Abstract

We are interested in studying an unsteady fluid-structure interaction problem in a three-dimensional space. We consider a homogeneous Newtonian fluid which is modeled by the Navier-Stokes equations. Whereas the motion of the structure is described by the quasi-incompressible non-linear Saint Venant-Kirchhoff model. We establish the local in time existence and uniqueness of solution for this model. For this sake, first we rewrite the non-linearity of the elastodynamic equation in an explicit way. Then, a linearized problem is introduced in the Lagrangian reference configuration and we prove that it admits a unique solution. Based on the \textit{a priori} estimates on the solution of this problem together with the fixed point theorem we prove that the non-linear problem admits a unique local in time solution. At last, by the inf-sup condition we reach to the existence of the fluid pressure.

\textit{Keywords:} Fluid-structure interaction, Navier-Stokes, elastodynamic equations, Saint Venant-Kirchhoff model, fixed point theorem.

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Introduction

Fluid-structure interaction (FSI) problem is a wide spread subject which has gain a lot of concern and interest among mathematicians. This is due to the fact that many real-world problems consider the analysis of FSI problems as an essential tool to avoid failure. For example, they are considered in the design of many engineering systems such as aircrafts, engines and bridges, where the FSI oscillations are studied. Also, in biological field, fluid-structure interaction problems play an important role in the analysis of aneurysms and blood flow in stenosed arteries. Various kinds of fluid-structure interaction problems have been studied by modeling the fluid by either Stokes or Navier-Stokes equations coupled with an equation modeling the structure. Some deal with incompressible fluids \cite{2, 3, 19}, others with compressible fluids \cite{4, 3}. Structures modeled with plate equations or shell equations were treated in \cite{14}. The Stokes equations coupled with beam equation were analyzed in \cite{17}. The case of a free boundary FSI with the flow being incompressible and coupled with a linear Kirchhoff elastic material has been treated in \cite{19}, where the existence and uniqueness locally in time of such motion has been proved. In \cite{19}, the existence locally in time of a
weak solution for an incompressible fluid with a rigid structure has been proved. Similar model has been studied in \cite{11} considering a variable density where the global existence of the solution has been proved, that is, the existence of the solution until collisions occur between either the structure and boundaries or between two structures. For the coupling of an incompressible fluid with elastic structure, the existence of global weak solutions has been proved in \cite{2} when adding a regularizing term to the structure motion. In 3D, the work in \cite{18} has proved the existence of steady solutions of the incompressible Navier-Stokes equations when coupled with the non-linear Saint Venant-Kirchhoff model. Whereas, the existence and uniqueness of a regular solution has been proved in the case of compressible Navier-Stokes equations coupled with the non-linear Saint Venant-Kirchhoff model in \cite{4}, and with linear elastic model in \cite{3}.

In our work we consider the interaction between an incompressible homogeneous Newtonian fluid modeled by the Navier-Stokes equations surrounded by a hyperelastic quasi-incompressible structure modeled by the non-linear Saint Venant-Kirchhoff model. We couple them in a one domain, by considering a common boundary and imposing some conditions on it. First, we introduce the coupled system at time $t$, which consists of the incompressible homogeneous Navier-Stokes equations with the elastodynamic equations modeled by the non-linear Saint Venant-Kirchhoff model. From mathematical point of view, Navier-Stokes equations are studied in the Eulerian (spatial) framework, whereas elastic structures are studied in the Lagrangian (material) framework. In order to be able to study the coupled system we use the deformation mappings of both the fluid and the structure domains to rewrite the coupled system in the Lagrangian framework, in particular, in the reference configuration corresponding to the time $t = 0$. Indeed, since we are working with a problem involving a free moving boundary, the Lagrangian frame allows us to consider working on a fixed domain. As for the Saint Venant-Kirchhoff model we rewrite it in an explicit form which enables us to easily deal with it when applying the fixed point theorem as well as to find some bounds on it. In a second step we partially linearize our system by considering the deformations to be given for given fluid velocity $\tilde{v}$ and structure’s displacement $\tilde{\xi}$, such that the couple $(\tilde{v}, \tilde{\xi})$ is in some fixed point space. The third step consists of formulating an auxiliary problem, which comes from the classical system by changing slightly the coupling conditions coming from the elastodynamic equations associated to the structure. The weak formulation is derived by considering a transformation of a divergence-free setting, so that the fluid pressure term will disappear. Using Faedo-Galerkin approach we define Galerkin approximations of the solutions and derive a priori estimates for the Galerkin sequence. By passing to the limit, and using compactness results with Aubin-Lions-Simon Theorem we prove the existence and uniqueness of a solution for the auxiliary problem. Based on the results concerning the auxiliary problem, and using the fixed point theorem we prove the existence and uniqueness of the solution of the partially linearized problem. Coming back to the non-linear problem, we use the fixed point theorem approach to prove the existence of a solution for the non-linear fluid-structure interaction problem. Finally, we establish the existence of an $L^2$ fluid pressure by verifying the inf-sup condition.

1. Fluid-Structure Interaction Problem

The fluid is governed by the homogeneous incompressible Navier-Stokes equations. Let $T > 0$ be given. At time $t$, let $\Omega_f(t) \subset \mathbb{R}^3$ denotes a regular (enough) bounded connected domain representing the lumen of the artery. Denote
by \( \partial \Omega_f(t) = \Gamma_{in}(t) \cup \Gamma_{out}(t) \cup \Gamma_f(t) \) its smooth boundary. The incompressible Navier-Stokes equations formulated in the Eulerian coordinates are

\[
\begin{aligned}
\rho_f \left( \partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right) - \nabla \cdot \mathbf{\sigma}_f(\mathbf{v}, p_f) &= 0 \quad \text{in } \Omega_f(t) \times (0, T), \\
\nabla \cdot \mathbf{v} &= 0 \quad \text{in } \Omega_f(t) \times (0, T), \\
\mathbf{\sigma}_f(\mathbf{v}, p_f) \cdot \mathbf{n}_f &= \mathbf{g}_f \quad \text{on } \Gamma_f(t) \times (0, T), \\
\mathbf{v} &= \mathbf{v}_{\text{in}} \quad \text{on } \Gamma_{\text{in}}(t) \times (0, T), \\
\mathbf{\sigma}_f(\mathbf{v}, p_f) \cdot \mathbf{n}_f &= 0 \quad \text{on } \Gamma_{\text{out}}(t) \times (0, T), \\
\mathbf{v} &= \mathbf{v}_0 \quad \text{in } \Omega_f(t) \text{ at } t = 0,
\end{aligned}
\]

where \( \mathbf{v} = (v_1, v_2, v_3)^t \) is the fluid velocity, \( p_f \) is its pressure and \( \rho_f > 0 \) is its density. We denote by \( \mathbf{g}_f \) an external load on \( \Gamma_f(t) \). The term \( \mathbf{\sigma}_f(\mathbf{v}, p_f) \) is the Cauchy stress tensor of the fluid whose expression is

\[
\mathbf{\sigma}_f(\mathbf{v}, p_f) = 2\mu \mathbf{D}(\mathbf{v}) - p_f \mathbf{I}.
\]

with \( \mu \) is its dynamic viscosity and \( \mathbf{D}(\mathbf{v}) = \frac{\nabla \mathbf{v} + (\nabla \mathbf{v})^t}{2} \) is the symmetric gradient. On the other hand, the structure is considered to be a quasi-incompressible homogeneous hyperelastic material modeled by the non-linear Saint Venant-Kirchhoff model \cite{6}. We denote by \( \Omega_s(t) \subset \mathbb{R}^3 \) a regular enough domain that represents the structure at any time \( t > 0 \) and by \( \partial \Omega_s(t) \) its smooth boundary such that \( \partial \Omega_s(t) = \Gamma_1(t) \cup \Gamma_2(t) \). The structure displacement \( \mathbf{\xi}_s \) satisfies the following equations

\[
\begin{aligned}
\rho_s \partial_t^2 \mathbf{\xi}_s - \nabla \cdot \mathbf{\sigma}_{\text{Qinc}}^s(\mathbf{\xi}_s) &= 0 \quad \text{in } \Omega_s(t) \times (0, T), \\
\mathbf{\sigma}_{\text{Qinc}}^s(\mathbf{\xi}_s) \cdot \mathbf{n}_s &= \mathbf{g}_s \quad \text{on } \Gamma_1(t) \times (0, T), \\
\mathbf{\xi}_s &= 0 \quad \text{on } \Gamma_2(t) \times (0, T),
\end{aligned}
\]

where \( \mathbf{\sigma}_{\text{Qinc}}^s \) is the Cauchy stress tensor characterizing the quasi-incompressible property of the structure and \( \mathbf{g}_s \) is a surface external force applied on \( \Gamma_1(t) \). To set up the FSI system, the domains \( \Omega_f(t) \) and \( \Omega_s(t) \) are coupled by considering \( \Gamma_1(t) \equiv \Gamma_f(t) \). Here and after the common boundary will be denoted by \( \Gamma_c(t) \). To ensure the global energy balance of the system some coupling conditions representing the continuity of the velocities and stresses must be imposed on the boundary \( \Gamma_c(t) \). These coupling conditions are given as

\[
\begin{aligned}
\mathbf{v} &= \partial_t \mathbf{\xi}_s, \quad \text{on } \Gamma_c(t) \times (0, T), \\
\mathbf{\sigma}_f(\mathbf{v}, p_f) \cdot \mathbf{n} &= \mathbf{\sigma}_{\text{Qinc}}^s(\mathbf{\xi}_s) \cdot \mathbf{n} \quad \text{on } \Gamma_c(t) \times (0, T),
\end{aligned}
\]

where \( \mathbf{n} \) is the outward normal from \( \Omega_f(t) \) to \( \Gamma_c(t) \).

Finally, we introduce the initial conditions

\[
\begin{aligned}
&\mathbf{v}(., 0) = \mathbf{v}_0 \quad \text{in } \Omega_f(0), \\
&\mathbf{\xi}_s(., 0) = \mathbf{\xi}_0 \quad \text{in } \Omega_s(0), \\
&\partial_t \mathbf{\xi}_s(., 0) = \mathbf{\xi}_1 \quad \text{in } \Omega_s(0), \\
&p_f(., 0) = p_{f_0} \quad \text{in } \Omega_f(0),
\end{aligned}
\]
which satisfy

\[ v_0 \in H^6(\Omega_f(0)), \quad \xi_0 \in H^4(\Omega_s(0)), \quad \xi_1 \in H^3(\Omega_s(0)) \quad \text{and} \quad p_{f0} \in H^3(\Omega_f(0)). \]

(4)

Let \( \Omega(t) = \left[ \Omega_f(t) \cup \overline{\Omega_s(t)} \right]^0 \) and \( \partial \Omega(t) = [\partial \Omega_f(t) \cup \partial \Omega_s(t)] \setminus [\partial \Omega_f(t) \cap \partial \Omega_s(t)] \).

At time \( t > 0 \), the coupled system is given by

\[
\begin{align*}
\rho_f \left( \partial_t v + (v \cdot \nabla)v \right) - \nabla \cdot \mathbf{\sigma}_f(v, p_f) &= 0 \quad \text{in} \quad \Omega_f(t) \times (0, T), \\
\nabla \cdot v &= 0 \quad \text{in} \quad \Omega_f(t) \times (0, T), \\
v &= v_{in} \quad \text{on} \quad \Gamma_{in}(t) \times (0, T), \\
\mathbf{\sigma}_f(v, p_f) \cdot n &= 0 \quad \text{on} \quad \Gamma_{out}(t) \times (0, T), \\
\rho_s \partial_t^2 \xi_s - \nabla \cdot \mathbf{\sigma}^{\text{Qinc}}(\xi_s) &= 0 \quad \text{in} \quad \Omega_s(t) \times (0, T), \\
\xi_s &= 0 \quad \text{on} \quad \Gamma_2(t) \times (0, T), \\
v &= \partial_t \xi_s \quad \text{on} \quad \Gamma_c(t) \times (0, T), \\
\mathbf{\sigma}_f(v, p_f) \cdot n &= \mathbf{\sigma}^{\text{Qinc}}(\xi_s) \cdot n \quad \text{on} \quad \Gamma_c(t) \times (0, T), \\
v(., 0) &= v_0 \quad \text{and} \quad p_f(., 0) = p_{f0} \quad \text{in} \quad \Omega_f(0), \\
\xi_s(., 0) &= \xi_0 \quad \text{and} \quad \partial_t \xi_s(., 0) = \xi_1 \quad \text{in} \quad \Omega_s(0).
\end{align*}
\]

(5a) \( \ldots \) (5j)

The Navier-Stokes equations are defined on the domain \( \Omega_f(t) \) which evolves over time from the initial configuration \( \Omega_f(0) \) according to a position function

\[
\mathcal{A}(., t) : \Omega_f(0) \rightarrow \Omega_f(t) \\
\tilde{x} \rightarrow \mathcal{A}(\tilde{x}, t) = x
\]

that associates to the Lagrangian coordinate of a fluid particle its Eulerian coordinate. For all \( \tilde{x} \in \Omega_f(0) \) the function \( \mathcal{A}(\tilde{x}, .) \) satisfies

\[
\begin{align*}
\partial_t \mathcal{A}(\tilde{x}, t) &= v(\mathcal{A}(\tilde{x}, t), t) \quad \text{for} \quad t \in (0, T), \\
\mathcal{A}(\tilde{x}, 0) &= \tilde{x}.
\end{align*}
\]

The function \( \mathcal{A} \) is called the Arbitrary Lagrangian-Eulerian (ALE) map.

Similarly, the elastodynamic equations in the displacement \( \xi_s \) are defined on the domain \( \Omega_s(t) \) which evolves over time from the initial configuration \( \Omega_s(0) \) according to a position function

\[
\varphi_s(., t) : \Omega_s(0) \rightarrow \Omega_s(t) \\
\tilde{y} \rightarrow \varphi_s(\tilde{y}, t) = y
\]

and we have

\[
\varphi_s(\tilde{y}, t) = \tilde{y} + \xi_s(\varphi_s(\tilde{y}, t), t).
\]

(6)

Notice that, using (6) we have

\[
\varphi_s(\tilde{y}, 0) = \tilde{y} + \xi_s(\tilde{y}, 0), \quad \text{that is} \quad \tilde{y} = \tilde{y} + \xi_0
\]
which yields $\xi_0 = 0$.

In the sequel, we omit the subscript $s$ of the structure displacement and deformation, that is, we write $\xi_s = \xi$ and $\varphi_s = \varphi$. Further, we refer to the space elements in $\Omega_f^0$ and $\Omega_s^0$ by $\bar{x}$.

The definition of these two mappings enables us to write System (5a)-(5j) on the domain $\Omega(0)$. To do so, we consider the following change of variables in terms of the deformation mappings $A$ and $\varphi$. For all $\tilde{x}$ in $\Omega_f(0)$ and $\Omega_s(0)$ and $t$ in $(0,T)$ set

$$\tilde{v}(\tilde{x}, t) = v(A(\tilde{x}, t), t), \quad \tilde{\xi}(\tilde{x}, t) = \xi(\varphi(\tilde{x}, t), t) \quad \text{and} \quad \tilde{p}_f(\tilde{x}, t) = p_f(A(\tilde{x}, t), t).$$

(7)

On the reference domain $\Omega_f(0)$, the fluid stress tensor is given by [20, Section 2.1.7] as

$$\tilde{\sigma}_f^0(\tilde{v}, \tilde{p}_f) = \left( \mu(\nabla \tilde{v}(\nabla A)^{-1} + (\nabla A)^{-1}(\nabla \tilde{v})^t) - \tilde{p}_f I \right) \text{cof}(\nabla A)$$

(8)

$$= \tilde{\sigma}_f^0(\tilde{v}) - \tilde{p}_f \text{cof}(\nabla A).$$

As for the quasi-incompressible structure, the Cauchy stress tensor is given in terms of the first Piola-Kirchhoff stress tensor $P$ as [20, Lemma 2.12]

$$P_{\text{Qinc}} = \det(\nabla \varphi)(\sigma_{\text{Qinc}}(\xi) \circ \varphi)(\nabla \varphi)^{-t}$$

$$= P + C(\det(\nabla \varphi) - 1)\text{cof}(\nabla \varphi)$$

(9)

with $C > 0$ a sufficiently large constant and

$$P = \nabla \varphi S(\nabla \varphi)$$

(10)

where

$$S(\nabla \varphi) = 2\mu_s E(\nabla \varphi) + \lambda_s \text{tr}(E(\nabla \varphi)) I$$

is the second Piola-Kirchhoff stress tensor and

$$E(\nabla \varphi) = \frac{1}{2}( (\nabla \varphi)^t \nabla \varphi - I)$$

is the Green-Lagrange strain tensor with $(\mu_s, \lambda_s) \in \mathbb{R}_+^* \times \mathbb{R}_+$ are the Lamé coefficients.

In particular, when considering the Saint Venant-Kirchhoff stress tensor, Expression (10) can be rewritten in terms of the displacement $\tilde{\xi}$ as

$$P = (I + \nabla \tilde{\xi}) \left( \mu_s \left( \nabla \tilde{\xi} + (\nabla \tilde{\xi})^t + (\nabla \tilde{\xi})^t \nabla \tilde{\xi} \right) + \frac{\lambda_s}{2} \left( 2\nabla \cdot \tilde{\xi} + |\nabla \tilde{\xi}|^2 \right) I \right).$$

(11)

Using relations (7)-(11) we reformulate the Navier-Stokes equations and the elastodynamic equations in the Lagrangian
coordinates. Hence, we can rewrite the coupled System (1)-(5) on $\Omega_f(0)$ and $\Omega_s(0)$ as

\[
\begin{align*}
\rho_f \text{det}(\nabla A) \partial_t \tilde{v} - \nabla \cdot \tilde{\sigma}_f(\tilde{v}, \tilde{p}_f) &= 0 \quad \text{in } \Omega_f(0) \times (0, T), \\
\nabla \cdot (\text{det}(\nabla A)(\nabla A)^{-1} \tilde{\sigma}) &= 0 \quad \text{in } \Omega_f(0) \times (0, T), \\
\tilde{v} &= v_{\text{in}} \circ A \quad \text{on } \Gamma_{\text{in}}(0) \times (0, T), \\
\tilde{\sigma}_f(\tilde{v}, \tilde{p}_f) \tilde{n} &= 0 \quad \text{on } \Gamma_{\text{out}}(0) \times (0, T), \\
\rho_s \text{det}(\nabla \varphi) \partial_t \tilde{\xi} - \nabla \cdot P - \nabla \cdot [C(\text{det}(\nabla \varphi) - 1) \text{cof}(\nabla \varphi)] &= 0 \quad \text{in } \Omega_s(0) \times (0, T), \\
\tilde{\xi} &= 0 \quad \text{on } \Gamma_2(0) \times (0, T), \\
\tilde{v} &= \tilde{\partial}_t \tilde{\xi} \quad \text{on } \Gamma_c(0) \times (0, T), \\
\tilde{\sigma}_f(\tilde{v}, \tilde{p}_f) \tilde{n} &= [P + C(\text{det}(\nabla \varphi) - 1) \text{cof}(\nabla \varphi)] \tilde{n} \quad \text{on } \Gamma_c(0) \times (0, T), \\
\tilde{v}(., 0) &= v_0 \quad \text{and } \quad \tilde{p}_f(., 0) = p_{f_0} \quad \text{in } \Omega_f(0), \\
\tilde{\xi}(., 0) &= \xi_0 = 0 \quad \text{and } \quad \tilde{\partial}_t \tilde{\xi}(., 0) = \xi_1 \quad \text{in } \Omega_s(0),
\end{align*}
\]

where $\nabla \varphi = \text{Id} + \nabla \tilde{\xi}$ is the gradient of the deformation and $\tilde{n}$ is the outward normal of $\Omega_s(0)$ on $\Gamma_c(0)$.

In order to deal with the structure model, we write the elasticity model in the spirit of [13], that is, we define

\[
c_{i\alpha j\beta} = \frac{\partial P_{ij}}{\partial (\delta_{\beta} \xi_j)}. \tag{13}
\]

Let us set

\[
c_{i\alpha j\beta}(\nabla \tilde{\xi}) = \mu_s(\delta_{ij} \delta_{\alpha j} + \delta_{\alpha j} \delta_{ij}) + \lambda_s(\delta_{\alpha j} \delta_{ij}) + c_{i\alpha j\beta}^l(\nabla \tilde{\xi}) + c_{i\alpha j\beta}^q(\nabla \tilde{\xi}), \tag{14}
\]

where $c_{i\alpha j\beta}^l(\nabla \tilde{\xi})$ is the linear part given by

\[
c_{i\alpha j\beta}^l(\nabla \tilde{\xi}) = \mu_s(\delta_{ij} \partial_{\beta} \tilde{\xi} + \delta_{\alpha j} \partial_{\beta} \tilde{\xi}_i + \delta_{ij} \partial_{\alpha} \tilde{\xi}_j + \delta_{\alpha j} \partial_{\alpha} \tilde{\xi}_j + \delta_{\beta j} \partial_{\alpha} \tilde{\xi}_i + \delta_{\alpha j} \partial_{\beta} \tilde{\xi}_i)
\]

\[
+ \lambda_s(\delta_{\alpha j} \partial_{\alpha} \tilde{\xi}_j + \delta_{\alpha j} \partial_{\beta} \tilde{\xi}_j + \nabla \cdot \tilde{\xi} + \delta_{\beta j} \partial_{\alpha} \tilde{\xi}_i)
\tag{15}
\]

and $c_{i\alpha j\beta}^q$ is the quadratic part written as

\[
c_{i\alpha j\beta}^q(\nabla \tilde{\xi}) = \mu_s \left( \delta_{ij} (\partial_{\beta} \tilde{\xi} \cdot \partial_{\alpha} \tilde{\xi}) + \partial_{\beta} \xi_i \partial_{\alpha} \tilde{\xi}_j + \delta_{\alpha j} (\nabla \tilde{\xi}_j \cdot \nabla \tilde{\xi}_i) \right) + \lambda_s \left( \frac{1}{2} \delta_{ij} \delta_{\beta} \nabla \tilde{\xi}_i^2 + \partial_{\alpha} \tilde{\xi}_i \partial_{\beta} \tilde{\xi}_i + \right). \tag{16}
\]

Hence, $c_{i\alpha j\beta}$ can be rewritten as

\[
c_{i\alpha j\beta}(\nabla \tilde{\xi}) = Cst + L(\nabla \tilde{\xi}) + Q(\nabla \tilde{\xi}), \tag{17}
\]

where $Cst$ is a constant, $L$ is a linear function in $\nabla \tilde{\xi}$ and $Q$ is a quadratic function in $\nabla \tilde{\xi}$.

Remark that the coefficients $c_{i\alpha j\beta}$ are symmetric, that is,

\[
c_{i\alpha j\beta} = c_{j\beta i\alpha} \quad \forall \ i, \alpha, j, \beta \in \{1, 2, 3\}. \tag{18}
\]

**Lemma 1.1.** For $k = i, \alpha, j, \beta \in \{1, 2, 3\}$, we denote by $\partial_k$ the partial derivative in space and by $\partial_t$ and $\partial_s$ the partial derivatives with respect to time. Some consequences of the relation [18] are the following.
1- For \( i, \alpha \in \{1, 2, 3\} \), the partial derivatives of \( P \) with respect to time and space are respectively

\[
\partial_t P_{\alpha} = \sum_{j, \beta=1}^{3} c_{\alpha j \beta} (\nabla \tilde{\xi}) \partial^2_{s j} \tilde{\xi}_j \quad \text{and} \quad \partial_s P_{\alpha} = \sum_{j, \beta=1}^{3} c_{\alpha j \beta} (\nabla \tilde{\xi}) \partial^2_{s j} \tilde{\xi}_j.
\]

2- The \( i \)-th component of the divergence of \( P \) is given by

\[
(\nabla \cdot P)_i = \sum_{\alpha, j, \beta=1}^{3} c_{\alpha j \beta} (\nabla \tilde{\xi}) \partial^2_{s j} \tilde{\xi}_j \quad \forall \ i = 1, 2, 3.
\]  

(19)

3- Assuming that \( P(\tilde{\xi}(.,0)) = 0 \) on \( \Gamma_1(0) \), the normal component of the stress tensor \( P \) on the boundary \( \Gamma_1(0) \) is

\[
\sum_{\alpha=1}^{3} P_{\alpha} \hat{n}_\alpha = \sum_{\alpha, j, \beta=1}^{3} \left( \int_0^t c_{\alpha j \beta} (\nabla \tilde{\xi}) \partial^2_{s j} \tilde{\xi}_j \ ds \right) \hat{n}_\alpha \quad \forall \ i = 1, 2, 3.
\]  

(20)

4- The \( i\alpha \)-th component of \( P \) is given by

\[
P_{i\alpha} = \sum_{\alpha, j, \beta=1}^{3} \int_0^t c_{i j \beta} (\nabla \tilde{\xi}) \partial^2_{s j} \tilde{\xi}_j \ ds \quad \forall \ i = 1, 2, 3.
\]

Proof.

