REMARKS ON ONE-COMPONENT INNER FUNCTIONS

ATTE REIJONEN

Abstract. A one-component inner function Θ is an inner function whose level set
\[ \Omega_\Theta(\varepsilon) = \{ z \in \mathbb{D} : |\Theta(z)| < \varepsilon \} \]
is connected for some \( \varepsilon \in (0, 1) \). We give a sufficient condition for a Blaschke product with
zeros in a Stolz domain to be a one-component inner function. Moreover, a sufficient condition
is obtained in the case of atomic singular inner functions.

We study also derivatives of one-component inner functions in the Hardy and Bergman
spaces. For instance, it is shown that, for \( 0 < p < \infty \), the derivative of a one-component inner
function Θ is a member of the Hardy space \( H^p \) if and only if \( \Theta^2 \) belongs to the Bergman
space \( A^p_{p-1} \), or equivalently \( \Theta^2 \in A^p_{p-1} \).

1. Examples of one-component inner functions

A bounded and analytic function is an inner function if it has unimodular radial limits
almost everywhere on the boundary \( \Gamma \) of the open unit disc \( \mathbb{D} \) of the complex plane \( \mathbb{C} \).
In this note, we study so-called one-component inner functions [12], which are inner functions Θ
whose level set
\[ \Omega_\Theta(\varepsilon) = \{ z \in \mathbb{D} : |\Theta(z)| < \varepsilon \} \]
is connected for some \( \varepsilon \in (0, 1) \). In particular, Blaschke products in this class are of interest.
For a given sequence \( \{z_n\} \subset \mathbb{D}\setminus\{0\} \) satisfying \( \sum_n (1 - |z_n|) < \infty \), the Blaschke product with
zeros \( \{z_n\} \) is defined by
\[ B(z) = \prod_n \frac{|z_n|}{z_n} \frac{z_n - z}{1 - \overline{z_n}z}, \quad z \in \mathbb{D}. \]
Here each zero \( z_n \) is repeated according to its multiplicity. In addition, we assume that \( \{z_n\} \)
is ordered by non-decreasing moduli.

Recently several authors have studied one-component inner functions in the context of
operator theory. See for instance [5] [7] [8]. In addition, A. B. Aleksandrov’s paper [4] is worth
mentioning. These references do not offer any concrete examples of infinite one-component
Blaschke products. In recent paper [11] by J. Cima and R. Mortini, one can find some
examples. However, all one-component Blaschke products constructed in [11] have some heavy
restrictions. Roughly speaking, zeros of all of them are at least uniformly separated. Recall
that \( \{z_n\} \subset \mathbb{D} \) is called uniformly separated if
\[ \inf_{n \in \mathbb{N}} \prod_{k \neq n} \frac{|z_k - z_n|}{1 - \overline{z_k}z_n} > 0. \]
As a concrete example, we mention that the Blaschke product with zeros \( z_n = 1 - 2^{-n} \) for
\( n \in \mathbb{N} \) is a one-component inner function [11]. In addition, it is a well-known fact that every
finite Blaschke product is a one-component inner function.

For \( \gamma \geq 1, \xi \in \mathbb{T} \) and \( C > 0 \), we define
\[ R(\gamma, \xi, C) = \{ z \in \mathbb{D} : |1 - \xi z|^{\gamma} \leq C(1 - |z|) \}. \]
The region $R(1, \xi, C)$ is a Stolz domain with vertex at $\xi$. Note that in the case $\gamma = 1$ we have to assume $C > 1$. For $\gamma > 1$, $R(\gamma, \xi, C)$ is a tangential approaching region in $\mathbb{D}$, which touches $\mathbb{T}$ at $\xi$. Denote the family of all Blaschke products whose zeros lie in some $R(\gamma, C, \xi)$ with a fixed $\gamma$ by $\mathcal{R}_\gamma$. References related to $\mathcal{R}_\gamma$ are for instance [3, 9, 18]. With these preparations we are ready to state our first main result.

**Theorem 1.** Let $B$ be a member of $\mathcal{R}_1$ with zeros $\{z_n\}_{n=1}^\infty$. If

$$\liminf_{n \to \infty} \frac{\sum_{|z_j| > |z_n|} (1 - |z_j|)}{1 - |z_n|} > 0,$$

(1.1)

then $B$ is a one-component inner function.

As a consequence of Theorem 1 we obtain the affirmative answer to the following question posed in [11]: Is the Blaschke product $B$ with zeros $z_n = 1 - n^{-2}$ for $n \in \mathbb{N}$ a one-component inner function? Some other examples of one-component inner functions are listed below. All of these examples can be verified by using the fact that condition (1.1) is valid if $\{z_n\}$ is ordered by strictly increasing moduli and

$$\liminf_{n \to \infty} \frac{1 - |z_{n+1}|}{1 - |z_n|} > 0.$$

**Example 2.** Let $1 < \alpha < \infty$ and $B$ be a Blaschke product with zeros

(a) $z_n = 1 - n^{-\alpha}$ for $n \in \mathbb{N}$, or

(b) $z_n = 1 - \frac{1}{n(\log n)^\alpha}$ for $n \in \mathbb{N}\setminus\{1\}$, or

(c) $z_n = 1 - \alpha^{-n}$ for $n \in \mathbb{N}$.

Then $B$ is a one-component inner function.

