On the error control at numerical solution
of reaction-diffusion equations *

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Abstract

We suggest guaranteed, robust a posteriori error bounds for approximate solutions of the reaction-diffusion equations, modeled by the equation $-\Delta u + \sigma u = f$ in $\Omega$ with any $\sigma = \text{const} \geq 0$. We also term our bounds consistent due to one specific property. It assumes that their orders of accuracy in respect to mesh size $h$ are the same with the respective not improvable in the order a priori bounds. Additionally, it assumes that the pointed out equality of the orders is provided by the testing flaxes not subjected to equilibration. For any $\sigma \in [0, \sigma^*]$, the right part of the new general bound of the paper contains, besides the usual diffusion term, the $L^2$ norm of the residual with the factor $1/\sqrt{\sigma^*}$, where $\sigma^*$ is some critical value. For the solutions by the finite element method, it is estimated as $\sigma^* \geq ch^{-2}$, $c = \text{const}$, if $\partial \Omega$ is sufficiently smooth and the finite element space is of the 1st order of accuracy at least. In general, at the derivation of a posteriori bounds, consistency is achieved by taking adequately into account the difference of the orders of the $L^2$ and $H^1$ error norms, that can be done in various ways with accordingly introduced $\sigma^*$. Two advantages of the obtained consistent posteriori error bounds deserve attention. They are better accuracy and the possibility to avoid the use of the equilibration in the flax recovery procedures, that may greatly simplify these procedures and make them much more universal. The technique of obtaining the consistent a posteriori bounds was briefly exposed by the author in [arXiv:1702.00433v1 [math.NA], 1 Feb 2017] and [DokladyMathematics, 96 (1), 2017, 380-383].

1 Introduction

For the successful error control of approximate solutions to the boundary value problems, the guaranteed a posteriori error majorant must be sufficiently accurate and cheap in a sense of the computational work. In particular, it is natural to expect that the computational cost of an

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error estimate does not exceed the cost of the numerical solution of the problem. For the elliptic
equations of the second order, most often the FEM’s (finite element methods) of the class $C$ are
used with the solutions belonging to finite-dimensional subspaces of $C(\Omega) \cap H^1(\Omega)$. The second
derivatives of such approximate solutions, which are needed to calculate the residuals, are not
defined. Therefore, the majorants of the error are calculated by using more smooth testing flows,
which are found with the help of FEM flows by means of special procedures, termed *flax recovery
procedures*. They significantly influence the accuracy of a posteriori error bounds.

The need to improve the smoothness of numerical flows without losses in accuracy motivated
the development of the flax recovery procedures, which currently are numerous. There are several
types of such procedures attempting as the reduction of the right parts of the error bounds, so
producing the fluxes, equilibrated in a weak or a strong form, and, for the result, allowing to
reduce the coefficients before the residuals or even to remove them from the bounds. For making
the equilibration simpler, it is usually done not for the right part $f$ of the elliptic equation,
but for its approximation $\hat{f}$, which on every finite element is defined, e.g., as the orthogonal
$L_2$-projection on the space of the finite element. Considerable attention was given to procedures
in which the evaluation of the smoothed and equilibrated fluxes was reduced to the resolving of
auxiliary local problems of approximation/interpolation. Among great number of works devoted
to the mentioned topics, we are able to refer to a few, in particular, to Zienkiewicz and Zhu
\[36\], Ainsworth and Oden \[1\], Babuska *et al.* \[6, 7\], Ern *et al.* \[17\], Cheddadi *et al.* \[12\], Kai
and Zhang \[9\] and Ainsworth and Vejchodsky \[3\], where additional extensive bibliography can be
found.

At the creation of flax recovery procedures, three requirements are paid attention:

α) preserving orders of accuracy, e.g., in the energy and other norms, the same as the nu-
merical fluxes determined by the approximate solution of the boundary value problem,
β) obtaining the balanced or weakly balanced recovered fluxes, which element wise provide
smallness of the residual type terms or make them equal to zero,
γ) providing linear or almost linear computational cost.

Obviously, the requirement (β) complicates the procedures, makes them directly dependent
on the specific boundary value problem to be solved and, therefore, less universal. In a number of
the a posteriori error majorants the orders of smallness of the residual type component and of
the others, evidently, differ. The implementation of (β) gives the possibility to reduce the order
of the first component and make it comparable with the error of the approximate solution. A
consistent and efficient implementation of this approach is found in Ainsworth and Oden \[1\] and
Ainsworth and Vejchodsky \[3\].

In the present work, we explore the possibility of obtaining the a posteriori error majorants
all terms of which would have the same order of smallness with respect to the mesh size $h$
independently of the equilibration. In this relation, we use the term *consistent a posteriori error
estimate/majorant* attributed to the a posteriori estimates/majorants, possessing the following
property: on the test fluxes, that meet the requirements (α), (γ), they provide the same orders
of accuracy with the corresponding a priori error estimates. Obviously, consistent majorant is
unimprovable in the order of accuracy, if the a priori error estimate of the numerical method is
unimprovable in the same sense.\[2\]

\[1\] In the literature, the term "consistency" is often understood in a weaker sense as turning majorant in zero
The typical summand of the majorants of the error energy norms is \( \theta^{1/2} \| \hat{f} - \sigma u_h + z \|_{L_2(\Omega)} \), where \( \sigma \) is the nonnegative constant reaction coefficient in the equation of reaction-diffusion, \( u_h \) is the approximate solution, and \( z \) is the testing flax vector function. Different authors come to different expressions for \( \theta \) and, in particular, to a) \( \theta = 1 / \sqrt{\sigma} \) for all \( \sigma > 0 \), b) \( \theta = \text{const} \) for \( \sigma \equiv 0 \), c) \( \theta = (ch)^2 \) for \( \sigma \leq (ch)^{-2} \), and d) \( \theta = 0 \) for \( \sigma \leq (ch)^{-2} \). Several examples of majorants of these types are given in the next section. It is not difficult to come to the conclusion that in the cases a) and b) the a posteriori bounds can be larger in \( h^{-1} \) times, and more in the case a), than the energy norm of the error, if the test flaxes satisfy only the requirements a) and \( \gamma \). This is a consequence of the fact that these bounds are nonconsistent. The orders of smallness of a posteriori majorants, related to c) and d), are equal to the orders provided by the corresponding a priori error bounds. At least this is true for the methods with the linear tetrahedral finite elements and the problems with the exact solutions belonging to \( H^2(\Omega) \). However, in a strict sense, the a posteriori bounds related to c) and d) can be also inconsistent when they are based, as it happens most often, on the use of the equilibrated testing flaxes. In other words, they can produce not satisfactory bounds, if testing flaxes satisfy a) and \( \gamma \), but are not equilibrated.

There exists other option to improve the accuracy of a posteriori error majorants and at the same time to simplify the procedures of flax recovery. Such an option is provided by the consistent a posteriori error bounds which are derived in this paper, following techniques, briefly presented in \[22, 23, 24\]. The above mentioned \( L_2 \)-norms of the residual type enter right parts of these bounds with the multiplier \( \theta^{1/2} = ch \), as in the case c), but without the assumption of the testing flax equilibration. Therefore, not only the accuracy of the a posteriori bounds is improved, but simultaneously the flax recovery procedures can be noticeably simplified.

It is worth noting that the structure of modern a posteriori error bounds for approximations of the solutions of the reaction-diffusion equations is met in the works of Aubin, see \[5\]. His majorant corresponds to the case a), does not assume the equilibration and is accurate for sufficiently big values of \( \sigma \). However, its accuracy drops when \( \sigma \to 0 \) and at \( \sigma = 0 \) it losses it sense. A number of later majorants, having similar structures, resulted from the attempts to improve accuracy by different remedies. In order to come to our bounds, we use the new way of their derivation, which adequately takes into account the difference in the orders of \( L_2 \) and \( H^1 \) error norms of approximate solutions. The majorants proposed in this work are defined for all \( \sigma \geq 0 \), for \( \sigma \geq (ch)^{-2} \) coincide with the majorant of Aubin, do not lose precision for \( \sigma \in [0, ch^{-2}] \), \( c = \text{const} \), and are consistent.

Simplicity of evaluation of constants in a posteriori error majorants is very important for the practice. In the paper, we suggest consistent majorants of two types with differently defined constants. In one of them, the constants depend on the constants in local bounds of approximation in \( L_2(\tau_r) \) and stability in \( H^1(\delta^{(r)}) \) of the quasinterpolation operator \( H^1(\delta^{(r)}) \to \mathbb{V}_h(\tau_r) \), where \( \mathbb{V}_h(\tau_r) \) is the space induced by the finite element \( \tau_r \), \( \delta^{(r)} \) is the smallest patch of the finite elements neighbouring \( \tau_r \). In particular, the quasiinterpolation operator of Scott and Zhang \[31\] can be used. As was mentioned, consistency with a priori error estimates, unimprovable in the order of accuracy, implies that a posteriori error majorant is accurate in the same sense. The exactness in the order of the majorant can be also confirmed by the inverse bounds and by the so called bounds of local effectiveness. Due to the discussed above properties of our majorants, they
are majorated by some known majorants and, as a consequence, some known inverse bounds can be easily adapted to our majorants. We consider an example of such bound.

The paper is organized as follows. Section 2 contains the formulation of the boundary value problem of reaction-diffusion, examples of known error majorants, similar in structure to the ones suggested in the paper, and their brief discussion from the point of view of consistency. In Section 3 we propose the new general a posteriori majorant for the error of approximation of the exact solution to the boundary value problem by an arbitrary sufficiently smooth function \( v \) that satisfies the essential boundary conditions. It is defined for \( \forall \sigma \geq 0 \) and coincides with the majorant of Aubin \([5]\) in the case of \( \sigma \), exceeding a certain critical value \( \sigma_\ast \). Therefore, it can be considered as the improved Aubin’s majorant. Several versions of the majorant are discussed, related to different ways of defining \( \sigma \ast \) and, respectively, coefficients of the majorant. The easiest one corresponds to the majorant applicable to the approximate solutions \( v \) by Galerkin method with coordinate functions belonging to the space \( H^2(\Omega) \). This means that it is directly applicable in the isogeometric analysis, see Cottrel etc. \([14]\), which makes use of coordinate functions of a higher smoothness. Majorants of Section 3 are quite general, they do not address properties of the mesh methods.

