RATIONAL TORSION OF GENERALISED MODULAR JACOBIANS OF LEVEL DIVISIBLE BY TWO PRIMES

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ABSTRACT. We consider the generalised Jacobian $J_0(N)_m$ of the modular curve $X_0(N)$ of level $N$, with respect to the modulus $m$ consisting of all cusps on the modular curve. When $N = p'q'$, for $p$ and $q$ odd prime numbers, we determine the group structure of the rational torsion $J_0(N)_m(Q)_{tor}$ up to 2-primary, $p$-primary and $q$-primary torsion. Our results extend those of Wei–Yamazaki for squarefree levels and Yamazaki–Yang for prime power levels.

1. INTRODUCTION

Given a positive integer $N$, let $X_0(N)$ denote the modular curve of level $N$ for the congruence subgroup $\Gamma_0(N) = \{(a \ b \ c \ d) \in SL_2(\mathbb{Z}) : c \equiv 0 (mod\ N)\}$. This is a projective nonsingular curve defined over $\mathbb{Q}$. Modular curves are geometric and algebraic objects relating the theory of elliptic curves to that of modular forms, and they have been widely studied over the last century, see [6] for an overview. The points of $X_0(N)$ parametrise isomorphism classes of elliptic curves over $\mathbb{Q}$ together with a rational cyclic subgroup of order $N$. After base change to $\mathbb{C}$, the curve $X_0(N)/\mathbb{C}$ is isomorphic to the compact Riemann surface $\mathcal{H}/\Gamma_0(N)$, where $\mathcal{H}$ denotes the complex upper-half plane. The open subset $\mathcal{H}/\Gamma_0(N)$ parametrises isomorphism classes of complex elliptic curves together with a cyclic subgroup of order $N$ and the closed points $\text{Cusps}(X_0(N)) := \mathbb{P}^1(\mathbb{Q})/\Gamma_0(N)$ form the set of cusps of $X_0(N)$. All points $P \in \text{Cusps}(X_0(N))$ are algebraic points defined over the cyclotomic extension $\mathbb{Q}(\zeta_N)$, cf. [32, Theorem 2.17].

Let $J_0(N)$ denote the Jacobian of the modular curve $X_0(N)$. The interest in the study of the rational torsion of $J_0(N)$ goes back to Levi’s Torsion Conjecture, which established a classification of the possible torsion subgroups of the group of rational points of an elliptic curve over $\mathbb{Q}$, cf. [22]. Almost seventy years later, Ogg [14, Conjecture 2] reformulated and generalised Levi’s Torsion Conjecture into what is now known as Ogg’s Conjecture. Writing $C_N$ for the image of the degree zero divisors with support at the cusps of $X_0(N)$ in $J_0(N)$ and defining $C_N(Q) := C_N \cap J_0(N)(Q)$, the statement made by Ogg and proved later in a renowned paper by Mazur [13, Theorem 1] was the following: let $N \geq 5$ be a prime number and $n = \text{num}(\frac{N-1}{12})$. The cyclic group $C_N(Q)$ of order $n$ is the full torsion subgroup $J_0(N)(Q)_{tor}$ of $J_0(N)(Q)$. Here, we write $\text{num}(q) := a$ for $q = a/b \in \mathbb{Q}$, with $a$ and $b$ coprime and $a > 0$. 

2020 Mathematics Subject Classification. 11G18, (14H40, 11G45, 11G16, 14G05).

Key words and phrases. generalised Jacobians, modular curves, rational points, eta-quotients, cuspidal divisor class group.

The author would like to thank Valentijn Karemaker for introducing her to the topic of the paper, the many insightful discussions, and for her wonderful support throughout the whole project. Thanks also to Gunther Cornelissen for his valuable comments on the introduction. Thanks to Jakub Byczewski for his help with the proof of Lemma 5.5. Finally, thanks to Takao Yamazaki, Yifan Yang, Fu-Tsun Wei, Hwajong Yoo and a referee for their feedback and corrections on earlier versions of the paper.
A natural generalisation of this conjecture states that the finite torsion group \( J_0(N)(\mathbb{Q})_{\text{tor}} \) is equal to \( C_N(\mathbb{Q}) \) for all positive integers \( N \); this is called the generalised Ogg’s Conjecture – see [10], [11], [15], [19] and [33] for progress on the proof of the conjecture. Moreover, if \( C(N) \) is the subgroup formed by the equivalence classes of \( \mathbb{Q} \)-rational divisors in \( C_N(\mathbb{Q}) \), i.e., the divisor classes fixed under the action of the absolute Galois group of \( \mathbb{Q} \), a further conjecture states that \( C_N(\mathbb{Q}) \) is equal to \( C(N) \) – see [20], [25] and [27] for progress on the proof of this conjecture. Recently, analogous work has been done for the case of Drinfeld modular curves – see [17], [18] – and for the modular curve \( X_1(N) \) – see [8, p. 153], [2, Table 1].

On a further note, understanding the torsion on the Jacobian \( J_0(N) \) allows us to study unramified abelian covers of the modular curve \( X_0(N) \), as every unramified abelian cover \( X \to X_0(N) \) is the pullback of a unique isogeny \( J \to J_0(N) \), see [23, Chapter I, Theorem 2]. Hence, we have a bijection

\[
\{ \text{finite unramified abelian covers } X \to X_0(N) \} \leftrightarrow \{ \text{isogenies } J \to J_0(N) \text{ with } \iota^{-1}(J) \simeq X \},
\]

where \( \iota \) is the Abel-Jacobi map \( X_0(N) \to J_0(N) \), see [23, Chapter I, Corollary]. Furthermore, the rational covers in this bijection correspond to rational isogenies, i.e., geometrically connected covers \( X \) over \( \mathbb{Q} \) correspond to isogenies whose kernel is a constant finite group scheme. By duality, the latter are in bijection with rational isogenies from \( J_0(N) \) whose kernel is given by the Cartier dual of these constant finite group schemes. Cartier duals of constant finite group schemes are said to be of \( \mu \)-type, cs.[13]. In [26], computes the odd \( \mu \)-type subgroups of \( J_0(N) \) for \( N \) squarefree.

Let \( m = \sum_{P \in \text{Cusps}(X_0(N))} P \) be the reduced rational divisor of \( X_0(N) \) with \( \text{Cusps}(X_0(N)) \) as support. We can view the rational effective divisor \( m \) as a modulus, i.e., as the formal product of the places of the global function field \( K = \mathbb{Q}(X_0(N)) \) corresponding to the points in \( \text{Cusps}(X_0(N)) \), and consider the generalised Jacobian \( J_0(N)_m \) in the Rosenlicht-Serre sense, see [21], [23, Chapters I, V]. By studying generalised Jacobians, Rosenlicht and Lang extended the geometric interpretation of class field theory for global function fields of curves over finite fields to curves over perfect fields. Just like the usual Jacobian \( J_0(N) \) allows us to study unramified abelian covers of the modular curve \( X_0(N) \), the generalised Jacobian \( J_0(N)_m \) allows us to study finite abelian covers of \( X_0(N) \) ramified at the support of \( m \). Hence, again via pullback, we now have the bijection

\[
\{ \text{finite abelian covers } X \to X_0(N) \text{ ramified at } m \} \leftrightarrow \{ \text{isogenies } J \to J_0(N)_m \text{ with } \iota^{-1}_m(J) \simeq X \},
\]

where \( \iota_m \) is a (canonical) rational map \( X_0(N) \to J_0(N)_m \), see [23, Chapter I, Corollary]. In (2), rational covers again correspond to rational isogenies. To classify the latter, we would like to use duality as we did before. However, since \( J_0(N)_m \) is not an abelian variety, the appropriate way to make sense of its dual is by considering the dual 1-motive of \( J_0(N)_m \) in the Deligne sense. The rational isogenies in (2) are now classified by \( \mu \)-type subgroups of this dual [30, Section 2]. In [30] the authors compute the latter \( \mu \)-type subgroups for prime levels \( N \). Bridging the gaps, studying the rational torsion of \( J_0(N)_m \) is a first step towards understanding ramified abelian covers of \( X_0(N) \).

In a different direction, recent papers of Gross [5] and Bruinier-Li [1] relate traces of singular moduli on modular curves to Heegner divisors in generalised Jacobians with cuspidal modulus. In [1, Theorem 1.1], the authors extend the Gross-Johnen-Zagier formula for classes of Heegner divisors of \( J_0(N) \) by showing that the generating series of these classes is a weakly holomorphic modular form of weight 3/2 with values in \( J_0(N)_m \).
Motivated by its inherent interest and applications, in this paper we study the structure of the group $J_0(N)_m(\mathbb{Q})_{\text{tor}}$. The group $J_0(N)_m(\mathbb{Q})$ is not finitely generated, but its torsion subgroup is finite, cf. [23], [29]. As an analogue of the generalised Ogg's Conjecture, Yamazaki–Yang [29, Theorem 1.1.3] and Wei–Yamazaki [28, Theorem 1.2.3] describe $J_0(N)_m(\mathbb{Q})_{\text{tor}}$ for $N$ a power of a prime number and $N$ squarefree respectively. We build upon their work and use results in [32, Theorem 1.6, 1.7] on the description of $C(N)$ and in [33, Theorem 1.4] on the generalised Ogg's Conjecture to prove the following result.

**Theorem A** (Theorem 4.2). Let $N = p^r q^s$ where $p$ and $q$ are two distinct odd prime numbers. For any odd prime $l$ with $l^2$ not dividing $3N$ we have

$$J_0(N)_m(\mathbb{Q})_{\text{tor}}[l^\infty] \cong \bigoplus_{1 \leq a \leq r, \ 1 \leq b \leq s} \mathbb{Z}/l^{\text{ord}_m(a,b)}\mathbb{Z},$$

for the integer $m(a, b)$ given by

$$m(a, b) = \begin{cases} 
\text{num} \left( \frac{a^r - 1 - b^r - 1}{24} \right) & \text{if } a, b \geq 2; \\
\text{num} \left( \frac{a^r - 1 - (q^2 - 1)(p - 1)}{24} \right) & \text{if } a = 1, b > 2; \\
\text{num} \left( \frac{a^r - 1 - (2^2 - 1)(q - 1)}{24} \right) & \text{if } a > 2, b = 1; \\
\text{num} \left( \frac{(q^2 - 1)(p - 1)}{24} \right) & \text{if } a = 1, b = 2; \\
\text{num} \left( \frac{(p^2 - 1)(q - 1)}{24} \right) & \text{if } a = 2, b = 1; \\
\text{num} \left( \frac{(p - 1)(q - 1)}{24} \right) & \text{if } a = b = 1; 
\end{cases}$$

where $i$ and $j$ are given by $i = \left\lfloor \frac{r+1-a}{2} \right\rfloor$ and $j = \left\lfloor \frac{s+1-b}{2} \right\rfloor$.

Our results coincide with those of Wei-Yamazaki in [28, Theorem 1.2.3] and with those of Yamazaki-Yang in [29, Proposition 1.3.2] when $N = pq$. In forthcoming work, we generalise the results obtained here for $N = p^r q^s$ to any odd level $N$.

In [28, Theorem 1.2.3], Wei and Yamazaki show that in the case of squarefree level $N = \prod_{i=1}^{t} p_i$ the size of the rational torsion of $J_0(N)_m$ – up to 2-torsion and 3-torsion – increases exponentially with the number of primes $s$ and linearly with each prime $p_i$, just like the rational torsion of $J_0(N)$ for these same levels. This contrasts with the case where $N = p^r$ discussed by Yamazaki and Yang in [29, Theorem 1.1.3]; for prime power, levels the rational torsion of $J_0(N)_m$ is trivial up to 2-torsion and $p$-torsion, no matter how big $p$ and $r$ are, which differs from what happens to the usual Jacobian for these levels. Contrary to the case in [29] we observe here that the size of $J_0(p^r q^s)_m(\mathbb{Q})_{\text{tor}}$ increases linearly with $r$ and $s$ since $J_0(p^r q^s)_m(\mathbb{Q})_{\text{tor}}[l^\infty]$ is given by the product of $r \cdot s$ cyclic subgroups. On the other hand, it follows from Theorem A that the size of $J_0(p^r q^s)_m(\mathbb{Q})_{\text{tor}}$ increases linearly with $p$ and $q$, following the trend observed in [29].

Finally, notice that under the assumption that the generalised Ogg's Conjecture and the conjecture $C_N(\mathbb{Q}) = C(N)$ hold, we could use the results in Theorem A for $l = p, q$ to obtain a complete explicit description of the rational torsion of the generalised Jacobian for $N = p^r q^s$. This would provide an analogue of generalised Ogg's conjecture for generalised Jacobians with cuspidal modulus and level $N = p^r q^s$. 
The structure of the article is as follows. In Section \(2\) we fix some notation and definitions. In Section \(3\) we review known results on \(J_0(N) / \mathbb{Q}\) and \(C_N(\mathbb{Q})\). For this, we revisit classical results on the cusps of \(X_0(N)\) and eta quotients – see Sections \(3.2\) and \(3.4\) –, and we explain the construction of certain divisors \(Z(d) \in \text{Div}_0(X_0(N))\) given in [32, Theorem 1.6] – see Section \(3.4.1\). These divisors generate \(C(N)\) and play a key role in the proof of Theorem A. Finally, in Proposition 3.26 we compute certain eta quotients related to the divisors \(Z(d)\). In Section \(4\) we recall some results on the generalised Jacobian \(J_0(N)_m\). We introduce a group homomorphism denoted by \(\delta\) and reduce the proof of Theorem A to a computation of the kernel of \(\delta\) on \(C(N)\) – see Theorem 4.2. Finally, in the remaining sections we spell out the results for the computation of \(\ker(\delta)\).

2. Notation

In this section we introduce some notation and definitions that will be used throughout the paper.

**Notation 2.1.** Let \(a, b, d \in \mathbb{Z}\) and \(p\) denote a prime number. We adopt the following notation in this paper, unless stated otherwise:

- \(\varphi(d)\) is Euler’s phi function;
- \(\text{ord}_p(d)\) denotes the \(p\)-adic valuation of \(d\);
- \((a, b) := \gcd(a, b)\) denotes the greatest common divisor of \(a\) and \(b\);
- given \(\frac{a}{b} \in \mathbb{Q}\), we denote the numerator of \(\frac{a}{b}\) by \(\text{num}(\frac{a}{b})\).

**Definition 2.2.** Given an odd prime number \(p\), we define \(B(p) := \text{num}(\frac{p^{-1}}{12})^{24}\). That is,

\[
B(p) = \begin{cases} 
2 & \text{if } p \equiv 1 \pmod{12}, \\
6 & \text{if } p \equiv 5 \pmod{12}, \\
4 & \text{if } p \equiv 7 \pmod{12}, \\
12 & \text{if } p \equiv 11 \pmod{12}.
\end{cases}
\]

We also define the positive integer

\[
A(p) = \begin{cases} 
1 & \text{if } p \neq 3, \\
3 & \text{if } p = 3.
\end{cases}
\]

**Definition 2.3.** Let \(p\) denote a prime number. Given a positive integer \(N\), we define the radical of \(N\) by

\[
\text{rad}(N) := \prod_{p \mid N} p,
\]

where the product runs over all the different prime factors of \(N\). We also define the integer \(k(N)\) by

\[
k(N) := N \prod_{p \mid N} (p - p^{-1}) = \frac{N}{\text{rad}(N)} \prod_{p \mid N} (p^2 - 1).
\]
Definition 2.4. Let \( N \) be a positive integer. We give the following definitions related to the set of divisors of \( N \):

- \( D_N \): the set of all positive divisors of \( N \), except for 1.
- \( \sigma_0(N) := \#D_N + 1 \): the number of divisors of \( N \), including 1.
- \( D_N^{\text{nsf}} \): the set of non-squarefree divisors of \( N \).
- \( D_N^{\text{sf}} = D_N \setminus D_N^{\text{nsf}} \): the set of all squarefree divisors of \( N \), except for 1.

Definition 2.5. Given a prime number \( p \), a positive integer \( r \), and an integer \( f \) such that \( 0 \leq f \leq r \), we define

\[
\mathcal{G}_p(r, f) := \begin{cases} 
    p^{r-1}(p^2 - 1) & \text{if } f = 0, \\
    1 & \text{if } f = 1, \\
    p^2 - 1 & \text{if } f = 2, \\
    p^{r-i-1}(p^2 - 1) & \text{if } 3 \leq f \leq r,
\end{cases}
\]

where \( i = \left\lfloor \frac{r+1-f}{2} \right\rfloor \).

Definition 2.6. Let \( N \) be an odd positive integer.

(a) We fix an ordering \( p < q \) on the set \( \{p|N : p \text{ prime}\} \) of primes appearing in the prime factorisation of \( N \), and we label the elements in this set accordingly: we let \( N = \prod_{i=1}^{\sigma_0(N)} p_i^{r_i} \) with \( i < j \) for \( p_i < p_j \). We could use the ordering defined in [32, Assumption 1.14], which depends on a prime number \( l \). However, for our purposes, this order can be completely arbitrary; see Remark 3.21. Hence, we will use the standard ordering of the natural numbers. That is, whenever we write \( p < q \), such as in Definition 3.18, we mean that \( p < q \).

(b) Let \( d = \prod_{i=1}^{n} p_i^{f_i} \) be a divisor of \( N \) under the chosen ordering. We define

\[
\mathcal{G}(N, d) := \begin{cases} 
    \prod_{i=1}^{n} \mathcal{G}_p(r_i, f_i) & \text{if } f_i \geq 2 \text{ for some } i, \\
    \prod_{i=1, i \neq m}^{n} \mathcal{G}_p(r_i, f_i)(p_m - 1) & \text{if } 0 \leq f_i \leq 1 \text{ for all } i,
\end{cases}
\]

where \( m \) is the smallest index \( i \) such that \( f_i = 1 \). Finally, we define

\[
n(N, d) := \text{num}\left( \frac{\mathcal{G}(N, d)}{24} \right).
\]

Notation 2.7. Given a field \( k \), we denote the group \((k^\times)_\text{tor}\) of the torsion units in \( k \), by \( \mu(k) \). Further, given a positive integer \( m \), \( \mu_m \) denotes the group of \( m \)-th roots of unity in \( \mathbb{C} \).

Notation 2.8. We use the following standard notation for the groups of divisors of a smooth projective curve \( C \) over a field \( k \):

- \( \text{Div}(C) \): the group of divisors of the curve \( C \);
- \( \text{Div}^0(C) \): the group of degree-zero divisors of the curve \( C \);
- \( k(C) \): the function field of the curve \( C \);
- We say that \( D \in \text{Div}(C) \) is a principal divisor if \( D = \text{div}(f) \) for some \( f \in k(C) \);
- When working with \( \text{Div}(C) \), \( \sim \) denotes the linear equivalence relation on \( \text{Div}(C) \). I.e., \( D_1 \sim D_2 \) if and only if \( D_1 - D_2 \) is a principal divisor.
3. $J_0(N)(\mathbb{Q})_{\text{tor}}$ and the Cuspidal Subgroup

This section is divided into five parts. In the first part, we will present the subgroup $J_0(N)(\mathbb{Q})_{\text{tor}}$ and introduce the classical statement of Ogg’s Conjecture. In the second part, we will explore the rational divisor class group $C(N)$ and in the third part we will explain how this is related to the rational torsion subgroup of $J_0(N)$ through reviewing main conjectures and results over these groups. In the fourth part, we discuss eta quotients and we explain the link of these complex functions with the divisors in $C(N)$, and finally, in the fifth and last part, we describe a set of divisors on $X_0(N)$ that generate $C(N)$ “optimally”. The main goal of this section is to explain the relevant theoretical background that leads to the main results of this section Theorem 3.1 and Proposition 3.26. In Theorem 3.2 we explain how to compute the group structure of the rational torsion of the modular Jacobian $J_0(N)$ through divisors; more specifically, we find divisors whose linear equivalence classes generate $J_0(N)(\mathbb{Q})_{\text{tor}}$ in an optimal way and we compute their order. As usual, by the order of a divisor $D$ on $X_0(N)$ we mean the order of the linear equivalence class $[D]$ of $D$ in $J_0(N)$; i.e., the smallest $n \in \mathbb{Z}$ such that there exists a modular function $f$ in $k(X_0(N))$ with
\[
\text{div}(f) = n \cdot D;
\]
we denote the order of a divisor $D$ by order$(D)$. Furthermore, in Proposition 3.26 we find a function $f \in k(X_0(N))$ for each of the divisors $D$ constructed in Theorem 3.2, such that $\text{div}(f) = n \cdot D$, where $n$ is the order of $D$.

