The vacuum energy density of QCD with \( n_f = 3 \) Quark Flavors

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ABSTRACT

An equivariant BRST-construction is used to define the continuum \( SU(3) \) gauge theory on a finite torus. I corroborate previous results using renormalization group techniques by explicitly computing the measure on the moduli-space of the model with 3 quark flavors to two loops. I find that the correction to the maximum of the one-loop effective action is indeed of order \( g^2 \) in the critical covariant gauge. The leading logarithmic corrections from higher loops are also shown to be suppressed by at least one order of \( g^2 \). I therefore can relate the expectation value \( \kappa^4 = - \langle \text{Tr} \, \bar{\gamma}^2 \rangle \) of the moduli \( \bar{\gamma} \) to the asymptotic scale parameter of the modified minimal subtraction scheme. An immediate consequence is the non-perturbative result \( \langle \Theta_{\mu\mu} \rangle = -\frac{3}{8\pi^2} \kappa^4 = -\frac{(3 \exp 2)}{4\pi^4} \Lambda_{\text{MS}}^4 = -0.2228 \ldots \Lambda_{\text{MS}}^4 \) for the expectation value of the trace of the energy momentum tensor of QCD with three quark flavors. This relation compares favorably with phenomenological estimates of \( \langle \Theta_{\mu\mu} \rangle \) from QCD sumrules for the charmonium system and \( \Lambda_{\text{MS}}^{(3)} \) from \( \tau \)-decay.

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1 Introduction

The perturbative regime of ordinary SU(n) gauge theory with $n_f$ quarks in the fundamental representation of the group has been thoroughly investigated in covariant gauges. The ultraviolet divergences of the loop expansion can be regularized by analytic continuation in the dimension of Euclidean space-time. The infrared divergences of ordinary covariant perturbation theory due to the massless gauge bosons and ghosts are however an indication of the limited validity of this asymptotic expansion. The infrared problem is partially circumvented by the Operator Product Expansion (OPE) [1] which is parametrized by non-perturbative matrix elements of certain local operators. Values for these matrix elements generally have to be extracted from experiment or from lattice simulations. I will here calculate one such non-perturbative quantity, the vacuum expectation value of the trace of the energy momentum tensor. The computation of this expectation value in terms of the asymptotic scale parameter requires little more than an asymptotic analysis of the theory.

The basic idea for this calculation was presented in ref. [2]. One uses the fact that infrared divergences are regulated in a gauge invariant fashion by considering the theory on a compact space-time manifold such as a torus. Asymptotic freedom asserts that perturbation theory is reasonably accurate for sufficiently small physical volume of the compact Euclidean manifold and one computes the deviations that arise as the volume of the torus becomes large [2,3].

The main challenge is to define the model on a compact Euclidean space-time in a covariant manner. This separates (dynamical) degrees of freedom that propagate from those which do not (moduli). In a reasonably gauge-fixed theory on a torus, the latter are a finite set of parameters and the loop expansion for the dynamical degrees of freedom depends on their values. In covariant gauges, that is gauges which preserve all the isometries of the compact space-time, the moduli are generally space-time independent constants which are related to vacuum expectation values of the basic fields and their composites. They differ from (dimensionful) couplings in that they take values in a certain (moduli) space with a measure which has to be calculated. The volume dependence of this measure essentially determines the transition to the thermodynamic limit of the theory: at sufficiently small space-time volumes the measure on the moduli space is almost flat but it constrains the relevant moduli-space to a submanifold in the thermodynamic limit that is characterized by a few constants.

The problem of constructing a sensible covariantly gauge-fixed SU(n) theory with quarks in the fundamental representation in a compact Euclidean space-time was solved in ref. [3] using an equivariant BRST-quantization that eliminates the constant zero-modes of the Faddeev-Popov ghosts. The argument for this construction can be summarized as follows. With fermions in the fundamental representation of the gauge group on a torus, the gauge field $A_\mu$ and Faddeev-Popov ghosts in covariant gauges are periodic fields [3]. Global gauge invariance of the action implies that the constant (anti-) ghost in ordinary covariant gauges decouples. One can show that this leads to a vanishing partition function.
of such a continuum model. [One verifies\cite{2,4} that the Witten-index vanishes when the gauge fixing is viewed as a Topological Quantum Field Theory (TQFT) on the gauge group]. The equivariant BRST-quantization eliminates this problem without destroying the covariance of the gauge-fixed model. Generic zero-modes of the ghosts in this case are absent for the finite torus. The construction introduces constant ghosts with ghost number $0, \pm 1$ and $\pm 2$ that form part of the moduli space. There are no zero-modes of the antiperiodic fermions to contend with and the remaining zero-modes of the gauge field are bosonic and do not give a vanishing partition function. These zero modes are additional moduli, but as observed in\cite{6}, zero-modes of the gauge field on a torus with fermions in the fundamental representation are innocuous and can be neglected for sufficiently large volumes.

Let me also justify considering the theory on a symmetric torus instead of some other compact manifold. The predominant reason is that this manifold preserves translational invariance and that the thermodynamic limit of interest is thus probably approached smoothly and uniformly. A torus of finite volume breaks the rotational invariance of Euclidean space, but this symmetry is not associated with any scale and generally believed to be recovered in the thermodynamic limit. A torus with periodic boundary conditions for the gauge fields also admits anti-periodic fermions in the fundamental representation, i.e. quarks. In addition, the space of gauge orbits on any finite torus with periodic boundary conditions is connected and the gauge dependence of topological sectors\cite{4} is therefore absent. Because the orbit space is connected, there is no strong CP-violation on any finite torus with periodic boundary conditions for the gauge fields. Instantons and anti-instantons appear (and disappear) pairwise and the Pontryagin number vanishes. Due to translational invariance there is some hope that the net effect of such metastable configurations can be summarily described by constant moduli. We will see that the moduli in fact saturate the vacuum energy density completely and that there are no “other” non-perturbative contributions to this quantity. A resolution of the $U_A(1)$-problem in this context is on the other hand still outstanding. [The solution proposed by Veneziano\cite{7} appears promising if the axial vector ghost has an effective mass that is inversely related to the volume of the torus. The susceptibility would vanish for any finite torus but the $U_A(1)$ problem could nevertheless be solved in the thermodynamic limit where this (conjectured) ghost becomes massless. Such an axial (Goldstone) vector ghost has however not been identified.] Let me note here that the $U_A(1)$ “problem” is peculiar to covariant gauges, since its formulation requires the use of Goldstone’s theorem to saturate Ward identities of the conserved, but gauge dependent, anomalous $U_A(1)$ current.

A technical argument for using a symmetric torus as compact Euclidean space-time is that perturbation theory on this manifold is relatively straightforward and that the thermodynamic limit of the expressions is readily taken. The mode expansion can be explicitly performed and an analytic continuation in the dimension $D$ of the torus regulates the UV-divergent mode-sums of the loop expansion as it regulates the loop-integrals in ordinary (covariant) perturbation theory.

In II the model proposed in I was qualitatively analyzed in detail. It was argued that perturbation theory for gauge dependent quantities is reliable in the vicinity of fixed
points of the parameter and moduli space only. Technically one observes that “dangerous” leading log corrections which tend to invalidate the perturbative expansion are absent at certain points of the parameter and moduli space. As we will see explicitly, the problem in perturbative QCD generally is that some of the leading log corrections of the loop expansion arise because generic gauge parameters and moduli flow to certain nontrivial fixed points. Using the renormalization group (RG) equation, these corrections can in principle be resummed (in practice this is approximately possible only if the gauge parameters and moduli are already sufficiently close to a fixed point), leading to a nonanalytic dependence on the coupling of the gauge parameters and moduli. The mess can be partially avoided by perturbatively expanding at the fixed points for these parameters from the outset. The remaining non-analytic dependence of the perturbative expansion is then due to the scale dependence of the loop expansion parameter $g$, asymptotically described by $\Lambda_{\text{ASP}}$, the only physical parameter of the theory.

There is another (non-perturbative) argument for expanding at the fixed points of gauge parameters and moduli. For generic values of the parameters and moduli, the loop expansion is sensitive to small changes in their values. One may imagine that such a change could effectively be induced by taking certain non-perturbative configurations into account. An example in point is the scale dependence of the covariant gauge parameter $\alpha$. Non-perturbative arguments\[3\] suggest that $\alpha$ is scale independent, since it is a coupling parameter of a TQFT. In a perturbative calculation, $\alpha$ does, however, generically show an asymptotic scale dependence. It is suggestive to consider this apparent scale dependence as due to Gribov copies of (perturbative) configurations that have been neglected in the loop expansion. At the fixed point, the combined effect of the neglected Gribov copies is at least asymptotically not scale dependent. Since this scale dependence of $\alpha$ is absent for any Green function, one may argue that the influence of other Gribov regions cancels at the fixed point and that the perturbative answer in this case is asymptotically correct. A similar argument can be used to infer that contributions from neglected non-perturbative configurations do not change the value of the moduli at fixed points. A perturbative evaluation of the model is thus asymptotically self-consistent only in the vicinity of a fixed point in the parameter and moduli space. The one-loop calculations in $II$ revealed nontrivial fixed points in the moduli and parameter space of that model for $n_f \leq n$, and the corresponding gauges were called Critical Covariant Gauges (CCG). The 1-loop effective potential for the moduli implied that the global $SU(n)$ symmetry of the gauge-fixed theory is spontaneously broken to $U(1)^{n-1}$ in CCG and the (planar) contribution to the Wilson loop from the corresponding Goldstone bosons showed confining behavior.

In section 2 of this article I give a concise description of the $SU(n)$-model discussed in $II$ and also present a possible extension for $SU(n > 2)$. The action and moduli space one has to consider simplifies considerably in the special case $n_f = n$. The CCG with $\delta = 1, \alpha = 3$ for $n_f = n$ is a subset of the class of gauges with $\delta = 1$ in which some of the moduli decouple. The simplified action we will subsequently consider is given by (2.13). In section 3 the two-loop effective action for the remaining moduli is calculated to two loops. It is expressed in terms of renormalized quantities in section 4 where I also argue that higher order corrections are suppressed in the CCG with $\alpha = 3$. In section 5 the scale $\kappa$
of the spontaneously broken $SU(n)$ symmetry is finally related to $\Lambda_{\overline{MS}}$ and section 6 uses this result to obtain the expectation value of the trace of the energy momentum tensor in terms of $\Lambda_{\overline{MS}}$. I conclude by comparing with the phenomenology of QCD sumrules.

