New interpretation of slave boson mean-field theory of the $t-J$ model: short-range antiferromagnetic and $d$-wave pairing correlations

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The $t-J$ Hamiltonian is studied in a mean-field approximation by taking into account antiferromagnetic and $d$-wave pairing correlations. Considering the presence of antiferromagnetic fluctuations, the weaknesses of a mean-field approximation and the limitation of the $t-J$ model near half-filling, we give a new interpretation to the slave boson mean-field theory of the $t-J$ model. We argue that due to phase coherence-breaking antiferromagnetic fluctuations and quantum fluctuations, superconducting long-range order does not appear strictly in two dimensions. $T_c$ resulting from interlayer pairing hopping can lead to a universal relation, when $T_c$ is scaled by $T_c^\text{max}$. Systematic reduction of superfluid density and increase of $(\Delta s) \text{max} / \kappa_B T_c$ ratio below and near optimal doping have their natural explanation in our picture. A crossover temperature $T^*$ found in some of magnetic experiments such as NMR is also easily understood in the present framework.

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I. INTRODUCTION

Since the discovery of high temperature superconductivity in Ln$_{2-x}$Ba$_x$CuO$_4$ by Bednorz and Müller, many anomalous features found in these materials have attracted considerable attention from condensed matter physicists. Not to mention the quantitative understanding of various puzzling experiments, even the qualitative understanding of overall picture of cuprate superconductors has been a challenge. Let us start by discussing a generic phase diagram of a hole-doped cuprate Ln$_{2-x}$Sr$_x$CuO$_4$ in the doping ($x = 1 - n$) and temperature ($T$) plane. Near half-filling and at low temperature, antiferromagnetic (AF) long-range order appears with $T_N = 250-300$ K at $x = 0$. It is destroyed by 2% doping concentration. When $x$ reaches 0.06, superconducting (SC) long-range order starts to appear, and it is also destroyed by 30% doping. In between them, $T_c$ reaches a maximum value of 40 K at $x \simeq 0.16$. It appears that for most hole-doped cuprates, a universal relation $T_c / T_c^\text{max} = 1 - 82.6(x - 0.16)^2$ is satisfied. The SC gap was found to have mainly $d$-wave character with possibility of a small mixture of other angular momentum states, in contrast to conventional BCS superconductors with an isotropic $s$-wave gap.

Various recent experiments also show the existence of a crossover temperature $T^*$ larger than $T_c$ in a doping range of $x = 0$ to $x \simeq 0.18-0.19$. Below this pseudogap temperature $T^*$, the low frequency spectral weight begins to be strongly suppressed. Surprisingly the doping dependences of $T^*$ and $T_c$ are completely different in spite of their close relationship suggested by angle resolved photoemission (ARPES), tunneling and NMR experiments. At optimal doping where $T_c$ is maximum, various non-Fermi liquid (NFL) properties are observed in the normal state. These include the linear temperature dependence of ab-plane resistivity, the quadratic $T$ dependence of Hall angle and so on up to 1000 K. Far beyond optimal doping, the normal state properties are well described by the conventional Landau Fermi liquid. Near optimal doping and underdoping, superfluid density $n_s$ is systematically suppressed with decreasing doping in spite of increasing SC gap amplitude. The resulting $(\Delta s) \text{max} / \kappa_B T_c$ ratio is strongly violated from the universal BCS value. In the overdoped regime, however, the SC properties appear to be well explained by the conventional weak coupling BCS theory.

Right after the discovery of high temperature superconductors, Anderson first proposed the one-band Hubbard model as the simplest Hamiltonian which might capture the correct low energy physics of copper oxides. In that seminal paper, he conjectured that the ground state of cuprates at half-filling and presumably away from half-filling as well may be described by a resonating valence bond (RVB) state. Subsequently Anderson and his co-workers applied a mean-field approximation to the strong coupling limit of the Hubbard model, namely, the $t-J$ model. The $t-J$ Hamiltonian was already known to be the large $U$ limit of the Hubbard Hamiltonian under certain assumptions. The phase diagram obtained by these authors and by others has been a starting point for further development of the theory such as $1/N$ expansion theory and gauge theory of the $t-J$ model.

In order to more conveniently handle the no-double-occupancy constraint imposed by the $t-J$ model, a slave boson representation of an original electron is often used. In this representation, an electron is decomposed into a
spinon (fermion) and a holon (boson), \( c^+_i \sigma = f^+_i \sigma b_i \). In slave boson theory of the \( t-J \) model, typically two mean-field order parameters are considered

\[
\chi_{ij} = \langle f^+_i \sigma f_j \sigma \rangle ,
\Delta_{ij} = \langle f_{j,↓} f_{i,↑} - f_{j,↑} f_{i,↓} \rangle ,
\]

together with \( \{ b_i \} \). Depending on the vanishing or nonvanishing of \( \Delta_{ij} \) and \( \langle b_i \rangle \), the doping and temperature plane is divided into four regions. \cite{25} Region I with \( \Delta_{ij} = 0 \) and \( \langle b_i \rangle \neq 0 \) is a Fermi liquid phase. Region II with \( \Delta_{ij} \neq 0 \) and \( \langle b_i \rangle = 0 \) is the spin gap phase, in which a \( d \)-wave gap appears in the fermion spectrum without Bose condensation of holons. Region III with \( \Delta_{ij} \neq 0 \) and \( \langle b_i \rangle \neq 0 \) indicates SC long-range order in physical electrons. Region IV with \( \Delta_{ij} = 0 \) and \( \langle b_i \rangle = 0 \) is designated as the strange metal phase, because it shows various non-Fermi liquid features.

