Star flows and multisingular hyperbolicity

Christian Bonatti, Adriana da Luz

January 12, 2018

Abstract

A vector field $X$ is called a star flow if every periodic orbit, of any vector field $C^1$-close to $X$, is hyperbolic. It is known that the chain recurrence classes of a generic star flow $X$ on a 3 or 4 manifold are either hyperbolic, or singular hyperbolic (see [MPP] for 3-manifolds and [GLW] on 4-manifolds).

In higher dimension (i.e. $\geq 5$) another phenomena can happen: singularities of different indices may be robustly in the same chain recurrence class of a star flow. We present a form of hyperbolicity (called multi-singular hyperbolic) which makes compatible the hyperbolic structure of regular orbits together with the one of singularities even if they have different indexes. We show that multisingular hyperbolicity implies that the flow is star, and conversely there is a $C^1$-open and dense subset of the open set of star flows which are multisingular hyperbolic.

1 Introduction

1.1 General setting and historical presentation

Considering the infinite diversity of the dynamical behaviors, it is natural to have a special interest on the robust properties that is, properties that are impossible to break by small perturbations of a system; in other words, a dynamical property is robust if its holds on a (non-empty) open set of diffeomorphisms or flows.

One important starting point of the dynamical systems as a mathematical field has been the characterization of the structural stability (i.e. systems whose topological dynamics are unchanged under small perturbations) by the hyperbolicity (i.e. a global structure expressed in terms of transversality and of uniform expansion and contraction). This characterization, first stated by the stability conjecture [PaSm], was proven for diffeomorphisms in the $C^1$ topology by Robin and Robinson in [R1], [R2] (hyperbolic systems are structurally stable) and Mañé [Ma2] (structurally stable systems are hyperbolic). The equivalent result for flows (also for the $C^1$ topology) leads to extra difficulties and was proven by [H2].

We can see in this case how the robustness of the properties is related with the structure in the tangent space: in this case, a very strong robust property is
related to a very strong uniform structure. However, hyperbolic systems are not dense in the set of diffeomorphisms or flows; instability and non-hyperbolicity may be robust. In order to describe a larger set of systems, one is lead to consider less rigid robust properties, and to try to characterize them by (weaker) structures that limit the effect of the small perturbations.

In this spirit there are several results for diffeomorphisms in the $C^1$ topology:

1. A system is robustly transitive if every $C^1$-close system is transitive. Ma proves that robustly transitive surface diffeomorphisms are globally hyperbolic (i.e. are Anosov diffeomorphism). This is no more true in higher dimensions (see examples in Sh, Ma1). DPU, BDP show that robustly transitive diffeomorphisms admits a structure called dominated splitting, and their finest dominated splitting is volume partially hyperbolic. This result extends to robustly transitive sets, and to robustly chain recurrent sets.

2. One says that a system is star if all periodic orbits are hyperbolic in a robust fashion: every periodic orbit of every $C^1$-close system is hyperbolic. For a diffeomorphism, to be star is equivalent to be hyperbolic (an important step is done in Ma and has been completed in H).

Now, what is the situation of these results for flows? The dynamics of flows seems to be very related with the dynamics of diffeomorphisms. Even more, the dynamics of vector fields in dimension $n$ looks like the one of diffeomorphisms in dimension $n - 1$. Several results can be translated from one setting to the other, for instance by considering the suspension. For example, D proved that robustly transitive flows on 3-manifolds are Anosov flows, generalizing Mañe’s result for surface diffeomorphisms. However, there is a phenomenon which is really specific to vector fields: the existence of singularities (zero of the vector field) accumulated, in a robust way by regular recurrent orbits. Then, some of the previously mentioned results may fail to be translated to the flow setting.

The first example with this behavior has been indicated by Lorenz in Lo under numerical evidences. Then GW constructs a $C^1$-open set of vector fields in a 3-manifold, having a topological transitive attractor containing periodic orbits (that are all hyperbolic) and one singularity. The examples in GW are known as the geometric Lorenz attractors.

The Lorenz attractor is also an example of a robustly non-hyperbolic star flow, showing that the result in H is no more true for flows. In dimension 3 the difficulties introduced by the robust coexistence of singularities and periodic orbits is now almost fully understood. In particular, Morales, Pacífico and Pujals (see MPP) defined the notion of singular hyperbolicity, which requires some compatibility between the hyperbolicity of the singularity and the hyperbolicity of the regular orbits. They prove that, for $C^1$-generic star flows on 3-manifolds, every chain recurrence class is singular hyperbolic. It was conjectured in GWZ that the same result could hold without the generic assumption. However BaMo built a star flow on a 3-manifold having a chain recurrence class...
which is not singular hyperbolic, contradicting the conjecture. We exhibit a very simple such example in Section 7.

In higher dimension we are far from understanding the relation between robust properties and hyperbolic structures for singular flows. There are very little examples, illustrating what are the possibilities. Let us mention [BLY] which build a flow having a robustly chain recurrent attractor containing saddles of different indices.

In this paper we propose a general way for extending the usual notions of hyperbolicity (hyperbolicity, partial hyperbolicity, volume hyperbolicity, etc.) which are well defined on compact invariant sets far from the singularities, to the case of regular recurrent orbits accumulating singularities. In particular, we will propose a notion of (multi)singular hyperbolicity which generalizes the ones already defined, and we will illustrate the power of this notion by paying a special attention to star flows.

There are already many results on the hyperbolic structure of the star flows, in dimensions larger than 3. The notion of singular hyperbolicity defined by [MPP] in dimension 3 admits a straightforward generalization in higher dimension: each chain recurrence class admits a dominated, partially hyperbolic splitting in two bundles, one being uniformly contracted (resp. expanded) and the other is sectionally area expanded (resp. sectionally area contracted). Far from the singular points, the singular hyperbolicity is equivalent to the hyperbolicity, as the direction spanned by the vector field is neither contracted nor expanded: the uniform area expansion means the uniform expansion of the normal directions.

If the chain recurrence set of a vector field $X$ can be covered by filtrating sets $U_i$ in which the maximal invariant set $Λ_i$ is singular hyperbolic, then $X$ is a star flow. Conversely, [GLW] and [GWZ] prove that this property characterizes the generic star flows on 4-manifolds. In [GSW] the authors prove the singular hyperbolicity of generic star flows in any dimensions assuming an extra property: if two singularities are in the same chain recurrence class then they must have the same $s$-index (dimension of the stable manifold). Indeed, the singular hyperbolicity implies directly this extra property.

However, in [dL], it is announced that the an example of a star flow in dimension 5 admitting singularities of different indices which belong robustly to the same chain recurrence class. This example cannot satisfy the singular hyperbolicity used in [GSW].

The example in [dL] can be done in such a way that it does not admit any dominated splitting of the tangent space for the flow: therefore the hyperbolic structure we defined does not lie on the tangent bundle. By itself, this fact is not a surprise: many hyperbolic structures for flow are not expressed in terms of the differential of the flow, but on its transverse structure (called the linear Poincaré flow). For instance, there exists a robustly transitive flow without dominated splitting, but its linear Poincaré flow needs to carry a dominated volume partially hyperbolic splitting. However the linear Poincaré flow is only defined far from the singularities, and therefore it cannot be used directly for understanding our example.
In [GLW], the authors define the notion of \emph{extended linear Poincaré flow} defined on some sort of blow-up of the singularities. Our notion of \emph{multi-singular hyperbolicity} will be expressed as the hyperbolicity of a well chosen reparametrization of this extended linear Poincaré flow, over a well chosen extension of the chain recurrence set.

Theorem 4 proves that this multisingular hyperbolicity characterizes the star flows in any dimensions: the multi-singular hyperbolic flows are star flows, and an open and dense subset of the star flows consists in multi-singular hyperbolic flows. We notice that the example in Section 7 as well as the ones in [BaMo] are multisingular hyperbolic.

The multisingular hyperbolicity is a way of making compatible the hyperbolic structure of the regular orbits with the one of the singularities. The same idea holds for weaker (uniform) forms of hyperbolicity. In this paper we show that the notions of dominated splitting, partial hyperbolicity, volume partial hyperbolicity (...) can be adapted for singular flows by multiplying the extended linear Poincaré flow by some well chosen cocycles. This will define the corresponding (multi)singular structures.

1.2 Rough presentation of our results on star flows

Just this once, the main result of this paper is the statement of a definition. We want to exhibit a definition of hyperbolicity which allows two singularities \( \sigma_1, \sigma_2 \) of different indices to be robustly in the same chain recurrent class \( C \), as such example as been announced in [dL].

1.2.1 The extended linear Poincaré flow

We deal with a vector field whose flow does not preserve any dominated splitting. As said before, it is natural to look for the hyperbolic structure on the normal bundle by considering the linear Poincaré flow (see the precise definition in Section 3.1).

However, as we deal with singular flows, the linear Poincaré flow is not defined everywhere: it is not defined on the singularities. A way proposed by [GLW] for bypassing this difficulty is the so called \emph{extended linear Poincaré flow} (see the precise definition in Section 3); we present it roughly below.

- The linear Poincaré flow is the natural linear cocycle over the flow, on the normal bundle to the flow. The dynamics on the fibers is the quotient of the derivative of the flow by the direction of the flow: this is possible as the direction of the flow is invariant.

- The vector field \( X \) provides an embedding of \( M \setminus \text{Sing}(X) \) into the projective tangent bundle \( PM \): to a point \( x \) one associates the line directed by \( X(x) \). Recall that one point \( L \) of \( PM \) corresponds to a line of the tangent space at a point of \( M \). The flow \( \phi^t \) of \( X \) induces (by the action of its derivative \( D\phi^t \)) a topological flow \( \phi^t_p \) on \( PM \) which extends the flow of \( X \).
The projective tangent bundle $\mathbb{P}M$ admits a natural bundle called the normal bundle $N$: the fiber over $L \in \mathbb{P}M$ (corresponding to a line $L \subset T_x M$) is the quotient $N_L = T_x M/L$. The derivative of the flow $D\phi^t$ of $X$ passes to the quotient on the normal bundle $N$ in a linear cocycle over $\phi^t$, called the extended linear Poincaré flow and denoted by $\psi^t_N$.

1.2.2 The extended maximal invariant set

The next difficulty is to define the set on which we would like to define an hyperbolic structure. We are interested in the dynamics of $X$ in a compact region $U$ on $M$, that is, to describe the maximal invariant set $\Lambda(X,U)$ in $U$. An important property is that the maximal invariant set depends upper-semicontinuously on the vector field $X$. This property is fundamental in the fact that “having a hyperbolic structure” is a robust property.

Therefore we needs to consider a compact part of $\mathbb{P}M$, as small as possible, such that

- it contains all the direction spanned by $X(x)$ for $x \in \Lambda(X,U) \setminus \text{Sing}(X)$,
- it varies upper semi-continuously with $X$

In Section 4.2 we define the notion of central space of a singular point $\sigma \in U$. Then we call extended maximal invariant set and we denote by $B(X,U) \subset \mathbb{P}M$ the set of all the lines $L$ contained

- either in the central space of a singular point in $\bar{U}$
- or directed by the vector $X(x)$ at a regular point $x \in \Lambda(X,U) \setminus \text{Sing}(X)$.

Proposition 20 proves that $B(X,U)$ varies upper semi-continuously with the vector field $X$.

In particular, the existence of a dominated splitting $N_L = E_L \oplus F_L$ of the normal bundle $N$ over $B(X,U)$ is a robust property.

1.2.3 The usual singular hyperbolicity

The usual notion of singular hyperbolicity consists in a dominated splitting of the flow on the maximal invariant set, so that one bundle is uniformly contracted and the other is uniformly sectionally area expanded. One can reformulate it in term of the extended linear Poincaré flow $\psi^t_N$ as follows: the cocycle $\psi^t_N$ admits a dominated splitting $E \oplus F$ over $B(X,U)$ so that

- $E$ is uniformly contracted by $\psi^t_N$,
- consider the reparametrized linear Poincaré flow obtained by multiplying the extended linear Poincaré flow $\psi^t_N$, on the normal space $N_L$, $L \in B(X,U)$, by the expansion of the derivative of the flow of $X$ in the direction $L$.

Then the reparametrized linear Poincaré flow expands uniformly the vectors in $F$. 

5
The singular hyperbolicity is not symmetric: one reparametrize the extended linear Poincaré flow only on one bundle of the dominated splitting.

1.2.4 The multisingular hyperbolicity

In our situation, the extended linear Poincaré flow $\psi^t_N$ admits a dominated splitting $E \oplus \prec F$ over $B(X,U)$, and the singular set $\text{Sing}(X) \cap U$ is divided in two sets:

- the set $S_E$ of singular points whose stable space has the same dimension as the bundle $E$,
- and the set $S_F$ of singular points whose unstable space has the same dimension as the bundle $F$.

We want:

- to reparametrize the cocycle $\psi^t_N$ in restriction to $E_L$ by the expansion in the direction $L$ if and only if the line $L$ is based at a point close to $S_E$;
- to reparametrize the cocycle $\psi^t_N$ in restriction to $F_L$ by the expansion in the direction $L$ if and only if the line $L$ is based at a point close to $S_F$.

This leads to a difficulty: the reparametrizing function needs to satisfying a cocycle relation.

In Section 5.1 we prove that there is a cocycle $\{h^t_E\}_{t \in \mathbb{R}}$, called a center-stable cocycle so that:

- $h^t_E(L)$ and $\frac{1}{h^t_E(L)}$ are uniformly bounded (independently of $t$), if $L$ is based on a point $x$ so that $x$ and $\phi^t(x)$ are out of a small neighborhood of $S_E$, where $\phi^t$ denotes the flow of $X$;
- $h^t_E(L)$ is in a bounded ratio with the expansion of $\phi^t$ in the direction $L$, if $L$ is based at a point $x$ so that $x$ and $\phi^t(x)$ are out of a small neighborhood of $S_F$.
- $h^t_E$ depends continuously on $X$.

Analogously we get the notion of center-unstable cocycles $\{h^t_F\}$ by exchanging the roles of $S_E$ and $S_F$ in the properties above.

We are now ready to define our notion of multisingular hyperbolicity.

**Definition 1.** Let $X$ be a $C^1$-vector field on a closed manifold $M$. Let $U$ be a compact set. We say that $X$ is **multisingular hyperbolic in $U$** if:

- the extended linear Poincaré flow admits a dominated splitting $E \oplus F$ over the extended maximal invariant set $B(U,X)$.
- every singular point in $U$ is hyperbolic.
• the set of singular points in $U$ is the disjoint union $S_E \cup S_F$ where the stable space of the points in $S_E$ has the same dimension as the bundle $E$ and the unstable space of the points in $S_F$ has the same dimension has the bundle $F$.

• the reparametrized linear Poincaré flow $h^t_E \psi^t_N$ is uniformly contracted on $E$, where $h^t_E$ is a center-stable cocycle,

• the reparametrized linear Poincaré flow $h^t_F \psi^t_N$ is uniformly expanded on $F$, where $h^t_F$ is a center-unstable cocycle,

Using the upper semi-continuous dependence of the set $B(U,X)$ on $X$, and of the continuous dependence of the center-stable and center unstable cocycles $h^t_F$ and $h^t_E$ on $X$, one proves

**Proposition 1.** The multisingular hyperbolicity of $X$ in $U$ is a $C^1$-robust property.

**Remark 2.** If all the singular points in $U$ have the same index, that is, if $S_E$ or $S_F$ is empty, then the multisingular hyperbolicity is the same notion as the singular hyperbolicity.

### 1.3 Star flows and multisingular hyperbolicity

We are now ready to state our results

**Theorem 3.** If $X$ is multisingular in $U$ then $X$ is a star flow on $U$: more precisely, for any vector field $Y$ $C^1$-close enough from $X$ any periodic orbit contained in $U$ is hyperbolic.

This follows from the robustness of the multisingular hyperbolicity and the fact that it induces the usual hyperbolic structure on the periodic orbits.

We only get the converse for generic star flows:

**Theorem 4.** There is a $C^1$-open and dense subset $U$ of $\mathcal{X}^1(M)$ so that is $X \in U$ is a star flow then the chain recurrent set $R(X)$ is contained in the union of finitely many pairwise disjoint filtrating regions in which $X$ is multisingular hyperbolic.

