Characterizing maximal families of mutually unbiased bases

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Abstract

We show that maximal families of mutually unbiased bases are characterized in all dimensions by partitioned unitary error bases, up to a choice of a family of Hadamards. Furthermore, we give a new construction of partitioned unitary error bases, and thus maximal families of mutually unbiased bases, from a finite field, which is simpler and more direct than previous proposals. We introduce new tensor diagrammatic characterizations of maximal families of mutually unbiased bases, partitioned unitary error bases, and finite fields as algebraic structures defined over Hilbert spaces.

1 Introduction

In this paper we present the following results:

• an equivalence between partitioned unitary error bases (partitioned UEBs) and maximal families of MUBs equipped with families of Hadamards;

• a construction of maximal families of MUBs from finite fields that is simpler than those proposed previously;

• a new tensor diagrammatic axiomatisation of maximal families of MUBs.

It has been shown that the largest family of $d$-dimensional mutually unbiased bases that can exist is $d + 1$ [3]. In light of this result we will refer to a family of $d + 1$ MUBs as a maximal family of MUBs. Maximal families of MUBs represent $d + 1$ measurements that are, in some sense ‘as far apart as possible’, and can perfectly distinguish any density operator on a $d$-dimensional Hilbert space [26]. Maximal families of MUBs are fundamental to areas such as quantum tomography [14] and quantum key distribution [7] and are as such of great importance to quantum information. In spite of this much is still to be discovered about maximal families of MUBs. In general dimension, it is not even known whether maximal families of MUBs exist although the existence of maximal families of MUBs in prime power dimension has however long been established [3, 6, 12]. The results in this paper build on previous work establishing a new diagrammatic
framework to tackle a much researched area. We have utilised this approach to clarify the exact nature of the relationship between UEBs and maximal families of MUBs. This is just the beginning for this framework and we expect further progress to follow. We briefly give definitions of the key structures which our theorems relate.

**Definition 1 (Mutually unbiased bases [26]).** A pair of orthonormal bases $|a_i\rangle$ and $|b_i\rangle$ with $i \in \{0, ..., d-1\}$ are mutually unbiased if $|\langle a_i|b_j\rangle|^2 = 1/d$ for all $i, j \in \{0, ..., d-1\}$.

**Definition 2 (Family of mutually unbiased bases).** A family of bases of a $d$-dimensional Hilbert space are a mutually unbiased family if they are pairwise mutually unbiased.

**Definition 3 (Unitary error basis [16]).** A unitary error basis on a $d$-dimensional Hilbert space is a family of $d^2$ unitary operators $U_{ij}$ where $i,j \in \{0, ..., d-1\}$ such that:

$$\text{Tr}(U_{ij}^\dagger \circ U_{mn}) = \delta_{im}\delta_{jn}d$$ (1)

Throughout this paper we make use of Penrose tensor diagrams closely related to those, which are used by the tensor networks community. We give a very brief introduction here in order to introduce one of the main results of the paper, a characterization of maximal families of MUBs using tensor diagrams. Refer to Section 2 for an in depth introduction to the tensor diagrams and other necessary background material.

We use wires to represent Hilbert spaces and boxes and nodes to represent linear maps. Tensor products are given by horizontal composition with composition of linear maps represented by connecting wires vertically. Direct sums are given by summation. We use the convention that our diagrams are read from bottom to top. Bra-ket notation can be translated into tensor diagrams as follows: $|i\rangle := \downarrow_\psi$ and $\langle i | := \uparrow_\psi$ with the inner product given by connecting the wires.

**Definition 4 (Basis tensors).** Given an orthonormal basis for a Hilbert space, $|i\rangle, 0 \leq i < n - 1$, we canonically define the following four linear maps:

Connected diagrams made up of these four linear maps are basis tensors, and are uniquely determined by the number of input and output wires.

We make extensive use of the basis tensor corresponding to the computational orthonormal basis which we will denote with black dots $\bullet$. In Section 3 we start by proving that the following diagrammatic equation completely characterizes maximal families of MUBs using the computational basis tensor, where $M$ is a linear map of type
Unitary error bases (UEBs) are fundamental to protocols such as teleportation and dense coding as well as finding application in quantum error correction \[15, 20, 22, 25\].

Partitioned UEBs \[3\] are unitary error bases containing the identity operator equipped with a partition into the identity and \(d+1\) disjoint classes each containing \(d-1\) commuting operators. From a partitioned UEB we can obtain a maximal family of MUBs by taking the common eigenbases of each of the commuting classes of operators \[3\]. We denote the map taking a partitioned UEB to its eigenbases by \(\theta\).

Later in Section 3, making use of our diagrammatic maximal families of MUBs we introduce the following converse map \(\phi_H\) : Maximal family of MUBs \(\rightarrow\) Partitioned UEB given extra data in the form of a family of Hadamards \(G\) and \(H\) (here \(\ast\) is a projector defined using \(\chi\)):

\[
\phi_H(M) := \frac{1}{d} \begin{pmatrix}
M & -M \\
M & M
\end{pmatrix} + H \\
G
\]

We show that given family of Hadamards \(H\) the composition \(\theta \circ \phi_H\) is the identity and thus conclude that all maximal families of MUBs can be obtained in this way up to a choice of \(H\) which we identify with a choice of eigenvalues. Each maximal MUB is associated with an infinite family of partitioned UEBs which are not necessarily equivalent.

In Section 4 we introduce a diagrammatic axiomatisation for finite fields as algebras over Hilbert spaces. We show that, given a finite field, the following unitary operators form a partitioned UEB:

\[
U_{FF} := \chi + \ast
\]
for the additive group. We finish by giving an example of our construction in dimension $d = 4$.

Related work. In their 2002 paper, Bandyopadhyay et al [3] introduced partitioned UEBs and showed how to obtain maximal families of MUBs from them. Gogioso and Zeng gave a diagrammatic axiomatisation of complex group algebras in their 2015 paper [13]. We have extended this to finite fields in order to give our diagrammatic construction of partitioned UEBs and thus, maximal families of MUBs.

2 Background

We begin with the definitions of mutually unbiased bases and unitary error bases (UEBs), we will then review the necessary categorical quantum mechanics material using Penrose tensor diagrams to present our main results diagrammatically in Section 3 and Section 4.

Basic definitions. The following result was given a particularly simple and elegant proof by Bandyopahyay et al [3].

**Theorem 5.** The largest family of MUBs that can exist on a $d$-dimensional Hilbert space is a family of $d + 1$ MUBs.

In Section 3 we prove the equivalent tensor diagrammatic characterisation of maximal families of MUBs given in the introduction.

We now define the equivalence of pairs of UEBs.

**Definition 6** (Equivalent UEBs [25]). Given UEBs $X_{ij}$ and $Y_{ij}$ they are equivalent if there exist unitary operators $U$ and $V$, complex numbers with unit absolute value $c_{ij}$ and permutation $p$ such that:

$$X_{ij} = c_{ij}UY_{p(i,j)}V$$  \hspace{1cm} (5)

Later, we will formally define partitioned UEBs in Definition 26, and give a tensor diagrammatic characterization of UEBs [19] in Proposition 25, with an additional tensor diagrammatic axiom for partitioned UEBs given in Lemma 27.

**Definition 7** (Hadamard). A Hadamard matrix of order $d$ is a $d \times d$ matrix $H$, such that $|H_{ij}| = 1$ and $HH^\dagger = H^\dagger H = dI_d$ [4].

