Viscosity of a two-dimensional Fermi liquid

D. S. Novikov*

Department of Electrical Engineering and Department of Physics, Princeton University, Princeton, New Jersey 08544

(Dated: September 24, 2018)

Shear viscosity of a two-dimensional Fermi liquid is found to be a nonanalytic function of temperature. In contrast to the quasiparticle lifetime that is determined by the forward-scattering processes, the main contribution to the viscosity arises from the quasiparticle scattering in the Cooper channel. The viscosity is enhanced by the logarithmic singularity of the vertex part. This singular behavior can manifest itself in the two-dimensional electron transport, and in the momentum relaxation of fermions in atomic traps.

PACS numbers: 71.10.Ay, 73.40.-c, 03.75.Kk, 66.20.+d

Viscosity characterizes momentum relaxation in a fluid. In the presence of a weakly inhomogeneous flow \( \mathbf{u}(r) \), the linear relation between the stress tensor of the \( d \)-dimensional fluid and the flow gradients is conventionally written as [1]

\[
\Pi_{ij} = -\eta \left( \partial_i u_j + \partial_j u_i - \frac{2}{d} \delta_{ij} \nabla \mathbf{u} \right) - \zeta \nabla \mathbf{u}.
\] (1)

The coefficients \( \eta \) and \( \zeta \) are the first (shear) and the second (bulk) viscosities, out of which \( \eta \) usually dominates: \( \eta \gg \zeta \). The linear relation (1), complemented by the momentum conservation law \( \rho(\partial_t + \mathbf{u} \nabla) \mathbf{u}_j = -\partial_j \Pi_{ij} \approx \eta \nabla^2 u_j \), where \( \rho(r) = m \mathcal{N}(r) \) is the fluid mass density, leads to the Navier-Stokes description on the length scales exceeding the molecular mean free path \( \ell \). Such a hydrodynamic approach bears great predictive power, since a single parameter \( \eta \) accounts for the details of the molecular interactions on the scale \( \ell \). The dependence of the viscosity on the temperature and density directly determines the fluid’s relaxation and transport properties. Calculating \( \eta(T, \mathcal{N}) \) is in general a difficult task.

Early developments of the kinetic theory by Chapman and Enskog have lead to the understanding of the viscosity of dilute gases. The latter can be estimated as [1, 2]

\[
\eta \sim \rho \bar{v} \ell, \tag{2}
\]

where \( \bar{v} \) is the typical velocity of a molecule in a gas, and the mean free path \( \ell \) is determined by the molecular density and the scattering cross-section. Assuming weak energy dependence of the scattering, one readily obtains the universal temperature dependence of the gas viscosity \( \eta \propto \sqrt{T} \) that stems from that of \( \bar{v} \) according to the Maxwell distribution.

Later on, the notion of viscosity was generalized for hydrodynamic modes in quantum fluids [3]. In particular, the shear viscosity has been studied in the context of 3d Fermi liquids (FL), with applications to liquid \(^3\)He. Remarkably, the simple estimate (2) qualitatively applies in the degenerate case, since at \( T \ll \epsilon_F \), where \( \epsilon_F \) is the Fermi energy, the FL is a dilute gas of quasiparticles [4]. Moreover, Eq. (2) yields a singular temperature dependence \( \eta \propto 1/T^2 \). The latter originates entirely from that of the quasiparticle inelastic mean free path \( \ell = v_F \tau_{qp}(T) \), with \( \hbar/\tau_{qp} \propto T^2/\epsilon_F \) the scattering rate at the Fermi surface in 3d [4, 5], while \( \bar{v} = v_F \) is \( T \)-independent.

Physically, the \( T \to 0 \) divergence signifies the increasing resistance to the shear flow \( \mathbf{u}(r) \) that causes distortions of the Fermi surface.