2 The Model

I will consider a special case of the covariant model of Euclidean QCD on a finite $D$-dimensional symmetric hypertorus $T_D = L \times L \times \ldots \times L$ proposed in II. The boundary conditions for the gauge field $A_\mu(x)$ and Faddeev-Popov ghosts $c(x)$ and $\bar{c}(x)$ are periodic and the model is described by the tree-level action

$$S_0 = S_C + 2 \int_{T_D} dx \text{Tr} \left[ \frac{1}{2\alpha} (\partial \cdot A(x))^2 + \delta \bar{c}(x) D^4 \cdot \partial c(x) + (1 - \delta) \bar{c}(x) \partial \cdot D^4 c(x) + \alpha \delta (1 - \delta) g^2 \bar{c}(x) c(x) c(x) - \delta [\bar{c}(x), c(x)] \gamma + \frac{1}{2\alpha g^2} \gamma^2 - \frac{1}{\alpha g} \partial \cdot A - \sigma c(x) c(x) - \alpha (1 - \delta) \phi \bar{c}(x) \bar{c}(x) - \frac{1}{g^2} \sigma \phi + \frac{1}{g} \gamma \bar{c}(x) + \frac{1}{g} \bar{c}(x) \right], \quad (2.1)$$

where $S_C$ is the Yang-Mills action of an $SU(n)$ gauge theory with $n_f$ quark flavors in the fundamental representation

$$S_C = \int_{T_D} \frac{1}{2} \text{Tr} F_{\mu\nu}(A) F_{\mu\nu}(A) + \sum_{i=1}^{n_f} \bar{\psi}_i (\not{D} - m_i) \psi_i. \quad (2.2)$$

The Euclidean Dirac operator here is

$$\not{D} \psi_i = \gamma_\mu (\partial_\mu + g A_\mu) \psi_i \quad (2.3)$$

and the field strength $F_{\mu\nu}(A)$ is

$$F_{\mu\nu}(A) = \partial_\mu A_\nu - \partial_\nu A_\mu + g [A_\mu, A_\nu] \quad (2.4)$$

Note that a CP-violating term proportional to the Pontryagin number $\int_T \text{Tr} F \wedge F$ vanishes on a torus with periodic boundary conditions\footnote{The Euclidean Dirac matrices $\gamma_\mu$ satisfy $\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\delta_\mu^\nu \mathbf{1}$. I will generally suppress color indices for notational clarity. Except for $\psi_i, \bar{\psi}_i$, vectors of the fundamental representations of $SU(n)$, all fields are traceless, anti-hermitian $n \times n$ matrices in this notation and transform under the adjoint representation of the group. The commutator $[\cdot, \cdot]$ is graded by the ghost number.}. I therefore did not include such a term in (2.2). As discussed in the introduction, there is no strong CP violation on a finite torus with periodic gauge fields.

The action (2.1) is on-shell invariant under the equivariant BRST-symmetry generated by the operator $s$. Its action on the fields is

$$s A_\mu(x) = D^4_\mu c(x) - [\omega, A_\mu(x)]$$

$^2$The Euclidean Dirac matrices $\gamma_\mu$ satisfy $\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\delta_\mu^\nu \mathbf{1}$. I will generally suppress color indices for notational clarity. Except for $\psi_i, \bar{\psi}_i$, vectors of the fundamental representations of $SU(n)$, all fields are traceless, anti-hermitian $n \times n$ matrices in this notation and transform under the adjoint representation of the group. The commutator $[\cdot, \cdot]$ is graded by the ghost number.

$^3$"Torus" sectors with non-vanishing Pontryagin number exist on a torus with twisted boundary conditions. These however can only be imposed in the absence of quarks in the fundamental representation\footnote{\textsuperscript{3}}.
\[ sc(x) = -[\omega, c(x)] - \frac{g}{2} [c(x), c(x)] - \frac{1}{g} \phi \]

\[ s\omega = -\frac{1}{2} [\omega, \omega] + \phi \]

\[ s\phi = -[\omega, \phi] \]

\[ s\bar{c}(x) = -[\omega, \bar{c}(x)] + b(x) \]

\[ sb(x) = -[\omega, b(x)] + [\phi, \bar{c}(x)] \]

\[ s\sigma = -[\omega, \sigma] + \bar{\sigma} \]

\[ s\bar{\sigma} = -[\omega, \bar{\sigma}] + [\phi, \sigma] \]

\[ s\bar{\gamma} = -[\omega, \bar{\gamma}] + \gamma \]

\[ s\gamma = -[\omega, \gamma] + [\phi, \bar{\gamma}] \]

\[ s\psi_i(x) = -\omega \psi_i(x) - gc(x) \psi_i(x) \]

\[ s\bar{\psi}_i(x) = -\bar{\psi}_i(x) \omega - g \bar{\psi}_i(x) c(x) \]  

\[(2.5)\]

where

\[ D^A_{\mu} c(x) = \partial_{\mu} c(x) + g[A_{\mu}(x), c(x)] \]  

\[(2.6)\]

is the usual covariant derivative in the adjoint representation. In (2.1) the Nakanishi-Lautrup field \( b(x) \) has been eliminated using the equation of motion

\[ b(x) = (\bar{\gamma}/g - \partial \cdot A(x))/\alpha - \delta g[\bar{c}(x), c(x)]. \]  

\[(2.7)\]

It is straightforward to show that the BRST-operator defined above is nilpotent on any element of the graded algebra constructed from the fields of Table 1:

\[ s^2 = 0 \]  

\[(2.8)\]

| field | \( A_{\mu}(x) \) | \( \psi_i(x) \& \bar{\psi}_i(x) \) | \( c(x) \) | \( \bar{c}(x) \) | \( b(x) \) | \( \phi \) | \( \omega \) | \( \sigma \) | \( \bar{\sigma} \) | \( \bar{\gamma} \) | \( \gamma \) |
|-------|----------------|----------------|--------|--------|--------|------|------|------|------|------|------|
| dim   | 1              | 3/2            | 0      | 2      | 2      | 0    | 4    | 4    | 2    | 2    |
| \( \phi \Pi \) | 0              | 0              | 1      | -1     | 0      | 2    | 1    | -2   | -1   | 0    | 1    |

\textbf{Table 1.} Dimensions and ghost numbers of the fields.

The constant ghost \( \omega \) generates global gauge transformations of all the fields in the BRST algebra, except itself. The action does not depend on it and one restricts observables to the equivariant cohomology \( \Sigma \),

\[ \Sigma = \{ \mathcal{O} : \frac{\partial \mathcal{O}}{\partial \omega} = 0; s\mathcal{O} = 0, \mathcal{O} \neq s\mathcal{F} \} \]  

\[(2.9)\]

where \( \mathcal{F} \) is itself \( \omega \)-independent. Since one is eventually interested only in expectation values of gauge invariant functionals of \( A, \bar{\psi} \) and \( \psi \), the notion of \textit{physical} observables can be further sharpened to functionals in the equivariant cohomology with vanishing ghost number\[\mathbb{Z}^\infty\].
The equation of motion of the Nakanishi-Lautrup field $b(x)$ is then modified to
\[ b(x) = \left( \gamma / g - \partial \cdot A(x) \right) / \alpha - \delta g [\bar{c}(x), c(x)] - \rho g A^2(x). \] (2.11)

Upon elimination of $b(x)$ using (2.11) the most general power counting renormalizable tree-level action of a covariantly gauge fixed $SU(n > 2)$ model depends on three gauge parameters $\alpha, \delta$ and $\rho$ and has the form
\[ S_1 = S_0 + 2\rho \int_{T_D} dx \Tr g(\partial \cdot A(x)) A^2(x) + g^2 \alpha(\delta - 1) [\bar{c}(x), c(x)] A^2(x) \]
\[ + g\alpha c(x) [A_\mu(x), \partial_\mu c(x)] + \frac{1}{2} \rho g^2 \alpha A^2(x) A^2(x) - \gamma A^2(x), \] (2.12)

where $\{\cdot, \cdot\}$ denotes the anticommutator. For $SU(n > 2)$ the analysis of $II$ in this sense is incomplete and should be repeated with the more general action (2.12). Let us only note here that a non-trivial value of the bosonic ghost $\bar{\gamma}$ would simultaneously regulate the infrared behavior of the tree level ghost- and gluon- propagators if a stable fixed point with $\delta \neq 0$ and $\rho \neq 0$ exists for the $SU(n > 2)$ model. The additional parameter $\rho$ could also soften the constraints found in $II$ and lead to non-trivial fixed points of the moduli- and parameter- space of an $SU(n > 2)$ gauge theory also for $n_f > n$. Since $\bar{\gamma}$ also couples to $A^2$ in (2.12), the analysis of $II$ becomes quite involved in this more general setting.

My objective here is the relation between the perturbative asymptotic scale $\Lambda_{UV}$ and the fixed points of the moduli space in the simplest possible scenario and I will not embark on an analysis of the rather complicated model described by (2.12). For the purist the following considerations apply only to an $SU(2)$ gauge theory, since the tree level action (2.11) in this case is the most general one.

I will in fact restrict myself to the RG-stable subset of covariant gauges with $\delta = 1, \rho = 0$. Using the result of $II$, one can expect a nontrivial fixed point in the moduli space of this restricted class of models for $n_f = n$ in $\alpha = 3$ gauge. Choosing $\delta = 1, \rho = 0$ greatly facilitates the calculations, because the moduli $\phi$ and $\sigma$ can be integrated out and there is no $\bar{\gamma} A^2$-vertex in this class of gauges. At $\delta = 1$ the coupling of $\phi$ to $[\bar{c}(x), c(x)]$ in (2.11) vanishes, and the integration over the $\phi$-moduli just sets $\sigma$ to zero. The quartic ghost coupling also vanishes at $\delta = 1$, further simplifying the calculation.

For $\delta = 1, \rho = 0$ the tree-level action (2.11) thus effectively is
\[ S = S_C + 2 \int_{T_D} dx \quad \Tr \left[ \frac{1}{2\alpha} (\partial \cdot A(x))^2 + \bar{c}(x) D^A \partial c(x) - [\bar{c}(x), c(x)] \bar{\gamma} + \frac{1}{2ag^2} \bar{\gamma}^2 \right. \]
\[ - \frac{1}{ag} \bar{\gamma} \partial \cdot A(x) + \frac{1}{g} \bar{\gamma} \bar{c}(x) + \frac{1}{g} \bar{\sigma} c(x) \left. \right] . \] (2.13)
The perturbative expansion depends only on the moduli $\bar{\gamma}$ of vanishing ghost number, since the moduli $\bar{\sigma}$ and $\gamma$ just serve to eliminate the (on a finite torus normalizable) constant modes of the ghost and anti-ghost. The action (2.13) is stable under renormalization and is the starting point of this calculation.

3 The Measure on the Moduli Space

The expectation value of an observable $O \in \Sigma$ is formally given by the path integral

$$\langle O \rangle = N \int d\bar{\gamma} d\bar{\sigma} d\gamma \int [D\psi D\bar{\psi} D\mathcal{A} Dc D\bar{c}] O \ exp S,$$

which defines the perturbative loop expansion. It was shown in I and II that (3.1) is generally normalizable, i.e. that $\langle 1 \rangle \neq 0$ for an SU(2) group. The global bosonic ghosts $\bar{\gamma}$ introduced by the equivariant BRST-algebra are the only relevant moduli parameters of the model[3]. We indicated in (3.1) that the integration over this finite dimensional moduli-space should be performed after the path integral over dynamical fields. As far as the dynamical fields are concerned, the moduli $\bar{\gamma}$ are thus parameters of the action. On a torus, the integrations over constant ghosts $\bar{\sigma}$ and $\gamma$ can be performed explicitly. They eliminate the zero-momentum modes of the Faddeev-Popov ghosts. The bosonic ghost $\bar{\gamma}$ in principle can also be eliminated, since $S$ depends on $\bar{\gamma}$ only quadratically. This would result in a nonlocal 4-point interaction of the dynamical ghosts that gives rise to infrared divergences in the loop expansion. A resummation of the perturbation series is therefore necessary and the term $\text{Tr} \bar{\gamma} [\bar{c}(x),c(x)]$ in (2.13) should be treated as an unconventional mass term for the dynamical ghosts. The ghost “masses”, $\bar{\gamma}$, are finally to be integrated with a certain measure. The measure, $d\mu(\bar{\gamma})$, on the moduli space

$$d\mu(\bar{\gamma}) = e^{\Gamma(\bar{\gamma})} \prod_{a=1}^{n^2-1} d\bar{\gamma}^a$$

is described by the effective action $\Gamma(\bar{\gamma})$. We will calculate $\Gamma(\bar{\gamma})$ order by order in the loop-expansion for the dynamical fields. Note that $\Gamma(\bar{\gamma})$ is proportional to the volume of Euclidean space-time to leading order in $L$. In the thermodynamic limit the measure (3.2) therefore effectively constrains the moduli space to the absolute maxima $\tilde{\bar{\gamma}}$ of $\Gamma(\bar{\gamma})$. The following calculation will determine this space.