1- Let \( r \) be the index that represents either the time derivative or the space derivative. For the \( i\alpha \)-th component of \( P(\tilde{\xi}) \) we have

\[
\partial_r (P(\tilde{\xi}))_{i\alpha} = \sum_{j, \beta=1}^{3} \frac{\partial (P(\tilde{\xi}))_{i\alpha}}{\partial (\partial_\beta \tilde{\xi}_j)} \frac{\partial (\partial_\beta \tilde{\xi}_j)}{\partial_r} = \sum_{j, \beta=1}^{3} c_{i\alpha j \beta} (\nabla \tilde{\xi}) \partial^2_{r \beta} \tilde{\xi}_j.
\]

2- Considering \( r = \alpha \) in the first part yields

\[
\partial_\alpha (P(\tilde{\xi}))_{i\alpha} = \sum_{j, \beta=1}^{3} c_{i\alpha j \beta} (\nabla \tilde{\xi}) \partial^2_{\alpha \beta} \tilde{\xi}_j.
\]

But for \( i = 1, 2, 3 \) we have

\[
(\nabla \cdot P(\tilde{\xi}))_i = \sum_{\alpha=1}^{3} \partial_\alpha (P(\tilde{\xi}))_{i\alpha} = \sum_{\alpha, j, \beta=1}^{3} c_{i j \beta} (\nabla \tilde{\xi}) \partial^2_{\beta} \tilde{\xi}_j.
\]

3- For any \( \tilde{\xi} \) in \( \Omega_s(0) \) we have

\[
P_{\alpha}(\tilde{\xi}(., t)) - P_{\alpha}(\tilde{\xi}(., 0)) = \int_0^t \partial_s P_{\alpha}(\tilde{\xi}(., s)) \ ds \quad \forall \ i, \alpha = 1, 2, 3.
\]

Substituting \( \partial_s P(\tilde{\xi}(., s)) \) by its expression from the first part gives

\[
P_{\alpha}(\tilde{\xi}(., t)) - P_{\alpha}(\tilde{\xi}(., 0)) = \sum_{j, \beta=1}^{3} \int_0^t c_{i j \beta} (\nabla \tilde{\xi}) \partial^2_{s j} \tilde{\xi}_j \ ds \quad \forall \ i, \alpha = 1, 2, 3.
\]

In particular, on \( \Gamma_1(0) \) we have \( P(\tilde{\xi}(., 0)) = 0 \). Consequently, taking the summation over \( \alpha \) yields
\[
\sum_{\alpha=1}^{3} P_{\alpha} (\xi (., t)) \tilde{n}_\alpha = \sum_{\alpha, j, \beta=1}^{3} \int_{0}^{t} (c_{\alpha j \beta}(\nabla \tilde{\xi}) \partial_{x j}^{2} \tilde{\xi}_j \, ds) \tilde{n}_\alpha \quad \forall \, i = 1, 2, 3.
\]

To deal with the quasi-incompressibility condition, we express it in a way similar to that of the first Piola-Kirchhoff stress tensor \[14\]. To do so, we use the notation introduced in \[8\, p.\, 5\] by defining the third-order orientation tensor \((\varepsilon_{ijk})\) whose components are the Levi-Civita symbol \(\{\varepsilon_{ijk}\}_{ijk}\). Using the Einstein summation convention on the indices, we define the \(ij\)-th element of the matrix \(\text{cof}(\nabla \varphi)\) by

\[
(cof(\nabla \varphi))_{ij} = \frac{1}{2} \varepsilon_{mni} \varepsilon_{pqj} \partial_{p} \varphi_{m} \partial_{q} \varphi_{n}.
\]

Further, the determinant of the 3-by-3-matrix \(\nabla \varphi\) is

\[
\det(\nabla \varphi) = \frac{1}{6} \varepsilon_{ijk} \varepsilon_{pqr} \partial_{i} \varphi_{j} \partial_{j} \varphi_{k}.
\]

We define

\[
d_{i\alpha j\beta}(\nabla \tilde{\xi}) = \left. \frac{\partial}{\partial(\partial_{\beta} \tilde{\xi}_j)} \left[ (\det(\nabla \varphi) - 1) \text{cof}(\nabla \varphi) \right] \right|_{\alpha}.
\]

Clearly, \(d_{i\alpha j\beta}(\nabla \tilde{\xi})\) is a polynomial in \(\nabla \tilde{\xi}\) of degree at most 4. Moreover, for \(i = \alpha\) and \(j = \beta\) we get the constant terms of this polynomial. Then we can write

\[
d_{i\alpha j\beta}(\nabla \tilde{\xi}) = Cst + d_{i\alpha j\beta}^{L}(\nabla \tilde{\xi}) + d_{i\alpha j\beta}^{Q}(\nabla \tilde{\xi}) + d_{i\alpha j\beta}^{T}(\nabla \tilde{\xi}) + d_{i\alpha j\beta}^{F}(\nabla \tilde{\xi})
\]

where \(d_{i\alpha j\beta}^{L}, d_{i\alpha j\beta}^{Q}, d_{i\alpha j\beta}^{T}\) and \(d_{i\alpha j\beta}^{F}\) stand for polynomials in \(\nabla \tilde{\xi}\) with respective degree 1, 2, 3 and 4. This writing enables us to give the \(i - th\) component of \(\nabla \cdot [C(\det(\nabla \varphi) - 1)\text{cof}(\nabla \varphi)]\). In fact,

\[
\left[ \nabla \cdot \left( C(\det(\nabla \varphi) - 1)\text{cof}(\nabla \varphi) \right) \right]_{i} = C \sum_{\alpha, j, \beta=1}^{3} d_{i\alpha j\beta}(\nabla \tilde{\xi}) \partial_{\alpha \beta}^{2} \tilde{\xi}_j \quad \text{for} \quad i = 1, 2, 3.
\]

In a way similar to \[20\], for \(i = 1, 2, 3\) the normal component of the quasi-incompressible condition on the boundary \(\Gamma_1(0)\) is

\[
\sum_{\alpha=1}^{3} \left[ (\det(\nabla \varphi) - 1)\text{cof}(\nabla \varphi) \right]_{\alpha} \tilde{n}_\alpha = \sum_{\alpha, j, \beta=1}^{3} \left( \int_{0}^{t} d_{i\alpha j\beta}(\nabla \tilde{\xi}) \partial_{x j}^{2} \tilde{\xi}_j \, ds \right) \tilde{n}_\alpha,
\]

provided that \(\left( \det(\nabla \varphi) - 1\right)\text{cof}(\nabla \varphi)(., 0) = 0\) on \(\Gamma_1(0)\).

In what follows, for simplicity we set

\[
b_{i\alpha j\beta} = c_{i\alpha j\beta} + C d_{i\alpha j\beta}.
\]
Using Relations (19), (20), (24) and (25), System (12) can be rewritten as

\[
\begin{align*}
\rho_f \text{det}(\nabla \cdot \mathbf{A}) \partial_t \tilde{v} - \nabla \cdot \tilde{\sigma}^f_i (\tilde{v}, \tilde{p}_f) &= 0 & \text{in } \Omega_f(0) \times (0,T), \\
\nabla \cdot (\text{det}(\nabla \cdot \mathbf{A})(\nabla \cdot \mathbf{A})^{-1}) \tilde{v} &= 0 & \text{in } \Omega_f(0) \times (0,T), \\
\tilde{v} &= v_\text{in} \circ \mathbf{A} & \text{on } \Gamma_{\text{in}}(0) \times (0,T), \\
\tilde{\sigma}^f_i (\tilde{v}, \tilde{p}_f) \tilde{n} &= 0 & \text{on } \Gamma_{\text{out}}(0) \times (0,T), \\
\rho_f \text{det}(\nabla \tilde{\xi} + \text{Id}) \partial_t^2 \tilde{\xi}_i - \sum_{\alpha,j=1}^3 b_{i\alpha j\beta} (\nabla \tilde{\xi}) \partial^2_{\alpha j} \tilde{\xi}_j &= 0 & \text{in } \Omega_s(0) \times (0,T), \\
\tilde{\xi} &= 0 & \text{on } \Gamma_2(0) \times (0,T), \\
\tilde{v} &= \partial_t \tilde{\xi} & \text{on } \Gamma_3(0) \times (0,T), \\
\left[ \tilde{\sigma}^f_i (\tilde{v}, \tilde{p}_f) \tilde{n} \right]_i &= \sum_{\alpha,j=1}^3 \left( \int_0^l b_{i\alpha j\beta} (\nabla \tilde{\xi}) \partial^2_{\alpha j} \tilde{\xi}_j ds \right) \tilde{n}_\alpha, & \text{on } \Gamma_c(0) \times (0,T), \\
\tilde{v}(.,0) = v_0 & \text{and } \tilde{p}_f(.,0) = p_{f_0} & \text{in } \Omega_f(0), \\
\tilde{\xi}(.,0) = 0 & \text{and } \partial_t \tilde{\xi}(.,0) = \xi_1 & \text{in } \Omega_s(0).
\end{align*}
\]

(27)

for \(i = 1, 2, 3\).

Notice that, unlike System (12), in this system the boundary condition related to the elastodynamic equation is incompatible with it. Indeed, for Equations (27) and (27) to combine we must have

\[
\left[ \tilde{\sigma}^f_i (\tilde{v}, \tilde{p}_f) \tilde{n} \right]_i = \sum_{\alpha,j=1}^3 \left( \int_0^l b_{i\alpha j\beta} (\nabla \tilde{\xi}) \partial^2_{\alpha j} \tilde{\xi}_j ds \right) \tilde{n}_\alpha, & \text{on } \Gamma_c(0) \times (0,T).
\]

(28)

This rewriting (27) of the elasticity equation is efficient when performing the fixed point theorem on the system. In fact, it helps to get over the difficulties emerging from the non-linearity of the Saint Venant-Kirchhoff model and the hyperbolic type of the equation. Due to this disagreement issue between Equations (27) and (27), the first step of the work is to consider an auxiliary problem including the natural boundary condition (28).

By considering the boundary and initial conditions we assume that the following compatibility conditions hold on the initial values

\[
\begin{align*}
v_0 &= \xi_1 & \text{on } \Gamma_c(0), \\
\sigma_f(v_0, p_{f_0}) &= 0 & \text{on } \Gamma_c(0), \\
p_{f_0} &= 2\mu D(v_0) & \text{in } \Omega_f(0), \\
\nabla p_{f_0} &= \mu \Delta v_0 & \text{in } \Omega_f(0), \\
\nabla \cdot \sigma_f(v_0, p_{f_0}) &= 0 & \text{on } \Gamma_c(0), \\
\partial_t p_f |_{t=0} &= S_1 n + E_1 n & \text{on } \Gamma_c(0), \\
\nabla \cdot \rho_s \left( S_1 + \partial_t p_f |_{t=0} \text{Id} \right) &= \rho_f \nabla \cdot E_1 & \text{on } \Gamma_c(0), \\
\rho_f \left( 2(\nabla \cdot v_0) \nabla \cdot E_1 + \nabla \cdot E_2 \right) &= \nabla \cdot S_2 & \text{on } \Gamma_c(0), \\
\left( E_2 - 2((\nabla \cdot v_0)S_3 + S_4) \right) n &= \rho_s S_2 n & \text{on } \Gamma_c(0).
\end{align*}
\]

where
Let us define the following spaces

\[ S_1 = -\mu (D(v_0))^2 - 2(\nabla v_0) \cdot \nabla v_0 + \sigma_f(v_0, p_{f_0})S_3. \]

- \[ E_1 = 2\mu \varepsilon(\xi_1) + \lambda_s(\nabla \cdot \xi_1)Id + \nabla \cdot v_0 Id. \]
- \[ S_2 = \partial_t^2 p_f|_{t=0}Id + 2\partial_t p_f|_{t=0}S_3 + p_{f_0}S_4 + 2\mu \varepsilon(\nabla \cdot E_1) - 2(D(v_0))^2 - 2(\nabla v_0)^t \nabla v_0)S_3 + 2D(v_0)S_4. \]
- \[ E_2 = 2\nabla \xi_1 E_1 + 2\mu_s(\nabla \xi_1)^t \nabla \xi_1 + \lambda_s \nabla \xi_1 + 2((\nabla \cdot v_0)S_3 + S_4). \]
- \[ S_3 = (\nabla \cdot v_0)Id - (\nabla v_0)^t. \]

These conditions are obtained from (12) by considering \( t = 0 \), differentiating in time once and twice \((12)_1, (12)_5, (12)_7\) and \((12)_8\) then considering \( t = 0 \) and taking into consideration the following identities

\[ \partial_t((\nabla A)^{-1})(., 0) = -\nabla v_0 \quad \text{and} \quad \partial_t(\det(\nabla A))(., 0) = \nabla \cdot v_0 \quad \text{in} \quad \Omega_f(0). \]

**Definition 1.1.** Let us define the following spaces

\[ S_m^T = L^\infty(0, T; H^m(\Omega_s(0))) \cap W^{m, \infty}(0, T; L^2(\Omega_s(0))) \quad 0 \leq m \leq 4, \]

\[ F_1^T = L^\infty(0, T; H^2(\Omega_f(0))) \cap L^2(0, T; H^1(\Omega_f(0))), \]

\[ F_2^T = L^\infty(0, T; H^2(\Omega_f(0))) \cap H^1(0, T; H^1(\Omega_f(0))) \cap W^{1, \infty}(0, T; L^2(\Omega_f(0))), \]

\[ F_4^T = L^\infty(0, T; H^4(\Omega_f(0))) \cap H^3(0, T; H^1(\Omega_f(0))) \cap W^{2, \infty}(0, T; H^2(\Omega_f(0))) \cap W^{3, \infty}(0, T; L^2(\Omega_f(0))), \]

\[ P_3^T = L^\infty(0, T; H^3(\Omega_f(0))) \cap H^3(0, T; L^2(\Omega_f(0))) \cap W^{1, \infty}(0, T; H^2(\Omega_f(0))) \cap W^{2, \infty}(0, T; H^1(\Omega_f(0))), \]

\[ H_1^T(0, T; L^2(\Gamma_e(0))) := \{ \psi \in H^1(0, T; (\Gamma_e(0))); \psi(0) = 0 \}. \]

Then, for \( M > 1 \) and \( T > 0 \) we define the following fixed point space

\[ A^T_M = \{ (\tilde{\xi}, \tilde{\xi}_1) \in F_4^T \times S_1^T, \tilde{\xi}(., 0) = 0, \partial_t \tilde{\xi}(., 0) = \xi_1 \text{ in } \Omega_s(0) \text{ and } ||\tilde{\xi}||_{F_4^T} \leq M, ||\tilde{\xi}||_{S_1^T} \leq M \} := A_{M_1}^T \times A_{M_2}^T. \]

After introducing the spaces needed, we are ready to state the main result of the work.

**Theorem 1.1 (Main Theorem).** Let \((v_0, \xi_1, p_{f_0})\) satisfy \((4)\) and \((29)\). Then, there exists \( T > 0 \) such that System \((27)\) admits a unique solution defined on \((0, T)\) satisfying

\[ (v \circ \mathcal{A}, \xi \circ \varphi, p_f \circ \mathcal{A}) \in F_4^T \times S_1^T \times P_3^T \]

\[ \mathcal{A} \in W^{1, \infty}(0, T; H^2(\Omega_f(0))) \times W^{2, \infty}(0, T; L^2(\Omega_f(0))) \]

and

\[ \varphi \in S_2^T. \]
For simplicity, for all \( m, r \geq 0 \) and \( p, q \in [1, +\infty] \), we denote the spaces \( W^{m,p}(0, T; W^{r,q}(\Omega_f(0))) \) and \( W^{m,p}(0, T; W^{r,q}(\Omega_s(0))) \) by \( W^{m,p}(W^{r,q}(\Omega_f(0))) \) and \( W^{m,p}(W^{r,q}(\Omega_s(0))) \), respectively.

Also the domain’s notation is simplified by writing \( \Omega_f(0) = \Omega_0^f \), \( \Omega_s(0) = \Omega_0^s \) and \( \Omega(0) = \Omega_0 \). Further, For all \( t > 0 \), define

\[
\Sigma_t = \Gamma_\varepsilon(0) \times (0, t).
\]

2. A Partially Linear System

Let \((v_0, \xi_1, p_{f_0})\) satisfy (4) and (29). Let \( 0 < T < 1 \) and consider \((\bar{v}, \bar{\xi}) \in A_M^T\) to be given. For these given functions we define the associated fluid flow \( \bar{A} \) and structure deformation \( \bar{\varphi} \) by

\[
\bar{A}(\bar{x}, t) = \bar{x} + \int_0^t \bar{v}(\bar{x}, s) \, ds \quad \forall \bar{x} \in \Omega_0^f, \tag{33}
\]

and

\[
\bar{\varphi}(\bar{x}, t) = \bar{x} + \bar{\xi}(\bar{x}, t) \quad \forall \bar{x} \in \Omega_0^s. \tag{34}
\]

We use the given \((\bar{v}, \bar{\xi})\) to partially linearize the non-linear system. Indeed, we consider the non-linear terms to be given in terms of \((\bar{v}, \bar{\xi})\). Let \( T \leq 1/M^4 \) and \( M > 1 \). We shall repeatedly use the following two lemmas which provide bounds on various norms of the deformation maps \( \bar{A} \) and \( \bar{\varphi} \). We omit the proof, for, the bounds are obtained by simple calculations using the generalized Poincaré inequality [5, Proposition III.2.38], Grönwall inequality and the embedding theorems [6, Corollary 9.13].

**Lemma 2.1.** For the fluid flow \( \bar{A} \) given by (33) for a given \( \bar{v} \in A_M^T \), there exists a constant \( C = C(\Omega_0^f) > 0 \) and a constant \( \kappa > 0 \) such that

1. \( ||\bar{A}||_{W^{1,\infty}(H^4) \cap W^{3,\infty}(H^2) \cap W^{4,\infty}(L^2) \cap H^4(H^1)} \leq C(1 + M). \)

2. \( ||\nabla \bar{A} - \text{Id}||_{W^{1,\infty}(H^4) \cap W^{3,\infty}(H^2) \cap W^{4,\infty}(L^2)} \leq CM. \)

3. \( ||\nabla \bar{A}||_{L^{\infty}(H^3)} \leq C. \)

4. \( ||\text{cof}(\nabla \bar{A})||_{L^{\infty}(H^3)} \leq C. \)

5. \( ||\partial_t \text{cof}(\nabla \bar{A})(t)||_{L^2(H^3)} \leq CT^{1/2}M. \)

6. \( ||(\nabla \bar{A})^{-1}(t)||_{L^\infty} \leq C||\nabla \bar{A}(t)||_{L^\infty}^{1/\kappa} \quad \text{for } t \in [0, T]. \)

7. \( ||\text{cof}(\nabla \bar{A}) - \text{Id}||_{L^{\infty}(H^3)} + ||(\nabla \bar{A})^{-1} - \text{Id}||_{L^{\infty}(H^3)} \leq CT^\kappa M. \)

8. \( ||\partial_t (\nabla \bar{A})^{-1}(t)||_{L^r} \leq C||\nabla \bar{v}(t)||_{L^r}, \quad \text{for } r \in [1, +\infty] \text{ and } t \in [0, T]. \)

9. \( ||\text{det}(\nabla \bar{A})||_{L^{\infty}(H^3)} \leq CM \quad \text{and } ||\partial_t \text{det}(\nabla \bar{A})||_{L^{\infty}(H^3)} \leq CM. \)

10. \( ||\text{det}(\nabla \bar{A}) - 1||_{L^{\infty}(H^3)} \leq CT^\kappa M. \)
Remark 2.1. The last part of Lemma 2.1 gives
\[ ||\det(\nabla \tilde{\mathbf{A}}) - 1||_{L^\infty} \leq C T^n M. \]
That is, for all \( t \in (0, T) \) and \( \tilde{x} \in \Omega^f_0 \) we have
\[ -CT^n M \leq \det(\nabla \tilde{\mathbf{A}}(\tilde{x}, t)) - 1 \leq CT^n M. \]
This gives
\[ \det(\nabla \tilde{\mathbf{A}}(\tilde{x}, t)) \geq 1 - CT^n M \quad \forall (\tilde{x}, t) \in \Omega^f_0 \times (0, T). \]

Remark 2.2. For \( 0 < n \leq 4 \), the quantity \( CT M^n \) can be approximated by \( CT^n M \), with \( \kappa > 0 \). Indeed, as \( TM^4 < 1 \), then we can find \( \kappa > 0 \) such that \( TM^4 \leq T^n \).

Lemma 2.2. Let \( M > 1 \), \( T > 0 \) and \( \tilde{\xi} \in A^T_{M^2} \) be given. There exists \( C > 0 \) such that for all \( i, \alpha, j, \beta \in \{1, 2, 3\} \), we have:

1. \[ ||c_{i\alpha j\beta}^l (\nabla \tilde{\phi}) + c_{i\alpha j\beta}^q (\nabla \tilde{\phi})|| \leq C(M + M^2) \] (35)
   where \( c_{i\alpha j\beta}^l \) and \( c_{i\alpha j\beta}^q \) are defined by the expressions (15) and (16) respectively.

2. For any matrix \( \mathbf{A} \in M_3(\mathbb{R}) \), we have
   \[ \sum_{i, \alpha, j, \beta = 1}^3 c_{i\alpha j\beta}^l (\nabla \tilde{\phi}) \mathbf{A}_{j\beta} \mathbf{A}_{i\alpha} \geq \frac{\mu}{2} |\mathbf{A} + \mathbf{A}^T|^2 - \lambda_\alpha |\text{tr}(\mathbf{A})|^2 - CT(M + M^2)|\mathbf{A}|^2. \] (36)

3. \[ ||d_{i\alpha j\beta}^l (\nabla \tilde{\phi}) + d_{i\alpha j\beta}^q (\nabla \tilde{\phi}) + d_{i\alpha j\beta}^T (\nabla \tilde{\phi}) + d_{i\alpha j\beta}^F (\nabla \tilde{\phi})|| \leq C(M + M^2 + M^3 + M^4). \] (37)

4. For any matrix \( \mathbf{A} \in M_3(\mathbb{R}) \) we have
   \[ \sum_{i, \alpha, j, \beta = 1}^3 d_{i\alpha j\beta}^l (\nabla \tilde{\phi}) \mathbf{A}_{j\beta} \mathbf{A}_{i\alpha} \geq \text{tr}(\mathbf{A})|^2 - CT(M + M^2 + M^3 + M^4)|\mathbf{A}|^2. \] (38)

5. \[ ||\nabla \tilde{\phi}||_{L^\infty(\Omega^f_0)} \leq C. \] (39)

6.
   \[ ||\text{cof}(\nabla \tilde{\phi})||_{L^\infty(\Omega^f_0)} \leq C \quad \text{and} \quad ||\text{cof}(\nabla \tilde{\phi})||_{L^2(\Omega^f_0)} \leq CT^{1/2}. \] (40)

7. We have
   \[ ||\text{det}(\nabla \tilde{\phi})||_{L^\infty(\Omega^f_0)} \leq C \quad \text{and} \quad ||\partial_t \text{det}(\nabla \tilde{\phi})||_{L^\infty(\Omega^f_0)} \leq CM. \] (41)
\[ \| (\nabla \varphi)^{-1} \|_{L^\infty(H^2(\Omega_0^c))} \leq C. \]  

(42)

9- For \( \xi \in A_M^T \) we have

\[ \| \det(\nabla \varphi) - 1 \|_{L^\infty(H^2(\Omega_0^c))} \leq CT. \]  

(43)

The main step to establish the local in time existence and uniqueness of solution of the coupled problem is to partially linearize it. This is achieved by considering the non-linear terms to be given, thus the flow map and deformation are given by (33) and (34) respectively. For the given \( \tilde{A}, \tilde{\varphi} \) and \( (\tilde{\mathbf{v}}, \tilde{\xi}) \in A_M^T \) we denote \( \tilde{b}_{\alpha j \beta} \tilde{\varphi} \) by \( \tilde{b}_{\alpha j \beta} \) and the fluid shear stress is denoted by \( \tilde{\sigma}_{f}^0(\tilde{\mathbf{v}}, \tilde{p}_f) \) when considering \( \tilde{A} \) in the expression (8). Now we write the system (27) in the reference configuration at time \( t = 0 \). Equation (27)\(_1\) is replaced by

\[ \rho_f \det(\nabla \tilde{A}) \partial_t \tilde{\mathbf{v}} - \nabla \cdot \tilde{\sigma}_{f}^0(\tilde{\mathbf{v}}, \tilde{p}_f) = 0 \quad \text{in } \Omega_0^f \times (0, T) \]

and Equation (27)\(_5\) is replaced by

\[ \rho_s \det(\nabla \varphi) \partial_t \tilde{\xi}_i - \sum_{\alpha, \beta = 1}^{3} \tilde{b}_{\alpha j \beta} \partial_{\alpha \beta}^2 \tilde{\xi}_j = 0, \quad i = 1, 2, 3 \quad \text{in } \Omega_0^s \times (0, T). \]

The coupling conditions on \( \Sigma_T \) are given by

\[
\begin{aligned}
\tilde{\mathbf{v}} &= \partial_t \tilde{\xi}, \\
\tilde{\sigma}_{f}^0(\tilde{\mathbf{v}}, \tilde{p}_f) \hat{n}_i &= \sum_{\alpha, \beta = 1}^{3} \left( \int_0^t \tilde{b}_{\alpha j \beta} \partial_{\alpha \beta}^2 \tilde{\xi}_j ds \right) \hat{n}_\alpha 
\end{aligned}
\]  

for \( i = 1, 2, 3. \)  

(44)

For \( (\tilde{\mathbf{v}}, \tilde{\xi}) \) being given in \( A_M^T \), we introduce the following mapping

\[ \Psi : (\tilde{\mathbf{v}}, \tilde{\xi}) \longrightarrow (\tilde{\mathbf{v}}, \tilde{\xi}) \]

where \( (\tilde{\mathbf{v}}, \tilde{\xi}) \) together with \( \tilde{p}_f \) form the solution of the partially linearized system.

