Long Time Behavior of Stochastic NLS with a Small Multiplicative Noise

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Abstract: We prove the global space-time bound for the defocusing mass critical nonlinear Schrödinger equation on \( \mathbb{R}^3 \) perturbed by a small multiplicative noise. The associated scattering behavior is also obtained. In addition to techniques from Fan and Xu (Global well-posedness for the defocusing mass-critical stochastic nonlinear Schrödinger equation on \( \mathbb{R} \) at \( L^2 \) regularity, 2018) and Fan and Zhao (On long time behavior for stochastic nonlinear Schrödinger equations with a multiplicative noise, 2020), the main new ingredients are the decomposition of the solution tailored for the bootstrap argument in this problem, and the incorporation of local smoothing norms to close the argument. We also prove the global space-time Strichartz estimate for the linear stochastic equation. It is a toy model of our nonlinear problem, but the bound itself is new and of its own interest. Furthermore, the proof we give for the linear model is more direct, and also illustrates the proof strategy for the nonlinear problem.

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1. Introduction

1.1. The problem and statement of the main result. The aim of this article is to study the long-time behavior of the solution \( u \) to the equation (in \( \mathbb{R}^+ \times \mathbb{R}^3 \))

\[
i \partial_t u + \Delta u = |u|^{4/3} u + \varepsilon V u \circ \dot{B}, \quad u(0, \cdot) = u_0 \in L^\infty_w L^2_\omega(\mathbb{R}^3).
\]  

(1.1)

Here, \( V \in \mathcal{S}(\mathbb{R}^3) \) is a real-valued Schwartz function independent of time, \( \dot{B} \) is the derivative of the standard Brownian motion \( B \), and \( \varepsilon > 0 \) is a small parameter. Finally, \( \circ \) denotes the Stratonovich product which preserves the \( L^2 \)-norm of the solution \( u \). This is the stochastic perturbation of mass-critical defocusing nonlinear Schrödinger equation in \( d = 3 \).

We write (1.1) in its Itô form as

\[
i \partial_t u + \Delta u = |u|^{4/3} u - \frac{i\varepsilon^2}{2} V^2 u + \varepsilon V u \cdot \dot{B}, \quad u(0, \cdot) \in L^\infty_w L^2_\omega,
\]  

(1.2)

where now the product between \( u \) and \( dB_t/dt \) is Itô, and the extra term \(-\frac{i\varepsilon^2}{2} V^2 u\) is the Itô-Stratonovich correction, which keeps the \( L^2 \) norm conserved.

We will work with deterministic initial data \( u_0 \in L^2_\omega \), and the extension to random initial data in \( L^\infty_w L^2_\omega \times \mathbb{R}^3 \) is straightforward as long as the randomness in \( u_0 \) is independent of the Brownian motion. Our main theorem is the following.

**Theorem 1.1.** Suppose \( V \in \mathcal{S}(\mathbb{R}^3) \) is a real-valued, time-independent Schwartz function. Let \( \varepsilon \) be sufficiently small (depending on \( V \) and \( \|u_0\|_{L^2_\omega} \)). Let \( (\alpha, \beta) \) be an admissible pair in \( \mathbb{R}^3 \) in the sense of Definition 2.2.

Then there exists \( C = C(\alpha, \beta, \|u_0\|_{L^2_\omega}) \) such that the solution \( u \) to (1.2) satisfies the space-time bounds:

\[
\|u\|_{L^\infty_w L^{\alpha/2}_t L^\beta_\omega(\Omega \times \mathbb{R}^+ \times \mathbb{R}^3)} \leq C, \quad \alpha \geq \frac{7}{3},
\]  

(1.3)

and

\[
\|u\|_{L^\infty_w L^{\alpha}_t L^\beta_\omega(\Omega \times \mathbb{R}^+ \times \mathbb{R}^3)} \leq C, \quad \alpha > 2.
\]  

(1.4)

Furthermore, we have the scattering asymptotic in the sense that there exists \( u^+ \in L^\infty_w L^2_\omega \) such that

\[
\|u(t) - e^{it\Delta} u^+\|_{L^2_w L^2_\omega} \xrightarrow{t \to \infty} 0.
\]  

(1.5)

We also have \( u(t) - e^{it\Delta} u^+ \to 0 \) in \( L^2_\omega \) almost surely.

We first prove the space-time bound (1.3) for admissible pairs \( (\alpha, \beta) \) with \( \alpha \in [\frac{7}{3}, 4) \). The threshold \( \alpha \geq \frac{7}{3} \) arises from the modified nonlinear stability in Proposition 5.9. The threshold \( \alpha < 4 \) will be explained in Sect. 3. With the bound for \( \alpha \in [\frac{7}{3}, 4) \), interpolation with the \( L^\infty_w L^2_\omega \) bound (mass conservation) gives (1.3) for all \( \alpha \in [\frac{7}{3}, +\infty] \).

Next, we use (1.3) to prove the scattering behavior (1.5). Similar as [27] but different from the usual deterministic situation, scattering is not a direct consequence of the global space-time bound (1.3). Extra efforts are needed to obtain it.
Finally, the $L^2_tL^2_x$-type bounds we established in the scattering behavior enable us to establish (1.4). This completes the global space-time bound for all admissible pairs except the endpoint $(2, 6)$. The difference between (1.3) and (1.4) is a minor technical issue – for $\alpha \in (2, \frac{7}{3})$, the integrability exponent in probability is 2 instead of $\alpha$.

As a toy case of the above theorem, we also prove analogous bounds for the linear model, known as the stochastic Strichartz estimates. This is the following theorem.

**Theorem 1.2.** Let $u$ satisfy the linear equation (in Itô form)

$$i \partial_t u + \Delta u = \epsilon Vu \cdot \dot{B} - \frac{i \epsilon^2}{2} V^2 u, \quad u(0, \cdot) = u_0 \in L^2_x,$$

where $V \in S(\mathbb{R}^3)$ is a real-valued Schwartz function, and $\epsilon > 0$ is a sufficiently small parameter depending on $V$. Then one has the global space-time Strichartz estimate

$$\|u\|_{L^\alpha_tL^\beta_x(\Omega \times \mathbb{R}^+ \times \mathbb{R}^3)} \lesssim \|u_0\|_{L^2_x}$$

where $(\alpha, \beta)$ is any admissible pair for the free Schrödinger operator with $\alpha \in (2, +\infty]$.

**Remark 1.1.** The scattering behavior for the linear equation was proved in [24].

Note that the smallness of $\epsilon$ here depends on $V$ but not on $\|u_0\|_{L^2_x}$. Also, since (1.6) is linear, Theorem 1.2 extends to random initial data $u_0 \in L^\alpha_tL^2_x(\Omega \times \mathbb{R}^3)$ immediately, as long as the randomness is independent of the Brownian motion.

Similar as the nonlinear case, we prove Theorem 1.2 first for admissible pairs $(\alpha, \beta)$ with $\alpha \in (2, 4)$, and then interpolate with mass conservation to extend to $\alpha \in (2, +\infty]$. Although Theorem 1.2 is a toy case of Theorem 1.1, we will give a direct and relatively independent proof, which also illustrates the proof strategy of Theorem 1.1. Furthermore, the bound (1.7) is new and of its own interest. We will give more discussions in the next subsection.

**Remark 1.2.** For both linear and nonlinear models, we cover the whole range of admissible pairs in $\mathbb{R}^3$ except the endpoint $(2, 6)$. Besides its importance in the deterministic case, the pair $(2, 6)$ is also of particular interest in our stochastic problem (see Sect. 1.3).

**Remark 1.3.** Some key ingredients in this article are of perturbation nature, and hence it is not clear how to extend our results without smallness assumption on $\epsilon$. On the other hand, it is not hard to generalize the noise to infinite dimensional Wiener processes as long as there is enough spatial regularity. But since our main focus is the effect of the noise on the long-time behavior of the dynamics, we focus on the simplest possible case and consider one Brownian motion only.

1.2. Background and motivation. The main aim of this article is to investigate the effect of a multiplicative noise (without decay in time) in the long time behavior of a dispersive system.

The nonlinear Schrödinger equation is one of the most typical dispersive equations. The global well-posedness of the $d$-dimensional deterministic defocusing nonlinear Schrödinger equation

$$i \partial_t v + \Delta v = |v|^{p-1}v, \quad v(0, \cdot) \in L^2_{x}(\mathbb{R}^d)$$

(1.8)
with $L^2_x$ initial data has been an important topic of study in dispersive equations. The equation is called mass-subcritical when $p - 1 < \frac{4}{d}$, and mass-critical when $p - 1 = \frac{4}{d}$. The local well-posedness to (1.8) for $p \in [1, 1 + \frac{4}{d}]$ is based on Strichartz estimates and is well understood classically. When $p < 1 + \frac{4}{d}$, the local existence time depends on the initial data via its $L^2$-norm only, so global well-posedness of the solution is a direct consequence of the local well-posedness and $L^2$ conservation law.

When $p = 1 + \frac{4}{d}$ (the mass-critical case), the global existence of the solution to (1.8) with general $L^2$ initial data becomes much subtler. It has been outstanding for a long time, and was finally resolved by Dodson in a series of works [18,20,21]. Furthermore, the global space-time bound of the solution in terms of the $L^2$-norm of the initial data was established, and scattering of the solution comes as a consequence.

The stochastic version of (1.8) with a Stratonovich multiplicative noise is given by

$$i \partial_t u + \Delta u = |u|^{p-1}u + u \circ \dot{W}, \quad (1.9)$$

where the noise $\dot{W}$ is the time derivative of a Wiener process $W$, and is white in time and coloured in space. This is a natural model for stochastic perturbation of (1.8). We also refer to the introductions in [15,29] for the physical background of (1.9). The main aim is to understand the impact of randomness on the dispersion.

The model (1.9) has been studied under various spatial assumptions on $W$, most of which could roughly be formulated by

$$W(t, x) = \sum_k \lambda_k V_k(x) B_t^{(k)}, \quad (1.10)$$

where $B^{(k)}$ are independent standard Brownian motions, $V_k$ are sufficiently nice functions with estimate uniform in $k$, and $\lambda_k$ satisfies proper decay assumptions. To the best of our knowledge, de Bouard and Debussche [15] were the first to construct a global solution to (1.9) for $p < 1 + \frac{4}{d}$ (mass-subcritical case). Subsequent refinements and extensions (also in subcritical cases) include [1,6,7,16], etc.

**Remark 1.4.** We also remark that our results still hold if one considers the general noise (1.10) assuming certain summability of the $\{\lambda_k\}_k$ and smallness like in our main theorems. Also note that infinite dimensional noises are sometimes useful to match certain non-degeneracy conditions in the study of blow up problems (see [17]).

When $\{\lambda_k\}_k$ is $\ell^1$-summable and $\{V_k\}_k$ are Schwartz functions with uniformly bounded smooth norms, the generalization is almost straightforward by triangle inequality. From this perspective, it may be helpful to focus on noise of the simple form as in this article to make the presentation of techniques more illustrative and accessible (see also [22,35]).

We will explain in the Appendix how to systematically transfer the result in the current article (or those within similar frameworks in [25,27]) to more general infinite dimensional noises when $\{\lambda_k\}_k$ is only $\ell^2$ summable in certain sense.

To end this remark, we also point out that although results in this article as well as in [15,17,23,28,42] have varying assumptions on noise, all have assumed enough spatial regularity (or certain $\ell^2$-summability on $\{\lambda_k\}$). However, it is not clear what the optimal assumption on the noise should be. It is a very different and interesting open problem to study singular noise, in particular the space-time white noise where $\{V_k\}$ forms an orthonormal bases of $L^2(\mathbb{R}^d)$ and $\lambda_k \equiv 1$ for all $k$.

In [23], the first two authors of the current article established global well-posedness for the stochastic mass-critical case in dimension 1 ($p = 5$). The main ingredients
are careful perturbation analysis around the global well-posedness results of Dodson as well as a martingale type control. The arguments also work for other dimensions, at least when the nonlinearity is not too irregular. Zhang [42] independently constructed global solutions to the stochastic defocusing equations in both mass and energy critical regimes.

On the other hand, very little was known about the long time behavior of the solutions. To start, one may introduce a parameter $\gamma > 0$, and consider the case where the noise $W$ in (1.9) is replaced by the time-decay version

$$W(t, x) = \sum_k \lambda_k V_k(x) \cdot \frac{B_k(t)}{\langle t \rangle^{\gamma}}.$$  

When $\gamma > 1$, the time-decay of the noise is strong enough, and it is not hard to see that the long time dynamics can be reduced to local dynamics. The next natural threshold, from the viewpoint of Burkholder inequality, is $\gamma > \frac{1}{2}$. In this case, the noise $W$ has finite quadratic variation over the whole time line. In fact, in [29], the authors established the scattering of the solution to (1.9) for noise with finite quadratic variation over the whole timeline (and sufficiently regular in space), which covers all $\gamma > \frac{1}{2}$. But their techniques break down at $\gamma = \frac{1}{2}$, where now the noise has infinite quadratic variation. In the recent work [27], the first and third authors of the current article established a global space-time bound and scattering of the solution to (1.9) in dimension 3 with $p - 1 = \frac{4}{3}$ (critical in dimension 3) and arbitrary $\gamma > 0$. The techniques also work for general $d \geq 2$.

The natural question now is whether one can get quantitative information on the long time behavior of the solution when $\gamma = 0$ (not allowing asymptotic decay of the noise in time), and this is the main aim of the current article. Our main result is to finally establish the global space-time bound and scattering for (1.9) with a small multiplicative constant $\varepsilon$ but no time-decay factor (see also [24] for the dispersive estimate for the linear equation with a small but non-decaying multiplicative noise).

Although the main interest here is to understand long time behaviors of the solution when the noise itself does not decay, our theorem also extends the range of exponents (in solution space) allowed for local well-posedness compared to the previous study in [23], where the value of the spatial integrability exponent $\beta$ is restricted.

