Strong Interactions, (De)coherence and Quarkonia

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Abstract. Quarkonia are the central objects to explore the non-perturbative nature of non-abelian
gauge theories. We describe the confinement-deconfinement phases for heavy quarkonia in a hot QCD
medium and thereby the statistical nature of the inter-quark forces. In the sense of one-loop quantum
effects, we propose that the “quantum” nature of quark matters follows directly from the thermody-
namic consideration of Richardson potential. Thereby we gain an understanding of the formation of hot
and dense states of quark gluon plasma matter in heavy ion collisions and the early universe. In the case
of the non-abelian theory, the consideration of the Sudhakov form factor turns out to be an efficient tool
for soft gluons. In the limit of the Block-Nordsieck resummation, the strong coupling obtained from the
Sudhakov form factor yields the statistical nature of hadronic bound states, e.g., kaons and Ds particles.

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1. Introduction
In this work, we study the geometric nature of the quark matter formation. Specifically, we shall
illustrate that the components of the vacuum fluctuations define a set of local pair correlations against
the vacuum parameters, e.g. charge, mass and angular momentum. Our consideration follows from the
notion of the thermodynamic geometry [1–3]. Importantly, this framework provides a mathematical
platform to exactly understand the nature of the pair local correlations and underlying geometric
structures pertaining to the global phase transitions in quarkonium systems. This perspective yields
a well-known understanding for the phase structures of mixtures of gases, black holes in string
theory [4–7] and in other diverse contexts, as well.

The main purpose of the present investigation is to determine the thermodynamic properties of
the quarkonium configurations, in general. Quantum chromodynamics (QCD), as the theory of strong
interactions, celebrates physics [8–10] at both the high and low temperature domains. Thereby, our
consideration plays a crucial role in understanding the phases and stability of the matter formation.
In the soft gluon limit, viz., for a small transverse momentum $k_\perp$, the behavior of the abelian theory
follows directly from the Poisson distribution and its numerical counterparts. On the other hand,
the statistical nature of the non-abelian soft gluons is understood in terms of the Sudhakov form factor \[12\]. Let us recall that the QCD coupling \(\alpha_s(k_{\perp})\) never lies near the limit \(k_{\perp} \to 0\), and so the QCD effects are limited for the bound state thus formed after the resummation. As mentioned in the Ref. \[15\], an integration over \(k_{\perp}\) requires the Sudhakov form factor
\[
h(b) = \int \frac{dk_i}{2k} \frac{dk_{\perp}}{k_{\perp}} (1 - \exp(-i k_{\perp} \cdot \vec{V})),
\]
where \(k_{\perp} \in (0, m_P/2)\) is due to a physical reason, i.e. that on average each soft gluon can take as much as half of the initial center of mass energy. Towards the determination of the index \(p\), an interesting argument follows from the work of Polyakov \[11\] which we shall explore further from the perspective of thermodynamic geometry in the subsequent consideration. Before doing so, let us consider the joint effects of the (i) confinement and (ii) rotation, and thus make a platform to describe the thermodynamic geometry of rotating quarkonia. Such a simplest configuration is described by Regge trajectories with the leading order effective potential
\[
V(r, J) = \frac{J(J+1)}{r^2} + Cr^{2p-1}.
\]
In this case, the effective theory \[13,14\] is inspired from the limiting QCD strong coupling
\[
\alpha_s(Q^2) = b^{-1} \frac{p}{\ln(1 + p(Q^2/\Lambda_{QCD})^p)},
\]
where \(b := (33 - 2N_f)/12\pi\). In the sense of one-loop quantum effects, we propose that the “quantum” nature of quark matters follows directly from the thermodynamic intrinsic geometry of the Richardson type potentials. In the momentum space, the net effective potential offers the right quarkonium bound states, after taking account of the one-loop exchange terms.