Consistent a posteriori error estimates for solutions by the finite element method are presented in Sections 4 and 5. Theorems 4.1 and 4.2 proved in Section 4 suggest different approaches to evaluation of constants. In the majorant of Theorem 4.2 they are expressed through the constants \( h, p \) in Sections 4 and 5. Theorems 4.1 and 4.2, proved in Section 4 suggest different approaches to the mesh methods.

In what follows \( \|\phi\|_{H^k(\mathcal{G})} \) is the norm in the Sobolev space \( H^k(\mathcal{G}) \) on the domain \( \mathcal{G} \)

\[
\|\phi\|_{H^k(\mathcal{G})}^2 = \|\phi\|_{L^2(\mathcal{G})}^2 + \sum_{l=1}^{k} |\phi|_{H^l(\mathcal{G})}^2 = \sum_{|\alpha|=l} \int_{\mathcal{G}} \frac{\partial^l \phi}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_m^{\alpha_m}} dx,
\]

where \( \|\cdot\|_{L^2(\Omega)} \) and \( \|\cdot\|_{H^k(\Omega)} \) will also be used simpler symbols \( \|\cdot\|_0 \), \( \|\cdot\|_1 \) and \( \|\cdot\|_k \), respectively. If \( \mathcal{G} = \Omega \), for \( \|\cdot\|_{L^2(\Omega)} \) and \( \|\cdot\|_{H^k(\Omega)} \) will also be used simpler symbols \( \|\cdot\|_0 \), \( \|\cdot\|_k \) and \( \|\cdot\|_k \), respectively.

Everywhere below it is assumed that on \( \Omega \subset \mathbb{R}^m \), \( m = 2, 3 \), the assemblage of compatible and, generally speaking, curvilinear finite elements is given with each finite element occupying domain \( \tau_r \), \( r = 1, 2, \ldots, \mathcal{R} \). Sometimes, we use the notation \( \mathcal{R} \) also for the set of numbers of finite elements. The finite elements are defined by sufficiently smooth mappings \( x = \mathcal{X}(\xi) : \tau_\xi \rightarrow \tau_r \) of the reference element, defined on the standard triangle or tetrahedron \( \tau_\xi \). The span of the coordinate functions of the reference element is the space \( \mathcal{P}_p \) of polynomials of degree \( p \in \mathbb{N}^+ \). If \( p > 1 \), sometimes we use also the notation \( \mathcal{V}_h(\Omega) = \mathcal{V}_{h,p}(\Omega) \). If other is not mentioned,
it is always assumed that the finite element assemblage satisfies the \textit{generalized conditions of quasiuniformity} with the mesh parameter $h > 0$, which can be understood as the maximum of diameters of finite elements. The \textit{generalized conditions of quasiuniformity} for the mappings, defining finite elements (and finite element mesh), as well as the \textit{generalized shape quasiuniformity (regularity) conditions} for curvilinear finite elements can be found, for instance, in Korneev and Langer [25, Section 3.2].

As a rule in applications $V_h(\Omega) \subset C(\Omega) \cap H^1(\Omega)$. At the same time, in the isogeometric analysis, more smooth finite dimensional spaces $V_h(\Omega) = V^l_h(\Omega) \subset C^1(\Omega) \cap H^2(\Omega)$, see Kortell et al. [14], are in the use in computational schemes for solving elliptic equations of 2\textsuperscript{nd} order. Superscript $l$ in the notations $V^l_h(\Omega)$, $V^l_{h,p}(\Omega)$ assumes inclusion of these spaces in $C^l(\Omega) \cap H^{l+1}(\Omega)$.

\section{Model problem, examples of a posteriori error majorants}

One of the earliest is the a posteriori error majorant of Aubin [5]. We illustrate it on the model problem

\begin{equation}
\mathbf{L}u \equiv -\text{div}(A \text{grad} u) + \sigma u = F(x), \quad x = (x_1, x_2, \ldots, x_m) \in \Omega \subset \mathbb{R}^m, \\
u |_{\Gamma_D} = \psi_D, \quad -A \nabla u \cdot v |_{\Gamma_N} = \psi_N,
\end{equation}

where $\Gamma_D$, $\Gamma_N$ are disjoint, for simplicity, simply connected parts of the boundary $\partial \Omega = \Gamma_D \cup \Gamma_N$, $m \Gamma_D > 0$, $v$ is the internal normal to $\partial \Omega$, $A$ - symmetric $m \times m$ matrix that satisfies the inequalities

$$\mu_1 \xi \cdot \xi \leq A \xi \cdot \xi \leq \mu_2 \xi \cdot \xi, \quad 0 < \mu_1, \mu_2 = \text{const},$$

for any $x \in \Omega$ and $\xi \in \mathbb{R}^m$. The reaction coefficient $\sigma \geq 0$ is assumed to be constant and, in some cases, element wise constant. The boundary of $\Omega$, the coefficients of the matrix $A$, and right part $f$ are always considered as sufficiently smooth, in particular, $f \in L_2(\Omega)$, if the requirements on the smoothness are not formulated differently.

Our primal interest will be the error estimates in the energy norm

$$|u| = \left( \|u\|_A^{-2} + \|u\|_{L_2(\Omega)}^2 \right)^{1/2}, \quad \|u\|_A^{-2} = \int_\Omega \nabla u \cdot A \nabla u.$$

For vectors $y \in \mathbb{R}^m$, we introduce also the spaces $L^2_1(\Omega) = (L^2(\Omega))^m$, $H(\Omega, \text{div}) = \{ y \in L^2(\Omega) : \text{div} y \in L_2(\Omega) \}$, $W_{h,p}(\Omega, \text{div}) = \{ y \in H(\Omega, \text{div}) : y_k|_{\tau_r} \in \mathcal{P}_p, \forall r \in \mathcal{R}, k = 1, 2, \ldots, m \}$ and the norm $\|y\|_{A^{-1}} = (\int_\Omega A^{-1} y \cdot y)^{1/2}$.

\textbf{Theorem 2.1.} Let $f \in L_2(\Omega)$, $0 < \sigma = \text{const}$, $\psi_D \in H^1(\Omega)$, $\psi_N \in L_2(\Gamma_N)$, $v$ be any function of $H^1(\Omega)$ that satisfies the boundary condition on $\Gamma_D$. Then, for any $z \in H(\Omega, \text{div})$, satisfying on $\Gamma_N$ the boundary condition $z \cdot v = \psi_N$, we have

$$|v - u|^2 \leq \|A \nabla u + z\|_{A^{-1}}^2 + \frac{1}{\sigma} \|f - \sigma v - \text{div} z\|_{L_2(\Omega)}^2.$$

\textbf{Proof.} Estimate (2.4) is a special case of the results of Aubin [5], see, e.g., Theorem 22 in Introduction and Theorems 1.2, 1.4, 1.6 in Chapter 10. \hfill $\square$
Obviously, if \( \sigma \to 0 \) the majorant of Aubin loses precision and with \( \sigma = 0 \) makes no sense. If \( \sigma \equiv 0 \), one can use the majorant of Repin and Frolov \(^{[31]}\). Let for simplicity, \( \Gamma_D = \partial \Omega \), \( \psi_D \equiv 0 \), \( \mathbf{A} = \mathbf{I} \), where \( \mathbf{I} \) – the identity matrix, and \( \sigma \equiv 0 \). Then

\[
\| \nabla(v - u) \|_{L^2(\Omega)} \leq (1 + \epsilon) \| \nabla v + z \|_{L^2(\Omega)} + c_\Omega(1 + \frac{1}{\epsilon}) \| \nabla \cdot z - f \|_{L^2(\Omega)}, \quad \forall \epsilon > 0 ,
\]

where \( v \) and \( z \) are a function and an arbitrary vector-function from \( \tilde{H}^1(\Omega) \) and \( \mathbf{H}(\Omega, \text{div}) \) respectively, and \( c_\Omega \) is the constant from the Friedricks inequality.

It was shown in \(^{[4]}\) \(^{[21]}\) that the correction of arbitrary vector-function \( z \in \mathbf{H}(\Omega, \text{div}) \) into the vector-function \( \tau \), satisfying the balance/equilibrium equations, can be done by quite a few rather simple techniques. In particular, it is true for the correction of the flux vector-function \( \nabla u_{\text{fem}} \) into \( \tau \). This allows to implement the a posteriori bound

\[
\| v - u \| \leq \| \mathbf{A} \nabla \mathbf{u}_{\text{fem}} + \tau(\mathbf{u}_{\text{fem}}) \|_{\mathbf{A}^{-1}},
\]

see, e.g., Mikhlin \(^{[29]}\), or the bound with the additional free vector-function in the right part, which we present below. For simplicity, we restrict considerations to the same homogeneous Dirichlet problem for the Poisson equation in a two-dimensional convex domain. Let \( T_k \) be the projection of the domain \( \Omega \) on the axis \( x_{3-k} \) and the equations of the left and lower parts of the boundary be \( x_k = a_k(x_{3-k}) \), \( x_{3-k} \in T_k \). If \( \beta_k \) are arbitrary bounded functions and \( \beta_1 + \beta_2 \equiv 1 \), then according to \(^{[4]}\) \(^{[21]}\)

\[
\| \nabla(v - u) \|_{L^2(\Omega)} \leq \| \nabla v + z \|_{L^2(\Omega)} + \sum_{k=1,2} \| \int_{x_k} \beta_k(f - \nabla \cdot z)(\vec{\eta}_k, x_{3-k}) \ d\vec{\eta}_k \|_{L^2(\Omega)}. \tag{2.6}
\]

In (2.6) on the right we have integrals from the residual and this hopefully will make the majorant more accurate. Besides there is an additional free function \( \beta_1 \) or \( \beta_2 \) and it’s right choice (for instance, with the use of the found approximate solution \( v \)) can accelerate the process of the minimization of the right part, if such a process is implemented. If to estimate one-dimensional integrals under the sign of the \( L_2 \)-norm, then we come to the bound similar to (2.5).

