Wide class of logarithmic potentials with power-tower kink tails

Avinash Khare$^1$ and Avadh Saxena$^2$

$^1$ Physics Department, Savitribai Phule Pune University, Pune 411007, India
$^2$ Theoretical Division and Center for Nonlinear Studies, Los Alamos National Laboratory, Los Alamos, New Mexico 87545, United States of America

E-mail: khare@physics.unipune.ac.in and avadh@lanl.gov

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Abstract

We present a wide class of potentials which admit kinks and corresponding mirror kinks with either a power law or an exponential tail at the two extreme ends and a power-tower form of tails at the two neighbouring ends. We analyse kink stability equation in all these cases and show that there is no gap between the zero mode and the beginning of the continuum. Finally, we provide a recipe for obtaining logarithmic potentials with power-tower kink tails and estimate kink–kink interaction strength.

Keywords: solitons, kink interaction, kink asymptotes, soliton stability, triple well potentials

(Some figures may appear in colour only in the online journal)

1. Introduction

A vast majority of kink solutions obtained during the last four decades for a variety of field theory potentials, e.g. sine-Gordon, double sine-Gordon, $\phi^4$, $\phi^6$, etc harbour kinks with an exponential tail [1]. Recently we and others have presented a wide class of kink-bearing potentials for which one has a power law kink tail [2–6]. Very recently we have also presented a model with a super-exponential profile with one of the tails also being either super-exponential [7–10] or super-super-exponential [11]. Thus by now we have models where one has a variety of kink tails such as of power law, exponential or super-exponential (or super-super-exponential) form. The obvious question is if there are models with still different types of kink tails.

The purpose of this paper is to present an entirely different and novel class of potentials with power-tower kink tails, thus further expanding the type of kink asymptotes one could realize. A power-tower function is defined by repeated $k$ exponentiations of itself, i.e., $f^{[1]}$ [12]. In the literature this operation is also called tetration [13]. For example $f^{[1]}$ is a power-tower...
function of order $k = 3$. One of the logarithmic potentials with super-exponential kink tails arises in the context of infinite order phase transitions [7] and, therefore, conceivably the family of potentials considered here may have similar physical relevance.

The plan of the paper is as follows. In section 2 we discuss a one-parameter family of potentials with kink tails of the form $et et$, where $e, t$ denote exponential and power-tower tail, respectively. By $et et$ we mean a mirror kink with two tails $et$ facing a kink with two tails $te$. We discuss the stability analysis of these kink solutions, which are expressed in terms of the exponential integral function $Ei(x)$ [14, 15] and show that there is no gap between the zero mode and the beginning of the continuum. In section 3 we consider a two-parameter family of potentials which lead to kink tails of the form $pt tp$, where $p, t$ denote power law and power-tower tail, respectively. We also discuss the stability analysis of these solutions and show that even in this case there is no gap between the zero mode and the beginning of the continuum. In section 4 we describe briefly the recipe for constructing kink solutions with power-tower type of tail. In section 5 we discuss the nature of the kink interaction in models discussed in the paper. Our main conclusions are summarized in section 6.

2. Models with tails of the form $et et$

In this section we consider a one parameter family of potentials of the form

$$V(\phi) = \frac{1}{2} \phi^2 m + \frac{1}{2}(1/2) \ln(\phi^2)^2, \quad m \geq 1.$$  \hspace{1cm} (1)

These potentials have degenerate minima at $\phi = 0, \pm 1$ with $V_{\text{min}} = 0$ while they have degenerate maxima at

$$\phi_{\text{max}} = \pm e^{-1/(m+1)}, \quad V_{\text{max}} = \frac{1}{2e^x(m+1)^x}.$$  \hspace{1cm} (2)

Thus notice that while $\phi_{\text{max}}(m = 1) = \pm e^{-1/2}$, as $m$ becomes larger, $\phi_{\text{max}}$ moves towards $\pm 1$. On the other hand while, $V_{\text{max}}(m = 1) = \frac{1}{8e^x}$, as $m$ becomes larger, $V_{\text{max}}$ decreases progressively towards zero. All these models for any integer $m$ admit a kink from 0 to 1 and a mirror kink from $-1$ to 0 (and corresponding antikinks) with tails of the form $et et$. Here $et et$ denotes that a mirror kink with two tails of the type $e$ and $t$ facing a kink with two tails of the type $t$ and $e$. The potential given by equation (1) is shown in figure 1 for different values of $m$.

