Sub-Riemannian (2, 3, 5, 6)-Structures

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Abstract—We describe all Carnot algebras with growth vector (2, 3, 5, 6), their normal forms, an invariant that separates them, and a change of basis that transforms such an algebra into a normal form. For each normal form, Casimir functions and symplectic foliations on the Lie coalgebra are computed. An invariant and normal forms of left-invariant (2, 3, 5, 6)-distributions are described. A classification, up to isometries, of all left-invariant sub-Riemannian structures on (2, 3, 5, 6)-Carnot groups is obtained.

Keywords: sub-Riemannian geometry, Carnot algebras, Carnot groups, left-invariant sub-Riemannian structures

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Sub-Riemannian structures [1] are stratified in depth, i.e., with respect to the minimum order of Lie brackets required for generating a tangent space from basis vector fields. The complexity of sub-Riemannian (SR) structures grows substantially with increasing step. At present, SR structures of step at most three have been examined in detail [2–8]. Accordingly, a task of great interest is to systematically investigate SR structures of step 4. The study of the simplest of these structures, namely, a nilpotent SR structure with growth vector (2, 3, 5, 8) (see Example 2 below) was initiated in [9–11]. Below, we obtain a complete classification of nilpotent SR structures and distributions with growth vector (2, 3, 5, 6). It is shown that all such structures are quotient structures of a nilpotent SR structure with growth vector (2, 3, 5, 8).

1. SUB-RIEMANNIAN QUOTIENT STRUCTURES

In this paper, all Lie algebras are considered over the field \( \mathbb{R} \).

Definition 1. A nilpotent Lie algebra \( \mathfrak{g} \) is called a Carnot algebra if

(i) \( \mathfrak{g} \) is graded:

\[ \mathfrak{g} = \mathfrak{g}^{(1)} \oplus \cdots \oplus \mathfrak{g}^{(s)}, \]

\[ [\mathfrak{g}^{(i)}, \mathfrak{g}^{(j)}] \subset \mathfrak{g}^{(i+j)}, \quad \mathfrak{g}^{(k)} = \{0\} \text{ for } k > s, \] (1)

(ii) \( \mathfrak{g} \) is generated by the first component:

\[ \operatorname{Lie}(\mathfrak{g}^{(1)}) = \mathfrak{g}. \] (2)

The corresponding connected simply connected Lie group is called a Carnot group.

Conditions (i) and (ii) are equivalent to the condition

\[ [\mathfrak{g}^{(i)}, \mathfrak{g}^{(j)}] = \mathfrak{g}^{(i+j)}, \quad \mathfrak{g}^{(k)} = \{0\} \text{ for } k > s. \]

Definition 2. The growth vector of a Carnot algebra \( \mathfrak{g} \) is defined as

\[ (n_1, \ldots, n_s), \quad n_j = \sum_{i=1}^{j} \dim \mathfrak{g}^{(i)}. \]

Here, \( (n_1, \ldots, n_s) \) is the growth vector of a left-invariant distribution on the Lie group of the Lie algebra \( \mathfrak{g} \) generated by the subspace \( \mathfrak{g}^{(1)} \subset \mathfrak{g} \).

Let \( M \) be a smooth manifold. A sub-Riemannian structure on \( M \) [1] is a pair \((\Delta, g)\) consisting of a vector distribution \( \Delta \subset TM \) and a scalar product \( g \) in \( \Delta \).

Let \( G \) be a Lie group and \( \mathfrak{g} \) be its Lie algebra. A left-invariant SR structure on \( G \) consists of a left-invariant distribution on \( G \) and a left-invariant scalar product in the distribution. This structure is specified by a subspace \( \Delta \subset \mathfrak{g} \) and a scalar product \( g \) in \( \Delta \). In this case, we say that \((\Delta, g)\) is an SR structure in the algebra \( \mathfrak{g} \).

Left-invariant SR structures on Carnot groups arise as nilpotent approximations of general SR structures on smooth manifolds [1].

