We introduce a new model of financial market with stochastic volatility driven by an arbitrary Hölder continuous Gaussian Volterra process. The distinguishing feature of the model is the form of the volatility equation which ensures the solution to be “sandwiched” between two arbitrary Hölder continuous functions chosen in advance. We discuss the structure of local martingale measures on this market, investigate integrability and Malliavin differentiability of prices and volatilities as well as study absolute continuity of the corresponding probability laws. Additionally, we utilize Malliavin calculus to develop an algorithm of pricing options with discontinuous payoffs.

Keywords: stochastic volatility, sandwiched process, Hölder continuous noise, option pricing, Malliavin calculus

MSC 2020: 91G30; 60H10; 60H35; 60G22

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1 Introduction

1.1 Motivation and background

Despite its fundamental role in the development of mathematical finance, the classic Black-Scholes approach does not reflect numerous phenomena observed in real financial markets. In particular, numerous empirical arguments clearly indicate that the volatility is far from being constant; for example, it is well known that the amplitude of variation in stock prices is prone to clustering \cite{29, 30} and there is a notable negative correlation between variance and returns of an asset \cite{21, 23, 24, 44}. However, the most prominent argument against the constant volatility is the behavior of the empirically observed Black-Scholes implied volatility surface \((T, \kappa) \mapsto \hat{\sigma}_{\text{emp}}(T, \kappa)\) with \(T\) denoting the time to maturity, \(\kappa := \log \frac{K}{e^{-rT}S_0}\) being...
the log-moneyness, \( K \) the strike, \( S_0 \) the current price of an underlying asset and \( r \) the interest rate. Contrary to the prediction of the classical Black-Scholes framework, \( \tilde{\sigma}_{\text{emp}}(T, \kappa) \) is not flat, heavily varies in both variables and, in addition, \( \kappa \mapsto \tilde{\sigma}_{\text{emp}}(T, \kappa) \) produces convex “smiley” patterns for any fixed \( T \) \[32, 33, 35, 37, 58\].

One of the common ways to modify the vanilla log-normal framework so to take such effects into account is to model the instantaneous volatility as a separate stochastic process itself. It is safe to say that this idea has evolved into a full-fledged area of research on its own, see e.g. books \[11, 50, 68, 71\] as well as review articles \[38, 45, 53, 59\]. The first models in continuous time \[43, 63, 59\] based on Brownian diffusions indeed had the ability to reproduce the “smiley” patterns of implied volatility (see e.g. \[30\] or \[50\] Section 2.8.2) but, as indicated in \[58\], the accuracy of reproduction was imperfect. For example, as noted in \[28\], the smile generated by most classical Brownian models levels out too quickly in comparison to actual implied volatilities. To address this issue, Comte and Renault \[28\] proposed to model volatility via fractional Brownian motion (fBm) with Hurst index \( H > 1/2 \). This process allows for mimicking the behavior of the volatility smile amplitude for longer maturities (see also a simulation study \[56\]) and, furthermore, it is consistent with the stylized fact stating the presence of long memory in volatility \[9, 10, 22, 43\]. We also mention \[25, 27, 81\] among other prominent works utilizing fBm with \( H > 1/2 \).

In addition to inconsistencies for large \( T \), Brownian diffusion models also have problems with shorter maturities. As reported in \[36, 52\], the empirically observed smile at-the-money becomes steeper as the time to maturity \( T \to 0 \) with the rule-of-thumb behavior

\[
\left| \tilde{\sigma}_{\text{emp}}(T, \kappa) - \tilde{\sigma}_{\text{emp}}(T, \kappa') \right| \propto T^{-\frac{1}{2} + H}, \quad \kappa, \kappa' \approx 0, \quad H \in \left(0, \frac{1}{2}\right).
\]

Actually, it turns out that the power law \(1.1\) is hard to reproduce by either Brownian diffusions or fractional models with \( H > 1/2 \) (see e.g. the analysis in \[7\] Section 7). In order to replicate this behavior of implied volatility, \[7\] Section 7.2.2 suggested using fBm with \( H < 1/2 \). This methodology, further popularized by \[53\], is known now as rough volatility, and its efficiency in reproducing \(1.1\) can be explained from the two perspectives:

- on the one hand, the theoretical result of \[54\] shows that continuous semimartingale price on a market with power law \(1.1\) and no arbitrage implies that the paths of the market volatility must have low Hölder regularity – just like fBms with \( H < 1/2 \);
- on the other hand, \[7\] proves that, in Malliavin differentiable models, the explosive behavior \(1.1\) in fact comes from the explosion in the Malliavin derivative of the volatility process. Conveniently, fBm with \( H < 1/2 \) exhibits such a property.

The field of rough volatility has since evolved into a vast area of research, with a plethora of papers having been published over the years. A comprehensive and regularly updated list of literature on the subject can be found in \[11\] but we mention separately rough Bergomi \[16\], SABR \[53\], Stein-Stein \[2, 62\] and Heston \[46, 47, 48\] models. However, rough volatility models are also not perfect and tend to have the following issues.

(P1) “Roughness” is usually introduced by incorporating fBm-like structures with \( H < 1/2 \). However, as noted in \[56, 57\], long memory as well as the shape of \( \hat{\sigma}(T, \kappa) \) for large \( T \) hint that the preferable choice should be \( H > 1/2 \). While there are stochastic models that can simultaneously be rough and very persistent (see e.g. \[35\]), in the specific context of modeling volatility with fBm, there seems to be a peculiar paradox with the choice of \( H \).

(P2) In the stochastic volatility framework, the transition between the physical measure \( P \) and the pricing measure \( Q \) usually involves the term of the type \( \int_0^T \frac{1}{\sigma(t)^2} dt \), where \( \sigma = \{\sigma(t), \ t \in [0, T]\} \) is the corresponding volatility process \[20\]. However, in several important rough volatility models (e.g. rough Stein-Stein \[2, 62\] or rough Heston \[46, 47, 48\]), it is not clear whether such integral is well-defined. Therefore, in such models, there is no transparent procedure for the change of measure.

(P3) As shown in e.g. \[61\], rough volatility may produce moment explosions in price \( S = \{S(t), \ t \geq 0\} \). In fact, this phenomenon is not exclusive to the rough models only and is a prevalent feature within
the stochastic volatility framework in general. In particular, as demonstrated in [8, Section 3], an “unlucky” choice of coefficients may cause $E_P[S^2(T)] = \infty$ and $E_Q[S^2(T)] = \infty$ for all big enough $T$ even in the standard Heston model. For more details, we refer the reader to [8, 70] or the article “Moment Explosions” in [31]. Models with moment explosions tend to have a number of issues of both technical and modeling nature.

- As noted in [8, Section 8], “several actively traded fixed-income derivatives require at least $L^2$ solutions to avoid infinite model prices”. As examples of such assets, [8, Appendix A] mentions CMS swaps and Eurodollar futures, but, in principle, problems may occur even for the standard European call options. Indeed, $S(T) \notin L^2(\mathbb{P})$ normally implies that $(S(T) - K)_+ \notin L^2(\mathbb{P})$ and hence it is generally not clear whether $E_P\left[\frac{dQ}{dP}(S(T) - K)_+\right] < \infty$.
- Moment explosions pose significant challenges for the related stochastic optimization problems. For instance, the infinite second moment of $(S(T) - K)_+$ entirely precludes the application of quadratic hedging techniques. This limitation significantly impacts stochastic volatility models that usually produce incomplete markets [20]. In addition, in utility maximization problems, moment explosions can result in infinite expected utility, see e.g. [69].
- Finally, moment explosions lead to some issues with numerical procedures: for example, as discussed in [3, Section 4.2], they may invalidate error estimates in discretization schemes for the corresponding models. The Malliavin integration-by-parts technique for numerical option pricing from [5] also implicitly relies on the existence of the second moment of the payoff.

1.2 The SVV model

In this paper, we introduce a modeling framework, which we call the *Sandwiched Volterra Volatility (SVV)* model, that accounts for the problems (P1)–(P3) mentioned above. Namely, we consider

$$S_i(t) = S_i(0) + \int_0^t \mu_i(s)S_i(s)ds + \int_0^t Y_i(s)S_i(s)dB^S_i(s), \quad (1.2)$$

$$Y_i(t) = Y_i(0) + \int_0^t b_i(s, Y_i(s))ds + \int_0^t \kappa_i(t, s)dB^Y_i(s), \quad (1.3)$$

$i = 1, ..., d$, where $S_i$ and $Y_i$ are price and volatility processes respectively, $\mu_i$ are deterministic continuous functions, $B^S_i$, $B^Y_i$ are correlated standard Brownian motions, $\kappa_i$ are arbitrary square integrable Volterra kernels such that each process $Z_i(t) := \int_0^t \kappa_i(t, s)dB^Y_i(s)$ is Hölder continuous up to the order $H_i \in (0, 1)$. The main distinguishing property of the model is related to the drift functions $b_i$: for each $i = 1, ..., d$ there exist two deterministic continuous functions $0 < \varphi_i < \psi_i$ and constants $c > 0$, $\gamma > \frac{1}{H_i} - 1$ and $y_* > 0$ such that

$$b_i(t, y) \geq \frac{c}{(y - \varphi_i(t))^{\gamma}}, \quad y \in (\varphi_i(t), \psi_i(t) + y_*),$$

$$b_i(t, y) \leq -\frac{c}{(\psi_i(t) - y)^{\gamma}}, \quad y \in (\psi_i(t) - y_*, \psi_i(t)). \quad (1.4)$$

Processes of the type (1.3) were extensively studied in [60] and will be referred to as *sandwiched processes* due to the fact that the above-mentioned shape of the drift ensures

$$\varphi_i(t) \leq Y_i(t) \leq \psi_i(t), \quad (1.5)$$

i.e. the volatility process $Y_i$ is “sandwiched” between $\varphi_i$ and $\psi_i$, which, to some extent, links our model to the uncertain volatility approaches [12, 49].

\[1\] As a side note, we acknowledge that moment explosions may have a meaningful interpretation from the modeling point of view. More precisely, the celebrated moment formula of Lee [72] (see also its refinement [17]) connects the asymptotic behavior of implied volatility surface when $\kappa \to \pm \infty$ with the values

$$\tilde{p} := \sup\{p > 0, \ E_Q[S^4+p(T)] < \infty\}, \quad \tilde{q} := \sup\{q > 0, \ E_Q[S^{-q}(T)] < \infty\}.$$

However, we emphasize that it is very hard to use this relationship as an argument either in favor of or against models with moment explosions. In practice, the observed range of strikes is always finite and hence it is not possible to determine the “correct” asymptotics of empirical implied volatility $\tilde{\sigma}_{emp}(T, \kappa)$ as $\kappa \to \pm \infty$. 3
Note that the choice of the drift with (1.4) is not as exotic as it may seem at first glance. For example, as described in [40, Section 4], sandwiched processes can be used to define generalizations of some common for stochastic volatility models such as Cox-Ingersoll-Ross (CIR) or Chan–Karolyi–Longstaff–Sanders (CKLS) processes. Here we also mention [64] as well as the series of papers [74, 75, 76], where a process of the sandwiched type with $\phi_i \equiv 0, \psi_i \equiv \infty$ was used in the context of CIR model driven by fractional Brownian motion with Hurst index $H > 1/2$.

The SVV model (1.2)–(1.3) has several substantial advantages addressing the problems (P1)–(P3) mentioned above.

- In line with the recent literature (see e.g. [67] or [73]), we choose Hölder continuous Gaussian Volterra noises as drivers for our volatility processes. Such a choice is aimed at addressing the problem (P1): as discussed in [57] (see also [4, Section 7.7]), a simple linear combination of two fractional Brownian motions $B^{H_1}$ and $B^{H_2}$ with $H_1 > 1/2$ and $H_2 < 1/2$ can be sufficient to get both roughness/power law and long memory simultaneously. And indeed, as we prove in a separate paper [42], taking $K_{i}(t, s) := \theta_1, i(t-s)^{H_1,i-\frac{1}{2}} + \theta_2, i(t-s)^{H_2,i-\frac{1}{2}} 1_{s<t}$ (1.6) in the SVV model (1.2)–(1.3) with $H_{1,i} > 1/2$ and $H_{2,i} < 1/2$ does give the power law of the implied volatility skew, despite the seemingly exotic choice of the drift and presence of the long memory component.

In addition to a fairly straightforward kernel (1.6), the SVV model also covers more involved covariance structures of the volatility noise. For example, one may choose

$$K_{i}(t, s) := (t-s)^{h_i(t)-\frac{1}{2}} 1_{s<t},$$

where $h_i : [0, T] \to (0, 1)$ is some Hölder continuous function, i.e. the process

$$Z_i(t) := \int_0^t (t-s)^{h_i(t)-\frac{1}{2}} dB_Y(s)$$

is a multifractional Brownian motion (see e.g. [79]). Such a choice can be supported by econometric evidence: in [74, Section 2.2], the regularity of SPX volatility is found to vary over time between 0.1 and 0.9. For more details on multifractional volatility, see also [11].

- In this paper, we choose $\varphi_i > 0$, so (1.5) implies that each process $Y_i$ is bounded away from zero and the integrals $\int_0^T \frac{1}{Y_i(t)} ds$ are well-defined. This allows us to provide a clear description of equivalent local martingale measures on the market completely solving the problem (P2).

- The boundedness of $Y_i$ from above guaranteed by (1.5) ensures that, for any $r \in \mathbb{R}$,

$$\mathbb{E} \left[ \sup_{t \in [0,T]} S_r^T(t) \right] < \infty,$$

which eliminates the moment explosion problem described in (P3). This allows us to avoid any issues with infinite model prices as well as aids in numerical algorithms: in particular, we provide $L^r$-discretization schemes for price processes $S_i, r \geq 1$, as well as utilize the methodology of [5] for Monte Carlo pricing of options with discontinuous payoffs. Finally, (1.7) for $r = 2$ allows to perform mean-variance hedging within the SVV model which is analyzed in a separate paper [41].

In addition to the advantages mentioned above, we also note that (1.5) is a very convenient technical property that is often present in the literature (see e.g. [51, 78, 82]). Moreover, the functions $\varphi_i, \psi_i$ can be regarded as another calibration parameters similar to the minimal instantaneous variance in recent papers [60, 82].
1.3 Structure of the paper and main results

The results of this paper can be roughly divided into three parts.

I. Description of the model. In the first part contained in Section 2 we provide detailed specifications of the SVV market model as well as characterize its properties from the financial viewpoint. Namely,

- in Theorem 2.6, we give the moment bounds for price processes $S_i$, $i = 1, \ldots, d$;
- in Subsection 2.2, we give the full description of the set of equivalent local martingale measures and prove that the market generated by (1.2)–(1.3) is arbitrage-free and incomplete;
- in Subsection 2.3 we provide some simulations to illustrate implied volatility surfaces generated by the SVV model.

II. Malliavin differentiability. The second part detailed in Section 3 is fully devoted to the problem of Malliavin differentiability of the SVV model (1.2)–(1.3). The goal of our analysis is twofold.

- On the one hand, the results of [7] allow to connect the Malliavin derivative of volatility with the power law of the corresponding implied volatility skew. In other words, Malliavin differentiability of (1.3) can be used to analytically prove the power law within the SVV model. We perform this analysis in a separate paper [42]; it turns out that, with the right choice of the Volterra kernel, the SVV model indeed reproduces the power law (1.1).
- On the other hand, Malliavin techniques are useful for numerical pricing of options with discontinuous payoffs in the spirit of [5, 19, 76].

In Section 3, we prove the Malliavin differentiability of both volatility (1.3) and price (1.2) using the method similar to [64, Theorem 3.3] based on characterization of Sobolev spaces over an abstract Wiener space from [87]. As a consequence, we obtain absolute continuity of the law of the price processes (1.2).

III. Malliavin integration-by-parts pricing of options with discontinuous payoffs. In the last part, contained in Section 4–6, we suggest an immediate numerical application of the Malliavin differentiability results from Section 3. Namely, we consider the computation of $E[f(S(T))]$, where $f$ is a discontinuous function. Usually, it is impossible to compute $E[f(S(T))]$ analytically, so one must apply numerical methods for that; for instance, one can take an approximation $\hat{S}(T)$ of $S(T)$ using some numerical scheme and perform some Monte Carlo-type simulation. This works well for Lipschitz payoffs $f$; however, according to [13], any discontinuities in $f$ lead to a substantial deterioration in convergence speed. For instance, the rate of convergence of $E[|f(S(T)) - f(\hat{S}(T))|^2]$ to zero is halved in comparison to the order of the scheme for $\hat{S}$. This aggravation is additionally worsened by typically low convergence rates of numerical schemes for models with rough volatility and it significantly limits the application of some advanced Monte Carlo methods such as multilevel Monte Carlo. In order to overcome this problem, [5] suggested using Malliavin integration-by-parts to replace the discontinuous $f$ under the expectation with some Lipschitz functional, and, as a result, the initial convergence rate was preserved. A similar technique was applied in [19, 76] for models driven by a fractional Brownian motion. In the present paper, we utilize the same approach for the SVV model (1.2)–(1.3) providing some useful extensions. In particular, unlike [76], we utilize the pathwise bounds for sandwiched processes derived in [40] which allows to obtain the quadrature formulas without any limitations on the time horizon or regularity of the noise driving the volatility. In addition, all [5, 19, 76] take $E[f(\hat{S}(T))]$ with respect to the physical measure (i.e. it is the expected payoff rather than the price) whereas we provide a quadrature formula under the change of measure as well. The analysis is performed both in 1-dimensional setting (e.g. for digital options) and for multidimensional basket options with discontinuous payoffs. This third part of our paper is organized as follows:

- Section 4 adapts the Malliavin integration-by-parts quadrature method from [5] to the SVV setting (1.2)–(1.3);
- in Section 5 we give error estimates the mentioned method provided that the volatility is discretized using the drift-implicit Euler scheme from [39];
- Section 6 contains simulation results.
2 Model description

In this Section, we define the Sandwiched Volterra Volatility (SVV) model as well as provide some basic results regarding its properties.

**Notation 2.1.** Throughout this paper, \( C \) denotes any positive deterministic constant the exact value of which is not relevant. Note that \( C \) may change from line to line (or even within one line).

2.1 Preliminaries and assumptions

**Probability space, filtration and correlation structure of Brownian motions.** Let \((\Omega, \mathcal{F}, \mathbb{P})\) be the canonical 2d-dimensional Wiener space, i.e. \( \Omega := C_{0, L^2}(\mathbb{R}) \), \( \mathbb{P} \) is the classical Wiener measure, \( \mathcal{F} \) is the \( \mathbb{P} \)-augmented Borel sigma-algebra. Let also \( W \) be the corresponding 2d-dimensional Brownian motion, i.e. \( W : [0, T] \times \Omega \to \mathbb{R}^{2d}, W(t, \omega) = \omega(t), \) and \( \mathcal{F} = \{ F_t, t \in [0, T] \} \) be the \( \mathbb{P} \)-augmented filtration generated by \( W \).

Now, let \( \Sigma = (\sigma_{i,j})_{i,j=1}^{d} \) be an arbitrary non-degenerate \( 2 \times 2 \) real correlation matrix, i.e. \( \Sigma \) is symmetric, positive definite, \( \sigma_{i,j} \in [-1,1] \) and \( \sigma_{i,i} = 1 \) for all \( i = 1, ..., 2d \). Since \( \Sigma \) is symmetric and positive definite, it admits a Cholesky decomposition of the form \( \Sigma = \mathcal{L} \mathcal{L}^T \), where \( \mathcal{L} = (l_{i,j})_{i,j=1}^{2d} \) is a lower triangular matrix with strictly positive diagonal entries. Define \( B(t) = (B_1(t), ..., B_{2d}(t))^T := \mathcal{L} W(t) \) and note that components of the 2d-dimensional stochastic process \( \{ B(t), t \in [0,T] \} \) are 1-dimensional Brownian motions with

\[ \mathbb{E}[B_i(t)B_j(s)] = (t \wedge s)\sigma_{i,j}, \quad t, s \in [0,T], \quad i, j = 1, ..., 2d. \]

In what follows, we will need to distinguish the first \( d \) components of \( B \) (that will generate the “outer” noise for the price) from the other \( d \) components (that will generate the noise for the volatility) and thus we introduce the notation

\[ B^S(t) = (B_1^S(t), ..., B_{2d}^S(t))^T := (B_1(t), ..., B_d(t))^T, \]
\[ B^Y(t) = (B_1^Y(t), ..., B_{2d}^Y(t))^T := (B_{d+1}(t), ..., B_{2d}(t))^T. \]

Clearly, since the matrix \( \mathcal{L} \) is not degenerate, the filtration generated by the 2d-dimensional process \( B \) coincides with \( \mathcal{F} \).

**Stochastic volatility process.** Consider now \( d \) measurable functions \( K_i : [0,T]^2 \to \mathbb{R}, \quad i = 1, ..., d \), that satisfy the following assumption.

**Assumption A.** For any \( i = 1, ..., d \)

(i) \( K_i \) is a square integrable Volterra kernel, i.e. \( K_i(t,s) = 0 \) whenever \( t < s \),

\[ \int_0^T \int_0^T K_i^2(t,s)dsdt = \int_0^T \int_0^d K_i^2(t,s)dsdt < \infty \]

and

\[ \int_0^T K_i^2(t,s)ds < \infty, \quad \int_0^T K_i^2(t,s)dt < \infty, \quad \forall s, t \in [0,T]; \]

(ii) there exists a constant \( H_i \in (0,1) \) such that for any \( \lambda \in (0, H_i) \)

\[ \int_0^t (K_i(t,u) - K_i(s,u))^2du \leq C_\lambda |t-s|^{2\lambda}, \quad 0 \leq s \leq t \leq T, \quad (2.1) \]

where \( C_\lambda > 0 \) is a constant, possibly dependent on \( \lambda \).