2.1. Kink solution

In general, for a given potential $V(\phi)$, the static field equation is

$$\frac{d^2 \phi}{dx^2} = \frac{dV}{d\phi}.$$  \hspace{1cm} (3)

In this paper we consider potentials which are nonnegative and have more than one global minima at which $V(\phi) = 0$. On integrating once, as is well known [1], this leads to the self-dual first order equation

$$\frac{d\phi}{dx} = \pm \sqrt{2 V(\phi)},$$  \hspace{1cm} (4)

since for the kink solution the constant of integration is zero. This is because for the kink solution, both $V(\phi)$ as well as $d\phi/dx$ vanish in the limit $x \to \pm \infty$. 


For the potential \( V(\phi) \) for \( m = 1, m = 2 \) and \( m = 4 \) (see equation (1)).

\[
\frac{d\phi}{dx} = \pm \phi^{m+1}[(1/2) \ln(\phi^2)].
\]

For the kink solution between 0 and 1 we need to solve the self-dual equation (5) with negative sign. This is easily integrated by making the substitution \( t = (1/2) \ln(\phi^2) \) and we obtain the implicit kink solution

\[
-x = \int e^{-mt} \frac{dt}{t} = Ei(-mt),
\]

where \( Ei(x) \) denotes the exponential integral function \([14, 15]\). Unfortunately, we do not know how to invert this function analytically \([16]\) and obtain \( t \) and hence \( \phi \) as a function of \( x \). However, using the Taylor series expansion of \( Ei(x) \) as given in \([14]\)

\[
Ei(x) = \gamma + \ln |x| + x + \frac{x^2}{2 \cdot 2!} + \ldots,
\]

as well as the asymptotic formula \([14]\)

\[
Ei(x) = e^x \left[ \frac{1}{x} + \frac{1}{x^2} + \frac{2!}{x^3} + \frac{3!}{x^4} + \ldots \right],
\]

we can estimate the tail behaviour around \( \phi = 0 \) as \( x \to -\infty \) and around \( \phi = 1 \) as \( x \to +\infty \). Here \( \gamma = 0.577 \) is Euler’s constant.

We find that

\[
\lim_{x \to -\infty} \phi_K^m(x) \ln[\phi_K(x)] = \frac{1}{m\gamma}, \quad \lim_{x \to +\infty} \phi_K(x) = 1 - \frac{e^{-(x+\gamma)} m}{m}.
\]
Figure 2. Kink solution $\phi(x)$ for $m = 1$, $m = 2$ and $m = 4$ (see equation (6)).

It is worth pointing out that the asymptotic behaviour around $\phi = 0$ (as $x \to -\infty$) in equation (9) can also be written as

$$\lim_{x \to -\infty} \phi_K(x)^m = e^{1/m},$$

which is known in the literature as the power-tower function of order two [12] or tetration [13].

Proceeding in the same way, we can immediately write down the corresponding mirror kink solution as well as the corresponding antikink and mirror antikink solutions.

For example, the corresponding mirror antikink solution is given by

$$\lim_{x \to -\infty} \phi_{MK}(x) = -\frac{1}{e^{(x-\gamma)/m}}, \quad \lim_{x \to +\infty} \phi_{MK}(x) = \frac{1}{e^{(x-\gamma)/m}}.$$  

Note that as $x$ goes from $-\infty$ to $+\infty$, $\phi_{MK}$ goes from 1 to 0.

