**K-Theory from a Physical Perspective**

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This is an expository paper which aims at explaining a physical point of view on the $K$-theoretic classification of $D$-branes. We combine ideas of renormalization group flows between boundary conformal field theories, together with spacetime notions such as anomaly cancellation and $D$-brane instanton effects. We illustrate this point of view by describing the twisted $K$-theory of the special unitary groups $SU(N)$.

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1. Introduction

This is an expository paper devoted to explaining some aspects of the $K$-theoretic classification of D-branes. Our aim is to address the topic in ways complementary to the discussions of [1] [2]. Reviews of the latter approaches include [3] [4] [5] [6]. Our intended audience is the mathematician who is well-versed in conformal field theory and $K$-theory, and has some interest in the wider universe of (nonconformal) quantum field theories.

Our plan for the paper is to begin in section 2 by reviewing the relation of D-branes and $K$-theory at the level of topological field theory. Then in section 3 we will move on to discuss D-branes in conformal field theory. We will advocate a point of view emphasizing 2-dimensional conformal field theories as elements of a larger space of 2-dimensional quantum field theories. “D-branes” are identified with conformal quantum field theories on 2-dimensional manifolds with boundary. From this vantage, the topological classification of D-branes is the classification of the connected components of the space of 2-dimensional theories on manifolds with boundary which only break conformal invariance through their boundary conditions.

In section 4 we will turn to conformal field theories which are used to build string theories. In this case, there is a spacetime viewpoint on the classification of D-branes. We will present a viewpoint on D-brane classification, based on anomaly cancellation and “instanton effects,” that turns out to be closely related to the Atiyah-Hirzebruch spectral sequence.

In section 5 we examine a detailed example, that of branes in WZW models, and show how, using the approach explained in sections 3 and 4 we can gain an intuitive understanding of the twisted K-theory of $SU(N)$. The picture is in beautiful harmony with a rigorous computation of M. Hopkins.

Let us warn the reader at the outset that in this modest review we are only attempting to give a broad brush overview of some ideas. We are not attempting to give a detailed and rigorous mathematical theory; nor are we attempting to give a comprehensive review of the subject.

2. Branes in 2-dimensional topological field theory

The relation of D-branes and K-theory can be illustrated very clearly in the extremely simple case of 2-dimensional (2D) topological field theory. This discussion was developed in collaboration with Graeme Segal [7].
We will regard a “field theory” along the lines of Segal’s contribution to this volume. It is a functor from a geometric category to some linear category. In the simple case of 2D topological field theory the geometric category has as objects disjoint collections of circles and as morphisms diffeomorphism classes of oriented cobordisms between the objects. The target category is the category of vector spaces and linear transformations. Recall that to give a 2D topological field theory of closed strings is to give a commutative, finite dimensional Frobenius algebra $C$. For example, the algebra structure follows from Fig. 1.

![Figure 1](image1.png)

**Figure 1.** The 3-holed surface corresponds to the basic multiplication of the Frobenius algebra.

Let us now enlarge our geometric category to include open as well as closed strings. Now there are ingoing/outgoing circles and intervals, while the morphisms are surfaces with two kinds of boundaries: ingoing/outgoing boundaries as well as “free-boundaries,” traced out by the endpoints of the in/outgoing intervals. These free boundary must be labelled by “boundary conditions” which, for the moment, are merely labels $a, b, \ldots$.

![Figure 2](image2.png)

**Figure 2.** Multiplication defining the nonabelian Frobenius algebra of open strings.
Because we have a functor, to any pair of boundary conditions we associate a vector space ("a statespace") $\mathcal{H}_{ab}$. Moreover, there is a coherent system of bilinear products

$$\mathcal{H}_{ab} \otimes \mathcal{H}_{bc} \rightarrow \mathcal{H}_{ac}$$

(2.1)
defined by Figure 2. This leads us to ask the key question: What boundary conditions are compatible with $\mathcal{C}$? “Compatibility” means coherence with “sewing” or “gluing” of surfaces; more precisely, we wish to have a well-defined functor. Thus, just the way Fig. 1 defines an associative commutative algebra structure on $\mathcal{C}$, Fig. 2, in the case $a = b = c$, defines a (not necessarily commutative) algebra structure on $\mathcal{H}_{aa}$. Moreover $\mathcal{H}_{aa}$ is a Frobenius algebra. Next there are further sewing conditions relating the open and closed string structures. Thus for example, we require that the operators defined by figures 3 and 4 be equal, a condition sometimes referred to as the “Cardy condition.”

![Figure 3](image3.png)

Figure 3. In the open string channel this surface defines a natural operator $\pi : \mathcal{H}_{aa} \rightarrow \mathcal{H}_{bb}$ on noncommutative Frobenius algebras.

![Figure 4](image4.png)

Figure 4. In the closed string channel this surface defines a composition of open-closed and closed-open transitions $\iota_{c-o} \circ \iota_{o-c} : \mathcal{H}_{aa} \rightarrow \mathcal{H}_{bb}$ that factors through the center.

As observed by Segal some time ago [8], the proper interpretation of (2.1) is that the boundary conditions are objects $a, b, \ldots$ in an additive category with:

$$\mathcal{H}_{ab} = \text{Mor}(a, b).$$

(2.2)

Therefore, we should ask what the sewing constraints imply for the category of boundary conditions. This question really consists of two parts: First, coherence of sewing is
equivalent to a certain algebraic structure on the target category. Once we have identified that structure we can ask for a classification of the examples of such structures. The first part of this question has been completely answered: The open/closed sewing conditions were first analyzed by Cardy and Lewellen \[9]\[10], and the resulting algebraic structure was described in \[7]\[11]. The result is the following:

**Proposition** To give an open and closed 2D oriented topological field theory is to give

1. A commutative Frobenius algebra $\mathcal{C}$.
2. Frobenius algebras $\mathcal{H}_{aa}$ for each boundary condition $a$.
3. A homomorphism $\iota_a : \mathcal{C} \to Z(\mathcal{H}_{aa})$, where $Z(\mathcal{H}_{aa})$ is the center, such that $\iota_a(1) = 1$, and such that, if $\iota^a$ is the adjoint of $\iota_a$ then

$$\pi^a_b = \iota_b \iota^a.$$  

(2.3)

Here $\pi^a_b : \mathcal{H}_{aa} \to \mathcal{H}_{bb}$ is the morphism, determined purely in terms of open string data, described by Fig. 3. When $\mathcal{H}_{ab}$ is the nonzero vector space it is a Morita equivalence bimodule and $\pi^a_b$ can be written as $\pi^a_b(\psi) = \sum \psi^\mu \psi^\mu$ where $\psi^\mu$ is a basis for $\mathcal{H}_{ab}$ and $\psi^\mu$ is a dual basis for $\mathcal{H}_{ba}$. More invariantly,

$$\theta_b(\pi^a_b(\psi) \chi) = \text{Tr}_{\mathcal{H}_{ab}} (L(\psi) R(\chi))$$  

(2.4)

where $\theta_b$ is the trace on $\mathcal{H}_{bb}$ and $L(\psi), R(\chi)$ are the left- and right- representations of $\mathcal{H}_{aa}$, $\mathcal{H}_{bb}$ on $\mathcal{H}_{ab}$, respectively.

The second step, that of finding all examples of such structures was analyzed in \[7\] in the case where $\mathcal{C}$ is a semisimple Frobenius algebra. The answer turns out to be very crisp:

**Theorem 1** Let $\mathcal{C}$ be semisimple. Then the set of isomorphism classes of objects in the category of boundary conditions is

$$K^0(\text{Spec}(\mathcal{C})) = K_0(\mathcal{C}).$$ 

(2.5)

There are important examples of the above structure when $\mathcal{C}$ is not semisimple, such as the topological $A$-and $B$-twisted $\mathcal{N} = 2$ supersymmetric sigma models. As far as we are aware, the classification of examples for non-semisimple $\mathcal{C}$ is an open problem.
Even in this elementary setting, there are interesting and nontrivial generalizations. When a 2D closed topological field theory has a symmetry $G$ it is possible to “gauge it.” The cobordism category is enhanced by considering cobordisms of principal $G$-bundles. In this case the closed topological field theory corresponds to a choice of “Turaev algebra,” a $G$-equivariant extension of a Frobenius algebra which, in the semisimple case, is characterized by a “spacetime” consisting of a discrete set of points (corresponding to the idempotents of the algebra), a “dilaton,” encoding the trace of the Frobenius algebra on the various idempotents, and a “$B$-field.” In this case we have

**Theorem 2** The isomorphism classes of objects in the category of boundary conditions for a $G$-equivariant open and closed theory with spacetime $X$ and “$B$-field” $[b] \in H^2_G(X; C^*)$ are in 1-1 correspondence with the K-group of $G$-equivariant, $b$-twisted $K$-theory classes: $K_{G,b}(X)$.

These results are, of course, very elementary. What I find charming about them is precisely the fact that they are so primitive: they rely on nothing but topological sewing conditions and a little algebra, and yet K-theory emerges ineluctably.

A more sophisticated category-theoretic approach to the classification of branes in rational conformal field theories has been described in [13, 14].

3. **K-theory and the renormalization group**

3.1. **Breaking conformal invariance on the boundary**

Let us now consider the much more difficult question of the topological classification of D-branes in a full conformal field theory (CFT). This immediately raises the question of what we even mean by a “D-brane.” Perhaps the most fruitful point of view is that D-branes are *local* boundary conditions in a 2D CFT $\mathcal{C}$ which preserve conformal symmetry. While there is an enormous literature on the subject of D-branes, the specific branes which have been studied are really a very small subset of what is possible.

One way to approach the classification of D-branes is to consider the space of 2D quantum field theories (QFT’s), defined on surfaces with boundary, which are *not* conformal, but which only break conformal invariance via their boundary conditions. Formally, there is a space $\mathcal{B}$ of such boundary QFT’s compatible with a fixed “bulk” CFT, $\mathcal{C}$. The
tangent space to $B$ is the space of local operators on the boundary because a local operator $O$ can be used to deform the action on a surface $\Sigma$ by:

$$S_{\text{worldsheet}} = S_{\text{bulk CFT}} + \int_{\partial \Sigma} ds O$$  \hspace{1cm} (3.1)

Here $ds$ is a line element. Note that in general we have introduced explicit metric-dependence in this term, thus breaking conformal invariance on the boundary.

