Solutions of the Einstein equations with matter

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Abstract
Recent results on solutions of the Einstein equations with matter are surveyed and a number of open questions are stated. The first group of results presented concern asymptotically flat spacetimes, both stationary and dynamical. Then there is a discussion of solutions of the equations describing matter in special relativity and Newtonian gravitational theory and their relevance for general relativity. Next spatially compact solutions of the Einstein-matter equations are presented. Finally some remarks are made on the methods which have been used, and could be used in the future, to study solutions of the Einstein equations with matter.

1. Introduction
The aim of this paper is to give an overview of recent results on solutions of the Einstein equations with matter and to present a number of open questions. The first thing which needs to be done is to delimit the area to be surveyed. The choice of which subjects are discussed is of course subjective. The word ‘matter’ indicates, in the context of this paper, that at least one of the fields which model the matter content of spacetime in a particular solution can be physically interpreted as describing matter made up of massive particles; in other words spacetimes which are empty or which only contain radiation are not included. Furthermore, only exact solutions of the Einstein equations are considered and approximate solutions (analytic or numerical) are only mentioned insofar as they are directly relevant to the main topic. The solutions discussed satisfy reasonable spatial boundary conditions and are smooth in the sense that they do not contain any distributional matter sources. The reason for concentrating on solutions of this type is the wish to consider situations where the Cauchy problem is known to be well-posed so that solutions can, if desired, be specified in terms of initial data. Finally, the phrase ‘exact solutions’ is used here in the literal sense and not in the sense of ‘explicit solutions’, as is common in general relativity.

There are two main reasons for studying solutions of the Einstein equations with matter. The first is the possibility of using them to model astrophysical phenomena. The second is to obtain insight into questions of principle in general relativity such as the nature of spacetime singularities and cosmic censorship. Both of these aspects are discussed in the following.

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It should be noted that in the case of the second of these motivations, the study of the Einstein equations with matter may even provide insights into the dynamics of the vacuum Einstein equations. On the one hand there is the possibility of thinking of a solution of the vacuum equations as approximately represented by a large scale geometry generated by a ‘source’ which is the effective energy-momentum of gravitational waves. On the other hand the inclusion of matter provides a wider range of examples where it is possible in a relatively simple context of highly symmetric solutions to gain intuition about the dynamics of the Einstein equations and develop mathematical tools which can be used to understand this dynamics. Spherically symmetric spacetimes are the most obvious examples of the latter possibility.

In order to talk about a solution of the Einstein equations with matter it is necessary to specify what the matter fields are (i.e. what kind of geometrical objects are used to describe them), how the energy-momentum tensor is built from these fields and the metric and what the matter field equations are. A solution of the Einstein equations with matter (or a solution of the Einstein-matter equations) is then a solution of the coupled system of partial differential equations consisting of the Einstein equations and the matter field equations. The specific matter models occurring in this paper are the following:

1) perfect fluids described by the Euler equation
2) kinetic theory, where the matter field equation is the Boltzmann equation (including the special case of collisionless matter where it reduces to the Vlasov equation)
3) elastic solids
4) any of 1)-3) coupled to an electromagnetic field in an appropriate manner

All of these matter models may be described as phenomenological. There is no mention here of matter fields described by fermions for the simple reason that the author is not aware of any relevant mathematical results in that case. The subject of viscous fluids is commented on briefly.

Section 2 presents solutions describing bodies in equilibrium while section 3 is concerned with dynamical asymptotically flat solutions. After an interlude on the relations to Newtonian theory and special relativity in section 4, section 5 describes results on cosmological solutions. Section 6 contains some reflections on the methods which have been used to prove theorems about the Einstein-matter equations up to now.

2. Solutions describing bodies in equilibrium

One of the most fundamental types of solutions of the Einstein equations with matter are those which describe an isolated body which is at rest or steadily rotating. These are asymptotically flat and stationary. The simplest case is that of spherically symmetric static solutions of the Einstein equations with a perfect fluid as matter model. The existence question for this case was studied in [1]. Consider a perfect fluid with energy density $\rho$
and pressure $p$ related by an equation of state $p = f(\rho)$. Here $f$ is a continuous non-negative real-valued function on an interval $[\rho_0, \infty)$, where $\rho_0 \geq 0$. Suppose that $f$ is $C^1$ for $\rho > 0$ and $df/d\rho > 0$ there. In [1] it was shown that for each positive real number $p_e$ there exists a unique inextendible spherically symmetric static solution of the Einstein-Euler equations with a given equation of state and central pressure $p_e$ which is global in the sense that the area radius $r$ takes on arbitrarily large values. The field equations are in this situation ordinary differential equations and the theorem is proved by integrating these equations starting at the centre. There are two essential points in the proof. The first is that the equations have a singularity at the centre so that the standard existence theory for ordinary differential equations is not applicable there and a replacement must be found. This is provided by a modification of the contraction mapping argument used to prove existence and uniqueness in the regular case. The second is that it is necessary to show that the quantity $2m/r$, where $m$ is the mass function, remains bounded away from unity, since otherwise the solution could break down at a finite radius.

This theorem is not the end of the story since there is no guarantee that the solutions whose existence it asserts are asymptotically flat; it could be that $m \to \infty$ as $r \to \infty$. In [1] two cases were exhibited where the solutions are asymptotically flat. The first is the case $\rho_0 > 0$. There it is easy to show that the pressure becomes zero at a finite radius $r_0$ and an exterior Schwarzschild solution can be matched there to give a model of a fluid body with finite radius and mass. Another class of asymptotically flat solutions was obtained by a rigorous perturbation argument based on the Newtonian limit. Newtonian models with finite radius for a polytropic fluid were used as seeds to produce solutions of the Einstein equations with perfect fluid source using the continuous dependence of the solution of an ordinary differential equation on parameters. The existence of the Newtonian solutions is classical [2].