Hadamards with the addition of a normalization constant, are precisely change of basis matrices between pairs of mutually unbiased bases [5]. To see how mutually unbiased bases can be recovered from Definition 7, consider the following. Given a Hadamard $H$, the matrix $\frac{1}{\sqrt{d}}H$ is unitary, since $HH^\dagger = H^\dagger H = dI_d$. So $\frac{1}{\sqrt{d}}H$ is a change of basis matrix between two orthonormal bases. Let $H' := \frac{1}{\sqrt{d}}H$, and represent the computational basis states by $|a_i\rangle$ and define $H'|a_i\rangle := |b_i\rangle$. The other Hadamard condition gives us that for all $i, j$ we have $|H_{ij}| = 1$ thus $|H_{ij}|^2 = 1$ and so:

$$|\langle a_j|b_i\rangle|^2 = |\langle a_j|H'|a_i\rangle|^2 = |\langle a_j|\frac{1}{\sqrt{d}}H|a_i\rangle|^2 = \frac{1}{d}|\langle a_j|H|a_i\rangle|^2 = 1$$

This gives us back Definition 1.

After we have established the necessary tensor diagrammatic notation below we introduce a tensor diagrammatic axiomatisation of Hadamards in Lemma 18 and controlled Hadamards which are indexed families of Hadamard in Definition 19.
Penrose tensor diagrams. We now introduce some results from categorical quantum mechanics and its graphical calculus for tensors [1, 2, 9] which is different although closely related to the diagrams used in the tensor network community [21, 23, 24]. For those with knowledge of category theory we will be working in FHilb, the category of linear maps and finite dimensional Hilbert spaces which is a †-symmetric monoidal category. Much of the following could be interpreted in a general †-symmetric monoidal category, however the main results of this paper use structure particular to FHilb.

In our formalism wires represent Hilbert spaces and boxes and nodes represent linear maps between Hilbert spaces. Different Hilbert spaces are represented by different coloured wires. We take the convention that diagrams are read from bottom to top. Composite linear maps are represented by vertical composition along the wires. Tensor products are represented by horizontal composition. Let $A$, $B$, $C$ and $D$ be Hilbert spaces represented by black, red, blue and green wires respectively. The following diagram therefore represents the linear map $F \otimes G$ for $F : A \to B$ and $G : C \to D$.

\[
\begin{array}{c}
\begin{array}{c}
F \\
\end{array} \\
\begin{array}{c}
G \\
\end{array}
\end{array}
\]

We will occasionally have to make use of different coloured wires but most of the time we will just use black wires to represent a single Hilbert space. We will use the convention that reflection in a horizontal axis represents adjunction or ‘taking the †’ and reflection in a vertical axis represents complex conjugation. Since linear algebraic equations remain true under taking the adjoint of both sides all diagrammatic equations will remain true under reflection about a horizontal axis. We will use the asymmetry of boxes representing linear maps to make it clear when we have taken an adjoint or complex conjugate of a given linear map. This works as follows:

\[
\begin{array}{c}
\begin{array}{c}
F \\
\end{array} \\
\begin{array}{c}
\end{array}
\end{array}^\dagger = \begin{array}{c}
\begin{array}{c}
\end{array} \\
\begin{array}{c}
F \\
\end{array}
\end{array} \quad \begin{array}{c}
\begin{array}{c}
F \\
\end{array} \\
\begin{array}{c}
\end{array}
\end{array}^* = \begin{array}{c}
\begin{array}{c}
\end{array} \\
\begin{array}{c}
F \\
\end{array}
\end{array}
\]

We represent states as boxes with wires going out but no wires coming in, and effects as boxes with wires coming in but none going out. Boxes with no wires in or out are complex scalars. The correspondence with bra ket notation is therefore as follows:

\[
|j\rangle := \begin{array}{c}
\end{array} \\
\begin{array}{c}
j
\end{array} \quad \langle i| := \begin{array}{c}
\end{array}i \\
\begin{array}{c}
i
\end{array} \quad \langle i|j\rangle := \begin{array}{c}
\end{array}i \\
\begin{array}{c}
j
\end{array}
\]

In order to capture the structure of maximal families of MUBs and partitioned UEBs using tensor diagrams in the next section, we use the computational basis tensor as an indexing set by making use of certain algebras on Hilbert space. We now introduce those algebras known as †-special commutative Frobenius algebras (†-SCFAs) on Hilbert spaces that are equivalent to orthonormal bases as we will show. First we use tensor diagrams to define the properties that make up the †-SCFA axioms.
Definition 8 (Associativity). The linear map $\triangleright$ is associative if the following equation holds:

\[
\begin{array}{c}
\triangleleft \triangleright \triangleright \\
\end{array}
= 
\begin{array}{c}
\triangleleft \triangleright \\
\end{array}
\tag{6}
\]

Definition 9 (Unitality). The linear map $\triangleright$ is unital if there exists a state $\bullet$ such that:

\[
\begin{array}{c}
\triangleleft \triangleright = \\
\end{array}
\begin{array}{c}
\bullet \\
\end{array}
\tag{7}
\]

The state $\bullet$ is called the unit.

Definition 10 (Commutativity). The linear map $\triangleright$ is commutative if the following equation holds:

\[
\begin{array}{c}
\triangleleft \triangleright = \\
\end{array}
\begin{array}{c}
\triangleright \\
\triangleright \\
\end{array}
\tag{8}
\]

If we take the adjoint of both sides of equations (6), (7) and (8) we obtain the definitions of coassociativity, counitality and cocommutativity respectively.

Definition 11 ((Co)monoid). The linear map $\triangleright$ together with $\bullet$ is a monoid if it is associative and unital. The linear map $\triangleright$ together with $\bullet$ is a comonoid if it is coassociative and counital.

Definition 12 (Comonoid homomorphism). A linear map $F$ is a comonoid homomorphism for the comonoid $\triangleright$ if the following equations hold:

\[
\begin{array}{c}
\bullet \triangleright F = \\
\end{array}
\begin{array}{c}
\bullet \\
\end{array}
\begin{array}{c}
\triangleright F \\
\end{array}
\tag{9}
\]

Definition 13 (Special, quasi-special). The linear maps $\triangleright$ and $\triangleright$ are special if the left hand side equation holds and quasi-special if the right hand side equation holds with $d$ the dimension of the Hilbert space:

\[
\begin{array}{c}
\bullet \triangleright = \\
\end{array}
\begin{array}{c}
\bullet \\
\end{array}
\begin{array}{c}
\triangleright \\
\end{array}
= 
\begin{array}{c}
\bullet \\
\bullet \\
\end{array}
= 
\begin{array}{c}
\bullet \\
\end{array}
\tag{10}
\]

Definition 14 ($\dagger$-Frobenius law). The linear map $\triangleright$ and its adjoint $\triangleright$ obey the $\dagger$-Frobenius law if the following equations hold:

\[
\begin{array}{c}
\bullet \triangleright = \\
\end{array}
\begin{array}{c}
\bullet \\
\end{array}
\begin{array}{c}
\triangleright \triangleright \\
\end{array}
\tag{11}
\]

Definition 15 ($\dagger$-special commutative Frobenius algebra). The linear map $\triangleright$ and state $\bullet$ together with their adjoints are a $\dagger$-special commutative Frobenius algebra ($\dagger$-SCFA) if they are a commutative monoid, special and obey the $\dagger$-Frobenius law.
The following result due to Coecke, Pavlović, and Vicary will prove important throughout this paper.

