Multiple $SO(5)$ isovector pairing and seniority $Sp(2Ω)$ multi-$j$ algebras with isospin

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Abstract.

With nucleons occupying several shell model $j$ orbits, the isovector pair creation operator $A^{j}_\mu_1$ (creates a two particle state with angular momentum $J = 0$ and isospin $T = 1$) is no longer unique. Choosing it to be a sum of single-$j$ isovector pair creation operators each with a phase, there will be multiple pair $SO(5)$ algebras with isospin; with $r$ number of $j$ orbits, there will be $2^r - 1$ $SO(5)$ algebras each with a corresponding complementary $Sp(2Ω)$ algebra [$2Ω = \sum j (2j + 1)$] that gives seniority and reduced isospin quantum numbers. Three applications of multiple $SO(5)$ algebras are presented demonstrating the usefulness of considering $SO(5)$ pairing algebras with general sign factors.

KEY WORDS: pairing, multiple algebras, multi-$j$, $SO(5)$, $Sp(2Ω)$.

1 Introduction

With identical nucleons (protons or neutrons) in a single-$j$ shell, the pair creation and annihilation operators ($S_+$ and $S_-$ respectively) and the number operator ($\hat{n}$) generate remarkably the quasi-spin $SU(2)$ algebra. On the other hand, the $Sp(2j + 1)$ subalgebra of the Spectrum Generating Algebra (SGA) $U(2j + 1)$ is 'complementary' to the quasi-spin-$SU(2)$ algebra and the seniority quantum number $v$ that labels the states w.r.t. $Sp(2j + 1)$ algebra, i.e. irreducible representation (irreps), corresponds to the quasi-spin quantum number and similarly, particle number $m$ that labels the irreps of $U(2j + 1)$ corresponds to the $z$-component of quasi-spin. More importantly, this solves the pairing Hamiltonian $H_p = -S_+S_-$ and allows one to extract $m$ dependence of many particle matrix elements of a given operator; see [1] for full details and applications.

All the single-$j$ shell results extend to the multi-$j$ shell situation i.e. for identical nucleons occupying several-$j$ orbits, with $(2j + 1)$ replaced by $2Ω = \sum j (2j + 1)$.
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In this situation, \( v \) is called generalized seniority. Now, a new result is that with \( r \) number of \( j \)-orbits there will be \( 2^{r-1} \) number of quasi-spin \( SU(2) \) and the corresponding \( Sp(2\Omega) \) algebras (\( S_+ = \sum_j \alpha_j S_+(j); \alpha_j = \pm 1 \)). These multiple quasi-spin algebras (one for each \( \alpha_j \) choice) play an important role in deciding selection rules for electric and magnetic multipole operators. See [2] for details regarding these multiple pairing algebras and [1, 3–5] for the goodness of multi-\( j \) seniority in certain nuclei. Also, multi-\( j \) seniority provides a framework for shell model theory of the interacting boson model [6, 7].

With isospin (\( T \)) degree of freedom, the algebra changes to the more complex \( SO(5) \) algebra. With \( m \)-nucleons in a single-\( j \) orbit the SGA is \( U(2(2j + 1)) \) with the 2 coming from isospin. An isospin conserving subalgebra chain is \( U(2(2j + 1)) \supset [U(2j + 1) \supset Sp(2j + 1)] \otimes SU_T(2) \). Particle number \( m \) labels \( U(2(2j + 1)) \), \((m, T)\) label \( U(2j + 1) \) and \( T \) labels \( SU_T(2) \) irreps. It is recognized in very early years of shell model that the isovector pair creation and annihilation operators, isospin and the number operator generate a \( SO(5) \) algebra, which is ‘complementary’ to the above \( Sp(2j + 1) \) algebra. The irreps of \( SO(5) \) contain two labels and they can be written in terms of the seniority \( v \) and reduced isotopic spin \( t \) quantum numbers with \((v, t)\) uniquely labeling the \( Sp(2j + 1) \) irreps. Another important result is that the isovector pairing Hamiltonian is simply related to the quadratic Casimir invariants of \( SO(5) \) and \( Sp(2j + 1) \). However, an unsatisfactory aspect of the \( SO(5) \) algebra of the shell model is that it does not contain isoscalar pair operators in its algebra. For the first papers on single-\( j \) shell pairing \( Sp(2j + 1) \) algebra and the corresponding \( SO(5) \) algebra see [8–13]. Similarly, for technical work on these algebras (for example deriving analytical formulas for the Wigner coefficients of \( SO(5) \)) see [12, 14–18] and for recent applications see [19–23] and references therein. Although many of the single-\( j \) shell results extend to the multi-\( j \) shell systems, for the multi-\( j \) shell situation a crucial aspect is that there will be multiple \( SO(5) \) algebras as the isovector pair creation operator here is no longer unique. The purpose of this paper is to introduce and analyze these multiple \( SO(5) \) isovector pairing algebras with isospin.

