Quantum Teichmüller space

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Abstract

We describe explicitly a noncommutative deformation of the ∗-algebra of functions on the Teichmüller space of Riemann surfaces with holes equivariant w.r.t. the mapping class group action.

1 Introduction

The main goal of the present paper is to describe the Hilbert space and the space of observables of quantum gravity in 2+1 dimensions. For this purpose, we use the following scheme. According to E.Verlinde and H.Verlinde [1] and Witten [2] (see also the monograph by Carlip [3]), the classical phase space of Einstein gravity on a 3D manifold is the Teichmüller space of its 2D boundary. (This is analogous to the fact that the classical phase space for 3D Chern–Simons theory is the moduli space of flat connections on the boundary manifold.) The Teichmüller space possesses the canonical (Weil–Petersson) Poisson structure and the mapping class group as a symmetry group. According to the correspondence principle (1) the algebra of observables of the corresponding quantum theory is a noncommutative deformation of the ∗-algebra of functions on the classical phase space in the direction given by the Poisson structure; (2) the Hilbert spaces of the theory is the ∗-representation space of this algebra; (3) the symmetry group acts on the algebra of observables by automorphisms. To solve this quantization problem it suffices to construct a family of ∗-algebras, which depends on the quantization parameter \( \hbar \), determine the action of the mapping class group by outer automorphisms, and show that the algebra and the action reproduces the classical algebra, the classical action, and the Poisson structure in the limit \( \hbar \to 0 \) (provided we believe in the existence and uniqueness of the quantization of a Poisson manifold).

In the present paper, we solve a very closely related problem, however not exactly the problem described above. We consider open 2D surfaces (although this case can be also interpreted in the spirit of the (2+1)D gravity, we skip the discussion to avoid long definitions). The corresponding Teichmüller space has a degenerate Weil–Petersson Poisson structure and the mapping class group is a symmetry group. We describe the deformation quantization of the corresponding Teichmüller space, the action of the mapping
class group by outer automorphisms, the unitary representations of the algebra, and the induced action of the mapping class group on the representation space.

We show that the action of the mapping class group on the algebra of observables depends on the quantization parameter $\hbar$ but remains unchanged if we replace $\hbar$ by $1/\hbar$, which is in agreement with a similar symmetry observed in [12].

Note that according to [1], the representation space of the observable algebra can be also interpreted as the space of conformal blocks of the Liouville conformal field theory. The construction described in this paper can be interpreted therefore as the construction of certain conformal block spaces and the mapping class group actions for this conformal field theory.

Briefly, the structure of the paper is the following. We give a very simple description of the mapping class group in terms of generators and relations of a certain groupoid called the *modular groupoid* (Sec. 2). We introduce coordinates on the Teichmüller space, such that the mapping class group action and the Weil–Petersson form takes an especially simple form simple (Sec. 3). Then, we deform the algebra of functions on the Teichmüller space into a Weyl algebra, define the action of the generators of the modular groupoid, and verify that the relations between the generators are indeed satisfied (Sec. 4).

The main mathematical ingredient of the construction is a version of the quantum logarithm and dilogarithm by Faddeev and Kashaev [4]. We interpret the corresponding five-term relation as the only nontrivial relation in the modular groupoid. The properties of the quantum logarithm and dilogarithm functions are proven in Sec. 4.1.

A similar construction was done independently and simultaneously by Kashaev [5]. However, our construction seems to be more elementary and universal.

2 Mapping class group. Graph description.

Recall that the mapping class group $\mathcal{D}(S)$ of a 2D surface $S$ is the group of homotopy classes of diffeomorphisms of the surface $S$. In this section, we give a simple combinatorial description $\mathcal{D}(S)$ for any open surface $S$.

A *fat graph* is a graph with a given cyclic ordering of ends of edges entering each vertex. Any fat graph have the distinguished set of closed paths called *faces*. These are the paths turning left at each vertex. Gluing annuli to a graph along its faces produces a 2D surface with the graph embedded into it. Therefore, we have a correspondence between the fat graphs and topological types of open 2D surfaces. Any open surface corresponds to a nonempty finite set of graphs (these are exactly the graphs that can be embedded into the surface in such a way that the surface can be shrunken onto it and the cyclic order of ends of edges coincides with the one induced by the orientation of the surface).

Any fat graph has its dual — the fat graph with the vertices of the dual graph being the faces of the original one. Each edge of the original graph corresponds to the dual graph edge that connects the faces the initial edge is incident to.

Denote by $|\Gamma|(S)$ the set of combinatorial types of three-valent graphs corresponding to a given surface. For any element of $|\Gamma|(S)$ fix a *marking*, i.e., a numeration of the edges. Denote by $\Gamma(S)$ the set of isotopy classes of embeddings of marked fat graphs into $S$. The presence of the marking changes the set of embeddings since some graphs may have nontrivial symmetry group. Introducing the marking is a tool to remove this
symmetry. (Here and below the vertical lines \(| \cdot |\) indicate the diffeomorphism class.)

The mapping class group \( \mathcal{D}(S) \) obviously acts freely on the space of embedded marked graphs having the space of combinatorial graphs as a quotient,

\[
\Gamma(S)/\mathcal{D}(S) = \lvert \Gamma \rvert(S).
\]

Recall that a group can be thought of to be a category with only one object and with all morphisms being invertible. Analogously, a groupoid is just a category such that all morphisms are invertible and such that any two objects are related by at least one morphism. Since the automorphism groups of different objects of a groupoid are obviously isomorphic to each other, we can associate a group to a groupoid in the canonical way.

We are going to construct the groupoid giving the mapping class group and admitting a simpler description in terms of generators and relations than the mapping class group itself.

