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Author(s): Radchenko, Danylo; Rodriguez Villegas, Fernando

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Independence polynomials and hypergeometric series

Danylo Radchenko and Fernando Rodriguez Villegas

Abstract

Let $\Gamma$ be a simple graph and $I_{\Gamma}(x)$ its multivariate independence polynomial. The main result of this paper is the characterization of chordal graphs as the only $\Gamma$ for which the power series expansion of $I_{\Gamma}^{-1}(x)$ is Horn hypergeometric.

1.

The motivation for this paper came from the desire to extend to several variables the study of families of motives (known as hypergeometric motives) attached to one variable hypergeometric series \cite{17, 18}. The connection between the classical complex function theory (hypergeometric series) and its arithmetic counterpart (hypergeometric motives) could be thought of having started with Schwarz’s \cite{24} celebrated determination of when the classical $2F_1$ series is algebraic (later extended to all one-variable hypergeometric series by Beukers–Heckman \cite{3}). Indeed, their combined work completely classifies weight zero hypergeometric motives in one variable.

The key connection between these two worlds (classical complex function theory and arithmetic) is the monodromy representation arising from analytic continuation. It seemed natural to try to understand the most basic possible case: when the series is the expansion of a rational function and hence the monodromy action is trivial. (For comparison, the algebraic, weight zero cases correspond to the monodromy group being finite as opposed to trivial.)

In the one-variable case we are essentially limited to the geometric series

$$1 + x + x^2 + \cdots = \frac{1}{1 - x},$$

which is not really interesting from this point of view. It turns out that in several variables a lot more happens. To start with, it is not even entirely clear how to determine in advance which hypergeometric series are rational (though there is a conjectural characterization \cite{7, 8}) and coming up with interesting examples is not that straightforward.

In working out some simple cases, we noticed that they often had the form $1/I(x)$ for some polynomial $I$, which on close inspection only involved very special monomials. We further noticed that these monomials seemed to correspond to independent sets of vertices in certain graphs. In this paper, we make this observation precise and prove it (see Theorem 2.2).

A key ingredient in the proof of our main result is the beautiful fact pointed out by de Bruijn (see §6) that a one-variable power series with rational coefficients cannot be hypergeometric if the exponential constant of growth of its coefficients is an irrational number.

In Section 7 we include some miscellaneous results and observations that arose in the process of proving our main theorem. Somewhat mysteriously we encounter some classical varieties like Cayley’s surface, Igusa’s quartic, Segre’s primal and Coble’s variety as well as a connection to wild character varieties.

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In this paper by a graph we will mean a simple graph; i.e. a usual graph with no multiple edges or loops. Let \( \Gamma \) be a graph on \( n \) vertices. We label the vertices of \( \Gamma \) and attach to the \( i \)th vertex an independent variable \( x_i \). The independence polynomial \([2, \text{Chapter 6}]\) of \( \Gamma \) is a polynomial in the variables \( x = (x_1, \ldots, x_n) \) defined as follows:

\[
I_\Gamma(x) = \sum_I x^I,
\]

where \( I \subseteq \{1, \ldots, n\} \) runs over the independent sets of vertices of \( \Gamma \) and

\[
x^I := \prod_{i \in I} x_i.
\]

An independent set \( I \subseteq \{1, \ldots, n\} \) is a subset of vertices of \( \Gamma \) such that no pair of elements of \( I \) are connected by an edge in \( \Gamma \). Note that \( I_\Gamma \) has constant term 1 for every graph \( \Gamma \).

The independent polynomial plays a role in statistical mechanics: it is the partition function of a lattice gas in the hardcore case; its vanishing locus is also important because of its connection to the Lovász local lemma in probability theory (see \([23]\)).

For example, if \( \Gamma := L_n \) is the line graph

\[
L_n \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet
\]

then \( I_\Gamma(x) = \sum_I x^I \), where \( I \subseteq \{1, \ldots, n\} \) runs over the subsets containing no consecutive numbers \( i, i+1 \) for \( i = 1, \ldots, n-1 \). The first few values of \( I_{L_n} \) are

\[
I_{L_1} = 1 + x_1, \\
I_{L_2} = 1 + x_1 + x_2, \\
I_{L_3} = 1 + x_1 + x_2 + x_3 + x_1x_3, \\
I_{L_4} = 1 + x_1 + x_2 + x_3 + x_4 + x_1x_3 + x_2x_4 + x_1x_4.
\]

These polynomials are in fact, up to re-indexing, the multivariate Fibonacci polynomials defined by the recursion

\[
F_n = F_{n-1} + x_{n-2}F_{n-2}, \quad n > 1, \quad F_0 = 0, \quad F_1 = 1.
\]

We have \( I_{L_n} = F_{n+2} \).

A graph \( \Gamma \) is called chordal if it has no induced subgraph isomorphic to the cycle graph \( C_n \) with \( n \geq 4 \) \([13, \text{Chapter 4, \S 1}]\). By induced subgraph defined by a subset \( J \) of vertices of \( \Gamma \) we mean the subgraph \( \Gamma(J) \subseteq \Gamma \) obtained by deleting from \( \Gamma \) the vertices not in \( J \) and all their attached edges. The cycle graph \( C_n \) consists of \( n > 2 \) vertices \( 1, 2, \ldots, n \) with an edge joining \( i \) with \( i+1 \), where the indices are read modulo \( n \).