In many respects, the slave boson mean-field theory of the \( t-J \) model \cite{10,22} has shed some important insight into the microscopic understanding of the cuprate superconductors. This is because the predicted phase diagram is, at least, qualitatively consistent with experiments, and the pseudogap is closely related to a spin gap, and furthermore it starts from the microscopic model as opposed to other phenomenological models. However, there are also some serious problems with the slave boson mean-field theory, as noted by Ubbens and Lee. \cite{25}

One of them is that the temperature scale for Bose condensation of holons is too high. Furthermore the maximum \( T_c \), which is determined by the two lines \( \Delta_{ij} \neq 0 \) and \( \langle b_i \rangle \neq 0 \), occurs at too small doping concentration \( x < 0.06 \). At this doping level, the SC long-range order even does not appear in cuprate superconductors. Close to half-filling, various exotic phases have been reported to be stable such as mixed phases \cite{18} (equivalently \( \pi \)-flux phases \cite{15}), dimerized phases, \cite{19,20} and staggered flux phases. \cite{21,22} It is unclear whether these states are realized or not in cuprates. In this paper we argue that these problems can be naturally resolved, when AF correlations, the weaknesses of a mean-field approximation and the limitation of the \( t-J \) model near half-filling are properly taken into account. In addition to that, we show that the universal relation of \( T_c/T^-_{c max} \), systematic reduction of superfluid density and increase of \( \langle \Delta_d \rangle_{max} / K_B T_c \) ratio below and near optimal doping can be naturally explained in our picture.

**II. FORMULATION**

The \( t-J \) model is described by the Hamiltonian where \( c^+_i \sigma \) destroys an electron at site \( i \) with spin \( \sigma \) on a two-dimensional square lattice

\[
H = -t \sum_{\langle i,j \rangle,\sigma} \left( (1 - n_{i,-\sigma})c^+_i \sigma c_j \sigma (1 - n_{j,-\sigma}) + H.c. \right) + J \sum_{\langle i,j \rangle} (\vec{S}_i \cdot \vec{S}_j - \frac{1}{4} n_i n_j) - \mu_0 \sum_{i,\sigma} c^+_i \sigma c_{i,\sigma} .
\]

t is a hopping parameter between nearest neighbors \( < i,j > \) and \( J \) denotes superexchange coupling. Double occupancy of two electrons at the same lattice site is forbidden by a projection operator \( (1 - n_{i,-\sigma}) \) in the hopping term. \( \mu_0 \) is the chemical potential controlling the electron density \( n \). \( \vec{S}_i \) and \( n_i \) are spin and charge density operators, respectively, and they are defined as

\[
\vec{S}_i = \frac{1}{2} \sum_{\alpha,\beta} c^+_i \sigma \vec{\sigma}_{\alpha,\beta} c_{i,\beta} ,
\]

\[
n_i = \sum_{\sigma} c^+_i \sigma c_{i,\sigma} .
\]

where \( \vec{\sigma} \) is a \( 2 \times 2 \) Pauli spin matrix. Through decomposition of an electron into a spinon (fermion) and a holon (boson), \( c^+_i \sigma = f^+_i \sigma b_i \), the \( t-J \) model becomes

\[
H = -t \sum_{\langle i,j \rangle,\sigma} \left( b_i \sigma b^+_j \sigma f^+_j \sigma f_{j,\sigma} + b_j \sigma b^+_i \sigma f^+_i \sigma f_{i,\sigma} \right) - \frac{J}{2} \sum_{\langle i,j \rangle} b^+_i \sigma b^+_j \sigma (f^+_i \sigma f^{+\dagger}_j \sigma - f^+_j \sigma f^{+\dagger}_i \sigma) - f^+_j \sigma f_{j,\sigma} - f^+_i \sigma f_{i,\sigma} - \mu_0 \sum_{i,\sigma} b^+_i \sigma b_{i,\sigma} + \sum_i \lambda_i (b^+_i \sigma b_i + \sum_{\sigma} f^+_i \sigma f_{i,\sigma} - 1) .
\]

Now the no-double-occupancy constraint \( n_i \leq 1 \) becomes an equality condition in terms of a spinon and a holon, \( b^+_i \sigma b_i + \sum_{\sigma} f^+_i \sigma f_{i,\sigma} = 1 \). The last term is to impose this condition through a Lagrange multiplier \( \lambda_i \).

In the spirit of a mean-field approximation, the terms with more than two operators should be decoupled in all possible ways. In principle, we may consider infinitely many species of order parameters. (infinitely) Many of them are irrelevant, namely, they do not have a stable mean-field solution, while (infinitely many) the others are relevant. In this situation, guidance from our physical intuition or more likely from experiments is extremely helpful to find most important leading correlations. The phase diagram found in high temperature superconductors has given us a clear answer \cite{26} to this question: AF and \( d \)-wave pairing correlations. Thus, in this paper we consider two order parameters with broken symmetry, \( m \) and \( s \) for AF and \( d \)-wave orders, respectively. A similar decoupling was previously considered by other groups. \cite{23,25}

