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Zero Excess and Minimal Length in Finite Coxeter Groups

S.B. Hart and P.J. Rowley

Abstract

Let $W$ be the set of strongly real elements of $W$, a Coxeter group. Then for $w \in W$, $e(w)$, the excess of $w$, is defined by $e(w) = \min\{\ell(x) + \ell(y) - \ell(w) \mid w = xy, x^2 = y^2 = 1\}$. When $W$ is finite we may also define $E(w)$, the reflection excess of $w$. The main result established here is that if $W$ is finite and $X$ is a $W$-conjugacy class, then there exists $w \in X$ such that $w$ has minimal length in $X$ and $e(w) = 0 = E(w)$. (MSC2000: 20F55)

1 Introduction

Suppose that $W$ is a Coxeter group and let $W$ denote the set of strongly real elements of $W$. So

$$W = \{w \in W \mid w = xy \text{ where } x, y \in W \text{ and } x^2 = 1 = y^2\}.$$ 

Let $w \in W$. Then the excess of $w$, $e(w)$, is defined by

$$e(w) = \min\{\ell(x) + \ell(y) - \ell(w) \mid w = xy, x^2 = y^2 = 1\}.$$ 

In [7] it is shown that any element $w \in W$ is $W$-conjugate to an element whose excess equals zero. Or, in other words, $w$ is $W$-conjugate to an element $xy$ where $x$ and $y$ are involutions or the identity element and $\ell(xy) = \ell(x) + \ell(y)$. The present paper explores this theme further in the case when $W$ is finite. When $W$ is finite it is well-known that $W = W$. From Carter’s seminal paper [3], it follows that, in fact, each $w \in W$ can be expressed in the form $w = xy$ where $x^2 = y^2 = 1$ and

$$V_{-1}(x) \cap V_{-1}(y) = \{0\}.$$ 

Here $V$ is a reflection module for $W$ (to be defined in Section 2) and $V_\lambda(x)$ denotes the $\lambda$-eigenspace of $x$ on $V$. We note [3] only considers the Weyl groups – for a proof covering all finite Coxeter groups see Lemma 2.4. Writing $L(w)$ for the reflection length of $w$, this means that

$$L(w) = L(x) + L(y).$$

With this in mind we define the reflection excess of $w$, denoted $E(w)$, to be

$$E(w) = \min\{\ell(x) + \ell(y) - \ell(w) \mid w = xy, x^2 = y^2 = 1, L(w) = L(x) + L(y)\}.$$ 

It is clear that $E(w) \geq e(w)$. However, it is not always the case that $E(w) = e(w)$. Consider the element $w = (145)(236)$ of $\text{Sym}(6) \cong W(A_5)$. We have $L(w) = 4$, $\ell(w) = 10$ and Table

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\[ x \quad y \quad L(x) + L(y) \quad \ell(x) + \ell(y) \]

\begin{tabular}{|c|c|c|c|}
\hline
(14)(23) & (15)(26) & 4 & 6 + 12 = 18 \\
(14)(36) & (15)(23) & 4 & 8 + 8 = 16 \\
(14)(26) & (15)(36) & 4 & 10 + 10 = 20 \\
(15)(23) & (45)(26) & 4 & 8 + 8 = 16 \\
(15)(36) & (45)(23) & 4 & 10 + 2 = 12 \\
(15)(26) & (45)(36) & 4 & 12 + 6 = 18 \\
(45)(23) & (14)(26) & 4 & 2 + 10 = 12 \\
(45)(36) & (14)(23) & 4 & 6 + 6 = 12 \\
(45)(26) & (14)(36) & 4 & 8 + 8 = 16 \\
(12)(46)(35) & (13)(24)(56) & 6 & 5 + 5 = 10 \\
(13)(24)(56) & (16)(25)(34) & 6 & 5 + 15 = 20 \\
(16)(25)(34) & (12)(35)(46) & 6 & 15 + 5 = 20 \\
\hline
\end{tabular}

Table 1: \((145)(236) = xy\)

1 gives the possibilities for \(x\) and \(y\). As can be seen, \(E(w) = 2\) but \(e(w) = 0\). Surprisingly, the difference between \(E(w)\) and \(e(w)\) can be arbitrarily large, as is shown in Proposition 3.3.

The main theorem in this paper is

**Theorem 1.1** Suppose that \(W\) is a finite Coxeter group. If \(X\) is a conjugacy class of \(W\), then there exists \(w \in X\) of minimal length in \(X\) such that \(e(w) = E(w) = 0\).

We do not know if it is true that for an arbitrary Coxeter group and a strongly real conjugacy class \(X\), there exists \(w \in X\) with \(w\) of minimal length in \(X\) and \(e(w) = 0\). However we note that it does hold for an arbitrary Coxeter group when \(X\) is a strongly real conjugacy class whose elements have finite order. This may be seen by combining a theorem of Tits (Chap V, Section 4, Ex 2d of [1]) with Lemma 2.3 (which holds in general) and Theorem 1.1.