Definition. Let the set \( \lvert \Gamma \rvert(S) \) be the set of objects. For any two graphs \( \lvert \Gamma \rceil, \lvert \Gamma_1 \rceil \in \lvert \Gamma \rvert(S) \) let a morphism from \( \lvert \Gamma \rceil \) to \( \lvert \Gamma_1 \rceil \) be a homotopy class of marked embeddings of both \( \lvert \Gamma \rceil \) and \( \lvert \Gamma_1 \rceil \) into \( S \) modulo the diagonal mapping class group action; we denote this morphism by \( \lvert \Gamma, \Gamma_1 \rceil \). If we have three embedded marked graphs \( \Gamma, \Gamma_1, \Gamma_2 \), then by definition the composition of \( \lvert \Gamma, \Gamma_1 \rceil \) and \( \lvert \Gamma_1, \Gamma_2 \rceil \) is \( \lvert \Gamma, \Gamma_2 \rceil \). The above described category is called the modular groupoid.

One can easily verify that (1) the multiplication of morphisms is unambiguously defined; (2) the class of the diagonal embedding \( \lvert \Gamma, \Gamma \rceil \) is the identity morphism and the inverse of the morphism \( \lvert \Gamma, \Gamma_1 \rceil \) is \( \lvert \Gamma_1, \Gamma \rceil \); (3) the group of automorphisms of an object is the mapping class group \( \mathcal{D}(S) \).

To give a description of the modular groupoid by generators and relations we need to introduce the distinguished sets of morphisms called flips and graph symmetries. We call a morphism \( \lvert \Gamma, \Gamma_\alpha \rceil \) a flip if the embedding \( \Gamma_\alpha \) is obtained from the embedding \( \Gamma \) by shrinking an edge \( \alpha \) and blowing up the obtained four-valent vertex in the other direction (see Fig. 3 below). We use the notation \( \Gamma_\alpha \) in order to emphasize the relation of this graph to the graph \( \Gamma \). Note that for the given graph \( \Gamma \), several marked embedded graphs may be denoted by \( \Gamma_\alpha \) because no marking of \( \Gamma_\alpha \) is indicated.

To each symmetry \( \sigma \) of a graph \( \Gamma \) we associate an automorphism, which is just \( \lvert \Gamma, \Gamma_\sigma \rceil \).

There is no canonical identification of edges of different graphs even if a morphism between them is given. However, for two graphs related by a flip, we can introduce such an identification. It is especially transparent in the dual picture where a flip just replaces one edge by another. Hence, one can identify the set of edges of two graphs as far as a representation of a morphism between the graphs as a sequence of flips is given. We exploit this identification and denote the corresponding edges of different graphs by the same letter if it is clear which sequence of flips relating these graphs is considered. To avoid confusion, note that this identification has nothing to do with the marking.

In this notation, the graph \( \Gamma_{\alpha_1 \cdots \alpha_n} \) is the graph obtained as a result of consecutive flips \( \alpha_n, \ldots, \alpha_1 \) of edges of a given graph \( \Gamma \).

There are three kinds of relations between flips, which are satisfied for any choice of marking for the graphs entering the relations.

**Proposition 0.1** A square of a flip is a graph symmetry: if \( \lvert \Gamma_\alpha, \Gamma \rceil \) is a flip in an edge
\[ \alpha, \text{ then } |\Gamma, \Gamma_\alpha| \text{ is also a flip and} \]

**R.2.** \[ |\Gamma, \Gamma_\alpha||\Gamma_\alpha, \Gamma| = 1. \]

Flips in disjoint edges commute: if \( \alpha \) and \( \beta \) are two edges having no common vertices, then

**R.4.** \[ |\Gamma_\alpha\beta, \Gamma_\alpha||\Gamma_\alpha, \Gamma| = |\Gamma_\alpha\beta, \Gamma_\beta||\Gamma_\beta, \Gamma|. \]

Five consecutive flips in edges \( \alpha \) and \( \beta \) having one common vertex is the identity: for such \( \alpha \) and \( \beta \), the graphs \( \Gamma_\alpha\beta \) and \( \Gamma_\beta\alpha \) are related by a flip and

**R.5.** \[ |\Gamma, \Gamma_\alpha||\Gamma_\alpha, \Gamma_\beta\alpha||\Gamma_\beta\alpha, \Gamma_\alpha\beta||\Gamma_\alpha\beta, \Gamma_\beta||\Gamma_\beta, \Gamma| = 1. \]

The proofs of relations **R.2** and **R.4** are obvious. Relation **R.5** can be seen more transparently in the dual graph picture. Indeed, a graph dual to a three-valent graph is a graph having triangular faces. A flip of the original graph corresponds to removing an edge on the dual graph and inserting another diagonal of the appearing quadrilateral. Figure 1 shows that the combination of the five flips is the identity. \[ \textbf{q.e.d.} \]

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**Fig. 1**

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**Theorem 1**

1. Flips and graph symmetries generate the modular groupoid.
2. The only relations between the generators are **R.2**, **R.4**, **R.5**, and the natural relations between flips and graph symmetries.

Replacing the mapping class group by the modular groupoid, we can simply express the latter through generators and relations.

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\(^{1}\text{The notation } \textbf{R.}n \text{ indicates the number } n \text{ of graphs entering this relation.} \)
Note that a graph symmetry can be represented as a ratio of two flips in a given edge and the modular groupoid is therefore generated only by the flips. We do not describe relations between flips and graph symmetries in details because they are quite obvious. In fact, the symmetry groups of $\Gamma$ and $\Gamma_\alpha$ act transitively on the set of flips $|\Gamma, \Gamma_\alpha|$, and this action can be considered as relations between flips and graph symmetries.

Theorem 1 can be proved using direct combinatorial methods of the simplicial geometry (cf. Viro [4]). However, we give the main idea of another proof, which is more specific for the 2D situation.

**Proof of Theorem 1.** To any connected simplicial complex $S$ we can associate a groupoid by taking a point in each top-dimensional simplex for objects and the homotopy classes of oriented paths connecting the chosen points as morphisms. The corresponding group is the fundamental group of the topological space given by the complex.

