On Lagrangians of Hypergraphs Containing Dense Subgraphs

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Abstract Motzkin and Straus established a remarkable connection between the maximum clique and the Lagrangian of a graph in 1965. This connection and its extensions were successfully employed in optimization to provide heuristics for the maximum clique number in graphs. It is useful in practice if similar results hold for hypergraphs. In this paper, we provide upper bounds on the Lagrangian of a hypergraph containing dense subgraphs when the number of edges of the hypergraph is in certain ranges. These results support a pair of conjectures introduced by Y. Peng and C. Zhao (2012) and extend a result of J. Talbot (2002).

Keywords Cliques of hypergraphs · Colex ordering · Lagrangians of hypergraphs · Polynomial optimization

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1 Introduction

In 1941, Turán [1] provided an answer to the following question: What is the maximum number of edges in a graph with n vertices not containing a complete subgraph of order k, for a given k? This is the well-known Turán theorem. Later, in another classical paper, Motzkin and Straus [2] provided a new proof of Turán theorem based on the continuous characterization of the clique number of a graph using Lagrangians of graphs.

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The Motzkin-Straus result basically says that the Lagrangian of a graph which is the maximum of a homogeneous quadratic multilinear function (determined by the graph) over the standard simplex of the Euclidean plane is connected to the maximum clique number of this graph (the precise statement is given in Theorem 2.1). This result provides a solution to the optimization problem for a class of homogeneous quadratic multilinear functions over the standard simplex of an Euclidean plane. The Motzkin-Straus result and its extension were successfully employed in optimization to provide heuristics for the maximum clique problem [3,6]. It has been also generalized to vertex-weighted graphs [6] and edge-weighted graphs with applications to pattern recognition in image analysis [3–9]. The Lagrangian of a hypergraph has been a useful tool in hypergraph extremal problems. For example, Sidorenko [10] and Frankl-Furedi [11] applied Lagrangians of hypergraphs in finding Turán densities of hypergraphs. Frankl and Rödl [12] applied it in disproving Erdős long standing jumping constant conjecture. In most applications, we need an upper bound for the Lagrangian of a hypergraph.

An attempt to generalize the Motzkin-Straus theorem to hypergraphs is due to Sós and Straus [13]. Recently, in [14,15], Rota Buló and Pelillo generalized the Motzkin and Straus’ result to r-graphs in some way using a continuous characterization of maximal cliques other than Lagrangians of hypergraphs. The obvious generalization of Motzkin and Straus’ result to hypergraphs is false. In fact, there are many examples of hypergraphs that do not achieve their Lagrangian on any proper subhypergraph. We attempt to explore the relationship between the Lagrangian of a hypergraph and the order of its maximum cliques for hypergraphs when the number of edges is in certain ranges though the obvious generalization of Motzkin and Straus’ result to hypergraphs is false.

The results presented in Sect. 3 and 4 in this paper provide substantial evidence for two conjectures in [16] and extend some known results in the literature [16,17]. The main results provide solutions to the optimization problem of a class of homogeneous multilinear functions over the standard simplex of the Euclidean space. The main results also give connections between a continuous optimization problem and the maximum clique problem of hypergraphs. Since practical problems such as computer vision and image analysis are related to the maximum clique problems, this type of results opens a door to such practical applications. The results in this paper can be applied in estimating Lagrangians of some hypergraphs, for example, calculations involving estimating Lagrangians of several hypergraphs in [11] can be much simplified when applying the results in this paper.

The rest of the paper is organized as follows. In Sect. 2, we state a few definitions, problems, and preliminary results. In Sect. 3 and Sect. 4, we provide upper bounds on the Lagrangian of a hypergraph containing dense subgraphs when the number of edges of the hypergraph is in a certain range. Then, as an application, using the main result in Sect. 3, we extend a result in [17] in Sect. 5. In Sect. 6, we give the proofs of some lemmas. Conclusions are given in Section 7.

2 Definitions and Preliminary Results

For a set $V$ and a positive integer $r$ we denote by $V^{(r)}$ the family of all $r$-subsets of $V$. An $r$-uniform graph or $r$-graph $G$ consists of a set $V(G)$ of vertices and a set $E(G) \subseteq V(G)^{(r)}$ of edges. An edge $e := \{a_1, a_2, \ldots, a_r\}$ will be simply denoted by $a_1 a_2 \ldots a_r$. An $r$-graph $H$ is a subgraph of an $r$-graph $G$, denoted by $H \subseteq G$ if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Let $K^{(r)}_r$ denote the complete $r$-graph on $r$ vertices, that is the $r$-graph on $r$ vertices containing all possible edges. A complete $r$-graph on $t$ vertices is also called a clique with order $t$. Let $\mathbb{N}$ be the set of all positive integers. For any integer $n \in \mathbb{N}$, we denote the set $\{1, 2, 3, \ldots, n\}$ by $[n]$. Let $[n]^{(r)}$ represent the complete $r$-uniform graph on the vertex set $[n]$. When $r = 2$, an $r$-uniform graph is a simple graph. When $r \geq 3$, an $r$-graph is often called a hypergraph.

For an $r$-graph $G = (V,E)$ and $i \in V$, let $E_i := \{A \in V^{(r-1)} : A \cup \{i\} \subseteq E\}$. For a pair of vertices $i, j \in V$, let $E_{ij} := \{B \in V^{(r-2)} : B \cup \{i,j\} \subseteq E\}$. Let

$E_i^{c} := \{A \in V^{(r-1)} : A \cup \{i\} \not\subseteq V^{(r)} \setminus E\}, E_{ij}^{c} := \{B \in V^{(r-2)} : B \cup \{i,j\} \not\subseteq V^{(r)} \setminus E\}$, and $E_{i,j} := E_i \cap E_j^{c}$. 


**Definition 2.1** For an $r$-uniform graph $G$ with the vertex set $[n]$, edge set $E(G)$, and a vector $\mathbf{x} := (x_1, \ldots, x_n) \in \mathbb{R}^n$, we associate a homogeneous polynomial in $n$ variables, denoted by $\hat{\lambda}(G, \mathbf{x})$ as follows:

$$
\hat{\lambda}(G, \mathbf{x}) := \sum_{i_1i_2\ldots i_r \in E(G)} x_{i_1}x_{i_2}\ldots x_{i_r}.
$$

Let $S := \{ \mathbf{x} := (x_1, x_2, \ldots, x_n) : \sum_{i=1}^{n} x_i = 1, x_i \geq 0 \text{ for } i = 1, 2, \ldots, n \}$. Let $\hat{\lambda}(G)$ represent the maximum of the above homogeneous multilinear polynomial of degree $r$ over the standard simplex $S$. Precisely

$$
\hat{\lambda}(G) := \max \{ \hat{\lambda}(G, \mathbf{x}) : \mathbf{x} \in S \}.
$$

The value $x_i$ is called the **weight** of the vertex $i$. A vector $\mathbf{x} := (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ is called a feasible weighting for $G$ if $\mathbf{x} \in S$. A vector $\mathbf{y} \in S$ is called an **optimal weighting** for $G$ if $\hat{\lambda}(G, \mathbf{y}) = \hat{\lambda}(G)$.

**Remark 2.1** Since $\hat{\lambda}(G)$ is the maximum of a polynomial function in $n$ variables $x_1, x_2, \ldots, x_n$ under the constraint $\sum_{i=1}^{n} x_i = 1$ and the theory of Lagrange function and multipliers is often used in evaluating the Lagrangian of a hypergraph, it is well-known that Lagrangians of hypergraphs have been studied in several papers [11, 12, 13, 18]. Throughout this paper, we also call $\hat{\lambda}(G)$ the Lagrangian of $G$.

The following fact is easily implied by Definition 2.1.

**Fact 2.1** Let $G_1, G_2$ be $r$-uniform graphs and $G_1 \subseteq G_2$. Then $\hat{\lambda}(G_1) \leq \hat{\lambda}(G_2)$.

In [2], Motzkin and Straus provided the following simple expression for the Lagrangian of a 2-graph.

**Theorem 2.1** (See [2], Theorem 1) If $G$ is a 2-graph with $n$ vertices in which a largest clique has order $t$ then

$$
\hat{\lambda}(G) = \hat{\lambda}(K_t^{(2)}) = \frac{t}{2}(1 - \frac{1}{t}).
$$

Furthermore, the vector $\mathbf{x} := (x_1, x_2, \ldots, x_n)$ given by $x_i := \frac{1}{t}$ if $i$ is a vertex in a fixed maximum clique and $x_i = 0$ otherwise is an optimal weighting.

This result provides a solution to the optimization problem of this type of homogeneous quadratic functions over the standard simplex of an Euclidean plane. It is well-known that Lagrangians of hypergraphs have been proved to be a useful tool in hypergraph extremal problems, for example, it has been applied in finding Turán densities of hypergraphs in [16, 17, 18]. In order to explore the relationship between the Lagrangian of a hypergraph and the order of its maximum cliques for hypergraphs when the number of edges is in certain ranges, the following two conjectures are proposed in [17].

**Conjecture 2.1** (See [16], Conjecture 1.3) Let $m$ and $t$ be positive integers satisfying $\binom{t-1}{r} \leq m \leq \binom{t-1}{r} + \binom{t-2}{r-1}$. Let $G$ be an $r$-graph with $m$ edges and contain a clique of order $t-1$. Then $\hat{\lambda}(G) = \hat{\lambda}(t-1)^{(r)}$.

**Conjecture 2.2** (See [16], Conjecture 1.4) Let $m$ and $t$ be positive integers satisfying $\binom{t-1}{r} \leq m \leq \binom{t-1}{r} + \binom{t-2}{r-1}$. Let $G$ be an $r$-graph with $m$ edges and contain no clique of order $t-1$. Then $\hat{\lambda}(G) < \hat{\lambda}(t-1)^{(r)}$.

In [16], we proved that Conjecture 2.1 holds for $r = 3$.

**Theorem 2.2** (See [16], Theorem 1.8) Let $m$ and $t$ be positive integers satisfying $\binom{t-1}{3} \leq m \leq \binom{t-1}{3} + \binom{t-2}{2}$. Let $G$ be a 3-graph with $m$ edges and contain a clique of order $t-1$. Then $\hat{\lambda}(G) = \hat{\lambda}(t-1)^{(3)}$.

For distinct $A, B \in \mathbb{N}^{(r)}$ we say that $A$ is less than $B$ in the colex ordering iff $\max(A \triangle B) \in B$, where $A \triangle B := (A \setminus B) \cup (B \setminus A)$. For example we have $246 < 156$ in $\mathbb{N}^{(3)}$ since $\max(\{2, 4, 6\} \triangle \{1, 5, 6\}) \in \{1, 5, 6\}$. In colex ordering, $123 < 124 < 123 < 234 < 125 < 135 < 235 < 145 < 245 < 345 < 126 < 136 < 236 < 146 < 246 < 346 < 156 < 256 < 356 < 456 < 127 < \cdots$. Note that the first $\binom{t}{r}$ $r$-tuples in the colex ordering of $\mathbb{N}^{(r)}$ are the edges of $[t]^{(r)}$.