As a simple example of what we have in mind, consider a massless scalar field $x^\mu : \Sigma \to \mathbb{R}^n$ with action

$$S_{\text{worldsheet}} = \int_{\Sigma} \partial x^\mu \bar{\partial} x^\mu + \int_{\partial \Sigma} ds T(x^\mu(\tau))$$  \hspace{1cm} (3.2)

where $T(x^\mu)$ is "any function" on $\mathbb{R}^n$ and $\tau$ is a coordinate on $\partial \Sigma$. Then the boundary interaction in (3.1) can be expanded

$$O = T(x) + A_\mu(x) \frac{dx^\mu}{d\tau} + B_\mu(x) \frac{d^2 x^\mu}{d\tau^2} + C_{\mu\nu}(x) \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} + \cdots$$  \hspace{1cm} (3.3)

The coefficients $T(x), A_\mu(x), \ldots$ are viewed as spacetime fields on the target space $\mathbb{R}^n$.

We expect that $B$ can be given a topology such that renormalization group flow (see below) is a continuous evolution on this space. In this topology $B$ is a disconnected space. The essential idea is that the connected components of this space are classified by some kind of $K$-theory. For example, if the conformal field theory is supersymmetric and has a target space interpretation in terms of a nonlinear $\sigma$ model, we expect the components of $B$ to correspond to the $K$-theory of the target space $X$

$$\pi_0(B) = K(X).$$  \hspace{1cm} (3.4)

Remarks:

1. In equation (3.4) we are being deliberately vague about the precise form of $K$-theory (e.g. $K$, vs. $KO, KR, K_\pm$ etc.). This depends on a discrete set of choices one makes in formulating the 2D field theory.

2. From this point of view the importance of some kind of supersymmetry on the worldsheet is clear. As an example in the next section makes clear, the RG flow corresponding to taking $O$ to be the unit operator always flows to a trivial fixed point with "no

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1 G. Segal points out to me that the proper formulation of the tangent space to a boundary CFT would naturally use the theory of jets.
boundary.” Therefore, unless the unit operator can be projected out, there cannot be interesting path components in $B$. In spacetime terms, we must cancel the “zero momentum tachyon.”

3. One might ask what replaces (3.4) when the CFT does not have an obvious target space interpretation. One possible answer is that one should define some kind of algebraic K-theory for an open string vertex operator algebra. There has been much recent progress in understanding more deeply Witten’s Chern-Simons open string field theory (see, e.g. [15,16,17,18,19]). This holds out some hope that the $K$ theory of the open string vertex algebra could be made precise. In string field theory D-branes are naturally associated to projection operators in a certain algebra, so the connection between $K$-theory and branes is again quite natural. See also sec. 3.5 below.

4. The space $B$ appeared in a proposal of Witten’s for a background-independent string field theory [20]. (Witten’s “other” open string field theory.)

5. It is likely that the classification of superconformal boundary conditions in the supersymmetric Gaussian model is complete [21,22,23]. The classification is somewhat intricate and it would be interesting to see if it is compatible with the general proposal of this paper.

3.2. Boundary renormalization group flow

One way physicists explore the path components of $B$ is via “renormalization group (RG) flow.” Since conformal invariance is broken on the boundary, we can ask what happens as we scale up the size of the boundary. This scaling defines 1-dimensional flows on $B$. These are the integral flows of a vector field $\beta$ on $B$ usually referred to as the “beta function.” A D-brane, or conformal fixed point, corresponds to a zero of $\beta$. Two D-branes which are connected by RG flow are in the same path component, and therefore have the same “K-theory charge.”

Let us recall a few facts about boundary RG flow. For a good review see [24]. For simplicity we will consider the bosonic case. A boundary condition $a \in B$ is a zero of $\beta$. At such an RG fixed point the theory is conformal, and hence the Virasoro algebra acts on the tangent space $T_aB$. We may choose a basis of local operators such that $L_0 O_i = \Delta_i O_i$. Here $L_0$ is the scaling operator in the Virasoro algebra. We may then choose coordinates $O = \sum_i \lambda^i O_i$ such that, in an open neighborhood of $a \in B$,

$$\beta \cong - \sum_i (1 - \Delta_i) \lambda^i \frac{d}{d\lambda^i}$$

(3.5)
Thus, as usual, perturbations by operators with $\Delta_i < 1$ correspond to unstable flows in the infrared (IR). It turns out there is an analog of Zamolodchikov’s $c$-theorem. Boundary RG flow is gradient flow with respect to an “action functional.” To construct it one introduces the natural function on $\mathcal{B}$ given by the disk partition function. Then set

$$g := (1 + \beta)Z_{\text{disk}}. \quad (3.6)$$

Next one introduces a metric on $\mathcal{B}$. Recalling that the local operators are to be identified with the tangent space we write

$$G(\mathcal{O}_1, \mathcal{O}_2) = \oint d\tau_1 d\tau_2 \sin^2\left(\frac{\tau_1 - \tau_2}{2}\right)\langle \mathcal{O}_1(\tau_1)\mathcal{O}_2(\tau_2) \rangle_{\text{disk}} \quad (3.7)$$

Then, the “$g$-theorem” states that

$$\dot{g} = -\beta^i \beta^j G_{ij} \quad (3.8)$$

The main nontrivial statement here is that $\iota(\beta)G$ is a locally exact one-form.

Remarks:

1. The $g$-theorem was first proposed by Affleck and Ludwig [25][26], who verified it in leading order in perturbation theory. An argument for the $g$-theorem, based on string field theory ideas, was proposed in [27][28].

2. In the Zamolodchikov theorem, the $c$-function at a conformal fixed point is the value of the Virasoro central charge of the fixed point conformal field theory. It is therefore natural to ask: “What is the meaning of $g$ at a conformal fixed point?” The answer is the “boundary entropy.” For example, when the CFT $\mathcal{C}$ is an RCFT with irreps $\mathcal{H}_i$, $i \in I$ of the chiral algebra, the boundary CFT’s preserving the symmetry are labelled by $i \in I$ and the $g$-function for these conformal fixed points is expressed in terms of the modular $S$-matrix via

$$g = \frac{S_{0i}}{\sqrt{S_{00}}}. \quad (3.9)$$

where 0 denotes the unit representation. It is notable that this can also be interpreted as a regularized dimension of the open string statespace $\sqrt{\dim \mathcal{H}_{ii}}$. If the CFT is part of a string theory with a target space interpretation then we can go further. In a string theory we have gravity and in this context the value of $g$ at a conformal fixed point is the brane tension, or energy/volume of the brane [29].

3. In the case of $\mathcal{N} = 1$ worldsheet supersymmetry we should take instead [28][30][31]:

$$g := Z_{\text{disk}}. \quad (3.10)$$

4. In an interesting series of papers A. Connes and D. Kreimer have re-interpreted perturbative renormalization of field theory and the renormalization group in terms of the structure of Hopf algebras [32][33]. We believe that the case of boundary RG flow in two-dimensions might be a very interesting setting in which to apply their ideas.
3.3. Tachyon condensation from the worldsheet viewpoint

Here is a simple example of the $g$ theorem. Consider a single scalar field on the disk $x : D \to \mathbb{R}$, where the disk $D$ has radius $r$. Then,

$$Z_{\text{disk}} = \int [dx] e^{-\int_D \partial x \partial x + \oint_{\partial D} T(x)}$$

(3.11)

Let’s just take $T(x) = t = \text{constant}$. Then, trivially, $Z_{\text{disk}}(t) = Z_{\text{disk}}(0)e^{-2\pi rt} = Z_{\text{disk}}(0)e^{-2\pi t(r)}$. Then

$$\beta^t := -\frac{\partial t(r)}{\partial \log r} = -t \Rightarrow \beta = -t \frac{d}{dt}$$

(3.12)

and an easy computation shows the metric is

$$ds^2 = e^{-t}(dt)^2.$$  

(3.13)

The $g$-function, or action, in this case is

$$g(t) = (1 + \beta)Z_{\text{disk}} = (1 + 2\pi rt)e^{-2\pi rt}g(0) = (1 + 2\pi t(r))e^{-2\pi t(r)}g(0)$$

(3.14)

At $t = 0$, $Z_{\text{disk}}$ is $r$-independent (hence conformally invariant) if we choose, say, Neumann boundary conditions for $x$. Thus at $t = 0$ we begin with an open/closed CFT consisting of a “D1 brane” wrapping the target $\mathbb{R}$ direction. Under RG flow to the IR, $t \to \infty$. At $t = \infty$ all boundary amplitudes are infinitely suppressed and “disappear.” We are left with a theory only of closed strings!

Remarks:

1. The RG flow (3.14) is unusual in that we can give exact formulae. This is due to its rather trivial nature. Moreover, note that this boundary interaction cancels out of all normalized correlators. Nevertheless, we feel that the above example nicely captures the essential idea. A less trivial example based on the boundary perturbation $\oint uX^2$ is analyzed in [34,35,27].

2. Let us return to remark 1 of section 3.1. It is precisely the zero-momentum tachyon (i.e. the unit operator) whose flow we wish to suppress in order to define a space $B$ with interesting path components.

3. The example of this section is essentially the “boundary string field theory” (BSFT) interpretation of Sen’s tachyon condensation [36]. In [20] Witten introduced an alternative formulation of open string field theory, in which, (at least when ghosts
decouple), the function $g$ is the spacetime action. This theory was further developed by Witten and Shatashvili in [34,37,38,39]. Interest in the theory was revived by [40,41,35,27,28]. These papers showed, essentially using the above example, that the dependence of the spacetime effective potential on the tachy on field is

$$V(T) \sim (T + 1)e^{-T}$$

for the bosonic string and

$$V(T) \sim e^{-T^2}$$

for the type IIA string (on an unstable D9 brane). The tachyon potential is minimized by $T \to \infty$, and at its minimum the open strings “disappear.”

### 3.4. $g$-function for the nonlinear sigma model

Suppose the closed CFT $C$ is a $\sigma$-model with spacetime $X$, dilaton $\Phi$, metric $g_{\mu\nu}$ and “gerbe connection” $B_{\mu\nu}$. A typical boundary condition involves, first of all, a choice of topological $K$-homology cycle [42], that is, an embedded subvariety $\iota : W \hookrightarrow X$ with Spin$^c$ structure (providing appropriate Dirichlet boundary conditions for the open strings) together with a choice of (complex) vector bundle

$$E \to W,$$

modulo some equivalence relations. We say a “ D-brane wraps $W$ with Chan-Paton bundle $E$.”