Some authors attempted to modify the majorant of (2.4) with the aim of achieving acceptable accuracy for all \( \sigma \geq 0 \), see e.g., Repin and Sauter \(^{[32]}\) and Churilova \(^{[33]}\). The majorant of the latter, defined for \( \forall \sigma = \text{const} \geq 0 \), has the form

\[
|v - u|^2 \leq (1 + \epsilon) \| \mathbf{A} \nabla v + z \|_{\mathbf{A}^{-1}}^2 + \frac{1}{\epsilon \sigma} \| f - \sigma \text{div } z \|_{L^2(\Omega)}^2 . \tag{2.7}
\]

One of the efficient majorants for the finite element solutions was developed by Ainsworth and Vejchodsky \(^{[2]}, [3] \). For its record, we need additional notations: \( h_\tau \) is the diameter \( \tau_\rho \), \( \Pi_\rho : L^2(\tau_\rho) \to P_p(\tau_\rho) \) is the operator of orthogonal projection in \( L^2(\tau_\rho) \), and \( \sigma_\tau = \text{const} \) is the value of \( \sigma \) on \( \tau_\rho \). The dependence of the constants on the data of the boundary value problem and finite element assemblage is much simpler, if the following condition is satisfied:

A) The domain \( \Omega \) is a polygon in \( \mathbb{R}^m \), \( m = 2, 3 \), \( \tau_\rho \) are compatible \( m \)-dimensional simplices (with flat faces and, respectively, straight edges) forming triangulation of \( \Omega \), satisfying the conditions of shape regularity.

For simplicity in Theorems \(^{[2]}\) \(^{[2]}\) and \(^{[3]}\) below, we additionally assume \( \Gamma_D = \partial \Omega \), \( \psi_D \equiv 0 \), \( \mathbf{A} = \mathbf{I} \).
Theorem 2.2. Let \( u \in \tilde{H}^1(\Omega) \) be the weak solution of the problem and \( u_{\text{fem}} \in \tilde{V}(\Omega) \) be the solution by the finite element method. Then there exists \( z \in W_{h,2}(\Omega, \text{div}) \) with the following properties:

i) \( z \) is evaluated by the patch wise numerical procedure of linear numerical complexity,

ii) for all \( x \in \tau_r \) and \( r \in \mathcal{R}_* = \{ r : \sqrt{\sigma_r} h_r < 1 \} \) satisfies the equalities

\[
\Pi^1 f - \sigma_r u_{\text{fem}} + \text{div} z = 0, \tag{2.8}
\]

iii) for the error \( e_{\text{fem}} = u - u_{\text{fem}} \) and the error indicator \( \eta_{\tau_r}(z) \), defined as

\[
\eta^2_{\tau_r}(z) = \| z - \nabla u_{\text{fem}} \|^2_{L^2(\tau_r)}, \quad \forall r \in \mathcal{R}_*,
\]

\[
\eta^2_{\tau_r}(z) = \| z - \nabla u_{\text{fem}} \|^2_{L^2(\tau_r)} + \frac{1}{\sigma_r} \| \Pi_K f - \sigma_r u_{\text{fem}} + \text{div} z \|^2_{L^2(\tau_r)}, \quad \forall r \in \mathcal{R} \setminus \mathcal{R}_*, \tag{2.9}
\]

there hold the bounds

\[
\| e_{\text{fem}} \|^2 \leq \sum_{r \in \mathcal{T}_h} \left[ \eta_{\tau_r}(z) + \text{osc}_{\tau_r}(f) \right]^2, \tag{2.10}
\]

\[
\eta^2_{\Omega}(z) = \sum_{r \in \mathcal{R}} \eta^2_{\tau_r}(z) \leq C \left[ \| e_{\text{fem}} \|^2 + \sum_{r \in \mathcal{R}} \text{osc}^2_{\tau_r}(f) \right], \tag{2.11}
\]

where \( \text{osc}_{\tau_r}(f) = \min \left\{ \frac{r}{\pi}, \frac{1}{\sqrt{\sigma_r}} \right\} \| f - \Pi^1 f \|_{L^2(\tau_r)} \).

Proof. See Ainsworth and Vejchodsky [3] for the proof. We note that in this work the bounds (2.10), (2.11) are derived in a more general form under more general conditions. In particular, \( \Gamma_N \neq \emptyset \), the bound (2.11) is proved in the local version, i.e., with \( \eta^2_{\tau_r}(z) \) on the left and with the restriction of the right part to the patch \( \delta^{(v)} \).

Let us present one more majorant obtained by Cheddadi et al. [12] for approximate solutions of the singularly perturbed reaction-diffusion problem by the method of vertex-centered finite volumes. Let us introduce the notations: \( \mathcal{D}_h \) is the dual in respect to \( \mathcal{T}_h \) partition of \( \Omega \); \( \mathcal{S}_h \) is the finer mesh, induced by the partition \( \mathcal{D}_h \); \( \mathcal{D} \) is the polygon with the center in the vertex of triangulation \( \mathcal{T}_h \) and containing all simplices of the finer mesh with this vertex, \( h_D \) is its diameter; \( \mathcal{D}_h^{\text{int}} \) is the set of all polygons \( D \), for which \( \partial D \cap \partial \Omega = \emptyset \). For additional information about these objects we refer to [12].

Theorem 2.3. Let \( u_h \) be the solution by the method of vertex-centered finite volumes, \( e_h = u - u_h \), vector-function \( z \in H(\Omega, \text{div}) \) satisfy the equalities

\[
(f - \nabla z - \sigma u_h, 1)_D = 0, \quad \forall D \in \mathcal{D}_h^{\text{int}}, \tag{2.12}
\]

and \( \theta_D = \min(C_D h_D^2, \sigma_D^{-1}) \), where \( C_D \) is the constant from the Poincaré inequality for the polygon \( D \). Then

\[
\| e_h \|^2 \leq \eta^2_{\Omega}(z) = \sum_{D \in \mathcal{D}_h} \left[ \| \nabla u_h + z \|_{L^2(D)} + \sqrt{\theta_D} \| f - \sigma u_h - \nabla \cdot z \|_{L^2(D)} \right]^2. \tag{2.13}
\]

Proof. Theorem is one of the results of [12], see Theorem 4.5.
Majorants in \((2.5) - (2.7)\) have definite merits, but are not consistent at the application, for instance, to solutions by the finite element and other mesh methods. If \(v = u_{\text{fem}}\) is the finite element solution to the problem \((2.1)\) at \(\Gamma_D = \partial \Omega, \psi_D \equiv 0, A = I,\) at \(\sigma = 0,\) then we can use \((2.5)\). For our purpose it is sufficient to consider the approximate solutions from the space \(V_h(\Omega) = V^1_{h,p}(\Omega) \subset C^1(\Omega) \cap H^2(\Omega).\) Under the assumption \(u \in H^l(\Omega),\) we have a priori error bounds
\[
\|u - v\|_{H^k(\Omega)} \leq ch^{l-k}\|u\|_{H^l(\Omega)}, \quad k = 1, 2, \quad k \leq l \leq p + 1.
\]
In particular, if \(f \in L_2(\Omega)\) and consequently \(l = 2,\) see below \((1.38),\) then according to \((2.14)\) the left part of \((2.5)\) is estimated as \(O(h^2).\) At the same time at the choice \(z = -\nabla v\) the first term of the right part of \((2.5)\) vanishes, but \(c_\Omega(1 + \frac{1}{\epsilon})\|\nabla \cdot z - f\|_{L_2(\Omega)}\) at any \(\epsilon > 0\) can be bounded only by a constant. The bounds \((2.14)\) are not impovable in the order for \(u \in H^2(\Omega).\) For \(\Omega \in \mathbb{R}^2\) this was proved by Oganesian and Ruhovets \([30]\) by estimating the corresponding Kolmogorov’s width. From their results and the results on the regularity of the solutions of \((2.1),\) it follows existence of such \(f \in L_2(\Omega)\) that \(u \in H^2(\Omega)\) and the second summand on the right of \((2.5)\) is estimated by the constant from below. In other words the orders of smallness of the left and the right parts of the a posteriori bound are different, and the value \(O(h^2)\) on the left is estimated by the right part only with the order of unity. If \(l > 2,\) the left and the right parts are estimated with not equal orders \(O(h^{2(l-k)})\) and \(O(h^{2(l-k-1)}).\)

Inconsistency of the majorant \((2.7)\) at \(\sigma \leq ch^{-\alpha}, 0 \leq \alpha < 2, c = \text{const},\) is established in a similar way. Majorant \((2.6)\) is in general also inconsistent. The inconsistency of the above mentioned majorants, obviously, is retained, if finite elements of the class \(C\) are used and the test flux is found by some recovery procedure, satisfying only the requirements \(\alpha, \gamma).\)

The equality of the orders of smallness of the left and right parts of the majorants \((2.6)\) and \((2.13)\) is well provided, as follows, e.g., from \((2.11)\) and similar bound, proved in \([12]\). However, it is achieved only for the test fluxes, satisfying the additional conditions reflecting the requirement \(\beta),\) see \((2.8)\) and \((2.12).\) For this reason these majorants might be called conditionally consistent.

### 3 Modified Aubin’s a posteriori error majorant robust for all \(\sigma \geq 0\)

In this Section we derive the guaranteed, reliable and robust majorant\(^{2}\) which is well defined for all \(\sigma \geq 0.\) More over, it will be shown that it is consistent for the finite element solutions of the problems with sufficiently smooth data, see Section \([5]\). The new majorant will coincide with the Aubin’s majorant for \(\sigma \geq \sigma_*,\) where \(\sigma_*\) is some critical value, which can be differently defined for different numerical methods and different ways of the derivation of a posteriori bounds. In general, when \(v \in \tilde{H}^1_{\Gamma_D}(\Omega)\) is any approximation for \(u,\) \(\sigma_*\) can be defined from the inequality
\[
\frac{\|u - v\|^2_{L_2(\Omega)}}{\|u - v\|^2_{L_2(\Omega)}} \geq \sigma_* > 0.
\]

In some situations this inequality can be relaxed. Suppose that \(v = u_G\) is the approximate solution by the Galerkin method in the subspace \(V(\Omega) \subset \tilde{H}^1_{\Gamma_D}(\Omega, \Delta).\) Then it can be adopted

\(^2\)Definitions of the terms guaranteed, reliable, locally effective etc., used in relation to the a posteriori error estimates, can be found in \([3]\).
that \( \sigma_* \) satisfies
\[
\frac{\|u - v\|^2}{\|u - Qu\|^2_{L_2(\Omega)}} \geq \sigma_* > 0.
\] (3.16)

where \( Q \) is the operator of orthogonal projection \( L_2(\Omega) \to V(\Omega) \), i.e., such that for \( \forall \phi \in L_2(\Omega) \) we have
\[
(Q\phi, \psi)_\Omega = (\phi, \psi)_\Omega, \quad \forall \psi \in V(\Omega).
\]

Inequality (3.16) can be also used for \( v \) from any subspace \( V(\Omega) \subset H^1_D(\Omega) \), but in this case additional conditions on the test flax \( z \), arise. It is sufficient, for instance, that \( z \) satisfied to the equalities (3.32).

