I. INTRODUCTION: THE MAXIMUM FORCE CONJECTURE AND THERMODYNAMICS

The maximum force conjecture proposed by Gibbons [1, 2] and Schiller [3, 4] argues that in general relativity any physically attainable force or tension between two bodies is bounded above by a quarter of the Planck force, $F_{pl}$. In 4-dimensions, the value in SI units is

$$F \leq \frac{c^4}{4G} = \frac{1}{4} F_{pl} \approx 3.25 \times 10^{13} \text{ N.} \quad (1)$$

Note that this is a classical quantity free of $\hbar$. Hereinafter we will work in the Planck units in which $\hbar = G = c = 1$ for simplicity of notation (so $F_{pl} = 1$), though one must be extra careful of the dependence of $\hbar$ in various quantities.

The inequality above is the strong form of the conjecture, whilst the weak form allows the maximum force to be some $O(1)$ multiple of the Planck force, not necessarily $1/4$ [5]. The conjecture in either form has been shown to be incorrect by Jowsey and Visser [6], who provided a few counterexamples (such as fluid spheres on the verge of gravitational collapse). Nevertheless, as mentioned therein, the conjecture could be correct under some specific conditions. That is, perhaps the discussion must be restricted to some classes of forces. Subsequently Faraoni [7] proposed that the idea of maximum force does apply in the context of black holes, but argued that such a bound on any force acting on black hole horizons is inherently tied to cosmic censorship (and therefore not a new, independent conjecture). See also [8].

In [9] a “Hookean force” of the form $F_1 = kx$ was introduced in the context of 4-dimensional asymptotically flat Kerr black hole, in which $k$ is a “spring constant” defined by $k = M\Omega_+^2$, where $M$ is the black hole mass and $\Omega_+$ the angular velocity of the event horizon (we will review this in Sec.(II)), while $x$ is identified with the horizon $r_+$. It was shown that the “Hookean force” tends to $1/4$ in the extremal limit, which seems to suggest that the maximum force conjecture is indeed related to the cosmic censorship conjecture, as argued by Faraoni. However, as we shall see below, when generalized to spacetime dimension $d$ above 4 the picture is quite different. Let us note that applying the maximum force conjecture to higher dimension is actually quite subtle due to the result in [8], an issue we will defer to the Discussion section.

In higher dimensions, there are more than one angular momenta, but we will focus mainly on the singly rotating case in which there is no extremal limit in $d \geq 6$. We find that the “Hookean force” is bounded by a dimensional dependent value less than $1/4$, which corresponds to the well-known Emparan-Myers fragmentation [10, 11] – black holes that are spinning too fast break into two because the latter configuration has higher entropy and is therefore thermodynamically preferred. Thus, remarkably the “Hookean force” detects thermodynamic instability: in some sense the black hole breaks much like a spring would if stretched too much.
In addition, we can define a “force” by taking the ratio \( F_2 := J^2/S^2 \), where \( J \) is the angular momentum and \( S \) the Bekenstein-Hawking entropy of the black hole. These two thermodynamical variables can be used to define the so-called Ruppeiner geometry [13] – a two-dimensional Riemannian metric in thermodynamic phase space – which has recently gained some attention in the context of black hole microstructure [14–21] and applications to black hole shadows [22–24]. Interestingly, the region in the thermodynamic phase space with positive Ruppeiner scalar curvature is marked by the boundary \( F_2 = \frac{1}{12} \left( \frac{d-5}{d-3} \right) \) and \( F_2 = \frac{1}{3} \left( \frac{d-3}{d-5} \right) \). The latter value corresponds to the minimum temperature of the black hole (applicable for \( d > 5 \)). As pointed out by Emparan and Myers, at this point the temperature of the black hole starts to behave like a black brane instead of a Kerr-type black hole. Due to the rapid rotation the black hole geometry is also highly flattened. Like a black brane, it suffers from the well-known Gregory-Laflamme instability [25, 26]. The relation between the sign of Ruppeiner geometry and the maximum force is thermore related to the cosmic censorship\(^3\), it is important to check with more examples to see whether this is just a coincidence. Thus, let us consider higher dimensional rotating black holes (the Myers-Perry solution [29]) with one angular momentum and use Eq.(3) to define \( k \).