In the supersymmetric case the most important boundary interaction is a choice of a (unitary) connection $A_\mu$ on $E$ and a (nonabelian) section of the normal bundle. In this paper we will set the normal bundle scalars to zero (although they are very interesting). Thus the $g$-function becomes

$$g = \langle \operatorname{Tr}_E P \exp \left( \oint_{\partial D} d\tau A_\mu(x(\tau)) \dot{x}^\mu(\tau) + F_{\mu\nu} \psi^\mu \psi^\nu + \cdots \right) \rangle$$

where $\psi^\mu$ are the susy partners of $x^\mu$. When $E$ is a line bundle $g$ can be computed for a variety of backgrounds and turns out to be the Dirac-Born-Infeld (DBI) action [43]:

$$g = \int_W e^{-\Phi} \sqrt{\det(g_{\mu\nu} + B_{\mu\nu} + F_{\mu\nu})} + \mathcal{O}((DF)^2)$$
If $E$ has rank $\geq 1$ to get a nice formula we need to add the condition $F_{\mu\nu} \ll 1$. In this case we have:

$$g = \text{rank}(E) \int e^{-\Phi} \sqrt{\det(g + B)} + \int_X e^{-\Phi} \text{Tr}(F \wedge *F) + \cdots$$

(3.20)

Remarks:

1. It follows from (3.20) that in the long-distance limit the gradient flows of the “$g$-theorem” generalize nicely some flows which appeared in the work of Donaldson on the Hermitian-Yang-Mills equations [14].

Let $X$ be a Calabi-Yau manifold. To $X$ we associate an $\mathcal{N} = (2, 2)$ superconformal field theory $C$. The boundary interaction (3.18) preserves $\mathcal{N} = 2$ supersymmetry iff $F$ is of type $(1, 1)$, i.e., iff $F^{2,0} = 0$ [15] [16]. RG flow preserves $\mathcal{N} = 2$ susy, and hence preserves the $(1, 1)$ condition on the field-strength. A boundary RG fixed point is defined (in the $\alpha' \to 0$ limit) by an Hermitian Yang-Mills connection. The RG flow is precisely the flow:

$$\frac{dA_\mu}{dt} = D_\nu F^{\nu}_\mu.$$  

(3.21)

Thus, one can view the flow from a perturbation of an unstable bundle to a stable one as an example of tachyon condensation. It might be interesting to think through systematically the implications for tachyon condensation of Donaldson’s results on the convergence of these flows.

2. The tachyon condensation from unstable $D9$ branes (or $D9\bar{D9}$ branes) to lower dimensional branes involves the Atiyah-Bott-Shapiro construction and Quillen’s superconnection in an elegant way. This has been demonstrated in the context of BSFT advocated in this section in [28] [17] [18]. Given the boundary data in (3.18) one is naturally tempted to see a role for the “differential K-theory” described in [19]. However, the nonabelian nature of the normal bundle scalars show that this is only part of the story. See [20] [21] [22] for some relevant discussions.

3.5. The Dirac-Ramond operator and the topology of $\mathcal{B}$

Let us now make some tentative remarks on how one might try to distinguish different components of $\mathcal{B}$. There are many indications that $K$-homology is a more natural framework for thinking about the relation of D-branes and K-theory [23] [24] [25] [26] [27] [28] [29] [30] [31] [32] [33].

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2 This remark is based on discussions with M. Douglas.
It was pointed out some time ago by Atiyah that the Dirac operator defines a natural K-homology class \([58]\). Indeed, abstracting the crucial properties of the Dirac operator leads to the notion of a Fredholm module \([59]\).

Now, in string theory, the Dirac operator is generalized to the Dirac-Ramond operator, that is, the supersymmetry operator \(Q\), (often denoted \(G_0\)) acting in the Ramond sector of a superconformal field theory. \(Q\) is a kind of Dirac operator on loop space as explained in \([60,61,62,63]\).

In the case of open strings it is still possible to define \(Q\) in the Ramond sector, and \(Q\) still has an interpretation as a Dirac operator on a path space. For example, suppose the \(\mathcal{N} = 1\) CFT has a sigma model interpretation with closed string background data \(g_{\mu\nu} + B_{\mu\nu}\). Suppose that the open string boundary conditions are \(x(0) \in \mathcal{W}_1, x(\pi) \in \mathcal{W}_2\), where the submanifolds \(\mathcal{W}_i\) are equipped with vector bundles \(E_i\) with connections \(A_i\). The supersymmetry operator will take the form:

\[
Q = \int_0^\pi d\sigma \psi^\mu(\sigma) \left( \frac{\delta}{\delta x^\mu(\sigma)} + g_{\mu\nu}(x(\sigma)) \frac{dx^\nu}{d\sigma} + (\omega_{\mu\nu\lambda} + H_{\mu\nu\lambda}) \psi^\nu \psi^\lambda \right) + \psi^\mu(0) A_{1,\mu}(x(0)) - \psi^\mu(\pi) A_{2,\mu}(x(\pi))
\]

where \(\omega_{\mu\nu\lambda}\) is the Riemannian spin connection on \(X\), and \(H_{\mu\nu\lambda}\) is the fieldstrength of the \(B\)-field. Just as in the closed string case, \(Q\) can be understood more conceptually as a Dirac operator on a bundle over the path space:

\[
\mathcal{P}(\mathcal{W}_1, \mathcal{W}_2) = \{x : [0, \pi] \to X | x(0) \in \mathcal{W}_1, x(\pi) \in \mathcal{W}_2\}
\]

(Preservation of supersymmetry imposes further boundary conditions on \(x\). See, for example, \([64,65,66]\) for details.) Quantization of \(\psi^\mu(\sigma)\) for fixed \(x^\mu(\sigma)\) produces a Fermionic Fock space. This space is to be regarded as a spin representation of an infinite dimensional Clifford algebra. These Fock spaces fit together to give a Hilbert bundle \(S\) over \(\mathcal{P}(\mathcal{W}_1, \mathcal{W}_2)\). The data \(g_{\mu\nu} + B_{\mu\nu}\) induce a connection on this bundle, as indicated in \((3.22)\). The effect of the boundaries is merely to change the bundle to

\[
S \to \text{ev}_0^*(E_1) \otimes (\text{ev}_\pi^*(E_2))^* \otimes S
\]

where \(\text{ev}\) is the evaluation map. The connections \(A_1, A_2\) induce connections on \((3.24)\). In the zeromode approximation \(Q\) becomes the Dirac operator on \(E_1 \otimes E_2^* \to \mathcal{W}_1 \cap \mathcal{W}_2\):

\[
Q \to \slashed{D}_{E_1 \otimes E_2^*} + \cdots
\]
Now let us consider RG flow. If RG flow connects boundary conditions $a$ to $a'$ then the target space interpretation of the superconformal field theories $\mathcal{H}_{ab}$ and $\mathcal{H}_{a'b'}$ can be very different. For example, tachyon annihilation can change the dimensionality of $\mathcal{W}$. Another striking example is the decay of many D0 branes to a single D2 brane discussed in section 5.5 below. It follows that any formulation of an RG invariant involving geometrical constructions such as vector bundles over path space is somewhat unnatural. However, what does make sense throughout the renormalization group trajectory is the supersymmetry operator $Q$, (so long as we restrict attention to $\mathcal{N} = 1$ supersymmetry-preserving flows). Moreover, it is physically “obvious” that $Q$ changes continuously under RG flow. This suggests that the components of $\mathcal{B}$ should be characterized by some kind of “homotopy class” of $Q$.

The conclusion of the previous paragraph immediately raises the question of where the homotopy class of $Q$ should take its value. We need to define a class of operators and define what is meant by continuous deformation within that class. While we do not yet have a precise proposal we can again turn to the zero-slope limit for guidance. In this limit, as we have noted, $Q \to \mathcal{D}_E \otimes \mathcal{E}_2^*$, and $\mathcal{D}$ defines, in a well-known way, a “$\theta$-summable $K$-cycle” for $\mathcal{A}$, the $C^*$ algebra completion of $C^\infty(\mathcal{W}_1 \cap \mathcal{W}_2)$, acting on the Hilbert space of $L^2$ sections of $S \otimes E_1 \otimes E_2^*$ over $\mathcal{W}_1 \cap \mathcal{W}_2$. That is $[(\mathcal{H}, \mathcal{D})] \in K^0(\mathcal{A})$. What is the generalization when we do not take the zeroslope limit? One possibility, in the closed string case, has been discussed in [67,68,69,59]. Another possibility is that one can define a notion of Fredholm module for vertex operator algebras. This has the disadvantage that it is tied to a particular conformal boundary condition $a$. It is possible, however, that the open string vertex operator algebras $\mathcal{A}_{aa}$ for different boundary conditions $a$ are “Morita equivalent” and that the homotopy class of $Q$ defines an element of some $K$-theory (which remains to be defined) “$K^0(\mathcal{A}_{aa})$.” This group should be independent of $a$ and only depend on $\mathcal{C}$. (See section 6.4 of [70] and [7] for some discussion of this idea.)

Remarks:
1. For some boundary conditions $a$ it is also possible to introduce a “tachyon field.” In this case the connection term $\psi^\mu(0)A_\mu(x(0))$ is replaced by Quillen’s superconnection. This happens, for example, if $a$ represents a $D - \overline{D}$ pair with $\mathbb{Z}_2$-graded bundle $E^+ \oplus E^-$. If the tachyon field $T \in \text{End}(E^+, E^-)$ is everywhere an isomorphism then boundary conditions with $a$ are in the same component as the trivial boundary condition, essentially by the example of section 3.3.
2. One strong constraint on the above considerations is that the Witten index

\[ \text{Tr}_{\mathcal{H}_a^b} (-1)^F e^{-\beta Q^2} \]  \hspace{1cm} (3.26)

must be a renormalization group invariant. In situations where we have the limit (3.25) we can use the index theorem to classify, in part, the components of \( \mathcal{B} \). Of course, this will miss the torsion elements of the \( K \)-theory.

4. \( K \)-theory from anomalies and instantons

In this section we consider the question of understanding the connected components of \( \mathcal{B} \) in the case where there is a geometrical target space interpretation of the CFT. We will be shifting emphasis from the worldsheet to the target space. We will use an approach based on a spacetime picture of branes as objects wrapping submanifolds of \( X \) to give an argument that (twisted) \( K \)-theory should classify components of \( \mathcal{B} \).