Definition 3. Let \((\Delta, g)\) be an SR structure in a Lie algebra \( \mathfrak{g} \), and let \( i \subset \mathfrak{g} \) be an ideal such that \( \Delta \cap i = \{0\} \). Let \( \tilde{\mathfrak{g}} = \mathfrak{g}/i \) be a quotient algebra and \( \pi: \mathfrak{g} \to \tilde{\mathfrak{g}} \) be a
canonical projection. Define $\tilde{\Delta} = \pi(\Delta)$ and $\tilde{g}(\pi(X), \pi(Y)) = g(X, Y)$ for $X, Y \in \Delta$. Then $(\tilde{\Delta}, \tilde{g})$ is an SR structure in $\tilde{g}$, which will be called a quotient structure of the SR structure $(\Delta, g)$.

Example 1. Let $g^3$ be a free nilpotent Lie algebra of step 3 with two generators (Cartan algebra); this is the Carnot algebra with growth vector $(2, 3, 5)$. There exists a basis $g^3 = \text{span}(X_1, \ldots, X_8)$ in which the nonzero Lie brackets are

$$\begin{align*}
[X_1, X_2] &= X_3, & [X_1, X_3] &= X_4, & [X_2, X_3] &= X_5, \\
[X_1, X_4] &= X_6, & [X_1, X_5] &= [X_2, X_4] &= X_7, \\
[X_2, X_5] &= X_8.
\end{align*}$$

Consider an SR structure $(\Delta, g)$ in $g^3$ with an orthonormal frame $(X_1, X_2)$ [8]. Sequentially choosing the subspaces $\mathbb{R}X_5$, $\text{span}(X_4, X_5)$, and $\text{span}(X_3, X_4, X_5)$ as an ideal $i \subset g^3$, we obtain SR quotient structures in the Engel algebra (growth vector $(2, 3, 4)$) [7], Heisenberg algebra (growth vector $(2, 3)$) [2], and the two-dimensional commutative algebra $\mathbb{R}^2$ (growth vector $(2)$).

Example 2. Let $g^8$ be a free nilpotent Lie algebra of step 4 with two generators; this is the Carnot algebra with growth vector $(2, 3, 5, 8)$, which will be called a nilpotent $(2, 3, 5, 8)$-algebra. There exists a basis $g^8 = \text{span}(X_1, \ldots, X_8)$ in which the nonzero Lie brackets are given by

$$\begin{align*}
[X_1, X_2] &= X_3, & [X_1, X_3] &= X_4, & [X_2, X_3] &= X_5, \\
[X_1, X_4] &= X_6, & [X_1, X_5] &= [X_2, X_4] &= X_7, \\
[X_2, X_5] &= X_8.
\end{align*}$$

Consider an SR structure $(\Delta, g)$ in $g^8$ with an orthonormal frame $(X_1, X_2)$ [9–11]. It is easy to see that this SR structure is unique, up to a Lie algebra automorphism. SR structure in $g^8$ of rank 2 satisfying the condition $\text{Lie}(\Delta) = g^8$, we call it a nilpotent SR $(2, 3, 5, 8)$-structure. In what follows, we will use a dual basis in the Lie coalgebra $(g^8)^*$:

$$\omega_1, \ldots, \omega_8 \in (g^8)^*, \quad \omega_i(X_j) = \delta_{ij}, \quad i, j = 1, \ldots, 8.$$

The goal of this paper is, given a structure $(\Delta, g)$, to describe all SR quotient structures for two-dimensional ideals $i \subset g^8$. These are exactly nilpotent SR structures with growth vector $(2, 3, 5, 6)$.

It is easy to see that a two-dimensional subspace $i \subset g^8$ is an ideal if and only if $i \subset Z(g^8) = \text{span}(X_6, X_7, X_8)$. Therefore, any quotient algebra $g^8/i$ by the two-dimensional ideal $i$ is a Carnot algebra with growth vector $(2, 3, 5, 6)$. Let us describe such algebras.