3.1. The group $J_0(N)(\mathbb{Q})_{\text{tor}}$. Let $N$ be a positive integer and consider the congruence subgroup $\Gamma_0(N) \subseteq \text{SL}_2(\mathbb{Z})$, which consists of the matrices in $\text{SL}_2(\mathbb{Z})$ whose lower-left entry is congruent to zero modulo $N$. This group defines an action on the complex upper-half plane $\mathcal{H}$ given by
\[
\gamma z = \frac{az + b}{cz + d}, \quad \text{for all } z \in \mathcal{H} \text{ and } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N).
\]

The modular curve $X_0(N) \backslash \mathbb{C}$ is the result of compactifying the quotient $\text{SL}_2(\mathbb{Z})/\mathcal{H}$ by adding a finite number of points called cusps. These cusps are defined by the quotient
\[
\text{Cusps}(X_0(N)) := \text{SL}_2(\mathbb{Z})/\mathbb{P}^1(\mathbb{Q}),
\]
where $\mathbb{P}^1(\mathbb{Q})$ is the rational projective line and the action of $\text{SL}_2(\mathbb{Z})$ is given as in (4), extending the action on $\mathcal{H}$. Hence, we denote a representative of a cusp of $X_0(N)$ by $[\frac{a}{b}]$, with $(a, b) = 1$. Furthermore, we denote the cusp $[\frac{1}{0}]$ by 0 and the cusp $[\frac{0}{1}]$ by $\infty$.

Rational torsion points on $X_0(N)$ are of big interest, in particular because they parametrise elliptic curves with rational torsion. However, $X_0(N)(\mathbb{Q})$ is hard to compute straight from the modular curve. To remedy this, we can embed $X_0(N)$ into its Jacobian variety $J_0(N)$ through the Torelli map. The latter is an abelian variety, so its group of rational points is a bit easier to work with as it satisfies Mordell’s theorem. Hence, we are interested in understanding the torsion subgroup $J_0(N)(\mathbb{Q})_{\text{tor}}$ for any given $N$. The following result gives an explicit description of this group for the case of $N$ prime.

Theorem 3.1. Let $N \geq 5$ be a prime number. Let $n = \text{num}(N - 1)/12$ and let $\{0\}, \{\infty\}$ be the two cusps of $X_0(N)$. The cyclic group of order $n$ generated by the class of the divisor $(0) - (\infty)$ is the full torsion subgroup of $J_0(N)(\mathbb{Q})$. 

This theorem was first conjectured by Ogg in 1995, [14, Conjecture 2], and proved by Mazur two years later, [13, Theorem 1]. For an arbitrary $N$, it is still not known how to compute the full group $J_0(N)(\mathbb{Q})_{\text{tor}}$, yet a number of recent articles have worked on generalising Ogg’s Conjecture. In order to write down the generalised statement as well as some advances on the topic, it is convenient to first introduce some definitions.

3.2. Cuspidal Subgroups. If $\varphi : J_0(N) \xrightarrow{\sim} \text{Pic}^0(X_0(N))$ is any isomorphism, we let $C_N$ be the \textit{cuspidal subgroup of} $J_0(N)$, that is, the subgroup of $J_0(N)(\overline{\mathbb{Q}})$ generated by the cusps of $X_0(N)$. In terms of divisors, by a slight abuse of notation, this can be written as

$$C_N := \{ \varphi^{-1}([D]) : [D] \in \text{Pic}^0(X_0(N))(\overline{\mathbb{Q}}), \text{ } D \text{ a cuspidal divisor} \},$$

where \textit{cuspidal divisors} are those divisors of $X_0(N)$ that are only supported at the cusps of $X_0(N)$. Let $C_N(\mathbb{Q})$ denotes the subgroup of rational points of $J_0(N)$ in $C_N$, i.e.,

$$C_N(\mathbb{Q}) = \{ P \in C_N : P \text{ is } \mathbb{Q}\text{-rational} \} = \{ \varphi^{-1}([D]) \in C_N : \sigma(D) \sim D \forall \sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \}.$$ 

The subgroup $C_N(\mathbb{Q})$ is called the \textit{rational cuspidal subgroup of} $J_0(N)$. By theorems from Manin [12, Corollary 3.6] and Drinfeld [4] we have that $C_N(\mathbb{Q}) \subseteq J_0(N)(\mathbb{Q})_{\text{tor}}$. Moreover, if $A_N := \text{Div}^0_{\text{cusp}}(X_0(N)) = \{ D \in \text{Div}^0(X_0(N)) : D \text{ cuspidal} \}$ and $u_N := \{ f \in \mathbb{C}(X_0(N)) : \text{div}(f) \in A_N, f \neq 0 \}/\mathbb{C}^\times$, we have the short exact sequence

$$0 \rightarrow U_N \rightarrow A_N \xrightarrow{F} C_N \rightarrow 0,$$

where $F : D \mapsto [D]$. The group $U_N$ is called the group of \textit{modular units}. Taking the group $A_N(\mathbb{Q}) = \{ D \in A_N : \sigma(D) = D \forall \sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \}$ of \textit{rational cuspidal divisors}, and $F|_{A_N(\mathbb{Q})} : A_N(\mathbb{Q}) \rightarrow C_N(\mathbb{Q})$ the restricted map from the exact sequence above, we write

$$C(\mathbb{Q}) := \{ F(D) : D \in A_N(\mathbb{Q}) \}$$

for the \textit{rational divisor class group} of $X_0(N)$. In [32], Yoo completely determines the structure of the group $C(\mathbb{Q})$ for any $N$. In particular, he shows the following.

**Theorem 3.2.** [32, Theorem 1.5] Let $N$ be a positive integer. For each $d \in D_N$, there exists a divisor $Z(d) \in A_N(\mathbb{Q})$ such that

$$C_N \simeq \langle [Z(d)] : d \in D_N^{\text{sf}} \rangle \oplus \bigoplus_{d \in D_N^{\text{nsf}}} \langle [Z(d)] \rangle \simeq \langle [Z(d)] : d \in D_N^{\text{sf}} \rangle \oplus \bigoplus_{d \in D_N^{\text{nsf}}} \mathbb{Z}/n(N,d)\mathbb{Z},$$

where $n(N,d)$ is the order of $Z(d)$.

In his paper, Yoo defines explicit divisors $Z(d)$ for each $d \in D_N$ such that Theorem 3.2 holds, and computes the orders $n(d,N)$. The description of $C(\mathbb{Q})$ in Equation (6) becomes fully explicit when we look at $C(\mathbb{Q})[1^\infty]$ for a given prime number $l$.

**Theorem 3.3.** [32, Theorem 1.6] Let $N$ be a positive integer. For each $d \in D_N$, let $Z(d) \in A_N(\mathbb{Q})$ be as in Theorem 3.2. Given any prime number $l$, there exists a divisor $Y(d)$ for each $d \in D_N^{\text{sf}}$ such that locally

$$\langle [Z(d)] : d \in D_N^{\text{sf}}[1^\infty] \rangle \simeq \bigoplus_{d \in D_N^{\text{sf}}} \langle [Y(d)] \rangle.$$
Hence,

\[ C(N)[T^\infty] \simeq \bigoplus_{d \in D_N^{sf}} \langle [Y_l(d)] \rangle \oplus \bigoplus_{d \in D_N^{ad \text{c}}} \langle [Z_l(d)] \rangle \simeq \bigoplus_{d \in D_N} \mathbb{Z}/n_l(N, d)\mathbb{Z}, \]

where \( n_l(N, d) \) is the \( l \)-primary part of \( n(N, d) \) for \( d \in D_N^{sf} \), and the \( l \)-part of the order of \( Y_l(d) \) if \( d \in D_N^{sf} \), and where \( Z_l(d) = (n(N, d)/n_l(N, d)) \cdot Z(d) \) and \( Y_l(d) = (\text{order}(Y(d))/n_l(N, d)) \cdot Y(d) \).

The subgroup \( C(N) \) will play a fundamental role in our results. However, while we can completely decompose the \( l \)-primary subgroup of \( C(N) \) into cyclic subgroups for all primes \( l \), it remains an open problem to find a global decomposition of \( \langle [Z(d)] : d \in D_N^{sf} \rangle \), and hence of \( C(N) \). Nonetheless, the decomposition in given Theorem 3.2 will be good enough for our purposes.

Next, we further develop the notion of cusps of \( X_0(N) \) and establish some notation to describe \( C(N) \) according to the decomposition given in Theorem 3.2.

By [32, Theorem 2.2], we can identify the set of cusps of \( X_0(N) \) with

\[ \left\{ \left\lfloor \frac{c}{d} \right\rfloor : d \in \mathbb{N}, d|N, (c, d) = 1 \text{ and } c \in \mathbb{Z}/(d, N/d)\mathbb{Z} \right\}, \]

cf. [32, Corollary 2.3]. Hence, for each cusp \( \omega \in \text{cusps}(X_0(N)) \) we can find a representative \( \left\lfloor \frac{c}{d} \right\rfloor \) such that \( d \) is a divisor of \( N \). More precisely, given \( \omega = \left\lfloor \frac{c}{d} \right\rfloor \) we can set \( d = (b, N) > 0 \) and find \( c \in \mathbb{Z} \) such that \((c, d) = 1 \) and \( \left\lfloor \frac{c}{d} \right\rfloor = \left\lfloor \frac{c}{d} \right\rfloor \). We define \( d \) as the level of the cusp \( \omega \). It follows from Equation (8) that we have \( \phi((d, N/d)) \) cusps of level \( d \).

The modular curve \( X_0(N)/\mathbb{C} \) has a canonical nonsingular projective model defined over \( \mathbb{Q} \). In this model, a cusp \( \omega \) of level \( d \) is defined over \( \mathbb{Q}(\zeta_{M_d}) \), where \( M_d = (d, N/d) \), \( \zeta_{M_d} \) is a primitive root of unity of order \( M_d \), and the action of \( \text{Gal}(\mathbb{Q}(\zeta_{M_d})/\mathbb{Q}) \) on the set of cusps of level \( d \) is transitive, cf. [32, Theorem 2.17]. Hence, if \( P(N)_d \) is the cuspidal divisor given by the sum of all cusps of level \( d \) in \( X_0(N) \), we have that \( P(N)_d \) is a rational divisor. Recall from Equation (5) that the group \( C(N) \) is the image of the group \( A_N(\mathbb{Q}) \) in the Jacobian \( J_0(N) \). By [32, Lemma 2.19], the group \( A_N(\mathbb{Q}) \) is generated by the classes of the divisors

\[ C(N)_d := \varphi(M_d) \cdot P(N)_1 - P(N)_d. \]

We will drop the level \( N \) from the notation of these divisors and just write \( C_d \) and \( P_d \) unless more clarity is needed. Let \( S(N)_{\mathbb{Q}} \) be the \( \mathbb{Q} \)-vector space of dimension \( \sigma_0(N) \), whose vectors have coordinates indexed by the divisors of \( N \) in a fixed ordering, e.g. the lexicographic order. We define the standard basis elements

\[ e(N)_d := (0, \cdots, 1_d, \cdots, 0), \]

for \( d \) a divisor of \( N \), with 1 at the \( d \)-th position.

Consider the \( \mathbb{Z} \)-lattice

\[ S(N) := \left\{ \sum_{d|N} a_d e(N)_d : a_d \in \mathbb{Z} \right\} \]

inside \( S(N)_{\mathbb{Q}} \) and its sublattice

\[ S(N)^0 := \left\{ \sum_{d|N} a_d e(N)_d : a_d \in \mathbb{Z}, \sum_{d|N} a_d \phi(M_d) = 0 \right\}. \]
If we consider the map that sends the divisor $P_d$ to the vector $e(N)_d$, we get an isomorphism

$$\Phi_N : \text{Div}_{\text{cusp}}(X_0(N))(\mathbb{Q}) \xrightarrow{\cong} S(N).$$

Notice also that, with this description, $A_N(\mathbb{Q}) \simeq S(N)^0$. Further, for $S(N)_Q := S(N) \otimes_{\mathbb{Z}} \mathbb{Q}$ we obtain

$$\text{Div}_{\text{cusp}}(X_0(N))(\mathbb{Q}) \otimes \mathbb{Q} \simeq S(N)_Q = \left\{ \sum_{d|N} a_d e(N)_d : a_d \in \mathbb{Q} \right\}.$$

**Remark 3.4.** Let $N = \prod_{i=1}^{t} p_i^{r_i}$ be the prime decomposition of the level $N$. Then we can identify $S(N)_Q$ with $\bigotimes_{i=1}^{t} S(p_i^{r_i})_Q$. If $d|N$ has prime decomposition $d = \prod_{i} p_i^{r_i}$, the divisor $P(N)_d$ is identified with $\bigotimes_{i} P(p_i^{r_i})_p^{r_i}$ by writing $e(N)_d = \bigotimes_{i} e(p_i^{r_i})$, cf. [32, Lemma 2.6, Remark 2.20]. In particular, if $N = p^r q^s$, we have $S(N)_Q \simeq S(p^r)_Q \otimes S(q^s)_Q$, where $p$ and $q$ are primes.

The spaces $S(N)$ will prove to be very useful for encoding the description of cuspidal divisors on $X_0(N)$ and doing further computations – see Section 3.4.

### 3.3. Conjectures

From the definitions in Subsection 3.2, we have the three inclusion of groups

$$C(N) \subseteq C_N(\mathbb{Q}) \subseteq J_0(N)(\mathbb{Q})_{\text{tor}}. \hspace{1cm} (10)$$

Notice that if $N$ is a prime number then $X_0(N)$ only has two cusps $-0$ and $-\infty$, which are rational points in $X_0(N)$. Hence, $C_N(\mathbb{Q}) = \langle [0-\infty] \rangle$ and Theorem 3.1 states $C_N(\mathbb{Q}) = J_0(N)(\mathbb{Q})_{\text{tor}}$. The natural generalisation of Theorem 3.1 is as follows.

**Conjecture A.** (generalised Ogg’s conjecture $^1$) – Let $N$ be a positive integer. Then

$$C_N(\mathbb{Q}) = J_0(N)(\mathbb{Q})_{\text{tor}}.$$

While the generalised Ogg’s conjecture is about the second inclusion in Equation (10), it is also an open and interesting question whether the first inclusion is also an equality. Intuitively, it would not necessarily be the case, as a divisor $D$ might not be $\mathbb{Q}$-rational itself but satisfy $\sigma(D) \sim D$ for all $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, and hence represent a rational point. However, in all the cases computed so far, it is the case that both groups are the same, which points to the following conjecture.

**Conjecture B.** (Rational cuspidal groups conjecture) – Let $N$ be a positive integer. Then

$$C_N(\mathbb{Q}) = C(N).$$

If both Conjecture A and Conjecture B would hold, we could use Theorems 3.2 and 3.3 to obtain a full description of the rational torsion of the Jacobian $J_0(N)$. The following list outlines some of the progress made towards proving Conjectures A and B.

(a) When $N = p^r$, the generalised Ogg’s Conjecture (Conjecture A) is true up to $6p$-torsion. This is due to results of Ling [10] and Lorenzini [11].

---

$^1$This is also referred to as Stein’s Conjecture, cf.[24]
(b) Wang-Yang [27] prove the equality of the groups \( C(N) \) and \( C_N(\mathbb{Q}) \) (Conjecture B) for all levels \( N = n^2M \), where \( M \) is squarefree and \( n \) is an integer dividing 24.

(c) When \( N \) is squarefree, then \( C_N(\mathbb{Q}) = C(N) \) (Conjecture B) and the generalised Ogg’s Conjecture (Conjecture A) is true up to 6-torsion. Furthermore, if \( 3 \nmid N \) the conjectures hold up to 2-torsion.

These results were proved by Ohta \[15\].

As portrayed in Theorem 3.3, we can also break the study of the rational torsion of the Jacobian into its \( l \)-primary parts, where \( l \) is a prime number. In this direction, the equality

\[
C(N)[l^{\infty}] = J_0(N)(\mathbb{Q})_{tor}[l^{\infty}]
\]

is proved in the following cases:

(d) If \( N = 3p \) for a prime \( p \) such that either \( p \not\equiv 1 \pmod{9} \) or \( 3\frac{p-1}{2} \not\equiv 1 \pmod{p} \), and \( l = 3 \); due to Yoo \[31\].

(e) If \( N \) is any positive integer and \( l \nmid 6N\prod_{p|N}(p^2-1) \); due to Ren \[19\].

(f) Recent major advances in the proof of Equation (11) are given in the next result; due to Yoo \[33\] and based on previous work by Ohta \[16\].

**Theorem 3.5.** \[33, Theorem 1.4\] For any positive integer \( N \), we have

\[
C(N)[l^{\infty}] = J_0(N)(\mathbb{Q})_{tor}[l^{\infty}]
\]

for any odd prime \( l \) such that \( l^2 \) does not divide \( 3N \).

For our purposes, the most relevant results are the construction of divisors \( Z(d) \) as generators of \( C(N) \) in \[32\] and the partial proof of Conjectures A and B in \[33\]. We will see in Section 4 that to compute the full rational torsion of the generalised modular Jacobian \( J_0(N)_m \) it is important to understand the rational torsion on \( J_0(N) \), since, up to 2-torsion, \( J_0(N)_m(\mathbb{Q})_{tor} \) is given by the kernel \( \ker(\delta|_{J_0(N)(\mathbb{Q})_{tor}}) \) of a certain homomorphism \( \delta \). Since we would like to use Equation (11) whenever it is known to hold, in the main result we will work with \( \ker(\delta|_{J_0(N)(\mathbb{Q})_{tor}[l^{\infty}]}) \), whenever \( l \notin \{p, q, 2\} \) is a prime number. In the following subsections we will follow the constructions in \[32\] to find the generating divisors \( Z(d) \) and understand \( C(N) \). These generators are given in Definition 3.20.

### 3.4. Cuspidal divisors and eta quotients

In this subsection we will see how to construct the divisors \( Z(d) \) appearing in Theorem 3.2 and how to compute the order of a given divisor \([D] \in C(N)\). Hence, this subsection aims to explain how to approach the problem of describing the group \( C(N) \).

Recall that by the order of a divisor \( D \) on \( X_0(N) \) we mean the smallest \( n \in \mathbb{Z}_{>0} \) such that there exists a modular function \( f \) in \( \mathcal{C}(X_0(N)) \) with

\[
\text{div}(f) = n \cdot D.
\]

Notice that being able to describe the functions \( f \) is central to our topic not only for computing the order of a given divisor \( D \) but also for finding relations between generators: a set of divisors is linearly dependent in \( J_0(N) \) if we can find a linear combination of them such that the resulting divisor is the divisor of a modular function \( f \in k(X_0(N)) \).
If $D$ is a degree-zero cuspidal divisor, then the function $f$ in Equation (12) does not have any zeroes or poles on $\mathcal{H}$ and has the same order of vanishing at all cusps of the same level. We can construct this kind of functions using Dedekind’s eta function.

**Definition 3.6.** Let $q = e^{2\pi i z}$ with $z$ a variable on the complex upper-half plane $\mathcal{H}$. Dedekind’s eta function $\eta : \mathcal{H} \to \mathbb{C}$ is defined by

$$\eta(z) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n).$$

It is well known that $\Delta(z) := \eta(z)^{24}$ is a modular form of weight 12 for $SL_2(\mathbb{Z})$. Furthermore, $\eta(z)$ is a holomorphic function on $\mathcal{H} \cup \mathbb{P}^1(\mathbb{Q})$ and $\eta(z) \neq 0$ for all $z \in \mathcal{H}$. Hence, Dedekind’s eta function can only have potential zeroes at the cusps of $X_0(N)$. For a divisor $d'$ of $N$ we can define the functions $\eta_{d'}(z) := \eta(d'z)$ and $\Delta_{d'}(z) := \Delta(d'z)$, and the following result holds.

