GRADED DECORATED MARKED SURFACES: CALABI-YAU-X CATEGORIES OF GENTLE ALGEBRAS

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Abstract. Let S be a graded marked surface. We construct a string model for Calabi-Yau-X category \( D_X(S_\Delta) \), via graded DMS (=decorated marked surfaces) \( S_\Delta \). We prove the isomorphism between the braid twist group of the surface and the spherical twist group of the category and \( q \)-intersection formulas for \( D_X(S_\Delta) \). We also give a topological realization of Lagrangian immersion \( D_\infty(S) \to D_X(S_\Delta) \), where \( D_\infty(S) \) is the topological Fukaya category associated to S. This generalizes previous work in the Calabi-Yau-3 case and also unifies the usual case (Calabi-Yau-\( \infty \) case, i.e. for \( D_\infty(S) \)).

Introduction

This paper generalizes the results of topological realization of Calabi-Yau-3 gentle type dg algebras in [17, 19] to the Calabi-Yau-X version, which also unifies the construction for the Calabi-Yau-\( \infty \) case (i.e. the usual derived categories of graded gentle algebras, cf. [5, 16, 24]) via topological Lagrangian immersion.

Stability conditions on Fukaya type categories. In the seminal works [3, 5], Bridgeland-Smith (BS) and Haiden-Katzarkov-Kontsevich (HKK) show, respectively, that the stability conditions on two Fukaya type categories \( D \) from a surface S can be identified with meromorphic quadratic differentials with corresponding presribed singularities. Here, a stability condition consists of a heart (an abelian category, or equivalently, the heart of a bounded t-structure) and a central charge (a group homomorphism) \( Z: K(D) \to \mathbb{C} \), where \( K(D) \) is the Grothendieck group of \( D \). The key ingredients on their approach are the following:

1°. Certain arc \( \eta \) on S corresponds to an object \( X_\eta \) in \( D \).
2°. A quadratic differential \( \phi \) determines a family of special arcs \( \eta_j \) on S (the saddle trajectories) and the corresponding objects \( X_{\eta_j} \) determines a heart in \( D \).
3°. The length of \( \eta_j \) w.r.t. \( \phi \) gives the central charge of \( X_{\eta_j} \), i.e. via the formula

\[
Z(X_{\eta_j}) = \int_{\eta_j} \sqrt{\phi}.
\]

Key words and phrases. Calabi-Yau-X categories, topological Fukaya categories, string model, decorated marked surface, \( q \)-intersections.

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In fact, [3] dexterously bypassed 1° to get 2° using cluster theory and [17] filled the gap with further applications on understanding the symmetries of spaces of stability conditions in [14].

One interesting question is about the relation between these two works. In [6, 7], we introduce the \(q\)-deformation of stability conditions and quadratic differentials to give an answer. On the categorical level, HKK’s topological Fukaya category \(D_\infty(S)\) can be embedded into a Calabi-Yau category \(D_X(S)\) with distinguish automorphism \(X\) as another (grading) shift functor. When specifying \(X\) to be 3, i.e. taking the orbit quotient of \(D_X(S)\) by the automorphism \(X - 3\), one recovers BS’s category \(D_3(S)\) (which is the subcategory of certain derived Fukaya category [23]).

In this sequel, we focus on ingredient 1° of the story, i.e. the topological realization of (objects and morphisms in) categories, that we construct a string model for \(D_X(S)\).

**Previous works on string models.** The previous related works on various categories of (graded) gentle algebras are summarized as follows:

- The derived category of Calabi-Yau-X type A algebra was constructed and investigated in [15] via a decorated disk, where it was shown that there is a faithful action of the (classical) braid group on the category and the total rank of the morphism space between two certain objects is equal to the bi-graded intersection number between the corresponding arcs.
- The derived categories of Calabi-Yau-3 gentle algebras were studied in [17, 19] via decorated marked surfaces (without punctures), where it was shown that closed arcs correspond to spherical objects (up to shift), the braid twist group is isomorphic to the spherical group and the double of intersection number between two closed arcs equals the dimension of total homomorphism space between the corresponding spherical objects.
- The derived categories of graded gentle algebras appeared in [5] as topological Fukaya categories of surfaces, where the indecomposable objects were proved to correspond to curves with local systems. Later, it was shown in [24] (resp. [16]) that the derived categories of finite dimensional gentle algebras (resp. homological smooth graded gentle algebras) are obtained in this way. The formula connecting dimensions of homomorphism space between objects and graded intersection numbers was also given.
- The cluster categories of Jacobian gentle algebras were studied in [1, 26] via triangulated marked surfaces without punctures, where the correspondence between curves and valued closed curves and indecomposable objects, the interpretation of Auslander-Reiten translation the via rotation, and the equality between the intersection of two curves and the dimension of \(\text{Ext}^1\) of the corresponding objects were given.
- Gentle algebras were realized as tiling algebras associated to partial triangulations of marked surfaces in [2], where it was shown that there is a bijection between indecomposable modules and permissible curves. An interpretation of the Auslander-Reiten translation via the pivot elementary move and a method on how to read morphisms from relative positions of curves were also given.
String model with double grading. The input data in our story is a graded marked surface \((S, \lambda, Y)\), where \(S\) is a marked surface, \(\lambda\) is a line field/grading and \(Y\) a set of marked points on \(\partial S\). This is the model for the derived category of graded gentle algebras. We construct the graded DMS \(S_\Delta\) from \(S\) by pulling certain marked points into the interior of \(S\), which naturally introduces another \(X\)-grading, which is usually known as the Adams grading in topology. To realize this grading, one can using the log surface model (cf. § 2.4).

Then we construct the double graded version of string model on the Calabi-Yau-\(X\) categories \(D_X(S)\) associated to \(S_\Delta\) and prove the following main result (cf. Theorem 5.5).

**Theorem 0.** Let \(S_\Delta\) be a graded DMS. There is a full formal arc system \(A\) with associated Calabi-Yau-\(X\) category \(D_X(S) = D_{fd}(\Gamma_A)\), such that

- There is a bijection \(X : \tilde{\eta} \mapsto X\tilde{\eta}\) from the set of double graded closed arcs on \(S_\Delta\) to the set of reachable spherical objects (Theorem 4.16).
- \(X\) induces an isomorphism \(\iota : \text{BT}(S_\Delta) \cong \text{ST}_*(\Gamma_A)\) (0.1) between the braid twist group and the spherical twist group, sending a braid twist \(b_{\tilde{\eta}}\) to the spherical twist \(\phi_{X\tilde{\eta}}\) (Theorem 4.11).
- Formula

\[
\text{Int}^q(\tilde{s}, \tilde{\eta}) = \text{dim}^q \text{Hom}^{\mathbb{Z}^2}(X\tilde{s}, X\tilde{\eta})
\]  

holds (Corollary 4.19), where \(\text{Int}^q\) is the \(\mathbb{Z}^2\)-graded \(q\)-intersection in (2.9) and \(\text{dim}^q \text{Hom}^{\mathbb{Z}^2}\) is the \(q\)-dimension of double graded \(\text{Hom}s\) defined in (3.4).

The main difficulty on generalizing the previous work \([15, 17, 18]\) lies on the lack of corresponding cluster theory (which is specific to the Calabi-Yau-3 case).

One of the applications is that we obtain the topological realization of Lagrangian immersion (Theorem 6.6), that our model contains the string model for the Calabi-Yau-\(\infty\) category \(D_{fd}(\Gamma_0 A)\) associated to a graded marked surface \(S\), which is triangle equivalent to the derived category of a graded gentle algebra associated to \(A\).

**Contents.** The paper is organized as follows. In § 1, we review the basics of double dg algebras and their derived categories. In § 2, we introduce the graded decorated marked surfaces (DMS) as our topological model. In § 3, we describe the string model on graded DMS and in § 4 we prove the main result with technical assumptions. Then we remove such assumptions in § 5 and finish the paper with describing topological realization of Lagrangian immersion in § 6.

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1. Derived categories of differential graded (dg) \(X\)-graded algebras

Throughout this paper, \(k\) denotes an algebraically closed field. We review the categorical preliminaries on dg stuffs.

1.1. Differential graded \(X\)-graded algebras. A \(\mathbb{Z}X\)-graded \((X\text{-graded for short})\) \(k\)-module is a \(k\)-module \(V\) which decomposes into a direct sum of the form

\[
V = \bigoplus_{i \in \mathbb{Z}} V_i,
\]

where each \(V_i\) is a \(k\)-module. The shift \(V[\varsigma X]_i\), for \(\varsigma \in \mathbb{Z}\), is defined to be the \(X\)-graded \(k\)-module whose components \(V[\varsigma X]_i = V_{i+\varsigma}\), \(i \in \mathbb{Z}\). A morphism \(f : V \to V'\) between two \(X\)-graded \(k\)-modules is a \(k\)-linear map such that \(f(V_i) \subset V'_j\) for any \(i \in \mathbb{Z}\). A complex of \(X\)-graded \(k\)-modules is a \(k\)-linear map such that \(f(V_i) \subset V'_{j+1}\) for any \(i \in \mathbb{Z}\).

A \(\mathbb{Z} \otimes \mathbb{Z}X\)-graded \((\mathbb{Z}^2\text{-graded for short})\) \(k\)-module is a \(k\)-module \(V\) which decomposes into a direct sum

\[
V = \bigoplus_{j \in \mathbb{Z}} V^j
\]

doing of \(X\)-graded \(k\)-modules \(V^j = \bigoplus_{i \in \mathbb{Z}} V^j_i\). Each element \(x\) in \(V^j_i\) is called to have bi-degree \(j + iX\). We also denote by \(|x|_1 = j\) and \(|x|_2 = i\). The shift \(V[\varsigma + \varsigma X]_i = V^{j+\varsigma}_i\), \(i, j \in \mathbb{Z}\). A morphism \(f : V \to V'\) of \(\mathbb{Z}^2\)-graded \(k\)-modules of bi-degree \(\varphi + \varsigma X\) is a \(k\)-linear map such that \(f(V^j_i) \subset (V')^{j+\varphi}_i\) for all \(i, j \in \mathbb{Z}\). Thus, for any \(\mathbb{Z}^2\)-graded \(k\)-modules \(V\) and \(V'\), we have a \(\mathbb{Z}^2\)-graded \(k\)-module \(\text{Hom}_{\mathbb{Z}^2 - \varphi}(V, V')\) consisting of the morphisms from \(V\) to \(V'\) of any bi-degrees.

A differential graded \((=dg)\) \(\mathbb{Z}X\)-graded \(k\)-module is a \(\mathbb{Z}^2\)-graded \(k\)-module \(V\) endowed with a differential \(d_V\), i.e. a morphism \(d_V : V \to V\) of bi-degree 1 such that \(d_V^2 = 0\). Equivalently, a dg \(\mathbb{Z}X\)-graded \(k\)-module is a complex of \(X\)-graded \(k\)-modules. The shift \(V[\varphi + \varsigma X]_i\) of a dg \(\mathbb{Z}X\)-graded \(k\)-module \(V\), for \(\varsigma, \varphi \in \mathbb{Z}\), is the shift endowed with the differential \((-1)^{\varphi}d_V\). For a dg \(\mathbb{Z}X\)-graded \(k\)-module \(V\), denote by \(H^*(V)\) the homology of \(V\) with respect to the differential \(d_V\). Note that \(H^*(V)\) is a \(\mathbb{Z}^2\)-graded \(k\)-module.

Let \(C_{dg}(k)\) be the category whose objects are dg \(\mathbb{Z}X\)-graded \(k\)-modules and whose morphism space from \(V\) to \(V'\) is a dg \(\mathbb{Z}X\)-graded \(k\)-module whose underlying \(\mathbb{Z}^2\)-graded \(k\)-module is \(\text{Hom}_{\mathbb{Z}^2 - \varphi}(V, V')\) and whose differential \(d\) is given by

\[
d(f) = d_{V'} \circ f - (-1)^{\varphi} f \circ d_V
\]

for \(f\) a morphism of bi-degree \(\varphi + \varsigma X\).

A dg \(\mathbb{Z}X\)-graded \(k\)-algebra is a dg \(\mathbb{Z}X\)-graded \(k\)-module \((\Gamma, d_\Gamma)\) endowed with a multiplication

\[
\Gamma^j_i \otimes \Gamma^{j'}_{i'} \to \Gamma^{j+j'}_{i+i'}
\]

\[
x \otimes y \mapsto xy
\]

such that the Leibniz rule holds:

\[
d_\Gamma(xy) = d_\Gamma(x)y + (-1)^j x d_\Gamma(y)
\]
for all \( x \in \Gamma \) and all \( y \in \Gamma \).

A dg \( \mathbb{X} \)-graded module of a dg \( \mathbb{X} \)-graded \( \mathbb{k} \)-algebra \( \Gamma \) is a dg \( \mathbb{X} \)-graded \( \mathbb{k} \)-module \((M, d_M)\) endowed with a \( \Gamma \)-action from the right

\[
M_i^j \otimes \Gamma_u^v \rightarrow M_{i+u}^{j+v}
\]

such that the Leibniz rule holds

\[
d_M(ma) = d_M(m)a + (-1)^j d_\Gamma(a),
\]

for all \( m \in M_i^j \) and all \( a \in \Gamma \).

For two dg \( \mathbb{X} \)-graded \( \Gamma \)-modules \( M \) and \( N \), we define \( \text{Hom}_\Gamma(M, N) \) to be the dg \( \mathbb{X} \)-graded \( \mathbb{k} \)-submodule of \( \text{Hom}_{C_{dg\mathbb{X}}(\mathbb{k})}(M, N) \) as follows:

\[
\text{Hom}_\Gamma(M, N) = \{ f \in \text{Hom}_{C_{dg\mathbb{X}}(\mathbb{k})}(M, N) \mid f(ma) = f(m)a \text{ for any } m \in M, a \in \Delta \}.
\]

The category \( \mathcal{C}(\Gamma) \) of dg \( \mathbb{X} \)-graded \( \Gamma \)-module is the category whose objects are the dg \( \mathbb{X} \)-graded \( \Gamma \)-modules, and whose morphisms are \( Z^0 \text{Hom}_\Gamma(M, N)_0 \), which consisting of the morphisms \( f \in \text{Hom}_\Gamma(M, N) \) of bi-degree 0 satisfying \( d(f) = 0 \).

The \text{homotopy category} \( \mathcal{H}(\Gamma) \) is a triangulated category whose suspension functor is the given by \( M \mapsto M[1] \). The \text{derived category} \( \mathcal{D}(\Gamma) \) of dg \( \mathbb{X} \)-graded \( \Gamma \)-modules is the localization of \( \mathcal{H}(\Gamma) \) at the full subcategory of acyclic dg \( \mathbb{X} \)-graded \( \Gamma \)-modules. Note that in each of the categories \( \mathcal{C}(\Gamma) \), \( \mathcal{H}(\Gamma) \) and \( \mathcal{D}(\Gamma) \), the map \( M \mapsto M[\mathbb{X}] \) induces an exact/triangle equivalence.

**Remark 1.1.** Let \( A = \bigoplus_{i \in \mathbb{Z}} A_i \) be an ordinary graded algebra. Regard it as a dg \( \mathbb{X} \)-graded algebra with \( A^0 = A \) and \( A^j = 0, \forall j \neq 0 \). Then the derived category of graded \( A \)-modules coincides with \( \mathcal{D}(A) \).

A morphism \( s : L \rightarrow M \) in \( \mathcal{C}(\Gamma) \) is called a quasi-isomorphism if its induced map \( H^*(s) : H^*(L) \rightarrow H^*(M) \) is an isomorphism. A dg \( \mathbb{X} \)-graded \( \Gamma \)-module \( P \) is cofibrant if

\[
\text{Hom}_{\mathcal{C}(\Gamma)}(P, L) \xrightarrow{\text{Hom}_{\mathcal{C}(\Gamma)}(P, s)} \text{Hom}_{\mathcal{C}(\Gamma)}(P, M)
\]

is surjective for each quasi-isomorphism \( s : L \rightarrow M \) which is surjective in each component. Let \( P \) be a cofibrant dg \( \mathbb{X} \)-graded \( \Gamma \)-module. Then the canonical map

\[
\text{Hom}_{\mathcal{H}(\Gamma)}(P, N) \rightarrow \text{Hom}_{\mathcal{D}(\Gamma)}(P, N)
\]

is bijective for all dg \( \mathbb{X} \)-graded \( \Gamma \)-module \( N \). The canonical projection from \( \mathcal{H}(\Gamma) \) to \( \mathcal{D}(\Gamma) \) admits a left adjoint functor \( p \) which sends a dg \( \mathbb{X} \)-graded \( \Gamma \)-module \( M \) to a cofibrant dg \( \mathbb{X} \)-graded \( \Gamma \)-module \( pM \) quasi-isomorphic to \( M \). Thus, we have

\[
\text{Hom}_{\mathcal{D}(\Gamma)}(M, N) = \text{Hom}_{\mathcal{H}(\Gamma)}(pM, N) = H^0 \text{Hom}_\Gamma(pM, N)_0.
\]

The perfect derived category \( \text{per}(\Gamma) \) is the smallest full subcategory of \( \mathcal{D}(\Gamma) \) containing \( \Gamma \) and which is stable under taking shifts (i.e. \([\varnothing + \mathbb{X}], \varnothing, \varsigma \in \mathbb{Z}\), extensions and direct summands. The finite dimensional derived category \( \mathcal{D}_{fd}(\Gamma) \) is the full subcategory
of \( D(\Gamma) \) consisting of those dg \( \mathbb{X} \)-graded \( \Gamma \)-modules whose homology is of finite total dimension.

For any dg \( \mathbb{X} \)-graded \( \Gamma \)-modules \( X \) and \( Y \), define \( \mathbf{R} \text{Hom}(X,Y) = \mathcal{H}\text{om}_\Gamma(pX,pY) \).

