \sum_{i \in T} c(i, j) x_{ij}\) denote the service/connection cost paid by client \(j\) for getting served by facilities in \(T\), \(Cost_s(\sigma, T) = \sum_{j \in C} Cost_s(j, \sigma, T)\) denote the total service cost paid by all the clients for getting served by facilities in \(T\), \(Cost_f(y, T) = \sum_{i \in T} f_i y_i\) denote the total facility opening cost of facilities in \(T\). Let \(Cost_{kFLP}(x, y) = Cost_f(\sigma, T) + Cost_s(\sigma, T)\) denote the total opening cost and the connection cost paid by all clients for getting service from the facilities in \(T\) under solution \(\sigma\).

The crux of this section is that we are able to open some extra facilities in a meta-cluster congregating the remaining openings in the dense clusters. We could not leverage these openings for knapsack median as we did not know how to open them integrally. Meta-clusters are formed in the same way as in knapsack median, however the approach to open the facilities in a meta-cluster is slightly different. For dense clusters, we define the notion of cluster instances. In a (dense) cluster \(j\), facilities \(i \in N_j\) are considered in non-decreasing order of \(f_i + uc(i, j)\) and, openings and assignments are shifted to them in a greedy manner. This gives us a solution with at most one fractional facility in each cluster. A covering knapsack problem is solved to round the fractionally opened facilities in a meta cluster. For sparse clusters, we define an LP similar to \(LP_2\). We could have used the same LP (involving both sparse and dense clusters) and built onto the work done in the previous section but the disposition becomes a
little tedious when congregating the remaining openings in the dense clusters. The approach (solving covering knapsack problem) used for dense cluster can not be used for sparse clusters as it may leave several sparse clusters at the top of a meta cluster without an opened facility. We cannot afford to do that.

3.1 Stage II: Obtaining a pseudo-integral solution

In this section, we define a new LP which provides us with a solution having at most two fractionally opened facilities in sparse clusters, that is, in $N_{CS}$. Meta-clusters are defined in the same way and Lemmas (5), (6) and (7) continue to hold. $p_j$, $q_j$, $t_j$ are defined in the same way as in knapsack median.

Let $k' = \sum_{G_j, j' \in CS \cap G_j, i' \in N_{j'}} y_{i'}^{*}$. be the total openings in all the sparse clusters by $LP_{opt}$. We now write another LP, denoted by $LP_4$ with cost function,

$$Cost_{kFLP_{sp}}(w) = \sum_{j \in CS} d_j \left[ \sum_{j \in N_j} c(i, j)w_i + c(j, \sigma(j)) \left( 1 - \sum_{i \in N_j} w_i \right) \right] + \sum_{i \in N_{CS}} f_i w_i$$

$$LP_4: \text{Minimize } Cost_{kFLP_{sp}}(w)$$

subject to $\sum_{i \in \tau(j)} w_i \leq 1 \forall j \in CS$ (16)

$\sum_{j' \in CS \cap G_j} \sum_{i \in \tau(j')} w_i \geq q_j - 1 \forall j: G_j$ is a MC (17)

$\sum_{i \in N_{CS}} w_i \leq k'$ (18)

$0 \leq w_i \leq 1$ (19)

**Lemma 13.** A feasible solution $w'$ to $LP_4$ can be obtained such that $Cost_{kFLP_{sp}}(w') \leq 12LP_{opt}$.

**Proof.** $w'$ is defined in the same way as in Lemma (8). Feasibility w.r.t constraints (16) and (17) holds similar to knapsack median. We need to show that constraint (18) is feasible.

$$\sum_{i \in N_{CS}} w_i' = \sum_{j \in CS} \sum_{j' \in CS \cap G_j} \sum_{i' \in N_{j'}} x_{i'j'}^{*} \leq \sum_{G_j} \sum_{j' \in CS \cap G_j} \sum_{i' \in N_{j'}} y_{i'}^{*} = k'$$

Bound on cost follows from Lemma (7), adding $\sum_{i \in N_{CS}} f_i w_i'$ to both the sides and observing that $w_i' \leq y_i^{*}$.

A pseudo-integral solution $\tilde{w}$, of cost at most $12LP_{opt}$, is obtained using the iterative algorithm in the same manner as for knapsack median.

