Teleportation, Bell’s inequalities and inseparability

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Relations between teleportation, Bell’s inequalities and inseparability are investigated. It is shown that any mixed two spin-$\frac{1}{2}$ state which violates the Bell-CHSH inequality is useful for teleportation. The result is extended to any Bell’s inequalities constructed of the expectation values of products of spin operators. It is also shown that there exist inseparable states which are not useful for teleportation within the standard scheme.

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Recently Bennett at al. [1] have discovered a new aspect of quantum inseparability – teleportation. It involves a separation of an input state into classical and quantum part from which the state can be reconstructed with perfect fidelity $F = 1$. The basic idea is to use a pair of particles in singlet state shared by sender (Alice) and receiver (Bob). Quite recently Popescu [2] noticed that the pairs in a mixed state could be still useful for (imperfect) teleportation. There was a question what value of fidelity of the transmission of an unknown state can ensure us about nonclassical character of the state forming the quantum channel. It has been shown [3] that the purely classical channel can give at most $F = \frac{2}{3}$ (see also Ref. [4] in this context). Then Popescu raised basic questions concerning a possible relation between teleportation, Bell’s inequalities and inseparability: “What is the exact relation between Bell’s inequalities violation and teleportation? Is every mixed state that can not be expressed as a mixture of product states useful for teleportation?” [2]

The problem is rather complicated, as these questions concern the mixed states which apparently possess the ability to behave classically in some respect but quantum mechanically in others [3]. Fortunately for $2 \times 2$ systems two basic questions concerning violation of Bell’s inequalities and inseparability of mixed states, have been solved completely. In particular, in Ref. [6] the effective criterion for violation of Bell’s inequalities has been obtained. Quite recently the problem of inseparability have been investigated in detail by Peres [6] and the authors [7]. In particular, the necessary [6] and sufficient [7] condition for separability of mixed states for $2 \times 2$ systems has been provided.

The main purpose of the present Letter is to present the effective criterion for teleportation via mixed two spin-$\frac{1}{2}$ states and discuss it in the context of Bell’s inequalities and inseparability. Using the results contained in Refs. [6, 7] we will show further that if a mixed two spin-$\frac{1}{2}$ state violates any in Bell’s inequality constructed of the expectation values of products of spin operators (in particular if it violates original Bell-CHSH one), then it is also useful for teleportation. We will also demonstrate that there are inseparable states which are not useful for teleportation within the standard scheme.

I. MAXIMAL FIDELITY FOR THE STANDARD TELEPORTATION SCHEME

We start with the representation of the state in the Hilbert-Schmidt space

$$\varrho = \frac{1}{4} [I \otimes I + r \cdot \sigma \otimes I + I \otimes s \cdot \sigma + \sum_{n,m=1}^{3} t_{nm} \sigma_n \otimes \sigma_m]$$

(1)

where $\varrho$ acts on Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 = C^2 \otimes C^2$, $I$ stands for identity operator, $\{ \sigma_n \}_{n=1}^{3}$ are the standard Pauli matrices, $r, s$ are vectors in $R^3$, $r \cdot \sigma = \sum_{i=1}^{3} r_i \sigma_i$. The coefficients $t_{nm} = \text{Tr}(\varrho \sigma_n \otimes \sigma_m)$ form a real matrix

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1 We call a state inseparable if it cannot be written as convex combination of product states

2 By the standard teleportation scheme we mean here that Alice uses Bell operator basis in her measurement while Bob is allowed to apply any unitary transformation.
which we shall denote by $T$. Note that the representation appears to be very convenient in the investigation of some aspects of inseparability of the mixed states. Indeed, all the parameters fall into two different classes: first ($r$ and $s$) — describing the local behaviour of the state, second ($T$ matrix) — responsible for correlations. It is compatible with the fact that the mean value of the Bell-CHSH observable depends only on the correlation parameters $T$ of the state $\rho$.

