EXPONENTIAL STABILIZATION OF A STRUCTURE WITH INTERFACIAL SLIP

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Abstract. Two exponential stabilization results are proved for a vibrating structure subject to an interfacial slip. More precisely, the structure consists of two identical beams of Timoshenko type and clamped together but allowing for a longitudinal movement between the layers. We will stabilize the system through a transverse friction and also through a viscoelastic damping.

1. Introduction. The structure under investigation is formed by two identical beams attached together on top of each other. It is subject to transversal and rotational vibrations like the ones produced by transverse bending and torsion. In principle there is always some slip between the layers in a vibrating structure. In many cases this slip is ignored. In some other cases, this slip is annihilated by fastening the layers very tightly (by using bolts, for instance) in order to avoid negative effects such as corrosion. However, fastening very tightly the components may alter negatively the accomplishments of the structure. This structure is known under the name 'laminated beams' and are widely used in many fields of engineering. The beams are continuously clamped together but allowing for a longitudinal displacement (slip) together with the transversal and rotational vibrations. The slip between the components is sometimes intentionally meant in order to produce a significant amount of damping. This damping is capable of bringing back the structure to rest.

In [10], the authors derived the following model

\[
\begin{align*}
&\rho w_{tt} + G (\psi - w_x) - D (3s_{xx} - \psi_{xx}) = 0, \\
&I_p (3s_{tt} - \psi_{xx}) - G (\psi - w_x) - D (3s_{xx} - \psi_{xx}) = 0, \\
&3I_p s_{tt} + 3G (\psi - w_x) + 4\gamma s + 4\beta s_t - 3Ds_{xx} = 0,
\end{align*}
\]

where \( x \in (0, 1) \) and \( t > 0 \), with initial data

\[
(w, \psi, s) (x, 0) = (w_0, \psi_0, s_0), \quad (w_t, \psi_t, s_t) (x, 0) = (w_1, \psi_1, s_1), \quad x \in (0, 1)
\]

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and the cantilever boundary conditions. Here w, ψ, ρ, G, I_p, D, γ, β denote the transverse displacement, rotation angle, density, shear stiffness, mass moment of inertia, flexural rigidity, adhesive stiffness, adhesive damping parameter, respectively and s is proportional to the amount of slip along the interface.

They investigated the following cases:
(a) The limiting case G → ∞, i.e. when the shear stiffness is ‘very’ large.
(b) The limit as I_p → 0 in (a)
(c) The cases β = 0, β ≠ 0 and γ = 0 and γ > 0, β > 0 in (b)
(d) The general case I_p > 0 and 0 < G < ∞.

The authors concluded that in case (a) and (b) the model reduces to a simple one providing a good approximation in case of harmonic motions with small frequencies.

In [32] the authors transformed the system into
\[ \begin{align*}
\rho w_{tt} + G (3s - \xi - w_x)_x &= 0, \\
I_p \psi_{tt} - G (3s - \xi - w_x) - DS_{xx} &= 0, \\
3I_p s_{tt} + 3G (3s - \xi - w_x) + 4\gamma s + 4\beta s_t - 3Ds_{xx} &= 0
\end{align*} \]

via the change of variable ξ = 3s - ψ (the effective rotation angle) subjected to the boundary conditions
\[ \begin{align*}
w(0, t) &= \xi(0, t) = s(0, t) = 0, \quad t > 0, \\
\xi_x(1, t) &= w_2(t), \quad s_x(1, t) = 0, \quad 3s(1, t) - \xi(1, t) - w_x(1, t) = u_1(t), \quad t > 0.
\end{align*} \]

They succeeded in proving an exponential stabilization result through the boundary control
\[ u_1(t) = k_1 w_1(1, t), \quad u_2(t) = -k_2 \xi_t(1, t) \]

using spectral theory provided that \( r_1 := \frac{G}{\rho} \neq \frac{D}{\varepsilon} := r_2 \) and \( k_i \neq r_i, \quad i = 1, 2 \).

It is worth noting here that it has been shown that the slip is able to stabilize asymptotically the structure but it cannot stabilize it exponentially (see Corollary 2.3 and Note 2.1 in [32]).

In [3], the same system is considered under the boundary conditions
\[ \psi(0, t) - w_x(0, t) = v_1(t), \quad 3s_x(1, t) - \psi_x(1, t) = v_2(t), \quad t > 0. \]

The authors established the exponential stabilization of the system through the boundary control
\[ v_1(t) = -k_1 w_1(0, t) - w(0, t), \quad v_2(t) = -k_1 \xi_t(1, t) - \xi(1, t) \]

provided that the ’dominant’ part of the closed loop system is itself exponentially stable.

In case s = 0, then the system reduces to the well known Timoshenko model [1,8,11,16,17,20-25,29-31,33,34]. Therefore the system gives prominence to the dynamics of the slip described by the third equation. We would like to revive this issue as the slip is usually unavoidable and in many cases it cannot be simply ignored.

