The modular form of the Barth-Nieto quintic

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0 Introduction

Barth and Nieto showed in [BN] that the quintic threefold

\[ N = \left\{ \sum_{i=0}^{5} u_i = \sum_{i=0}^{5} \frac{1}{u_i} = 0 \right\} \subset \mathbb{P}^5. \]

parametrizes birationally the space of Kummer surfaces associated to abelian surfaces with a \((1,3)\)-polarization and a level 2 structure. The quintic \(N\) has a smooth model which is a Calabi-Yau threefold. From this Barth and Nieto deduced that the Siegel modular variety \(A_3(2)\) parametrizing abelian surfaces with a \((1,3)\)-polarization and a level 2 structure also has a smooth model which is a Calabi-Yau manifold whose Euler number is 80. As a consequence this implies that there is exactly one weight 3 cusp form (up to a scalar) with respect to the modular group \(\Gamma_3(2)\) which defines the space \(A_3(2)\). It is a natural question to ask to determine this cusp form. In this paper we give the answer to this problem by showing that the modular form in question is \(\Delta_3\) where \(\Delta_1\) is a remarkable cusp form of weight 1 with respect to \(\Gamma_3\) (the paramodular group of a \((1,3)\)-polarization) with a character of order 6. This function was discovered in [GN2] and it determines a generalized Lorentzian Kac–Moody superalgebra of Borcherds type. The form \(\Delta_1\) is also the automorphic discriminant of the moduli space of \(K3\) surfaces whose lattice of transcendental cycles is contained in \(U(12) \oplus U(12) \oplus <2>\) (see [GN2, Theorem 5.2.1]).

One of the main features of \(\Delta_1\) is that it vanishes precisely along the diagonal

\[ \mathcal{H}_1 = \{ \tau_2 = 0 \} = \left\{ \tau = \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_3 \end{pmatrix}, \tau_1, \tau_3 \in \mathbb{H}_1 \right\} \subset \mathbb{H}_2 \]

which parametrizes split abelian surfaces. The vanishing order along the diagonal is 1. In fact it turns out (see Proposition 1.1) that there is only a list of four modular forms with respect to the paramodular group \(\Gamma_t\) with this property. All of these modular forms are particularly interesting (see remarks at the end of Section 1). This paper is mostly concerned with the geometric consequences which can be derived from the form \(\Delta_1\). In section
we give a straightforward construction of a smooth, projective Calabi-Yau model of the variety $A_3(2)$, resp. its Voronoi compactification $A^*_3(2)$ (cf. Theorem 2.7). Our method is entirely within the framework of Siegel modular varieties, i.e. it uses the toroidal compactification (we describe its properties in 2.4), but is independent of the arguments of Barth and Nieto who go via the embeddings of the Kummer surfaces. In section 3 we study the modular variety $A_3(3)$ where the level 2 structure is replaced by a level 3 structure. Here we can use the existence of the modular form $\Delta_1$ to show that this space is of general type and to construct a minimal model (see Theorems 3.1 and 3.2).

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1 Modular forms vanishing along the diagonal

For an integer $t \geq 1$ the paramodular group $\Gamma_t$ is the subgroup

$$\Gamma_t = \left\{ g \in \text{Sp}(4, \mathbb{Q}); g \in \begin{pmatrix} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & t\mathbb{Z} \\ t\mathbb{Z} & \mathbb{Z} & t\mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & t\mathbb{Z} & \mathbb{Z} & t\mathbb{Z} \\ \mathbb{Z} & t^{-1}\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \end{pmatrix} \right\}$$

of $\text{Sp}(4, \mathbb{Q})$. Its geometric meaning is that the quotient

$$A_t = \Gamma_t \backslash \mathbb{H}_2$$

of the Siegel space $\mathbb{H}_2$ by $\Gamma_t$ is the moduli space of $(1, t)$-polarized abelian surfaces. Let $\pi : \mathbb{H}_2 \to A_t$ be the quotient map.

The diagonal

$$\mathcal{H}_1 = \{ \tau_2 = 0 \} = \left\{ \tau = \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_3 \end{pmatrix}, \tau_1, \tau_3 \in \mathbb{H}_1 \right\} \subset \mathbb{H}_2$$

resp. its image $\pi(\mathcal{H}_1)$ in $A_t$ parametrizes split polarized abelian surfaces. The surface $\pi(\mathcal{H}_1)$ is a component of the Humbert surface $H_1$ of discriminant 1. For the theory of Humbert surfaces we refer the reader to [vdG], [GH1]. The equality $\pi(\mathcal{H}_1) = H_1$ holds if and only if the equation $b^2 \equiv 1 \mod 4t$ has mod $2t$ only the solution $\pm 1$. There is a remarkable series of modular forms (with a character) whose zero locus in $A_t$ consists exactly of $\pi(\mathcal{H}_1)$. We list these modular forms in the following table.
These forms were discussed in some detail in [GN2]. They arise from Jacobi forms by arithmetical lifting and are denominator functions of generalized Kac-Moody superalgebras. These functions vanish on $H_1$ (and its $\Gamma_t$-translates) of order 1 and nowhere else. Such forms are very special, in fact we have

**Proposition 1.1** If $F$ is a modular form of integral (or half-integral) weight with a character (or a multiplier system) with respect to $\Gamma_t$ such that $F$ vanishes exactly on the $\Gamma_t$-translates of $H_1$ and with vanishing order 1, then $t = 1, 2, 3$ or 4 and $F$ is equal, up to a constant, to $\Delta_5, \Delta_2, \Delta_1$ or $\Delta_{1/2}$.

**Proof.** In the theory of automorphic forms it is sometimes more natural to deal with a group conjugated to $\Gamma_t$, namely

$$\Gamma'_t = I_t \Gamma_t I_t^{-1},$$

where $I_t = \text{diag}(1, t^{-1}, 1, t)$. This is again a subgroup of $\text{Sp}(4, \mathbb{Q})$. Note that $A'_t = \Gamma'_t \backslash \mathbb{H}_2 \cong \Gamma_t \backslash \mathbb{H}_2 = A_t$. In order to avoid confusion we shall denote all objects which refer to the group $\Gamma'_t$ by a $'$.

Let $F$ be a modular form of weight $k$ with respect to $\Gamma'_t$ and $d \neq t$ be an integer. We shall use the following operator of multiplicative symmetrisation

$$[F]_d = \prod_{M \in \Gamma'_t \cap \Gamma'_d \backslash \Gamma'_t} F|_{kM}$$

where $(F|_{kM})(Z) := \det (C \tau + D)^{-k} F(M < Z >)$ ($M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$) is the standard slash-operator. (Compare (1) with the operators of symmetrisation studied in [G2] and [GN2, §3]). It is clear that $[F]_d$ is a modular form with respect to $\Gamma'_d$.

