A parafermionic hypergeometric function 
and supersymmetric $6j$-symbols

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Abstract

We study properties of a parafermionic generalization of the hyperbolic hypergeometric function appearing as the most important part in the fusion matrix for Liouville field theory and the Racah-Wigner symbols for the Faddeev modular double. We show that this generalized hypergeometric function is a limiting form of the rarefied elliptic hypergeometric function $V(r)$ and derive its transformation properties and a mixed difference-recurrence equation satisfied by it. At the intermediate level we describe symmetries of a more general rarefied hyperbolic hypergeometric function. An important $r = 2$ case corresponds to the supersymmetric hypergeometric function given by the integral appearing in the fusion matrix of $N = 1$ super Liouville field theory and the Racah-Wigner symbols of the quantum algebra $U_q(osp(1|2))$. We indicate relations to the standard Regge symmetry and prove some previous conjectures for the supersymmetric Racah-Wigner symbols by establishing their different parametrizations.

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1 Introduction

It is well known that in order to solve some problems in physics one should first master the corresponding mathematical tools. There are now many topics in modern mathematical physics where properties of the elliptic gamma functions [21, 27] and the Faddeev modular quantum dilogarithm [8, 9] (or the hyperbolic gamma function) play an important role. In particular, they are needed in the investigations of 4D \( N = 1 \) superconformal field theories [7], 3D \( N = 2 \) supersymmetric field theories [34], Liouville field theory [32], Ruijsenaars-Schneider and van Diejen integrable models [5, 22, 29], etc. In all these applications a crucial role is played by integrals of various products of these generalized gamma functions, which are called elliptic/hyperbolic hypergeometric integrals. In the case of 4D \( N = 1 \) supersymmetric theories elliptic hypergeometric integrals [26, 30] define superconformal indices [7], and in the case of 3D \( N = 2 \) supersymmetric theories hyperbolic hypergeometric integrals define partition functions [34].

For the Liouville field theory and Ruijsenaars-Schneider model a special integral of a product of eight hyperbolic gamma functions (given by formula (16) below) introduced by Ruijsenaars [22] is particularly important. It enters as the most essential part in the fusion matrix of the former model and gives eigenfunctions of the Hamiltonian in the latter case. From the group-theoretical point of view, it describes 6j-symbols for the Faddeev modular double derived by Ponsot and Teschner [17]. Various generalizations of the mentioned problems, such as studies of 4D \( N = 1 \) supersymmetric field theories on the \( L(r, 1) \times S^1 \) space-time, where \( L(r, 1) \) is the simplest lens space [19], 3D \( N = 2 \) field theories on \( L(r, 1) \) [34], and supersymmetric [11] or parafermionic extensions of the Liouville field theory [11], require consideration of the so-called rarefied or lens elliptic [19, 31] and hyperbolic [6, 10, 13, 23, 24] gamma functions and corresponding generalized hypergeometric functions.
In this paper we study a parafermionic generalization of the Ruijsenaars hyperbolic hypergeometric function (see expression (46) below) given by a sum of integrals of the product of eight rarefied hyperbolic gamma functions, as defined in [23]. We show that this function can be derived as a degeneration of the rarefied elliptic hypergeometric $V^{(r)}$-function introduced in [31]. Applying a specific limiting procedure to the transformations of the $V^{(r)}$-function found in [31], we deduce various symmetry properties of the parafermionic hypergeometric function and show that they can be considered as generalizations of the well known Regge symmetries of $6j$-symbols for the $SU(2)$ group.

We consider in detail the supersymmetric case corresponding to $r = 2$. Specializing the so-called second symmetry transformation to $r = 2$, we derive in a straightforward way a supersymmetric analogue of the formula obtained earlier for the Racah-Wigner symbols of the Faddeev modular double [33]. After that we analyze the supersymmetric Racah-Wigner symbols suggested in [12,16]. First we check that corresponding expressions indeed represent particular cases of the supersymmetric hypergeometric function. In particular, we show that all the weird sign factors that appear in these expressions come simply from the change of signs in the definition of the rarefied hyperbolic gamma functions entering the supersymmetric hypergeometric function (see equation (82)). Then we check that the restriction to the Neveu-Schwarz sector of the general supersymmetric Racah-Wigner symbol, suggested in [16], is in agreement with the Neveu-Schwarz sector Racah-Wigner symbols derived in [12]. And, finally, we give general expression for the supersymmetric Racah-Wigner symbols, considered in [16], in all available parametrizations for both the Neveu-Schwarz and Ramond sectors.

The paper is organized as follows. In Sect. 2, we describe necessary properties of the rarefied hyperbolic gamma function. In Sect. 3, we deduce transformation properties of the hyperbolic hypergeometric integral entering $6j$-symbols of the Faddeev modular double from the symmetry properties of the elliptic $V$-function. In Sect. 4, we review difference-recurrence equations for hyperbolic hypergeometric integrals. In Sect. 5, we define the parafermionic hypergeometric integral and derive its symmetries from the corresponding transformation properties of the rarefied elliptic $V^{(r)}$-function, whereas in Sect. 6 we obtain the difference-recurrence equation for it. In Sect. 7 we study in detail the supersymmetric case. Appendices contain some auxiliary material.

2 Properties of the rarefied elliptic and hyperbolic gamma functions

The standard elliptic gamma function $\Gamma(z;p,q)$ can be defined as a double infinite product:

$$\Gamma(z;p,q) = \prod_{j,k=0}^{\infty} \frac{1 - z^{-1}p^{j+1}q^{k+1}}{1 - zp^jq^k}, \quad |p|, |q| < 1, \quad z \in \mathbb{C}^\ast. \quad (1)$$

It is symmetric in bases $p$ and $q$, $\Gamma(z;p,q) = \Gamma(z;q,p)$, and satisfies the equations

$$\Gamma(qz;p,q) = \theta(z;p)\Gamma(z;q,p), \quad \Gamma(pz;p,q) = \theta(z;q)\Gamma(z;q,p),$$

where $\theta(z;p)$ is a short Jacobi theta-function

$$\theta(z;q) = (z;q)_\infty(qz^{-1};q)_\infty, \quad (z;q)_\infty := \prod_{j=0}^{\infty} (1 - zq^j),$$
which is related to the standard Jacobi $\theta_1$-function as follows

$$\theta_1(u|\tau) = -\theta_{11}(u) = -\sum_{\ell \in \mathbb{Z}+1/2} e^{\pi i \ell^2} e^{2\pi i \ell(u+1/2)}$$

$$= i q^{1/8} e^{-\pi i u(q;q)_{\infty}} \theta(e^{2\pi i u}; q), \quad q = e^{2\pi i \tau}.$$  

The elliptic gamma function associated with the simplest lens space is defined as a product of two standard elliptic gamma functions with different bases [19],

$$\gamma_e(z,m;p,q) = \Gamma(zp^m;p^r,pq)\Gamma(zq^{-m};q^r,pq)$$

$$= \prod_{j,k=0}^{\infty} \frac{1 - z^{-1}p^{-m}(pq)^{j+1}p^{r(k+1)}}{1 - z^{-1}q^{-m}(pq)^{j+1}q^{r(k+1)}} \frac{1 - z^{-1}q^m(pq)^{j+1}q^{r(k+1)}}{1 - z^{-1}q^m(pq)^{j+1}q^{r(k+1)}}, \quad m \in \mathbb{Z}. \tag{2}$$

As shown in [31], function (2) can be written as a special product of the standard elliptic gamma functions with bases $p^r$ and $q^r$. For the fundamental region $0 \leq m \leq r$ it has the form:

$$\gamma_e(z,m;p,q) = \prod_{k=0}^{m-1} \Gamma(q^{-m}z(pq)^k;p^r,q^r) \prod_{k=0}^{r-m-1} \Gamma(p^mz(pq)^k;p^r,q^r).$$

The quasiperiodicity property:

$$\frac{\gamma_e(z,m+kr;p,q)}{\gamma_e(z,m;p,q)} = \left( -\frac{\sqrt{pq}}{z} \right)^{mk+r(k-1)/2} \left( \frac{q}{p} \right)^k \left( \frac{m^2+mr+k-1+r^2(k-1)(2k-1)}{12} \right), \quad k \in \mathbb{Z}, \tag{3}$$

established in [31], implies, that for all other values of $m$, the function $\gamma_e(z,m;p,q)$ is determined by its form in the fundamental interval of $m$. For further use, it is also convenient to introduce the function:

$$\Gamma^{(r)}(z,m;p,q) = (-z)^{m(m-1)/2} p^{m(m-1)(m-2)/6} q^{m(m-1)(m+1)/6} \gamma_e(z,m;p,q). \tag{4}$$

The elliptic gamma function has the following asymptotic behaviour [18][21]:

$$\Gamma(e^{-2\pi vy}, e^{-2\pi v\omega_1}, e^{-2\pi v\omega_2}) = e^{-\pi(2y-\omega_1-\omega_2)/12\omega_1\omega_2} \gamma_2^{(2)}(y;\omega_1,\omega_2), \quad v \to 0$$

where $\gamma_2^{(2)}(y;\omega_1,\omega_2)$ is the Faddeev modular quantum dilogarithm [8,9] or the hyperbolic gamma function [21][22]. The parameter $v$ approaches 0 along the positive real axis and parameters $\omega_1$ and $\omega_2$ have positive real parts, Re($\omega_{1,2}$) > 0.

We use the shorthand notation $\gamma_2^{(2)}(y;\omega) := \gamma_2^{(2)}(y;\omega_1,\omega_2)$ when the quasiperiods coincide with $\omega_1$ and $\omega_2$. Otherwise the quasiperiods will be written explicitly. The function $\gamma_2^{(2)}(y;\omega)$ has the integral representation

$$\gamma_2^{(2)}(y;\omega) = \exp \left( - \int_0^\infty \left( \frac{\sinh(2y-\omega_1-\omega_2)x}{2 \sinh(\omega_1x) \sinh(\omega_2x)} - \frac{2y-\omega_1-\omega_2}{2\omega_1\omega_2x} \right) \frac{dx}{x} \right) \tag{6}$$

and obeys the equations:

$$\frac{\gamma_2^{(2)}(y+\omega_1;\omega)}{\gamma_2^{(2)}(y;\omega)} = 2 \sin \frac{\pi y}{\omega_2}, \quad \frac{\gamma_2^{(2)}(y+\omega_2;\omega)}{\gamma_2^{(2)}(y;\omega)} = 2 \sin \frac{\pi y}{\omega_1}. \tag{7}$$
It has the following asymptotics [14]:

\[
\lim_{y \to \infty} \gamma^{(2)}(y; \omega) = e^{-iB_{2,2}(y;\omega)/r}, \quad \text{for arg } \omega_1 < \arg y < \arg \omega_2 + \pi,
\]

\[
\lim_{y \to \infty} \gamma^{(2)}(y; \omega) = e^{iB_{2,2}(y;\omega)/r}, \quad \text{for arg } \omega_1 - \pi < \arg y < \arg \omega_2,
\]

where \(B_{2,2}(y; \omega)\) is the second order Bernoulli polynomial:

\[
B_{2,2}(y; \omega) = \frac{1}{\omega_1 \omega_2} \left( (y - \omega_1 + \omega_2)^2 - \frac{\omega_1^2 + \omega_2^2}{12} \right).
\]

Using asymptotics [5] one can show that [23]:

\[
\gamma_{e} \left( e^{-\frac{2\pi y \omega}{r}}; m; e^{-\frac{2\pi \omega_1}{r}}; e^{-\frac{2\pi \omega_2}{r}} \right) = e^{-\pi(2y-\omega_1-\omega_2)/12\omega_1\omega_2} \Lambda(y; m; \omega),
\]

where the function \(\Lambda(y, m; \omega)\) is defined as follows. For the fundamental region \(0 \leq m \leq r\) one has

\[
\Lambda(y, m, \omega) = \prod_{k=0}^{m-1} \gamma^{(2)} \left( \frac{y}{r} + \omega_2 \left( 1 - \frac{m}{r} \right) + \left( \omega_1 + \omega_2 \right) \frac{k}{r}; \omega \right)
\times \prod_{k=0}^{r-m-1} \gamma^{(2)} \left( \frac{y}{r} + \frac{m}{r} \omega_1 + \left( \omega_1 + \omega_2 \right) \frac{k}{r}; \omega \right).
\]

Quasiperiodicity relation [3] in the limit [9] yields

\[
\Lambda(y, m + kr; \omega) = (-1)^{mk+r\frac{k(k-1)}{2}} \Lambda(y, m; \omega).
\]