Let us briefly recall the standard teleportation scheme. It involves two particle source producing pairs in a given mixed state $\rho$ which forms the quantum channel (originally formed by pure singlet state $\parallel$). One of the particles is given to Bob while the other one and a third particle in an unknown state $\phi$ are subjected to Alice’s joint measurement. The latter is given by a family of projectors

$$P_k = |\psi_k\rangle\langle\psi_k| \quad k = 0, 1, 2, 3, \quad (2)$$

where $\psi_k$ constitute the so-called Bell basis

$$\psi_{1,2} = \frac{1}{\sqrt{2}} (e_1 \otimes e_1 \mp e_2 \otimes e_2)$$

$$\psi_{3,0} = \frac{1}{\sqrt{2}} (e_1 \otimes e_2 \pm e_2 \otimes e_1) \quad (3)$$

with $e_1, e_2$ being standard basis in $C^2$. Then using two bits Alice sends to Bob the number of outcome $k$ and Bob applies some unitary transformation $U_k$ obtaining in this way his particle in a state $\varrho_k$.

Then the fidelity of a transmission of the unknown state is given by formula $\parallel$

$$F = \int_S dM(\phi) \sum_k p_k \text{Tr}(\varrho_k P_\phi) \quad (4)$$

where the integral is taken over all $\phi$ belonging to the Bloch sphere with uniform distribution $M$, $p_k = \text{Tr}[(P_k \otimes I)(P_\phi \otimes \varrho)]$ denotes the probability of the $k$-th outcome. Now the task is to find such $U_k$’s that produce the highest fidelity (a choice of a quadruple of $U_k$’s we shall call strategy). In this purpose, let us compute the integral $\parallel$.

$$\varrho_k = \frac{1}{p_k} \text{Tr}_{1,2} \left[ (P_k \otimes U_k)(P_\phi \otimes \varrho)(P_k \otimes U_3) \right] \quad (5)$$

Here the partial trace is taken over the states of unknown particle and Alice’s one. Putting $P_\phi = \frac{1}{2} (I + a \cdot \sigma)$ one obtains

$$p_k \varrho_k = \frac{1}{8} [(1 + (a, T_k r)] I + O_k^\dagger [s + T^\dagger T_k a] \cdot \sigma \quad (6)$$

Here $T_k$’s and $r$, $s$, $T$ correspond to $P_k$’s and $\rho$ respectively via formula $\parallel$ (we have: $T_0 = \text{diag}(-1, -1, -1)$, $T_1 = \text{diag}(-1, 1, 1)$, $T_2 = \text{diag}(1, -1, 1)$, $T_3 = \text{diag}(1, 1, -1)$, $r_k = s_k = 0$, for $k = 0, 1, 2, 3$); $O_k$’s are rotations in $R^3$ obtained from $U_k$’s by

$$U \hat{n} \cdot \sigma U^\dagger = (O_k \hat{n}) \cdot \sigma \quad (7)$$

($O$ is here determined uniquely as the group of rotations $O^+(3)$ is a homomorphic image of $U(2)$ group $\parallel$). Omitting the terms which do not contribute to the integral $\parallel$ and using the formula

$$\int_S (a, Aa)dM(a) = \frac{1}{3} \text{Tr}A \quad (8)$$

one obtains

$$F = \frac{1}{8} \sum_k (1 + \frac{1}{3} \text{Tr}T_k^\dagger TO_k) \quad (9)$$

Now we shall maximize $F$ under all strategies. Clearly, as $-T_k$ is a rotation we see that the maxima of the terms in the above formula do not depend on $k$ any longer so that

$$F_{\text{max}} = \max_\rho \frac{1}{2} (1 - \frac{1}{3} \text{Tr}T^\dagger T) \quad (10)$$

where the maximum is taken over all rotations. Note that we have

$$F_{\text{max}} \leq \frac{1}{2} (1 + \frac{1}{3} \text{Tr}T^\dagger T) \quad (11)$$

