The equation of state for scalar-tensor gravity

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We show that the field equation of Brans-Dicke gravity and scalar-tensor gravity can be derived as the equation of state of Rindler spacetime, where the local thermodynamic equilibrium is maintained. Our derivation implies that the effective energy can not feel the heat flow across the Rindler horizon.

Motivated by black hole mechanics and Hawking radiation, it has become increasingly clear that there is a deep connection between gravity and thermodynamics. A key evidence of this connection was discovered by Jacobson,\textsuperscript{1)} who derived the Einstein equation as an equation of state of local Rindler spacetime by constructing the equilibrium thermodynamics $\delta Q = T dS$. This derivation suggests a gravitational theory built upon the principle of equivalence must be thought of as the macroscopic limit of some underlying microscopic theory.

The relation between gravity and thermodynamics has been found also in other spacetimes, including a general static spherically symmetric spacetime,\textsuperscript{2)} the quasi-de Sitter geometry of inflationary universe,\textsuperscript{3)} the expanding universe,\textsuperscript{4)} the quintessence dominated accelerating universe,\textsuperscript{5)} and the spacetime with any spatial curvature.\textsuperscript{6)} The relation has been further disclosed in gravity theories beyond Einstein gravity, including Lovelock gravity,\textsuperscript{6), 7)} braneworld gravity,\textsuperscript{8)} nonlinear gravity,\textsuperscript{7), 9), 10)} and scalar-tensor gravity\textsuperscript{7), 10), 11)} etc. However, in the nonlinear gravity and scalar-tensor gravity, it was argued that the non-equilibrium thermodynamics instead of the equilibrium thermodynamics should be taken into account to build the relation to gravity.\textsuperscript{7), 9), 11)}

An alternative treatment to reinterpretate of the nonequilibrium correction was introduced in,\textsuperscript{12)} where a mass-like function is introduced to connect the first law of thermodynamics and Friedmann equations in some gravity theories. Recently, using the idea of “local-boost-invariance” introduced in,\textsuperscript{13)} as well as a more general definition of local entropy, the nonlinear $f(R)$ gravity field equation is derived from the Rindler space-time thermodynamics, maintaining the local thermodynamic equilibrium.\textsuperscript{14)} In the present letter, we want to know whether the method developed in\textsuperscript{14)} holds or not in more general scalar-tensor gravity. Compared with the $f(R)$ gravity, there is the similar extended local entropy, but a new degree of freedom, that is an additional scalar field mediating the gravitational interaction.

Let us begin with briefly introducing (see the detail review in\textsuperscript{15)}) the local Rindler

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spacetime where we will construct the thermodynamics. In the vicinity of any spacetime point \( p \), a free-falling observer can use the equivalence principle to describe his local coordinate system as flat. Then the observer can locally define a causal horizon as follows. Choose a spacelike surface patch \( B \) including \( p \) and choose one side of the boundary of the past of \( B \). Near the point \( p \), this boundary is a congruence of the null geodesics orthogonal to \( B \). These comprise the horizon. The called "local Rindler horizon", is adjusted so that the generators of the causal horizon have vanishing expansion \( \theta \) and shear \( \sigma \).

Jacobson\(^1\) pointed out that we can construct equilibrium thermodynamics of local Rindler horizon at \( p \). The thermodynamics is described by Clausius relation \( \delta Q = T dS \). Here the entropy \( S \) is associated with the causal horizon, suggested by the observation that they hide information. Jacobson suspected the entropy measuring the “many degrees of freedom outside”, what presumably results in entanglement entropy just at the horizon. He then assumed that the entropy is proportional to horizon area, namely the proportionality between entropy and the horizon area, formulated by \( S = \eta A \). Jacobson takes \( \eta \) as an unknown proportionality constant in Einstein gravity. In \( f(R) \) gravity, it is interesting to find that,\(^{14}\) through replacing the proportionality constant with a field-dependent effective constant and using the boost-invariant truncation, similar equilibrium thermodynamics can be constructed. The field-dependent effective constant is motivated by a result that, for a static black hole in \( f(R) \) gravity, entropy can be re-expressed as the 1/4 area of the bifurcation surface in units of an effective Newton constant.\(^{16}\) In the present letter, we will also use field-dependent \( \eta \) because the black hole entropy in scalar-tensor gravity is also 1/4 of horizon area in units of the corresponding effective Newton constant.\(^{17}\)

