Algebraic Construction of Spherical Harmonics

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The angular wave functions for a hydrogen atom are well known to be spherical harmonics, and are obtained as the solutions of a partial differential equation. However, the differential operator is given by the Casimir operator of the $SU(2)$ algebra and its eigenvalue $l(l + 1)\hbar^2$, where $l$ is non-negative integer, is easily obtained by an algebraic method. Therefore the shape of the wave function may also be obtained by extending the algebraic method. In this paper, we describe the method and show that wave functions with different quantum numbers are connected by a rotational group in the cases of $l = 0, 1$ and $2$.

Key words: Mirror Operator, Spherical Harmonics

1. Introduction—Representation of Angular Moment

Spherical harmonics (hereafter abbreviated to SHs), $Y_{l,m}(\theta, \phi)$, are usually obtained by solving the following partial differential equation using the Laplacian on $S_2$ [1]–[6]:

$$\left[ -\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) - \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right] Y(\theta, \phi) = \lambda Y(\theta, \phi),$$

(1)

where $\lambda$ is the real eigenvalue, which is later shown to be $l(l + 1)$. However, there is another method of solving this equation by using algebra, that does not depend on the choice of coordinates. In this algebraic method the eigenvalues are easily obtained but eigenfunctions (angular wave functions: SHs) cannot be calculated. The purpose of this manuscript is to show how to obtain SHs by algebraic method.

In this section, we briefly sketch the conventional algebraic method to provide a self-contained explanation [1]–[7]. Then using the ideas, tools, and notations expressed here, we will show how to obtain the wave function. Hereafter, we utilize the natural unit $\hbar = 1$ for simplicity.

Note that the differential operator in the l.h.s. of Eq. (1) is given by

$$\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2,$$

(2)

with

$$\hat{L}_x = -i \left( y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y} \right),$$

$$\hat{L}_y = -i \left( z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z} \right),$$

$$\hat{L}_z = -i \left( x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) = -i \frac{\partial}{\partial \phi},$$

where $\hat{L}_i$ ($i = x, y, z$) satisfies the $SU(2)$ algebra:

$$[\hat{L}_i, \hat{L}_j] = i \epsilon_{ijk} \hat{L}_k,$$

(6)

$\epsilon_{ijk}$ is the absolute antisymmetric unit tensor with $\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1$, $\epsilon_{213} = \epsilon_{132} = \epsilon_{321} = -1$, and other terms equal to 0. We find that $\hat{L}_z$ is a Casimir operator:

$$[\hat{L}_z, \hat{L}_i] = 0 \ (i = x, y, z).$$

(7)

We can diagonalize one of the three angular momentum operators. Usually we select $\hat{L}_z$ to be diagonal. Furthermore, by using the notation

$$\hat{L}_\pm = \hat{L}_x \pm i \hat{L}_y,$$

(8)

we obtain

$$[\hat{L}_z, \hat{L}_\pm] = \pm \hat{L}_\pm.$$

(9)

Let us consider the following eigenvalue equation of $\hat{L}_z$:

$$\hat{L}_z |m\rangle = m|m\rangle,$$

(10)
where the ket vector $|m\rangle$ denotes the eigenstate with eigenvalue $m$ [2]. For consistency with Eq. (9), we obtain
\[
\hat{L}_z(\hat{L}_\pm|m\rangle) = (m \pm 1)(\hat{L}_\pm|m\rangle).
\]
This means we have a new ket with an eigenvalue that differs by $\pm 1$.
\[
|m\pm 1\rangle \simeq \hat{L}_\pm|m\rangle,
\]
where $\simeq$ denotes the ambiguity of the constant coefficient, which should be determined from the normalization conditions except total phase. We consider the maximum state $m_{\text{max}} = l$ that satisfies
\[
\hat{L}_+|l\rangle = 0
\]
with the unit norm
\[
<l|l\rangle = 1.
\]
Then, from the relation
\[
\hat{L}^2 = \hat{L}_-\hat{L}_+ + \hat{L}_z(1 + \hat{L}_z),
\]
we obtain
\[
\hat{L}^2|l\rangle = l(l + 1)|l\rangle.
\]
By applying $\hat{L}_-$ to the highest-weight state $|l\rangle$ several times, we obtain lower-lying states such as,
\[
|l, m\rangle \simeq (\hat{L}_-)^{-m}|l\rangle,
\]
where $\simeq$ denotes the ambiguity of the constant coefficient, which should be determined from the normalization condition:
\[
<l, m|l, m\rangle = 1.
\]
Then we have the important results
\[
\hat{L}^2|l, m\rangle = l(l + 1)|l, m\rangle,
\]
where the first relation follows from
\[
[\hat{L}^2, \hat{L}_\pm] = 0.
\]
Note that we obtain
\[
\hat{L}_-|l, -l\rangle = 0,
\]
from the zero norm condition:
\[
||\hat{L}_-|l, -l\rangle\|^2 \equiv <l, -l|\hat{L}_+\hat{L}_-|l, -l\rangle = 0.
\]
This equality follows from
\[
[\hat{L}_+, (\hat{L}_-)^{2+l}]|l\rangle = 0.
\]
Then we obtain
\[
-l \leq m \leq +l.
\]
We finally write down the explicit form of the normalized states as
\[
|l, m\rangle = \frac{1}{\sqrt{2l+1}} \left( (l+m)! \sqrt{(l-m)!} \hat{L}_-^{l-m}|l\rangle \right).
\]
Another representation is
\[
|l, \pm m\rangle = C(l, |m\rangle)(\hat{L}_\pm)^{m}|l, 0\rangle,
\]
where $C(l, |m\rangle) \equiv \sqrt{(l-m)!/(l+m)!}$.