As we see, the function \(\Lambda(y, m; \omega)\) for all the values of \(m\) is determined by its form in the fundamental region. We also have from the asymptotics [9] and definition [4]:

\[
\Gamma^{(r)} \left( e^{-\frac{2\pi y \omega}{r}}; m; e^{-\frac{2\pi \omega_1}{r}}; e^{-\frac{2\pi \omega_2}{r}} \right) = e^{-\pi(2y-\omega_1-\omega_2)/12\omega_1\omega_2} \left( -1 \right)^{\frac{m(m-1)}{2}} \Lambda(y, m; \omega).
\]

Applying the limit [9] to definition [2] one can derive another expression for the function \(\Lambda(y, m; \omega)\):

\[
\Lambda(y, m; \omega) = \gamma^{(2)} \left( \frac{y + m \omega_1}{r}; \omega_1; \omega_1 + \omega_2 \right) \gamma^{(2)} \left( \frac{y + (r-m) \omega_2}{r}; \omega_2; \omega_1 + \omega_2 \right).
\]

Using this expression one can show that the function \(\Lambda(y; m; \omega)\) has the following asymptotics [15][23]:

\[
\lim_{y \to \infty} \Lambda(y, m; \omega) = e^{-i\frac{B_{2,2}(y;\omega)}{r}} \left( \frac{1}{r} B_{2,2}(y;\omega) + \frac{m^2}{r} \omega_1 + \frac{m}{r} \omega_2 + \frac{1}{2r} \right), \quad \text{for arg } \omega_1 < \arg y < \arg \omega_2 + \pi,
\]

\[
\lim_{y \to \infty} \Lambda(y, m; \omega) = e^{i\frac{B_{2,2}(y;\omega)}{r}} \left( \frac{1}{r} B_{2,2}(y;\omega) + \frac{m^2}{r} \omega_1 + \frac{m}{r} \omega_2 - \frac{1}{2r} \right), \quad \text{for arg } \omega_1 - \pi < \arg y < \arg \omega_2.
\]

We also would like to note that below we use the following shorthand notation:

\[
\Gamma(az^{\pm k}; p, q) \equiv \Gamma(az^{k}; p, q)\Gamma(az^{-k}; p, q),
\]

\[
\Gamma^{(r)}(az^{\pm k}; m \pm n; p, q) \equiv \Gamma^{(r)}(az^{k}; m + n; p, q)\Gamma^{(r)}(az^{-k}; m - n; p, q),
\]

\[
\gamma^{(2)}(y \pm x; \omega) \equiv \gamma^{(2)}(y + x; \omega)\gamma^{(2)}(y - x; \omega),
\]

\[
\Lambda(y \pm x; m \pm n; \omega) \equiv \Lambda(y + x; m + n; \omega)\Lambda(y - x; m - n; \omega),
\]

\[
\gamma^{(2)}(y, x; \omega) \equiv \gamma^{(2)}(y; \omega)\gamma^{(2)}(x; \omega),
\]

\[
\theta(x, y, z; p) \equiv \theta(x; p)\theta(y; p)\theta(z; p).
\]
3 Symmetries of $6j$-symbols for the Faddeev modular double

In this section we review known symmetry properties of the following hyperbolic hypergeometric function introduced by Ruijsenaars in 1994 \cite{21}

$$J_h(\beta, \gamma) = \int_{-i\infty}^{i\infty} \prod_{a=1}^{4} \gamma^{(2)}(\beta_a - z; \omega) \gamma^{(2)}(\gamma_a + z; \omega) \frac{dz}{i\sqrt{\omega_1\omega_2}} \quad (16)$$

with the parameters $\beta_a, \gamma_a$ satisfying the balancing condition

$$\sum_{a=1}^{4}(\gamma_a + \beta_a) = 2(\omega_1 + \omega_2), \quad (17)$$

and explain how they emerge from symmetries of the elliptic analogue of the Euler–Gauss hypergeometric function. The contour of integration in (16) is of the Mellin-Barnes type. Namely, it separates the sequences of poles going to infinity in the right half-plane from those lying in the left half-plane. For generic values of $\beta_a$ and $\gamma_a$ such a contour always exists. With the conventional choice $\omega_1 = b$ and $\omega_2 = b^{-1}$, the function (16) defines a key element in the construction of $6j$-symbols of the Faddeev modular quantum double and fusion matrix of the Liouville field theory with the central charge $c = 1 + 6(b + b^{-1})^2$ \cite{17,32}.

3.1 Symmetry $I_a$

Consider the $V$-function, an elliptic analogue of the Euler–Gauss hypergeometric function introduced in \cite{30},

$$V(g_1, \ldots, g_8; p, q) = \frac{(p, p)^\infty (q, q)^\infty}{4\pi i} \int_{T} \prod_{a=1}^{8} \frac{\Gamma(g_a z^{\pm 1}; p, q)}{\Gamma(z^{\pm 2}; p, q)} \frac{dz}{z}, \quad (18)$$

where $T$ is the unit circle with positive orientation, the parameters satisfy constraints $|g_a| < 1$ and the balancing condition $\prod_{a=1}^{8} g_a = p^2 q^2$. Here we used the shorthand notation (15). This function has the $W(E_7)$ Weyl group of symmetry transformations, whose key generating relation has been established in \cite{27}:

$$V(g_1, \ldots, g_8; p, q) = \prod_{1 \leq j < k \leq 4} \Gamma(g_j g_k; p, q) \prod_{5 \leq j < k \leq 8} \Gamma(g_j g_k; p, q) V(\tilde{g}_1, \ldots, \tilde{g}_8; p, q), \quad (19)$$

where

$$\tilde{g}_j = \rho^{-1} g_j, \quad \tilde{g}_{j+4} = \rho g_{j+4}, \quad j = 1, 2, 3, 4, \quad \rho = \frac{\sqrt{g_1 g_2 g_3 g_4}}{p q} = \sqrt{\frac{p q}{g_5 g_6 g_7 g_8}}$$

with $|\tilde{g}_a| < 1$.

Introduce a hyperbolic hypergeometric function $I_h(\mathbf{s})$ defined by the integral

$$I_h(\mathbf{s}) = \int_{-i\infty}^{i\infty} \prod_{j=1}^{8} \gamma^{(2)}(s_j \pm z; \omega) \gamma^{(2)}(\pm 2z; \omega) \frac{dz}{2i\sqrt{\omega_1\omega_2}}. \quad (20)$$
with \( s_j \) satisfying the conditions \( \text{Re}(s_j) > 0 \) and
\[
\sum_{j=1}^{8} s_j = 2Q, \quad Q := \omega_1 + \omega_2
\] (21)
(for the notation, see (15)). Applying the hyperbolic degeneration limit (5) to the transformation rule (19), one comes to the following identity [2]
\[
I_h(\tilde{s}) = \prod_{1 \leq j < k \leq 4} \gamma(2)(s_j + s_k; \omega) \prod_{5 \leq j < k \leq 8} \gamma(2)(s_j + s_k; \omega) I_h(\tilde{s}),
\] (22)
where
\[
\tilde{s}_j = s_j + \eta, \quad \tilde{s}_{j+4} = s_{j+4} - \eta, \quad j = 1, 2, 3, 4, \quad \eta = \frac{1}{2}(\omega_1 + \omega_2 - \sum_{j=1}^{4} s_j)
\]
with \( \text{Re}(\tilde{s}_j) > 0 \). Let us parametrize
\[
s_{1,2,5,6} = \gamma_{1,2,3,4} + i\mu, \quad s_{3,4,7,8} = \beta_{1,2,3,4} - i\mu
\] (23)
in (22), shift the integration variable \( z \rightarrow z - i\mu \) on both sides of the equality, and take the limit \( \mu \rightarrow -\infty \). Then we obtain [2]:
\[
J_h(\beta, \gamma) = \prod_{j,k=1}^{2} \gamma(2)(\beta_j + \gamma_k; \omega) \prod_{j,k=3}^{4} \gamma(2)(\beta_j + \gamma_k; \omega)
\]
\[
\times J_h(\beta_1 + \eta, \beta_2 + \eta, \beta_3 - \eta, \beta_4 - \eta, \gamma_1 + \eta, \gamma_2 + \eta, \gamma_3 - \eta, \gamma_4 - \eta),
\] (24)
where \( J_h(\beta, \gamma) \) is the Ruijsenaars function (16), (17), and \( \eta = \frac{1}{2}(\omega_1 + \omega_2 - \beta_1 - \beta_2 - \gamma_1 - \gamma_2) \).

3.2 Symmetry \( I_h \)
Define another hyperbolic hypergeometric integral:
\[
E_h(\rho) = \int_{-\infty}^{\infty} \prod_{i=1}^{6} \gamma(2)(\rho_i \pm z; \omega) \frac{dz}{\gamma(2)(\pm 2z; \omega) 2i\sqrt{\omega_1 \omega_2}}
\] (25)
(for notation, see (15)). Applying in equality (22) a different parametrization
\[
s_{1,2,3,8} = \beta_{1,2,3,4} + i\mu, \quad s_{4,5,6,7} = \gamma_{1,2,3,4} - i\mu,
\]
shifting the integration variable \( z \rightarrow z + i\mu \) only in the left-hand side integral, and taking the limit \( \mu \rightarrow \infty \), one finds [2]:
\[
J_h(\beta, \gamma) = \prod_{i=1}^{3} \gamma(2)(\beta_i + \gamma_4; \omega) \gamma(2)(\gamma_i + \beta_4; \omega)
\]
\[
\times E_h(\beta_1 + \xi, \beta_2 + \xi, \beta_3 + \xi, \gamma_1 - \xi, \gamma_2 - \xi, \gamma_3 - \xi),
\] (26)
where \( 2\xi = Q - \gamma_4 - \sum_{i=1}^{3} \beta_i \).
3.3 Symmetry II

The second type of identities follows from the key generating relation (19) after a group action composition (a repetition of (19) after a permutation of parameters),

\[ V(g_1, \ldots, g_8; p, q) = \prod_{j,k=1}^{4} \Gamma(g_j g_{k+4}; p, q)V\left(\frac{T^{1/2}}{g_1}, \ldots, \frac{T^{1/2}}{g_4}, \frac{U^{1/2}}{g_5}, \ldots, \frac{U^{1/2}}{g_8}; p, q\right), \]

(27)

where \( T = g_1 g_2 g_3 g_4 \), \( U = g_5 g_6 g_7 g_8 \), and \(|T^{1/2}/g_j|, |U^{1/2}/g_{j+4}| < 1, j = 1, \ldots, 4\). The hyperbolic degeneration limit (8) for integrals described in the previous sections reduces relation (27) to the following identity for \( I_h(s) \) function (29):

\[ I_h(s) = 4 \prod_{j,k=1}^{4} \gamma^{(2)}(s_j + s_{k+4}; \omega)I_h(G - s_1, \ldots, G - s_4, Q - G - s_5, \ldots, Q - G - s_8), \]

(28)

where \( G := \frac{1}{4} \sum_{j=1}^{4} s_j \) and \( Q = \omega_1 + \omega_2 \). Now, for the parametrization (28), the \( \mu \rightarrow \infty \) limit yields

\[ J_h(\beta, \gamma) = \prod_{j,k=1}^{4} \gamma^{(2)}(\gamma_j + \beta_{k+2}; \omega)\gamma^{(2)}(\gamma_{j+2} + \beta_k; \omega)J_h(G - \gamma_1, G - \gamma_2, Q - G - \gamma_3, Q - G - \gamma_4; G - \beta_1, G - \beta_2, Q - G - \beta_3, Q - G - \beta_4), \]

(29)

where \( G = \frac{1}{2}(\gamma_1 + \gamma_2 + \beta_1 + \beta_2) \).