3.1 The Tree-level Measure

To lowest order in $\hbar$ the measure on the moduli-space is Gaussian and apparently given by the quadratic term $\frac{L^D}{\alpha_q^2} \text{Tr} \bar{\gamma}^2$ of the action (2.13). Note that the term $\int \text{Tr} \bar{\gamma} \partial A$ in (2.13) vanishes on a torus with periodic boundary conditions for the gauge field $A_{\mu}(x)$. Since we eventually wish to consider the thermodynamic limit of the symmetric torus, $1/L$ is not a very useful mass scale. Introducing an arbitrary, but in the limit $L \to \infty$ finite,
renormalization mass $\mu$, the tree-level effective action in $D = 4 - 2\varepsilon$ dimensions can be expressed in terms of the dimensionless quantities that also appear in the 1- and 2-loop contributions

$$\Gamma^{\text{tree}}(\bar{\gamma}) = \frac{L^D \text{Tr} \bar{\gamma}^2}{\alpha g^2} = -\frac{L^D}{\alpha g^2} \sum_{i=1}^{n} e_i^2 = -(\mu L)^D \frac{16\pi^2}{n\alpha \hat{g}^2} \sum_{1 \leq i < j \leq n} v_{ij}^2, \quad (3.3)$$

where $\hat{g} = g\mu^{-\varepsilon}$ is the dimensionless coupling constant and the $e_i, i = 1, \ldots, n$, are the real eigenvalues of the traceless hermitian matrix $i\bar{\gamma} = i\bar{\gamma}a^a$. Since $\sum_i e_i = 0$, the effective action is a function of the dimensionless differences

$$v_{ij} := \frac{e_i - e_j}{4\pi \mu^2}, \quad (3.4)$$

of the eigenvalues only.

Note that (3.3) is the effective action in terms of bare parameters $\alpha = Z_\alpha \alpha_R, \hat{g} = Z_g \hat{g}_R$ and $\bar{\gamma} = Z_{\bar{\gamma}} \bar{\gamma}_R$. By calculating the (regularized) effective action to two loops, I will determine the first few coefficients in the loop-expansion of some of these renormalization constants in the $\overline{\text{MS}}$-scheme. The coupling $g$ and gauge parameter $\alpha$ are also related to vertices of the dynamical fields and one can thus check the validity of Ward-identities to some extent explicitly.

Before calculating loop corrections to the effective action note the following interesting fact. Let us for the moment ignore the constraints imposed by the BRST-algebra (2.5) and regard $\bar{\gamma}$ as the zero-momentum mode of a background field $\gamma(x)$. The coupling $\bar{\gamma}(x) \partial \cdot A(x)$ of $\bar{\gamma}(x)$ to the longitudinal gauge field in (2.13) in this case precisely compensates the $\bar{\gamma}^2(x)$-term of the tree-level effective action for all non-zero momentum modes of $\bar{\gamma}(x)$. The effective action for non-constant modes of $\bar{\gamma}(x)$ in fact would vanish to all orders in the loop expansion due to a supersymmetric compensation between ghosts and longitudinal gluons. This supersymmetric compensation is absent for the constant part $\bar{\gamma}$ of $\bar{\gamma}(x)$. The effective action for constant $\bar{\gamma}$ does not vanish, and the corresponding measure on the moduli space is therefore not flat. In this sense the non-trivial measure on the moduli space we will find is a consequence of the lack of supersymmetric compensation between ghost and (unphysical) gluonic degrees of freedom with vanishing momentum. The same imbalance ensures that the partition function, proportional to the Witten index of the TQFT on the gauge group $[3, 4]$, does not vanish. This mechanism is reminiscent of the lack of supersymmetric compensation for the ground state in models with unbroken supersymmetry and thus perhaps worth noting. The following will determine the ground state configuration of the interacting theory.

### 3.2 The One-Loop Measure

The measure on the moduli space is Gaussian to lowest order in $\hbar$ only. The dynamics depends on the value of $\bar{\gamma}$ and the one-loop radiative corrections to the effective action are depicted in Fig. 1. These contributions to the effective action were evaluated in $II$
(and for $SU(2)$ also in$^3$). For a $D = 4 - 2\varepsilon$ dimensional torus, the leading contribution for $L\mu \sim \infty$ is$^2$

$$\Gamma^{1-\text{loop}}(\bar{\gamma}) = (\mu L)^D \sum_{1 \leq i < j \leq n} 2v^2_{ij} - \varepsilon \cos(\frac{1}{2}\pi \varepsilon)\Gamma(\varepsilon - 2) \ . \quad (3.5)$$

![Diagram](image)

Legend:

- $\bar{\gamma}$ - insertion
- $\rightarrow$ - ghost-propagation

Fig. 1: Diagrammatic representation of 1-loop contributions to the effective action of the moduli.

Defining

$$t_{ij} := \gamma_E - 1 + \frac{1}{2} \ln \left( v^2_{ij} \right) , \quad (3.6)$$

where $\gamma_E = 0.577216 \ldots$ is Euler's constant, (3.3) for $\varepsilon \sim 0$ has the expansion

$$\Gamma^{1-\text{loop}}(\bar{\gamma}) = (\mu L)^D \sum_{1 \leq i < j \leq n} v^2_{ij} \left( \frac{1}{\varepsilon} + \frac{1}{2} - t_{ij} + O(\varepsilon) \right) , \quad (3.7)$$

Note that $\Gamma^{1-\text{loop}}$ does not depend on $\alpha$ nor on $\hat{g}$. Since $\bar{\gamma} = Z_{\bar{\gamma}} \bar{\gamma}_H$ with $Z_{\bar{\gamma}} - 1$ of $O(\hat{g}_R^2/\varepsilon)$, $\Gamma^{1-\text{loop}}$ gives rise to a term of order $\hat{g}_R^2$ in the renormalized two-loop effective action. This contribution is proportional to

$$\left. \frac{\partial}{\partial s} \Gamma^{1-\text{loop}}(s\bar{\gamma}) \right|_{s=1} = (\mu L)^D \sum_{1 \leq i < j \leq n} (-2)(v^2_{ij})^{1-\varepsilon/2} \cos(\frac{1}{2}\pi \varepsilon)\Gamma(\varepsilon - 1) \ . \quad (3.8)$$

where the term of $O(\varepsilon)$ still gives a finite contribution to the renormalized two-loop effective action, since the coefficient $Z_{\bar{\gamma}} - 1$ is of $O(\hat{g}_R^2/\varepsilon)$.

### 3.3 The Two-Loop Measure

The $\bar{\gamma}$-dependent part of the (unrenormalized) 2-loop effective action is depicted in Fig. 2. With the methods of $II$, the leading contribution for $\mu L \sim \infty$ can be computed for a $D$-dimensional torus. The zero-modes of the ghosts are absent in the mode expansion and the zero-modes of the gluon field give a subleading contribution for large $\mu L$. For the leading contribution to the effective action the mode-sums can be replaced by integrals
since they are IR-finite in $D > 2$ for $\bar{\gamma} \neq 0$. In appendix A these integrals are evaluated using conventional Feynman techniques with the result \((8.11)\), valid for $2 < D < 4$. For $\varepsilon \sim 0$ the expansion of \((8.11)\) has the form

$$
\Gamma^{2-\text{loop}}(\bar{\gamma}) \stackrel{\varepsilon \to 0}{=} (\mu L)^D \frac{n g^2}{32\pi^2} \sum_{1 \leq i < j \leq n} v_{ij}^2 \left\{ \frac{3 - \alpha}{2\varepsilon^2} + \frac{2 - (3 - \alpha)[1/4 + t_{ij}]}{\varepsilon} + 3 - 4t_{ij} \right\} + (\alpha - 3)\{\text{finite terms}\} + O(\varepsilon) \right\}.
$$

(3.9)

Fig. 2: a) Diagrammatic representation of the $\bar{\gamma}$-dependent 2-loop contributions to the effective action of the moduli. b) The contribution with the tree-level ghost propagator (hatched) for fixed moduli $\bar{\gamma} \neq 0$. c) The $\bar{\gamma}$-independent subtraction at $\bar{\gamma} = 0$.

Note that the terms of \((3.9)\) proportional to $t_{ij}/\varepsilon$ and $1/\varepsilon^2$ vanish for $\alpha = 3$. The two-loop contribution to the effective action is therefore of order $\hat{g}^2$ instead of order $\hat{g}^2 t_{ij}$ compared to the one-loop effective action \((3.7)\) in this gauge. We will see that this suppression of the two loop corrections in $\alpha = 3$ gauge is not accidental and that higher order terms of the form $\hat{g}^{2n}s_{ij}^{n+1}$ with $n \geq 1$ are also absent in this gauge.

4 The Renormalized Two-Loop Effective Action

We have so far computed the dependence of the effective action on the bare variables $\bar{\gamma}, \hat{g}$ and $\alpha$ in $D < 4$ dimensions to two loops. To take the limit $\varepsilon \to 0$, the effective action has to be reexpressed in terms of renormalized quantities that remain finite in this limit. We could of course have computed $\Gamma$ as a function of $\bar{\gamma}_R, \hat{g}_R$ and $\alpha_R$ from the outset, by writing the action \((2.13)\) in terms of renormalized fields and couplings and including counterterm contributions in the computation of the effective action. The two methods are entirely equivalent, but the approach I chose requires less computational effort. It avoids the computation of an infinite set of (counterterm) diagrams which just give the derivative \((3.8)\) of the infinite set of 1-loop diagrams already computed.

