Communication Using Eigenvalues of Higher Multiplicity of the Nonlinear Fourier Transform

Javier García

Abstract—A generalized Nonlinear Fourier Transform (GNFT), which includes eigenvalues of higher multiplicity, is considered for information transmission over fiber optic channels. Numerical algorithms are developed to compute the direct and inverse GNFTs. For closely-spaced eigenvalues, examples suggest that the GNFT is more robust than the NFT to the practical impairments of truncation, discretization, attenuation and noise. Communication using a soliton with one double eigenvalue is numerically demonstrated, and its information rates are compared to solitons with one and two simple eigenvalues.

Index Terms—Inverse Scattering Transform, Nonlinear Fourier Transform, optical fiber, higher multiplicity eigenvalues, spectral efficiency

I. INTRODUCTION

Current optical transmission systems exhibit a peak in the achievable rate due to the Kerr nonlinearity of the Nonlinear Schrödinger Equation (NLSE) [1]. Several techniques have been proposed to attempt to overcome this limit, of which the Inverse Scattering Transform (IST) [2], or the Nonlinear Fourier Transform (NFT) [3], has attracted considerable attention, see [4] for an overview of the advances and perspectives of the NFT for optical communications.

Information transmission using the NFT has been demonstrated both numerically and experimentally in several works, such as [5], [6], [7]. For purely discrete spectrum modulation, the spectral efficiencies obtained so far are not very high [8]. In this paper, eigenvalues of higher multiplicity in the discrete spectrum are considered for communication. The theory for these eigenvalues has been developed in [9], [10], but its application to communications have to the best of our knowledge not been explored yet. We develop a generalized NFT (GNFT) approach to communications. The GNFT applies to a larger class of signals than the NFT, and thereby provides additional degrees of freedom that might help to improve communications systems. Our simulations also show that our generalized NFT (GNFT) processing seems to be more robust than NFT for signals with closely-spaced simple eigenvalues, even if they do not perfectly coincide.

The paper is organized as follows. In Section II we introduce the NLSE model. Section III briefly describes the NFT. In Section IV we explain the theory of higher multiplicity eigenvalues from [9], [10], and we prove some properties of the GNFT. In Section V we show how to compute the direct and inverse GNFT. Section VI evaluates the effect of practical impairments for closely-spaced eigenvalues. Section VII numerically demonstrates information transmission using the GNFT, and Section VIII concludes the paper.

Notation: the subscripts $t$, $z$, and $\lambda$ (and only these) denote partial derivatives with respect to the corresponding variable, e.g., $a_\lambda$ denotes $\partial a/\partial \lambda$. Repeated subscripts and parenthesis-sized superscripts denote higher-order derivatives, e.g., $a_{\lambda\lambda} = a^{(2)} = \partial^2 a/\partial \lambda^2$ and $a^{(\ell)} = \partial^\ell a/\partial \lambda^\ell$. In the latter case, the derivative is taken with respect to $\lambda$.

II. SYSTEM MODEL

Assuming perfect attenuation compensation, the slowly varying component $Q(Z,T)$ of an electrical field propagating along an optical fiber obeys the NLSE [11] Eq. (2.3.46):

$$\frac{\partial}{\partial Z} Q(Z,T) = -j \frac{\beta_2}{2} \frac{\partial^2}{\partial T^2} Q(Z,T) + j\gamma |Q(Z,T)|^2 Q(Z,T) + N(Z,T)$$

(1)

where $Z$ is distance, $T$ is time, $\beta_2$ is the group velocity dispersion (GVD) parameter, and $\gamma$ is the nonlinear coefficient. The distributed noise $N(Z,T)$ satisfies

$$\int_Z^Z N(Z',T) dZ' = \sqrt{N_{\text{ASE}} W(Z,T)}$$

(2)

where $N_{\text{ASE}}$ is the noise spectral density. Note that, unlike [11], we do not include the distance in the definition of $N_{\text{ASE}}$. The Wiener process $W(Z,T)$ may be defined as

$$W(Z,T) = \lim_{K \to \infty} \frac{1}{\sqrt{K}} \sum_{k=1}^{KZ} W_k(T)$$

(3)

where the $W_k(T)$ are independent and identically distributed (i.i.d.) circularly symmetric complex Gaussian processes with zero mean, bandwidth $B$, and autocorrelation

$$E[W_k(T)W_k'(T')] = B \text{sinc} (B(T-T'))$$

(4)

where $\text{sinc}(x) \equiv \sin(\pi x)/(\pi x)$.

III. THE NONLINEAR FOURIER TRANSFORM

In this section, we briefly introduce the steps involved in the NFT. For more detail, the reader is referred to [3].

By applying the following change of variables:

$$T = T_0 t, \quad Z = z \frac{T_0^2}{|\beta_2|} \quad Q(Z,T) = \frac{1}{T_0} \sqrt{\frac{|\beta_2|}{\gamma}} q(z,t)$$

(5)

the NLSE [11], ignoring noise, is normalized to

$$q(z,t) = -j \text{sign}(\beta_2) q_0(z,t) + j 2 |q(z,t)|^2 q(z,t)$$

(6)

and we choose $\beta_2 < 0$ to focus on the case of anomalous GVD [11] p. 131]. The parameter $T_0$ can be freely chosen.

