Entanglement measures and the Hilbert-Schmidt distance

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Abstract

In order to construct a measure of entanglement on the basis of a “distance” between two states, it is one of desirable properties that the “distance” is nonincreasing under every completely positive trace preserving map. Contrary to a recent claim, this letter shows that the Hilbert-Schmidt distance does not have this property.

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As classical information arises from probability correlation between two random variables, quantum information arises from entanglement [1, 2]. Motivated by the finding of an entangled state which does not violate Bell’s inequality, the problem of quantifying entanglement has received an increasing interest recently.

Vedral et. al. [3] proposed three necessary conditions that any measure of entanglement has to satisfy and showed that if a “distance” between two states has the property that it is nonincreasing under every completely positive trace preserving map (to be referred to as the CP nonexpansive property), the “distance” of a state to the set of disentangled states satisfies their conditions. It has been shown that the quantum relative entropy and the Bures metric have the CP nonexpansive property [3], and it has been conjectured that so does the Hilbert-Schmidt distance [4].

In the interesting Letter [5], Witte and Trucks claimed that the Hilbert-Schmidt distance really has the CP nonexpansive property and conjectured that the distance generates a measure of entanglement satisfying even the stronger condition posed later by Vedral and Plenio [4]. However, it can be readily seen that their suggested proof includes a serious gap.

In this Letter, it will be shown that, contrary to their claim, the Hilbert-Schmidt distance does not have the CP nonexpansive property by presenting a counterexample.

Let \( \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \) be the Hilbert space of a quantum system consisting of two subsystems with Hilbert spaces \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \). We assume that \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) have the same finite dimension. We shall consider the notion of entanglement with respect to the above two subsystems. Let \( \mathcal{T} \) be the set of density operators on \( \mathcal{H} \). The set \( \mathcal{D} \) of disentangled states is the set of all convex combinations of pure tensor product states. There are several requirements that every measure of entanglement, \( E \), should satisfy [3, 4]:

\[(E1) \quad E(\sigma) = 0 \text{ for all } \sigma \in \mathcal{D}.\]
(E2) For any family of bounded operators \{V_i\} of the form \(V_i = A_i \otimes B_i\) such that \(\sum_i V_i V_i^\dagger = I\),

(a) \(E(\sum_i V_i \sigma V_i^\dagger) \leq E(\sigma)\),

(b) \(\sum_i \text{Tr}[V_i \sigma V_i^\dagger] E(V_i \sigma V_i^\dagger / \text{Tr}[V_i \sigma V_i^\dagger]) \leq E(\sigma)\).

Condition (E1) ensures that disentangled states have a zero value of entanglement. Condition (E2) ensures that the amount of entanglement does not increase totally or in average by so-called purification procedures. Note that (E2-a) implies the following condition:

(E3) \(E(\sigma) = E(U_1 \otimes U_2 \sigma U_1^\dagger \otimes U_2^\dagger)\) for all unitary operators \(U_i\) on \(H_i\) for \(i = 1, 2\).

Condition (E3) ensures that a local change of basis has no effect on the amount of entanglement.

Vedral et. al. \[3\] proposed the following general construction of the measure of entanglement \(E\). Let \(D : \mathcal{T} \times \mathcal{T} \rightarrow \mathbb{R}\) be a function satisfying the following conditions:

(D1) \(D(\sigma, \rho) \geq 0\) and \(D(\sigma, \sigma) = 0\) for any \(\sigma, \rho \in \mathcal{T}\).

(D2) \(D(\Theta \sigma, \Theta \rho) \leq D(\sigma, \rho)\) for any \(\sigma, \rho \in \mathcal{T}\) and for any completely positive trace preserving map \(\Theta\) on the space of operators on \(\mathcal{H}\).

Condition (D1) ensures that \(D\) has some properties of “distance”. Condition (D2) ensures that the “distance” does not increase by any nonselective operations. Then, it is shown that the “distance” \(E(\sigma)\) of a state \(\sigma\) to the set \(\mathcal{D}\) of disentangled states defined by

\[
E(\sigma) = \inf_{\rho \in \mathcal{D}} D(\sigma, \rho) \tag{1}
\]

satisfies conditions (E1) and (E2-a). It is shown that the quantum relative entropy and the Bures metric satisfy (D1) and (D2) \[3\], and it is conjectured that the Hilbert-Schmidt distance is a reasonable candidate of a “distance” to generate an entanglement measure \[4\]. Here, the Hilbert-Schmidt distance is defined by

\[
D_{HS}(\sigma, \rho) = \|\sigma - \rho\|_{HS}^2 = \text{Tr}[(\sigma - \rho)^2]
\]

for all \(\sigma, \rho \in \mathcal{T}\), which satisfies (D1) since \(\|\sigma - \rho\|_{HS}\) is a true metric.