Let $C_{r,m}$ denote the $r$-graph with $m$ edges formed by taking the first $m$ sets in the colex ordering of $\mathbb{N}^{(r)}$. The following result in [17] states that the value of $\hat{\lambda}(C_{r,m})$ can be easily figured out when $m$ is in a certain range.
Lemma 2.1 (See [17], Lemma 2.4) For any integers \( m, t, \) and \( r \) satisfying \((t-1)^{-1} \leq m \leq \left(\frac{t}{r}\right)^{-1} + \left(\frac{t^2}{r^2}\right)^{-1}\), we have \( \lambda(C_{r,m}) = \lambda([r-1]^{(r)}) \).

Note that Conjectures 2.1 and 2.2 refine the following open conjecture of Frankl and Füredi.

Conjecture 2.3 (See [11], Conjecture 4.1) The \( r \)-graph with \( m \) edges formed by taking the first \( m \) sets in the colex ordering of \( \mathbb{N}^{(r)} \) has the largest Lagrangian of all \( r \)-graphs with \( m \) edges. In particular, the \( r \)-graph with \( \binom{t-1}{r} \) edges and the largest Lagrangian is \( [r]^{(r)} \).

Note that the upper bound \((t-1)^{-1} + \left(\frac{t^2}{r^2}\right)^{-1}\) in Conjecture 2.1 is the best possible. For example, if \( m = (t-1)^{-1} + \left(\frac{t^2}{r^2}\right)^{-1} + 1 \) then \( \lambda(C_{r,m}) > \lambda([r-1]^{(r)}) \). To see this, take \( x := (x_1, \ldots, x_t) \in S \), where \( x_1 = x_2 = \cdots = x_{r-2} = \frac{1}{r-1} \) and \( x_{r-1} = x_r = \frac{1}{3(r-1)} \), then \( \lambda(C_{r,m}, x) > \lambda([r-1]^{(r)}) \).

In [17], Talbot proved the following.

Theorem 2.3 (See [17], Theorem 2.1) Let \( m \) and \( t \) be integers satisfying \((t-1)^{-1} \leq m \leq \left(\frac{t}{r}\right)^{-1} + \left(\frac{t^2}{r^2}\right)^{-1} - (t-1) \). Then \( \lambda(G) \leq \lambda([r-1]^{(r)}) \).

Theorem 2.4 (See [17], Theorem 3.1) For any \( r \geq 4 \) there exists constants \( \gamma_r \) and \( k_0(r) \) such that if \( m \) satisfies
\[
\left(\frac{t-1}{r}\right) \leq m \leq \left(\frac{t-1}{r}\right) + \left(\frac{t^2}{r-1}\right) - \gamma_r(t-1)^{r-2}
\]
with \( t \geq k_0(r) \) and \( G \) is an \( r \)-graph on \( t \) vertices with \( m \) edges, then \( \lambda(G) \leq \lambda([r-1]^{(r)}) \).

Note that, Theorems 2.3 and 2.4 in this paper are equivalent to Theorems 2.1 and 3.1 in [18] after shifting \( t \) to \( t-1 \).

Some evidence of Conjectures 2.1 and 2.2 can be found in [19, 20]. In particular, we proved

Theorem 2.5 (See [19], Theorem 1.10) (a) Let \( m \) and \( t \) be positive integers satisfying
\[
\left(\frac{t-1}{r}\right) \leq m \leq \left(\frac{t-1}{r}\right) + \left(\frac{t^2}{r-1}\right) - (2^{r-3} - 1)\left(\frac{t-2}{r-2}\right) - 1.
\]
Let \( G \) be an \( r \)-graph on \( t \) vertices with \( m \) edges and contain a clique of order \( t-1 \). Then \( \lambda(G) = \lambda([r-1]^{(r)}) \).

(b) Let \( m \) and \( t \) be positive integers satisfying \((t-1)^{-1} \leq m \leq \left(\frac{t^2}{r^2}\right)^{-1} - (t-2) \). Let \( G \) be a 3-graph with \( m \) edges and without containing a clique of order \( t-1 \). Then \( \lambda(G) < \lambda([r-1]^{(r)}) \).

In this paper, we provide upper bounds on the Lagrangian of a 3-graph, a 4-graph, and an \( r \)-graph, respectively, when the hypergraph contains dense subgraphs and the number of edges of the hypergraph is in a certain range. These results support Conjectures 2.1 and 2.2, and extend Theorem 2.3.

We will impose one additional condition on any optimal weighting \( x := (x_1, x_2, \ldots, x_n) \) for an \( r \)-graph \( G \):
\[
|\{i : x_i > 0\}| \text{ is minimal, i.e. if } y \text{ is a feasible weighting for } G \text{ satisfying } |\{i : y_i > 0\}| < |\{i : x_i > 0\}|, \text{ then } \lambda(G, y) < \lambda(G). \quad (1)
\]

When the theory of Lagrange multipliers is applied to find the optimum of \( \lambda(G) \), subject to \( \sum_{i=1}^{n} x_i = 1 \), note that \( \lambda(E_i, x) \) corresponds to the partial derivative of \( \lambda(G, x) \) with respect to \( x_i \). The following lemma gives some necessary conditions of an optimal weighting of \( \lambda(G) \).

Lemma 2.2 (See [12], Theorem 2.1) Let \( G := (V, E) \) be an \( r \)-graph on the vertex set \([n]\) and \( x := (x_1, x_2, \ldots, x_n) \) be an optimal feasible weighting for \( G \) with \( k \leq n \) non-zero weights \( x_1, x_2, \ldots, x_k \) satisfying condition (1). Then for every \( \{i, j\} \in [k]^{(2)} \), (a) \( \lambda(E_i, x) = \lambda(E_j, x) = r \lambda(G) \), (b) there is an edge in \( E \) containing both \( i \) and \( j \).
The following definition is also needed.

**Definition 2.2** An r-graph $G := (V, E)$ on the vertex set $[n]$ is left-compressed if $j_1 j_2 \cdots j_r \in E$ implies $i_1 i_2 \cdots i_r \in E$ provided $i_p \leq j_p$ for every $p, 1 \leq p \leq r$. Equivalently, an r-graph $G := (V, E)$ is left-compressed iff $E_{j_i} = \emptyset$ for any $1 \leq i < j \leq n$.

**Remark 2.2** (a) In Lemma 2.2, part(a) implies that

$$x_j \lambda(E_{i,j}, x) + \lambda(E_{i,j}, x) = x_i \lambda(E_{i,j}, x) + \lambda(E_{j,i}, x).$$

In particular, if $G$ is left-compressed, then

$$(x_i - x_j) \lambda(E_{i,j}, x) = \lambda(E_{j,i}, x)$$

for any $i, j$ satisfying $1 \leq i < j \leq k$ since $E_{j,i} = \emptyset$.

(b) If $G$ is left-compressed, then for any $i, j$ satisfying $1 \leq i < j \leq k$,

$$x_i - x_j = \frac{\lambda(E_{i,j}, x)}{\lambda(E_{j,i}, x)}$$

holds. If $G$ is left-compressed and $E_{i,j} = \emptyset$ for $i, j$ satisfying $1 \leq i < j \leq k$, then $x_i = x_j$.

(c) By (3), if $G$ is left-compressed, then an optimal feasible weighting $x := (x_1, x_2, \ldots, x_n)$ for $G$ must satisfy

$$x_1 \geq x_2 \geq \cdots \geq x_n \geq 0.$$  

In the proofs of our results, we need to consider various left-compressed 3-graphs on vertex set $[t]$, which can be obtained from a Hessian diagram as follows.

A triple $i_1 i_2 i_3$ is called a descendant of a triple $j_1 j_2 j_3$ if $i_s \leq j_s$ for each $1 \leq s \leq 3$, and $i_1 + i_2 + i_3 < j_1 + j_2 + j_3$. In this case, the triple $j_1 j_2 j_3$ is called an ancestor of $i_1 i_2 i_3$. The triple $i_1 i_2 i_3$ is called a direct descendant of $j_1 j_2 j_3$ if $i_1 i_2 i_3$ is a descendant of $j_1 j_2 j_3$ and $j_1 + j_2 + j_3 = i_1 + i_2 + i_3 + 1$. We say that $j_1 j_2 j_3$ has lower hierarchy than $i_1 i_2 i_3$ if $j_1 j_2 j_3$ is an ancestor of $i_1 i_2 i_3$. This is a partial order on the set of all triples. Fig.1 is a Hessian diagram on all triples on vertex set $[t]$. In this diagram, $i_1 i_2 i_3$ and $j_1 j_2 j_3$ are connected by an edge if $i_1 i_2 i_3$ is a direct descendant of $j_1 j_2 j_3$.

**Remark 2.3** A 3-graph $G$ is left-compressed iff all descendants of an edge of $G$ are edges of $G$. Equivalently, if a triple is not an edge of $G$, then none of its ancestors will be an edge of $G$.

### 3 The Lagrangians of 3-graphs Containing Subgraph $K^{(3)}_{t-1}$

Let $K^{(3)}_{t-1}$ denote the hypergraph obtained by $K^{(3)}_{t-1}$ with one edge removed, where $K^{(3)}_{t-1}$ stands for a complete 3-graph with $t - 1$ vertices. Denote $\lambda_{(m,t-1)} := \max \{ \lambda(G) : G$ is a 3-graph with $m$ edges and $G$ containing $K^{(3)}_{t-1}$ but not containing $K^{(3)}_{t-1} \}$. We now prove Theorem 3.1.

**Theorem 3.1** Let $m$ and $t$ be positive integers satisfying $(t-1) \leq m \leq \binom{t-1}{3} + \binom{t-2}{2}$. Let $G$ be a 3-graph with $m$ edges containing $K^{(3)}_{t-1}$ but not containing $K^{(3)}_{t-1}$. Then $\lambda(G) < \lambda([t - 1]^{(3)})$ for $t \geq 6$.

In the proof of Theorem 3.1 we need several lemmas.
Lemma 3.1 Let \( m \) and \( t \) be positive integers satisfying \( \binom{t-1}{3} \leq m \leq \binom{t}{3} - \binom{t-2}{2} \). Then there exists a left-compressed 3-graph \( G \) with \( m \) edges containing \( |t-1|^3 \setminus \{ (t-3)(t-2)(t-1) \} \) but not containing \( |t-1|^3 \) such that \( \lambda(G) = \lambda^{(3)}_{(m,t-1)} \) and there exists an optimal weighting \( \mathbf{x} := (x_1, x_2, \ldots, x_n) \) of \( G \) satisfying \( x_i \geq x_j \) when \( i < j \).

The proof of Lemma 3.1 is similar to the proof of Lemma 3.1 in [20]. However Lemma 3.1 in [20] cannot be used directly here. For completeness, we give the proof in Sect. 6.

Lemma 3.2 (See [20], Proposition 3.7 ) Let \( G \) be a 3-graph on \( t \) vertices with at most \( \binom{t}{3} + \binom{t-1}{2} \) edges. If \( G \) does not contain \( K^{(3)}_{m,t-1} \), then \( \lambda(G) < \lambda(|t-1|^3) \) for \( 6 \leq t \leq 12 \).