Of course, we need to derive the expression for $F_{\text{max}}$ only if the latter is greater than $\frac{4}{9}$ which is the upper bound for the classical teleportation $\parallel$. If $F_{\text{max}} > \frac{4}{9}$ we say that the state forming the quantum channel is useful for teleportation. Clearly, $F_{\text{max}}$ can exceed $\frac{4}{9}$ only if $\text{Tr}T^\dagger T > 1$. Now basing on the results contained in Ref. $\parallel$ one can see that the latter condition implies $\text{det} T < 0$. But then the inequality $\parallel$ passes into equality. Consequently, defining function $N(\rho) := \text{Tr}\sqrt{T^\dagger T}$ one has

**Proposition 1** Any mixed spin-$\frac{1}{2}$ state is useful for (standard) teleportation iff $N(\rho) > 1$. Then the fidelity amounts to

$$F_{\text{max}} = \frac{1}{2} (1 + \frac{1}{3} N(\rho)) \quad (12)$$

Now, if $N(\rho) > 1$ then there exist rotations $O_1$ and $O_2$ such that $O_1 TO_2$ is diagonal with $a_{ii} < 0$ for $i = 1, 2, 3$. Then the best strategy is given by unitaries $U_k = U \sigma_k$ where $U$ is determined (up to an irrelevant phase factor) by $O = O_1 O_2$ via formula $\parallel$.

**Example.** Consider pure state of the form

$$|\psi\rangle = ae_1 \otimes e_2 - be_2 \otimes e_1 \quad (13)$$

One obtains

$$F_{\text{max}} = \frac{2a^3 - b^3}{3 a - b} \quad (14)$$

which is compatible with Ref. $\parallel$. 

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II. RELATION BETWEEN BELL’S INEQUALITIES AND TELEPORTATION

Note that $N(\varrho)$ is a function of the correlation parameters $T$ only. It allows to establish a relation between teleportation and Bell’s inequality due to Clauser, Horne, Shimony and Holt (Bell-CHSH). As one knows the necessary and sufficient condition for violating the Bell-CHSH inequality involves a real valued function $M(\varrho) = \max_{i>j}(u_i + u_j)$ where $u_i$ are eigenvalues of matrix $T_iT_j$. Then the inequality $M(\varrho) \leq 1$ is equivalent to the Bell-CHSH one. Now as $u_i \leq 1$ for $i = 1, 2, 3$ and $N(\varrho) = \sum_{i=1}^{3} \sqrt{u_i}$ we obtain a relation

$$N(\varrho) \geq M(\varrho). \quad (15)$$

Note that for any state which violates the Bell-CHSH inequality we have $M(\varrho) > 1$. Then, according to the relation (15) and Prop. 1 we get the estimate

$$F_{max} \geq \frac{1}{2}(1 + \frac{1}{3}M(\varrho)) > \frac{2}{3}. \quad (16)$$

As the maximal mean value of the CHSH-Bell observable is $B_{max} = 2\sqrt{M(\varrho)}$ we have also

$$F_{max} \geq \frac{1}{2}(1 + \frac{1}{12}B_{max}^2). \quad (17)$$

The inequalities (16), (17) are valid for an arbitrary mixed two spin-$\frac{1}{2}$ state which violate the Bell-CHSH inequality and they say us that any such a state is useful for teleportation.

Now we shall see that even a stronger statement is valid. For this purpose consider generalized Bell-CHSH inequalities i.e. all the Bell’s inequalities which can be constructed of the expectations of products of spin operators $a \cdot \sigma \otimes b \cdot \sigma$ where $a$ and $b$ are unit vectors [1]. Of course, the expectations (or correlation functions)

$$E(a, b) \equiv \text{Tr}(\varrho a \cdot \sigma \otimes b \cdot \sigma) = (a, Tb) \quad (18)$$

depend only on the $T$ matrix. Hence the generalized Bell-CHSH inequalities can be violated only if $N(\varrho) > 1$. Indeed, if $N(\varrho) \leq 1$, there always exists some separable state that has the same $T$ matrix as the state $\varrho$ (see Ref. 3). In this way we have obtained

**Proposition 2** Every mixed two spin-$\frac{1}{2}$ state which violates any generalized Bell-CHSH inequality is useful for teleportation.

