Local solvability on $\mathbb{H}_1$: non-homogeneous operators

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Abstract Local solvability and non-solvability are classified for left-invariant differential operators on the Heisenberg group $\mathbb{H}_1$ of the form $L = P_n(X, Y) + Q(X, Y)$ where the $P_n$ are certain homogeneous polynomials of order $n \geq 2$ and $Q$ is of lower order with $X = \partial_x$, $Y = \partial_y + x \partial_w$ on $\mathbb{R}^3$. We extend previous studies of operators of the form $P_n(X, Y)$ via representations involving ordinary differential operators with a parameter.

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1 Introduction

We continue our study of the solvability of operators of the form $P(X, Y)$ for certain left-invariant vector fields on the Heisenberg group with underlying space $\mathbb{R}^3$. We choose the realization of the corresponding Lie algebra using $X \overset{\text{def}}{=} \partial_x$, $Y \overset{\text{def}}{=} \partial_y + x \partial_w$ for (group) variables $x, y, w$. Note that our operators include all generators of $\mathfrak{h}_{\mathbb{C}}^1$ since $\partial_w = [X, Y]$.

Local solvability for such a class of operators is well researched and we defer to [8, 9, 14] and the references therein for an introduction to the present research. Our work closely follows those techniques used in [2, 14–16] to study related operators.

The study operators of order $n \geq 2$ which can be expressed as polynomials (in operator notation) of the form

$$L = P(X, Y) = \sum_{l=0}^{n} P_l(X, Y)$$

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where each $P_l$ is homogeneous of degree $l$ in the non-commuting variables $X, Y$ with complex (constant) coefficients. Moreover, the highest-order terms $P_n$ form a so-called \textit{generic} operator by which we mean the following: In the complex variable $z$, $P_n(i z, 0) = z^n$ and $P_n(i z, 1)$ has distinct complex (characteristic) roots $\{y_j\}_{j=1}^n$. In this article we will characterize local solvability of operators $L$ in terms of related ordinary differential operators of the form $L_\pm^\mu = \sum_{l=0}^n \mu^{n-l} P_l(i \partial_t, \pm t)$ and $L_\pm^\infty = P_n(i \partial_t, \pm t)$ along their respective adjoint operators.

The major object of the present work is to examine the effect, if any, that the inclusion of lower-order terms has to the solvability of a homogenous left-invariant operator. Some of the more famous results in local (non-) solvability are characterized in terms of the principle symbol defined on $T^*(\mathbb{R}^m)$ given by $p_n(\vec{x}, \vec{\xi}) \equiv \sum_{|\alpha|=n} a_\alpha(\vec{x})(i \vec{\xi})^\alpha$ for an operator $L = \sum_{|\alpha|=n} a_\alpha(\vec{x}) \partial_\alpha^\vec{\xi}$.

Necessary and sufficient conditions for operators of principle type appear in the works [11,12]. More general criteria appear in [6] (Theorem 6.1.1): For $\partial_\xi$ to be locally solvable, $p_n(\vec{x}, \vec{\xi})$ must satisfy $p_n(\vec{x}, \vec{\xi}) = 0 \implies \sum_{l=1}^m \partial_{x_l} p_n(\vec{x}, \vec{\xi}) \beta_{\xi_l} \partial_{\xi_l} \tilde{p}_n(\vec{x}, \vec{\xi}) - \partial_{\xi_l} p_n(\vec{x}, \vec{\xi}) \beta_{x_l} p_n(\vec{x}, \vec{\xi}) = 0$. In these results the highest-order derivative terms determine (non-)solvability; and, the inclusion of any smooth lower-order terms do not alter this property. The latter result has been applied to various left-invariant operators on $\mathbb{H}_m$ of various dimensions $m$ [10]. Our operators, however, are at least doubly characteristic (see [14]). In contrast, our approach is, for the most part, to study solvability of operators $P(X, Y)$ as compared to the solvability of the operator $P_n(X, Y)$, formed by highest-order terms of $P(X, Y)$ in the subalgebra of $h^2_C$ generated by $X$ and $Y$—not necessarily those of highest order in differentiation: Note, for instance, that $\partial_{\mu}$ is of order two in the subalgebra, but has a symbol of order one.

We elaborate on our motivation for using the particular representations $L_\pm^\mu$ of $L$: For $f(x, y, w) \in S(\mathbb{R}^3)$ and $\hat{\cdot}$ denoting Fourier (inverse) transform with respect to the second and third variables, we write $L f(x, y, w) = L(\hat{f})(x, y, w) =$

\[
\frac{1}{2\pi} \int_{\mathbb{R}^2} e^{-i(\xi y + \eta w)} P(\partial_x, -i(\xi + \eta x)) \hat{f}(x, \xi, \mu) \, d\xi \, d\eta
\]  

(1.2)

The change of variables $t = x\mu \pm \xi/\mu$ and $\mu = \sqrt{|\eta|}$ leads us to the representations $\Lambda_\pm^\mu \mu > 0$ given by $X \to \mu \frac{d}{dt}$, $Y \to \mp i \mu t$ (resp.) whereby we obtain

\[
\Lambda_\pm^\mu (L) = \sum_{l=0}^n \Lambda_\mu (P_l(X, Y)) = (-i \mu)^n \sum_{l=0}^n \left( \frac{i}{\mu} \right)^{n-l} P_l \left( i \frac{d}{dt}, t \right)
\]  

(1.3)

The realizations $L_\pm^\mu$ lend themselves to analysis of ordinary differential equations involving a parameter (with singularity at $\mu = 0$). In turn, our positive results on solvability occur in the cases where solutions to $L_\pm^\mu f = g$ can be used to construct parametrices for $L$. We introduce

\textbf{Definition 1.4} The operator $L_\mu$ has as a \textit{regular parametrix} on $\Omega$ if for every bounded function $g \in C^\infty(\mathbb{R})$ there is a function $F(t, \mu)$ satisfying the following:

\begin{enumerate}
\item $F$ is a smooth function in the variables $(t, \mu)$ in domain $\Omega$;
\item $L_\mu F = g$ on $\Omega$; and,
\item For every $m, \exists a, C > 0$ so that $|\partial^j_t F(t, \mu)| \leq C(1 + |t| + |\mu|)^a$ on $\Omega$ for each $j : 0 \leq j \leq m$.
\end{enumerate}
A general result which we are ready to present is the following

**Lemma 1.5** An operator $L$ as in (1.1) is locally solvable if each of the associated operators $\mathcal{L}_\mu^\pm$ have regular parametrices on $\mathbb{R} \times (\mu_0, \infty)$ for some $\mu_0 > 0$.

From the above lemma we will obtain the following results:

**Theorem 1.6** For the operator $L$ as in (1.1) suppose that the (generic) polynomial $P_n$ has characteristic roots $\gamma_j$ all with non-zero real parts. Then $L$ is locally solvable if $P_n(X, Y)$ is locally solvable.

From [14] we have immediately

**Corollary 1.7** The operator $L$ of Theorem 1.6 is locally solvable if $\ker(L^{\pm}_\mu)^* \cap \mathcal{S}(\mathbb{R}) = \{0\}$ for both choices of $\pm$ sign.

Results on non-solvability we are ready to state are as follows:

**Theorem 1.8** Suppose $L$ is as in (1.1) for some generic $P_n$. Then $L$ is not locally solvable if, for some choice of $\pm$ sign, the set of parameters $\mu \in \mathbb{R}^+: \ker(L^{\pm}_\mu)^* \cap \mathcal{S}(\mathbb{R}) \setminus \{0\} \neq \emptyset$ has a limit point in $\mathbb{R}^+$.

**Theorem 1.9** An operator as in Theorem 1.6 is not locally solvable if the cardinality of either $\{\gamma_j | \text{Re}\gamma_j > 0\}$ or $\{\gamma_j | \text{Re}\gamma_j < 0\}$ is greater than $n/2$.

The outline of the article is as follows: In Sect. 2 we establish estimates of bases for $\ker L_\mu$ for $(t, \mu)$ in real domains as in Lemma 1.5. In Sect. 3 we establish estimates with $(t, \mu)$ extended to certain complex domains. In Sect. 4 we construct solutions to $Lu = f$ to prove Lemma 1.5 and Theorem 1.6. In Sect. 5 we provide particular results on non-solvability, including proofs of Theorems 1.8 and 1.9. In Sect. 6 we develop criteria for non-solvability by which we develop exact conditions for some subclass of operators $L$. In Sect. 7 we apply our general results to various classes of second-order operators and compare our results to those of a well-known subclass.

### 2 Canonical bases for $\ker L_\mu$

We will introduce some notation which we will use throughout the remainder of the article: Given functions $f$ and $g$, the expression $f \lesssim g$ will mean $\exists C > 0$ (fixed) so that $|f| \leq C|g|$ holds on the specified domain; and, the expression $f \asymp g$ will mean that $f \lesssim g$ and $g \lesssim f$ both hold.

We now derive asymptotic estimates for certain bases of $\ker L_\mu$ for real $\mu$ following a diagonalization procedure in [2]. We note that for each $l$ $P_l \left( i \frac{\partial}{\partial t}, t \right)$ can be written in the form

$$P_l \left( i \frac{\partial}{\partial t}, t \right) = \sum_{j=0}^{l} t^{n-j} q_{l,j}(t) \frac{\partial^j}{\partial t^j}$$

for $q_{l,j}(t) = d_{l,j} + \tilde{q}_{l,j}(t)$ where the $d_{l,j}$ are complex constants (vanishing for $j > l$) and $\tilde{q}_{l,j}(t) = \sum_{0 \leq m \leq n-j} e_{l,j,m} t^{2m}$ for some complex constants $e_{l,j,m}$ (see Eq. (2.4) of [2]); in fact, $d_{n,n-1} = -i \sum_{j=0}^{n} \gamma_j$.  

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By way of rearrangement, we write
\[ \mathcal{L}_\mu = \sum_{l=0}^{n} \frac{1}{\mu^{n-l}} P_l \left( i \frac{\partial}{\partial t}, t \right) = \sum_{j=0}^{n} t^{n-j} Q_j(t, \mu) \frac{\partial^{j}}{\partial t^{j}} \]
for \( Q_j(t, \mu) = q_n,j(t) + \sum_{l=1}^{n-j} \frac{1}{(\mu l)^j} q_{n-l,j}(t) \) (a vacuous sum is taken to be zero). Therefore,
\[ Q_j(t, \mu) = d_{n,j} + \frac{d_{n-1,j}}{\mu t} + \left( \epsilon_{n,j,1} + \frac{d_{n-2,j}}{\mu^2} \right) \frac{1}{t^2} + \epsilon_j(t, \mu) \]
where \( \epsilon_j(t, \mu) \) may be expressed as a linear combination with complex coefficients of monomials of the form \( \frac{1}{t^{a+b}} \mu^b \) for integers \( a \geq 0 \) and \( 3 \leq b \leq l \). We note that, for \( t \) restricted to any compact subset of \( \mathbb{R} \), \( Q_j(t, \mu) \) converges to \( q_j(t, \mu) \) uniformly as \( \mu \to \infty \).

Let us formulate the differential equation \( \mathcal{L}_\mu f = 0 \) if and only if \( u' = Au \) where \( u \) is the column vector \(( f, f', \ldots, f^{(n-1)})^\dagger \), with \( \dagger \) denoting transpose, and where \( A = \)
\[
\begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & & \vdots \\
-t^n Q_0(t, \mu) & -t^{n-1} Q_1(t, \mu) & -t^{n-2} Q_2(t, \mu) & \cdots & -t Q_{n-1}(t, \mu)
\end{pmatrix}
\]

We now move to diagonalize \( A \) modulo appropriate error terms. We single out the lower-order terms of \( A \) as we define its principal part \( A_0 \) to be the same as \( A \) except that the \( n-th \) row entries satisfy \( [A_0]_{n,k} = -t^{n-k-1} \). Let us also define \( S_0 \) by \([S_0]_{j,k} = (\gamma_{jk} t)^{j-1} \) and define \( A_0 \) to be the diagonal matrix with \([A_0]_{j,j} = \gamma_{jj} t \). We note that \( S_0 \) diagonalizes \( A_0 \) in that \( A_0 S_0 = S_0 A_0 \). We consider higher-order terms (in \( t \)) by setting (formally)
\[ A = A_0 + E_1 + E_2 + E_3 \]
where the \( E_i \)'s satisfy the following: \([E_i]_{j,k} = 0 \) for \( j < n \); \([E_1]_{n,k} = a_k t^{n-k-1} \); \([E_2]_{n,k} = b_k t^{n-k-1} \); and, \([E_3]_{n,k} = -\epsilon_k(t, \mu) t^{n-k-2} \). (\( \epsilon_k \equiv 0 \) for \( k \geq n - 2 \)). Here the non-vanishing coefficients are given by \( a_j = -\frac{d_{n-1,j-1}}{\mu} \), \( b_j = -(\epsilon_{n,j,1} + \frac{d_{n-2,j-1}}{\mu^2}) \) and \( \epsilon_j(t, \mu) \) as above for \( 1 \leq j \leq n \).