Lemma 3.3 Let \( G \) be a left-compressed 3-graph containing \( |t-1|^3 \setminus \{ (t-3)(t-2)(t-1) \} \) but not containing \( |t-1|^3 \) with \( m \) edges such that \( \lambda(G) = \lambda^{(3)}_{(m,t-1)} \). Let \( \mathbf{x} := (x_1, x_2, \ldots, x_n) \) be an optimal weighting of \( G \) and \( k \) be the number of positive weights in \( \mathbf{x} \), then \( \lambda(G) < \lambda(|t-1|^3) \) or \( |k-1|^3 \setminus \mathbf{E} \leq k-2 \).

The proof of Lemma 3.3 is similar to Lemma 3.2 in [20]. However Lemma 3.2 in [20] cannot be used directly here. For completeness, we give the details of the proof in Sect. 6.

Proof of Theorem 3.7 Let \( m \) and \( t \) be positive integers satisfying \( \binom{t-1}{3} \leq m \leq \binom{t}{3} - \binom{t-2}{2} \). Let \( G := (V,E) \) be a 3-graph with \( m \) edges containing \( K^{(3)}_{t-1} \) but not containing \( K^{(3)}_{t-1} \) such that \( \lambda(G) = \lambda^{(3)}_{(m,t-1)} \). Let \( \mathbf{x} := (x_1, x_2, \ldots, x_n) \) be an optimal weighting of \( G \) and \( k \) be the number of non-zero weights in \( \mathbf{x} \). By Lemma 3.1 we can assume that \( G \) is left-compressed and contains \( |t-1|^3 \setminus \{ (t-3)(t-2)(t-1) \} \) but not contain \( |t-1|^3 \) and \( x_1 \geq x_2 \geq \ldots \geq x_k \geq x_{k+1} = \ldots = x_n = 0 \). Since \( \mathbf{x} \) has only \( k \) positive weights, we can assume that \( G \) is on \( [k] \).

Now we proceed to show that \( \lambda(G) < \lambda(|t-1|^3) \). By Lemma 3.2 Theorem 3.1 holds when \( t \leq 12 \). Next we assume \( t \geq 13 \). If \( \lambda(G) \geq \lambda(|t-1|^3) \), then \( k \geq t \). Otherwise \( k \leq t-1 \), since \( G \) does not contain \( |t-1|^3 \), then \( \lambda(G) < \lambda(|t-1|^3) \).

Since \( G \) is left-compressed and \( 1(k-1)k \in E \), then \( |k-2|^2 \cap E_k | \geq 1 \). If \( k \geq t+1 \), then applying Lemma 3.3 we have

\[
m = |E| = |E \cap |k-1|^3| + |k-2|^2 \cap E_k | + |E_{(k-1)k}|
\]
\[
\begin{align*}
\geq & \left( \binom{t}{3} \right) - (t-1) + 2 \\
\geq & \left( \binom{t-1}{3} \right) + \left( \binom{t-2}{2} \right) + 1,
\end{align*}
\]
which contradicts the assumption that \( m \leq \binom{t-1}{3} + \binom{t-2}{2} \). Recall that \( k \geq t \), so we have
\[ k = t. \]

Since \( \hat{\lambda}^{3-} \) does not decrease as \( m \) increases, it is sufficient to show the case that \( m = \binom{t-1}{3} + \binom{t-2}{2} \). Let \( G' := G \cup \{ (t-3)(t-2)(t-1) \} \backslash \{ (t-1)t \} \). If we can prove that \( \hat{\lambda}(G, x) < \hat{\lambda}(G', x) \), then since \( G' \) contains \( \binom{t-1}{3} + \binom{t-2}{2} \) edges, we have \( \hat{\lambda}(G', x) \leq \hat{\lambda}(G') = \lambda(t-1)^3 \)). Consequently, \( \hat{\lambda}(G) < \lambda(t-1)^3 \). Now we show that \( \hat{\lambda}(G, x) < \hat{\lambda}(G', x) \). Note that
\[ \hat{\lambda}(G', x) - \hat{\lambda}(G, x) = x_{t-3}x_{t-2}x_{t-1} - x_1x_{t-1}x_t. \]

By Remark 2.2b, we have
\[ x_1 = x_{t-3} + \frac{\hat{\lambda}(E_{1\setminus(t-3)}, x)}{\hat{\lambda}(E_{(t-3)}, x)} \]
and
\[ x_{t-2} = x_t + \frac{\hat{\lambda}(E_{(t-2)x}, x)}{\hat{\lambda}(E_{(t-3)}, x)}. \]

Combining equations 5, 6 and 7, we get
\[ \lambda(G', x) - \lambda(G, x) = x_{t-3}(x_t + \frac{\hat{\lambda}(E_{1\setminus(t-3)}, x)}{\hat{\lambda}(E_{(t-3)}, x)})x_{t-1} - (x_{t-3} + \frac{\hat{\lambda}(E_{1\setminus(t-3)}, x)}{\hat{\lambda}(E_{(t-3)}, x)})x_{t-1}x_t \]
\[ = \frac{\hat{\lambda}(E_{(t-2)x}, x)}{\hat{\lambda}(E_{(t-3)}, x)}x_{t-3}x_{t-1} - \frac{\hat{\lambda}(E_{1\setminus(t-3)}, x)}{\hat{\lambda}(E_{(t-3)}, x)}x_{t-1}x_t. \]

By Remark 2.2b
\[ x_1 = x_{t-2} + \frac{\hat{\lambda}(E_{1\setminus(t-3)}, x)}{\hat{\lambda}(E_{(t-3)}, x)} \leq x_{t-2} + \frac{x_{t-3}x_{t-1} + (x_2 + \ldots + x_3)x_t}{x_2 + \ldots + x_{t-3} + x_t} < x_{t-2} + x_{t-1} + x_t. \]

Hence \( \lambda(E_{1\setminus(t-3)}, x) - \lambda(E_{(t-2)x}, x) \geq x_{t-3}x_{t-1} + x_t - x_1 > 0 \) and \( \lambda(E_{1\setminus(t-3)}, x) > \lambda(E_{(t-2)x}, x) \). Clearly \( x_{t-3} > x_t \) since \((t-5)(t-1) \in E_{(t-3)}\). Therefore to show that \( \lambda(G, x) < \lambda(G', x) \), it is sufficient to show that
\[ \hat{\lambda}(E_{(t-2)x}, x) \geq \hat{\lambda}(E_{1\setminus(t-3)}, x). \]

If \((t-6)(t-1) \in E\), then all triples in \([t]^3 \backslash \{(t-3)(t-2)(t-1)\} \) are edges in \( G \) since \( G \) is left-compressed. If \( E \not= [t]^3 \backslash \{(t-3)(t-2)(t-1)\} \), then \( m > \binom{t}{3} - 1 \leq \binom{t-1}{3} + \binom{t-2}{2} \), which is a contradiction. Therefore, either
\[ E = [t]^3 \backslash \{(t-3)(t-2)(t-1)\}, \]
and
\[ \hat{\lambda}(E_{(t-2)x}, x) = x_{t-5}x_{t-1} + x_{t-5}x_{t-3} + x_{t-5}x_{t-4} + x_{t-4}x_{t-3} + x_{t-4}x_{t-1}, \]
and
\[ \hat{\lambda}(E_{1\setminus(t-3)}, x) = x_{t-2}x_{t-1} + x_{t-5}x_{t-1} + x_{t-4}x_{t-3} + x_{t-2}x_{t-1} + x_{t-1}x_t. \]
Clearly (10) holds in this case.

If \((t - 6)(t - 1) \not\in E\), then

\[
\lambda(E_{(t-2)|r}, x) \geq x_{t-3}\lambda(E_{(t-3)|r}, x) + x_{t-4}x_{t-1} + x_{t-5}x_{t-1} + x_{t-6}x_{t-1} - x_{t-3}x_{t-2} - x_{t-3}x_{t-1} = x_{t-3}\lambda(E_{(t-3)|r}, x) + x_{t-4}x_{t-1} + x_{t-5}x_{t-1} - x_{t-3}x_{t-2} - x_{t-3}x_{t-1}
\]

\[
\geq x_{t-3}\lambda(E_{(t-3)|r}, x) + x_{t-5}x_{t-1} + x_{t-6}x_{t-1} - x_{t-3}x_{t-2} - x_{t-3}x_{t-1}
\]

\[
= x_{t-3}(\lambda(E_{(t-3)|r}, x) - x_{t-2}) + x_{t-5}x_{t-1} + x_{t-6}x_{t-1} - x_{t-3}x_{t-2}
\]

and

\[
\lambda(E_{1\setminus(t-3)|r}, x) = x_{t-3}\lambda(E_{(t-3)|r}, x) + x_{t-2}x_{t-1}
\]

\[
= x_{t-3}(\lambda(E_{(t-3)|r}, x) - x_{t-2}) + x_{t-2}x_{t-1} + x_{t-2}x_{t-1}
\]

Clearly (10) holds in this case.

This completes the proof of Theorem 3.1.

\[\square\]

Remark 3.1 Note that for \(t \leq 5\), the left-compressed 3-graph with \(\binom{t-1}{3} + \binom{t-2}{2}\) edges always contains \(K_{t-1}^{(3)}\).

Combining Theorems 2.3 and 3.1, we have that, if \(G\) is a 3-graph containing \(K_{t-1}^{(3)}\) with at most \(\binom{t-1}{3} + \binom{t-2}{2}\) edges, then \(\lambda(G) \leq \lambda([t-1]^{(3)})\).

Also, applying Theorem 3.1, we derive two easy corollaries that support Conjecture 2.2.

Corollary 3.1 Let \(m\) and \(t\) be positive integers satisfying \(\binom{t-1}{3} \leq m \leq \binom{t-1}{3} + \binom{t-2}{2}\). Let \(G := (V, E)\) be a left-compressed 3-graph on the vertex set \([t]\) with \(m\) edges and not containing a clique of order \(t - 1\). If \(|E_{(t-1)|r}| \leq 3\), then \(\lambda(G) < \lambda([t-1]^{(3)})\).

Proof Because \(\lambda_{3-\{m, t-1\}}^{3}\) doesn’t decrease as \(m\) increases, we can assume that \(m = \binom{t-1}{3} + \binom{t-2}{2}\). Since \(G := (V, E)\) does not contain \([t-1]^{(3)}\) and \(G\) is left-compressed, then \((t-3)(t-2)(t-1) \not\in E\). If \(|E_{(t-1)|r}| = 1\), then \(G\) must contain \([t-1]^{(3)}\). Therefore, \(|E_{(t-1)|r}| = 2\) or \(3\).

If \(t \leq 5\), Theorem 3.1 clearly holds. Next, we assume \(t \geq 6\) and distinguish two cases.

