TURING-HOPF BIFURCATION OF A CLASS OF MODIFIED LESLIE-GOWER MODEL WITH DIFFUSION

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Abstract. In this paper, the dynamics of a class of modified Leslie-Gower model with diffusion is considered. The stability of positive equilibrium and the existence of Turing-Hopf bifurcation are shown by analyzing the distribution of eigenvalues. The normal form on the centre manifold near the Turing-Hopf singularity is derived by using the method of Song et al. Finally, some numerical simulations are carried out to illustrate the analytical results. For spruce budworm model, the dynamics in the neighbourhood of the bifurcation point can be divided into six categories, each of which is exactly demonstrated by the numerical simulations. Then according to this dynamical classification, a stable spatially inhomogeneous periodic solution has been found, which can be used to explain the phenomenon of periodic outbreaks of spruce budworm.

1. Introduction. Since introduced by Leslie and Gower[13, 14], the Leslie-Gower (LG) model
\begin{align}
\frac{dN}{dt} &= AN - BN^2 - CNP, \\
\frac{dP}{dt} &= DP - \frac{EP^2}{N},
\end{align}
and its various modifications have received great attention [18, 27, 7, 17, 2, 3, 1, 5, 23]. Model (1) assumes that the carrying capacity of the prey $N$ is limited by a fixed value $A$ and the carrying capacity for the predator $P$ is directly proportional to the prey $N$.

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In [19], Murray proposed a class of modified Leslie-Gower model:
\[
\frac{dN(x,t)}{dt} = rN(x,t) \left( 1 - \frac{N(x,t)}{K} \right) - R(N(x,t))P(x,t),
\]
\[
\frac{dP(x,t)}{dt} = \delta P(x,t) \left( 1 - \frac{hP(x,t)}{N(x,t)} \right),
\]
where \(R(N)\) can be one of the predation terms below
\[
R(N) = AN, \quad R(N) = ANB + N, \quad R(N) = AN^2B + N^2, \quad R(N) = A(1 - e^{\alpha N}).
\]
Obviously, model (2) can become into the Leslie-Gower model (1) when \(R(N) = AN\).

In the paper, taking into account the inhomogeneous distribution of the preys and predators in different spatial locations, we consider the following modified Leslie-Gower model with diffusion and Neumann boundary conditions

\[
\begin{cases}
\frac{\partial N(x,t)}{\partial t} = d_1 \Delta N(x,t) + rN(x,t) \left( 1 - \frac{N(x,t)}{K} \right) - R(N(x,t))P(x,t), \\
\frac{\partial P(x,t)}{\partial t} = d_2 \Delta P(x,t) + \delta P(x,t) \left( 1 - \frac{hP(x,t)}{N(x,t)} \right), \\
\frac{\partial N(x,t)}{\partial \nu} = \frac{\partial P(x,t)}{\partial \nu} = 0, & x \in \partial \Omega, t > 0,
\end{cases}
\]

where \(d_1, d_2 > 0\) is the diffusion coefficient characterizing the rate of the spatial dispersion of the prey and predator population, respectively, \(r, \delta > 0\) is the linear birth rate of prey and predator, respectively, \(K > 0\) is the carrying capacity of prey population, \(h > 0\) is the proportionality coefficient of prey density to the carrying capacity for the predator, \(\nu\) is the outward unit normal vector on \(\partial \Omega, \Omega = (0,l\pi)\) and the function \(R\) satisfies

\((H_0)\) : \(R(0) = 0, R(s)\) is strictly monotone increasing and adequately smooth for \(s \in [0, \infty)\).

Clearly, the assumption \((H_0)\) holds when \(R(N)\) is one of the predation terms in (3).

As is well known, Turing-Hopf singularity is a degenerate case where Hopf and Turing bifurcations occur simultaneously. At the moment, the corresponding characteristic equation has a pair of simple purely imaginary roots and a simple zero root. There exist very rich dynamics near Turing-Hopf singularity, which includes stable constant equilibrium, nonconstant steady state, spatially homogeneous and inhomogeneous periodic solutions. Due to the increasing interests in the studying of the bifurcation phenomena in the reaction-diffusion systems arising from the biology, chemistry and physics [4, 9, 10, 11, 15, 16, 21, 25, 26], and also the fact that the existence of stable spatially inhomogeneous periodic solution can be used to explain the periodic fluctuation of biological populations, we will focus our thoughts on the degenerate case in this paper.

We would also like to mention that Faria presented an approach in reference [6], by which an explicit normal form on the centre manifold near an equilibrium of
partial functional differential equations can be calculated, especially for the investigation of generic Hopf bifurcations. However, Faria did not address approaches of calculating normal forms for partial functional differential equations with Turing-Hopf singularity. Although, Song et al. presented a rigorous procedure for calculating the normal form associated with the Turing-Hopf bifurcation of partial functional differential equations in [20]. There are still very few studies on Turing-Hopf bifurcation of the model with practical significance (see [22, 24]).

The rest of the paper is organized as follows. In Section 2, we investigate the existence and stability of positive equilibrium and the existence of Turing-Hopf bifurcation. In Section 3, we compute the normal forms on the centre manifold for Turing-Hopf bifurcation by using the method in [20]. In Section 4, we carry out some numerical simulations to support and extend our analytical results.

2. Stability and Turing-Hopf bifurcation. In this section, we consider the stability and Turing-Hopf bifurcation of the following system

\begin{align*}
\frac{\partial u(x,t)}{\partial t} &= d_1 \Delta u(x,t) + ru(x,t)(1-u(x,t))-f(u(x,t))v(x,t), \\
\frac{\partial v(x,t)}{\partial t} &= d_2 \Delta v(x,t) + \delta v(x,t)\left(1 - \frac{v(x,t)}{ru(x,t)}\right), \\
\frac{\partial u(x,t)}{\partial \nu} &= \frac{\partial v(x,t)}{\partial \nu} = 0, \quad x \in \partial \Omega, t > 0,
\end{align*}

(5)

where \( f \) satisfies the assumption \((H_0)\). Clearly, system (4) can be written as system (5) by setting

\[ u(x,t) = \frac{1}{K} N(x,t), \quad v(x,t) = \frac{hr}{K} P(x,t). \]

The positive equilibrium \( E^* = (u_*, v_*) \) of system (5) satisfies

\[ 1 - u_* = f(u_*), \quad v_* = ru_. \]

(6)

Since the function

\[ g(u) = 1 - u - f(u), \]

is strictly monotone decreasing with \( g(0) = 1 \) and \( g(\infty) = -\infty \), we have the following conclusion.

**Theorem 2.1.** For any \( r, \delta, d_1, d_2, l > 0 \) and the function \( f \), Eq.(5) has and only has one positive constant equilibrium \( E^*(u_*, v_*) \), where \( u_* \) and \( v_* \) satisfy (6).

