Tuning topological phase and quantum anomalous Hall effect by interaction in quadratic band touching systems

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Interaction driven topological phases can significantly enrich the class of topological materials and thus are of great importance. Here, we study the phase diagram of interacting spinless fermions filling the two-dimensional checkerboard lattice with a quadratic band touching (QBT) point. By developing new diagnosis based on the state-of-the-art density-matrix renormalization group and exact diagonalization, we determine accurate quantum phase diagram for such a system at half-filling with three distinct phases. For weak nearest-neighboring interactions, we demonstrate the instability of the QBT towards an interaction-driven spontaneous quantum anomalous Hall (QAH) effect. For strong interactions, the system breaks the rotational symmetry realizing a nematic charge-density-wave (CDW) phase. Interestingly, for intermediate interactions we discover a symmetry-broken bond-ordered critical phase sandwiched in between the QAH and CDW phases, which splits the QBT into two Dirac points driven by interaction. Instead of the direct transition between QAH and CDW phases, our identification of an intermediate phase shed new light on the theoretical understanding of the interaction-driven phases in QBT systems.

Recently topological phases of matter in the band structure with nontrivial topological Berry phase become an exciting research area of modern physics, culminating in the experimental observations of Haldane-honeycomb insulator [1] and the quantum anomalous Hall (QAH) effect in topological insulator [2]. Generally speaking, in such kind of systems with a nontrivial topological invariant Chern number [3], exemplified by the integer quantum Hall effect in the absence of magnetic field [4], time-reversal symmetry (TRS) breaking of band structure is typically necessary. For topologically trivial band structures with zero Chern number, it was proposed that the strong correlation between electrons can also induce spontaneous TRS breaking at mean field level [6, 7] and lead to QAH effect in Dirac semimetals. These interaction-induced topological phases are interesting, since they can significantly enrich the class of topological materials [8–10]. However such a mechanism is challenged by subsequent numerical simulations [11–14], which do not support the interaction driven QAH effect. Alternatively, it has been identified that a quadratic band touching (QBT) point has topologically feature, and can be driven towards a QAH phase with TRS breaking even under arbitrary weak interactions, while strong interactions may lead to other competing phases [15, 16].

The theoretical prediction of such interaction-driven topological phases [17–25], stimulates extensive studies by more rigorously theoretical and numerical methods, including low energy renormalization group approach in C4 symmetric checkerboard lattice [26–29] and bilayer graphene [30, 31], first-principle calculations in spin-dependent optical square lattice [32] and halogenated hematite nanosheets [33], and recently the unbiased numerical exact diagonalization (ED) diagnosis in both C4 symmetric checkerboard lattice [34] and C6 symmetric Kagome lattice [35]. However, the stability of the QAH effect in the presence of weak interaction has not been settled. Based on the mean-field theory, the gap protecting the QAH state is exponentially small for weak interaction, making such a phase difficult to be identified. A recent density-matrix renormalization group (DMRG) study finds a semimetal phase for kagome QBT systems with nearest neighboring interactions [36], while adding further long-ranged interactions (V1 ~ V2 ~ V3) favors a robust gapped QAH state [35]. Thus, the weak interaction effect on the QBT point remains elusive, because finite-size calculations are incapable of detecting extremely small energy gap. Furthermore, the issue of the quantum phase transition between the interaction-driven topological phase and other phases is hardly touched. While ED analysis points to a first-order phase transition...
from the QAH phase to the charge density wave phase without intermediate phase [34, 35] the interesting scenario of intermediate Dirac liquid phase in such QBT systems deserves to be explored by applying controlled numerical methods.

In this work, we develop accurate numerical diagnosis for the topological phases to address these challenge issues through the state-of-the-art DMRG and ED simulations. A schematic diagram of our main results is shown in Fig. 1 for interacting spinless fermions occupying the topologically trivial checkerboard lattice with a QBT. For weak interactions $V < V_1$, by applying the Hellmann–Feynman theorem, we demonstrate the emergence of interaction-driven QAH phase whose topological properties are featured by two degenerate TRS breaking ground states arising from a pair of Kramers degeneracy. For weak interactions $V < V_2$, the Haldane gap $\Delta$ opens as a function of $V$. The gapless particle-hole symmetric state persists for the topological phases to address these challenge issues.