**Lemma 3.7.** [32, Lemma 3.2] Let $N$ be a positive integer and $d'$ a divisor of $N$. The function $\Delta_{d'}(z)$ is a modular form of weight 12 for $\Gamma_0(N)$. Moreover, if $\omega \in \text{cusps}(X_0(N))$ is a cusp of level $d$, the order of vanishing of $\Delta_{d'}(z)$ at $\omega$ is

$$a_N(d, d') := \frac{N}{(d, N/d)} \times \frac{(d, d')^2}{dd'}.$$

Hence, we can use the functions $\eta_{d'}(z)$ and $\Delta_{d'}(z)$ to construct modular functions on $X_0(N)$ with prescribed poles and zeroes with specific orders at cusps. This construction uses the so-called eta quotients:

**Definition 3.8.** Let $N$ be a positive integer. For each divisor $d'$ of $N$ we choose an integer $r_{d'} \in \mathbb{Z}$. A function $g_r : \mathcal{H} \to \mathbb{C}$ of the form

$$g_r = \prod_{d' \mid N} \eta_{d'}(z)^{r_{d'}}$$

is called an **eta quotient of level $N$**. If we allow the $r_{d'}$ to be an element in $\mathbb{Q}$, then the function resulting from this construction is called a **generalised eta quotient of level $N$**. These generalised eta quotients are considered as power series with rational coefficients. For a fixed $N$, we denote the set of generalised eta quotients of level $N$ by $E(N)$.

Similarly to the space $S(N)$, which we use to encode cuspidal divisors, we will define spaces to encode information about eta quotients. For this, let $S'(N)$ a copy of the $\mathbb{Z}$-lattice $S(N)$ and $S'(N)_\mathbb{Q}$ be a copy of the $\mathbb{Q}$-vector space $S(N)_\mathbb{Q}$. We use the element $r = \sum_{d' \mid N} r_{d'} \cdot e(N)_{d'} \in S'(N)_\mathbb{Q}$ to denote the eta quotient $g_r = \prod_{d' \mid N} \eta_{d'}(z)^{r_{d'}}$.

With this notation and using Definition 3.8 and Lemma 3.7, we can determine when a generalised eta quotient is a modular function on $X_0(N)$ and control the divisor of this function. This is made explicit through the following two results.

**Proposition 3.9.** [9, Section 3.2] Let $g_r = \prod_{d' \mid N} \eta_{d'}(z)^{r_{d'}}$ be a generalised eta quotient with $r = \sum_{d' \mid N} (r_{d'} \cdot e(N)_{d'}) \in S'(N)_\mathbb{Q}$. Then $g_r$ is a modular function on $X_0(N)$ if and only if all the following conditions hold:

1. for all $d'$ we have $r_{d'} \in \mathbb{Z}$, i.e., $r \in S'(N)$;
(2) we have $\sum_{d \mid N} r_{d'} \cdot d' \equiv 0 \pmod{24}$;
(3) we have $\sum_{d' \mid N} r_{d'} \cdot N / d' \equiv 0 \pmod{24}$;
(4) the sum of the coefficients satisfies $\sum_{d' \mid N} r_{d'} = 0$;
(5) we have $\prod_{d' \mid N} (d')^{r_{d'}} \in \mathbb{Q}^2$.

**Proposition 3.10.** [9, Proposition 3.2.8] Let $r = \sum_{d' \mid N} r_{d'} \cdot e(N)_{d'} \in S'(N)_{\mathbb{Q}}$ and consider the function $g_r(z)$ on the upper-half plane. With notation as in Equation (13), it holds that

$$\text{div}(g_r) = \sum_{d \mid N} \left( \sum_{d' \mid N} \frac{a_N(d, d')}{24} \cdot r_{d'} \right) \cdot P_d.$$  

**Remark 3.11.** Notice from Propositions 3.9 and 3.10 that we can construct a linear map $\Lambda(N)$ from $S(N)_{\mathbb{Q}}'$ to $S(N)_{\mathbb{Q}}$ such that under this map, the image of a vector $r \in S(N)_{\mathbb{Q}}'$ with $g_r(z)$ a modular function on $X_0(N)$ gives the element in $S^0(N)$ that encodes the divisor of the function $g_r$. It follows that the representation of this map as a matrix is

$$\Lambda(N) := \left( \frac{a_N(d, d')}{24} \right)_{d, d' \mid N} \in M_{\sigma_0(N)}(\mathbb{Q}),$$

a square matrix indexed by the divisors of $N$, and

$$\text{div}(g_r) = \Phi_N^{-1}(\Lambda(N) \times (r)),$$

where $\Phi_N$ is the map described in Equation (9). Furthermore, the matrix $\Lambda(N)$ is invertible (over $\mathbb{Q}$) – see [9, Lemma 3.2.9] or [32, Lemma 3.7]. Hence we see that, given any divisor $D \in C(N)$, the generalised eta quotient $g(z)$ corresponding to the vector $r(D) := \Lambda(N)^{-1} \times \Phi_N(D)$ satisfies $\text{div}(g) = D$. The criteria in Proposition 3.9 tell us when this function is a function on $X_0(N)$, and hence, when $D$ is a trivial element in $J_0(N)$.

The following diagram summarises the information collected so far in the results of this subsection:

$$\begin{array}{ccccccc}
\text{Div}^0_{\text{Cusp}}(X_0(N))_{\mathbb{Q}} & \xrightarrow{\Phi_N} & S^0(N) & \xrightarrow{\Lambda(N)^{-1}} & S(N)_{\mathbb{Q}}' & \xrightarrow{g} & E(N) \\
D = \sum a_d P_d & \overset{\text{a}}{\longrightarrow} & \sum a_d e(N)_{d} & \overset{\text{b}}{\longrightarrow} & r(D) = \sum r_{d'} e(N)_{d'} & \overset{\text{c}}{\longleftarrow} & g(r(D))
\end{array}$$

(14)

Here, $g(r(D)) := g_r(D)(z)$.

Next, we give two definitions that, together with the information collected Diagram (14), are used to compute the order of a given divisor $D \in A_N(\mathbb{Q})$ in $J_0(N)$.

**Definition 3.12.** Let $D \in A_N(\mathbb{Q})$ and $k(N)$ as in Definition 2.3. We define the vector $V(D)$ in $S'(N)$ as $V(D) := \frac{k(N)}{24} \cdot r(D)$. In this notation, we define the greatest common divisor of the divisor $D$, denoted $\text{GCD}(D)$, as the greatest common divisor of the entries $\frac{k(N)}{24} r_{d'}$ of the vector $V(D)$. Finally, we define the vector $V(D) = (V(D)_{d'})_{d' \mid N}$ in $S'(N)$ as $V(D) := \text{GCD}(D)^{-1} \cdot V(D)$.
**Definition 3.13.** Let $D \in A_N(\mathbb{Q})$. For each prime number $l$, we define

$$P_w(l) := \sum_{\text{ord}_{l}(d') \neq 2\mathbb{Z}} \mathbb{V}(D)_{d'}.$$ 

Moreover, we let

$$h(D) := \begin{cases} 
1 & \text{if } P_w(l) \in 2\mathbb{Z} \text{ for all primes } l \\
2 & \text{if } P_w(l) \notin 2\mathbb{Z} \text{ for some prime } l.
\end{cases}$$

Next, we introduce a well-known result based on Remark 3.11.

**Theorem 3.14.** cf. [32, Proposition 3.10, Theorem 3.13] Let $D$ be a rational cuspidal divisor on the modular curve $X_0(N)$. The order of $D$ is the smallest positive integer $n$ such that $g(n \cdot r(D))$ is a modular function on $X_0(N)$ and we have

$$\text{order}(D) = \text{num} \left( \frac{k(N) \cdot h(D)}{24 \cdot \text{GCD}(D)} \right).$$

From this theorem we see that expressing the divisors $D \in A_N(\mathbb{Q})$ through their associated generalised eta quotient is very useful for describing the group $C(N)$. The full description of the proof of the computation of the order of the divisor $D$ can be found in [32, Section 3]; in the same section, the reader can find a criterion for linear independence of divisor classes that is used later to prove the results written in Equations (6) and (7). Theorem 3.14 leads to the following theorem, which will be used in Proposition 3.26 to compute the eta quotients corresponding to the divisors $Z(d)$ from Theorem 3.2.

**Theorem 3.15.** cf. [32, Theorem 3.15] Let $N$ be a positive integer. Let $D \in A_N(\mathbb{Q})$. If there exist two positive integers $N_1, N_2 \geq 2$ such that $N = N_1N_2$, $(N_1, N_2) = 1$ and $\Phi_{N}(D) = V_1 \otimes Z V_2$ with $V_i \in S(N_i)$, we say that $D$ is defined by tensors. In this case,

$$V(D) = V(D_1) \otimes Z V(D_2) \text{ and } \mathbb{V}(D) = \mathbb{V}(D_1) \otimes Z \mathbb{V}(D_2),$$

where $D_i = \Phi_{N_i}^{-1}(V_i)$.

### 3.4.1. The divisors $Z(d)$

Recall from Theorem 3.2 that we can describe $C(N)$, the rational divisor class group of $X_0(N)$, by finding divisors $Z(d)$ such that $C(N) \cong \{ [Z(d)] : d \in D_N^{sf} \otimes \bigoplus_{d \in D_N^{ad}} ([Z(d)]) \}$. In this subsection we give the explicit construction of the divisors $Z(d)$ in the case $N = p^r q^s$; the general case can be found in [32].

Let $p$ and $q$ be two odd primes. We aim to determine the divisors $Z(d)$ generating $C(p^r q^s)$. We will do this through first constructing vectors in $S^0(p^r)$ for any odd prime $p$, whose preimages under $\Phi$ correspond to the divisors $Z(d)$ generating $C(p^r)$, and later giving vectors encoding the divisors $Z(d)$ for $N = p^r q^s$ by means of the equality $S^0(p^r q^s) = S^0(p^r) \otimes S^0(q^s)$. The following vectors in $S(p^r)$ will be useful in these constructions, cf. [32, Proposition 6.14].

**Definition 3.16.** Let $p$ be an odd prime. For any $r \geq 1$ and any $0 \leq f \leq r$, we define $r + 1$ vectors $A_p(r, f)$ in $S(p^r)$ as follows:
A_p(r,0)_p^k = \begin{cases} 
1 & \text{if } k = 0, \\
0 & \text{otherwise};
\end{cases}

A_p(r,1)_p^k = p^{\max(r-2k,0)};

\text{if } r \geq 3 \text{ is odd, } A_p(r,2)_p^k = \begin{cases} 
K_p \left( \frac{r-3}{2} \right) & \text{if } k = 0, \\
K_p \left( \frac{r-1-2k}{2} \right) & \text{if } 0 < k \leq \frac{r-1}{2}, \\
0 & \text{if } k = \frac{r+1}{2}, \\
-p \cdot K_p \left( \frac{2k-r-3}{2} \right) & \text{otherwise};
\end{cases}

\text{if } r \geq 2 \text{ is even, } A_p(r,2)_p^k = \begin{cases} 
K_p \left( \frac{r-2-2k}{2} \right) & \text{if } 0 \leq k < \frac{r}{2}, \\
0 & \text{if } k = \frac{r}{2}, \\
-K_p \left( \frac{2k-r-2}{2} \right) & \text{otherwise};
\end{cases}

\text{where } K_p(j) = \sum_{i=0}^{j} p^{2i};

\text{if } 3 \leq f = r - 2j \leq r, \quad A_p(r,f)_p^k = \begin{cases} 
p^j & \text{if } k = 0, \\
-1 & \text{if } r - a \leq k \leq r, \\
0 & \text{otherwise};
\end{cases}

\text{if } 3 \leq f = r - 2j + 1 \leq r, \quad A_p(r,f)_p^k = \begin{cases} 
1 & \text{if } 0 \leq k \leq j, \\
-p^j & \text{if } k = r, \\
0 & \text{otherwise};
\end{cases}

In the following definition, we describe the vectors in \(S(p^r)^0\) which correspond to the divisors generating \(C(N)\) when \(N = p^r\) for any odd prime \(p\), cf. [32, Theorem 1.6].

**Definition 3.17.** For any \(r \geq 1\) and \(0 \leq f \leq r\) we further define \(r\) vectors \(B_p(r,f)\) in \(S(p^r)^0\), as follows:

\[ B_p(r,1) = p^{r-1}(p+1)A_p(r,0) - A_p(r,1); \]

otherwise

\[ B_p(r,f) = A_p(r,f). \]

Now we use the vectors in Definition 3.16 together with Remark 3.4 to produce vectors in \(S(p^r q^j)^0\). These will be the vectors encoding the divisors that generate \(C(p^r q^j)\).
Definition 3.18. For \( N = p^rq^s \), we take the ordering \( p < q \) on the set \( \{ p, q \} \) fixed in Definition 2.6. Given a divisor \( d = p^aq^b \in D_N \), we define the vectors

\[
Z(d) := \begin{cases} 
A_p(r,a) \otimes A_q(s,b) & \text{if } d \in D_N^{\text{nsf}}, \\
B_p(r,1) \otimes A_q(s,1) & \text{if } d = pq, \\
A_p(r,0) \otimes B_q(s,1) & \text{if } d = q, \\
B_p(r,1) \otimes A_q(s,0) & \text{if } d = p.
\end{cases}
\]

Notice that the definition of the vector \( Z(pq) \) depends on the ordering of the primes \( p \) and \( q \). However, as we will see in Remark 3.21, this is not a problem for our purposes: we can fix any ordering to define \( Z(pq) \).

We now give an example of these constructions.

Example 3.19. Let \( p = 5 \), \( q = 7 \), \( r = 1 \) and \( s = 2 \). For \( N = 5 \cdot 7^2 \) and the ordering \( 5 < 7 \), the vectors \( Z(d) \) defined in Definition 3.18 are

\[
\begin{align*}
\cdot Z(5) &= (1,1,-1_5) \otimes (1,0,0,0_7^2) = (1, -1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0); \\
\cdot Z(7) &= (1,1,0,0) \otimes (7,1,1,1,1,1,1) = (7, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0); \\
\cdot Z(5 \cdot 7) &= (1,1,-1_5) \otimes (7^2,1,1,1,1,1,1) = (7^2, -1, 1, 1, 1, 1, 1); \\
\cdot Z(7^2) &= (1,1,0,0) \otimes (1,0,0, -1_7) = (1, 0, 0, 0, 0, -1, 0, 0, 0, 0); \\
\cdot Z(5 \cdot 7^2) &= (5,1,1) \otimes (1,0,0, -1_7) = (5, 1, 0, 0, 0, -1, 0, 0, 0, 0).
\end{align*}
\]

Notice that with the ordering \( 5 < 5 \) we would have

\[
Z(5 \cdot 7) = (5,1,1) \otimes (7,1,1,1,1,1,1) = (5 \cdot 7, 1, 7, 5, 7, 7, 5, 7, 5, 7, 7, 5).
\]

Definition 3.20. With notation as in Definition 3.18, for any \( d \in D_N \) we define the divisor

\[
Z(d) := \Phi^{-1}_N(Z(d)) \in \text{Div}^0_{\text{cusp}}(X_0(N))(\mathbb{Q}).
\]

Furthermore, we denote the order of the divisor \( Z(d) \) by \( n_d \).

The proof of Theorem 3.2 relies on the definition of the divisors \( Z(d) \). In [32], the author proves that these divisors generate \( C(N) \) and he develops criteria to determine if a set of divisors in \( C(N) \) is linear independent and uses these to conclude that \( \langle \{ Z(d) : d \in D_N \} \rangle = \langle \{ Z(d) : d \in D_N^{\text{nsf}} \} \otimes \bigoplus_{d \in D_N^{\text{nsf}}} \langle \{ Z(d) \} \rangle \). Furthermore, using Theorem 3.14 we can compute the order of each \( Z(d) \). Theorem 3.2 is one of the key results for the proof of Theorem A.

Remark 3.21. As mentioned in Definition 3.18, the divisor \( Z(pq) \) is the only divisor whose definition depends on the ordering of \( p \) and \( q \). This is not a problem, however: we are interested in studying the space

\[
\langle \{ Z(d) : d \in D_N^{\text{nsf}} \} \rangle \otimes \bigoplus_{d \in D_N^{\text{nsf}}} \langle \{ Z(d) \} \rangle
\]

that these divisors generate. One can see that, considering \( Z'(pq) := \Phi^{-1}_N(A_p(r,1) \otimes B_q(s,1)) \), the other potential definition for \( Z(pq) \), we have

\[
\langle Z(p), Z(q), Z(pq) \rangle = \langle Z(p), Z(q), Z'(pq) \rangle,
\]

as we have \( Z'(pq) = p^{-1}(p + 1)Z(q) - (q^{-1}(q + 1)Z(p) - Z(pq)) \) from Definitions 3.16 and 3.18.
Next, we give vectors in $S'(p^r)$ that are going to be useful to compute $r(Z(d))$ in Proposition 3.26.

**Definition 3.22.** [32, Definition 6.15] Let $p$ be an odd prime number and $r$ a positive integer. For any $0 \leq f \leq r$ we define

$$
\mathcal{A}_p(r, f) := \begin{cases} 
(p_1, -1_p, 0, \cdots, 0) & \text{if } f = 0, \\
(1_1, 0, \cdots, 0) & \text{if } f = 1, \\
(1_1, 0, \cdots, 0, -1_{p^r}) & \text{if } f = 2 \text{ and } r \in 2\mathbb{Z}, \\
(0_1, 1_2, 0, \cdots, 0, -1_{p^r}) & \text{if } f = 2 \text{ and } r \notin 2\mathbb{Z}, \\
(p_1, -1_p, 0, \cdots, 0, 1_{p^{r-j} - 1}, -p_{p^{r-j}}, 0, \cdots, 0) & \text{if } 3 \leq f = r - 2j \leq r, \\
(0, \cdots, 0, p_{p^j}, -1_{p^{j+1}}, 0, \cdots, 0, 1_{p^{r-j-1}}, -p^j) & \text{if } 3 \leq f = r - 2j + 1 \leq r - 1.
\end{cases}
$$

Further, for $1 \leq f \leq r$ we define

$$
\mathcal{B}_p(r, f) := \begin{cases} 
(1_1, -1_p, 0, \cdots, 0) & \text{if } f = 1, \\
\mathcal{A}_p(r, f) & \text{otherwise}.
\end{cases}
$$

**Definition 3.23.** Let $p$ be an odd prime number and $r$ a positive integer. We define the linear map $\Upsilon(p^r) : S^{0}(N) \to S(N)_{Q}^{r}$ by

$$
\Upsilon(p^r) := \frac{k(p^r)}{24} \Lambda^{-1}(p^r).
$$

We use the same notation, $\Upsilon(p^r)$, for the representation of this linear map as a matrix in $\text{SL}_2(\mathbb{Z})$.

**Lemma 3.24.** [32, Lemma 6.16] Let $p$ be an odd prime number and $r$ a positive integer. For any $0 \leq f \leq r$, we have

$$
\Upsilon(p^r) \times \mathcal{A}_p(r, f) = g_p(r, f) \cdot \mathcal{A}_p(r, f);
$$

and for any $1 \leq f \leq r$, we get

$$
\Upsilon(p^r) \times \mathcal{B}_p(r, f) = \begin{cases} 
p^{r-1}(p + 1) \cdot \mathcal{B}_p(r, f) & \text{if } f = 1, \\
g_p(r, f) \cdot \mathcal{B}_p(r, f) & \text{otherwise},
\end{cases}
$$

where

$$
g_p(r, f) := \begin{cases} 
1 & \text{if } f = 0, \\
p^{r-1}(p^2 - 1) & \text{if } f = 1, \\
p^{r-1} & \text{if } f = 2, \\
p^j & \text{if } 3 \leq f \leq r \text{ and } j = \left\lfloor \frac{r+1}{2} \right\rfloor.
\end{cases}
$$

**Definition 3.25.** Let $N = p^r q^s$ be a positive odd integer. For any divisor $d \in D_N$ of $N$, let $Z(d)$ be the divisor described in Definition 3.20, whose order is $n_d = n(N, d)$. We define the eta quotient $h_d(z)$ by

$$
h_d(z) := g(r(n(N, d) \cdot Z(d))).
$$

By Theorem 3.14, this is a modular function on $X_0(N)$. 
The following result explicitly describes \( h_d(z) \) for all \( d \in D_N \). The importance of these computations will become clear in Lemma 4.1; in particular, we will need to compute the leading Fourier coefficients of these eta quotients, which is done in Proposition 5.6.