For concreteness, suppose our CFT is part of a background in type II string theory in a spacetime

\[ X_9 \times \mathbb{R}, \]  \hspace{1cm} (4.1)

where the \( \mathbb{R} \) factor is to be thought of as time, while \( X_9 \) is compact and spin. Suppose moreover that spacetime is equipped with a \( B \)-field with fieldstrength \( H \). This can be used to introduce a twisted \( K \)-group \( K_H(X_9) \). We’ll show how \( K_H(X_9) \), arises naturally in answering the question: \textit{What subvarieties of} \( X_9 \text{ } \text{can a D-brane wrap?} \) The answer involves \textit{anomaly cancellation} and \textit{instanton effects}, and leads to the slogan: “\( K \)-theory = anomalies modulo instantons.” As an example of this viewpoint, we apply it to compute the twisted \( K \)-theory \( K_H(SU(N)) \) for \( N = 2, 3 \). We will be following the discussion of [74]. For other discussions of the relation of twisted \( K \)-theory to Dbranes see [2, 72, 73, 74, 75, 49]. The point of view presented here has been further discussed in [76, 77].

\[ ^3 \text{In fact, } H \text{ should be refined to a 3-cocycle for integral cohomology. In the examples considered in detail below this refinement is not relevant.} \]
4.1. What subvarieties of \( X_9 \) can a D-brane wrap?

Since we are discussing topological restrictions and classification, we will identify D-brane configurations which are obtained via continuous deformation. Traditionally, then, we would replace the cycle \( \mathcal{W} \) wrapped by a D-brane by its homology class. This leads to the “cohomological classification of D-branes.” In the cohomological classification of branes we follow two rules:

(A) Free branes\(^4\) can wrap any nontrivial homology cycle, if \( i^* (H_{DR}) \) is exact.

(B) A brane wrapping a nontrivial homology cycle is absolutely stable.

In the K-theoretic classification of branes we have instead the modified rules:

\( (A') \) D-branes can wrap \( \mathcal{W} \subset X_9 \) only if \( W_3(\mathcal{W}) + [H]|_\mathcal{W} = 0 \) in \( H^3(\mathcal{W}, \mathbb{Z}) \)

\( (B') \) Branes wrapping homologically nontrivial \( \mathcal{W} \) can be unstable if, for some \( \mathcal{W}' \subset X_9 \), \( PD(\mathcal{W} \subset \mathcal{W}') = W_3(\mathcal{W}') + [H]|_{\mathcal{W}'} \).

Here and below, \( W_3(\mathcal{W}) := W_3(N\mathcal{W}) \) is the Stiefel-Whitney class of the normal bundle of \( \mathcal{W} \) in \( X_9 \).

We will first explain the physical reason for \( (A') \) and \( (B') \) and then explain the relation of \( (A') \) and \( (B') \) to twisted K-theory.

\begin{center}
\textbf{Figure 5.} A disk string worldsheet ends on a D-brane worldvolume \( \mathcal{W} \).
\end{center}

\(^4\) i.e., branes considered in isolation, with no other branes ending on them
To begin, condition \((A')\) is a condition of anomaly cancellation. Consider a string worldsheet \(D\) with boundary on a D-brane wrapping \(\mathcal{W} \times \mathbb{R}\) as in Fig. 5. The \(g\)-function is, schematically

\[
g = \int [Dx][D\psi] \ e^{-\int_D \bar{\partial}_x \partial_x + \psi \partial \psi + \cdots} \ e^{i \int_D B} \ \text{Pfaff}(\bar{\partial}_D) \ \text{Tr} P e^{i \oint_{\partial D} A} \tag{4.2}
\]

The measure of the path integral must be well-defined on the space of all maps

\[
\{x : D \to X_9 : x(\partial D) \subset \mathcal{W}\} \tag{4.3}
\]

By considering a loop of paths such that \(\partial D\) sweeps out a surface in \(\mathcal{W}\) it is easy to see that, at the level of the DeRham complex,

\[
\iota^*(H_{DR}) = dF \tag{4.4}
\]

must be trivialized. Heuristically \(F := F + \iota^*(B)\), although neither \(F\) nor \(\iota^*(B)\) is separately well-defined. Note that it is the combination \(F\) which appears in the \(g\)-function \((3.19)\), and hence must be globally well-defined on the brane worldvolume \(\mathcal{W} \times \mathbb{R}\). The equation \(dF = \iota^*(H_{DR})\) means \(H_{DR}\) is a magnetic source for \(F\) on the brane worldvolume \(\mathcal{W} \times \mathbb{R}\).

A more subtle analysis of global anomaly cancellation by Freed and Witten [78] shows that

\[
\iota^*[H] + W_3(T\mathcal{W}) = 0 \tag{4.5}
\]

at the level of integral cohomology. (See also the discussion of [72].)

![Figure 6](image_url)

**Figure 6.** A D-brane wrapping spatial cycle \(\mathcal{W}\) propagates in time and terminates on a configuration \(\mathcal{W}'\) localized in time. This configuration of D-branes is anomaly-free.

Let us now turn to the stability condition \((B')\). Suppose there is a cycle \(\mathcal{W}' \subset X_9\) on which

\[
W_3(\mathcal{W}') + [H]_{|\mathcal{W}} \neq 0. \tag{4.6}
\]
As we have just seen, anomaly cancellation implies that we cannot wrap a D-brane on $\mathcal{W}'$. However, while a free brane wrapping $\mathcal{W}'$ is anomalous, we can cancel the anomaly by adding a magnetic source for $\mathcal{F}$. A D-brane ending on a codimension 3 cycle $\mathcal{W} \subset \mathcal{W}'$ provides such a magnetic source. Hence, we can construct an anomaly free configuration by adding a D-brane wrapping a cycle $\mathbb{R}^- \times \mathcal{W}$ that *ends* on $\mathcal{W} \subset \mathcal{W}'$, where $\mathcal{W}$ is such that

$$PD(\mathcal{W} \subset \mathcal{W}') = W_3(\mathcal{W}') + [H]|_{\mathcal{W}'.}$$

(4.7)

Here $\mathbb{R}^-$ should be regarded as a semiinfinite interval in the time-direction as in Fig. 6.

Fig. 6 suggests a clear physical interpretation. A brane wraps a spatial cycle $\mathcal{W}$, propagates in time, and terminates on a D-“instanton” wrapping $\mathcal{W}'$. This means the brane wrapping a spatial cycle $\mathcal{W}$ can be unstable, and decays due to the configuration wrapping $\mathcal{W}'$. The basic mechanism is closely related to the “baryon vertex” discussed by Witten in the AdS/CFT correspondence [79].

4.2. Relation to K-theory via the Atiyah-Hirzebruch spectral sequence

$(A')$ and $(B')$ are in fact conditions of K-theory. In order to understand this, let us recall the Atiyah-Hirzebruch spectral sequence (AHSS). Let $X$ be a manifold. A K-theory class $x \in K^0(X)$ determines a system of integral cohomology classes: $c_i(x) \in H^{2i}(X, \mathbb{Z})$, while $x \in K^1(X)$ determines $\omega_{2i+1}(x) \in H^{2i+1}(X, \mathbb{Z})$. Let us ask the converse. Given a system of cohomology classes, $(\omega_1, \omega_3, \ldots)$ does there exist an $x \in K^1(X)$? The AHSS is a successive approximation scheme: $E_1^*, E_3^*, E_5^*, \ldots$ for describing when such a system of cohomology classes $(\omega_1, \omega_3, \ldots)$ arises from a K-theory class.

In order to relate the AHSS to D-branes we regard $PD(\omega_k)$ in $X$ as the spatial cycle of a (potentially unstable) brane of spatial dimension $\dim X - k$. In this way, a system $(\omega_1, \omega_3, \ldots)$ determines a collection of branes, and hence the AHSS helps us decide which subvarieties of $X$ can be wrapped. Now let us look at the AHSS in more detail:

The first approximation is the cohomological classification of D-branes:

$$K^0(X) \sim E_1^0(X) := H^{\text{even}}(X, \mathbb{Z})$$

$$K^1(X) \sim E_1^1(X) := H^{\text{odd}}(X, \mathbb{Z})$$

(4.8)

While we use the term “instanton” for brevity, the process illustrated in figure 6 need not be nonperturbative in string theory. Indeed, the example of section 5.5 below is a process in *classical* string theory. The decay process is simply localized in the time direction.
The first nontrivial approximation is

\[ K_0(X) \sim E_0^0(X) := (\text{Ker } d_3|_{H_{\text{even}}})/(\text{Im } d_3|_{H_{\text{odd}}}) \]

\[ K_1(X) \sim E_1^1(X) := (\text{Ker } d_3|_{H_{\text{odd}}})/(\text{Im } d_3|_{H_{\text{even}}}) \]  \hspace{1cm} (4.9)

with

\[ d_3(a) := Sq^3(a) + [H] \sim a. \]  \hspace{1cm} (4.10)

Let us pause to define \( Sq^3(a) \). Let us suppose, for simplicity, that the Poincaré dual \( PD(a) \) can be represented by a manifold \( \mathcal{W} \) and let \( \iota : \mathcal{W} \hookrightarrow X_9 \) be the inclusion. Then we let

\[ Sq^3(a) = \iota_*(W_3(\mathcal{W})) \]  \hspace{1cm} (4.11)

where \( \iota_* \) is a composition of three operations: first take the Poincaré dual of \( W_3(\mathcal{W}) \) within \( \mathcal{W} \), then push forward the homology cycle, and then take the Poincaré dual in \( X_9 \). Equivalently, regard \( a \) as a class compactly supported in a tubular neighborhood of \( \mathcal{W} \) and consider the class \( W_3(\mathcal{W}) \sim a \) where \( W_3(\mathcal{W}) \) is pulled back to the tubular neighborhood.

Returning to the AHSS, in general one must continue the approximation scheme. This is true, for example, when computing the twisted K-theory of \( SU(N) \) for \( N \geq 3 \).