A Blaschke product $B$ is said to be thin if its zeros $\{z_n\}_{n=1}^\infty$ satisfy

$$\lim_{n \to \infty} (1 - |z_n|^2) |B'(z_n)| = 1.$$

We interpret that finite Blaschke products are not thin. By [11, Corollary 21], any thin Blaschke product is not a one-component inner function. Using this fact and [10, Proposition 4.3(i)], we can give an example which shows that condition (1.1) in Theorem 1 is essential.

**Example 3.** Let $B$ be the Blaschke product with zeros $\{w_n\}_{n=1}^\infty$ ordered by strictly increasing moduli and satisfying

$$\frac{1 - |w_{n+1}|}{1 - |w_n|} \to 0, \quad n \to \infty.$$

Then, by [10, Proposition 4.3(i)], $B$ is a thin Blaschke product (with uniformly separated zeros). Consequently, for instance, the Blaschke product with zeros $z_n = 1 - 2^{-2^n}$ for $n \in \mathbb{N}$ is not a one-component inner function. Note that zeros $\{z_n\}$ lie in $R(1,1,C)$ for every $C > 1$ but they do not satisfy (1.1).

Let us recall a classical result of O. Frostman [15]: The Blaschke product $B$ with zeros $\{z_n\}$ has a unimodular radial limit at $\xi \in \mathbb{T}$ if and only if

$$\sum_n \frac{1 - |z_n|}{|\xi - z_n|} < \infty.$$

(1.2)

A Blaschke product is called a Frostman Blaschke product if it has a unimodular radial limit at every point on $\mathbb{T}$. It is a well-known fact that an infinite Frostman Blaschke product cannot be a one-component inner function; see for instance [3, Theorem 1.11] or Theorem 3 in Section 3. Using this fact, we show that any $\mathcal{R}_\gamma$ with $\gamma > 1$ contains a member which is not a one-component inner function but its zeros $\{z_n\}$ satisfy (1.1). This means that the hypothesis $B \in \mathcal{R}_1$ in Theorem 1 is essential.
Example 4. Fix $\gamma > 1$ and choose $\alpha > 1$ such that $\alpha > \frac{1}{\gamma - 1}$. Let $\{z_n\}$ be such that
\[ |z_n| = 1 - n^{-\alpha} \quad \text{and} \quad |1 - z_n| = n^{-a/\gamma}, \quad n \in \mathbb{N}. \]
Since the sequence $\{z_n\}$ is a subset of $R(\gamma, 1, 1)$, all points of $\{z_n\}$ lie in $\mathbb{D}$. Moreover, it is clear that $\{z_n\}$ satisfies the Blaschke condition $\sum_n (1 - |z_n|) < \infty$ and (1.1) in Theorem 1. Hence the Blaschke product with zeros $\{z_n\}$ is well-defined. Furthermore,
\[ \sum_{n=1}^{\infty} \frac{1 - |z_n|}{|1 - z_n|} = \sum_{n=1}^{\infty} n^{a/\gamma - \alpha} < \infty; \]
and thus, the Blaschke product $B$ has a unimodular radial limit at 1 by Frostman’s result. Since condition (1.2) is trivially valid for every $\xi \in \mathbb{T}\{1\}$, $B$ is an infinite Frostman Blaschke product. Consequently, it is not a one-component inner function.

Recall that a singular inner function takes the form
\[ S_{\sigma}(z) = \exp \left( \int_{\mathbb{T}} \frac{z + \xi}{z - \xi} d\sigma(\xi) \right), \quad z \in \mathbb{D}, \]
where $\sigma$ is a positive measure on $\mathbb{T}$, singular with respect to the Lebesgue measure. If the measure $\sigma$ is atomic, then this definition reduces to the form
\[ S(z) = \exp \left( \sum_{n=1}^{\infty} \gamma_n \frac{z + e^{i\theta_n}}{z - e^{i\theta_n}} \right), \quad z \in \mathbb{D}, \]
where $\theta_n \in [0, 2\pi)$ are distinct points and $\gamma_n > 0$ satisfy $\sum_{n=1}^{\infty} \gamma_n < \infty$. These functions are known as atomic singular inner functions associated with $\{e^{i\theta_n}\}$ and $\{\gamma_n\}$.

An atomic singular inner function associated with a measure having only finitely many mass points is a one-component inner function; see [11 Corollary 17]. In the literature, one cannot find any example of a one-component singular inner function associated with a measure having infinitely many mass points. However, the following result gives a way to construct such functions.

Theorem 5. Let $S$ be the atomic singular inner function associated with $\{e^{i\theta_n}\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$. Moreover, assume that the following conditions are valid:

(i) $\theta_0 = 0, \{\theta_n\}_{n=1}^{\infty} \subset (0, 1)$ is strictly decreasing and $\lim_{n \to \infty} \theta_n = 0$.

(ii) There exists a constant $C = C(S) > 0$ such that $|\theta_n - \theta_{n+1}| \leq C \gamma_n^2$ for all sufficiently large $n \in \mathbb{N}$.

Then $S$ is a one-component inner function.

Next we give a concrete example of a one-component singular inner function. This example is a direct consequence of Theorem 4.

Example 6. Let $\theta_0 = 0, \theta_n = 2^{-n}, \gamma_0 = 1$ and $\gamma_n = n^{-2}$ for $n \in \mathbb{N}$. Then the singular inner function $S$ associated with $\{e^{i\theta_n}\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ is a one-component inner function.