Before proceeding further, let us add that the general mathematical theory describing complementarity between identical nucleon number non-conserving quasi-spin \( SU(2) \) and the number conserving \( Sp(2\Omega) \), complementarity between \( SO(5) \) and \( Sp(2\Omega) \) with isospin and similar complementarity between many other algebras (for example proton-neutron pairing \( SO(8) \) algebra with \( LST \) coupling and the corresponding number conserving algebras [24, 25]) including those for boson systems (see for example [26, 27]) is due to Neergard based on Howe’s general duality theorem [28, 29]; first proof of complementarity is due to Helmers [10] and later work is due to Rowe et al. [30]. Now we will give a preview.

In Section 2, multiple pairing \( SO(5) \) and \( Sp(2\Omega) \) algebras for the multi-\( j \) situation are introduced. Section 3 gives formulas for constructing many-particle
following the results in \([2, 11, 15]\), it is easy to recognize the generators of the subalgebra \(\sum_{j} \langle \beta \rangle\) algebras. Then, with two \(j\) orbits we have four \(SO(5)\) and \(SO(\pm, \pm)\) algebras. Further, the ten operators \(\sum_{j} \langle \beta \rangle\) and \(\sum_{j} \langle \beta \rangle\) with three \(j\) orbits there will be eight \(SO(5)\) algebras. Then, with two \(j\) orbits we have two \(SO(5)\) algebras \(SO^{(\pm, \pm)}(5)\) and \(SO^{(+, -)}(5)\), with three we have four \(SO(5)\) algebras \(SO^{(\pm, +)}(5)\), \(SO^{(+, -)}(5)\), \(SO^{(+, +)}(5)\) and \(SO^{(-, -)}(5)\), with four \(j\) orbits there will be eight \(SO(5)\) algebras and so on. Significantly, the isovector pairing Hamiltonians \(H_\mu(\beta) = -G \sum_\mu A^\mu_\beta(\beta) A^{\dagger \mu}_{-\beta}(\beta)\) with \(G\) the pairing strength, is simply related to \(C_2(\lambda(\beta)(5))\), the quadratic Casimir invariant of \(SO(\beta)(5)\):

\[
C_2(\lambda(\beta)(5)) = 2 \sum_\beta A^\beta_\beta(\beta) A_{-\beta \dagger}^{\beta}(\beta) + T^2 + Q_0(Q_0 - 3).
\]

Further, \(SO(5)\) irreps are labeled by \((\omega_1, \omega_2)\) with \(\omega_1\) and \(\omega_2\) both integers or half integers and \(\omega_1 \geq \omega_2 \geq 0\). Then, the eigenvalues of \(C_2(\lambda(\beta)(5))\) are \(\langle C_2(\lambda(\beta)(5)) \rangle^{(\omega_1, \omega_2)} = \omega_1(\omega_1 + 3) + \omega_2(\omega_2 + 1)\). Expressions for Casimir invariants are given by Racah very early \([32]\). We will now turn to the complementary \(Sp^{\beta}(2\Omega)\) algebras.