In the following, we analyze the stability and the existence of Turing-Hopf bifurcation for the positive equilibrium \( E^* \). Notice that the value of \( u_* \) only depends on the function \( f \) and has nothing to do with the parameters \( r \) and \( \delta \). Therefore, we take the parameters \( r \) and \( \delta \) as bifurcation parameters. The characteristic equations for the positive equilibrium \( E^* \) are the following sequence of quadratic equations

\[ \lambda + d_1 \left(\frac{n}{T}\right)^2 - r(1 - 2u_* - u_* f'(u_*)) = 0, n \in \mathbb{N}_0, \]

\[ -\delta r \lambda + d_2 \left(\frac{n}{T}\right)^2 + \delta = 0, n \in \mathbb{N}_0, \]

(7)

where \( \mathbb{N}_0 = \{0\} \cup \mathbb{N} \) and \( \mathbb{N} \) is the positive integer set. That is,

\[ \Delta_n(\lambda) = \lambda^2 + T_n \lambda + h_n = 0, n \in \mathbb{N}_0, \]
where
\[ T_n = (d_1 + d_2)\left(\frac{n}{l}\right)^2 + T_0, \]
\[ h_n = d_1d_2\left(\frac{n}{l}\right)^4 + \left(d_1\delta - d_2r(1 - 2u_\ast - u_\ast f'(u_\ast))\right)\left(\frac{n}{l}\right)^2 + h_0, \]
(8)
with
\[ T_0 = \delta - r(1 - 2u_\ast - u_\ast f'(u_\ast)), \]
\[ h_0 = \delta ru_\ast(1 + f'(u_\ast)). \]
(9)
Denote
\[ \alpha(u_\ast) = 1 - 2u_\ast - u_\ast f'(u_\ast), \]
\[ \beta(u_\ast) = u_\ast(1 + f'(u_\ast)) > 0. \]

Then, we have that \( T_0 = \delta - \alpha(u_\ast)r \) and \( h_0 = \beta(u_\ast)r\delta \). Therefore, if \( \alpha(u_\ast) \leq 0 \), we obtain \( T_0 > 0 \) and \( h_0 > 0 \) for any \( \delta, r > 0 \). Which implies that the positive equilibrium \( E^* \) of system (5) without diffusion is asymptotically stable for any \( \delta, r > 0 \). If \( \alpha(u_\ast) > 0 \), we know that \( h_0 > 0 \) such that for system (5) without diffusion, Hopf bifurcation occurs when \( T_0 = 0 \). That is Hopf bifurcation occurs at the line of \( \delta = \mathcal{H}_0(r) \), where
\[ \mathcal{H}_0(r) = \alpha(u_\ast)r, \quad r > 0, \]
(10)
It is easy to verify that the following transversality condition holds:
\[ \frac{d\text{Re}\{\lambda(r)\}}{dr}\bigg|_{\delta=\mathcal{H}_0(r)} = \frac{\alpha(u_\ast)}{2} > 0. \]
(11)
Therefore, the result on the system (5) without diffusion follows immediately.

**Lemma 2.2.** Assume that \( \mathcal{H}_0(r) \) is defined by Eq. (10). For system (5) without diffusion \( (d_1 = d_2 = 0) \), we have
(i) if \( \alpha(u_\ast) \leq 0 \) holds, then the positive equilibrium \( E^* \) is asymptotically stable for any \( \delta, r > 0 \).
(ii) if \( \alpha(u_\ast) > 0 \) holds, then the positive equilibrium \( E^* \) is asymptotically stable for \( \delta > \mathcal{H}_0(r) \) and unstable for \( \delta < \mathcal{H}_0(r) \); the Hopf bifurcation line is defined by \( \delta = \mathcal{H}_0(r) \).

If \( \alpha(u_\ast) \leq 0 \), one can easily obtain that \( T_n > 0 \) and \( h_n > 0 \) for \( n \in \mathbb{N}_0 \). Therefore, we have the following result.

**Theorem 2.3.** If \( \alpha(u_\ast) \leq 0 \), then the positive equilibrium \( E^* \) of (5) is asymptotically stable for any \( \delta, r, d_1, d_2, l > 0 \).

In the following, we focus on the diffusion-driven instability and the bifurcation analysis for system (5) under the case of \( \alpha(u_\ast) > 0 \). For convenience, we define
\[ \Lambda = \alpha^2(u_\ast)(d_2 - d_1)^2 - 4\alpha(u_\ast)\beta(u_\ast)d_1d_2, \]
(12)
and make some hypotheses as follows:
\[ (C_1) \; d_2 \leq d_1, \]
\[ (C_2) \; d_2 > d_1 \text{ and } \Lambda < 0, \]
\[ (C_3) \; d_2 > d_1 \text{ and } \Lambda > 0. \]

First it is the case for no Turing instability.
Proof. It is easy to see from (8) that for any \( n \in \mathbb{N}_0, T_n > 0 \) can be satisfied provided that \( \delta > H_0(r) \) and \( h_n = 0 \) is equivalent to

\[
\delta = S_n(r) = \frac{\alpha(u_*)d_2\left(\frac{n}{T}\right)^2 - d_1d_2\left(\frac{n}{T}\right)^4}{d_1\left(\frac{n}{T}\right)^2 + \beta(u_*)r}, \quad r > 0, n \in \mathbb{N}_0.
\]

(13)

Clearly, if all lines determined by \( \delta = S_n(r) \) lie below the Hopf bifurcation line defined by \( \delta = H_0(r) \) for \( r > 0 \), then there is no Turing instability. After a simple calculation, we obtain that

\[
H_0(r) - S_n(r) = \frac{1}{d_1\left(\frac{n}{T}\right)^2 + \beta(u_*)r}P_n(r),
\]

where

\[
P_n(r) = \alpha(u_*)\beta(u_*)r^2 - \alpha(u_*)(d_2 - d_1)\left(\frac{n}{T}\right)^2r + d_1d_2\left(\frac{n}{T}\right)^4.
\]

Obviously, if \( d_2 \leq d_1 \), we have \( P_n(r) > 0 \) for any \( n \in \mathbb{N}_0 \) and \( r > 0 \), which means that \( H_0(r) > S_n(r) \) for any \( n \in \mathbb{N}_0 \) and \( r > 0 \). Nextly, we consider the case of \( d_2 > d_1 \) and \( \Lambda < 0 \). Firstly, we have \( S_0(r) = 0 \) for any \( r > 0 \). This leads to \( H_0(r) > S_0(r) \) for any \( r > 0 \). For \( n \in \mathbb{N} \), we can obtain that \( P_n(r) > 0 \) for any \( r > 0 \) if \( \Lambda < 0 \). Thus, \( H_0(r) > S_n(r) \) for any \( n \in \mathbb{N} \) and \( r > 0 \). The proof is complete.

Then it is the case for the existence of Turing instability.

Theorem 2.5. If \( \alpha(u_*) > 0 \) and \( (C_3) \) hold, then there will be diffusion-driven Turing instability and Turing-Hopf bifurcation at some Turing-Hopf singularity. More precisely, denote

\[
\gamma_0 = p_1^-; \quad \gamma_n = \begin{cases} q^+(n, n+1), & \text{if } p_{n+1}^- \leq p_n^+; \\ p_{n+1}^-, & \text{if } p_{n+1}^- > p_n^+. \end{cases} \quad n \in \mathbb{N},
\]

and

\[
\mathcal{L}_0(r) = H_0(r), \quad r \in (0, \gamma_0],
\]

\[
\mathcal{L}_n(r) = \begin{cases} S_n(r), & \text{if } p_{n+1}^- \leq p_n^+; \\ \hat{\mathcal{L}}_n(r), & \text{if } p_{n+1}^- > p_n^+. \end{cases} \quad n \in \mathbb{N}, \quad r \in (\gamma_{n-1}, \gamma_n],
\]

where \( p_n^+ \) and \( q^+(n, m) \) are defined below and

\[
\hat{\mathcal{L}}_n(r) = \begin{cases} S_n(r), & r \in (\gamma_{n-1}, p_n^+]; \\ H_0(r), & r \in [p_n^+, \gamma_n], \end{cases} \quad n \in \mathbb{N},
\]

then we have the following results:

(i) The positive equilibrium \( E^* \) is asymptotically stable for any \( r > 0, \delta > \mathcal{L}_n(r), n \in \mathbb{N}_0 \).