**Proposition 16.** [10] In FHilb †-SCFAs are in one to one correspondence with orthonormal bases. All †-SCFAs can be written in terms of the corresponding orthonormal bases as follows:

\[
\begin{align*}
\begin{array}{c}
\sum_i \end{array} \end{align*}
\]

Equation (12) can be used to show that the following very useful result holds:

**Corollary 17.** [10] For a †-SCFA any two connected diagrams of black dots with the same numbers of inputs and outputs are equal. For example:

\[
\begin{align*}
\begin{array}{c}
\end{array} = \begin{array}{c}
\end{array}
\end{align*}
\]

Given a †-SCFA , we will refer to the corresponding basis as the *black basis*, the basis states as *black states* and their adjoints as *black effects* (we also extend this terminology to other colours). We can also see by combining equations (12) and (13) that, for a †-SCFA , any black state or effect composed with a connected diagram of black dots will be *copied* in the following sense:

\[
\begin{align*}
\begin{array}{c}
\end{array} = \begin{array}{c}
\end{array}
\end{align*}
\]

This allows us to use the basis states of a †-SCFA as an indexing set. For the rest of this paper black wires will represent the d dimensional Hilbert space \( \mathcal{H} \cong \mathbb{C}^d \). We will take the black †-SCFA , to be the †-SCFA corresponding to the computational basis of \( \mathcal{H} \). As a first example, we now define Hadamards and controlled Hadamards using tensor diagrams.

**Lemma 18** (Hadamard). Given a †-SCFA , on a d-dimensional Hilbert space \( \mathcal{H} \), a linear map of type \( H : \mathcal{H} \to \mathcal{H} \) is a Hadamard if and only if the following equations hold:

\[
\begin{align*}
\begin{array}{c}
\end{array} = \begin{array}{c}
\end{array} = \begin{array}{c}
\end{array} = \begin{array}{c}
\end{array}
\end{align*}
\]
Proof. The left hand side of equation (15) is simply a tensor diagrammatic translation of $HH^\dagger = H^\dagger H = dI_d$. We now show the equivalence of the right hand side of equation (15) and the other condition of Definition 7, that for all $i, j$, $|H_{ij}| = 1$. We have for all $i, j$:

$$
|H_{ij}| = 1
\iff |\langle i|H|j\rangle| = 1
\iff \langle i|H|j\rangle\langle j|H^\dagger|i\rangle = 1
\iff \langle i|H|j\rangle\langle j|H^\dagger|i\rangle = \langle i\rangle\langle j\rangle
$$

We now translate this final equation into the graphical calculus, for all $i, j$:

$$
\begin{array}{c}
\begin{array}{c}
\includegraphics{proof15.png}
\end{array}
\end{array}
= _{i} _{i} _{j} _{j}
$$

Rearranging the left hand side we have for all $i, j$:

$$
\begin{array}{c}
\begin{array}{c}
\includegraphics{proof16.png}
\end{array}
\end{array}
= \includegraphics{proof17.png}
$$

Thus we have that for all $i, j$:

$$
\begin{array}{c}
\begin{array}{c}
\includegraphics{proof18.png}
\end{array}
\end{array}
\iff \begin{array}{c}
\begin{array}{c}
\includegraphics{proof19.png}
\end{array}
\end{array}
\iff \begin{array}{c}
\begin{array}{c}
\includegraphics{proof20.png}
\end{array}
\end{array}
\iff \begin{array}{c}
\begin{array}{c}
\includegraphics{proof21.png}
\end{array}
\end{array}
$$

Since $\langle i|H|j\rangle\langle j|H^\dagger|i\rangle = \langle i\rangle\langle j\rangle \iff \langle j|H^\dagger|i\rangle\langle i|H|j\rangle = \langle i\rangle\langle j\rangle$, the other part of the right hand side of equation (15) follows similarly. 

We now introduce a mathematical object which captures the idea of an indexed family of Hadamards. We introduce another Hilbert space which we will represent with a red wire, equipped with a $^\dagger$-SCFA. The states copyable by this $^\dagger$-SCFA will index the Hadamards in the family.

**Definition 19** (Controlled Hadamard). Given a $^\dagger$-SCFA, $\bullet$, on a $d$-dimensional Hilbert space $\mathcal{H}$ and another $^\dagger$-SCFA, $\clubsuit$, on a, possibly different, Hilbert space $\mathcal{G}$, a linear map
$H : \mathcal{G} \otimes \mathcal{H} \rightarrow \mathcal{G}$ is a \textit{controlled Hadamard} if the following equations hold.

\begin{align*}
H \otimes H = d \quad \text{and} \quad H \otimes H = d \quad \text{for all red states} \quad i 
\end{align*}

Controlled Hadamards are indexed families of Hadamards in the following sense.

\textbf{Lemma 20.} Given a controlled Hadamard $H$ and some red state $i$, define $H_i$ as follows:

\begin{align*}
H_i := 
\end{align*}

For all red states $i$, $H_i$ as defined above is a Hadamard.

\textit{Proof.} If we compose equations (16) with $i$ we have:

\begin{align*}
\leftrightarrow H_i \otimes H_i = d \quad \text{and} \quad H_i \otimes H_i = d \quad \text{for all red states} \quad i
\end{align*}

So by Lemma 18 for all $i$, $H_i$ is a Hadamard.

Thus given a controlled Hadamard the number of Hadamards in the family is equal to the dimension of the red Hilbert space which in practice, for our purposes is often the same as the black Hilbert space. Considering the above lemma and the discussion below Definition 7 we have the following corollary.
Corollary 21. Given a controlled Hadamard $H$ with black $\dagger$-SCFA $\blacktriangledown$, and red $\dagger$-SCFA $\blacktriangleleft$, define the following bases $B^i := |b^i_j\rangle$; for each red state $i$, and black state $j$:

$$|b^i_j\rangle := \frac{1}{\sqrt{d}} H_i^{\dagger} j$$

(18)

Then each basis $B^i$ is mutually unbiased to the black basis.

Definition 22 (Permutation). A permutation with respect to a $\dagger$-SCFA is a comonoid homomorphism (of the comonoid part of the $\dagger$-SCFA) which is unitary.

Remark 23. In $\text{FHilb}$ Definition 22 gives the usual notion of a permutation matrix where the $\dagger$-SCFA is a choice of basis.

We will later require a tensor diagrammatic characterisation for a permutation $P$ of type $\mathcal{H} \otimes \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}$ with respect to the tensor product of the standard black $\dagger$-SCFA, $\blacktriangledown$, with itself. The condition that $P$ must be unitary becomes:

$$P \circ P = P \circ P =$$

(19)

Referring to Definition 12 we require the following equations:

$$P \circ P = P \circ P =$$

(20)

This ensures that given black basis states $|i\rangle$ and $|j\rangle$ we have $P(|i\rangle \otimes |j\rangle) = |m\rangle \otimes |n\rangle$ for some $n, m$ also black basis states.

A well known result which is not difficult to prove and will be useful is that isometric operators on finite dimensional Hilbert spaces are always unitary ([18], page 130). Since we will mainly be working with finite dimensional Hilbert spaces we will make use of this to shorten proofs of unitarity.

