A BILLINGSLEY-TYPE THEOREM FOR THE PRESSURE OF AN ACTION OF AN AMENABLE GROUP

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Abstract. This paper extends the definition of Bowen topological entropy of subsets to Pesin-Pitskel topological pressure for the continuous action of amenable groups on a compact metric space. We introduce the local measure theoretic pressure of subsets and the relation between local measure theoretic pressure of Borel probability measures and Pesin-Pitskel topological pressure on an arbitrary subset of a compact metric space.

1. Introduction. Entropy is undoubtedly among the most essential characteristics of dynamical systems. The classical measure-theoretic entropy for an invariant measure and the topological entropy were introduced in [13] and [1] respectively. The basic relation between measure theoretic entropy and topological entropy is the variational principle [10, 9]. Since then the subjects involving definition of new measure-theoretic and topological notions of entropy and studying the relation between them have gained a lot of attention in the study of dynamical system.

Topological pressure is a nontrivial and natural generalization of topological entropy. One of the most fundamental dynamical invariants that associate to a continuous map is the topological pressure with a potential function. It roughly measures the orbit complexity of the iterated map on the potential function. The notion of topological pressure was brought to the theory of dynamical systems by Ruelle [22] and Walters [24]. Ruelle [21] introduced topological pressure of a continuous function for actions of the groups $\mathbb{Z}^n$ on compact spaces in this context when the action is expansive and satisfies the specification condition. Later, the variational principle was formulated by Walters in [24].

The theory related to the topological pressure and variational principle plays a fundamental role in statistical mechanics, ergodic theory and dynamical systems [4, 22, 24, 8, 12]. Since the works of Bowen [5] and Ruelle [22], the topological pressure turned into a basic tool in the dimension theory related to dynamical systems. From a viewpoint of dimension theory, Pesin and Pitskel [20] introduced another way to define topological pressure for continuous functions on noncompact sets in the case of $\mathbb{Z}$-actions, which was based on the Carathéodory structure [7],

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which we call the Pesin-Pitskel topological pressure. In [20], Pesin and Pitskel proved the variational principle under some supplementary conditions.

Brin and Katok showed in [6] the interrelations between a measure-theoretic entropy and dimension-like characteristics of smooth dynamical systems. Ma and Wen [16] applied a dimensional type characteristic of the entropy to obtain the following relation between local measure entropy and the dimensional type entropy:

**Theorem A.** (Theorem 1 in [16]) Let $f$ be a continuous map on a compact metric space $X$. Let $\mu$ be a Borel probability measure on $X$, $E$ be a Borel subset of $X$ and $0 < s < \infty$.

1. If $h_\mu(f, x) \leq s$ for all $x \in E$, then $h(f, E) \leq s$.
2. If $h_\mu(f, x) \geq s$ for all $x \in E$ and $\mu(E) > 0$, then $h(f, E) \geq s$.

Ma and Wen’s result was an analogue of the Billingsley’s Theorem [2]. In 2012, B. Liang and K. Yan [15] introduced the topological pressure for any sub-additive potentials of a countable discrete amenable group action and any given open cover, and established a local variational principle for it. In 2013, A. Bis [3] generalized the notion of local measure entropy for the case of a group or a pseudogroup of homeomorphism of a metric space. He obtained an analogue of the variational principle for group and pseudogroup actions. Later, Tang, Cheng and Zhao [23] proved that Bowen topological pressure is bounded by measure theoretic pressure of Borel probability measures, which extended the result in [16] for Bowen topological pressure of integer group action.

In this paper, we generalize Ma-Wen’s result [16] to dynamical systems acting by a countable discrete amenable group. We define the Pesin-Pitskel topological pressure and the local measure theoretic pressure for amenable group action and establish an analogue of the Billingsley’s Theorem between local measure theoretic pressure and the topological pressure.

We organize the paper as follows: we begin in Section 3 by setting up our Pesin-Pitskel topological pressure definition and giving some important properties for amenable group actions. In Section 4 we give the local measure theoretic pressure of arbitrary subsets for amenable group action and we prove that the Pesin-Pitskel topological pressure can be bounded by the local measure theoretic pressure for actions of amenable groups. Finally, in Section 5 we calculate the Pesin-Pitskel topological pressure of some subsets of Bernoulli shifts for amenable groups.

2. **Preliminaries.** In this section, we recall some basic properties of amenable groups.

Let $G$ be a discrete infinitely countable group. Denote by $\mathcal{F}(G)$ the set of all nonempty finite subsets of $G$. For $K \in \mathcal{F}(G)$ and $\delta > 0$, denote by $\mathfrak{B}(K, \delta)$ the set of all $F \in \mathcal{F}(G)$ satisfying $|KF \setminus F| < \delta|F|$. The group $G$ is called amenable if $\mathfrak{B}(K, \delta)$ is nonempty for every $(K, \delta)$. This is equivalent to the existence of a sequence of nonempty finite subsets $\{F_n\}$ of $G$ which are asymptotically invariant, i.e.,

$$\lim_{n \to \infty} \frac{|F_n \Delta g F_n|}{|F_n|} = 0 \quad \text{for all} \quad g \in G.$$  

Such sequences are called Folner sequences. For details on actions of amenable groups, one may refer to Ornstein and Weiss’s pioneering paper [18] or Kerr and Li’s book [12].

The collection of pairs $\Lambda = \{(K, \delta) : K \in \mathcal{F}(G), \delta > 0\}$ forms a net where $(K', \delta') > (K, \delta)$ means $K' \supseteq K$ and $\delta' \leq \delta$. For an $\mathbb{R}$-valued function $\varphi$ defined on
\( \mathcal{F}(G) \), we define
\[
\limsup_{F} \varphi(F) := \lim_{(K, \delta) \in \Lambda} \sup_{F \in \mathcal{B}(K, \delta)} \varphi(F)
\]
(2.1)
and
\[
\liminf_{F} \varphi(F) := \lim_{(K, \delta) \in \Lambda} \inf_{F \in \mathcal{B}(K, \delta)} \varphi(F).
\]
(2.2)

**Remark 2.1.** From the definition of the partial order ‘\( \succ \)’, it is clear that \( \mathcal{B}(K', \delta') \subseteq \mathcal{B}(K, \delta) \) if \( (K', \delta') \succ (K, \delta) \). Thus it follows that
\[
\limsup_{F} \varphi(F) = \inf_{(K, \delta) \in \Lambda} \sup_{F \in \mathcal{B}(K, \delta)} \varphi(F),
\]
(2.3)
\[
\liminf_{F} \varphi(F) = \sup_{(K, \delta) \in \Lambda} \inf_{F \in \mathcal{B}(K, \delta)} \varphi(F).
\]

Finally, we state the Følner property of amenable groups which is one of fundamental characterizations of amenability. The Følner property is useful for exhibiting amenable groups which are not locally finite.

**Definition 2.2.** Let \( \psi \) be a real-valued function on the set of all nonempty finite subsets of \( G \). We say that \( \psi(F) \) converges to a limit \( L \) as \( F \) becomes more and more invariant if for every \( \epsilon > 0 \) there are a nonempty finite set \( K \subset G \) and a \( \delta > 0 \) such that \( |\psi(F) - L| < \epsilon \) for every \( F \in \mathcal{B}(K, \delta) \).

**Fact 2.3.** [11, Fact 2.2] If the limit \( \lim_{F} \psi(F) \) exists as \( F \) becomes more and more invariant, then
\[
\lim_{F} \psi(F) = \limsup_{F} \psi(F) = \liminf_{F} \psi(F).
\]

A Følner sequence \( \{F_n\} \) is called nested if \( F_n \subseteq F_{n+1} \) for all \( n \geq 1 \).

In [17] (Corollary 5.3), Namioka proved the following theorem.

**Theorem 2.4.** Let \( G \) be a countable amenable group. Then there is a nested Følner sequence \( \{F_n\} \) such that
\[
\bigcup_{n=1}^{\infty} F_n = G.
\]

At the end of this section, we give some notations that will appear in the later paper.

In this paper, we always assume that the group \( G \) is a discrete infinitely countable amenable group and \( X \) is a compact metrizable space. Denote by \( G \acts X \) a \( G \)-action topological dynamical system. Meanwhile, we denote by \( \mathcal{M}(X) \) the space of all Borel probability measures on \( X \) and by \( C(X) \) the Banach space of all continuous functions from \( X \) into \( \mathbb{R} \) equipped with the supremum norm \( \| \cdot \| \).

3. Topological pressures of the subsets for the amenable group action.

3.1. Pesin-Pitskel topological pressure for subsets. Let \( G \) act on a compact metrizable space \( X \) continuously. Consider a finite open cover \( \mathcal{U} \) of \( X \). For \( F \in \mathcal{F}(G) \), we call a map \( U : F \to \mathcal{U} \) a string of length \( m(U) = |F| \). We denote by \( \text{dom}(U) \) the domain of the string \( U : F \to \mathcal{U} \), i.e., \( \text{dom}(U) = F \).

For a given string \( U \) we associate the set
\[
X(U) = \bigcap_{g \in \text{dom}(U)} g^{-1}U(g)
\]
\[
= \{ x \in X : gx \in U(g) \text{ for all } g \in \text{dom}(U) \}.
\]
For each \( f \in C(X) \) and each string \( U \), we define
\[
(S_U f)(x) = \sum_{g \in \text{dom}(U)} f(gx) \quad \text{and} \quad S_{\text{dom}(U)} f(U) = \sup_{x \in X(U)} (S_U f)(x).
\]
If \( X(U) = \emptyset \) then we define \( \sup_{x \in X(U)} (S_U f)(x) = -\infty \).

Given a set \( F \in \mathcal{F}(G) \), we denote by \( \mathcal{U}(\mathcal{F}(G)) \) the set of all strings \( U : F \to \mathcal{U} \).

Let \( Z \subseteq X \). We say that a collection \( \Omega \subset \bigcup_{F \in \mathcal{F}(G)} \mathcal{U}(\mathcal{F}(G)) \) covers \( Z \) if \( \bigcup_{U \in \Omega} X(U) \supseteq Z \).

If \( \delta > 0 \) and \( F \in \mathcal{F}(G) \). For \( s \in \mathbb{R} \), we define
\[
\mathcal{M}_{s(K,\delta)}^s(G,\mathcal{U},f,Z) = \inf_{\Omega \subset \bigcup_{F \in \mathcal{F}(G)} \mathcal{U}(\mathcal{F}(G))} \left\{ \sum_{U \in \Omega} \exp \left( -s \min_{\mathcal{U}(U)} + \sup_{x \in X(U)} (S_U f)(x) \right) \right\},
\]
where the infimum is taken over all collections of strings \( \Omega \subset \bigcup_{F \in \mathcal{F}(G)} \mathcal{U}(\mathcal{F}(G)) \) that covers \( Z \).

This quantity was first introduced by Ruelle in [22]. It is well known (see, e.g. ([19], Proposition 1.1, p.13)) that \( \mathcal{M}_{s(K,\delta)}^s(G,\mathcal{U},f,:) \) is an outer measure on \( X \).

Note that the quantity \( \mathcal{M}_{s(K,\delta)}^s(G,\mathcal{U},f,Z) \) does not decrease as \( (K,\delta) \) increases.

Thus the limit of the net \( \{\mathcal{M}_{s(K,\delta)}^s(G,\mathcal{U},f,Z)\}_{(K,\delta) \in \Lambda} \) exists. So we define
\[
\mathcal{M}^s(G,\mathcal{U},f,Z) := \lim_{(K,\delta) \in \Lambda} \mathcal{M}_{s(K,\delta)}^s(G,\mathcal{U},f,Z)
= \sup_{(K,\delta) \in \Lambda} \mathcal{M}_{s(K,\delta)}^s(G,\mathcal{U},f,Z).
\]

(3.1)

Similar to Proposition 1.2 (p.13) in [19], we now describe a crucial property of the function \( \mathcal{M}^t(G,\mathcal{U},f,Z) \) for a fixed set \( Z \). For the completeness, we give a proof here.

**Fact 3.1.** We have:

1. If \( \mathcal{M}^s(G,\mathcal{U},f,Z) < \infty \) and \( t > s \), then \( \mathcal{M}^t(G,\mathcal{U},f,Z) = 0 \).
2. Also, if \( \mathcal{M}^s(G,\mathcal{U},f,Z) > 0 \) and \( t < s \), then \( \mathcal{M}^t(G,\mathcal{U},f,Z) = \infty \).

**Proof.** Let \( \epsilon > 0 \) and \( M = \mathcal{M}^s(G,\mathcal{U},f,Z) + 1 \). So there is a positive real number \( 0 < \delta_0 < 1 \) and a nonempty finite set \( K_0 \in \mathcal{F}(G) \) with \( \frac{2M}{(t-s)|K_0|} < \epsilon \) which satisfy that, for each \( (K,\delta) \in \Lambda \) with \( (K,\delta) \succ (K_0,\delta_0) \), there is a collection
\[
\Gamma_{(K,\delta)} \subset \bigcup_{F \in \mathcal{F}(K,\delta)} \mathcal{U}(\mathcal{F}(G))
\]
which covers \( Z \) and satisfies
\[
\sum_{U \in \Gamma_{(K,\delta)}} \exp \left( -s \min_{\mathcal{U}(U)} + \sup_{x \in X(U)} (S_U f)(x) \right) < M.
\]

Note that \( (K,\delta) \succ (K_0,\delta_0) \) implies \( \delta < 1 \) and
\[
\frac{2M}{(t-s)|K|} < \epsilon.
\]

Briefly, for a string \( U \in \Gamma_{(K,\delta)} \), set \( F = \text{dom}(U) \). Then one has
\[
|KF| - |F| \leq |KF\setminus F| \leq \delta |F| < |F|.
\]
Thus, it follows that

$$|K|/2 \leq |KF|/2 \leq |F| = m(U).$$

So $m(U) \geq |K|/2$ for each $U \in \Gamma_{(K,\delta)}$.