2. CARNOT ALGEBRAS WITH GROWTH VECTOR $(2, 3, 5, 6)$

**Theorem 1.** (i) In any Carnot algebra $g$ with growth vector $(2, 3, 5, 6)$, there is a basis $g = \text{span}(X_1, \ldots, X_6)$ in which all nonzero Lie brackets have the form

$$\begin{align*}
[X_1, X_2] &= X_3, & [X_1, X_3] &= X_4, & [X_2, X_3] &= X_5, \\
[X_1, X_4] &= \alpha X_6, & [X_1, X_5] &= [X_2, X_4] &= \beta X_6, \\
[X_2, X_5] &= \gamma X_6, \\
(\alpha, \beta, \gamma) &\in \mathbb{R}^3 \setminus \{0\}.
\end{align*}$$

(ii) Any two Carnot algebras with growth vector $(2, 3, 5, 6)$ that have proportional triples $(\alpha, \beta, \gamma)$ in (6) are isomorphic to each other. Accordingly, such algebras are denoted by $g^6_{\alpha, \beta, \gamma}$ $(\alpha : \beta : \gamma) \in \mathbb{R}P^2$.

(iii) The following isomorphism of Lie algebras holds:

$$g^6_{\alpha, \beta, \gamma} \cong g^8/(\ker(\alpha \circ Z(g^8))), \quad \omega = \alpha \omega_0 + \beta \omega_7 + \gamma \omega_8 \neq 0.$$

Each basis in the algebra $g^6_{\alpha, \beta, \gamma}$ with multiplication table (5), (6) is associated with a quadratic form $Q(x, y) = \alpha x^2 + 2\beta xy + \gamma y^2$. Depending on the sign of its discriminant $s = \text{sgn}(\alpha \gamma - \beta^2) \in \{0, \pm 1\}$, the algebra $g^6_{\alpha, \beta, \gamma}$ is called

- parabolic if $s = 0$,
- elliptic if $s = 1$,
- hyperbolic if $s = -1$.

**Remark 1.** Depending on the number $s = \text{sgn}(\alpha \gamma - \beta^2)$, the topology of the sets of triples $(\alpha : \beta : \gamma) \in \mathbb{R}P^2$ is as follows:

$$\begin{align*}
A_p &= \{ (\alpha : \beta : \gamma) \in \mathbb{R}P^2 \mid s = 0 \}, \\
A_E &= \{ (\alpha : \beta : \gamma) \in \mathbb{R}P^2 \mid s = 1 \}, \\
A_H &= \{ (\alpha : \beta : \gamma) \in \mathbb{R}P^2 \mid s = -1 \}.
\end{align*}$$

Topologically, $A_p$ is a circle, $A_E$ is an open disk, and $A_H = \mathbb{R}P^2 \setminus (A_p \cup A_E)$ is a projective plane with a cutout hole, i.e., a Möbius strip.

Below, we give several examples of Carnot algebras with growth vector $(2, 3, 5, 6)$, together with nonzero Lie brackets in the corresponding basis. Recall that the notation $N_{6,2,*}$ for these algebras was used in [12, 13]. A classification of nilpotent Lie algebras of dimension ≤7 was obtained in [12], and all Carnot algebras of dimension ≤7 were described in [13].

**Example 3.** The parabolic algebra $g^6_{1,0,0} = N_{6,2,7}$ is spanned by $(X_1, \ldots, X_6)$:

$$\begin{align*}
[X_1, X_2] &= X_3, & [X_1, X_3] &= X_4, \\
[X_2, X_3] &= X_5, & [X_1, X_4] &= X_6.
\end{align*}$$
Example 4. The hyperbolic algebra $\mathfrak{g}_{1:0(-1)} = N_{6,2,5} = \text{span}(X_1, ..., X_6)$,

$$[X_1, X_2] = X_3, \quad [X_1, X_3] = X_4, \quad [X_2, X_3] = X_5, \quad [X_1, X_4] = [X_2, X_5] = X_6.$$ 