Regarding \( S_0 = S_0(t) \) as a matrix-valued function of \( t \), it is not difficult to show
\[ \det S_0(t) = t^{\frac{n(n-1)}{2}} \det S_0(1) \]
Since \( S_0(1) \) is a VanderMonde Matrix, \( S_0(t) \) is invertible for all \( t \neq 0 \): Indeed,
\[ [S_0^{-1}(t)]_{j,k} = \frac{1}{t^{k-1}} [S_0^{-1}(1)]_{j,k} \]

We define \( S = S_0(I + A + \Delta) \) where \( A \) is a matrix-valued function \( \alpha_{i,j} \) and \( \delta_{i,j} \) to be determined. We set the diagonal elements \( \alpha_{j,j} = \delta_{j,j} = 0 \). Formally, we write
\[ S^{-1} = (I + A + \Delta)^{-1} S_0^{-1} = I - (A + \Delta)(I + A + \Delta)^{-1} = I - (A + \Delta) + (A + \Delta)^2(I + A + \Delta)^{-1} \]
To appraise the error terms in the diagonalization \( S^{-1}AS \) we introduce the notation \( A = O(t^p) \), meaning that all entries of the matrix \( A \) are majorized by \( t^p \) uniformly for all sufficiently large \( t \) and \( \mu \). Let us define \( D_i \) \( \equiv \) \( S_0^{-1} \varepsilon_i S_0 \) for \( i = 1, 2, 3 \). We compute

\[
[D_1]_{j,k} = [S_0^{-1}(t)]_{j,n} \sum_{l=1}^{n} a_l y_k^{l-1} t^{n-1} = [S_0^{-1}(1)]_{j,n} \sum_{l=1}^{n} a_l y_k^{l-1}
\]

so that \( D_1 = O(1) \) is constant with respect to \( t \); in fact, \( D_1 \to 0^{n \times n} \) as \( \mu \to \infty \). Also,

\[
[D_2]_{j,k} = [S_0^{-1}(t)]_{j,n} \left( \sum_{l=1}^{n} b_l y_k^{l-1} \right) t^{n-2} = \frac{1}{t} [S_0^{-1}(1)]_{j,n} \sum_{l=1}^{n} b_l y_k^{l-1}
\]

so that \( D_2 = O(t^{-1}) \). It follows similarly that \( D_3 = O(t^{-2}) \).

We proceed with the diagonalization of \( A \) as we compute

\[
S^{-1}AS = [I - (A + \Delta) + (\Delta^2 + A \Delta + \Delta A + \Delta^2)(I + A + \Delta)^{-1}] S_0^{-1}
\]

\[
\circ (A_0 + \varepsilon_1 + \varepsilon_2 + \varepsilon_3) S_0 [I + A + \Delta]
\]

\[
= [I - (A + \Delta) + (\Delta^2 + A \Delta + \Delta A + \Delta^2)(I + A + \Delta)^{-1}]
\]

\[
\circ (\Lambda_0 + D_1 + D_2 + D_3) [I + A + \Delta]
\]

(2.1)

Given fixed \( \alpha_{i,j} \) and \( \delta_{i,j} \), the matrix \( I + A + \Delta \) is invertible for all sufficiently large \( t \) for (say) \( \mu \geq 1 \) and \( (I + A + \Delta)^{-1} = O(1) \). Upon multiplying out (2.1) and collecting terms up to \( O(t^{-2}) \), one obtains

\[
S^{-1}AS = \Lambda_0
\]

\[+ D_1 - \sigma A, \Lambda_0 \]

(2.2)

\[+ D_2 - \sigma A, D_1 + \sigma A, \Lambda_0 - \sigma A, \Lambda_0 \]

(2.3)

\[+ O(t^{-2}) \]

(2.4)

The elements of (2.2) are known from above; each summand in (2.3) is \( O(1) \), indeed constant with respect to \( t \); and, each summand in (2.4) is \( O(t^{-1}) \) and, more precisely, is a product of \( \frac{1}{t} \) times a matrix that is constant with respect to \( t \).

We make the substitution \( v = S^{-1}u \) so that the differential equation becomes \( v' = Bu \) for \( B = S^{-1}AS - S^{-1}S' \). We first estimate \( S^{-1}S' \) according to

\[
S^{-1}S' = S_0^{-1}S_0' - (I + A + \Delta)^{-1}(A + \Delta) S_0^{-1} S_0'
\]

\[
+ S^{-1}S' \Delta + S^{-1}S' A + S^{-1}S_0' A' + S^{-1}S_0 A' + \Delta'.
\]

To estimate \( S^{-1}S' \) modulo terms of order \( O(t^{-2}) \), we find that it suffices to estimate \( S_0^{-1}S_0' \): with \( [S_0']_{j,k} = (j - 1) y_k^{j-1} t^{j-2} \), we compute

\[
[S_0^{-1}S_0']_{j,k} = \sum_{l=1}^{n} [S_0^{-1}(t)]_{j,l} (l - 1) y_k^{l-1} t^{l-2}
\]

\[
= \frac{1}{t} \sum_{l=1}^{n} [S_0^{-1}(1)]_{j,l} (l - 1) y_k^{l-1}
\]
and \( S_0^{-1} S'_0 = O(t^{-1}) \). Now,
\[
S^{-1} S' = (I + A + \Delta)^{-1} S_0^{-1} S'_0
= (I - (A + \Delta)(I + A + \Delta)^{-1}) S_0^{-1} S'_0
= S_0^{-1} S'_0 - (A + \Delta)(I + A + \Delta)^{-1} S_0^{-1} S'_0.
\]

It is now clear that \( S^{-1} S' - S_0^{-1} S'_0 = O \left( \frac{1}{t^2} \right) \). Therefore,
\[
\begin{align*}
S^{-1} AS - S^{-1} S'_0 &= \Lambda_0 \\
+ &\mathcal{D}_1 - [A, \Lambda_0] \\
+ &\mathcal{D}_2 + [\mathcal{D}_1, A] + A[A, \Lambda_0] - S_0^{-1} S'_0 - [\Delta, \Lambda_0] \\
+ &O(t^{-2}).
\end{align*}
\]

Since the roots \( \gamma_j \) are distinct, we may define the elements of \( \mathcal{A} \) uniquely by setting its diagonal elements \( \delta_{ij} \) to zero and by setting \( [\mathcal{A}, \Lambda_0] = \mathcal{D}_1 \) off the diagonal.

We then set
\[
[\Delta, \Lambda_0] = \mathcal{D}_2 + [\mathcal{D}_1, A] + A[A, \Lambda_0] + S_0^{-1} S'_0 \text{ off the diagonal}
\]

and set the diagonal elements \( \alpha_{ij} \) to zero to define the matrix \( \Delta \). We now set \( \forall j \)
\[
\beta_{\mu,j} \overset{\text{def}}{=} [\mathcal{D}_1]_{j,j}
\]
and
\[
\frac{\rho_{\mu,j}}{t} \overset{\text{def}}{=} [\mathcal{D}_2 + [\mathcal{D}_1, A] + A[A, \Lambda_0] - S_0^{-1} S'_0]_{j,j}.
\]

We note that the \( \beta_j \)'s are complex constants depending only on \( P_n \) and \( P_{n-1} \), specifically on the coefficients \( d_{n-1,k} \) and the roots \( \gamma_k \). Clearly, \( \beta_{\mu,j} \to 0 \) as \( \mu \to \infty \) and, moreover, \( \rho_{\mu,j} \to \rho_j \) as \( \mu \to \infty \) \( \forall j \) where the limits \( \rho_j \) depend only on \( P_n \).

We now have \( B = \Lambda + \mathcal{R} \) for
\[
\Lambda \overset{\text{def}}{=} \begin{pmatrix}
\gamma_1 t + \beta_{\mu,1} + \frac{\rho_{\mu,1}}{t} & 0 & \cdots & 0 \\
0 & \gamma_2 t + \beta_{\mu,2} + \frac{\rho_{\mu,2}}{t} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \gamma_n t + \beta_{\mu,n} + \frac{\rho_{\mu,n}}{t}
\end{pmatrix}
\]

where the matrix \( \mathcal{R} \) satisfies \( \mathcal{R} = O(t^{-2}) \). We now define
\[
\Phi_j(t, \mu) \overset{\text{def}}{=} \gamma_j \frac{t^2}{2} + \beta_{\mu,j} t + \rho_{\mu,j} \ln|t|.
\]

Since the coefficients \( \beta_{\mu,j} \) and \( \rho_{\mu,j} \) depend only on \( \mu \) and have definite limits as \( \mu \to +\infty \), we may rearrange the rows of \( B \), if necessary, to suppose that \( \exists \mu_0, t_1 \geq 0 \) so that \( \Re \Phi_j(t, \mu) - \Re \Phi_{j+1}(t, \mu) \geq 0 \) for \( t > t_1 \) and \( \mu > \mu_0 \).
For \( v \) as above, we define \( w = e^{-\Phi_1}v \): So, \( w \) satisfies

\[
\begin{align*}
w' &= -\Phi'_1 e^{-\Phi_1}v + e^{-\Phi_1}v' \\
&= -\Phi'_1 e^{-\Phi_1}v + e^{-\Phi_1} (w + R) \\
&= (\Phi'_1 I + \Lambda) w + R w
\end{align*}
\]

where \( \Lambda \) is a diagonal matrix with elements

\[
[\Lambda]_{j,j} = - (\gamma_1 - \gamma_j) t - (\beta_{\mu,1} - \beta_{\mu,j}) - (\rho_{\mu,1} - \rho_{\mu,j}) \frac{1}{t}.
\]

We now determine estimates for \( w \) for sufficiently large positive \( t \), estimates for negative \( t \) will be similar. Choose \( t > t_1 \) so large that the estimates of \( A \) hold and that

\[
\text{Re}(\Phi_1(t, \mu) - \Phi_j(t, \mu)) \geq 0 \quad \forall t \geq 0 \text{ and } \forall \mu \geq \mu_0 \text{ for } j > 1.
\]

We choose \( w(t; y) \) to be the unique solution such that

\[
w(t; y) = (0, \ldots, 0, 1)^t \text{ for } y \geq x_1.
\]

It follows as in [2] that

\[
\frac{d}{dt} |w(t, y)|^2 \geq -2|R(t)||w(t, y)|^2
\]

for \( t \geq t_1 \) so that

\[
|w(t; y)| \lesssim \exp \left( \int_t^y 2|R(s)| \, ds \right) \cdot |w(y; y)| \lesssim 1.
\]

uniformly for \( t, y \geq x_1 \) and for \( \mu \geq \mu_0 \). Further, we then bootstrap as we apply (2.13) to obtain more accurate estimates for \( w_j(t, y) \). For \( j < n \)

\[
w'_j(t; y) = (\Phi'_j(t, u) - \Phi'_1(t, u)) w_j(t; y) + O(|R(t)||w(t; y)|)
\]

so that

\[
|w_j(t)| \lesssim \int_t^y |\exp(\Phi_1(t, \mu) - \Phi_j(t, \mu) - (\Phi_1(s, \mu) - \Phi_j(s, \mu)))| |R(s)| \, ds
\]

Since the assignment \( t \to \Re(\Phi_1(t, \mu) - \Phi_j(t, \mu)) \) is an increasing function for \( \mu > 0 \), we have

\[
|w_j(t, \mu)| \lesssim \int_t^y |R(s)| \, ds \lesssim \frac{1}{t}
\]

uniformly in \( t \) and \( \mu \). Also, from (2.13) and (2.14) we have

\[
|w_n(t; y) - 1| \lesssim |R(t)||w(t; y)|
\]

so that

\[
|w_n(t, y) - 1| \lesssim \int_t^y |R(s)| \, ds \lesssim \frac{1}{t}
\]
We now address the convergence of the function \( w(\cdot, y) \), as \( y \to \infty \). For \( x_1 \leq t \leq y_1 \leq y_2 \) the function \( w(t; y_1) - w(t; y_2) \) is a solution of Eq. (2.10); and, hence, (2.13) and (2.14) hold for \( w(t; y) \) replaced by \( w(t; y_1) - w(t; y_2) \) so that

\[
|w(t; y_1) - w(t; y_2)| \lesssim \int_{t}^{y_1} |\mathcal{R}(s)| \cdot |w(y_1; y_1) - w(y_1; y_2)|
\]

\[
\lesssim |w_n(y_1; y_2) - 1| + \sum_{j=1}^{n-1} |w_j(y_1, y_2)| \lesssim \frac{1}{y_1}
\]

uniformly in \( t, y_1 \) and \( y_2 \). So, \( w(t; y) \) converges uniformly on compact subsets of \([t_1, \infty)\) as \( y \to \infty \) to a function \( w(t) \) which satisfies

\[
|w_n(t) - 1| + \sum_{j=1}^{n-1} |w_j(t)| \lesssim \frac{1}{t}
\]

and satisfies \( \lim_{t \to \infty} w(t) = (0, \ldots, 0, 1) \). We have shown that for \( v = e^{\Phi_1}w \),

\[
|v(t)| \lesssim |e^{\Phi_1}|; \quad |v_n(t) - e^{\Phi_1}| \lesssim |e^{\Phi_1}|t^{-1}; \quad |v_j(t)| \lesssim |e^{\Phi_1}|t^{-1} \quad \text{for } j < n : \quad (2.15)
\]

These estimates hold for all \( t \geq t_0 \) and \( \mu \geq \mu_0 \).

