Bäcklund Transformations for
Darboux Integrable Differential Systems:
Examples and Applications

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July 14, 2014
1 Introduction

This article can be viewed as a continuation of the article Bäcklund Transformations for Darboux Integrable Differential Systems [5]. In that paper we established a general group-theoretical approach to the construction of Bäcklund transformations. We then showed how this construction can be applied to construct Bäcklund transformation between equations which are Darboux integrable. Here we give a number of detailed examples and new applications which demonstrate the theory. In particular our final example demonstrates how our group theoretical approach produces all the Bäcklund transformations in [11].

Let $\Delta_1(z) = 0$ and $\Delta_2(u) = 0$ be two systems of partial differential equations in the unknown functions $z$ and $u$. A Bäcklund transformation between these two systems of equations is a third system of equations $\Delta_3(u, z) = 0$ with the following property. If a solution for $\Delta_1(z) = 0$ is given, then $\Delta_3(u, z) = 0$ becomes a system of total differential equations for $u$ which is then a solution to $\Delta_2(u) = 0$. Likewise, if a solution to $\Delta_2(u) = 0$ is given, then $\Delta_3(u, z) = 0$ becomes a system of total differential equations for $z$ which is then a solution to $\Delta_1(z) = 0$.

Our approach to the construction of Bäcklund transformations is formulated within the differential-geometric setting of symmetry reduction of exterior differential systems (EDS). From this geometric viewpoint, two differential systems $I_1 \subset \Omega^*(M_1)$ and $I_2 \subset \Omega^*(M_2)$, defined on manifolds $M_1$ and $M_2$, are said to be related by a Bäcklund transformation if there exists a differential system $B \subset \Omega^*(N)$ on a manifold $N$ and maps

$$\begin{align*}
\left( \begin{array}{c}
B, N \\
I_1, M_1 \\
I_2, M_2
\end{array} \right)
\end{align*}$$

which define $B$ as integrable extensions for both $I_1$ and $I_2$. Here $I_1$ and $I_2$ play the role of the differential equations $\Delta_1(z) = 0$ and $\Delta_2(u) = 0$ while the differential system $B$ plays the role of the third equation or Bäcklund transformation $\Delta_3(u, z) = 0$.

To describe our method in some detail, let us first recall a few definitions. Let $G$ be a Lie group which acts on a manifold $M$. We say that $G$ acts regularly on $M$ whenever the orbit space $M/G$ admits the structure of a differentiable manifold for which the canonical projection map $q_G : M \to M/G$ is a surjective submersion. We say that $G$ is a symmetry group of a differential system $I \subset \Omega^*(M)$ if the diffeomorphisms of $M$ defined by $G$ preserve $I$, and we say the $G$ acts transversely to $I$ if the tangent spaces to the orbits of $G$ are pointwise transverse to the space of 1-forms in $I$ (that is, no non-zero vector in the tangent space to the orbits is annihilated by all the 1-forms in $I$, see equation (2.13)). As we shall see below in Theorem A, this transversality condition plays an important role in the construction of integrable extensions and hence Bäcklund transformations.
Let \( p : M \to N \) be a smooth submersion. Then we define the \textit{reduced differential system} \( \mathcal{I}/p \) on \( N \) by
\[
\mathcal{I}/p = \{ \theta \in \Omega^*(N) | p^*(\theta) \in \mathcal{I} \}.
\] (1.2)

In the special case where \( G \) is a regularly acting symmetry group of \( \mathcal{I} \), we shall write \( \mathcal{I}/G \) in place of \( \mathcal{I}/q_G \). Finally, we say that the diagram
\[
\begin{array}{ccc}
(I, M) & \xrightarrow{p} & (J, N) \\
\downarrow{r} & & \downarrow{q} \\
(K, L)
\end{array}
\]
is a \textit{commutative diagram} of EDS if \( q \circ p = r \), \( J = \mathcal{I}/p \) and \( K = J/q = \mathcal{I}/r \). Additional information on these definitions can be found in Section 2 in [1].

With these definitions in place we now state our main theorem on the construction of Bäcklund transformations by symmetry group reduction whose proof is given in [5].

**Theorem A.** Let \( \mathcal{I} \) be a differential system on \( M \) with symmetry groups \( G_1 \) and \( G_2 \). Let \( H \) be a common subgroup of \( G_1 \) and \( G_2 \) and assume that the actions of \( G_1 \), \( G_2 \) and \( H \) are all regular on \( M \). Then the orbit projection maps \( p_1 : M/H \to M_1/G_1 \) and \( p_2 : M/H \to M_2/G_2 \) are smooth surjective submersions and
\[
\begin{array}{ccc}
(I, M) & \xrightarrow{q_{G_1}} & (I/H, M/H) \\
\downarrow{q_H} & & \downarrow{q_{G_2}} \\
(I/G_1, M/G_1) & \xleftarrow{p_1} & (I/G_2, M/G_2)
\end{array}
\] (1.3)
is a commutative diagram of EDS. Furthermore, if the actions of \( G_1 \) and \( G_2 \) are transverse to \( \mathcal{I} \), then the maps in (1.3) are all integrable extensions and the diagram
\[
\begin{array}{ccc}
(I/H, M/H) & \xrightarrow{p_1} & (I/G_1, M/G_1) \\
\downarrow{p_2} & & \downarrow{}
\end{array}
\] (1.4)
defines \( (I/H, M/H) \) as a Bäcklund transformation between \( (I/G_1, M/G_1) \) and \( (I/G_2, M/G_2) \).

The transversality hypothesis guarantees that the maps \( q_{G_i} : I \to I/G_i \) are integrable extensions (see Theorem 2.1 in Section 2.3 or Theorem 2.1 in [5]). The proof that the maps \( p_i \) are integrable extensions is given by Theorem 3.4 in [5].
Our first remark about diagram (1.4) is that the maps $p_i$ in the diagram (1.4) are in general not group quotients. Secondly, the double fibration in (1.4) is, at least in terms of the manifolds which are constructed, a simple generalization of the double fibration construction for homogeneous spaces presented by Baston and Eastwood [8] (page 69) in the context of the Penrose transform.

In Section 3 we shall use the group theoretical methods provided by Theorems A to construct Bäcklund transformations for a variety of Darboux integrable differential systems. In each example we shall begin with a differential system $\mathcal{I}$ given as a direct sum $\mathcal{K}_1 + \mathcal{K}_2$ on a product manifold $M_1 \times M_2$. We define a group action $H$ acting diagonally on $M_1 \times M_2$ which is a symmetry group of $\mathcal{I}$ and which acts transversely and, from this action, calculate the quotient differential system $\mathcal{B} = \mathcal{I}/H$. We then pick two more Lie symmetry groups $G_1$ and $G_2$ of $\mathcal{I}$, with $H \subset G_1 \cap G_2$ and calculate the quotient differential systems $\mathcal{I}_1 = \mathcal{I}/G_1$ and $\mathcal{I}_2 = \mathcal{I}/G_2$. The orbit projection maps $p_i : \mathcal{B} \rightarrow \mathcal{I}_i$ define the sought after Bäcklund transformation. Theorem 6.1 in [5] shows that all three differential systems in diagram (1.4) are Darboux integrable. We then use Theorem 6.1 and Theorem 7.7 in [5] to determine the Darboux invariants and the Vessiot algebra which are fundamental invariants for each of these Darboux integrable system.

With Theorem A in hand as a method for constructing Bäcklund transformations we are immediately lead to the converse problem. Given a Bäcklund transformation (1.1) can it be constructed by Theorem A? In the context of Bäcklund transformations between Darboux integrable systems $\mathcal{I}_1$ and $\mathcal{I}_2$ we have identified in [5] sufficient conditions for this to be the case. In order to describe these conditions we first note that the proof of Theorem A is obtained by two applications of the following Lemma proved in Lemma 3.2 and Theorem 3.4 in [5].

**Lemma A.** Let $G$ be a symmetry group of an EDS $\mathcal{I}$ on a manifold $M$ which acts regularly on $M$. Let $H \subset G$ be a closed subgroup of $G$. Then the orbit mapping

$$p : M/H \rightarrow M/G \quad \text{defined by} \quad p(Hx) = Gx$$

(1.5)

is a surjective submersion which gives rise to the following commutative diagram of EDS

$$
\begin{array}{ccc}
\mathcal{I} & \xrightarrow{q_H} & \mathcal{I}/H \\
\downarrow{q_G} & & \downarrow{p} \\
\mathcal{I}/G & & \\
\end{array}
$$

(1.6)

that is, $(\mathcal{I}/H)/p = \mathcal{I}/G$. If $G$ acts transversely to $\mathcal{I}$ then $\mathcal{I}/H$ is an integrable extension of $\mathcal{I}/G$.

In order to prove that a Bäcklund transformation $\mathcal{B}$ is obtained using Theorem A, we first need to decide when an integrable extension $\mathcal{E}$ of a Darboux integrable system $\mathcal{I}$ can be obtained by an application of Lemma A. The condition on $\mathcal{E}$ is that it be maximally compatible (see Section 2.5 or Section 8 in [5]). Roughly speaking this means $\mathcal{E}$ has the maximum number of possible Darboux invariants. This results in the following theorem.
**Theorem B.** Let \( p : (\mathcal{E}, N) \to (\mathcal{I}, M) \) be an integrable extension of Darboux integrable systems \( \mathcal{E} \) and \( \mathcal{I} \) and suppose the pair \((\mathcal{E}, \mathcal{I})\) is maximally compatible. Let \( K = K_1 + K_2, P = M_1 \times M_2 \) with Vessiot group \( G \) be the (local) canonical quotient representation of \((\mathcal{I}, M)\). There exists a subgroup \( H \subset G \) such that
\[
(\mathcal{E}, N) \cong_{\text{loc}} (K/H_{\text{diag}}, P/H_{\text{diag}})
\]
and the submersion \( p \) is locally identified with the orbit projection map from \( P/H_{\text{diag}} \) to \( P/G_{\text{diag}} \).

Theorem B is summarized by the commutative diagram of differential systems.
\[
\begin{array}{ccc}
(K_1 + K_2, M_1 \times M_2) & \rightarrow & (\mathcal{I}, M) \\
q_{H_{\text{diag}}} & \downarrow & \q_{G_{\text{diag}}} \\
(\mathcal{E}, N) & \rightarrow & P \\
\end{array}
\]

The following result is given in [3] (Theorems 1.3 and 1.4 in [3]). Let \( \mathcal{I}_2 \) be a Monge Ampère system on a 5-manifold which is Darboux integrable system but not Monge integrable and where the Vessiot algebra is not \( \mathfrak{so}(3, \mathbb{R}) \). A Bäcklund transformation having 1-dimensional fibres can be constructed using Theorem A (or Theorem 9.1 in [5] between \( \mathcal{I}_2 \) and the wave equation \( I_3=0 \) as a Monge-Ampère system on a 5-manifold. In Section 4 we to prove the converse. We show in Theorem 4.1 that every Bäcklund transformation with 1-dimensional fibre between such a Darboux integrable system \( \mathcal{I}_2 \) Monge Ampère system on a 5-manifold which is Darboux integrable system but not Monge integrable and the wave equation \( \mathcal{I}_1 \) can always be obtained using the group methods of Theorem A as in [3]. In particular, this shows that all the Bäcklund transformations explicitly constructed in [1] between hyperbolic Monge-Ampère systems and the hyperbolic Monge-Ampère system for the wave equation \( I_1 \) can as symmetry reductions of differential systems using Theorem A (or Theorem 9.1 in [5]). As a further consequence Theorem 4.7 shows that for a Darboux integrable system \( \mathcal{I}_2 \) whose Vessiot group is \( \mathfrak{so}(3, \mathbb{R}) \) then no such Bäcklund transformation can exist. An explicit example of a Darboux integral Monge-Ampère equation with \( \mathfrak{so}(3, \mathbb{R}) \) Vessiot algebra is given which provides a specific counter example of part 2 of Theorem 1 in [1].

## 2 Preliminaries

In this section we gather together a number of definitions and basic results on integrable extensions, reductions of exterior differential systems and Darboux integrability. Conventions and some results from [11] are used, see also [1], [5].
2.1 Exterior Differential Systems

We assume that an EDS $\mathcal{I}$, defined on a manifold $M$, has constant rank in the sense that each one of its homogeneous components $\mathcal{I}^p \subset \Omega^p(M)$ coincides with the sections $\mathcal{S}(\mathcal{I}^p)$ of a constant rank subbundle $\mathcal{I}^p \subset \Lambda^p(M)$. If $\mathcal{A}$ is a subset of $\Omega^*(M)$, we let $\langle \mathcal{A} \rangle_{\text{alg}}$ and $\langle \mathcal{A} \rangle_{\text{diff}}$ be the algebraic and differential ideals generated by $\mathcal{A}$. If $\mathcal{I}$ and $\mathcal{J}$ are differential systems on the same manifold, we let

$\mathcal{I} + \mathcal{J} = \langle \mathcal{I} \cup \mathcal{J} \rangle_{\text{alg}}$. \hspace{1cm} (2.1)

Note that $\mathcal{I} + \mathcal{J}$ is differentially closed. As usual, a differential system $\mathcal{I}$ is called a Pfaffian system if there is a constant rank subbundle $\mathcal{I} \subset \Lambda^1(M)$ such that $\mathcal{I} = \langle S(\mathcal{I}) \rangle_{\text{diff}}$. As is customary, the subbundle $\mathcal{I}$ shall also be referred to as a Pfaffian system.

Let $\mathcal{I} \subset \Lambda^1(M)$ be a Pfaffian system. A local first integral of $\mathcal{I}$ is a smooth function $f: U \to \mathbb{R}$, defined on an open set $U$, such that $df \in \mathcal{I}$. For each point $x \in M$ we define

$\mathcal{I}_x^\infty = \text{span}\{ df_x | f \text{ is a local first integral, defined about } x \}$. \hspace{1cm} (2.2)

We shall always assume that $\mathcal{I}^\infty = \cup_{x \in M} \mathcal{I}_x^\infty$ is a constant rank bundle on $M$. It is easy to check that $\mathcal{I}^\infty$ is the (unique) maximal, completely integrable, Pfaffian subsystem of $\mathcal{I}$.

The bundle $\mathcal{I}^\infty$ can be computed algorithmically from the derived flag of $\mathcal{I}$. The derived system $\mathcal{I}' \subset \mathcal{I}$ is defined pointwise by

$\mathcal{I}'_p = \text{span}\{ \theta_p | \theta \in S(\mathcal{I}) \text{ such that } d\theta \equiv 0 \mod \mathcal{I} \}$.

Letting $\mathcal{I}^{(0)} = \mathcal{I}$ and assuming $\mathcal{I}^{(k)}$ is constant rank we define $\mathcal{I}^{(k+1)}$ inductively by $\mathcal{I}^{(k+1)} = (\mathcal{I}^{(k)})'$ for $k = 0, 1, \ldots, N$, where $N$ is the smallest integer where $\mathcal{I}^{(N+1)} = \mathcal{I}^{(N)}$. Then $\mathcal{I}^{(N)}$ is completely integrable and $\mathcal{I}^\infty = \mathcal{I}^{(N)}$. More information about the derived flag of a Pfaffian system can be found in [7].

2.2 Integrable Extensions of Differential Systems

An EDS $\mathcal{E}$ on a manifold $N$ is called an integrable extension (see [9]) of $\mathcal{I}$ on a manifold $M$ if there exists a submersion $p: N \to N$ and a subbundle $\mathcal{J} \subset \Lambda^1(N)$ such that

[i] rank $J = \dim N - \dim M$, \hspace{0.5cm} (ii) $\text{ann}(J) \cap \ker(p_\ast) = 0$ \hspace{0.5cm} and \hspace{0.5cm} (iii) $\mathcal{E} = \langle S(J) \cup p_\ast(\mathcal{I}) \rangle_{\text{alg}}$. \hspace{1cm} (2.3)

Here $\text{ann}(J)$ is the subbundle of vectors in $TM$ which annihilate the 1-forms in $J$. The second condition in (2.3) states that $J$ is transverse to $p$. A subbundle $J$ satisfying the three properties is called an admissible subbundle for the extension $\mathcal{E}$.