The heat \( Q \) is interpreted as the mean flux of (boost) energy across the horizon. In Einstein gravity, Jacobson assumed that all the heat flow across the horizon is the energy carried by matter. In \( f(R) \) gravity, similar perspective is adopted.\(^{14}\) For scalar-tensor gravity, here we simply assume the heat flow across the horizon is the energy carried by matter and scalar field. However, in the later of this letter, we will point out that this definition of the energy (and heat) is a little coarse and an important implication of this definition. To give the expression of heat flux, we consider an approximate boost killing vector \( \chi \) which is future pointing on the causal horizon and vanishes at the space-time point \( p \). Killing vector \( \chi \) has a relation with the tangent vector of causal horizon \( K \): \( \chi = -k\lambda K + O(\lambda^2) \), where \( k \) is the acceleration of \( \chi \) and \( \lambda \) is the affine parameter of the corresponding null geodesic line such that \( \lambda = 0 \) at \( p \). Thus, the heat flux is given by \( \delta Q = \int T_{ab} \chi^a d\Sigma^b \), where \( T_{ab} \) denotes the energy–momentum tensor of matter and scalar field, and the integral is taken over a small region of pencil of generators of the horizon terminating at \( p \). If the area element of horizon is \( dA \) then \( d\Sigma^b = -K^b d\lambda dA \). Thus, the final expression for the variation of heat, at leading order is

\[
\delta Q = - \int k\lambda T_{ab} K^a K^b d\lambda dA. \tag{1}
\]

The quantum vacuum in flat spacetime for the generator of Lorentz boosts could be treated as a Gibbs ensemble with temperature \( T_0 = 1/2\pi \) in units of Planck constant.
Note that a uniformly accelerated observer behaves as if immersed in a thermal bath at the Unruh temperature\textsuperscript{18}

\[ T = kT_0 = \frac{k}{2\pi} \]  

(2)

Now we consider the variation of entropy expression \( S = \eta A \). Change in the horizon area is given in terms of the expansion of the congruence of null geodesics generating the horizon \( \delta A = \int \theta d\lambda dA \). Since the equation of geodesic deviation for null geodesic congruence is given by the Raychaudhuri equation \( d\theta/d\lambda = -\theta^2/2 - \sigma_{ab}\sigma^{ab} - R_{ab}K^aK^b \). Considering vanishing shear \( \sigma \) and expansion \( \theta \) terms in local Rindler coordinate system we get the solution \( \theta = -\lambda R_{ab}K^aK^b \) at leading order in \( \lambda \). Therefore, the relevant expression for the variation of entropy is evaluated

\[ \delta S = -\lambda \int (\eta R_{ab} - \nabla_a \nabla_b \eta) K^aK^b d\lambda dA. \]  

(3)

As noticed in,\textsuperscript{14} in this expression \((\eta, K^a\nabla_a \eta)\) are to be evaluated at its leading contribution in \( \lambda \). Its boost-invariant part at first order in \( \lambda \) has been used to effectively incorporate the boost invariant notion of creating an “approximated bifurcation point at the first order in \( \lambda \) at p.\textsuperscript{13} Considering the condition that Clausius relation \( \delta Q = T \delta S \) with heat (1), temperature (2), and entropy (3) is satisfied for all vectors \( K \), one can obtain

\[ T^{ab} = \frac{\eta}{2\pi} R^{ab} - \nabla^a \nabla^b \eta \frac{\eta}{2\pi} + g^{ab} H, \]  

(4)

where the new term about \( H \) is added since the tangent vector \( K \) generating horizon has null norm. Hence, at this point we have a local equation with two unknown functions \((\eta, H)\). We will show the functions may be determined by imposing the divergence free for the energy–momentum tensor of matter.