A rational quadratic divisor $H'_l$ with respect to $\Gamma'_t$ is defined by

$$H'_l = \{ \left( \begin{array}{c} \tau_1 \\ \tau_2 \\ \tau_3 \end{array} \right) \in \mathbb{H}_2 \mid f(\tau_2^2 - \tau_1 \tau_3) + c \tau_3 + b \tau_2 + a \tau_1 + e = 0 \},$$

where $l = (e, a, b, c, f) \in \mathbb{Z}^5 \times (t\mathbb{Z})^2$ such that $(e, a, b, \frac{c}{t}, \frac{f}{t}) = 1$. The integer $D(l) = b^2 - 4ac - 4ef$ is called the discriminant of $H'_l$. The surface $H'_l$ in $A'_t$ is defined as the image of $H'_l$ under the natural projection map $\pi'_l: \mathbb{H}_2 \to A'_t$. It is a component of the Humbert surface $H'_{D(l)}$ and by abuse of notation we shall also sometimes refer to $H'_l$ as an Humbert surface.
Let $d$ be a divisor of $t$. Then $\mathcal{H}'_{d}$ is a rational quadratic divisor of the same discriminant $D(l)$ with respect to $\Gamma'_{d}$, if $(\epsilon, a, b, c, \frac{d}{e}) = 1$. For example, this is the case if $(D(l), t) = 1$. Let us assume, that $\text{div}_{\mathcal{A}'_{d}} F = m \mathcal{H}'_{l}$ and $(D(l), t) = 1$. It follows from the consideration above that $\text{div}_{\mathcal{A}'_{d}} ([F]_{d})$ is a sum of some irreducible components of the Humbert surface of discriminant $D(l)$.

We are interested in the case $d = 1$ and $H'_{1} = H'_{1} = H_{1}$. Note that the Humbert surface $H_{1}$ is irreducible in $\mathcal{A}_{1}$. We set $\gamma'_{l} = \Gamma_{1} \cap \Gamma'_{l}$. By standard arguments we see that for an element $g \in \Gamma_{1}$ the class $\gamma'_{l} g$ is determined by the last line of $g$ considered as an element of $\mathbb{P}^{3}(\mathbb{Z}_{l}^{*})$. More precisely,

$$|\gamma'_{l} \setminus \Gamma_{1}| = |\mathbb{Z}_{l}^{*} \setminus \{(a, b, c, d) \text{ mod } t, (a, b, c, d) = 1\}|.$$  

Therefore

$$[\Gamma_{1} : \gamma'_{l}] = \varphi(t)^{-1} t^{4} \prod_{p | t} (1 - p^{-4}) = t^{3} \prod_{p | t} (1 + p^{-1})(1 + p^{-2}).$$

Here $\varphi$ as usual denotes the Euler $\varphi$-function. The weight of $[F]_{1}$ equals the weight of $F$ multiplied by the above index.

The order of zero of $[F]_{1}$ along the Humbert surface $H_{1}$ is equal to the number of left cosets $\gamma'_{l} M$ in $\Gamma_{1}$ such that there exists an element $\gamma \in \Gamma'_{l}$ with $\gamma^{-1} M < \mathcal{H}_{1} \supseteq \mathcal{H}_{1}$. By $\text{Sp}(\mathbb{R})$ the stabilizer of $\mathcal{H}_{1}$ in $\text{Sp}_{4}(\mathbb{R})$ is the group generated by

$$\text{SL}_{2}(\mathbb{R}) \times \text{SL}_{2}(\mathbb{R}) \cong \left\{ \begin{pmatrix} a & b & 0 & 0 \\ 0 & a & b & 0 \\ c & d & 0 & 0 \\ 0 & c & 0 & d \end{pmatrix} \in \text{Sp}(\mathbb{R}) \right\}$$

and the involution

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} < \begin{pmatrix} \tau_{1} & \tau_{2} \\ \tau_{2} & \tau_{3} \end{pmatrix} = \begin{pmatrix} \tau_{3} & \tau_{2} \\ \tau_{2} & \tau_{1} \end{pmatrix}. \quad (2)$$

It follows that $M < \mathcal{H}_{1}$ coincides with a $\Gamma'_{l}$-translate of $\mathcal{H}_{1}$ if and only if $\gamma'_{l} M$ contains an element with the last line $(0, *, 0, *)$ or $(*, 0, *, 0)$ mod $t$.

The number of such classes is equal to

$$2 \varphi(t)^{-1} t^{2} \prod_{p | t} (1 - p^{-2}) = 2t \prod_{p | t} (1 + p^{-1}).$$

As a result we have proved the following: if $F$ is of weight $k$ and its divisor in $\mathbb{P}^{2}$ is exactly the $\Gamma'_{l}$-orbit of $m \mathcal{H}_{1}$, then $[F]_{1}$ has weight $k t^{3} \prod_{p | t} (1 + p^{-1})(1 + p^{-2})$ and divisor

$$\text{div}_{\mathcal{A}_{1}} ([F]_{1}) = 2m t \prod_{p | t} (1 + p^{-1}) \mathcal{H}_{1}.$$
It is well known (see [F1]), that for the Siegel modular group \( \Gamma_1 \) there is a cusp form \( \Delta_5 \) of weight 5 with a character of order 2 whose divisor is \( H_1 \). Using the Koecher principle we conclude that \([F]_1\) is a power of \( \Delta_5 \). Thus

\[
kt^3 \prod_{p \mid t} (1 + p^{-1})(1 + p^{-2}) = 10mt \prod_{p \mid t} (1 + p^{-1}). \tag{3}
\]

If \( m = 1 \), then we have \( kt^2 \prod_{p \mid t} (1 + p^{-2}) = 10 \). Since \( t^2 \) must be smaller than 20, there are only four numerical possibilities, namely \((t, k) = (1, 5), (2, 2), (3, 1), (4, 1/2)\). Again by Koecher’s principle the corresponding modular forms are, if they exist, unique up to a scalar, hence it remains to recall the existence of the modular forms in question. We have already used the famous modular form \( \Delta_5 \), which is the product of all even Siegel theta-constants (see [F1]). For \( t = 4 \) we consider the even Siegel theta-constant with characteristic \( 1/2(1, 1) \):

\[
\Delta_{1/2}(Z) = \frac{1}{2} \sum_{n, m \in \mathbb{Z}} \left( -\frac{1}{n} \right) \left( -\frac{1}{m} \right) \exp \left( \pi i \left( \frac{n^2}{4} \tau_1 + nm \tau_2 + m^2 \tau_3 \right) \right)
= \sum_{m > 0} \left( -\frac{1}{m} \right) \vartheta(\tau_1, m \tau_2) \exp \left( \pi i m^2 \tau_3 \right).
\]

Here \( \vartheta(\tau_1, \tau_2) \) is the Jacobi theta-constant with characteristic \( (1/2, 1/2) \), which is a Jacobi modular form of index \( 1/2 \). This representation implies (see [GN2, Theorem 1.11]) that \( \Delta_{1/2} \) is a modular form with respect to \( \Gamma'_4 \) since \( \Gamma'_4 \) is generated by the Jacobi group and the involution \((\tau_1 \tau_2 \tau_3) \mapsto (\tau_3/4 \tau_2 \tau_2/4 \tau_1)\).