Finally, the corresponding mirror kink solution is given by

$$\lim_{x \to -\infty} \phi_M(x) = 1 - \frac{e^{(x-\gamma)/m}}{m}, \quad \lim_{x \to +\infty} \phi_M(x) = 1 + \frac{e^{(x-\gamma)/m}}{m}.$$  

Note that as $x$ goes from $-\infty$ to $+\infty$, $\phi_M$ goes from 0 to 1.
Note that as $x$ goes from $-\infty$ to $+\infty$, $\phi_{MK}$ goes from $-1$ to $0$. Also, note the relationship 
\begin{equation}
\phi_{AK}(x) = -\phi_{MK}(x). \tag{15}
\end{equation}

We can invert equation (6) numerically and obtain the kink solution $\phi_K$ as a function of $x$. Kink profiles for three different values of $m$ are depicted in figure 2. Note that with increasing $m$, the approach to $\phi = 0$ for large negative $x$ becomes progressively slower in accordance with the power-tower function. In other words, for large $m$ the kink profile tends to become symmetric.

### 2.2. Kink mass

One can easily calculate the kink mass for the entire family of potentials. In particular, for the kink potential as given by equation (1), the kink mass is given by
\begin{equation}
M_K = \pm \int_0^1 \sqrt{2V(\phi)} d\phi, \tag{16}
\end{equation}
which in our case is given by
\begin{equation}
M_K = -\int_0^1 d\phi \phi^{m+1} \ln(\phi)d\phi = \frac{1}{(m+2)^2}. \tag{17}
\end{equation}
Note that we must choose the $-ve$ sign in equation (16) since $\ln(\phi)$ is negative between $\phi = 0$ to $\phi = 1$. Also observe that the kink mass decreases as $m$ increases.

### 2.3. Stability analysis

Next, we perform the stability analysis of the above kink solution and show that, akin to the kinks with the power law tail \cite{4}, for all the above kink solutions, there is no gap between the zero mode and the beginning of the continuum. First we perturb around the kink solution $\phi(x,t) = \phi_K(x) + \eta(x)e^{-i\omega t}$. Then, one can show that to the lowest order in $\eta$ one gets a Schrödinger-like equation with kink potential $V_K(x)$, see equation (19).

In this case, since the self-dual first order equation is as given by (5) (with minus sign) hence the kink zero mode is given by
\begin{equation}
\eta_0(x) = \frac{d\phi_K}{dx} \propto [\phi_K(x)]^{m+1} \ln[\phi_K(x)], \tag{18}
\end{equation}
where $\phi_K$ is the kink solution. The above zero mode $\eta_0$ is clearly nodeless and vanishes as $x \to \pm \infty$ since as $x$ goes from $-\infty$ to $+\infty$, $\phi$ varies from 0 to 1.

Finally, we also calculate the corresponding kink potential $V_K(x)$ which appears in the stability equation
\begin{equation}
-\frac{d^2\eta}{dx^2} + V_K(x)\eta = \omega^2 \eta, \tag{19}
\end{equation}
where $V_K(x) = \frac{d^2V(\phi)}{d\phi^2}$, evaluated at $\phi = \phi_K(x)$. On using the kink potential as obtained from equation (1) we find that
\begin{equation}
V_K(x) = \frac{d^2V(\phi_K)}{d\phi^2} = [\phi_K(x)]^{2m} [m + 1](2m + 1)[\ln(\phi_K)]^2 \\
+ (4m + 3) \ln(\phi_K) + 1, \tag{20}
\end{equation}
and hence it is clear that $V_K(x = +\infty) = 1$, $V_K(x = -\infty) = 0$ so that the continuum begins at $\omega^2 = 0$, i.e. there is no gap between the zero mode and the beginning of the continuum.