3 Maximal families of MUBs and partitioned UEBs

We now move on to the discussion of maximal MUBs. For this section we will use only black wires and all wires will represent the Hilbert space $\mathcal{H} \cong \mathbb{C}^d$ as usual. We consider the black basis states of the $\dagger$-SCFA $\blacktriangledown$, as the computational basis denoted by $\downarrow$, $i \in \{0, ..., d - 1\}$ and use them as an indexing set in the way described in the previous section.
We now give a tensor diagrammatic characterisation of a maximal MUB, which we will show to be equivalent to Definition 1. We characterise a maximal family of MUBs as a linear map $M$ of type $H \otimes H \rightarrow H$ together with the computational basis $\dagger$-SCFA. Let the $d + 1$ bases of a maximal MUB be denoted $B^i, i \in \{\ast, 0, 1, \ldots d - 1\}$, where the $k$th basis state of $B^\ast$ is denoted $|b^k\rangle$.

We take the states of the basis $B^\ast$ to be those copyable by $\dagger$, so $\downarrow_i := |b^i\rangle$. The linear map $M$ encodes the $d^2$ basis states of the remaining $d$ bases in the following way.

$$|b^k_i\rangle = \begin{array}{c}
\begin{array}{c}
M \\
\downarrow_i \\
\downarrow_k
\end{array}
\end{array}$$ (21)

**Theorem 24** (Tensor diagrammatic maximal MUBs). Given a $\dagger$-SCFA on a $d$-dimensional Hilbert space $H$, a linear map $M$ of type $H \otimes H \rightarrow H$ is a maximal family of MUBs iff $\sqrt{d}M$ is a controlled Hadamard and the following equation holds.

$$\begin{array}{c}
\begin{array}{c}
M \\
\downarrow_i \\
\downarrow_j
\end{array}
\begin{array}{c}
M \\
\downarrow_m \\
\downarrow_n
\end{array} = \frac{1}{d}
\begin{array}{c}
\begin{array}{c}
\bullet \\
\bullet \\
- \end{array}
\end{array}
\begin{array}{c}
\bullet \\
\bullet \\
\bullet
\end{array} + \begin{array}{c}
\begin{array}{c}
\bullet \\
\bullet \\
\bullet
\end{array}
\end{array}
\begin{array}{c}
\bullet \\
\bullet \\
\bullet
\end{array}
\end{array}$$ (22)

**Proof.** Consider composition by arbitrary black basis states $\downarrow_i, \downarrow_j$ and effects $\uparrow_i, \uparrow_j$ on both sides of the equation (22).

$$\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\uparrow_i \\
\bullet
\end{array}
\begin{array}{c}
\bullet
\end{array}
\begin{array}{c}
\bullet \\
\bullet
\end{array}
\begin{array}{c}
\bullet
\end{array} = \frac{1}{d}
\begin{array}{c}
\begin{array}{c}
\bullet \\
\bullet \\
- \end{array}
\end{array}
\begin{array}{c}
\bullet \\
\bullet \\
\bullet
\end{array} + \begin{array}{c}
\begin{array}{c}
\bullet \\
\bullet \\
\bullet
\end{array}
\end{array}
\begin{array}{c}
\bullet \\
\bullet \\
\bullet
\end{array}
\end{array} \\ \Leftrightarrow \begin{array}{c}
\begin{array}{c}
\downarrow_i \\
\bullet
\end{array}
\begin{array}{c}
\bullet
\end{array}
\begin{array}{c}
\bullet \\
\bullet
\end{array}
\begin{array}{c}
\bullet
\end{array}
\end{array}^2 = \frac{1}{d}(1 - \delta_{im}) + \delta_{im}\delta_{jn}
\end{array}$$

Since $i, j, m$ and $n$ were chosen arbitrarily this holds for all values of $i, j, m$ and $n$. So our tensor diagrammatic axiom is equivalent to the following; for all $i, j, m, n$:

$$|\langle b^i_j|b^m_n\rangle|^2 = \frac{1}{d}(1 - \delta_{im}) + \delta_{im}\delta_{jn}$$ (23)

For $i = m$ we have $|\langle b^i_j|b^m_n\rangle|^2 = \delta_{jn}$, which indicates that for all $i$, $B^i$ is an orthonormal basis. For $i \neq m$ we have $|\langle b^i_j|b^m_n\rangle|^2 = 1/d$, in other words $B^i$ and $B^m$ are mutually unbiased.
The requirement that $\sqrt{d} M$ is a controlled Hadamard ensures that each basis is mutually unbiased to black basis by Corollary \[21\].

We now give a tensor diagrammatic characterisation of unitary error bases which first appeared in the author’s masters thesis \[19\] and is equivalent to Definition \[3\] as we show.

**Proposition 25** (Tensor diagrammatic unitary error bases). Given a $d$-dimensional Hilbert space $\mathcal{H}$ with a $\dagger$-SCFA $\mathcal{H}$, and linear map $U : \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$, define the following family of linear maps $U_{ij} | i, j \in \{0, \ldots, d-1\}$:

$$U_{ij} := U_{i}^{j}$$

(24)

The linear maps $U_{ij}$ are a unitary error basis iff the following equations hold:

$$U_{ij} U_{ij} = U_{ij} U_{ij} = d$$

(25)

**Proof.** We first show that the left hand equation of equation (25) is equivalent to each $U_{ij}$ being unitary.

We compose the left hand equation with the black states $\downarrow$, $\downarrow$, and effects $\uparrow$, $\uparrow$, as follows; for all $i, j, m, n$:

$$U_{ij} U_{mn} = d$$

(25)

So it is equivalent to all $U_{ij}$ being isometric operators and thus unitary operators. We now show that the right hand side equation of (25) is equivalent to equation (1). We again compose by black states and effects to obtain; for all $i, j, m, n$:

$$\text{Tr}(U_{ij}^\dagger \circ U_{mn}) = \delta_{im} \delta_{jn} d$$

This completes the proof. \[\square\]
We now introduce notation for a projector which we will require in our description of partitioned UEBs:

\[
\begin{array}{c}
\ast \\
\downarrow_0
\end{array}
:=-
\begin{array}{c}
0 \\
\downarrow_0
\end{array}
\quad (26)
\]

Note that:

\[
\begin{array}{c}
\ast \\
\downarrow_0
\end{array} =
\begin{array}{c}
0 \\
\downarrow_0
\end{array} -
\begin{array}{c}
0 \\
\downarrow_0
\end{array} = 0 \quad (27)
\]

Also note that for \( n \neq 0 \):

\[
\begin{array}{c}
\ast \\
\downarrow_n
\end{array} =
\begin{array}{c}
n \\
\downarrow_n
\end{array} -
\begin{array}{c}
0 \\
\downarrow_0
\end{array} =
\begin{array}{c}
n \\
\downarrow_n
\end{array} \quad (28)
\]

It can easily be shown that there exists a maximum of \( d \) commuting unitary operators in dimension \( d \). We now define partitioned UEBs (partitioned UEBs).

**Definition 26** (Partitioned unitary error basis \([3]\)). A partitioned unitary error basis (partitioned UEB), is a \( d \)-dimensional UEB containing the identity, with a partition \( \{ \text{id}_d \} \sqcup C_* \sqcup C_0 \sqcup ... \sqcup C_{d-1} \), such that each class \( C_i, i \in \{ *, 0, ..., d - 1 \} \) contains exactly \( d - 1 \) matrices, which together with \( \mathbb{I}_d \) form maximal classes of \( d \) commuting operators.