Taking homology, we have

\[
H^*(\mathbf{R} \text{Hom}(X,Y)) = \text{Ext}^Z(X,Y) := \bigoplus_{\rho, \varsigma \in \mathbb{Z}} \text{Hom}_{D(\Gamma)}(X,Y[\rho + \varsigma \mathbb{X}]).
\]

For any object \( T \in D(\Gamma) \), denote by \( \langle T \rangle \) the closure of \( T \) in \( D(\Gamma) \) under extensions, shifts (i.e. \([\rho + \varsigma \mathbb{X}], \rho, \varsigma \in \mathbb{Z}\) and direct summands (e.g. \( \langle \Gamma \rangle = \text{per} \Gamma \)). We have the following derived Morita equivalence (an analogue of [8, Theorem in Section 4.3]).

**Theorem 1.2.** There is a \( \mathbf{k} \)-linear triangle equivalence

\[
\mathbf{R} \text{Hom}(T, -) : \langle T \rangle \simeq \text{per}(\mathbf{R} \text{Hom}(T,T)).
\]

### 1.2. Dg \( \mathbb{X} \)-graded quiver algebras

A quiver \( Q \) consists of the set \( Q_0 \) of vertices, the set \( Q_1 \) of arrows, and two functions \( s, t : Q_1 \to Q_0 \) sending an arrow to its starting and ending, respectively. We denote an arrow by \( a : s(a) \to t(a) \).

A (nontrivial) path \( \rho \) of length \( s \) in \( Q \) is a sequence \( a_1a_2 \cdots a_s \) of arrows with \( t(a_i) = s(a_{i+1}) \) for \( 1 \leq i < s \). The starting and ending of a path \( \rho = a_1a_2 \cdots a_s \) are \( s(\rho) = s(a_1) \) and \( t(\rho) = t(a_s) \), respectively. The composition of paths \( \rho \) and \( \rho' \) is \( \rho\rho' \) if it is again a path (i.e. \( t(\rho) = s(\rho') \)), or zero otherwise. To each vertex \( v \in Q_0 \), there is an associated trivial path \( e_v \) of length 0 with \( s(e_v) = t(e_v) = v \). The path algebra \( \mathbf{k}Q \) of \( Q \) is the \( \mathbf{k} \)-algebra whose basis is the family of (trivial or nontrivial) paths and whose multiplication is given by the composition of paths.

A \( \mathbb{Z} \oplus \mathbb{X} \)-graded (or \( \mathbb{Z}^2 \)-graded for short) quiver is a triple \((Q, | \cdot |_1, | \cdot |_2)\), where \( Q \) is a quiver and \(| \cdot |_1, | \cdot |_2\) are maps from \( Q_1 \) to \( \mathbb{Z} \). An arrow \( a \in Q_1 \) is called to have bi-degree \( \deg(a) = \rho + \varsigma \mathbb{X} \) if \(|a|_1 = \rho, |a|_2 = \varsigma \). Any non-trivial path \( \rho = a_1a_2 \cdots a_s \) has bi-degree \((\sum_{i=1}^s |a_i|_1) + (\sum_{i=1}^s |a_i|_2) \mathbb{X}\) and any trivial path has bi-degree 0. Thus, the path algebra \( \mathbf{k}Q \) becomes a \( \mathbb{Z}^2 \)-graded algebra.

A differential on a \( \mathbb{Z}^2 \)-graded \( Q \) is a map \( d : Q_1 \to \mathbf{k}Q \) such that for any \( a \in Q_1 \), \( d(a) \) is a linear combination of paths \( p \) of bi-degree \((|a|_1 + 1) + |a|_2 \mathbb{X}\) with \( s(a) = s(p) \) and \( t(a) = t(p) \), and such that if we extend \( d \) to a map \( \mathbf{k}Q \to \mathbf{k}Q \) linearly and by the Leibniz rule, then \( d^2 = 0 \). Then \( \Gamma := (\mathbf{k}Q, d) \) is a dg \( \mathbb{X} \)-graded algebra.

Let \( S_i \) be the simple \( \Gamma \)-module corresponding to a vertex \( i \) in \( Q_0 \). Let \( S = \oplus_{i \in Q_0} S_i \).

Denote by

\[
\text{Ext}^{Z_2}(S,S) = \text{Ext}^{Z_2 \mathbb{X}}(S,S) = \bigoplus_{\rho, \varsigma \in \mathbb{Z}} \text{Hom}_{D(\Gamma)}(S,S[\rho + \varsigma \mathbb{X}]).
\]

Then we have \( \text{Ext}^{Z_2}(S,S) \cong H^*(\mathbf{R} \text{Hom}(S,S)) \) as \( Z_2 \)-graded \( \mathbf{k} \)-algebras. Following [10, Appendix A.15], we have the following result (cf. also [12, Lemma 2.15]).

**Theorem 1.3.** A basis of \( \text{Ext}^{Z_2}(S,S) \) is formed by \( \pi_a : S_i \to S_j[\rho + \varsigma \mathbb{X}] \), for \( a \) an arrow of bi-degree \((1 - \rho) + \varsigma \mathbb{X}\) or a trivial path at \( i \) when \( i = j \) and \( \rho = \varsigma = 0 \). Moreover,
there is an $\mathbb{X}$-graded $A_\infty$ structure on $\text{Ext}^Z_2(S, S)$ given by

$$m_r(\pi_{a_1}, \pi_{a_2}, \ldots, \pi_{a_r}) = \pi_b$$

whenever $a_1a_2\cdots a_r$ appears in the expression of $d(b)$, and such that there is a quasi-isomorphism $R\text{Hom}(S, S) \to \text{Ext}^Z_2(S, S)$ of $\mathbb{X}$-graded $A_\infty$ algebras.

We have the following useful consequence of Theorem 1.2 and Theorem 1.3.

**Corollary 1.4.** There is a triangle equivalence

$$\mathcal{D}_{fd}(\Gamma) \simeq \text{per} \text{Ext}^Z_2(S, S)$$

where $\text{Ext}^Z_2(S, S)$ has the $A_\infty$ structure in Theorem 1.3.

2. Calabi-Yau-$\mathbb{X}$ categories from graded decorated marked surfaces

In this section, we introduce the topological model for Calabi-Yau-$\mathbb{X}$ version of graded gentle algebras.

2.1. Marked surfaces with line field. We partially follow [5] and [15]. A marked surface $S$ is a compact oriented surface with non-empty boundary $\partial S$ and with two finite sets $M$ and $Y$ of marked points on $\partial S$ satisfying that each connected component of $\partial S$ contains the same number of marked points (at least one) in $M$ and $Y$, and they are alternative.

Let $\mathbb{P}TS^0$ the real projectivization of the tangent bundle of $S$. Take a line filed, or grading $\lambda$ of $S$, that is, a section $\lambda: S \to \mathbb{P}TS^0$. Then the projection $\mathbb{P}TS^0 \to S$ with $\mathbb{RP}^1 \cong S^1$-fiber gives a short exact sequence

$$0 \to \pi_1(S^1) \to \pi_1(\mathbb{P}TS^0) \to \pi_1(S) \to 0,$$

or

$$0 \to H_1(S^1) \to H_1(\mathbb{P}TS^0) \to H_1(S) \to 0,$$

or

$$0 \to H^1(S) \to H^1(\mathbb{P}TS^0) \xrightarrow{\pi_*} H^1(S^1) = \mathbb{Z} \to 0.$$

The grading $\lambda$ is determined by a class in $H^1(\mathbb{P}TS^0)$ ([16, Lemma 1.2]), denoted by $[\lambda]$, induced from a split of $\pi_S$ (i.e. $\pi_S([\lambda]) = 1$ in $H^1(S^1)$). Such a class is equivalent to a split of $H_1(S^1) \to H_1(\mathbb{P}TS^0)$ or a split of $\pi_1(S^1) \to \pi_1(\mathbb{P}TS^0)$, as $H_1$ is the abelization of $\pi_1$. Thus $\lambda$ also determines (and is determined by) a $\mathbb{Z}$-covering $\mathbb{R}TS^0 \to \mathbb{P}TS^0$, where $\mathbb{R}TS^0$ is the $\mathbb{R}$-bundle of $S$ that can be constructed by gluing $\mathbb{Z}$ copies of $\mathbb{P}TS^0$ cut by $\lambda$.

**Definition 2.1.** A graded marked surface $(S, \lambda)$ consists of a marked surface $S$ and a line filed $\lambda$.

A morphism $f: (S, \lambda) \to (S', \lambda')$ between two graded marked surfaces is a map $f: S \to S'$ such that it preserves the marked points and $[\lambda] = f^* [\lambda']$. There is a natural automorphism, called the grading shift [1] on $(S, \lambda)$, which is given by rotating $\lambda: S \to \mathbb{P}^1$ by $\pi$. 
2.2. Graded DMS. Let $(S, \lambda)$ be a graded marked surface. Now we introduce the decorated version.

The \textit{decorated marked surface} (DMS) $S_\Delta$ of $S$ is obtained from $S$ by decorating a set $\Delta$ of points in the interior of $S$, where $|\Delta| = |Y| = |M|$.

A \textit{cut} $c$ is a set of curves on $S$, pairing (connecting) points in $\Delta$ and $Y$ with no intersections or self-intersections. Let $\mathbb{P}T(S \setminus \Delta)$ the real projectivization of the tangent bundle of $S \setminus \Delta$.

A \textit{grading} $\Lambda$ on $S_\Delta$ is a class in $H^1(\mathbb{P}T(S \setminus \Delta), \mathbb{Z}^2)$, with values $(1, 0)$ on each loop $\{p\} \times \mathbb{P}^1$ on $T_p(S \setminus \Delta)$ for $p \notin \Delta$ and values $(-2, 1)$ on each loop $l_2 \times \{x\}$ on $S$ around any $Z \in \Delta, x \in \mathbb{P}^1$.

The grading $\Lambda$ is called compatible with $\lambda$ and $c$ if the projection of $\Lambda(\alpha)$ on the first $\mathbb{Z}$ coordinate agrees with $\lambda(\alpha)$, for any simple loop on $S$ that does not intersect with $c$. Note that the projection of $\Lambda(\alpha)$ on the first $\mathbb{Z}$ corresponds to a section $\Lambda_1 : S \setminus \Delta \to \mathbb{P}T(S \setminus \Delta)$.

\textbf{Definition 2.2.} Let $S$ be a marked surface with grading $\lambda$ and a cut $c$. A \textit{graded DMS} $(S_\Delta, \Lambda, \lambda, c)$ consists of a DMS $S_\Delta$ with a grading $\Lambda$ that is compatible with the grading $\lambda$ on $S$ and the cut $c$. For simplicity, we will omit $(\lambda, c)$ sometimes and only write $(S_\Delta, \Lambda)$.

2.3. Graded curves. Let $(S_\Delta, \Lambda, \lambda, c)$ be a graded DMS. Denote by $S^0_\Delta = S \setminus (\partial S \cup \Delta)$.

For a curve $c : [0, 1] \to S$, we always assume $c(t) \in S^0_\Delta$ for any $t \in (0, 1)$.

\textbf{Notions 2.3.} We have the following types of curves.

- A curve $c$ is called \textit{open} if $c(0)$ and $c(1)$ are in $M$.
- A \textit{open arc} is a open curve without self-intersections in $S^0_\Delta$. We call two open arcs do not cross each other if they do not have intersections in $S^0_\Delta$.
- An \textit{open boundary segment} is the closure of a component of $\partial S \setminus M$.
- A curve $c$ is called \textit{closed} if $c(0)$ and $c(1)$ are in $\Delta$.
- A \textit{closed arc} is a closed curve without self-intersections in $S^0_\Delta$ and whose two endpoints are not the same decoration. Denote by $CA(S_\Delta)$ the set of closed arcs.
- A closed curve is called \textit{admissible} if it does not cut out a once-decorated monogon by one of its self-intersections in $S^0_\Delta$. See the upper right picture in Figure 3. Denote by $AC(S_\Delta)$ the set of admissible closed curves.

A grading $\tilde{c}$ on a curve $c$ is given by a family of (homotopy classes of) paths in $\mathbb{P}(T_c(t)S^0_\Delta)$ from $\Lambda_1(c(t))$ to $\tilde{c}(t)$, varying continuously with $t \in (0, 1)$. Pairs $(c, \tilde{c})$ are called \textit{graded} curves, and are simply denoted by $\tilde{c}$ usually. Equivalently, a graded curve $\tilde{c}$ is one of $\mathbb{Z}$ lifts, in $\mathbb{R}TS^0$, of an usual curve $c$ on $S$. Denote by $CA(S_\Delta)$ (resp. $AC(S_\Delta)$) the set of graded closed arcs (resp. graded admissible closed curves).

For any graded curves $\tilde{\sigma}$ and $\tilde{\tau}$, let $p = \tilde{\sigma}(t_1) = \tilde{\tau}(t_2) \in S^0_\Delta$ be a point where $\tilde{\sigma}$ and $\tilde{\tau}$ intersect transversally. The \textit{intersection index} of $\tilde{\sigma}$ and $\tilde{\tau}$ at $p$ is defined to be

$$\text{ind}_p(\tilde{\sigma}, \tilde{\tau}) = \tilde{\sigma}(t_1) \cdot \kappa \cdot \tilde{\tau}^{-1}(t_2) \in \pi_1(\mathbb{P}(T_pS^0)) \cong \mathbb{Z}$$
where $\kappa$ is the (homotopy class of) path in $\mathbb{P}(T_pS^\triangle)$ from $\check{\sigma}(t_1)$ to $\check{\tau}(t_2)$ given by clockwise rotation by an angle smaller than $\pi$. We have the following equality (see [5, Equation (2.5)])

$$ind_p(\check{\sigma}, \check{\tau}) + ind_p(\check{\tau}, \check{\sigma}) = 1. \quad (2.1)$$

Equivalently, the intersection index is the shift $[ind_p(\check{\sigma}, \check{\tau})]$ such that the lift $\check{\sigma}$ intersects the lift $\check{\tau}[ind_p(\check{\sigma}, \check{\tau})]$ on $\mathbb{R}TS^\triangle$.

The notion of intersection index can be generalized to the case $p \in \triangle$ as follows. Fix a small circle $l \subset S \setminus \triangle$ around $p$. Let $\alpha : [0, 1] \to l$ be an embedded arc which moves clockwise around $l$, such that $\alpha$ intersect $\check{\sigma}$ and $\check{\tau}$ at $\alpha(0)$ and $\alpha(1)$, respectively. Then $\alpha$ is unique up to a change of parametrization. Fixing an arbitrary grading $\hat{\alpha}$ on $\alpha$, the intersection index $ind_p(\check{\sigma}, \check{\tau})$ is defined to be

$$ind_p(\check{\sigma}, \check{\tau}) := ind_{\alpha(0)}(\check{\sigma}, \hat{\alpha}) - ind_{\alpha(1)}(\check{\tau}, \hat{\alpha}). \quad (2.2)$$

2.4. Log DMS and $q$-intersection number. Next, we introduce the log DMS to unwind the second grading of DMS. Let $(S_{\triangle}, \Lambda, \lambda, c)$ be a decorated DMS. Denote by $\Lambda_1$ the section $S \setminus \triangle \to FT(S \setminus \triangle)$ corresponding to the projection of $\Lambda(\alpha)$ on the first $\mathbb{Z}$.

**Definition 2.4.** The log DMS log $S_{\triangle}$ is obtained from $S_{\triangle}$ by copying $S_{\triangle}$ for $\mathbb{Z}$ times, denoting by $S^m_{\triangle}$, $m \in \mathbb{Z}$, cutting each sheet along all arcs $c^m_i$, where $c^m_i$ is the $m$-th copy of the cut $c$, denoting by $c^m_i \pm$ the cut marks, and identifying $c^m_i$ with $c^{m+1}_i$. The grading of log $S_{\triangle}$ inherits from the section $\Lambda_1$. Each copy $S^m_{\triangle}$ is called the $m$-th sheet of log $S_{\triangle}$.

Denote by $\pi_{\triangle} : \log S_{\triangle} \to S_{\triangle}$ the covering map. For any graded curve $\check{\eta} \in \mathcal{C}(S_{\triangle})$ in a minimal position with respect to the cut $c$ and an integer $m \in \mathbb{Z}$, there is a curve $\overline{\eta}^m$ in log $S_{\triangle}$, called a lift of $\check{\eta}$, such that $\pi_{\triangle}(\overline{\eta}^m) = \check{\eta}$ and whose starting segment is in the sheet $S^m_{\triangle}$. We will use $[mX]$ for the $X$-grading shift, i.e. $\overline{\eta}^m[m'X] = \overline{\eta}^{m+m'}$.

Denote by $\overline{\eta}^m$ the inverse of $\overline{\eta}^m$.

For any $\check{\eta} \in \mathcal{CA}(S_{\triangle})$, we call any of its lifts in log $S_{\triangle}$ a double graded closed arc; for any $\check{\eta} \in \mathcal{AC}(S_{\triangle})$, we call any of its lifts in log $S_{\triangle}$ a double graded admissible closed curve in log $S_{\triangle}$. Denote by $\mathcal{CA}^X(S_{\triangle})$ (resp. $\mathcal{AC}^X(S_{\triangle})$) the set of double graded closed arcs (resp. double graded admissible closed curves) in log $S_{\triangle}$.

For any lifts $\check{\sigma}^m$ and $\check{\tau}^{m'}$ in log $S_{\triangle}$ of gradings of curves $\sigma$ and $\tau$, we call an intersection $p$ of $\sigma$ and $\tau$ is an intersection of $\check{\sigma}^m$ and $\check{\tau}^{m'}$ with bi-index

$$ind_p^\square(\check{\sigma}^m, \check{\tau}^{m'}) = ind_p(\check{\sigma}^m, \check{\tau}^{m'}) + ind_p^X(\check{\sigma}^m, \check{\tau}^{m'})X \quad (2.3)$$

where $ind_p(\check{\sigma}^m, \check{\tau}^{m'}) := ind_p(\check{\sigma}, \check{\tau})$ and $ind_p^X(\check{\sigma}^m, \check{\tau}^{m'}) = \varsigma$ if

- $p \notin \triangle$ and some lift of $p$ is an intersection between $\overline{\sigma}^m$ and $\overline{\tau}^{m'+\varsigma}$;
- $p \in \triangle$ and
– either the segments of $\tilde{\sigma}^m$ and $\tilde{\tau}^m+\varsigma$ near $p$ are in the same sheet of $\log S_{\Delta}$ (i.e. $\varsigma = m - m'$) and the angle from $\tilde{\sigma}^m$ to $\tilde{\tau}^m+\varsigma$ is clockwise around $p$ that does not cross the cut $c$ (see the left picture of Figure 1);
– or the segments of $\tilde{\sigma}^m$ and $\tilde{\tau}^m+\varsigma+1$ near $p$ are in the same sheet of $\log S_{\Delta}$ (i.e. $\varsigma = m - m' + 1$) and the angle from $\tilde{\sigma}^m$ to $\tilde{\tau}^m+\varsigma+1$ clockwise around $p$ that crosses the cut $c$ (see the right picture of Figure 1).

