Generalizations of two theorems of Ritt on decompositions of polynomial maps

V. V. Bavula

Dedicated to F. van Oystaeyen on the occasion of his 60'th birthday

Abstract

In 1922, J. F. Ritt [13] proved two remarkable theorems on decompositions of polynomial maps of \( \mathbb{C}[x] \) into irreducible polynomials (with respect to the composition \( \circ \) of maps). Briefly, the first theorem states that in any two decompositions of a given polynomial function into irreducible polynomials the number of the irreducible polynomials and their degrees are the same (up to order). The second theorem gives four types of transformations of how to obtain all the decompositions from a given one. In 1941, H. T. Engstrom [7] and, in 1942, H. Levi [11] generalized respectively the first and the second theorem to polynomial maps over an arbitrary field \( K \) of characteristic zero. The aim of the paper is to generalize the two theorems of J. F. Ritt to a more general situation: for, so-called, reduction monoids (\( (K[x], \circ) \) and \( (K[x^2] \times x, \circ) \) are examples of reduction monoids). In particular, analogues of the two theorems of J. F. Ritt hold for the monoid \( (K[x^2], \circ) \) of odd polynomials. It is shown that, in general, the two theorems of J. F. Ritt fail for the cusp \( (K + K[x^2] \times x^2, \circ) \) but their analogues are still true for decompositions of maximal length of regular elements of the cusp.

Key Words: the two theorems of Ritt, Ritt transformations, composition of polynomial maps, cusp transformations, irreducible map, the length and defect of a polynomial.

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1 Introduction

In this paper, \( K \) is a field of characteristic zero and \( K[x] \) is a polynomial algebra over the field \( K \) in a single variable \( x \). The polynomial algebra \( K[x] \) is a monoid, \( (K[x], \circ) \), where
is the composition of polynomial functions, \((a \circ b)(x) := a(b(x))\), and \(x\) is the identity element of the monoid \(K[x]\). An element \(u\) of the monoid \(K[x]\) is a unit iff \(\deg(u) = 1\). The group of units of the monoid \((K[x], \circ)\) is denoted by \(K[x]^*\).

A polynomial \(a \in K[x]\) is said to be irreducible (or prime or indecomposable) if \(\deg(a) > 1\) and the polynomial \(a\) is not a composition of two non-units, i.e. \(a\) is an irreducible element of the monoid \((K[x], \circ)\). This concept of irreducibility should not be confused with the concept of irreducibility of the multiplicative monoid \((K[x], \cdot)\) which is not used in the paper. A polynomial which is not irreducible is said to be reducible or composite. When \(K = \mathbb{C}\) composite polynomials were studied by J. F. Ritt [13]. He proved two theorems that completely describe the decompositions composite polynomials may possess. His first theorem states: any two decompositions of a given polynomial of \(\mathbb{C}[x]\) into irreducible polynomials contain the same number of polynomials; the degrees of the polynomials in one decomposition are the same as those in the other, except, perhaps, for the order in which they occur.

Two decompositions of a polynomial \(a\) into irreducible polynomials

\[
a = p_1 \circ \cdots \circ p_r = q_1 \circ \cdots \circ q_r
\]

are called equivalent if there exist \(r - 1\) polynomials of the first degree \(u_1, \ldots, u_{r-1}\) such that

\[
q_1 = p_1 \circ u_1, \quad q_2 = u_1^{-1} \circ p_2 \circ u_2, \ldots, q_{r-1} = u_{r-2}^{-1} \circ p_{r-1} \circ u_{r-1}, \quad q_r = u_{r-1}^{-1} \circ p_r.
\]

Suppose that in a decomposition of the polynomial \(a\) into irreducible polynomials

\[
a = p_1 \circ \cdots \circ p_r\tag{1}
\]

there is an adjacent pair of irreducible polynomials

\[
p_i = \lambda_1 \circ \pi_1 \circ \lambda_2, \quad p_{i+1} = \lambda_2^{-1} \circ \pi_2 \circ \lambda_3
\]

where \(\lambda_1, \lambda_2\) and \(\lambda_3\) are polynomials of degree 1 and where \(\pi_1\) and \(\pi_2\), of unequal degrees \(m\) and \(n\), respectively, are of any of the following three types:

\[
\begin{align*}
(a) \quad & \pi_1 = T_m, \quad \pi_2 = T_n, \\
(b) \quad & \pi_1 = x^m, \quad \pi_2 = x^r g(x^m), \\
(c) \quad & \pi_1 = x^r g^n, \quad \pi_2 = x^n,
\end{align*}
\]

where \(g = g(x)\) is a polynomial, \(T_n\) is the trigonometric polynomial, \(T_n(\cos t) := \cos(n t)\).

Then, for the polynomial \(a\) we have a decomposition distinct from (1),

\[
a = p_1 \circ \cdots \circ p_{i-1} \circ p_i^* \circ p_{i+1}^* \circ p_{i+2} \circ \cdots \circ p_r\tag{2}
\]

where respectively to the three cases above the polynomials \(p_i^*\) and \(p_{i+1}^*\) are as follows:
(a) \[ p^*_i = \lambda_1 \circ T_n, \quad p^*_{i+1} = T_m \circ \lambda_3, \]
(b) \[ p^*_i = \lambda_1 \circ [x^rg^m], \quad p^*_{i+1} = x^m \circ \lambda_3, \]
(c) \[ p^*_i = \lambda_1 \circ x^n, \quad p^*_{i+1} = [x^rg(x^n)] \circ \lambda_3. \]

Clearly, \( \deg(p^*_i) = \deg(p^*_{i+1}) = n \) and \( \deg(p^*_{i+1}) = \deg(p_i) = m. \)

The second theorem of J. F. Ritt states: \textit{if} \( a \in \mathbb{C}[x] \) \textit{has two distinct decompositions into irreducible polynomials, we can pass from either to a decomposition equivalent to the other by repeated steps of the three types just indicated.}

He writes in his paper, p. 53: \textit{“The analogous problem for fractional rational functions is much more difficult. There is a much greater variety of possibilities, as one sees, without going far, on considering the formulas for the transformation of the periods of the elliptic functions. There are even cases in which the number of prime functions in one decomposition is different from that in another.”} We will see later in the paper that the situation is similar for the cusp.

J. F. Ritt’s approach is based on the monodromy group associated with the equation \( f(x) - y = 0. \)

Later H. T. Engstrom [7] and H. Levi [11] proved respectively the first and the second theorem of J. F. Ritt for the polynomial algebra \( K[x] \) where \( K \) is a field of characteristic zero. Their methods are algebraic.

It is known that the theorems of J. F. Ritt are false in prime characteristic [3], [10], but the first theorem is true for, so-called, tame polynomials [9], [15]. For some generalizations, applications and connections with the two theorems of J. F. Ritt the reader is referred to [11] [3] [4] [6] [10] [12] [15] [14] [16] [17].

The goal of this paper is to generalize the two theorems of J. F. Ritt to a more general situation (for, so-called, \textit{reduction} monoids - see Section 2 for a definition; \( (K[x], \circ) \) and \( (K[x^2|x, \circ) \) are reduction monoids). The advantage of our method is that generalizations of the two theorems are proved in one go.

For a natural number \( r \), let \( S_r \) be the symmetric group. For reduction monoids (the definition is given in Section 2), the first and the second statement of the following theorem are generalizations of the first and the second theorem of J. F. Ritt, respectively. The first statement is precisely the same as the first theorem of J. F. Ritt, but the second statement contains only ‘half’ of the second theorem of J. F. Ritt, as the second part of the second theorem of J. F. Ritt classifies all the situations \( p_ip_{i+1} = p'_ip'_{i+1} \) for the monoid \( \mathcal{C}[x], \circ). \)

\textbf{Theorem 1.1} Let \( \mathcal{M} \) be a reduction monoid, \( \mathcal{M}^* \) be its group of units, \( a \in \mathcal{M} \) with \( |a| > 1 \), and \( a = p_1 \cdots p_r = q_1 \cdots q_s \) be two decompositions of the element \( a \) into irreducible factors. Then

1. \( r = s \) and \( |p_1| = |q_{\sigma(1)}|, \ldots, |p_r| = |q_{\sigma(r)}| \) for a permutation \( \sigma \in S_r; \) and
2. if the decompositions are distinct then one can be obtained from the other by finitely many transformations on adjacent irreducible factors of the following two types:

(a) \( p_1 \cdots p_i p_{i+1} \cdots p_r = p_1 \cdots (p_i u)(u^{-1} p_{i+1}) \cdots p_r \) where \( u \in \mathcal{M}^* \),

(b) \( p_1 \cdots p_i p_{i+1} \cdots p_r = p_1 \cdots p'_i p'_{i+1} \cdots p_r \) where \( p_i p_{i+1} = p'_i p'_{i+1} \), the numbers \( |p_i| \) and \( |p_{i+1}| \) are co-prime, \( |p_i| = |p'_{i+1}| \) and \( |p_{i+1}| = |p'_i| \).

Consider the submonoid \( (\mathcal{O} := K[x^2], \circ) \) of odd polynomials of the monoid \( (K[x], \circ) \).

**Theorem 1.2** Let \( K \) be a field of characteristic zero. Then the monoid \( \mathcal{O} \) is a reduction monoid where \( | \cdot | = \deg \).

The group \( \mathcal{O}^* \) of units of the monoid \( \mathcal{O} \) is equal to the group \( \{ \lambda x \mid \lambda \in K^* \} \) where \( K^* := K \setminus \{0\} \). The first two statements of the next corollary follow at once from Theorems \[1.1\] and \[1.2\] statement 3 follows from the second theorem of J. F. Ritt but not in a straightforward way as many additional results are used in its proof: Theorem \[2.6\] Lemma \[2.3\] Lemma \[2.8\] (see Section \[2\] for detail).

**Corollary 1.3** Let \( K \) be a field of characteristic zero, \( a \in \mathcal{O} \) with \( \deg(a) > 1 \), and \( a = p_1 \circ \cdots \circ p_r = q_1 \circ \cdots \circ q_s \) be two decompositions of the element \( a \) into irreducible factors of the monoid \( \mathcal{O} \). Then

1. \( r = s \) and \( \deg(p_1) = \deg(q_{\sigma(1)}), \ldots, \deg(p_r) = \deg(q_{\sigma(r)}) \) for a permutation \( \sigma \in S_r \); and

2. if the decompositions are distinct then one can be obtained from the other by finitely many transformations on adjacent irreducible factors of the following two types:

(a) \( p_1 \circ \cdots \circ p_i \circ p_{i+1} \circ \cdots \circ p_r = p_1 \circ \cdots \circ (p_i \circ u) \circ (u^{-1} \circ p_{i+1}) \circ \cdots \circ p_r \) where \( u \in \mathcal{O}^* \),

(b) \( p_1 \circ \cdots \circ p_i \circ p_{i+1} \circ \cdots \circ p_r = p_1 \circ \cdots \circ p_i \circ p_i^* \circ p_{i+1} \circ \cdots \circ p_r \) where \( p_i \circ p_{i+1} = p_i^* \circ p_{i+1}^* \),

the degrees \( \deg(p_i) \) and \( \deg(p_{i+1}) \) are co-prime, \( \deg(p_i) = \deg(p_i^*) \) and \( \deg(p_{i+1}) = \deg(p_{i+1}^*) \).

3. There are only the following options for the pairs \( P := (p_i, p_{i+1}) \) and \( P^* := (p_i^*, p_{i+1}^*) \):

(a) \( P = (T_n, T_m) \) and \( P^* = (T_m, T_n) \) where \( n \) and \( m \) are odd distinct primes,

(b) \( P = (x^s, x^t \alpha(x^2)) \) and \( P^* = (x^s, x^t \alpha(x^2)) \),

(c) \( P = (x^s, x^t \alpha(x^2)) \) and \( P^* = (x^t, x^s \alpha(x^2)) \),

where \( s \) is an odd prime number, \( t \) is an odd number, and \( \alpha \in K[x] \setminus K \) with \( \alpha(0) \neq 0 \).
Up to my knowledge, the monoid $O$ is the only example distinct from $K[x]$ for which (analogues of) the two theorems of J. F. Ritt hold. It would be interesting to find more examples (the definition of reduction monoid is very arithmetical). It is a curious fact that the monoid $O$, in fact, comes from non-commutative situation. The monoid $O$ is the monoid of all central algebra endomorphisms of a certain localization of the quantum plane which is a non-commutative algebra (see Section 2 for detail). It would be interesting to find more reduction monoids coming from non-commutative situation (and as a result to obtain analogues of the two theorems of J. F. Ritt for them). Notice that in the definition of reduction monoid $M$ is not necessarily a commutative algebra, it is just an abelian group. Moreover, in the case of the odd polynomials, $O$ is not even an algebra.

The cusp submonoid $(K + K[x]x^2, \circ)$ of $(K[x], \circ)$ looks similar to the monoid $O$ but for it situation is completely different. In particular, the cusp submonoid is not a reduction monoid.