As a by-product of the proof, we also establish a global space-time bound for the linear stochastic equation (1.6), which is a global-in-time stochastic Strichartz estimate. Although there have been various types of stochastic Strichartz estimates (for example [6,7,15,30]), all of them are on a fixed time interval. To the best of our knowledge, Theorem 1.2 is the first global-in-time one.

We refer to Sects. 3 and 4 for a description of the strategy in proving Theorems 1.2 and 1.1.

1.3. A brief discussion on $T T^*$ and endpoint Strichartz estimate. We give a brief discussion on our linear estimate (1.7). Strichartz estimates play a fundamental role in the study of dispersive equations. Historically, deterministic Strichartz estimates follow from the

---

1 Conceptually, the methods works for $d \geq 3$. Somehow, $d = 2$ is not forbidden thanks to the extra time decay.

2 [23] is written for $d = 1$, but the same techniques can be used in other dimensions.
dispersive estimates via a $TT^*$ argument. For our model (1.6), the corresponding dispersive estimate was proved in [24]. But defining the operator $T^*$ here requires running Brownian motion and solving the equation backwards in time. Since the Burkholder inequality and martingale structure is used essentially in our proof, it is not clear at this stage how to apply similar scheme to the nonlinear model. We expect rough path theory might be of help here. On the other hand, our proof for the linear problem here also serves as a toy model for the nonlinear problem.

We remark again that we miss the endpoint $(2, 6)$ in our Strichartz estimate. This pair is highly non-trivial even in the free Schrödinger case (see [32]). One key idea in [32] is to treat certain dual estimate as bi-linear estimates rather than linear ones.

The endpoint $(2, 6)$, except its usual importance in deterministic problem, also has particular interest our multiplicative noise model. This is because from the view point of Burkholder inequality, it is of interest to establish $L^2_t$ time bound for $V u d B_t$ (this is also why the finite quadratic variation condition is important in the work of [29]). We hope to be able to treat the end point in future works.

**Remark 1.5.** One can see from the proof, in particular in the last section, that in both linear and nonlinear case, the solution $u$ can be split into several parts, all satisfying an $L^2_t L^6_x$ bound except for a term of the form $e^{it\Delta} f(t)$, where $f(t)$ is a martingale in $L^2_x$ which converges to some $f \in L^\infty_\omega L^6_x$. We expect this may be of some help but it is not clear for us whether it is enough to conclude the $L^2_t L^6_x$ bound.

**1.4. Comparison to the deterministic model.** We end the introduction with several comparison to deterministic model. To approach results of type as Theorem 1.1 for deterministic models, there is now a systematic approach called Kenig–Merle road map, after the work [33, 34]. And for 3d mass critical model, the road map has been well implemented in the seminal work, [19]. The idea, which is also called concentration compactness rigidity method, is the following proof by contradiction: If scattering or space time bound does not hold, then one can apply certain (concentration) compactness argument to derive some minimal counter example, which will be so rigid and can not exist. It is not clear to us how to implement such a road map to our model, but this scheme definitely gives us the intuition that our solution will not blow up, which is already proved in [23]. One crucial element missing to apply Kenig-Merle road map to our model, is that we do not understand the linear model (1.6) enough to understand certain compactness.

We also mention it is also possible to approach the (linear) problem in the following way, i.e. view $V u \circ \dot{B}$ as some $\tilde{V}(x, t)$, and view the linear problem as understand the following linear Schrödinger equation,

$$i \partial_t u + \Delta u + \tilde{V}(t, x) u = 0. \quad (1.11)$$

It is not easy to handle dispersive estimate or Strichartz estimate handle potential has a $t$ dependence. It is possible (but highly non-trivial) to handle relatively rough $\tilde{V}$ in a deterministic way, see [36]. But if one wants to completely forget the stochastic structure in our $\tilde{V}$, and approach (1.6) as deterministic version of (1.11), then our $\tilde{V}$ would be very rough and too hard to handle from our perspective since the derivative of Brownian motion is only a distribution.

Note that this also suggests it will be very interesting but also challenging to understand a concentration compactness type property for (1.6).
1.5. Notations. Throughout the article, we fix the potential \( V \in S(\mathbb{R}^3) \) and \( \varepsilon \) sufficiently small depending on \( V \) (and also on \( \|u_0\|_{L^2_x} \) in the nonlinear model). We also fix \( m_0 > 0 \) such that \( \|u_0\|_{L^2_x} \leq m_0 \) for all initial data considered in this article.

We write \( S(t) = e^{it\Delta} \) for the propagator of the free Schrödinger equation, and \( H(t) \) for the propagator of the linear damped equation; that is, \( H(t)\psi_0 \) solves the equation

\[
i\partial_t \psi + \Delta \psi = -\frac{i\varepsilon^2}{2} V^2 \psi, \quad \psi(0, \cdot) = \psi_0. \tag{1.12}
\]

Note that \( H(t) = H_\varepsilon(t) \) depends on the parameter \( \varepsilon \). But since \( V \) and \( \varepsilon \) are fixed throughout the article, we will simply write \( H(t) \). We also write \( \langle x \rangle = (1 + |x|^2)^{\frac{1}{2}} \), and \( p' \) for the conjugate of \( p \in [1, +\infty] \) in the sense that \( \frac{1}{p} + \frac{1}{p'} = 1 \).

Throughout, \( (\alpha, \beta) \) is an admissible pair in \( \mathbb{R}^3 \) (see Definition 2.2). Our theorem covers the range \( \alpha \in (2, +\infty) \), but in different stages of the proof, we will make restrictions on the range of \( \alpha \). These will be specified in relevant contexts.

For fixed admissible pair \( (\alpha, \beta) \), we let \( \tilde{\beta} \) be such that

\[
\frac{1}{\tilde{\beta}} = \frac{1}{\beta} + \frac{1}{3\alpha} = \frac{1}{2} - \frac{1}{3\alpha}, \tag{1.13}
\]

which gives the embedding \( W^{\frac{1}{2}, \tilde{\beta}} \hookrightarrow L^\beta \) in \( \mathbb{R}^3 \). We also define the norms

\[
\|f\|_{X^{\alpha, p}} := \|\langle \cdot \rangle^{-10} f\|_{W^{\alpha, p}}, \quad \|f\|_{X} := \|f\|_{L^\beta} + \|f\|_{X^{\frac{1}{2}, \tilde{\beta}}}. \tag{1.14}
\]

The integrability exponent in the second norm of consisting of \( X \) is \( \tilde{\beta} \), the conjugate of \( \beta \). The appearance of these exponents is explained in Lemmas 5.6 and 5.7. The norm \( X \) depends on \( (\alpha, \beta) \), but we omit it for notational simplicity whenever no confusion arises.

We write \( A \lesssim B \) to say that there is a constant \( C \) such that \( A \leq CB \). We use \( A \simeq B \) when \( A \lesssim B \lesssim A \). Particularly, we write \( A \lesssim_{\gamma} B \) to express that \( A \leq C(\gamma) B \) for some constant \( C(\gamma) \) depending on \( \gamma \). Without special clarification, the implicit constant \( C \) can vary from line to line.

1.6. Structure of the paper. The rest of the article is organized as follows. In Sect. 2, we state some preliminary lemmas, including the classical dispersive, Strichartz and local smoothing estimates for the free Schrödinger operator, and the Burkholder inequality for martingales in proper Banach spaces. In Sects. 3 and 4, we give an overview of the strategies in proving Theorems 1.2 and 1.1. In particular, we show how the global bounds (1.3) and (1.7) can be reduced to certain intermediate bounds.

Section 5 is devoted to establishing properties of the damped linear Schrödinger operator \( H \), including the corresponding dispersive, Strichartz, and local smoothing estimates. In Sects. 6 and 7, we prove the intermediate bounds for the damped operator and complete Theorem 1.2 and the global space-time bound (1.3) with a restricted range of exponents \( (\alpha, \beta) \).

In Sect. 8, the space-time bound (1.3) was used to establish the scattering behavior of the solution. Finally in Sect. 9, we combine all previous ingredients (including the scattering bound) to conclude the space-time bound (1.3) for the full range of Strichartz pairs \( (\alpha, \beta) \) except the endpoint \((2, 6)\).
2. Preliminaries

2.1. Estimates for the free Schrödinger operator. We state the standard dispersive, Strichartz, and the local smoothing estimates for the free Schrödinger operator $e^{it\Delta}$ in $d = 3$. We refer to [11,39] for details.

**Lemma 2.1** (Dispersive estimate). The linear operator $e^{it\Delta}$ in $\mathbb{R}^3$ satisfies the bound

$$\|e^{it\Delta}f\|_{L^\infty_t L^1_x} \lesssim t^{-\frac{3}{2}} \|f\|_{L^1_x}. \tag{2.1}$$

Moreover, by interpolation with the unitary relation $\|e^{it\Delta}f\|_{L^2_t L^2_x} = \|f\|_{L^2_x}$, we have

$$\|e^{it\Delta}f\|_{L^p_t L^p_x} \lesssim t^{-d(\frac{1}{2} - \frac{1}{p})} \|f\|_{L^{p'}_x}, \tag{2.2}$$

for every $p \geq 2$.

**Definition 2.2.** Let $q, r \in [2, +\infty]$. We say $(q, r)$ is an admissible Strichartz pair in $\mathbb{R}^3$ if

$$\frac{2}{q} + \frac{3}{r} = \frac{5}{2}.$$

**Lemma 2.3** (Strichartz estimate). Let $(q, r)$ be an admissible Strichartz pair in $\mathbb{R}^3$. Then we have the bound

$$\|e^{it\Delta}f\|_{L^q_t L^r_x(\mathbb{R} \times \mathbb{R}^3)} \lesssim \|f\|_{L^2(\mathbb{R}^3)}. \tag{2.3}$$

Also, for any two Strichartz pairs $(q_1, r_1)$ and $(q_2, r_2)$, we have

$$\left\| \int_0^t e^{i(t-s)\Delta} F(s)ds \right\|_{L^{q_1}_{t}L^{r_1}_{x}(\mathbb{R} \times \mathbb{R}^3)} \lesssim \|F\|_{L^{q'_2}_{t}L^{r'_2}_{x}(\mathbb{R} \times \mathbb{R}^3)}, \tag{2.4}$$

where $q'_2$ and $r'_2$ are conjugates of $q_2$ and $r_2$.

We state the standard local smoothing estimate as follows (see [13,38,40] and also [39] for details).

**Lemma 2.4.**

$$\int \int \langle x \rangle^{-1-\delta} |\nabla^{1/2} e^{it\Delta} f|^2 dx dt \lesssim \|f\|_{L^2_x}^2. \tag{2.4}$$

Note that (2.4) implies the associated estimate for the Duhamel part, via the standard $TT^*$ argument,

$$\left\| \langle x \rangle^{-\frac{1}{2}-\delta} \nabla \int_0^t e^{i(t-s)\Delta} F(s)ds \right\|_{L^2_t L^2_x} \lesssim \|\langle x \rangle^{\frac{1}{2}+\delta} F\|_{L^2_{t,x}}. \tag{2.5}$$

Next, we state a pointwise version of local smoothing estimate.

**Lemma 2.5.** For every $R > 0$ and $t \geq 1$, we have

$$\|\nabla e^{it\Delta} f\|_{L^2_t(B_R)} \lesssim \frac{R}{t} \|\langle x \rangle f\|_{L^2_x}. \tag{2.6}$$

Lemma 2.5 follows from the conservation of conformal energy $\|(x + it\nabla)u\|_{L^2_x}^2$. See [39] for a detailed proof.
2.2. Global well-posedness for the defocusing mass critical nonlinear Schrödinger equation. Consider the defocusing mass critical NLS in $\mathbb{R}^3$

$$i \partial_t u + \Delta u = |u|^{4/3} u, \quad u(0, \cdot) = u_0 \in L^2_x(\mathbb{R}^3).$$ (2.7)

The following theorem is a seminal result by Dodson [18].

**Theorem 2.1.** Let $u$ solve (2.7) with initial data $u_0 \in L^2_x$. Then, $u$ is global, and there exists $C > 0$ depending on $\|u_0\|_{L^2}$ only such that

$$\|u\|_{L^2_t L^6_x(\mathbb{R}^+ \times \mathbb{R}^3)} \cap L^\infty_t L^2_x(\mathbb{R}^+ \times \mathbb{R}^3) \leq C.$$ (2.8)

Furthermore, $u$ scatters to a linear (free Schrödinger) solution.

2.3. The Burkholder inequality. We will frequently use the following version of Burkholder inequality (for martingales in Banach space).

**Lemma 2.6.** Let $B$ be the standard Brownian motion. Let $\rho \in (1, +\infty)$, $p \in [2, +\infty)$, and $\Phi$ be an $L^p(\mathbb{R}^d)$-valued process adapted to the filtration generated by $B_t$. Then, we have

$$\left\| \sup_{0 \leq a \leq b \leq t} \int_a^b \Phi(s) dB_s \right\|_{L^p_x(L^\rho)} \lesssim_{p, \rho} \left( \int_0^t \|\Phi(s)\|_{L^p_x}^2 ds \right)^{1/2}.$$ (2.9)

We refer to [3,4,9,10] for proofs and also statements in more general situations.

3. Overview of the Proof of Theorem 1.2 and Some Reductions

In this section, we give an overview of the proof of Theorem 1.2, and show that it reduces to Lemmas 3.3 and 3.4. The proofs of these two lemmas will be given in Sect. 6 later.