In particular $\mathcal{E}$ is an integrable extension of $\mathcal{I}$ if there exists $\dim N - \dim M$ 1-forms $\{ \xi^u \}$ on $N$ which define a local basis for $S(J)$ and satisfy

$d \xi^u \equiv 0 \mod \{ p_\ast(\mathcal{I}), \xi^u \}$ or $d \xi^u \equiv 0 \mod \{ E^1, p_\ast(I^2) \}$. \hspace{1cm} (2.4)
We then have

\[ E = (\xi^u, p^*\mathcal{I})_{\text{alg}}. \] (2.5)

The definition of integral extension allows us to relate the integral manifolds of \( E \) and \( \mathcal{I} \) as follows. Let \( s : P \to N \) be an immersed integral manifold of \( E \) and let \( \tilde{s} = p \circ s : P \to M \). Then condition [ii] of (2.3) implies that \( \tilde{s} \) is an immersion and hence [iii] implies that \( \tilde{s} \) is an integral manifold of \( \mathcal{I} \). Conversely, if \( \tilde{s} : P \to M \) is an integral manifold of \( \mathcal{I} \) and \( Q = p^{-1}(\tilde{s}(P)) \), then \( Q \) is a submanifold of \( N \) and the third condition in (2.3) implies that \( E|_Q \) is a Frobenius system. Consequently integral manifolds of \( \mathcal{I} \) can be lifted locally to integrable manifolds of \( E \) by integrating a system of ordinary differential equations.

Further properties of integrable extensions can be found in Section 2.1 of [5].

### 2.3 Reduction of Differential Systems

Recall from the introduction that if \( p : N \to M \) is a smooth surjective submersion and \( \mathcal{I} \) an EDS on \( N \), then the reduction of \( \mathcal{I} \) with respect to \( p \), denoted by \( \mathcal{I}/p \), is the EDS on \( M \) defined by

\[ \mathcal{I}/p = \{ \theta \in \Omega^*(M) \mid p^*\theta \in \mathcal{I} \}. \] (2.6)

Note that \( \mathcal{I}/p \) is not necessarily constant rank without additional hypotheses. If the fibres of \( p \) are connected, then the reduction \( \mathcal{I}/p \) can be computed using Corollary II.2.3 of [7] which states that \( \theta \in \mathcal{I} \) satisfies \( \theta = p^*\tilde{\theta} \) for some \( \tilde{\theta} \in \mathcal{I}/p \) if and only if,

\[ X \cdot \theta = 0 \quad \text{and} \quad X \cdot d\theta = 0 \quad \text{for all} \quad X \in \ker(p_*) . \] (2.7)

Similarly, if \( \mathcal{I} \subset \Lambda^p(N) \), then the reduction of the bundle \( I \) with respect to \( p \) is

\[ I/p = \{ \theta \in \Lambda^p(M) \mid p^*\theta \in I \}. \] (2.8)

It is easy to check that if \( p : N \to M \) is a submersion and \( E \) on \( N \) is an integrable extension of \( \mathcal{I} \) on \( M \) then \( E/p = \mathcal{I} \).

We now specialize (2.6) to the case of reduction by a Lie group. Let \( G \) be a finite dimensional Lie group acting on \( M \) with left action \( \mu : G \times M \to M \). Define maps \( \mu_x : G \to M \) and \( \mu_g : M \to M \) by

\[ \mu_x(g) = \mu(g, x) = g \cdot x = \mu_g(x) \quad \text{for} \quad x \in M \text{ and } g \in G. \]

For each \( Z_e \in T_eG \), the corresponding infinitesimal generator for the action \( \mu \) is the vector field \( X \) on \( M \) defined pointwise by

\[ X_x = (\mu_x)_*(Z_e) \quad \text{for all} \quad x \in M. \] (2.9)

The set of all infinitesimal generators for the action \( \mu \) is a Lie algebra of vector fields on \( M \) which we denote by \( \Gamma_G \). If \( g \) is the Lie algebra of right invariant vector fields on \( G \) and \( Z \in g \), then the map \( \tilde{\mu} : g \to \Gamma_G \) defined by (2.9) is a Lie algebra homomorphism.
Let $\Gamma_G \subset TM$ be the integrable distribution generated by the point-wise span of $\Gamma_G$. We will assume that all actions are regular in the sense that the orbit space $M/G$ has a smooth manifold structure such that the canonical projection $q_G : M \to M/G$ is a smooth submersion and hence $\Gamma_G = \ker(q_{G*})$. Since each orbit of $G$ is the inverse image of a point in $M/G$, the implicit function theorem implies that the orbits are imbedded submanifolds.

The group $G$ acting on $M$ is a symmetry group of $\mathcal{I}$ if, for each $g \in G$ and $\theta \in \mathcal{I}$, $\mu_g^*(\theta) \in \mathcal{I}$. Under these circumstances we define the symmetry reduction of $\mathcal{I}$ by $G$ to be

$$\mathcal{I}/G = \mathcal{I}/q_G = \{ \bar{\theta} \in \Omega^*(M/G) \mid q_{G*}^*(\bar{\theta}) \in \mathcal{I} \}. \quad (2.10)$$

In other words $\mathcal{I}/G$ is the reduction given by equation (2.6) with respect to the submersion $q_G : M \to M/G$. However, by utilizing the $G$-invariance of $\mathcal{I}$, the computation of a local basis of sections for $\mathcal{I}/G$ is now algebraic [1]. This fact is not necessarily true for the reduction in (2.6) for generic $p$. In analogy with (2.7), a form $\theta \in \Omega^p(M)$ satisfies $\theta = q_{G*}^*(\bar{\theta})$ for some $\bar{\theta} \in \Omega^p(M/G)$ if and only if $\theta$ is $G$-basic, that is, $G$ semi-basic and $G$ invariant so that

$$X \cdot \theta = 0 \quad \text{and} \quad \mu_g^*(\theta) = \theta \quad \text{for all} \quad X \in \Gamma_G \quad \text{and} \quad g \in G. \quad (2.11)$$

Likewise, if $A \subset \Lambda^p(M)$ is a $G$-invariant subbundle, then

$$A/G = A/q_G = \{ \bar{\theta} \in \Lambda^p(M/G) \mid q_{G*}^*(\bar{\theta}) \in A \}. \quad (2.12)$$

As a cautionary remark, we observe that if $\mathcal{I}$ is a $G$-invariant Pfaffian differential system with $I = \langle S(I) \rangle_{\text{diff}}$, then it is generally not true that $\mathcal{I}/G$ is the Pfaffian system for $I/G$, that is

$$\mathcal{I}/G \neq \langle S(I/G) \rangle_{\text{diff}}. \quad (2.13)$$

However, it is true that $S(I/G)_{\text{diff}} \subset \mathcal{I}/G$. See [1] for examples.

A symmetry group $G$ of an EDS $\mathcal{I}$ is said to be transverse to $\mathcal{I}$ if

$$\text{ann}(I^1) \cap \Gamma_G = 0. \quad (2.13)$$

This transversality condition holds for all the examples and application we consider. A fundamental consequence of transversality is the following (see Theorem 2.1 in [5]).

**Theorem 2.1.** Let $G$ be a symmetry group of an EDS $\mathcal{I}$ on a manifold $M$. Assume $G$ acts regularly on $M$ and transversely to $\mathcal{I}$. Then $\mathcal{I}$ is an integrable extension of the reduced differential system $\mathcal{I}/G$.

It is useful to summarize a few relevant facts from [1] about symmetry reduction and the role of transverse actions for differential systems generated by 1-forms and 2-forms.

If $G$ acts transversely then locally there exists a $G$-invariant open set $U$ and a set of generators

$$\mathcal{I}|_U = \langle \theta^i, \theta_{G,ab}^a, \tau_{G,ab}^a \rangle_{\text{alg}}, \quad (2.14)$$
where \( \theta_i, \theta_{G, sb}, \tau_{\alpha, G, sb} \in \Omega^1(U), \tau_{G, sb}^0 \in \Omega^2(U) \) and \( \theta_{G, sb}, \tau_{G, sb}^0 \) are \( G \)-semi-basic. Let \( \sigma : U/G \to U \) be a cross-section, then
\[
\left( \mathcal{I}/G \right)|_{U/G} = \langle \sigma^* \theta_{G, sb}^0, \sigma^* \tau_{G, sb}^0 \rangle_{\text{alg}}.
\] (2.15)

From equations (2.14) and (2.15) we deduce that a new set of generators for \( \mathcal{I} \) are given by
\[
\mathcal{I}|_U = \langle \theta_i, \theta_{G}, \tau_{\alpha, G} \rangle_{\text{alg}}
\] (2.16)

where \( \theta_G = q_G^* \sigma^* \theta_{G, sb}^0 \) and \( \tau_{G} = q_G^* \sigma^* \tau_{G, sb}^0 \) are \( G \)-basic.

**Theorem 2.2.** If \( \mathcal{I} \) is a Pfaffian system and \( G \) acts transversely to the derived system \( \mathcal{I}' \) then \( \mathcal{I}/G \) is a constant rank Pfaffian system (see [1]).

**Remark 2.3.** We will work locally in our examples starting with a Lie algebra \( \Gamma_G \) of infinitesimal symmetries of \( \mathcal{I} \). A Lie algebra \( \Gamma_G \) of vector fields is regular on \( M \) if the canonical projection \( q_{\Gamma} : M \to M/\Gamma \) to the leaf space \( M/\Gamma \) is a smooth submersion. This regularity condition will hold at least locally in all our examples. We shall write \( \mathcal{I}/\Gamma_G \) in place of \( \mathcal{I}/q_{\Gamma_G} \) for the reduction of \( \mathcal{I} \).

In the computations leading to local generators of \( \mathcal{I}/G \) as in equations (2.16), the \( G \)-basic forms are replaced by \( \Gamma_G \)-basic forms
\[
X \to \theta = 0 \quad \text{and} \quad \mathcal{L}_X \theta = 0 \quad \text{for all} \ X \in \Gamma_G,
\] (2.17)

where \( \mathcal{L}_X \) is the Lie derivative along \( X \).

**Remark 2.4.** The orbit projection maps \( p_i : M/\Gamma_H \to M/\Gamma_{G_i} \) in diagram (1.4) can be written in local coordinates by expressing the independent \( \Gamma_{G_i} \) invariants on \( M \) in terms of the independent \( \Gamma_H \) invariants on \( M \). See Section 3 where this is demonstrated.

### 2.4 Darboux Integrable Differential Systems

In this section we review the definition of Darboux integrable differential systems and we state a result from [6] which simplifies the verification of this definition. We begin with the definition of a decomposable differential system and the notion of a Darboux pair as given in [6].

**Definition 2.5.** An exterior differential system \( \mathcal{I} \) on \( M \) is **decomposable of type** \([p, \rho]\), where \( p, \rho \geq 2 \), if about each point \( x \in M \) there is a local coframe
\[
\hat{\theta}^1, \ldots, \hat{\theta}^r, \hat{\sigma}^1, \ldots, \hat{\sigma}^p, \check{\sigma}^1, \ldots, \check{\sigma}^\rho,
\] (2.18)

such that \( \mathcal{I} \) is algebraically generated by 1-forms and 2-forms
\[
\mathcal{I} = \langle \hat{\theta}^1, \ldots, \hat{\theta}^r, \hat{\Omega}^1, \ldots, \hat{\Omega}^s, \check{\Omega}^1, \ldots, \check{\Omega}^\tau \rangle_{\text{alg}},
\] (2.19)

where \( s, \tau \geq 1, \hat{\Omega}^s \in \Omega^2(\check{\sigma}^1, \ldots, \check{\sigma}^\rho) \), and \( \check{\Omega}^\tau \in \Omega^2(\check{\sigma}^1, \ldots, \check{\sigma}^\rho) \).
Equation (2.19) implies that the 1-forms $\tilde{\theta}^e$ satisfy structure equations of the form

$$d\tilde{\theta}^e \equiv A_{ab}^e \tilde{\sigma}^a \wedge \tilde{\sigma}^b + B_{\alpha\beta}^e \tilde{\sigma}^\alpha \wedge \tilde{\sigma}^\beta \mod \{ \tilde{\theta}^e \}$$ (2.20)

where

$$\text{span}\{ A_{ab}^e \tilde{\sigma}^a \wedge \tilde{\sigma}^b \} \subset \text{span}\{ \tilde{\Omega}^a \} \quad \text{and} \quad \text{span}\{ B_{\alpha\beta}^e \tilde{\sigma}^\alpha \wedge \tilde{\sigma}^\beta \} \subset \text{span}\{ \tilde{\Omega}^a \}. \quad (2.21)$$

In the special case that $\mathcal{I}$ is a Pfaffian system, these inclusions then become equalities.

An important example of a decomposable system is given by a class $r$ hyperbolic differential system, as defined in [10]. These systems are decomposable differential system of type $[2, 2]$ on an $r + 4$ manifold. In Section 4 our focus is on hyperbolic Monge-Ampère systems which are a special class of $r = 1$ hyperbolic systems.

**Definition 2.6.** Let $\mathcal{I}$ be a decomposable differential system. The bundles $\hat{V}, \check{V} \subset T^*M$ defined by

$$\hat{V} = \text{span}\{ \hat{\theta}^e, \hat{\sigma}^a \} \quad \text{and} \quad \check{V} = \text{span}\{ \check{\theta}^e, \check{\sigma}^\alpha \}$$ (2.22)

are called the associated singular Pfaffian systems with respect to the decomposition (2.19). The differential systems generated by

$$\hat{V} = \langle \hat{\theta}^e, \hat{\sigma}^a, \hat{\Omega}^a \rangle_{\text{diff}} \quad \text{and} \quad \check{V} = \langle \check{\theta}^e, \check{\sigma}^\alpha, \check{\Omega}^a \rangle_{\text{diff}}$$ (2.23)

are called the associated singular differential systems for $\mathcal{I}$.

Note that

$$I^1 = \hat{V} \cap \check{V} \quad \text{and} \quad T^*M = \hat{V} + \check{V}. \quad (2.24)$$

For further information on the relationship between a decomposable differential system and its singular systems see Theorem 2.6 of [8].

The characteristic systems defined in [10] for a class $r$ hyperbolic differential system coincide with the singular Pfaffian systems (2.22). In this special case, the decomposition in Definition 2.5 is unique up to an interchange in the singular Pfaffian systems. However for a general decomposable system the decomposition in Definition 2.5 may not be unique.

Two decomposable differential systems $\mathcal{E}$ and $\mathcal{I}$, defined on manifolds $N$ and $M$, and with singular Pfaffian systems $\{ \hat{Z}, \check{Z} \}$ and $\{ \hat{V}, \check{V} \}$ are defined to be equivalent if there is a diffeomorphism $\phi : N \to M$ such that $\phi^*(\mathcal{I}) = \mathcal{E}$, $\phi^*(\hat{V}) = \hat{Z}$ and $\phi^*(\check{V}) = \check{Z}$. Under these circumstances, it follows that $\phi^*(\hat{V}) = \hat{Z}$ and $\phi^*(\check{V}) = \check{Z}$.

The definition of a Darboux integrable differential system is given in terms of its singular Pfaffian systems.

**Definition 2.7.** A pair of Pfaffian systems $\hat{V}$ and $\check{V}$ on a manifold $M$ define a Darboux pair if

$$\hat{V} + \check{V}^\infty = T^*M \quad \text{and} \quad \check{V} + \hat{V}^\infty = T^*M, \quad \text{and} \quad \check{V}^\infty = T^*M, \quad \text{and}$$ (2.25)
A decomposable differential system \( \mathcal{I} \) is \textbf{Darboux integrable} if its singular Pfaffian systems (2.22) define a Darboux pair.

The test for Darboux integrability can be simplified if \( \mathcal{I} \) admits no first integrals.

**Theorem 2.8.** Let \( \mathcal{I} \) be decomposable differential system with singular Pfaffian systems \( \Hat{V} \) and \( \tilde{V} \) and suppose that \( (I^1)^\infty = (\Hat{V} \cap \tilde{V})^\infty = 0 \). If \( \Hat{V} \) and \( \tilde{V} \) satisfy conditions [i] in the definition of a Darboux pair, then condition [ii] is automatically satisfied and \( \mathcal{I} \) is Darboux integrable.

The proof of this theorem is given in Appendix 3 of [5].

**Definition 2.9.** The Darboux invariants for the Darboux pair \( \{\Hat{V}, \tilde{V}\} \) or for a Darboux integrable differential system \( \mathcal{I} \) are the first integrals for \( \Hat{V} \) or \( \tilde{V} \), that is, \( C^\infty(M) \) functions \( f \) such that \( df \in S(\Hat{V}) \) or \( df \in S(\tilde{V}) \).

