A GENERATING PROBLEM FOR SUBFACTORS

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ABSTRACT. Bisch and Jones proposed the classification of planar algebras by simple generators and relations. In this paper, we study the generating problem for a family of group-subgroup subfactors associated with the Kneser graphs, namely, to determine the generators with minimal size. In particular, we prove that this family of subfactors are generated by 2-boxes and this provides an affirmative answer to a question of Vaughan Jones. This generator problem is also related to the theory of quantum permutation groups, and the main theorem also provides an infinite family of strongly regular graphs with no quantum symmetry.

1. Introduction

Vaughan Jones initiated modern subfactor theory by his remark index theorem [Jon83]. Since then, there are many different formalisms to understand the central object, namely, the standard invariants for subfactors [Bis97, Pop95, Ocneanu88]. Later on, Vaughan Jones introduced the subfactor planar algebras as a topological axiomatization of standard invariants [Jon99]. A planar algebra $\mathcal{P}$ consists of a sequence of finite-dimensional $C^*$-algebras $\mathcal{P}_m'$ (which are called the $m$-box spaces) and a natural action of the operad of planar tangles. This perspective displays that the standard invariants is a representation of fully labeled planar tangles in the flavor of topological quantum field theory [Atiyah88].

From the perspective of planar algebras, Bisch and Jones proposed the classification of subfactors by simple generators and relations [BJ00, BJ03, BJLe06]. The motivating examples are the Birman-Murakami-Wenzl algebras [BW89, Mur90], which admits the Yang-Baxter relations. Such Yang-Baxter relation planar algebras are completely classified in [Liu15], and a new family of subfactor planar algebras were discovered there which has a deep connection to conformal field theory. For Yang-Baxter relations, the critical dimension of the 3-box space is 15: the dimension of the 3-box space of singly-generated Yang-Baxter relation planar algebras is less than or equal to 15. Moreover, when the dimension is less than 15, Yang-Baxter relations will always hold. However, when the dimension is 15, Yang-Baxter relations are not automatic anymore. Therefore, do there exists singly-generated planar algebras beyond Yang-Baxter relations? In particular, Jones asked whether the subfactor planar algebra for $S_2 \times S_3 \subset S_5$ is generated by its 2-boxes in the late nineties. In [Ren17, Ren19], we provide an affirmative answer to this question and a skein theory from the perspective of group-action models. Later on, Jones asked the following question.

Question 1.1 (Jones, 2017). Are the subfactor planar algebras for $S_2 \times S_{n-2} \subset S_n$ generated by their 2-boxes?

This question is also closely related to the classification of spin models for Kauffman polynomial from self-dual strongly regular graphs by Jaeger [Jaeger92]. In particular, he discovered a new spin model based on the Higman-Sims graph [HS68]. The spin model is described by a spin model planar algebra, and the adjacency matrix is a 2-box. Question 1.1 can be asked in this general setup: given
a strongly regular graph $\Gamma$, the associated group-action model $\mathcal{P}_\Gamma^\bullet$ is defined to be the fixed-point planar subalgebra of the spin model planar algebra. The 2-box space is spanned by Temperley-Lieb diagrams and the adjacency matrix $A_\Gamma$. Therefore, one can ask whether the planar algebra $\mathcal{P}_\Gamma^\bullet$ is generated by the adjacency matrix $A_\Gamma$. Since the spin model planar algebra $\mathcal{P}_\Gamma^\bullet$ is defined by the combinatorial data of the graph $\Gamma$, the generating property in Question 1.1 is intrinsically determined by $\Gamma$.

**Definition 1.2.** Let $\Gamma$ be a strongly regular graph. We say $\Gamma$ has property $(G)$ if the associated planar algebra $\mathcal{P}_\Gamma^\bullet$ has the generating property, namely, it is generated by its adjacency matrix $A_\Gamma$.

The referred subfactor planar algebra in Question 1.1 can be obtained from the Kneser graph $KG_{n,2}$. Therefore, Question 1.1 is equivalent to ask whether $KG_{n,2}$ has property $(G)$ for $n \geq 5$. In Ren17, we provide an affirmative answer to Question 1.1 in the case when $n = 5$, namely, the Petersen graph $KG_{5,2}$ has property $(G)$.

In this paper, by exploiting the universal skein theory for group-action models [Ren19], we first give constructions of generators for the planar algebras $\mathcal{P}_{KG_{n,2}}^\bullet$ under the assumption that the transposition $R$ is generated by 2-boxes, namely, $R \in \langle \mathcal{P}_{KG_{n,2}}^2 \rangle$. Then we confirm the validity of the assumption provided with a universal construction, and thus we prove the main theorem, namely,

**Main Theorem.** The Kneser graph $KG_{n,2}$ has property $(G)$ for $n \geq 5$.

We first remark that the relation between the transposition and the generating property was first revealed independently by Jones [Jon] and Curtin [Cur03]. They showed that any planar subalgebra $\mathcal{Q}_\Gamma^\bullet$ of some spin model, $\mathcal{Q}_\Gamma$ has the generating property if and only if $R \in \langle \mathcal{Q}_\Gamma^2 \rangle$. We enhance the statement by dropping the assumption that $\mathcal{Q}_\Gamma$ is a planar subalgebra of some spin model. In this case, the transposition $R$ is characterized by skein relations.