We also consider fermionic and bosonic exchange couplings, \( \langle f^+_i \sigma f_{j,\sigma} \rangle \) and \( \langle b^+_i \sigma b_j \rangle \), respectively. In order to simplify calculations as well as to avoid any possible double counting problem, we do not give a dynamical aspect to holons. The presence of holons, which are introduced to keep track of empty sites, is taken into account as enforcing correlated hopping of spinons in the hopping term. In a mean-field approximation, \( \langle b^+_i \sigma b_j \rangle \) is approximated as \( x = 1 - n \) and \( \lambda_i \) is replaced by \( \lambda \). Now the
three order parameters and the holon hopping amplitude are defined as
\[ m = \frac{1}{2} \langle 1 \rangle (f_{i_1} f_{i_1} - f_{i_1} f_{i_1}) \]
\[ = \frac{1}{2N} \sum_{k} \langle f_{k+\vec{Q},\uparrow} f_{k,\uparrow} - f_{k+\vec{Q},\downarrow} f_{k,\downarrow} \rangle, \]
\[ s = \frac{1}{N} \sum_{k} \phi_d(\vec{k})(f_{k,\uparrow} f_{k,\downarrow}), \]
\[ \Delta_f = (f_{k,\sigma} f_{\sigma}) = \frac{1}{2N} \sum_{k} \phi_s(\vec{k})(f_{k,\sigma} f_{k,\sigma}), \]
\[ \Delta_b = \langle b_i^+ b_i \rangle \approx x = 1 - n, \]
where
\[ \phi_d(\vec{k}) = \cos k_x - \cos k_y, \]
\[ \phi_s(\vec{k}) = \cos k_x + \cos k_y. \]
\( \vec{Q} \) is the AF wave vector \((\pi, \pi)\) in two dimensions and \( N \) the total number of lattice sites.

In terms of the above parameters, the mean-field Hamiltonian is written
\[ H_{MF} = \sum_{\vec{k},\sigma} \varepsilon(\vec{k}) f_{\vec{k},\sigma}^\dagger f_{\vec{k},\sigma} \]
\[ -2Jm \sum_{\vec{k}} \langle f_{k+\vec{Q},\uparrow} f_{k,\uparrow} - f_{k+\vec{Q},\downarrow} f_{k,\downarrow} \rangle \]
\[ -Js \sum_{\vec{k}} \phi_d(\vec{k})(f_{k,\uparrow} f_{k,\downarrow} + f_{k,\downarrow} f_{k,\uparrow}) + F_0 \]
where
\[ \varepsilon(\vec{k}) = \phi_s(\vec{k})(J(\Delta_f + 2t\Delta_b) - \mu), \]
\[ F_0 = N(2Jm^2 + Js^2 + 8t\Delta_b\Delta_f + 2J\Delta_f^2 - \mu), \]
\[ \mu = Jn + \mu_0 - \lambda. \]
By introducing a four component field operator \( \Psi_k^\dagger = (f_{k,\uparrow}^\dagger f_{k,\downarrow}^\dagger f_{k+\vec{Q},\uparrow}^\dagger f_{k+\vec{Q},\downarrow}^\dagger) \),
\[ \Psi_k^\dagger = (f_{k,\uparrow}^\dagger f_{k,\downarrow}^\dagger f_{k+\vec{Q},\uparrow}^\dagger f_{k+\vec{Q},\downarrow}^\dagger), \]
Eq. (3) may be written in a more compact form
\[ H_{MF} = \sum_{\vec{k}} \Psi_k^\dagger M_{\vec{k}} \Psi_k + F_0. \]

The prime symbol on the summation requires the summation of wave vectors in half of the first Brillouin zone, in order to take into account the doubling of a magnetic unit cell in the presence of (commensurate) AF order. The matrix \( M_{\vec{k}} \) is given as
\[
M_{\vec{k}} = \begin{pmatrix}
\varepsilon(\vec{k}) & -Js\phi_d(\vec{k}) & -2Jm & 0 \\
-Js\phi_d(\vec{k}) & \varepsilon(\vec{k}) & 0 & -2Jm \\
-2Jm & 0 & \varepsilon(\vec{k}+\vec{Q}) & Js\phi_d(\vec{k}) \\
0 & -2Jm & Js\phi_d(\vec{k}) & -\varepsilon(\vec{k}+\vec{Q})
\end{pmatrix}.
\]
The energy eigenvalues of \( M_{\vec{k}} \) yield four energy dispersions \( \pm E_{\pm}(\vec{k}) \),
\[ E_{\pm}(\vec{k}) = [(\varepsilon_{\vec{k}}^2 + \varepsilon_{\vec{k}+\vec{Q}}^2)/2 + (2Jm)^2 + (Js\phi_d(\vec{k}))^2]^{1/2}, \]
where \( g(\vec{k}) \) is given as
\[ g(\vec{k}) = [(\varepsilon_{\vec{k}}^2 - \varepsilon_{\vec{k}+\vec{Q}}^2)^2/4 + ((\varepsilon_{\vec{k}} + \varepsilon_{\vec{k}+\vec{Q}})(2Jm))^2]^{1/2}. \]

The free energy is easily obtained either from the trace formula or from the Feynman theorem
\[ F = -2T \sum_{\vec{k}} \log(2 \cosh \frac{E_{\vec{k}}(\vec{k})}{2T}) + F_0. \]