The characteristic dependence \( \eta \propto 1/T^2 \), first estimated by Pomeranchuk in 1950 [1, 6] on dimensional grounds, was later confirmed by the calculations based on the FL kinetic equation [2, 3, 7]. It was also shown that the second viscosity \( \zeta \) is practically irrelevant, \( \zeta \sim (T/\epsilon_F)^2 \eta \) [3]. Subsequent \(^4\)He measurements [8] confirmed the relation \( \eta T^2 = \text{const.} \)

In the present work we consider the shear viscosity of the two-dimensional Fermi liquid. From a practical standpoint, the problem is relevant to a variety of Fermi systems, ranging from the 2d electrons in heterostructures, to the trapped Fermi gases. Theoretically, the problem is compelling since in a 2d FL one generally expects nonanalytic energy- and temperature-dependence of response functions due to strong restrictions on the quasiparticle scattering [9]. Indeed, a naive estimate of the kind (2) should give a logarithmic suppression \( \eta \propto 1/T^2 \ln(\epsilon_F/T) \) due to the well-known enhancement of the quasiparticle scattering rate \( \tau_{qp}^{-1} \propto (T^2/\epsilon_F) \ln(\epsilon_F/T) \) [10–19]. Quite unexpectedly, we find that the 2d viscosity is enhanced by the square of the large logarithm \( \ln(\epsilon_F/T) \), as compared to that of the 3d FL:

\[
\eta \sim \frac{3 N \hbar}{\pi \epsilon_F^2} \left( \frac{\epsilon_F}{T} \right)^2 \ln^2 \left( \frac{\epsilon_F}{T} \right). \tag{3}
\]

The behavior (3) originates from the Cooper-channel processes in which the sum \( s = p_1 + p_2 \) of the colliding momenta is much smaller than the Fermi momentum, \( s \ll p_F \). The logarithm arises from the corresponding singularity of the vertex part \( \Gamma \propto F_\pi/\ln(p_F/s) \) [4, 20], while the quasiparticle collisions at other angles appear to be less relevant.

The temperature dependence (3) can manifest itself in the form of the interaction correction to the transport properties of 2d electron systems in the presence of a smooth disorder potential. The temperature-dependent hydrodynamic contributions to conductivity, \( \sigma \propto 1/\eta(T) \), caused by the resistance to the laminar flow of an electron liquid, have been anticipated since 1960’s [21]. Recently the role of hydrodynamic modes in electron transport was revealed by observing the switching from Knudsen to Poiseuille flow of the electrons in 2d wires [22, 23]. Assuming the laminar flow in the bulk 2d samples, the result (3) suggests that in \( d = 2 \) the hydrodynamic modes may lead to a singular temperature dependence.
of the resistivity $R_{xx} \propto \ln^2 T/T^2$. This prediction can be relevant for transport measurements in clean 2d heterostructures in the metal-insulator transition regime: The interaction between the delocalized carriers may cause the apparently “insulating” correction to transport [24]. Viscosity may also limit the functionality of the FET devices suggested as a means of plasma waves generation [25]. Finally, hydrodynamic modes can play a role in the dynamics of the trapped atomic Fermi gases [26, 27]. The result (3) points at a possibility of measuring the Cooper-channel amplitude $F_c$ (defined below) as a functional of the repulsive interaction between the fermions.

In what follows, we consider the 2d FL Boltzmann equation and draw a distinction between the scattering contributions to the quasiparticle lifetime and to the viscosity.

The Boltzmann equation. — The kinetic equation for the FL

$$
\partial_t n + \partial_{\xi} n \partial_{p\xi} - \partial_{p} n \partial_{p} = 0
$$

(4)

where the quasiparticle energy $\epsilon_p = \epsilon_p^{(0)} + \delta \epsilon$, $\delta \epsilon = \int f_{pp'} \partial \rho_{pp'} \partial \mu$, is the functional of the distribution function $n = n_0(\epsilon^{(0)}) + \delta n$, and $f$ is the Landau function [$d\tau_p \equiv 2d^2p/(2\pi h)^2$]. We assume the presence of a small nonuniform velocity flow $u(r)$ such that the equilibrium distribution function is

$$
n_0 = f \left[ (\epsilon_p^{(0)} - pu - \epsilon_F)/T \right], \quad f(x) = (e^{x} + 1)^{-1}.
$$

(5)

The linearized LHS of Eq. (4) is obtained by setting $\delta n \approx n_0$ and keeping the flow gradients $\partial_i u_j$ in the second term:

$$
-\frac{\partial n_0}{\partial \epsilon} \left( p_j \partial \epsilon/\partial p_i - \delta_{ij} \frac{d}{dp} \frac{\partial \epsilon}{\partial p} \right) \frac{1}{2} \left( \partial_i u_j + \partial_j u_i - \delta_{ij} \nabla u \right).
$$

(6)

The first and the third terms of Eq. (4) almost cancel each other. Their difference $\sim (T/\epsilon_F)^2$ yields $\zeta \sim (T/\epsilon_F)^2 \eta$ [3].