The remainder of this note is organized as follows. In the next section, we study one-component inner functions whose derivatives belong to the Hardy or Bergman spaces. In particular, we give partial improvements for [11 Theorem 6.2] and [17 Theorem 3.10]. Sections 3 and 4 consist of the proofs of Theorems 2 and 3, respectively.

2. Derivatives of One-Component Inner Functions in Functions Spaces

We begin by fixing the notation. Let $H(\mathbb{D})$ be the space of all analytic functions in $\mathbb{D}$. For $0 < p < \infty$, the Hardy space $H^p$ consists of those $f \in H(\mathbb{D})$ such that
\[ \|f\|_{H^p} = \sup_{0 \leq r < 1} M_p(r, f) < \infty, \quad \text{where} \quad M_p^p(r, f) = \frac{1}{2\pi} \int_{0}^{2\pi} |f(re^{i\theta})|^p \ d\theta. \]
For $0 < p < \infty$ and $-1 < \alpha < \infty$, the Bergman space $A^p_\alpha$ consists of those $f \in H(D)$ such that
\[
\|f\|_{A^p_\alpha}^p = \int_D |f(z)|^p |1 - |z||^\alpha \, dA(z) < \infty,
\]
where $dA(z) = dx \, dy$ is the Lebesgue area measure on $D$. Write $f \leq g$ if there exists a constant $C > 0$ such that $f \leq Cg$, while $f \geq g$ is understood in an analogous manner. If $f \leq g$ and $f \geq g$, then the notation $f \asymp g$ is used.

By [20, Theorem 5] and [23, Lemma 1.4], we have
\[
\{ f : f' \in A^p_{p-1} \} \subset H^p, \quad 0 < p \leq 2.
\]
and
\[
H^p \subset \{ f : f' \in A^p_{p-1} \}, \quad 2 \leq p < \infty.
\]
It is clear that $\{ f : f' \in A^2_1 \} = H^2$, while otherwise the inclusions are strict. For instance, an example showing the strictness of inclusions (2.1) and (2.2) can be given by using gap series; see details in [6]. Nevertheless, we have the following result, which is essentially a consequence of [1, Theorem 6.2] and [17, Theorem 3.10].

**Theorem 7.** Let $\frac{1}{2} < p < \infty$ and $\Theta$ be an inner function. Then the following statements are equivalent:

(a) $\Theta' \in H^p$,
(b) $\Theta' \in A^p_{p-1}$,
(c) $\Theta'' \in A^p_{p-1}$.

Before the proof of Theorem 7, we note that $M_p(r, f^{(n)})$ and $M_p(r, D^n f)$ are comparable for every $f \in H(D)$ and $n \in \mathbb{N}$ [14]. Here $D^n f$ is the fractional derivative of order $n$. This fact is exploited when we apply some results in the literature.

**Proof.** The equivalence (a) $\iff$ (c) is a consequence of [14, Theorem 3.10]. For $\frac{1}{2} < p < 1$, the equivalence (a) $\iff$ (b) can be verified, for instance, using [1, Theorem 6.2] together with [22, Corollary 7]. It is a well-known fact that the only inner functions whose derivative belongs to $H^p$ for some $p \geq 1$ are finite Blaschke products. Using this fact together with [14, Theorem 7(c)] and the equivalence (a) $\iff$ (c), it is easy to deduce that an inner function $\Theta$ is a finite Blaschke product if it satisfies any of conditions (a)–(c) for some $p \geq 1$. In addition, it is clear that every finite Blaschke product $\Theta$ satisfies conditions (a)–(c) for all $p > 0$. Finally the assertion follows by combining the above-mentioned facts. \hfill $\square$

It is an open question whether the statement of Theorem 7 is valid also for $0 < p \leq \frac{1}{2}$. The next result shows that the answer is affirmative if $\Theta$ is a one-component inner function. This result is significant because we have only a limited amount of information about inner functions whose derivative belongs to $H^p$ or $A^p_\alpha$ for some $p \leq \frac{1}{2}$ and $\alpha \leq p - 1$.

**Theorem 8.** Let $0 < p < \infty$ and $\Theta$ be a one-component inner function. Then conditions (a)–(c) in Theorem 7 are equivalent.

Let us recall [1, Theorem 1.9], which consists of a strengthened Schwarz-Pick lemma for one-component inner functions. This result plays a key role in the proof of Theorem A.

**Theorem A.** Let $n \in \mathbb{N}$ and $\Theta$ be a one-component inner function. Then there exists $C = C(n, \Theta) > 0$ such that
\[
|\Theta^{(n)}(z)| \leq C \left( \frac{1 - |\Theta(z)|}{1 - |z|} \right)^n
\]
for all $z \in D$.

For the proof of Theorem A, we need also a generalization of [1, Theorem 6.1]. Before it we recall that the spectrum $\rho(\Theta)$ of an inner function $\Theta$ is the set of all point on $T$ in which $\Theta$ does not have an analytic continuation. It is a well-known fact that the spectrum of a Blaschke product consists of the accumulation points of zeros. By [11, Chapter 2, Theorem 6.2], the
Proof of Theorem 8.

□

the first assertion, Hardy’s convexity and the mean convergence theorem [13], the proof is choice □

statements are equivalent:

is worth noting that \( \rho(\Theta) \) has a Lebesgue measure zero if \( \Theta \) is a one-component inner function. This is due to [4] Theorem 1.11] or Theorem 13 in Section 5.