The usual wave function of the angle (SHs) can be given by specifying the representation such as $(\theta, \phi)$:
\[
|Y_{\ell m}(\theta, \phi)\rangle = <\theta, \phi|m\rangle.
\]
The “bra” $<\theta, \phi|m\rangle$ denotes the representation and the “ket” $|l, m\rangle$ denotes the state. The inner product of these two vectors gives the usual wave function.

2. Directional Parity (Mirror) Operator

We define the space inversion (mirror) operator for each direction, $\hat{P}_x$, $\hat{P}_y$, $\hat{P}_z$ in the following [8]:
\[
\hat{P}_x(x, p_x, \text{other})\hat{P}_x^{-1} = (-x, -p_x, \text{other}),
\]
\[
\hat{P}_y(y, p_y, \text{other})\hat{P}_y^{-1} = (-y, -p_y, \text{other}),
\]
\[
\hat{P}_z(z, p_z, \text{other})\hat{P}_z^{-1} = (-z, -p_z, \text{other}.
\]

Therefore, from Eqs. (3), (4) and (5), we obtain
\[
\hat{P}_x\hat{L}_+\hat{P}_x^{-1} = \hat{L}_x, \hat{P}_y\hat{L}_y\hat{P}_y^{-1} = \hat{L}_y, \hat{P}_z\hat{L}_z\hat{P}_z^{-1} = \hat{L}_z.
\]

Then the relation $[\hat{L}^2, \hat{P}_x] = 0$ follows, as is expected. Furthermore, we have the trivial condition
\[
\hat{P}_x^2 = \hat{P}_y^2 = \hat{P}_z^2 = 1.
\]

The product of two different mirror operators is a rotation operator, for example,
\[
\hat{P}_x\hat{P}_y = e^{i\hat{L}_z\pi},
\]
\[
\hat{P}_y\hat{P}_z = e^{i\hat{L}_x\pi},
\]
\[
\hat{P}_z\hat{P}_x = e^{i\hat{L}_y\pi},
\]
from which the following interesting property is obtained:
\[
\hat{P}_x\hat{L}_x|l, m\rangle = \hat{P}_x\hat{L}_x\hat{P}_x^{-1}|l, m\rangle = \hat{L}_x|\hat{P}_x|l, m\rangle = m|\hat{P}_x|l, m\rangle.
\]

A similar relation also holds for $\hat{P}_y$. Therefore, we have
\[
\hat{L}_x(\hat{P}_x|l, m\rangle) = -m|\hat{P}_x|l, m\rangle,
\]
\[
\hat{L}_z(\hat{P}_y|l, m\rangle) = -m|\hat{P}_y|l, m\rangle.
\]
However for $\hat{P}_z$, we have
\begin{equation}
\hat{L}_z(\hat{P}_z |l, m \rangle) = +m(\hat{P}_z |l, m \rangle).
\end{equation}

Thus, we can assume the following three equations:
\begin{align}
\hat{P}_x |l, m \rangle &= \alpha_x (l, m) |l, -m \rangle, \\
\hat{P}_y |l, m \rangle &= \alpha_y (l, m) |l, -m \rangle, \\
\hat{P}_z |l, m \rangle &= \alpha_z (l, m) |l, m \rangle,
\end{align}
where $\alpha_x (l, m)$ are unknown c-numbers but with the properties
\begin{align}
\alpha_x (l, m) \alpha_x (l, -m) &= 1, \\
\alpha_y (l, m) \alpha_y (l, -m) &= 1, \\
\alpha_z (l, m) \alpha_z (l, -m) &= 1.
\end{align}