Recall now the Ponsot-Teschner formula (17) for 6j-symbols of the Faddeev modular quantum double \( U_q(sl(2, \mathbb{R}) \otimes U_q(sl(2, \mathbb{R})) \), \( q = e^{\pi i b^2} \) and \( \bar{q} = e^{\pi i b^{-2}} \):

\[ \{ \alpha_1 \alpha_2 | \alpha_3 \alpha_4 \}_{b} = \frac{S_b(\alpha_3 + \alpha_2 - \alpha_1)S_b(\alpha_1 + \alpha_4 - \alpha_3)}{S_b(\alpha_3 + \alpha_2 - \alpha_1)S_b(\alpha_1 + \alpha_4 - \alpha_3)}|S_b(2\alpha_t)|^2 J_h(\beta_a^0, \gamma_a^0; b), \]

(30)

where

\[ J_h(\beta_a^0, \gamma_a^0; b) = \int_{-\infty}^{\infty} \prod_{a=1}^{4} S_b(z + \gamma_a^0)S_b(-z + \beta_a^0) \frac{dz}{i}, \quad S_b(x) := \gamma^{(2)}(x, b, b^{-1}), \]

and \( \gamma_a^0, \beta_a^0, a = 1, \ldots, 4 \), are the Ponsot-Teschner parameters:

\[
\begin{align*}
\gamma_1^0 &= -Q/2 + \alpha_3 - \alpha_4, \quad \beta_1^0 = Q/2 + \alpha_s, \\
\gamma_2^0 &= -Q/2 + \alpha_1 - \alpha_2, \quad \beta_2^0 = Q/2 - \alpha_t + \alpha_4 + \alpha_2, \\
\gamma_3^0 &= Q/2 - \alpha_3 - \alpha_4, \quad \beta_3^0 = -Q/2 + \alpha_t + \alpha_4 + \alpha_2, \\
\gamma_4^0 &= Q/2 - \alpha_1 - \alpha_2, \quad \beta_4^0 = 3Q/2 - \alpha_s.
\end{align*}
\]

Expression (30) gives also, up to some normalization factors, the fusion matrix of the Liouville field theory with the central charge \( c = 1 + 6Q^2 \). In fact, in this form the parameters (31) appeared in (33), but they can be easily derived from the ones used in (17) after shifting the integration variable by \( -Q/2 - \alpha_s \). Here and below when we shift the integration variable in such a way, we assume that the integration contour is deformed in an appropriate way.
Using relation $2G = \alpha_q - \alpha_t + \alpha_1 + \alpha_3$ and also shifting the integration variable $z$ by $G + Q + \beta_3$ in the integral standing on the right-hand side of relation (29), we obtain the identity

$$J_h(\beta^0, \gamma^0) = \Omega(\alpha)J_h(\beta^0, \gamma^0),$$

(32)

where

$$\gamma_1 = \alpha_{1234}, \quad \gamma_2 = 2Q, \quad \beta_1^0 = -\alpha_{23t}, \quad \beta_3^0 = -\alpha_{12s},$$

(33)

and

$$\Omega(\alpha) = \gamma(2)(Q + \alpha_s - \alpha_3 - \alpha_4; \omega)\gamma(2)(Q + \alpha_s - \alpha_1 - \alpha_2; \omega) \times \gamma(2)(Q - \alpha_t + \alpha_2 - \alpha_3; \omega)\gamma(2)(Q - \alpha_t + \alpha_4 - \alpha_1; \omega) \times \gamma(2)(-Q + \alpha_t + \alpha_2 + \alpha_3; \omega)\gamma(2)(Q - \alpha_s + \alpha_3 - \alpha_4; \omega) \times \gamma(2)(-Q + \alpha_t + \alpha_4 + \alpha_3; \omega)\gamma(2)(Q - \alpha_s + \alpha_1 - \alpha_2; \omega).$$

(34)

Equality (32) was derived in (33) in a substantially more complicated way, and it was used for finding of the hyperbolic volume of a non-ideal tetrahedron in the quasi-classical limit of 6j-symbols for the Faddeev modular double.

3.4 Symmetry III

The third form of the symmetry transformation for the $V$-function follows from equating right-hand side expressions in (19) and (27),

$$V(g_1, \ldots, g_8; p, q) = \prod_{1 \leq j < k \leq 8} \Gamma(g_j g_k; p, q)V\left(\frac{\sqrt{pq}}{g_1}, \ldots, \frac{\sqrt{pq}}{g_8}, p, q\right).$$

(35)

The hyperbolic degeneration limit (5) results in the following symmetry transformation for the $I_h(\mathbf{z})$ function

$$I_h(\mathbf{z}) = \prod_{1 \leq j < k \leq 8} \gamma(2)(s_j + s_k; \omega)I_h\left(\Delta\right), \quad \lambda_j = \frac{\omega_1 + \omega_2}{2} - s_j.$$

(36)

For reducing this relation, we replace in (36)

$$s_j \rightarrow \gamma_j + i\mu, \quad s_{j+4} = \beta_j - i\mu, \quad j = 1, \ldots, 4, \quad z \rightarrow z - i\mu,$$

where $z$ denotes integration variable on both sides of the equality. The balancing condition takes the form $\sum_{j=1}^{4} (\beta_j + \gamma_j) = 2Q$. After going to the limit $\mu \rightarrow -\infty$, we come to the identity (25)

$$\int_{-\infty}^{\infty} \prod_{j=1}^{4} \gamma(2)(\gamma_j + z; \omega)\gamma(2)(\beta_j - z; \omega)dz = \prod_{j,k=1}^{4} \gamma(2)(\gamma_j + \beta_k; \omega) \times \int_{-\infty}^{\infty} \prod_{j=1}^{4} \gamma(2)(\frac{1}{2}Q - \beta_j + z; \omega)\gamma(2)(\frac{1}{2}Q - \gamma_j - z; \omega)dz.$$

(37)
3.5 Relation to the Regge symmetry

The standard Regge symmetry for the $SU(2)$ group Racah-Wigner symbols corresponds to a particular reflection transformation invariance \cite{20}:

$$\{ \alpha_1 \alpha_2 | \alpha_3 \alpha_4 \} = \{ \frac{S-\alpha_3}{s_3} \frac{S-\alpha_2}{s_2} | \alpha_3 \alpha_4 \},$$  \hspace{1cm} (38)

where $S = \frac{1}{2}(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)$. Consider the effect of this reflection of $\alpha_i$, $i = 1, \ldots, 4$, for the parameters entering Ponsot-Teschner 6j-symbols in both the original definition \cite{30} and the transformation \cite{32}.

The transformation $\alpha_i \to S - \alpha_i$ brings parameters \cite{31} to the form

$$\begin{align*}
\gamma_1^r &= -Q/2 - \alpha_3 + \alpha_4, & \beta_1^r &= Q/2 + \alpha_s, \\
\gamma_2^r &= -Q/2 - \alpha_1 + \alpha_2, & \beta_2^r &= Q/2 - \alpha_t + \alpha_1 + \alpha_3, \\
\gamma_3^r &= Q/2 - \alpha_1 - \alpha_2, & \beta_3^r &= -Q/2 + \alpha_t + \alpha_1 + \alpha_3, \\
\gamma_4^r &= Q/2 - \alpha_3 - \alpha_4, & \beta_4^r &= 3Q/2 - \alpha_s.
\end{align*}$$  \hspace{1cm} (39)

The same form of parameters follows from the permutation symmetry and transformation $L_a$ \cite{24}. Indeed, shifting the integration variable for $J_h$ in the right-hand side of \cite{24} by $\eta$, we can write

$$J_h(\beta_1 + \eta, \beta_2 + \eta, \beta_3 - \eta, \beta_4 - \eta, \gamma_1 + \eta, \gamma_2 + \eta, \gamma_3 - \eta, \gamma_4 - \eta)$$

$$= J_h(\beta_1 + 2\eta, \beta_2 + 2\eta, \beta_3 + \beta_4, \gamma_1 + 2\eta, \gamma_2 - 2\eta, \gamma_3 + 2\eta, \gamma_4 - 2\eta).$$  \hspace{1cm} (40)

Since the left-hand side expression in \cite{24} is symmetric in $\beta_a$ and $\gamma_a$, we can permute indices of the latter parameters in the right-hand side as well. If we take a partner of \cite{24} with the following function on the right-hand side

$$J_h(\beta_1^r, \beta_2^r + 2\eta, \beta_3^r + 2\eta, \beta_4^r, \gamma_1^r - 2\eta, \gamma_2^r - 2\eta, \gamma_3^r + 2\eta, \gamma_4^r)$$

where $2\eta = Q - (\beta_2^r + \beta_3^r + \gamma_3^r + \gamma_4^r) = \alpha_1 - \alpha_2 + \alpha_3 - \alpha_4$, we obtain parameters \cite{39} in a permuted order.

If we apply Regge reflection to parameters \cite{33}, which we recall were obtained from \cite{31} by the second symmetry transformation, we just permute them with each other. Therefore the latter parametrization has the advantage to make $J_h(\beta_1^r, \gamma_1^r)$ explicitly Regge invariant. One can check, that in this parametrization the prefactors of $J_h$-function in the expression for 6j-symbols are also invariant under this transformation. In the original parametrization \cite{31} the invariance of 6j-symbols follows from the nontrivial additional $S_0$-function factors emerging in the transformation \cite{24}.

This analysis shows that we can consider symmetries of the generalized hypergeometric functions as extensions of the Regge and permutation group symmetries of 6j-symbols for the $SU(2)$ group to the 6j-symbols of the Faddeev modular quantum double. It is interesting to note that from the latter 6j-symbols one can obtain by various limiting procedures 6j-symbols of the unitary principal series representations of the $SL(2, \mathbb{C})$ group \cite{4}, and of the $U_q(su(2))$ quantum group \cite{16}. So, the
work [4] contains a generalization of the Regge and reflection symmetries to the 6j-symbols for the unitary principal series representations of the $SL(2, \mathbb{C})$ group. Taking particular limits one can rederive symmetries for the 6j-symbols of $U_q(su(2))$ quantum group and the classical $SU(2)$ group. In the last section, we obtain analogues of the Regge and reflection symmetries for 6j-symbols of the quantum supergroup $U_q(osp(1\vert 2))$.

4 Difference equations

As shown in [28, 29], the $V$-function satisfies the following finite-difference equation called the elliptic hypergeometric equation

\[ \mathcal{L}(g)(U(qg_6, q^{-1}g_7) - U(g)) + (g_6 \leftrightarrow g_7) + U(g) = 0, \quad (41) \]

where

\[ U(g) = \frac{V(g_1, \ldots, g_8; p, q)}{\Gamma(g_6g_\pm_1; p, q)\Gamma(g_7g_\pm_1; p, q)} \]

and

\[ \mathcal{L}(g) = \frac{\theta\left(\frac{qg}{g_6}; p\right)\theta\left(g_6g_8; p\right)\theta\left(\frac{qg_8}{g_6}; p\right)}{\theta\left(\frac{qg_8}{g_7}; p\right)\theta\left(g_7g_8; p\right)\theta\left(\frac{q}{g_7}; p\right)} \prod_{k=1}^{5} \frac{\theta\left(\frac{qg}{g_k}; p\right)}{\theta\left(g_kg_8; p\right)}. \]

In (41) $(g_6 \leftrightarrow g_7)$ means that there stands the previous expression with the parameters $g_6$ and $g_7$ permuted.

Using the asymptotic relations (5) and

\[ \theta(e^{-2\pi v y}, e^{-2\pi v \omega_1}) = e^{-\pi v \frac{y}{\omega_1}}2\sin \pi y \omega_1, \quad (42) \]

one can derive the following difference equation for the corresponding hyperbolic hypergeometric function [2]:

\[ \mathcal{A}(s; \omega_1, \omega_2)(Y(s_6 + \omega_2, s_7 - \omega_2) - Y(s)) + (s_6 \leftrightarrow s_7) + Y(s) = 0, \quad (43) \]

where

\[ \mathcal{A}(s; \omega_1, \omega_2) = \frac{\sin \frac{\pi}{\omega_1}(s_6 - s_8 - \omega_2)\sin \frac{\pi}{\omega_1}(s_6 + s_8)\sin \frac{\pi}{\omega_1}(s_8 - s_6)}{\sin \frac{\pi}{\omega_1}(s_6 - s_7)\sin \frac{\pi}{\omega_1}(s_7 - s_6 - \omega_2)\sin \frac{\pi}{\omega_1}(s_7 + s_6 - \omega_2)} \times \prod_{k=1}^{5} \frac{\sin \frac{\pi}{\omega_1}(s_7 + s_k - \omega_2)}{\sin \frac{\pi}{\omega_1}(s_8 + s_k)}, \quad (44) \]

and

\[ Y(s) = \frac{I_h(s)}{\gamma^{(2)}(s_6 \pm s_8, s_7 \pm s_8; \omega)}. \]

We now reparametrize $s_a$ in the identity (43) in the following asymmetric way

\[ s_a = \gamma_a + i\mu, \quad s_{a+4} = \beta_a - i\mu, \quad a = 1, 2, 3, 4. \]
Then the balancing condition takes the form $\sum_{a=1}^{4}(\gamma_a + \beta_a) = 2(\omega_1 + \omega_2)$. Now we shift in (21) the integration variable $z \to z - i\mu$ and take the limit $\mu \to -\infty$. As a result, we obtain from the above equation
\[
\mathcal{D}(\beta, \gamma; \omega_1, \omega_2)(U(\beta_2 + \omega_2, \beta_3 - \omega_2) - U(\beta, \gamma)) + (\beta_2 \leftrightarrow \beta_3) + U(\beta, \gamma) = 0,
\]
where
\[
\mathcal{D}(\beta, \gamma; \omega_1, \omega_2) = \frac{\sin \frac{\pi}{\omega_1}(\beta_2 - \beta_4 - \omega_2) \sin \frac{\pi}{\omega_1}(\beta_4 - \beta_2)}{\sin \frac{\pi}{\omega_1}(\beta_2 - \beta_3) \sin \frac{\pi}{\omega_1}(\beta_3 - \beta_2 - \omega_2)} \prod_{k=1}^{4} \frac{\sin \frac{\pi}{\omega_1}(\beta_3 + \gamma_k - \omega_2)}{\sin \frac{\pi}{\omega_1}(\beta_4 + \gamma_k)},
\]
\[
U(\beta, \gamma) = \frac{J_h(\beta, \gamma)}{\gamma(2)(\beta_2 - \beta_4, \beta_3 - \beta_4; \omega)}.
\]
Equation (45) was derived in [4].