Recently, Witte and Trucks \[5\] claimed that the Hilbert-Schmidt distance also satisfies (D2) and that the prospective measure of entanglement, \(E_{HS}\), defined by

\[
E_{HS}(\sigma) = \inf_{\rho \in \mathcal{D}} D_{HS}(\sigma, \rho)
\]

satisfies (E1) and (E2-a).

It should be pointed out first that their suggested proof of condition (D2) for \(D_{HS}\) is not justified. Let \(f\) be a convex function on \((0, \infty)\) and let \(f(0) = 0\). Let \(\Phi\) be a trace preserving positive map on the space of operators such that \(\|\Phi\| \leq 1\). Then, Lindblad’s theorem \[6\] asserts that for every positive operator \(A\) we have

\[
\text{Tr}[f(\Phi A)] \leq \text{Tr}[f(A)], \tag{2}
\]

where \(f(A)\) is defined as usual through the spectral resolution of \(A\). It is suggested that with the help of the above theorem it can be shown that

\[
D_{HS}(\Theta \sigma, \Theta \rho) \leq D_{HS}(\sigma, \rho) \tag{3}
\]
by regarding $D_{HS}$ as a convex function on $T_+(\mathcal{H}) \oplus T_+(\mathcal{H})$ for all positive mappings $\Theta$. However, it is not clear at all how $D_{HS}$ and $\Theta$ satisfy the assumptions of Lindblad’s theorem.

Now, we shall show a counterexample to the claim that $D_{HS}$ satisfies condition (D2). Let $A$ and $B$ be $4 \times 4$ matrices defined by

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. $$

Then we have

$$A^\dagger A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. $$

It follows that $A^\dagger A + B^\dagger B = I_4$ and hence

$$\Theta \sigma = A\sigma A^\dagger + B\sigma B^\dagger, $$

where $\sigma$ is arbitrary, defines a completely positive trace preserving map. Let $\sigma$ and $\rho$ be density matrices defined by

$$\sigma = \begin{pmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \rho = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 \end{pmatrix}. $$

Then we have

$$(\sigma - \rho)^2 = \begin{pmatrix} 1/4 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 0 \\ 0 & 0 & 1/4 & 0 \\ 0 & 0 & 0 & 1/4 \end{pmatrix}$$

and hence

$$D_{HS}(\sigma, \rho) = \text{Tr}[(\sigma - \rho)^2] = 1.$$ 

On the other hand, we have

$$A\sigma A^\dagger = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B\sigma B^\dagger = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A\rho A^\dagger = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 \end{pmatrix}, \quad B\rho B^\dagger = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 \end{pmatrix}. $$
It follows that

\[(\Theta \sigma - \Theta \rho)^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\]

and hence

\[D_{HS}(\Theta \sigma, \Theta \rho) = \text{Tr}[(\Theta \sigma - \Theta \rho)^2] = 2.\]

We conclude therefore

\[D_{HS}(\Theta \sigma, \Theta \rho) > D_{HS}(\sigma, \rho).\]

From the above counterexample, we conclude that the inequality

\[D_{HS}(\Theta \sigma, \Theta \rho) \leq D_{HS}(\sigma, \rho)\]

is not generally true for completely positive trace preserving maps \(\Theta\). Therefore, it is still quite open whether \(E_{HS}\) is a good candidate for an entanglement measure or not.

In order to obtain a tight bound for \(D_{HS}(\Theta \sigma, \Theta \rho)\), we take advantage of Kadison’s inequality \([\ref{7}]\). If \(\Phi\) is a positive map, then we have

\[\Phi(A)^2 \leq \|\Phi\| \Phi(A^2)\]

for all Hermitian \(A\). Applying the above inequality to the positive trace preserving map \(\Phi = \Theta\) and \(A = \sigma - \rho\), we have

\[(\Theta \sigma - \Theta \rho)^2 \leq \|\Theta\| \Theta[(\sigma - \rho)^2].\]

By taking the trace of the both sides we obtain the following conclusion: For any trace preserving positive map \(\Theta\) and any states \(\sigma\) and \(\rho\), we have

\[D_{HS}(\Theta \sigma, \Theta \rho) \leq \|\Theta\| D_{HS}(\sigma, \rho).\]

The previous example shows that the bound can be attained with \(\|\Theta\| = 2\).

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