Let \( f(x) = \Pi^1 f \) for \( x \in \tau_r, \ r = 1, 2, \ldots, R \). In Theorem below domains \( \tau_r \) can be understood as arbitrary convex subdomains of some partition of the domain
\[
\Omega = \text{interior}\left\{ \bigcup_{1}^{R} \tau_r \right\}, \quad \tau_r \cup \tau_r' = \emptyset, \ r \neq r', \quad \text{diam}[\tau_r] = h_r,
\]
for which the Poincaré inequalities hold, see, e.g., Nazarov and Poborchi [28],
\[
\inf_{c \in \mathbb{R}} \|\phi - c\|_{L_2(\tau_r)} \leq \frac{h_r}{\pi}|\phi|_{H^1(\tau_r)}, \quad \phi \in H^1(\tau_r).
\]

**Theorem 3.1.** Let \( \Gamma_D = \partial \Omega \), conditions of Theorem [2.1] be fulfilled, and \( \sigma_* \) satisfies the inequality (3.15). Then
\[
|v - u|^2 \leq \Theta \mathcal{M}(\sigma, \sigma_*, f, v, z),
\] (3.17)

where
\[
\mathcal{M}(\sigma, \sigma_*, f, v, z) = \|A\nabla v + z\|^2_{H^{-1}[\Omega]} + \|f - \sigma v - \text{div} z\|^2_{L_2(\Omega)}.
\] (3.18)

and
\[
\Theta = \left\{ \begin{array}{l}
2/(1 + \kappa), \quad \forall \sigma \in [0, \sigma_*] \\
1, \quad \forall \sigma > \sigma_*
\end{array} \right\}, \quad \theta = \left\{ \begin{array}{l}
1/\sigma_*, \quad \forall \sigma \in [0, \sigma_*] \\
1/\sigma, \quad \forall \sigma > \sigma_*
\end{array} \right\}.
\] (3.19)

with \( \kappa = \sigma/\sigma_* \). Besides, for \( \sigma \in [0, \sigma_*] \) and \( \sigma \geq \sigma_* \), respectively, we have the bounds
\[
|v - u|^2 \leq \Theta_1 \mathcal{M}(\sigma, \sigma_*, \hat{f}, v, z) + \sum_{r} \frac{h_r^2}{\sigma_r} \int_{\tau_r} (f - \Pi^1 f)^2 dx, \quad \forall \varepsilon > 0,
\] (3.20)

\[
|v - u|^2 \leq \Theta_2 \mathcal{M}(\sigma, \sigma_*, \hat{f}, v, z) + \sum_{r} \frac{1}{\sigma_r} \int_{\tau_r} (f - \Pi^1 f)^2 dx,
\] (3.21)

where
\[
\Theta_1 = \left\{ \begin{array}{l}
(2 + \varepsilon)/(1 + \kappa), \quad 0 \leq \sigma \leq \sigma_*/(1 + \varepsilon), \\
1 + \varepsilon, \quad \sigma_*/(1 + \varepsilon) \leq \sigma \leq \sigma_*
\end{array} \right\},
\]

and
\[
\Theta_2 = 1 + \frac{1}{1 + \kappa^{-1}}.
\]

If \( v = u_G \) is the approximate solution by the method of Galerkin in the space \( V(\Omega) \subset H^1_D(\Omega, \Delta) \), then the bound (3.17)-(3.19) takes place with \( \sigma_* \), satisfying to the inequality (3.16) and \( z = z_G := -A\nabla u_G \), i.e.,
\[
|u_G - u|^2 \leq \Theta \mathcal{M}_G(\sigma, \sigma_*, f, u_G, z_G),
\]
\[
\mathcal{M}_G(\sigma, \sigma_*, f, u_G, z_G) = \theta \|f - \sigma u_G - \text{div} z_G\|^2_{L_2(\Omega)}.
\] (3.22)
Proof. Obviously, for \( \sigma \geq \sigma_* \) the majorant (3.17)-(3.19) coincides with the majorant (2.1), whereas for \( \sigma \in [0, \sigma_*] \) the majorants (3.17)-(3.19) and (2.1) are significantly different. Therefore, it is necessary to consider only the case \( \sigma \leq \sigma_* \). In order not to encumber the proof with secondary details, we assume in the proof that \( A = I \) and \( \psi_D \equiv 0 \).

For the solution \( u \) of the problem and arbitrary \( v \in H^1(\Omega) \) and \( z \in H(\Omega, \text{div}) \), we have
\[
|v - u|^2 = \int_\Omega [\nabla (v-u) \cdot \nabla (v-u) + \sigma (v-u) (v-u)] = 
\int_\Omega [(\nabla v + z) \cdot \nabla (v-u) - (z + \nabla u) \cdot \nabla (v-u) + \sigma (v-u) (v-u)].
\] (3.23)

Integrating by parts the second term on the right and using the inequality
\[
a_1 b_1 + a_2 b_2 \leq (a_1^2 + \frac{1}{\sigma_*}) (b_1^2 + \sigma_* b_2^2)^{1/2},
\] (3.24)
we find that
\[
|v - u|^2 = \|\nabla (u-v)\|_{L^2(\Omega)}^2 + \sigma \|u-v\|_{L^2(\Omega)}^2 \leq \left[ \|\nabla v - z\|_{L^2(\Omega)}^2 + \frac{1}{\sigma_*} \|f - \sigma v \text{div} z\|_{L^2(\Omega)}^2 \right]^{1/2} \times \left[ \|\nabla (u-v)\|_{L^2(\Omega)}^2 + \sigma_* \|u-v\|_{L^2(\Omega)}^2 \right]^{1/2}.
\] (3.25)

At \( \beta \in (0, 1] \) the inequality (3.15) allow us to transform the second multiplier in the right part of (3.25) to the form
\[
\|\nabla (u-v)\|_{L^2(\Omega)}^2 + \sigma_* \|u-v\|_{L^2(\Omega)}^2 = |u - v|^2 + (\sigma_* - \sigma) \|u-v\|_{L^2(\Omega)}^2 \leq
\]
\[
[u - v]^2 + (\sigma_* - \sigma) \left[ \frac{\beta}{\sigma_*} \|\nabla (u-v)\|_{L^2(\Omega)}^2 + (1 - \beta) \|u-v\|_{L^2(\Omega)}^2 \right] =
\]
\[
[1 + (\sigma_* - \sigma) \frac{\beta}{\sigma_*}] \|\nabla (u-v)\|_{L^2(\Omega)}^2 + [(1 - \beta) (\sigma_* - \sigma) + \sigma] \|u-v\|_{L^2(\Omega)}^2.
\] (3.26)

The choice \( \beta = 1/(1 + \kappa) \) makes the relation of the multipliers before second and first norms on the right of (3.26) equal to \( \sigma \). Substituting such \( \beta \) in (3.26) and then (3.26) in (3.25) leads to the inequality
\[
|v - u|^2 \leq \frac{2}{1 + \kappa} \left[ \|\nabla v - z\|_{L^2(\Omega)}^2 + \frac{1}{\sigma_*} \|f - \sigma v \text{div} z\|_{L^2(\Omega)}^2 \right]^{1/2} |v - u|^2,
\] (3.27)
which is equivalent to (3.17) at \( A = I \).

In order to prove (3.20), we transform (3.23) to the form
\[
|v - u|^2 = \int_\Omega [\nabla (v-u) \cdot \nabla (v-u) + \sigma (v-u) (v-u)] dx =
\]
\[
= \int_\Omega \left\{ (\nabla v + z) \cdot \nabla (v-u) + [\nabla \cdot z + \Delta u + \sigma (v-u)] (v-u) \right\} dx =
\]
\[
= \int_\Omega \left\{ (\nabla v + z) \cdot \nabla (v-u) + [\nabla \cdot z + \hat{f} + \sigma v] (v-u) + (\hat{f} - f) (v-u) \right\} dx.
\] (3.28)

We represent the integral of the last summand in the right part of (3.28) by the sum of the integrals over subdomains \( \tau_r, r = 1, 2, \ldots, R \), and each estimate with the help of Poincaré inequality:
\[
\int_{\tau_r} (\hat{f} - f) (v-u) dx = \int_{\tau_r} (\hat{f} - f) (v-u - c) dx \leq
\]
\[
\leq \|\hat{f} - f\|_{L^2(\tau_r)} \inf_{c \in \mathbb{R}} \|v - u - c\|_{L^2(\tau_r)} \leq
\]
\[
\leq \frac{d_{\tau_r}}{\varepsilon} \|\hat{f} - f\|_{L^2(\tau_r)} (\varepsilon |v - u|_{H^1(\tau_r)}) .
\] (3.29)
Now for getting (3.20) it is sufficient to substitute (3.29) in (3.28), to apply to the right part the Cauchy inequality and than to use the inequality (3.26) with $\beta$ satisfying the condition

$$
\sigma [1 + \varepsilon + (\sigma_* - \sigma)\frac{\beta}{\sigma_*}] = (1 - \beta)(\sigma_* - \sigma) + \sigma.
$$

The inequality (3.21) follows for $\beta = \frac{1}{2}\left(1 + \frac{1}{1 + \kappa^{-1}}\right)$,

from the estimates

$$
\|v - u\|^2 \leq \left[\|\nabla v - z\|_{L^2(\Omega)}^2 + \frac{1}{\sigma}\|f - \sigma v - \text{div} z\|_{L^2(\Omega)}^2 + \frac{1}{\sigma}\|\hat{f} - f\|_{L^2(\Omega)}^2\right]^{1/2} \times
$$

$$
\times \left[\|\nabla (u - v)\|_{L^2(\Omega)}^2 + 2\sigma\|u - v\|_{L^2(\Omega)}^2\right]^{1/2},
$$

$$
\|u - v\|^2_{L^2(\Omega)} \leq \beta\|u - v\|^2_{L^2(\Omega)} + (1 - \beta)\frac{1}{\sigma^2}\|\nabla (u - v)\|^2_{L^2(\Omega)},
$$

second of which is obtained with the use of (3.15).