The number of (functionally) independent Darboux invariants is therefore given by the sum of the ranks of the completely integrable systems \( \Hat{V}^\infty \) and \( \tilde{V}^\infty \).

Theorem 6.1 in [5] shows how Darboux integrable systems can be constructed using the group reduction of pairs of differential systems generate by 1-forms and 2-forms. It is a remarkable fact, established in [6], that the converse is true locally, that is, every Darboux integrable hyperbolic system can be realized locally as a non-trivial quotient of a pair of exterior differential systems with a common symmetry group. This theorem will be the starting point for the examples and its precise formulation is as follows.

**Theorem 2.10.** Let \( \mathcal{I} \) be a Darboux integrable system on a manifold \( M \) and let \( \Hat{V} \) and \( \tilde{V} \) be the singular systems as in (2.22). Fix a point \( x_0 \) in \( M \) and let

[a] \( M_1 \) and \( M_2 \) be the maximal integral manifolds of \( \Hat{V}^\infty \) and \( \tilde{V}^\infty \) through \( x_0 \), and

[b] \( K_1 \) and \( K_2 \) be the restrictions of \( \Hat{V} \) to \( K_1 \) and \( \tilde{V} \) to \( K_2 \) respectively.

Then there are open sets \( U \subset M \), \( U_1 \subset M_1 \), \( U_2 \subset M_2 \), each containing \( x_0 \), and a Lie algebra of vector fields \( \Gamma_G \) acting freely on \( U_1 \) and \( U_2 \) and which are transverse symmetries of \( K_1 \) and \( K_2 \) so that the following is satisfied,

[i] \( U = (U_1 \times U_2)/\Gamma_{G_{\text{diag}}} \) and \( \mathcal{I}|_U = ((K_1 + K_2)_{U_1 \times U_2})/\Gamma_{G_{\text{diag}}} \),

[ii] \( \Hat{V} = (K_1 + T^*M_2)/\Gamma_{G_{\text{diag}}} \) and \( \tilde{V} = (T^*M_1 + K_2)/\Gamma_{G_{\text{diag}}} \).

[iii] Let \( p_a : U \rightarrow U_a \) be the projections, then the Darboux invariants are computed by

\[
\Hat{V}^\infty = p_2^*(T^*(U_2/\Gamma_G)), \quad \text{and} \quad \tilde{V}^\infty = p_1^*(T^*(U_1/\Gamma_G)).
\] (2.27)

The Lie algebra of vector fields \( \Gamma_G \) in Theorem 2.10 is called the \textbf{Vessiot algebra} of the Darboux integrable system \( \mathcal{I} \) and denoted by \( \text{vess}(\mathcal{I}) \). We refer to part [i] in Theorem 2.10 as \textbf{the (local)}
canonical quotient representation for a Darboux integrable system \( I \). The vector-fields \( \Gamma_G \) are the infinitesimal generators of a Lie group \( G \) acting on \( M \) which we call the Vessiot group of the Darboux integrable system \( I \). Part [iii] in Theorem 2.10 will be used to compute the Darboux invariants in Example 3.3.

Remark 2.11. It is a non-trivial but algorithmic process to find the Lie algebra of vector-fields \( \Gamma_G \). A useful relationship between the number of independent Darboux invariants of a Darboux integrable system \( I \) and the dimension of \( \text{vess} I \) is

\[
\dim \text{vess}(I) = \dim \Gamma_G = \dim M - \text{rank}(\hat{V}^\infty) - \text{rank}(\tilde{V}^\infty). \tag{2.28}
\]

2.5 Integrable Extensions of Darboux Integrable Systems

The following theorem relates the singular systems and the Darboux integrability of a decomposable differential system \( I \) and an integrable extension \( E \).

**Theorem 2.12.** Let \( p : (E, N) \to (I, M) \) be an integrable extension with \( J \) an admissible subbundle of \( T^*N \) for \( (E, I) \).

[i] If \( I \) is decomposable of type \([p, \rho]\) with singular Pfaffian systems \( \hat{V} \) and \( \tilde{V} \), then \( E \) is decomposable of type \([p, \rho]\) with singular Pfaffian systems

\[
\hat{Z} = J \oplus p^*(\hat{V}) \quad \text{and} \quad \tilde{Z} = J \oplus p^*(\tilde{V}). \tag{2.29}
\]

[ii] If \( I \) is Darboux integrable and \((E^1)^\infty = 0\), then \( E \) is Darboux integrable.

The relations in (2.29) imply that the number of Darboux invariants for the singular Pfaffian systems \( \hat{Z} \) and \( \tilde{Z} \) for the extension \( E \) are bounded by

\[
\text{rank}(p^*(\hat{V}^\infty)) \leq \text{rank}(\hat{Z}^\infty) \leq \text{rank}(p^*(\hat{V}^\infty)) + \text{rank}(J) \quad \text{and} \quad \text{rank}(p^*(\tilde{V}^\infty)) \leq \text{rank}(\tilde{Z}^\infty) \leq \text{rank}(p^*(\tilde{V}^\infty)) + \text{rank}(J).
\]

The case where the number of Darboux invariants for \( E \) exceeds the number of Darboux invariants for \( I \) by the maximal number possible, that is, the case where

\[
\text{rank}(\hat{Z}^\infty) = \text{rank}(\hat{V}^\infty) + \text{rank}(J) \quad \text{and} \quad \text{rank}(\tilde{Z}^\infty) = \text{rank}(\tilde{V}^\infty) + \text{rank}(J) \tag{2.30}
\]

will be implied by part [ii] in the following definition.

**Definition 2.13.** Let \( p : (E, N) \to (I, M) \) be an integrable extension of decomposable systems \( E \) and \( I \) with singular Pfaffian systems \( \{\hat{Z}, \tilde{Z}\} \) and \( \{\hat{V}, \tilde{V}\} \) respectively.

\[\text{If } (E^1)^\infty \neq 0, \text{ then one can always let } P \text{ be an integral manifold } N' \text{ of } (E^1)^\infty = 0, \text{ let } F = E_N \text{ and replace } (E, N) \text{ in the statement of the theorem by } (F, P), \text{ where now } F^\infty = 0\]
The extension $(E, I)$ is said to be compatible with respect to the singular Pfaffian systems $\{\mathring{Z}, \mathring{\bar{Z}}\}$ and $\{\mathring{V}, \mathring{\bar{V}}\}$ if
\[
p^*(\mathring{V}) \subset \mathring{Z} \quad \text{and} \quad p^*(\mathring{\bar{V}}) \subset \mathring{\bar{Z}}.
\] (2.31)

The integrable extension $(E, I)$ is called maximally compatible with respect to the singular Pfaffian systems $\{\mathring{Z}, \mathring{\bar{Z}}\}$ and $\{\mathring{V}, \mathring{\bar{V}}\}$ if there exists an admissible subbundle $\mathring{J} \subset \mathring{Z} \cap \mathring{\bar{Z}}$ for $(E, I)$ such that
\[
\mathring{Z} = \mathring{J} \oplus p^*(\mathring{\bar{V}}) \quad \text{and} \quad \mathring{\bar{Z}}^\infty = \mathring{J} \oplus p^*(\mathring{\bar{V}}^\infty),
\] (2.32)
and, similarly, an admissible subbundle $\mathring{J} \subset \mathring{Z} \cap \mathring{\bar{Z}}$ for $(E, I)$ such that
\[
\mathring{Z} = \mathring{J} \oplus p^*(\mathring{\bar{V}}) \quad \text{and} \quad \mathring{\bar{Z}}^\infty = \mathring{J} \oplus p^*(\mathring{\bar{V}}^\infty).
\] (2.33)

The following theorem relates equation (2.30) and part [ii] of Definition 2.13 and improves Theorem 5.3 in [5] when the additional hypothesis of $I$ being Darboux integrable is made.

**Theorem 2.14.** Let $p : (E, N) \to (I, M)$ be an integrable extension of the Darboux integrable system $I$ which is compatible with respect to the singular Pfaffian systems $\{\mathring{Z}, \mathring{\bar{Z}}\}$ and $\{\mathring{V}, \mathring{\bar{V}}\}$. If condition (2.30) holds then the extension is maximally compatible.

**Proof.** On account of compatibility we have
\[
J + p^*(\mathring{\bar{V}}^\infty) + p(\mathring{\bar{V}}) \subset \mathring{\bar{Z}}^\infty + \mathring{\bar{Z}},
\] (2.34)
while Darboux integrability of $I$ implies
\[
\dim M = \text{rank}(\mathring{\bar{V}}^\infty + \mathring{\bar{V}}) = \text{rank}(\mathring{\bar{V}}^\infty) + \text{rank}(\mathring{\bar{V}}) - \text{rank}(\mathring{\bar{V}}^\infty \cap \mathring{\bar{V}}).
\] (2.35)
Therefore equation (2.34) gives
\[
\text{rank}(\mathring{\bar{Z}}^\infty + \mathring{\bar{Z}}) = \text{rank}(J) + \dim M = \dim N.
\] (2.36)

Now using (2.30) in equation (2.36) we find
\[
\dim N = \text{rank}(\mathring{\bar{Z}}^\infty + \mathring{\bar{Z}}) = \text{rank}(\mathring{\bar{Z}}^\infty) + \text{rank}(\mathring{\bar{Z}}) - \text{rank}(\mathring{\bar{Z}}^\infty \cap \mathring{\bar{Z}}) = \text{rank}(\mathring{\bar{V}}^\infty) + \text{rank}(J) + \text{rank}(\mathring{\bar{V}}) + \text{rank}(J) - \text{rank}(\mathring{\bar{Z}}^\infty \cap \mathring{\bar{Z}}).
\] (2.37)
Combining equation (2.37) with (2.35) we have
\[
\dim N = 2 \text{rank}(J) + \dim M + \text{rank}(\mathring{\bar{V}}^\infty \cap \mathring{\bar{V}}) - \text{rank}(\mathring{\bar{Z}}^\infty \cap \mathring{\bar{Z}}),
\]
and so
\[
\text{rank}(\mathring{\bar{Z}}^\infty \cap \mathring{\bar{Z}}) = \text{rank}(\mathring{\bar{V}}^\infty \cap \mathring{\bar{V}}) + \text{rank}(J).
\]
Therefore there exists $\mathring{J} \subset \mathring{Z} \cap \mathring{\bar{Z}}$ so that
\[
\mathring{Z}^\infty \cap \mathring{\bar{Z}} = \mathring{J} \oplus p^*(\mathring{\bar{V}}^\infty \cap \mathring{\bar{V}}).
\]
where \( \hat{J} \) is an admissible subbundle and (2.32) holds. A similar argument shows the existence of a subbundle \( \tilde{J} \) so that
\[
\tilde{\mathcal{Z}}^\infty \cap \tilde{\mathcal{Z}} = \tilde{J} \oplus \mathbf{p}^*(\tilde{V}^\infty \cap \tilde{V}),
\]
and (2.33) holds.

### 3 Examples

In Examples 3.1 and 3.2, we show how our approach effortlessly gives the Bäcklund transformations between various Darboux integrable \( f \)-Gordon equations constructed in [11] and [16]. Our constructions hold for \( C^\infty \) differential systems while the methods in [11] are based on the Cartan-Kähler theorem and hold only for \( C^\omega \) EDS.

Examples 3.4 and 3.5 show that our method also gives Bäcklund transformations for non-Monge-Ampère equations and for PDE which are Darboux integrable at higher jet order. Bäcklund transformations are constructed for the \( A_2 \) Toda lattice systems in Example 3.7. A Bäcklund transformation for an over-determined system of PDE is given in Example 3.8.

**Example 3.1.** In this first example we spell out the details of how Theorem A leads to the group theoretical construction of a well known Bäcklund transformation. We start by taking for \( \mathcal{I} \) the standard contact system \( K_1 + K_2 \) on \( M = J^2(R, R) \times J^2(R, R) \). In coordinates \((x, v, v_x, v_{xx}, y, w, w_y, w_{yy})\) on \( M \) we have
\[
K_1 = \langle \theta_v = dv - v_x dx, \theta_{v_x} = dv_x - v_{xx} dx \rangle_{\text{diff}} \quad \text{and} \quad K_2 = \langle \theta_w = dw - w_y dy, \theta_{w_y} = dw_y - w_{yy} dy \rangle_{\text{diff}}.
\]
(3.1)
(3.2)

The total derivative vector fields defined by \( \mathcal{I} \) are
\[
D_x = \partial_x + v_x \partial_v + v_{xx} \partial_{v_x} \quad \text{and} \quad D_y = \partial_y + w_y \partial_w + w_{yy} \partial_{w_y}.
\]
(3.3)

We consider the infinitesimal group action \( \Gamma_H \) on \( M \) defined
\[
\Gamma_H = \text{span}\{ X_1 = \partial_v - \partial_w, \ X_2 = v \partial_v + v_x \partial_{v_x} + v_{xx} \partial_{v_{xx}} + w \partial_w + w_y \partial_{w_y} + w_{yy} \partial_{w_{yy}} \}.
\]
(3.4)

The vector-fields in \( \Gamma_H \) are symmetries of \( \mathcal{I} \) and the quotient system \( \mathcal{B} = \mathcal{I}/\Gamma_H \) will be a rank 2 Pfaffian system (see Theorem 2.2) on a 6-dimensional manifold \( N = M/\Gamma_H \).

The lowest order invariants of \( \Gamma_H \) acting on \( M \) are \((x, y, V, W)\) where
\[
V = \log \frac{v_x}{v + w} \quad \text{and} \quad W = \log \frac{w_y}{v + w}.
\]
(3.5)

The total vector-fields \( D_x \) and \( D_y \) in equation (3.3) commute with the vector-fields in \( \Gamma_H \) and therefore \( D_x V \) and \( D_y W \) are also \( \Gamma_H \) invariants. The projection map \( q_H : M \to M/\Gamma_H \) can then
be written in coordinates \((x, y, V, V_x, W, W_y)\) on \(M/\Gamma_H\) as

\[
\mathbf{q}_H = [x = x, y = y, V = \log \frac{v_x}{v + w}, W = \log \frac{w_y}{v + w}, V_x = D_x V = \frac{v_{xx}}{v_x} - \frac{v_x}{v + w}, W_y = D_y W = \frac{w_{yy}}{w_y} - \frac{w_y}{v + w}].
\]

(3.6)

Here we take the open set where \(v + w > 0, v_x > 0\) and \(w_y > 0\) for the domain of \(\mathbf{q}_H\).

Note that if we compute \(D_y V\) and \(D_x W\) using (3.5), then we find the following syzygies on the differential invariants,

\[
D_y V = -\frac{w_y}{v + w} = -e^W, \quad D_x W = -\frac{v_x}{v + w} = -e^V.
\]

(3.7)

Equations (3.7) are the quotient differential equations representing the Bäcklund transformation \(B\).

We now show how equations (3.7) arise by computing the quotient differential system \(I/\Gamma_H\).

This Pfaffian system is determined by computing the \(\Gamma_H\) semi-basic one-forms in \(I\). A simple linear algebra computation gives,

\[
I^1_{H,\text{sb}} = \text{span}\{ \theta^1_{H,\text{sb}} = \frac{1}{v_x} \theta_{v_x} - \frac{1}{w_y} \theta_{w_y}, \theta^2_{H,\text{sb}} = \frac{1}{v_x} \theta_{v_x} + \frac{1}{w_y} \theta_{w_y} - \frac{2}{v + w} (\theta_v + \theta_w) \}.
\]

(3.8)

Pulling back the one-forms in (3.8) with the cross-section

\[
\sigma(x, y, V, W, V_x, W_y) = (x = x, y = y, v = 1, v_x = e^V, v_{xx} = e^V V_x + e^{2V}, w = 0, w_y = e^W, w_{yy} = e^W W_y + e^{2W})
\]

we obtain the rank 2 quotient Pfaffian system \(B\),

\[
B = \text{span}\{ \beta^1 = \frac{1}{2} (\sigma^* \theta^1_{H,\text{sb}} + \sigma^* \theta^2_{H,\text{sb}}) = dV - V_x dx + e^W dy, \quad \beta^2 = \frac{1}{2} (\sigma^* \theta^1_{H,\text{sb}} - \sigma^* \theta^2_{H,\text{sb}}) = dW + e^V dx - W_y dy \}.
\]

(3.9)

The associated PDE system for (3.9) is easily seen to be

\[
V_y = -e^W, \quad W_x = -e^V
\]

which agrees with (3.7).