The proof of Theorem 4 follows closely the proof in [GSW] that star flows with only singular points of the same index are singular hyperbolic.

**Question 1.** Can we remove the generic assumption, at least in dimension 3, in Theorem 4? In other word, is it true that, given any star $X$ flow (for instance on a 3-manifold) every chain recurrence class of $X$ is multisingular hyperbolic?
2 Basic definitions and preliminaries

2.1 Chain recurrent set

The following notions and theorems are due to Conley [Co] and they can be found in several other references (for example [AN]).

- We say that pair of sequences \( \{ x_i \}_{0 \leq i \leq k} \) and \( \{ t_i \}_{0 \leq i \leq k-1} \), \( k \geq 1 \), are an \( \varepsilon \)-pseudo orbit from \( x_0 \) to \( x_k \) for a flow \( \phi \), if for every \( 0 \leq i \leq k - 1 \) one has
  \[
  t_i \geq 1 \quad \text{and} \quad d(x_{i+1}, \phi^{t_i}(x_i)) < \varepsilon.
  \]

- A compact invariant set \( \Lambda \) is called chain transitive if there for any \( \varepsilon > 0 \), for any \( x, y \in \Lambda \) there is an \( \varepsilon \)-pseudo orbit from \( x \) to \( y \).

- We say that \( x \in M \) is chain recurrent if for every \( \varepsilon > 0 \), there is an \( \varepsilon \)-pseudo orbit from \( x \) to \( x \). We call the set of chain recurrent points, the Chain recurrent set and we note it \( \mathcal{R}(M) \).

- We say that \( x, y \in \mathcal{R}(M) \) are chain related if, for every \( \varepsilon > 0 \), there are \( \varepsilon \)-pseudo orbits form \( x \) to \( y \) and from \( y \) to \( x \). This is an equivalence relation. The equivalent classes of this equivalence relation are called chain recurrence classes.

![Diagram of an \( \varepsilon \)-pseudo orbit](image)

Figure 1: an \( \varepsilon \)-pseudo orbit

**Definition 2.** An attracting region (also called trapping region by some authors) is a compact set \( U \) so that \( \phi^t(U) \) is contained in the interior of \( U \) for every \( t > 0 \). The maximal invariant set in an attracting region is called an attracting set. A repelling region is an attracting region for \( -X \), and the maximal invariant set is called a repeller.
• A filtrating region is the intersection of an attracting region with a repelling region.

• Let $C$ be a chain recurrent class of $M$ for the flow $\phi$. A filtrating neighborhood of $C$ is a (compact) neighborhood which is a filtrating region.

![Figure 2: A trapping region or attracting region.](image)

**Definition 3.** Let $\{\phi^t\}$ by a flow on a Riemannian manifold $M$. A complete Lyapunov function is a continuous function $\mathcal{L}: M \to \mathbb{R}$ such that

- $\mathcal{L}(\phi^t(x))$ is decreasing for $t$ if $x \in M \setminus \mathcal{R}(M)$
- Two points $x, y \in M$ are chain related if and only if $\mathcal{L}(x) = \mathcal{L}(y)$
- $\mathcal{L}(\mathcal{R}(M))$ is nowhere dense.

Next result is called the fundamental theorem of dynamical systems by some authors:

**Theorem 5** (Conley [Co]). Let $X$ be a $C^1$ vector field on a compact manifold $M$. Then its flow $\{\phi^t\}$ admits a complete Lyapunov function.

Next corollary will be often used in this paper:

**Corollary 6.** Let $\phi$ be a $C^1$-vector field on a compact manifold $M$. Every chain class $C$ of $X$ admits a basis of filtrating neighborhoods, that is, every neighborhood of $C$ contains a filtrating neighborhood of $C$.

### 2.2 Linear cocycle

Let $\phi = \{\phi^t\}_{t \in \mathbb{R}}$ be a topological flow on a compact metric space $K$. A linear cocycle over $(K, \phi)$ is a continuous map $A^t: E \times \mathbb{R} \to E$ defined by

$$A^t(x, v) = (\phi^t(x), A_t(x)v),$$

where $E$ and $\mathbb{R}$ are Euclidean spaces.
where

- $\pi: E \to K$ is a $d$ dimensional linear bundle over $K$;
- $A_t: (x,t) \in K \times \mathbb{R} \mapsto GL(E_x, E_{\phi^t(x)})$ is a continuous map that satisfies the cocycle relation:
  \[ A_{t+s}(x) = A_t(\phi^s(x))A_s(x), \text{ for any } x \in K \text{ and } t, s \in \mathbb{R} \]

Note that $A = \{A_t\}_{t \in \mathbb{R}}$ is a flow on the space $E$ which projects on $\phi^t$.

If $\Lambda \subset K$ is a $\phi$-invariant subset, then $\pi^{-1}(\Lambda) \subset E$ is $A$-invariant, and we call the restriction of $A$ to $\Lambda$ the restriction of $\{A_t\}$ to $\pi^{-1}(\Lambda)$.

2.3 Hyperbolicity, dominated splitting on linear cocycles

**Definition 4.** Let $\phi$ be a topological flow on a compact metric space $\Lambda$. We consider a vector bundle $\pi: E \to \Lambda$ and a linear cocycle $A = \{A_t\}$ over $(\Lambda, X)$.

We say that $A$ admits a dominated splitting over $\Lambda$ if

- there exists a splitting $E = E_1 \oplus \cdots \oplus E_k$ over $\Lambda$ into $k$ subbundles
- The dimension of the subbundles is constant, i.e. $dim(E_x^i) = dim(E_y^i)$ for all $x, y \in \Lambda$ and $i \in \{1 \ldots k\}$,
- The splitting is invariant, i.e. $A_t(x)(E_x^i) = E_{\phi^t(x)}^i$ for all $i \in \{1 \ldots k\}$,
- there exists a $t > 0$ such that for every $x \in \Lambda$ and any pair of non vanishing vectors $v \in E_x^i$ and $u \in E_x^j$, $i < j$ one has
  \[ \frac{||A^t(u)||}{||u||} \leq \frac{1}{2} \frac{||A^t(v)||}{||v||} \quad (1) \]

We denote $E_1 \ominus \cdots \ominus E_k$, the splitting is $t$-dominated.

A classical result (see for instance [BDV, Appendix B]) asserts that the bundles of a dominated splitting are always continuous. A given cocycle may admit several dominated splittings. However, the dominated splitting is unique if one prescribes the dimensions $dim(E^i)$.

Associated to the dominated splitting we define a family of cone fields $C_{a}^{iu}$ around each space $E^i \ominus \cdots \ominus E^k$ as follows. Let us write the vectors $v \in E$ as $v = (v_1, v_2)$ with $v_1 \in E^1 \ominus \cdots \ominus E^{i-1}$ and $v_2 \in E^i \ominus \cdots \ominus E^k$. Then the cone field $C_{a}^{iu}$ is the set

\[ C_{a}^{iu} = \{ v = (v_1, v_2) \text{ such that } ||v_1|| < a ||v_2|| \} \]
These are called the **family of unstable conefields** and the domination gives us that they are strictly invariant for times larger than \( t \): i.e. the cone \( C^u_a \) at \( T_x M \) is taken by \( A^t \) to the interior of the cone \( C^u_a \) at \( T_{f^t x} M \).

Analogously we define the **stable family of conefields** \( C^s_a \) around \( E^1 \oplus \cdots \oplus E^i \) and the domination gives us that they are strictly invariant for negative times smaller than \(-t\).

One says that one of the bundle \( E^i \) is **(uniformly) contracting** (resp. **expanding**) if there is \( t > 0 \) so that for every \( x \in \Lambda \) and every non vanishing vector \( u \in E^i_x \) one has \( \|A^t(u)\| < \frac{1}{2} \) (resp. \( \|A^t(u)\| < \frac{1}{2} \)). In both cases one says that \( E^i \) is **hyperbolic**.
Notice that if \( E^j \) is contracting (resp. expanding) then the same holds for any \( E^i, i < j \) (reps. \( j < i \)) as a consequence of the domination.

**Definition 5.** We say that the linear cocycle \( A \) is hyperbolic over \( \Lambda \) if there is a dominated splitting \( E = E^s \oplus \ldots \oplus E^u \) over \( \Lambda \) into 2 hyperbolic subbundles so that \( E^s \) is uniformly contracting and \( E^u \) is uniformly expanding.

One says that \( E^s \) is the stable bundle, and \( E^u \) is the unstable bundle.

The existence of a dominated splitting or of a hyperbolic splitting is an open property in the following sense,

**Proposition 7.** Let \( K \) be a compact metric space, \( \pi: E \to K \) be a \( d \)-dimensional vector bundle, and \( A \) be a linear cocycle over \( K \). Let \( \Lambda_0 \) be a \( \phi \)-invariant compact set. Assume that the restriction of \( A \) to \( \Lambda_0 \) admits a dominated splitting \( E^1 \oplus \ldots \oplus E^k \), for some \( t > 0 \).

Then there is a compact neighborhood \( U \) of \( \Lambda_0 \) with the following property. Let \( \Lambda = \bigcap_{t \in \mathbb{R}} \phi(t)(U) \) be the maximal invariant set of \( \phi \) in \( U \). Then the dominated splitting admits a unique extension as a \( 2t \)-dominated splitting over \( \Lambda \). Furthermore if one of the subbundle \( E^i \) is hyperbolic over \( \Lambda_0 \) it is still hyperbolic over \( \Lambda \).

As a consequence, if \( A \) is hyperbolic over \( \Lambda_0 \) then (up to shrink \( U \) if necessary) it is also hyperbolic over \( \Lambda \).

### 2.4 Robustness of hyperbolic structures

The aim of this section is to explain that Proposition 7 can be seen as a robustness property.

Let \( M \) be a manifold and \( \phi_n \) be a sequence of flows tending to \( \phi_0 \) as \( n \to +\infty \), in the \( C^0 \)-topology on compact subsets: for any compact set \( K \subset M \) and any \( T > 0 \), the restriction of \( \phi^t_n \) to \( K \), \( t \in [-T,T] \), tends uniformly (in \( x \in K \) and \( t \in [-T,T] \)) to \( \phi^t_0 \).

Let \( \Lambda_n \) be compact \( \phi_n \)-invariant subsets of \( M \), and assume that the upper limit of the \( \Lambda_n \) for the Hausdorff topology is contained in \( \Lambda_0 \): more precisely, any neighborhood of \( \Lambda_0 \) contains all but finitely many of the \( \Lambda_n \). Let us present another way to see this property:

Consider the subset \( I = \{0\} \cup \left\{ \frac{1}{n}, n \in \mathbb{N} \setminus \{0\} \right\} \subset \mathbb{R} \) endowed with the induced topology. Consider \( M_\infty = M \times I \). Let \( \Lambda_\infty \) denote

\[
\Lambda_\infty = \Lambda_0 \times \{0\} \cup \bigcup_{n>0} \Lambda_n \times \left\{ \frac{1}{n} \right\} \subset M_\infty.
\]

With this notation, the upper limit of the \( \Lambda_n \) is contained in \( \Lambda_0 \) if and only if \( \Lambda_\infty \) is a compact subset.

Let \( \pi: E \to M \) be a vector bundle. We denote \( E_\infty = E \times I \) the vector bundle \( \pi_\infty: E \times I \to M \times I \). We denote by \( E_\infty|_{\Lambda_\infty} \) the restriction of \( E_\infty \) on the compact subset \( \Lambda_\infty \).
Assume now that $A_n$ are linear cocycles over the restriction of $E$ to $\Lambda_n$. We denote by $A_\infty$ the map defined on the restriction $E_\infty|\Lambda_\infty$ by:

$$A_\infty^t(x,0) = (A_0^t(x),0), \text{ for } (x,0) \in \Lambda_0 \times \{0\} \text{ and }$$

$$A_\infty^t(x,\frac{1}{n}) = (A_n^t(x),\frac{1}{n}), \text{ for } (x,0) \in \Lambda_n \times \{\frac{1}{n}\}.$$  

**Definition 6.** With the notation above, we say that the family of cocycles $A_n$ tends to $A_0$ as $n \to \infty$ if the map $A_\infty$ is continuous, and therefore is a linear cocycle.

As a consequence of Proposition 7 we get:

**Corollary 8.** Let $\pi: E \to M$ be a linear cocycle over a manifold $M$ and let $\phi_n$ be a sequence of flows on $M$ converging to $\phi_0$ as $n \to \infty$. Let $\Lambda_n$ be a sequence of $\phi_n$-invariant compact subsets so that the upper limit of the $\Lambda_n$, as $n \to \infty$, is contained in $\Lambda_0$.

Let $A_n$ be a sequence of linear cocycles over $\phi_n$ defined on the restriction of $E$ to $\Lambda_n$. Assume that $A_n$ tend to $A_0$ as $n \to \infty$.

Assume that $A_0$ admits a dominated splitting $E = E^1 \oplus \cdots \oplus E^k$ over $\Lambda_0$. Then, for any $n$ large enough, $A_n$ admits a dominated splitting with the same number of sub-bundles and the same dimensions of the sub-bundles. Furthermore, if $E^i$ was hyperbolic (contracting or expanding) over $\Lambda_0$ it is still hyperbolic (contracting or expanding, respectively) for $A_n$ over $\Lambda_n$.

The proof just consist in applying Proposition 7 to a neighborhood of $\Lambda_0 \times \{0\}$ in $\Lambda_\infty$.

### 2.5 Reparametrized cocycles, and hyperbolic structures

Let $A = \{A^t(x)\}$ and $B = \{B^t(x)\}$ be two linear cocycles on the same linear bundle $\pi: E \to \Lambda$ and over the same flow $\phi^t$ on a compact invariant set $\Lambda$ of a manifold $M$. We say that $B$ is a reparametrization of $A$ if there is a continuous map $h = \{h^t\}: \Lambda \times \mathbb{R} \to (0, +\infty)$ so that for every $x \in \Lambda$ and $t \in \mathbb{R}$ one has

$$B^t(x) = h^t(x)A^t(x).$$

The reparametrizing map $h^t$ satisfies the cocycle relation

$$h^{r+s}(x) = h^r(x)h^s(\phi^r(x)),$$

and is called a cocycle.

One easily check the following lemma:

**Lemma 9.** Let $A$ be a linear cocycle and $B$ be a reparametrization of $A$. Then any dominated splitting for $A$ is a dominated splitting for $B$.

**Remark 10.**  
- If $h^t$ is a cocycle, then for any $\alpha \in \mathbb{R}$ the power $(h^t)^\alpha: x \mapsto (h^t(x))^\alpha$ is a cocycle.
• If $f^t$ and $g^t$ are cocycles then $h^t = f^t \cdot g^t$ is a cocycle.

A cocycle $h^t$ is called a coboundary if there is a continuous function $h : \Lambda \to (0, +\infty)$ so that

$$h^t(x) = \frac{h(\phi^t(x))}{h(x)}.$$ 

A coboundary cocycle is uniformly bounded. Two cocycles $g^t, h^t$ are called cohomological if $g^t \cdot h^t$ is a coboundary.

Remark 11. The cohomological relation is an equivalence relation among the cocycle and is compatible with the product: if $g_1^t$ and $g_2^t$ are cohomological and $h_1^t$ and $h_2^t$ are cohomological then $g_1^t h_1^t$ and $g_2^t h_2^t$ are cohomological.

Lemma 12. Let $A = A^t$ be a linear cocycle, and $h = h^t$ be a cocycle which is bounded. Then $A$ is uniformly contracted (resp. expanded) if and only if the reparametrized cocycle $B = h \cdot A$ is uniformly contracted (resp. expanded).

As a consequence one gets that the hyperbolicity of a reparametrized cocycle only depends on the cohomology class of the reparametrizing cocycle:

Corollary 13. if $g$ and $h$ are cohomological then $g \cdot A$ is hyperbolic if and only if $h \cdot A$ is hyperbolic.

3 The extended linear Poincaré flow

3.1 The linear Poincaré flow

Let $X$ be a $C^1$ vector field on a compact manifold $M$. We denote by $\phi^t$ the flow of $X$.

Definition 7. The normal bundle of $X$ is the vector sub-bundle $N_X$ over $M \setminus Sing(X)$ defined as follows: the fiber $N_X(x)$ of $x \in M \setminus Sing(X)$ is the quotient space of $T_x M$ by the vector line $\mathbb{R} \cdot X(x)$.

