TENSOR PRODUCTS OF LEAVITT PATH ALGEBRAS

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Abstract. We compute the Hochschild homology of Leavitt path algebras over a field $k$. As an application, we show that $L_2$ and $L_2 \otimes L_2$ have different Hochschild homologies, and so they are not Morita equivalent; in particular they are not isomorphic. Similarly, $L_\infty$ and $L_\infty \otimes L_\infty$ are distinguished by their Hochschild homologies and so they are not Morita equivalent either. By contrast, we show that $K$-theory cannot distinguish these algebras; we have $K_* (L_2) = K_* (L_2 \otimes L_2) = 0$ and $K_* (L_\infty) = K_* (L_\infty \otimes L_\infty) = K_*(k)$.

1. Introduction

Elliott’s theorem [21] stating that $O_2 \otimes O_2 \cong O_2$ plays an important role in the proof of the celebrated classification theorem of Kirchberg algebras in the UCT class, due to Kirchberg [14] and Phillips [19]. Recall that a Kirchberg algebra is a purely infinite, simple, nuclear and separable C*-algebra. The Kirchberg-Phillips theorem states that this class of simple C*-algebras is completely classified by its topological $K$-theory. The analogous question whether the algebras $L_2$ and $L_2 \otimes L_2$ are isomorphic has remained open for some time. Here $L_2$ is the Leavitt algebra of type $(1, 2)$ over a field $k$ (see [17]), that is, the $k$-algebra with generators $x_1, x_2, x_1^*, x_2^*$ and relations given by $x_i^* x_j = \delta_{i,j}$ and $\sum_{i=1}^2 x_i x_i^* = 1$.

In this paper we obtain a negative answer to this question. Indeed, we analyze a much larger class of algebras, namely the tensor products of Leavitt path algebras of finite quivers, in terms of their Hochschild homology, and prove that, for $1 \leq n < m \leq \infty$, the tensor products $E = \bigotimes_{i=1}^n L(E_i)$ and $F = \bigotimes_{j=1}^m L(F_j)$ of Leavitt path algebras of non-acyclic finite quivers $E_i, F_j$, are distinguished by their Hochschild homologies (Theorem 5.1). Because Hochschild homology is Morita invariant, we conclude that $E$ and $F$ are not Morita equivalent for $n < m$. Since $L_2$ is the Leavitt path algebra of the graph with one vertex and two arrows, we obtain that $L_2 \otimes L_2$ and $L_2$ are not Morita equivalent; in particular they are not isomorphic.

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Recall that, by a theorem of Kirchberg [15], a simple, nuclear and separable $C^*$-algebra $A$ is purely infinite if and only if $A \otimes \mathcal{O}_\infty \cong A$. We also show that the analogue of Kirchberg’s result is not true for Leavitt algebras. We prove in Proposition 5.3 that if $E$ is a non-acyclic quiver, then $L_\infty \otimes L(E)$ and $L(E)$ are not Morita equivalent, and also that $L_\infty \otimes L_\infty$ and $L_\infty$ are not Morita equivalent.

Using the results in [6] we prove that the algebras $L_2$ and $L_2 \otimes L(F)$, for $F$ an arbitrary finite quiver, have trivial $K$-theory: all algebraic $K$-theory groups $K_i$, $i \in \mathbb{Z}$, vanish on them (this follows from Lemma 6.1 and Proposition 6.2). We also compute $K_*(L(F)) = K_*(L_\infty \otimes L(F))$ and that $K_*(L_\infty) = K_*(L_\infty \otimes L_\infty) = K_*(k)$ is the $K$-theory of the ground field (see Proposition 6.3 and Corollary 6.4). This implies in particular that, in contrast with the analytic situation, no classification result, in terms solely of $K$-theory, can be expected for a class of central, simple $k$-algebras, containing all purely infinite simple unital Leavitt path algebras, and closed under tensor products. It is worth mentioning that an important step towards a $K$-theoretic classification of purely infinite simple Leavitt path algebras of finite quivers has been achieved in [2].

We refer the reader to [4], [8] and [20] for the basics on Leavitt algebras, Leavitt path algebras and graph $C^*$-algebras, and to [22] for a nice survey on the Kirchberg-Phillips Theorem.

Notations. We fix a field $k$; all vectorspaces, tensor products and algebras are over $k$. If $R$ and $S$ are unital $k$-algebras, then by an $(R,S)$-bimodule we understand a left module over $R \otimes S^{op}$. By an $R$-bimodule we shall mean an $(R,R)$ bimodule, that is, a left module over the enveloping algebra $R^e = R \otimes R^{op}$. Hochschild homology of $k$-algebras is always taken over $k$; if $M$ is an $R$-bimodule, we write

$$HH_n(R,M) = \text{Tor}_n^{R^e}(R,M)$$

for the Hochschild homology of $R$ with coefficients in $M$; we abbreviate $HH_n(R) = HH_n(R,R)$.