Therefore, for $s < t$, we have

$$M > \sum_{U \in \Gamma_{(K,\delta)}} \exp \left( -s m(U) + \sup_{x \in X(U)} (S_U f)(x) \right)$$

$$= \sum_{U \in \Gamma_{(K,\delta)}} \exp \left( -t m(U) + \sup_{x \in X(U)} (S_U f)(x) \right) \cdot \exp((t-s)m(U))$$

$$\geq M_{(K,\delta)}^t(G,\mathcal{U},f,Z) \exp \left( \frac{(t-s)|K|}{2} \right)$$

$$\geq M_{(K,\delta)}^t(G,\mathcal{U},f,Z) \frac{(t-s)|K|}{2}.$$

It follows that

$$M_{(K,\delta)}^t(G,\mathcal{U},f,Z) < \frac{2M}{(t-s)|K|} < \epsilon.$$

The above inequality leads to

$$M_{(K,\delta)}^t(G,\mathcal{U},f,Z) = 0.$$

So the statement of (1) is proved. This contradiction gives $M_{(K,\delta)}^t(G,\mathcal{U},f,Z) = \infty$ whenever $M_{(K,\delta)}^t(G,\mathcal{U},f,Z) > 0$ and $t < s$. \hfill \Box

Given a subset $Z$ of $X$, by Fact 3.1, there exists a unique $s$ such that

$$M_{(K,\delta)}^t(G,\mathcal{U},f,Z) = \begin{cases} \infty & \text{if } t < s, \\ 0 & \text{if } t > s. \end{cases}$$

So we define

$$h_{top}^P(G,\mathcal{U},f,Z) = \sup \{ t : M_{(K,\delta)}^t(G,\mathcal{U},f,Z) = \infty \} = \inf \{ t : M_{(K,\delta)}^t(G,\mathcal{U},f,Z) = 0 \}. \quad (3.2)$$

Let $d$ be a compatible metric on the compact metrizable space $X$ and $\mathcal{U}$ be a finite open cover of $X$. Thus we can consider the diameter of the subset of $X$. Let $\text{diam}(\mathcal{U}) = \max\{\text{diam}(U_i) : U_i \in \mathcal{U}\}$ be the diameter of the cover $\mathcal{U}$. So we define

$$\limsup_{\text{diam}(\mathcal{U}) \to 0} h_{top}^P(G,\mathcal{U},f,Z) := \lim_{\epsilon \to 0} \sup_{\text{diam}(\mathcal{U})<\epsilon} \left\{ h_{top}^P(G,\mathcal{U},f,Z) \right\}$$

$$= \inf_{\epsilon>0} \sup_{\text{diam}(\mathcal{U})<\epsilon} \left\{ h_{top}^P(G,\mathcal{U},f,Z) \right\} \quad (3.3)$$

and

$$\liminf_{\text{diam}(\mathcal{U}) \to 0} h_{top}^P(G,\mathcal{U},f,Z) := \lim_{\epsilon \to 0} \inf_{\text{diam}(\mathcal{U})<\epsilon} \left\{ h_{top}^P(G,\mathcal{U},f,Z) \right\}$$

$$= \sup_{\epsilon>0} \inf_{\text{diam}(\mathcal{U})<\epsilon} \left\{ h_{top}^P(G,\mathcal{U},f,Z) \right\} \quad (3.4)$$

If

$$\limsup_{\text{diam}(\mathcal{U}) \to 0} h_{top}^P(G,\mathcal{U},f,Z) = \liminf_{\text{diam}(\mathcal{U}) \to 0} h_{top}^P(G,\mathcal{U},f,Z),$$

then we say that the limit $\lim_{\text{diam}(\mathcal{U}) \to 0} h_{top}^P(G,\mathcal{U},f,Z)$ exists.
Remark 3.2. Let \( d, \rho \) be two compatible metrics on \( X \). It is clear that the identity map
\[
Id : (X, d) \to (X, \rho)
\]
is a homeomorphism.

So the identity map \( Id \) is uniformly continuous from the metric space \( (X, d) \) onto the metric space \( (X, \rho) \). The following fact shows that the limits in definitions (3.3) and (3.4) do NOT depend on the choice of the compatible metric on \( X \).

Fact 3.3. Let \( d, \rho \) be two compatible metrics on \( X \). Then
\[
\limsup_{\text{diam}(\mathcal{U}) \to 0} h_{\text{top}}^P(G, \mathcal{U}, f, Z) = \limsup_{\text{diam}(\mathcal{U}) \to 0} h_{\text{top}}^P(G, \mathcal{U}, f, Z);
\]
\[
\liminf_{\text{diam}(\mathcal{U}) \to 0} h_{\text{top}}^P(G, \mathcal{U}, f, Z) = \liminf_{\text{diam}(\mathcal{U}) \to 0} h_{\text{top}}^P(G, \mathcal{U}, f, Z).
\]

Proof. We only need to prove the first equality. By the symmetry of the compatible metrics and the equality (3.3), it is sufficient to prove that
\[
\inf_{\eta > 0} \sup_{\text{diam}(\mathcal{U}) < \eta} \left\{ h_{\text{top}}^P(G, \mathcal{U}, f, Z) \right\} \leq \inf_{\varepsilon > 0} \sup_{\text{diam}(\mathcal{U}) < \varepsilon} \left\{ h_{\text{top}}^P(G, \mathcal{U}, f, Z) \right\}. \quad (3.5)
\]

Let \( \varepsilon > 0 \). By Remark 3.2, we know that there is \( \delta_0 > 0 \) such that, for any subset \( E \subseteq X \), if \( \text{diam}_d(E) < \delta_0 \) then \( \text{diam}_\rho(E) < \varepsilon \). Thus it follows that
\[
\inf_{\eta > 0} \sup_{\text{diam}(\mathcal{U}) < \eta} \left\{ h_{\text{top}}^P(G, \mathcal{U}, f, Z) \right\} \leq \sup_{\text{diam}(\mathcal{U}) < \delta_0} \left\{ h_{\text{top}}^P(G, \mathcal{U}, f, Z) \right\} \leq \sup_{\text{diam}(\mathcal{U}) < \varepsilon} \left\{ h_{\text{top}}^P(G, \mathcal{U}, f, Z) \right\}.
\]

The arbitrariness of \( \varepsilon \) implies the inequality (3.5). Hence the fact is obtained. \( \square \)

According to the above definitions and arguments, we have the following limit existence result for dynamical systems acting by a countable discrete amenable group. Our result is a generalization of Theorem 11.1 (p.69) in [19] or Proposition 1 (p.309) in [20].

Proposition 3.4. Let \( X \) be a compact metrizable space and \( f \in C(X) \). Then the limit
\[
\lim_{\text{diam}(\mathcal{U}) \to 0} h_{\text{top}}^P(G, \mathcal{U}, f, Z)
\]
exists.

Proof. Let \( d \) be a compatible metric on \( X \). It is clear that we only need to show
\[
\limsup_{\text{diam}(\mathcal{U}) \to 0} h_{\text{top}}^P(G, \mathcal{U}, f, Z) \leq \liminf_{\text{diam}(\mathcal{Y}) \to 0} h_{\text{top}}^P(G, \mathcal{Y}, f, Z).
\]

Let \( \mathcal{U} \) be a finite open cover of \( X \) and \( \mathcal{Y} \) be a finite open cover of \( X \) with diameter smaller than the Lebesgue number \( L(\mathcal{U}) \) of \( \mathcal{U} \). Thus \( \mathcal{Y} \) is finer than \( \mathcal{U} \). In what follows, we will prove that
\[
h_{\text{top}}^P(G, \mathcal{U}, f, Z) \leq h_{\text{top}}^P(G, \mathcal{Y}, f, Z) + M^f_{\mathcal{U}} \quad (3.6)
\]
where
\[
M^f_{\mathcal{U}} = \max_{U \in \mathcal{U}} \left\{ M^f_U \right\} \quad \text{and} \quad M^f_U = \sup \left\{ |f(x) - f(y)| : x, y \in U \right\} \quad \text{for} \quad U \in \mathcal{U}.
\]

We may assume that \( h_{\text{top}}^P(G, \mathcal{Y}, f, Z) < \infty \). Let \( s \) be any real number with \( s > h_{\text{top}}^P(G, \mathcal{Y}, f, Z) \). Thus, one has
\[
M^s(G, \mathcal{Y}, f, Z) = 0. \quad (3.8)
\]
Let $K \in \mathcal{F}(G)$ and $\delta > 0$.

**Claim.** $\mathcal{M}^{+M^{f}_{\mathcal{V}}}_{(K,\delta)}(G,\mathcal{U},f,Z) \leq \mathcal{M}^{\ast}_{(K,\delta)}(G,\mathcal{V},f,Z)$.

Let $\Omega_{(K,\delta)} \subseteq \bigcup_{F \in \mathcal{B}(K,\delta)} \mathcal{W}(\mathcal{V})$ be any collection of strings which covers $Z$.

Now, we construct a collection of strings $\Gamma_{(K,\delta)} \subseteq \bigcup_{F \in \mathcal{B}(K,\delta)} \mathcal{W}(\mathcal{V})$ which covers $Z$.

For each string $V \in \Omega_{(K,\delta)}$, we know that $V$ is a mapping from $F$ into $\mathcal{V}$, where $F = \text{dom}(V) \in \mathcal{B}(K,\delta)$. For every $g \in F$, $V(g)$ is an open subset of $X$. Owing to $\mathcal{V}$ is finer than $\mathcal{U}$, we can find an open subset $U_{V}(g) \in \mathcal{U}$ which satisfies that $V(g) \subseteq U_{V}(g)$. Therefore, we get a string $U_{V}$ from $F$ into $\mathcal{U}$. Furthermore, it is clear that $m(U_{V}) = |F| = m(V)$ and

$$X(V) = \bigcap_{g \in F} g^{-1}V(g) \subseteq \bigcap_{g \in F} g^{-1}U_{V}(g) = X(U_{V}). \quad (3.9)$$

Consequently, we define a collection of strings as follows

$$\Gamma'_{(K,\delta)} = \{U_{V} : V \in \Omega_{(K,\delta)}\}.$$ 

Then (3.9) implies that the collection $\Gamma'_{(K,\delta)}$ covers $Z$.

We define $\Omega'_{(K,\delta)}$ to be the set of $V$ in $\Omega_{(K,\delta)}$ satisfying $X(V) \neq \emptyset$. Then we define $\Gamma_{(K,\delta)}$ to be $\{U_{V} : V \in \Omega'_{(K,\delta)}\}$. It is clear that the collection $\Gamma_{(K,\delta)}$ also covers $Z$.

For each $V \in \Omega'_{(K,\delta)}$, we denote $F = \text{dom}(U_{V}) = \text{dom}(V)$. Note that $X(U_{V}) \subseteq g^{-1}U_{V}(g)$ for each $g \in F$, and $X(V) \neq \emptyset$. So we have

$$\sup_{x \in X(U_{V})} \left\{ \sum_{g \in F} f(gx) \right\} - \sup_{y \in X(V)} \left\{ \sum_{g \in F} f(gy) \right\} \leq \sup \left\{ \left| \sum_{g \in F} f(gx) - \sum_{g \in F} f(gy) \right| : x, y \in X(U_{V}) \right\} \quad \text{(since (3.9))}$$

$$\leq \sum_{g \in F} \sup \{ |f(gx) - f(gy)| : x, y \in X(U_{V}) \}$$

$$\leq \sum_{g \in F} \sup \{ |f(gx) - f(gy)| : x, y \in g^{-1}U_{V}(g) \}$$

$$= \sum_{g \in F} \sup \{ |f(gx) - f(gy)| : gx, gy \in U_{V}(g) \}$$

$$\leq \sum_{g \in F} \sup \{ |f(u) - f(v)| : u, v \in U_{V}(g) \}$$

$$\leq |F|M_{W} \quad \text{(by the definition of $M_{W}$, (see (3.7)))}$$

$$= m(V)M_{W}.$$ 

That is

$$\sup_{x \in X(U_{V})} \left\{ \sum_{g \in \text{dom}(U_{V})} f(gx) \right\} \leq \sup_{y \in X(V)} \left\{ \sum_{g \in \text{dom}(V)} f(gy) \right\} + m(V)M_{W}.$$
Using the fact that \( m(U_V) = m(V) \), we have
\[
\sum_{U_V \in \Gamma(K, \delta)} \exp \left( - \left( s + M^f_{\mathcal{V}} \right) m(U_V) + \sup_{x \in X(U_V)} \sum_{g \in \text{dom}(U_V)} f(gx) \right) \\
\leq \sum_{V \in \Omega(K, \delta)} \exp \left( - \left( s + M^f_{\mathcal{V}} \right) m(V) + \sup_{y \in X(V)} \sum_{g \in \text{dom}(V)} f(gy) + m(V)M^f_{\mathcal{V}} \right) \\
= \sum_{V \in \Omega(K, \delta)} \exp \left( -s m(V) + \sup_{y \in X(V)} \sum_{g \in \text{dom}(V)} f(gy) \right) .
\]

By the definition of \( \mathcal{M}^{s+M^f_{\mathcal{V}}}_{(K, \delta)}(G, \mathcal{V}, f, Z) \), it follows that
\[
\mathcal{M}^{s+M^f_{\mathcal{V}}}_{(K, \delta)}(G, \mathcal{V}, f, Z) \\
\leq \sum_{U_V \in \Gamma(K, \delta)} \exp \left( - \left( s + M^f_{\mathcal{V}} \right) m(U_V) + \sup_{x \in X(U_V)} \sum_{g \in \text{dom}(U_V)} f(gx) \right) \\
\leq \sum_{V \in \Omega(K, \delta)} \exp \left( -s m(V) + \sup_{y \in X(V)} \sum_{g \in \text{dom}(V)} f(gy) \right) .
\]

The arbitrariness of the collection \( \Omega(K, \delta) \) which covers \( Z \) implies that
\[
\mathcal{M}^{s+M^f_{\mathcal{V}}}_{(K, \delta)}(G, \mathcal{V}, f, Z) \leq \mathcal{M}^s_{(K, \delta)}(G, \mathcal{V}, f, Z).
\]

Taking the limit for the net \((K, \delta) \in \Lambda, \) we get
\[
\mathcal{M}^{s+M^f_{\mathcal{V}}}(G, \mathcal{V}, f, Z) \leq \mathcal{M}^s(G, \mathcal{V}, f, Z). \tag{3.10}
\]

Combining with (3.8), it follows that
\[
\mathcal{M}^{s+M^f_{\mathcal{V}}}(G, \mathcal{V}, f, Z) = 0
\]
which implies that
\[
h^P_{\text{top}}(G, \mathcal{V}, f, Z) \leq s + M^f_{\mathcal{V}} .
\]

Since \( s \) is any real number with \( s > h^P_{\text{top}}(G, \mathcal{V}, f, Z) \), we have that
\[
h^P_{\text{top}}(G, \mathcal{V}, f, Z) \leq h^P_{\text{top}}(G, \mathcal{V}, f, Z) + M^f_{\mathcal{V}},
\]
that is, the inequality (3.6) is obtained.

Note that \( \mathcal{V} \) is any finite open cover of \( X \) with diameter smaller than the Lebesgue number \( L(\mathcal{V}) \) of \( \mathcal{V} \). Therefore, one has
\[
h^P_{\text{top}}(G, \mathcal{V}, f, Z) \leq \inf_{\text{diam}(\mathcal{V}) < L(\mathcal{V})} \{ h^P_{\text{top}}(G, \mathcal{V}, f, Z) \} + M^f_{\mathcal{V}} \\
\leq \sup_{\varepsilon > 0} \inf_{\text{diam}(\mathcal{V}) < \varepsilon} \{ h^P_{\text{top}}(G, \mathcal{V}, f, Z) \} + M^f_{\mathcal{V}} \\
= \lim \inf_{\text{diam}(\mathcal{V}) \to 0} \{ h^P_{\text{top}}(G, \mathcal{V}, f, Z) \} + M^f_{\mathcal{V}} . \tag{since (3.4)}
\]

Taking the limsup as the diameter of the open cover \( \mathcal{V} \) tends to zero, we get that
\[
\lim \sup_{\text{diam}(\mathcal{V}) \to 0} h^P_{\text{top}}(G, \mathcal{V}, f, Z) \leq \lim \inf_{\text{diam}(\mathcal{V}) \to 0} h^P_{\text{top}}(G, \mathcal{V}, f, Z) + \lim \sup_{\text{diam}(\mathcal{V}) \to 0} M^f_{\mathcal{V}} .
\]
Since $X$ is compact and $f$ is uniformly continuous on $X$, it is easy to see that
\[ \limsup_{\text{diam}(\mathcal{U}) \to 0} M^f_\mathcal{U} = 0. \]
Thus we have
\[ \limsup_{\text{diam}(\mathcal{U}) \to 0} h_{\text{top}}^P(G, \mathcal{U}, f, Z) \leq \liminf_{\text{diam}(\mathcal{V}) \to 0} h_{\text{top}}^P(G, \mathcal{V}, f, Z) \]
as desired.