(ii) The critical line of Turing instability is \( \delta = \mathcal{L}_n(r)(\mathcal{L}_n(r) > H_0(r)), n \in \mathbb{N} \) and Turing instability occurs for \( r > 0, H_0(r) < \delta < \mathcal{L}_n(r), n \in \mathbb{N} \).

(iii) System (5) undergoes Turing-Hopf bifurcation at the point \( (p_1^-, \alpha(u_*)p_1^-) \); If \( p_{n+1}^- > p_n^+, n \in \mathbb{N}, \) then system (5) undergoes Turing-Hopf bifurcation at the point \( (p_n^+, \alpha(u_*)p_n^+), (p_{n+1}^+, \alpha(u_*)p_{n+1}^+) \).
Proof. From the proof of Theorem 2.4, we know that Turing instability occurs for the condition (C3) holds. In this case, \( H_0(r) - S_0(r) = 0 \) has no positive root and \( H_0(r) - S_n(r) = 0, n \in \mathbb{N} \) has two positive roots

\[
p_{n}^{\pm} = \frac{\alpha(u_\ast)(d_2 - d_1) \pm \sqrt{\Lambda}}{2\alpha(u_\ast)\beta(u_\ast)}\left(\frac{n}{r}\right)^2,
\]

such that

\[
\begin{cases}
H_0(r) < S_n(r), & \text{if } p_n^- < r < p_n^+,
\end{cases}
\]

\[
\begin{cases}
H_0(r) > S_n(r), & \text{if } 0 < r < p_n^- \text{ or } r > p_n^+.
\end{cases}
\]

Assume that \( m,n \in \mathbb{N}, m > n \). Then we have

\[
S_m(r) - S_n(r) = \frac{\alpha(u_\ast)rd_2\left(\frac{n}{r}\right)^2 - d_1d_2\left(\frac{n}{r}\right)^4}{d_1\left(\frac{n}{r}\right)^2 + \beta(u_\ast)r} = \frac{d_2(m^2 - n^2)}{(d_1\left(\frac{n}{r}\right)^2 + \beta(u_\ast)r)(d_1\left(\frac{n}{r}\right)^2 + \beta(u_\ast)r)^2}Q_{nm}(r),
\]

where

\[
Q_{nm}(r) = \frac{\alpha(u_\ast)\beta(u_\ast)r^2 - d_1\beta(u_\ast)m^2 + n^2}{d_1\left(\frac{n}{r}\right)^2 + \beta(u_\ast)r}.r - d_1^2m^2n^2 - \frac{d_2(m^2 - n^2)}{(d_1\left(\frac{n}{r}\right)^2 + \beta(u_\ast)r)^2}.
\]

It is easy to see that for any \( m,n \in \mathbb{N} \), \( Q_{nm}(r) \) has and only has one positive zero

\[
q^+(n,m) = \frac{d_1\beta(u_\ast)m^2 + n^2}{d_1\left(\frac{n}{r}\right)^2 + \beta(u_\ast)r} + \sqrt{\frac{d_1^2\beta^2(u_\ast)\left(m^2 + n^2\right)}{d_1\left(\frac{n}{r}\right)^2 + \beta(u_\ast)r} + \frac{4d_1^2\alpha(u_\ast)\beta(u_\ast)m^2n^2}{d_1\left(\frac{n}{r}\right)^2 + \beta(u_\ast)r}}.
\]

such that

\[
\begin{cases}
S_m(r) < S_n(r), & \text{if } 0 < r < q^+(n,m),
\end{cases}
\]

\[
\begin{cases}
S_m(r) > S_n(r), & \text{if } r > q^+(n,m).
\end{cases}
\]

It follows that \( \delta = S_m(r) \) and \( \delta = S_n(r) \) have and only have one intersecting point for any \( m,n \in \mathbb{N}, m > n \).

From (14) and (15), we know that \( p_n^\pm \) increases and tends towards positive infinity with the increase of \( n \) and \( q^+(n,m) \) increases and tends towards positive infinity with the increase of \( m \) for a fixed \( n \in \mathbb{N} \).

Furthermore, we can obtain the transversality condition below

\[
\left. \frac{d\text{Re}\{\lambda(r)\}}{dr} \right|_{\delta = S_n(r)} = \frac{d_1d_2\left(\frac{n}{r}\right)^4 + d_1\delta\left(\frac{n}{r}\right)^2}{T_n} > 0, \text{ if } \delta > H_0(r).
\]

According to the above analysis, it is clear that for \( 0 < r \leq \gamma_0 = p_1^- \), the boundary of stable region of the positive equilibrium \( E^+ \) is \( H_0(r) \). If \( p_2^- \leq p_1^- \), then for \( \gamma_0 < r \leq \gamma_1 = q^+(1,2) \), the boundary of stable region is \( S_1(r) \) (see Fig.1(a)). Otherwise, when \( \gamma_0 < r \leq \gamma_1 = p_2^- \), the boundaries consist of \( S_1(r) \) with \( \gamma_0 < r \leq p_1^+ \) and \( H_0(r) \) with \( p_1^+ < r \leq \gamma_1 \) (see Fig.1(b)). By using the mathematical induction, we can obtain that the whole boundaries of stable region consist of \( L_n(r), n \in \mathbb{N}_0 \). This, together with the transversality condition (16), completes the proof of (i) and (ii).

Obviously, when \( (r,\delta) = (p_1^-,\alpha(u_\ast)p_1^-) \), \( \Delta_0(\lambda) = 0 \) has a pair of purely imaginary roots \( \pm i \sqrt{T_0} \) and \( \Delta_1(\lambda) = 0 \) has a root \( \lambda = 0 \) with a negative real root \( \lambda = -T_1 \) and all other roots have negative real parts. Together with the transversality conditions (11) and (16), we can arrive at the conclusion that system (5) undergoes Turing-Hopf bifurcation at \( (p_1^-,\alpha(u_\ast)p_1^-) \). If \( p^{n+1}_{n} > p^n_{n} \) holds, \( \Delta_n(\lambda) = 0 \) and
\( \Delta_{n+1}(\lambda) = 0 \) has a zero root under the condition \((r, \delta) = (p_1^+, \alpha(u_*)p_1^+)\) and \((r, \delta) = (p_{n+1}, \alpha(u_*)p_{n+1}^-)\), respectively. Meanwhile all other roots, except a pair of purely imaginary roots \( \pm i\sqrt{n_0} \) of \( \Delta_0(\lambda) = 0 \), have negative real parts. Thus, \((p_1^+, \alpha(u_*)p_1^+)\) and \((p_{n+1}, \alpha(u_*)p_{n+1}^-)\) are in the same situation. This completes the proof of (iii).

\[ \Delta_{n+1}(\lambda) = 0 \]

3. Normal forms for Turing-Hopf bifurcation. According to Theorem 2.5, we know that for given parameters \( d_1, d_2, l \) and function \( f \) such that \( \alpha(u_*) > 0 \) and \( (C_3) \) hold, there is at least one Turing-Hopf bifurcation point in the \( r - \delta \) plane. Denote the Turing-Hopf bifurcation point as \( (r_*, \delta_*) \), where \( \delta_* = \alpha(u_*)r_* \) and \( (r_*, \delta_*) \) is an intersecting point of the lines \( \delta = H_0(r) \) and \( \delta = S_{n_1}(r) \). Next we shall compute the normal forms for Turing-Hopf bifurcation by using the method in [20]. For convenience, we introduce the perturbation parameters \( \mu_1, \mu_2 \) by setting \( r = r_* + \mu_1 \) and \( \delta = \delta_* + \mu_2 \) such that \( (\mu_1, \mu_2) = (0, 0) \) is the Turing-Hopf bifurcation point in the perturbation plane of \( \mu_1 \) and \( \mu_2 \). Then system (5) becomes

\[
\begin{align*}
\frac{\partial u}{\partial t} &= d_1 \Delta u + (r_* + \mu_1)u(1 - u) - f(u)v, \\
\frac{\partial v}{\partial t} &= d_2 \Delta v + (\delta_* + \mu_2)v \left( 1 - \frac{v}{(r_* + \mu_1)u_*} \right).
\end{align*}
\] (17)

The positive equilibrium of system (17) becomes \((u_*, (r_* + \mu_1)u_*)\).