Let us now focus our attention on the thermodynamic geometry of quarkonium bound states with finitely many parameters of the effective field theory. We consider here a framework of the intrinsic Riemannian geometry whose covariant metric tensor is defined as the Hessian matrix of the QCD coupling, with respect to a finite number of arbitrary parameters carried by the soft gluons and quarks. Such a consideration yields the space spanned by \(n\) parameters of the strong QCD coupling \(\alpha_s\), which, in the present treatment, exhibits a \(n\)-dimensional intrinsic Riemannian manifold \(M_n\). As per the definition of the thermodynamic geometry \[1–7\], the components of the covariant metric tensor are given by
\[
g_{ij} := \partial^2 \alpha_s(\vec{x}) / \partial x^i \partial x^j,
\]
where the vector \(\vec{x} \in M_n\). In the strongly coupled quarkonium effective configuration, there are only a few physical parameters, what makes the analysis fairly simple. As mentioned in the Ref. \[15\], the variables of the interest of the above quarkonium configurations are the momentum scale parameter, \(Q^2 := q\), the mass \(M\), and the angular momentum \(J\), if any.

2. Massless Quarkonia
Let us first examine the massless non-rotating quarkonia and study the thermodynamic stability properties as emphasized earlier, and subsequently include the rotation. Considering Eqn.(3), one obtains the following expression
\[
A(q, p) := \frac{p}{b\ln(1 + p(q/L)^p)},
\]
for the strong QCD coupling, where \(L := A_{QCD}^2\). To compute the thermodynamic metric tensor in the space of the parameters \(\{q, p\}\), we employ the Eqn.(4), which leads to the following expression for the components of the metric tensor
\[
g_{qq} = \frac{p^3}{b^2} \frac{n_{11}^Q}{r_{11}^Q}, \quad g_{qp} = \frac{p^2}{b} \frac{n_{12}^Q}{r_{12}^Q}, \quad g_{pp} = \frac{1}{b} \frac{n_{22}^Q}{r_{22}^Q}.
\]
In order to simplify the subsequent notations, let us define the logarithmic factor as

$$l(p) := \ln(1 + p(q/L)^3).$$

Thus, the factors in the numerator of the local pair correlations can be expressed as

$$n_{11}^Q := 2p^2(q/L)^{2p} + l(p)((q/L)^3 - p(q/L)^3 + (q/L)^{2p}),$$

$$n_{12}^Q := 2(q/L)^{2p} + l(p)(2(q/L)^{2p}q^2 \ln(q/L) - 3(q/L)^3 - 2(q/L)^{2p}p - (q/L)^{2p}p \ln(q/L)),$$

$$n_{22}^Q := 2(q/L)^{2p}(p + 2p^2 \ln(q/L) + p^2 \ln(q/L)^2 - l(p)(q/L)^3 + (q/L)^{2p}$$

$$+ 4(q/L)^{2p}p \ln(q/L) + 2(q/L)^{2p}p^2 \ln(q/L) + (q/L)^{2p}ln^3(q/L)^2).$$

In the case of the massless non-rotating quarkonia, we find that the factors \{\(r_{ij}^Q\) | \(i = 1, 2, 3\)} take the uniform value \(l(p)^3 \exp(2l(p))\) in all denominators of the local pair correlation functions. In a given QCD phase, this happens when the parameters \{\(q, p\)\} are confined in the domain

$$\{D := (q, p) \in M_2 | n_{11}^Q > 0, n_{22}^Q > 0\}. \quad (9)$$

Over the above domain of \(\{q, p\}\), the massless non-rotating quarkonia is well-behaved and locally stable. From the definition of the thermodynamic geometry, we find further for the generic value of the parameters that the Gaussian fluctuations form a stable set of correlations over \(\{q, p\}\), if the determinant of the metric tensor

$$\|g\| = \frac{p^3}{\sigma^2 q^2 l(p)^3 \exp(3l(p))} n_{Q}^Q$$

remains a positive function on the intrinsic \(qp\)-surface \((M_2(R), g)\). Explicitly, we obtain that the numerator of the determinant of the metric tensor can be expressed as

$$n_{Q}^Q(q, p) := 2(q/L)^{3p}(p + 3p^2 + 2p^2 \ln(q/L) + 2p^3 \ln(q/L) + p^3 \ln(q/L)^2)$$

$$- l(p)(2(q/L)^{2p} + p(q/L)^3 + 2p(q/L)^{2p} \ln(q/L) + 4p(q/L)^{2p}p \ln(q/L))$$

$$+ 2p(q/L)^{2p}p \ln(q/L) + 4p^2(q/L)^{2p}p^2 \ln(q/L) + 7p(q/L)^{2p}). \quad (11)$$