Consider next spherically symmetric static solutions of the Einstein equations with collisionless matter. Here the matter is thought of as made up of a cloud of test particles which interact with other only by the gravitational field which they generate collectively. They are described statistically by a distribution function $f$ which is the number density of particles with given position and momentum at a given time. It satisfies the Vlasov (collisionless Boltzmann) equation. (For information on this equation and general relativistic kinetic theory in general the reader is referred to [3].) Each particle travels along a geodesic and as a result of the assumed symmetries of the spacetime the energy $E$ and modulus of the angular momentum $F$ of each particle (defined in terms of the Killing vectors) are conserved. Any distribution function of the form $f = \phi(E, F)$ for some function $\phi$ is a solution of the Vlasov equation. This provides a useful method of searching for spherically symmetric solutions. In [4] the case where $\phi$ is a function of the
energy $E$ alone was studied. This problem is similar to the fluid problem discussed already and the difficulties which have to be overcome to obtain an existence and uniqueness theorem are the same. Once this has been done there remains the question, for what functions $\phi$ the solutions obtained are asymptotically flat. A class of solutions with finite radius was obtained in [4] by perturbing Newtonian solutions, as in the fluid case. The existence of appropriate Newtonian solutions follows from the results of [5]. The more general case where $\phi$ also depends on $F$ was studied by Rein [6]. He proved the existence of solutions with regular centre and finite radius as in the simpler case. He also produced solutions representing a shell of matter surrounding a Schwarzschild black hole. Interestingly in that case the finiteness of the radius could be shown directly without using the Newtonian limit. This paper also contains a method of bounding $2m/r$ which is simpler and applicable in more general situations than that given in [1]. (Yet another method can be found in [7].) There is a conjecture (‘Jeans’ Theorem’) that all static spherically symmetric solutions of the Einstein-Vlasov equations are such that $f$ depends only on $E$ and $F$. The corresponding theorem in Newtonian theory has been proved [5]. The solutions of [6] with black holes are counterexamples to the naive generalization of this statement to general relativity.

**Problem** Prove an analogue of Jeans’ theorem in general relativity. Is the naive generalization true in the absence of black holes?

Under rather general circumstances static solutions of the Einstein equations with a perfect fluid as matter model are spherically symmetric. No such general constraint exists on solutions with collisionless matter. In fact it is probably the case that the astrophysical objects where a description by collisionless matter is most accurate are elliptical galaxies since on the one hand they contain little gas (which would require a hydrodynamic treatment) and on the other hand they contain sufficiently many stars to make the statistical model better than in the case of globular clusters [8].

**Problem** Investigate the existence of static solutions of the Einstein-Vlasov system which are not spherically symmetric. Are there solutions of this type containing tidally deformed black holes (i.e. ones not locally isometric to the Schwarzschild solution)?

The equations for spherically symmetric elastic bodies have been studied in [9] but there is no existence theory for general constitutive relations known in that case.

**Problem** Study the existence of spherically symmetric static elastic bodies in general relativity

It can be seen from the above that quite a lot is known about asymptotically flat static solutions of the Einstein-matter equations. Much less is known about solutions which are stationary but not static. In fact there is only one existence theorem available in this case, which is due to Heilig [10]. He proves the existence of rigidly rotating solutions with compactly
supported density for a perfect fluid having an equation of state belonging to a certain class which includes equations of state of the form \( p = K \rho^\gamma \) with \( 1 < \gamma < 2 \). The method of proof is once again a perturbation argument, starting from Newtonian theory. The idea is to start with a non-rotating Newtonian solution and perturb simultaneously in the parameter \( \lambda = 1/c^2 \), where \( c \) is the speed of light, and the angular velocity \( \omega \) of rotation. Technically, this is done by applying the implicit function theorem in a subspace of a weighted Sobolev space. The subspace is defined by symmetry conditions and is necessary to get rid of zero eigenvalues of the linearized operator. This argument is a generalization of the classic work of Lichtenstein [11] on the existence of rotating fluid bodies in Newtonian theory. (For a modern treatment of this, see Heilig [12].) The solutions produced by the perturbation argument are slowly rotating. It would be interesting to produce more extreme solutions of the kind which can be observed in numerical computations[13].

**Problem** Prove the existence of rapidly rotating solutions of the Einstein-Euler system.

It is also natural to examine possible analogues of Heilig’s theorem for collisionless matter.

**Problem** Prove the existence of stationary solutions of the Einstein-Vlasov system which are not static.

Now we turn to a discussion of results which prove, under certain hypotheses, that a static solution of the Einstein-Euler system must be spherically symmetric. The recent progress on this topic is based on an idea of Masood-ul-Alam to use the positive mass theorem to prove the spherical symmetry in a way similar to that in which that theorem had previously been used to prove uniqueness results for black holes. A given static solution of the Einstein-Euler system is compared with a reference solution, which is spherically symmetric and has the same equation of state and surface potential as the given solution. This idea was used to prove a theorem in [14] under rather restrictive conditions on the equation of state. This was improved by Beig and Simon [15], who extended it to equations of state \( p = f(\rho) \) satisfying the inequality

\[
\frac{\rho + p}{f'} \frac{d\kappa}{d\rho} + 2\kappa + \frac{1}{5} \kappa^2 \leq 0
\]

where \( \kappa = (f')^{-1}(\rho + p)/(\rho + 3p) \). However the statement was still only a relative one, namely that if, for a given solution, a corresponding reference solution exists, then the given solution must coincide with the reference solution and hence be spherically symmetric. The question of the existence of reference solutions remained open. It was answered by Lindblom and Masood-ul-Alam [16]. They did not exactly show the existence of a smooth reference solution but they did produce a solution of the equations with good enough properties to make the comparison argument go through. Thus the
proof of spherical symmetry was completed for equations of state satisfying the above inequality.

3. Dynamical asymptotically flat solutions

Once we have solutions describing equilibrium situations, it is natural to go on to look for dynamical solutions which describe the motion of bodies, at least on a short time interval. The appropriate tool for doing this is the Cauchy problem for the Einstein-matter equations. While the local Cauchy problem for these equations is well understood for most types of matter provided there is a strictly positive lower bound for the density, there are difficulties in describing bodies, where the density should tend to zero at infinity. The partial results which are known will now be described. In the case of a perfect fluid two results are known, both of which only cover restricted classes of initial data. The first of these is a theorem of Kind\[17,18\] in the spherically symmetric case for equations of state with $\rho_0 > 0$. The equations are written in Lagrange coordinates and the method of integration along characteristics is used. The use of Lagrange coordinates reduces a problem which a priori involves a (moving) free boundary to one with a fixed boundary, thus making the problem mathematically more tractable. Local in time existence and uniqueness of smooth solutions is shown for initial data with compactly supported density. This problem is essentially one-dimensional, except for the centre of symmetry, which has to be handled separately. In higher dimensions the equations for a perfect fluid described in Lagrange coordinates are degenerate and it is not known how to handle the initial value problem directly in this form.