A special case occurs when there are exactly two clusters in a MC, one of them being dense (say $j_d$) and other being sparse (say $j_s$), i.e. $p_j = q_j = 1$ and
$LP_4$ did not open any facility in $j_s$. (Note that when both clusters are sparse, $LP_4$ must have opened a facility in the MC). Let the set of these sparse clusters belonging to such MC’s be denoted as $C'_S$. Clusters in $C'_S$ are handled in the same manner as dense clusters. Set $C_S = C_S \setminus C'_S$.

For dense clusters and sparse clusters in $C'_S$, we obtain what we call an almost integral solution, within each cluster. A solution is said to be almost integral if at most one facility is opened fractionally. We first define the notion of cluster instances. Let $g \in C_D \cup C'_S$. Define the following linear program for cluster instance:

$$LP_5: \text{Minimize} \quad \text{Cost}_{FLP}(g) = \sum_{i \in N_g} (f_i + uc(i, g))z_i$$

subject to

$$u \sum_{i \in N_g} z_i = d_g$$

$$0 \leq z_i \leq 1$$

where $z$ denotes a solution to the cluster instance.

Let

$$b^f_g = \sum_{i \in N_g} f_i y^*_i$$

and

$$b^c_g = \sum_{i \in N_g} \sum_{j' \in c} x^*_i j'[c(i, j') + 2lC'_j']$$

be the budget on connection cost and facility opening cost, respectively, for the cluster centered at $g$.

**Lemma 14.** Let $g \in C_D \cup C'_S$, for cluster instance $S_g(g, N_g, d_g)$, a feasible solution $z$ can be obtained such that

1. $\sum_{i \in N_g} z_i \leq \sum_{i \in N_g} y^*_i$ i.e. $\text{Size}(z, N_g) \leq \text{Size}(y^*, N_g)$ and,

2. $\sum_{i \in N_g} (f_i + uc(i, g))z_i \leq b^f_g + b^c_g$

**Proof.** Let $i \in N_g$. Set $z_i = \sum_{j' \in c} x^*_i j'/u = \frac{l_i}{u} \leq y^*_i$. We will show that $z$ so defined is a feasible solution to the cluster instance. $u \sum_{i \in N_g} z_i = \sum_{i \in N_g} l_i = d_g$. Also, $u \sum_{i \in N_g} c(i, g)z_i = u \sum_{i \in N_g} c(i, g)(\frac{\sum_{j' \in c} x^*_i j'}{u}) = \sum_{i \in N_g} \sum_{j' \in c} c(i, g)x^*_i j' \leq$
Lemma 15. For a feasible solution $z$ to a cluster instance centered at $g \in C_D \cup C'_S$, we can construct an almost integral solution $z'$ with $\text{Size}(z', N_g) = \text{Size}(z, N_g)$ without increasing the cost.

Proof. Arrange the fractionally opened facilities in $z$ in non-decreasing order of $f_i + c(i, g)u$ and greedily transfer the total opening $\text{Size}(z, N_g)$ to them. Let $z'$ denote the new openings. Let $l'_i = z'_i u$. Note that $\sum_{i \in N_g} l'_i = u \sum_{i \in N_g} z'_i = d_g$. Clearly, $\sum_{i \in N_g} (f_i + c(i, g)u) z'_i \leq \sum_{i \in N_g} (f_i + c(i, g)u) z_i = b_g f_i + b_g$. □

3.2 Stage III: Obtaining an integrally open solution

In this section, we round the fractionally opened facilities obtained in the previous section. In the following lemma, we first handle the two fractionally opened facilities (if any) in sparse clusters.

Lemma 16. Given an optimal pseudo-integral $\tilde{w}$ for LP$_4$, an integrally open solution $\tilde{w}$ can be obtained such that

1. $\sum_{i \in N_{C_S}} \tilde{w}_i \leq k' + 1$.
2. $\text{CostkFLP}_{sp}(\tilde{w}) \leq \text{CostkFLP}_{sp}(\tilde{w}) + f_{max} \leq 13\text{LP}_{opt}$.