Till the end of this section let $K$ be an algebraically closed field of characteristic zero and let $A$ be the subalgebra of the polynomial algebra $K[x]$ generated by the monomials $x^2$ and $x^3$. The algebra $A = K + K[x]x^2$ is isomorphic to the algebra of regular functions on the cusp $s^2 = t^3$. It is obvious that $(A, \circ)$ is a sub-semi-group of $(K[x], \circ)$. For a polynomial $a \in K[x]$ of degree $\deg(a) > 1$, let $\Dec(a)$ be the set of all decompositions of the polynomial $a$ into irreducible polynomials of $K[x]$ (with respect to $\circ$). The length $l(a)$ of the polynomial $a \in K[x]$ is the number of irreducible polynomials in any decomposition of $\Dec(a)$. Similarly, for a polynomial $a \in A \setminus K$, let $\Dec_A(a)$ be the set of all decompositions of the polynomial $a$ into irreducible polynomials of $A$. The natural number

$$l_A(a) := \max\{r \mid p_1 \circ \cdots \circ p_r \in \Dec_A(a)\}$$

is called the $A$-length of the element $a$. It is obvious that

$$l_A(a) \leq l(a).$$

In general, this inequality is strict (Corollary 3.4). An element $a \in A$ is called regular (respect. irregular) if $l_A(a) = l(a)$ (resp. $l_A(a) < l(a)$). The are plenty of elements of both types. Moreover, if $a$ is irregular then $a \circ (x + \lambda)$ is regular for some $\lambda \in K$. A decomposition

$$p_1 \circ \cdots \circ p_{l_A(a)} \in \Dec_A(a)$$

is called a decomposition of maximal length or a maximal decomposition for the element $a$. Let $\Max(a)$ be the set of all maximal decompositions for $a$. Clearly, $\Max(a) \subseteq \Dec_A(a)$, but, in general, $\Max(a) \neq \Dec_A(a)$, see [14]. Lemma 3.7 describes the set $\Max(a)$.

In general, the number of irreducible polynomials in decomposition into irreducible polynomials of an element of $A$ is non-unique (Lemma 3.5); moreover, it can vary greatly. So, for the cusp the two theorems of J. F. Ritt do not hold. Therefore, the cusp is not a reduction monoid. Nevertheless, for decompositions of maximal length of each regular element $a$ of $A$ analogues of the two theorems do hold – Theorem 1.4 and Theorem 1.5 if $K$ is algebraically closed (if $K$ is not algebraically closed then, in general, Theorem 1.5 does not hold).
Theorem 1.4 Let $K$ be a field of characteristic zero, $a$ be a regular element of $A$ such that $a \not\in K$, and

$$a = p_1 \circ \cdots \circ p_r = q_1 \circ \cdots \circ q_r$$

be two decompositions of maximal length of the element $a$ into irreducible polynomials of $A$. Then

$$\deg(p_1) = \deg(q_{\sigma(1)}), \ldots, \deg(p_r) = \deg(q_{\sigma(r)})$$

for a permutation $\sigma \in S_r$.

Theorem 1.4 follows from the first theorem of J. F. Ritt (or from Theorem 1.3). In general, for irregular elements Theorem 1.4 is not true (Proposition 3.6), i.e. the invariance of degrees (up to permutation) does not hold. The next theorem is an analogue of the second theorem of J. F. Ritt for regular elements. A new moment is that the transformations $(\text{Adm})$, $(\text{Ca})$, $(\text{Cb})$ and $(\text{Cc})$ are defined on three adjacent elements rather than two as in the second theorem of J. F. Ritt.

Theorem 1.5 Let $K$ be an algebraically closed field of characteristic zero, $a$ be a regular element of $A$ such that $a \not\in K$, and $X, Y \in \text{Max}(a)$. Then the decomposition $Y$ can be obtained from the decomposition $X$ by finitely many transformations of the following four types: $(\text{Adm})$, $(\text{Ca})$, $(\text{Cb})$ and $(\text{Cc})$, see below.

For a non-scalar polynomial $f$ of $K[x]$, a polynomial $\lambda + \mu x$ of degree 1 is called an $f$-admissible polynomial if $\lambda$ is a root of the derivative $f' := \frac{df}{dx}$ of $f$.

Let $a \in A \setminus K$ with $r := l_A(a) = l(a)$, and $Z := p_1 \circ \cdots \circ p_i \circ p_{i+1} \circ \cdots \circ p_r \in \text{Max}(a)$. Consider the following four types of transformations of the decomposition $Z$ that produce a new decomposition $Z^* \in \text{Max}(a)$ where

$$Z^* := \begin{cases} 
  p_1 \circ \cdots \circ p_{i-1} \circ p_i^* \circ p_{i+1} \circ \cdots \circ p_r & \text{if } i + 1 < r, \\
  p_1 \circ \cdots \circ p_{i-1} \circ p_i^* & \text{if } i + 1 = r.
\end{cases}$$

(Adm) In both cases, $p_i^* := p_i \circ u$ and $p_{i+1}^* := u^{-1} \circ p_{i+1}$ where $u \in K[x]^*$ is $p_i$-admissible, and $p_{i+2}^* = p_{i+2}$ if $i + 1 < r$ ($u^{-1}$ is the inverse of the element $u$ in the monoid $(K[x], \circ)$, i.e. $u^{-1}$ is the inverse map of $u$).

In the remaining three cases below, $\gcd(\deg(p_i), \deg(p_{i+1})) = 1$, all $\lambda_i \in K[x]^*$, $p$ is a prime number, polynomials $x^a g^p(x)$ and $x^a g(x^p)$ satisfy the condition that $g(0) \neq 0$, $\lambda_i^{-1}$ is the inverse of the element $\lambda_i$ in the monoid $(K[x], \circ)$.

(Ca) If $i + 1 < r$, $p_i = \lambda_l \circ T_k \circ \lambda_2$ and $p_{i+1} = \lambda_2^{-1} \circ T_l \circ \lambda_3$ where $k$ and $l$ are distinct odd prime numbers, $\lambda_2$ is $T_k$-admissible and $\lambda_3$ is $T_l$-admissible, then

$$p_i^* := \lambda_1 \circ T_l \circ \lambda_4, \quad p_{i+1}^* := \lambda_4^{-1} \circ T_k \circ \lambda_3 \circ \lambda_5 \quad \text{and} \quad p_{i+2}^* := \lambda_5^{-1} \circ p_{i+2},$$

where $\lambda_4$ is $T_l$-admissible and $\lambda_5$ is $T_k \circ \lambda_3$-admissible.
(Cb) If \( i + 1 < r \), \( p_i = \lambda_1 \circ x^p \) and \( p_{i+1} = [x^s g(x^p)] \circ \lambda_2 \) where \( \lambda_2 \) is \( x^s g(x^p) \)-admissible, then
\[
p_i^* := \lambda_1 \circ [x^s g^p] \circ \lambda_3, \quad p_{i+1}^* := \lambda_3^{-1} \circ x^p \circ \lambda_2 \circ \lambda_4 \quad \text{and} \quad p_{i+2}^* := \lambda_4^{-1} \circ p_{i+2},
\]
where \( \lambda_3 \) is \( x^s g^p \)-admissible and \( \lambda_2 \circ \lambda_4 \) is \( x^p \)-admissible.
If \( i + 1 = r \), \( p_{r-1} = \lambda_1 \circ x^p \) and \( p_r = [x^s g(x^p)] \circ \lambda_2 \) where \( s \geq 2 \) and \( \lambda_2 \in K^*x \), then
\[
p_{r-1}^* := \lambda_1 \circ [x^s g^p] \quad \text{and} \quad p_r^* := x^p \circ \lambda_2.
\]

(Cc) If \( i + 1 < r \), \( p_i = \lambda_1 \circ [x^s g^p] \circ \lambda_2 \) and \( p_{i+1} = \lambda_2^{-1} \circ x^p \circ \lambda_3 \) where \( \lambda_2 \) is \( x^s g^p \)-admissible and \( \lambda_3 \) is \( x^p \)-admissible, then
\[
p_i^* := \lambda_1 \circ x^p, \quad p_{i+1}^* := [x^s g(x^p)] \circ \lambda_3 \circ \lambda_4 \quad \text{and} \quad p_{i+2}^* := \lambda_4^{-1} \circ p_{i+2},
\]
where \( \lambda_3 \circ \lambda_4 \) is \( x^s g(x^p) \)-admissible.
If \( i + 1 = r \), \( p_{r-1} = \lambda_1 \circ x^s g^p \), \( s \geq 2 \), and \( p_r = x^p \circ \lambda_2 \) where \( \lambda_2 \) is \( x^p \)-admissible, then
\[
p_{r-1}^* := \lambda_1 \circ x^p \quad \text{and} \quad p_r^* := [x^s g(x^p)] \circ \lambda_2.
\]

Decompositions of polynomials with coefficients in a commutative ring were studied by the author in \[2\].

2 Generalizations of the two theorems of J. F. Ritt

In this section, the two theorems of J. F. Ritt are generalized to a more general situation. They are proved for reduction monoids (Theorem 1.1). The polynomial algebra \( K[x] \) is a reduction monoid with respect to the composition of functions. These generalizations are inspired by the paper of H. T. Engstrom \[7\] and we follow some of his ideas. Proofs of Theorem 1.1, Theorem 1.2 and Corollary 1.3 are given.

Natural numbers \( i \) and \( j \) are called co-prime (or relatively prime) if \( \gcd(i, j) = 1 \).

Definition. A multiplicative monoid \( \mathcal{M} \) is called a reduction monoid if the following axioms hold for all elements \( a, b, c \in \mathcal{M} \) (where \( \mathcal{M}^* \) is the group of units of the monoid \( \mathcal{M} \)):

(A1) \( \mathcal{M} \) is a \( \mathbb{Z} \)-module (i.e. \( \mathcal{M} \) is an abelian group under \( + \)) such that
\[
(a + b)c = ac + bc.
\]

(A2) There exists a map \( | \cdot | : \mathcal{M} \rightarrow \mathbb{N} := \{0, 1, \ldots \} \) such that
\[
|ab| = |a||b| \quad \text{and} \quad |a + b| \leq \max \{|a|, |b|\}.
\]

(A3) \( a \in \mathcal{M}^* \) iff \( |a| = 1 \).
(A4) If $ac = bc$ then $a = b$ provided $|c| > 1$.

(A5) For any elements $a, b \in \mathcal{M}$ with $|a| > 1$ and $|b| > 1$ and, in addition, there exists an element $x \in \mathcal{M}a \cap \mathcal{M}b$ such that $|x| \neq 0$, there exists an element $c \in \mathcal{M}$ such that $\mathcal{M}a \cap \mathcal{M}b = \mathcal{M}c$ and $|c| = \text{lcm}(|a|, |b|)$.

(A6) If $\alpha a = \beta b$ with $|\alpha| = i, |\alpha| = jk, |\beta| = j, |b| = ik, ijk \geq 1$, and the natural numbers $i$ and $j$ are co-prime then $a = a_1c$ and $b = b_1c$ for some elements $a_1, b_1$ and $c$ of $\mathcal{M}$ such that $|c| = k$.

Example. $(K[x], \circ)$ is the reduction monoid where $| \cdot | := \deg$. The axioms (A1)-(A4) are obvious. The axioms (A5) and (A6) follow respectively from Theorems 2.2 and 3.1 of the paper [7].

If $p$ is an irreducible element of the monoid $\mathcal{M}$ then so are the elements $up$ and $pu$ for all units $u \in \mathcal{M}^*$.

- Each element $a$ of $\mathcal{M}$ with $|a| > 1$ is a product of irreducible elements.

To prove this statement we use induction on $|a|$. By (A2) and (A3), each element $a$ with $|a| = 2$ is irreducible. Suppose that $|a| > 2$ and the result holds for all elements $a'$ of $\mathcal{M}$ with $1 < |a'| < |a|$. Then either the element $a$ is irreducible or, otherwise, it is a product, say $bc$, of two non-units $b$ and $c$. Since $|a| = |b||c|$, $|b| > 1$ and $|c| > 1$ (see (A2) and (A3)), we have $1 < |b| < |a|$ and $1 < |c| < |a|$. By induction, the elements $b$ and $c$ are products of irreducible elements, then so is the element $a$. $\square$

Corollary 2.1 Let $\mathcal{M}$ be a reduction monoid, $p$ and $q$ be irreducible elements of $\mathcal{M}$ such that $\mathcal{M}^*p \neq \mathcal{M}^*q$ and there exists an element $a \in \mathcal{M}p \cap \mathcal{M}q$ with $|a| > 1$. Then the natural numbers $|p|$ and $|q|$ are co-prime.

Proof. Suppose that the natural numbers $|p|$ and $|q|$ are not co-prime, i.e. $k := \gcd(|p|, |q|) > 1$, we seek a contradiction. Then $|p| = ki, |q| = kj$ for some co-prime natural numbers $i$ and $j$. By (A5), $\mathcal{M}p \cap \mathcal{M}q = \mathcal{M}c$ for some element $c$ of $\mathcal{M}$ with $|c| = \text{lcm}(|p|, |q|) = ijk$. Then $c = \alpha p = \beta q$ for some elements $\alpha$ and $\beta$ of $\mathcal{M}$ with $|\alpha| = j$ and $|\beta| = i$. By (A6), there exist elements $p_1, q_1, d \in \mathcal{M}$ such that $p = p_1d$, $q = q_1d$, $|d| = k > 1$, $|p_1| = i$, $|q_1| = j$.

If $i = j = 1$ then $|\alpha| = |\beta| = 1$, and so $\alpha, \beta \in \mathcal{M}^*$, by (A3). The equality $\alpha p = \beta q$ implies that $\mathcal{M}^*p = \mathcal{M}^*q$. This contradicts to the assumption of the corollary.

Therefore, either $i > 1$ or $j > 1$ or both $i$ and $j$ are strictly greater than 1. These mean that either the element $p$ is reducible (since $p = p_1d$, $|p_1| = i > 1$, $|d| > 1$) or the element $q$ is reducible (since $q = q_1d$, $|q_1| = j > 1$, $|d| > 1$) or both elements $p$ and $q$ are reducible. These contradictions prove the corollary. $\square$

Proof of Theorem [1,1]
1. The first statement is an easy corollary of the second (since in the case (a): $|p_i u| = |p_i|$ and $|u^{-1}p_{i+1}| = |p_{i+1}|$, by (A2) and (A3)).