Assumption [A(i)] allows to define a \( d \)-dimensional Gaussian Volterra process \( Z = (Z_1, ..., Z_d) \) such that

\[ Z_i(t) := \int_0^t K_i(t,s)dB^Y_i(s), \quad t \in [0,T]. \quad (2.2) \]

Moreover, Assumption [A(ii)] ensures that the paths of each \( Z_i \) are regular a.s., which is summarized in the following proposition.
Proposition 2.2 (\cite{15}, Theorem 1 and Corollary 4). The Gaussian process \( \{Z_i(t), t \in [0,T]\} \) defined by (2.2) has a modification that is a.s. Hölder continuous up to the order \( H_i \in (0,1) \) if and only if (2.1) holds for any \( \lambda \in (0,H_i) \). Moreover, for any \( \lambda \in (0,H_i) \), the random variable
\[
\Lambda_{\lambda,i} := \sup_{0 \leq s < t \leq T} \frac{|Z_i(t) - Z_i(s)|}{|t - s|^\lambda}
\] (2.3)
has moments of all orders.

Remark 2.3. By Proposition 2.2, it is evident that one can choose the modification of \( Z = (Z_1, ..., Z_d) \) such that for all \( \omega \in \Omega \)
\[
|Z_i(\omega,t) - Z_i(\omega,s)| \leq \Lambda_{\lambda,i}(\omega)|t - s|^\lambda,
\]
where \( \lambda \in (0,H_i) \). In what follows, this Hölder continuous modification will be used.

Now, let \( \varphi_i, \psi_i : [0,T] \rightarrow \mathbb{R}, i = 1, ..., d \), be \( d \) pairs of functions such that

1) \( \varphi_i, \psi_i \) are Hölder continuous up to the order \( H_i \), where \( H_i \) is from Assumption \( \Lambda \)(ii);
2) for each \( i = 1, ..., d \) and \( t \in [0,T] \):
\[
0 < \varphi_i(t) < \psi_i(t).
\] (2.4)

For any \( a_1, a_2 \in [0, \min_{i=1,...,d} \frac{1}{p}||\varphi_i - \varphi_i||_\infty] \), where \( ||.||_\infty \) is the standard sup-norm, denote
\[
D_{a_1,a_2} := \{(t,y) \in [0,T] \times \mathbb{R}_+, y \in (\varphi_i(t) + a_1, \psi_i(t) - a_2) \}
\] (2.5)
and consider \( d \) functions \( b_i : D_{0,0} \rightarrow \mathbb{R}, i = 1, ..., d \), that satisfy the following assumptions.

Assumption B.

(i) The functions \( b_i \in C(D_{0,0}), i = 1, ..., d \);

and there exist constants \( c > 0, p > 1, y_* \in (0, \frac{1}{p} \min_{i=1,...,d}||\varphi_i - \varphi_i||_\infty] \) such that for any \( i = 1, ..., d \) the following conditions hold:

(ii) for any \( \varepsilon \in (0, \min \{1, \min_{i=1,...,d} \frac{1}{p}||\varphi_i - \varphi_i||_\infty\} \})
\[
|b_i(t_1,y_1) - b_i(t_2,y_2)| \leq \frac{c}{\varepsilon^p} \left((y_1 - y_2) + |t_1 - t_2|^\lambda\right), \ (t_1,y_1),(t_2,y_2) \in D_{\epsilon,c}^i;
\]
(iii) for some constant \( \gamma_i > \frac{1}{p_i} - 1 \), where \( H_i \) is from Assumption \( \Lambda \)(ii),
\[
b_i(t,y) \geq \frac{c}{(y - \varphi_i(t))^{\gamma_i}}, \ (t,y) \in D_{0,0} \setminus D_{y_*,0}^i,
\]
\[
b_i(t,y) \leq -\frac{c}{(\psi_i(t) - y)^{\gamma_i}}, \ (t,y) \in D_{0,0}^i \setminus D_{y,y_*}^i;
\]
(iv) there exists a continuous partial derivative \( \frac{\partial b_i}{\partial y} \) with respect to the spatial variable and
\[
\frac{\partial b_i}{\partial y}(t,y) < c, \ (t,y) \in D_{0,0}^i.
\]

Next, consider the \( d \)-dimensional sandwiched process \( (Y_1, ..., Y_d) \) defined by
\[
Y_i(t) = Y_i(0) + \int_0^t b_i(s,Y_i(s))ds + Z_i(t), \ t \in [0,T], \ i = 1, ..., d,
\] (2.6)
where
- \( Y_i(0) \in (\varphi_i(0), \psi_i(0)) \) are deterministic constants;
- \( Z_i, i = 1, ..., d \), are defined by (2.2) with \( K_i \) satisfying Assumption \( \Lambda \).
Remark 2.4. Each equation in \((2.6)\) is treated pathwisely and, according to \[32\], has a unique solution for the given \(\omega \in \Omega\) provided that the corresponding path \(Z_t(\cdot, \omega)\) of the noise is H"older continuous up to the order \(H_i\) (and this happens a.s. by Remark 2.3). Moreover, in this case, according to \[32\] Theorem 3.2, for each \(i = 1, \ldots, d\) and \(\lambda \in \left(\frac{1}{\gamma_i}, H_i\right)\), where \(\gamma_i\) is from Assumption \([B](ii)\), one can find deterministic constants \(L_{1,i}, L_{2,i} > 0\) and \(\alpha_i > 0\) that depend only on the shape of \(b_i\) and \(\lambda\) such that for all \(t \in [0, T]\):

\[
\varphi(t) + \frac{L_{1,i}}{(L_{2,i} + \Lambda_{\lambda,i})^\alpha_i} \leq Y_i(t) \leq \psi(t) - \frac{L_{1,i}}{(L_{2,i} + \Lambda_{\lambda,i})^\alpha_i},
\]

where \(\Lambda_{\lambda,i}\) is a random variable defined by \((2.3)\). In particular, this means that \(0 < \varphi_i(t) < Y_i(t) < \psi_i(t)\) a.s., \(t \in [0, T]\), \(i = 1, \ldots, d\) (which justifies the name “sandwiched”). Note that \(Z_t\) does not have to be Gaussian for \((2.7)\) to hold: it is sufficient to assume \(\lambda\)-H"older continuity with \(\lambda > \frac{1}{\gamma_i}\).

Since \(\varphi_i, i = 1, \ldots, d,\) are continuous positive functions, it is clear that for any \(r > 0\):

\[
\mathbb{E}\left[\sup_{t \in [0,T]} \frac{1}{Y_i^r(t)}\right] \leq \frac{1}{\min_{t \in [0,T]} \varphi_i(t)} < \infty.
\]

Furthermore, since \(\Lambda_{\lambda,i}\) has moments of all orders, it is easy to see that for any \(r > 0\):

\[
\mathbb{E}\left[\sup_{t \in [0,T]} \frac{1}{(Y_i(t) - \varphi_i(t))^r}\right] < \infty, \quad \mathbb{E}\left[\sup_{t \in [0,T]} \frac{1}{(\psi_i(t) - Y_i(t))^r}\right] < \infty.
\]

Remark 2.5. Note that item (iv) of Assumption \([B]\) are not required for existence and uniqueness of the solution to \((2.6)\). It will be exploited later for the numerical scheme.

Price process. The components of the \(d\)-dimensional price process \(S = (S_1, \ldots, S_d)\) will be defined as solutions to the SDEs of the form

\[
S_i(t) = S_i(0) + \int_0^t \mu_i(s) S_i(s) ds + \int_0^t Y_i(s) S_i(s) dB_i^S(s), \quad t \in [0, T], \quad i = 1, \ldots, d,
\]

where

- \(S_i(0) > 0, i = 1, \ldots, d,\) are deterministic constants;
- \(\mu_i: [0, T] \to \mathbb{R}, i = 1, \ldots, d,\) are \(H_i\)-H"older continuous functions, where each \(H_i\) is from Assumption \([B](ii)\);
- \(Y_i, i = 1, \ldots, d\) are sandwiched volatility processes defined above.

We now list some simple properties of processes \(S_i, i = 1, \ldots, d\).

Theorem 2.6. For any \(i = 1, \ldots, d\), equation \((2.8)\) has a unique solution of the form

\[
S_i(t) = S_i(0) \exp\left\{ \int_0^t \left( \mu_i(s) - \frac{Y_i^2(s)}{2} \right) ds + \int_0^t Y_i(s) dB_i^S(s) \right\}.
\]

Furthermore, for any \(r \in \mathbb{R}\):

\[
\mathbb{E}\left[\sup_{t \in [0,T]} S_i^r(t)\right] < \infty.
\]

Proof. Each \(Y_i, i = 1, \ldots, d,\) is a bounded process and thus existence and uniqueness of solution to the corresponding equation follows from e.g. \[20\] Theorem 16.1.2 while the explicit form of the solution can be checked straightforwardly via Itô’s formula.
Next, using the explicit form of $S_t$ as well as boundedness of $Y_t$ one can see that there exists a constant $C > 0$ such that

\[
S_t^i(t) = S_t^i(0) \exp \left\{ r \int_0^t \mu_i(s) ds - \frac{r}{2} \int_0^t Y_i^2(s) ds + r \int_0^t Y_i(s) dB_i^S(s) \right\}
\]

\[
\leq S_t^i(0) \exp \left\{ |r| T \max_{s \in [0,T]} |\mu_i(s)| + \frac{|r| T}{2} \max_{s \in [0,T]} \psi_i^2(s) \right\} \exp \left\{ r \int_0^t Y_i(s) dB_i^S(s) \right\}
\]

\[
= S_t^i(0) \exp \left\{ |r| T \max_{s \in [0,T]} |\mu_i(s)| + \frac{|r| T}{2} \max_{s \in [0,T]} \psi_i^2(s) \right\} \exp \left\{ \frac{r^2}{2} \int_0^t Y_i^2(s) ds \right\} \times \tag{2.10}
\]

\[
\leq c_i \exp \left\{ \int_0^t r Y_i(s) dB_i^S(s) - \frac{1}{2} \int_0^t (r Y_i(s))^2 ds \right\}
\]

\[
:= c_i \mathcal{E}_t \{ r Y_i \cdot B_t^S \},
\]

where

\[
c_i := S_t^i(0) \exp \left\{ |r| T \max_{s \in [0,T]} |\mu_i(s)| + \frac{|r| (|r| + 1) T}{2} \max_{s \in [0,T]} \psi_i^2(s) \right\}
\]

and

\[
\mathcal{E}_t \{ r Y_i \cdot B_t^S \} := \exp \left\{ \int_0^t r Y_i(s) dB_i^S(s) - \frac{1}{2} \int_0^t (r Y_i(s))^2 ds \right\}, \quad t \in [0,T].
\]

Note that the Novikov’s criterion immediately yields that the process $\mathcal{E}_t \{ r Y_i \cdot B_t^S \}, t \in [0,T]$, is a uniformly integrable martingale such that

\[
\mathbb{E} \left[ \mathcal{E}_t \{ r Y_i \cdot B_t^S \} \right] = 1, \quad t \in [0,T],
\]

and, moreover,

\[
\mathcal{E}_t \{ r Y_i \cdot B_t^S \} \leq 1 + \int_0^t \mathcal{E}_s \{ r Y_i \cdot B_t^S \} r Y_i(s) dB_i^S(s), \quad t \in [0,T].
\]

By the Burkholder–Davis–Gundy inequality, there exists a constant $C_1 > 0$ such that

\[
\mathbb{E} \left[ \sup_{t \in [0,T]} \mathcal{E}_t \{ r Y_i \cdot B_t^S \} \right] \leq 1 + \mathbb{E} \left[ \sup_{t \in [0,T]} \int_0^t \mathcal{E}_s \{ r Y_i \cdot B_t^S \} r Y_i(s) dB_i^S(s) \right]
\]

\[
\leq 1 + C_1 \mathbb{E} \left[ \left( \int_0^T \mathcal{E}_s \{ r Y_i \cdot B_t^S \} Y_i^2(s) ds \right)^{\frac{1}{2}} \right]
\]

\[
\leq 1 + C_1 \sup_{s \in [0,T]} \psi_i(s) \left( \int_0^T \mathbb{E} \left[ \mathcal{E}_s^2 \{ r Y_i \cdot B_t^S \} \right] ds \right)^{\frac{1}{2}} \tag{2.11}
\]

By Novikov’s criterion, $\mathbb{E} \left[ \mathcal{E}_s \{ 2 r Y_i \cdot B_t^S \} \right] = 1$, so

\[
\mathbb{E} \left[ \mathcal{E}_s^2 \{ r Y_i \cdot B_t^S \} \right] = \mathbb{E} \left[ \mathcal{E}_s \{ 2 r Y_i \cdot B_t^S \} \exp \left\{ \int_0^s (r Y_i(u))^2 du \right\} \right]
\]

\[
\leq \exp \left\{ r^2 T \max_{s \in [0,T]} \psi_i^2(s) \right\} \mathbb{E} \left[ \mathcal{E}_s \{ 2 r Y_i \cdot B_t^S \} \right]
\]

\[
= \exp \left\{ r^2 T \max_{s \in [0,T]} \psi_i^2(s) \right\} < \infty
\]

and hence, taking into account (2.10), we obtain that

\[
\mathbb{E} \left[ \sup_{t \in [0,T]} S_t^i(t) \right] \leq c_i \left( 1 + C_1 \sqrt{T} \sup_{s \in [0,T]} \psi_i(s) \exp \left\{ \frac{r^2 T}{2} \max_{s \in [0,T]} \psi_i^2(s) \right\} \right),
\]

which yields the required result.
Numérale and discounted price process. As a numéraire, we will use the function $e^{\int_0^t \nu(s)ds}$, $t \in [0, T]$, where $\nu : [0, T] \to \mathbb{R}_+$ denotes a Hölder continuous function of order $\min_{l=1,...,d} H_l$ representing an instantaneous interest rate. The discounted price process denoted as

$$\tilde{S}_i(t) := e^{-\int_0^t \nu(s)ds}S_i(t), \quad t \in [0, T], \quad i = 1, ..., d,$$

has thus dynamics of the form

$$\tilde{S}_i(t) = S_i(0) + \int_0^t \tilde{\mu}_i(s)\tilde{S}_i(s)ds + \int_0^t Y_i(s)\tilde{S}_i(s)d\tilde{B}_i^S(s), \quad t \in [0, T], \quad i = 1, ..., d,$$

where $\tilde{\mu}_i := \mu_i - \nu$.

2.2 Incompleteness of the market and martingale measures

Recall that our model is driven by the $2d$-dimensional Gaussian process

$$\begin{pmatrix} B^S(t) \\ B^Y(t) \end{pmatrix} = \mathcal{L} W(t), \quad t \in [0, T],$$

where $W = \{W(t), \ t \in [0, T]\}$ is a standard $2d$-dimensional Brownian motion and $\mathcal{L} = (\ell_{ij})_{i,j=1}^{2d}$ is nondegenerate lower triangular $2d \times 2d$ real matrix with positive entries on a diagonal such that $\mathcal{L} \mathcal{L}^T = \Sigma$. Let also

$$\mathcal{L} = \begin{pmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} \\ \mathcal{L}_{21} & \mathcal{L}_{22} \end{pmatrix},$$

where $\mathcal{L}_{ij}$ we denote the $d \times d$ blocks of matrix $\mathcal{L}$. Note that $\mathcal{L}_{12}$ is a zero matrix and $\mathcal{L}_{11}$ is a lower triangular matrix with positive elements on the diagonal. In particular, $\mathcal{L}_{11}$ is invertible, so we can define its inverse $\mathcal{L}_{11}^{-1} = (\ell_{-1}^{(j,k)})_{j,k=1}^d$. Finally, for a given semimartingale $X$, we denote the corresponding Doléans-Dade exponential by

$$\mathcal{E}_t \{ X \} := \exp \left\{ X_t - X_0 - \frac{1}{2} [X]_t \right\}.$$

The next lemma is in the spirit of Lemma 2.10 in [20].

Lemma 2.7. Let $M$ be a positive local $\mathbb{P}$-martingale w.r.t. $\mathbb{F}$ with $M(0) = 1$ a.s. Then $M\tilde{S}_i$, $i = 1, ..., d$, are all local martingales if and only if

$$M(t) = \mathbb{E}_t \left\{ -\sum_{j=1}^d \int_0^t \left( \sum_{k=1}^d \ell_{j,k}^{(-1)} \tilde{\mu}_k(s) \right) dW_j(s) \right\} \mathcal{E}_t \left\{ \sum_{l=d+1}^{2d} \int_0^t \xi_l(s)dW_l(s) \right\}$$

$$= \mathbb{E}_t \left\{ -\sum_{j=1}^d \int_0^t \left( \sum_{k=1}^d \ell_{j,k}^{(-1)} \tilde{\mu}_k(s) \right) dW_j(s) + \sum_{l=d+1}^{2d} \int_0^t \xi_l(s)dW_l(s) \right\}$$

(2.12)

where $\left( \ell_{j,k}^{(-1)} \right)_{j,k=1}^d$ are the elements of $\mathcal{L}_{11}^{-1}$ and $\xi_l$, $l = d + 1, ..., 2d$, are predictable processes such that $\int_0^t \xi_l(s)dW_l(s)$ are well-defined local martingales.

Proof. By the virtue of the local martingale representation theorem, $M$ is a local $\mathbb{P}$-martingale w.r.t. $\mathbb{F}$ if and only if there exist predictable processes $\zeta_j = \{ \zeta_j(t), \ t \in [0, T] \}$, $j = 1, ..., 2d$, such that

$$\mathbb{P} \left( \int_0^T \zeta_j^2(s)ds < \infty \right) = 1, \quad j = 1, ..., 2d,$$

and

$$M(t) = 1 + \sum_{j=1}^{2d} \int_0^t \zeta_j(s)dW_j(s).$$

(2.13)
By Itô’s formula,
\[
d(M(t)\tilde{S}_t(t)) = \left( \tilde{\mu}_i(t)M(t) + Y_i(t) \sum_{j=1}^{2d} \ell_{i,j} \zeta_j(t) \right) \tilde{S}_t(t)dt
+ \sum_{j=1}^{2d} (\ell_{i,j} M(t) Y_i(t) + \zeta_j(t)) \tilde{S}_t(t)dW_j(t),
\]
and thus \(M\tilde{S}_t\) is a local martingale if and only if
\[
\int_0^t \left( \tilde{\mu}_i(s)M(s) + Y_i(s) \sum_{j=1}^{2d} \ell_{i,j} \zeta_j(s) \right) \tilde{S}_t(s)ds = 0, \quad t \in [0,T].
\]
This, in turn, implies that
\[
\sum_{j=1}^{2d} \ell_{i,j} \zeta_j(t) = -M(t) \frac{\tilde{\mu}_i(t)}{Y_i(t)}, \quad i = 1, \ldots, d,
\]
almost everywhere w.r.t. \(dt \otimes d\mathbb{P}\). Since \(\mathcal{L}\) is lower-triangular and each \(Y_i(t)\) is positive a.s., we have that
\[
\mathcal{L}_{11} \begin{pmatrix} \zeta_1(t) \\ \vdots \\ \zeta_d(t) \end{pmatrix} = -M(t) \begin{pmatrix} \frac{\tilde{\mu}_1(t)}{Y_1(t)} \\ \vdots \\ \frac{\tilde{\mu}_d(t)}{Y_d(t)} \end{pmatrix}
\]
(2.14)
almost everywhere w.r.t. \(dt \otimes d\mathbb{P}\) and hence
\[
\begin{pmatrix} \zeta_1(t) \\ \vdots \\ \zeta_d(t) \end{pmatrix} = \mathcal{L}_{11}^{-1} \begin{pmatrix} -M(t) \frac{\tilde{\mu}_1(t)}{Y_1(t)} \\ \vdots \\ -M(t) \frac{\tilde{\mu}_d(t)}{Y_d(t)} \end{pmatrix}
\] (2.15)
almost everywhere w.r.t. \(dt \otimes d\mathbb{P}\). Using notation \(\mathcal{L}_{11}^{-1} = \left( \ell_{j,k}^{(-1)} \right)_{j,k=1}^d\) and substituting (2.15) to (2.13), we see that the process \(M\) should satisfy the SDE of the form
\[
dM(t) = -M(t) \sum_{j=1}^d \sum_{k=1}^d \ell_{j,k}^{(-1)} \frac{\tilde{\mu}_k(t)}{Y_k(t)} dW_j(t) + \sum_{l=d+1}^{2d} \zeta_l(t)dW_l(t).
\]
This SDE has a unique solution (see e.g. [26]) of the form
\[
M(t) = \mathcal{E}_t \left\{ -\sum_{j=1}^d \int_0^t \left( \sum_{k=1}^d \ell_{j,k}^{(-1)} \frac{\tilde{\mu}_k(s)}{Y_k(s)} \right) dW_j(s) \right\} \\
\times \left( 1 + \sum_{l=d+1}^{2d} \int_0^t \mathcal{E}_u^{-1} \left\{ -\sum_{j=1}^d \int_0^u \left( \sum_{k=1}^d \ell_{j,k}^{(-1)} \frac{\tilde{\mu}_k(s)}{Y_k(s)} \right) dW_j(s) \right\} \zeta_l(u)dW_l(u) \right) (2.16)
\]
Furthermore, since the process \(M\) is positive, the continuous process \(\widetilde{M}\) defined by
\[
\widetilde{M}(t) := 1 + \sum_{l=d+1}^{2d} \int_0^t \zeta_l(s)dW_l(s)
\]
must also be positive, thus it can be represented as

\[ M(t) = \mathcal{E}_t \left( \sum_{i=d+1}^{2d} \int_0^T \xi_i(s) dW_i(s) \right) \tag{2.17} \]

with \( \xi_l, l = d+1, \ldots, 2d \), being some predictable processes such that the corresponding stochastic integrals are well-defined. It remains to notice that \( \xi_l, l = d+1, \ldots, 2d \), can be arbitrary predictable stochastic processes such that the corresponding stochastic integrals are well-defined, thus \( \xi_l, l = d+1, \ldots, 2d \), can also be chosen arbitrarily. Taking into account (2.16) and (2.17), we get the representation (2.12).