This paper is arranged as follows. Our next section gathers together relevant background material while reviewing much of the standard notation used for Coxeter groups. Section 3, apart from proving Proposition 3.3, focusses on the proof of Theorem 1.1. Part of the proof involves checking, with the aid of Magma [2], all the cuspidal classes of the exceptional finite irreducible Coxeter groups. The data resulting from these calculations is documented in the appendix. At present, when studying minimal elements in conjugacy classes, this case-by-case approach is often the best we can do –see for example Chapter 3 of Geck and Pfeiffer [6].

## 2 Preliminary Results and Notation

From now on we assume that \(W\) is a finite Coxeter group and quickly review standard notation and facts about Coxeter groups. So \(W\) has a presentation of the form

\[ W = \langle R \mid (rs)^{m_{rs}} = 1, r, s \in R \rangle \]
where \( m_{rs} = m_{sr} \in \mathbb{N}, m_{rr} = 1 \) and \( m_{rs} \geq 2 \) for \( r, s \in R, r \neq s \). The length of an element \( w \) of \( W \), denoted by \( \ell(w) \), is defined to be

\[
\ell(w) = \begin{cases} 
\min\{l \mid w = r_1 \cdots r_l, r_i \in R\} & \text{if } w \neq 1 \\
0 & \text{if } w = 1.
\end{cases}
\]

Taking \( V \) to be a real vector space with basis \( \Pi = \{ \alpha_r \mid r \in R \} \), for \( r, s \in R \)

\[
\langle \alpha_r, \alpha_s \rangle = -\cos \left( \frac{\pi}{m_{rs}} \right).
\]

defines a symmetric bilinear form \( \langle \ , \ \rangle \) on \( V \). Letting \( r, s \in R \) we define

\[
r \cdot \alpha_s = \alpha_s - 2 \langle \alpha_r, \alpha_s \rangle \alpha_r.
\]

This then extends to an action of \( W \) on \( V \) which is both faithful and respects the bilinear form \( \langle \ , \ \rangle \) (see [8]). The elements of \( R \) act as reflections upon \( V \) and \( V \) is referred to as a reflection module for \( W \). The subset \( \Phi = \{ w \cdot \alpha_r \mid r \in R, w \in W \} \) of \( V \) is the root system of \( W \), and \( \Phi^+ = \{ \sum_{r \in R} \lambda_r \alpha_r \in \Phi \mid \lambda_r \geq 0 \text{ for all } r \} \) and \( \Phi^- = -\Phi^+ \) are, respectively, the positive and negative roots of \( \Phi \). For \( w \in W \), let \( N(w) = \{ \alpha \in \Phi^+ \mid w \cdot \alpha \in \Phi^- \} \) – it is an important fact that \( \ell(w) = |N(w)| \). In a similar vein we have

**Lemma 2.1** Let \( g, h \in W \). Then

\[
N(gh) = N(h) \setminus [-h^{-1}N(g)] \cup h^{-1}[N(g) \setminus N(h^{-1})].
\]

Hence \( \ell(gh) = \ell(g) + \ell(h) - 2 \mid N(g) \cap N(h^{-1}) \mid. \)

**Proof** See Lemma 2.2 in [7].

For \( J \) a subset of \( R \) define \( W_J \) to be the subgroup generated by \( J \). Such a subgroup of \( W \) is referred to as a standard parabolic subgroup. Standard parabolic subgroups are Coxeter groups in their own right with root system

\[
\Phi_J = \{ w \cdot \alpha_r \mid r \in J, w \in W_J \}
\]

(see Section 5.5 of [8] for more on this). A conjugate of a standard parabolic subgroup is called a parabolic subgroup of \( W \). Finally, a *cuspidal* element of \( W \) is an element which is not contained in any proper parabolic subgroup of \( W \). Equivalently, an element is cuspidal if its \( W \)-conjugacy class has empty intersection with all the proper standard parabolic subgroups of \( W \).

**Theorem 2.2** Let \( 0 \neq v \in V \). Then the stabilizer of \( v \) in \( W \) is a proper parabolic subgroup of \( W \).

**Proof** Consult Ch V §3 Proposition 2 of [1].

**Lemma 2.3** Suppose that \( J \subseteq I \) and \( w \in W_J \). Then

\[
\min\{\ell(h^{-1}wh) \mid h \in W_J\} = \min\{\ell(g^{-1}wg) \mid g \in W\}.
\]
Proof The proof is by induction on the rank of \( W \), the rank 1 case being trivial. Let \( x, y \in W \) be such that \( w = xy \) with \( x^2 = y^2 = 1 \). (For \( W \) a Weyl group, this is possible by [3].) The case when \( W \) is a dihedral group is straightforward to verify while types \( H_3 \) or \( H_4 \) may be checked using [2]). If \( V_1(x) \cap V_1(y) = \{0\} \), we are done. So suppose \( 0 \neq v \in V_1(x) \cap V_1(y) \). By Theorem 2.2, \( w \) is contained in a proper parabolic subgroup of \( W \). Hence \( w \) is conjugate to an element \( u \) of some proper standard parabolic subgroup \( W_J \) of \( W \). By induction \( u = ab \) for some \( a, b \in W_J \) where \( a^2 = b^2 = 1 \) and \( V_1(a) \cap V_1(b) = \{0\} \). The appropriate conjugates of \( a \) and \( b \) will have the same properties with respect to \( w \), so proving the lemma.