Lemma 9. Let \( 0 < p < 1 \), \( -1 < \alpha < \infty \) and \( \Theta \) be an inner function. Then there exists \( C = C(p, \alpha) > 0 \) such that

\[
\int_0^1 |\Theta'(re^{i\theta})|^{p+\alpha+1} (1-r)^\alpha \, dr \leq C |\Theta'(e^{i\theta})|^p, \quad \theta \in \mathbb{T} \setminus \rho(\Theta).
\]

In particular, \( \|\Theta\|_{A_{p+\alpha+1}}^{p+\alpha+1} \leq 2\pi C \|\Theta\|_{H^p}^p \).

Proof. Let \( \theta \in \mathbb{T} \setminus \rho(\Theta) \). By [1] Lemma 6.1, we know that \( |\Theta'(re^{i\theta})| \leq 4|\Theta'(e^{i\theta})| \) for all \( r \in [0,1) \). Using this fact together with the Schwarz-Pick lemma, we obtain

\[
\int_0^1 |\Theta'(re^{i\theta})|^{p+\alpha+1} (1-r)^\alpha \, dr \leq \int_0^x (1-r)^{-p-1} \, dr + 4 \int_x^1 (1-r)^\alpha \, dr
\]

\[
\leq (1-x)^{-p-1} + (1-x)^{\alpha+1} |\Theta'(e^{i\theta})|^{p+\alpha+1}
\]

for every \( x \in [0,1) \). Now it suffices to show that

\[(1-x)^{-p+1} + (1-x)^{\alpha+1} |\Theta'(e^{i\theta})|^{p+\alpha+1} \leq |\Theta'(e^{i\theta})|^p \quad (2.3)
\]

for some \( x \). If \( |\Theta'(e^{i\theta})| \leq 1 \), then this true for \( x = 0 \). In the case where \( |\Theta'(e^{i\theta})| > 1 \), the choice \( x = 1 - 1/|\Theta'(e^{i\theta})| \) implies (2.3). Since the last assertion is a direct consequence of the first assertion, Hardy’s convexity and the mean convergence theorem [13], the proof is complete.

Now we are ready to prove Theorem 8.

Proof of Theorem 8. By Theorem 5 we may assume \( 0 < p < 1 \) (or even \( p = 1/2 \)). Using Theorem 10 with \( n = 2 \), [2] Theorem 6] and Lemma 9 with \( \alpha = p - 1 \), we obtain

\[
\|\Theta\|^p_{A_{p-1}} \leq \int_D \left( \frac{1 - |\Theta(z)|}{1 - |z|} \right)^{2p} (1 - |z|)^{-p} \, dA(z) = \|\Theta\|^p_{H^p} \leq \|\Theta\|^p_{H^p}.
\]

The assertion follows from (2.1) and (2.4).

It is a well-known fact that, for \( 0 < p < \infty \) and \(-1 < \alpha < \infty \), the Bergman space \( A^p_{\alpha} \) coincides with \( \{ f : f' \in A_{\alpha+p}^p \} \). Using this result, it is easy to generalize condition (c) in Theorem 8 to the form \( \Theta(n) \in A^p_{\alpha(n)-1} \) for any/every \( n \in \mathbb{N} \setminus \{1\} \). For \( 0 < p < 1 \), we can show this also by modifying the proof of Theorem 8 and as a substitute of this process we obtain the following result.

Corollary 10. Let \( 0 < p < \infty \) and \( \Theta \) be a one-component inner function. Then the following statements are equivalent:

(a) \( \Theta' \in H^p \),
(b) \( \Theta' \in A^p_{\alpha+1} \) for some \( \alpha \in (-1, \infty) \),
(c) \( \Theta' \in A^p_{\alpha+1} \) for every \( \alpha \in (-1, \infty) \).

Proof. By the proof of Theorem 8 we know that, for \( 1 \leq p < \infty \), \( \Theta \) satisfies any/all of conditions (a)–(c) if and only if it is a finite Blaschke product. Hence we may assume \( 0 < p < 1 \). Moreover, let \(-1 < \alpha < \infty \) and \( n \in \mathbb{N} \setminus \{1\} \). Then [14] Theorem 3, Theorem 10, Theorem 12, Theorem 6], the Schwarz-Pick lemma and Lemma 9 yield

\[
\|\Theta\|^p_{H^p} \leq \|\Theta(n)\|^p_{A^p_{\alpha(n)-1}} \leq \int_D \left( \frac{1 - |\Theta(z)|}{1 - |z|} \right)^{np} (1 - |z|)^{p(n-1)-1} \, dA(z)
\]

\[
\leq \|\Theta\|^p_{A^p_{\alpha(n)-1}} \leq \|\Theta\|^p_{H^p} \leq \|\Theta\|^p_{H^p}
\]

when \( p + \alpha + 1 \leq np \). Since we may choose \( n = n(p, \alpha) \) such that \( n \geq (p + \alpha + 1)/p \), the assertion follows from (2.4). □
It is worth noting that the statement of Corollary 10 for $p > \frac{1}{2}$ is valid also if $\Theta$ is an arbitrary inner function. This is due to [11 Theorem 6.2] and [22 Corollary 7].