Another way of trying to avoid the difficulties of a moving boundary is to ignore the boundary and treat the region with fluid and the exterior vacuum region on the same footing. The problem with this is the following. In a situation where the density is bounded below by a positive constant the Euler equations can be written in symmetric hyperbolic form and this immediately leads to a local existence and uniqueness theorem. However, when the equations are put into symmetric hyperbolic form in the usual way, the hyperbolicity breaks down at any point where the density vanishes. If the density is everywhere positive but tends to zero at infinity then the equations are hyperbolic but not uniformly so, which means that the standard theory does not guarantee a time of existence of a solution which is uniform in space. There is a case where the difficulty of the boundary can be got round. This is a generalization of a trick which was first used for fluid bodies in Newtonian theory by Makino \[19\]. The density is replaced by a new variable $w$ which is in general a non-smooth function of $\rho$ at the boundary. After this change of variables the equations can be written in a form which is symmetric hyperbolic even when the density vanishes. A local existence theorem for suitable initial data is an immediate consequence \[20,21\]. The problem with this is that ‘suitable’ initial data must be such that
$w$ is sufficiently differentiable and this represents a rather strange restriction when thought of in terms of the energy density $\rho$, which is the variable with a straightforward physical interpretation. That this restriction is a serious one can be seen from the fact [20] that the boundary of the body is freely falling (i.e. the flow lines of the fluid in the boundary are geodesic) for all these solutions. Intuitively this means that the matter is behaving almost like dust near the surface of the body.

**Problem** Prove a local in time existence and uniqueness theorem for solutions of the Einstein-Euler equations with spatially compact support and less stringent restrictions on the data than required for the solutions of Makino type (e. g. small perturbations of the data coming from a static solution)

It is probable that the difficulty of this problem is not just one of mathematical technique but that there is a physical reason why it is so delicate. This is the Rayleigh-Taylor instability which causes fluid interfaces to be violently unstable under certain circumstances. (For a discussion of this point see Beale et. al. [22].)

In Newtonian physics it has been shown by Secchi[23] that the Cauchy problem for the Navier-Stokes equations coupled to gravity is well-posed locally in time for initial data of compact support provided the equation of state satisfies $\rho_0 > 0$ and both coefficients of viscosity are non-zero. It is well-known that there are difficulties in describing a relativistic viscous fluid and, in particular, there does not seem to exist a preferred relativistic generalization of the Navier-Stokes equation. (For a recent discussion of relativistic models for a viscous fluid see [24].) The relativistic models of a viscous fluid which have been proposed and which have a well-posed Cauchy problem are hyperbolic, in contrast to the Navier-Stokes equation, which is parabolic in character. It is thus not clear whether the incorporation of viscosity in this way leads to an improvement in the Cauchy problem for a fluid body, as it does in the Newtonian case.

In kinetic theory there is no qualitative change in the equations in going from non-vacuum to vacuum and a local existence and uniqueness theorem for the Einstein-Boltzmann equations with initial density of compact support is known [25]. For elasticity theory, there are theorems for the interior of an elastic medium [26] but no existence theorems for initial data corresponding to an elastic body in general relativity are known. As in the fluid case it is necessary to face a moving boundary problem.

**Problem** Prove a local existence and uniqueness theorem for an elastic body in general relativity

Since the only description of matter for which a satisfactory general local existence theorem for a localized concentration of matter is known is that given by kinetic theory, it is natural to start with this kind of matter when looking for global results. Moreover, within the kinetic description the simplest case is that of collisionless matter described by the Vlasov equation. The first result on the global behaviour of asymptotically flat solutions of
the Einstein-Vlasov equations was obtained in [27]. It was shown that the maximal Cauchy development of small, spherically symmetric initial data for these equations is geodesically complete. The asymptotic behaviour of the solution at large times could also be described in some detail. The proof was carried out using Schwarzschild coordinates. In the same paper a criterion was obtained for when a solution of the equations (written in Schwarzschild coordinates) on a given time interval can be continued to a larger time interval. This criterion was strengthened in [28] where it was shown that if a solution of the equations in Schwarzschild coordinates develops a singularity then the first singularity must be at the centre of symmetry. This essentially means that the entire problem of showing global existence of solutions of these equations for large initial data reduces to controlling the behaviour of the solution near the centre. This control has not yet been achieved; if it could be then it should furnish a proof of the weak cosmic censorship hypothesis for spherically symmetric spacetimes with this matter model.

Recently, Christodoulou [29] has obtained some interesting results on the global behaviour of spherically symmetric solutions of the Einstein-Euler equations in the case of a very special equation of state. The pressure is zero up to a certain critical density and for densities above this value the matter is stiff \( \frac{dp}{d\rho} = 1 \). This equation of state is highly idealized but can be used as a first approximation in the study of the dynamics of a supernova. Christodoulou is able to obtain global control of the solution of the Cauchy problem in this case and he finds solutions whose global behaviour resembles qualitatively that of a supernova explosion. There is a collapse followed by the formation of a shock wave which blows matter off to infinity. The fact that shock waves are included makes it clear that the solutions involved are not classical solutions of the Einstein-Euler equations but weak solutions. However, in contrast to many cases where the existence of weak solutions of systems of partial differential equations has been proved, the solutions here are shown to be uniquely determined by initial data.

The global dynamical solutions mentioned up to now are spherically symmetric. Unfortunately it seems that once spherical symmetry is abandoned the largest possible dimension of the isometry group of an asymptotically flat spacetime without singularities or distributional matter sources is one. Moreover in this case, which is that of axisymmetry, the single Killing vector field has fixed points. In our present state of knowledge, axisymmetric solutions of the Einstein equations do not seem to be significantly easier to handle than the general case. Note that there are no results on asymptotically flat solutions of the Einstein equations with matter approaching in generality the theorem of Christodoulou and Klainerman [30] concerning the vacuum field equations. In fact we know nothing about the global dynamics of solutions of the Einstein-matter equations without symmetry. It should be mentioned that there is one case which, while not asymptotically flat in the usual sense, does share some features of asymptotic flatness. This is
the case of cylindrically symmetric spacetimes which are asymptotically flat in all directions where this is consistent with the symmetry. Solutions of the source-free Einstein-Maxwell equations with this symmetry have been studied by Berger, Chruściel and Moncrief[31]. Solutions of the Einstein equations with matter having this symmetry should also be investigated.

4. Newtonian theory and special relativity

To put the global results just mentioned in context, it is useful to consider the question of solutions of the equations for matter in special relativity and in the Newtonian theory of gravity. (One of the advantages of the definition of ‘matter’ used here is that it is typical that each matter model in general relativity has a reasonable analogue in Newtonian theory.) It is important to realize that open questions abound even in these simpler contexts but that there are a number of interesting new results concerning the Euler equation and the Boltzmann equation. Some of these will now be reviewed.