Consider one-body operators \(u^\beta_{m_{1}, m_{2}}(j_{1}, j_{2})\) defined in terms of the single particle creation and annihilation operators in \((jt)\) space, \(u^{\dagger}_{m_{1}, m_{2}}(j_{1}, j_{2}) = (a^{\dagger}_{j_{1}} \tilde{a}_{j_{2}})_{m_{1}, m_{2}}\) where \(\tilde{a}_{j_{2} - m_{1}} = (-1)^{j_{2} - m_{1} + 1} a_{j_{2} - m_{1}}\). Now, it is easy to prove that the operators \(u^{\dagger}_{m_{1}, m_{2}}(j_{1}, j_{2})\) generate the \(U(4\Omega)\) GGA. Moreover, we have the subalgebra \(U(4\Omega) \supset [U(2\Omega) \supset Sp(2\Omega)] \supset SU_{c}(2)\) with \(u^{k, 0}_{m_{1}, m_{2}}(j_{1}, j_{2})\) operators generating \(U(2\Omega)\) and \(SU_{c}(2)\) generating isospin. Following the results in \([2, 11, 15]\), it is easy to recognize the generators of \(Sp^{\beta}(2\Omega)\).
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and they are,
\[ u^k_{\mu,0}(j_1,j_2) ; k = \text{odd} \]
\[ v^k_{\mu,0}(j_1,j_2) = u^k_{\mu,0}(j_1,j_2) + X(j_1,j_2,k) u^k_{\mu,0}(j_2,j_1) ; \ j_1 > j_2 \ . \] (2)

Now, the most important result is that for every $SO^{(3)}(5)$, there will be a complementary $Sp^{(3)}(2\Omega)$ algebra with generators given by Eq. (2) provided
\[ X(j_1,j_2,k) = (-1)^{j_1+j_2+k} \beta_1 \beta_2 . \] (3)

Proof for the complementarity is given first by Helmers [10]. Thus, the multiple $SO^{(3)}(5)$ algebras with number non-conserving generators and $Sp^{(3)}(2\Omega)$ algebras with only number conserving generators are complementary provided Eq. (3) is satisfied along with Eqs. (1) and (2).

Turning to the irreps, all $m$ nucleon states transform as the antisymmetric irrep $\{1^m\}$ of $U(4\Omega)$ and the irreps of $U(2\Omega)$ will be two columned irreps $\{2^{m_1}1^{m_2}\}$ in Young tableaux notation with $2m_1 + m_2 = m$ and $T = m_2/2$. Similarly, the $Sp(2\Omega)$ irreps are two columned denoted by $\{2^{v_1}1^{v_2}\}$ giving $v = 2v_1 + v_2$ the seniority quantum number and $t = v_2/2$ the reduced isospin. Group theory allows us to obtain $(m,T) \rightarrow (v,t)$ reductions or $(v,t) \rightarrow T$ for a given $m$ [31].

More importantly, it can be shown that $(\omega_1,\omega_2)$ is equivalent to $(t,v)$ giving $\omega_1 = \Omega - (v/2)$ and $\omega_2 = t$. With all these, the eigenvalues of $H_{p}^{(\beta)}$ are [11],
\[ \langle H_{p}^{(\beta)} \rangle^{m,T,v,t} = -\frac{G}{4} \left[ (m-v)(2\Omega + 3 - \frac{m+v}{2}) - 2T(T+1) + 2t(t+1) \right] . \] (4)

Note that $SO^{(3)}(5) \supset [SO(3) \supset SO(2)] \otimes U(1)$ with $SO(3)$ generating $T$, $SO(2)$ generating $M_T$ ($T_z$ quantum number $(N-Z)/2$) and $U(1)$ generating particle number or $H_1 = (m - 2\Omega)/2$. Then, the eigenstates of $H_{p}^{(\beta)}$ are
\[ |\Psi_{H_{p}^{(\beta)}}\rangle \Rightarrow |(\Omega - \frac{v}{2},t), H_1 = \frac{m - 2\Omega}{2}, M_T = \frac{N-Z}{2} \rangle . \] (5)

and the labels do not depend on $\beta$. However, explicit structure of the wavefunctions do depend on $\beta$; see Section 4. Thus, they will effect various selection rules and matrix elements of certain transition operators (see Section 4).

Finally, with $SO(5)$ algebra it is possible to factorize $(m,T)$ dependence of various matrix elements [1,15,21]. In order to enumerate the irrep labels in Eq. (5), used is $U(4\Omega) \supset [U(2\Omega) \supset Sp(2\Omega)] \otimes SU_T(2)$ reductions starting with the $\{1^m\}$ irrep of $U(4\Omega)$. All the rules for these are known [24,31,33].