Let us turn to the proof of the bound (3.22). For $e = v - u$ and $z \in H(\Omega, \text{div})$ we have

$$
\|e\|^2 = \int_{\Omega} \{(\nabla v + z) \cdot \nabla e + [f - \sigma v - \nabla \cdot z] e\} \, dx. \tag{3.31}
$$

Suppose, $z$ satisfies the identity

$$
\int_{\Omega} (f - \sigma v - \nabla \cdot z) \psi \, dx, \quad \forall \psi \in V(\Omega), \tag{3.32}
$$

on the finite element space $V(\Omega)$. If $v \in V(\Omega)$, then for $e_o = (Qu - u)$ the equalities

$$
e - Qe = v - u - Qv + Qu = Qu - u = e_o$$

hold and

$$
\int_{\Omega} [f - \sigma v - \nabla \cdot z] e \, dx = \beta \int_{\Omega} [f - \sigma v - \nabla \cdot z] (e - Qe) \, dx +
$$

$$
(1 - \beta) \int_{\Omega} [f - \sigma v - \nabla \cdot z] e \, dx = \beta \int_{\Omega} [f - \sigma v - \nabla \cdot z] e_o \, dx +
$$

$$
(1 - \beta) \int_{\Omega} [f - \sigma v - \nabla \cdot z] e \, dx. \tag{3.33}
$$

Substituting (3.33) in (3.31), and applying the Cauchy inequality and the inequality (3.16), we get

$$
\|e\|^2 \leq \{\|\nabla v + z\|^2 + \left[\frac{\beta}{\sigma_*} + \frac{1 - \beta}{\sigma}\right] \|f - \sigma v - \nabla \cdot z\|^2\}^{1/2} \times
$$

$$
\times \{1 + \beta\}\|e\|^2_{B} + (1 - \beta)c\|e\|^2_{L^2(\Omega)}\}^{1/2}. \tag{3.34}
$$

The use of $\beta = (1 - \kappa)/(1 + \kappa)$ leads to the estimate coinciding formally with (3.17)-(3.19), but with $\sigma_*$, satisfying the inequality (3.16). Now we note that when $V(\Omega) \subset H^1_{\Gamma_0}(\Omega, \Delta)$ the identity (3.32) holds. Besides, for $z = -A\nabla u_G$, now well defined, the norm $\|A\nabla v + z\|_{A^{-1}}$ equals zero, that proves the bound (3.22).
Remark 3.1. There are other ways of obtaining majorants similar to (3.17)–(3.19) and (3.20)–(3.21). For simplicity, let $A \equiv \mathbf{I}$ and $\Gamma D = \partial \Omega$. We can consider the subsidiary problem

$$- \Delta u + k u = f_{\lambda, \sigma}, \quad x \in \Omega, \quad u|_{\partial \Omega} = 0, \quad (3.35)$$

with an arbitrary $k \geq \sigma^*$ and $f_{\lambda, \sigma} = f + (k - \sigma)u$, whose solution is the same with the problem (2.1) at the specified $A$ and $\Gamma_D$. For the approximation $v$ of the solution to the problem (3.35) in Aubin’s majorant related to (3.35) one can use the approximation of the problem (2.1). After substitution of $f_{\lambda, \sigma} = f + (k - \sigma)u$ in the Aubin’s majorant, application of the Cauchy inequality with $\varepsilon$, and some manipulations, we come to the subsidiary majorant. By minimization of it in the respect of $k$, $\beta$ and $\varepsilon$, we come to the set majorants including similar to those in Theorem 3.1.

Obviously, by changing the choice of $\lambda$, $\beta$, and $\varepsilon$, we can change the weights before the first and second norms in the right parts of the majorants.

4 Consistent a posteriori majorants for finite element method errors

For specific classes of approximate solutions and, in particular, for solutions by the finite element method, the critical values $\sigma^*$ of the reaction coefficient in the derived error majorants can be estimated.

Lemma 4.1. Let $\Gamma_D = \partial \Omega$, $\sigma \equiv \text{const}$, $\psi_D \equiv 0$, $u \in \dot{H}^1(\Omega)$, $f \in H^{-1}(\Omega)$, the finite element assemblage generates the space $\mathcal{V}_{h,p}(\Omega)$, $p \geq 1$, and $e_{\text{fem}} = u_{\text{fem}} - u$. Then

$$\|e_{\text{fem}}\|_0 \leq c_1 h \|e_{\text{fem}}\|_A, \quad c_1 = \frac{\sqrt{\mu_2}}{\mu_1} c_\circ c_{\text{ap}}, \quad (4.36)$$

with the constants $c_\circ$, $c_{\text{ap}}$, defined below, see. (4.38), (4.45).

Proof. Let us consider the problem of finding the solution $\chi \in \dot{H}^1(\Omega)$ of the integral identity

$$a_\Omega(\chi, v) + \sigma(\chi, v)_\Omega = \langle F, v \rangle, \quad \forall v \in \dot{H}^1(\Omega), \quad (4.37)$$

where

$$a_\Omega(v, w) = \int_\Omega \nabla v \cdot A \nabla w \, dx.$$ 

If $\Omega$ is sufficiently smooth and $\sigma \geq 0$, then

$$\|\chi\|_2 \leq c_\circ \|F\|_0, \quad c_\circ = c_\circ(\Omega) = \text{const}, \quad (4.38)$$

with any $F \in L_2(\Omega)$. To prove this, we note that for $\sigma \leq 1$ the inequality holds, see Ladyzhenskaya and Uraltseva [27]. Obviously, at $\sigma \geq 1$ we have

$$\|\chi\|_0 \leq \sigma^{-1} \|F\|_0, \quad (4.39)$$

and by (4.38) for the problem

$$- \text{div}(A \text{grad} \chi) = F_\sigma, \quad F_\sigma = F - \sigma \chi, \quad \chi|_{\partial \Omega} = 0, \quad (4.40)$$
it follows that
\[ \| \chi \|_2 \leq c_\sigma \| F \|_0 \leq c_\sigma (\| F \|_0 + \sigma \| \chi \|_0) \leq 2c_\sigma \| F \|_0. \] (4.41)

It is left only to redefine the constant in (4.38).

Let us introduce the notations \( u_o, u_{fe} \) and \( u_s \) for the functions minimizing \( \| u - \phi \|_0^2, \| u - \phi \|^2_A \), and \( h^{-2}\| u - \phi \|_{0,\Omega}^2 + \| u - \phi \|^2_A \), respectively, among all \( \phi \in \tilde{V}_h(\Omega) \) and the notations for the respective errors \( e_o = u_o - u, u_{fe} = u_{fe} - u \) and \( e_s = u_s - u \). Since \( u_{fem} \) minimizes \( \| u - \phi \|^2, \phi \in \tilde{V}_h(\Omega) \), we conclude that
\[ \| e_{fem} \|^2_A + \sigma \| e_{fem} \|_0^2 \leq \| u - \tilde{u} \|^2_A + \sigma \| u - \tilde{u} \|_0^2, \] (4.42)
where \( \tilde{u} \) can be any from functions \( \tilde{u} = u_o, u_{fe}, u_s \). If to take into account the inequalities \( \| e_{fem} \|_0 \geq \| e_o \|_0 \) and \( \| e_{fem} \|_A \geq \| e_{fe} \|_A \), following from the definitions of fun \( u_o \) and \( u_{fe} \), then
\[ \| e_{fem} \|_0 \leq \| e_{fe} \|_0, \]
\[ \| e_{fem} \|_A \leq \| e_o \|_A. \] (4.43)

Let \( \phi \in \tilde{H}^1(\Omega) \) be the solution of the problem
\[ a_\Omega(v, \phi) = (v, e_{fe})_\Omega, \quad \forall v \in \tilde{H}^1(\Omega). \] (4.44)

Obviously, \( e_{fe} \in L_2(\Omega) \) and as a consequence of (4.38) and symmetry of the bilinear form \( a_\Omega(\cdot, \cdot) \), we have \( \phi \in H^2(\Omega) \) and
\[ \| \phi \|_2 \leq c_o \| e_{fe} \|_0. \]

let us approximate \( \phi \) by some function \( \phi_{ap} \in \tilde{V}_{h,p}(\Omega) \). We can use \( \phi_{ap} \in \tilde{V}_{h,1}(\Omega) \subset \tilde{V}_{h,p}(\Omega) \), and, in the case when the condition \( A \) is fulfilled, obtain it by means of the quasi-interpolation operator of Scott and Zhang [34]. In general, \( \phi_{ap} \) can be understood as the finite element solution, or \( L_2 \)-projection, or interpolation from \( \tilde{V}_{h,1}(\Omega) \) or \( \tilde{V}_{h,p}(\Omega) \). Let us underline that since in Lemma we only use the constant \( c_{ap} \) in the inequality
\[ | \phi - \phi_{ap} |^2 \leq c_{ap}^2 h^2 \| \phi \|_2^2 \leq c_o^2 c_{ap}^2 h^2 \| e_{fe} \|_0^2, \] (4.45)
but not the very function \( \phi_{ap} \), we can imply by \( \phi_{ap} \) any of the listed approximations which provides the better value of the constant.