Now three mean-field equations are obtained by the stationary condition of \( F \) with respect to the corresponding order parameters, \( \frac{\partial F}{\partial m} = \frac{\partial F}{\partial \mu} = \frac{\partial F}{\partial \lambda} = 0 \), and one more unknown constant \( \mu \) is determined by the thermodynamic relation \( n = 1 - x = -\frac{\partial F}{\partial \mu} \). The resulting four equations are
\[ m = \frac{1}{2N} \sum_{\vec{k}} \sum_{\alpha=\pm} \left\{ (2Jm) + \alpha \frac{(\varepsilon_{\vec{k}} + \varepsilon_{\vec{k}+\vec{Q}})^2(2Jm)}{2g(\vec{k})} \right\} \]
\[ \times \frac{1}{E_{\vec{k}}(\vec{k})} \tan(\beta E_{\vec{k}}(\vec{k})), \]
\[ s = \frac{1}{2N} \sum_{\vec{k}} \sum_{\alpha=\pm} \phi_d^2(\vec{k})(Js) \frac{1}{E_{\vec{k}}(\vec{k})} \tan(\beta E_{\vec{k}}(\vec{k})), \]
\[ \Delta_f = \frac{1}{4N} \sum_{\vec{k}} \sum_{\alpha=\pm} \phi_s^2(\vec{k})(J\Delta_f + 2t\Delta_b) \left\{ 1 + \alpha \frac{2\mu^2}{g} \right\} \]
\[ \times \frac{1}{E_{\vec{k}}(\vec{k})} \tan(\beta E_{\vec{k}}(\vec{k})), \]
\[ n = 1 - \frac{1}{2N} \sum_{\vec{k}} \sum_{\alpha=\pm} \left\{ (\varepsilon_{\vec{k}} + \varepsilon_{\vec{k}+\vec{Q}}) \right\}
\[ \left( \frac{(\varepsilon_{\vec{k}} + \varepsilon_{\vec{k}+\vec{Q}})^2}{2g(\vec{k})} \right) \frac{1}{E_{\vec{k}}(\vec{k})} \tan(\beta E_{\vec{k}}(\vec{k})). \]

Before presenting our results, several comments are in order concerning the mean-field Hamiltonian and equations (1)–(10)ons. First, \( b_i b_i^\dagger b_j b_j^\dagger \) in the second term and \( b_i b_i^\dagger \) in the
third term of Eq. (4) are replaced by unity. Extra decoupling of these factors may double count what spinons have already taken care of. Anyhow replacing those factors by unity is expected to be a reasonable approximation, as long as doping $x = 1 - n$ is not high. Second, in this paper a uniform bond order of $\Delta_1$ is assumed, namely, $\langle f_{i,\sigma}^+ f_{j,\sigma} \rangle$ is independent of a relative direction of a bond $(i, j)$. $\pi$-flux and staggered flux phases whose $\langle f_{i,\sigma}^+ f_{j,\sigma} \rangle$ depends on a relative direction of a bond, were found to be stable only at very small doping for $t/J > 1$. Third, the mean-field decoupling $\langle f_{i,\uparrow} f_{j,\uparrow} - f_{i,\downarrow} f_{j,\downarrow} \rangle$ in general induces superconductivity with extended $s$-wave symmetry as well as with $d$-wave symmetry. As noted by Inui et al. [27], the former is strongly suppressed in the presence of a finite staggered magnetization. As will be shown shortly, mean-field AF order is also found to exist in a large phase space near half-filling. This may justify neglecting superconductivity with extended $s$-wave symmetry in the important region of the phase space. Last, in this paper spin-triplet order parameter $\langle f_{\pi,\uparrow}^+ f_{\pi,\downarrow} f_{-\pi,\downarrow} f_{-\pi,\uparrow} \rangle$ is not explicitly considered. In a mean-field decoupling scheme, there is no such term which directly induces spin-triplet order parameter. However, it is dynamically generated when two mean-field orders $m$ and $s$ coexist, without affecting $m$ and $s$. Although spin-triplet order parameter is not explicitly mentioned here, it appears in the coexistence region of mean-field AF and SC phases.

III. RESULTS AND DISCUSSIONS

In Fig. 1 our calculated mean-field phase diagram is presented for $t/J = 4$. Its variation to $t/J (=3-4)$ is negligible. Near half-filling mean-field AF and SC orders coexist, while far away from half-filling only the latter prevails. The overall result is qualitatively similar to what other groups [27-28] obtained before. It is also similar to our previous result [29] based on a phenomenological model in which SC order and AF order come from different interaction terms. At this point, we should point out reasons for rejecting the unphysical result (dashed curve) of the calculated mean-field SC order near half-filling. Other groups [27-28] also obtained a similar result for the mean-field $T_c$ near half-filling. Due to the following reasons (both theoretical and experimental), we take the solid curve as more appropriate mean-field $T_c$, which is obtained by setting $m$ to zero. In this respect, the solid curve may be the upper bound of the correct mean-field $T_c$.

Theoretically there are two reasons why the reducing of SC order parameter near half-filling is unphysical. First, as also noted by Inui et al. [27], in a (uniform) mean-field approximation holes have direct overlap with a staggered AF order and suffer from strong time reversal symmetry breaking. It causes a rapid destruction of the mean-field SC order near half-filling. However, the local deformation of AF order in the immediate vicinity of holes, which is absent in a typical mean-field approximation, enables the holes to avoid direct overlap with the AF order. Consequently mean-field SC and AF orders can coexist without sacrificing the energy gain through the local deformation of AF order or through a microscopic separation of the two orders like in stripes. [20,23]