In this case, we find that the scalar curvature reduces to the following specific form

$$R(q, p) = \frac{bl(p)}{2p^2(n_{Q}^Q)^2} (n_R^{(0)Q} + n_R^{(1)Q}l(p) + n_R^{(2)Q}l(p)^2 + n_R^{(3)Q}l(p)^3). \quad (12)$$

We find that the factors of the numerator of the scalar curvature take a set of interesting expressions [15]. In the case when the Ricci scalar curvature \(R(q, p)\) vanishes, the underlying quarkonium system is found to be in equilibrium. Such a state of the configuration can arise with \(\{n^{(i)Q} = 0, i = 0, 1, 2, 3\}\) if the other factors of the scalar curvature remain non-zero. In the other case, when the scalar curvature \(R(q, p)\) diverges, the configuration goes over a transition. Such an extreme behavior of the quarkonia is expected to happen, when either the index \(p\) or the numerator of the determinant of the metric tensor \(n_Q\) vanish. We further observe that the stability of massless non-rotating quarkonia exists in certain bands. A closer view shows that the quarkonia are decaying and interacting particles in the Coulombic limit. For the regime of the rising potential, the respective determinant of the metric tensor and scalar curvature shows that the limiting rising potential quarkonia are stable and non-interacting particles. In the Regge trajectory model [15], we find interestingly that all possible local and global thermodynamic stability behavior of the three parameter \(\{q, p, J\}\) massless quarkonia remains the same up to the sign of \(b1\), as if there were no effects of the rotation in the underlying configuration. The fact that the efficiency of the rotation induces a mass to the quarkonium is analyzed by considering the Bloch-Nordsieck resummation of the angular phases.

3. Massive Quarkonia

Let us firstly illustrate the cases for the two parameter configurations with either \(\{q, J\}\) or \(\{q, m\}\) fluctuating and then systematically extend the consideration for the generic three parameter quarkonia.
3.1. Two Parameter Quarkonia

For the of massive quarkonia, the resummed strong QCD coupling takes the following form

\[ A(q, J) := \frac{1}{k} \ln \left( \frac{1 - J(0, a\sqrt{q})}{1 + p(q/L)^p} \right) \ln \left( \frac{\sqrt{J + \sqrt{J - q}}}{\sqrt{J - \sqrt{J - q}}} \right) \]

(13)

where \( J(\nu, x) \) is the Bessel function of the first kind of the order \( \nu \). In order to describe fluctuations in the QJ-plane, let the logarithmic factor concerning the Bloch-Nordsieck rotation be defined as

\[ f(q, J) := \ln \left( \frac{\sqrt{J + \sqrt{J - q}}}{\sqrt{J - \sqrt{J - q}}} \right) \]

(14)

From the Eqn.(4), we find that the components of the metric tensor are

\[ g_{qq} = - \frac{p(n_{11}^{(0)J} + n_{11}^{(1)J}l(p) + n_{11}^{(2)J}(p)^2)}{4b(p)^2 \exp(2l(p)/q)^{(J - q)/2}} \]

\[ g_{pp} = \frac{p(n_{12}^{(0)J} + n_{12}^{(1)J}l(p))}{2b(p)^2 \exp(l(p)/q)^{(J - q)/2}} \]

\[ g_{pp} = - \frac{p(2J - q)(1 - J(0, a\sqrt{q}))}{2b(p)^2 \exp(l(p)/q)^{(J - q)/2}} \]

(15)

Without any approximation, the factors in the numerator of the pure qq and qJ-components can be expressed as linear combinations of the integer powers of the scaling \((q/L)^{np}\), where \( n \in Z \). As a result, we see that the geometric nature of the parametric pair correlation functions turns out to be remarkably interesting, viz., the domains of the local stability of the fluctuating quarkonia may be easily described in terms of the parameter \( q \) and \( J \). Under the Gaussian fluctuations of \( \{q, J\} \), the local stability of the system requires that (i) \( qq \)-fluctuations satisfy the constraint

\[ n_{11}^{(0)J} + n_{11}^{(1)J}l(p) + n_{11}^{(2)J}(p)^2 < 0 \]