To any codimension one simplex we can associate two classes of paths (differing by orientation and having the identity morphism as their product) connecting adjacent top-dimensional simplices. It is natural to call them flips. We can associate a relation between the flips to any codimension two simplex. It is obvious that this set of flips generates the groupoid and that the only relation between the flips are given by codimension two simplices.

The same is true for an orbifold simplicial complex, where we replace simplices by quotients of simplices by finite groups. In this case, we must choose one generic point per each top-dimensional simplex as an object and orbifold homotopy classes of paths as morphisms. The corresponding group is the orbifold fundamental group of the orbifold given by the complex. The groupoid is now generated by flips and groups of top dimension simplices and still the only nontrivial relations are those given by codimension two simplices.

Consider now the Strebel [7] orbifold simplicial decomposition of the moduli space of complex structures on $S$. The orbifold fundamental group of the moduli space $\mathcal{M}$ is just the mapping class group $\mathcal{D}(S)$. Recall that Strebel orbisimplices are enumerated by fat graphs corresponding to $S$ and the dimension of a simplex is equal to the number of its edges. One can easily see that the groupoid of the Strebel complex coincides with the modular groupoid. Moreover, the flips of the former correspond to the flips of the latter. The relations between flips are given by codimension two cells, which correspond either to graphs with two four-valent vertices (which produces relation $R.4$) or to graphs with one five-valent vertex (which produces relation $R.5$). Relation $R.2$ holds true for any simplicial complex.  

q.e.d.

### 3 Teichmüller space. Graph description.

Recall briefly the combinatorial description of the Teichmüller spaces of complex structures on Riemann surfaces with holes. The Teichmüller space $\mathcal{T}^h(S)$ is the space of complex structures on a (possibly open) surface $S$ modulo diffeomorphisms homotopy equivalent to the identity. There are two types of behaviour of a complex structure in a vicinity of a hole. The vicinity can be isomorphic (as a complex manifold) either to an annulus or to a punctured disc. The holes of the second kind are called punctures.

For technical reasons, instead of the Teichmüller space $\mathcal{T}^h(S)$, we consider its finite covering $\mathcal{T}^H(S)$. A point of $\mathcal{T}^H(S)$ is determined by a point of $\mathcal{T}^h(S)$ and by orientations
of all holes of $S$ which are not punctures. This covering is obviously ramified over the subspace of surfaces with punctures.

The Poincaré uniformization theorem states that any complex surface $S$ is a quotient of the upper half-plane $H$ under the action of a discrete (Fuchsian) subgroup $\Delta(S)$ of the group $PSL(2, \mathbb{R})$ of the automorphisms of $H$,

$$S = H/\Delta(S).$$

The upper half plane possesses the $PSL(2, \mathbb{R})$-invariant Hermitian metric with curvature $-1$ given in the standard coordinates $z, \tau$ by $(\text{Im } z)^{-2} dz d\tau$. Hence on any complex surface $S$ there exists a canonical Hermitian metric of the curvature $-1$.

Any homotopy class of closed curves $\gamma$ in $S$ (except the curves surrounding one puncture) contains a unique closed geodesic of the length

$$l(\gamma) = |\log \lambda_1/\lambda_2|,$$

where $\lambda_1$ and $\lambda_2$ are different eigenvalues of the element of $PSL(2, \mathbb{R})$ that corresponds to $\gamma$. (If we assume that geodesics’ surrounding punctures have zero length, then formula (1) is valid for all classes of curves.)

Since the work of Penner [8], the fat graphs are used for describing not only moduli, but also Teichmüller spaces. We use a version of this description, which is rather explicit and simple.

**Theorem 2** Given a three-valent marked embedded graph $\Gamma \in \Gamma(S)$, there exists a canonical isomorphism between the set of points of the Teichmüller space $T^H(S)$ and the set $\mathbb{R}^{\# \text{edges}}$ of assignments of real numbers to the edges of the graph.

For details of the proof see [9]. The construction of the numbers on edges starting from a point of $T^H(S)$ and vice versa is rather explicit and elementary. As an illustration, we construct the Fuchsian group $\Delta(S) \subset PSL(2, \mathbb{R})$ corresponding to a given set of numbers on edges of a graph $\Gamma \in |\Gamma|(S)$. To describe the Fuchsian group, we must associate an element $X_\gamma \in PSL(2, \mathbb{R})$ to any element of the fundamental group $\gamma \in \pi_1(S)$.

Starting from a fat graph $\Gamma \in \Gamma(S)$, consider another fat graph $\tilde{\Gamma}$ in which a vicinity of each vertex is removed and the arising three ends of edges are connected by three new edges forming a triangle. Orient all edges of the inserted triangles clockwise as in Fig. 2.

![Fig. 2](image)

We now introduce the matrices $X(z), z \in \mathbb{R}$ and $L$ from $PSL(2, \mathbb{R})$,

$$X(z) = \begin{pmatrix} 0 & e^{z/2} \\ -e^{-z/2} & 0 \end{pmatrix},$$

$$L = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix},$$
To any path $\gamma$ on the graph $\tilde{\Gamma}$ consisting of $N$ edges we associate an element of $X_\gamma \in \text{PSL}(2, \mathbb{R})$ by the rule

$$X_\gamma = X_N \ldots X_1,$$

where $X_i = X(z_\alpha)$, $L$ or $L^{-1}$ depending on whether the $i$th segment of the path goes along the long edge or along a side of a small triangle in positive or negative direction, respectively.

Choosing one vertex and considering all elements of $\text{PSL}(2, \mathbb{R})$ corresponding to paths starting and ending at this vertex, we obtain the desired Fuchsian group $\Delta(S)$.

Because $X(z)^2 = 1$ in $\text{PSL}(2, \mathbb{R})$ we indeed obtain a homomorphism of $\pi_1(\tilde{\Gamma}) \rightarrow \text{PSL}(2, \mathbb{R})$. Note also that since $L^3 = 1$, this homomorphism factorises as $\pi_1(\tilde{\Gamma}) \rightarrow \pi_1(\Gamma) \rightarrow \text{PSL}(2, \mathbb{R})$ and the constructed group $\Delta(S)$ is in fact isomorphic to $\pi_1(\Gamma) = \pi_1(S)$ (in other words, the product along $\gamma$ does not depend on how $\gamma$ goes around small triangles).