2. Hochschild homology

Let $k$ be a field, $R$ a $k$-algebra and $M$ an $R$-bimodule. The Hochschild homology $HH_*(R,M)$ of $R$ with coefficients in $M$ was defined in the introduction; it is computed by the Hochschild complex $HH(R,M)$ which is given in degree $n$ by

$$HH(R,M)_n = M \otimes R^e \otimes R.$$

It is equipped with the Hochschild boundary map $b$ defined by

$$b(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n + (-1)^n a_n a_0 \otimes \cdots \otimes a_{n-1}.$$
Lemma 2.1. Let $I$ be a filtered ordered set and let \{(R_i, M_i) : i \in I\} be a directed system of pairs $(R_i, M_i)$ consisting of an algebra $R_i$ and an $R_i$-bimodule $M_i$, with algebra maps $R_i \to R_j$ and $R_i$-bimodule maps $M_i \to M_j$ for each $i \leq j$. Let $(R, M) = \text{colim}_i(R_i, M_i)$. Then $HH_n(R, M) = \text{colim}_i HH_n(R_i, M_i)$ ($n \geq 0$).

Let $R_i$ be a $k$-algebra and $M_i$ an $R_i$-bimodule ($i = 1, 2$). The Künneth formula establishes a natural isomorphism ([23, 9.4.1])
\[
HH_n(R_1 \otimes R_2, M_1 \otimes M_2) \cong \bigoplus_{p=0}^{n} HH_p(R_1, M_1) \otimes HH_{n-p}(R_2, M_2).
\]

Another fundamental fact about Hochschild homology which we shall need is Morita invariance. Let $R$ and $S$ be Morita equivalent algebras, and let $P \in R \otimes S^{\text{op}} - \text{mod}$ and $Q \in S \otimes R^{\text{op}} - \text{mod}$ implement the Morita equivalence. Then ([23, Thm. 9.5.6])
\[
(2.2) \quad HH_n(R, M) = HH_n(S, Q \otimes_R M \otimes_R P).
\]

Lemma 2.3. Let $R_1, \ldots, R_n$ and $S_1, \ldots, S_m$ be a finite and an infinite sequence of algebras, and let $R = \bigotimes_{i=1}^{n} R_i$, $S_{\leq m} = \bigotimes_{j=1}^{m} S_j$ and $S = \bigotimes_{j=1}^{\infty} S_j$. Assume:

1. $HH_q(R_i) \neq 0 \neq HH_q(S_j)$ (for $0 \leq q \leq n$, $1 \leq i \leq n$, $1 \leq j$).
2. $HH_p(R_i) = HH_p(S_j) = 0$ for $p \geq 2$, $1 \leq i \leq n$, $1 \leq j$.
3. $n \neq m$.

Then no two of $R, S_{\leq m}$ and $S$ are Morita equivalent.

Proof. By the Künneth formula, we have
\[
HH_n(R) = \bigotimes_{i=1}^{n} HH_1(R_i) \neq 0, \quad HH_p(R) = 0 \quad p > n.
\]

By the same argument, $HH_p(S_{\leq m})$ is nonzero for $p = m$, and zero for $p > m$. Hence if $n \neq m$, $R$ and $S_{\leq m}$ do not have the same Hochschild homology and therefore they cannot be Morita equivalent, by (2.2). Similarly, by Lemma 2.1, we have
\[
HH_n(S) = \bigoplus_{J \subseteq \{1, \ldots, m\}, |J| = n} \left( \bigotimes_{j \in J} HH_1(S_j) \right) \otimes \left( \bigotimes_{j \not\in J} HH_0(S_j) \right).
\]
so that $HH_n(S)$ is nonzero for all $n \geq 1$, and thus it cannot be Morita equivalent to either $R$ or $S_{\leq m}$. □

3. HOCHSCHILD HOMOLOGY OF CROSSED PRODUCTS

Let $R$ be a unital algebra and $G$ a group acting on $R$ by algebra automorphisms. Form the crossed-product algebra $S = R \rtimes G$, and consider the Hochschild complex $HH(S)$. For each conjugacy class $\xi$ of $G$, the graded submodule $HH^\xi(S) \subset HH(S)$ generated in degree $n$ by the elementary tensors $a_0 \rtimes g_0 \otimes \cdots \otimes a_n \rtimes g_n$ with $g_0 \cdots g_n \in \xi$ is a subcomplex, and we have a direct sum decomposition $HH(S) = \bigoplus \xi HH^\xi(S)$. The following theorem of Lorenz describes the complex $HH^\xi(S)$ corresponding to the conjugacy class $\xi = \langle g \rangle$ of an element $g \in G$ as hyperhomology over the centralizer subgroup $Z_g \subset G$.