The Hawking temperature of a Myers-Perry black hole in \( d \)-dimensional spacetime is given by

\[
T = \frac{1}{4\pi} \left( \frac{2r_+^{d-5}}{\mu} + \frac{d - 5}{r_+} \right),
\]

where \( \mu \) is essentially a normalized mass parameter\(^4\):

\[
\mu := \frac{16\pi G}{(d-2)\Omega_{d-2} M}, \quad \Omega_{d-2} := \frac{2\pi^{\frac{d-1}{2}}}{\Gamma \left( \frac{d-1}{2} \right)}.
\]

The horizon is the solution of the expression

\[
r_+^2 + a^2 - \frac{\mu}{r_+} = 0.
\]

In \( d = 5 \), the extremal limit corresponds to \( r_+ = 0 \); it is an example of the so-called “extremal vanishing horizon” (EVH) spacetime [30]. Since the horizon overlaps with the central ring singularity, the extremal solution is essentially a naked singularity [10]. In \( d \geq 6 \) dimensions, singly rotating Myers-Perry black holes can possess arbitrarily large angular momentum since there is no extremal limit. The case \( a \gg r_+ \) is called the “ultra-spinning limit”. As such we expect that the case for \( d \geq 6 \) would be substantially different from the \( d = 5 \) and \( d = 4 \) cases.

In the ultra-spinning limit, we have, from Eq.(6),

\[
\mu = r_+^{d-5} a^2.
\]

We can substitute this into Eq.(5) and obtain:

\[
T = \frac{1}{2\pi} \left( \frac{r_+}{a^2} + \frac{d - 5}{2r_+} \right).
\]

II. SPRINGY BLACK HOLES AND THE HOOKEAN FORCE

It was discovered in [9] that the Hawking temperature of an asymptotically flat Kerr black hole in 4 dimensions can be expressed succinctly as

\[
T = \frac{1}{2\pi} (g - k),
\]

in which \( g := (4M)^{-1} \) is the surface gravity of a Schwarzschild black hole of the same mass. The “spring constant” \( k \) is defined by \( k := M\Omega^2 \), in analogy with the familiar expression \( k = m\omega^2 \) for a mass \( m \) attached to the end of a spring in elementary classical mechanics\(^2\).

Hooke’s law \( F_1 = kx \) for a spring, seemingly naively applied by identifying \( x \) with the event horizon \( r_+ \), would yield, in the extremal limit [9],

\[
\lim_{J \to M^2} F_1 = \frac{1}{4M} \left( \lim_{J \to M^2} r_+ \right) = \frac{1}{4M} \cdot M = \frac{1}{4}.
\]

While this seems to suggest that the Hookean force is in agreement with the maximum force conjecture and furthermore related to the cosmic censorship\(^3\), it is important to check with more examples to see whether this is just a coincidence. Thus, let us consider higher dimensional rotating black holes (the Myers-Perry solution [29]) with one angular momentum and use Eq.(3) to define \( k \).

\(^2\) To keep the analogy we prefer to keep (the hidden) \( h \) outside the bracket in Eq.(3), so that \( k \) – and consequently the Hookean force – is a classical quantity. We will also do this in higher dimensions.

\(^3\) In the sense that if the maximum force is exceeded, the black hole parameters exceed the censorship bound and we get a naked singularity.

\(^4\) Note that \( \Omega_{d-2} \) is the area of the unit \( (d-2) \)-dimensional sphere, not to be confused with the angular velocity of the horizon \( \Omega_+ \).
The Hookean force is then expressible as
\[ T_* = \frac{d - 3}{4\pi r_+} = \frac{d - 3}{4\pi \mu \pi^2}. \] (9)

Following [9] and using the fact that the surface gravity is \( \kappa = 2\pi T \), we want to rewrite Eq.(8) in the form of Eq.(3). Using Eq.(7), (9) and (8), we have:

\[
T = \frac{1}{2\pi} \left[ \frac{d - 3}{2\mu \pi^2} - \left( \frac{d - 3}{2\mu \pi^2} - \frac{d - 5}{2r_+} \frac{r_+}{a^2} \right) \right] \\
= \frac{1}{2\pi} \left[ \frac{d - 3}{2\mu \pi^2} - \left( \frac{d - 3}{2r_+} \frac{r_+}{a} - \frac{d - 5}{2r_+} - \frac{r_+}{a^2} \right) \right] \quad \text{where we identify the quantity in the square brackets of the second line as the spring constant. Next, we define a Hookean force as before by \( F_1 := kx \). If we again identify \( x = r_+ \), then:} \\
F_1 = \frac{d - 3}{2} \left( \frac{r_+}{a} \right) \frac{s_+}{2} - \frac{d - 5}{2} - \frac{r_+^2}{a^2} \lesssim 0.21, (11)\]

where the value was obtained numerically for \( r_+/a < 1 \) and \( d > 5 \), which are the conditions required by the ultra-spinning limit and the positiveness of the temperature.