2. For each element $b$ of the monoid $\mathcal{M}$ with $|b| > 1$, let $\text{Dec}(b)$ be the set of all decompositions of the element $b$ into irreducible components. Two such decompositions, say $X$ and $Y$, are equivalent, $X \sim Y$, if one can be produced from the other by finitely many transformations of the types (a) and (b). Clearly, this is an equivalence relation on the set $\text{Dec}(b)$. Let $X, Y \in \text{Dec}(b)$ and $X', Y' \in \text{Dec}(b')$. If $X \sim Y$ then $XX' \sim YY'$ in $\text{Dec}(bb')$ and $X'X \sim Y'Y$ in $\text{Dec}(bb')$. If $X \sim Y$ and $X' \sim Y'$ then $XX' \sim YY'$ in $\text{Dec}(bb')$.

To finish the proof of statement 2 we have to show that $p_1 \cdots p_r \sim q_1 \cdots q_s$. To prove this fact we use induction on $|a|$. Note that if the element $a$ is irreducible then $\text{Dec}(a) = \{a\}$, and there is nothing to prove. The base of the induction, $|a| = 2$, is obvious since the element $a$ is irreducible, by (A2) and (A3). Suppose that $|a| \geq 3$ and the result is true for all elements $a'$ of $\mathcal{M}$ with $1 < |a'| < |a|$. We may assume that the element $a$ is reducible, i.e. $r \geq 2$ and $s \geq 2$. The proof consists of considering several possibilities.

Suppose that $\mathcal{M}^*p_r = \mathcal{M}^*q_s$, i.e. $p_r = uq_s$ for some element $u \in \mathcal{M}^*$. By (A4), we can delete the element $q_s$ in the equality

$$p_1 \cdots p_{r-1}uq_s = q_1 \cdots q_{s-1}q_s.$$  

As a result, there are two decompositions of the element

$$a' := p_1 \cdots p_{r-1}u = q_1 \cdots q_{s-1}$$

into irreducible components with $1 < |a'| = \frac{|a|}{|q_s|} < |a|$ (note that $p_{r-1}u$ is the irreducible element). By induction, these two decompositions are equivalent in $\text{Dec}(a')$. In particular, $r = s$. Now,

$$p_1 \cdots p_r \sim p_1 \cdots (p_{r-1}u)(u^{-1}p_r) \sim p_1 \cdots (p_{r-1}u) \cdot q_s \sim q_1 \cdots q_{r-1} \cdot q_s,$$

as required.

Suppose that $\mathcal{M}^*p_r \neq \mathcal{M}^*q_s$. Then, by Corollary 2.1, the natural numbers $|p_r|$ and $|q_s|$ are co-prime since $a = p_1 \cdots p_r = q_1 \cdots q_s \in \mathcal{M}p_r \cap \mathcal{M}q_s$ and the elements $p_r$ and $q_s$ are irreducible. By (A6),

$$\mathcal{M}p_r \cap \mathcal{M}q_s = \mathcal{M}c$$

for some element $c$ of the monoid $\mathcal{M}$ with $|c| = \text{lcm}(|p_r|, |q_s|) = |p_r||q_s|$ since the numbers $|p_r|$ and $|q_s|$ are co-prime. Since $a \in \mathcal{M}c$ and $c \in \mathcal{M}p_r \cap \mathcal{M}q_s$, there exist elements $d, \alpha, \beta \in \mathcal{M}$ such that

$$a = dc, \quad c = \alpha p_r = \beta q_s. \quad (3)$$

We can write the equality $dc = a$ in two different ways:

$$d \alpha p_r = p_1 \cdots p_{r-1}p_r \quad \text{and} \quad d \beta q_s = q_1 \cdots q_{s-1}q_s.$$
By (A4), we can delete the element \( p_r \) in the first equality and the element \( q_s \) in the second:

\[
d\alpha = p_1 \cdots p_{r-1} \quad \text{and} \quad d\beta = q_1 \cdots q_{s-1}.
\]

(4)

Note that \( 1 < |p_1| \leq |d\alpha| = \frac{|a|}{|p_r|} < |a| \) and \( 1 < |q_1| \leq |d\beta| = \frac{|a|}{|q_s|} < |a| \) since \( r, s \geq 2 \). Then induction yields the equivalence relations

\[
d\alpha \sim p_1 \cdots p_{r-1} \quad \text{and} \quad d\beta \sim q_1 \cdots q_{s-1}.
\]

There are two options: either \(|d| > 1\) or \(|d| = 1\).

If \(|d| > 1\) then \( 1 < |p_r| \leq |c| = \frac{|a|}{|d|} < |a| \) (see (3)), and so, by induction, \( \alpha p_r \sim \beta q_s \).

Now,

\[
p_1 \cdots p_{r-1} p_r \sim d\alpha p_r \sim d\beta q_s \sim q_1 \cdots q_{s-1} q_s.
\]

Finally, suppose that \(|d| = 1\). By (A3), the element \( d \) is a unit of the monoid \( \mathcal{M} \) since \(|d| = 1\). Then \( \mathcal{M}c = \mathcal{M}da = \mathcal{M}a \) (since \( c = da \)). Without loss of generality we may assume that \( c = a \) and \( d = 1 \). Then the equations (4) mean that

\[
\alpha = p_1 \cdots p_{r-1} \quad \text{and} \quad \beta = q_1 \cdots q_{s-1}.
\]

(5)

Recall that we have the equality \(|c| = |p_r||q_s|\). In combination with (3), i.e. \( a = c = \alpha p_r = \beta q_s \), it yields the equalities

\[
|\alpha| = |q_s| \quad \text{and} \quad |\beta| = |p_r|.
\]

In particular, the numbers \(|\alpha|\) and \(|\beta|\) are co-prime. Recall that \( r \geq 2 \) and \( s \geq 2 \). Now, the case \( r = s = 2 \) is trivially true, \( p_1 p_2 \sim q_1 q_2 \), since \( a = p_1 p_2 = q_1 q_2 \) and the numbers \(|p_1| = |q_2|\) and \(|p_2| = |q_1|\) are co-prime. This is a transformation of the type (b).

It remains to consider the case \((r, s) \neq (2, 2)\). In a view of symmetry, we may assume that \( r \geq 3 \) and \( s \geq 2 \). We prove that this case is not possible, i.e. we seek a contradiction. In order to get a contradiction, the axiom (A6) will be applied to the equality

\[
p_1 \cdot (p_2 \cdots p_r) = \beta \cdot q_s.
\]

(6)

First, note that the numbers

\[
i := |p_1| = \frac{|p_1 \cdots p_{r-1}|}{|p_2 \cdots p_{r-1}|} = \frac{|\alpha|}{|p_2 \cdots p_{r-1}|} = \frac{|q_s|}{|p_2 \cdots p_{r-1}|} \quad \text{and} \quad j := |\beta| = |p_r|
\]

are co-prime since the numbers \(|q_s|\) and \(|p_r|\) are co-prime; \( i > 1 \) and \( j > 1 \). Clearly, \( k := |p_2 \cdots p_{r-1}| > 1 \) since \( r \geq 3 \); \( |p_2 \cdots p_r| = k j \) and \(|q_s| = k i \). Applying the axiom (A6) to the equality (6), we obtain the equalities

\[
p_2 \cdots p_r = AC \quad \text{and} \quad q_s = BC
\]

for some elements \( A, B \) and \( C \) of the monoid \( \mathcal{M} \) with \(|C| = k > 1\). Then \(|B| = \frac{|q_s|}{|C|} = \frac{k i}{k} = i > 1\), and so the elements \( B \) and \( C \) are not units. Therefore, the element \( q_s = BC \) is reducible, a contradiction. The proof of Theorem 1.1 is complete. \( \square \)
Proof of Theorem 1.2

In the proof of Theorem 1.2 we use the Theorem of Lüroth and the fact that $\mathcal{O}$ is a submonoid of the reduction monoid $(K[x], \circ)$. The axioms (A1)–(A4) are obvious for the monoid $\mathcal{O}$.

Let us prove that the axiom (A5) holds for $\mathcal{O}$. Let $a$ and $b$ be elements of the monoid $\mathcal{O}$ such that $\deg(a) > 1$, $\deg(b) > 1$, and there exists an element $x' \in \mathcal{O} \circ a \cap \mathcal{O} \circ b$ with $\deg(x') \geq 1$. Note that $x' \in \mathcal{O}$. Then $x' \in K[x] \circ a \cap K[x] \circ b$, and so $K[x] \circ a \cap K[x] \circ b = K[x] \circ c$ for some element $c$ of $K[x]$, by the axiom (A5) for the reduction monoid $K[x]$. Moreover, $\deg(c) = \text{lcm}(\deg(a), \deg(b))$.

It suffices to show that $c + \nu \in \mathcal{O}$ for some element $\nu \in K$. For, we introduce the $K$-algebra automorphism $\omega$ of the polynomial algebra $K[x]$ given by the rule $x \mapsto -x$. Then

$$K[x] = K[x^2] \oplus K[x^2]x = K[x^2] \oplus \mathcal{O},$$

where $K[x^2]$ is the fixed ring for the automorphism $\omega$, and $\mathcal{O}$ is the eigen-space for $\omega$ that corresponds to the eigenvalue $-1$, i.e. $\mathcal{O} = \ker(\omega + 1)$. Note that the equality $K[x] \circ a \cap K[x] \circ b = K[x] \circ c$ simply means that

$$K[a] \cap K[b] = K[c],$$

and so the element $c$ is uniquely defined up to an affine transformation. By (7), the element $c$ is a unique sum $c_0 + c_1 x$ for some elements $c_0, c_1 \in K[x^2]$. Note that $c_1 \neq 0$ since, otherwise, $c = c_0 \in K[x^2]$, and then

$$x' \in \mathcal{O} \circ a \cap \mathcal{O} \circ b \subseteq K[x] \circ a \cap K[x] \circ b = K[c] \subseteq K[x^2].$$

Now, $x' \in \mathcal{O} \cap K[x^2] = 0$, a contradiction (recall that $\deg(x') \geq 1$, by the assumption). This contradiction proves the claim that $c_1 \neq 0$. Note that

$$\omega(K[c]) = \omega(K[a] \cap K[b]) = \omega(K[a]) \cap \omega(K[b]) = K[-a] \cap K[-b] = K[a] \cap K[b] = K[c].$$

This means that $\omega(c) = \lambda c + \mu$ for some scalars $\lambda \neq 0$ and $\mu$ of $K$. In combination with the equality $\omega(c) = c_0 - c_1 x$ and the fact that $c_1 \neq 0$, it gives that $\lambda = -1$, i.e. $\omega(c) = -c + \mu$. Then changing $c$ to $c - \frac{\mu}{2}$ we may assume that $\mu = 0$, i.e. $\omega(c) = -c$. This means that $c \in \mathcal{O}$, as required. This proves that the axiom (A5) holds for the monoid $\mathcal{O}$.

To finish the proof of Theorem 1.2 it remains to establish the axiom (A6) for the monoid $\mathcal{O}$.

Suppose that elements $a$, $b$, $\alpha$ and $\beta$ of the monoid $\mathcal{O}$ satisfy the following conditions: $\alpha \circ a = \beta \circ b$ with $\deg(\alpha) = i$, $\deg(a) = jk$, $\deg(\beta) = j$, $\deg(b) = ik$, $ijk \geq 1$, and the natural numbers $i$ and $j$ are co-prime. We have to show that $a = a_1 \circ d$ and $b = b_1 \circ d$ for some elements $a_1$, $b_1$ and $d$ of the monoid $\mathcal{O}$ such that $\deg(d) = k$. In the proof of the axiom (A5) for the monoid $\mathcal{O}$, we found the element $c \in \mathcal{O}$ such that

$$K[c] = K[a] \cap K[b], \ \deg(c) = \text{lcm}(\deg(a), \deg(b)) = ijk.$$
Then, it is easy to show that
\[ K(c) = K(a) \cap K(b). \] (8)
Indeed, by the Theorem of Lüroth, \( K(a) \cap K(b) = K(c^*) \) for some element \( c^* \in K(x) \) which can be chosen from the polynomial algebra \( K[x] \), by Lemma 3.1, [7]. Then
\[ K[c^*] = K[x] \cap K(c^*) = (K[x] \cap K(a)) \cap (K[x] \cap K(b)) = K[a] \cap K[b] = K[c], \]
and so the equality (8) follows.