Using this construction of the Fuchsian group, we can compute the length of any closed geodesic using (1). In particular, the length of the geodesics surrounding a hole (face) $f$ is

$$l_f = \left| \sum_{\alpha \in f} z_\alpha \right| .$$

(2)

The sign of the sum determines the mutual orientation of a hole and the surface. (For a generic path, the length is obviously nonlinear in $z_\alpha$'s.)

![Fig. 3](image-url)

Therefore, for each embedded graph $\Gamma \in \Gamma(S)$, we obtain a global coordinate system on $\mathcal{T}H(S)$ and transition maps between coordinate systems corresponding to different embedded graphs must exist. In other words, there exists a transition map for any morphism of the modular groupoid. As follows from Theorem 1, any transition map can be expressed as a composition of the transition maps corresponding to flips. For a flip the transition map is given by the following rule: the numbers on all edges except the five involved in the flip remain unchanged and the numbers on the remaining five edges are changed as shown in Fig. 3, while

$$\phi(z) = \log(e^z + 1).$$

(3)

Although it follows directly from the geometrical setting, it can be straightforwardly verified that this transformation indeed does not change the numbers on edges when the sequences of flips $\text{R.2}$, $\text{R.4}$, and $\text{R.5}$ are applied.

A canonical Poisson structure called the Weil–Peterson structure exists on $\mathcal{T}H(S)$ (cf. Goldman [10]). This structure is degenerate, and the Casimir functions are just the lengths of geodesics surrounding holes (2).
Theorem 3. In the graph coordinates, the Weil–Petersson bracket $B_{WP}$ has a very simple form:

$$B_{WP} = \sum_{\alpha} \frac{\partial}{\partial z_\alpha} \wedge \frac{\partial}{\partial z_{\alpha^r}},$$

(4)

where the sum is taken over all oriented edges $\alpha$, while $\alpha^r$ is the edge next to the right with respect to the end of $\alpha$ and $z_\alpha$ is the number assigned to the (nonoriented) edge $\alpha$.

The idea of the proof was given in [9]. In fact, for our purposes it is sufficient to check that the form (4) and the set of Casimir functions are independent on the choice of the embedded graph, what can be easily verified explicitly.

4 Quantization

Recall that a quantization of a Poisson manifold equivariant w.r.t. a discrete group action is a family of $\ast$-algebras $A^h$ depending smoothly on a positive real parameter $\hbar$, acting on $G$ by outer automorphisms and having the following relation to the Poisson manifold.

1. For $\hbar = 0$, the algebra is isomorphic as a $G$-module to the $\ast$-algebra of complex-valued function on the Poisson manifold.

2. The Poisson bracket on $A^0$ given by $\{a_1, a_2\} = \lim_{\hbar \to 0} \frac{[a_1, a_2]}{\hbar}$ coincides with the one generated by the Poisson structure of the manifold.

We now quantize a Teichmüller space $T^h(S)$ equivariantly w.r.t. the mapping class group $D(S)$.

Remark. By a smooth one-parameter family of $\ast$-algebras we mean just that all these algebras can be identified as vector spaces and the multiplication rule varies smoothly under changing the parameter $\hbar$. We do not require the identification between algebras to be equivariant with respect to the group action.

Let $T^h(\Gamma)$ where $\Gamma \in [\Gamma](S)$ is a $\ast$-algebra generated by the generator $Z^h_\alpha$ (one generator per one unoriented edge $\alpha$) and relations

$$[Z^h_\alpha, Z^h_\beta] = 2\pi i \hbar \{z_\alpha, z_\beta\}$$

(5)

with the $\ast$-structure

$$(Z^h_\alpha)^* = Z^h_\alpha.$$  

(6)

Here $z_\alpha$ and $\{\cdot, \cdot\}$ stand for the respective coordinate functions on the classical Teichmüller space and the Weil–Petersson Poisson bracket on it. Note that according to formula (4), the right-hand side of (5) is merely a constant which may take only five values $0, \pm 2\pi i \hbar, \pm 4\pi i \hbar$.

We have constructed one $\ast$-algebra per object of the modular groupoid. Now in order to describe the equivariance, we must associate a homomorphism of the corresponding $\ast$-algebras to any morphism of the modular groupoid. For this, we must associate a unitary morphism of $\ast$-algebras to any flip and any graph symmetry and verify that the relations R.2, R.4, and R.5 are satisfied by these morphisms.

The action of a graph symmetry can be obviously reduced to permuting the generators.

We now define the flip morphisms by the rule

$$\{A, B, C, D, Z\} \to \{A + \phi^h(Z), B - \phi^h(-Z), C + \phi^h(Z), D - \phi^h(-Z), -Z\},$$  

(7)
where $A$, $B$, $C$, $D$, and $Z$ are as in Fig. 4 and $\phi^h(x)$ is the real function of one real variable,

$$\phi^h(z) = -\frac{\pi \hbar}{2} \int_{\Omega} \frac{e^{-ipz}}{\sinh(\pi p) \sinh(\pi \hbar p)} dp,$$

while the contour $\Omega$ goes along the real axis bypassing the singularity at the origin from above. (The generators on the edges not shown on the Fig. 4 do not change)

![Diagram](image)

Fig. 4

We now formulate the main theorem of the paper.

**Theorem 4** The family of algebras $T^h(\Gamma)$ is a quantization of the space $T^H(S)$. Relation (7) defines the action of the modular groupoid (and of the mapping class group $D(S)$, in particular) on $T^h(\Gamma)$ by external $*$-morphisms.

Theorem 4 implies, in particular, that the algebras corresponding to different graphs are isomorphic, which is emphasized by using the notation $T^h(S)$ instead of $T^h(\Gamma)$.