The upper bound 0.21, in fact rather surprisingly, corresponds to the Emparan-Myers fragmentation, as can be checked explicitly. First we take Eq.(5) and leave \( \mu \) implicit:

\[
T = \frac{1}{2\pi} \left[ \frac{d - 3}{2\mu \pi^2} - \left( \frac{d - 3}{2\mu \pi^2} - \frac{d - 5}{2r_+} - \frac{r_+^d}{\mu} \right) \right].
\]
The Hookean force is then expressible as

\[
F_1 = kr_+ = \frac{d - 3}{2} \frac{r_+}{\mu \pi^2} - \frac{d - 5}{2} - \frac{r_+^d}{\mu}.
\]
In 6 dimensions, the approximate result in [10] is that the black hole decays into two Schwarzschild fragments when \( a/r_+ > 1.36 \). Thus requiring that

\[
\frac{a}{r_+} \lesssim 1.36 \quad \Rightarrow \quad \frac{a^3}{\mu} \lesssim 0.88, (12)
\]

and substituting this into the equation for the force yields:

\[
F_1 = \frac{3}{2} \mu \pi^2 - \frac{r_+^3}{\mu} - \frac{1}{2} = \frac{3}{2} \frac{r_+ a}{\mu^2} - \frac{r_+^3 a^3}{\mu^2} - \frac{1}{2} \lesssim 0.21.
\]

This agrees with Eq.(11) despite \( a/r_+ \lesssim 1.36 \) is not too ultra-spinning. Similarly, in 7 and 8 dimensions we have \( F_1 \lesssim 0.19 \) and \( F_1 \lesssim 0.18 \), respectively, which are less than the 0.21 value obtained in the ultra-spinning limit in Eq.(11). It can be shown that, in fact, the upper bound decreases monotonically with \( d \), and tends to 0.1534 in the limit \( d \to \infty \).

In five dimensions, we can solve Eq.(6) exactly and obtain

\[
r_+ = \sqrt{\mu - a^2}, (13)
\]
The temperature in Eq.(5) and the temperature in the Schwarzschild limit in Eq.(9) become, respectively,

\[
T = \frac{r_+}{2\pi \mu}, \quad T_* = \frac{1}{2\pi r_+} = \frac{1}{2\pi} \sqrt{\mu}.
\]
The temperature is always positive, provided the cosmic censorship bound \( \mu \geq a^2 \) holds. Using the expression for the horizon in Eq.(13), we obtain:

\[
T = \frac{1}{2\pi} \left[ \frac{1}{\sqrt{\mu}} - \left( \frac{\sqrt{\mu} - \sqrt{\mu - a^2}}{\mu} \right) \right] = \frac{1}{2\pi} (g - k).
\]
Therefore, the Hookean force is given by:

\[
F_1 = kr_h = \left( \frac{\sqrt{\mu} - \sqrt{\mu - a^2}}{\mu} \right) \sqrt{\mu - a^2}
\]
\[
= \sqrt{1 - \frac{a^2}{\mu}} \left( 1 - \sqrt{1 - \frac{a^2}{\mu}} \right) \leq \frac{1}{4}, (14)
\]

where the last inequality is obtained by maximizing the function \( F_1 \) with respect to \( a^2/\mu \).

We can therefore conclude that: in dimensions 4 and 5, in which there exists an extremal limit for singly spinning black holes, the maximum Hookean force is 1/4. However this maximum value does not correspond to the extremal limit in dimension 5 (as it does in dimension 4), since in that case \( \mu = a^2 \) and \( F_1 = 0 \). This makes sense because \( r_+ = 0 \) for the extremal 5-dimensional solution. The maximum value of the Hookean force is attained at \( a^2/\mu = 3/4 \) instead. For dimension 6 and above there is no extremal limit, and the exact bound on the Hookean force is dimensional dependent and decreasing with number of dimensions (they are all bounded above by approximately 0.21). Furthermore the bound corresponds to the Emparan-Myers fragmentation point in each given dimension. Thus the Hookean force reveals thermodynamical instability in black holes. This is despite its seemingly naive definition.