Quasiparticle collisions conserve energy and momentum. The kinetic term $\{ n \}$ vanishes for the equilibrium distribution $n_0(\epsilon) = n - \delta n$, where $\epsilon$ is the true quasiparticle energy, and the deviation $\delta n = n_0 - (n_0/\partial \epsilon) \partial \epsilon$ includes the FL corrections. As usual, we identify the smooth part of $\delta n$ as $\delta n = -((n_0/\partial \epsilon) \epsilon) \psi$, where $\partial n_0/\partial \epsilon = -n_0(1 - n_0)/T$. The resulting linearized collision term written in terms of $\psi$ has the form similar to that of a weakly-interacting Fermi gas:

$$
\text{St} \{ n \} = -\frac{1}{T} \int d\tau_{12} d\tau_{1'} d\tau_{1''} w(\theta) n_0 n_0(1 - n_0') (1 - n_0'') \frac{\delta (\epsilon_1 + \epsilon_2 - \epsilon_1' - \epsilon_1'') (\psi_1 + \psi_2 - \psi_1' - \psi_2' )}{T}.
$$

(7)

Here $p_1 + p_2 = p_1' + p_2'$ is assumed (see Fig. 1), $\theta = \angle(p_1, p_2)$, and the scattering probability $w(\theta) \simeq (2\pi h)^2 \Gamma(p_1, p_2) \rightarrow \Gamma(p_1, p_2)'^2$ is determined by the quasi-particle interaction vertex $\Gamma$ and the quasiparticle weight $z$ [4].

We look for a solution $\psi$ of the linearized Eq. (4) in the form of (6), $\psi(p) = -q(p) Y(p, u)$, where the angular part

$$
Y(p, u) = \left( p_j \frac{\partial \epsilon}{\partial p_i} - \delta_{ij} \frac{d}{dp} \frac{\partial \epsilon}{\partial p} \right) \frac{1}{2} \left( \partial_i u_j + \partial_j u_i - \delta_{ij} \nabla u \right)
$$

(8)

has the form of the second angular harmonic $Y \propto \cos 2(\theta - \phi_u)$, $\phi_u = \arg(p_x + ip_y)$, with respect to the angle $\phi_u = 1/2\tan^{-1} \frac{\partial_x u_y + \partial_y u_x}{\partial_x u_y - \partial_y u_x}$ defined by the flow $u(x, y)$. Substituting $\psi$ into the collision term (7), we aim at the equation for the scalar deviation $q$ that has a meaning of the viscous relaxation time. The integration measure $d\tau_p d\tau_{1'} = \nu T^2 d\xi d\xi' (d\theta/d2\pi)(d\theta'/d2\pi)$, where $\xi = \epsilon - \epsilon_F$, and $\nu T = \nu T^2/(\pi h)^2$ is the 2d density of states. The energy delta-function sets the value for the angle $\theta_1 = \angle(p_1, p_1')$ via

$$
\delta(\epsilon_1 + \epsilon_2 - \epsilon_1' - \epsilon_2') = |\partial \epsilon_2')/\partial \theta_{1'}|^{-1} \delta(\theta_{1'} - \theta_1^{(0)}).
$$

(9)

with the Jacobian [10]

$$
\partial \epsilon_2'/\partial \theta_{1'} = -2 \left( \epsilon_1 \epsilon_2 \sin^2 \theta + A \right)^{1/2} \approx -2\epsilon_2 |\sin \theta'|,
$$

(10)

where $\theta' = \angle(p_1', p_2')$ (Fig. 1), and

$$
A \equiv \xi_1 \xi_2 - \xi_1 \xi_2' = \xi_1 (1 - \xi_1') (\xi_1' - \xi_2).
$$

(11)

In the last part of Eq. (10) we used $\cos \theta' \approx (1 - \frac{2}{3} \cos \theta)$, $A \approx A/\epsilon_F^2$, that follows from $p_1 p_2 \rightarrow p_1' p_2'$, where the relation between $\theta$ and $\theta'$ is illustrated in Fig. 2.