From \( u = Sv \) we obtain a solution to \( \mathcal{L}_\mu f = 0 \) from the component \( \psi_1(t, \mu) \) \( \equiv \) \( u_1 \) with \( u_j = \frac{d^{j-1}}{dt^{j-1}} \psi_1 \) for \( 1 \leq j \leq n \). Moreover,

\[
u = Sv = S_0(I + A + \Delta)u = e^{\Phi_1} \cdot \left[ \begin{array}{c} 0 \\ \vdots \\ 0 \\ 1 \end{array} \right] + O \left( \frac{1}{t} \right).
\]

We therefore have \( \forall j \leq n - 1 \) that

\[
\frac{d^j}{dt^j} \psi_1(x, \mu) = (\gamma_1 t)^j e^{\Phi_1(t, \mu)} (1 + o(1))
\]

as \( t \to \infty \) where constants implicit in the estimates hold uniformly for \( \mu \geq \mu_0 \). Moreover, it follows from the construction that \( \psi_1 \) is of class \( C^\infty(\mathbb{R}) \) as a function of \( t \) for \( t \geq t_0 \) and holomorphic as a function of \( \mu \) for \( \mu \geq \mu_0 \). Then, \( \psi_1 \) extends to a function \( C^\infty(\mathbb{R}) \) in \( t \) and holomorphic in \( \mu \) away from 0.

The above procedure may be carried out inductively and does not involve much alteration of the cited work. Hence, we defer the remainder of the proof of the following result to the Appendix:

**Proposition 2.16** There are bases \( \{ \psi_k^\pm(t, \mu) \}_{k=1}^{n} \) of \( \ker \mathcal{L}_\mu \) of functions, \( C^\infty(\mathbb{R}) \) as functions of \( t \) and holomorphic as functions of \( \mu \) on \( \Re \mu > 0 \), which for each \( 1 \leq k \leq n \) and \( 0 \leq j \)

satisfy

\[
\frac{d^j}{dt^j} \psi_k^\pm(t, \mu) \lesssim (1 + |t|)^j e^{\Phi_k^\pm(t, \mu)} \quad (2.17)
\]
for $\pm t > 0$ (resp.) for $\Phi_k^\pm$'s as in (2.9).

For ease of reference, we end this section with the following remark which follows by inspection from Eqs. (2.1) and (2.8):

**Remark 2.18** The transformation $P(i \frac{\partial}{\partial t}, t) \rightarrow P(i \frac{\partial}{\partial t}, -t)$ leads to the transformations $\gamma_j \rightarrow -\gamma_j$ and $d_{n-1,k} \rightarrow (-1)^{n-k} d_{n-1,k}$.

### 3 Further estimates: analytic extensions, Wronskians and adjoints

We start with the more novel techniques beyond those of [14–16]. In those articles, as in this one, singularities in the parametrices are a concern as the Fourier parameter $\eta$ tends to 0. However, in the previous articles the singularities were canceled by solving $P_n(i \frac{\partial}{\partial t}, t) f = g$ for functions $g$ with zeros of appropriate order so as to cancel such singularities in $f$. Here, we cannot avoid the singularity of $L_\mu$ at $\mu = 0$ by such procedures; but, we can bypass it as we pass the parameter $\mu$ to the complex plain.

We now form estimates of bases of ker$L_\mu$ but now with complex-valued $t$, $\mu$. We note that the solutions can be analytically continued [3] and, as we shall show, the estimates as in Proposition 2.16 hold if (2.12) holds, perhaps for some reordering of the characteristic roots $\gamma_j$. Such estimates turn out to hold for $t$ contained in certain complex sets of the form $\mathcal{K} + S$ where $\mathcal{K}$ is compact and $S$ is conic.

**Proposition 3.1** Let $\Omega$ be a simply connected, compact subset of $\mathbb{C}$ which is a positive distance from the origin. For any given $\alpha \in [0, \pi/2]$ there is a compact set $\mathcal{K} \subset \mathbb{C}$ where the operator ker$L_\mu$ has bases $\{\psi_j^\pm(t, \mu)\}_{j=1}^n$ for which estimates as in (2.17) hold for $\mu \in \Omega$ and $t$ on a complex set of the form

\[
\Upsilon^\pm_{\alpha} = \{z_0 + se^{-i\theta} | \pm s > 0, \ u_1 < \theta - \alpha < u_2, \ z_0 \in \mathcal{K} \} \ (\text{resp.}) \quad (3.2)
\]

We note that the choice of $\mathcal{K}$ may require that $\Re z_0 e^{-i\alpha}$ and $\Im z_0 e^{-i\alpha}$ are sufficiently large and positive; and, our choices of $u_1$, $u_2$ may depend only on $\alpha$. Furthermore, we may adjust the domains $\Upsilon^\pm_{\alpha}$ to obtain $\mathcal{K}$, $\Omega$, $u_1$ and $u_2$ common to each $\alpha$ and $\pm$ sign.

**Proof** We will first prove the case for $\alpha = 0$. Let us (re)arrange the characteristic roots so that the following hold for each pair of indices $(j, l): 1 \leq j < l \leq n$:

1. $\Re(\gamma_j - \gamma_l) \geq 0$.
2. $\Re(\gamma_j - \gamma_l) = 0 \implies \Im(\gamma_j - \gamma_l) > 0$

Then set $\Delta_{j,l} \overset{\text{def}}{=} \gamma_j - \gamma_l$, $\beta_{j,l} \overset{\text{def}}{=} \beta_j - \beta_l$, and $\rho_{j,l} \overset{\text{def}}{=} \rho_j - \rho_l$ defined as in Proposition 2.17, but with $\gamma_j$’s in the present arrangement.

We now set

\[
\Phi_{j,l}(z, \mu) \overset{\text{def}}{=} \Delta_{j,l} z^2/2 + \beta_{j,l} z/\mu + \rho_{j,l} / z
\]

and compute

\[
\Re\Phi_{j,l}(z_0 + se^{-i\theta}, \mu) = \frac{s^2}{2} (\Re\Delta_{j,l} \cos(2\theta) + \Im\Delta_{j,l} \sin(2\theta))
\]

\[
+ s \left( \cos \theta \Re \left( z_0 \Delta_{j,l} + \frac{\beta_{j,l}}{\mu} \right) \sin \theta \Im \left( z_0 \Delta_{j,l} + \frac{\beta_{j,l}}{\mu} \right) \right)
\]

\[
+ \Re(z_0^2 \Delta_{j,l} + z_0 \beta_{j,l} / \mu) + \Re(\rho_{j,l} / (z_0 + se^{-i\theta}))
\]
We estimate \( w(z) \) defined as in Proposition 2.17 but extended to complex domains: there are positive \( \theta_1, \theta_2, s_0 \) so that
\[
-\Phi_{j,t}(z_0 + se^{-i\theta}) + \Phi_{j,t}(z_0 + s_0e^{-i\theta}) < -c_2s + c_3
\]
for some positive constants \( C, c_1, c_2 \) uniformly for \(-\theta_1 < \theta < \theta_2\) and \( s \geq s_0 \). Likewise to (2.10) we find that the function \( w \) satisfies
\[
|w(z_0 + se^{-i\theta})| \leq C_\theta |w(z_0 + s_0e^{-i\theta})|
\]
for some constant \( C_\theta \) (depending on \( \theta \)) for each such \( \theta \). It now follows from an application of the Phragmen–Lindelöf Theorem (cf. Section 5.5 [3]) that, for a possibly smaller region \(-\theta_1 \leq \nu_1 < \theta < \nu_2 \leq \theta_2\), \( w(z) \) is bounded. The various induction arguments as in Proposition 2.16 then follow to complete the proof in this case.

For \( \alpha \neq 0 \) we set \( \xi = te^{-i\alpha} \) and apply the above arguments to
\[
e^{ni\alpha} P(ie^{-i\alpha} \partial \xi, \xi e^{i\alpha}) = \sum_{l=0}^{n} \left( \frac{e^{i\alpha}}{\mu} \right)^{n-l} P_l(i \partial \xi, \xi e^{2i\alpha}).
\]
We replace the characteristic roots \( \gamma_j \) by \( \tilde{\gamma}_j \equiv \gamma_j e^{2i\alpha} \forall j \) and rearrange them so that \( \text{Re}(\tilde{\gamma}_j - \tilde{\gamma}_l) \geq 0 \) again for indices \( 1 \leq j < l \leq n \). \( \square \)

Using the variation of constants formula to form solutions to \( \mathcal{L}_\mu f = g \) [3], we need to analyze certain Wronskians and related determinants: Given a basis \( \tilde{\psi} \) of \( \ker \mathcal{L}_\mu \) we set \( W(\tilde{\phi})(t, \mu) = \det A \) with \( A \) the \( n \times n \) matrix \( A \) given by \( [A]_{k,j} = \frac{d^{k-1}}{dt^{k-1}} \phi_j \) and \( W_j(t, \mu) \) defined likewise but with the \( j \)th column of \( A \) replaced by \( (0, \ldots, 0, 1)^T \). We will apply superscript \( \pm \) to the \( W_j \)’s and to \( W \) to indicate their corresponding basis pairs \( \tilde{\psi}^\pm \); or, we may simply drop the superscript when the basis is clearly implied.

**Proposition 3.3** Suppose \( \tilde{\psi}^\pm \) is a basis of \( \mathcal{L}_\mu \) as in Proposition 2.16 defined for \( (t, \mu) \) in \( \mathbb{R}^\pm \times (\mu_0, \infty) \). Then, for some real constant \( a \), the functions \( h_j^\pm \equiv W_j^\pm / W^\pm \) satisfy
\[
\frac{d^k}{dt^k} h_j^\pm(t, \mu) \leq e^{-Re(\gamma_j^2 + \beta_j^1)} (1 + |t|)^{a+k} \quad (\text{resp.})
\]
Moreover, such estimates likewise hold for bases as in Proposition 3.1 on their associated domains (3.2).

**Proof** We will first prove the estimates for \( h_j = h_j^+ \), temporarily dropping the superscript. In the case with domain \( \mathbb{R}^+ \times (\mu_0, \infty) \) we note that for any sequence \( \mu_1 : l = 1, 2, \ldots, \) tending to \( +\infty \) there is a subsequence (say \( \mu_1 \)) so that each function \( \frac{d^k}{dt^k} \tilde{\psi}^j(\cdot, \mu_1) \) converges uniformly on compact sets of \( \mathbb{R} \) (ucs) to functions \( \frac{d^k}{dt^k} \xi_j(t) \) respectively for each \( 0 \leq k < n \) and where \( \tilde{\xi} \) is a bases for \( \ker P_{\tilde{\gamma}}(i \partial_t, t) \). Here \( \tilde{\xi} \) satisfies \( W(\tilde{\xi}) = Ce^{\gamma t^2/2} \) for some constant \( C \) (cf. [2,14]) where \( \gamma \equiv \sum_{j=1}^{n} \gamma_j \). Therefore, using Abel’s formula along with the above estimate, \( W(\tilde{\psi})(t, \mu) \asymp e^{\gamma t^2/2 + (a_0 - \beta_j^1)t/\mu} \) for sufficiently large \( \mu_0 \) where \( a_n = [E_1]_{n,n} \).