For a field extension \( \Delta \subseteq \Gamma \), let \([\Gamma : \Delta] := \dim_{\Delta}(\Gamma)\). Consider the fields \( K(c) \subseteq K(a) \subseteq K(x) \). Then
\[
ijk = \deg(c) = [K(x) : K(c)] = [K(x) : K(a)] \cdot [K(a) : K(c)] \\
= \deg(a) \cdot [K(a) : K(c)] = jk \cdot [K(a) : K(c)],
\]
hence \([K(a) : K(c)] = i\). By symmetry, \([K(b) : K(c)] = j\). By the Theorem of Lüroth, the composite field \( K(a)K(b) = K(a,b) \subseteq K(x) \) is equal to \( K(d) \) for some rational function \( d \in K(x) \) which can be chosen to be a polynomial of \( K[x] \) since \( a, b \in K[x] \). Let us show that
\[
[K(d) : K(c)] = ij. \] (9)
Clearly,
\[
[K(d) : K(c)] = [K(a,b) : K(c)] = [K(a)(b) : K(a)][K(a) : K(c)] \\
\leq [K(c)(b) : K(c)][K(a) : K(c)] \\
= [K(b) : K(c)][K(a) : K(c)] = ji.
\]
To prove the reverse inequality note that
\[
[K(d) : K(c)] = [K(d) : K(a)][K(a) : K(c)] = [K(d) : K(a)] \cdot i, \\
[K(d) : K(c)] = [K(d) : K(b)][K(b) : K(c)] = [K(d) : K(b)] \cdot j,
\]
and so \([K(d) : K(c)] \geq \text{lcm}(i, j) = ij \) since the numbers \( i \) and \( j \) are co-prime. This proves the equality (9). Now,
\[
\deg(d) = \frac{[K(x) : K(c)]}{[K(d) : K(c)]} = \frac{ijk}{ij} = k.
\]
Note that
\[
K(\omega(d)) = \omega(K(d)) = \omega(K(a,b)) = K(\omega(a),\omega(b)) = K(-a,-b) = K(a,b) = K(d). \]
This means that \( \omega(d) = \lambda d + \mu \) for some scalars \( \lambda \neq 0 \) and \( \mu \) of \( K \) since \( d \in K[x] \) and \( \omega(K[x]) = K[x] \). By (7), the polynomial \( d \) is a unique sum \( d_0 + d_1 x \) for some polynomials \( d_0, d_1 \in K[x^2] \). We must have \( d_1 \neq 0 \) since, otherwise, \( d = d_0 \in K[x^2] \). Since \( a = a_1 \circ d \) for
some polynomial \( a_1 \in K[x] \), we would have \( a \in a_0 \circ K[x^2] \subseteq K[x^2] \), and so \( a \in \mathcal{O} \cap K[x^2] = 0 \), a contradiction (since \( a \neq 0 \)). Therefore, \( d_1 \neq 0 \). Then the equalities

\[
d_0 - d_1 x = \omega(d) = \lambda d + \mu = \lambda d_0 + \mu + \lambda d_1 x
\]
yield \( \lambda = -1 \), and so \( \omega(d) = -d + \mu \). Then changing \( d \) for \( d - \frac{d}{2} \) we may assume that \( \mu = 0 \), that is \( \omega(d) = -d \), i.e. \( d \in \mathcal{O} \). We claim that the polynomial \( a_1 \in K[x] \) in the equality \( a = a_1 \circ d \) above belongs to \( \mathcal{O} \). To prove this we write the polynomial \( a_1 \) as a unique sum \( u + v x \) for some polynomials \( u, v \in K[x^2] \). Note that \( u \circ d, v \circ d \in K[x^2] \) and \((v \circ d) \cdot d \in \mathcal{O} \). The inclusion

\[
a = a_1 \circ d = u \circ d + (v \circ d) \cdot d \in \mathcal{O}
\]
yields \( u \circ d = 0 \), i.e. \( u = 0 \). This proves that \( a_1 = v x \in \mathcal{O} \). By symmetry, we have \( b = b_1 \circ d \) for some element \( b_1 \in \mathcal{O} \). This means that the axiom (A6) holds for the monoid \( \mathcal{O} \). The proof of Theorem 1.2 is complete. □

**Definition.** A Ritt transformation of the decomposition (1) is either one of the decompositions (a), (b) or (c) with \( \lambda_2 = 1 \) and \( \gcd(\deg(p_i), \deg(p_{i+1})) = 1 \) (in all three cases) and with the numbers \( m \) and \( n \) being odd prime numbers in the case (a) (see (2)) or a decomposition of the type

\[
(d) \quad p_1 \circ \cdots \circ (p_i \circ u) \circ (u^{-1} \circ p_{i+1}) \circ \cdots \circ p_r
\]

for some polynomial \( u \in K[x]^* \).

In his paper, J. F. Ritt wrote (page 52, the last line): “Case (a) with \( m = 2 \) can be reduced to Case (b) by linear transformation.” In more detail, for each natural number \( k \geq 1 \),

\[
T_2 = -1 + 2x^2 = (-1 + 2x) \circ x^2 = \alpha \circ x^2, \quad \alpha := -1 + 2x,
\]

\[
T_{2k+1} = \sum_{i=0}^{k} \binom{2k+1}{2i} x^{2k+1-2i} (1 - x^2)^i = xt_{2k+1}(x^2),
\]

\[
t_{2k+1}(x) := \sum_{i=0}^{k} \binom{2k+1}{2i} x^{k-i} (1 - x)^i.
\]

Let \( n = 2k + 1 \). Then

\[
T_2 \circ T_n = \alpha \circ x^2 \circ [x t_n(x^2)] = \alpha \circ [x t_n^2] \circ x^2 = \alpha \circ [x t_n^2] \circ \alpha^{-1} \circ \alpha \circ x^2 = \alpha \circ [x t_n^2] \circ \alpha^{-1} \circ T_2,
\]

and the remark of J. T. Ritt is obvious. Note that \( T_n \circ T_2 = T_2 \circ T_n = \alpha \circ [x t_n^2] \circ \alpha^{-1} \circ T_2 \), and so (by (A4))

\[
T_n = \alpha \circ [x t_n^2] \circ \alpha^{-1}.
\]

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Now, it is obvious that also the case (a) with \( n = 2 \) can be reduced to the case (c) by linear transformation. This is the reason why in the definition of Ritt transformation \( m \) and \( n \) are odd primes (in the case (a)).

All trigonometric polynomials \( T_l = xt_l(x^2) \) do not belong to the algebra \( A \) where \( l \) runs through all odd prime numbers (since \( T_l'(0) = l \neq 0 \)). But \( T_2 \in A \).

The next corollary follows from Theorem 1.1 and the second theorem of Ritt(-Levi), it is implicit in the papers [13] and [11].

**Corollary 2.2** If \( a \in K[x] \) has two decompositions into irreducible polynomials then one can be obtained from the other by Ritt transformations.

**Proof of Corollary 1.3.(3).**

The idea of the proof of Corollary 1.3.(3) is to use the second theorem of Ritt-Levi in combination with Lemma 2.3, Theorem 2.6 and Lemma 2.8. We first prove all these preliminary results that are interesting on their own.

**Lemma 2.3** Let \( K \) be a field of characteristic zero, \( a \) and \( b \) be non-scalar polynomials of \( K[x] \) such that \( a \circ b \in \mathcal{O} \). If one of the polynomials \( a \) or \( b \) belongs to the set \( \mathcal{O} \), then so does the other.

**Proof. Case (i):** \( a \in \mathcal{O} \). The polynomial \( a \) is a non-scalar polynomial, and so

\[
a = \sum_{n=0}^{N} \lambda_n x^{2n+1}, \quad \lambda_n \in K, \quad \lambda_N \neq 0.
\]

Due to the decomposition \( K[x] = K[x^2] \oplus K[x^2]x \), each polynomial \( p \) of \( K[x] \) is a unique sum \( p = p^{ev} + p^{od} \) of an even \( p^{ev} \in K[x^2] \) and odd \( p^{od} \in K[x^2]x \) polynomials. Then \( b = b_0 + b_1 \) where \( b_0 := b^{ev} \) and \( b_1 := b^{od} \). We have to show that \( b_0 = 0 \). Suppose that \( b_0 \neq 0 \), we seek a contradiction. Clearly, \( b_1 \neq 0 \) since otherwise we would have the inclusion \( c \in K[x^2]x \cap K[x^2] = 0 \), a contradiction. Let us consider the even part of the polynomial \( c \),

\[
c^{ev} = (a \circ b)^{ev} = \left( \sum_{n=0}^{N} \lambda_n (b_0 + b_1)^{2n+1} \right)^{ev} = \sum_{n=0}^{N} \lambda_n \sum_{m=0}^{n} \binom{2n + 1}{2m + 1} b_0^{2m+1} b_1^{2(n-m)}.
\]

The degrees of the nonzero polynomials \( b_0 \) and \( b_1 \) are even and odd numbers respectively. Therefore, either \( \deg(b_0) > \deg(b_1) \) or, otherwise, \( \deg(b_0) < \deg(b_1) \). The leading coefficient of the polynomial \( c^{ev} \) is equal to

\[
\begin{cases}
\lambda_N b_0^{2N+1} & \text{if } \deg(b_0) > \deg(b_1), \\
\lambda_N (2N+1) b_0 b_1^{2N} & \text{if } \deg(b_0) < \deg(b_1).
\end{cases}
\]
The first case is obvious; the second case follows from the inequalities: for all natural numbers \( m \) and \( n \) such that \( 0 \leq m \leq n \),

\[
\text{deg}(b_0^{2m-1}b_1^{n-m+1}) - \text{deg}(b_0^{2m+1}b_1^{n-m}) = 2(\text{deg}(b_1) - \text{deg}(b_0)) > 0.
\]

Since in both cases the leading term of the polynomial \( c^\mu \) is non-zero, we have \( c^\mu \neq 0 \). This contradicts to the assumption that \( c \in K[x^2]x \), i.e. \( c^\mu = 0 \). The contradiction finishes the proof of the case (i).

Case (ii): \( b \in \mathcal{O} \). Then \( \omega(b) = -b \). Similarly, \( \omega(c) = -c \) since \( c \in K[x^2]x \). The polynomial \( a \) is a unique sum \( a^e + a^o \) of even and odd polynomials. Comparing both ends of the following series of equalities

\[
-(a^e \circ b + a^o \circ b) = -c = \omega(c) = \omega(a \circ b) = a \circ \omega(b) = a \circ (-b)
\]

we conclude that \( a^e \circ b = 0 \), hence \( a^e = 0 \) since \( b \) is a non-scalar polynomial, and so \( a = a^o \in \mathcal{O} \), as required. The proof of Lemma 2.3 is complete. \( \square \)

Let \( f = f_0 + f_1 \in K[x] \) where \( f_0 := f^e \) and \( f_1 := f^o \). Let \( f^{(k)} := \frac{d^k f}{dx^k} \) and \( f^{(k)}(g) := \frac{d^k f}{dx^k} \circ g \). Then \( f^{(2n)} = f_0^{(2n)} + f_1^{(2n)} \) and \( f^{(2n+1)} = f_1^{(2n+1)} + f_0^{(2n+1)} \) where \( f_0^{(2n)}, f_1^{(2n+1)} \in K[x^2] \) and \( f_1^{(2n)}, f_0^{(2n+1)} \in \mathcal{O} \).

**Lemma 2.4** Let \( f = f^e + f^o \in K[x] \) and \( \mu \in K^* \). Then \( (x + \mu) \circ f \in \mathcal{O} \) iff \( f^e = -\mu \).

**Proof.** \( (x + \mu) \circ f = \mu + f = \mu + f^e + f^o \in \mathcal{O} \) iff \( f^e = -\mu \). \( \square \)

**Lemma 2.5** Let \( a \in \mathcal{O} \) and \( f = f_0 + f_1 \in K[x] \) where \( f_0 := f^e \) and \( f_1 := f^o \). Then

\[
(a \circ f)^e = \sum_{k \geq 0} a^{(2k+1)}(f_1) \cdot \frac{f_0^{2k+1}}{(2k+1)!} \quad \text{and} \quad (a \circ f)^o = \sum_{k \geq 0} a^{(2k)}(f_1) \cdot \frac{f_0^{2k}}{(2k)!}.
\]

**Proof.** The result is an easy consequence of the Taylor’s formula,

\[
a \circ f = a(f_1 + f_0) = \sum_{i \geq 0} a^{(i)}(f_1) \cdot \frac{f_0^i}{i!},
\]

and the following two facts: \( a^{(2k+1)}(f_1) \in K[x^2] \) and \( a^{(2k)}(f_1) \in \mathcal{O} \). \( \square \)

**Theorem 2.6** Suppose that \( a \in \mathcal{O} \) with \( \text{deg}(a) > 1 \), \( \mu \in K^* \), and \( f \in K[x] \setminus K \). Then \( (x + \mu) \circ a \circ f \notin \mathcal{O} \).

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Proof. Suppose that \((x + \mu) \circ a \circ f \in \mathcal{O}\), we seek a contradiction. Then
\[
-\mu = (a \circ f)^{ev} \quad \text{(by Lemma 2.4)}
\]
\[
= \sum_{k \geq 0} a^{2(k+1)}(f_1) \cdot \frac{f_0^{2k+1}}{(2k+1)!} \quad \text{(by Lemma 2.5)}
\]
\[
= f_0 \cdot \sum_{k \geq 0} a^{(2k+1)}(f_1) \cdot \frac{f_0^{2k}}{(2k+1)!}.
\]
Comparing the degrees of both ends of the series of equalities above, we conclude that \(f_0 \in K^*\) since \(\mu \neq 0\). Let \(\partial := \frac{d}{dx}\). Then \(-\mu = \Delta \partial(f_1)\) where the linear map
\[
\Delta := \sum_{k \geq 0} \frac{f_0^{2k+1}}{(2k+1)!} \partial^{2k} : K[x] \to K[x]
\]
is equal to \(f_0(1 - n)\) where \(n := -\sum_{k \geq 1} \frac{f_0^{2k}}{(2k+1)!} \partial^{2k}\) is a locally nilpotent map, that is \(K[x] = \bigcup_{i \geq 1} \ker(\Delta^i)\). The map \(\Delta\) is invertible and \(\Delta^{-1} = f_0^{-1}(1 + n + n^2 + \cdots)\). Then \(\partial(f_1) = -\Delta^{-1}(\mu) = -f_0^{-1}\), and so \(\deg(f_1) \leq 1\), that is \(f_1 = \gamma x\) for some \(\gamma \in K^*\) since \(f \notin K\). We claim that \(f_0 \neq 0\) since otherwise we would have the inclusion \((x + \mu) \circ a \circ f_1 = a \circ f_1 + \mu \in \mathcal{O}\), which would have implied that \(\mu = 0\) (since \(a \circ f_1 \in \mathcal{O}\), a contradiction. Changing, if necessary, the element \(a\) to \(a \circ f_1 = a \circ [\gamma x] \in \mathcal{O}\), we may assume that \(\gamma = 1\). Then \(\mathcal{O} \ni (x + \mu) \circ a \circ (f_0 + x)\) iff
\[
-\mu = (a \circ (f_0 + x))^{ev} = \sum_{k \geq 0} a^{(2k+1)} \frac{f_0^{2k+1}}{(2k+1)!} \quad \text{(see above)}.
\]
This implies that \(\deg(a) \leq 1\), a contradiction (since \(\deg(a) > 1\), by the assumption). This contradiction finishes the proof of the theorem. \(\square\)

The next corollary follows at once from Theorem 2.6.