As a consequence of Theorem 8, we can also give sufficient conditions for higher order derivatives of one-component inner functions to be in the Hardy space $H^p$ or the Nevanlinna class $\mathcal{N}$. Recall that $f \in \mathcal{H}(\mathbb{D})$ belongs to the Nevanlinna class $\mathcal{N}$ if

$$\sup_{0 < r < 1} \int_{0}^{2\pi} \log^+ |f(re^{i\theta})| \, d\theta < \infty,$$

where $\log^+ 0 = 0$ and $\log^+ x = \max\{0, \log x\}$ for $0 < x < \infty$.

**Corollary 11.** Let $0 < p < \infty$, $n \in \mathbb{N}$ and $\Theta$ be a one-component inner function. Then the following statements are valid:

(a) If $\Theta' \in H^p$, then $\Theta^{(n)} \in H^{p/n}$.

(b) If $\Theta' \in \mathcal{N}$, then $\Theta^{(n)} \in \mathcal{N}$.

**Proof.** As a consequence of Theorem 11, we find $C = C(n, \Theta)$ such that

$$|\Theta^{(n)}(\xi)| \leq C|\Theta'(\xi)|^n, \quad \xi \in \mathbb{T} \setminus \rho(\Theta).$$

(2.6)

Since the spectrum $\rho(\Theta)$ has a Lebesgue measure zero, inequality (2.6), Hardy’s convexity and the mean convergence theorem yield

$$\|\Theta'|_{H^p}^p = \frac{1}{2\pi} \int_{0}^{2\pi} |\Theta'(e^{i\theta})|^p \, d\theta \geq \frac{1}{2\pi} \int_{0}^{2\pi} |\Theta^{(n)}(e^{i\theta})|^{p/n} \, d\theta = \|\Theta^{(n)}|_{H^{p/n}}^p.$$ 

Hence assertion (a) is proved. Since case (b) can be verified in a similar manner, the proof is complete. \qed

We close this section with two results regarding certain one-component singular inner functions.

**Corollary 12.** Let $0 < p < \infty$ and $S$ be the one-component atomic singular inner function associated with $\{e^{\theta_n}\}$ and $\{\gamma_n\} \in l^{1/2}$. Then $S$ satisfies any/all of conditions (a)–(c) in Theorem 7 if and only if $p < \frac{1}{2}$.

**Proof.** By [22 Theorem 3], for $\frac{1}{2} \leq p < \infty$, the derivative of $S$ belongs to $A^{2p-1}$ if and only if $p < \frac{1}{2}$. Since $H^{P_1} \subset H^{P_2}$ for $0 < p_2 \leq p_1 < \infty$, the assertion follows from this result and Theorem 7. \qed

The following result shows that Corollary 11(a) is sharp.

**Corollary 13.** Let $0 < p < \infty$, $m \in \mathbb{N}$ and $S$ be the one-component atomic singular inner function associated with $\{e^{i\theta_n}\}$ and $\{\gamma_n\} \in l^{1/2}$. Moreover, assume that there exist an index $j = j(S)$ and $\varepsilon = \varepsilon(j) > 0$ such that $|\theta_j - \theta_n| > \varepsilon$ for all $n \neq j$. Then $S^{(m)} \in H^p$ if and only if $p < \frac{1}{2m}$.

**Proof.** By Corollary 12, $S' \in H^{mp}$ if and only if $p < \frac{1}{2m}$. Consequently, Corollary 11(a) implies $S^{(m)} \in H^p$ for $p < \frac{1}{2m}$. Hence it suffices to show that $S^{(m)} \in H^p$ only if $p < \frac{1}{2m}$.

Fix $j = j(S)$ to be the smallest index such that $|\theta_j - \theta_n| > \varepsilon$ for all $n \neq j$ and some $\varepsilon = \varepsilon(j) > 0$. Let us represent $S$ in the form $S = S_1S_2$, where

$$S_1(z) = \exp \left( \sum_{k=0}^{m} \binom{m}{k} \left( \sum_{k=0}^{m-k} \binom{m-k}{k} \left( \gamma_j \frac{z - e^{i\theta_j}}{\bar{z} - e^{-i\theta_j}} \right)^{m-k} \binom{m-k}{k} \left( \gamma_j \frac{e^{i\theta_j}}{\bar{z} - e^{-i\theta_j}} \right)^{k} \right) \right), \quad z \in \mathbb{D},$$

and $S_2 = S/S_1$. Using this factorization, we obtain

$$|S^{(m)}(e^{i\theta})| = \sum_{k=0}^{m-k} \binom{m-k}{k} S_1^{m-k}(e^{i\theta}) S_2^k(e^{i\theta}) \leq |S_1^{(m)}(e^{i\theta}) S_2(e^{i\theta})| = |S^{(m)}(e^{i\theta})|.$$
when \( \theta \) (which is not \( \theta_j \)) is close enough to \( \theta_j \) depending on \( S \) and \( m \). Consequently, we find a sufficiently small \( \alpha = \alpha(p, S, m) > 0 \) such that

\[
\int_0^{2\pi} |S^{(m)}(e^{j\theta})|^p \, d\theta = \int_{\theta_j - \alpha}^{\theta_j + \alpha} |S^{(m)}(e^{j\theta})|^p \, d\theta \leq \int_0^{2\pi} |S^{(m)}(e^{j\theta})|^p \, d\theta,
\]

where the comparison constants depend only on \( p, S \) and \( m \). It follows that \( S^{(m)} \in H^p \) only if \( S^{(m)}_1 \in H^p \). Moreover, a simple modification of the main result of \[21\] shows that \( S^{(m)}_1 \in H^p \) if and only if \( p < \frac{1}{2m} \). Combining these facts, we deduce that \( S^{(m)} \in H^p \) (if and) only if \( p < \frac{1}{2m} \). This completes the proof.

\[\Box\]

3. Proof of Theorem 1

We begin by stating a modification of \[1\] Theorem 1.11, which is due to \[2\] p. 2915, Remark 2.