To express the two-loop effective action by the renormalized quantities, we simply
substitute the expansions
\[
\bar{\gamma} = \gamma_R Z_{\bar{\gamma}} = \gamma_R (1 + \frac{\hat{g}_R^2}{16\pi^2} Z_{\bar{\gamma}}^{(1)} + \frac{\hat{g}_R^4}{(16\pi^2)^2} Z_{\bar{\gamma}}^{(2)} + O(\hat{g}_R^6))
\]
\[
\bar{g}^2 = \hat{g}_R^2 Z_{\bar{g}}^2 = \hat{g}_R^2 (1 + \frac{\hat{g}_R^2}{16\pi^2} Z_{\bar{g}}^{(1)} + \frac{\hat{g}_R^4}{(16\pi^2)^2} Z_{\bar{g}}^{(2)} + O(\hat{g}_R^6))
\]
\[
\alpha = \alpha_R Z_{\alpha} = \alpha_R (1 + \frac{\hat{g}_R^2}{16\pi^2} Z_{\alpha}^{(1)} + \frac{\hat{g}_R^4}{(16\pi^2)^2} Z_{\alpha}^{(2)} + O(\hat{g}_R^6)),
\]
for the bare quantities in in the previous expressions. In the \(MS\)-scheme the \(Z_j^{(i)}\) are determined by requiring that they cancel only the \(1/\varepsilon^k\) terms. To the order we calculated,
\[
\Gamma^{2\text{-}\text{loop}}(\gamma_R Z_{\bar{\gamma}}; \bar{g}_R^2 Z_{\bar{g}}^2, \alpha_R Z_{\alpha}) = \Gamma^{2\text{-}\text{loop}}(\gamma_R; \bar{g}_R, \alpha_R) + O(\hat{g}_R^4),
\]
and
\[
\Gamma^{1\text{-}\text{loop}}(\gamma_R Z_{\bar{\gamma}}) = \Gamma^{1\text{-}\text{loop}}(\gamma_R) + Z_{\bar{\gamma}}^{(1)} \frac{\hat{g}_R^2}{16\pi^2} \frac{\partial}{\partial s} \Gamma^{1\text{-}\text{loop}}(s\gamma_R) \bigg|_{s=1} + O(\hat{g}_R^4).
\]
Comparing (3.9) and (3.8), the \(v_{ij}^2 t_{ij}/\varepsilon\) term in (3.9) requires that we choose
\[
Z_{\bar{\gamma}}^{(1)} = \frac{n(\alpha_R - 3)}{4\varepsilon},
\]
in the \(MS\)-scheme. The remaining divergent terms of,
\[
\Gamma^{1\text{-}\text{loop}} + \Gamma^{2\text{-}\text{loop}} = (\mu L)^D \sum_{1 \leq i < j \leq n} v_{Rij}^2 \left( \frac{1}{\varepsilon} + \frac{\hat{g}_R^2 n}{16\pi^2} \left( \frac{3 - \alpha_R}{4\varepsilon^2} + \frac{5 + \alpha_R}{8\varepsilon} \right) \right) + \text{finite terms}
\]
have to be canceled by the tree-level effective action (3.3) expressed in terms of \(\gamma_R; \alpha_R\) and \(\hat{g}_R\)
\[
\Gamma^{\text{tree}}(\gamma_R Z_{\bar{\gamma}}; \hat{g}_R^2 Z_{\bar{g}}^2; \alpha_R Z_{\alpha}) = - (\mu L)^D \sum_{1 \leq i < j \leq n} \frac{16\pi^2 v_{Rij}^2}{n\alpha_R \hat{g}_R^2} \left( 1 + \frac{\hat{g}_R^2}{16\pi^2} (2Z_{\gamma}^{(1)} - Z^{(1)} - Z_{\alpha}^{(1)} + 2Z_{\gamma}^{(2)} - Z^{(2)} - Z_{\alpha}^{(2)}) \right) + O(\hat{g}_R^4)
\]
Comparing with (4.3) one obtains two relations for the \(Z\)'s in (4.1):
\[
Z^{(1)} + Z_{\alpha}^{(1)} = 2Z_{\gamma}^{(1)} - \frac{n\alpha_R}{\varepsilon} = - \frac{n(\alpha_R + 3)}{2\varepsilon}
\]
\[
2Z_{\gamma}^{(2)} = Z^{(2)} + Z_{\alpha}^{(2)} + Z^{(1)} Z_{\alpha}^{(1)} - \frac{n^2 (13\alpha_R^2 + 6\alpha_R + 9)}{16\varepsilon^2} + \frac{n^2 \alpha_R (5 + \alpha_R)}{8\varepsilon}
\]
The first of these is a check of the two-loop calculation, since \(Z^{(1)}\) and \(Z_{\alpha}^{(1)}\) can also be determined from the 1-loop radiative corrections to dynamical vertices. The second
relation in (4.7) gives $Z^{(2)}$ in terms of the renormalization constants for the couplings. With

\[
Z^{(1)} = -\frac{\beta_0}{\varepsilon} = \left(\frac{2}{3} n_f - \frac{11}{3} n\right)/\varepsilon
\]

\[
Z^{(1)}_\alpha = \left(\frac{13 - 3\alpha_R}{6} n - \frac{2}{3} n_f\right)/\varepsilon
\]

one verifies that the first relation in (4.7) indeed holds.

The terms proportional to $1/\varepsilon$ and $1/\varepsilon^2$ thus are seen to vanish through order $\hat{g}_R^2$ when the effective action is expressed in terms of $\bar{\gamma}_R$, $\hat{g}_R$ and $\alpha_R$. In the MS-scheme we have

\[
\Gamma(\bar{\gamma}_R, \hat{g}_R, \alpha_R) = (\mu L)^D \sum_{1 \leq i < j \leq n} v_{Rij}^2 \left\{-\frac{16\pi^2}{n\alpha_R \hat{g}_R^2} + 1/2 - t_{Rij}\right\}
\]

\[
+ \frac{n\hat{g}_R^2}{16\pi^2} (3/2 - 2t_{Rij} + (\alpha_R - 3)\text{finite terms}) + O(\hat{g}_R^4)
\]

To simplify expressions allow me to sometimes abuse notation in the following and denote renormalized quantities without an index $(R)$ whenever confusion with bare quantities is unlikely.

### 4.1 Higher Loop Corrections to the Effective Action

Since the coefficient $Z^{(1)}_\gamma$, given by (4.4), vanishes for $\alpha_R = 3$, the anomalous dimension of $\bar{\gamma}_R$ is of order $\hat{g}_R^4$ in this gauge. In II it was observed that $\alpha_R = 3$ is the stable non-trivial UV fixed point of the gauge parameter $\alpha_R$ for $n_f = n$ because $Z^{(1)}_\alpha$, given by (1.8), then vanishes as well. Using RG-arguments, it was shown\(^2\) that $\alpha = 3, \delta = 1, (\rho = 0)$ is the CCG for $n_f = n$. Remarkably the leading asymptotic correction of order $\hat{h}$ in the effective action (1.3) is of the form $\hat{g}_R^2 t_{Rij}$ rather than $\hat{g}_R^2 s_{Rij}^2$ for $\alpha_R = 3$. The two-loop contribution to the effective action in this gauge is thus suppressed by a factor $\hat{g}^2$ compared to the one-loop contribution. We will see explicitly that this is no accident and that in fact all higher loop contributions are similarly suppressed asymptotically in $\alpha_R = 3$ gauge.

It is not difficult to show that a term of order $\bar{\gamma}^2_{Rij}$ in the renormalized effective action is associated with an unrenormalized $k$-loop contribution that diverges as $\bar{\gamma}^2/\varepsilon^k$. I will now argue that a generic 1-Particle Irreducible (1PI) $k \geq 2$-loop vacuum diagram that depends on $\bar{\gamma}$ at most diverges like $\bar{\gamma}^2/\varepsilon^{k-1}$ for $\alpha = 3$. The vacuum diagram we consider obviously must have at least one ghost loop and there must be at least two insertions of $\hat{g}$ for it to depend on $\bar{\gamma}$. We select a ghost loop with at least one insertion of $\hat{g}$. For $n \geq 2$ the diagram is then generically of the form shown in Fig. 3, since a gluon must be emitted somewhere before the insertion of $\bar{\gamma}$ on the ghost loop and eventually must be reabsorbed by the same ghost loop if the diagram is 1PI.
Fig. 3: Diagrammatic representation of the skeleton of a 1PI contribution to the $\bar{\gamma}$-dependent effective action with two or more loops. Shaded areas denote generic parts of a multi-loop diagram that can be considered as a correction to the ghost-gluon vertex or of the two-point functions themselves. See the text for details.

The leading divergence of all the $k-2$ loop integrals apart from $l_1$ and $l_2$, is furthermore at most $1/\varepsilon^{k-2}$. The main point of the argument is that such a divergence (if it occurs) cannot depend on $\bar{\gamma}$ and is compensated by counterterms for the subdiagrams that do not depend on the mass-scale $\bar{\gamma}$. The reason is that an insertion of $\bar{\gamma}$ renders the ghost-gluon vertex and ghost selfenergy superficially convergent (since one of the ghost momenta factorizes in the gauge $\delta = 1$). The insertion of $\bar{\gamma}$ thus reduces the leading divergence of these sub-diagrams by at least one power of $\varepsilon$. A similar argument holds for the gluon polarization, since a single insertion of $(\tilde{\bar{\gamma}})_{ab} = f^{abc}\bar{\gamma}^c$ gives a vanishing contribution due to Bose symmetry. As far as the leading divergence of order $1/\varepsilon^{k-2}$ for the subdiagrams is concerned, the diagram of Fig. 3 is thus equivalent to the two-loop diagram of Fig. 2 multiplied by $(const./\varepsilon^{k-2})$. We have explicitly shown that the two-loop contribution (3.3) diverges only as $\bar{\gamma}^2/\varepsilon$ when $\alpha = 3$. The leading divergence of the $k \geq 2$-loop contribution is thus at most $\bar{\gamma}^2/\varepsilon^{k-1}$. The leading dependence on $\ln(\mu^2)$, viz. $t_{Rij}$, of the renormalized effective action in $D = 4$ dimensions evaluated to $N$-loops in $\alpha_R = 3, \delta = 1$ gauge in the $MS$-scheme is therefore of the form

$$\Gamma^{(N)}(\bar{\gamma}) = \frac{L^4}{16\pi^2} \sum_{1 \leq i < j \leq n} (e_i - e_j)^2 \left\{ -\frac{16\pi^2}{3ng^2} + \frac{1}{2} - t_{ij} + \sum_{k=1}^{N-1} c_k \left( g^2 t_{ij} \right)^k \right\}, \quad (4.10)$$

where the $c_k$ are $\mu$-independent coefficients determined by the $(k+1)$-loop contribution, $e_i, i = 1, \ldots, n$ are the $n$ real eigenvalues of the traceless hermitian matrix $i\bar{\gamma}R$, and $g = g_{MS}$ is the renormalized coupling constant in the minimal subtraction ($MS$) scheme. For $g^2 t_{ij}$ of $O(1)$ asymptotically, contributions to the effective action from two and more loops are thus suppressed relative to the one-loop contribution by at least one order in $g^2$. The perturbative calculation of the effective action in the limit $g \to 0$ thus gives a sensible asymptotic expansion in the gauge $\alpha = 3, \delta = 1, \rho = 0$. 

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5 The RG-invariant maximum of the effective action

Since the effective action $\Gamma(\bar{\gamma})$ in (4.2) is essentially proportional to the space-time volume for $(\mu L)^4 \sim \infty$, only absolute maxima $\bar{\gamma}$ of $\Gamma(\bar{\gamma})$ are of relevance in the evaluation of expectation values (3.1) of observables in the thermodynamic limit. We now determine the absolute maxima $\bar{\gamma}$ of $\Gamma(\bar{\gamma})$ in $\alpha = 3, \delta = 1$ gauge.