Estimating \( \| e_{fe} \|_0 \) by means of the Aubin-Nitsche trick [4] for the problem (4.44) and using the bound (4.45), we get:
\[ \| e_{fe} \|_0^2 = \frac{1}{\mu_1} a_\Omega(e_{fe}, \phi) \leq \frac{1}{\mu_1} \inf_{w \in \tilde{V}_h(\Omega)} | a_\Omega(e_{fe}, \phi - w) | \leq \frac{\sqrt{p_2}}{\mu_1} \| e_{fe} \|_A \inf_{w \in \tilde{V}_h(\Omega)} | \phi - w | \leq \frac{\sqrt{p_2}}{\mu_1} \| e_{fe} \|_A | \phi - \phi_{ap} |_2 \leq \frac{\sqrt{p_2}}{\mu_1} c_o c_{ap} h \| e_{fe} \|_A \| e_{fe} \|_0. \] (4.46)
This bound together with the inequality (4.43) and the definitions of functions \( e_{fe}, e_{fem} \) results in the bound (4.36).
Theorem 4.1. Let $\Gamma_D = \partial \Omega$, $\psi_D \equiv 0$, and $u \in \bar{H}^1(\Omega, \Delta)$. Let also the finite element assembly generate the space $\bar{V}_0^0(\Omega) \subset H^1(\Omega)$, $p \geq 1$, and $u_{\text{fem}}$ be the solution by the finite element method. Then for $\sigma$ satisfying $0 \leq \sigma \leq \sigma_* = 1/(c_1 h)^2$, where $c_1 = \sqrt{\text{cap}}_1 c_\text{cap}$, and any $z \in H(\Omega, \text{div})$

$$|e_{\text{fem}}|^2 \leq \frac{2}{1 + c_1^2 h^2 \sigma} \mathcal{M}^{(1)}_{\text{fem}}(\sigma, \hat{f}, z),$$

$$\mathcal{M}^{(1)}_{\text{fem}}(\sigma, \hat{f}, z) = \left[ |A \nabla u_{\text{fem}} + z|^2_{L^2(\Omega)} + c_1^2 h^2 \| f - \sigma u_{\text{fem}} - \text{div} z \|^2_{L^2(\Omega)} \right].$$

Under the condition $A)$, for $\sigma \leq \sigma_*/(1 + \varepsilon)$ it holds also the bound

$$|e_{\text{fem}}|^2 \leq \frac{2 + \varepsilon}{1 + c_1^2 h^2 \sigma} \mathcal{M}^{(1)}_{\text{fem}}(\sigma, \hat{f}, z) + \sum_{i=1}^m \frac{h_{\tau_i}^2}{\varepsilon \pi^2} \int_{\tau_i} (f - \Pi^1_{\text{fem}} f)^2 \, dx, \quad \forall \varepsilon > 0.$$  

Proof. Since $\mathcal{M}^{(1)}_{\text{fem}}(\sigma, f, z) = \mathcal{M}(\sigma, \sigma_*, f, v, z)$ with $\sigma_*$ defined according to $(4.36)$, theorem is a direct consequence of Theorem 3.1 and Lemma 4.1.

The way of evaluation of the constant $c_1$, presented in the proof of Lemma 4.1 is rather general and can be expanded on the analogous a posteriori error bounds of the finite element method solutions for the $2n$th-order elliptic equations, $n \geq 1$, see Korneev [24]. The most complicated in it is evaluation of the constant $c_0$. However, in many cases such estimates are well known. For instance, if the domain is convex, then $|v|_2 \leq \| \Delta v \|_0$, see Ladyzhenskaya [26] (6.5) in ch. II]. Therefore, at $A = I$ and $\sigma \equiv 0$ we have $c_0 \leq 1$, and from $(4.41)$ we conclude that at least $c_0 \leq 2$ for any $\sigma \geq 0$. Existence of the constant $c_0$ poses some conditions on smoothness of the boundary and coefficients of the equation (if they are not constant). At the same time, there is the possibility to avoid the mentioned additional restrictions, except for those related to the suitable approximation operator. If there exists some interpolation type or other approximation operator with locally defined approximations for functions from $H^1(\Omega)$, then it is possible to show that the constants in the a posteriori bounds depend only on the local approximation properties of the finite element space. A good example of such an operator is the quasi-interpolation operator of Scott and Zhang [34], which will be used below to illustrate the statement. We start from the description of the properties of this operator needed for our purpose.

Let $\Omega \subset \mathbb{R}^m$, $m \geq 2$, be a bounded Lipschitz domain, which is the domain of the quasiuniform triangulation $T_h$ with vertices $x^{(i)}$, $i = 1, 2, \ldots, I$, and simplices $\tau_r$ of diameters not greater $h$. For simplicity it is assumed that faces of simplices are plain and that the following quasiuniformity conditions are fulfilled:

$$0 < c_{\Delta} \leq p_r / h_r, \quad c^{(1)} h \leq h_r \leq h,$$

(4.49)

where $p_r$ and $h_r$ are the radius of the largest inscribed sphere and the diameter of simplex $\tau_r$, respectively. To each vertex $x^{(i)}$, we relate $(m - 1)$-dimentional simplex $\tau^{(m-1)}_i$, which is the face of one of the simplices $\tau_r$ having $x^{(i)}$ for the vertex. For $m$ vertices of the simplex $\tau^{(m-1)}_i$ we will use also notations $z^{(i)}_l$, $l = 1, 2, \ldots, m$, assuming for definiteness that $z^{(i)}_1 = x^{(i)}$. Clearly the choice of the face $\tau^{(m-1)}_i$ is not unique, but for $x^{(i)} \in \partial \Omega$ we always take one of the faces $\tau^{(m-1)}_i \subset \partial \Omega$. We will formulate the result of Scott and Zhang using the simpler notations $V_{\Delta}(\Omega)$,
\[ V_{\text{tr}}(\partial \Omega), \text{ and } \bar{V}_\Delta(\Omega) \text{ for the space of continuous piece wise linear functions } V^0_{h,1}(\Omega), \text{its trace on the boundary, and its subspace of functions, vanishing on the boundary, respectively.} \]

We define functions \( \theta_i \in P_1(\tau_i^{(m-1)}) \), satisfying equations
\[
\int_{\tau_i^{(m-1)}} \theta_i \lambda_l^{(i)} \, dx = \delta_{i,l}, \quad l = 1, 2, \ldots, m, \tag{4.50}
\]
where \( \lambda_l^{(i)} \) are the barycentric coordinates in \( \tau_i^{(m-1)} \), corresponding to the vertices \( z_i^{(i)} \), and \( \delta_{i,l} \) is the Kronecker’s symbol. If \( \phi_i \in V_\Delta(\Omega) \) are the basis functions in \( V_\Delta(\Omega) \), defined by the equalities \( \phi_i(x_j) = \delta_{i,j} \), \( i, j = 1, 2, \ldots, I \), then for any \( v \in H^1(\Omega) \) the quasi-interpolation \( I_h v \) is the function
\[
I_h v = \sum_{i=1}^I \left( \int_{\tau_i^{(m-1)}} \theta_i v \, dx \right) \phi_i(x). \tag{4.51}
\]

**Lemma 4.2.** The quasi-interpolation operator \( I_h : H^1(\Omega) \to V_\Delta(\Omega) \) is a projection and has the following properties:

a) \( I_h v : H^1(\Omega) \to V_\Delta(\Omega) \) and, if \( v \in V_\Delta(\Omega) \), then \( I_h v = v \),

b) \( (v - I_h v) \in H^1(\Omega) \), if \( v|_{\partial \Omega} \in V_{\text{tr}}(\partial \Omega) \),

c) \( \|v - I_h v\|_{H^1(\Omega)} \leq c_{sz}(t, s) h^{s-1} \|v\|_{L^2(\Omega)} \) for \( t = 0, 1 \), \( s = 1, 2 \), and \( \forall v \in H^s(\Omega) \),

d) \( \|I_h v\|_{H^1(\Omega)} \leq \tilde{c}_{sz} \|v\|_{H^1(\Omega)} \) \( \forall v \in H^1(\Omega) \),

where \( c_{sz}(t, s) \), \( \tilde{c}_{sz} \), and \( \tilde{c} \) are positive constants, depending on \( c_\Delta \).

**Proof.** Scott and Zhang [31] proved a more general result. In a given form the lemma was formulated and proved by Xu and Zou [32]. \( \square \)

**Theorem 4.2.** Let \( \Gamma_D = \partial \Omega \), \( \psi_D = 0 \), \( u \in H^1(\Omega, \Delta) \). Let also the finite element assemblage satisfies the condition \( A \) and generates the space \( V^0_{h,1}(\Omega) \subset H^1(\Omega) \) and \( z \in H(\Omega, \text{div}) \). Then for \( v = u_{\text{fem}} \) at any \( \sigma \in [0, 1/(c_{sz}(0, 1) h^2)] \) holds the bound
\[
|v - u|^2 \leq \Theta_{sz} M_{\text{fem}}^{(2)}(\sigma, f, z), \quad M_{\text{fem}}^{(2)}(\sigma, f, z) = M(\sigma, \theta_{sz}^{-1}, f, u_{\text{fem}}, z), \tag{4.52}
\]
where
\[
\Theta_{sz} = \frac{1 + \tilde{c}_{sz}(1, 1)}{1 + \tilde{c}_{sz}(0, 1) h^2 \sigma}, \quad \theta_{sz} = c_{sz}(0, 1)^2 h^2. \tag{4.53}
\]
and \( \tilde{c}_{sz}(1, 1) \) is the constant, depending only upon \( c_\Delta \) and \( \hat{c}^{(1)} \), given in \( [4, 57] \).

