Grassmann integral and Balian-Brézin decomposition in Hartree-Fock-Bogoliubov matrix elements

Takahiro Mizusaki$^1$, Makito Oi$^2$, Fang-Qi Chen$^2$, Yang Sun$^{2,3}$

$^1$ Institute of Natural Sciences, Senshu University, 3-8-1 Kanda-Jinbocho, Chiyoda-ku, Tokyo 101-8425, Japan
$^2$Department of Physics and Astronomy, Shanghai Jiao Tong University, Shanghai 200240, People’s Republic of China
$^3$Institute of Modern Physics, Chinese Academy of Sciences, Lanzhou 730000, People’s Republic of China

We present a new formula to calculate matrix elements of a general unitary operator with respect to Hartree-Fock-Bogoliubov states allowing multiple quasi-particle excitations. The Balian-Brézin decomposition of the unitary operator (Il Nuovo Cimento B 64, 37 (1969)) is employed in the derivation. We found that this decomposition is extremely suitable for an application of Fermion coherent state and Grassmann integrals in the quasi-particle basis. The resultant formula is compactly expressed in terms of the Pfaffian, and shows the similar bipartite structure to the formula that we have previously derived in the bare-particles basis (Phys. Lett. B 707, 305 (2012)).

I. INTRODUCTION

In nuclear many-body physics, evaluations of matrix elements of many-body operators have been a major obstacle to implementations of sophisticated methods and theories beyond the mean-field approximation. Nuclear physicists have put effort into finding convenient formulae \[1–3\] to calculate matrix elements (and overlaps) with respect to Hartree-Fock-Bogoliubov (HFB) states. But difficulties remained in the attempts to circumvent difficulties associated with the known formulae \[4, 5\], there had not been any significant progress for decades in an analytical attempt to make a breakthrough. Recently such a breakthrough was achieved by Robledo who was successful in deriving a new formula in terms of the Pfaffian \[6\] with Fermion coherent states and Grassmann integral \[7\]. After his pioneering work, many studies followed by exploiting these mathematical tools in the HFB matrix elements \[8–11\].

The latest focus in this research field is to find a formula to evaluate HFB matrix elements with multiple quasi-particle excitations \[9–11\], \[\langle \phi | a_{\nu_1}^\dagger \cdots a_{\nu_n}^\dagger | \theta | a_{\nu_1} \cdots a_{\nu_n} | \phi \rangle\]. The quasi-particle basis \(a, a^\dagger\) is obtained through a canonical transformation (called the Bogoliubov transformation) of the bare-particle basis \(c, c^\dagger\),

\[a_i^\dagger = \sum_{j=1}^{M} (U_{i,j} c_j^\dagger + V_{i,j} c_j),\]

\(M\) is the dimension of the single-particle model space, which is taken to be an even integer. Coefficients \(U\) and \(V\) in the expression above define the Bogoliubov transformation. HFB state \(|\phi\rangle\) is also obtained through the Bogoliubov transformation applied to the bare-particle vacuum \(|0\rangle\), hence \(|\phi\rangle\) corresponds to the vacuum for the quasi-particles, or \(a_{\nu} |0\rangle = 0\). Induces \(v_1, \cdots, v_n\) and \(v'_1, \cdots, v'_n\), attached to the creation and annihilation operators, specify quantum states in the quasi-particle basis. The symbol \(|\theta\rangle\) stands for a unitary operator

\[|\theta\rangle \equiv \frac{e^{-i\hat{S}} |\phi\rangle}{\langle \phi | e^{-i\hat{S}} | \phi \rangle},\]

where \(\hat{S}\) is a one-body operator expressed in the quasi-particle basis \[13\], and \(\theta\) is a parameter to specify an element in a group produced by the generator \(\hat{S}\). Symbol \(|\theta\rangle\) for unitary operators is indebted to Ref.\[2\].