Example 5. The elliptic algebra $\mathfrak{g}_{1:0:1} = N_{6,2,5a} = \text{span}(X_1, ..., X_6)$,

$$[X_1, X_2] = X_3, \quad [X_1, X_3] = X_4, \quad [X_2, X_3] = X_5, \quad [X_1, X_4] = [X_2, X_5] = X_6.$$ 

Theorem 2. (i) The algebras $\mathfrak{g}_{1:0:0}$, $\mathfrak{g}_{1:0(-1)}$, and $\mathfrak{g}_{1:0:1}$ are pairwise nonisomorphic. Any algebra $\mathfrak{g}_{s:0:1}$, $(\alpha : \beta : \gamma) \in \mathbb{R} P^2$, is isomorphic to one of these algebras.

(ii) The number $s = \text{sgn}(\alpha \gamma - \beta^2) \in \{0, \pm 1\}$ is an invariant of the Carnot algebra $\mathfrak{g}$; $s$ is called the signature of $\mathfrak{g}$.

The algebras $\mathfrak{g}_{s:0:1}$ are separated by the signature $s$:

$$\mathfrak{g}_{s:0:1} \cong \mathfrak{g}_{1:0:0} \iff s = 0 \quad (\text{parabolic algebra}),$$
$$\mathfrak{g}_{s:0:1} \cong \mathfrak{g}_{1:0(-1)} \iff s = -1 \quad (\text{hyperbolic algebra}),$$
$$\mathfrak{g}_{s:0:1} \cong \mathfrak{g}_{1:0:1} \iff s = 1 \quad (\text{elliptic algebra}).$$

Thus, the signature $s \in \{0, \pm 1\}$ is an invariant separating three classes of isomorphism of the algebras $\mathfrak{g}_{s:0:1}$.

Remark 2. Item (i) in Theorem 2 was first proved in [13]. We proved it independently, together with an algorithm for reducing the multiplication tables in the $(2, 3, 5, 6)$-algebra to the normal form in Examples 3–5.

To find a basis change in the algebra $\mathfrak{g} = \mathfrak{g}_{s:0:1}$ that transforms the basis into one of the normal forms indicated in Examples 3–5, it suffices to reduce the quadratic form $Q(x, y) = \alpha x^2 + 2\beta xy + \gamma y^2$ to a sum of squares, to apply this change to the basis $(X_1, X_2)$ of the space $\mathfrak{g}^{(1)}$, and to normalize the vector $X_6$ generating the space $\mathfrak{g}^{(4)}$.

3. NILPOTENT (2, 3, 5, 6)-DISTIBUTIONS AND SUB-RIEMANNIAN STRUCTURES

Definition 4. Let $\mathfrak{g}$ be a Carnot algebra with growth vector $(2, 3, 5, 6)$. A nilpotent sub-Riemannian $(2, 3, 5, 6)$-structure is a sub-Riemannian structure $(\Delta, g)$ in $\mathfrak{g}$ satisfying the conditions

$$\text{dim} \Delta = 2, \quad \text{Lie}(\Delta) = \mathfrak{g},$$

or, equivalently, the equality $\Delta \oplus \mathfrak{g}^{(2)} \oplus \mathfrak{g}^{(3)} \oplus \mathfrak{g}^{(4)} = \mathfrak{g}$. The corresponding plane $\Delta \subset \mathfrak{g}$ is called a nilpotent $(2, 3, 5, 6)$-distribution in $\mathfrak{g}$.

The vector $(2, 3, 5, 6)$ is the growth vector of the left-invariant distribution on the Lie group $G$ of the Lie algebra $\mathfrak{g}$ defined by the plane $\Delta \subset \mathfrak{g}$.