To construct a Bäcklund transformations Theorem A we first add to the infinitesimal group action \(\Gamma_H\) the vector field

\[
Z_1 = \partial_v + \partial_w.
\]

(3.10)

The action \(\Gamma_{G_1} = \text{span}\{X_1, X_2, Z_1\} = \text{span}\{\partial_v, \partial_w, X_2\}\) is not a (local) diagonal action on \(J^2(\mathbb{R}, \mathbb{R}) \times J^2(\mathbb{R}, \mathbb{R})\). Secondly, we add to the infinitesimal group action \(\Gamma_H\) the vector field

\[
Z_2 = v^2 \partial_v + D_x (v^2) \partial_{v_x} + D_x^2 (v^2) \partial_{v_{xx}} - w_2 \partial_w - D_y (w^2) \partial_{w_y} + D_y^2 (w^2) \partial_{w_{yy}}.
\]

(3.11)
The action $\Gamma_G = \{X_1, X_2, Z_2\}$ is the (local) diagonal action on $J^2(\mathbb{R}, \mathbb{R}) \times J^2(\mathbb{R}, \mathbb{R})$ of the standard (local) $\mathfrak{s}(2, \mathbb{R})$ action on each of the dependent variables.

Since $\Gamma_H \subset \Gamma_G$, we can use the $\Gamma_H$ invariants in equation (3.3) to compute the $\Gamma_G$ invariants. Since $Z_1(V) = (v + w)^{-2}$ and $Z_1(W) = (v + w)^{-2}$ we find that

$$u_1 = V - W = \log \frac{v_x}{v + w} - \log \frac{w_y}{v + w} = \log v_x - \log w_y$$

is $Z_1$ invariant and hence $\Gamma_G$ invariant. Again $D_x$ and $D_y$ are invariant with respect to every vector-field in $\Gamma_G$, and we have the other $\Gamma_G$ invariants $u_{1x} = D_x u_1$ and $u_{1y} = D_y u_1$ giving the quotient map

$$q_{\Gamma_G} = [x = x, y = y, u_1 = \log v_x - \log w_y, u_{1x} = D_x (\log v_x - \log w_y) = \frac{v_{xx}}{v_x}, u_{1y} = D_y (\log v_x - \log w_y) = -\frac{w_{yy}}{w_y}]$$

where $(x, y, u_1, u_{1x}, u_{1y})$ are the coordinates on $M/\Gamma_G$. In this case we have the syzygy

$$u_{1xy} = D_x D_y (u_1) = D_x D_y (\log v_x - \log w_y) = 0,$$

and so the quotient is the wave equation.

To complete our description of the left hand side of diagram (1.3) we need to calculate $p_1$ and $\mathcal{I}/\Gamma_G$. Using Remark 2.4 and equation (3.6) we have, for example,

$$u_{1x} = \frac{v_{xx}}{v_x} = V_x - e^V$$

so that the projection map $p_1 : M/\Gamma_H \to M/\Gamma_G$ can be written in coordinates as

$$p_1 = [x = x, y = y, u_1 = \log v_x - \log w_y = V - W, u_{1x} = \frac{v_{xx}}{v_x} = V_x - e^V, u_{1y} = -\frac{w_{yy}}{w_y} = -W_y + e^W].$$

In order to compute the reduced system $\mathcal{I}/\Gamma_G$, we need the $\Gamma_G$ semi-basic 1-forms and 2-forms in $\mathcal{I}$. We find the generators in the form of equation (2.14) to be

$$\mathcal{I} = \left\langle \theta_v, \theta_w, \frac{1}{v_x} \theta_{v_x} + \frac{1}{w_y} \theta_{w_y}, \theta_{sb} = \frac{1}{v_x} \theta_{v_x} - \frac{1}{w_y} \theta_{w_y}, \tau_{sb}^1 = \frac{1}{v_x} d \theta_{v_x} - \frac{v_{xx}}{v_x} \theta_v, \tau_{sb}^2 = \frac{1}{w_y} d \theta_{w_y} - \frac{w_{yy}}{w_y} \theta_w \right\rangle_{alg}.$$

The coordinate representations for the semi-basic forms in (3.15) are

$$\theta_{sb} = \frac{1}{v_x} \theta_{v_x} - \frac{1}{w_y} \theta_{w_y} = \frac{1}{v_x} d v_x - \frac{1}{w_y} d w_y - \frac{v_{xx}}{v_x} d x + \frac{w_{yy}}{w_y} d y,$$

$$\tau_{sb}^1 = \frac{1}{v_x} d \theta_{v_x} - \frac{v_{xx}}{v_x} \theta_v \wedge d x = -\frac{1}{v_x} d v_{xx} + \frac{v_{xx}}{v_x} d x \wedge d x,$$

$$\tau_{sb}^2 = \frac{1}{w_y} d \theta_{w_y} - \frac{w_{yy}}{w_y} \theta_w \wedge d y = -\frac{1}{w_y} d w_{yy} + \frac{w_{yy}}{w_y} d y \wedge d y.$$
The quotient is then easily computed using the invariants in equation \((3.13)\) in equation \((3.16)\), and we find

\[\theta_{sb} = q^*_G (du_1 - u_{1x} dx - u_{1y} dy), \quad \tau_{sb} = q^*_G (-du_1x) \wedge dx, \quad \tau^2_{sb} = q^*_G (du_{1y}) \wedge dy, \quad (3.17)\]

so that

\[\mathcal{I}/\Gamma_{G_1} = \langle du_1 - u_{1x} dx - u_{1y} dy, \ du_{1x} \wedge dx, \ -du_{1y} \wedge dy \rangle_{\text{alg}}. \quad (3.18)\]

This is the Monge-Ampère representation of the wave equation on the 5-dimensional manifold \(M/\Gamma_{G_1}\).

Finally it is now a simple matter to check directly that \(B\) is an integrable extension of \(\mathcal{I}_1 = \mathcal{I}/\Gamma_{G_1}\). Indeed using equations \((3.18)\) and \((3.14)\) we have

\[P^*_1 (du_1 - u_{1x} dx - u_{1y} dy) = dV - V_x dx - dW - W_y dy = \beta_1 - \beta_2,\]

\[P^*_1 (du_{1x} \wedge dx) = d(V_x + e^V) \wedge dx = -d\beta_1 + e^V \beta_1 \wedge dx + e^W \beta_2 \wedge dy,\]

\[P^*_1 (-du_{1y} \wedge dy) = d(W_y + e^W) \wedge dy = -d\beta_2 + e^W \beta_2 \wedge dy + e^V \beta_1 \wedge dx. \quad (3.19)\]

Therefore from these equations and the fact that \(B\) is a Pfaffian system we have \(B = \langle \beta_2, \ P^*\mathcal{I}_1 \rangle_{\text{alg}},\) as required.

We now compute the reduction of \(\mathcal{I}\) and \(M\) by the action of \(\Gamma_{G_2} = \text{span}\{X_1, X_2, Z_2\}\). The lowest order invariant of \(\Gamma_{G_2}\) on \(M\) can be computed using the invariants for \(\Gamma_H\) in equation \((3.5)\) by requiring them to be invariant under \(Z_2\). We find

\[u_2 = \log \frac{2v_x w_y}{(v + w)^2} \quad (3.20)\]

with further \(\Gamma_{G_2}\) invariants given by \(D_x\) and \(D_y\) total differentiation of \(u_2\). Hence the quotient map \(q_{G_2}\) is given by

\[q_{G_2} = [x = x, y = y, u_2 = \log \frac{2v_x w_y}{(v + w)^2}],\]

\[u_{2x} = D_x(u_2) = \frac{v_x}{v} - \frac{2v_x}{v + w}, \quad u_{2y} = D_y(u_2) = \frac{w_y}{w} - \frac{2w_y}{w + w}], \quad (3.21)\]

where \((x, y, u_2, u_{2x}, u_{2y})\) are coordinates on \(M\Gamma_{G_2}\). The syzygy is then computed to be the Liouville equation

\[D_x D_y u_2 = \frac{2v_x w_y}{(v + w)^2} = e^{u_2}.\]

The map \(p_2 : M/\Gamma_H \to M/\Gamma_{G_2}\) is given in coordinates by using equations \((3.21)\), \((3.6)\) and Remark \(2.21\) by

\[p_2 = [x = x, y = y, u_2 = \log \frac{2v_x w_y}{(v + w)^2} = V + W + \log 2,\]

\[u_{2x} = D_x (\log \frac{2v_x w_y}{(v + w)^2}) = V_x - e^V, \quad u_{2y} = D_y (\log \frac{2v_x w_y}{(v + w)^2}) = W_y - e^W]. \quad (3.22)\]
In order to compute the reduced system $\mathcal{I}_2 = \mathcal{I}/\Gamma_{G_2}$ we need the $\Gamma_{G_2}$ semi-basic 1-forms and 2-forms in $\mathcal{I}$. We find generators for $\mathcal{I}$ in the form of equation (2.14) to be

$$\mathcal{I} = \left\{ \theta_v, \theta_w, \theta_{v_x}, \theta_{w_x}, \theta_{sb} = \frac{1}{v_x} \theta_{v_x} + \frac{1}{w_y} \theta_{w_y} + \frac{2}{v+w} (\theta_v + \theta_w) \right\},$$

$$\tau_{sb}^1 = \left( dv_{xx} + \frac{v_x}{w_y(v+w)} \theta_{w_x} - \left( \frac{v_{xx}}{v_x} + \frac{v_x}{v+w} \right) \theta_{v_x} \right) \wedge dx,$$

$$\tau_{sb}^2 = \left( dw_{yy} + \frac{w_y}{v_x(v+w)} \theta_{v_x} - \left( \frac{w_{yy}}{w_y} + \frac{w_y}{v+w} \right) \theta_{w_x} \right) \wedge dy, \tag{3.23}$$

Using the quotient map in (3.21), it is easy to verify equation (2.6) where the quotient system is

$$\mathcal{I}_2 = \mathcal{I}/\Gamma_{G_2} = \langle du_v - u_{xx}dx - u_{yy}dy, (du_{2x} - e^{u_2}dy) \wedge dx, (du_{2y} - e^{u_2}dx) \wedge dy \rangle_{\text{alg}}. \tag{3.24}$$

In fact equation (3.21) we have

$$q_{G_2}^* (du_v - u_{xx}dx - u_{yy}dy) = \theta_{sb},$$

$$q_{G_2}^* ((du_{2x} - e^{u_2}dy) \wedge dx) = \frac{1}{v_x} \tau_{sb}^1 - \frac{v_x}{v+w} \theta_{sb} \wedge dx,$$

$$q_{G_2}^* ((du_{2y} - e^{u_2}dx) \wedge dy) = \frac{1}{w_y} \tau_{sb}^2 - \frac{w_y}{v+w} \theta_{sb} \wedge dy, \tag{3.25}$$

from which we conclude that $\mathcal{I}_2/\Gamma_{G_2} = \mathcal{I}_2$.

From here it is now also easy to check that $\mathcal{B}$ is an integral extension of $\mathcal{I}_2$. By using equation (3.22) and the forms in equations (3.9) and (3.24), we find,

$$p_2^* (du_v - u_{xx}dx - u_{yy}dy) = \beta_1 + \beta_2,$$

$$p_2^* ((du_{2x} - e^{u_2}dy) \wedge dx) = -d\beta_1 + e^W \beta_2 \wedge dy - e^V \beta_1 \wedge dx,$$

$$p_2^* ((du_{2y} - e^{u_2}dx) \wedge dy) = -d\beta_2 + e^V \beta_1 \wedge dx - e^W \beta_2 \wedge dy, \tag{3.26}$$

and therefore $\mathcal{B} = \langle \beta_2, p_2^* \mathcal{I}_2 \rangle_{\text{alg}}$.

In summary, we have used the local group actions $\Gamma_H$, $\Gamma_{G_1}$ and $\Gamma_{G_2}$ to produce the following diagram of differential systems,

$$\begin{align*}
\mathcal{K}_1 + \mathcal{K}_2 & \xrightarrow{q_{\Gamma_H}} \mathcal{K}_1 \\
\mathcal{K}_1 & \xrightarrow{q_{\Gamma_{G_1}}} \mathcal{K}_2 \\
\mathcal{I}_1 & \xrightarrow{q_{\Gamma_{G_2}}} \mathcal{I}_2 \tag{3.27}
\end{align*}$$

where

$$u_{1xy} = 0, \quad u_{2xy} = e^{u_2}$$

and

$$u_1 = V - W, \quad u_2 = V + W + \log 2. \tag{3.28}$$
Equations (3.28) are to be supplemented with their total derivatives, see equations (3.13), (3.14), (3.21), and (3.22). Theorem A insures that all the maps \( q_H, q_{G_1}, q_{G_2}, p_1 \) and \( p_2 \) define integrable extensions.

Finally note that if we eliminate the variables \( V \) and \( W \) (and their derivatives) from the equations defining \( p_1 \) and \( p_2 \) (see (3.14) and (3.22)) we arrive at
\[
\begin{align*}
  u_{1x} - u_{2x} &= \sqrt{2} \exp \frac{u_1 + u_2}{2}, \\
  u_{1y} + u_{2y} &= -\sqrt{2} \exp \frac{-u_1 + u_2}{2},
\end{align*}
\]
which is the usual Bäcklund transformation relating the wave equation to the Liouville equation.

**Example 3.2.** We continue with Example 3.1. Again we take for \( I \) the standard contact system \( K_1 + K_2 \) in (3.2) on \( M = J^2(\mathbb{R}, \mathbb{R}) \times J^2(\mathbb{R}, \mathbb{R}) \). We construct other Bäcklund transformations using Theorem A by adding to the infinitesimal group action \( \Gamma_H \) the vector-fields
\[
Z_3 = y \partial_w + \partial_{w_y}, \quad Z_4 = x \partial_v + \partial_{v_x} - y \partial_w - \partial_{w_y}.
\]
Let \( \Gamma_{G_i} = \text{span}\{ \Gamma_H, Z_i \}, i = 3, 4 \). The action \( \Gamma_{G_2} \) in Example 3.1 and \( \Gamma_{G_4} \) are diagonal actions on \( J^2(\mathbb{R}, \mathbb{R}) \times J^2(\mathbb{R}, \mathbb{R}) \) while \( \Gamma_{G_3} \) and \( \Gamma_{G_4} \) are not.

We calculate the reductions \((I_i, M_i)\) of \( I \) by \( \Gamma_{G_i} \) and the orbit projection maps \( p_i : N \to M_i \). Again each \( M_i \) has dimension 5 and the \( I_i \) are all type 1 hyperbolic systems (that is, generated by a single 1-form and a pair of decomposable 2-forms). The result is a 4-fold collection of integrable extensions \( p_i : \mathcal{B} \to I_i \) extending diagram (3.27). In terms of the associated partial differential equations, these are given explicitly by
\[
\begin{align*}
  \mathcal{K}_1 + \mathcal{K}_2 \\
  \downarrow q_{\Gamma_H} \\
  V_y &= -e^W, \quad W_x = -e^V \\
  p_1: & \\
  u_{1xy} &= 0, \quad u_{2xy} = e^{u_4} \\
  p_2: & \\
  u_{3xy} &= \frac{1}{u_3 - x} u_{3x} u_{3y} \\
  p_3: & \\
  u_{4xy} &= \frac{2u_4 - x - y}{(u_4 - x)(u_4 - y)} u_{4x} u_{4y}
\end{align*}
\]
where
\[
\begin{align*}
  u_1 &= V - W, \quad u_2 = V + W + \log 2, \quad u_3 = \frac{ye^W + xe^V - 1}{e^V}, \quad u_4 = \frac{ye^W + xe^V - 1}{e^W + e^V}.
\end{align*}
\]
These equations are to be supplemented with their total derivatives.

Each of the projections \( p_i \) defines \( \mathcal{B} \) as an integrable extension so that any two of the equations in (3.30) are Bäcklund equivalent via the common Pfaffian system \( \mathcal{B} \). For example if we eliminate
the variables $V$ and $W$ (and their derivatives) from the equations defining $p_3$ and $p_4$ the Bäcklund transformation relating the third and forth equation in (3.30) can be written

$$u_4(u_4 - x)u_{3x} - u_3(u_3 - x)u_{4x} = 0, \quad u_4(u_4 - y)u_{3y} + yu_3u_{4y} = 0.$$ 

The systems $I_2, I_3, I_4$ are all well-known examples of Monge-Ampère equations which become Darboux integrable after one prolongation [13]. In this regard, the calculations leading to (3.30) could also be done starting with the canonical differential system on $J^3(R, R) \times J^3(R, R)$ as done in [3] and leading to $B^{[1]}$ (the prolongation of $B$) as a common Bäcklund transformation between the rank 3 Pfaffian systems $I_i^{[1]}, i = 1, \ldots, 4$.