Case 1. \(|E_{(t-1)|r}| = 2\). Note that \(G\) is left-compressed, in view of Fig.1,

\[
E = [t]^{(3)} \setminus \{3(t-1)r, 4(t-1)r, \ldots, (t-2)(t-1)r, (t-3)(t-2)r, (t-4)(t-2)r\}.
\]

Case 2. \(|E_{(t-1)|r}| = 3\). In this case, since \(G\) is left-compressed, in view of Fig.1, we only need to consider \(E = [t]^{(3)} \setminus \{4(t-1)r, \ldots, (t-2)(t-1)r, (t-3)(t-2)r, (t-4)(t-3)r\} \setminus \{3(t-1)r, 4(t-1)r, \ldots, (t-2)(t-1)r, (t-3)(t-2)r, (t-4)(t-2)r\}\).

In both cases, left-compressed 3-graph \(G\) does not contain the edge \((t-3)(t-2)(t-1)\). Thus, the conditions in Theorem 3.1 are satisfied. Therefore, we are done.

The next corollary states that if 3-graph \(G\) contains a dense subgraph close to the structure in \(C_{3,m}\), then we have \(\lambda(G) < \lambda([t-1]^{(3)})\).

Corollary 3.2 Let \(m\) and \(t\) be positive integers satisfying \(\binom{t-1}{3} \leq m \leq \binom{t-1}{3} + \binom{t-2}{2}\). Let \(G := (V, E)\) be a left-compressed 3-graph on the vertex set \([t]\) with \(m\) edges and not containing a clique of size \(t - 1\), and \(|E(G)| \Delta E(C_{3,m})| \leq 6\). Then, \(\lambda(G) < \lambda([t-1]^{(3)})\).

Proof If \(m \leq \binom{t-1}{3} + \binom{t-2}{2}\), then \(|E_{(t-1)|r}| \leq 3\), since otherwise \(|E(G)| \Delta E(C_{3,m})| > 6\). Applying Corollary 3.2 we have \(\lambda(G) < \lambda([t-1]^{(3)})\).

\[\square\]
4 The Lagrangians of Hypergraphs Containing A Clique of Order \( t - 2 \) or \( t - 1 \)

In this section, we prove the following.

Theorem 4.1 Let \( m \) and \( t \) be positive integers satisfying \( \left( \frac{t-1}{3} \right) \leq m \leq \left( \frac{t-1}{3} \right) + \left( \frac{t-2}{2} \right) - \frac{t-2}{2} \). Let \( G \) be a 3-graph with \( m \) edges and \( G \) contain the maximum clique of order \( t - 2 \). Then \( \lambda(G) < \lambda([t-1]^{(3)}) \).

Theorem 4.2 Let \( m \) and \( t \) be positive integers satisfying \( \left( \frac{t-1}{3} \right) \leq m \leq \left( \frac{t-1}{3} \right) + \left( \frac{t-2}{2} \right) - 2^{t-2} \left( \frac{t-2}{2} - 1 \right) \). Let \( G \) be an \( r \)-graph on \( t \) vertices with \( m \) edges and with a clique of order \( t - 2 \). Then \( \lambda(G) \leq \lambda([t-1]^{(r)}) \).

Theorem 4.3 Let \( m \) and \( t \) be positive integers satisfying \( \left( \frac{t-1}{4} \right) \leq m \leq \left( \frac{t-1}{4} \right) + \left( \frac{t-2}{3} \right) \). Let \( G \) be a 4-graph with \( m \) edges and a clique of order \( t - 1 \). Then \( \lambda(G) = \lambda([t-1]^{(4)}) \).

Here \( \left( \frac{t-1}{4} \right) + \left( \frac{t-2}{3} \right) \) is not the best upper bound that we can obtain. This bound is for simplicity of the proof.

Denote \( \lambda_{(m,p)} := \max \{ \lambda(G) : G \) is an \( r \)-graph with \( m \) edges and \( G \) contains a maximum clique of order \( p \) \}.

Similar to the proof of Lemma 3.1 in [20], we can prove the following lemma. We will give the proof in Sect. 6.

Lemma 4.1 Let \( m \) and \( t \) be positive integers satisfying

\[ \left( \frac{t-1}{3} \right) \leq m \leq \left( \frac{t-1}{3} \right) + \left( \frac{t-2}{2} \right) - \frac{t-2}{2}. \]

Then there exists a left-compressed 3-graph \( G \) with \( m \) edges containing the maximum clique \( [t-2]^{(3)} \) such that \( \lambda(G) = \lambda_{(m,t-2)}^{3} \).

Similar to the proof of Lemma 3.2 in [20], we have the following lemma. For completeness, we will give the proof in Sect. 6.

Lemma 4.2 Let \( G \) be a left-compressed 3-graph containing the maximum clique \( [t-2]^{(3)} \) with \( m \) edges such that \( \lambda(G) = \lambda_{(m,t-2)}^{3} \). Let \( x := (x_{1}, x_{2}, \ldots, x_{n}) \) be an optimal weighting of \( G \) and \( k \) be the number of positive weights in \( x \). Then \( \lambda(G) < \lambda([t-1]^{(3)}) \) or \( ||k-1||^{(3)} \cap E || \leq k - 2 \).

We also need the following lemma whose proof is similar to Lemma 2.7 in [17] and Lemma 3.3 in [16]. We will give it in Sect. 6.

Lemma 4.3 Let \( m \) and \( t \) be positive integers satisfying \( \left( \frac{t-1}{3} \right) \leq m \leq \left( \frac{t-1}{3} \right) + \left( \frac{t-2}{2} \right) - \frac{t-2}{2} \). Let \( G \) be a left-compressed 3-graph on the vertex set \( [t] \) and contain the maximum clique \( [t-2]^{(3)} \) with \( m \) edges such that \( \lambda(G) = \lambda_{(m,t-2)}^{3} \). Assume \( b := |E_{(t-1)}|, \) then \( \lambda(G) < \lambda([t-1]^{(3)}) \) or

\[ ||t-2||^{(2)} \setminus E || \leq b. \]

Proof of Theorem 4.1 Let \( m \) and \( t \) be positive integers satisfying \( \left( \frac{t-1}{3} \right) \leq m \leq \left( \frac{t-1}{3} \right) + \left( \frac{t-2}{2} \right) - \frac{t-2}{2} \). Clearly we can assume that \( t \geq 5 \). Let \( G := (V,E) \) be a 3-graph with \( m \) edges containing a maximum clique of order \( t - 2 \) such that \( \lambda(G) = \lambda_{(m,t-2)}^{3} \). Let \( x := (x_{1}, x_{2}, \ldots, x_{n}) \) be an optimal weighting of \( G \) and \( k \) be the number of non-zero weights in \( x \). By Lemma 4.1, we can assume that \( G \) is left-compressed with the maximum clique \( [t-2]^{(3)} \) and \( x_{1} \geq x_{2} \geq \ldots \geq x_{k} > x_{k+1} = \ldots = x_{n} = 0 \). Since \( x \) has only \( k \) positive weights, we can assume that \( G \) is on \( [k] \).

Now we proceed to show that \( \lambda(G) < \lambda([t-1]^{(3)}) \). If \( \lambda(G) \geq \lambda([t-1]^{(3)}) \), then \( k \geq t \). Otherwise \( k \leq t - 1 \), since \( G \) does not contain \( [t-1]^{(3)} \), then \( \lambda(G) < \lambda([t-1]^{(3)}) \).
Because \( E \) is left-compressed, \( E_{ij} = \emptyset \) for \( 1 \leq i < j \leq b \). Hence, by Remark 2.2(a), we have \( x_1 = x_2 = \cdots = x_b \). Clearly, \( b \leq k - 5 \).

Since \( G \) is left-compressed and \( 1(k-1)k \in E \), then \( |(k-2)^2 \cap E_k| \geq 1 \). So applying Lemma 4.2 similar to (4), we have \( k = t \).

Since \( k = t \), we can assume that \( G \) is on \([t]\). By Remark 2.2(b), we have

\[
x_1 = x_{t-3} + \frac{\lambda(E_1\setminus (t-3), x)}{\lambda(E_1(t-3), x)}.
\]

Recall that \( G \) contains a clique order of \( t - 2 \), we have

\[
\lambda(E_1\setminus (t-3), x) = x_{t-1} \lambda(E_{t-3}(t-1), x) + x_t \lambda(E_{t-3}(t), x) - x_{t-1} x_t.
\]

Hence

\[
x_1 < x_{t-3} + \frac{x_{t-1} \lambda(E_{t-3}(t-1), x) + x_t \lambda(E_{t-3}(t), x)}{\lambda(E_1(t-3), x)}.
\]

Since for \( i \neq t \), \( 1 \), \( t-2 \), \( t-3 \), \( i \in E_{t-3}^c \) implies that \( i(t-3) \in [t-2]^2 \setminus E_t \) and \( i \in E_{t-2}^c \) implies that \( i(t-2) \in [t-2]^2 \setminus E_t \). \( t - 1 \in E_{t-2}^c \), \( t - 1 \in E_{t-2}^c \), and \( t - 2 \in E_{t-3}^c \), \( t - 3 \in E_{t-3}^c \). \( (t-2) - (t-3) \in [t-2]^2 \setminus E_t \). Applying Lemma 4.3 then

\[
|E_{t-3}^c| + |E_{t-2}^c| \leq |[t-2]^2 \setminus E_t| + 3 \leq b + 3.
\]

Note that \( b \leq t - 5 \) and

\[
|E_{t-3}^c| \leq |E_{t-2}^c|,
\]

So

\[
|E_{t-3}^c| \leq \frac{b + 3}{2} \leq \frac{t - 2}{2}.
\]

Since \( G \) is left-compressed, then

\[
|E_{t-3}(t-1)| \leq |E_{t-3}^c| \leq \frac{t - 2}{2}.
\]

So

\[
x_1 < x_{t-3} + \frac{x_{t-1} \lambda(E_{t-3}(t-1), x) + x_t \lambda(E_{t-3}(t), x)}{\lambda(E_1(t-3), x)}
\]

\[
\leq x_{t-3} + \frac{x_{t-1} \lambda(E_{t-3}(t-1), x)}{\lambda(E_1(t-3), x)} + \frac{x_t \lambda(E_{t-3}(t), x)}{\lambda(E_1(t-3), x)}
\]

\[
\leq 2x_{t-3}.
\]

This implies

\[
2x_{t-3}x_{t-2}x_{t-1} - x_1 x_{t-1} x_t > 0.
\]

Let \( C := [t-1]^3 \setminus E \) be all triples containing \( t-1 \) not in \( E \),

\[
E' := E \cup C \setminus \{(b - 1)^3, (t - 1)^3, (b - 1)^3, (t - 1)^3, \ldots, b(t - 1)^3 \} \text{ and } G' := ([t]^3, E')\).
\]

Then

\[
\lambda(G', x) - \lambda(G, x) = \lambda(C, x) - \frac{|C|}{2} x_1 x_{t-1} x_t
\]

\[
\geq |C| x_{t-3} x_{t-2} x_{t-1} - \frac{|C|}{2} x_1 x_{t-1} x_t
\]

\[
\geq \frac{|C|}{2} (2x_{t-3}x_{t-2}x_{t-1} - x_1 x_{t-1} x_t) > 0.
\]
Remark 4.1

Lemma 4.5(b) and Theorem 4.2 imply that if $G$ contains a clique of order $t - 1$, we have $\lambda(G, x) < \lambda(G', x)$ by Theorem 4.2. Hence $\lambda(G, x) < \lambda(G', x) \leq \lambda([t - 1]^{(3)})$. This proves Theorem 4.1.