For example, the following graph is not chordal
since removing the central vertex leaves the graph $C_4$

![Graph Image](image)

The following graph on the other hand is chordal

![Graph Image](image)

Finally, a power series

$$ F(x) = \sum_{m \geq 0} c_m x^m, \quad m = (m_1, \ldots, m_n), \quad x^m := x_1^{m_1} \cdots x_n^{m_n} $$

is called Horn hypergeometric [14] if $c_m$ is nonzero for all $m \geq 0$ and

$$ \frac{c_{m+e_i}}{c_m}, \quad e_i := (0, \ldots, 1, \ldots, 0) $$

is a rational function of $m_1, \ldots, m_n$ for every $i = 1, \ldots, n$.

Remark 2.1. In the definition of Horn hypergeometric the assumption that $c_m$ is nonzero could be relaxed (see [1] for a general discussion) but it simplifies the arguments and is all we will need.

We can now state our main result.

Theorem 2.2. The following are equivalent.

1) The graph $\Gamma$ is chordal.
2) The power series expansion

$$ \frac{1}{I_\Gamma(x)} = \sum_{m \geq 0} (-1)^{|m|} c_m x^m, \quad |m| := m_1 + \cdots + m_n, $$

is Horn hypergeometric.
3) The power series expansion

$$ I_\Gamma(x)^{-s} = \sum_{m \geq 0} (-1)^{|m|} c_m(s) x^m, \quad |m| := m_1 + \cdots + m_n, $$

is Horn hypergeometric for all $s \notin \mathbb{Z}_{\leq 0}$.

The proof of the main theorem is spread over the next several sections. In Corollary 4.3 we prove that 1) $\Rightarrow$ 3). We then prove that 2) $\Rightarrow$ 1), which takes longer and is completed in Proposition 6.3. This finishes the proof as the remaining implication 3) $\Rightarrow$ 2) is trivial.

We should mention that by a theorem of Cartier–Foata [6] the coefficients $c_m$ in Theorem 2.2, expression 2), have a combinatorial interpretation and are in particular positive integers. Indeed, consider the algebra $A_\Gamma$ generated over $\mathbb{Q}$ by elements $w_1, \ldots, w_n$ with relations

$$ w_i w_j = w_j w_i, $$
if and only if $i$ and $j$ are not connected by an edge in $\Gamma$. Then Cartier–Foata [6] prove that
\[
\sum_j (-1)^{|I|} w^I = \sum_j w^J,
\]
(3)
where the sum on the left runs over subsets $I \subseteq \{1, \ldots, n\}$ such that all $w_i$ with $i \in I$ commute with each other, whereas the sum on the right runs over distinct monomials $w^J$ in the algebra.

Now consider the abelianization map
\[ \Phi : A_\Gamma \to \mathbb{Q}[x_1, \ldots, x_n] \]
\[ w_i \mapsto x_i. \]
Applied to the left hand side of (3) we obtain $I_\Gamma(-x)^{-1}$. Hence we deduce that
\[ c_m = \# \{ J \mid \Phi(w^J) = x^m \}. \]
In other words, $c_m$ counts all the rearrangements of the monomial $w_1^{m_1} \cdots w_n^{m_n}$ that give distinct monomials in $A_\Gamma$.

For example, if $\Gamma = K_n$ is the complete graph on $n$ vertices then $I_{K_n}(x) = 1 + x_1 + \cdots + x_n$ and
\[
\frac{1}{1 - x_1 - \cdots - x_n} = \sum_{m \geq 0} (m_1 + \cdots + m_n)! \frac{m_1! \cdots m_n!}{m_1! \cdots m_n!} x^m.
\]
In fact, the right-hand side is Horn hypergeometric and this is a simple instance of the main theorem since $K_n$ is clearly chordal.

3.

To any integral matrix $A \in \mathbb{Z}^{n \times n}$ we associate the following Nahm system of equations:
\[
1 - z_i = x_i \prod_{j=1}^n z_j^{a_{i,j}}, \quad i = 1, \ldots, n.
\]
(4)
We call it a Nahm system, because it specializes (for $A$ symmetric and positive-definite) to the system considered by Nahm in his conjecture on the modularity of certain associated $q$-hypergeometric series when $x_i = 1$ (see [19, p. 42], [26, equation (25)]). We think of the system as expressing the $z$ as algebraic functions of the $x$ and we are interested in the corresponding power series expressions for $z_i$. Note that $z_i = 1$ when $x_i = 0$, so these power series have constant term equal to 1.

It follows from the multivariate Lagrange inversion (see [22] for details) that for any $s_1, \ldots, s_n$ we have
\[
z_1^{s_1} \cdots z_n^{s_n} = \frac{1}{D} \sum_{m \geq 0} (-1)^{|m|} \prod_{j=1}^n \binom{s_j + a_j(m)}{m_j} x^m,
\]
(5)
where
\[ D := \sum_{m \geq 0} (-1)^{|m|} \prod_{j=1}^n \binom{a_j(m)}{m_j} x^m, \]
(6)
and
\[ a_j(m) = \sum_{i=1}^n a_{i,j} m_i. \]
are the linear forms determined by the columns of $A$. Here we interpret the binomial coefficients as polynomials of the top entry
\[
\binom{x}{m} := \frac{x(x-1)\cdots(x-m+1)}{m!},
\]
for any non-negative integer $m$. We also have (see [22])
\[
D^{-1} = \det \left( I_n + \text{diag} \left( \frac{1-z_1}{z_1}, \ldots, \frac{1-z_n}{z_n} \right) A \right),
\]
where $I_n$ is the identity matrix of size $n$.