Now, let us interpret the procedure of taking \( d_3 \) cohomology in physical terms. To interpret \( \text{Ker } d_3 \) note that from (4.11) it follows that

\[ d_3(a) = 0 \iff \left(W_3(\mathcal{W}) + [H]\right) \sim a = 0. \]  \hspace{1cm} (4.12)

Recall that global anomaly cancellation for a D-brane wrapping \( \mathcal{W} \) implies

\[ W_3(\mathcal{W}) + [H]|_{\mathcal{W}} = 0 \]  \hspace{1cm} (4.13)

and this in turn implies \( d_3(a) = 0 \). Thus, the physical condition \( (A') \) implies \( PD(\mathcal{W}) \in \text{Ker } d_3 \).

Next, let us interpret the quotient by the image of \( d_3 \) in (4.9). Suppose \( a = d_3(a') = (Sq^3 + [H])(a') \). Then, choose representatives

\[ PD(a) = \mathcal{W} \quad PD(a') = \mathcal{W}', \]  \hspace{1cm} (4.14)

where \( \mathcal{W} \) is codimension 3 in \( \mathcal{W}' \). A D-brane terminating on \( \mathcal{W} \) can be the magnetic source for the D-brane gauge field on \( \mathcal{W}' \) and

\[ PD(\mathcal{W} \hookrightarrow \mathcal{W}') = W_3(\mathcal{W}') + [H]|_{\mathcal{W}'} \quad \Rightarrow \quad a = d_3(a') \]  \hspace{1cm} (4.15)

Therefore, the physical process of D-instanton induced brane instability implies one should take the quotient by the image of \( d_3 \). (In fact, conditions \( (A'), (B') \) contain more information than \( d_3 \).)
4.3. Examples: Twisted $K$-groups of $SU(N)$

As an illustration of the above point of view let us consider the twisted $K$-groups of $SU(2)$ and $SU(3)$.

Consider first $K_H(SU(2))$. Then $H = k\omega$ where $\omega$ generates $H^3(SU(2); \mathbb{Z})$. In the cohomological model of branes we have $H^{\text{even}}(SU(2)) = H^0 \cong H_3 = \mathbb{Z}$, corresponding to “D3-branes” (or D2-instantons) while $H^{\text{odd}}(SU(2)) = H^3 \cong H_0 = \mathbb{Z}$ corresponding to “D0-branes.” Now condition $(A')$ shows that we can only have D0 branes. Indeed, D2 instantons wrapping $SU(2) = S^3$ violate D0-brane charge by $k$ units as in Fig. 7. For this reason if we take $SU(2)$ as the cycle $W'$ in condition $(B')$ then it follows that a system with $k$ D0 branes is in the same connected component of $B$ as a system with no D0 branes at all. In section 5.5 we will explain in more detail how this can be.

![Figure 7. $k$ D0 branes terminate on a wrapped D2-brane instanton in the $SU(2)$ level $k$ theory.](image)

In this way we conclude that

$$
K^0_H(SU(2)) = 0
$$
$$
K^1_H(SU(2)) = \mathbb{Z}/k\mathbb{Z}
$$

as is indeed easily confirmed by rigorous mathematical arguments.

Let us now consider $K_H(SU(3))$. Here the AHSS is not powerful enough to determine the $K$-group. However, it is important to bear in mind that the physical conditions $A', B'$ contain more information, and are stronger, than the $d_3$-cohomology. Once again we take $H = k\omega$, where $\omega$ generates $H^3(SU(3); \mathbb{Z})$.

In the cohomological model we have $H^{\text{odd}} \cong H_3 \oplus H_5$. Now, 3-branes cannot wrap $SU(2) \subset SU(3)$ since $[H]_{DR} \neq 0$. But 5-branes can wrap the cycle $M_5 \subset SU(3)$, where $M_5$ is Poincaré dual to $\omega$. Now,

$$
\int_{SU(3)} \omega S q^2 \omega = 1
$$

(4.17)
and hence

\[ \iota^*(\omega) = W_3(M_5) \]  

(4.18)
is nonzero. (In fact, it turns out that the cycle \( M_5 \) can be represented by the space of symmetric \( SU(3) \) matrices. This space is diffeomorphic to \( SU(3)/SO(3) \) and is a simple example of a non-Spin\(^c \) manifold.)

It follows from (4.18) that if \( M_5 \) is wrapped \( r \) times, anomaly cancellation implies

\[ r(k + 1)W_3 = 0 \]  

(4.19)
The D-brane instantons relevant to condition \((B')\) are just the \( D2 \)-branes wrapping \( SU(2) \). We thus conclude that

\[ L^1_{H=\kappa \omega}(SU(3)) = \begin{cases} \mathbb{Z}/k\mathbb{Z} & k \text{ odd} \\ 2\mathbb{Z}/k\mathbb{Z} & k \text{ even} \end{cases} \]  

(4.20)

Let us now turn to the even-dimensional branes, \( H^{\text{even}} \cong H_0 \oplus H_8 \). \( 8 \)-branes are anomalous because \([H]_{DR} \neq 0\), but 0-branes are anomaly-free.

**Figure 8.** \( k \) \( D0 \) branes terminate on a wrapped \( D2 \)-brane instanton in an \( SU(2) \) subgroup of \( SU(3) \)

**Figure 9.** When \( k \) is even \( \frac{1}{2}k \) \( D0 \) branes can terminate on a hemisphere of \( SU(2) \) which terminates on a generator of \( H_5(SU(3), \mathbb{Z}) \).

Now, D0-brane charge is not conserved because of the standard process of Fig. 8. There is, however, a more subtle instanton, illustrated in 9, in which a 3-chain ends on
a nontrivial element in $H_2(M_5; \mathbb{Z}) \cong \mathbb{Z}_2$. This instanton violates D0 charge by $\frac{1}{2}k$ units, when $k$ is even. In this way we conclude that

$$K^0_H(SU(3)) = \begin{cases} \mathbb{Z}/k\mathbb{Z} & k \text{ odd} \\ \mathbb{Z}/\frac{1}{2}\mathbb{Z} & k \text{ even} \end{cases}$$

(4.21)

One could probably extend the above procedure to compute the twisted K-theory of higher rank groups using (at least for $SU(N)$), Steenrod’s cell-decomposition, but this has not been done. In part inspired by the above results (and the result for D0 charge quantization explained in the next section) M. Hopkins computed the twisted $K$-homology of $SU(N)$ rigorously. He finds that, for $H = k\omega$:

$$K_{H,*}(SU(N)) = (Z/d_{k,N}Z) \otimes \Lambda_Z[w_3, \ldots, w_{2N-1}]$$

(4.22)

where

$$d_{k,N} = \gcd\left[\binom{k}{1}, \binom{k}{2}, \ldots, \binom{k}{N-1}\right].$$

(4.23)

We find perfect agreement for $G = SU(2), SU(3)$ above.

It is interesting to compare (4.22) with

$$H_*(SU(N)) = \Lambda_Z[w_3, w_5, \ldots, w_{2N-1}].$$

(4.24)

Recall that $SU(N) \sim S^3 \times S^5 \times \cdots S^{2N-1}$, rationally. Evidently, the topologically distinct D-branes can be pictured as wrapping different cycles in $SU(N)$, subject to certain decay processes. In the next section we will return to the worldsheet RG point of view to explain the most important of these decay processes. We will also give a simple physical argument (which in fact predated Hopkins’ computation) for why the group of charges should be torsion of order $d_{k,N}$.

5. The example of branes in $SU(N)$ WZW models

In this section we will use the theory of “symmetry-preserving branes” to determine the order $d_{k,N}$ of the D0 charge group for $SU(N)$ level $k$ WZW model. Different versions of the argument are given in [81,82,71,83]. For reviews with further details on the material of this section see [84,85,86] and references therein.

Let us summarize the strategy of the argument here:
1. We define the “elementary” or “singly-wrapped” symmetry-preserving boundary conditions algebraically using the formalism of boundary conformal field theory. These boundary conditions are labelled by the unitary irreps \( \lambda \in P_k^+ \) of the centrally extended loop group.

2. We give a semiclassical picture of these boundary conditions as branes wrapping special regular conjugacy classes with a nontrivial Chan-Paton line bundle. See equations (5.14) and (5.24) below.

3. We then discuss how it is that “multiply-wrapped” symmetry preserving branes can lie in components of \( B \) corresponding to certain singly-wrapped branes. For example, a “stack of \( L \) D0 branes” can be continuously connected by RG flow to a symmetry-preserving brane labelled by \( \lambda \), provided the number of D0 branes \( L \) is equal to the dimension \( d(\lambda) \) of the representation \( \lambda \) of the group \( G \).

4. This implies that the symmetry-preserving brane \( \lambda \) has, in some sense, D0 charge \( L \). On the other hand, as we have seen in the previous section, the D0 charge must be finite and cyclic. Thus the D0 charge is \( d(\lambda) \bmod d_{k,N} \), for some integer \( d_{k,N} \).

5. Finally we note that symmetry-preserving branes for different values of \( \lambda \) can sometimes be related by a rigid rotation continuously connected to 1. Such branes are obviously in the same component of \( B \), and this suffices to determine the order \( d_{k,N} \) of the torsion group.

5.1. WZW Model for \( G = SU(N) \)

Let us set our notation. The WZW field \( g : \Sigma \to G \) has action

\[
S = \frac{k}{8\pi} \int_\Sigma \text{Tr}_N[(g^{-1}\partial g)(g^{-1}\bar{\partial} g)] + 2\pi k \int \omega
\]

where the trace is in the fundamental representation. The target space \( G = SU(N) \) has a metric

\[
ds^2 = -\frac{k}{2} \text{Tr}_N(g^{-1}dg \otimes g^{-1}dg)
\]

and a “B-field” with fieldstrength

\[
H = k\omega, \quad \omega := -\frac{1}{24\pi^2} \text{Tr}(g^{-1}dg)^3
\]

where \([\omega]\) generates \( H^3(G;\mathbb{Z}) \cong \mathbb{Z} \). The CFT state space is

\[
\mathcal{H}^{\text{closed}} \cong \bigoplus_{P_k^+} \mathcal{H}_\lambda \otimes \bar{\mathcal{H}}_\lambda^*. \tag{5.4}
\]
where \( \mathcal{H}_\lambda, \tilde{\mathcal{H}}_{\lambda^*} \) are the left- and right-moving unitary irreps of the loop group \( \tilde{LG}_k \), as described in [87].