(16)

and (ii) \( JJ \)-fluctuations be constrained to the following limiting values of the Bessel function

\[ J(0, a\sqrt{q}) < 1, \quad q > 2J \]

\[ J(0, a\sqrt{q}) > 1, \quad q < 2J \]

(17)

In this case, we find that the determinant of the metric tensor reduces to the following expression

\[ \|q\| = - \frac{p^2(n_{12}^{(0)J} + n_{12}^{(1)J}l(p) + n_{12}^{(2)J}(p)^2)}{8b^2(p)^4 \exp(2l(p)/q)^{(J - q)/2}} \]

(18)

where \( n_{12}^{(1)J} = 4p^2(q/L)^p(n_{12}^{(1)J} + n_{12}^{(2)J}l(p)^2) \) and \( n_{12}^{(2)J} = n_{12}^{(20)J} + 2p(n_{12}^{(21)J}l(p) + n_{12}^{(22)J}(q/L)^2p^2) \). Furthermore, it turns out that all factors appearing in the numerator of the determinant of the metric tensor, e.g., \( \{n_{12}^{(0)J}, n_{12}^{(1)J}, n_{12}^{(2)J}, n_{12}^{(20)J}, n_{12}^{(21)J}, n_{12}^{(22)J}\} \), can be presented as linear combinations of the scaling \((q/L)^{np}\), where \( n \in Z \). Combining the effects of all fluctuations of the \( \{q, J\} \), we observe that the quarkonia are stable for \( q := Q^2 \in (1, 4) \). In general, the global stability requires that the determinant of the metric tensor must be positive definite, which in the present case transform as

\[ n_{12}^{(0)J} + n_{12}^{(1)J}l(p) + n_{12}^{(2)J}(p)^2 < 0 \]

(19)

It turns out that the thermodynamic curvature may be written as the series of the charmion logarithmic factor \( l(p) \) and Bloch-Nordsieck logarithmic factor \( f(q, J) \) of rotation as the coefficient of the expansion. Systematically, the exact expression for the scalar curvature takes the form

\[ R(q, J) = \frac{k(l(p))}{2p^2(n_{12}^{(0)J})^2 \sum \lambda_n \times l(p)^n} \]

(20)

where the \( B_n \) in the numerator of the scalar curvature are polynomials in \( p \), whose coefficients are the functions of the Bloch-Nordsieck logarithmic factor \( f(q, J) \). In sequel, we find that the quantitative properties of the scalar curvature and the Riemann curvature tensor remain similar [15], when we consider the variable \( Q \) as the transverse momentum with the understanding that \( k = k_\perp \) and the mass as the other variable of the arbitrary two parameter real quarkonium system. The present consideration expalines the regions of the thermodynamic (in)stability for the massive quarkonia and the (un)stable phases of the one loop QCD. As the non-linear effects become stronger and stronger, it turns out that the thermodynamic instability and correlations grow further. This motivates us to extend our analysis to the case of more general quarkonium configurations.
3.2. Three Parameter Quarkonia

Finally, let us analyze the general massive rotating quarkonia, when all parameters, viz., the scale $q$, index $p$ and angular momentum $J$ of the theory are allowed to fluctuate. In the framework of the Bloch-Nordsieck resummation, the strong QCD coupling takes the following form

$$A(q,p,J) = \frac{1}{b} \frac{p(1 - J(0,a\sqrt{q}))}{\ln(1 + p(q/L)^p)} \ln\left(\frac{\sqrt{J} + \sqrt{J - q}}{\sqrt{J} + \sqrt{J - J^2}}\right).$$

(21)