3 Construction of many-particle matrix for pairing Hamiltonian generating multiple $SO(5)$ algebras

In order to probe the role of multiple pair $SO^{(3)}(5)$ algebras with isospin, we need to obtain the eigenstates of the pairing Hamiltonian $H_p$ as a func-
Hamiltonian \[ \{, \beta \} \}\)’s. A convenient basis for constructing the \( \mathcal{H}_p \) matrix is the product basis defined by the single-\( j \) shell \( SO(5) \) basis. We will illustrate this using two \( j \)-orbits say \( j_1 \) and \( j_2 \). Hereafter, we call the corresponding spaces \( a \) and \( b \) respectively (or 1 and 2). Then, the basis states are, \( \Psi_{ab}(T \mathcal{M}_T) = \{ (ω_a^1 ω_b^2) H^a T^a (ω_b^1 ω_a^2) H^b T^b ; T \mathcal{M}_T \} \) or equivalently \( \{ (v_1, t_1) m_1 T_{11}, (v_2, t_2) m_2 T_{22} ; T \mathcal{M}_T \} \). Given \( m \) number of nucleons, with \( m_1 \) in number in the first orbit and \( m_2 \) in the second orbit, \( m = m_1 + m_2 \). Note that \( \Omega_1 = j_1 + \frac{1}{2}, \Omega_2 = j_2 + \frac{1}{2}, H^a = \frac{m_1}{2} - \Omega_1 \) and \( H^b = \frac{m_2}{2} - \Omega_2 \). Similarly \( T^a \) and \( T^b \) are the isospins in the two spaces respectively. Now, a general pairing Hamiltonian [with \( A_{\mu}^i(\alpha) = A_{\mu}^i(j_1) + \alpha A_{\mu}^i(j_2) \)] is,

\[
\mathcal{H}_p(\xi, \alpha) = \frac{1 - \xi}{m} \hat{n}_2 - \frac{\xi}{m^2} \left\{ 4 \sum_{\mu} A_{\mu}^i(\alpha) [A_{\mu}^i(\alpha)]^\dagger \right\} \ .
\]

Here, \( \hat{n}_2 \) is the number operator for the second orbit and \( \xi \) and \( \alpha \) are parameters changing from 0 to 1 and +1 to −1 respectively. Note that for \( \xi = 1 \) and \( \alpha = +1 \) we have a \( SO(+)^+(5) \) algebra in the total two-orbit space and similarly for \( \xi = 1 \)
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and \( \alpha = -1 \) the \( SO^{(-)}(5) \) algebra. Diagonal matrix elements of \( H_\mu \) in our basis follow easily from Eq. (4); note that \( n_2 \) gives \( m_2 \) and the other part giving diagonal matrix elements is \( \sum_\mu \{(A_{\mu}^1(j_1)|A_{\mu}^1(j_1)^\dagger\} + \alpha^2 (A_{\mu}^1(j_2)|A_{\mu}^1(j_2)^\dagger\}. \)

The off-diagonal matrix elements involve \( SO(5) \supset SO(3) \otimes U(1) \) reduced Wigner coefficients and using Eq. (14) of \[15\] will give,

\[
\langle \omega_1^a \omega_2^b | H^\mu_\alpha T^\mu_i, (\omega_1^a \omega_2^b) H^\mu_\alpha T^\mu_i; T M_T | H_\rho(\zeta, \alpha) \rangle
\]

\[
\frac{4\kappa}{m^2} (\alpha)
\]

\[
\times [(\omega_1^a (\omega_1^a + 3) + \omega_2^b (\omega_2^b + 1)]^{1/2} \left[ (\omega_1^b (\omega_1^b + 3) + \omega_2^b (\omega_2^b + 1)]^{1/2}
\]

\[
\times (-1)^{T_f^\mu + T_i^\mu + T_i^\mu + 1} (\sqrt{2T_f^\mu + 1}) (2T_i^\mu + 1)
\]

\[
\times \left\{ \frac{T}{T^\mu_i} \right\}
\]

\[
= \left\{ \frac{T}{T^\mu_i} \right\}
\]

\[
\times \left\{ \frac{T}{T^\mu_i} \right\}
\]

\[
(11) 1, 1 \text{ } || \text{ } (\omega_1^a \omega_2^b) H^\mu_\alpha T^\mu_i
\]

\[
(11) -1, 1 \text{ } || \text{ } (\omega_1^a \omega_2^b) H^\mu_\alpha T^\mu_i
\]

for \( H^\mu_\alpha = H^\mu_\alpha + 1 \) and \( H^\mu_\alpha = H^\mu_\alpha - 1; H^\mu_\alpha = \omega_1^a - \Omega_1, H^\mu_\alpha = \omega_2^b - \Omega_2 \). The \( \{ - - - + \text{ } || \text{ } \} \) factors above are the \( SO(5) \supset SO(3) \otimes U(1) \) reduced Wigner coefficients. For the \( m = 6 \) system considered in the next Section, the needed Wigner coefficients follow from Tables III in [15] and Table A.1 in [16].