**Theorem B.** Let \( \Theta \) be an inner function. Then the following statements are equivalent:

(a) \( \Theta \) is a one component inner function.

(b) There exists a constant \( C = C(\Theta) > 0 \) such that

\[
|\Theta''(\zeta)| \leq C|\Theta'(\zeta)|^2, \quad \zeta \in \mathbb{T}\backslash \rho(\Theta),
\]

and

\[
\liminf_{r \to 1^-} |\Theta(r\xi)| < 1, \quad \xi \in \rho(\Theta).
\]

(c) There exists a constant \( C = C(\Theta) > 0 \) such that \[1.1\] holds for every \( \zeta \in \mathbb{T}\backslash \rho(\Theta) \), the Lebesgue measure of \( \rho(\Theta) \) is zero and \( \Theta' \) is not bounded on any arc \( \Gamma \subset \mathbb{T}\backslash \rho(\Theta) \) with \( \mathbb{T} \cap \rho(\Theta) \neq \emptyset \).

Now we are ready to prove Theorem 1.

**Proof of Theorem 1.** If \( B \) is an arbitrary Blaschke product with zeros \( \{z_n\} \), then

\[
\frac{B'(z)}{B(z)} = \sum_{n=1}^{\infty} \frac{|z_n|^2 - 1}{(1 - z_n \bar{z})(z_n - z)} \quad \text{and} \quad |B(z)| \leq \frac{|z_n - z|}{|1 - z_n \bar{z}|}.
\]

Using these estimates, one can easily verify

\[
|B''(z)| \leq \frac{|B'(z)|^2}{|B(z)|} + 2|B(z)|^{-1} \sum_{n=1}^{\infty} \frac{1 - |z_n|^2}{|1 - \bar{z}_n z|^2}, \quad z \in \mathbb{D}.
\]

In particular,

\[
|B''(\zeta)| \leq |B'(\zeta)|^2 + 2 \sum_{n=1}^{\infty} \frac{1 - |z_n|^2}{|1 - \bar{z}_n \zeta|^2} \leq |B'(\zeta)|^2 + \sum_{n=1}^{\infty} \frac{1 - |z_n|^2}{|z_n - \zeta|^2}
\]

for every \( \zeta \in \mathbb{T}\backslash \rho(B) \). Using \[3\] Theorem 2, we deduce that \[3.1\] (with \( \Theta = B \)) is valid if

\[
\sum_{n=1}^{\infty} \frac{1 - |z_n|^2}{|z_n - \zeta|^2} \lesssim \left( \sum_{n=1}^{\infty} \frac{1 - |z_n|^2}{|z_n|^2} \right)^2
\]

holds for every \( \zeta \in \mathbb{T}\backslash \rho(B) \).

Assume without loss of generality that zeros \( \{z_n\} \) of \( B \) lie in a Stolz domain \( R(1, 1, C) \), and remind that \( \{z_n\} \) is ordered by non-decreasing moduli. Then \( \rho(B) = 1 \), and the functions \( f \) and \( g \), defined by

\[
f(x) = \begin{cases} 
1, & 0 \leq x < 1, \\
\min \{ n \leq x (1 - |z_n|) \}, & 1 \leq x < \infty,
\end{cases}
\]

and

\[
g(\theta) = \inf \{ x : f(x) \leq \theta \}, \quad 0 < \theta \leq 1,
\]

are non-increasing. Since \( f(w) = 1 - |z_n| \) for \( n \in \mathbb{N} \) and \( n \leq w < n + 1 \), it is clear that \( g : [0, 1] \to \mathbb{N} \cup \{ 0 \} \), \( f(x) > \theta \) for \( x < g(\theta) \), and \( f(x) \leq \theta \) for \( x \geq g(\theta) \). Write \( \zeta = e^{i\theta} \), and
assume without loss of generality that \( \theta > 0 \) is close enough to zero. Using [3, Lemma 3] together with some standard estimates, we obtain

\[
\left( \sum_{n=1}^{\infty} \frac{1 - |z_n|^2}{|z_n - \zeta|^2} \right)^{1/2} \geq \left( \sum_{n=1}^{\infty} \frac{1 - |z_n|^2}{|z_n|^2 + \theta^2} \right)^{1/2} 
\]

Using estimates (3.4) and (3.5), it is easy to see that condition (3.3) is valid for

\[
\sum_{n=1}^{\infty} \frac{1 - |z_n|^2}{|z_n - \zeta|^2} \leq \left( \sum_{n=1}^{\infty} \frac{1 - |z_n|^2}{(1 - |z_n|^2 + \theta^2)} \right)^{1/2}
\]

\[
\leq \left( \sum_{n<\varrho} f(n)^{-2} + \theta^{-3} \sum_{n \geq \varrho} f(n) \right)^{1/2}
\]

\[
\leq \left( \sum_{n<\varrho} f(n)^{-2} + \theta^{-3/2} \left( \sum_{n \geq \varrho} f(n) \right) \right)^{1/2} 
\]