3. Models with tails of the form ptp

In this section we present a two-parameter family of potentials of the form

$$V(\phi) = (1/2)\phi^2 (n+2)(1/2 \ln(\phi^2))^{2n+2}, \quad m, n \geq 1. \quad (21)$$

These potentials have degenerate minima at $\phi = 0$, $\pm 1$ with $V_{\min} = 0$ while they have degenerate maxima at

$$\phi_{\text{max}} = \pm e^{-(n+1)/(m+1)}, \quad V_{\text{max}} = \frac{1}{2e^{2(n+1)}} \left[ \frac{(n+1)}{(m+1)} \right]^{2(n+1)}. \quad (22)$$

Notice that both $\phi_{\text{max}}$ and $V_{\text{max}}$ depend on two parameters $m$ and $n$. Notice also that for a fixed $m$, as $n \to \infty$, $\phi_{\text{max}} \to 0$ and $V_{\text{max}} \to 0$. On the other hand, for a fixed $n$, as $m \to \infty$, $\phi_{\text{max}} \to 1$ and $V_{\text{max}} \to 0$. Finally, for $m = n$, $\phi_{\text{max}} = \pm 1/e$ and the corresponding $V_{\text{max}} = 1/2e^{2(n+1)}$. It is interesting to note that for a given $m$, all the potentials as given by equation (21) with arbitrary integer $n$ have the same value $V(\phi) = 1/2e^{2(n+1)}$ in case $\phi = \pm 1/e$ or $V(\phi) = 1/2e^{2(n+1)}$ in case $\phi = \pm 1$. The potential in equation (21) is shown in figure 3 for $n = 1$ and three different values of $m$. In contrast, in figure 4 the potential in equation (21) is shown for $m = 1$ and $n = 1, 2, 3, 4$.

All these models, for any integers $m$ and $n$ admit a kink from 0 to 1 and a mirror kink from $-1$ to 0 (and corresponding antikinks) with tails of the form ptp. Here ptp denotes that a mirror kink with two tails of the type $p$ and $t$ facing a kink with two tails of the type $t$ and $p$.

In order to obtain the kink solution from 0 to 1, we need to solve the self-dual equation

$$\frac{d\phi}{dx} = \pm e^{m+1}\left[ (1/2 \ln(\phi^2)) \right]^{n+1}. \quad (23)$$

This is easily done by making the substitution $t = (1/2 \ln(\phi^2))$ and we obtain

$$\pm x = \int e^{-mt} t^{n+1} \, dt. \quad (24)$$

This is readily integrated using [15]

$$\int e^{ax} x^n \, dx = -e^{ax} \sum_{k=1}^{k=n-1} \frac{a^{k-1}}{(n-1)(n-2) \ldots (n+1-k)x^{n-k} + a^{n-1}(n-1)! Ei(ax)}. \quad (25)$$

We thus obtain

$$\pm x = -e^{-mt} \sum_{k=1}^{k=n} \frac{(-m)^{k-1}}{n(n-1) \ldots (n+1-k)x^{n+1-k} + (-m)^n n! Ei(-mt)}. \quad (26)$$

Using equations (7) and (8) we can now find the kink tail around both $\phi = 0$ and $\phi = 1$. We note that in order to find the self-dual kink solution between 0 and 1, we need to take $+x (-x)$
in equation (26) depending on whether \( n \) is an odd (or even) integer. Using equation (8) we then find that for any integer \( n \)

\[
\lim_{x \to -\infty} \phi_K(x) = \left( -\frac{1}{m} \right)^{\frac{1}{n+1}}.
\]

On the other hand, for both odd and even integer \( n \) using equation (7) we find that

\[
\lim_{x \to -\infty} \phi_K(x) = 1 - \frac{1}{\left( n x + \frac{e^{nx}}{(n-1)^2} \right)^{\frac{1}{n}}}
\]

It is worth pointing out that for even integer \( n \), the asymptotic behaviour around \( \phi = 0 \) (as \( x \to -\infty \)) in equation (27) can also be written as

\[
\lim_{x \to -\infty} \phi_K(x) = e^{1/(mx)}^{1/(n+1)}.
\]

On the other hand, for odd integer \( n \), the asymptotic behaviour around \( \phi = 0 \) (as \( x \to -\infty \)) in equation (27) can be written as

\[
\lim_{x \to -\infty} \phi_K(x) = e^{1/(-mx)}^{1/(n+1)},
\]

which is the power-tower function of order two [12]. It is also related to the iterated or repeated exponentiation, i.e. tetration [13].