III. INSEPARABILITY AND TELEPORTATION

Let us now turn back to the question: “Is every mixed state that can not be expressed as a mixture of product states useful for teleportation?” Generally, the problem is rather complicated as it requires to obtain the maximum of the fidelity over all possible teleportation procedures. Here we will see that within the standard teleportation scheme the answer is “no”. For this purpose consider the following class of the states

$$\varrho = p_1|\psi_1\rangle\langle\psi_1| + p_2|\psi_2\rangle\langle\psi_2| \quad (19)$$

where $|\psi_1\rangle = ae_1 \otimes e_1 + be_2 \otimes e_2$ \quad (20)

$|\psi_2\rangle = ae_1 \otimes e_2 + be_2 \otimes e_1$ \quad (21)

with $a, b > 0$, $\{e_i\}$ being standard basis in $C^2$, $0 < (p_1 - p_2)^2 \leq (a^2 - b^2)^2$. The above states have interesting properties. First, note that as $M(\varrho) = 1 + (p_1 - p_2)^2 - (a^2 - b^2)^2 \leq 1$ they do not violate the Bell-CHSH inequality. In addition, it is possible to choose the parameters $p_1$ and $a$ so that the maximal absolute value of the expectation of products of spin operators is arbitrarily close to zero. Then it follows from Prop. 1 that many of the states (14) are not useful for teleportation. But what can we say about the above states in the context of the inseparability? As it was mentioned in the introduction, the effective criterion for inseparability of the states of $2 \times 2$ systems have been found [16-18]. Namely a two spin-$\frac{1}{2}$ state is inseparable if and only if its partial transposition is not a positive operator. The matrix elements of partial transposition $\varrho^{T_2}$ of a state $\varrho$ is given by

$$\varrho_{m\mu,n\nu}^{T_2} \equiv \varrho_{m\nu,n\mu}, \quad (22)$$

where

$$\varrho_{m\mu,n\nu} = \langle e_m \otimes e_{\mu}|\varrho|e_n \otimes f_{\nu}\rangle. \quad (23)$$

Now it is easy to see that all the states (19) are inseparable [1]. In fact, one can show that the inseparability of the above states manifests itself via “hidden” nonlocality (see Ref. 16) which can be revealed (15) by means of Gisin’s filtering method (17). Thus we have provided an example of states which are inseparable and nonlocal but still are not useful for teleportation within the standard scheme.

IV. CONCLUSION

In conclusion, we have considered the questions concerning possible relations between teleportation, violation of Bell’s inequalities and inseparability. In particular, we have obtained the maximal fidelity for the

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3In Ref. 14 the states 19 were shown to be inseparable by means of entropic criterion (see in this context Ref. 3). The latter appears to be equivalent to inseparability for the considered states, but it is not the case in general 16-18.
standard teleportation scheme with the quantum channel formed by any mixed two spin-$\frac{1}{2}$ state. It involves only the correlation parameters of the state. Then it was possible to compare the two different aspects of quantum inseparability: teleportation and Bell’s inequalities. More precisely, we have shown that if a mixed two spin-$\frac{1}{2}$ state violates any generalized Bell-CHSH inequality (in particular if it violates the original Bell-CHSH one) then it is also useful for teleportation.

We have also considered the states which are inseparable, but are not useful for the standard teleportation. Here the inseparability is due to the relation between the local and correlation parameters. Then there is a question: what would happen if we allowed Alice to use any projectors – not only the maximally entangled ones? In fact she may perform any generalized measurements. In the formula for the fidelity the local parameters could then also appear. It is not clear whether a higher fidelity can be obtained within so generalized scheme. Thus, the problem of a relation between the widely understood Bell’s inequalities (e.g. involving nonstandard measurements [1,2,3]) and more general teleportation schemes needs further investigations.

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