**Proposition 3.26.** Let \( N = p^q r^s \) be a positive odd integer. For any \( d \in D_N \) divisor of \( N \), let \( h_d(z) \) be the modular function on \( X_0(N) \) described in Definition 3.25. The eta quotient \( h_d(z) \) is

\[
\begin{align*}
\frac{\eta(1z)^{s}\eta(pqz)}{\eta(qz)^{s}\eta(p\eta z)}, & \quad \text{for } d = q; \\
\frac{\eta(1z)^{s}\eta(pqz)}{\eta(qz)^{s}\eta(q\eta z)}, & \quad \text{for } d = p; \\
\left( \frac{h(1z)}{h(qz)} \right)^{B(p)}, & \quad \text{for } d = pq, \text{ where } B(p) \text{ is as in Definition 2.2}; \\
\begin{cases}
\frac{\eta(qz)^{s}\eta(pqz)}{\eta(qz)^{s}\eta(p\eta z)} & \text{if } s \text{ odd}, \\
\frac{\eta(1z)^{s}\eta(pqz)}{\eta(qz)^{s}\eta(p\eta z)} & \text{if } s \text{ even},
\end{cases} & \quad \text{for } d = q^2; \\
\frac{\eta(1z)^{s}\eta(q^{r-1}z)^{r}\eta(pq^{r-1}z)^{s}\eta(pqz)}{\eta(qz)^{s}\eta(q^{r-1}z)^{r}\eta(pqz)^{s}\eta(q^{r-1}z)}, & \quad \text{for } d = q^b \text{ with } 3 \leq b = s - 2j \leq s; \\
\frac{\eta(qz)^{s}\eta(q^{r-1}z)^{r}\eta(pq^{r-1}z)^{s}\eta(pqz)}{\eta(qz)^{s}\eta(q^{r-1}z)^{r}\eta(pq^{r-1}z)^{s}\eta(pqz)}, & \quad \text{for } d = q^b \text{ with } 3 \leq b = s - 2j + 1 \leq s; \\
\begin{cases}
\left( \frac{\eta(qz)}{\eta(q^{r-1}z)} \right)^{A(q)} & \text{if } s \text{ odd}, \\
\left( \frac{\eta(1z)}{\eta(q^{r-1}z)} \right)^{A(q)} & \text{if } s \text{ even},
\end{cases} & \quad \text{for } d = pq^2, \text{ where } A(q) \text{ is as in Definition 2.2}; \\
\frac{\eta(1z)^{s}\eta(q^{r-1}z)^{r}\eta(pq^{r-1}z)^{s}\eta(qz)}{\eta(qz)^{s}\eta(q^{r-1}z)^{r}\eta(pqz)^{s}\eta(q^{r-1}z)}, & \quad \text{for } d = pq^b \text{ with } 3 \leq b = s - 2j \leq s; \\
\frac{\eta(qz)^{s}\eta(q^{r-1}z)^{r}\eta(pq^{r-1}z)^{s}\eta(qz)}{\eta(qz)^{s}\eta(q^{r-1}z)^{r}\eta(pq^{r-1}z)^{s}\eta(qz)}, & \quad \text{for } d = pq^b \text{ with } 3 \leq b = s - 2j + 1 \leq s; \\
\begin{cases}
\frac{\eta(pqz)^{s}\eta(pq^{r-1}z)^{r}\eta(qz)}{\eta(q^{r-1}z)^{r}\eta(pq^{r-1}z)^{s}\eta(qz)}, & \text{if } r, s \text{ odd}, \\
\frac{\eta(q^{r-1}z)^{r}\eta(pq^{r-1}z)^{s}\eta(qz)}{\eta(q^{r-1}z)^{r}\eta(pq^{r-1}z)^{s}\eta(qz)}, & \text{if } r, s \text{ even}, \\
\frac{\eta(1z)^{s}\eta(pq^{r-1}z)^{s}\eta(1z)}{\eta(qz)^{s}\eta(pq^{r-1}z)^{s}\eta(qz)}, & \text{if } r \text{ odd}, s \text{ even}, \\
\frac{\eta(qz)^{s}\eta(pq^{r-1}z)^{s}\eta(qz)}{\eta(qz)^{s}\eta(pq^{r-1}z)^{s}\eta(qz)}, & \text{if } r \text{ even}, s \text{ odd},
\end{cases} & \quad \text{for } d = p^2q^2.
\end{align*}
\]
\[
\begin{align*}
\begin{cases}
\left( \frac{n(p^2 q^2)}{n(p^q q^r)} \right)^p \frac{n(p^q q^r)}{n(p^q q^r)} & \text{if } r \text{ odd}, \\
\left( \frac{n(p^2 q^2)}{n(p^q q^r)} \right)^p \frac{n(p^{q+1} q^r)}{n(p^q q^r)} & \text{if } r \text{ even},
\end{cases}
\end{align*}
\]

for \( d = p^2 q^b \) with \( b = s - 2j \leq s \).

\[
\begin{align*}
\begin{cases}
\left( \frac{n(p^2 q^2)}{n(p^q q^r)} \right)^p \frac{n(p^q q^r)}{n(p^q q^r)} & \text{if } r \text{ odd}, \\
\left( \frac{n(p^2 q^2)}{n(p^q q^r)} \right)^p \frac{n(p^{q+1} q^r)}{n(p^q q^r)} & \text{if } r \text{ even},
\end{cases}
\end{align*}
\]

for \( d = p^2 q^b \) with \( b = s - 2j + 1 \leq s \).

\[
\begin{align*}
\begin{cases}
\left( \frac{n(p^2 q^2)}{n(p^q q^r)} \right)^p \frac{n(p^q q^r)}{n(p^q q^r)} & \text{if } r \text{ odd}, \\
\left( \frac{n(p^2 q^2)}{n(p^q q^r)} \right)^p \frac{n(p^{q+1} q^r)}{n(p^q q^r)} & \text{if } r \text{ even},
\end{cases}
\end{align*}
\]

Similarly, from the table here, we obtain \( h_4(z) \) for the divisors \( d \in \{ p^a, p^a q, p^a q^2 \} \) with \( a \geq 3 \text{ odd}, b \geq 3 \text{ even} \), by using Remark 3.4, we can compute the entries of the vectors \( \Lambda(p^r)^{-1}(Z(d')) \) using Definition 3.16 and Lemma 3.24, taking into account that \( \Upsilon(N) = \frac{k(N)}{24} \Lambda^{-1}(N) \). Then, recalling Remark 3.4, we use Theorem 3.15 and apply these computations together with Definition 3.20 to compute \( \Lambda^{-1}(Z(d')) \) for any \( d \in D_N \). \( \square \)

4. The generalised Jacobians \( J_q(N)_m \)

generalised Jacobians – as defined by Rosenlicht [21] and Serre [23] – provide a generalisation of the usual Jacobian suitable for curves with singularities at specific points – i.e., for ramified coverings. If \( C \) is a smooth projective curve and \( m = \sum_{P \in C} a_P P \in \text{Div}(C) \) is a modulus (that is, an effective divisor), we consider the curve \( C_m \) that results after the identification of all the points in \( m \). The generalised Jacobian \( J(C_m) \) is the group of invertible line bundles on the singular curve \( C_m \). It is an abelian variety which is the extension of the usual Jacobian variety \( J(C) \) by a commutative affine algebraic group.
Following the interest in the rational torsion of the modular Jacobian arising from the statements in the generalised Ogg’s conjecture – see Conjecture A – and Conjecture B, recent works by Yamazaki-Yang [29] and Wei-Yamazaki [28] consider similar questions for the generalised Jacobian of the curve $X_0(N)$ with modulus given by the sum of the cusps of the modular curve. In the present article, we are going to work with the same set-up and from now on we will fix $N = p^r q^s$, where $p$ and $q$ are two odd primes. Hence, recall that we take $C = X_0(N)$ and $m = \sum_{p \in \text{Cusps}(X_0(N))} P_p$. As in Section 3.2, for each $d’|N$ we define $P_{d’}$ as the divisor in $\text{Div}(X_0(N))$ given by the sum of all the level-$d’$ cusps of $X_0(N)/C$. Recall that the action of $\text{Gal}(\mathbb{Q}(\zeta_{M_{d’}})/\mathbb{Q})$ is transitive on the set of cusps of level $d’$, cf. [32, Theorem 2.17]. Hence, $\sigma(P_{d’}) = P_{d’}$ for all $\sigma \in \text{Gal}(\mathbb{Q}(\zeta_{M_{d’}})/\mathbb{Q})$ and so $P_{d’} \in J_0(N)(\mathbb{Q})$ is a rational point in $X_0(N)$. Moreover, we denote the residue field of the point $P_{d’}$ by $\mathbb{Q}(P_{d’})$. We will denote the generalised modular Jacobian with respect to the modulus $m$ by $J_0(N)_m$; it is the extension of $J_0(N)$ by a commutative affine algebraic group; namely, there is a short exact sequence

$$0 \to L_m \to J_0(N)_m \to J_0(N) \to 0,$$

where $L_m$ is given by the quotient

$$0 \to \mathbb{G}_m \to \prod_{d’|N} \text{Res}_{\mathbb{Q}(P_{d’})/\mathbb{Q}} \mathbb{G}_m \to L_m \to 0.$$

We can apply the functor $H^1(\mathbb{Q}, -)$ to the short exact sequence in (15) and get the following exact sequence:

$$0 \to \bigoplus_{d’|N, d’\neq N} \mathbb{Q}(P_{d’})^\times \to J_0(N)_m(\mathbb{Q}) \to J_0(N)(\mathbb{Q}) \to 0. \tag{16}$$

Applying the left-exact functor $\text{Hom}(\mathbb{Q}/\mathbb{Z}, -)$ to the short exact sequence in Equation (16), we obtain the exact sequence

$$0 \to \bigoplus_{d’|N, d’\neq N} (\mathbb{Q}(P_{d’})^\times)_{\text{tor}} \to J_0(N)_m(\mathbb{Q})_{\text{tor}} \to J_0(N)(\mathbb{Q})_{\text{tor}} \xrightarrow{\delta} \bigoplus_{d’|N, d’\neq N} (\mathbb{Q}(P_{d’})^\times \otimes \mathbb{Q}/\mathbb{Z}, \tag{17}$$

where the connecting map $\delta$ is a group homomorphism. In their papers, Yamazaki-Yang and Wei-Yamazaki determine the structure of $J_0(N)_m(\mathbb{Q})_{\text{tor}}$ for $N$ a power of a prime and squarefree, respectively, by using known cases of the generalised Ogg’s Conjecture and computing the kernel of the group homomorphism $\delta$ when restricted to $C_N(\mathbb{Q})$

The key lemma in both papers – and also central to our paper – is Lemma 2.3.1 in [29], which we present here.

**Lemma 4.1.** [29, Lemma 2.3.1] Let $D = \sum_{d’|N} a_{d’} \cdot P_{d’} \in \text{Div}^0(X_0(N))$ be a degree zero divisor supported on $m$ such that $[D] \in J_0(N)(\mathbb{Q})_{\text{tor}}$. Let $n \in \mathbb{Z}_{>0}$ so that $n \cdot [D] = 0$, i.e., so that there exists $f \in \mathbb{Q}(X_0(N))$ such that $\text{div}(f) = n \cdot D$. We have

$$\delta([D]) = \left( \frac{f}{a_{d’}} \right)_n (P_N)^{t_{P_{d’}}}_{P_N} \otimes \mathbb{Q}(P_{d’})^\times \otimes \mathbb{Q}/\mathbb{Z},$$

where $t_{P_{d’}}$ is a uniformizer of $D$ at a cusp of level $d’$. Notice that this description does not depend on the choice of $f$ nor on that of $t_{P_{d’}}$. 


Our strategy follows that of Yamazaki-Yang and Wei-Yamazaki: in the case \( N = p^r q^s \), we compute the \( l \)-primary part of the kernel of \( \delta \) for \( l \) any odd number such that \( l^2 \nmid 3N \). The main tool – and reason – for this are the results in Theorem 3.5; namely, given a positive integer \( N \) we have

\[
C(N)[l^\infty] = J_0(N)(\mathbb{Q})_{\text{tor}}[l^\infty]
\]

for any odd prime \( l \) such that \( l^2 \) does not divide \( 3N \). This is one of the proven cases of the generalised Ogg’s conjecture and it allows us to use the divisors constructed in the previous section for our purposes.

We know from Theorem 3.2 that

\[
C(N) \simeq \langle [Z(d)] : d \in D_N^{\text{sf}} \rangle \oplus \bigoplus_{d \in D_N^{\text{nsf}}} \langle [Z(d)] \rangle.
\]

Therefore, the image of \( C(N) \) under the homomorphism \( \delta \) satisfies

\[(18) \quad \text{im}(\delta) \simeq \langle \delta([Z(d)]) : d \in D_N \rangle.\]

Notice from the description obtained in Lemma 4.1 that to compute \( \delta([Z(d)]) \) for each \( d \in D_N \), it is sufficient to compute \( \left( \frac{h_d}{m_{d'}} \right)(P_{d'}) \) for each \( d' \) dividing \( N \), i.e., the leading Fourier coefficient of the modular function \( h_d(z) \) at \( P_{d'} \). After base change to \( \mathbb{C} \), these Fourier coefficients can be computed, as they are given by the Fourier coefficients of the functions at the \( \mathbb{C} \)-valuated points of each \( P_{d'} \). However, since cusps of the same level are Galois conjugate, it will be sufficient to pick a cusp \( Q_{d'} \) on \( X_0(N) \) for each level \( d' \) and compute the leading Fourier coefficient of \( h_d(z) \) at each \( Q_{d'} \); this will be carried out in Section 5, and more precisely in Proposition 5.8. Furthermore, since \( \delta \) is, a homomorphism of abelian groups, from Theorem 3.5 we have

\[(19) \quad \ker(\delta|_{J_0(N)(\mathbb{Q})_{\text{tor}}[l^\infty]}) = \ker(\delta|_{C(N)[l^\infty]}) = \ker(\delta|_{C(N)})[l^\infty]\]

for \( l \) an odd prime such that \( l^2 \nmid 3N \) – that is \( l \neq 2 \) and possibly \( l \neq p, q \). This means that we can conduct our computations by first looking into \( \ker(\delta|_{C(N)}) \) using Equation (18), and then look into its \( l \)-primary part for any \( l \). As a result of this process, we obtain the main result of this paper.

**Theorem 4.2.** Let \( N = p^r q^s \) where \( p \) and \( q \) are two distinct odd prime numbers (not 3). For any odd prime \( l \) with \( l^2 \) not dividing \( 3N \) we have

\[
J_0(N)(\mathbb{Q})_{\text{tor}}[l^\infty] \simeq \bigoplus_{1 \leq a \leq r, \ 1 \leq b \leq s} \mathbb{Z}/l^{\text{ord}_l(m(a, b))}\mathbb{Z}.
\]
where

$$m(a, b) = \begin{cases} \text{num} \left( \frac{x^{r-1} - y^{r-1}}{24} \right) & \text{if } a, b \geq 2; \\ \text{num} \left( \frac{x^{r-1} - y^{r-1}}{24} \right) & \text{if } a = 1, b > 2; \\ \text{num} \left( \frac{x^{r-1} - y^{r-1}}{24} \right) & \text{if } a > 2, b = 1; \\ \text{num} \left( \frac{x^{r-1} - y^{r-1}}{24} \right) & \text{if } a = 1, b = 2; \\ \text{num} \left( \frac{x^{r-1} - y^{r-1}}{24} \right) & \text{if } a = 2, b = 1; \\ \text{num} \left( \frac{x^{r-1} - y^{r-1}}{24} \right) & \text{if } a = b = 1; \\ \end{cases}$$

where $i$ and $j$ are given by $i = \left\lceil \frac{r+1-a}{2} \right\rceil$ and $j = \left\lfloor \frac{a+1-b}{2} \right\rfloor$.

Proof. In the following sections we prove this theorem in parts. More precisely, in Section 5 we describe the image of $C(N)$ under the homomorphism $\delta$ using the divisors $Z(d)$ and Theorem 3.2 – see Proposition 5.8. Next, in Section 6 we describe the kernel of $\delta|_{C(N)}$, again using some of the results in Section 5 – see Theorem 6.4. Using Theorem 3.5 we can identify $\ker(\delta|_{J_0(N)(\mathbb{Q})_{\text{tor}}[l]} = \ker(\delta|_{C(N)}[l])$; therefore, the proof of the theorem follows from the description of $\ker(\delta|_{N}(\mathbb{Q})_{\text{tor}}[l])$ in Theorem 6.4 and the description of $J_0(N)|_{\text{tor}}$ given in the exact sequence in Equation (17).

In [29, Proposition 1.3.2], Yamazaki–Yang show that whenever $p$ and $q$ are both congruent to 1 modulo 12, then $J_0(pq)|_{\text{tor}} = Z/(p-1)(q-1)Z$; and Wei-Yamazaki in [28, Theorem 1.2.3] extend their results by showing that for any odd $p$ and $q$ we have $J_0(pq)|_{\text{tor}} = Z/(p-1)(q-1)Z$ up to 2-primary and 3-primary torsion. The results in Theorem 4.2 coincide with both computations when $N = pq$ and give an explicit description of $J_0(p^r q^s)|_{\text{tor}}[l^\infty]$ for $l$ not equal to $p$ nor $q$.

The results in Theorem 4.2 show that – away from 2-primary, $p$-primary and $q$-primary torsion – the size of $J_0(p^r q^s)|_{\text{tor}}[l^\infty]$ increases (linearly) with $r$ and $s$. This results contrasts with the case of $N = p^r$ computed in [29, Theorem 1.1.3], where one has that the rational torsion of $J_0(p^r)|_{\text{tor}}$ is trivial up to 2-torsion and $p$-torsion for any $r$. Hence, the case $N = p^r q^s$ resembles more the situation of squarefree level computed in [28, Theorem 1.2.3], where for $N = \prod_{i=1}^{s} p_i$ the size of $J_0(N)|_{\text{tor}}$ – up to 2-primary torsion and 3-primary torsion – increases (exponentially) with $s$.

Moreover, it follows from Theorem 4.2 that, for $l \neq 2, p, q$, the size of $J_0(p^r q^s)|_{\text{tor}}[l^\infty]$ also increases linearly with $p$ and $q$, just like observed in [28], where the size of $J_0(N)|_{\text{tor}}[l^\infty]$ for $N$ squarefree increases linearly with each $p_i$. This is, again, different from the situation in the prime-power case, where we have that $J_0(p^r)|_{\text{tor}}$ is trivial up to 2-torsion and $p$-torsion for any $p$ [29, Theorem 1.1.3].

The changes observed in the torsion of the generalised Jacobian diverge from the study of the usual Jacobian, where the behaviour of $J_0(N)|_{\text{tor}}$ is more even throughout the different types of level. The size of $J_0(N)|_{\text{tor}}[l^\infty]$ for $N = \prod_{i=1}^{s} p_i$ squarefree and $l \neq 2, 3$, increases linearly with each prime $p_i$ and (exponentially) with $s$ – see [15]; for $N = p^r$ and $l \neq 2, p$, it also increases linearly with $p$ and $r$ – see [11] or [32] –; and for $N = p^r q^s$ and $l \neq 2, p, q$, it increases linearly with the $p$, $q$, $r$ and $s$ – see [33, Theorem 1.7].
Furthermore, the results in [29, 28] and Theorem 4.2 point towards the idea that the size of the rational torsion of $J_0(N)$ is always bigger than that of $J_0(N)_m$. For squarefree level $N = \prod_{i=1}^{s} P_i$, up to 2-primary torsion and 3-primary torsion, $J_0(N)(\mathbb{Q})_{\text{tor}}$ is given by the product of $2^i - 1$ cyclic subgroups, while of $J_0(N)_m(\mathbb{Q})_{\text{tor}}$ is isomorphic to the product of $2^i - (s + 1)$ of those cyclic subgroups. For prime-power levels $N = p^r$, up to 2-primary torsion and $p$-primary torsion, $J_0(p^r)(\mathbb{Q})_{\text{tor}}$ is given by the product of $2r - 1$ cyclic groups, while the rational torsion of $J_0(N)_m$ is trivial up to 2-primary torsion and $p$-primary torsion. For odd levels $N = p^r q^s$, $J_0(N)(\mathbb{Q})_{\text{tor}}[\ell^\infty]$ is the product of $(r + 1) \cdot (s + 1)$ cyclic groups – see [32, Theorem 1.7] –, while $J_0(N)_m(\mathbb{Q})_{\text{tor}}[\ell^\infty]$ is the product of $r \cdot s$ subgroups of those cyclic groups.