\( W_j(\tilde{\psi}) \) is a finite linear combination of \( (n-1) \)-fold products of the form \( \prod_{l \neq j} \beta_l^1 \tilde{\psi}_l \) for distinct \( l : 1 \leq l \leq n \) and distinct \( \alpha_l : 0 \leq \alpha_l < n \). Since \( \sum \beta_j = \text{tr} D_1 = \text{tr} E_1 = a_n \) we find
\[
W_j(\tilde{\psi})(t, \mu) \leq e^{(\gamma - \gamma_j) t^2/2 + (a_n - \beta_j^1)t/\mu} (1 + |t|)^a
\]
\( \square \)
for some constant $\alpha > 0$. From [14] we recall that the functions $h_j$ form a basis for $L^*_\mu$ and, arguing by matching asymptotics, the result of the proposition holds $\forall k$ in the present case.

The proof for the $h^-_j$'s follows in exactly the same way as above, using the corresponding estimates on $\mathbb{R}^n \times (\mu_0, \infty)$ for some large $\mu_0 > 0$. Finally, the proof for bases defined as in Proposition 3.1 follows similarly, in fact more readily since $\mu$ is restricted to compact subset of $\mathbb{C}$, and we are done. $\square$

4 Parametrices

We start this section with more definitions and notation. We will order the bases functions

$\Phi^\pm := \{ \phi_j^+ (t, \mu) \}_{j=1}^n, \{ \phi_j^- (t, \mu) \}_{j=1}^n \}$

admissible (or an admissible pair) if they satisfy $\phi^\pm = U^\pm \tilde{\phi}^\pm$ (resp.) with $n \times n$ matrices $U^\pm$ satisfying the following: $U = U(\mu)$ has $C^\omega$ entries on $(\mu_0, \infty)$; $[U]_{j,k} \lesssim \mu^a \forall j, k$ for some fixed $a > 0$; $[U]_{j,k} = 0$ for $j < k$ (upper-triangular); and, $[U]_{j,j} \gtrsim \mu^{-b} \forall j$ for some fixed $b > 0$. In this case, it is easy to show

$$\partial_t^k \phi_j^\pm (t) \ll pol \mu e^{\Phi_j^\pm} (1 + |t|^k)$$

$t \geq 0$ (resp.) for all sufficiently large $\mu > 0$. We further denote by $J_\alpha$ the least index whereby $j \geq J_\alpha$ implies that either $\text{Re} \gamma_j < 0$ or $\text{Re} \beta \approx 0$ and $\text{Re} \beta^\pm \gtrsim 0$ for $(\gamma_j, \beta_j^\pm)$ (resp.). We simply distinguish the basis functions in decreasing order according to their exponential growth for large $t$ in their respective domains. We note that for a given admissible pair the associated functions

$\mathcal{H}_j^\pm \defeq W_j (\tilde{\phi}^\pm) / W (\tilde{\phi}^\pm)$

satisfy

$$\mathcal{H}_j = h_j / [U]_{j,j} \ll pol \{ t, \mu \} e^{-\gamma_j t^2 / 2 - \beta_j t} / \mu$$

on the corresponding domains.

To characterize the global behavior of our bases we introduce definitions regarding transition matrices associated with admissible $\phi^\pm$. Given a permutation $\sigma$ of $\{1, 2, \ldots, n\}$, denote by $I_\sigma$ the $n \times n$ matrix with elements defined by $I_\sigma_{j,k} = \delta_{j,\sigma(k)}$, with $\delta$ denoting the Kronecker delta function. We characterize transition (scattering) matrices $\tilde{\phi}^\pm = A \tilde{\phi}^-$ as follows: Given an $n \times n$ matrix, $A$, the expression $A \leftrightarrow I_\sigma$ will mean that $A = U I_\sigma V$ for some invertible, upper-triangular $n \times n$ matrices $U$ and $V$. And, with slight abuse of notation $A \leftrightarrow B$ will mean that $A \leftrightarrow I_\sigma$ and $B \leftrightarrow I_\sigma$ both hold. We note that “$\leftrightarrow$” is an equivalence relation on $GL(n, \mathbb{C})$. Given $J^\pm$ as above, we will say that the permutation $\sigma$ is resolving if $\sigma(j) > J^- \forall j \lesssim J^+$. 

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We will say the operator $\mathcal{L}_\mu$ is regular if on a real interval $(\mu_0, \infty)$ there is some admissible pair of bases $\varphi^\pm$ and smooth functions $a^\pm_{j,l}(\mu) \lesssim \mu^r$ on $(\mu_0, \infty)$ so that $\forall j \leq J^\pm$

$$\varphi^\pm_j(t, \mu) = \sum_{l > J^\pm} a^\pm_{j,l}(\mu) \varphi^\pm_l(t, \mu) \ (resp.)$$

for some fixed $r > 0$.

We now proceed with steps of constructing a parametrix under the hypothesis of Lemma 1.5 where we need only to consider the parameter $\mu$ in a (perhaps large) compact complex neighborhood of 0. More precisely, we will suppose $\mu \in \{re^{i\theta}/2 \mid 0 \leq \theta \leq \pi\}$ for some fixed $r > 0$. Let us restrict real $x, \xi$ so that $x$ is bounded, say $|x| < M$, and $\xi > 0$; and, fix $0 \leq \alpha \leq \pi$ and $\lambda > 0$ as we set

$$t(x, \xi, \mu) = x\mu + (\rho e^{i\alpha/2} + \xi)/\mu; \ z_0 = x\mu + \lambda e^{i\alpha/2}/\mu. \hspace{1cm}(4.1)$$

Given $\alpha$ there is a positive $\delta$ and a sufficiently large $R$ so that $z_0$ and $t$ take values in sets $\mathcal{U}$ and $\mathcal{U}_{a/2}$ as in (3.2), respectively, with (say) $\delta = \nu_1 = \nu_2$ and, hence, the estimates of Proposition 3.1 hold for some basis of $\ker\mathcal{L}_\mu$ for such values of $t$, $\mu$ and $z_0$.

We set out to choose a finite collection of such bases as follows: We apply the Heine–Borel theorem to choose finitely many such intervals $\mathcal{U}_l = \{\theta : |\theta - \alpha| < \delta_l\} : l = 1, \ldots, N$ (say) with small positive $\delta_l$’s to form a refined open cover of $[0, \pi/2]$. To each $l$ we may find corresponding constants $\lambda_l$’s so large that the associated $z_0, t$ of (4.1) lie in the respective domains of those bases in Proposition 3.1.

For each given $\mathcal{U}_l^\pm$ let us reorder the corresponding pairs $(\gamma_{l,j}, \beta_{l,j}^\pm)$ associated with bases $\tilde{\psi}_l^\pm$ in the fashion as the $\tilde{\psi}_l^\pm$ in Proposition 3.1; and, let us likewise define the indices $J_l^\pm$. Then for bounded $g \in C^\infty(\mathbb{R})$ set $z_l^{\pm \text{def}} = \lambda_l e^{-i\alpha}, \ x_l^{\pm \text{def}} = \xi + z_l$ and regard $t = t(x, \xi, \mu)$ as in (4.1). Then we define for $[\tilde{\psi}_l^\pm]_{j,l}^\pm \text{def} = \psi_{j,l}^\pm$

$$F_l^\pm(t, \mu) \text{def} = \sum_{j=1}^n \psi_{j,l}^\pm(t, \mu) \int_{C_{l,j}^\pm(t, \mu)} \mathcal{H}_{l,j}(\xi) g(\xi/\mu - \bar{\xi}_l/\mu^2) d\xi \hspace{1cm}(4.2)$$

with respective contours $C_{l,j}^\pm$ given by $C_{l,j}^\pm(t, \mu) = \{s\mu + \bar{\xi}_l/\mu \mid 0 \leq s \leq \xi, \pm \xi > 0\}$ if $j > J_l^\pm$ and $C_{l,j}^\pm(t, \mu) = \{s\mu + \bar{\xi}_l/\mu \mid |x| \leq s < \infty, \pm \xi > 0\}$ if $j \leq J_l^\pm$. Note that the contours are chosen so that $\xi/\mu - \bar{\xi}_l/\mu^2$ give only real values.

For each $l$ and choice of $\pm$ sign we have solutions to $\mathcal{L}_\mu f = g(x)$ for $(t, \mu)$ on the corresponding domains. Since $\partial_x^k t \lesssim 1 \forall j$, we find that for any given $\kappa$ there is an $a > 0$ so that, on their domains,

$$\partial_x^k F_l^\pm \lesssim (1 + |t(x, \xi, \mu)|)^a \lesssim (1 + |\xi|)^a$$

is satisfied for $\forall k \leq \kappa$ and $\forall l$. The proof follows the analysis in [14] and details are deferred to the Appendix (see Proposition 8.1 and the comments that follow). Define the $F_l^\pm$’s to be zero outside their corresponding domains and now let $\chi_l(\theta)$ (for $\theta = \arg \mu$) be a partition of unity subordinate to the $\mathcal{U}_l^\pm$’s and let $c_l$ be quantities that are constant with respect to $x, \mu$ and $\xi$. Then we set

$$G(x, \xi, \mu) = \sum_{l=1}^m c_l \chi_l(\theta) (\Theta(\xi) F_l^+(t(x, \xi, \mu)) + \Theta(-\xi) F_l^-(t(x, \xi, \mu))) \hspace{1cm}(4.3)$$
where $\Theta$ is the unit (Heaviside) step function to create a smooth function for $x \in [-M, M]$, $\xi \in \mathbb{R}$, $\mu \in \{re^{i\theta} | 0 \leq \theta \leq \pi/2\}$.

We have not shown $L_{\mu}^{-}$ to have regular parametrices for complex $\mu$, as per Definition 1.4, but we can establish partial construction of solutions to $Lu = f$ with, as yet, no additional conditions. We state

**Proposition 4.4** Given $g \in C^\infty(\mathcal{I})$ for a neighborhood of $0 \ni \mathcal{I} \subset \mathbb{R}$, there is a function $f(x, y, w, \xi, \theta) \in C^\infty(\mathcal{I} \times \mathbb{R}^3 \times [0, \pi/2])$ satisfying

$$P(X, Y) f = g(x)e^{-i(y\xi+w\mu^2)} \sum_{l=1}^{m} \chi_l(\theta)e^{-iyz_l}$$

with $\mu = \lambda e^{i\theta}$ for fixed $\lambda > 0$.

**Proof** We employ (4.3) setting $c_l = e^{-iz_l y}$ for each $l$ and we set $f = (i/\mu)^{n}e^{-i(y\xi+w\mu^2)} g$. We note that, on the respective domains $\mathcal{U}_l^{\pm}$, the $F_l^{\pm}$’s as in (4.2) now satisfy

$$L(e^{-i(y\xi+\mu^2w)}F) = e^{-i(y(\xi+z_l)+w\mu^2)} P(\partial_x, -i(\xi + z_l + \mu^2 x))F$$

$$= e^{-i(y(\xi+z_l)+w\mu^2)} (-i\mu)^n L_{\mu} F_l$$

$$= e^{-i(y(\xi+z_l)+w\mu^2)} g(x).$$

So, we obtain our desired result as we compute

$$L f = \sum_{l=1}^{m} \chi_l(\theta)e^{-i(y(\xi+z_l)+w\mu^2)} (\Theta(\xi)L_{\mu} F^+ + \Theta(-\xi)L_{\mu} F^-)$$

$$= \sum_{l=1}^{m} \chi_l(\theta)e^{-i(y(\xi+z_l)+w\mu^2)} (\Theta(\xi) + \Theta(-\xi))g(x)$$

$$= \sum_{l=1}^{m} \chi_l(\theta)e^{-i(y(\xi+z_l)+w\mu^2)} g(x).$$

\[\square\]

We now present the

**Proof of Lemma 1.5** We replace $g(x)$ by functions $\hat{g}(x, \xi, \eta)$ for $g \in C_0(\mathbb{R}^3)$, involving the other Fourier variable $\eta$ as we substitute $\mu = \sqrt{\eta}$ into (4.1). First we show that for $F$ as in Definition 1.4

$$\mathcal{F}(x, \xi, \eta) \overset{\text{def}}{=} F(t(x, \xi, \sqrt{\eta}), \sqrt{\eta})$$

defines a function in $C^\infty(\mathbb{R}) \times S(\mathbb{R}) \times S((\mu_0^2, \infty))$. By linearity of the operator $L_{\mu}$ we have that for any $r > 0$ and index $m$,

$$\partial_x^k \mathcal{F} \lesssim |\eta|^{-n/2}(1 + |\sqrt{\eta}|)^m(1 + |\xi| + |\eta|)^{-r}(1 + |x|\sqrt{|\eta|} + |\xi/\eta|)^{a}$$

for $|x| < M, |\eta| > \mu_0^2, \xi \in \mathbb{R} \forall k \leq m$. So $\forall \alpha > 0, \partial_x^k \mathcal{F} \lesssim (1 + |\xi| + |\eta|)^{-\alpha}$ by setting $r > \alpha + a + m/2$.