**Theorem 3.1.** [16]. Let $R$ be a unital $k$-algebra, $G$ a group acting on $R$ by automorphisms, $g \in G$ and $Z_g \subset G$ the centralizer subgroup. Let $S = R \rtimes G$ be the crossed product algebra, and $HH_\langle g \rangle(S) \subset HH(S)$ the subcomplex described above. Consider the $R$-submodule $S_g = R \rtimes g \subset S$. Then there is a quasi-isomorphism

$$HH_\langle g \rangle(S) \sim \rightarrow H(Z_g, HH(R, S_g)).$$

In particular we have a spectral sequence

$$E^2_{p,q} = H_p(Z_g, HH_q(R, S_g)) \Rightarrow HH_{p+q}(R, S_g).$$

**Remark 3.2.** Lorenz formulates his result in terms of the spectral sequence alone, but his proof shows that there is a quasi-isomorphism as stated above.

Let $A$ be a not necessarily unital $k$-algebra, write $\tilde{A}$ for its unitalization. Recall from [24] that $A$ is called $H$-unital if the groups $\operatorname{Tor}_n^\tilde{A}(k, A)$ vanish for all $n \geq 0$. Wodzicki proved in [24] that $A$ is $H$-unital if and only if for every embedding $A \triangleleft R$ of $A$ as a two-sided ideal of a unital ring $R$, the map

$$HH(A) \rightarrow HH(R : A) = \ker (HH(R) \rightarrow HH(R/A))$$

is a quasi-isomorphism.

**Lemma 3.3.** Theorem 3.1 still holds if the condition that $R$ be unital is replaced by the condition that it be $H$-unital.

**Proof.** Follows from Theorem 3.1 and the fact, proved in [11, Prop. A.6.5], that $R \rtimes G$ is $H$-unital if $R$ is. □

Let $R$ be a unital algebra, and $\phi : R \rightarrow pRp$ a corner isomorphism. As in [7], we consider the skew Laurent polynomial algebra $R[t_+, t_-, \phi]$; this is the $R$-algebra
generated by elements $t_+$ and $t_-$ subject to the following relations.

$$
t_+a = \phi(a)t_+ \\
\phi(a)(t_-) = \phi(a)t_-
$$

$t_-t_+ = 1$

$t_+t_- = \phi$. 

Observe that the algebra $S = \mathbb{R}[t_+, t_-] \subseteq \mathbb{R}$ is graded by $deg(r) = 0$, $deg(t_\pm) = \pm 1$. The homogeneous component of degree $n$ is given by

$$
R[t_+, t_-, \phi]_n = \begin{cases} 
\mathbb{R} & n = 0 \\
\mathbb{R}^n & n > 0
\end{cases}
$$

Proposition 3.4. Let $R$ be a unital ring, $\phi: R \rightarrow pRp$ a corner isomorphism, and $S = \mathbb{R}[t_+, t_-, \phi]$. Consider the weight decomposition $HH(S) = \bigoplus_{m \in \mathbb{Z}} mHH(S)$. There is a quasi-isomorphism

$$
(3.5) 
\sim Cone(1 - \phi: HH(R, S_m) \rightarrow HH(R, S_m)).
$$

Proof. If $\phi$ is an automorphism, then $S = \mathbb{R} \times_{\phi} \mathbb{Z}$, the right hand side of (3.5) computes $HH(\mathbb{Z}, HH(R, S_m))$, and the proposition becomes the particular case $G = \mathbb{Z}$ of Theorem 3.1. In the general case, let $A$ be the colimit of the inductive system

$$
R \xrightarrow{\phi} R \xrightarrow{\phi} R \xrightarrow{\phi} \cdots
$$

Note that $\phi$ induces an automorphism $\hat{\phi}: A \rightarrow A$. Now $A$ is $H$-unital, since it is a filtering colimit of unital algebras, and thus the assertion of the proposition is true for the pair $(A, \hat{\phi})$, by Lemma 3.3. Hence it suffices to show that for $B = A \times_{\phi} \mathbb{Z}$ the maps $HH(S) \rightarrow HH(B)$ and $Cone(1 - \phi: HH(R, S_m) \rightarrow HH(R, S_m)) \rightarrow Cone(1 - \phi: HH(A, B_m) \rightarrow HH(A, B_m))$ ($m \in \mathbb{Z}$) are quasi-isomorphisms. The analogous property for $K$-theory is shown in the course of the third step of the proof of [6, Thm. 3.6]. Since the proof in loc. cit. uses only that $K$-theory commutes with filtering colimits and is matrix invariant on those rings for which it satisfies excision, it applies verbatim to Hochschild homology. This concludes the proof. \qed