Second, the unphysical result comes from the limitation of the $t-J$ model near half-filling. At half-filling, the kinetic energy term of the $t-J$ model collapses. In fact, this is directly responsible for the degeneracy of superconductivity with $d$-wave symmetry and extended $s$-wave symmetry at half-filling, as first noticed by Kotliar. [18] However, the collapsing of the kinetic energy term does not happen in the Hubbard model at any filling, as long as $t/U$ is kept finite. As noted in our previous work, [32] the energy dispersion of a hole is given by

$$\sqrt{(2t)^2 (\phi_d(\vec{k}))^2 + (U/2)^2 + \Delta^2 (\phi_d(\vec{k}))^2}$$

(19)

for a half-filled Hubbard band with $U > W = 8t$. $\Delta \phi_d(\vec{k})$ is the $d$-wave mean-field SC gap. For a realistic strength of $U \sim 1.5W$, $\Delta$ was found to be $0.07t$. Even for $U \gg t$ it saturates to be $0.69t$. Thus the characteristic energy scale for the hopping term, $2t$, is never much smaller than $\Delta$. As a result, the degeneracy of superconductivity with $d$-wave symmetry and extended $s$-wave symmetry at half-filling does not happen in the Hubbard model. This limitation of the $t-J$ model near half-filling is ascribed to an inconsistent treatment of the hopping and the superexchange terms in the $t-J$ model. The former is obtained in the $U = \infty$ limit, while the latter in the finite $U = 4t^2/J$ limit. When the same limit is applied to the Hubbard model, the hopping term in Eq. (19) vanishes when divided by $U$, but the last term survives in the finite $U$ limit. This leads to the same result as the $t-J$ model at half-filling. The consequence of this excessive reducing of the kinetic term of the $t-J$ model near half-filling is to enhance localized AF correlations. It causes SC order to lose in competition with the AF order, even after the local deformation of AF order is properly taken into account.

One strong evidence supporting this argument is provided by failure in capturing pairing correlations in the exact diagonalization (ED) study of the $t-J$ model at half-filling. [33] In spite of its exact and unbiased nature, the obtained energy dispersion is similar to what would be found only by AF correlations. This is contrasted with the experimentally observed energy dispersion in insulating cuprates, [34,35] namely, a $d$-wave-like modulation of the insulating gap. As a result, a reasonable agreement with the experimental result is achieved only by introducing unjustified fitting parameters such as the $t'$ and $t''$ terms into the $t-J$ model. In fact these terms act like...
enhancing the kinetic energy term. Yet another evidence comes from the U(1) gauge theory of the $t-J$ model by Ubbens and Lee. \[25\] These authors found that the spin-gap (or pseudogap) phase is completely destroyed by gauge-field fluctuations near half-filling. In this respect, going to the SU(2) formulation may not help to resolve the problem. This limitation of the $t-J$ model near half-filling may be overcome by using a fully systematic $t/U$ expansion in the Hamiltonian, the wave function and all the operators from which the single particle Green’s function, optical conductivity and so on are defined. \[36,37\] If this problem of the $t-J$ model near half-filling is corrected, we believe that the exotic phases found near half-filling do not exist.

From experimental point of view, two recent ARPES experiments \[34,35\] dictate the presence of strong pairing correlations at half-filling. An ARPES experiment for an insulating cuprate Sr$_2$CuO$_2$Cl$_2$ \[34\] clearly shows that the near isotropy and the overall band dispersion along ($\pi/2$, $\pi/2$) -- ($\pi$, 0) and ($\pi/2$, $\pi/2$) -- (0, 0) cannot be explained by considering only AF order or its fluctuations. Furthermore a $d$-wave-like modulation of the insulating gap in Cu$_2$CuO$_2$Cl$_2$ \[35\] is totally mysterious from that point of view. Furthermore numerous experiments \[35\] have shown that the pseudogap temperature $T^*$ increases with decreasing doping all the way down to half-filling. It is believed that mean-field $T_c$ has a similar doping dependence with $T^*$.

Now we are in a position to point out the weaknesses of a mean-field approximation in low dimensions. Based on this observation and some exact results in the Hubbard model, we will draw useful information from the slave boson mean-field theory of the $t-J$ model. First of all it is of great importance to note that in a mean-field approximation long-range order already sets in, when the corresponding correlation length reaches roughly one lattice spacing. This forces the above mean-field phase line to be interpreted as the onset of the corresponding short-range correlations. The question that mean-field order can become truly long-range order or not, depends on whether the correlation length diverges or not with lowering temperature. In this respect, a potential location of AF long-range order is only at half-filling where the AF correlation length logarithmically diverges at low temperature. Away from half-filling, the AF correlation length initially grows below $T_N^{MF}$ and then saturates. This tells that the phase diagram of mean-field AF order can be viewed as the presence of short-range AF correlations for $x \leq x_c \simeq 0.18$ -- $0.19$ at low temperature. If there were AF long-range order in the model, it would be at half-filling and at zero temperature due to the Mermin-Wagner theorem. \[39\]

Since the paring correlation length logarithmically diverges at any filling for $x \simeq 0.35$ with decreasing temperature, in principle SC long-range order may appear from half-filling all the way up to $x \simeq 0.35$. In this paper, however, we argue that SC long-range order does not occur at any filling strictly in two dimensions. Below $x = x_c$ where the short-range AF correlations are present, the AF fluctuations create locally the spin density wave (SDW) state. It causes the breaking of time-reversal symmetry and thus of SC long-range phase coherence. Above $x = x_c$ where the paring correlations are relatively weak, \[10\] quantum fluctuations can easily destroy the SC long-range order. This interpretation is consistent with the rigorous result by Su and Suzuki. \[11\] These authors proved the nonexistence of $d_{x^2-y^2}$ superconductivity (long-range order) at any nonzero temperature in the two-dimensional Hubbard model. Note that the proof by Su and Suzuki does not exclude increasing pairing correlations with decreasing temperature just like the AF correlations for a half-filled Hubbard band. The readers should not be confused with SC long-range order $\langle \Delta_g(0) \rangle \neq 0$ and with pairing correlations $\langle \Delta^+_g(0) \Delta_g(0) \rangle > \langle \Delta^+_g(0) \Delta_g(0) \rangle_0$. $\langle \cdot \rangle_0$ means an expectation value evaluated in the ground state of noninteracting electrons. \[12\]