Letting $\phi_l(\theta)$ denote the characteristic function of the associated set $\mathcal{U}_l$, we replace $g$ in Proposition 4.4 by $\sum_{l=1}^{m} \phi_l(\theta)\hat{g}(x, \xi + z_l, \pm \mu^2)$ to redefine $F_l^{\pm}$ (resp.) and set $\lambda = \mu_0^2$. \[\circled{Springer}\]
Likewise, the various partial derivatives $\frac{\partial^k f}{\partial x} f(x, y, w, \xi, \theta)$ are each majorized by $(1 + |\xi|)^{-s}$ for any $s > 0$. We apply $L$ to $F = f_1 + f_2$ where

$$f_1 \equiv \frac{1}{2\pi} \int \int (i / \sqrt{|\eta|})^n e^{-i(y\xi + w\eta)} (F^+(x, \xi, \eta) + F^-(x, \xi, \eta)) d\eta d\xi$$

$$L f_1 = \frac{1}{2\pi} \int \int e^{-i(y\xi + w\eta)} \hat{g}(x, \xi, \eta) d\eta d\xi$$

$$f_2 \equiv \frac{1}{2\pi} \int \int f(x, y, w, \xi, \theta) \lambda^2 i e^{2i\theta} d\theta d\xi$$

$$L f_2 = \frac{1}{2\pi} \int \int \hat{g}(x, \xi + z_l, \zeta)e^{-i(y\xi + w\zeta)} \sum_{l=1}^{m} \chi_l \left( \frac{\arg \zeta}{2} \right) e^{-iyz_l} d\zeta d\xi$$

where $\zeta = \mu^2$ and $C$ is the complex contour given by boundary of the upper half of the disc centered at 0 of radius $\mu_0^2$. By the analyticity and integrability of $\hat{g}$ in the variable $\xi$ we may apply Cauchy’s integral theorem to obtain

$$\int e^{-iy(\xi + z_l)} \hat{g}(x, \xi + z_l, \zeta) d\xi = \int e^{-iy\xi} \hat{g}(x, \xi, \zeta) d\xi$$

for $(\arg \zeta)/2 \in \text{supp} \chi_l$. We now apply the Fubini–Tonelli theorem along with Cauchy’s integral theorem in the variable $\zeta$ to obtain

$$L f_2 = \frac{1}{2\pi} \int \int \sum_{l=1}^{m} \chi_l \left( \frac{\arg \zeta}{2} \right) \hat{g}(x, \xi, \zeta)e^{-i(y\xi + w\zeta)} d\zeta d\xi$$

$$= \frac{1}{2\pi} \int \int \hat{g}(x, \xi, \zeta)e^{-i(y\xi + w\zeta)} d\zeta d\xi$$

$$= \frac{1}{2\pi} \int \int e^{-i(y\xi + w\eta)} \hat{g}(x, \xi, \eta) d\zeta d\eta$$

$$L F = \frac{1}{2\pi} \int e^{-i(y\xi + w\eta)} \hat{g}(x, \xi, \eta) d\eta d\xi = (\hat{g})(x, y, w) = g(x, y, w).$$

We are now prepared to state a main result from which we may determine solvability through the representations $\Lambda^\pm_\mu$ for large $\mu$:

**Theorem 4.5** The operator $L$ as in (1.1) is locally solvable if both $\mathcal{L}^\pm_\mu$ are regular.

**Proof** Since our proof repeats content of previous works, we will give a sketch of proof and defer details to these works. We will show that an operator $\mathcal{L}_\mu$ has a regular parametrix if it is regular:

\[ \hat{g}(x, \xi, \eta) = g(x, y, w). \]
where

\[ K(t, \mu) = \sum_{l=0}^{n} \phi_{l}(t, \mu) \int_{0}^{t} \mathcal{H}_{l}^{+}(\tau, \mu) g(\tau) d\tau = \sum_{l=0}^{n} \phi_{l}^{-}(t, \mu) \int_{0}^{t} \mathcal{H}_{l}^{-}(\tau, \mu) g(\tau) d\tau \]

(cf. (12) and (13) of [16]). We set

\[ \alpha_{l}^{\pm}(\mu) \overset{\text{def}}{=} \int_{\pm \infty}^{0} \mathcal{H}_{l}^{\pm}(\tau, \mu) g(\tau) d\tau \]

(resp.) so that we can write

\[ F(t, \mu) = \sum_{l=1}^{n} \alpha_{l}(\mu) \phi_{l}(t, \mu) t \int_{+\infty}^{t} \mathcal{H}_{l}(\tau, \mu) g(\tau) d\tau + \sum_{l=1}^{n} \alpha_{l}(\mu) \phi_{l}(t, \mu) \]

\[ F(t, \mu) = \sum_{l=1}^{n} \alpha_{l}(\mu) \phi_{l}(t, \mu) t \int_{-\infty}^{t} \mathcal{H}_{l}^{-}(\tau, \mu) g(\tau) d\tau + \sum_{l=1}^{n} \alpha_{l}(\mu) \phi_{l}^{+}(t, \mu). \]

The estimates on \( K \) and its derivatives \( \frac{\partial^{k}}{\partial t^{k}} K \) follow as in Section 1 of [14], applying the Chain Rule along with integral estimates as in Proposition 8.1, to obtain \( \frac{\partial^{k}}{\partial t^{k}} F(t, \mu) \lesssim \text{pol}(t, \mu) \) on \( \mathbb{R} \times (\mu, \infty) \) for each \( k \).

5 Solvability

We see that when \( \text{Re} \gamma_{j} = 0 \) the corresponding bases function \( \psi_{j}(t, \mu) \), for finite \( \mu \), may have asymptotic growth far different than that of a corresponding limit \( \zeta_{j}(t) \). For instance, a limit function \( \zeta_{j}(\text{say}) \) may have polynomial growth where every sequence \( \psi(\cdot, \mu_{l}) : l \to \infty \) tending to it may have exponential growth or decay in the variable \( t \). We distinguish a particular subclass of operators where this discrepancy does not take place. We introduce

\[ \text{Definition 5.1} \]

We will say that the polynomial \( P \) has property \( \mathcal{G} \) if \( P_{n} \) is generic and the \( \gamma_{j}, \beta_{j} : 1 \leq j \leq n \) associated with \( P \) satisfy

\[ \text{Re} \gamma_{j} = 0 \implies \text{Re} \beta_{j} = 0. \]

We note that this property depends only on the coefficients of \( P_{n} \) and \( P_{n-1} \). We are ready to state

\[ \text{Theorem 5.2} \]

Suppose that \( L = P(X, Y) \) where \( P \) has property \( \mathcal{G} \). Then, the operator \( L \) is locally solvable if \( \ker(L_{\infty}^{\pm})^{*} \cap \mathcal{S}(\mathbb{R}) \) contains only the zero function for each choice of sign.

\[ \text{Proof} \]

We will show that an operator \( \mathcal{L}_{\mu} \) is regularizable when \( \mathcal{L}_{\infty}^{*} \) satisfies the hypothesis. Here, \( J^{+} = J^{-} \overset{\text{def}}{=} J \) and our result follows as in proof of Corollary 4.3 of [16] since the associated transition matrix \( A \) has real analytic coefficients and tends to a finite limit as \( \mu \to \infty \). We deduce that there is an admissible pair of bases \( \hat{\phi}^{\pm} \) for which \( \hat{\phi}^{\pm} = \Lambda^{\pm} I_{\sigma} \hat{\phi}^{\mp} \) (resp.) for square matrices \( \Lambda^{\pm} \) and \( I_{\sigma} \) satisfying the following for each choice of sign.
and sufficiently large $\mu > 0$: $\Lambda$ is lower triangular with ones on the main diagonal where $\Lambda(\mu) \to I$ as $\mu \to \infty$; and, $[I]_{j,k} = \delta_{j,\sigma(j)}$ where $\delta$ is the Kröniker delta function and $\sigma$ is a resolving permutation. By carrying out the matrix multiplication we find that $L_\mu$ is regularizable.

Since both $L_\mu^\pm$ are regularizable, the proof is complete by applying Theorem 4.5. \qed

From [14] the hypotheses on $L_\infty^\pm$ are equivalent to the local solvability of $P_n(X, Y)$, whereby we immediately conclude the following:

**Corollary 5.3** Suppose that $L = P(X, Y)$ is locally solvable where $P$ is a generic polynomial of order $n \geq 2$. Then, the operator $L + K$ is locally solvable for any $K$ contained in the subalgebra of $\mathfrak{h}_1^C$ generated by $X$ and $Y$ of order less than or equal to $n - 2$.

**Corollary 5.4** Let $L$ and $K$ be as in Corollary 5.3 except that the characteristic roots of $P$ each have non-zero real parts. Then the operator $L + K$ is locally solvable for any such $K$ of order less than or equal to $n - 1$.

We demonstrate that regularity is not a precise condition for local solvability of our operators as an even weaker condition on $L_\mu^\pm$ still assures solvability. We introduce

**Definition 5.5** An operator $L_\mu$ will be called quasi-regular if there is a finite, refined open cover $\mathcal{I}_l: l = 1, 2, \ldots, m$ of a semi-infinite interval $[\mu_0, \infty)$ along with admissible pairs $\vec{\phi}_l^\pm$, with components $[\vec{\phi}_l^\pm]_j \overset{\text{def}}{=} \phi_{j,l}^\pm$ (resp.) that satisfy estimates as in (4.5) but with $\mu$ restricted to $\mathcal{I}_l$.

We note that the $\mathcal{I}_l$’s can each be assigned a unique resolving permutation $\sigma_l$ associated with the transition matrix $A_l$ defined on $\mathcal{I}_l$. We note further that it is trivial to show that a regularizable operator is also quasi-regular. We will see (in Sect. 6) that for a large subclass of our operators the aforementioned condition is exact.

**Theorem 5.6** An operator $L = P(X, Y)$ is locally solvable if the operators $L_\mu^\pm$ are both quasi-regular.

**Proof** Via a change of variables, if necessary, it will suffice to show that an operator $L_\mu$ has a regular parametrix if it is quasi-regular. We can construct functions $F_l(t, \mu)$ as in (4.6) but with $F$ restricted to $\mathbb{R} \times \mathcal{I}_l$. Then for a smooth partition of unity $g_l(\mu)$ subordinate to the associated $\mathcal{I}_l: l = 1, \ldots, m$ we set $F(t, \mu) \overset{\text{def}}{=} \sum_{l=1}^m g_l(\mu) F_l(t, \mu)$. Local solvability of $L$ then follows as in the proof of Theorem 4.5 and the cited references therein. \qed

**6 Non-solvability**

We will develop criteria for local non-solvability of $L$ in terms of representations $\Lambda_\mu^\pm$—particularly through the adjoints of $L_\mu^\pm$. We first need to verify that our approach to solving (1.2) has an analogue as applied to adjoints of our partial differential operators. For constructing functions in $\ker L^*$ the method is clear via

**Proposition 6.1** Under $\Lambda_\mu^\pm$, the operator $L^*$ has representations given by $\Lambda_\mu^\pm(L^*) = (i\mu)^n (L_\mu^\pm)^*$ (resp.).
Proof We will drop ± the superscript. It is not difficult to show that the conclusion of our proposition holds for operators homogenous in \( X \) and \( Y \) (cf. Proposition 2.3 [14]) so that \( \Lambda_\mu(P_l(X, Y)^*) = (\Lambda_\mu(P_l(X, Y)))^* \forall l \). It follows that \( \Lambda_\mu(L^*) = (\Lambda_\mu(L))^* \) and our result is immediate.