Date of current version October 16, 2018. J. García is with the Institute for Communications Engineering (LNT), Technical University of Munich, Munich 80333, Germany (e-mail: javier.garcia@tum.de). His work was supported by the German Research Foundation under Grant KR 3517/8-1.
The NFT is based on the existence of a Lax pair \((L,M)\) of operators that satisfies
\[
L_z = ML - LM.
\]
As shown in [2] Section 1.4, the eigenvalues \(\lambda\) of \(L\) are invariant in \(z\). For the NLSE, the eigenvectors \(v\) of \(L\) satisfy
\[
v_z = M v
\]
\[
v_k = \left(\begin{array}{cc}
-j\lambda & q \\
-q^* & j\lambda
\end{array}\right) v
\]
where
\[
L(z,t) = j \left(\begin{array}{cc}
\frac{\partial}{\partial t} & -q \\
-q^* & \frac{\partial}{\partial t}
\end{array}\right)
\]
\[
M(z,t,\lambda) = \left(\begin{array}{cc}
2j\lambda^2 - j|q|^2 & -2\lambda q - jq_t \\
2q^* - jq^* & -2j\lambda^2 + j|q|^2
\end{array}\right).
\]
The NFT is calculated by solving the Zakharov-Shabat system [9]. We will often drop the dependence on \(z\) to simplify notation. A solution \(v^2(t,\lambda)\), bounded in the upper complex half plane \((\lambda \in \mathbb{C}^+)\), is obtained using the boundary condition
\[
v^2(t,\lambda) \rightarrow \left(\begin{array}{c}
1 \\
0
\end{array}\right) e^{-j\lambda t}, \quad t \rightarrow -\infty.
\]
The spectral functions \(a(\lambda)\) and \(b(\lambda)\) are obtained as
\[
a(\lambda) = \lim_{t \rightarrow \infty} v^2(t,\lambda)e^{j\lambda t}.
\]
\[
b(\lambda) = \lim_{t \rightarrow \infty} v^2(t,\lambda)e^{-j\lambda t}.
\]
The NFT of the signal \(q(z,t)\) is made up of two spectra:
- the continuous spectrum \(Q_c(\lambda) = \frac{b(\lambda)}{a(\lambda)}\), for \(\lambda \in \mathbb{R}\);
- the discrete spectrum \(Q_d(\lambda_k) = \frac{b(\lambda_k)}{a(\lambda_k)}\), for the \(K\) eigenvalues \(\{\lambda_k \in \mathbb{C}^+: a(\lambda_k) = 0\}\).

The usefulness of the NFT lies in the fact that, given a signal \(q(z,t)\) propagating according to the noise-free, lossless NLSE [6], its NFT evolves in \(z\) according to the following multiplicative relations:
\[
Q_c(z,\lambda) = Q_c(0,\lambda)e^{4j\lambda^2 t} \quad (14a)
\]
\[
\lambda_k(z) = \lambda_k(0) \quad (14b)
\]
\[
Q_d(z,\lambda_k) = Q_d(0,\lambda_k)e^{4j\lambda_k^2 t} \quad (14c)
\]

IV. EIGENVALUES OF HIGHER MULTIPLICITY

If \(\lambda_k\) is a multiple zero of \(a(\lambda)\), then \(a(\lambda_k) = 0\), and the above definition of the discrete spectrum is not valid. To the best of our knowledge, all the work on NFT-based communication assumes that all zeros of \(a(\lambda)\) are simple, i.e., that all eigenvalues \(\lambda_k\) have multiplicity 1. There has been, however, some work [9], [10] on the mathematical theory of higher multiplicity eigenvalues.

If the multiplicity of the eigenvalue \(\lambda_k\) is \(L_k\), we need \(L_k\) constants \(Q_{k0}, \ldots, Q_{k,(L_k-1)}\) to determine the discrete spectrum. In [10], these norming constants are defined from the coefficients \(r_{k,\ell}\) of the principal part of the Laurent series of \(b(\lambda)/a(\lambda)\) around \(\lambda_k\):
\[
\frac{b(\lambda)}{a(\lambda)} = \frac{r_{k,L_k-1}}{(\lambda - \lambda_k)^{L_k}} + \cdots + \frac{r_{k,0}}{(\lambda - \lambda_k)} + \mathcal{O}(1).
\]
The norming constants \(Q_{k,\ell}\) can be calculated as
\[
Q_{k,\ell} = \frac{j^\ell}{(L_k - \ell - 1)!} \lim_{\lambda \rightarrow \lambda_k} \frac{d^{L_k-\ell-1}}{d\lambda^{L_k-\ell-1}} \left[ (\lambda - \lambda_k)^L \frac{b(\lambda)}{a(\lambda)} \right].
\]
The generalization of the distance evolution equation [14c] to the case \(L_k \geq 1\) is given by [10] Eq. (4.9):
\[
[Q_{k,(L_k-1)}(z) \ldots Q_{k0}(z)] = [Q_{k,(L_k-1)}(0) \ldots Q_{k0}(0)] e^{-4j\Lambda k z} \quad (17)
\]
for all \(k \in \{1,\ldots, K\}\), where
\[
\Lambda_k = \left(\begin{array}{cc}
-j\lambda_k & -1 \\
0 & -j\lambda_k
\end{array}\right) \quad (18)
\]
We write the GNFT as
\[
\text{GNFT} \{q(t)\} = (Q_c(\lambda), \{\lambda_k\}, \{Q_{k,\ell}\}).
\]