Theorem 3. Let $\mathfrak{g}$ be a Carnot algebra with growth vector $(2, 3, 5, 6)$ of signature $s \in \{0, \pm 1\}$. For any nilpotent $(2, 3, 5, 6)$-distribution $\Delta \subset \mathfrak{g}$, there exists a frame $\Delta = \text{span}(X_1, X_2)$ such that

$$[X_1, X_2] = X_3, \quad [X_1, X_3] = X_4, \quad [X_2, X_3] = X_5,$$

$$[X_1, X_4] = X_6, \quad [X_1, X_5] = [X_2, X_4] = 0, \quad [X_2, X_5] = sX_6.$$ 

Theorem 4. Let $\mathfrak{g}$ be a Carnot algebra with growth vector $(2, 3, 5, 6)$. For any nilpotent SR $(2, 3, 5, 6)$-structures in $\mathfrak{g}$, there exists an orthonormal frame $(X_1, X_2)$ and a number $v \in [-1, 1]$ for which

$$[X_1, X_2] = X_3, \quad [X_1, X_3] = X_4, \quad [X_2, X_3] = X_5,$$

$$[X_1, X_4] = X_6, \quad [X_1, X_5] = [X_2, X_4] = 0, \quad [X_2, X_5] = vX_6.$$ 

The number $v \in [-1, 1]$ in (12) is called the canonical parameter of the SR structure $(\Delta, g)$. It is easy to see that $v$ is equal to the ratio of the smaller (in absolute value) eigenvalue of $Q$ to the larger eigenvalue. Moreover, $s = \text{sgn} v$.

Theorem 5. The canonical parameter $v \in [-1, 1]$ is an invariant of the nilpotent SR $(2, 3, 5, 6)$-structure.

Let $(\Delta, g)$ be an SR structure on a manifold $M$. The corresponding SR distance $d$ [1] transforms $M$ into a metric space. Recall that an isometry between metric spaces $(M, d)$ and $(\bar{M}, \bar{d})$ is a mapping $F: M \to \bar{M}$ such that

$$d(x, y) = \bar{d}(F(x), F(y)), \quad x, y \in M.$$ 

The metric group [15] is a Lie group with a left-invariant distance inducing a manifold topology on this group. In particular, any Carnot group is a metric group. It was proved in [15] that any isometry between metric groups is an analytic mapping;

any isometry between connected nilpotent metric groups is an affine mapping (i.e., the composition of a left shift and an isomorphism).

The following theorem provides a classification of left-invariant SR $(2, 3, 5, 6)$-structures up to isometries of corresponding Carnot groups.

Theorem 6. Let $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ be nilpotent Carnot $(2, 3, 5, 6)$-algebras, and let $G$ and $\tilde{G}$ be corresponding Carnot groups. Let $(\Delta, g)$ and $(\tilde{\Delta}, \tilde{g})$ be nilpotent SR $(2, 3, 5, 6)$-structures in $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$, and $\nu$ and $\tilde{\nu}$ be corresponding canoni-
cal parameters, and \( d \) and \( \tilde{d} \) be corresponding left-invariant SR metrics on \( G \) and \( \tilde{G} \).

The metric spaces \((G, d)\) and \((\tilde{G}, \tilde{d})\) are isometric if and only if \( \nu = \tilde{\nu} \).

**Theorem 7.** Let \( \mathfrak{g} = \mathfrak{g}^6 \) be the \((2, 3, 5, 8)\)-algebra from Example 2 with a basis \((X_1, \ldots, X_8)\) according to (3), (4), and let \((\Delta, g)\) be an SR structure in \( \mathfrak{g} \) with orthonormal frame \((X_1, X_2)\). Let \( i = \text{ker} \omega \cap Z(\mathfrak{g}) \) with \( \omega = \alpha_0 \omega_0 + \beta_0 \omega_0 + \gamma_0 \neq 0 \) be any 2-dimensional subspace of \( Z(\mathfrak{g}) \). Then the SR quotient structure \((\tilde{\Delta}, \tilde{g})\) is one of the \( \text{SR} (2, 3, 5, 6)\)-structures in \( \tilde{\mathfrak{g}} = \mathfrak{g}/i \):

(i) in the parabolic case \( s = 0 \), we have \( \tilde{\mathfrak{g}} \cong \mathfrak{g}^{6:0:6} \);
(ii) in the hyperbolic case \( s = -1 \), we have \( \tilde{\mathfrak{g}} \cong \mathfrak{g}^{6:1:0:6} \);
(iii) in the elliptic case \( s = 1 \), we have \( \tilde{\mathfrak{g}} \cong \mathfrak{g}^{-6:0:6} \).