In general the eigenvalues of $\bar{\gamma}$ will be functions of the renormalization point $\mu$, and will either vanish or approach infinity as $\mu \to \infty$. For $n_f = n$, they however approach a finite value proportional to $\Lambda_{MS}$. Maxima $\bar{\gamma}$ of the effective action (4.10) for large $\mu^2/(\bar{e}_i - \bar{e}_j)$ satisfy

$$0 = \frac{8\pi^2}{\mathcal{L}^4} \frac{\partial}{\partial s} \Gamma^{(N)}(s\bar{\gamma}, g, \mu) \bigg|_{s=1} = \sum_{1 \leq i < j \leq n} (\bar{e}_i - \bar{e}_j)^2 \left\{ -\frac{16\pi^2}{3n g^2} + \frac{c_1 g^2}{2} - \bar{t}_{ij} \left[ 1 - g^2 \sum_{k=0}^{N-2} \tilde{c}_k \left( g^2 \bar{t}_{ij} \right)^k \right] \right\} ,$$

(5.1)

where

$$\tilde{c}_k(\mu) = c_{k+1} + \frac{1}{2} g^2 (k + 2) c_{k+2}$$

(5.2)

and $g^2$ is the running coupling constant of the $MS$-scheme. Its dependence on the renormalization point $\mu$ to two loops for sufficiently large $\mu/\Lambda_{MS}$ is\[8\]

$$\frac{g^2}{16\pi^2} = \frac{1}{\beta_0 \ln \mu^2/\Lambda_{MS}^2} - \frac{\beta_1}{\beta_0} \frac{\ln \mu^2/\Lambda_{MS}^2}{\ln \mu^2/\Lambda_{MS}^2} ,$$

(5.3)

where

$$\beta_0 = \frac{11n - 2n_f}{3} \quad \text{and} \quad \beta_1 = \frac{34n^2 - 13n_f + 3n_f/n}{3}$$

(5.4)

are the first two coefficients of the $\beta$-function and $\Lambda_{MS}$ is the so defined $\mu$-independent asymptotic scale parameter of the $MS$-scheme. For $n_f < \frac{1}{3}n$ quark flavors $g^2$ vanishes logarithmically as $\mu \to \infty$, implying asymptotic freedom. For (5.1) to hold in the limit $\mu \to \infty$, $\bar{t}_{ij}$ must be of order $1/g^2$ and we see that the sum in (5.1) is a correction of order $g^2$ to $\bar{t}_{ij}$. Since the perturbation series diverges for $N \to \infty$, this is more precisely true in the sense of an asymptotic expansion: for any fixed, but not necessarily small, order in the loop expansion, the contributions from two and more loops become negligible in the vicinity of the maximum $t_{ij} \sim \bar{t}_{ij}$ of the effective action as $\mu \to \infty$.

The condition (5.1) for the maximum of the effective action can be regarded as an asymptotic definition of the coupling $g$ in terms of the renormalization point $\mu$ and the scale $\kappa^4 = -\text{Tr} \bar{\gamma}^2$. The situation is similar to comparing the asymptotic scales of two perturbative renormalization schemes\[3\]. Using (5.3) to eliminate the coupling $g^2$ in favor of the asymptotic scale $\Lambda_{MS}$ of the $MS$-scheme, (5.1) in the limit $\mu \to \infty$ implies that

$$\frac{3n}{\sum_{1 \leq i < j \leq n} (\bar{e}_i - \bar{e}_j)^2} \left( 1 - \gamma_E - \frac{1}{2} \ln \left( \frac{\bar{e}_i - \bar{e}_j}{4\pi \mu^2} \right)^2 \right) \sum_{1 \leq i < j \leq n} (\bar{e}_i - \bar{e}_j)^2 = \frac{\beta_0 \ln \mu^2/\Lambda_{MS}^2}{\sum_{1 \leq i < j \leq n} (\bar{e}_i - \bar{e}_j)^2} ,$$

(5.5)
Eq. (5.5) gives the asymptotic scale dependence of the eigenvalues $\tilde{\epsilon}_i(\mu)$ corresponding to a maximum of the effective action. We see that the maximum approaches a finite limit as $\mu \to \infty$ only if $3n = \beta_0$. From (5.4), this is the case for

$$n_f = n$$

(5.6)
quark flavors.

For an $SU(2)$ group and two quark flavors, (5.5) can be explicitly solved for the one independent eigenvalue $\tilde{\epsilon} = \tilde{\epsilon}_1 = -\tilde{\epsilon}_2$ of $\tilde{\gamma}$,

$$\tilde{\epsilon} = \pm 2\pi e^{1-\gamma} \Lambda_{MS}^2 = \pm \frac{e}{2} \Lambda_{MS}^2 \sim \pm (1.1658 \Lambda_{MS})^2$$

(5.7)

We thus have that

$$\kappa^4(n = n_f = 2) = -\text{Tr} \tilde{\gamma}^2 = \tilde{\epsilon}_1^2 + \tilde{\epsilon}_2^2 = \frac{e^2}{2} \Lambda_{MS}^4 \sim (1.3864 \Lambda_{MS})^4$$

(5.8)

In the case of $SU(3)$ with three quark flavors, (5.5) by itself is not sufficient to determine the texture of the eigenvalues of $\tilde{\gamma}$. However, if the maxima of (4.10) are unique up to permutations of the eigenvalues $\tilde{\epsilon}_i$, the discrete symmetry $\tilde{\gamma} \rightarrow -\tilde{\gamma}$ of the effective action must be equivalent to a permutation of the eigenvalues $\tilde{\epsilon}_i$. For an $SU(3)$ group this requires that the absolute maxima of the effective action are permutations of

$$\tilde{\epsilon}_1 = -\tilde{\epsilon}_2 = \tilde{\epsilon}, \quad \tilde{\epsilon}_3 = 0$$

(5.9)

For $n_f = 3$, (5.5) then determines the value of $\tilde{\epsilon}$ in terms of $\Lambda_{MS}$

$$\tilde{\epsilon} = 2^{-2/3} e (4\pi \Lambda_{MS}^2 e^{-\gamma}) = 2^{-2/3} e \Lambda_{MS}^2 \sim (1.3086 \Lambda_{MS})^2$$

(5.10)

and the corresponding value for the scale

$$\kappa^4(n_f = n = 3) = -\text{Tr} \tilde{\gamma}^2 = 2^{-1/3} e^2 \Lambda_{MS}^4 \sim (1.5562 \Lambda_{MS})^4$$

(5.11)

For $SU(n > 3)$ groups and $n_f = n$, symmetry arguments and (5.5) alone are not sufficient to completely determine the absolute maxima of the effective action. The relation (5.5) however implies that one can find the fixed points in the limit $\mu \to \infty$ by (numerically) maximizing

$$\Gamma^{eff}(\tilde{\gamma}; \Lambda_{MS}) = -\frac{L^4}{32\pi^2} \sum_{1 \leq i < j \leq n} (e_i - e_j) \ln \frac{(e_i - e_j)^2}{(\exp 3) \Lambda_{MS}^4}$$

(5.12)

This is not to say that (5.12) is the effective potential for $\tilde{\gamma}$. It cannot be, since the anomalous dimensions of $\tilde{\gamma}$ and $\alpha$ only vanish to order $\hbar$ for $\alpha = 3$ and $n_f = n$. $\Gamma^{eff}$ therefore does not satisfy the correct renormalization group equation. We have however seen that higher loop corrections become negligible in the vicinity of maxima for $\mu/\Lambda_{MS} \sim \infty$, since (5.5) shows that they are of order $g^2$. $\Gamma^{eff}$ correctly describes the variation of the
effective action near its maxima in the limit \( g^2 \to 0 \) and thus correctly reproduces (5.3) in this limit. Higher loop corrections to (4.10) in general are not negligible sufficiently far from the maxima when \( |\ln \left| (e_i - e_j)/\mu^2 \right| | \gg 1/g^2 \). \( \Gamma^{\text{eff}} \) of (5.12) in this case differs significantly from the true effective action.

It was argued in II that the deviation of the true effective action from \( \Gamma^{\text{eff}} \) is of no importance in the thermodynamic limit, since the measure on the moduli space, proportional to \( e^{\Gamma(\bar{\gamma})} \), constrains the moduli space of \( \bar{\gamma} \) to the immediate vicinity of absolute maxima of the effective action. Fluctuations of the moduli vanish as \( 1/L^2 \). In the thermodynamic-(\( L \to \infty \)) and critical- (\( g^2 \to 0 \)) limit, the measure on the moduli space is thus effectively given by \( \Gamma^{\text{eff}} \).

The following remark is perhaps of interest: our perturbative evaluation is uncontrolled near \( \bar{\gamma} = 0 \) and the fact that \( \Gamma^{(N)}(0) \) and \( \Gamma^{\text{eff}}(0) \) vanish does not imply that the true effective action also vanishes for \( \bar{\gamma} = 0 \), since \( |\ln \left| (e_i - e_j)/\mu^2 \right| | \to \infty \gg 1/g^2 \) for any finite value of \( \mu \) in this case. Higher orders in the loop expansion are therefore much more important than the computed ones and an asymptotic expansion of the effective action is pretty useless near \( \bar{\gamma} = 0 \). One could imagine that after a resummation of the perturbative expansion, the factor \( \bar{\gamma}^2 \) of the effective action cancels for \( \bar{\gamma} \to 0 \) and that the true effective action has a finite or perhaps even divergent limit for \( \bar{\gamma} \to 0 \). In II it was argued that an absolute maximum of the true effective action at \( \bar{\gamma} = 0 \) would imply that nonperturbative effects dominate even asymptotically, since the absolute maximum of the effective action sets the asymptotic physical scale of the model – as we have seen, this maximum does not occur at \( \bar{\gamma} = 0 \) in any finite order of the loop expansion. This argument however does not a priori imply that the true effective action vanishes at \( \bar{\gamma} = 0 \), i.e. that \( \Gamma^{\text{eff}}(\bar{\gamma}) \) not only reproduces the true effective action near its absolute maximum, but also gives the difference in energy density between the broken and unbroken phases.

The next section shows that \( \Gamma^{\text{eff}}(\bar{\gamma}) \) does indeed reproduce the correct energy difference.

6 \( \langle \Theta_{\mu\mu} \rangle \) in terms of \( \Lambda_{\overline{\text{MS}}} \)

We can also compute the energy density difference between the broken phase with \( \langle \bar{\gamma}^2 \rangle = \bar{\gamma}^2 \neq 0 \) and the \( SU(n) \)-symmetric one with \( \langle \bar{\gamma}^2 \rangle = 0 \) by a change of scale, since Euclidean invariance relates the vacuum expectation value of a component of the energy momentum tensor \( \Theta_{\mu\nu} \) to the vacuum expectation value of its trace

\[
\langle \Theta_{\mu\nu} \rangle = \frac{\delta_{\mu\nu}}{4} \langle \Theta_{\rho\rho} \rangle \tag{6.1}
\]

and an insertion of \( \Theta_{\mu\mu} \) is the response to a change in scale. In the chiral limit with \( m_f = 0 \) the action (2.13) in \( D = 4 \) dimensions is invariant under an infinitesimal dilation and a corresponding change of integration variables,

\[
\delta x = -x \delta \lambda \quad \Rightarrow \delta L = -L \delta \lambda \\
\delta \Phi_i(x) = (d_i \Phi_i(x) - x_{\mu} \partial_{\mu} \Phi_i(x)) \delta \lambda \tag{6.2}
\]
where $\Phi_i$ is a generic field variable and $d_i$ is its canonical dimension as given in Table 1. Note that the change of variables also includes the moduli, which are integration variables and in this sense differ from mass parameters and coupling constants. The reason for the formal invariance of the massless theory under a change of scale of course is that there are no dimensionful parameters in $D = 4$. This is no longer true for the regularized model in $D < 4$ (or for that matter any other regularization of the model) and (6.2) is not a symmetry of the renormalized theory (which does depend on a physical mass scale such as $\Lambda_{\text{MS}}$ that does not vanish in the chiral limit). One can compute the scale anomaly from the fact that the (bare) coupling constants and fields of the dimensionally regularized model in $D < 4$ effectively change when the fields are transformed according to (6.2). This anomalous contribution to the trace of the energy momentum tensor of QCD was first derived by Collins, Duncan and Joglekar [10]. They found that the dilation (6.2) is equivalent to a zero-momentum insertion of the renormalized operator

$$\Theta_{\mu\nu} = \frac{\beta(g)}{2g} F_{\mu \nu} F_{\mu \nu}^a + \sum_f m_f \bar{\Psi}_f \Psi_f (1 + \gamma_m(g))$$  \hspace{1cm} (6.3)$$

in gauge invariant correlation functions. In (6.3) $\beta(g)$ is the $\beta$-function describing the scale dependence of the coupling constant and $\gamma_m(g)$ is the anomalous dimension of the quark masses. Naively one expects only an insertion of $\sum_f m_f \bar{\Psi}_f \Psi_f$ in the massive case. The rest of (6.3) is the anomalous contribution to the trace of the energy momentum tensor. The arguments of [10] also hold for the action (2.13) and the trace anomaly need not be recalculated here. It suffices to note that the action (2.13) differs from the model considered by [10] in the gauge fixing sector only. Since BRST-exact insertions in gauge invariant correlation functions vanish in any dimension, the net effect of a dilation on gauge invariant observables is an insertion of (6.3) also for our model. Taking the vacuum expectation value of (6.3)

$$\langle \Theta_{\mu\nu} \rangle = \frac{\beta(g)}{2g} \langle F_{\mu \nu} F_{\mu \nu}^a \rangle + \sum_f m_f \langle \bar{\Psi}_f \Psi_f \rangle (1 + \gamma_m(g))$$  \hspace{1cm} (6.4)$$

we see that non-vanishing gluon and quark condensates imply a reduction of the vacuum energy compared to a hypothetical scale invariant phase in which these condensates vanish.