4. Hochschild homology of the Leavitt path algebra

Let $E = (E_0, E_1, r, s)$ be a finite quiver and let $\hat{E} = (E_0, E_1 \sqcup E_1^*, r, s)$ be the double of $E$, which is the quiver obtained from $E$ by adding an arrow $\alpha^*$ for each arrow $\alpha \in E_1$, going in the opposite direction. The Leavitt path algebra of $E$ is the algebra $L(E)$ with one generator for each arrow $\alpha \in E_1$ and one generator $p_i$
for each vertex \( i \in E_0 \), subject to the following relations

\[
p_{i}p_{j} = \delta_{i,j}p_{i}, \quad (i, j \in E_0)
\]

\[
p_{s(\alpha)}\alpha = \alpha = \alpha p_{r(\alpha)}, \quad (\alpha \in \hat{E}_1)
\]

\[
\alpha^*\beta = \delta_{\alpha,\beta}p_{r(\alpha)}, \quad (\alpha, \beta \in E^1)
\]

\[
p_{i} = \sum_{\alpha \in E^1, s(\alpha) = i} \alpha\alpha^*, \quad (i \in E_0 \setminus \text{Sink}(E)).
\]

The algebra \( L = L(E) \) is equipped with a \( \mathbb{Z} \)-grading. The grading is determined by \( |\alpha| = 1 \), \( |\alpha^*| = -1 \), for \( \alpha \in E^1 \). Let \( L_{0,n} \) be the linear span of all the elements of the form \( \gamma\nu^* \), where \( \gamma \) and \( \nu \) are paths with \( r(\gamma) = r(\nu) \) and \( |\gamma| = |\nu| = n \). By [8, proof of Theorem 5.3], we have \( L_{0} = \bigcup_{n=0}^{\infty} L_{0,n} \). For each \( i \in E^0 \), and each \( n \in \mathbb{Z}^+ \), let us denote by \( P(n,i) \) the set of paths \( \gamma \) in \( E \) such that \( |\gamma| = n \) and \( r(\gamma) = i \). The algebra \( L_{0,0} \) is isomorphic to \( \prod_{i \in E^0} k \). In general the algebra \( L_{0,n} \) is isomorphic to

\[
(4.1) \quad \left[ \prod_{m=0}^{n-1} \left( \prod_{i \in \text{Sink}(E)} M_{|P(m,i)|}(k) \right) \right] \times \left[ \prod_{i \in E^0} M_{|P(n,i)|}(k) \right].
\]

The transition homomorphism \( L_{0,n} \to L_{0,n+1} \) is the identity on the factors

\[
\prod_{i \in \text{Sink}(E)} M_{|P(m,i)|}(k),
\]

for \( 0 \leq m \leq n - 1 \), and also on the factor

\[
\prod_{i \in \text{Sink}(E)} M_{|P(n,i)|}(k)
\]

of the last term of the displayed formula. The transition homomorphism

\[
\prod_{i \in E^0 \setminus \text{Sink}(E)} M_{|P(n,i)|}(k) \longrightarrow \prod_{i \in E^0} M_{|P(n+1,i)|}(k)
\]

is a block diagonal map induced by the following identification in \( L(E)_0 \): A matrix unit in a factor \( M_{|P(n,i)|}(k) \), where \( i \in E^0 \setminus \text{Sink}(E) \), is a monomial of the form \( \gamma\nu^* \), where \( \gamma \) and \( \nu \) are paths of length \( n \) with \( r(\gamma) = r(\nu) = i \). Since \( i \) is not a sink, we can enlarge the paths \( \gamma \) and \( \nu \) using the edges that \( i \) emits, obtaining paths of length \( n + 1 \), and the last relation in the definition of \( L(E) \) gives

\[
\gamma\nu^* = \sum_{\{\alpha \in E_1 | s(\alpha) = i\}} (\gamma\alpha)(\nu\alpha)^*.
\]

Assume \( E \) has no sources. For each \( i \in E_0 \), choose an arrow \( \alpha_i \) such that \( r(\alpha_i) = i \). Consider the elements

\[
t_+ = \sum_{i \in E_0} \alpha_i, \quad t_- = t_+^*.
\]
One checks that \( t_- t_+ = 1 \). Thus, since \( |t_\pm| = \pm 1 \), the endomorphism

\[
\phi: L \longrightarrow L, \quad \phi(x) = t_+ x t_-
\]

is homogeneous of degree 0 with respect to the \( \mathbb{Z} \)-grading. In particular it restricts to an endomorphism of \( L_0 \). By [7, Lemma 2.4], we have

\[
L = L_0[t_+, t_- , \phi].
\]

Consider the matrix \( N'_E = [n_{i,j}] \in M_{e_0} \mathbb{Z} \) given by

\[
n_{i,j} = \#\{ \alpha \in E_1 : s(\alpha) = i, \ r(\alpha) = j \}.
\]