First, we start by defining an auxiliary problem that considers the boundary condition (28). Choosing a suitable functional space we write the variational formulation where the pressure term disappears. Uniqueness and existence of solution of the auxiliary problem are established in the next section.

3. An Auxiliary Problem

As we mentioned before, there is a disagreement between the elasticity equation and the stress coupling condition on \( \Sigma_T \) attributed to it. Thus, we set up an auxiliary problem in which the natural boundary condition (28) is used. This problem constitutes the first tool in establishing the existence and uniqueness of the strong solution of the FSI problem. We start by introducing the auxiliary problem. Let \( g = [g_1, g_2, g_3]^T \) be a function in \( H^1(\Omega_0^c) \), \( L^2(\Omega_0^C(0)) \), \( \mathbb{R}^3 \).
and consider the following system:

\[
\begin{align*}
\rho_f \det(\nabla \vec{A}) \partial_t \vec{v} - \nabla \cdot \sigma^0_f(\vec{v}, \vec{p}_f) &= 0 & \text{in} & \Omega_0 \times (0, T), \\
\nabla \cdot (\det(\nabla \vec{A})(\nabla \vec{A})^{-1}) \vec{v} &= 0 & \text{in} & \Omega_0 \times (0, T), \\
\vec{v} &= v_{in} \circ \vec{A} & \text{on} & \Gamma_{in}(0) \times (0, T), \\
\vec{p}_f(\vec{v}, \vec{p}_f) \vec{n} &= 0 & \text{on} & \Gamma_{out}(0) \times (0, T), \\
\rho_s \det(\nabla \vec{A}) \partial_t^2 \vec{\xi} - \sum_{\alpha, j, \beta = 1}^3 \tilde{b}_{i\alpha j\beta} \partial_{\alpha}^2 \vec{\xi}_j &= 0 & \text{in} & \Omega_0 \times (0, T), \\
\vec{\xi} &= 0 & \text{on} & \Gamma_2(0) \times (0, T), \\
\vec{v}_0 &\partial_t \vec{\xi}_i & \text{on} & \Gamma_c(0) \times (0, T), \\
\left[ \sigma_f^0(\vec{v}, \vec{p}_f) \vec{n} \right]_i &\sum_{\alpha, j, \beta = 1}^3 \left( \tilde{b}_{i\alpha j\beta} \partial_{\alpha} \vec{\xi}_j \right) \vec{n}_\alpha + g_i & \text{in} & \Gamma_c(0) \times (0, T), \\
\vec{v}(\cdot, 0) &= \vec{v}_0, \quad \text{and} \quad \vec{p}_f(\cdot, 0) = p_{f_0} & \text{in} & \Omega_0, \\
\vec{\xi}(\cdot, 0) &= 0 \quad \text{and} \quad \partial_t \vec{\xi}(\cdot, 0) = \vec{\xi}_1 & \text{in} & \Omega_0.
\end{align*}
\]

(45)

The following lemma states the existence and uniqueness of solution for the auxiliary problem.

**Lemma 3.1.** Let \((\vec{v}, \vec{\xi}) \in A_T^{\| \cdot \|}, v_0 \in L^2(\Omega_0^T), \vec{\xi}_1 \in L^2(\Omega_0^T)\) and \(p_{f_0} \in L^2(\Omega_0^T)\). For \(T\) small with respect to \(M\) and the initial conditions, there exists a unique weak solution \((\vec{v}, \vec{\xi}) \in F_1^T \times S_1^T\) of (45). In addition, this solution satisfies the following a priori estimate

\[
||\vec{v}||_{L^2}^2 + ||\vec{\xi}||_{L^2}^2 \leq C \left[ \frac{\rho_f}{2} ||v_0||_{L^2(\Omega_0^T)}^2 + \frac{\rho_s}{2} ||\vec{\xi}_1||_{L^2(\Omega_0^T)}^2 + ||g||_{H^1(\Gamma_c(0)))}^2 \right].
\]

(46)

**Remark 3.1.** Taking \(T\) small with respect to \(M\) and the initial conditions, means that there exists \(n_0 > 0\) and \(\varepsilon\) positive such that

\[
T \leq \left\{ \varepsilon \left( \frac{M^{n_0}}{h(||v_0||_{H^2(\Omega_0^T)} \cdot ||\vec{\xi}_1||_{H^2(\Omega_0^T)} \cdot ||p_{f_0}||_{L^2(\Omega_0^T)} \cdot ||g||_{H^1(\Gamma_c(0)))}} \right) \right\}.
\]

From here on, we simplify the notation for all the norms by omitting the indication for the domain as it is always clear from the context. For instance, we write \(||\vec{v}||_{L^2} = ||\vec{v}||_{L^2(\Omega_0^T)}\) and \(||\vec{\xi}||_{L^2} = ||\vec{\xi}||_{L^2(\Omega_0^T)}\).

In order to prove Lemma 3.1, we proceed as follows. First, we write the variational formulation corresponding to the coupled system using a divergence-free functional space. Then, we use a Faedo-Galerkin approach to find an approximation of the solution, which enables us to find some a priori estimates on the Galerkin sequences. Using the estimates and compactness results we prove the existence and uniqueness of the solution.

### 3.1. Variational Formulation

Consider the following divergence-free functional space

\[
\vec{W} = \left\{ \vec{\eta} \in H^1(\Omega_0) | \nabla \cdot (\det(\nabla \vec{A})(\nabla \vec{A})^{-1}) \vec{\eta} = 0 \text{ on } \Omega_0 \text{ and } \vec{\eta} = 0 \text{ on } \Omega_0 \setminus \tilde{\Gamma}_{out}(0) \right\}.
\]

Let \([., .]\) denote the weighted \(L^2\) inner product defined by

\[
[\vec{\gamma}, \vec{\eta}] = \int_{\Omega_0} \rho_f \vec{\gamma} \cdot \vec{\eta} \, d\vec{x} + \int_{\Omega_0} \rho_s \vec{\gamma} \cdot \vec{\eta} \, d\vec{x} \quad \forall \, \vec{\gamma}, \vec{\eta} \in \vec{W}.
\]
This norm is equivalent to the norm $|| \cdot ||_{L^2(\Omega_0)}$.

In order to derive the variational formulation of (45), we multiply Equations (45) and (45) by a test function $\tilde{\eta} \in \widetilde{W}$, integrate by parts and take into consideration the boundary and the coupling conditions to get

$$
\begin{aligned}
\rho_f \int_{\Omega_0^f} \det(\nabla \mathbf{A}) \partial_t \tilde{v} \cdot \tilde{\eta} \, d\tilde{x} + \int_{\Omega_0^f} \tilde{\sigma}^0(\tilde{v}) : \nabla \tilde{\eta} \, d\tilde{x} + \rho_s \int_{\Omega_0^s} \det(\nabla \phi) \partial_t \tilde{\xi} \cdot \tilde{\eta} \, d\tilde{x} \\
+ \sum_{i,\alpha,j,\beta=1}^3 \int_{\Omega_0^f} b_{i\alpha j\beta} \partial_\beta \tilde{\xi}_j \partial_\alpha \tilde{\eta}_i \, d\tilde{x} + \sum_{i,\alpha,j,\beta=1}^3 \int_{\Omega_0^s} \partial_\alpha b_{i\alpha j\beta} \partial_\beta \tilde{\xi}_j \tilde{\eta}_i \, d\tilde{x} = \int_{\Gamma_c(0)} g \cdot \tilde{\eta} \, d\tilde{\Gamma} \quad \forall \, \tilde{\eta} \in \widetilde{W}.
\end{aligned}
$$

(47)

Note that, the space $\widetilde{W}$ is the transformation of the space

$$
\mathcal{W} = \left\{ \eta \in H^1(\Omega(t)) \mid \nabla \cdot \eta = 0 \text{ on } \Omega_f(t) \text{ and } \eta = 0 \text{ on } \partial \Omega(t) \setminus \Gamma_{out}(t) \right\}.
$$

This explains the disappearance of the pressure term $\tilde{p}_f$ from the weak formulation.

**Remark 3.2.** ": this corresponds to the Hadamard product of matrices defined by

$$
A : B = \sum_{i,j=1}^n A_{i,j} B_{i,j}, \text{ for } A, B \in \mathbb{M}_n(\mathbb{R}).
$$

In order to derive the weak formulation we consider a global test function $\tilde{\eta}$ in $\widetilde{W}$. This will simplify the work. In fact, rather than looking for two solutions using two independent test functions on each sub-domain, we search for one solution $\tilde{\gamma}$ over the domain $\Omega_0$. By considering a global test function we are able to embed the stress condition into the formulation in such a way that it would cancel out on the entire domain. Further, we will guarantee the existence of a weak solution $\tilde{\gamma}$ in $\widetilde{W}$. Consequently, $\tilde{v}$ and $\tilde{\xi}$ are considered to be the restriction of $\tilde{\gamma}$ on the sub-domains $\Omega_0^f$ and $\Omega_0^s$, respectively. Note that, if we consider the restriction of $\tilde{\eta}$ on the two sub-domains $\Omega_0^f$ and $\Omega_0^s$, we cannot guarantee the existence of the weak solutions in the restriction of $\widetilde{W}$ on each sub-domain. Thus, we introduce the auxiliary function $\tilde{\gamma}$ defined by

$$
\tilde{\gamma} = \begin{cases} 
\tilde{v} & \text{in } \Omega_0^f, \\
\partial_t \tilde{\xi} & \text{in } \Omega_0^s,
\end{cases} \quad \text{and} \quad \tilde{\gamma}_0 = \begin{cases} 
\nu_0 & \text{in } \Omega_0^f, \\
\xi_1 & \text{in } \Omega_0^s,
\end{cases}
$$

(48)

which is a continuous function on $\Omega_0$, due to the continuity of velocities across the interface $\Gamma_c(0)$ which is given by the condition (45). By this definition, we can write $\tilde{v}(t) = \tilde{\gamma}(t)$ on $\Omega_0^f$, and $\tilde{\xi}(t) = \int_0^t \tilde{\gamma}(s) \, ds$ on $\Omega_0^s$, based on the fact that $\tilde{\xi}(0) = \xi_0 = 0$. Then, for all test functions $\tilde{\eta}$ in $\widetilde{W}$, the weak formulation (47) is equivalent to

$$
\begin{aligned}
\rho_f \int_{\Omega_0^f} \det(\nabla \mathbf{A}) \partial_t \tilde{v} \cdot \tilde{\eta} \, d\tilde{x} + \int_{\Omega_0^f} \tilde{\sigma}^0(\tilde{v}) : \nabla \tilde{\eta} \, d\tilde{x} \\
+ \int_{\Omega_0^f} \tilde{\sigma}^0(\tilde{\gamma}) : \nabla \tilde{\eta} \, d\tilde{x} + \sum_{i,\alpha,j,\beta=1}^3 \int_{\Omega_0^f} b_{i\alpha j\beta} \partial_\beta \tilde{\xi}_j \partial_\alpha \tilde{\eta}_i \, d\tilde{x} \\
+ \sum_{i,\alpha,j,\beta=1}^3 \int_{\Omega_0^s} \partial_\alpha b_{i\alpha j\beta} \partial_\beta \tilde{\xi}_j \tilde{\eta}_i \, d\tilde{x} = \int_{\Gamma_c(0)} g \cdot \tilde{\eta} \, d\tilde{\Gamma},
\end{aligned}
$$

(49)

$$
\tilde{\gamma}(0) = \tilde{\gamma}_0, \quad \int_0^t \left( \tilde{\gamma}(s) \big|_{\Omega_0^f} \right)_{\Gamma_c(0)} \, ds = \int_0^t \left( \tilde{\gamma}(s) \big|_{\Omega_0^s} \right)_{\Gamma_c(0)} \, ds, \quad \forall \, t \in [0,T].
$$
3.2. Galerkin Approximation

In order to show that the system admits a unique solution we will use a Faedo-Galerkin approach. Let \( \{ \psi_l \}_{l=1}^n \) be a basis of \( \widetilde{W} \) in \( L^2(\Omega_0) \) which is orthogonal for the \( H^1 \)-Norm and orthonormal for the \( L^2 \)-Norm.

Take \( \widetilde{W}_n = \operatorname{span}\{\psi_1, \ldots, \psi_n\} \). We seek to find a Galerkin approximation \( \{ \tilde{\chi}_n \}_{n=1}^\infty \in C^1(0, T; \widetilde{W}_n) \) of the form

\[
\tilde{\chi}_n = \sum_{l=1}^n f_l^n(t) \psi_l(\tilde{x})
\]

satisfying

\[
\begin{aligned}
\rho_f \int_{\Omega_0} \det(\nabla \tilde{A}) \partial_l \tilde{\chi}_n \cdot \tilde{n}_n \ d\tilde{x} + \rho_s \int_{\Omega_0} \det(\nabla \tilde{\varphi}) \partial_l \tilde{\varphi} \cdot \tilde{n}_n \ d\tilde{x} \\
+ \int_{\Omega_0} \tilde{\sigma}_l^0(\tilde{\chi}_n) : \nabla \tilde{n}_n \ d\tilde{x} + \sum_{i, \alpha, j, \beta = 1}^3 \int_{\Omega_0} \bar{b}_{i\alpha j\beta} \partial_\beta (\int_0^t \tilde{\chi}_n(s) ds)_j \partial_\alpha \tilde{n}_n,i \ d\tilde{x} \\
+ \sum_{i, \alpha, j, \beta = 1}^3 \int_{\Omega_0} \partial_\alpha \bar{b}_{i\alpha j\beta} \partial_\beta (\int_0^t \tilde{\chi}_n(s) ds)_j \tilde{n}_n,i \ d\tilde{x} = \int_{\Gamma_c(0)} g \cdot \tilde{n}_n \ d\tilde{\Gamma}, \quad \forall \tilde{n}_n \in \widetilde{W}_n,
\end{aligned}
\]

and

\[
[[\tilde{\chi}_n(0), \tilde{n}_n]] = [[\tilde{\chi}_0, \tilde{n}_n]], \quad \forall \tilde{n}_n \in \widetilde{W}_n.
\]

Notice that, trivially \( \tilde{\chi}_n \) defined in (50) satisfies

\[
\int_0^t \left( \tilde{\chi}_n(s) |_{\Omega_0} \right) ds = \int_0^t \left( \tilde{\chi}_n(s) |_{\Omega_0} \right) ds, \quad \forall t \in [0, T].
\]

We can write (51), (52) as an equivalent system of first-order, linear ordinary differential equation (ODE) for \( \{ f_l^n \}_{l=1}^n \).

Set \( h_l^n(t) = \int_0^t f_l^n(s) ds \) for \( l = 1, \cdots, n \). For \( 1 \leq k \leq n \), the problem (51), (52) is equivalent to the following ODE initial value problem

\[
\begin{aligned}
\sum_{l=1}^n \frac{d}{dt} f_l^n(t) \left[ \rho_f \int_{\Omega_0} \det(\nabla \tilde{A}) \psi_l \cdot \psi_k \ d\tilde{x} + \rho_s \int_{\Omega_0} \det(\nabla \tilde{\varphi}) \psi_l \cdot \psi_k \ d\tilde{x} \\
+ \int_{\Omega_0} \tilde{\sigma}_l^0(\psi_l) : \nabla \psi_k \ d\tilde{x} + \sum_{i, \alpha, j, \beta = 1}^3 \int_{\Omega_0} \bar{b}_{i\alpha j\beta} \partial_\beta (\int_0^t \psi_l(s) ds)_j \partial_\alpha \psi_k,i \ d\tilde{x} \\
+ \sum_{i, \alpha, j, \beta = 1}^3 \int_{\Omega_0} \partial_\alpha \bar{b}_{i\alpha j\beta} \partial_\beta (\int_0^t \psi_l(s) ds)_j \psi_k,i \ d\tilde{x} \right] \left[ D_{i,\alpha,j,\beta} \right] k \\
= \int_{\Gamma_c(0)} g \cdot \psi_k \ d\tilde{\Gamma}, \\
\frac{d}{dt} h_l^n(t) = f_l^n(t) \quad \forall 1 \leq l \leq n, \\
\sum_{l=1}^n [\psi_l, \psi_k] f_l^n(0) = [[\tilde{\chi}_0, \psi_k]], \\
h_l^n(0) = 0 \quad \forall 1 \leq l \leq n.
\end{aligned}
\]
System \([54]\) can be rewritten in the following matrix form

\[
\begin{bmatrix}
\begin{bmatrix}
[\psi_l, \psi_k]_{l,k=1}^n \\
0_n
\end{bmatrix}
& 0_n \\
0_n & I_n
\end{bmatrix}
\begin{bmatrix}
d\frac{d}{dt} \gamma_n \\
I_n
\end{bmatrix}
= \begin{bmatrix}
f^n_1(t) \\
\vdots \\
f^n_{n}(t)
\end{bmatrix}
\begin{bmatrix}
S_n \\
0_n
\end{bmatrix}
+ \begin{bmatrix}
h^n_1(t) \\
\vdots \\
h^n_{n}(t)
\end{bmatrix}
\begin{bmatrix}
D_n \\
0_n
\end{bmatrix}
+ \begin{bmatrix}
\int_{\Gamma_\epsilon(0)} g \cdot \psi_k \, d\Gamma
\end{bmatrix}_{k=1}^n
\]

\((55)\)

with

\[
[\psi_l, \psi_k]_{l,k=1}^n = \left[ \rho_f \int_{\Omega_0} \det(\nabla \tilde{\mathbf{A}}) \psi_l \cdot \psi_k \, d\tilde{x} + \rho_s \int_{\Omega_0} \det(\nabla \tilde{\varphi}) \psi_l \cdot \psi_k \, d\tilde{x} \right]_{l,k=1}^n,
\]

\[
S_n = \left[ \int_{\Omega_0} \sigma^0(\psi_l) : \nabla \psi_k \, d\tilde{x} \right]_{l,k=1}^n \quad \text{and} \quad [D_n]_{l,k=1}^n = [D_{l,\alpha,\beta}]_{l,k=1}^n.
\]

The matrix \(A\) is a positive definite matrix as the function set \([\psi_l]_{l=1}^n\) is linearly independent. Moreover, \(A\) is bounded on \((0,T)\). Further, matrices \(B\) and \(C\) are bounded on \((0,T)\). Hence by theory for systems of linear first order ODEs, we get that system \([54]\) admits a unique \(C^1\)-solution \([f^n_1, \ldots, f^n_n, h^n_1, \ldots, h^n_n]\) which yields the existence of a unique Galerkin approximation \([\tilde{\gamma}_n]_n\) of \([51]-[52]\) such that \(\tilde{\gamma}_n \in W^{1,\infty}(0,T; H^1(\Omega_0))\).

Now we proceed to derive a priori estimates on \(\tilde{\gamma}_n\).

3.3. A Priori Estimates

**Step 1: Estimates on \(\tilde{\gamma}_n\)**

We aim to find some estimates on \(\tilde{\gamma}_n\). For this sake we set \(\tilde{\gamma}_n = \tilde{\gamma}_n\) in \([51]\) to get

\[
\rho_f \int_{\Omega_0} \det(\nabla \tilde{\mathbf{A}}) \partial_t \tilde{\gamma}_n \cdot \tilde{\gamma}_n \, d\tilde{x} + \int_{\Omega_0} \sigma^0(\tilde{\gamma}_n) : \nabla \tilde{\gamma}_n \, d\tilde{x} + \rho_s \int_{\Omega_0} \det(\nabla \tilde{\varphi}) \partial_t \tilde{\gamma}_n \cdot \tilde{\gamma}_n \, d\tilde{x}
\]

\[
+ \sum_{i,\alpha,\beta=1}^3 \int_{\Omega_0} \tilde{b}_{\alpha\beta} \partial_j \left( \int_0^t \tilde{\gamma}_n(s) \, ds \right) \partial_i \tilde{\gamma}_n(s) \, d\tilde{x}
\]

\[
+ \sum_{i,\alpha,\beta=1}^3 \int_{\Omega_0} \partial_i \tilde{b}_{\alpha\beta} \partial_j \left( \int_0^t \tilde{\gamma}_n(s) \, ds \right) \tilde{\gamma}_n(s) \, d\tilde{x} = \int_{\Gamma_\epsilon(0)} g \cdot \tilde{\gamma}_n \, d\Gamma.
\]

\((56)\)
Then, integrating over \((0, t)\) and applying integration by parts yield
\[
\begin{align*}
\frac{\partial t}{2} \int_{\Omega_0^t} \det(\nabla \mathbf{A})(t) |\mathbf{\tilde{a}}_n(t)|^2 \, d\bar{x} + \frac{\partial s}{2} \int_{\Omega_0^s} \det(\nabla \mathbf{\tilde{A}})(t) |\mathbf{\tilde{a}}_n(t)|^2 \, d\bar{x} \\
+ \int_0^t \int_{\Omega_0^t} \frac{\mu}{2} \det(\nabla \mathbf{A}) |\nabla \mathbf{\tilde{a}}_n(\nabla \mathbf{A})^{-1} + (\nabla \mathbf{\tilde{a}}_n)^{-1} (\nabla \mathbf{\tilde{a}}_n)^{t} |^2 \, d\bar{x} \, ds \\
+ \frac{1}{2} \sum_{i,\alpha, j, \beta = 1}^3 \int_{\Omega_0^t} \bar{b}_{\alpha j \beta}(t) \partial_i (\int_0^s \mathbf{\tilde{a}}_n(s) \, ds)_j \partial_{\alpha} (\int_0^s \mathbf{\tilde{a}}_n(s) \, ds)_i \, d\bar{x} \\
- \frac{\rho_s}{2} \int_0^t \int_{\Omega_0^s} \partial_t \det(\nabla \mathbf{\tilde{A}})|\mathbf{\tilde{a}}_n|^2 \, d\bar{x} \, ds \\
- \frac{1}{2} \sum_{i,\alpha, j, \beta = 1}^3 \int_0^t \int_{\Omega_0^t} \partial_i \bar{b}_{\alpha j \beta}(t) \partial_j (\int_0^s \mathbf{\tilde{a}}_n(s) \, ds)_j \partial_{\alpha} (\int_0^s \mathbf{\tilde{a}}_n(s) \, ds)_i \, d\bar{x} \, ds \\
- \frac{\rho_t}{2} \int_0^t \int_{\Omega_0^t} \partial_t \det(\nabla \mathbf{A})|\mathbf{\tilde{a}}_n|^2 \, d\bar{x} \, ds \\
+ \sum_{i,\alpha, j, \beta = 1}^3 \int_0^t \int_{\Omega_0^t} \partial_i \bar{b}_{\alpha j \beta}(t) \partial_j (\int_0^s \mathbf{\tilde{a}}_n(s) \, ds)_j \partial_{\alpha} (\int_0^s \mathbf{\tilde{a}}_n(s) \, ds)_i \, d\bar{x} \, ds \\
= \int_0^t \int_{\Gamma_{\varnothing}(0)} \mathbf{g} \cdot \mathbf{\tilde{a}}_n \, d\Gamma \, ds + \frac{\partial t}{2} \int_{\Omega_0^t} |\mathbf{\tilde{a}}_n(0)|^2 \, d\bar{x} + \frac{\partial s}{2} \int_{\Omega_0^s} |\mathbf{\tilde{a}}_n(0)|^2 \, d\bar{x}.
\end{align*}
\]

(57)

We start by deriving estimates on the terms of (57).