Proceeding in the same way as in section 2, we can immediately write down the corresponding mirror kink solution as well as the corresponding antikink and mirror antikink solutions.

Figure 3. Potential \( V(\phi) \) for \( n = 1 \) and \( m = 1, m = 2 \) and \( m = 4 \) (see equation (21)).
Figure 4. Potential $V(\phi)$ for $m = 1$, and $n = 1, 2, 3, 4$ (see equation (21)).

For example, the corresponding mirror antikink solution is given by

$$\lim_{x \to -\infty} \phi_{\text{MAK}}^m(x) \left( \frac{1}{2} \ln[\phi_{\text{MAK}}^2(x)] \right)^{n+1} = \frac{(-1)^{n+m}}{m},$$

$$\lim_{x \to +\infty} \phi_{\text{MAK}}(x) = -1 + \frac{1}{\left[ n + \frac{\mu^n}{(n-1)s} \right]^{1/n}}.$$  \hfill (31)

Note that as $x$ goes from $-\infty$ to $+\infty$, $\phi_{\text{MAK}}$ goes from 0 to $-1$. As in section 2, in this case too the kink and the mirror antikink solutions satisfy the relationship

$$\phi_{\text{MAK}}(x) = -\phi_{K}(x).$$  \hfill (32)

On the other hand, the corresponding antikink solution is given by

$$\lim_{x \to -\infty} \phi_{\text{AK}}^m(x) = 1 - \frac{1}{\left[ -nx + \frac{\mu^n}{(n-1)s} \right]^{1/n}},$$

$$\lim_{x \to +\infty} \phi_{\text{AK}}^m(x)[\ln[\phi_{\text{AK}}^m(x)] + 1]^{n+1} = \frac{(-1)^{n+1}}{m}. $$ \hfill (33)

Note that as $x$ goes from $-\infty$ to $+\infty$, $\phi_{\text{AK}}$ goes from 1 to 0.

Finally, the corresponding mirror kink solution is given by

$$\lim_{x \to -\infty} \phi_{\text{MK}}(x) = -1 + \frac{1}{\left[ -nx + \frac{\mu^n}{(n-1)s} \right]^{1/n}}.$$
Kink solution \( \phi(x) \) for \( n = 1 \) and \( m = 1, 2, 4 \) (see equation (24)). Inset shows in detail how the three profiles cross each other for \( x > 0 \).

\[ \lim_{x \to +\infty} \phi_{MK}^m(x) \left( \frac{1}{2} \ln[\phi_{MK}^2(x)] \right)^{n+1} = \frac{(-1)^{n+m+1}}{m}. \] (34)

Note that as \( x \) goes from \( -\infty \) to \( +\infty \), \( \phi_{MK} \) goes from \(-1\) to \(0\). As in section 2, the mirror kink and the antikink solutions given above also satisfy the relationship

\[ \phi_{AK}(x) = -\phi_{MK}(x). \] (35)

Equation (24) can be inverted numerically and the kink profile is depicted in figure 5 in case \( n = 1 \) and \( m = 1, 2, 4 \). On the other hand, in figure 6 we depict the kink profile in case \( m = 1 \) and \( n = 1, 2, 3, 4 \).

For \( x \to +\infty \) the kink tails approach \( \phi = 1 \) as a power-law whereas for \( x \to -\infty \) the kink tails approach \( \phi = 0 \) as a power-tower function. With increasing \( m \) for large negative \( x \) the tails approach \( \phi = 0 \) progressively slowly. In other words, for large \( m \) the kink profile tends to become symmetric.

3.1. Kink mass

One can easily calculate the kink mass for the entire family of potentials. Specifically, for the kink potential given by equation (21), the kink mass is given by

\[ M_K = \pm \int_0^1 d\phi \phi^{m+1} [\ln(\phi)]^{n+1} d\phi = \frac{(n+1)!}{(m+2)^{n+2}}, \] (36)

where the + sign is for \( n \) odd and the − sign is for \( n \) even. Note again that the kink mass decreases as \( m \) increases keeping \( n \) fixed. On the other hand, the kink mass increases as \( n \) increases keeping \( m \) fixed.
3.2. Stability analysis

Next, we perform the stability analysis of the kink solutions discussed in this section and show that like in the previous section (as well as the kinks with the power law tail), for all the kink solutions of this section, there is no gap between the zero mode and the onset of the continuum.