This feature of pairing correlations also dictates the mean-field SC phase line to be interpreted as the onset ($T^*$) of short-range pairing correlations (pseudogap) \[13\] instead of as true long-range order. The pairing correlations extend all the way up to $x \simeq 0.35$. It makes the crossover region of pairing correlations broad with respect to $x = x_c$ at low temperature. We further argue that strictly in two dimensions SC and AF long-range orders are absent even at zero temperature, since the two correlations may act like frustrating the long-range coherence of the other correlations. \[14\]

**IV. UNIVERSAL CURVE OF $T_c/T^{MAX}_c$ AND REDUCED SUPERFLUID DENSITY**

Therefore, the absence of SC long-range order strictly in two dimensions strongly suggests that true SC long-range order is driven by a pair hopping process along the c-axis due to interlayer coupling. This is consistent with the general trend that the cuprate compounds with more CuO$_2$ planes in a unit cell show higher $T_c$. In the present paper, the interlayer coupling means not only the coupling between CuO$_2$ planes in a unit cell, but also between other CuO$_2$ planes in different unit cells. When the interlayer coupling is turned on, a potential location with the highest $T_c$ is near $x = x_c$ in which the phase coherence-breaking AF correlations nearly vanish but the pairing correlations are still robust. The resulting phase diagram (Fig. 2) will look like one where $T^*$ falls from a high value onto the $T_c$ line rather than the other where $T^*$ smoothly merges with $T_c$ in the slightly overdoped region, as recently argued by Tallon and Loram. \[14\] In our
scenario $T_c$ is never part of the $T^*$ line. For different compounds, different strength of interlayer pairing hopping drives SC long-ranger order in the same background of the two-dimensional electron system. This leads to a universal relation \[ T_c/T_c^{\text{max}} = 1 - 82.6(x - 0.16)^2, \]
when $T_c$ is scaled by $T_c^{\text{max}}$. Away from half-filling, the AF correlations manifest their existence most strongly in the SC state, because they easily destroy the SC long-range phase coherence. Our scenario also predicts that due to scatterings with AF fluctuations, quasiparticle scattering rate remains finite for $x \leq x_c$ even in a clean sample and at $T = 0$. For $x \leq x_c$, in our picture, the Landau quasiparticle with nonvanishing quasiparticle residue does not exist in the normal state due to strong scatterings with pairing and AF fluctuations. But it can be stabilized in the SC state ($T < T_c$) where its coherence is restored through a pair hopping process along the c-axis.

The crucial point in proper understanding of the SC state is not the validity of the weak coupling BCS theory, but more importantly whether Cooper pairs are constructed from antiferromagnetically correlated electrons or not. In fact depending on the doping concentration with respect to $x = x_c$, the SC state can be qualitatively different. For $x \leq x_c$ the SC state has significant AF correlations, while for $x > x_c$ it has virtually no AF correlations, thus justifying the conventional BCS theory based on the noninteracting electrons. With decreasing doping, the pairing correlations as well as the phase coherence-breaking AF correlations increase in underdoped and optimally doped samples ($x \leq x_c$). As a consequence, with decreasing doping the SC gap amplitude increases, because it is determined by the pseudogap size below $T^*$. On the other hand, superfluid density or $T_c$ decreases due to the increasing phase coherence-breaking AF correlations. The resulting $(\Delta_d)^{\text{max}}/K_BT_c$ ratio is strongly doping dependent, monotonically increasing with decreasing doping for $x \leq x_c$. Above $x_c$ (overdoping), however, it is expected that the ratio approaches more or less the BCS mean-field value. This feature cannot be understood in the absence of AF correlations which are allowed in the model.

The effective strength for the SC long-range order is also strongly doping dependent and is largest near $x_c$. It decreases below $x_c$ due to the increasing phase coherence-breaking AF correlations and also above $x_c$ owing to the decreasing pairing correlations. In this respect, it is not surprising to find that the superfluid density and the SC condensation energy have their maximum values near $x_c$, and decrease below and above $x_c$. In the present scenario, the pseudogap is virtually unchanged by an applied magnetic field, because the characteristic energy scale for the pseudogap, $\Delta$, is much larger than the Zeeman energy. On the other hand, the SC long-range order is relatively easily destroyed by it due to its phase coherence-breaking nature.

In some of magnetic experiments such as NMR, another crossover temperature $T^0$ (larger than $T^*$) is often identified, at which Knight shift shows its maximum. This feature can be easily understood on the basis of the competing nature of pairing and AF correlations as well as of the phase diagram (Fig. 2). For $T^* < T < T^0$, the two correlations compete, while they grow with decreasing temperature. In spin-lattice relaxation rate which picks up strongly the $\vec{q} = \vec{Q}$ component, for $T^* < T < T^0$ the contribution from the AF correlations dominates that from the pairing correlations. It makes $1/T_1T$ keep increasing until it starts to decrease at $T^*$. On the other hand, Knight shift which picks up the $\vec{q} = 0$ component and thus is unaware of the growing AF correlations, is more strongly influenced by the increasing pairing correlations. Consequently Knight shift reaches its maximum at $T^0$, and starts to slowly decrease below it and then rapidly decrease below $T^*$.

It is also worthwhile to comment on the SC condensation energy in the present picture. The SC long-range order is stabilized only through a pair hopping process along the c-axis (due to the interlayer coupling). It forces the SC condensation energy to come from the lowering of the c-axis kinetic energy in the SC state. This interlayer coupling theory was already proposed by Anderson and others, and many features are consistent with c-axis optical measurements.