Lemma 2.5 Let \( w = r_1r_2 \ldots r_n \) and let \( X \) denote the \( W \)-conjugacy class of \( w \). Then

(i) the minimal length in \( X \) is \( n \); and
(ii) the product of \( r_1, \ldots, r_n \) in any order is an element of \( X \).

Proof See Proposition 3.1.6 of [6].

Note that the minimal length elements of \( X \) in Lemma 2.5 are known as the Coxeter elements of \( W \). The famous classification of irreducible finite Coxeter groups obtained by Coxeter [4] (see also [8]) states

Theorem 2.6 An irreducible finite Coxeter group is either of type \( A_n(n \geq 1) \), \( B_n(n \geq 2) \), \( D_n(n \geq 4) \), \( \text{Dih}(2m) \), \( E_6 \), \( E_7 \), \( E_8 \), \( F_4 \), \( H_3 \) or \( H_4 \).

We next discuss concrete descriptions of the Coxeter groups of types \( A_n, B_n \) and \( D_n \) which will feature in a number of our proofs. First, \( W(A_n) \) may be viewed as being \( \text{Sym}(n + 1) \) with the set of fundamental reflections given by \( \{(12), (23), \ldots, (n n + 1)\} \). The elements of \( W(B_n) \) can be thought of as signed permutations of \( \text{Sym}(n) \). We say a cycle in an element of \( W(B_n) \) is of negative sign type if it has an odd number of minus signs, and positive sign type otherwise. The set of fundamental reflections in \( W(B_n) \) can be taken to be \( \{(12), (23), \ldots, (n 1 \ n), (n)\} \). An element \( w \) expressed as a product \( g_1g_2 \cdots g_k \) of disjoint signed cycles is positive if the product of all the sign types of the cycles is positive, and negative otherwise. The group \( W(D_n) \) consists of all positive elements of \( W(B_n) \). The fundamental reflections of \( W(D_n) \) can be taken to be the set \( \{(12), (23), \ldots, (n 1 \ n), (n)\} \). The positive roots of \( B_n \) are of the form \( e_i \pm e_j \) for \( 1 \leq i < j \leq n \) and \( e_i \) for \( 1 \leq i \leq n \). The positive roots of \( D_n \) are of the form \( e_i \pm e_j \) for \( 1 \leq i < j \leq n \). Therefore the root system of \( D_n \) consists of the long roots of the root system for \( B_n \). Even if \( w \) is positive, it may contain negative cycles, which we wish on occasion to consider separately, so when considering elements of \( W(D_n) \) we often work in the environment of \( W(B_n) \) to avoid ending up with non-group elements. We draw the reader's attention to the fact that when regarding \( W \) as a group of permutations or signed permutations we act on the right, as is customary for permutations. We further note that, in the case of a type \( B_n \) Coxeter group described as a group of signed permutations, the roots do not all have the same length. This doesn’t accord with our earlier description of \( \Phi \) where all the roots have length 1. However this causes no problems here.
The following result is obtained from Propositions 3.4.6, 3.4.7, 3.4.11, 3.4.12 of [6]. The length of minimal length elements in conjugacy classes is not given explicitly, but expressions for elements of minimal length are given in Section 3.4.2 and from these the length can be easily calculated.

Proposition 2.7 Let $W$ be of type $B_n$ or $D_n$. Then the following hold.

(i) Conjugacy classes in $W$ are parameterized by signed cycle type, with one class for each signed cycle type except in the case where all cycles are even length and positive, and $W$ is of type $D_n$. In this case there are two conjugacy classes, which can be interchanged by the length-preserving graph automorphism.

(ii) Cuspidal conjugacy classes in $W$ are those whose signed cycle type consists only of negative cycles.

(iii) Each cuspidal conjugacy class $X$ corresponds to a non-increasing partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of $n$, where each $\lambda_i$ is the length of a negative cycle of $w \in X$. Let $\mu_i = \sum_{j=i+1}^k \lambda_i$. The minimal length of an element of $X$ is

$$\ell_{\min}(X) = \begin{cases} \sum_{i=1}^k (\lambda_i + 2\mu_i) & \text{if } W = W(B_n) \\ \sum_{i=1}^k (\lambda_i - 1 + 2\mu_i) & \text{if } W = W(D_n) \end{cases}.$$  

We close this section with a few pieces of notation. For a subset $X$ of $W$ we define $\ell_{\min}(X) = \min\{\ell(x) \mid x \in X\}$. The longest element of $W$ will be denoted by $w_0$. When $W = W(A_{n-1}) \cong \text{Sym}(n)$, $W$ acts in the usual way on the set $\Omega = \{1, 2, \ldots, n\}$. In this case, for $w \in W$, the support of $w$, $\text{supp}(w)$, is given by $\text{supp}(w) = \{\delta \in \Omega \mid \delta^w \neq \delta\}$. Finally, the dihedral group of order $2m$ will be denoted by $\text{Dih}(2m)$.