The cusp forms \( \Delta_2 \) and \( \Delta_1 \) for \( \Gamma'_2 \) and \( \Gamma'_3 \) can be defined as the arithmetic lifting of the Jacobi forms \( \varphi_1(\tau_1, \tau_2) = \eta(\tau_1)^3 \vartheta(\tau_1, \tau_2) \) and \( \varphi_1(\tau_1, \tau_2) = \eta(\tau_1) \vartheta(\tau_1, \tau_2) \) of index \( 1/2 \) respectively (see [GN2, Theorem 1.12]):

\[
\Delta_2(Z) = \sum_{m \equiv 1 \mod 4 \quad m > 0} m \sum_{ad = m \mod b} d^{-2} \left( \frac{-1}{a} \right) \varphi_1 \left( \frac{a^3 + 4b}{d}, a \tau_2 \right) \exp \left( \pi i m \tau_3 \right)
\]

\[
\Delta_1(Z) = \sum_{m \equiv 1 \mod 6 \quad m > 0} \sum_{ad = m \mod b} d^{-1} \left( \frac{-1}{a} \right) \varphi_2 \left( \frac{a^3 + 6b}{d}, a \tau_2 \right) \exp \left( \pi i m \tau_3 \right).
\]

\( \Delta_{1/2}, \Delta_1, \Delta_2 \) vanish along \( H_1 \), because \( \vartheta(\tau_1, 0) \equiv 0 \). The consideration with \([F]_1\) above implies that the order of zero along the diagonal for each modular form is one and that \( H_1 \) is its full divisor. (In [GN2] this fact was proved using the Borcherds lifting.) Representations of the modular forms \( \Delta_1, \Delta_2 \) as lifting give us elementary formulae for the Fourier coefficients of...
these functions (see [GN2, Example 1.14]) which imply, for example, that \( \Delta_1 \) and \( \Delta_2 \) are cusp forms:

\[
\Delta_1(Z) = \sum_{M \geq 1} \sum_{n, m > 0, l \in \mathbb{Z}} \left( \frac{4}{l} \right) \left( \frac{12}{M} \right) \left( \frac{6}{a} \right) \exp \left( \pi i \left( \frac{n}{3} \tau_1 + l \tau_2 + m \tau_3 \right) \right)
\]

and

\[
\Delta_2(Z) = \sum_{N \geq 1} \sum_{n, m > 0, l \in \mathbb{Z}} \left( \frac{4}{N} \right) \left( \frac{4}{a} \right) \exp \left( \pi i \left( \frac{n}{2} \tau_1 + l \tau_2 + m \tau_3 \right) \right).
\]

The symmetrisation (1) gives us some formulae for \( \Delta_5 \) and \( \Delta_2 \) in terms of the theta-constant \( \Delta_1/2 \).

1. **\( \Delta_5 \) in terms of \( \Delta_1/2 \).** First we have seen in the above proof that

\[
\Delta_5^2 = (\text{const}) \prod_{M \in \Gamma_1' \cap \Gamma_1} \Delta_1/2|_M M.
\]

One can get a simpler representation (see [GN2, Theorem 1.11]) considering a subgroup \( \Gamma_{1,2} \) of \( \Gamma_1 \) conjugated to \( \Gamma_4' \), namely

\[
\Gamma_{1,2} = \text{diag}(1, 2, 1, 2^{-1}) \Gamma_4' \text{diag}(1, 2^{-1}, 1, 2).
\]

The modular form \( \tilde{\Delta}_1/2(Z) = \Delta_1/2(\frac{\tau_1}{\tau_2/2}, \frac{\tau_2}{\tau_3/4}) \) is \( \Gamma_{1,2}^- \)-modular form, where \( \Gamma_{1,2}^+ \) is the double extension of \( \Gamma_{1,2} \) defined by the involution (2). It is easy to check that \( [\Gamma_1 : \Gamma_{1,2}^+] = 10 \). Thus

\[
\Delta_5(Z) = (\text{const}) \prod_{M \in \Gamma_{1,2}^- \setminus \Gamma_1} \tilde{\Delta}_1/2|_M(Z).
\]

This is a new variant of the classical representation of \( \Delta_5(Z) \) as product of 10 theta-constants.

2. **\( \Delta_2 \) in terms of \( \Delta_1/2 \).** Here we can use the symmetrisation for the pair \( (\Gamma_2', \Gamma_4') \). We fix a system of representatives \( 0 \Gamma_4' \setminus 0 \Gamma_2 = \{M_1, \ldots, M_8\} \). It is easy to see that

\[
0 \Gamma_2' \setminus \Gamma_2' = \{E_4, S_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, J_2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -2 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix} \}.
\]

Since \( S_2 \in \Gamma_4' \) it follows that

\[
\Gamma_2' \cap \Gamma_4' \setminus \Gamma_2' = \{M_1, \ldots, M_8, M_1 J_2, \ldots, M_8 J_2\}.
\]
As in the proof of Proposition 1.1, it follows that
\[ [\Delta_{1/2}]_2 = \prod_{M \in \Gamma'_2 \cap \Gamma'_2 \backslash \Gamma'_2} \Delta_{1/2} M = (\text{const}) \Delta^4_2. \]

We note that \( \Delta^4_2 \) is the cusp form of minimal weight with trivial character for \( \Gamma_2 \) (see [F2], [G2]).