\section{V. Conclusion}

In summary, we have studied the $t-J$ Hamiltonian in a mean-field approximation by taking into account AF and $d$-wave pairing correlations. Considering the presence of AF fluctuations, the weaknesses of a mean-field approximation and the limitation of the $t-J$ model near half-filling, we gave a new interpretation to the slave boson mean-field theory of the $t-J$ model. We argued that due to phase coherence-breaking AF fluctuations and quantum fluctuations, SC long-range order does not appear strictly in two dimensions. $T_c$ resulting from the interlayer pairing hopping can lead to a universal relation, when $T_c$ is scaled by $T_c^{\text{max}}$. Systematic reduction of superfluid density and increase of $(\Delta_d)^{\text{max}}/K_BT_c$ ratio below and near optimal doping have their natural explanation in our picture. Another crossover temperature $T^0$ found in some of magnetic experiments such as NMR is also easily understood in the present framework.

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[1] J. G. Bednorz and K. A. Müller, Z. Phys. B 64, 189 (1986).
[2] C. Almasan and M. B. Maple in Chemistry of High Temperature Superconductors, edited by C. N. R. Rao (World Scientific, Singapore, 1991).
[3] H. Zhang and H. Sato, Phys. Rev. Lett. 70, 1697 (1993).
[4] B. G. Levi, Phys. Today, 46, 17 (1993); ibid. 49, 19 (1996).
[5] Z. -X. Shen, D. S. Dessau, B. O. Wells, D. M. King, W. E. Spicer, A. J. Arko, D. Marshall, L. W. Lombardo, A. Kapitulnik, P. Dickinson, S. Doniach, J. DiCarlo, T. Loeser, and C. H. Park, Phys. Rev. Lett. 70, 1553 (1993).
[6] D. S. Marshall, D. S. Dessau, A. G. Loeser, C-H. Park, A. Y. Matsurua, J. N. Eckstein, I. Bozovic, P. Fournier, A. Kapitulnik, W. E. Spicer, and Z.-X. Shen, Phys. Rev. Lett. 76, 4841 (1996).
[7] J. Bardeen, L. N. Cooper, and J. R. Schrieffer, Phys. Rev. 108, 1175 (1957).
[8] H. Ding, T. Yokoya, J. C. Campuzano, T. Takahashi, M. Randeria, M. R. Norman, T. Mochiku, K. Kadawaki, and J. Giapintzakis, Nature 382, 51 (1996).
[9] A. G. Loeser, Z. -X. Shen, D. S. Dessau, D. S. Marshall, C. H. Park, P. Fournier, A. Kapitulnik, Science 273, 325 (1996).
[10] J. W. Loram, K. A. Mirza, J. R. Cooper, and W. Y. Liang, Phys. Rev. Lett. 71, 1740 (1993).
[11] Ch. Renner, B. Revaz, J. -Y. Genoud, K. Kadawaki, and O Fischer, Phys. Rev. Lett. 80, 149 (1998).
[12] M. Takigawa, P. C. Hammel, R. H. Heffner, and Z. Fisk, Phys. Rev. B 43, 247 (1991).
[13] C. C. Homes and T. Timusk, R. Liang, D. A. Bonn, and W. N. Hardy, Phys. Rev. Lett. 71, 1645 (1993).
[14] J. L. Tallon and J. W. Loram, cond-mat/0005063; C. Bernhard, J. L. Tallon, T. Blasius, A. Golinik, and C. Niedermayer, cond-mat/0006285.
[15] P. W. Anderson, Science 235, 1196 (1987).
[16] G. Baskaran, Z. Zou, and P. W. Anderson, Solid State Commun. 69, 973 (1987); P. W. Anderson, G. Baskaran, Z. Zou, and T. Hsu, Phys. Rev. Lett. 58, 2790 (1987).
[17] A. E. Ruckenstein, P. J. Hirshfeld, and J. Appel, Phys. Rev. B 36, 857 (1987); G. Kotliar and J. Liu, Phys. Rev. B 38, 5142 (1988); Y. Suzukiura, Y. Hasegawa, and H. Fukuyama, J. Phys. Soc. Jpn. 57, 401 (1988); Y. Suzukiura, Y. Hasegawa, and H. Fukuyama, J. Phys. Soc. Jpn. 57, 2768 (1988).
[18] G. Kotliar, Phys. Rev. B 37, 3664 (1988).
[19] I. Affleck and J. B. Marston, Phys. Rev. B 37, 3774 (1988); J. B. Marston and I. Affleck, Phys. Rev. B 39, 11 538 (1989); I. Affleck, Z. Zou, T. Hsu, and P. W. Anderson, Phys. Rev. B 38, 745 (1998).
[20] T. Dombre and G. Kotliar, Phys. Rev. B 39, 855 (1989); N. Read and S. Sachdev, Phys. Rev. Lett. 62, 1694 (1989).
[21] P. Lederer, D. Poilblanc, and T. M. Rice, Phys. Rev. Lett. 63, 1519 (1989); F. C. Zhang, Phys. Rev. Lett. 64, 974 (1990).
[22] M. U. Ubbens and P. A. Lee, Phys. Rev. B 46, 8434 (1992).
[23] M. Grilli and G. Kotliar, Phys. Rev. Lett. 64, 1170 (1990); Z. Wang, Y. Bang, and G. Kotliar, Phys. Rev. Lett. 67, 2733 (1991); A. Tandon, Z. Wang, and G. Kotliar, Phys. Rev. Lett. 83, 2046 (1999).
[24] G. Baskaran and P. W. Anderson, Phys. Rev. B 37, 580 (1988); L. Ioffe and A. Larkin, Phys. Rev. B 39, 8988 (1989); P. A. Lee and N. Nagaosa, Phys. Rev. B 45, 966 (1992); R. Hlubina, W. O. Putikka, T. M. Rice, and D. V. Khveshekenko, Phys. Rev. B 46, 11 224 (1992); R. B. Laughlin, J. Low Temp. Phys. 99, 443 (1995).
[25] M. U. Ubbens and P. A. Lee, Phys. Rev. B 49, 6853 (1994).
[26] In this paper we restrict ourselves to a uniform state. On general grounds, solutions with an inhomogeneous stripe structure are possible.
[27] M. Inui, S. Doniach, J. P. Hirschfeld, and A. E. Ruckenstein, Phys. Rev. B 37, 2320 (1988).
[28] M. Inaba, H. Matsukawa, M. Saitoh, and H. Fukuyama, Physica C 257, 299 (1996).
[29] B. Kyung, cond-mat/0004280 and to appear in Phys. Rev. B.
[30] J. Zaanen and O. Gunnarsson, Phys. Rev. B 40, 7391 (1989); D. Poilblanc and T. M. Rice, Phys. Rev. B 39, 9749 (1989); H. J. Schulz, J. Physique 50, 2833 (1989); K. Machida, Physica C 158, 192 (1989).
[31] J. M. Tranquada, J. D. Axe, N. Ichikawa, A. R. Moodenbaugh, Y. Nakamura, and S. Uchida Phys. Rev. Lett. 78, 338 (1997).
[32] B. Kyung, cond-mat/0005262.
[33] P. W. Leung, B. W. Wells, and R. J. Gooding, Phys. Rev. B 56, 6320 (1997).
[34] B. O. Wells, Z.-X. Shen, A. Matsurua, D. M. King, M. A. Kastner, M. Greven, and R. J. Birgeneau, Phys. Rev. Lett. 74, 964 (1995).
[35] F. Ronning, C. Kim, D. L. Feng, D. S. Marshall, A. G. Loeser, L. L. Miller, J. N. Eckstein, I. Bozovic, Z. -X. Shen, Science, 282, 2067 (1998).
[36] W. Stephan (private communication).
[37] H. Eskes, A. M. Oleś, M. B. Meinders, and W. Stephan, Phys. Rev. B 50, 17 980 (1994).
[38] T. Timusk and B. Statt, Rep. Prog. Phys. 62, 61 (1999).
[39] N. D. Mermin and H. Wagner, Phys. Rev. Lett. 17, 1133 (1966).
[40] Since at a large doping level $\langle b^*_i b_j \rangle \approx x$ underestimates the hopping between spinons, the obtained mean-field $T_c$ is overestimated. In a more realistic situation, the mean-field $T_c$ should be more strongly suppressed near and beyond optimal doping.
[41] G. Su and M. Suzuki, Phys. Rev. B 40, 506 (1998).
[42] $\Delta_{g}(i)$ is a pair operator with an appropriate gap symmetry. Generally $\Delta_{g}(i)$ is defined as $\Delta_{g}(i) = \frac{1}{N} \sum_{k \notin \mathbb{Z}} g(\delta)(c_{i+\delta,k} c_{i+\delta,k}^\dagger - c_{i+\delta,k} c_{i+\delta,k}^\dagger)$, where $g(\delta)$ is an appropriate gap structure factor in real space.
[43] More precisely speaking, the pseudogap temperature $T^*$ appears when the pairing correlation length becomes a