Making the change of variables by the translation

\[
\bar{u} = u - u_* \quad \text{and} \quad \bar{v} = v - (r_* + \mu_1)u_*,
\]

and dropping the bars, (17) is transformed into

\[
\begin{align*}
\frac{\partial u}{\partial t} &= d_1 \Delta u + (r_* + \mu_1)(u + u_*)(1 - (u + u_*)) - f(u + u_*)(v + (r_* + \mu_1)u_*), \\
\frac{\partial v}{\partial t} &= d_2 \Delta v + (\delta_* + \mu_2)(v + (r_* + \mu_1)u_*) \left( 1 - \frac{v + (r_* + \mu_1)u_*}{(r_* + \mu_1)(u + u_*)} \right).
\end{align*}
\] (18)

It follows from Section 2 that for system (18), when \( \mu_1 = \mu_2 = 0 \), \( \Delta_0(\lambda) = 0 \) has a pair of purely imaginary roots \( \pm i\omega_c \) with \( \omega_c = \sqrt{\beta(u_*)r_*}\delta_* \), \( \Delta_{n_*}(\lambda) = 0 \) has a zero root \( \lambda = 0 \) and a negative real root \( \lambda = -T_{n_*} \), and if \( n \neq 0, n_* \), all roots of \( \Delta_n(\lambda) = 0 \) have negative real parts.
Define the real-valued Sobolev space
\[ \mathcal{X} = \{ (u, v) \in H^2(0, l\pi), \frac{\partial u}{\partial r} = \frac{\partial v}{\partial r} = 0 \text{ at } x = 0, l\pi \}, \]
with the inner product
\[ [U, V] = \int_0^{l\pi} (u_1 v_1 + u_2 v_2) dx, \text{ for } U = (u_1, u_2)^T, V = (v_1, v_2)^T \in \mathcal{X}. \]
Then system (18) can be written as the following differential equation in the abstract space \( \mathcal{C} = C(\mathbb{R}, \mathcal{X}) \)
\[ \dot{U} = \mathcal{L}U + \tilde{F}(U, \mu), \]
where \( U = (u, v)^T, \mu = (\mu_1, \mu_2), D = \text{diag}(d_1, d_2), \mathcal{L}U = D\Delta U + L_0(U), \tilde{F}(U, \mu) = L(\mu)(U) - L_0(U) + F(U, \mu), \)
and
\[ L(\mu) = \begin{pmatrix} \alpha(u_*) (r_* + \mu_1) & -f(u_*) \\ (r_* + \mu_1) (\delta_* + \mu_2) & -(\delta_* + \mu_2) \end{pmatrix}, \]
and
\[ F(U, \mu) = \begin{pmatrix} F^{(1)}(u, v, \mu_1, \mu_2) \\ F^{(2)}(u, v, \mu_1, \mu_2) \end{pmatrix}, \]
with
\[ F^{(1)}(u, v, \mu_1, \mu_2) = -(r_* + \mu_1) u^2 - f(u + u_*) (v + (r_* + \mu_1) u_* + f(u_*) v) + (r_* + \mu_1) u_* f(u_*) + (r_* + \mu_1) u_* f'(u_*) u, \]
\[ F^{(2)}(u, v, \mu_1, \mu_2) = (\delta_* + \mu_2) (v + (r_* + \mu_1) u_*) (1 - \frac{v + (r_* + \mu_1) u_*}{r_* (r_* + \mu_1) (u + u_*)}) + (\delta_* + \mu_2) v - (r_* + \mu_1) (\delta_* + \mu_2) u. \]
For the formal Taylor expansions of \( L, \)
\[ L(\mu) = L_0 + \mu_1 L_1^{(1,0)} + \mu_2 L_1^{(0,1)} + \frac{1}{2} \mu_1^2 L_2^{(2,0)} + 2 \mu_1 \mu_2 L_2^{(1,1)} + \mu_2^2 L_2^{(0,2)} + \cdots \]
we have
\[ L_0 = \begin{pmatrix} \alpha(u_*) r_* & -f(u_*) \\ r_* \delta_* & -\delta_* \end{pmatrix}, \quad L_1^{(1,0)} = \begin{pmatrix} \alpha(u_*) & 0 \\ \delta_* & 0 \end{pmatrix}, \quad L_1^{(0,1)} = \begin{pmatrix} 0 & 0 \\ r_* & -1 \end{pmatrix}. \]
It is well known that the eigenvalues of \( D\Delta \) on \( \mathcal{X} \) are
\[ \delta_n^{(j)} = -d_j \left( \frac{n}{l} \right)^2, \quad j = 1, 2, n \in \mathbb{N}_0, \]
and the corresponding normalized eigenfunctions are given by
\[ \beta_n^{(j)} = \gamma_n(x) e_j, \quad \gamma_n(x) = \frac{\cos \left( \frac{n \pi x}{l} \right)}{\| \cos \left( \frac{n \pi x}{l} \right) \|_{2,2}}, \]
where \( e_j \) is the unit coordinate vector of \( \mathbb{R}^2 \) and \( n \) is often called wave number.
Let
\[ \mathcal{M}_n(\lambda) = \begin{pmatrix} \lambda + d_4 \left( \frac{n}{l} \right)^2 - \alpha(u_*) r_* & f(u_*) \\ -r_* \delta_* & \lambda + d_2 \left( \frac{n}{l} \right)^2 + \delta_* \end{pmatrix}. \]
Then, by a straightforward calculation, we can obtain that $\xi_0 \in \mathbb{C}^2$ and $\xi_{n_*} \in \mathbb{R}^2$ are the eigenvectors associated with the eigenvalues $i\omega_c$ and 0 respectively, and $\eta_0 \in \mathbb{C}^2$ and $\eta_{n_*} \in \mathbb{R}^2$ are the corresponding adjoint eigenvectors, where
\[
\xi_0 = \frac{1}{f(u_*)} \begin{pmatrix} f(u_*) \\ \delta_* - i\omega_c \end{pmatrix}, \quad \eta_0 = \frac{1}{2\omega_c} \begin{pmatrix} \omega_c - i\delta_* \\ if(u_*) \end{pmatrix},
\]
\[
\xi_{n_*} = \frac{1}{f(u_*)} \begin{pmatrix} f(u_*) \\ \delta_* - d_1 \left( \frac{n_*}{T} \right)^2 \end{pmatrix}, \quad \eta_{n_*} = \frac{1}{(d_1 + d_2)\left( \frac{n_*}{T} \right)^2} \begin{pmatrix} \delta_* + d_2 \left( \frac{n_*}{T} \right)^2 \\ - f(u_*) \end{pmatrix},
\]
such that
\[
\langle \Psi_1, \Phi_1 \rangle = I_2, \quad \langle \Psi_2, \Phi_2 \rangle = 1,
\]
where $I_2$ is an $2 \times 2$ identity matrix and
\[
\Phi_1 = (\xi_0, \xi_0), \quad \Phi_2 = \xi_{n_*}, \quad \Psi_1 = \text{col}(\eta_0^T, \eta_0^T), \quad \Psi_2 = \eta_{n_*}^T.
\]
Then the phase space $\mathcal{X}$ for (19) can be decomposed as
\[
\mathcal{X} = \mathcal{P} \oplus \mathcal{Q}, \quad \mathcal{P} = \text{Im} \pi, \quad \mathcal{Q} = \text{Ker} \pi,
\]
where $\dim \mathcal{P} = 3$ and $\pi : \mathcal{X} \rightarrow \mathcal{P}$ is the projection defined by
\[
\pi(U) = \left( \Phi_1 \left( \Psi_1, ([U, \beta_0^{(1)}], [U, \beta_0^{(2)}])^T \right) \right)^T \beta_0
\]
\[
+ \left( \Phi_2 \left( [U, \beta_{n_*}^{(1)}], [U, \beta_{n_*}^{(2)}])^T \right) \right)^T \beta_{n_*},
\]
with $\beta_0 = \text{col}(\beta_0^{(1)}, \beta_0^{(2)})$ and $\beta_{n_*} = \text{col}(\beta_{n_*}^{(1)}, \beta_{n_*}^{(2)})$.