After some simplification, we find that the components of the metric tensor are

$$g_{qq} = \frac{p(n_{11}^{(0)G} + n_{11}^{(1)G}(p) + n_{11}^{(2)G}(l(p))^2)}{4b(p)^3 \exp(2b(p)q/L^2(J - q))},$$

$$g_{qJ} = \frac{p(n_{12}^{(0)G} + n_{12}^{(1)G}(p))}{2b(p)^3 \exp(2b(p)q/L^2(J - q))},$$

$$g_{pJ} = \frac{(1 + J(0,a\sqrt{q}))(n_{22}^{(0)G} + n_{22}^{(1)G}(p))}{2b(p)^3 \exp(2b(p)(J - q))},$$

where the coefficients $\{n_{11}^{(1)G}, n_{11}^{(2)G}, n_{12}^{(1)G}, n_{12}^{(2)G}\}$ appearing in the components of the metric tensor are shown to factorize as before. In the powers of $p$, the exact factorizations are given as follows:

(i) the $qq$-component

$$n_{11}^{(1)G} = 4p^2(q/L)^p n_{11}^{(12)G} + 4p^3(q/L)^2 p n_{11}^{(12)G},$$

$$n_{11}^{(2)G} = n_{11}^{(20)G} + 2p(q/L)^p n_{11}^{(21)G} + p^2(q/L)^2p,$$

(22)

(ii) the $qp$-component

$$n_{12}^{(1)G} = p(q/L)^p n_{12}^{(11)G} + p^2(q/L)^2 p n_{12}^{(12)G},$$

$$n_{12}^{(2)G} = n_{12}^{(20)G} + 2p(q/L)^p n_{11}^{(21)G} + p^2(q/L)^2 p n_{12}^{(22)G}.$$  

(23)

Interestingly, the factors appearing in the numerator of the $qq$, $qp$ and $qJ$-components can be cascaded as a linear combination in the integer powers of the scaling $(q/L)^n$, where $n \in Z$. Similarly, the corresponding factors of numerator of the $pp$-components take the following expressions

$$n_{22}^{(0)G} = -2p(q/L)^p (1 + 2p \ln(q/L) + p^2 \ln(q/L)^2),$$

$$n_{22}^{(1)G} = (q/L)^p (2 + 4p \ln(q/L) + p^2 \ln(q/L)^2 + p(q/L)^2 (1 + 2p \ln(q/L))).$$

(24)

Finally, the factors in the numerator of the $pJ$-components are

$$n_{23}^{(0)G} = -p(q/L)^p - p^2 \ln(q/L)(q/L)^p,$$

$$n_{23}^{(1)G} = 1 + p(q/L)^p.$$  

(25)

The local stability of the configuration requires a set of constraints on the domains of the parameters. Specifically, for the same sign of $\{p,b\}$, we find that the local stability of the fluctuating quarkonia enforces the following simultaneous requirements

(i) the $qq$- fluctuations satisfy

$$n_{11}^{(0)G} + n_{11}^{(1)G}(p) + n_{11}^{(2)G}(l(p))^2 < 0,$$

(26)

(ii) the $JJ$- fluctuations remain within the limiting values of the Bessel function

$$J(0,a\sqrt{q}) > 1, \quad 2J < q,$$

$$< 1, \quad 2J > q,$$

(27)

(iii) the $pp$- fluctuations satisfy

$$n_{22}^{(0)G} + n_{22}^{(1)G}(p) > 0, \quad J(0,a\sqrt{q}) > 1,$$

$$< 0, \quad J(a\sqrt{q}) > 1.$$  

(28)
for the same sign of \( \{ f, b \} \). The silent feature of the \( J J \)-fluctuations is that we find the two distinct local behaviors for \( J > 0 \) and \( J < 0 \). Subsequently, we find that the \( qp \)-surface is stable, if there exits a positive surface minor

\[
p_{G}^2 := - \frac{(n_{G}^{(0)})^2 + n_{G}^{(1)} I(p) + n_{G}^{(2)} I(p)^2 + n_{G}^{(3)} I(p)^3)}{4b^2 (p^2 \exp(3(p)(p^2)/2(J - \eta^2))}. \tag{30}
\]