Let \( e'_0 = |\text{Sink}(E)| \). We assume that \( E_0 \) is ordered so that the first \( e'_0 \) elements of \( E_0 \) correspond to its sinks. Accordingly, the first \( e'_0 \) rows of the matrix \( N'_E \) are 0. Let \( N_E \) be the matrix obtained by deleting these \( e'_0 \) rows. The matrix that enters the computation of the Hochschild homology of the Leavitt path algebra is

\[
\begin{pmatrix} 0 \\ 1_{e_0-e'_0} \end{pmatrix} - N'_E: \mathbb{Z}^{e_0-e'_0} \longrightarrow \mathbb{Z}^{e_0}.
\]

By a slight abuse of notation, we will write \( 1 - N'_E \) for this matrix. Note that \( 1 - N'_E \in M_{e_0 \times (e_0-e'_0)}(\mathbb{Z}) \). Of course \( N_E = N'_E \) in case \( E \) has no sinks.

**Theorem 4.4.** Let \( E \) be a finite quiver without sources, and let \( N = N_E \). For each \( i \in E_0 \setminus \text{Sink}(E) \), and \( m \geq 1 \), let \( V_{i,m} \) be the vectorspace generated by all closed paths \( c \) of length \( m \) with \( s(c) = r(c) = i \). Let \( Z = \langle \sigma \rangle \) act on

\[
V_m = \bigoplus_{i \in E_0 \setminus \text{Sink}(E)} V_{i,m}
\]

by rotation of closed paths. We have:

\[
mHH_n(L(E)) = \begin{cases} 
\text{coker} (1 - \sigma: V_{[m]} \to V_{[m]}) & n = 0, m \neq 0 \\
\text{ker} (1 - N') & n = m = 0 \\
\text{coker} (1 - N') & n = 1, m \neq 0 \\
\text{ker} (1 - N') & n = 1, m = 0 \\
0 & n \notin \{0,1\}.
\end{cases}
\]

**Proof.** Let \( L = L(E) \), \( P = P(E) \subset L \) the path algebra of \( E \) and \( W_m \subset P \) be the subspace generated by all paths of length \( m \). For each fixed \( n \geq 1 \), and \( m \in \mathbb{Z} \), consider the following \( L_{0,n} \)-bimodule

\[
L_{m,n} = \begin{cases} 
L_{0,n} W_m L_{0,n} & m > 0 \\
L_{0,n} W^*_m L_{0,n} & m < 0.
\end{cases}
\]

Write \( L = L(E) \), and let \( _m L \) be the homogeneous part of degree \( m \); we have

\[
_m L = \bigcup_{n \geq 1} L_{m,n}.
\]
If \( m \) is positive, then there is a basis of \( L_{m,n} \) consisting of the products \( \alpha \theta \beta^* \) where each of \( \alpha, \beta \) and \( \theta \) is a path in \( E \), \( r(\alpha) = s(\theta) \), \( r(\beta) = r(\theta) \), \(|\alpha| = |\beta| = n\) and \( |\theta| = m \). Hence the formula

\[
\pi(\alpha \theta \beta) = \begin{cases} 
\theta & \text{if } \alpha = \beta \\
0 & \text{else}
\end{cases}
\]
defines a surjective linear map \( L_{m,n} \to V_m \). One checks that \( \pi \) induces an isomorphism

\[
HH_0(L_{0,n}, L_{m,n}) \cong V_m \quad (m > 0).
\]
Similarly if \( m < 0 \), then

\[
HH_0(L_{0,n}, L_{m,n}) = V_{|m|}^* \cong V_{-m}.
\]
Next, by (4.1), we have

\[
HH_0(L_{0,n}) = k[E \setminus \text{Sink}(E)] \oplus \bigoplus_{i \in \text{Sink}(E)} k^{r(i,n)}.
\]
Here

\[
r(i, n) = \max\{ r \leq n : P(r, i) \neq \emptyset \}.
\]
Now note that, because \( L_{0,n} \) is a product of matrix algebras, it is separable, and thus \( HH_1(L_{0,n}, M) = 0 \) for any bimodule \( M \). As observed in (4.3), for the automorphism \( (4.2) \), we have \( L = L_0[t_+, t_-, \varphi] \). Hence in view of Proposition 3.4 and Lemma 2.1, it only remains to identify the maps \( HH_0(L_{0,n}, L_{m,n}) \to HH_0(L_{0,n + 1}, L_{m,n + 1}) \) induced by inclusion and by the homomorphism \( \varphi \). One checks that for \( m \neq 0 \), these are respectively the cyclic permutation and the identity \( V_{|m|} \to V_{|m|} \). The case \( m = 0 \) is dealt with in the same way as in [6, Proof of Theorem 5.10].