According to (23), $U \in \mathcal{X}$ can be decompose as
\[
U = \left( \Phi_1 \left( z_1 \right) \right)^T \begin{pmatrix} \beta_0^{(1)} \\ \beta_0^{(2)} \end{pmatrix} + (z_3 \Phi_2)^T \begin{pmatrix} \beta_{n_*}^{(1)} \\ \beta_{n_*}^{(2)} \end{pmatrix} + w
\]
\[
= (z_1 \xi_0 + z_2 \xi_0)\gamma_0(x) + z_3 \xi_{n_*} \gamma_{n_*}(x) + \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}
\]
\[
= (\Phi_1, \Phi_2) \begin{pmatrix} z_1 \gamma_0(x) \\ z_2 \gamma_0(x) \\ z_3 \gamma_{n_*}(x) \end{pmatrix} + \begin{pmatrix} u_1 \\ u_2 \end{pmatrix},
\]
where
\[
\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \left( \Psi_1, ([U, \beta_0^{(1)}], [U, \beta_0^{(2)}])^T \right), \quad z_3 = \left( \Psi_2, ([U, \beta_{n_*}^{(1)}], [U, \beta_{n_*}^{(2)}])^T \right).
\]
Letting $\Phi = (\Phi_1, \Phi_2)$, $z_x = (z_1 \gamma_0(x), z_2 \gamma_0(x), z_3 \gamma_{n_*}(x))^T$, (24) can be rewritten as
\[
U = \Phi z_x + w.
\]
In order to simply the notation, let
\[
\begin{pmatrix} \tilde{F}, \beta_0^{(1)} \\ \tilde{F}, \beta_0^{(2)} \end{pmatrix}_{v=n_*} = \text{col} \left( \begin{pmatrix} \tilde{F}, \beta_0^{(1)} \\ \tilde{F}, \beta_0^{(2)} \end{pmatrix}, \begin{pmatrix} \tilde{F}, \beta_{n_*}^{(1)} \\ \tilde{F}, \beta_{n_*}^{(2)} \end{pmatrix} \right),
\]
\[
\begin{pmatrix} \tilde{F}, \beta_0^{(1)} \\ \tilde{F}, \beta_0^{(2)} \end{pmatrix}_{v=0} = \text{col} \left( \begin{pmatrix} \tilde{F}, \beta_0^{(1)} \\ \tilde{F}, \beta_0^{(2)} \end{pmatrix}, \begin{pmatrix} \tilde{F}, \beta_{n_*}^{(1)} \\ \tilde{F}, \beta_{n_*}^{(2)} \end{pmatrix} \right),
\]
(26)
Then, denoting by $\mathcal{L}_1$ the restriction of $\mathcal{L}$ to $Q$, system (19) is equivalent to a system of abstract ODEs in $R^n \times Q$, with finite and infinite dimensional variables also separated in the linear term.

\[
\begin{cases}
\dot{z} = Bz + \Psi \begin{bmatrix} \tilde{F}(z, w, \mu), \beta^{(1)}_0 \\ \tilde{F}(z, w, \mu), \beta^{(2)}_0 \end{bmatrix}^v = 0, \\
\dot{w} = \mathcal{L}_1(w) + H(z, w, \mu),
\end{cases}
\]  

(27)

where

\[
z = (z_1, z_2, z_3)^T, \quad B = \text{diag}\{i\omega_c, -i\omega_c, 0\},
\]

\[
\Psi = \text{diag}\{\Psi_1, \Psi_2\}, \quad \tilde{F}(z, w, \mu) = \tilde{F}(\Phi z_x + w, \mu)
\]

and

\[
H(z, w, \mu) = \tilde{F}(z, w, \mu) - \left(\begin{bmatrix} \eta^T_0, \tilde{F}(z, w, \mu), \beta^{(1)}_0 \\ \tilde{F}(z, w, \mu), \beta^{(2)}_0 \end{bmatrix} \right) \xi_0
\]

\[
+ \left(\begin{bmatrix} \eta^T_0, \tilde{F}(z, w, \mu), \beta^{(1)}_0 \\ \tilde{F}(z, w, \mu), \beta^{(2)}_0 \end{bmatrix} \right) \bar{\xi}_0 \gamma_0(x)
\]

\[
- \left(\begin{bmatrix} \eta^T_n, \tilde{F}(z, w, \mu), \beta^{(1)}_n \\ \tilde{F}(z, w, \mu), \beta^{(2)}_n \end{bmatrix} \right) \bar{\xi}_n \gamma_n(x).
\]

(29)

According to [20], by a recursive transformation, we can obtain that the normal form for Turing-Hopf bifurcation reads as

\[
\dot{z} = Bz + \begin{bmatrix} B_{11}\mu_1 z_1 + B_{22}\mu_2 z_1 \\ B_{11}\mu_1 z_1 + B_{22}\mu_2 z_1 + B_{210}\tilde{z}_1^2 z_2 + B_{102}\tilde{z}_1 z_3^2 \\ B_{13}\mu_1 z_3 + B_{23}\mu_2 z_3 \end{bmatrix} + O(||z||^2),
\]

(30)

where

\[
B_{11} = \eta^T_0 L^{(1,0)}_1(\xi_0) = \frac{\alpha(u_*)}{2} + \frac{i\delta_s(f(u_*) - \alpha(u_*))}{2\omega_c},
\]

\[
B_{21} = \eta^T_0 L^{(0,1)}_1(\xi_0) = -\frac{1}{2} + \frac{i(r_f(u_*) - \delta_s)}{2\omega_c},
\]

\[
B_{13} = \eta^T_n L^{(1,0)}_1(\xi_n) = \frac{\delta_s(f(u_*) - \delta_s)}{(d_1 + d_2)(\frac{\alpha}{\omega_c})^2},
\]

\[
B_{23} = \eta^T_n L^{(0,1)}_1(\xi_n) = \frac{\delta_s - r_f(u_*) - d_1}{(d_1 + d_2)(\frac{\alpha}{\omega_c})^2},
\]

and

\[
B_{210} = C_{210} + \frac{3}{2}(D_{210} + E_{210}), \quad B_{102} = C_{102} + \frac{3}{2}(D_{102} + E_{102}),
\]

\[
B_{111} = C_{111} + \frac{3}{2}(D_{111} + E_{111}), \quad B_{003} = C_{003} + \frac{3}{2}(D_{003} + E_{003}).
\]