Since the fragmentation happens prior to the conjectured maximum value 1/4, however, one cannot infer any significance of the purported maximum value in \( d \geq 6 \) other than the bound is always satisfied. One may wonder whether there are other suitably defined forces that mark thermodynamical instability or phase transition of some kind at the value 1/4. This motivates the next topic we wish to explore: the thermodynamic geometry of Ruppeiner.
The scalar curvature of the Ruppeiner metric, $\mathcal{R}$, has positive curvature in the shaded regions of the plots shown in Fig. 1. Its boundaries are marked by yellow lines. The red line corresponds to $F_2 = 1/4$ along which $\mathcal{R}$ is divergent (phase transition). The black line always marks the lower boundary of the positive $\mathcal{R}$ region. For $d = 6$, the red line coincides with this line. As $d$ is increased, the $F_2 = 1/4$ line shifts into the shaded region (more accurately, the region changes while the line stays fixed), so that in the large $d$ limit the red line and black line coincide, as shown in the right most plot for $d = 10^5$.

III. THERMODYNAMIC GEOMETRY AND THE MAXIMUM FORCE

The Ruppeiner metric $^5 [13]$ is essentially the Hessian of the entropy

$$g^R_{ij} = -\partial_i \partial_j S(M, N^n), \quad \text{(15)}$$

taken as a Riemannian metric (thus endowed with the usual Levi-Civita connection) in the phase space. Here $M$ is the energy (black hole mass for our case), and $N^n$ are other extensive variables of the system (such as black hole charge and angular momentum). It has been applied to study the thermodynamics of black holes, in particular thermodynamic instability and phase transitions $^{35–47}$. The scalar curvature of the Ruppeiner metric, $\mathcal{R}$, plays a crucial role in these analyses. The scalar curvature is positive if the underlying statistical interactions of the thermodynamical system is repulsive, and likewise it is negative for attractive interactions $^{36, 48}$. These are referred to as being “Fermi-like” and “Bose-like”, respectively, in $^6 [49, 50]$. Furthermore the larger the value of $\mathcal{R}$, the less stable the system $^{51, 52}$. A phase transition can be expected when $\mathcal{R}$ diverges$^7$, while $\mathcal{R} = 0$ corresponds to an “ideal gas”-like behavior.

For the singly rotating Myers-Perry black holes, the Ruppeiner metric is 2-dimensional (its explicit form is not useful for our discussion), and we have $^{39}$

$$\mathcal{R} = -\frac{1}{S} \left( \frac{1 - 12 \cdot \frac{d-5}{d-3} \frac{J^2}{S^2}}{1 + 4 \cdot \frac{d-5}{d-3} \frac{J^2}{S^2}} \right). \quad \text{(16)}$$

The properties of this scalar curvature and its relation to black hole thermodynamics have been investigated rather thoroughly in the aforementioned literature. In particular, as pointed out in $^{39}$, in $d = 4$ the scalar curvature diverges along the curve $J^2/S^2 = 1/4$ in the $(S, T)$-plane, which corresponds to the extremal limit.

In view of the appearance of 1/4, we are motivated to define another force by $F_2 := J^2/S^2$ and explore its behavior in higher dimensions. Again, we note that $d = 5$ is special because $\mathcal{R}$ simplifies to $-1/S$, which becomes divergent in the extremal limit. Note that $\mathcal{R} < 0$ for both $d = 4$ and $d = 5$. However, for $d \geq 6$ the curvature diverges not in the extremal limit (which does not exist), but rather along the curve

$$F_2 := \frac{J^2}{S^2} = \frac{1}{4} \left( \frac{d-3}{d-5} \right). \quad \text{(17)}$$

It is even more illuminating if we plot the region that has positive $\mathcal{R}$ in the $(S, T)$-plane, as shown in Fig.(1). The line $F_2 = 1/4$ that would correspond to the maximum force (if indeed the conjecture was correct and applicable to $F_2$) marks the boundary of the positive $\mathcal{R}$ region in $d = 6$, as opposed to marking the extremal limit in $d = 4$. As dimensionality increases, the line shifts into the shaded region (or rather the line stays fixed but the shaded region changes). In the large $d$ limit $^{35–57}$, it coincides with the phase transition line that marks the lower boundary of the positive $\mathcal{R}$ region. Thus we see that the maximum force bound $F = 1/4$ is relevant for black hole thermodynamics.

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$^5$ Ruppeiner metric is a special case of Fisher information metric. See, e.g., $^{32–34}$. For a brief introduction to its mathematical structures, see $^{35}$.

$^6$ Note the definition of their Ruppeiner metric – hence also the curvature scalar – differs by an overall sign from our convention.

$^7$ However, phase transition can also occur when there is no such divergence. See $^{53}$ and the discussion in $^{54}$.
IV. DISCUSSION: MAXIMUM FORCE AND BLACK HOLE MICROSTRUCTURE?