Note that, if $M$ is endowed with a Riemannian metric, then $N_X(x)$ is canonically identified with the orthogonal space of $X(x)$:

$$N_X = \{(x, v) \in TM, v \perp X(x)\}$$

Consider $x \in M \setminus Sing(X)$ and $t \in \mathbb{R}$. Thus $D\phi^t(x) : T_x M \to T_{\phi^t(x)}$ is a linear automorphism mapping $X(x)$ onto $X(\phi^t(x))$. Therefore $D\phi^t(x)$ passes to the quotient as an linear automorphism $\psi^t(x) : N_X(x) \to N_X(\phi^t(x))$:

$$
\begin{array}{ccc}
T_x M & \xrightarrow{D\phi^t(x)} & T_{\phi^t(x)} M \\
\downarrow & & \downarrow \\
N_X(x) & \xrightarrow{\psi^t(x)} & N_X(\phi^t(x))
\end{array}
$$

where the vertical arrow are the canonical projection of the tangent space to the normal space parallel to $X$. 

14
Proposition 14. Let $X$ be a $C^1$ vector field on a manifold and $\Lambda$ be a compact invariant set of $X$. Assume that $\Lambda$ does not contained any singularity of $X$. Then $\Lambda$ is hyperbolic if and only if the linear Poincaré flow over $\Lambda$ is hyperbolic.

Notice that the notion of dominated splitting for non-singular flow is sometimes better expressed in term of Linear Poincaré flow: for instance, the linear Poincaré flow of a robustly transitive vector field always admits a dominated splitting, when the flow by itself may not admit any dominated splitting. See for instance the suspension of the example in [BV].

3.2 The extended linear Poincaré flow

We are dealing with singular flows and the linear Poincaré flow is not defined on the singularity of the vector field $X$. However we can extend the linear Poincaré to a flow, as in [GLW], on a larger set, and for which the singularities of $X$ do not play a specific role. We call this the extended linear Poincaré flow.

This flow will be a linear co-cycle define on some linear bundle over a manifold, that we define now.

Definition 8. Let $M$ be a $d$ dimensional manifold.

- We call the projective tangent bundle of $M$, and denote by $\Pi_P: \mathbb{P}M \to M$, the fiber bundle whose fiber $\mathbb{P}_x$ is the projective space of the tangent space $T_xM$: in other words, a point $L_x \in \mathbb{P}_x$ is a 1-dimensional vector subspace of $T_xM$.

- We call the tautological bundle of $\mathbb{P}M$, and we denote by $\Pi_T: \mathcal{T}M \to \mathbb{P}M$, the 1-dimensional vector bundle over $\mathbb{P}M$ whose fiber $\mathcal{T}_L$, $L \in \mathbb{P}M$, is the line $L$ itself.
• We call normal bundle of $\mathbb{P}M$ and we denote by $\Pi_N: N \to \mathbb{P}M$, the $(d - 1)$-dimensional vector bundle over $\mathbb{P}M$ whose fiber $N_L$ over $L \in \mathbb{P}x$ is the quotient space $T_xM/L$.

If we endow $M$ with riemannian metric, then $N_L$ is identified with the orthogonal hyperplane of $L$ in $T_xM$.

Let $X$ be a $C^r$ vector field on a compact manifold $M$, and $\phi^t$ its flow. The natural actions of the derivative of $\phi^t$ on $\mathbb{P}M$, $T_M$ and $N$ define $C^{r-1}$ flows on these manifolds. More precisely, for any $t \in \mathbb{R}$,

- We denote by $\phi^t_L: \mathbb{P}M \to \mathbb{P}M$ the $C^{r-1}$ diffeomorphism defined by $\phi^t_L(L_x) = D\phi^t(L_x) \in \mathbb{P}\phi^t(x)$.

- We denote by $\phi^t_T: T_M \to T_M$ the $C^{r-1}$ diffeomorphism whose restriction to a fiber $T_L$ is the linear automorphisms onto $T_{\phi^t(L)}$ which is the restriction of $D\phi^t$ to the line $T_L$.

- We denote by $\psi^t_N: N \to N$ the $C^{r-1}$ diffeomorphism whose restriction to a fiber $N_L$, $L \in \mathbb{P}x$, is the linear automorphisms onto $N_{\phi^t(L)}$ defined as follows: $D\phi^t(x)$ is a linear automorphism from $T_xM$ to $T_{\phi^t(x)}M$, which maps the line $T_L \subset T_xM$ onto the line $T_{\phi^t(L)}$. Therefore it passe to the quotient in the announced linear automorphism.

$\begin{array}{ccc}
T_xM & \xrightarrow{D\phi^t} & T_{\phi^t(x)}M \\
\downarrow & & \downarrow \\
N_L & \xrightarrow{\psi^t_N} & N_{\phi^t(L)}
\end{array}$

Note that $\phi^t_L$, $t \in \mathbb{R}$ defines a flow on $\mathbb{P}M$ which is a co-cycle over $\phi^t$ whose action on the fibers is by projective maps.

The one-parameter families $\phi^t_T$ and $\psi^t_N$ define flows on $T_M$ and $N$, respectively, are linear co-cycles over $\phi^t$. We call $\phi^t_T$ the tautological flow, and we call $\psi^t_N$ the extended linear Poincaré flow. We can summarize by the following diagrams:

$\begin{array}{ccc}
N & \xrightarrow{\psi^t_N} & N \\
\downarrow & & \downarrow \\
\mathbb{P}M & \xrightarrow{\phi^t} & \mathbb{P}M \\
\downarrow & & \downarrow \\
M & \xrightarrow{\phi^t} & M
\end{array}$

$\begin{array}{ccc}
T_M & \xrightarrow{\phi^t_T} & T_M \\
\downarrow & & \downarrow \\
\mathbb{P}M & \xrightarrow{\phi^t} & \mathbb{P}M \\
\downarrow & & \downarrow \\
M & \xrightarrow{\phi^t} & M
\end{array}$

**Remark 15.** The extended linear Poincaré flow is really an extension of the linear Poincaré flow defined in the previous section; more precisely:

Let $S_X: M \setminus \text{Sing}(X) \to \mathbb{P}M$ be the section of the projective bundle defined as $S_X(x)$ is the line $(X(x)) \in \mathbb{P}x$ generated by $X(x)$. Then
• the fibers $N_X(x)$ and $N_{S_X}(x)$ are canonically identified and
• the linear automorphisms $\psi^t: N_X(x) \to N_X(\phi^t(x))$ and $\psi^t_{S_X}: N_{S_X}(x) \to N_{S_X}(\phi^t(x))$ are equal (under the identification of the fibers)

### 3.3 Maximal invariant set and lifted maximal invariant set

Let $X$ be a vector field on a manifold $M$ and $U \subset M$ be a compact subset. The **maximal invariant set** $\Lambda = \Lambda_U$ of $X$ in $U$ is the intersection

$$\Lambda_U = \bigcap_{y \in \mathbb{R}} \phi^t(U).$$

We say that a compact $X$-invariant set $K$ is **locally maximal** if there exist an open neighborhood $U$ of $K$ so that $X = \Lambda_U$.

**Definition 9.** We call **lifted maximal invariant set in** $U$, and we denote by $\Lambda_{P,U} \subset \mathbb{P}M$, (or simply $\Lambda_P$ if one may omit the dependence on $U$), the closure of the set of lines $\langle X(x) \rangle$ for regular $x \in \Lambda_U$:

$$\Lambda_{P,U} = \overline{S_X(\Lambda_U \setminus \text{Sing}(X))} \subset \mathbb{P}M,$$

where $S_X: M \setminus \text{Sing}(X) \to \mathbb{P}M$ is the section defined by $X$.

Next remark is very useful for getting robust properties:

**Remark 16.** Let $U$ be a compact subset of $M$.

• The maximal invariant set $\Lambda_{X,U}$ depends upper semi-continuously on $X$: if $Y$ is close to $X$ then $\Lambda_{Y,U}$ is contained in an arbitrarily small neighborhood of $\Lambda_{X,U}$.

• If $X$ has no singularities in $U$ (that is $\text{Sing}(X) \cap U = \emptyset$), then the lifted maximal invariant set $\Lambda_{U,P}(Y)$ depends upper-semi continuously on the vector field $Y$ in a small neighborhood of $X$: for $Y$ close to $X$ the intersection $\text{Sing}(Y) \cap U$ is empty and $\Lambda_{U,P}(Y)$ is just the image $S_Y(\Lambda_{Y,U})$, and $S_Y$ depends continuously on $Y$. Therefore, the semi-continuity of $\Lambda_{U,P}(Y)$ is a straightforward consequence of the one of $\Lambda_{Y,U}$.

• One difficulty that we need to deal with when one considers the lifted maximal invariant set for flows with singularities is that it does no longer depends semi-continuously on the flow, in general.

### 4 The extended maximal invariant set

#### 4.1 Strong stable, strong unstable and center spaces associated to a hyperbolic singularity.

Let $X$ be a vector field and $\sigma \in \text{Sing}(X)$ be a hyperbolic singular point of $X$. Let $\lambda^s_1 < \lambda^u_1 < \lambda^s_2 < \lambda^u_2 \ldots \lambda^s_l$ be the Lyapunov exponents of $\phi_t$ at $\sigma$
and let $E_{k} \oplus \cdots \oplus E_{2} \oplus \langle E_{1} \rangle \oplus \langle E_{2} \rangle \oplus \cdots \oplus \langle E_{l} \rangle$ be the corresponding (finest) dominated splitting over $\sigma$.

A subspace $F$ of $T_{\sigma}M$ is called a center subspace if it is of one of the possible form below:

- Either $F = E_{1} \oplus \langle E_{2} \rangle \oplus \cdots \oplus \langle E_{k} \rangle$
- Or $F = E_{k} \oplus \langle E_{2} \rangle \oplus \cdots \oplus \langle E_{l} \rangle$
- Or else $F = E_{k} \oplus \cdots \oplus E_{2} \oplus \langle E_{1} \rangle \oplus \langle E_{2} \rangle \oplus \cdots \oplus \langle E_{l} \rangle$

A subspace of $T_{\sigma}M$ is called a strong stable space, and we denote it $E_{\text{ss}}(\sigma)$, is there in $i \in \{1, \ldots, k\}$ such that:

$$E_{i}^{\text{ss}}(\sigma) = E_{k} \oplus \cdots \oplus E_{j+1} \oplus \langle E_{l} \rangle$$

A classical result from hyperbolic dynamics asserts that for any $i$ there is a unique injectively immersed manifold $W_{i}^{\text{ss}}(\sigma)$, called a strong stable manifold tangent at $E_{i}^{\text{ss}}(\sigma)$ and invariant by the flow of $X$.

We define analogously the strong unstable spaces $E_{j}^{\text{uu}}(\sigma)$ and the strong unstable manifolds $W_{j}^{\text{uu}}(\sigma)$ for $j = 1, \ldots, l$.

### 4.2 The lifted maximal invariant set and the singular points

Let $U$ be a compact region, and $X$ a vector field. Let $\sigma \in \text{Sing}(X) \cap U$ be a hyperbolic singularity of $X$ contained in $U$. We are interested on $\Lambda_{\text{P},U}(X) \cap \mathbb{P}_{\sigma}$. More precisely the aim of this section is to add to the lifted maximal invariant set $\Lambda_{\text{P},U}$, some set over the singular points in order to recover some upper semi-continuity properties.

We define the escaping stable space $E_{\text{ss},U}(\sigma)$ as the biggest strong stable space $E_{i}^{\text{ss}}(\sigma)$ such that the corresponding strong stable manifold $W_{i}^{\text{ss}}(\sigma)$ is escaping, that is:

$$\Lambda_{X,U} \cap W_{j}^{\text{ss}}(\sigma) = \emptyset$$

We define the escaping unstable space analogously.

We define the central space $E_{c,U}(\sigma)$ as the center space such that

$$T_{\sigma}M = E_{\text{ss},U}^{c} \oplus E_{\text{uu},U}^{c} \oplus E_{\text{uu},U}^{c}$$

We denote by $\mathbb{P}_{i,U}$ the projective space of $E^i(\sigma,U)$ where $i = \{\text{ss, uu, c}\}$.

**Lemma 17.** Let $U$ be a compact region and $X$ a vector field whose singular points are hyperbolic. Then, for any $\sigma \in \text{Sing}(X) \cap U$, one has :

$$\Lambda_{\text{P},U} \cap \mathbb{P}_{\sigma,U}^{\text{ss}} = \Lambda_{\text{P},U} \cap \mathbb{P}_{\sigma,U}^{\text{uu}} = \emptyset.$$

**Proof.** Suppose (arguing by contradiction) that $L \in \Lambda_{\text{P},U} \cap \mathbb{P}_{\sigma,U}^{\text{ss}}$. This means that there exist a sequence $x_{n} \in \Lambda_{X,U} \setminus \text{Sing}(X)$ converging to $\sigma$, such that $L_{x_{n}}$ converge to $L$, where $L_{x_{n}}$ is the line $\mathbb{R}X(x_{n}) \in \mathbb{P}_{x_{n}}$. 

18
We fix a small neighborhood $V$ of $\sigma$ endowed with local coordinates so that the vector field is very close to its linear part in these coordinates: in particular, there is a small cone $C^{ss} \subset V$ around $W^{ss}_{\sigma,U}$ so that the complement of this cone is strictly invariant in the following sense: the positive orbit of a point out of $C^{ss}$ remains out of $C^{ss}$ until it leaves $V$.

For $n$ large enough the points $x_n$ belong to $V$. As $RX(x_n)$ tend to $L$, this implies that the point $x_n$, for $n$ large, are contained in the cone $C^{ss}$.

In particular, the point $x_n$ cannot belong to $W^{u}(\sigma)$. Therefore they admits negative iterates $y_n = \phi^{-t_n}(x_n)$ with the following property.

- $\phi^{-t}(x_n) \in V$ for all $t \in [0, t_n]$,
- $\phi^{-t_n-1}(x_n) \notin V$,
- $t_n \to +\infty$.

Up to consider a subsequence one may assume that the point $y_n$ converge to a point $y$, and one easily check that the point $y$ belongs to $W^{s}(\sigma) \setminus \sigma$. Furthermore all the points $y_n$ belong to $\Lambda_X,U$ so that $y \in \Lambda_X,U$.

We conclude the proof by showing that $y$ belongs to $W^{ss}_{\sigma,U}$, which is a contradiction with the definition of the escaping strong stable manifold $W^{ss}_{\sigma,U}$. If $y \notin W^{ss}_{\sigma,U}$ then it positive orbit arrive to $\sigma$ tangentially to the weaker stable spaces: in particular, there is $t > 0$ so that $\phi^t(y)$ does not belong to the cone $C^{ss}$.

Consider $n$ large, in particular $t_n$ is larger than $t$ and $\phi^t(y_n)$ is so close to $y$ that $\phi^t(y_n)$ does not belong to $C^{ss}$: this contradict the fact that $x_n = \phi^{t_n}(y_n)$ belongs to $C^{ss}$.

We have proved $\Lambda_{P,U} \cap P^{ss}_{\sigma,U} = \emptyset$; the proof that $\Lambda_{P,U} \cap P^{uu}_{\sigma,U}$ is empty is analogous.

As a consequence we get the following characterization of the central space of $\sigma$ in $U$:

**Lemma 18.** The central space $E^c_{\sigma,U}$ is the smallest center space containing $\Lambda_{P,U} \cap P_{\sigma}$.

**Proof.** The proof that $E^c_{\sigma,U}$ contains $\Lambda_{P,U} \cap P_{\sigma}$ is very similar to the end of the proof of Lemma 17 and we just sketch it: by definition of the strong escaping manifolds, they admit an neighborhood of a fundamental domain which is disjoint from the maximal invariant set. This implies that any point in $\Lambda_X,U$ close to $\sigma$ is contained out of arbitrarily large cones around the escaping strong direction. Therefore the vector $X$ at these points is almost tangent to $E^c_{\sigma,U}$.