**Corollary 2.7** Suppose that \(a \in \mathcal{O}\) with \(\deg(a) \geq 1\), \(\mu \in K^*\), and \(f \in K[x] \setminus K\). If \((x + \mu) \circ a \circ f \in \mathcal{O}\) then \(\deg(a) = 1\).

**Example.** \((x + \mu) \circ [\lambda x] \circ (x - \lambda^{-1} \mu) \in \mathcal{O}\) for all \(\lambda, \mu \in K^*\).

**Lemma 2.8** Let \(f \in K[x]\) with \(\deg(f) \geq 1\) and \(u \in K[x]^*\). Then \(f \circ x^2 \circ u \notin \mathcal{O}\).

**Proof.** Let \(u = \lambda x + \mu\) for some \(\lambda \in K^*\), and \(f = \sum_{i=0}^n \lambda_i x^i\) where \(n := \deg(f)\), and so \(\lambda_n \neq 0\). Then \(f \circ x^2 \circ (\lambda x + \mu) = \sum_{i=0}^n (\lambda x + \mu)^{2i} = \lambda_n \lambda^n x^{2n}\) + smaller terms, and so \(f \circ x^2 \circ u \notin \mathcal{O}\). \(\square\)
The proof of Corollary 1.3 (3) continued. Let us continue with the proof of Corollary 1.3 (3). Recall that $O^* = \{ x \in K^* \}$. We have to show that if there is an equality $p \circ q = p^* \circ q^*$ where $p, q, p^*$ and $q^*$ are irreducible elements of the monoid $O$ then modulo basic transformations of the pairs $P := (p, q)$ and $P^* := (p^*, q^*)$:

$$(p, q) \mapsto (u \circ p \circ v, v^{-1} \circ q \circ w), \quad (p^*, q^*) \mapsto (u \circ p^* \circ \tilde{v}, \tilde{v}^{-1} \circ q \circ w),$$

where $u, v, \tilde{v}, w \in O^*$, we have either the equality $P = P^*$ or, otherwise, $P$ and $P^*$ as in Corollary 1.3 (3).

If $(p^*, q^*) = (p \circ v, v^{-1} \circ q)$ for some element $v \in K[x]^*$ then, by Lemma 2.3, $v \in O^*$, and there is nothing to prove, the result is obvious. So, suppose that $(p^*, q^*) \neq (p \circ v, v^{-1} \circ q)$ for all element $v \in K[x]^*$. Then by the second theorem of Ritt-Levi the pair $P^*$ can be obtained from the pair $P$ by finitely many Ritt transformations

$$P = P_1 \sim_R P_2 \sim_R \cdots \sim_R P_s = P^*,$$

and necessarily some of the Ritt transformations are of the types (a), (b) or (c). It might happen that the elements $p$ and $q$ are reducible in the monoid $K[x]$ (but the essence of the proof is to show that they are, in fact, irreducible in $K[x]$).

Each Ritt transformation $P_i := (p_i, q_i) \sim_R P_{i+1} := (p_{i+1}, q_{i+1})$ may transform either the irreducible factors (in $(K[x], \circ)$) of $p_i$ or of $q_i$ or simultaneously the last irreducible factor, say $l_i$, of $p_i$ and the first irreducible factor, say $f_i$, of $q_i$. The first two types of Ritt transformations do not change the elements $p_i$ and $q_i$. So, there exists an index $i$ such that the Ritt transformation $P_i \sim_R P_{i+1}$ is of the third type and, necessarily, of one of the types (a), (b) or (c) as in the definition of Ritt transformations since, for given $u \in K[x]^*$ and $a \in O^*$, the inclusion $u \circ a \in O^* \implies u \in O^*$ (Lemma 2.3). Let $i$ be the least such an index. For each $j$, let $Q_j := (l_j, f_j)$. Then $p_j = a_j \circ l_j$ and $q_j = f_j \circ \beta_j$ for some polynomials $\alpha_j, \beta_j \in K[x]$. There are the following three options for the pairs $Q_i = (l_i, f_i)$ and $Q_{i+1} = (l_{i+1}, f_{i+1})$ (where $u, v, w, \tilde{w} \in K[x]^*$):

(a) $Q_i = (u \circ T_n \circ o, w^{-1} \circ T_m \circ o)$ and $Q_{i+1} = (u \circ T_m \circ \tilde{w}, \tilde{w}^{-1} \circ T_n \circ o)$ where $n$ and $m$ are odd primes,

(b) $Q_i = (u \circ [x^t \beta^s] \circ o, w^{-1} \circ x^s \circ o)$ and $Q_{i+1} = (u \circ x^s \circ \tilde{w}, \tilde{w}^{-1} \circ [x^t \beta(x^s)] \circ o)$,

(c) $Q_i = (u \circ x^s \circ o, w^{-1} \circ [x^t \beta(x^s)] \circ o)$ and $Q_{i+1} = (u \circ [x^t \beta^s] \circ \tilde{w}, \tilde{w}^{-1} \circ x^s \circ o)$,

where $s$ is a prime number, $t \geq 0$, and $\beta \in K[x]$ with $\beta(0) \neq 0$. In the cases (b) and (c), $s$ is an odd prime number since, otherwise, by Lemma 2.8 the polynomials $p_{i+1} \notin O$ (the case (b)) and $p_i \notin O$ (the case (c)), which are contradictions.

Let us consider the case (a). Note that $T_m, T_n \in O$. Applying Theorem 2.6 to the inclusion $w^{-1} \circ T_m \circ (v \circ \beta) = q_i \in O$, we see that $w^{-1} \in O^*$. Then we have the inclusion $T_m \circ (v \circ \beta) \in O$ which yields the inclusion $v \circ \beta \in O$, by Lemma 2.3 (since $T_m \in O$). Since $q_i$ is an irreducible element of the monoid $O$, we must have $v \circ \beta \in O^*$.

Since $w \in O^*$ and $(\alpha_i \circ u \circ T_n) \circ w = p_i \in O$, we have the inclusion $\alpha_i \circ u \in O$, hence $\alpha_i \circ u \in O$ (by Lemma 2.3 since $T_n \in O$). Moreover, $\alpha_i \circ u \in O^*$ since $p_i$ is an irreducible element of the monoid $O$. As a result, we have the case (a) of Corollary 1.3 (3).
Let us consider the case (b). Since $x^s \in \mathcal{O}$ and $w^{-1} \circ x^s \circ (v \circ \beta_i) = q_i \in \mathcal{O}$, we have $w^{-1} \in \mathcal{O}^*$ (by Theorem 2.4). Then $x^s \circ (v \circ \beta_i) \in \mathcal{O}$, hence $v \circ \beta_i \in \mathcal{O}$, by Lemma 2.3. The element $q_i$ is an irreducible element of the monoid $\mathcal{O}$, and so $v \circ \beta_i \in \mathcal{O}^*$. By replacing the element $v$ with $v \circ \beta_i$, we may assume that $\beta_i = 1$ and $v \in \mathcal{O}^*$. Now, it follows from the inclusion

$$\mathcal{O} \ni q_{i+1} = \tilde{w}^{-1} \circ \left[x^t \beta(x^s)\right] \circ v \circ \beta_i = \tilde{w}^{-1} \circ \left[x^t \beta(x^s)\right] \circ v$$

that $\tilde{w}^{-1} \circ \left[x^t \beta(x^s)\right] \in \mathcal{O}$.

If $t \neq 0$ then $\tilde{w}^{-1} \in \mathcal{O}^*$, and so $x^t \beta(x^s) \in \mathcal{O}$, hence $t$ is odd (since $\beta(0) \neq 0$), and $\beta = \alpha(x^2)$ for some polynomial $\alpha(x) \in K[x]$. Since $[x^t \beta(x^s)] \circ w \in \mathcal{O}$ and $(\alpha_i \circ u) \circ [x^t \beta(x^s)] \circ w = p_i \in \mathcal{O}$, we have $\alpha_i \circ u \in \mathcal{O}$, by Lemma 2.3. Therefore, $\alpha_i \circ u \in \mathcal{O}^*$ since $p_i$ is an irreducible element of the monoid $\mathcal{O}$ and $x^t \beta(x^s) \notin \mathcal{O}^*$. This means that we have the case (b) of Corollary 1.3 (3) (if $t \neq 0$).

To finish with the case b it suffices to show that the remaining subcase when $t = 0$ is impossible. Suppose that $t = 0$, we seek a contradiction. Then the inclusion $\tilde{w}^{-1} \circ \beta(x^s) \in \mathcal{O}$ yields $\beta = \tilde{w} \circ x^t \alpha_1(x^2)$ for some odd natural number $T$ and a polynomial $\alpha_1(x) \in K[x]$ with $\alpha_1(0) \neq 0$. Note that $w \in \mathcal{O}^*$ and

$$\mathcal{O} \ni p_i = \alpha_i \circ u \circ \beta^s \circ w = \alpha_i \circ u \circ x^s \circ \tilde{w} \circ \left[x^t \alpha_1(x^2)\right] \circ w.$$ 

Since $x^t \alpha_1(x^2) \circ w \in \mathcal{O}$ and the element $p_i \in \mathcal{O}$ is irreducible, we must have $\alpha_i \circ u \circ x^s \circ \tilde{w} \in \mathcal{O}^*$, by Lemma 2.3, hence $s = 1$, a contradiction ($s$ is a prime number).

The remaining case (c) follows from the case (b) by interchanging the roles of the pairs (and repeating the proof of the case (b)).

Therefore, the pairs $P_i$ and $P_{i+1}$ are as in Corollary 1.3 (3). By the minimality of $i$, we have $p = p_1 = \cdots = p_i$ and $q = q_1 = \cdots = q_i$, and so $P = P_i$. Now, the result is obvious. The proof of Corollary 1.3 (3) is complete. 

**Remark.** Let us explain the remark made in the Introduction that the monoid $\mathcal{O}$ has non-commutative origin. Let $\lambda$ be a nonzero scalar. The algebra

$$\Lambda = \langle x, y \mid xy = \lambda yx \rangle$$

is called the quantum plane. The algebra $\Lambda$ is the skew polynomial algebra $K[y][x; \sigma]$ where $\sigma$ is the $K$-algebra automorphism of the polynomial algebra $K[y]$ which is given by the rule $\sigma(y) = \lambda y$. The localization $\Lambda' := S^{-1}\Lambda$ of the algebra $\Lambda$ at the Ore set $S := K[y] \setminus \{0\}$ is the skew polynomial algebra $\Lambda' = K(y)[x; \sigma]$. Let $\lambda = -1$. The centre $Z'$ of the algebra $\Lambda$ is the polynomial algebra $K(y^2)[x^2]$ with coefficients from the field $K(y^2)$. Clearly,

$$\Lambda' = K(y)[x^2] \oplus K(y)[x^2]x$$

where the algebra $K(y)[x^2]$ is the fixed ring of the inner automorphism $\omega_y : u \mapsto yuy^{-1}$ of $\Lambda'$, and $K(y)[x^2]x = \ker(\omega_y + 1)$. Then it follows that the monoid $\mathcal{E}$ of all the $K$-algebra endomorphisms of $\Lambda'$ elements of which fix the element $y$ is equal to the set $\{\tau_\alpha : x \mapsto \alpha x \mid \alpha \in K(y)[x^2]\}$. The endomorphism $\tau_\alpha$ is called a central endomorphism if $\alpha \in Z'$. 

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The submonoid \( Z := \{ \tau_\alpha \mid \alpha \in Z' \} \) of all central endomorphisms of \( \Lambda' \) is isomorphic to the monoid \( \mathcal{O} \) of odd polynomials in \( x \) where the base field is \( K(y^2) \) rather than \( K \).

The set \( \text{Irr}(K[x]) \) of all the irreducible elements of the monoid \( (K[x], \circ) \) is the union of its three subsets,

\[
\text{Irr}(K[x]) = \mathcal{P} \cup \mathcal{Q} \cup \mathcal{R}
\]

where an irreducible polynomial \( p \) is an element of the set \( \mathcal{P} \) iff \( p \in K[x]^* \circ x^l \circ K[x]^* \) for some prime number \( l \); an irreducible polynomial \( p \) belongs to \( \mathcal{Q} \) iff either

\[
p \in K[x]^* \circ [x^s g(x')] \circ K[x]^* \quad \text{or} \quad p \in K[x]^* \circ [x^s g'] \circ K[x]^*
\]

for some prime number \( l \), \( s \geq 1 \), \( g(x) \in K[x] \setminus K \) with \( g(0) \neq 0 \); \( \mathcal{R} := \text{Irr}(K[x]) \setminus \mathcal{P} \cup \mathcal{Q} \).

**Proposition 2.9**

1. The union \((10)\) is a disjoint union.

2. The set \( \mathcal{P} \cup \mathcal{Q} \) contains precisely all the irreducible polynomials of \( K[x] \) that are involved in all the Ritt transformations.

**Proof.**

1. By Lemma 2.10 the union \( \mathcal{P} \cup \mathcal{Q} \) is disjoint. Now, statement 1 is obvious.

2. For a prime number \( l \), a polynomial \( f \) of the form \( g(x') = g(x) \circ x^l \) (resp. \( g' = x^l \circ g \)) is irreducible iff \( f \in \mathcal{P} \) (then, necessarily, \( g \) is a unit). By Lemma 2.11 and the explicit formula for \( T_i \) (see above), for each odd prime number \( l \),

\[
K[x]^* \circ T_i \circ K[x]^* \subseteq \mathcal{Q}.
\]

But \( T_2 \in \mathcal{P} \). Now, statement 2 follows from the definitions of Ritt transformations and of the sets \( \mathcal{P} \) and \( \mathcal{Q} \). \( \square \)

**Lemma 2.10** Let \( f(x) \) be a non-scalar polynomial of \( K[x] \) such that \( f(0) \neq 0 \), \( s \) and \( p \) be natural numbers such that \( s \geq 1 \) and \( p \geq 2 \). Then the polynomials \( x^s f(x^p) \) and \( x^s f^p \) do not belong to the set \( \mathcal{N} := \bigcup_{n \geq 2} K[x]^* \circ x^n \circ K[x]^* \).