If $A$ is upper triangular with 1 along the diagonal then we can recursively solve for the $z$ in terms of the $x$. In particular, $z_i$ is a rational function of $x_1, \ldots, x_n$. It also follows easily from (7) that in this case
\[
D = z_1 \cdots z_n.
\]

It appears to be rare for non-upper-triangular matrices $A$ (more precisely, for matrices that are not permutation-similar to an upper triangular matrix) to give rise to rational $z$, but it does happen. A simple but interesting example (related to the 5-term relation for the dilogarithm, see the forthcoming paper by F. Rodriguez Villegas) is the following. Take $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ then one easily checks that
\[
D = \frac{1}{1-x_1x_2}, \quad z_1 = \frac{1-x_1}{1-x_1x_2}, \quad z_2 = \frac{1-x_2}{1-x_1x_2}.
\]

We have the following recursion for $D$.

**Proposition 3.1.** Let $A$ be upper-triangular with 1 along the diagonal. Let $A^*$ be the $(n-1) \times (n-1)$ matrix obtained by removing the $n$th row and column of $A$ and let $D^*$ be the corresponding value of $D$ as in (6) for $A^*$. Then
\[
D(x_1, \ldots, x_n) = \frac{1}{1+x_n} D^* \left( \frac{x_1}{(1+x_n)^{a_{1.n}}}, \ldots, \frac{x_{n-1}}{(1+x_n)^{a_{n-1,n}}} \right).
\]

**Proof.** The claim follows from
\[
\sum_{m \geq 0} (-1)^m \binom{a + m}{m} x^m = \frac{1}{(1+x)^{a+1}}.
\]
We leave the details to the reader. \qed

The following corollary is immediate.

**Corollary 3.2.** If $A$ is an upper-triangular matrix with 1 along the diagonal, then $D$ is a rational function of $x_1, \ldots, x_n$.

We associate to a graph $\Gamma$ with $n$ labelled vertices the following upper-triangular matrix $A = (a_{i,j})$ with 1 along the diagonal:
\[
a_{i,j} = \begin{cases} 
1, & i = j, \\
1, & i \sim j, \quad i < j, \\
0, & \text{otherwise},
\end{cases}
\]
where $i \sim j$ means that the two vertices $i$ and $j$ are connected by an edge in $\Gamma$. In other words, $A$ is basically the top half of the adjacency matrix of $\Gamma$. 


A (reverse) perfect elimination ordering of the vertices of $\Gamma$ is a labelling of the vertices such that for each $1 \leq k \leq n$ the subgraph $\Gamma_k \subseteq \Gamma$ induced by the set of vertices with labels $1 \leq i < k$ connected to the $k$th vertex is a complete graph [13, Chapter 4, §2], [12].

For example, the following is a perfect elimination ordering of the graph $\Gamma$:

These are the corresponding subgraphs $\Gamma_k$.

**Proposition 4.1.** Let $\Gamma$ be a graph with a given perfect elimination ordering of its vertices. Let $A$ be the corresponding upper triangular matrix defined above. Then

$$D(x_1, \ldots, x_n) = \frac{1}{I_\Gamma(x_1, \ldots, x_n)},$$

(11)

where $I_\Gamma$ is the independence polynomial of $\Gamma$ and $D$ is defined in (6).

**Proof.** We prove the claim by induction in $n$ with the recursion (9) as the key step, the case of one vertex being trivial. We identify the vertices of $\Gamma$ with $\{1, \ldots, n\}$ using the given perfect elimination ordering. Let $\Gamma^*$ be the graph obtained from $\Gamma$ by deleting the vertex $n$ and all of its attached edges.

Let $I^* \subseteq \{1, \ldots, n-1\}$ be an independent set of $\Gamma^*$. It can contain at most one vertex connected to $n$ in $\Gamma$ since by definition of perfect elimination ordering any two such vertices are connected by an edge. Moreover, $I = I^* \cup \{n\}$ is an independent set of $\Gamma$ if and only if no vertex in $I^*$ is connected to $n$. In terms of the independence polynomial this can be formulated as follows. Let

$$y_i := \begin{cases} x_i/(1 + x_n), & i \sim n, \\ x_i, & \text{otherwise.} \end{cases}$$

Then

$$I_\Gamma(x_1, \ldots, x_n) = (1 + x_n)I_{\Gamma^*}(y_1, \ldots, y_{n-1}).$$

This is precisely the recursion satisfied by $\frac{1}{D}$ in terms of $\frac{1}{D^*}$ by (9) and the claim follows. □

To prove that 1) implies 3) in Theorem 2.2 we need the following (compare with [21], §12.4).
Corollary 4.2. With the hypothesis of the proposition we have for all \( s \)
\[
I_{\Gamma}(x_1, \ldots, x_n)^{-s} = \sum_{m \geq 0} (-1)^{|m|} \prod_{j=1}^{n} \left( s - 1 + a_j(m) \right)^{m_j} x^m.
\] (12)

In particular, if \( s \) is not an integer \( \leq 0 \), then \( I_{\Gamma}(x_1, \ldots, x_n)^{-s} \) is Horn hypergeometric.