Secondly, it was pointed out by Snyder and Reutter that the generating property in Definition 1.2 is also studied in the theory of quantum permutation groups [LMR17, Cha19, MRV19]. Quantum automorphism groups were defined by Banica (See e.g. [Ban05]). A graph $\Gamma$ is said to have no quantum symmetry if its quantum automorphism group coincides with its automorphism group. Moreover, we have that

$$\Gamma \text{ has property } (G) \iff \Gamma \text{ has no quantum symmetry.}$$

(1)

In the theory of quantum permutation groups, it is an important task to determine graphs having no quantum symmetry. In [BB07], Banica and Bichon computed the quantum automorphism groups for strongly regular graphs with vertices less than or equal to 11, except for the Petersen graph. Therefore, Main theorem confirms that the Petersen graph has no quantum symmetry and provide an infinite family of strongly regular graphs with no quantum symmetry, namely,

**Corollary 1.3.** The Kneser graphs $KG_{n,2}$ has no quantum symmetry for $n \geq 5$.

In the end, Main Theorem confirms that the simplest generator for the planar algebra for $\mathcal{P}_{KG_{n,2}}^\bullet$ is a single 2-box $A_\Gamma$. However, Universal skein theory for group actions tells us that in the simplest skein theory, the generators are a 2-box and an $n$-box; and one of the relations appears in the $2n$-box space. This phenomenon gives us a hint that the complexity of skein theory might be more subtle than the sizes of generators and relations.

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2. Preliminary

In this section, we recall the basics of spin models and group-action models. We refer the readers for more details to [Jon99] for spin model planar algebras and [Ren19] for group-action models.

Definition 2.1 (Spin models). Let $X$ be a finite set with size $d$. The spin model ${\mathcal{P}}_\bullet$ associated to $X$ is a family of vector spaces $\{\mathcal{P}_n : n \geq 0\}$, where $\mathcal{P}_n = \mathcal{F}(X^k, \mathbb{C})$, namely, the complexed-valued functions on $X^k$. Moreover, there are three basic operations on $\mathcal{P}_\bullet$:

- **Tensor product:**
  \[
  f \otimes g(x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_m) = f(x_1, x_2, \cdots, x_n)g(y_1, y_2, \cdots, y_m).
  \]
- **Contraction:** for $1 \leq k \leq n$,
  \[
  C_{k,k+1}(f)(x_1, x_2, \cdots, x_n) = \delta_{x_k,x_{k+1}} \cdot (x_1, x_2, \cdots, x_{k-1}, x_{k+2}, \cdots, x_n),
  \]
  where $\delta$ is the Kronecker delta.
- **Permutation:** for $1 \leq k \leq n$,
  \[
  S_{k,n}(f)(x_1, x_2, \cdots, x_n) = (x_1, x_2, \cdots, x_{k-1}, x_k, x_{k+2}, \cdots, x_n).
  \]

Definition 2.2 (Group-action models). Let $X$ be a finite set of size $d$ and $\mathcal{P}_\bullet$ be the spin model associated to $X$. Suppose there exists an action $\beta$ of a finite group $G \leq S_d$ on the set $X$ and thus the action $\beta$ can be extended diagonally on $X^k$ by

\[
\beta(g)(x_1, x_2, \cdots, x_n) = (\beta(g)x_1, \beta(g)x_2, \cdots, \beta(g)x_n).
\]

Therefore, this induces an action of $G$ on the spin model $\mathcal{P}_\bullet$, still denoted by $\beta$. The group $G$ is indeed the gauge symmetry of the spin model. The group-action model, denoted by $\mathcal{P}_G$, is defined to be fixed points under the group action $\beta$, namely,

\[
\mathcal{P}_G^n = \{ f \in \mathcal{P}_n : \beta(g)f = f, \forall g \in G \}.
\]

It is straightforward to verify that the group-action model is closed under the three operations defined in Definition 2.1.

In [Ren19], we provide a universal skein theory for group-action models. Here, we recall the generators in the following proposition.

Proposition 2.3. Let $\mathcal{P}_G$ be a group-action model associated to the group action $G \curvearrowright X$ where $X$ is a set of size $d$. Suppose $f \in \mathcal{P}_G^n$ for some $n \in \mathbb{N}$. We represent $f$ as the following diagram

\[
\begin{array}{c}
\text{n} \\
\stackrel{\uparrow}{\cdots} \\
\text{f} \\
\end{array}
\]

The group-action model $\mathcal{P}_G$ is generated by

- **The GHZ tensor:** let $I_3 = \{(x,x,x) : x \in X\}$. The GHZ tensor is defined to be $\chi_{I_3}$, namely, the characteristic function of $I_3$. Moreover, the GHZ tensor is represented by the following diagram.

\[
\text{GHZ} = \begin{array}{c}
\bullet \\
\end{array}
\]
• The transposition $R$: For an arbitrary point $(x_1,x_2,x_3,x_4) \in X^4$, we define the transposition $R$ by $R(x_1,x_2,x_3,x_4) = \delta_{x_1,x_3}\delta_{x_2,x_4}$. Moreover, $R$ is represented by the following diagram.

\[ R = \begin{array}{c}
\end{array} \] \hspace{1cm} (8)

• The molecule $M$: let $S_M = \{ (\beta(g)1,\beta(g)2,\cdots,\beta(g)d) : g \in G \}$. The molecule $M$ is defined to be $\chi_{S_M}$, namely, the characteristic function of $S_M$.

**Remark.** Let $H$ be the stabilizer of a single point $x \in X$. Then the group-action model $\mathcal{G}_H^G$ is the even part of the group-subgroup subfactor planar algebra for $H \leq G$. In the language of subfactor planar algebras, the GHZ tensor is exactly the Temperley-Lieb diagram,

\[ \begin{array}{c}
\end{array} \]

**Remark.** In general, the GHZ$_n$ tensor is defined to be the characteristic function on the set

\[ \{ (x,x,\cdots,x) : x \in X \} \] \hspace{0.5cm} (9)

Moreover, it is represented by the following diagram:

\[ \begin{array}{c}
\end{array} \]

3. THE GENERATING PROPERTY OF SUBFACTORS

The 2-box space $\mathcal{P}_2$ is spanned by $\{ I,J,A \}$, where $I$ is the identity matrix, $J$ is the matrix with each entry being 1 and $A$ is the adjacent matrix of $KG_{n,2}$. We represent these rank-2 tensors by the following diagrams:

\[ \begin{array}{c}
\end{array} \]