It is worth mentioning that \( \{ n_{G}^{(0)}, n_{G}^{(1)}, n_{G}^{(2)}, n_{G}^{(3)} \} \) have the following factorizations

\[
\begin{align*}
 n_{G}^{(0)} & = (q/L)^3 p_{G}^{(0)}, \\
 n_{G}^{(1)} & = (q/L)^2 p_{G}^{(1)} + (q/L)^3 p_{G}^{(2)}, \\
 n_{G}^{(2)} & = (q/L)^3 p_{G}^{(2)} + (q/L)^2 p_{G}^{(2)} + (q/L)^3 p_{G}^{(3)}, \\
 n_{G}^{(3)} & = n_{G}^{(0)} + (q/L)^p n_{G}^{(1)} + (q/L)^2 p_{G}^{(1)} + (q/L)^3 p_{G}^{(2)} + (q/L)^3 p_{G}^{(3)}.
\end{align*}
\tag{31}
\]

In this identification, the factors appearing in the powers of \( (q/L)^p \) have the following structures

\[
\begin{align*}
 n_{G}^{(0)} & = 8 f^2 n_{G}^{(0)}, \\
 n_{G}^{(1)} & = n_{G}^{(10)} + n_{G}^{(11)} f + n_{G}^{(12)} f^2, i = 2, 3, \\
 n_{G}^{(2)} & = n_{G}^{(20)} + n_{G}^{(21)} f + n_{G}^{(22)} f^2, i = 1, 2, 3, \\
 n_{G}^{(3)} & = n_{G}^{(30)} + n_{G}^{(31)} f + n_{G}^{(32)} f^2, i = 0, 1, 2, 3.
\end{align*}
\tag{32}
\]

Further, the factors 032, 12i, 13i, 21i, 22i and 23i take the following explicit expressions

\[
\begin{align*}
 n_{G}^{(0)} & = 8 f^2 n_{G}^{(0)} + 8 p_{G}^{(0)} n_{G}^{(2)}, \\
 n_{G}^{(1)} & = 8 f^2 n_{G}^{(1)} + 8 p_{G}^{(1)} n_{G}^{(1)}, \\
 n_{G}^{(2)} & = 8 f^2 n_{G}^{(2)} + 8 p_{G}^{(2)} n_{G}^{(2)}, \\
 n_{G}^{(3)} & = 8 f^2 n_{G}^{(3)} + 8 p_{G}^{(3)} n_{G}^{(3)} + 4 p_{G}^{(3)} n_{G}^{(3)} n_{G}^{(3)},
\end{align*}
\tag{33}
\]

Interestingly, the sub-factors of the factors 032, 120, 121, 122, 130, 131, 211, 212, 220, 221, 222, 230, 231 and 232 can be expressed as the linear combinations of the integral powers of the scaling \( \ln(q/L) \) and the integral powers of the zeroth order Bessel function. As per the expectation, we observe that the 3j-i-factors that appear are relatively simpler than their foregoing counterparts. In fact, we find that the 30-factors can be seen as a two variable polynomial in \( J, g \). For the general quarkonia with \( \{ q, p, J \} \) fluctuating, the stability of the \( qp \)-surface requires that the principle minor \( p_{G}^2 \) remains positive on \( (M_3, g) \). Correspondingly, this leads to the constraint that the \( \{ q, p, J \} \) satisfy

\[
n_{G}^{(0)} + n_{G}^{(1)} I(p) + n_{G}^{(2)} I(p)^2 + n_{G}^{(3)} I(p)^3 < 0.
\tag{34}
\]

As per the generic fluctuations of the strongly coupled massive rotating quarkonia, we obtain the following general expressions for the determinant of the metric tensor

\[
\|g\| = - \frac{p f(1 + J(0, a\sqrt{q}))}{8b^2 (p^2 \exp(3(p)(p^2)/2(J - \eta^2)))} (n_{G}^{(0)} + n_{G}^{(1)} I(p) + n_{G}^{(2)} I(p)^2 + n_{G}^{(3)} I(p)^3).
\tag{35}
\]