Using the self-dual first order equation (23), the kink zero mode is given by

\[ \eta_0(x) = \frac{d \phi_K}{dx} = [\phi_K(x)]^{m+1}(\ln[\phi_K(x)])^{n+1}, \]  

(37)

which is clearly nodeless. The corresponding kink stability potential is given by

\[ V_K(x) = \frac{d^2 V(\phi_K)}{d\phi^2} = [\phi_K(x)]^{2m} [(m + 1)(2m + 1)\ln(\phi_K)]^{2n+2} + (4m + 3)(n + 1)\ln(\phi_K)]^{2n+1} + (n + 1)(2n + 1)\ln(\phi_K)]^{2n}. \]  

(38)

Thus \( V(x = +\infty) = V(x = -\infty) = 0 \) and hence the continuum begins from \( \omega^2 = 0 \) so that there is no gap between the zero mode and the onset of the continuum.

4. Recipe for obtaining power-tower kink tails

By now we have a large number of kink bearing models which admit kinks with a variety of tails such as exponential [1], power-law [2, 3, 5, 6], super-exponential [7–10], super-super-exponential [11] and power-tower. It is then natural to inquire if there is a recipe for constructing models which admit such a diverse variety of kink tails. In this context we might add that the
recipe for constructing kink solutions with an exponential or a power law tail is well known. For completeness we mention it first and then give the recipe for constructing the kink solutions with either super-exponential or power-tower type of tail.

Since a kink has finite energy it implies that the solution must approach one of the minima (vacua), say \( \phi_0 \), of the theory as \( x \to \pm \infty \). If the lowest non-vanishing derivative of the potential at the minimum has order \( m \), then by Taylor series expansion of the potential at the minimum and writing the field close to it as \( \phi = \phi_0 + \eta \), one finds that the self-dual first order equation in \( \eta \) implies that (assuming that the potential vanishes at the minimum)

\[
\frac{d\eta}{dx} \propto \eta^{m/2}. \tag{39}
\]

Thus if \( m = 2 \) then \( \eta \propto e^{-\alpha x} \) (i.e. exponential tail) while if \( m > 2 \) then \( \eta \propto 1/x^{2/(m-2)} \) (i.e. power law tail).

In our recent paper about the super-exponential tail \[7\], we have shown that if instead

\[
\frac{d\eta}{dx} \propto \eta \ln(\eta^2), \tag{40}
\]

then \( \eta \propto e^{-e^{-\alpha x}} \), so that there is a super-exponential tail.

On the other hand using the results of this paper it is clear that if

\[
\frac{d\eta}{dx} \propto \eta^{m+1}[\ln(\eta^2)]^{n+1}, \quad m, n \geq 1, \tag{41}
\]

then \( \eta \) is a solution of the equation

\[
\eta^m[\ln(\eta^2)]^{n+1} = (-1)^n \frac{1}{mx}, \tag{42}
\]

which leads to power-tower kink tails.

5. Behaviour of power-tower tails

We will now discuss the possible power law behaviour and kink interactions corresponding to the various power-tower type of tails obtained here.

(a) In this paper we have not been able to calculate the force \[17\] between the \((-1, 0)\) mirror kink (which has exponential and power-tower type of tails around \(-1\) and 0, respectively) and the \((0, 1)\) kink (which has power-tower type and exponential tails around 0 and 1, respectively) since the two ends facing each other have power-tower type of tail and in this case it is not straightforward to invert and obtain the behaviour of the tail as a function of \( x \) when \( x \to -\infty \). As an illustration, consider the asymptotic behaviour around \( \phi = 0 \) in case \( x \to -\infty \) as given by equation (9) with \( m = 1 \), i.e.