**Proof.** Suppose that \( x^s f(x^p) \in \mathcal{N} \), that is \( x^s f(x^p) = u \circ x^n \circ v \) for some elements \( u \) and \( v \) of the set \( K[x]^* \) and \( n \geq 2 \). We seek a contradiction. The derivative \( (u \circ x^n \circ v)' \) of the polynomial \( u \circ x^n \circ v \) has a single root with multiplicity \( n - 1 \geq 1 \). The same is true for the derivative of the polynomial \( x^s f(x^p) \) which is equal to

\[
(x^s f(x^p))' = x^{s-1}(sf(x^p) + px^p f'(x^p)) = x^{s-1}L(x^p) \neq 0
\]

where \( L(x) := sf(x) + px f'(x) \). If \( s \geq 2 \) then zero must be a root of the polynomial \( L(x^p) \), but \( L(0) = sf(0) \neq 0 \), a contradiction. If \( s = 1 \) then the polynomial \( L(x^p) \) must have a single root, say \( \lambda \), which is not equal to zero since \( L(0) \neq 0 \). Let \( e \) be a \( p \)’th root of 1 which is not equal to 1. Then \( e \lambda \) is another root of \( L(x^p) \) distinct from \( \lambda \), a contradiction. Therefore, \( x^s f(x^p) \notin \mathcal{N} \).
Suppose that \( x^af^p(x) \in N \), that is \( x^af^p(x) = u \circ x^n \circ v \) for some elements \( u \) and \( v \) of the set \( K[x]^* \) and \( n \geq 2 \). We seek a contradiction. By the same argument as in the previous case, the derivative \((x^af^p)'\) of the polynomial \( x^af^p \) must have a single root with multiplicity \( n - 1 \geq 1 \). Clearly,

\[
0 \neq (x^af^p)' = x^{a-1} \cdot f^{p-1} \cdot (sf + pxf').
\]

Note that the polynomial \( f^{p-1} \) has a nonzero root since \( f(0) \neq 0 \). Hence, \( s = 1 \) and the polynomials \( f^{p-1} \) and \( f + pxf' \) have the same root, say \( \lambda \), but may be with different multiplicities. The root \( \lambda \) is a nonzero one since \( f(0) \neq 0 \). Then \( f = \mu(x - \lambda)^m \) for some \( 0 \neq \mu \in K \) and \( m \geq 1 \), and so

\[
f + pxf' = \mu(x - \lambda)^{m-1}(x - \lambda + pmx).
\]

Hence, \( \lambda = \lambda(1 + pm)^{-1} \), and so \( 1 = 1 + pm > 1 \), a contradiction. Therefore, \( x^af^p(x) \notin N \).

\[\square\]

**Lemma 2.11** Let \( p \) be an odd natural number such that \( p \geq 3 \). Then the trigonometric polynomial \( T_p \) does not belong to the set \( N := \cup_{n \geq 2}K[x]^* \circ x^n \circ K[x]^* \).

**Proof.** The derivative \( T_p' \) of the polynomial \( T_p \) has at least two distinct roots (Lemma 2.12) since \( p \geq 3 \), and so the result. \( \square \)

The next result will be used in the proof of Theorem 1.5.

**Lemma 2.12** Let \( p \) be a natural number such that \( p \geq 2 \). Then

1. The derivative \( T_p' \) of the trigonometric polynomial \( T_p \) is a polynomial of degree \( p - 1 \) which has \( p - 1 \) distinct roots: \( \cos(\pi p), i = 1, 2, \ldots, p - 1 \).

2. If \( k \) and \( l \) are distinct prime numbers then the polynomials \( T_k' \) and \( T_l' \) have no common roots.

**Proof.** 1. By the very definition, the numbers \( \cos(\pi p), i = 1, 2, \ldots, p - 1 \), are distinct. Note that \( \sin(\pi p) \neq 0 \) and \( \sin(p \cdot \pi p) = 0 \) for all \( i = 1, 2, \ldots, p - 1 \). Since

\[
T_p'(\cos(x)) \sin(x) = p \sin(px),
\]

we have \( T_p'(\cos(\pi p)) = 0 \) for all \( i = 1, 2, \ldots, p - 1 \). Now, statement 1 is obvious since \( \deg(T_p') = \deg(T_p) - 1 \leq p - 1 \).

2. Statement 2 follows from statement 1. \( \square \)

Let \( a \) be a polynomial of \( K[x] \) with \( \deg(a) > 1 \) and \( X = p_1 \circ \cdots \circ p_r \in \text{Dec}(a) \) be a decomposition of the polynomial \( a \) into irreducible polynomials of \( K[x] \). Let \( n_\mathcal{P}(X), n_\mathcal{Q}(X) \) and \( n_\mathcal{R}(X) \) be the numbers of irreducible factors \( p_i \) of the types \( \mathcal{P}, \mathcal{Q} \) and \( \mathcal{R} \) respectively. For each prime number \( l \), let \( n_\mathcal{P,l}(X) \) be the number of irreducible factors \( p_i \) such that \( p_i \in K[x]^* \circ x^l \circ K[x]^* \).
Theorem 2.13 The numbers \( n_P(X), n_Q(X), n_R(X) \) and \( n_{P,l}(X) \) do not depend on the decomposition \( X \).

Proof. Recall that (10) is a disjoint union, and the set \( P \cup Q \) contains precisely all the irreducible polynomials that are involved in all the Ritt transformations (Proposition 2.9). Then it follows from the definition of Ritt transformations that the numbers \( n_P(X), n_Q(X), n_{P,l}(X) \) do not depend on the decomposition \( X \). Then the number

\[
 n_R = l(a) - n_P(X - n_Q(X)
\]
does not depend on the decomposition \( X \) either. \( \Box \)

Definition. The common value of all the numbers \( n_P(X), X \in \text{Dec}(a) \), is denoted by \( n_P(a) \). Similarly, the numbers \( n_Q(a), n_R(a) \) and \( n_{P,l}(a) \) are defined.

3 Analogues of the two theorems of J. F. Ritt for the cusp

In this section, Theorems 1.4 and 1.5 are proved. It is shown that, in general, the first theorem of J. F. Ritt does not hold for the cusp, i.e., in general, the number of irreducible polynomials in decomposition of element of \( A \) into irreducible polynomials is not unique (Lemma 3.5). For each element \( a \) of \( A \), the set \( \text{Max}(a) \) is found (Lemma 3.7).

In this section, \( K \) is an algebraically closed field of characteristic 0 if it is not stated otherwise.

The algebra \( K[s, t]/(s^2 - t^3) \) of regular functions on the cusp \( s^2 = t^3 \) is isomorphic to the subalgebra \( A := K[x^2, x^3] \) of the polynomial algebra \( K[x] \) (via \( s \mapsto x^3, t \mapsto x^2 \)). For a polynomial \( a \in K[x] \), let \( a' := \frac{da}{dx} \) and \( a'(0) := \frac{da}{dx}(0) \). Then

\[
 A = \{ a \in K[x] | a'(0) = 0 \}.
\]

The polynomial algebra \( K[x] \) is a monoid with respect to the composition \( \circ \) of functions. It follows from the chain rule, \( (a \circ b)' = a'(b)b' \), that

\[
 K[x] \circ A \subseteq A \text{ and } A \circ (x) \subseteq A
\]

where \( (x) \) is the ideal of the polynomial algebra \( K[x] \) generated by the element \( x \). In particular, \( (A, \circ) \) is a semigroup but not a monoid. Indeed, suppose that \( e \) is an identity of \( A \) then \( \deg(a) = \deg(e \circ a) = \deg(e) \deg(a) \) for all elements \( a \in A \), and so \( \deg(e) = 1 \). But the semigroup \( A \) contains no element of degree 1, a contradiction.

Note that \( A \cap K[x]^* = \emptyset \). So, each element of \( A \) is not a unit of the monoid \( (K[x], \circ) \).

The next lemma gives a necessary and sufficient condition for a composition of two polynomials to be an element of \( A \).
Lemma 3.1 Let $K$ be a field of characteristic zero and $a,b \in K[x]$. Then $a \circ b \in A$ iff either $b \in A$ or $b \notin A$ and the value $b(0)$ of the polynomial $b(x)$ at $x = 0$ is a root of the derivative $\frac{da}{dx}$ of $a$.

Proof. $a \circ b \in A$ iff $0 = (a \circ b)'(0) = a'(b(0))b'(0)$ iff either $b'(0) = 0$ or, otherwise, $a'(b(0)) = 0$ iff either $b \in A$ or, otherwise, $b(0)$ is a root of $a'$. □

Let $\text{Irr}(A)$ and $\text{Irr}(K[x])$ be the sets of irreducible elements of the semi-groups $A$ and $K[x]$ respectively. The set $\text{Irr}(A)$ is the disjoint union of its two subsets $C$ and $D$ where

\[ C := \text{Irr}(A) \cap \text{Irr}(K[x]) = \{ p \in \text{Irr}(K[x]) \mid p'(0) = 0 \} \]

and $D := \text{Irr}(A) \setminus C$. So, the set $C$ contains precisely all the irreducible elements of $K[x]$ that belong to the semi-group $A$, and the set $D$ contains precisely all the irreducible elements of $A$ which are reducible in $K[x]$. Below, Proposition 3.2 states a necessary and sufficient condition for an irreducible element of $A$ to belong to the set $C$ or $D$. First, let us give some definitions.

For a polynomial $a \in K[x]$, let $\mathcal{R}(a)$ and $\text{Dec}(a)$ be, respectively, the set of its roots and the set of all possible decompositions into irreducible factors in $K[x]$. For an element $a \in A$, let $\text{Dec}_A(a)$ be the set of all possible decompositions into irreducible factors in $A$. If $p_1 \circ \cdots \circ p_r \in \text{Dec}(a)$ then

\[ a' = (p_1 \circ \cdots \circ p_r)' = p'_1(p_2 \circ \cdots \circ p_r) \cdot p'_2(p_3 \circ \cdots \circ p_r) \cdots p'_{r-1}(p_r) \cdot p'_r, \]

and so

\[ \mathcal{R}(a') = \mathcal{R}(p'_1(p_1 \circ \cdots \circ p_{r-1})) \cup \cdots \cup \mathcal{R}(p'_{r-1}(p_r)) \cup \mathcal{R}(p'_r). \] (13)

Let

\[ \mathcal{E}(a) := \cup_{p_1 \circ \cdots \circ p_r \in \text{Dec}(a)} \mathcal{R}(p'_r). \]

By the very definition, the set $\mathcal{E}(a)$ is a subset of $\mathcal{R}(a')$. In particular, the set $\mathcal{E}(a)$ is a finite set. In general, $\mathcal{E}(a) \neq \mathcal{R}(a')$. For each element $p \in \text{Irr}(K[x])$, $q \in \text{Irr}(A)$ and $\lambda \in \mathcal{R}(q')$, we have the inclusions (where $K^* := K \setminus \{0\}$)

\[ K[x]^* \circ p \circ K[x]^* \subseteq \text{Irr}(K[x]) \quad \text{and} \quad K[x]^* \circ q \circ (\lambda + K^* x) \subseteq \text{Irr}(A). \]

In particular, $K[x]^* \circ q \circ K[x] \subseteq \text{Irr}(A)$ and $K[x]^* \circ q \circ (\lambda + x) \subseteq \text{Irr}(A)$.

Proposition 3.2 Let $p \in A \setminus K$. Then

1. $p \in C$ iff $p \in \text{Irr}(K[x])$ and $p'(0) = 0$.

2. $p \in D$ iff $p \notin C$ and, for each decomposition $p_1 \circ \cdots \circ p_r \in \text{Dec}(p)$, $(p_2 \circ \cdots \circ p_r)'(0) \neq 0$. 

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Proof. 1. This is obvious.

2. \((\Rightarrow)\) Suppose that \(p \in \mathcal{D}\). Then, obviously, \(p \not\in \mathcal{C}\). Suppose that \((p_2 \circ \cdots \circ p_r)'(0) = 0\) for some decomposition \(p_1 \circ \cdots \circ p_r \in \text{Dec}(p)\), we seek a contradiction. Let \(\lambda\) be a root of the polynomial \(p_1'\). The elements

\[
q_1 := p_1 \circ (x + \lambda_1) \quad \text{and} \quad q_2 := (x - \lambda_1)^{-1} \circ p_2 \circ \cdots \circ p_r
\]

belong to the semi-group \(A\), and

\[p = q_1 \circ q_2.\]

This contradicts to the irreducibility of the element \(p\). Therefore, \((p_2 \circ \cdots \circ p_r)'(0) \neq 0\).

\((\Leftarrow)\) Suppose that \(p \not\in \mathcal{C}\) and, for each decomposition \(p_1 \circ \cdots \circ p_r \in \text{Dec}(p)\), \((p_2 \circ \cdots \circ p_r)'(0) \neq 0\). Suppose that the element \(p\) is reducible, i.e. \(p = a \circ b\) for some elements \(a, b \in A \setminus K\), we seek a contradiction. Fix decompositions \(p_1 \circ \cdots \circ p_s \in \text{Dec}(a)\) and \(p_{s+1} \circ \cdots \circ p_r \in \text{Dec}(b)\). Then \(p = p_1 \circ \cdots \circ p_r\) and \((p_{s+1} \circ \cdots \circ p_r)'(0) = 0\) since \(b \in A\), and so \((p_2 \circ \cdots \circ p_r)'(0) = 0\) (by the chain rule), a contradiction. So, the element \(p\) is irreducible in \(A\), hence \(p \in \mathcal{D}\) since \(p \not\in \mathcal{C}\). \(\square\)

The following two corollaries give a method of construction of elements of the set \(\mathcal{D}\). In particular, they show that the set \(\mathcal{D}\) is a non-empty set.