5 Symmetries of the parafermionic hypergeometric integral

For a positive integer $r$ we define the function
\[
J_{\epsilon}(\beta_2, \gamma; \omega_1, \omega_2) = \int_{-\infty}^{\infty} \sum_{m \in \mathbb{Z} + \epsilon} \prod_{a=1}^{4} \Lambda(-y + \beta_a; l_a - m; \omega) \prod_{a=1}^{4} \Lambda(y + \gamma_a; t_a + m; \omega) \frac{dy}{ir\sqrt{\omega_1\omega_2}},
\]
where the rarefied hyperbolic gamma function $\Lambda$ is defined in (10) or (13). Here $t_a, l_a \in \mathbb{Z} + \epsilon$ and $\epsilon = 0, \frac{1}{2}$. Parameters $\gamma_j, \beta_j$ and $l_j, t_j$ satisfy the constraints:
\[
\sum_{j=1}^{4}(\gamma_j + \beta_j) = 2Q, \quad \sum_{j=1}^{4}(l_j + t_j) = 0.
\]

Function (46) is a parafermionic generalization of the integral (16). In fact, the case when $\epsilon = 1/2$ does not define new integral, since by the simultaneous shifts $l_a \to l_a - \epsilon, t_a \to t_a + \epsilon, m \to m - \epsilon$ one can eliminate the parameter $\epsilon$. But we keep it, because it allows us in coming sections to write parafermionic generalizations of the symmetry relations described above in a symmetric way. A relation of function (46) to the two-dimensional conformal field theory is discussed below in section 7.

5.1 Parafermionic symmetry I

Let us consider the rarefied elliptic hypergeometric function [31]:
\[
V_{\epsilon}^{(r)}(g, p; p, q) = \frac{(p^r; p^r)_{\infty}(q^r; q^r)_{\infty}}{4\pi i} \sum_{m \in \mathbb{Z} + \epsilon} \int_{T} \prod_{a=1}^{8} \frac{\Gamma^{(r)}(g_a z^{\pm 1}, n_a \pm m; p, q) d\zeta}{\Gamma^{(r)}(z^{\pm 2}, \pm 2m; p, q)} z,
\]
where $g_a \in \mathbb{C}^*, |g_a| < 1, n_a \in \mathbb{Z} + \epsilon, \epsilon = 0, \frac{1}{2}$, and
\[
\prod_{a=1}^{8} g_a = (pq)^2, \quad \sum_{a=1}^{8} n_a = 0
\]
integer, or otherwise.

The basic symmetry transformation of the $V_\epsilon^{(r)}$-function is a direct generalization of the relation [19]:

$$V_\epsilon^{(r)}(g, \tilde{g}; p, q) = V_\delta^{(r)}(g, \tilde{g}; p, q) \times \prod_{1 \leq b < c \leq 4} \Gamma^{(r)}(g_b g_c, n_b + n_c; p, q) \Gamma^{(r)}(g_b + g_c + 4, n_b + 4 + n_c + 4; p, q),$$

where

$$\left\{ \begin{array}{ll}
\tilde{g}_a = f g_a, & a = 1, 2, 3, 4, \\
\tilde{g}_a = f^{-1} g_a, & a = 5, 6, 7, 8 ;
\end{array} \right. \quad f = \sqrt{pq} = \sqrt{g_1 g_2 g_3 g_4},$$

$$\left\{ \begin{array}{ll}
\tilde{n}_a = n_a - \frac{1}{2}(\sum_{b=1}^{4} n_b), & a = 1, 2, 3, 4, \\
\tilde{n}_a = n_a + \frac{1}{2}(\sum_{b=1}^{4} n_b), & a = 5, 6, 7, 8,
\end{array} \right.$$  

and it is assumed that $|g_a|, |	ilde{g}_a| < 1$. Here $\delta = 0, \frac{1}{2},$ and one should take $\delta = \epsilon$, if the sum of $n_a$ is an even integer, or otherwise $\delta \neq \epsilon$.

Define the following rarefied hyperbolic hypergeometric function:

$$W_\epsilon^{(r)}(g, \tilde{g}; \omega) = \int_{-i\infty}^{i\infty} \sum_{m \in \mathbb{Z} + \epsilon} \prod_{a=1}^{8} \Lambda(s_a + y; n_a + m; \omega) \Lambda(\pm 2y; \pm 2m; \omega) \frac{dy}{2i \sqrt{\omega_1 \omega_2}},$$

where $\Lambda(x + y; n + m; \omega) = \Lambda(x + y; n + m; \omega)\Lambda(x - y; n - m; \omega)$ and the following balancing constraints on the parameters $s_j$ and $n_j$ hold true:

$$\sum_{j=1}^{8} s_j = 2Q, \quad \sum_{j=1}^{8} n_j = 0.$$

Using the limit [12], one can show that the transformation [49] reduces to a symmetry relation for function [51] of the form:

$$W_\epsilon^{(r)}(g, \tilde{g}; \omega) = W_\delta^{(r)}(\tilde{g}, \tilde{g}; \omega) \times \prod_{1 \leq j < k \leq 4} \Lambda(s_j + s_k; n_j + n_k; \omega) \prod_{5 \leq j < k \leq 8} \Lambda(s_j + s_k; n_j + n_k; \omega),$$

$$\tilde{s}_j = s_j + \xi, \quad \tilde{s}_{j+4} = s_{j+4} - \xi, \quad j = 1, 2, 3, 4, \quad \xi = \frac{1}{2}(\omega_1 + \omega_2 - \sum_{j=1}^{4} s_j)$$

with the same $\tilde{n}_a$ as defined in [50]. Now we parametrize the variables $n_a$ and $s_a$ in this identity in the following way:

$$s_{1,2,5,6} = \gamma_{1,2,3,4} + i\mu, \quad n_{1,2,5,6} = t_{1,2,3,4},$$
$$s_{3,4,7,8} = \beta_{1,2,3,4} - i\mu, \quad n_{3,4,7,8} = t_{1,2,3,4}.$$
where γ_j, β_j and l_j, t_j satisfy the constraints (17). Shifting the integration variable y → y - iμ on both sides and taking the limit μ → -∞, we obtain from (52):

\[ J_\varepsilon(\beta_\varepsilon, \tilde{l}; \gamma_\varepsilon, \tilde{t}; \omega) = e^{i\pi [\sum_{j=1}^{3} (t_j - 2(\varepsilon + \delta) \Lambda(\gamma_j + \beta_k; j + l_k; \omega) \prod_{j,k=3}^{4} \Lambda(\gamma_j + \beta_k; j + l_k; \omega) ,} \right]

where \( J_\varepsilon(\beta_\varepsilon, \tilde{l}; \gamma_\varepsilon, \tilde{t}; \omega) \) is defined in equation (46) and \( J_\delta(\beta_\delta, \tilde{l}; \gamma_\delta, \tilde{t}; \omega) \) has the following arguments

\[
\beta_a = \beta_a + \Theta(a) \eta, \quad \tilde{t}_a = l_a - \Theta(a) N, \\
\gamma_a = \gamma_a + \Theta(a) \eta, \quad \tilde{t}_a = t_a - \Theta(a) N, \quad a = 1, 2, 3, 4,
\]

where \( \Theta(a) \) is the sign function taking the values

\[
\Theta(a) = 1, \quad a = 1, 2 \quad \text{and} \quad \Theta(a) = -1, \quad a = 3, 4
\]

and

\[ N = \frac{1}{2} (t_1 + t_2 + l_1 + l_2), \quad \eta = \frac{1}{2} (Q - \gamma_1 - \gamma_2 - \beta_1 - \beta_2). \]

As before, the discrete variable \( \delta \) should be determined from the requirement for \( N + \varepsilon + \delta \) be integer (recall that in (54) \( t_a, l_a \in \mathbb{Z} + \varepsilon \) and \( t_a, \tilde{t}_a \in \mathbb{Z} + \delta \)).

In fact, one can eliminate \( \varepsilon \) and \( \delta \) from (54) by shifting parameters \( l_a \) and \( t_a \), as described after formula (46), and shifting the summation variable \( m \to m + N - \varepsilon \) in \( J_\delta(\beta_\delta, \tilde{l}; \gamma_\delta, \tilde{t}; \omega) \), which is allowed by the periodicity of the summand \( m \to m + r \) in (46). This yields the identity

\[
J_0(\beta_\varepsilon, \tilde{l}; \gamma_\varepsilon, \tilde{t}; \omega) = e^{i\pi [t_3 + t_1 + l_1 + l_2]} J_0(\tilde{\beta}_\varepsilon, \tilde{t}^*; \tilde{\gamma}_\varepsilon, \tilde{t}^*; \omega) \\
\times \prod_{j,k=1}^{2} \Lambda(\gamma_j + \beta_k; j + l_k; \omega) \prod_{j,k=3}^{4} \Lambda(\gamma_j + \beta_k; j + l_k; \omega) ,
\]

(56)

where \( l_a^* = l_a - (\Theta(a) + 1) N, \quad t_a^* = t_a - (\Theta(a) - 1) N, \quad a = 1, 2, 3, 4, \)

which is a generalization of the relation (24) for the parafermionic hypergeometric integral (46).

### 5.2 Parafermionic symmetry I_b

Define a parafermionic analogue of the function (25):

\[
E_\varepsilon(\rho, \omega; \omega) = \int_{-i\infty}^{i\infty} \sum_{m \in \mathbb{Z}_+, \varepsilon} \prod_{a=1}^{6} \Lambda(\rho_a \pm y; v_a \pm m; \omega) \frac{dy}{2i \sqrt{\omega_1 \omega_2}}
\]

where \( v_a \in \mathbb{Z} + \varepsilon, a = 1, \ldots, 6, \) and the shorthand is fixed in (15). In order to reduce relation (52), we use the parametrization

\[
s_{1,2,3,8} = \beta_{1,2,3,4} + i\mu, \quad n_{1,2,3,8} = l_{1,2,3,4}, \\
s_{4,5,6,7} = \gamma_{4,1,2,3} - i\mu, \quad n_{4,5,6,7} = t_{4,1,2,3},
\]

14
with the balancing condition (17), shift the integration variable \( y \rightarrow y + i \mu \) only in the left-hand side integral, and take the limit \( \mu \to \infty \). This procedure leads to the identity

\[
J_\epsilon(\beta, \ell; \gamma, l) = e^{i \pi (t_4 + t_4)} \prod_{i=1}^{3} \Lambda(\beta_i + \gamma_4; l_i + t_4; \omega)\Lambda(\gamma_i + \beta_4; t_i + l_4; \omega)E_8(\rho, \nu; \omega),
\]

where \( t_a, l_a \in \mathbb{Z} + \epsilon, v_a \in \mathbb{Z} + \delta \), and

\[
\rho_a = \beta_a + \xi, \quad v_a = l_a - L, \quad a = 1, 2, 3, \quad L = \frac{1}{2}(l_1 + l_2 + l_3 + t_4),
\]

\[
\rho_a = \gamma_a - \xi, \quad v_a = t_a + L, \quad a = 4, 5, 6, \quad \xi = \frac{1}{2}(\omega_1 + \omega_2 - \gamma_4 - \sum_{j=1}^{3} \beta_j).
\]

The parameter \( \delta \) again should be taken equal to \( \epsilon \), if \( L \) is an integer, and not equal to \( \epsilon \), if \( L \) is a half-integer. The identity (57) is a generalization of relation (26) to the rarefied hyperbolic hypergeometric functions.

### 5.3 Parafermionic symmetry II

The second identity for the rarefied elliptic hypergeometric function is obtained from (49) by a group composition and it is a generalization of relation (27):

\[
V_{\epsilon}^{(r)}(g, \nu; p, q) = \prod_{1 \leq b, c \leq 4} \Gamma^{(r)}(g_b g_{c+4}, n_b + n_{c+4}; p, q) V_{\rho}^{(r)}(\hat{g}, \hat{\nu}; p, q),
\]

where

\[
\hat{g}_a = \frac{\sqrt{l_1 l_2 l_3 l_4}}{g_a}, \quad a = 1, 2, 3, 4, \quad \hat{g}_a = \frac{\sqrt{l_5 l_6 l_7 l_8}}{g_a}, \quad a = 5, 6, 7, 8,
\]

with \( |g_a|, |\hat{g}_a| < 1 \) and

\[
\begin{align*}
\hat{n}_a &= -n_a + \frac{1}{2} \left( \sum_{\ell=1}^{4} n_\ell \right), \quad a = 2, 3, 4, \\
\hat{n}_a &= -n_a - \frac{1}{2} \left( \sum_{\ell=1}^{4} n_\ell \right), \quad a = 5, 6, 7, 8.
\end{align*}
\]

As before, the discrete parameters are chosen as \( \rho = \epsilon \), if \( \sum_{\ell=1}^{4} n_\ell \) is an even integer, or \( \rho \neq \epsilon \) otherwise.