First of all, as \(\det(\nabla \mathbf{A}) - 1 \geq -\text{CT}^n M\), then we have
\[
\begin{align*}
\frac{\partial t}{2} \int_{\Omega_0^t} \det(\nabla \mathbf{A}) |\mathbf{\tilde{a}}_n(t)|^2 \, d\bar{x} - \frac{\partial t}{2} \int_{\Omega_0^t} \partial_t \det(\nabla \mathbf{A}) |\mathbf{\tilde{a}}_n| \, d\bar{x} \, ds \\
\geq \frac{\partial t}{2} (1 - \text{CT}^n M) ||\mathbf{\tilde{a}}_n||_{L^\infty(L^2)}^2 - \frac{\partial t}{2} ||\partial_t \det(\nabla \mathbf{A})||_{L^2(H^2)} ||\mathbf{\tilde{a}}_n||_{L^\infty(L^2)}^2 \\
\geq \frac{\partial t}{2} (1 - \text{CT}^n M) ||\mathbf{\tilde{a}}_n||_{L^\infty(L^2(\Omega_0^t))}^2.
\end{align*}
\]

(58)

The fluid stress term is decomposed as follows
\[
\int_0^t \int_{\Omega_0^t} \frac{\mu}{2} |\nabla \mathbf{\tilde{a}}_n(\nabla \mathbf{A})^{-1} + (\nabla \mathbf{A})^{-1} (\nabla \mathbf{\tilde{a}}_n)^{t} |^2 \, d\bar{x} \, ds = N_1 + N_2.
\]

For \(N_1\) we use Korn’s inequality and Lemma [2,1] then there exists \(C_k > 0\) such that
\[
N_1 = \int_0^t \int_{\Omega_0^t} \frac{\mu}{2} |\nabla \mathbf{\tilde{a}}_n(\nabla \mathbf{A})^{-1} + (\nabla \mathbf{A})^{-1} (\nabla \mathbf{\tilde{a}}_n)^{t} |^2 \, d\bar{x} \, ds \geq \int_0^t \int_{\Omega_0^t} \mu |\epsilon(\tilde{a}_n)|^2 - \mu |\nabla \mathbf{\tilde{a}}_n ((\nabla \mathbf{A})^{-1} - 1\mathbf{d})|^2 \, d\bar{x} \, ds.
\]

\[
\geq \mu (C_k - \text{CT}^n M) ||\tilde{a}_n||_{L^2(H^1(\Omega_0^t))}^2.
\]

Similarly for \(S_2\) we use Lemma [2,1] which yields
\[
N_2 = \int_0^t \int_{\Omega_0^t} \frac{\mu}{2} |\det(\nabla \mathbf{\tilde{A}}) - 1| |\nabla \mathbf{\tilde{a}}_n(\nabla \mathbf{A})^{-1} + (\nabla \mathbf{A})^{-1} (\nabla \mathbf{\tilde{a}}_n)^{t} |^2 \, d\bar{x} \, ds \geq -||\det(\nabla \mathbf{\tilde{A}}) - 1||_{L^\infty(H^3)} ||\tilde{a}_n||_{L^2(H^1)}^2 ||(\nabla \mathbf{\tilde{a}}_n)^{-1}||_{L^\infty(H^3)}^2
\]

\[
\geq \mu \text{CT}^n M ||\tilde{a}_n||_{L^2(H^1(\Omega_0^t))}^2.
\]

Therefore,
\[
N_1 + N_2 \geq \mu (C_k - \text{CT}^n M) ||\tilde{a}_n||_{L^2(H^1(\Omega_0^t))}^2.
\]

(59)

As for the integrals on the domain \(\Omega_0^t\), first of all we have
\[
\int_{\Omega_0^t} \det(\nabla \mathbf{\tilde{A}}) |\mathbf{\tilde{a}}_n(t)|^2 \, d\bar{x} \geq (1 - \text{CT}^n M) ||\tilde{a}_n||_{L^\infty(L^2(\Omega_0^t))}^2.
\]

(60)
Thanks to \((61)\) it holds
\[
\int_0^T \int_{\Omega_0} \left| \partial_t \det(\nabla \varphi) \tilde{\gamma}_n(t) \right| \, d\tilde{x} \, ds \leq T \left| \partial_t \det(\nabla \varphi) \right|_{L^\infty(L^2(\Omega_0^c))} \|\tilde{\gamma}_n\|_{L^\infty(L^2(\Omega_0^c))}^2 \leq CT^n M \|\tilde{\gamma}_n\|_{L^\infty(L^2(\Omega_0^c))}^2.
\]

Using \((36)\) and \((38)\) together with Korn’s Inequality give
\[
\frac{1}{2} \sum_{i,\alpha,\beta=1}^3 \int_{\Omega_0^c} \tilde{b}_{\alpha i \beta} \partial_i \left( \int_0^T \tilde{\gamma}_n(s) \, ds \right) \partial_{\alpha \beta} \left( \int_0^T \tilde{\gamma}_n(s) \, ds \right) \, d\tilde{x} \, ds \leq \mu_c C_k \|f_0^T \tilde{\gamma}_n(s)\|_{H^1(\Omega_0^c)}^2 + \frac{C + \lambda_s}{2} \left| \nabla \cdot (f_0^T \tilde{\gamma}_n(s)) \right|_{L^2(\Omega_0^c)}^2
\]
\[\quad - CT^n M \|f_0^T \tilde{\gamma}_n(s)\|_{H^1(\Omega_0^c)}^2.\]

On the other hand, using \((35)\) and \((37)\) in addition to Young’s inequality \((4)\) Proposition II.2.16] and the Sobolev embeddings yield
\[
\left| \frac{1}{2} \sum_{i,\alpha,\beta=1}^3 \int_{\Omega_0^c} \partial_s \tilde{b}_{\alpha i \beta} \partial_\beta \left( \int_0^T \tilde{\gamma}_n(s) \, ds \right) \partial_i \left( \int_0^T \tilde{\gamma}_n(s) \, ds \right) \, d\tilde{x} \, ds \right|
\]
\[\quad + \frac{1}{2} \sum_{i,\alpha,\beta=1}^3 \int_{\Omega_0^c} \partial_s \tilde{b}_{\alpha i \beta} \partial_\beta \left( \int_0^T \tilde{\gamma}_n(s) \, ds \right) \partial_i \left( \int_0^T \tilde{\gamma}_n(s) \, ds \right) \, d\tilde{x} \, ds \leq CT^n M \left[ \|f_0^T \tilde{\gamma}_n(s)\|_{L^\infty(H^1(\Omega_0^c))}^2 + \|\tilde{\gamma}_n\|_{L^\infty(L^2(\Omega_0^c))}^2 \right].
\]

Finally, applying integration by parts then using the trace inequality \([3, \text{ Theorem III.2.19.]}\] and Young’s inequality we get
\[
\left| \int_0^T \int_{\Gamma_c(0)} \mathbf{g} \cdot \tilde{\gamma}_n \, d\Gamma \, ds \right| \leq C \delta T \|f_0^T \tilde{\gamma}_n(s)\|_{L^\infty(H^1(\Omega_0^c))}^2 + C_4 \|\mathbf{g}\|_{H^1(L^2(\Gamma_c(0))))}^2.
\]

Where we have used Hölder’s \([3, \text{ Proposition II.2.18}]\) inequality with the fact that \(\mathbf{g}(., 0) = 0\) which gives
\[
\|\mathbf{g}(., 0)\|_{L^\infty(0, T)}^2 \leq T \|\mathbf{g}(., 0)\|_{H^1(0, T)}^2 \quad \text{on } \Gamma_c(0)
\]

In order to deal with \(\int_{\Omega_0} \tilde{\gamma}_n(0)^2 d\tilde{x}\), we use Lemma \([12, \text{ Lemma 2.2}]\) which yields
\[
\|\tilde{\gamma}_n(0)\|_{L^2(\Omega_0)}^2 = \|\pi_n \tilde{\gamma}_0\|_{L^2(\Omega_0)}^2 \leq \|\tilde{\gamma}_0\|_{L^2(\Omega_0)}^2 = \|\mathbf{v}_0\|_{L^2(\Omega_0)}^2 + \|\mathbf{\xi}_1\|_{L^2(\Omega_0)}^2.
\]

Combining \((58)-(64)\) and using \((63)\) we obtain
\[
\left( \mu C_k - \mu C T^n M \right) \|\tilde{\gamma}_n\|_{L^2(H^1(\Omega_0^c))}^2 + \frac{\rho f_T}{2} (1 - C T^n M) \|\tilde{\gamma}_n\|_{L^\infty(L^2(\Omega_0^c))}^2 + \frac{\rho f_T}{2} (1 - C T^n M - C T^n M) \|\tilde{\gamma}_n\|_{L^2(L^2(\Omega_0^c))}^2
\]
\[\quad + \left( \mu C_k - C T^n M - C \delta T \right) \|f_0^T \tilde{\gamma}_n(s)\|_{L^\infty(H^1(\Omega_0^c))}^2 \leq C \left[ \frac{\rho f_T}{2} \|\mathbf{v}_0\|_{L^2(\Omega_0)}^2 + \frac{\rho f_T}{2} \|\mathbf{\xi}_1\|_{L^2(\Omega_0)}^2 \right] + C_4 \|\mathbf{g}\|_{H^1(L^2(\Gamma_c(0))))}^2.
\]

Remark that, the constants \(\mu_s, \lambda_s\) and \(\mu\) are given as large values by the constitutive laws of the structure and the fluid. Moreover, \(\delta\) is a negligible positive real number, hence norms that are factored by the term \(\delta\) are being absorbed by larger terms. Finally, we take \(T\) small with respect to \(M\) and the initial values, that is, the factor \(C T^n M\) is negligible. These assumptions lead to the following estimate
\[
\|\tilde{\gamma}_n\|_{L^2(H^1(\Omega_0^c))}^2 + \|\tilde{\gamma}_n\|_{L^\infty(L^2(\Omega_0))}^2 + \|f_0^T \tilde{\gamma}_n(s)\|_{L^\infty(H^1(\Omega_0^c))}^2 \leq C \left[ \frac{\rho f_T}{2} \|\mathbf{v}_0\|_{L^2(\Omega_0)}^2 + \frac{\rho f_T}{2} \|\mathbf{\xi}_1\|_{L^2(\Omega_0)}^2 + \|\mathbf{g}\|_{H^1(L^2(\Gamma_c(0))))}^2 \right].
\]

**Step 2: Estimates on \(\partial_t \tilde{\gamma}_n\)**

The next step is to derive some estimates on \(\partial_t \tilde{\gamma}_n\). Consider a function \(\tilde{n}\) in \(\tilde{W}\) such that \(\|\tilde{n}\|_{L^2(H^1(\Omega_0))} \leq 1\). The function \(\tilde{n}\) can be written as
\[
\tilde{n} = \pi_n \tilde{n} + (\tilde{n} - \pi_n \tilde{n}).
\]
where $\pi_n$ is the projection from $L^2(\Omega_0)$ into $\bar{W}_n$. Notice that, as we have $\partial_t \tilde{\gamma}_n \in \bar{W}_n$, then

$$[[\partial_t \tilde{\gamma}_n(t), \tilde{\eta}]] = [[\partial_t \tilde{\gamma}_n(t), \pi_n \tilde{\eta}]] + [[\partial_t \tilde{\gamma}_n(t), \tilde{\eta} - \pi_n \tilde{\eta}]] = [[\partial_t \tilde{\gamma}_n(t), \pi_n \tilde{\eta}]].$$

Set $\tilde{\eta}_n = \pi_n \tilde{\eta}$ in \(\text{(51)}\). By integrating over \((0, t)\) we obtain

$$\rho_t \int_0^t \int_{\Omega_0} \det(\nabla \vec{X}) \partial_t \tilde{\gamma}_n \cdot \pi_n \tilde{\eta} \, d\vec{x} \, ds + \rho_s \int_0^t \int_{\Omega_0} \det(\nabla \varphi) \partial_t \gamma_n \cdot \pi_n \tilde{\eta} \, d\vec{x} \, ds + \int_0^t \int_{\Omega_0} \bar{\sigma}_j^0(\gamma_n) : \nabla \pi_n \tilde{\eta} \, d\vec{x} \, ds$$

$$+ \sum_{i, \alpha, j, \beta = 1} \int_0^t \int_{\Omega_0} \bar{b}_{i\alpha j\beta} \partial_\beta(\int_0^\tau \bar{\gamma}_n(\tau) \, d\tau) \partial_\alpha(\pi_n \tilde{\eta}) \, d\vec{x} \, ds$$

$$+ \sum_{i, \alpha, j, \beta = 1} \int_0^t \int_{\Omega_0} \partial_\alpha \bar{b}_{i\alpha j\beta} \partial_\beta(\int_0^\tau \bar{\gamma}_n(\tau) \, d\tau) \partial_\alpha(\pi_n \tilde{\eta}) \, d\vec{x} \, ds + \int_0^t \int_{\Gamma_{c,0}} g \cdot \pi_n \tilde{\eta} \, d\Gamma \, ds. \quad (68)$$

This is equivalent to say,

$$(1 - CT^M) \int_0^t [[\partial_t \tilde{\gamma}_n(s), \pi_n \tilde{\eta}(s)]] \, ds - \int_0^t \int_{\Omega_0} \bar{\sigma}_j^0(\gamma_n) : \nabla \pi_n \tilde{\eta} \, d\vec{x} \, ds - \sum_{i, \alpha, j, \beta = 1} \int_0^t \int_{\Omega_0} \bar{b}_{i\alpha j\beta} \partial_\beta(\int_0^\tau \bar{\gamma}_n(\tau) \, d\tau) \partial_\alpha(\pi_n \tilde{\eta}) \, d\vec{x} \, ds$$

$$- \sum_{i, \alpha, j, \beta = 1} \int_0^t \int_{\Omega_0} \partial_\alpha \bar{b}_{i\alpha j\beta} \partial_\beta(\int_0^\tau \bar{\gamma}_n(\tau) \, d\tau) \partial_\alpha(\pi_n \tilde{\eta}) \, d\vec{x} \, ds + \int_0^t \int_{\Gamma_{c,0}} g \cdot \pi_n \tilde{\eta} \, d\Gamma \, ds. \quad (69)$$

Bounding the terms of the right hand side of the above equality yields

$$(1 - CT^M) \int_0^t [[\partial_t \tilde{\gamma}_n(s), \pi_n \tilde{\eta}(s)]] \, ds \leq \int_0^t ||\bar{\sigma}_j^0(\gamma_n)||_{L^2(\Omega_0')} ||\nabla \pi_n \tilde{\eta}||_{L^2(\Omega_0')} \, ds$$

$$+ \sum_{i, \alpha, j, \beta = 1} \int_0^t ||\bar{b}_{i\alpha j\beta}||_{L^\infty(\Omega_0')} ||\partial_\beta(\int_0^\tau \bar{\gamma}_n(\tau) \, d\tau)||_{L^2(\Omega_0')} ||\partial_\alpha(\pi_n \tilde{\eta})||_{L^2(\Omega_0')} \, ds$$

$$+ \sum_{i, \alpha, j, \beta = 1} \int_0^t ||\partial_\alpha \bar{b}_{i\alpha j\beta}||_{L^\infty(\Omega_0')} ||\partial_\beta(\int_0^\tau \bar{\gamma}_n(\tau) \, d\tau)||_{L^2(\Omega_0')} ||\pi_n \tilde{\eta}||_{L^2(\Omega_0')} \, ds$$

Using Hölder’s inequality with the Sobolev embeddings \((H^2 \subset L^\infty)\), the right hand side of \(\text{(69)}\) is bounded above by

$$C ||\tilde{\gamma}_n||_{L^2(H^1(\Omega_0'))} + CT^M ||\int_0^t \gamma_n(s) \, ds||_{L^\infty(H^1(\Omega_0'))} + ||g||_{H^1(L^2(\Gamma_{c,0}))} ||\pi_n \tilde{\eta}||_{L^2(H^1(\Omega_0'))}. \quad (70)$$

Then using the previous estimate \(\text{(51)}\) we get

$$\int_0^t [[\partial_t \tilde{\gamma}_n(s), \pi_n \tilde{\eta}(s)]] \, ds \leq C \left[ \frac{\rho_t}{2} ||v_0||_{L^2} + \frac{\rho_s}{2} ||\xi_1||_{L^2} + C ||g||_{H^1(L^2(\Gamma_{c,0}))} \right] ||\pi_n \tilde{\eta}||_{L^2(H^1(\Omega_0'))}.$$
Existence of the Weak Solution

Now passing to the limit as \( n \to \infty \) in (71) and (71) gives us the estimates on \( \tilde{\gamma} \). To show that \( \tilde{\gamma} \) satisfies (49) we proceed as follows. We fix an integer \( N \) and choose a function \( \tilde{\eta} \in \mathcal{C}^1([0, T], \mathcal{W}) \) of the form

\[
\tilde{\eta} = \sum_{i=1}^{N} d_i(t) \psi_i(\tilde{x}).
\]

(72)

For \( n > N \), we integrate (71) with respect to \( t \) to get

\[
\left\{ \begin{array}{l}
\rho_f \int_0^T \int_{\Omega_0^s} \text{det}(\nabla \mathbf{A}) \partial_t \tilde{\gamma}_n \cdot \tilde{\eta} \, d\mathbf{x} \, dt + \int_0^T \int_{\Omega_0^s} \tilde{\sigma}_n^0(\tilde{\gamma}_n) : \nabla \tilde{\eta} \, d\mathbf{x} \, dt + \rho_s \int_0^T \int_{\Omega_0^s} \text{det}(\nabla \tilde{\phi}) \partial_t \tilde{\gamma}_n \cdot \tilde{\eta} \, d\mathbf{x} \, dt \\
+ \sum_{i=a,j,b=1}^3 \int_0^T \int_{\Omega_0^s} b_{i\alpha \beta} \partial_\beta (\int_0^t \gamma_0(s) \, ds) \, \partial_\alpha \tilde{\eta} \, d\mathbf{x} \, dt \\
+ \sum_{i=a,j,b=1}^3 \int_0^T \int_{\Omega_0^s} \partial_\alpha \tilde{b}_{i\alpha \beta} \partial_\beta (\int_0^t \gamma_0(s) \, ds) \, \tilde{\eta} \, d\mathbf{x} \, dt = \int_0^T \int_{\Gamma_{\gamma}(0)} \mathbf{g} \cdot \tilde{\eta} \, d\Gamma \, dt,
\end{array} \right.
\]

holds true for all \( \tilde{\eta} \in L^2([0, T], \mathcal{W}) \) due to the fact that the space spanned by the functions of the form (72) is dense in \( L^2([0, T], \mathcal{W}) \). Hence, (73) implies (49).

To show that the initial conditions are satisfied we will consider \( \tilde{\eta} \in \mathcal{C}^1([0, T], \mathcal{W}) \) in (72) and integrate by parts to get

\[
\left\{ \begin{array}{l}
\rho_f \int_0^T \int_{\Omega_0^s} \tilde{\gamma} \cdot \partial_t (\tilde{\eta} \text{det}(\nabla \mathbf{A})) \, d\mathbf{x} \, dt - \int_0^T \int_{\Omega_0^s} \tilde{\sigma}_n^0(\tilde{\gamma}) : \nabla \tilde{\eta} \, d\mathbf{x} \, dt + \rho_s \int_0^T \int_{\Omega_0^s} \tilde{\gamma} \cdot \partial_t (\tilde{\eta} \text{det}(\nabla \tilde{\phi})) \, d\mathbf{x} \, dt \\
- \sum_{i,a,j,b=1}^3 \int_0^T \int_{\Omega_0^s} b_{i\alpha \beta} \partial_\beta (\int_0^t \gamma_0(s) \, ds) \, \partial_\alpha \tilde{\eta} \, d\mathbf{x} \, dt - \sum_{i,a,j,b=1}^3 \int_0^T \int_{\Omega_0^s} \partial_\alpha \tilde{b}_{i\alpha \beta} \partial_\beta (\int_0^t \gamma_0(s) \, ds) \, \tilde{\eta} \, d\mathbf{x} \, dt \\
= \int_0^T \int_{\Gamma_{\gamma}(0)} \mathbf{g} \cdot \tilde{\eta} \, d\Gamma \, dt - \rho_f \int_{\Omega_0^s} \tilde{\eta}(0) \, d\mathbf{x} \, \mathbf{d} - \rho_s \int_{\Omega_0^s} \partial_t \tilde{\xi}(0) \cdot \tilde{\eta}(0) \, d\mathbf{x}.
\end{array} \right.
\]

(75)

On the other hand, integrating by parts in time Equation (73) and passing to the limit we get

\[
\left\{ \begin{array}{l}
\rho_f \int_0^T \int_{\Omega_0^s} \tilde{\gamma} \cdot \partial_t (\tilde{\eta} \text{det}(\nabla \mathbf{A})) \, d\mathbf{x} \, dt - \int_0^T \int_{\Omega_0^s} \tilde{\sigma}_n^0(\tilde{\gamma}) : \nabla \tilde{\eta} \, d\mathbf{x} \, dt + \rho_s \int_0^T \int_{\Omega_0^s} \tilde{\gamma} \cdot \partial_t (\tilde{\eta} \text{det}(\nabla \tilde{\phi})) \, d\mathbf{x} \, dt \\
- \sum_{i,a,j,b=1}^3 \int_0^T \int_{\Omega_0^s} b_{i\alpha \beta} \partial_\beta (\int_0^t \gamma_0(s) \, ds) \, \partial_\alpha \tilde{\eta} \, d\mathbf{x} \, dt - \sum_{i,a,j,b=1}^3 \int_0^T \int_{\Omega_0^s} \partial_\alpha \tilde{b}_{i\alpha \beta} \partial_\beta (\int_0^t \gamma_0(s) \, ds) \, \tilde{\eta} \, d\mathbf{x} \, dt \\
= - \int_0^T \int_{\Gamma_{\gamma}(0)} \mathbf{g} \cdot \tilde{\eta} \, d\Gamma \, dt - \rho_f \int_{\Omega_0^s} \tilde{\eta}(0) \, d\mathbf{x} \, \mathbf{d} - \rho_s \int_{\Omega_0^s} \partial_t \tilde{\xi}(0) \cdot \tilde{\eta}(0) \, d\mathbf{x}.
\end{array} \right.
\]

(76)

Comparing (75) and (76) yields

\[
[[\tilde{\eta}, \tilde{\eta}(0)]] = [[\tilde{\gamma}(0), \tilde{\eta}(0)]].
\]

Since \( \tilde{\eta}(0) \in \mathcal{W} \) is arbitrary, then the initial conditions are verified.