Before going any further, it is time to introduce the concrete gravity theory. We first consider the simple Brans-Dicke gravity. Its Lagrangian is\textsuperscript{19}

\[ L = \phi R - \frac{w}{\phi} g^{ab} \partial_a \phi \partial_b \phi + L_m, \]  

(5)

where Brans-Dicke scalar field \( \phi \) plays the role of the effective gravitational constant \( (\phi = \frac{1}{8\pi G_{\text{eff}}}) \), and \( L_m \) is the Lagrangian of ordinary matter. Varying the action, we have the equation of motion of scalar field

\[ 2w \nabla^2 \phi - \frac{w}{\phi} (\nabla \phi)^2 + \phi R = 0, \]  

(6)

and gravitational field equation

\[ G_{ab} = \frac{1}{\phi} \left( T^{m}_{ab} + T^{\phi}_{ab} + T^{e}_{ab} \right), \]  

(7)

where

\[ T^{\phi}_{ab} = \frac{w}{\phi} \left[ \nabla_a \phi \nabla_b \phi - \frac{1}{2} g_{ab} (\nabla \phi)^2 \right], \]  

(8)
\[ T_{ab}^\phi = -g_{ab} \nabla^2 \phi + \nabla_a \nabla_b \phi. \quad (9) \]

As aforesaid, we have simply assumed the heat flow across the horizon is the energy carried by matter and scalar field. Now we will discuss it carefully. It is clear that \( T_{ab}^m \) denotes the energy–momentum tensor of matter as it comes from the variation of the matter action with respect to metric. But what is the energy-momentum tensor of scalar field? We only know when the first term in (5) vanishes, \( T_{ab}^\phi \) is the energy–momentum tensor of scalar field, since it comes from varying the purely scalar field parts of the action with respect to metric, but do not know how to explain the remained \( T_{ab}^e \). In Ref., (20) it was argued that the appropriate definition of the scalar field energy-momentum tensor should be formulated in Einstein frame. Obviously, it is not helpful in our situation. One is tempted to identify all scalar field terms \( T_{ab}^\phi + T_{ab}^e \) with the energy-momentum tensor of the scalar field by comparing the field equation (7) with Einstein field equation. Naively, \( T_{ab}^e \) can be explained as a kind of effective energy. But we will show the unknown functions \((\eta, H)\) and the correct field equation (7) can be obtained if we take \( T_{ab}^\phi \) as the energy–momentum tensor of scalar field instead of the \( T_{ab}^\phi + T_{ab}^e \).

Taking \( T_{ab}^\phi \) as the energy–momentum tensor of scalar field, the local equation (4) reads

\[ T_{ab}^m + T_{ab}^\phi = \frac{\eta}{2\pi} R^a_b - \nabla_a \nabla^b \frac{\eta}{2\pi} + g^{ab} H. \]

Imposing the matter conservation \( \nabla_a T_{ab}^m = 0 \), the above equation can be expanded as

\[ \nabla_a T_{ab}^\phi = \frac{\eta}{4\pi} \nabla^b R + (\nabla_a \nabla^b \nabla \nabla - \nabla^a \nabla^2) \frac{\eta}{2\pi} + \nabla^b H - \frac{1}{2} (\nabla^2 \nabla^b + \nabla_a \nabla^b \nabla_a) \frac{\eta}{2\pi}. \quad (10) \]

Setting \( H = h + \nabla^2 \frac{\eta}{2\pi} \), Eq. (10) can be written as

\[ \nabla_a T_{ab}^\phi = \frac{\eta}{4\pi} \nabla^b R + \nabla^b h. \quad (11) \]

Taking covariant divergence of \( T_{ab}^\phi \) (8)

\[ \nabla_a T_{ab}^\phi = \nabla_a \left[ \frac{w}{\phi} \nabla^a \phi \nabla^b \phi \right] - \frac{1}{2} g^{ab} \nabla^2 \phi \]

\[ = \frac{1}{2\phi} \nabla^b \phi \left[ 2w \nabla^2 \phi - \frac{w}{\phi} (\nabla \phi)^2 \right], \]

we have

\[ \frac{1}{2\phi} \nabla^b \phi \left[ 2w \nabla^2 \phi - \frac{w}{\phi} (\nabla \phi)^2 \right] = \frac{\eta}{4\pi} \nabla^b R + \nabla^b h. \]