Proof. The first claim follows from the above proposition by using (5) and (8). Since \( a_j(m) \) is an integer and \( a_j(m) \geq m_j \), the binomial coefficient in (12) can vanish only if \( s \) is a non-positive integer. \( \square \)

As an example of (11) we have the following expansion. For any positive integer \( n \)
\[
\frac{1}{F_{n+2}(x_1, \ldots, x_n)} = \sum_{m \geq 0} (-1)^{|m|} \prod_{j=2}^{n} \left( m_j + m_j - 1 \right)^{m_j} x^m,
\]
where \( F_n \) is the Fibonacci polynomial (2). It is clear that the labeling

\[
\begin{array}{ccccccc}
 & & & & & & \\
 & & & & & & \\
 & & & & & & \\
1 & \cdots & 2 & \cdots & n-1 & n \\
& & & & & &
\end{array}
\]

of the vertices of \( L_n \) is a perfect elimination ordering.

Note that Corollary 4.2 implies, in particular, that if a graph \( \Gamma \) has a perfect elimination ordering then its independence polynomial \( I_{\Gamma} \) satisfies that the expansion of \( I_{\Gamma}^{-s} \) in power series is Horn hypergeometric for all \( s \not\in \mathbb{Z}_{\leq 0} \). Not every graph has a perfect elimination ordering. It is a remarkable fact [13, Theorem 4.1] that a graph has a perfect elimination ordering if and only if it is chordal. We conclude the following.

Corollary 4.3. Let \( \Gamma \) be a chordal graph. Then its independence polynomial \( I_{\Gamma} \) satisfies that the expansion of \( I_{\Gamma}^{-s} \) in a power series is Horn hypergeometric for all \( s \not\in \mathbb{Z}_{\leq 0} \).

This is one direction in our main theorem. To prove the reverse direction will take a bit more work.

The first observation is that if \( \Gamma(J) \subseteq \Gamma \) is the subgraph induced by a subset \( J \) of its vertices then \( I_{\Gamma(J)} \) is obtained from \( I_{\Gamma} \) by setting \( x_j = 0 \) for every \( j \) not in \( J \). It follows that the independence polynomial \( I_{\Gamma} \) of a non-chordal graph \( \Gamma \) specializes to the independence polynomial \( I_n \) of the cycle graph \( C_n \) for some \( n \geq 4 \) by setting appropriate variables equal to zero.

The second observation is that for a power series the property of being Horn hypergeometric is preserved by the specialization to zero of any number of its variables. Hence, to finish the proof of the main theorem it is enough to show that \( I_n \) is not Horn hypergeometric for any \( n \geq 4 \).

Notice that \( I_3^{-1}(x) \) is Horn hypergeometric. Indeed, we have
\[
I_3(x_1, x_2, x_3) = 1 + x_1 + x_2 + x_3,
\]
\[
\frac{1}{I_3(x_1, x_2, x_3)} = \sum_{m_1, m_2, m_3 \geq 0} (-1)^{|m|} \frac{(m_1 + m_2 + m_3)!}{m_1!m_2!m_3!} x_1^{m_1} x_2^{m_2} x_3^{m_3}.
\]

In fact, we have a case of the strong law of small numbers: the cycle graph \( C_n \) and the complete graph \( K_n \) coincide for \( n = 3 \) but not for any \( n \geq 4 \).
Recall that \( I_n(x_1, \ldots, x_n) \) denotes the independence polynomial of the cycle graph \( C_n \) for \( n \geq 3 \). It will be convenient to extend the definition and include
\[
I_1(x) := 1 + x_1, \quad I_2(x_1, x_2) := 1 + x_1 + x_2.
\]

We would like to describe the coefficients in the power series expansion of \( I_n(x) - 1 \). We will make use of the Nahm system (4) associated to the following matrix:
\[
A := \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 & 1 \\
1 & 1 & 0 & \cdots & 0 & 0 \\
0 & 1 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & 0 \\
0 & 0 & 0 & \cdots & 1 & 1 \\
\end{pmatrix}, \quad a_{i,j} := \begin{cases} 
1, & j = i \text{ or } j \equiv i - 1 \text{ mod } n, \\
0, & \text{otherwise.}
\end{cases}
\]

Namely, consider the system
\[
\begin{cases}
1 - z_1 &= x_1 z_1 z_n, \\
1 - z_2 &= x_2 z_2 z_1, \\
\vdots &= \vdots \\
1 - z_n &= x_n z_n z_{n-1}.
\end{cases}
\]

Then
\[
D = \sum_{m \geq 0} (-1)^{|m|} \binom{m_1 + m_2}{m_1} \binom{m_2 + m_3}{m_2} \cdots \binom{m_n + m_1}{m_n} x^m.
\]

**Proposition 5.1.** Let
\[
u := z_1 \cdots z_n, \quad v := (-1)^n x_1 \cdots x_n
\]
and
\[
M := \begin{pmatrix}
1 & -1 \\
-x_1 & 0 \\
0 & 1 \\
-x_2 & 0 \\
\vdots & \vdots \\
-x_n & 0 \\
\end{pmatrix}.
\]

Then the following statements hold:

(i) \( \text{tr}(M) = I_n(x_1, \ldots, x_n) \), \( \det(M) = v. \)

(ii) \( M \binom{z_n}{1} = uv \binom{z_n}{1}. \)

(iii) The polynomial \( X^2 - I_n(x_1, \ldots, x_n)X + v \) has roots \( u^{-1} \) and \( uv \).