Figure 1. $\chi$-index $\text{ind}_p^\chi(\tilde{\sigma}^m, \tilde{\tau}^m')$ at a decoration

Next, we give couple of lemmas on basic properties of bi-index of intersections.

**Lemma 2.5.** Let $\tilde{\sigma}^m, \tilde{\tau}^m'$ be two lifts in $\log S_{\Delta}$ of gradings of curves $\sigma$ and $\tau$. Let $p$ be an intersection of $\sigma$ and $\tau$ in $S \setminus \partial S$. If $p \notin \Delta$, we have
$$\text{ind}_p^E(\tilde{\sigma}^m, \tilde{\tau}^m') + \text{ind}_p^E(\tilde{\tau}^m', \tilde{\sigma}^m) = 1. \tag{2.4}$$
If $p \in \Delta$, we have
$$\text{ind}_p^E(\tilde{\sigma}^m, \tilde{\tau}^m') + \text{ind}_p^E(\tilde{\tau}^m', \tilde{\sigma}^m) = \chi. \tag{2.5}$$

**Proof.** By definition, we have
$$\text{ind}_p^\chi(\tilde{\sigma}^m, \tilde{\tau}^m') + \text{ind}_p^\chi(\tilde{\tau}^m', \tilde{\sigma}^m) = \begin{cases} 0 & \text{if } p \notin \Delta \\ \chi & \text{if } p \in \Delta. \end{cases}$$
Combining this with (2.1), we have the formula (2.4). The second equation (2.5) follows from the fact that the grading $\Lambda$ takes value $(-2, 1)$ for a loop around any decoration. \hfill $\square$

**Lemma 2.6.** Let $\tilde{\sigma}^m, \tilde{\tau}^m'$ be two lifts in $\log S_{\Delta}$ of gradings of curves $\sigma$ and $\tau$. Let $p \in \Delta$ be an intersection of $\sigma$ and $\tau$. Fix a small circle $l \subset S \setminus \Delta$ around $p$. Let $\alpha : [0, 1] \to l$ be an embedded arc which moves clockwise around $l$, such that $\alpha$ intersect $\sigma$ and $\tau$ at $\alpha(0)$ and $\alpha(1)$, respectively. Then for any lift $\tilde{\alpha}''$ in $\log S_{\Delta}$ of a grading of $\alpha$, we have
$$\text{ind}_p^E(\tilde{\sigma}^m, \tilde{\tau}^m') = \text{ind}_p^E(\tilde{\sigma}^m, \tilde{\alpha}''') - \text{ind}_p^E(\tilde{\tau}^m', \tilde{\alpha}'''). \tag{2.6}$$

**Proof.** If $\tilde{\sigma}^m$ and $\tilde{\tau}^m'$ are in the case of the left picture of Figure 1, we have
$$\text{ind}_{\alpha(0)}^\chi(\tilde{\sigma}^m, \tilde{\alpha}''') = m - m'', \text{ ind}_{\alpha(1)}^\chi(\tilde{\tau}^m', \tilde{\alpha}''') = m' - m''.$$
So we have \( \text{ind}_{\alpha(0)}(\tilde{\sigma}^m, \tilde{\alpha}^{m''}) - \text{ind}_{\alpha(1)}(\tilde{\tau}^m, \tilde{\alpha}^{m''}) = m'' - m' = \text{ind}_{\rho}(\tilde{\sigma}^m, \tilde{\tau}^m) \). If \( \tilde{\sigma}^m \) and \( \tilde{\tau}^m \) are in the case of the right picture of Figure 1, we have
\[
\text{ind}_{\alpha(0)}(\tilde{\sigma}^m, \tilde{\alpha}^{m''}) = m - m'' \quad \text{and} \quad \text{ind}_{\alpha(1)}(\tilde{\tau}^m, \tilde{\alpha}^{m''}) = m' - m'' - 1.
\]
So we have \( \text{ind}_{\alpha(0)}(\tilde{\sigma}^m, \tilde{\alpha}^{m''}) - \text{ind}_{\alpha(1)}(\tilde{\tau}^m, \tilde{\alpha}^{m''}) = m'' - m' + 1 = \text{ind}_{\rho}(\tilde{\sigma}^m, \tilde{\tau}^m) \). Combining this with (2.2), we get the formula (2.6).

Lemma 2.7. Let \( \tilde{\sigma}^m, \tilde{\gamma}^m, \tilde{\rho}^{m''} \) be lifts of gradings of \( \sigma, \gamma, \rho \) such that \( \sigma, \gamma, \tau \) intersect at \( p \) in clockwise order as in Figure 2, then we have
\[
\text{ind}_{\rho}(\tilde{\sigma}^m, \tilde{\rho}^{m''}) = \text{ind}_{\rho}(\tilde{\sigma}^m, \tilde{\gamma}^m) + \text{ind}_{\rho}(\tilde{\gamma}^m, \tilde{\rho}^{m''}).
\] (2.7)

Proof. This following directly from definition.

![Figure 2](image-url)

For any \( \varrho, \varsigma \in \mathbb{Z} \), denote by \( \cap^{\varrho+\varsigma X}(\tilde{\sigma}^m, \tilde{\tau}^m) \) the set of intersections between \( \tilde{\sigma}^m \) and \( \tilde{\tau}^m \) with bi-index \( \varrho + \varsigma X \). We will use the notations
\[
\text{Int}_{\Delta}^{\varrho+\varsigma X}(\tilde{\sigma}^m, \tilde{\tau}^m) := \left| \cap^{\varrho+\varsigma X}(\tilde{\sigma}^m, \tilde{\tau}^m) \cap \Delta \right|,
\]
\[
\text{Int}_{S_{\Delta}}^{\varrho+\varsigma X}(\tilde{\sigma}^m, \tilde{\tau}^m) := \left| \cap^{\varrho+\varsigma X}(\tilde{\sigma}^m, \tilde{\tau}^m) \cap S_{\Delta} \right|,
\]
\[
\text{Int}_{S_{\Delta}}^{\varrho+\varsigma X}(\tilde{\sigma}^m, \tilde{\tau}^m) := \frac{1}{2} \cdot \text{Int}_{\Delta}^{\varrho+\varsigma X}(\tilde{\sigma}^m, \tilde{\tau}^m) + \text{Int}_{S_{\Delta}}^{\varrho+\varsigma X}(\tilde{\sigma}^m, \tilde{\tau}^m),
\]
for the index-(\( \varrho + \varsigma X \)) intersection numbers at decorations, in the interior, and at all of \( S_{\Delta} \setminus \partial S_{\Delta} \), respectively. The total intersection
\[
\text{Int}_{\mathcal{T}}(\tilde{\sigma}^m, \tilde{\tau}^m) = \sum_{\varrho, \varsigma \in \mathbb{Z}} \text{Int}_{\mathcal{T}}^{\varrho+\varsigma X}(\tilde{\sigma}^m, \tilde{\tau}^m)
\]
is the sum over all indices, where \( \mathcal{T} = \Delta, S_{\Delta} \) or \( S_{\Delta} \). Define the \( \mathbb{Z}^2 \)-graded \( q \)-intersection of \( \tilde{\sigma}^m, \tilde{\tau}^m \in \tilde{\text{AC}}^{X}(S_{\Delta}) \) to be
\[
\text{Int}_{q}(\tilde{\sigma}^m, \tilde{\tau}^m) = \sum_{\varrho, \varsigma \in \mathbb{Z}} q^{\varrho+\varsigma X} \cdot \text{Int}_{\mathcal{T}}^{\varrho+\varsigma X}(\tilde{\sigma}^m, \tilde{\tau}^m) + (1 + q^{-1}) \sum_{\varrho, \varsigma \in \mathbb{Z}} q^{\varrho+\varsigma X} \cdot \text{Int}_{S_{\Delta}}^{\varrho+\varsigma X}(\tilde{\sigma}^m, \tilde{\tau}^m). \quad \text{(2.8)}
\]
Note that we have \( \text{Int}_{q}(-, -) \big|_{q=1} = 2 \text{Int}_{S_{\Delta}}(-, -) \).
Define the $\mathbb{Z}^2$-graded $q$-intersection of a lift $\tilde{\gamma}^m$ of a graded open curve and $\tilde{\tau}'^m \in \tilde{\mathcal{AC}}^X(S_\Delta)$ to be
\[
\text{Int}^q(\tilde{\gamma}^m, \tilde{\tau}'^m) = \sum_{\hat{\rho}, \varsigma \in \mathbb{Z}} q^{\hat{\rho} + \varsigma X} \cdot \text{Int}_{S_\Delta}^{\hat{\rho} + \varsigma X}(\tilde{\gamma}^m, \tilde{\tau}'^m).
\] (2.9)

The following notation is useful in the paper.

**Definition 2.8** (Extension of curves). Let $\sigma, \tau$ be two curves in $S_\Delta$ with $\sigma(0) = \tau(0) \in \Delta$. The (positive) extension $\tau \land \sigma$ of $\tau$ by $\sigma$ (w.r.t. the common starting point) is defined in Figure 3, which consists of the operation so-called smoothing out at such an intersection and possibly (at most) one operation–the unknotted a non-admissible intersection.

![Figure 3. The extension as smoothing out and unknotting](image)

Let $\tilde{\sigma}^m$ and $\tilde{\tau}'^m$ be lifts of gradings of $\sigma$ and $\tau$, respectively. The extension $\tilde{\tau}'^m \land \tilde{\sigma}^m$ is the lift of the grading of $\tau \land \sigma$ inherited from $\tilde{\tau}'^m$.

2.5. **Differential graded $X$-graded quiver algebras from surfaces.**

**Definition 2.9.** An (open) full formal arc system $\mathbf{A}$ of a graded DMS $(S_\Delta, \Lambda, \lambda, c)$ is a collection of pairwise non-isotopic and non-crossing graded open arcs which do not intersect the cut $c$, such that it divides the surface $S$ into polygons, called $\mathbf{A}$-polygons, satisfying that each $\mathbf{A}$-polygon contains exactly one decoration in $\Delta$ and has exactly one open boundary segment as its edge.

The number of graded open arcs in any $\mathbf{A}$ only depends on the numerical data of $S$, which is the analogue of the well-known fact that the number of arcs in a triangulation only depends on the numerical data of a marked surface. Such a number is called the rank of $S$, which is given by the following formula. We leave the proof to the readers.

**Lemma 2.10.** Let $\mathbf{A}$ be a full formal open arc system of $S$. Then
\[
|\mathbf{A}| = 2g + b + |\mathbf{M}| - 2
\]
where $g$ is the genus of $S$ and $b$ is the number of components of $\partial S$. 
Definition 2.11 (Arc segments). Let $A$ be a full formal arc system of $S$. An arc segment is a curve $\rho : [0, 1] \to S_\Delta$ without self-intersections such that its interior is in the interior of an $A$-polygon $P$ and whose endpoints are on the edges of $P$.

An arc segment $\rho$ in $P$ is called positive (resp. negative), if the decoration in $P$ is on the right (resp. left) hand side of $\rho$. Note that the inverse of an arc segment $\rho$ in $P$ has the opposite sign of $\rho$. See green arc segments in Figure 4.

Any arc segment $\rho$ will be considered up to isotopy with respect to the interior of the edges of $P$. Define an equivalent relation on positive arc segments, that $\rho_1 \sim \rho_2$ if and only if $\rho_1$ and $\rho_2$ are in two neighbor $A$-polygons and form a digon (see Figure 5). Denote by $[\rho]$ the class of lifts $\tilde{\rho}_m$ of gradings of $\rho$ with $\rho' \sim \rho$. Let $AS(A)$ be the set of classes $[\rho]$ of arc segments $\rho$ and $AS^+(A)$ the set of classes $[\rho]$ of positive arc segments $\rho$. For any $[\rho_1], [\rho_2] \in AS^+(A)$, if there are representatives $\tilde{\rho}_1^m \in [\rho_1]$ and $\tilde{\rho}_2^m \in [\rho_2]$ such that the composition $\tilde{\rho}_1^m \cdot \tilde{\rho}_2^m$ is again a positive arc segment, we define $[\rho_1] \cdot [\rho_2] = [\rho_1 \cdot \rho_2]$. A class $[\rho]$ is called trivial if $\rho$ is isotopy to a segment of an arc in $A$, i.e. the ones in the left picture of Figure 5. A class $[\rho]$ is called loop-type if its endpoints are in the same arc of $A$, and together with that arc, it encloses a decoration as shown in the right picture of Figure 5.

Let $A = \{\tilde{\gamma}_i \mid 1 \leq i \leq n\}$ be a full formal arc system of $S_\Delta$. We denote by $A_Z := \cup_{m \in \mathbb{Z}} A^m$, where $A^m = \{\tilde{\gamma}_i^m \mid 1 \leq i \leq n, \ m \in \mathbb{Z}\}$. Note that any arc in $A$ does not cross any arc in the cut $c$. So each $\tilde{\gamma}_i^m$ is in the sheet $S_\Delta^m$. We regard $\tilde{\gamma}_i \in A$ as $\tilde{\gamma}_i^0 \in A_Z$ sometimes.
Lemma 2.13. For a full formal arc system $A = \{\gamma_i \mid 1 \leq i \leq n\}$ of $S_A$, there is an associated $(\mathbb{Z} \oplus \mathbb{Z} \mathbb{X}=)\mathbb{Z}^2$-graded quiver $Q_A$ with a map $d_A : (Q_A)_1 \rightarrow kQ_A$ given by the following data.

- The vertices of $Q_A$ are (indexed by) the open arcs in $A$: $i := \gamma_i$ for $1 \leq i \leq n$;
- Each non-trivial $[\rho] \in AS^+(A)$ with $\rho(0) \in \gamma_i$ and $\rho(1) \in \gamma_j$ induces a $\mathbb{Z}^2$-graded arrow $b_{[\rho]} : i \rightarrow j$ of bi-degree

$$\deg(b_{[\rho]}): = \text{ind}_{\rho(1)}^\mathbb{Z}(\gamma_j, \tilde{\rho}^m) - \text{ind}_{\rho(0)}^\mathbb{Z}(\gamma_i, \tilde{\rho}^m),$$

(2.10)

where $\tilde{\rho}^m$ is an arbitrary representative in $[\rho]$.
- The map $d_A$ is given by

$$d_A(b_{[\rho]}) = \sum_{[\rho_1],[\rho_2]=[\rho]} (-1)^{|b_{[\rho_1]|}^1} b_{[\rho_1]} b_{[\rho_2]},$$

(2.11)

where the sum runs over all pairs $([\rho_1],[\rho_2])$ of positive arc segments $[\rho_1],[\rho_2] \in AS^+(A)$ satisfying $[\rho] = [\rho_1] : [\rho_2]$.

Lemma 2.13. The map $d_A$ is a differential. That is, $d_A^2(b_{[\rho]}) = 0$ for any $[\rho] \in AS^+(A)$.

Proof.

$$d_A^2(b_{[\rho]}) = d_A \left( \sum_{[\rho_1],[\rho_2]=[\rho]} (-1)^{|b_{[\rho_1]|}^1} b_{[\rho_1]} b_{[\rho_2]} \right)
= \sum_{[\rho_1],[\rho_2]=\rho} (-1)^{|b_{[\rho_1]|}^1} \left( d_A(b_{[\rho_1]}) b_{[\rho_2]} + (-1)^{|b_{\rho_1}|^1} b_{[\rho_1]} d_A(b_{[\rho_2]}) \right)
= \sum_{[\rho_1],[\rho_2]=\rho} (-1)^{|b_{\rho_1}|^1} \left( b_{[\rho_1]} b_{[\rho_2]} + \sum_{[\rho_2],[\rho_2']=\rho} (-1)^{|b_{\rho_2'}|^1} b_{[\rho_1]} b_{[\rho_2]} b_{[\rho_2']}ight)
= \sum_{[\rho_1],[\rho_2],[\rho_3]=\rho} (-1)^{|b_{\rho_1}|^1} \left( b_{[\rho_1]} b_{[\rho_2]} b_{[\rho_3]} + \sum_{[\rho_1],[\rho_2],[\rho_3]=\rho} (-1)^{|b_{\rho_2}|^1} b_{[\rho_1]} b_{[\rho_2]} b_{[\rho_3]} \right)
= 0.$$

So $(kQ_A,d_A)$ gives a dg $\mathbb{X}$-graded algebra $\Gamma_A$, after extending $d_A$ linearly to a map from $kQ_A$ to $kQ_A$ of bi-degree 1 via the Leibniz rule.

Remark 2.14. Note that, for any arc segment $[\rho] \in AS^+(A)$ of loop-type, we have $\deg(b_{[\rho]}) = 1 - \mathbb{X}$. Moreover, arcs in $AS^+(A)$ comes in pairs, namely,

- a loop-type/trivial arc segment will be paired with a trivial/loop-type arc segment, where their endpoints are in the same arc of $A$.
- any other (non-trivial, non-loop-type) $[\rho] \in AS^+(A)$ with $b_{[\rho]} : i \rightarrow j$ can be uniquely paired with $[\rho]^* \in AS^+(A)$ with $b_{[\rho]^*} : j \rightarrow i$, such that

$$\deg(b_{[\rho]}) + \deg(b_{[\rho]^*}) = 2 - \mathbb{X}.$$
For example, the green arcs in the first two pictures in Figure 4 are paired together.