\[
\lim_{x \to -\infty} \phi_K(x) \ln(\phi_K(x)) = \frac{1}{x}. \tag{43}
\]

If \( \ln(\phi(x)) \) were not there then we know that for large negative \( x \), \( \phi(x) \propto -1/x \). In this connection we notice that in a recent publication \[7\] we have shown that in the case of the potential \( V(\phi) = (1/2)\phi^2[(1/2) \ln(\phi^2)]^2 \), while the kink tail around \( \phi = 0 \) would have been an exponential tail in case there were no \( \ln(\phi^2) \) term present, because of the \( \ln(\phi^2) \) term, the kink tail actually gets even weaker and is in fact super-exponential. Taking
Figure 7. $\phi(x)$ vs $x$ obtained by inverting equation (43) and compared with $\phi = 1/x$ and $\phi = 1/x^{1.25}$. Inset shows the comparison at large values of $x$.

Figure 8. $\phi(x)$ vs $X = 4x$ obtained by inverting equation (45) for the case $m = 4$.

... this as a guide, we speculate that corresponding to the power-tower form as given by equation (43), the behaviour of $\phi_K(x)$ for large negative $x$ should be of the form

$$\lim_{x \to -\infty} \phi_K(x) = \frac{1}{(-x)^{\epsilon_{1.0}}}$$

$\epsilon_{1.0} > 0$.

(44)
Figure 9. φ(x) vs x obtained by inverting equation (27) for the case \( m = 1 \) and \( n = 2 \), and compared with \( φ = 1/x \), \( φ = 1/x^{1.2} \) and \( φ = 1/x^{1.5} \). Inset shows the comparison for large values of x.

Here by \( \epsilon_{1,0} \) one means \( \epsilon_{m=1,n=0} \) corresponding to \( φ^m \) and \( [\ln(φ)]^{n+1} \) in equation (21) with \( m = 1, n = 0 \). We have inverted equation (43) numerically and from figure 7 we see that it can be fitted in the form of equation (44) with \( \epsilon_{1,0} \) approximately equal to 0.25. This would imply that the potential around \( φ = 0 \) is of the form \( φ^k \) with \( k = 9/5 \). We might add here that the new Manton formalism [18–20], even though developed for integer \( k \) is also valid for any real number \( k \). Using this information, one can estimate the force between the \((-1, 0)\) mirror kink and the \((0, 1)\) kink using the new Manton formalism and show that the kink–kink force would vary like \( R^{-9/2} \), where \( R \) is the distance between the two kinks.

In the same way, one can numerically invert for any \( m \) the equation around \( φ = 0 \) for large negative x as given by (9), i.e.

\[
\lim_{x \to -\infty} φ_κ(x)^m \ln(φ_κ(x)) = \frac{1}{mx},
\]

and try to numerically estimate the corresponding exponent. As an illustration, in figure 8 we have inverted equation (45) for the case of \( m = 4 \). We would like to point out here that in case there were no \( \ln(φ^2) \) term around \( φ = 0 \) in equation (1), then using the recent Manton formalism [18–20] one would have immediately predicted that the force between \((-1, 0)\) mirror kink and \((0, 1)\) kink would vary like \( R^{-2(m+1)/m} \), where \( R \) is the distance between the two kinks.

(b) Generalization of the above discussion in the case of the power-tower equation (27) with arbitrary \( m \) and \( n \) is now straightforward. In particular, one can numerically invert for any \( m, n \) equation (27) around \( φ = 0 \) for large negative x and try to numerically estimate the corresponding exponent. We speculate that the behaviour of \( φ_κ(x) \) for large negative x
should be of the form

\[ \lim_{x \to -\infty} \phi_{K}(x) = \frac{1}{(-mx)^{1/m+\epsilon_{m,n}}}, \quad \epsilon_{m,n} > 0. \]  

(46)

As an illustration, in Figure 9 we have numerically inverted equation (27) in case \( m = 1, n = 1 \) and we find that \( \epsilon_{1,1} \) is approximately equal to 0.5 which is larger than \( \epsilon_{1,0} \); the latter is approximately 0.25.