We consider the spinless fermions in the topologically trivial checkerboard lattice model,

$$
H = -t \sum_{\langle \mathbf{r}, \mathbf{r}' \rangle} a_{\mathbf{r}}^\dagger b_{\mathbf{r}'} - \sum_{\langle \langle \mathbf{r}, \mathbf{r}' \rangle \rangle} (t_{\mathbf{r}, \mathbf{r}'}^a a_{\mathbf{r}}^\dagger a_{\mathbf{r}'} + t_{\mathbf{r}, \mathbf{r}'}^b b_{\mathbf{r}}^\dagger b_{\mathbf{r}'}) 
+ h.c. + V \sum_{\mathbf{r}} n_{\mathbf{r}}^a n_{\mathbf{r}}^b + m \sum_{\mathbf{r}} (n_{\mathbf{r}}^a - n_{\mathbf{r}}^b).
$$

Here $a_{\mathbf{r}}, b_{\mathbf{r}'}$ are the particle annihilation operators for sublattices A,B respectively, and $n_{\mathbf{r}}^a = a_{\mathbf{r}}^\dagger a_{\mathbf{r}}, n_{\mathbf{r}}^b = b_{\mathbf{r}}^\dagger b_{\mathbf{r}}$ the particle number operators at site $\mathbf{r}$. $\langle \langle \ldots \rangle \rangle$ denote the nearest-neighbor and the next-nearest-neighbor pairs of sites on a checkerboard lattice, respectively. As shown in Fig. 1 the hopping amplitudes $t_{\mathbf{r}, \mathbf{r}'}^a = t'$ for solid lines and $t_{\mathbf{r}, \mathbf{r}'}^b = -t'$ for dashed lines in sublattice A, while $t_{\mathbf{r}, \mathbf{r}'}^b = t'$ for solid lines and $t_{\mathbf{r}, \mathbf{r}'}^b = -t'$ for dashed lines in sublattice B. The single-particle dispersion hosts a QBT at $(k_x, k_y) = (\pi, \pi)$ with Berry flux $\pm 2\pi$, protected by time-reversal symmetry and $C_4$ rotational symmetry. Note that by adding a nonzero opposite shift $\delta \neq 0$ to the hopping $t_{\mathbf{r}, \mathbf{r}'}^{a,b} = t_{\mathbf{r}, \mathbf{r}'}^{a,b} + (-\delta)\delta$ of both sublattices, the system $H$ breaks $C_4$ symmetry down to $C_2$ symmetry, and the quadratic band touching splits into two Dirac points with gapless particle-hole symmetric dispersions.

We study the many-body ground state of $H$ at half-filling in a finite system of $N_x \times N_y$ unit cells (the total number of sites $N_s = 2 \times N_x \times N_y$), based on ED and DMRG methods. In the ED study, the energy eigenstate is labeled by a total momentum $K = (K_x, K_y)$ in units of $(2\pi/N_x, 2\pi/N_y)$ in the Brillouin zone. For larger systems we exploit both finite and infinite DMRG on the cylindrical geometry, and keep the dimension of DMRG kept states up to 3600 to obtain accurate results (the truncation error is of the order $10^{-6}$), for cylinders with open boundary conditions in the $x$-direction and periodic boundary conditions in the $y$-direction.

**QAH Phase.**— We first address the critical issue if a robust QAH effect can be stabilized with the presence of weak and nearest interaction. We start from studying the ground state properties up to a maximum system sizes $N_s = 32$ based on ED. The geometry hosts both $C_4$ point-group symmetry and time-reversal symmetry, and we find an exact two-fold ground state degeneracy $|\psi_{\pm}\rangle$ for momentum $K = (0,0)$ below excitation continuum for systems with finite interaction $V$. This pair of degenerating eigenstates are also eigenstates of $C_4$ rotation with eigenvalues $\pm i$, respectively, which serves as important evidence of time-reversal symmetry breaking as long as the energy excitation gap remains open for large systems. In order to illustrate two TRS spontaneously breaking states with opposing chiralities, we consider the system response to the TRS breaking perturbation hopping phase $e^{i\phi}a_{\mathbf{r}}^\dagger b_{\mathbf{r}'} + h.c.$ to the nearest-neighbor A and B sites (the phase $\phi$ is a tiny detecting flux per plaquette for detecting QAH order). From the Hellmann–Feynman theorem [38, 39], we can derive the TRS breaking chiral bond current $J_r = i(a_{\mathbf{r}}^\dagger b_{\mathbf{r}} - b_{\mathbf{r}}^\dagger a_{\mathbf{r}})$ between nearest-neighboring sites from the linear response of the ground state energy, as