Proof. If there is no fractionally opened facility, then set $\tilde{w}_i = \tilde{w}_i \forall i \in F$. Else, let $\{i_1, i_2\}$ denote the two fractionally opened facilities in $\tilde{w}$. Set $\tilde{w}_i = \tilde{w}_i \forall i \in F \setminus \{i_1, i_2\}$ and $\tilde{w}_i = \tilde{w}_{i_2} = 1$. Clearly, $\sum_{i \in N_{C_S}} \tilde{w}_i = \sum_{i \in N_{C_S}} \tilde{w}_i + 1 = \leq k' + 1$. As in Knapsack median $f_{i_1} \tilde{w}_{i_1} + f_{i_2} \tilde{w}_{i_2} \leq f_{i_1} \tilde{w}_{i_1} + f_{i_2} \tilde{w}_{i_2} + f_{max} \Rightarrow \sum_{i \in N_{C_S}} f_i \tilde{w}_i \leq \sum_{i \in N_{C_S}} f_i \tilde{w}_i + f_{max}$. Hence, $\text{CostkFLP}_{sp}(\tilde{w}) \leq \text{CostkFLP}_{sp}(\tilde{w}) + f_{max} \leq 13\text{LP}_{opt}$. □

To round the fractionally opened facilities in $C_D \cup C'_S$, we write a meta-cluster instance. Let $G_j$ be a MC. Let $I_j$ be the set of fractionally opened facilities in $(C_D \cup C'_S) \cap G_j$ under solution $z'$. Further, let $B_j = \sum_{g \in (C_D \cup C'_S) \cap G_j, i \in I_j \cap N_g} (f_i + uc(i, g)) z'_i$ be the cost paid by facilities in $I_j$ including the opening cost and the
connection cost of the assignment coming onto them, under \( z' \). Let \( \mu_j = u \sum_{i \in I_j} z'_i \).

Then \( z' \) provides a feasible solution of cost no more than \( B \) to the following instance:

\[
\text{Minimize } \sum_{g \in (C_D \cup C'_G) \cap G_j, i \in I_j \cap N_g} (f_i + wc(i, g)) z_i \\
\text{subject to } \sum_{i \in I_j} z_i \geq \lfloor \mu_j / u \rfloor \quad (22)
\]

It’s a (covering) knapsack problem and an optimal integral solution \( \hat{z} \) to this can be obtained in polynomial time.

Let \( B = \sum_{G_j} B_j \). Note that, \( B = \sum_{G_j} B_j = \sum_{G_j} \sum_{g \in (C_D \cup C'_G) \cap G_j} (f_i + wc(i, g)) z'_{i} \leq \sum_{G_j} \sum_{g \in (C_D \cup C'_G) \cap G_j} (b^c_g + b^l_g) \leq (2l + 1) LP_{opt} \). A covering knapsack problem is solved for each meta-cluster \( G_j \) and an integrally open solution \( \hat{z} \) of total cost no more than \( B \) is obtained.

**Lemma 17.** Consider a meta-cluster \( G_j \). The demand of \( G_j \) and the demand coming on \( G_j \) from the children meta-clusters can be assigned to the facilities opened in \( G_j \) such that

1. Capacity is violated at most by a factor of \( 2 + \frac{1}{l} \) for \( l \geq 2 \).
2. Each dense cluster is self-sufficient i.e. its demand can be completely assigned within the cluster itself.
3. Demands are assigned only to facilities above them in the meta-cluster except for the root cluster of a root meta-cluster.
4. At most \( u \) units of demand in \( G_j \) remain un-assigned.
5. Demand coming from all the children meta-clusters are assigned to facilities within \( G_j \).
6. Total distance traveled by demand \( d_g \) of \( g \in C' \) to reach the centers of the clusters in which they are served is bounded by \( 2ld_p(c(g), \sigma(g)) \).

**Proof.** We first prove claim (1). Let \( G_j \) be a non leaf meta-cluster. Let \( \alpha_j \) denote the number of facilities integrally opened in all dense clusters of \( G_j \) \( \forall j \in C_D \) by Lemma (15). Let \( \alpha_j = \max\{q_j - 1, 0\} + \beta_j + \lfloor \mu_j / u \rfloor \). The total demand of \( G_j \) is at most \( u(q_j + \beta_j + \mu_j / u) + u(t_j + 1) \leq (\alpha_j + 2)u + (t_j + 1)u \) whereas the total available capacity is at least \( \alpha_j u \). Thus the capacity violation is bounded by \( \frac{(\alpha_j + 2)u + (t_j + 1)u}{\alpha_j u} \leq \frac{(\alpha_j + 2)u + (\alpha_j + 2)u}{\alpha_j u} \) (as \( \alpha_j + 1 \geq t_j \)) = \( 2 + 4/\alpha_j \leq 2 + 4/(l - 1) \) (as \( \beta_j \geq p_j \) and hence \( \alpha_j \geq l - 1 \)).