Using the equations of motion (6), we obtain an integrability condition

\[ \frac{\eta}{4\pi} \nabla^b R + \nabla^b h + \frac{1}{2} \nabla^b \phi R = 0. \quad (12) \]

Obviously, this condition can be solved if

\[ \eta = 2\pi \phi, \quad h = -\frac{\phi R}{2}. \quad (13) \]
Substituting expressions (13) into equation (4), we finally give the equation of state

\[ T_{mab} + T^{\phi ab} = \phi R^{ab} - \nabla^a \nabla^b \phi + g^{ab} \nabla^2 \phi - \frac{g^{ab} \phi R}{2} \]

\[ = \phi G^{ab} - \left( \nabla^a \nabla^b \phi - g^{ab} \nabla^2 \phi \right) \]

which is just the correct field equation (7). Also \( \eta = 2\pi \phi \) results that the entropy expression \( S = \eta A \) is identical with the expected black hole entropy \( S = \frac{A}{4 \sigma_{\text{eff}}} \) in Brans-Dicke gravity.\(^{17}\)

The thermodynamic derivation for the equation of state can be easily extended to general scalar-tensor theory of gravity with the Lagrangian

\[ L = \frac{1}{2} F(\phi) R - \frac{1}{2} (\nabla \phi)^2 - V(\phi) + L_m, \]

In this case, gravitational field equation and \( T^{\phi}_{ab} \) have similar forms as (7) and (9), instead that \( F(\phi) \) plays the role of the gravitational constant. The equation of motion is

\[ \nabla^2 \phi - V'(\phi) + \frac{1}{2} F'(\phi) R = 0, \quad (14) \]

and the energy–momentum tensor of scalar field

\[ T^{\phi}_{ab} = \nabla_a \phi \nabla_b \phi - g_{ab} \left( \frac{1}{2} (\nabla \phi)^2 + V(\phi) \right). \quad (15) \]

Assuming conservation of matter, and using the covariant divergence of \( T^{\phi}_{ab} \) (15)

\[ \nabla_a T^{\phi ab} = \nabla^b \phi \left( \nabla^2 \phi - V' \right) \]

and the equation of motion (14), the integrability condition (11) can be solved if \( \eta = 2\pi F, \ h = -\frac{ER}{2} \). Substituting these functions into equation (4), one can obtain the correct field equation.

In summary, we have studied the Rindler space-time thermodynamics in the scalar-tensor gravity. We have shown that the gravitational field equation can be derived as an equation of state of local equilibrium thermodynamics. It is an alternative treatment to reinterpretate of the nonequilibrium correction introduced in scalar-tensor gravity.\(^{7,11}\) This work has generalized the approach developed in\(^{14}\) for \( f(R) \) gravity. It is known that \( f(R) \) gravity can be treated as a special scalar-tensor theory by introducing the scalar field \( \phi = R \) and potential \( V = \phi f' - f \) and choosing the Brans-Dicke parameter \( \omega = 0 \) (see\(^{21}\) for a review). We have shown the consistence between two theories on the thermodynamic aspect, which is not affected by the concrete potential and parameter. We have introduced a more general local entropy expression as done in \( f(R) \) gravity,\(^{14}\) where the proportionality constant between the horizon area and entropy is replaced with a field depended function. Since this is motivated by the black hole entropy expression in scalar-tensor gravity, we suspect that the entropy of local Rindler horizon for the more generalized gravity should be modified as corresponding black hole entropy. This is consistent
with the entropy expressions of cosmological apparent horizon in different gravity theories.\(^7\),\(^12\) Our equilibrium thermodynamics is constructed by assuming the conserved energy of matter, which suggests the energy exchange between the matter and scalar field may lead to the nonequilibrium Rindler spacetime. We have also assumed the energy–momentum tensor of scalar field with nonminimal coupling to gravitation is just the one with minimal coupling, which means that the effective energy \(T^{\alpha\beta}_{\text{eff}}\) can not feel the heat flow across the horizon. This implies that we can not simply deal with thermodynamicas of effective energy as the one of normal matter. We expect that it may be helpful to understand the negative entropy of phantom dark energy\(^22\) which may be an effective energy in modified gravity theories.

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