Due to Proposition 3.4, we can define the Pesin-Pitskel topological pressure of the action $G \curvearrowright X$ as follows:
\[ h_{\text{top}}^P(G, f, Z) := \lim_{\text{diam}(\mathcal{U}) \to 0} h_{\text{top}}^P(G, \mathcal{U}, f, Z). \] (3.11)

For $f \in C(X)$ and each string $U$, we write
\[ m^f_U = \inf_{x \in X(U)} \sum_{g \in \text{dom}(U)} f(gx) \quad \text{and} \quad M^f_U = \sup_{x \in X(U)} \sum_{g \in \text{dom}(U)} f(gx). \]
For each string $U$, we choose any real number
\[ f^*(U) \in \left[ m^f_U, M^f_U \right] \subseteq \mathbb{R}. \] (3.12)
Let $Z \subseteq X$. We define
\[ M^a_{(K, \delta)}(G, \mathcal{U}, f^*, Z) = \inf_{\Omega \in \mathcal{W}_F(\mathcal{U})} \left\{ \sum_{U \in \Omega} \exp(-\alpha m(U) + f^*(U)) \right\} \]
where the infimum is taken over all collections
\[ \Omega \subset \bigcup_{F \in \mathcal{B}(K, \delta)} \mathcal{W}_F(\mathcal{U}) \quad \text{that covers } Z. \]
Similarly to the definitions of (3.1) and (3.2), we can define
\[ M^a(G, \mathcal{U}, f^*, Z) = \lim_{(K, \delta) \in \Lambda} M^a_{(K, \delta)}(G, \mathcal{U}, f^*, Z) \]
and
\[ h_{\text{top}}^P(G, \mathcal{U}, f^*, Z) = \inf \{ \alpha : M^a(G, \mathcal{U}, f^*, Z) = 0 \} \]
\[ = \sup \{ \alpha : M^a(G, \mathcal{U}, f^*, Z) = \infty \}. \]
For the relation between the numbers $h_{\text{top}}^P(G, \mathcal{U}, f^*, Z)$ and $h_{\text{top}}^P(G, \mathcal{U}, f, Z)$, we have the following theorem.

**Theorem 3.5.** For any map $f^*$, the limit $\lim_{\text{diam}(\mathcal{U}) \to 0} h_{\text{top}}^P(G, \mathcal{U}, f^*, Z)$ exists and
\[ \lim_{\text{diam}(\mathcal{U}) \to 0} h_{\text{top}}^P(G, \mathcal{U}, f^*, Z) = h_{\text{top}}^P(G, f, Z). \]

**Proof.** Let $d$ be a compatible metric on $X$ and $\mathcal{U}$ be a finite open cover of $X$. According to the definitions, it is clear that
\[ h_{\text{top}}^P(G, \mathcal{U}, f^*, Z) \leq h_{\text{top}}^P(G, \mathcal{U}, f, Z). \]
Combining with Proposition 3.4 we get that
\[ \limsup_{\text{diam}(\mathcal{U}) \to 0} h_{\text{top}}^P(G, \mathcal{U}, f^*, Z) \leq \limsup_{\text{diam}(\mathcal{U}) \to 0} h_{\text{top}}^P(G, \mathcal{U}, f, Z) = h_{\text{top}}^P(G, f, Z). \] (3.13)
In what follows, we will show that
\[ h^p_{\text{top}}(G, \mathcal{U}, f, Z) \leq h^p_{\text{top}}(G, \mathcal{U}, f^*, Z) + M^f_{\mathcal{U}} \]  
where
\[ M^f_{\mathcal{U}} = \max_{U \in \mathcal{U}} \left\{ M_U^f \right\} \quad \text{and} \quad M_U^f = \sup \{ |f(x) - f(y)| : x, y \in U \} \quad \text{for} \ U \in \mathcal{U}. \]

We may assume that \( h^P_{\text{top}}(G, \mathcal{U}, f^*, Z) < \infty \). Let \( s > h^P_{\text{top}}(G, \mathcal{U}, f^*, Z) \). Let \( K \in \mathcal{F}(G) \) and \( \delta > 0 \). Suppose that \( \Omega(K, \delta) \subseteq \bigcup_{F \in \mathcal{B}(K, \delta)} \mathcal{F}_F(\mathcal{U}) \) is any collection of the strings which covers \( Z \). For each string \( U \in \Omega(K, \delta) \), by the definition of string, it is clear that
\[ g(X(U)) \subseteq U(g) \quad \text{for all} \quad g \in \text{dom}(U). \]

Recall that \( f^*(U) \) is a real number satisfying
\[ f^*(U) \in \left[ m_U^f, M_U^f \right]. \]

Thus, for any \( U \in \Omega(K, \delta) \), we have
\[ \sup_{y \in X(U)} \sum_{g \in \text{dom}(U)} f(gy) - f^*(U) \]
\[ \leq M_U^f - m_U^f \]
\[ \leq \sup \left\{ \left| \sum_{g \in \text{dom}(U)} f(gx) - \sum_{g \in \text{dom}(U)} f(gy) \right| : x, y \in X(U) \right\} \]
\[ \leq \sum_{g \in \text{dom}(U)} \sup \{ |f(gx) - f(gy)| : x, y \in X(U) \} \]
\[ \leq \sum_{g \in \text{dom}(U)} \sup \{ |f(x) - f(y)| : x, y \in U(g) \} \] (since \( g(X(U)) \subseteq U(g) \))
\[ \leq m(U)M^f_{\mathcal{U}}. \]

It follows that
\[ \sum_{U \in \Omega(K, \delta)} \exp \left( -(s + M^f_{\mathcal{U}})m(U) + \sup_{y \in X(U)} \sum_{g \in \text{dom}(U)} f(gy) \right) \]
\[ \leq \sum_{U \in \Omega(K, \delta)} \exp(-s m(U) + f^*(U)). \]

So, we deduce that
\[ \mathcal{M}^{s+M^f_{\mathcal{U}}}(G, \mathcal{U}, f, Z) \leq \mathcal{M}^{s}(G, \mathcal{U}, f^*, Z) \]
for any \((K, \delta) \in \Lambda\), which means
\[ \mathcal{M}^{s+M^f_{\mathcal{U}}}(G, \mathcal{U}, f, Z) \leq \mathcal{M}^{s}(G, \mathcal{U}, f^*, Z). \]

Since \( s \) is any positive real number with \( s > h^P_{\text{top}}(G, \mathcal{U}, f^*, Z) \), one has \( \mathcal{M}^{s}(G, \mathcal{U}, f^*, Z) = 0 \), i.e.,
\[ \mathcal{M}^{s+M^f_{\mathcal{U}}}(G, \mathcal{U}, f, Z) = 0. \]

Thus, it follows that
\[ h^P_{\text{top}}(G, \mathcal{U}, f, Z) \leq h^P_{\text{top}}(G, \mathcal{U}, f^*, Z) + M^f_{\mathcal{U}}. \]  
(3.15)
Since $X$ is compact and $f$ is uniformly continuous on $X$, we have
\[ \limsup_{\text{diam}(\mathcal{U}) \to 0} M_{\mathcal{U}}^f = 0. \]

Taking the liminf as the diameter of the open cover $\mathcal{U}$ tends to zero, one has
\[ h_{\text{top}}^p(G, f, Z) \leq \liminf_{\text{diam}(\mathcal{U}) \to 0} h_{\text{top}}^p(G, \mathcal{U}, f^*, Z). \quad (3.16) \]

Therefore, combining the inequalities (3.13) and (3.16), we get that
\[ \lim_{\text{diam}(\mathcal{U}) \to 0} h_{\text{top}}^p(G, \mathcal{U}, f^*, Z) = h_{\text{top}}^p(G, f, Z). \]

Hence the theorem is proved. \[ \square \]

By the above theorem, we denote
\[ h_{\text{top}}^p(G, f^*, Z) = \lim_{\text{diam}(\mathcal{U}) \to 0} h_{\text{top}}^p(G, \mathcal{U}, f^*, Z). \]

In order to get the following results, we need to define the infimum function $f_*$ as follows:

For each string $U$, we define
\[ f_*(U) := m^f_U = \inf_{x \in X(U)} \sum_{g \in \text{dom}(U)} f(gx). \]

**Corollary 3.6.** For the above function $f_*$, one has
\[ \lim_{\text{diam}(\mathcal{U}) \to 0} h_{\text{top}}^p(G, \mathcal{U}, f^*, Z) = \sup_{\mathcal{V}} h_{\text{top}}^p(G, \mathcal{V}, f^*, Z) \]
where $\mathcal{V}$ runs over all finite open covers of $X$.

**Proof.** Let $d$ be a compatible metric on $X$. By Theorem 3.5, we know that the limit
\[ \lim_{\text{diam}(\mathcal{U}) \to 0} h_{\text{top}}^p(G, \mathcal{U}, f^*, Z) \]
exists. Furthermore, according to the definitions it is clear that
\[ \lim_{\text{diam}(\mathcal{U}) \to 0} h_{\text{top}}^p(G, \mathcal{U}, f^*, Z) \leq \sup_{\mathcal{V}} h_{\text{top}}^p(G, \mathcal{V}, f^*, Z). \]

Thus it suffices to prove that
\[ \sup_{\mathcal{V}} h_{\text{top}}^p(G, \mathcal{V}, f^*, Z) \leq \lim_{\text{diam}(\mathcal{U}) \to 0} h_{\text{top}}^p(G, \mathcal{U}, f^*, Z). \]

Let $\mathcal{V}, \mathcal{U}$ be two finite open covers of $Z$ with $\mathcal{V} \succ \mathcal{U}$ (i.e. for each element $V$ of $\mathcal{V}$ there is an element $U$ of $\mathcal{U}$ such that $V \subset U$).

In what follows, we show that
\[ h_{\text{top}}^p(G, \mathcal{U}, f^*, Z) \leq h_{\text{top}}^p(G, \mathcal{V}, f^*, Z). \quad (3.17) \]

We may assume that $h_{\text{top}}^p(G, \mathcal{V}, f^*, Z) < \infty$. Let $s$ be any real number with $s > h_{\text{top}}^p(G, \mathcal{V}, f^*, Z)$. Thus, one has
\[ \mathcal{M}^*(G, \mathcal{V}, f^*, Z) = 0. \quad (3.18) \]

By (3.1), we know that
\[ \mathcal{M}^*(G, \mathcal{V}, f^*, Z) = \sup_{(K, \delta) \in \Lambda} \mathcal{M}^*_{(K, \delta)}(G, \mathcal{V}, f^*, Z). \]

So we get that
\[ \mathcal{M}^*_{(K, \delta)}(G, \mathcal{V}, f^*, Z) = 0 \quad \text{for all} \quad K \in \mathcal{F}(G) \quad \text{and} \quad \delta > 0. \]
Let $K \in \mathcal{F}(G)$ and $\delta > 0$.

**Claim.** $\mathcal{M}^{*}_{(K, \delta)}(G, \mathcal{W}, f_{*}, Z) = 0$.

Let $\Omega_{(K, \delta)} \subseteq \bigcup_{F \in \mathcal{B}(K, \delta)} \mathcal{W}_{F}(\mathcal{V})$ be any collection of strings which covers $Z$.

Now we construct a collection of strings $\Gamma_{(K, \delta)} \subseteq \bigcup_{F \in \mathcal{B}(K, \delta)} \mathcal{W}_{F}(\mathcal{V})$ which covers $Z$.

For each string $V \in \Omega_{(K, \delta)}$, we know that $V$ is a mapping from $F$ into $\mathcal{V}$, where $F = \text{dom}(V) \in \mathcal{B}(K, \delta)$.

For every $g \in F$, $V(g)$ is an element of $\mathcal{V}$. Thus, by $\mathcal{V} \supset \mathcal{W}$, there is an open subset $U_{V}(g) \in \mathcal{W}$ such that $V(g) \subseteq U_{V}(g)$. Therefore, we get a string $U_{V}$ from $F$ into $\mathcal{W}$. Furthermore, it is clear that $m(U_{V}) = |F| = m(V)$ and

$$X(V) = \bigcap_{g \in F} g^{-1}V(g) \subseteq \bigcap_{g \in F} g^{-1}U_{V}(g) = X(U_{V}). \quad (3.19)$$

Consequently, we define a collection of strings as follows

$$\Gamma_{(K, \delta)} = \{U_{V} : V \in \Omega_{(K, \delta)}\}.$$ 

The (3.19) implies that the collection $\Gamma_{(K, \delta)}$ covers $Z$.

For each $V \in \Omega_{(K, \delta)}$, we write $F = \text{dom}(U_{V}) = \text{dom}(V)$.

Note that $X(V) \subseteq X(U_{V})$. So we have

$$f_{*}(U_{V}) = \inf_{x \in X(U_{V})} \sum_{g \in \text{dom}(U_{V})} f(gx)$$

$$= \inf_{x \in X(U_{V})} \sum_{g \in F} f(gx)$$

$$\leq \inf_{x \in X(V)} \sum_{g \in F} f(gx)$$

$$= \inf_{x \in X(V)} \sum_{g \in \text{dom}(V)} f(gx)$$

$$= f_{*}(V).$$

Hence we get that

$$\mathcal{M}^{*}_{(K, \delta)}(G, \mathcal{W}, f_{*}, Z) \leq \sum_{U_{V} \in \Gamma_{(K, \delta)}} \exp (-s \cdot m(U_{V}) + f_{*}(U_{V}))$$

$$\leq \sum_{V \in \Omega_{(K, \delta)}} \exp (-s \cdot m(V) + f_{*}(V)).$$

Since $\Omega_{(K, \delta)} \subseteq \bigcup_{F \in \mathcal{B}(K, \delta)} \mathcal{W}_{F}(\mathcal{V})$ is any collection of strings which covers $Z$, it follows that

$$\mathcal{M}^{*}_{(K, \delta)}(G, \mathcal{W}, f_{*}, Z) \leq \mathcal{M}^{*}_{(K, \delta)}(G, \mathcal{V}, f_{*}, Z) = 0.$$ 

Thus we obtain that $\mathcal{M}^{*}(G, \mathcal{W}, f_{*}, Z) = 0$ which implies that

$$h_{\top}^{P}(G, \mathcal{W}, f_{*}, Z) \leq s.$$ 

Since $s$ is any real number with $h_{\top}^{P}(G, \mathcal{V}, f_{*}, Z) < s$, we have that

$$h_{\top}^{P}(G, \mathcal{W}, f_{*}, Z) \leq h_{\top}^{P}(G, \mathcal{V}, f_{*}, Z). \quad (3.20)$$

Let $\epsilon > 0$. Then there is an open cover $\mathcal{W}'$ of $X$ such that

$$\sup_{\mathcal{V}} h_{\top}^{P}(G, \mathcal{V}, f_{*}, Z) < h_{\top}^{P}(G, \mathcal{W}', f_{*}, Z) + \epsilon.$$
Let $\delta = L(\mathcal{W}')$ be the Lebesgue number of the finite open cover $\mathcal{W}'$. For any finite open cover $\mathcal{W}$ of $X$ with $\text{diam}(\mathcal{W}) < \delta$, it follows that $\mathcal{W} \succ \mathcal{W}'$. By the inequality (3.20), one has

$$\sup_{\mathcal{Y}} h^p_{\text{top}}(G, \mathcal{Y}, f, \mathcal{Z}) \leq h^p_{\text{top}}(G, \mathcal{W}', f, \mathcal{Z}) + \epsilon$$

$$\leq h^p_{\text{top}}(G, \mathcal{W}, f, \mathcal{Z}) + \epsilon.$$  

The arbitrariness of the finite open cover $\mathcal{W}$ implies that

$$\sup_{\mathcal{Y}} h^p_{\text{top}}(G, \mathcal{Y}, f, \mathcal{Z}) \leq \inf_{\text{diam}(\mathcal{W}) < \delta} h^p_{\text{top}}(G, \mathcal{W}, f, \mathcal{Z}) + \epsilon$$

$$\leq \inf_{\eta > 0} \sup_{\text{diam}(\mathcal{W}) < \eta} h^p_{\text{top}}(G, \mathcal{W}, f, \mathcal{Z}) + \epsilon$$

$$= \lim_{\text{diam}(\mathcal{W}) \to 0} h^p_{\text{top}}(G, \mathcal{W}, f, \mathcal{Z}) + \epsilon. \quad \text{(since (3.4))}$$

Letting $\epsilon \to 0$, we get

$$\sup_{\mathcal{Y}} h^p_{\text{top}}(G, \mathcal{Y}, f, \mathcal{Z}) \leq \lim_{\text{diam}(\mathcal{W}) \to 0} h^p_{\text{top}}(G, \mathcal{W}, f, \mathcal{Z}).$$

Hence the proof is completed. \(\square\)

**Remark 3.7.** If $f^*(U) > m_U^f$ for some string $U$, we can NOT obtain that

$$\lim_{\text{diam}(\mathcal{W}) \to 0} h^p_{\text{top}}(G, \mathcal{W}, f^*, Z) = \sup_{\mathcal{Y}} h^p_{\text{top}}(G, \mathcal{Y}, f^*, Z)$$

where $\mathcal{Y}$ runs over all finite open covers of $X$.