5. IMAGE OF THE MAP $\delta$

In this section we compute the image of $C(N)$ under the connecting map $\delta$. In order to do so, we first generalise [29, Proposition 6.1.1] in Lemma 5.5 and give the main result in Proposition 5.8. Recall from Theorem 3.2 that $C(N) = \{ [Z(d)] : d \in D_N^0 \oplus \bigoplus_{d \in D_N^0} \{ [Z(d)] \} \}$, where the divisors $Z(d)$ are as described in Definition 3.20. Recall also that we fix $N = p^r q^s$, where $p$ and $q$ are two odd primes and $r, s \geq 1$.

Recall from previous sections that for each divisor $d'$ of $N$ we regard $P_{d'}$ as the divisor given by the sum of all the cusps of level $d'$ and that to compute the image of the homomorphism $\delta$ we need to compute the leading Fourier coefficient at each $P_{d'} \in X_0(N)(\mathbb{Q})$ of the eta quotient $h_{\lambda}(z)$ associated to the divisor $Z(d)$ for each $d \in D_N$. On the other hand, recall that the Fourier coefficients of a modular function $f(z)$ at $P_{d'}$ lie in the residue field $\mathbb{Q}(P_{d'})$. For a suitable choice of model of the modular curve the latter is equal to $\mathbb{Q}(Q_i)$, the field of definition of the cusp $Q_d$, under the geometric point $P_{d'}$, which is given by the cyclotomic extension $\mathbb{Q}(Q_{d'}) = \mathbb{Q}(\zeta_{M_{d'}})$ with $M_{d'} = (d', N/d')$. Moreover, cusps of the same level are Galois conjugates, hence, so are the Fourier coefficients at these cusps, so we only need pick one $Q_{d'} \in \text{Cusp}(X_0(N))(\mathbb{C})$ for each level $d'$ and compute the Fourier coefficients at these chosen points. In our case, for each divisor $d'$ of $N$ we choose the cusp $Q_{d'} = [\frac{-1}{d'}]$. To do so, first we need to choose a uniformizer at each cusp $Q_{d'}$. Namely, given a reduced fraction $\omega = \frac{-a}{c} \in \mathbb{Q}$, we can take a matrix $\gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{SL}_2(\mathbb{Z})$ such that $\gamma \cdot \omega = \omega$, and choose the minimal $h \in \mathbb{N}_{>0}$ such that the matrix $\left( \begin{array}{cc} 1 & h \\ 0 & 1 \end{array} \right)$ lies in $\gamma^{-1} \Gamma_0(N) \gamma$. With this notation, $e^{2\pi i y^{-1}z/h}$ is a uniformizer at the cusp $[\omega]$; see [3, p. 16] for a complete argument. With this choice of uniformizer, the Fourier expansion of a modular function $f(z)$ at the cusp $[\omega]$ is given by the expansion at infinity of $f(z) \gamma = f(\gamma z)$. This can be observed in the following result, which will be used in Proposition 5.6 to compute the Fourier coefficient of $h_{\lambda}(z)$ at $Q_{d'}$.

**Proposition 5.1.** cf. [7, Prop. 2.1] For $m \in \mathbb{Z}$, let $\eta_m(z) := \eta(mz)$ with $m \in \mathbb{Z}$ and let $\omega = \frac{-a}{c} \in \mathbb{Q}$ be a reduced fraction with $c \neq 0$. Let $a$ and $b$ be chosen such that $\gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{SL}_2(\mathbb{Z})$. The Fourier expansion of $\eta_m(z)$ at the cusp $[\omega]$ is given by the Fourier expansion at infinity of $\eta_m(\gamma^{-1}z)$, which is of the form

\begin{equation}
\nu_{\gamma}(\gamma, m) \sqrt{\frac{(c, m)}{m} (-cz + a)} \sum_{n=1}^{\infty} \left( \frac{n}{12} \right) e^{2\pi i \left( \frac{n}{2} \right)^2 (\gamma(c, m) z + u(\gamma, z)(c, m))},
\end{equation}

(20)
where

(a) \( v(\gamma, m) \) is an integer;
(b) \( \nu_{\eta}(\gamma, m) \) is some 24th root of unity;
(c) \( \left( \frac{n}{12} \right) \) denotes the Jacobi-Kronecker symbol.

**Definition 5.2.** Following [29], for a given odd prime \( p \) we define

\[ p^\ast := e^{2\pi i(p-1)/4} p \]

and

\[ \sqrt{p^\ast} := e^{2\pi i(p-1)/8} \sqrt{p}. \]

We denote by \( \mathcal{O}_{d'} \) the following subgroup of \( \mathbb{Q}(\zeta_{M_{d'}}) \):

\[ \mathcal{O}_{d'} := \begin{cases} \langle p, q \rangle & \text{if } d' = 1, \\ \langle \sqrt{p^\ast}, q \rangle & \text{if } d' = p^n \text{ and } 1 \leq n \leq r, \\ \langle p, \sqrt{q^\ast} \rangle & \text{if } d' = q^m \text{ and } 1 \leq m \leq s, \\ \langle \sqrt{p^\ast}, \sqrt{q^\ast} \rangle & \text{otherwise.} \end{cases} \]

**Remark 5.3.** Notice that the factor \((-cz + a)\) in the square root of Equation (20) does not depend on \( m \). On the other hand, if an eta quotient \( g(z) = \prod_{m \mid N} \eta(mz)^{r_m} \) defines a modular function on \( X_0(N) \), then in particular \( \sum_m r_m = 0 \). So if we use Proposition 5.1 to compute the leading coefficient of \( g(z) \) at a cusp, the factor \( \sqrt{(-cz + a)} \) will appear as \( \frac{\sqrt{(-cz + a)}}{\sum_m r_m} = 1 \). Hence, the leading coefficient of \( g(z) \) at the cusp \( \omega = \left[ \frac{-d}{c} \right] \) will be given by

\[ e^{2\pi i \kappa_\ell} \cdot \prod_{m \mid N} \left( \sqrt{\frac{c, m}{m}} \right)^{r_m}, \]

where \( e^{2\pi i \kappa_\ell} \) is the product of the \( \nu_{\eta}(\gamma, m) \), the factors \( e^{2\pi i(v(\gamma, m)/(c, m)/24m)} \), and \( \left( \frac{1}{12} \right)^{\sum_m r_m} = 1 \).

In the case \( N = p^r q^s \), the \( m \)'s appearing in the factors of the eta quotients \( h_d(z) \) are divisors of \( N \) – see Proposition 3.26 –, and the \( c \)'s are the denominators of the cusps \( Q_{d'} \), so \( \frac{c, m}{m} = \frac{1}{p^u q^v} \) for some \( 0 \leq a \leq r \) and \( 0 \leq b \leq s \). In particular, the leading Fourier coefficient of the function \( h_d(z) \) at \( Q_{d'} \) is in the subgroup \( \mathcal{O}_{d'} \) defined in Notation 5.2.

**Definition 5.4.** With the notation as in Definition 5.2, consider the subgroup \( \Omega_{d'} \) given by the image of \( \mathcal{O}_{d'} \) in the quotient \( \mathbb{Q}(Q_{d'})^\ast / \mu(\mathbb{Q}(Q_{d'})) \). We denote

\[ \Omega := \bigoplus_{d' \mid N, d' \neq N} \mathcal{O}_{d'} \subseteq \bigoplus_{d' \mid N, d' \neq N} \mathbb{Q}(Q_{d'})^\ast / \mu(\mathbb{Q}(Q_{d'})). \]

By Remark 5.3 and since all the torsion units \( \mu(\mathbb{Q}(Q_d)) \) map into the identity in \( \bigoplus_{d' \mid N, d' \neq N} \mathbb{Q}(Q_{d'}) \otimes \mathbb{Q}/\mathbb{Z} \), the map \( \delta \) factors as
Lemma 5.5. The map $\phi$ in Equation (21) is an injection.

Proof. The proof builds on that of [29, Proposition 6.1.1]. Let $p, q \geq 3$ be two distinct prime numbers and let $n, m \in \mathbb{Z}$. We consider the following maps on the tensor products of $\mathbb{Z}$–modules:

\begin{align*}
\phi_1 : \mathbb{Q}/\mathbb{Z} &\rightarrow \mathbb{Q}^x \otimes \mathbb{Q}/\mathbb{Z}, \quad x \mapsto p \otimes x; \\
\phi_2 : \mathbb{Q}/\mathbb{Z} &\rightarrow \mathbb{Q}^x \otimes \mathbb{Q}/\mathbb{Z}, \quad x \mapsto q \otimes x; \\
\phi_3 : \mathbb{Q}/\mathbb{Z} &\rightarrow \mathbb{Q}^x \otimes \mathbb{Q}/\mathbb{Z}, \quad x \mapsto pq \otimes x; \\
\phi_4 : \mathbb{Q}/\mathbb{Z} &\rightarrow \mathbb{Q}(\zeta_{p^nq^m})^x \otimes \mathbb{Q}/\mathbb{Z}, \quad x \mapsto \sqrt{p^x} \otimes x. \\
\phi_5 : \mathbb{Q}/\mathbb{Z} &\rightarrow \mathbb{Q}(\zeta_{p^nq^m})^x \otimes \mathbb{Q}/\mathbb{Z}, \quad x \mapsto \sqrt{q^x} \otimes x. \\
\phi_6 : \mathbb{Q}/\mathbb{Z} &\rightarrow \mathbb{Q}(\zeta_{p^nq^m})^x \otimes \mathbb{Q}/\mathbb{Z}, \quad x \mapsto \sqrt{pq^x} \otimes x.
\end{align*}

Notice that since $\mathbb{Q}/\mathbb{Z}$ is an injective module, splitting follows automatically if we prove injectivity. The maps described in (22), (23) and (24) are all injections since up to the torsion part $\{\pm 1\}$, the set of primes generates all of $\mathbb{Q}^x$ as a free $\mathbb{Z}$-module. Consider the map

(28) \quad \varphi : \mathbb{Q}^x \otimes \mathbb{Q}/\mathbb{Z} \rightarrow \mathbb{Q}(\zeta_{p^nq^m})^x \otimes \mathbb{Q}/\mathbb{Z}, \quad y \otimes x \mapsto y \otimes x.

We claim that if

(29) \quad \ker(\varphi) = \{0, \quad p^x \otimes \frac{1}{2}, \quad q^x \otimes \frac{1}{2}, \quad p^x q^x \otimes \frac{1}{2}\}

then the maps described in (25), (26) and (27) are also injective. We write down the proof for $\phi_4$, but the proofs for $\phi_5$ and $\phi_6$ are entirely analogous. Indeed, for any $x \in \mathbb{Q}/\mathbb{Z}$ such that $x \in \ker(\varphi_4)$ we have $\sqrt{p^x} \otimes x = 0 \in \mathbb{Q}(\zeta_{p^nq^m})^x \otimes \mathbb{Q}/\mathbb{Z}$. For each $x \in \mathbb{Q}/\mathbb{Z}$, let $y$ be an element in $\mathbb{Q}/\mathbb{Z}$ with $2y = x$. Notice that since $\sqrt{p^x} \in \mathbb{Q}(\zeta_{p^nq^m})^x$ we have $p^x \otimes y = \sqrt{p^x} \otimes x$ in $\mathbb{Q}(\zeta_{p^nq^m})^x \otimes \mathbb{Q}/\mathbb{Z}$. Now, since by definition $p^x \in \mathbb{Q}^x$ and $y \in \mathbb{Q}/\mathbb{Z}$, we can also view $p^x \otimes y$ as an element of $\mathbb{Q}^x \otimes \mathbb{Q}/\mathbb{Z}$, and $p^x \otimes y \in \varphi^{-1}(\mathbb{Q}(\zeta_{p^nq^m})^x \otimes \mathbb{Q}/\mathbb{Z})$. But then also $p^x \otimes y \in \ker(\varphi)$. Consider the map $\phi_4^* : \mathbb{Q}/\mathbb{Z} \rightarrow \mathbb{Q}^x \otimes \mathbb{Q}/\mathbb{Z}$ given by $\phi_4^*(x) := p^x \otimes x$. Since $\phi_4$ is an injection, $\phi_4^*$ is also an injection. So, from $p^x \otimes y \in \ker(\varphi)$ together with the fact that $\ker(\varphi) = \{0, \quad p^x \otimes \frac{1}{2}, \quad q^x \otimes \frac{1}{2}, \quad p^x q^x \otimes \frac{1}{2}\}$, we have that $y \in \{0, \frac{1}{2}\}$ in $\mathbb{Q}/\mathbb{Z}$ due to the explicit structure of $\mathbb{Q}^x \otimes \mathbb{Q}/\mathbb{Z}$. It follows then that $x \in \mathbb{Z}$ and so $x = 0 \in \mathbb{Q}/\mathbb{Z}$.
We will now prove (29). Notice that by the definitions of \( p^* \) and \( q^* \), we have
\[
\ker(\varphi) \supseteq \{0, \ \ p^* \otimes \frac{1}{2}, \ q^* \otimes \frac{1}{2}, \ p^* q^* \otimes \frac{1}{2}\}.
\]
Thus, to show the other inclusion we will see that \( \ker(\varphi) \cong \mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z} \). Notice that we can write \( \mathbb{Q}^* \otimes \mathbb{Q}/\mathbb{Z} = H^1(G_{\mathbb{Q}}, \mu(\overline{\mathbb{Q}})) \) and \( \mathbb{Q}(\zeta_{p^*q^*})^* \otimes \mathbb{Q}/\mathbb{Z} = H^1(G_{\mathbb{Q}}(\zeta_{p^*q^*}), \mu(\overline{\mathbb{Q}})) \), where \( G_{\mathbb{Q}} \) and \( G_{\mathbb{Q}}(\zeta_{p^*q^*}) \) are the absolute Galois groups of \( \mathbb{Q} \) and \( \mathbb{Q}(\zeta_{p^*q^*}) \) respectively. Hence,
\[
\ker(\varphi) = \ker(H^1(G_{\mathbb{Q}}, \mu(\overline{\mathbb{Q}})) \rightarrow H^1(G_{\mathbb{Q}}(\zeta_{p^*q^*}), \mu(\overline{\mathbb{Q}}))),
\]
and using the inflation-restriction sequence we obtain \( \ker(\varphi) = H^1(G, H^0(G_{\mathbb{Q}}(\zeta_{p^*q^*}), \mu(\overline{\mathbb{Q}}))), \) where \( G = \text{Gal}(\mathbb{Q}(\zeta_{p^*q^*})/\mathbb{Q}) \). By definition \( H^0(G_{\mathbb{Q}}(\zeta_{p^*q^*}), \mu(\overline{\mathbb{Q}})) = \mu(\overline{\mathbb{Q}})^{G_{\mathbb{Q}}(\zeta_{p^*q^*})} = \mu_{2p^*q^*} \), so we are left with computing \( \ker(\varphi) = H^1(G, \mu_{2p^*q^*}) \). Write \( N' = 2p^*q^* \) and \( L := \mathbb{Q}(\zeta_{N'}) = \mathbb{Q}(\zeta_{p^*q^*}) \), and consider the Kummer sequence
\[
1 \rightarrow \mu_{N'} \rightarrow L^* \xrightarrow{\cdot \delta'} (L^*)^{N'} \rightarrow 1.
\]
By the long exact sequence in cohomology we have
\[
1 \rightarrow H^0(G, \mu_{N'}) \rightarrow H^0(G, L^*) \xrightarrow{\cdot \delta'} H^0(G, (L^*)^{N'}) \rightarrow H^1(G, \mu_{N'}) \rightarrow H^1(G, L^*) = 1,
\]
where the last equality follows from Hilbert’s Theorem 90. Hence,
\[
H^1(G, \mu_{N'}) = H^0(G, (L^*)^{N'}/\text{im}(N')) = ((L^*)^{N'} \cap \mathbb{Q}^*)/(\mathbb{Q}^*)^{N'},
\]
which is the group consisting of rational numbers whose \( N' \)-th roots are in \( L^* \) up to the \( N' \)-th powers in \( \mathbb{Q}^* \). Now, take \( \alpha \in \mathbb{Q}^* \) such that \( N' \sqrt{\alpha} \in L^* \) and \( N' \sqrt{\alpha} \notin \mathbb{Q} \). Since \( N' \geq 3 \), the Galois closure \( \mathbb{Q}(N' \sqrt{\alpha})^{\text{Gal} | Q} \) is a non-abelian extension unless it is quadratic, i.e., unless \( (N' \sqrt{\alpha})^2 \in \mathbb{Q} \). But \( L/\mathbb{Q} \) is abelian, hence we need \( \alpha = b^{N'/2} \) for some \( b \in \mathbb{Q}^* \setminus (\mathbb{Q}^*)^{N'} \) and \( H^1(G, \mu_{N'}) \) is an elementary abelian 2-group. In particular it corresponds to the multiplicative group generated by square roots of rational numbers contained in \( L \), i.e., generated by the set of prime numbers \( \rho \) such that \( \sqrt{\rho^2} \in L^* \). In our case, this is the group generated by \( \{p^*, q^*\} \) and taken modulo the multiplicative group of rational numbers. It follows that \( H^1(G, \mu_{N'}) \cong \mathbb{Z}/2\mathbb{Z} \otimes \mathbb{Z}/2\mathbb{Z} \) as we wanted. \( \square \)

Thanks to this result, computing the kernel of \( \delta \) is equivalent to computing the kernel of the map \( \overline{\delta} \), which is relatively easier, since we no longer need to take the torsion units into account. The following proposition brings us a step closer to computing the image of the generators of \( C(N) \) under the map \( \overline{\delta} \). We let \( d' = p^n q^m \) be any divisor of the positive integer \( N = p^r q^s \), and consider \( Q_{d'} \), the cusp at the curve \( X_0(N) \) of level \( d' \) given by \( Q_{d'} := \lfloor \frac{1}{p^n q^m} \rfloor \).

**Proposition 5.6.** Set \( N = p^r q^s \) with \( p \) and \( q \) two distinct odd primes. Let \( d = p^aq^b \neq 1 \) and \( d' = p^n q^m \) be two divisors of \( N \). Consider the eta quotient \( h_d(z) \) attached to the divisor \( Z(d) \) as described in Proposition 3.26. Let \( c_d(n, m) \) be the leading Fourier coefficient of \( h_d(z) \) at the cusp \( Q_{d'} \). The class of \( c_d(n, m) \) in \( \Omega_{d'} \) is given by the list below. Hence, \( c_d(n, m) = e(d, n, m) \cdot c_d(n, m) \) where \( e(d, n, m) \) is an element of \( \mu(\mathbb{Q}(Q_{d'})^*) \).