We now give a tensor diagrammatic characterization of partitioned UEBs. We assume that the partitioned UEB has been ordered such that \( U_{00} = \mathbb{I}_d, C_* = \{ U_{a0} | a \in \{ 1, ..., d-1 \} \} \) and for \( i \in \{ 0, ..., d-1 \} \), \( C_i = \{ U_{ik} | k \in \{ 1, ..., d-1 \} \} \). Up to equivalence (see Definition \([5]\) any partitioned UEB can be written in this way. We also choose a computational basis \( \dagger\text{-SCFA} \), such that the class \( C_* \) is diagonal with respect to it.

**Lemma 27.** A unitary error basis \( U \), with \( U_{00} \) equal to the identity, is a partitioned UEB iff the following tensor diagrammatic equation holds.

\[
\begin{array}{c}
U \\
\downarrow_0
\end{array} +
\begin{array}{c}
U \\
\downarrow_0
\end{array} =
\begin{array}{c}
U \\
\downarrow_0
\end{array} +
\begin{array}{c}
U \\
\downarrow_0
\end{array} \quad (29)
\]

**Proof.** To show the equivalence with Definition \([\text{26}]\) we compose with black states \( \downarrow, \downarrow_j, \downarrow_n, \downarrow_n \).
in the following way; for all $i, j, m, n$:

\[
\begin{align*}
U_i^{m} U_j^{n} + U_i^{m} U_j^{n} &= U_i^{n} U_j^{m} + U_i^{m} U_j^{n} \\
U_i^{m} U_j^{n} &= U_i^{n} U_j^{m} + U_i^{m} U_j^{n}
\end{align*}
\]

If $j = 0$ and $n \neq 0$ we obtain $0 + 0 = 0 + 0$, the first zero in each summand being due to equation (27), the second summands are multiplied by $\langle 0 | n \rangle = 0$. Similarly if $j \neq 0$ and $n = 0$ we obtain $0 + 0 = 0 + 0$. So no condition is imposed by equation (29) unless either, case one $j = n = 0$ or case two $j \neq 0$ and $n \neq 0$.

**Case one.** For $j = n = 0$, again by equation (27) we obtain for all $i, m$:

\[
0 + \langle 0 | 0 \rangle^2 U_{m0} U_{i0} = 0 + \langle 0 | 0 \rangle^2 U_{i0} U_{m0}
\]

\[
\Leftrightarrow U_{m0} U_{i0} = U_{i0} U_{m0}
\]

This shows that the class $C_*$ together with the identity, $U_{00}$ form a maximal class of commuting operators.

**Case two.** For $j \neq 0$ and $n \neq 0$ by equation (28) we obtain for all $i, m$:

\[
\langle i | m \rangle U_{m} U_{ij} + \langle 0 | j \rangle \langle 0 | n \rangle U_{m0} U_{i0} = \langle i | m \rangle U_{ij} U_{m} + \langle 0 | j \rangle \langle 0 | n \rangle U_{i0} U_{m0}
\]

\[
\Rightarrow \delta_{im} U_{m} U_{ij} = \delta_{im} U_{ij} U_{m}
\]

For $i \neq m$ this gives $0 = 0$, for $i = m$ we have that for each $i$ the $d - 1$ operators $U_{ik}$ with $k \in \{1, ..., d - 1\}$ pairwise commute. Thus the classes $C_i$ with $i \in \{0, ..., d - 1\}$ together with the identity $U_{00}$ form maximal classes of commuting operators. This completes the proof.

**Main results.** We first present the following theorem due to Bandyopadhyay et al [3].

**Theorem 28.** Given $U$, a partitioned UEB, the common eigenbases $| b_i^k \rangle$ for each class $C_i | i \in \{0, ..., d - 1\}$ form a maximal family of MUBs.

As a notational point we introduce $\theta$ to represent the map from partitioned UEBs to maximal families of MUBs given by Theorem 28. In their paper Banyopadhyay et al also give a construction which takes a maximal MUB in dimension $d$ and the Fourier matrix for the cyclic group $\mathbb{Z}_d$ and gives a partitioned UEB. The following construction generalises theirs.
In the following construction we will use a Hadamard $G$ and a controlled Hadamard $H$ (see Definition 19). Every Hadamard is equivalent to a Hadamard with ones along the first column and first row \cite{5}. We assume that each Hadamard in the controlled family as well as $G$ are in this form. This gives us the following axioms.

\begin{align}
H_0 &= H_0 = 0 \\
G_0 &= G_0 = 0
\end{align}

We now provide the main result of this section, a converse to Theorem 28 taking a maximal MUB and a controlled Hadamard to construct a partitioned UEB. We will later show in Theorem 30 that if we start with a partitioned UEB and then obtain a maximal MUB by taking the eigenbases (see Theorem 28) and then perform the following construction we recover the partitioned UEB we started with.

**Theorem 29.** Given a maximal MUB on a $d$ dimensional Hilbert space, a controlled Hadamard $H$ and an additional Hadamard $G$ the following map $\phi_H$ gives a partitioned UEB:

\begin{align}
\phi_H(M) := M + H + G
\end{align}

**Proof.** We show that $\phi_H(M)$ is a UEB. First the left hand equation of \cite{25}.

\begin{align}
\phi_H(M) \phi_H(M) := M + H + G
\end{align}
Now the right hand equation of (3):
We now show that equation (29) holds:

Let $\theta$ be the map that takes a partitioned UEB and gives the corresponding maximal family of MUBs according to Theorem 28. We now investigate the map $\theta$ and the infinite family of maps $\phi_H$ each taking a maximal family of MUBs and giving a partitioned UEB, given by Theorem 29 above. Given a controlled Hadamard, $H$ and a maximal family of MUBs we now consider the effect of the composition $\theta \circ \phi_H$ on the maximal family of MUBs:

**Theorem 30.** Given a maximal family of MUBs $M$ and a controlled Hadamard $H^i$:

$$\theta \circ \phi_H(M) = M$$

(32)
Proof.

\[ \phi_H(M) := \]

By design we have a partition \( \{ U_{00} \} \sqcup C_* \sqcup C_0 \sqcup \ldots \sqcup C_{d-1} \), where \( C_* = \{ U_{i0} | i \in \{ 1, \ldots, d-1 \} \} \) and for \( k \in \{ 0, \ldots, d-1 \}, C_k = \{ U_{aj} | j \in \{ 1, \ldots, d-1 \} \} \). Clearly the eigenbasis of \( C_* \) is the black basis. Let \( |b^i_j\rangle \) be the \( k \)th state of the \( i \)th basis of \( M \). We claim that \( |b^i_j\rangle \) is the \( k \)th eigenstate of \( C_i \). To see this consider the following composite linear map.

\[ \phi_H(M) \]

If we input black states \( i, j, k \) with \( k \neq 0 \) the above equation becomes \( \phi_H(M)_{ik} |b^i_j\rangle = [H]_{jk} |b^i_k\rangle \). Thus the bases of the original maximal family of MUBs are the eigenbases of \( \phi_H(M) \) as required.