In the literature, positive densities \( M(T) \) associated with local \( \mathbb{P} \)-martingales \( M \) such that \( M\tilde{S}_i \) all become local martingales are commonly called \textit{strict local martingale densities} (see [33] for more detail) and Lemma 2.7 thus gives the description of the set of all strict local martingale densities on the market. However, note that the measure defined by \( d\tilde{Q} := M(T) d\mathbb{P} \) is not necessarily a probability measure and one has to check that \( \tilde{Q}(\Omega) = \mathbb{E}[M(T)] = 1 \) (or, equivalently, that \( M \) is a uniformly integrable martingale) separately. Moreover, exploiting the Bayes’ theorem for conditional expectations, it is easy to see that if \( \tilde{Q} \) defined above is a probability measure, then \( \tilde{S}_i, i = 1, \ldots, d \), are local \( \tilde{Q} \)-martingales, which would imply the no-arbitrage property of the market. It turns out that the market model in consideration indeed has this property which is formulated in the next theorem.

\textbf{Theorem 2.8.} The market defined in Subsection 2.1 is arbitrage-free and incomplete.

\textbf{Proof.} Each \( Y_k, k = 1, \ldots, d \), is bounded away from zero by the corresponding \( \min_{t \in [0,T]} \varphi_k(t) > 0 \), thus, by Novikov’s condition,

\[ M(t) = \mathcal{E}_t \left\{ - \sum_{j=1}^d \int_0^T \left( \sum_{k=1}^d \tilde{\mu}_j(s) \tilde{X}_k(s) \right) dW_j(s) \right\} \]

is a uniformly integrable martingale (this case corresponds to \( \xi_l \equiv 0 \) in (2.12), \( l = d+1, \ldots, 2d \)). Therefore the associated \( \tilde{Q} \) is a probability measure, i.e. the market is arbitrage free.

Furthermore, again by Novikov’s condition, any \( \xi_l \) such that

\[ \mathbb{E} \left[ \exp \left\{ \frac{1}{2} \sum_{l=d+1}^{2d} \int_0^T \xi_l(t)^2 dt \right\} \right] < \infty \]

leads to \( M \) being a uniformly integrable martingale, thus the market is incomplete. \( \square \)

\section{2.3 Implied volatility surface generated by the SVV model}

One of the accepted benchmarks for the quality of a financial model is the shape of the implied volatility generated by it (see e.g. the discussion in [33]). In this subsection, we provide some results regarding the shape of the implied volatility surfaces generated by the SVV model as well as illustrate these shapes with simulations.

Assume, for simplicity, that the instantaneous interest rate \( \nu \) is constant and consider the SVV model of the form

\[ S(t) = S(0) + \nu \int_0^t S(s) ds + \int_0^t Y(s) S(s) \left( \sqrt{1 - \rho^2} dB_1(s) + \rho dB_2(s) \right), \]

\[ Y(t) = Y(0) + \int_0^t b(s, Y(s)) ds + \int_0^t \mathcal{K}(t, s) dB_2(s), \tag{2.18} \]

where \( B_1, B_2 \) are independent Brownian motions, \( \rho \in (-1, 1), \mathcal{K} \) satisfies Assumption \text{[A]} and \( b \) satisfies Assumption \text{[B]} so that

\[ 0 < \min_{t \in [0,T]} \varphi(t) < \varphi(t) \quad < Y(t) \quad < \psi(t) \quad < \max_{t \in [0,T]} \psi(t) < \infty. \tag{2.19} \]
Remark 2.9. It is easy to check that, in the particular case of the SVV model (2.18), the discounted price process $\tilde{S} = \{\tilde{S}(t), t \in [0,T]\} = \{e^{-\nu t}S(t), t \in [0,T]\}$ satisfies the SDE

$$\tilde{S}(t) = S(0) + \int_0^t Y(s)\tilde{S}(s) \left(\sqrt{1-\rho^2}dB_1(s) + \rho dB_2(s)\right),$$

and hence is a martingale. This means that, under the model (2.18), the non-arbitrage price $\Pi$ of a European call-option with maturity $T$ and payoff $K$ is equal to

$$\Pi_{SVV}(T,K) = e^{-\nu T}\mathbb{E}[(S(T) - K)_+].$$

Such a choice allows to simplify the arguments in this section and increase the numerical stability of the simulations.

Consider also the standard Black-Scholes price of a European call-option with payoff $K$ and maturity $T$ at a given volatility level $\sigma > 0$:

$$\Pi_{BS}(T,K;\sigma) := S(0)\Phi\left(\frac{\log \frac{S(0)}{K} + \frac{\sigma^2 T}{\sigma}\sqrt{T}}{\sigma\sqrt{T}}\right) - K e^{-\nu T}\Phi\left(\frac{\log \frac{S(0)}{K} - \frac{\sigma^2 T}{\sigma}\sqrt{T}}{\sigma\sqrt{T}}\right),$$

where $\Phi$ is the CDF of the standard normal distribution. We are interested in the behavior of the implied volatility surface $(T,K) \mapsto \tilde{\sigma}(T,K)$ generated by the SVV model defined as

$$\Pi_{BS}(T,K;\tilde{\sigma}(T,K)) = \Pi_{SVV}(T,K).$$

Remark 2.10. It is easy to check that $\tilde{\sigma}(T,K)$ exists and is uniquely defined for all $T,K > 0$ since the function $\sigma \mapsto \Pi_{BS}(T,K;\sigma)$ is strictly increasing for all $T,K > 0$, and, moreover,

$$\lim_{\sigma \to 0^+} \Pi_{BS}(T,K;\sigma) = (S(0) - e^{-\nu T}K)_+$$

$$= (\mathbb{E}[e^{-\nu T}S(T)] - e^{-\nu T}K)_+$$

$$\leq \mathbb{E}[(e^{-\nu T}S(T) - e^{-\nu T}K)_+] = \Pi_{SVV}(T,K).$$

Boundedness of the implied volatility. First of all, let us prove that the SVV implied volatility $(T,K) \mapsto \tilde{\sigma}(T,K)$ is bounded.

Theorem 2.11. For all $T,K > 0$,

$$\min_{t \in [0,T]} \varphi(t) \leq \tilde{\sigma}(T,K) \leq \max_{t \in [0,T]} \psi(t).$$

Proof. By the seminal result [12] of Avellaneda, Levy and París, the bounds (2.19) imply that

$$\Pi_{BS}(T,K;\min_{t \in [0,T]} \varphi(t)) \leq \Pi_{SVV}(T,K) \leq \Pi_{BS}(T,K;\max_{t \in [0,T]} \psi(t)),$$

which immediately yields the claim. \hfill \blacksquare

Remark 2.12. Theorem 2.11 indicates that one should choose the upper bound function $\psi$ such that $\max_{t \in [0,T]} \psi(t)$ exceeds the maximum value of the realized implied volatility observed on the market. Under the normal market conditions, it is usually sufficient to take $\max_{t \in [0,T]} \psi(t) = 1$.

At-the-money skew. For the notational convenience, denote $\kappa := \log \frac{K}{e^{\nu T}S(0)}$ and consider a reparametrization $\tilde{\sigma}_{log-m} = \tilde{\sigma}_{log-m}(T,\kappa)$ of $\tilde{\sigma} = \tilde{\sigma}(T,K)$ defined as

$$\tilde{\sigma}_{log-m}(T,\kappa) := \tilde{\sigma}(T,S(0)e^{\kappa + \nu T}).$$

It is well-known (see e.g. a detailed discussion in [38] Section 2.2) or [39]), that smiles at-the-money of empirically observed implied volatilities $\tilde{\sigma}_{emp}(T,\kappa)$ become progressively steeper as $T \to 0$ with a rule-of-thumb approximation

$$\left| \tilde{\sigma}_{emp}(T,\kappa) - \tilde{\sigma}_{emp}(T,\kappa') \right| \propto T^{-\frac{1}{2}+H}, \quad \kappa, \kappa' \approx 0, \quad H \in \left(0, \frac{1}{2}\right).$$

(2.20)
In order to replicate the empirical behavior (2.20), one may want a model generating the implied volatility \( \hat{\sigma}_{\log,m} = \sigma_{\log,m}(T, \kappa) \) with

\[
\left| \frac{\partial \hat{\sigma}_{\log,m}(T, \kappa)}{\partial \kappa} \right|_{\kappa=0} = O(T^{-\frac{1}{2} + H}), \quad T \to 0.
\]  

(2.21)

Luckily, the SVV model (2.18) does possess the property (2.21) with the right choice of the kernel \( K \).

Theorem 2.13. Let \( H \in \left( \frac{1}{3}, \frac{1}{2} \right) \) and \( \rho < 0 \) in (2.18). Take

\[
b(t, y) = \frac{\theta_1(t)}{(y - \varphi(t))^{\gamma_1}} - \frac{\theta_2(t)}{(\psi(t) - y)^{\gamma_2}} + a(t, y),
\]  

(2.22)

where

- \( \gamma_1 > \frac{1}{H} - 1, \gamma_2 > \frac{1}{H} - 1; \)
- the functions \( \theta_1, \theta_2 : [0, T] \to \mathbb{R} \) are strictly positive and continuous;
- the function \( a : [0, T] \times \mathbb{R} \to \mathbb{R} \) is locally Lipschitz in \( y \) uniformly in \( t \), i.e. for any \( N > 0 \) there exists a constant \( C_N \) that does not depend on \( t \) such that

\[
|a(t, y_2) - a(t, y_1)| \leq C_N |y_2 - y_1|, \quad t \in [0, T], \quad y_1, y_2 \in [-N, N];
\]
- \( a : [0, T] \times \mathbb{R} \to \mathbb{R} \) is two times differentiable w.r.t. the spatial variable \( y \) with \( a, a', a'' \) all being continuous on \([0, T] \times \mathbb{R}\)

and assume that the kernel \( K \) is such that

- for any \( 0 \leq s < t \leq T \), there is a constant \( C > 0 \) such that

\[
|K(t, s)| \leq C |t - s|^{-\frac{1}{2} + H};
\]
- there exists a constant \( K_Y > 0 \) such that

\[
\frac{1}{\tau^{\frac{1}{2} + H}} \int_0^\tau \int_0^{\tau} K(t, s) \, dt \, ds \to K_Y, \quad \tau \to 0 + .
\]

(2.23)

Then the power law (2.21) holds for the SVV model (2.18).

Proof. The proof of this claim can be found in the dedicated paper [42]. \( \square \)

Remark 2.14. The claim of Theorem 2.13 holds for drifts that are more general than (2.22). For more details, see Remark [43, Remark 2.6].

Remark 2.15. The condition \( H > \frac{1}{3} \) in Theorem 2.13 is consistent with the recent empirical estimate \( H \approx 0.19 \) for the SPX implied volatility obtained in [39].

Simulations. For simulations, we take \( \nu = 0 \) and the SVV model of the form

\[
S(t) = 1 + \int_0^t Y(s) S(s) \left( \sqrt{0.75} d B_1(s) - 0.5 d B_2(s) \right),
\]

\[
Y(t) = 0.5 + \int_0^t \left( \frac{0.005}{(Y(s) - 0.05)^5} - \frac{0.005}{(1 - Y(s))^5} + 0.05(0.5 - Y(s)) \right) ds + 0.3 \int_0^t K(t, s) dB_2(s),
\]  

(2.24)

where the choice of \( K \) is varied. In order to plot the implied volatility surface, we

- estimate the call option prices \( \mathbb{E} [(S(T) - K)_+] \) under the model (2.24) for \( T = \frac{n}{200}, \ n = 1, \ldots, 200, \) and \( K = 0.5 + m/100, \ m = 0, \ldots, 150, \) using the standard Monte Carlo method, i.e. average over 1500000 realizations of the payoff \( (S(T) - K)_+ \) (the algorithm for the simulation of the random variable \( S(T) \) is described in full detail further in Subsection 5.1);
• calculate the corresponding Black-Scholes implied volatility \( (T, \kappa) \mapsto \tilde{\sigma}(T, \kappa) \), where \( \kappa \) denotes the log-moneyness, using the standard procedure (see e.g. [38, Section 2.2] for more details).

**Remark 2.16.** Note that the parameters in (2.24) are illustrational and are not calibrated to the real market data.

**Example 2.17.** Let us consider the rough fractional kernel

\[
\mathcal{K}(t, s) := \frac{1}{\Gamma(0.7)} (t - s)^{-0.3} \mathbb{1}_{s < t}
\]

in (2.24). In this case, the implied volatility surface produced by the SVV model (2.24) is depicted on Figure 1.

Note that Figure 1(b) visually demonstrates how that the absolute at-the-money implied volatility skew

\[
\left| \frac{\partial \tilde{\sigma}_{\text{log-m}}}{\partial \kappa}(T, 0) \right|
\]

increases as \( T \to 0 \). Figure 2 contains the variation of (2.26) with respect to \( T \) on the standard (Fig. 2(a)) and logarithmic (Fig. 2(b)) scales and shows that (2.26) indeed follows the power law with

\[
\left| \frac{\partial \tilde{\sigma}_{\text{log-m}}}{\partial \kappa}(T, 0) \right| = O(T^{-0.3}), \quad T \to 0.
\]

(a) Implied volatility surface  
(b) Implied volatility smiles for different maturities

Figure 1: Implied volatility surface (a) and implied volatility smiles (b) generated by the SVV model (2.24) with the rough kernel (2.25). Note that the smile becomes steeper as the time to maturity \( T \to 0 \), which reproduces a similar effect happening on real markets (for more details, see e.g. [36, 52] or [38, Section 2.2]).

As a final remark, observe that the points on Fig. 2(b) are not located on a single line; in fact, the linear fit performed over the short maturities substantially differs from the fit over long maturities (see Fig. 3). Such behavior agrees with the recent empirical study [36] reporting the same phenomenon for SPX implied volatility skews (c.f. [36, Figure 2]).

**Example 2.18.** Let us consider the mixed fractional kernel

\[
\mathcal{K}(t, s) = \left( \frac{\sqrt{0.4}}{\Gamma(0.7)} (t - s)^{-0.3} + \frac{\sqrt{1.8}}{\Gamma(1.4)} (t - s)^{0.4} \right) \mathbb{1}_{s < t}
\]

(2.27)
Figure 2: Absolute at-the-money implied volatility skew Equation (2.26) generated by the SVV model Equation (2.24) with the rough kernel Equation (2.25) on standard (a) and logarithmic (b) scales.

Figure 3: At-the-money implied volatility skew Equation (2.26) generated by the SVV model Equation (2.24) with the rough kernel Equation (2.25) on a logarithmic scale. The lines depict two linear fits: over the short maturities (red line, the slope is $-0.3006433$ which is consistent with [42, Theorem 4.8]) and over the long maturities (green line, the slope is $-0.9920939$). Such a behavior is consistent with the recent paper [36].

in Equation (2.24). The corresponding implied volatility surface and implied volatility smiles are presented on Figure 4. Comparing the results with the purely rough case from Example 2.17, we note that the presence of a “long memory” component $(t - s)^{0.4}$ in Equation (2.27) keeps the smiles “less flat” and more pronounced for larger maturities, which is consistent with the results of [56, 57]. In addition, just like in the purely rough case from Example 2.17, the explosive behavior of the skew as $T \to 0$ is guaranteed by
Figure 4: Implied volatility surface (a) and implied volatility smiles (b) generated by the SVV model (2.24) with the mixed fractional kernel (2.27). Just as in the purely rough case, the absolute at-the-money skew increases for small maturities, which is consistent with the real-life market behavior. Note that the presence of the term \((t - s)^{0.4}\) in the kernel ensures that the smile “flattens out” for larger maturities much slower than in the purely rough case, which agrees with the results of [56, 57].

Theorem 2.13 which implies

\[
\left| \frac{\partial \sigma_{\text{log-m}}}{\partial \kappa} (T, 0) \right| = O(T^{-0.3}), \quad T \to 0,
\]

see Figure 5 below. In other words, the mixed kernel indeed allows to keep the power law of the at-the-money skew as \(T \to 0\), and increases the curvature of the smile for larger values of \(T\).

Figure 5: Absolute at-the-money implied volatility skew (2.26) generated by the SVV model (2.24) with the mixed kernel (2.27) on standard (a) and logarithmic (b) scales.
3 Malliavin differentiability of volatility and price processes and absolute continuity of their laws

3.1 Malliavin differentiability of volatility and price w.r.t. the initial Brownian motion $W$

This Subsection is dedicated to the Malliavin differentiability of the volatility and price processes within the SVV model. Our strategy is as follows: first of all, we use the classical characterization of the Malliavin derivative in terms of the stochastic Gateaux derivative and prove the differentiability of sandwiched processes $Y_i, i = 1, \ldots, d$. Then, we prove an appropriate Malliavin chain rule and apply it to the representation (2.9) to obtain the differentiability of $S_i, i = 1, \ldots, d$.

Malliavin differentiability of volatility. In order to introduce the concept of the Malliavin differentiability, one must first fix the underlying isonormal Gaussian process and an obvious choice is the initial Brownian motion $W$. Recall that $(\Omega, \mathcal{F}, \mathbb{P})$ is the canonical $2d$-dimensional Wiener space, $W = (W_1, \ldots, W_{2d})$ is the associated $2d$-dimensional Brownian motion and stochastic volatility processes $Y_i = \{Y_i(t), t \in [0, T]\}, i = 1, \ldots, d$, are given by

$$Y_i(t) = Y_i(0) + \int_0^t b_i(s, Y_i(s))ds + Z_i(t), \quad t \in [0, T],$$

where $Y_i(0) \in (\varphi_i(0), \psi_i(0))$, each $b_i$ satisfies Assumption $B$; $Z_i(t) = \int_0^t \mathcal{K}_i(t, s)dB_i^Y$ is a Gaussian Volterra process with the kernel $\mathcal{K}_i$ satisfying Assumption $A$ and

$$B_i^Y := \sum_{j=1}^{2d} \ell_{d+i,j}W_j.$$

Denote also by $\mathcal{H}$ the associated Cameron-Martin space, i.e. the space of all continuous functions $F = (F_1, \ldots, F_{2d}) \in C_0([0,T];\mathbb{R}^{2d})$ such that each $F_i, i = 1, \ldots, 2d$, can be represented as

$$F_i(\cdot) = \int_0^\cdot f_i(s)ds,$$

where $f = (f_1, \ldots, f_{2d}) \in L^2([0,T];\mathbb{R}^{2d})$. For the relation between $F$ and $f$ specified above, we will use the notation $F = \int_0^\cdot f(s)ds$. By $\mathbb{D}^{1,2}$, we denote the space of Malliavin differentiable functions (w.r.t. the $L^2$-closure).

Before proceeding to the main results of this Subsection, let us give an important auxiliary theorem.

**Theorem 3.1 (R7, Theorem 3.1).** A random variable $\eta \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$ belongs to $\mathbb{D}^{1,2}$ w.r.t. $W$ if and only if the following two conditions are satisfied:

(i) $\eta$ is ray absolutely continuous, i.e. for any $F \in \mathcal{H}$ there exists a version of the process $\{\eta(\omega + \varepsilon F), \varepsilon \geq 0\}$ that is absolutely continuous;

(ii) $\eta$ is stochastically Gateaux differentiable, i.e. there exists a random vector

$$D\eta \in L^2(\Omega; L^2([0,T];\mathbb{R}^{2d}))$$

such that for any $F = \int_0^\cdot f(s)ds \in \mathcal{H}$, $f \in L^2([0,T];\mathbb{R}^{2d})$,

$$\frac{1}{\varepsilon} \left( \eta(\omega + \varepsilon F) - \eta(\omega) \right) \xrightarrow{\mathbb{P}} (D\eta, f)_{L^2([0,T];\mathbb{R}^{2d})}, \quad \varepsilon \to 0.$$

In this case the random vector $D\eta$ from (ii) is the Malliavin derivative of $\eta$.

In order to apply Theorem 3.1 to obtain the Malliavin differentiability of $Y_i(t), i = 1, \ldots, d, t \in [0, T]$, we first need to formally identify

$$Y_i(\omega + \varepsilon F, t), \quad i = 1, \ldots, d, \quad t \in [0, T],$$

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for any $F = \int_0^T f(s)ds \in \mathcal{H}$. For the given $\omega$, it is reasonable to associate these expressions with random variables $Y^{\varepsilon,F}_i(\omega,t)$ given by

$$Y^{\varepsilon,F}_i(\omega,t) = Y(0) + \int_0^t b_i \left( s, Y^{\varepsilon,F}_i(\omega,s) \right) ds + Z^{\varepsilon,F}_i(\omega,t), \quad t \in [0,T],$$

where $Z^{\varepsilon,F}_i(\omega,t) := Z_i(\omega + \varepsilon F, t)$. Indeed, equations (2.6) are treated pathwisely and each $Z^{\varepsilon,F}_i(\omega,t)$ is Hölder continuous w.r.t. $t$ for all $\omega \in \Omega$. This means that the equation above has a unique solution for all $\omega \in \Omega$ and

$$Y_i^{\varepsilon,F}(\omega,t) = Y_i(\omega + \varepsilon F, t), \quad \omega \in \Omega, \quad t \in [0,T].$$

However, in order to apply Theorem 3.1, one needs to get a more convenient representation of $Z_i^{\varepsilon,F}(\omega,t)$ that “splits” the pieces corresponding to $\omega$ and $F$. Intuitively, it is clear that a natural candidate for such a representation is

$$Z_i^{\varepsilon,F}(\cdot,t) = Z_i(\cdot,t) + \varepsilon \sum_{j=1}^{2d} \ell_{d+i,j} \int_0^t K_i(t,s) f_j(s) ds, \quad t \in [0,T],$$

and this guess is formalized in the next simple proposition.