Let $Q^0_A$ be the $X$-degree zero part of $Q_A$, i.e., the quiver with the same vertices and with arrows $b$ such that $\deg(b) \in \mathbb{Z} \subset \mathbb{Z} \oplus \mathbb{Z}X$. The property above basically means $Q_A$ is Calabi-Yau-$X$ double of $Q^0_A$, in the sense of [13, Def. 6.2]. Equivalently, for any pair of arc segments $([\rho], [\rho]^*)$, where neither of them is trivial nor loop-type, exactly one of $b[\rho]$ and $b[\rho]^*$ is in $Q^0_A$.

Let $\Gamma^0_A$ be the differential graded subalgebra of $\Gamma_A$ given by $kQ^0_A$ with induced differential $d^0_A$. Then $\Gamma_A$ is the Calabi-Yau-$X$ completion of $\Gamma^0_A$ (see [10]). So $D_{fd}(\Gamma_A)$ is $X$-Calabi-Yau, i.e., $X$ is the Serre functor on $D_{fd}(\Gamma_A)$.

**Definition 2.15.** Let $\text{Sim} \\Gamma_A = \{S_i \mid 1 \leq i \leq n\}$ be the set of simple $\Gamma_A$-modules, where $S_i$ corresponds to $\gamma_i$. Denote by $E_A$ the $\mathbb{Z}^2$-graded algebra $\text{Ext}_{\mathbb{Z}}^{\mathbb{Z}}(S,S)$, where $S = \oplus_{i=1}^n S_i$. So the simples $S_i, 1 \leq i \leq n,$ can be regarded as the indecomposable direct summands of $E_A$. Denote by $\text{add}E_A$ the $\mathbb{Z}^2$-graded category consisting of the indecomposable direct summands of $E_A$.

By the construction (2.11) of $d_A$, the $A_\infty$ structure introduced in Theorem 1.3 is an ordinary associated multiplication. So by Theorem 1.3 and Corollary 1.4, we have the following result.

**Proposition 2.16.** There is a triangle equivalence

$$D_{fd}(\Gamma_A) \simeq \text{per} \ E_A. \quad (2.12)$$

The morphisms in the $\mathbb{Z}^2$-graded category $E_A$ can be described in the following way.

- Each non-trivial $[\rho] \in \text{AS}^+(A)$ with $\rho(0) \in \gamma_i$ and $\rho(1) \in \gamma_j$ induces a morphism $\pi_{[\rho]} := \pi_{b[\rho]} : S_i \to S_j$ of bi-degree

$$\deg(\pi_{[\rho]}) = 1 - \text{ind}^{\mathbb{Z}^2}_{\rho(1)}(\gamma_i, \gamma_j) + \text{ind}^{\mathbb{Z}^2}_{\rho(0)}(\gamma_i, \gamma_j) \quad (2.13)$$

where $\gamma^m$ is an arbitrary representative in $[\rho]$; each trivial $[\rho] \in \text{AS}^+(A)$ with $\rho(0) \in \gamma_i$ and $\rho(1) \in \gamma_j$ induces a morphism $\pi_{[\rho]} := \pi_{b[\rho]} : S_i \to S_j$ of bi-degree 0. All of $\pi_{[\rho]}$, $[\rho] \in \text{AS}^+(A)$ with $\rho(0) \in \gamma_i$ and $\rho(1) \in \gamma_j$, form a basis of the $\mathbb{Z}^2$-graded $k$-vector space $\text{Hom}_{\mathbb{Z}^2}(S_i, S_j)$.

- The compositions are given by

$$\pi_{[\rho_2]} \circ \pi_{[\rho_1]} = \begin{cases} \pi_{[\rho]} & \text{if } [\rho_1] \cdot [\rho_2] = [\rho] \\ 0 & \text{otherwise} \end{cases}. \quad (2.14)$$

In particular, each representative $\gamma^m \in [\rho] \in \text{AS}^+(A)$ with $\rho(0) \in \gamma_i$ and $\rho(1) \in \gamma_j$ gives a morphism of bi-degree 1 between shifts of indecomposable direct summands of $E_A$:

$$\pi_{\gamma^m} : S_i[- \text{ind}^{\mathbb{Z}^2}_{\rho(0)}(\gamma_i, \gamma_j)] \to S_j[- \text{ind}^{\mathbb{Z}^2}_{\rho(1)}(\gamma_j, \gamma_i)]. \quad (2.15)$$
3. String model on log DMS

Throughout this section, \((S_\Delta, \Lambda)\) is a graded DMS and \(A = \{\gamma_1, \ldots, \gamma_n\}\) is a full arc system of \((S_\Delta, \Lambda)\). We still use the notations in the previous section. That is, \(\Gamma_A\) is the dg \(\mathbb{X}\)-graded quiver associated to \(A\), defined by the \(\mathbb{Z}^2\)-graded quiver \(Q_A\) and the differential \(d_A, S_1, \ldots, S_n\) are the simple \(\Gamma_A\) modules corresponding to \(\gamma_1, \ldots, \gamma_n\), respectively, and \(E_A\) is the \(\mathbb{Z}^2\)-graded algebra \(\text{Ext}^{\mathbb{Z}^2}(\mathbb{S}, \mathbb{S})\), where \(\mathbb{S} = \oplus_{i=1}^n S_i\). Note that each \(S_i\) can be regarded as an indecomposable direct summand of \(E_A\). We have already shown that there is a triangle equivalence

\[
D_{fd}(\Gamma_A) \simeq \per E_A
\]

In this section, we shall associate to each curve \(\sigma^m \in \operatorname{AC}^X(S_\Delta)\) an object \(X_{\sigma^m}\) in \(E_A\), which can then be regarded as an object in \(D_{fd}(\Gamma_A)\).

3.1. String model. A string in \(E_A\) is a sequence of morphisms \(f_1, f_2, \cdots, f_p\) of bi-degree 1 of shifts of indecomposable direct summands of \(E_A\):

\[
\omega : S_{k_0}[\theta_0 + \zeta_0 \mathbb{X}] \xrightarrow{f_1} S_{k_1}[\theta_1 + \zeta_1 \mathbb{X}] \xrightarrow{f_2} \cdots \xrightarrow{f_p} S_{k_p}[\theta_p + \zeta_p \mathbb{X}] (3.1)
\]

such that

- each \(f_i\) is from left to right, or from right to left;
- if both \(f_i\) and \(f_{i+1}\) point to the right, then \(f_{i+1} \circ f_i = 0\);
- if both \(f_i\) and \(f_{i+1}\) point to the left, then \(f_i \circ f_{i+1} = 0\).

This string \(\omega\) gives a dg \(\mathbb{X}\)-graded \(E_A\)-module \(X_\omega\) whose underlying \(\mathbb{Z}^2\)-graded \(E_A\)-module is

\[
|X_\omega| = \bigoplus_{i=0}^p S_{k_i}[\theta_i + \zeta_i \mathbb{X}]
\]

and whose differential is given by the \(f_i, 1 \leq i \leq p\). By definition, \(X_\omega \in \per E_A\).

Construction 3.1. Let \(\sigma^m \in \operatorname{AC}^X(S_\Delta)\) with its underlying admissible closed curve \(\sigma\) in a minimal position with respect to \(A\). Suppose that \(\sigma\) intersects \(A\) at \(V_0, V_1, \cdots, V_p\) in order, where \(V_j\) is in \(\gamma_{k_j}\), and denote by \(V_{i-1}\) and \(V_{i+1}\) its starting and ending points, respectively. We denote by \(\sigma_{ij}\) the segment of \(\sigma\) from \(V_i\) to \(V_j\) for any \(-1 \leq i < j \leq p+1\). Then \(A\) divides \(\sigma\) into arc segments \(\sigma_{i-1,i}, 0 \leq i \leq p+1\). We call an arc segment \(\sigma_{i-1,i}\) an interior arc segment, if \(1 \leq i \leq p\). Denote by \(\sigma^m\) the segments of \(\sigma^m\) divided by \(A\) in order. Then we have that each \(\sigma^m_{i-1,i}\) is a lift of a grading of \(\sigma_{i-1,i}\). Define

\[
\omega(\sigma^m) : S_{k_0}[\chi_0] \xrightarrow{f_1} S_{k_1}[\chi_1] \xrightarrow{f_2} \cdots \xrightarrow{f_p} S_{k_p}[\chi_p], (3.2)
\]

where

\[
f_i = \begin{cases} 
\pi_{\sigma^m_{i-1,i}} & \text{if } \sigma^m_{i-1,i} \text{ is positive,} \\
\pi_{\sigma^m_{i-1,i}} & \text{if } \sigma^m_{i-1,i} \text{ is negative,}
\end{cases}
\]

\(1 \leq i \leq p\), and \(\chi_j = - \text{ind}_{V_j}^\mathbb{Z}2(\gamma_{k_j}, \sigma^m), 0 \leq j \leq p\), see Figure 6.
Lemma 3.2. \( \omega(\tilde{\sigma}^m) \) is a string.

Proof. This follows from that if both \( f_i \) and \( f_{i+1} \) points to the same direction, then the corresponding positive arc segments \( \sigma_{i-1,i} \) and \( \sigma_{i,i+1} \) have no composition.  \( \square \)

Notations 3.3. Denote by \( X_{\tilde{\sigma}^m} = X_{\omega(\tilde{\sigma}^m)} \). By the construction, we have \( X_{\tilde{\sigma}^m} = X_{\bar{\sigma}^m} \).

For any \( \tilde{\sigma}^m \in \tilde{AC}^X(\mathcal{S}_\Delta) \), we denote by \( X_\sigma \) the isoclass of the object \( X_{\tilde{\sigma}^m} \) in the orbit category \( \mathcal{D}_fd(\Gamma_A)/(\langle [X], [1]\rangle) \). So for any \( \tilde{\sigma}^m, \tilde{\tau}^m \in \tilde{AC}^X(\mathcal{S}) \), we have

\[
\text{Hom}_{\mathcal{D}_fd(\Gamma_A)/(\langle [X], [1]\rangle)}(X_\sigma, X_\tau) = \bigoplus_{q, \varsigma \in \mathbb{Z}} \text{Hom}_{\mathcal{D}_fd(\Gamma_A)}(X_{\tilde{\sigma}^m}, X_{\tilde{\tau}^m}[[q + \varsigma X]])
\]  

(3.3)

Whenever we say a triangle in \( \mathcal{D}_fd(\Gamma_A)/(\langle [X], [1]\rangle) \), we always mean the image of a triangle in \( \mathcal{D}_fd(\Gamma_A) \).

Define the \( q \)-dim of \( \text{Hom}^Z \) for any objects \( X_1, X_2 \) in \( \mathcal{D}(\Gamma_A) \) as follows

\[
\dim^q \text{Hom}^Z(X_1, X_2) := \sum_{q, \varsigma \in \mathbb{Z}} q^{\theta+\varsigma X} \cdot \dim \text{Hom}^{\theta+\varsigma X}(X_1, X_2).
\]  

(3.4)

We have the following result as a graded version of [17, Lemma 5.7 and Corolary 5.8]. Here \( \Gamma_A^i = e_i \Gamma_A \) with \( e_i \) the trivial path at \( i \) in \( Q_A \).

Lemma 3.4. Let \( \tilde{\sigma}^m \in \tilde{AC}^X(\mathcal{S}_\Delta) \) and \( \tilde{\gamma}_i \in \mathcal{A} \). For any \( q, \varsigma \in \mathbb{Z} \), we have

\[
\dim^q \text{Hom}^Z(\Gamma_A^i, X_{\tilde{\sigma}^m}) = \text{Int}^q(\tilde{\gamma}_i, \tilde{\sigma}^m).
\]  

(3.5)

Proof. Let \( \omega(\tilde{\sigma}^m) \) be the one in (3.2). Since \( \dim^q \text{Hom}^Z(\Gamma_A^i, S_k) = \delta_{i,k} \), we have \( \text{Hom}(\Gamma_A^i, f_j) = 0 \) for any \( 1 \leq i \leq n \) and \( 1 \leq j \leq p \), due to that each \( f_j \) corresponds to a non-trivial arc segment. Thus, the required formula follows from the construction of \( X_{\tilde{\sigma}^m} \).  \( \square \)

Definition 3.5. Let \( \sigma \in AC(\mathcal{S}_\Delta) \) without self-intersections in \( \mathcal{S}_\Delta^0 \). A grading \( \tilde{\sigma} \) is called dual to \( \tilde{\gamma}_i \in \mathcal{A} \) with respect to \( \mathcal{A} \) if \( \tilde{\gamma}_i \) intersects it once with intersection index \( 0 \) and \( \tilde{\gamma}_j, j \neq i, \) does not intersect it. Denote by \( \bar{s}_i \) the dual to \( \gamma_i \) and by \( \mathcal{A}_\Delta = \{ \bar{s}_i \mid 1 \leq i \leq n \} \).
By construction, we have $X_{\tilde{s}_j} = S_j$ for any $1 \leq j \leq n$. Conversely, we have the following.

**Lemma 3.6.** Let $\tilde{\sigma}^m \in \widetilde{AC}^X(S_\Delta)$. If $X_{\tilde{\sigma}^m} = S_j$ for some $1 \leq j \leq n$, then $m = 0$ and $\tilde{\sigma} = \tilde{s}_j$.

**Proof.** By Lemma 3.4, we have

$$\dim_q(\tilde{\gamma}_i, \tilde{\sigma}^m) = \dim_q \text{Hom}^Z_\Gamma(\Gamma_A^i, X_{\tilde{\sigma}^m}) = \dim_q \text{Hom}^Z_\Gamma(\Gamma_A^i, S_j) = \delta_{i,j}$$

So $\tilde{\gamma}_j$ crosses $\tilde{\sigma}^m$ once and the intersection index is 0, while any other $\tilde{\gamma}_i, i \neq j$, does not cross $\tilde{s}_j$. This implies that $\tilde{\gamma} = \tilde{s}_j$. Moreover, since the $X$-index of the intersection between $\tilde{\gamma}_j$ and $\tilde{\sigma}^m$ is zero, we have $m = 0$. 

---

### 3.2. Morphisms induced by angles at decorations.

Let $\tilde{\tau}^m' \in \widetilde{AC}^X(S_\Delta)$ whose underlying $\tau$ is in a minimal position with respect to $A$ and $\sigma$. Note that $\tau$ may be homotopic to $\sigma$. Suppose that $\tilde{\tau}$ intersects $A$ at $W_0, W_1, \ldots, W_q$ in order, where $W_j$ is in $\tilde{\gamma}_{t_j} \in A$. Denote by $W_{-1}$ and $W_{q+1}$ its starting and ending points, respectively, and $\tilde{\tau}_{t_j}$ the segment of $\tilde{\tau}$ from $W_i$ to $W_j$ for any $-1 \leq i < j \leq q+1$. Denote by $\tilde{\tau}_{-1,0}, \tilde{\tau}_{0,1}, \ldots, \tilde{\tau}_{q,q+1}$ the segments of $\tilde{\tau}^m'$ divided by $A^Z$ in order. Then there is a string associated to $\tilde{\tau}^m'$:

$$\omega(\tilde{\tau}^m') : S_{l_0}[\chi'_0] \xrightarrow{g_1} S_{l_1}[\chi'_1] \xrightarrow{g_2} \cdots \xrightarrow{g_q} S_{l_q}[\chi'_q],$$

(3.6)

where

$$g_i = \begin{cases} 
\pi_{\tilde{\tau}_{t_{i-1},t_i}} & \text{if } \tau_{t_{i-1},t_i} \text{ is positive,} \\
\pi_{\tilde{\tau}_{t_{i-1},t_i}} & \text{if } \tau_{t_{i-1},t_i} \text{ is negative,}
\end{cases}$$

and $\chi'_j = -\text{ind}_{W_j}(\tilde{\gamma}_{t_j}, \tilde{\tau}^m')$ (see Figure 7). Then we have an $X_{\tilde{\tau}^m'} := X_{\omega(\tilde{\tau}^m')} \in \text{per } E_A$.

---

**Figure 7.** Segments of $\tilde{\tau}^m'$, cut out by $A^Z$.
Construction 3.7. Assume that $\sigma(0) = \tau(0)$. There is an angle $\theta(\sigma, \tau)$ from $\sigma$ to $\tau$ clockwise at the decoration $\sigma(0) = \tau(0)$. We construct a morphism $\varphi(\bar{\sigma}, \bar{\tau}) : X_{\bar{\sigma}} \to X_{\bar{\tau}}[\nu(\bar{\sigma}, \bar{\tau})]$, for
\[\nu(\bar{\sigma}, \bar{\tau}) = \deg(\pi_{[\tau(1,0) \setminus \rho]}(\rho(\sigma))) + \chi_0 - \chi_0',\]
(see (2.13) for the notation $\deg(\pi_{[\rho]}(\rho(\sigma)))$ induced by $\theta(\sigma, \tau)$ via the sequence of morphisms
\[
\{\varphi_s : S_{k_s}[\chi_s] \to S_{l_s}[\chi_s' + \nu(\bar{\sigma}, \bar{\tau})]\}_{s \geq 0},
\]
where $\varphi_s = \varphi(\bar{\sigma}, \bar{\tau})_s = \pi_{[\rho(\sigma)]}[\chi_s]$ if $\tau_{1,s+1} \wedge \sigma_{1,s+1}$ is a positive arc segment, or 0 otherwise.

We now give explicit descriptions of $\varphi(\bar{\sigma}, \bar{\tau})$ in different cases. By the construction, if $\sigma_{1,0}$ is not isotopic to $\tau_{1,0}$ (i.e. $\sigma$ and $\tau$ separate at the beginning), the only non-zero component in $\varphi(\bar{\sigma}, \bar{\tau})$ is $\varphi_0 = \pi_{[\rho(\sigma)]}[\chi_0]$. See Figure 8, where the green arc segment $\rho = \tau_{1,0} \wedge \sigma_{1,0}$ is positive.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{diagram.png}
\caption{Cases of $\sigma_{1,0} \sim \tau_{1,0}$}
\end{figure}

If $\sigma_{1,s}$ is isotopic to $\tau_{1,s}$ for some $s \geq 0$ then $\nu(\bar{\sigma}, \bar{\tau}) = \chi_0 - \chi_0'$ and $\varphi_0, \ldots, \varphi_s$ are the identities; and if in addition $\sigma_{1,s+1}$ is not isotopic to $\tau_{1,s+1}$ any more, then there are the following cases, where $P$ denotes the $\mathbf{A}$-polygon containing $\sigma_{1,s+1}$ and $\tau_{1,s+1}$.