Solutions of the classical (compressible) Euler equations typically develop singularities in finite time [32]. Shock waves are the best known kind of singularity which occurs. If we wish to study the global dynamics it is necessary to consider weak solutions. There are various results known on the existence of global weak solutions in one space dimension for certain equations of state but there are no uniqueness theorems in one dimension and no global existence theorems in higher dimensions. This shows how difficult the study of the compressible Euler equations is from a mathematical point of view and emphasizes how remarkable it is that Christodoulou has been able to obtain the results mentioned in the previous section. Until recently even less was known in the special relativistic case than in the non-relativistic case. However Smoller and Temple [33] have now proved a global existence theorem for the special relativistic Euler equation with a linear equation of state, so that the difference is no longer so great. Another question which may be asked is that of the nature of the singularities which occur when classical solutions of the Euler equation break down. A partial answer in the case of classical hydrodynamics was obtained by Chemin[34] who showed that when a classical solution breaks down the expansion or rotation of the fluid or the gradient of the density must blow up at some point. Generalizations of this result to the cases of a self-gravitating fluid in Newtonian theory and that of a special relativistic fluid have been obtained by Brauer[35].

In classical hydrodynamics viscous fluids are more tractable than perfect fluids from a mathematical point of view. In particular, it is known that given initial data close to equilibrium data there exist corresponding global smooth solutions of the Navier-Stokes equations. The proof uses the fact that the Navier-Stokes equations are essentially parabolic in character. If similar results hold for relativistic models of viscous fluids then the mechanism which prevents singularity formation must be more subtle, since the equations are hyperbolic.
Turning now to kinetic theory, the situation is much better. Self-gravitating collisionless matter is described in Newtonian theory by the Vlasov-Poisson system. Many years of study of solutions of these equations with asymptotically flat initial data culminated recently in the proof of global existence of classical solutions for general initial data [36, 37]. The proof can also be adapted to apply to Newtonian cosmology [38]. Less is known for the Boltzmann equation but there are nevertheless a number of significant results. For the classical Boltzmann equation global existence of classical solutions has been proved for spatially homogeneous initial data and for data which are small or close to equilibrium. For general data with finite energy and entropy global existence of weak solutions (without uniqueness) was proved by DiPerna and Lions [39]. For information on these results and on the classical Boltzmann equation in general see [40,41]. In the last few years some of these results have been extended to the special relativistic Boltzmann equation. Glassey and Strauss have proved a global existence theorem for initial data close to equilibrium [42] while Dudyński and Ekiel-Jezewska [43] have proved a relativistic analogue of the DiPerna-Lions result on weak solutions. Interestingly the oldest of the theorems concerning the classical Boltzmann equation has apparently not yet been extended to the relativistic case, which leads to the following

**Problem** Prove a global in time existence theorem for classical solutions of the Boltzmann equation in special relativity with spatially homogeneous initial data or, even better, an appropriate analogue for spatially homogeneous solutions of the Einstein-Boltzmann equations.

If a particular matter model is such that it develops singularities when considered as a test field in a given smooth spacetime (and in particular in special relativity) then it is not surprising if the system obtained by coupling this matter model to the Einstein equations develops singularities which have little to do with gravitational collapse. The best-known example is that of the shell-crossing and shell-focussing singularities of dust. It is simply that the matter variables lose differentiability and cause the geometry to lose differentiability as well. Hence if matter models of this kind are permitted, it seems overoptimistic to expect that good global properties of spacetime such as strong cosmic censorship or the existence of a global foliation by hypersurfaces of constant mean curvature will hold. Therefore a good strategy in studying these problems is to choose only matter models which always have global smooth solutions corresponding to smooth initial data on a Cauchy surface in a smooth globally hyperbolic spacetime. Such matter models will be called tame. A model which is obviously tame is that of collisionless matter since in that case the field equation in a given background spacetime is linear. Of course it is easy to think of field theoretic matter models which are tame because the equations are linear, e.g. a Maxwell field or a massless scalar field. Evidence that the restriction to tame matter models is useful in global problems is provided by the results on crushing singularities[44] which
will be discussed in the next section. The comparison with Newtonian theory can also be helpful. An example is the insight provided into the numerical results of Shapiro and Teukolsky [45] by comparing them with the analytic results for the Vlasov-Poisson system mentioned above (see [46]). More information on the issue of the choice of matter model in general relativity can be found in [47].

5. Spatially compact spacetimes

Next spatially compact spacetimes will be discussed, i.e. spacetimes which possess a compact Cauchy surface. These will be referred to in what follows as cosmological spacetimes since this boundary condition is appropriate for cosmological models. Of course it is not claimed that it is the only appropriate one. However the importance of imposing some spatial boundary condition in mathematical studies of cosmological models should be emphasized. For without an assumption of this kind anything can happen; it is impossible to say anything reasonable about the nature of singularities since any kind of singularity can be built into the initial data. The assumption of a compact Cauchy surface is the simplest possibility and therefore a reasonable starting point. Here the variety of symmetry types which permit the Einstein-matter equations to be studied under relatively simple conditions is much richer than in the asymptotically flat case.

The most symmetric cosmological spacetimes are those which are spatially locally homogeneous. This means by definition that the universal covering of the spacetime admits a group of isometries with three-dimensional spacelike orbits. In other word the universal covering spacetime is spatially homogeneous. The reason for wishing to include spacetimes which are spatially locally homogeneous rather than just those which are spatially homogeneous is that it allows a much greater variety of symmetry types for a spatially compact spacetime. If a spatially compact spacetime is spatially homogeneous then it must have Bianchi I, Bianchi IX or Kantowski-Sachs symmetry. Generalizing to local homogeneity permits in addition Bianchi types II, III, V, VI₀, VII₀ and VIII. The literature on the subject of spatially homogeneous spacetimes is vast and only a few recent developments will be mentioned here. In [48] the singularities of spatially compact spatially locally homogeneous solutions of the Einstein equations were investigated for a rather general class of phenomenological matter models. The results were that if the spacetime is not vacuum then for Bianchi IX and Kantowski-Sachs spacetimes there is a curvature singularity in both time directions while for the other Bianchi types there is a curvature singularity in one time direction and the spacetime is causally geodesically complete in the other time direction. In particular this proves strong cosmic censorship in this class of spacetimes. It is interesting to note that in this case the Einstein equations with matter are easier to analyse than the vacuum Einstein equations. In the vacuum case the geodesic completeness statement still holds but the
statement about curvature singularities does not. It fails for spacetimes
with Cauchy horizons, such as the Taub-NUT solution. In the vacuum case
it is possible to determine which spacetimes admit an extension through a
Cauchy horizon and which do not [49,50] but the question of whether those
spacetimes which do not admit extensions through a Cauchy horizon do have
curvature singularities remains open.