Our next result follows as in Sect. 2 [14] and, hence, we provide here only a sketch of proof of the following

**Theorem 6.2** Suppose that \( \exists \psi \in \ker \mathcal{L}_\mu^* \) such that \( \psi \cdot \chi \in \mathcal{S}(\mathbb{R} \times \mathcal{I}) \setminus \{0\} \) for \( \chi \in \mathcal{C}_c^\infty(\mathcal{I}) \) where \( \mathcal{I} \) is a bounded open subinterval of \( \mathbb{R}^+ \). Then, the associated operator \( L \) is not locally solvable.

**Proof** We suppose (perhaps after a change of variables) that \( \mathcal{I} = (1, 2) \). It follows as in Proposition 2.4 [14] that there is a non-trivial function \( \Psi \in \ker L^* \cap \mathcal{S}(\mathbb{R}^3) \) with \( \mu = \sqrt{\xi_3} \) given by:

\[
\hat{\Psi}(x, \xi_2, \xi_3) = \psi(t(x, \xi_2, \mu)) F(\xi_2, \mu),
\]

where \( F \in \mathcal{C}_0(\mathbb{R}^2) \) and \( \text{supp} F(\xi_2, \mu) \subseteq \{ |\xi_2| < 1 \& \xi_3 \geq 1 \} \).

We set \( v_\tau \overset{\text{def}}{=} \phi(\xi)\Psi(t, \tau y, \tau^2 w) \) and \( F_\tau \overset{\text{def}}{=} \phi(x/\tau, y/\tau, w/\tau^2)\Psi(\xi) \) for some \( \phi \in \mathcal{C}_0(\mathbb{R}^3) \) such that \( \phi = 1 \) on a neighborhood of the origin. These functions are constructed likewise to Propositions 2.5 and 2.7 of [14], so that in the Sobolev norm \( v_\tau \) satisfies \( ||L^* v_\tau||_2 \lesssim \tau^{-a} \) for all \( \tau > 1 \) for any fixed \( v \) and positive \( a \).

To complete our result, it suffices to show that for any integer \( N > 0 \) there are constants \( C_1, C_2 > 0 \) so that \( ||v_\tau||_{(-N)} > C_1 \tau^{-N-2} \) and \( ||v_\tau||_{(-N-3)} < C_2 \tau^{-N-4} \), thereby showing that for sufficiently large \( \tau > 1 \) Hörmander’s criteria from Lemma 26.4.5 [7] is violated (see also (2.19) [14]). The desired estimates follow in the same manner as (2.13) through (2.18) [14], considering Remark 8.2 (Appendix).

We apply the above results to develop a priori criteria for non-solvability. Given (distinct) characteristic roots \( \{ \gamma_j \}_{j=1}^\mu \) and associated \( \beta_j \) as in (2.8), let us denote \( B^\pm \overset{\text{def}}{=} \{ j | \pm \text{Re} \gamma_j > 0 \} \), and \( \pm^\pm \overset{\text{def}}{=} \{ j | \text{Re} \gamma_j = 0 \& \pm \text{Re} \beta_j > 0 \} \) (resp.). We state our result in terms of the cardinality (card) of these sets:

**Theorem 6.3** \( L = P(X, Y) \) is not locally solvable if either of the following holds:

1. \( \text{card}(B^+ \cup \mathcal{E}^+) + \text{card}(B^+ \cup \mathcal{E}^-) > n; \) or
2. \( \text{card}(B^- \cup \mathcal{E}^+) + \text{card}(B^- \cup \mathcal{E}^-) > n. \)

**Proof** By choosing the appropriate representation, we may suppose that case (1) holds: By Remark 2.18 along with (2.1) we see that \( \mathcal{E}^+ \cup \mathcal{E}^- \) is the same under each such representation. We note further that \( \phi(t) \in \ker P(i \partial_t, \mu t)^* \iff \phi(-t) \in \ker P(-i \partial_t, -t)^* \) where the associated parameters \( \gamma_j, \beta_j \) transform as \( (\gamma_j, \beta_j) \rightarrow (\gamma_j, -\beta_j) \) for each \( j \).

By dimensional arguments it follows that for any admissible pair of bases \( \tilde{\gamma}^\pm \) of \( \ker \mathcal{L}_\mu \) the associated transition matrix \( \Lambda \) satisfies \( \Lambda \leftrightarrow I_\sigma \) for a non-resolving \( \sigma \) for \( \mu \) on a non-empty open interval. Then there is a pair of bases \( \tilde{h}_j^\pm \) of \( \ker \mathcal{L}_\mu \) given by \( \tilde{h}_j^- = (A^{-1})^\dagger \tilde{h}_j^+ \) (admissible after reordering \( h_j \rightarrow h_{n+1-j} \forall j \)) where \( (A^{-1})^\dagger \leftrightarrow I_\sigma \). It follows as in Corollary 4.3 [16] that \( \exists l \in B^+ \cup \mathcal{E}^+ \) so that

\[
b_j^- (\cdot, \mu) = \sum_{j \leq J^+} a_j(\mu) h_j^+ (\cdot, \mu) \overset{\text{def}}{=} \tilde{\Psi}(\cdot, \mu) \quad (6.4)
\]
where the $a_j$ are real analytic functions, not all trivial. By Proposition 2.16 and analyticity arguments on the $a_j$’s, there is a bounded, non-empty interval $I \subset R^+$ so that \( \forall \epsilon > 0 \) there exists a bounded, non-empty interval $I \subset R^+$ so that $\psi(t, \mu) < e^{-\epsilon |t|}$ holds on $\mathbb{R} \times I$ for some $\epsilon > 0$. The result now follows by applying Proposition 6.2.

Proof of Theorem 1.8 Let $\tilde{\mu} > 0$ be the accumulation point. By analyticity of transition matrices $A(\mu)$ we find that $A \leftrightarrow I_\sigma$ for all $\mu \in I$ containing $\tilde{\mu}$ for some fixed, non-resolving $\sigma$. It then follows that a function $\psi(\cdot, \mu)$ can be constructed as done in (6.4).

Proposition 6.5 If $L_\mu$ is not quasi-regular, then there is a sequence $\mu_j : j = 1, 2, \ldots$ with $\mu_j \to +\infty$ and functions $Y_j^\pm(t, \mu) \approx_{pol \mu} h_j(t, \mu)$ for $h_j$’s as in Proposition 3.3 for which the following hold: $\exists \kappa \leq J^-$ and coefficients $\alpha_j(l)$ so that

$$F(t) \overset{\text{def}}{=} Y_k^-(t, \mu_i) = \sum_{j=1}^n \alpha_j(l) Y_j^+(t, \mu_i)$$

where $\forall j \alpha_j(l) \leq C \mu_i^a$ for some fixed $C, a > 0$ and where for any $B > 0$ we find $\alpha_j(l) < C \mu_i^{-B}$ is satisfied for some constant $C > 0$ for each $j > J^+$. Moreover, the $F_j$’s may be chosen so as to converge (ucs) to a non-trivial function in $ker L_\mu^\infty$.

The proof follows as that of Proposition 3.2 [16] and further elaboration is deferred to the Appendix (see Remark 8.3).

Theorem 6.6 Suppose that for an operator $L = P(X, Y)$ one of operators $L^\pm$ (say $L$) is not quasi-regular. Then $L$ is not locally solvable if either of the following additional conditions hold:

1. $P$ has property $G$;
2. Functions $F_j \in ker L_\mu^*$ as in Proposition 6.5 can be chosen to satisfy $F_j(t) \approx_{pol \ t} e^{bt}$ for some $b > 0$ with implied bounds uniform in $l$.

Proof We may, perhaps after a change of variables, suppose that it is the operator $L_\mu = L_\mu^+$ that is not piecewise regularizable. We will show in this case that necessary criteria, in the form of an inequality involving $L^*$, will be violated. From [7] we find that if $L$ is locally solvable near 0, then the following holds: $\forall \epsilon > 0, \exists N > 0$ such that

$$\left| \int \phi \nabla \right| \leq N ||\phi||_{L^N} ||L^*\Psi||_{L^N} \quad (6.7)$$

for every $\phi, \Psi \in C^\infty(\mathbb{R}^3)$ supported in $|\langle x, y, w \rangle| < \epsilon$.

We begin our construction of functions $\phi_l, \Psi_l : l = 1, 2, \ldots$ which for any given $N$ violates this inequality for sufficiently large $l$. Under our hypothesis we conclude from Proposition 6.5 that there is an increasing sequence $1 < \mu_1 \to +\infty$ and non-trivial functions $F_j \in ker L_\mu^*$ $\forall l$ satisfying the following: There are real constants $M_1 < M_2$ and a positive $\delta > 0$ so that $F_j(t) > 1 \forall l$ on the interval $(M_1 - \delta, M_2 + \delta)$; and, for any given $k$ there is an exponent $a \geq 0$ so that for some constants $r, s > 0$

$$\frac{d^j}{dt^j} F_j(t) \leq (1 + |t|)^a (C(l)e^{Q(t)} + e^{-rQ(t)}) \quad (6.8)$$

where $Q(t) \overset{\text{def}}{=} t^2$ and where, for any $\alpha > 0$, $C(l) \leq \mu_i^{-\alpha}$ holds uniformly $\forall l$. Letting $v_l \overset{\text{def}}{=} \sqrt{\ln \mu_i}$, we choose a non-negative $h \in C^\infty(\mathbb{R}^3)$ supported in $|x| \leq \lambda_2, |y|, |w| \leq \epsilon/2$
so that \( h(\bar{x}) = 1 \) on \(|x| \leq \lambda_1, |y| & |w| < \epsilon/3\); here, \( 1 \leq \lambda_1 < \lambda_2 \) are constants yet to be determined. We set \( h_l(x) \) \( \equiv h(x, \mu_l, \nu_l, y, w) \) (noting that \( \mu_l/\nu_l \) tends to \( \infty \) as \( l \) tends to \( \infty \)). Finally, we choose a non-negative \( \chi \in C_0^\infty(\mathbb{R}) \) supported in \((M_1, M_2)\) so that \( \int_{\mathbb{R}} \chi(s) ds = 1 \).

We now introduce functions

\[
\Psi_l(\bar{x}) \equiv h_l(\bar{x}) \int_{\mathbb{R}} e^{-i(\mu_l^2 w + \mu_l y)} \mu_l^{-1} \chi(\xi/\mu_l) F_l(x, \mu_l + \xi/\mu_l) d\xi \equiv h_l(\bar{x}) f_l(\bar{x})
\]

We see that \( f_l(\bar{x}) \) \( \equiv \) \( \int_{M_1} e^{-i(\mu_l^2 w + \mu_l y)} \chi(u) F_l(x, \mu_l + u) du \) \( \in C^\infty(\mathbb{R}^3) \). Choose a non-negative \( \phi(\bar{x}) \) \( \in C_0^\infty(\mathbb{R}) \) supported in \(|\bar{x}| < \epsilon/2\) so that \( \phi(\bar{x}) = 1 \) on \(|\bar{x}| < \epsilon/3\) and set \( \phi_l \equiv \phi(x, y, \mu_l, w, \mu_l^2) \).