**Proposition 3.2.** Let $\varepsilon \geq 0$ and $F = \int_0^T f(s)ds \in \mathcal{H}$. Then there exists $\Omega_{\varepsilon,F}, \mathcal{P}(\Omega_{\varepsilon,F}) = 1$, such that for any $i = 1, \ldots, d$ and $\omega \in \Omega_{\varepsilon,F}$

$$Z_i^{\varepsilon,F}(\omega,t) = Z_i(\omega,t) + \varepsilon \sum_{j=1}^{2d} \ell_{d+i,j} \int_0^t K_i(t,s) f_j(s) ds, \quad t \in [0,T].$$

**Proof.** By definition, for any $t \in [0,T]$,

$$Z_i(\cdot,t) = L^2 - \lim_{n \to \infty} \sum_{l=0}^{k_n-1} K_i^{n,l}(t) \left( B^Y_1(\cdot, t_{n,l+1}) - B^Y_1(\cdot, t_{n,l}) \right)$$

$$= L^2 - \lim_{n \to \infty} \sum_{j=1}^{2d} \ell_{d+i,j} \sum_{l=0}^{k_n-1} K_i^{n,l}(t) \left( W_j(\cdot, t_{n,l+1}) - W_j(\cdot, t_{n,l}) \right)$$

$$=: L^2 - \lim_{n \to \infty} Z_i^n(\cdot,t),$$

where $\pi^n = \{ 0 = t_{n,0} < t_{n,1} < \ldots < t_{n,k_n} = t \}$, $n \geq 1$, is a system of partitions such that $\sup_{l=0, \ldots, k_n-1} (t_{n,l+1} - t_{n,l}) \to 0$ as $n \to \infty$ and $K_i^n(t,s) := \sum_{l=0}^{k_n-1} K_i^{n,l}(t) \mathbb{1}_{[t_{n,l}, t_{n,l+1})}(s)$ is such that $\int_0^t (K_i(t,s) - K_i^n(t,s))^2 ds \to 0$, $n \to \infty$.

By the Riesz theorem, there exist $\Omega'_i \subset \Omega$, $\mathcal{P}(\Omega'_i) = 1$, and a subsequence of partitions that will again be denoted by $\{ \pi^n, n \geq 1 \}$ such that for all $i = 1, \ldots, d$ and $\omega \in \Omega'_i$,

$$Z_i(\omega,t) = \lim_{n \to \infty} Z_i^n(\omega,t). \quad (3.1)$$

Moreover, note that, by the definition of the probability space and Brownian motion $W$, for any $\omega = (\omega_1, \ldots, \omega_{2d}) \in \Omega = C_0([0, T]; \mathbb{R}^{2d})$

$$Z_i^n(\omega,t) = \sum_{j=1}^{2d} \ell_{d+i,j} \sum_{l=0}^{k_n-1} K_i^{n,l}(t) \left( W_j(\omega, t_{n,l+1}) - W_j(\omega, t_{n,l}) \right)$$

$$= \sum_{j=1}^{2d} \ell_{d+i,j} \sum_{l=0}^{k_n-1} K_i^{n,l}(t) \left( \omega_j(t_{n,l+1}) - \omega_j(t_{n,l}) \right),$$

whence, for all $\omega \in \Omega$,

$$Z_i^n(\omega + \varepsilon F, t) = Z_i^n(\omega) + \varepsilon \sum_{j=1}^{2d} \ell_{d+i,j} \sum_{l=0}^{k_n-1} K_i^{n,l}(t) \left( F_j(t_{n,l+1}) - F_j(t_{n,l}) \right)$$
and thus, for all \( \omega \in \Omega' \),
\[
\lim_{n \to \infty} Z_i^n(\omega + \varepsilon F, t) = \lim_{n \to \infty} Z_i^n(\omega) + \varepsilon \sum_{j=1}^{2d} \sum_{l=0}^{b_n-1} K_i^n(l)(F_j(t_n, t_{l+1}) - F_j(t_n, t_l)) + Z_i(\omega, t) + \varepsilon \sum_{j=1}^{2d} \int_0^t K_i(t, s)f_j(s)ds.
\]

Now, denote \( \Omega_{\varepsilon,F,t} := \{ \omega \in \Omega' : \omega + \varepsilon F \in \Omega' \} \). Then \( \mathbb{P}(\Omega_{\varepsilon,F,t}) = 1 \) since, by the Cameron-Martin theorem,
\[
\mathbb{P}(\Omega_{\varepsilon,F,t}) = \int_\Omega \mathbb{P}(\omega + \varepsilon F) d\mathbb{P}(\omega) = \mathbb{E} \left[ \mathbb{P}^\varepsilon(\omega) \exp \left\{ \varepsilon I_f - \frac{\varepsilon^2}{2} ||I_f||_{L^2}(\Omega) \right\} \right] = 1,
\]
where \( I_f := \sum_{j=1}^{2d} \int_0^T f_j(s)dw_j(s) \), and therefore, taking into account (3.1) and (3.2), for any \( \omega \in \Omega_{\varepsilon,F,t} \)
\[
Z_i(\omega + \varepsilon F, t) = Z_i(\omega, t) + \varepsilon \sum_{j=1}^{2d} \int_0^t K_i(t, s)f_j(s)ds.
\]

Finally, it remains to notice that, for the fixed \( \varepsilon \geq 0 \) and \( F \in \mathcal{H} \), both left- and right-hand sides of (3.3) are continuous w.r.t. \( t \), whence we can put
\[
\Omega_{\varepsilon,F} := \bigcap_{t \in \mathbb{Q} \cap [0, T]} \Omega_{\varepsilon,F,t}.
\]

Now we are ready to prove the key result of the Subsection. We will use the same approach as in [64, Theorem 3.3] and check the conditions of Theorem [3.1]

**Theorem 3.3.** Let Assumptions [A] and [B] hold. Then, for all \( i = 1, \ldots, d \) and \( t \in [0, T] \), \( Y_i(t) \in \mathbb{D}_{1,2} \) and
\[
DY_i(t) = (D^1Y_i(t), \ldots, D^{2d}Y_i(t)) \in L^2(\Omega \times [0, T]; \mathbb{R}^{2d}),
\]
\[
D_i^h Y(t) := \ell_d+i,j \left( K_i(t, u) + \int_u^t K_i(s, u) \frac{\partial b_i}{\partial y}(s, Y_i(s)) e^{\int_u^s \frac{b_i}{\sigma}(v, Y_i(v))dv}ds \right) \mathbb{P}(0),
\]
with \( \ell_d+i,j, i = 1, \ldots, d, j = 1, \ldots, 2d \) being the corresponding elements of the matrix \( \mathcal{L} \).

**Proof.** We will split the proof into three steps. First, we verify that \( DY_i(t) \) given by (3.4) indeed satisfies
\[
\sum_{j=1}^{2d} \mathbb{E} \int_0^T (D_i^h Y(t))^2 du < \infty
\]
and then check the conditions of Theorem 3.1

**Step 1: correctness of the form of derivative.** By Remark 2.4 for all \( s \in [0, T] \): \( (s, Y_i(s)) \in \mathcal{D}^{1,0}, \frac{\partial b_i}{\partial y}(s, Y_i(s)) \leq c \mathcal{C}^r(0), \ s \in [0, T] \).

Moreover, by Assumption [B iv],
\[
e^{\int_0^s \frac{b_i}{\sigma}(v, Y_i(v))dv} \leq e^T.
\]
Therefore

\[ \mathbb{E} \int_0^T (D_t^\epsilon Y_i(t))^2 \, du \leq 2T^2 \ell^2 \int_0^t \gamma_i^2(t, u) \, du \]

\[ + 2T^2 e^{2T} \ell^2 \int_0^t \mathbb{E} \left[ \gamma_i^2(0) \right] \int_0^t \gamma_i^2(s, u) \, ds \, du \]

\[ < \infty, \]

because \( \int_0^t \gamma_i^2(t, s) \, ds < \infty \) by Assumption [Aii] and the random variable \( \xi_i(0) \) has moments of all orders.

**Step 2: ray absolute continuity.** Let \( F \in \mathcal{H} \) be fixed. Then, as explained above, the random process \( \{Y_i^{\epsilon,F}(t), \epsilon \geq 0\} \) defined for every \( \epsilon \geq 0 \) by the equation

\[ Y_i^{\epsilon,F}(t) = Y_0 + \int_0^t b_i(s, Y_i^{\epsilon,F}(s)) \, ds + Z_i(t) + \epsilon \sum_{j=1}^{2d} \ell_{d+i,j} \int_0^t \gamma_i(t, s) \, ds \]

\[ + \gamma_i(0) \int_0^t \gamma_i(s, u) \, ds \, du \]

is a version of the process \( \{Y_i(\omega + \epsilon F, t), \epsilon \geq 0\} \). Let us prove that this version is Lipschitz w.r.t. \( \epsilon \) and hence absolutely continuous. First, fix \( \lambda \in \big( \frac{1}{\gamma_i+1}, H_i \big) \), where \( H_i \) is from Assumption [Aii] and \( \gamma_i \) is from Assumption [Biii], and take an arbitrary \( \epsilon^* > 0 \). Then, by Remark [2,3] and Assumption [Aii], for any \( \epsilon \in [0, \epsilon^*] \)

\[ |Z_i(t_1) - Z_i(t_2)| + \epsilon \sum_{j=1}^{2d} \ell_{d+i,j} \int_0^{t_1} \gamma_i(t_1, s) \, ds + \sum_{j=1}^{2d} \ell_{d+i,j} \int_0^{t_2} \gamma_i(t_2, s) \, ds \]

\[ \leq \lambda \epsilon_{\lambda,\theta} |t_1 - t_2|^\lambda + \epsilon \left( \int_0^{t_1} \gamma_i(t_1, s) - \gamma_i(t_2, s) \right) |t_1 - t_2|^\lambda \]

\[ \leq \left( \lambda \epsilon_{\lambda,\theta} + \epsilon^* \gamma_i \right) \sum_{j=1}^{2d} \ell_{d+i,j} \left\| \gamma_i \right\|_{L^2([0,T])} |t_1 - t_2|^\lambda, \]

which implies, by Remark [2,4] that \( (t, Y_i^{\epsilon,F}(t)) \in \mathcal{D}_i^{\epsilon \lambda} \) for any \( \epsilon \in [0, \epsilon^*] \) and \( t \in [0, T] \), where \( \mathcal{D}_i^{\epsilon \lambda} \) is defined by [2,5] and

\[ \xi_i(\epsilon^*) := \left( \frac{\left( L_{2,i} + \lambda \epsilon_{\lambda,\theta} + \epsilon^* \gamma_i \right) \sum_{j=1}^{2d} \ell_{d+i,j} \left\| \gamma_i \right\|_{L^2([0,T])}}{L_{1,i}} \right)^{\alpha_i} \]

for some deterministic positive constants \( L_{1,i}, L_{2,i} \) and \( \alpha_i \) that depend only on \( \lambda \) and the shape of the drift \( b_i \). This implies, by Assumption [Biii], that, for all \( s \in [0,T] \) and \( 0 \leq \epsilon_1 \leq \epsilon_2 \leq \epsilon^* \),

\[ |b_i(s, Y_i^{\epsilon_2,F}(s)) - b_i(s, Y_i^{\epsilon_1,F}(s))| \leq c \xi_i^p(\epsilon^*) \left| Y_i^{\epsilon_2,F}(s) - Y_i^{\epsilon_1,F}(s) \right|, \]

whence

\[ \left| Y_i^{\epsilon_2,F}(t) - Y_i^{\epsilon_1,F}(t) \right| \leq \int_0^t \left| b_i(s, Y_i^{\epsilon_2,F}(s)) - b_i(s, Y_i^{\epsilon_1,F}(s)) \right| \, ds \]

\[ + |\epsilon_2 - \epsilon_1| \sum_{j=1}^{2d} \ell_{d+i,j} \int_0^t \gamma_i(t, s) \, ds \]

\[ \leq c \xi_i^p(\epsilon^*) \int_0^t \left| Y_i^{\epsilon_2,F}(s) - Y_i^{\epsilon_1,F}(s) \right| \, ds \]

\[ + |\epsilon_2 - \epsilon_1| \left( \int_0^T \gamma_i^2(t, s) \, ds \right)^{\frac{1}{2}} \sum_{j=1}^{2d} \ell_{d+i,j} \left\| \gamma_i \right\|_{L^2([0,T])}^{\alpha_i}, \]

\[ \text{whence} \]

\[ \frac{\left( L_{2,i} + \lambda \epsilon_{\lambda,\theta} + \epsilon^* \gamma_i \right) \sum_{j=1}^{2d} \ell_{d+i,j} \left\| \gamma_i \right\|_{L^2([0,T])}}{L_{1,i}} \]

\[ \text{for some deterministic positive constants} \ L_{1,i}, L_{2,i} \ \text{and} \ \alpha_i \ \text{that depend only on} \ \lambda \ \text{and the shape of the drift} \ b_i. \]
where \( f_{i}^{T} \mathcal{K}^{2}_{i}(t, s)ds < \infty \) by Assumption [A.i]. Now, it follows from Gronwall’s inequality that, for all \( 0 \leq \varepsilon_{1} \leq \varepsilon_{2} \leq \varepsilon^{*} \),

\[
|Y^{\varepsilon^{*}, F}_{i}(t) - Y^{\varepsilon_{1}, F}_{i}(t)| \leq \left( \int_{0}^{T} \mathcal{K}^{2}_{i}(t, s)ds \right)^{\frac{1}{2}} \left\| \sum_{j=1}^{2d} \mathcal{C}_{d+i,j} f_{j}(s) \right\|_{L^{2}([0, T])} e^{CT|\varepsilon^{*} - \varepsilon_{2} - \varepsilon_{1}|},
\]

which implies the desired absolute continuity w.r.t. \( \varepsilon \).

**Step 3: stochastic Gateaux differentiability.** Again, it is sufficient to compute the limit in probability of the form

\[
P_{\varepsilon} \lim_{\varepsilon \to 0} \frac{Y^{\varepsilon, F}_{i}(t) - Y^{0, F}_{i}(t)}{\varepsilon},
\]

where \( Y^{\varepsilon, F}_{i}(t) \) is defined by (3.5).

Using the mean-value theorem, we obtain that

\[
Y^{\varepsilon, F}_{i}(t) - Y^{0, F}_{i}(t) = \int_{0}^{t} \left( b_{i}(s, Y^{\varepsilon, F}_{i}(s)) - b_{i}(s, Y^{0, F}_{i}(s)) \right) ds + \varepsilon \sum_{j=1}^{2d} \mathcal{C}_{d+i,j} \int_{0}^{t} \mathcal{K}_{i}(t, s) f_{j}(s) ds
\]

(3.7)

where \( \Theta_{i, \varepsilon}(s) := \frac{\partial}{\partial \theta_{i}} \left( s, Y^{0, F}_{i}(s) + \theta_{i, \varepsilon}(s)(Y^{\varepsilon, F}_{i}(s) - Y^{0, F}_{i}(s)) \right) \) with \( \theta_{i, \varepsilon}(s) \) being some value between 0 and 1. It is easy to verify that (3.7) implies the representation of the form

\[
Y^{\varepsilon, F}_{i}(t) - Y^{0, F}_{i}(t) = \varepsilon \sum_{j=1}^{2d} \int_{0}^{t} \exp \left\{ \int_{s}^{t} \Theta_{i, \varepsilon}(v) dv \right\} dG_{i,j}(s),
\]

where

\[
G_{i,j}(t) := \mathcal{C}_{d+i,j} \int_{0}^{t} \mathcal{K}_{i}(t, s) f_{j}(s) ds.
\]

Note that, just like in Step 1, it follows from Remark 2.4 that for all \( \varepsilon \in [0, 1] \) and \( s \in [0, T] \) \( (s, Y^{\varepsilon, F}_{i}(s)) \in \mathcal{D}^{\alpha} \), where \( \xi_{i}(1) \) is defined via (3.6). This, together with continuity of \( \frac{\partial}{\partial \theta_{i}} \) due to Assumption [B.iv] and compactness of \( \mathcal{D}^{\alpha} \), implies that there exists a finite positive random variable \( \eta_{i} \) such that \( |\Theta_{i, \varepsilon}(s)| \leq \eta_{i} \) for all \( \varepsilon \in [0, 1] \) and \( s \in [0, T] \). Moreover, the mapping \( s \mapsto \exp \left\{ \int_{s}^{t} \Theta_{i, \varepsilon}(v) dv \right\} \), \( s \in [0, t] \), is Lipschitz and whence the integrals \( \int_{0}^{t} \exp \left\{ \int_{s}^{t} \Theta_{i, \varepsilon}(v) dv \right\} dG_{i,j}(s) \), \( j = 1, ..., 2d \), are well-defined as pathwise limits of Riemann-Stieltjes integral sums due to the classical results of Young ([90, Section 10], see also [33, Chapter 6] for a more recent overview on the subject).

Now, applying the integration by parts formula, we obtain

\[
\frac{Y^{\varepsilon, F}_{i}(t) - Y^{0, F}_{i}(t)}{\varepsilon} = \sum_{j=1}^{2d} \int_{0}^{t} \exp \left\{ \int_{s}^{t} \Theta_{i, \varepsilon}(v) dv \right\} dG_{i,j}(s)
\]

\[
= \sum_{j=1}^{2d} \left( G_{i,j}(t) - \int_{0}^{t} G_{i,j}(s) d \left( \exp \left\{ \int_{s}^{t} \Theta_{i, \varepsilon}(v) dv \right\} \right) \right)
\]

\[
= \sum_{j=1}^{2d} \left( G_{i,j}(t) + \int_{0}^{t} G_{i,j}(s) \Theta_{i, \varepsilon}(s) \exp \left\{ \int_{s}^{t} \Theta_{i, \varepsilon}(v) dv \right\} ds \right),
\]

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and, taking into account the definition of $G_{i,j},$

$$\frac{Y_{i}^{ε,F}(t) - Y_{i}^{0,F}(t)}{ε} = \sum_{j=1}^{2d} \int_{0}^{T} D_{i,j}(ε, t, u) f_{j}(u) du$$

with

$$D_{i,j}(ε, t, u) := \ell_{d+i,j} \left( K_{i}(t, u) + \int_{u}^{t} K_{i}(s, u) \Theta_{i,ε}(s) \exp \left\{ \int_{s}^{t} \Theta_{i,ε}(v) dv \right\} ds \right) 1_{[0,T]}(u).$$

Now, note that for all $s \in [0, T]$ and $ε \in [0, 1]$

$$|\Theta_{i,ε}(s)| \exp \left\{ \int_{s}^{t} \Theta_{i,ε}(v) dv \right\} < η \exp{cT},$$

where $c$ is from Assumption [B.iv], and therefore we can apply the dominated convergence theorem to obtain that

$$\frac{Y_{i}^{ε,F}(t) - Y_{i}^{0,F}(t)}{ε} \to (DY_{i}(t), f)_{L^{2}([0,T], \mathbb{R}^{2d})} \ a.s., \ ε \to 0,$$

where $DY_{i}(t)$ has exactly the form (3.4), which finishes the proof.

\[\square\]

**Remark 3.4.**

1. It is possible to prove higher order Malliavin differentiability of $Y$ provided that the drift coefficient $b$ is regular enough. For more details, see [22].

2. Theorem 3.3 can be used to prove that the SVV model can reproduce the power law of the implied volatility skew provided an appropriate choice of the Volterra kernels in volatility. We prove this result in [43, Theorem 4.8].

**Malliavin differentiability of price.** An obvious way to obtain the Malliavin differentiability of the price processes $(S_{1}, ..., S_{d})^{T}$ would be to notice that each $S_{i}$ has the form

$$S_{i}(t) = e^{X_{i}(t)}, \quad t \in [0, T],$$

where $X_{i}$ is the log-price process that, due to Theorem 2.6, can be written as

$$X_{i}(t) := \log S_{i}(t) = X_{i}(0) + \int_{0}^{t} \left( \mu_{i}(s) - \frac{Y_{i}^{2}(s)}{2} \right) ds + \int_{0}^{t} Y_{i}(s) dB_{i}(s). \quad (3.8)$$

Then, in order to prove that $S_{i}(t) \in D^{1,2}$, one can first establish that $X_{i}(t) \in D^{1,2}$ and then apply an appropriate chain rule. However, the classical chain rule results (see e.g. [78, Proposition 1.2.3, Proposition 1.2.4]) require either all partial derivatives $\frac{∂f}{∂x_{j}}$ to be bounded (which is not the case for such functions as $f(x) = x^{2}$ or $f(x) = e^{x}$) or the law of $η$ to be absolutely continuous (which we have not established yet for the log-price $X_{i}(t)$). Therefore, we give a version of the Malliavin chain rule that fits our needs below.

**Proposition 3.5.** Let $η = (η_{1}, ..., η_{k}) \in (D^{1,2})^{k}$ and $f \in C^{(1)}(\mathbb{R}^{k}; \mathbb{R})$ be a real-valued function such that

1) if $\frac{∂f}{∂x_{j}}$ is unbounded, then there exist sequences $\{r_{n}, n \geq 1\}, \{s_{j,n}, n \geq 1\}, r_{n}, s_{j,n} \to ∞$ as $n \to ∞,$ such that

$$\left| \frac{∂f}{∂x_{j}}(x_{1}, ..., x_{k}) \right| \geq s_{j,n}$$

implies that $|f(x_{1}, ..., x_{k})| \geq r_{n};$

2) $E \left[ |f(η)|^{2} + \left\| \sum_{j=1}^{k} \frac{∂f}{∂x_{j}}(η) D_{Δj}(0, T) \right\|^{2}_{L^{2}([0,T], \mathbb{R}^{2d})} \right] < ∞.$

Note that our version of the chain rule is essentially based on [78, Lemma A.1], but the proof of the latter contains a mistake in the argument.
Then \( f(\eta) \in \mathbb{D}^{1,2} \) and \( Df(\eta) = \sum_{j=1}^{k} \frac{\partial f}{\partial x_j}(\eta)D\eta_j \).

**Proof.** According to [77 Proposition 1.2.3], if all partial derivatives \( \frac{\partial f}{\partial x_j} \) are bounded, then the claim of our Proposition is true and the chain rule holds. For the general case, let \( \phi \in C^\infty(\mathbb{R}) \) be compactly supported function such that \( \phi(x) = x \) whenever \( |x| \leq 1 \) and \( |\phi(x)| \leq |x| \) for all \( x \in \mathbb{R} \).