3 Minimal Length and Zero Excess and Reflection Excess

We begin with an elementary lemma.

Lemma 3.1 For $w \in W$, the following hold.

(i) Both $e(w)$ and $E(w)$ are non-negative and even.

(ii) If $w$ is an involution or the identity element, then $e(w) = E(w) = 0$.

(iii) $\ell(w)$ is the sum of the lengths, $e(w)$ is the sum of the excesses, and $E(w)$ is the sum of the reflection excesses of the irreducible direct factors of $W$.

Proof For (iii), suppose $x^2 = y^2 = 1$ and $xy = w$. Then, using Lemma 2.1, $\ell(w) = \ell(x) + \ell(y) - 2|N(x) \cap N(y)|$ and hence $\ell(x) + \ell(y) - \ell(w)$ is even and (iii) follows. The remaining parts of the lemma are straightforward.  

In the next lemma we encounter the following two subsets of $W$:

$$\mathcal{I}_w = \{x \in W \mid x^2 = 1, w^x = w^{-1}\}$$  

and

$$\mathcal{J}_w = \{x \in W \mid x^2 = 1, w^x = w^{-1}, V_1(w) \subseteq V_1(x)\}.$$  

5
Lemma 3.2 Suppose that \( w \in W \)

(i) \( \mathcal{J}_w \) is the set of \( x \) such that \( w = xy \) where \( x^2 = y^2 = 1 \) and \( L(w) = L(x) + L(y) \).

(ii) If \( w \) is cuspidal, then \( \mathcal{I}_w = \mathcal{J}_w \) and, in particular, \( e(w) = E(w) \).

Proof Suppose \( x \in \mathcal{J}_w \). Set \( y = xw \). Clearly \( y^2 = 1 \). Let \( v \in V_1(x) \cap V_1(y) \). Then \( y \cdot v = x \cdot v \) and hence \( w \cdot v = v \). That is, \( v \in V_1(w) \). But \( V_1(w) \subseteq V_1(x) \) as \( x \in \mathcal{J}_w \). Thus \( v \in V_1(x) \cap V_1(y) \) and so \( v = 0 \). Therefore \( V_1(x) \cap V_1(y) = 0 \), whence \( L(w) = L(x) + L(y) \).

Next we consider \( \mathcal{I}_w \). Set \( x, y \in \mathcal{J}_w \). Then \( x^2 = y^2 = 1 \). Hence \( x \cdot (x + y) = x \cdot (x - y) = 0 \). This allows us to use [Proposition 2.7(iv), [7]] and Lemma 3.2(ii) to deduce that \( e(w) = E(w) \).

Theorem 2.2, that \( V_1(w) = 0 \). So \( \mathcal{I}_w = \mathcal{J}_w \), and clearly \( e(w) = E(w) \).

We now give examples of elements in the symmetric group which have arbitrarily large reflection excess, while having zero excess, showing that these quantities can differ by arbitrarily large amounts.

Proposition 3.3 Let \( W = W(A_n - 1) = \text{Sym}(n) \) and assume \( 0 < 4k \leq n \). Define

\[
\begin{align*}
w_1 &= (1 \ 4 \ 6 \ 8 \cdots 4k - 2 \ 4k - 1), \\
w_2 &= (2 \ 4k \ 4k - 3 \cdots 7 \ 5 \ 3)
\end{align*}
\]

and \( w = w_1w_2 \). Then \( e(w) = 0 \) but \( E(w) \geq 4(k-1)^2 \).

Proof We define two involutions \( x, y \) as follows.

\[
\begin{align*}
x &= (13)(2 \ 4k - 1)(45)(67) \cdots (4k - 4 \ 4k - 3 \ 4k - 2 \ 4k) \\
y &= (12)(34) \cdots (4k - 1 \ 4k)
\end{align*}
\]

A simple check shows \( w = xy \). Now \( N(y) = \{e_{2i-1} - e_{2i} : 1 \leq i \leq 2k\} \). To find \( N(x) \cap N(y) \), note that

\[
\begin{align*}
x \cdot (e_{1} - e_{2}) &= e_{3} - e_{4k-1} \in \Phi^+ \\
x \cdot (e_{3} - e_{4}) &= e_{1} - e_{5} \in \Phi^+ \\
x \cdot (e_{4k-3} - e_{4k-2}) &= e_{4k-4} - e_{4k} \in \Phi^+ \\
x \cdot (e_{4k-1} - e_{4k}) &= e_{2} - e_{4k-2} \in \Phi^+ \\
x \cdot (e_{2i-1} - e_{2i}) &= e_{2i-2} - e_{2i+1} \in \Phi^+(3 \leq i \leq 2k-2).
\end{align*}
\]

Hence \( N(x) \cap N(y) = \emptyset \), which means \( e(w) = 0 \) by Lemma 2.1.