Claim: \( t_j \leq \alpha_j + 1 \). We have \( \alpha_j = (q_j - 1) + \beta_j + \lfloor \mu_j / u \rfloor \implies \alpha_j + 1 = q_j + \beta_j + \lfloor \mu_j / u \rfloor \implies \alpha_j + 1 \geq q_j + p_j + \lfloor \mu_j / u \rfloor \implies \alpha_j + 1 \geq q_j + p_j \implies \alpha_j + 1 \geq t_j \).
A leaf meta-cluster with only one cluster in it is handled in the same way as in knapsack median. Next consider a leaf meta-cluster with at least two clusters in it. First consider the special case, when a leaf MC has \( \alpha_j = 1 \). Note that \( \lfloor \mu_j / u \rfloor \) must be 0 in this case. This meta-cluster contains exactly two clusters (both sparse or one dense and one sparse; it cannot have both dense). In case when there are two sparse clusters, then at least one facility is opened in this MC by \( LP_4 \) and total demand is no more than \( 2u \) ensuring that capacity violation is no more than 2. Next consider the case when there is one dense cluster (centered at \( j_d \)) and one sparse cluster (centered at \( j_s \)). If \( j_s \not\in \mathcal{C}_S' \), then we have one facility opened in each cluster, total demand is no more than \( 3u \) and total available capacity is at least \( 2u \), hence the violation of capacity is within factor 2. Else, (i.e. \( j_s \in \mathcal{C}_S' \)) then total demand in the meta-cluster is \( (\beta_j + \mu_j / u)u < 2u \) and with one facility opened in \( \tau(j_d) \), capacity violation is no more than 2.

For a leaf meta-cluster with \( \alpha_j \geq 2 \), the capacity violation is at most \( \frac{(\alpha_j + 2)u}{\alpha_j u} = 1 + \frac{2}{\alpha_j} \leq 2 \) for \( \alpha_j \geq 2 \).

Next, we prove claims (2), (3), (4) (5) and (6). As in the case of knapsack median, there is at most one cluster with no facility opened in it and it is a sparse cluster. Assignments are also done in the same manner with the following observation:

Let \( g \in \mathcal{C}_S \) such that no facility is opened in \( \tau(g) \). If \( g \) is not the center of a root cluster, \( \sigma(g) \in \mathcal{C}_P \) and no facility was opened by covering knapsack in \( \sigma(g) \), it may be required to serve a little less than \( 3u \) units of the demand which we do not allow. In such a case, demand of \( g \) is moved up in the tree until it reaches the root. If it remains at root as well, it is served by the parent meta-cluster. It is guaranteed that the demand will be served in the parent meta-cluster. The demand of \( g \) has to travel at most \( l \) edges to reach the root and since the edge costs are monotonically non-increasing as we go up the tree, it travels a distance of at most \( l c_g(g, \sigma(g)) \). Thus, the total distance traveled by demand \( d_g \) of a \( g \) to reach the centers of the clusters in which they are served is bounded by \( 2ld_g c_g(g, \sigma(g)) \).

**Lemma 18.** An integrally open solution \( \bar{\sigma} = < \bar{x}, \bar{y}> \) to Capacitated k-Facility Location Problem instance \((\mathcal{C}, \mathcal{F}, c, f, u, k)\) can be obtained with capacity violation at most \((2 + \epsilon)\) and using at most \( k + 1 \) facilities such that \( \text{Cost}(\bar{\sigma}, \mathcal{F}) = O(1/\epsilon)LP_{opt} \) for a fixed \( \epsilon > 0 \).

**Proof.** Choose \( l \) such that \( 4/(l-1) < \epsilon \). The capacity violation then follows from Lemma (17.1). Let \( I = \cup_{\mathcal{C}_i} I_j \). Set \( \bar{y}_i = \hat{w}_i \forall i \in \mathcal{N}_C, \bar{y}_i = \hat{z}_i \forall i \in \mathcal{N}_C \setminus I, \bar{y}_i = \hat{z}_i \forall i \in I. \) Let \( j' \in \mathcal{C} \) and \( j \in \mathcal{C}' \) be such that \( \mathcal{A}_{\sigma_x}(j', \mathcal{N}_j) > 0 \). For \( j'' \in \lambda(j), i \in \tau(j'') \), set \( \bar{x}_{ij''} = \sum_{i' \in \mathcal{N}_{j''}} \bar{y}_{i'} \theta(i, j'', j') \mathcal{A}_{\sigma_x}(j', \mathcal{N}_j) \). Borrowing notations from Lemma (12),
Finally, the bound \((6)\) is used to bound the cost of assigning a part of the demand \(d_j\) in a facility opened in \(\lambda(j)\cap \mathcal{C}_3\) is bounded by the cost of \(LP_4\). Lemma (15) is used to bound the cost of assignment to the facilities opened in \(\mathcal{N}_{C_2} \setminus I\) by the greedy approach. Finally, the bound \(B\) of the covering knapsack problem is used to bound the cost of assignments to facilities opened in \(I\).