(iv) \( I_n(x_1, \ldots, x_n) = u^{-1} + uv \)

(v) \( D^{-1} = u^{-1} - uv \)

(vi) \( D^{-2} = I_n(x_1, \ldots, x_n)^2 - (-1)^n 4x_1 \cdots x_n. \)

(vii) \[
\frac{1}{\sqrt{I_n(x_1, \ldots, x_n)^2 - (-1)^n 4x_1 \cdots x_n}} = \sum_{m \geq 0} (-1)^{|m|} \binom{m_1 + m_2}{m_1} \binom{m_2 + m_3}{m_2} \cdots \binom{m_n + m_1}{m_n} x^m.
\]
Proof. (i) The second identity is immediate. For the first identity we expand the trace as

$$\text{tr}(M) = \sum_{i_1, \ldots, i_n} a^{(1)}_{i_1} \cdots a^{(n)}_{i_n},$$

where $a^{(k)}_{i_j}$ are the entries of the $k$-th matrix in the product defining $M$, and note that if we encode $(i_1, \ldots, i_n)$ by $I = \{j \in \{1, \ldots, n\} : i_j = 2\}$, then the $I$th term vanishes if $I$ contains two indices consecutive modulo $n$ (since $a^{(k)}_{22} = 0$) and is equal to $\prod_{i \in I} x_i$ otherwise.

(ii) This follows by a direct calculation from (14) and (13).

(iii) It follows from (ii) that $uv$ is an eigenvalue of $M$. From the determinant value in (i) the other eigenvalue is $u^{-1}$. The quadratic polynomial is the characteristic polynomial of $M$ by (i).

(iv) Follows immediately from (iii).

(v) By (7) using the system equations $D^{-1}$ is the determinant of the $n \times n$ matrix $W = (w_{i,j})$ with $w_{i,j} = z_i^{-1}$ for $i = \pi$ and $x_1 z_i$ for $j = i - 1 \mod n$. Consequently,

$$D^{-1} = \prod_{i=1}^n z_i^{-1} - (-1)^n \prod_{i=1}^n x_i z_i,$$

which is what we wanted to prove.

(vi) From (iii) and (v) we see that $D^{-2}$ is the discriminant of the quadratic polynomial in (iii) and the claim follows.

(vii) This is just a restatement of (vi).

The expansion (vii) was proved earlier by Carlitz [5, 20, §4.4].

6.

As mentioned, we are interested in the coefficients of the power series expansion of $I_n(x_1, \ldots, x_n)^{-1}$. To obtain these we will extend the results of the previous section. Let

$$c_{m,j} := \frac{(m_1 + m_2)! \cdots (m_{n-1} + m_n)! (m_n + m_1)!}{(m_1 + j)! (m_1 - j)! \cdots (m_n + j)! (m_n - j)!},$$

Note that

$$c_{m,-j} = c_{m,j}.$$ Let

$$R(z; x_1, \ldots, x_n) := \sum_{m \geq 0} \sum_{|j| \leq \min(m)} (-1)^{|m|} c_{m,j} z^j x^m$$

be the generating series of these coefficients. Note that the coefficient of $z^0$ of $R(z; x_1, \ldots, x_n)$ is equal to $D$.

Fix some non-negative integer $j$. The coefficient of $z^j$ in $R$ can be expressed in the form

$$v^j \sum_{m \geq 0} (-1)^{|m|} \binom{m_1 + m_2 + 2j}{m_1} \binom{m_2 + m_3 + 2j}{m_2} \cdots \binom{m_n + m_1 + 2j}{m_n} x^m$$

after writing $m_i$ for $m_i - j$, where recall that $v = (-1)^n x_1 \cdots x_n$. By Lagrange inversion (5) we find that this in turn equals $D v^j (z_1 \cdots z_n)^j$.  \[\square\]
To simplify the notation let \( w := v(z_1 \ldots z_n)^2 \). From the coefficients of \( R \) in powers of \( z \) we can reconstruct the series; summing the geometric series we find that
\[
R = D \left( 1 + \frac{wz}{1 - wz} + \frac{wz^{-1}}{1 - wz^{-1}} \right).
\]
Alternatively,
\[
R^{-1} = \frac{1}{D} \left( \frac{1 + w}{1 - w} - \left( \frac{1}{z^2} + \frac{1}{z^{-2}} \right)^2 \frac{w}{1 - w^2} \right) = \frac{1}{D} \frac{(w - z)(w - z^{-1})}{1 - w^2}.
\] (16)

**Proposition 6.1.** The power series \( R(z; x_1, \ldots, x_n) \) is the Taylor expansion of a rational function. More precisely,
\[
R(z; x_1, \ldots, x_n) = \frac{I_n(x_1, \ldots, x_n)}{I_n(x_1, \ldots, x_n)^2 - (-1)^n \left( \frac{1}{z^2} + \frac{1}{z^{-2}} \right)^2 x_1 \cdots x_n}.
\] (17)

**Proof.** Using (16) it is enough to show that
\[
I_n(x_1, \ldots, x_n) = \frac{1}{D} \left( \frac{1 + w}{1 - w} \right) = Dv \left( \frac{1 - w^2}{w} \right)
\]
and this follows easily from Proposition 5.1 noting that \( w = vu^2 \).