A. Properties of the GNFT

We prove the following properties in Appendix A
1) Phase shift:
\[
\text{GNFT} \{q(t)e^{j\phi_0}\} = (Q_c(\lambda)e^{-j\phi_0}, \{\lambda_k\}, \{Q_{k,\ell}e^{-j\phi_0}\}).
\]
2) Time shift: if \(q'(t) = q(t - t_0)\) then
\[
\text{GNFT} \{q'(t)\} = (Q_c'(\lambda), \{\lambda'_k\}, \{Q'_{k,\ell}\}) \quad (21)
\]
satisfies
\[
Q'_c(\lambda) = Q_c(\lambda)e^{-2j\lambda t_0} \quad (22a)
\]
\[
\lambda'_k = \lambda_k \quad (22b)
\]
\[
[Q'_c(L_k-1) \ldots Q'_{k0}] = [Q_{k,(L_k-1)} \ldots Q_{k0}] e^{2\Lambda_k t_0} \quad (22c)
\]
3) Frequency shift:
\[
\text{GNFT} \{q(t)e^{-2j\omega t}\} = (Q_c(\lambda - \omega_0), \{\lambda_k + \omega_0\}, \{Q_{k,\ell}\}).
\]
4) Time dilation: for \(T > 0\)
\[
\text{GNFT} \left\{ \frac{1}{T} q \left( \frac{t}{T} \right) \right\} = \left( Q_c(T\lambda), \left\{ \frac{\lambda_k}{T} \right\}, \left\{ Q_{k,\ell} \frac{T}{T+1} \right\} \right).
\]
5) Parseval’s theorem:
\[
\int_{-\infty}^{\infty} |q(t)|^2 \, dt = \frac{1}{\pi} \int_{-\infty}^{\infty} \log \left( 1 + |Q_c(\lambda)|^2 \right) \, d\lambda
\]
\[+ \sum_{k=0}^{K} \sum_{\lambda \in \{\lambda_k\}} L_k \cdot \lambda_k. \quad (25)
\]

V. NUMERICAL COMPUTATION OF THE (I)GNFT

We extend existing numerical algorithms that compute the (I)NFT to include multiple eigenvalues.
A. Direct GNFT

Most algorithms that compute the direct NFT discretize the Zakharov-Shabat system (9) to find $a(\lambda)$ and $b(\lambda)$ from (13). Let $u = (u_1, u_2)^T$, where $u_1(t, \lambda) = v_1^2(t, \lambda)e^{j\lambda t}$ and $u_2(t, \lambda) = v_2^2(t, \lambda)e^{-j\lambda t}$. Then from (9) and (13) we have

$$
\begin{align*}
  u_1(t, \lambda) &= \begin{pmatrix} q(t)e^{2j\lambda t} \\ 0 \end{pmatrix}u(t, \lambda) \\
  (a(\lambda) - b(\lambda)) &= \lim_{t \to \infty} \frac{u_1(t, \lambda)}{u_2(t, \lambda)}.
\end{align*}
$$

(26)

(27)

To compute the GNFT of $q(t)$, we discretize the time axis for $t \in [t_1, t_2]$. Let $t_n = t_1 + n\epsilon$, $q_n = q(t_n)$, where $n \in \{0, \ldots, N-1\}$, $N$ is the number of samples, and $\epsilon = (t_2 - t_1)/(N-1)$ is the step size. Similarly, let $u[n] = u(t_1 + n\epsilon, \lambda)$. Starting at $u[0] = (1, 0)^T$ (see (12)), the following update step is applied iteratively:

$$
u[n + 1] = A[n]u[n], \quad n \in \{0, \ldots, N-2\}
$$

and we have $a(\lambda) = u_1[N-1]$ and $b(\lambda) = u_2[N-1]$. The kernel $A[n]$ depends on the discretization algorithm. We consider the trapezoidal kernel proposed in (12):

$$
A[n] = \begin{pmatrix} \cos \left( \frac{q_n}{\epsilon} \right) & \sin \left( \frac{q_n}{\epsilon} \right) e^{2j\lambda_n} \\ -\sin \left( \frac{q_n}{\epsilon} \right) e^{-j\lambda_n} & \cos \left( \frac{q_n}{\epsilon} \right) \end{pmatrix}
$$

(28)

where $\theta_n = \arg(q_n)$. However, the following analysis is valid for any kernel $A[n]$. To obtain the norming constants $Q_{k\ell}$, we need to calculate higher order $\lambda$-derivatives of $a(\lambda)$ and $b(\lambda)$. We obtain bounds on the order of the required derivatives.

**Lemma 1.** The value of $q_{k\ell}$ in (16) depends on $\lambda_k$ only through $a^{(m)}(\lambda_k)$ for $m \in \{L_k, \ldots, 2L_k-\ell-1\}$ and $b^{(m)}(\lambda_k)$ for $n \in \{0, \ldots, L_k-\ell-1\}$.