The $m = 0$ state has rotational symmetry around the $z$-axis since
\[
L_z |l, 0 \rangle = 0, \quad \text{and} \quad e^{-iL_\phi |l, 0 \rangle} = |l, 0 \rangle.
\]
Thus, the state should have $x$- and $y$-axis mirror symmetry:
\begin{align}
\hat{P}_x |l, 0 \rangle &= |l, 0 \rangle, \quad \hat{P}_y |l, 0 \rangle = |l, 0 \rangle.
\end{align}

Then we obtain
\[
\hat{P}_x \hat{L}_\pm \hat{P}_x^{-1} = \hat{L}_\mp.
\]
On the other hand,
\[
\hat{P}_y |l, \pm |m \rangle = C(l, |m|) \hat{P}_y (\hat{L}_\mp)^{|m|} \hat{P}_y^{-1} |l, 0 \rangle,
\]
where we utilized
\[
\hat{P}_x \hat{L}_\pm \hat{P}_x^{-1} = \hat{L}_\mp.
\]
Therefore, we obtain
\[
\alpha_x (l, m) = 1, \quad \alpha_y (l, m) = (-1)^m.
\]

We obtain $\alpha_z (l, m)$ as follows. From $\hat{P}_x \hat{P}_z = e^{i\hat{L}_z}$, and an explicit form of $e^{i\hat{L}_z}$ obtained from the $SU(2)$ representation (see Eq. (62) for $l = 1$ and Eq. (84) for $l = 2$, and see also Appendix Eq. (A.5)), we have
\[
e^{i\hat{L}_z} |l, m \rangle = (-1)^{|l|+m} \hat{P}_z |l, m \rangle.
\]
Thus, we obtain $\alpha_z (l, m) = (-1)^{|l|+m}$.

To summarize,
\begin{align}
\hat{P}_x |l, m \rangle &= |l, -m \rangle, \\
\hat{P}_y |l, m \rangle &= (-1)^{|m|} |l, -m \rangle, \\
\hat{P}_z |l, m \rangle &= (-1)^{|l|+m} |l, m \rangle.
\end{align}

The mirror operator in arbitrary direction is discussed in Appendix.

### 3. $l = 0$ (s-state) Case
We start with the trivial case $l = 0$.

We have
\[
\hat{L}_z |s \rangle = 0
\]
for s-state $|s \rangle$. Then the eigenvalue of $\hat{L}_z$ should be zero from Eq. (21). Therefore, we only have the $m = 0$ state, which means that
\[
\hat{L}_z |s \rangle = \hat{L}^- |s \rangle = 0.
\]

Then we obtain
\[
\hat{L}_z |s \rangle = \hat{L}_+ |s \rangle = 0.
\]

These equations imply the following rotational invariance:
\[
e^{-i\hat{L}_\phi} |s \rangle = |s \rangle, \quad e^{-i\hat{L}_\phi} |s \rangle = |s \rangle, \quad e^{-i\hat{L}_\phi} |s \rangle = |s \rangle,
\]
where $\theta_x$, $\theta_y$, and $\theta_z$ are arbitrary independent angles. Then the state $|s \rangle$ should satisfy
\[
Y_{00}(\theta, \phi) = < \theta, \phi |s \rangle = \text{const.}
\]

Note that when we illustrate the form of the angle wave function, we take the radial length $r = |Y_{lm}(\theta, \phi)|$ as the magnitude of wave function, and show the wave function as the surface $r = r(\theta, \phi)$. Thus, the form of wave function given by $r = Y_{00}(\theta, \phi)$ is a sphere.

### 4. $l = 1$ (p-state) Case
Let us start to find the form of the $l = 1$ states. First we define the state vectors,
\[
|1, 1 \rangle := \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad |1, 0 \rangle := \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad |1, -1 \rangle := \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.
\]

Then the representation of the angular momentum
\[
|L_j \rangle_{mn} = \langle m |L_j |1, n \rangle, \quad (j = x, y, z)
\]
takes the following form (In the matrix representation, we write operators in bold):
\[
L_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix},
\]
\[
L_y = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 \end{pmatrix},
\]
\[
L_z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
\]