For any \( m \geq 1 \) define \( f_m(x) := m\phi \left( \frac{f(x)}{m} \right) \) and observe that \( f_m \) has bounded partial derivatives. Indeed, if \( \frac{\partial f}{\partial x_j} \) is bounded, so is \( \frac{\partial f_m}{\partial x_j} \). Otherwise, let \( n^* \) be such that \( \frac{m}{n^*} > \max \{ -\inf \text{supp} \phi, \text{supp} \phi \} \). Then, for all \((x_1, \ldots, x_k) \in \mathbb{R}^k\) such that \( |\frac{\partial f}{\partial x_j}(x_1, \ldots, x_k)| \geq s_{j,n^*} \), it holds that \( \frac{f(x_1, \ldots, x_k)}{m} \notin \text{supp} \phi \) and therefore

\[
\frac{\partial f_m}{\partial x_j}(x_1, \ldots, x_k) = \frac{\partial f}{\partial x_j}(x_1, \ldots, x_k)\phi' \left( \frac{f(x_1, \ldots, x_k)}{m} \right) = 0.
\]

Whence

\[
\left| \frac{\partial f_m}{\partial x_j}(x_1, \ldots, x_k) \right| = \left| \frac{\partial f}{\partial x_j}(x_1, \ldots, x_k) \right| \phi' \left( \frac{f(x_1, \ldots, x_k)}{m} \right) \leq s_{j,n^*} \max_{z \in \mathbb{R}} |\phi'(z)|.
\]

Note that \( |f_m(\eta)| \leq |f(\eta)| \) for all \( m \geq 1 \) and \( \lim_{m \to \infty} f_m(\eta) = f(\eta) \) a.s. Moreover,

\[
|Df_m(\eta)| = \left| \sum_{j=1}^{k} \frac{\partial f_m}{\partial x_j}(\eta)D\eta_j \right| \leq \max_{z \in \mathbb{R}} |\phi'(z)| \left| \sum_{j=1}^{k} \frac{\partial f}{\partial x_j}(\eta)D\eta_j \right|
\]

for all \( m \geq 1 \) and

\[
\lim_{m \to \infty} Df_m(\eta) = \sum_{j=1}^{k} \frac{\partial f}{\partial x_j}(\eta)D\eta_j \quad \text{a.s.}
\]

Therefore, by the dominated convergence theorem,

\[
\lim_{m \to \infty} \mathbb{E} \left[ |f_m(\eta) - f(\eta)|^2 + \left| Df_m(\eta) - \sum_{j=1}^{k} \frac{\partial f}{\partial x_j}(\eta)D\eta_j \right|^2 \right]_{L^2([0,T];\mathbb{R}^{2d})} = 0,
\]

and the result follows by the closedness of the operator \( D \). \( \square \)

Now we can get the following simple corollary for the latter direct application.

**Corollary 3.6.** For any \( i = 1, \ldots, d \) and \( t \in [0,T] \), \( X_i(t) \in \mathbb{D}^{1,2} \) and its Malliavin derivative \( DX_i(t) = (D^1X_i(t), \ldots, D^{2d}X_i(t)) \) has the form

\[
D^j_iX_i(t) = -\int_0^t Y_i(s)D^j_iY_i(s)ds + \ell_{i,j}Y_i(u) + \int_0^t D^j_iY_i(s)d\mathbb{S}(s)1_{[0,t]}(u).
\]

**Proof.** First, note that if \( u > t \), then the claim is true due to [77 Corollary 1.2.1]. Let now \( u \leq t \). By the definition of the Malliavin derivative,

\[
D^j_u \left[ \int_0^t \mu_i(s)ds \right] = 0. \tag{3.9}
\]

Next, note that for any \( s \in [0,T] \) \( Y_i^2(s) \in \mathbb{D}^{1,2} \). Indeed, by Proposition 9.5, it is sufficient to check that

\[
\mathbb{E} \left[ Y_i^2(s) + \|Y_i(s)DY_i(s)\|_{L^2([0,T];\mathbb{R}^{2d})}^2 \right] < \infty
\]

which is true since \( Y_i(s) \) is bounded. Whence \( \frac{1}{2} \int_0^t D^j_u Y_i^2(s)ds = \int_0^t Y_i(s)D^j_iY_i(s)ds \), \( j = 1, \ldots, 2d \), these integrals are well defined and so, using the chain rule and closedness of the operator \( D \), we can write (see e.g. [88 Theorem 1.2.2] or [66 Theorem 1.2.4])

\[
D^j_u \left[ \frac{1}{2} \int_0^t Y_i^2(s)ds \right] = \int_0^t Y_i(s)D^j_u Y_i(s)ds, \quad j = 1, \ldots, 2d. \tag{3.10}
\]
Finally, by adaptedness and boundedness of \(Y_i\) and [77 Proposition 1.3.2],
\[
D_u^2 \left[ \int_0^t Y_i(s) dB^S_i(s) \right] = D_u^2 \left[ \sum_{k=1}^{2d} \int_0^t \ell_{i,k} Y_i(s) dB_k(s) \right] = \ell_{i,j} Y_i(u) + \sum_{k=1}^{2d} \ell_{i,k} \int_0^t D_u^2 Y_i(s) dB_k(s)
\]
(3.11)
as required.

\[\square\]

\section{Malliavin differentiability w.r.t. the transformed Brownian motion. Absolute continuity of probability laws}

In order to proceed, we will require absolute continuity of the law of \(X_i(t)\) w.r.t. the Lebesgue measure. It is well known (see e.g. [77 Chapter 2]) that a sufficient condition for that is
\[
\|D X_i(t)\|_{L^2([0,T];\mathbb{R}^{2d})} > 0 \quad \text{a.s.,}
\]
(3.12)
thus it would be sufficient to check \((3.12)\). However, the form of \(D X_i(t)\) obtained in Subsection \(3.1\) is not very convenient for this matter. In order to get the desired properties, we will transform the original Brownian motion \(W\) and consider the Malliavin derivative w.r.t. this transformation.

Namely, recall that \(B = (B_1, ..., B_{2d})^T\) is a \(2d\)-dimensional Gaussian process and
\[
B(t) = \begin{pmatrix} B^S(t) \\ B^Y(t) \end{pmatrix} = L W(t),
\]
where by \(B^S = (B^S_1, ..., B^S_{2d})^T\) and \(B^Y = (B^Y_1, ..., B^Y_{2d})^T\) we denote the first and the last \(d\) components of \(B\) correspondingly and \(L = (\ell_{i,j})_{i,j=1}^{2d}\) is a lower-triangular matrix such that \(L L^T = \Sigma = (\sigma_{ij})_{i,j=1}^{2d}\). Now, define \(U = (u_{i,j})_{i,j=1}^{2d}\) an upper \(2d \times 2d\)-triangular matrix with positive values on the diagonal such that
\[
\overline{W} := U^{-1} B = U^{-1} L W
\]
is a \(2d\)-dimensional Brownian motion. Let now \(\overline{D} = (\overline{D}^1, ..., \overline{D}^{2d})\) be the Malliavin derivative with respect to the new Brownian motion \(\overline{W}\) and \(\overline{D}^{1,2}\) be its domain. Clearly, the matrix \(U^{-1} L\) is non-degenerate and hence \(\overline{W}\) and \(\overline{W}\) generate the same filtration \(\mathcal{F}\) and, moreover, it is easy to see that \(\overline{D}^{1,2}\) coincides with \(D^{1,2}\). Finally, by construction, \(\overline{W}_1\) is independent of the processes \(B^S_2, ..., B^S_d, B^Y_1, ..., B^Y_d\) and therefore the following result holds.

\begin{thm}
\textbf{Theorem 3.7.} \textit{Let Assumptions [A] and [B] hold. Then, for any } t \in (0,T],
\[
\overline{D}^i_0 X_i(t) = \begin{cases} u_{1,1} Y_1(v) 1_{[0,t]}(v), & \text{if } i = 1, \\ 0, & \text{if } i \neq 1, \end{cases}
\]
and
\[
\overline{D}^i_0 S_i(t) = \begin{cases} u_{1,1} S_i(v) Y_1(v) 1_{[0,t]}(v), & \text{if } i = 1, \\ 0, & \text{if } i \neq 1, \end{cases}
\]
where \(u_{1,1}\) is the upper left element of \(U\). In particular,
\[
\|\overline{D} X_i(t)\|_{L^2([0,T];\mathbb{R}^{2d})} > 0 \quad \text{a.s.}
\]
\end{thm}

\textit{Proof.} The expression for \(\overline{D}^1 X_i(t)\) is obtained straightforwardly by differentiating the right-hand side of \(3.8\) and taking into account that
\[
B^S_1(t) = \sum_{k=1}^{2d} u_{1,k} \overline{W}_k(t)
\]
and \(\overline{D}^1 Y_i(t) = 0\) due to independence of \(\overline{W}_1\) and \(B^Y_i\). The expression for \(\overline{D}^1 S_i(t)\) can be easily obtained using the chain rule from Proposition \(3.5\) \(\square\)
Corollary 3.8. For each $i = 1, \ldots, d$ and $t \in (0, T]$, the law of $X_i(t)$ (and therefore of $S_i(t) = e^{X_i(t)}$) has continuous and bounded density.

Proof. By Theorem 3.7
\[ \|D X_i(t)\|_{L^2([0, T]; \mathbb{R}^d)} > 0 \quad \text{a.s.} \]
and, moreover, $D X_i(t)$ is adapted to the filtration generated by $B^i$. Thus the result for $X_i(t)$ follows from e.g. [77, Proposition 2.1.1]. The existence of the density for $X_i(t)$, $i = 2, \ldots, d$, can be obtained in the same manner by interchanging the order of indices.

As we can see, each of $X_i(t)$, $i = 1, \ldots, d$, has a density with respect to the Lebesgue measure. However, in what follows we will require a stronger result, namely the existence of the joint density of $X(t) = (X_1(t), \ldots, X_d(t))$.

Theorem 3.9. For all $t \in (0, T]$, the law of the random vector $X(t)$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$.

Proof. We will follow the approach developed in [5, Theorem 5.3] for the standard Brownian Heston model. Namely, we will check that for any $f \in C^\infty(\mathbb{R}^d; \mathbb{R})$ such that $f$ and all its partial derivatives are bounded the following relation holds:
\[ \left| \mathbb{E} \left[ \frac{\partial f}{\partial x_i}(X(t)) \right] \right| \leq c_i \sup_{x \in \mathbb{R}^d} |f(x)|, \quad i = 1, \ldots, d. \]

Then the result will follow from [77, Lemma 2.1.1].

Note that, by applying sequentially the chain rule for the Malliavin derivative and the duality relation between Malliavin derivative and the Skorokhod integral, we can obtain
\[ \left| \mathbb{E} \left[ \frac{\partial f}{\partial x_1}(X(t)) \right] \right| = \frac{1}{t} \left| \mathbb{E} \left[ \int_0^t \frac{\partial f}{\partial x_1}(X(u)) D X_1(t) \int_0^t \frac{1}{D X_1(t)} du \right] \right| \\
= \frac{1}{t} \left| \mathbb{E} \left[ \int_0^t D X_1(t) \frac{1}{D X_1(t)} du \right] \right| \\
= \frac{1}{t} \left| \mathbb{E} \left[ f(X(t)) \int_0^t \frac{1}{u_1 Y_1(u)} d\tilde{W}_1(u) \right] \right| \\
\leq \frac{1}{tu_1} \mathbb{E} \left[ \left| \int_0^t \frac{1}{Y_1(u)} d\tilde{W}_1(u) \right| \sup_{x \in \mathbb{R}^d} |f(x)| \right] \\
eq c_i \sup_{x \in \mathbb{R}^d} |f(x)|, \]
i.e. the required property holds for $\frac{\partial f}{\partial x_1}(X(t))$. As for $\frac{\partial f}{\partial x_i}(X(t))$ with $i = 2, \ldots, d$, one can reorder the elements of $B$ so that by swapping $B^i$ is the first one and then apply the same arguments.

4 Smooth payoff functionals by Malliavin integration by parts

In the remaining part of the paper, we tackle directly the computation of option prices with a focus on discontinuous payoffs. As explained earlier in the Introduction, discontinuities in the payoff function $f$ lead to deterioration of the convergence rate of $\mathbb{E} f(S(T))$ to $\mathbb{E} f(S(T))$, where $\tilde{S}$ denotes some discretization of $S$, which causes an undesirable bias in Monte Carlo computations. In order to overcome this issue, we will apply the results of Section 3 and utilize the quadrature technique via Malliavin integration by parts proposed initially in [5] and adapted to fBm-driven stochastic volatility in [19, 76]. The idea of the method is as follows: using the Malliavin calculus techniques, one can replace the discontinuous functional $f$ under the expectation with a locally Lipschitz one. The latter expression is far more convenient for numerical algorithms and guarantees a better convergence rate. In this Section, we explain this smoothing technique whereas the numerical algorithm and the corresponding convergence results will be provided in Section 5. Note that, in the papers [19, 76] mentioned above, $\mathbb{E} f(S_1(T), \ldots, S_d(T))$ is computed w.r.t. the physical measure and not the martingale one, i.e. the numerical algorithms
described there concern the expected payoff rather than the option price. However, thanks to the results of Subsection 2.2, we have a detailed description of equivalent local martingale measures on the market and hence are able to modify the method to account for a measure change. We explain this modification described there concern the arbitrary H"older continuous function and \( L = (\ell_{i,j})_{i,j=1}^2 \) is such that \( L^T = \Sigma = (\sigma_{i,j})_{i,j=1}^2 \). Using the transformation from Subsection 3.2, one can rewrite (4.1) as

\[
S(t) = S(0) + \int_0^t \mu(s)S(s)ds + \int_0^t Y(s)S(s)dW_1(s), \\
Y(t) = Y(0) + \int_0^t b(s,Y(s))ds + \int_0^t K(t,s) (\ell_{2,1}dW_1(s) + \ell_{2,2}dW_2(s)),
\]

where \( K = K_t \) satisfies Assumption A with \( H_t = H, b = b_1 \) satisfies Assumption B, \( \mu = \mu_1 \) is an arbitrary \( H \)-H"older continuous function and \( \ell = (\ell_{i,j})_{i,j=1}^2 \) is such that \( L^T = \Sigma = (\sigma_{i,j})_{i,j=1}^2 \). Using the transformation from Subsection 3.2, one can rewrite (4.1) as

\[
S(t) = S(0) + \int_0^t \mu(s)S(s)ds + \int_0^t Y(s)S(s) \left( u_{1,1}d\bar{W}_1(s) + u_{1,2}d\bar{W}_2(s) \right), \\
Y(t) = Y(0) + \int_0^t b(s,Y(s))ds + \int_0^t K(t,s)d\bar{W}_2(s),
\]

where \( \bar{W} = (\bar{W}_1, \bar{W}_2) \) is a 2-dimensional Brownian motion such that

\[
\begin{pmatrix} u_{1,1} & u_{1,2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \bar{W}_1 \\ \bar{W}_2 \end{pmatrix} = \begin{pmatrix} B_1^S \\ B_1^Y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \ell_{2,1} & \ell_{2,2} \end{pmatrix} \begin{pmatrix} W_1 \\ W_2 \end{pmatrix}.
\]

Let now \( f \) be a measurable payoff function that satisfies the following assumption.

**Assumption C.** Function \( f: \mathbb{R} \to \mathbb{R} \) is locally Riemann integrable function (not necessarily continuous) of polynomial growth, i.e. there exist constants \( q > 0 \) and \( c_f > 0 \) such that

\[
|f(s)| \leq c_f (1 + |s|^q), \quad s > 0.
\]

Denote now \( F(s) := \int_0^s f(u)du, g(x) := f(e^x), G(x) := F(1) + \int_0^x g(v)dv, \) i.e. \( f(S(T)) = g(X(T)) \), where \( X(t) := \log S(t), t \in [0,T] \). By Theorem 3.7, \( X(t) \in \tilde{D}^{1,2} \) with

\[
\tilde{D}^1X(t) = u_{1,1}Y(v)\mathbb{I}_{[0,t]}(v)
\]

and, by Theorem 3.9, \( X(t) \) has density. Thus one can use Proposition 1.2.4 from [77] and remark after it to claim that

\[
\tilde{D}^1G(X(t)) = g(X(t))\tilde{D}^1X(t), \quad t \in (0,T],
\]

if \( G \) turns out to be globally Lipschitz continuous. However, we will require the Malliavin chain rule for \( G(X(T)) \) in the setting provided by Assumption C, i.e. when \( G \) is only locally Lipschitz continuous. To do this, we will extend [77] Proposition 1.2.4 in a manner similar to [19] Remark 9.

**Proposition 4.1.** For \( G \) defined above and any \( t \in (0,T], G(X(t)) \in \tilde{D}^{1,2} \) and

\[
\tilde{D}G(X(t)) = g(X(t))\tilde{D}X(t), \quad t \in (0,T],
\]

**Proof.** For any \( n \in \mathbb{N} \), denote

\[
g_n(x) := g(x)\mathbb{I}_{[-n,n]}(x)
\]

and

\[
G_n(x) := F(1) + \int_0^x g_n(v)dv = G(x)\mathbb{I}_{[-n,n]}(x) + G(n)\mathbb{I}_{[n,\infty]}(x) + G(-n)\mathbb{I}_{(-\infty,-n]}(x).
\]
It is clear that $G_n$ is globally Lipschitz and thus $G_n(X(t)) \in \hat{D}^{1,2}$ and

$$\hat{DG}_n(X(t)) = g_n(X(t))\hat{D}X(t).$$

Thus, taking into account the closedness of the operator $\hat{D}$, it is sufficient to check that

$$g_n(X(t)) \xrightarrow{L^2(\Omega)} g(X(t)), \quad n \to \infty,$$

and

$$g_n(X(t))\hat{D}X(t) \xrightarrow{L^2([0,T]\times\Omega)} g(X(t))\hat{D}X(t), \quad n \to \infty.$$

Observe that $g_n(X(t)) \to g(X(t))$ a.s., $n \to \infty$. By Assumption C, $f$ is of polynomial growth and thus, by Theorem 2.6, $Ef^2(S(t)) = Eg^2(X(t)) < \infty$ and (4.4) follows from the dominated convergence theorem.

Finally, using polynomial growth of $f$, explicit shape of $\hat{D}X(t)$ given by Theorem 3.7 as well as estimates similar to the ones from Step 1 of the proof of Theorem 3.3, one can deduce that

$$\mathbb{E} \left[g^2(X(t)) \int_0^T (\hat{D}_n^1 X(t))^2 du \right] < \infty,$$

and therefore (4.5) also holds due to dominated convergence.

Now, having Proposition 4.1, we are ready to proceed to the main result of this Subsection in the spirit of [5, Theorem 4.2] or [19, Lemma 11].

**Theorem 4.2.** Let $S(T)$ be given by (4.2), $f: \mathbb{R} \to \mathbb{R}$ satisfy Assumption C, $X(t) := \log S(t)$, $F(s) := \int_0^s f(u) du$, $g(x) := f(e^x)$, $G(x) := F(1) + \int_0^x g(v) dv$. Then

$$\mathbb{E} f(S(T)) = \frac{1}{Tu_{1,1}} \mathbb{E} \left[G(X(T)) \int_0^T \frac{1}{Y(u)} d\bar{W}_1(u) \right].$$

Alternatively,

$$\mathbb{E} f(S(T)) = \mathbb{E} \left[ F(S(T)) \frac{S(T)}{S(T)} \left(1 + \frac{1}{Tu_{1,1}} \int_0^T \frac{1}{Y(u)} d\bar{W}_1(u) \right) \right].$$

**Proof.** In order to obtain representation (4.6), observe that

$$\mathbb{E} f(S(T)) = \mathbb{E} g(X(T)) = \frac{1}{T} \mathbb{E} \left[ \int_0^T g(X(T))\hat{D}_n^1 X(T) \frac{1}{\hat{D}_n^1 X(T)} du \right] = \frac{1}{Tu_{1,1}} \mathbb{E} \left[ \int_0^T \hat{D}_n^1 X(T) \frac{1}{Y(u)} du \right] = \frac{1}{Tu_{1,1}} \mathbb{E} \left[ G(X(T)) \int_0^T \frac{1}{Y(u)} d\bar{W}_1(u) \right].$$

Here we used the chain rule from Proposition 4.1 and duality relation between the Malliavin derivative and Skorokhod integral (which coincides in our case with the Itô integral).

As for the representation (4.7), note that

$$G(x) = F(1) + \int_0^x f(e^v)e^v e^v dv = F(e^x) + \int_0^x F(e^v) e^v dv.$$

Moreover, using the same argument as in Proposition 4.1, it is easy to prove that

$$\hat{D}_n^1 \int_0^T \frac{F(e^v)}{e^v} dv = \frac{F(S(T))}{S(T)} \hat{D}_n^1 X(T),$$

and

$$\hat{D}_n^1 \int_0^T \frac{F(e^v)}{e^v} dv \xrightarrow{L^2([0,T])} \frac{F(S(T))}{S(T)} \hat{D}_n^1 X(T), \quad n \to \infty.$$
therefore, again by the Malliavin duality, we obtain that
\[
\mathbb{E}\left[\int_0^T \frac{F(e^u)}{e^u} du \frac{1}{Y(u)} \int_0^T \frac{1}{Y(u)} d\tilde{W}_1(u) \right] = \mathbb{E}\left[\tilde{D}_u \int_0^T \frac{F(e^u)}{e^u} du \frac{1}{Y(u)} \int_0^T \frac{1}{Y(u)} d\tilde{W}_1(u) \right] = \mathbb{E}\left[\int_0^T \frac{F(S(t))}{S(t)} \frac{1}{\tilde{D}_u X(T)} \frac{1}{Y(u)} d\tilde{W}_1(u) \right]
\]
which implies (4.7). \qed

4.2 Multidimensional case
The approach described in Subsection 4.1 can also be utilized for basked options in \(d\)-dimensional SVV models. Namely, consider a \(d\)-dimensional volatility process \(Y = (Y_1, \ldots, Y_d)\), where each \(Y_i\) is defined by (2.6), a \(d\)-dimensional price process \(X = (X_1, \ldots, X_d)\) given by (2.8) and the corresponding log-price process \(X = (X_1, \ldots, X_d)\), \(X_i = \log S_i\).