- The decoration in $P$ is on the right hand side of both $\sigma_{1,s+1}$ and $\tau_{1,s+1}$. See the first picture of Figure 9, where the green arc segment $\tau_{1,s+1} \wedge \sigma_{1,s+1}$ is positive.
- The decoration in $P$ is on the left hand side of both $\sigma_{1,s+1}$ and $\tau_{1,s+1}$. See the second picture of Figure 9, where the green arc segment $\tau_{1,s+1} \wedge \sigma_{1,s+1}$ is positive.
- The decoration in $P$ is on the different hand sides of $\sigma_{1,s+1}$ and $\tau_{1,s+1}$. There are the following subcases.
  - $\sigma_{1,s+1}$ does not intersect $\tau_{1,s+1}$, and none of $\sigma_{1,s+1}$ and $\tau_{1,s+1}$ has endpoints in the same edge of $P$. See the the first picture in the first row of Figure 10, where the green arc segment $\tau_{1,s+1} \wedge \sigma_{1,s+1}$ is negative.
  - $\sigma_{1,s+1}$ intersects $\tau_{1,s+1}$ in $P$. Then the green curve $\tau_{1,s+1} \wedge \sigma_{1,s+1}$ is not an arc segment. See the second picture in the first row of Figure 10.
\[ \sigma_{s,s+1} \text{ does not intersect } \tau_{s,s+1}, \text{ and (at least) one of } \sigma_{s,s+1}, \tau_{s,s+1} \text{ satisfies that its endpoints are in the same edge of } P. \text{ See the pictures in the second row of Figure 10, where the green curve } \tau_{-1,s+1} \wedge \sigma_{-1,s+1} \text{ is not an arc segment.} \]

- At least one of \( \sigma_{s,s+1} \) and \( \tau_{s,s+1} \) connects to the decoration in \( P \). Then the green curve \( \tau_{-1,s+1} \wedge \sigma_{-1,s+1} \) is not an arc segment. The possible cases are shown in the figures in the third row of Figure 10.

So in each case in Figure 9, we have \( \varphi_s = \pi[\tau(-1,s)\wedge\sigma(-1,s)][\chi_s] \), while in each case in Figure 10, we have \( \varphi_s = 0 \).

**Lemma 3.8.** \( \varphi(\tilde{\sigma}^m, \tilde{\tau}^{m'}) \) is a morphism from \( X_{\tilde{\sigma}^m} \) to \( X_{\tilde{\tau}^{m'}} \).

**Proof.** The proof of [19, Lemma A.6] works here. That is, in the cases in Figure 8 and 9, this follows from (2.14); in the cases in Figure 10, this follows from that \( f_s \) points to the right (or does not exist) while \( g_s \) points to the left (or does not exist). \( \square \)

**Lemma 3.9.** \( \varphi(\tilde{\sigma}^m, \tilde{\tau}^{m'}) \) is not zero in \( \mathcal{D}_{fd}(\Gamma_A) \).

**Proof.** The proof of [19, Lemma A.6] works here. That is, for the case \( \tilde{\sigma}_{-1,0} \sim \tilde{\tau}_{-1,0} \) (i.e. in Figures 9 and 10), \( \varphi_0 \) is the identity by definition, which does not factor through \( f_1 \) and \( g_1 \). So \( \varphi(\tilde{\sigma}^m, \tilde{\tau}^{m'}) \) is not zero. The case \( \tilde{\sigma}_{-1,0} \sim \tilde{\tau}_{-1,0} \) (i.e. in Figure 8) is a little complicated, we refer to the proof of [19, Lemma A.6]. \( \square \)

Hence we have the following result.

**Proposition 3.10.** \( \varphi(\tilde{\sigma}^m, \tilde{\tau}^{m'}) \) is a well-defined non-zero morphism from \( X_{\tilde{\sigma}^m} \) to \( X_{\tilde{\tau}^{m'}} \) of bi-degree \( \text{ind}^{\mathbb{Z}}(\tilde{\sigma}^m, \tilde{\tau}^{m'}) \).
Figure 10. Cases of $\tau_{-1, s+1} \land \sigma_{-1, s+1}$ that are negative or not arc segments

**Proof.** By the above two lemma, we only need to calculate the bi-degree. Let $p = \sigma(0) = \tau(0)$ and $\rho = \tau_{-1,0} \land \sigma_{-1,0}$. Take an arbitrary representative $\tilde{\rho} \in [\rho]$. Then we have

\[
\begin{align*}
\text{ind}_{Z^2}^p(\tilde{\sigma}, \tilde{\tau}) & \overset{(2.6)}{=} \text{ind}_{W_0}^2(\tilde{\sigma}, \tilde{\rho}) - \text{ind}_{W_0}^2(\tilde{\tau}, \tilde{\rho}) \\
& \overset{(2.7)}{=} \text{ind}_{V_0}^2(\tilde{\sigma}, \tilde{\gamma}_k) + \text{ind}_{V_0}^2(\tilde{\gamma}_k, \tilde{\rho}) - (\text{ind}_{W_0}^2(\tilde{\tau}, \tilde{\gamma}_l) - \text{ind}_{W_0}^2(\tilde{\rho}, \tilde{\gamma}_l)) \\
& \overset{(2.4)}{=} (1 + \chi_0) - (1 + \chi_0') + (1 + \text{ind}_{W_0}(\tilde{\gamma}_k, \tilde{\rho}) - \text{ind}_{V_0}(\tilde{\gamma}_l, \tilde{\rho}')) \\
& \overset{(2.13)}{=} \chi_0 - \chi_0' + \text{deg}(\pi_{[\beta]}) \\
& \overset{(3.7)}{=} \nu(\tilde{\sigma}, \tilde{\tau})
\end{align*}
\]

We have the following special case.
Lemma 3.11. If \( \tilde{\sigma}^m = \tilde{\tau}^{m'} \), then
\[
\varphi(\tilde{\sigma}^m, \tilde{\tau}^{m'}) = \varphi(\tilde{\tau}^{m'}, \tilde{\sigma}^m), \quad \varphi(\tilde{\tau}^{m'}, \tilde{\sigma}^m) = \varphi(\tilde{\sigma}^m, \tilde{\tau}^{m'}). 
\]
In particular, one of the above pairs is the identity.

Proof. This is the last case in Figure 10, where \( \tilde{\sigma}^m \) and \( \tilde{\tau}^{m'} \) might be switched. \( \square \)

The above result shows that for any \( \tilde{\sigma}^m \in \widetilde{AC}^X(S_\triangle) \), the 2\( \pi \) angle from \( \sigma \) to itself at its starting point induces a non-identity morphism \( \varphi(\tilde{\sigma}^m, \tilde{\sigma}^m) \), which coincides with \( \varphi(\tilde{\sigma}^m, \tilde{\tau}^{m}) \), induced by the 2\( \pi \) angle from \( \sigma \) to itself at its ending point. Moreover, it is in fact the Calabi-Yau dual of the identity map of \( X_{\tilde{\sigma}^m} \). From now on, \( \varphi(?, ?) \) always denote the non-identify angle-2\( \pi \) map instead of the identify angle-0 one.

We simply denote by \( \varphi(\sigma, \tau) \) the morphism \( \varphi(\tilde{\sigma}^m, \tilde{\tau}^{m'}) \) in the orbit category \( D_{fd}(\Gamma_A)/(\{1\}, X) \).

Proposition 3.12. Let \( \sigma_1, \sigma_2, \sigma_3 \in AC(S_\triangle) \) with \( \sigma_1(0) = \sigma_2(0) = \sigma_3(0) \). If the start segments of \( \sigma_1, \sigma_2 \) and \( \sigma_3 \) are in clockwise order at the starting point, see the left picture in Figure 11, then we have
\[
\varphi(\sigma_2, \sigma_3) \circ \varphi(\sigma_1, \sigma_2) = \varphi(\sigma_1, \sigma_3)
\]
in the orbit category \( D_{fd}(\Gamma_A)/(\{1\}, X) \).

Proof. This can be checked case by case, using the construction and the composition formula (2.14). \( \square \)

Let \( \tilde{\sigma}^m, \tilde{\tau}^{m'} \in \widetilde{AC}^X(S_\triangle) \) with \( \sigma(0) = \tau(0) \). Let \( \tilde{\eta}^{m''} = \tilde{\tau}^{m'} \wedge \tilde{\sigma}^m \). When \( \tilde{\eta}^{m''} \) is a union of two curves \( \tilde{\eta}^{m_1} \) and \( \tilde{\eta}^{m_2} \) (c.f. pictures on the right in Figure 3, where \( \tilde{\eta}^{m_1} \) denotes the left curve in the lower picture, while \( \tilde{\eta}^{m_2} \) denotes the right one), we denote
\[
X_{\tilde{\eta}^{m''}} = X_{\tilde{\eta}^{m_1}} \oplus X_{\tilde{\eta}^{m_2}}, \quad \begin{cases} 
\varphi(\tilde{\tau}^{m'}, \tilde{\eta}^{m''}) = \begin{pmatrix} 0 & \varphi(\tilde{\tau}^{m'}, \tilde{\eta}^{m_2}) \end{pmatrix}^T, \\
\varphi(\tilde{\eta}^{m''}, \tilde{\sigma}^m) = \varphi(\tilde{\eta}^{m_1}, \tilde{\sigma}^m). 
\end{cases}
\]

Similarly, in the orbit category \( D_{fd}(\Gamma_A)/(\{1\}, X) \), we denote
\[
X_\eta = X_{\eta_1} \oplus X_{\eta_2}, \quad \begin{cases} 
\varphi(\tau, \eta) = \begin{pmatrix} 0 & \varphi(\tau, \eta_2) \end{pmatrix}^T, \\
\varphi(\eta, \sigma) = \varphi(\eta_1, \sigma). 
\end{cases}
\]

Here \((-)^T\) denotes the transpose of a matrix.

Proposition 3.13. Let \( \sigma, \tau \in AC(S_\triangle) \) with \( \sigma(0) = \tau(0) \). Let \( \eta = \tau \wedge \sigma \). Then we have a triangle
\[
X_\eta \xrightarrow{\varphi(\eta, \sigma)} X_\sigma \xrightarrow{\varphi(\sigma, \tau)} X_\tau \xrightarrow{\varphi(\tau, \eta)} X_\eta
\]
in the orbit category \( D_{fd}(\Gamma_A)/(\{1\}, X) \). In particular, we have
\[
\varphi(\tau, \eta) \circ \varphi(\sigma, \tau) = \varphi(\sigma, \tau) \circ \varphi(\eta, \sigma) = \varphi(\eta, \sigma) \circ \varphi(\tau, \eta) = 0.
\]

Proof. This follows from checking the mapping cone case by case. \( \square \)
Corollary 3.14. Let $\tilde{\sigma}^m, \tilde{\tau}^m \in AC^X(S_\triangle)$ with $\sigma(0) = \tau(0)$. Let $\tilde{\eta}^m'' = \tilde{\tau}^m \wedge \tilde{\sigma}^m$. Then we have a triangle
\[
X_{\tilde{\eta}^m''[-\nu'']} \xrightarrow{\varphi(\tilde{\eta}^m'', \tilde{\sigma}^m)} X_{\tilde{\sigma}^m} \xrightarrow{\varphi(\tilde{\sigma}^m, \tilde{\tau}^m)} X_{\tilde{\tau}^m}[\nu] \xrightarrow{\varphi(\tilde{\tau}^m, \tilde{\eta}^m'')} X_{\tilde{\eta}^m''[\nu + \nu']}
\]
where
\[\nu = \text{ind}^Z_{\sigma(0)}(\tilde{\sigma}^m, \tilde{\tau}^m), \quad \nu' = \text{ind}^Z_{\tau(1)}(\tilde{\tau}^m, \tilde{\eta}^m''), \quad \nu'' = \text{ind}^Z_{\eta(0)}(\tilde{\eta}^m'', \tilde{\sigma}^m).
\]

Proof. This follows directly from Proposition 3.13 by carefully writing down the bidegrees, using Proposition 3.10. □

The following is a generalization of [19, Lemma 3.3], with a different approach.

Corollary 3.15. Let $\sigma_1, \sigma_2, \sigma_3$ be admissible closed curves in $AC(S_\triangle)$ which share the same starting point. If the start segments of $\sigma_1, \sigma_2$ and $\sigma_3$ are in counterclockwise order at the starting point, see the right picture in Figure 11, then
\[\varphi(\sigma_2, \sigma_3) \circ \varphi(\sigma_1, \sigma_2) = 0
\]
in the orbit category $\mathcal{D}_{fd}(\Gamma_A)/([1], X)$.

Proof. By using repeatedly Proposition 3.12, we have
\[
\varphi(\sigma_2, \sigma_3) \circ \varphi(\sigma_1, \sigma_2) = \varphi(\sigma_1, \sigma_3) \circ \varphi(\sigma_2, \sigma_1) = \varphi(\sigma_1, \sigma_1)
\]
\[= \varphi(\sigma_1, \sigma_3) \circ \varphi(\sigma_3 \wedge \sigma_1, \sigma_1)
\]
\[= \varphi(\sigma_1, \sigma_3) \circ \varphi(\sigma_3 \wedge \sigma_1, \sigma_1)
\]
where the last one is zero due to Proposition 3.13 and $\sigma_3 \wedge \sigma_1$ might be the union of two curves. □

Corollary 3.16. Let $\sigma_1, \sigma_2, \sigma_3 \in AC(S_\triangle)$ with $\sigma_1(0) = \sigma_2(0)$ and $\sigma_2(1) = \sigma_3(1)$, see Figure 12. Then
\[\varphi(\sigma_2, \sigma_3) \circ \varphi(\sigma_1, \sigma_2) = 0.
\]

Proof. By Proposition 3.13, we have $\varphi(\sigma_2, \sigma_2 \wedge \sigma_1) \circ \varphi(\sigma_1, \sigma_2) = 0$. Using Proposition 3.12, we have
\[
\varphi(\sigma_2, \sigma_3) \circ \varphi(\sigma_1, \sigma_2) = \varphi(\sigma_2 \wedge \sigma_1, \sigma_3) \circ \varphi(\sigma_2 \wedge \sigma_1, \sigma_1)
\]
where $\sigma_2 \wedge \sigma_1$ might be the union of two curves. □

3.3. Braid twists and spherical twists. Recall that any closed arc $\alpha \in CA(S_\triangle)$ induces an element $B_\alpha$ in the mapping class group $\text{MCG}(S_\triangle)$, the braid twist along $\alpha$, cf. [17, Fig. 5].

Definition 3.17. The braid twist group $\text{BT}(S_\triangle)$ is the subgroup of the mapping class group $\text{MCG}(S_\triangle)$ generated by the braid twists along closed arcs in $\text{CA}(S_\triangle)$. And $\text{BT}(A)$ is the one generated by the braid twists along closed arcs in $\text{A}_\triangle \cap \text{CA}(S_\triangle)$.
For any $\alpha \in \text{CA}(S_\Delta)$ and any $\tilde{\sigma}^m \in \tilde{\text{AC}}^X(S_\Delta)$, the braid twist $B_\alpha(\tilde{\sigma}^m)$ of $\tilde{\sigma}^m$ along $\alpha$ is the lift of grading of $B_\alpha(\sigma)$ inherited from $\tilde{\sigma}^m$. Here, inherited means that we can apply the braid twists on each sheet of log $S_\Delta$.

Recall that $\mathcal{D}_{fd}(\Gamma_A)$ is a Calabi-Yau-$X$ triangulated category.

**Definition 3.18.** An object $M$ in $\mathcal{D}_{fd}(\Gamma_A)$ is called $X$-spherical if $$\text{Hom}_{\mathcal{D}}(M, M[l + mX]) = \begin{cases} k & l = 0 \text{ and } m = 0 \text{ or } 1, \\ 0 & \text{otherwise}. \end{cases}$$ An $X$-spherical object $M$ in $\mathcal{D}_{fd}(\Gamma_A)$ induces an auto-equivalence $\phi_M \in \text{Aut}(\mathcal{D}_{fd}(\Gamma_A))$, called spherical twist [22], by $$\text{RHom}(M, X) \otimes M \to X \to \phi_M(X) \to (\text{RHom}(M, X) \otimes M)[1]$$ for any $X \in \mathcal{D}_{fd}(\Gamma_A)$.

**Definition 3.19.** Denote by $\text{Sph}^{22}(\Gamma_A)$ the set of reachable spherical objects, which consists of spherical objects in $X \left(\tilde{\text{CA}}^X(S_\Delta)\right) := \{X_{\tilde{\sigma}^m} \mid \tilde{\sigma}^m \in \tilde{\text{CA}}^X(S_\Delta)\}$. Let $\text{Sph}(\Gamma_A) = \text{Sph}^{22}(\Gamma_A)/\langle [1], [X] \rangle$. Define $\text{ST}(\Gamma_A)$ to be the subgroup of $\text{Aut}(\mathcal{D}_{fd}(\Gamma_A))$ generated by $\phi_S$ for any $S \in \text{Sph}^{22}(\Gamma_A)$. 
In fact, after we prove the intersection formulae (4.5), one can check that \( \text{Sph}^Z(\Gamma_A) \) consists of all spherical objects in \( X(\overline{\text{AC}^X(S_\Delta)}) \).

4. Intersection Formulae

In this section, we prove our results under some assumptions first, and will and remove such assumptions in Section 5.

**Assumption 4.1.** We impose the following assumptions in this section:

1°. For any \( A \)-polygon \( P \), there is no self-folded edges, i.e. when going around its edges, no arc in \( A \) will be count twice.

2°. Any two \( A \)-polygons share at most one arc in \( A \).