Next we consider \( E(w) \). Since \( w_1 = (1 \ 4 \ 6 \ 8 \cdots 4k - 2 \ 4k - 1) \) and \( 1 < 4 < \cdots < 4k - 1 \), we can compare \( w_1 \) seen as an element of \( \text{Sym}(\text{supp}(w_1)) \) with the element \( (1 \ 2 \ \cdots \ 2k) \) as an element of \( \text{Sym}(2k) \). This allows us to use [Proposition 2.7(iv), [7]] and Lemma 3.2(ii) to deduce that \( E(w_1) \geq E((12 \cdots 2k)) = 2(k-1)^2 \). This is because whenever \( w_1 = \sigma_1 \tau_1 \) where \( \sigma_1, \tau_1 \) are involutions in \( \text{Sym}(\text{supp}(w_1)) \), we have that

\[
|N(\sigma_1) \cap N(\tau_1)|_{\text{Sym}(\text{supp}(w_1))} \subseteq N(\sigma_1) \cap N(\tau_1).
\]
Similarly, by comparing \(w_2 = (2 4k \ 4k-3 \ldots 7 \ldots 3)\) to \((1 2k \ 2k-1 \ldots 3 2) = (1 2 \ldots 2k)^{-1}\) we deduce that \(E(w_2) \geq 2(k-1)^2\).

Suppose \(w = \sigma \tau\) with \(L(\sigma) + L(\tau) = L(w)\). Then \(\sigma = \sigma_1 \sigma_2, \ \tau = \tau_1 \tau_2\) where \(w_1 = \sigma_1 \tau_1, \ w_2 = \sigma_2 \tau_2\), \(\text{supp}(\sigma_1) \cup \text{supp}(\tau_1) \subseteq \text{supp}(w_1)\) and \(\text{supp}(\sigma_2) \cup \text{supp}(\tau_2) \subseteq \text{supp}(w_2)\). Any \(\{i, j\} \subseteq \text{supp}(w_1)\) (and hence \(w_1(i), w_1(j)\)) will thus be fixed by \(\sigma_2\) and \(\tau_2\). So if \(e_i - e_j \in N(\sigma_1) \cap N(\tau_1)\), then \(e_i - e_j \in N(\sigma) \cap N(\tau)\). Hence

\[
(N(\sigma_1) \cap N(\tau_1))_{\text{Sym(supp}(w_1))} \subseteq N(\sigma) \cap N(\tau).
\]

Similarly

\[
(N(\sigma_2) \cap N(\tau_2))_{\text{Sym(supp}(w_2))} \subseteq N(\sigma) \cap N(\tau).
\]

This implies that \(E(w) \geq E(w_1) + E(w_2) \geq 4(k-1)^2\).

**Lemma 3.4** Let \(W\) be a finite Coxeter group, and \(X\) be the conjugacy class of Coxeter elements of \(W\). Then there exists \(w \in X\) of minimal length in \(X\) such that \(e(w) = E(w) = 0\).

**Proof** Since the Coxeter graph of every finite Coxeter group is a forest (that is, a disjoint union of trees), the Coxeter graph of \(W\) is 2-colourable. Therefore we may divide the fundamental reflections into two sets \(R = R_1 \cup R_2\), such that for \(r, s \in R_i\), \(rs = sr\). Now let \(x = \prod_{r \in R_1} r\) and \(y = \prod_{r \in R_2} r\). Then \(w = xy\) is an element of \(X\) of minimal length, and \(e(w) = E(w) = 0\).

We remark that if \(W\) is irreducible, its Coxeter graph can only be 2-coloured in one way and consequently there are exactly two Coxeter elements for which \(e(w) = E(w) = 0\).

**Proposition 3.5** Let \(W\) be irreducible of type \(A_{n-1}, B_n\) or \(D_n\) and \(X\) be a conjugacy class of \(W\). Then there exists \(w \in X\) such that \(e(w) = E(w) = 0\) and \(w\) has minimal length in \(X\).

**Proof** It suffices to consider the case where \(X\) is cuspidal. Suppose \(W = W(A_{n-1})\). The only cuspidal class is the class of Coxeter elements, and by Lemma 3.4 Proposition 3.5 holds. It remains to deal with the case where \(W\) is type \(B_n\) or type \(D_n\) and \(X\) is cuspidal. Here, by Proposition 2.7, the elements of \(X\) contain only negative cycles of lengths \(\lambda_1 \geq \ldots \geq \lambda_k\), where \(\sum \lambda_i = n\). We will construct an element \(w \in X\) of minimal length. Let \(\nu_1 = 0\) and for \(2 \leq i \leq k\) let \(\nu_i = \sum_{j=1}^{i-1} \lambda_j\). As in Proposition 2.7, for \(1 \leq i \leq k\), let \(\mu_i = \sum_{j=i+1}^{k} \lambda_j\). Note that \(n = \mu_i + \lambda_i + \nu_i\). By Proposition 2.7, the minimal length of elements of \(X\), \(l_{\min}(X)\), is given by

\[
l_{\min}(X) = \begin{cases} 
\sum_{i=1}^{k} (\lambda_i + 2\mu_i) & \text{if } W = W(B_n); \\
\sum_{i=1}^{k} (\lambda_i - 1 + 2\mu_i) & \text{if } W = W(D_n).
\end{cases}
\]