Similar to the $\mathbb{Z}$-action case (see, e.g. Theorem 11.2, p.70 in Pesin’s book [19]), we now give some properties of the Pesin-Pitskel topological pressure of $G$ on the set $Z \subseteq X$.

**Proposition 3.8.** Let $G \curvearrowright X$ be a continuous action. Then

1. For $Z \subseteq Z'$, $h^p_{\text{top}}(G, f, Z) \leq h^p_{\text{top}}(G, f, Z')$.

2. For $Z = \bigcup_{i=1}^{\infty} Z_i$, we have

$$h^p_{\text{top}}(G, f, Z) = \sup_{i \geq 1} h^p_{\text{top}}(G, f, Z_i).$$

3. For $Z \subseteq X$ and $g \in G$, we have

$$h^p_{\text{top}}(G, f, gZ) = h^p_{\text{top}}(G, f, Z).$$

**Proof.** (1) follows directly from the definition of the Pesin-Pitskel topological pressure.

For (2), by Theorem 3.5 we only need to show that

$$h^p_{\text{top}}(G, f, Z) \leq \sup_{i \geq 1} h^p_{\text{top}}(G, f, Z_i).$$

We assume that $\sup_{i \geq 1} h^p_{\text{top}}(G, f, Z_i) < \infty$ and $s$ is any real number with

$$s > \sup_{i \geq 1} h^p_{\text{top}}(G, f, Z_i).$$

By Corollary 3.6, we know that, for each $i \geq 1$, $h^p_{\text{top}}(G, f, Z_i) = \sup_{\mathcal{W}} h^p_{\text{top}}(G, \mathcal{W}, f, Z_i)$, where $\mathcal{W}$ runs over all finite open covers of $X$. Therefore, for any $i \geq 1$ and any finite open cover $\mathcal{W}$ of $X$, we have $h^p_{\text{top}}(G, \mathcal{W}, f, Z_i) \leq s$.

In what follows, we will show that $h^p_{\text{top}}(G, f, Z) \leq s$. 
Let $\mathcal{U}$ be a finite open cover of $X$. So we only need to prove $h_{\text{top}}^P(G, \mathcal{U}, f_*, Z) \leq s$. Since $h_{\text{top}}^P(G, \mathcal{U}, f_*, Z_i) < s$ for $i \geq 1$, by (3.1), we know that $\mathcal{M}_{(K, \delta)}^s(G, \mathcal{U}, f_*, Z_i) = 0$ for all $K \in \mathcal{F}(G)$ and $\delta > 0$.

Let $K \in \mathcal{F}(G)$ and $\delta > 0$.

**Claim.** $\mathcal{M}_{(K, \delta)}^s(G, \mathcal{U}, f_*, Z) = 0$.

Let $\epsilon > 0$. For each $i$, there is a collection of strings $\Omega_{i, (K, \delta)}(G, \mathcal{U}, f_*, Z_i)$ which covers $Z_i$ and satisfies

$$\sum_{U \in \Omega_{i, (K, \delta)}} \exp(-s m(U) + f_*(U)) < \frac{\epsilon}{2^{i+1}}.$$ 

Now we define a collection of strings as follows

$$\Gamma_{(K, \delta)} = \left\{ U : U \in \bigcup_{i=1}^{\infty} \Omega_{i, (K, \delta)} \right\} \subseteq \bigcup_{F \in \mathcal{B}(K, \delta)} \mathcal{W}_F(\mathcal{U}).$$

It is clear that the collection $\Gamma_{(K, \delta)}$ covers $Z$. Furthermore, one has

$$\mathcal{M}_{(K, \delta)}^s(G, \mathcal{U}, f_*, Z) \leq \sum_{U \in \Gamma_{(K, \delta)}} \exp(-s m(U) + f_*(U))$$

$$\leq \sum_{i \geq 1} \sum_{U_i \in \Omega_{i, (K, \delta)}} \exp(-s m(U_i) + f_*(U_i))$$

$$< \sum_{i \geq 1} \frac{\epsilon}{2^{i+1}}$$

$$< \epsilon.$$ 

It follows that

$$\mathcal{M}_{(K, \delta)}^s(G, \mathcal{U}, f_*, Z) = 0,$$

which means

$$\mathcal{M}^s(G, \mathcal{U}, f_*, Z) = 0.$$ 

Thus we have that

$$h_{\text{top}}^P(G, \mathcal{U}, f_*, Z) \leq s.$$ 

So we deduce that

$$h_{\text{top}}^P(G, \mathcal{U}, f_*, Z) = \sup_{\mathcal{W}} h_{\text{top}}^P(G, \mathcal{U}, f_*, Z) \leq s.$$ 

Since $s$ is any real number with $\sup_{i \geq 1} h_{\text{top}}^P(G, f_*, Z_i) < s$, we get that

$$h_{\text{top}}^P(G, f_*, Z) \leq \sup_{i \geq 1} h_{\text{top}}^P(G, f_*, Z_i).$$ 

For (3), it suffices to prove $h_{\text{top}}^P(G, f, gZ) \leq h_{\text{top}}^P(G, f, Z)$ by the symmetries of two sets $Z$ and $gZ$. We may assume that $h_{\text{top}}^P(G, f, Z) < +\infty$.

Let $s = h_{\text{top}}^P(G, f, Z)$ and $\kappa > 0$. Let $\mathcal{U}$ be a finite open cover of $X$, $K \in \mathcal{F}(G)$ and $\delta > 0$. Since $\mathcal{M}^{s+\kappa}(G, \mathcal{U}, f, Z) = 0$, there exist a set $K_1 \in \mathcal{F}(G)$ with $K_1 \supseteq K$ and a real number $\delta_1$ with $0 < \delta_1 < \delta$ such that

$$\mathcal{M}_{(K_1, \delta_1)}^{s+\kappa}(G, \mathcal{U}, f, Z) < 1.$$
From the definition of $\mathcal{M}^{s+\kappa}_{(K_1,\delta_1)}(G,\mathcal{W},f,Z)$, there exists a collection $\Omega = \{U_i : i \in I\}$ of strings which satisfies that
\[
\Omega \subseteq \bigcup_{F \in \mathfrak{B}(K_1,\delta_1)} \mathcal{W}_F(\mathcal{W}), \quad Z \subseteq \bigcup_{i \in I} X(U_i)
\]
and
\[
\sum_{i \in I} e^{-(s+\kappa)} m(U_i) + S_{\text{dom}(U_i)} f(U_i) < 1. \tag{3.21}
\]
The collection $\Omega$ covering $Z$ means that $Z \subseteq \bigcup_{i \in I} \bigcap_{t \in \text{dom}(U_i)} t^{-1} U_i(t)$. Since $g : X \to X$ is a homeomorphism, it follows that
\[
gZ \subseteq g \left( \bigcup_{i \in I} \bigcap_{t \in \text{dom}(U_i)} t^{-1} U_i(t) \right) = \bigcup_{i \in I} \bigcap_{t \in \text{dom}(U_i)} (tg^{-1})^{-1} U_i(t). \tag{3.22}
\]
Consequently, we define the new string $\tilde{U}_i : \text{dom}(U_i)g^{-1} \to \mathcal{W}$ by means of $U_i$ as follows:
\[
\tilde{U}_i : \text{dom}(U_i)g^{-1} \to \mathcal{W} \quad \text{and} \quad \tilde{U}_i(tg^{-1}) = U_i(t),
\]
namely, $\tilde{U}_i(tg^{-1}) = U_i(t)$. From (3.22) we find
\[
gZ \subseteq \bigcup_{i \in I} X(\tilde{U}_i). \tag{3.23}
\]
Denote $F_i = \text{dom}(U_i)$. For the nonempty finite set $F_i g^{-1}$, we have
\[
\frac{|K F_i g^{-1} \setminus F_i g^{-1}|}{|F_i g^{-1}|} = \frac{|K F_i \setminus F_i|}{|F_i|} \leq \frac{|K_i F_i \setminus F_i|}{|F_i|} \quad (\text{since } K \subseteq K_1)
\]
\[
\leq \delta_1 \quad (\text{since } F_i \in \mathfrak{B}(K_1,\delta_1))
\]
\[
< \delta \quad (\text{since } \delta_1 < \delta)
\]
which implies that $F_i g^{-1} \in \mathfrak{B}(K,\delta)$.
Meanwhile, it is easy to see that
\[
S_{\text{dom}(U_i)} f(\tilde{U}_i) = \sup \left\{ \sum_{h \in \text{dom}(\tilde{U}_i)} f(hx) : x \in X(\tilde{U}_i) \right\}
\]
\[
= \sup \left\{ \sum_{h \in \text{dom}(U_i)g^{-1}} f(hx) : x \in g(X(U_i)) \right\}
\]
\[
= \sup \left\{ \sum_{t \in \text{dom}(U_i)} f(ty) : y \in X(U_i) \right\}
\]
\[
= S_{\text{dom}(U_i)} f(U_i).
\]
Combining (3.21), (3.23), \(m(\tilde{U}_i) = m(U_i)\) and the above equality, we know that the collection \(\tilde{\Omega} = \{\tilde{U}_i : i \in I\}\) covers \(gZ\) and satisfies
\[
\Omega \subseteq \bigcup_{F \in \mathcal{B}(K,\delta)} \mathcal{W}_F(\mathcal{U}) \quad \text{and} \quad \sum_{i \in I} \exp \left( -(s + \kappa) m(\tilde{U}_i) + S_{\text{dom}(\tilde{U}_i)} f(\tilde{U}_i) \right) < 1.
\]

Thus, we deduce that
\[
\mathcal{M}^{s+\kappa}(G, \mathcal{U}, f, gZ) < 1 \quad \text{for any} \quad (K, \delta) \in \Lambda
\]
which implies that
\[
h_{\text{top}}(G, \mathcal{U}, f, gZ) \leq s + \kappa.
\]
The arbitrariness of \(\kappa\) implies the desired conclusion.

3.2. Bowen pseudometric pressure of subsets. In mathematics, a pseudometric is a generalized metric space in which the distance between two distinct points can be zero.

A pseudometric space \((X, \rho)\) is a set \(X\) together with a non-negative real-valued function \(\rho : X \times X \to \mathbb{R} \geq 0\) (called a pseudometric) such that, for every \(x, y, z \in X\),
\[
(1) \quad \rho(x, x) = 0, \\
(2) \quad \rho(x, y) = \rho(y, x), \\
(3) \quad \rho(x, z) \leq \rho(x, y) + \rho(y, z).
\]

Unlike a metric space, points in a pseudometric space need not be distinguishable; that is, one may have \(\rho(x, y) = 0\) for distinct points \(x \neq y\).

Let \((X, T)\) be a topological space and \(\rho\) be a pseudometric on \(X\). Then \(\rho\) is said to be continuous if the map
\[
\rho : X \times X \to \mathbb{R} \geq 0
\]
is continuous. Here the topology of the space \(X \times X\) is the product topology.

Throughout \(G \curvearrowleft X\) is a continuous action on a compact metrizable space and \(\rho\) is a continuous pseudometric on \(X\).

**Definition 3.9.** Let \(F\) be a nonempty finite subset of \(G\). Define on \(X\) the pseudometric
\[
\rho_F(x, y) = \max_{s \in F} \rho(sx, sy).
\]

For every \(\epsilon > 0\) we denote by \(B^\rho_F(x, \epsilon)\) the Bowen ball of radius \(\epsilon\) in the pseudometric \(\rho_F\) around \(x\), i.e.,
\[
B^\rho_F(x, \epsilon) = \{y \in X : \rho_F(x, y) < \epsilon\}.
\]

It is not hard to see that the Bowen ball \(B^\rho_F(x, \epsilon)\) is an open subset of \(X\).

**Definition 3.10.** The pseudometric \(\rho\) is said to be dynamically generating if for all distinct \(x_1, x_2 \in X\) there is an \(s \in G\) for which \(\rho(sx_1, sy_2) > 0\).

It is well known that one can obtain a compatible metric from any dynamically generating continuous pseudometric \(\rho\), see for example [14, 12]. So we have the following result and omit the details here.

**Lemma 3.11.** Let \(\rho\) be a dynamically generating continuous pseudometric on \(X\). Enumerate the elements of \(G\) as \(s_1 = e, s_2, \ldots\). Define a new continuous pseudometric \(\tilde{\rho}\) on \(X\) by \(\tilde{\rho}(x, y) = \sum_{j=1}^{\infty} 2^{-j+1} \rho(s_j x, s_j y)\) for all \(x, y \in X\). Then \(\tilde{\rho}\) is a compatible metric on \(X\).
For the dynamically generating pseudometric, from a simple compactness argument one has the following result (see p.250 in [12]).

**Lemma 3.12.** Let $\rho$ be a dynamical generating continuous pseudometric on $X$ and $\widetilde{\rho}$ be the compatible metric determined by $\rho$ as in Lemma 3.11. Let $\epsilon > 0$. Then there exist a finite set $K \subseteq G$ and a $\delta > 0$ such that $\widetilde{\rho}(x, y) < \epsilon$ if $\rho(sx, sy) < \delta$ for all $s \in K$.

The Pesin-Pitskel topological pressure can be defined in the following alternative way.

Let $f \in C(X)$, $x \in X$ and $F \in \mathcal{F}(G)$. Denote $f_F(x) = \sum_{g \in F} f(gx)$.

Let $s \in \mathbb{R}$ and $(K, \delta) \in \Lambda$. For $\epsilon > 0$, put

$$h^\epsilon_{(K, \delta), e}(\rho, G, f, Z) = \inf_\Gamma \sum_i \exp \left( -s|F_i| + \sup_{y \in B^\rho_{F_i}(x, \epsilon)} f_{F_i}(y) \right)$$

where the infimum is taken over all collections

$$\Gamma = \{B^\rho_{F_i}(x_i, \epsilon)\}_{i \in I} \quad \text{with} \quad x_i \in X, \; F_i \in \mathcal{B}(K, \delta)$$

and

$$\bigcup_{i \in I} B^\rho_{F_i}(x_i, \epsilon) \supseteq Z.$$ Note that the quantity $h_{(K, \delta), e}^\epsilon(\rho, G, f, Z)$ does not decrease as $(K, \delta)$ increases.