Assume now for instance that:

- $E^c_{\sigma,U} = E^s_1 \oplus E^s_{i-1} \oplus \cdots \oplus E^s_1 \oplus E^u_1 \oplus \cdots \oplus E^u_j$: in particular $W^{ss}_{i+1}(\sigma)$ is the escaping strong stable manifold, and
\[ \Lambda_P \cap \mathbb{P}_\sigma \text{ is contained in the smaller center space} \]
\[ E_{i-1}^s \oplus \cdots \oplus E_i^s \oplus E_i^u \oplus \cdots \oplus E_j^u. \]

We will show that the strong stable manifold \( W^{ss}_i(\sigma) \) is escaping, contradicting the maximality of the escaping strong stable manifold \( W^{ss}_{i+1}(\sigma) \). Otherwise, there is \( x \in W^{ss}_i(\sigma) \setminus \{\sigma\} \cap \Lambda_{X,U} \). The positive orbit of \( x \) tends to \( \sigma \) tangentially to \( E_i^s \oplus \cdots \oplus E_{i-1}^s \) and thus \( X(\phi^t(x)) \) for \( t \) large is almost tangent to \( E_i^s \oplus \cdots \oplus E_{i-1}^s \); this implies that \( \Lambda_P \cap \mathbb{P}_\sigma \) contains at least a direction in \( E_i^s \oplus \cdots \oplus E_{i-1}^s \) contradicting the hypothesis.

\[ \Box \]

**Lemma 19.** Let \( U \) be a compact region. Given \( \sigma \) be a hyperbolic singular point in \( U \), the point \( \sigma \) has a continuation \( \sigma_Y \) for vector fields \( Y \) in a \( C^1 \)-neighborhood of \( X \). Then both escaping strong stable and unstable spaces \( E^{ss}_{\sigma_Y,U} \) and \( E^{uu}_{\sigma_Y,U} \) depend lower semi-continuously on \( Y \).

As a consequence the central space \( E^c_{\sigma_Y,U} \) of \( \sigma_Y \) in \( U \) for \( Y \) depends upper semi-continuously on \( Y \), and the same happens for its projective space \( \mathbb{P}^a_{\sigma_Y,U} \).

**Proof.** We will make only the proof for the escaping strong stable space, as the proof for the escaping strong unstable space is identical.

As \( \sigma \) is contained in the interior of \( U \), there is \( \delta > 0 \) and a \( C^1 \)-neighborhood \( \mathcal{U} \) of \( X \) so that, for any \( Y \in \mathcal{U} \), one has:

- \( \sigma \) has a hyperbolic continuation \( \sigma_Y \) for \( Y \);
- the finest dominated splitting of \( \sigma_X \) for \( X \) has a continuation for \( \sigma_Y \) which is a dominated splitting (but maybe not the finest);
- the local stable manifold of size \( \delta \) of \( \sigma_Y \) in contained in \( U \) and depends continuously on \( Y \);
- for any strong stable space \( E^{ss}(\sigma) \) the corresponding local strong stable manifold \( W^{ss}(\sigma_Y) \) varies continuously with \( Y \in \mathcal{U} \).

Let denote \( E^{ss} \) denote the escaping strong stable space of \( \sigma \) and \( W^{ss}_i(\sigma) \) be the corresponding local strong stable manifold. We fix a sphere \( S_X \) embedded in the interior of \( W^{ss}_i(\sigma) \), transverse to \( X \) and cutting every orbit in \( W^{ss}_i(\sigma) \setminus \{\sigma\} \). By definition of escaping strong stable manifold, for every \( x \in S_X \) there is \( t(x) > 0 \) so that \( \phi^{t(x)}(x) \) is not contained in \( U \).

As \( S_X \) is compact and the complement of \( U \) is open, there is a finite family \( t_i, i = 0, \ldots, k \), an open covering \( V_0, \ldots, V_k \) and a \( C^1 \)-neighborhood \( \mathcal{U}_1 \) of \( X \) so that, for every \( x \in U_i \) and every \( Y \in \mathcal{U}_1 \), the point \( \phi^{t_i}(x) \) does not belong to \( U \).

For \( Y \) in a smaller neighborhood \( \mathcal{U}_2 \) of \( X \), the union of the \( V_i \) cover a sphere \( S_Y \subset W^{ss}_i(\sigma_Y,Y) \) cutting every orbit in \( W^{ss}_i(\sigma_Y,Y) \setminus \{\sigma_Y\} \).

This shows that \( W^{ss}_i(\sigma_Y,Y) \) is contained in the escaping strong stable manifold of \( \sigma_Y \), proving the lower semi-continuity.

\[ \Box \]
4.3 The extended maximal invariant set

We are now able to define the subset of $\mathbb{P}M$ which extends the lifted maximal invariant set and depends upper-semicontinuously on $X$.

**Definition 10.** Let $U$ be a compact region and $X$ a vector field whose singular points are hyperbolic. Then the set

$$B(X, U) = \Lambda_{P, U} \cup \bigcup_{\sigma \in Sing(X) \cap U} P_{\sigma, U} \subset \mathbb{P}M$$

is called the extended maximal invariant set of $X$ in $U$.

**Proposition 20.** Let $U$ be a compact region and $X$ a vector field whose singular points are hyperbolic. Then the extended maximal invariant set $B(X, U)$ of $X$ in $U$ is a compact subset of $\mathbb{P}M$, invariant under the flow $\phi_P$. Furthermore, the map $X \mapsto B(X, U)$ depends upper semi-continuously on $X$.

**Proof.** First notice that the singular points of $Y$ in $U$ consists in finitely many hyperbolic singularities varying continuously with $Y$ in a neighborhood of $X$. The extended maximal invariant set is compact as being the union of finitely many compact sets.

Let $Y_n$ be a sequence of vector fields tending to $X$ in the $C^1$-topology, and let $(x_n, L_n) \in B(Y_n, U)$. Up to considering a subsequence we may assume that $(x_n, L_n)$ tends to a point $(x, L) \in \mathbb{P}M$ and we need to prove that $(x, L)$ belongs to $B(X, U)$.

First assume that $x \notin Sing(X)$. Then, for $n$ large, $x_n$ is not a singular point for $Y_n$ so that $L_n = < Y_n(x_n) >$ and therefore $L = < X(x) >$ belongs to $B(X, U)$, concluding.

Thus we may assume $x = \sigma \in Sing(X)$. First notice that, if for infinitely many $n$, $x_n$ is a singularity of $Y_n$ then $L_n$ belongs to $P_{\sigma, Y_n, U}$. As $P_{\sigma, Y, U}$ varies upper semi-continuously with $Y$, we conclude that $L$ belongs to $P_{\sigma, X, U}$, concluding.

So we may assume that $x_n \notin Sing(Y)$. We fix a neighborhood $V$ of $\sigma$ endowed with coordinates, so that $X$ (and therefore $Y_n$ for large $n$) is very close to its linear part in $V$. Let $S_X \subset W_{loc}^u(\sigma)$ be a sphere transverse to $X$ and cutting every orbit in $W_{loc}^u(\sigma) \setminus \{\sigma\}$, and let $W$ be a small neighborhood of $S_X$.

First assume that, for infinitely many $n$, the point $x_n$ does not belong $W^u(\sigma)$. There is a sequence $t_n > 0$ with the following property:

- $\phi_{Y_n}^{-t_n}(x_n) \in V$ for all $t \in [0, t_n]$
- $\phi_{Y_n}^{-t_n}(x_n) \in W$
- $t_n$ tends to $+\infty$ as $n \to \infty$.

Up to considering a subsequence, one may assume that the points $y_n = \phi_{Y_n}^{-t_n}(x_n)$ tend to a point $y \in W^s(\sigma)$.
Claim. The point $y$ does not belong to $W_{\sigma,U}^{ss}$.

Proof. By definition of the escaping strong stable manifold, for every $y \in W_{\sigma,U}^{ss}$ there is $t$ so that $\phi^t(y) \notin U$; thus $\phi^t_{|_{Y_n}}(y_n)$ do not belong to $U$ for $y_n$ close enough to $y$; in particular $y_n \notin \Lambda_{Y_n,U}$. □

Thus $y$ do not belong to $W_{\sigma,U}^{ss}$. Choosing $T > 0$ large enough, one gets that the line $\langle X(z) \rangle, z = \phi^T(y)$, is almost tangent to $E^c = E_{\sigma,U}^s \oplus E_{\sigma,U}^{uu}$. As a consequence, for $n$ large, one gets that $\langle Y_n(z_n) \rangle$, where $z_n = \phi^T_{|_{Y_n}}(y_n)$, is almost tangent to the continuation $E_{n}^{cu}$ of $E^c$ for $\sigma_n,Y_n$. Let $x_n = \phi^T_{|_{Y_n}}(y_n)$, and as $t_n - T \to +\infty$, the dominated splitting implies that $L_n = \langle Y_n(x_n) \rangle$ is almost tangent to $E_{\sigma,U}^{cu}$.

This shows that $L$ belongs to $E^c$. Notice that this also holds if $x_n$ belong to the unstable manifold of $\sigma_{V_n}$.

Arguing analogously we get that $L$ belongs to $E^{cs} = E_{\sigma,U}^s \oplus E_{\sigma,U}^{ss}$. Thus $L$ belongs to $E_{\sigma,U}^{c}$ concluding. □

5 Multisingular hyperbolicity

5.1 Reparametrizing cocycle associated to a singular point

Let $X$ be a $C^1$ vector field, $\phi^t$ its flows, and $\sigma$ be a hyperbolic zero of $X$. We denote by $\Lambda_X \subset PM$ the union

$$\Lambda_X = \overline{\{RX(x), x \notin Zero(X)\}} \cup \bigcup_{x \in Zero(X)} PT_x M.$$  

It can be shone easily that this set is upper semi-continuous, as in the case of $B(X,U)$ (see [20]).

Lemma 21. $\Lambda_X$ is a compact subset of $PM$ invariant under the flow $\phi^t_{|_{\sigma}}$, and the map $X \mapsto \Lambda_X$ is upper semi-continuous. Finally, if the zeros of $X$ are hyperbolic then, for any compact regions one has $B(X,U) \subset \Lambda_X$.

Let $U_\sigma$ be a compact neighborhood of $\sigma$ on which $\sigma$ is the maximal invariant.

Let $V_\sigma$ be a compact neighborhood of $Zero(X) \setminus \{\sigma\}$ so that $V_\sigma \cap U_\sigma = \emptyset$. We fix a $(C^1)$ Riemannian metric $\|\cdot\|$ on $M$ so that

$$\|X(x)\| = 1$$

for all $x \in M \setminus (U_\sigma \cup V_\sigma)$.

Consider the map $h: \Lambda_X \times \mathbb{R} \to (0, +\infty), h(L,t) = h^t(L)$, defined as follows:

- if $L \in PT_x M$ with $x \notin U_\sigma$ and $\phi^t(x) \notin U_\sigma$, then $h^t(L) = 1$;
- if $L \in PT_x M$ with $x \in U_\sigma$ and $\phi^t(x) \notin U_\sigma$ then $L = RX(x)$ and $h^t(L) = \frac{1}{\|X(x)\|}$;
- if $L \in PT_x M$ with $x \notin U_\sigma$ and $\phi^t(x) \in U_\sigma$ then $L = RX(x)$ and $h^t(L) = \|X(\phi^t(x))\|$. 

22
• if $L \in \mathbb{P}_{T_x}M$ with $x \in U_\sigma$ and $\phi^t(x) \in U_\sigma$ but $x \neq \sigma$ then $L = \mathbb{R}X(x)$ and $h^t(L) = \frac{\|X(\phi^t(x))\|}{\|X(x)\|}$;

• if $L \in \mathbb{P}_{T_\sigma}M$ then $h^t(L) = \frac{\|\phi^t(u)\|}{\|u\|}$ where $u$ is a vector in $L$.

Note that the case in which $x$ is not the singularity and $x \in U_\sigma$ can be written as in the last item by taking $u = X(x)$.

Figure 4: the local cocycle $h^t_\sigma$ associated to the singularity $\sigma = \sigma_0$

**Lemma 22.** With the notation above, the map $h$ is a (continuous) cocycle on $\Lambda_X$.

**Proof.** The continuity of $h$ comes from the continuity of the norm and the fact that the neighborhoods $U$ and $V$ do not intersect. Now we aim to show that $h$ verifies the cocycle relation:

$$h^t(\phi^s(L))h^s(L) = h^{t+s}(L)$$

• if $L \in \mathbb{P}_{T_x}M$ with $x \notin U_\sigma$, $\phi^s(x) \notin U_\sigma$, $\phi^{s+t}(x) \notin U_\sigma$, then $h^{t+s}(L) = h^t(\phi^s(L))h^s(L) = 1$;

• Let $L \in \mathbb{P}_{T_x}M$ with $x \notin U_\sigma$, $\phi^s(x) \notin U_\sigma$, $\phi^{s+t}(x) \in U_\sigma$. Since $\|X(\phi^s(u))\| = 1$ then $h^s(L) = 1$ and,

$$h^t(\phi^s(L))h^s(L) = \|X(\phi^t(\phi^s(u)))\|$$

$$= \|X(\phi^{t+s}(u))\|$$

$$= h^{t+s}(L),$$

23
\begin{itemize}

- if \( L \in \mathcal{PT}_x M \) with \( x \notin U_\sigma \), \( \phi^t(x) \in U_\sigma \) and \( \phi^{t+s}(x) \notin U_\sigma \) then \( L = \mathbb{R}X(x) \),

\[ h^t(\phi^t_0(L))h^s(\phi^s) = \frac{1}{\|X(\phi^s(x))\|}\|X(\phi^s(x))\| = 1 = h^{t+s}(L) \]

- if \( L \in \mathcal{PT}_x M \) with \( x \notin U_\sigma \), \( \phi^t(x) \in U_\sigma \) and \( \phi^{t+s}(x) \in U_\sigma \) then \( L = \mathbb{R}X(x) \),

\[ h^t(\phi^t_0(L))h^s(\phi^s) = \frac{\|X(\phi^t(\phi^s(x)))\|}{\|X(\phi^s(x))\|}\|X(\phi^s(x))\| = \|X(\phi^{t+s}(x))\| = 1 = h^{t+s}(L) \]

- if \( L \in \mathcal{PT}_x M \) with \( x \in U_\sigma \), \( \phi^t(x) \notin U_\sigma \) and \( \phi^{t+s}(x) \notin U_\sigma \), then \( h^t(\phi^t_0(L)) = 1 \)

\[ h^t(\phi^t_0(L))h^s(\phi^s) = \frac{1}{\|X(x)\|} = h^{t+s}(L) \]

- Let \( L \in \mathcal{PT}_x M \) with \( x \in U_\sigma \), \( \phi^t(x) \notin U_\sigma \), \( \phi^{t+s}(x) \in U_\sigma \). Since \( h^s(L) = \frac{1}{\|X(x)\|} \) then,

\[ h^t(\phi^t_0(L))h^s(\phi^s) = \frac{\|X(\phi^t(\phi^s(x)))\|}{\|X(x)\|} = \frac{1}{\|X(x)\|} = h^{t+s}(L) \]

- if \( L \in \mathcal{PT}_x M \) with \( x \in U_\sigma \), \( \phi^t(x) \in U_\sigma \) and \( \phi^{t+s}(x) \notin U_\sigma \) then \( L = \mathbb{R}X(x) \),

\[ h^t(\phi^t_0(L)) = \frac{1}{\|X(\phi^t(\phi^s(x)))\|} \]

\[ h^t(\phi^t_0(L))h^s(\phi^s) = \frac{\|X(\phi^t(x))\|}{\|X(x)\|} \frac{1}{\|X(\phi^s(x))\|} = \frac{1}{\|X(x)\|} = h^{t+s}(L) \]

\end{itemize}
• if \( L \in \mathbb{P}T_xM \) with \( x \in U_\sigma, \phi^s(x) \in U_\sigma \) and \( \phi^{t+s}(x) \in U_\sigma \) then \( L = \mathbb{R}X(x) \),

\[
h'_t(\phi^s_x(L))h^s(L) = \frac{\|X(\phi^t(\phi^s_x(x)))\| \|X(\phi^s(x))\|}{\|X(x)\|},
\]

\[
= \frac{\|X(\phi^t(\phi^s_x(x)))\|}{\|X(x)\|},
\]

\[
= \frac{\|X(\phi^{t+s}(x))\|}{\|X(x)\|},
\]

\[
= h^{t+s}(L)
\]

• if \( L \in \mathbb{P}T_\sigma M \), let \( u \) be a vector in \( L \), then

\[
h^{t+s}(L) = \frac{\|D\phi^{t+s}_x(u)\|}{\|u\|};
\]

\[
= \frac{\|D\phi^{t+s}_x(D\phi^s_x(u))\| \|D\phi^s_x(u)\|}{\|u\|},
\]

\[
= h^t(\phi^s_x(L))h^s(L)
\]

\[\square\]

Lemma 23. The cohomology class of a cocycle \( h \) defined as above, is independent from the choice of the metric \( \|\cdot\| \) and of the neighborhoods \( U_\sigma \) and \( V_\sigma \).