Conversely, any nilpotent \((2, 3, 5, 6)\)-structure in each of these algebras is realized as a quotient structure of a nilpotent \((2, 3, 5, 8)\)-structure \((\Delta, g)\).

**Remark 3.** An SR structure in the elliptic algebra \( \mathfrak{g}^{6:1:0:6} \) with canonical parameter \( \nu = 1 \) was considered in [14]. For this structure, it was proved that the vertical subsystem of the Hamiltonian system in the Pontryagin maximum principle [1] is Liouville integrable. This Hamiltonian system was integrated in [11].

## 4. CASIMIR FUNCTIONS AND SYMPLECTIC FOLIATIONS

Let us recall some basic concepts of symplectic geometry. Let \( \mathfrak{g} \) be a Lie algebra and \( \mathfrak{g}^* \) be the Lie coalgebra (the dual of the space \( \mathfrak{g} \)). Any function \( h \in C^\infty(\mathfrak{g}^*) \) is called a Hamiltonian. For any Hamiltonians \( f \) and \( g \), the Poisson bracket is defined as

\[ \{f, g\}(\lambda) = \langle f_* [df, dg], \lambda \rangle, \quad \lambda \in \mathfrak{g}^*. \]

The Casimir function (on a submanifold \( M \subset \mathfrak{g}^* \)) is any Hamiltonian \( f \) commuting with all Hamiltonians in the sense of the Poisson bracket:

\[ \{f, h\}(\lambda) = 0, \quad h \in C^\infty(\mathfrak{g}^*), \]
\[ \lambda \in \mathfrak{g}^* \quad (\text{accordingly} \quad \lambda \in M). \]

A symplectic foliation on \( \mathfrak{g}^* \) is a partition of \( \mathfrak{g}^* \) into symplectic leaves \( L(\lambda) \), which are orbits of the coadjoint action of the Lie group \( G \) of the Lie algebra \( \mathfrak{g} \) in \( \mathfrak{g}^* \):

\[ L(\lambda) = \{(Ad^\lambda_x)\}(\lambda) \mid x \in G, \quad \lambda \in \mathfrak{g}^*. \]

The rank of the Poisson bracket \( \{\cdot, \cdot\} \) at a point \( \lambda \in \mathfrak{g}^* \) is defined as the dimension of the symplectic leaf passing through \( \lambda \) and is denoted by \( \text{rank}(\lambda) \). The symplectic leaves and Casimir functions are invariants of the Hamiltonian vector field

\[ \tilde{\lambda} = h(\lambda), \quad \lambda \in \mathfrak{g}^*. \]

for any Hamiltonian \( h \), so they are important for analyzing left-invariant Hamiltonians on the Lie group of the Lie algebra \( \mathfrak{g} \), in particular, for studying left-invariant optimal control problems on this Lie group.

For coalgebras \( \mathfrak{g}^* \) of all Carnot algebras \( \mathfrak{g} \) with growth vector \((2, 3, 5, 6)\), we describe their Casimir functions and symplectic foliations. In each of the algebras \( \mathfrak{g}^* = \mathfrak{g}^{6:0:6}, \mathfrak{g}^{6:1:0:6}, \text{and} \mathfrak{g}^{-6:0:6} \), we use a canonical basis \( \mathfrak{g} = \text{span}(X_1, \ldots, X_8) \) according to Examples 3–5 and corresponding linear Hamiltonians \( h_1, \ldots, h_8 \in \mathfrak{g}^* \), \( h_i(\lambda) = \langle \lambda, X_i \rangle \). In each of the cases presented below, the symplectic leaf \( L(\lambda) \) is a connected component of common level surfaces of the Casimir functions on corresponding submanifolds in \( \mathfrak{g}^* \).