As explained in Ref.\[2\], such matrix elements of unitary operators shown in Eq.\[1\] are essential ingredients in the beyond-mean-field theories, such as quantum number projection. In the case of angular momentum projection, the parameter \(\theta\) corresponds to the Euler angles and \(\hat{S}\) to the angular momentum operator \(\hat{J}\). In order to derive a formula for this matrix element Eq.\[1\], we will apply the Fermion coherent state and Grassmann integral. As elucidated in previous studies, these two mathematical entities show a close affinity with the Pfaffian, and they can simplify calculations involving many anti-commuting operators to a great extent. It should be noted here that Hara and Iwasaki previously investigated the mathematical structure of the matrix elements Eq.\[1\] \[2\], in connection to the Projected Shell Model (PSM) \[14\]. PSM is basically a configuration mixing method with multi quasi-particle states based on the
II. THE BALIAN-BRÉZIN DECOMPOSITION

Following Ref. [13], a unitary operator $[\theta]$ in Eq. (3) can be expressed as a product of three operators in the quasi-particle basis,

$$[\theta] = e^{B(\theta)} e^{C(\theta)} e^{A(\theta)},$$  \hspace{1cm} (4)

with

$$A(\theta) = \frac{1}{2} A(\theta)_{\nu',\nu} a_{\nu'} a_{\nu},$$

$$B(\theta) = \frac{1}{2} B(\theta)_{\nu',\nu} a_{\nu'}^* a_{\nu},$$

$$C(\theta) = \sum (ln C(\theta))_{\nu',\nu} a_{\nu'} a_{\nu}.$$

We call Eq. (4) the Balian-Brézin decomposition. Matrices $A(\theta), B(\theta)$, and $C(\theta)$ in Eq. (5) correspond to contractions and can be written with the help of the Bogoliubov transformation matrices [2]

$$A_{\nu',\nu}(\theta) \equiv \langle \theta | a_{\nu'}^* a_{\nu} \rangle = (V^* (\theta) U^{-1}(\theta))_{\nu',\nu},$$

$$B_{\nu',\nu}(\theta) \equiv \langle a_{\nu'} a_{\nu} | \theta \rangle = (U^{-1}(\theta) V(\theta))_{\nu',\nu},$$

$$C_{\nu',\nu}(\theta) \equiv \langle a_{\nu'} | a_{\nu} | \theta \rangle = (U^{-1}(\theta))_{\nu',\nu}.$$

By inserting Eq. (4) into the matrix elements Eq. (1), we have

$$\mathcal{M}_I = \langle \phi | a_{\nu_1} \cdots a_{\nu_I} e^{B(\theta)} e^{C(\theta)} e^{A(\theta)} a_{\nu_1}^* \cdots a_{\nu_I}^* | \theta \rangle.$$  \hspace{1cm} (7)

Hereafter, we omit symbol $[\theta]$ for the sake of brevity. For subsequent notations, we introduce a shorthand notation $J$ for the indices of quasi-particle operators, as $J = \{v_1, \cdots, v_J\}, J' = \{v_1', \cdots, v_{J'}\}, \ (v_1 < \cdots < v_J \ \text{and} \ v_1' < \cdots < v_{J'}).$ These indices $J$ and $J'$ are subsets of a set $|M| = \{1, 2, \cdots, M\},$ in which $M$ represents the number of elements in $|M|$ and corresponds to the dimension of the single-particle model space. Index $I$ in Eq. (7) is defined as a set $I = \{v_1', \cdots, v_{J'}', v_1 + M \cdots v_J + M\}$ and corresponds to a subset of $[2M] = \{1, 2, \cdots, 2M\}.$ With these notations, the matrix elements Eq. (7) are expressed as

$$\mathcal{M}_I = \langle \phi | (a \cdots a) \ e^{B(\theta)} e^{C(\theta)} e^{A(\theta)} (a^\dagger \cdots a^\dagger) \ | \theta \rangle.$$  \hspace{1cm} (8)

where $(a \cdots a)^{\dagger}$ and $(a^\dagger \cdots a^\dagger)^{\dagger}$ stand for $a_{\nu_1} \cdots a_{\nu_I}$ and $a_{\nu_1}^\dagger \cdots a_{\nu_I}^\dagger$, respectively.