So we define

$$h^\epsilon(\rho, G, f, Z) = \lim_{(K, \delta) \in \Lambda} h^\epsilon_{(K, \delta), e}(\rho, G, f, Z)$$

(3.25)

With a similar argument as in Fact 3.1, we have the following propositions.

**Proposition 3.13.** For any $s \in \mathbb{R}$, the function $h^\epsilon(\rho, G, f, \cdot)$ satisfies the following properties:

1. $h^\epsilon(\rho, G, f, \emptyset) = 0$ for $s \geq 0$;
2. $h^\epsilon_{(K, \delta), e}(\rho, G, f, Z_1) \leq h^\epsilon_{(K, \delta), e}(\rho, G, f, Z_2)$ for $Z_1 \subseteq Z_2 \subseteq X$;
3. $h^\epsilon_{(K, \delta), e}(\rho, G, f, \bigcup_{i \geq 1} Z_i) \leq \sum_{i \geq 1} h^\epsilon_{(K, \delta), e}(\rho, G, f, Z_i)$ where $Z_i \subseteq X$, $i = 1, 2, \ldots$.

**Proposition 3.14.** For any $Z \subseteq X$, one has

$$\inf\{s : h^\epsilon(\rho, G, f, Z) = 0\} = \sup\{s : h^\epsilon(\rho, G, f, Z) = \infty\}.$$  

So, there exists a critical value, which we will denote by $h_\epsilon(\rho, G, f, Z)$, where $h^\epsilon(\rho, G, f, Z)$ jumps from $\infty$ to 0, i.e.

$$h^\epsilon(\rho, G, f, Z) = \begin{cases} 0 & \text{if } h_\epsilon(\rho, G, f, Z) > s, \\ \infty & \text{if } h_\epsilon(\rho, G, f, Z) < s. \end{cases}$$

In other words, we define the critical value

$$h_\epsilon(\rho, G, f, Z) = \inf\{s : h^\epsilon(\rho, G, f, Z) = 0\}$$

$$= \sup\{s : h^\epsilon(\rho, G, f, Z) = \infty\}.$$  

We call

$$h^P(\rho, G, f, Z) = \lim_{\epsilon \to 0} h_\epsilon(\rho, G, f, Z)$$
the Bowen pseudometric Pesin-Pitskel pressure of $Z$.

Set

$$f_\ast(B^\rho_F(x_F, \epsilon)) := \inf_{y \in B^\rho_F(x_F, \epsilon)} f_F(y).$$

We define

$$h^\ast_{(K, \delta), \epsilon}(\rho, G, f_\ast, Z) = \inf \sum_i \exp \left( -s|F_i| + f_\ast(B^\rho_{F_i}(x_i, \epsilon)) \right)$$

(3.26)

where the infimum is taken over all collections $\Gamma = \{B^\rho_{F_i}(x_i, \epsilon)\}_{i \in I}$ with $F_i \in \mathfrak{B}(K, \delta)$, $x_i \in X$ and $\bigcup_{i \in I} B^\rho_{F_i}(x_i, \epsilon) \supseteq Z$.

**Remark 3.15.** Let $0 < \epsilon_1 < \epsilon_2$. If a family of Bowen balls with radius $\epsilon_1$ covers $Z$, then the family of Bowen balls with the same centers which have radius $\epsilon_2$ also covers $Z$. At the same time, it is clear that

$$\inf_{y \in B^\rho_F(x_F, \epsilon_1)} f_F(y) \geq \inf_{y \in B^\rho_F(x_F, \epsilon_2)} f_F(y).$$

Thus one has

$$h^\ast_{(K, \delta), \epsilon_1}(\rho, G, f_\ast, Z) \geq h^\ast_{(K, \delta), \epsilon_2}(\rho, G, f_\ast, Z).$$

Similar to topological pressure, we also define

$$h^\ast_\epsilon(\rho, G, f_\ast, Z) = \lim_{(K, \delta) \in \Lambda} h^\ast_{(K, \delta), \epsilon}(\rho, G, f_\ast, Z) = \sup_{(K, \delta) \in \Lambda} h^\ast_{(K, \delta), \epsilon}(\rho, G, f_\ast, Z)$$

(3.27)

and

$$h_\epsilon(\rho, G, f_\ast, Z) = \inf\{s : h^\ast_\epsilon(\rho, G, f_\ast, Z) = 0\}$$

$$= \sup\{s : h^\ast_\epsilon(\rho, G, f_\ast, Z) = \infty\}.$$  

Due to Remark 3.15, we know that $h_\epsilon(\rho, G, f_\ast, Z)$ does not decrease when $\epsilon$ decreases. So we set

$$h^P(\rho, G, f_\ast, Z) := \lim_{\epsilon \to 0} h_\epsilon(\rho, G, f_\ast, Z).$$

It is easy to see from the monotonicity of the function $h_\epsilon(\rho, G, f_\ast, Z)$ that

$$h^P(\rho, G, f_\ast, Z) := \sup_{\epsilon > 0} h_\epsilon(\rho, G, f_\ast, Z).$$

(3.28)

The proof idea of the following theorem comes from oral communication with Professor Hanfeng Li.

**Theorem 3.16.** Let $\rho$ be a dynamically generating continuous pseudometric on $X$. Then

$$h^P(\rho, G, f, Z) = h^P(\rho, G, f_\ast, Z).$$

**Proof.** It is clear that $h^P(\rho, G, f_\ast, Z) \leq h^P(\rho, G, f, Z)$. Thus it suffices to prove that

$$h^P(\rho, G, f, Z) \leq h^P(\rho, G, f_\ast, Z).$$

We may assume that $h^P(\rho, G, f_\ast, Z) < \infty$. Let $\kappa > h^P(\rho, G, f_\ast, Z)$ be any real number. Due to the equation (3.28), one has

$$h_\epsilon(\rho, G, f_\ast, Z) < \kappa \quad \text{for all} \quad 0 < \epsilon.$$  

(3.29)

Furthermore, we get

$$h^\ast_\epsilon(\rho, G, f_\ast, Z) = 0.$$  

(3.30)

Let $\tilde{\rho}$ be the compatible metric determined by $\rho$ as in Lemma 3.11. So $f$ is uniformly continuous on compact metric space $(X, \tilde{\rho})$. Thus there exist a constant $M_f$ for which $|f(x)| \leq M_f$ with all $x \in X$. 

Let $\gamma > 0$. Then there is $\theta > 0$ such that
\[ |f(x) - f(y)| < \frac{\gamma}{2} \quad \text{if} \quad \tilde{\rho}(x, y) < \theta. \]  
(3.31)

By Lemma 3.12, for above positive number $\theta$ and the dynamically generating pseudometric $\rho$, there exist a nonempty finite subset $K_0 \subset G$ and $\eta > 0$ such that
\[ \tilde{\rho}(x, y) < \theta \quad \text{if} \quad \rho(sx, sy) < \eta \quad \text{for all} \quad s \in K_0. \]  
(3.32)

Choose a nonempty finite subset $K \subset G$ with
\[ K_0 \subseteq K, \quad K^{-1} = K \quad \text{and} \quad \frac{M_I}{|K|} < \frac{\gamma}{4}. \]  
(3.33)

Let $0 < \epsilon < \eta/2$ and $0 < \delta < \gamma/(4|K|M_I)$.

**Claim.** We have
\[ h^{(K, \delta)}_{i, (\rho, G, f, Z)}(\rho, G, f, Z) \leq h^{(K, \delta)}_{i, e}(\rho, G, f, Z). \]

Let $\Gamma_{(K, \delta)} = \{B^\rho_{F_i}(x_i, \epsilon)\}_{i \in I}$ be a collection of Bowen balls such that $\bigcup_{i \in I} B^\rho_{F_i}(x_i, \epsilon) \supseteq Z$ with $x_i \in X$ and $F_i \in \mathfrak{B}(K, \delta)$.

Fix $i \in I$. Denote
\[ F'_i := \{g \in F_i : Kg \subseteq F_i\}. \]

Note that
\[ \{g \in F_i : Kg \subseteq F_i\} = \bigcap_{s \in K} (F_i \cap s^{-1}F_i). \]

Thus, since $K^{-1} = K$, one has
\[ F_i \setminus F'_i = \bigcup_{s \in K} (F_i \setminus s^{-1}F_i) \quad \text{(since} \quad K^{-1} = K) \]
\[ = \bigcup_{t \in K} (F_i \setminus tF_i). \]

Thus we have
\[ |F_i \setminus F'_i| \leq \sum_{t \in K} |F_i \setminus tF_i| \]
\[ = \sum_{t \in K} |F_i \setminus F_i| \quad \text{(since} \quad |F_i \setminus F_i| = |F_i \setminus tF_i|) \]
\[ \leq |K||KF_i \setminus F_i|. \]

So we get
\[ \frac{|F_i \setminus F'_i|}{|F_i|} \leq |K||KF_i \setminus F_i| = |K|\delta < \frac{\gamma}{4M_I}. \]  
(3.34)

Let $u, v \in B^\rho_{F_i}(x_i, \epsilon)$. Then, by (3.34), one has
\[ \frac{1}{|F_i|} \sum_{g \in F_i} |f(gu) - f(gv)| = \frac{1}{|F_i|} \sum_{g \in F'_i} |f(gu) - f(gv)| + \frac{1}{|F_i|} \sum_{g \in F_i \setminus F'_i} |f(gu) - f(gv)| \]
\[ \leq \frac{1}{|F_i|} \sum_{g \in F'_i} |f(gu) - f(gv)| + \frac{2M_I|F_i \setminus F'_i|}{|F_i|} \]
\[ \leq \frac{1}{|F_i|} \sum_{g \in F'_i} |f(gu) - f(gv)| + \frac{\gamma}{2}. \]  
(3.35)
Let \( g' \in F_i' \). For any \( s \in K \) we know that \( sg' \in F_i \). Thus
\[
\rho(s(g'u), s(g'v)) = \rho((sg')u, (sg')v) < 2\epsilon < \eta.
\]
From (3.32) and (3.33), we get
\[
\tilde{\rho}(g'u, g'v) < \theta.
\]
By (3.31) it follows that
\[
|f(g'u) - f(g'v)| < \frac{\gamma}{2}.
\]
Combining with (3.35) one has, for any \( u, v \in B^p_{F_i}(x_i, \epsilon) \),
\[
\frac{1}{|F_i|} \sum_{g \in F_i} |f(gu) - f(gv)| \leq \frac{1}{|F_i|} \sum_{g \in F_i} |f(gu) - f(gv)| + \frac{\gamma}{2} \leq \frac{\gamma}{2} \frac{|F_i|}{|F_i|} + \frac{\gamma}{2} \leq \gamma.
\]
(3.36)

Note that \( B^p_{F_i}(x_i, \epsilon) \subseteq g^{-1} B^p(gx_i, \epsilon) \) for every \( g \in F_i \). Then one has
\[
\sup_{y \in B^p_{F_i}(x_i, \epsilon)} \sum_{g \in F_i} f(gy) - \inf_{x \in B^p_{F_i}(x_i, \epsilon)} \sum_{g \in F_i} f(gx)
\]
\[
\leq \sup \left\{ \left| \sum_{g \in F_i} f(gx) - \sum_{g \in F_i} f(gy) \right| : x, y \in B^p_{F_i}(x_i, \epsilon) \right\}
\]
\[
\leq \sup \left\{ \sum_{g \in F_i} |f(gx) - f(gy)| : x, y \in B^p_{F_i}(x_i, \epsilon) \right\}
\]
\[
\leq |F_i| \gamma.
\]
Therefore, we get
\[
-(\kappa + \gamma)|F_i| + \sup_{y \in B^p_{F_i}(x_i, \epsilon)} \sum_{g \in F_i} f(gy) \leq -\kappa|F_i| + \inf_{x \in B^p_{F_i}(x_i, \epsilon)} \sum_{g \in F_i} f(gx).
\]
It follows that
\[
h^{k+\gamma}_{(K, \delta), \epsilon}(\rho, G, f, Z) \leq \sum_{i \in I} \exp \left( -(\kappa + \gamma)|F_i| + \sup_{y \in B^p_{F_i}(x_i, \epsilon)} \sum_{g \in F_i} f(gy) \right)
\]
\[
\leq \sum_{i \in I} \exp \left( -\kappa|F_i| + f_s(B^p_{F_i}(x_i, \epsilon)) \right).
\]
The arbitrariness of the collection \( \Gamma_{(K, \delta)} \) which covers \( Z \) implies that
\[
h^{k+\gamma}_{(K, \delta), \epsilon}(\rho, G, f, Z) \leq h^{k+\gamma}_{(K, \delta), \epsilon}(\rho, G, f_s, Z).
\]
So the claim is obtained.

From (3.25) we know that
\[
h^{k+\gamma}(\rho, G, f, Z) = \sup_{(K', \delta') \in A} h^{k+\gamma}_{(K', \delta'), \epsilon}(\rho, G, f, Z),
\]
\[
h^{k}_\epsilon(\rho, G, f_s, Z) = \sup_{(K', \delta') \in A} h^{k}_{(K', \delta'), \epsilon}(\rho, G, f_s, Z).
\]
Let \((K', \delta') \in \Lambda\). We choose the \((K, \delta)\) satisfying

\[
K_0 \cup K' \subseteq \overline{K}, \quad K^{-1} = \overline{K}, \quad \frac{2M_f}{\overline{K}} < \gamma \quad \text{and} \quad 0 < \delta < \min\{\delta', \frac{\gamma}{4M_f}\}.
\]

By Claim we have

\[
h_{(K', \delta'), \epsilon}(\rho, G, f, Z) \leq h_{(\overline{K}, \delta), \epsilon}(\rho, G, f, Z) \leq \sup_{(\hat{K}, \hat{\delta}) \in \Lambda} h_{\hat{K}, \hat{\delta}, \epsilon}(\rho, G, f_*, Z) = h^*_\epsilon(\rho, G, f_*, Z).
\]

The arbitrariness \((K', \delta') \in \Lambda\) implies that

\[
h^{\kappa + \gamma}_\epsilon(\rho, G, f, Z) \leq h^*_\epsilon(\rho, G, f_*, Z).
\]

Combining with (3.30), it follows that

\[
h^{\kappa + \gamma}_\epsilon(\rho, G, f, Z) = 0
\]

which implies that

\[
h_\epsilon(\rho, G, f, Z) \leq \kappa + \gamma.
\]

Taking the upper limit as \(\epsilon \to 0\) we get \(h^P(\rho, G, f, Z) \leq \kappa + \gamma\). The arbitrariness of \(\gamma\) and \(\kappa > h^P(\rho, G, f_*, Z)\) imply that

\[
h^P(\rho, G, f, Z) \leq h^P(\rho, G, f_*, Z)
\]

which is our desired. \(\square\)

Theorem 9.38 in [12] said not only that we may compute topological entropy using separated sets, but that we can do it using dynamically generating continuous pseudometrics, and not merely compatible metrics. Kerr and Li studied deeply the properties of dynamically generating continuous pseudometric and obtained many interesting results in [12]. Similar to Theorem 9.38 in [12], we have the following result.