Now for \(1 \leq i \leq k\) define two involutions, \(\tau_i\) and \(\sigma_i\), as follows.

\[
\tau_i = \begin{cases} 
(\nu_i + 1 \ \nu_i + 2)(\nu_i + 3 \ \nu_i + 4) \cdots (\nu_i + \lambda_i - 2 \ \nu_i + \lambda_i - 1)(\nu_i - \lambda_i) & \text{if } \lambda_i \text{ is odd} \\
(\nu_i + 2 \ \nu_i + 3)(\nu_i + 4 \ \nu_i + 5) \cdots (\nu_i + \lambda_i - 2 \ \nu_i + \lambda_i - 1)(\nu_i - \lambda_i) & \text{if } \lambda_i \text{ is even}
\end{cases}
\]

\[
\sigma_i = \begin{cases} 
(\nu_i + 2 \ \nu_i + 3)(\nu_i + 4 \ \nu_i + 5) \cdots (\nu_i + \lambda_i - 1 \ \nu_i + \lambda_i) & \text{if } \lambda_i \text{ is odd} \\
(\nu_i + 1 \ \nu_i + 2)(\nu_i + 3 \ \nu_i + 4) \cdots (\nu_i + \lambda_i - 1 \ \nu_i + \lambda_i) & \text{if } \lambda_i \text{ is even}
\end{cases}
\]
Define $w_i = τ_i σ_i$, $τ = τ_1 \cdots τ_k$, $σ = σ_1 \cdots σ_k$ and $w = τ σ = w_1 \cdots w_k$. Note that if $w \in W(D_n)$, then $k$ is even and hence $τ$ and $σ$ are both elements of $W(D_n)$. If $λ_i$ is odd, then

$$w_i = (ν_i^+ + 1 \ ν_i^+ + 2 \ ν_i^+ + 4 \ \cdots \ ν_i^+ + λ_i − 3 \ ν_i^+ + λ_i − 1 \ ν_i^+ + λ_i \ ν_i^+ + λ_i − 2 \ ν_i^+ + 4 \ ν_i^+ + 3)$$

whereas if $λ_i$ is even, then

$$w_i = (ν_i^+ + 1 \ ν_i^+ + 3 \ ν_i^+ + 5 \ \cdots \ ν_i^+ + λ_i − 3 \ ν_i^+ + λ_i − 1 \ ν_i^+ + λ_i \ ν_i^+ + λ_i − 2 \ ν_i^+ + 4 \ ν_i^+ + 2).$$

In either case, $w_i$ is a negative $λ_i$-cycle. Hence $w \in X$. We now consider $ℓ(τ)$ and $ℓ(σ)$. Firstly, note that $σ$ is a product of distinct mutually commuting fundamental reflections. Hence $ℓ(σ) = \sum_{i=1}^{k} [\frac{λ_i}{2} − 1]$. Now $τ$ can be split into a product $τ = τ' τ''$ where $τ'$ consists of the 2-cycles of $τ$ and $τ'' = (ν_1)(ν_2) \cdots (ν_k)$. Now $ℓ(τ') = \sum_{i=1}^{k} [\frac{λ_i}{2} − 1]$ and $N(τ)$ consists of the roots of the form $e_a − e_b$ where for some $i$, $\{a, b\} \subseteq \{ν_i + 1, \ldots, ν_i + λ_i\}$. Also it is not hard to see that, thinking of $τ''$ as an element of $W(B_n)$,

$$N(τ'') = \cup_{i=1}^{k} \{([e_{ν_i + λ_i}] \cup \{e_{ν_i + λ_i} \pm e_b : ν_i + λ_i < b \leq n}\}.$$ 

Therefore $N(τ') \cap N(τ'') = \emptyset$. Hence $ℓ(τ) = ℓ(τ') + ℓ(τ'')$. If $W = W(B_n)$ we get $ℓ(τ'') = k + 2 \sum_{i=1}^{k} λ_i$ and hence $ℓ(τ) = \sum_{i=1}^{k} [\frac{λ_i}{2} − 1] + k + 2 \sum_{i=1}^{k} λ_i$. If $W = W(D_n)$ then $ℓ(τ'') = 2 \sum_{i=1}^{k} λ_i$ and hence $ℓ(τ) = \sum_{i=1}^{k} [\frac{λ_i}{2} − 1] + 2 \sum_{i=1}^{k} λ_i$.