As mentioned before, we find that the \( \{ n_{G}^{(0)}, n_{G}^{(1)}, n_{G}^{(2)}, n_{G}^{(3)} \} \)-terms factorize as follows

\[
\begin{align*}
 n_{G}^{(0)} & = (q/L)^3 p_{G}^{(0)} (8 p_{G}^{(0)} n_{G}^{(0)} + 8 p_{G}^{(0)} n_{G}^{(0)} + 8 p_{G}^{(0)} n_{G}^{(0)}), \\
 n_{G}^{(1)} & = (q/L)^2 p_{G}^{(1)} + (q/L)^3 p_{G}^{(1)}, \\
 n_{G}^{(2)} & = (q/L)^3 p_{G}^{(2)} + (q/L)^2 p_{G}^{(2)} + (q/L)^3 p_{G}^{(2)}, \\
 n_{G}^{(3)} & = (q/L)^3 p_{G}^{(3)} + (q/L)^2 p_{G}^{(3)} + (q/L)^3 p_{G}^{(3)}.
\end{align*}
\tag{36}
\]
After a direct simplification, we obtain the sub-factorizations

\begin{align}
    n^{(0)|G}_g &= n^{(00)|G}_g + f n^{(01)|G}_g, \quad i = 4, 5, 6, \\
    n^{(1)|G}_g &= n^{(10)|G}_g + f n^{(11)|G}_g, \quad i = 2, 3, \\
    n^{(2)|G}_g &= n^{(20)|G}_g + f n^{(21)|G}_g, \quad i = 1, 2, 3, \\
    n^{(3)|G}_g &= n^{(30)|G}_g + f n^{(31)|G}_g, \quad i = 0, 1, 2, 3.
\end{align}

(37)

As per our computation, we observe that the terms appearing in the various powers of \( l(p) \) have a sub-factorization in the index \( p \). Specifically, we find that the factors of 12, 13, 21, 22 and 23-components are given by

\begin{align}
    n^{(120)|G}_g &= n^{(1202)|G}_g p^2 + n^{(1203)|G}_g p^3 + n^{(1204)|G}_g p^4, \\
    n^{(121)|G}_g &= n^{(1212)|G}_g p^2 + n^{(1213)|G}_g p^3 + n^{(1214)|G}_g p^4 + n^{(1215)|G}_g p^5, \\
    n^{(130)|G}_g &= n^{(1303)|G}_g p^3 + n^{(1304)|G}_g p^4 + n^{(1305)|G}_g p^5, \\
    n^{(131)|G}_g &= n^{(1313)|G}_g p^3 + n^{(1314)|G}_g p^4 + n^{(1315)|G}_g p^5 + n^{(1316)|G}_g p^6, \\
    n^{(210)|G}_g &= n^{(2103)|G}_g p^3 + n^{(2104)|G}_g p^4 + n^{(2105)|G}_g p^5, \\
    n^{(211)|G}_g &= n^{(2112)|G}_g p^3 + n^{(2113)|G}_g p^4 + n^{(2115)|G}_g p^5, \\
    n^{(220)|G}_g &= n^{(2203)|G}_g p^3 + n^{(2204)|G}_g p^4, \\
    n^{(230)|G}_g &= n^{(2303)|G}_g p^3 + n^{(2304)|G}_g p^4, \\
    n^{(231)|G}_g &= n^{(2313)|G}_g p^3 + n^{(2314)|G}_g p^4, \\
    n^{(232)|G}_g &= n^{(2323)|G}_g p^3 + n^{(2324)|G}_g p^4, \\
    n^{(233)|G}_g &= n^{(2333)|G}_g p^3 + n^{(2334)|G}_g p^4, \\
    n^{(311)|G}_g &= n^{(3113)|G}_g p^3 + n^{(3114)|G}_g p^4 + n^{(3115)|G}_g p^5 + n^{(3116)|G}_g p^6, \\
    n^{(320)|G}_g &= n^{(3203)|G}_g p^3 + n^{(3204)|G}_g p^4, \\
    n^{(321)|G}_g &= n^{(3213)|G}_g p^3 + n^{(3214)|G}_g p^4, \\
    n^{(330)|G}_g &= n^{(3303)|G}_g p^3 + n^{(3304)|G}_g p^4, \\
    n^{(331)|G}_g &= n^{(3313)|G}_g p^3 + n^{(3314)|G}_g p^4 + n^{(3315)|G}_g p^5 + n^{(3316)|G}_g p^6, \\
    n^{(332)|G}_g &= n^{(3323)|G}_g p^3 + n^{(3324)|G}_g p^4, \\
    n^{(333)|G}_g &= n^{(3333)|G}_g p^3 + n^{(3334)|G}_g p^4.
\end{align}

(38)

As in the case of the surface minor, we observe that all the individual sub-factorizations, (e.g. 040, 041, 050, 051, 060, 061, 120i, 121j, 130i, 130j, 230j, 2304, 2313, 2314), can be expressed as the linear combination over the integral powers of the zeroth order Bessel function and the scaling \( \ln (q/L)^n \).