The modular forms considered above have some applications to algebraic geometry and physics. It is known (see [GN2, Theorem 5.2.1] and [GN3]) that they are the automorphic discriminant of a moduli space of \( K3 \) surfaces with the lattice of transcendental cycles of type \( U(n)^2 \oplus <2> \), where \( U(n) = \begin{pmatrix} 0 & -n \\ -n & 0 \end{pmatrix} \) and \( n = 1 \) for \( \Delta_5 \), \( n = 8 \) for \( \Delta_2 \), \( n = 12 \) for \( \Delta_1 \) and \( n = 16 \) for \( \Delta_{1/2} \). In [GN1]–[GN2] it was proved that the three cusp forms considered above determine the first members of the main series of generalized Lorentzian Kac–Moody superalgebras of rank 3 and they have an interesting infinite product expansion. In the case of \( \Delta_1 \) we have the following formula (see [GN2, Theorem 2.6]):
\[ \Delta_1(Z) = q^{\frac{1}{8}} r^{\frac{1}{8}} s^{\frac{1}{8}} \prod_{n \geq 0, m \geq 0, l \in \mathbb{Z}} (1 - q^n r^l s^{3m}) f(nm, l) \]
where \( q = e^{2\pi i \tau_1}, r = e^{2\pi i \tau_2}, s = e^{2\pi i \tau_3} \) and
\[ \sum_{n \geq 0, l} f(n, l) q^n r^l = \left( \frac{\vartheta(\tau_1, 2\tau_2)}{\vartheta(\tau_1, \tau_2)} \right)^2 \]
\[ = r^{-1} \left( \prod_{n \geq 1} (1 + q^{n-1}r)(1 + q^{n}r^{-1})(1 - q^{2n-1}r^2)(1 - q^{2n-1}r^{-2}) \right)^2. \]

In physics the modular form \( \Delta_5 \) appears in the two-loops vacuum amplitude of bosonic strings (see, for example, [BK], [M1]). We remark also that \( \Delta_2 \) and \( \Delta_5 \) are related to the perturbative prepotential and the perturbative Wilsonian gravitational coupling of some four parameter \( D = 4, N = 2 \) string models (see [K2], [CCL], [C]). Moreover in [DVV] it was shown that \( \Delta_{2/5}^{-2} \) can be interpreted as the second quantized elliptic genus of a K3 surface (see also [K2], [M2]).

In this note we discuss the geometric significance of the form \( \Delta_1 \) which is very closely connected with the Barth-Nieto quintic [BN] and we shall briefly comment on the forms \( \Delta_2 \) and \( \Delta_5 \).
2 The moduli space $\mathcal{A}_3(2)$

Let $\Gamma_3(2)$ be the subgroup of the paramodular group $\Gamma_3$ which defines the moduli space $\mathcal{A}_3(2) = \Gamma_3(2)\backslash \mathbb{H}_2$ of (1, 3)-polarized abelian surfaces with a level 2 structure. Then

$$\Gamma_3(2) = \left\{ g \in \Gamma_3 : g - 1 \in \begin{pmatrix} 2\mathbb{Z} & 2\mathbb{Z} & 2\mathbb{Z} & 6\mathbb{Z} \\ 6\mathbb{Z} & 2\mathbb{Z} & 6\mathbb{Z} & 6\mathbb{Z} \\ 2\mathbb{Z} & 2\mathbb{Z} & 2\mathbb{Z} & 6\mathbb{Z} \\ 2\mathbb{Z} & \frac{3}{2}\mathbb{Z} & 2\mathbb{Z} & 2\mathbb{Z} \end{pmatrix} \right\}.$$  

The group $\Gamma_3$ is conjugate via $\text{diag}(1, 1, 1, 3)$ to the symplectic group $\text{Sp}(\Lambda_3, \mathbb{Z})$ where $\Lambda_3$ is the following symplectic form

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \\ -1 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \end{pmatrix}. $$

Under this isomorphism $\Gamma_3(2)$ is identified with the group $\text{Sp}^{(2)}(\Lambda_3, \mathbb{Z})$ consisting of all elements $g \in \text{Sp}(\Lambda_3, \mathbb{Z})$ with $g = 1 \mod 2$.

Lemma 2.1 The modular form $\Delta^3_1$ is a weight 3 cusp form with respect to $\Gamma_3(2)$.

Proof. The form $\Delta_1$ is a cusp form with respect to $\Gamma_3$ with a character $\chi_6$ of order 6. Hence $\Delta^3_1$ has a character $\chi_2 = \chi_6^3$ of order 2. It follows from [3H2, Theorem 2.1] that there is exactly one such character and that this character arises in the following way

$$1 \rightarrow \Gamma_3(2) \rightarrow \Gamma_3 \rightarrow \text{Sp}(4, \mathbb{Z}/2) \cong S_6 \rightarrow 1.$$

In particular $\chi_2|_{\Gamma_3(2)} \equiv 1$ and hence $\Delta^3_1$ is a modular form with respect to $\Gamma_3(2)$.  

Remark 2.2 It follows from the results of Barth and Nieto [BN] that $\mathcal{A}_3(2)$ has a smooth projective model $\tilde{\mathcal{A}}_3(2)$ which is a Calabi-Yau 3-fold. By Freitag’s extension result the space $S_3(\Gamma_3(2))$ of weight 3 cusp forms is isomorphic to $\Gamma(\tilde{\mathcal{A}}_3(2), K_{\tilde{\mathcal{A}}_3(2)})$ for every smooth projective model. It follows that $\Delta^3_1$ is the unique weight 3 cusp form with respect to $\Gamma_3(2)$. We shall, however, not use the result of Barth and Nieto in what follows. In fact our methods will allow us to construct a smooth projective Calabi-Yau model of $\mathcal{A}_3(2)$ in an easy way.
For what follows we have to determine the singularities of the moduli space $A_3(2)$ and of a suitable toroidal compactification. An important role is played by the involution

$$I := \begin{pmatrix} 1 & -1 \\ -1 & 1 \\ -1 & 1 \end{pmatrix}$$

Note that $I \in \Gamma_3(2)$ and that $\text{Fix } I = \mathcal{H}_1$.

**Lemma 2.3** Up to conjugation with elements in $\Gamma_3$ the only elements of finite order in $\Gamma_3(2)$ are $\pm 1, \pm I$.

**Proof.** We work in $\text{Sp}(\Lambda_3, \mathbb{Z})$. Then every element $g \in \text{Sp}^{(2)}(\Lambda_3, \mathbb{Z})$ is of the form

$$g = \begin{pmatrix} 1 + A & B \\ C & 1 + D \end{pmatrix} \text{ with } A, B, C, D \equiv 0 \mod 2.$$ 

Hence

$$g^2 = \begin{pmatrix} 1 + 2A + A^2 + BC & 2B + AB + BD \\ 2C + CA + DC & 1 + CB + 2D + D^2 \end{pmatrix} \equiv 1 \mod 4.$$

This shows that $g^2 = 1$ and hence $g$ is an involution. By a result of Brasch [Br, Folgerung 2.9] the only involutions in $\Gamma_3$ are

$$\pm 1, \pm I, \pm \begin{pmatrix} -1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ 0 & 0 & -1 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Since the last involution is not in $\Gamma_3(2)$, the claim follows. \hfill \Box

We consider the toroidal compactification $A_3^*(2)$ which belongs to the second Voronoi decomposition. We shall refer to this compactification either as Voronoi or as in [IKW2] as Igusa compactification of $A_3(2)$.

**Theorem 2.4** The variety $A_3^*(2)$ is Gorenstein. It has exactly 15 isolated singularities of type $V_{\frac{1}{2}}(1,1,1)$.