Clearly, we still need to compute $C_{ijk}$, $D_{ijk}$ and $E_{ijk}$. Letting

\[
F_{jkj} = (F^{(1)}_{jkj}, F^{(2)}_{jkj})^T,
\]

with

\[
F^{(k)}_{jkj} = \frac{\partial^{j_1+j_2} F^{(k)}(0, 0, 0, 0)}{\partial u^{j_1} \partial v^{j_2}}, \quad k = 1, 2, j_1 + j_2 = 2, 3.
\]
Then from (20), we have

\[ F_{20} = \begin{pmatrix} r_s u_* f'''(u_*) - 2r_* \\ -2\alpha(u_*) \frac{r_*^2}{u_*} \\ -2\alpha(u_*) \frac{1}{u_*} \end{pmatrix}, \quad F_{11} = \begin{pmatrix} -f'(u_*) \\ 2\alpha(u_*) \frac{r_*}{u_*} \end{pmatrix}, \]

\[ F_{02} = \begin{pmatrix} 0 \\ -2\alpha(u_*) \frac{1}{u_*} \end{pmatrix}, \]

and

\[ F_{30} = \begin{pmatrix} -r_s u_* f'''(u_*) \\ 6\alpha(u_*) \frac{r_*^2}{u_*^2} \end{pmatrix}, \quad F_{21} = \begin{pmatrix} -f''(u_*) \\ -4\alpha(u_*) \frac{r_*}{u_*^2} \end{pmatrix}, \]

\[ F_{12} = \begin{pmatrix} 0 \\ 2\alpha(u_*) \frac{1}{u_*^2} \end{pmatrix}, \quad F_{03} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \]

Thus, we can obtain

\[ A_{200} = F_{20} + 2\xi_{02} F_{11} + \xi_{02}^2 F_{02} = \bar{A}_{020}, \]

\[ A_{002} = F_{20} + 2\xi_{n,2} F_{11} + \xi_{n,2}^2 F_{02}, \]

\[ A_{110} = 2(F_{20} + 2\text{Re}(\xi_{02}) F_{11} + |\xi_{02}|^2 F_{02}), \]

\[ A_{101} = 2(F_{20} + (\xi_{02} + \xi_{n,2}) F_{11} + \xi_{02} \xi_{n,2} F_{02}) = \bar{A}_{011}, \]

\[ A_{210} = 3(F_{30} + (2\xi_{02} + \bar{\xi}_{02}) F_{21} + (\xi_{02}^2 + 2|\xi_{02}|^2) F_{12}), \]

\[ A_{102} = 3(F_{30} + (\xi_{02} + 2\xi_{n,2}) F_{21} + (\xi_{02}^2 + 2\xi_{02} \xi_{n,2}) F_{12}), \]

\[ A_{111} = 6(F_{30} + (\xi_{n,2} + 2\text{Re}(\xi_{02})) F_{21} + (|\xi_{02}|^2 + 2\xi_{n,2} \text{Re}(\xi_{02})) F_{12}), \]

\[ A_{003} = F_{30} + 3(\xi_{n,2} F_{21} + \xi_{n,2}^2 F_{12}), \]

and

\[ h_{0200} = \frac{1}{\sqrt{1/\pi}} (\mathcal{M}_0(2i\omega_c))^{-1} (A_{200} - (\check{\eta}_{0}^T A_{200} \xi_0 + \check{\eta}_{0}^T A_{200} \xi_0)), \]

\[ h_{0020} = \frac{1}{\sqrt{1/\pi}} (\mathcal{M}_0(-2i\omega_c))^{-1} (A_{020} - (\check{\eta}_{0}^T A_{020} \xi_0 + \check{\eta}_{0}^T A_{020} \xi_0)), \]

\[ h_{0002} = \frac{1}{\sqrt{1/\pi}} (\mathcal{M}_0(0))^{-1} (A_{002} - (\check{\eta}_{0}^T A_{002} \xi_0 + \check{\eta}_{0}^T A_{002} \xi_0)), \]

\[ h_{0110} = \frac{1}{\sqrt{1/\pi}} (\mathcal{M}_0(0))^{-1} (A_{110} - (\check{\eta}_{0}^T A_{110} \xi_0 + \check{\eta}_{0}^T A_{110} \xi_0)), \]

\[ h_{n,101} = \frac{1}{\sqrt{1/\pi}} (\mathcal{M}_{n_1}(i\omega_c))^{-1} (A_{101} - \check{\eta}_{n_1}^T A_{101} \xi_{n_1}), \]

\[ h_{n,011} = \frac{1}{\sqrt{1/\pi}} (\mathcal{M}_{n_2}(-i\omega_c))^{-1} (A_{011} - \check{\eta}_{n_2}^T A_{011} \xi_{n_2}), \]

\[ h_{2n,002} = \frac{1}{\sqrt{2\pi}} (\mathcal{M}_{2n_2}(0))^{-1} A_{002}, \]

\[ h_{2n,110} = (0, 0)^T. \]
From [20], $C_{ijk}$, $D_{ijk}$ and $E_{ijk}$ can be computed out as follows:

\[ C_{210} = \frac{1}{6\pi} \eta_0^T A_{210}, \quad C_{102} = \frac{1}{6\pi} \eta_0^T A_{102}, \quad C_{111} = \frac{1}{6\pi} \eta_n^T A_{111}, \quad C_{003} = \frac{1}{4\pi} \eta_n^T A_{003}, \]

\[ D_{210} = \frac{1}{6\pi \omega_c^2} \left( - (\eta_0^T A_{200})(\eta_0^T A_{110}) + |\eta_0^T A_{200}|^2 + \frac{2}{3} |\eta_0^T A_{200}|^2 \right), \]

\[ D_{102} = \frac{1}{6\pi \omega_c^2} \left( - 2(\eta_0^T A_{200})(\eta_0^T A_{002}) + (\eta_0^T A_{110})(\eta_0^T A_{002}) + 2(\eta_0^T A_{002})(\eta_0^T A_{110}) \right), \]

\[ D_{111} = - \frac{1}{3\pi \omega_c} \text{Im}(\eta_n^T A_{101})(\eta_0^T A_{110}), \]

\[ D_{003} = - \frac{1}{3\pi \omega_c} \text{Im}(\eta_n^T A_{101})(\eta_0^T A_{002}). \]

\[ E_{210} = \frac{1}{3\sqrt{3\pi}} \eta_0^T((\xi_0 F_{11}) h^{(1)}_{0110} + (\xi_0 F_{02} + F_{11}) h^{(2)}_{0110}) \]

\[ + (\xi_0 F_{11}) h^{(1)}_{0220} + (\xi_0 F_{02} + F_{11}) h^{(2)}_{0220}), \]

\[ E_{102} = \frac{1}{3\sqrt{3\pi}} \eta_0^T((\xi_0 F_{11}) h^{(1)}_{0002} + (\xi_0 F_{02} + F_{11}) h^{(2)}_{0002}) \]