When the order of a product is completely reversed, such an order is denoted as $\mathcal{M}_I.$ The relation between $\mathcal{M}_I$ and $\mathcal{M}_J$ is given as

$$\mathcal{M}_I = (a \cdots a) (a \cdots a)^{\dagger} (-)^{\Sigma_{J}(n-1)},$$  \hspace{1cm} (9)

where $(a \cdots a)^{\dagger} = a_{\nu_1}^\dagger \cdots a_{\nu_I}^\dagger.$ Note that an additional phase emerges in the right-hand side of the above equation due to anti-commutation.

III. FERMION COHERENT STATE AND GRASSMANN INTEGRAL

In the present paper, we exclusively rely on Grassmann numbers $\xi^+$ and $\xi.$ They satisfy the anti-commutation rules,

$$\xi^+ \xi^+ + \xi^+ \xi = 0,$$  \hspace{1cm} (10)

$$\xi^+ \xi^+ + \xi^+ \xi^+ = 0,$$  \hspace{1cm} (11)

$$\xi^+ \xi^+ + \xi^+ \xi^+ = 0,$$  \hspace{1cm} (12)

where indices $\nu, \nu'$ run from 1 to $M (1, \cdots, M).$ With these Grassmann numbers, Fermion coherent states [7] in the quasi-particle basis are defined as

$$|\xi\rangle = e^{-\Sigma_{\nu} \xi^+_\nu a_{\nu}} |\phi\rangle.$$  \hspace{1cm} (13)

where the HFB state is normalized $\langle \phi | \phi \rangle = 1.$ This definition is slightly different from the one introduced in Ref. [8], where the operator and the vacuum are replaced as $c_i \rightarrow a_v$ and $|0\rangle \rightarrow |\phi\rangle.$ By definition, Fermion coherent states are eigenstates of the annihilation operator,

$$a_{\nu} |\xi\rangle = \zeta_{\nu} |\xi\rangle.$$  \hspace{1cm} (14)

The adjoint variable $\xi^+_\nu$ is also introduced in the eigenvalue equation,

$$\langle \xi | a_{\nu}^\dagger = \langle \xi | \xi^+_\nu.$$

The overlap between the HFB vacuum and the Fermion coherent state is $\langle \phi | \xi \rangle = 1.$ The closure relation [7] is expressed as

$$\int \mathcal{D}(\xi^+ \xi) e^{-\Sigma_{\nu} \xi^+_\nu \xi} |\xi\rangle \langle \xi| = 1,$$  \hspace{1cm} (16)

where $\mathcal{D}(\xi^+ \xi) = \prod a^\dagger d^{\xi^+} d\xi a.$ Differential elements $d\xi$ and $d^{\xi^+}$ are anti-commuting. Although this ordering for $\mathcal{D}(\xi^+ \xi)$ given in the above closure relation is widely employed, we use other ordering for the differential elements in the present study, which is

$$\mathcal{D}(\xi^+ \xi) = d^{\xi^+}_{[M]} d\xi_{[M]}^a d^{\xi^+}_{[M]} d\xi_{[M]}^a$$  \hspace{1cm} (17)

where $(-)^M = 1$ is used because $M$ is even. “Half products” in the above expression are defined as

$$d\xi_{[M]}^a = d\xi_1 \cdots d\xi_{2M},$$

$$d^{\xi^+}_{[M]} = d^{\xi^+}_{[M]} \cdots d^{\xi^+}_{[M]}.$$  \hspace{1cm} (18)
where \([M] = \{1, 2, \cdots, M\}\). This definition of the ordering is also applied to Grassmann numbers \(\xi^*\).