Since a dilation (6.2) is equivalent to an insertion of $\Theta_{\mu\nu}$ at zero momentum, we can compute (6.4) from the effective action $\Gamma(\varphi)$ for the constant ghost $\varphi$ after all dynamical fields have been integrated out. The finite dimensional integral over the moduli $\varphi$ itself does not diverge and the limit $D \to 4$ of the (renormalized) effective action can be taken before performing these integrals. The trace of the energy momentum tensor is furthermore RG-invariant and therefore does not depend on the renormalization scale $\mu$. We can take the limit $\mu \to \infty$ to compute it. In this limit the change of scale (6.2) effectively results in an insertion of $-L^4 n \text{Tr} \gamma^2/(8\pi^2)$ since $\Gamma^{\text{eff}}(\gamma)$ describes the measure accurately in the vicinity of a maximum of the effective action. Since the anomalous dimension of $\gamma$ and $\alpha$ vanish to order $g^2$ in $\alpha = 3$ gauge, we obtain for $n_f = n$ flavors,

$$L^4 \langle \Theta_{\mu\nu} \rangle = \lim_{\mu \to \infty} \left. \left( 2s \frac{\partial}{\partial s} - 4L \frac{\partial}{\partial L} \right) \Gamma(s \gamma; g(\mu), \mu) \right|_{s=1}$$
\[ \begin{align*}
&= \left\langle (2s \frac{\partial}{\partial s} - 4L \frac{\partial}{\partial L}) \Gamma^{\text{eff}}(s\tilde{\gamma}; \Lambda_{\overline{\text{MS}}}) \right|_{s=1} \\
&= -\left\langle \frac{L^4}{8\pi^2} \sum_{1 \leq i < j \leq n} (e_i - e_j)^2 \right\rangle \\
&= \frac{L^4 n}{8\pi^2} \text{Tr} \tilde{\gamma}^2 = -\frac{L^4 n}{8\pi^2} \kappa^4(n_f = n)
\end{align*} \] (6.5)

The expectation value of the trace of the energy momentum tensor can thus be expressed in terms of the fixed point \( \tilde{\gamma} \) of the maximum of the effective action \( \Gamma(\tilde{\gamma}) \). Since the left hand side of (6.3) is the expectation value of a gauge invariant operator, it is worth emphasizing that the fixed point \( \tilde{\gamma} \) of the effective action is not gauge dependent although our perturbative evaluation of \( \tilde{\gamma} \) in terms of \( \Lambda_{\overline{\text{MS}}} \) for \( n_f = n \) flavors apparently was restricted to the gauge \( \alpha_R = 3, \delta = 1, \rho = 0 \). As emphasized in II, \( \alpha_R = 3 \) is however not just any particular gauge. It is the stable fixed point of the gauge parameter for \( n_f = n \). As shown in II this is related to the fact that \( Z_\alpha = 1 + O(g^4) \) at \( \alpha_R = 3 \) for \( n_f = n \). In this sense \( \lim_{\mu \to \infty} \alpha_R(\mu) = 3 \) for whatever gauge \( (\alpha_R \neq 0) \) one starts from at finite \( \mu \). [To have \( \lim_{\mu \to \infty} a_R(\mu) \neq 3 \) for \( n_f = n \), the bare gauge parameter \( a_B(\mu) \) would have to be \( \mu \)-dependent – invalidating the RG-analysis. Note that Landau gauge, \( \alpha_R = 0 \), is an unstable fixed point for \( n_f < \frac{13}{4} n \) \( \mathbb{C} \).] The “dangerous” leading logs in gauges \( \alpha_R \neq 3 \) are related to the running of \( \alpha_R \) and the anomalous dimension of \( \tilde{\gamma} \) in these gauges. In the limit \( \mu \to \infty \) in which \( \alpha_R(\mu) \to 3 \), the contribution to the fixed point of the effective action from “dangerous logs” cancels (which can be inferred from the RG-equation). In a loop expansion of the effective action to finite order this cancellation is not seen in gauges \( \alpha_R(\mu) \neq 3 \) because the perturbative expansion is analytic in \( g^2 \). Choosing the fixed point \( \alpha_R(\mu) = 3 \) is not a choice of gauge in the \( \mu \to \infty \) limit, but rather presents a simplification of the asymptotic analysis. The value \( \alpha_R(\mu) \) at finite \( \mu \) only determines how the fixed point \( \alpha_R = 3 \) is approached \( \mathbb{C} \).

The asymptotic evaluation of the expectation value of the trace of the energy momentum tensor in terms of \( \kappa \) for \( n_f = n \) flavors in (6.5) thus does not depend on the choice of covariant gauge at finite \( \mu \) (with the possible exception of Landau gauge, \( \alpha_R = 0 \), for which our asymptotic analysis is not valid). We can combine (5.5) and (5.11) to the physically interesting relation

\[ \langle \Theta_{\mu \nu} \rangle = -\frac{3}{8\pi^2} \kappa^4(n_f = n = 3) = -\frac{3e^2}{4^{5/3}3\pi^2} \Lambda_{\overline{\text{MS}}}^4 \sim -(0.687\Lambda_{\overline{\text{MS}}})^4 \] (6.6)

between the vacuum expectation value of the trace of the energy momentum tensor and the asymptotic scale parameter \( \Lambda_{\overline{\text{MS}}} \) of the modified minimal subtraction scheme of an \( SU(3) \) model with three quark flavors.

Using (6.4) the difference \( \Delta \varepsilon \) between the energy density \( \varepsilon_{\text{true}} \) of the true ground state and that of the perturbative vacuum with unbroken \( SU(3) \)-symmetry is

\[ \Delta \varepsilon = \varepsilon_{\text{true}} - \varepsilon_{\text{pert}} = \frac{1}{4} \left( \langle \Theta_{\mu \nu} \rangle_{\tilde{\gamma}} - \langle \Theta_{\mu \nu} \rangle_{\tilde{\gamma}_0} \right) = -\frac{3}{32\pi^2} \kappa^4 = -\frac{3e^2}{16^{4/3}3\pi^2} \Lambda_{\overline{\text{MS}}}^4 \sim -(0.486\Lambda_{\overline{\text{MS}}})^4, \] (6.7)
for $SU(3)$ with three quark flavors. The difference (5.7) is precisely $\Gamma_{eff}(\tilde{\gamma}) - \Gamma_{eff}(0)$ of (5.12).

7 Discussion and a comparison with phenomenology

I wish to stress that the final result (6.6) is a determination rather than an estimate of the expectation value of the trace of the energy momentum tensor for an $SU(3)$ gauge theory with three quark flavors. It implies that the unbroken phase with global $SU(n)$ color symmetry, and thus $\tilde{\gamma} = \langle \bar{\gamma} \rangle = 0$, corresponds to the scale invariant phase with $\langle \bar{\psi}_f \psi_f \rangle = 0$ and $\langle F_{\mu\nu} F^{\mu\nu} \rangle = 0$. The relation (6.1) furthermore shows that there are no other non-perturbative contribution to the expectation value of the trace of the energy momentum tensor. More precisely: if a nontrivial fixed point $\tilde{\gamma}$ of the moduli $\bar{\gamma}$ exists, its magnitude gives the expectation value of the trace of the energy momentum tensor. As I have argued, this fixed point of the moduli space in fact does not depend on the gauge parameter $\alpha$. Eq. (6.6) gives it an even broader, completely gauge invariant, meaning. The contribution to the vacuum energy density from non-perturbative field configurations such as instantons, monopoles and whatever else there might be is subsumed by the expectation value of the moduli $\bar{\gamma}$. The perturbative expansion around a particular ground state with $\tilde{\gamma} \neq 0$ makes sense because Eq. (6.4) implies that this vacuum has the correct energy density. It is furthermore clear that the modified perturbative expansion automatically includes power corrections to physical correlation functions at large Euclidean momenta. These power corrections first appear at the three-loop level in correlation functions of gauge invariant quark currents. They arise due to insertions of $\tilde{\gamma}$ in ghost loops. Due to global gauge invariance of physical correlation functions, the power corrections in the chiral limit are of order $\tilde{\gamma}^2 / p^4, \tilde{\gamma}^3 / p^6, \ldots$ etc. as one also expects from the OPE [1]. The normalization of the coefficients is, however, no longer arbitrary. Eq. (5.9) and (5.10) relate the asymptotic expectation values of $\bar{\gamma}$ to the fundamental scale $\Lambda_{\overline{MS}}$ of the model. We furthermore know that corrections to these asymptotic values are analytic in $g^2$ in the gauge $\alpha = 3, \delta = 1, \rho = 0$. They can thus be reliably computed in the framework of perturbation theory. In this asymptotic sense the $SU(3)$ model with $n_f = 3$ flavors is solved, since the non-analytic “non-perturbative” corrections of the OPE have been determined in terms of the asymptotic scale parameter. One could consider this an explicit realization of the program first proposed by Stingl [11].

We have here obtained the fixed point of the moduli space only for the special case of an $SU(n)$-symmetric model with $n_f = n$ flavors. The analysis in II suggests that a fixed point $\tilde{\gamma} \neq 0$ can also be found for $n_f < n$. The perturbative analysis in this case is complicated by the fact that the CCG occurs for $\delta \neq 1$. The moduli $\phi$ and $\sigma$ then have to be taken into account and cannot be trivially integrated out. A non-trivial fixed point of the moduli space of an $SU(3)$-model with $n_f \sim 6 > 3$ quark flavors, if it exists, would be of greater phenomenological interest. In the restricted space of covariant gauges with $\rho = 0$ no such fixed point was ascertained [3]. We are currently investigating whether there is a fixed point in the moduli space of the $SU(3)$-model for $n_f > 3$ in the enlarged
set of covariant gauges with $\rho \neq 0$.

By performing a two-loop calculation of the effective potential, we have explicitly verified that there is a non-trivial fixed point $\tilde{\gamma}$ in the moduli-space of an $SU(3)$-model with $n_f = 3$ quark flavors. The eigenvalues of $i\tilde{\gamma}$ have been related (see (5.3) and (5.11)) to the asymptotic scale parameter $\Lambda_{\overline{MS}}$. The reader probably noticed that the one-loop effective action actually suffices for this determination. The two-loop calculation in this special case, however, explicitly demonstrates the suppression of higher order corrections to the fixed point of the moduli space in CCG – which in $\cite{2}$ was inferred from the RG-equation. The methods of appendix A can furthermore be used to calculate the loop expansion of $SU(n)$-invariant (physical) correlators in terms of the eigenvalues of the fixed point $i\tilde{\gamma}$. Although these calculations are still somewhat cumbersome, we can in principle calculate all power corrections due to the moduli in this model order by order in the modified perturbative expansion.