Then the rotational matrix can be calculated by the Taylor expansion of the following matrix-valued exponent:
\[
e^{-i\hat{L}_\phi} = \sum_{n=0}^{\infty} \frac{1}{n!} (-i\hat{L}_\phi)^n.
\]
To carry out this calculation, we predict the form of the matrix \((L_z)^n\) and prove it by mathematical induction. Then we calculate the sum of the series. We obtain

\[
e^{-iL_x\phi} = \frac{1}{2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \cos \phi - \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \sin \phi, \quad (61)
\]

\[
e^{-iL_y\phi} = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 0 \end{pmatrix} \cos \phi - \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \sin \phi, \quad (62)
\]

\[
e^{-iL_x\phi} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \cos \phi - i \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \sin \phi. \quad (63)
\]

We construct the real spherical harmonics (hereafter abbreviated to RSHs) to show their form graphically. The SH itself is a complex function, and the method of constructing the RSH from the SH is well known [7]. In accordance with this construction, we define the following states:

\[
|x\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} (|1, -1\rangle - |1, 1\rangle), \quad (64)
\]

\[
|y\rangle = \frac{i}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \frac{i}{\sqrt{2}} (|1, 1\rangle + |1, -1\rangle), \quad (65)
\]

\[
|z\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = |1, 0\rangle. \quad (66)
\]

Then we discuss the form of the states. We start with the \(|z\rangle\) state. First, this state has rotational symmetry around the \(z\)-axis:

\[
e^{-iL_x\phi}|z\rangle = |z\rangle. \quad (67)
\]

Second, the \(|z\rangle\) state has parity odd for \(\hat{P}_z\), from Eq. (50), which means that the state has \(xy\)-plane as the node plane.

\[
\hat{P}_z|z\rangle = -|z\rangle. \quad (68)
\]

Then the form of \(|z\rangle\) \((r = |<\theta, \phi|z\rangle|)\) is considered to be that in Fig. 1, where the different contrasts show the phase inversion. Figure 1 is a rough sketch and the precise form will be discussed later.

Next we consider the forms of the other two states. First, from Eqs. (61), (65) and (66), we obtain

\[
e^{-iL_x(-\pi/2)}|z\rangle = |y\rangle. \quad (69)
\]

Second, from Eqs. (62), (64) and (66), we obtain

\[
e^{-iL_y(\pi/2)}|z\rangle = |x\rangle. \quad (70)
\]

Therefore, these three states have the same form but different orientations. We obtain the form of \(|z\rangle\) more precisely as follows. The rotation of the \(|z\rangle\) state around the \(y\) axis by angle \(\alpha\) gives the following relation from Eqs. (62), (64) and (66):

\[
e^{-iL_y\alpha}|z\rangle = \cos \alpha |z\rangle + \sin \alpha |x\rangle. \quad (71)
\]

Then we examine the \(z\) direction. For this purpose, we multiply "bra" \(<\theta = 0|\) from the left. We obtain

\[
<\theta = 0|e^{-iL_y\alpha}|z\rangle = \cos \alpha <\theta = 0|z\rangle + \sin \alpha <\theta = 0|x\rangle. \quad (72)
\]

The l.h.s. can be calculated as

\[
e^{iL_y\alpha}|\theta = 0\rangle = |\theta = \alpha, \phi = \pi\rangle. \quad (73)
\]

By taking the Hermitian conjugate, we obtain

\[
<\theta = 0|e^{-iL_y\alpha}|z\rangle = <\theta = \alpha, \phi = \pi|z\rangle = <\theta = \alpha|z\rangle, \quad (74)
\]

where the final equality originates from the rotational symmetry of \(|z\rangle\) about the \(z\) axis. This situation is shown in Fig. 2. \(<\theta = \alpha, \phi = \pi|z\rangle\) is shown by the arrow in the left figure. To obtain the length of this arrow, we rotate the state \(|z\rangle\) around the \(y\) axis by angle \(\alpha\) and examine the \(z\) direction.

Furthermore, from the form of \(|x\rangle\), we have

\[
<\theta = 0|x\rangle = 0. \quad (74)
\]

Fig. 1. Schematic diagram of \(|z\rangle\) state.

Fig. 2. Rotation of \(|z\rangle\) around \(y\) axis by angle \(\alpha\).
From Eqs. (72), (73) and (74), we have

This comes from Eq. (70) and Fig. 1, later explicitly shown in Fig. 4, as obtained from Eqs. (69), (70) and (77).