For the sake of convenience in later discussions, we need to define several other orderings for “partial products” of the differential elements. Firstly, we define products relevant to the indices \(J = \{v_1, \cdots, v_n\}\) appearing in the matrix element Eq. (8),

\[
\begin{align*}
    d \xi^*_J &= d \xi^*_{v_1} \cdots d \xi^*_{v_n}, \\
    d \xi^-_J &= d \xi_{v_1} \cdots d \xi_{v_n}, \quad (19)
\end{align*}
\]

This partial product was originally introduced in Ref. [1]. The second type of partial products is defined for the complement set \(\mathcal{J} = [M] - J\). For an example, \(d \xi^*_J d \xi^*_{\mathcal{J}} = d \xi^*_J d \xi^-_{\mathcal{J}}\). Combining two types of partial products, we would like to rewrite a half product \(d \xi^*_{[M]}\) as well as \(d \xi^-_{[M]}\) as,

\[
\begin{align*}
    d \xi^*_{[M]} &= d \xi^*_J d \xi^*_\mathcal{J}(-)^{|J| + n(M - 1)}, \\
    d \xi^-_{[M]} &= d \xi^-_J d \xi^-_{\mathcal{J}}(-)^{|J| + n(M - 1)), \quad (21)
\end{align*}
\]

where \(|J| = \sum v_k\). The latter relation can be obtained by reversing the arrow directions of \(\xi^*\) in terms of \(\xi^*_{[M]}\) and \(J\) in the former relation. The relevant phase emerging due to the reordering can be calculated by referring to Eq. (9). Noting that the complement set \(\mathcal{J}\) comes back to \(J = \mathcal{J}\), it is also possible to obtain the following ordering from Eq. (21).

\[
\begin{align*}
    d \xi^*_{[M]} &= d \xi^*_J d \xi^*_\mathcal{J}(-)^{|J| - n(M - 1)} , \\
    d \xi^-_{[M]} &= d \xi^-_J d \xi^-_{\mathcal{J}}(-)^{|J| + n(M - 1)} . \quad (22)
\end{align*}
\]

The number of the elements of \(\mathcal{J}\) is denoted as \(n = M - n\). The relations Eqs. (21,22) also hold for \(\xi^*\).

In summary, by these relations Eqs. (21,22) obtained above, the differential elements in total can be expressed as

\[
\begin{align*}
    d \xi^*_{[M]} d \xi^-_{[M]} &= d \xi^*_J d \xi^*_\mathcal{J} d \xi^-_J d \xi^-_{\mathcal{J}}, \\
    d \xi^-_{[M]} d \xi^*_{[M]} &= d \xi^-_J d \xi^-_{\mathcal{J}} d \xi^*_J d \xi^*_\mathcal{J} , \quad (23)
\end{align*}
\]

where we used \(|J| + |\mathcal{J}| = \frac{1}{2} M(M + 1)\).

IV. MATRIX ELEMENT IN TERMS OF GRASSMANN INTEGRAL

Now, let us introduce two sets of Grassmann numbers \(\{\xi^*, \xi\}\) and \(\{\eta^*, \eta\}\). These Grassmann numbers are anti-commuting each other. We then insert two kinds of the closure relations Eq. (15) for \(\xi^*\) and \(\eta\) to the matrix element Eq. (7). The closure relation for \(\xi^*\) \((\eta)\) is inserted in the position between \(a^*\) and \(e^\beta\) \((e^\alpha\) and \(a^*\)) in the above relation, as \(\eta\).

\[
M_I = \int \mathcal{D}((\xi^*\xi)\langle \phi|\langle a^* \cdots a^* |\xi^-\rangle e^{-S_\xi A_{\xi^*} A_{\eta^*}} \int \mathcal{D}(\eta^* \), \langle \eta|\langle a^* \cdots a^* |\xi^-\rangle \phi) . \quad (24)
\]

At this stage, all the operators are replaced with Grassmann numbers. First of all, the following relations hold:

\[
\phi\langle a^* \cdots a^* |\xi^-\rangle = (\xi^* \xi^-) \phi \quad (25)
\]

and

\[
\langle \eta|\langle a^* \cdots a^* |\xi^-\rangle = (\eta^* \eta^-) \phi . \quad (26)
\]

Here, we used \(\phi\xi^* = 1\) and \(\langle \phi|\eta\rangle = 1\), as well as Eqs. (14,15). We also introduced short-hand notations \(\xi^*\) and \(\eta\) in the above relation, as \(\xi^* \xi^- = \xi^*_1 \cdots \xi^*_{\mathcal{J}}\) and \(\eta^* \eta^- = \eta^*_1 \cdots \eta^*_\mathcal{J}\).