A comparison with phenomenology is complicated by the fact that more than three quark flavors are dynamical at momentum scales above $\mu^2 \sim m_c^2 \sim (2\text{GeV})^2$. An $SU(3)$-model with only three flavors can thus only be expected to have reasonable accuracy at energies where the effective coupling $\alpha_s$ is still rather large, $\alpha_s(2\text{GeV}) \sim 0.35$. Let us nevertheless attempt a comparison with the phenomenology of QCD sum-rules while keeping this restriction in mind.

From moment sum rules in the $J/\Psi$ channel and chiral perturbation theory, the LHS of (6.6) can be estimated $\cite{12}$. Any reasonable $\cite{12, 13, 14}$ value for the gluon condensate, $\langle \bar{\Psi}\Psi \rangle$ and the light quark masses gives that the quark contribution to $\langle \Theta_{\mu\nu} \rangle$ is at most 20%. It is thus comparable with the systematic error in the determination of the gluon condensate. For a rough comparison with QCD-sumrules (and that is unfortunately all we can hope for), (6.6) essentially states that

$$\left( \frac{2}{3} \frac{\Lambda_{\overline{MS}}^{(3)}}{\Lambda_{\overline{MS}}} \right)^4 \sim \frac{8}{9} \langle \Theta_{\mu\nu} \rangle \sim \frac{\alpha_s}{\pi} F_{\mu\nu}^a F_{\mu\nu}^a$$  \hspace{1cm} (7.1)$$

where $\Lambda_{\overline{MS}}^{(3)}$ is the asymptotic scale parameter of the modified minimal subtraction scheme for a three-flavor model. The best determination of the gluon condensate at the highest energies for which only three flavors are dynamical is obtained using QCD-sumrules for the charmonium system. These balance perturbative contributions against power corrections. Using ratios of moments, Shifman, Vainshtein and Zakharov (SVZ)$\cite{12}$ originally estimated

$$\left( \frac{\alpha_s}{\pi} F_{\mu\nu}^a F_{\mu\nu}^a \right)_{SVZ} \sim (330 \pm 30\text{MeV})^4 \text{ with } \alpha_s(2.5\text{GeV}) = 0.2 \text{ and } m_c \sim 1.26\text{GeV}$$  \hspace{1cm} (7.2)$$

Most recent high-energy data indicate that their value for $\alpha_s$ is a bit low and that $\alpha_s(2.5\text{GeV}) \sim 0.3$ would be more appropriate (see Fig. 4). Since $\alpha_s$ determines the perturbative contribution to the moments, the extracted condensate value depends somewhat on $\alpha_s$. In the method used by $\cite{12}$ it is also relatively sensitive to $m_c$. 

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Fig. 4: The running strong coupling constant $\alpha_s$ of the modified minimal subtraction ($\overline{MS}$) scheme. The experimental data and best fit (solid line) with its 1\,\sigma error (grainy area) are reproduced from figure 9.2 of the summary by the Particle Data Group[15]. I have highlighted the experimental datapoint from $\tau$-decay at $\sim 1.8\,\text{GeV}$ in solid and also included with a solid point the value of $\alpha_s(2.5\,\text{GeV}) \sim 0.2$ used in the determination of the gluon condensate by[12]. Also shown are two bands derived from estimates of the gluon condensate. The upper band corresponds to Narison’s[14], whereas the lower band is obtained from Shifman, Vainshtein and Zakharov’s[12] value for the gluon condensate. Using (7.1) these estimates imply $\Lambda^{(3)}_{\overline{MS}} = 580 \pm 20\,\text{MeV}$, respectively $\Lambda^{(3)}_{\overline{MS}} = 500 \pm 40\,\text{MeV}$. To appreciate the quality of the fit note that data above $Q \sim 20\,\text{GeV}$, for which power corrections are negligible, are compatible with a somewhat larger value of $\Lambda^{(5)}_{\overline{MS}}$.

A recent analysis by Narison using double ratios of moments[14] reaches the conclusion that

$$\left< \frac{\alpha_s}{\pi} F^a_{\mu
u} F^a_{\mu\nu} \right>_{\text{Narison}} \sim (375 - 400\,\text{MeV})^4 \quad \text{with} \quad \alpha_s(1.3\,\text{GeV}) = 0.64^{+0.36}_{-0.18} \quad (7.3)$$

The advantage of this method compared to the previous one is that the double ratios of moments do not depend on the charm mass $m_c$ in leading order. The estimate (7.3) for the gluon condensate is about twice that of[12]. The strong coupling constant $\alpha_s$ in this analysis was obtained at the optimal scale for the $\chi_c(P_1^1) - \chi_c(P_1^3)$ mass-splitting. If a perturbative evolution from $\mu \sim 1.3\,\text{GeV}$ to $\mu \sim M_Z$ is justified, the value for the coupling in (7.3) corresponds to[14] $\alpha_s(M_Z) \sim 0.127 \pm 0.011$ (see also Fig.4).

With (7.1) the SVZ-value (7.2) for the gluon condensate gives $\Lambda^{(3)}_{\overline{MS}} \sim 500 \pm 40\,\text{MeV}$. Using the three-loop evolution[13] of the corresponding coupling constant (for $n_f = n =
3) one obtains $\alpha_s(2.5\text{GeV}) \sim 0.325 \pm 0.015$. Although compatible with other recent determinations of $\alpha_s(2.5\text{GeV})$ (see Fig. 4), this value is not consistent with $\alpha_s(2.5\text{GeV}) \sim 0.32$ that was used for the sum-rule estimate of the gluon condensate.

The more recent determination (7.3) for the gluon condensate with (7.1) corresponds to $\Lambda^{(3)}_{\overline{\text{MS}}} \sim 580 \pm 20\text{MeV}$. Again using the three-loop evolution [15] this implies $\alpha_s(1.3\text{GeV}) = 0.70 \pm 0.04$, or equivalently, $\alpha_s(2.5\text{GeV}) = 0.356 \pm 0.008$. While consistent with the coupling that was used [14] to extract the gluon condensate, this value for $\alpha_s(2.5\text{GeV})$ is somewhat large compared to other determinations of $\alpha_s$ (see Fig. 4). Perhaps one should note in this context that the second term of the $\beta$-function is almost 40% of the first at $\alpha_s = 0.7$. Radiative corrections to the gluon condensate itself (which after all give $\beta(g)/2g$ as the coefficient of $\langle F^2 \rangle$) and higher loop corrections on the perturbative side of the sum-rules do not appear to be entirely negligible at these large values for $\alpha_s$. The quoted error of only 15%[14] for the gluon condensate is therefore perhaps not a very conservative estimate of these systematic errors.

Using (7.1) and the three-loop evolution [13] for $\alpha_s$, one can also proceed in the opposite direction and estimate $\Lambda^{(3)}_{\overline{\text{MS}}}$ and thus the gluon condensate from $\alpha_s$ at scales $\mu < 2.5\text{GeV}$. This determination of the gluon condensate is only as accurate as the estimate for the asymptotic scale, which suffers from the logarithmic relation between the latter and $\alpha_s$. Probably one of the best experimental determinations of $\alpha_s$ at relatively low energies is the one from $\tau$-decay. The central value of $\alpha_s(m_\tau) = 0.370 \pm 0.033$ includes an estimate of hadronic power corrections to the decay. Neglecting these would, however, change $\alpha_s(m_\tau)$ by less than the quoted error[15].

$\alpha_s(m_\tau) = 0.370 \pm 0.033$ corresponds to $\Lambda^{(3)}_{\overline{\text{MS}}} = 450 \pm 80\text{MeV}$ and with (7.1) implies a gluon condensate of $\langle \frac{\alpha_s}{\pi} F^2 \rangle \sim 300 \pm 60\text{MeV}$. Although only a very rough value for the gluon condensate, it agrees within errors with the one obtained by Shifman, Vainstein and Zacharov (but for a considerably larger coupling). It is also compatible with Narison’s lowest estimate. As Fig. 4 shows, this state of affairs is not conclusive, but I consider it encouraging in view of the fact that the errors on $\alpha_s$ and $\langle \frac{\alpha_s}{\pi} F^2 \rangle$ are almost wholly systematic.

In summary: the exact relation (5.6) between the asymptotic scale $\Lambda_{\overline{\text{MS}}}$ of the modified minimal subtraction scheme and $\langle \Theta_{\mu\nu} \rangle$ for an $SU(3)$ gauge theory with three quark flavors is in reasonable agreement with current estimates of the gluon-condensate and $\alpha_s$.

Note finally that $\Lambda^{(3)}_{\overline{\text{MS}}} \sim 500\text{MeV}$ (compatible with Shifman, Vainstein and Zakharov’s value for the gluon condensate and $\alpha_s$ from $\tau$-decay) in (7.7) corresponds to a vacuum energy density of about $-455\text{MeV/fm}^3$. Assuming that the $SU(3)$-symmetry is essentially restored within a sphere of radius $\sim 0.8\text{fm}$ inside a nucleon, one naively obtains $M \sim 970\text{MeV}$ for the mass of such a “bubble”.

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8 Appendix A: Calculation of the two-loop contribution to the effective action

In the gauge \( \delta = 1, \rho = 0 \) the only two-loop contribution to the effective action is the one shown graphically in Fig. 2. For a sufficiently large symmetric \( D = 4 - 2\varepsilon \) dimensional torus Fig. 2 is the diagrammatic representation for the expression

\[
\Gamma^{2\text{-loop}}(\bar{\gamma}) = \frac{\hat{g}^2}{2} \mu^4 (\mu L)^D \int \frac{d^D p d^D q}{(2\pi \mu)^{2D}} \, p \mu q \nu D^{ab}_{\mu\nu} (p-q) f^{acd} f^{be\delta} \times \left( D^{cf}(p; \bar{\gamma}) D^{ed}(q; \bar{\gamma}) - D^{cf}(p; 0) D^{ed}(q; 0) \right)
\]

(8.1)

where \( \hat{g} = g \mu^{-\varepsilon} \) is the dimensionless coupling constant and \( L \) is the linear dimension of the symmetric torus. In (8.1) \( D^{ab}_{\mu\nu}(k) \) is the tree level gluon propagator

\[
D^{ab}_{\mu\nu}(k) = \delta^{ab}(\delta_{\mu\nu} k^2 + (\alpha - 1)k_\mu k_\nu)/k^4
\]

(8.2)

and \( D^{ab}(k; \bar{\gamma}) \) denotes the tree-level ghost-antighost correlator for fixed moduli \( \bar{\gamma} \). Defining the matrix \( (\bar{\gamma})_{ab} = f^{abc} \bar{\gamma}^c \) and using the integral parametrization

\[
D^{ab}(k; \bar{\gamma}) = (\delta^{ab} k^2 + f^{abc} \bar{\gamma}^c)^{-1} = \int_0^\infty d\lambda e^{-\lambda k^2} \left( e^{-\lambda \bar{\gamma}} \right)_{ab},
\]