While the questions of geodesic completeness and curvature singularities
are of capital importance, it is also of interest to have more detailed information
about the dynamics near the singularity. In the case where the matter
model considered is a perfect fluid quite a lot is known [51]. On the other
hand the complexity of the dynamics depends significantly on the matter
model chosen. This will be illustrated by two examples. The first is the case
of solutions of the Einstein-Vlasov system of Bianchi type I. Note that this is
the Bianchi type which is a priori the simplest. This problem was considered
in [52] but only with limited success. The possible types of dynamical behaviour
near the singularity were reduced to a short list but the question of
which types on this list are actually realized was not answered. In particular
the possibility was left open that complicated oscillatory behaviour might
occur in this situation. A numerical investigation of the dynamics would be
desirable. With luck it would reveal the answer and this answer could then
be proved to be correct. The other example is due to Leblanc, Kerr and
Wainwright [53] and concerns spacetimes of Bianchi type VI\(_0\) with perfect
fluid and a magnetic field. It was found that these spacetimes display a behaviour very similar to that of the Bianchi IX spacetimes with a fluid alone
(Mixmaster spacetimes). This is very interesting for the following reason. A
fascinating open question is whether the complicated Mixmaster behaviour
is stable or whether it would be destroyed by a small inhomogeneous perturbation. It would be very convenient for numerical or analytical studies if the
Mixmaster spacetime could be perturbed in the class of spacetimes with two
local Killing vectors. Unfortunately any small perturbation of a Mixmaster spacetime has either no more than one local Killing vector or it has three, in
which case it is again a Mixmaster solution. The Leblanc-Kerr-Wainwright
solutions do not suffer from this problem. A Bianchi VI\(_0\) spacetime can be
perturbed to a spacetime with two local Killing vectors. These generalized
Bianchi VI\(_0\) spacetimes have Cauchy surfaces whose topology is that of a
bundle over a circle with fibre a torus. The part of the bundle over a small
part of the circle admits a \(U(1) \times U(1)\) symmetry group but there is no corre-
spanding global action. One problem which does remain is that perturbing
a fluid solution is likely to lead to problems with shocks, so that the matter
model used in [53] would need to be replaced by something else. Neverthe-
less, this appears a promising approach to understanding more about the
stability of the Mixmaster solution.

Our understanding of the global properties of inhomogeneous cosmo-
logical solutions of the Einstein-matter equations is still very limited. With
one exception, to be mentioned at the end of this section, theorems are only available in cases with three local Killing vectors. Note that more is known in the better-studied vacuum case, where there are results for spacetimes with only two Killing vectors [54]. The existence of Killing vectors in a spacetime allows the study of the dynamics of the Einstein equations to be reduced to an effective problem in lower dimensions. If, however, the dimension of the orbits is not constant, this leads to singularities in the equations in the lower dimensional space. In this way a large part of the benefit of the reduction is lost. It is thus natural to begin by studying the case where the dimension of the orbits is constant. Spherical symmetry on $S^2 \times S^1$ and plane and hyperbolic symmetry have this property. In the latter two cases the situation is similar to that encountered for spatially locally homogeneous spacetimes. If a compact Cauchy surface is required then the group defining the full symmetry acts only on the universal cover of spacetime and not on spacetime itself. Note that it is also possible to have spherically symmetric spacetimes with spatial topology $S^3$ but there the orbits are of variable dimension (there are two ‘centres’). The most precise result on the dynamics of spacetimes of this type was obtained by Rein [55] who showed that certain kinds of initial data for the Einstein-Vlasov equations with spherical, plane or hyperbolic symmetry necessarily lead to curvature singularities where the Kretschmann scalar $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ blows up uniformly. These data include an open set of all data with the given symmetry.

Another type of result concerns the question of crushing singularities. Recall that a crushing singularity in a cosmological spacetime is one where a neighbourhood of the singularity can be covered by a foliation by compact hypersurfaces whose mean curvature tends uniformly to infinity. Under mild assumptions on the matter content of spacetime this is equivalent to the condition that a neighbourhood of the singularity can be covered by a foliation by constant mean curvature (CMC) hypersurfaces, whose mean curvature tends to infinity. The significance of crushing singularities has been discussed in [56] and [57], where it was suggested that singularities in cosmological spacetimes should have this property, provided the matter content of spacetime is sufficiently well-behaved. In [44] and [58] this was confirmed in the context of solutions of the Einstein-Vlasov equations with spherical, plane and hyperbolic symmetry which admit at least one CMC hypersurface. In the spherical case (on $S^2 \times S^1$) the whole spacetime can be covered by a CMC foliation where the mean curvature takes on all real values. In the cases of plane and hyperbolic symmetry the CMC foliation obtained was only shown to cover a neighbourhood of the initial singularity. In a spacetime with one of these symmetries it is possible to define an ‘area radius’ as the square root of the area of an orbit. (In the plane and hyperbolic cases ‘orbit’ should be interpreted as denoting a subset of spacetime whose inverse image under the projection from the universal covering space is an orbit.) In Rein’s work [55] this area radius was used as a time coordinate and the
results obtained were stronger than those of [44] and [58]. In fact the use of $r$ as a time coordinate has a major advantage and a major disadvantage. The advantage is that this coordinate is optimally adapted to the symmetry and hence is a powerful tool. The disadvantage is that this procedure has no obvious analogue in general spacetimes, or even spacetimes with less than two Killing vectors. The CMC approach, on the other hand, is potentially applicable to any cosmological spacetime. For this reason it would be very desirable to prove an analogue of Rein’s result working directly with the CMC approach.

Yet another class of results is those which have been obtained by Burnett [59], [60] on the closed universe recollapse conjecture. He shows that in a globally hyperbolic spherically symmetric cosmological spacetime satisfying the dominant energy and non-negative pressures conditions there is a finite upper bound to the length of all causal curves. In other words, under these assumptions the universe has a finite lifetime. In contrast to the other cases mentioned above, his theorems cover the case of spherical symmetry on $S^3$. It is nevertheless the case that the proof for $S^3$ is significantly harder than that for $S^2 \times S^1$. This kind of theorem does not distinguish between spacetime singularities and matter singularities such as shell-crossing or shocks. However this is a distinction which it is difficult to make precise in our present state of knowledge [47].