Consider now a basket option with a payoff \(f\left(\sum_{i=1}^d \alpha_i S_i(T)\right)\) with \(f: \mathbb{R} \to \mathbb{R}\) satisfying Assumption \(\mathbb{C}\) and \(\alpha_i > 0, i = 1, \ldots, d\). As before, define \(F(s) := \int_0^s f(u) du\) and observe that the function
\[
G_1(x_1, \ldots, x_d) := \frac{\sum_{i=1}^d \alpha_i e^{x_i}}{\alpha_1 e^{x_1}}, \quad (x_1, \ldots, x_d) \in \mathbb{R}^d,
\]
is locally Lipschitz continuous: if \(\sum_{i=1}^d \alpha_i e^{x_i} > \sum_{i=1}^d \alpha_i e^{x_i}\), then
\[
|G_1(x_1, \ldots, x_d) - G_1(x_1', \ldots, x_d')| \leq \frac{e^{-x_1}}{\alpha_1} \int_0^{\sum_{i=1}^d \alpha_i e^{x_i}} |f(s)| ds + \frac{1}{\alpha_1} \int_0^{\sum_{i=1}^d \alpha_i e^{x_i}} |f(s)| ds |e^{-x_1} - e^{-x_i}|
\]
\[
\leq c f e^{-x_1} \left(1 + \sum_{i=1}^d \alpha_i e^{x_i} \right)^{q} \sum_{j=1}^d \alpha_i e^{x_i \vee x_j} |x_j - x_j'| \\
+ c \frac{1}{\alpha_1} \left(\sum_{i=1}^d \alpha_i e^{x_i} \right)^{q+1} \sum_{j=1}^d \alpha_i e^{x_i} |x_j - x_j'|
\]
Note that polynomial growth of \(f\) (and hence \(F\)) together with Theorem 2.6 imply that random variables \(F\left(\sum_{i=1}^d \alpha_i S_i(T)\right)\) and \(G_1(X_1(T), \ldots, X_d(T))\) have moments of all orders. Moreover, Theorem 3.9 implies that the law of \(X\) (and hence of \(S\)) is absolutely continuous w.r.t. the Lebesgue measure on \(\mathbb{R}^d\). Therefore, by approximating \(G_1(X(t))\) with Lipschitz functions just like in Proposition 4.1, it is straightforward to get the following chain rule for \(G_1(X(t))\).

**Proposition 4.3.** For \(G_1\) defined above and any \(t \in (0, T]\), \(G_1(X(t)) \in \mathbb{D}^{1,2}\) and
\[
\tilde{D}G_1(X(t)) = \sum_{i=1}^d \frac{\partial G_1(X(t))}{\partial x_i} \tilde{D}X_i(t), \quad t \in (0, T].
\]

We are now ready to formulate an analogue of Theorem 4.2 for the basket option.

**Theorem 4.4.** Let \(S = (S_1, \ldots, S_d)\) be given by (2.8), \(X = (X_1, \ldots, X_d)\) be the corresponding log-price process, \(X_i = \log S_i\), \(f: \mathbb{R} \to \mathbb{R}\) satisfy Assumption \(\mathbb{C}\), \(F(s) := \int_0^s f(u) du\) and \(\alpha_i > 0, i = 1, \ldots, d\). Then
\[
\mathbb{E}f\left(\sum_{i=1}^d \alpha_i S_i(T)\right) = \mathbb{E}\left[\frac{F\left(\sum_{i=1}^d \alpha_i S_i(T)\right)}{\alpha_1 S_1(T)} \left(1 + \frac{1}{Y_{1,1}} \int_0^T \frac{1}{Y_1(u)} d\tilde{W}_1(u) \right)\right].
\]
Proof. Note that
\[
\frac{\partial G_1(x_1, \ldots, x_d)}{\partial x_1} = f \left( \sum_{i=1}^{d} \alpha_i e^{x_i} \right) - \frac{1}{\alpha_1 e^{x_1}} F \left( \sum_{i=1}^{d} \alpha_i e^{x_i} \right)
\]
hence, taking into account Proposition 4.3 and Theorem 3.7
\[
\begin{align*}
\mathbb{E} \left[ f \left( \sum_{i=1}^{d} \alpha_i S_i(T) \right) \right] &= \mathbb{E} \left[ \int_{0}^{T} \left( f \left( \sum_{i=1}^{d} \alpha_i e^{X_i(T)} \right) - \frac{1}{\alpha_1 e^{X_1(T)}} F \left( \sum_{i=1}^{d} \alpha_i e^{X_i(T)} \right) \right) \, dv \right] \\
&\quad + \mathbb{E} \left[ \int_{0}^{T} \frac{1}{\alpha_1 e^{X_1(T)}} F \left( \sum_{i=1}^{d} \alpha_i e^{X_i(T)} \right) \, dv \right] \\
&= \mathbb{E} \left[ \int_{0}^{T} \frac{1}{\alpha_1 e^{X_1(T)}} \frac{\partial G_1(X(T))}{\partial x_1} \, dv \right] + \mathbb{E} \left[ G_1(X(T)) \right] \\
&= \mathbb{E} \left[ \int_{0}^{T} \frac{1}{\alpha_1 e^{X_1(T)}} \frac{1}{D_1^i X_1(T)} \, dv \right] + \mathbb{E} \left[ G_1(X(T)) \right] \\
&= \mathbb{E} \left[ \int_{0}^{T} \frac{1}{\alpha_1 e^{X_1(T)}} \frac{1}{u_{1,1} Y_1(v)} \, dv \right] + \mathbb{E} \left[ G_1(X(T)) \right] \\
&= \mathbb{E} \left[ \frac{F \left( \sum_{i=1}^{d} \alpha_i S_i(T) \right)}{\alpha_1 S_1(T)} \left( 1 + \frac{1}{T u_{1,1}} \int_{0}^{T} \frac{1}{Y_1(v)} \, d\tilde{W}_1(v) \right) \right],
\end{align*}
\]
as required. \qed

4.3 Quadrature via Malliavin integration under a change of measure

Theorems 4.2 and 4.4 concern the expectation w.r.t. the physical measure, i.e. the obtained representation of the expected payoff is not the option price. In this Section, we will modify the quadrature technique to account for a measure change.

Consider a martingale measure with the density
\[
\mathcal{E}_T := \mathcal{E}_T \left\{ - \sum_{j=1}^{d} \int_{0}^{T} \left( \sum_{k=1}^{d} \ell_{j,k}^{(-1)} \tilde{\mu}_k(s) \right) \frac{d\tilde{W}_j(s)}{Y_k(s)} \right\},
\]
where \( \ell_{j,k}^{(-1)} \) and \( \tilde{\mu}_k(s) \) are given in Subsection 2.2.

Remark 4.5. In the literature, the measure with the density (4.10) is commonly referred to as the minimal martingale measure, see e.g. [8].

Recall that
\[
W = \mathcal{L}^{-1} u \tilde{W},
\]
so denote \( \rho_{j,i} \) the values such that
\[
W_j(t) = \sum_{i=1}^{2d} \rho_{j,i} \tilde{W}_i(t),
\]
i.e.
\[
\mathcal{E}_T = \mathcal{E}_T \left\{ - \sum_{j=1}^{d} \sum_{i=1}^{2d} \rho_{j,i} \left( \sum_{k=1}^{d} \ell_{j,k}^{(-1)} \tilde{\mu}_k(s) \right) \frac{d\tilde{W}_i(s)}{Y_k(s)} \right\}
\]

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Proposition 3.5, it is easy to establish that $E$ where $\beta$

Theorem 4.4 is true. Theorem 4.6.

The function $f$ and $h$

In particular, this implies that $Y_i$ is bounded away from zero by $\varphi_i^* := \min_{t \in [0, T]} \varphi_i(t)$ and hence $\frac{1}{Y_i(s)}$ can be represented as follows:

$$\frac{1}{Y_i(s)} = h_i(Y_i(s)), \quad h_i(y) := \begin{cases} 1 & y > \varphi_i^*, \\ \frac{1}{\varphi_i^*} y + \frac{2}{\varphi_i^*}, & y \leq \varphi_i^*. \end{cases}$$

The function $h_i$ is bounded, continuously differentiable and has bounded derivative, hence by the classical chain rule $\frac{1}{Y_i(s)} \in \mathbb{D}_1^{1,2}$ and

$$\tilde{D}^1 \left[ \frac{1}{Y_i(s)} \right] = 0.$$ 

In particular, this implies that

$$\tilde{D}^1 \left[ -\frac{1}{2} \sum_{i=1}^{2d} \int_0^T \left( \sum_{j=1}^d \rho_{j,i} \left( \sum_{k=1}^d \ell_{j,k}^{(-1)} \frac{\tilde{\mu}_k(s)}{Y_k(s)} \right) \right)^2 ds \right] = 0$$

and

$$\tilde{D}^1 \left[ -\sum_{i=1}^{2d} \int_0^T \left( \sum_{j=1}^d \rho_{j,i} \left( \sum_{k=1}^d \ell_{j,k}^{(-1)} \frac{\tilde{\mu}_k(s)}{Y_k(s)} \right) \right) dW_i(s) \right]$$

where $\beta_k := \left( -\sum_{j=1}^d \rho_{j,k} \ell_{j,k}^{(-1)} \right)$. Furthermore, using the boundedness of $Y$ and the chain rule from Proposition 3.5 it is easy to establish that $E_T \in \tilde{\mathbb{D}}^{1,2}$ and

$$\tilde{D}^1 E_T = E_T \sum_{k=1}^d \beta_k \frac{\tilde{\mu}_k(v)}{Y_k(v)}$$

Let now $f: \mathbb{R} \to \mathbb{R}$ satisfy Assumption $\alpha_i > 0$, $i = 1, ..., d$. Then the following modification of Theorem 4.4 is true.

**Theorem 4.6.** Let $S = (S_1, ..., S_d)$ be given by (2.8), $X = (X_1, ..., X_d)$ be the corresponding log-price
process, $X_i = \log S_i$, $f : \mathbb{R} \to \mathbb{R}$ satisfy Assumption \textcolor{red}{C} $F(s) := \int_0^s f(u)du$ and $\alpha_i > 0$, $i = 1, \ldots, d$. Then

$$
\mathbb{E}\left[ \mathcal{E}_T f \left( \sum_{i=1}^d \alpha_i S_i(T) \right) \right] = \mathbb{E}\left[ \mathcal{E}_T \frac{F \left( \sum_{i=1}^d \alpha_i S_i(T) \right)}{\alpha_1 S_1(T)} \left( 1 + \frac{1}{T \mu_{i,1}} \left( \int_0^T \frac{1}{Y_1(v)} d\tilde{W}_1(v) - \sum_{k=1}^d \int_0^T \beta_k \frac{\tilde{\mu}_k(v)}{Y_k(v)Y_1(v)} dv \right) \right) \right].
$$

(4.11)

Proof. Denote

$$
G_1(x_1, \ldots, x_d) := \frac{F \left( \sum_{i=1}^d \alpha_i e^{x_i} \right)}{\alpha_1 e^{x_1}}, \quad (x_1, \ldots, x_d) \in \mathbb{R}^d.
$$

Using polynomial growth of $f$, boundedness of $Y_i$, existence of all moments of $S_i$ and Proposition \textcolor{red}{3.5} it is easy to prove that

$$
\mathcal{E}_T G_1(X(T)) \in \mathbb{D}^{1,2}
$$

and

$$
\tilde{D}_i \left[ \mathcal{E}_T G_1(X(T)) \right] = \mathcal{E}_T \tilde{D}_i G_1(X(T)) + G_1(X(T)) \tilde{D}_i \mathcal{E}_T
$$

$$
= \mathcal{E}_T \tilde{D}_i G_1(X(T)) + \mathcal{E}_T G_1(X(T)) \sum_{k=1}^d \beta_k \frac{\tilde{\mu}_k(v)}{Y_k(v)}.
$$

Next, observe that

$$
\mathbb{E}\left[ \mathcal{E}_T f \left( \sum_{i=1}^d \alpha_i S_i(T) \right) \right] = \mathbb{E}\left[ \mathcal{E}_T \left( f \left( \sum_{i=1}^d \alpha_i e^{X_i(T)} - G_1(X(T)) \right) \right) \right] + \mathbb{E}\left[ \mathcal{E}_T G_1(X(T)) \right]
$$

and

$$
\mathbb{E}\left[ \mathcal{E}_T \left( f \left( \sum_{i=1}^d \alpha_i e^{X_i(T)} - G_1(X(T)) \right) \right) \right] = \mathbb{E}\left[ \mathcal{E}_T \frac{\partial G_1(X(T))}{\partial x_1} \right]
$$

$$
= \mathbb{E}\left[ \frac{1}{T} \int_0^T \mathcal{E}_T \frac{\partial G_1(X(T))}{\partial x_1} \tilde{D}_1 X_1(T) \frac{1}{\tilde{D}_i X_1(T)} dv \right]
$$

$$
= \mathbb{E}\left[ \frac{1}{T \mu_{i,1}} \int_0^T \mathcal{E}_T G_1(X(T)) \frac{1}{Y_1(v)} dv \right]

= \frac{1}{T \mu_{i,1}} \left( \mathbb{E}\left[ \int_0^T \tilde{D}_i \left[ \mathcal{E}_T G_1(X(T)) \right] \frac{1}{Y_1(v)} dv \right] \right) - \mathbb{E}\left[ \int_0^T G_1(X(T)) \tilde{D}_i \mathcal{E}_T \frac{1}{Y_1(v)} dv \right]

= \frac{1}{T \mu_{i,1}} \left( \mathbb{E}\left[ \mathcal{E}_T G_1(X(T)) \int_0^T \frac{1}{Y_1(v)} d\tilde{W}_1(v) \right] \right) - \mathbb{E}\left[ \mathcal{E}_T G_1(X(T)) \int_0^T \frac{1}{Y_1(v)} d\tilde{W}_1(v) \right]

= \mathbb{E}\left[ \mathcal{E}_T \frac{F \left( \sum_{i=1}^d \alpha_i S_i(T) \right)}{\alpha_1 S_1(T)} \left( \int_0^T \frac{1}{Y_1(v)} d\tilde{W}_1(v) - \sum_{k=1}^d \beta_k \frac{\tilde{\mu}_k(v)}{Y_k(v)Y_1(v)} dv \right) \right].
$$

Taking into account all of the above, we can finally write:

$$
\mathbb{E}\left[ \mathcal{E}_T f \left( \sum_{i=1}^d \alpha_i S_i(T) \right) \right] = \mathbb{E}\left[ \mathcal{E}_T \frac{F \left( \sum_{i=1}^d \alpha_i S_i(T) \right)}{\alpha_1 S_1(T)} \left( 1 + \frac{1}{T \mu_{i,1}} \left( \int_0^T \frac{1}{Y_1(v)} d\tilde{W}_1(v) - \sum_{k=1}^d \int_0^T \beta_k \frac{\tilde{\mu}_k(v)}{Y_k(v)Y_1(v)} dv \right) \right) \right]
$$

which ends the proof. \hfill \Box
Remark 4.7. Note that the martingale measure should not necessarily be the minimal one: it is possible to obtain a representation of the type (4.11) for other martingale measures provided that the corresponding densities are Malliavin differentiable and one can use the chain rules as in the proof of Theorem 4.6.

5 Price approximation for options with discontinuous payoffs

In this Section, we provide a numerical algorithm for computation of \( E \left[ f \left( \sum_{i=1}^{d} \alpha_i S_i(T) \right) \right] \), both with respect to physical and minimal martingale measures. We will exploit the representations obtained in Theorems 4.2, 4.4 and 4.6: absence of discontinuities under the expectation sign allows to avoid deterioration of the convergence speed.

The Section is structured as follows. In Subsection 5.2, we describe the numerical scheme used to simulate \( Y, X \) and \( S \) (the corresponding numerical approximations will be denoted by \( \hat{Y}, \hat{X} \) and \( \hat{S} \)). In Subsection 5.2, we consider a straightforward way of computing \( E \left[ f \left( \sum_{i=1}^{d} \alpha_i S_i(T) \right) \right] \) by plugging in \( \hat{Y} \) and \( \hat{S} \) into (4.6) or (4.11). An estimate of the approximation error is provided. In Subsection 5.3, we discuss another algorithm for computation of the expectation for a one-dimensional case: the vector \( \left( X(T), f_0^T \frac{1}{\psi(t)} dt \right) \) turns out to be conditionally Gaussian w.r.t. an appropriate \( \sigma \)-field and one can use this fact to approximate (4.6) without simulating \( \hat{X}(T) \) at all.

5.1 Numerical schemes

Numerical scheme for the volatility process. For the stochastic volatility processes \( Y_i, i = 1, \ldots, d \), we will use the drift-implicit Euler scheme described in [39]. For reader’s convenience, we briefly describe it below as well as give the necessary convergence results.

Let \( \{0 = t_0 < t_1 < \cdots < t_N = T\} \) be a uniform partition of \([0, T]\), \( t_k := \frac{kT}{N} \), \( k = 0, 1, \ldots, N \), with the mesh \( \Delta_N := \frac{T}{N} \) such that
\[
\Delta_N \max_{i=1,\ldots,d} \sup_{(t,y) \in D_{0,y}} \left| \frac{\partial b_i}{\partial y} (t, y) \right| < 1. \tag{5.1}
\]

Remark 5.1. Note that such \( \Delta_N \) exists due to Assumption B(iv).

Define \( \hat{Y}_i(t) \) as follows:
\[
\hat{Y}_i(0) = Y_i(0), \\
\hat{Y}_i(t_{k+1}) = \hat{Y}_i(t_k) + b_i(t_{k+1}, \hat{Y}_i(t_{k+1})) \Delta_N + (Z_i(t_{k+1}) - Z_i(t_k)), \\
\hat{Y}_i(t) = \hat{Y}_i(t_k), \quad t \in [t_k, t_{k+1}),
\]
where \( Z_i(t) := \int_0^t K_i(t, s) dB_i^Y (s) \). Note that the second expression in (5.2) is considered as an equation with respect to \( \hat{Y}_i(t_{k+1}) \) and, under Assumption B, it has a unique solution such that \( \hat{Y}_i(t_{k+1}) \in (\varphi_i(t_{k+1}), \psi_i(t_{k+1})) \), see [39] Section 2 for more details. The convergence results from [39] for this scheme are summarized in the following theorem.

Theorem 5.2. Let Assumptions A and B hold and the mesh of the partition \( \Delta_N \) satisfy (5.1). Fix \( i = 1, \ldots, d \), take an arbitrary \( \lambda_i \in (0, H_i) \), \( H_i \) from Assumption A, and fix \( r \geq 1 \). Then there exists a constant \( C > 0 \) that does not depend on the partition such that
\[
E \left[ \sup_{t \in [0, T]} \left| Y_i(t) - \hat{Y}_i(t) \right|^r \right] \leq C \Delta_N^{r/2}
\]
and
\[
E \left[ \sup_{t \in [0, T]} \left| \frac{1}{Y_i(t)} - \frac{1}{\hat{Y}_i(t)} \right|^r \right] \leq C \Delta_N^{r/2}.
\]
Numerical scheme for the price process. Recall that the log-price $X_i$ has the form

$$X_i(t) = X_i(0) + \int_0^t \left( \mu_i(s) - \frac{Y_i^2(s)}{2} \right) ds + \int_0^t Y_i(s) dB^S_i(s),$$

where $\mu_i$ is $H_i$-Hölder continuous with $H_i$ being from Assumption A(ii). For a given uniform partition $\{0 = t_0 < t_1 < ... < t_N = T\}$ satisfying (5.1), consider the random process $\hat{X}_i = \{\hat{X}_i(t), t \in [0, T]\}$ such that for any point $t_n$ of the partition

$$\hat{X}_i(t_n) = X_i(0) + \frac{T}{N} \sum_{k=0}^{n-1} \left( \mu_i(t_k) - \frac{Y_i^2(t_k)}{2} \right) + \frac{T}{N} \sum_{k=0}^{n-1} \hat{Y}_i(t_k) \left(B^S_i(t_{k+1}) - B^S_i(t_k)\right),$$

where $\hat{Y}_i$ is defined by (5.2), and for any $t \in (t_n, t_{n+1})$

$$\hat{X}_i(t) = \hat{X}_i(t_n).$$

**Theorem 5.3.** Let Assumptions A and B hold and the mesh of the partition $\Delta_N$ satisfy (5.1). Fix $i = 1, ..., d$, take an arbitrary $\lambda_i \in (0, H_i)$, where $H_i$ is from Assumption A and fix $r \geq 1$. Then there exists a constant $C > 0$ that does not depend on the partition such that

$$E \left[ \sup_{n=0,1, ..., N} |X_i(t_n) - \hat{X}_i(t_n)|^r \right] \leq C \Delta_N^{\lambda_i}.$$

In particular,

$$E \left[ |X_i(T) - \hat{X}_i(T)|^r \right] \leq C \Delta_N^{\lambda_i}.$$

**Proof.** Observe that for any $n = 1, ..., N$

$$|X(t_n) - \hat{X}(t_n)|^r \leq C \left( \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} |\mu_i(s) - \mu_i(t_k)| ds \right)^r + C \left( \int_0^T |Y_i^2(s) - \hat{Y}_i^2(s)| ds \right)^r$$

$$+ C \left( \int_0^T (Y_i(s) - \hat{Y}_i(s)) dB^S_i(s) \right)^r \leq C \left( \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} |s - t_k|^\lambda_i ds \right)^r + C \left( \int_0^T |Y_i(s) - \hat{Y}_i(s)| |Y_i(s) - \hat{Y}_i(s)| ds \right)^r$$

$$+ C \left( \sup_{t \in [0, T]} \left| \int_0^t (Y_i(s) - \hat{Y}_i(s)) dB^S_i(s) \right| \right)^r \leq C \Delta_N^{\lambda_i} + C \sup_{t \in [0, T]} |Y_i(t) - \hat{Y}_i(t)|^r + C \left( \sup_{t \in [0, T]} \left| \int_0^t (Y_i(s) - \hat{Y}_i(s)) dB^S_i(s) \right| \right)^r.$$

Hence, by the Burkholder-Davis-Gundy inequality and Theorem 5.2

$$E \left[ \sup_n |X(t_n) - \hat{X}(t_n)|^r \right] \leq C \Delta_N^{\lambda_i} + CE \left[ \sup_{t \in [0, T]} |Y_i(t) - \hat{Y}_i(t)|^r \right]$$

$$+ CE \left[ \left( \sup_{t \in [0, T]} \left| \int_0^t (Y_i(s) - \hat{Y}_i(s)) dB^S_i(s) \right| \right)^r \right] \leq C \Delta_N^{\lambda_i} + CE \left[ \left( \int_0^T (Y_i(s) - \hat{Y}_i(s))^2 ds \right)^{\frac{r}{2}} \right] \leq C \Delta_N^{\lambda_i}.$$
Next, denote $\hat{S}_i(t) := e^{\hat{X}_i(t)}$.