**Theorem 3.17.** Let \(\rho\) be a dynamically generating continuous pseudometric on \(X\). Then

\[
h^P_{\text{top}}(G, f, Z) = h^P(\rho, G, f, Z).
\]

**Proof.** To establish the proof of Theorem 3.17, it suffices to carry out the following two steps:

**Step 1.** We will show

\[
h^P(\rho, G, f, Z) \leq h^P_{\text{top}}(G, f, Z).
\]

Due to Theorem 3.16 we know that \(h^P(\rho, G, f, Z) = h^P(\rho, G, f_*, Z)\). So we only need to prove \(h^P(\rho, G, f_*, Z) \leq h^P_{\text{top}}(G, f, Z)\). Let \(\epsilon > 0\), \(\mathcal{U}\) be a finite open cover of \(X\) such that \(\text{diam}(\mathcal{U}) < \epsilon\). In what follows, we will show that

\[
h_\epsilon(\rho, G, f_*, Z) \leq h^P_{\text{top}}(G, \mathcal{U}, f, Z).
\]

We may assume that \(h^P_{\text{top}}(G, \mathcal{U}, f, Z) < +\infty\). Let \(s\) be any real number with \(s > h^P_{\text{top}}(G, \mathcal{U}, f, Z)\). Thus, one has

\[
\mathcal{M}^s(G, \mathcal{U}, f, Z) = 0.
\]
Let \((K, \delta) \in \Lambda\).

**Claim.**
\[ h_{s, \varepsilon}^\rho(K, \delta, (\rho, G, f_*, Z)) \leq M_s^\rho(K, \delta, (G, \mathcal{U}, f, Z)). \]

Let \(\Gamma_{(K, \delta)} \subseteq \bigcup_{F \in \mathcal{B}(K, \delta)} \mathcal{U}(F)\) be a collection of strings which covers \(Z\). We may assume that \(X(U) \neq \emptyset\) for each string \(U \in \Gamma_{(K, \delta)}\). Fix a point \(x_U \in X(U)\). Note that
\[ X(U) = \bigcap_{g \in \text{dom}(U)} g^{-1}U(g). \]

Thus for any point \(y \in X(U)\), one has \(\rho(gx_U, gy) \leq \text{diam}(\mathcal{U}) < \varepsilon\) (for any \(g \in \text{dom}(U)\)), that is,
\[ X(U) \subseteq B_{\text{dom}(U)}^\rho(x_U, \varepsilon). \quad (3.39) \]

So we get a collection of Bowen balls
\[ \Omega_{(K, \delta)} = \{ V_U = B_{\text{dom}(U)}^\rho(x_U, \varepsilon) \}_{U \in \Gamma_{(K, \delta)}}. \]

Clearly, the collection \(\Omega_{(K, \delta)}\) of Bowen balls covers \(Z\). Note that \(|\text{dom}(U)| = m(U)\) and (3.39), we have
\[ h_{s, \varepsilon}^\rho(K, \delta, (\rho, G, f_*, Z)) \leq \sum_{U \in \Gamma_{(K, \delta)}} \exp \left( -s m(U) \right) \sup_{y \in X(U)} \sum_{g \in \text{dom}(U)} f(gy), \]

From the arbitrariness of the collection \(\Gamma_{(K, \delta)}\) which covers \(Z\), it follows that
\[ h_{s, \varepsilon}^\rho(K, \delta, (\rho, G, f_*, Z)) \leq M_s^\rho(K, \delta, (G, \mathcal{U}, f, Z)). \]

Taking the limit for the net \((K, \delta) \in \Lambda\), we get
\[ h_{s}^\rho(K, \delta, (\rho, G, f_*, Z)) \leq M_s^\rho(K, \delta, (G, \mathcal{U}, f, Z)). \]

Combining with (3.38), it follows that
\[ h_{s}^\rho(K, \delta, (\rho, G, f_*, Z)) = 0 \]
which implies that
\[ h_{\varepsilon}^\rho(K, \delta, (\rho, G, f_*, Z)) \leq s. \]

Since \(s\) is any real number with \(s > h_{\text{top}}^P(G, \mathcal{U}, f, Z)\), we have that
\[ h_{\varepsilon}^\rho(K, \delta, (\rho, G, f_*, Z)) \leq h_{\text{top}}^P(G, \mathcal{U}, f, Z). \]

Thus we get
\[ h_{\varepsilon}^\rho(K, \delta, (\rho, G, f_*, Z)) \leq \lim_{\text{diam}(\mathcal{U}) \rightarrow 0} h_{\text{top}}^P(G, \mathcal{U}, f, Z) = h_{\text{top}}^P(G, f, Z). \]

Letting \(\varepsilon \rightarrow 0\) one has
\[ h_{\rho}^P(K, \delta, (\rho, G, f_*, Z)) \leq h_{\text{top}}^P(G, f, Z). \]

Hence the conclusion of Step 1 is obtained.
Step 2. We will show

\[ h_{\text{top}}^p(G, f, Z) \leq h^p(\rho, G, f, Z). \]

By Theorem 3.5 we only need to prove

\[ h_{\text{top}}^p(G, f_*, Z) \leq h^p(\rho, G, f, Z). \] (3.40)

Let \( \mathcal{U} \) be a finite open cover of \( X \) and \( \kappa > 0 \). Clearly it suffices to prove that

\[ h_{\text{top}}^p(G, \mathcal{U}, f_*, Z) \leq \limsup_{\epsilon \to 0} h_\epsilon(\rho, G, f, Z) + 2\kappa. \] (3.41)

Let \( \tilde{\rho} \) be the compatible metric on \( X \) which is determined by the dynamically generating continuous pseudometric \( \rho \) as in Lemma 3.11. Denote \( L = L_\rho(\mathcal{U}) \) by the Lebesgue number of \( \mathcal{U} \) under the metric \( \tilde{\rho} \).

Let \( 0 < \epsilon_0 < L/6 \). Since \( \rho \) is continuous and \( X \) is compact, there is a constant \( M \) with \( \rho(x, y) \leq M \) for all \( x, y \in X \). Consequently, there exists \( N \in \mathbb{N} \) such that

\[ \sum_{j=N+1}^\infty \frac{M}{2^j-1} < \epsilon_0. \] (3.42)

Set

\[ K_0 = \{g_1, g_2, \cdots, g_N\} \quad \text{and} \quad \delta_0 = \frac{\kappa}{N \log |\mathcal{U}|} \] (3.43)

where \( |\mathcal{U}| \) denotes the number of the elements of \( \mathcal{U} \). Take \( s = h_{\epsilon_0}(\rho, G, f, Z) \). Thus

\[ \lim_{(K, \delta) \in \Lambda} h_{s, \epsilon_0}^{s+\kappa}(\rho, G, f, Z) = 0. \]

So there exist \( K_1 \in \mathcal{F}(G) \) and \( \delta_1 > 0 \) with \( (K_1, \delta_1) \succ (K_0, \delta_0) \) such that, for any \( (K, \delta) \succ (K_1, \delta_1) \), there is a countable collection

\[ \{B_{F_i}(x_i, \epsilon_0)\}_{i \in I} \quad \text{with} \quad x_i \in X, \quad F_i \in \mathcal{B}(K, \delta), \quad \bigcup_{i \in I} B_{F_i}(x_i, \epsilon_0) \supseteq Z \] (3.44)

and

\[ \sum_{i \in I} \exp \left( -(s + \kappa)|F_i| + \sup_{y \in B_{F_i}(x_i, \epsilon_0)} f_{F_i}(y) \right) < 1. \] (3.45)

For each \( i \in I \), we denote \( A_i = \bigcap_{j=1}^N g_j^{-1} F_i \). Due to \( (K, \delta) \succ (K_0, \delta_0) \) we know that \( g_j \in K \) for \( j = 1, \cdots, N \). Since \( F_i \in \mathcal{B}(K, \delta) \) and \( g_1 = e \), one has \( A_i \subseteq F_i \) and

\[
|F_i \setminus A_i| = \left| \bigcup_{j=1}^N (F_i \setminus g_j^{-1} F_i) \right| \leq \sum_{j=1}^N \left| F_i \setminus g_j^{-1} F_i \right|
\]

\[
= \sum_{j=1}^N \left| g_j F_i \setminus F_i \right| \leq N \left| K F_i \setminus F_i \right| \leq N \delta |F_i|.
\] (3.46)

Note that, for every \( t \in A_i \) and \( g_j \) \((1 \leq j \leq N)\), one has \( g_j t \in F_i \). For any pair of points \( y_1, y_2 \in B_{F_i}(x_i, \epsilon_0) \) \((i \in I)\) and any \( t \in A_i \), we have

\[
\tilde{\rho}(ty_1, ty_2) = \sum_{j=1}^\infty \frac{\rho(g_j ty_1, g_j ty_2)}{2^{j-1}}
\]
\[
\leq \sum_{j=1}^{N} \frac{\rho(g_j t y_1, g_j t y_2)}{2^{j-1}} + \epsilon_0 \quad \text{(using (3.42))}
\]
\[
\leq \sum_{j=1}^{N} \frac{\rho(g_j t y_1, g_j t x_i)}{2^{j-1}} + \sum_{j=1}^{N} \frac{\rho(g_j t x_i, g_j t y_2)}{2^{j-1}} + \epsilon_0 \leq \sum_{j=1}^{N} \frac{\rho_F(y_1, x_i)}{2^{j-1}} + \sum_{j=1}^{N} \frac{\rho_F(x_i, y_2)}{2^{j-1}} + \epsilon_0 \quad \text{(since } g_j t \in F_i \text{ for } 1 \leq j \leq N)\]
\[
\leq 5\epsilon_0 < L.
\]
This fact shows that the diameter of the set \(tB^0_{F_i}(x_i, \epsilon_0)\) under the metric \(\bar{\rho}\) is less than \(L\), i.e.
\[
diam_{\bar{\rho}}(tB^0_{F_i}(x_i, \epsilon_0)) < L.
\]
Therefore, for each \(t \in A_i\), \(tB^0_{F_i}(x_i, \epsilon_0)\) is contained in an element of \(\mathcal{W}\), that is, there exists an open set \(U_t \in \mathcal{W}\) such that
\[
B^0_{F_i}(x_i, \epsilon_0) \subseteq t^{-1}U_t. \tag{3.47}
\]
Let \(\text{Map}_i = \{\varphi \mid \varphi : F_i \setminus A_i \to \mathcal{W}\}\) be the set of all maps from \(F_i \setminus A_i\) into \(\mathcal{W}\). It is clear that
\[
|\text{Map}_i| = |\mathcal{W}|^{|F_i \setminus A_i|}.
\]
For any \(i \in I\) and each \(\varphi \in \text{Map}_i\), we define the string \(U_{i, \varphi} \in \mathcal{W}_{F_i}(\mathcal{W})\) as follows:
\[
U_{i, \varphi}(t) := \begin{cases} 
U_t & t \in A_i \\
\varphi(t) & t \in F_i \setminus A_i.
\end{cases}
\]
For fixed \(i \in I\), from (3.47) it is easy to see that
\[
B^0_{F_i}(x_i, \epsilon_0) \subseteq \bigcap_{t \in A_i} t^{-1}U_t = \bigcup_{\varphi \in \text{Map}_i} X(U_{i, \varphi}). \tag{3.48}
\]
For \(i \in I\), we denote
\[
\Omega_i := \left\{ U_{i, \varphi} \mid \varphi \in \text{Map}_i \text{ and } X(U_{i, \varphi}) \cap B^0_{F_i}(x_i, \epsilon_0) \neq \emptyset \right\}. \tag{3.49}
\]
Note that \(F_i \in \mathfrak{B}(K, \delta)\) for each \(i \in I\). Now we define a set of strings \(\Omega_{(K, \delta)}\) as follows:
\[
\Omega_{(K, \delta)} := \bigcup_{i \in I} \Omega_i \subseteq \bigcup_{F \in \mathfrak{B}(K, \delta)} \mathcal{W}_F(\mathcal{W}).
\]
It follows that
\[
\bigcup_{U \in \Omega_{(K, \delta)}} X(U) = \bigcup_{i \in I} \bigcup_{U_{i, \varphi} \in \Omega_i} X(U_{i, \varphi}) \supseteq \bigcup_{i \in I} B^0_{F_i}(x_i, \epsilon_0) \quad \text{(since (3.48))}
\]
\[
\supseteq Z. \quad \text{(since (3.44))}
\]
Note that \(m(U_{i, \varphi}) = |F_i|\) for \(U_{i, \varphi} \in \Omega_i\),
\[
|\Omega_i| \leq |\text{Map}_i| \leq |\mathcal{W}|^{|F_i \setminus A_i|} \quad \text{and} \quad \delta \leq \delta_0 = \frac{\kappa}{N \log |\mathcal{W}|}.
\]

4. Local measure pressures of subsets for actions of amenable groups. Let $X$ be a compact metrizable space and $\mathcal{M}(X)$ denote the set of all Borel probability measures on $X$ which is equipped with weak* topology.

Recall that $G$ is a discrete countable infinite amenable group and $e \in G$ is its unit. Fix a family of finite subsets $\{G_n\}_{n=1}^{\infty}$ of $G$ which satisfies that $e \in G_1 \subseteq G_2 \subseteq \cdots$ and $G = \bigcup_{n=1}^{\infty} G_n$.

For the sake of our following proofs, we present some facts.

**Fact 4.1.** For an $\mathbb{R}$-valued function $\varphi$ defined on $\mathcal{F}(G)$ one has

$$\lim_{n \to \infty} \inf_{F \in \mathcal{B}(G_n,1/n)} \varphi(F) = \lim_{n \to \infty} \inf_{F \in \mathcal{B}(G_n,1/n)} \varphi(F),$$

$$\lim_{n \to \infty} \sup_{F \in \mathcal{B}(G_n,1/n)} \varphi(F) = \lim_{n \to \infty} \sup_{F \in \mathcal{B}(G_n,1/n)} \varphi(F).$$

**Proof.** Here we only prove the first equation. From Remark 2.1 in Section 2 it follows that

$$\lim_{n \to \infty} \inf_{F \in \mathcal{B}(G_n,1/n)} \varphi(F) \geq \inf_{F \in \mathcal{B}(G_n,1/n)} \varphi(F)$$

for each $n \in \mathbb{N}$ and hence that

$$\lim_{n \to \infty} \inf_{F \in \mathcal{B}(G_n,1/n)} \varphi(F) \geq \lim_{n \to \infty} \inf_{F \in \mathcal{B}(G_n,1/n)} \varphi(F).$$
Let $K \in \mathcal{F}(G)$ and $\delta > 0$. Then there exists $N = N(K, \delta) \in \mathbb{N}$ such that $K \subseteq G_n$ and $1/n < \delta$ for all $n \geq N$, i.e. $\mathcal{B}(G_n, 1/n) \subseteq \mathcal{B}(K, \delta)$ ($n \geq N$). Thus,

$$\inf_{F \in \mathcal{B}(K, \delta)} \varphi(F) \leq \lim_{n \to \infty} \inf_{F \in \mathcal{B}(G_n, 1/n)} \varphi(F)$$

for all $n \geq N$.

which implies

$$\inf_{F \in \mathcal{B}(K, \delta)} \varphi(F) \leq \lim_{n \to \infty} \inf_{F \in \mathcal{B}(G_n, 1/n)} \varphi(F).$$

Hence the fact is proved. \qed

**Definition 4.2.** Let $\rho$ be a dynamically generating continuous pseudometric on $X$.

Let $F \in \mathcal{F}(G)$, $f \in C(X)$, $\delta > 0$ and $\mu \in \mathcal{M}(X)$. For $x \in X$, we define

$$h_{\mu}(G, \rho, f, x, \delta, F) = \frac{1}{|F|} \log \left( e^{f(x)} \mu(B^\rho_f(x, \delta))^{-1} \right),$$

$$h_{\mu}(G, \rho, f, x, \delta) = \liminf_{F \to \infty} h_{\mu}(G, \rho, f, x, \delta, F).$$

The local lower measure pressure is defined by

$$h_{\mu}^L(G, \rho, f, x) = \lim_{\delta \to 0} h_{\mu}(G, \rho, f, x, \delta).$$

Similarly, we define

$$\overline{h}_{\mu}(G, \rho, f, x, \delta) = \limsup_{F \to \infty} h_{\mu}(G, \rho, f, x, \delta, F).$$

The local upper measure pressure is defined by

$$\overline{h}_{\mu}^U(G, \rho, f, x) = \lim_{\delta \to 0} \overline{h}_{\mu}(G, \rho, f, x, \delta).$$

**Remark 4.3.** If $\delta_1 < \delta_2$ then

$$h_{\mu}(G, \rho, f, x, \delta_1) \geq h_{\mu}(G, \rho, f, x, \delta_2) \quad \text{and} \quad \overline{h}_{\mu}(G, \rho, f, x, \delta_1) \geq \overline{h}_{\mu}(G, \rho, f, x, \delta_2).$$

**Fact 4.4.** Let $F \in \mathcal{F}(G)$ and $r > 0$. For any $a \in \mathbb{R}$, the set $E = \{ x \in X : \mu(B^\rho_f(x, r)) > a \}$ is an open subset of $X$.