Let \(j'' \in \mathcal{C}_3 \cap \lambda(j), i \in \sigma(j'')\). Then, \(c(j'', i) \leq c(j'', \sigma(j'')) \leq c(j, \sigma(j))\).

\[
\sum_{j \in J_2} \sum_{j'' \in \mathcal{C}_3 \cap \lambda(j)} \sum_{i \in \mathcal{N}_{J''}} \sum_{j' \in \mathcal{C}} \hat{x}_{ij'} c(i, j'') \leq \sum_{j \in J_2} d_j (1 - \sum_{i \in \mathcal{N}_{J'}} \hat{\hat{w}}_i) c(j, \sigma(j)) \leq 13LP_{opt} - (6)
\]

From Lemma (17) we have
\[
\hat{g}_i \leq (2 + 4/(l - 1))u \hat{y}_i \quad \forall i \in \mathcal{F} - (7),
\]

Multiplying both sides of the above equation by \(c(i, j'')\) and summing over all \(j'' \in \mathcal{C}_D, i \in \mathcal{N}_{J''} \setminus I\),
\[
\sum_{j'' \in \mathcal{C}_D} \sum_{i \in \mathcal{N}_{J''} \setminus I} \hat{g}_i c(i, j'') \leq (2 + 4/(l - 1))u \sum_{j'' \in \mathcal{C}_D} \sum_{i \in \mathcal{N}_{J''} \setminus I} z'_i c(i, j'')
\]

Adding \(\sum_{i \in \mathcal{N}_{J''} \setminus I} f_i z'_i\) to both the sides, we get
\[
\sum_{j'' \in \mathcal{C}_D} \sum_{i \in \mathcal{N}_{J''} \setminus I} \hat{g}_i c(i, j'') + \sum_{i \in \mathcal{N}_{J''} \setminus I} f_i z'_i \leq (2 + 4/(l - 1)) \sum_{j'' \in \mathcal{C}_D} \sum_{i \in \mathcal{N}_{J''} \setminus I} (f_i + u c(i, j'')) z'_i - (8)
\]
Again multiplying both sides of equation (7) by \( c(i, j'') \), summing over all \( j'' \in C_D, i \in I \) and adding \( \sum_{i \in I} f_i \hat{z}_i \) to both the sides, we get,

\[
\sum_{j'' \in C_D} \sum_{i \in I} \hat{g}_i c(i, j'') + \sum_{i \in I} f_i \hat{z}_i \leq (2 + 4/(l-1)) \sum_{j'' \in C_D, i \in I} (f_i + u c(i, j'')) \hat{z}_i
\]

\[
\leq (2 + 4/(l-1)) \sum_{j'' \in C_D} \sum_{i \in I} (f_i + u c(i, j'')) z'_i - (9)
\]

Last inequality holds as \( z' \) is a feasible solution to Knapsack

Adding equations (8) and (9), we get

\[
\sum_{j'' \in C_D} \sum_{i \in N_{j''}} \hat{g}_i c(i, j'') + \sum_{i \in N_{C_D}} f_i \hat{z}_i \leq (2 + 4/(l-1)) \sum_{j'' \in C_D} \sum_{i \in N_{j''} \setminus I} (f_i + u c(i, j'')) z'_i + \sum_{j'' \in C_D} \sum_{i \in N_{C_D}} (f_i + u c(i, j'')) z'_i
\]

\[
(2 + 4/(l-1)) B \leq (2l + 1)(2 + 4/(l-1)) LP_{opt} - (10) \text{ (by Lemma (8) and (14))}
\]

Equation (10) bounds the cost of assigning the total demand accumulated at \( j'' \) to \( i \) including its own demand and the demand coming from \( J_2 \).