Before proving Theorem 4, we discuss its corollaries and the properties of the constructed algebra and the mapping class group action.

**Corollary 1 C.1.** In the limit $\hbar \to 0$, the morphism (7) coincides with the classical morphism in Fig 3.

C.2 The morphisms $T^h(\Gamma) \to T^{1/h}(\Gamma)$ given by the mapping $Z^h_\alpha \mapsto Z^{1/h}/\hbar$ commute with the modular morphisms.

C.3 The center of the algebra $T^h(S)$ is generated by the sums $Z^h_f = \sum_{\alpha \in f} Z^h_\alpha$ ranging all edges $\alpha$ surrounding a given face $f$. The mapping class group acts on the generators $Z^h_f$ by permutations, i.e., exactly as it acts on the holes of the surface $S$.

C.4 For any set of real numbers $l_1, \ldots, l_s$ assigned to the holes of the surface $S$, there exists a unique unitary representation $\pi^h(S, l_1, \ldots, l_s)$, of $T^H(S)$ in a Hilbert space $H^h(S, l_1, \ldots, l_s)$, such that the central elements are represented by multiplications by these constants, $Z^h_f = l_f$.

C.5 Let $D(S, l_1, \ldots, l_s) \subset D(S)$ be a subgroup of the mapping class group preserving the assignment of the numbers $l_1, \ldots, l_s$ to the holes. Then there exists a unique projective representation $\rho$ of $D(S, l_1, \ldots, l_s)$ in $H^h(S, l_1, \ldots, l_s)$, determined by the condition that $\rho(x)\pi(t) = \pi(xt)\rho(x)$ for any $x \in D(S, l_1, \ldots, l_s)$ and $t \in T^H(S)$.

C.6 The obstacle for the representation $\pi(S, l_1, \ldots, l_s)$ to be a true representation (and not only a projective one) lies in the second cohomology group $H^2(D(S, l_1, \ldots, l_s), U(1))$, or, equivalently, in the second homology group of the orbifold moduli space $H^2(T^H(S)/D(S, l_1, \ldots, l_s), U(1))$. 


The main technical tool used in the proof of Theorem 4 is a simple observation, which we formulate as a separate lemma.

**Lemma 1** Consider an algebra (quantum torus) generated by two generators \( U \) and \( V \) satisfying the commutation relation \( qUV = q^{-1}VU \), where \( q \) is a nonvanishing complex number. Then the automorphism of the algebra given by \( U \rightarrow (1 + qU)V \), \( V \rightarrow U^{-1} \) has order five.

We formulate few more corollaries of Theorem 4 and Lemma 1. Although being technical, they seem to be sufficiently elegant to be stated separately.

**Corollary 2 C.7** Let \( K \) be an operator acting in the Hilbert space \( L^2(\mathbb{R}) \) and having the integral kernel
\[
K(x, z) = F^h(z)e^{-\frac{z^2}{4\hbar}},
\]
where
\[
F^h(z) = \exp \left( -\frac{1}{4} \int_\Omega \frac{e^{-ipz}}{p \sin(\pi p) \sinh(\pi \hbar p)} dp \right).
\]
Then the operator \( K \) is unitary up to a multiplicative constant and satisfies the identity
\[
K^5 = \text{const}.
\]

**C.8** Let \( h = m/n \) be a rational number and assume that both \( m \) and \( n \) are odd. Introduce a linear operator \( L(u) \) acting in the space \( \mathbb{C}^n \) and depending on one positive real parameter \( u \) through its matrix
\[
L(u)^j_i = F^h(j, u)q^{2ij},
\]
where
\[
F^h(j, u) = (1 + u)^{j/n} \prod_{k=0}^{j-1} (1 + q^{2k-1}u^{1/n})^{-1}.
\]
Then the following identity holds:
\[
L(v^{-1})L(u^{-1}v^{-1} + u^{-1})L(v + vu^{-1} + u^{-1})L(v + uv)L(u) = \text{const}.
\]

**Remarks.**

1. In accordance with the intuition, the set of irreducible representations of \( \mathcal{T}^h(S) \) is in the one-to-one correspondence with the symplectic leaves of \( \mathcal{T}^H(S) \).

2. The group \( \mathcal{D}(S, l_1, \ldots, l_s) \) actually depends only on the distribution of values of the central generators \( l_1, \ldots, l_s \). If all \( l_i \)'s are equal, then, obviously, \( \mathcal{D}(S, l_1, \ldots, l_s) = \mathcal{D}(S) \).

3. The described constructions can be considered as the generalization of the Shale–Weil representation of the symplectic group (cf. [11]). In the Shale–Weil case, the Poisson manifold under quantization is the first cohomology group \( H^1(S) \) with compact support and with the natural mapping class group action. This analogy becomes even more transparent if one represents the space \( H^1(S) \) as the space of representations of \( \pi_1(S) \) in \( \mathbb{R} \), and \( \mathcal{T}^h(S) \) as an open subset of the space of representations of \( \pi_1(S) \) in \( \text{PSL}(2, \mathbb{R}) \).

4. There is a natural question whether the described set of mapping class group representations constitutes a modular functor. In other words, since for any embedding
S₁ → S₂ of one surface into another there exists the induced homomorphism of the mapping class groups \( D(S₁) \rightarrow D(S₂) \), the question arises whether a constructed unitary representation of \( D(S₂) \) always decomposes into a direct sum of the constructed representations of \( D(S₁) \). The answer is presumably negative.