Denote by \( \text{Sim} H_{\Gamma A} \) the set of simple \( \Gamma_A \)-modules. Then Assumption 4.1 implies that \( \text{Sim} H_{\Gamma A} \in \text{Sph}^Z(\Gamma_A) \) and \( \text{Sph}(\Gamma_A) = \text{ST}(\Gamma_A) \cdot \text{Sim} H_{\Gamma A}/([1], [\Xi]) \) as in [17, (2.1)].

4.1. The case that all intersections are at decorations. Recall we have the notations \( \text{CA}(S_\Delta) \) and \( \text{AC}(S_\Delta) \) from Notion 2.3. Recall that \( \text{CA}^X(S_\Delta) \) denotes the set of lifts of graded closed arcs in \( \text{CA}(S_\Delta) \).

Let \( A \) be a full formal arc system of \( S_\Delta \).

**Lemma 4.2.** For any \( \tilde{\alpha}^m \in \overline{\text{AC}^X(S_\Delta)} \) and any decoration \( Z \) in an \( A \)-polygon which \( \sigma \) crosses, there are \( \tilde{\alpha}^{m'}, \tilde{\beta}^{m''} \in \overline{\text{AC}^X(S_\Delta)} \), such that

\[
\text{Int}(\tilde{\alpha}_i^0, \tilde{\alpha}^{m'}) + \text{Int}(\tilde{\gamma}_i^0, \tilde{\alpha}^{m'}) = \text{Int}(\tilde{\gamma}_i^0, \tilde{\eta}^m) \quad \text{for any } 1 \leq i \leq n;
\]
\[
\tilde{\alpha}^{m'}(0) = \tilde{\beta}^{m''}(0) = Z \quad \text{and} \quad \tilde{\alpha}^m = \tilde{\beta}^{m''} \land \tilde{\alpha}^{m'}.
\]

In the case when \( \tilde{\alpha}^m \) is in \( \overline{\text{CA}^X(S_\Delta)} \) and \( Z \) is not an endpoint of \( \eta \), the second is equivalent to that \( \text{Int}(\tilde{\alpha}^{m'}, \tilde{\beta}^{m''}) = \frac{1}{2} \) and \( \tilde{\eta}^m = B_\alpha(\tilde{\beta}^{m''}) \).

**Proof.** Let \( P \) be the \( A \)-polygon which \( Z \) lives in. Take a line segment \( l \) in \( P_i \) from \( Z \) to a point \( p \) in an interior arc segment of \( \sigma \) such that \( l \) does not cross any other arc segment of \( \sigma \) in \( S_\Delta^2 \). Smoothing the intersection \( p \) of \( \sigma \) and \( l \), we get two admissible closed curves \( \alpha \) and \( \beta \), where \( \alpha \) is the composition of \( l \) and the segment of \( \eta \) from \( P \) to \( \sigma(0) \) and \( \beta \) is the composition of \( l \) and the segment of \( \eta \) from \( p \) to \( \sigma(1) \). Then some lifts of \( \alpha, \beta \) are as required (cf. [17, Fig. 8, Fig. 9 and Lem. 3.14]).

**Proposition 4.3.** For any admissible closed curves \( \eta_1, \eta_2 \) satisfying \( \text{Int}_{S_\Delta^2} \eta_1 \eta_2 = 0 \), the morphisms \( \varphi(\sigma, \tau) \) with \( \sigma \in \{\eta_1, \overline{\eta_1}\} \) and \( \tau \in \{\eta_2, \overline{\eta_2}\} \) and \( \sigma(0) = \tau(0) \), form a basis of \( \text{dim Hom}(X_{\eta_1}, X_{\eta_2}) \) in the orbit category \( D_{fd}(\Gamma_A)/([1], [\Xi]) \). In particular, we have
\[
\text{Int}_\Delta(\eta_1, \eta_2) = \text{dim Hom}(X_{\eta_1}, X_{\eta_2}).
\]

**Proof.** Use the induction on
\[
I = \text{Int}(A, \eta_1) + \text{Int}(A, \eta_2).
\]
The starting case is $I = 2$, where both $\eta_1$ and $\eta_2$ are in $A_\Delta^\times$. So the formula follows directly from the structure of $E(A)$. Now suppose that the proposition holds for any pair $(\eta_1, \eta_2)$ with $I \leq r$ for some $r \geq 2$ and consider the case when $I = r + 1$.

The arcs in $A$ divide $\eta_1$ and $\eta_2$ into the arc segments $\eta_1(-1,0), \cdots, \eta_1(p,p + 1)$ and $\eta_2(-1,0), \cdots, \eta_2(q,q + 1)$, respectively in order. Since $r \geq 2$, we have $p + 1 > 1$ or $q + 1 > 1$. Then there is a decoration $Z$ which is in the same $A$-polygon as an interior arc segment of $\eta_1$ or $\eta_2$. Take a line segment $l$ from $Z$ to an interior arc segment of $\eta_1$ or $\eta_2$ such that it does not cross any other arc segments of $\eta_1$ or $\eta_2$. Without loss of generality, we assume $l$ intersects $\eta_1$. Then $\eta_1$ decomposes into $\alpha$ and $\beta$ by $l$, which induces a triangle in $D_{fd}(\Gamma_A)/\langle [1], [\X] \rangle$:

$$X_\alpha \xrightarrow{\varphi(\alpha,\beta)} X_\beta \xrightarrow{\varphi(\beta,\eta_2)} X_{\eta_2}$$

Applying $\text{Hom}(-, X(\eta_2))$ to the triangle, we have an exact sequence

$$\text{Hom}(X_\alpha, X_{\eta_2}) \xrightarrow{\varphi(\alpha,\beta)} \text{Hom}(X_{\eta_1}, X_{\eta_2}) \xrightarrow{\varphi(\beta,\eta_2)} \text{Hom}(X_\beta, X_{\eta_2})$$

Since the line segment does not cross $\eta_2$ and $\text{Int}_{S \setminus \triangle}(\eta_1, \eta_2) = 0$, we have $\text{Int}_{S \setminus \triangle}(\alpha, \eta_2) = 0$ and $\text{Int}_{S \setminus \triangle}(\beta, \eta_2) = 0$. So using Corollary 3.12, we have that

- $\eta_2(0) = \eta(0) \iff \eta_2(0) = \alpha(1)$, and $\varphi(\beta, \eta_2) \circ \varphi(\eta_1, \eta_2) = \varphi(\eta_1, \eta_2)$;
- $\eta_2(1) = \eta(0) \iff \eta_2(1) = \alpha(1)$, and $\varphi(\beta, \eta_2) \circ \varphi(\eta_1, \eta_2) = \varphi(\eta_1, \eta_2)$;
- $\eta_2(0) = \eta(1) \iff \eta_2(0) = \beta(1)$, and $\varphi(\beta, \eta_2) \circ \varphi(\eta_1, \eta_2) = \varphi(\beta, \eta_2)$;
- $\eta_2(1) = \eta(1) \iff \eta_2(1) = \beta(1)$, and $\varphi(\beta, \eta_2) \circ \varphi(\eta_1, \eta_2) = \varphi(\beta, \eta_2)$;
- $\eta_2(0) = \beta(0) \iff \eta_2(0) = \alpha(0)$, and $\varphi(\beta, \eta_2) \circ \varphi(\alpha, \beta) = \varphi(\alpha, \eta_2)$; and
- $\eta_2(1) = \beta(0) \iff \eta_2(1) = \alpha(0)$, $\varphi(\beta, \eta_2) \circ \varphi(\alpha, \beta) = \varphi(\alpha, \eta_2)$.

By inductive assumption, the proposition holds for $\alpha$ and $\beta$. Then by the above correspondences and formulae, the proposition holds for $\eta$. □

**Corollary 4.4.** For any $\tilde{\eta}_1^{m_1}, \tilde{\eta}_2^{m_2} \in \widetilde{AC}_\Delta^\times(S_\Delta)$ satisfying $\text{Int}_{S \setminus \triangle}(\tilde{\eta}_1^{m_1}, \tilde{\eta}_2^{m_2}) = 0$, we have $\text{Int}^g(\tilde{\eta}_1^{m_1}, \tilde{\eta}_2^{m_2}) = \dim^g \text{Hom}^{\mathbb{Z}^2}(X_{\tilde{\eta}_1^{m_1}}, X_{\tilde{\eta}_2^{m_2}})$ and each $\text{Hom}^{\mathbb{Z}^2}(X_{\tilde{\eta}_1^{m_1}}, X_{\tilde{\eta}_2^{m_2}})$ has a basis $\varphi(\tilde{\alpha}^m, \tilde{\alpha}^m')$ with $\tilde{\alpha}^m \in \{\tilde{\eta}_1^{m_1}, \tilde{\eta}_1^{m_1}\}$ and $\tilde{\alpha}^m' \in \{\tilde{\eta}_2^{m_2}, \tilde{\eta}_2^{m_2}\}$, $\tilde{\alpha}^m(0) = \tilde{\alpha}^m(0)$ and $\text{ind}_{\phi(0)}(\tilde{\alpha}^m, \tilde{\alpha}^m') = \varphi + \zeta_\X$.

**Corollary 4.5.** Let $\tilde{\eta}_m \in \widetilde{CA}_\Delta^\times(S_\Delta)$. Then $X_{\tilde{\eta}_m}$ is $\X$-spherical.

**Proof.** Since $\text{Int}_{S \setminus \triangle}(\tilde{\eta}_m, \tilde{\eta}_m) = 0$ and $\text{Int}_{S \setminus \triangle}(\tilde{\eta}_m, \tilde{\eta}_m) = 1$, by Corollary 4.4, we have that the dimension $\dim^g \text{Hom}^{\mathbb{Z}^2}(X_{\tilde{\eta}_m}, X_{\tilde{\eta}_m}) = 1 + q^\X$, which implies that $X_{\tilde{\eta}_m}$ is $\X$-spherical. □

Note that the twist functor $\phi_{X_{\tilde{\eta}_m}}$ only depend on $\eta \in CA(S)$. Thus we will write $\phi_{X_{\eta}}$ instead.
**Corollary 4.6.** Let $\alpha \in CA(S_{\Delta})$ and $\beta \in AC(S_{\Delta})$. If $\text{Int}_{S_{\Delta}}(\alpha, \beta) = 0$, then

$$X_{B_a(\beta)} = \phi X_{\alpha}(X_{\beta})$$

**Proof.** For the case $\text{Int}(\alpha, \beta) = 0$, use Proposition 4.3.

For the case that $\beta$ is closed arc, follow the proof of [19, Proposition 3.1].

The last case is that both of the endpoints of $\beta$ and the starting of $\alpha$ coincide. Without loss of generality, we assume that the starting segments of $\alpha$, $\beta$ and $\overline{\beta}$ are in clockwise order. So we have $\beta \wedge \alpha \in AC(S_{\Delta}).$

It suffices to show that we have a triangle

$$X_{\alpha} \oplus X_{\alpha} \xrightarrow{\varphi(\alpha, \beta)} X_{\beta} \to X_{B_a(\beta)} \to .$$

To see this, note that the composition $\varphi(\beta, \beta \wedge \alpha) \circ \varphi(\alpha, \beta) = \varphi(\alpha, \beta \wedge \alpha)$ (see Figure 13).

![Figure 13.](image-url)

So by the octahedral axiom, we have the following commutative diagram of triangles

```
\begin{align*}
X(\beta \wedge \alpha) & \rightarrow X_{\beta \wedge \alpha} \\
* & \rightarrow X_{B_a(\beta)} \rightarrow X_{\alpha} \rightarrow X_{\beta} \rightarrow * \\
* & \rightarrow X_{\alpha} \rightarrow X_{\beta} \rightarrow * \\
\varphi(\alpha, \beta) & \rightarrow \varphi(\alpha, \beta) \\
\varphi(\alpha, \beta \wedge \alpha) & \rightarrow \varphi(\alpha, \beta \wedge \alpha) \\
X_{\beta \wedge \alpha} & \rightarrow X_{\beta \wedge \alpha}
\end{align*}
```
Then we get the required triangle, which implies the required equality.

**Corollary 4.7.** Let \( \alpha \in \text{CA}(S_\Delta) \) and \( \tilde{\beta}^m \in \tilde{\text{AC}}^X(S_\Delta) \). If \( \text{Int}_{S_\Delta}(\alpha, \beta) = 0 \), then

\[
X_{B_{m}(\tilde{\beta}^m)} = \phi_{X_{\alpha}}(X_{\tilde{\beta}^m}).
\]

4.2. **Main results.** In Assumption 4.1, the first condition (no self-folded edge) is equivalent to the condition that any close curve in \( A^*_\Delta \) has different endpoint. Thus we have \( A^*_\Delta = A^*_\Delta \cap \text{CA}(S_\Delta) \).

**Lemma 4.8.** \( \text{CA}(S_\Delta) = \text{BT}(A) \cdot A^*_\Delta \) and \( \text{BT}(S_\Delta) = \text{BT}(A) \).

**Proof.** A similar proof with [17, Lemma 4.2] works as follows.

Let \( \eta \in \text{CA}(S_\Delta) \). Use the induction on \( l(\eta) = \text{Int}(A, \eta) \). When \( l = 1, \eta \in A^* \). Suppose any closed arc with length smaller than \( l \) is in \( \text{BT}(A) \cdot A^*_\Delta \). Now consider \( \eta \in \text{CA}(S_\Delta) \) with \( l(\eta) = n \). By Lemma 4.2 and Assumption 4.1, we have closed arcs \( \alpha, \beta \in \text{CA}(S_\Delta) \) such that \( \eta = B_{s}(\beta) \) and \( l(\eta) = l(\alpha) + l(\beta) \). So \( \alpha = b(s) \) and \( \beta = b'(s') \) for some \( b, b' \in \text{BT}(A) \) and \( s, s' \in A^*_\Delta \). So \( \eta = B_{s}(\beta) = b \circ B_{s} \circ b' \circ b^{-1}(s') \in \text{BT}(A) \cdot A^*_\Delta \) as required.

For the second equality, for any \( \eta \in \text{CA}(S_\Delta) \), there is a \( b \in \text{BT}(A) \) and an \( s \in A^*_\Delta \) such that \( \eta = b(s) \). Then we have \( B_{\eta} = b \circ B_{s} \circ b^{-1} \in \text{BT}(A) \).

A lifted graded version of the above lemma is the following.

**Lemma 4.9.** \( \tilde{\text{CA}}^X(S_\Delta) = \text{BT}(A) \cdot (A^*_\Delta)^{2^2} \)

**Lemma 4.10.** For any \( s \in A^*_\Delta \) and any \( \tilde{\eta}^m \in \tilde{\text{CA}}^X(S_\Delta) \), we have

\[
\phi_{X_{\eta}}(X_{\tilde{\eta}^m}) = X_{\tilde{\eta}^m}, \quad \varepsilon = -1, 1. \tag{4.1}
\]

**Proof.** Without loss of generality, we only deal the case for \( \varepsilon = 1 \). Use the induction on \( l(\eta) = \text{Int}(\eta, A) \). When \( l(\eta) = 1 \), we have \( \eta \in A^*_\Delta \). Then \( \text{Int}_{S_\Delta}(s, \eta) = 0 \), which implies the required equality by Corollary 4.6.

Suppose the equality holds for \( l(\eta) < x \), for some \( x \geq 2 \). Consider the case \( l(\eta) = x \). By Lemma 4.2 and Assumption 4.1, there are \( \tilde{\alpha}^{m'}, \tilde{\beta}^{m''} \in \tilde{\text{CA}}^X(S_\Delta) \) with \( \text{Int}_{S_\Delta}(\tilde{\alpha}^{m'}, \tilde{\beta}^{m''}) = 0 \) and \( \tilde{\eta}^m = B_{\eta}(\tilde{\beta}^{m''}) \). By Corollary 4.7, we have

\[
X_{\tilde{\eta}^m} = \phi_{X_{\tilde{\alpha}^{m'}}}(X_{\tilde{\beta}^{m''}}).
\]

Twisted by \( B_{s} \), we have \( \text{Int}_{S_\Delta}(B_{s}(\tilde{\alpha}^{m'}), B_{s}(\tilde{\beta}^{m''})) = 0 \). By Corollary 4.7, we have

\[
X_{B_{s}(\tilde{\eta}^m)} = \phi_{X_{B_{s}^{\tilde{\alpha}^{m'}}}}(X_{B_{s}^{\tilde{\beta}^{m''}}}).
\]

By the inductive assumption, we have

\[
X_{B_{s}(\tilde{\alpha}^{m'})} = \phi_{X_{s}}(X_{\tilde{\alpha}^{m'}}), \quad \text{and} \quad X_{B_{s}^{\tilde{\beta}^{m''}}} = \phi_{X_{s}}(X_{\tilde{\beta}^{m''}}).
\]
So
\[ X_{B_t(\overline{\gamma}^m)} = \phi X_{B_s(\overline{\gamma}^m')} (X_{B_t(\overline{\gamma}^m')}) \]
\[ = \phi \phi X(s) (X_{\overline{\gamma}^m'}) \]
\[ = (\phi X(s) \circ \phi X_{\overline{\gamma}^m'} \circ \phi^{-1} X(s)) (\phi X(s)(X_{\overline{\gamma}^m'}) \]
\[ = (\phi X(s) \circ \phi X_{\overline{\gamma}^m'})(X_{\overline{\gamma}^m'}) \]
\[ = \phi_X(s)(X_{\overline{\gamma}^m'}) \]

\[ \square \]

Let \( Z^ST_0 = ST(\Gamma_A) \cap \langle [1], X \rangle \) and
\[ ST_s(\Gamma_A) = ST(\Gamma_A)/Z^ST_0 \subset Aut^o K \] in the orbit category \( D_{fd}(\Gamma_A)/\langle [1], X \rangle \).

The group \( ST^*(\Gamma_A) \) is a subgroup of the auto-equivalence group of the orbit category \( D_{fd}(\Gamma_A)/\langle [1], X \rangle \).

Denote by \( b_i = B_s \) for any \( s_i \in \Lambda_A^* \) and by \( \phi_i = \phi S_i \) for any simple \( S_i \). Now we process to prove the Calabi-Yau-X version of the main results in [17, 19].