Adding (2), (5), (6) and (10), we get that the cost of serving the demand of any client \( j' \in C \) by appropriate facilities is bounded by \((2l + 1 + 26l + 13 + (2l + 1)(2 + 4/(l-1)) LP_{opt} = O(l) LP_{opt} = O(1/\epsilon) LP_{opt}.
\]

\[
\square
\]

4 Capacitated Facility Location Problem

Capacitated Facility Location Problem is a special case of Capacitated \( k \)-Facility Location Problem for which \( k = |F| \). The natural LP-relaxation for instance (\( C, F, c, f, u \)) of unif-FLP is given as follows.

\[
LP_6 : \text{Minimize } \text{CostFLP}(x, y) = \sum_{i \in F} f_i y_i + \sum_{i \in F} \sum_{j \in C} c(i, j) x_{ij}
\]

subject to

\[
\sum_{i \in F} x_{ij} = 1 \quad \forall j \in C \quad (23)
\]

\[
\sum_{j \in C} x_{ij} \leq u y_i \quad \forall i \in F \quad (24)
\]

\[
x_{ij} \leq y_i \quad \forall i \in F, \ j \in C \quad (25)
\]

\[
x_{ij} \geq 0, \ y_i \geq 0
\]

Let \( \sigma^* = < x^*, y^* > \) denote the optimal solution of \( LP_6 \). Let \( \text{CostFLP}(x, y) = \text{Cost}_f(\sigma, T) + \text{Cost}_s(\sigma, T) \) denote the total opening cost and the connection cost paid by all clients for getting service from a given subset of facilities \( T \) under solution \( \sigma \). Clusters are formed the same way as in knapsack median with \( l = 2 \). Sparse clusters are easy to handle in this case as there is no limit on the
count on the opened facilities. For a sparse cluster centered at \( j \), we open the cheapest facility \( i^* \) in \( \text{ball}(j) \), close all other facilities in the cluster and shift their demands to \( i^* \). Let \( \hat{\sigma} = < \hat{x}, \hat{y} > \) be the solution so obtained.

**Lemma 19.** The solution \( \hat{\sigma} = < \hat{x}, \hat{y} > \) satisfies the following:

1. it is integrally open and feasible
2. \( \text{Cost}(\hat{\sigma}, N_{C_\sigma}) \leq O(1) LP_{opt} \).

**Proof.** For each \( j \in C_S \), claim (1) follows as \( i^* \) is fully open and \( d_j < u \). Next, we prove claim (2). Let \( j \in C_S \), \( i \in N_j \) and \( j' \in C \) be such that \( x^*_i j' > 0 \). Then, by Lemma (13), \( c(j, j') \leq 2c(i, j') + 2R_{j'} \). Also, since \( i^* \in \text{ball}(j) \), we have \( c(i^*, j) \leq R_j \). Thus, \( c(i^*, j') \leq c(j, j') + c(i^*, j) \leq c(j, j') + R_j \). If \( R_j < c(j, j') \) then \( c(i^*, j') < 2c(i, j') \leq 4c(i, j') + 4R_{j'} \) else \( c(i^*, j') \leq R_j \leq 4R_j \). Adding \( \sigma \) and \( \hat{\sigma} \) to get

\[
\sum_{i \in N_j} x^*_i j' (4c(i, j') + 4R_{j'}) \leq \text{Cost}_{\text{opt}}(\text{ball}(j))
\]

Thus,

\[
\text{Cost}_{\text{opt}}(j', \hat{\sigma}, N_{C_{\hat{\sigma}}}) \leq 4\text{Cost}_{\text{opt}}(j', \sigma^*, N_{C_{\sigma}}) + 8\hat{C}_{j'} A_{\sigma^*}(j', N_{C_{\sigma}})
\]

Also,

\[
\text{Cost}_{\text{opt}}(\hat{\sigma}, N_{C_{\hat{\sigma}}}) \leq 2\text{Cost}_{\text{opt}}(\sigma^*, N_{C_{\sigma}})
\]

Equation (2) follows as \( i^* \) is cheapest and \( \text{Size}(y^*, \text{ball}(j)) \geq 1/2 \). Adding equation (1) over all \( j' \in C \), we get

\[
\text{Cost}_{\text{opt}}(\hat{\sigma}, N_{C_{\hat{\sigma}}}) \leq 4\text{Cost}_{\text{opt}}(\sigma^*, N_{C_{\sigma}}) + 8\sum_{j' \in C} \hat{C}_{j'} A_{\sigma^*}(j', N_{C_{\sigma}})
\]

Adding equation (2) and (3) we get claim (2).

\[\square\]

Dense clusters are dealt in the same manner as in section 3.1. Lemma (14) and (15) continue to hold.