**Proof.** It follows from Lemma 2.3 that the branch locus of the projection map $\mathbb{H}_2 \to A_3(2)$ are the translates of $\mathcal{H}_1$. Since $I$ acts locally like a reflection the space $A_3(2)$ is smooth. It remains to determine the singularities on
the boundary. Here we can proceed along the same lines as in [HKW1] and [Br]. We shall first treat the corank 1 boundary components $D(l)$. Up to the action of $\Gamma_3/\Gamma_3(2) \cong S_6$ there are two types, namely $D(l_0)$ and $D(l_{(0,1)})$ where $l_0 = (0,0,1,0)$ and $l_{(0,1)} = (0,0,0,1)$. Here we shall treat the boundary surface $D(l_0)$ in detail, the other boundary surface $D(l_{(0,1)})$ can be treated in exactly the same way. The parabolic subgroup associated to $l_0$ is

$$P_{l_0}(\Gamma_3(2)) = \left\{ \begin{pmatrix} \varepsilon & m & q & 3n \\
0 & a & * & 3b \\
0 & 0 & \varepsilon & 0 \\
0 & c/3 & * & d \end{pmatrix} ; m, n, q, \in \mathbb{Z}, \varepsilon = \pm 1, \begin{pmatrix} a & b \\
c & d \end{pmatrix} \in \Gamma_1(2) \right\}.$$

Here the entries $*$ are determined by the condition that the matrix is symplectic. This description follows from [HKW2, Proposition I.3.87] and the description of the group $\Gamma_3(2)$. Dividing out by the rank 1 lattice $P'_{l_0}(\Gamma_3(2))$ which is given by the elements with $\varepsilon = 1, m = n = 0, \begin{pmatrix} a & b \\
c & d \end{pmatrix} = 1$, we obtain the quotient group $P''_{l_0}(\Gamma_3(2)) = P_{l_0}(\Gamma_3(2))/P'_{l_0}(\Gamma_3(2))$ which can be identified with the following matrix group:

$$P''_{l_0}(\Gamma_3(2)) \cong \left\{ \begin{pmatrix} 1 & \varepsilon m & \varepsilon n \\
0 & \varepsilon a & \varepsilon b \\
0 & \varepsilon c & \varepsilon d \end{pmatrix} ; \begin{pmatrix} a & b \\
c & d \end{pmatrix} \in \Gamma_1(2), m, n \in \mathbb{Z}, \varepsilon = \pm 1 \right\}.$$

This acts on $\mathbb{C}^* \times \mathbb{C} \times \mathbb{H}_1$ with coordinates $t_1 = e^{2\pi i \tau_1/2}, \tau_2, \tau_3$ as follows:

$$t'_1 = t_1 e^{\pi i [m\tau_2 - \tau'_2(\varepsilon \tau_2 + 3n)]}, \quad \tau'_2 = (\varepsilon \tau_2 + m\tau_3 + 3n)(\varepsilon \tau_3 + d)^{-1}, \quad \tau'_3 = 3(\varepsilon \tau_3 + b)(\varepsilon \tau_3 + d)^{-1}.$$

We have to find all points $P = (t_1, \tau_2, \tau_3)$ with $g(P) = g$ for some $1 \neq g \in P''_{l_0}(\Gamma_3(2))$. Invariance of the third component implies that $\tau_3/3$ is a fixed point for $\begin{pmatrix} a & b \\
c & d \end{pmatrix}$. Hence $\begin{pmatrix} a & b \\
c & d \end{pmatrix} = 1_2$ or $\begin{pmatrix} a & b \\
c & d \end{pmatrix} = -1_2$. In the first case we have the possibilities $\varepsilon = 1$ or $-1$. If $\varepsilon = 1$ then $g = 1$. If $\varepsilon = -1$ then

$$g = \begin{pmatrix} 1 & m & n \\
0 & -1 & 0 \\
0 & 0 & -1 \end{pmatrix}$$

with fixed locus

$$\tau_2 = \frac{1}{2}(m\tau_3 + 3n).$$
Since the involution $g$ acts locally like a reflection in the neighbourhood of this point this leads to smooth points in $A_3(2)$. (Note that $g$ is induced by the involution
\[
\begin{pmatrix}
1 & m & 0 & 3n \\
0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix} \in \Gamma_3(2).
\]
This involution is conjugate to $I$ and hence the curve given by $\tau_2 = (m\tau_3 + n)/2$ is the intersection of a translate of $H_1$ with the boundary component $D(l_0)$. If \[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
\] = \(-1\) the case $\varepsilon = -1$ gives $g = 1$ and $\varepsilon = 1$ again leads to the curve $\tau_2 = (m\tau_3 + n)/2$.

It remains to consider the corank 2 boundary components $E(h)$. Since all of these are equivalent under $\Gamma_3/\Gamma_3(2) \cong \mathbb{S}$ it suffices to consider the case $h = (0,0,1,0) \land (0,0,0,1)$. Here
\[
P(h) = \left\{ \begin{pmatrix} A & 0 \\ 0 & tA^{-1} \end{pmatrix} ; A \in \text{GL}(2,\mathbb{Z}), A - 1_2 \in \begin{pmatrix} 2\mathbb{Z} & 2\mathbb{Z} \\ 6\mathbb{Z} & 2\mathbb{Z} \end{pmatrix} \right\} \cdot P'(h)
\]
where $P'(h)$ is the lattice
\[
P'(h) = \left\{ \begin{pmatrix} 1_2 & B \\ 0 & 1_2 \end{pmatrix} ; B = tB, B \in \begin{pmatrix} 2\mathbb{Z} & 6\mathbb{Z} \\ 6\mathbb{Z} & 6\mathbb{Z} \end{pmatrix} \right\}.
\]
Let $N \subset \text{Sym}(2,\mathbb{Z})$ be the lattice generated by the matrices $B$ and let $\Sigma_N \subset N_{\mathbb{R}}$ be the fan induced by the Legendre decomposition (which is here equal to the second Voronoi decomposition). Then we have the partial quotient
\[
e(h) : \mathbb{H}_2 \to T_n \cong (\mathbb{C}^*)^3 \\
\begin{pmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{pmatrix} \mapsto (e^{2\pi i \tau_1/2}, e^{2\pi i \tau_2/6}, e^{2\pi i \tau_3/6}).
\]
The fan $\Sigma_N$ defines a torus embedding $T_N \subset X_{\Sigma_N} =: X$. Here, however, the situation differs from [HKW1]. The main difference is that $N$ is not equivalent to the lattice $\text{Sym}(2,\mathbb{Z})$. We can, however, compare the torus embedding $T_N \subset X_{\Sigma_N}$ with the standard torus embedding $T_{\text{Sym}(2,\mathbb{Z})} \subset X_{\Sigma_N}$.
To do this let $N'$ be the lattice spanned by matrices $B'$ with
\[
B' = tB'; \quad B' \in \begin{pmatrix} 6\mathbb{Z} & 6\mathbb{Z} \\ 6\mathbb{Z} & 6\mathbb{Z} \end{pmatrix}
\]
Then $N'$ is equivalent to $\text{Sym}(2,\mathbb{Z})$ and $N/N' \cong \mathbb{Z}/3$. This defines a commutative diagram
\[
\begin{array}{ccc}
T_{N'} & \subset & X' = X_{\Sigma_N'} \\
\downarrow & & \downarrow \\
T_N & \subset & X = X_{\Sigma_N}
\end{array}
\]
where the vertical maps are quotients by the cyclic group $N/N'$. The variety $X'$ is covered by affine sets $X'_\sigma \cong \mathbb{C}^3$ and hence smooth. This is no longer the case for $X$. In the analogous situation without a level 2 structure Brasch [Br, Satz(III.5.21)] has shown that $X_{\Sigma N}$ has, up to the action of $P''(h)$, exactly one singularity $P$ which is of type $V_{3(1,1,1)}^{13}$. This singularity arises as follows: Let $\sigma_1$ be the cone 