Given that the composite map \( \theta \circ \phi_H \) is the identity we conclude that \( \theta \) is surjective and for all controlled Hadamards \( H \), the map \( \phi_H \) is injective. Given some maximal family of MUBs \( M \) and controlled Hadamard \( H \), the proof to Theorem 30 allows us to identify the eigenvalues of the partitioned UEB \( \phi_H(M) \) with the entries of the controlled Hadamard. This is precisely the information lost by the map \( \theta \). So every maximal family of MUBs corresponds to an infinite family of partitioned UEBs for different choices of eigenvalues. These UEBs are in general inequivalent. Also every partitioned UEB corresponds to a maximal family of MUBs with a particular choice of controlled Hadamard. This holds in any dimension, so the existence problem for maximal families of MUBs in arbitrary dimension can be phrased in terms of partitioned UEBs. Similarly for non-maximal families of MUBs we have corresponding UEBs with partial partition into maximally commuting sub-families.

4 Tensor diagrammatic finite fields

We now present a construction of partitioned UEBs from a finite field. In order to achieve this we first introduce a Tensor diagrammatic characterisation of finite fields, so that we can interpret finite fields as algebraic structures in Hilbert space. We begin this section by reviewing the Tensor diagrammatic properties of abelian groups in Hilbert space [13]. We recall how the character theory of abelian groups can be reconstructed using tensor
diagrams. This gives us a graphical representation of complex group algebras, and the usual complex character theory \cite{8,13}. We will then build on this framework to discuss finite fields as algebraic structures defined over Hilbert spaces.

4.1 Abelian groups

First we define abelian groups.

**Definition 31 (Abelian group).** \[17\] A set together with a binary operation is an abelian group if it is closed, unital, associative, commutative and every element has an inverse.

We continue with the convention that $\blacklozenge$ is a $\dagger$-SCFA and use the black states as an indexing set. In this case we are indexing the elements of the abelian group, and later the elements of the finite field.

**Unitality, associativity and commutativity.** We introduce another Frobenius algebra which will represent the binary operator of the group. Let $\blacklozenge$ be a $\dagger$-quasi-special commutative Frobenius algebra ($\dagger$-qSCFA). Since $\blacklozenge$ is a commutative Frobenius algebra it is unital, associative and commutative by definition.

**Closure.** It is the interaction of red and black that gives us the structure of a group. We require that $\blacklozenge$ is closed with respect to $\blacklozenge$, this means that we need $\blacklozenge$ to take pairs of black basis states to black basis states. This is equivalent to $\blacklozenge$ being a comonoid homomorphism for $\blacklozenge'$ (see Definition \[12\]). This is encapsulated by the following axiom:

**Definition 32 (Bialgebra).** A pair of unital associative algebras are a bialgebra if:

\[
\begin{align*}
\cdot & = \cdot \\
\cdot & = \cdot \\
\cdot & = \cdot
\end{align*}
\]

Note that the right hand side of the second equation is the empty diagram indicating the identity complex scalar $1$.

Given Frobenius algebras $\blacklozenge$ and $\blacklozenge'$, we call them Frobenius bialgebras if they obey the bialgebra laws. Note that the third bialgebra rule means that the red unit is copyable by, and thus a state of, the black basis. We will assume that the basis is ordered such that $\blacklozenge = \blacklozenge'$. Since we think of $\blacklozenge$ as a binary operation taking black states to other black states, the morphism $\blacklozenge$ should be real valued with respect to the black basis when considered as a linear map in Hilbert space. The following axiom gives this property in the general case:

**Definition 33 ( $\bullet$-real \[11\]).** In a $\dagger$-symmetric monoidal category given an object $A$ with a $\dagger$-SCFA $\blacklozenge$, a morphism $F : A^\otimes m \rightarrow A^\otimes m$ is $\bullet$-real if:

\[
\begin{align*}
\cdot & = \cdot
\end{align*}
\]
We require $\cdot$ to be $\circ$-real, which gives us the following equations:

\[
\begin{align*}
\circ & = \circ \\
\circ & = \circ \\
\end{align*}
\] (35)

**Inverses.** The following condition is equivalent to the binary operator having inverses, which gives us a Hopf algebra [11].

**Definition 34 (Strong complementarity).** Given two $\dagger$-commutative Frobenius algebras $\circ$ and $\circ$, they are *strongly complementary* if the following composite linear maps are both unitary:

\[
\begin{align*}
\circ & = \circ \\
\circ & = \circ \\
\end{align*}
\] (36)

The following theorem is useful in understanding how the character theory of an Abelian group can be derived using tensor diagrams.

**Theorem 35 ([27], Theorem 9).** A pair of $\dagger$-commutative Frobenius algebras are strongly complementarity if and only if the corresponding bases are mutually unbiased.

**Character group.** We shall now see that the basis states copyable by $\dagger$ form the character group, which has binary operator $\circ$. Let $\chi$ be the change of basis linear map that maps the red basis to the black basis up to a normalization factor, as follows:

\[
\begin{align*}
\circ & = \frac{1}{d} \\
\circ & = \frac{1}{d} \\
\end{align*}
\] (37)

Considering the discussion below Definition 7 and Theorem 35 we can see that $\chi$ is a Hadamard. This Hadamard is the Fourier transform of the group and its rows, which are the copyable states of the red basis, are the irreducible characters. Apart from the axioms for a Hadamard (see Definition 7), it can easily be shown that the following equations hold which gives us the expected character theory.

\[
\begin{align*}
\circ & = \circ \\
\circ & = \circ \\
\end{align*}
\] (38)

Let $\chi_i(x) := \langle i|\chi|x \rangle$ so $\chi_i(x)$ is the $i$th irreducible character applied to an element of the group $x$. The left hand equation is then equivalent to; for all $i, x, y \in \{0, ..., d-1\}$, $\chi_i(x + y) = \chi_i(x)\chi_i(y)$ which is the expected property of a character. For a more detailed discussion of the above please refer to the 2015 paper by Gogioso and Zeng [13]. We summarize the results of this subsection in the following theorem:
Theorem 36. [12] Given a $d$-dimensional Hilbert space with a $\dagger$-$q$SCFA $\bigcirc$, and a $\dagger$-SCFA $\blacktriangle$, the following are equivalent:

- The copyable states of $\bigcirc$, form an abelian complex group algebra under the linearly extended binary operator $\bigcirc$;
- The algebras $\bigcirc$ and $\blacktriangle$ form a strongly complementary bialgebra and $\blacktriangle$ is $\bullet$-real.

4.2 Finite fields

We now define a finite field.

Definition 37 (Finite field). [17] A finite set $A$ together with closed binary operators $\bullet$ and $\circ$ is a finite field if:

- Addition: The operator $\bullet$ is an abelian group on the set $A$ with unit $0 \in A$;
- Multiplication: The operator $\circ$ is an abelian group on the subset $A' := A \setminus \{0\}$;
- Distributivity: For all $a, b, c \in A$, $a \bullet (b \circ c) = (a \bullet b) \circ (a \bullet c)$.

Finite fields only exist in prime power dimensions [17] so we take $d = p^n$ for some prime $p$ and $n \in \mathbb{N}$, and as usual take the black wires to represent the Hilbert space $\mathcal{H} \cong \mathbb{C}^d$.

Addition. In formulating a diagrammatic notation for finite fields as algebraic structures defined over Hilbert spaces we start with an abelian group algebra representing addition. We therefore take $\bigcirc$ and $\blacktriangle$ to be a pair of strongly complementary $\dagger$-commutative Frobenius bialgebras with black special, red quasi-special and red $\bullet$-real. As seen in the last subsection $\bigcirc$ copies the additive characters which form the columns of a Fourier Hadamard matrix on $\mathcal{H}$ which we will again call $\chi$ with the formal definition given by equation (37).