Proof. Let \( \|\cdot\| \) and \( \|\cdot\|' \) be two different metrics and 2 different sets of neighborhoods of \( \sigma \) and \( \text{Zero}(X) \setminus \{\sigma\} \) such that:

• \( V_\sigma \cap U_\sigma = \emptyset \).
• \( V'_\sigma \cap U'_\sigma = \emptyset \).
• \( V'_\sigma \cap U_\sigma = \emptyset \) and \( V_\sigma \cap U'_\sigma = \emptyset \).
• \( \|X(x)\| = 1 \) for all \( x \in M \setminus (U_\sigma \cup V_\sigma) \),
• \( \|X(x)\|' = 1 \) for all \( x \in M \setminus (U'_\sigma \cup V'_\sigma) \).

We define \( h \) as above for the metric \( \|\cdot\| \) and \( h' \) as above for the metric \( \|\cdot\|' \). We define a function \( g: B(X, U) \to (0, +\infty) \)

• if \( L \in \mathbb{P}T_xM \) with \( x \notin V'_\sigma \cup V_\sigma \) then \( g(L) = \frac{\|u\|'}{\|u\|} \) with \( u \) a non vanishing vector in \( L \),
• and if \( L \in \mathbb{P}T_\sigma M \) with \( x \in V'_\sigma \cup V_\sigma \), then \( g(L) = 1 \).

Claim. The function \( g: B(X, U) \to (0, +\infty) \) defined above is continuous

25
Proof. Since $V'_\sigma \cap U_\sigma = \emptyset$ and $V_\sigma \cap U'_\sigma = \emptyset$, the continuity in the boundary of $V \cup V'$ comes from the fact that $\| \cdot \|$ and $\| \cdot \|'$ are 1 out of $U \cup U'$. Also since $V'_\sigma \cap U_\sigma = \emptyset$ and $V_\sigma \cap U'_\sigma = \emptyset$, The continuity of the norms $\| \cdot \|$ and $\| \cdot \|'$, and the fact that they are 1 out of $U \cup U'$, gives us the continuity in the boundary of $U \cup U'$.

The following claim will show us that the functions $h$ and $h'$ differ in a coboundary defined as $g^i(L) = \frac{g(D\phi^i(x))}{g(u)}$.

Claim. The functions $h$ and $h'$ are such that $h''(u) = h'(u)\frac{g(D\phi^i(x))}{g(u)}$.

Proof. 

• For the metric $\| \cdot \|$ and $L \in \mathbb{P}T_x M$ with $x \notin U_\sigma \cup U'_\sigma$ and $\phi^i(x) \notin U_\sigma \cup U'_\sigma$, $g^i(L) = 1$. On the other side $h''(L) = 1$ as desired.

• If $L \in \mathbb{P}T_x M$ with $x \notin U_\sigma \cup U'_\sigma$ and $\phi^i(x) \notin U_\sigma \cup U'_\sigma$, then $g^i(L) = \frac{1}{\|X(x)\|}$.

  Take $u = X(x)$, $h''(L) = \frac{1}{\|X(x)\|^n}$.

  $h''(L) = h'(L)\frac{\|X(x)\|}{\|X(x)\|^n}.$

• If $L \in \mathbb{P}T_x M$ with $x \notin U_\sigma \cup U'_\sigma$ and $\phi^i(x) \in U_\sigma \cup U'_\sigma$, then $L = \mathbb{R}X(x)$ and $g^i(L) = \frac{\|D\phi^i(x)\|}{\|D\phi^i(x)\|'}$. Then since $h''(L) = \|D\phi^i(x)\|$, then

  $h''(L) = h'(L)\frac{\|D\phi^i(x)\|'}{\|D\phi^i(x)\|}.$

• If $L \in \mathbb{P}T_x M \cap B(X, U)$ with $x \in U_\sigma$ and $\phi^i(x) \in U_\sigma$. Take $u = X(x)$, then

  $g^i(L)\frac{\|D\phi^i(x)\|'}{\|D\phi^i(x)\|}.$

  and $h''(L) = \frac{D\phi^i(x)}{\|u\|'}$. So $h''(L) = h'(L)g^i(L)$.

Now in order to finish the proof we need to show that assuming the condition that the norms where such that $V'_\sigma \cap U_\sigma = \emptyset$ and $V_\sigma \cap U'_\sigma = \emptyset$ does not make us loose generality. For this, suppose we started with any other norm $\| \cdot \|''$ and that there exist 2 neighborhood such that.

• $V'' \cap U'' = \emptyset$.

• $\|X(x)\|'' = 1$ for all $x \in M \setminus (U'' \cup V'')$.

26
Let us choose a smaller neighborhood $V''_\sigma \subset V''_\sigma$. This satisfies $V''_\sigma \cap U''_\sigma = \emptyset$. Analogously $U'_\sigma \subset U''_\sigma$ will satisfy $V''_\sigma \cap U'_\sigma = \emptyset$. Now if we choose this neighborhoods $V'$ and $U'$ as small as we want, and a norm norm $\|\cdot\|$ such that $\|X(x)\|' = 1$ for all $x \in M \setminus (U''_\sigma \cup V''_\sigma)$. the claims above implies that the corresponding $h''$ and $h'$ differ in a coboundary. Therefore $h'$ can be chosen so that $h''$ and $\tilde{h}$ differ in a coboundary. \hfill $\Box$

We denote by $[h(X,\sigma)]$ the cohomology class of any cocycle defined as $h$ above.

**Lemma 24.** Consider a vector field $X$ and a hyperbolic zero $\sigma$ of $X$. Then there is a $C^1$-neighborhood $U$ of $X$ so that $\sigma$ has a well defined hyperbolic continuation $\sigma_Y$ for $Y$ in $U$ and for any $Y \in U$ there is a map $h_Y : \Lambda_Y \times \mathbb{R} \to (0, +\infty)$ so that

- for any $Y$, $h_Y$ is a cocycle belonging to the cohomology class $[h(Y,\sigma_Y)]$
- $h_Y$ depends continuously on $Y$: if $Y_n \in U$ converge to $Z \in U$ for the $C^1$-topology and if $L_n \in \Lambda_Y$, converge to $L \in \Lambda_Z$ then $h_Y(L_n)$ tends to $h_Y(L)$ for every $t \in \mathbb{R}$; furthermore this convergence is uniform in $t \in [-1, 1]$. 

**Proof.** The manifold $M$ is endowed with a Riemannian metric $\|\cdot\|$. We fix the neighborhoods $U_\sigma$ and $V_\sigma$ for $X$ and $U$ is a $C^1$-neighborhood of $X$ so that $\sigma_Y$ is the maximal invariant set for $Y$ in $U_\sigma$ and $\text{Zero}(Y) \setminus \{\sigma_Y\}$ is contained in the interior of $V_\sigma$. Up to shrink $U$ if necessary, we also assume that there are compact neighborhoods $\bar{U}_\sigma$ of $\sigma_Y$ contained in the interior of $U_\sigma$ and $\bar{V}_\sigma$ of $\text{Zero}(Y) \setminus \{\sigma_Y\}$ contained in the interior of $V_\sigma$.

We fix a continuous function $\xi : M \to [0, 1]$ so that $\xi(x) = 1$ for $x \in M \setminus (U_\sigma \cup V_\sigma)$ and $\xi(x) = 0$ for $x \in \bar{U}_\sigma \cup \bar{V}_\sigma$.

For any $Y \in U$ we consider the map $\eta_Y : M \to (0, +\infty)$ defined by

$$\eta_Y(x) = \frac{\xi(x)}{\|X(x)\|} + 1 - \xi(x).$$

This map is a priori not defined on $\text{Zero}(Y)$ but extends by continuity on $y \in \text{Zero}(Y)$ by $\eta_Y(y) = 1$, and is continuous.

This maps depends continuously on $Y$. Now we consider the metric $\|\cdot\|_Y = \eta_Y \|\cdot\|$. Note that $\|Y(x)\|_Y = 1$ for $x \in M \setminus (U_\sigma \cup V_\sigma)$.

Now $h_Y$ is the cocycle built at lemma 23 for $U_\sigma, V_\sigma$ and $\|\cdot\|_Y$.

Notice that, according to Remark 11 if $\sigma_1, \ldots, \sigma_k$ are hyperbolic zeros of $X$ the homology class of the product cocycle $h^t_{\sigma_1} \cdots h^t_{\sigma_k}$ is well defined, and admits representatives varying continuously with the flow.

We are now ready for defining our notion of multisingular hyperbolicity.
5.2 Definition of multisingular hyperbolicity

**Definition 11.** Let $X$ be a $C^1$-vector field on a compact manifold and let $U$ be a compact region. One says that $X$ is *multisingular hyperbolic* on $U$ if

1. Every zero of $X$ in $U$ is hyperbolic. We denote $S = \text{Zero}(X) \cap U$.

2. The restriction of the extended linear Poincaré flow $\{\psi^t_{X}\}$ to the extended maximal invariant set $B(X, U)$ admits a dominated splitting $N_L = E_L \oplus F_L$.

3. There is a subset $S_E \subset S$ so that the reparametrized cocycle $h^t_{E} \psi^t_X$ is uniformly contracted in restriction to the bundles $E$ over $B(X, U)$ where $h_E$ denotes

$$h_E = \Pi_{\sigma \in S_E} h_\sigma.$$ 

4. There is a subset $S_F \subset S$ so that the reparametrized cocycle $h^t_{F} \psi^t_X$ is uniformly expanded in restriction to the bundles $F$ over $B(X, U)$ where $h_F$ denotes

$$h_F = \Pi_{\sigma \in S_F} h_\sigma.$$ 

**Remark 25.** The subsets $S_E$ and $S_F$ are not necessarily uniquely defined, leading to several notions of multisingular hyperbolicity. We can also modify slightly this definition allowing to consider the product of power of the $h_\sigma$. In that case $\tilde{h}_E$ would be on the form

$$h^t_{E} = \Pi_{\sigma \in S_E} (h^t_\sigma)^{\alpha_{E}(\sigma)}$$

where $\alpha_{E}(\sigma) \in \mathbb{R}$.

Our first main result is now

**Theorem 26.** Let $X$ be a $C^1$-vector field on a compact manifold $M$ and let $U \subset M$ be a compact region. Assume that $X$ is multisingular hyperbolic on $U$. Then $X$ is a star flow on $U$, that is, there is a $C^1$-neighborhood $U_0$ of $X$ so that every periodic orbit contained in $U$ of a vector field $Y \in U$ is hyperbolic. Furthermore $Y \in U$ is multisingular hyperbolic in $U$.

**Proof.** Recall that the extended maximal invariant set $B(Y, U)$ varies upper semicontinuously with $Y$ for the $C^1$-topology. Therefore, according to Proposition 7 there is a $C^1$-neighborhood $U_0$ of $X$ so that, for every $Y \in U_0$ the extended linear Poincaré flow $\psi^t_{X,Y}$ admits a dominated splitting $E \oplus F$ over $B(Y, U)$, whose dimensions are independent of $Y$ and whose bundles vary continuously with $Y$.

Now let $S_E$ and $S_F$ be the sets of singular point of $X$ in the definition of singular hyperbolicity. Now Lemma 24 allows us to choose two families of cocycles $h^t_{E,Y}$ and $h^t_{F,Y}$ depending continuously on $Y$ in a small neighborhood $U_1$ of $X$ and which belongs to the product of the cohomology class of cocycles.
associated to the singularities in $S_E$ and $S_F$, respectively. Thus the linear cocycles
\[ h^t_{E,Y} \cdot \psi^t_{N,Y} |_{E,Y}, \quad \text{over } B(Y,U) \]
varies continuously with $Y$ in $U$, and is uniformly contracted for $X$. Thus, it is uniformly contracted for $Y$ in a $C^1$-neighborhood of $X$.

One shows in the same way that
\[ h^t_{F,Y} \cdot \psi^t_{N,Y} |_{F,Y}, \quad \text{over } B(Y,U) \]
is uniformly expanded for $Y$ in a small neighborhood of $X$.

We just prove that there is a neighborhood $U$ of $X$ so that $Y \in U$ is multisingular hyperbolic in $U$.

Consider a (regular) periodic orbit $\gamma$ of $Y$ ant let $\pi$ be its period. Just by construction of the cocycles $h_E$ and $h_F$, one check that
\[ h^\pi_E(\gamma(0)) = h^\pi_F(\gamma(0)) = 1. \]
One deduces that the linear Poincaré flow is uniformly hyperbolic along $\gamma$ so that $\gamma$ is hyperbolic, ending the proof.

5.3 The multisingular hyperbolic structures over a singular point

The aim of this section is next proposition

Proposition 27. Let $X$ be a $C^1$-vector field on a compact manifold and $U \subset M$ a compact region. Assume that $X$ is multisingular hyperbolic in $U$ and let $i$ denote the dimension of the stable bundle of the reparametrized extended linear Poincaré flow.

let $\sigma$ be a zero of $X$. Then

\begin{itemize}
  \item either at least one entire invariant (stable or unstable) manifold of $\sigma$ is escaping from $U$.
  \item or $\sigma$ is Lorenz like, more precisely
    \begin{itemize}
      \item either the stable index is $i + 1$, the center space $E^c_{\pi U}$ contains exactly one stable direction $E^s_1$ ($\dim E^s_1 = 1$) and $E^c_{\pi} \oplus E^u(\sigma)$ is sectionally dissipative; in this case $\sigma \in S_F$.
      \item or the stable index is $i$, the center space $E^c_{\pi U}$ contains exactly one unstable direction $E^u_1$ ($\dim E^u_1 = 1$) and $E^c_{\pi} \oplus E^s(\sigma)$ is sectionally contracting; then $\sigma \in S_E$.
    \end{itemize}
\end{itemize}

Note that in the first case of this proposition the class of the singularity must be trivial. If it was not, the regular orbits of the class that accumulate on $\sigma$, entering $U$, would accumulate on an orbit of the stable manifold. Therefore the
stable manifold could not be completely escaping. The same reasoning holds for the unstable manifold.

Let \( E^s_k \oplus < \cdots \oplus < E^s_1 \oplus < E^u_1 \oplus < \cdots \oplus < E^u_\ell \) be the finest dominated splitting of the flow over \( \sigma \). For the proof, we will assume, in the rest of the section that the class of \( \sigma \) is not trivial, and therefore we are not in the first case of our previous proposition. In other word, we assume that there are \( a > 0, b > 0 \) so that

\[
E^c_{\sigma,U} = E^s_a \oplus < \cdots \oplus < E^s_1 \oplus < E^u_1 \oplus < \cdots \oplus < E^u_b.
\]

We assume that \( X \) is multisingular hyperbolic of \( s \)-index \( i \) and we denote by \( E \oplus < F \) the corresponding dominated splitting of the extended linear Poincaré flow over \( B(X,U) \).

**Lemma 28.** Let \( X \) be a \( C^1 \)-vector field on a compact manifold and \( U \subset M \) a compact region. Assume that \( X \) is multisingular hyperbolic in \( U \) and let \( i \) denote the dimension of the stable bundle of the reparametrized extended linear Poincaré flow.

let \( \sigma \) be a zero of \( X \). Then with the notation above,

- either \( i = \dim E \leq \dim E^s_a \oplus \cdots \oplus \dim E^s_{a+1} \) (i.e. the dimension of \( E \) is smaller or equal than the dimension of the biggest stable escaping space).

- or \( \dim M - i - 1 = \dim F \leq \dim E^u_1 \oplus \cdots \oplus \dim E^u_{b+1} \) (i.e. the dimension of \( F \) is smaller or equal than the dimension of the biggest unstable escaping space).