Though it is natural to write down the Duhamel formula for solution $u$ to (1.6) based on linear Schrödinger equations and view $u \circ dW_t$ as an input,

$$u(t) = e^{it\Delta} u_0 - i \varepsilon \int_0^t e^{i(t-s)\Delta} (Vu(s)) dB_s - \frac{\varepsilon^2}{2} \int_0^t e^{i(t-s)\Delta} (V^2u(s)) ds,$$ (3.2)

it turns out, as observed in [27], it may be of advantage to view (1.6) as a damped Schrödinger equation perturbed by the stochastic term. More precisely, let $H$ be the linear propagator for linear damped Schrödinger equation such that $\psi(t) = H(t)\psi_0$ solves

$$i \partial_t \psi + \Delta \psi = -\frac{i \varepsilon^2}{2} V^2 \psi, \quad \psi(0, \cdot) = \psi_0.$$ (3.1)

We write down the Duhamel formula for $u$ based on $H$ so that

$$u(t) = H(t)u_0 - i \varepsilon \int_0^t H(t-s)(Vu(s)) dB_s.$$ (3.2)

[27] studies the nonlinear model with a slowly decaying noise, while here the noise does not decay in time.
Note that both $H$ and $u$ depend on $\varepsilon$. But since $\varepsilon$ is a fixed parameter, we omit the dependence for notational convenience. Also, we write

$$\begin{align*}
(\Gamma u)(t) := -i \int_0^t H(t-s)(Vu(s))dB_s,
\end{align*}$$

and (3.2) now reads

$$u(t) = H(t)u_0 + \varepsilon(\Gamma u)(t).$$

We will focus on a priori estimates, and the desired estimate (1.7) then follows via a standard continuity argument when $\varepsilon$ is chosen small enough. Since the Strichartz estimate for $H$ in (5.2) directly handles the term $H(t)u_0$, Theorem 1.2 will follow from the following proposition and smallness of $\varepsilon$.

**Proposition 3.1.** Let $u$ be the solution to (1.6) with initial data $u_0 \in L^2_x$. Then for $\varepsilon$ sufficiently small (depending on $V$), one has the bound

$$
\|\Gamma u\|_{L^a_tL^\beta_x(\Omega \times \mathbb{R}^+ \times \mathbb{R}^3)} \lesssim \|u_0\|_{L^2_x} + \|u\|_{L^a_tL^\beta_x(\Omega \times \mathbb{R}^+ \times \mathbb{R}^3)}.
$$

**Remark 3.2.** Note that this proposition depends on estimates for $H$ established in Sect. 5, whose proof in turn relies on smallness of $\varepsilon$.

The proof of Proposition 3.1 relies on the fact that $u$ is a solution to (1.6). Indeed, we need to do one more expansion of (3.3) to get the expression

$$
(\Gamma u)(t) = -i \int_0^t H(t-s)(V H(s)u_0)dB_s - \varepsilon \int_0^t H(t-s)
\left(V \int_0^s (H(s-r)Vu(r))dB_r\right)dB_s.
$$

Note that since all the terms in (3.6) are stochastic integrals and will be estimated with the Burkholder inequality, our estimate indeed gives a stronger version of (3.5) as

$$
\|\sup_{a \leq b \leq t} \| \int_a^b H(t-s)(Vu(s))dB_s \|_{L^\beta_t} \|L^a_t L^\beta_x(\Omega \times \mathbb{R}^+ \times \mathbb{R}^3)} \lesssim \|u_0\|_{L^2_x} + \|u\|_{L^a_tL^\beta_x(\Omega \times \mathbb{R}^+ \times \mathbb{R}^3)}.
$$

Such maximal type estimates will play an important role in the nonlinear problem later (also in [23,26,27]).

We now give details on the bounds for the terms on the right hand side of (3.6). The first term will be estimated by the following lemma.

**Lemma 3.3.** When $\varepsilon$ is small enough, one has

$$
\| \int_0^t H(t-s)(V H(s)u_0)dB_s \|_{L^a_tL^\beta_x(\Omega \times \mathbb{R}^+ \times \mathbb{R}^3)} \lesssim \|u_0\|_{L^2_x}.
$$
We will see later that the above lemma should be understood as the left hand side of (3.8) being controlled by free linear solution \( H(t)u_0 \).

To estimate the second term in (3.6), we let

\[
A(t) := V \int_0^t H(t - s)(Vu(s))dB_s. \tag{3.9}
\]

Hence, we have

\[
\int_0^t \int_0^s H(t - s) \left( VH(s - r)(Vu(r)) \right) dB_r dB_s = \int_0^t H(t - s)A(s)dB_s. \tag{3.10}
\]

The estimates for the second term in (3.6) is the content of the following lemma.

**Lemma 3.4.** Recall the definition of \( \tilde{\beta} \) from (1.13). We have the bounds

\[
\|A\|_{L_\omega^\alpha L_t^\alpha W_x^\frac{1}{\alpha} X_\beta' (\Omega \times \mathbb{R}^+ \times \mathbb{R}^3)} \lesssim \|u\|_{L_\omega^\alpha L_t^\alpha L_x^\beta (\Omega \times \mathbb{R}^+ \times \mathbb{R}^3)}, \tag{3.11}
\]

and

\[
\|A\|_{L_\omega^\alpha L_t^\alpha L_x^\beta (\Omega \times \mathbb{R}^+ \times \mathbb{R}^3)} \lesssim \|u\|_{L_\omega^\alpha L_t^\alpha L_x^\beta (\Omega \times \mathbb{R}^+ \times \mathbb{R}^3)}. \tag{3.12}
\]

Here, \( \tilde{\beta}' \) and \( \beta' \) are conjugates of \( \tilde{\beta} \) and \( \beta \). Furthermore, we have the following two estimates

\[
\left\| \int_{(t-1)\land 0}^t H(t - s)A(s)dB_s \right\|_{L_\omega^\alpha L_t^\alpha L_x^\beta (\Omega \times \mathbb{R}^+ \times \mathbb{R}^3)} \lesssim \|A(t)\|_{L_\omega^\alpha L_t^\alpha L_x^\beta (\Omega \times \mathbb{R}^+ \times \mathbb{R}^3)}, \tag{3.13}
\]

and

\[
\left\| \int_0^t (t - s)A(s)dB_s \right\|_{L_\omega^\alpha L_t^\alpha L_x^\beta (\Omega \times \mathbb{R}^+ \times \mathbb{R}^3)} \lesssim \|A\|_{L_\omega^\alpha L_t^\alpha L_x^\beta (\Omega \times \mathbb{R}^+ \times \mathbb{R}^3)}. \tag{3.14}
\]

We refer to Lemma 5.6 for the natural appearance of the exponents \( \frac{1}{\alpha} \) and \( \tilde{\beta} \). Lemmas 3.3 and 3.4 will be proven in Sect. 6.

We end this section with an explanation why the admissible pair \((4, 3)\) is special in our analysis, and how local smoothing type estimates may help us handle \((4-, 3+)\). It is very tempting to directly prove, based on (3.3) and assumptions \( u \in L_\omega^\alpha L_t^\alpha L_x^\beta (\Omega \times \mathbb{R}^+ \times \mathbb{R}^3) \), that

\[
\int_0^t H(t - s)(Vu(s))dB_s \in L_\omega^\alpha L_t^\alpha L_x^\beta (\Omega \times \mathbb{R}^+ \times \mathbb{R}^3) = L_\omega^\alpha L_t^\alpha L_x^\beta (\mathbb{R}^+ \times \Omega \times \mathbb{R}^3). \tag{3.15}
\]

By Burkholder inequality, one has

\[
\left\| \int_0^t H(t - s)(Vu(s))dB_s \right\|_{L_\omega^\alpha L_t^\alpha} \lesssim \left( \int_0^t \|H(t - s)(Vu(s))\|^2_{L_\omega^\alpha L_t^\alpha} ds \right)^{1/2}. \tag{3.16}
\]
Let us neglect the integrability issue in the $\omega$ variable for the moment, so the question now becomes the following: for space-time function $F \in L_t^{\alpha} L_x^{\beta}(\mathbb{R}^+ \times \mathbb{R}^3)$, can one prove

$$\int_0^t \| H(t-s)(VF(s))\|_{L_x^{\beta}}^2 ds \in L_t^{\frac{\alpha}{2}}(\mathbb{R}^+) ?$$

(3.17)

Using the fact that $V$ is well localized and that the damped linear operator $H$ satisfies the same dispersive estimate as $e^{it\Delta}$ (Lemma 5.1 below), one has the pointwise (in time) bound

$$\| H(t-s)(VF(s))\|_{L_x^{\beta}}^2 \lesssim (t-s)^{-6(\frac{1}{2} - \frac{1}{\beta})} \| VF(s)\| \lesssim (t-s)^{-6(\frac{1}{2} - \frac{1}{\beta})} \| F\|_{L_x^{\beta}}^2 .$$

(3.18)

Plugging it into (3.17) and letting $(\alpha, \beta) = (4, 3)$, one sees that it reduces to estimate the quantity

$$\left\| \int_0^t (t-s)^{-1} \| F(s)\|_{L_x^{\beta}}^2 ds \right\|_{L_t^{\frac{\alpha}{2}}(\mathbb{R}^+)},$$

which has a non-integrable singularity at $s \approx t$, and a non-integrable tail at small values of $s$ (when $t \to +\infty$).

By choosing $(\alpha, \beta) = (4-, 3+)$, one overcomes the divergence for $s$ far away $t$, but paying the price that $(t-s)^{-3+\frac{6}{\beta}}$ is more singular at $|t-s| \ll 1$.

This is the place where one needs to use $F = u$ being the solution of the equation and the local smoothing estimate. From Duhamel formula, we expect $F(x) \sim H(s)u_0$ at a first order approximation, and $VH(s)u_0$ can be raised a half derivative via local smoothing estimate for $H$ (Lemma 5.3). Hence, by choosing $\tilde{\beta}$ according to (1.13) (in particular $\tilde{\beta} < 3$), one hopes to estimate for $s$ close to $t$ by

$$\| H(t-s)(VF(s))\|_{W_x^{\mu,\tilde{\beta}}}^2 \lesssim (t-s)^{-3+\frac{6}{\tilde{\beta}}} \| F\|_{L_x^{\tilde{\beta}}}^2 .$$

(3.19)

This can solve the problem for non-integrability of $(t-s)^{-3+\frac{6}{\tilde{\beta}}}$ for $s$ close to $t$, as far as $\tilde{\beta} = 3-$ and $W^{\mu,\tilde{\beta}}$ embeds $L_x^{\tilde{\beta}}$.

The bound (3.19) is in general not true since one usually only obtains a local smoothing. Thus we need to expand as in (3.6) and add a localization weight into the norm to incorporate local smoothing estimates. The detailed proof, including some extra technical issues, will be given in Sect. 6.

4. Overview of the Proof of Theorem 1.1

We now turn to an overview of the proof of our main result, Theorem 1.1. We will mainly focus on (1.3) for $(\alpha, \beta) = (4-, 3+)$. Given global space-time bound (1.3), the scattering behavior will follow by refining the analysis in [27], and we will explain that in Sect. 8.
4.1. The strategy. Recall from Proposition 3.1 that in the linear problem, the control of the quantity $\|\Gamma u\|_{L^\alpha_t L^\beta_x}$ is sufficient to establish the space-time bound. But in the nonlinear case, following [23, 27], we need to control a maximal version of $\Gamma u$ rather than itself (see Remark 3.2).

Also, directly estimating the terms on the right-hand side of Duhamel formula is not very suited for the nonlinear problem. The key extra idea here is to split $u = u_1 + u_2$ such that $u_1$ behaves like a linear solution, and in particular allows an application of local smoothing type estimate, and $u_2$ behaves like a maximal version of $\Gamma u$, so $u_2$ can also be estimated in a bootstrap scheme with the small multiplicative constant. This type of decomposition already appeared in [27, Lemma 3.6] in the study of scattering asymptotics (and also implicitly in [23] for a study of local theory). But we need to further refine and exploit its properties to suit the purpose of the current article. The existence of such a decomposition is nontrivial.

Remark 4.1. Behind such a decomposition is a modified stability, which we will present with more details in Sect. 5.4. This is similar in spirit to the decompositions in the pioneering works of Bourgain [2] on two-dimensional cubic nonlinear Schrödinger equation with $\phi_2^4$ initial data, and da Prato–Debussche [14] on the dynamical $\phi_2^4$ singular stochastic PDE, but also has subtle differences in the implementation.

In the works [2, 14], the decomposition of the solution into a linear part and a nonlinear remainder part is global in space and time, with the linear part being explicit, and the nonlinear remainder part satisfying an autonomous and explicit equation.

On the other hand, in our situation, such a clean decomposition is not available since the noise is multiplicative. Instead, we obtain an inexplicit decomposition $u = u_1 + u_2$ on a sequence of random intervals tailored to our problem at hand. The actual form of the decomposition vary on those intervals, and that the $u_1$ and $u_2$ on each interval depend on the solution $u$ itself in a nontrivial and rather inexplicit way.

We now go into more details.

4.2. Main technical reductions and bootstrap scheme. We will fix $V$ and $m_0 := \|u\|_{L^2_x}$. We will assume $\varepsilon$ is small enough so that for any universal constant $C$, (which may implicitly depending on $m_0$ and $V$), $C\varepsilon < 1$. We also fix a Strichartz pair $(\alpha, \beta) = (4-, 3+)$. Recall the definition of $\tilde{\beta}$ from (1.13).

Let $u$ be the solution to (1.1). We introduce the quantity $M^*_\varepsilon$ by

$$M^*_\varepsilon(t) := \varepsilon \sup_{0 \leq a < b \leq t} \left\| \int_a^b H(t - s)V(x)u(s)dB_s \right\|_{L^\beta_x} + \varepsilon \sup_{0 \leq a < b \leq t} \left\| \int_a^b \langle x \rangle^{-100} H(t - s)(Vu(s))dB_s \right\|_{W^{1, \tilde{\beta}}_x}.$$  

There are two differences compared with [27] (see Sect. 3.3 and in particular eqs. (3.8) and (3.15) in [27]): we choose $(\alpha, \beta) = (4-, 3+)$ rather than $(4+, 3-)$, and also add a second part in $M^*$ in order to incorporate the local smoothing estimate.

We write $M^*(t) = M^*_\varepsilon(t)$ whenever there is no confusion. Note that $M^*(t)$ is random since $u$ is. According to [27, Lemma 4.9], one expects a bound of the type

$$\|u\|_{L^\infty_t L^2_x L^\beta_t(\Omega \times [0, T] \times \mathbb{R}^3)} \lesssim m_0 1 + \|M^*(t)\|_{L^\infty_t L^2_t([0, T])}$$  

(4.2)
uniformly over $T > 0$. Actually, we will prove a stronger version of it in Proposition 4.2 below, which encodes the above mentioned decomposition.