The symmetry of the integrand in Eq. (7) with respect to $\theta \rightarrow -\theta$ allows us to substitute $Y_2 \rightarrow Y_1 \cos 2\theta$, $Y_1' \rightarrow Y_1 \cos 2\theta'$, and $Y_2' \rightarrow Y_1 \cos 2\theta$, and to cancel $Y_1 \equiv Y(p_1, u)$ from both sides of the Boltzmann equation. Introducing the dimensionless energy variables $x = \xi/T$, as a result we arrive at the integral equation for $q(x)$:

$$
\frac{f(-x_1) f(-x_1')}{\nu T^2} = \int f(x_2) f(-x_2') w(\theta) Q d\theta dx_2 dx_1
$$

$$
= \frac{(2\pi)^2 \times 2\epsilon_F |\sin \theta'|}{T},
$$

(12)

Here $x_2 = x_1 + x_2 - x_1'$, the integration in $\theta$ is between 0 and $\pi$ since the particles are indistinguishable, and

$$
Q \equiv q(x_1) + q(x_1')(\cos 2\theta - \cos 2\theta')
$$

$$
= q_1 - q_1' + 2q_1' \sin^2 \theta_1' + \sin^2 \theta_2' - \sin^2 \theta.
$$

(13)

The stress tensor $\Pi_{ij} = \int d\sigma \partial \rho_{ij}/\partial p_j$, combined with the definition (1), yields the viscosity in terms of $q(x)$:

$$
\eta^{(2d)} = \frac{1}{4} N \epsilon_F^2 \tau_{\eta}, \quad \tau_{\eta} = \frac{\int q(x) dx}{(2 \cos \frac{\pi}{2})^2}
$$

(14)
where $N = \nu F \epsilon_F$. Above we used $\langle \hat{p}_i \hat{p}_j \rangle = \frac{1}{2} \delta_{ij}$ and $\langle \hat{p}_i \hat{p}_j \hat{p}_k \hat{p}_l \rangle = \frac{1}{8} (\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}).$

**Quasiparticle lifetime.** The solution of the integral equation (12) is related to the problem of the 2d quasiparticle lifetime $\tau_{qp}$, that is determined via substituting $Q(x_1, x_1', x_2, \theta)$ by the isorithmic term $\tau_{qp}(x_1)$. Below we will focus on this simpler problem first, to underscore the different roles played by the Cooper channel scattering processes in the inelastic lifetime and in the viscosity.

To calculate $\tau_{qp}$, one specifies the scattering probability $w(\theta)$ entering Eq. (12). In Refs. [10–16, 18, 19] the scattering was assumed to be $\theta$-independent, $w = \text{const}$. In this case the forward scattering ($\theta = 0$) and the Cooper-channel ($\theta = \pi$) contribute equally to the logarithmic singularity in $\tau_{qp}$, via $\int_{\theta_0}^{\pi/2} \frac{d\theta'}{\sin \theta'} = \int_{\pi/2}^{\pi - \theta_0} \frac{d\theta'}{\sin \theta'} \simeq \ln (\theta / |\theta| - 1/2), |\theta| \ll 1.$

Here $\theta_0 = 0$ for $A > 0$ and $\sin \theta_0 = \sqrt{-A}$ for $A < 0$.

We now argue that the contributions of the forward scattering and of the backscattering are in fact parametrically different if one takes into account the Cooper ladder of diagrams that are logarithmically divergent to all orders in the interaction. Summing the ladder causes a singularity in $w(\theta \sim \pi; \theta_1) \simeq \frac{2\pi^2}{A} \Gamma(p_1, p_1'; s)^2$, $\theta_1 = \Gamma(p_1, p_1')$, due to the renormalization of the Cooper-channel interaction vertex

$$\nu_F^2 F' = F_F + F_F^* \Phi_2, F_F^* \Phi(\theta) = \sum_{\nu=m=0}^{\infty} F_F^* \cos m\theta,$$

where $\nu_F^2 w(\theta) \simeq \frac{2\pi}{h} (1 + \frac{1}{2} \cos \theta, |\Phi|) \ll 1.$

Here we approximated the probability $w(0)$ by its average over the Fermi surface (neglecting the $\theta = \pi$ suppression) and averaged over the spin polarizations [28]. As a result [29],

$$\frac{\hbar}{\tau_{qp}(\xi, T)} \simeq \frac{\xi^2 + \pi^2 T^2}{8\pi \epsilon_F F_0^2 \ln (\epsilon_F / \xi^2 + T^2)}.$$  (17)

The rate (17) is twice smaller than that of Refs. [15, 16, 19] as a consequence of the Cooper-channel renormalization [30].