**Example 3.3.** In this example we compute the Darboux invariants (2.9) for Examples 3.1 and 3.2 by applying the group theoretical methods in part [iii] in Theorem 2.10 (or part [iii] in Theorem 6.1 in [5] for the non-diagonal actions). We also determine the Vessiot algebra (see (2.28)) for each of these examples using Theorem 7.7 in [5].

The Darboux invariants for Examples 3.1 and 3.2 are computed by noting that the hypothesis of Theorem 6.1 in [5] are satisfied after one prolongation for all the actions $\Gamma_{G_i}$. Therefore we will work with the prolongations $\text{pr} \Gamma_{G_i}$ on $J^3(R, R) \times J^3(R, R)$. By an abuse of notation we will label the prolongations by $\Gamma_{G_i}$ and let $M = J^3(R, R) \times J^3(R, R)$. We then use part [iii] in Theorem 2.10 (or [iii] Theorem 6.1 in [5]) to compute the Darboux invariants for the Darboux integrable systems $(J^3(R, R) \times J^3(R, R))/\Gamma_{G_i}$.

Let $\Gamma_{i,a} = \rho_a(\Gamma_i)$ where $\rho_a : \Gamma_{G_i} \rightarrow \Gamma_{G_i,a}, a = 1, 2$ is the projection onto the $a$th factor where $\Gamma_{G_i,a}$ acts by contact transformations on $J^3(R, R)$. In the context of Examples 3.1 and 3.2 part [iii] in Theorem 2.10 or [iii] Theorem 6.1 in [5] states the following

*The first integrals of $\hat{V}$ are the differential invariants of $\Gamma_{G_i,2}$ on $J^3(R, R)$ expressed as functions on $M/\Gamma_{G_i}$, and the first integrals of $\hat{V}$ are the differential invariants of $\Gamma_{G_i,1}$ on $J^3(R, R)$ expressed as functions on $M/\Gamma_{G_i}$.*

We will use this remark to compute a set of functionally independent Darboux integrals for each of our examples. We begin with the case where the action $\Gamma_{G_i}$ is diagonal ($i = 2, 4$). Then $\rho_a : \Gamma_{G_i} \rightarrow \Gamma_{G_i,a}$ is an isomorphism and the number of independent first integrals (or Darboux invariants) for each of the singular systems $\{\hat{V}_i, \hat{V}_i\}, i = 2, 4$, is therefore $(\dim J^3(R, R) - \dim \Gamma_{G_i}) = 2, i = 2, 4$.

We start with the Lie algebra of vector fields $\Gamma_{G_2} = \text{span}\{X_1, X_2, Z_2\}$ given by equations (3.4) and (3.11) (or more precisely their prolongation). The action of the projections $\rho_a(\Gamma_{G_2})$ on $J^3(R, R)$ gives the prolongation of standard action of $\mathfrak{sl}(2, R)$ on the dependent variable,

$$\Gamma_{G_2,1} = \text{span}\{\partial_\nu, v \partial_\nu + v_x \partial_\nu + v_{xx} \partial_\nu v_{xxx}, v^2 \partial_\nu + D_x(v^2) \partial_\nu v_{xxx} + D_x^2(v^2) \partial_\nu v_{xxxx}\}.$$ 

As $\Gamma_{i,a}$ are prolongations of infinitesimal point transformations they preserve the contact system on $J^3(R, R)$ and local coordinates on the quotients consist of a set of independent differential
invariants. In the case of $\Gamma_{G_2}$ the two functionally independent differential invariants on $J^3(R, R)$ are well-known to be

$$f_1 = x, \quad f_2 = \frac{2u_x v_{xxx} - 3v_x^2}{2v_x^2}. \quad (3.32)$$

The invariants of $\Gamma_{G_2}$ on $J^3(R, R) \times J^3(R, R)$ that we need are

$$u_{2x} = D_x \left( \log \frac{2v_x w_y}{(v + w)^2} \right) = \frac{v_{xx}}{u_x} - 2 \frac{v_x}{v + w}, \quad u_{2xx} = D_x(u_{2x}) = \frac{v_{xxx}}{v_x} - \frac{v_x^2}{v^2} - 2 \frac{v_{xx}}{v + w} + \frac{v_x^2}{(v + w)^2}. \quad (3.33)$$

Expressing the invariant $f_2$ in equation (3.32) in terms of those in equation (3.33) (see italicized remark above in this section) we obtain

$$f_2 = \frac{2u_x v_{xxx} - 3v_x^2}{2v_x^2} = u_{2xx} - \frac{1}{2} u_{2x}^2.$$ 

Therefore $f_1 = x$ and $f_2 = u_{2xx} - \frac{1}{2} u_{2x}^2$ are two independent Darboux invariants for the Liouville equation $u_{2xy} = \exp(u_2)$. The function $f_2$ is checked to be a Darboux invariant in the traditional way by showing $D_y(u_{2xx} - \frac{1}{2} u_{2x}^2) = 0$ when $u_2$ is a solution to the Liouville equation. Similarly $g_1 = y$ and $g_2 = u_{2yy} - \frac{1}{2} u_{2y}^2$ are independent Darboux invariants. Of course, these Darboux invariants are well-known. The point here, which is not generally understood, is that the Darboux invariants can **always** be expressed in terms of differential invariants. In this case $f_2$ in (3.32) is just the Schwatizian derivative.

The action of $\Gamma_{G_2}$ is diagonal and hence Theorem 7.7 in [5] gives the Vessiot algebra as (isomorphic to) $\Gamma_{G_2}$ (or $\mathfrak{sl}(2, R)$).

The infinitesimal group action $\Gamma_{G_4} = \text{span}\{X_1, X_2, Z_4\}$ from equations (3.31) and (3.29) is again diagonal. One of the projections of $\Gamma_{G_4}$ (or more precisely its prolongation) is

$$\Gamma_{G_4,1} = \text{span}\{\partial_v, v \partial_v + v_x \partial_{vx} + v_{xx} \partial_{vxx} + v_{xxx} \partial_{vxxx}, x \partial_v + \partial_{vx}\}.$$ 

The differential invariants of $\Gamma_{G_4,1}$ acting on $J^3(R, R)$ expressed in terms of $u_4$ and it’s derivatives from equation (3.31) are

$$f_1 = x, \quad f_2 = \frac{v_{xxx}}{v_x} = \frac{u_{4xx}}{u_{4x}} + \frac{1 - 2u_{4x}}{u_4 - x}. \quad (3.34)$$

Thus four functionally independent Darboux invariants are $x, y$, the invariant in equation (3.31) and its $y$ analogue. The action of $\Gamma_{G_4}$ is diagonal and hence Theorem 7.7 in [5] gives the Vessiot algebra as (isomorphic to) $\Gamma_{G_4}$.

We now examine the actions $\Gamma_{G_1}$ and $\Gamma_{G_2}$ on $J^3(R, R) \times J^3(R, R)$ which are not diagonal. The projections for the prolongation of $\Gamma_{G_1} = \text{span}\{X_1, X_2, Z_1\}$ from equation (3.4) and (3.10) are

$$\Gamma_{G_1,1} = \text{span}\{\partial_v, v \partial_v + v_x \partial_{vx} + v_{xx} \partial_{vxx} + v_{xxx} \partial_{vxxx}\}, \quad \text{and}$$

$$\Gamma_{G_1,2} = \text{span}\{\partial_w, w \partial_w + w_x \partial_{wx} + w_{xx} \partial_{wxx} + w_{xxx} \partial_{wxxx}\}.$$ 

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Therefore by equation 6.10 in Theorem 6.1 in [1], the singular systems for the wave equation \( J^1_3 \) each have 3 independent first integrals. In fact the differential invariants of \( \Gamma_{G_3} \) on \( J^3(R, R) \) are easily computed in terms of \( u_3 \) in equation (3.13) and it’s derivatives giving the Darboux invariants, 

\[
  f_1 = x, \quad f_2 = \frac{v_{xx}}{v_x} = u_{1x}, \quad f_3 = \frac{v_{xxx}}{v_{xx}} = u_{1xx}
\]

and their \( y \) analogues. In this case the action of \( \Gamma_{G_1} \) is not diagonal but can be reduced to a diagonal action by Theorem 7.7 (also see Lemma 6.8) in [1]. Furthermore equation (2.28) shows that the dimension of the Vessiot algebra is one.

Alternatively to equation (2.28) we can appeal to Theorem 7.7 in [1] to compute the Vessiot algebra. This theorem requires that we compute the subalgebra \( \Gamma_{A_1} = \text{span} \{ \partial_u \} \) and \( \Gamma_{A_2} = \text{span} \{ \partial_v \} \) of \( \Gamma_{G_1} \) (see Section 6.2 in [1]). We then obtain \( \Gamma_{G_1, \text{diag}} = \Gamma_{G_1}/(\Gamma_{A_1} + \Gamma_{A_2}) = \{ u\partial_u + v\partial_v \} \).

Since the wave equation is Darboux integrable on the 5-manifold (without prolongation) we work with the quotient map \( q_{\Gamma_{G_1}} \) on \( J^2(R, R) \times J^2(R, R) \) which by equation (6.31) of Lemma 6.8 in [1] factors as

\[
  \begin{array}{ccc}
  (J^2(R, R) \times J^2(R, R), K_1 + K_2) & \xrightarrow{q_{\Gamma_{A_1}} \times q_{\Gamma_{A_2}}} & (J^1(R, R) \times J^1(R, R), \tilde{K}_1 + \tilde{K}_2) \\
  & \downarrow q_{\Gamma_{G_1}} & \\
  & (\tilde{M}, \tilde{Z}_1) \ . 
  \end{array}
\]

where \( \tilde{K} \) is the contact system on \( J^1(R, R) \) and \( \tilde{M} \) is the 5-dimensional quotient manifold. Theorem 7.7 in [1] then states that the Vessiot algebra of the wave equation is (isomorphic to) the 1-dimensional algebra \( \Gamma_{G_1, \text{diag}} \) acting on \( J^1(R, R) \times J^1(R, R) \).

For the (prolongation of) the Lie algebra of vector fields \( \Gamma_{G_3} = \text{span} \{ X_1, X_2, Z_3 \} \) defined by equations (3.21) and (3.29) we have the projections

\[
  \Gamma_{G_{3,1}} = \text{span} \{ \partial_u, v\partial_v + v_x\partial_{v_x} + v_{xx}\partial_{v_{xx}} + v_{xxx}\partial_{v_{xxx}} \}, \quad \text{and} \quad \Gamma_{G_{3,2}} = \text{span} \{ \partial_w, w\partial_u + w_y\partial_{v_y} + w_{yy}\partial_{v_{yy}} + w_{yyy}\partial_{v_{yyy}}, y\partial_w + \partial_{v_y} \}.
\]

It is worth emphasizing that unlike in the cases \( i = 1, 2, 4 \) the algebras in the projections in equation (3.36) have different dimensions.

Again by the italicized remark above in this section we deduce that the singular system \( \tilde{V}_3 \) for \( J^1_3 \) has 2 independent first integrals because \( \Gamma_{G_{3,2}} \) has 3-dimensional orbits on \( J^3(R, R) \). Expressed in terms of \( u_3 \) and its derivatives from equation (3.31) these become the following Darboux invariants for the \( u_3 \) equation in (3.30),

\[
  g_1 = y, \quad g_2 = \frac{w_{yyy}}{w_{yy}} = \frac{u_{3yy}}{u_{3y}} - \frac{1}{y}.
\]

The singular system \( \tilde{V}_3 \) has 3 first independent first integrals because \( \Gamma_{G_{3,1}} \) in equation (3.36) has 2-dimensional orbits on \( J^3(R, R) \). For \( \Gamma_{G_{3,1}} \) the differential invariants in terms of \( u_3 \) and its derivatives are found, using \( (3.31) \), to be

\[
  f_1 = x, \quad f_2 = \frac{v_{xx}}{v_x} = -\frac{u_{3x}}{u_3 - x}, \quad f_3 = \frac{v_{xxx}}{v_{xx}} = \frac{u_{3xx}}{u_{3x}} + \frac{1 - 2u_{3x}}{u_3 - x}.
\]
This gives 3 independent first integrals for $\tilde{V}$ (or Darboux invariants for the $u_3$ equation in (3.30)).

In this case the action of $\Gamma_{G_3}$ can be reduced to a diagonal action by Theorem 7.7 in [5]. In particular note that equation (2.28) shows that the dimension of the Vessiot algebra is two. The subalgebras in Theorem 7.7 of [5] for this case are $\Gamma_{A_1} = \{0\}$ and $\Gamma_{A_2} = \{y\partial_w + \partial_w y\}$ of $\Gamma_{G_2}$. In this case $\Gamma_{G_3,\mathrm{diag}} = \Gamma_{G_3} / (\Gamma_{A_1} + \Gamma_{A_2}) = \Gamma_{G_3} = \text{span}\{X_1, X_2\}$.

Finally we also note that Theorem 6.1 in [5] applies to the diagonal action $\Gamma_H$ in (3.4) and $B = \mathcal{I}/\Gamma_H$ where the generators are given in (3.9). In fact the structure equations for $B$ in (3.9) are

$$d\beta^1 = -\hat{\pi} \wedge \hat{\omega}, \quad d\beta^2 = -\tilde{\pi} \wedge \tilde{\omega} \mod B,$$

where

$$\hat{\omega} = dx, \quad \hat{\pi} = dV_x - e^{V+W} dy, \quad \tilde{\omega} = dy, \quad \tilde{\pi} = dW_y - e^{V+W} dx.$$

From these equations it is easily checked that $B$ is an $s = 2$ hyperbolic system where the singular Pfaffian systems are

$$\hat{V} = \text{span}\{\beta^1, \beta^2, \hat{\pi}, \hat{\omega}\} \quad \text{and} \quad \tilde{V} = \text{span}\{\beta^1, \beta^2, \tilde{\pi}, \tilde{\omega}\}.$$

It is also easily checked that $B$ is a Darboux integrable with first integrals for the singular system $\hat{V}$ being $\{x, V_x + e^V\}$ and $\{y, W_y + e^W\}$ for $\hat{V}$. These are easily obtained from the differential invariants of $\Gamma_{H,i}$ on the individual jet spaces as done for $G_{i}^2$ and $G_{i}^4$ above. The action of $\Gamma_H$ is also diagonal and hence coincides with the Vessiot algebra of $B$.

We conclude this example by making a few remarks which describe the relationships between the invariants of $B^{[1]}$ and those of $\mathcal{I}/\Gamma_{G_i}$. We also examine a basic relationship between the Darboux invariants of $\mathcal{I}/\Gamma_{G_i}$.

The differential system $B$ representing the Bäcklund transformation is Darboux integrable without prolongation and as we have seen above $B$ has 4 independent Darboux invariants. The prolongation $B^{[1]}$ admits 6 Darboux invariants.

As noted above the Vessiot algebra of $B$ is the 2-dimensional solvable Lie algebra $\Gamma_H$, while Theorem 7.5 in [5] shows that this is isomorphic to the Vessiot algebra of the prolongation $B^{[1]}$ of $B$. In fact the prolongation of the vector fields $\Gamma_H$ is the Vessiot algebra of $B^{[1]}$. Theorem 7.3 in [5] implies that the Vessiot algebra of $B^{[1]}$ is a sub-algebra of each of the Vessiot algebras for $I_2^{[1]}$ and $I_4^{[1]}$. This same theorem shows that the induced homomorphism on the Vessiot algebras defined by $p_1$ is not injective, while for $I_3^{[1]}$ the algebras are isomorphic.