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few lattice spacing. This makes $T^*$ smaller than the mean-field $T_c$. In fact experiments \cite{38,14} show that $T^*$ falls to a small value near $x = x_c$.

\cite{44} Unlike the SC long-range order, the absence of AF long-range order at zero temperature is not so crucial for our argument.

\cite{45} A. P. Reyes, D. E. MacLaughlin, M. Takigawa, P. C. Hammel, R. H. Heffner, J. D. Thompson, and J. E. Crow, Phys. Rev. B 43, 2989 (1991); R. E. Walstedt, R. F. Bell, and D. Mitzi, Phys. Rev. B 44, 7760 (1991); M. Bankay, M. Mali, J. Roos, and D. Brinkmann, Phys. Rev. B 50, 6416 (1994).

\cite{46} J. Wheatley, T. Hsu, and P. W. Anderson, Nature 333, 121 (1988); A. J. Leggett, Science 274, 587 (1996); P. W. Anderson, Science 279, 1196 (1998).

\cite{47} D. N. Basov, T. Timusk, B. Dabrowski, and J. D. Jorgensen Phys. Rev. B 50, 3511 (1994).

\begin{figure}
\centering
\includegraphics[width=0.8\textwidth]{phase_diagram1}
\caption{Calculated phase diagram in doping ($x = 1 - n$) and temperature ($T$) plane for $t/J = 4$. $T_N^{\text{MF}}$ and $T_c^{\text{MF}}$ are mean-field AF and SC ordering temperatures, respectively.}
\end{figure}

\begin{figure}
\centering
\includegraphics[width=0.8\textwidth]{phase_diagram2}
\caption{Schematic phase diagrams in doping ($x = 1 - n$) and temperature ($T$) plane. $T^0 = T_N^{\text{MF}}$ and $T^*$ are crossover temperatures of AF and pairing correlations, respectively. $T_c$ is a temperature in which SC long-range order sets in. $x = x_c$ is a doping concentration where $T^0 = T_N^{\text{MF}}$ and $T^*$ nearly vanish in the absence of interlayer coupling.}
\end{figure}