Let \( \text{Cl-Cost}_f(z, N_{C_D}) = \sum_{i \in N_{C_D}} f_i z_i \) denote the total facility opening cost paid by all \( i \in N_{C_D} \) under a solution \( z \). Further, let \( l_i \) be the assignment of the demand of \( j \) to facility \( i \) in \( N_j \) with \( z_i \) opening, then, \( \text{Cl-Cost}_s(z, N_{C_D}) = \sum_{j \in C_D} \sum_{i \in N_j} c(i, j) l_i \), denote the total service cost paid by all \( j \in C_D \) for getting served by facilities in \( N_{C_D} \) under solution \( z \). Then, \( \text{Cost}_{\text{opt}}(z, N_{C_D}) = \text{Cl-Cost}_f(z, N_{C_D}) + \text{Cl-Cost}_s(z, N_{C_D}) \).

**Lemma 20.** Let \( d_j \geq u \) and \( 0 < \epsilon < 1/2 \) be fixed. Given an almost integral solution \( \bar{z}' \) and assignment \( \bar{l}' \) as obtained in Lemma (13), an integrally open solution \( \bar{z} \) and assignment \( \bar{l} \) (possibly fractional) can be obtained such that

1. \( \bar{l}_i \leq (1 + \epsilon) \bar{z}_i u \forall i \in N_j \), and \( \sum_{i \in N_j} \bar{l}_i = \sum_{i \in N_j} l'_i = d_j \)
2. Cost_{C1}(\hat{z}, N'_j) \leq (1/\epsilon) Cost_{C1}(z', N'_j) \leq (1/\epsilon)(b'_j + b''_j)

Proof. We will construct solution \hat{z} and assignment \hat{l} such that they satisfy the following equations along with the claims (1) and (2):

\[ \sum_{i \in N_j} f_i \hat{z}_i \leq \frac{1}{\epsilon} \sum_{i \in N_j} f_i z'_i \quad \forall j \quad \forall i \in C_D, \quad (26) \]

\[ \sum_{i \in N_j} c(i, j) \hat{l}_i \leq (1 + \epsilon) \sum_{i \in N_j} c(i, j) l'_i \quad \forall j \quad \forall i \in C_D \quad (27) \]

Adding (26) and (27) we get claim (2). We now proceed to prove claims (1), (26) and (27). If there is no fractionally open facility, we do nothing, i.e. set \hat{z} = z'. Else, there is exactly one fractional facility, say \hat{i}_1, and at least one integral facility, say \hat{i}_2, as \text{Size}(z', N'_j) \geq 1. There are two possibilities w.r.t \hat{i}_1,

1. \hat{z}'_{\hat{i}_1} < \epsilon \\
2. \hat{z}'_{\hat{i}_1} \geq \epsilon

In first case, close \hat{i}_1 and shift its demand to \hat{i}_2 at a loss of factor (1 + \epsilon) in its capacity. Note that \ell'_{\hat{i}_1} < \epsilon u while \ell'_{\hat{i}_2} = u. Thus, set \hat{z}_{\hat{i}_1} = 0, \hat{l}_{\hat{i}_1} = 0 and \hat{l}_{\hat{i}_2} = \ell'_{\hat{i}_1} + \ell'_{\hat{i}_2} < (1 + \epsilon)u = (1 + \epsilon)u \hat{z}_{\hat{i}_2}. Also, \hat{l}_{\hat{i}_2} \leq (1 + \epsilon) \ell'_{\hat{i}_2}. Then, c(\hat{i}_2, j) \hat{l}_{\hat{i}_2} \leq (1 + \epsilon) c(\hat{i}_2, j) \ell'_{\hat{i}_2}. There is no loss in facility cost in this case.

In second case, simply open \hat{i}_1, at a loss of 1/\epsilon in facility cost. Set \hat{z}_{\hat{i}_1} = 1. Then, \hat{f}_{\hat{i}_1} \hat{z}_{\hat{i}_1} \leq (1/\epsilon) \hat{f}_{\hat{i}_1} \hat{z}'_{\hat{i}_1}. There is no loss in connection cost in this case.

\[ \sum_{i \in N_j} \hat{l}_i = \sum_{i \in N_j} l'_i = d_j \quad \forall j \quad \forall i \in C_D \quad \text{holds clearly.} \]

Lemma 21. Let 0 < \epsilon < 1/2 be fixed. An integrally open solution \bar{\sigma} = \langle \bar{x}, \bar{y} \rangle to (C, F, c, f, u) can be obtained such that,