**Figure 1.** The rank-2 tensors $I$, $J$ and $A$.

Moreover, the adjacent matrix $T$ for the complement of $KG_{n,2}$ is given by $T = J - I - A$ and we represent it as \[ \begin{array}{c}
\end{array} \]. Therefore the equation $J = I + T + A$ tells us that

\[ \begin{array}{c}
\end{array} \] \hspace{0.5cm} (10)

**Proposition 3.1.** Let $R \in \mathcal{P}_4^G$ be the transposition. We define

• The element $R_A$ as follows:

\[ \begin{array}{c}
\end{array} \] \in \mathcal{A}_4. \]
The element $R_T$ as follows:

\[
\begin{array}{c}
\begin{array}{cc}
& \\
& \\
& \\
\end{array}
\end{array}
\in \mathcal{A}_4.
\]

Then we have that

\[
\begin{array}{c}
\begin{array}{cc}
& \\
& \\
& \\
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{cc}
& \\
& \\
& \\
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{cc}
& \\
& \\
& \\
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{cc}
& \\
& \\
& \\
\end{array}
\end{array}
\]

(11)

Proof. It follows directly by computation. □

Notation 3.2. We denote the submodel of $\mathcal{S}^{P_n}$ generated by $P_2$ by $\mathcal{A}_4$.

Theorem 3.3. The strongly regular graph $KG_{n,2}$ has property $(G)$ if and only if

\[
\begin{array}{c}
\begin{array}{cc}
& \\
& \\
& \\
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{cc}
& \\
& \\
& \\
\end{array}
\end{array} \in \mathcal{A}_4.
\]

(12)

Proof. Suppose $R \in \mathcal{A}_4$. We construct an element $X_n$ in $\mathcal{A}_{(\frac{n}{2})}$ with respect to $KG_{n,2}$ as follows:

(1) Draw $KG_{n,2}$ on $\mathbb{R}^2$ such that every vertex lies on the line $y = 1$ and the remaining part of $KG_{n,2}$ are below the line $y = 1$. Label the vertices from left to right by $1, 2, \cdots, \binom{n}{2}$.

(2) Each vertex is replaced by $\text{GHZ}(n-2)$; each edge is replaced by $A$; each crossing is replaced by $R$. This defines an element in $\mathcal{A}_{(\frac{n}{2})}$, and we denote it by $X_n$.

Suppose $\vec{i} = (i_1, i_2, \cdots, i_{\binom{n}{2}})$ is an arbitrary point in $V_{(\frac{n}{2})}$. By the construction, we have that $X_n(\vec{i})$ must take a value in $\{0, 1\}$. Set $M = \{\vec{i} : X_n(\vec{i}) = 1\} \subset V_{(\frac{n}{2})}$. Let $\vec{i} \in M$. First we show that for any $1 \leq k, l \leq \binom{n}{2}$, we must have $i_k \neq i_l$: This will be discussed in two cases by contradiction:

Assume that there exists $1 \leq k, l \leq \binom{n}{2}$ such that $i_k = i_l$. By definition of $A$, we know that $k$ and $l$ are not connected by an edge in $KG_{n,2}$.

(1) When $n$ is odd: By the construction of $KG_{n,2}$, there exists a subset $W \subset V$, such that

- The induced subgraph on $W$ is the complete graph on $\frac{n-1}{2}$ vertices.
- There exists $a, b \in W$ such that $k$ is connected to every vertex in $W \setminus \{b\}$ and $l$ is connected to every one in $W \setminus \{a\}$.

The assumption that $i_k = i_l$ implies that the induced subgraph on $\{i_k\} \cup \{i_w : w \in W\}$ is the complete graph on $\frac{n+1}{2}$ vertices. This leads to a contradiction.

(2) When $n$ is even: By the construction of $KG_{n,2}$, there exists $W \subset V$ and $c \in V$ such that

- The induced subgraph on $W \cup \{a\}$ is the complete graph on $\frac{n-2}{2}$ vertices.
- There exists $a, b \in W$ such that $k$ is connected to every vertex in $W \setminus \{b\}$ and $l$ is connected to every one in $W \setminus \{a\}$.
- The vertex $c$ is neither connected to $k$ nor $l$.

Similarly, the assumption that $i_k = i_l$ implies that the induced subgraph on $\{i_k\} \cup \{i_w : w \in W\}$ is the complete graph on $\frac{n}{2}$ vertices. Therefore, the two induced subgraphs
\{i_k\} \sqcup \{i_w : w \in W\} and \{i_w : w \in W\} are the complete graph on n vertices and \(i_k\) and \(i_l\) are two different vertices. However, this is a contradiction by the Kneser construction of \(KG_{n,2}\).

Therefore, for every \(\vec{i} \in M\), we have that \(i_j\)s are distinct. Let \(g_{\vec{i}}\) be the permutation on \(V\) defined by sending \(i_j\) to \(j\). By the construction of \(X_n\), we know that \(g_{\vec{i}}\) is an automorphism of \(KG_{n,2}\). Moreover, for every automorphism \(g\), we have that \(X_n(\beta(g)1, \cdots, \beta(g)n) = 1\). This implies that

\[ X_n = \chi_S = M. \tag{13} \]

Since the assumption of this lemma asserts that GHZ and \(R\) belongs to \(\mathcal{A}_\bullet\), it follows from Proposition 2.3 that

\[ \mathcal{P}_S \subset \mathcal{A}_\bullet \subset \mathcal{P}_S, \tag{14} \]

namely, the graph \(KG_{n,2}\) has property \((G)\).

