Automatic analysis of the stress-strain response of rheological models

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Viscoelastic material behaviour can be represented by rheological models combining elastic and viscous rheological elements. These combinations can be serial and parallel connections of such elements as well as combinations of both. A method based on transfer functions formulated in the Laplace domain allows the calculation of constitutive equations of serial and parallel connections by a simple summation.

Furthermore the use of a breadth-first search algorithm for the analysis of complex systems of rheological elements is explained. By using this search algorithm a hierarchy of nested connections in a system can be found. This hierarchy guarantees dependencies are satisfied during the calculation of constitutive equations and signal propagation through the system.

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1 Introduction

The representation of viscoelastic materials, using rheological models consisting of elastic and viscous elements, is illustrative because it is graphic and easy to understand. Nevertheless, the calculation of the stress or strain response to a given input signal may require a complex constitutive equation. This is true even for rather simple one dimensional systems of elements like the four element Burgers model [1]. To deal with this problem an algorithm based on the Laplace transform is formulated. This algorithm can be used to obtain the constitutive equation of any one dimensional viscoelastic rheological model and all of the model’s subsystems. It allows the user to calculate stress and strain of all elements and subsystems of such a model. The simplicity of the process allows the efficient analysis of even complex rheological models with a lot of elements. The algorithm was implemented in a program allowing the user to understand one dimensional rheological models representing viscoelastic materials.

2 Rheological models as transmission systems

In system theory a transfer function characterizes a system. The output of a system is given by an input signal applied to the transfer function of the system. Using the Laplace transform for the description of input and output signals renders the characterization of so called transmission systems easy [2,3].

\[ G_{sp}(s) = \frac{\sigma_{sp}(s)}{\varepsilon_{sp}(s)} = E \frac{1}{1} \quad , \quad G_{da}(s) = \frac{\sigma_{da}(s)}{\varepsilon_{da}(s)} = \eta \frac{s}{1} \]

The left term in equation (1) shows the transfer function of a linear elastic spring in the transfer domain \( s \) of the Laplace transform. It can be derived from Hooke’s law. The stress of the spring is given by \( \sigma_{sp} \) while \( \varepsilon_{sp} \) is its strain and \( E \) represents the Young’s modulus of the spring. The right hand side of equation (1) is the transfer function of a linear viscous dashpot, derived from the ideal viscous flow law. The viscosity is \( \eta \).

Rheological models of viscoelastic materials can be constructed by combining elastic and viscous elements. Such combinations can be serial or parallel connections as well as combinations of both. Fig. 1 illustrates how the transfer functions of such combinations can be derived from the transfer functions of the individual elements. The shown scheme also works for combinations of more than two elements. The transfer functions of basic elements and combinations are formulated in form of rational functions (see equation (1)). This formulation allows an easy handling of the needed reciprocal sums. Nominator and denominator of a transfer function are saved separate as list like data structures. A simple switch of variables leads to the reciprocal of the rational function. The calculation of the transfer functions of parallel and serial connections can be simplified to a summation and therefore to a single operation.

3 Model analysis

Elements of one dimensional rheological models representing viscoelastic materials can be combinations of two or more elements themselves. An example is the four element Burgers model [1] as shown in Fig. 2. It consists of a serial connection of one elastic element, one viscous element and a parallel connection of another elastic and another viscous element. To
calculate the transfer function of this model, as described in section 2, the transfer function of the inner parallel connection has to be known. A hierarchy has to be established. The transfer function of the model’s parallel connection must be calculated before the transfer function of the outer serial connection.

To do so, a breadth-first search algorithm, well known from graph theory, is used. Every elastic and viscous element as well as every serial and parallel connection is handled like a node in a graph. The breadth-first search algorithm defines an order of all elements and connections. The reversed order is used to establish the hierarchy of connections required to calculate all transfer functions from inside out.

4 Example of the Burgers model

As an example the transfer function of the four element Burgers model [1] (Fig. 2) shall be derived. In the following variables $N$ and $D$ are describing nominators and denominators of transfer functions in form of a rational function. The superscript expression $(\cdot)^p$ indicates variables of elements of a serial connection. Variables marked with $(\cdot)^p$ are part of a parallel connection. The remaining indices follow the numbering established in Fig. 2.

Performing a breadth-first search from the beginning of the system to its end will find the elements in the following order: first spring 1, dashpot 2 and the inner parallel connection and, later on during the search, spring 3 and dashpot 4. This indicates that the parallel connection is part of the serial connection. It has to be analysed first. Equation (2) illustrates this.

\[
\frac{N^p}{D^p} = \frac{N_3^p}{D_3^p} + \frac{N_4^p}{D_4^p} = \frac{E_3}{1} + \frac{\eta_4 s}{1} = \frac{(\eta_4 s + E_3)}{1}
\]  

(2)

The transfer functions of all elements and combinations of elements can be expressed as rational functions. The transfer function of the parallel connection is the sum of the rational functions representing the transfer functions of spring 3 and dashpot 4 ($E_3$ and $\eta_4$ in equation (1), see also Fig. 2). In equation 3 the same procedure is applied to the serial connection of the system. It leads to the transfer function of the Burgers model.

\[
\frac{1}{G^B(s)} = \frac{D^p}{N^p} = \frac{D_1^p}{N_1^p} + \frac{D_2^p}{N_2^p} = \frac{1}{E_1} + \frac{1}{\eta_2 s} + \frac{1}{(\eta_4 + E_3 s)} = \frac{\eta_2 \eta_4 s^2 + (E_3 \eta_2 + E_1 \eta_4 + E_1 \eta_2) s + E_1 E_3}{E_1 \eta_2 \eta_4 s^2 + E_1 E_3 \eta_2 s}
\]  

(3)

As shown in Fig. 1, the reciprocal transfer function of a serial connection is the reciprocal sum of the transfer functions characterizing it’s elements. In this example these elements are spring 1 and the dashpot 2, characterized by $E_1$ and $\eta_2$, and the parallel connection analysed before. As all transfer functions are represented by rational functions, this process can be simplified to a summation. Transfer functions of parallel and serial connections can be calculated by the same operation. The resulting transfer function can be used to calculated the response to a given input signal.

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