Then the wave function can be written as

\[ e^{-iL_y(n/2)}|z> = |y>, \]

Then the rotation matrices around the \( x, y, \) and \( z \) axes can be calculated as

\[
\begin{align*}
 L_x &= \begin{pmatrix}
 0 & 1 & 0 & 0 \\
 1 & 0 & \sqrt{6}/2 & 0 \\
 0 & \sqrt{6}/2 & 0 & 1 \\
 0 & 0 & 1 & 0
\end{pmatrix}, \\
 L_y &= i \begin{pmatrix}
 0 & 1 & 0 & 0 \\
 1 & 0 & -\sqrt{6}/2 & 0 \\
 0 & \sqrt{6}/2 & 0 & -1 \\
 0 & 0 & 1 & 0
\end{pmatrix}, \\
 L_z &= \begin{pmatrix}
 2 & 0 & 0 & 0 \\
 0 & 1 & 0 & 0 \\
 0 & 0 & 1 & 0 \\
 0 & 0 & 0 & -1
\end{pmatrix}.
\end{align*}
\]

5. \( l = 2 \) (d-state) Case

Next we consider the \( l = 2 \) case. The matrix elements of the angular moment can be calculated as

\[ [L_3]_{mn} \equiv <x, m|\hat{L}_3|x, n>, \quad (j = x, y, z) \]  

with the notation

\[
\begin{align*}
 |2, 2> &= \begin{pmatrix}
 1 \\
 0 \\
 0 \\
 0
\end{pmatrix}, & |2, 1> &= \begin{pmatrix}
 1 \\
 0 \\
 0 \\
 0
\end{pmatrix}, & |2, 0> &= \begin{pmatrix}
 0 \\
 0 \\
 1 \\
 0
\end{pmatrix}, \\
 |2, -1> &= \begin{pmatrix}
 0 \\
 0 \\
 0 \\
 1
\end{pmatrix}, & |2, -2> &= \begin{pmatrix}
 0 \\
 0 \\
 0 \\
 1
\end{pmatrix}.
\end{align*}
\]

Then we have the explicit matrix forms

\[ e^{-iL_{3x}\phi} = \begin{pmatrix}
 A & iB & C & iD & E \\
 iB & F & iG & H & iD \\
 C & iG & J & iG & C \\
 iD & H & iG & F & iB \\
 E & iD & C & iB & A
\end{pmatrix}, \]

and (82).

\[ e^{-iL_{3y}\phi} = \begin{pmatrix}
 A & B & -C & -D & E \\
 B & F & G & H & D \\
 -C & G & J & G & -C \\
 D & -H & -G & F & B \\
 E & D & -C & B & A
\end{pmatrix}, \]

Now we consider the \( l = 3 \) case. The angular moment can be calculated as

\[ [L_3]_{m} \equiv <x|\hat{L}_3|x>, \quad (j = x, y, z) \]  

with the notation

\[
\begin{align*}
 |3, 3> &= \begin{pmatrix}
 1 \\
 0 \\
 0 \\
 0
\end{pmatrix}, & |3, 2> &= \begin{pmatrix}
 0 \\
 1 \\
 0 \\
 0
\end{pmatrix}, & |3, 1> &= \begin{pmatrix}
 0 \\
 0 \\
 1 \\
 0
\end{pmatrix}, & |3, 0> &= \begin{pmatrix}
 0 \\
 0 \\
 0 \\
 1
\end{pmatrix}, \\
 |3, -1> &= \begin{pmatrix}
 0 \\
 0 \\
 0 \\
 1
\end{pmatrix}, & |3, -2> &= \begin{pmatrix}
 0 \\
 0 \\
 0 \\
 1
\end{pmatrix}, & |3, -3> &= \begin{pmatrix}
 0 \\
 0 \\
 0 \\
 1
\end{pmatrix}.
\end{align*}
\]

Then we have the explicit matrix forms

\[ e^{-iL_{3x}\phi} = \begin{pmatrix}
 A & B & -C & -D & E \\
 B & F & G & H & D \\
 -C & G & J & G & -C \\
 D & -H & -G & F & B \\
 E & D & -C & B & A
\end{pmatrix}, \]

Then we conclude this section, we show all the forms of the \( l = 1 \) members i.e., the \( |x>, |y>, \) and \( |z> \) states, and their relations in Fig. 4, as obtained from Eqs. (69), (70) and (77).
where
\[ A = \frac{3}{8} + \frac{1}{8} \cos 2\phi + \frac{1}{2} \cos \phi, \]
\[ B = -\frac{1}{2} \sin \phi - \frac{1}{4} \sin 2\phi, \]
\[ C = \frac{\sqrt{6}}{8} \cos 2\phi - 1, \]
\[ D = -\frac{1}{4} \sin 2\phi + \frac{1}{2} \sin \phi, \]
\[ E = \frac{3}{8} + \frac{1}{8} \cos 2\phi - \frac{1}{2} \cos \phi, \]
\[ F = \frac{1}{2} (\cos \phi + \cos 2\phi), \]
\[ G = -\frac{\sqrt{6}}{4} \sin 2\phi, \]
\[ H = \frac{1}{4} (\cos 2\phi - \cos \phi), \]
\[ J = \frac{1}{4} + \frac{3}{4} \cos 2\phi. \] (86)