**Proposition 5.4.** Let Assumptions $A$ and $B$ hold and the mesh of the partition $\Delta_N$ satisfy (5.1). Then, for any $r \in \mathbb{R}$

$$\sup_{N \geq 1} \sup_{n=0,1,\ldots,N} \mathbb{E} \left[ \hat{S}_i(t_n) \right] < \infty.$$ 

**Proof.** Since both $\mu_i$ and $\tilde{Y}_i$ are bounded and the bound does not depend on the partition, it is sufficient to prove that

$$\sup_{N \geq 1} \sup_{n=0,1,\ldots,N} \mathbb{E} \left[ \exp \left\{ r \int_0^{t_n} \tilde{Y}_i(s) dB_i(s) \right\} \right] < \infty.$$ 

Consider a stochastic process

$$\exp \left\{ r \int_0^t \tilde{Y}_i(s) dB_i(s) \right\}, \quad t \in [0, T].$$

Then, since $\tilde{Y}_i$ is bounded, Novikov’s condition implies that

$$\mathbb{E} \left[ \exp \left\{ \int_0^t r\tilde{Y}_i(s) dB_i(s) - \frac{r^2}{2} \int_0^t \tilde{Y}_i^2(s) ds \right\} \right] = 1$$

and hence there exist a constant $C$ that does not depend on the partition such that

$$\mathbb{E} \left[ \exp \left\{ r \int_0^t \tilde{Y}_i(s) dB_i(s) \right\} \right] = \mathbb{E} \left[ \exp \left\{ r \int_0^t \tilde{Y}_i(s) dB_i(s) - \frac{r^2}{2} \int_0^t \tilde{Y}_i^2(s) ds \right\} \right] \leq C \mathbb{E} \left[ \exp \left\{ r \int_0^t \tilde{Y}_i(s) dB_i(s) - \frac{r^2}{2} \int_0^t \tilde{Y}_i^2(s) ds \right\} \right] = C < \infty,$$

which implies the required result.

\[ \Box \]

**Theorem 5.5.** Let Assumptions $A$ and $B$ hold and the mesh of the partition $\Delta_N$ satisfy (5.1). Fix $i = 1, \ldots, d$, take an arbitrary $\lambda_i \in (0, H_i)$, where $H_i$ is from Assumption $A$ and fix $r \geq 1$. Then there exists a constant $C > 0$ that does not depend on the partition such that

$$\sup_{N \geq 1} \sup_{n=0,1,\ldots,N} \mathbb{E} \left[ |S_i(t_n) - \hat{S}_i(t_n)|^r \right] \leq C \Delta_N^{\lambda_i}.$$ 

In particular,

$$\mathbb{E} \left[ |S_i(T) - \hat{S}_i(T)|^r \right] \leq C \Delta_N^{\lambda_i}.$$

**Proof.** For any $n = 0, 1, \ldots, N$, we use Theorem 2.6, Proposition 5.4 and Theorem 5.3 to write:

$$\mathbb{E} \left[ |S_i(t_n) - \hat{S}_i(t_n)|^r \right] = \mathbb{E} \left[ |e^{X_i(t_n)} - e^{\hat{X}_i(t_n)}|^r \right] \leq \mathbb{E} \left[ \left( e^{X_i(t_n)} + e^{\hat{X}_i(t_n)} \right)^r |X_i(t_n) - \hat{X}_i(t_n)|^r \right] \leq \left( \mathbb{E} \left[ \left( e^{X_i(t_n)} + e^{\hat{X}_i(t_n)} \right)^{2r} \right] \right)^{\frac{r}{2}} \left( \mathbb{E} \left[ |X_i(t_n) - \hat{X}_i(t_n)|^{2r} \right] \right)^{\frac{1}{2}} \leq C \left( \mathbb{E}[e^{2rX_i(t_n)}] + \mathbb{E}[e^{2r\hat{X}_i(t_n)}] \right)^{\frac{1}{2}} \left( \mathbb{E} \left[ |X_i(t_n) - \hat{X}_i(t_n)|^{2r} \right] \right)^{\frac{1}{2}} \leq C \Delta_N^{\lambda_i},$$

where $C$ does not depend on the partition, which implies the required result.

\[ \Box \]
Approximation of the martingale density. In order to get an approximation of option price (i.e. under the change of measure), we need a result concerning approximation of the corresponding density. Let \( \mathcal{E}_T \) be the minimal martingale measure defined by (4.10). Fix a uniform partition \( \{0 = t_0 < t_1 < \ldots < t_N = T\} \), \( t_n = \frac{n}{N} \), with the mesh satisfying (5.1) and put
\[
\hat{\mathcal{E}}_T := \exp \left\{ - \frac{d}{N} \sum_{j=1}^{N-1} \left( \sum_{k=1}^{d} \ell_{j,k}^{(-1)} \frac{\tilde{\mu}_k(t_n)}{Y_k(t_n)} \right) (W_j(t_{n+1}) - W_j(t_n)) - \frac{1}{2} \sum_{j=1}^{d} T \sum_{n=0}^{N-1} \left( \sum_{k=1}^{d} \ell_{j,k}^{(-1)} \frac{\tilde{\mu}_k(t_n)}{Y_k(t_n)} \right)^2 \right\}.
\]

**Theorem 5.6.** Let Assumptions A and B hold and the mesh of the partition \( \Delta_N \) satisfy (5.1). Let also \( \lambda < \min_{i=1,\ldots,d} H_i \) with \( H_i \) being from Assumption A. Then, for any \( r \geq 1 \), there exists a constant \( C > 0 \) that does not depend on the partition such that
\[
\mathbb{E} \left[ |\mathcal{E}_T - \hat{\mathcal{E}}_T|^r \right] \leq C \Delta_N^\lambda.
\]

**Proof.** The proof is straightforward, so we will provide an outline omitting all the details.

First, observe that for every \( j = 1, \ldots, d \), using boundedness of each \( \frac{1}{Y_k} \), \( \lambda \)-Hölder continuity of each \( \tilde{\mu}_k \) as well as Theorem 5.2, we can write
\[
\mathbb{E} \left[ \sum_{n=0}^{N-1} \left( \sum_{k=1}^{d} \ell_{j,k} \frac{\tilde{\mu}_k(t_n)}{Y_k(t_n)} \right) (W_j(t_{n+1}) - W_j(t_n)) - \frac{1}{2} \sum_{k=1}^{d} T \sum_{n=0}^{N-1} \left( \sum_{k=1}^{d} \ell_{j,k} \frac{\tilde{\mu}_k(t_n)}{Y_k(t_n)} \right)^2 \right]^r
\]
\[
\leq C \mathbb{E} \left[ \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \left( \sum_{k=1}^{d} \ell_{j,k} \frac{\tilde{\mu}_k(t_n)}{Y_k(s)} \right)^2 ds \right]^{\frac{r}{2}}
\]
\[
\leq C \mathbb{E} \left[ \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \left( \sum_{k=1}^{d} \ell_{j,k} \frac{\tilde{\mu}_k(t_n)}{Y_k(s)} - \tilde{\mu}_k(s) \right)^2 ds \right]^{\frac{r}{2}}
\]
\[
+ C \mathbb{E} \left[ \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \left( \frac{1}{Y_k(s)} - \frac{1}{Y_k(t_n)} \right)^2 ds \right]^{\frac{r}{2}}
\]
\[
\leq C \mathbb{E} \left[ \sum_{k=1}^{d} \frac{1}{Y_k(t_n)} \int_{t_n}^{t_{n+1}} (\tilde{\mu}_k(t_n) - \tilde{\mu}_k(s))^2 ds \right]^{\frac{r}{2}}
\]
\[
+ C \mathbb{E} \left[ \sum_{k=1}^{d} \left( \frac{1}{Y_k(s)} - \frac{1}{Y_k(t_n)} \right)^2 ds \right]^{\frac{r}{2}}
\]
\[
\leq C \mathbb{E} \left[ \sum_{k=1}^{d} \sup_{s \in [0,T]} \left( \frac{1}{Y_k(s)} - \frac{1}{Y_k(t_n)} \right)^2 \right]^{\frac{r}{2}}
\]
\[
\leq C \Delta_N^\lambda
\]

where \( C \) is, as always, a constant that does not depend on the partition and which may vary from line to line. Similarly, for each \( j = 1, \ldots, d \),
\[
\mathbb{E} \left[ \sum_{j=1}^{d} T \sum_{n=0}^{N-1} \left( \sum_{k=1}^{d} \ell_{j,k} \frac{\tilde{\mu}_k(t_n)}{Y_k(t_n)} \right)^2 \right]^r
\]
\[
\leq C \Delta_N^\lambda,
\]
therefore we conclude that
\[ \mathbb{E} \left[ \left| \log \mathcal{E}_T - \log \hat{\mathcal{E}}_T \right| \right] \leq C \Delta_N^\lambda. \]
Using the arguments similar to Proposition 5.3, one can see that for all \( r \geq 1 \) there exists a constant \( C \) that does not depend on the partition such that
\[ \mathbb{E} [\mathcal{E}_T^r] + \max_{N \geq 1} \mathbb{E} \left[ \hat{\mathcal{E}}_T^r \right] < C \]
and hence
\[ \mathbb{E} [\mathcal{E}_T - \hat{\mathcal{E}}_T]^r \leq C \mathbb{E} \left[ \left( \mathcal{E}_T - \hat{\mathcal{E}}_T \right)^r \right] \leq C \Delta_N^\lambda. \]
\[ \square \]

5.2 Approximation by discretizing both \( Y \) and \( S \)
After introducing the numerical schemes for \( Y \), \( X \) and \( S \), we are ready to proceed to main results of this Section. We begin with a theorem that is in the spirit of [19, Theorem 15].

**Theorem 5.7.** Let \( S = (S_1, \ldots, S_d) \) be given by (2.8), \( X = (X_1, \ldots, X_d) \) be the corresponding log-price process, \( X_i = \log S_i \), \( f: \mathbb{R} \to \mathbb{R} \) satisfy Assumption C \[ F(s) := \int_0^s f(u)du \] and \( \alpha_i > 0 \), \( i = 1, \ldots, d \). Fix a uniform partition \( \{0 = t_0 < t_1 < \ldots < t_N = T\} \) with \( t_n = \frac{Tn}{N} \), with the mesh satisfying (5.1) and take \( \tilde{Y} = (\tilde{Y}_1, \ldots, \tilde{Y}_d) \), \( \tilde{X} = (\tilde{X}_1, \ldots, \tilde{X}_d) \) and \( \tilde{S} = (\tilde{S}_1, \ldots, \tilde{S}_d) \) as described in Subsection 5.2. Finally, let \( \lambda < \min_{i=1,\ldots,d} \lambda_i \) with \( \lambda_i \) being from Assumption [10]. Then

1) for any \( r \geq 1 \), there exists a constant \( C > 0 \) such that
\[ \mathbb{E} \left[ \frac{F \left( \sum_{i=1}^d \alpha_i S_i(T) \right)}{\alpha_1 S_1(T)} \left( 1 + \frac{\int_0^T \frac{1}{Y_i(u)} d\tilde{W}_1(u)}{T u_{1,1}} \right) \right] - \mathbb{E} \left[ \frac{F \left( \sum_{i=1}^d \alpha_i \tilde{S}_i(T) \right)}{\alpha_1 \tilde{S}_1(T)} \left( 1 + \frac{\int_0^T \frac{1}{Y_i(u)} d\tilde{W}_1(u)}{T u_{1,1}} \right) \right] \leq C \Delta_N^\lambda, \]

2) there exists a constant \( C > 0 \) such that
\[ \left\| \mathbb{E} \left[ f \left( \sum_{i=1}^d \alpha_i S_i(T) \right) \right] - \mathbb{E} \left[ \frac{F \left( \sum_{i=1}^d \alpha_i \tilde{S}_i(T) \right)}{\alpha_1 \tilde{S}_1(T)} \left( 1 + \frac{\int_0^T \frac{1}{Y_i(u)} d\tilde{W}_1(u)}{T u_{1,1}} \right) \right] \right\| \leq C \Delta_N^\lambda. \]

**Proof.** By Theorem 4.4, it is enough to prove item 1. We provide the proof only for \( d = 1 \) (in this case we can put \( \alpha_1 = 1 \) without any loss in generality). For the general case of \( d > 1 \), the arguments are the same but require a more notation-heavy presentation. For reader’s convenience, we split the proof into three steps.

**Step 1.** We begin with noting that, by the Burkholder-Davis-Gundy inequality and Theorem 5.2
\[ \mathbb{E} \left[ \left| \int_0^T \left( \frac{1}{Y(u)} - \frac{1}{Y(u)} \right) d\tilde{W}_1(u) \right|^r \right] \leq C \mathbb{E} \left[ \left( \int_0^T \left( \frac{1}{Y(u)} - \frac{1}{Y(u)} \right)^2 du \right)^{\frac{r}{2}} \right] \leq C \Delta_N^\lambda. \]

**Step 2.** Next, let us show that for any \( r \geq 1 \)
\[ \mathbb{E} \left[ \left| \frac{F(S(T))}{S(T)} - \frac{F(\tilde{S}(T))}{\tilde{S}(T)} \right|^r \right] \leq C \Delta_N^\lambda. \]
It is clear that
\[ \mathbb{E} \left[ \left| \frac{F(S(T))}{S(T)} - \frac{F(\hat{S}(T))}{\hat{S}(T)} \right|^r \right] \leq C \mathbb{E} \left[ \left| \frac{F(S(T))}{S(T)} - \frac{F(\hat{S}(T))}{\hat{S}(T)} \right|^{2r} \right] \]
\[ \quad + C \mathbb{E} \left[ \left| \frac{F(S(T))}{S(T)} - \frac{F(\hat{S}(T))}{\hat{S}(T)} \right| \right]. \] 
(5.6)

Now we estimate both terms in the right-hand side separately. Observe that \( F \) has polynomial growth and thus, by Theorem 2.6, \( \mathbb{E}[|F(S(T))|^{2r}] < \infty \). Therefore
\[ \mathbb{E} \left[ |F(S(T))|^r \left| \frac{1}{S(T)} - \frac{1}{\hat{S}(T)} \right|^r \right] \leq \left( \mathbb{E} \left[ |F(S(T))|^{2r} \right] \mathbb{E} \left[ \left| \frac{1}{S(T)} - \frac{1}{\hat{S}(T)} \right|^{2r} \right] \right)^{\frac{1}{2}} \]
\[ \quad \leq C \left( \mathbb{E} \left[ \left| \frac{1}{S(T)} - \frac{1}{\hat{S}(T)} \right|^{2r} \right] \right)^{\frac{1}{2}}. \]

Moreover, observe that, by Theorem 2.6 and Proposition 5.3, there exists constant \( C > 0 \) that does not depend on the partition such that
\[ \mathbb{E} \left[ e^{-4r(X(T)+\tilde{X}(T))}(e^{X(T)} + e^{\tilde{X}(T)})^{4r} \right] < C, \]
thus, using Theorem 5.3 the inequality \(|e^x - e^y| \leq (e^x + e^y)|x - y|, x, y \in \mathbb{R}\), we can write
\[ \mathbb{E} \left[ \left| \frac{1}{S(T)} - \frac{1}{\hat{S}(T)} \right|^{2r} \right] = \mathbb{E} \left[ e^{-2r(X(T)+\tilde{X}(T))} \left| e^{X(T)} - e^{\tilde{X}(T)} \right|^{2r} \right] \]
\[ \leq \mathbb{E} \left[ e^{-2r(X(T)+\tilde{X}(T))}(e^{X(T)} + e^{\tilde{X}(T)})^{2r} \left| X(T) - \tilde{X}(T) \right|^{2r} \right] \]
\[ \leq \left( \mathbb{E} \left[ e^{-4r(X(T)+\tilde{X}(T))}(e^{X(T)} + e^{\tilde{X}(T)})^{4r} \right] \right)^{\frac{1}{2}} \left( \mathbb{E} \left[ \left| X(T) - \tilde{X}(T) \right|^{4r} \right] \right)^{\frac{1}{2}} \]
\[ \leq C \Delta_N^{2r\lambda}, \]
where \( C > 0 \) is again a constant that does not depend on the partition. Therefore
\[ \mathbb{E} \left[ \left| \frac{F(S(T))}{S(T)} - \frac{1}{\hat{S}(T)} \right|^r \right] \leq C \Delta^{r\lambda}. \]
(5.7)

Next, observe that for any \( s_1, s_2 > 0 \)
\[ |F(s_1) - F(s_2)| = \left| \int_{s_1 \wedge s_2}^{s_1 \vee s_2} f(u) du \right| \leq C(1 + s_1^q + s_2^q)|s_1 - s_2|, \]
so
\[ \mathbb{E} \left[ \left| \frac{1}{S^r(T)} \left| F(S(T)) - F(\hat{S}(T)) \right| \right|^r \right] \]
\[ \leq \left( \mathbb{E} \left[ \left| \frac{1}{S^{2r}(T)} \right|^r \right] \right)^{\frac{1}{r}} \left( \mathbb{E} \left[ \left| F(S(T)) - F(\hat{S}(T)) \right|^{2r} \right] \right)^{\frac{1}{2r}} \]
\[ \leq C \left( \mathbb{E} \left[ \left| F(S(T)) - F(\hat{S}(T)) \right|^{2r} \right] \right)^{\frac{1}{2}} \]
\[ \leq C \left( \mathbb{E} \left[ (1 + S^q(T) + \hat{S}^q(T))^{2r} \left| S(T) - \hat{S}(T) \right|^{2r} \right] \right)^{\frac{1}{2}}. \]
\[ \leq C \left( \mathbb{E} \left[ (1 + S^q(T) + \hat{S}^q(T))^{|4r|} \right] \right)^{\frac{1}{2}} \left( \mathbb{E} \left[ |S(T) - \hat{S}(T)|^{4r} \right] \right)^{\frac{1}{2}} \]
\[ \leq C \left( \mathbb{E} \left[ |S(T) - \hat{S}(T)|^{4r} \right] \right)^{\frac{1}{2}}, \]

where we used Theorem 2.6 together with Proposition 5.4 to estimate \( \mathbb{E} \left[ \hat{S}^{-2r}(T) \right] \) together with \( \mathbb{E} \left[ (1 + S^q(T) + \hat{S}^q(T))^{|4r|} \right] \).

Finally, using the same argument as in the proof of (5.7), we can obtain that
\[ \left( \mathbb{E} \left[ |S(T) - \hat{S}(T)|^{4r} \right] \right)^{\frac{1}{2}} \leq C \Delta_N^{\lambda}, \]

and thus
\[ \mathbb{E} \left[ \frac{1}{\hat{S}^r(T)} \left| F(S(T)) - F(\hat{S}(T)) \right|^r \right] \leq C \Delta_N^{\lambda}. \quad (5.8) \]

Now (5.5) follows from (5.6), (5.7) and (5.8).

\textbf{Step 3.} Now we proceed to the claim of the theorem. Since
\[ \mathbb{E} \left[ \left| \frac{F(S(T))}{S(T)} \right|^{2r} \right] < \infty \]
due to the polynomial growth of the mapping \( x \mapsto \frac{F(x)}{x} \), we can deduce by Step 1, Step 2 and Theorem 4.2 that
\[
\begin{align*}
\mathbb{E} & \left[ \left( \frac{F(S(T))}{S(T)} \left( 1 + \int_0^T \frac{1}{Y(u)} d\hat{W}_1(u) \right) \right) - F(\hat{S}(T)) \left( 1 + \int_0^T \frac{1}{Y(u)} d\bar{W}_1(u) \right) \right] \\
& \leq C \mathbb{E} \left[ \left| \frac{F(S(T))}{S(T)} - \frac{F(\hat{S}(T))}{\hat{S}(T)} \right|^r \right] \\
& \quad + C \mathbb{E} \left[ \left| \frac{F(S(T))}{S(T)} \int_0^T \left( \frac{1}{Y(u)} - \frac{1}{Y(u)} \right) d\hat{W}_1(u) \right|^r \right] \\
& \quad + C \mathbb{E} \left[ \int_0^T \frac{1}{Y(u)} d\hat{W}_1(u) \left( \frac{F(S(T))}{S(T)} - \frac{F(\hat{S}(T))}{\hat{S}(T)} \right) \right]^r \\
& \leq C \Delta_N^{\lambda} + C \left( \mathbb{E} \left[ \left| \frac{F(S(T))}{S(T)} \right|^{2r} \right] \right)^{\frac{1}{2}} \left( \mathbb{E} \left[ \left| \int_0^T \left( \frac{1}{Y(u)} - \frac{1}{Y(u)} \right) d\hat{W}_1(u) \right|^{2r} \right] \right)^{\frac{1}{2}} \\
& \quad + C \left( \mathbb{E} \left[ \int_0^T \frac{1}{Y(u)} d\hat{W}_1(u) \right|^{2r} \right] \right)^{\frac{1}{2}} \left( \mathbb{E} \left[ \left| \frac{F(S(T))}{S(T)} - \frac{F(\hat{S}(T))}{\hat{S}(T)} \right|^{2r} \right] \right)^{\frac{1}{2}} \\
& \leq C \Delta_N^{\lambda},
\end{align*}
\]

which ends the proof.