Using the limit (12) one can show that relation (58) implies that the function (51) satisfies the following second symmetry relation:

\[
W_{\epsilon}(\mathbf{g}, \nu; \omega) = W_{\rho}(\mathbf{\hat{g}}, \mathbf{\hat{\nu}}; \omega) \prod_{1 \leq b, c \leq 4} \Lambda(s_b + s_{c+4}; n_b + n_{c+4}; \omega),
\]

\[
\hat{s}_j = G - s_j, \quad \hat{s}_{j+4} = Q - G - s_{j+4}, \quad j = 1, 2, 3, 4, \quad G = \frac{1}{2} \sum_{j=1}^{4} s_j,
\]

and \( \hat{n}_a \) are given in (59). Applying to (60) the limiting procedure (58) we obtain:

\[
J_\epsilon(\beta, \ell; \gamma, l) = e^{2\pi i M} J_\rho(\hat{\beta}, \hat{\ell}; \hat{\gamma}, \hat{l})
\]

\[
\times \prod_{j,k=1}^{2} \Lambda(\gamma_j + \beta_{k+2}; t_j + l_{k+2}; \omega)\Lambda(\gamma_{j+2} + \beta_k; t_{j+2} + l_k; \omega),
\]

15
where the arguments of function $J_{\rho}(\hat{\beta}, \hat{\gamma}, \hat{\delta})$ can be written as

$$
\hat{\beta}_a = Q/2 + \Theta(a)(G - Q/2) - \gamma_a, \quad \hat{\gamma}_a = Q/2 + \Theta(a)(G - Q/2) - \beta_a, \quad \hat{\delta}_a = -t_a + \Theta(a)M,
$$

with $\Theta(a)$ defined in (53) and

$$
M = \frac{1}{2}(t_1 + t_2 + l_1 + l_2), \quad G = \frac{1}{2}(\gamma_1 + \gamma_2 + \beta_1 + \beta_2).
$$

Again, here $\rho$ is equal to $\epsilon$ if $M$ is an integer, or $\rho \neq \epsilon$ otherwise. Recall that in (61) $t_a, l_a \in \mathbb{Z} + \epsilon$ and $\hat{t}_a, \hat{l}_a \in \mathbb{Z} + \rho$.

As in the previous section, one can exclude from formula (61) both $\epsilon$ and $\delta$ parameters by the shifts of parameters $t_a, l_a$, and of the summation variable, as indicated after formula (65), and obtain the equality

$$
J_0(\hat{\beta}, \hat{\gamma}, \hat{\delta}) = e^{2i\pi M} J_0(\hat{\beta}', \hat{\gamma}', \hat{\delta}')
\times \prod_{j,k=1}^2 \Lambda(\gamma_j + \beta_{j+2}; t_j + l_{j+2}; \omega) \Lambda(\gamma_j + \beta_{j+2}; t_{j+2} + l_k; \omega),
$$

(64)

$$
\hat{\beta}_a = Q/2 + \Theta(a)(G - Q/2) - \gamma_a, \quad \hat{\gamma}_a = Q/2 + \Theta(a)(G - Q/2) - \beta_a, \quad \hat{\delta}_a = -t_a + (\Theta(a) - 1)M,
$$

(65)

Here $t_a, t'_a, l_a, l'_a \in \mathbb{Z}$.

Let us rewrite formula (64) in the parametrization (31). Inserting $\gamma_\alpha^\circ$ and $\beta_\alpha^\circ$, $a = 1, 2, 3, 4$, defined in (63), we obtain:

$$
J_0(\alpha, \beta, \gamma, \delta) = e^{2i\pi M} \Omega(\alpha, \beta, \gamma, \delta) J_0(\alpha', \beta', \gamma', \delta'),
$$

(66)

where $\beta_\alpha^\circ$ and $\gamma_\alpha^\circ$ are defined in (53), $l'_a, t'_a$ are the same as in (65), and

$$
\Omega(\alpha, \beta, \gamma, \delta) = \Lambda(Q + \alpha_s - \alpha_3 - \alpha_4; l_1 + t_3; \omega) \Lambda(Q + \alpha_s - \alpha_1 - \alpha_2; l_1 + t_4; \omega)
\times \Lambda(Q - \alpha_t - \alpha_1; l_2 + t_3; \omega) \Lambda(Q - \alpha_t - \alpha_4; l_2 + t_4; \omega)
\times \Lambda(-Q + \alpha_t + \alpha_2 + \alpha_3; l_3 + t_1; \omega) \Lambda(-Q + \alpha_s + \alpha_2 - \alpha_3; l_4 + t_1; \omega)
\times \Lambda(-Q + \alpha_t + \alpha_4 + \alpha_1; l_3 + t_2; \omega) \Lambda(-Q + \alpha_s + \alpha_1 - \alpha_2; l_4 + t_2; \omega).
$$

(67)

Formula (66) is a parafermionic generalization of formula (32).

### 5.4 Parafermionic symmetry III

The third symmetry transformation for rarefied elliptic hypergeometric function is obtained after equating the right-hand side expressions in (49) and (58):

$$
V_{c}^{(r)}(g_a, g_b, p, q) = \prod_{1 \leq b < c \leq 8} \Gamma^{(r)}(g_b g_c, n_b + n_c; p, q) V_{c}^{(r)} \left( \frac{\sqrt{pq}}{g}, -w; p, q \right). \quad (68)
$$
Applying the limit (12) to (68) one can deduce for function (51) the third symmetry transformation

\[ W_\epsilon(x, y; \omega) = W_\epsilon(Q/2 - x, -y; \omega) \prod_{1 \leq j < k \leq 8} \Lambda(s_j + s_k; n_j + n_k; \omega). \]  

(69)

Parametrize \( s_a \) and \( n_a \) as:

\[ s_j = \gamma_j + i\mu, \quad s_{j+4} = \beta_j - i\mu, \quad n_j = t_j, \quad n_{j+4} = l_j \quad j = 1, \ldots, 4. \]

(70)

Shifting now the integration variables \( y \rightarrow y - i\mu \) on both sides of (69) and taking the limit \( \mu \rightarrow -\infty \), we come to the symmetry transformation

\[ J_\epsilon(\beta, \mu; t, l) = e^{i\pi \sum_{j=1}^{4} t_j} \prod_{j,k=1}^{4} \Lambda(\gamma_j + \beta_k; t_j + l_k; \omega) \]

\[ \times \int_{-i\infty}^{i\infty} \sum_{m \in \mathbb{Z}_r + c} \prod_{m=1}^{4} \Lambda(Q/2 - y - \gamma_a; -t_a - m; \omega) \Lambda(Q/2 + y - \beta_a; -l_a + m; \omega) \frac{dy}{ir\sqrt{\omega_1\omega_2}}. \]

This is a parafermionic extension of the standard hyperbolic identity (37).

6 Parafermionic difference-recurrence equation

As shown in (31), the function

\[ U_\epsilon(g, n) := \frac{V_\epsilon^{(r)}(g, n)}{\prod_{k=1}^{2} \Gamma(gk_3, n_k \pm n_3; p, q)} \]

satisfies the following mixed difference-recurrence equation:

\[ \mathcal{A} \left( \frac{g_1}{q_1}, \frac{g_2}{q_2}, \ldots, \frac{g_8}{q_8}; p; q^r \right) \left( U_\epsilon(pg_1, p^{-1}g_2, n_1 - 1, n_2 + 1) - U_\epsilon(g_a, n_a) \right) + \mathcal{A} \left( \frac{g_2}{q_2}, \frac{g_3}{q_1}, \ldots, \frac{g_8}{q_8}; p; q^r \right) \left( U_\epsilon(p^{-1}g_1, pg_2, n_1 + 1, n_2 - 1) - U_\epsilon(g_a, n_a) \right) + U_\epsilon(g_a, n_a) = 0, \]

(72)

where the \( \mathcal{A} \)-potential has the form

\[ \mathcal{A}(g_1, \ldots, g_8; p; q^r) := \frac{\theta \left( g_1, g_3, q_1, g_1, q_1^{-1}, g_3, q_1, q^r \right)}{\theta \left( g_2, g_2, q_2, g_2, q_2^{-1}, g_2, q_2, q^r \right)} \prod_{a=1}^{8} \frac{\theta \left( g_2g_a, q^r \right)}{\theta \left( g_3g_a, q^r \right)} \]

(for notation, see (15)). Applying now to equation (72) the limits (12) and (42), we obtain:

\[ \mathcal{B}(s, n; \omega_1, \omega_2)(Z(s_1 + \omega_2, s_2 - \omega_2, n_1 - 1, n_2 + 1) - Z(s)) + (s_1, n_1 \leftrightarrow s_2, n_2) + Z(s) = 0, \]

(73)

where

\[ \mathcal{B}(s, n; \omega_1, \omega_2) = \frac{\sin \frac{\pi}{\omega_1}(s_1 - s_3 - \omega_2 + (n_3 - n_1 + r - 1)\omega_1)}{\sin \frac{\pi}{\omega_1}(s_1 - s_2 + (n_2 - n_1)\omega_1)} \]

\[ \times \frac{\sin \frac{\pi}{\omega_1}(s_1 + s_3 - (n_1 + n_3)\omega_1)}{\sin \frac{\pi}{\omega_1}(s_2 - s_1 - \omega_2 + (n_1 - n_2 + r - 1)\omega_1)} \frac{\sin \frac{\pi}{\omega_1}(s_3 - s_1 + (n_1 - n_3)\omega_1)}{\sin \frac{\pi}{\omega_1}(s_2 + s_k - \omega_2 + (n_2 + n_k + 1 - r + \epsilon)\omega_1)} \]

\[ \times \prod_{k=4}^{8} \frac{\sin \frac{\pi}{\omega_1}(s_2 + s_k - \omega_2 - (n_2 + n_k + 1 - r + \epsilon)\omega_1)}{\sin \frac{\pi}{\omega_1}(s_3 + s_k - (n_3 + n_k)\omega_1)}. \]
\[ Z(s, n) = \frac{W_{\epsilon}(s_i, n_i)}{\Lambda(s_1 \pm s_3, n_1 \pm n_3)\Lambda(s_2 \pm s_3, n_2 \pm n_3)}, \]

with the balancing condition:

\[ \sum_{i=1}^{8} s_i = 2(\omega_1 + \omega_2), \quad \sum_{i=1}^{8} n_i = 0. \]

Applying to (73) the limiting procedure (70), we obtain

\[ D(\gamma, \beta, L, t)(K(\gamma_1 + \omega_2, \gamma_2 - \omega_2, t_1 - 1, t_2 + 1) - K(\gamma, \beta, L, t) + (\gamma_1 \leftrightarrow \gamma_2, t_1 \leftrightarrow t_2) + K(\gamma, \beta, L, t) = 0, \]

where

\[ K(\gamma, \beta, L, t) = \frac{J_{\epsilon}(\beta, L; \gamma; t)}{\Lambda(\gamma_1 - \gamma_3, t_1 - t_3)\Lambda(\gamma_2 - \gamma_3, t_2 - t_3)}, \]

and

\[ D(\gamma, \beta, L, t) = \frac{\sin \frac{\pi}{rot}(\gamma_1 - \gamma_3 - \omega_2 + (t_3 - t_1 + r - 1)\omega_1)\sin \frac{\pi}{rot}(\gamma_3 - \gamma_1 + (t_1 - t_3)\omega_1)}{\sin \frac{\pi}{rot}(\gamma_1 - \gamma_2 + (t_2 - t_1)\omega_1)\sin \frac{\pi}{rot}(\gamma_2 - \gamma_1 - \omega_2 + (t_1 - t_2 + r - 1)\omega_1)} \times \prod_{k=1}^{4} \frac{\sin \frac{\pi}{rot}(\gamma_2 + \beta_k - \omega_2 - (t_2 + l_k + 1 - r + \epsilon)\omega_1)}{\sin \frac{\pi}{rot}(\gamma_3 + \beta_k - (t_3 + l_k + \epsilon)\omega_1)}. \]

Equation (74) is a parafermionic extension of the difference equation (45).