**Proof.** For any \( w \in \hat{V}^0_{h,1}(\Omega) \) we have the equality
\[
|e_{\text{fem}}|^2 = \int_{\Omega} \left[ \nabla(e_{\text{fem}}) \cdot \nabla(e_{\text{fem}}) + \sigma e_{\text{fem}} e_{\text{fem}} \right] = \\
= \int_{\Omega} \left[ (\nabla u_{\text{fem}} + z) \cdot \nabla(e_{\text{fem}} + w) - (z + \nabla u) \cdot \nabla(e_{\text{fem}} + w) + \sigma(u_{\text{fem}} - u)(e_{\text{fem}} + w) \right]. \tag{4.54}
\]

Integration by parts of the second summand in square brackets of the right part and application of the Cauchy inequality with \( \epsilon > 0 \) result in the inequality
\[
|e_{\text{fem}}|^2 = \int_{\Omega} \left[ (\nabla u_{\text{fem}} + z) \cdot \nabla(e_{\text{fem}} + w) + (\text{div } z + \Delta u + \sigma(u_{\text{fem}} - u))(e_{\text{fem}} + w) \right] \leq \\
\leq \left\{ \|\nabla u_{\text{fem}} + z\|_0^2 + \frac{1}{\epsilon} \|f - \sigma u_{\text{fem}} - \text{div } z\|_0^2 \right\}^{1/2} \left\{ \|\nabla(e_{\text{fem}} + w)\|_0^2 + \epsilon \|e_{\text{fem}} + w\|_0^2 \right\}^{1/2} \tag{4.55}
\]
According to Lemma \[4.2\] and the definition of the operator \(Q\) of \(L_2\)-projection upon \(V_{h,1}^0(\Omega)\), for \(\phi - Q\phi\) with any \(\phi \in H^1(\Omega)\), there are valid the bounds
\[
\|\phi - Q\phi\|_0 \leq \|\phi\|_0 , \\
\|\phi - Q\phi\|_0 \leq c_{sz}(0,1)h\|\nabla\phi\|_0 , \\
\|\nabla(\phi - Q\phi)\|_0 \leq \tilde{c}_{sz}(1,1)h\|\nabla\phi\|_0 , \\
\tag{4.56}
\]
in which the constant \(\tilde{c}_{sz}(1,1)\) depends only on \(c_\Delta\) and \(\hat{\alpha}^{(1)}\). The proof is needed only for the last bound, and it follows from the relations
\[
\|\nabla(\phi - Q\phi)\|_0 \leq \|\nabla(\phi - I_h\phi)\|_0 + \|\nabla(I_h\phi - Q\phi)\|_0 \leq \tilde{c}_{sz}\|\nabla\phi\|_0 + c_{1,0}h^{-1}\left[\|I_h\phi - \phi\|_0 + \|\phi - Q\phi\|_0\right] \leq \\
\left(\tilde{c}_{sz} + 2c_{1,0}c_{sz}(0,1)\right)\|\nabla\phi\|_0 ,
\]
where \(c_{1,0}\) is the constant in the inverse inequality
\[
\|\nabla(I_h\phi - Q\phi)\|_0 \leq c_{1,0}h^{-1}\|I_h\phi - Q\phi\|_0 .
\]
Therefore,
\[
\tilde{c}_{sz}(1,1) = \tilde{c}_{sz} + 2c_{1,0}c_{sz}(0,1) .
\tag{4.57}
\]

It is worth noting, that the third inequality \(4.56\), indicating the stability in \(H^1(\Omega)\) of \(L_2\)-projection, was proved by Bramble and Xu \[8\] in a different way with a different way of evaluanging the constant \(\tilde{c}_{sz}(1,1)\).

For the reason that \(w = Qe_{fem} \in V_{h,1}^0(\Omega)\), it can be adopted \(w = Qe_{fem}\). Combining with \(4.56\) and setting \(\epsilon = \sigma_{sz} := (c_{sz}(0,1)h)^{-1}\) leads the bound
\[
\|\nabla(e_{fem} + w)\|_0^2 + c_{sz}\|e_{fem} + w\|_0^2 = \|\nabla(e_{fem} + w)\|_0^2 + \beta\|e_{fem} + w\|_0^2 + \\
+ (\sigma_{sz} - \beta\sigma)\|e_{fem} + w\|_0^2 \leq \tilde{c}_{sz}^2(1,1)\|\nabla e_{fem}\|_0^2 + \beta\|e_{fem}\|_0^2 + \frac{\sigma_{sz} - \beta\sigma}{\sigma_{sz}}\|\nabla e_{fem}\|_0^2 .
\tag{4.58}
\]

On the basis of \(4.58\) we conclude that
\[
\|\nabla(e_{fem} + w)\|_0^2 + \sigma_{sz}\|e_{fem} + w\|_0^2 \leq \frac{1 + \tilde{c}_{sz}^2(1,1)}{1 + \kappa}\left[\|\nabla e_{fem}\|_0^2 + \sigma\|e_{fem}\|_0^2\right],
\tag{4.59}
\]
with \(\kappa = \sigma / \sigma_{sz}\). Now from \(4.55\) and \(4.59\) the theorem follows.

\[\square\]

**Remark 4.1.** Quasi-interpolation operator \(I_h\) is defined in \[33\] on triangulations, satisfying the conditions of the shape regularity
\[
0 < c_\Delta \leq \rho_r / h_r , \quad h_r \leq h
\]
with preserving the properties a), b) and the properties c), d) taking the form
\[
\|v - I_h v\|_{1,\tau} \leq c_{sz}(t,s)h^{s-1}\|v\|_{s,\delta_r}, \quad t = 0, 1, s = 1, 2, \quad \forall v \in H^1(\delta_r) , \\
\|I_h v\|_{1,\tau} \leq \hat{c}_{sz}\|v\|_{1,\delta_r} \quad \text{and} \quad \|I_h v\|_{1,\tau} \leq \hat{c}_{sz}\|v\|_{1,\delta_r} \quad \forall v \in H^1(\delta_r) ,
\tag{4.60}
\]

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where \(c_{sz}(t, s) = \text{const}, \delta_r = \text{interior}\{\cup_{\Gamma_r} \Gamma_r : \Gamma_r \cap \tau_r \neq \emptyset\}\) and \(r = 1, 2, \ldots, \mathcal{R}\). More over, these authors designed also the quasi-interpolation operators \(I_{h_p} : H^1(\Omega) \rightarrow \mathbb{V}_{h,p}(\Omega)\), for which again the properties a), b) are preserved, but in (4.60) \(s = 1, 2, \ldots, p + 1\). These facts lead to some important generalizations of Theorem 4.2. One of them is the expansion of the a posteriori error bounds (4.52)-(4.53) on the solutions by the finite element methods from the spaces \(\mathbb{V}_{h,p}(\Omega)\), \(p > 1\). In this case, the constants related to \(I_h\) must be replaced by the respective constants related to the operator \(I_{h,p}\). Other possibility is the generalization on the case of piece wise constant reaction coefficient \(\sigma|_{\tau_r} = \sigma_r = \text{const} \geq 0\).

5  Consistency with a priori bounds, inverse like bound

This Section is dedicated to the properties of the suggested a posteriori error majorants, which are important for the construction of the convergent adaptive algorithms.

4.1. Consistency

More accurate a posteriori majorants (4.47), (4.48) and (4.53) of the finite element methods errors are consistent with the unimprovable a priori error estimates. In order to become convinced in this, let us first present the a priori approximation error estimates.

If \(v \in H^l(\Omega)\) and its finite element approximation \(\tilde{v}\) belongs to \(\mathbb{V}_{h}(\Omega) = \mathbb{V}_{h,p}(\Omega)\), \(1 \leq l \leq p+1\), then there are several ways to find such \(\tilde{v}\) that

\[
\|v - \tilde{v}\|_{k,\Omega} \leq c_{k,l} h^{l-k} \|v\|_{l,\Omega}, \quad k = 0, 1, \quad c_{k,l} = \text{const},
\]

(5.61)

where \(c_{k,l} = c_{k,l}(k, l, c_{s}, \alpha^{(1)})\), if the condition \(A)\) is fulfilled. In a more general case of the curvilinear finite elements \(c_{s}, \alpha^{(1)}\) can be understood as the corresponding characteristics from the conditions of generalized quasiumiformity. The bounds (5.61) can be found, e.g., in [13] [20] [34]. At \(p = 1, l = 1, 2\), and \(\tilde{u} = I_h u\) they are a concenuece, for instance, of Lemma 4.2. In general, if \(v = u\) is the solution of the reaction-diffusion problem, then for \(\tilde{v}\), as it follows from (4.43), one can take any of functions \(u_o, u_{fe}, \) or \(u_s\).

Lemma 5.1. Let \(\Gamma_D = \partial \Omega, \sigma \equiv \text{const} \geq 0, \psi_D \equiv 0, u \in \dot{H}^1(\Omega) \cap H^l(\Omega)\). Let also finite element assemblage generates the space \(\mathbb{V}_{h,p}(\Omega), \ p \geq 1\). Then for \(1 \leq l \leq p+1\) there are hold the bounds

\[
\|e_{\text{fem}}\|_0 \leq c_{0,l} h^l \|u\|_l,
\]

(5.62)

\[
\|e_{\text{fem}}\|_A \leq \sqrt{\mu_2 c_{1,l} h^{l-1}} \|u\|_l.
\]

(5.63)

with the constants \(c_{k,l}\) from (5.61). Besides, for \(\sigma \leq c_1^2 h^{-2}\) and for \(\sigma \geq c_1^2 h^{-2}\), \(f \in L_2(\Omega)\), respectively we have the bounds

\[
\|e_{\text{fem}}\|^2 \leq (c_{0,l}^2 + \mu_2 c_{1,l}^2) h^{2(l-1)} \|u\|_l^2,
\]

(5.64)

\[
\|e_{\text{fem}}\|^2 \leq \mu_2 c_{1,l}^2 h^{2(l-1)} \|u\|_l^2 + \sigma^{-1} \|f\|^2_0.
\]

(5.65)

Proof. Lemma follows from (4.43), (5.61), (4.39) and Lemmas 4.1 and 4.2.