If \( a \geq 2, b \geq 2 \) : \( e(d, n, m) \) for all \( n, m \)
If $a = 1, b = 2$:

\[
\begin{cases}
\left(\sqrt{(q^*)^{m-1}}\right)^{\delta(q)} & \text{for } m = 0, \\
\left(\sqrt{(q^*)^{m}}\right)^{\delta(q)} & \text{for } m \neq 0;
\end{cases}
\]

If $a = 1, b = s - 2j$:

\[
\begin{cases}
\sqrt{(q^*)^{q(s-j)-(s-j-j-2)}} & \text{for } m = 0, \\
\sqrt{(q^*)^{q(s-j-m)-(s-j-m-1)}} & \text{for } 0 < m < s - j, \\
1 & \text{for } s - j \leq m \leq s;
\end{cases}
\]

If $a = 1, b = s - 2j + 1$:

\[
\begin{cases}
\sqrt{(q^*)^{q(s-j)-(s-j-j-2)}} & \text{for } 0 \leq m \leq j, \\
\sqrt{(q^*)^{q(s-m)-(s-m-1)}} & \text{for } j < m \leq s - 1, \\
1 & \text{for } m = s;
\end{cases}
\]

If $b = 1, a = 2$:

\[
\begin{cases}
\left(\sqrt{(p^*)^{r-1}}\right)^{\delta(p)} & \text{for } n = 0, \\
\left(\sqrt{(p^*)^{r-n}}\right)^{\delta(p)} & \text{for } n \neq 0;
\end{cases}
\]

If $b = 1, a = r - 2i$:

\[
\begin{cases}
\sqrt{(p^*)^{p(r-i)-(r-i-j-2)}} & \text{for } n = 0, \\
\sqrt{(p^*)^{p(r-i-n)-(r-i-n-1)}} & \text{for } 0 < n < r - i, \\
1 & \text{for } r - i \leq n \leq s;
\end{cases}
\]

If $b = 1, a = r - 2i + 1$:

\[
\begin{cases}
\sqrt{(p^*)^{p(r-i)-(r-i-j-2)}} & \text{for } 0 \leq n \leq i, \\
\sqrt{(p^*)^{p(r-n)-(r-n-1)}} & \text{for } i < n \leq r - 1, \\
1 & \text{for } n = r;
\end{cases}
\]

If $a = 0, b = 2$:

\[
\begin{cases}
\sqrt{(q^*)^{(s-1)(p-1)}} & \text{for } m = 0, \\
\sqrt{(q^*)^{(s-m)(p-1)}} & \text{for } 0 < m \leq s;
\end{cases}
\]

If $a = 0, b = s - 2j$:

\[
\begin{cases}
\sqrt{(q^*)^{(p-1)(q(s-j)-(s-j-j-2))}} & \text{for } m = 0, \\
\sqrt{(q^*)^{(p-1)(q(s-j-m)-(s-j-m-1))}} & \text{for } 0 < m < s - j, \\
1 & \text{for } s - j \leq m \leq s;
\end{cases}
\]

If $a = 0, b = s - 2j + 1$:

\[
\begin{cases}
\sqrt{(q^*)^{(p-1)(q(s-j)-(s-j-j-2))}} & \text{for } 0 \leq m \leq j, \\
\sqrt{(q^*)^{(p-1)(q(s-m)-(s-m-1))}} & \text{for } j < m \leq s - 1, \\
1 & \text{for } m = s;
\end{cases}
\]
If $b = 0, a = 2$:
\[
\begin{align*}
\sqrt{(p^n)^{(r-1)(q-1)}} & \quad \text{for } n = 0, \\
\sqrt{(p^n)^{(r-n)(q-1)}} & \quad \text{for } 0 < n \leq r;
\end{align*}
\]

If $b = 0, a = s-2i$:
\[
\begin{align*}
\sqrt{(p^n)^{(q-1)(p(r-i) - (r-1))}} & \quad \text{for } n = 0, \\
\sqrt{(p^n)^{(q-1)(p(r-n) - (r-n-1))}} & \quad \text{for } 0 < n < r - i, \\
1 & \quad \text{for } r - i \leq n \leq r;
\end{align*}
\]

If $b = 0, a = r-2i + 1$:
\[
\begin{align*}
\sqrt{(p^n)^{(q-1)(p(r-i) - (r-1))}} & \quad \text{for } 0 \leq n \leq i, \\
\sqrt{(p^n)^{(q-1)(p(r-n) - (r-n-1))}} & \quad \text{for } i < n \leq r - 1, \\
1 & \quad \text{for } n = r;
\end{align*}
\]

If $a = 1, b = 1$:
\[
\begin{align*}
\sqrt{(p^n)^{B(p)}} & \quad \text{for } n = 0, \\
1 & \quad \text{for } n \neq 0,
\end{align*}
\]
where $B(p)$ is as in Definition 2.2.

If $a = 0, b = 1$:
\[
\begin{align*}
\sqrt{(q^n)^{(p-1)}} & \quad \text{for } m = 0, \\
1 & \quad \text{for } m \neq 0;
\end{align*}
\]

If $a = 1, b = 0$:
\[
\begin{align*}
\sqrt{(p^n)^{(q-1)}} & \quad \text{for } n = 0, \\
1 & \quad \text{for } n \neq 0.
\end{align*}
\]

**Proof.** We will write down the proof of the case $a = 1, b = 0$. The other cases are computed analogously. For the divisor $d = p$ we have from Proposition 3.26 that the eta quotient $h_p(z)$ such that $\text{div}(h_p(z)) = n_p \cdot Z(p)$ is given by
\[
h_p(z) = \left( \frac{\eta(z)^4 \eta(pqz)}{\eta(pz)^4 \eta(qz)} \right)^{\frac{24}{\theta(p,q)}}.
\]

We now apply Proposition 5.1 and we obtain that the leading Fourier coefficient at the cusp $\frac{1}{p^n q^m}$ is given by
\[
c_d(n, m) := e'(p, n, m) \left( \frac{\sqrt{(1,p^n q^m)}}{1} \right)^q \frac{\sqrt[p]{(p, p^n q^m)}}{p} \left( \frac{\sqrt[q]{(p, p^n q^m)}}{q} \right).
\]

where $e'(p, n, m)$ is a root of unity. Now we take
\[
e(p, n, m) = e'(p, n, m)/(\varepsilon^2 \pi i (p-1)/8)^{a_{n,m}}(\varepsilon^2 \pi i (q-1)/8)^{b_{n,m}},
\]
where $\varepsilon$ is a root of unity.
where \( a_{n,m} \) and \( b_{n,m} \) are the powers of \( \sqrt{p} \) and \( \sqrt{q} \) respectively in \( c_d(n,m) \). By an explicit computation by cases we get that

\[
c_d(n,m) = \begin{cases} 
  e(p,n,m)\sqrt{(p^n)(q^m)} & \text{for } n = 0, \\
  e(p,n,m) & \text{for } n \neq 0,
\end{cases}
\]

where \( e(p,n,m) \) is a torsion element of \( \mathbb{Q}(Q_d)^* \).

\[\square\]

**Definition 5.7.** Let \( p, q \) be two distinct odd primes. Let \( N = p^r q^s \) and let \( d = p^a q^b \neq 1 \) be a divisor of \( N \). We define \( \nu_{a,b} \in \Omega \) to be the tuple appearing in the expression

\[
\overline{\delta}([Z(d)]) := \nu_{a,b} \otimes \frac{1}{n_d}
\]

for the image of the divisor \( Z(d) \) defined in Definition 3.20 under the map \( \overline{\delta} \) given in Equation (17), where \( n_d \) is the order of \( Z(d) \). Note that \( \nu_{a,b} \) is well-defined since \( n_d \) is well-defined – see Definition 2.6.

The next proposition is the main result of this section.

**Proposition 5.8.** Let \( p, q \) be two odd distinct primes. Set \( N = p^r q^s \). Let \( d = p^a q^b \neq 1 \) be a divisor of \( N \). The image of the generator \( Z(d) \) defined in Equation (3.20) under the map \( \overline{\delta} \) described in (17) is given in Table 5. The sub-index \((n,m)\) of the tuples appearing on the table denotes the entry of the tuple corresponding to the divisor of \( N \) given by \( p^n q^m \). In other words, if \( \nu_{a,b} \) is the tuple in \( \Omega \) corresponding to the image of \( Z(p^a q^b) \) under \( \overline{\delta} \), the entry indexed by \((n,m)\) corresponds to the quotient of the leading Fourier coefficient of \( h_{p^a q^b}(z) \) at the cusp \( \frac{1}{p^a q^b} \) divided by the leading Fourier coefficient of \( h_{p^a q^b}(z) \) at the cusp \( \frac{1}{p^r q^s} \).

**Proof.** The result follows from Proposition 5.6, which shows that the leading Fourier coefficient of \( h_d(z) \) at the cusp \( \frac{1}{p^r q^s} \) is 1 for all \( d \), Lemma 5.5, and the description of the map \( \delta \) given in Lemma 4.1. \[\square\]
\[ \overline{\delta}([Z(d)]) = v_{a,b} \otimes \frac{1}{n_d} = \]

\[ (1, \ldots, 1) \otimes \frac{1}{n_d} \quad \text{if } a \geq 2, b \geq 2; \quad (5.8.1) \]

\[ \left( (\sqrt{q^s})^{s-1}_{(0,0)}, (\sqrt{q^s})^{s-m}_{(0,1 \leq m \leq s)}, \ldots, (\sqrt{q^s})^{s-1}_{(r-1,0)}, (\sqrt{q^s})^{s-m}_{(r-1,1 \leq m \leq s)}, (\sqrt{q^s})^{s-1}_{(r,0)}, (\sqrt{q^s})^{s-m}_{(r,1 \leq m \leq s-1)} \right)^{A(q^2)} \otimes \frac{1}{n_d} \quad \text{if } a = 1, b = 2; \quad (5.8.2) \]

\[ \left( (\sqrt{q^s})^{q(s-j)-(s-j-2)}_{(0,0)}, (\sqrt{q^s})^{q(s-j-m)-(s-j-m-1)}_{(0,1 \leq j \leq m \leq s)}, 1_{(0,1 \leq j \leq m \leq s)} \right), \ldots, \left( (\sqrt{q^s})^{q(s-j)-(s-j-2)}_{(r-1,0)}, (\sqrt{q^s})^{q(s-j-m)-(s-j-m-1)}_{(r-1,1 \leq j \leq m \leq s)}, 1_{(r-1,1 \leq j \leq m \leq s)} \right) \otimes \frac{1}{n_d} \quad \text{if } a = 1, b = s-2j; \quad (5.8.3) \]

\[ \left( (\sqrt{q^s})^{q(s-j)-(s-j-2)}_{(0,0)}, (\sqrt{q^s})^{q(s-m)-(s-m-1)}_{(0,1 \leq j \leq m \leq s)} \right), \ldots, \left( (\sqrt{q^s})^{q(s-j)-(s-j-2)}_{(r,0)}, (\sqrt{q^s})^{q(s-m)-(s-m-1)}_{(r,1 \leq j \leq m \leq s-1)} \right) \otimes \frac{1}{n_d} \quad \text{if } a = 1, b = s-2j+1; \quad (5.8.4) \]

\[ \left( (\sqrt{q^s})^{(s-1)(p-1)}_{(0,0)}, (\sqrt{q^s})^{(s-m)(p-1)}_{(0,1 \leq m \leq s)}, \ldots, (\sqrt{q^s})^{(s-1)(p-1)}_{(r-1,0)}, (\sqrt{q^s})^{(s-m)(p-1)}_{(r-1,1 \leq m \leq s)}, (\sqrt{q^s})^{(s-1)(p-1)}_{(r,0)}, (\sqrt{q^s})^{(s-m)(p-1)}_{(r,0 \leq m \leq s-1)} \right) \otimes \frac{1}{n_d} \quad \text{if } a = 0, b = 2; \quad (5.8.5) \]

\[ \left( (\sqrt{q^s})^{(p-1)(q(s-j)-(s-j-2))}_{(0,0)}, (\sqrt{q^s})^{(p-1)(q(s-j-m)-(s-j-m-1))}_{(0,1 \leq j \leq m \leq s)}, 1_{(0,1 \leq j \leq m \leq s)} \right), \ldots, \left( (\sqrt{q^s})^{(p-1)(q(s-j)-(s-j-2))}_{(r,0)}, (\sqrt{q^s})^{(p-1)(q(s-j-m)-(s-j-m-1))}_{(r,1 \leq j \leq m \leq s-1)} \right) \otimes \frac{1}{n_d} \quad \text{if } a = 0, b = s-2j; \quad (5.8.6) \]

2This meant that we power each entry of the tuple to \( A(q) \)
\[
\begin{aligned}
\left(\sqrt{q^s}\right)_{(0,0)}^{(p-1)(q(s-j)-(s-j-2))},
\left(\sqrt{q^s}\right)_{(0,j+1)}^{(p-1)(q(s-m)-(s-m-2))},
\ldots,
\left(\sqrt{q^s}\right)_{(r-1,0)}^{(p-1)(q(s-j)-(s-j-2))},
\left(\sqrt{q^s}\right)_{(r-1,j+1)}^{(p-1)(q(s-m)-(s-m-2))},
\left(\sqrt{q^s}\right)_{(r,0)}^{(p-1)(q(s-j)-(s-j-2))},
\left(\sqrt{q^s}\right)_{(r,j+1)}^{(p-1)(q(s-m)-(s-m-2))} \otimes \frac{1}{n_d}
\end{aligned}
\] 
if \(a = 0, b = s - 2j + 1\); \ (5.8.7)

\[
\begin{aligned}
\left(\sqrt{p^s}\right)_{(0,0)}^{r-1},
\left(\sqrt{p^s}\right)_{(1,1)}^{r-n},
\ldots,
\left(\sqrt{p^s}\right)_{(0,s-1)}^{r-1},
\left(\sqrt{p^s}\right)_{(1,s-1)}^{r-n},
\left(\sqrt{p^s}\right)_{(1,n-1)}^{r-1},
\left(\sqrt{p^s}\right)_{(1,n-1,s-1)}^{r-n}
\end{aligned}
\] 
\(A(p) \otimes \frac{1}{n_d}\) 
if \(a = 2, b = 1\); \ (5.8.8)

\[
\begin{aligned}
\left(\sqrt{p^s}\right)_{(0,0)}^{p(r-i)-(r-i-2)},
\left(\sqrt{p^s}\right)_{(1,1)}^{p(r-i)-(r-i-1)}
\end{aligned}
\] 
\(1_{(r-i+1)\leq n\leq r} \otimes \frac{1}{n_d}\) 
if \(a = r - 2i, b = 1\) \ (5.8.9)

\[
\begin{aligned}
\left(\sqrt{p^s}\right)_{(0,s-1)}^{p(r-i)-(r-i-2)},
\left(\sqrt{p^s}\right)_{(1,s-1)}^{p(r-i)-(r-i-1)}
\end{aligned}
\] 
\(1_{(r-i+1)\leq n\leq r-1} \otimes \frac{1}{n_d}\) 
if \(a = r - 2i + 1, b = 1\); \ (5.8.10)

\[
\begin{aligned}
\left(\sqrt{p^s}\right)_{(0,0)}^{p(r-i)-(r-i-2)},
\left(\sqrt{p^s}\right)_{(i+1)}^{p(r-n)-(r-n-1)}
\end{aligned}
\] 
\(1_{i\leq n\leq r} \otimes \frac{1}{n_d}\) 
if \(a = r - 2i + 1, b = 1\); \ (5.8.10)

\[
\begin{aligned}
\left(\sqrt{p^s}\right)_{(0,0)}^{(r-1)(q-1)},
\left(\sqrt{p^s}\right)_{(0,s-1)}^{(r-n)(q-1)}
\end{aligned}
\] 
\(1_{(r-1)(q-1), (r-n)(q-1)} \otimes \frac{1}{n_d}\) 
if \(a = 2, b = 2\); \ (5.8.11)
\[
\left(\frac{p}{\sqrt{p^2}}\right)_{(0,0)}^{(q-1)(p(r-i) - (r-i-2))}, \left(\frac{p}{\sqrt{p^2}}\right}_{(1\leq n \leq r-i,0)}^{(q-1)(p(r-i-n) - (r-i-n-1))}, 1_{(r-i+1 \leq n \leq r,0)}, \\
\vdots, \\
\left(\frac{p}{\sqrt{p^2}}\right)_{(0,s-1)}^{(q-1)(p(r-i) - (r-i-2))}, \left(\frac{p}{\sqrt{p^2}}\right)_{(1\leq n \leq r-i,s-1)}^{(q-1)(p(r-i-n) - (r-i-n-1))}, 1_{(r-i+1 \leq n \leq r,s-1)}, \\
\left(\frac{p}{\sqrt{p^2}}\right)_{(0,i)}^{(q-1)(p(r-i) - (r-i-2))}, \left(\frac{p}{\sqrt{p^2}}\right)_{(1\leq n \leq r-i,i)}^{(q-1)(p(r-i-n) - (r-i-n-1))}, 1_{(r-i+1 \leq n \leq r-1,i)} \otimes \frac{1}{n_d}
\]

\[
\left(\frac{p}{\sqrt{p^2}}\right)_{(0\leq n \leq r,s)}^{B}, 1_{(1,0 \leq m \leq s)}, \cdots, 1_{(r-1,0 \leq m \leq s)}, 1_{(r,0 \leq m \leq s-1)} \otimes \frac{1}{n_d}
\]  
if \(a = 1, b = 1\); (5.8.14)

\[
\left(\frac{q}{\sqrt{q^2}}\right)_{(0\leq n \leq r,0)}^{(p-1)}, 1_{(0 \leq n \leq r,s-1)}, \cdots, 1_{(0 \leq n \leq r,s-1)}, 1_{(0 \leq n \leq r-1,s-1)} \otimes \frac{1}{n_d}
\]  
if \(a = 0, b = 1\); (5.8.15)

\[
\left(\frac{p}{\sqrt{p^2}}\right)_{(0,m)}^{(q-1)}, 1_{(1,0 \leq m \leq s)}, \cdots, 1_{(r-1,0 \leq m \leq s)}, 1_{(r,0 \leq m \leq s-1)} \otimes \frac{1}{n_d}
\]  
if \(a = 1, b = 0\). (5.8.16)
6. KERNEL OF $\delta$

To complete the description of $J_0(N)_m(\mathbb{Q})_{\text{tor}}$ it remains to compute the kernel of the map $\delta : J_0(N)(\mathbb{Q}) \to \bigoplus \mathbb{Q}(P_d) \otimes \mathbb{Q}/\mathbb{Z}$ introduced in the exact sequence of Equation (17). Furthermore, in Lemma 5.5 we reduced the computation of $\ker(\delta)$ to that of $\ker(\overline{\delta})$, where $\overline{\delta}$ is the map resulting from the factorisation of $\delta$ given in Diagram (21). On the other hand, using Equation (19) we reduce the computation of $\ker(\delta(J_0(N)(\mathbb{Q})_{\text{tor}}[I^\infty]))$ into that of $\ker(\overline{\delta}|_{C(N)})[I^\infty]$. Hence, the purpose of this section will be to build up towards the proof of Theorem 6.4, which describes the kernel of $\overline{\delta}|_{C(N)}$ in terms of divisors $D \in C(N)$. This proof will be split into two main parts:

(a) Firstly we will find a set of divisors generating the kernel of $\overline{\delta}$ and show that they form a basis — see Corollary 6.13.
(b) Secondly, we will compute the orders of these divisors — see Proposition 6.14.

To state the main result of this section, first we need the following definitions.

**Definition 6.1.** Let $p, q$ be two odd primes and let $N = p^r q^s$. For each pair $(a, b) \in \mathbb{Z}^2$ with $1 \leq a \leq r$ and $1 \leq b \leq s$, we define the divisor $D(a, b) \in \text{Div}_0^\text{cusp}(X_0(N))(\mathbb{Q})$ as follows:

\[
D(a, b) := \begin{cases}
Z(p^a q^b) & \text{if } a \geq 2, b \geq 2; \\
p^{r-1}(p+1)Z(p^b) - Z(pq^b) & \text{if } a = 1, b \geq 2; \\
q^{s-1}(q+1)Z(p^a) - Z(p^aq) & \text{if } a \geq 2, b = 1; \\
q^{s-1}(q+1)Z(p) - Z(pq) & \text{if } a = b = 1.
\end{cases}
\]

If $p = 3$, $D(a, b)$ is defined as in (30) for $(a, b) \neq (2, 1)$ and $D(2, 1) := (p^2 - 1)q^{s-1}(q+1) \cdot Z(p^2)$. If $q = 3$, $D(a, b)$ is defined as in (30) for $(a, b) \neq (1, 2)$ and $D(1, 2) := (q^2 - 1)p^{r-1}(p+1) \cdot Z(q^2)$.

**Remark 6.2.** For the case $p = 3$, we have that $\text{ord}(Z(p^2q)) = 1$, and so $[Z(p^2q)] = 0$. Hence, we need to take $D(2, 1) = (p^2 - 1)q^{s-1}(q+1) \cdot Z(p^2)$. Similarly, for $q = 3$ we take $D(1, 2) = (q^2 - 1)p^{r-1}(p+1) \cdot Z(q^2)$.