The following lemma implies that we only need to consider left-compressed $r$-graphs when Theorem 4.2 is proved. The proof is given in Sect. 6.

**Lemma 4.4** Let $m$ and $t$ be positive integers satisfying

$$\left(\frac{t - 1}{r}\right) \leq m \leq \left(\frac{t}{r}\right) - 1.$$

Then there exists a left-compressed $G$ with $m$ edges containing the clique $[t - 2]^{(r)}$ such that $\lambda(G) = \lambda'(m, t - 2)$ and there exists an optimal weighting $x := (x_1, x_2, \ldots, x_n)$ of $G$ satisfying $x_i \geq x_j$ when $i < j$.

We also need the following in the proof of Theorem 4.2 and Theorem 4.3.

**Lemma 4.5** (See [19], Theorem 3.4) Let $r \geq 3$ and $t \geq r + 2$ be positive integers. Let $G$ be a left-compressed $r$-graph on $t$ vertices satisfying $||t - 2|^{(r-1)} \setminus E| \geq 2^{r-3}|E_{(t-1)}|$. Then

(a) If $G$ contains $[t - 1]^{(r)}$, then $\lambda(G) = \lambda([t - 1]^{(r)})$.

(b) If $G$ does not contain $[t - 1]^{(r)}$, then $\lambda(G) < \lambda([t - 1]^{(r)})$.

**Proof of Theorem 4.2** Let $m$ and $t$ be positive integers satisfying $\left(\frac{t - 1}{r}\right) \leq m \leq \left(\frac{t}{r}\right) - 1$. Let $G$ be an $r$-graph with $m$ edges and $t$ vertices with a clique order of $t - 2$. By Lemma 4.3 we can assume $G$ is left-compressed. By Lemma 4.5 it is sufficient to show that $||t - 2|^{(r-1)} \setminus E| \geq 2^{r-3}|E_{(t-1)}|$. If not, then $||t - 2|^{(r-1)} \setminus E| < 2^{r-3}|E_{(t-1)}|$ and $||t - 2|^{(r-1)} \setminus E_{(t-1)}| \leq ||t - 2|^{(r-1)} \setminus E| < 2^{r-3}|E_{(t-1)}|$. Since $G$ contains the clique $[t - 2]^{(r)}$, then

$$m = \left(\frac{t - 2}{r}\right) + 2\left(\frac{t - 2}{r - 1}\right) - ||t - 2|^{(r-1)} \setminus E| - ||t - 2|^{(r-1)} \setminus E_{(t-1)}| + |E_{(t-1)}| \geq \left(\frac{t - 1}{r}\right) + \left(\frac{t - 2}{r - 1}\right) - 2^{r-2}\left(\frac{t - 2}{r - 2}\right) - 1.$$ 

since $|E_{(t-1)}| \leq \left(\frac{t - 2}{r - 2}\right) - 1$, this is a contradiction. Note that, if $|E_{(t-1)}| = \left(\frac{t - 2}{r - 2}\right)$, then $E = [t]^{(r)}$ since $G$ is left-compressed and $m = \left(\frac{t}{r}\right)$, which results in a contradiction too. This proves Theorem 4.2. \(\Box\)

**Remark 4.1** Lemma 4.5(b) and Theorem 4.2 imply that if $m$ and $t$ are positive integers satisfying

$$\left(\frac{t - 1}{r}\right) \leq m \leq \left(\frac{t - 1}{r}\right) + \left(\frac{t - 2}{r - 1}\right) - 2^{r-2}\left(\frac{t - 2}{r - 2}\right) - 1$$

and $G$ is a $r$-graph on $t$ vertices with $m$ edges and with a maximum clique of order $t - 2$. Then

$$\lambda(G) < \lambda([t - 1]^{(r)}).$$
Proof of Theorem 2.3 Let \( m \) and \( t \) be positive integers satisfying \( \binom{t-1}{4} \leq m \leq \binom{t-1}{4} + \left(\frac{\binom{4}{2}}{3}\right) \). Let \( G \) be a 4-graph with \( m \) edges and a clique of order \( t-1 \). Since it contains a clique of order \( t-1 \), without loss of generality, we may assume that it contains \([t-1]\)\(^4\). Since \( G \) contains \([t-1]\)\(^4\), we have \( \lambda(G) \geq \lambda([t-1]\)\(^4\)\). Next we prove that \( \lambda(G) \leq \lambda([t-1]\)\(^4\)\).

Let \( x := (x_1, x_2, \ldots, x_n) \) be an optimal weighting of \( G \) and \( k \) be the number of non-zero weights in \( x \). If \( k \leq t-1 \), clearly \( \lambda(G) \leq \lambda([t-1]\)\(^4\)\). Assume that \( k \geq t \). Recall that \( \binom{t-1}{4} \leq m \leq \binom{t-1}{4} + \left(\frac{\binom{4}{2}}{3}\right) \) and \( G \) contains \([t-1]\)\(^4\), hence \( |E_k| \leq \left(\frac{\binom{4}{2}}{3}\right) \). By Fact 2.1, Lemma 2.2(a) and Theorem 2.3, we have

\[
\lambda(G, x) = \frac{1}{4} \lambda(E_k, x) \leq \frac{1}{4} \left(\frac{\binom{4}{2}}{3}\right) \left(\frac{1}{\binom{k-2}{2}}\right)^3 \leq \frac{(t-4)(t-6)}{24(t-2)^2} < \frac{(t-2)(t-3)(t-4)}{24(t-1)^3} = \lambda([t-1]\)\(^4\)\).
\]

\( \square \)

Remark 4.2 Also, note that Theorem 3.1, Theorem 4.1 and Remark 4.1 provide further evidence for Conjecture 2.1. Theorem 5.1 also provides further evidence for Conjecture 2.1.

5 Remarks

Frankl and Füredi [11] asked the following question: Given \( r \geq 3 \) and \( m \in \mathbb{N} \) how large can the Lagrangian of an \( r \)-graph with \( m \) edges be? Conjecture 2.3 proposes a solution to the question mentioned above.

Denote

\[
\lambda^r_m := \max \{ \lambda(G) : G \text{ is an } r \text{-graph with } m \text{ edges} \}.
\]

The following lemma implies that we only need to consider left-compressed \( r \)-graphs when Conjecture 2.3 is explored.

Lemma 5.1 (See [17], Lemma 2.3) There exists a left-compressed \( r \)-graph \( G \) with \( m \) edges such that

\[
\lambda(G) = \lambda^r_m.
\]

We extend Theorem 2.3 in Theorem 5.1 which is a corollary of Theorem 3.1.

Theorem 5.1 Let \( m \) and \( t \) be positive integers satisfying \( \binom{t-1}{3} \leq m \leq \binom{t-1}{3} + \left(\frac{\binom{2}{2}}{3}\right) \). Then Conjecture 2.3 is true for \( r = 3 \) and this value of \( m \).

Proof Let \( x := (x_1, x_2, \ldots, x_n) \) be an optimal weighting for \( G \) and \( k \) be the number of positive weights in \( x \). We can assume that \( G \) is left-compressed by Lemma 5.1. So \( x_1 \geq x_2 \geq \ldots \geq x_k > x_{k+1} = \ldots = x_n = 0 \) by Remark 2.2(c). Since \( x \) has only positive weights, we can assume that \( G \) is on vertex set \([k]\).

Now we proceed to show that \( \lambda(G) \leq \lambda([t-1]\)\(^3\)\). If \( \lambda(G) > \lambda([t-1]\)\(^3\)\), then \( k \geq t \) since otherwise \( k \leq t-1 \) and then \( \lambda(G) \leq \lambda([t-1]\)\(^3\)\). Next we apply the following lemma.

Lemma 5.2 (See [17], Lemma 2.5) Let \( m \) be a positive integer. Let \( G \) be a left-compressed 3-graph with \( m \) edges such that \( \lambda(G) = \lambda^3_m \). Let \( x := (x_1, x_2, \ldots, x_n) \) be an optimal weighting for \( G \) and \( k \) be the number of non-zero weights in \( x \), then

\[
|\{k-1\}\)\(^3\} \setminus E| \leq k - 2.
\]

So similar to [4], we have \( k = t \). Next we need the following lemma whose proof follows the lines of Lemma 2.5 in [17]. For completeness, we give the proof in Sect. 6.
Lemma 5.3 Let \( G \) be a left-compressed 3-graph on the vertex set \([t]\) with \( m \) edges where
\[
\left( \frac{t-1}{3} \right) \leq m \leq \left( \frac{t-1}{3} \right) + \left( \frac{t-2}{2} \right),
\]
and \( \lambda(G) = \lambda_m^3 \). Let \( x := (x_1, x_2, \ldots, x_t) \) be an optimal weighting for \( G \). Then
\[
|\{t-1\}^3 \setminus E| \leq t-3, \text{ or } \lambda(G) \leq \lambda(|\{t-1\}^3|).
\]
Assume Lemma 5.3 holds, we continue the proof of Theorem 5.1. If \( \lambda(G) > \lambda(|\{t-1\}^3|) \), then
\[
|\{t-1\}^3 \setminus E| \leq t-3 \text{ by Lemma 5.3}
\]
we add any \(|\{t-1\}^3 \setminus E| - 1\) triples in \(|\{t-1\}^3 \setminus E| \) to \( E \) and let the new 3-graph be \( G' \). Then \( G' \) contains \( K^3_{t-1} \), the number of edges in \( G' \) is at most \( \binom{t-1}{3} + \binom{t-2}{2} \) and \( \lambda(G') \geq \lambda(G) \).
Applying Theorem 2.5 and Theorem 3.1, \( \lambda(G') \leq \lambda(|\{t-1\}^3|) \). Therefore \( \lambda(G) \leq \lambda(|\{t-1\}^3|) = \lambda(C_{3,m}) \) by Lemma 2.1. This completes the proof of Theorem 5.1. \( \square \)

6 Proofs of Some Lemmas

Proof techniques of lemmas in this section follow from proof techniques of some lemmas in [17, 19, 20]. As mentioned earlier, lemmas in those papers cannot be applied directly to situations in this paper. For completeness, we give the proof of these lemmas in this section.

Proof of Lemma 5.7 Let \( G \) be a 3-graph on the vertex set \([n]\) with \( m \) edges containing \( K^3_{t-1} \) but not containing \( K^3_t \). Let \( x := (x_1, x_2, \ldots, x_n) \) be an optimal weighting of \( G \). We can assume that \( x_i \geq x_j \) when \( i < j \) since otherwise we can just relabel the vertices of \( G \) and obtain another extremal 3-graph for \( m \) and \( t-1 \). Let \( x := (x_1, x_2, \ldots, x_n) \) satisfying \( x_i \geq x_j \) when \( i < j \). Next we obtain a new 3-graph \( G' \) from \( G \) by performing the following:

1. If \((t-3)(t-2)(t-1) \in E(G)\), then there is at least one triple in \(|\{t-1\}^3 \setminus E(G)|\), we replace \( (t-3)(t-2)(t-1) \) by this triple.
2. If an edge in \( G \) has a descendant other than \((t-3)(t-2)(t-1) \) that is not in \( E(G) \), then replace this edge by a descendant other than \((t-3)(t-2)(t-1) \) with the lowest hierarchy. Repeat this until there is no such an edge.