\[ + (\xi_0 F_{11}) h^{(1)}_{1101} + (\xi_0 F_{02} + F_{11}) h^{(2)}_{1101}, \]

\[ E_{111} = \frac{1}{3\sqrt{3\pi}} \eta_0^T((\xi_0 F_{11}) h^{(1)}_{111} + (\xi_0 F_{02} + F_{11}) h^{(2)}_{111}) \]

\[ + (\xi_0 F_{11}) h^{(1)}_{22} + (\xi_0 F_{02} + F_{11}) h^{(2)}_{22} \]

\[ + \eta_n^T(F_{20} + \xi_0 F_{11}) \left( \frac{1}{3\sqrt{3\pi}} h^{(1)}_{0110} + \frac{1}{3\sqrt{2\pi}} h^{(1)}_{(2n,1)10} \right) \]

\[ + (F_{11} + \xi_0 F_{02}) \left( \frac{1}{3\sqrt{3\pi}} h^{(2)}_{0110} + \frac{1}{3\sqrt{2\pi}} h^{(2)}_{(2n,1)10} \right) \],

\[ E_{003} = \eta_n^T(F_{20} + \xi_0 F_{11}) \left( \frac{1}{3\sqrt{3\pi}} h^{(1)}_{0002} + \frac{1}{3\sqrt{2\pi}} h^{(1)}_{(2n,0)02} \right) \]

\[ + (F_{11} + \xi_0 F_{02}) \left( \frac{1}{3\sqrt{3\pi}} h^{(2)}_{0002} + \frac{1}{3\sqrt{2\pi}} h^{(2)}_{(2n,0)02} \right) \].

The normal form Eq.(30) can now be written in real coordinates $v$ through the change of variables $z_1 = v_1 - i v_2$, $z_2 = v_1 + i v_2$, $z_3 = v_3$, and then changing to cylindrical coordinates by $v_1 = \rho \cos \Theta$, $v_2 = \rho \sin \Theta$, $v_3 = \varsigma$, we obtain, truncating at third order terms and removing the azimuthal term

\[ \dot{\rho} = \alpha_1(\mu) \rho + \kappa_{11} \rho^3 + \kappa_{12} \rho \varsigma^2, \]

\[ \dot{\varsigma} = \alpha_2(\mu) \varsigma + \kappa_{21} \rho^2 \varsigma + \kappa_{22} \varsigma^3, \]  

where \(\alpha_1(\mu) = \text{Re}(B_{11}) \mu_1 + \text{Re}(B_{21}) \mu_2, \ \alpha_2(\mu) = B_{13} \mu_1 + B_{23} \mu_2, \ \kappa_{11} = \text{Re}(B_{210}), \ \kappa_{12} = \text{Re}(B_{102}), \ \kappa_{21} = B_{111}, \ \kappa_{22} = B_{003} \).

For all possible dynamics of the normal form (31) in the sufficiently small neighborhood of the origin, refer to the books [8, 12]. The truncated normal form (31) is
TURING-HOPF BIFURCATION OF LG MODEL

An example and simulations. In this section, we make some numerical simulations to support and extend our analytical results. Choose \( R(N) = \frac{AN^2}{B + N^2} \), then (4) becomes

\[
\begin{aligned}
\frac{\partial N(x,t)}{\partial t} &= d_1 \Delta N(x,t) + rN(x,t) \left( 1 - \frac{N(x,t)}{K} \right) - \frac{AN^2(x,t)P(x,t)}{B + N^2(x,t)}, \\
\frac{\partial P(x,t)}{\partial t} &= d_2 \Delta P(x,t) + \delta P(x,t) \left( 1 - \frac{hP(x,t)}{N(x,t)} \right).
\end{aligned}
\]  

(32)

We would like to mention that the system (32) without the diffusion was proposed as a prey-predator model to describe the interaction of spruce budworm and birds, see [19, p71]. In fact, spruce budworm is one of the most destructive insect in North American forests and there are periodic outbreaks of spruce budworm (every 30-40 years lasting for about 10 years) causing billions of dollars loss to forest industry. Therefore, understanding the reason why spruce budworm population can outbreak periodically, is very important to control the growth of spruce budworm and protect spruce and fir forests.

After the variable substitution at the beginning of Section 2, (32) can be written as

\[
\begin{aligned}
\frac{\partial u(x,t)}{\partial t} &= d_1 \Delta u(x,t) + ru(x,t)(1 - u(x,t)) - \frac{au^2(x,t)v(x,t)}{b + u^2(x,t)}, \\
\frac{\partial v(x,t)}{\partial t} &= d_2 \Delta v(x,t) + \delta v(x,t) \left( 1 - \frac{v(x,t)}{ru(x,t)} \right),
\end{aligned}
\]  

(33)

where \( a = \frac{AK}{KN} \) and \( b = \frac{B}{KN} \).

Let \( \Omega = (0, \pi) \) and \( a = 1, b = 0.0025, d_1 = 0.05, d_2 = 0.33 \). For this set of parameter values, we obtain that system (33) has a unique positive steady state \( E^*(0.1296, 0.1296r) \). By calculation, we have that \( \alpha(u_*) = 0.5152 > 0 \) and \( A = 0.0087 > 0 \). Then from Theorem 2.5, system (33) at least undergoes Turing-Hopf bifurcation at the point \( P^*(p_1, \alpha(u_*)p_1) = (0.1388, 0.0705) \) (see Fig.1(a)) and

\[ H_0(r) : \delta = 0.5152r; \quad S_1(r) : \delta = \frac{0.0258r - 0.0165}{0.05 + 0.3552r}. \]

According to the procedure in Section 3 with \( n_* = 1 \), the normal form truncated to the third order terms is

\[
\begin{aligned}
\dot{\rho} &= (0.2576\mu_1 - 0.5\mu_2)\rho - 0.4877\rho^3 - 1.6228\rho\varsigma^2, \\
\dot{\varsigma} &= (0.3806\mu_1 - 0.2613\mu_2)\varsigma - 2.0709\rho^2\varsigma - 0.2273\varsigma^3,
\end{aligned}
\]  

(34)

where \( \mu_1 \) and \( \mu_2 \) are perturbation parameters for the Turing-Hopf point \( P^* \).

Notice that \( \rho \geq 0 \) and \( \varsigma \) is an arbitrary real number. System (34) has a zero equilibrium \( A_0 = (0, 0) \) for all \( \mu_1, \mu_2 \); three trivial equilibria

\[ A_1 = (\sqrt{0.5282\mu_1 - 1.0251\mu_2}, 0), \text{ for } \mu_2 < 0.5152\mu_1, \]

\[ A_2^+ = (0, \pm\sqrt{1.6741\mu_1 - 1.1494\mu_2}, 0), \text{ for } \mu_2 < 1.4564\mu_1, \]
and two nontrivial equilibria

\[ A_3^\pm = (\sqrt{0.1720\mu_1 - 0.0955\mu_2}, \pm \sqrt{0.1070\mu_1 - 0.2794\mu_2}), \]

for \( \mu_2 < 0.3831\mu_1 \) and \( \mu_2 < 1.8011\mu_1 \).