Finally, by passing to the limit in (73), we obtain (49). This yields the existence of the weak solution \( \tilde{\gamma} \) of System (19).
Using (59)-(63) we get

\[ \text{4. Existence of Solution for the Linearized System} \]

Continuously setting \( \tilde{\gamma} \) for all \( \tilde{\eta} \in C^0(0,T; \tilde{W}) \), the solution \( \tilde{\gamma} \) satisfies the following variational formulation

\[
\begin{aligned}
\rho_f & \int_{\Omega_0^f} \det(\nabla \tilde{\mathbf{A}}) \partial_t \tilde{\gamma} \cdot \tilde{\eta} \, d\tilde{x} + \int_{\Omega_0^f} \tilde{\sigma}(\tilde{\gamma}) : \nabla \tilde{\eta} \, d\tilde{x} + \rho_s \int_{\Omega_0^s} \det(\nabla \tilde{\varphi}) \partial_t \tilde{\gamma} \cdot \tilde{\eta} \, d\tilde{x} \\
+ & \sum_{i \beta} b_{i \alpha j \beta} \partial_t \tilde{\gamma} (I_{\tilde{\gamma}}(s)ds)_j \partial_o \tilde{\eta} \, d\tilde{x} + \sum_{i \beta} b_{i \alpha j \beta} \partial_t (I_{\tilde{\gamma}}(s)ds)_j \tilde{\eta} \, d\tilde{x} = 0
\end{aligned}
\]  

Taking \( \tilde{\eta} = \tilde{\gamma} \) and integrating over \((0, t)\) we get

\[
\begin{aligned}
\rho_f & \int_0^t \int_{\Omega_0^f} \det(\nabla \tilde{\mathbf{A}}) \tilde{\gamma}(t)^2 \, d\tilde{x} + \int_0^t \int_{\Omega_0^f} \tilde{\sigma}(\tilde{\gamma}) : \nabla \tilde{\gamma} \, d\tilde{x} + \frac{\rho_s}{2} \int_0^t \int_{\Omega_0^s} \det(\nabla \tilde{\varphi}) \tilde{\gamma}(t)^2 \, d\tilde{x} \\
- & \frac{\rho_f}{2} \int_0^t \int_{\Omega_0^f} \tilde{\gamma}(t)^2 \partial_t \det(\nabla \tilde{\mathbf{A}}) \, d\tilde{x} - \frac{\rho_s}{2} \int_0^t \int_{\Omega_0^s} \tilde{\gamma}(t)^2 \partial_t \det(\nabla \tilde{\varphi}) \, d\tilde{x} \\
+ & \frac{1}{2} \sum_{i \beta} b_{i \alpha j \beta} \partial_t \tilde{\gamma} (I_{\tilde{\gamma}}(s)ds)_j \partial_o \tilde{\gamma}(t) \, d\tilde{x} - \frac{1}{2} \sum_{i \beta} b_{i \alpha j \beta} \partial_t (I_{\tilde{\gamma}}(s)ds)_j \partial_o \tilde{\gamma}(t) \, d\tilde{x} \\
+ & \sum_{i \beta} b_{i \alpha j \beta} \partial_t (I_{\tilde{\gamma}}(s)ds)_j \partial_o \tilde{\gamma}(t) \, d\tilde{x} = 0.
\end{aligned}
\]  

Using (59)-(63) we get

\[
||\tilde{\gamma}(t)||_{L^2(H^1(\Omega_0^f))}^2 + ||\tilde{\gamma}(t)||_{L^\infty(L^2(\Omega_0^f))}^2 + ||\tilde{\gamma}(s)ds)||_{L^2(\Omega_0^s)}^2 + ||\tilde{\gamma}(s)ds)||_{L^\infty(H^1(\Omega_0^s))}^2 \leq 0,
\]

which yields that \( \tilde{\gamma}_1 = \tilde{\gamma}_2 \). Therefore, \( \tilde{\gamma} \) is a unique solution of (69). In addition, we have

\[
\tilde{\gamma}|_{\Omega_0^s} \in L^\infty(L^2(\Omega_0^s)) \cap L^2(H^1(\Omega_0^s)), \quad \tilde{\gamma}|_{\Omega_0^s} \in L^\infty(L^2(\Omega_0)) \text{ and } \int_0^T \tilde{\gamma}(s)|_{\Omega_0^s} \, ds \in L^\infty(H^1(\Omega_0^s)).
\]

Consequently, setting \( \tilde{\varphi} = \tilde{\gamma}|_{\Omega_0^s} \) and \( \tilde{\xi} = \xi_0 + \int_0^t \tilde{\gamma}(s)|_{\Omega_0^s} \, ds \), we obtain the existence and uniqueness of the weak solution \( (\tilde{\varphi}, \tilde{\xi}) \in F_t \times S_t \) for the System (65).
which is nothing but the auxiliary problem (45) when considering
\[ g_i = - \sum_{\alpha,j,\beta=1}^3 \left( \int_0^t \partial_s b_{\alpha j \beta} \partial_\beta \xi_j \, ds \right) \tilde{n}_\alpha, \quad i = 1, 2, 3. \]

**Proposition 4.1.** Let \((\tilde{v}, \tilde{\xi}) \in A^T_M, v_0 \in H^1(\Omega^f_0), \) and \(\xi_1 \in H^1(\Omega^s_0)\) satisfying (1) and (29). For \(T\) small with respect to \(M\) and the initial conditions, there exists a unique weak solution \((\tilde{v}, \tilde{\xi}) \in F^T_2 \times S^T_2\) of (51). Moreover we get a priori estimate on the solution given by
\[ \|\tilde{v}\|_{F^T_2}^2 + \|\tilde{\xi}\|_{S^T_2}^2 \leq C \|v_0\|_{H^1}^2 + C \|\xi_1\|_{H^1}^2. \]  

Notice that, increasing the regularity of the initial data by considering \(v_0 \in H^1(\Omega^f_0)\) and \(\xi_1 \in H^1(\Omega^s_0)\) will lead to a more regular solution [13, 6]. Using the regularity results we achieve a solution \((\tilde{v}, \tilde{\xi}) \in F^T_2 \times S^T_2\). Now we prove Proposition 4.1.

The proof is based on the fixed point theorem. Indeed, the first step is to find estimates on \(\partial_t \tilde{\gamma}\) in \(F^T_2 \times S^T_2\) then we prove the existence and uniqueness of a weak solution of (81).

First, we consider System (45) with the function
\[ g_i = \hat{h}_i = - \sum_{\alpha,j,\beta=1}^3 \left( \int_0^t \partial_s b_{\alpha j \beta} \partial_\beta \hat{\xi}_j \, ds \right) \tilde{n}_\alpha, \quad i = 1, 2, 3. \]

Observe that \(\hat{h} \in H^1_l(0,T; L^2(\Gamma_c(0))).\) Thanks to (83) and the trace inequality we have
\[ \|\hat{h}\|_{H^1_l(0,T; L^2(\Gamma_c(0)))} \leq CT^\alpha M \|\hat{\xi}\|_{S^T_2}. \]  

Therefore, as \(\hat{\xi}\) is fixed, by Lemma 3.1 we get the existence and uniqueness of \((\tilde{v}, \tilde{\xi}) \in F^T_2 \times S^T_2\) satisfying
\[ \|\tilde{v}\|_{F^T_2}^2 + \|\tilde{\xi}\|_{S^T_2}^2 \leq C \left[ \frac{\rho_f}{2} ||v_0||_{L^2}^2 + \frac{\rho_s}{2} ||\xi_1||_{L^2}^2 + T^\alpha M \|\tilde{\xi}\|_{S^T_2} \right]. \]  

To prove that the solution \((\tilde{v}, \tilde{\xi})\) is in the space \(F^T_2 \times S^T_2\) we use the fixed point theorem. To this end we introduce the map \(\Psi_0\) from \(S^T_2\) to \(S^T_2\) defined as
\[ \Psi_0 : \tilde{\xi} \longmapsto \tilde{\xi}. \]

As we mentioned previously, we ensure the existence of a weak solution \((\tilde{v}, \tilde{\xi}) \in F^T_2 \times S^T_2\). In order to prove its uniqueness it is sufficient to prove that \(\Psi_0\) is a contraction on \(S^T_2\). This is achieved by deriving some a priori estimates on \(\partial_t \tilde{\gamma}\).
4.1. Estimates on $\partial_t \tilde{\gamma}$ and $\partial_t^2 \tilde{\xi}$

We proceed to derive a priori estimates on $\partial_t \tilde{\gamma}$. Differentiating in time the weak formulation \[69\]. Taking $\tilde{\eta} = \partial_t \tilde{\gamma}$ yields

$$
\begin{align*}
\frac{\partial \tilde{\gamma}}{2} \int_{\Omega_0^t} \det(\nabla \tilde{\xi})(t) |\partial_t \tilde{\gamma}(t)|^2 \, d\tilde{x} + \int_0^t \int_{\Omega_0^t} \partial_s \tilde{\gamma}^T(\tilde{\gamma}) : \partial_s \nabla \tilde{\gamma} \, d\tilde{x} \, ds \\
+ \frac{\partial s}{2} \int_{\Omega_0^t} \det(\nabla \tilde{\gamma})(t) |\partial_t \tilde{\gamma}(t)|^2 \, d\tilde{x} + \frac{\partial s}{2} \int_0^t \int_{\Omega_0^t} \partial_s \det(\nabla \tilde{\xi}) |\partial_t \tilde{\gamma}|^2 \, d\tilde{x} \, ds \\
+ \frac{\partial s}{2} \int_{\Omega_0^t} \int_{\Omega_0^t} \partial_s \det(\nabla \tilde{\gamma}) |\partial_t \tilde{\gamma}|^2 \, d\tilde{x} \, ds + \frac{1}{2} \sum_{i=1}^3 \int_{\Omega_0^t} \left[ b_{i\alpha\beta} \partial_t \tilde{\gamma} \partial_s \tilde{\gamma}_i \right] (t) \, d\tilde{x} \\
\geq \frac{1}{2} \sum_{i=1}^3 \int_{\Omega_0^t} \int_{\Omega_0^t} \partial_s b_{i\alpha\beta} \partial_s \tilde{\gamma}_i (\int_0^s \tilde{\gamma}(\tau) d\tau)_j \partial_s \tilde{\gamma}_j (\int_0^s \tilde{\gamma}(\tau) d\tau)_i \, d\tilde{x} \, ds \\
\text{ for } i \neq j \\
+ \frac{1}{2} \sum_{i=1}^3 \int_{\Omega_0^t} \int_{\Omega_0^t} \partial_s b_{i\alpha\beta} \partial_s \tilde{\gamma}_i (\int_0^s \tilde{\gamma}(\tau) d\tau)_j \partial_s \tilde{\gamma}_j (\int_0^s \tilde{\gamma}(\tau) d\tau)_i \, d\tilde{x} \, ds \\
\end{align*}
$$

As for the stress term in \[85\] we have

$$
A_1 = \frac{1}{2} \int_{\Omega_0^t} \det(\nabla \tilde{\xi}) \left| \partial_s \nabla \tilde{\gamma} \right|^2 \, d\tilde{x} \, ds = A_1 + A_2 + A_3.  \tag{86}
$$

For $A_2$ we use Young’s inequality with Hölder’s inequality to obtain

$$
|A_2| = \mu \int_{\Omega_0^t} \int_{\Omega_0^t} \left( \nabla \tilde{\gamma} \partial_s (\nabla \tilde{\xi})^{-1} + \partial_s (\nabla \tilde{\xi})^{-1} \nabla \tilde{\gamma} \right) \det(\nabla \tilde{\xi}) \partial_s \nabla \tilde{\gamma} \, d\tilde{x} \, ds \\
\leq C \mu^2 M \left[ C \left| \tilde{\gamma} \right|_{H^2(\Omega_0^t)}^2 + \delta \right] \left| \tilde{\gamma} \right|_{H^1(\Omega_0^t)}^2.  \tag{87}
$$

Similarly, for $A_3$ we have

$$
|A_3| = \mu \int_{\Omega_0^t} \int_{\Omega_0^t} \left( \nabla \tilde{\gamma} \partial_s (\nabla \tilde{\xi})^{-1} + \partial_s (\nabla \tilde{\xi})^{-1} \nabla \tilde{\gamma} \right) \partial_s \det(\nabla \tilde{\xi}) \partial_s \nabla \tilde{\gamma} \, d\tilde{x} \, ds \\
\leq C \mu^2 M \left[ C \left| \tilde{\gamma} \right|_{L^\infty(\Omega_0^t)}^2 + \delta \right] \left| \tilde{\gamma} \right|_{H^1(\Omega_0^t)}^2.  \tag{88}
$$

Therefore, the summation of Equations \[85\] and \[88\] is bounded above by

$$
C \mu^2 M \left[ C \left| \tilde{\gamma} \right|_{L^\infty(\Omega_0^t)}^2 + \delta \right] \left| \tilde{\gamma} \right|_{H^1(\Omega_0^t)}^2.  \tag{89}
$$

As for the integrals over $\Omega_0^t$, first we have

$$
\begin{align*}
\frac{\partial s}{2} \int_{\Omega_0^t} \det(\nabla \tilde{\xi})(t) |\partial_t \tilde{\gamma}(t)|^2 \, d\tilde{x} + \frac{\partial s}{2} \int_0^t \int_{\Omega_0^t} \partial_s \det(\nabla \tilde{\xi}) |\partial_t \tilde{\gamma}|^2 \, d\tilde{x} \, ds \\
\geq \frac{\partial s}{2} \left( 1 - C \mu^2 M \right) \left| \tilde{\gamma} \right|_{L^\infty(\Omega_0^t)}^2.  \tag{91}
\end{align*}
$$
On the contrary, using (36) and (38) with Korn’s inequality gives

$$
\frac{1}{2} \sum_{i, \alpha, j, \beta = 1}^{3} \int_{\Omega_0} \left[ \overline{b_{\alpha j \beta}} \partial_{\alpha j} \gamma_{\beta} \partial_{\gamma} \gamma \right] (t) \, d \overline{\xi} \\
\geq \mu CT \| \overline{\gamma} \|^2_{L^\infty(H^1(\Omega_0^\varepsilon))} + \frac{c + \lambda^2}{2} \| \nabla \cdot \overline{\gamma} \|^2_{L^\infty(L^2(\Omega_0^\varepsilon))} - CT^\mu M \| \overline{\gamma} \|^2_{L^\infty(H^1(\Omega_0^\varepsilon))}.
$$

(92)

On the other hand, using (35) and (37) we have

\[
\left| \sum_{i, \alpha, j, \beta = 1}^{3} \left[ - \frac{1}{2} \int_{0}^{t} \int_{\Omega_0^\varepsilon} \partial_{\alpha} \overline{b_{\alpha j \beta}} \partial_{\gamma} \gamma (f_{\alpha}^\varepsilon \overline{\gamma} (\tau) \, d \tau, \, d \overline{\xi} \, d s \\
+ \int_{0}^{t} \int_{\Omega_0^\varepsilon} \partial_{\alpha} \overline{b_{\alpha j \beta}} \partial_{\gamma} \gamma (f_{\alpha}^\varepsilon \overline{\gamma} (\tau) \, d \tau, \, d \overline{\xi} \, d s \\
- \int_{0}^{t} \int_{\Omega_0^\varepsilon} \partial_{\alpha} \overline{b_{\alpha j \beta}} \partial_{\gamma} \gamma (f_{\alpha}^\varepsilon \overline{\gamma} (\tau) \, d \tau, \, d \overline{\xi} \, d s \right] \right| \\
\leq CT^\mu M \left[ C_\delta \| f_{\alpha}^\varepsilon \overline{\gamma} (s) \, d s \|_{L^\infty(H^2(\Omega_0^\varepsilon))} + \delta \| \partial_{\gamma} \overline{\gamma} \|^2_{L^\infty(H^1(\Omega_0^\varepsilon))} + \| \overline{\gamma} \|^2_{L^\infty(H^1(\Omega_0^\varepsilon))} \right].
\]

For the integrals across the boundary we use Young’s inequality in addition to the trace inequality to obtain

\[
\left| \int_{0}^{t} \int_{\Gamma_\varepsilon(0)} \partial_{n} \overline{\mathbf{h}} \cdot \partial_{n} \overline{\gamma} \, d \overline{\xi} \, d s \right| \leq CT^\mu M \left[ C_\delta \| \overline{\xi} \|^2_{L^2} + \delta \| \overline{\gamma} \|^2_{L^\infty(H^1(\Omega_0^\varepsilon))} \right],
\]

(93)

and for $i, \alpha, j, \beta = 1, 2, 3$, we have

\[
\left| \sum_{i, \alpha, j, \beta = 1}^{3} \int_{0}^{t} \int_{\Gamma_\varepsilon(0)} \partial_{\alpha} \overline{b_{\alpha j \beta}} \partial_{\gamma} \gamma (f_{\alpha}^\varepsilon \overline{\gamma} (\tau) \, d \tau, \, d \overline{\xi} \, d s \right| \\
\leq CT^{1/2}(M + M^2 + M^3 + M^4) \left[ C_\delta \| f_{\alpha}^\varepsilon \overline{\gamma} (s) \, d s \|_{L^\infty(H^2(\Omega_0^\varepsilon))} + \delta \| \overline{\gamma} \|^2_{L^\infty(H^1(\Omega_0^\varepsilon))} \right].
\]

(94)

Therefore, considering the restriction of $\overline{\gamma}$ on each sub-domain, and for $T$ small with respect to $M$ and the initial conditions, i.e., the factors $CT^\mu M$ and $CT^{1/2}M^4$ are negligible, and using (33) we get

\[
\| \overline{\xi} \|^2_{L^1 \cap (L^2)} + \| \overline{\xi} \|^2_{L^1 \cap (H^1)} + \| \overline{\xi} \|^2_{L^2 \cap (L^2)} + \| \overline{\xi} \|^2_{L^2 \cap (H^1)} \\
\leq C \| \nu_0 \|^2_{H^1} + C \| \xi_1 \|^2_{H^1} + CT^\mu M \left( \| \overline{\xi} \|^2_{L^2} + \| \overline{\xi} \|^2_{L^2} \right).
\]

(95)

4.2. Estimates Using Spatial Regularity

We have proved that the linear system has a strong solution $(\overline{\mathbf{v}}, \overline{\xi}) \in F^T_2 \times S^T_2$. Therefore, for all $t \in (0, T)$, the fluid velocity $\overline{\mathbf{v}}$ satisfies the following equation

\[
\nabla \cdot \overline{\mathbf{v}}^\gamma (\overline{\mathbf{v}}) = \rho_f \text{det}(\nabla \overline{\xi}) \partial_t \overline{\mathbf{v}} \quad \text{in} \quad \Omega_0^\varepsilon,
\]

which can be rewritten as

\[
\mu \nabla \cdot (\nabla \overline{\mathbf{v}} + (\nabla \overline{\mathbf{v}})^T) = \rho_f \text{det}(\nabla \overline{\xi}) \partial_t \overline{\mathbf{v}} + F_\overline{\mathbf{v}} \quad \text{in} \quad \Omega_0^\varepsilon,
\]

with

\[
F_\overline{\mathbf{v}} = -\mu \nabla \cdot f_\overline{\mathbf{v}},
\]

where

\[
f_\overline{\mathbf{v}} = \left( \nabla \overline{\mathbf{v}}((\nabla \overline{\xi})^{-1} - \text{Id}) + ((\nabla \overline{\xi})^{-T} - \text{Id}) \nabla \overline{\mathbf{v}}^T \right) \text{cof}(\nabla \overline{\xi}) - (\nabla \overline{\mathbf{v}} + (\nabla \overline{\mathbf{v}})^T) \text{cof}(\nabla \overline{\xi}) \text{Id}.
\]
Using Lemma 2.1 we have
\[ ||F_{\hat{\vartheta}}||_{L^\infty(L^2)} \leq ||f_{\hat{\vartheta}}||_{L^\infty(H^1)} \leq 2\mu CT^{\alpha} M ||\hat{\vartheta}||_{L^\infty(H^2)}. \]
Hence, we obtain
\[ \mu ||\hat{\vartheta}||_{L^\infty(H^2)} \leq \rho_j CT^{\alpha} M ||\partial_t \hat{\vartheta}||_{L^\infty(L^2)} + 2\mu CT^{\alpha} M ||\hat{\vartheta}||_{L^\infty(H^2)}. \] (96)

Besides, the structure displacement \( \hat{\xi} \) satisfies the following equation
\[ -\nabla \cdot \left( 2\mu_4 \varepsilon(\hat{\xi}) + \lambda_4 (\nabla \cdot \hat{\xi}) \mathbf{I} d \right) = -\rho_4 \det(\nabla \varphi) \partial^2_{\hat{\xi}} \hat{\xi} + H^\xi_{\hat{\xi}} + H^d_{\hat{\xi}}, \]
with
\[ H^\xi_{\hat{\xi},i} = \sum_{\alpha,j,\beta=1}^3 \left( \partial^2_{\hat{\xi}} \hat{\xi}_{\alpha j} + \partial^2_{\hat{\xi}} \hat{\xi}_{\alpha j} \right) \partial^2_{\hat{\xi}} \hat{\xi}_j, \quad \text{for } i = 1, 2, 3, \]
and
\[ H^d_{\hat{\xi},i} = C \sum_{\alpha,j,\beta=1}^3 \left( \partial^2_{\hat{\xi}} \hat{\xi}_{\alpha j} + \partial^2_{\hat{\xi}} \hat{\xi}_{\alpha j} + \partial^2_{\hat{\xi}} \hat{\xi}_{\alpha j} \right) \partial^2_{\hat{\xi}} \hat{\xi}_j, \quad \text{for } i = 1, 2, 3. \]

Using elliptic estimates and thanks to (35) we get
\[ ||\hat{\xi}||_{L^\infty(H^2)} \leq \rho_5 CT^{\alpha} ||\partial^2_{\hat{\xi}} \hat{\xi}||_{L^\infty(L^2)} + CT (M + M^2 + M^3 + M^4) ||\hat{\xi}||_{L^\infty(H^2)}. \] (98)

To bound \( ||\partial_t \hat{\vartheta}||_{L^\infty(L^2(\Omega^M_0))} \) and \( ||\partial^2_{\hat{\xi}} \hat{\xi}||_{L^\infty(L^2(\Omega^M_0))} \) we use (35). Finally, taking \( T \) small with respect to \( M \) and the initial conditions in (96) and (98), then combining them with (35), we achieve the following estimate
\[ ||\hat{\vartheta}||_{L^\infty(S^T)} + ||\hat{\xi}||_{S^T_2} \leq CT^{\alpha} M ||\hat{\xi}||_{S^T_2} + C ||\hat{\vartheta}_0||_{H^1} + C ||\xi_0||_{H^1}. \] (99)

4.3. Fixed Point Theorem for the Linearized System

Based on the estimate (99) on the solution \( (\hat{\vartheta}, \hat{\xi}) \) of the linear system (81), we proceed to prove that the function \( \Psi_0 \) is a contraction on \( S^T_2 \). Let \( \hat{\xi}_1, \hat{\xi}_2 \in S^T_2 \). For \( a = 1, 2 \), we denote by \( (\hat{\vartheta}_a, \hat{\xi}_a) \) the solution of (45) with
\[ g_i = \hat{h}^a_i = - \sum_{\alpha,j,\beta=1}^3 \left( \int_0^t \partial_s \hat{h} \partial_{\beta}(\hat{\xi}_a) \right) d s, \quad i = 1, 2, 3. \]
Since \( (\hat{\vartheta}_1, \hat{\xi}_1) \) and \( (\hat{\vartheta}_2, \hat{\xi}_2) \) satisfy System (45), then we can say that \( (\hat{\vartheta}_1 - \hat{\vartheta}_2, \hat{\xi}_1 - \hat{\xi}_2) \) satisfies System (45) with \( g_i = \hat{h}^1_i - \hat{h}^2_i \) and null initial data. Hence, applying (99) to \( (\hat{\vartheta}_1 - \hat{\vartheta}_2, \hat{\xi}_1 - \hat{\xi}_2) \) and noticing that the right hand side of the estimate contains only a norm on \( S^T_2 \) given by \( ||\hat{\xi}_1 - \hat{\xi}_2||_{S^T_2} \) added to some constants, consequently we get
\[ ||\hat{\xi}_1 - \hat{\xi}_2||_{S^T_2} = ||\Psi_0(\hat{\xi}_1) - \Psi_0(\hat{\xi}_2)||_{S^T_2} \leq CT^{\alpha} M ||\hat{\xi}_1 - \hat{\xi}_2||_{S^T_2}. \] (100)

Taking \( T \) small enough with respect to \( M \), gives that \( \Psi_0 \) is a contraction on \( S^T_2 \). Therefore, we assure the existence and uniqueness of a fixed point \( \hat{\xi} \in S^T_2 \). Consequently, we obtain the existence and uniqueness of a solution \( (\hat{\vartheta}, \hat{\xi}) \) for the system (81). Finally, with the assumption of \( T \) being small with respect to \( M \) and denoting \( C ||\hat{\vartheta}_0||_{H^1} + C ||\xi_0||_{H^1} \) by \( C_0 \) we obtain
\[ ||\hat{\vartheta}||_{L^\infty(S^T_2)} + ||\hat{\xi}||_{S^T_2} \leq C_0. \] (101)
5. Regularity of Solution of the Linearized System

5.1. Regularity of the solution

**Proposition 5.1.** Let \((\tilde{v}, \tilde{\xi}) \in A^T_M\), with the assumption that \(v_0 \in H^6(\Omega_0^T)\) and \(\xi_1 \in H^3(\Omega_0^T)\) and satisfies (29). For \(T\) small with respect to \(M\) and the initial conditions, the solution \((\tilde{v}, \tilde{\xi})\) is in the space \(F^T_4 \times S^T_4\). Further, it satisfies

\[
\|\tilde{v}\|_{F^T_4} + \|\tilde{\xi}\|_{S^T_4} \leq C_0,
\]

where \(C_0\) denotes a constant in the norms \(\|v_0\|_{H^6(\Omega_0^T)}\) and \(\|\xi_1\|_{H^3(\Omega_0^T)}\).