**Corollary 4.5.** Let \( E \) be a finite quiver with at least one non-trivial closed path.

i) \( HH_n(L(E)) = 0 \) for \( n \notin \{0, 1\} \).

ii) \( _mHH_*(L(E)) \cong _{-m}HH_*(L(E)) \) (\( m \in \mathbb{Z} \)).

iii) There exist \( m > 0 \) such that \( _mHH_0(L(E)) \) and \( _mHH_1(L(E)) \) are both nonzero.

**Proof.** We first reduce to the case where the graph does not have sources. By the proof of [6, Theorem 6.3], there is a finite complete subgraph \( F \) of \( E \) such that \( F \) has no sources, \( F \) contains all the non-trivial closed paths of \( E \), \( \text{Sink}(F) = \text{Sink}(E) \), and \( L(F) \) is a full corner in \( L(E) \) with respect to the homogeneous idempotent \( \sum_{v \in F^0} p_v \). It follows that \( HH_*(L(E)) \) and \( HH_*(L(F)) \) are graded-isomorphic. Therefore we can assume that \( E \) has no sources.

The first two assertions are already part of Theorem 4.4. For the last assertion, let \( \alpha \) be a primitive closed path in \( E \), and let \( m = |\alpha| \). Let \( \sigma \) be the cyclic permutation; then \( \{ \sigma^i \alpha : i = 0, \ldots, m - 1 \} \) is a linearly independent set. Hence \( N(\alpha) = \sum_{i=0}^{m-1} \sigma^i \alpha \) is a nonzero element of \( V_m^\sigma = _mHH_1(L(E)) \). Since on the other hand \( N \) vanishes on the image of \( 1 - \sigma : V_m \to V_m \), it also follows that the class of \( \alpha \) in \( _mHH_0(L(E)) \) is nonzero. \( \square \)
5. Applications

**Theorem 5.1.** Let $E_1, \ldots, E_n$ and $F_1, \ldots, F_m$ be finite quivers. Assume that $n \neq m$ and that each of the $E_i$ and the $F_j$ has at least one non-trivial closed path. Then the algebras $L(E_1) \otimes \cdots \otimes L(E_n)$ and $L(F_1) \otimes \cdots \otimes L(F_m)$ are not Morita equivalent.

**Proof.** Immediate from Lemma 2.3 and Corollary 4.5(iii). \qed

**Example 5.2.** It follows from Theorem 5.1 that $L_2$ and $L_2 \otimes_k L_2$ are not Morita equivalent. There is another way of proving this, due to Jason Bell and George Bergman [9]. By Theorem 3.3 of [3], $\text{l.gl.dim} L_2 \leq 1$. Using a module-theoretic construction, Bell and Bergman show that $\text{l.gl.dim}(L_2 \otimes_k L_2) \geq 2$, which forces $L_2$ and $L_2 \otimes_k L_2$ to be not Morita equivalent. Bergman then asked Warren Dicks whether general results were known about global dimensions of tensor products and was pointed to Proposition 10(2) of [12], which is an immediate consequence of Theorem XI.3.1 of [10], and says that if $k$ is a field and $R$ and $S$ are $k$-algebras, then $\text{l.gl.dim} R + \text{w.gl.dim} S \leq \text{l.gl.dim}(R \otimes_k S)$. Consequently, if $\text{l.gl.dim} R < \infty$ and $\text{w.gl.dim} S > 0$, then $\text{l.gl.dim} R < \text{l.gl.dim}(R \otimes_k S)$; in particular, $R$ and $R \otimes_k S$ are then not Morita equivalent. To see that $\text{w.gl.dim} L_2 > 0$, write $x_1$, $x_2$, $x_1^*$, $x_2^*$ for the usual generators of $L_2$ and use normal-form arguments to show that \( \{ a \in L_2 \mid ax_1 = a + 1 \} = \emptyset \) and $\{ b \in L_2 \mid x_1b = b \} = \{ 0 \}$. Hence, in $L_2$, $x_1 - 1$ does not have a left inverse and is not a left zero divisor (or see [4]); thus, $\text{w.gl.dim} L_2 > 0$.

We denote by $L_\infty$ the unital algebra presented by generators $x_1, x_1^*, x_2, x_2^*, \ldots$ and relations $x_i^* x_j = \delta_{i,j}1$.

**Proposition 5.3.** Let $E$ be any finite quiver having at least one non-trivial closed path. Then $L_\infty \otimes L(E)$ and $L(E)$ are not Morita equivalent. Similarly $L_\infty \otimes L_\infty$ and $L_\infty$ are not Morita equivalent.