The other direction follows directly and therefore the lemma is proved. \(\square\)

Lemma 3.4. The strongly regular graph \(KG_{n,2}\) has property \((G)\) if and only if

\[ \begin{array}{c}
\includegraphics[width=0.2\textwidth]{figure.png}
\end{array} \in \mathcal{A}_4. \tag{15} \]

Proof. \((\Rightarrow)\) It follows directly.

\((\Leftarrow)\) Suppose \(R_A \in \mathcal{A}_\bullet\). By Lemma 3.3 and Equation 11, we only need to show \(R_T \in \mathcal{A}_4\). We define a sequence of elements \(\gamma_k \in \mathcal{A}_{3k}\) recursively as follows:

\[ \gamma_1 = \begin{array}{c}
\includegraphics[width=0.1\textwidth]{figure.png}
\end{array}. \tag{16} \]

\[ \gamma_{k+1} = \begin{array}{c}
\includegraphics[width=0.2\textwidth]{figure.png}
\end{array}. \tag{17} \]

Claim. Let \(B_k = \{(i_1, i_2, \cdots, i_k, i_1, i_2, \cdots, i_k, i_k, i_{k-1}, \cdots, i_1) : S(s, i_t) = 1 \forall 1 \leq s \leq t \leq k\}\). Then we have that \(\gamma_k = \chi_{B_k}\).

Proof of Claim. We prove this by induction on \(k\). It is straightforward to see that the claim is true when \(k = 1\). Now assume it is true for \(k\). Let \((i_1, i_2, \cdots, i_{k+1}, j_1, j_2, \cdots, j_{k+1}, m_{k+1}, m_k, \cdots, m_1) \in\)
We denote
\[ \gamma_{k+1}(i_1, i_2, \cdots, i_{k+1}, j_1, j_2, \cdots, j_{k+1}, m_{k+1}, \cdots, m_1) \] (18)
\[ = \gamma_k(i_1, \cdots, i_k, j_1, \cdots, j_k, m_k, m_{k-1}, \cdots, m_1) \delta_{i_{k+1}, j_{k+1}} \delta_{i_{k+1}, m_{k+1}} \prod_{t=1}^k S(i_{k+1}, j_t) \] (19)
\[ = \left( \prod_{t=1}^k \delta_{i_t, m_t} \delta_{i_t, m_t} \right) \left( \prod_{1 \leq s \leq t \leq k} S(i_s, i_t) \right) \delta_{i_{k+1}, j_{k+1}} \delta_{i_{k+1}, m_{k+1}} \prod_{t=1}^k S(i_{k+1}, j_t) \] (20)
\[ = \chi_{B_{k+1}}. \] (21)

This proves the claim. □

Now we construct \( R_T \) explicitly in the following two cases:

(1) Suppose \( n \) is odd. Let \( m = \frac{n-3}{2} \). Consider the following element \( Y \) in \( A_4 \).

Now we evaluate \( Y \) on an arbitrary point \((v_1, v_2, v_3, v_4) \in V^4 \). By definition, we have that

\[ \frac{Y(v_1, v_2, v_3, v_4)}{T(v_1, v_2)T(v_3, v_4)T(v_1, v_4)} = \sum_{v_5 \in V; i \in V_m} \left( \prod_{t=1}^4 T(v_t, v_5) \right) \left( \prod_{1 \leq s \leq m, 1 \leq t \leq 5} S(i_s, v_t) \right). \] (23)

Note that \( Y \) is invariant under the action \( \beta \) of \( S_n \). One can assume \( v_1 = \{1, 2\} \) and \( v_2 = \{1, 3\} \) without loss of generality. By definition of \( \gamma_m \), we know that

\[ \bigcup_{t=1}^m i_t = \{4, 5, \cdots, n\}. \] (24)
This implies that \( v_5 = \{2, 3\} \) and \( v_4, v_3 \subset \{1, 2, 3\} \). Since there are terms \( T(v_3, v_5) \), \( T(v_4, v_5) \) and \( T(v_1, v_2)T(v_3, v_4)T(v_1, v_4) \), we know that \( v_3 = \{1, 2\} \) and \( v_4 = \{1, 3\} \) if \( Y(v_1, v_2, v_3, v_4) \neq 0 \). Moreover, by counting the possible choice of \( \vec{i} \), we know that

\[
Y(v_1, v_2, v_3, v_4) = \frac{(n - 3)!}{2^m} \delta_{v_1, v_3} \delta_{v_2, v_4} T(v_1, v_2),
\]

namely, \( Y = \frac{(n - 3)!}{2^m} R_T \).

(2) Suppose \( n \) is even. Let \( m = \frac{n - 4}{2} \). Consider the following element \( Y \) in \( \mathcal{A}_4 \).

Now we evaluate \( Y \) on an arbitrary point \((v_1, v_2, v_3, v_4) \in V^4\). By definition, we have that

\[
\frac{Y(v_1, v_2, v_3, v_4)}{T(v_1, v_2)T(v_3, v_4)T(v_1, v_4)} = \sum_{v_5, v_6 \in V: v : v_5 \in V^m} \left( \prod_{j=1}^2 S(v_5, v_j)T(v_6, v_j) \right) \frac{S(v_5, v_6)T(v_5, i_m)}{S(v_5, i_m)} \left( \prod_{t=1}^m S(v_5, i_t)S(v_6, i_t) \right)
\]

Without loss of generality, one can assume that \( v_1 = \{1, 2\} \) and \( v_2 = \{1, 3\} \). By definition of \( \gamma_m \), we know that

\[
\bigcup_{i=1}^m i_t \bigcup v_5 = \{4, 5, \ldots, n\}.
\]

This implies that \( v_6 = \{2, 3\} \). The rest of the computation is exactly similar to that in the previous case and thus \( Y \) is a multiple of \( R_T \). To be more precise, we have that

\[
Y = \frac{(n - 3)!}{2^{m-1}} \delta_{v_1, v_3} \delta_{v_2, v_4} T(v_1, v_2) = \frac{(n - 3)!}{2^{m-1}} R_T.
\]
Therefore, in both cases we have that
\[\in \mathcal{A}_{4,+}.\] (29)

Combined with Equation (15), we know that the transposition \(R\) belongs to \(\mathcal{A}\), since \(R = R_T + R_A + \text{GHZ}_4\). By Theorem 3.3, we have that \(\mathcal{A} = \mathcal{B}^{S_n}\), namely, \(KG_{n,2}\) has property (G). \(\square\)

Now we return to prove the main theorem.