Then \( G' \) satisfies the following properties:

1. The number of edges in \( G' \) is the same as the number of edges in \( G \).
2. \( \lambda(G) = \lambda(G,x) \leq \lambda(G') \leq \lambda(G') \).
3. \((t-3)(t-2)(t-1) \notin E(G')\).
4. \(|\{t-1\}^3 \setminus \{(t-3)(t-2)(t-1)\}| \in E(G')\).
5. For any edge in \( E(G') \), all its descendants other than \((t-3)(t-2)(t-1) \) will be in \( E(G') \).

If \( G' \) is not left-compressed, then there is an ancestor \( uvw \) of \((t-3)(t-2)(t-1) \) such that \( uvw \in E(G') \). We claim that \( uvw \) must be \((t-3)(t-2)t \). If \( uvw \) is not \((t-3)(t-2)t \), then since all descendants other than \((t-3)(t-2)(t-1) \) of \( uvw \) will be in \( E(G') \), then all descendants of \((t-3)(t-2)t \) (other than \((t-3)(t-2)(t-1) \)) or all descendants of \((t-3)(t-2)(t+1) \) (other than \((t-3)(t-2)(t-1) \)) will be in \( E(G') \). So all triples in \(|\{t-1\}^3 \setminus \{(t-3)(t-2)(t-1)\}| \), all triples in the form of \( ijt \) (where \( ij \in [t-2] \)), and all triples in the form of \( ijt+1 \) (where \( ij \in [t-2] \)) or all triples in the form of \( i(t-1)l \), \( 1 \leq i \leq t-3 \) will be in \( E(G') \), then
\[
m \geq \left( \frac{t-1}{3} \right) - 1 + \left( \frac{t-2}{2} \right) + (t-3) > \left( \frac{t-1}{3} \right) + \left( \frac{t-2}{2} \right).
\]
which is a contradiction. So \(uvw\) must be \((t-3)(t-2)t\). Since \(m \leq \binom{t}{2} + \binom{t}{3}\) and all the descendants other than \((t-3)(t-2)(t-1)\) of an edge in \(G'\) will be an edge in \(G'\), there are two possibilities.

Case 1. \(E(G') = \{(t-1)^{(3)} \setminus \{(t-1)(t-2)(t-3)\}) \cup \{ijt, ij \in [t-2]\} \cup \{12(t+1)\}\).

Case 2. \(E(G') = \{(t-1)^{(3)} \setminus \{(t-1)(t-2)(t-3)\}) \cup \{ijt, ij \in [t-2]\}\).

Let \(y := (y_1, y_2, \ldots, y_n)\) be an optimal weighting of \(G'\), where \(n = t+1\) or \(n = t\). We claim that if Case 1 happens, then \(y_i = y_{i+1} = 0\), since \(E_{t-12} = E_{t-13} = \emptyset\) (by Lemma 3.3). If Case 2 happens, then \(y_i = 0\) since \(E_{t-32} = \emptyset\) (by Lemma 3.3). Hence we can assume that \(G\) is left-compressed.

**Proof of Lemma 3.3** Since \(G\) contains the clique of \([t-1]^{(3)} \setminus \{(t-3)(t-2)(t-1)\}\), it is true for \(k \leq t\). Next we assume that \(k \geq t+1\).

Since \(G\) is left-compressed, \(1(k-1)k \in E\). Let \(b := \max\{i: (k-1)i \in E\}\). Since \(E\) is left-compressed, then \(E_i := \{1, \ldots, i-1, i+1, \ldots, k\}\), for \(1 \leq i \leq b\), and \(E_{k+1} = \emptyset\) for \(1 \leq i < j \leq b\). Hence, by Remark 2.2(a), we have \(x_1 = x_2 = \cdots = x_b\).

We define a new feasible weighting \(y\) for \(G\) as follows. Let \(y_i = x_i\) for \(i \neq k-1, k\), \(y_{k-1} = x_{k-1} + x_k\), and \(y_k = 0\).

By Lemma 2.2(a), \(\lambda(E_{k-1}, x) = \lambda(E_k, x)\), so
\[
\lambda(G, y) - \lambda(G, x) = x_k(\lambda(E_{k-1}, x) - x_k(\lambda(E_k, x)))
\]
\[
= -x_k(\lambda(E_k, x) - x_{k-1}\lambda(E_{k-1}, x)) - x_{k-1}x_k(\lambda(E_k, x))
\]
\[
= -bx_kx_{k-1}^2.
\]

Since \(y_k = 0\) we may remove all edges containing \(k\) from \(E\) to form a new 3-graph \(\bar{G} := ([k], E)\) with \(|E| := |E| - |E_k|\) and \(\lambda(G, y) = \lambda(G, y)\). We will show that if Lemma 3.3 fails to hold then there exists a set of edges \(F \subset [k-1]^{(3)} \setminus E\) satisfying
\[
\lambda(F, y) > bx_kx_{k-1}^2
\]
and
\[
|F| \leq |E_k|.
\]

Then, using (11), (12), and (13), the 3-graph \(G' := ([k], E')\), where \(E' := E \cup F\), satisfies \(|E'| \leq |E|\) and
\[
\lambda(G', y) = \lambda(\bar{G}, y) + \lambda(F, y)
\]
\[
> \lambda(G, y) + bx_kx_{k-1}^2
\]
\[
= \lambda(G, x).
\]

Hence \(\lambda(G') > \lambda(G)\). Note that \(G'\) still contains \([t-1]^{(3)} \setminus \{(t-3)(t-2)(t-1)\}\) since \(G'\) contains all edges in \(E \cap [k-1]^{(3)} \supseteq E \cap [t-1]^{(3)}\). If \(G'\) does not contain a clique of size \(t-1\), note that \(G'\) still contain \([t-1]^{(3)} \setminus \{(t-3)(t-2)(t-1)\}\), it contradicts to \(\lambda(G) = \lambda_{3, t-1}\). If \(G'\) contains a clique of size \(t-1\), then by Theorem 2.5 \(\lambda(G') = \lambda([t-1]^{(3)})\) and consequently \(\lambda(G') < \lambda([t-1]^{(3)})\).

We must now construct the set of edges \(F\) satisfying (12) and (13). Applying Remark 2.2(a) by taking \(i = 1, j = k - 1\), we have
\[
x_1 = x_{k-1} + \lambda(E_{1(k-1)}, x) - \lambda(E_{(k-1)}, x).
\]
Let \(C := [k-2] \setminus E_{k-1}\). Then \(\lambda(E_{1(k-1)}, x) = x_k \sum_{i=2}^{k-1} x_i + \lambda(C, x)\). Applying this and multiplying \(bx_kx_{k-1}^2\) to the above equation (note that \(\lambda(E_{1(k-1)}, x) = \sum_{i=2, i \neq k-1} x_i\)), we have
\[
bx_kx_{k-1}^2 = bx_kx_{k-1}^2 + \frac{bx_kx_{k-1}^2 \sum_{i=2, i \neq k-1} x_i}{\sum_{i=2, i \neq k-1} x_i} + \frac{bx_kx_{k-1}^2 \lambda(C, x)}{\sum_{i=2, i \neq k-1} x_i}.
\]
Since \( x_1 \geq x_2 \geq \cdots \geq x_k \), then
\[
bx_1 x_k^2 \leq bx_{k-1} x_k^2 \left( 1 + \frac{k - (b + 2)}{k - 3} \right) + \frac{bx_k \lambda(C, x)}{k - 2},
\]
(14)

Define \( \alpha := \frac{b(C)}{k - 2} \) and \( \beta := \frac{b(1 + \frac{k - (b + 2)}{k - 3})}{k - 2} \). Note that \( \left[ b(1 + \frac{k - (b + 2)}{k - 3}) \right] \leq k - 2 \) since \( b \leq k - 2 \). Let the set \( F_1 \subset [k - 1]^{(3)} \setminus E \) consist of the \( \alpha \) heaviest edges in \([k - 1]^{(3)} \setminus E\) containing the vertex \( k - 1 \) (note that \( |[k - 2]^{(2)} \setminus E_{k-1}| = |C| \geq \alpha \)). Recalling that \( y_{k-1} = x_{k-1} + x_k \) we have
\[
\lambda(F_1, y) \geq \frac{bx_k \lambda(C, x)}{k - 2} + \alpha x_{k-1} x_k^2.
\]
So using (14)
\[
\lambda(F_1, y) - bx_{k-1} x_k^2 \geq x_{k-1} x_k^2(\alpha - \beta).
\]
(15)

We now distinguish two cases.

Case 1. \( \alpha > \beta \).

In this case \( \lambda(F_1, y) - bx_{k-1} x_k^2 > 0 \) so defining \( F := F_1 \) satisfies (12). We need to check that \( |F| \leq |E_k| \). Since \( E \) is left-compressed, then \( \{b\}^{(2)} \cup \{1, \ldots , b\} \times \{b + 1, \ldots , k - 1\} \subset E_k \). Hence
\[
|E_k| \geq \frac{b[b - 1 + 2(k - 1 - b)]}{2} \geq \frac{b(k - 1)}{2}
\]
(16)
since \( b \leq k - 2 \). Recall that \( |F| = \alpha = \frac{b(C)}{k - 2} \). Since \( C \subset [k - 2]^{(2)} \), we have \( |C| \leq \binom{k - 2}{2} \). So using (16) we obtain
\[
|F| \leq \frac{b(k - 3)}{2} \leq \frac{b(k - 1)}{2} \leq |E_k|.
\]
So both (12) and (13) are satisfied.

Case 2. \( \alpha \leq \beta \).