**Proof.** One argues by contradiction. One consider \( L^s, L^u \in P^c_{\sigma,U} \) so that \( L^s \) corresponds to a line in \( E^s_a \) and \( L^u \) a line in \( E^u_b \). Assuming that the conclusion of the lemma is wrong, one gets that the projection of \( E^u_b \) on the normal space \( N_{L^s} \) is contained in \( F(L^s) \) and the projection of \( E^s_a \) on the normal space \( N_{L^u} \) is contained in \( E(L^u) \).

There is \( L \in P^c_{\sigma,U} \), corresponding to a line in \( E^s_a \oplus E^u_b \) and there are times \( r_n, s_n \) tending to +\( \infty \) so that \( L_{-n} = \phi_{-r_n}^{-1}(L) \to L^s \) and \( L_n = \phi_{s_n}^r \to L^u \).

**Claim 29.** Given any \( T > 0 \) there is \( n \) and there are vectors \( u_n \) of the normal space \( N_{L_{-n}} \) to \( L_{-n} \) so that the expansion of \( u_n \) by \( \psi_T^{L_{-n}} \) is larger that \( \frac{1}{7} \) times the minimum expansion in \( F(L_{-n}) \) and the contraction of the vector \( \psi_T^{L_{-n}}(u_n) \) by \( \psi_T^{L_n} \) is less than \( 2 \) times the maximal expansion in \( E(L_n) \).

The existence of such vectors \( u_n \) contradicts the domination \( E \oplus < F \) ending the proof.

According to Lemma 28 we now assume that \( i \leq \dim E^s_k + \oplus + \dim E^s_{a+1} \) (the other case is analogous, changing \( X \) by \(-X\)).

**Lemma 30.** With the hypothesis above, for every \( L \in P^c_{\sigma,U} \) the projection of \( E^c_{\sigma,U} \) on the normal space \( N_L \) is contained in \( F(L) \).
Proof. It is because the projection of $E_k^s \oplus \cdots \oplus E_{a+1}^s$ has dimension at least the dimension $i$ of $E$ and hence contains $E(L)$. Thus the projection of $E_{a,U}^s$ is transverse to $E$. As the projection of $E_{a,U}^s$ on $N_L$ defines a $\phi_l^t$-invariant bundle over the $\phi_l^t$-invariant compact set $\mathbb{F}_{σ,U}^c$, one concludes that the projection is contained in $F$.

As a consequence the bundle $F$ is not uniformly expanded on $\mathbb{F}_{σ,U}^c$ for the extended linear Poincaré flow. As it is expanded by the reparametrized flow, this implies $σ ∈ S_F$.

Consider now $L ∈ E_a^s$. Then $ψ_t^N$ in restriction to the projection of $E_{a,U}^c$ on $N_L$ consists in multiplying the natural action of the derivative by the exponential contraction along $L$. As it is included in $F$, the multisingular hyperbolicity implies that it is a uniform expansion: this means that

- $L$ is the unique contracting direction in $E_{σ,U}^s$: in other words, $a = 1$ and $\text{dim} E^s - a = 1$.
- the contraction along $a$ is less than the expansion in the $E^u_j$, $j > 1$. In other words $E_{σ,U}^c$ is sectionally expanding.

For ending the proof of the Proposition 27 it remains to check the $s$-index of $σ$: at $L ∈ E_a^s$ one gets that $F(L)$ is isomorphic to $E_1^u \oplus \cdots \oplus E_ℓ^u$ so that the $s$-index of $σ$ is $i + 1$, ending the proof.

6 The multisingular hyperbolicity is a necessary condition for star flows: Proof of theorem 4

The aim of this section is to prove Lemma 31 below:

Lemma 31. Let $X$ be a generic star vector field on $M$. Consider a chain-recurrent class $C$ of $X$. Then there is a filtrating neighborhood $U$ of $C$ so that the extended maximal invariant set $B(X,U)$ is multisingular hyperbolic.

Notice that, as the multisingularity of $B(X,U)$ is a robust property, Lemma 31 implies Theorem 4.

As already mentioned, the proof of Lemma 31 consists essentially in recovering the results in [GSW] and adjusting few of them to the new setting. So we start by recalling several of the results from or used in [GSW].

To start we state the following properties of star flows:

Lemma 32 ([L Ma2]). For any star vector field $X$ on a closed manifold $M$, there is a $C^1$ neighborhood $U$ of $X$ and numbers $η > 0$ and $T > 0$ such that, for any periodic orbit $γ$ of a vector field $Y ∈ U$ and any integer $m > 0$, let $N = N_s \oplus N_u$ be the stable-unstable splitting of the normal bundle $N$ for the linear Poincaré flow $ψ_t^Y$ then:
• Domination: For every \( x \in \gamma \) and \( t \geq T \), one has
\[
\frac{\| \psi_\gamma^Y |_{N_x} \|}{\min(\psi_\gamma^Y |_{N_x})} \leq e^{-2\eta t}
\]
• Uniform hyperbolicity at the period: at the period If the period \( \pi(\gamma) \) is larger than \( T \) then, for every \( x \in \gamma \), one has:
\[
\Pi_{i=0}^{(m\pi(\gamma)/T) - 1} \| \psi_\gamma^Y |_{N_x} (\phi_{i+T}^Y(x)) \| \leq e^{-m\eta\pi(\gamma)}
\]
and
\[
\Pi_{i=0}^{(m\pi(\gamma)/T) - 1} \min(\psi_\gamma^Y |_{N_x} (\phi_{i+T}^Y(x))) \geq e^{m\eta\pi(\gamma)}.
\]
Here \( \min(A) \) is the mini-norm of \( A \), i.e., \( \min(A) = \| A^{-1} \|^{-1} \).

Now we need some generic properties for flows:

**Lemma 33** ([C] [BGY]). There is a \( C^1 \)-dense \( G_\delta \) subset \( \mathcal{G} \) in the \( C^1 \)-open set of star flows of \( M \) such that, for every \( X \in \mathcal{G} \), one has:

• Every critical element (zero or periodic orbit) of \( X \) is hyperbolic and therefore admits a well defined continuation in a \( C^1 \)-neighborhood of \( X \).

• For every critical element \( p \) of \( X \), the Chain Recurrent Class \( C(p) \) is continuous at \( X \) in the Hausdorff topology;

• If \( p \) and \( q \) are two critical elements of \( X \), such that \( C(p) = C(q) \) then there is a \( C^1 \) neighborhood \( U \) of \( X \) such that the chain recurrent class of \( p \) and \( q \) still coincide for every \( Y \in U \)

• For any nontrivial chain recurrent class \( C \) of \( X \), there exists a sequence of periodic orbits \( Q_n \) such that \( Q_n \) tends to \( C \) in the Hausdorff topology.

**Lemma 34** (lemma 4.2 in [GSW]). Let \( X \) be a star flow in \( M \) and \( \sigma \in \text{Sing}(X) \). Assume that the Lyapunov exponents of \( \phi_\tau(\sigma) \) are
\[
\lambda_1 \leq \cdots \leq \lambda_{s-1} \leq \lambda_s < 0 < \lambda_{s+1} \leq \lambda_{s+2} \leq \cdots \leq \lambda_d
\]
If the chain recurrence class \( C(\sigma) \) of \( \sigma \), is nontrivial, then:

• either \( \lambda_{s-1} \neq \lambda_s \) or \( \lambda_{s+1} \neq \lambda_{s+2} \).

• if \( \lambda_{s-1} = \lambda_s \), then \( \lambda_s + \lambda_{s+1} < 0 \).

• if \( \lambda_{s+1} = \lambda_{s+2} \), \( \lambda_s + \lambda_{s+1} > 0 \).

• if \( \lambda_{s-1} \neq \lambda_s \) and \( \lambda_{s+1} \neq \lambda_{s+2} \), then \( \lambda_s + \lambda_{s+1} \neq 0 \).
We say that a singularity $\sigma$ in the conditions of the previous lemma is Lorenz like of index $s$ and we define the saddle value of a singularity as the value

$$sv(\sigma) = \lambda_s + \lambda_{s+1}.$$ 

Consider a Lorenz like singularity $\sigma$, then:

- if $sv(\sigma) > 0$, we consider the splitting

  $$T_\sigma M = G^{ss}_\sigma \oplus G^{cu}_\sigma$$

  where (using the notations of Lemma 34) the space $G^{ss}_\sigma$ corresponds to the Lyapunov exponents $\lambda_1$ to $\lambda_{s-1}$, and $G^{cu}_\sigma$ corresponds to the Lyapunov exponents $\lambda_s, \ldots, \lambda_d$.

- if $sv(\sigma) < 0$, we consider the splitting

  $$T_\sigma M = G^{cs}_\sigma \oplus G^{uu}_\sigma$$

  where the space $G^{cs}_\sigma$ corresponds to the Lyapunov exponents $\lambda_1$ to $\lambda_{s+1}$, and $G^{uu}_\sigma$ corresponds to the Lyapunov exponents $\lambda_{s+2}, \ldots, \lambda_d$.

**Corollary 35.** Let $X$ be a vector field and $\sigma$ be a Lorenz-like singularity of $X$ and let $h_\sigma : \Lambda_X \times \mathbb{R} \to (0, +\infty)$ be a cocyle in the cohomology class $[h_\sigma]$ defined in Section 5.1.

1. First assume that $\text{Ind}(\sigma) = s + 1$ and $sv(\sigma) > 0$. Then the restriction of $\psi_N$ over $\mathbb{P}G^{cu}_\sigma$ admits a dominated splitting $N_L = E_L \oplus F_L$, with $\dim(E_L) = s$, for $L \in \mathbb{P}G^{cu}_\sigma$. Furthermore,

   - $E$ is uniformly contracting for $\psi_N$
   - $F$ is uniformly expanding for the reparametrized extended linear Poincaré flow $h_\sigma \cdot \psi_N$.

2. Assume now that $\text{Ind}(\sigma) = s$ and $sv(\sigma) < 0$. One gets a dominated splitting $N_L = E_L \oplus F_L$ for $L \in \mathbb{P}G^{cs}_\sigma$ so that $\dim(E_L) = s$, the bundle $F$ is uniformly expanded under $\psi_N$ and $E$ is uniformly contracted by $h_\sigma \cdot \psi_N$.

**Proof.** We only consider the first case $\text{Ind}(\sigma) = s + 1$ and $sv(\sigma) > 0$, the other is analogous and can be deduced by reversing the time.

We consider the restriction of $\psi_N$ over $\mathbb{P}G^{cu}_\sigma$, that is, for point $L \in \Lambda_X$ corresponding to lines contained in $G^{cu}_\sigma$. Therefore the normal space $N_L$ can be identified, up to a projection which is uniformly bounded, to the direct sum of $G^{ss}_\sigma$ with the normal space of $L$ in $G^{cu}_\sigma$.

Now we fix $E_L = G^{ss}_\sigma$ and $F_L$ is the normal space of $L$ in $G^{cu}_\sigma$. As $G^{ss}_\sigma$ and $G^{cu}_\sigma$ are invariant under the derivative of the flow $\phi_t$, one gets that the splitting $N_L = E_L \oplus F_L$ is invariant under the extended linear Poincaré flow over $\mathbb{P}G^{cu}_\sigma$.

Let us first prove that this splitting is dominated:
By Lemma 34 if we choose a unit vector $v$ in $E_L$ we know that for any $t > 0$ one has
\[ \| \psi_t^N(v) \| \leq Ke^{t\lambda_{s-1}}. \]

Now let us choose a unit vector $u$ in $F_L$, and consider $w_t = \psi_t^N(u) \in F_{\phi_t^L}(L)$. Then for any $t > 0$, one has
\[ \| D\phi_t^{-1}(w_t) \| \leq K'e^{t(-\lambda_s)} \| w_t \|. \]

The extended linear Poincaré flow is obtained by projecting the image by the derivative of the flow on the normal bundle. Since the projection on the normal space does not increase the norm of the vectors, one gets
\[ \| \psi_t^{-1}(w_t) \| \leq K'e^{t(-\lambda_s)} \| w_t \|, \]

is other words
\[ \frac{1}{\| \psi_t^N(u) \|} \leq K'e^{t(-\lambda_s)} \]

Putting together these inequalities one gets:
\[ \frac{\| \psi_t^N(v) \|}{\| \psi_t^N(u) \|} \leq KK'e^{t(\lambda_{s-1}-\lambda_s)}. \]

This provides the domination as $\lambda_{s-1} - \lambda_s < 0$.

Notice that $E_L = G^s_{\sigma}$ is uniformly contracted by the extended linear Poincaré flow $\psi_N$, because it coincides, on $G^s_{\sigma}$ and for $L \in PG^u_{\sigma}$, with the differential of the flow $\phi^t$. For concluding the proof, it remains to show that the reparametrized extended linear Poincaré flow $h_{\sigma} \cdot \psi_N$ expands uniformly the vectors in $F_L$, for $L \in PG^u_{\sigma}$.

Notice that, over the whole projective space $P_{\sigma}$, the cocycle $h_{\sigma,t}(L)$ is the rate of expansion of the derivative of $\phi_t$ in the direction of $L$. Therefore $h_{\sigma} \cdot \psi_N$ is defined as follows: consider a line $D \subset N_L$. Then the expansion rate of the restriction of $h_{\sigma} \cdot \psi_N$ to $D$ is the expansion rate of the area on the plane spanned by $L$ and $D$ by the derivative of $\phi_t$.

The hypothesis $\lambda_s + \lambda^{s+1} > 0$ implies that the derivative of $\phi_t$ expands uniformly the area on the planes contained in $G^u_{\sigma}$, concluding.

\[ \square \]

Lemma 36 (Lemma 4.5 and Theorem 5.7 in [GSW]). Let $X$ be a $C^1$ generic star vector field and let $\sigma \in Sing(X)$. Then there is a filtrating neighborhood $U$ of $C(\sigma)$ so that, for every two periodic points $p, q \subset U$,

\[ Ind(p) = Ind(q), \]

Furthermore, for any singularity $\sigma'$ in $U$,

\[ Ind(\sigma') = Ind(q) \quad \text{if} \; sv(\sigma) < 0, \]

or

\[ Ind(\sigma') = Ind(q) + 1 \quad \text{if} \; sv(\sigma) > 0. \]
Lemma 37. There is a dense $G_δ$ set $G$ in the set of star flows of $M$ with the following properties: Let $X$ be in $G$, let $C$ be a chain recurrent class of $X$. Then there is a (small) filtrating neighborhood $U$ of $C$ so that the lifted maximal invariant set $\tilde{Λ}(X,U)$ of $X$ in $U$ has a dominated splitting $N = E \oplus F$ for the extended linear Poincaré flow, so that $E$ extends the stable bundle for every periodic orbit $γ$ contained in $U$.

Proof. According to Lemma 36, the class $C$ admits a filtrating neighborhood $U$ in which the periodic orbits are hyperbolic and with the same index. On the other hand, according to Lemma 33, every chain recurrence class in $U$ is accumulated by periodic orbits. Since $X$ is a star flow, Lemma 32 asserts that the normal bundle over the union of these periodic orbits admits a dominated splitting for the linear Poincaré flow, corresponding to their stable/unstable splitting. It follows that the union of the corresponding orbits in the lifted maximal invariant set have a dominated splitting for $N$. Since any dominated splitting defined on an invariant set extends to the closure of this set, we have a dominated splitting on the closure of the lifted periodic orbits, and hence on the whole $Λ(X,U)$.

Lemma 37 asserts that the lifted maximal invariant set $\tilde{Λ}(X,U)$ admits a dominated splitting. What we need now is extend this dominated splitting to the extended maximal invariant set

$$B(X,U) = \tilde{Λ}(X,U) \cup \bigcup_{σ_i ∈ Zero(X) \cap U} \mathbb{P}_{σ_i,U}.$$ 

Now we need the following theorem to have more information on the projective center spaces $\mathbb{P}_{σ_i,U}$.

Lemma 38 (lemma 4.7 in [GSW]). Let $X$ be a star flow in $M$ and $σ$ be a singularity of $X$ such that $C(σ)$ is nontrivial. Then:

- if $sv(σ) > 0$, one has:
  $$W^{ss}(σ) \cap C(σ) = \{σ\},$$
  where $W^{ss}(σ)$ is the strong stable manifold associated to the space $G^{ss}_σ$.

- if $sv(σ) < 0$, then:
  $$W^{uu}(σ) \cap C(σ) = \{σ\},$$
  where $W^{uu}(σ)$ is the strong unstable manifold associated to the space $G^{uu}_σ$.