**Proof.** Let $C_n$ be the algebra presented by generators $x_1, x_1^*, \ldots, x_n, x_n^*$ and relations $x_i^* x_j = \delta_{i,j}1$, for $1 \leq i, j \leq n$. Then

\[
L_\infty = \lim \limits_{\rightarrow} C_n,
\]

and $C_n \cong L(E_n)$, where $E_n$ is the graph having two vertices $v, w$ and $2n$ arrows $e_1, \ldots, e_n, f_1, \ldots, f_n$, with $s(e_i) = r(e_i) = v = s(f_i)$ and $r(f_i) = w$ for $1 \leq i \leq n$. (The isomorphism $C_n \to L(E_n)$ is obtained by sending $x_i$ to $e_i + f_i$ and $x_i^*$ to $e_i^* + f_i^*$.) It follows from Theorem 4.4 and (5.4) that the formulas in Theorem 4.4 for $mHH_n(L_\infty)$, $m \neq 0$, hold taking as $V_{i,m}$ the vectorspace generated by all the words in $x_1, x_2, \ldots$ of length $m$, and that $\vartheta HH_0(L_\infty) = k$ and $\vartheta HH_n(L_\infty) = 0$ for $n \geq 1$. As before, Lemma 2.3 gives the result. \qed

**Theorem 5.5.** Let $E_1, \ldots, E_n$ and $F_1, \ldots, F_m, \ldots$ be a finite and an infinite sequence of quivers. Assume that the number of indices $i$ such that $F_i$ has at least
one non-trivial closed path is infinite. Then the algebras $L(E_1) \otimes \cdots \otimes L(E_n)$ and $\bigotimes_{i=1}^{\infty} L(F_i)$ are not Morita equivalent.

**Proof.** Immediate from Lemma 2.3 and Corollary 4.5(iii). \hfill \Box

**Example 5.6.** Let $L(\infty) = \bigotimes_{i=1}^{\infty} L_2$, and let $E$ be any quiver having at least one non-trivial closed path. Then $L(\infty) \otimes L(E)$ and $L(E)$ are not Morita equivalent.

It would be interesting to know the answer to the following question:

**Question 5.7.** Is there a unital homomorphism $\phi: L_2 \otimes L_2 \to L_2$?

Observe that, to build a unital homomorphism $\phi: L_2 \otimes L_2 \to L_2$, it is enough to exhibit a non-zero homomorphism $\psi: L_2 \otimes L_2 \to L_2$, because $eL_2e \cong L_2$ for every non-zero idempotent $e$ in $L_2$.

6. **$K$-theory**

To conclude the paper we note that algebraic $K$-theory cannot distinguish between $L_2$ and $L_2 \otimes L_2$ or between $L_\infty$ and $L_\infty \otimes L_\infty$. For this we need a lemma, which might be of independent interest. Recall that a unital ring $R$ is said to be regular supercoherent in case all the polynomial rings $R[t_1, \ldots, t_n]$ are regular coherent in the sense of [13].

**Lemma 6.1.** Let $E$ be a finite graph. Then $L(E)$ is regular supercoherent.

**Proof.** Let $P(E)$ be the usual path algebra of $E$. It was observed in the proof of [4, Lemma 7.4] that the algebra $P(E)[t]$ is regular coherent. The same proof gives that all the polynomial algebras $P(E)[t_1, \ldots, t_n]$ are regular coherent. This shows that $P(E)$ is regular supercoherent. By [4, Proposition 4.1], the universal localization $P(E) \to L(E) = \Sigma^{-1}P(E)$ is flat on the left. It follows that $L(E)$ is left regular supercoherent (see [6, page 23]). Since $L(E) \otimes k[t_1, \ldots, t_n]$ admits an involution, it follows that $L(E)$ is regular supercoherent. \hfill \Box

**Proposition 6.2.** Let $R$ be regular supercoherent. Then the algebraic $K$-theories of $L_2$ and of $L_2 \otimes R$ are both trivial.

**Proof.** Let $E$ be the quiver with one vertex and two arrows. Then $L_2 \cong L(E)$, and we have

$$L_2 \otimes R = L_R(E).$$

Applying [6, Theorem 7.6] we obtain that $K_*(L_R(E)) = K_*(L(E)) = 0$. The result follows. \hfill \Box

We finally obtain a $K$-absorbing result for Leavitt path algebras of finite graphs, indeed for any regular supercoherent algebra.

**Proposition 6.3.** Let $R$ be a regular supercoherent algebra. Then the natural inclusion $R \to R \otimes L_\infty$ induces an isomorphism $K_i(R) \to K_i(R \otimes L_\infty)$ for all $i \in \mathbb{Z}$.
Proof. Adopting the notation used in the proof of Proposition 5.3, we see that it is enough to show that the natural map $R \to R \otimes L(E_n)$ induces isomorphisms $K_i(R) \to K_i(R \otimes L(E_n))$ for all $i \in \mathbb{Z}$ and all $n \geq 1$. Since $R$ is regular supercoherent the $K$-theory of $R \otimes L(E_n) \cong L_R(E_n)$ can be computed by using [6, Theorem 7.6]. By the explicit form of the quiver $E_n$, we thus obtain that $K_i(R \otimes L(E_n)) \cong (K_i(R) \oplus K_i(R))/(-n, 1-n)K_i(R)$.