**Proof.** Let $x_0 \in E$ and take a positive integer $m > 1/r$. It is clear that $\mu(B^\rho_f(x_0, r)) > a$. Since the sequence $\{B^\rho_f(x_0, r - 1/n) : n = m, m+1, \ldots \}$ of open Bowen balls increases and

$$B^\rho_f(x_0, r) = \bigcup_{n=m}^{\infty} B^\rho_f\left(x_0, r - \frac{1}{n}\right),$$

we know that

$$\lim_{n \to \infty} \mu\left(B^\rho_f\left(x_0, r - \frac{1}{n}\right)\right) = \mu(B^\rho_f(x_0, r)) > a.$$ 

So there exists a positive real number $r_1$ with $0 < r_1 < r$ such that

$$\mu(B^\rho_f(x_0, r_1)) > a.$$ 

Set $\delta = r - r_1$. For any point $y \in B^\rho_f(x_0, \delta)$, it is easily checked that

$$B^\rho_f(y, r) \supseteq B^\rho_f(x_0, r_1).$$

Thus, we get $\mu(B^\rho_f(y, r)) > a$ which implies that $y \in E$, i.e.,

$$B^\rho_f(x_0, \delta) \subseteq E.$$ 

Therefore, $E$ is open as the set $B^\rho_f(x_0, \delta)$ is open. Hence the fact is proved. \qed
Fact 4.5. Let $F$ be a nonempty finite subset of $G$. Then the function

$$h(x) = \frac{1}{|F|} \log \mu(B^\rho_F(x, r))$$

is Borel measurable.

Proof. It is immediately obtained from Fact 4.4. \qed

Fact 4.6. The two functions $h^\mu_{\rho}(G, \rho, f, x)$ and $\overline{h}^\mu_{\rho}(G, \rho, f, x)$ are Borel measurable.

Proof. Here, we only prove the function $h^\mu_{\rho}(G, \rho, f, x)$ is Borel measurable. The proof of the measurability of $\overline{h}^\mu_{\rho}(G, \rho, f, x)$ is similar.

From the definition, it is easy to see that

$$h^\mu_{\rho}(G, \rho, f, x) = \lim_{m \to \infty} h_{\mu}(G, \rho, f, x, \frac{1}{m}).$$

Thus it suffices to prove that the function

$$g_m(x) = h_{\mu}(G, \rho, f, x, \frac{1}{m}) = \liminf_F h_{\mu}(G, \rho, f, x, \frac{1}{m}, F)$$

is Borel measurable. By Fact 4.1, one has

$$g_m(x) = \liminf_{n \to \infty} \inf_{F \in \mathcal{B}(G_n, 1/n)} h_{\mu}(G, \rho, f, x, \frac{1}{m}, F)$$

where $\{G_n\}_{n=1}^\infty$ is a family of finite subsets of $G$ which satisfies that $e \in G_1 \subseteq G_2 \subseteq \cdots$ and $G = \bigcup_{n=1}^\infty G_n$. Hence we only need to prove that the function

$$h_n(x) = \inf_{F \in \mathcal{B}(G_n, 1/n)} h_{\mu}(G, \rho, f, x, r, F)$$

is Borel measurable for given $r > 0$.

Since $G$ is countable group, the set $\mathcal{B}(G_n, 1/n)$ is countable. Thus we only need to show that $h_{\mu}(G, \rho, f, x, r, F)$ is Borel measurable. Furthermore,

$$h_{\mu}(G, \rho, f, x, r, F) = \frac{1}{|F|} \log \left( e^{f_F(x)} \mu(B^\rho_F(x, r))^{-1} \right)$$

$$= -\frac{1}{|F|} \log \mu(B^\rho_F(x, r)) + \frac{1}{|F|} f_F(x).$$

It is easy to see that the function $x \mapsto \frac{1}{|F|} f_F(x)$ is Borel measurable as it is a continuous function. So it is enough to show that the function

$$x \mapsto -\frac{1}{|F|} \log \mu(B^\rho_F(x, r))$$

is Borel measurable. Fact 4.5 shows that the above function is Borel measurable.

Hence the fact is obtained. \qed

In the following, we prove a lemma which is much like the classical covering lemma. For the $\mathbb{Z}$-action case of this lemma, please refer to [16] (see Lemma 1, p.506).

Lemma 4.7. Let $G \curvearrowright X$ be a continuous action and $\rho$ be a continuous pseudometric metric on $X$. Let $\{F_n\}$ be a nested sequence of finite subsets of $G$. Let $r > 0$ and $\mathcal{B}(r) = \{B^\rho_{F_n}(x, r) : x \in X, n \in \mathbb{N}\}$. For any family $\mathcal{F} \subseteq \mathcal{B}(r)$, there exists a countable sub-family $\mathcal{G} \subseteq \mathcal{F}$ consisting of pairwise disjoint Bowen balls such that

$$\bigcup_{B^\rho_{F_n}(x, r) \in \mathcal{F}} B^\rho_{F_n}(x, r) \subseteq \bigcup_{B^\rho_{F_m}(y, r) \in \mathcal{G}} B^\rho_{F_m}(y, 3r).$$
Proof. Let $\Omega = \{\omega : \omega \subseteq F\}$ denote the partial ordered (by inclusion) set consisting of all subfamilies $\omega$ of $F$ with the following properties:

1. $\omega$ consists of pairwise disjoint Bowen balls from $F$;
2. If a Bowen ball $B^p_{F_m}(x, r) \in F$ meets some Bowen ball from $\omega$, then there exists a Bowen ball $B^p_{F_m}(y, r) \in \omega$ such that $F_m \subseteq F_n$ and $B^p_{F_n}(x, r) \cap B^p_{F_m}(y, r) \neq \emptyset$.

Thus $\Omega$ is non-empty. Indeed, let

$$n_0 := \min\{n : B^p_{F_n}(x, r) \in F\}.$$ 

We choose a point $x_0$ such that $B^p_{F_{n_0}}(x_0, r) \in F$. Then the family consisting of only one Bowen ball

$$\omega_0 := \{B^p_{F_{n_0}}(x_0, r)\}$$

belongs to $\Omega$, that is, $\omega_0 \in \Omega$.

Let $\mathcal{G}$ be a totally ordered sub-collection of $\Omega$. Then we define

$$\omega^* := \cup\{\omega : \omega \in \mathcal{G}\}.$$ 

For each $\omega \in \mathcal{G}$, it is clear that $\omega \subseteq \omega^*$. Now we show that $\omega^* \in \Omega$. It is not hard to see that all Bowen balls of $\omega^*$ are pairwise disjoint as $\mathcal{G}$ is a totally ordered sub-collection of $\Omega$. Meanwhile, if a Bowen ball $B^p_{F_p}(z, r) \in F$ meets some Bowen ball from $\omega^*$, then $B^p_{F_p}(z, r)$ meets a Bowen ball from $\omega$ for some $\omega \in \mathcal{G}$. By the definition of $\omega$, there is a Bowen ball $B^p_{F_m}(y, r) \in \omega \subseteq \omega^*$ such that $F_m \subseteq F_p$ and $B^p_{F_m}(y, r) \cap B^p_{F_p}(z, r) \neq \emptyset$. Hence $\omega^* \in \Omega$.

By Zorn’s lemma, there exists a maximal element $\mathcal{G}$ in $\Omega$. We claim that

$$\bigcup_{B^p_{F_m}(x, r) \in F} B^p_{F_m}(x, r) \subseteq \bigcup_{B^p_{F_m}(y, r) \in \mathcal{G}} B^p_{F_m}(y, 3r).$$

First, we prove that the intersection of each Bowen ball of $F$ and the union of all Bowen balls from $\mathcal{G}$ is nonempty. Otherwise, there exists a Bowen ball $B^p_{F_m}(x, r) \in F$ such that

$$B^p_{F_m}(x, r) \cap \left(\bigcup_{B^p_{F_p}(y, r) \in \mathcal{G}} B^p_{F_p}(y, r)\right) = \emptyset.$$ 

So we can define

$$m_0 = \min\{m : B^p_{F_m}(x, r) \in F\}$$

does NOT meet any Bowen ball from $\mathcal{G}$. We choose a Bowen ball $B^p_{F_{m_0}}(x_0, r) \in F$ such that it does not meet any Bowen ball in $\mathcal{G}$. We write $\mathcal{G}^* = \mathcal{G} \cup \{B^p_{F_{m_0}}(x_0, r)\}$.

Now, we show that $\mathcal{G}^* \in \Omega$. It is clear that all Bowen balls of $\mathcal{G}^* \subseteq F$ are pairwise disjoint. Suppose that $B^p_{F_p}(z, r) \in F$ meets some Bowen ball from $\mathcal{G}^*$. Then we divide into two cases into consideration.

- If $B^p_{F_p}(z, r)$ meets some Bowen ball from $\mathcal{G}$, owing to $\mathcal{G} \in \Omega$, then there is $B^p_{F_n}(x, r) \in \mathcal{G}$ such that $F_n \subseteq F_p$ and $B^p_{F_n}(x, r) \cap B^p_{F_p}(z, r) \neq \emptyset$.
- If $B^p_{F_p}(z, r)$ does NOT meet any Bowen ball from $\mathcal{G}$, then $B^p_{F_p}(z, r) \cap B^p_{F_{m_0}}(x_0, r) = \emptyset$. By the definition of $m_0$, it follows that $m_0 \leq p$, i.e., $F_{m_0} \subseteq F_p$.

According to above arguments we deduce that $\mathcal{G} \cup \{B^p_{F_{m_0}}(x_0, r)\} \in \Omega$ which contradicts that $\mathcal{G}$ is a maximal element in $\Omega$. 




Hence, each Bowen ball $B_{F_n}^\rho(x,r) \in F$ must meet some element from $\mathcal{G}$. Since $\mathcal{G} \subseteq \Omega$, there is a Bowen ball $B_{F_p}^\rho(y,r) \in \mathcal{G}$ such that
\[
F_p \subseteq F_n \quad \text{and} \quad B_{F_n}^\rho(x,r) \cap B_{F_p}^\rho(y,r) \neq \emptyset.
\]
Since the metric $\rho_{F_p} \leq \rho_{F_n}$, one has
\[
B_{F_n}^\rho(x,r) \subseteq B_{F_p}^\rho(y,3r)
\]
which implies that
\[
\bigcup_{b_{F_n}^\rho(x,r) \in F} B_{F_n}^\rho(x,r) \subseteq \bigcup_{b_{F_m}^\rho(y,r) \in \mathcal{G}} B_{F_m}^\rho(y,3r).
\]

Since $X$ is a compact metrizable space, it satisfies the second axiom of countability. Furthermore, owing to the fact that all elements of $\mathcal{G}$ are open and pairwise disjoint, the set $\mathcal{G}$ is countable. Hence the lemma is proved. \(\square\)

The following theorem establishes the relation between Pesin-Pitskel topological pressure and the local pressure of a Borel probability measure. It is a generalization of Ma-Wen’s result [16] to dynamical systems acting by a countable discrete amenable group.

**Theorem 4.8.** Let $G \subset X$ be a continuous action, $f \in C(X)$ and $\rho$ a dynamically generating continuous pseudometric on $X$. Let $\mu$ be a Borel probability measure on $X$ and $Z \subseteq X$ a Borel subset. For $s \in \mathbb{R}$, the following properties hold:

1. If $h^P_\mu(G, \rho, f, x) \geq s$ for all $x \in Z$ and $\mu(Z) > 0$, then $h^P(\rho, G, f, Z) \geq s$;
2. If $h^P_\mu(G, \rho, f, x) \leq s$ for all $x \in Z$ then $h^P(\rho, G, f, Z) \leq s$.

**Proof.** (1) Let $\nu > 0$. We want to show that
\[
h^P(\rho, G, f, Z) \geq s - \nu.
\]
Let $k \in \mathbb{N}$, $\{G_k\}_{k=1}^\infty$ be a family of finite subsets of $G$ which satisfies that $e \in G_1 \subseteq G_2 \subseteq \cdots$ and $G = \bigcup_{k=1}^\infty G_k$. Set
\[
Z_k = \left\{ x \in Z : \mu \left( B_{F}^\rho(x, \frac{1}{k}) \right) \leq e^{f_F(x) - (s-\nu)\nu} \right\} \quad \text{for all} \quad F \in \mathfrak{B}(G_k, \frac{1}{k}).
\]
For each $F \in \mathfrak{B}(G_k, \frac{1}{k})$, we denote
\[
Z_F = \left\{ x \in X : \mu \left( B_F^\rho \left( x, \frac{1}{k} \right) \right) \leq e^{f_F(x) - (s-\nu)\nu} \right\}.
\]
To make the proof precise we proceed as follows.

**Step 1.** We will show that $Z_k$ is a Borel subset of $X$.

Note that
\[
Z_F = \left\{ x \in X : -\frac{1}{|F|} \log \mu \left( B_F^\rho \left( x, \frac{1}{k} \right) \right) + \frac{1}{|F|} f_F(x) \geq s - \nu \right\}.
\]
By Fact 4.5, we know that the function $x \mapsto -\frac{1}{|F|} \log \mu \left( B_F^\rho \left( x, \frac{1}{k} \right) \right)$ is Borel. At the same time, the function $x \mapsto \frac{1}{|F|} f_F(x)$ is continuous. Thus we get that the set $Z_F$ is a Borel subset of $X$. Meanwhile, it is easy to see that
\[
Z_k = Z \cap \left( \bigcap_{F \in \mathfrak{B}(G_k, \frac{1}{k})} Z_F \right).
\]
Furthermore, we know that the set $\mathcal{B}(G_k, \frac{1}{k})$ is countable. Hence we deduce that $Z_k$ is a Borel subset of $X$ since $Z$ is a Borel set.

**Step 2.** \{$Z_k$\} is an increasing sequence of Borel sets and $\bigcup_{k=1}^{\infty} Z_k = Z$.

The monotonicity of $G_m$ implies that

$$\mathcal{B}(G_{k+1}, \frac{1}{k+1}) \subseteq \mathcal{B}(G_k, \frac{1}{k})$$

which leads to $Z_k \subseteq Z_{k+1}$.

Now we prove $\bigcup_{k=1}^{\infty} Z_k = Z$.