1. \sum_{j \in C} \bar{x}_{ij} \leq (1 + \epsilon) \bar{y}_i u \quad \forall i \in F, \\
2. Cost_{FLP}(\bar{x}, \bar{y}) \leq O(1/\epsilon) LP_{opt}

Proof. Set \bar{y}_i = \bar{y}_i \forall i \in N_{C2}, \bar{y}_i = \bar{z}_i \forall i \in N_{C_D}, \text{ and } \bar{x}_{ij'} = \bar{x}_{ij'} \forall i \in N_{C_S}, j' \in C \text{ and for } j \in C_D, j' \in C, i \in N_j, \text{ let } \bar{x}_{ij'} = \frac{\bar{l}_i}{d_j} \sum_{i' \in N_j} \bar{x}_{i'j} = \frac{\bar{l}_i}{d_j} A_{\sigma}(j', N_j). \text{ Feasibility wrt constraints (22) and (25): } \bar{x}_{ij'} \leq \frac{\bar{l}_i}{d_j} \leq 1 = \bar{z}_i = \bar{y}_i, \text{ and } A_{\bar{\sigma}}(j', N_j) = \sum_{i \in N_j} \bar{x}_{ij'} = \sum_{i \in N_j} \frac{\bar{l}_i}{d_j} A_{\sigma}(j', N_j) = A_{\sigma}(j', N_j). \text{ Then, the following holds:}

a. \sum_{j' \in C} \bar{x}_{ij'} \leq (1 + \epsilon) \bar{y}_i u \forall i \in N_{C_D}, \\
b. Cost(\bar{\sigma}, N_{C_D}) \leq O(1/\epsilon) LP_{opt}
Claim (a) follows from Lemma (20) as follows: Let $j \in C_D, i \in N_j, j' \in C$.

Then, $\sum_{j' \in \mathcal{C}} x_{ij'} = \sum_{j' \in \mathcal{C}} \frac{l_i}{d_{j'}} A_{\sigma^*}(j', N_j) = \hat{l}_i \leq (1 + \epsilon) u \hat{z}_i = (1 + \epsilon) u \bar{y}_i$.

From Lemma (4), we have

$$\sum_{j \in \mathcal{C}} \sum_{j' \in \mathcal{C}} c(j, j') A_{\sigma^*}(j', N_j) \leq 6LP_{opt} - (1).$$

From the proof of Lemma (3), we also have

$$\sum_{j \in \mathcal{C}} \sum_{i \in \mathcal{N}_j} \sum_{j' \in \mathcal{C}} \left( c(i, j') + 4 \hat{C}_{j'} \right) x_{ij'}^* \leq 5LP_{opt}.$$

Summing claim (2) of Lemma (20) over all $j \in \mathcal{C}_D$, we get

$$\text{Cost}_{\mathcal{C}_D}(\hat{z}, \mathcal{N}_{\mathcal{C}_D}) \leq (1/\epsilon) \sum_{i \in \mathcal{N}_{\mathcal{C}_D}} \left( b_f j + b_c j \right) = (1/\epsilon) \sum_{i \in \mathcal{N}_{\mathcal{C}_D}} \sum_{j \in \mathcal{C}_D} \sum_{i \in \mathcal{N}_j} \sum_{j' \in \mathcal{C}} \left( c(i, j') + 4 \hat{C}_{j'} \right) x_{ij'}^* \leq (1/\epsilon) \sum_{i \in \mathcal{N}_{\mathcal{C}_D}} f_i y_i^* + 5LP_{opt} = O(1/\epsilon)LP_{opt} - (2).$$

Adding equation (1) and (2), we get claim (b), which along with claim (2) of Lemma (19) implies the desired bound.

$$\square$$

5 Conclusion

We presented first constant factor approximation algorithm for (uniform) capacitated knapsack median problem violating the budget by an additional cost of $f_{max}$ in the facility opening cost and a loss of factor $(3 + \epsilon)$ in capacity. We also gave a first deterministic constant factor $(O(1/\epsilon))$ approximation for uniform capacitated $k$ facility location at a loss of $(2 + \epsilon)$ in capacity and using at most $k + 1$ facilities. We also improve upon the approximation factor of $(O(1/\epsilon^2))$ by Byrka et al [5] which uses dependent rounding to round the fractional solution.

We also gave a constant factor $(O(1/\epsilon))$ approximation for uniform capacitated facility location at a loss of $(1 + \epsilon)$ in capacity. The result shows that the natural LP is not too bad.

It would be interesting to extend our results to non-uniform capacities. Conflicting requirement of facility costs and capacities makes the problem challenging.

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