**Corollary 3.3** Suppose that an element \(q\) of \(A\) is a composition \(p_1 \circ \cdots \circ p_r\) of irreducible factors \(p_i \in \text{Irr}(K[x])\) such that \(r \geq 2\), \((p_2 \circ \cdots \circ p_r)'(0) \neq 0\) and

\[
\text{Dec}(q) = \{(p_1 \circ u_1) \circ (u_1^{-1} \circ p_2 \circ u_2) \circ \cdots \circ (u_{r-1}^{-1} \circ p_r) | u_1, \ldots, u_{r-1} \in K[x]^*\}.
\]

Then \(q \in \mathcal{D}\).

**Proof.** Since \(r \geq 2\), \(q \not\in \mathcal{C}\). By the assumption, for each decomposition \(q_i \circ \cdots \circ q_r \in \text{Dec}(q)\), we can find elements \(u_1, \ldots, u_{r-1} \in K[x]^*\) such that

\[
q_1 = p_1 \circ u_1, \quad q_2 = u_1^{-1} \circ p_2 \circ u_2, \ldots, \quad q_r = u_{r-1}^{-1} \circ p_r.
\]

Now, \(\mathcal{R}((q_2 \circ \cdots \circ q_r)') = \mathcal{R}((u_1^{-1} \circ p_2 \circ \cdots \circ p_r)') = \mathcal{R}((p_2 \circ \cdots \circ p_r)') \neq 0\). By Proposition 3.3 (2), \(q \in \mathcal{D}\). \(\square\)

Note that any sufficiently generic irreducible polynomials \(p_1, \ldots, p_r \in \text{Irr}(K[x])\) \((r \geq 2)\) with \(p_1 \circ \cdots \circ p_r \in A\) satisfy the assumptions of Corollary 3.3. For example, take generic polynomials \(p_1, \ldots, p_r \in K[x]\) such that \((p_1 \circ \cdots \circ p_r)'(0) = 0\) and \((p_2 \circ \cdots \circ p_r)'(0) \neq 0\) then all \(p_i \in \text{Irr}(K[x])\) and \(p_1 \circ \cdots \circ p_r \in \mathcal{D}\).

**Corollary 3.4** Let \(r \geq 2\) be a natural number. For each natural number \(i = 1, \ldots, r\), let \(p_i = \sum_{j=0}^n a_{ij} x^j \in K[x]\) be a polynomial of prime degree \(n_i \geq 5\). Suppose that \(a_{11} := -\sum_{j=2}^n j u_{1j} (p_2 \circ \cdots \circ p_r(0))^{j-1}\) and that all the elements \(a_{ij}\) of the field \(K\) with \((i, j) \neq (1, 1)\) are algebraically independent over the field of rational numbers \(\mathbb{Q}\). Then \(p_1 \circ \cdots \circ p_r \in \mathcal{D}\). In particular, \(\mathcal{D} \neq \emptyset\).
Proof. The definition of the element \( a_{11} \) means that \( p_1'((p_2 \circ \cdots \circ p_r)(0)) = 0 \). This implies that \( (p_1 \circ \cdots \circ p_r)'(0) = 0 \), and so \( p_1 \circ \cdots \circ p_r \in A \). Next, we show that the assumption of Corollary 3.3 hold. The polynomials \( p_i \) are irreducible since their degrees are prime numbers. The elements \( a_{ij}, \ i = 2, \ldots, r, j = 1, \ldots, n_i \), are algebraically independent over \( \mathbb{Q} \), hence \( (p_2 \circ \cdots \circ p_r)'(0) \neq 0 \). Suppose that

\[
\text{Dec}(p_1 \circ \cdots \circ p_r) \neq \{(p_1 \circ u_1) \circ (u_1^{-1} \circ p_2 \circ u_2) \circ \cdots \circ (u_{r-1}^{-1} \circ p_r) \mid u_1, \ldots, u_{r-1} \in K[x]^* \},
\]

we seek a contradiction. Then, by the second theorem of Ritt-Levi, there exists a pair \((p_i, p_{i+1})\) and elements \( \alpha, \beta, \gamma \in K[x]^* \) such that the pair \((\alpha \circ p_i \circ \beta, \beta^{-1} \circ p_{i+1} \circ \gamma)\) is one of the three types:

\[
\begin{align*}
(a) & \quad (T_{n_i}, T_{n_{i+1}}), \\
(b) & \quad (x^{n_i} x^r g(x^{n_i})), \quad r + n_i \deg(g) = n_{i+1}, \\
(c) & \quad (x^r g^{n_{i+1}}, x^{n_{i+1}}), \quad r + n_{i+1} \deg(g) = n_i.
\end{align*}
\]

For each polynomial \( f \in K[x] \), let \( C(f) \) be the subfield of \( K \) generated by its coefficients over \( \mathbb{Q} \). In the case (a) (resp. (b)) \( p_i = \alpha^{-1} \circ T_{n_i} \circ \beta^{-1} \) (resp. \( p_i = \alpha^{-1} \circ x^{n_i} \circ \beta^{-1} \)). On the one hand, the transcendence degree \( \text{tr.deg} C(p_i) = n_i \geq 5 \), on the other hand, \( \text{tr.deg} C(\alpha^{-1} \circ T_{n_i} \circ \beta^{-1}) \leq 4 \) (resp. \( \text{tr.deg} C(\alpha^{-1} \circ x^{n_i} \circ \beta^{-1}) \leq 4 \)), a contradiction. Similarly, in the case (c), \( p_{i+1} = \beta \circ x^{n_{i+1}} \gamma^{-1} \), and so

\[
5 \leq \text{tr.deg} C(p_{i+1}) = \text{tr.deg} C(\beta \circ x^{n_{i+1}} \gamma^{-1}) \leq 4,
\]

a contradiction. These contradictions mean that the assumptions of Corollary 3.3 hold for the element \( p_1 \circ \cdots \circ p_r \), and so \( p_1 \circ \cdots \circ p_r \in \mathcal{D} \). In particular, \( \mathcal{D} \) is a non-empty set. \( \square \)

The next lemma shows that, in general, the first theorem of J. F. Ritt does not hold for the cusp.

**Lemma 3.5** In general, the number of irreducible polynomials in decomposition into irreducible polynomials of an element of \( A \) is non-unique. Moreover, it can vary greatly.

**Proof.** Let \( p \in \mathcal{D} \) and \( q \in \text{Irr}(A) \). Consider their composition \( a := p \circ q \). Fix a decomposition \( p_1 \circ \cdots \circ p_r \in \text{Dec}(p) \), and then, for each \( i = 1, \ldots, r \), fix a root, say \( \lambda_i \), of the polynomial \( p_i \). Consider the elements of \( \mathcal{C} \):

\[
a_1 := p_1 \circ (x + \lambda_1), a_2 := (x - \lambda_1)^{-1} \circ p_2 \circ (x + \lambda_2), \ldots, a_r := (x - \lambda_{r-1})^{-1} \circ p_r \circ (x + \lambda_r),
\]

Then \( a_{r+1} := (x - \lambda_r)^{-1} \circ q \in \text{Irr}(A) \) and

\[
a = p \circ q = a_1 \circ \cdots \circ a_r \circ a_{r+1}
\]

are two irreducible decompositions for the element \( a \) with distinct numbers of irreducible factors. \( \square \)
Lemma 3.5 means that both theorems of J. F. Ritt fail badly for the cusp. However, we can describe a procedure of how to obtain all irreducible decompositions of any given element of \( A \). Let \( a \in A \setminus K \). Take any decomposition \( p_1 \circ \cdots \circ p_r \in \text{Dec}(a) \). Suppose that it is possible to insert brackets

\[
\ldots \circ \ldots \circ \ldots 
\]

in such a way that inside the brackets are irreducible elements of \( A \) (in principal, this can be checked using Proposition 3.2). It gives an irreducible decomposition for the element \( a \) in \( A \). Moreover, all irreducible decompositions of the element \( a \) in \( A \) can be obtained in this way.

**Proof of Theorem 1.5**

We keep the notation of Theorem 1.5. So, \( a \in A \setminus K \) with \( l_A(a) = l(a) \), and \( X, Y \in \text{Max}(a) \). We have to show that the decomposition \( Y \) can be obtained from the decomposition \( X \) using some of the transformations \( (\text{Adm}) \), \( (Ca) \), \( (Cb) \) or \( (Cc) \). We call these transformations the cusp transformations. Note that \( \text{Max}(a) \subseteq \text{Dec}(a) \), and so \( X, Y \in \text{Dec}(a) \).

Let \( X', Y' \in \text{Max}(a) \). We write \( X' \sim_A Y' \) if the decomposition \( Y' \) can be obtained from the decomposition \( X' \) by using the cusp transformations. The relation \( \sim_A \) on the set \( \text{Max}(a) \) is an equivalence relation since the cusp transformations are reversible. This means that the inverse of a transformation of the type \( (\text{Adm}) \) or \( (Ca) \) is a transformation of the type \( (\text{Adm}) \) or \( (Ca) \) respectively; and the inverse of a transformation of the type \( (Cb) \) or \( (Cc) \) is a transformation of the type \( (Cb) \) or \( (Cc) \) respectively. We write \( X' \sim_C Y' \) if the decomposition \( Y' \) is obtained from the decomposition \( X' \) by a single cusp transformation. Theorem 1.5 means that the set \( \text{Max}(a) \) is an equivalence class under the equivalence relation \( \sim_A \), i.e. the equivalence relation \( \sim_A \) on \( \text{Max}(a) \) coincides with the equivalence relation \( \sim \), by the second theorem of Ritt-Levi (the equivalence relation \( \sim \) is defined in the proof of Theorem 1.1). We write \( X' \sim_R Y' \) if \( Y' \) is obtained from \( X' \) by a single Ritt transformation.

Let \( r := l_A(a) = l(a) \). Since \( X, Y \in \text{Max}(a) \), we have

\[
X = p_1 \circ \cdots \circ p_r \quad \text{and} \quad Y = q_1 \circ \cdots \circ q_r
\]

for some irreducible polynomials \( p_i, q_i \in \mathcal{C} \).

**Case (a)**: \( K[x]^* \cdot p_r = K[x]^* \cdot q_r \), i.e. \( q_r = \alpha \circ p_r \) for some polynomial \( \alpha \in K[x]^* \). Let \( b := p_1 \circ \cdots \circ p_{r-1} \). Then \( b \circ p_r = a = q_1 \circ \cdots \circ q_r = q_1 \circ \cdots \circ (q_{r-1} \circ \alpha) \circ p_r \). By (A4), we can delete \( p_r \) at both ends of the chain of equalities above, and the result is

\[
b = p_1 \circ \cdots \circ p_{r-1} = q_1 \circ \cdots \circ (q_{r-1} \circ \alpha).
\]

By Corollary 2.2, the decomposition \( V := q_1 \circ \cdots \circ (q_{r-1} \circ \alpha) \in \text{Dec}(b) \) can be obtained from the decomposition \( U := p_1 \circ \cdots \circ p_{r-1} \in \text{Dec}(b) \) by applying, say \( t \), Ritt transformations

\[
U = U_0 \sim_R U_1 \sim_R U_2 \sim_R \cdots \sim_R U_t = V.
\]
Then the decomposition \( Y = V \circ p_r \) can be obtained from the decomposition \( X = U \circ p_r \) by applying cusp transformations of the type \((\text{Adm})\) in the following way. First, we have the elements of the set \( \text{Dec}(a) \):

\[
X = W_0 := U_0 \circ p_r, \ldots, W_i := U_i \circ p_r, \ldots, W_t := U_t \circ p_r, \ W_{t+1} := Y.
\]

An important fact is that the last element of all decompositions, that is \( p_r \), is an element of \( A \). Let \( U_i := P_1 \circ \cdots \circ P_{r-1} \) where \( P_1, \ldots, P_{r-1} \in \text{Irr}(K[x]) \). For each polynomial \( P_j \), fix a \( P_j \)-admissible element, say \( u_{ij} \), of \( K[x]^* \), and consider the decomposition

\[
W_i^* = P_i^* \circ \cdots \circ P_r^* \in \text{Max}(a)
\]

where

\[
P_1^* := P_1 \circ u_{i1}, P_2^* := u_{i1}^{-1} \circ P_2 \circ u_{i2}, \ldots, P_{r-1}^* := u_{i,r-2}^{-1} \circ P_{r-1} \circ u_{i,r-1}, P_r^* := u_{i,r-1}^{-1} \circ p_r.
\]

It is obvious that the decomposition \( W_i^* \) is obtained from the decomposition \( W_i \) by applying \( r - 1 \) transformations of the type \((\text{Adm})\). Let \( \text{Adm}(u_{i1}, \ldots, u_{i,r-1}) \) denote their composition (in arbitrary order since the transformations commute). We assume that for \( i = 0, t + 1 \) all the \( u \)'s are equal to \( x \). This means that the transformation \( \text{Adm}(x, \ldots, x) \) is the identity transformation, and, obviously, \( W_0^* = W_0 = X \) and \( W_{t+1}^* = W_{t+1} = Y \). So, there is the chain of elements of the set \( \text{Max}(a) \):

\[
X = W_0^*, \ W_1^*, \ldots, W_t^*, \ W_{t+1}^* = Y.
\]

For each natural number \( i = 1, \ldots, t + 1 \), the decomposition \( W_i^* \) is obtained from the decomposition \( W_{i-1}^* \) by applying cusp transformations of the type \((\text{Adm})\):

\[
\text{Adm}(u_{i-1,1}^{-1} \circ u_{i1}, \ldots, u_{i-1,r-1}^{-1} \circ u_{i,r-1}).
\]

Therefore, \( X \sim_A Y \).