### 4.1. Parabolic Algebra \( \mathfrak{g} = \mathfrak{g}^{6:0:6} \)

1. If \( h_2 h_6 \neq 0 \), then \( \text{rank}(\lambda) = 4 \).
   
   Casimir functions: \( h_3, h_5 \).

2. If \( h_2 = 0 \) and \( h_2^2 + h_4^2 + h_6^2 \neq 0 \), then \( \text{rank}(\lambda) = 2 \).
   
   Casimir functions: \( h_4, h_5, h_2^2/2 + h_6 h_5 - h_2 h_6 \).

3. If \( h_2 = 0 \) and \( h_2^2 + h_4^2 + h_6^2 \neq 0 \), then \( \text{rank}(\lambda) = 2 \).
   
   Casimir functions: \( h_6, C_1 = h_6 h_5 - h_2^2/2, C_2 = h_6 h_5 - C_1 h_2 = h_4^2/6 \).

4. If \( h_2 = \cdots = h_6 = 0 \), then \( \text{rank}(\lambda) = 0 \).
   
   Casimir functions: \( h_1, h_2 \).

### 4.2. Hyperbolic Algebra \( \mathfrak{g} = \mathfrak{g}^{6:1:0:6} \)

1. If \( h_6 \neq 0 \), then \( \text{rank}(\lambda) = 4 \).
   
   Casimir functions: \( h_5, h_5 h_6 + h_2^2 - h_5^2/2 \).

2. If \( h_2 = 0 \) and \( h_2^2 + h_4^2 + h_6^2 \neq 0 \), then \( \text{rank}(\lambda) = 2 \).
   
   Casimir functions: \( h_4, h_5, h_2^2/2 + h_6 h_5 - h_2 h_6 \).

3. If \( h_2 = \cdots = h_5 = 0 \), then \( \text{rank}(\lambda) = 0 \).
   
   Casimir functions: \( h_1, h_2 \).

### 4.3. Elliptic Algebra \( \mathfrak{g} = \mathfrak{g}^{-6:0:6} \)

1. If \( h_6 \neq 0 \), then \( \text{rank}(\lambda) = 4 \).
   
   Casimir functions: \( h_5, h_5 h_6 - h_2^2/2 \).

2. If \( h_2 = 0 \) and \( h_2^2 + h_4^2 + h_6^2 \neq 0 \), then \( \text{rank}(\lambda) = 2 \).
Casimir functions: \( h_1, h_2, \frac{h_2^2}{2} + h_3 h_4 - h_2 h_5 \).

(3) If \( h_3 = \cdots = h_6 = 0 \), then \( \text{rank}(\lambda) = 0 \).

Casimir functions: \( h_1, h_2 \).

CONCLUSIONS

The following results were obtained in this paper.

We described the Carnot algebras with growth vector \((2, 3, 5, 6)\), i.e., quotient algebras of a free nilpotent Carnot algebra \( q^8 \) of step 4 with two generators. Previously, it was known that there are three normal forms of such algebras: parabolic, hyperbolic, and elliptic. An invariant separating these algebras was found, namely, the signature \( s \in \{0, \pm 1\} \). Additionally, a change of basis transforming the multiplication table into one of three normal forms was described.

We studied the left-invariant sub-Riemannian structures with growth vector \((2, 3, 5, 6)\), i.e., quotient structures of a (unique) sub-Riemannian structure with growth vector \((2, 3, 5, 8)\) by the two-dimensional subspace of the center of the algebra \( q^8 \). A classification of \((2, 3, 5, 6)\)-structures was obtained up to isometries: all of them are uniquely parametrized by a canonical parameter \( v \in \{0, \pm 1\} \) such that \( s = \text{sgn} v \). A classification of left-invariant \((2, 3, 5, 6)\)-distributions was obtained: in each of the \((2, 3, 5, 6)\)-algebras (parabolic, hyperbolic, elliptic) there exists a unique, up to isomorphism, distribution of this type.

For each Carnot algebra \( q \) with growth vector \((2, 3, 5, 6)\), the rank of the Poisson bracket, Casimir functions, and a symplectic foliation in the Lie coalgebra \( q^* \) were calculated.

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