**Definition 6.3.** Let $p, q$ be two odd primes and let $N = p^r q^s$. Let $d$ be a divisor of $N$ and write $n_d := n(N, d)$ for the order of the divisor $Z(d)$ — see 2.6 for the definition of $n(N, d)$. For each pair $(a, b) \in \mathbb{Z}^2$ with $1 \leq a \leq r$ and $1 \leq b \leq s$ we define the integer $m(a, b)$ as follows:

\[
m(a, b) := \begin{cases}
n_{p^aq^b} & \text{if } a \geq 2, b \geq 2; \\
\text{num}\left(\frac{p^{r-1}(p-1)(q^{s-1}q+1)}{24}\right) & \text{if } a = 1, b > 2; \\
\text{num}\left(\frac{(q^{s-1}q+1)}{24}\right) & \text{if } a = 1, b = 2; \\
\text{num}\left(\frac{p^{r-1}p^{r-1}(p-1)}{24}\right) & \text{if } a > 2, b = 1; \\
\text{num}\left(\frac{p^{r-1}p^{r-1}(p-1)}{24}\right) & \text{if } a = 1, b = 2; \\
\text{num}\left(\frac{p^{r-1}p^{r-1}(p-1)(q+1)}{24}\right) & \text{if } a = b = 1.
\end{cases}
\]

The integers $m(a, b)$ will be the respective orders of the divisors $D(a, b)$ described in Definition 6.1.

Now we are ready to state the main theorem of this section, whose proof will occupy the rest of the section.
Theorem 6.4. Let $N = p^r q^s$ with $p, q$ two odd primes. We have

$$\ker(\overline{\delta}|_{C(N)}) = \bigoplus_{1 \leq a \leq r, 1 \leq b \leq s} \langle [D(a, b)] \rangle \simeq \bigoplus_{1 \leq a \leq r, 1 \leq b \leq s} \mathbb{Z}/m(a, b)\mathbb{Z}. \]

For the proof of this theorem, we will first prove in Proposition 6.5 that the divisors $D(a, b)$ defined in Definition 6.1 generate $\ker(\overline{\delta}|_{C(N)})$ by using the results in Proposition 5.8. Then we will show in Lemma 6.12 that they are linearly independent classes of divisors by using the linear independent sets of the divisors $Z(d)$ given in Theorem 3.2. Finally, we will compute the orders of the divisors $D(a, b)$ by using Theorem 3.14.

Proposition 6.5. Let $N = p^r q^s$ with $p, q$ two odd primes. The divisors $[D(a, b)]$ generate $\ker(\overline{\delta}|_{C(N)})$. That is,

$$\ker(\overline{\delta}|_{C(N)}) = \langle [D(a, b)] : 1 \leq a \leq r, 1 \leq b \leq s \rangle.$$ 

Proof of Proposition 6.5. We write down the proof for $p, q \neq 3$. The proof for $p = 3$ or $q = 3$ follows similarly. We split up the proof into several claims.

Claim 6.6. For any $a \in \{1, \ldots, r\}$ and $b \in \{1, \ldots, s\}$, the divisor $D(a, b)$ satisfies $\overline{\delta}([D(a, b)]) = 0$.

Proof of Claim 6.6. We use the computations of the images of the generators $Z(d)$ of $C(N)$ under the map $\overline{\delta}$ given in Proposition 5.8.

Firstly, it follows from Equation (5.8.1) that $\overline{\delta}([D(a, b)]) = \overline{\delta}([Z(d)]) = 0$ for all $d = p^a q^b$ with $a \geq 2$ and $b \geq 2$. Recall that the tuples $v_{a,b}$ are elements of the $\mathbb{Z}$-module $\Omega$, where the module operation is given by exponentiation on the entries of the tuples in $\Omega$.

Secondly, from Equations (5.8.5)–(5.8.7), we see that we may write

$$\overline{\delta}([Z(q^2)]) = \left( (\sqrt[q^2]{(s-1)(p-1)}(0,0) , (\sqrt[q^2]{(s-1)(p-1)}(0,0 \leq m \leq s)) , \cdots , (\sqrt[q^2]{(s-1)(p-1)}(r-1,0) , (\sqrt[q^2]{(s-1)(p-1)}(r-1,0 \leq m \leq s)) \right) \otimes \frac{1}{p^{r-1}(p+1)(q^2-1)} =

\left( (\sqrt[q^2]{(s-1)}(0,0) , (\sqrt[q^2]{(s-1)}(0,0 \leq m \leq s)) , (\sqrt[q^2]{(s-1)}(1,0) , (\sqrt[q^2]{(s-1)}(1,1 \leq m \leq s-1)) , \cdots , (\sqrt[q^2]{(s-1)}(r-1,0) , (\sqrt[q^2]{(s-1)}(r-1,1 \leq m \leq s-1)) \right) \otimes \frac{1}{p^{r-1}(p+1)(q^2-1)}.

Similarly,

$$\overline{\delta}([Z(q^{r-2j})]) = \left( (\sqrt[q^{r-2j}]{(s-j)(s-j-2)}(0,0) , (\sqrt[q^{r-2j}]{(s-j)(s-j-2)}(0,0 \leq m \leq s-j) , 1(0,s-j+1 \leq m \leq s)) , \cdots , (\sqrt[q^{r-2j}]{(s-j)(s-j-2)}(r-1,0) , (\sqrt[q^{r-2j}]{(s-j)(s-j-2)}(r-1,1 \leq m \leq s-j)) , 1(0,s-j+1 \leq m \leq s)) \right) \otimes \frac{1}{p^{r-1}(p+1)(q^{r-2j}-1)(q^2-1)}$$

and
\[
\overline{\delta}([Z(q^{r-2j+1})]) = \left( \left( \sqrt{q^r} \right)_{(0,0 \leq m \leq s)}^{(q(s-j)-(s-j-2))}, \left( \sqrt{q^r} \right)_{(0,j+1 \leq m \leq s)}^{(q(s-m)-(s-m-2))}, \right.
\]
\[
\left. \ldots, \left( \sqrt{q^r} \right)_{(r-1,0 \leq m \leq s)}^{(q(s-j)-(s-j-2))}, \left( \sqrt{q^r} \right)_{(r,0 \leq m \leq s)}^{(q(s-m)-(s-m-2))}, \left( \sqrt{q^r} \right)_{(r,j+1 \leq m \leq s)}^{(q(s-m)-(s-m-2))} \right) \otimes \frac{1}{p^{r-1}q^{\pi_1(q-1)}}.
\]

\[
D(1,b) := p^{r-1}(p + 1) \cdot Z(q^b) - Z(pq^b)
\]

indeed satisfy \( \overline{\delta}([D(1,b)]) = 0 \). For all \( b \geq 2 \). From analogous arguments, we conclude that the divisors

\[
D(a,1) := q^{r-1} (q + 1) \cdot Z(p^a) - Z(p^aq)
\]

are also in \( \text{ker}(\overline{\delta}) \).

Consider now the divisor \( D(1,1) \). From Equations (5.8.14) and (5.8.16) we see that

\[
\overline{\delta}([Z(pq)]) = \left( \left( \sqrt{p^{r-1}} \right)_{(0,0 \leq m \leq s)}^{b}, 1_{(1,0 \leq m \leq s)}, \ldots, 1_{(r-1,0 \leq m \leq s)}, 1_{(r,0 \leq m \leq s-1)} \right) \otimes \frac{1}{\text{num}(\frac{r-1}{12})}
\]
\[
= \left( \left( \sqrt{p^{r-1}} \right)_{(0,0 \leq m \leq s)}^{b}, 1_{(1,0 \leq m \leq s)}, \ldots, 1_{(r-1,0 \leq m \leq s)}, 1_{(r,0 \leq m \leq s-1)} \right) \otimes \frac{24}{p^{r-1}}.
\]

Since \( p, q \geq 5 \), we have \( n_p = \text{num}\left( \frac{(p-1)q^{r-1}(q^2-1)}{24} \right) = \frac{(p-1)q^{r-1}(q^2-1)}{24} \), and

\[
\overline{\delta}([Z(p)]) = \left( \left( \sqrt{p^{r-1}} \right)_{(0,m)}^{(q-1)24}, 1_{(1,0 \leq m \leq s)}, \ldots, 1_{(r-1,0 \leq m \leq s)}, 1_{(r,0 \leq m \leq s-1)} \right) \otimes \frac{24}{(p-1)q^{r-1}(q+1)}.
\]

Hence, the divisor

\[
D(1,1) := q^{r-1} (q + 1) \cdot Z(p) - Z(pq),
\]

indeed satisfies \( \overline{\delta}([D(1,1)]) = 0 \). \( \square \)

Finally, from Equation (5.8.15), we have

\[
\overline{\delta}([Z(q)]) = \left( \left( \sqrt{q^{r-1}} \right)_{(0,m)}^{(p-1)}, 1_{(1,0 \leq m \leq s)}, \ldots, 1_{(r-1,0 \leq m \leq s)}, 1_{(r,0 \leq m \leq s-1)} \right) \otimes \frac{24}{(q-1)p^{r-1}(p^{r-1})}
\]
\[
\left( \left( \sqrt{q^{r-1}} \right)_{(0,m)}^{(p-1)}, 1_{(1,0 \leq m \leq s)}, \ldots, 1_{(r-1,0 \leq m \leq s)}, 1_{(r,0 \leq m \leq s-1)} \right) \otimes \frac{24}{(q-1)p^{r-1}(p^{r-1})}.
\]

Hence, \([p^{(r-1)}(p+1)(q-1)Z(q)]\) is in \( \text{ker}(\overline{\delta}) \). But using Theorem 3.14, we see that

\[
\text{ord}([q-1]D(1,1) - p^{(r-1)}(p+1)(q-1)Z(q)) = 1.
\]

It follows that \([q-1]D(1,1) = [p^{(r-1)}(p+1)(q-1)Z(q)], i.e., \([p^{(r-1)}(p+1)(q-1)Z(q)] \in \langle D(1,1) \rangle \). Recall from Theorem 3.2 that the divisors \([Z(d)]\), for \( d \) ranging over all the divisors of \( N \) not equal to 1, generate \( C(N) \). Since we saw in the proof of Claim 6.6 that \( \overline{\delta}([Z(d)]) = 0 \) for all \( a \geq 2 \) and \( b \geq 2 \), this implies that

\[
\text{im}(\overline{\delta}|_{C(N)}) = \langle \{v_{0,b} \otimes \frac{1}{n_q b}, v_{1,b} \otimes \frac{1}{p^{r+1}}, v_{a,0} \otimes \frac{1}{n_p}, v_{a,1} \otimes \frac{1}{n_p}, v_{b,0} \otimes \frac{1}{n_q}, v_{b,1} \otimes \frac{1}{n_q}, v_{1,0} \otimes \frac{1}{n_p}, v_{1,1} \otimes \frac{1}{n_p} \rangle : a, b \geq 2 \rangle,
\]
where we recall from Definition 5.7 that we write $\delta([Z(d)]) = v_{a,b} \otimes \frac{1}{n_d}$ when $n_d$ is the order of the divisor $Z(d)$.

We now define the following sets of tuples:

\[
S(0, b) := \{v_{0,b}\}_{b \geq 2}, \quad S(1, b) := \{v_{1,b}\}_{b \geq 2}, \\
S(a, 0) := \{v_{a,0}\}_{a \geq 2}, \quad S(a, 1) := \{v_{a,1}\}_{a \geq 2}, \quad \text{and} \quad S(sf) := \{v_{0,1}, v_{1,1}\}.
\]

It follows from Equations (5.8.14-5.8.16) that

\[
(31) \quad v_{1,b} = (p - 1) \cdot v_{0,b} \quad \text{and} \quad v_{a,1} = (q - 1) \cdot v_{a,0}
\]

for all $a, b \geq 2$, and that

\[
(32) \quad B(p) \cdot v_{1,0} = (q - 1) \cdot v_{1,1},
\]

where $B(p)$ is as in Definition 2.2. These are the underlying relations that describe the divisors $D(a,b)$ for $a = 1$ or $b = 1$. Indeed, the proof of Claim 6.6 relies on the fact that, by construction, the divisors $D(a,b)$ satisfy the relations induced from Equations (31) and (32).

Hence, to prove that the the divisors $D(a,b)$ generate $\ker(\delta)$, we need to show that the linear relations of the tuples $v_{a,b}$ described Equations (31) and (32) are the only relations; in other words, that there is no linear relation between the tuples $v_{a,b}$ with $a = 1$ or $b = 1$ independent from the relations displayed in those equations.

We can deduce from the nature of the relations in (31) and (32) that, to show that such a linear relation does not exist it is sufficient to see that the sets

\[
S(0, b), S(a, 0), S(sf), S(0, b) \cup S(a, 0), S(0, b) \cup S(sf) \quad \text{and} \quad S(a, 0) \cup S(sf)
\]

are sets of mutually linearly independent tuples.

**Claim 6.7.** Each of the sets $S(0, b), S(a, 0),$ and $S(sf)$ consists of $\mathbb{Z}$-linearly independent tuples. In other words, there is no linear combination of the divisors in the set $\{Z(q^b)\}_{b \geq 2}$ that lies in the kernel of $\delta$; and similarly for each of the sets of divisors $\{Z(p^s)\}_{s \geq 2}$, and $\{Z(q), Z(pq)\}$.

**Proof of Claim 6.7.** Let us start with showing the linear independence of the tuples in $\{v_{0,b}\}_{b \geq 2}$. Recall that all entries of the tuples $v_{0,b}$ for $b \geq 2$ lie in the subgroup of $O_d$ generated by $q$ if $\text{ord}_q(d) = 0$ and by $\sqrt{q^2}$ if $\text{ord}_q(d) > 0$. That is, when $\text{ord}_q(d) = 0$, picking $q$ (resp. $\sqrt{q^2}$) as a representative of its class in $O_d$, we have that $v_{0,b}(d)$ is of the form $q^k$ for some $k \in \mathbb{Z}$ (resp. $(\sqrt{q^2})^k$ when $\text{ord}_q(d) > 0$). Hence, when $\text{ord}_q(d) = 0$, let $\nu_{q,d} : O_d \to \mathbb{Z}$ be the $\mathbb{Z}$-module homomorphism defined by $\nu_{q,d}(q^k) = k$ (resp. $\nu_{q,d}((\sqrt{q^2})^k) = k$ when $\text{ord}_q(d) > 0$). We construct the $(s - 1) \times (s + 1)$ matrix $M$ such that the $(n,m)$-th entry of $M$, denoted $M(n,m)$, satisfies

\[
\begin{align*}
\text{1. } M(n, 1) \text{ is } & \nu_{q,1}(v_{0,n+1}(q^0)), \\
\text{2. for } 1 < m \leq s, M(n, m) \text{ is } & \nu_{q,m-1}(v_{0,n+1}(q^{m-1})),
\end{align*}
\]

Notice from Proposition 5.1 that, if $v_{a,b}(p^n q^m)$ denotes the entry of the tuple $v_{a,b}$ indexed by the divisor $p^n q^m$, the entries of $v_{0,b+1}$ with $b \geq 2$ satisfy $v_{0,b}(p^n q^m) = v_{0,b}(p^n' q^m')$ if $m = m'$; hence the index $n$ does not play a role in this case and the matrix $M$ contains all the necessary information from $v_{0,b+1}$ to prove the claim.

By using the usual matrix transformations, we proceed to do the following operations on $M$:
· **Step 1:** we subtract the \((s-1)\)-th row from each of the rows with a different parity from \((s-1)\), except the first row. In the bottom row of \(M\) we have the vector given by the exponents of \(q\) or \(\sqrt{q^r}\) at the entries of the tuple \(v_{0,s}\), labelled by \(q^m\) for \(0 \leq m \leq s\). That is, we have \(v_{0,s}\) with \(s = s - 2j\) when setting \(j = 0\); and from Proposition 5.6 we get

\[
M(s - 1, m) = \begin{cases} 
(p - 1)(q(s) - (s - 2)), & \text{for } m = 0; \\
(p - 1)(q(s - m) - (s - m - 1)), & \text{for } 0 < m \leq s - 1; \\
0, & \text{for } m = s.
\end{cases}
\]

(33)

On the other hand, each row of different parity from that of \(s - 1\) is the \((s - 2j)\)-th row of \(M\) for some \(j \in \{1, \ldots, \left\lfloor \frac{s}{2} \right\rfloor\}\), and corresponds to a tuple \(v_{0,s-2j+1}\). Hence, from Proposition 5.6 we have

\[
M(s - 2j, m) = \begin{cases} 
(p - 1)(q(s - j) - (s - j - 2)), & \text{for } 0 \leq m \leq j; \\
(p - 1)(q(s - m) - (s - m - 1)), & \text{for } j < m \leq s - 1; \\
0, & \text{for } m = s.
\end{cases}
\]

Hence, \(M(s - 1, m) = M(s - 2j, m)\) if and only if \(m \geq j + 1\). We also have \(M(s - 2j, j) = M(s - 1, j) = (p - 1) \cdot (q(s - j) - (s - j - 2) - (q(s - j) - (s - j - 1)) = (p - 1)\) for each \(j \in \{1, \ldots, \left\lfloor \frac{s}{2} \right\rfloor\}\); so by performing Step 1, we obtain a matrix \(M^{(1)} \sim M\) that at each row \((s - 1) - 2j + 1\) satisfies \(M^{(1)}(s - 2j, m) = 0\) for all \(m \geq j + 1\) and \(M^{(1)}((s - 1) - 2j + 1, j) = (p - 1) \neq 0\), and that equals \(M\) in the untouched rows.

· **Step 2:** we subtract the last row from \(q\) times the first row. In the first row, we have the vector corresponding to \(v_{0,2}\). Hence,

\[
M^{(1)}(1, m) = \begin{cases} 
(s - 1)(p - 1), & \text{for } m = 0; \\
(s - m)(p - 1), & \text{for } 0 < m \leq s.
\end{cases}
\]

(1) \(M^{(1)}(1, m)\)

So, using Equation (33) we have that \(q \cdot M^{(1)}(1, m) - M^{(1)}(s - 1, m) =

\[
(p - 1) \cdot (q(s - m) - (q(s - m) - (s - m - 1)) = (p - 1) \cdot \begin{cases} 
\frac{s - 2q}{2}, & \text{for } m = 0; \\
(s - m - 1), & \text{for } 1 \leq m \leq s - 1; \\
0, & \text{for } m = s.
\end{cases}
\]

(34)

Hence, by performing Step 2 we obtain a matrix \(M^{(2)} \sim M\) satisfying \(M^{(2)}(1, m) = (q \cdot M^{(1)}(1, m) - M^{(1)}(s - 1, m) = 0\) for \(m = s\) and \(s - 1\), and \(M^{(2)}(1, m) = q \cdot M^{(1)}(1, m) - M^{(1)}(s - 1, m) = (p - 1)\) for \(m = s - 2\), and that is equal to \(M^{(1)}\) in the untouched rows.

· **Step 3:** we subtract \(q\) times the first row from the \((s - 3)\)-th row. The latter represents the tuple \(v_{0,s-2}\), so from Proposition 5.6 we have

\[
M^{(2)}(s - 3, m) = (p - 1) \begin{cases} 
(q(s - 1) - (s - 1 - 2)), & \text{for } m = 0; \\
(q(s - 1 - m) - (s - 1 - m - 1)), & \text{for } 0 < m < s - 1; \\
0, & \text{for } s - 1 \leq m \leq s.
\end{cases}
\]
By Equation (34), by performing Step 3 we obtain a matrix $M^{(3)} \sim M$ such that

$$M^{(3)}(s - 3, m) = M^{(2)}(s - 3, m) - q \cdot M^{(2)}(1, m) = (p - 1) \cdot \begin{cases} 0 & \text{for } m \geq s - 2; \\ m - s + 2 & \text{for } 0 < m \leq s - 3. \end{cases}$$

**Step 4** This step is divided into sub-steps.
- **Step 4(1)**: we add $q$ times the $(s - 3)$-th row to the $(s - 5)$-th row.
- **Step 4(2)**: we add $q$ times the $(s - 5)$-th row to the $(s - 7)$-th row.

\[\vdots\]
- **Step 4(i)**: we add $q$ times row $(s - (2i + 1))$ to row $(s - (2i + 1 + 1))$.
- Stop the recursion of steps 4(i) when $s - (2i + 1) = 2$ if $s$ is odd or when $s - (2i + 1) = 3$ if $s$ is even.