The most obvious generalization of all these results to a context with less stringent symmetry assumptions would be to the case of $U(1) \times U(1)$ symmetry. For the closed universe recollapse conjecture the spatial topology should be chosen to be $S^3$ or $S^2 \times S^1$, which necessarily leads to a variable dimension for the group orbits. For the other two types of approach it is natural to stick to orbits of fixed dimension to start with and thus to choose the spatial topology $S^1 \times S^1 \times S^1$. There is also the possibility of taking spacetimes with some kind of local $U(1) \times U(1)$ symmetry, such as the generalized Bianchi VI$_0$ spacetimes mentioned above. These possibilities are now being investigated in detail. They represent the simplest situations in which the coupling of localized gravitational waves to matter (and to each other) can be studied.

The last example of a theorem which gives information about the global dynamics of solutions of the Einstein equations with matter which will be discussed here is due to Newman [61]. This is concerned with the existence of spacetimes with isotropic singularities, a concept which is related to Penrose’s Weyl curvature hypothesis. The matter model is a perfect fluid with the equation of state $p = \frac{1}{3} \rho$. It is shown that, without requiring any symmetry assumptions, it is possible to set up a well-posed Cauchy problem with initial data given on the singularity itself. The data which can be given is roughly speaking half the amount which can be given on a regular spacelike hypersurface and all the spacetimes produced have isotropic singularities. Since the solution is uniquely determined by the initial data, if the data are
homogeneous and isotropic then the solution will have Robertson-Walker symmetry.

6. Methods

The results discussed in this paper have been obtained by a wide variety of methods. Some of these have been specially produced to solve particular problems while others can be seen to be potentially of wider significance. This section contains some remarks on some of the latter. The Einstein-matter equations form in general a system of partial differential equations which is at least in part hyperbolic. The one general technique available at present for studying hyperbolic equations is the method of energy estimates. This has been pushed to the limit in the work of Christodoulou and Klainerman on the vacuum Einstein equations [30]. An introduction to this method accessible to relativists can be found in [62]. The method consists in obtaining inequalities for the time variation of the $L^2$ norm of the unknown quantity $u$ in the equation, defined by $\|u(t)\|_{L^2} = (\int (u(t,x))^2 dx)^{1/2}$, and the corresponding norms of derivatives of $u$. In the theory of nonlinear elliptic equations, which is much more developed than that of nonlinear hyperbolic equations, it is useful to consider in addition the $L^p$ norms for $p \neq 2$. Unfortunately it is known that, while all $L^p$ norms of solutions of elliptic equations have nice properties, this only holds for hyperbolic equations if $p = 2$. There are nevertheless possibilities of trying to improve on energy estimates by some limited use of $L^p$ norms. This idea has not yet borne fruit in the case of the Einstein equations but it has recently been used to obtain a new result concerning the Yang-Mills equations [63]. This is a local in time existence and uniqueness theorem for initial data which are only assumed to have finite energy. Since the energy is conserved, this immediately implies a global in time existence and uniqueness theorem. In particular the global existence result of Eardley and Moncrief [64] is reproduced by a quite new method.

The only one of the global theorems discussed in the above in which energy estimates play a role is that of Newman. The reason is that in all the other cases the symmetry assumed is so strong that there are no locally propagating gravitational waves and correspondingly no truly hyperbolic phenomena. In Newman’s theorem the global result for the Einstein-matter equations is translated into a local result for a conformally transformed system. This has some similarity to the regular conformal field equations used by Friedrich [65] to study the Einstein equations in vacuum or with certain massless fields. An important difference is that while Friedrich’s equations are regular, Newman’s contain a (mild) singularity, so that it is necessary to do energy estimates directly. One lower order term in the equations contains a factor $1/t$, where $t$ is a cosmic time parameter which vanishes at the singularity and it must be checked that the standard existence theorem for symmetric hyperbolic systems can be modified to accommodate this. Wave phenomena appear when the Einstein-matter equations with $U(1) \times U(1)$
symmetry are considered but even there, the fact that the hyperbolic equations are effectively only in one space dimension (at least if the orbit dimension is constant) means that energy estimates are not the only tool at our disposal.

In a number of the results above an important role is played by Hawking’s quasi-local mass. If $S$ is a compact surface in spacetime of area $A(S)$ the Hawking mass $m(S)$ is defined to be $C(A(S))^{1/2}(\chi(S)/2 + (1/4\pi) \int_S \rho \mu)$ where $\rho$ and $\mu$ are the expansions of the two families of future-pointing null geodesics which start orthogonal to $S$, $\chi(S)$ is the Euler characteristic of $S$ and $C$ is a constant. This constant is usually chosen so that the Hawking mass of a symmetric sphere in the Schwarzschild solution is equal to the Schwarzschild mass parameter. However the exact value of the constant is probably of little significance in general. An alternative expression for the mass is obtained by applying the Gauss-Bonnet theorem. When this is done the expression $\int (K + \rho \mu)$ comes up, where $K$ is the Gaussian curvature. In order to have a physical interpretation as a mass the above expression should be non-negative, at least under some restrictions. It is well known that the Hawking mass is negative for certain compact surfaces in flat space and this suggests limiting consideration to surfaces which are in some sense as symmetric as possible (cf. [66]). In spacetimes with spherical and plane symmetry surfaces spanned by the local Killing vector fields have Hawking masses which are non-negative and can only be zero if the spacetime is flat[44]. The optimism which this might cause is limited by the fact that in spacetimes with hyperbolic symmetry surfaces spanned by the local Killing vector fields can have negative mass. In this case the mass is still of some use for analysing the Einstein equations but perhaps as a general rule the Hawking mass is most useful in the case where the surface $S$ is a sphere.

**Problem** Find a way to obtain some control of solutions of the Einstein equations with less than two local Killing vectors using the Hawking mass (or some variant of it).

A technique which seems to be closely related to the Hawking mass and its positivity was introduced by Malec and Ó Murchadha [67]. They write the constraint equations in a special way in order to obtain estimates for an initial data set in terms of its mean curvature. Their technique is a priori limited to spacetimes with a preferred foliation by surfaces and initial data sets which are unions of these surfaces. However it may be asked whether it does not give a hint of some more general property of the Einstein constraint equations.

The central tool in the proof of Christodoulou and Klainerman of the nonlinear stability of Minkowski space is the Bel-Robinson tensor. It is worth noting that Horowitz and Schmidt[68] found, at least in vacuum, a close connection between the Bel-Robinson tensor and the Hawking mass of small spheres.