**Proof.** See Appendix [3] □

For an eigenvalue of multiplicity $L_k$, we compute the first $2L_k - 1$ derivatives of $u[N-1]$ by setting the following additional initial conditions and update steps

$$
\begin{align*}
u^{(m)}[0] &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad m \in \{1, \ldots, 2L_k-1\} \\
u^{(m)}[n + 1] &= \sum_{r=0}^{m} \begin{pmatrix} m \\ r \end{pmatrix} A^{(r)}[n]u^{(m-r)}[n]
\end{align*}
$$

(30a)

(30b)

where $A^{(r)}[n]$, the $r$-th order $\lambda$-derivative of $A[n]$, is obtained in closed form. Once we have the required values of $a, b$ and their derivatives, we use (16) to compute the norming constants. In (16), the derivative is evaluated in closed form, and then L’Hôpital’s rule is applied repetitively to obtain an expression for $Q_{k\ell}$ that depends only on nonzero derivatives of $a$. See [61]-[63] for details. For $L_k = 2$, this gives

$$
\begin{align*}
  Q_{k1} &= \frac{2b(\lambda_k)}{a_{\lambda\lambda}(\lambda_k)} \\
  Q_{k0} &= \frac{2b(\lambda_k)}{a_{\lambda\lambda}(\lambda_k)} - \frac{2b(\lambda_k)a_{\lambda\lambda\lambda}(\lambda_k)}{a_{\lambda\lambda}(\lambda_k)^2}.
\end{align*}
$$

(31a)

(31b)

**Forward-Backward Method:**

This technique was proposed in [12] to improve numerical stability. We write (28) as

$$
\begin{align*}
  (a(\lambda), b(\lambda)) &= A[N-1] \cdots A[1]A[0] \begin{pmatrix} 1 \\ 0 \end{pmatrix} = RL \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\end{align*}
$$

(32)

where $R = A[N-1] \cdots A[n_0]$ and $L = A[n_0-1] \cdots A[0]$, and $n_0$ is chosen according to some criterion to minimize the numerical error. The iterative procedure (28) is run forward up to $n_0 - 1$ to obtain

$$
\begin{align*}
  \left( \frac{l_1}{l_2} \right) &= L \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} L_{11} \\ L_{21} \end{pmatrix}
\end{align*}
$$

(33)

and backward from $r[N-1] = (0, 1)^T$ down to $r[n_0 - 1]$:

$$
\begin{align*}
  \left( \frac{r_1}{r_2} \right) &= R^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -R_{12} \\ R_{11} \end{pmatrix}.
\end{align*}
$$

(34)

The kernel $A[n]^{-1}$ is used to compute (34) for the trapezoidal case this amounts to replacing $\epsilon$ with $-\epsilon$ in (29). Note that (34) is valid only for kernels with unit determinant.

Using (30), we obtain $r_1, r_2, l_1, l_2$, and their derivatives up to order $2L_k - 1$. From (32) we have

$$
\begin{align*}
a(\lambda) &= R_{11} L_{11} + R_{12} L_{21}
\end{align*}
$$

(35)

and we compute

$$
\begin{align*}
a^{(\ell)}(\lambda_k) &= \sum_{m=0}^{\ell} \left( \frac{\ell}{m} \right) \left( r_2^{(m)}(\ell - m) - r_1^{(m)}(\ell - m) \right).
\end{align*}
$$

(36)

To obtain $b^{(\ell)}(\lambda_k)$, note that

$$
\begin{align*}
b(\lambda_k) &= R_{21} L_{11} + R_{22} L_{21} \\
 &= \frac{R_{21}}{R_{11}} (R_{11} L_{11} + R_{12} L_{21}) + \frac{L_{21}}{R_{11}} \\
 &= \frac{R_{21}}{R_{11}} a(\lambda_k) + \frac{L_{21}}{R_{11}}
\end{align*}
$$

(37)

where we used $R_{22} = (1 + R_{12} R_{21})/R_{11}$. The $\ell$-th derivative of the left summand in (37) is 0 for $\ell \leq L_k - 1$, because $a^{(\ell)}(\lambda_k) = 0$ for $\ell \leq L_k - 1$. Therefore, we have

$$
\begin{align*}
b^{(\ell)}(\lambda_k) &= \frac{d^\ell}{d\lambda^\ell} \frac{L_{21}}{R_{11}} |_{\lambda=\lambda_k}
\end{align*}
$$

(38)

which can be written in closed form using (58) below. Equations (38) and (36), together with (16) or (31), let us compute the GNFT from the forward-backward method.

B. Inverse GNFT

The inverse GNFT can be computed by solving the generalized Gelfand-Levitan-Marchenko equation (GLME) [13]:

$$
\begin{align*}
K(t, y) &= \Omega^*(t + y) \\
+ \int_t^y dx \int_{-\infty}^{\infty} ds K(t, s)\Omega(s + x)^*\Omega(x + y) = 0.
\end{align*}
$$

(39)

The kernel $\Omega(y)$ is given by [9]:

$$
\begin{align*}
\Omega(y) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} Q_e(\lambda)e^{j\lambda y} d\lambda + \sum_{k=1}^{K} \sum_{\ell=0}^{L_k-1} \frac{Q_{k\ell}}{\ell!} e^{j\lambda_k\ell} y.
\end{align*}
$$

(40)
The inverse GNFT is then obtained as

\[ q(t) = -2K(t, t). \]  

(41)

The derivation of (39-41) is given in [2] and is based on expressing \( v^2(t) = v^2(-\infty) + \int_{-\infty}^{t} K(t, s) e^{-j\lambda s} \, ds \) and substituting in (40). A numerical procedure to solve (39) is given in [13, Section 4.2]: it suffices to replace \( F(y) \) by \( \Omega(y) \).