Define the critical bifurcation lines are as follows:

\[ H_0: \mu_2 = 0.5152\mu_1; \ S_1: \mu_2 = 1.4564\mu_1; \]
\[ T_1: \mu_2 = 0.3831\mu_1, \mu_2 > 0; \ T_2: \mu_2 = 1.8011\mu_1, \mu_2 < 0. \]

Then, according to the results in [8], the bifurcation diagram in the \((\mu_1, \mu_2)\) parameter plane and the corresponding phase portraits of system (34) in the \((\rho, \varsigma)\) plane can be shown in Fig.2. The four straight lines \( H_0, S_1, T_1 \) and \( T_2 \) divide the \((\mu_1, \mu_2)\) parameter plane into six regions marked by \( 1-6 \). Noticing that the equilibria \( A_0, A_1, A_2^+ \) and \( A_2^- \) of the normal form system (34) correspond to the constant equilibrium, the spatially homogeneous periodic solution, the nonconstant steady state like \( \cos x \)-shape and spatially inhomogeneous periodic solution of system (33), respectively. So the dynamics of the original system (33) near the Turing-Hopf point \( P^* \) in the plane of parameters \( r \) and \( \delta \) can be identified in terms of the dynamics of the normal form system (34).

![Bifurcation Diagram](image)

**Figure 2.** Bifurcation diagrams and dynamical classification near the Turing-Hopf point \( P^* \)

In region \( \Phi \), system (34) has only one equilibrium \( A_0 \) and it is asymptotically stable. This leads to the positive constant equilibrium \( E^* \) of the original system (33) is asymptotically stable, as shown in Fig.3 for \( (\mu_1, \mu_2) = (-0.01, 0.02) \) and the initial value \( u(x, 0) = 0.1296 + 0.005 \cos x, v(x, 0) = 0.0167 + 0.01 \cos x \).

In region \( \Phi \), system (34) has three equilibria: \( A_0, A_1^+ \) and \( A_2^- \). The equilibrium \( A_0 \) is unstable and the equilibria \( A_1^+ \) and \( A_2^- \) are asymptotically stable. So, the original system (33) has an unstable constant equilibrium and two stable nonconstant steady states like \( \cos x \)-shape. For the fixed parameter \( (\mu_1, \mu_2) = (0.022, 0.014) \) and choosing different initial values, system (33) can converge to one of these two nonconstant steady states, as shown in Fig.4(a) and (b) for the initial value \( u(x, 0) = 0.1296 - 0.02 \cos x, v(x, 0) = 0.0208 + 0.01 \cos x \) and Fig.4(c) and (d) for the initial value \( u(x, 0) = 0.1296 + 0.02 \cos x, v(x, 0) = 0.0208 - 0.01 \cos x \).
Figure 3. When \((\mu_1, \mu_2) = (-0.01, 0.02)\) lies in region \(\odot\), the positive constant equilibrium \(E^*(0.1296, 0.0167)\) is asymptotically stable. The initial value is \(u(x, 0) = 0.1296 + 0.005 \cos x, v(x, 0) = 0.0167 + 0.01 \cos x\).

Figure 4. When \((\mu_1, \mu_2) = (0.022, 0.014)\) lies in region \(\odot\), the positive constant equilibrium \(E^*(0.1296, 0.0208)\) is unstable and there are two stable spatially inhomogeneous steady states like \(\cos x\). (a) and (b) The initial value is \(u(x, 0) = 0.1296 - 0.02 \cos x, v(x, 0) = 0.0208 + 0.01 \cos x\); (c) and (d) the initial value is \(u(x, 0) = 0.1296 + 0.02 \cos x, v(x, 0) = 0.0208 - 0.01 \cos x\).

In region \(\odot\), system (34) has four equilibria: \(A_0\), \(A_1\), \(A_2^+\) and \(A_2^-\). The equilibria \(A_0\) and \(A_1\) are unstable and the equilibria \(A_2^+\) and \(A_2^-\) are still asymptotically stable. So, the original system (33) has an unstable constant equilibrium, an unstable spatially homogeneous periodic solution and two stable nonconstant steady states like \(\cos x\)-shape. Choosing \((\mu_1, \mu_2) = (0.02, 0.01)\) and the initial value as \(u(x, 0) = 0.1576 - 0.002 \cos x, v(x, 0) = 0.0234\), the dynamics of system (33) evolves from
Figure 5. When $(\mu_1, \mu_2) = (0.02, 0.01)$ lies in region $\Phi$, the positive constant equilibrium $E^*(0.1296, 0.0206)$ is unstable and there is a heteroclinic orbit connecting the unstable spatially homogeneous periodic solution to stable spatially inhomogeneous steady state. The initial value is $u(x, 0) = 0.1576 - 0.002 \cos x, v(x, 0) = 0.0234$. (a) and (b) are transient behaviours for $u$ and $v$, respectively; (c) and (d) are middle-term behaviours for $u$ and $v$, respectively; (e) and (f) are long-term behaviours for $u$ and $v$, respectively.

the spatially homogeneous periodic solution, and spatially inhomogeneous periodic solution and finally to the nonconstant steady state, as shown in Fig.5.

In region $\Phi$, system (34) has six equilibria: $A_0, A_1, A_2^\pm$ and $A_3^\pm$. The equilibria $A_0, A_1$ and $A_2^\pm$ are unstable and the equilibria $A_3^+$ and $A_3^-$ are asymptotically stable. These two stable $A_3^+$ and $A_3^-$ implies that system (33) has two stable spatially inhomogeneous periodic solution. For $(\mu_1, \mu_2) = (0.4, 0.12)$ and the initial value $u(x, 0) = 0.1506 - 0.001 \cos x, v(x, 0) = 0.0691 + 0.001 \cos x$, close to the unstable
When \((\mu_1, \mu_2) = (0.4, 0.12)\) lies in region \(\Phi\), the positive constant equilibrium \(E^*(0.1296, 0.0698)\) is unstable and there are stable spatially inhomogeneous periodic solution. The initial value is \(u(x, 0) = 0.1306 - 0.001 \cos x, v(x, 0) = 0.0691 + 0.001 \cos x\). (a) and (b) are transient behaviours for \(u\) and \(v\), respectively; (c) and (d) are long-term behaviours for \(u\) and \(v\), respectively.

**Figure 6.** When \((\mu_1, \mu_2) = (0.4, 0.12)\) lies in region \(\Phi\), the positive constant equilibrium \(E^*(0.1296, 0.0698)\) is unstable and there are stable spatially inhomogeneous periodic solution. The initial value is \(u(x, 0) = 0.1306 - 0.001 \cos x, v(x, 0) = 0.0691 + 0.001 \cos x\). (a) and (b) are transient behaviours for \(u\) and \(v\), respectively; (c) and (d) are long-term behaviours for \(u\) and \(v\), respectively.

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When \((\mu_1, \mu_2) = (-0.01, -0.015)\) lies in region \(\Phi\), the positive constant equilibrium \(E^*(0.1296, 0.0167)\) is unstable and there are heteroclinic solution connecting the unstable spatially inhomogeneous steady state to stable spatially homogeneous periodic solution. The initial value is \(u(x, 0) = 0.1526 - 0.065 \cos x, v(x, 0) = 0.0189 - 0.0015 \cos x\). (a) and (b) are transient behaviours for \(u\) and \(v\), respectively; (c) and (d) are long-term behaviours for \(u\) and \(v\), respectively.

When \((\mu_1, \mu_2) = (-0.02, -0.022)\) lies in region \(\Phi\), the positive constant equilibrium \(E^*(0.1296, 0.0154)\) is unstable and there is a stable spatially homogeneous periodic solution. The initial value is \(u(x, 0) = 0.1296, v(x, 0) = 0.0154 - 0.001 \cos x\).

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