$$\sigma_1 = \mathbb{R}_{\geq 0} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \mathbb{R}_{\geq 0} \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} + \mathbb{R}_{\geq 0} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$ 

Then $X'_{\sigma_1} \cong \mathbb{C}^3$ and $N/N'$ acts on $X'_{\sigma_1}$ by $\langle \text{diag}(\rho) \rangle$ where $\rho = e^{2\pi i/3}$. Hence the origin gives rise to a singularity $P$ of type $V_{3(1,1,1)}^{13}$. A similar analysis applies in the presence of a level 2 structure. A straightforward calculation (see also [Br, Hilfssatz(III.5.23)]) shows that the stabilizer of $P$ in $P''(h)$ is trivial if we work with a level 2 structure, resp. isomorphic to the symmetric group $S_3$ if we work without a level structure. This implies that the singularities of $A_3^*(2)$ are of type $V_{3(1,1,1)}^{13}$. To count the number of singularities recall that the number of corank 2 boundary components is 15 (see e.g. [Fr]). It remains to show that every boundary component $E(h)$ has exactly one singularity. To count this number we consider the groups 

$$\Gamma_0(3) = \left\{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{SL}(2, \mathbb{Z}); c \equiv 0 \mod 3 \right\}.$$ 

and 

$$\Gamma_0^{(2)}(3) = \left\{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma_0(3); \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \equiv 1 \mod 2 \right\}.$$ 

The quotient $\Gamma_0(3)/\Gamma_0^{(2)}(3)$ is isomorphic to the symmetric group $S_3$ and this is exactly the stabilizer of the cone $\sigma_1$, resp. the point $P$. This gives the claim.

We have already introduced the Humbert surface $H_1 \subset A_3$. The number of components of $H_1$ in $A_3$ is given by $\#\{ \pm b \mod 6; b^2 \equiv 1 \mod 12 \} = 1$ (see [GH2]). Hence $H_1 = \pi(H_1)$. Recall that $H_1$ parametrizes split polarized abelian surfaces. Let $H_1(2) \subset A_3(2)$ be the inverse image of $H_1$ in $A_3(2)$. We denote the closure of $H_1$, resp. $H_1(2)$ in $A_3$, resp. $A_3^*(2)$ by $\bar{H}_1$, resp. $\bar{H}_1(2)$.

**Proposition 2.5** The surface $\bar{H}_1(2)$ has 20 components $\bar{H}_1^i(2)$ which are equivalent under the group $\Gamma_3/\Gamma_3^i(2) \cong S_3$. Every component $\bar{H}_1^i(2)$ is smooth and isomorphic to the product $X(2) \times X(2)$ of modular curves of level 2. Two different components $\bar{H}_1^i(2)$ and $\bar{H}_1^j(2)$ with $i \neq j$ do not intersect. The intersection of $\bar{H}_1^i(2)$ with the boundary is transversal and consists of the curves $\{\text{cusp}\} \times X(2)$ and $X(2) \times \{\text{cusp}\}$. The surface $\bar{H}_1(2)$ is disjoint from the singularities of $A_3^*(2)$.
Proof. The number of components was computed in [HSN, Theorem 3.2]. Clearly the intersection $\bar{H}_i(2)$ of $\bar{H}_1(2)$ with $A_3(2)$ is isomorphic to $X^o(2) \times X^o(2)$. To see that $\bar{H}_1(2)$ is still a product one can proceed as in the proof of [HKW1, Proposition (1.5.53)]. The crucial point is to study the points in the boundary components $D(l)$, resp. $E(h)$ which came from points with a non-trivial stabilizer. For $D(l)$ we did this explicitly in the proof of Theorem (2.4) which also gives us immediately that $\bar{H}_1(2)$ and $D(l)$ intersect transversally. The calculations for $E(h)$ are also straightforward although slightly more cumbersome. Since this follows from [Br, Hilfssatz(II.5.23)] we will not give the details here. At the same time we find that there are no points whose stabilizer contains $\mathbb{Z}/2 \times \mathbb{Z}/2$ and hence two different components $\bar{H}_i(2)$ and $\bar{H}_j(2)$ cannot meet. The singularities of $A^*(2)$ are in the deepest points. None of these points have a stabilizer which contains a group $\mathbb{Z}/2$ hence $\bar{H}_1(2)$ does not contain any of these points. \qed

The curve $X(2)$ is rational and hence the above proposition shows that $\bar{H}_1(2)$ is a disjoint union of 20 quadrics. Since $X(2)$ has 3 cusps the intersection of every such quadric with the boundary is a union of 6 lines which give a divisor of bidegree $(3, 3)$. On the other hand the intersection of $\bar{H}_1$ with a boundary surface consists of 4 sections, namely the fixed points of the Kummer involution. Altogether we have $15+15=30$ boundary surfaces and in this way we obtain a $(30, 20\times 6)$-configuration.

We now turn to the canonical bundle of $A^*(2)$. Since the map $\pi : \mathbb{H}_2 \to A_3(2)$ is branched of order 2 along $\bar{H}_1(2)$ it follows that

$$K_{A_3^*(2)} = 3L - D - \frac{1}{2}H_1. \quad (4)$$

Here $L$ is the $\mathbb{Q}$-line bundle of modular forms of weight 1 and $D$ denotes the boundary.