5. The function \( F^h(z) \) (see \((11)\)) asymptotically behaves as \( F(z) \rightarrow \exp\{\text{Li}_2(e^z) / \hbar\} \) when \( \hbar \rightarrow 0 \). Here \( \text{Li}_2(z) \) is the dilogarithm function

\[
\text{Li}_2(u) = \int_0^u \frac{\log(1 + x)}{x} dx.
\]

Both this function and matrix function \((12)\) are versions of quantum dilogarithms introduced by Faddeev and Kashaev \([4]\). Note that the celebrated five-term relation for \( \text{Li}_2 \) can be rewritten in the form

\[
\tilde{L}(v^{-1})\tilde{L}(u^{-1}v^{-1} + u^{-1})\tilde{L}(v + vu^{-1} + u^{-1})\tilde{L}(v + uv)\tilde{L}(u) = 1,
\]

where \( \tilde{L}(u) = e^{\text{Li}_2(u)} \).

6. Taking into account the previous remarks, the operators representing the mapping class group in the Hilbert space \( H^\hbar(S, l_1, \ldots, l_s) \) can be written as compositions of (1) Fourier transforms, (2) operators of multiplication by exponents of quadratic forms, and (3) operators of the multiplication by the function \( F^h \) (which is also very close to the Shale–Weil representation, where only the first two operations are involved).

7. Assuming the standard relation between deformation quantization and geometric quantization to be true, the algebra \( T^\hbar(S) \) can be represented in the space of sections of a certain linear bundle over the Teichmüller space. Analogously, the subalgebra \( M^\hbar(S) \subset T^\hbar(S) \) that is invariant under the action of the mapping class group \( D(S) \) must act on sections of a certain linear bundle over the moduli space. Modular forms are good candidates for these sections.

### 4.1 Properties of the function \( \phi^h(z) \)

for proving Theorem \([9]\) and its corollaries we present here the properties of the function

\[
\phi^h(z) = -\frac{\pi h}{2} \int_\Omega \frac{e^{-ipz}}{\sinh(\pi p) \sinh(\pi \hbar p)} dp.
\]

**Proposition 4.1** For the function \( \phi^h(z) \), we have

- **P.1** \( \lim_{\hbar \to 0} \phi^h(z) = \log(e^z + 1) \);
- **P.2** \( \phi^h(z) - \phi^h(-z) = z \);
- **P.3** \( \overline{\phi^h(z)} = \phi^h(\overline{z}) \);
- **P.4** \( \frac{1}{h} \phi^h(z) = \phi^{1/h}(z/\hbar) \);
- **P.5** \( \phi^h(z + i\pi \hbar) - \phi^h(z - i\pi \hbar) = \frac{2\pi i \hbar}{e^{-z/\hbar + 1}} \);
- **P.6** \( \phi^h(z + i\pi) - \phi^h(z - i\pi) = \frac{2\pi i}{e^{-z/\hbar + 1}} \);
- **P.7** the function \( \phi^h(z) \) is meromorphic with poles at the points \( \{\pi i(m+n\hbar) | m, n \in \mathbb{N}\} \) and \( \{-\pi i(m+n\hbar) | m, n \in \mathbb{N}\} \).
Proofs. To prove property P.1 note that $\lim_{z\to-\infty} \phi^h(z) = 0$. Therefore, it suffices to prove, that

$$
\lim_{h \to 0} \frac{\partial}{\partial z} \phi^h(z) = \frac{1}{e^{-z} + 1}.
$$

The left-hand side of this equality can be easily computed using residues:

$$
\lim_{h \to 0} \frac{\partial}{\partial z} \phi^h(z) = \lim_{h \to 0} \frac{\pi h}{2} \int_{\Omega} \frac{-ipe^{-ipz}}{\sinh(\pi p) \sinh(\pi hp)} dp =
$$

$$
= \frac{i}{2} \int_{\Omega} \frac{e^{-ipz}}{\sinh(\pi p)} dp = \frac{i}{2} \frac{1}{1 + e^z} \left( \int_{\Omega} - \int_{\Omega+i} \right) \frac{e^{-ipz}}{\sinh(\pi p)} dp = \frac{-1}{1 + e^z} \text{Res}_{p=i\pi} \frac{e^{-ipz}}{\sinh(\pi p)} = \frac{1}{e^{-z} + 1}.
$$

q.e.d.

Property P.2 can be verified by computing the right-hand side using residues:

$$
\phi^h(z) - \phi^h(-z) = \frac{\pi h}{2} \int_{\Omega} \frac{e^{-ipz} - e^{ipz}}{\sinh(\pi p) \sinh(\pi hp)} dp = \frac{\pi h}{2} \left( \int_{\Omega} + \int_{\Omega+i} \right) \frac{e^{-ipz}}{\sinh(\pi p) \sinh(\pi hp)} dp =
$$

$$
= \frac{\pi h}{2} 2\pi i \text{Res}_{z=0} \frac{e^{-ipz}}{\sinh(\pi p) \sinh(\pi hp)} = z.
$$

q.e.d.

Property P.3 can be obtained by changing the integration variable $q = -p$:

$$
\phi^h(z) = -\frac{\pi h}{2} \int_{\Omega} \frac{e^{ipz}}{\sinh(\pi p) \sinh(\pi hp)} dp =
$$

$$
= \frac{\pi h}{2} \int_{-\Omega} \frac{e^{ipz}}{\sinh(\pi p) \sinh(\pi hp)} dp = -\frac{\pi h}{2} \int_{\Omega} \frac{e^{-ipz}}{\sinh(\pi p) \sinh(\pi hp)} dp = \phi^h(\pi).
$$

q.e.d.

Property P.4 can be obtained by changing the integration variable $q = p/h$:

$$
\phi^{1/h}(z/h) = -\frac{\pi}{2h} \int_{\Omega} \frac{e^{-ipz/h}}{\sinh(\pi p) \sinh(\pi p/h)} dp =
$$

$$
= -\frac{\pi}{2h} \int_{\Omega} \frac{e^{-iqz}}{\sinh(\pi q) \sinh(\pi q)} dq = \phi^h(z)/h.
$$

q.e.d.

