Improved Sample Complexity Bounds for Branch-and-Cut

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Integer programming

- Integer program (IP) in standard form:

  \[
  \begin{align*}
  \text{Max } c \cdot x \\
  \text{s.t. } Ax & \leq b \\
  x & \in \mathbb{Z}^n
  \end{align*}
  \]

- One of the most useful and widely applicable optimization techniques

Scheduling  Routing  Combinatorial auctions  Clustering
Branch-and-cut

• Powerful tree-search algorithm used by fastest solvers to solve IPs in practice

• Our contribution: improved theory for using machine learning to tune (1) general model of tree search and (2) any-and-all aspects of branch-and-cut
Branch-and-bound

• Powerful tree-search algorithm used to solve IPs in practice

• Uses the linear programming (LP) relaxation to do an informed search through the set of feasible integer solutions

\[
\text{IP} \quad \begin{align*}
\text{Max } & \ c \cdot x \\
\text{s.t. } & \ Ax \leq b \\
& \ x \in \mathbb{Z}^n
\end{align*}
\]

\[
\text{LP relaxation} \quad \begin{align*}
\text{Max } & \ c \cdot x \\
\text{s.t. } & \ Ax \leq b \\
& \ x \in \mathbb{R}^n
\end{align*}
\]
Branch-and-bound: branching

- Choose variable \( i \) to branch on.
- Generate one subproblem with \( x[i] \leq \lfloor x_{LP}^*[i] \rfloor \) another with \( x[i] \geq \lceil x_{LP}^*[i] \rceil \)

\[
\begin{align*}
\text{Max } c \cdot x \\
\text{s.t. } Ax &\leq b \\
x &\in \mathbb{Z}^n
\end{align*}
\]

\[
\begin{align*}
\text{Max } c \cdot x \\
\text{s.t. } Ax &\leq b \\
x[i] &\leq 2 \\
x &\in \mathbb{Z}^n
\end{align*}
\]

\[
\begin{align*}
\text{Max } c \cdot x \\
\text{s.t. } Ax &\leq b \\
x[i] &\geq 3 \\
x &\in \mathbb{Z}^n
\end{align*}
\]
Branch-and-bound: pruning

- Prune subtrees if
  - LP relaxation at a node is integral, infeasible, or
  - (Bounding) LP optimal worse than best feasible integer solution found so far

\[
\begin{align*}
\text{Max } c \cdot x \\
\text{s.t. } Ax &\leq b \\
x[i] &\leq 2 \\
x &\in \mathbb{Z}^n
\end{align*}
\]
Branch-and-bound: node selection

- At every stage, need to choose a leaf to explore further
- Variety of heuristics (e.g. best-bound-first chooses the node with the smallest LP objective)

\[
\begin{align*}
\text{Max } c \cdot x \\
\text{s.t. } Ax &\leq b \\
x &\in \mathbb{Z}^n \\
\end{align*}
\]

\[
\begin{align*}
\text{Max } c \cdot x \\
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x[i] &\leq 2 \\
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\end{align*}
\]

\[
\begin{align*}
\text{Max } c \cdot x \\
\text{s.t. } Ax &\leq b \\
x[i] &\geq 3 \\
x &\in \mathbb{Z}^n \\
\end{align*}
\]
Branch-and-cut

• Branch-and-bound, but at each node may add cutting planes

• Method of getting tighter LP relaxation bounds, and thus pruning subtrees sooner
Cutting planes

- Constraint $\alpha x \leq \beta$ is a valid cutting plane if it does not cut off any integer feasible points.

Valid cutting planes

An invalid cutting plane
Cutting planes

• If $\alpha x \leq \beta$ is valid and separates the LP optimum, can speed up B&C by pruning nodes sooner

$x_{LP}^*$ after adding cut

Integer optimum $x_{IP}^*$
Tuning branch-and-cut

- Solvers like CPLEX, Gurobi have numerous parameters that control various aspects of the search (CPLEX has 170 page manual describing 172 parameters)
Abstracting away: tree search

- Select node Q that maximizes node selection rule $n_{score}(T, Q)$
  - Select action A that maximizes action score $a_{score}(T, Q, A)$
  - Either prune tree at Q, or add children
  - Continue until all nodes are pruned

Actions chosen using mixture of scoring rules:
$$a_{score} = \mu \cdot a_{score_1} + (1 - \mu) \cdot a_{score_2}$$