**Proposition 4.2.** Let $(\alpha, \beta)$ be admissible with the further restriction that $\alpha \in \left[\frac{1}{3}, \frac{14}{5}\right]$. For any $[0, T]$ ($T$ could be $+\infty$), one can find a decomposition\(^4\) of $u := u_1 + u_2$, such that

$$
\|u_1\|_{L^\alpha_{0,T} L^\beta_2(\Omega \times [0,T] \times \mathbb{R}^3)} \leq 1 + \|M^*(t)\|_{L^\alpha_{0,T} L^\beta_2(\Omega \times [0,T])},
$$

and

$$
\|u_2(t)\|_\mathbb{X} \leq M^*(t)
$$

almost surely for every $t \in [0, T]$. Both bounds are uniform in $T > 0$.

Note that Proposition 4.2 implies the bound (4.2). We want to highlight this decomposition and hopefully it will play more role in the future study of scattering without any smallness assumption. And one will see in the proof the first part $u_1$ is understood by studying (5.20) and its perturbation. And $u_2$ is estimated by the maximal object, $M^*$, itself. Hence it remains to prove

$$
\|M^*(t)\|_{L^\alpha_{0,T} L^\beta_2(\Omega \times \mathbb{R}^+)} < C_{m_0}
$$

for some constant $C$ depending on $m_0$ when $\epsilon$ is small enough.

From view point of standard continuity and bootstrap argument we will focus on a priori estimates for global solution $u$ to (1.1). Strictly speaking, one needs to do bootstrap for all solution $u$ in $[0, T]$, prove uniform bounds in $T$, and then extend $T$ to infinity. But for conciseness of the presentation, we will omit this limiting procedure, and only consider $T = +\infty$ in what follows.

We will prove, assuming $\|M^*(t)\|_{L^\alpha_{0,T} L^\beta_2(\Omega \times \mathbb{R}^+)} < +\infty$, that

$$
\|M^*(t)\|_{L^\alpha_{0,T} L^\beta_2(\Omega \times \mathbb{R}^+)} \lesssim_{m_0} 1 + \epsilon \|M^*(t)\|_{L^\alpha_{0,T} L^\beta_2(\Omega \times \mathbb{R}^+)}. \tag{4.6}
$$

Note that when $\epsilon$ is small enough, one derives the desired bootstrap type estimate

$$
\|M^*(t)\|_{L^\alpha_{0,T} L^\beta_2(\Omega \times \mathbb{R}^+)} \lesssim 1. \tag{4.7}
$$

Strictly speaking, one needs to do bootstrap for all solution $u$ in $[0, T]$, prove uniform estimates in $T$, and then extend to $T = +\infty$. We will only focus on the limit case $T = \infty$ for clarity, and leave the other details for interested readers, but the arguments for general $T$ are essentially same as $T = \infty$.

We now focus on the proof of (4.6). We will assume the finiteness of $\|M^*(t)\|_{L^\alpha_{0,T} L^\beta_2(\Omega \times \mathbb{R}^+)}$ in the whole article whenever we study the nonlinear problem.

Recall the norms $\mathbb{X}^{\frac{1}{\alpha}, \frac{\beta}{\beta'}}$ and $\mathbb{X}$ from (1.14). By Burkholder inequality, we have

$$
\|M^*(t)\|_{L^\alpha_{0,T} L^\beta_2(\Omega \times \mathbb{R}^+)} = \|M^*\|_{L^\alpha_{0,T} L^\beta_2(\Omega \times \mathbb{R}^+)}
$$

$$
\lesssim \epsilon \left\| \int_0^t \|H(t-s)Vu(s)\|^2_{\mathbb{X}} ds \right\|_{L^\alpha_{0,T} L^\beta_2(\Omega \times \mathbb{R}^+)}^{1/2}.
$$

\(^4\) This decomposition may depending on $T$, but all the implicit constants in the inequality does not depend on $T.$
Note that the factor $\varepsilon$ is crucial for closing of our estimates, and will be chosen small enough finally. (4.6) then follows from the following purely deterministic functional inequality.

**Lemma 4.3.** One has bound

$$\left\| \left( \int_0^t \| H(t-s)(VG(s)) \|_{L^2_x}^2 ds \right)^{1/2} \right\|_{L^q_t(\mathbb{R}^+)} \lesssim \| G \|_{L^q_x(\mathbb{R}^+ \times \mathbb{R}^3)}. \quad (4.8)$$

We will prove Proposition 4.2 and Lemma 4.3 in Sect. 7. The proof relies on the material in Sect. 5, which deals with properties for damped linear and nonlinear Schrödinger equations. We end this section by proving (4.6) assuming Lemma 4.3 and Proposition 4.2.

4.3. **Proof of (4.6) assuming Proposition 4.2 and Lemma 4.3.** We first check that (4.3) and (4.4) implies (4.2). This follows from

$$\| u_2 \|_{L^q_t L^p_x(\Omega \times \mathbb{R}^+ \times \mathbb{R}^3)} \lesssim 1 + \| M^* \|_{L^q_{\omega,t}(\Omega \times \mathbb{R}^3)}. \quad (4.9)$$

But this is follows by taking $L^q_{\omega,t}$ on both sides of (4.4). Now, plug (4.2) into (4.8), and (4.6) follows.

## 5. Properties for Damped Model and Modified Stability

In this section, we present the needed properties for damped model. The proof of Theorem 1.2 only relies on the material in Sect. 5.1. The proof of Theorem 1.1 crucially relies on the material in Sect. 5.4. We remark that, except for Lemma 5.4, the smallness assumptions on $\varepsilon$ is not necessary. We conjecture one may remove the smallness assumption also for this Lemma. However, the main bootstrapping argument in the article depends on the smallness of $\varepsilon$, unless one uses some extra idea.

### 5.1. Dispersive and Strichartz estimates for damped linear operator.

We start with the dispersive estimates and Strichartz estimates. Let $H$ be the linear propagator to (3.1).

Based on the important observation in [31], the term $-i\frac{\varepsilon}{2} \cdot V^2 \psi$ in (3.1) can be treated in a perturbative way (see also [37] for a nice survey). We also note that for general real-valued potential $V$ (without smallness assumption on $\varepsilon$), the estimate also holds (see [41]).

**Lemma 5.1.** One has the same dispersive estimate that

$$\| H(t) f \|_{L^p_x} \lesssim t^{-\frac{3}{2} - \frac{1}{p}_+} \| f \|_{L^p_x} \quad (5.1)$$

for every $p \geq 2$.

We also have the usual Strichartz estimate.
Lemma 5.2. For all Strichartz pair \((q, r)\), one has

\[
\|H(t)f\|_{L^q_t L^r_x(\mathbb{R}^+ \times \mathbb{R}^3)} \lesssim \|f\|_{L^2_x}.
\] (5.2)

Furthermore, for all Strichartz pairs \((q_1, r_1)\) and \((q_2, r_2)\), we have

\[
\left\| \int_a^t H(t-s)F(s)ds \right\|_{L^{q_1}_t L^{r_1}_x(\mathbb{R}^+ \times \mathbb{R}^3)} \lesssim \|F\|_{L^{q_2'}_t L^{r_2'}_x(\mathbb{R}^+ \times \mathbb{R}^3)},
\] (5.3)

where \(q'_2\) and \(r'_2\) are conjugates of \(q_2\) and \(r_2\).

We note that Lemmas 5.1 and 5.2 both hold without smallness assumption on \(\varepsilon\). The proofs of these two lemmas can be found in [27, Lemmas 4.1 and 4.2].

5.2. Local smoothing estimates for damped model. Next, we present two local smoothing Lemma for the damped model.

Lemma 5.3. Let \(H\) be the linear propagator to (3.1) with sufficiently small \(\varepsilon\). Then one has the bound

\[
\int_0^\infty \int \langle x \rangle^{-1-\delta} \|\nabla^{1/2} H(t)f\|^2 dx dt \lesssim \|f\|^2_{L^2_x}.
\] (5.4)

The proof of (5.4) follows from the Duhamel Formula, the (dual) local smoothing estimate, the (endpoint) Strichartz estimate associated with \(H\), and the fact that \(V\) is localized in space.

Lemma 5.4. For \(\varepsilon\) small enough, let \(H\) be the linear propagator to (3.1), one has

\[
\|\nabla H(t)f\|_{L^2_x(B_R)} \lesssim \frac{\langle R \rangle^{3/2}}{t} \|\langle x \rangle f\|_{L^2_x}, \quad t > 1,
\] (5.5)

Lemma 5.4 will be obtained based on the similar estimates for linear Schrödinger equation in a perturbative way. The lemma is probably not optimal but is sufficient for the purpose of this article. The proof of Lemma 5.4 relies on the following Lemma.

Lemma 5.5. Assuming

\[
\|\nabla H(t)f\|_{L^2_x(B_R)} \leq \frac{C \langle R \rangle^{3/2}}{t} \|\langle x \rangle f\|_{L^2_x},
\] (5.6)

then one has, for \(\tilde{C}\) only depending on \(C\), such that

\[
\|\nabla \int e^{i(t-s)\Delta} V^2 H(s)f\|_{L^2_x(B_R)} \leq \tilde{C} \frac{\langle R \rangle^{3/2}}{t} \|\langle x \rangle f\|_{L^2_x}.
\] (5.7)

Assuming Lemma 5.5 for the moment. We prove Lemma 5.4.
**Proof of Lemma 5.4.** By Duhamel formula, one has

\[ H(t)f = e^{it\Delta}f + c e^2 \int_0^t e^{i(t-s)\Delta} V^2 H(s)f \, ds \]  

(5.8)

(Here \( c = -\frac{1}{2} \), but we only need to know it is a constant.) With (5.5), and Lemma 5.5, we deduce, (as a priori estimate),

\[ \|\nabla H(t)f\|_{L^2_x(B_R)} \lesssim \frac{\langle R \rangle^{3/2}}{t} \|\langle x \rangle f\|_{L^2_x} + e^2 c \frac{\tilde{C} \langle R \rangle^{3/2}}{t} \|\langle x \rangle f\|_{L^2_x}. \]  

(5.9)

Using the fact that \( \epsilon \) is small, a standard bootstrap argument gives Lemma 5.4. \( \square \)

We now prove Lemma 5.5.

**Proof of Lemma 5.5.** We may only consider \( R \geq 1 \). Let us fix \( R \). We recall \( V \) is fast decaying. And we fix \( t \geq 1 \).

We first note that for any \( g \) with

\[ \|\nabla g\|_{L^2_x(B_R)} \lesssim C \langle R \rangle^{3/2} B \text{ for some } B > 0 \]  

(5.10)

we have, thanks to the fast decay of \( V \), that

\[ \|Vg\|_{H^1} \lesssim CB. \]  

(5.11)

We may only consider \( t \gg 1 \). Indeed, if \( t \lesssim 1 \), in \([t/2, t]\), we estimate via

\[ \left\| \int_{t/2}^t e^{i(t-s)\Delta} \nabla (V^2 H(s)f) \right\|_{L^2_x(B_R)} \lesssim t \sup_{s \in \left[\frac{t}{2}, t\right]} \|V^2 H(s)f\|_{H^1}. \]  

(5.12)

On \( s \in [0, t/2] \), since \(|t - s| \geq t/2\) and \( V^2 H(s)f \) is uniformly localized in space, we estimate via

\[ \left\| \int_0^{t/2} e^{i(t-s)\Delta} \nabla (V^2 H(s)f) \right\|_{L^2_x(B_R)} \lesssim \frac{1}{t/2} \|VH(s)f\|_{L^2_x} \lesssim \|\langle x \rangle f\|_{L^2_x}. \]  

(5.13)

We now plug in bootstrap assumption (5.6), so (5.12) and (5.13) can be controlled by \( t^{1/2} \|\langle x \rangle f\|_2 \) and the desired estimates follow. (Note that the \( R^{3/2} \) is absorbed by fast decay of \( V \).)

Now, let us fix \( t \gg 1 \), we consider three ranges of \( s \). When \( s \in [t-1, t] \), the estimate is similar to the previous, and we estimate via

\[ \left\| \int_{t-1}^t e^{i(t-s)\Delta} \nabla (V^2 H(s)f) ds \right\|_{L^2_x} \lesssim \sup_{s \in [t-1, t]} \|V^2 H(s)f\|_{H^1} \lesssim C \frac{1}{t} \|\langle x \rangle f\|_{L^2_x}. \]  

(5.14)

When \( s \in [0, 1] \), we simply using \( \|\langle x \rangle V^2 H(s)f\|_{L^2_x} \lesssim \|f\|_{L^2_x} \), and apply (2.6). When \( s \in [1, t-1] \), we first apply point-wise estimate
\[
\|e^{i(t-s)\Delta} \nabla (V^2 H(s))\|_{L_t^\infty} \lesssim (t-s)^{-3/2} \|\nabla V H(s)\|_{L_t^2} \\
\lesssim (t-s)^{-3/2} s^{-1} \|\langle x \rangle f\|_{L_t^1}.
\] (5.15)

Now, apply
\[
\int_1^{t-1} (t-s)^{-3/2} s^{-1} ds \lesssim t^{-1},
\] (5.16)

and using the fact locally \(L^\infty\) embeds \(L_x^2(B_R)\) with a constant \(R^{3/2}\). The desired estimates now follow. \qed

5.3. Some interpolation and embedding lemmas. We need the following lemmas.

**Lemma 5.6.** Let \((\alpha, \beta)\) be an admissible pair in \(\mathbb{R}^3\) with \(\alpha \in (2, 4)\). Then for every \(p \geq 1\), we have the bound
\[
\|F\|_{L_t^\alpha W_x^{1/p, \beta' \prime}} \lesssim \|F\|_{L_t^\alpha W_x^{1/p, \beta' \prime}}^{1/\beta - 1/\beta} \|F\|_{L_t^{\alpha', \beta' \prime}(\mathbb{R}^+ \times \mathbb{R}^3)},
\]
where all the norms are taken over the domain \(\mathbb{R}^+ \times \mathbb{R}^3\). Furthermore, for \(\tilde{\beta}\) such that \(\frac{1}{\tilde{\beta}} - \frac{1}{\beta} = \frac{1}{3\alpha}\), we have the embedding
\[
\|f\|_{L_t^{\alpha}(\mathbb{R}^3)} \lesssim \|f\|_{W_x^{1/\tilde{\beta}, \tilde{\beta}}}.\]

**Proof.** The first bound follows from interpolation and Hölder inequality, and the second bound is standard Sobolev embedding. \qed

**Lemma 5.7.** We have the Strichartz estimates
\[
\|e^{it\Delta} f\|_{L_t^a X^{1/\beta, \beta' \prime}(\mathbb{R}^+ \times \mathbb{R}^3)} \lesssim \|f\|_{L_t^2},
\] (5.17)

and
\[
\left\| \int_0^t e^{i(t-s)\Delta} F(s) ds \right\|_{L_t^a X^{1/\beta, \beta' \prime}(\mathbb{R}^+ \times \mathbb{R}^3)} \lesssim \|F\|_{L_t^{a', \beta' \prime}(\mathbb{R}^+ \times \mathbb{R}^3)},
\] (5.18)

where \((\alpha, \beta)\) and \((a, b)\) are admissible pairs, and \(a', b'\) are the conjugates of \(a\) and \(b\). The same bounds are true if the free Schrödinger operator is replaced by the damped operator \(H\).