When there is no continuous spectrum, a closed-form expression is given in [9] for the generalized \( K \)-solitons:

\[ q(z, t) = -2b^H e^{-\Lambda^H t} (I + M(z, t)N(t))^{-1} e^{-\Lambda^H t + i\Lambda^H z} e^c \]  

(42)

where

\[ \Lambda = \begin{pmatrix} \Lambda_1 & 0 & \cdots & 0 \\ 0 & \Lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \Lambda_K \end{pmatrix} \]  

(43)

\[ b = (b^T_1 \cdots b^T_K)^T \quad b_k = (0 \cdots 1)^T \in \{0, 1\}^{L_k \times 1} \]  

\[ c = (c^T_1 \cdots c^T_K)^T \quad c_k = (Q^*_k(L_k - 1) \cdots Q^*_k 0)^T \]

\[ M(z, t) = \int_{-\infty}^{\infty} e^{-\Lambda^H s + i\Lambda^H z} \times c^H e^{-\Lambda^H s - i\Lambda^H z} \, ds \]  

(44)

\[ N(t) = \int_{-\infty}^{\infty} e^{-\Lambda^H t} b^H e^{-\Lambda^H t} \, dx. \]  

(45)

\( \Lambda_k \) is given by (18), and \( I \) is an identity matrix of size \( \sum L_k \).

The integrals (44) and (45) must be computed numerically.

C. Example: Double Soliton (DS)

From (42), a soliton with a second order eigenvalue at \( \lambda = \xi + j\eta \) and norming constants \( Q_{11} \) and \( Q_{10} \) is given by

\[ q(z, t) = \frac{h(z, t)}{f(z, t)} \]  

(46)

where

\[ h(z, t) = -j4\eta e^{-j\arg Q_{11}} e^{-2\xi t} e^{-j4(z^2 - \eta^2)} z \left\{ \begin{array}{l} e^{-X} \left[ -|Q_{11}|^2 (2\eta t + 8\eta (\xi + j\eta) z + 2) - \eta Q_{11} Q_{10} \right] \\
+ e^X \left[ |Q_{11}|^2 (2\eta t + 8\eta (\xi - j\eta) z + 2) + \eta Q_{11} Q_{10}^* \right] \end{array} \right. \]  

(47)

\[ f(z, t) = |Q_{11}|^2 \left[ \cosh (2X) + 1 \right] + 2 |Q_{10}\eta + Q_{11} (2\eta t + 8\eta (\xi + j\eta) z + 1)|^2 \]  

(48)

\[ X = 2\eta t + 8\eta\xi z - \log \frac{|Q_{11}|}{4\eta^2}. \]  

(49)

We refer to this soliton as a double soliton (DS). The evolution of the norming constants \( \| \) reduces to

\[ Q_{11}(z) = Q_{11}(0) e^{4j\lambda^2 z} \]  

(50a)

\[ Q_{10}(z) = (Q_{10}(0) + 8\lambda z Q_{11}(0)) e^{4j\lambda^2 z}. \]  

(50b)

Note from (46) that a DS does not exhibit periodic (breathing) behavior in \( z \). The monotonic growth of some norming constants with \( z \) suggests that, generally, solitons with higher multiplicity eigenvalues do not breathe.

The following expression for the center time of the DS was obtained empirically and seems to be valid based on our simulations, but we have not found a proof:

\[ \frac{\int_{-\infty}^{\infty} t |q(z, t)|^2}{\int_{-\infty}^{\infty} |q(z, t)|^2} = \frac{1}{2\eta} \log \left( \frac{|Q_{11}(z)|}{4\eta^2} \right). \]  

(51)

This definition of the center time has proven useful for the analysis of the propagation of trains of ordinary solitons [14].

VI. EFFECT OF IMPLEMENTATION LIMITATIONS

Eigenvalues of higher multiplicity require the pulse to have the exact shape given by, e.g., (42) or (46). Any deviation caused by practical limitations such as discretization, truncation, attenuation or noise splits an eigenvalue \( \lambda_k \) of multiplicity \( L_k \) into \( L_k \) very closely spaced eigenvalues \( \lambda_{k,\ell} \). However, the computation of the spectral amplitudes

\[ Q_d(\lambda_{k,\ell}) = \frac{b(\lambda_{k,\ell})}{a(\lambda_{k,\ell})} \]  

(52)

is unstable, because the denominator is close to 0. The norming constants \( Q_{k,\ell} \) of the GNFT seem to be more stable, even when the actual signal has two closely spaced eigenvalues instead of one eigenvalue of multiplicity 2. Furthermore, the search algorithm might not be able to distinguish eigenvalues that are too close. We next illustrate these observations.