**The role of the Cooper channel in 2d.** The above calculation demonstrates that out of the two singularities (at $\theta = 0$ and $\pi$) of the integrand in Eq. (12), the former determines the lifetime $\tau_{qp}$, while the latter is suppressed by the Cooper logarithm. Below we argue that in the case of the viscosity, it is the $\theta = \pi$ scattering that dominates, while other collision angles $\theta$, including $\theta = 0$, are less relevant in $T/\epsilon_F$. Indeed, for a generic $\theta$, the energy and momentum conservation select the forward scattering, $\theta_1 \approx 0$ and $\theta_2 \approx \theta$ (Fig. 1). Due to the angular structure of Eq. (13), these contributions to the integrand are subdominant in the powers of $T/\epsilon_F$ since they originate from the deviations of the colliding momenta from $p_F$. This feature is specific to the 2d scattering. In contrast, in 3d, the collision term for the viscosity is acquired from all the scattering angles $\theta$ and $\phi$, where $\phi$ is the angle between the planes defined by the incoming and outgoing momenta. (The 3d collision term of Refs. [2, 3] indeed becomes small in $T/\epsilon_F$ for the in-plane scattering $\phi = 0$ or $\pi$.) For the head-to-head collisions ($\theta = \pi$), the phase volume for scattering rapidly increases (similar to the situation in BCS superconductivity), and there is no $T/\epsilon_F$ suppression of the integrand.

**Viscosity calculation.** The main difference of the collision term (12) from that for the inelastic lifetime is the complicated angular structure in Eq. (13), which we will focus on below. In accord with the above discussion, the main contribution comes from the region where $\sin^2 \theta_1 \sim 1$, in which case $|\theta|, |\theta'| \ll 1$, where $\theta' = \pi - \theta$ (right panel of Fig. 1). In this limit $\sin^2 \theta_1 \approx \sin^2 \theta_2 \approx |\theta|^2 \approx \bar{\theta}^2$, such that $Q \approx q(x_1) = q(x_1') + 4q(x_1') \sin^2 \theta_1$. To estimate $\theta_1$, in the limit $|\theta|, |\theta'| \ll 1$, we introduce the unit vector $\hat{s} \approx (p_1 - p_2, r_F \theta) / s$ in the direction of $p_1 + p_2$ (here we chose the first coordinate axis parallel to $p_1$), and the unit vector $\hat{s}' \approx (p_1' - p_2' + r_F \theta') / s'$ that is rotated relative to $s$ by $\theta_1 \approx \theta_2$. Here $s^2 = (p_1 - p_2)^2 + p_F^2 \bar{\theta}^2 = (p_1 - p_2)^2 + p_F^2 \bar{\theta}^2.$ Then $\cos \theta_1 \approx \hat{s} \cdot \hat{s}'$. Since $\bar{\theta}^2 \simeq \bar{A}$, it is clear that for $\sin^2 \theta_1 \sim 1$, one needs either $\bar{\theta}^2 \ll \bar{\theta}^2$ for $A > 0$, or $\bar{\theta}^2 \ll \bar{\theta}^2$ for $A < 0$. For such values of $\theta$ and $\theta'$, $s / p_F \approx \max \{|\theta|, |\theta'|\}$, and we estimate

$$\cos^2 \theta_1 \approx \min \left\{ (\bar{\theta} / \bar{\theta}')^2, (\bar{\theta}' / \bar{\theta})^2 \right\}.$$  (18)