In the case of the \( l(p)^4 \) terms, we find that the sub-factors pertaining to 30, 31 and 32-factors have no dependence on \( \ln (q/L)^n \) and thus they are expressible as polynomial expressions in the Bessel function only. From this observation, we predict that the regions of the thermodynamic stability are present for \( q, J \in (1, 4) \). Globally, the stability of \( (M_3, g) \) constrains the principle minors \( \{g_{ii}, p^G_{ij}, \|g\|\} \) to remain positive. Specifically, for the same sign of \( \{b, p, f\} \), the volume stability of the \( (M_3, g) \) imposes the following constraint

\begin{align}
    \sum_{i=0}^{3} n^{(i)|G}_g l(p)^i &> 0, \quad J(0, a\sqrt{q}) < 1, \\
    &< 0, \quad J(0, a\sqrt{q}) < 1.
\end{align}

(39)

Importantly, it is worth mentioning that both the limiting configurations with \( J = q \) and \( J(0, a\sqrt{q}) = 1 \) are abided from the thermodynamic stability constraints. To summarize the phases of generic quarkonia, the exact formula for the scalar curvature may analogously be deduced, as the one we have offered for the fluctuations in the QJ-plane. In this sense, we find that the summation over \( l(p) \) naturally arises with the \( B_n \) as the polynomials in \( p \), whose coefficients can be expressed as the functions of the Bloch-Nordsieck logarithmic factor \( f(q, J) \). Based on the analysis of the present paper, the thermodynamically stable index of the Bloch-Nordsieck rotating massive quarkonia is constrained by the following set

\begin{align}
    \mathcal{P}_{m \neq 0} := \{ p \mid n^G_{ij} > 0, \quad n^G_{ii} > 0, \quad n^G_{i} > 0 \}.
\end{align}

(40)

As the gluons become softer and softer, we find, in the limit of Bloch-Nordsieck resummation, that the underlying Sudakov form factor offers all possible thermodynamically stable phases of the strongly coupled quarkonia. As examined for the QJ-plane, we would like to explicitly understand properties of the set \( \mathcal{P}_{m \neq 0} \) and the associated Ricci scalar curvature. Up to a phase of QCD, the global properties of the three parameter quarkonia remain the same as we have exactly indicated for the two parameter quarkonia. As mentioned in the Ref. [15], we have computed the intrinsic geometric properties of the Bessel function of the first kind convoluted with two logarithmic functions of the respective weights \( (0, 2p) \). Further analysis of the geometric features of these exploitations is left for the future.
4. Conclusion and Outlook

We have examined the role of the thermodynamic intrinsic geometry for a class of quarkonium configurations. We have offered a geometric perspective to the confinement-deconfinement phase of (heavy) quarkonia in a hot QCD medium and thereby described the statistical nature of the inter-quark forces. Specifically, the intrinsic geometric analysis provides a set of physical indications encoded in the geometric quantities, e.g., the scalar curvature and possible geometrically non-trivial invariants offer the global correlation properties of an ensemble or subensemble of equilibrium configurations. In the sense of statistical mechanics, our analysis involving a Gaussian distribution of the particles ensures the thermodynamical properties of the underlying quarkonia in the late time limit. From the perspective of one-loop quantum effects, the nature of quark matter is shown to follow directly from the thermodynamic consideration of the Richardson potential.

Our study of the quarkonia could further be explored towards other configurations concerning the non-perturbative and non-abelian nature of the gauge theories. Such a consideration provides a unified description, encompassing all the regimes of QCD at finite temperature, i.e. the Coulombic, the linear rising and the Regge rotating regimes, for both massless and massive quarkonia. Phenomenologically, our results can be thus be used to investigate the statistical nature of soft gluons and the associated phenomenon at the LHC.

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