A. Effect of pulse truncation

We simulated three pulses: a DS, a 2S and a 1S, with the parameters in Table I. The three pulses have the same energy, which forces the 1S to have different duration and bandwidth than the 2S and DS. Heuristic optimization was used to obtain a small time-bandwidth product (TBP) for all signals. The duration \( T_{0.999} \) of the time interval that contains 99.9% of the signal energy, as well as the bandwidth \( B_{0.999} \), are listed in Table I. Using a sampling time of \( T_s = 0.0058s \), we varied the truncation interval \( T \). For a fairer comparison, the results of the simulations are plotted as a function of the ratio \( T/T_{0.999} \). For reference, the ratio of truncated energy for the different truncation intervals is plotted in Fig.2. The eigenvalues of the generated pulse were found using a Newton-Raphson search algorithm [3]. The first-order spectral amplitudes \( Q_1 \) and \( Q_2 \) and the second-order norming constants \( Q_{11} \) and \( Q_{10} \) of the found eigenvalues were computed using FBT: forward-backward computation [12] with the trapezoidal kernel [29].

| Pulse | Parameters of the pulses used in Fig.2 | \( T_{0.999} \) | \( B_{0.999} \) |
|-------|--------------------------------------|-------------|-------------|
| DS    | \( \lambda_1 = 1.25j \) \( Q_{11} = 6.25 \) \( Q_{10} = 40.10 \) | 5.25        | 12.67       |
| 2S    | \( \lambda_2 = 1.5j \) \( Q_{2} = 20.96956 \) | 5.41        | 12.67       |
| 1S    | \( \lambda_1 = 2.5j \) \( Q_{1} = 5 \) | 1.31        | 23.75       |

Table I
Figure 1. First row: effect of truncation on (1a) imaginary part of eigenvalues, (1b) norming constants of a DS, a 2-soliton (2S) and a 1-soliton (1S). The NFT spectral amplitudes of the DS are shown in (1c). Second row: effect of sampling period.

The 2S (red dashed curves) performs better than the DS, probably because the higher order derivatives of $a(\lambda)$ in (31) are less stable. However, a comparison of the blue solid curves in the second and third columns of Fig. 1 shows that GNFT processing is better than NFT when the transmit signal has very closely spaced eigenvalues or higher multiplicity eigenvalues. Note that, for the remainder of the paper, we will denote $a_\lambda(\lambda_k)$ as $k\ell$ for a 1S or a 2S.

Fig. 1 (a) shows the relative error $\Delta\eta/\eta$ in the imaginary part of the eigenvalues. For the DS (blue solid curves), truncation splits the eigenvalue into two closely spaced eigenvalues $\lambda_1$ (unmarked curve) and $\lambda_2$ (marked curve), which move closer as $T$ increases. For $T/T_{0.999} \geq 1.357$ (vertical blue solid line), the search algorithm does not distinguish two eigenvalues anymore. Even when two are found, their spectral amplitudes (1c), obtained with NFT processing, are unstable (highly dependent on $T$), because the denominator $a_\lambda(\lambda_k)$ is close to 0. The norming constants in (1b), obtained using the GNFT, are much more stable.

The second column shows the NMSE of eigenvalues: NMSE $\eta$.

The third column shows the error in GNFT norming constants: $|\Delta Q_k|/|Q_k|$.

The fourth column shows the DS with NFT processing: $|Q_k|$.
of Figure 1 shows that the 1S (yellow dash-dotted curve) is more robust than the 2S and DS to all impairments. This comes at the cost of spectral efficiency, as the 1S offers fewer degrees of freedom for communication.

B. Effect of pulse discretization

For the second row of Fig. 1, we chose a large truncation interval $T = 10.5866$ and varied the sampling time $T_s$. Due to the different bandwidths of the pulses, we plot the results as a function of the product $T_s \cdot T_{0,999}$. Again, with enough resolution ($T_s B_{0,999} \leq 0.318$), the search algorithm does not find two distinct eigenvalues anymore. The norming constants are much more stable than the spectral amplitudes even when two distinct, closely spaced eigenvalues are found. The error in the norming constants $|\Delta Q_{kl}|/|Q_{kl}|$ for the DS is almost the same as for the 2S (see plot (2b)).

C. Effect of attenuation

For the third row of Fig. 1 we simulated the propagation from $z = 0$ to $z = 1$ of the three pulses with $T_s = 0.0058$ and $T = 10.5866$ along the noise-free NLSE channel with normalized attenuation coefficient $\alpha = 0.4646$ (corresponding to 10-km propagation and attenuation 0.2 dB/km with $\beta_2$ and $\gamma$ from Table I). The eigenvalue of the DS splits into two eigenvalues that separate due to attenuation. The attenuation affects the spectral amplitudes (3c) much more strongly than it affects the norming constants (3b). The eigenvalues of the 2S behave similarly to the results in 15.

D. Effect of noisy NLSE propagation

We simulated the propagation of the three pulses along a 4000-km lossless link with $T_s = 0.0771$ and $T = 48.43$ and the parameters in Table I. The spectral density of the distributed noise was $N_{ASE} = 6.4893 \cdot 10^{-24}$ Ws/m. The signal power was varied by changing the free parameter $T_0$. The figure of merit is the normalized mean square error:

$$\text{NMSE}_x = \frac{\mathbb{E}[|x - \mathbb{E}[x]|^2]}{\mathbb{E}[x]^2}$$

(53)

where $\mathbb{E}[.]$ denotes the expectation operator. Again, the instability of the spectral amplitudes is clear from the results in Fig I (4c).

Simulations with different values of the soliton parameters $\lambda_k$, $Q_{kl}$ yield similar curves to those in Fig I for the three pulses, though the performance comparison between the pulses changes. Only the shape of the attenuation curves (third row) seems to strongly depend on the initial parameters.