The RSHs are given as follows \[7\]:
\[ |xy\rangle = -\frac{i}{\sqrt{2}} |(2, 2) - (2, -2)\rangle, \]
\[ |x^2 - y^2\rangle = \frac{1}{\sqrt{2}} |(2, 2) + (2, -2)\rangle, \]
\[ |yz\rangle = \frac{i}{\sqrt{2}} |(2, 1) + (2, -1)\rangle, \]
\[ |xz\rangle = -\frac{1}{\sqrt{2}} |(2, 1) - (2, -1)\rangle, \]
\[ |z^2\rangle = |2, 0\rangle. \] (87)

We start with the analysis of $|xy\rangle$. The first observation is the rotation of $|xy\rangle$ around the $z$ axis by $-\pi/2$:
\[ e^{iL_x(\pi/2)}|xy\rangle = -|xy\rangle. \] (88)

This means that the state is fourfold symmetric except for the phase change $e^{i\pi}$. Second, the state has two node planes,
\[ P_x|xy\rangle = -\frac{i}{\sqrt{2}} P_x|(2, 2) - (2, -2)\rangle \]
\[ = -\frac{i}{\sqrt{2}} |(2, -2) - (2, 2)\rangle = -|xy\rangle, \] (89)
\[ P_y|xy\rangle = -\frac{i}{\sqrt{2}} P_y|(2, 2) - (2, -2)\rangle \]
\[ = -\frac{i}{\sqrt{2}} |(2, -2) - (2, 2)\rangle = -|xy\rangle. \] (90)

This means that the $yz$ plane and the $xz$ plane are node planes. Then we have the form of $|xy\rangle$ shown in Fig. 5.

Another three states can be constructed easily from $|xy\rangle$ as follows:
\[ e^{iL_y(\pi/2)}|xy\rangle = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = |yz\rangle, \] (91)
\[ e^{-iL_x(\pi/2)}|xy\rangle = -\frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = |xz\rangle, \] (92)
\[ e^{iL_z(\pi/4)}|xy\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = |x^2 - y^2\rangle. \] (93)

Therefore, these $|yz\rangle$, $|xz\rangle$, and $|x^2 - y^2\rangle$ states have the same form as $|xy\rangle$ but different orientations.

6. $|z^2\rangle$ State

Finally, we consider the form of $|z^2\rangle$. We have two symmetries,
\[ e^{-iL_x\phi}|z^2\rangle = |z^2\rangle, \] (94)
\[ P_x|z^2\rangle = |z^2\rangle. \] (95)

These equations show rotational symmetry around the $z$ axis, and reflection (mirror) symmetry about the $xy$ plane, that are insufficient informations to construct the form of the state $|z^2\rangle$.

Let us rotate $|z^2\rangle$ around the $x$ axis by angle $-\alpha$.
\[ e^{iL_x\alpha}|z^2\rangle = \begin{pmatrix} A & iB & C & iD \\ iB & F & iG & H \\ C & iG & J & iG \\ iD & H & iG & F \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \]
\[ = \begin{pmatrix} C \\ iG \\ J \\ iG \\ C \end{pmatrix} = \sqrt{2}C(-\alpha)|x^2 - y^2\rangle \]
\[ + \sqrt{2}G(-\alpha)|yz\rangle + J(-\alpha)|z^2\rangle. \] (96)
Let us examine the \( z \) direction. We multiply the “bra” \( <\theta = 0| \) to both sides of Eq. (96).

\[
<\theta = 0| e^{iL_\alpha} |z^2 > = \sqrt{2}C(-\alpha) <\theta = 0|x^2 - y^2 > + \sqrt{2}G(-\alpha) <\theta = 0|yz > + J(-\alpha) <\theta = 0|z^2 >.
\]

(97)

Note that \( <\theta = 0|x^2 - y^2 > = <\theta = 0|yz > = 0 \) hold here. This comes from the following reasons. From Eq. (93) and Fig. 5, we have \( <\theta = 0|x^2 - y^2 > = 0 \). From Eq. (91) and Fig. 5, we have \( <\theta = 0|yz > = 0 \). Both are later shown in Fig. 9 explicitly. Furthermore, from

\[
e^{-iL_\alpha} |\theta = 0 > = |\theta = \alpha, \phi = -\pi/2 >,
\]

we obtain

\[
<\theta = 0| e^{iL_\alpha} |z^2 > = <\theta = \alpha, \phi = -\pi/2|z^2 > = <\theta = \alpha|z^2 >.
\]