**Theorem 3.5.** The Kneser graph \(KG_{n,2}\) has property (G) for \(n \geq 5\).

**Proof.** By Lemma 3.4, we need to show that
\[\in \mathcal{A}_{4,+}.\] (30)

To show this, we investigate the subspace \(Q \subset \mathcal{P}_{4}^{S_n}\) defined by the fixed points under the following tangle:

The idea is to show that \(Q \subset \mathcal{A}_{4}\). This implies that \(R_A \in \mathcal{A}\) and by Lemma 3.4 we have that the transposition \(R \in \mathcal{A}\). We discuss this in two cases:

1. **General case:** Suppose \(n \geq 8\). It follows that the subspace \(Q\) is 9-dimensional and has the following basis \(B:\)
   \[
   \begin{align*}
   b_1 &= \{\{1, 2\}, \{3, 4\}, \{1, 2\}, \{3, 4\}\}, \\
   b_2 &= \{\{1, 2\}, \{3, 4\}, \{1, 2\}, \{3, 5\}\}, \\
   b_3 &= \{\{1, 2\}, \{3, 4\}, \{1, 2\}, \{5, 6\}\}, \\
   b_4 &= \{\{1, 2\}, \{3, 4\}, \{1, 5\}, \{3, 4\}\}, \\
   b_5 &= \{\{1, 2\}, \{3, 4\}, \{1, 5\}, \{3, 6\}\}, \\
   b_6 &= \{\{1, 2\}, \{3, 4\}, \{1, 5\}, \{6, 7\}\}, \\
   b_7 &= \{\{1, 2\}, \{3, 4\}, \{5, 6\}, \{3, 4\}\}, \\
   b_8 &= \{\{1, 2\}, \{3, 4\}, \{5, 6\}, \{3, 7\}\}, \\
   b_9 &= \{\{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 8\}\}.
   \end{align*}
   \]

Let \(\mathcal{B}\) be the set of following elements in \(\mathcal{A}_{4} \cap \mathcal{D}\): 

\[
\begin{array}{cccc}
\includegraphics[width=0.1\textwidth]{image1} & \includegraphics[width=0.1\textwidth]{image2} & \includegraphics[width=0.1\textwidth]{image3} & \includegraphics[width=0.1\textwidth]{image4} \\
\includegraphics[width=0.1\textwidth]{image5} & \includegraphics[width=0.1\textwidth]{image6} & \includegraphics[width=0.1\textwidth]{image7} & \includegraphics[width=0.1\textwidth]{image8} \\
\end{array}
\]
In order to show the above diagrams form a basis of the subspace $Q$, we need to compute the inner product matrix of the above diagrams and the basis $\mathcal{B}$. It follows from a direct computation that the inner product matrix $M$ is given as

\[
M = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
(n-4)/2 & (n-5)/2 & (n-6)/2 & (n-7)/2 & (n-8)/2 & (n-9)/2 & (n-10)/2 & (n-11)/2 \\
(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9) & (y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8, y_9)
\end{bmatrix}.
\]

In the inner product matrix $M$, \(\{x_i : 1 \leq i \leq 9\}\) and \(\{y_i : 1 \leq i \leq 9\}\) are given by

\[
x_1 = \binom{n-4}{2} \binom{n-6}{2},
\]

\[
x_2 = \left(\binom{n-5}{2} + \binom{n-6}{2}\right) \binom{n-5}{2},
\]

\[
x_3 = \binom{n-4}{2} + 2(n-6) \binom{n-5}{2} + \binom{n-6}{2} \binom{n-6}{2},
\]

\[
x_4 = \binom{n-5}{2} \binom{n-7}{2},
\]

\[
x_5 = (n-6) \binom{n-6}{2} + \binom{n-6}{2} \binom{n-7}{2},
\]

\[
x_6 = \binom{n-5}{2} + 2(n-7) \binom{n-6}{2} + \binom{n-7}{2} \binom{n-7}{2},
\]

\[
x_7 = \binom{n-6}{2} \binom{n-8}{2},
\]

\[
x_8 = (n-7) \binom{n-7}{2} + \binom{n-7}{2} \binom{n-8}{2},
\]

\[
x_9 = \binom{n-6}{2} + 2(n-8) \binom{n-7}{2} + \binom{n-8}{2} \binom{n-8}{2},
\]

\[
y_1 = \frac{1}{16} (2919840 - 3704488n + 2039584n^2 - 637336n^3 + 123793n^4 - 15324n^5 + 1182n^6 - 52n^7 + n^8),
\]

\[
y_2 = y_4 = \frac{1}{16} (3699360 - 4400712n + 2297408n^2 - 688048n^3 + 129385n^4 - 15652n^5 + 1190n^6 - 52n^7 + n^8),
\]

\[
y_3 = y_7 = \frac{1}{16} (4579680 - 5154008n + 2567320n^2 - 739896n^3 + 135017n^4 - 15980n^5 + 1198n^6 - 52n^7 + n^8),
\]

\[
y_5 = \frac{1}{16} (4564896 - 5145064n + 2565280n^2 - 739688n^3 + 135009n^4 - 15980n^5 + 1198n^6 - 52n^7 + n^8),
\]

\[
y_6 = y_8 = \frac{1}{16} (5534816 - 5947448n + 2845304n^2 - 792464n^3 + 140673n^4 - 16308n^5 + 1206n^6 - 52n^7 + n^8),
\]

\[
y_9 = \frac{1}{16} (6600576 - 6800712n + 3135568n^2 - 846168n^3 + 146369n^4 - 16636n^5 + 1214n^6 - 52n^7 + n^8).
\]
The determinant of \( M \) is
\[
\det M = 8(2n^2 - 26n + 83).
\]