Properties P.5 and P.6 can be proven analogously to the property P.1. Both proofs are analogous, and we present only the first of them here

$$
\phi^h(z + i\pi h) - \phi^h(z - i\pi h) = -\frac{\pi h}{2} \int_{\Omega} \frac{e^{-ipz} e^{ip\pi h} - e^{-ipz} e^{-ip\pi h}}{\sinh(\pi p) \sinh(\pi hp)} dp =
$$

$$
= -\pi h \int_{\Omega} \frac{e^{-ipz}}{\sinh(\pi p)} dp = -\pi h \frac{\text{Res}_{p=\pi\pi} e^{-ipz}}{\sinh(\pi p)} = \frac{2\pi i h}{e^z + 1}.
$$

q.e.d.

Property P.7 follows from the obvious observation that integral $[\mathbb{N}]$ converges at $|\Im z| < \pi(1 + h)$ and, using properties P.5 and P.6, the function $\phi^h$ can be continued to the whole complex plane.
4.2 Proof of Theorem 4 and its corollaries.

Corollary C.1 follows from identity P.1.

For proving property C.2 we must verify that the morphism $T^h(\Gamma) \to T^{1/h}(\Gamma)$ commutes with a flip (this morphism obviously commutes with a graph symmetry). This implies that $(A + \phi^h(Z))/h = A/h + \phi^h(Z/h)$, $(B - \phi^{1/h}(-Z))/h = A/h - \phi^h(-Z/h)$, etc. Therefore, it suffices to prove that $\phi^h(z)/h = \phi^{1/h}(z/h)$, which is just property P.4.

For proving the corollary C.3 we must verify that a flip maps central generators to the corresponding central generators. This follows from property P.2 of the function $\phi^h(z)$.

Proof of Lemma 1.

One can reformulate Lemma 1 as follows. Construct a sequence of elements of the quantum torus using the recursion relation $U_{-1} = V^{-1}$, $U_0 = U$, $U_{i+1} = (1 + qU_i)U_{i-1}$. Then, $qU_{i+1}U_i = q^{-1}U_iU_{i+1}$ and $U_{i+5} = U_i$ for any $i$.

The lemma can be proved by a straightforward calculation:

$$U_{i+5} = (1 + qU_{i+4})U_{i+3}^{-1} = U_{i+3}^{-1} + qU_{i+2}^{-1}U_{i+3}^{-1} + U_{i+2}^{-1} = (1 + q^{-1}U_{i+1})U_{i+2} = U_i.$$

q.e.d.

The proofs of properties C.3, C.4, and C.5 are straightforward calculations; remark only that C.3 and C.4 follow directly from the Stone–von Neumann theorem.

The proof of Theorem 4 can be reduced to proving the following set of statements.

S.1 Morphisms (7) are indeed the algebra morphisms.

S.2 Morphism (7) preserve the $*$-structure.

S.3 Morphisms (7) satisfy relations R.2, R.4, and the graph symmetry relations.

S.4 Morphisms (7) satisfy the pentagon identity R.5.

Statement S.1 follows from the equalities $[A + \phi^h(Z), B - \phi^h(-Z)] = 0$, $[A + \phi^h(Z), D - \phi^h(-Z)] = 2\pi i h$ (the remaining relations are analogous or obvious). Using the property P.2 we can transform the commutators:

$$[A + \phi^h(Z), B - \phi^h(-Z)] = [A, B] - [A, \phi^h(-Z)] + [\phi^h(Z), B] =$$

$$= 2\pi i h \left( -1 - \frac{\partial}{\partial Z}(-\phi^h(-Z)) + \frac{\partial}{\partial Z} \phi^h(Z) \right) = 0.$$

Analogously,

$$[A + \phi^h(Z), D - \phi^h(-Z)] = [A, D] - [A, \phi^h(-Z)] + [\phi^h(Z), D] =$$

$$= 2\pi i h \left( \frac{\partial}{\partial Z}(-\phi^h(-Z)) + \frac{\partial}{\partial Z} \phi^h(Z) \right) = 2\pi i h.$$

The proof of the statement S.2, i.e., that morphisms (7) preserve the real structure is obviously equivalent to the realness condition for the function $\phi^h(z)$, i.e., to the property P.3.

Morphisms (7) agree with relation R.2 because of property P.2. The agreement with relation R.4 as well as with the graph symmetry relations is obvious by construction. The
only nontrivial verification is the proof that morphisms \( \mathbb{R}5 \) reproduce pentagon identity

There are seven generators involved in the sequence of flips \( \mathbb{R}5 \). Denote them by \( A_0, B_0, C_0, D_0, E_0, X_0, \) and \( Y_0 \) as shown in Fig. 1. Note that the flip results in the cyclical rotation of the piece of graph shown in Fig. 1. Denote by \( A_i, B_i, C_i, D_i, E_i, X_i, \) and \( Y_i \) the algebra elements associated to the edges of this piece of graph after performing \( i \) flips. The rules how these elements are changed by \( \mathbb{R}5 \) are

\[
\begin{align*}
X_{i+1} &= Y_i - \phi^h(-X_i) \\
Y_{i+1} &= -X_i \\
A_{i+1} &= D_i \\
B_{i+1} &= E_i \\
C_{i+1} &= A_i + \phi^h(X_i) \\
D_{i+1} &= B_i - \phi^h(-X_i) \\
E_{i+1} &= C_i + \phi^h(X_i)
\end{align*}
\]  \( \text{(14)} \)

Our aim is to prove that these sequences of operators are five-periodic.