Amongst the set of conformal boundary conditions (i.e. branes) there is a distinguished set of “symmetry-preserving boundary conditions” leaving the diagonal sum of left and right-moving currents \( J + \tilde{J} \) unbroken. See [84] [85] for more details. Since there is an unbroken affine symmetry the open string morphism spaces \( \mathcal{H}_{\text{open}}^{ab} \) are themselves representations of \( \tilde{LG}_k \). Accordingly, these are objects in the category of boundary conditions labelled by \( \lambda \in P_k^+ \). The decomposition of the morphism spaces as irreps of \( \tilde{LG}_k \) is given by

\[
\mathcal{H}_{\lambda_1, \lambda_2}^{\text{open}} = \bigoplus_{\lambda_3 \in P_k^+} N_{\lambda_1, \lambda_2}^{\lambda_3} \mathcal{H}_{\lambda_3}
\]

(5.5)

where \( N_{\lambda_1, \lambda_2}^{\lambda_3} \) are the fusion coefficients.

The most efficient way to establish (5.5) is via the “boundary state formalism.” In the 2D topological field theory of section 2, the boundary state associated to boundary condition \( a \) is defined to be \( \iota^a(1_a) \) where \( 1_a \) is the unit in the open string algebra \( \mathcal{H}_{aa} \). This is an element of the closed string algebra \( \mathcal{C} \) which “creates” a free boundary with boundary condition \( a \). Similarly, in boundary CFT, to every conformal boundary condition \( a \) one associates a corresponding “boundary state”

\[
|B(a))\rangle \in \mathcal{H}_{\text{closed}}
\]

(5.6)

For the symmetry-preserving WZW boundary conditions the corresponding boundary state is given by the Cardy formula:

\[
|B(\lambda))\rangle = \sum_{\lambda' \in P_k^+} \frac{S_{\lambda'}}{\sqrt{S_0}} 1_{\mathcal{H}_{\lambda'}} \in \mathcal{H}_{\text{closed}}
\]

(5.7)

where \( S_{\lambda'} \) is the modular \( S \)-matrix, \( 0 \) denotes the basic representation, and we think of the closed string statespace as:

\[
\mathcal{H}_{\text{closed}} \cong \bigoplus_{P_k^+} \mathcal{H}_\lambda \otimes \tilde{\mathcal{H}}_{\lambda^*} \cong \bigoplus_{P_k^+} \text{Hom}(\mathcal{H}_\lambda, \mathcal{H}_\lambda)
\]

(5.8)

Applying the Cardy condition to (5.7) we get (5.5).

Since the disk partition function is the overlap of the ground state with the boundary state, \( Z_{\text{disk}} = \langle 0 | B(\lambda) \rangle \), the \( g \)-function for these conformal fixed points follows immediately from (5.7):

\[
g(\lambda) = \frac{S_{\lambda,0}}{\sqrt{S_0}}.
\]

(5.9)
Finally, as we noted before, it is important to introduce worldsheet supersymmetry in order to have any stable branes at all. It suffices to introduce $\mathcal{N} = 1$ supersymmetry, although when embedded in a type II string background the full background can have $\mathcal{N} = 2$ supersymmetry. It is also important to have a well-defined action by $(-1)^F$ on the conformal field theory. This distinguishes the cases where the rank of $G$ is odd and even. When the rank is odd we can always add an $\mathcal{N} = 1$ Feigin-Fuks superfield, as indeed is quite natural when building a type II string background.

5.2. Geometrical interpretation of the symmetry-preserving branes

We would like to discuss the geometrical interpretation of the symmetry-preserving boundary condition labelled by $\lambda$. That is, we would like some semiclassical picture of the brane as an extended object in the group manifold. In this section we explain how that is derived.

Let us first recall how the geometry of the compact target space is recovered in the WZW model. In the WZW model the metric is proportional to $k$, so the path integral measure has weight factor

$$\sim e^{-kS}$$  \hspace{1cm} (5.10)

Thus, we expect semiclassical pictures to emerge in the limit $k \to \infty$. In this limit the vertex operator algebra “degenerates” to become the algebra of functions on the group $G$. For example, CFT correlators become integrals over the group manifold:

$$\langle \hat{F}_1(g(z_1, \bar{z}_1)) \cdots \hat{F}_n(g(z_n, \bar{z}_n)) \rangle \to \int_G d\mu(g) F_1(g) \cdots F_n(g)$$  \hspace{1cm} (5.11)

On the left hand side $\hat{F}_i$ are suitable vertex operators of dimension $\sim 1/k$. On the right hand side, $F_i$ are corresponding $L^2$ functions on $G$. Roughly speaking, the CFT statespace degenerates as

$$\mathcal{H}^{\text{closed}} \cong \bigoplus_{p_k} \mathcal{H}_\lambda \otimes \mathcal{H}_\lambda^* \to L^2(G) \otimes \mathcal{H}^{\text{string}}$$  \hspace{1cm} (5.12)

where $L^2(G)$ is the limit of the primary fields and $\mathcal{H}^{\text{string}}$ contains the “oscillator excitations.” In this limit the boundary state degenerates:

$$|B(\lambda)\rangle \to B_\lambda + \cdots$$  \hspace{1cm} (5.13)

where $B_\lambda \in L^2(G)$ and becomes a distribution in the $k \to \infty$ limit. While (5.12) is clearly heuristic, (5.13) has a well-defined meaning because the overlaps of $|B(\lambda)\rangle$ with primary fields of dimension $\sim 1/k$ have well-defined limits.
Figure 10. Distinguished conjugacy classes in $SU(2)$. These are the semiclassical worldvolumes of the symmetry-preserving branes.

Using equation (5.7), the formulae for the modular $S$-matrix, and the Peter-Weyl theorem one finds that the function $B_\lambda$ is concentrated on the regular conjugacy class

$$O_{\lambda,k} := \left[ \exp \left( 2\pi i \frac{\lambda + \rho}{k + h} \right) \right]$$

leading to the semiclassical picture of branes in Fig. 10 above. Here $\rho$ is the Weyl vector and $h$ is the dual Coxeter number. (As usual, replace $k \to k - h, \lambda \in P_{k-h}^+$ for the supersymmetric case.) See [88][89][71] for more details.

Remarks:

1. Since $k$ is the semiclassical expansion parameter we only expect to be able to localize the branes to within a length-scale $\ell_{\text{string}} \sim 1/\sqrt{k}$ when using closed string vertex operators [90]. Let $\mathfrak{t}$ be the the Lie algebra of the maximal torus, and let $\chi \in \mathfrak{t}$ parametrize conjugacy classes. Then the metric $ds^2 \sim k(d\chi)^2$ and hence vertex operators can only “resolve” angles $\delta\chi \geq \frac{1}{\sqrt{k}}$. This uncertainty encompasses many different conjugacy classes (5.14). Nevertheless, the semiclassical geometrical pictures give exact results for many important physical quantities. The reason for this is that the relevant exact CFT results are polynomials in $1/k$, and hence can be exactly computed in a semiclassical expansion.

2. The basic representation $\lambda = 0$ gives the “smallest” brane. We will refer to this as a “D0-brane.” In a IIA string compactification built with the WZW model this state is used to construct a D0 brane. Note, however, that in this description it is not pointlike, but rather has a size of order the string length $\sim 1/\sqrt{k}$.
5.3. Using CFT to measure the distance between branes

To lend further support to the geometrical picture advocated above, let us show that D-branes can be rotated in the group, and that the distance between them can then be measured using CFT techniques.

First, we explain how to “rotate” D-branes. $G_L \times G_R$ acts on $\mathcal{H}^{\text{closed}}$, and therefore we can consider the boundary state

$$g_L g_R |B(\lambda)\rangle\rangle.$$  \hfill (5.15)

In the $k \to \infty$ limit this state has a limit similar to (5.13). In particular, it is supported on the subset

$$g_L \mathcal{O}_{\lambda,k} g_R \subset G.$$  \hfill (5.16)

**Figure 11.** Using a D0 brane as boundary condition $a$ we can probe for the location of brane $b$ by studying the lowest mass of the stretched strings.

Now, let us consider the open string statespace $\mathcal{H}^{\text{open}}_{ba}$ with $a$ corresponding to a rotated D0 brane, $a = g_L \cdot |B(\lambda = 0)\rangle\rangle$ and $b = |B(\lambda)\rangle\rangle$. A typical string in this space may be pictured as in Fig. 11. This picture suggests a way to “measure the distance” between the two branes, and thereby to define the positions of branes in the spirit of [91,92,93]. The picture suggests that the open-string channel partition function has an expansion for small $q_o$

$$\text{Tr}_{\mathcal{H}^{\text{open}}_{b,a}} q_o^{L_0} \simeq q_o \left( T_f D \right)^2 + \cdots$$  \hfill (5.17)

where $D$ is the geodesic distance between the center of the D0 brane at $g_L$ and the brane $b$. $T_f$ is the fundamental string tension (we set $\alpha' = 1$ here, so $T_f = 1/(2\pi)$).

We can actually compute the $q_o$ expansion of (5.17) using the expression for the boundary state together with the Cardy condition:

$$\text{Tr}_{\mathcal{H}_{b,a}} q_o^{L_0 - c/24} = \langle \langle B(\lambda) | q_c^{(L_0 + L_0 - c/12)} \rho_L(g) | B(0) \rangle \rangle.$$  \hfill (5.18)
where \( q_c = e^{2\pi i \tau_c} \), \( q_o = e^{-2\pi i / \tau_o} = e^{2\pi i \tau_o} \). The computation is straightforward. Let us quote the result for \( SU(2) \). If \( g_L \) is conjugate to

\[
\begin{pmatrix}
e^{ix} & 0 \\
0 & e^{-ix}
\end{pmatrix}
\]

(5.19)

with \( 0 \leq \chi \leq \pi \), then, in the Ramond sector, the leading power of \( q_o \) in (5.17) is

\[ k \left( \frac{\hat{\chi}_j - \chi}{2\pi} \right)^2 \]

(5.20)

Here \( \hat{\chi}_j = \pi(2j + 1)/k \), and the brane \( b \) is labelled by \( j \in \{0, \frac{1}{2}, \ldots, \frac{k-2}{2}\} \). The formula (5.20) is precisely \( (T_f D)^2 \), as naively expected. Again, we see that the geometrical picture of the branes is beautifully reproduced from the conformal field theory. Very similar remarks hold for the D-branes in coset models [89].

5.4. Why are the branes stable?

The geometrical picture advocated in the previous sections raises an interesting puzzle. We will now describe this puzzle, and its beautiful resolution in [94] [95].