VII. INFORMATION TRANSMISSION USING THE GNFT

We simulated a communications system with the parameters in Table I. We compared DS, 2S and 1S with the same eigenvalues as the pulses in Section VI. The 1S uses multi-ring modulation on $Q_1 = Q_d(\lambda_1)$ with 32 rings and 128 phases per ring. The 2S has the two spectral amplitudes $Q_1 = Q_d(\lambda_1)$ and $Q_2 = Q_d(\lambda_2)$, while the DS has the two norming constants $Q_{11}$ and $Q_{10}$. Both the 2S and the DS have 4 rings and 16 phases per spectral amplitude. These parameters were heuristically optimized to obtain a small TBP for all the transmit signals and all positions in $z$. The ring amplitudes for the 1S are

$$|Q_0| \in \{0.088754 \cdot 1.6142^k : k \in \{0, \ldots, 31\}\}.$$  

(54)

The ring amplitudes for the 2S and DS are given in Table III. The phases are uniformly spaced in $[0, 2\pi]$, starting at 0 for $Q_0$, $Q_1$ and $Q_{11}$, and at $\pi/16$ for $Q_2$ and $Q_{10}$. The optimal criterion for choosing ring amplitudes is not known, but expressions such as (51) suggest that geometric progressions are better suited than arithmetic progressions.

The free parameter $T_0$ in (5) was used to obtain the desired powers. We used a sampling period of $T_s = 0.0771$ and a truncation interval of $T = 48.43$ in the simulations.

Lossless propagation according to (1) was simulated using the split-step Fourier method. In all systems, the transmitter used closed-form expressions to generate the solitons, and the receiver used FBT to obtain the norming constants. Equalization was performed by inverting (17). The mutual information of the transmitted and received symbols was measured and normalized by the TBP to obtain the spectral efficiency. In the 2S and DS systems, the joint mutual information $I(Q_1^{TX}, Q_2^{TX}, Q_1^{RX}, Q_2^{RX})$ was computed, where $Q_k^{TX}$ refers to the transmitted symbols and $Q_k^{RX}$ refers to the received and equalized symbols.

Figure 5 shows the spectral efficiency for the three systems. At their optimal power, the DS performs better than the 1S, but worse than the 2S. At this point, the DS has broadened in time at most by 14%, and the 2S by 11%. This small difference is not enough to account for the observed gap in spectral efficiency. The main reason for this gap is the lower stability of the DS: the higher order derivatives in (3) make the norming constants of the DS (especially $Q_{10}$) less stable than the spectral amplitudes of the 2S. The results of Section VI and Fig. 1 also support this view. However, Fig. 5 demonstrates that the generalized NFT with multiple zeros can be used to transmit information. Although the DS does not seem to offer any practical advantage with respect to the 2S, the use of an additional degree of freedom might bring improvements in systems with many eigenvalues, where close spacing is unavoidable.

VIII. CONCLUSION

Starting from the theory in [9], [10], we proved some properties of the GNFT that are useful for communications. We...
designed and implemented algorithms to compute the GNFT, and we numerically demonstrated information transmission using higher multiplicity eigenvalues. With this, we extend the class of signals that admit an NFT, providing additional degrees of freedom for NFT-based optical communications.

There are several directions for future work. Extending the Darboux algorithm to the IGNFT would speed its computation. More insight into the duration, bandwidth and robustness to the change of variable \( t \to t/T \) in \([22b]\) proves that \( a'(\lambda) = a(T \lambda) \) and \( b'(\lambda) = b(T \lambda) \). Using this in \([15]\) proves \([24]\).
and \( c(\lambda) = (\lambda - \lambda_k)^{L_k} b(\lambda) \). Note that \( a \) has a zero of order \( L_k \), and therefore \( a^{(m)}(\lambda_k) = 0 \) for \( m \in \{0, \ldots, L_k - 1\} \). To compute \( Q_{k\ell} \), we repeatedly apply L’Hôpital’s rule until the numerator and the denominator become nonzero in the limit:

\[
Q_{k\ell} = \frac{g^{(r)}(\lambda_k)}{[d^r a(\lambda) L_k^{-\ell} / d\lambda^r]_{\lambda = \lambda_k}}.
\]

The number \( r \) of times we need to differentiate is the order of the zero in the denominator:

\[
r = L_k (L_k - \ell). \tag{63}
\]

The summands in \( g^{(r)}(\lambda) \) are of the form

\[
g_s(\lambda) = K_s \frac{d^{(L_k - \ell)} L_k}{d(\lambda - \ell) L_k} e^{(L_k - \ell - m - 1) a L_k - \ell - |p| - 1} \prod_{i=1}^{m} (a^{(i)})^{p_i}, \tag{64}
\]

where \( s \) is an index, and \( K_s \) is a constant independent of \( \lambda \). If we apply the product rule to (64), any nonzero summand at \( \lambda = \lambda_k \) must differentiate the factor \( e^{(L_k - \ell - m - 1)} \) at least \( \ell + m + 1 \) times. Thus, the other factors are differentiated at most \( (L_k - \ell) L_k - \ell - m - 1 \) times. The derivative of

\[
a L_k - \ell - |p| - 1 \prod_{i=1}^{m} (a^{(i)})^{p_i} \tag{65}
\]

is a sum of terms of the same form. Each new term has the same amount \( \sum_i p_i \) of \( a \)-factors as the original (an \( a \)-factor here refers to \( a \) or one of its derivatives), and the number of differentiations \( \sum_i i p_i \) in the \( a \)-factors is increased by 1. We conclude that \( g_s(\lambda) \) is made up of summands that contain

- \( L_k - \ell - |p| - 1 + \sum_i p_i = L_k - \ell - 1 \) \( a \)-factors that have
- \( (L_k - \ell) L_k - \ell - m - 1 + \sum_i i p_i = (L_k - \ell) L_k - \ell - 1 \) differentiations.