By Proposition 5.1 we have proved the existence and uniqueness of \((\tilde{v}, \tilde{\xi}) \in F^T_4 \times S^T_4\). Increasing the regularity of the initial conditions results a more regular solution. The regularity of the solution in case of a linear fluid-structure interaction problem where the structure is considered to be quasi-incompressible have been proved in [10]. Hence \((\tilde{v}, \tilde{\xi})\) belongs to \(F^T_4 \times S^T_4\).

Next, we proceed to derive a priori estimates on the solution \((\tilde{v}, \tilde{\xi})\) in \(F^T_4 \times S^T_4\).

5.2. A Priori estimates on \(\gamma\) in \(A^T_M\)

**A Priori Estimates Using Time Regularity**

The solution \(\tilde{\gamma}\) satisfies (49) with

\[
g_i = - \sum_{\alpha,j,\beta=1}^{3} \left( \int_0^t \partial_s \bar{b}_{\alpha j \beta} \partial_t \tilde{\xi}_j \, ds \right) \bar{n}_a, \quad i = 1, 2, 3.
\]

Differentiating three times with respect to time and taking \(\tilde{n} = \partial^3_t \tilde{\gamma}\) yield

\[
\begin{aligned}
\rho_f \int_{\Omega_0^T} \partial_t (\nabla \bar{\mathcal{A}}) \partial^3_t \tilde{\gamma} \cdot \partial^3_t \tilde{\gamma} \, d\bar{x} + C_1 + \rho_s \int_{\Omega_0^T} \partial_t (\nabla \bar{\varphi}) \partial^3_t \tilde{\gamma} \cdot \partial^3_t \tilde{\gamma} \, d\bar{x} + C_2 \\
+ \mu \int_{\Omega_0^T} \partial^3_t \left( \nabla \tilde{\gamma}(\nabla \bar{\mathcal{A}})^{-1} + (\nabla \bar{\mathcal{A}})^{-t}(\nabla \tilde{\gamma})^t \right) \text{cof}(\nabla \bar{\mathcal{A}}) : \nabla \partial^3_t \tilde{\gamma} \, d\bar{x} + C_3 \\
+ \sum_{i,\alpha,j,\beta=1}^{3} \int_{\Omega_0^T} \bar{b}_{\alpha j \beta} \partial_t \partial^3_t \tilde{\gamma}_j \partial_\alpha \partial^3_t \tilde{\gamma}_i \, d\bar{x} + C_4 = \int_{\Gamma_{\varepsilon}(0)} \partial^3_t \bar{g} \cdot \partial^3_t \tilde{\gamma} \, d\Gamma,
\end{aligned}
\]

where,

\[
C_1 = 3 \rho_f \int_{\Omega_0^T} \partial_t \partial^2_t \left( \nabla \tilde{\gamma}(\nabla \bar{\mathcal{A}})^{-1} + (\nabla \bar{\mathcal{A}})^{-t}(\nabla \tilde{\gamma})^t \right) \text{cof}(\nabla \bar{\mathcal{A}}) : \nabla \partial^2_t \tilde{\gamma} \, d\bar{x}
\]

\[
C_2 = 3 \rho_s \int_{\Omega_0^T} \partial_t \partial^2_t \left( \nabla \bar{\varphi} \cdot \partial^2_t \tilde{\gamma} \cdot \partial^2_t \tilde{\gamma} \, d\bar{x} + 3 \rho_s \int_{\Omega_0^T} \partial^3_t \partial_t (\nabla \bar{\varphi}) \partial^3_t \tilde{\gamma} \cdot \partial^3_t \tilde{\gamma} \, d\bar{x} + \rho_f \int_{\Omega_0^T} \partial^3_t \partial_t (\nabla \bar{\mathcal{A}}) \partial_t \tilde{\gamma} \cdot \partial^3_t \tilde{\gamma} \, d\bar{x},
\]

\[
C_3 = -3 \mu \int_{\Omega_0^T} \partial^3_t \left( \nabla \tilde{\gamma}(\nabla \bar{\mathcal{A}})^{-1} + (\nabla \bar{\mathcal{A}})^{-t}(\nabla \tilde{\gamma})^t \right) \text{cof}(\nabla \bar{\mathcal{A}}) : \nabla \partial^2_t \tilde{\gamma} \, d\bar{x}
\]

\[
+ 3 \mu \int_{\Omega_0^T} \partial_t \left( \nabla \tilde{\gamma}(\nabla \bar{\mathcal{A}})^{-1} + (\nabla \bar{\mathcal{A}})^{-t}(\nabla \tilde{\gamma})^t \right) \partial^3_t \text{cof}(\nabla \bar{\mathcal{A}}) : \nabla \partial^2_t \tilde{\gamma} \, d\bar{x}
\]

\[
+ \mu \int_{\Omega_0^T} \left( \nabla \tilde{\gamma}(\nabla \bar{\mathcal{A}})^{-1} + (\nabla \bar{\mathcal{A}})^{-t}(\nabla \tilde{\gamma})^t \right) \partial^3_t \text{cof}(\nabla \bar{\mathcal{A}}) : \nabla \partial^3_t \tilde{\gamma} \, d\bar{x},
\]

\[
C_4 = -3 \sum_{i,\alpha,j,\beta=1}^{3} \int_{\Omega_0^T} \partial^2_t \bar{b}_{\alpha j \beta} \partial_t (f^{\gamma}_t(s) \, ds) \partial_\alpha \partial^3_t \tilde{\gamma}_j \, d\bar{x} + 3 \sum_{i,\alpha,j,\beta=1}^{3} \int_{\Omega_0^T} \partial^2_t \bar{b}_{\alpha j \beta} \partial_t \tilde{\gamma}_j \partial_\alpha \partial^3_t \tilde{\gamma}_i \, d\bar{x}
\]

\[
+ 3 \sum_{i,\alpha,j,\beta=1}^{3} \int_{\Omega_0^T} \partial_t \bar{b}_{\alpha j \beta} \partial_t \tilde{\gamma}_j \partial_\alpha \partial^3_t \tilde{\gamma}_i \, d\bar{x} + 3 \sum_{i,\alpha,j,\beta=1}^{3} \int_{\Omega_0^T} \partial_t \partial_\alpha \bar{b}_{\alpha j \beta} \partial_\beta (f^{\gamma}_t(s) \, ds) \partial^3_t \tilde{\gamma}_i \, d\bar{x}.
\]
First proceeding as in (58) we get
\[ \rho_f \int_0^t \int_{\Omega_0^t} \det(\nabla \tilde{A}) \partial_t^2 \tilde{\gamma} \cdot \partial_t^2 \tilde{\gamma} \ dx \ ds \geq \frac{\rho_L}{2} \left( 1 - \text{CT}^n M \right) \| \partial_t^2 \tilde{\gamma} \|_{L^2(\Omega_0^t)}^2 - \| \tilde{v}_0 \|_{H^0}^2. \] (104)

For the fluid stress term we proceed as in (86) to get
\[ \mu \int_0^t \int_{\Omega_0^t} \partial_t^3 \left( \nabla \tilde{\gamma} (\nabla \tilde{A})^{-1} + (\nabla \tilde{A})^{-1} (\nabla \tilde{\gamma})^t \right) \text{cof}(\nabla \tilde{A}) : \nabla \partial_t^2 \tilde{\gamma} \ dx \ ds \geq \mu \| \partial_t^2 \tilde{\gamma} \|_{L^2(H^1(\Omega_0^t))}. \] (105)

On the domain \( \Omega_0^t \), similarly as (101), we have
\[ \rho_2 \int_0^t \int_{\Omega_0^t} \det(\nabla \tilde{\varphi}) \partial_t^2 \tilde{\gamma} \cdot \partial_t^2 \tilde{\gamma} \ dx \ ds \geq \frac{\rho_L}{2} \left( 1 - \text{CT}^n M \right) \| \partial_t^2 \tilde{\gamma} \|_{L^2(\Omega_0^t)}^2 - \| \tilde{\xi}_1 \|_{H^3}^2. \] (106)

Using (55)-(56) with Korn’s inequality give
\[ \sum_{i, o, j, \beta = 1}^3 \int_0^t \int_{\Omega_0^t} \tilde{g} \partial_t^2 \tilde{\gamma} \cdot \partial_t^2 \tilde{\gamma} \ dx \ ds \geq \mu_s \| \partial_t^2 \tilde{\gamma} \|_{L^2(H^1(\Omega_0^t))} - \text{CT}^n M \| \int_0^t \tilde{\gamma}(s) ds \|_{S_4^T}. \] (107)

Further, proceeding as in (93) and (94) we get
\[ \int_0^t \int_{\Gamma_c(0)} \partial_t^2 g \cdot \partial_t^2 \tilde{\gamma} \ d\Gamma \ ds \leq \text{CT}^n M \| \int_0^t \tilde{\gamma}(s) ds \|_{S_4^T} \| \tilde{\gamma} \|_{F_4^T}. \] (108)

On the other hand, to deal with \( C_1, C_2, C_3 \) and \( C_4 \) we use the following bounds
\[ \| \partial_t^k \det(\nabla \tilde{A}) \|_{L^\infty(\Omega_0^t)} \leq CM^k, \quad k = 1, 2, 3. \] (109)
\[ \| \partial_t^k \det(\nabla \tilde{\varphi}) \|_{L^\infty(\Omega_0^t)} \leq CM^k, \quad k = 1, 2, 3. \] (110)
\[ \| \partial_t^k (\nabla \tilde{A})^{-1} \|_{L^\infty(\Omega_0^t)} \leq CM^k, \quad k = 1, 2, 3. \] (111)
\[ \| \partial_t^k \text{cof}(\nabla \tilde{A}) \|_{L^\infty(\Omega_0^t)} \leq CM^k, \quad k = 1, 2, 3. \] (112)

Then,
\[ \int_0^t C_1 \ ds \leq \text{CT}^n M \| \tilde{\gamma} \|_{F_4^T}^2. \] (113)

Similarly
\[ \int_0^t C_2 \ ds \leq \text{CT}^n M \| \tilde{\gamma} \|_{F_4^T}^2. \] (114)

On the other hand,
\[ \int_0^t C_3 \ ds \leq \text{CT}^n M \| \int_0^t \tilde{\gamma}(s) ds \|_{S_4^T}^2 \] (115)

and
\[ \int_0^t C_4 \ ds \leq \text{CT}^n M \| \int_0^t \tilde{\gamma}(s) ds \|_{S_4^T}^2. \] (116)

Combining (104)-(108) with (113)-(116) and considering the restriction of \( \tilde{\gamma} \) on each sub-domain give
\[ \| \partial_t^2 \tilde{\varphi} \|_{L^2(H^1(\Omega_0^t))}^2 + \| \partial_t^2 \tilde{\gamma} \|_{L^2(H^1(\Omega_0^t))}^2 + \| \partial_t^2 \tilde{\psi} \|_{L^2(H^1(\Omega_0^t))}^2 + \| \partial_t^2 \tilde{\xi} \|_{L^2(H^1(\Omega_0^t))}^2 \]
\[ \leq \text{CT}^n M (\| \tilde{\varphi} \|_{F_4^T}^2 + \| \tilde{\gamma} \|_{S_4^T}^2) + C(\| \tilde{\xi}_1 \|_{H^3} + \| \tilde{v}_0 \|_{H^0}). \] (117)

This estimate together with (101) lead to the following estimate
\[ \| \tilde{\varphi} \|_{W^{3, \infty}(L^2(\Omega_0^t))}^2 + \| \tilde{\gamma} \|_{W^{3, \infty}(L^2(\Omega_0^t))}^2 + \| \tilde{\psi} \|_{H^1(H^1(\Omega_0^t))}^2 + \| \tilde{\xi} \|_{W^{3, \infty}(H^1(\Omega_0^t))}^2 + \| \tilde{\tilde{\xi}} \|_{W^{4, \infty}(L^2(\Omega_0^t))}^2 \]
\[ \leq \text{CT}^n M (\| \tilde{\varphi} \|_{F_4^T}^2 + \| \tilde{\gamma} \|_{S_4^T}^2) + C(\| \tilde{\xi}_1 \|_{H^3} + \| \tilde{v}_0 \|_{H^0}). \] (118)
Spatial Regularity

**Step 1:** Estimates on \( \tilde{\mathbf{v}} \) in \( W^{2,\infty}(H^2(\Omega_0^i)) \) and \( \tilde{\mathbf{\xi}} \) in \( W^{2,\infty}(H^2(\Omega_0^i)) \).

The fluid velocity \( \tilde{\mathbf{v}} \) satisfies the elliptic equation

\[
\mu \nabla \cdot \left( \nabla \tilde{\mathbf{v}} + (\nabla \tilde{\mathbf{v}})^T \right) = \rho_f \text{det}(\nabla \tilde{\mathbf{A}}) \partial_t \tilde{\mathbf{v}} + F_{\tilde{\mathbf{v}}} \quad \text{in} \quad \Omega_0^i, \tag{119}
\]

with

\[
F_{\tilde{\mathbf{v}}} = -\mu \nabla \cdot \mathbf{f}_{\tilde{\mathbf{v}}},
\]

where

\[
\mathbf{f}_{\tilde{\mathbf{v}}} = \left( \nabla \tilde{\mathbf{v}} \left( (\nabla \tilde{\mathbf{A}})^{-1} - \mathbf{I} \right) + \left( (\nabla \tilde{\mathbf{A}})^{-1} - \mathbf{I} \right) \nabla \tilde{\mathbf{v}} \right) \text{cof}(\nabla \tilde{\mathbf{A}}) - \left( \nabla \tilde{\mathbf{v}} + (\nabla \tilde{\mathbf{v}})^T \right) \left( \text{cof}(\nabla \tilde{\mathbf{A}}) - \mathbf{I} \right).
\]

as defined in Subsection 4.2. First we have

\[
||\partial_t^2 \left( \text{det}(\nabla \tilde{\mathbf{A}}) \partial_t \tilde{\mathbf{v}} \right)||_{L^\infty(L^2)} \leq CM^2 ||\tilde{\mathbf{v}}||_{W^{3,\infty}(L^2)} \leq C ||\mathbf{v}_0||_{H^6} + C||\mathbf{\xi}_1||_{H^3} + CT^\alpha M(||\tilde{\mathbf{v}}||_{F^T} + ||\tilde{\mathbf{\xi}}||_{S^T}).
\]

First, let us estimate \( F_{\tilde{\mathbf{v}}} \) in \( W^{2,\infty}(L^2) \). In fact differentiating \( f_{\tilde{\mathbf{v}}} \) two times in time gives

\[
\left[ \partial_{tt}^2 \nabla \tilde{\mathbf{v}} \right] \left( (\nabla \tilde{\mathbf{A}})^{-1} - \mathbf{I} \right) + 2(\partial_t \nabla \tilde{\mathbf{v}}) (\partial_t (\nabla \tilde{\mathbf{A}})^{-1}) + (\nabla \tilde{\mathbf{v}}) (\partial_t^2 (\nabla \tilde{\mathbf{A}})^{-1}) \right] \text{cof}(\nabla \tilde{\mathbf{A}})
\]

Using (109) with the embedding of \( H^2 \) in \( L^\infty \) and taking into consideration

\[
||\tilde{\mathbf{v}}||_{L^\infty(H^1)} \leq C||\mathbf{v}_0||_{H^6} + T||\tilde{\mathbf{v}}||_{H^3(H^1)}
\]

yield

\[
||F_{\tilde{\mathbf{v}}}||_{W^{2,\infty}(L^2)} \leq CT^\alpha M ||\tilde{\mathbf{v}}||_{W^{2,\infty}(H^1)} + C||\mathbf{v}_0||_{H^6}. \tag{120}
\]

Therefore, using (118) and the elliptic estimates on \( \tilde{\mathbf{v}} \) we obtain

\[
||\tilde{\mathbf{v}}||_{W^{2,\infty}(H^2)} \leq C||\mathbf{v}_0||_{H^6} + C||\mathbf{\xi}_1||_{H^3} + CT^\alpha M(||\tilde{\mathbf{v}}||_{F^T} + ||\tilde{\mathbf{\xi}}||_{S^T}). \tag{121}
\]

On the other hand, the structure displacement \( \tilde{\mathbf{\xi}} \) satisfies (47). Differentiating two times in time yield

\[
-\partial_{tt}^2 \left[ \nabla \cdot \left( 2\mu_\varepsilon \varepsilon(\tilde{\mathbf{\xi}}) + \lambda_\varepsilon (\nabla \cdot \tilde{\mathbf{\xi}}) \mathbf{I} \right) \right] = -\rho_\varepsilon \partial_{tt}^2 \left( \text{det}(\nabla \varphi) \partial_{tt}^2 \tilde{\mathbf{\xi}} \right) + \partial_{tt}^2 \mathbf{H}_{\tilde{\mathbf{\xi}}} + \partial_{tt}^2 \mathbf{H}_\varepsilon,
\]

with

\[
\mathbf{H}_{\tilde{\mathbf{\xi}},i} = \sum_{\alpha,j=1}^{3} (\partial_{\alpha j}^L + \partial_{\alpha j}^T) \partial_{\alpha k}^L \tilde{\mathbf{\xi}}, \quad \text{for} \quad i = 1, 2, 3,
\]

and

\[
\mathbf{H}_\varepsilon = C \sum_{\alpha,j=1}^{3} \left( \partial_{\alpha j}^L + \partial_{\alpha j}^T \right) \partial_{\alpha k}^L \tilde{\mathbf{\xi}}, \quad \text{for} \quad i = 1, 2, 3.
\]

First, we have

\[
||\text{det}(\nabla \varphi) \partial_{tt}^2 \tilde{\mathbf{\xi}}||_{W^{2,\infty}(L^2)} \leq C||\tilde{\mathbf{\xi}}||_{H^6}. \tag{117}
\]

Then using (118) we get

\[
||\text{det}(\nabla \varphi) \partial_{tt}^2 \tilde{\mathbf{\xi}}||_{W^{2,\infty}(L^2)} \leq C||\mathbf{v}_0||_{H^6} + C||\mathbf{\xi}_1||_{H^3} + CT^\alpha M(||\tilde{\mathbf{v}}||_{F^T} + ||\tilde{\mathbf{\xi}}||_{S^T}). \tag{122}
\]
Further, for \( \partial_t^2 H^c_\xi \) we have
\[
\partial_t^2 H^c_\xi(\bar{x}, t) = \partial_t^2 H^c_\xi(\bar{x}, 0) + \int_0^t \partial_s^2 H^c_\xi(\bar{x}, s) \, ds \quad \forall \, \bar{x} \in \Omega_0^c. \tag{123}
\]

Simple calculation of \( \partial_t^2 H^c_\xi(\bar{x}, s) \) then setting \( t = 0 \) and using the fact that \( \partial_t \tilde{c}_{\alpha \beta}^l(\bar{x}, 0) \) is a function of \( \xi_1 \) give
\[
||\partial_t^2 H^c_\xi(\bar{x}, 0)||_{L^\infty(\Omega_0^c)} \leq C||\xi_1||_{H^3}. \tag{124}
\]
Moreover,
\[
\int_0^t \partial_s^2 H^c_\xi(\bar{x}, s) \, ds = \sum_{i, \alpha, \beta = 1}^3 \left[ \int_0^t \partial_s^2 (\tilde{c}_{\alpha \beta}^l(\bar{x}, s)) \partial_{s, \alpha \beta} \tilde{\xi}_j \, ds + 3 \int_0^t \partial_s (\tilde{c}_{\alpha \beta}^l(\bar{x}, s) + \tilde{c}_{\alpha \beta}^l(\bar{x}, s)) \partial^2_{s, \alpha \beta} \tilde{\xi}_j \, ds \right].
\]

As \( \tilde{\xi}(0) = 0 \), then
\[
\sum_{i, \alpha, \beta = 1}^3 \left[ (\tilde{c}_{\alpha \beta}^l(\bar{x}, 0) + \tilde{c}_{\alpha \beta}^l(\bar{x}, 0)) \partial^2_{s, \alpha \beta} \tilde{\xi}_j \right](0) = 0. \tag{125}
\]

As a result, combining (124), (125) and (126) an estimate on \( \tilde{\xi} \) in \( W^{2, \infty}(L^2(\Omega_0^c)) \) is given by
\[
||\tilde{\xi}||_{W^{2, \infty}(L^2(\Omega_0^c))} \leq C||\nu_0||_{H^5} + C||\xi_1||_{H^3} + CT^\alpha M(||\tilde{\nu}||_{F^T} + ||\tilde{\xi}||_{s^T}). \tag{127}
\]

Finally, combining (121) and (127) we get
\[
||\tilde{\xi}||_{W^{2, \infty}(L^2(\Omega_0^c))} + ||\tilde{\xi}||_{W^{2, \infty}(H^2(\Omega_0^c))} \leq C||\nu_0||_{H^6} + C||\xi_1||_{H^3} + CT^\alpha M(||\tilde{\nu}||_{F^T} + ||\tilde{\xi}||_{s^T}). \tag{128}
\]

(121) \leq CT^\alpha M ||\tilde{\xi}||_{s^T}.

On the other hand, integrating by parts in time in the last integral of the right hand side gives
\[
\sum_{i, \alpha, \beta = 1}^3 \left[ (\tilde{c}_{\alpha \beta}^l(\bar{x}, 0) + \tilde{c}_{\alpha \beta}^l(\bar{x}, 0)) \partial^2_{s, \alpha \beta} \tilde{\xi}_j \right](0) = 0. \tag{126}
\]

As a result, combining (122), (125) and (126) an estimate on \( \tilde{\xi} \) in \( W^{2, \infty}(L^2(\Omega_0^c)) \) is given by
\[
||\tilde{\xi}||_{W^{2, \infty}(L^2(\Omega_0^c))} \leq C||\nu_0||_{H^5} + C||\xi_1||_{H^3} + CT^\alpha M(||\tilde{\nu}||_{F^T} + ||\tilde{\xi}||_{s^T}). \tag{127}
\]

Finally, combining (121) and (127) we get
\[
||\tilde{\xi}||_{W^{2, \infty}(L^2(\Omega_0^c))} + ||\tilde{\xi}||_{W^{2, \infty}(H^2(\Omega_0^c))} \leq C||\nu_0||_{H^6} + C||\xi_1||_{H^3} + CT^\alpha M(||\tilde{\nu}||_{F^T} + ||\tilde{\xi}||_{s^T}). \tag{128}
\]

(121) \leq CT^\alpha M ||\tilde{\xi}||_{s^T}.

**Step 2:** Estimates on \( \tilde{v} \) in \( L^\infty(H^4(\Omega_0^c)) \) and \( \tilde{\xi} \) in \( L^\infty(H^4(\Omega_0^c)) \).