(98)

where the final equality originates from the rotational symmetry around the \( z \) axis from Eq. (94). Then we obtain

\[
<\theta = \alpha|z^2 > = J(-\alpha) <\theta = 0|z^2 >.
\]

(99)

In an explicit form, we have

\[
Y_z(\theta, \phi) = \frac{l_0}{4} (1 + 3 \cos 2\theta), \quad l_0 \equiv Y_z(\theta = 0).
\]

(100)

This method is graphically shown in Fig. 6. The dark gray ellipsoid shows the \( |z^2 > \) state and the light gray ellipsoid shows the state \( |z^2 > \equiv e^{iL_\alpha} |z^2 > \). Then, we easily find that

\[
<\theta = \alpha, \phi = -\pi/2|z^2 > = <\theta = 0|z^2 >.
\]

(The length of the dashed arrow of \( |z^2 > \) is the same as the length of \( |z^2 > \) in the \( z \) direction.) Furthermore, \( |z^2 > \) can be expanded in the form of Eq. (96). We therefore obtain Eq. (100). From this result, we have the form of \( <\theta|z^2 > \) shown in Fig. 7.

In Fig. 8, the dark gray part and light gray part (similar to a torus but with a point hole) have opposite phases. The node plane becomes two cones with \( \theta = 54.7^\circ \) and \( \theta = 125.3^\circ \).

The states comprising the members of \( l = 2 \) are shown in Fig. 9. One of the remaining problems is the relation between \( |z^2 > \) and the other states. From Fig. 9, we search for the states that may become elements to construct the \( |z^2 > \) state. The rotation of \( |xz > \) around the \( y \) axis by \(-\pi/4\) with the rotation of \( |yz > \) around the \( x \) axis by \( \pi/4 \) may have similar forms to \( |z^2 > \) as shown in Fig. 10.

This idea can be realized in the following calculation:

\[
e^{iL_\pi/4}|xz > + e^{-iL_\pi/4}|yz > \\
= (-\frac{1}{2} |x^2 - y^2 > + \frac{\sqrt{3}}{2} |z^2 >) \\
+ (\frac{1}{2} |x^2 - y^2 > + \frac{\sqrt{3}}{2} |z^2 >) \\
= \sqrt{3}|z^2 >.
\]
Or alternatively,\

\[ |z^2> = \frac{1}{\sqrt{3}} (e^{iL_y \pi/4} |xz> + e^{-iL_x \pi/4} |yz>). \] (101)

In this way, we obtain the relation between \(|z^2>\) and the other states.

7. Functional Form of \(|xy> State\)

To conclude the study of the \(l = 2\) state, we finally discuss the functional form of the \(|xy> state\). From the fourfold property of the \(|xy> state\), it is sufficient to study only one piece of four leaves of \(|xy>\). For this purpose, we focus on one leaf in the region \(x > 0, y > 0\) of \(|xy>\). To obtain the wave function on the xy plane with \(\phi = \alpha\), we consider the wave function

\[ l(\alpha) = < \theta = \pi/2, \phi = \alpha|xy>. \]

To obtain \(l(\alpha)\), we rotate the \(|xy> state\) around the z axis by angle \(-\alpha\), and examine the x direction as shown in Fig. 11.

Using Eq. (85), we obtain

\[ e^{iL_x \alpha}|xy> = \cos 2\alpha |xy> + \sin 2\alpha |x^2 - y^2>. \] (102)

Then we have

\[ < \theta = \pi/2, \phi = 0|e^{iL_x \alpha}|xy> \]
\[ = \cos 2\alpha < \theta = \pi/2, \phi = 0|xy> \]
\[ + \sin 2\alpha < \theta = \pi/2, \phi = 0|x^2 - y^2>. \] (103)

The l.h.s. can be calculated as

\[ e^{-iL_x \alpha}|\theta = \pi/2, \phi = 0> = |\theta = \pi/2, \phi = \alpha>. \] (104)

The Hermitian conjugation gives

\[ < \theta = \pi/2, \phi = 0|e^{iL_x \alpha} = < \theta = \pi/2, \phi = \alpha|. \] (105)

Furthermore, by using

\[ < \theta = \pi/2, \phi = 0|xy = 0, \]
we obtain

\[ < \theta = \pi/2, \phi = \alpha|xy > \]
\[ = \sin 2\alpha < \theta = \pi/2, \phi = 0|x^2 - y^2>. \] (106)