It follows that \( \det M \neq 0 \) when \( n \geq 8 \). Therefore, the set \( \mathcal{B} \) forms a basis for \( Q \), namely, \( Q \subset \mathcal{A}_4 \). Hence, we have that
\[

(31)
\]

(2) **Reduced case:** Now we proceed to the case when \( n = 7 \). In this case, the dimension of \( Q \) reduces to 8 and the basis \( \mathcal{B} \) reduces to
\[
\begin{align*}
\mathcal{B} &= \{ \{1, 2\}, \{3, 4\}, \{1, 2\}, \{3, 4\} \}, \\
\mathcal{B} &= \{ \{1, 2\}, \{3, 4\}, \{1, 2\}, \{3, 5\} \}, \\
\mathcal{B} &= \{ \{1, 2\}, \{3, 4\}, \{1, 2\}, \{5, 6\} \}, \\
\mathcal{B} &= \{ \{1, 2\}, \{3, 4\}, \{1, 5\}, \{3, 4\} \}, \\
\mathcal{B} &= \{ \{1, 2\}, \{3, 4\}, \{1, 5\}, \{3, 6\} \}, \\
\mathcal{B} &= \{ \{1, 2\}, \{3, 4\}, \{1, 5\}, \{6, 7\} \}, \\
\mathcal{B} &= \{ \{1, 2\}, \{3, 4\}, \{5, 6\}, \{3, 4\} \}, \\
\mathcal{B} &= \{ \{1, 2\}, \{3, 4\}, \{5, 6\}, \{3, 7\} \}.
\end{align*}
\]

It follows by direct computation that
\[
M = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
3 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 2 & 5 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 & 5 & 1
\end{bmatrix}.
\]

Moreover, \( \det M = 4 \neq 0 \). Now we proceed to the case when \( n = 6 \). In this case, the dimension of \( Q \) reduces to 6 and the basis \( \mathcal{B} \) reduces to
\[
\begin{align*}
\mathcal{B} &= \{ \{1, 2\}, \{3, 4\}, \{1, 2\}, \{3, 4\} \}, \\
\mathcal{B} &= \{ \{1, 2\}, \{3, 4\}, \{1, 2\}, \{3, 5\} \}, \\
\mathcal{B} &= \{ \{1, 2\}, \{3, 4\}, \{1, 2\}, \{5, 6\} \}, \\
\mathcal{B} &= \{ \{1, 2\}, \{3, 4\}, \{1, 5\}, \{3, 4\} \}, \\
\mathcal{B} &= \{ \{1, 2\}, \{3, 4\}, \{1, 5\}, \{3, 6\} \}, \\
\mathcal{B} &= \{ \{1, 2\}, \{3, 4\}, \{1, 5\}, \{6, 7\} \}, \\
\mathcal{B} &= \{ \{1, 2\}, \{3, 4\}, \{5, 6\}, \{3, 4\} \}.
\end{align*}
\]
It follows by direct computation that
\[
M = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 \\
3 & 1 & 0 & 1 & 0 \\
\end{bmatrix} \tag{33}
\]

Moreover, \(\det M = 1 \neq 0\).

Therefore, in both cases, we show that \(B\) is a basis for the subspace \(Q\). Since \(R_S \in Q\) and \(B \subset A_4\), we have that \(R_S \in \mathcal{A}\). By Lemma 3.4 and Theorem 3.3 we prove the theorem. \(\square\)

4. Appendix

Theorem 3.3 says that in a submodel \(\mathcal{A}\) of a spin model, the existence of the transposition \(R\) guarantees that the submodel \(\mathcal{A}\) carries the symmetry of a finite group. This theorem is first proved independently by Jones \cite{Jon} and Curtin \cite{Cur03}. In §3, we give an alternative proof for the spin model associated to \(KG_{n,2}\). In this section, we enhance the statement of Jones and Curtin for general planar algebras.

In \cite{Ren19}, the transposition \(R\) and the GHZ tensor are characterized by the following relations:

**Proposition 4.1.** Let \(\mathcal{P}\) be a spin model. Then the transposition \(R \in \mathcal{P}_4\) and \(\text{GHZ} \in \mathcal{P}_3\) satisfies the following relations:

1. **Reidemeister moves:**

   \[
   \begin{align*}
   &\begin{tikzpicture}[baseline=-.5ex]
   \fill (0,0) rectangle (1,1);
   \fill (0,0) rectangle (1,1);
   \end{tikzpicture} = \\
   &\begin{tikzpicture}[baseline=-.5ex]
   \fill (0,0) rectangle (1,1);
   \fill (0,0) rectangle (1,1);
   \end{tikzpicture} = \\
   &\begin{tikzpicture}[baseline=-.5ex]
   \fill (0,0) rectangle (1,1);
   \fill (0,0) rectangle (1,1);
   \end{tikzpicture} = \\
   \end{align*}
   \]

2. **Flatness:** For any \(x \in \mathcal{P}\), we have that

   \[
   \begin{tikzpicture}[baseline=-.5ex]
   \fill (0,0) rectangle (1,1);
   \fill (0,0) rectangle (1,1);
   \end{tikzpicture} = \\
   \begin{tikzpicture}[baseline=-.5ex]
   \fill (0,0) rectangle (1,1);
   \fill (0,0) rectangle (1,1);
   \end{tikzpicture}.
   \]