We now compute an upper bound of the RHS of (6.7), first by estimating the norms of \( L^* \Psi_l \). We compute \( L^* \Psi_l = P_l f_l \) where \( P_l \equiv [L^*, h_l] \) is a partial differential operator of order \( n \) supported in

\[
U_l \equiv \{ |x| \nu_l \lambda_1 \leq |x| \mu_l \leq \lambda_2 \nu_l \& |y|, |w| \leq \epsilon/2\}.
\]

By the Leibniz Rule we may write \( P_l = \sum_{|\alpha| = 1}^n a_{\alpha,l}(\bar{x}) \partial_x^\alpha \) where the coefficients satisfy \( a_{\alpha,l} \leq \mu_l^{-|\alpha|} \) uniformly on \( U_l \). The operator \( \partial_x^\alpha P_l \), of order \( n + |\beta| \), behaves likewise, but \( \forall \alpha \) the coefficient multiplying \( \partial_x^\alpha \) is dominated by \( \mu_l^{n+|\beta|-|\alpha|} \). Further, for \( \bar{x} \in U_l \), \( Q(\lambda_1 \nu_l + M_1) \leq Q(x \mu_l + \xi/\mu_l) \leq Q(\lambda_2 \nu_l + M_2) \). For any multi-index \( |\alpha| \leq k \) we have the following bounds (uniform also in \( l \))

\[
\partial_x^\alpha P_l \circ f_l(\bar{x}) \leq \mu_l^{2k+n}(1 + \nu_l)^{a+n+k}(R_l + \mathcal{E}_l)
\]

(6.9)

where we have set \( \mathcal{R}_l \equiv c(l)e^{\lambda_2^2 y^2} = c(l)\mu_l^{\lambda_2^2} \) and \( \mathcal{E}_l \equiv e^{-r\lambda_1 y^2} = \mu_l^{-\lambda_2^2 r} \). So by setting \( k = N, \lambda_2^2 > (3N + 2n + a)/r + B \) and fixing \( \lambda_2 > \lambda_1 \), we obtain \( \|L^* \Psi\|_{\mathcal{C}^N} \leq (\mu_l^{-B} + c(l)\mu_l^{\lambda_2^2}) \). Then, for any integers \( N, B \geq 0 \), \( 3c_{N,B} > 0 \) so that \( \|L^* \Psi\|_{\mathcal{C}^N} \leq c_{N,B} \mu_l^{-B} \).

We now estimate the norms of \( \phi_l \). For any \( k \) the derivatives \( \partial_x^\alpha \phi_l \) \( |\alpha| \leq k \) are compactly supported in \(|\bar{x}| < \epsilon\). Then, by the Chain Rule, \( \partial_x^\alpha \phi_l(\bar{x}) \leq \mu_l^{2k} \) for such \( \alpha \). Furthermore, we find that the RHS of (6.7) is majorized by \( \mu_l^{2N-B} \) uniformly in \( l \).

We now find a lower bound on LHS of (6.7). Let us restrict \( 0 \leq \epsilon \leq \pi/4 \) (say). Then, for \( \bar{x} \in \text{supp} \phi_l \), \( \Re e^{i(\mu_l^2 y + \mu_l w)} \geq \sqrt{2}/2 \). And, for sufficiently large \( l \) (st. \( \nu_l > \lambda_1/\delta \)) and \( \bar{x} \in \text{supp} h_l \), we have \( \chi''(x, \mu_l + \xi/\mu_l) \geq 1 \). Hence,

\[
\Re(\phi_l(\bar{x}) \tilde{\Psi}_l(\bar{x})) \geq \frac{\sqrt{2}}{2} \phi_l(\bar{x}) h_l(\bar{x}) \int_{\mathbb{R}} \chi(u) du;
\]

and, for sufficiently large \( l \) (as above and for \( \mu_l/\nu_l > 6\lambda_1/\epsilon \))

\[
\int_{\mathbb{R}} h_l(\bar{x}) \phi_l(\bar{x}) d\bar{x} \geq \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} 1 dy dw dx = \frac{\lambda_1 \sqrt{\ln \mu_l}}{\mu_l^4}.
\]

Therefore LHS of (6.7), for sufficiently large \( l \), is bounded below by \( \lambda_1 \mu_l^{-4}/\sqrt{2} \). The condition on \( L \) can therefore be shown not to hold as we choose \( \lambda_1 > 0 \) so large that we may
fix $B > 4 + 2N$. The condition (6.7) is then violated for all sufficiently small $\epsilon > 0$ for any index $N$ for all sufficiently large $l$; and, hence, the result of case 1 is shown.

The proof of case 2 is similar to that of case 1 by replacing $Q$ in (6.8) with $Q_l(t) \overset{\text{def}}{=} |l|/\mu_l$ and by redefining $v_l \overset{\text{def}}{=} \mu_l \ln \mu_l$, $h_l(\lambda) \overset{\text{def}}{=} h(x v_l/\mu_l, y, w)$. Then (6.9) holds for $\Re_l \overset{\text{def}}{=} \mathcal{E}(l)e^{b l^2 v_l/\mu_l} = \mathcal{E}(l)^{\mu_l} b_l^2$ and $\Theta_l \overset{\text{def}}{=} e^{-r l \sqrt{\mu_l}} = \mu_l^{-r l^1}$ for $l = 1, 2, \ldots$ We may choose $\lambda_1 < \lambda_2$ so large that RHS of (6.7) is likewise majorized by $\mu_l^{-2}$ for any desired $B > 0$ as $l \to \infty$. In this case we compute $\sqrt{2} \Re \int_{t_1}^{t_2} \phi \Psi d\lambda \overset{\text{def}}{=} \lambda_1 \mu_l^{-3}/\ln \mu_l$ for sufficiently large $l$, so that LHS has a lower bound the same as that of case 1. The criteria (6.7) are thus violated and the proof is complete. $\blacksquare$

We find that quasi-regularity of both $\mathcal{L}_\mu^+ \overset{\text{def}}{=} \mathcal{L}_\mu$ is an exact condition for various subclasses of our operators.

**Corollary 6.10** If $L = P(X, Y)$ where $P$ has property $\mathcal{G}$, then $L$ is locally solvable if and only if both $\mathcal{L}_\mu^\pm$ are quasi-regular.

**Corollary 6.11** Suppose that the characteristic roots of (generic) $P_n$ associated with $L = P(X, Y)$ satisfy either $\Re \gamma_j \not= 0 \forall j$ or $\Re \gamma_j = 0 \forall j$. Then $L$ is locally solvable if and only if both $\mathcal{L}_\mu^\pm$ are quasi-regular.

At this point one may suspect that local solvability of $P(X, Y)$ is not always assured whenever $P_n(X, Y)$ is solvable. Indeed, we give examples of non-solvability in such some such cases in the next section.

### 7 Some second-order examples

We now give some examples of solvable and non-solvable operators in the second-order case. We start with criteria for local non-solvability for the adjoints of the operators

$$L = -X^2 - i a_1 Y X + a_2 Y^2 - i a_1 X, Y] - i b_1 X + b_2 Y + c \quad (7.1)$$

where $a_k, b_k, \alpha, c$ complex numbers with $a_1^2 \not= 4 a_2$. We compute

$$\mathcal{L}_\mu = \frac{d^2}{dt^2} + a_1 t \frac{d}{dt} + a_2 t^2 + \alpha + \frac{b_1}{\mu} \frac{d}{dt} + \frac{b_2}{\mu} t + c \frac{1}{\mu^2}.$$ 

With $\gamma_j$ as the characteristic roots, $\Re \gamma_1 \geq \Re \gamma_2$, the corresponding $\beta_j$’s are given by (2.8) to be

$$\beta_1 = \frac{1}{\gamma_2 - \gamma_1} (b_1 + \gamma_1 b_2); \quad \beta_2 = \frac{-1}{\gamma_2 - \gamma_1} (b_1 + \gamma_2 b_2).$$

Thus, from Theorem 6.3, we immediately conclude the following

**Proposition 7.2** The adjoint operator $L^*$ for $L$ as in (7.1) is not locally solvable if one the cases hold:

1. $\Re \gamma_1$ and $\Re \gamma_2$ are non-zero and have the same sign;
2. $\Re \gamma_1 = 0 > \Re \gamma_2$ and

$$\Im(\gamma_1 - \gamma_2) \Im(b_1 + \gamma_1 b_2) > \Re \gamma_2 \Re(b_1 + \gamma_1 b_2);$$

$\square$ Springer
As an example, let us set \( a_2 = \alpha = 0 \) and let \( a_1 = -2\lambda \) for some real \( \lambda \neq 0 \). The characteristic roots are 1 and 2. Here, \( L_\infty \) is self-adjoint and bases \( \psi^\pm \) of \( \ker L_\infty \) are given:

\[
\psi_1^\pm = 1, \quad \psi_2^\pm = \int_0^\infty e^{\lambda s^2} \, ds \quad \text{(resp.) when } \lambda < 0; \quad \text{and, } \psi_1^\pm = \int_0^\infty e^{\lambda s^2} \, ds, \quad \psi_2^\pm = 1 \quad \text{for } \lambda > 0.
\]

So, \( \ker L_\infty \backslash S(\mathbb{R}) \} \) is empty for any such \( \lambda \). Thus, the associated operator \( L^* \) is locally solvable when \( b_1 = b_2 = c = 0 \). However, the operator \( L^* \) is not locally solvable for any \( b_2 \) and \( c \) when \( \text{Re} b_1 > 0 \), although the operator \( P_2(X, Y)^* \) is locally solvable.

We also note, conversely, that non-solvability of \( P_n(X, Y) \) is not generally a sufficient condition for non-solvability of \( P(X, Y) \). A class of operators, known as generalized Laplacians, serve to demonstrate this point: Let \( L_{\lambda, \alpha} \) be given by

\[
(\lambda^2 - 1)L_{\lambda, \alpha} = (1 - \lambda^2)X^2 + Y^2 + i\lambda(XY + YX) + i\alpha[X, Y] \tag{7.3}
\]

for constant \( \lambda, \alpha \) such that \(-1 < \lambda < 1\) is real and \( \alpha \in \mathbb{C} \). Here, \( L_{\lambda, \alpha} \) is not locally solvable when \( \alpha \in \mathbb{Z}^+ \) is odd; yet, for any constant \( c \neq 0 \), \( L_{\lambda, \alpha} + c \) is locally solvable for any \( \lambda, \alpha \) in their domains (see Theorem 3.3 [10]). We elaborate on operators (7.3) as we sketch an alternate proof (viz [4, 5, 13]) that the operator \( L_{\lambda, \alpha} + c \) is locally solvable for any \( c \neq 0 \):

We start with the case \( \lambda = 0 \). Since the characteristic roots are \( \pm 1 \), from Theorem 4.5 it suffices to show that \( L_\mu = \partial^2_t - t^2 + \alpha - z \mu^2 \) for any fixed \( \alpha \) and \( z \neq 0 \) is regular since the analysis for each associated \( L_\mu^\pm \) is essentially the same. In the case \( \alpha > 0 \) not an odd integer we see that \( \ker L_{+\infty} = \ker L_{\infty} \) which contains no \( S(\mathbb{R}) \)-class functions other than the zero function. It then follows from Theorem 5.2, that (7.3) is locally solvable.

Let us now continue with the case \( \lambda = 0 \) but now with \( \alpha > 0 \) an odd integer. We form (admissible) bases \( \tilde{\psi}^\pm \) of \( \ker L_\mu \) using the well-known parabolic cylinder functions \( U, V \) (cf. Chapter 19 [1]): We set \( a = a(\mu, \alpha, z) = \frac{\alpha + z\mu^2}{2} \), and write

\[
\phi_1^\pm(t, \mu) = V(a, \pm \sqrt{2}t) \times e^{-t^2/2}\left( \pm \frac{\alpha + z\mu^2}{2} \right) \\
\phi_2^\pm(t, \mu) = U(a, \pm \sqrt{2}t) \times e^{-t^2/2}\left( \pm \frac{\alpha + z\mu^2}{2} \right)
\]

for \(|\mu| > 2|z|, \pm t > 1\) (resp.). With sufficiently large \( \mu_0 > 2|z| \), the associated transition matrix \( A(\mu) \) satisfies

\[
[A]_{1,1} = \sin(\pi \alpha) \propto 1; \quad [A]_{1,2} = \pi^{-1} \Gamma(a + 1/2) \cos^2(\pi \alpha) \propto \mu; \\
[A]_{2,1} = \pi / \Gamma(a + 1/2) \propto \mu; \quad [A]_{2,2} = \sin(\pi \alpha) \propto 1
\]

on \((\mu_0, \infty)\); and, hence, \( L_\mu \) is regular.