**Proof.** The first one follows from local smoothing estimate interpolated with mass conservation. The latter follows from the following un-retarded estimate via Christ-Kiselev Lemma (12):
\[
\left\| \int_0^\infty e^{i(t-s)\Delta} F(s) ds \right\|_{L_t^a X^{1/\beta, \beta' \prime}(\mathbb{R}^+ \times \mathbb{R}^3)} \lesssim \|F\|_{L_t^{a', \beta' \prime}(\mathbb{R}^+ \times \mathbb{R}^3)}, \text{ where } a, b \text{ admissible.}
\] (5.19)

But this is a consequence of (5.17) and Strichartz estimate
\[
\left\| \int_0^\infty e^{-is\Delta} F(s) ds \right\|_{L_t^2} \lesssim \|F\|_{L_t^{a', \beta' \prime}(\mathbb{R}^+ \times \mathbb{R}^3)}.
\]

The arguments for the operator \(H\) are the same. This completes the proof. \qed
5.4. Modified stability. The material in this subsection is essentially Lemmas 4.3 and 4.4 and Proposition 4.6 in [27], but we have made those results stronger by taking the local smoothing into considerations.

As shown\(^5\) in [27, Lemma 4.3], one can apply Dodson’s result (Theorem 2.1) as a black box to establish the desired space time bound for the solution \( w \) to

\[
i \partial_t w + \Delta w = -\frac{i\varepsilon^2}{2} \cdot \nabla^2 w + |w|^{4/3} w, \quad w(0, \cdot) = w_0 \in L^2_x(\mathbb{R}^3).
\]

We need the following slightly stronger version of [27, Lemma 4.3].

**Lemma 5.8.** Let \( w \) solve (5.20) with initial data \( w_0 \in L^2_x \). Then \( w \) is global forward in time \( \mathbb{R}^+ \) with estimates

\[
\|w\|_{L^\infty_t \mathcal{X}(\mathbb{R}^+ \times \mathbb{R}^3)} \lesssim \|w_0\|_{L^2_x}^{-1}.
\]

**Proof.** We only need to prove the bound (5.21). An integration by parts argument (just computing the time derivative of \( \|w(t)\|_{L^2_x}^2 \)) gives

\[
\epsilon^2 \int_0^t \int V^2 |w|^2 \, dx \, ds = \|w_0\|_{L^2_x}^2 - \|w(t)\|_{L^2_x}^2.
\]

Thus, \( \epsilon^2 V^2 w \) is in the dual space \( L^2_x L_x^{6/5} \), and its size is controlled via \( \|w_0\|_{L^2_x} \).

We also know \( |w|^{4/3} w \in L^a_t L^b_x \) for some \( (a, b) \) admissible since \( w \) is in \( L^a_t L^b_x \), whose size is also controlled by \( \|w_0\|_{L^2_x} \) as proved in Lemma 4.3 in [27]. Via Duhamel formula based on the free Schrödinger operator, we have

\[
w(t) = e^{it\Delta} w_0 - i \int_0^t e^{i(t-s)\Delta} (|w(s)|^{4/3} w(s)) \, ds - \frac{\epsilon^2}{2} \int_0^t e^{i(t-s)\Delta} (V^2 w(s)) \, ds.
\]

Thus, the desired estimates then follow from Lemma 5.7. This completes the proof. \( \square \)

We are ready to present a more refined version of [27, Proposition 4.6]. Note that the main point is that one needs to state the stability in the form of Duhamel formula.

**Proposition 5.9.** Let \( u \) solve in some interval \([a, b]\) the equation

\[
u(t) = H(t - a) u(a) - i \int_a^t H(t - s)(|u(s)|^{4/3} u(s)) \, ds + \eta(t),
\]

with \( \|u\|_{L^\infty_a L^2_x} \leq m_0 \), and \( \eta(a) = 0 \). Then for \((\alpha, \beta)\) admissible and \( \frac{7}{3} \leq \alpha \leq \frac{14}{3} \), there exist \( e_{m_0}, B_{m_0} > 0 \) such that if \( \|\eta\|_{L^\infty_a L^2_x \cap L^a_t L^b_t \cap L^\alpha_t \mathcal{X} \cap L^\beta_t \mathcal{X}([a, b] \times \mathbb{R}^3)} \leq e_{m_0} \), then

\[
u = u - \eta
\]

satisfies the bound

\[
\|v\|_{L^\infty_a L^2_x \cap L^\infty_t L^2_x \cap L^\alpha_t \mathcal{X} \cap L^\beta_t \mathcal{X}([a, b] \times \mathbb{R}^3)} \leq \frac{B_{m_0}}{2},
\]

\(^5\) The setting were slightly different, but the proof works line by line the same if one plugs \( \varepsilon = 0 \) in Lemma 4.3 or formula (4.19) in [27].
As a consequence, we have

$$\|u\|_{L_t^\infty L_x^\beta((a,b)\times\mathbb{R}^3)} \lesssim B_{m_0}.$$  \hspace{1cm} (5.27)

**Proof.** The key to the proof, as in the proof of Proposition 4.6, [27], is to study the equation of $v$ rather than $u$. And the $L_t^2 L_x^\beta_\infty \cap L_t^\infty L_x^2$ bound was already proven there.\(^6\)

We only need to do the $L_t^\alpha X^{\frac{1}{2}, \tilde{\beta}}$. Note that $v$ solves

$$\begin{cases}
i v_t - \Delta v + i\varepsilon^2 V^2 v = (|v + \eta|^{4/3})(v + \eta), \\
u(t_1) = u(t_1).
\end{cases}$$  \hspace{1cm} (5.28)

Let $F_1 = -\frac{i\varepsilon^2}{2} \cdot V^2 v$, then $F_1 \in L_t^2 L_x^{6/5}$.

Let $F_2 = (|v + \eta|^{4/3})(v + \eta)$, then $F_2 \in L_t^{a'} L_x^{b'}$ for some $(a, b)$ admissible (since $v$, $\eta$ both in $L_t^a L_x^\beta$). By Duhamel formula, one has

$$v(t) = e^{i(t-a)\Delta} v(a) - i \int_a^t e^{i(t-s)\Delta} (F_1(s) + F_2(s))ds.$$  \hspace{1cm} (5.29)

Now, applying the bounds (5.17) and (5.18), the claim then follows from standard continuity argument.  \hfill \square

6. **Main Estimates for the Linear Model: Proof of Lemmas 3.3 and 3.4**

6.1. **Proof of Lemma 3.3.** We fix an admissible pair $(\alpha, \beta)$ with $\alpha \in (2, 4)$ (and hence $\beta \in (3, 6)$), and split the integral into

$$\int_0^t H(t-s)(VH(s)u_0)dB_s = \int_0^{(t-1)\vee 0} H(t-s)(VH(s)u_0)dB_s$$

$$+ \int_{(t-1)\vee 0}^t H(t-s)(VH(s)u_0)dB_s.$$  \hspace{1cm}

We first prove following two estimates regarding linear solutions:

$$\|VH(t)u_0\|_{L_t^\alpha W_x^{\frac{1}{2}, \tilde{\beta}'}(\mathbb{R}^+ \times \mathbb{R}^3)} \lesssim \|u_0\|_{L_x^2}$$  \hspace{1cm} (6.1)

and

$$\|VH(t)u_0\|_{L_t^\alpha L_x^{\beta'}(\mathbb{R}^+ \times \mathbb{R}^3)} \lesssim \|u_0\|_{L_x^2}.$$  \hspace{1cm} (6.2)

These are the quantities inside the operator $H(t-s)$. For the first one, by localization of $V$ and the local smoothing estimate in Lemma 5.3, we have

$$\|VH(t)u_0\|_{L_t^\alpha W_x^{\frac{1}{2}, \tilde{\beta}'}(\mathbb{R}^+ \times \mathbb{R}^3)} \lesssim \|u_0\|_{L_x^2}.$$  \hspace{1cm}

The requirement for $(\alpha, \beta)$ being an admissible pair and $\alpha \in (2, 4)$ implies $\tilde{\beta} \in (\frac{12}{5}, 3)$. Since $\tilde{\beta}' < \beta$, by localization of $V$ and the Strichartz estimate in Lemma 5.2, we have

$$\|VH(t)u_0\|_{L_t^\alpha L_x^{\beta'}} \lesssim \|H(t)u_0\|_{L_t^\alpha L_x^\beta} \lesssim \|u_0\|_{L_x^2}.$$  \hspace{1cm}

\(^6\) Again, one may take the $\varepsilon_0 = 0$ in (4.38), [27], and the proof will go through.
Combining the above two bounds and applying Lemma 5.6, we obtain (6.1). (6.2) is a direct consequence of the fact $\beta' < \beta$, $V$ is localized, and the Strichartz estimate in Lemma 5.2.

Now, let $F(t) := H(t)u_0$. With the bounds (6.1) and (6.2), it remains to estimate the quantities

$$\int_0^{(t-1)\vee 0} H(t-s)F(s)dB_s \quad \text{and} \quad \int_0^t H(t-s)F(s)dB_s.$$

For the first one, using Minkowski to change the order of $L^\alpha_{t_0}$ and $L^\alpha_t$ and then apply Burkholder, we get

$$\left\| \int_0^{(t-1)\vee 0} H(t-s)F(s)dB_s \right\|_{L^\alpha_t L_{t_0}^{\beta'}} \lesssim \left\| \int_0^{(t-1)\vee 0} \|H(t-s)F(s)\|_{L^\alpha_t}^{\beta} \right\|_{L^\alpha_{t_0} L_t^{\beta'}}.$$

Using the dispersive estimate and that $t-s \geq 1$, as well as Young’s integration inequality, we get

$$\left\| \int_0^{(t-1)\vee 0} H(t-s)F(s)dB_s \right\|_{L^\alpha_t L_{t_0}^{\beta'}} \lesssim \left\| F \right\|_{L^\alpha_{t_0} L_t^{\beta'}}^2,$$

where in the application of Young, we have used that $\beta > 3$ so $6\left(\frac{1}{2} - \frac{1}{\beta'}\right) > 1$. The desired bound then follows from (6.2).

We now turn to the estimate for $\int_{(t-1)\vee 0}^t H(t-s)F(s)dB_s$. We apply dispersive estimate and Sobolev embedding to obtain

$$\|H(t-s)F(s)\|_{L^\alpha_t} \lesssim \|H(t-s)F(s)\|_{W^{1,\beta}}$$

$$\lesssim (t-s)^{-3\left(\frac{1}{2} - \frac{1}{\beta'}\right)} \|F\|_{W^{1,\beta'}}. \quad (6.3)$$

Next, for all $t$ fixed, by Burkholder inequality, we have

$$\left\| \int_{(t-1)\vee 0}^t H(t-s)F(s)dB_s \right\|_{L^\alpha_{t_0} W^{1,\beta}_t} \lesssim \left\| \int_{(t-1)\vee 0}^t \|H(t-s)F(s)\|_{W^{1,\beta}} \frac{1}{2} ds \right\|_{L^\alpha_{t_0}}^{1/2}. \quad (6.4)$$

Summarizing (6.3) and (6.4), we obtain

$$\left\| \int_{(t-1)\vee 0}^t H(t-s)(VH(s)u_0)ds \right\|_{L^\alpha_{t_0} L_t^{\beta'}(\Omega \times \mathbb{R}^+ \times \mathbb{R}^3)} \lesssim \int_{\mathbb{R}^+} \left( \int_{(t-1)\vee 0}^t \|H(t-s)(VH(s)u_0)\|_{W^{1,\beta}}^2 ds \right)^{\frac{\beta}{2}} dt \quad (6.5)$$

$$\lesssim \int_{(t-1)\vee 0}^t (t-s)^{-6\left(\frac{1}{2} - \frac{1}{\beta'}\right)} \|VH(s)u_0\|_{W^{1,\beta}}^2 ds \leq \left\| \int_{(t-1)\vee 0}^t (t-s)^{-6\left(\frac{1}{2} - \frac{1}{\beta'}\right)} \|VH(s)u_0\|_{W^{1,\beta}}^2 ds \right\|_{L_t^{\beta'}}.$$

An application of Young inequality closes the estimate via (6.1) since $(t-s)^{-6\left(\frac{1}{2} - \frac{1}{\beta'}\right)}$ is locally integrable (as a function of $s$, localized around $0 \leq t - s \leq 1$).
6.2. Proof of Lemma 3.4. Now we turn to the proof Lemma 3.4. We first observe that (3.13) and (3.14) follow from (3.11) and (3.12) via arguments similar to the previous section. Indeed, now we are estimating \( \int H(t-s)A(s)dB_s \) while in the previous section we are estimating \( \int H(t-s)F(s)dB_s \), and we just need to replace (6.1) and (6.2) by (3.11) and (3.12). Since the arguments are almost the same line by line, we omit the details here.