Case \((\beta)\): \( K[x]^* p_r \neq K[x]^* q_r \). By Corollary 2.2, this means that \( p_r = \lambda_{r-1}^{-1} \circ \pi \circ \lambda_r \) for some units \( \lambda_{r-1}, \lambda_r \in K[x]^* \) such that \( \lambda_r \) is \( \pi \)-admissible and the polynomial \( \pi \) is one of the following types:

\[
\begin{align*}
(a) & \quad \pi = T_l, \text{ where } l \text{ is an odd prime number,} \\
(b) & \quad \pi = x^s g(x^p), \text{ where } s \geq 1, \ g(x) \in K[x] \setminus K, \ g(0) \neq 0, \ p \text{ is a prime number,} \\
(c) & \quad \pi = x^p, \text{ where } p \text{ is a prime number.}
\end{align*}
\]

Remark. We exclude the situation when \( s = 0 \) in the case \((b)\) since otherwise we would have the case \((c)\) due to irreducibility of the element \( \pi \) and the equality \( g(x^p) = g(x) \circ x^p \).

We consider the three cases separately and label them respectively as \((\beta a)\), \((\beta b)\) and \((\beta c)\).

Case \((\beta a)\): \( \pi = T_l \) where \( l \) is an odd prime number. By the second theorem of Ritt-Levi, the element \( q_r \) in the decomposition \( Y = q_1 \circ \cdots \circ q_r \) must be of the type \( \mu \circ T_m \circ \lambda_r \) for some
prime number $m$ such that $m \neq l$ (see Case $(\beta)$) where $\lambda_r$ is necessarily a $T_m$-admissible polynomial and $\mu \in K[x]^*$. If $\nu$ is the only root of the polynomial $\lambda_r$, then

$$\nu \in \mathcal{R}(T'_n) \cap \mathcal{R}(T'_m) = \emptyset \quad \text{(Lemma 2.12(2))},$$

a contradiction. Therefore, this case is impossible.

Case $(\beta b)$: $\pi = x^q g(x^q)$ (as in the case $(b)$ above). Then for the element $q_r$ there are two options either $q_r \in K[x]^* \circ T_k \circ \lambda_r$ for some prime number $k$ or, otherwise, $q_r \in K[x]^* \circ x^q \circ \lambda_r$ for some prime number $q$. For $k \neq 2$, the first option is not possible since by interchanging $X$ and $Y$ we would have the impossible Case $(\beta a)$ (recall that the cusp transformations are reversible). For $k = 2$, $T_2 = (-1 + 2x) \circ x^2$, and so we have, in fact, only the second option, i.e. $q_r = \mu \circ x^q \circ \lambda_r$ for some unit $\mu \in K[x]^*$. This means that the invariant number $n_{\mathcal{P}_q} \geq 1$.

Let $i$ be the greatest index such that $p_i \in K[x]^* \circ x^q \circ K[x]^*$. In this case, we call the element $p_i$ the largest $x^q$ in the decomposition $X$ denoted $L(X)$. The decompositions

$$H(X) := p_1 \circ \cdots \circ p_{i-1} \text{ and } T(X) := p_{i+1} \circ \cdots \circ p_r$$

are called the head and the tail of the decomposition $X$ respectively. The invariance of the number $n_{\mathcal{P}_q}$ means that we can control the largest $x^q$ under Ritt transformations. The largest $x^q$ remains unchanged under a Ritt transformation either of the head or the tail of $X$, and it moves to the right or left by one point if the largest $x^q$ is involved in the Ritt transformation of the type (b) or (c) from the Introduction respectively.

Let $p_i = \lambda_{i-1}^{-1} \circ x^q \circ \lambda_i$ for some units $\lambda_{i-1}, \lambda_i \in K[x]^*$. Then the tail $T(X)$ of $X$ has clear structure. We claim that there exist units $\lambda_{i+1}, \ldots, \lambda_{r-2} \in K[x]^*$ such that

$$p_j = \lambda_{j-1}^{-1} \circ \pi_j \circ \lambda_j, \quad j = i+1, \ldots, r-1,$$

where $\pi_j$ is either $x^n$ for a prime number $n$ or, otherwise, $x^t f(x^t)$ for some $t \geq 1$ and $f(x) \in K[x]$ such that $\deg(f) \geq 1$ and $f(0) \neq 0$. The decomposition $Y$ is obtained from the decomposition $X$ by several Ritt transformations

$$X = X_0 \sim_R X_1 \sim_R \cdots \sim_R X_k \sim_R \cdots \sim_R X_m = Y.$$ 

Using the explicit form of Ritt transformations the claim follows easily by the backward induction on $k$ starting with the obvious case $k = m - 1$.

Using the claim we can produce $r - i$ cusp transformations

$$X = Z_i \sim_{C} Z_{i+1} \sim_{C} \cdots \sim_{C} Z_r$$

such that on each step the largest $x^q$ moves one point to the right, and the last irreducible element in the decomposition $Z_r$ is $q_r = \mu \circ x^q \circ \lambda_r$. On the first step, $Z_i \sim_{C} Z_{i+1}$, the cusp transformation changes the triple

$$(p_i, p_{i+1}, p_{i+2}) = (\lambda_{i-1}^{-1} \circ x^q \circ \lambda_i, \lambda_{i-1}^{-1} \circ \pi_{i+1} \circ \lambda_{i+1}, p_{i+2})$$
as above by changing the index $i$ into the pair $\lambda^{-1}_{i-1} \circ x^n, x^q, \lambda_{i+1} \circ p_{i+2}$ if $\pi_{i+1} = x^n$, 
and $(\lambda^{-1}_{i-1} \circ [x^l f^q] \circ \nu, \nu \circ x^q, \lambda_{i+1} \circ p_{i+2})$ if $\pi_{i+1} = x^l f(x^q)$,
provided $i + 1 < r$ where $\nu \in K[x]^*$ is $x^l f^q$-admissible. If $i + 1 = r$, the cusp transformation $Z_{r-1} \sim_C Z_r$ changes the pair
$$(p_{r-1}, p_r) = (\lambda^{-1}_{r-2} \circ x^n \circ \lambda_{r-1}, \lambda^{-1}_{r-1} \circ [x^s h(x^q)] \circ \lambda_r)$$
into the pair
$$(p^*_{r-1}, p^*_r) = (\lambda^{-1}_{r-2} \circ [x^s h^q], x^n \circ \lambda_r)$$
where $h(x^q) = g(x^p)$. The remaining cusp transformations are defined by the same formulae as above by changing the index $i$ accordingly. Now, the decompositions $Z_r$ and $Y$ satisfy the assumption of the case $(\alpha)$, and so $Z_r \sim_A Y$. Now, $X \sim_A Z_r$ and $Z_r \sim_A Y$, and so $X \sim_A Y$.

Case $(\beta c)$: $\pi = x^p$ (as in the case $(c)$ above). The element $q_r$ has the form $\mu \circ \tilde{\pi} \circ \lambda_r$ where for the element $\tilde{\pi}$ we have the same three options $(a)$, $(b)$ or $(c)$ as for the element $\pi$. Interchanging $X$ and $Y$, we reduce the cases $(a)$ and $(b)$ for the element $\tilde{\pi}$ to the cases $(a)$ and $(b)$ for $\pi$ which have been considered already. For the last case, $\tilde{\pi} = x^q$, we repeat word for word the arguments of the case $(\beta b)$ starting from the claim there. The proof of Theorem 1.5 is complete. □

Proof of Theorem 1.4

Theorem 1.4 follows easily from the first theorem of J. F. Ritt (or from Theorem 1.5 and the definition of the cusp transformations, i.e. the transformations (Adm), (Cα), (Cβ) and (Cc)). □

Proposition 3.6 In general, Theorem 1.4 does not hold for irregular elements.

Proof. Let $m$ and $n$ be distinct prime numbers, $g(x)$ and $h(x)$ be non-scalar polynomials of $K[x]$ such that $h(0) \neq 0$, $k := s + n \deg(g)$ and $l := 1 + m \deg(h)$ are prime numbers for some natural number $s \geq 2$. Then the degrees of the polynomials $x^n, x^s g(x^n)$ and $xh(x^m)$ are prime numbers. Hence, the polynomials $x^n, x^s g(x^n)$ and $x^s g^n$ are elements of the set $\text{Irr}(A)$, and $xh(x^m) \in \text{Irr}(K[x]) \setminus A$. It is obvious that
$$p := [x^s g(x^n)] \circ [xh(x^m)], \quad q := x^n \circ [xh(x^m)] \in \mathcal{D},$$
and the element $a := x^n \circ [x^s g(x^n)] \circ [xh(x^m)] \in A$ is irregular since $h(0) \neq 0$. Then
$$a = x^n \circ p = x^s g^n \circ q \in \text{Dec}_A(a),$$

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(\deg(x^n), \deg(p)) = (n, kl) \text{ and } (\deg(x^g^n), \deg(g)) = (k, nl). \text{ Since } k > n, \text{ we have } (n, kl) \neq (k, nl) \text{ and } (n, kl) \neq (nl, k). \text{ This means that Theorem 1.4} \text{ does not hold for the irregular element } a. \qed

In general, for an element } a \text{ of } A \text{ there exists a decomposition } p_1 \circ \cdots \circ p_t \in \Dec_A(a) \text{ with } t < l_A(a), \text{ i.e. }

\begin{equation}
\text{Max}(a) \neq \Dec_A(a). \tag{14}
\end{equation}

Example. Let } k \text{ be an odd prime number, } g \text{ be a non-scalar polynomial of } K[x] \text{ such that } l := s + 2 \deg(g) \text{ is a prime number for some natural number } s \geq 2. \text{ Let } \lambda \text{ be a root of the trigonometric polynomial } T_k. \text{ Consider the element } a := [x^g]^2 \circ T_k \circ T_2 \in A. \text{ The elements }

p_1 := x^g, \quad p_2 := T_k \circ (x + \lambda) \text{ and } p_3 := (x - \lambda) \circ T_2

\text{of the algebra } A \text{ are irreducible since their degrees are prime numbers. Let } q_1 := T_2. \text{ Note that } q_2 := [x^g(x^2)] \circ T_k \in D \text{ since } T_k \in (x) \setminus (x^2) \text{ and } s \geq 2. \text{ Then }

a = p_1 \circ p_2 \circ p_3 = q_1 \circ q_2 \in \Dec_A(a). \qed

For an element } a \text{ of } A \text{, the number } \defect(a) := l(a) - l_A(a) \text{ is called the } \text{defect of the element } a. \text{ The element } a \text{ is irregular iff } \defect(a) > 0. \text{ For each root } \lambda \text{ of the derivative } a' \text{ of a polynomial } a \text{ of } K[x], \text{ the number }

\text{ind}_a(\lambda) := \max \{ i \mid \exists p_1 \circ \cdots \circ p_r \in \Dec(a) \text{ such that } p_i'(p_{i+1} \circ \cdots \circ p_r \circ x)(0) = 0 \}

\text{is called the index of } \lambda. \text{ If } a \in A \text{ then }

l_A(a) = \text{ind}_a(0). \tag{15}

To prove this fact note that it is obvious that } l_A(a) \leq \text{ind}_a(0). \text{ For } i := \text{ind}_a(0), \text{ let us fix a decomposition } p_1 \circ \cdots \circ p_r \in \Dec(a) \text{ with } p_i'(p_{i+1} \circ \cdots \circ p_r \circ x)(0) = 0. \text{ For each } j = 1, \ldots, i - 1, \text{ let } u_j \text{ be a } p_j\text{-admissible element of } K[x]^*. \text{ The elements }

q_1 := p_1 \circ u_1, \quad q_2 := u_1^{-1} \circ p_2 \circ u_2, \ldots, q_{i-1} := u_{i-2}^{-1} \circ p_{i-1} \circ u_{i-1}, \quad q_i := u_{i-1}^{-1} \circ p_i \circ \cdots \circ p_r

\text{belong to the algebra } A, \text{ and } a = q_1 \circ \cdots \circ q_i. \text{ Hence, } l_A(a) \geq \text{ind}_a(0). \text{ This establishes the equality } (15).

For each element } a \text{ of } A \text{ with } i := \text{ind}_a(0), \text{ let }

\text{Dec}(a, 0) := \{ p_1 \circ \cdots \circ p_r \in \Dec(a) \mid p_i'(p_{i+1} \circ \cdots \circ p_r \circ x)(0) = 0 \}.

\text{The next lemma gives all the decompositions of maximal length for each element of } A.

\textbf{Lemma 3.7} \text{ Let } a \text{ be an element of } A \text{ and } i := \text{ind}_a(0). \text{ Then }

\text{Max}(a) = \{ (p_1 \circ u_1) \circ (u_1^{-1} \circ p_2 \circ u_2) \circ \cdots \circ (u_{i-2}^{-1} \circ p_{i-1} \circ u_{i-1}) \circ (u_{i-1}^{-1} \circ p_i \circ \cdots \circ p_r) \mid p_1 \circ \cdots \circ p_r \in \Dec(a, 0), \; u_j \in K[x]^* \text{ is } p_j \text{-admissible} \}.
Proof. It is obvious that the RHS ⊆ Max(a). On the other hand, if \( q_1 \circ \cdots \circ q_i \in \text{Max}(a) \) then \( q_1 \circ \cdots \circ q_i \in \text{RHS} \). It suffices to put \( p_j = q_j \) and \( u_j = x \). □

By Lemma 3.7 if the element \( a \) of \( A \) is irregular and \( q_1 \circ \cdots \circ q_i \in \text{Max}(a) \) then necessarily \( q_1, \ldots, q_{i-1} \in \mathcal{C} \) and \( q_i \in \mathcal{D} \).

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Department of Pure Mathematics
University of Sheffield
Hicks Building
Sheffield S3 7RH
UK
email: v.bavula@sheffield.ac.uk

IHES
Le Bois-Marie
35, Route de Chartes
F-91440 Bures-sur-Yvette
France
email: bavula@ihes.fr