3. **Frobenius relations:**

   \[
   \begin{align*}
   &\begin{tikzpicture}[baseline=-.5ex]
   \fill (0,0) rectangle (1,1);
   \fill (0,0) rectangle (1,1);
   \end{tikzpicture} = \\
   &\begin{tikzpicture}[baseline=-.5ex]
   \fill (0,0) rectangle (1,1);
   \fill (0,0) rectangle (1,1);
   \end{tikzpicture} = \\
   \end{align*}
   \]

Let \(\mathcal{D}\) be a positive planar algebra such that there exists \(S \in \mathcal{D}_4\) and \(W \in \mathcal{D}_3\) satisfying relations (1), (2) and (3). Note that relations (1), (2) and (3) provide the skein theory for partition planar algebras \cite{Jon94}. This leads to the following corollaries:
(1) The circle parameter for $\mathcal{L}_\bullet$ is an integer, namely, there exists $d \in \mathbb{N}$ such that
\[
\circ = d.
\]

(2) Let $\mathcal{P}_{S_d}$ be the group-action model associated to $S_d \curvearrowright \{1, 2, \cdots, d\}$. Then there exists planar algebraic homomorphism $\alpha$ from $\mathcal{P}_{S_d}$ to $\mathcal{L}_\bullet$ such that $\alpha(R) = S$ and $\alpha(GHZ) = W$.

**Proposition 4.2.** For every $n \in \mathbb{N}$, there exists a homomorphism from $S_n$ to $\mathcal{L}_{2n}$ by sending the permutation $(k, k + 1)$ to $k - 1 \not< n - k - 1$.

**Proof.** It follows that $S$ is a symmetric braiding from Relation 1. \hfill \Box

**Remark.** Suppose $\sigma \in S_n$. We represent it by the diagram $\sigma$.

**Proposition 4.3.** Suppose $n \in \mathbb{N}$, we define the following operations on $\mathcal{L}_n$:

1. A binary operation $\circ$: given $f, g \in \mathcal{L}_n$,

\[
f \circ g = \begin{array}{c}
\begin{array}{c}
\circ\\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
f\\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
g\\
\end{array}
\end{array}
\]

2. The involution $^\dagger$: given $f \in \mathcal{L}_n$, let $\sigma$ be the permutation on $\{1, 2, \cdots, n\}$ such that $\sigma(j) = n - j$ for $1 \leq j \leq n$,

\[
f^\dagger = \begin{array}{c}
\begin{array}{c}
f^*\\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\sigma\\
\end{array}
\end{array}
\]

3. The norm $\| \cdot \|$: we equip $\mathcal{L}_n$ with the inner product $\langle \cdot, \cdot \rangle$: for $x, y \in \mathcal{L}_n$,

\[
\begin{array}{c}
\begin{array}{c}
y^*\\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\circ\\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
x\\
\end{array}
\end{array}
\]

We denote the norm on $\mathcal{L}_n$ by $\langle \cdot, \cdot \rangle$ by $\| \cdot \|_2$. Suppose $f \in \mathcal{L}_n$, we define the norm $\| \cdot \|$ by

\[
\| f \| = \sup\{\| f \circ x \|_2 : \| x \|_2 = 1\}.
\]
Then $(\mathcal{P}, \circ, ^{\dagger}, \| \cdot \|)$ is a commutative $C^*$-algebra and we call it the Hadamard algebra.

**Proof.** This proposition follows directly from the skein relations of $S$ and $W$. \hfill \Box

**Lemma 4.4.** Suppose $p$ is a minimal projection in $\mathcal{Q}_m$ for some $m \in \mathbb{N}$. Then for any $g \in S_m$, the following are minimal projections:

\[
\begin{array}{ccc}
\bullet \quad \bullet \quad \bullet \\
\hspace{1cm} m \\
\hspace{0.5cm} g \\
\hspace{0.5cm} P \\
\hspace{2cm} m \\
\hspace{0.5cm} P \\
\hspace{0.5cm} m - 1 \\
\hspace{2cm} P \\
\hspace{0.5cm} m - 1 \\
\end{array}
\]

**Proof.** This lemma follows directly from the skein relations of $S$ and $W$. We will only show the first one as an example. Let $x \in \mathcal{Q}_m$. Then we have that

\[
(g \cdot p) \circ x = \begin{array}{ccc}
\bullet \quad \bullet \\
\hspace{1cm} m \\
\hspace{0.5cm} g \\
\hspace{0.5cm} P \\
\hspace{2cm} m \\
\hspace{0.5cm} P \\
\hspace{0.5cm} x \\
\hspace{2cm} P \\
\hspace{0.5cm} m \\
\end{array} = \begin{array}{ccc}
\bullet \quad \bullet \\
\hspace{1cm} m \\
\hspace{0.5cm} g \\
\hspace{0.5cm} P \\
\hspace{2cm} m \\
\hspace{0.5cm} P \\
\hspace{0.5cm} x \\
\hspace{2cm} P \\
\hspace{0.5cm} m \\
\end{array} = g \cdot (p \circ (g^{-1} \cdot x)).
\]

Since $p$ is a minimal projection, we know the right hand side is nonzero if and only if $p = g^{-1} \cdot x$ if and only if $g \cdot p = x$. This implies that $g \cdot p$ is a minimal projection. \hfill \Box

**Theorem 4.5.** Let $\mathcal{Q}_\bullet$ be a planar algebra such that there exists $R \in \mathcal{Q}_4$ and $W \in \mathcal{Q}_3$ satisfying relations $[1]$, $[2]$ and $[3]$. Then there exists $d \in \mathbb{N}$ and a finite group $G \leq S_d$ such that $\mathcal{Q}_\bullet$ is isomorphic to $\mathcal{P}_G$.