**Theorem 4.11.** There is a canonical group homomorphism
\[ \iota : BT(S_{\Delta}) \to ST_s(\Gamma_A) \]
\[ \text{(4.2)} \]
\[ \text{sending the generator } b_i \text{ to the generator } \phi_i. \]

**Proof.** Let \( b = b_{i_1}^{e_1} \cdots b_{i_k}^{e_k} \) for some \( i_j \in \{1, \cdots , n\} \), \( e_j \in \{\pm 1\} \), \( 1 \leq j \leq k \) and \( k \in \mathbb{N} \).

Let \( \phi = \phi_{i_1}^{e_1} \cdots \phi_{i_k}^{e_k} \). If \( b = 1 \), then \( b(s_i) = s_i \), for any \( 1 \leq i \leq n \). By (repeatedly using) Lemma 4.10, we have
\[ X_{b(s_i)} = \phi(X_{s_i}). \]

Thus, \( S_i[Z + ZX] = \phi(S_i[Z + ZX]) \), i.e. \( \phi(S_i) = S_i[t_i + t'_iX] \) for some integers \( t_i, t'_i \). Since \( \phi \) is an equivalence, we deduce that all \( t_i \) (resp. \( t'_i \)) must be the same by computing the Ext-algebras. Therefore \( \phi = [t + tX] \) for some integers \( t \) and \( t' \). So \( \phi = 1 \) in \( ST_s(\Gamma_A) \). \[ \square \]

**Corollary 4.12.** For any \( \eta \in CA(S_{\Delta}) \), we have
\[ \iota(B_{\eta}^\circ) = \phi X_{\eta}. \]
\[ \text{(4.3)} \]

**Proof.** The proof of [17, Corollary 6.4] works here as follows. By Lemma 4.8, there is a \( b_{i_1}^{e_1} \cdots b_{i_k}^{e_k} \in BT(S_{\Delta}) \) and an \( s \in \Lambda_A^* \) such that \( \eta = b_{i_1}^{e_1} \cdots b_{i_k}^{e_k}(s) \). So
\[ \iota(B_s) = \iota(B_{b_{i_1}^{e_1} \cdots b_{i_k}^{e_k}}(s)) = \iota(b_{i_1}^{e_1} \cdots b_{i_k}^{e_k}) \]
\[ \text{Thm. 4.11} \quad \phi_{i_1}^{e_1} \cdots \phi_{i_k}^{e_k} \circ \phi_X(s) = \phi_{i_1}^{e_1} \cdots \phi_{i_k}^{e_k}. \]

\[ \text{Lem. 4.10} \quad \phi_X(s). \]

\[ \square \]

**Corollary 4.13.** For any \( b \in BT(S_{\Delta}) \) and any \( \overline{\gamma}^m \in \overline{AC}^X(S_{\Delta}) \), we have
\[ X_{b(\overline{\gamma}^m)} = \iota(b)(X_{\overline{\gamma}^m}). \]
Proof. By Lemma 4.8, \( b = b_{i_1}^c \circ \cdots \circ b_{i_k}^c \) for some \( 1 \leq i_j \leq n \) and \( \varepsilon_j \in \{ \pm 1 \} \). By using repeatedly (4.1), we get the required formula. \( \square \)

**Corollary 4.14.** The formula (4.1) holds for any \( s \in \mathbb{CA}(\mathbb{S}_\Delta) \) and any \( \tilde{\eta}^m \in \mathbb{CA}^X(\mathbb{S}_\Delta) \).

Proof. By Lemma 4.8, there is a \( b \in \mathbb{BT}(\mathbb{S}_\Delta) \) and an \( s' \in \mathbb{A}_\Delta^* \) such that \( b(s') \). Then by Corollary 4.12, we have \( \iota(B_s) = \phi_{X_s} \), and so

\[
X_{B_s(\tilde{\eta}^m)} = \iota(B_s)(X_{\tilde{\eta}^m}) = \phi_{X_s}(X_{\tilde{\eta}^m}).
\]

\( \square \)

**Lemma 4.15.** For any \( \eta \in \mathbb{CA}(\mathbb{S}_\Delta) \), we have that \( X_{\tilde{\eta}^m} \in \text{Sph}(\Gamma_{\mathbb{A}})([1], \mathbb{X}) \), where

\[
\text{Sph}(\Gamma_{\mathbb{A}})([1], \mathbb{X}) := \{ X[n + m\mathbb{X}] \mid X \in \text{Sph}(\Gamma_{\mathbb{A}}), m, n \in \mathbb{Z} \}.
\]

Proof. By Lemma 4.9, there is \( b \in \mathbb{BT}(\mathbb{S}_\Delta) \) and \( s_i \in \mathbb{A}_\Delta^* \) such that \( \tilde{\eta}^m = b(s^{m'}) \) for some grading \( \tilde{s} \) and some integer \( m' \). So by Corollary 4.13,

\[
X_{\tilde{\eta}^m} = X_{b(s^{m'})} = \iota(b)(X_{s^{m'}}) = \iota(b)(S_i[-\text{ind}^{\mathbb{Z}_2}(\gamma_0, \tilde{s}^{m'})]) = \iota(b)(S_i)[-\text{ind}^{\mathbb{Z}_2}(\gamma_0, \tilde{s}^{m'})] \in \text{Sph}(\Gamma_{\mathbb{A}})([1], \mathbb{X}).
\]

\( \square \)

**Theorem 4.16.** The map \( \tilde{\eta}^m \mapsto X_{\tilde{\eta}^m} \) gives a bijection

\[
X : \text{CA}^X(\mathbb{S}_\Delta) \rightarrow \text{Sph}^{\mathbb{Z}_2}(\Gamma_{\mathbb{A}}).
\]

Proof. For the injectivity, assume \( \eta, \eta' \in \text{CA}(\mathbb{S}_\Delta) \) with \( X_{\tilde{\eta}^m} = X_{\tilde{\eta'}^m} \). Let \( b \in \mathbb{BT}(\mathbb{S}_\Delta) \) with \( \tilde{\eta}^m = b(s^{m''}) \) for some \( s \in \mathbb{A}_\Delta^* \). So

\[
X_{\tilde{\eta}^m} = \iota(b)^{-1}X_{\tilde{\eta}^m} = \iota(b)^{-1}X_{\tilde{\eta'}^m} = X_{b^{-1}(s^{m'})}.
\]

By Lemma 3.6, we have \( s^{m''} = b^{-1}(s^{m'}) \), which implies that \( \tilde{\eta}^m = \tilde{\eta'}^m \) as required.

For the surjectivity, for any \( X \in \text{Sph}^{\mathbb{Z}_2}(\Gamma_{\mathbb{A}}) \), by definition, \( X = \phi_{i_1}^c \circ \cdots \circ \phi_{i_k}^c(S)[\varrho + \zeta \mathbb{X}] \) for some simple \( S \). Let \( \eta^m = b_{i_1}^c \circ \cdots \circ b_{i_k}^c(s^{m''}) \) with \( X_{s^{m''}} = S[\varrho + \zeta \mathbb{X}] \). Then we have \( X_{\tilde{\eta}^m} = X \).

\( \square \)

**Lemma 4.17.** Let \( s \in \mathbb{A}_\Delta^* \) and \( \eta \in \text{AC}(\mathbb{S}_\Delta) \) such that \( \eta \) is contained in the two \( \mathbb{A} \)-polygons that are incident with \( s \). Then \( \text{Int}_{\mathbb{S}_\Delta^+}(s, \eta) = 0 \).

Proof. Note that \( s \) and \( \eta \) will be represented by a curve in their isotopy class that is in general position and has minimal intersection with each other and with \( \mathbb{A} \). Suppose that there is an intersection \( p \) between \( s \) and \( \eta \) such that \( p \notin \Delta \). Starting from \( p \), there are two paths following \( \eta \). These two paths can not intersect before hitting \( s \) (cf. green circle in Figure 14) —otherwise they form a once-decorated monogon that is not admissible. Then they can either hit \( s \setminus \Delta \) or \( \Delta \). If they both hit \( \Delta \) first, \( \eta \) is isotopy to \( s \), which is a contradiction. Then one of them have to hit \( s \setminus \Delta \) first, cf. (cf. Figure 14). After that, such a path can not hit the previous footprint or \( s \) (cf. blue circle in Figure 14), which forces it winds the corresponding decoration before hitting it. However, this is a contradiction as it is not in a minimal position w.r.t. \( s \). \( \square \)
Lemma 4.18. For any $s^{m_0} \in (A_X^*)^{22}$ and any $\tilde{\eta} \in \tilde{AC}^X(S_\Delta)$, we have
\[
\text{Int}^q(s^{m_0}, \tilde{\eta}) = \dim^q \text{Hom}^{22}(X_{s^{m_0}}, X_{\tilde{\eta}}).
\] (4.5)

Proof. To be compatible with the notation before, let $\tilde{\eta} = \tilde{\eta}^{m''}$ for some $m'' \in \mathbb{Z}$.

Use the induction on $l(\tilde{\eta}^{m''}) := \text{Int}_{S_\Delta}(A_{22}^*, \tilde{\eta}^{m''})$. For the case $l(\tilde{\eta}^{m''}) = 1$, we have $\tilde{\eta}^{m''} \in (A_X^*)^{22}$. So $\text{Int}_{S_\Delta}(s^{m_0}, \tilde{\eta}^{m''}) = 0$. Then the formula (4.5) holds by Corollary 4.4.

Now suppose that the formula holds for $l(\tilde{\eta}^{m''}) < x$ for some $x \geq 2$. Consider the case $l(\tilde{\eta}^{m''}) = x$. Since the case that $\text{Int}_{S_\Delta}(s^{m_0}, \tilde{\eta}^{m''}) = 0$ is contained in Corollary 4.4, we assume that $\text{Int}_{S_\Delta}(s^{m_0}, \tilde{\eta}^{m''}) \neq 0$. Consider the following two cases:

(1) If each endpoint of $s^{m_0}$ is an endpoint of $\tilde{\eta}^{m''}$, then by Lemma 4.17, the condition $\text{Int}_{S_\Delta}(s^{m_0}, \tilde{\eta}^{m''}) \neq 0$ implies that there is a decoration $Z$ which is not an endpoint of $s^{m_0}$ or $\tilde{\eta}^{m''}$ and which is in an $A$-polygon crossing $\eta$. Then there are $\tilde{\sigma}^m, \tilde{\tau}^{m'} \in \tilde{AC}(S_\Delta)$ such that $\tilde{\eta}^{m''} = \tilde{\tau}^{m'} \wedge \tilde{\sigma}^m$, $l(\tilde{\eta}^{m''}) = l(\tilde{\sigma}^m) + l(\tilde{\tau}^{m'})$ and $\text{Int}^q(s^{m_0}, \tilde{\eta}^{m''}) = \text{Int}^q(s^{m_0}, \tilde{\sigma}^m) + \text{Int}^q(s^{m_0}, \tilde{\tau}^{m'})$.

(2) If $s^{m_0}$ has an endpoint $Z$ which is not an endpoint of $\tilde{\eta}^{m''}$. Then consider a segment $l \subset s$ from $Z$ to the closest intersection between $s$ and $\eta$, which decomposes $\tilde{\eta}^{m''}$ into $\tilde{\sigma}^m$ and $\tilde{\tau}^{m'}$ with $l(\tilde{\sigma}^m) \leq l(\tilde{\sigma}^m) + l(\tilde{\tau}^{m'}) \leq l(\tilde{\eta}^{m''}) + 2$. So $\tilde{\eta}^{m''} = \tilde{\tau}^{m'} \wedge \tilde{\sigma}^m$ and $\text{Int}^q(s^{m_0}, \tilde{\eta}^{m''}) = \text{Int}^q(s^{m_0}, \tilde{\sigma}^m) + \text{Int}^q(s^{m_0}, \tilde{\tau}^{m'})$.

In both cases, by Corollary 3.14, we have a triangle

\[
X_{\tilde{\eta}^{m''}}[\nu''] \xrightarrow{\varphi(\tilde{\eta}^{m''}, \tilde{\eta}^{m''})[-\nu'']} X_{\tilde{\tau}^{m'}} \xrightarrow{\varphi(\tilde{\sigma}^m, \tilde{\tau}^{m'})} X_{\tilde{\eta}^{m''}}[\nu'] \xrightarrow{\varphi(\tilde{\tau}^{m''}, \tilde{\eta}^{m''})[\nu']} X_{\tilde{\eta}^{m''}}[\nu + \nu']
\]

where

\[
\nu = \text{ind}_{\vartheta(0)}^Z(\tilde{\tau}^{m'}, \tilde{\tau}^{m''}), \quad \nu' = \text{ind}_{\vartheta(1)}^Z(\tilde{\tau}^{m''}, \tilde{\eta}^{m''}), \quad \nu'' = \text{ind}_{\vartheta(0)}^Z(\tilde{\eta}^{m''}, \tilde{\tau}^{m'}).
\]

Applying $\text{Hom}(X_{s^{m_0}}, -)$ to this triangle, we have an exact sequence, which implies that it is sufficient to show that the formula (4.5) holds for $\tilde{\sigma}^m$ and $\tilde{\tau}^{m'}$, and

\[
\text{Hom}(X_{s}, \varphi(\tau, \sigma)) = 0
\]
in the orbit category $\mathcal{D}_{fd}(\Gamma_A)/\langle [1], \mathbb{X} \rangle$. Since by the construction, $\varphi(\sigma, \tau)$ is given by the morphism induced by an angle of the $A$-polygon containing $Z$. But for case (1) the curve $\gamma \in A$ corresponding to $s$ is not an edge of this polygon and for case (2) the starting segments of $s, \sigma, \tau$ are in the counterclockwise order (see Figure 15). So by (2.14) and Corollary 3.15, respectively, we have that in both of these two cases, $\text{Hom}(X(s), \varphi(\sigma, \tau)) = 0$ holds.

Now we only need to show the formula (4.5) holds for $\tilde{s}^m$ and $\tilde{\tau}^{m'}$. In case (1), this follows from the inductive assumption. Next consider case (2). If both $l(\tilde{\sigma}^m)$ and $l(\tilde{\tau}^{m'})$ are smaller than $l(\eta)$, then this also follows from the inductive assumption. Otherwise, either $l(\tilde{\sigma}^m)$ or $l(\tilde{\tau}^{m'})$ is bigger than or equals $l(\tilde{\eta}^{m''})$, then the other one is not bigger then 2. In this case, we must have $\{l(\tilde{\sigma}^m), l(\tilde{\tau}^{m'})\} = \{2, l(\tilde{\eta}^{m''})\}$ and both of them has a less intersections with $s$ in $S_\Delta^\circ$. Use another induction on such an intersection number, one reduces the situation to the case that $\text{Int}_{S_\Delta^\circ}(\tilde{s}^{m_0}, \tilde{\eta}^{m''}) = 0$. This completes the proof.

![Figure 15. The composition of $\sigma$ and $\tau$](image-url)

**Corollary 4.19.** The formula (4.5) holds for any $\tilde{s}^{m_0} \in \widetilde{CA}^X(S_\Delta)$ and any $\tilde{\eta} \in \widetilde{AC}^X(S_\Delta)$.

*Proof.* By Lemma 4.9, there is an $b \in \text{BT}(S_\Delta)$ and $\tilde{s}_1^{m_1} \in A$ such that $\tilde{s}^{m_0} = b(\tilde{s}_1^{m_1})$. So we have

\[
\begin{align*}
\text{Int}^q(\tilde{s}^{m_0}, \tilde{\eta}) &= \text{Int}^q(\tilde{s}_1^{m_1}, b^{-1}(\tilde{\eta})) \\
&\xrightarrow{\text{Lem. 4.18}} \dim^q \text{Hom}^\mathbb{Z}(X_{\tilde{s}_1^{m_1}}, X_{b^{-1}(\tilde{\eta})}) \\
&\xrightarrow{\text{Cor. 4.13}} \dim^q \text{Hom}^\mathbb{Z}(X_{\tilde{s}_1^{m_1}}, \iota(b)^{-1}X_{\tilde{\eta}}) \\
&= \dim^q \text{Hom}^\mathbb{Z}(\iota(b)X_{\tilde{s}_1^{m_1}}, X_{\tilde{\eta}}) \\
&\xrightarrow{\text{Cor. 4.13}} \dim^q \text{Hom}^\mathbb{Z}(X_{b(\tilde{s}_1^{m_1})}, X_{\tilde{\eta}}) \\
&= \dim^q \text{Hom}^\mathbb{Z}(X_{\tilde{s}^{m_0}}, X_{\tilde{\eta}})
\end{align*}
\]
5. General case

Now let us consider the general case, i.e. show that all the results in Section 4 still hold when removing Assumption 4.1. Here we summarize the results above:

(X1) There is a bijection $X$ as (4.4) in Theorem 4.16.
(X2) $X$ induces an isomorphism (4.2) in Theorem 4.11 satisfying (4.3).
(X3) Formula (4.5) holds as in Corollary 4.19.

We will refer these results as Property-X. The strategy consists of the following two steps.

1°. For any graded marked surface $S_\Delta$, there is a graded marked surface $S_\Delta^+$ obtained from $S_\Delta$ by adding marked points on its boundary components such that $S_\Delta^+$ admits a full formal arc system $A_\Delta^+$ satisfying Assumption 4.1. Thus, $A_\Delta^+$ satisfies Property-X by Theorem 4.16, Theorem 4.11 and Corollary 4.19.

2°. $S_\Delta$ inherits Property-X from $S_\Delta^+$ by deleting extra marked points/decorations.

Note that this strategy was applied in [17] when dealing with the Calabi-Yau-3 case. However, there is no corresponding results from cluster theory that we can import.

5.1. Step 1°. Let us show the claim by proving the following lemma.

**Lemma 5.1.** Let $(g, b, m = \sum_{i=1}^{b} m_i)$ be the numerical data of $S$, where $g$ is the genus of $S$, $b$ is the number of boundary components of $S$ and $m_i$ is the number of marked points in $M$ on the $i$-th boundary component for any $1 \leq i \leq b$. If $m_i \geq 2$ for any $1 \leq i \leq b$ and $m_1 \geq 4g$, then $S_\Delta$ admits a full formal arc system satisfying Assumption 4.1.