Applying hypothesis (1.1) and the above-mentioned properties of \( f \) and \( g \), we find \( C = C(B) > 0 \) such that

\[
\sum_{n \geq \varrho} f(n) \geq C f(\varrho) - 1 \geq C \varrho.
\]

It follows that

\[
\sum_{n=1}^{\infty} \frac{1 - |z_n|^2}{|z_n - \zeta|^2} \geq \sum_{n=1}^{\infty} \frac{1 - |z_n|^2}{(1 - |z_n|^2 + \theta^2)} 
\]

\[
\geq \frac{1}{2} \sum_{n<\varrho} f(n)^{-1} + \frac{\theta^{-2}}{2} \sum_{n \geq \varrho} f(n) 
\]

\[
\geq \frac{1}{2} \sum_{n<\varrho} f(n)^{-1} + \frac{\sqrt{C} \theta^{-3/2}}{2} \left( \sum_{n \geq \varrho} f(n) \right)^{1/2} 
\]

Using estimates (3.4) and (3.5), it is easy to see that condition (3.3) is valid for \( \zeta \in \mathbb{T} \}\{1\}. Consequently, \( B \) satisfies (1.1).

Let \( B_0 \) be the Blaschke product with zeros \( \{|z_n|\} \). It is obvious that \( \lim \inf_{r \to 1^{-}} |B_0(r)| = 0 \). Hence, by the deduction above, it is clear that \( B_0 \) satisfies condition (b) in Theorem 11 and thus also the other conditions are valid. Since \( B_0 \) satisfies (c) in Theorem 3 also \( B \) satisfies it. This is due to [3, Lemma 3], which asserts that \( |B'(\xi)| = |B_0'(\xi)| \) for \( \xi \in \mathbb{T} \}\{1\}. Hence \( B \) is a one-component inner function by Theorem 11. This completes the proof.

4. PROOF OF THEOREM 5

Let us prove Theorem 5.

Proof of Theorem 5. Due to hypothesis (i), the set of mass points \( \{e^{i\theta_n} \}_{n=0}^{\infty} \) is closed. Consequently, the spectrum \( \rho(S) \) consists of points \( \{e^{i\theta_n} \}_{n=0}^{\infty} \). Hence, by [16, Chapter 2, Theorem 6.2], we have

\[
\lim_{r \to 1^{-}} |S(r\xi)| = 0, \quad \xi \in \rho(S).
\]

This means that \( S \) satisfies condition (3.2) (with \( \Theta = S \)) in Theorem 3. Consequently, it suffices to show that \( S \) fulfills also (3.1).

By a straightforward calculation, one can check that

\[
S^{\theta}(z) = 4 \left( \sum_{n=0}^{\infty} \frac{\gamma_n e^{i\theta_n}}{(z - e^{i\theta_n})^3} + \left( \sum_{m=0}^{\infty} \frac{\gamma_m e^{i\theta_m}}{(z - e^{i\theta_m})^2} \right)^2 \right) \exp \left( \sum_{k=0}^{\infty} \frac{\gamma_k z + e^{i\theta_k}}{z - e^{i\theta_k}} \right), \quad z \in \mathbb{D}.
\]
Since
\[ |S'(\zeta)| = 2 \sum_{n=0}^{\infty} \frac{\gamma_n}{|\zeta - e^{i\theta_n}|^2}, \quad \zeta \in \mathbb{T}\setminus \rho(S), \]
by \(3\) Theorem 2], we obtain
\[ |S''(\zeta)| \leq 4 \sum_{n=0}^{\infty} \frac{\gamma_n}{|\zeta - e^{i\theta_n}|^3} + |S'(\zeta)|^2, \quad \zeta \in \mathbb{T}\setminus \rho(S). \]
Consequently, it suffices to show
\[ \sum_{n=0}^{\infty} \frac{\gamma_n}{|\zeta - e^{i\theta_n}|^3} \leq \left( \sum_{n=0}^{\infty} \frac{\gamma_n}{|\zeta - e^{i\theta_n}|^2} \right)^2, \quad \zeta \in \mathbb{T}\setminus \rho(S). \quad (4.1) \]
Assume without loss of generality that \( \zeta \in \mathbb{T}\setminus \rho(S) \) is close enough to one, and write \( \zeta = e^{i\theta} \).
Choose \( j = j(\theta, S) \in \mathbb{N} \cup \{0\} \) such that \( \theta - \theta_j \) is as small as possible. Then standard estimates yield
\[ \left( \sum_{n=0}^{\infty} \frac{\gamma_n}{|\zeta - e^{i\theta_n}|^3} \right)^{1/2} \leq \left( \sum_{n=0}^{\infty} \frac{\gamma_n}{|\theta - \theta_n|^2} \right)^{1/2} \leq |\theta - \theta_j|^{-3/2} \left( \sum_{n=0}^{\infty} \gamma_n \right)^{1/2} = |\theta - \theta_j|^{-3/2} \quad (4.2) \]
and
\[ \sum_{n=0}^{\infty} \frac{\gamma_n}{|\zeta - e^{i\theta_n}|^2} = \sum_{n=0}^{\infty} \frac{\gamma_n}{|\theta - \theta_n|^2} \geq \frac{\gamma_j}{|\theta - \theta_j|^2}. \quad (4.3) \]
If \( \theta < 0 \), then \( j = 0 \); and hence, \((4.1)\) is a direct consequence of \((4.2)\) and \((4.3)\). Let \( \theta > 0 \).
By hypothesis (i), we have \( \theta_{j+1} < \theta < \theta_j \), where \( j \in \mathbb{N} \) is large enough. Consequently, hypothesis (ii) gives
\[ \frac{\gamma_j}{|\theta - \theta_j|^2} \geq \frac{\gamma_j}{|\theta - \theta_j|^{3/2}|\theta_j - \theta_{j+1}|^{1/2}} \geq |\theta - \theta_j|^{-3/2}. \]
According to this estimate, \((4.1)\) is a consequence of \((4.2)\) and \((4.3)\). Finally the assertion follows from Theorem \(3\).