**Proof.** Let $d$ be the circle parameter of $\mathcal{Q}_\bullet$. By Corollary $[1]$, we have that $d$ is an integer. By universal skein theory for group actions [Ren19], the group-action model $\mathcal{P}_G$ is generated by GHZ, the transposition $R$ and the molecule $Y = \chi_{S_d}(1,2,\ldots,d)$. By Universal skein theory for group-actions [Ren19], we know that

\[
\begin{array}{ccc}
\bullet \quad \bullet \\
\hspace{1cm} d \\
\hspace{0.5cm} Y \\
\hspace{2cm} Y \\
\hspace{0.5cm} Y^* \\
\hspace{1cm} Y \\
\hspace{0.5cm} Y \\
\end{array} = \sum_{g \in S_d} \begin{array}{ccc}
\bullet \quad \bullet \\
\hspace{1cm} d \\
\hspace{0.5cm} g \\
\hspace{2cm} Y \\
\hspace{0.5cm} Y \\
\hspace{1cm} d \\
\hspace{0.5cm} Y \\
\hspace{2cm} Y \\
\end{array}, \quad (35)
\]

\[
\begin{array}{ccc}
\bullet \quad \bullet \\
\hspace{1cm} d \\
\hspace{0.5cm} Y \\
\hspace{2cm} Y \\
\hspace{0.5cm} Y \\
\hspace{1cm} d \\
\hspace{0.5cm} Y \\
\hspace{2cm} Y \\
\end{array} = |S_d| \quad (36)
\]

\[
\begin{array}{ccc}
\bullet \quad \bullet \\
\hspace{1cm} d \\
\hspace{0.5cm} Y \\
\hspace{2cm} Y \\
\hspace{0.5cm} Y \\
\hspace{1cm} d \\
\hspace{0.5cm} Y \\
\hspace{2cm} Y \\
\end{array}
\]

\[
\begin{array}{ccc}
\bullet \quad \bullet \\
\hspace{1cm} d \\
\hspace{0.5cm} Y \\
\hspace{2cm} Y \\
\hspace{0.5cm} Y \\
\hspace{1cm} d \\
\hspace{0.5cm} Y \\
\hspace{2cm} Y \\
\end{array}
\]
It follows that the image of $Y$ under $\alpha$ is a projection in the Hadamard algebra $(\mathcal{Q}_d, \circ)$. We still denote it by $Y$ with abusing of notations. Since $(\mathcal{Q}_d, \circ)$ is commutative, there exist orthogonal minimal projections $Y_1, Y_2, \ldots, Y_m$ such that

$$Y = Y_1 + Y_2 + \cdots + Y_m.$$  \hfill (37)

Let $X = \{Y_1, Y_2, \ldots, Y_m\}$. For every $g \in S_d$ and $1 \leq j \leq m$, we have that $g \cdot Y_j$ is also a minimal subprojection of $Y$ by Lemma 4.4. Therefore, there exists an action of $S_d$ on $X$ which permutes these minimal projections. For every $1 \leq j \leq m$, let $G_j$ be the stabilizer of $Y_i$, namely,

$$G_j = \{g \in S_d : g \cdot Y_j = Y_j\}.$$ \hfill (38)

By Equation (35), we know that

$$Y_i Y_i = \sum_{g \in S_d} g \cdot Y_i Y_i = \sum_{g \in S_d} g \cdot Y_i Y_i.$$ \hfill (39)

By comparing the $x^*x$ where $x$ is the left- and right-hand side of Equation (39), we have that

$$(Y_i^* Y_i)^2 = |G_i| Y_i^* Y_i,$$

$$(\Rightarrow) Y_i^* Y_i = |G_i|.$$ \hfill (40)

This implies that

$$\sum_{i=1}^m |G_i| = \sum_{i=1}^m Y_i^* Y_i = Y^* Y = |S_d|.$$ \hfill (41)

Let $O_i$ be the orbit of $Y_i$. By the Orbit-Stabilizer Theorem, we have that

$$|G_i||O_i| = |S_d|.$$ \hfill (42)
By summing over $i$, this implies that
\[ \sum_{i=1}^{m} |G_i||O_i| = m|S_d| = m \sum_{i=1}^{m} |G_i|. \quad (44) \]

Note that $|O_i| \leq m$ for every $1 \leq i \leq m$. This forces that $|O_i| = m$ for every $1 \leq i \leq m$. Therefore, for every $2 \leq j \leq m$, there exists $g \in S_d$ such that $Y_j = g \cdot Y_1$. Moreover, this implies that all the $G_i$’s are isomorphic and we denote it by $G$.

Now we show that the planar algebra $\mathcal{P}_*$ is generated by $\{S, W, Y_1\}$. Let $p$ be an arbitrary minimal projection in $\mathcal{P}_m$ for some $m \in \mathbb{N}$. Note that $\mathcal{P}_{Sd} \subset \mathcal{P}_m$. There exists a projection in $\mathcal{P}_{Sd}$ such that $p$ is a subprojection of $S$. By Universal skein theory for group-action models [Ren19], there exist $T \in \mathcal{P}_{d+m}$ such that
\[ \sum_{i=1}^{m} Y_i = m Y_1. \quad (45) \]

Let $P_i$ be the $i$-th term in the right hand side of Equation (45) for $1 \leq i \leq m$. By Lemma 4.4 we know that each $P_i$ is a minimal projection. Since $p$ is also a minimal projection, there must exist $1 \leq i \leq m$ such that $p = P_i$. Note that for every $1 \leq i \leq m$, there exists $g \in S_d$ such that $Y_i = g \cdot Y_1$. This implies that $p$ is generated by $\{S, W, Y_1\}$. Moreover, the generators satisfy the universal skein theory for group-action models. Therefore, we have that $\mathcal{P}_*$ is isomorphic to $\mathcal{P}_G$. $\square$

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