It now remains to prove (3.11) and (3.12). We remark that we choose the space of \( L_{\omega,1}^\alpha \) type so we can switch the order of integral in \( t \) and \( \omega \) freely.

Proof of (3.11). By Burkholder inequality, for \( t \) fixed, one has

\[
\|A(t)\|_{L_{\omega,1}^\alpha} \lesssim \left( \int_0^t \|VH(t-s)(Vu(s))\|^2_{W_x^{1/2,\beta'}} \, ds \right)^{1/2} .
\] (6.6)

We are left with the proof of

\[
\left\| \int_0^t \|VH(t-s)(Vu(s))\|^2_{W_x^{1/2,\beta'}} \, ds \right\|_{L_{\omega,1}^\alpha} \lesssim \|u\|^2_{L_{\omega,1}^\alpha L_{t}^\beta}. \] (6.7)

We note that one should not worry too much about the integrability in \( x \) but focus rather on regularity in \( x \), since \( V \) is Schwartz. And one can also process in a deterministic way, estimate via fixing \( \omega \) and take \( L_{\omega,1}^\alpha \) at the end.

One may split the integral in (6.7) into \( |t-s| \leq 1 \) and \( |t-s| \geq 1 \). For \( |t-s| \leq 1 \), first applying Hölder inequality (relying on \( \alpha \geq 2 \)) and then switching the order of integrations in \( s \) and \( t \), we get

\[
\left\| \int_0^t \chi_{|t-s|\leq 1} \|VH(t-s)(Vu(s))\|^2_{W_x^{1/2,\beta'}} \, ds \right\|_{L_{t}^\alpha} \lesssim \int_{\mathbb{R}^+} \int_0^t \|VH(t-s)(Vu(s))\|^2_{W_x^{1/2,\beta'}} \, ds \, dt
\]

\[
= \int_{\mathbb{R}^+} \left( \int_s^{s+1} \|VH(t-s)(Vu(s))\|^2_{W_x^{1/2,\beta'}} \, dt \right) \, ds .
\] (6.8)

Applying the bound (6.1) and using localization of \( V \), one obtains

\[
\int_{\mathbb{R}^+} \left( \int_s^{s+1} \|VH(t-s)(Vu(s))\|_{W_x^{1/2,\beta'}} \, dt \right) \, ds \lesssim \int_{\mathbb{R}^+} \|Vu(s)\|^\alpha_{L_t^\beta} \, ds \lesssim \|u\|^\alpha_{L_{\omega,1}^\alpha L_t^\beta} .
\] (6.9)

Taking expectation in \( \omega \), the desired bound then follows.

For \( |t-s| \geq 1 \), applying the local smoothing estimate in Lemma 5.4 and using localization of \( V \), we get

\[
\|\langle \cdot \rangle^{-10}H(t-s)(Vu(s))\|_{\dot{H}^1} \lesssim \|u(s)\|_{L_t^\beta(t-s)^{-1}} .
\] (6.10)

Thus, we derive

\[
\|VH(t-s)(Vu(s))\|_{W_x^{1,\beta'}} \lesssim \|u(s)\|_{L_t^\beta(t-s)^{-1}} .
\] (6.11)
By Young’s inequality, we obtain
\[
\left\| \int_0^t \chi_{|t-s| \geq 1} \| u(s) \|_{L_t^\beta L_x^2}^2 (t-s)^{-2} ds \right\|_{\frac{1}{2}} \lesssim \| u \|_{L_t^\infty L_x^\delta}.
\] (6.12)

Taking \( L_\omega^\alpha \) on both sides and desired estimate will follow. This completes the proof for (3.11).

**Proof of (3.12).** By Burkholder inequality, for \( t \) fixed, one has
\[
\| A(t) \|_{L_\omega^\alpha L_x^{\beta'}} \lesssim \left\| \int_0^t V H(t-s)(V u(s)) \|_{L_x^{\beta'}}^2 \right\|_{L_\omega^2}^{1/2},
\] (6.13)
and it remains to estimate the right hand side above.

On one hand, using localization of \( V \), we have
\[
\| V H(t-s)(V u(s)) \|_{L_x^{\beta'}} \lesssim \| V u(s) \|_{L_x^2} \lesssim \| u(s) \|_{L_x^{\beta}}, \text{ for } t-s \leq 1.
\] (6.14)

On the other hand, by dispersive estimate, we have
\[
\| V H(t-s)(V u(s)) \|_{L_x^{\beta'}} \lesssim \| H(t-s)V(x)u(s) \|_{L_x^\infty} \lesssim (t-s)^{-3/2} \| u(s) \|_{L_x^{\beta}}, \text{ for } t-s \geq 1.
\] (6.15)

To summarize, we have
\[
\int_0^t \| V H(t-s)(V u(s)) \|_{L_x^{\beta'}}^2 ds \lesssim \int_0^t (1+|t-s|)^{-3} \| u(s) \|_{L_x^{\beta}}^2 ds.
\] (6.16)

Applying Young’s inequality in \( t \) and taking the \( L_\omega^\alpha \) norm, the desired estimates will follow.

7. Main Estimates for the Nonlinear Model—Proof of Proposition 4.2 and Lemma 4.3

In this section, we give proofs for Proposition 4.2 and Lemma 4.3, thus completing the proof of the global bound (1.3) for \( \alpha \in \left[ \frac{7}{3}, 4 \right) \). The bound for range \( \alpha \geq 4 \) follow from interpolating \( \alpha = \frac{7}{3} \) with the \( L_t^\infty L_x^2 \) bound from the mass conservation. We start with Proposition 4.2.

7.1. Proof of Proposition 4.2. The proof of Proposition 4.2 is schematically similar to the proofs of Lemmas 3.6 and 4.10 in [27], with extra focus on the local smoothing effect. It in particular relies on Proposition 5.9 in the current article, which is an enhancement of [27, Proposition 4.6].

Recall Proposition 5.9 and the constants \( e_{m_0} \) and \( B_{m_0} \). Also recall the discussion below Proposition 4.2 that it suffices to prove the bound for \( T = +\infty \). We fix a realization \( \omega \in \Omega \). Divide \( \mathbb{R}^+ \) into \( J \) intervals \( \left\{ [\tau_k, \tau_{k+1}) \right\}_{k=0}^{J-1} \) such that
\[
\| M^* \|_{L_t^\alpha L_x^\delta([\tau_k, \tau_{k+1}] \times \mathbb{R}^3)} \leq e_{m_0},
\]
and that the total number $J$ of intervals is bounded by

$$J \leq 1 + \frac{\|M^*\|_{L^q_t(R^+)}^\alpha}{e^{m_0}}. \quad (7.1)$$

On each interval $[\tau_k, \tau_{k+1}]$, $u$ satisfies

$$u(t) = H(t - \tau_k)u(\tau_k) - i \int_{\tau_k}^t H(t - s)(|u(s)|^4 u(s))ds + \eta_k(t), \quad t \in [\tau_k, \tau_{k+1}],$$

where

$$\eta_k(t) = -i \varepsilon \int_{\tau_k}^t H(t - s)(V u(s))dB_s$$

satisfies $\|\eta\|_{\mathcal{X}([\tau_k, \tau_{k+1}] \times \mathbb{R}^3)} \leq e^{m_0}$ by definition. We define $u_2$ on $t \in \mathbb{R}^+$ by

$$u_2(t) := \eta_k(t), \quad t \in [\tau_k, \tau_{k+1}],$$

and $u_1 := u - u_2$. The pointwise bound (4.4) follows directly from the definition of $M^*$ and $u_2$. As for $u_1$, by Proposition 5.9, we have

$$\|u_1\|_{L^\infty_t \mathcal{X}(\mathbb{R}^+ \times \mathbb{R}^3)} \leq B_m^{a} \cdot J.$$ 

Taking $L^\infty_t \mathcal{X}$-norm on both sides above and using (7.1), the bound (4.3). This completes the proof of Proposition 4.2.

### 7.2. Proof of Lemma 4.3.

In view of splitting the time interval, it suffices to consider two cases:

$$\left\| \left( \int_0^{(t-1)\vee 0} \|H(t - s)(VG(s))\|_{\mathcal{X}}^2 ds \right)^{\frac{1}{2}} \right\|_{L^q_t \mathbb{R}^+} \quad (7.2)$$

and

$$\left\| \left( \int_{(t-1)\vee 0}^t \|H(t - s)(VG(s))\|_{\mathcal{X}}^2 ds \right)^{\frac{1}{2}} \right\|_{L^q_t \mathbb{R}^+}. \quad (7.3)$$

Also, we note that estimate (4.8) is indeed two estimates noticing the definition of the $\mathcal{X}$-norm. We consider the $X^{\frac{1}{2}}_{\frac{1}{2}}\beta'^2$-norm case and the $L^\beta_X$-norm case respectively. Thus totally there are four sub-cases.

We consider $X^{\frac{1}{2}}_{\frac{1}{2}}\beta'^2$-norm. For the first case, we use the second local smoothing estimate (5.5) together with the Young’s inequality,

$$\left\| \left( \int_0^{t-1} \|H(t - s)(VG(s))\|_{X^{\frac{1}{2}}_{\frac{1}{2}}\beta'^2} ds \right)^{\frac{1}{2}} \right\|_{L^q_t \mathbb{R}^+} \lesssim \left\| \left( \int_0^{t-1} (t - s)^{-2} \|VG(s)\|_{L^2_s} ds \right)^{\frac{1}{2}} \right\|_{L^q_t \mathbb{R}^+}$$
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\[ \lesssim \left\| \left( \int_0^{t-1} (t - s)^{-\frac{2}{\beta}} \| G(s) \|^2_{L_x^\beta} ds \right)^{\frac{1}{2}} \right\|_{L_t^q \mathbb{R}^+} \]
\[ \lesssim \| G \|_{L_t^q \mathbb{X}}. \]

For the second case, we use the first local smoothing estimate Lemma 5.3 together with the observation that on finite interval \( L^2 \)-norm can be improved to \( L^\alpha \)-norm \((\alpha > 2)\)

\[ \left\| \left( \int_{(t_1) \vee 0}^t \| H(t) \|_{L_x^\beta} \right)^2 ds \right\|_{L_t^q \mathbb{R}^+} \]
\[ \lesssim \left\| \left( \int_{t_1}^t \| H(t) \|_{X^\frac{1}{\alpha} \beta^\prime} \right)^2 ds \right\|_{L_t^q \mathbb{R}^+} \]
\[ \lesssim \| H(t) \|_{L_x^\beta \mathbb{L}_t^q \mathbb{L}_x^\alpha} \]
\[ \lesssim \| G \|_{L_t^q \mathbb{X}}. \]

We now consider \( L_x^\beta \)-norm. For the first case, we use the dispersive estimate (5.1) together with the Young’s inequality,

\[ \left\| \left( \int_0^{t-1} (t - s)^{-\frac{3}{2} - \frac{3}{\beta'}} \| V G(s) \|^2_{L_x^\beta} ds \right)^{\frac{1}{2}} \right\|_{L_t^q \mathbb{R}^+} \]
\[ \lesssim \left\| \left( \int_0^{t-1} (t - s)^{-\frac{3}{2} - \frac{3}{\beta'}} \| V G(s) \|^2_{W_1^\beta} ds \right)^{\frac{1}{2}} \right\|_{L_t^q \mathbb{R}^+} \]
\[ \lesssim \left\| \left( \int_0^{t-1} (t - s)^{-\frac{3}{2} - \frac{3}{\beta'}} \| G(s) \|^2_{W_1^\beta} ds \right)^{\frac{1}{2}} \right\|_{L_t^q \mathbb{R}^+} \]
\[ \lesssim \| G \|_{L_t^q \mathbb{X}}. \]

For the second case, we use the dispersive estimate (5.1), Sobolev embedding and the Young’s inequality,

\[ \left\| \left( \int_{t_1}^t \| H(t) \|_{L_x^\beta} \right)^2 ds \right\|_{L_t^q \mathbb{R}^+} \]
\[ \lesssim \left\| \left( \int_{t_1}^t \| H(t) \|_{W_1^\beta} \right)^2 ds \right\|_{L_t^q \mathbb{R}^+} \]
\[ \lesssim \left\| \left( \int_{t_1}^t (t - s)^{-\frac{3}{2} - \frac{3}{\beta'}} \| V G(s) \|^2_{L_x^\beta} ds \right)^{\frac{1}{2}} \right\|_{L_t^q \mathbb{R}^+} \]
\[ \lesssim \left\| \left( \int_{t_1}^t (t - s)^{-\frac{3}{2} - \frac{3}{\beta'}} \| G(s) \|^2_{W_1^\beta} ds \right)^{\frac{1}{2}} \right\|_{L_t^q \mathbb{R}^+} \]
\[ \lesssim \| G \|_{L_t^q \mathbb{X}}. \]

This completes the proof.
8. On Scattering Behavior—Proof of (1.5)

As mentioned earlier, unlike the deterministic critical equation, the scattering behavior in the current stochastic setting is not a direct consequence of the global space-time bound (1.3), and extra efforts are needed to obtain it. The proof in this section is a refinement of the corresponding part in [27], using the ideas in the proof of the main bound (1.3).