We will show by induction that if, after performing Step 4(i-1) to the matrix $M^{(4)}$, whose $(s - (2i + 1) - 1)$-th row corresponds to the tuple $v_{0,s-2i}$, we obtain a matrix $M^{(4)}$ whose $(s - (2i + 1))$-th row is equal to

$$M^{(4)}(s - (2i + 1), m) = M^{(4-1)}(s - 2i + 1, m) - q \cdot M^{(4-1)}(s - 2(i + 1) + 1, m)$$

$$= (p - 1) \cdot \begin{cases} 0 & \text{for } m \geq s - (i + 1); \\ m - s + (i + 1) & \text{for } 0 < m \leq s - (i + 2); \end{cases}$$

then, after performing step 4(i) to $M^{(4)}$, we obtain a matrix $M^{(4+1)}$ whose $(s - (2i + 1 + 1))$-th row is equal to

$$M^{(4+1)} = M^{(4)}(s - (2(i + 1) + 1), m) - q \cdot M^{(4)}(s - 2i + 1, m) = (p - 1) \cdot \begin{cases} 0 & \text{for } m \geq s - (i + 2); \\ m - s + (i + 2) & \text{for } 0 < m \leq s - (i + 3). \end{cases}$$

**Base Case:** We set $4^{(0)} = \text{Step 3}$. Then, after performing step $4^{(0)}$, we see in Equation (35) that row $(s - 3)$ of the matrix $M^{(4)} = M^{(3)}$ satisfies Equation (36).

**Induction Step:** Assume that after performing Step 4(i-1) we obtain a matrix $M^{(4)}$ satisfying Equation (36).

Then, by Proposition 5.6 we have

$$M^{(4)}(s - 2(i + 1) + 1, m) =$$

$$= (p - 1) \cdot \begin{cases} 0 & \text{for } m = 0 \\ (q(s - (i + 1)) - (s - (i + 1) - 2)) & \text{for } 0 < m < s - (i + 1), \\ (q(s - (i + 1) - m) - (s - (i + 1) - m - 1)) & \text{for } s - (i + 1) \leq m. \end{cases}$$

Hence, it follows from Equations (36) and (37) that

$$M^{(4)}(s - 2(i + 1) + 1, m) + q \cdot M^{(4)}(s - 2i + 1, m) =$$
Let us illustrate the proof of Claim 6.7. We will use \(\sim\) to denote the usual equivalence relation of matrices up to elementary row transformations. Now we proceed with Step 1 of the proof: we subtract row 6 from row 5 and row 3, and we get

\[
M \sim (-1 + p) \cdot \begin{pmatrix}
3 & 6 & 5 & 4 & 3 & 2 & 1 & 0 \\
1/2(5q - 3) & (4q - 3) & (3q - 2) & (2q - 1) & q & 0 & 0 & 0 \\
1/2(5q - 3) & (5q - 3) & (5q - 3) & (4q - 3) & (3q - 2) & (2q - 1) & q & 0 \\
1/2(6q - 4) & (5q - 4) & (4q - 3) & (3q - 2) & (2q - 1) & q & 0 & 0 \\
1/2(6q - 4) & (6q - 4) & (5q - 4) & (4q - 3) & (3q - 2) & (2q - 1) & q & 0 \\
1/2(7q - 5) & (6q - 5) & (5q - 4) & (4q - 3) & (3q - 2) & (2q - 1) & q & 0
\end{pmatrix}.
\]

Example 6.8. Let us illustrate the proof of Claim 6.7 when \(N = p^3q^7\) and \(p, q \neq 3\). We will work out the proof for the set \(S(0, b)\). Using Proposition 5.8 and following the proof of Claim 6.6 we construct the matrix \(M\) as

\[
M = (-1 + p) \cdot \begin{pmatrix}
3 & 6 & 5 & 4 & 3 & 2 & 1 & 0 \\
1/2(5q - 3) & (4q - 3) & (3q - 2) & (2q - 1) & q & 0 & 0 & 0 \\
1/2(5q - 3) & (5q - 3) & (5q - 3) & (4q - 3) & (3q - 2) & (2q - 1) & q & 0 \\
1/2(6q - 4) & (5q - 4) & (4q - 3) & (3q - 2) & (2q - 1) & q & 0 & 0 \\
1/2(6q - 4) & (6q - 4) & (5q - 4) & (4q - 3) & (3q - 2) & (2q - 1) & q & 0 \\
1/2(7q - 5) & (6q - 5) & (5q - 4) & (4q - 3) & (3q - 2) & (2q - 1) & q & 0
\end{pmatrix}.
\]
Now we apply Step 2: we multiply the first row by \( q \), we subtract row 6 from it and we get
\[
\begin{pmatrix}
1/2(5-q) & 5 & 4 & 3 & 2 & 1 & 0 & 0 \\
1/2(5q-3) & (4q-3) & (3q-2) & (2q-1) & q & 0 & 0 & 0 \\
1/2(2-2q) & (2-q) & 1 & 0 & 0 & 0 & 0 & 0 \\
1/2(6q-4) & (q-3) & (3q-2) & (2q-1) & q & 0 & 0 & 0 \\
1/2(q-1) & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1/2(7q-5) & (6q-5) & (5q-4) & (4q-3) & (3q-2) & (2q-1) & q & 0
\end{pmatrix}
\]

We apply step 3: we subtract \( q \) times the first row from row 4 and we get
\[
\begin{pmatrix}
1/2(5-q) & 5 & 4 & 3 & 2 & 1 & 0 & 0 \\
1/2(5q-3) & (4q-3) & (3q-2) & (2q-1) & q & 0 & 0 & 0 \\
1/2(2-2q) & (2-q) & 1 & 0 & 0 & 0 & 0 & 0 \\
1/2(q^2 + q - 4) & -4 & -3 & -2 & -1 & 0 & 0 & 0 \\
1/2(q-1) & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1/2(7q-5) & (6q-5) & (5q-4) & (4q-3) & (3q-2) & (2q-1) & q & 0
\end{pmatrix}
\]

Next, we apply step 4\(^{(1)}\): we set \( i = 1 \) and we add \( q \) times row 4 to the row 2. Then
\[
\begin{pmatrix}
1/2(5q-3) & (4q-3) & (3q-2) & (2q-1) & q & 0 & 0 & 0 \\
1/2(2-2q) & (2-q) & 1 & 0 & 0 & 0 & 0 & 0 \\
1/2(q^2 + q - 4) & -4 & -3 & -2 & -1 & 0 & 0 & 0 \\
1/2(q-1) & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1/2(7q-5) & (6q-5) & (5q-4) & (4q-3) & (3q-2) & (2q-1) & q & 0
\end{pmatrix}
\]

Since for \( i = 2 \) we have \( 7 - (2 \cdot 2 + 1) = 2 \), we stop the process here. If we now reorder the rows of \( M \) according to the permutation \((15)(23)\) we obtain
\[
\begin{pmatrix}
1/2(q-1) & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1/2(2-2q) & (2-q) & 1 & 0 & 0 & 0 & 0 & 0 \\
1/2(q^3 + q^2 + q - 3) & -3 & -2 & -1 & 0 & 0 & 0 & 0 \\
1/2(q^2 + q - 4) & -4 & -3 & -2 & -1 & 0 & 0 & 0 \\
1/2(5q-3) & (4q-3) & (3q-2) & (2q-1) & q & 0 & 0 & 0 \\
1/2(7q-5) & (6q-5) & (5q-4) & (4q-3) & (3q-2) & (2q-1) & q & 0
\end{pmatrix}
\]

a matrix just as the one described at the end of the proof of Claim 6.7. This shows that the tuples in \( S(0, b) \) for \( N = p^3q^7 \) are linearly independent tuples in the module \( \Omega \).

**Claim 6.9.** The sets \( S(a, 0) \) and \( S(0, b) \) are linearly independent of each other. In other words, there is no linear combination of the divisors in the set \( \{ Z(q^b), Z(p^a) \}_{a \geq 2, b \geq 2} \) that lies in the kernel of \( \overline{\delta} \).

**Proof of Claim 6.9.** By considering Equations (5.8.2-5.8.13) we see that the tuples \( \nu_{0,b} \) appearing in Equations (5.8.2-5.8.7) only have entries that lie in the multiplicative group generated by \( q \in \Omega_p \) for \( 0 \leq n \leq r \) and by \( \sqrt{q^m} \in \Omega_{p^n} \) for \( 0 \leq n \leq r \), \( 1 \leq m \geq s \) and \( p^aq^b \neq p^r q^s \). On the other hand, the tuples \( \nu_{a,0} \) in (7.5.8-13)
only have entries generated by \( p \in \Omega_m \) for \( 0 \leq m \leq s \) and \( \sqrt{p^n} \in \Omega_{p^n m} \) for \( n \geq 1 \) and \( p^aq^b \neq p^r q^s \). Hence, the tuples in \( \{v_{a,0}\}_{a \geq 2} \) are also respectively linearly independent of \( \{v_{0,b}\}_{b \geq 2} \), and the set \( S(a,0) \cup S(0, b) \) is a set of linearly independent tuples.

\[ \square \]

**Claim 6.10.** The tuples in the set \( S(sf) \) are linearly independent from those in any of the sets \( S(0, b) \) and \( S(a, 0) \). In other words, there is no linear combination of the divisors \( \{Z(q^b)\}_{b \geq 2} \cup \{Z(pq), Z(q)\} \) that lies in the kernel of \( \Omega \), and the same holds for the sets or \( \{Z(p^a)\}_{a \geq 2} \cup \{Z(pq), Z(q)\} \).

**Proof of Claim 6.10.** As in the proof of Claim 6.7, we will focus on proving the linear independence of the tuples in the set \( \{v_{0,b}\}_{b \geq 2} \cup \{v_{1,1}, v_{0,1}\} \) and deduce the statement for the other sets from it. The tuple \( v_{0,1} \) is independent from the rest of the tuples in this set, as its entries are generated by \( q \in \Omega_p \) with \( \operatorname{ord}_q(d) = 0 \) and \( \sqrt{q} \in \Omega_p \operatorname{ord}_q(d) > 0 \), while the entries of the other tuples are generated by \( p \in \Omega_p \) with \( \operatorname{ord}_p(d) = 0 \) and \( \sqrt{p^m} \in \Omega_p \operatorname{ord}_p(d) > 0 \). Hence, if we construct the matrix \( M' = (M_{i,j})_{(i,j)} \) by considering the tuple corresponding to the exponents of \( p \) and \( \sqrt{p^m} \) in \( v_{1,1} \) and inserting it at the top of the matrix \( M \) constructed in the proof of Claim 6.7, we obtain an \( s \times (s+1) \) matrix. Since from Equation (5.8.14) we have that

\[ v_{1,1} = (\sqrt{p^m})_{(0,m)}^{(1,\cdots,1)}, \]

the matrix \( M' \) satisfies \( M'_{1,1} \neq 0 \) and \( M'_{1,j} = 0 \) for all \( j \neq 1 \). If we proceed with matrix transformations as in the proof of Claim 6.7, we obtain an \( s \times (s+1) \) matrix whose \((n,n)\)-th entry is non-zero and whose \((n,m)\)-th entry is zero for all \( m > n \). Hence, also \( v_{1,1} \) is linearly independent from the other tuples, as we wanted. By an analogous argument we see that the tuples in \( \{v_{a,0}, v_{1,1}, v_{0,1}\}_{a \geq 2} \) are also independent.

\[ \square \]

**Example 6.11.** Following Example 6.8, we set \( N = p^3 q^7 \) with \( p, q \neq 3 \), and we show explicitly how to carry out the construction of the proof of Claim 6.10 in this case. That is, we show that \( S(0, b) \cup S(sf) \) is a set of linearly independent tuples by constructing \( M' \) for this particular \( N \), i.e.,

\[ M' \sim (-1 + p) \cdot \begin{pmatrix}
 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 3 & 6 & 5 & 4 & 3 & 2 & 1 & 0 \\
 1/2(5q - 3) & (4q - 3) & (3q - 2) & (2q - 1) & q & 0 & 0 & 0 \\
 1/2(2q - 2) & (2 - q) & 1 & 0 & 0 & 0 & 0 & 0 \\
 1/2(q - 4) & (5q - 4) & (4q - 3) & (3q - 2) & (2q - 1) & q & 0 & 0 \\
 1/2(q - 1) & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 1/2(7q - 5) & (6q - 5) & (5q - 4) & (4q - 3) & (3q - 2) & (2q - 1) & q & 0
\end{pmatrix}. \]

So, after repeating the steps in Example 6.8 with the rows \((2) - (s) \) of \( M' \) we obtain

\[ M' \sim (-1 + p) \cdot \begin{pmatrix}
 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 1/2(q - 1) & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 1/2(2q - 2) & (2 - q) & 1 & 0 & 0 & 0 & 0 & 0 \\
 1/2(q^3 + q^2 + q - 3) & -3 & -2 & -1 & 0 & 0 & 0 & 0 \\
 1/2(q^2 + q - 4) & -4 & -3 & -2 & -1 & 0 & 0 & 0 \\
 1/2(5q - 5) & 5 & 4 & 3 & 2 & 1 & 0 & 0 \\
 1/2(7q - 5) & (6q - 5) & (5q - 4) & (4q - 3) & (3q - 2) & (2q - 1) & q & 0
\end{pmatrix}, \]
and the linear independence follows.

Hence, to conclude with the proof of Proposition 6.12, we combine the results obtained in Claims 6.7, 6.9 and 6.10 and obtain that there cannot be any other linear combination of the divisors $Z(d)$ in the kernel of $\overline{\delta}$ other than those in Equations (31) and (32). This fact together with Claim 6.6 show that the divisors $D(a, b)$ generate the kernel of $\overline{\delta}$. □

**Lemma 6.12.** The divisors $D(a, b)$ for $1 \leq a \leq r$ and $1 \leq b \leq s$ are all linearly independent. That is,

$$\langle [D(a, b)] : 1 \leq a \leq r, 1 \leq b \leq s \rangle = \bigoplus_{a, b} \langle [D(a, b)] \rangle.$$

**Proof.** By Definition 6.1 of the divisors $D(a, b)$ and Theorem 3.2 we have:

- $D(a, b) \in (Z(p^a q^b))$ for all $a, b \geq 2$;
- $D(1, b) \in (Z(p^a q^b)) \oplus (Z(p^a q^b))$ for all $b \geq 2$;
- $D(a, 1) \in (Z(p^a q^b)) \oplus (Z(p^a q^b))$ for all $a \geq 2$;
- $D(1, 1) \in (Z(d) : d \in D_N^N)$.

Hence, using Theorem 3.5 we obtain that

$$\langle [D(a, b)] : 1 \leq a \leq r, 1 \leq b \leq s \rangle = \bigoplus_{1 \leq b \leq s} \langle [D(a, b)] \rangle,$$

which proves the statement. □

**Corollary 6.13.** Let $N = p^r q^s$ with $p$ and $q$ two odd primes. The divisors $[D(a, b)]$, for $1 \leq a \leq r$ and $1 \leq b \leq s$, form a basis of $\ker(\overline{\delta}|_{C(N)})$. That is,

$$\ker(\overline{\delta}|_{C(N)}) = \bigoplus_{1 \leq a \leq r, 1 \leq b \leq s} \langle [D(a, b)] \rangle.$$

**Proof.** This proposition follows immediately from Proposition 6.5 and Lemma 6.12. □

**Proposition 6.14.** For any $1 \leq a \leq r$ and $1 \leq b \leq s$, the order of $[D(a, b)]$ in $C(N)$ is given by the integer $m(a, b)$ given in Definition 6.3.

**Proof.** We first write down the proof for the case $p, q \neq 3$.

If $a, b \geq 2$ we know that the order of $D(a, b)$ is the order of the divisor $Z(p^a q^b)$ given in Theorem 3.2. We will compute the order of the other divisors using Theorem 3.14. Thus, we will have to compute $\gcd(D(a, b))$ and $h(D(a, b))$ for each divisor with either $a = 1$ or $b = 1$ – see Definitions 3.12 and 3.13.

Let us start with $D(1, 2)$. From Theorem 3.15 and Lemma 3.24 we have that

$$V(D(1, 2)) = p^{-1}(p + 1) ((p, -1, 0, \ldots, 0) \otimes q^{(s-1)} : (1, 0, \ldots, 0, -1)) =$$

$$p^{-1}(p^2 - 1) \cdot (1, 0, \ldots, 0) \otimes q^{(s-1)} \cdot (1, 0, \ldots, 0, -1)) =$$

$$p^{-1}((p - 1)(1, 0, \ldots, 0) \otimes (1, 0, \ldots, 0, -1)) =$$

$$p^{-1}((p - 1)(1, 0, \ldots, 0, -1)) =$$

$$p^{-1}(p + 1) q^{(s-1)} \cdot (1, -1, 0, \ldots, 0, -1, 1, 0, \ldots, 0, 0, \ldots, 0).$$

Hence, $\gcd(D(1, 2)) = p^{-1} q^{(s-1)} (p + 1)$ and $V(D(1, 2)) = (1, -1, 0, \ldots, 0, -1, 0, \ldots, 0, 0, \ldots, 0)$. 


For any prime $l \neq p, q$ and any $d \mid N$ we have $\text{ord}_l(d') = 0$. Hence, from Definition 3.13 it follows

$$Pw_l(D(1,2)) = 1 + (-1)^{pq} + (-1)^q \equiv 0 \pmod{2}$$

for all $l \neq p, q$. On the other hand, $Pw_p(D(1,2)) = 1 + (-1)^q = 0$, and $Pw_q(D(1,2)) = 1 + (-1)^p = 0$ if $s$ is odd and $Pw_q(D(1,2)) = 1 + (-1)^p + pq = 0$ if $s$ is even. Hence, $h(D(1,2)) = 1$. Similarly, we obtain that

$$V(D(1,b)) = p^{r-1}(p + 1)q^i \cdot (q_1, -q, 0, \cdots, 0, -1, l_p, 0, \cdots, 0, 1, q^{j-1}, -1, p^{j-1}, 0, \cdots, -q, p^{j-1}, q^{p^{j-1}}, 0, \cdots, 0),$$

where $j = \left[\frac{s+1-b}{2}\right]$ and GCD$(D(1,2)) = p^{r-1}q^j(p + 1)$. A computation shows that $Pw_l(D(1,b)) \equiv 0 \pmod{2}$ and $h(D(1,b)) = 1$ for all primes $l$. By analogous arguments we obtain that

$$\text{GCD}(a, 1) = \begin{cases} p^{r-1}q^{s-1}(q + 1) & \text{if } a = 2; \\ q^{s-1}p^{i}(q + 1) & \text{if } a > 2 \text{ and } i = \left[\frac{r+1-q}{2}\right]; \end{cases}$$

and $h(D(a, 1)) = 1$ for all $a \geq 2$. Finally, for the divisor $(D, 1)$ we have

$$V(D(1,1)) = q^{s-1}(q + 1) \left(p^{r-1}(p + 1) \cdot (q_1, -q, 0, \cdots, 0, -1, l_p, 0, \cdots, 0) - \left(p^{r-1}(p + 1)q^{s-1}(q^2 - 1) \cdot (1, -1, 0, \cdots, 0) = p^{r-1}q^{s-1}(p + 1)(q + 1) \cdot (1, -1, 0, \cdots, 0, -1, l_p, 0, \cdots, 0) \right) \right);$$

so GCD$(D(1,1)) = p^{r-1}q^{s-1}(p + 1)(q + 1)$ and $h(D(1,1)) = 1$. Finally, if we let $m(a, b) = \text{order}(D(a, b))$ and we apply Theorem 3.14, we obtain

$$m(a, b) = \text{num} \left(\frac{k(N) \cdot h(D(a, b))}{24 \cdot \text{GCD}(D(a, b))}\right)$$

and the statement of the proposition follows, which concludes the proof of the lemma for $p, 1 \neq 3$. For the case $p = 3$, we use that the order of $D(2, 1)$ is then $\text{num} \left(\frac{(q-1)^2}{3}\right) = \text{num} \left(\frac{(p^2-1)(q-1)}{24}\right)$. Finally, for $q = 3$ we can use analogous argument.

\[\square\]

**Proof of Theorem 6.4.** This theorem now follows from Corollary 6.13 and Proposition 6.14. 

\[\square\]

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