**Corollary 6.2.** The following power series expansion holds
\[
I_n(x_1, \ldots, x_n)^{-1} = \sum_{m \geq 0} (-1)^{|m|} \sum_{|j| \leq \min(m)} (-1)^j \binom{m_1 + m_2}{m_1 + j} \binom{m_2 + m_3}{m_2 + j} \cdots \binom{m_n + m_1}{m_n + j} x^m.
\] (18)

**Proof.** It follows from the proposition by taking \( z = -1 \).

We are now ready to finish the proof of our main result.

**Proposition 6.3.** For \( n \geq 4 \) the power series expansion of \( I_n(x)^{-1} \) is not Horn hypergeometric.

**Proof.** Let \( c_m \) be the coefficients in the expansion of \( I_n(x)^{-1} \)
\[
I_n(x)^{-1} = \sum_{m \geq 0} (-1)^{|m|} c_m x^m.
\]
To prove the claim it is enough to show that if the one variable series (the main diagonal)
\[
H_n(x) := \sum_{k \geq 0} c_k x^k, \quad c_k := c_{(k, \ldots, k)}
\]
is Horn hypergeometric then \( n \leq 3 \).

The case \( n = 1 \) being trivial we may assume \( n > 1 \). By Corollary 6.2 we have
\[
c_k = \sum_{|j| \leq k} (-1)^j \binom{2k}{k + j}^n, \quad n > 1.
\]
These numbers are known as de Bruijn numbers in the literature and are denoted by \( S(n, k) \).
De Bruijn in his book [10] computed the asymptotic behaviour of \( S(n, k) \) for fixed \( n \) and
large $k$. It follows from his computation that
\[ c_{k+1}/c_k \to \kappa_n, \quad k \to \infty, \]
where
\[ \kappa_n := (2 \cos(\pi/2n))^{2n}. \]

We now apply de Bruijn’s argument: if $H_k(x)$ is Horn hypergeometric then $k \mapsto c_{k+1}/c_k$ is a rational function of $k$, and since $c_{k+1}/c_k \in \mathbb{Q}$ for all $k \geq 0$ (recall that by discussion following (3) $c_k \neq 0$ for $k \geq 0$) this rational function is defined over $\mathbb{Q}$ (since the coefficients of the rational function, after multiplying by the denominator, are determined by a system of linear equations over $\mathbb{Q}$). Therefore, $\kappa_n$ has to be rational. On the other hand, using the above formula for $\kappa_n$ one can show that it is rational only for $n \leq 3$, thus proving the claim. (Here is a short proof of the last statement: $2 \cos(\pi/2n)$ is an algebraic integer, it generates a real cyclotomic extension of $\mathbb{Q}$ of degree $\varphi(4n)/2$, and all of its $\varphi(4n)/2$ conjugates are real numbers in $(-2, 2)$. Therefore, if $\kappa_n$ is rational, then there can be at most two conjugates, since their absolute values have to be equal, and hence $\varphi(4n) \leq 4$, thus $n \leq 3$.)

\[ \square \]

7.

In this section we sketch very briefly several miscellaneous results stemming from the previous discussion; these will be expanded on in a later publication.

1) We can expand the right-hand side of (17) in the variable $t := \frac{1}{2}(z^{1/2} + z^{-1/2})$ and compare coefficients to the left-hand side to obtain some interesting identities. We will make this explicit for $n = 3$ where the identity generalizes that of Dixon [25, p. 156] (corresponding to the appropriate formulation for $k = 0$).

**Proposition 7.1.** For $k > 0$ and $m = (m_1, m_2, m_3)$ a triple of non-negative integers we have
\[
\frac{1}{k!} \sum_{j=0}^{\min(m)} (-1)^j \frac{(2j + k)(j + k - 1)!}{j!} \left( \frac{m_1 + m_2 + k}{m_2 + k + j} \right) \left( \frac{m_2 + m_3 + k}{m_3 + k + j} \right) \left( \frac{m_3 + m_1 + k}{m_1 + k + j} \right)
\]
\[ = \frac{(k + m_1 + m_2 + m_3)!}{k!m_1!m_2!m_3!}. \]

**Proof.** We give a sketch of the proof and leave the details to the reader. With the definition of $t$
\[ T_{2j}(t) = \frac{1}{2} \left( z^j + z^{-j} \right), \]
where $T_{2j}(t)$ is the $(2j)$th Chebyshev polynomial. We have
\[ T_{2j}(t) = j \sum_{l=0}^{j} (-1)^{l+j} \frac{(j + l - 1)!}{(j - l)! (2l)!} (2t)^{2l}, \quad j > 0. \]