Again, the fluid velocity satisfies (114). We estimate \( F_\xi \) in \( L^\infty(H^2(\Omega_0^c)) \). First,
\[
||F_\xi||_{L^\infty(H^2(\Omega_0^c))} \leq \mu ||f_\xi||_{L^\infty(H^3(\Omega_0^c))}.
\]

But,
\[
||f_\xi||_{L^\infty(H^3)} \leq 2||\nabla \tilde{v}||_{L^\infty(H^3)}||\nabla \tilde{\alpha}||^{-1} - \text{Id}||_{L^\infty(H^3)}||\text{cof}(\nabla \tilde{\alpha})||_{L^\infty(H^3)} + ||\text{cof}(\nabla \tilde{\alpha}) - \text{Id}||_{L^\infty(H^3)}||\nabla \tilde{v}||_{L^\infty(H^3)} \leq CT^\alpha M||\tilde{v}||_{L^\infty(H^4)}.
\]
Further, using Estimate (128) we have
\[ \|\det(\nabla \tilde{\mathbf{A}})\partial_t \tilde{\mathbf{u}}\|_{L^\infty(H^2)} \leq CM\|\tilde{\mathbf{u}}\|_{W^{2,\infty}(H^2)} \leq C\|v_0\|_{H^6} + C\|\xi_1\|_{H^3} + CT^\kappa M\left(\|\tilde{\mathbf{u}}\|_{E_T^1} + \|	ilde{\mathbf{e}}\|_{S_T^1}\right). \]

Hence, the elliptic estimates yield
\[ \|\tilde{\mathbf{u}}\|_{L^\infty(H^4)} \leq C\|v_0\|_{H^6} + C\|\xi_1\|_{H^3} + CT^\kappa M\left(\|\tilde{\mathbf{u}}\|_{E_T^1} + \|	ilde{\mathbf{e}}\|_{S_T^1}\right). \] (129)

Besides, the structure displacement \( \tilde{\mathbf{e}} \) satisfies (77). Then, by using the fact that \( \xi_0 = 0 \) with (35) we have
\[ \|\tilde{c}_{i\alpha j} + \tilde{c}_{i\alpha j}\|_{L^\infty(H^2(\alpha^0_0))} \leq T\|\partial_t \tilde{c}_{i\alpha j} + \partial_t \tilde{c}_{i\alpha j}\|_{L^\infty(H^2(\alpha^0_0))} \leq CT^\kappa M. \]

Thus, \( H E \tilde{\xi} \) can be estimated by
\[ C\|\tilde{\mathbf{u}}\|_{L^\infty(H^4)} + CT^\kappa M\|\tilde{\mathbf{e}}\|_{L^\infty(H^4)}. \] (130)

By a similar argument, we find that \( H E \tilde{\xi} \) can be estimated by
\[ CT^\kappa M\|\tilde{\mathbf{e}}\|_{L^\infty(H^4)}. \]

Thanks to the Estimate (128) on \( \tilde{\xi} \), (130) can be estimated by
\[ C\|v_0\|_{H^6} + C\|\xi_1\|_{H^3} + CT^\kappa M\left(\|\tilde{\mathbf{u}}\|_{E_T^1} + \|	ilde{\mathbf{e}}\|_{S_T^1}\right). \] (131)

Therefore, using the elliptic estimate we get
\[ \|\tilde{\mathbf{e}}\|_{L^\infty(H^4)} \leq C\|v_0\|_{H^6} + C\|\xi_1\|_{H^3} + CT^\kappa M\left(\|\tilde{\mathbf{u}}\|_{E_T^1} + \|	ilde{\mathbf{e}}\|_{S_T^1}\right). \] (132)

Combining (129) and (132) yield
\[ \|\tilde{\mathbf{u}}\|_{L^\infty(H^4(\alpha^0_0))} + \|\tilde{\mathbf{e}}\|_{L^\infty(H^4(\alpha^0_0))} \leq C\|v_0\|_{H^6} + C\|\xi_1\|_{H^3} + CT^\kappa M\left(\|\tilde{\mathbf{u}}\|_{E_T^1} + \|	ilde{\mathbf{e}}\|_{S_T^1}\right). \] (133)

Finally, Estimates (118), (128) and (133) give
\[ \|\tilde{\mathbf{u}}\|_{E_T^1} + \|\tilde{\mathbf{e}}\|_{S_T^1} \leq C\|v_0\|_{H^6} + C\|\xi_1\|_{H^3} + CT^\kappa M\left(\|\tilde{\mathbf{u}}\|_{E_T^1} + \|	ilde{\mathbf{e}}\|_{S_T^1}\right). \] (134)

Assuming that \( T \) small with respect to \( M \) and the initial values yield
\[ \|\tilde{\mathbf{u}}\|_{E_T^1} + \|\tilde{\mathbf{e}}\|_{S_T^1} \leq C\|v_0\|_{H^6} + C\|\xi_1\|_{H^3} = C_0. \] (135)

6. Existence of Solution of the Non-Linear Coupled Problem

From Proposition 1.1 there exists \( \tilde{C}_0 > 0 \) and \( \kappa > 0 \) such that for all \( M > 0 \) and \( (\tilde{\mathbf{u}}, \tilde{\mathbf{e}}) \in A^T_{M_1} \), there exists \( T_1 > 0 \) so that the solution of (81) satisfies
\[ \|\tilde{\mathbf{u}}\|_{E_T^1} + \|\tilde{\mathbf{e}}\|_{S_T^1} \leq \tilde{C}_0, \] (136)

for all \( T \leq T_1 \).

Taking \( M = \tilde{C}_0 \) we get
\[ \|\tilde{\mathbf{u}}\|_{E_T^1} + \|\tilde{\mathbf{e}}\|_{S_T^1} \leq \tilde{M}. \] (137)

We seek to prove the existence of a solution of the non-linear coupled problem (35). To establish this result we use the fixed point theorem. For this sake, for any \( T \leq \tilde{T} \), setting \( E = F_T^1 \times S_T^1 \) and \( W = A^T_{M_1} \). The set \( W \) is a closed subset of \( E \).
We define the function $\Psi : (\tilde{v}, \tilde{\xi}) \rightarrow (\tilde{v}, \tilde{\xi})$ that maps $(\tilde{v}, \tilde{\xi}) \in W$ into $(\tilde{v}, \tilde{\xi}) \in W$ which is the solution of the linear system \[31\]. An element $(a, b) \in \Psi(W)$ is written as $(a, b) = (\Psi(\tilde{v}, \tilde{\xi})$ where $(\tilde{v}, \tilde{\xi})$ belongs to $W$. But the definition of $\Psi$ gives that $\Psi(\tilde{v}, \tilde{\xi}) = (\tilde{v}, \tilde{\xi})$ which is the unique solution of the linear problem \[31\] in $W$, consequently $(a, b) = (\tilde{v}, \tilde{\xi}) \in W$. Therefore, $\Psi(W) \subset W$.

Consider two pairs $(\tilde{v}_1, \tilde{\xi}_1)$ and $(\tilde{v}_2, \tilde{\xi}_2) \in W$ and two solutions $(\tilde{v}_1, \tilde{\xi}_1), (\tilde{v}_2, \tilde{\xi}_2)$ of the linear system \[31\] associated to $(\tilde{v}_1, \tilde{\xi}_1)$ and $(\tilde{v}_2, \tilde{\xi}_2)$, respectively. Therefore $\tilde{v}_1, \tilde{v}_2, \tilde{\xi}_1$ and $\tilde{\xi}_2$ satisfy the variational formulations \[17\] and \[32\] with

$$
g_i = - \sum_{\alpha, \beta, \gamma = 1}^3 \left( \int_0^t \partial_\beta \tilde{b}_{\alpha \gamma j} \partial_\gamma \tilde{\xi}_j \, ds \right) \tilde{n}_\alpha, \quad i = 1, 2, 3.
$$

Set $\tilde{\xi} = \tilde{\gamma}_1 - \tilde{\gamma}_2$, then $\tilde{\xi}(0) = 0$. The main work in this section is to find estimates on $\tilde{\xi}$ and $\partial_\xi \tilde{\xi}$. These estimates will enable us to apply the fixed point theorem for a suitable choice of $T$ to be precised later.

### 6.1. Estimates on $\tilde{\xi}$

Consider $\tilde{\xi} = \tilde{\gamma}_1 - \tilde{\gamma}_2$ in \[10\] then $\tilde{\xi}$ satisfies the following variational formulation

$$
\begin{align*}
\rho \int_{\Omega_b} \det(\nabla \tilde{A}_1) \partial_\xi \tilde{\xi} \cdot \tilde{n} \, d\tilde{x} &+ \int_{\Omega_b} \tilde{\alpha}(\tilde{\xi}) : \nabla \tilde{n} \, d\tilde{x} + \rho \int_{\Omega_b} \det(\nabla \tilde{\varphi}_1) \partial_\xi \tilde{\xi} \cdot \tilde{n} \, d\tilde{x} \\
+ \rho \int_{\Omega_b} \partial_\xi \tilde{\gamma}_2 \cdot [\det(\nabla \tilde{A}_1) - \det(\nabla \tilde{A}_2)] \tilde{n} \, d\tilde{x} \quad &+ \rho \int_{\Omega_b} \partial_\xi \tilde{\gamma}_2 \cdot [\det(\nabla \tilde{\varphi}_1) - \det(\nabla \tilde{\varphi}_2)] \tilde{n} \, d\tilde{x} \\
+ \sum_{i, \alpha, \beta, \gamma = 1}^3 \int_{\Omega_b} b_{i \alpha \beta \gamma}(\nabla \tilde{\xi}_1) \partial_\beta(\int_0^t \tilde{\gamma}_1(s) \, ds) \partial_\gamma \tilde{n}_i \, d\tilde{x} \quad &+ \sum_{i, \alpha, \beta, \gamma = 1}^3 \int_{\Omega_b} \partial_\alpha b_{i \alpha \beta \gamma}(\nabla \tilde{\xi}_1) \partial_\beta(\int_0^t \tilde{\gamma}_1(s) \, ds) \partial_\gamma \tilde{n}_i \, d\tilde{x} \\
+ \sum_{i, \alpha, \beta, \gamma = 1}^3 \int_{\Omega_b} \partial_\alpha b_{i \alpha \beta \gamma}(\nabla \tilde{\xi}_2) \partial_\beta(\int_0^t \tilde{\gamma}_2(s) \, ds) \partial_\gamma \tilde{n}_i \, d\tilde{x} \quad &+ \sum_{i, \alpha, \beta, \gamma = 1}^3 \int_{\Omega_b} \partial_\alpha b_{i \alpha \beta \gamma}(\nabla \tilde{\xi}_2) \partial_\beta(\int_0^t \tilde{\gamma}_2(s) \, ds) \partial_\gamma \tilde{n}_i \, d\tilde{x} \\
+ \int_{\Omega_b} F_0 : \nabla \tilde{n} \, d\tilde{x} &\equiv \int_{\Gamma^\varepsilon(0)} G \cdot \tilde{n} \, d\tilde{\Gamma} \quad \forall \, \tilde{n} \in \tilde{W},
\end{align*}
$$

where

$$
\tilde{\alpha}(\tilde{\xi}) = \mu \left[ (\nabla \tilde{\xi} (\nabla \tilde{A}_1))^{-1} + (\nabla \tilde{A}_1)^{-t} (\nabla \tilde{\xi})^t \right] \text{col}(\nabla \tilde{A}_1)
$$

and

$$
F_0 = \mu [\nabla \tilde{\gamma}_2 ((\nabla \tilde{A}_1)^{-1} \text{col}(\nabla \tilde{A}_1) - (\nabla \tilde{A}_2)^{-t} \text{col}(\nabla \tilde{A}_2))] \\
+ \mu [(\nabla \tilde{A}_1)^{-t} (\nabla \tilde{\gamma}_2)^t \text{col}(\nabla \tilde{A}_1) - (\nabla \tilde{A}_2)^{-t} (\nabla \tilde{\gamma}_2)^t \text{col}(\nabla \tilde{A}_2)].
$$

Further, for $i = 1, 2, 3$,

$$
G_i = \sum_{\alpha, \beta, \gamma = 1}^3 \left( \int_0^t \partial_\beta b_{i \alpha \beta \gamma}(\nabla \tilde{\xi}_1) \partial_\gamma \tilde{n}_\alpha \, d\tilde{\tau} \right) \tilde{n}_\beta + \sum_{\alpha, \beta, \gamma = 1}^3 \left( \int_0^t \partial_\beta b_{i \alpha \beta \gamma}(\nabla \tilde{\xi}_2) \partial_\gamma \tilde{n}_\alpha \, d\tilde{\tau} \right) \tilde{n}_\beta.
$$

Moreover, for simplicity, in what follows we set

$$
L_0 = \rho_j \partial_\xi \tilde{\gamma}_2 [\det(\nabla \tilde{A}_1) - \det(\nabla \tilde{A}_2)] \quad \text{and} \quad L_1 = \rho_j \partial_\xi \tilde{\gamma}_2 [\det(\nabla \tilde{\varphi}_1) - \det(\nabla \tilde{\varphi}_2)].
$$
Taking $\bar{\eta} = \bar{\zeta}$ and using the fact that $\bar{\zeta}(0) = 0$, then proceeding as in (145) yield

$$\frac{\rho_s}{2} \int_{\Omega_0^s} \det(\nabla \bar{A}_0)(\bar{\zeta}(t)) d\bar{x} - \frac{\rho_s}{2} \int_{0}^{t} \int_{\Omega_0^s} \partial_t \det(\nabla \bar{A}_0)(\bar{\zeta}(t)) d\bar{x} ds$$

$$+ \int_{0}^{t} \int_{\Omega_0^s} \bar{\sigma}_1(\bar{\zeta}) : \nabla \bar{\zeta} d\bar{x} ds + \frac{\rho_s}{2} \int_{\Omega_0^s} \det(\nabla \bar{\varphi}_1)(\bar{\zeta}(t)) d\bar{x}$$

$$- \frac{\rho_s}{2} \int_{0}^{t} \int_{\Omega_0^s} \partial_s \det(\nabla \bar{\varphi}_1)(\bar{\zeta}(t)) d\bar{x} ds + \int_{0}^{t} \int_{\Omega_0^s} F_0 : \nabla \bar{\zeta} d\bar{x} ds$$

$$+ \int_{0}^{t} \int_{\Omega_0^s} \bar{L}_0 \cdot \bar{\eta} d\bar{x} ds + \int_{0}^{t} \int_{\Omega_0^s} \bar{L}_1 \cdot \bar{\eta} d\bar{x} ds$$

$$+ \frac{1}{2} \sum_{i, \alpha, j, \beta = 1}^{3} \int_{\Omega_0^s} b_{i\alpha j\beta}(\nabla \bar{\xi}_1) \partial_\beta (\int_{0}^{t} \bar{\zeta}(s) ds) \partial_\alpha (\int_{0}^{t} \bar{\zeta}(s) ds) d\bar{x}$$

$$+ \frac{1}{2} \sum_{i, \alpha, j, \beta = 1}^{3} \int_{0}^{t} \int_{\Omega_0^s} \partial_s b_{i\alpha j\beta}(\nabla \bar{\xi}_1) \partial_\beta (\int_{0}^{t} \bar{\zeta}(s) ds) \partial_\alpha (\int_{0}^{t} \bar{\zeta}(s) ds) d\bar{x} ds$$

$$+ \sum_{i, \alpha, j, \beta = 1}^{3} \int_{0}^{t} \int_{\Omega_0^s} \partial_s b_{i\alpha j\beta}(\nabla \bar{\xi}_1) \partial_\beta (\int_{0}^{t} \bar{\zeta}(s) ds) \partial_\alpha (\int_{0}^{t} \bar{\zeta}(s) ds) d\bar{x} ds$$

(143)

We proceed to estimate the terms of (143) in the spirit of (144) by using the fact that

$$||\text{cof}(\nabla \bar{A}_1) - \text{cof}(\nabla \bar{A}_2)||_{L^\infty(H^1)} \leq C||\bar{\xi}_1 - \bar{\xi}_2||_{H^2}$$

$$||\nabla \bar{A}_1||_{L^\infty(H^1)} - ||\nabla \bar{A}_2||_{L^\infty(H^1)} \leq C||\bar{\xi}_1 - \bar{\xi}_2||_{H^2}$$

$$||\det(\nabla \bar{A}_1) - \det(\nabla \bar{A}_2)||_{L^\infty(H^1)} \leq C||\bar{\xi}_1 - \bar{\xi}_2||_{H^2}$$

(144)

which can be established in the similar manner used in Lemma 241.

First, using Lemma 222 we have

$$\frac{\rho_s}{2} \int_{\Omega_0^s} \det(\nabla \bar{\varphi}_1)(\bar{\zeta}(t)) d\bar{x} - \frac{\rho_s}{2} \int_{0}^{t} \int_{\Omega_0^s} \partial_t \det(\nabla \bar{\varphi}_1)(\bar{\zeta}(t)) d\bar{x} ds$$

$$\geq \rho_s (1 - CT^\alpha M) ||\bar{\zeta}||_{L^\infty(L^2(\Omega_0^s))}^2$$

Using (35) and (31), for all $i, \alpha, j, \beta \in \{1, 2, 3\}$ we have

$$||b_{i\alpha j\beta}(\nabla \bar{\xi}_1) - b_{i\alpha j\beta}(\nabla \bar{\xi}_2)||_{L^\infty(H^1)} \leq C||\bar{\xi}_1 - \bar{\xi}_2||_{L^\infty(H^2)}$$

$$||\partial_s b_{i\alpha j\beta}(\nabla \bar{\xi}_1) - \partial_s b_{i\alpha j\beta}(\nabla \bar{\xi}_2)||_{L^\infty(L^2)} \leq C||\bar{\xi}_1 - \bar{\xi}_2||_{L^\infty(H^2)}$$

(145)

and

$$||\partial_\alpha b_{i\alpha j\beta}(\nabla \bar{\xi}_1) - \partial_\alpha b_{i\alpha j\beta}(\nabla \bar{\xi}_2)||_{L^\infty(L^2)} \leq C||\bar{\xi}_1 - \bar{\xi}_2||_{H^1}$$

Then an estimate on $G$ is given by

$$||G||_{H^1(L^2(\Gamma_{\varepsilon}(0)))} \leq CT^\alpha M \left(||\bar{\xi}_1 - \bar{\xi}_2||_{H^2} + ||\bar{\zeta}||_{L^\infty(H^1(\Omega_0^s))}\right)$$

(146)
Hence, proceeding similarly as in (144) we get
\[
\int_{0}^{t} \int_{\Gamma_{\varnothing}(0)} \mathbf{G} \cdot \partial_{\text{a}}(f_{0}^{e}(\xi)(\tau)) d\mathbf{\xi} ds \leq C T^{n} M ||\xi_{1} - \xi_{2}||_{S_{T}^{2}}^{2} + \delta C T^{n} M ||\tilde{\xi}||_{S_{T}^{2}}^{2}.
\] (147)

Taking into consideration (145) and the embedding \( H^{1} \subset L^{6} \) [Theorem 9.9] we obtain
\[
\left| - \sum_{i, \alpha, j, \beta = 1}^{3} \int_{0}^{t} \int_{\Omega_{0}^{e}} \left( b_{\alpha j} \beta(\nabla \xi_{1})(t) \partial_{\beta}(f_{0}^{e}(\xi)(s)) \partial_{\alpha}(f_{0}^{e}(\xi)(s)) d\mathbf{\xi} d\tau \right) \right|
\leq CT M ||\xi_{1} - \xi_{2}||_{L^{\infty}(\Omega_{0}^{e})}^{2} + C T M ||\tilde{\xi}||_{L^{\infty}(H^{1}(\Omega_{0}^{e}))}^{2}.
\]

On the contrary, using (36) and (38) we have
\[
\frac{1}{2} \sum_{i, \alpha, j, \beta = 1}^{3} \int_{0}^{t} \int_{\Omega_{0}^{e}} b_{\alpha j} \beta(\nabla \xi_{1})(t) \partial_{\beta}(f_{0}^{e}(\xi)(s)) \partial_{\alpha}(f_{0}^{e}(\xi)(s)) d\mathbf{\xi} d\tau \geq \mu_{s} ||f_{0}^{e}(\xi)(s)||_{L^{\infty}(\Omega_{0}^{e})}^{2} + \frac{\lambda_{s} + C}{2} \sum_{i, \alpha, j, \beta = 1}^{3} \left| \partial_{\beta}(f_{0}^{e}(\xi)(s)) \partial_{\alpha}(f_{0}^{e}(\xi)(s)) \right|^{2} d\mathbf{\xi} ds \geq \frac{\mu_{s}}{2} (1 - C T^{n} M) ||\tilde{\xi}||_{L^{\infty}(L^{2}(\Omega_{0}^{e}))}^{2}.
\] (148)

Whereas, for the integrals on the fluid domain \( \Omega_{0}^{f} \) we have
\[
\frac{\rho_{T}}{2} \int_{\Omega_{0}^{f}} \det(\nabla \tilde{\mathbf{A}}_{1})(\xi(t))^{2} d\tilde{\xi} + \frac{\rho_{T}}{2} \int_{\Omega_{0}^{f}} \partial_{t} \det(\nabla \tilde{\mathbf{A}}_{1}) d\tilde{\xi} ds \geq \frac{\rho_{T}}{2} (1 - C T^{n} M) ||\tilde{\xi}||_{L^{\infty}(L^{2}(\Omega_{0}^{f}))}^{2}.
\]

On the other hand, for \( F_{0} \) we have
\[
||F_{0}||_{L^{2}(L^{2}(\Omega_{0}^{f}))}^{2} \leq C T ||\tilde{v}_{1} - \tilde{v}_{2}||_{F_{T}^{2}}^{2}.
\] (149)

Then, using Young’s inequality we bound the integral \( \int_{0}^{t} \int_{\Omega_{0}^{f}} F_{0} : \nabla \tilde{\xi} d\tilde{\xi} ds \) as
\[
\left| \int_{0}^{t} \int_{\Omega_{0}^{f}} F_{0} : \nabla \tilde{\xi} d\tilde{\xi} ds \right| \leq C_{T} C T ||\tilde{v}_{1} - \tilde{v}_{2}||_{F_{T}^{2}}^{2} + \delta ||\tilde{\xi}||_{L^{2}(H^{1}(\Omega_{0}^{f}))}^{2}.
\] (150)

In order to deal with the integral in \( L_{0} \) we use (144) and Young’s inequality to get
\[
\left| \int_{0}^{t} \int_{\Omega_{0}^{f}} \partial_{t} \gamma_{2}[\det(\nabla \tilde{\mathbf{A}}_{1}) - \det(\nabla \tilde{\mathbf{A}}_{2})] \tilde{\xi} d\tilde{\xi} ds \right| \leq C_{3} C T ||\tilde{v}_{1} - \tilde{v}_{2}||_{F_{T}^{2}}^{2} + \delta ||\tilde{\xi}||_{L^{2}(H^{1}(\Omega_{0}^{f}))}^{2}.
\] (151)

Similarly, for the integral in \( L_{1} \) we have
\[
\left| \int_{0}^{t} \int_{\Omega_{0}^{f}} \partial_{t} \gamma_{2}[\det(\nabla \tilde{\varnothing}_{1}) - \det(\nabla \tilde{\varnothing}_{2})] \tilde{\xi} d\tilde{\xi} ds \right| \leq C_{3} C T ||\tilde{v}_{1} - \tilde{v}_{2}||_{F_{T}^{2}}^{2} + \delta ||\tilde{\xi}||_{L^{2}(H^{1}(\Omega_{0}^{f}))}^{2}.
\] (152)

Finally, proceeding in a similar manner as Subsection 3.3 with the use of (139)-(152) and taking into consideration that \( T \) is small with respect to \( M \) we get
\[
||\tilde{\xi}||_{F_{T}^{2}}^{2} + \int_{0}^{t} \int_{\Omega_{0}^{f}} \tilde{\xi} d\tilde{\xi} ds \leq C T^{n} M \left[ ||\tilde{\xi}_{1} - \tilde{\xi}_{2}||_{S_{T}^{2}}^{2} + ||\tilde{v}_{1} - \tilde{v}_{2}||_{F_{T}^{2}}^{2} \right].
\] (153)
6.2. Estimates on $\partial_t \bar{\zeta}$

The weak solution $\bar{\zeta}$ satisfies (138). Differentiating (138) in times gives the following variational formulation:

\[
\left\{ \begin{array}{l}
\rho_f \int_{\Omega_0^t} \det(\nabla \tilde{A}_1)|\partial_t \bar{\zeta}(t)|^2 \, d\tilde{x} + \rho_s \int_{\Omega_0^t} |\partial_t \bar{\zeta}(t)|^2 \, d\tilde{x} + \rho_s \int_{\Omega_0^t} |\partial_t (\nabla \tilde{A}_1)|\partial_t \bar{\zeta}(t)|^2 \, d\tilde{x} \\
+ \rho_s \int_{\Omega_0^t} \det(\nabla \tilde{A}_1) \partial_t^2 \bar{\zeta} \cdot \tilde{n} \, d\tilde{x} + \rho_s \int_{\Omega_0^t} \partial_t \det(\nabla \tilde{A}_1) \partial_t \bar{\zeta} \cdot \tilde{n} \, d\tilde{x} + \int_{\Omega_0^t} \partial_t \tilde{\sigma}^0_1(\bar{\zeta}) : \nabla \tilde{n} \, d\tilde{x} \\
+ \partial_t \mathbf{F}_0 : \nabla \tilde{n} \, d\tilde{x} + \int_{\Omega_0^t} \partial_t \mathbf{L}_0 \cdot \tilde{n} \, d\tilde{x} - \int_{\Omega_0^t} \partial_t \mathbf{L}_1 \cdot \tilde{n} \, d\tilde{x} + \int_{\Omega_0^t} \partial_t \tilde{\sigma}_1(\bar{\zeta}) : \partial_t \nabla \bar{\zeta} \, d\tilde{x} \\
+ \int_{\Omega_0^t} \partial_t \mathbf{F}_0 : \partial_t \nabla \bar{\zeta} \, d\tilde{x} + \int_{\Omega_0^t} \partial_t \mathbf{H}_0 : \partial_t (f_0^1 \bar{\zeta}(s)) \, d\tilde{x} - \int_{\Omega_0^t} \partial_t \mathbf{H}_0 : \partial_t \tilde{\zeta} \, d\tilde{x} - \int_{\Omega_0^t} \partial_t \mathbf{L}_0 \cdot \partial_t (f_0^1 \bar{\zeta}(s)) \, d\tilde{x} \\
- \int_{\Omega_0^t} \partial_t \mathbf{L}_1 \cdot \partial_t (f_0^1 \bar{\zeta}(s)) \, d\tilde{x} - \int_{\Omega_0^t} \partial_t \mathbf{F}_0 : \partial_t (f_0^1 \bar{\zeta}(s)) \, d\tilde{x} - \int_{\Omega_0^t} \partial_t \mathbf{F}_0 : \partial_t (f_0^1 \tilde{\zeta}(s)) \, d\tilde{x} \\
+ \int_{\Gamma_s(0)} \left( b_{\alpha \alpha \beta}(\nabla \tilde{\xi}_1) - b_{\alpha \alpha \beta}(\nabla \tilde{\xi}_2) \right) \partial_t^2 (f_0^1 \tilde{\zeta}(s)) \, ds + \int_{\Omega_0^t} \partial_t \tilde{\sigma}_1(\bar{\zeta}) : \partial_t \nabla \bar{\zeta} \, d\tilde{x} \\
\end{array} \right.
\]