Finally, we consider Theorem 9.1 and Corollary 9.2 in [5] for these examples. These theorems state that for the Bäcklund transformation $B^{[1]}$ constructed between the pairs $\mathcal{I}_i$ given in equation (3.31), that the systems $\mathcal{I}_i$ arising from diagonal quotients have strictly fewer independent Darboux invariants than those constructed from non-diagonal quotients. This applies to the pairs $(\Gamma_{G_1}, \Gamma_{G_2})$, $(\Gamma_{G_3}, \Gamma_{G_4})$, $(\Gamma_{G_3}, \Gamma_{G_4})$ and is verified by the computations in this example.
Example 3.4. Our next example is based upon results found in the PhD thesis of Francesco Strazzullo [14]. We start with the canonical Pfaffian systems $\mathcal{K}_1$ and $\mathcal{K}_2$ for the Monge equations
\[
\frac{du}{ds} = F\left(\frac{d^2 v}{ds^2}\right) \quad \text{and} \quad \frac{dy}{dt} = G\left(\frac{d^2 z}{dt^2}\right) \quad \text{where} \quad F_{v''} \neq 0 \quad \text{and} \quad G_{z''} \neq 0. \tag{3.37}
\]
On $\mathbb{R}^5[s, u, v, v_x, v_{xx}]$ and $\mathbb{R}^8[t, y, z, z_t, z_{tt}]$ these are given by
\[
\mathcal{K}_1 = \langle dv - v_x ds, dv_x - v_{xx} ds, du - F(v_{xx}) ds \rangle_{\text{diff}} \quad \text{and} \quad \mathcal{K}_2 = \langle dz - z_t dt, dz_t - z_{tt} dt, dy - G(z_{tt}) dt \rangle_{\text{diff}}
\]
and we take $\mathcal{I} = \mathcal{K}_1 + \mathcal{K}_2$ on $M = \mathbb{R}^{10}$. The conditions $F_{v''} \neq 0$ and $G_{z''} \neq 0$ imply that the derived flags for $\mathcal{K}_1$ and $\mathcal{K}_2$ have ranks $[3, 2, 0]$.

We shall calculate reductions of $\mathcal{I}$ using the infinitesimal symmetries
\[
X_1 = \partial_v, \quad X_2 = s\partial_v + \partial_{v_x}, \quad X_3 = \partial_u, \quad Y_1 = \partial_z, \quad Y_2 = t\partial_z + \partial_{z_t}, \quad Y_3 = \partial_y, \quad Z_1 = X_1 - Y_1, \quad Z_2 = X_2 + Y_2, \quad Z_3 = X_3 - Y_3. \tag{3.38}
\]
Let $\Gamma_G = \{X_1, Y_1, Z_2\}$, $\Gamma_G = \{Z_1, Z_2, Z_3\}$ and $\Gamma_H = \Gamma_G \cap \Gamma_{G_2} = \{Z_1, Z_2\}$. These infinitesimal actions all satisfy the conditions of Theorem 6.1 in [5], and hence the quotients are Darboux integrable. The actions $\Gamma_H$ and $\Gamma_{G_2}$ are diagonal actions while $\Gamma_{G_1}$ is not.

Let $N = \mathbb{R}^8[x_1, \ldots, x_7, x_8]$, $M_1 = \mathbb{R}^7[x_1, \ldots, x_6, x_7]$ and $M_2 = \mathbb{R}^7[x_1, \ldots, x_6, x_8]$. Then projection maps $q_H : M \to N$, $q_{G_1} : M \to M_1$ and $q_{G_2} : M \to M_2$ for the reduction by these 3 actions are
\[
q_H = [x_1 = t, x_2 = s, x_3 = z_{tt}, x_4 = v_{xx}, x_5 = z_t - v_x, x_6 = u, x_7 = y, x_8 = z + v - (s + t)v_x],
\]
\[
q_{G_1} = [x_1 = t, x_2 = s, x_3 = z_{tt}, x_4 = v_{xx}, x_5 = z_t - v_x, x_6 = u, x_7 = y],
\]
\[
q_{G_2} = [x_1 = t, x_2 = s, x_3 = z_{tt}, x_4 = v_{xx}, x_5 = z_t - v_x, x_6 = u + y, x_7 = z + v - (s + t)v_x]
\]
and the commutative diagram of Pfaffian systems [13] is easily computed to be
Here $F = F(x_4)$ and $G = G(x_3)$. All the quotient differential systems here are Darboux integrable, all the maps are integrable extensions. The first integrals for each of the singular systems are easily calculated from the group invariants for each action as described by italicized remark in the previous example. The Vessiot algebras are all abelian (Theorem 7.7 in [5]) while from (2.28) we have the dimension of these algebras are $\dim \text{vess}(\mathcal{I}/\Gamma_{G_1}) = 3$, $\dim \text{vess}(\mathcal{I}/\Gamma_H) = 2$ and $\dim \text{vess}(\mathcal{I}/\Gamma_{G_2}) = 1$.

These properties are all easily verified explicitly. For $\mathcal{B} = \mathcal{I}/\Gamma_H$ the structure equations are

$$d\beta^1 = G' \hat{x}^1 \wedge \hat{x}^2,$$
$$d\beta^2 = F' \hat{x}^1 \wedge \hat{x}^2,$$
$$d\beta^3 = -\hat{x}^1 \wedge \hat{x}^2, \quad d\beta^4 = -\hat{x}^1 \wedge \hat{x}^2,$$

where $\hat{x}^1 = dx_1, \hat{x}^2 = dx_3, \hat{x}^3 = dx_2, \hat{x}^4 = dx_4$. This is a class $r = 4$ hyperbolic Pfaffian system. The singular Pfaffian systems $\{\beta^1, \beta^2, \hat{x}^1, \hat{x}^2\}$ and $\{\beta^1, \beta^2, \hat{x}^1, \hat{x}^2\}$ have first integrals $\{x_1, x_3, x_7\}$ and $\{x_2, x_4, x_6\}$. The derived system is

$$B' = \text{span}\{\beta^4 + \frac{x_1 + x_2}{F'} \beta^2, \beta^3 - \frac{1}{G'\beta^1} + \frac{1}{F'}\beta^2\}.$$

These pair of 1-forms in $B'$ each define admissible sub-bundles for $\mathcal{B}$ as integrable extensions of $\mathcal{I}/\Gamma_{G_1}$ and $\mathcal{I}/\Gamma_{G_2}$ respectively.

Just as in the previous example, the quotient map for the non-diagonal action factors through the reduction by using Lemma 6.7 in [5] with the product action $\{\partial_v, \partial_z\}$. For the system $\mathcal{I}_1 = \mathcal{I}/\Gamma_{G_1}$, the structure equations are

$$d\alpha^1 = \hat{x}^1 \wedge \hat{x}^2 - \hat{x}^1 \wedge \hat{x}^2, \quad d\alpha^2 = G' \hat{x}^1 \wedge \hat{x}^2, \quad d\alpha^3 = F' \hat{x}^1 \wedge \hat{x}^2.$$

There are 3 first integrals for each singular Pfaffian system and hence, by a classical theorem of Lie,
This implies that $\mathcal{I}$ is (contact) equivalent to the wave equation. The change of variables

\[ X' = -x_6 + x_2 F, \quad Y' = x_7 - x_1 G, \quad U' = x_5 - x_1 x_3 + x_2 x_4, \]

\[ P' = \frac{1}{F'}, \quad Q' = \frac{1}{G'}, \quad R' = -\frac{F''}{x_2 F'^3}, \quad T' = \frac{G''}{x_1 G'^3} \]

transforms $\mathcal{I}$ to the standard Pfaffian system for $U'_{X',Y'} = 0$.

To calculate the structure equations for $\mathcal{I}_2 = \mathcal{I}/\Gamma_{G_2}$ we define a new co-frame by

\[ \theta^1 = (x_1 + x_2)G' \gamma_1 - (x_1 + x_2) \gamma_2 - (F' + G') \gamma_3, \quad \theta^2 = F' \gamma_1 + \gamma_2, \quad \theta^3 = -G' \gamma_1 + \gamma_2, \]

\[ \hat{\theta}^1 = dx_1 - \mu_2(x_1 + x_2) dx_3, \quad \hat{\theta}^2 = dx_3, \quad \hat{\theta}^3 = dx_2 - \mu_1(x_1 + x_2) dx_4, \]

where $\mu_1 = \frac{F''}{F' + G'}$ and $\mu_2 = \frac{G''}{F' + G'}$. The resulting structure equations are

\[ d\theta^1 = \theta^2 \wedge \hat{\theta}^1 + \theta^3 \wedge \hat{\theta}^1, \quad \text{mod } \theta^1 \]

\[ d\theta^2 = (F' + G')\hat{\theta}^1 \wedge \hat{\theta}^2 - \mu_1(\theta^2 - \theta^3) \wedge \hat{\theta}^2, \]

\[ d\theta^3 = (F' + G')\hat{\theta}^1 \wedge \hat{\theta}^2 + \mu_2(\theta^2 - \theta^3) \wedge \hat{\theta}^2. \]

These structure equations show that $\mathcal{I}_2$ defines a hyperbolic second order PDE. From the equations for $d\theta^2$ and $d\theta^3$ one can determine that the Monge-Ampère invariants for $\mathcal{I}_2$ are proportional to $F''$ and $G''$. Thus $\mathcal{I}_2$ is of generic type (type $(7, 7)$ in the terminology of [12]) when $F'' \neq 0$ and $G'' \neq 0$ and Monge-Ampère type when $F'' = G'' = 0$. The functions $\{x_1, x_3\}$ and $\{x_2, x_4\}$ are independent first integrals for the singular systems of $\mathcal{I}_2$.

In the special case of the Hilbert-Cartan equations

\[ \frac{du}{ds} = \left(\frac{d^2 u}{ds^2}\right)^2 \quad \text{and} \quad \frac{dy}{dt} = \left(\frac{d^2 y}{dt^2}\right)^2 \]

we can use a recent theorem of D. The [15] to explicitly identify the PDE defined by $\mathcal{I}_2$. This theorem asserts that the equation $3U_{X,Y} + 1 = 0$ is uniquely characterized as the PDE in the plane with a 9-dimensional symmetry algebra with Levi decomposition $\mathfrak{sl}(2) \ltimes \mathfrak{r}$ and such that the derived series for the radical $\mathfrak{r}$ has dimensions $[6, 5, 2]$. Indeed, we find that the symmetry algebra for $\mathcal{I}_2$ is the 9-dimensional Lie algebra defined by

\[ X_1 = x_3 \partial_{x_1} + x_4 \partial_{x_2} + \frac{1}{2}(x_3 - x_4) \partial_{x_5} + \frac{1}{3}(x_4^3 + x_5^3) \partial_{x_6} + (-\frac{1}{2}x_4x_1 - \frac{1}{2}x_4x_2 - \frac{1}{2}x_6 + x_3x_5) \partial_{x_8}, \]

\[ X_2 = \frac{1}{2}x_1 \partial_{x_1} + \frac{1}{2}x_2 \partial_{x_2} - \frac{1}{2}x_3 \partial_{x_3} - \frac{1}{2}x_4 \partial_{x_4} - \frac{1}{2}x_6 \partial_{x_6} + \frac{1}{2}x_8 \partial_{x_8}, \]

\[ X_3 = -\frac{1}{2}x_1 \partial_{x_1} - \frac{1}{2}x_2 \partial_{x_2} + \frac{1}{4}(x_5^2 - x_7^2) \partial_{x_5} + (-x_1x_5 + x_8) \partial_{x_6} + (-\frac{1}{12}x_1^3 + \frac{1}{12}x_2^2x_1 + \frac{1}{12}x_3^2) \partial_{x_8}, \]

\[ X_4 = x_1 \partial_{x_1} + x_2 \partial_{x_2} + x_3 \partial_{x_3} + x_4 \partial_{x_4} + 2x_5 \partial_{x_5} + 3x_6 \partial_{x_6} + 3x_8 \partial_{x_8}, \]

\[ X_5 = \partial_{x_5} + x_1 \partial_{x_5}, \quad X_6 = \frac{1}{2} \partial_{x_3} - \frac{1}{2} \partial_{x_4} + \frac{1}{2}(x_1 + x_2) \partial_{x_5} + x_5 \partial_{x_6} + \frac{1}{4}(x_1 + x_2)^2 \partial_{x_8}, \]

\[ X_7 = -\partial_{x_1} + \partial_{x_2}, \quad X_8 = \partial_{x_6}, \quad X_9 = \partial_{x_8}. \]

(3.44)
The radical of this Lie algebra is \{X_4, X_5, X_6, X_7, X_8, X_9\} with second derived algebra \{X_8, X_9\}. This algebraic data, together with the fact that \mathcal{I}_2 is of generic type, suffices to characterize \mathcal{I}_2 as the standard Pfaffian system for 3U_{XY}U_{3XY}^3 + 1 = 0 \(\text{[15], section 7}\). By finding the diffeomorphism which maps the symmetries \{X_1, X_2, \ldots, X_9\} to the symmetries of 3U_{XY}U_{3XY}^3 + 1 = 0, we are able to produce the following explicit contact equivalence:

\[
X = -2 \frac{x_3 + x_4}{x_1 + x_2}, \quad Y = x_5 - \frac{1}{2} (x_3 - x_4) (x_1 + x_2), \quad S = \frac{1}{2} (x_2 - x_1), \quad T = \frac{2}{x_1 + x_2},
\]

\[
U = 2 \frac{x_3 + x_4}{x_1 + x_2} (-x_8 + x_1 x_5) - x_6 + \frac{1}{3} (2x_1 - x_2) x_2^2 - \frac{2}{3} (x_1 + x_2) x_3 x_4 - \frac{1}{3} (x_1 - 2x_2) x_3^2, \quad (3.45)
\]

\[
P = x_8 - x_1 x_5 + \frac{1}{6} (x_1 + x_2) ((2x_1 - x_2) x_3 - (x_1 - 2x_2) x_4), \quad Q = \frac{2 x_4 x_1 - x_3 x_2}{x_1 + x_2}.
\]

The change of variables (3.40) and (3.45) allows us to rewrite the Bäcklund transformation (3.39) as

\[
dx_5 - x_3 \, dx_1 + x_4 \, dx_2 \\
dx_6 - x_4^2 \, dx_2, \quad dx_7 - x_3^2 \, dx_1 \\
dx_8 - x_3 \, dx_1 + x_4 (x_1 + x_2) \, dx_2
\]

(3.46)

It is interesting to remark that the identification of the quotient \mathcal{I}_2 = \mathcal{I}/G_2 as the canonical Pfaffian system for \(3u_{xx}u_{yy}^3 + 1 = 0\) can actually be accomplished without explicitly calculating the reduction \mathcal{I}/G_2 or its symmetry algebra. Indeed, we know that the symmetry algebra for \mathcal{I} is \(\mathfrak{g} = \mathfrak{g}_2 + \mathfrak{g}_2\), two copies of the split real form of the exceptional Lie algebra \(\mathfrak{g}_2\). For a basis for \(\mathfrak{g}\), take

\[
\{ y_6, \ldots, y_1, h_1, h_2, x_1, \ldots, x_6, \tilde{y}_6, \ldots, \tilde{y}_1, \tilde{h}_1, \tilde{h}_2, \tilde{x}_1, \ldots, \tilde{x}_6 \},
\]

where the \(y_i\) are the negative roots, the \(h_i\) the Cartan sub-algebra and the \(x_i\) the positive roots. In terms of this basis, the sub-algebra defined by the infinitesimal generators \(\Gamma_{G_2}\) is

\[
\mathfrak{h} = \{ x_2 - \tilde{x}_2, x_3 - \tilde{x}_3, x_6 + \tilde{x}_6 \}
\]

and we find the normalizer of \(\mathfrak{h}\) in \(\mathfrak{g}\) to be the 12-dimensional algebra

\[
\text{nor}(\mathfrak{h}) = \{ y_5 - \tilde{y}_5, y_1 + \tilde{y}_1, h_1 + \tilde{h}_1, h_2 + \tilde{h}_2, x_2, x_3, x_4 - \tilde{x}_4, x_5 - \tilde{x}_5, x_6, \tilde{x}_2, \tilde{x}_3, \tilde{x}_6 \}.
\]

From this we deduce that the symmetry algebra of \(\mathcal{I}/G_2\) is at least 9-dimensional. But because the equation is of generic type, the symmetry algebra is at most 9-dimensional and therefore the abstract Lie algebra for the symmetry algebra of \(\mathcal{I}_2\) is precisely \(\text{nor}(\mathfrak{h})/\mathfrak{h}\).
Example 3.5. For each $n = 0, 1, 2 \ldots$ the equation
\[ u_{xy} = 2n \sqrt{pq} \]  
\[ \frac{x + y}{x} \]  
(3.47)
is Darboux integrable at the $n + 2$ jet level. We use this equation to show that the group theoretic approach to the construction of Bäcklund transformations is not limited to equations which are Darboux integrable on the 2-jet level as in Examples 3.1 and 3.2.