It is important to mention that for simplicity, in Eq. (7) we are not showing the additional label that is required as discussed in [15, 16]. This label is called \( \beta \) in [16]. Finally, Eq. (7) can be extended to three or more orbits by using isospin \( T \) couplings. Thus, \( H_\mu \) construction is possible with multiple \( SO^{(\beta)}(5) \) algebras provided all the needed Wigner coefficients in Eq. (7) are known.

4 Applications of multiple \( SO(5)/Sp(2\Omega) \) algebras

Electromagnetic transition operators \( T^{EL} \) and \( T^{M\Lambda} \) are one-body operators and their \( SO(5) \supset [SO(3) \supset SO(2)] \otimes U(1) \) tensorial structure is \( T^{(\omega_1^a \omega_2^b)}_{H_{\mu}^i,T,M_F} \) with \( (\omega_1^a \omega_2^b) = (11) \oplus (10) \oplus (00). \) More importantly, Eqs. (2) and (3) show that it is possible for \( T^{EL} \) and \( T^{M\Lambda} \) to be generators of \( Sp^{(\beta)}(2\Omega) \) giving selection rules under certain conditions. We have the following results: (i) isovector parts of \( T^{EL} \) and \( T^{M\Lambda} \) will not be \( Sp^{(\beta)}(2\Omega) \) scalars as the generators of these algebras are only isoscalar operators; (ii) the isoscalar part of \( T^{M\Lambda} \) with \( L \) odd (they preserve parity) or even can be \( Sp^{(\beta)}(2\Omega) \) scalars provided \( \beta_{j_1(t)} = (-1)^t \) for the \( j_1(t) \) orbits; (iii) the \( T^{EL} \) with \( L \) even or odd will not be generators of any \( Sp^{(\beta)}(2\Omega) \) as the \( \Xi(j_1,t,l) \) (see Eqs. (2) and (3)) given by the isoscalar part will not lead to a formula for \( \beta_{j_1} \) real. With the phase choice \( \beta_{j_1(t)} = (-1)^t \), the selection rule from the generators that they will not change \( (v,t) \) or \( (\omega_1^a \omega_2^b) \) irreps can be used in experimental tests of this phase choice. Besides this, EL and ML transitions can change seniority only by units of 2, i.e. transition for \( v \rightarrow v, v \pm 2 \) states are only allowed. In addition, the \( (m,T) \) dependence of say quadrupole moments and \( B(E2) \)'s can be written down using \( SO(5) \) algebra.
In the second application, let us consider a two level system with first level having $\Omega_1 = 6$ with $-ve$ parity and the second level having $\Omega_2 = 5$ with $+ve$ parity. This is appropriate for nuclei in $A=56-80$ region so that the $(1p_{1/2}, 0f_{5/2}, 1p_{1/2})$ orbits with degenerate single particle levels give the $\Omega_1 = 6$ orbit (we will call it orbit #1 or $a$) and $0g_{9/2}$ gives the $\Omega_2 = 5$ orbit (we will call it orbit #2 or $b$). In our numerical calculations we use the system with six nucleons in the above two orbits. Then, the number of $+ve$ parity basis states for $m = 6$ and $T = 0$ will be 24 as shown in Table 1. Using these basis states, the matrix for $H_p$ defined by Eq. (6) is constructed following the formulation in Section 3. Diagonalization of $H_p(\xi = 1, \alpha = \pm 1)$ will give eigenvalues that must be same as those given by Eq. (4) with $(m = 6, T = 0)$ and $(v, t) = (6, 0), (4, 1)$ and $(2, 0)$. The eigenvalues are $0$, $-44/m^2$ and $-84/m^2$ respectively with degeneracies 13, 9 and 2 respectively. It is easy to see that the wavefunctions are of the form $|(v_1, t_1)(v_2, t_2)(v, t)\gamma, m = 6, T = 0\rangle$ where $\gamma$ are additional labels. Therefore, a sum of $C_2(SO^{(a)}(5))$ and $C_2(SO^{(b)}(5))$ will remove some of the degeneracies in the spectrum without changing the eigenvectors. By adding a
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term $- (\xi/m^2) | C_2SO(\alpha) \rangle + C_2SO(\beta) \rangle$ to \( H_p(\xi, \alpha) \) we have calculated the eigenvalues for the \((m = 6, T = 0)\) system and shown in Fig. 1 are the energies of the 24 states as a function of \( \xi \) for nine \( \alpha \) values. For example, wavefunctions for the lowest two degenerate states are,