Remark 39. Consider a vector field $X$ and a hyperbolic singularity $σ$ of $X$. Assume that $W^{ss}(σ) \cap C(σ) = \{σ\}$, for a strong stable manifold $W^{ss}(σ)$, where $C(σ)$ is the chain recurrence class of $σ$.

Then there is a filtrating neighborhood $U$ of $C(σ)$ on which the strong stable manifold $W^{ss}(σ)$ is escaping from $U$ (see the definition in Section 4.3).

35
Proof. Each orbit in $W^{ss}(\sigma) \setminus \{\sigma\}$ goes out some filrating neighborhood of $C(\sigma)$ and the nearby orbits go out of the same filrating neighborhood. Notice that the space of orbits in $W^{ss}(\sigma) \setminus \{\sigma\}$ is compact, so that we can consider a finite cover of it by open sets for which the corresponding orbits go out a same filrating neighborhood of $C(\sigma)$. The announced filrating neighborhood is the intersection of these finitely many filrating neighborhoods. 

Remark 39 allows us to consider the escaping strong stable and strong unstable manifold of a singularity $\sigma$ without refering to a specific filrating neighborhood $U$ of the class $C(\sigma)$: these notions do not depend on $U$ small enough. Thus the notion of the center space $E_c^\sigma = E^c(\sigma, U)$ is also independent of $U$ for $U$ small enough. Thus we will denote

$$P_c^\sigma = P_c^\sigma, U$$

for $U$ sufficiently small neighborhood of the chain recurrence class $C(\sigma)$.

Remark 40. Lemma 38 together with Remark 39 implies that:

- if $sv(\sigma) > 0$, then the center space $E_c^\sigma$ is contained in $G^{cu}$
- if $sv(\sigma) < 0$, then $E_c^\sigma \subset G^{cs}$.

Lemma 41. Let $X$ be a generic star vector field on $M$. Consider a chain recurrent class $C$ of $X$. Then there is a neighborhood $U$ of $C$ so that the extended maximal invariant set $B(X, U)$ has a dominated splitting for the extended linear Poincaré flow

$$N_{B(X, U)} = E \oplus F$$

which extends the stable-unstable bundle defined on the lifted maximal invariant set $\tilde{\Lambda}(X, U)$.

Proof. The case where $C$ is not singular is already done. According to Lemma 36 there an integer $s$ and a neighborhood $U$ of $C$ so that every periodic orbit in $U$ has index $s$ and every singular point $\sigma$ in $U$ is Lorenz like, furthermore either its index is $s$ and $sv(\sigma) < 0$ or $\sigma$ has index $s + 1$ and $sv(\sigma) > 0$.

According to Remark 39 one has:

$$B(X, U) \subset \tilde{\Lambda}(X, U) \cup \bigcup_{sv(\sigma_i) < 0} \mathcal{F}^{cs}_{\sigma_i} \cup \bigcup_{sv(\sigma_i) > 0} \mathcal{F}^{cs}_{\sigma_i}$$

By Corollary 35 and Lemma 37 each of this set admits a dominated splitting $E \oplus F$ for the extended linear Poincaré flow $\psi_X$ with $dim E = s$.

The uniqueness of the dominated splittings for prescribed dimensions implies that these dominated splitting coincides on the intersections concluding. 

We already proved the existence of a dominated splitting $E \oplus F$, with $dim(E) = s$, for the extended linear Poincaré flow over $B(X, U)$ for a small filrating neighborhood of $C$, where $s$ is the index of any periodic orbit in $U$. It
remains to show that the extended linear Poincaré flow admits a reparametrization which contracts uniformly the bundle $E$ and a reparametrization which expands the bundle $F$.

Lemma 34 divides the set of singularities in 2 kinds of singularities, the ones with positive saddle value and the ones with negative saddle value. We denote

$$S_E := \{ x \in \text{Zero}(X) \cap U \text{ such that } sv(x) < 0 \} \quad \text{and},$$

$$S_F := \{ x \in \text{Zero}(X) \cap U \text{ such that } sv(x) > 0 \} .$$

Recall that Section 5.1 associated a cocycle $h_\sigma : \Lambda_X \to \mathbb{R}$, whose cohomology class is well defined, to every singular point $\sigma$.

Now we define

$$h_E = \prod_{\sigma \in S_E} h_\sigma \quad \text{and} \quad h_F = \prod_{\sigma \in S_F} h_\sigma .$$

Now Lemma 31 and therefore Theorem 4 are a direct consequence of the next lemma:

**Lemma 42.** Let $X$ be a generic star vector field on $M$. Consider a chain recurrent class $C$ of $X$. Then there is a neighborhood $U$ of $C$ so that the extended maximal invariant set $B(X,U)$ is such that the normal space has a dominated splitting $N_{B(X,U)} = E \oplus F$ such that the space $E$ (resp. $F$) is uniformly contracting (resp. expanding) for the reparametrized extended linear Poincaré flow $h_t \cdot \psi_t N$.

The proof uses the following theorem by Gan Shi and Wen, which describes the ergodic measures for a star flow. Given a $C^1$ vector field $X$, an ergodic measure $\mu$ for the flow $\phi_t$, is said to be hyperbolic if either $\mu$ is supported on a hyperbolic singularity or $\mu$ has exactly one zero Lyapunov exponent, whose invariant subspace is spanned by $X$.

**Theorem 43** (Lemma 5.6 [GSW]). Let $X$ be a star flow. Any invariant ergodic measure $\mu$ of the flow $\phi_t$ is a hyperbolic measure. Moreover, if $\mu$ is not the atomic measure on any singularity, then $\text{supp}(\mu) \cap H(P) \neq \emptyset$, where $P$ is a periodic orbit with the index of $\mu$, i.e., the number of negative Lyapunov exponents of $\mu$ (with multiplicity).

of lemma 42. We argue by contradiction, assuming that the bundle $E$ is not uniformly contracting for $h_E \cdot \psi_t^X$ over $B(X,U)$ for every filtrating neighborhood $U$ of the class $C$.

One deduces the following claim:

**Claim.** Let $\tilde{C}(\sigma) \subset \tilde{\Lambda}(X)$ be the closure in $PM$ of the lift of $C(\sigma) \setminus \text{Zero}(X)$. Then, for every $T > 0$, there exists an ergodic invariant measure $\mu_T$ whose support is contained in $P_\sigma \cup \tilde{C}(\sigma)$ such that

$$\int \log \| h_E^T \cdot \psi^X_t |_E \| \, d\mu(x) \geq 0 .$$
Proof. For all $U$, there exist an ergodic measure $\mu_T$ whose support is contained in $B(X,U)$ such that

$$\int \log \| h_T \psi^T \|_E \, d\mu_T(x) \geq 0.$$

But note that the class $C$, needs not to be a priori a maximal invariant set in a neighborhood $U$. We fix this by observing the fact that

$$\mathbb{P}_\sigma^c \cup \tilde{C}(\sigma) \subset B(X,U)$$

for any $U$ as small as we want and actually we can choose a sequence of neighborhoods $\{U_n\}_{n \in \mathbb{N}}$ such that $U_n \to C$ and therefore

$$\mathbb{P}_\sigma^c \cup \tilde{C}(\sigma) = \bigcap_{n \in \mathbb{N}} B(X,U_n).$$

This defines a sequence of measures $\mu^n_T \to \mu^0_T$ such that

$$\int \log \| h_T \psi^T \|_E \, d\mu^n_T(x) \geq 0,$$

and with supports converging to $\mathbb{P}_\sigma^c \cup \tilde{C}(\sigma)$. The resulting limit measure $\mu^0_T$, whose support is contained in $\mathbb{P}_\sigma^c \cup \tilde{C}(\sigma)$, might not be hyperbolic but it is invariant. We can decompose it in sum of ergodic measures, and so if

$$\int \log \| h_T \psi^T \|_E \, d\mu^0_T(x) \geq 0,$$

There must exist an ergodic measure $\mu_T$, in the ergodic decomposition of $\mu^0_T$,

$$\int \log \| h_T \psi^T \|_E \, d\mu_T(x) \geq 0,$$

and the support of $\mu_T$ is contained in $\mathbb{P}_\sigma^c \cup \tilde{C}(\sigma)$. \qed

Recall that for generic star flows, every chain recurrence class in $B(X,U)$ is Hausdorff limit of periodic orbits of the same index and that satisfy the conclusion of Lemma 32. Let $\eta > 0$ and $T_0 > 0$ be given by Lemma 32. We consider an ergodic measure $\mu = \mu_T$ for some $T > T_0$.

Claim. Let $\nu_n$ be a measure supported on a periodic orbits $\gamma_n$ with period $\pi \gamma_n$ bigger than $T$ , then $\int \log h_T^E d\nu_n(x) = 0$.

Proof. By definition of $h_T^E$

$$\log h_T^E d\nu_n(x) = \log \Pi_{\sigma_i \in \mathcal{S}_E} \| h_{\sigma_i}^T \| \, d\nu_n(x),$$

so it suffices to prove the claim for a given $h_{\sigma_i}^T$. For every $x$ in $\gamma$ by the cocycle condition in lemma 22 we have that

38
\[ \Pi_{i=0}^{(m\pi(\gamma)/T)-1} h_{i\gamma}^T (\phi_{iT}(x)) = h_{\sigma_i}^{(m\pi(\gamma)/T)-1}(x) \]

The norm of the vector field restricted to \( \gamma \) is bounded, and therefore \( h_{\sigma_i}^{(m\pi(\gamma)/T)-1}(x) \) is bounded for \( m \in \mathbb{N} \) going to infinity. Then this is also true for \( h_T^E \). Since \( \nu_n \) is an ergodic measure, we have that

\[ \int \log h_E^T d\nu_n(x) = \lim_{m \to \infty} \frac{1}{m} \sum_{i=0}^{(m\pi(\gamma)/T)-1} \log (h_E^T(\phi_{iT}(x))) \]

\[ = \lim_{m \to \infty} \frac{1}{m} \log \left( \Pi_{i=0}^{(m\pi(\gamma)/T)-1} h_E^T(\phi_{iT}(x)) \right) \]

\[ = \lim_{m \to \infty} \frac{1}{m} \log \left( h_E^{(m\pi(\gamma)/T)-1}(x) \right) \]

\[ = 0 \]

\[ \square \]

**Claim.** There is a singular point \( \sigma_i \) so that \( \mu \) is supported on \( \mathbb{F}_{\sigma_i} \).

**Proof.** Suppose that \( \mu \) weights 0 on \( \bigcup_{\sigma_i \in \mathbb{F}_{\sigma_i}} \mathbb{P} \), then \( \mu \) projects on \( M \) on an ergodic measure \( \nu \) supported on the class \( C(\sigma) \) and such that ut weights 0 in the singularities, for which

\[ \int \log \| h_E^{T} \psi^T \| d\nu(x) \geq 0. \]

Recall that \( \psi^T \) is the linear Poincaré flow, and \( h_E^{T} \) can be defined as a function of \( x \in M \) instead of as a function of \( L \in \mathbb{P} M \) outside of an arbitrarily small neighborhood of the singularities.

However, as \( X \) is generic, the ergodic closing lemma implies that \( \nu \) is the weak*-limit of measures \( \nu_n \) supported on periodic orbits \( \gamma_n \) which converge for the Hausdorff distance to the support of \( \nu \). Therefore, for \( n \) large enough, the \( \gamma_n \) are contained in a small filtrating neighborhood of \( C(\sigma) \) therefore satisfy

\[ \int \log \| h_E^{T} \psi^T \| d\nu(x) \leq -\eta. \]

The map \( \log \| h_E^{T} \psi^T \| \) is not continuous. Nevertheless, it is uniformly bounded and the unique discontinuity points are the singularities of \( X \). These singularities have (by assumption) weight 0 for \( \nu \) and thus admit neighborhoods with arbitrarily small weight. Out of such a neighborhood the map is continuous. One deduces that

\[ \int \log \| h_E^{T} \psi^T \| d\nu(x) = \lim \int \log \| h_E^{T} \psi^T \| d\nu_n(x) \]

and therefore is strictly negative, contradicting the assumption. This contradiction proves the claim.

\[ \square \]
On the other hand, Corollary 35 asserts that \( h_E \cdot \psi_N \) contracts uniformly the bundle \( E \):

- over the projective space \( \mathbb{P}G^{cs}_{\sigma_i} \), for \( \sigma_i \) with \( sv(\sigma_i) < 0 \): note that, in this case, \( \sigma_i \in S_E \) so that \( h_E \) coincides with \( h_{\sigma_i} \) on \( \mathbb{P}G^{cs}_{\sigma_i} \);
- over \( \mathbb{P}G^{cu}_{\sigma_i} \) for \( \sigma_i \) with \( sv(\sigma_i) > 0 \): note that, in this case \( \sigma_i \notin S_E \) so that \( h_E \) is constant equal to 1 on \( \mathbb{P}G^{cu}_{\sigma_i} \times \mathbb{R} \).

Recall that \( \mathbb{P}^e_{\sigma_i} \) is contained in \( \mathbb{P}G^{cs}_{\sigma_i} \) (resp. \( \mathbb{P}G^{cu}_{\sigma_i} \) if \( sv(\sigma_i) < 0 \) (resp. \( sv(\sigma_i) > 0 \)). One deduces that there is \( T_1 > 0 \) and \( \varepsilon > 0 \) so that

\[
\log \left\| h_E \cdot \psi_N \right\|_{E_L} \leq -\varepsilon, \quad \forall L \in \mathbb{P}^e_{\sigma_i}, \; T > T_1.
\]

Therefore the measures \( \mu_T \), for \( T > \sup T_0, T_1 \) cannot be supported on \( \mathbb{P}^e_{\sigma_i} \), leading to a contradiction.

The expansion for \( F \) is proved analogously.

And this finishes the proof of Lemma 42 and therefore the proof of Lemma 31 and Theorem 4.

\section{A multisingular hyperbolic set in \( \mathbb{R}^3 \)}

This section will be dedicated to the building a chain recurrence class in \( M^3 \) containing 2 singularities of different indexes, that will be multisingular hyperbolic. However this will not be a robust class, and the singularities will not be robustly related. Other examples of this kind are exhibited in \cite{BaMo}. A robust example is announced in \cite{dL} this example is built in a 5-dimensional manifold, since the results in \cite{MPP} and \cite{GLW} show us that in dimension 3 and 4 the star flows are, open and densely, singular hyperbolic. This structure forbids the coexistence of singularities of different indexes in the same class.

We add this example all the same since it illustrates de situation in the simplest way we could.

**Theorem 44.** There exists a vector fields \( X \) in \( S^2 \times S^1 \) with an isolated chain recurrent class \( \Lambda \) such that :

- There are 2 singularities in \( \Lambda \). They are Lorenz like and of different index.
- a cycle between the singularities. The cycle and the singularities are the only other orbits in \( \Lambda \).
- The set \( \Lambda \) is multisingular hyperbolic.

To begin with the proof of the theorem, let us start with the construction of a vector field \( X \), that we will later that it has the properties of the Theorem.

We consider a vector field in \( S^2 \) having:

- A source \( f_0 \) such that the basins of repulsion of \( f_0 \) is a disc bounded by a cycle \( \Gamma \) formed by the unstable manifold of a saddle \( s_0 \) and a sink \( \sigma_0 \).
• A source $\alpha_0$ in the other component limited by $\Gamma$.

• We require that the tangent at $\sigma_0$ splits into 2 spaces, one having a stronger contraction than the other.

Note that the unstable manifold of $s_0$, is formed by two orbits. This two orbits have their $\omega$-limit in $\sigma_0$, and as they approach $\sigma_0$, they become tangent to the weak stable direction, (see figure 7).

Now we consider $S^2$ embedded in $S^3$, and we define a vectorfield $X_0$ in $S^3$ that is normally hyperbolic at $S^2$, in fact we have $S^2$ times a strong contraction, and 2 extra sinks that we call $\omega_0$ and $P_0$ completing the dynamics (see figure 7).

Note that $\sigma_0$ is now a saddle and the weaker contraction at $\sigma_0$ is weaker than the expansion. So $\sigma_0$ is lorenz like.