Let $\mathcal{G}_n$ be the standard differential system for (3.47). To construct Bäcklund transformations between $\mathcal{G}_{n-1}$ and $\mathcal{G}_n$, we start with $I = H_{n+1}^{1} + H_{n+1}^{2}$, where $H_{n+1}^{1}$ and $H_{n+1}^{2}$ are the standard rank $n + 1$ differential systems for the $n$-th order Monge equations
\[ \frac{du}{ds} = (\frac{dn v}{ds})^2 \quad \text{and} \quad \frac{dy}{dt} = (\frac{dn z}{dt})^2. \]

The vector fields
\[ U = \partial_x, \quad V_i = s^i \partial_y, \quad Y = \partial_y, \quad Z_i = t^i \partial_z, \quad i = 0, 1, \ldots, 2n - 1 \]  
(3.48)
all lift to symmetries of $I$. To apply Theorem A in the introduction, we let
\[ \Gamma_{\mathcal{G}_1} = \{ U + V, V_0, Z_0, V + Z_i \}_{1 \leq i \leq 2n-2}, \quad \Gamma_{\mathcal{G}_2} = \{ U + V, V_i + Z_i \}_{0 \leq i \leq 2n-1} \quad \text{and} \quad \Gamma_H = \Gamma_{\mathcal{G}_1} \cap \Gamma_{\mathcal{G}_2} = \{ U + V, V_i + Z_i \}_{0 \leq i \leq 2n-2}. \]  
(3.49)  
(3.50)

Then we have the following commutative diagram of Pfaffian systems

\[ \mathcal{H}_n^{n+1} + \mathcal{H}_2^{n+1} \]
\[ \mathcal{G}_1 \]
\[ \mathcal{G}_2 \]
\[ B_n \]
\[ \Gamma_{\mathcal{G}_1} \]
\[ \Gamma_{\mathcal{G}_2} \]
\[ \Gamma_H \]
\[ \mathcal{H}_1^{n+1} \]
\[ \mathcal{H}_2^{n+1} \]
(3.51)

leading to the following theorem.

**Theorem 3.6.** The differential equations
\[ U_{xy} = \frac{2(n-1) \sqrt{U_x U_y}}{x + y} \quad \text{and} \quad V_{xy} = \frac{2n \sqrt{V_x V_y}}{x + y} \]  
(3.52)
are related by the Bäcklund transformation
\[ (\sqrt{U_x} - \sqrt{V_x})^2 = (2n - 1)(U - V) \quad \text{and} \quad (\sqrt{U_y} + \sqrt{V_y})^2 = 0. \]  
(3.53)
Note that for \( n = 1 \) this coincides with the Bäcklund transformation given in [16]. Detailed formulas for the various projection maps and quotients for (3.47) can be found in [2].

**Example 3.7.** As our next example we consider the \( A_2 \) Toda lattice system

\[
\begin{align*}
    u_{xy} &= 2e^u - e^v, \\
    v_{xy} &= -e^u + 2e^v.
\end{align*}
\]  

This system is Darboux integrable and a Bäcklund transformation shall be constructed using Theorem A in the introduction. By a simple linear change of variables, the \( A_2 \) Toda lattice system can be rewritten as

\[
\begin{align*}
    u_{xy} &= e^{2u-v}, \\
    v_{xy} &= e^{-u+2v}.
\end{align*}
\]  

This latter formulation proves to be more amenable to our analysis.

The construction of the canonical quotient representation (see Theorem 2.10), for the \( A_2 \) Toda lattice, has heretofore not appeared in the literature. We summarize the main steps here. The standard Pfaffian system \( \mathcal{I} \) for (3.54) is a rank 6 Pfaffian system on a 12-dimensional submanifold \( M \) of \( J^2(\mathbb{R}^2, \mathbb{R}^2) \). To calculate the Vessiot group for (3.54) we must work at the prolongation order for which the system is Darboux integrable. However, the quotient representation and the Bäcklund transformation will be calculated as projections to \( M \).

The first prolongation \( I^{[1]} \) is a rank 10 Pfaffian system defined on a 16-dimensional submanifold \( M^{[1]} \) of \( J^3(\mathbb{R}^2, \mathbb{R}^2) \). The associated singular Pfaffian system \( \hat{\mathcal{V}} \) is generated by \((I^{[1]})^\prime\), together with the three 1-forms \( dx \),

\[
    du_{xxx} - e^{2u-v}(2u_{xx} - v_{xx} + (2u_x - v_x)^2) dy, \\
    dv_{xxx} - e^{2v-u}(2v_{xx} - u_{xx} + (-u_x + 2v_x)^2) dy.
\]

The first integrals for \( \hat{\mathcal{V}} \) are \( \hat{I}_1 = x, \hat{I}_2 = u_{xx} + v_{xx} - u_x^2 + u_x v_x - v_x^2, \)

\[
    \hat{I}_3 = u_{xxx} + u_x^2 v_x + (-2u_{xx} + v_{xx} - v_x^2)u_x, \quad \text{and} \quad \hat{I}_4 = v_{xxx} + u_x v_x^2 + (u_{xx} - 2v_{xx} - u_x^2)v_x.
\]

By interchanging \( x \) with \( y \) in the above equations, we obtain the formulas for the other singular Pfaffian system \( \hat{\mathcal{V}} \) and its first integrals. We conclude that

\[
\dim \hat{\mathcal{V}} = \dim \hat{\mathcal{V}} = 13, \quad \dim \hat{\mathcal{V}}(\infty) = \dim \hat{\mathcal{V}}(\infty) = 4, \quad \text{and} \quad \dim(\hat{\mathcal{V}}(\infty) \cap \mathcal{V}) = \dim(\hat{\mathcal{V}} \cap \mathcal{V}(\infty)) = 4
\]

from which it then follows that the prolonged \( A_2 \) Toda lattice system \( I^{[1]} \) is Darboux integrable. By Theorem 2.10 the prolonged \( A_2 \) Toda lattice system is the quotient of the product of two, rank 9 Pfaffian systems \( W_1 \) and \( W_2 \) on 12-dimensional manifolds by the diagonal action of an 8-dimensional Lie group \( G \). The calculation of the 4-adapted co-frame for \( I^{[1]} \) shows that \( G \) is the special linear group \( A_2 = SL(3) \). The Pfaffian system \( W_1 \) is given by the restriction of \( \hat{\mathcal{V}} \) to the zero set of the first integrals \( \hat{I}_a \) (see Theorem 2.10) and hence

\[
W_1 = \{du - u_x dx, dv - v_x dx, du_x - u_{xx} dx, dv_y - e^{2u-v} dx, dv_x - v_{xx} dx, dv_y - e^{2v-u}(2v_y - u_y) dx\}.
\]  

(3.56)
The dimensions of the derived systems of $W_1$ and the dimensions of their Cauchy characteristics are

$$[\dim W'_1, \ldots, \dim W'_1] = [7, 5, 3, 1, 0] \text{ and } [\dim C(W'_1), \ldots, \dim C(W'_1)] = [2, 4, 6, 9, 12].$$

These numerical invariants suffice to characterize $W_1$ as the canonical contact system on the mixed jet space $J^{4,5}(\mathbb{R}, \mathbb{R}^2)$. At this point it remains to identify the precise form of the action of the Vessiot group $SL(3)$ on $J^{4,5}(\mathbb{R}^2)$. Once this is done, the joint differential invariants for the diagonal action of $G = SL(3)$ on $J^{3,4}(\mathbb{R}, \mathbb{R}^2) \times J^{3,4}(\mathbb{R}, \mathbb{R}^2)$ then determines the quotient representation

$$q_G : J^{3,4}(\mathbb{R}, \mathbb{R}^2) \times J^{3,4}(\mathbb{R}, \mathbb{R}^2) \rightarrow \frac{J^{3,4}(\mathbb{R}, \mathbb{R}^2) \times J^{3,4}(\mathbb{R}, \mathbb{R}^2)}{SL(3)} \cong_{\text{loc}} M.$$

However, the standard coordinates for $J^{3,4}(\mathbb{R}, \mathbb{R}^2)$ lead to very long explicit formulas for $q_G$ which make our subsequent derivation of the Bäcklund transformation for the $A_2$ Toda lattice quite complicated. A much better choice of coordinates for $J^{3,4}(\mathbb{R}, \mathbb{R}^2)$ is $[x, u, u_1, u_2, u_3, v, v_1, v_2, v_3, v_4]$, with

$$W_1 = \{du - u_1 dx, du_1 - u_2 dx, du_2 - u_3 dx, dv - v_1 du, dv_1 - v_2 du, dv_2 - v_3 du, dv_3 - v_4 du\}. \quad (3.57)$$

We leave it to the reader to check that (3.56) and (3.57) are equivalent Pfaffian systems. For our second copy of $J^{3,4}(\mathbb{R}, \mathbb{R}^2)$ we use coordinates $[y, w, w_1, w_2, z, z_1, z_2, z_3, z_4]$, with

$$W_2 = \{dw - w_1 dx, dw_1 - w_2 dx, dw_2 - w_3 dx, dz - z_1 dw, dz_1 - z_2 dw, dz_2 - z_3 dw, dz_3 - z_4 dw\}.$$

The integral manifolds for $W_1$ and $W_2$ are given by

$$u = f(x), \quad u_1 = f'(x), \quad v = g(f(x)), \quad v_1 = g'(f(x)), \quad w = h(y), \quad z = k(h(y)), \ldots \quad (3.58)$$

Note that the formulas for the total derivative vector fields and the prolongation formula for vector fields on $J^{3,4}(\mathbb{R}, \mathbb{R}^2)$ are slightly different in these coordinates.

The desired infinitesimal diagonal action of $SL(3)$ on $J^{3,4}(\mathbb{R}, \mathbb{R}^2) \times J^{3,4}(\mathbb{R}, \mathbb{R}^2)$ is given by the prolongation of the vector fields

$$X_1 = \partial_u + \partial_w, \quad X_2 = \partial_v + \partial_z, \quad X_3 = u \partial_u + w \partial_w, \quad X_4 = v \partial_v + z \partial_w, \quad X_5 = u \partial_v + w \partial_z, \quad X_6 = v \partial_v + z \partial_z,$$

$$Z_1 = u^2 \partial_u + u v \partial_v + w^2 \partial_w + w z \partial_z, \quad Z_2 = u v \partial_u + v^2 \partial_v + w z \partial_w + z^2 \partial_z.$$

The lowest order joint differential invariants for this action are

$$U^1 = \frac{1}{3} \log \frac{u^1 v^1 w^1 z^2}{(u - w)z_1 + z - v^1} \quad \text{and} \quad U^2 = \frac{1}{3} \log \frac{-u^1 v^2 w^1 z^2}{(u - w)v_1 + z - v^1}. \quad (3.59)$$

These equations, together with their total derivatives, give the quotient representation $q_G$ for the $A_2$ Toda lattice.
The construction of the Bäcklund transformation for \((3.55)\) now follows the construction of the Bäcklund transformation for Liouville’s equation given in Example 10.1. Let
\[ \Gamma_G = \{ X_1, X_2, X_3, X_4, X_5, X_6, Z_3 = \partial_u - \partial_w, Z_4 = \partial_v - \partial_z \}, \]
\[ \Gamma_G = \Gamma_G = \{ X_1, X_2, X_3, X_4, X_5, X_6, Z_1, Z_2 \}, \]
\[ \Gamma_H = \Gamma_G \cap \Gamma_G = \{ X_1, X_2, X_3, X_4, X_5, X_6 \}. \]

The lowest order joint differential invariants for the action of \(\Gamma_G\) on \(J^{3,4}(\mathbb{R}, \mathbb{R}^2) \times J^{3,4}(\mathbb{R}, \mathbb{R}^2)\) are
\[ V^1 = \log \frac{-2u_1w_1v_2}{(v_1 - z_1)^2} \quad \text{and} \quad V^2 = \log \frac{w_3}{u_1v_2} \quad (3.60) \]
which imply that \(I/G\) is the differential system for the decoupled Liouville-wave
\[ V^1_{xy} = e^{V^1}, \quad V^2_{xy} = 0. \quad (3.61) \]

We next calculate the symmetry reduction of \(J^{3,4}(\mathbb{R}, \mathbb{R}^2) \times J^{3,4}(\mathbb{R}, \mathbb{R}^2)\) by the 6-dimensional Lie group with generators \(\Gamma_H\). This will be a rank 8 Pfaffian system \(B\) on a 14-dimensional manifold. For this action the lowest order joint differential invariants are the first order invariants
\[ W^1 = \log \frac{u_1(v_1 - z_1)}{(w - u)v_1 + v - z} \quad \text{and} \quad W^2 = \log \frac{u_1(z_1 + v_1)}{(w - u)v_1 + v - z}. \quad (3.62) \]
It then follows that the Pfaffian system \(B\) is a partial prolongation of the standard Pfaffian system for the equations
\[ W^1_{xy} = -W^1_y(W^2_x + e^{W^1}) \quad \text{and} \quad W^2_{xy} = -W^2_x(e^{W^2} + W^1). \quad (3.63) \]

At this point we have constructed the Pfaffian systems for all of the equations in the following commutative diagram, as well as the quotient maps for the group actions \(\Gamma_H\), \(\Gamma_G\), and \(\Gamma_G\).

We calculate the expressions for \(U^1\), \(U^2\), \(V^1\), \(V^2\) in terms of \(W^1\) and \(W^2\) and their derivatives to be
\[ U^1 = \frac{2}{3}W^1 + \frac{1}{3}W^2 + \frac{1}{3}\log(-(W^1_y)^2W^2_x), \quad U^2 = \frac{1}{3}W^1 + \frac{2}{3}W^2 + \frac{1}{3}\log(-(W^1_yW^2_x)^2), \]
\[ V^1 = \log(2W^1_yW^2_x), \quad V^2 = 2W^2 - 2W^1 + \log\left(\frac{W^1_y}{W^2_x}\right). \quad (3.64) \]
These formulas, together with their $x$ and $y$ derivatives to order 2, define the projection maps $p_1$ and $p_2$. It is a simple matter to check directly that $p_1$ and $p_2$ define integrable extensions. Finally, the elimination of the variables $W^1$ and $W^2$ from (3.64) lead to following first order PDE

$$V_x^1 - \frac{1}{3}V_y^2 - 2U_x^2 = \sqrt{2}e^{(U^2 - \frac{1}{6}V^2 - U^1 + \frac{1}{6}V^1)}, \quad V_x^2 + 3U^1_x = -3\sqrt{2}e^{(-\frac{1}{2}V^1 + U^1 - \frac{1}{6}V^2)},$$

$$V_y^1 + \frac{1}{3}V_y^2 - 2U_y^1 = \sqrt{2}e^{(U^1 - U^2 + \frac{1}{2}V^1 + \frac{1}{6}V^2)}, \quad -V_y^2 + 3U_y^2 = -3\sqrt{2}e^{(U^2 - \frac{1}{2}V^1 + \frac{1}{6}V^2)}$$

for $U^1, U^2, V^1, V^2$, which give a Bäcklund transformation in the classical sense.

**Example 3.8.** In [6], several simple examples of Darboux integrable systems in 3 independent variables were given. Bäcklund transformations for all of these are easily constructed. For example, the Darboux integrable system

$$u_{xz} = uu_x, \quad u_{yz} = uu_y \tag{3.65}$$

can be obtained as the quotient of $J(R^2, R) \times J^2(R, R)$ with coordinates $(x, y, v; z, w, w_z, w_{zz})$ by taking the diagonal action $\Gamma_G$ of $\mathfrak{sl}(2, R)$ acting on the dependent variable $v$ and $w$ given by

$$X_1 = \partial_v + \partial_w, \quad X_2 = v\partial_v + w\partial_w + w_z\partial_{w_z} + w_{zz}\partial_{w_{zz}}, \quad X_3 = v^2\partial_v + w^2\partial_w + 2ww_z\partial_{w_z} + (2w_z^2 + 2ww_{zz})\partial_{w_{zz}}.$$

The differential invariant is

$$u = \frac{w_{zz}}{w_z} + \frac{w_z}{v - w} \tag{3.66},$$

which leads to the general solution to the (3.65). Working in a manner similar to Example 3.1 with $\Gamma_H = \{X_1, X_2\}$ we have the invariants

$$S = \frac{w_z}{w - v}, \quad T = \frac{w_{zz}}{w_z} + \frac{2w_z}{v - w} \tag{3.67}$$

The system

$$2S_x + T_x = 0, \quad 2S_y + T_y = 0, \quad S_z = TS - S^2 \tag{3.68}$$

are the syzygies, while $u = T$ in equation (3.66).