Consistency of the a posteriori error bounds (4.47), (4.48) and (4.52)-(4.53) with the a priori bound (5.64) is established in the same way, and for the finite element solutions of an increased
smootheness \( u_{\text{fem}} \in \mathbb{V}_{h,p}^1(\Omega) \subset H^2(\Omega) \) it is practically evident. Note, that for such solutions additionally
\[
\|u - u_{\text{fem}}\|_2 \leq \tilde{c}_2 l h^{l-2} \|u\|_{l,\Omega}, \quad l \geq 2, \quad \tilde{c}_2 = \text{const}.
\] (5.66)
For the proof it is sufficient to implement the inverse inequality, the inequalities (2.2), the second inequality (4.43) and approximation estimates (5.61):
\[
\|e_{\text{fem}}\|_2 \leq \|u - u_{\text{int}}\|_2 + \|u_{\text{int}} - u_{\text{fem}}\|_2 \leq \|u - u_{\text{int}}\|_2 + ch^{-1} \|u_{\text{int}} - u_{\text{fem}}\|_1 \leq \|e_{\text{int}}\|_2 + c h^{-1} (\|e_{\text{int}}\|_1 + \sqrt{\frac{1}{\mu_1}} \|e_{\text{c}}\|_{\mathcal{A}}) \leq h^{l-2} [c_2 l + c_1 c_1 (1 + \sqrt{\frac{2}{\mu_1}})] \|u\|_{l,\Omega} = \tilde{c}_2 l h^{l-2} \|u\|_{l,\Omega}.
\]
Let us introduce for the right part of (4.47) the notation
\[
\eta_1^2 (e_{\text{fem}}) = \frac{2}{1 + c_1^2 h^2 \sigma} M_1^{(1)} (\sigma, f, z).
\]
In case \( u_{\text{fem}} \in \mathbb{V}_{h,p}^1(\Omega) \) it can be adopted \( z = z_{\text{fem}} = -A \nabla z_{\text{fem}}, \) and the diffusion component in \( \eta_1 (e_{\text{fem}}) \) vanishes, i.e., \( A \nabla u_{\text{fem}} + z_{\text{fem}} \big|_{\mathcal{A}^{-1}} = 0. \) Therefore, taking into account the inequality \( \sigma \leq 1/(c_1 h)^2, \) the first inequality (4.43), and the estimates (5.62), (5.66), we will have
\[
\eta_1^2 (e_{\text{fem}}) \leq \sqrt{2} c_1 h \|f - \sigma u_{\text{fem}} - \nabla z\|_0 \leq \sqrt{2} c_1 h \|\mathcal{L} e_{\text{fem}}\|_0 \leq \sqrt{2} \left[ \frac{1}{c_1 h} \|e_{\text{fe}}\|_0 + c_1 \mu_2 h |e_{\text{fem}}|_2 \right] \leq \sqrt{2} h^{l-2} \left[ \frac{\alpha_l}{c_1^2} + c_1 c_2 \mu_2 \right] \|u\|_{l,\Omega}.
\] (5.67)
In the order of \( h \) this bound is the same as the unimprovable in the order a priori bound (5.64).
Suppose \( u_{\text{fem}} \in \mathbb{V}_{h,p}^1(\Omega) \subset H^1(\Omega), \mathbb{V}_{h,p}^1(\Omega) \not\subset H^2(\Omega). \) Then it is natural to require that for the recovered flax \( z \) the same in the order estimates of convergence, as the ones reflected for the flax \( z_{\text{fem}} \) in Lemma 5.1 were hold alongside with the estimate
\[
\|\nabla (-A \nabla u - z)\|_0 \leq \tilde{c}_2 h^{l-2} \|u\|_{l,\Omega}, \quad \tilde{c}_2 = \text{const},
\]
corresponding to (5.66). Note, that the latter estimate follows from the former in the same way as (5.66) follows from (5.61). If the pointed out requirements are fulfilled the bound
\[
\eta_1 (e_{\text{fem}}) \leq c_1 h^{l-1} \|u\|_{l,\Omega}, \quad c_1 = \text{const},
\] (5.68)
is proved similarly to the similar bound (5.67).
It is worth noting that the a posteriori bounds, derived in this work, differ from a number of known bounds only in the coefficients before the norms in their right parts. Our coefficients have smaller or the same as earlier known orders. For this reason the efficient flax recovery algorithms, suggested earlier, are efficient for our bounds as well.

As was mentioned above, an a posteriori bound is unimprovable in the order, if it is consistent with the a priori bound unimprovable in the order. This means that in the class of solutions, for which the a priori bound is unimprovable in the order, there exist such that
\[
\eta_1 (e_{\text{fem}}) \leq C \|e_{\text{fem}}\|.
\] (5.69)
Indeed, let \( f \in L_2(\Omega), \ d = 2, \) and the inequality (4.38) is fulfilled. Then the inequalities (5.64) and (5.68) are fulfilled as well. At the same time, such \( f \in L_2(\Omega) \) exists that

\[
\|u\|_2 \leq c_2 \inf_{\phi \in \mathbb{V}_h(\Omega)} \|\nabla (u - \phi)\|_0 = c_2 \|\nabla e_{\text{fem}}\|_0 \leq c_2 \|\nabla e_{\text{fem}}\|_0 \leq c_2 \|\nabla e_{\text{fem}}\|_0, \quad c_2 = \text{const}.
\]

The first of these inequalities follows from the estimates of the \( N \)-width of the compact of functions \( v \in H^1(\Omega), \ |v| = 1, \) for \( N = h^{-2}, \) see [30, Гл. 4, п. 4.1]. In turn, from (5.68), (5.70) follows (5.69), which together with (4.47) yield the two sided bound

\[
|e_{\text{fem}}|^2 \leq \eta_i(e_{\text{fem}}) \leq C_i|e_{\text{fem}}|^2.
\]

4.2. Inverse like inequality

Derivation of a posteriori error majorants for numerical solutions was often accompanied by creation of the flux recovery algorithms, including algorithms with the equilibration, and by the proofs with their help of the inverse like and the local effectiveness bounds, see [17, 9, 15, 21, 30, 10, 11]. As was already mentioned, some of these results, accompanying majorants obtained by other authors, are straightforwardly expandable on the majorants, suggested in this paper. Moreover, the range of admissible fluxes, for which, \( e.g. \), the inverse like bounds hold, can be widened, because the equilibration of fluxes can became unnecessary due to a more adequate representation of the residual terms in the majorants. We illustrate these inferences by one example.

**Theorem 5.1.** Let the conditions of Theorem 4.1 are fulfilled and \( A = I, \ u_{\text{fem}} \in \mathbb{V}_{h,1}(\Omega). \) Let also the vector-function \( z \in \mathbb{W}_{h,2}(\Omega, \text{div}) \) be defined as \( \text{L}^2 \)-projection of the vector-function \( z_{\text{fem}} \) on the space \( \mathbb{W}_{h,2}(\Omega, \text{div}) \). Then for \( k = 1, 2 \)

\[
\mathcal{M}_{\text{fem}}^{(k)}(\sigma, f, z) \leq C \left[ |e_{\text{fem}}|^2 + \sum_{\kappa = 1}^R \frac{h_{\kappa}^2}{\pi^2} \int_{\tau_{\kappa}} (f - \Pi_{\kappa}^1 f)^2 dx \right]
\]

with the constant \( C = C(\Omega, c_\omega) \).

**Proof.** Let us consider the majorant \( \mathcal{M}^{(1)} \). Suppose, there exists \( y_\sigma \in \mathbb{W}_{h,\kappa}(\Omega, \text{div}), \ \kappa \geq 1, \) for which the estimate (5.72) holds with \( z = y_\sigma \), \( i.e., \)

\[
\mathcal{M}^{(k)}_{\text{fem}}(\sigma, f, y_\sigma) \leq C_\sigma \left[ |e_{\text{fem}}|^2 + \sum_{\kappa = 1}^R \frac{h_{\kappa}^2}{\pi^2} \int_{\tau_{\kappa}} (f - \Pi_{\kappa}^1 f)^2 dx \right].
\]

Let also \( y \) be orthogonal \( \text{L}^2 \)-projection of vector function \( z_{\text{fem}} \) upon the space \( \mathbb{W}_{h,\kappa}(\Omega, \text{div}) \). If to use the inequality

\[
\|\nabla u_{\text{fem}} - y\|_{\text{L}^2(\Omega)}^2 \leq \|\nabla u_{\text{fem}} - y_\sigma\|_{\text{L}^2(\Omega)}^2,
\]

and then the inequality (5.73), we obtain

\[
\mathcal{M}^{(1)}_{\text{fem}}(\sigma, f, y) \leq 2 \left\{ \mathcal{M}^{(1)}_{\text{fem}}(\sigma, f, y_\sigma) + (c_1 h)^2 \|\text{div}(y_\sigma - y)\|_{\text{L}^2(\Omega)}^2 \right\} \leq 2 \left\{ C_\sigma \left[ |e_{\text{fem}}|^2 + \sum_{\kappa = 1}^R \frac{h_{\kappa}^2}{\pi^2} \int_{\tau_{\kappa}} (f - \Pi_{\kappa}^1 f)^2 dx \right] + (c_1 h)^2 \|\text{div}(y_\sigma - y)\|_{\text{L}^2(\Omega)}^2 \right\}
\]

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The difference \( y_\circ - y \) belongs to the finite element space \( W_{h,\kappa}(\Omega, \text{div}) \) and, therefore, the inverse inequality is fulfilled. The use of the inverse inequality and (5.74), (5.73) yields the bound

\[
c_i^1 h \| \text{div}(y_\circ - y) \|_{L^2(\Omega)} \leq c_i^1 c_\| y_\circ - y \|_{L^2(\Omega)} \leq c_i^1 \| y_\circ - \nabla u_{\text{fem}} \|_{L^2(\Omega)} + \| \nabla u_{\text{fem}} - y \|_{L^2(\Omega)} \leq 2c_i^1 c_\| y_\circ - \nabla u_{\text{fem}} \|_{L^2(\Omega)} \leq 2c_i^1 c_\| y_\circ - y \|_{L^2(\Omega)}
\]

(5.76)

which together with (5.75) means validity of (5.72) for \( z = y \) under the condition of existence of \( y_\circ \) with the pointed out property.

To complete the proof for \( k = 1 \) and \( \kappa = 2 \) we note that Ainsworth and Vejchodský suggested the algorithm for evaluation of such a flux \( z_{AV} \in W_{h,2}(\Omega, \text{div}) \), that for \( y_\circ = z_{AV} \) the inequality (5.73), obviously, holds. The proof of the bound (5.72) for \( k = 2 \) is only slightly different.

Let \( u \) and \( u_{\text{fem}} \) are the exact and the finite element solutions to the problem in Theorem 5.1 and \( \hat{u} \) and \( \hat{u}_{\text{fem}} \) be the respective solutions of the same problem, but with the right part \( \hat{f} \) instead of \( f \). It is easy to notice that \( u_{\text{fem}} = u_{\text{fem}} \). Therefore, as a consequence of (4.47) and (5.72) for \( \hat{e}_{\text{fem}} = \hat{u}_{\text{fem}} - \hat{u} \), we have

\[
\| \hat{e}_{\text{fem}} \|^2 \leq \frac{2}{1 + c_i^2 h^2 \sigma} \mathcal{M}_{\text{fem}}^{(1)}(\sigma, \hat{f}, z) \leq C \| \hat{e}_{\text{fem}} \|^2,
\]

with the same \( z \) as in Theorem 5.1. Similar inequalities hold for \( \mathcal{M}_{\text{fem}}^{(2)} \).

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