There are various indications that self-similarity emerges in certain sin-
gularities of solutions of the Einstein equations. An illustration of this is the use of dimensionless variables in the analysis of Bianchi models [51] which puts the equations in a tractable form, where self-similar solutions correspond to fixed points of the dimensionless dynamical system. Another is the privileged position held by self-similar spherically symmetric spacetimes in the study of naked singularities in asymptotically flat spacetimes.

**Problem** Give a rigorous analysis of spherically symmetric self-similar solutions of the Einstein-matter equations for a variety of matter models.

The problem to be solved here is to prove the existence of solutions satisfying reasonable boundary conditions and this in turn comes down to the qualitative analysis of a system of ordinary differential equations. A model for an analysis of this type can be found in the paper [69] of Christodoulou.

The question of cutting off a self-similar solution at a finite radius to get an asymptotically flat solution is also important.

It would be difficult to make this list of available techniques which might be used to study general solutions of the Einstein-matter equations much longer. The conclusion is that all of these should be studied intensively and that at the same time new techniques should be sought actively. Therein lies the best hope of the study of the Einstein-matter equations developing beyond its present embryonic stage in the coming years. The many new results on global solutions of partial differential equations obtained recently make it realistic to expect that a lot of progress can be made in understanding the global dynamics of solutions of the Einstein equations with matter.

**References**

[1] Rendall, A. D., Schmidt, B. G.: Existence and properties of spherically symmetric static fluid bodies with given equation of state. Class. Quantum Grav. 8, 985-1000 (1991).

[2] Chandrasekhar, S.: An introduction to the study of stellar structure. Dover, New York, 1957.

[3] Ehlers, J.: Survey of general relativity theory. In: W. Israel (ed.) Relativity, Astrophysics and Cosmology. Reidel, Dordrecht, 1973.

[4] Rein, G., Rendall, A. D.: Smooth static solutions of the spherically symmetric Vlasov-Einstein system. Ann. Inst. H. Poincaré (Physique Théorique) 59, 383-397 (1993).

[5] Batt, J., Faltenbacher, W. and Horst, E.: Stationary spherically symmetric models in stellar dynamics. Arch. Rat. Mech. Anal. 93, 159-183 (1986).

[6] Rein, G.: Static solutions of the spherically symmetric Vlasov-Einstein system. Math. Proc. Camb. Phil. Soc. 115, 559-570 (1994).

[7] Baumgarte, T., Rendall, A. D.: Regularity of spherically symmetric static solutions of the Einstein equations. Class. Quantum Grav. 10, 327-332 (1993).