The natural map $R \to L_R(E_n)$ factors as $R \to Rw \oplus Rw \to L_R(E_n)$. The first map induces the diagonal homomorphism $K_i(R) \to K_i(R) \oplus K_i(R)$ sending $x$ to $(x, x)$. The second map induces the natural surjection $K_i(R) \oplus K_i(R) \to (K_i(R) \oplus K_i(R))/(-n, 1-n)K_i(R)$.

Therefore the natural homomorphism $R \to L_R(E_n)$ induces an isomorphism $K_i(R) \cong K_i(L_R(E_n))$. This concludes the proof. \hfill \square

Corollary 6.4. The natural maps $k \to L_\infty \to L_\infty \otimes L_\infty$ induce $K$-theory isomorphisms $K_*(k) = K_*(L_\infty) = K_*(L_\infty \otimes L_\infty)$.

Proof. A first application of Proposition 6.3 gives $K_*(k) = K_*(L_\infty)$. A second application shows that for $E_n$ as in the proof above, the inclusion $L(E_n) \to L(E_n) \otimes L_\infty$ induces a $K$-theory isomorphism; passing to the limit, we obtain the corollary. \hfill \square

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REFERENCES

[1] G. Abrams, G. Aranda Pino. The Leavitt path algebra of a graph. J. Algebra 293 (2005), 319–334.
[2] G. Abrams, A. Louly, E. Pardo, C. Smith. Flow invariants in the classification of Leavitt path algebras. J. Algebra 333 (2011), 202–231.
[3] G. M. Bergman and Warren Dicks, Universal derivations and universal ring constructions. Pacific J. Math. 79 (1978), 293-337.
[4] P. Ara, M. Brustenga. Module theory over Leavitt path algebras and $K$-theory. J. Pure Appl. Algebra 214 (2010), 1131-1151.
[5] P. Ara, M. Brustenga. The regular algebra of a quiver. J. Algebra 309 (2007), 207–235.
[6] P. Ara, M. Brustenga, G. Cortiñas. K-theory of Leavitt path algebras. Münster Journal of Mathematics, 2 (2009), 5–34.
[7] P. Ara, M.A. González-Barroso, K.R. Goodearl, E. Pardo. Fractional skew monoid rings. J. Algebra 278 (2004), 104–126.
[8] P. Ara, M. A. Moreno, E. Pardo. Nonstable $K$-theory for graph algebras. Algebr. Represent. Theory 10 (2007), 157-178.
[9] J. Bell, G. Bergman. Private communication, 2011.
[10] H. Cartan and S. Eilenberg, *Homological Algebra*. Princeton University Press, Princeton, N. J., 1956.
[11] G. Cortiñas, E. Ellis. *Isomorphism conjectures with proper coefficients*. Preprint, 2011.
[12] S. Eilenberg, A. Rosenberg and D. Zelinsky, *On the dimension of modules and algebras, VIII. Dimension of tensor products*. Nagoya Math. J. 12 (1957), 71–93.
[13] S. M. Gersten. *K-theory of free rings*. Comm. Algebra 1 (1974), 39-64.
[14] E. Kirchberg. *The classification of purely infinite C*-algebras using Kasparov theory*. Preprint.
[15] E. Kirchberg, N.C. Phillips. *Embedding of exact C*-algebras into O2*. J. Reine Angew. Math. 525 (2000), 637–666.
[16] M. Lorenz. *On the homology of graded algebras*. Comm. Algebra. 20 (1992), 489–507.
[17] W. G. Leavitt. *The module type of a ring*. Trans. Amer. Math. Soc. 103 (1962), 113–130.
[18] J. L. Loday. *Cyclic homology*, 1st ed. Grund. math. Wiss. 301. Springer-Verlag Berlin, Heidelberg 1998.
[19] N. C. Phillips. *A classification theorem for nuclear purely infinite simple C*-algebras*. Doc. Math. 5 (2000), 49—114.
[20] I. Raeburn. *Graph algebras*. CBMS Regional Conference Series in Mathematics, 103. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2005.
[21] M. Rørdam. *A short proof of Elliott’s theorem*. C. R. Math. Rep. Acad. Sci. Canada 16 (1994), 31–36.
[22] M. Rørdam. *Classification of nuclear, simple C*-algebras*. Classification of nuclear C*-algebras. Entropy in operator algebras, Encyclopaedia Math. Sci. 126, 1–145, Springer, Berlin, 2002.
[23] C. Weibel. *An introduction to homological algebra*, Cambridge Univ. Press, 1994.
[24] M. Wodzicki. *Excision in cyclic homology and in rational algebraic K-Theory*. Ann. of Math. 129 (1989), 591–639.