### 7 Supersymmetric Racah-Wigner symbols

In this section we specialize some of the above results to the case \( r = 2 \). We show that the \( r = 2 \) rarefied hyperbolic gamma function \( \Lambda(y, m; \omega) \) is a building block for the fusion matrix of the \( N = 1 \) supersymmetric Liouville theory and, closely related to it, Racah-Wigner symbols of the quantum supergroup \( U_q(osp(1|2)) \) for the continuous series representations. The fusion matrix of the Neveu-Schwarz sector of \( N = 1 \) supersymmetric Liouville theory was studied in [3, 11, 12]. There were suggested some integrals of linear combinations of the product of eight, so-called, Neveu-Schwarz and Ramond hyperbolic gamma functions, \( S_{NS}(x) \) and \( S_R(x) \), in the parametrization (31). First, we review some basics of the \( N = 1 \) super Liouville theory. Then describe relations of \( S_{NS}(x) \) and \( S_R(x) \) to \( \Lambda(y, m; \omega) \) for \( r = 2 \) and show that all the suggested integrals coincide with the parafermionic hypergeometric function (46) for \( r = 2 \) with the corresponding match of parameters. In [16], an expression for the universal supersymmetric Racah-Wigner symbols was suggested in the parametrization (31). Using formula (66) we show that the restriction of these symbols to the Neveu-Schwarz sector is in agreement with the expression proposed in [12] and also find its general expression in parametrization (51).
7.1 Some basics on $N = 1$ super Liouville field theory

$N = 1$ super Liouville field theory is defined on a two-dimensional surface with the metric $g_{ab}$ by the local Lagrangian density

$$\mathcal{L} = \frac{1}{2\pi} g_{ab} \partial_a \varphi \partial_b \varphi + \frac{1}{2\pi} (\psi \partial \bar{\psi} + \bar{\psi} \partial \psi) + 2\mu b^2 \bar{\psi} \psi e^{b\varphi} + 2\pi \mu b^2 e^{2b\varphi}. \quad (75)$$

The energy-momentum tensor and the superconformal current are

$$T = -\frac{1}{2} (\partial \varphi \partial \varphi - Q \partial^2 \varphi + \psi \partial \bar{\psi}), \quad G = i(\psi \partial \varphi - Q \partial \psi), \quad (76)$$

where $Q = b + b^{-1}$. The modes $L_n = \oint dz 2\pi i z^{n+1} T(z)$ and $G_k = \oint dz 2\pi i z^{k+1/2} G(z)$ satisfy the superconformal algebra

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12} m(m^2 - 1)\delta_{m+n},$$

$$[L_m, G_k] = \frac{m - 2k}{2} G_{m+k},$$

$$\{G_k, G_l\} = 2L_{l+k} + \frac{c}{3} \left( k^2 - \frac{1}{4} \right) \delta_{k+l}, \quad (77)$$

with the central charge $c = c_{SL} = \frac{3}{2} + 3Q^2$. Here $k$ and $l$ take integer values for the Ramond algebra and half-integer values for the Neveu-Schwarz algebra.

Since in the Neveu-Schwarz sector we have supercurrent generators $G_k$ with half-integer $k$, descendant fields are broken into two representations of the plain Virasoro algebra of integer and half-integer level. Thus in the Neveu-Schwarz sector there are two types of Virasoro primary fields. The first type is associated with the vertex operator $N_\alpha = e^{\alpha \varphi}$ and has the conformal dimension $\Delta_\alpha = \frac{1}{2} \alpha (Q - \alpha)$. The physical states have $\alpha = \frac{Q}{2} + iP$.

The second type primary field is given by the supercurrent descendant $\tilde{N}_\alpha = G_{-1/2} N_\alpha$ and has the conformal dimension

$$\Delta_{\tilde{\alpha}} = \frac{1}{2} \alpha (Q - \alpha) + \frac{1}{2}.$$ 

The Ramond vertex operators and their dimensions are

$$R_\alpha^\pm = \sigma^\pm e^{\alpha \varphi}, \quad \Delta_{R_\alpha^\pm} = \frac{1}{16} + \frac{1}{2} \alpha (Q - \alpha),$$

where $\sigma^\pm$ are the spin fields satisfying the property:

$$\psi(z) \sigma^\pm(0) \sim \frac{\sigma^\mp(0)}{\sqrt{z}}.$$

The Neveu-Schwarz and Ramond operators with the same conformal dimensions are proportional to each other, namely, one has $N_\alpha \propto N_{Q-\alpha}$, $R_\alpha \propto R_{Q-\alpha}.$

In the paper [12], $6j$-symbols of continuous series representation of the quantum supergroup $U_q(osp(1|2))$ were denoted as $\{ \alpha_1 \alpha_2 \mid \alpha_3 \alpha_4 \}^\nu_\nu_2 (\nu_3 \nu_4)$ (we drop the subindex $b$ indicating that we deal with the
quantum group). Here $\alpha_j, j = 1, \ldots, 4$, $\alpha_s, \alpha_t$ are the continuous spins fixing corresponding representations, and $\nu_k = 0, 1$, $k = 1, \ldots, 4$, take into account a doubling of the Clebsch-Gordan coefficients following from the existence of two different intertwining operators for the decomposition of tensor products of two representations with fixed $\alpha_1$ and $\alpha_2$. The following dictionary is established between the $6j$-symbols of $U_q(osp(1|2))$ and the fusion matrix of $N = 1$ supersymmetric Liouville field theory in the Neveu-Schwarz sector [12]:

$$F_{N_{\alpha_s}, N_{\alpha t}}[N_{\alpha_3}, N_{\alpha_2}] \propto \{ \alpha_1 \alpha_2 | \alpha_3 \alpha_4 \}_{11}, \quad F_{N_{\alpha_s}, N_{\alpha t}}[N_{\alpha_3}, N_{\alpha_1}] \propto \{ \alpha_1 \alpha_2 | \alpha_3 \alpha_4 \}_{00}.$$

This dictionary shows that if one considers $6j$-symbols as fusion matrices, then $\alpha_j$ acquire the meaning of momenta and $\nu_j$ distinguish between $N_{\alpha_s}$ and $\tilde{N}_{\alpha}$ primary field entries.

The general supersymmetric Racah-Wigner symbols were introduced in [16]. They were denoted as $$\left\{ \begin{array}{ccc} \alpha_1 & \alpha_3 & \alpha_5 \\
\alpha_2 & \alpha_4 & \alpha_6 \\
\nu_1 & \nu_2 & \nu_3 \end{array} \right\}_{\nu_1 \nu_2 \nu_3},$$ where $\alpha^a$ does not mean a power of $\alpha$, but denotes a pair of variables—continuous $\alpha$ and discrete $a$. The choice $a_j = 0$ corresponds to the Neveu-Schwarz operator, and $a_j = 1$ is connected to the Ramond operator. For the Ramond operator $\nu_k = 0, 1$ distinguish between $R^+$ and $R^-$ operators. When all $a_j = 0$, one gets the $6j$-symbols of $U_q(osp(1|2))$.

The general parafermion hypergeometric function [16] is expected to be a key ingredient of the fusion matrix of the parafermionic Liouville field theory discussed in [1] with the central charge $c = \frac{3r}{r+2} + \frac{6}{r}(b+b^{-1})^2$. For $r = 2$, this is indeed so for the $N = 1$ supersymmetric Liouville field theory, as follows from the comparison with the results of [12].

### 7.2 Supersymmetric hypergeometric function and $6j$-symbols

We start by describing the relation between $S_{NS}(x), S_R(x)$ and the function $\Lambda(y, m; \omega)$ for $r = 2$ [23]. Setting $\omega_2 = b$ and $\omega_1 = b^{-1}$, $Q = b + b^{-1}$, and using the notation accepted in conformal field theory literature $\gamma^{(2)}(z; b, 1/b) =: S_b(z)$, we obtain

$$\Lambda(y, 0; b^{-1}, b) = S_b \left( \frac{y}{2} \right) S_b \left( \frac{y}{2} + \frac{Q}{2} \right) \equiv S_{NS}(y) \equiv S_1(y),$$

$$\Lambda(y, 1; b^{-1}, b) = S_b \left( \frac{y}{2} + \frac{b}{2} \right) S_b \left( \frac{y}{2} + \frac{b^{-1}}{2} \right) \equiv S_R(y) \equiv S_0(y).$$

The subscript $a$ of $S_a(y)$ is defined modulo 2: $S_{a+2}(y) \equiv S_a(y)$. Using formula (11), it is easy to see the relation between $\Lambda(y, m; b^{-1}, b)$ for $r = 2$ and $S_a(y)$ for arbitrary $m$:

$$\Lambda(y, m + 2k; \omega) = (-1)^m \Lambda(y, m; \omega).$$

Formulae (79), (80), and (81) imply

$$\Lambda(y, K; b^{-1}, b) = S_{NS}(y) \equiv S_1(y), \quad \text{if } K \text{ is even,}$$

$$\Lambda(y, K; b^{-1}, b) = (-1)^{\frac{K+1}{2}} S_R(y) \equiv S_0(y), \quad \text{if } K \text{ is odd,}$$

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or, combining these equalities, we have

\[ \Lambda(y, K; b^{-1}, b) = (-1)^{F(K)} S_{K+1}(y), \quad F(K) = \frac{1 - (-1)^K K - 1}{2}. \]  

(82)

Now we are ready to discuss supersymmetric Racah-Wigner symbols for the supergroup \(U_q(\text{osp}(1|2))\) which have the following expression [12]:

\[
\{a_1 a_2 \mid a_3 \alpha_4 \}_{\nu_1 \nu_2}^{\nu_3 \nu_4} = \frac{S_{\nu_2}(\alpha s_2 + \alpha t - \alpha t_1) S_{\nu_1}(\alpha t_1 + \alpha t - \alpha s)}{S_{\nu_2}(\alpha t_1 + \alpha s - \alpha s_4) S_{\nu_3}(\alpha t_3 + \alpha s - \alpha s_4)} I_{\alpha s, \alpha t}^{\nu_3 \nu_4}, \]

(83)

where \(\alpha_4 = Q - \alpha_4\), \(\nu_j = 0, 1, j = 1, 2, 3, 4\), \(\sum_{i=1}^{4} \nu_i = 0 \mod 2\), and

\[
I_{\alpha s, \alpha t}^{\nu_3 \nu_4} = (-1)^{\nu_3 \nu_2 + \nu_4} \int_{-\infty}^{\infty} \sum_{\nu = 0}^{1} (-1)^{\nu_1 (\nu_2 + \nu_4)} S_{1 + \nu + \nu_1}(y + \gamma_1 + \gamma_2) S_{1 + \nu + \nu_4}(y + \gamma_2)
\times S_{1 + \nu_1 + \nu_3}(y + \gamma_3) S_{1 + \nu_1 + \nu_4}(y + \gamma_3)
\times S_{\nu_1 + \nu_2 + \nu_3}(y + \gamma_4) S_{\nu_1 + \nu_2 + \nu_4}(y + \gamma_4)
\times S_{\nu_1 + \nu_2 + \nu_3}(y + \gamma_4) S_{\nu_1 + \nu_2 + \nu_4}(y + \gamma_4)
\times \frac{dy}{2 i}, \]

(84)

We recall that parameters \(\beta_\gamma^0\) and \(\gamma_\gamma^0\) are defined in [31]. In fact, in order to rewrite 6j-symbols expression of [12] in the form (84), we have replaced \(\alpha_4\) by its reflected value, \(\alpha_4 \rightarrow \alpha_4\) (which corresponds to the equivalent representation), and we also shifted the integration variable by \(-Q/2 - \alpha s\).

One can check that the integral (84) is a special case of the supersymmetric hypergeometric function [46] for \(r = 2\) for an appropriate parametrization. Without loss of the generality, we take \(\epsilon = 0\) and set

\[
t_1 = \nu_3, \quad t_3 = \nu_3, \quad l_1 = -1, \quad l_3 = 1 - \nu_2 - \nu_3, \\
t_2 = \nu_4, \quad t_4 = -\nu_4, \quad l_2 = 1 + \nu_2 - \nu_3, \quad l_4 = -1,
\]

so that the condition \(\sum_{i=1}^{4} (t_i + l_i) = 0\) is satisfied. Originally, there were \(6\) independent discrete variables among \(t_j\) and \(l_j\), and only \(3\) variables among \(\nu_j\) are independent, i.e. we have given to three discrete variables some fixed values. One thus obtains

\[
J_{0}(\beta_1, -1, \beta_2, 1 + \nu_2 - \nu_3, \beta_3, 1 - \nu_2 - \nu_3, \beta_4, -1; \gamma_1, \gamma_1, \gamma_2, \gamma_2, \gamma_3, \gamma_3, \gamma_4, \nu_4)
\]

\[
= \int_{-\infty}^{\infty} \sum_{\nu = 0}^{1} \Lambda(y + \gamma_1, \nu_3 + \nu) \Lambda(y + \gamma_2, \nu + \nu_4) \Lambda(y + \gamma_3, \nu + \nu_3)
\times \Lambda(y + \gamma_4, \nu - \nu_4) \Lambda(-y + \beta_2, -\nu - 1) \Lambda(-y + \beta_2, -\nu + 1 + \nu_2 - \nu_3)
\times \Lambda(-y + \beta_4, -\nu + 1 + \nu_2 - \nu_3) \Lambda(-y + \beta_4, -\nu - 1) \frac{dy}{2i},
\]

(85)

where we drop the dependence on \(b\) in the notation. Expression (85) coincides with (84). To verify that, we should check only the sign factor, the rest is obvious. It is clear that the pair of the terms with the same spin structure will not produce any sign, so we can expect some sign factor only from the products

\[ \Lambda(y + \gamma_2, \nu + \nu_4) \Lambda(y + \gamma_4, \nu - \nu_4) \]  

(86)
Noting that factors in (84) are coming from the sign differences in \( \Lambda(y, M) \) and similar ones described below show that the use of our \( \Lambda \)-function is more convenient.