Suppose that Lemma 5.3 fails to hold. So \( |[k - 1]^{(3)} \setminus E| \geq k - 1 \geq \beta + 1 \) (recall that \( \beta \leq k - 2 \)). Let \( F_2 \) consist of any \( \beta + 1 - \alpha \) edges in \([k - 1]^{(3)} \setminus (E \cup F_1) \) and define \( F := F_1 \cup F_2 \). Then since \( \lambda(F_2, y) \geq (\beta + 1 - \alpha) x_{k-1}^2 \) and using (15),
\[
\lambda(F, y) - bx_{k-1} x_k^2 = \lambda(F_1, y) - bx_{k-1} x_k^2 + \lambda(F_2, y) \geq (\beta + 1 - \alpha) x_{k-1}^2 - x_{k-1} x_k^2(\beta - \alpha) > 0.
\]
So (12) is satisfied. What remains is to check that \( |F| \leq |E_k| \). In fact,
\[
|F| = \beta + 1 \leq k - 1 \leq \frac{b(k - 1)}{2} \leq |E_k|
\]
when \( b \geq 2 \). If \( b = 1 \), then,
\[
|F| = \beta + 1 = 3 \leq k - 2 = \frac{b[b - 1 + 2(k - 1 - b)]}{2} \leq |E_k|
\]
since \( k \geq t \geq 5 \). \( \square \)

Proof of Lemma 7.7 Let \( G \) be a 3-graph on the vertex set \([n]\) with \( m \) edges containing a maximal clique of order \( t - 2 \) such that \( \lambda(G) = \lambda_{(m,t-2)}^3 \). We call such a \( G \) an extremal 3-graph for \( m \) and \( t - 2 \). Let \( x := (x_1, x_2, \ldots , x_n) \) be an optimal weighting of \( G \). We can assume that \( x_i \geq x_j \) when \( i < j \) since otherwise we can just relabel the vertices of \( G \) and obtain another extremal 3-graph for \( m \) and \( t - 2 \) with an optimal weighting \( x := (x_1, x_2, \ldots , x_n) \) satisfying \( x_i \geq x_j \) when \( i < j \). Next we obtain a new 3-graph \( G' \) from \( G \) by performing the followings

1. If \( (t - 3)(t - 2)(t - 1) \in E(G) \), then there is at least one triple in \([t - 1]^{(3)} \setminus E(G) \), we replace
(t − 3)(t − 2)(t − 1) by this triple;
2. If an edge in \( G \) has a descendant other than \((t − 3)(t − 2)(t − 1)\) that is not in \( E(G) \), then replace this edge by a descendant other than \((t − 3)(t − 2)(t − 1)\) with the lowest hierarchy. Repeat this until there is no such an edge.

Then \( G' \) satisfies the followings
1. The number of edges in \( G' \) is the same as the number of edges in \( G \);
2. \( G \) contains the clique \([t − 2]^{(3)}\);
3. \( \lambda(G) = \lambda(G, x) \leq \lambda(G', x) \leq \lambda(G') \);
4. \((t − 3)(t − 2)(t − 1) \notin E(G')\);
5. For any edge in \( E(G) \), all its descendants other than \((t − 3)(t − 2)(t − 1)\) will be in \( E(G') \).

If \( G' \) is not left-compressed, then there is an ancestor \( uvw \) of \((t − 3)(t − 2)(t − 1)\) such that \( uvw \in G' \) and all the descendant of \( uvw \) other than \( uvw \) are in \( G' \). Hence \( E(G') \supseteq ((t − 1) \setminus \{(t − 3)(t − 2)(t − 1)\}) \cup \{ijt, ij \in [t − 2]\} \), and
\[
m \geq \left( t - 1 \right) - 1 + \left( t - 2 \right) > \left( t - 1 \right) + \left( t - 2 \right) - t - 2.
\]
which is a contradiction. Hence \( G' \) is left-compressed. \( \square \)

**Proof of Lemma 2.2** Since \( G \) contains the clique \([t − 2]^{(3)}\), it is true for \( k \leq t - 1 \). Assume that \( k > t \).

Since \( G \) is left-compressed, \( 1(k − 1) − k \in E \). Let \( b := \text{max}\{i : (k − 1)i \in E\} \). Since \( E \) is left-compressed, \( E_i = \{1, \ldots, i − 1, i + 1, \ldots, k\}^{[2]} \), for \( 1 \leq i \leq b \), and \( E_i \cup E_j = \emptyset \) for \( 1 \leq i < j \leq b \). Hence, by Remark 2.2(a), we have \( x_1 = x_2 = \cdots = x_b \).

We define a new feasible weighting \( y \) for \( G \) as follows. Let \( y_i := x_i \) for \( i \neq k − 1, k \), \( y_{k−1} := x_{k−1} + x_k \), and \( y_k := 0 \).

By Lemma 2.2(a), \( \lambda(E_{k−1}, x) = \lambda(E_k, x) \), so
\[
\lambda(G, y) − \lambda(G, x) = x_k(\lambda(E_k, x) − x_k \lambda(E_{k−1}, x)) − x_k − 1 \lambda(E_{k−1}, x) = x_k \lambda(E_k, x) − x_k \lambda(E_{k−1}, x) − x_k \lambda(E_{k−1}, x) = −b x_k x_k^2.
\]
(17)

Since \( y_k = 0 \) we may remove all edges containing \( k \) from \( E \) to form a new 3-graph \( \overline{G} := ([k], \overline{E}) \) with \( |\overline{E}| := |E| - |E_k| \) and \( \lambda(\overline{G}, y) = \lambda(G, y) \). We will show that if Lemma 4.2 fails to hold then there exists a set of edges \( F \subset [k - 1]^{(3)} \setminus E \) satisfying
\[
\lambda(F, y) > bx_1 x_k^2,
\]
(18)
and
\[
|F| \leq |E_k|.
\]
(19)

Then, using (17), (18), and (19), the 3-graph \( G' := ([k], E') \), where \( E' := \overline{E} \cup F \), satisfies \( |E'| \leq |E| \) and
\[
\lambda(G', y) = \lambda(\overline{G}, y) + \lambda(F, y) > \lambda(G', y) + bx_1 x_k^2 = \lambda(G, x).
\]

Hence \( \lambda(G') > \lambda(G) \). Note that \( G' \) still contains the clique \([t − 2]^{(3)}\) since \( G' \) contains all edges in \( E \cap [k − 1]^{(3)} \supset [t − 2]^{(3)} \). If \( G' \) does not contain a clique of size \( t − 1 \), it contradicts to \( \lambda(G) = \lambda_{m,t−2}^3 \). If \( G' \)
contains a clique of size \( t - 1 \), then by Theorem 2.2, \( \lambda(G') = \lambda((t - 1)^{(3)}) \) and consequently \( \lambda(G') < \lambda((t - 1)^{(3)}) \).

We must now construct the set of edges \( F \) satisfying (18) and (19). Applying Remark 2.2(a) by taking \( i = 1, j = k - 1 \), we have

\[
x_1 = x_{k-1} + \frac{\lambda(E_{1(k-1)}, x)}{\lambda(E_{1(k-1)}, x)}.
\]

Let \( C := [k-2]^{(2)} \setminus E_{k-1} \). Then \( \lambda(E_{1(k-1)}, x) = x_k \sum_{i=1}^{k-2} x_i + \lambda(C, x) \). Applying this and multiplying \( b x_1^2 \) to the above equation (note that \( \lambda(E_{1(k-1)}, x) = \sum_{i=2, i \neq k-1} x_i \)), we have

\[
b x_1 x_1^2 = b x_{k-1} x_k^2 + \frac{b x_1^3 \sum_{i=2, i \neq k-1} x_i}{\sum_{i=2, i \neq k-1} x_i} - 1 \geq \frac{b x_1^2 \lambda(C, x)}{k - 2}.
\]

Since \( x_1 \geq x_2 \geq \cdots \geq x_k \), then

\[
b x_1 x_1^2 \leq b x_{k-1} x_k^2 (1 + \frac{k - (b + 2)}{k - 3}) + \frac{b x_2 \lambda(C, x)}{k - 2}.
\]

Define \( \alpha := \frac{b(C)}{k-2} \) and \( \beta := \frac{b(1 + \frac{k-(b+2)}{k-3})}{k-2} \). Note that \( [b(1 + \frac{k-(b+2)}{k-3})] \leq k - 2 \) since \( b \leq k - 2 \). So \( \beta \leq k - 2 \).

Let the set \( F_1 \subset [k-1]^{(3)} \setminus E \) consist of the \( \alpha \) heaviest edges in \( [k-1]^{(3)} \setminus E \) containing the vertex \( k - 1 \) (note that \( |[k-2]^{(2)} \setminus E_{k-1}| = |C| \geq \alpha \)). Recalling that \( y_{k-1} = x_{k-1} + x_k \) we have

\[
\lambda(F_1, y) \geq bx_1 x_k^2 \geq x_{k-1} x_k^2 (\alpha - \beta).
\]

(20)

We now distinguish two cases.

Case 1. \( \alpha > \beta \).

In this case \( \lambda(F_1, y) - bx_{k-1} x_k^2 > 0 \) so defining \( F := F_1 \) satisfies (18). We need to check that \( |F| \leq |E_k| \). Since \( E \) is left-compressed, then \( [b]^{(2)} \cup \{1, \ldots, b\} \times \{b+1, \ldots, k-1\} \subset E_k \). Hence

\[
|E_k| \geq \frac{b[|b| + 2(k - 1 - b)]}{2} \geq \frac{b(k-1)}{2}
\]

(22)

since \( b \leq k - 2 \). Recall that \( |F| = \alpha = \frac{b(C)}{k-2} \). Since \( C \subset [k-2]^{(2)} \), we have \( |C| \leq \binom{k-2}{2} \). So using (20) we obtain

\[
|F| \leq \frac{b(k-3)}{2} \leq \frac{b(k-1)}{2} \leq |E_k|.
\]

So both (18) and (19) are satisfied.

Case 2. \( \alpha \leq \beta \).

Suppose that Lemma 4.2 fails to hold. So \( |[k-1]^{(3)} \setminus E| \geq k - 1 \geq \beta + 1 \) (recall that \( \beta \leq k - 2 \)). Let \( F_2 \) consist of any \( \beta + 1 - \alpha \) edges in \( [k-1]^{(3)} \setminus (E \cup F_1) \) and define \( F := F_1 \cup F_2 \). Then since \( \lambda(F_2, y) \geq (\beta + 1 - \alpha)x_{k-1}^2 \) and using (21),

\[
\lambda(F, y) - bx_{k-1} x_k^2 = \lambda(F_1, y) - bx_{k-1} x_k^2 + \lambda(F_2, y) \geq (\beta + 1 - \alpha)x_{k-1}^2 - x_{k-1} x_k^2 (\beta - \alpha) > 0.
\]

(21)

So (18) is satisfied. What remains is to check that \( |F| \leq |E_k| \). In fact,

\[
|F| = \beta + 1 \leq k - 1 \leq \frac{b(k-1)}{2} \leq |E_k|
\]
when \( b \geq 2 \). If \( b = 1 \), then applying (21),

\[
|F| = \beta + 1 = 3 \leq k - 2 = \frac{b(b - 1 + 2(k - 1 - b))}{2} \leq |E_k|
\]

since \( k \geq t \geq 5 \). \( \square \)

**Proof of Lemma 4.3** Let \( b := \max \{ i : i(t - 1)t \in E \} \). Since \( E \) is left-compressed, then \( E_i = \{ 1, \ldots , i - 1, i + 1, \ldots , t \} \) for \( 1 \leq i \leq b \) and \( E_i, j = \emptyset \) for \( 1 \leq i < j \leq b \).

Hence, by Remark 2.2(a), we have \( x_1 = x_2 = \cdots = x_b \). Consider a new weighting for \( G \), \( z := (z_1, z_2, \ldots , z_t) \) given by \( z_i := x_i \) for \( i \neq t - 1, t \), \( z_{t - 1} := 0 \) and \( z_t := x_{t - 1} + x_t \). By Lemma 2.2(a), \( \lambda(G_{t - 1}, \lambda) = \lambda(G, \lambda) \), so

\[
\lambda(G, z) - \lambda(G, \lambda) = x_{t - 1}(\lambda(E_t, \lambda) - \lambda(E_{t - 1}, \lambda)) - x_t^2 \sum_{i = 1}^b x_i = -bx_1x_{t - 1}.
\]

\[
(23)
\]

Since \( z_{t - 1} = 0 \) we may remove all edges containing \( t - 1 \) from \( E \) to form a new 3-graph \( \overline{G} := ([t], \overline{E}) \) with \( |\overline{E}| := |E| - |E_{t - 1}| \) and \( \lambda(\overline{G}, \lambda) = \lambda(G, \lambda) \).