(8.3)

for the ghost propagators, the \( D \)-dimensional momentum integrals in (8.1) can be separated from the color-summations. The expression for \( \Gamma^{2\text{-loop}} \) becomes,

\[
\Gamma^{2\text{-loop}}(\bar{\gamma}) = \frac{\hat{g}^2}{2} \mu^4 (\mu L)^D \int_0^\infty d\lambda_1 \int_0^\infty d\lambda_2 \left\{ f^{d a c} (e^{-\lambda_1 \bar{\gamma}})_{c f} f^{f a e} (e^{-\lambda_2 \bar{\gamma}})_{e d} - f^{d a c} f^{f a e} \right\}
\]

\[
\times \int \frac{d^D p d^D q}{(2\pi \mu)^{2D}(p-q)^4} \left( (p-q)^2 pq + (\alpha - 1)(p^2 - pq)(pq - q^2) \right) e^{-\lambda_1 p^2 - \lambda_2 q^2}
\]

(8.4)

The momentum space integrals can be performed in \( D > 2 \) dimensions

\[
\int \frac{d^D p d^D q}{(2\pi \mu)^{2D}(p-q)^4} \left( (p-q)^2 pq + (\alpha - 1)(p^2 - pq)(pq - q^2) \right) e^{-\lambda_1 p^2 - \lambda_2 q^2} = \int_0^\infty d\lambda \frac{\lambda^2}{2} \times
\]

\[
\left\{ \frac{\partial^2}{\partial \lambda_1 \partial \lambda} + \frac{\partial^2}{\partial \lambda_2 \partial \lambda} + \frac{\alpha - 1}{2} \left( \frac{\partial}{\partial \lambda_1} - \frac{\partial}{\partial \lambda_2} \right)^2 - \frac{\alpha + 1}{2} \frac{\partial^2}{\partial \lambda^2} \right\}
\]

\[
= \frac{D - 1 - \alpha(D - 3)}{D - 2} \frac{\lambda_1 \lambda_2}{(4\pi \mu^2)^D(\lambda_1 \lambda_2)^{D/2}(\lambda_1 + \lambda_2)^2}
\]

(8.5)

I next evaluate the color trace in (8.4) in terms of the (real) eigenvalues \( e_i, i = 1, \ldots, n \) of the hermitian matrix \( i\bar{\gamma} = it^a \bar{\gamma}^a \). Note that

\[
(e^{-\lambda \bar{\gamma}})_{ab} = -2\text{Tr} t^a e^{\lambda \bar{\gamma}} t^b e^{-\lambda \bar{\gamma}}
\]

(8.6)
with the convention that the anti-hermitian generators $t^a$ of the fundamental representation of $SU(n)$ are normalized to $\text{Tr} t^a t^b = -\frac{1}{2} \delta^{ab}$. Repeatedly using the completeness relation
\[
t^a_{ij} t^b_{kl} = -\frac{1}{2} \left( \delta_{il} \delta_{kj} - \frac{1}{n} \delta_{ij} \delta_{kl} \right)
\] of the generators in the fundamental representation of $su(n)$, the color trace in (8.4) can be rewritten as follows:
\[
f^{dac} (e^{-\lambda_1 \gamma})_{cf} f^{fae} (e^{-\lambda_2 \gamma})_{ed} = 4 \text{Tr} [t^d, t^e] e^{\lambda_1 \gamma t^f} e^{-\lambda_1 \gamma} \text{Tr} [t^f, t^a] e^{\lambda_2 \gamma t^d} e^{-\lambda_2 \gamma}
= -2 \text{Tr} [e^{\lambda_1 \gamma t^f} e^{-\lambda_1 \gamma}, t^f] [e^{\lambda_2 \gamma t^d} e^{-\lambda_2 \gamma}, t^f]
= \text{Tr} e^{-\lambda_2 \gamma} \text{Tr} e^{-\lambda_1 \gamma t^f} + \text{Tr} e^{-\lambda_1 \gamma} \text{Tr} e^{-\lambda_2 \gamma} - 2 \text{Tr} e^{-\lambda_2 \gamma} \text{Tr} e^{-\lambda_1 \gamma}
= \text{Tr} 1 - \frac{1}{2} \text{Tr} e^{-\lambda_2 \gamma} \text{Tr} e^{-\lambda_1 \gamma} - \frac{1}{2} \text{Tr} e^{-\lambda_2 \gamma} \text{Tr} e^{-\lambda_1 \gamma}
= - \sum_{ijk} \cos (\lambda_1 (e_i - e_k) + \lambda_2 (e_j - e_k))
\] (8.8)

where the symbol $\sum'_{ijk}$ denotes the sum over all triplets $(i, j, k)$ except those with $i = j = k$. The first term in the expansion of the cosine does not depend on the eigenvalues and corresponds to the diagram shown in Fig.2c. This $\gamma$-independent term of the effective action can be absorbed in the normalization $\mathcal{N}$ of (3.4) and has been subtracted in (8.1) since it is of no physical interest. Using (8.8), the color trace in (8.4) expressed by the eigenvalues of $\tilde{\gamma}$ is,
\[
f^{dac} (e^{-\lambda_1 \tilde{\gamma}})_{cf} f^{fae} (e^{-\lambda_2 \tilde{\gamma}})_{ed} - f^{dac} f^{cad} = \sum_{ijk} (1 - \cos (\lambda_1 (e_i - e_k) + \lambda_2 (e_j - e_k)))
= 2 \sum_{ijk} \sin^2 ((\lambda_1 (e_i - e_k) + \lambda_2 (e_j - e_k))/2)
\] (8.9)

When none of the eigenvalues $e_i, i = 1, \ldots, n$ coincide, the argument of the sine does not vanish generically for arbitrary values of $\lambda_1$ and $\lambda_2$ and regulates the infrared behaviour of the integrand in (8.4). Inserting (8.9) and (8.3) in (8.2), the parametric expression for $\Gamma^{2-\text{loop}}$ is
\[
\Gamma^{2-\text{loop}}(\tilde{\gamma}) = \hat{g}^2 (\mu L)^D \frac{D - 1 - \alpha (D - 3)}{16 \pi^2 (D - 2)} \sum_{ijk} \int_0^\infty \frac{\lambda_1 \lambda_1}{\lambda_1^{D/2}} \int_0^\infty \frac{\lambda_2 \lambda_2}{\lambda_2^{D/2}} \sin^2 \left( \frac{1}{2} \lambda (v_{ik} + (1 - x)v_{jk}) \right) \frac{(\lambda_1 + \lambda_2)^2}{(\lambda_1 + \lambda_2)^2}
\]
\[
= \hat{g}^2 (\mu L)^D \frac{D - 1 - \alpha (D - 3)}{16 \pi^2 (D - 2)} \sum_{ijk} \int_0^1 dx \int_0^\infty d\lambda \frac{\lambda^{D-1}(x(1-x))^{(D-2)/2}}{(\lambda_1 + \lambda_2)^2}
\] (8.10)

where $v_{km}$ is the dimensionless difference of eigenvalues defined in (3.3). Feynman's parametrization for the integrals shows that the overall degree of divergence of the graph is logarithmic, as could have been expected. The integration over $\lambda$ in (8.10) is readily performed in $D < 4$ dimensions and we are left with an integral over a single Feynman parameter
\[
\Gamma^{2-\text{loop}}(\tilde{\gamma}) = \hat{g}^2 (\mu L)^D \frac{D - 1 - \alpha (D - 3)}{16 \pi^2 (D - 2)} \cos \left( \frac{D\pi}{2} \right) \Gamma(2 - D) \]
\[ \times \sum_{ijk}^{'} \int_0^{1/2} dx \left( \frac{(xv_{ik} + (1 - x)v_{jk})^2}{x(1-x)} \right)^{(D-2)/2}, \quad (8.11) \]

where the symmetry \( i \leftrightarrow j \) under the sum was used. Changing the integration variable to \( y = x/(1 - x) \) and introducing \( \varepsilon = (4 - D)/2 \), the integral in (8.11) can also be written

\[ \int_0^{1/2} dx \left( \frac{(xv_{ik} + (1 - x)v_{jk})^2}{x(1-x)} \right)^{(D-2)/2} = \int_0^1 y^{1-\varepsilon} dy \left[ \left( \frac{v_{ik} + v_{jk}}{y} \right)^2 \right]^{1-\varepsilon}. \quad (8.12) \]

For \( \varepsilon \to 0_+ \), (8.12) diverges as \( v_{jk}^2/\varepsilon \). To isolate this divergence of (8.12), consider

\[ \int_0^1 \frac{dy}{(1+y)^2} \left( \frac{v_{jk}^2}{y} \right)^{1-\varepsilon} = v_{jk}^2 \left( \frac{1}{\varepsilon} - \frac{1}{2} - \ln(2v_{jk}) \right) + \varepsilon \left( \frac{\pi^2}{2} - 1 + \ln(4) + (\ln(v_{jk}^2))^2 \right) + \ldots \quad (8.13) \]

and define the for \( \varepsilon > 1 \) and arbitrary real values of \( a \) and \( b \) regular integral

\[ I_{\varepsilon}(a,b) := \int_0^1 \frac{dy}{(1+y)^2} \left\{ (ya^2 + b^2/y + 2ab)^{1-\varepsilon} - (b^2/y)^{1-\varepsilon} \right\} \]

\[ = \int_0^1 \frac{dy}{(1+y)^2} (ya^2 + 2ab) - \varepsilon \int_0^1 \frac{dy}{(1+y)^2} \left\{ (ya^2 + 2ab) \ln \left( \frac{y(y + b/y)}{a + b/y} \right) + b^2 \frac{\ln \left( \frac{y + b/y}{b} \right)}{y} \right\} + O(\varepsilon^2) \]

\[ = ab + (\ln 2 - \frac{1}{2})a^2 + O(\varepsilon) \quad (8.14) \]

In gauges \( \alpha \neq 3 \) the term of order \( \varepsilon \) in (8.14) gives a finite contribution to \( \Gamma_{2-loop}^2 \) in \( D = 4 \) and would have to be computed. For \( \alpha = 3 \), the critical gauge that will ultimately interest us -- the two-loop effective action simplifies greatly: the factor \( D - 1 - \alpha(D - 3) \) in (8.14) in this gauge is of order \( \varepsilon \). The term of order \( \varepsilon \) in (8.14) thus gives a finite contribution to the 4-dimensional effective action proportional to \( (\alpha - 3) \). This (finite) contribution will be of no further interest and for the sake of brevity will not be given in the form of more elementary integrals.

Making use of the symmetric summation over eigenvalues, (8.14) and (8.13) together lead to,

\[ \sum_{ijk}^{'} \int_0^1 \frac{y^{1-\varepsilon} dy}{(1+y)^2} \left[ \left( \frac{v_{ik} + v_{jk}}{y} \right)^2 \right]^{1-\varepsilon} = \sum_{ijk}^{'} I_{\varepsilon}(v_{ik},v_{jk}) + \int_0^1 \frac{dy}{(1+y)^2} (v_{jk}^2/y)^{1-\varepsilon} \]

\[ = 2n \sum_{1 \leq i < j \leq n} v_{ij}^2 \left( \frac{1}{\varepsilon} - \frac{1}{2} - \ln(v_{ij}^2) + O(\varepsilon) \right) \quad (8.15) \]

Expanding (8.11) for \( \varepsilon = (4 - D)/2 \to 0_+ \) using (8.14), the 2-loop contribution to the effective action for a symmetric torus with volume \( (\mu L)^D \to \infty \) gives (3.9) of the main text.

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4 Algebric programs such as MATHEMATICA readily express the term proportional to \( \varepsilon \) in (8.14) in the form of dilogarithms and products of logs.
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