Expanding the right-hand side of (17) in the variable $t$ we find in general for any $l > 0$
\[
\frac{v^l}{I_n(x)^{2l+1}} = \frac{1}{2(2l)!} \sum_{m \geq 0} (-1)^{|m|} \sum_{j=l}^{\min(m)} (-1)^{l+j} \frac{(j + l - 1)!}{(j - l)!} \left( \frac{m_1 + m_2}{m_1 + j} \right) \left( \frac{m_2 + m_3}{m_2 + j} \right) \ldots \left( \frac{m_n + m_1}{m_n + j} \right) x^m. 
\]
Specializing to \( n = 3 \), expanding both sides in and comparing coefficients yields the claim for \( k \) even. A similar argument works for \( k \) odd. Alternatively, since both sides of the identity are polynomial functions in \( k \), we obtain the case of odd \( k \) by interpolation. \( \square \)

It is curious that the visible fourfold symmetry on the right-hand side is far from clear on the left-hand side.

2) We state a few intriguing elementary identities that arise from the Horn singularity analysis in the case of the cyclic graphs.

The Horn–Kapranov [16] parametrization determined by the hypergeometric series in (15) is the following:

\[
\begin{align*}
\phi_0 &= (\lambda_1 + \lambda_0) \cdots (\lambda_n + \lambda_0) \\
\phi_1 &= \frac{(\lambda_1 - \lambda_0)(\lambda_1 + \lambda_0)}{(\lambda_1 + \lambda_2)(\lambda_1 + \lambda_n)} \\
\vdots & \vdots \\
\phi_n &= \frac{(\lambda_n - \lambda_0)(\lambda_n + \lambda_0)}{(\lambda_n + \lambda_1)(\lambda_n + \lambda_{n-1})}
\end{align*}
\]

(19)

Since the singularities of the series occur at the points of vanishing of the denominator of

\[
\Delta(z; x_1, \ldots, x_n) := I_n(x_1, \ldots, x_n)^2 - (-1)^n \left( z^{\frac{1}{2}} - z^{-\frac{1}{2}} \right)^2 x_1 \cdots x_n
\]

(20)

we have \( \Delta(\phi_0; \phi_1, \ldots, \phi_n) = 0 \). It follows that

\[
I_n(\phi_1, \ldots, \phi_n) = \frac{\prod_{i=1}^n (\lambda_i + \lambda_0) + \prod_{i=1}^n (\lambda_i - \lambda_0)}{\prod_{i=1}^n (\lambda_i + \lambda_i+1)},
\]

where the indices are read modulo \( n \). This identity follows from part (iv) of Proposition 5.1, since for \( x_i = \phi_i \) we can solve the Nahm system explicitly by taking

\[
z_i = \frac{\lambda_i + \lambda_{i+1}}{\lambda_i + \lambda_0}.
\]

If we set \( \lambda_0 = 0 \) and \( u_i := \lambda_{i+1}/\lambda_i \) then

\[
\begin{align*}
\phi_0 &= 1 \\
\phi_1 &= -1/(1 + u_1)(1 + u_n^{-1}) \\
\vdots & \vdots \\
\phi_n &= -1/(1 + u_n)(1 + u_{n-1}^{-1})
\end{align*}
\]

(21)

If we relax the condition that \( u_1 \cdots u_n = 1 \) that is a consequence of their definition and treat them as independent variables then plugging in the rational map (21) we obtain

\[
I_n \left( \frac{1}{(1 + u_1)(1 + u_n^{-1})}, \ldots, \frac{1}{(1 + u_n)(1 + u_{n-1}^{-1})} \right) = \frac{1 + u_1 \cdots u_n}{\prod_{i=1}^n (1 + u_i)}.
\]

(22)

Again, this identity follows from part (iv) of Proposition 5.1 by taking \( z_i = 1 + u_i^{-1} \). Writing this identity explicitly for \( n = 2 \) and \( n = 3 \) we find

\[
1 - \frac{1}{(1 + u_1)(1 + u_2^{-1})} - \frac{1}{(1 + u_2)(1 + u_1^{-1})} = \frac{1 + u_1 u_2}{(1 + u_1)(1 + u_2)}
\]
and

\[ 1 - \frac{1}{(1 + u_1)(1 + u_2)} - \frac{1}{(1 + u_2)(1 + u_1)} = \frac{1}{(1 + u_1)(1 + u_2)(1 + u_3)}. \]

The naive analogue of this identity does not hold for more than three variables as the left-hand side no longer is the specialization of \( I_n \).

3) The varieties defined by the vanishing of \( \Delta(z; x_1, \ldots, x_n) \) seem to be quite interesting. Here we discuss a few cases in the special case of \( z = 1 \) for small \( n \), where the varieties are classical. Let

\[ \Delta(x_1, \ldots, x_n) := \Delta(1; x_1, \ldots, x_n) = I_n(x_1, \ldots, x_n)^2 - (-1)^n 4x_1 \cdots x_n. \]

It is a polynomial of degree \( n \). Let \( \Delta_n(x_0, x_1, \ldots, x_n) \) be the homogenization of \( \Delta \) and \( X_n \subseteq \mathbb{P}^n \) its zero locus.

For \( n = 2 \) we have

\[ \Delta_2(x_0, x_1, x_2) = x_0^2 + x_1^2 + 2x_0x_1 + 2x_0x_2 - 2x_1x_2 \]

and \( X_2 \subseteq \mathbb{P}^2 \) is a smooth conic.