Using formulae from Appendix A for \( a_j \) and \( b_j \), and denoting it as \( \alpha \), we easily see from (82) that the minus sign appears only when \( \nu = 0 \). Collecting all together, we obtain:

\[
\Lambda \left( y + \gamma_2 \nu + \nu_4 \right) \Lambda \left( y + \gamma_4 \nu - \nu_4 \right) = (-1)^{\nu_4(\nu + 1)} S_{1 + \nu + \nu_4} \left( y + \gamma_2 \nu \right) S_{1 + \nu + \nu_4} \left( y + \gamma_4 \nu \right).
\]

Similarly, it is obvious that the product (87) yields negative sign only if \( \nu_2 = 1 \). Putting in (87) \( \nu_2 = 1 \), one can check that the minus sign appears only when \( \nu + \nu_3 = 1 \). Therefore we can write

\[
\Lambda \left( y + \beta_2 \nu + \nu_2 + \nu_3 \right) \Lambda \left( y + \beta_3 \nu - \nu_2 - \nu_3 \right)
= (-1)^{\nu_2(\nu + \nu_3)} S_{\nu + \nu_2 + \nu_3} \left( y + \beta_2 \nu \right) S_{\nu + \nu_2 + \nu_3} \left( y + \beta_3 \nu \right).
\]

Noting that \((-1)^{\nu_4(\nu + 1)}(-1)^{\nu_2(\nu + \nu_3)} = (-1)^{\nu_2(\nu + \nu_4)}(-1)^{\nu(\nu_2 + \nu_4)} \), we see that all the awkward sign factors in (84) are coming from the sign differences in \( \Lambda(y, M) \) and \( S_\nu(y) \) functions (82). This fact and similar ones described below show that the use of our \( \Lambda \)-function is more convenient.

Now we employ relation (66) for function (85) to give another form of the integral (84). From definition (63), we find \( M = \frac{1}{2}(\nu_2 + \nu_4) \) and all spin structures in (65):

\[
\begin{align*}
\ell'_1 &= -t_1 = -\nu_3, \quad \ell'_3 = -t_3 - 2M = -\nu_3 - \nu_2 - \nu_4, \\
\ell'_2 &= -t_2 = -\nu_4, \quad \ell'_4 = -t_4 - 2M = -\nu_2.
\end{align*}
\]

Shifting there additionally the summation variable \( m \) by \(-\nu_2\) and denoting it as \( \nu \), we obtain for the integral standing on the right-hand side of relation (66):

\[
\tilde{J}_0(\alpha, \nu) \equiv J_0(\beta^0, \ell'_1, \gamma^0, \ell'_2) = \int_{-\infty}^{\infty} \sum_{\nu=0}^{1} \Lambda(-y - \alpha_{23\ell}, \nu_2 - \nu_3 - \nu) \Lambda(-y - \alpha_{14\ell}, \nu_2 - \nu_4 - \nu) \times \Lambda(-y - \alpha_{12s}, -\nu_3 - \nu_4 - \nu) \Lambda(-y - \alpha_{34s}, -\nu) \times \Lambda(y + \alpha_{12s}, \nu_4 + \nu + 1) \Lambda(y + \alpha_{13s}, -\nu_2 + \nu_3 + \nu_4 + \nu + 1) \times \Lambda(y + 2Q, \nu_3 + \nu - 1) \Lambda(y + \alpha_{24s}, -\nu_2 + \nu + 1) \frac{dy}{2i}. \quad (88)
\]

Using formulae from Appendix A for \( a_j = 0 \), we come to the expression:

\[
\tilde{J}_0(\alpha, \nu) = (-1)^D \int_{-\infty}^{\infty} \sum_{\nu=0}^{1} S_{\nu_2 - \nu_3 - \nu + 1}(-y - \alpha_{23\ell}) S_{\nu_2 - \nu_4 - \nu + 1}(-y - \alpha_{14\ell}) \times S_{-\nu_4 - \nu_3 - \nu + 1}(-y - \alpha_{12s}) S_{-\nu_2 + \nu + 1}(-y - \alpha_{34s}) S_{\nu_2 + \nu} \times S_{-\nu_2 + \nu_3 + \nu_4 + \nu}(y + \alpha_{13s}) S_{\nu_3 + \nu}(y + 2Q) S_{-\nu_2 + \nu}(y + \alpha_{24s}) \frac{dy}{2i}. \quad (89)
\]
where \( D = \nu_2 \nu_3 \nu_4 + \nu_2 \nu_3 + \nu_2 \nu_4 + \nu_3 \nu_4 + \nu_2 + \nu_4 \). Compute also the prefactor \( \Omega \) [67]:

\[
\Omega(\alpha, \nu) = (-1)^{\nu_3 (\nu_2 + \nu_4)} S_{\nu_3} (Q + \alpha_s - \alpha_3 - \alpha_4) S_{\nu_4} (Q + \alpha_s - \alpha_1 - \alpha_2) \\
\times S_{\nu_2} (Q - \alpha_t + \alpha_2 - \alpha_3) S_{\nu_1} (Q - \alpha_t + \alpha_4 - \alpha_1) S_{\nu_2} (-Q + \alpha_t + \alpha_2 + \alpha_3) \\
\times S_{\nu_1} (Q - \alpha_s + \alpha_3 - \alpha_4) S_{\nu_1} (-Q + \alpha_t + \alpha_4 + \alpha_1) S_{\nu_2} (Q - \alpha_s + \alpha_1 - \alpha_2),
\]

so that one can write

\[
I_{a_s, a_t} \left[ \begin{array}{ccc} \alpha_3 & \alpha_2 & \alpha_1 \\ \alpha_4 & \alpha_4 & \alpha_2 \\ \end{array} \right]^{\nu_3 \nu_4} = (-1)^{\nu_2 + \nu_4} \Omega(\alpha, \nu) J_0(\alpha, \nu),
\]

where \( J_0 \)-function is fixed in [89].

Now we can compare our results with the formula suggested in [16] for the universal supersymmetric Racah-Wigner symbols in the parametrization [33]:

\[
\left\{ \begin{array}{cccc} \alpha_1 & \alpha_3 & \alpha_3 & \alpha_4 \\ \alpha_2 & \alpha_4 & \alpha_4 & \alpha_2 \\ \end{array} \right\}^{\nu_3 \nu_4} = \mathcal{P}(\alpha_j, \nu_j) \int d\nu \sum_{\nu_0=0}^{1} \left( (-1)^X S_{1+\nu_1+\nu_4+\nu} (u - \alpha_{1234}) \right. \\
\times S_{1+\nu} (u - \alpha_{s34}) S_{1+\nu_1+\nu_4+\nu} (u - \alpha_{234}) S_{1+\nu_2+\nu_4+\nu} (u - \alpha_{134}) \\
\left. \times S_{\nu_4+\nu} (\alpha_{1234} - u) S_{\nu_1+\nu+a_1} (\alpha_{st13} - u) S_{\nu_2+\nu+a_2} (\alpha_{st24} - u) S_{\nu_3+\nu+a_s} (2Q - u) \right),
\]

where

\[
(-1)^X = (-1)^{\nu_1 \nu_2 \nu_4 + \nu_1 \nu_3 + \nu_2 \nu_4 + \nu_1 \nu_4 + \nu_2 \nu_4 + \nu_2 \nu_4 + \nu_2 \nu_4},
\]

and the following selection rules hold true

\[
\sum_{j=1}^{4} \nu_j = a_s + a_t \pmod{2},
\]

\[
a_s = a_1 + a_2 = a_3 + a_4 \pmod{2}, \quad a_t = a_1 + a_4 = a_2 + a_3 \pmod{2}.
\]

The prefactor \( \mathcal{P}(\alpha_j, \nu_j) \) is a normalization of 6j-symbols, it is represented by a complicated combination of square roots of various \( S_t \)-functions and is given explicitly in [16]. For brevity we will not discuss it here. Parameters \( a_i = 0, 1 \) distinguish between the NS and the Ramond sectors: \( a_i = 0 \) corresponds to primary fields of the NS-sector, and \( a_i = 1 \) to those of the R-sector. Expression (92) was suggested without proof, but it was tested using the limit when it reduces to 6j-symbols of the finite-dimensional representations of \( U_q(osp(1|2)) \).

As clear from the definition, in the case, when all \( a_j = 0 \), i.e. when all primaries lie in the NS sector and (92) gives 6j-symbols of \( U_q(osp(1|2)) \), this expression should match with (83). This was not proved in [16], and now we can do it.

First, let us resolve the constraint (94) for \( \nu_1 \):

\[
\nu_1 = a_s + a_t - \nu_2 - \nu_3 - \nu_4 \pmod{2},
\]
and insert in the integrand of (92), which yields
\[
\left\{ \begin{array}{c}
\alpha_{a_1}', 
\alpha_{a_2}', 
\alpha_{a_4}', 
\alpha_{a_1}, 
\alpha_{a_2}, 
\alpha_{a_4}
\end{array} \right\}^{\nu_3 \nu_4} = \mathcal{P}(\alpha_i, \nu_4) \int du \sum_{\nu=0}^{1} (-1)^X S_{1+\nu_3+\nu_4+\nu}(u - \alpha_{12s}) \\
\times S_{1+\nu}(u - \alpha_{s4}) S_{1+\nu_2+\nu_3-\alpha_4 - \alpha_t + \nu}(u - \alpha_23k) S_{1+\nu_2+\nu_4+\nu}(u - \alpha_{14t}) S_{\nu_4+\nu}(\alpha_{1234} - u) \\
\times S_{\nu_2+\nu_3+\nu_4+\nu_2+\nu_4}(2Q - u) \right). 
\] (97)

Here we used also the constraints (95). Setting all \( a_j = 0 \) and identifying \( u = -y \), one can see that integral in the right-hand side of (97) coincides with (88) up to the \((-1)^D\) sign factor.

Now we would like to show that the function standing in (92) after the \( \mathcal{P}(\alpha_j, \nu_j) \) factor coincides with the special \( r = 2 \) case of the parafermionic hypergeometric integral (96). Formula (88) suggests the following generalization of the function \( \tilde{J}_0(\alpha, \nu, u) \) including the variables \( a_j \):
\[
\tilde{J}_0(\alpha, \nu, u) = \int_{-\infty}^{\infty} \sum_{\nu_1=0}^{1} \Lambda(-y - \alpha_{23t}, \nu_1 - \nu - a_s - a_t) \Lambda(-y - \alpha_{34s}, -\nu) \\
\times \Lambda(-y - \alpha_{14t}, -\nu_1 - \nu - a_s - a_t) \Lambda(-y - \alpha_{12s}, -\nu_2 - \nu - a_s - a_t) \\
\times \Lambda(y + \alpha_{1234}, \nu_4 + \nu + 1) \Lambda(y + \alpha_{s13}, -\nu_2 + \nu_3 + \nu_4 + \nu - 1 + a_2 + a_t) \\
\times \Lambda(y + 2Q, \nu_3 + \nu + 1 + a_s) \Lambda(y + \alpha_{s24}, -\nu_2 + \nu + 1 - a_2) \frac{dy}{2^1}, 
\] (98)

Clearly the spin structures of (97) and (88) match and we should check only the sign appearing in passage from \( \Lambda(x, K) \) to \( S_{K+1}(x) \).