If \(|t - 2| \overline{E}_t > b\), we will show that there exists a set of edges \( F \subset \{ 1, \ldots , t - 2, t \} \setminus E \) satisfying

\[
\lambda(F, z) > bx_1x_{t - 1}.
\]

Then using (23) and (24), the 3-graph \( G' := ([t], E') \), where \( E' := \overline{E} \cup E \), satisfies \( \lambda(G', z) > \lambda(G) \). Since \( z \) has only \( t - 1 \) positive weights, then \( \lambda(G', z) \leq \lambda([t - 1]^{(3)}) \), and consequently

\[
\lambda(G) < \lambda([t - 1]^{(3)}).
\]

We must now construct the set of edges \( F \). Since \( G \) is left-compressed, applying Remark 2.2(a) by taking \( i = 1 \), \( j = t \), we get

\[
x_1 = x_t = \frac{\lambda(E_1, \lambda)}{\lambda(E_t, \lambda)}.
\]

Let \( D := [t - 2]^{(2)} \setminus E_t \). Then \( \lambda(E_1, \lambda) = x_{t - 1} \sum_{i = b + 1}^{t - 2} x_i + \lambda(D, \lambda) \). Applying this and multiplying \( bx_1x_{t - 1} \) to the above equation (note that \( \lambda(E_t, \lambda) = \sum_{i = t - 2}^{t - 1} x_i \), we have

\[
bx_1x_{t - 1} = bx_1x_{t - 1}^2 + bx_1x_{t - 1}^2 \sum_{i = b + 1}^{t - 2} x_i + bx_1x_{t - 1}^2 + bx_1x_{t - 1} \lambda(D, \lambda).
\]

(25)

Let \( c := \frac{\sum_{i = b + 1}^{t - 2} x_i}{\sum_{i = t - 2}^{t - 1} x_i} \) and \( d := \frac{bx_1x_{t - 1}}{\sum_{i = t - 2}^{t - 1} x_i} \). Then

\[
bx_1x_{t - 1} = bx_1x_{t - 1}^2 + bx_1x_{t - 1}^2 + dx_1x_{t - 1} \lambda(D, \lambda).
\]

Let \( F \) consist of those edges in \( \{ 1, \ldots , t - 2, t \}^{(3)} \setminus E \) containing the vertex \( t \). Then

\[
\lambda(F, z) = (x_{t - 1} + x_t) \lambda(D, \lambda).
\]

(26)

Since \(|t - 2| \overline{E}_t > b\), then

\[
\lambda(D, \lambda) > bx_1x_{t - 1}.
\]

(27)
Applying equations (25), (26), and (27), we get
\[
\lambda(F, z) - bx_1x_{j-1}^2 = \lambda(D, x) - bx_1x_{j-1}^2 - bxc_j^2d - dx_j \lambda(D, x) \\
= [(1 - d)x_{j-1} + x_j] \lambda(D, x) - bx_1x_{j-1}^2 - bxc_j^2d \\
> [(1 - d)x_{j-1} + x_j]bx_1x_{j-1} - bx_1x_{j-1}^2 - bxc_j^2d \\
= bx_1x_{j-1}(1 - d - c) \geq 0,
\]
since
\[
c + d = \frac{\sum_{l=i+1}^{j-2} x_l + bx_{j-1}}{\sum_{l=1}^{j-2} x_l} \leq 1.
\]

Let \( G' := ([t], \bar{E} \cup F) \), then \( \lambda(G', z) = \lambda(G, x) + \lambda(F, z) = \lambda(G, x) - bx_1x_{j-1}^2 + \lambda(F, z) > \lambda(G, x) \). On the other hand, since \( z \) has only \( t - 1 \) positive weights, then \( \lambda(G', z) < \lambda([t-1]^{(3)}) \).\)

**Proof of Lemma 4.2** Let \( m \) and \( t \) be positive integers satisfying \( \binom{t-1}{r-1} \leq m \leq \binom{t}{r} - 1 \). Let \( G := (V, E) \) be an \( r \)-graph on vertex set \( V := [n] \) with \( m \) edges containing a clique of size \( t - 2 \) such that \( \lambda(G) = \lambda'_m(3) \). We call such a \( G \) an extremal \( r \)-graph for \( m \) and \( t - 2 \). Let \( G = (V, E) \) be an optimal weighting of \( G \). We can assume that \( x_i \geq x_j \) when \( i < j \) since otherwise we can just relabel the vertices of \( G \) and obtain another extremal \( r \)-graph for \( m \) and \( t - 2 \) with an optimal \( \lambda := (x_1, x_2, \ldots, x_n) \) satisfying \( x_i \geq x_j \) when \( i < j \). If \( G \) is not left-compressed, then there is an edge whose ancestor is not an edge. Replace all those edges by its available ancestor with the highest hierarchy, then we get a left-compressed \( r \)-graph \( G' \) which contains the clique \([t-2]^{(r)} \) and \( \lambda(G', z) \geq \lambda(G, x) \).\)

**Proof of Lemma 5.3**\( \lambda_e := \max\{i : i(t - 1) \in E\} \). Since \( E \) is left-compressed, then \( E_t = \{1, \ldots, i - 1, i + 1, \ldots, t\} \), for \( 1 \leq i \leq b \), and \( E_{\lambda(i)} = \emptyset \) for \( 1 \leq i < j \leq b \).

Hence, by Remark 2.2(a), we have \( x_1 = x_2 = \cdots = x_b \). We define a new feasible weighting \( y \) for \( G \) as follows.

Let \( y_i := x_i \) for \( i \neq t - 1, t, y_{t-1} := x_{t-1} + x_t \) and \( y_t := 0 \).

By Lemma 5.2(a), \( \lambda(E_{t-1}, x) = \lambda(E_t, x) \), so
\[
\lambda(G, y) - \lambda(G, x) = x_i(\lambda(E_{t-1}, x) - x_i \lambda(E_{(i-1)}, x)) \\
\quad - x_i(\lambda(E_t, x) - x_i \lambda(E_{(i-1)}, x)) - x_{t-1} x_t \lambda(E_{(t-1)}, y) \\
= x_i(\lambda(E_{t-1}, x) - \lambda(E_t, x)) - x_i^2 \sum_{i=1}^{b} x_i \\
= -bx_1x_{j-1}^2.
\] (28)

Since \( y_t = 0 \) we may remove all edges containing \( t \) from \( E \) to form a new 3-graph \( \bar{G} := ([t], \bar{E}) \) with \( |\bar{E}| := |E| - E_t \) and \( \lambda(\bar{G}, y) = \lambda(G, y) \).

We will show that if \( |\[t-1]^{(3)} \setminus E| \geq t - 2 \) then there exists a set of edges \( F \subset \[t-1]^{(3)} \setminus E \) satisfying
\[
\lambda(F, y) \geq bx_1x_{j-1}^2.
\] (29)

Then, using (28), (29), the 3-graph \( G' := ([t], \bar{E} \cup F) \), where \( E' := \bar{E} \cup F \), satisfies
\[
\lambda(G', y) = \lambda(\bar{G}, y) + \lambda(F, y) \\
\geq \lambda(G, y) + bx_1x_{j-1}^2 \\
= \lambda(G, x).
\]

Since \( y \) has only \( t - 1 \) positive weights, then \( \lambda(G') \leq \lambda([t-1]^{(3)}) \), and consequently \( \lambda(G) \leq \lambda([t-1]^{(3)}) \).
We must now construct the set of edges $F$ satisfying (29). Applying Remark 2.2(a) by taking $i = 1, j = t - 1$, we have
\[ x_1 = x_{t-1} + \frac{\lambda(E_{t-1})}{\lambda(E_{t-1})} x_1. \]

Let $C := [t-2] \setminus E_{t-1}$. Then $\lambda(E_{t-1}) = x_t \sum_{i=2}^{t-1} x_i + \lambda(C, x)$. Applying this and multiplying $bx_1^2$ to the above equation (note that $\lambda(E_{t-1}) = \sum_{i=2, j \neq t-1} x_i$), we have
\[ bx_1^2 = bx_{t-1}x_1^2 + \frac{bx_1^2 \sum_{i=2}^{t-2} x_i + bx_1^2 \lambda(C, x)}{\sum_{i=2, j \neq t-1} x_i}. \]

Since $x_1 \geq x_2 \geq \ldots \geq x_r$, then
\[ bx_1x_2^2 \leq bx_{t-1}x_1^2 (1 + \frac{t - (b + 2)}{t - 2}) + bx_1 \lambda(C, x) \frac{t - 2}{t - 2}. \]

Define $\alpha := \frac{\lambda(C)}{t-2}$ and $\beta := \frac{\lambda(1 + (b+2))}{t-2}$. Note that since $b \leq t - 2$, so $\beta \leq t - 2$. Let the set $F_1 \subseteq [t-1] \setminus E$ consist of the $\alpha$ heaviest edges in $[t-1] \setminus E$ containing the vertex $t - 1$ (note that $|t-2| \setminus E_{t-1} = |C| \geq \alpha$). Recalling that $y_{t-1} = x_{t-1} + x_t$ we have
\[ \lambda(F_1, y) \geq \frac{bx_1 \lambda(C, x)}{t-2} + \alpha x_{t-1}x_1^2. \]

So using (30)
\[ \lambda(F_1, y) - bx_1x_2^2 \geq x_{t-1}x_1^2 (\alpha - \beta). \]

If $\alpha > \beta$, then $\lambda(F_1, y) - bx_1x_2^2 > 0$. So defining $F := F_1$ satisfies (29).

Assume $\alpha \leq \beta$. Suppose that $|t-1| \setminus E \geq t - 2$. So $|t-1| \setminus E \geq t - 2 \geq \beta$ (recall that $\beta \leq t - 2$). Let $F_2$ consist of any $\beta - \alpha$ edges in $[t-1] \setminus (E \cup F_1)$ and define $F := F_1 \cup F_2$. Then since $\lambda(F_2, y) \geq (\beta - \alpha)x_{t-1}^3$ and using (30)
\[ \lambda(F, y) - bx_1x_2^2 = \lambda(F_1, y) - bx_{t-1}x_1^2 + \lambda(F_2, y) \geq (\beta - \alpha)x_{t-1}^3 - x_{t-1}x_1^2 (\beta - \alpha) \geq 0. \]

This proves Lemma 5.3.

\[ \square \]

7 Conclusions

At this moment, we are not able to extend the arguments in this paper to verify Conjectures 2.1, 2.2 and 2.3 for more general cases. When $r \geq 4$, the computation is more complex. If there is some technique to overcome this difficulty, then the idea used in proving Theorem 3.1 can be used to improve our results much further.

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