\[
\begin{align*}
|\psi_{m=6, T=0}^2\rangle &= \sqrt{\frac{3}{4}} |0(0, 0); 2(20)0\rangle - \alpha \sqrt{\frac{3}{4}} |2(0, 0); 2(20)1\rangle + \sqrt{\frac{3}{4}} |0(0, 0); 6(20)0\rangle , \\
|\psi_{m=6, T=0}^1\rangle &= \sqrt{\frac{3}{4}} |0(0, 0); 0(00)0\rangle - \alpha \sqrt{\frac{3}{4}} |4(2, 0); 2(00)1\rangle + \sqrt{\frac{3}{4}} |2(2, 0); 4(00)0\rangle .
\end{align*}
\]

(8)

Here the notation used is \(|m_1(v_1, t_1); m_2(v_2, t_2)\rangle\). Eq. (8) shows the role of \( \alpha \), i.e. the two \( SO(5) \) algebras. As seen from Fig. 1, clearly by changing \((\xi, \alpha)\) it is possible to study order-chaos-order transitions. Detailed analysis of this including all \( T \)'s will be reported elsewhere.

Two-particle transfer strengths form the third application. As an example let us consider removal of an isovector pair from the lowest two states [these are \( \Psi_1 \) and \( \Psi_2 \) in Eq. (8)] of the \((m = 6, T = 0)\) system generating the states of \((m = 4, T = 1)\) system. To study the transfer strengths, we have diagonalized \( H_p(\xi = 1, \alpha = \pm 1) \) in \((m = 4, T = 1)\) space and the basis states here are 14 in number. Then, the eigenstates belong to \((v, t) = (21), (20)\) and \((41)\) irreps in the 4 nucleon space. There are three, two and nine states respectively with these irreps and the corresponding eigenvalues are $-44/m^2$, $-40/m^2$ and 0 respectively. The transition operator for example can be chosen to be \( P = [A_v^\dagger(j_i)]^\dagger \) or it can be \([A_v^\dagger(\alpha)]^\dagger \) with \( \alpha = +1 \) or $-1$. These will not change \((v_1t_1)\) and \((v_2t_2)\) of the states. From Eq. (8) it is easy to see that the transition is allowed to the two states with \((v, t) = (2, 0)\) and these are

\[
\begin{align*}
|\phi_{m=4, T=1}^1\rangle &= \sqrt{\frac{3}{4}} |0(0, 0); 2(00)1\rangle + \alpha \sqrt{\frac{3}{4}} |2(2, 0); 2(00)1\rangle , \\
|\phi_{m=4, T=1}^2\rangle &= \sqrt{\frac{3}{4}} |0(0, 0); 4(20)1\rangle + \alpha \sqrt{\frac{3}{4}} |2(0, 0); 2(20)0\rangle .
\end{align*}
\]

(9)

With the \( \alpha \) dependence in both the six and four particle states [see Eqs. (8) and (9)], clearly, the two-particle transfer strengths depend on \( \alpha \). In practice we need to add a term in the Hamiltonian that mixes the states \( \Psi_1 \) and \( \Psi_2 \) and similarly \( \Phi_1 \) and \( \Phi_2 \). This is being investigated and explicit formulas for the transfer strengths will be reported elsewhere.

5 Conclusions

Extending the previous results [2] on multiple \( SU(2) \) pairing algebras for identical nucleons occupying several \( j \)-orbits, in this paper it is shown that there are multiple \( SO(5) \) pairing algebras for nucleons (with isospin) occupying several \( j \)-orbits. Further, a method to analyze the results, based on the algebra in [15], due to multiple \( SO(5) \) (or the equivalent \( Sp(2\Omega) \)) algebras is described. Finally, in three applications are briefly discussed. More detailed investigations of multiple \( SO(5) \) algebras and their applications will be reported elsewhere. Present
work complements the corresponding investigations without isospin in [2] and on multiple $SU(3)$ algebras in [34, 35].

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