We elaborate further on the odd \( \alpha \) case above to give a partial demonstration of our parametrix, regarding the steps involving (4.2). The estimates on the \( \phi_1^\pm \) hold on complex sectors of the form \( |\pm \arg t - \pi/4| \leq \delta < \pi/4 \) (resp.) for \( \mu \) sufficiently large along complex arcs \( \text{Im} \mu \geq 0, \mu \) fixed. Hence, we have clear choices for \( U_1^\pm \) as in (4.1), using \( \alpha_1 = \pi/4, \delta = \pi/2 \) and \( \lambda_1 = \mu_1^2 \). Moreover, there are constants \( c_1, c_2 \neq 0 \) so that \( \mathcal{H}_1^\pm = c_1 \phi_1^\pm \) and \( \mathcal{H}_2^\pm = c_2 \phi_1^\pm \) (resp.), whereby the integral formula along with contours \( C_1^\pm \) are readily determined.
We now suppose that $\lambda \neq 0$. The operation $P_{\mu,\lambda}: P_{\mu,\lambda}(\phi(t)) = e^{\mu t^2/2} e^{\pm j(t/\sqrt{1-\lambda^2})}$ gives $	ext{ker}\Lambda_\mu(L_{\lambda,c}) = P_{\mu,\lambda}(\text{ker}\Lambda_\mu(L_{0,c}))$ (see Proposition 7.2 [10]). Moreover, $\forall \phi \in \text{ker}\Lambda_\mu$, $P_{\mu,\lambda}(\phi) \in \mathcal{S}(\mathbb{R})$ if and only if $\phi \in \mathcal{S}(\mathbb{R})$. Since $|\lambda| < 1$, the polynomial $P$ associated with (7.3) has property $G$. It now follows from Theorem 5.2 that (7.3) is locally solvable in the specified domains of $\alpha$ and $\lambda$.

Appendix

Proposition 8.1 Given $\alpha \in \mathbb{R}$, and $a, \gamma > 0$,

$$\int_0^t e^{\gamma s^2 + \alpha s} (1 + s)^a \, ds \lesssim e^{\gamma t^2 + \alpha t} (1 + t)^{a+1}$$

$$\int_t^\infty e^{-\gamma s^2 + \alpha s} (1 + s)^a \, ds \lesssim e^{-\gamma t^2 + \alpha t} (1 + t)^a$$

for $t \geq 0$. Moreover, for any positive $\alpha_0$ constants implicit in the asymptotics may be chosen so that the estimates are uniform for $|\alpha| \leq \alpha_0$.

Proof The results hold for $\alpha = 0$ [14]; so, employing the change of variables $u = s + \frac{\alpha}{2\gamma}$ we compute

$$\int_0^t e^{\gamma s^2 + \alpha s} (1 + s)^a \, ds = \int_0^a e^{\gamma u^2 - \frac{\alpha^2}{4\gamma^2}} \left(1 + \left|u - \frac{\alpha}{2\gamma}\right|\right)^a \, du$$

$$\leq e^{-\frac{\alpha^2}{4\gamma}} \left(1 + \frac{\alpha}{2\gamma}\right)^a \int_0^a e^{\gamma u^2} (1 + |u|)^a \, du$$

$$\leq e^{-\frac{\alpha^2}{4\gamma}} \left(1 + \frac{\alpha}{2\gamma}\right)^a \left[ \int_0^{-|\alpha|/2\gamma} e^{\gamma u^2} (1 + |u|)^a \, du + \int_{-|\alpha|/2\gamma}^t e^{\gamma u^2} (1 + u)^a \, du \right]$$

$$\leq e^{-\frac{\alpha^2}{4\gamma}} \int_0^t e^{\gamma u^2} (1 + u)^a \, du \lesssim e^{-\frac{\alpha^2}{4\gamma}} e^{\gamma(t + \frac{\alpha}{2\gamma})^2} \left(1 + t + \frac{|\alpha|}{2\gamma}\right)^{a+1}$$

$$\lesssim e^{\gamma t^2 + \alpha t} \left(1 + \frac{\alpha_0}{2\gamma}\right)^{a+1} (1 + t)^{a+1} \lesssim e^{\gamma t^2 + \alpha t} (1 + t)^{a+1}$$

which holds uniformly for $t \geq 0$ and for $|\alpha| \leq \alpha_0$. 

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By the same substitution we likewise compute for \( t \geq 0 \),

\[
\int_{t}^{\infty} e^{-\gamma s^2 + \alpha s} (1 + s)^a \, ds = e^{\frac{a^2}{2\gamma}} \int_{t+\frac{a}{2\gamma}}^{\infty} e^{-\gamma u^2} \left( 1 + |u - \frac{\alpha}{2\gamma}| \right)^a \, du
\]

\[
\leq e^{\frac{a^2}{2\gamma}} \left( 1 + \frac{\alpha_0}{2\gamma} \right)^a \left[ \int_{\frac{a}{2\gamma}}^{\infty} e^{\gamma u^2} (1 + |u|)^a \, du + \int_{t+\frac{a}{2\gamma}}^{\infty} e^{\gamma u^2} (1 + |u|)^a \, du \right]
\]

\[
\lesssim e^{\frac{a^2}{2\gamma}} e^{-\gamma \left( t - \frac{\alpha}{2\gamma} \right)^2} (1 + t)^a = e^{-\gamma t^2 + \alpha t} (1 + t)^a
\]

uniformly for \(|\alpha| \leq \alpha_0\) and \( t \geq 0 \); and, the result follows.

\(\square\)

Remark 8.2 We make a correction to equation 2.3 [14]. For each \( \tau > 0 \) the support of \( \tilde{F}_\tau \) (the full Fourier transform of \( F_\tau \)) contains a neighborhood of the origin and thus the result should appear as

\[
||v_\tau||_{(s)}^2 \leq c_s \tau^{-4} \int_{\xi_3 \geq 1} (1 + (\xi_1^2 + \xi_2^2 \tau)^2 + (\xi_3^2 \tau^2)^2)^{-s} |\tilde{F}_\tau(\xi)|^2 d\xi
\]

\[
\leq c_s \tau^{-4-2s} \int_{\xi_3 \geq 1} |\tilde{F}_\tau(\xi)|^2 d\xi \leq c_s \tau^{-4-2s} \int_{\xi_3 \geq 1} |\tilde{F}_\tau(\xi)|^2 d\xi
\]

for some constant \( c_s > 0 \). This follows by the Paley-Wiener Theorem and the \( S(\mathbb{R}) \)-mode of convergence of \( \tilde{F}_\tau \) to its limit \( \tilde{\Psi} \), supported in \(|\xi| \xi_3 \geq 1\).

Remark 8.3 The statements of Propositions 4.1, 4.4 and Lemma 4.5 [16] (here \( z = \mu \)) each hold for operators \( \mathcal{L}_\mu \) of the present article under our definition of resolving permutations but restricting the domain of \( \mathcal{A} \) to semi-infinite intervals \((R, \infty)\) (here \( R = \mu_0 \)) for sufficiently large, fixed \( R > 0 \).

Let us elaborate on the above remark to point out modifications necessary to apply the aforementioned results to the present work: We may interpret the classifications \( \mathcal{P}_R \), \( \mathcal{F}_R \) and \( \mathcal{B}_R \) in the manner of [16] (Appendix) regarding only large \( z > 0 \) (ignoring meromorphicity near the origin). Furthermore, in the context of the present article, we recast (19) [16] as follows: If \( \mathcal{L}_\mu \) is regular, then there exists an admissible pair of bases for \( \ker \mathcal{L}_\mu \) such that \( \tilde{\phi}_i^- = A(\mu) \tilde{\phi}_i^+ \) where \( A \) in an \( n \times n \) matrix which likewise takes the block form

\[
A = \left( \begin{array}{ccc}
0^{J^- \times J^-} & M^{J^- \times J^+} & 0 \\
N^{J^+ \times J^-} & 0^{J^+ \times J^+} & 0 \\
A & B & Q
\end{array} \right).
\]

(8.4)

Here, \( Q = Q^{(n-J^+ - J^-) \times (n-J^+ - J^-)} \) and the dimensions of \( A \) and \( B \) are thus obvious. Except for their dimensions, the corresponding blocks have the same properties as those in Proposition 4.2 [16]. Moreover, if \( \mathcal{L}_\mu \) is quasi-regular, there is a transition matrix \( A(\mu) \) which takes such block forms on open sets \( U_i \) (corresponding to various resolving \( \sigma_i \)), whereby equation (20) and Proposition 4.4 therein also hold.

\(^1\) The author thanks the anonymous reviewer of said article for having pointed out the error in the original manuscript.
Let us say that $A$ has a degenerate row when there is an index $K > J^+$ whereby $[A(\mu_\ell)]_{j,k} : k < K$ are each rapidly decreasing on a common sequence $\mu_\ell \to \infty$ (sr). From the proofs of Corollaries 3.2 and 4.2 of [16] we find that if $\mathcal{L}_\mu$ is irregular, in the present context, then $(A^{-1})^\dagger$ has a degenerate row and the result of Lemma 4.5 therein also follows here. We note that the order of the admissible pairs according to $\pm$ sign is arbitrary. Remark 8.3 still holds upon interchange of bases, redefining $A$, $\sigma$ and the $J^\pm$ accordingly.

**Proof of Proposition 2.16** It follows from Section 4 of [2] that given the solution $v$ as in (2.10), we may find another solution by solving the equation

$$
\vec{v}' = \vec{A} \vec{v} + (\vec{R} + \vec{C}) \vec{v}.
$$

(8.5)

Here $\vec{A}$ and $\vec{R}$ are obtained by deleting the last row and last column from $\Lambda$ and $\mathcal{R}$, respectively and where $\vec{C}$ is an $(n - 1) \times (n - 1)$ matrix with components $[\vec{C}]_{i,j} = \frac{v_i}{v_n} \frac{B}{B_{i,j}}$, supposing (perhaps by increasing $t_0$ if necessary) that $v_n(t) \neq 0$ for $t \geq t_0$. Then, the vector

$$
w = g \begin{pmatrix}
v_1 \\
\vdots \\
v_{n-1} \\
v_n
\end{pmatrix} + \begin{pmatrix}
\vec{v}_1 \\
\vdots \\
\vec{v}_{n-1} \\
0
\end{pmatrix}
$$

yields a solution to $\mathcal{L}_\mu y = 0$ where

$$
g' = \frac{1}{v_n} \sum_{j=1}^{n-1} [B]_{n,j} \vec{v}_j.
$$

(8.6)

To obtain estimates for $w$, we have from analysis as in (2.15) that

$$
|\vec{v}| \lesssim |e^{\Phi_2}|; \quad |\vec{v}_{n-1} - e^{\Phi_2}| \lesssim |t^{-1}| |e^{\Phi_2}|; \quad |\vec{v}_i| \lesssim |t^{-1}| |e^{\Phi_2}|
$$

(8.7)

for $j < n - 1$. Since the right-hand side of (8.6) is majorized by $|t^{-2}| |\exp(\Phi_2 - \Phi_1)(t)|$ uniformly in $t$ and $\mu$, we may uniquely define $g$ by setting $g(t_0) = 0$ to obtain

$$
|g(t)| \lesssim |t^{-1}| |\exp(\Phi_2 - \Phi_1)(t)|.
$$

It then follows that $w$ and its components also satisfy estimates (8.7) for $j \leq n$. We then obtain another solution $\psi_2 = [Sw]_{1,1}$, with $\gamma_2$ replacing $\gamma_1$ in $S$, this one satisfying

$$
\frac{d^j}{dt^j} \psi_2 = [Sw]_{1,j} = (\gamma_2 t + \beta_{\mu,2})^{j-1} e^{\Phi_2(t,\mu)} (1 + o(1)) \quad \text{as} \quad t \to \infty
$$

with bounds uniform for $\mu > \mu_0$. Moreover, from the construction of $g$, we have that $\psi_2(t, \mu)$ is a holomorphic function of $\mu$ provided $t$ is sufficiently large. It is clear that $\psi_2(t, \mu)$ extends to a holomorphic function of $\mu$ for all $t$. We then obtain estimates for linearly independent functions $\psi_j : 1 \leq j \leq n$ by induction each being holomorphic functions of $\mu$. Estimates and holomorphic properties for large $t < 0$ follow similarly: By the same analysis on $P(i \partial_s, s)$ with $s = -t$, the $\gamma_j$'s remain the same; yet, by (2.1) and (2.8), the $\beta_j$'s change sign. The main result, having been shown up to derivatives of order $j = n - 1$, follows inductively by applying (2.17) to $\frac{d^{j+n}}{dt^{j+n}} \psi = \frac{d^j}{dt^j} \circ \mathcal{L}_\mu \psi$ for any $\psi \in \ker \mathcal{L}_\mu$ (cf. [14]); here,

$$
\frac{d^l}{dt^l} \circ \mathcal{L}_\mu = \sum_{j=0}^{n+l} a_j(t, \mu) \frac{d^j}{dt^j}
$$

whose coefficients satisfy $a_j(t, \mu) \lesssim (1 + |t|)^{n+l-j}$.  

□
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