Assume for a moment that the five-periodicity of \( X_i \) is proved. Then the five-periodicity of \( Y_i \) is obvious, because \( Y_{i+1} = -X_i \). The five-periodicity of, say, \( A_i \) follows from the calculation

\[
X_{i+1} = Y_i - \phi^h(-X_i) = -X_{i-1} - \phi^h(-X_i).
\]

Therefore,

\[
\phi^h(-X_i) = -X_{i+1} - X_{i-1}.
\]

Taking into account \( \mathbb{P}2 \), we have

\[
\phi^h(X_i) = X_i - X_{i+1} - X_{i-1}
\]

Now we can use these identities to transform \( A_{i+5} \):

\[
A_{i+5} = D_{i+4} = B_{i+3} = \phi^h(-X_{i+3}) = E_{i+2} = \phi^h(-X_{i+3}) = C_{i+1} = A_i + \phi^h(X_i) + \phi^h(X_{i+1}) - \phi^h(-X_{i+3}) = A_i + (X_i - X_{i-1} - X_{i+1}) + (X_{i+1} - X_i - X_{i+2}) + (X_{i+4} + X_{i+2}) = A_i + X_{i+4} - X_{i-1} = A_i.
\]

We have shown that the five-periodicity of \( A_i \) (and therefore of \( B_i, C_i, D_i, E_i, \) and \( Y_i \)) follows from the five-periodicity of \( X_i \).

For proving the five-periodicity of \( X_i \), we introduce the algebraic elements

\[
U_i = e^{-X_i}, \quad \bar{U}_i = e^{-X_{i/\hbar}},
\]

which satisfy the following commutation relations:

\[
q U_i U_{i+1} = q^{-1} U_{i+1} U_i; \quad \bar{q} \bar{U}_i \bar{U}_{i+1} = \bar{q}^{-1} \bar{U}_{i+1} \bar{U}_i; \quad U_i \bar{U}_j = \bar{U}_j U_i; \quad \bar{U}_i = (U_i)^h,
\] \( \text{(15)} \)

where

\[
q = e^{\pi i h}, \quad \bar{q} = e^{\pi i / h}.
\]

These algebraic elements transform in an especially simple way:

\[
U_{i+1} = (1 + q U_i) U_{i-1}^{-1} \quad (16)
\]

\[
\bar{U}_{i+1} = (1 + q \bar{U}_i) \bar{U}_{i-1}^{-1} \quad (17)
\]
Indeed,

\[ U_{i+1} = e^{-X_{i+1}} = e^{-Y_i + \phi^h(-X_i)} = \exp \left( \frac{1}{2\pi i h} \int_{-X_i}^{-X_i + 2\pi i h} \phi^h(z) dz \right) e^{-Y_i} = \]

\[ = \exp \left( \frac{1}{2\pi i h} \int_{-X_i}^{-X_i} \left( \phi^h(z + 2\pi i h) - \phi^h(z) \right) dz \right) e^{X_{i-1}} = \]

\[ = \exp \left( \int_{-\infty}^{X_i} \frac{dz}{e^{-z - \pi i h} + 1} \right) U_{i-1}^{-1} = (1 + qU_i)U_{i-1}^{-1}, \]

where we have used the standard formula

\[ e^{A + \Phi(B)} = \exp \left\{ \frac{1}{[A, B]} \int_B^{B+[A,B]} \Phi(z) dz \right\} e^A, \]

which is valid for all \( A \) and \( B \) such that the commutator \([A, B]\) is a nonzero scalar.

The proof of (L7) is analogous.

Now in order to prove that \( X_i \) is five-periodic it suffices to verify, that both \( U_i \) and \( \bar{U}_i \) are five-periodic. Indeed, the five-periodicity of \( U_i \) only does not suffice because the logarithm of an operator is ambiguously defined. However, if we have two families of operators \( U \) and \( \bar{U} \) depending on \( \hbar \) continuously, then if there exists an operator \( X \) (also depending on \( \hbar \) continuously) such that \( U = e^X \) and \( \bar{U} = e^{X/\hbar} \), then this operator is uniquely determined. It can be found as the limit

\[ \lim_{(m+n/\hbar) \to 0} (U^m \bar{U}^n) / (m + n/\hbar) \]

for any irrational value of \( \hbar \).

Relations (L3) and (L7) coincide with the relations from the assertion of Lemma [1], and the sequences \( U_i \) and \( \bar{U}_i \) are therefore five-periodic. \( \text{q.e.d.} \)

To prove relation C.7, we consider the representation of the algebra generated by two real generators \( X \) and \( Y \) with the relation \([X, Y] = -2\pi i \hbar\). Any unitary representation of this algebra is equivalent to the representation on the space \( L^2(\mathbb{R}) \), where \( X = 2\pi i \hbar \partial / \partial z \) and \( Y = z \), while \( z \) is the parameter on the real line. Now one can easily verify that \( K^{-1}XK = Y - \phi^h(-X) \) and \( K^{-1}YK = -X \). As follows from Theorem [4], \( K^{-5}XK^5 = X \) and \( K^{-5}YK^5 = Y \), and the operator \( K^5 \) is therefore a scalar. \( \text{q.e.d.} \)

Relation C.8 can be proved analogously. The quantum torus \( qUV = q^{-1}VU \) (where \( q = e^{\pi i m/n} \) and both \( m \) and \( n \) are odd) has a center generated by \( U^n \) and \( V^n \). For each nonzero constants \( u \) and \( v \) there exists a unique (up to the conjugation) finite-dimensional irreducible representation for which \( U^n = u \) and \( V^n = v \). In this case, \( (1 + qU)V^n = (1 + u)v \) and the action of the automorphism on the set of representations has therefore the order 5 (as follows from the lemma). We choose the explicit matrix realization:

\[ U\{u\}_j = u^{1/n} \delta^{j+1}_j, \quad V\{v\}_j = v^{1/n} q^{2j} \delta^j_j, \quad i, j \in \mathbb{Z}/n\mathbb{Z}. \]

We now can easily verify that \( L(u)U\{u\}L^{-1}(u) \sim (1 + qU\{(1 + u)v\})V\{u^{-1}\} \) and \( L(u)V\{v\}L^{-1}(u) \sim U\{(1 + u)v^{-1}\} \), where the sign \( \sim \) means the equality up to a scalar factor. If we denote by \( L \) the left-hand side of equality (L2), then \( LU\{u\}L^{-1} \sim U\{u\} \) and \( LV\{v\}L^{-1} \sim V\{v\} \). Therefore, \( L = \text{const} \) because the representation with the given value of the center is unique. \( \text{q.e.d.} \)
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