Multiplication. The multiplication of a finite field also forms an abelian group on the non-zero elements. We introduce another Hilbert space, $\mathcal{H}' \cong \mathbb{C}^{d-1}$ which we represent as green wires, and a $\dagger$SCFA $\bigcirc$. We also introduce linear maps to relate the green and black Hilbert spaces:

$$p := \begin{array}{c}
\bullet \\
\end{array} \quad \iota := \begin{array}{c}
\circ \\
\end{array}$$  (39)

We require the following relationships between $p, \iota, \bigcirc, \blacktriangle$, and $\bullet$:

$$= \begin{array}{c}
\bullet \\
\end{array} \quad = \begin{array}{c}
\circ \\
\end{array} \quad = \begin{array}{c}
\bigcirc \\
\end{array} \quad = \begin{array}{c}
\blacktriangle \\
\end{array} \quad = \begin{array}{c}
\bullet \\
\end{array}$$  (40)

We will assume that the black basis has been ordered such that $\downarrow_0 := \bullet$. This makes $p$ and $\iota$ an isomorphism between $\mathcal{H}'$ and the $d - 1$-dimensional subspace of $\mathcal{H}$ spanned by
the non-zero black states. This isomorphism takes the green basis states to the non-zero black states. The Hilbert space $\mathcal{H}'$ is the analogue of the set $A'$ in Definition 37.

The following lemma shows that the linear map given by $\iota \circ p$ is equal to the projector defined by equation (26), we will make use of this projector.

**Lemma 38.** The following equation holds.

$$= -$$

(41)

**Proof.** By the 4th equation of (40):

$$= - \Rightarrow = -$$

In light of this lemma we will again use $*$ to denote this projector.

$$:=$$

We now introduce the multiplication acting on the subspace $\mathcal{H}'$ which we represent as $\circ$. We require that $\circ$ is a $\hat{\mathcal{T}}$-qSCFA, and that $\circ$, and $\circ$ are a strongly complementary bialgebra. Thus $\circ$ is an abelian group on the green basis states. This also tells us that the yellow unit is a green basis state and thus isomorphic to a black basis state not equal to the red unit. We corrupt notation slightly to represent this state as $\circ$. We denote the multiplicative character Fourier Hadamard matrix as $\psi$ formally defined as follows:

$$= \frac{1}{d-1}$$

(42)

We now introduce the multiplication on the whole Hilbert space $\mathcal{H}$, which we denote $\otimes$. We define $\otimes$, and $\otimes$ as follows:

$$:= + + +$$

(43)
This ensures that \( \langle \rangle \) agrees with \( \langle \rangle \) on the subspace isomorphic to \( \mathcal{H}' \). The linear map \( \langle \rangle \) is associative, commutative and unital with unit \( \langle \rangle \), as can easily be proven from the axioms and the definition of \( \langle \rangle \). We also require that \( \langle \rangle \) and \( \langle \rangle \) form a bialgebra (this implies the requirement already made that \( \langle \rangle \) and \( \langle \rangle \) form a bialgebra). Although \( \langle \rangle \) and \( \langle \rangle \) are not strongly complementary following condition can easily be derived from the definitions:

\[
\begin{align*}
\text{Distributivity.} \quad \text{Finally we relate } \langle \rangle \text{ and } \langle \rangle \text{ as follows.}

\text{Definition 39 (Left distributivity). Let } \langle \rangle \text{ and } \langle \rangle \text{ each form a bialgebra with } \langle \rangle \text{. Yellow left distributes over red if the following equation holds:}

\begin{align*}
\begin{array}{c}
\begin{array}{c}
\text{\langle \rangle left distributes over } \langle \rangle \text{ if}
\end{array}
\end{array}
\end{align*}

\[\text{(46)}\]

Right distributivity is defined by reflecting both sides of equation (46) in a vertical axis. We now show that right distributivity follows from left distributivity and commutativity.

\text{Lemma 40. If } \langle \rangle, \langle \rangle, \text{ and } \langle \rangle \text{ are commutative and yellow left distributes over red, then yellow right distributes over red; so the following equation holds:}

\begin{align*}
\begin{array}{c}
\begin{array}{c}
\text{\langle \rangle left distributes over } \langle \rangle \text{ if}
\end{array}
\end{array}
\end{align*}

\[\text{(47)}\]

\text{Proof.}

\[\text{(44)}\]
Additive characters. We also require the following interaction between the yellow unit and $\chi$:

\[
\begin{align*}
\begin{array}{c}
\text{LHS} \\
\text{RHS}
\end{array}
\end{align*}
\]

This corresponds to the following algebraic equation which can be recovered by composing by computational basis states: $\chi_a(b) = \chi_1(a \cdot b)$.

Definition 41 (Complex finite field). Given a $d$-dimensional Hilbert space represented by black wires and a $d-1$-dimensional Hilbert space represented by green wires, a complex finite field is a pair of †-SCFAs $\downarrow$, a pair of †-qSCFAs $\uparrow$, as well as $\downarrow$, as defined by equation (14), linear maps $\chi$ and $\psi$ defined by equations (37) and (42), linear maps $p$ and $\iota$ defined by equation (39) and obeying equations (26) such that equations (46) and (48) hold. We denote a complex finite field $(\downarrow, \uparrow, \chi, \psi)$.

We summarize the results of this subsection in the following theorem.

Theorem 42. Given a complex finite field $(\downarrow, \uparrow, \chi, \psi)$, $\downarrow$ and $\uparrow$ are the linear extension of the addition and multiplication respectively of a finite field with the underlying set of elements given by the states copyable by $\downarrow$. $\chi$ and $\psi$ are the complex Fourier Hadamards for the additive and multiplicative groups respectively.

Proof. The binary operator $\downarrow$ forms an abelian group on the states copyable by $\downarrow$, which is the first axiom of Definition 37. On the subspace of $\mathcal{H}$ isomorphic to $\mathcal{H}'$ which is spanned by the non-zero black states $\downarrow$, agrees with $\downarrow$, and thus forms an abelian group, thus fulfilling the second axiom of Definition 37. Equation (46) is precisely the linear extension of distributivity, the third axiom of Definition 37. The properties of $\chi$ and $\psi$ were proven by Gogioso and Zeng [13].

4.3 A construction of $d+1$ MUBs

We now give an application of the complex finite fields developed in the previous subsection to the problem of constructing maximal families of MUBs. First we present two lemmas which will be necessary to proving the main result of this section.

Lemma 43. Given a complex finite field $(\downarrow, \uparrow, \chi, \psi)$, the following equation holds:

\[
\begin{align*}
\begin{array}{c}
\text{LHS} \\
\text{RHS}
\end{array}
\end{align*}
\]

Proof.
Lemma 44. Given a complex finite field \((\chi, \psi, \alpha, \beta, \gamma, \delta, \chi, \psi)\), the following equation holds:

\[
\begin{align*}
\text{LHS} & \hspace{2cm} \text{RHS} \\
\end{align*}
\]

Proof.