All the \( a \)-factors of nonzero summands must be at least \( L_k \)-order derivatives. In the worst case, there are \( L_k - \ell - 2 \) \( a \)-factors with an \( L_k \)-th order derivative. The remaining \( a \)-factor must have a derivative of order \( [(L_k - \ell) L_k - \ell - 1] - [(L_k - \ell) L_k - \ell - 2] = 2L_k - \ell - 1 \).

The product of \( a \)-factors in (64) has a zero at \( \lambda_k \) of order

\[
L_a = (L_k - \ell - |p| - 1) L_k + \sum_i p_i (L_k - i) = (L_k - \ell - 1) L_k - m.
\]

This means that any nonzero summand after applying the product rule will differentiate \( e^{(L_k - \ell - m - 1)} \) at most

\[
(L_k - \ell) L_k - L_a = L_k + m
\]

times. This yields a term with \( c^{2(L_k - \ell - 1)} \). As \( c = (\lambda - \lambda_k)^{L_k} b \), the highest order derivative on \( b \) is \( b^{(L_k - \ell - 1)} \).

ACKNOWLEDGMENT

The author wishes to thank Prof. G. Kramer and B. Leible for useful comments and proofreading the paper.

REFERENCES

[1] R. J. Essiambre, G. Kramer, P. J. Winzer, G. J. Foschini, and B. Goebel, "Capacity limits of optical fiber networks," J. Lightw. Technol., vol. 28, no. 4, pp. 662–701, Feb 2010.
[2] M. Ablowitz and H. Segur, Solitons and the Inverse Scattering Transform. Society for Industrial and Applied Mathematics, 1981.
[3] M. I. Yousefi and F. R. Kschischang, “Information transmission using the Nonlinear Fourier Transform, Part I-III,” IEEE Trans. Inf. Theory, vol. 60, no. 7, pp. 4312–4369, July 2014.
[4] S. K. Turitsyn, J. E. Prilepsky, S. T. Le, S. Wahls, L. L. Frumin, M. Kamalian, and S. A. Derevyanko, “Nonlinear Fourier transform for optical data processing and transmission: advances and perspectives,” Optica, vol. 4, no. 3, pp. 307–322, Mar 2017.
[5] Z. Dong et al., “Nonlinear frequency division multiplexed transmissions based on NFT,” IEEE Photon. Technol. Lett., vol. 27, no. 15, pp. 1621–1623, Aug 2015.
[6] S. T. Le, H. Buelow, and V. Aref, “Demonstration of 64 x 0.5Gbaud nonlinear frequency division multiplexed transmission with 3QAM,” in 2013 Optical Fiber Commun. Conf. and Exhib. (OFC), March 2013.
[7] S. T. Le, I. D. Philips, J. E. Prilepsky, P. Harper, N. J. Doran, A. D. Ellis, and S. K. Turitsyn, “First experimental demonstration of nonlinear inverse synthesis transmission over transoceanic distances,” in 2016 Optical Fiber Commun. Conf. and Exhib. (OFC), March 2016.
[8] S. Hari, M. I. Yousefi, and F. R. Kschischang, “Multieigenvalue communication,” J. Lightw. Technol., vol. 34, no. 13, pp. 3110–3117, July 2016.
[9] T. Aktosun, F. Demontis, and C. van der Meer, “Exact solutions to the focusing Nonlinear Schrödinger Equation,” Inverse Problems, vol. 23, no. 5, p. 2171, 2007.
[10] T. N. B. Martin, “Generalized Inverse Scattering Transform for the Nonlinear Schrödinger Equation for bound states with higher multiplicities,” Electronic J. Differential Equations, vol. 2017, no. 179, pp. 1–15, July 2017.
[11] G. P. Agrawal, Nonlinear Fiber Optics, 4th ed. Academic Press, Oct 2012.
[12] V. Aref, “Control and detection of discrete spectral amplitudes in nonlinear Fourier spectrum,” ArXiv e-prints, May 2016.
[13] S. T. Le, J. E. Prilepsky, and S. K. Turitsyn, “Nonlinear inverse synthesis for high spectral efficiency transmission in optical fibers,” Opt. Express, vol. 22, no. 22, pp. 26720–26741, Nov 2014.
[14] J. E. Prilepsky, S. A. Derevyanko, and S. K. Turitsyn, “Lattice approach to the dynamics of phase-coded soliton trains,” J. of Physics A: Math. and Theoretical, vol. 45, no. 2, p. 025202, 2012.
[15] J. E. Prilepsky and S. A. Derevyanko, “Breakup of a multisoliton state of the linearly damped Nonlinear Schrödinger Equation,” Phys. Rev. E, vol. 75, p. 036616, Mar 2007.
[16] L. Arbogast, Du calcul des dérivarions, 1800.