For \( n = 3 \) we have

\[ \Delta_3(x_0, x_1, x_2, x_3) = x_0( x_0 + x_1 + x_2 + x_3)^2 + 4x_1x_2x_3. \]

We find that \( X_3 \subseteq \mathbb{P}^3 \) is a cubic surface with the four double points

\[ (-1: 1: 1: 1), \quad (-1: 0: 0: 1), \quad (-1: 0: 1: 0), \quad (-1: 1: 0: 0). \]

It follows that \( X_3 \) is projectively isomorphic to the Cayley surface \( [11, \text{p. 500}], [15, \text{p. 75}] \).

For \( n = 4 \) we find that \( X_4 \subseteq \mathbb{P}^4 \) is a quartic threefold non-singular except for 15 lines. These lines meet in appropriate groups of three lines at 15 points. The resulting configuration is known as the Cremona–Richmond configuration \([9, \S 9]\). The variety \( X_4 \) is isomorphic to the Castelnuovo–Richmond quartic \([11, \text{p. 532}]\) (also known as the Igusa quartic \([15, \S 3.3]\)).

4) Let us analyze the Nahm system \((13)\) a bit more closely. We have the following.

**Proposition 7.2.** (i) Let \( n > 1 \). The Nahm system \((13)\) has the following solution in \( K := F(\sqrt{\Delta}) \), where \( F := \mathbb{Q}(x_1, \ldots, x_n) \) and

\[ \Delta(x_1, \ldots, x_n) := I_n(x_1, \ldots, x_n)^2 - (-1)^n 4x_1 \cdots x_n. \]

For \( i = 1, \ldots, n \)

\[ z_i = -b_i + \frac{\Delta}{2a_i}, \quad a_i := x_iF_n(x_{i+2}, x_{i+3}, \ldots, x_{i-1}), \quad b_i := I_n(x_1, \ldots, -x_i, \ldots, x_n), \quad \text{where} \quad F_n \text{ is the Fibonacci polynomial (2) in } n - 2 \text{ variables.} \]

(ii) For \( i = 1, \ldots, n \) let \( c_i := -F_n(x_i, x_{i+1}, \ldots, x_{i-3}) \) then

\[ \Delta = b_i^2 - 4a_i c_i. \]

We see that \( z_1, \ldots, z_n \) are rational functions on the double cover \( Z_n \) of \( \mathbb{P}^n \) ramified at \( X_n \). These double covers are also classical varieties: \( Z_3 \) is Segre’s primal \([11, \text{p. 530}], [15, \S 3.2] \) and \( Z_4 \) is Coble’s variety \([15, \S 3.5]\).

5) There is a connection between the varieties \( X_n \) of 4) and wild character varieties \([4, \S 5]\). Consider the following matrix in invertible variables \( y_1, \ldots, y_n \)

\[ Y := \begin{pmatrix} y_{i-1}^{-1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_{i-1}^{-1} & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} y_{i-1}^{-1} & 1 \\ 1 & 0 \end{pmatrix}. \]
If we insert in between each pair of factors the diagonal matrices
\[(\begin{pmatrix} -y_i & 0 \\ 0 & 1 \end{pmatrix}) \begin{pmatrix} 1 & 0 \\ 0 & -y_i \end{pmatrix}, \quad i = 1, \ldots, n \]
using that
\[
\begin{pmatrix} 1 & 0 \\ 0 & -x \end{pmatrix} \begin{pmatrix} y^{-1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -y & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ xy & 0 \end{pmatrix}
\]
we obtain
\[
\begin{pmatrix} 1 & 0 \\ 0 & -y_n \end{pmatrix} Y \begin{pmatrix} 1 & 0 \\ 0 & -y_n \end{pmatrix}^{-1} = \begin{pmatrix} -1 & 1 \\ xy & 0 \end{pmatrix} \begin{pmatrix} y^{-1} & 0 \\ 0 & y^{-1} \end{pmatrix} \begin{pmatrix} -1 & 1 \\ y_1 y_n & 0 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ y_2 y_1 & 0 \end{pmatrix} \cdots \begin{pmatrix} -1 & 1 \\ y_n y_{n-1} & 0 \end{pmatrix},
\]
where \( y := (-1)^n y_1 \cdots y_n \).

Assume now that \( n = 2k \) is even. We may insert \( P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) appropriately in the definition of \( Y \) (24) and find that
\[
Y = \begin{pmatrix} 1 & y_1^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ y_2^{-1} & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & 0 \\ y_{n-1}^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

It follows by (25) that the matrices \( Y \) and \( M \) of (14) are related and hence by (26) \( M \) is related to the equations involved in the definition of certain wild character varieties (see (26)). The difference is that we impose the condition that the characteristic polynomial of \( M \) has a double root instead of prescribing its entries. This should correspond to taking the Zariski closure of the \( 2 \times 2 \) Jordan block instead of a torus element as the target of the moment map.

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Danylo Radchenko
ETH Zurich
Mathematics Department
Zürich 8092
Switzerland
danradchenko@gmail.com

Fernando Rodriguez Villegas
The Abdus Salam International Centre for Theoretical Physics
Strada Costiera 11
Trieste 34151
Italy
villegas@ictp.it

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