To obtain scattering behavior, it is more natural to write down the Duhamel formula based on the free Schrödinger operator:

\[
e^{-it\Delta} u(t) = u_0 - i \int_0^t e^{-is\Delta} (|u(s)|^{4/3} u(s)) \, ds - i\varepsilon \int_0^t e^{-is\Delta} (Vu(s)) \, dB_s - \frac{\varepsilon^2}{2} \int_0^t e^{-is\Delta} (V^2 u(s)) \, ds.
\]

(8.1)

We need to show that all terms on the right hand side above converge in \( L^2_\omega L^2_t \) as \( t \to +\infty \). Same as in [27], the term with the nonlinearity can be handled with the space-time bound (1.3) via the Strichartz estimate

\[
\left\| \int_0^t e^{-is\Delta} (|u(s)|^{4/3} u(s)) \, ds \right\|_{L^2_t L^2_\omega} \lesssim \| u \|^7/3 \| L^{3\alpha/7}_{t\omega} L^{\beta/7}_{x\omega} (|t'| \times \mathbb{R}^3) \lesssim \| u \|^{7/3} L^{3\alpha}_{t\omega} L^{\beta}_{x\omega} (|t'| \times \mathbb{R}^3),
\]

where \((\frac{3\alpha}{7}, \frac{3\beta}{7})\) is dual of a Strichartz pair if \((\alpha, \beta)\) is admissible with \( \alpha \geq \frac{7}{3} \). The other two terms on the right hand side will converge if we can prove that

\[
\| u \|_{L^2_t L^2_\omega W(\Omega \times \mathbb{R}^+ \times \mathbb{R}^3)} < +\infty,
\]

(8.2)

where

\[
\| f \|_W := \| \langle \cdot \rangle^{-100} f \|_{L^5_x}.
\]

(8.3)

Note that \( \| f \|_W \lesssim \| f \|_{L^6_x} \). We will actually prove a stronger norm than \( W \). Let \( Z \) be defined as

\[
\| f \|_Z := \| \langle \cdot \rangle^{-10} f \|_{H^{1/2}}.
\]

(8.4)

We will see that this norm appears naturally in the bootstrap scheme below. Let \( m_0 := \| u_0 \|_{L^\infty_t L^2_x} \), and \( \varepsilon \) be chosen small enough (and fixed).

In order to prove the bound (8.2) (with the stronger \( Z \)-norm instead of \( W \)-norm), we write down the Duhamel formula based on the damped operator \( H \):

\[
u(t) = H(t)u_0 - i\varepsilon \int_0^t H(t-s)(Vu(s)) \, dB_s - i \int_0^t H(t-s) (|u(s)|^{4/3} u(s)) \, ds.
\]

(8.5)

Via (8.5), the crucial space time bound (1.3), we can obtain (8.2) if we can prove

\[
\left\| \int_0^t H(t-s)(Vu(s)) \, dB_s \right\|_{L^2_t L^2_\omega Z} \lesssim m_0 1.
\]

(8.6)
In order to obtain this bound, we will need to work with a stronger norm (than $M^*$ as defined in (4.1)) of the stochastic term. More precisely, we introduce

$$M^*_1(t) := \varepsilon \sup_{0 \leq a \leq b \leq t} \left\| \int_a^b H(t-s)(V u(s)) dB_s \right\|_{L^2}.$$  \hspace{1cm} (8.7)

Clearly (8.6) follows if we can prove

$$\|M^*_1\|_{L^2_{\omega}L^2_t(\Omega \times \mathbb{R}^+)} \lesssim m_0 1.$$  \hspace{1cm} (8.8)

Standard bootstrap technique will yield (8.8) once we have a priori estimate

$$\|M^*_2\|_{L^2_{\omega}L^2_t(\Omega \times \mathbb{R}^+)} \lesssim m_0 1 + \varepsilon \|M^*_1(t)\|_{L^2_{\omega}L^2_t(\Omega \times \mathbb{R}^+)}.$$  \hspace{1cm} (8.9)

The key to prove (8.9) is the following lemma, which is an enhanced version of Proposition 4.2 suited for the bootstrap scheme for $L^2_{\omega}Z$-norm.

**Lemma 8.1.** Let $u$ be the solution to (1.1). Then we can split $u = u_1 + u_2$ such that\(^7\)

$$\|u_1\|_{L^2_{\omega}L^2_t Heroes(\Omega \times \mathbb{R}^+) \lesssim 1 + \|M^*\|_{L^2_{\omega}L^2_t}^2.$$  \hspace{1cm} (8.10)

and

$$\|u_2\|_{L^2_{\omega}L^2_t Heroes(\Omega \times \mathbb{R}^+) \lesssim \|M^*_1\|_{L^2_{\omega}L^2_t(\Omega \times \mathbb{R}^+)},$$  \hspace{1cm} (8.11)

**Proof.** The argument is essentially the same as the proof of [27, Lemma 3.6] and Proposition 4.2, except that we need to apply a stronger version of Proposition 5.9 with $Z$-norm included. But the arguments remain essentially the same. \hspace{1cm} \Box

The important in Lemma 8.1 is that $u_1$ is controlled by $\|M^*\|_{L^2_{\omega}L^2_t}$, which has been bounded already in Sects. 4 and 7.

Applying Burkholder inequality, for fixed $t$, we have

$$\|M^*_1(t)\|_{L^2_{\omega}} \lesssim 2^2 \int_0^t \left\| \langle x \rangle^{-10}(\nabla)^{1/2}(H(t-s)(V u(s))) ds \right\|_{L^2_{\omega}L^2_t}.$$  \hspace{1cm} (8.12)

Now, split $u = u_1 + u_2$ as in Lemma 8.1, and arguing similarly as in the proof of Lemma 4.3, we have (for $i = 1, 2$) that

$$\int_0^t \left\| H(t-s)(V u_i(s)) \right\|_{L^2_{\omega}} ds \lesssim \int_{|t-s|<1, s\in[0,t]} \left\| \langle \nabla \rangle^{1/2}(V u_i(s)) \right\|_{L^2_t} ds$$

$$+ \int_{|t-s|>1, s\in[0,t]} (t-s)^{-3} \left\| \langle D \rangle^{1/2} V u_i(s) \right\|_{L^1_t} ds.$$  \hspace{1cm} (8.13)

Taking $L^2_{\omega}L^2_t$-norm on both sides, using localization of $V$ to reduce the right hand side to $\|u_1\|_{L^2_{\omega}L^2_t Heroes(\Omega \times \mathbb{R}^+) \lesssim 1}$ and $\|u_2\|_{L^2_{\omega}L^2_t Heroes(\Omega \times \mathbb{R}^+) \lesssim 1}$, and applying Lemma 8.1 give the desired bootstrap bound (8.9). This completes the proof of the scattering behavior (1.5).

The above sure scattering behavior, the previous $L^2_{\omega}$ bounds imply that the relevant quantities are almost surely finite. These in particular imply the almost sure convergence (in $L^2_\omega$) of all the terms on the right hand side of (8.1) except the one involving stochastic integration. The stochastic term also converges almost surely in $L^2_\omega$ since it is $L^2_{\omega}$-bounded martingale in $L^2_\omega$. Hence $u(t) - e^{it\Delta}u^* \rightarrow 0$ almost surely in $L^2_\omega$.

\(^7\) Roughly speaking, $u_1$ satisfies all good properties of the solution to the free linear Schrödinger equation, and recall that $\|e^{it\Delta}f\|_{L^2_{\omega}L^2_t} \lesssim \|f\|_{L^2_\omega}$. 

---

Long Time Behavior of Stochastic NLS
9. Proof of \((1.4)\)—Extending to All Non-endpoint \((\alpha, \beta)\)

In this section, we will finally complete the bound \((1.4)\) for all \(\alpha > 2\). It suffices to prove it for \(\alpha \in (2, 4)\). So we fix \((\alpha, \beta)\) admissible with \(\alpha \in (2, 4)\). Recalling \((1.13)\), this in particular implies \(\beta > 3\) and \(\tilde{\beta} < 3\).

First, observe that by Lemma 8.1 and the estimate \((8.8)\), we have
\[
\|x^{-100}(\nabla)^{1/2} u\|_{L^2_t L^2_x} \lesssim 1. \tag{9.1}
\]

Now, we write \(u\) in terms of the Duhamel formula for \(S(t)\) so that
\[
u(t) = e^{it} u_0 - i \int_0^t e^{i(t-s)\Delta} (|u(s)|^{4/3} u(s)) ds
- i \varepsilon \int_0^t e^{i(t-s)\Delta} (V u(s)) dB_x - \frac{\varepsilon^2}{2} \int_0^t e^{i(t-s)\Delta} (V^2 u(s)) ds. \tag{9.2}
\]

Except for the stochastic integral term, all other terms can be directly handle by Strichartz estimate (using the bound \((1.3)\) with \((a, b)\) admissible and \(a \geq \frac{1}{2}\)), and bounded in \(L^2_\omega L^\alpha_t L^\beta_x\). We only need to handle the stochastic term. We will indeed control it in \(L^\alpha_\omega L^\alpha_t L^\beta_x\).

Note that by Lemma 5.6 and the mass conservation, \((9.1)\) implies the bound
\[
\|Vu\|_{L^\alpha_\omega L^\alpha_t \dot{W}^{\frac{1}{\tilde{\beta}}}_t} + \|Vu\|_{L^\alpha_\omega L^\alpha_t L^{\tilde{\beta}}_x} \lesssim_{m_0} 1 \tag{9.3}
\]
where we let \(\dot{W}^{\frac{1}{\tilde{\beta}}}_t\) embeds \(L^{\tilde{\beta}}\), noting that \(\tilde{\beta} < 3\). Similar as before, We split the stochastic integral into \(|t-s| < 1\) and \(|t-s| > 1\).

For \(|t-s| > 1\), we apply Burkholder, and observe
\[
\int_0^t \chi_{|t-s| > 1} \|S(t-s)Vu(s)\|_{L^\beta_x} ds \lesssim \int_0^t \langle t-s \rangle^{-6(1/2-1/\beta)} \|Vu(s)\|_{L^\beta_x}^2 ds \tag{9.4}
\]
and an application of Young inequality will end the proof, noting that \(6(1/2 - 1/\beta) > 1\).

For \(|t-s| < 1\), using Burkholder, we have
\[
\int_0^t \chi_{|t-s| < 1} \|S(t-s)Vu(s)\|_{L^\beta_x}^2 ds \lesssim \int_0^t \chi_{|t-s| < 1} \langle t-s \rangle^{-6(\frac{1}{2} - \frac{1}{\beta})} \|D^{1/2} Vu(s)\|_{L^\beta_x}^2 ds \tag{9.5}
\]
and \(6(\frac{1}{2} - \frac{1}{\beta}) < 1\) since \(\tilde{\beta} < 3\), and we use Young to conclude. This completes the proof of \((1.4)\).

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Declarations

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Appendix A: Generalization to Infinite Dimensional Noise Case and Further remarks

In this section, we explain how to generalize the results in this article (and those in [25,27]) to certain infinite dimensional noise setting. The material in this appendix is essentially contained in the pioneering work of de Bouard and Debussche ([15]), though maybe in an implicit way. We explain it with details for the convenience of the readers.

Note that in this section, we are focusing on how to transfer a result to noise of infinite dimensions for (A.2), assuming a result for (A.4) has already obtained for Wiener process of simple form \( W(t, x) = V(x) B_t \). So the smallness of noise will not be relevant here (or one can always absorb the smallness into \( W \) by multiplying an extra \( \epsilon \)), and we will omit it.

Before we get into more details, we would like to emphasize again that while our results can be generalized to certain infinite dimensional setting, to remove smallness assumption of the current article even in the linear case with the simplest noise \( W(t, x) = V(x) B(t) \) is highly nontrivial, and it is not clear to us now how one could achieve it. One may compare this situation to the seminal work of Rodnianski-Schlag ([36]) for dispersive estimates with a time dependent potential.

Instead of (1.10), it is more convenient to consider the more general form

\[
W(t, x) = \sum_k (\Phi e_k)(x) B^{(k)}_t,
\]

(A.1)

where \( \{B^{(k)}_t\}_k \) are independent standard Brownian motions, \( \{e_k\}_k \) is an orthonormal basis of \( L^2(\mathbb{R}^d) \), and \( \Phi \) is an operator with assumptions to be specified in Sect. A.1 below. The exact dimension \( d \) is not important in this section.

The \( \ell^2 \)-summability of the coefficients in (1.10) is contained in the assumptions on \( \Phi \) (see Sect. A.1 below).

The associated stochastic nonlinear Schrödinger equation corresponding to the noise (A.1) is of form (see also (1.5) in [15])

\[
i \partial_t u + \Delta u = \mathcal{N}(u) - \frac{i}{2} F_{\Phi} u + u \cdot \dot{W},
\]

(A.2)
where \( \cdot \) denotes the Itô product, \( \mathcal{N}(u) \) is the nonlinear part (or simply 0 in the linear case), and
\[
F_\Phi(x) := \sum_k (\Phi e_k)^2(x)
\]
(A.3)
is the Itô-Stratonovich correction. Note that the exact form of \( \mathcal{N}(u) \) is also not relevant in this section.

When one considers the special case with \( \Phi e_1 = V, \Phi e_k = 0 \) for all other \( k \), then \( F_\Phi = V^2 \), and associated stochastic PDE is
\[
i \partial_t u + \Delta u = \mathcal{N}(u) - \frac{i}{2} V^2 u + V u \cdot \dot{B}.
\]
(A.4)
This recovers the situation in our main text.

A.1. \( \gamma \)-radonifying operator and assumptions on \( \Phi \) for general noise. The key notion with the assumption of \( \Phi \) is the \( \gamma \)-radonifying operator (\cite[Sect. 2]{15}).

Recall an operator \( K \) is \( \gamma \)-radonifying from a Hilbert space \( H \) to a Banach space \( B \) (denoted by \( K \in R(H,B) \)) if
\[
\|K\|_{R(H,B)} := \left( \mathbb{E}(\| \sum_k \gamma_k K \tilde{e}_k \|_B^2) \right)^{1/2} < +\infty,
\]
where \( \gamma_k \) are i.i.d standard normal random variables, and \( \{\tilde{e}_k\} \) is an orthonormal basis of \( H \). The exact choice of the basis does not affect the value \( \|K\|_{R(H,B)} \), which we call the \( \gamma \)-radonifying norm of \( K \) (from \( H \) to \( B \)).

Note that when \( B \) is also a Hilbert space, then \( \|K\|_{R(H,B)} \) is just the Hilbert-Schmidt norm \( (\sum_k \|K e_k\|_B^2)^{1/2} \). This is why one often expresses an operator be \( \gamma \)-radonifying with \( \ell^2 \)-summability condition on the coefficients.