[ Dropping the $(p_1 - p_2)(p_1' - p_2')$ terms in $s' \cdot \hat{s}'$ is justified since their contribution will be smaller in powers of $T/\epsilon_F$ after the energy integration.] To perform the angular integration
in Eq. (12) we keep the lowest singlet and triplet harmonics in the series (15), and assume \( F_0^s \approx F_0^s / \ln(1/|\theta|) \) and \( F_1^s \approx F_1^s / \ln(1/|\theta|) \) [20]. Then their contributions are

\[
\int_0^{\theta_0} \frac{(F_0^s)^2 d\theta \sin^2 \theta}{\ln^2 A + \sin^2 \theta} \approx \frac{(F_0^s)^2}{\ln^2 A} - \frac{(F_0^s)^2}{\ln^2 A} \bigg|_{\theta=0}
\]

(19)

\[
\int_0^{\theta_0} \frac{(F_1^s)^2 \cos^2 \theta \sin^2 \theta}{\ln^2 A + \sin^2 \theta} \approx \frac{(F_1^s)^2}{3 \ln^2 A}
\]

(20)

(here \( A > 0 \)). For \( A < 0 \), the calculation is similar by interchanging \( \theta \leftrightarrow \theta_0 \), \( \theta \leftrightarrow \theta_0 \), yielding the above results with \( A \rightarrow |A| \) under the logarithms. Thus the net angular contribution is \( \frac{1}{2} F_0^s / \ln^2 |A| \), where we introduced the spin-averaged dimensionless coupling \( \frac{1}{2} F_0^s \), with \( F_0^s = (F_0^s)^2 + (F_1^s)^2 \). We plug Eqs. (19) and (20) into Eq. (12), use the standard integrals \( \int dx_1 dx_2 x f(x_1) f(-x_1) f(x_2) f(-x_2) = (\cosh(x_1/2) \cosh(x_2/2) - 1) \delta(x_1 - x_2) \), and obtain the integral equation

\[
\frac{1}{\cosh(\pi}) \equiv \frac{1}{4 \pi} \left( \frac{1 + 1}{1 + x_1^2} \right) y(x_1) - \int dx_1 \tilde{K}(x_1, x_2) y(x_2)
\]

(21)

for the function \( y(x_1) = \frac{q(x_1)}{2 \theta_0 \cosh(x_1/2)} \ln \frac{\epsilon_F}{T} \), where

\[
\tilde{K}(x) = K(x) \times 2 \left( 1 - \left( \frac{F_0^s}{F_0^s} \right)^2 \right) \ln^3 \frac{\epsilon_F}{T} \left( 1 + x_1^2 \right)
\]

(22)

and \( \eta / \tau_0 \equiv T^2 / (4 \pi \epsilon_F) \) [29].

To solve Eq. (21), we note that the viscosity (14) is determined by \( \xi \lesssim T \), which allows us to set \( x = 0 \) under the logarithms, in \( 1 / \sqrt{|A|} \approx \ln(\epsilon_F/T) \), reducing it to the standard problem [2, 7], whose solution yields

\[
\tau_0 = \frac{1}{2 \ln \epsilon_F} \left( \frac{1}{3} + \frac{4 \alpha}{\pi^2} \sum_{l=1,3,\ldots} \frac{2n+1}{n^2(n+1)^2} \left[ \frac{1}{n(n+1) - \alpha} \right] \right)
\]

(23)

where \( \alpha = 2 \left[ 1 - \left( \frac{F_0^s}{F_0^s} \right)^2 \right] / \ln^3(\epsilon_F/T) \). Since \( 2 - \alpha < 1 \), from Eq. (14) we obtain the result (3). As advertized, the viscosity is fully determined by the quasiparticle scattering probability \( \nu_0^2 m^2 \simeq \frac{2 \pi}{F_0^s} / \ln^2(\epsilon_F/T) \) in the Cooper channel.

In summary, we found a nonanalytic temperature behavior of the 2d Fermi liquid viscosity and related it to the logarithmic singularity of the quasiparticle scattering amplitude in the Cooper channel. The ratio between the viscous and the inelastic scattering times is enhanced by the factor \( \sim \ln^3(\epsilon_F/T) \) due to the restrictions on the 2d quasiparticle scattering.

It is a pleasure to thank I. Aleiner and B. Altshuler for helpful discussions. This work was supported by NSF MRSEC grant DMR 02-13706.