(154)

where $\tilde{\sigma}^0_1(\bar{\zeta})$, $\mathbf{F}_0$, $\mathbf{L}_0$ and $\mathbf{L}_1$ are defined in (139) and (140)–(142), respectively. Take $\tilde{n} = \partial_t \bar{\zeta}$ in (154) to get

\[
\left\{ \begin{array}{l}
\frac{\rho_f}{2} \frac{d}{dt} \int_{\Omega_0^t} \det(\nabla \tilde{A}_1)|\partial_t \bar{\zeta}(t)|^2 \, d\tilde{x} + \frac{\rho_s}{2} \frac{d}{dt} \int_{\Omega_0^t} |\partial_t \bar{\zeta}(t)|^2 \, d\tilde{x} + \frac{\rho_s}{2} \frac{d}{dt} \int_{\Omega_0^t} |\partial_t (\nabla \tilde{A}_1)|\partial_t \bar{\zeta}(t)|^2 \, d\tilde{x} \\
+ \frac{1}{2} \frac{d}{dt} \int_{\Omega_0^t} \sum_{i, \alpha, j, \beta=1}^3 b_{\alpha \alpha \beta}(\nabla \tilde{\xi}_1) \partial_t^2 (f_0^1 \bar{\zeta}(s)) \, ds \, d\tilde{x} \\
- \frac{1}{2} \int_{\Omega_0^t} \partial_t b_{\alpha \alpha \beta}(\nabla \tilde{\xi}_1) \partial_t^2 (f_0^1 \bar{\zeta}(s)) \, ds \, d\tilde{x} \\
+ \frac{3}{2} \int_{\Omega_0^t} \partial_t b_{\alpha \alpha \beta}(\nabla \tilde{\xi}_1) \partial_t^2 (f_0^1 \bar{\zeta}(s)) \, ds \, d\tilde{x} \\
- \frac{3}{2} \int_{\Omega_0^t} \partial_t b_{\alpha \alpha \beta}(\nabla \tilde{\xi}_1) \partial_t^2 (f_0^1 \bar{\zeta}(s)) \, ds \, d\tilde{x} \\
= \int_{\Gamma_s(0)} \left( b_{\alpha \alpha \beta}(\nabla \tilde{\xi}_1) - b_{\alpha \alpha \beta}(\nabla \tilde{\xi}_2) \right) \partial_t^2 (f_0^1 \tilde{\zeta}(s)) \, ds \\
\end{array} \right.
\]

(155)

where

\[
H_{0,i} = \sum_{\alpha, \beta=1}^3 \left( b_{\alpha \alpha \beta}(\nabla \tilde{\xi}_1) - b_{\alpha \alpha \beta}(\nabla \tilde{\xi}_2) \right) \partial_t^2 (f_0^1 \tilde{\zeta}(s)) \quad \text{for } i = 1, 2, 3.
\]

(156)

**Step 1**

Now we proceed to derive some estimates on

\[
\bar{\zeta}|_{\Omega_0^t} \in H^1(H^1) \cap W^{1,\infty}(L^2), \quad \bar{\zeta}|_{\Omega_0^t} \in W^{1,\infty}(L^2) \cap L^{\infty}(H^1) \quad \text{and} \quad \int_0^t \bar{\zeta}(s)|_{\Omega_0^t} \, ds \in L^{\infty}(H^1).
\]

First, we have

\[
\int_{\Omega_0^t} \det(\nabla \tilde{A}_1)|\partial_t \bar{\zeta}(t)|^2 \, d\tilde{x} + \int_0^t \int_{\Omega_0^t} \partial_t \det(\nabla \tilde{A}_1)|\partial_t \bar{\zeta}|^2 \, d\tilde{x} \, ds \geq (1 - C T^\alpha M)\|\bar{\zeta}\|_{W^{1,\infty}(\Omega_0^t)}^2 - CT^\alpha M\|\bar{\zeta}\|_{L^2}^2.
\]

(157)
Whereas, for the fluid stress term, proceeding as in (158) and (160) we get
\[ \int_0^t \int_{\Omega_0^s} \partial_t \tilde{\sigma}_1(\tilde{\zeta}) : \partial_n \nabla \tilde{\zeta} \, d\tilde{x} \, ds \geq \mu C_k \| \partial_t \tilde{\zeta} \|_{L^2(H^1(\Omega_0^s))}^2 - \mu C T^\alpha M \| \partial_t \tilde{\zeta} \|_{L^2(H^1(\Omega_0^s))}^2. \]  
(158)

For \( \int_{\Omega_0^s} \partial_t F_0 : \partial_t \nabla \tilde{\zeta} \, d\tilde{x} \), we argue as in (159) to obtain
\[ \int_{\Omega_0^s} \partial_t F_0 : \partial_t \nabla \tilde{\zeta} \, d\tilde{x} \leq C_\delta \| \tilde{\vartheta}_1 - \tilde{\vartheta}_2 \|_{F^T}^2 + \int_{\Omega_0^s} \delta \| \partial_t \nabla \tilde{\zeta} \|_2^2 \, d\tilde{x}. \]
(159)

Similarly, we have
\[ \int_{\Omega_0^s} \partial_t L_0 : \partial_t \tilde{\zeta} \, d\tilde{x} \leq C_\delta \| \tilde{\vartheta}_1 - \tilde{\vartheta}_2 \|_{F^T}^2 + \delta \| \partial_t \nabla \tilde{\zeta} \|_{L^2(\Omega_0^s)}^2. \]
(160)

Combining (159) - (160) and integrating over \((0, t)\) we get
\[ \rho_f \| \tilde{\zeta} \|_{W^{1, \infty}(L^2)} + \mu \| \tilde{\zeta} \|_{H^1(H^1)} - C T^\alpha M \| \partial_t \tilde{\zeta} \|_{L^2(H^1)} \leq C T \| \tilde{\vartheta}_1 - \tilde{\vartheta}_2 \|_{F^T}^2 + \delta \| \partial_t \nabla \tilde{\zeta} \|_{L^2(\Omega_0^s)}^2. \]
(161)

As for the integrals on the domain \(\Omega_0^s\), first we have
\[ \int_{\Omega_0^s} \det(\nabla \tilde{\varphi}_1) |\partial_t \tilde{\zeta}(t)|^2 \, d\tilde{x} + \int_0^t \int_{\Omega_0^s} \partial_t \det(\nabla \tilde{\varphi}_1) |\partial_t \tilde{\zeta}|^2 \, d\tilde{x} \, ds \geq (1 - C T^\alpha M) \| \tilde{\zeta} \|_{W^{1, \infty}(L^2(\Omega_0^s))} - C T^\alpha M \| \tilde{\zeta} \|_{L^2(\Omega_0^s)}^2. \]
(162)

Further, using (26) then taking supremum over \((0, T)\) yield
\[ \frac{1}{2} \sum_{i, \alpha, j, \beta = 1} \int_{\Omega_0^s} \left[ b_{i\alpha\beta}(\nabla \tilde{\xi}_1) \partial_\beta \tilde{\zeta}_j, \partial_\alpha \tilde{\zeta}_i \right](t) \, d\tilde{x} \]
(163)
\[ \geq \mu_s \| \tilde{\zeta} \|_{L^\infty(H^1(\Gamma_0^s))}^2 + \frac{\lambda_s + \zeta}{2} \| \nabla \cdot \tilde{\zeta} \|_{L^\infty(L^2(\Omega_0^s))}^2 - C T^\alpha M \| \tilde{\zeta} \|_{L^2(\Omega_0^s)} \]

On the other hand, using (15) we have
\[ \left| - \frac{1}{2} \sum_{i, \alpha, j, \beta = 1} \int_{\Omega_0^s} \partial_\beta b_{i\alpha\beta}(\nabla \tilde{\xi}_1) \partial_\beta \tilde{\zeta}_j, \partial_\alpha \tilde{\zeta}_i \, d\tilde{x} + \sum_{i, \alpha, j, \beta = 1} \int_{\Omega_0^s} \partial_\beta b_{i\alpha\beta}(\nabla \tilde{\xi}_1) \partial_\beta \tilde{\zeta}_j, \partial_\alpha \tilde{\zeta}_i \, d\tilde{x} \right| \]
(164)
\[ \leq C T^\alpha M \| \tilde{\zeta} \|_{L^2(\Omega_0^s)}^2. \]

In order to estimate \( \int_{\Omega_0^s} \partial_t H_0 \cdot \partial_t^2 (\int_0^t \tilde{\zeta}(s) \, ds) \, d\tilde{x} \) we use the following two inequalities
\[ \| b_{i\alpha\beta}(\nabla \tilde{\xi}_1) - b_{i\alpha\beta}(\nabla \tilde{\xi}_2) \|_{L^\infty(H^1)} \leq C \| \tilde{\xi}_1 - \tilde{\xi}_2 \|_{L^\infty(H^2(\Omega_0^s))} \]
(165)

and
\[ \| \partial_t b_{i\alpha\beta}(\nabla \tilde{\xi}_1) - \partial_t b_{i\alpha\beta}(\nabla \tilde{\xi}_2) \|_{L^\infty(L^2)} \leq C \| \tilde{\xi}_1 - \tilde{\xi}_2 \|_{W^{1, \infty}(H^1(\Omega_0^s))}. \]
(166)

These inequalities together with Young’s inequality give
\[ \int_{\Omega_0^s} \partial_t H_0 \cdot \partial_t^2 (\int_0^t \tilde{\zeta}(s) \, ds) \, d\tilde{x} \leq C \delta \left( \| \tilde{\xi}_1 - \tilde{\xi}_2 \|_{W^{1, \infty}(H^1)} + \| \tilde{\xi}_1 - \tilde{\xi}_2 \|_{L^\infty(H^2)}^2 + \delta \| \partial_t \tilde{\zeta} \|_{L^\infty(L^2(\Omega_0^s))} \right). \]
(167)

Further,
\[ \int_{\Omega_0^s} \partial_t L_1 \cdot \partial_t^2 (\int_0^t \tilde{\zeta}(s) \, ds) \, d\tilde{x} \leq C \delta \| \tilde{\xi}_1 - \tilde{\xi}_2 \|_{S^2}^2 + \delta \| \partial_t \tilde{\zeta}(t) \|_{L^2(\Omega_0^s)}^2. \]
(168)
Finally, thanks to the trace inequality and (145), for $i, \alpha, j, \beta \in \{1, 2, 3\}$ we have
\[
\left\| \int_{\Gamma_c(0)} \left( b_{i\alpha j\beta} (\nabla \tilde{\xi}_1) - b_{i\alpha j\beta} (\nabla \tilde{\xi}_2) \right) \partial_i \tilde{\zeta} (\int_0^1 \tilde{\gamma}_2 (s) ds) \partial_j \tilde{\zeta} (\int_0^1 \tilde{\zeta} (s) ds) d\Gamma \right\|_{L^1(\Gamma_c(0))} \leq \| b_{i\alpha j\beta} (\nabla \tilde{\xi}_1) (t) - b_{i\alpha j\beta} (\nabla \tilde{\xi}_2) (t) \|_{H^1(\Gamma_c(0))} \| \nabla \tilde{\gamma}_2 (t) \|_{H^2(\Omega_1^0)} \| \partial_t \tilde{\zeta} (t) \|_{H^1(\Omega_1^0)}.
\]
Hence, after combining (162) - (164) and (167) - (169) then integrating over $(0, t)$ we obtain
\[
\frac{\mu}{2} \| \tilde{\zeta} \|_{L_\infty(L^2(\Omega_1^0))} + \mu_s \| \tilde{\zeta} \|_{L^2(H^1(\Omega_1^0))} \leq CT \left[ \| \tilde{\xi}_1 - \tilde{\xi}_2 \|_{W^{1, \infty}(H_1)} + \| \tilde{\xi}_1 - \tilde{\xi}_2 \|_{L_\infty(H_2)} \right].
\]

**Step 2**

Our next step is to estimate $\tilde{\zeta}|_{\Omega_1^0} \in L_\infty(H^2(\Omega_1^0))$ and $\int_0^1 \tilde{\zeta} (s) |_{\Omega_1^0} ds \in L_\infty(H^2(\Omega_1^0))$. The fluid velocity $\tilde{\zeta}|_{\Omega_1^0}$ satisfies the following elliptic equation
\[
- \nabla \cdot \left( (\nabla \tilde{\zeta}) |_{\Omega_1^0} + (\nabla \tilde{\zeta}) |_{\Omega_1^0} \right) = \nabla \cdot F_0 + \nabla \cdot F_1 + \nabla \cdot L_0 - \det(\nabla A) \partial_t \tilde{\zeta}|_{\Omega_1^0} \text{ in } \Omega_1^0
\]
where $F_0$ is defined in (140) and
\[
F_1 = \nabla \tilde{\zeta} \left( \mathbb{I}_d - (\nabla A)^{-1} \text{cof}(\nabla A) \right) + \left( \nabla \tilde{\zeta} \right)^t - \det(\nabla A) \partial_t \tilde{\zeta}|_{\Omega_1^0}.
\]
We have
\[
\| \nabla \cdot F_0 \|_{L_\infty(L^2(\Omega_1^0))} \leq \| F_0 \|_{L_\infty(H^1(\Omega_1^0))} \leq CT \| \tilde{v}_1 - \tilde{v}_2 \|_{F_2^T}
\]
and
\[
\| \nabla \cdot L_0 \|_{L_\infty(L^2(\Omega_1^0))} \leq \| L_0 \|_{L_\infty(H^1(\Omega_1^0))} \leq CT \| \tilde{v}_1 - \tilde{v}_2 \|_{F_2^T}.
\]
For $F_1$ we have
\[
\| \nabla \cdot F_1 \|_{L_\infty(L^2(\Omega_1^0))} \leq \| \nabla \cdot \left[ \nabla \tilde{\zeta} \left( \mathbb{I}_d - (\nabla A)^{-1} \text{cof}(\nabla A) \right) \right] \|_{L_\infty(L^2(\Omega_1^0))} + \| \nabla \cdot \left[ \left( \nabla \tilde{\zeta} \right)^t - \det(\nabla A) \partial_t \tilde{\zeta} \right] \|_{L_\infty(L^2(\Omega_1^0))}. \]
For the term $B_1$ we use the embedding of $H^3 \subset L_\infty$ and Lemma (2.1) to get
\[
B_1 \leq \left\| \nabla \cdot \left[ \nabla \tilde{\zeta} \left( \mathbb{I}_d - (\nabla A)^{-1} \text{cof}(\nabla A) \right) \right] \right\|_{L_\infty(L^2)} \leq CT^n M \| \tilde{\zeta} \|_{L_\infty(H^2)}.
\]
On the other hand,
\[
\| B_2 \|_{L_\infty(L^2)} \leq CT^n M \| \tilde{\zeta} \|_{L_\infty(H^2)}.
\]
Consequently, we obtain
\[
\| \nabla \cdot F_1 \|_{L_\infty(L^2)} \leq CT^n M \| \tilde{\zeta} \|_{L_\infty(H^2)}.
\]
Therefore, $\tilde{\zeta}|_{\Omega_1^0} \in L_\infty(H^2(\Omega_1^0))$ and
\[
\mu \| \tilde{\zeta} \|_{L_\infty(H^2(\Omega_1^0))} \leq C \| \partial_t \tilde{\zeta} \|_{L_\infty(L^2(\Omega_1^0))} + CT \| \tilde{v}_1 - \tilde{v}_2 \|_{F_2^T}.
\]
Besides, the displacement \( f_0^t \zeta(s)ds \), satisfies the following equation

\[-\mu_1 \nabla \cdot (\nabla \int_0^t \zeta(s)ds + (\nabla \int_0^t \zeta(s)ds)^T) = H_0 + H_1 + H_2 - \nabla \cdot L_1, \tag{178}\]

where \( H_0 \) is defined by (156). As for \( H_1 \), it is given by

\[H_{1,i} = - \sum_{\alpha,j=1}^3 b_{\alpha j} b_{\alpha j} (\nabla \xi) + b_{\alpha j} \nabla \xi \] \(\partial_{\alpha j}(\int_0^t \zeta(s)ds) \) \( \text{for} \ i = 1, 2, 3, \tag{179}\]

and the expression of \( H_2 \) is

\[H_2 = -\text{det}(\nabla \varphi_1)\partial_t \zeta.\]

Using (165) and (166) we have

\[||H_0||_{L^\infty(L^2(\Omega_0^5))} \leq \sum_{i,\alpha,j=1}^3 ||b_{\alpha j}(\nabla \xi) - b_{\alpha j}(\nabla \xi)||_{L^\infty(L^2(\Omega_0^5))} ||\int_0^t \zeta(s)ds||_{L^\infty(H^2(\Omega_0^5))} \]

\[\leq CT^n M ||\tilde{\xi}_1 - \tilde{\xi}_2||_{L^\infty(H^1(\Omega_0^5))}. \tag{180}\]

For \( H_1 \) we use (35) to obtain

\[||H_1||_{L^\infty(L^2(\Omega_0^5))} \leq \sum_{i,\alpha,j=1}^3 ||b_{\alpha j}(\nabla \xi) + b_{\alpha j}(\nabla \xi)||_{L^\infty(L^2)} ||\zeta||_{L^\infty(H^2(\Omega_0^5))} \]

\[\leq CT(M + M^2 ||\int_0^t \zeta(s)ds||_{L^\infty(H^2(\Omega_0^5))}). \]

In addition, we have

\[||H_2||_{L^\infty(L^2(\Omega_0^5))} \leq CT||\text{det}(\nabla \varphi_1)\partial_t \zeta||_{L^\infty(L^2(\Omega_0^5))} \leq CT M ||\int_0^t \zeta(s)ds||_{W^{2,\infty}(L^2(\Omega_0^5))}. \]

Finally, for \( L_1 \) it holds

\[||\nabla \cdot L_1||_{L^\infty(L^2(\Omega_0^5))} \leq ||L_1||_{L^\infty(H^1(\Omega_0^5))} \leq CT||\tilde{\xi}_1 - \tilde{\xi}_2||_{S^2}. \]

Whence, \( \int_0^t \zeta(s)ds \in L^\infty(H^2(\Omega_0^5)) \) and a priori estimate is given as

\[\mu_1 ||\int_0^t \zeta(s)ds||_{L^\infty(H^2(\Omega_0^5))} \leq CT||\tilde{\xi}_1 - \tilde{\xi}_2||_{S^2}. \tag{181}\]

Therefore, combining estimates (164), (170), (177) and (181) we arrive to

\[||\tilde{\xi}_1||_{H^2} + ||\int_0^t \zeta(s)ds||_{S^2} \leq CT \times \tilde{\xi}_1 - \tilde{\xi}_2||_{S^2}. \tag{182}\]

Taking \( T \) small with respect to \( M \) gives that \( \Psi \) is a contraction on \( A^2_{S^2} \). This yields the existence of a unique solution \((\tilde{\nu}, \tilde{\xi})\) in \( A^2_{S^2} \) of the non-linear coupled system (53)-(55).

7. Existence and Uniqueness of the Fluid Pressure

7.1. Existence and Uniqueness of an \( L^2 \)-Pressure

After we have proved the existence and uniqueness of the fluid velocity \( \nu \) and the structure displacement \( \xi \), we need to prove the existence of the fluid pressure \( p_f \) so that the proof of the existence of the weak solution for the coupled system (53)-(55) is complete. The proof of existence of the \( L^2 \) function \( p_f \) is based on Lemma [16, p.58, Lemma 4.1] [1] that reduces the proof to showing that the following inf-sup condition holds for the functional spaces \( \{W, L^2(\Omega_f(t))\} \):

\[\inf_{q \in L^2(\Omega_f(t))} \sup_{z \in W} \frac{b(z, q)}{||z||_{H^2(\Omega_f(t))} ||q||_{L^2(\Omega_f(t))}} \geq C_1 > 0, \tag{183}\]

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with
\[ b(z, q) = -\int_{\Omega_f(t)} q \, \text{div} \, z \, dx \quad \text{and} \quad z \in W, \; q \in L^2(\Omega_f(t)). \] (184)

\textbf{Theorem 7.1.} The inf-sup condition \[183\] holds for the functional spaces \( \{W, L^2(\Omega_f(t))\} \).

\textbf{Proof.} We will proceed in a similar manner as \[1, Lemma 3.1\]. To show that the condition holds, it suffices to show that
\[ \forall \; q \in L^2(\Omega_f(t)), \exists \; z \in W \; \text{such that} \; \text{div} z|_{\Omega_f(t)} = q \; \text{in} \; \Omega_f(t) \] (185)
and
\[ ||z||_{H^1(\Omega(t))} \leq C_1 ||q||_{L^2(\Omega_f(t))}. \] (186)

Let \( \overline{q} \in L^2(\Omega(t)) \) be the extension of \( q \) obtained by defining
\[ \overline{q} = \frac{1}{|\Omega_\Lambda(t)|} \int_{\Omega_\Lambda(t)} q \, dx. \] (187)

Note that \( \int_{\Omega(t)} \overline{q} \, dx = \int_{\Omega_f(t)} \overline{q} \, dx + \int_{\Omega_s(t)} \overline{q} \, dx = 0 \), this gives \( \overline{q} \in L^2_0(\Omega(t)) \). Hence, by the virtue of \[5, Theorem IV.3.1\], there exists a unique \( z \in H^1_0(\Omega(t)) \) such that
\[ \text{div} z = \overline{q} \; \text{on} \; \Omega(t) \; \text{and} \; ||z||_{H^1(\Omega(t))} \leq C_1 ||\overline{q}||_{L^2(\Omega(t))}. \] (188)

Since \( H^1_0(\Omega(t)) \subset W \), then \( z \in W \). Moreover, by restricting \( \text{div} z = \overline{q} \) to \( \Omega_f(t) \) we get that \( \text{div} z|_{\Omega_f(t)} = q \). Therefore \[185\] is proved, consequently the inf-sup Condition \[183\] is verified.

By the end of this proof, we get the existence of a pressure \( p_f \in L^\infty(L^2(\Omega_f(t))) \) which is unique due to \[5, Theorem IV.2.4\].

7.2. Regularity of the Fluid Pressure

The fluid pressure \( p_f \) is related to the fluid velocity \( v \) by the Navier-Stokes equations. Indeed, at \( t = 0 \) we have
\[ \rho_f \text{det}(\nabla \sigma) \partial_\tau \tilde{v} - \nabla \cdot \tilde{\sigma}^0(\tilde{v}, \tilde{p}_f) = 0 \quad \text{in} \; \Omega_f(0) \times (0, T), \]
As a result, the regularity of \( \tilde{p}_f \) is linked to the regularity of \( \tilde{v} \) which is proved straightforward using Ne\v{c}as inequality \[4, Theorem IV.1.1\]. Therefore, as \( \tilde{v} \in F^T_3 \) then \( \tilde{p}_f \in P^T_3 \). Again using \[20, Lemma 2.56\], we get the existence and uniqueness of a fluid pressure \( p_f \) in the set \( Q^T_3 \) which is equivalent to \( P^T_3 \) where the functions of \( Q^T_3 \) are defined over \( \Omega_f(t) \). To this end, we have proved the existence and uniqueness locally in time of a solution \((v, \xi, p_f)\) of the non-linear coupling problem of an incompressible fluid with a quasi-incompressible structure.

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