In the context of singly rotating Myers-Perry black holes, the Hookean force \( F_1 = kx \) satisfies the maximum force conjecture. In \( d = 4 \) and \( d = 5 \) the maximum force is attained – while in the former case this corresponds to the extremal limit, this is not true for the latter. In \( d \geq 6 \), the Hookean force is further bounded by a dimensional dependent value (all smaller than 1/4) which corresponds to Emparan-Myers fragmentation. The force \( F_2 = J^2/S^2 \) is also related to black hole thermodynamics via Ruppeiner geometry. Let us attempt to give a coherent unified picture of what these results imply.

Once we know the force value at which the fragmentation occurs, we can set \( F_2 \) to this value and then plot the line in the \((S,T)\)-plane. It turns out that the fragmentation line is above the maximum force line \( F = 1/4 \) (See Fig. (2) for the \( d = 6 \) case). Except in \( d = 6 \) in which the maximum force line corresponds to the upper boundary of \( R > 0 \) region (and thus the fragmentation line lies in the \( R < 0 \) region), in all dimensions \( d > 6 \) the fragmentation line also lies in the \( R > 0 \) region. From the interpretation of the Ruppeiner scalar curvature, this means that generically the fast-spinning black holes tend to break when the interaction between the underlying microstructures (perhaps “spacetime atoms” [58]) is sufficiently repulsive. Intuitively, if the interaction is attractive, it tends to restore any stretching that occurs to the horizon. As the interaction becomes repulsive, the flattening of the horizon due to large angular momentum eventually breaks the black hole. As the number of dimension increases, a stronger repulsion is required before fragmentation can occur. A possible explanation – at least intuitively – is that in higher dimension, a spacetime atom would have more neighbors that it bonds to in the additional spatial directions, hence requiring more force to break the bond.

In all cases, fragmentation occurs before the maximum force is attained, so in this sense the maximum force conjecture holds. We remark that Emparan and Myers [10] conjectured that even in 5 dimensions, fragmentation could occur at \( a^2/\mu \approx 0.72 \), which is slightly less than the value the Hookean force attains its maximum \( a^2/\mu = 0.75 \). This is consistent with the situation in higher dimensions.

Even if we do not consider fragmentation, it is interesting to note that the lower boundary of the positive scalar curvature region \((R \to +\infty)\) corresponds to the line along which \( F = \frac{1}{d-3} < 1/4 \), which tends to the maximum force 1/4 in the large \( d \) limit\(^8\) as shown in the right plot of Fig.(1). In the Ruppeiner geometry approach, a divergent \( R \) often implies a phase transition. Here it coincides with the onset of Gregory-Laflamme instability [11, 60], which is a dynamical instability, but is related to thermodynamical instability by the extension of the result of Gubser and Mitra [60–62].

To conclude, our findings suggest that the concept of maximum force appears to hold at least in some thermodynamical contexts of black hole physics. Furthermore the maximum force is not necessarily related to the cosmic censorship conjecture. Finally, let us remark that in [8] Barrow and Gibbons argued that there is no maximum force in dimensions above 4, much less the universal significance for the value 1/4. Nevertheless, we see a clear sign of the significance of the maximum force 1/4 in generic dimensions in the examples discussed in this work.

We propose a possible explanation for this discrepancy: perhaps the maximum force conjecture with a universal upper bound 1/4 applies to some classes of forces in the context of black hole thermodynamics\(^9\). Specifically it might have to do with the forces between spacetime atoms that underlie black hole horizon, whatever that may be. The fact that the same maximum force conjecture happens to hold in the other contexts explored in 4-dimensions, including the gravitational force between two masses studied in [8], may be a coincidence (thus explaining why in many other contexts forces are unbounded [6]). If it is not a coincidence, perhaps this is a hint that gravity in 4 dimensions is in some sense more “thermodynamical” than gravity in higher dimensions. If so, this could be relevant to the emergent gravity program [63–66].

It is perhaps interesting to note that Robert Hooke had wanted to describe the behavior of not just actual springs, but also “springy bodies”, which include both solid and fluid [67]. Perhaps it would not surprise him that black holes – which admit hydrodynamical descriptions [68–71] – exhibit properties that can be described by his law.

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\(^8\) More aspects of Ruppeiner geometry in the large \( d \) limit can be found in [59].

\(^9\) Incidentally, we have defined the spring constant in \( F_1 = kx \) for higher dimensions using the temperature expression in Eq.(3). This \( k \) does not satisfy \( k = m\Omega_0^2 \) in higher dimensions, despite this equation inspired the term “spring constant” in 4-dimensions.
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