Consider a D-brane wrapping \( O_{\lambda,k} \subset G \) once, as in Fig. 10. In the context of the type II string theory, the brane has a nonzero tension \( T \), with units of energy/volume. Hence, wrapping a submanifold \( W \) with a brane costs energy \( E \sim T \operatorname{vol}(W) \). However, the regular conjugacy classes \( O_{\lambda,k} \subset G \) are homologically trivial. For example, for \( SU(2) \), \( O = S^2 \subset S^3 \). We therefore expect the brane to be unstable and to contract to a point.

This leads to a paradox: We know from conformal field theory that the brane is absolutely stable. From the expression for the boundary state we can compute the spectrum of operators in the open string state from

\[
\operatorname{Tr}_{\mathcal{H}_{\lambda,\lambda}} q^{L_0 - c/24}
\]

(5.21)

and we find all \( \Delta_i \geq 1 \). According to (3.5) it follows that there are no unstable flows under \( \beta \) away from this point!

The resolution of the paradox lies in the fact that D-branes also have gauge theory degrees of freedom on them. The brane carries a \( U(1) \) line bundle \( \mathcal{L} \rightarrow O \) with connection. If this bundle is twisted then there is a stabilizing force opposing the tension.

\[ k(\delta \chi)^2 \gg 1 \] that the conclusion holds.\[ ^6 \]

In the bosonic case (5.20) turns out to be

\[ \frac{k+2}{4} \left( \frac{\hat{\chi}_j - \chi}{\pi} \right)^2 + \frac{1}{2} \frac{\chi}{\pi} (1 - \frac{\chi}{\pi}) - \frac{1}{(k+2)} \]

so it is only for \( k(\delta \chi)^2 \gg 1 \) that the conclusion holds.
To illustrate the resolution in the simplest terms, consider the example of $SU(2)$ with conjugacy class $O = S^2$, of radius $R = \sqrt{k} \sin \chi$. If the Chan-Paton line bundle of the brane has Chern class $n \in \mathbb{Z}$, then $\int_O F = 2\pi n$. It follows that the Yang-Mills action is

$$\int_{S^2} F \wedge *F \sim \frac{n^2}{R^2}$$

and hence we can evaluate the $g$-function (3.20)

$$g(\chi) \sim R^2 + \frac{n^2}{R^2}$$

This has a minimum at $\chi \sim \pi n/k$, and hence we expect an RG flow in the sector of $B$ determined by $n$ to evolve to this configuration.

The above arguments have been generalized from $SU(2)$ to higher rank groups in [71]. The result that emerges is that $|B(\lambda)|$ can be pictured, semiclassically, as wrapping the conjugacy class $O_{\lambda,k}$. The brane is singly wrapped, and its Chan-Paton line bundle $L_{\lambda} \to O_{\lambda,k}$ has first Chern class

$$c_1(L_{\lambda}) = \lambda + \rho \in H^2(G/T; \mathbb{Z}) \cong \Lambda_{\text{weight}}.$$  

(For further details see [71].)

It is interesting to study the $g$-function and its approximation by the DBI action in this problem. We restrict attention to the bosonic WZW model. Let $\chi$ parametrize the conjugacy classes in $G$. For the Chan-Paton line bundle (5.24) the DBI action

$$g_{DBI}(\chi) := \int_{O_\chi} \sqrt{\det(g + F + B)}$$

as a function of $\chi$ is minimized at $\chi_* = 2\pi (\lambda + \rho)/(k + h)$, where it takes the value:

$$g_{DBI}(\chi_*)/g_{DBI}(0) = \prod_{\alpha > 0} \left( \frac{k \sin \frac{1}{2} \alpha \cdot \chi}{\pi \alpha \cdot \rho} \right).$$

Here the product is over positive roots. This compares remarkably well with the exact CFT answer:

$$g(\lambda)/g(0) = \prod_{\alpha > 0} \left( \frac{\sin \pi \alpha \cdot (\lambda + \rho)/(k + h)}{\sin \pi \alpha \cdot \rho/(k + h)} \right).$$

Note that the right-hand side is the quantum dimension $d_q(\lambda)$, in harmony with (5.9) above.
5.5. How collections of D0 branes evolve to symmetry-preserving branes

The semiclassical picture of the symmetry-preserving branes we have just described raises an important new point. In type II string compactification, if a brane carries a topologically nontrivial Chan-Paton bundle then it carries nontrivial induced D-brane charge. In the present case since the Chan-Paton line bundle has \( \int_W e^{c_1(L)} \neq 0 \) it carries D0 charge. This suggests that the conformal fixed point characterized by \( |B(\lambda)\rangle\rangle \) is in the same component of \( \mathcal{B} \) as the fixed point corresponding to a “collection of D0 branes.” In this section we review why that is true.\(^7\)

By a “stack of D0 branes” physicists mean the boundary state \( L|B(0)\rangle\rangle \) for some positive integer \( L \). By definition, the open string sectors for such a stack of D0 branes have state spaces

\[
\mathcal{H}_{LB(0),b}^{\text{open}} = \mathbb{C}^L \otimes \mathcal{H}_{B(0),b}^{\text{open}}
\]

for any boundary condition \( b \).

**Claim:** If \( \lambda \in P_k^+ \) and \( L = d(\lambda) \), then \( L|B(0)\rangle\rangle \) is in the same component of \( \mathcal{B} \) as \( |B(\lambda)\rangle\rangle \).

Note that

\[
\frac{g(B(\lambda))}{g(L\lambda(0))} = \frac{S_{\lambda,0}}{L S_{00}} = \frac{d_q(\lambda)}{d(\lambda)} < 1
\]

so the claim is nicely consistent with the \( g \)-theorem. In particular, if these fixed points can be connected by RG flow then \( L \) D0’s are unstable to \( \lambda \), and not vice versa. We sometimes refer to this instability as the “blowing up effect.”

The RG flow in question arises in the theory of the Kondo effect and was studied by Affleck and Ludwig \[25\] \[26\]. Their results were applied in the present context by Schomerus and collaborators. See \[84\], and references therein. Kondo model trajectories are obtained by perturbing a conformal fixed point by the holonomy of the unbroken current algebra in some representation. The flow, which should take \( L|B(0)\rangle\rangle \) to \( |B(\lambda)\rangle\rangle \), is given by considering the disk partition function

\[
Z(u) = \langle \text{Tr}_\lambda \left( \text{P \exp} \int d\tau u J(\tau) \right) \rangle.
\]

---

\(^7\) Actually, the most obvious embedding of the \( SU(2) \) WZW model into a type IIA background using a Feigin-Fuks superfield produces a background for which the definition of RR D0 charge is in fact subtle. The relevant \( U(1) \) RR gauge group is spontaneously broken to \( \mathbb{Z}_k \) due to the condensation of a spacetime scalar field of charge \( k \). See \[97\] for more discussion.
As explained in [84], the results of Affleck and Ludwig lend credence to the main claim. Actually, it is important to take into account $N = 1$ supersymmetry in this problem. In the supersymmetric WZW model we have a superfield

$$
J_a(z) = \psi_a(z) + \theta I_a(z),
$$

(5.31)

where we have chosen an orthonormal basis for the Lie algebra, labelled by $a = 1, \ldots, \dim G$. The OPE's are [99,100,101]

$$
\begin{align*}
I_a(z)I_b(w) & \sim \frac{k\delta_{ab}}{(z-w)^2} + f_{ab}^c I_c(w) + \cdots \\
I_a(z)\psi_b(w) & \sim \frac{f_{ab}^c\psi_c(w)}{z-w} + \cdots \\
\psi_a(z)\psi_b(w) & \sim \frac{k\delta_{ab}}{(z-w)} + \cdots
\end{align*}
$$

(5.32)

By a standard argument the currents $J_a = I_a + \frac{1}{2k}f_{abc}\psi_b\psi_c$ decouple from the fermions and satisfy a current algebra with level $k - h$. The Hamiltonian and supersymmetry charge are given (in the Ramond sector) by

$$
\begin{align*}
Q &= \oint dz \left( \frac{1}{k}J_a\psi_a - \frac{1}{6k^2}f_{abc}\psi_a\psi_b\psi_c \right) \\
H &= \frac{1}{2k} \oint dz \left( :J_a J_a: + \partial \psi_a \psi_a \right)
\end{align*}
$$

(5.33)

The supersymmetry transformations are $[Q,\psi_a] = I_a$ and $[Q,I_a] = \partial \psi_a$.

Now, let us add a Kondo-like boundary perturbation preserving $N = 1$ supersymmetry. This is given by choosing a representation $\lambda$ of $G$ and taking

$$
g(u) = \langle \text{Tr}_\lambda \left( P \exp \oint d\tau u I(\tau) \right) \rangle
$$

(5.34)

Using $[Q,I_a] = \partial \psi_a$ to vary the perturbed action in (5.34) we may compute the perturbed supercharge $Q_u$. This operator acts on $C^L \otimes H_{\text{open}}$ as

$$
Q_u = Q + w\psi^a(0)S^a = Q + u \sum_{n \in \mathbb{Z}} \psi_n^a S^a
$$

(5.35)

---

8 The following argument combines elements from [84,98,71].
In the first line we have passed to a Hamiltonian formalism for the open string on a space 
\([0, \pi]\) (and we are only modifying the boundary condition at \(\sigma = 0\)), and we have introduced 
explicit generators \(S^a\) for the finite dimensional representation \(C^L\) of the group \(G\). In the 
second line we have used the doubling trick to express the result in terms of modes of a 
single-valued chiral vertex operator on the plane \(C\). We can now compute the perturbed 
Hamiltonian

\[
H_u = Q_n^2 = H + uI^a(0)S^a + u^2(\psi^a(0)S^a)^2
\]

The third term in (5.36) is singular, but the renormalization of this term is fixed by the 
requirement of supersymmetry. The Hamiltonian can be written as

\[
H_u = \frac{1}{2k} \sum_n \left( : (J_n + ukS^a)(J_{-n} + ukS^a) + n\psi^a_n\psi^a_{-n} \right) \\
+ \frac{1}{2} u(u - \frac{1}{k}) \sum_{n,m} f^{abc}_n \psi^a_n\psi^b_m S^c
\]

so that the vacuum of the theory evolves in a complicated way as a function of \(u\). Note 
that, exactly for \(u = u_* = 1/k\), the Hamiltonian simplifies into

\[
H_* = \frac{1}{2k} \sum_n \left( : J_n^a J_{-n}^a : + n\psi^a_n\psi^a_{-n} \right)
\]

where

\[
J_n^a := J_n^a + S^a
\]

also satisfy a current algebra with level \(k - h\). Thus, at \(u = u_*\), we can build a new 
superconformal algebra with these currents.