[8] Binney, J., Tremaine, S.: Galactic dynamics. Princeton University Press, Princeton, 1987.
[9] Kijowski, J., Magli, G.: A generalization of the relativistic equilibrium equation for a non-rotating star. Gen. Rel. Grav. 24, 139-158 (1992).
[10] Heilig, U.: On the existence of rotating stars in general relativity. Commun. Math. Phys. 166, 457-493 (1995).
[11] Lichtenstein, L.: Gleichgewichtsfiguren rotierender Flüssigkeiten. Springer, Berlin. 1933.
[12] Heilig, U.: On Lichtenstein’s analysis of rotating Newtonian stars. Ann. Inst. H. Poincaré (Physique Théorique) 60, 457-487 (1994).
[13] Herold, H., Neugebauer, G.: Gravitational fields of rapidly rotating neutron stars: numerical results. In: Ehlers, J., Schäfer, G. (eds.) Relativistic gravity research. Lecture Notes in Physics 410, Springer, Berlin, 1992.
[14] Masood-ul-Alam, A. K. M.: On spherical symmetry of static perfect fluid spacetimes and the positive mass theorem. Class. Quantum Grav. 4, 625-633 (1987).
[15] Beig, R., Simon, W.: On the uniqueness of perfect fluid solutions in general relativity. Commun. Math. Phys. 144, 373-390 (1992).
[16] Lindblom, L., Masood-ul-Alam, A. K. M.: On the spherical symmetry of static stellar models. Commun. Math. Phys. 162, 123-145 (1994).
[17] Kind, S.: Anfangs-Randwertprobleme für die Einsteingleichungen zur Beschreibung von Flüssigkeitskugeln und deren linearisierten Störungen. Thesis, Munich University.
[18] Kind, S., Ehlers, J.: Initial boundary value problem for the spherically symmetric Einstein equations for a perfect fluid. Class. Quantum Grav. 18, 2123-2136 (1993).
[19] Makino, T.: On a local existence theorem for the evolution equation of gaseous stars. In: Nishida, T., Mimura, M., and Fujii, H. (eds.) Patterns and Waves. North Holland, Amsterdam, 1986.
[20] Rendall, A. D.: The initial value problem for a class of general relativistic fluid bodies. J. Math. Phys. 33, 1047-1053 (1992).
[21] Rendall, A. D.: The initial value problem for self-gravitating fluid bodies. In: K. Schmūdgen (ed.) Mathematical Physics X. Springer, Berlin, 1992.
[22] Beale, J. T., Hou, T. Y. and Lowengrub, J. S.: Growth rates for the linearized motion of fluid interfaces away from equilibrium. Commun. Pure Appl. Math. 46, 1269-1301 (1993).
[23] Secchi, P.: On the equations of viscous gaseous stars. Ann. Scuola Norm. Sup. Pisa 18, 295-318 (1991).
[24] Geroch, R., Lindblom, L.: Dissipative relativistic fluid theories of divergence type. Phys. Rev. D41, 1855-1861 (1990).
[25] Bancel, D., Choquet-Bruhat, Y.: Existence, uniqueness and local stability for the Einstein-Maxwell-Boltzmann system. Commun. Math. Phys. 33, 83-96 (1973).
[26] Choquet-Bruhat, Y., Lamoureux-Brousse, L.: Sur les équations de l’élasticité relativiste. C. R. Acad. Sci. Paris 276, 1317-1320 (1973).
[27] Rein, G., Rendall, A. D.: Global existence of solutions of the spherically
symmetric Vlasov-Einstein system with small initial data. Commun. Math. Phys. 150, 561-583 (1992).
[28] Rein, G., Rendall, A. D. and Schaeffer, J.: A regularity theorem for solutions of the spherically symmetric Vlasov-Einstein system. Commun. Math. Phys. 168, 467-478 (1995).
[29] Christodoulou, D.: Self-gravitating fluids: a two-phase model. Arch. Rat. Mech. Anal. 130, 343-400 (1995) and further papers to appear.
[30] Christodoulou, D., Klainerman, S.: The global nonlinear stability of the Minkowski space. Princeton University Press, Princeton. 1993.
[31] Berger, B. K., Chruściel, P. T. and Moncrief, V.: On ‘asymptotically flat’ spacetimes with $G_2$-invariant Cauchy surfaces. Ann. Phys. 237, 322-354 (1995).
[32] Sideris, T.: Formation of singularities in three-dimensional compressible fluids. Commun. Math. Phys. 101, 475-485 (1979).
[33] Smoller, J., Temple, B.: Global solutions of the relativistic Euler equations. Commun. Math. Phys. 156, 65-100 (1993).
[34] Chemin, J.-Y.: Remarques sur l’apparition de singularités dans les écoulements Euleriens compressibles. Commun. Math. Phys. 133, 323-339 (1990).
[35] Brauer, U.: Unpublished.
[36] Pfaffelmoser, K: Global classical solutions of the Vlasov-Poisson system in three dimensions for general initial data. J. Diff. Eq. 95, 281-303 (1992).
[37] Lions, P.-L., Perthame, B.: Propagation of moments and regularity for the three-dimensional Vlasov-Poisson system. Invent. Math. 105, 415-430 (1991).
[38] Rein, G., Rendall, A. D.: Global existence of classical solutions to the Vlasov-Poisson system in a three dimensional, cosmological setting. Arch. Rat. Mech. Anal. 126, 183-201 (1994).
[39] DiPerna, R. J., Lions, P.-L.: On the Cauchy problem for Boltzmann equations: global existence and weak stability. Ann. Math. 130, 321-366 (1989).
[40] Cercignani, C.: The Boltzmann equation and its applications. Springer, New York, 1988.
[41] Cercignani, C., Illner, R. and Pulvirenti, M.: The mathematical theory of dilute gases. Springer, New York, 1994.
[42] Glassey, R. T., Strauss, W.: Asymptotic stability of the relativistic Maxwellian. Publ. Math. RIMS Kyoto 29, 167-233 (1993).
[43] Dudyński, M., Ekiel-Jeżewska, M.: Global existence proof for relativistic Boltzmann equation. J. Stat. Phys. 66, 991-1001 (1992).
[44] Rendall, A. D.: Crushing singularities in spacetimes with spherical, plane and hyperbolic symmetry. Class. Quantum Grav. 12, 1517-1533 (1995).
[45] Shapiro, S. L., Teukolsky, S. A.: Formation of naked singularities: the violation of cosmic censorship. Phys. Rev. Lett. 66, 994-997 (1991).
[46] Rendall, A. D.: Cosmic censorship and the Vlasov equation. Class.
Quantum Grav. 9, L99-L104 (1992).
[47] Rendall, A. D.: On the choice of matter model in general relativity. In R. d’Inverno (ed.) Approaches to Numerical Relativity. Cambridge University Press, Cambridge, 1992.
[48] Rendall, A. D.: Global properties of locally spatially homogeneous cosmological models with matter. Preprint gr-qc/9409009 (to appear in Math. Proc. Camb. Phil. Soc.)
[49] Siklos, S. T. C.: Occurrence of whimper singularities. Commun. Math. Phys. 58, 255-272 (1978).
[50] Chruściel, P. T., Rendall, A. D.: Strong cosmic censorship in vacuum space-times with compact locally homogeneous Cauchy surfaces. Ann. Phys. 242, 349-385 (1995).
[51] Wainwright, J., Hsu, L.: A dynamical systems approach to Bianchi cosmologies: orthogonal models of class A. Class. Quantum Grav. 6, 1409-1431 (1989).
[52] Rendall, A. D.: The initial singularity in solutions of the Einstein-Vlasov system of Bianchi type I. Preprint IHES/P/95/23, gr-qc/9505017.
[53] Leblanc, V. G., Kerr, D. and Wainwright, J.: Asymptotic states of magnetic Bianchi VI0 cosmologies. Class. Quantum Grav. 12, 513-541 (1995).
[54] Chruściel, P. T., Isenberg, J. and Moncrief, V. Strong cosmic censorship in polarised Gowdy spacetimes. Class. Quantum Grav. 7, 1671-1680 (1990).
[55] Rein, G.: Cosmological solutions of the Vlasov-Einstein system with spherical, plane and hyperbolic symmetry. Preprint gr-qc/9409041 (to appear in Math. Proc. Camb. Phil. Soc.)
[56] Eardley, D., Smarr, L.: Time functions in numerical relativity: marginally bound dust collapse. Phys. Rev. D19, 2239-2259 (1979).
[57] Eardley, D., Moncrief, V.: The global existence problem and cosmic censorship in general relativity. Gen. Rel. Grav. 13, 887-892 (1981).
[58] Burnett, G., Rendall, A. D.: Existence of maximal hypersurfaces in some spherically symmetric spacetimes. Preprint gr-qc/9507001.
[59] Burnett, G.: Incompleteness theorems for the spherically symmetric spacetimes. Phys. Rev. D43, 1143-1149 (1991).
[60] Burnett, G.: Lifetimes of spherically symmetric closed universes. Phys. Rev. D51, 1621-1631 (1995).
[61] Newman, R. P. A. C.: On the structure of conformal singularities in classical general relativity. Proc. R. Soc. London A443, 473-492; 493-515 (1993).
[62] Klainerman, S. On the mathematical theory of fields and general relativity. GR13 Proceedings, IOP, Bristol, 1993.
[63] Klainerman, S., Machedon, M.: Finite energy solutions of the Yang-Mills equations in $\mathbb{R}^{3+1}$. Ann. Math. 142, 39-119 (1995).
[64] Eardley, D., Moncrief, V.: The global existence of Yang-Mills fields in $M^{3+1}$. Commun. Math. Phys. 83, 171-212 (1982).
[65] Friedrich, H.: On the global existence and asymptotic behaviour of solutions to the Einstein-Yang-Mills equations. J. Diff. Geom. 34, 275-345 (1991).

[66] Christodoulou, D., Yau, S.-T.: Some remarks on the quasi-local mass. Contemporary Mathematics 71, 9-14 (1988).

[67] Malec, E., Ó Murchadha, N.: Optical scalars and singularity avoidance in spherical spacetimes. Phys. Rev. D50, 6033-6036 (1994).

[68] Horowitz, G. T., Schmidt, B. G.: Note on gravitational energy. Proc. R. Soc. Lond. A381, 215-224 (1982).

[69] Christodoulou, D.: Examples of naked singularity formation in the gravitational collapse of a scalar field. Ann. Math. 140, 607-653 (1994).