Now, given that certain results\(^8\) are true for the model (A.4) for \( V \in X \) and \( V^2 \in Y \), where \( X \) and \( Y \) are Banach spaces which only involve (weighted) \( L^2 \) and \( L^p \) based spaces and that the norm of \( X \) is strictly stronger than \( L^2 \), to transfer the same results to (A.2), the proper assumptions to set for \( \Phi \) is that \( \Phi \) is \( \gamma \)-radonifying from \( L^2(\mathbb{R}^d) \) to \( X \), and \( F_\Phi \in Y \) (note this is again an \( \ell^2 \) assumption for \( \Phi e_k \)).

It should also be noted and will be explained in Sect.A.3 below that in practice, the assumption \( F_\Phi \in Y \) can often be obtained from the assumption that \( \Phi \) is \( \gamma \)-radonifying from \( L^2(\mathbb{R}^d) \) to \( X \).

For example, for the current article, let \( \|f\|_X := \sup_{|\alpha|+|\beta| \leq N} \|\langle x \rangle^\alpha \langle D \rangle^\beta f\|_{L^2} \) for sufficiently large \( N \) (\( N = 1000 \) is enough),\(^9\) then one can follow our proof line by line to see for all \( V \) such that \( \|V\|_X < \infty \), there exists an \( \epsilon \) depending on initial data \( u_0 \) and \( V \) such that the associated solution to (A.4) scatters. Then one will have, by rewriting the estimates using the Language of radonifying operator as we explained below, that, for all \( \Phi \) be \( \gamma \)-radonifying from \( L^2 \) to \( X \), there exists an \( \epsilon \) depending on initial data \( u_0 \) and \( \|\Phi\|_{R(L^2,X)} \) only so all the associated solution to (A.2) scatters.

\(^8\) We note that although the current article is written for Schwartz \( V \), only finitely many derivatives and a finite order of decay is needed in the proof. In particular, it is enough to assume \( \sup_{|\alpha|+|\beta| \leq N} \|\langle x \rangle^\alpha D^\beta V\|_{L^2} < \infty \) with sufficiently large \( N \).

\(^9\) Such a choice is for simplicity only and far from being optimal.
We explain below how to handle estimates involving the stochastic integral and $F_\Phi$ with the above assumptions, and hence one can generalize the results for model (A.4) to (A.2).

A.2. Burkholder type estimate for the stochastic integral. We first recall [15, Lemma 2.1]. Let $H$ be a separable Hilbert space and $E, F$ be separable Banach spaces. If $K \in R(H, E)$ and $L \in L(E, F)$ is a bounded linear operator from $E$ to $F$, then one has

$$
\| L \circ K \|_{R(H, F)} \lesssim \| L \|_{E \to F} \| K \|_{R(H, E)}. \quad (A.5)
$$

We can now explain how to generalize Burkholder type estimates for stochastic integrals involved in the model (A.4) to those in (A.2). We focus on a concrete example to for illustration.

Assume, for some Banach space $X_1$, $X_2$, and $X_3$, one has the functional inequalities

$$
u(t) V \|_{X_2} \lesssim \| u(t) \|_{X_1} \| V \|_{X},$$

$$
\| e^{i(t-s)\Delta} v(s) V \|_{X_3} \lesssim \| e^{i(t-s)\Delta} x_2 \rightarrow x_3 \| u(s) V \|_{X_2}.
$$

(A.6)

Then for the model (A.4), one applies the Burkholder inequality to the quantity $\int_0^t e^{i(t-s)\Delta} (uV) dB_s$ and use (A.6) to obtain

$$
\left\| \int_0^t e^{i(t-s)\Delta} (uV) dB_s \right\|_{L^p_{\omega} X_3} \lesssim \left( \int_0^t \left( \| e^{i(t-s)\Delta} x_2 \rightarrow x_3 \| u(s) \|_{X_2} \| V \|_{X_3} \|^2 ds \right)^{\frac{1}{2}} \right)^{\frac{1}{2}} \quad (A.7)
$$

Then for (A.2), one just applies the Burkholder inequality ([5, 8] and [16, page 168]) to $\int_0^t e^{i(t-s)\Delta} (udW_s)$ to obtain

$$
\left\| \int_0^t e^{i(t-s)\Delta} (udW_s) \right\|_{L^p_{\omega} X_3} \lesssim \left( \int_0^t \| e^{i(t-s)\Delta} u(s) \Phi \|_{R(L^2, X_3)}^2 ds \right)^{\frac{1}{2}} \quad (A.8)
$$

Using (A.5) and (A.6), we see the left hand side above can be controlled by

$$
\left\| \int_0^t \left( \| e^{i(t-s)\Delta} x_2 \rightarrow x_3 \| u(s) \|_{X_2} \Phi \|_{R(L^2, X_3)}^2 ds \right)^{\frac{1}{2}} \right\|_{L^p_{\omega}} \quad (A.9)
$$

Thus the assumption that $V \in X$ for some Banach space $X$ for the model (A.4) can be replaced by the assumption $\| \Phi \|_{R(H, X)}$ for model (A.2).

A.3. Estimates involved with $F_\Phi$. In this article and also in the works [23, 25, 27], the estimates for model (A.4) involving $-\frac{1}{2} V^2 u$ or associated estimates involving $-\frac{1}{2} F_\Phi u$ for (A.2) are purely deterministic, and the latter depends on $\| V^2 \|_{Y}$ only. Hence it is enough to assume $F_\Phi \in Y$ so that all the estimates with terms involving $V^2$ automatically carry to $F_\Phi$.

What we would like to emphasize here is that in practice, the assumption for $F_\Phi \in Y$ is often contained in the $\gamma$-radonifying assumptions of $\Phi$. This is similar to that $V \in L^p$ implies $V^2 \in L^\frac{p}{2}$.

Parallel estimates for $F_\Phi$ as the following hold. Let $\Phi$ be $\gamma$-radonifying from $L^2$ to $L^2 \cap L^p$, then $F_\Phi \in L^{p/2}$ ( [16, page 173]). Since $X$ and $Y$ only involves (weighted) $L^2$ and $L^p$ based norms, one can absorb the assumptions $F_\Phi \in Y$ into $\| \Phi \|_{R(L^2, X)} < \infty$ when $X$ be chosen properly.
A.4. Conclusion. From the above discussions, one deduces that results for the model (A.4) with $V \in X$ can be systematically transferred to the model (A.2) if $\Phi$ is $\gamma$-radonifying from $L^2(\mathbb{R}^d)$ to $X$ for a large class of choices of $X$. The seemingly simple model (A.4) indeed captures the essential key features of (A.2) under the above assumption on noise, at least when one is working on well-posedness type problems.

References

1. Barbu, V., Röckner, M., Zhang, D.: Stochastic nonlinear Schrödinger equations with linear multiplicative noise: rescaling approach. J. Nonlinear Sci. 24(3), 383–409 (2014)
2. Bourgain, J.: Invariant measures for the 2D-defocusing nonlinear Schrödinger equation. Commun. Math. Phys. 176(2), 421–445 (1996)
3. Brzeźniak, Z.: On stochastic convolution in Banach spaces and applications. Stoch. Stoch. Rep. 61(3–4), 245–295 (1997)
4. Brzeźniak, Z., Peszat, S.: Space-time continuous solutions to SPDE’s driven by a homogeneous Wiener process. Studia Math. 137(3), 261–299 (1999)
5. Brzeźniak, Z.: On stochastic convolution in Banach spaces and applications. Stoch.: Int. J. Probab. Stoch. Process. 61(3–4), 245–295 (1997)
6. Brzeźniak, Z., Liu, W., Zhu, J.: The stochastic Strichartz estimates and stochastic nonlinear Schrödinger equations driven by Lévy noise. J. Funct. Anal. 281(1) (2021)
7. Brzeźniak, Z., Millet, A.: On the stochastic Strichartz estimates and the stochastic nonlinear Schrödinger equation on a compact Riemannian manifold. Potential Anal. 41, 269–315 (2014)
8. Brzeźniak, Z., Peszat, S.: Space-time continuous solutions to spde’s driven by a homogeneous wiener process. Stud. Math. 137(3), 261–299 (1999)
9. Burkholder, D.L.: Distribution function inequalities for martingales. Ann. Probab. 1, 19–42 (1973)
10. Burkholder, D.L., Davis, B.J., Gundy, R.F.: Integral inequalities for convex functions of operators on martingales. In: Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability (Univ. California, Berkeley, Calif., 1970/1971), Vol. II: Probability Theory, pp. 223–240 (1972)
11. Cazenave, T.: Semilinear Schrödinger Equations, vol. 10. American Mathematical Society (2003)
12. Christ, M., Kiselev, A.: Maximal functions associated to filtrations. J. Funct. Anal. 179(2), 409–426 (2000)
13. Constantin, P., Saut, J.-C.: Effets régularisants locaux pour des équations dispersives générales. Comptes rendus de l’Académie des Sci. Série I, Math. 304(14), 407–410 (1987)
14. Da Prato, G., Debussche, A.: Strong solutions to the stochastic quantization equations. Ann. Probab. 31(4), 1900–1916 (2003)
15. de Bouard, A., Debussche, A.: A stochastic nonlinear Schrödinger equation with multiplicative noise. Commun. Math. Phys. 205 (1999)
16. de Bouard, A., Debussche, A.: A stochastic nonlinear Schrödinger equation in $H^1$. Stoch. Anal. Appl. 21(1), 97–126 (2003)
17. de Bouard, A., Debussche, A.: Blow-up for the stochastic nonlinear schrödinger equation with multiplicative noise (2005)
18. Dodson, B.: Global well-posedness and scattering for the defocusing, $L^2$-critical nonlinear Schrödinger equation when $d \geq 3$. J. Am. Math. Soc. 25(2), 429–463 (2012)
19. Dodson, B.: Global well-posedness and scattering for the defocusing, mass-critical nonlinear Schrödinger equation when $d \geq 3$. J. Am. Math. Soc. 25(2), 429–463 (2012)
20. Dodson, B.: Global well-posedness and scattering for the defocusing, $L^2$-critical, nonlinear Schrödinger equation when $d = 1$. Am. J. Math. 138(2), 531–569 (2016)
21. Dodson, B.: Global well-posedness and scattering for the defocusing, $L^2$-critical, nonlinear Schrödinger equation when $d = 2$. Duke Math. J. 165(18), 3435–3516 (2016)
22. Fan, C., Yiming, S., Zhang, D.: A note on log-log blow up solutions for stochastic nonlinear Schrödinger equations. Stoch. Partial Differ. Equ.: Anal. Comput. pp. 1–15 (2021)
23. Fan, C., Xu, W.: Global well-posedness for the defocusing mass-critical stochastic nonlinear Schrödinger equation on $\mathbb{R}$ at $L^2$ regularity, to appear in Analysis & PDE
24. Fan, C., Weijun, X.: Decay of the stochastic linear Schrödinger equation in $d \geq 3$ with small multiplicative noise. Stoch. Partial Differ. Equ.: Anal. Comput. (2020)
25. Fan, C., Weijun, X.: Decay of the stochastic linear schrödinger equation in $d \geq 3$ with small multiplicative noise. Stoch. Partial Differ. Equ.: Anal. Comput. 9(2), 472–490 (2021)
26. Fan, C., Xu, W.: A Wong–Zakai theorem for mass critical NLS. SIAM J. Math. Anal. (2021)
27. Fan, C., Zhao, Z.: On long time behavior for stochastic nonlinear Schrödinger equations with a multi-
  plicative noise. arXiv preprint arXiv:2010.11045 (2020)
28. Herr, S., Röckner, M., Spitz, M., Zhang, D.: The three dimensional stochastic Zakharov system. arXiv
  preprint arXiv:2301.02089 (2023)
29. Herr, S., Röckner, M., Zhang, D.: Scattering for stochastic nonlinear Schrödinger equations. Commun.
  Math. Phys. 368 (2019)
30. Hornung, F.: The nonlinear stochastic Schrödinger equation via stochastic Strichartz estimates. J. Evol.
  Equ. 18(3), 1085–1114 (2018)
31. Journé, J.-L., Soffer, A., Sogge, C.D.: Decay estimates for Schrödinger operators. Commun. Pure Appl.
  Math. 44(5), 573–604 (1991)
32. Keel, M., Tao, T.: Endpoint strichartz estimates. Am. J. Math., 955–980 (1998)
33. Kenig, C.E., Merle, F.: Global well-posedness, scattering and blow-up for the energy-critical, focusing,
  non-linear Schrödinger equation in the radial case. Invent. Math. 166(3), 645–765 (2006)
34. Kenig, C.E., Merle, F.: Global well-posedness, scattering and blow-up for the energy-critical focusing
  non-linear wave equation. Acta Math. 201(2), 147–212 (2008)
35. Röckner, M., Su, Y., Zhang, D.: Multi solitary waves to stochastic nonlinear Schrödinger equations.
  Probab. Theory Related Fields, pp. 1–64 (2023)
36. Rodnianski, I., Schlag, W.: Time decay for solutions of Schrödinger equations with rough and time-
  dependent potentials. Invent. Math. 155, 451–513 (2004)
37. Schlag, W.: Dispersive Estimates for Schrödinger Operators: A Survey, Mathematical Aspects of Non-
  linear Dispersive Equations. Ann. of Math. Stud., vol. 163, Princeton University Press, Princeton, NJ, pp.
  255–285 (2007)
38. Sjölin, P.: Regularity of solutions to the Schrödinger equation. Duke Math. J. 55(3), 699–715 (1987)
39. Tao, T.: Nonlinear Dispersive Equations: Local and Global Analysis, vol. 106. American Mathematical
  Society (2006)
40. Vega, L.: Schrödinger equations: pointwise convergence to the initial data. Proc. Am. Math. Soc. 102(4),
  874–876 (1988)
41. Wang, X.P.: Time-decay of semigroups generated by dissipative Schrödinger operators. J. Differ. Equ.
  253(12), 3523–3542 (2012)
42. Zhang, D.: Stochastic nonlinear Schrödinger equations in the defocusing mass and energy critical cases,
  arXiv preprint arXiv:1811.00167 (2018)

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