Now $w = τ σ$, so $ℓ(w) ≤ ℓ(τ) + ℓ(σ)$. If $W = W(B_n)$, then

$$ℓ(w) \leq ℓ(τ) + ℓ(σ) = \sum_{i=1}^{k} [\frac{λ_i}{2} − 1] + k + 2 \sum_{i=1}^{k} λ_i + \sum_{i=1}^{k} [\frac{λ_i}{2} − 1] = \sum_{i=1}^{k} λ_i + 2 μ_i = l_{min}(X) \leq ℓ(w).$$

A similar calculation when $W = W(D_n)$ shows again that $ℓ(w) ≤ ℓ(τ) + ℓ(σ) = l_{min}(X) \leq ℓ(w)$. Therefore in each case, $ℓ(w) = ℓ(τ) + ℓ(σ) = l_{min}(X)$. Hence $e(w) = 0$ and $w$ has minimal length in $X$. Since $w$ is cuspidal, $E(w) = e(w) = 0$ and we have therefore proved the proposition.

**Proof of Theorem 1.1** By Lemma 3.1(v) we only need establish the theorem for $W$ an irreducible Coxeter group. This has been done for $W$ of type $A_{n−1}, B_n$ and $D_n$ in Proposition 3.5. The case when $W$ is isomorphic to $\text{Dih}(2m)$ is easy to handle. For the remaining exceptional irreducible Coxeter groups we had to resort to using Magma [2]. The sizes of $X_{0\text{min}}$, where $X_{0\text{min}}$ denotes the set of elements in $X$ which have minimal length in $X$ and excess zero, are itemized in the appendix as well as giving a representative element of this set. In all cases $X_{0\text{min}}$ is non-empty, whence Theorem 1.1 is proven.

□
Appendix

Tables 3–6 give cuspidal classes in exceptional Weyl groups. Each class is labeled using the system in Carter [3]. Detailed information about these classes is given in [6], Tables B.1 – B.6. For a conjugacy class $X$ let $X_{\text{min}}$ be the set of elements of minimal length in $X$ and $X_{0\text{min}}^0$ be the set of $w \in X_{\text{min}}$ for which $e(x) = E(x) = 0$. For each class $X$ we give $|X_{\text{min}}|$, $|X_{0\text{min}}^0|$ and a representative $w \in X_{0\text{min}}^0$. Tables 7 and 8 give cuspidal classes in $H_3$ and $H_4$. For these tables the label of the class is its address in the CHEVIE list [5]. See also [6]. And finally, to improve readability, we write the element, say, $r_1 r_3 r_2 r_4$ as $1324$.

| Class $X$ | $|X_{\text{min}}|$ | $|X_{0\text{min}}^0|$ | $w \in X_{0\text{min}}^0$ |
|----------|-----------------|-----------------|-----------------|
| $F_4$    | 8               | 2               | 1324            |
| $B_4$    | 14              | 2               | 124323          |
| $F_4(a_1)$ | 16           | 16              | 12132343        |
| $D_4$    | 8               | 6               | 1232343234      |
| $C_3 + A_1$ | 8          | 6               | 1213213234      |
| $D_4(a_1)$ | 12           | 12              | 121321343234    |
| $A_3 + A_1$ | 16           | 12              | 12132132343234  |
| $A_2 + A_2$ | 16           | 16              | 1213213432132343 |
| $4A_1$   | 1               | 1               | $w_0$          |

Table 2: Cuspidal Classes in $F_4$

| Class $X$ | $|X_{\text{min}}|$ | $|X_{0\text{min}}^0|$ | $w \in X_{0\text{min}}^0$ |
|----------|-----------------|-----------------|-----------------|
| $E_6$    | 32              | 2               | 142365          |
| $E_6(a_1)$ | 80           | 4               | 12542346        |
| $E_6(a_2)$ | 144          | 36              | 231423154654    |
| $A_5 + A_1$ | 48           | 10              | 23423454231465  |
| $3A_2$   | 80              | 80              | 123142314542314565423456 |

Table 3: Cuspidal Classes in $E_6$

| Class $X$ | $|X_{\text{min}}|$ | $|X_{0\text{min}}^0|$ | $w \in X_{0\text{min}}^0$ |
|----------|-----------------|-----------------|-----------------|
| $E_7$    | 64              | 2               | 1423657         |
| $E_7(a_1)$ | 160         | 4               | 125423476       |
| $E_7(a_2)$ | 280         | 18              | 12465423457     |
| $E_7(a_3)$ | 366         | 40              | 1231542365476   |
| $D_6 + A_1$ | 96          | 10              | 123423465423457 |
| $A_7$    | 316             | 58              | 12365423476542345 |
| $E_7(a_4)$ | 800         | 422             | 234354231435465765431 |
| $D_6(a_2) + A_1$ | 708     | 190             | 13542345654231435765423 |
| $A_5 + A_2$ | 420        | 194             | 1234231436542314576542345 |
| $D_4 + 3A_1$ | 32         | 20              | 123423454234565423456754234567 |
| $2A_3 + A_1$ | 360        | 326             | 423454365423143542654317654234567 |
| $7A_1$   | 1               | 1               | $w_0$          |