$$
\langle \psi_{\pm}|J_r|\psi_{\pm}\rangle = \frac{1}{2N_s} \frac{\partial E_{\pm}(\phi)}{\partial \phi} \bigg|_{\phi=0},
$$

where $E_{\pm}(\phi) = \langle \psi_{\pm}|H(\phi)|\psi_{\pm}\rangle$. As indicated in Fig. 2(a), we can see $\langle \psi_{+}|J_r|\psi_{+}\rangle = -\langle \psi_{-}|J_r|\psi_{-}\rangle \simeq 0.067$, implying the opposite chiralities of TRS breaking for weak interactions. To extract the topological invariants of the doublet ground states for any value $\phi$, we utilize the twisted boundary conditions $\psi(\phi; \mathbf{r} + N_x \hat{e}_x) = \psi(\phi; \mathbf{r}) \exp(i\theta_\alpha)$ where $\theta_\alpha$ is the twisted angle in the $\alpha$ ($x$ or $y$)-direction. The system is periodic when one flux quantum $\theta_\alpha = 0 \rightarrow 2\pi$ is inserted. Meanwhile, the many-body Chern number of the ground state wavefunction $\psi_{\pm}(\phi)$ is defined as [40, 41]

$$
C_{\pm}(\phi) = \int \frac{d\theta_x d\theta_y}{2\pi i} \left[ \frac{\partial \psi_{\pm}}{\partial \theta_x} \frac{\partial \psi_{\pm}}{\partial \theta_y} - \frac{\partial \psi_{\pm}}{\partial \theta_y} \frac{\partial \psi_{\pm}}{\partial \theta_x} \right].
$$

We identify the topological invariants $C_{\pm}(\phi) = \pm 1$ for the two-fold ground states $|\psi_{\pm}(\phi)\rangle$ under an infinitesimal TRS breaking phase $\phi \ll 1$. Due to the adiabatic connection between the $\psi_{\pm}(\phi)$ and $\psi_{\pm}(0)$ as shown in Fig. 2(a), we can obtain the topological invariants for these doublet states $C_{\pm} = C_{\pm}(\phi \rightarrow 0) = \pm 1$. In the contrary, for
strong interactions, we find that the expectation value of $J_r$ in the ground state vanishes precisely, and the topological invariants $C_{\pm}(\phi) = 0$, signalling a topologically trivial nematic CDW phase. Indeed, the density structure factor $S = \frac{1}{V} \sum_{\alpha,\beta} \sum_{r,r'} (-1)^{\alpha} (-1)^{\beta} \langle n^\alpha_r n^\beta_{r'} \rangle$ shows a strong peak, where $\alpha, \beta \in \{A, B\}$ denote sublattice indices, $(-1)^\alpha = 1$ for $\alpha = A(B)$ for such a CDW phase.