**Proof.** If $b = 1$, take a polygon-presentation of $S_\Delta$ where the vertex is one of the marked points. Then the left picture in Figure 16 shows a required full formal arc system (for the genus 2 case). For $b > 1$, we can perform adding a boundary component operation, cf. the right picture in Figure 16, to produce the required full formal arc system inductively. □

5.2. Step 2°. Now, suppose that $S_\Delta^+$ is obtained from $S_\Delta$ by adding marked points on its boundary components (and correspondingly decorated points in its interior).

**Definition 5.2.** A slide of a full formal arc system $A$, with respect to one of its arc $\gamma = AB$ and a chosen endpoint $B$ of $\gamma$, is an operation by replacing $\gamma$ in $A$ with $\gamma'$ to form another full formal arc system $A'$. Here, $\gamma' = AC$ is a diagonal in the $A$-polygon $P$ on the right of $\gamma$ when going from $A$ to $B$, where $C$ is the adjacent vertex in $P$ of $B$ other than $A$. See the left picture in Figure 17.

**Lemma 5.3.** Let $A$ be a full formal arc system of $S_\Delta^+$ satisfies Property-X. If $A'$ is a slide of $A$, then $A'$ also satisfies Property-X.
Figure 16. A full formal arc system on polygon-presentation of a genus 2 marked surface and adding a boundary component

Figure 17. The slide and removing

Proof. The dual $A'^* = (A^* \cap A') \cup \{\tilde{s}'\}$, where $\tilde{s}'$ is the dual of $BC$ w.r.t. $A'$. Denote by $X$ and $X'$ the maps defined in Construction 3.1 from $\tilde{AC}^+(S_\Delta)$ to $D_{fd}(\Gamma_{A^+})$ and $D_{fd}(\Gamma_{A'})$, respectively. Let $S'$ be the direct sum of the simple $\Gamma_{A'}$-modules. Then

$$S' = X'(A'^*) := \oplus_{\tilde{\sigma} \in A'^*} X_{\tilde{\sigma}}$$

Denote by

$$X(A'^*) = \oplus_{\tilde{\sigma} \in A'^*} X_{\tilde{\sigma}}.$$

Since $\text{Int}(S_\Delta^{\tilde{\sigma}})(\tilde{\sigma}_1, \tilde{\sigma}_2) = 0$ for any $\tilde{\sigma}_1, \tilde{\sigma}_2 \in A'^*$, by Corollary 4.4, both of the $\mathbb{Z}^2$-graded algebras $\text{Hom}_{\mathbb{Z}^2}(S', S') = \text{Hom}_{\mathbb{Z}^2}(X'(A'^*), X'(A'^*))$ and $\text{Hom}_{\mathbb{Z}^2}(X(A'^*), X(A'^*))$ have a basis of the form $\varphi(-, -)$, whose compositions are given by the formulae in Proposition 3.12 and Corollaries 3.15 and 3.16. Hence we have an isomorphism of $\mathbb{Z}^2$-graded algebras

$$\text{Hom}_{\mathbb{Z}^2}(S', S') \cong \text{Hom}(X(A'^*), X(A'^*)).$$
which induces a triangle equivalence
\[ F : \mathcal{D}_{fd}(\Gamma_{A'}) \rightarrow \mathcal{D}_{fd}(\Gamma_A). \]

By the constructions of \( X \) and \( X' \), we have \( F(X_{\tilde{\eta}^m}) = X_{\tilde{\eta}^m} \) for any \( \tilde{\eta}^m \in \tilde{AC}^X(S_\Delta) \). This implies that \( A' \) also satisfies Property-X. □

Let \( A^+ \) be a full formal arc system of \( S^+_\Delta \) satisfying Assumption 4.1. Thus, \( A^+ \) satisfies Property F. Let \( A \) be a marked point and \( M_1, M_2 \) be the adjacent marked points of \( A \). Suppose that \( M_i \neq A \) and the boundary arc \( AM_2 \) is in a \( A^+ \)-polygon \( P \), which contains the open arc \( \gamma = AB \) as in the left picture of Figure 17. Now repeated the operation slide to \( A^+ \) w.r.t. \( \gamma \) and the endpoint other than \( A \), we will turn \( P \) into a digon, whose edges are \( AM_2 \) and \( \hat{\gamma} \), as shown in the right picture of Figure 17. Denote the resulting full formal arc system by \( A_0 \). So the DMS \( S^-_\Delta \) obtained from \( S^+_\Delta \) by removing \( A \) and a decoration can be also obtained from \( S^+_\Delta \) by cutting along \( \hat{\gamma} \) and removing \( A \). Notice that \( S^-_\Delta \) will inherits a full formal arc system from \( A_0 \), which is denoted by \( A^- \).

**Lemma 5.4.** \( S^-_\Delta \) and \( A^- \) satisfies Property-X.

**Proof.** For the corresponding differential graded \( X \)-graded algebras and associated categories satisfy the following relations:

- \( Q_{A^-} \) is a full subquiver of \( Q_{A^+} \);
- \( \Gamma_{A^-} \) is a (differential graded \( X \)-graded) subalgebra of \( \Gamma_{A^+} \);
- \( \mathcal{D}_{fd}(\Gamma_{A^-}) \) is subcategory of \( \mathcal{D}_{fd}(\Gamma_{A^+}) \), generated (as full thick subcategory) by the simples corresponding to the vertices of \( Q_{A^-} \).

□

5.3. **Final statement.** Use the lemma above and induction on the marked points we add to obtain \( S^+_\Delta \) from \( S_\Delta \), we have the following statement.

**Theorem 5.5.** Given a graded DMS \((S_\Delta, \Lambda, \lambda, c)\), there is a full formal arc system \( A \), satisfying Property-X.

6. **Topological realization of Lagrangian immersion**

Recall from Definition 2.9 and Definition 3.5 that we have an open full formal arc system \( A \) on a graded DMS \((S_\Delta, \Lambda, \lambda, c)\) with its dual arc system \( A^\_\Delta \).

6.1. **Derived categories of gentle algebras and Lagrangian immersion.**

**Definition 6.1.** For a full formal arc system \( A = \{\hat{\gamma}_i \mid 1 \leq i \leq n\} \) of \( S_\Delta \), the associated \((\mathbb{Z})\)-graded quiver \( Q^0_A \) is the \( X \)-degree zero part of \( Q_A \) in Definition 2.12 with induced differential \( d^0_A \).
Then we obtain a dg algebra $\Gamma^0_A$, which is the $X$-degree zero part of $\Gamma_A$. Denote by $D_{fd}(\Gamma^0_A)$ the bounded/finite-dimensional derived category of $\Gamma^0_A$. Note that, $D_{fd}(\Gamma^0_A)$ is in fact the topological Fukaya category introduced in [5].

Let $Q$ be a graded quiver (i.e. its arrows are $\mathbb{Z}$-graded) and $I$ a set of paths of length at least 2, such that $kQ/I$ is finite dimensional, where $\langle I \rangle$ is the ideal generated by $I$. The pair $(Q, I)$ is called gentle provided that the following hold.

1°. for any vertex $i \in Q_0$, there are at most two arrows in and at most two arrows out;
2°. for any arrow $\alpha \in Q_1$, there is at most one arrow $\beta \in Q_1$ with $\alpha\beta \in I$ and at most one arrow $\gamma \in Q_1$ with $\alpha\gamma \notin I$; and there is at most one arrow $\beta' \in Q_1$ with $\beta\alpha \in I$ and at most one arrow $\gamma' \in Q_1$ with $\gamma\alpha \notin I$.

A graded gentle algebra is (Morita equivalent to) the algebra $kQ/I$ for a (graded) gentle pair $(Q, I)$.

**Lemma 6.2** ([24, 25]). $H^*(\Gamma^0_A)$ is a graded gentle algebra and there is a triangle equivalence $D_{fd}(\Gamma^0_A) \simeq D_{fd}(H^*(\Gamma^0_A))$.

Recall that $S_i, 1 \leq i \leq n$ are the simple $\Gamma_A$-modules corresponding to $\tilde{\gamma}_i$. Let $S^0_i, 1 \leq i \leq n$ are the simple $\Gamma^0_A$-modules corresponding to $\tilde{\gamma}_i$.

By [10, Lem. 4.4], there is a Lagrangian immersion (cf. [13, Def. 7.2])

$$\mathcal{L}_c : D_{fd}(\Gamma^0_A) \to D_{fd}(\Gamma_A),$$

(6.1)

sending $S^0_i$ to $S_i$ and such that there is a canonical isomorphism

$$\text{RHom}_{D_{fd}(\Gamma_A)}(\mathcal{L}_c(X_1), \mathcal{L}_c(X_2)) = \text{RHom}_{D_{fd}(\Gamma^0_A)}(X_1, X_2) \oplus \text{RHom}_{D_{fd}(\Gamma^0_A)}(X_2, X_1)^*[-X],$$

for any pair of objects $X_1, X_2$ in $D_{fd}(\Gamma^0_A)$. We denote by $\mathcal{E}^0_A = \text{Ext}^Z(S^0, S^0)$, where $S^0 = \oplus_{i=1}^n S^0_i$.

**Proposition 6.3.** Let $\tilde{\sigma}^m \in \tilde{\mathcal{AC}}^X(S_\Delta)$. Then $X_{\tilde{\sigma}^m}$ is in $\mathcal{L}_c(D_{fd}(\Gamma^0_A))$ if and only if $\tilde{\sigma}^m$ is in the zero-sheet $S^0_\Delta$ of $\log \mathcal{S}_\Delta$, or equivalently, $m = 0$ and $\sigma$ does not intersect the cut $c$.

**Proof.** The curve $\tilde{\sigma}^m$ is in the zero sheet if and only if $\text{Int}_{S^0_\Delta}(\tilde{\gamma}_i, \tilde{\sigma}^m) = 0$ for any $1 \leq i \leq n$ and $\zeta \neq 0$. By Lemma 3.4, this is equivalent to $\text{dim}\text{Hom}(\Gamma^0_A, X_{\tilde{\sigma}^m}[\zeta X]) = 0$ for any $\zeta \neq 0$. By the simple-projective duality, this is equivalent to that $X_{\tilde{\sigma}^m}$ is in the triangulated subcategory of $D_{fd}(\Gamma_A)$ generated by the simples $S_i, 1 \leq i \leq n$, which is $\mathcal{L}_c(D_{fd}(\Gamma^0_A))$. \hfill $\Box$

**Definition 6.4.** We call a graded curve in the zero-sheet $S^0_\Delta$ of $\log \mathcal{S}_\Delta$ a zero-level curve. Denote by $\tilde{\mathcal{AC}}^0(\log \mathcal{S}_\Delta)$ the set of zero-level curves.

Note that $\tilde{\mathcal{AC}}^0(\log \mathcal{S}_\Delta)$ is a subset of $\tilde{\mathcal{AC}}^X(S_\Delta)$.
6.2. Dual arc systems on graded marked surfaces. On $S$, we also have the notion of full formal arc system as follows.

**Definition 6.5.** An open (resp. closed) full formal arc system $A$ (resp. $A^*$) on $(S, M, Y)$ is a collection of pairwise non-intersect simple curves connecting points in $M$ (resp. $Y$), such that it divides $S$ into $A$-polygons (resp. $A^*$-polygons), each of which contains exactly one open (resp. closed) boundary segment. Here, a open (resp. closed) boundary segment is a connected component of $\partial S \setminus M$ (resp. $\partial S \setminus Y$).

Denote by $\tilde{\text{Str}}(S)$ the set of graded closed curves in $S$, which are graded curves connecting points in $Y$. There is a natural operation $\text{im}_c$ (w.r.t. the cut $c$) on the set $\tilde{\text{Str}}(S)$ to $\tilde{\text{AC}}^0(\log S_\Delta)$, sending a curve $\tilde{\sigma}$ to a curve $\text{im}_c(\tilde{\gamma})$ by pulling its both endpoints on $Y$ to $\Delta$ via the corresponding curves in the cut $c$ (see Figure 18 where the greed one is sent to the red one) with that the grading of $\text{im}_c(\tilde{\gamma})$ inherits from $\tilde{\sigma}$. This (the grading) is well-defined because $\text{im}_c(\tilde{\sigma})$ does not cross the cut $c$ and the grading $\Lambda$ of $S_\Delta$ is compatible with the grading $\lambda$ of $S$ and the cut $c$. Clearly, $\text{im}_c$ is an bijection. It follows that $\text{im}_c$ preserves intersection indices.

![Figure 18. Closed arcs on $S_\Delta$ (in red) and in $S$ (in green)](image)

Then data $(S_\Delta, \lambda, A, A^*_\Delta, c)$ one-one corresponds to the data $(S, \lambda, A, A^*)$, which consists of

- the underlying graded marked surface $(S, \lambda)$;
- the induced full formal arc system $A$ on $S$ (when forgetting the decoration set $\Delta$ and
- the dual arc system $A^* = \text{im}_c^{-1}(A^*_\Delta)$ on $S$. See Figure 19:
  - $A$ consists of blue arcs;
  - $A^*_\Delta$ consists of red ones,
  - $c$ consists of dashed gray one and
  - $A^*$ consists of green ones.

On the other hand, $A^*$ also determines $(A, A^*_\Delta, c)$ in the sense that there is exactly one decorated in an $A$-polygon; thus the cut (paring of $Y$ and $\Delta$) is determined uniquely; and then $A^*_\Delta = \text{im}_c(A^*)$. Moreover, $A^*$ is in fact a closed full formal arc system on $(S, Y, M)$.
6.3. **Topological realization.** There is an analogous string model for \((S, \lambda, Y)\) as follows:

- each positive arc segment in an \(A\)-polygon of \(S\) corresponds to an arrow in \(Q_0^A\), which induces a \(\mathbb{Z}_2\)-graded morphism between the simples \(S_i^0, 1 \leq i \leq n\);
- each \(\tilde{\sigma} \in \text{Str}(S)\) gives a string \(\omega(\tilde{\sigma})\), which is a sequence of morphisms between simples induced by the arc segments of \(\tilde{\sigma}\) divided by \(A\);
- Define \(X_0^\sigma\) to be \(X_{\omega(\tilde{\sigma})}\), which is the dg module of \(\Gamma_0^A\) associated to the string \(\omega(\tilde{\sigma})\).

Thus, we get a map

\[ X^0: \text{Str}(S) \to D_{fd}(\Gamma_0^A). \]  
(6.2)

Therefore, the operation \(\text{im}_c\) can be thought as a topological realization of the Lagrangian immersion \(L_c\), in the sense that we have the following.

**Theorem 6.6.** For any \(\tilde{\sigma} \in \text{Str}(S)\), we have

\[ L_c(X_0^\tilde{\sigma}) = X_{\text{im}_c(\tilde{\sigma})}. \]  
(6.3)

**Proof.** Since \(\text{im}_c(\tilde{\sigma})\) does not cross the cut \(c\), the string model for it does not contain any morphism whose associated arrow is not \(X\)-free. So this model is the same as the string model for \(\tilde{\sigma}\). Hence we are done. \(\square\)

Let \(\tilde{CA}(S)\) be the set of graded closed arc in \(S\) and recall that \(\tilde{AC}(S)\) is the set of graded closed curve on \(S\). An immediate corollary is that the following \(\mathbb{Z}\)-graded \(q\)-intersection formula from Corollary 4.19, that non-zero \(X\)-degree terms all vanish.

**Corollary 6.7.** For any \(\tilde{\sigma} \in \tilde{CA}(S)\) and any \(\tilde{\eta} \in \tilde{AC}(S)\), we have

\[ \text{Int}^q(\tilde{\sigma}, \tilde{\eta}) = \dim^q \text{Hom}^\bullet(X_0^{\tilde{\sigma}}, X_0^{\tilde{\eta}}). \]  
(6.4)

We can generalize the formula above to the case when \(\tilde{\sigma}, \tilde{\eta}\) are both graded closed curves (recall that the difference is that an arc is a simple curve, i.e. without interior
self-intersections). But we leave this small refinement to [21], where we will prove such a formula for graded skew gentle algebras.

6.4. Further study: Koszul duality via graph duality. One can construct a string model for \((S, \lambda, M)\), by taking the closed full formal arc system \(A^*\) as reference. Namely, there is a map

\[
\rho: \widetilde{\text{AO}}(S) \to \text{per} \Gamma^0_A,
\]

(6.5)
sending open arcs \(\tilde{\gamma}_i\) in \(A\) to the projective indecomposables of \(\Gamma^0_A\), where \(\widetilde{\text{AO}}(S)\) is the set of graded open curves on \(S\) (that is, curves connecting points in \(M\)). People expect that (6.5) and (6.2) are topological realization of Koszul duality, in the sense that

\[
\text{Int}(\tilde{\gamma}, \tilde{\sigma}) = \dim \text{Hom}^\bullet(\rho(\tilde{\gamma}), X^0(\tilde{\sigma})),
\]

(6.6)
for any \(\tilde{\gamma} \in \widetilde{\text{AO}}(S), \tilde{\sigma} \in \text{Str}(S)\). In particular, given a silting set \(\{\rho(\tilde{\gamma}_j)\}\), the corresponding curves \(\{\tilde{\gamma}_j\}\) will form a full formal arc system \(A_0\) with dual full form arc system \(A^*_0 = \{\tilde{\sigma}_j\}\), satisfying that \(\{X^0(\tilde{\sigma}_j)\}\) is the set of simples of the heart dual to \(\{\rho(\tilde{\gamma}_j)\}\). The set \(\{X^0(\tilde{\sigma}_j)\}\) is also known as a simple minded collection. Such type of results for Calabi-Yau-3 version of \(\text{per} \Gamma_A\) and \(D_{\text{fd}}(\Gamma_A)\) can be found in Table 1 in either [17] or [18] with corresponding formula (6.6) in [19].

We plan to further investigate this for derived categories of any graded gentle algebras (without restriction of finite dimension or finite global dimension) in [20].

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