This is the same as

\[ l(\alpha) \equiv < \theta = \pi/2, \phi = \alpha|xy \equiv L_0 \sin 2\alpha, \]
\[ L_0 \equiv < \theta = \pi/2, \phi = 0|x^2 - y^2>. \] (107)

The same discussion can be generalized to a fixed \(\theta\) (i.e., a cone surface with \(\theta = \text{const.}\))

\[ l(\theta, \alpha) = < \theta, \phi = \alpha|xy \equiv L(\theta) \sin 2\alpha, \]
\[ L(\theta) \equiv < \theta, \phi = 0|x^2 - y^2>, \] (108)

where the quantity \(L(\theta)\) is shown in Fig. 12.

\(L(\theta)\) is obtained by the method shown in Fig. 13. To obtain \(L(\theta) \equiv < \theta = \alpha, \phi = 0|x^2 - y^2>\), we rotate \(|x^2 - y^2>\) around y axis by \(-\alpha\), and examine the z direction. First, the rotation of state \(|x^2 - y^2>\) around the y axis by \(-\alpha\) is given by

\[ e^{iL_y \alpha}|x^2 - y^2> = -\sqrt{2}C(-\alpha)|x^2> \]
\[ + (B(-\alpha) + E(-\alpha))|xz> \]
\[ + (A(-\alpha) + E(-\alpha))|x^2 - y^2>. \] (109)

Using

\[ < \theta = 0|xz \equiv < \theta = 0|x^2 - y^2> = 0, \]
we obtain
\[
< \theta = 0|e^{i L_y \alpha}|x^2 - y^2 > = -\sqrt{2} C(-\alpha) < \theta = 0|z^2 > = \frac{\sqrt{3}}{4} (1 - \cos 2\alpha) < \theta = 0|z^2 > .
\] (110)

Furthermore from the relation
\[
e^{-i L_y \alpha}|\theta = 0 >= |\theta = \alpha, \phi = 0 > ,
\]
we obtain
\[
L(\alpha) = < \theta = \alpha, \phi = 0|x^2 - y^2 > = \frac{\sqrt{3}}{4} (1 - \cos 2\alpha) l_0 ,
\]
where \( l_0 \equiv < \theta = 0|z^2 > . \) From Eqs. (108) and (111), we obtain
\[
Y_{s_0}(\theta, \phi) = \frac{\sqrt{3}}{4} l_0 (1 - \cos 2\theta) \sin 2\phi , \] (112)
with
\[
L_0 = \frac{\sqrt{3}}{2} l_0 .
\]

8. Conclusion
We have shown a new method of obtaining spherical harmonics without solving the partial differential equation. This involves using the SU(2) algebra and the directional space inversion (mirror) operator, where the latter was introduced in Sec. 2. The node plane is expressed in simple manner using this new operator. Second, we have shown that the same \( l \) states but different \( m \) states are related to each other by the rotational group SU(2). Solving the partial differential equation Eq. (1) is the simplest way to obtain the form of spherical harmonics; however, the physical relations between the solutions with different quantum numbers can simply be understood using this method.

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Appendix A.
We define the mirror operator in the \((\theta, \phi)\) direction as
\[
\hat{P}(\theta, \phi) .
\] (A.1)

The method of rotating \((\theta, \phi)\) in the +\( x \) direction is as follows.
1) \(-\phi\) rotation around z axis,
2) \(\pi/2 - \theta\) rotation around y axis.

Therefore the space inversion into \((\theta, \phi)\) direction is given by the following steps.
1) rotation around z axis by \(-\phi\),
2) rotation around y axis by \(\pi/2 - \theta\),
3) space inversion in \( x \) direction,
4) rotation around y axis by \(-\pi/2 + \theta\),
5) rotation around z axis by +\( \phi\).

Then we obtain the following general formula for the mirror operator in an arbitrary direction:
\[
\hat{P}(\theta, \phi) = e^{-iL_y\phi}e^{iL_z(<\pi/2-\theta)}\hat{P}_x e^{-iL_y(<\pi/2-\theta)}e^{iL_z\phi} . \] (A.2)
Using the matrix form of \( \hat{P}_x \),
\[
(P_x)_{m,n} = \delta_{m,-n} . \] (A.3)
we can directly verify the relation by using the general formula
\[
(P_x)_{m,n} = (-1)^m \delta_{m,-n} , \] (A.4)
\[
(P_x)_{m,n} = (-1)^{m+1} \delta_{m,n} . \] (A.5)

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