To access larger system sizes to establish the stability of the QAH, we exploit an unbiased DMRG approach using a cylindrical geometry up to a maximum width $L_y = 16$ ($N_y = 8$). By randomly choosing different initial states in DMRG simulations, we can obtain two different ground states $|\psi_{\pm}\rangle$ with degenerate energies $E_+ \simeq E_-$ and opposite chiral circulating loop currents per plaquette $\langle \psi_{\pm} | J_r | \psi_{\pm} \rangle$, from a DMRG algorithm allowing complex wavefunctions. In Fig. 2(b), we measure the current-current correlation functions $\langle \psi_+ | J_r | \psi_+ \rangle$ between nearest-neighbouring bonds $\langle r, r' \rangle$ and $\langle r_0, r'_0 \rangle$, with the distance $|r - r_0|$. For different system sizes, the bond current long-range order parameter $\langle \psi_+ | J_r | \psi_+ \rangle = \lim_{r \rightarrow \infty} \langle \psi_+, J_r | \psi_+ \rangle$ for weak interactions persists to a finite value at the large distance $|r - r_0|$ limit. Meanwhile in Fig. 2(b), we also characterize the topological nature of the ground state from its topological charge pumping by inserting one $U(1)$ charge flux quantum $\theta_y = \theta$ from $\theta = 0$ to $\theta = 2\pi$ in the periodic $y$-direction of the cylinder system based on the adiabatic DMRG [12] in connection to the quantized Hall conductance. Here we partition the lattice system on the cylinder along the $x$-direction into two halves with equal lattice sites. The transverse transfer of the total charge from the right side to the left side in the $x$-direction is encoded by the expectation value $Q(\theta) = tr[\hat{\rho}_L(\theta)N_L]$. $N_L$ is the particle number in the left cylinder part, and $\hat{\rho}_L$ the reduced density matrix of the corresponding left part. Under the inserting of the flux $\theta_y = \theta$ in the $y$-direction, the change of $Q(\theta)$ indicates the transverse charge transfer from the right side to the left side in the $x$-direction, induced by the topological Hall conductances of the state $|\psi_{\pm}\rangle$. From Fig. 2(c), we obtain a nearly quantized transverse Hall conductance $C_{\pm} = \Delta Q = Q(2\pi) - Q(0) = \pm 1$ for these two degenerate ground states $|\psi_{\pm}\rangle$. Again, for strong interactions, the current-current correlation functions are vanishingly small, and the charge pumping disappears, signaling the absence of a QAH phase. Significantly, our results establish that any weak interaction will drive the system into the QAH phase.

Intermediate Phase.— We turn to analyze the emergent intermediate phase between the QAH phase and the nematic CDW phase. Figure 3(a1) depicts the evolution of the interacting many-body low energy spectrum. Near the point $V = V_1$, the doublet ground states of the QAH phase undergo level crossing with the two-fold degenerate excited levels. When the interaction $V$ increases further, there appears another level crossing near the transition around $V = V_2$. For $V_1 < V < V_2$, the doublet ground states host bond order $\Delta_r = \frac{1}{2N_c} \langle \psi_+ | e^{i\theta} | \psi_+ \rangle$ vanishes, $\langle \psi_+ | e^{i\theta} | \psi_+ \rangle = 0$, signaling the absence of a QAH phase. In contrast, for strong interactions, the current-current correlation functions are vanishingly small, and the charge pumping disappears, signaling the absence of a QAH phase. Significantly, our results establish that any weak interaction will drive the system into the QAH phase.

Meanwhile, we plot the evolutions of the current order parameter $\langle J_r \rangle$, sublattice bond order parameter $\langle \Delta_r \rangle$ and nematic CDW density structure factor $S/N_y$ as a function of $V$ in Fig. 3(b). For weak interactions $V < V_1$, $\langle J_r \rangle$ has a finite expectation value, which increases with the increase of the interaction strength. On the other hand, both $\langle \Delta_r \rangle$ and $S/N_y$ take small values consistent with the properties of a gapped QAH phase. When inter-
action $V$ goes across $V_1$, $\langle J_r \rangle$, $\langle \Delta_r \rangle$, and $S/N_s$ experience a sudden jump, where $\langle J_r \rangle$ drops down to a vanishingly small value of the order $10^{-4}$, signaling the collapsing of a topological phase, while $\langle \Delta_r \rangle$, and $S/N_s$ jump to a finite large value. When interaction further increases beyond the critical value $V_2$, $\langle \Delta_r \rangle$ drops down to a vanishingly small value of the order $10^{-5}$ while $S/N_s$ undergoes another step jump to a larger value, where the system becomes an insulating nematic CDW phase with a large excitation gap as shown in Fig. 3(a1).