Using formulae from appendix A, we can write
\[
\tilde{J}_0(\alpha, \nu, u) = \mathcal{C} (-1)^B \int_{-\infty}^{\infty} \sum_{\nu=0}^{1} \left((-1)^C S_{1+\nu_3+\nu_4+\nu}(u - \alpha_{12s}) S_{1+\nu}(u - \alpha_{s43}) \\
\times S_{1+\nu_2+\nu_3+\nu_4+\nu_2}(u - \alpha_{23t}) S_{1+\nu_2+\nu_4+\nu}(u - \alpha_{14t}) S_{\nu_4+\nu}(\alpha_{1234} + y) \\
\times S_{\nu_2+\nu_3+\nu_4+\nu_2+\nu_4}(\alpha_{s13} + y) S_{\nu_2+\nu_4+\nu_2}(\alpha_{s24} + y) S_{\nu_3+\nu_4+\nu}(2Q + y) \frac{dy}{2^1}, \right) 
\] (99)
where
\[
(-1)^C = (-1)^{\nu(a_s a_t + a_s a_t + a_s a_t + a_s a_t + a_s a_t + a_s a_t + a_s a_t + a_s a_t)}, 
\] (100)
and
\[
B = a_s a_t (\nu_3 + \nu_2) + a_s (\nu_2 \nu_3 + \nu_2 + \nu_3) + a_t (\nu_2 \nu_4 + \nu_3 \nu_4 + \nu_2 + \nu_3 + \nu_4) \\
+ a_2 a_t (\nu_2 + \nu_3 + \nu_4 + 1) + a_2 (\nu_2 \nu_3 + \nu_2 \nu_4 + \nu_3 \nu_4 + \nu_2 + \nu_3 + \nu_4 + 1) \\
+ \nu_2 \nu_3 \nu_4 + \nu_2 \nu_4 + \nu_2 \nu_4 + \nu_3 \nu_4 + \nu_2 + \nu_4. 
\]

Taking \( \nu_1 \) as in (96) and expressing \( a_1, a_4 \) in terms of \( a_2, a_s \) and \( a_t \) with the help of relations (95), we find that the sign \((-1)^C (100)\) coincides with \((-1)^X (93)\). So, identifying \( y = -u \), we see that function (88) coincides with (97) up to the indicated \( \mathcal{P}(\alpha_j, \nu_j) \) factor and \((-1)^B\) sign. Note that
the sign \((-1)^X\) was found in \[16\] only by analyzing a degeneration limit of \([92]\). Here it is obtained from the condition that the Racah-Wigner symbols are written as the parafermionic hypergeometric integral \([46]\) for \(r = 2\). An important fact is that now we have in formula \([98]\) six independent discrete variables, say, \(\nu_j, a_j, j = 1, 2, 3\), as it should be for the parafermionic hypergeometric function.

We can apply also relation \([66]\) in the opposite direction to the integral \([98]\) and derive the generalization of integral \([81]\), giving, up to some prefactors, the Racah-Wigner symbols in the Ponsot-Teschner type parametrization \([31]\), for non-zero \(a_i\):

\[
\tilde{J}_0(\alpha, \nu, \mathbf{a}) = (-1)^{\nu_2 + \nu_4 + a_2 - a_4} \Omega^{-1}(\alpha, \nu, \mathbf{a}) I_{a_2, a_4}^{\nu_2, \nu_4} \left[ \begin{array}{cc} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{array} \right]_{\nu_1 \nu_2} \nu_3 \nu_4 ,
\]

where

\[
I_{a_2, a_4}^{\nu_2, \nu_4} = \int_{-\infty}^{\infty} \sum_{\nu = 0}^{\infty} \Lambda(y + \gamma_1^0, \nu_3 + a_2 + a_s + a_t + \nu) \\
\times \Lambda(y + \gamma_2^0, \nu + \nu_4 + a_2) \Lambda(y + \gamma_3^0, \nu + \nu_3 + a_s) \Lambda(y + \gamma_4^0, -\nu - \nu_4 + a_s) \\
\times \Lambda(\nu + \beta_1^0, -\nu - 1 - a_s) \Lambda(\nu + \beta_2^0, -\nu - 1 + \nu - \nu_4 - a_s) \\
\times \Lambda(\nu + \beta_3^0, -\nu + 1 - \nu_4 - a_2 - a_s) \Lambda(\nu + \beta_4^0, -\nu - 1) \frac{dy}{2i}.
\]

Using formulae in Appendix B, we can rewrite this expression in terms of the \(S_K(x)\)-functions:

\[
I_{a_2, a_4}^{\nu_2, \nu_4} = (-1)^F \int_{-\infty}^{\infty} \sum_{\nu = 0}^{\infty} (-1)^E S_{1+\nu+\nu_3+a_s}(y + \gamma_1^0) \\
\times S_{1+\nu+\nu_4+a_2}(y + \gamma_2^0) S_{1+\nu_3+a_s}(y + \gamma_3^0) S_{1+\nu_4+a_s}(y + \gamma_4^0) S_{-\nu-a_s}(-y + \beta_2^0) \\
\times S_{\nu+\nu_2+\nu_4-a_2}(-y + \beta_2^0) S_{\nu+\nu_2\nu_4-a_4}(-y + \beta_2^0) S_{-\nu_4}(-y + \beta_4^0) \frac{dy}{2i},
\]

where \(E = \nu(\nu_2 + \nu_4) + \nu(\nu_3a_2 + \nu_2a_t + \nu_4a_2 + \nu_4a_s + a_2a_s + a_s + a_t)\) and

\[
F = \nu_3\nu_2 + \nu_4 + \nu_2\nu_3a_t + \nu_2a_2a_t + \nu_2a_2a_t + \nu_3a_2a_2 + \nu_3a_2a_2 + \nu_2a_s + \nu_2a_2 + \nu_2a_2 + a_2a_2 + a_2a_2 + a_s,
\]

\[
\Omega(\alpha, \nu, \mathbf{a}) = (-1)^T S_{\nu_3}(Q + a_s - a_3 - a_4) S_{\nu_4}(Q + a_s - a_1 - a_2) \\
\times S_{\nu_2+a_3}(Q - a_t + a_2 - a_3) S_{\nu_4}(Q - a_t + a_2 - a_3) \\
\times S_{\nu_2+a_3}(Q - a_t + a_2 + a_3) S_{\nu_4+a_3}(Q - a_s + a_3 - a_4) \\
\times S_{\nu_4+a_3}(Q - a_t + a_4 + a_1) S_{\nu_4+a_2}(Q - a_s + a_1 - a_2),
\]

with the integer \(T\) of the form

\[
T = \nu_3(\nu_2 + \nu_4) + (a_s + a_t + a_2)(\nu_2\nu_3 + \nu_2\nu_4 + \nu_3\nu_4) + a_s(\nu_2 + 1) \\
+ a_t(\nu_4 + 1) + a_2a_s(\nu_3 + 1) + a_2a_t(\nu_4 + 1) + a_1a_s(\nu_3 + 1) + a_1a_2a_2.
\]

To conclude, the parafermionic hypergeometric function \([16]\) for \(r = 2\) coincides with the key functional part of the universal Racah-Wigner symbols in \(N = 1\) supersymmetric Liouville theory.
determined in [16]. It is natural to expect that for arbitrary \( r \) it will determine key functional ingredient of the fusion matrices of the general parafermionic Liouville field theory of [1] and corresponding generalization of the Racah-Wigner symbols.

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### A Sign calculations I

Here we compute the signs appearing in passing from the \( \Lambda(x, K) \)-functions to \( S_{K+1}(x) \)-functions in the derivation of the integrals (89) and (99). After cumbersome calculations we obtain:

\[
\Lambda(-y - \alpha_{23t}, \nu_2 - \nu_3 - \nu - a_s - a_t) = S_{\nu_2 - \nu_3 - \nu - a_s - a_t + 1}(-y - \alpha_{23t}) \\
\times (-1)^{\nu_2 a_1 + (\nu_2 + \nu_3 + 1) + \nu_2 \nu_4 + \nu_3 \nu_4 + \nu_3 + \nu_3 + \nu_4 + \nu_4 + 1},
\]

\[
\Lambda(-y - \alpha_{14t}, \nu_2 - \nu_4 - \nu) = S_{\nu_2 - \nu_4 - \nu + 1}(-y - \alpha_{14t})(-1)^{\nu_2 \nu_4 + (\nu_2 + 1) + (\nu_4 + 1)},
\]

\[
\Lambda(-y - \alpha_{12s}, -\nu_3 - \nu_4 - \nu) = S_{-\nu_3 - \nu_4 - \nu + 1}(-y - \alpha_{12s})(-1)^{\nu_3 \nu_4 + (\nu_3 + \nu_4 + 1)},
\]

\[
\Lambda(-y - \alpha_{34s}, -\nu) = S_{\nu + 1}(-y - \alpha_{34s})(-1)^\nu,
\]

\[
\Lambda(y + \alpha_{12s}, \nu_4 + \nu + 1) = S_{\nu_4 + \nu}(-1)^{\nu_4 \nu},
\]

\[
\Lambda(y + \alpha_{13st}, -\nu_2 + \nu_3 + \nu_4 + \nu - 1 + a_2 + a_1) = S_{-\nu_2 + \nu_3 + \nu_4 + \nu - 1 + a_2 + a_1}(y + \alpha_{13st}) \\
= (-1)^{\nu_2 \nu_3 + (\nu_2 + \nu_3 + 1) + \nu_2 \nu_4 + \nu_3 \nu_4 + \nu_2 + \nu_3 + \nu_4 + 1 + 1},
\]

\[
A = a_2 a_1 (\nu_2 + \nu_3 + \nu_4 + 1) + (a_2 + a_1)(\nu_2 \nu_3 + \nu_2 \nu_4 + \nu_3 \nu_4 + \nu_3 + \nu_4 + 1)
\]

\[
+ \nu_2 \nu_3 \nu_4 + \nu_2 \nu_4 + \nu_3 \nu_4 + \nu_3 + \nu_4 + 1,
\]

\[
\Lambda(y + 2Q, \nu_3 + \nu - 1 + a_4) = S_{\nu + \nu + a_4}(y + 2Q)(-1)^{(\nu_3 + 1)(\nu + 1)(a_4 + 1)},
\]

\[
\Lambda(y + \alpha_{24st}, -\nu_2 + \nu + 1 - a_2) = S_{-\nu_2 + \nu - a_2}(y + \alpha_{24st})(-1)^{(\nu + 1)(\nu + 1)a_2}.
\]

### B Sign calculations II

Here we compute the signs that emerge after the replacements of \( \Lambda(x, K) \)-functions by \( S_{K+1}(x) \) during the derivation of the expression (103). After lengthy calculations we obtain:

\[
\Lambda(y + \gamma_1^0, \nu_3 + a_2 + a_s + a_t + \nu) = S_{\nu_3 + a_2 + a_s + a_t + \nu + 1}(y + \gamma_1^0) \\
\times (-1)^{\nu_3 (a_s + a_2) + (\nu + \nu_3)} + \nu_3 (a_s + a_2 + a_s + a_t) + a_2 a_s a_t,
\]

\[
\Lambda(y + \gamma_2^0, \nu + \nu_4 + a_2) = S_{\nu + \nu_4 + a_2 + 1}(y + \gamma_2^0)(-1)^{\nu_4 a_2},
\]

\[
\Lambda(y + \gamma_3^0, \nu + \nu_3 + a_s) = S_{\nu + \nu_3 + a_s + 1}(y + \gamma_3^0)(-1)^{\nu_3 a_s},
\]
\[ \Lambda(y + \gamma_4^\alpha, \nu - \nu_4 + a_4) = S_{\nu - \nu_4 + a_4 + 1}(y + \gamma_4^\alpha)(-1)^{\nu_4(\nu_4 + 1)(a_4 + 1)}, \]
\[ \Lambda(-y + \beta_1^\nu, -\nu - 1 - a_4) = S_{-\nu - a_4}(-y + \beta_1^\nu)(-1)^{\nu_1(\nu_1 + 1)(a_1 + 1)}, \]
\[ \Lambda(-y + \beta_2^\nu, -\nu + 1 + \nu_2 - \nu_3 - a_2 - a_3 - a_4 - a_5) = S_{-\nu + \nu_2 - \nu_3 - a_2 - a_3 - a_5}(-y + \beta_2^\nu) \]
\[ \times (-1)^{\nu_3(a_2 + a_3 + a_5 + 1)(a_2 + a_3 + a_5 + 1)(a_2 + a_3 + a_5 + 1)} S_{a_2 + a_3 + a_5}(-y + \beta_2^\nu), \]
\[ \Lambda(-y + \beta_3^\nu, -\nu + 1 - \nu_2 - \nu_3 - a_2 - a_3 = S_{-\nu + \nu_2 - \nu_2 - a_2 - a_3}(-y + \beta_3^\nu), \]
\[ \times (-1)^{\nu_3(a_2 + a_3 + 1)(a_2 + a_3 + 1)} S_{a_2 + a_3}(-y + \beta_3^\nu), \]
\[ \Lambda(-y + \beta_4^\nu, -\nu - 1) = (-1)^{\nu + 1} S_{-\nu}(-y + \beta_4^\nu). \]

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