Secondly, we deal with a product of exponential operators \(e^\beta c^\alpha\). Because \(\hat{A}\) and \(\hat{B}\) do not contain any \(a^* a\) terms, the expectation value of \(e^\beta c^\alpha\) is transformed as

\[
\langle \xi|e^\beta c^\alpha|\eta\rangle = e^{\Sigma_{\nu} \phi_{\nu}^\prime \phi_{\nu}^\prime \mu \Sigma_{\nu} \phi_{\nu}^\prime \phi_{\nu}^\prime} \langle \xi|e^\beta|\eta\rangle e^{\Sigma_{\nu} \phi_{\nu}^\prime \phi_{\nu}^\prime} \langle \xi|c^\alpha|\eta\rangle , \quad (27)
\]

which is derived in Appendix. Thus, \(\langle \xi|e^\beta|\eta\rangle\) term becomes

\[
\langle \xi|e^\beta|\eta\rangle = \langle \xi|c^\alpha|\eta\rangle = \Sigma_{\nu} \phi_{\nu}^\prime \phi_{\nu}^\prime \phi_{\nu}^\prime \phi_{\nu}^\prime , \quad (29)
\]

where we use Eq. (47) in Appendix.

Together with Eqs. (17,23,26,27,29), Eq. (24) is represented in terms of the Grassmann integrals,

\[
M_I = \int d \xi^*_J d \xi^-_\mathcal{J} \langle \xi^* \xi^- |\nu| \phi \rangle e^{-S_\xi A_{\xi^*} A_{\eta^*}} \int d \eta^*_J d \eta^-_\mathcal{J} \langle \eta^* \eta^- |\nu| \phi \rangle e^{-S_\eta A_{\eta^*} A_{\eta^*}} B(\xi^*, \eta) \quad (30)
\]

where we use Eq. (47) for the differential elements and

\[
B(\xi^*, \eta) \equiv e^{\Sigma_{\nu} \phi_{\nu}^\prime \phi_{\nu}^\prime \mu \Sigma_{\nu} \phi_{\nu}^\prime \phi_{\nu}^\prime} e^{\Sigma_{\nu} \phi_{\nu}^\prime \phi_{\nu}^\prime} e^{\Sigma_{\nu} \phi_{\nu}^\prime \phi_{\nu}^\prime} . \quad (31)
\]

We should note that \(B(\xi^*, \eta)\) is a function of only \(\xi^*\) and \(\eta\), but not of \(\xi\) and \(\eta^*\).

V. EVALUATION OF GRASSMANN INTEGRAL

Now we carry out these Grassmann integrals in Eq. (30). Part of Eq. (30) concerning Grassmann integral over \(\eta^*\)
is rewritten as

\[
\int d\eta_{n}^{*} d\eta_{n} e^{-\Sigma_{n}^{*} \eta_{n} \eta_{n}} B(\xi_{n}^{*}, \eta_{n}) = \int d\eta_{J} d\eta_{J} B(\xi_{J}^{*}, \eta_{J}) \int d\eta_{n}^{*} d\eta_{n} e^{-\Sigma_{n}^{*} \eta_{n} \eta_{n}}.
\]

(32)

where we use Eq. (22). The integral over \(\eta_{n}^{*} (v \in J)\) gives rise to unity because quadratic forms of Grassmann variables vanish by definition \((\xi_{n}^{*} \theta_{n}^{*} = 0)\). As a result,

\[
\int d\eta_{J}^{*} e^{-\Sigma_{n}^{*} \eta_{n} \eta_{n}} = 1.
\]

(33)

The integral over \(\eta_{v}^{*} (v \in J)\) gives rise to a product of \(\eta\)'s as

\[
\int d\eta_{n}^{*} e^{-\Sigma_{n}^{*} \eta_{n} \eta_{n}} = (-\mathbf{1}) \eta_{n} \eta_{n},
\]

(34)

where \(\mathbf{1} = M - n\). Then the integral over \(\eta_{v}^{*} (v \in J)\) in the Eq. (32) can be carried out as

\[
\int d\eta_{J} d\eta_{J} B(\xi_{J}^{*}, \eta_{J}) = (-\mathbf{1}) \int d\eta_{J} B(\xi_{J}^{*}, \eta_{J}),
\]

(35)

where \(\{ \eta \}_J\) stands for the Grassmann variables \(\eta_{v_{1}}, \ldots, \eta_{v_{n}}\).