\textbf{Theorem 5.8.} Let \( S \) be given by (1.2), \( X \) be the corresponding log-price process, \( X = \log S, f: \mathbb{R} \to \mathbb{R} \) satisfy Assumption (G) \( F(s) := \int_0^s f(u) du, g(x) := f(e^x), G(x) := F(1) + \int_0^x g(r) dr \). Fix a uniform partition \( 0 = t_0 < t_1 < \ldots < t_n = T \), \( t_i = \frac{T}{n} \), with the mesh satisfying (5.1) and take \( \hat{Y}, \hat{X} \) and \( \hat{S} \) as described in Subsection 5.7. Finally, let \( \lambda < H \) with \( H := H_1 \) being from Assumption (H). Then
1) for any \( r \geq 1 \), there exists a constant \( C > 0 \) such that
\[
\mathbb{E} \left[ \frac{1}{T u_{1,1}} G(X(T)) \int_0^T \frac{1}{Y(u)} d\tilde{W}_1(u) - \frac{1}{T u_{1,1}} G(\tilde{X}(T)) \int_0^T \frac{1}{Y(u)} d\tilde{W}_1(u) \right] \leq C \Delta N^\lambda .
\]

2) there exists a constant \( C > 0 \) such that
\[
\left| \mathbb{E} \left[ f(S(T)) \right] - \mathbb{E} \left[ \frac{1}{T u_{1,1}} G(\tilde{X}(T)) \int_0^T \frac{1}{Y(u)} d\tilde{W}_1(u) \right] \right| \leq C \Delta N^\lambda .
\]

Proof. Once again, item 1 implies item 2 by Theorem [12] and we sketch the proof of the former omitting the details. First, observe that for any \( x \)
\[
\text{Due to existence of exponential moments of } X \text{ as well as polynomial growth of } f, \text{ it is also clear that for any } r \geq 1
\[
\mathbb{E} \left[ \left| G(X(T)) - G(\tilde{X}(T)) \right|^r \right] < \infty .
\]
Finally, as noted in Step 1 of the proof of Theorem [5.7]
\[
\mathbb{E} \left[ \left| \int_0^T \left( \frac{1}{Y(u)} - \frac{1}{Y(u)} \right) d\tilde{W}_1(u) \right|^r \right] \leq C \Delta N^\lambda .
\]

Therefore
\[
E \left[ \left| G(X(T)) \int_0^T \frac{1}{Y(u)} d\tilde{W}_1(u) - G(\tilde{X}(T)) \right| \right] \\
\leq CE \left[ \left| G(X(T)) \int_0^T \frac{1}{Y(u)} d\tilde{W}_1(u) - \frac{1}{T u_{1,1}} G(\tilde{X}(T)) \int_0^T \frac{1}{Y(u)} d\tilde{W}_1(u) \right| \right] \\
+ CE \left[ \left| G(X(T)) - G(\tilde{X}(T)) \right| \int_0^T \frac{1}{Y(u)} d\tilde{W}_1(u) \right] \\
\leq C \left( \mathbb{E} \left[ \left| G(X(T)) \right|^{2r} \right] \right)^{1/2} \left( \mathbb{E} \left[ \left| \int_0^T \left( \frac{1}{Y(u)} - \frac{1}{Y(u)} \right) d\tilde{W}_1(u) \right|^{2r} \right] \right)^{1/2} \\
+ C \left( \mathbb{E} \left[ \left| \int_0^T \frac{1}{Y(u)} d\tilde{W}_1(u) \right|^{2r} \right] \right)^{1/2} \left( \mathbb{E} \left[ \left| G(X(T)) - G(\tilde{X}(T)) \right|^{2r} \right] \right)^{1/2} \\
\leq C \Delta N^\lambda .
\]
We conclude the Subsection with an approximation result in line with Theorem 5.7 involving a change of measure. We omit the proof since it is straightforward and utilizes the same machinery as the one of Theorem 5.7 (one just has to additionally use Theorem 5.6 in order to estimate the type of 
\[ \mathbb{E} \left[ |E_T - \hat{E}_T|^r \right] \].

**Theorem 5.9.** Let \( S = (S_1, \ldots, S_d) \) be given by (2.8), \( X = (X_1, \ldots, X_d) \) be the corresponding log-price process, \( X_i = \log S_i, f: \mathbb{R} \to \mathbb{R} \) satisfy Assumptions [5.4], \( F(s) := \int_0^s f(u) du \) and \( \alpha_i > 0, i = 1, \ldots, d \), and \( E_T \) be the minimal martingale measure defined by (4.10). Fix a uniform partition \( \{ 0 = t_0 < t_1 < \ldots < t_N = T \} \), \( t_n = \frac{T}{N} \), with the mesh satisfying (5.1) and take \( \tilde{Y} = (\tilde{Y}_1, \ldots, \tilde{Y}_d), \tilde{X} = (\tilde{X}_1, \ldots, \tilde{X}_d), \tilde{S} = (\tilde{S}_1, \ldots, \tilde{S}_d) \) and \( \tilde{E}_T \) as described in Subsection 5.4. Finally, let \( \lambda < \min_{i=1,\ldots,d} H_i \) with \( H_i \) being from Assumption 4. Then

1) for any \( r \geq 1 \), there exists a constant \( C > 0 \) such that

\[
\mathbb{E} \left[ \left| E_T F \left( \sum_{i=1}^d \alpha_i S_i(T) \right) \left( 1 + \int_0^T \frac{1}{Y_i(v)} d\tilde{W}_1(v) - \sum_{k=1}^d \int_0^T \beta_k \frac{\tilde{\mu}_k(v)}{Y_i(v)Y_i(v)} dv \right) \right] \right| \leq C \Delta_N^{3r},
\]

2) there exists a constant \( C > 0 \) such that

\[
\left| \mathbb{E} \left[ E_T f \left( \sum_{i=1}^d \alpha_i S_i(T) \right) \right] \right| - \mathbb{E} \left[ \tilde{E}_T F \left( \sum_{i=1}^d \alpha_i \tilde{S}_i(T) \right) \left( 1 + \int_0^T \frac{1}{Y_i(v)} d\tilde{W}_1(v) - \sum_{k=1}^d \int_0^T \beta_k \frac{\tilde{\mu}_k(v)}{Y_i(v)Y_i(v)} dv \right) \right] \right| \leq C \Delta_N^{3r}.
\]

### 5.3 Approximation via conditional Gaussianity for one-dimensional model

Assume now that \( d = 1 \) and the price \( S \) is given by (4.2). The core idea of Subsection 5.2 was to plug in the simulated paths \( \tilde{Y}, \tilde{S} \) and \( \tilde{X} \) into representations (4.6) or (4.7). However, this requires simulation of both volatility and the price which inevitably leads to higher variance in Monte Carlo estimations. However, it is possible to avoid simulation of the price and in this Subsection we describe the corresponding approach.

First note that the random vector \( (\tilde{X}(T), \int_0^T \frac{1}{Y(u)} d\tilde{W}_1(u))^T \) is conditionally Gaussian with respect to the \( \sigma \)-field \( \tilde{F}_T := \sigma(\tilde{W}_2(t), t \in [0, T]) \). The next lemma presents the parameters of the joint conditional distribution of the random vector \( (\tilde{X}(T), \int_0^T \frac{1}{Y(u)} d\tilde{W}_1(u))^T \).

**Lemma 5.10.** The random vector \( (\tilde{X}(T), \int_0^T \frac{1}{Y(u)} d\tilde{W}_1(u))^T \) is conditionally Gaussian with respect to the \( \sigma \)-field \( \tilde{F}_T := \sigma(\tilde{W}_2(t), t \in [0, T]) \) with conditional mean

\[
\begin{pmatrix}
X(0) + \frac{T}{N} \sum_{k=0}^{N-1} \left( \mu(t_k) - \frac{1}{2} \tilde{Y}^2(t_k) \right) + u_{1,2} \sum_{k=0}^{N-1} \tilde{Y}(t_k)(\tilde{W}_2(t_{k+1}) - \tilde{W}_2(t_k)) \\
0
\end{pmatrix} := \begin{pmatrix} m(\tilde{Y}) \end{pmatrix}
\]

and conditional covariance matrix

\[
\tilde{C} = \begin{pmatrix}
u_{1,1}^2 \frac{T}{N} \sum_{k=0}^{N-1} \tilde{Y}^2(t_k) \\
\frac{T}{N} \sum_{k=0}^{N-1} \tilde{Y}^2(t_k) & u_{1,1} T \sum_{k=0}^{N-1} \tilde{Y}^2(t_k)
\end{pmatrix} = \begin{pmatrix}
u_{1,1}^2 V_1^2 & u_{1,1}^2 T \nu_{1,1} V_2^2 \\
\nu_{1,1}^2 T \nu_{1,1} V_2^2 & V_2^2
\end{pmatrix},
\]

where \( V_1^2 := \frac{T}{N} \sum_{k=0}^{N-1} \tilde{Y}^2(t_k) \) and \( V_2^2 := \frac{T}{N} \sum_{k=0}^{N-1} \tilde{Y}^2(t_k) \). Moreover, if additionally

\[
\det \tilde{C} = \nu_{1,1}^2(V_1^2V_2^2 - T^2) > 0
\]
with probability 1, then the random vector \( \left( \hat{X}(T), \int_0^T \frac{1}{Y(u)} d\hat{W}_1(u) \right)^T \) has a conditional density of the form

\[
\phi(x, y) = \frac{1}{2\pi \sqrt{\det \mathcal{C}}} \exp \left\{ -\frac{V_2^2(x - m(\hat{Y}))^2 + u_{1,1}^2 V_1^2 y^2 - 2T u_{1,1} (x - m(\hat{Y})) y}{2 \det \mathcal{C}} \right\}.
\]

**Proof.** Recall that the random variable \( \hat{X}(T) \) can be represented in the form

\[
\hat{X}(T) = X(0) + \frac{T}{N} \sum_{k=0}^{N-1} \left( \mu(t_k) - \frac{\hat{Y}^2(t_k)}{2} \right) + u_{1,1} \sum_{k=0}^{N-1} \hat{Y}(t_k) \left( \hat{W}_1(t_{k+1}) - \hat{W}_1(t_k) \right)
+ u_{1,2} \sum_{k=0}^{N-1} \hat{Y}(t_k) \left( \hat{W}_2(t_{k+1}) - \hat{W}_2(t_k) \right).
\]

Now the explicit form of conditional mean, conditional covariance matrix and conditional density (provided that the conditional covariance matrix is non-degenerate) can be computed directly. \( \square \)

**Theorem 5.11.** Let \( \det \hat{\mathcal{C}} > 0 \) with probability 1, where \( \hat{\mathcal{C}} \) is the conditional covariance matrix from Lemma 5.10. Then

\[
\frac{1}{T u_{1,1}} \mathbb{E} \left[ G(\hat{X}(T)) \int_0^T \frac{1}{Y(u)} d\hat{W}_1(u) \right] = \frac{1}{\sqrt{2\pi u_{1,1}^2}} \int_{\mathbb{R}} G(x) \mathbb{E} \left[ \frac{x - m(\hat{Y})}{V_1} \exp \left\{ -\frac{\left( x - m(\hat{Y}) \right)^2}{2u_{1,1}^2 V_1^2} \right\} \right] dx.
\]

**Proof.** Using Lemma 5.10, we can write

\[
\mathbb{E} \left[ G(\hat{X}(T)) \int_0^T \frac{1}{Y(u)} d\hat{W}_1(u) \right] = \mathbb{E} \left[ \frac{1}{T u_{1,1}} \mathbb{E} \left[ G(\hat{X}(T)) \int_0^T \frac{1}{Y(u)} d\hat{W}_1(u) \mid \hat{\mathcal{F}}_T \right] \right]
= \mathbb{E} \left[ \int_{\mathbb{R}} G(x) \mathbb{E} \left[ \int_{\mathbb{R}} \phi(x, y) dy \right] dx \right]
= \int_{\mathbb{R}} G(x) \mathbb{E} \left[ \int_{\mathbb{R}} \phi(x, y) dy \right] dx.
\]

The integral \( \int_{\mathbb{R}} \phi(x, y) dy \) can be simplified. Namely, denote for simplicity \( \bar{x} := x - m(\hat{Y}) \). Then

\[
\int_{\mathbb{R}} \phi(x, y) dy = \frac{1}{2\pi \sqrt{\det \mathcal{C}}} \int_{\mathbb{R}} y \exp \left\{ -\frac{V_2^2 \bar{x}^2 + u_{1,1}^2 V_1^2 y^2 - 2 u_{1,1} T \bar{x} y}{2 \det \mathcal{C}} \right\} dy
= \frac{1}{2\pi \sqrt{\det \mathcal{C}}} \int_{\mathbb{R}} y \exp \left\{ -\frac{1}{2 \det \mathcal{C}} \left( u_{1,1} V_1 y - \frac{T \bar{x}}{V_1} \right)^2 + V_2^2 \bar{x}^2 - \frac{T^2 \bar{x}^2}{V_1^2} \right\} dy
= \frac{1}{2\pi \sqrt{\det \mathcal{C}}} \exp \left\{ -\frac{\bar{x}^2}{2u_{1,1}^2 V_1^2} \right\} \sqrt{\frac{T \bar{x} \sqrt{\det \mathcal{C}}}{u_{1,1} V_1}}
= \frac{T \bar{x}}{\sqrt{2\pi u_{1,1}^2 V_1}} \exp \left\{ -\frac{\bar{x}^2}{2u_{1,1}^2 V_1^2} \right\}.
\]

This yields the result. \( \square \)
Equation (5.10) from Theorem 5.11 essentially relies on the condition \( \det \hat{C} > 0 \) with probability 1. A simple sufficient condition is provided by the following proposition.

**Proposition 5.12.** Let \( Z(t) = \int_0^t K(t,s)d\tilde{W}_2(s), \ t \in [0,T], \) and \( \{0 = t_0 < t_1 < \ldots < t_N = T\} \) be a uniform partition of \([0,T]\) with the mesh \( \Delta_N := \frac{T}{N} \) having the following property: \( c\Delta_N < 1, \) where \( c \) is an upper bound for \( \frac{\partial \phi}{\partial y} \) from Assumption B(iv).

If the random vector
\[
(Z(t_1) - Z(t_0), Z(t_2) - Z(t_1), \ldots, Z(t_N) - Z(t_{N-1}))^T
\]
has absolutely continuous distribution w.r.t. the Lebesgue measure on \( \mathbb{R}^N, \) then \( \det \hat{C} > 0 \) with probability 1 and (5.10) holds.

**Proof.** Note that \( \det \hat{C} \geq 0 \) a.s., because \( \hat{C} \) is a conditional covariance matrix, i.e. we need to check that \( \det \hat{C} \neq 0 \) with probability 1. Taking into account that
\[
\det \hat{C} \neq 0 \iff \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} \hat{Y}^2(t_k)\hat{Y}^{-2}(t_l) \neq 1,
\]
it is sufficient to check that the random vector \( (\hat{Y}(t_1), \ldots, \hat{Y}(t_N))^T \) has density with respect to the Lebesgue measure. Denote
\[
b_n(y) := y - b(t_n,y)\Delta_N, \quad n = 1, \ldots, N,
\]
and observe that for any \( n = 1, \ldots, N \) is a 1-to-1 mapping of the interval \((\varphi(t_n), \psi(t_n))\) onto \( \mathbb{R} \) with \( b'_n > 0. \) By (5.2), for any \( n = 1, \ldots, N \)
\[
\hat{Y}(t_n) = b_n^{-1}\left(\hat{Y}(t_{n-1}) + Z(t_n) - Z(t_{n-1})\right)
\]
and thus
\[
\begin{pmatrix}
\hat{Y}(t_1) \\
\hat{Y}(t_2) \\
\vdots \\
\hat{Y}(t_N)
\end{pmatrix}
= \begin{pmatrix}
b_1^{-1}(Y(0) + Z(t_1) - Z(t_0)) \\
b_1^{-1}(Y(0) + Z(t_1) - Z(t_0)) + Z(t_2) - Z(t_1)) \\
\vdots \\
b_N^{-1}(b_{N-1}^{-1}(\cdots) + Z(t_N) - Z(t_{N-1}))
\end{pmatrix}
=: I(Z(t_1) - Z(t_0), Z(t_2) - Z(t_1), \ldots, Z(t_N) - Z(t_{N-1})).
\]

It is straightforward to check that function \( I: \mathbb{R}^N \rightarrow \prod_{n=1}^N(\varphi(t_n), \psi(t_n)) \) from (5.11) is a bijection and its Jacobian \( \det J_I(z) \) is strictly positive for any \( z \in \mathbb{R}^N. \) This implies that the random vector \( (\hat{Y}(t_1), \ldots, \hat{Y}(t_N))^T \) has density, i.e. is absolutely continuous w.r.t. the Lebesgue measure, which ends the proof. \( \square \)

### 6 Simulations

Let us illustrate the algorithms described in Section 5 with numerical simulations. For simplicity, we assume that

- \( T = 1, \) \( d = 1; \)
- an explicit shape of the SVV model is
\[
S(t) = 1 + \int_0^t Y(s)S(s)\left(\sqrt{0.75}dB_1(s) - 0.5dB_2(s)\right),
\]
\[
Y(t) = 0.5 + \int_0^t \left(\frac{0.005}{(Y(s) - 0.05)^5} - \frac{0.05}{(1 - Y(s))^5} + 0.05(0.5 - Y(s))\right)ds + 0.3\int_0^t K(t,s)dB_2(s),
\]
(6.1)

where \( B_1, B_2 \) are independent Brownian motions and \( K(t,s) := \frac{1}{(H+1/2)}(t-s)^{H-1/2} \mathbb{1}_{s<t} \) with \( H = 0.2; \)
the instantaneous interest rate $\nu(t) \equiv 0$;

- the option payoff function is a discontinuous function given by
  \[ f(s) = \mathbb{1}_{(0.5, \infty)}(s) + \mathbb{1}_{(1, \infty)}(s) + \mathbb{1}_{(1.5, \infty)}(s). \]

**Remark 6.1.** Comparing (6.1) with the representation (4.2) from Section 4, we have that $B_1 = \tilde{W}_1$, $B_2 = \tilde{W}_2$, $u_{1,1} = \sqrt{0.75}$ and $u_{1,2} = -0.5$.

We want to compute the value $\mathbb{E}[f(S(T))]$ and compare the following three numerical methods:

1) **Standard Monte Carlo (Std. MC)**: we simply simulate 10000 independent realizations of
\[ f(\tilde{S}(T)) \] and average over them;

2) **Malliavin Monte Carlo with double discretization (MMCDD)**: we simulate 10000 independent realizations of
\[ \frac{F(\tilde{S}(T))}{S(T)} \left( 1 + \int_0^T \frac{1}{\hat{Y}(u)} dW_1(u) \right) \] and then average over them;

3) **Malliavin Monte Carlo via conditional Gaussianity (MMCCG)**: we exploit the method described in Subsection 5.3 and average over 10000 independent realizations of
\[ \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} G(x) \left( \frac{x - m(\hat{Y})}{V_1^2} \right) \exp \left\{ -\frac{(x - m(\hat{Y}))^2}{2V_1^2} \right\} dx. \]

For all three methods, we consider the uniform partitions of sizes $N = 10, 100, 1000$, and 10000. Table 1 contains information on means (being also the estimators of $\mathbb{E}[f(S(T))]$) and standard deviations of (6.2), (6.3) and (6.4) for different partition sizes and Hurst indices.

**Remark 6.2.** In this type of simulation, there are two sources of error:

(I) the error between $\mathbb{E}[f(S(T))]$ and

1) $\mathbb{E}[f(\tilde{S}(T))]$ for Std. MC,
2) $\mathbb{E} \left[ \frac{F(\tilde{S}(T))}{S(T)} \left( 1 + \int_0^T \frac{1}{\hat{Y}(u)} dW_1(u) \right) \right]$ for MMCDD and
3) $\mathbb{E} \left[ \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} G(x) \left( \frac{x - m(\hat{Y})}{V_1^2} \right) \exp \left\{ -\frac{(x - m(\hat{Y}))^2}{2V_1^2} \right\} dx \right]$ for MMCCG;

(II) the error coming from the Monte Carlo estimation of the corresponding expectations.

The error of type (I) is generally controlled by the size of the partition $N$, and the Malliavin integration-by-parts technique is used specifically to reduce it. The error of type (II) is reduced by increasing the number of Monte Carlo samples. In this context, standard deviations (Sd) of random variables (6.2), (6.3) and (6.4) evaluate how many Monte Carlo simulations are required to achieve a certain level of accuracy of the mean estimation.

As we can see, the estimators of $\mathbb{E}[f(S(T))]$ by the standard Monte Carlo for small $N$ differ substantially from the ones for larger $N$ whereas both MMCDD and MMCCG show better stability with respect to the size of the partitions (thanks to the Malliavin integration-by-parts regularization technique). In addition, the MMCCG method showed by far the lowest standard deviations out of all three algorithms: this happens because there is no additional error from simulating $\tilde{S}(T)$ which results in a far more stable behavior of (6.4) (c.f. [19]).
Table 1: Computation of $\mathbb{E}[f(S(T))]$: standard Monte Carlo vs Malliavin MC with double discretization vs Malliavin MC via conditional Gaussianity. Columns entitled “Mean” contain the corresponding estimates of $\mathbb{E}[f(S(T))]$ whereas the “Sd” columns present the standard deviations of \[6.2\] for Std. MC, \[6.3\] for MMCDD and \[6.4\] for MMCCG

| Partition size, $N$ | Std. MC | MMCDD | MMCCG |
|--------------------|---------|--------|--------|
|                    | Mean    | Sd     | Mean   | Sd     | Mean   | Sd     |
| 10                 | 1.5194  | 0.9212 | 1.3961 | 1.3863 | 1.4111 | 0.3205 |
| 100                | 1.4559  | 0.9162 | 1.4152 | 1.4150 | 1.4075 | 0.3168 |
| 1000               | 1.3963  | 0.9405 | 1.4101 | 1.3846 | 1.4082 | 0.3178 |
| 10000              | 1.4036  | 0.9315 | 1.4061 | 1.4002 | 1.4084 | 0.3197 |

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