Nodes chosen using mixture of scoring rules:
$$n_{score} = \lambda \cdot n_{score_1} + (1 - \lambda) \cdot n_{score_2}$$
Cut scoring rule example

**Efficacy:**

distance between cut and $x^*_{LP}$

\[
\text{score}_1(\alpha^T x \leq \beta) = \frac{\alpha x^*_{LP} - \beta}{\|\alpha\|_2}
\]
Cut scoring rule example

**Parallelism:**

angle between cut and objective

\[
\text{score}_2(\alpha^T x \leq \beta) = \frac{|c\alpha|}{\|\alpha\|_2 \|c\|_2}
\]

Better parallelism

Worse parallelism
Cut scoring rule example

**Directed cutoff:**

distance between cut and $x^*_\text{LP}$, in direction of current best integer solution

$$\text{score}_3(\alpha^T x \leq \beta) = \frac{\alpha x^*_\text{LP} - \beta}{|\alpha(\bar{x} - x^*_\text{LP})|} \cdot \|\bar{x} - x^*_\text{LP}\|_2$$
Pathwise scoring rules

• All the previous scoring rules are *pathwise*: they only depend on the LP information accumulated along the path from the root to the node in question

• Open source solver SCIP uses hard-coded mixture of scores to choose cuts

\[
\frac{3}{5} \text{score}_1 + \frac{1}{10} \text{score}_2 + \frac{1}{2} \text{score}_3 + \frac{1}{10} \text{score}_4
\]
Generalization guarantees for

tree search and branch-and-cut

Distribution-dependent parameter

selection of $\mu, \lambda$
Parameterized tree search

- Select node $Q$ that maximizes node selection rule $nscore(T, Q)$
  - Select action $A$ that maximizes action score $ascore(T, Q, A)$
  - Either prune tree at $Q$, or add children
  - Continue until all nodes are pruned

Actions chosen using mixture of pathwise scoring rules:
\[
ascore = \mu \cdot ascore_1 + (1 - \mu) \cdot ascore_2
\]

Nodes chosen using mixture of pathwise scoring rules:
\[
nscore = \lambda \cdot nscore_1 + (1 - \lambda) \cdot nscore_2
\]
Learning to tune tree search

Best parameters for airline-scheduling IPs...

...might not be useful for combinatorial-auction IPs solved by a sourcing firm
Learning to tune branch-and-cut

If a certain set of parameters yields small average branch-and-cut tree size over IP samples...

\[
\text{Max } c_1 \cdot x \quad \text{s.t. } A_1 x \leq b_1 \\
x \in \mathbb{Z}^n
\]

\[
\text{Max } c_N \cdot x \quad \text{s.t. } A_N x \leq b_N \\
x \in \mathbb{Z}^n
\]

\[\sim D\]

...is it likely to yield a small branch-and-cut tree on a fresh IP?

\[
\text{Max } c \cdot x \quad \text{s.t. } A x \leq b \\
x \in \mathbb{Z}^n
\]

\[\sim D\]
Sample complexity

- $Q$ – domain of input root nodes to tree search
- $F = \{f_{\mu,\lambda}: Q \to \mathbb{R} | \mu, \lambda\}$ class of functions (e.g. tree size)
- Sample complexity $N_F(\varepsilon, \delta)$ is the minimum $N_0 \in \mathbb{N}$ such that for any $N \geq N_0$:

$$\Pr_{Q_1,\ldots,Q_N \sim D} \left( \sup_{f \in F} \left| \frac{1}{N} \sum_{i=1}^{N} f(Q_i) - \mathbb{E}_{Q \sim D}[f(Q)] \right| \leq \varepsilon \right) \geq 1 - \delta$$

for any distribution $D$ on $Q$. 
Sample complexity of tuning tree search

**Theorem [BPSV CP’22]:** For all $\mu, \lambda$, the number of samples so that the difference between average training performance and expected performance when $\mu, \lambda$ is used to select actions and nodes throughout the tree is (whp) at most $\varepsilon$ is

$$\tilde{O} \left( \frac{H^2}{\varepsilon^2} (\Delta^2 \log k + \Delta \log b) \right)$$

$\Delta$ = tree depth  
$k$ = tree branching factor  
$b$ = # actions available at each node  
$H$ = cap on size of tree

First guarantee that handles multiple critical aspects of branch-and-cut:  
Node selection, branching, and cutting plane selection
Generalization guarantee for tree search