After the integrals over \(\eta_{v}^{*} (v \in J)\) and \(\eta_{n}^{*}\) are performed, the integral over \(\eta_{v}^{*} (v \in J)\) remains. Then, Eq. (30) is represented by the integrals over \(\xi, \xi^{*}\) and \(\eta_{v}^{*} (v \in J)\) as

\[
\mathcal{M}_1 = (-\mathbf{1}) \int d\xi_{[M]}^{*} d\xi_{[M]} (\xi \cdots \xi) \frac{1}{\mathbf{P}^{-1}} d\eta_{J} B(\xi_{J}^{*}, \eta_{J}),
\]

(36)

In the same way, the integral over \(\xi\) in the above integrals is carried out as

\[
\int d\xi_{\mathbf{P}}^{*} d\xi_{\mathbf{P}} (\xi \cdots \xi) \frac{1}{\mathbf{P}} d\eta_{J} B(\xi_{J}^{*}, \eta_{J}).
\]

(37)

Thus,

\[
\mathcal{M}_1 = (-\mathbf{1}) \int d\xi_{\mathbf{P}}^{*} d\xi_{\mathbf{P}} (\xi \cdots \xi) \frac{1}{\mathbf{P}} d\eta_{J} B(\xi_{J}^{*}, \eta_{J}),
\]

(38)

where \(\{ \xi_{J}^{*} \}_J\) stands for the Grassmann variables \(\xi_{v_{1}}, \ldots, \xi_{v_{n}}\).

In the last line, an order of differential elements is reversed and the sign factor is calculated to be \(\mathbf{1} = \mathbf{1} + \frac{1}{2}(n + n') (n + n' - 1)\).

We define a new Grassmann vector \(z\) as \(z = (\xi_{v_{1}}, \ldots, \xi_{v_{n}}, \eta_{v_{1}}, \ldots, \eta_{v_{n}})\). With a relation Eq. (41), connecting the Pfaffian with the Grassmann integral, we obtain

\[
\mathcal{M}_1 = (-\mathbf{1}) \int d\xi_{z}^{*} d\xi_{z} (\xi \cdots \xi) \frac{1}{\mathbf{P}} d\eta_{J}^{*} B(\xi_{J}^{*}, \eta_{J}),
\]

(39)

where \(\mathbf{M}'\) is a sub-matrix of the skew-symmetric matrix \(\mathbf{M}'\)

\[
\mathbf{M}' = \begin{pmatrix} -B & C \\ -C'^{T} & -A \end{pmatrix},
\]

(40)

concerning the index set \(I\). Block matrices \(A, B\) and \(C\) in \(\mathbf{M}'\) are the contractions in Eq. (3). The dimensions of \(\mathbf{M}'\) and \(\mathbf{M}'\) are given to \((n + n')\) and \(2M\), respectively.

The Pfaffian in Eq. (39) can be transformed further as

\[
Pf(\mathbf{M}'_I) = Pf \begin{pmatrix} -B & C \\ -C'^{T} & -A \end{pmatrix},
\]

(41)

Finally we obtain a new and compact Pfaffian formula for the matrix elements \(\mathcal{M}_1\) as

\[
\mathcal{M}_1 = Pf(\mathbf{M}_I),
\]

(42)

where

\[
\mathbf{M} = \begin{pmatrix} -B & C \\ -C'^{T} & -A \end{pmatrix}.
\]

(43)

Matrix \(\mathbf{M}\) has a bipartite structure, consisting of \(A, B\) and \(C\) in the Balian-Brézin decomposition Eq. (3). This structure is quite similar to that of the previous one obtained in Ref. [9], but in the present formula, the definition of the contraction is different due to the presence of the unitary operator \([\theta]\).