The previous paragraph strongly suggests that \(u_* = 1/k\) is a second critical point 
for the boundary conformal field theory. Now, we can use an observation of Affleck and 
Ludwig. If \(\mathcal{H}_{\lambda'}\) is a representation of \(J_n^a\), then with respect to a new current algebra \(J_n^a\) 
we can decompose:

\[
C^L \otimes \mathcal{H}_{\lambda'} \cong \bigoplus_{\lambda''} N_{\lambda',\lambda''}^{\lambda'} \mathcal{H}_{\lambda''}
\]

(An easy way to prove (5.40) is to consider the cabling of Wilson lines in 3D Chern-Simons 
theory, and use the Verlinde algebra.) Therefore it follows that

\[
\text{Tr}_{\lambda_1} \left( P \exp \oint u_* I \right) |B(\lambda_2)\rangle = \sum_{\lambda_3} N_{\lambda_1,\lambda_2}^{\lambda_3} |B(\lambda_3)\rangle
\]
where the boundary states on the RHS are constructed using $\mathcal{J}_n^a$. It would be worthwhile to give a direct proof of (5.41). The identity has been verified at large $k$ in [98].

Let us close this subsection with a number of remarks.

1. Note that when there is more than one term on the right-hand side of (5.40) a local boundary condition has evolved into a (mildly) nonlocal boundary condition. Regrettably, this muddies the proposed definition of D-branes as local boundary conditions preserving conformal invariance.

2. The instability of a stack of $L$ D0-branes to decay to a symmetry-preserving brane has been much discussed in the literature in the framework of noncommutative gauge theory. See [84] and references therein. The arguments show that the “D-brane instantons” of the previous section should be viewed as real-time processes taking place in classical string theory.

3. The Kondo flows are integrable flows. The $g$ function has been studied in [102,103,104,105,106,107] in several examples and for certain boundary conditions related to the free fermion construction of current algebras. It is possible that the techniques of [102,103,104] can be used to give exact results for how the boundary state evolves along the RG trajectory. This could be very interesting indeed.

4. The “blowing up effect” is closely related to some work of [108,109]. These authors study families of Fredholm operators over the space of gauge fields on $S^1$. The perturbed supersymmetry operator along the RG flow is related to the family of Fredholm operators studied in [108,109].

5. One of the most remarkable aspects of the blowing-up effect is the disappearance of $k$ D0-branes “into nothing.” Let us stress that this is an effect studied in the laboratory! One studies electrons coupled to a magnetic “impurity.” Translating this system into conformal field theory terms [110,111] reveals the boundary $SU(2)$ model with $k = 1$; the presence of the magnetic impurity, in the high temperature regime, translates into the presence of a single D0 brane. The RG flow parameter is the temperature, and, as $T \to 0$, the magnetic impurity is screened and “disappears.” The absence of the magnetic impurity corresponds to the disappearance of the D0 brane.

6. The effect we are discussing can be related, by U-duality, to the Myers effect [111]. (Apply $S$-duality to a IIB solitonic 5-brane.)

7. Actual evaluation of the standard D0-brane charge formula $\int_{O_{\lambda,k}} e^{F+B}$ yields a quantum dimension for the group. As far as we know, this curious fact has not yet been properly understood.

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5.6. The D0 charge group

At this point we have two notions of D0-brane charge. On the one hand, we have naive D0 charge \( L = d(\lambda) \). On the other hand, as we explained in section 4, due to D-brane instantons, the true D0 charge, which we denote by \( q(\lambda) \), must be a torsion. Indeed, we know that D-brane instanton effects will impose a relation:

\[
q(\lambda) = d(\lambda) \mod d_{k,N}
\]

for some integer \( d_{k,N} \), but, (without Hopkins), it is hard to account for all possible D-brane instantons. So, we will now determine the order, \( d_{k,N} \), of the torsion group using the blowing up effect and a simple observation regarding rotated branes.

![Figure 12. Two symmetry-preserving branes related by a rigid rotation.](image)

Recall from section 5.3 that we can rotate our branes by \( G_L \times G_R \). Sometimes it can happen that the special conjugacy classes can be rotated into one another

\[
g_L \mathcal{O}_{\lambda,k} g_R = \mathcal{O}_{\lambda',k}.
\]

For example, if \( G = SU(2) \) the conjugacy classes \( \mathcal{O}_{j,k} \) and \( \mathcal{O}_{k/2-j,k} \) can be rotated into each other:

\[
(-1) \cdot \mathcal{O}_{j,k} = \mathcal{O}_{\frac{k}{2} - j,k}
\]

as in Fig. 12. Let us ask which representations are related in this way.

In order to answer this question we use the well-known relation between the center of a compact connected, simply-connected Lie group \( G \) and the automorphisms of the extended Dynkin diagram. For example, if \( G = SU(N) \), \( Z(G) \cong Z_N \), and \( Z_N \) acts on the extended Dynkin diagram by rotation. Next, the automorphisms of the extended Dynkin diagram act on the space of level \( k \) integrable representations \( P^+_k \). For example, for \( SU(2) \)

\[
j \rightarrow j' = \frac{1}{2} k - j.
\]
while for $SU(N)_k$ the generator of $Z(G)$ acts on the Dynkin labels by

$$\lambda = (a_1, \ldots, a_{N-1}) \rightarrow \lambda' = (k - \sum a_i, a_1, \ldots, a_{N-2}). \quad (5.46)$$

A beautiful result of group theory is that if $z \in Z(G)$ then

$$zO_{\lambda,k} = O_{z\cdot\lambda,k} \quad (5.47)$$

Now we can use (5.47) to determine the order of the D0 charge group. To see this note first that two branes related by a rigid rotation must have the same D0 charge! On the other hand, $\lambda$ and $z \cdot \lambda$ have different dimensions, and hence have different naive D0 charge. Therefore, we seek an integer $d_{k,N}$ such that

$$d(z \cdot \lambda) = \pm d(\lambda) \mod d_{k,N} \quad \forall z \in Z(G), \lambda \in P_k^+ \quad (5.48)$$

where the sign $\pm$ depends on $z$ and the rank of $G$, and accounts for orientation (see [71] for more details). It turns out that this condition determines $d_{k,N}$:

$$d_{k,N} = \gcd\left[k \begin{pmatrix} 1 \\ 1 \end{pmatrix}, k \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \ldots, k \begin{pmatrix} N-1 \\ N-1 \end{pmatrix}\right]. \quad (5.49)$$

in perfect agreement with Hopkins’ result! (See [82], [71] for details of some of the arithmetic involved.)

Remarks:

1. The generalization of $d_{k,N}$ to other compact simple Lie groups has been discussed in [112].

2. After my talk at the conference, M. Hopkins made a curious remark which I would like to record here. There is a simple mathematical relation between twisted equivariant K-theory and twisted K-theory of $G$ which has several of the same ingredients as the physical discussion we have just given. If $\pi_1(G)$ is torsion free the Kunneth formula of [113], [114] suggests that the two twisted K-theories are related by

$$\mathbb{Z} \otimes_{R(G)} K_{G,H}(G) = K_H(G) \quad (5.50)$$

Here the representation ring $R(G)$ is to be thought of as the ring of functions on the representation variety $G/T$ with $T$ acting by conjugation. $R(G)$ acts on $1 \in \mathbb{Z}$ by the dimension of the representation, while $K_{G,H}(G)$ is the Verlinde algebra, thanks to the theorem of Freed, Hopkins, and Teleman [115], [116], [117]. Curiously, from the point of view of algebraic geometry this means that the special conjugacy classes have an intersection with the identity element, when considered as varieties over $\mathbb{Z}$.
5.7. Comment on Cosets

The point of view explained above has potentially interesting applications to branes in coset models. Roughly speaking, if $L \subset G$ is a subgroup then the branes in the $\mathcal{N} = 1$ supersymmetric coset model $G/L$ should be classified by the twisted equivariant K-theory $K_{L,H}(G)$, where the twisting comes from the WZW $G$-theory. The branes in such coset models have been studied in many papers. See $[118, 89, 86, 119, 121, 122]$ for a sampling. For the $SU(2)/U(1)$ model the stable $A$-banes described in $[89]$ are in perfect accord with the twisted equivariant $K$-theory. The higher rank situation is somewhat more subtle and is currently under study by S. Schafer-Nameki $[123]$.

6. Conclusion

Our goal in this talk was not to establish rigorous mathematical theorems but to explain how physics can suggest some intuitions for K-theory which are complementary to the more traditional (and rigorous!) approaches to the subject. Such alternative viewpoints and heuristics can sometimes suggest new and surprising directions for enquiry, or can suggest simple heuristics for already-known results. The above “derivation” of the twisted K-theory of $SU(N)$ is just one example, but there are others. For example, the symmetry-preserving branes are precisely the branes which descend to branes in the $G/G$ gauged WZW model. The reason is that the gauge group acts on $G$ by conjugation, and only the symmetry-preserving boundary conditions preserve this gauge symmetry. Now, the $G/G$ WZW model is a topological field theory whose Frobenius algebra is the Verlinde algebra. This provides a simple perspective on the physics underlying the result of Freed, Hopkins, and Teleman $[115][116][117]$. (This remark is also related to the discussion of $[115]$.)

Let us conclude by mentioning some future directions which might prove to be interesting to the mathematics community, and which are suggested by the more physical approach to K-theory advocated in this paper.

First, in the context of spacetime supersymmetric models a special class of boundary conditions, the so-called “BPS states” might have an interesting product structure $[124][125]$. Thus, perhaps the category of boundary conditions (or an appropriate subcategory) can also be given the structure of a tensor category.

Second, the RG approach to D-branes suggests an interesting generalization of the McKay correspondence to non-crepant toric resolutions of orbifold singularities $[126]$. 

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Finally, the K-theoretic classification of D-branes in type II string theory must somehow be compatible with the U-duality symmetries these theories enjoy, and must somehow be compatible with 11-dimensional M-theory. Only bits and pieces of this story are at present understood. It is possible that the full resolution will be deep and will have interesting mathematical applications.

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