**Theorem [BPSV CP’22]:** For all $\mu, \lambda$, difference between average training performance and expected performance when $\mu, \lambda$ is used to select actions and nodes throughout the tree is (whp)

$$\tilde{O}\left(H \sqrt{\frac{\Delta^2 \log k + \Delta \log b}{N}}\right)$$

- $\Delta =$ tree depth
- $k =$ tree branching factor
- $b =$ # actions available at each node
- $H =$ cap on size of tree

Holds for any (unknown) distribution over tree-search problem instances

First guarantee that handles multiple critical aspects of branch-and-cut: Node selection, branching, and cutting plane selection
Tree search guarantees

• Main challenge: performance functions (e.g. size of tree) are highly discontinuous
  – Miniscule change in parameters can lead to exponential difference in tree size

• We prove that parameterized tree search is structured

• Allows us to bound the intrinsic complexity (pseudo-dimension from learning theory) of the class of performance functions parameterized by $(\mu, \lambda)$, which implies our sample complexity bounds
Tree search structure

**Theorem [BPSV CP’22]:**

Fix path-wise node selection scores $\text{nscore}_1, \text{nscore}_2$ and path-wise action selection scores $\text{ascore}_1, \text{ascore}_2$, and the input node $Q$.

There are $\leq k^{\Delta(9+\Delta)}b^\Delta$ rectangles partitioning $[0,1]^2$ such that for any rectangle $R$, the node-selection score $\lambda \cdot \text{nscore}_1 + (1 - \lambda) \cdot \text{nscore}_2$ and action selection score $\mu \cdot \text{ascore}_1 + (1 - \mu) \cdot \text{ascore}_2$ result in the same tree for all $(\mu, \lambda) \in R$.

$\Delta =$ tree depth

$k =$ tree branching factor
Back to branch-and-cut

• Our result implies polynomial bounds for:
  – Branching: single-variable, multi-variable, branching on general disjunctions with bounded coefficients,…
  – Cutting planes: cover cuts, clique cuts, any cuts derived from simplex tableau (Chvátal cuts, Gomory mixed integer cuts)
  – Allows node selection to be tuned simultaneously

• Prior work
  – [Balcan et al. ICML’18] studied single-variable branching with pathwise scoring rules (our result recovers theirs)
  – [Balcan, Prasad, Vitercik, Sandholm NeurIPS’21] studied Chvátal cuts, but obtained a much weaker bound when these are applied throughout the tree due to not using pathwise assumption
Knapsack cover cuts – an experiment

- Set of items $N$, item $i \in N$ has value $p_i \geq 0$ and weight $w_i \geq 0$
- Set of knapsacks $K$, knapsack $k \in K$ has capacity $W_k \geq 0$
- **Goal:** find feasible packing of maximum weight

\[
\begin{align*}
\text{maximize} & \quad \sum_{i \in N} \sum_{k \in K} p_i x_{k,i} \\
\text{subject to} & \quad \sum_{i \in N} w_i x_{k,i} \leq W_k \quad \forall k \in K \\
& \quad \sum_{k \in K} x_{k,i} \leq 1 \quad \forall i \in N \\
& \quad x_{k,i} \in \{0,1\} \quad \forall i \in N, k \in K
\end{align*}
\]
Knapsack cover cuts – an experiment

• Cover cut for knapsack $k$: if $w_1 + w_2 + w_3 \geq W_k$ (items 1, 2, 3 are jointly too heavy for knapsack $k$), can enforce the constraint $x_{k,1} + x_{k,2} + x_{k,3} \leq 2$

• We tune convex combinations of cut scoring rules to control the addition of cover cuts* throughout the branch-and-cut tree

*actually a special kind of cover cut: extended minimal cover cuts
Knapsack cover cuts – an experiment

Figure 1 Chvátal distribution with 35 items and 2 knapsacks.

Figure 2 Chvátal distribution with 35 items and 3 knapsacks.
Knapsack cover cuts – an experiment

(a) $\mu \cdot E + (1 - \mu) \cdot P$.  
(b) $\mu \cdot E + (1 - \mu) \cdot D$.  
(c) $\mu \cdot D + (1 - \mu) \cdot P$.

**Figure 3** Reverse Chvátal distribution with 100 items and 10 knapsacks.

(a) $\mu \cdot E + (1 - \mu) \cdot P$.  
(b) $\mu \cdot E + (1 - \mu) \cdot D$.  
(c) $\mu \cdot D + (1 - \mu) \cdot P$.

**Figure 4** Reverse Chvátal distribution with 100 items and 15 knapsacks.