Let $x_0 \in Z$. The assumption $h^P_\rho(G, \rho, f, x_0) \geq s > \nu$ shows that

$$\lim_{\delta \to 0} \liminf_{F} \frac{1}{|F|} \log \left( e^{f_F(x_0)} \mu(B^0_F(x_0, \delta))^{-1} \right) > s - \nu.$$ 

So there exists $\delta_0 > 0$ such that

$$\liminf_{F} \frac{1}{|F|} \log \left( e^{f_F(x_0)} \mu(B^0_F(x_0, \delta_0))^{-1} \right) > s - \nu.$$ 

Combining with Fact 4.1 we have

$$\lim_{n \to \infty} \inf_{F \in \mathcal{B}(G_n, \frac{1}{n})} \frac{1}{|F|} \log \left( e^{f_F(x_0)} \mu(B^0_F(x_0, \delta_0))^{-1} \right) > s - \nu.$$ 

Thus, we can find an $N \in \mathbb{N}$ such that

$$\exp \left( f_F(x_0) - (s - \nu)|F| \right) > \mu(B^0_F(x_0, \delta_0)) \quad \text{for each} \quad F \in \mathcal{B}(G_N, \frac{1}{N}), \quad (4.1)$$

We choose a positive integer $m \in \mathbb{N}$ with

$$m > N \text{ and } \frac{1}{m} < \delta_0. \quad (4.2)$$

The facts of $m > N$ and $G_m \supseteq G_N$ imply that

$$\mathcal{B}(G_m, \frac{1}{m}) \subseteq \mathcal{B}(G_N, 1/N).$$

Therefore, for every $F \in \mathcal{B}(G_m, 1/m)$, we know that

$$\mu \left( B^0_F \left( x_0, \frac{1}{m} \right) \right) \leq \mu(B^0_F(x_0, \delta_0)) < \exp \left( f_F(x_0) - (s - \nu)|F| \right) \quad (4.3)$$

which implies $x_0 \in Z_m$ as desired. Hence Step 2 is proved.

From Steps 1 and 2, we know that $\lim_{n \to \infty} \mu(Z_n) = \mu(Z)$. It follows that there is $m \in \mathbb{N}$ satisfying

$$\mu(Z_m) \geq \frac{1}{2} \mu(Z) > 0. \quad (4.4)$$

**Step 3.** We will show that

$$h^P(\rho, G, f, Z) \geq s - \nu.$$ 

Let $K$ be a nonempty finite subset of $G$ and $\delta > 0$ with $(K, \delta) \succ (G_m, \frac{1}{m})$. Let $0 < r < \frac{1}{m}$. Suppose that $\Gamma = \{B^0_F(x_i, r)\}_{i \in \mathcal{I}}$ is a collection of Bowen balls of $X$ with $x_i \in X$ and satisfies $F_i \in \mathcal{B}(K, \delta)$ and

$$\bigcup_{i \in \mathcal{I}} B^0_F(x_i, r) \supseteq Z.$$ 

To finish the proof, we now define the index set $\mathcal{I}' \subseteq \mathcal{I}$ by

$$\mathcal{I}' = \{i \in \mathcal{I} : B^0_F(x_i, r) \cap Z_m \neq \emptyset\}.$$
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It is clear that
\[ \bigcup_{i \in I'} B_{F_i}^\rho(x_i, r) \supseteq Z_m. \tag{4.5} \]
We now choose a point \( y_i \in B_{F_i}^\rho(x_i, r) \cap Z_m \) for each \( i \in I' \). Therefore, one has
\[ B_{F_i}^\rho(x_i, r) \subseteq B_{F_i}^\rho(y_i, 2r) \subseteq B_{F_i}^\rho\left(y_i, \frac{1}{m}\right) \quad \text{for every} \quad i \in I'. \tag{4.6} \]
From (4.5) and (4.6), we then have
\[ \bigcup_{i \in I'} B_{F_i}^\rho\left(y_i, \frac{1}{m}\right) \supseteq Z_m. \tag{4.7} \]
So we get
\[ \mu\left(B_{F_i}^\rho\left(y_i, \frac{1}{m}\right)\right) \leq \exp\left(f_{F_i}(y_i) - (s - \nu)|F_i|\right). \tag{4.8} \]
Consequently, we deduce that
\[
\sum_{i \in I} \exp\left(- (s - \nu)|F_i| + \sup_{y \in B_{F_i}^\rho(x_i, r)} \sum_{g \in F_i} f(gy)\right)
\geq \sum_{i \in I'} \mu\left(B_{F_i}^\rho\left(y_i, \frac{1}{m}\right)\right) \quad (\text{since (4.8)})
\geq \mu\left(\bigcup_{i \in I'} B_{F_i}^\rho\left(y_i, \frac{1}{m}\right)\right) \quad (\text{since (4.7)})
\geq \mu(Z_m) \quad (\text{since (4.4)})
\geq \frac{1}{2} \mu(Z).
\]
The arbitrariness of the collection \( \Gamma \) leads to
\[ h_{(K, \delta), r}^r(\rho, G, f, Z) \geq \frac{1}{2} \mu(Z) > 0. \]
We deduce by the definition of \( h_r(\rho, G, f, Z) \) that
\[ h_r(\rho, G, f, Z) \geq s - \nu \quad \text{for all} \quad 0 < r < \frac{1}{2m}. \]
Hence
\[ h_{(\rho, G, f, Z)}^r = \limsup_{r \to 0} h_r(\rho, G, f, Z) \geq s - \nu. \]
The arbitrariness of \( \nu \) implies that
\[ h_{(\rho, G, f, Z)}^r \geq s. \]
Hence we complete the proof.

(2) We assume that \( \overline{h}_\mu^P(G, \rho, f, x) \leq s \) for all \( x \in Z \).
Let \( \epsilon > 0 \) and \( \nu > 0 \). To begin with, we prove the following fact.

**Fact 4.9.** Let \( z_0 \in Z \). If \( \overline{h}_\mu^P(G, \rho, f, z_0) \leq s \), then
\[
\limsup_{F} \frac{1}{|F|} \log \left( e^{f_F(z_0)} \mu(B_{F}(z_0, \epsilon))^{-1} \right) < s + \nu.
\]
We denote
\[ \psi(z_0, r) = \limsup_F \frac{1}{|F|} \log \left( e^{f_F(z_0)} \mu(B^0_F(z_0, r))^{-1} \right). \]

It is easy to see that
\[ \psi(z_0, r_1) \geq \psi(z_0, r_2) \quad \text{when} \quad 0 < r_1 < r_2. \]

Since \( \lim_{r \to 0} \psi(z_0, r) = h^0_\mu(G, \rho, f, z_0) < s + \nu \), there exists \( 0 < r_0 < \epsilon \) such that \( \psi(z_0, r_0) < s + \nu \) which implies that
\[ \psi(z_0, \epsilon) \leq \psi(z_0, r_0) < s + \nu. \]

So the fact is proved.

Using the fact that \( G \) is a countably infinite amenable group and Theorem 2.5 holds, there exists a nested Følner sequence \( \{F_n\} \) of \( G \). Let \( K \in \mathcal{F}(G) \) and \( \delta > 0 \).

For each \( z \in Z \) with \( h^0_\mu(G, \rho, f, z) \leq s \), by Fact 4.9, we have
\[ \limsup_F \frac{1}{|F|} \log \left( e^{f_F(z)} \mu(B^0_F(z, \epsilon))^{-1} \right) < s + \nu. \]

Thus, there exist \( K(z) \in \mathcal{F}(G) \) and \( \delta(z) > 0 \), which may depend on \( z \), such that
\[ (K(z), \delta(z)) > (K, \delta) \quad \text{i.e.} \quad K(z) \supseteq K, \delta(z) \leq \delta, \]

and
\[ \frac{1}{|F|} \log \left( e^{f_F(z)} \mu(B^0_F(z, \epsilon))^{-1} \right) < s + \nu. \quad (4.9) \]

As \( \{F_n\} \) is a Følner sequence of \( G \), there exists \( n(z) \in \mathbb{N} \) such that
\[ F_{n(z)} \in \mathcal{B}(K(z), \delta(z)) \subseteq \mathcal{B}(K, \delta). \quad (4.10) \]

It is immediate from (4.9) and (4.10) that
\[ \mu(B^0_{F_{n(z)}}(z, \epsilon))^{-1} < \exp \left( \left( (s + \nu)|F_{n(z)}| - f_{F_{n(z)}}(z) \right) \right). \quad (4.11) \]

Define a collection of Bowen balls by
\[ \mathcal{F} = \{B^0_{F_{n(z)}}(z, \epsilon) : z \in Z \}. \]

Recalling that \( \{F_n\} \) is nested, by Lemma 4.7, there is a subcollection
\[ \mathcal{G} = \{B^0_{F_{n_i}}(z, \epsilon) \}_{i \in I} \subseteq \mathcal{F} \quad (n_i = n(z_i)) \]

of pairwise disjoint Bowen balls satisfying
\[ Z \subseteq \bigcup_{i \in I} B^0_{F_{n_i}}(z_i, 3\epsilon). \quad (4.12) \]

According to the inequality (4.11), we have
\[ \mu(B^0_{F_{n_i}}(z_i, \epsilon)) > \exp \left( -(s + \nu)|F_{n_i}| + f_{F_{n_i}}(z_i) \right) \quad \forall \quad i \in I. \quad (4.13) \]

Recall the definition of \( h^{s+\nu}_{(K, \delta), 3\epsilon}(\rho, G, f_*, Z) \) as follows (see (3.26)):
\[ h^{s+\nu}_{(K, \delta), 3\epsilon}(\rho, G, f_*, Z) = \inf_{\Gamma} \sum_j \exp \left( -(s + \nu)|F_j| + f_*(B^0_{F_j}(x_j, 3\epsilon)) \right) \]

where the infimum is taken over all collections
\[ \Gamma = \{B^0_{F_j}(x_j, 3\epsilon) \}_{j \in J} \quad \text{with} \quad x_j \in X, \quad F_j \in \mathcal{B}(K, \delta) \]
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and

\[ \bigcup_{j \in J} B_{F_j}^p(x_j, 3\epsilon) \supseteq Z. \]

Here

\[ f_*(B_{F_j}^p(x_j, 3\epsilon)) = \inf_{y \in B_{F_j}^p(x_j, 3\epsilon)} f_{F_j}(y). \]

Since \( x_j \in B_{F_j}^p(x_j, 3\epsilon) \), it follows that

\[ f_*(B_{F_j}^p(x_j, 3\epsilon)) \leq f_{F_j}(x_j). \]  \hspace{1cm} (4.14)

So the definition of \( h^\nu_{(K, \delta), 3\epsilon}(\rho, G, f_*, Z) \) and the disjointness of \( \{B_{F_{n_i}}^p(z_i, 3\epsilon)\}_{i \in I} \) yield that

\[ h^\nu_{(K, \delta), 3\epsilon}(\rho, G, f_*, Z) \leq \sum_{i \in I} \exp \left( -(s + \nu)|F_{n_i}| + f_*(B_{F_{n_i}}^p(z_i, 3\epsilon)) \right) \]  \hspace{1cm} (since (4.12))

\[ \leq \sum_{i \in I} \exp \left( -(s + \nu)|F_{n_i}| + f_{F_{n_i}}(z_i) \right) \]  \hspace{1cm} (since (4.14))

\[ < \sum_{i \in I} \mu(B_{F_{n_i}}^p(z_i, \epsilon)) \]  \hspace{1cm} (since (4.13))

\[ \leq \mu(X) = 1. \]

From the above inequality, it follows that

\[ h_{3\epsilon}(\rho, G, f_*, Z) \leq s + \nu. \]

Taking \( \epsilon \to 0 \), we get

\[ h^P(\rho, G, f_*, Z) \leq s + \nu. \]

The arbitrariness of \( \nu \) implies that

\[ h^P(\rho, G, f_*, Z) \leq s. \]

From Theorem 3.16 we get \( h^P(\rho, G, f, Z) = h^P(\rho, G, f_*, Z) \leq s \). Hence the theorem is proved. \( \Box \)

5. **An example of Bowen topological pressure.** Let \( G \) be a countably infinite amenable group and \( A = \{1, \cdots, k\} \) be a finite set. We equip \( A \) with the discrete topology and \( A^G \) with the associated product topology. Let \( G \curvearrowright A^G \) be the left action of \( G \) on \( A^G \). This left action of \( G \) on \( A^G \) is called the \textit{G-shift} on \( A^G \). Define on \( A^G \) the continuous pseudometric

\[ \rho(x, y) = \begin{cases} 0, & \text{if } x(e) = y(e) \\ 1, & \text{if } x(e) \neq y(e). \end{cases} \]

Then it is easy to see \( \rho \) is a dynamically generating pseudometric.

Let \( \mu \) be the uniform distribution on \( A \), i.e., \( \mu(i) = \frac{1}{k} \) for \( i \in A \). By Daniell-Kolmogorov extension theorem, there is a unique Borel probability measure \( \mu^G \) on \( A^G \) which behaves as an ordinary product measure on Borel cylinder, i.e., if \( F \) is a nonempty finite subset of \( G \), \( \{A_s\}_{s \in F} \) is a collection of Borel subsets of \( A \), and \( \pi_F : A^G \to A^F \) is the coordinate restriction map then

\[ \mu^G \left( \pi_F^{-1} \left( \prod_{s \in F} A_s \right) \right) = \prod_{s \in F} \mu(A_s). \]
Theorem 5.1. Let $G$ be a countably infinite amenable group, $k$ a positive integer and $A = \{1, \cdots, k\}$. Let $\rho$ be the dynamically generating continuous pseudometric on $A^G$ defined as above. Let $\mu$ be the uniform distribution on $A$ and $\mu^G$ be the Borel probability measure on $A^G$ which is determined by $\mu$. Let $f \equiv C$ be a constant function on $A^G$.

For any Borel set $Z \subseteq A^G$, if $\mu^G(Z) > 0$, then
$$h_{\text{top}}^P(G, C, Z) = \log k + C.$$ 

Proof. Let $x \in A^G$ and $F$ be a nonempty finite subset of $G$ and $0 < \epsilon < 1$. It can easily be verified that
$$B^\rho_F(x, \epsilon) = \{y \in A^G : y|_{F^{-1}} = x|_{F^{-1}}\}.$$

Thus, we have
$$\mu^G(B^\rho_F(x, \epsilon)) = \left(\frac{1}{k}\right)^{|F|}$$

which implies that
$$h_{\mu^G}^P(G, \rho, C, x) = \lim_{\epsilon \to 0} \liminf_{F} \frac{1}{|F|} \log \mu^G(B^\rho_F(x, \epsilon)) + C = \log k + C,$$

$$h_{\mu^G}^P(G, \rho, C, x) = \lim_{\epsilon \to 0} \limsup_{F} \frac{1}{|F|} \log \mu^G(B^\rho_F(x, \epsilon)) + C = \log k + C.$$

Since the Borel set $Z$ satisfies $\mu^G(Z) > 0$ and Theorem 4.8 holds, we deduce that
$$h^P(\rho, G, C, Z) = \log k + C.$$

From Theorem 3.17 we have
$$h_{\text{top}}^P(G, C, Z) = \log k + C.$$

This complete the proof of Theorem 5.1.

Example 5.2. Define a Borel subset $Z_0$ of $A^G$ as follows: Fix $g \in G$.
$$Z_0 := \{x \in A^G : x(e) = x(g) = 1\}.$$

It is clear that $Z_0$ is not $G$-invariant and $\mu^G(Z_0) = (1/k)^2 > 0$. From Theorem 5.1 we have
$$h_{\text{top}}^P(G, C, Z_0) = \log k + C.$$

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