**Lemma 2.6** For every component of $\bar{H}_1(2)$ the normal bundle has bidegree $(-1, -1)$.

**Proof.** Recall that for every component $\bar{H}_1(2) \cong X(2) \times X(2)$ is a quadric and hence $K_{\bar{H}_1(2)}$ has bidegree $(-2, -2)$. By adjunction

$$K_{\bar{H}_1(2)} = \left(K_{A^*_2(2)} + \bar{H}_1(2)\right) |_{\bar{H}_1(2)}.$$  

Using formula (1) for $K_{A^*_2(2)}$ we can deduce from this that

$$\bar{H}_1(2) |_{\bar{H}_1(2)} = 2(K_{\bar{H}_1(2)} - 3L + D) |_{\bar{H}_1(2)}.$$

The line bundle $L$ restricts to $X(2) \times X(2)$ as $L_X(2) \boxtimes L_X(2)$ which has bidegree $(-1, -1)$. Since $X(2)$ has 3 cusps we had already seen that the boundary...
intersects $\tilde{H}_i^1(2)$ in a divisor of bidegree $(3,3)$. Finally $K_{\tilde{H}_i^1(2)}$ has bidegree $(-2,-2)$ and the claim follows immediately by adding the bidegrees in the right hand side of formula (2).

\textbf{Theorem 2.7} There exists a commutative diagram

\[
\begin{array}{ccc}
\tilde{Z} & \xrightarrow{f} & Z \\
\tilde{\pi} & \downarrow & \pi \\
A_3^*(2) & \xrightarrow{f'} & Z'
\end{array}
\]

where the vertical maps $\pi$ and $\tilde{\pi}$ are blow-ups of the singularities and where $f$ and $f'$ contract each of the 20 quadrics $\tilde{H}_i^1(2)$ to a line such that $Z$ is a smooth projective Calabi-Yau threefold.

\textit{Proof.} We first remark that the singularities are harmless: If $X$ is a singularity of type $V_{\frac{1}{4}(1,1,1)}$ and $f: Y \rightarrow X$ is the blow-up of $X$ in the singular point then $Y$ is smooth and $f^*K_X = K_Y$. Since the normal bundle of every component $\tilde{H}_i^1(2)$ has bidegree $(1,1)$ we can contract each set of rulings and obtain a manifold. The crucial point is to show that this can be done in such a way that the resulting manifold is projective. We shall do this using Cornalba’s criterion \cite[Theorem 2]{Co}. Let $\Phi$ and $\Psi$ be the rulings of a component $\tilde{H}_i^1(2)$. We choose an ample line bundle $A$ on $A_3^*(2)$ and assume that $A.\Psi \geq A.\Phi$. Since the boundary components cut out rulings on $\tilde{H}_i^1(2)$ of the form $\{\text{cusp}\} \times X(2)$ or $X(2) \times \{\text{cusp}\}$ we can choose a component $D_k$ such that $D_k.\Phi = 0$ and $D_k.\Psi = 1$. Choose $\alpha \gg 0$ such that $\alpha A + D_k$ is ample. Then we set

\[s := (\alpha A + D_k).\Phi = \alpha A.\Phi > 0.\]

With this choice of $s$ we find that

\[
\begin{align*}
(\alpha A + D_k + s\tilde{H}_i^1(2)).\Phi & = \alpha A.\Phi - s = 0 \\
(\alpha A + D_k + s\tilde{H}_i^1(2)).\Psi & = \alpha A.\Psi + 1 - s > 0.
\end{align*}
\]

It then follows from Cornalba’s theorem that the manifold which is obtained by contracting the rulings homologous to $\Phi$ is smooth projective. We can do this for every component of $\tilde{H}_i^1(2)$ separately.

The modular form $\Delta_3^3$ defines a canonical form

\[\omega = \Delta_3^3 d\tau_1 \wedge d\tau_2 \wedge d\tau_3\]

which descends to $A_3^*(2)$ and vanishes of order 1 along $\tilde{H}_i^1$. It has no other zeroes in $A_3^*(2)$. Since the Fourier expansion of $\Delta_3^3$ starts with $e^{2\pi i \tau_1/6}$, resp. $e^{2\pi i \tau_3/18}$ it follows that $\Delta_3^3$ vanishes of order 1 along the boundary
D and hence \(\omega\) has no zeroes along a boundary component. This proves that \(f'_{*}(\omega)\) defines a nowhere vanishing canonical form outside \(f'(\bar{H}_1(2))\). Since \(f'(\bar{H}_1(2))\) is a union of curves we have in fact obtained a nowhere vanishing form on \(Z'\), i.e. \(K_{Z'} = \mathcal{O}_{Z'}\) and hence also \(K_Z = \mathcal{O}_Z\) since the only singularities present are of type \(V_{\frac{1}{3}}(1,1,1)\). Finally \(q(Z) = h^1(\mathcal{O}_Z) = 0\) since this holds for every smooth projective model of a Siegel modular variety \(\mathbb{F}_3\).

\[\square\]

**Remark 2.8** The Calabi-Yau manifold \(Z\) is clearly birational to the Calabi-Yau model constructed by Barth and Nieto in \([BN]\), but it is not clear whether the two models are isomorphic. By recent results of Batyrev \([Ba]\) and Huybrechts \([Hu]\) it follows, however, that the two models have the same Betti numbers, in particular \(e(Z) = 80\).

### 3 The moduli space \(\mathcal{A}_3(3)\)

We consider the group

\[
\text{Sp}^{(3)}(\Lambda_3, \mathbb{Z}) = \{ g \in \text{Sp}(\Lambda_3, \mathbb{Z}); g \equiv 1 \text{ mod } 3 \}
\]

resp. the conjugated group

\[
\Gamma_3(3) = R_3^{-1} \text{Sp}(\Lambda_3, \mathbb{Z}) R_3
\]

where \(R_3 = \text{diag } (1,1,1,3)\). Then \(\mathcal{A}_3(3) = \Gamma_3(3) \setminus \mathbb{H}_2\) is the moduli space of \((1,3)\)-polarized abelian surfaces with a level 3 structure. (For the definition of generalized level \(n\) structure see also \([LB, \text{chapter 8}]\).) Again we denote by \(\mathcal{A}_3^*(3)\) the Voronoi compactification of \(\mathcal{A}_3(3)\).

**Theorem 3.1** (i) The variety \(\mathcal{A}_3^*(3)\) is Gorenstein and its singular locus consists of a finite number of points of type \(V_{\frac{1}{3}}(1,1,1)\).

(ii) The variety \(\mathcal{A}_3^*(3)\) is of general type.

**Proof.** (i) By \([LB, \text{Corollary 5.1.10}]\) the group \(\Gamma_3(3)\) acts freely on \(\mathbb{H}_2\). To compute the singularities of \(\mathcal{A}_3^*(3)\) we can then proceed as in the proof of Theorem 2.4. These arguments show that for \(l = l_0\) and \(l = l_{(0,1)}\) the group \(P'_l(\Gamma_3(3))\) acts freely. The analysis for the corank 2 boundary components is also analogous. In this case the lattice \(P'(h)\) is

\[
P'(h) = \left\{ \left( \begin{array}{cc} 12 & B \\ 0 & 12 \end{array} \right); \; B = \tau B, \; B \in \left( \begin{array}{cc} 3\mathbb{Z} & 9\mathbb{Z} \\ 9\mathbb{Z} & 9\mathbb{Z} \end{array} \right)\right\}
\]

and we find again a finite number of isolated singularities of type \(V_{\frac{1}{3}}(1,1,1)\).