In case the exponent is 0.5, this would imply that the potential around \( \phi = 0 \) is of the form \( \phi^{2k} \) with \( k = 5/3 \). Using this information, one can estimate the force between \((-1, 0)\) mirror kink and \((0, 1)\) kink using the new Manton formalism [18–20] and show that the kink–kink force would vary like \( R^{-5} \), where \( R \) is the distance between the two kinks.

(c) Looking at the two examples of \( m = 1, n = 0 \) and \( m = 1, n = 1 \) it is immediately clear that \( \epsilon_{1,0} < \epsilon_{1,1} \). We speculate that in general for arbitrary \( m, n \) we will have the inequality \( \epsilon_{m,n_1} < \epsilon_{m,n_2} \) in case \( n_1 < n_2 \).

(d) In the same way, in Figure 10 we have numerically inverted equation (27) in case \( m = 1, n = 9 \). It is clear from the figure that the exponent \( \epsilon_{1,9} \) is much bigger than 0.5. It thus appears that as \( n \) becomes progressively larger, effectively the kink tail around \( \phi = 0 \) for large negative \( x \) will approach an exponential tail.

We first elaborate our argument in the case of \( m = 1 \) and arbitrary \( n \). For the \( m = 1 \) and arbitrary \( n \) case, the corresponding exponent is \( \epsilon_{1,n} \). This would imply that the potential around \( \phi = 0 \) is of the form \( \phi^{2k} \) with

\[ k = \frac{2 + \epsilon_{1,n}}{1 + \epsilon_{1,n}}. \]  

(47)

Now we have seen from the examples of \( m = 1, n = 0 \); \( m = 1, n = 1 \) and \( m = 1, n = 9 \) that as \( n \) increases \( \epsilon_{1,n} \) becomes progressively larger. In other words, for very large \( n \) we

\[ \phi(x) vs x obtained by inverting equation (27) with m = 1 and n = 9. \]
expect that $\epsilon_{1,n} \gg 2$ and hence for very large $n$, the factor $k$ as defined by equation (47) tends to 1 which corresponds to an exponential tail. However, we would like to emphasize that no matter how large $n$ is, so long as it is finite, $k$ is strictly greater than one such that for all finite $n$, the kink tail has power law fall off thereby justifying the name power-tower. Thus kinks with power-tower-type of tails provide a bridge between kinks with power law tail and kinks with exponential tail.

Generalization to arbitrary $m$ is now straightforward. We surmise that for very large $n$, no matter what $m$ is, $m_{e,n} \gg m + 1$, so that even in this case $k$ would tend to one which corresponds to an exponential tail, although for any large but finite $m,n$, it will strictly be greater than one.

We hope to address some of these issues in the near future.

6. Summary and open questions

In this paper we have considered a continuous one-parameter family of potentials as given by equation (1), all of which have kink tails of the form $e^t$ where $e$ and $t$ correspond to exponential and power-tower type of tail, respectively. Similarly, in section 3 we have constructed a two-parameter family of potentials given by equation (21), all of which admit kink tails of the form $pt^p$ where $p$ corresponds to power law type of tail. For all these cases we have calculated the corresponding kink masses. Furthermore, we have shown that the kink stability equation in all these cases is such that there is no gap between the zero mode and the beginning of the continuum in the relevant Schrödinger-like equation. Based on the new Manton formalism [18–20] we have also provided qualitative estimates about the kink–kink force in case the tails facing each other are of power-tower type.

One obvious question is, can one similarly construct at least a one-parameter family of potentials which give tails of the form $e^t e^t$, $pt^p t^p$ and $p e^t e^t$, $e^t p t^p$ and $p t^p t^p$ as well as the mixed tails of the form $e^t p t^p$, $e^t p t^p$ and $e^t p t^p$? Finally, can one construct models with an admixture of super-exponential tails and exponential and/or power law and/or power-tower type of tails?

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ORCID iDs

Avadh Saxena © https://orcid.org/0000-0002-3374-3236

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