Table 4: Cuspidal Classes in $E_7$
| Class $X$ | $|X_{\text{min}}|$ | $|X^0_{\text{min}}|$ | $w \in X^0_{\text{min}}$ |
|-----------|-----------------|-----------------|------------------|
| $E_8$     | 128             | 2               | 14682357         |
| $E_8(a_1)$ | 320             | 4               | 1425423768       |
| $E_8(a_2)$ | 624             | 10              | 435423145768     |
| $E_8(a_3)$ | 732             | 40              | 12316542376548   |
| $E_8(a_5)$ | 1516            | 66              | 1231654237687654 |
| $E_7 + A_1$ | 192            | 10              | 2423454231437658 |
| $D_8$     | 852             | 18              | 24235423165438765|
| $E_8(a_3)$ | 2696            | 238             | 23423546542314768765 |
| $D_8(a_1)$ | 2040            | 178             | 1245423654765423876543 |
| $E_8(a_7)$ | 2360            | 316             | 35423145642317654231418 |
| $E_8(a_6)$ | 3370            | 422             | 2354231435467654231458765 |
| $E_7(a_2) + A_1$ | 1758    | 100             | 1342345423143542654238765|
| $E_6 + A_2$ | 840             | 194             | 123454231435465423143548765 |
| $D_8(a_2)$ | 4996            | 362             | 314235476542358765423143548765 |
| $A_8$     | 2816            | 592             | 12342314354654231435426542314354265478 |
| $D_8(a_3)$ | 7748            | 910             | 231423145423657654231435487654231435426543768 |
| $D_6 + 2A_1$ | 256          | 48              | 3435423143542654234568765423456765 |
| $A_7 + A_1$ | 2080            | 422             | 2314254236542345676542314354265478 |
| $E_8(a_8)$ | 4480            | 4480            | 123143154234565423145676542314354678654231435426543768 |
| $E_7(a_4) + A_1$ | 11592      | 5158            | 123435423456542317654231435426543765876543 |
| $2D_4$    | 4070            | 1262            | 12425423456542765423456787654231435426543768 |
| $E_6(a_2) + A_2$ | 16374      | 7438            | 342314542365423143546576542876542314354265765 |
| $A_8 + A_2 + A_1$ | 3752     | 1910            | 314234542365423567654234567876542314354265478 |
| $D_5(a_1) + A_3$ | 15134     | 4900            | 425423143546542314354265423143542654317687654231435426543768 |
| $2A_4$    | 7952            | 4058            | 134542314354265423143546765423143542654317687654231435426543768 |
| $2D_4(a_1)$ | 15120        | 15120           | 1423542654231765423143542654876542314354265431765423487654231435426543768 |
| $D_4 + 4A_1$ | 56             | 42              | 1242315423143654231435427654231435426543187654231435426543768 |
| $2A_3 + 2A_1$ | 1260         | 1090            | 123142315431654765423143542654387654231435426543187654231435426543768 |
| $A_2$     | 4480            | 4480            | 1234123454231435426543765423143542654317687654231435426543768 |
| $8A_1$    | 1               | 1               | $w_0$            |

Table 5: Cuspidal Classes in $E_8$

| Class $X$ | $|X_{\text{min}}|$ | $|X^0_{\text{min}}|$ | $w \in X^0_{\text{min}}$ |
|-----------|-----------------|-----------------|------------------|
| 6         | 4               | 2               | 312              |
| 8         | 6               | 4               | 21231            |
| 9         | 6               | 6               | 121232123        |
| 10        | 1               | 1               | $w_0$            |

Table 6: Cuspidal Classes in $H_3$
| Class $X$ | $|X_{\text{min}}|$ | $|X^0_{\text{min}}|$ | $w \in X^0_{\text{min}}$ |
|-----------|----------------|----------------|-----------------|
| 11        | 8              | 2              | 1324            |
| 14        | 12             | 4              | 212431          |
| 15        | 18             | 8              | 32434121        |
| 16        | 22             | 10             | 2321234121      |
| 18        | 24             | 24             | 121213212343    |
| 19        | 34             | 26             | 32121321234121  |
| 21        | 12             | 10             | 2132121321234121|
| 22        | 24             | 20             | 2321234321234121|
| 23        | 38             | 30             | 213212134321234121|
| 24        | 40             | 40             | 12121321213214321234|
| 25        | 36             | 28             | 2132121321234321234121|
| 26        | 24             | 24             | 121213212134321213212343|
| 27        | 56             | 52             | 12123212132123432121321234|
| 28        | 40             | 38             | 1212132123432121321234321234|
| 29        | 60             | 60             | 121213212132124321213214321234|
| 30        | 24             | 24             | 121213212132432121321234321213212343|
| 31        | 36             | 34             | 12121321213212343212132123432121321234|
| 32        | 40             | 40             | 1212132121321243212132124321213214321234|
| 33        | 24             | 24             | 1212132121321432121321234321213212343212·13212343|
| 34        | 1              | $w_0$          |                 |

Table 7: Cuspidal Classes in $H_4$
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