We construct a partitioned UEB as follows:

Theorem 45. Given a complex finite field \((\chi, \psi, \alpha, \beta, \gamma, \delta, \chi, \psi)\) the following is a partitioned UEB:

\[
U_{FF} := \chi + \rho \quad \hfill (51)
\]
Proof. We first prove that $U_{FF}$ is a UEB. We do this by showing that $U_{FF}$ is equivalent to a shift and multiply basis. First we rearrange equation (51).

\[ U_{FF} := \chi^* + (7) \]

We now prove that the following linear map $P$, as defined below, is a permutation.

\[ P := \chi^* + \chi \]

First we show that equation (19) holds for $P$.

\[ P \cdot P := \chi^* \chi^* + (45) \]

Now we show that equation (20) holds for $P$.

\[ P \cdot P := \chi \chi^* + (33) \]

\[ = \chi \chi^* + (13) \]

\[ = \chi^* + (13) \]

\[ = P \cdot P \]
So $P$ is a permutation and so $U_{FF}$ is equal to $V_{P(i,j)}$, where $V_{ij}$ is given by the following:

\[ V_{ij} := \chi_{ij} \]  

(52)

Since $\chi$ is a finite abelian group it is a finite quasigroup and thus a Latin square. $\chi$ is a Hadamard and so $V$ is a shift and multiply basis, and therefore a UEB \[10, 20, 25\]. $V$ and $U_{FF}$ are equivalent by equation (51), and so $U_{FF}$ is a UEB.

**Commuting property.** We now prove the following:

\[ U_{FF} \circ U_{FF} = U_{FF} \]  

(53)

\[ U_{FF} \circ U_{FF} := \chi^{\chi} + \chi^{\chi} + \chi^{\chi} + \chi^{\chi} \]  

\[ = 0 + 0 + 0 + 0 \]  

Red (13)

\[ U_{FF} \circ U_{FF} := \chi^{\chi} + \chi^{\chi} + \chi^{\chi} + \chi^{\chi} \]  

\[ = 0 + 0 + 0 + 0 \]  

Red (13)
We now show that:

\[
\begin{align*}
U_F F & = U_F F \\
U_F F & := U_F F \\
& + 0 + 0 + 0
\end{align*}
\]
We can now combine equations (53) and (54) to obtain the following:

\[
\begin{align*}
    \chi &= \begin{pmatrix}
        1 & 1 & 1 & 1 \\
        1 & 1 & -1 & -1 \\
        1 & -1 & -1 & 1 \\
        1 & -1 & 1 & -1 \\
    \end{pmatrix}
\end{align*}
\]

This concludes the proof.

We now present an example of a partitioned UEB in dimension \(d = 4\) constructed from the finite field \(\mathbb{F}_4\).

**Example 46.** The Fourier transform Hadamard for the additive group of \(\mathbb{F}_4\) is given by the following matrix.

\[
\chi := \begin{pmatrix}
    1 & 1 & 1 & 1 \\
    1 & 1 & -1 & -1 \\
    1 & -1 & -1 & 1 \\
    1 & -1 & 1 & -1 
\end{pmatrix}
\]

Let \(M_{ij} := \begin{pmatrix} U_{FF} \end{pmatrix}_{ij} \) with \(U_{FF}\) as defined in equation (51), then the partitioned UEB,
with partitions $C_x, x \in \{*, 0, ..., 3\}$, is as follows:

\[
\mathcal{M}_{00} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad \mathcal{M}_{01} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad \mathcal{M}_{02} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad \mathcal{M}_{03} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad \mathcal{M}_{10} = \begin{pmatrix}
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}, \quad \mathcal{M}_{11} = \begin{pmatrix}
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}, \quad \mathcal{M}_{12} = \begin{pmatrix}
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}, \quad \mathcal{M}_{13} = \begin{pmatrix}
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}, \quad \mathcal{M}_{20} = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad \mathcal{M}_{21} = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad \mathcal{M}_{22} = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad \mathcal{M}_{23} = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad \mathcal{M}_{30} = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0
\end{pmatrix}, \quad \mathcal{M}_{31} = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0
\end{pmatrix}, \quad \mathcal{M}_{32} = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0
\end{pmatrix}, \quad \mathcal{M}_{33} = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0
\end{pmatrix}
\]

The partitions are:

- $C_* := \{\mathcal{M}_{10}, \mathcal{M}_{20}, \mathcal{M}_{30}\}$
- $C_0 := \{\mathcal{M}_{01}, \mathcal{M}_{02}, \mathcal{M}_{03}\}$
- $C_1 := \{\mathcal{M}_{11}, \mathcal{M}_{12}, \mathcal{M}_{13}\}$
- $C_2 := \{\mathcal{M}_{21}, \mathcal{M}_{22}, \mathcal{M}_{23}\}$
- $C_3 := \{\mathcal{M}_{31}, \mathcal{M}_{32}, \mathcal{M}_{33}\}$

Thus since $\mathcal{M}_{00} = I_4$, we have:

$$U_{FF} = \{I_4\} \cup C_* \cup C_0 \cup C_1 \cup C_2 \cup C_3$$

It can easily be verified that this is a partition into maximal commuting sub-families and that $U_{FF}$ is a UEB.

## 5 conclusion

We have introduced a tensor diagrammatic characterization of maximal families of MUBs, partitioned unitary error bases, Hadamards and controlled Hadamards. As an application of these tensor diagrammatic characterizations we have introduced a new construction for a partitioned UEB from a maximal family of MUBs extending work by Bandyopadhyay [3], which makes clear the exact nature of the correspondence between partitioned UEBs and maximal families of MUBs. Each partitioned UEB gives rise to a unique maximal family of MUBs. Each maximal family of MUBs gives rise to an infinite family of possibly inequivalent partitioned UEBs each partitioned UEB corresponding to a choice of controlled Hadamard. Further work in this direction is to investigate whether the property of monomiality of UEBs, introduced by Wocjan et al [4], is invariant under the choice of controlled Hadamard in our construction to ensure the property is well defined.

We have also introduced a tensor diagrammatic characterization of finite fields as algebraic structures defined over Hilbert spaces, extending existing characterizations of abelian groups [13]. As an application of this and a further application of the tensor diagrammatic characterizations of partitioned UEBs we introduced a new construction of partitioned UEBs and thus maximal families of MUBs from a finite field. This is different from the construction due to Bandyopadhyay et al [3], with the partition being easier to calculate. Further work is necessary to investigate whether this construction could be adapted to one requiring less structure than that of a finite field.
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A Minor lemmas

In this section we present a number of minor diagrammatic lemmas that are essential to the proofs of the main Theorems. We assume throughout that all wires are $d$-dimensional Hilbert spaces and the operators are as defined in Section I.

Lemma 47. Given a controlled Hadamard $H^i$ and $\dagger$-SCFA $\blacklozenge$, the following equation holds.

\[
\begin{align*}
\text{Lemma 47. Given a controlled Hadamard } H^i \text{ and } \dagger\text{-SCFA } \blacklozenge, \text{ the following equation holds.}
\end{align*}
\]
Proof. We take the LHS of equation (55):

\[
H^* (7) = H^* (30) = 0 \quad (56)
\]

Lemma 48. The following equation holds.

\[
= 0 \quad (57)
\]

Proof. By Lemma 38:

\[
= - \Rightarrow = - \Rightarrow = 0
\]

Lemma 49. The following equation holds.

\[
= \quad (58)
\]

Proof. By Lemma 48 and Definition 32:

\[
= \quad = \quad = \quad = 0
\]