Now we remove a neighborhood of $f_0$ and $P_0$. The resulting manifold is diffeomorphic to $S^2 \times [-1, 1]$ and the vector field $X_0$ will be entering at $S^2 \times \{1\}$ and outing at $S^2 \times \{-1\}$ (see figure 7).
Figure 6: $S^2$ normally hyperbolic in $S^3$

Now we consider an other copy of $S^2 \times [-1, 1]$ with a vector field $X_1$ that is the reverse time of $X_0$. Therefore $X_1$ has a saddle called $\sigma_1$ that has a strong expansion, a weaker expansion and a contraction, and is lorenz like. It also has a sink called $\alpha_1$ a source called $\omega_1$ and saddle called $s_1$.

The vector field $X_1$ is outing at $S^2 \times \{1\}$ and entering at $S^2 \times \{-1\}$.

We can now paste $X_1$ and $X_0$ along their boundaries ($S^2 \times \{-1\}$ with $S^2 \times \{-1\}$ and the other two). Since both vector fields are transversal to the boundaries we can obtain a $C^1$ vector field $X$ in the resulting manifold that is diffeomorphic to $S^2 \times S^1$.

We do not paste $S^2 \times \{-1\}$ with $S^2 \times \{-1\}$ by the identity but with a rotation so that

$$\left( w^u(\alpha_0) \cap (S^2 \times \{-1\}) \right)^c$$

and

$$w^u(s_0) \cap S^2 \times \{-1\}$$
are mapped to
\[ w^s(\alpha_1) \cap (S^2 \times \{ -1 \}) . \]
We require as well that
\[ w^u(\sigma_0) \cap (S^2 \times \{ -1 \}) \]
is mapped to
\[ w^s(\sigma_1) \cap (S^2 \times \{ -1 \}) , \]
We will later require an extra condition on this gluing map, which is a generic condition, and that will guarantee the multisingular hyperbolicity.

To glue \( S^2 \times \{ 1 \} \) with \( S^2 \times \{ 1 \} \), let us first observe that \( w^s(\alpha_0) \cap (S^2 \times \{ 1 \}) \) is a circle that we will call \( C_0 \). We can also define the corresponding \( C_1 \). We paste \( S^2 \times \{ 1 \} \) with \( S^2 \times \{ 1 \} \), mapping \( C_0 \) to cut transversally \( C_1 \).

Note that the resulting vector field \( X \) has a cycle between two lorenz like singularities \( \sigma_0 \) and \( \sigma_1 \).

![Figure 8: Pasting \( S^2 \times \{ -1 \} \) with \( S^2 \times \{ -1 \} \).](image)

**Lemma 45.** The vector field \( X \) defined above is such that the cycle and the singularities are the only chain recurrent points.

**Proof.**
- All the recurrent orbits by \( X_0 \) in \( S^3 \) are the singularities. Once we remove the neighborhoods of the 2 singularities obtaining the manifold with boundary \( S^2 \times [-1,1] \), the only other orbits with hopes of being recurrent need to cut the boundaries.

- The points in \( S^2 \times \{ -1 \} \) that are not in \( \overline{w^u(\alpha_0)}^c \) are wondering since they are mapped to the stable manifold of the sink \( \alpha_1 \)

43
The points in $w^u(\alpha_0)$ that are not in $w^u(\sigma_0)$ are in $w^u(s_0)$ or in $w^u(\sigma_0)$.

The points in $S^2 \times \{-1\} \cap w^u(s_0)$ are mapped to the stable manifold of $\alpha_1$.

As a conclusion, the only point in $S^2 \times \{-1\}$ whose orbit could be recurrent is the one in

$$(S^2 \times \{-1\}) \cap w^u(\sigma_0).$$

Let us now look at the points in $S^2 \times \{1\}$. There is a circle $C_0$, corresponding to $w^s(\sigma_0) \cap (S^2 \times \{1\})$ that divides $S^2 \times \{1\}$ in 2 components. One of this components is the basin of the sink $\omega_0$ and the other is what used to be the basin of $P_0$. So we have the following options:

- The points that are in the basin of the sink $\omega_0$ are not chain recurrent.

- The points that are in what used to be the basin of $P_0$ are either mapped into the basin of $\omega_1$ or are sent to what used to be the basin of $P_1$. Note that this points cross $S^2 \times \{-1\}$ for the past, and since they are not in the stable manifold of $\sigma_1$ they are wondering.

- Some points in $C_0$ will be mapped to the basin of $\omega_1$, others to what used to be the basin of $P_1$, and others to $C_1$. In the two first cases those points are wondering.

![Figure 9: Pasting $S^2 \times \{1\}$ with $S^2 \times \{1\}$](image)

As a conclusion,

- The only recurrent orbits that cross $S^2 \times \{1\}$, are in the intersection of $C_0$ with $C_1$.  

44
• The only recurrent orbits that cross $S^2 \times \{-1\}$, are in the intersection of $w^u(\sigma_0)$ with $w^s(\sigma_1)$.

• The only recurrent orbits that do not cross the boundaries of $S^2 \times [-1,1]$ are singularities.

This proves our lemma.

For the Lorenz singularity $\sigma_0$ of $X$ which is of positive saddle value and such that $T_{\sigma_0}M = E^s \oplus E^u \oplus E^{wu}$, we define $B_{\sigma_0} \subset PM$ as

$$B_{\sigma_0} = \pi_P (E^s \oplus E^{wu}) .$$

For the Lorenz singularity $\sigma_1$ of $X$ which is of positive saddle value and such that $T_{\sigma_1}M = E^s \oplus E^u \oplus E^{wu}$, we define $B_{\sigma_1} \subset PM$ as

$$B_{\sigma_1} = \pi_P (E^s \oplus E^u) .$$

Let $a$, $b$ and $c$ be points that are one in each of the 3 regular orbits forming the cycle between the two singularities of $X$. We call $a$ to the one such that the $\alpha$-limit of $a$ is $\sigma_0$. We define $L_a = S_X(a)$, $L_b = S_X(b)$ and $L_c = S_X(c)$. We also note $O(L_a)$, $O(L_b)$ and $O(L_c)$ as the orbits of $L_a$, $L_b$ and $L_c$ by $\phi^t$.

**Proposition 46.** Suppose that $X$ is a vector field defined above. Then there exist an open set $U$ containing the orbits of $a$, $b$ and $c$ and the saddles $\sigma_0$ and $\sigma_1$, such that the extended maximal invariant set $B(U, X)$ is

$$B(U, X) = B_{\sigma_0} \cup B_{\sigma_1} \cup O(L_a) \cup O(L_b) \cup O(L_c) .$$

**Proof.** The 2 orbits of strong stable manifold of $\sigma_0$ go by construction to $\alpha_0$ for the past. Therefore it is escaping. The fact that there is a cycle tells us that there are no other escaping directions, therefore the center space is formed by the weak stable and the unstable spaces. By definition $B_{\sigma_0} = \mathbb{P}_{\sigma_0}$. Analogously we see that $B_{\sigma_1} = \mathbb{P}_{\sigma_1}$. Since the cycle formed by the orbits of $a$, $b$ and $c$ and the saddles $\sigma_0$ and $\sigma_1$ is an isolated chain recurrence class, we can chose $U$ small enough so that this chain-class is the maximal invariant set in $U$. This proves our proposition.

**Lemma 47.** We can choose a vector field $X$ defined above is Multisingular hyperbolic in $U$.

**Proof.** The reparametrized linear Poincaré flow is Hyperbolic in restriction to the bundle over $B_{\sigma_0} \cup B_{\sigma_1}$ and of index one. We consider the set $B_{\sigma_0} \cup B_{\sigma_1} \cup O(L_a)$.

The strong stable space at $\sigma_0$ is the stable space for the reparametrized linear Poincaré flow. There is a well defined stable space in the linearized neighborhood of $\sigma_0$ and since the stable space is invariant for the future, there is a one dimensional stable flag that extends along the orbit of $a$. We can reason analogously with the strong unstable manifold of $\sigma_1$ and conclude that
there is an unstable flag extending through the orbit of \( a \), and they intersect transversally. This is because this condition is open and dense in the possible gluing maps of \( S^2 \times \{-1\} \) to \( S^2 \times \{-1\} \), with the properties mentioned above. Therefore the set \( B_{\sigma_0} \cup B_{\sigma_1} \cup O(L_a) \) is hyperbolic for the reparametrized linear Poincaré flow.

Analogously we prove that \( B_{\sigma_0} \cup B_{\sigma_1} \cup O(L_a) \cup O(L_b) \cup O(L_c) \) is hyperbolic for the reparametrized linear Poincaré flow, and since from proposition \([46]\) there exist a \( U \) such that,

\[
B_{\sigma_1} \cup B_{\sigma_2} \cup O(L_a) \cup O(L_b) \cup O(L_c) = B(U, X).
\]

Then \( X \) is Multisingular hyperbolic in \( U \).

The example in \([BaMo]\) consists on two singular hyperbolic sets (negatively and positively) \( H^- \) and \( H^+ \) of different indexes, and wondering orbits going from one to the other. Since they are singular hyperbolic \( H^- \) and \( H^+ \) are multisingular hyperbolic sets of the same index. Moreover, the stable and unstable flags (for the reparametrized linear Poincare flow) along the orbits joining \( H^- \) and \( H^+ \) intersect transversally. This is also true for \( H^- \).

With all this ingredients we can prove (In a similar way as we just did with the more simple example above) that the chain recurrence class containing \( H^- \) and \( H^+ \) in \([BaMo]\) is multisingular hyperbolic, while it was shone by the authors that it is not singular hyperbolic.

References

[AN] John M. Alongi, Gail Susan Nelson *Recurrence and Topology*, Volume 85 of Graduate studies in mathematics, ISBN 0821884050, 9780821884058

[BC] C. Bonatti and S. Crovisier, *Récurrence et généricité*, Invent. Math., 158 (2004), 33–104.

[BDP] C. Bonatti, L. J. Díaz and E. R. Pujals, *A C1-generic dichotomy for diffeomorphisms: Weak forms of hyperbolicity or infinitely many sinks or sources*, Annals of Math. (2), 158 (2003), 355–418.

[BV] C. Bonatti, Christian; M. Viana, *SRB measures for partially hyperbolic systems whose central direction is mostly contracting.*

[BDV] C. Bonatti, L. J. Díaz, and M. Viana *Dynamics beyond uniform hyperbolicity. A global geometric and probabilistic perspective.* Encyclopaedia of Mathematical Sciences, 102. Mathematical Physics, III. Springer-Verlag, Berlin, (2005). xviii+384 pp.

[BGY] C. Bonatti, S. Gan and D. Yang, *Dominated chain recurrent classes with singularities*, arXiv:1106.3905.
[BLY] C. Bonatti, M. Li, and D. Yang, *A robustly chain transitive attractor with singularities of different indices*. J. Inst. Math. Jussieu 12 (2013), no. 3, 449–501.

[BaMo] S. Bautista, CA Morales *On the intersection of sectional-hyperbolic sets* arXiv preprint arXiv:1410.0657

[BPV] C. Bonatti, A. Pumariño, and M. Viana, *Lorenz attractors with arbitrary expanding dimension*. C. R. Acad. Sci. Paris Sér. I Math. 325 (1997), no. 8, 883–888.

[Co] C. Conley, *Isolated Invariant Sets and the Morse Index*, CBMS Regional Conference Series in Mathematics, vol. 38, American Mathematical Society, Rhode Island, 1978.

[C] S. Crovisier, *Periodic orbits and chain-transitive sets of C1-diffeomorphisms*, Publ. Math. Inst. Hautes Études Sci., 104 (2006), 87–141.

[D] C.I. Doering, *Persistently transitive vector fields on three-dimensional manifolds*. Dynamical systems and bifurcation theory (Rio de Janeiro, 1985), 59–89, Pitman Res. Notes Math. Ser., 160, Longman Sci. Tech., Harlow, 1987.

[dL] A. da Luz *Star flows with singularities of different indices* work in progress.

[DPU] L. J. Díaz, E. Pujals, and R. Ures, *Partial hyperbolicity and robust transitivity*. Acta Math. 183 (1999), no. 1, 1–43.

[GY] S. Gan and D. Yang, *Morse-Smale systems and horseshoes for three-dimensional singular flows*, arXiv:1302.0946.

[G] J. Guckenheimer, *A strange, strange attractor*, in The Hopf Bifurcation Theorems and its Applications, Applied Mathematical Series, 19, Springer-Verlag, 1976, 368–381.

[H] S. Hayashi, *Diffeomorphisms in F^1(M) satisfy Axiom A*, Ergod. Th. Dynam. Sys., 12 (1992), 233–253.

[H2] S. Hayashi, *Connecting invariant manifolds and the solution of the C^1 stability conjecture and ω-stability conjecture for flows*, Ann. of Math., 145 (1997), 81–137.

[K] A. Katok, *Lyapunov exponents, entropy and periodic orbits for diffeomorphisms*, Inst. Hautes Études Sci. Publ. Math., 51 (1980), 137–173.

[GLW] M. Li, S.Gan, L.Wen, *Robustly Transitive singular sets via approach of an extended linear Poincaré flow*. Discrete and Continuous Dynamical System. Volume 13, Number 2, July 2005
[GSW] Y. Shi, S. Gan, L. Wen, On the singular hyperbolicity of star flows. Journal Of Modern Dynamics. October 2013.

[GWZ] S. Gan, L. Wen and Zhu, Indices of singularities of robustly transitive sets, Discrete Contin. Dyn. Syst., 21 (2008), 945–957.

[GW] J. Guckenheimer and R. Williams, Structural stability of Lorenz attractors, Inst. Hautes Etudes Sci. Publ. Math. ’, 50 (1979), 59–72.

[L] S. Liao, On $(\eta, d)$-contractible orbits of vector fields, Systems Sci. Math. Sci., 2 (1989), 193–227.

[Lo] E. N. Lorenz, Deterministic nonperiodic flow, J. Atmosph. Sci., 20 (1963), 130–141.

[Ma] R. Mañé, An ergodic closing lemma, Ann. Math. (2), 116 (1982), 503–540.

[Ma1] R. Mañé, Contributions to the stability conjecture. Topology 17 (1978), no. 4, 383–396.

[Ma2] R. Mañé, A proof of the $C^1$ stability Conjecture, Publ. Math. IHES, 66 (1988), 161–210.

[MM] R. Metzger and C. Morales, On sectional-hyperbolic systems, Ergodic Theory and Dynamical Systems, 28 (2008), 1587–1597.

[MPP] C. Morales, M. Pacifico and E. Pujals, Robust transitive singular sets for 3-flows are partially hyperbolic attractors or repellers, Ann. Math. (2), 160 (2004), 375–432.

[PaSm] J. Palis and S. Smale, Structural stability theorems, in 1970 Global Analysis (Proc. Sympos. Pure Math., Vol. XIV, Berkeley, Calif., 1968), Amer. Math. Soc., Providence, R.I, 1970, 223–231.

[Pl] V. Pliss, A hypothesis due to Smale, Diff. Eq., 8 (1972), 203–214.

[PuSh] C. Pugh and M. Shub, $\omega$-stability for flows, Invent. Math., 11 (1970), 150–158.

[PuSh2] C. Pugh and M. Shub, Ergodic elements of ergodic actions, Compositio Math., 23 (1971), 115–122.

[R1] J. W. Robbin, A structural stability theorem. Ann. of Math. (2) 94 (1971) 447–493.

[R2] C. Robinson, Structural stability of $C^1$ diffeomorphisms. J. Differential Equations 22 (1976), no. 1, 28–73.

[Sh] M. Shub, Topologically transitive diffeomorphisms on $T^4$ In Dynamical Systems, volume 206 of Lecture Notes in Math., lecture (16) page 28-29 and lecture (22) page 39, Springer Verlag, 1971.
[Sm] S. Smale, *The $\omega$-stability theorem*, in 1970 Global Analysis (Proc. Sympos. Pure Math., Vol. XIV, Berkeley, Calif., 1968), Amer. Math. Soc., Providence, R.I., 1970, 289–297.

[W] L. Wen, *On the C1 stability conjecture for flows*, J. Differential Equations, 129 (1996), 334–357.

[WX] L. Wen and Z. Xia, C1 connecting lemmas, Trans. Am. Math. Soc., 352 (2000), 5213–5230.

[YZ] D. Yang and Y. Zhang, *On the finiteness of uniform sinks*, J. Diff. Eq., 257 (2014), 2102–2114. [32] S.