If we let

$$\Gamma_G = \{\partial_v, \partial_z, X_2\}$$

then $\Gamma_H = \Gamma_G \cap \Gamma_G$. The $\Gamma_G$ invariants

$$P = \log v_y - \log v_x, \quad Q = \log w_z - \log v_x$$

satisfy

$$P_z = 0, \quad P_x = Q_x + e^Q y \tag{3.69}.$$ 

Note that from (3.67) that $P = \log(S_y S_x^{-1})$ and $Q = \log(-S^2 S_x^{-1})$. The differential equation (3.68) define a Bäcklund transformation between the over-determined system (3.65) and (3.69).
4 An Application to Monge-Ampère systems in the plane

Let \( \mathcal{I}_2 \) be a hyperbolic Monge-Ampère system on a 5-manifold \( M_2 \) (see [10] and [11]). Then \( \mathcal{I}_2 \) is an \( r = 1 \) hyperbolic EDS on a 5-manifold as defined in Section 2.4. Let \( \mathcal{I}_2^{[1]} \) be the prolongation of \( \mathcal{I}_2 \) to the space of regular 2-dimensional integral elements. Then \( \mathcal{I}_2^{[1]} \) is an \( r = 3 \) hyperbolic system [10]. Let \( \hat{V} \) and \( \check{V} \) denote the singular Pfaffian systems of \( \mathcal{I}_2 \) and let \( \hat{V}_{2,[1]} \) and \( \check{V}_{2,[1]} \) denote the singular Pfaffian systems for \( \mathcal{I}_2^{[1]} \).

In this section we will assume that \( \mathcal{I}_2^{[1]} \) is Darboux integrable but not Monge integrable [11]. This is equivalent to the following rank hypotheses on the singular systems.

\[
\text{rank}(\hat{V}_2^{\infty}) = 1, \quad \text{rank}(\check{V}_2^{\infty}) = 1, \quad \text{and} \quad \text{rank}(\hat{V}_{2,[1]}^{\infty}) = 2, \quad \text{rank}(\check{V}_{2,[1]}^{\infty}) = 2.
\]

Equation (2.28) shows that for the Darboux integrable system \( \mathcal{I}_2^{[1]} \) the dimension of the Vessiot algebra \( \text{vess}(\mathcal{I}_2^{[1]}) \) is 3.

Let \( \mathcal{I}_1 \) be the standard Monge-Ampère hyperbolic system on the 5-manifold \( M_1 \) for the wave equation \( u_{xy} = 0 \). Then \( \mathcal{I}_1 \) is Darboux integrable (without prolongation) and the singular Pfaffian systems \( \hat{V}_1 \) and \( \check{V}_1 \) satisfy

\[
\text{rank}(\hat{V}_1^{\infty}) = 2 \quad \text{and} \quad \text{rank}(\check{V}_1^{\infty}) = 2.
\]

Equations (2.28) and (4.2) show that the Vessiot algebra for the Darboux integrable system \( \mathcal{I}_1 \) (and hence the wave equation) has dimension 1.

Suppose that \( \mathcal{B} \) is a Bäcklund transformation relating \( \mathcal{I}_1 \) and \( \mathcal{I}_2 \), that is,

\[
\begin{align*}
(\mathcal{B}, N) & \quad \text{p}_1 \quad \text{p}_2 \\
(\mathcal{I}_1, M_1) & \quad \text{p}_1 \quad (\mathcal{I}_2, M_2)
\end{align*}
\]

where the fibres of \( \text{p}_1 \) and \( \text{p}_2 \) each have dimension 1. Our goal in this section is to prove that \( \mathcal{B} \) is Darboux integrable and that the Vessiot algebra \( \text{vess}(\mathcal{B}) \) can be identified with a subalgebra of the Vessiot algebra \( \text{vess}(\mathcal{I}_2^{[1]}) \) of the Darboux integrable system \( \mathcal{I}_2^{[1]} \).

The existence of the Bäcklund transformation (4.3) implies that \( \mathcal{B} \) has the following properties.

**Theorem 4.1.** Let \( (\mathcal{B}, N) \) be a (local) Bäcklund transformation, with 1-dimensional fibers, between a hyperbolic Monge-Ampère system \( (\mathcal{I}_2, M_2) \) satisfying (4.1) and the Monge-Ampère system \( (\mathcal{I}_1, M_1) \) satisfying (4.2).

[i] The differential system \( \mathcal{B} \) is an \( s = 2 \) hyperbolic system.

Let \( \hat{W} \) and \( \check{W} \) be the corresponding singular Pfaffian systems for \( \mathcal{B} \).

[ii] The space of Darboux invariants for \( \mathcal{B} \) satisfies \( \hat{W}^{\infty} = \text{p}_1^{\ast}(\hat{V}_1^{\infty}), \check{W}^{\infty} = \text{p}_1^{\ast}(\check{V}_1^{\infty}) \).

[iii] The differential system \( \mathcal{B} \) is Darboux integrable.

[iv] The Vessiot algebra \( \text{vess}(\mathcal{B}) \) is 2-dimensional.
Proof of Theorem 4.2. Part [i]: It follows immediately from part [i] in Theorem 5.1 of [5] that an integral extension with 1-dimensional fibre of a decomposable system of type [2, 2] on a 5-manifold (or an \( r = 1 \) hyperbolic) is again a type [2,2] system on a 6-manifold and hence an \( r = 2 \) hyperbolic system. This conclusion applies whether we view \( B \) as an integrable extension of \( I_1 \) or \( I_2 \).

Part [ii]: Let \( \hat{W} \) and \( \check{W} \) be the singular systems of \( B \). Since these are unique up to being interchanged and since this uniqueness property also holds for both \( \hat{\mathcal{V}}_a \) and \( \check{\mathcal{V}}_a \) we may assume that parts [i] and [ii] of Theorem 5.1 in [5] hold simultaneously. In particular this implies \( \pi^a_{\hat{\mathcal{V}}_a} \subset \hat{W} \) and \( \pi^a_{\check{\mathcal{V}}_a} \subset \check{W} \). If we apply equation \( IB \) in Section 2.1 from [5] to the integrable extensions \( \pi_a : \hat{W} \to \hat{\mathcal{V}}_a \) and \( \pi_a : \check{W} \to \check{\mathcal{V}}_a \), then \( \pi_a^1 \) and \( \pi_a^\infty \) are \( 3 \)-dimensional hyperbolic systems which are the prolongations of \( I_1 \). Let \( \hat{\mathcal{V}}_a \) be the singular systems of \( I_1 \). By the hypothesis in (4.1) we have that \( \hat{\mathcal{V}}_a \) is Darboux integrable.\( \hat{\mathcal{V}}_a \) is an \( 3 \)-dimensional hyperbolic system that is Darboux integrable.

Part [iii]: The singular systems \( \hat{\mathcal{V}}_1 \) and \( \check{\mathcal{V}}_1 \) of \( I_1 \) satisfy \( \text{rank} \ (\hat{\mathcal{V}}_1) = 3 \) and \( \text{rank} \ (\check{\mathcal{V}}_1) = 3 \). If we apply equation (2.28) to the integrable extensions \( \pi_a : \hat{W} \to \hat{\mathcal{V}}_a \) and \( \pi_a : \check{W} \to \check{\mathcal{V}}_a \), then \( \pi_a^1 \) and \( \pi_a^\infty \) are \( 3 \)-dimensional hyperbolic systems which are the prolongations of \( I_1 \). Finally let \( \pi_a : M_a^{[1]} \to M_a \) and \( \pi : N^{[1]} \to N \) be the projection maps on the prolongation spaces.

By the hypothesis in [4,1] we have that \( I_2 \) is Darboux integrable, but we also have following properties of \( I_1 \) and \( I_2 \).

Lemma 4.2. The differential system \( I_1 \) is

[i] Darboux integrable.

[ii] The singular system \( \hat{\mathcal{V}}_1 \) and \( \check{\mathcal{V}}_1 \) of \( I_1 \) satisfy \( \text{rank} \ (\hat{\mathcal{V}}_1) = 3 \) and \( \text{rank} \ (\check{\mathcal{V}}_1) = 3 \).

[iii] \( \text{vess}(I_1) \) is 1-dimensional.

Lemma 4.3. The differential system \( B^{[1]} \) has the following properties.

[i] It is an \( r = 4 \) hyperbolic system that is Darboux integrable.

[ii] The singular systems \( \hat{\mathcal{V}}_1 \) and \( \check{\mathcal{V}}_1 \) of \( B^{[1]} \) satisfy \( \text{rank} \ (\hat{\mathcal{V}}_1) = 3 \) and \( \text{rank} \ (\check{\mathcal{V}}_1) = 3 \).

[iii] The Vessiot algebra \( \text{vess}(B^{[1]}) \) is 2-dimensional and is isomorphic (as an abstract Lie algebra) to \( \text{vess}(B) \).
These properties of $B^{[1]}$ follow by the same arguments used to prove Theorem 4.1, or by utilizing Theorem 7.5 in [5].

We have the following theorem which summarizes the effect of prolongation.

**Theorem 4.4.** There exists unique projection maps $p_1^{[1]} : N^{[1]} \to M_a^{[1]}$ giving rise to a commutative diagram

\[
\begin{array}{ccc}
(I_1^{[1]}, M_1^{[1]}) & \overset{\pi_1}{\longrightarrow} & (I_2^{[1]}, M_2^{[1]}) \\
\downarrow p_1 & & \downarrow p_2 \\
(I_1, M_1) & \overset{\pi}{\longrightarrow} & (I_2, M_2)
\end{array}
\]

of EDS where the top portion

\[
\begin{array}{ccc}
(I_1^{[1]}, M_1^{[1]}) & \overset{\pi_1}{\longrightarrow} & (I_2^{[1]}, M_2^{[1]}) \\
\downarrow p_1 & & \downarrow p_2 \\
(I_1^{[1]}, M_1) & \overset{\pi}{\longrightarrow} & (I_2^{[1]}, M_2)
\end{array}
\]

is a Bäcklund transformation with 1-dimensional fibres between the Darboux integrable systems $I_1^{[1]}$ and $I_2^{[1]}$.

**Proof.** The entire theorem and diagram (4.5) is a direct consequence of property IE [iv] in Section 2.1 of [5].

Note that as in Theorem 4.1 we have $\hat{W}_{\infty}^{[1]} = p_1^{[1]}(\hat{V}_{\infty}^{[1]}(I_1^{[1]}))$ and $\tilde{W}_{\infty}^{[1]} = p_1^{[1]}(\tilde{V}_{\infty}^{[1]}(I_1^{[1]}))$.

With these facts in mind our goal for this section is easily obtained.

**Theorem 4.5.** The integrable extension $p_2^{[1]} : (B^{[1]}, N^{[1]}) \to (I_2^{[1]}, M_2^{[1]})$ in diagram (4.5) is maximally compatible.

**Proof of Theorem 4.5.** First the hypothesis that $I_2^{[1]}$ is Darboux integrable but not Monge integrable means that the rank conditions in equation (4.1) are satisfied. Part [ii] in Lemma 4.3 gives the equalities

\[
\text{rank}(\hat{W}_{\infty}^{[1]}) = 3 = \text{rank}(\hat{V}_{\infty}^{[1]}(I_2^{[1]})) + 1 \quad \text{and} \quad \text{rank}(\tilde{W}_{\infty}^{[1]}) = 3 = \text{rank}(\tilde{V}_{\infty}^{[1]}(I_2^{[1]})) + 1.
\]

Therefore by Theorem 2.14, $p_2^{[1]} : B^{[1]} \to I_2^{[1]}$ is maximally compatible.

We emphasize that Theorem 4.5 is particular to the extension $p_2^{[1]} : B^{[1]} \to I_2^{[1]}$ and that maximal compatibility does not hold for $B^{[1]}$ as an extension of the wave equation $I_1^{[1]}$. This follows immediately by [ii] in Lemma 4.2 and [ii] in Lemma 4.3 which shows that equation (2.30) does not hold (with $\text{rank}(J) = 1$) for the integrable extension $B^{[1]}$ of $I_1^{[1]}$.

Our final result is a corollary of Theorem 4.5.
Corollary 4.6. There is a Lie algebra monomorphism from the algebra $\text{vess} (\mathcal{B})$ to the algebra $\text{vess} (\mathcal{I}_2^{[1]})$ and the Bäcklund transformation $\mathcal{B}^{[1]}$ between $\mathcal{I}_1^{[1]}$ and $\mathcal{I}_2^{[1]}$ can be constructed locally as a group quotient (using the monomorphism above) in accordance with Theorem B (or Theorem 8.2 in [5]).

This corollary implies that $\mathcal{B}$ can be constructed in a similar manner by deprolongation [3]. Our final result is at odds with part 2 of Theorem 1 in [11].

Theorem 4.7. Let $\mathcal{I}_2$ be a hyperbolic Monge-Ampère system satisfying (4.1). If the Vessiot algebra $\text{vess} (\mathcal{I}_2^{[1]})$ is $\mathfrak{so}(3)$, then $\mathcal{I}_2$ is not locally Bäcklund equivalent (with a Bäcklund transformation having 1-dimensional fibers) to the wave equation. The Vessiot algebra for the equation

$$u_{xy} = \frac{\sqrt{1-u_x^2} \sqrt{1-u_y^2}}{\sin u}$$

is $\mathfrak{so}(3)$ and therefore (4.6) is not Bäcklund equivalent (in the sense of (4.3)) to the wave equation.

Proof of Theorem 4.7. The Lie algebra $\mathfrak{so}(3)$ has no two dimensional subalgebra. Therefore by Corollary 4.6 no Bäcklund transformation $\mathcal{B}$ of $\mathcal{I}_2$ with one-dimensional fibre exists.

To finish the proof of the theorem we now show equation (4.6) has $\mathfrak{so}(3)$ as its Vessiot algebra. Let $(x, y, u, u_x, u_y, u_{xx}, u_{yy})$ be the standard jet coordinates on the open set $M^{[1]} \subset \mathbb{R}^7$, where $|u_x| < 1$ and $|u_y| < 1$. The Darboux invariants for equation (4.6) are

$$\xi = \frac{u_{xx}}{\sqrt{1-u_x^2}} \cot u, \quad \hat{\xi} = \frac{u_{yy}}{\sqrt{1-u_y^2}} \cot u.$$ (4.7)

In terms of coordinates $(x, y, u, \alpha = \arcsin u_x, \beta = \arcsin u_y, \xi, \hat{\xi})$, where $-\pi/2 < \alpha, \beta < \pi/2$ the differential system $\mathcal{I}_1^{[1]}$ is the Pfaffian system given by the 1-forms

$$\theta^1 = \hat{\xi} dx - (\hat{\xi} \cos u - \sin u \cos \beta) dy - \alpha \cos u \, d\beta$$

$$\theta^2 = (\xi \sin u \sin \alpha + \sin \alpha \cos u \cos \beta - \cos \alpha \sin \beta) dy + \cos \alpha \, du - \sin \alpha \cos u \, d\beta$$

$$\theta^3 = -dx - (\sin \beta \sin \alpha + \xi \cos \alpha \sin u + \cos \alpha \cos u \cos \beta) dy + \sin \alpha \, du + \cos \alpha \sin u \, d\beta.$$ (4.8)

The structure equations for the co-frame $(\theta^1, \theta^2, \theta^3, dx, d\xi, dy, d\hat{\xi})$ are

$$d\theta^1 = -\theta^2 \wedge \theta^3 - \theta^2 \wedge dx - dx \wedge d\xi + \cos u \, dy \wedge d\xi$$

$$d\theta^2 = \theta^1 \wedge \theta^3 + \theta^1 \wedge dx + \hat{\xi} \theta^3 \wedge dx - \sin \alpha \cos u \, dy \wedge d\xi$$

$$d\theta^3 = -\theta^1 \wedge \theta^2 - \hat{\xi} \theta^2 \wedge dx + \cos \alpha \sin u \, dy \wedge d\xi.$$ and one sees by inspection that this is a 4-adapted coframe (see [5] or [6]). The structure equations therefore imply that the Vessiot algebra for (4.6) is $\mathfrak{so}(3)$.
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