(ii) By (i) every modular form of weight 3\(k\) with respect to the group \(\Gamma_3(3)\)
which vanishes of order $k$ along the boundary gives rise to a $k$-fold differential form on $A_3^*(3)$. The form $\Delta_1^6$ is a modular form with respect to $\Gamma_3(3)$ and vanishes of order 3 along the boundary. Hence the space

$$W_{9k} := \Delta_1^k M_{3k}(\Gamma_{1,3}(3)) \subset M_{9k}(\Gamma_3(3))$$

is a space of forms of weight $9k$ vanishing of order 3 along the boundary. Since the dimension of $W_{9k}$ grows as $\text{const} \cdot k^3$ it follows that the 3-fold $A_3^*(3)$ is of general type. 

In this case we can again contract the components $\bar{H}_i^1(3)$ of the Humbert surface of discriminant 1 to curves to obtain a projective minimal model of $A_3^*(3)$.

**Theorem 3.2** The components $\bar{H}_i^1(3)$ of the Humbert surface $\bar{H}_1^1(3)$ of discriminant 1 can be contracted to curves such that the resulting variety $\hat{A}_3^3(3)$ is projective and has finitely many isolated singularities of type $V_{\frac{1}{4}}^{(1,1,1)}$. The variety $\hat{A}_3^3(3)$ is a minimal model of $A_3^*(3)$.

**Proof.** As before every component $\bar{H}_i^1(3)$ is isomorphic to the product $X(3) \times X(3)$ of modular curves of level 3, and hence to a quadric. Since $\Gamma_3(3)$ acts freely on $\mathbb{H}_2$ we find for the canonical bundle of $A_3^*(3)$ that

$$K_{A_3^*(3)} = 3L - D.$$ 

Using adjunction for the surfaces $\bar{H}_1^1(3)$ one obtains

$$\bar{H}_1^1(3)|_{\bar{H}_1^1(3)} = K_{\bar{H}_1^1(3)} - (3L - D)|_{\bar{H}_1^1(3)}.$$ 

The line bundle $L_{X(3)}$ has degree 1 and $X(3)$ has 4 cusps. It follows that the normal bundle of $\bar{H}_1^1(3)$ has bidegree $(-1, -1)$. We can now argue as in the proof of Theorem 2.7 and contract one set of rulings for each of the quadrics $\bar{H}_1^1(3)$ to obtain a projective 3-fold $\hat{A}_3^3(3)$. Since the singularities of $A_3^*(3)$ are disjoint from the Humbert surface $\bar{H}_1^1(3)$ the variety $\hat{A}_3^3(3)$ has the same singularities as $A_3^*(3)$. In particular $\hat{A}_3^3(3)$ is Gorenstein.

It remains to prove that $K_{\hat{A}_3^3(3)}$ is nef. Let $f : A_3^3(3) \to \hat{A}_3^3(3)$ be the contraction map. Then

$$f^* K_{\hat{A}_3^3(3)} = K_{A_3^3(3)} - \sum_i \bar{H}_i^1(3) = K_{A_3^3(3)} - \bar{H}_1^1(3) =: K'.$$ 

We claim that $K' \cdot C \geq 0$ for every curve $C$ in $A_3^5(3)$. Since $K'$ is trivial on $\bar{H}_1^1(3)$ we can assume that $C$ is not contained in the Humbert surface of discriminant 1. The form $\Delta_1^6$ is a modular form (without a character) of weight 6 with respect to the group $\Gamma_3(3)$. It vanishes of order 6 along the components $\bar{H}_i^1(3)$ of $\bar{H}_1^1(3)$ and of order 3 along the boundary. Hence

$$6L = 6\bar{H}_1^1(3) + 3D.$$
resp. \[ D = 2L - 2\bar{H}_1(3). \]

This implies that
\[ K_{A_3(3)} = L + 2\bar{H}_1(3) \]
resp.
\[ K' = L + \bar{H}_1(3). \]

Since \( L \) is nef (a multiple of \( L \) is free) it follows that \( K'.C \geq 0 \) for every curve \( C \) not contained in \( \bar{H}_1(3) \).

\[ \square \]

4 Concluding remarks

In this section we want to remark briefly on the geometric relevance of the modular forms \( \Delta_2 \) and \( \Delta_5 \).

The modular form \( \Delta_2 \) is a modular form of weight 2 with a character of order 4 with respect to the group \( \Gamma_2 \). The restriction of this character to the congruence subgroup \( \Gamma_2(4) \) is trivial. In particular \( \Delta_2 \) is a weight 2 cusp form with respect to \( \Gamma_2(4) \) and arguing along the same lines as in the proof of Theorem 3.1 this implies, under the assumption that all singularities of \( A_2^{*}(4) \) are canonical, that this space is of general type. The normal bundle of the components \( \bar{H}_1^{*}(4) \) of the Humbert surface of discriminant 1 has bidegree \((-2, -2)\) and the restriction of \( K_{A_2^{*}(4)} \) to these components is trivial. Since \( 2L = \bar{H}_1(4) - D \) it follows that \( K_{A_2^{*}(4)} = L + \bar{H}_1(4) \) and, again modulo checking that all singularities of \( A_2^{*}(4) \) are canonical, this shows that the variety \( A_2^{*}(4) \) is a minimal model.

The modular form \( \Delta_5 \) has weight 5 and a character of order 2 with respect to the integer symplectic group \( \Gamma_1 \). Since \( \Delta_5 \) vanishes on the Humbert surface \( H_1 \) we find that \( 10L = 2H_1 \) on \( A_1 \). The divisors \( 5L \) and \( H_1 \) differ by 2-torsion (cf. [GH2]). On the moduli space \( A_1^{*}(n) \) we have the relation \( 10L = 2\bar{H}_1(n) + nD \). Since \( K_{A_1^{*}(n)} = 3L - D \) it follows that
\[
K_{A_1^{*}(n)} = (3 - \frac{10}{n})L + \frac{2}{n}D.
\]

Note also that the normal bundle of the components \( \bar{H}_1(n) \) is negative for \( n \leq 3 \), trivial for \( n = 4 \) and positive for \( n \geq 5 \). This can be used to show that \( K_{A_1^{*}(n)} \) is nef for \( n = 4 \) and ample for \( n \geq 5 \) (see also [I], [E]).
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