VI. CONCLUSION

In this paper, we presented a compact Pfaffian formula for matrix elements of a general unitary operator between any multi quasi-particle HFB states. To obtain this new Pfaffian formula, we use the Fermion coherent state and Grassmann integral.

This kind of matrix elements has been conventionally evaluated by means of the extended Wick’s theorem [15]. The evaluation of matrix elements by the Wick theorem was studied extensively for multi quasi-particle HFB states by Hara and Iwasaki in Ref. [2], which is the essential component in the calculations of the Projected Shell Model (PSM) [14]. The
obtained result by Hara and Iwasaki, however, suffers from the problem of combinatorial complexity arising from the Wick’s theorem. Standard DSM calculations for more than four quasiparticle HFB states are practically difficult.

In the present work, we were successful to find a general expression for the matrix elements in terms of the Pfaffian, which has a more compact and closed form than the result obtained by Hara and Iwasaki. Recently, Ref. [8] shows a Pfaffian formula to matrix elements similar to Eq. (1), but our Pfaffian formula is physically intuitive due to the presence of the contractions.

The matrix in our formula has a bipartite structure and consists of the matrices $A$, $B$ and $C$ appeared in the Balian-Brézin decomposition. This structure is quite similar to the previous one obtained in Ref. [9] where the matrix in the formula is expressed in a bipartite form of the skew-symmetric contraction matrix with density and pairing tensor. The both formulaes are found to be expressed in terms of the Pfaffian and the generalized contraction matrices.

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Appendix

The Pfaffian is defined as

$$ Pf(A) = \frac{1}{2^n n!} \sum_{\sigma \in S_{2n}} \text{sgn}(\sigma) \prod_{i=1}^{n} a_{\sigma(2i-1) \sigma(2i)} $$

for a skew-symmetric matrix $A$ with dimension $2n \times 2n$, of which matrix elements are $a_{ij}$. The $\sigma$ is a permutation of $\{1, 2, 3, \cdots, 2n\}$, sgn($\sigma$) is its sign, and $S_{2n}$ represents a symmetry group. For an $n \times n$ ($n$ odd) matrix, $Pf(A) = 0$. For a $2 \times 2$ matrix, $Pf(A) = a_{12}$. For a $4 \times 4$ matrix, $Pf(A) = a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}$.

For a matrix $P$ with dimension $2n \times 2n$, a following relation holds as

$$ Pf(P^TAP) = detQPf(A). $$

A relation between the Pfaffian and Grassmann integral is presented in Refs. [10] [16] as

$$ \int d\theta_1 \cdots d\theta_n \exp \left( \frac{1}{2} \theta^T A \theta \right) = Pf(A), $$

where $\theta_1, \theta_2, \cdots, \theta_n$ are Grassmann variables and $\theta^T = (\theta_1, \theta_2, \cdots, \theta_n)$ is a Grassmann vector. Matrix $A$ is a skew-symmetric matrix with $2n \times 2n$ dimension.

An overlap between two Fermion coherent states $|\xi\rangle$ and $|\eta\rangle$ is given by [7]

$$ \langle \xi | \eta \rangle = \langle \phi | e^{-\sum_\nu \xi_\nu a_\nu^\dagger} e^{-\sum_\nu \eta_\nu a_\nu^\dagger} |\phi \rangle = \langle \phi | e^{-\sum_\nu \xi_\nu a_\nu^\dagger} e^{\sum_\nu \eta_\nu a_\nu} \sum_\nu \eta_\nu |\phi \rangle. $$

Finally we prove a relation $e^\sum_{\nu} a_\nu^\dagger a_\nu |\xi\rangle = |C_{\xi}\rangle$. By the definition Eq. (33) and $C_{\xi} = \sum_{\nu} (ln C_{\nu}) a_\nu^\dagger a_\nu^\dagger$, $a_\nu^\dagger a_\nu^\dagger = e^{\sum_\nu \xi_\nu a_\nu^\dagger} C_{\nu} a_\nu^\dagger a_\nu^\dagger$.}

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