Interestingly, the intermediate phase between $V_1$ and $V_2$ is associated with a finite bond order $\langle \Delta_r \rangle$. To inspect properties of the phase in the thermodynamic limit, we carry out a finite size scaling of bond order $\Delta_r$ and nematic order $\delta_{CDW} = \langle n_r^a - n_r^b \rangle = \frac{1}{N} \frac{\partial E(m)}{\partial m}$. In Fig. 4, it is found that $\Delta_r$ extrapolates to a finite value, but $\delta_{CDW}$ gradually decreases down to zero in the limit $1/N^2 \rightarrow 0$. Our ED and DMRG methods provide a very strong evidence for the existence of the intermediate topologically trivial bond-ordered phase. Physically, the doublet states of the QAH phase host opposite eigenvalues $\pm i$ in the presence of $C_4$ rotation symmetry and time-reversal symmetry, while both bond-ordered and nematic CDW phase breaks $C_4$ symmetry down to $C_2$ symmetry. In Ref. [54], such an argument is used to claim a first-order phase transition between $C_4$ symmetric phases and $C_2$ symmetric phases. Our ED and DMRG study access much larger systems, which lead to the discovery of a time-reversal symmetric intermediate bond-ordered phase sandwiched in between QAH and CDW phases. At mean-field level, one can show that this phase is gapless with the QBT splits into two Dirac points. At numerical side, we confirm that the intermediate phase is gapless characterized by a finite central charge from the entanglement entropy.

Phase Transition.— To further study phase transitions, we exploit the finite DMRG calculation on a cylindrical geometry up to a maximum width $L_y = 16$ ($N_y = 8$) and length $L_x = N_x = 20$. We measure five different physical quantities of the ground state as a function of $V$: the wavefunction overlap $F(V) = |\langle \psi(V)|\psi(V + \delta V)\rangle|^2$ ($\delta V$ is as small as 0.1$\delta$), the entanglement entropy $S_L$ in the middle of the cylinder, in addition to order parameters $\Delta_r$, $\delta_{CDW}$ and $J_r$. The first-order transition is characterized by the discontinuous behavior of these physical quantities.

As shown in Figs. 5(a-c), for $V < V_1$, $F(V)$ has a large value close to 1, both $S_L$ and $\delta_{CDW}$ exhibit featureless properties, but $J_r$ grows slightly with the increase of $V$, implying the robustness of the QAH phase. When $V$ approaches a transition point $V_1$, $F(V)$ suddenly drops down to a very small value close to zero, and both order parameters $\Delta_r$, $J_r$ exhibit a sharp discontinuous jump near the transition point. Similarly, $S_L$ starts to drop at $V \approx V_1$, with a discontinuous derivative $\partial S_L/\partial V$. These results are consistent with a first order transition between the QAH and intermediate phases [44, 45]. When the interaction $V$ increases further close to $V_2$, $\delta_{CDW}$ undergoes a jump to a large saturated value close to 1, while $\Delta_r$ gradually drops down to zero in the strongly interacting regime. $F(V)$ exhibits a minimum, sharpening as the cylinder width increases from $N_y = 4$ to $N_y = 8$, while entanglement entropy $S_L$ also drops down to a smaller value consistent with an insulating phase at $V > V_2$. These results are consistent with a first-order transition into a CDW phase at $V = V_2$.

In summary, we have numerically presented a solid diagnosis of an interaction-driven spontaneous QAH phase in the checkerboard lattice with a quadratic band touch-
ing by turning on any weak interaction. Such a diagnosis relies on finite size scaling up to wide systems \((L_y = 20\) lattice spacing) and use detecting flux for the QAH. The QAH phase hosts two-fold ground state degeneracies with opposite spontaneous TRS breaking behaviors, and a quantized Hall conductance measured by Laughlin argument of charge pumping. In particular, we demonstrate the existence of a bond-ordered phase sandwiched between the QAH phase and the nematic CDW phase, characterized by the sublattice bond order. We believe that this work would open a new route for the study of the possible competing intermediate phases under the interplay of interaction and frustration, and excite a more extensive investigation of the fate of the QAH phases in many other systems, such as bilayer graphene [30, 31], and \(\mathbb{C}_6\) symmetric Kagome lattice [17] where an intermediate gapless CDW phase was also proposed. Other future directions include a study of the interplay of nearest-neighboring and next-nearest-neighboring interactions, which may lead to rich possibility and other competing phases driven by interactions [46]. At experimental side, our work also suggests a practical way of opening and closing the QBT gap, inducing quantized dissipationless transport currents by interactions.

Note add.—In the preparation of this work, we become aware of a parallel work from [46].

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