Acknowledgements. The author thanks Toshiyuki Sugawa for valuable comments.

References

[1] P. Ahern, The mean modulus and the derivative of an inner function, Indiana Univ. Math. J. 28 (1979), no. 2, 311–347.
[2] P. Ahern, The Poisson integral of a singular measure, Canad. J. Math. 35 (1983), no. 4, 735–749.
[3] P. Ahern and D. N. Clark, On inner functions with \( H^p \) derivative, Michigan Math. J. 21 (1974), 115–127.
[4] A. B. Aleksandrov, Embedding theorems for coinvariant subspaces of the shift operator, II, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 262 (1999), Issled. po Linein. Oper. i Teor. Funkts. 27, 5–48, 231; translation in J. Math. Sci. (New York) 110 (2002), no. 5, 2907–2929.
[5] A. Aleman, Y. Lyubarskii, E. Malinnikova and K.–M. Perfekt, Trace ideal criteria for embeddings and composition operators on model spaces, J. Funct. Anal. 270 (2016), no. 3, 861–883.
[6] A. Baernstein, D. Girela and J. A. Peláez, Univalent functions, Hardy spaces and spaces of Dirichlet type, Illinois J. Math. 48 (2004), no. 3, 837–859.
[7] A. Baranov, R. Bessonov and V. Kapustin, Symbols of truncated Toeplitz operators, J. Funct. Anal. 261 (2011), no. 12, 3437–3456.
[8] R. V. Bessonov, Fredholmness and compactness of truncated Toeplitz and Hankel operators, Integral Equations Operator Theory 82 (2015), no. 4, 451–467.
[9] G. T. Cargo, Angular and tangential limits of Blaschke products and their successive derivatives, Canad. J. Math. 14 (1962), 334–348.
[10] I. Chalendar, E. Fricain and D. Timotin, Functional models and asymptotically orthonormal sequences, Ann. Inst. Fourier (Grenoble) 53 (2003), no. 5, 1527–1549.
[11] J. Cima and R. Mortini, One-component inner functions, Complex Anal. Synerg. 3 (2017), no. 1, Paper No. 2, 15 pp.
[12] B. Cohn, Carleson measures for functions orthogonal to invariant subspaces, Pacific J. Math. 103 (1982), no. 2, 347–364.
[13] P. Duren, Theory of $H^p$ spaces, Academic Press, New York–London, 1970.
[14] T. M. Flett, The dual of an inequality of Hardy and Littlewood and some related inequalities, J. Math. Anal. Appl. 38 (1972), 746–765.
[15] O. Frostman, Sur les produits de Blaschke, Kungl. Fysiografiska Sällskapets i Lund Förhandlingar 12 (1942), no. 15, 109–182.
[16] J. Garnett, Bounded analytic functions. Revised 1st edition, Springer, New York, 2007.
[17] D. Girela, C. González and M. Jevtić, Inner functions in Lipschitz, Besov, and Sobolev spaces, Abstr. Appl. Anal. 2011, Art. ID 626254, 26 pp.
[18] D. Girela, J. A. Peláez and D. Vukotić, Interpolating Blaschke products: Stolz and tangential approach regions, Constr. Approx. 27 (2008), no. 2, 203–216.
[19] A. Gluchoff, On inner functions with derivative in Bergman spaces, Illinois J. Math. 31 (1987), no. 3, 518–528.
[20] J. E. Littlewood and R. E. A. C. Paley, Theorems on Fourier series and power series (II), Proc. London Math. Soc. (2) 42 (1936), no. 1, 52–89.
[21] M. Mateljević and M. Pavlović, On the integral means of derivatives of the atomic function, Proc. Amer. Math. Soc. 86 (1982), no. 3, 455–458.
[22] F. Pérez-González, J. Rättyä and A. Reijonen, Derivatives of inner functions in Bergman spaces induced by doubling weights, Ann. Acad. Sci. Fenn. Math. 42 (2017), no. 2, 735–753.
[23] S. A. Vinogradov, Multiplication and division in the space of analytic functions with an area-integrable derivative, and in some related spaces (Russian), Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 222 (1995), Issled. po Linein. Oper. i Teor. Funktsii 23, 45–77; English J. Math. Sci. (New York) 87 (1997), 3806–3827.

Graduate School of Information Sciences, Tohoku University, Aoba-ku, Sendai 980-8579, Japan

E-mail address: atte.reijonen@uef.fi