We discuss here two models. In the first one we stabilize the system by a frictional damping acting on the transverse displacement. Namely, we will consider the system
\[ \begin{align*}
\rho w_{tt} + G (\psi - w_x)_x + aw_t &= 0, \\
I_p (3s_{tt} - \psi_{tt}) - G (\psi - w_x) - (3s - \psi)_x &= 0, \\
I_p s_{tt} + G (\psi - w_x) + \frac{3}{2} \gamma s - s_{xx} + \frac{3}{2} \beta s_t &= 0
\end{align*} \]

for x ∈ (0, 1), t > 0, some a > 0 (frictional damping coefficient), with the boundary conditions
\[ \begin{align*}
w(0, t) &= \psi(0, t) = s(0, t) = 0, \quad t \geq 0, \\
s_x(1, t) &= \psi_x(1, t) = 0, \quad w_x(1, t) = \psi(1, t), \quad t \geq 0.
\end{align*} \]
In the second one we shall be concerned with the adoption of viscoelastic dampings. These kinds of dampings may be caused by the utilization of a special material capable of driving the structure to rest in a very fast manner without any boundary or other types of control (having in mind that viscoelastic material are sensitive to changes in frequency and temperature [2]). We will investigate the system

\[
\begin{align*}
\rho w_{tt} + G(\psi - w_x)_x + G \int_0^t h(t-r)w_{xx}(r)dr &= 0, \\
I_\rho (3s_{tt} - \psi_{tt}) - G(\psi - w_x) - (3s - \psi)_{xx} + \int_0^t h(t-r)(3s - \psi)_{xx}(r)dr &= 0, \\
I_\rho s_{tt} + G(\psi - w_x) + \frac{4}{3} \gamma s - s_{xx} + \frac{5}{3} \beta s_t &= 0, \\
\end{align*}
\]

(3)

for \( x \in (0,1), t > 0 \), a positive (relaxation function) \( h \) satisfying \( \xi = 1 - \int_0^\infty h(r)dr > 0 \) and \( -\beta_0 h \leq h'(t) \leq -\beta_1 h \) for some \( \beta_0, \beta_1 \) positive (guaranteeing the hyperbolicity of the system) with the boundary conditions

\[
\begin{align*}
w(0,t) &= \psi(0,t) = s(0,t) = 0, \quad t \geq 0, \\
s_x(1,t) &= \psi_x(1,t) = 0, \quad w_x(1,t) = \psi(1,t), \quad t \geq 0. \\
\end{align*}
\]

(4)

The well-posedness of our system has been discussed in part in [32,3] (see also references in [10]). It suffices to combine these results with the ones in [4,5,7,9,18] to cope with the (nonlocal) viscoelastic term. We have weak solutions in \( (V^1_s \times L^2)^3 \) and strong solutions in \( V^1_s \times H^1 \)

\[
V^k_s = \{ v : v \in H^k(0,1) : v(0) = 0 \}, \quad k = 1,2.
\]

We shall focus here on the asymptotic behavior of solutions and in particular with the exponential stabilization of the system.

Note that, for the first problem, the last two equations imply that

\[
I_\rho \psi_{tt} + 4G(\psi - w_x) - \psi_{xx} + 4\gamma s + 4\beta s_t = 0, \quad t > 0.
\]

(5)

2. Frictional damping case (Problem (1)). The energy of the system (1)-(2) is given by

\[
E_1(t) = \frac{1}{2} \left[ 3I_\rho \|s_t\|^2 + 4\gamma \|s\|^2 + 3 \|s_x\|^2 + I_\rho \|3s_t - \psi_{tt}\|^2 + \|(3s - \psi)_x\|^2 \right] + \rho \|w_t\|^2 + G \|\psi - w_x\|^2, \quad t \geq 0
\]

(6)

where \( \|\cdot\| \) denotes the norm in \( L^2(0,1) \).

Proposition 1. The energy \( E_1(t) \) given by (6) satisfies

\[
E'_1(t) = -a \|w_t\|^2 - 4\beta \|s_t\|^2, \quad t \geq 0.
\]

Proof. Multiplying the first equation of (1) by \( w_t \) and integrating over \( (0,1) \) we obtain

\[
\frac{\rho}{2} \frac{d}{dt} \|w_t\|^2 + G((\psi - w_x)_x, w_t) + a \|w_t\|^2 = 0, \quad t \geq 0
\]

or

\[
\frac{\rho}{2} \frac{d}{dt} \|w_t\|^2 - G(\psi - w_x, w_{xt}) + [G(\psi - w_x) w_t]_0 + a \|w_t\|^2 = 0, \quad t \geq 0
\]

and by our boundary conditions

\[
\frac{\rho}{2} \frac{d}{dt} \|w_t\|^2 - G(\psi - w_x, w_{xt}) + a \|w_t\|^2 = 0, \quad t \geq 0.
\]
Note that
\[ G(\psi - w_x, w_{xt}) = -G(\psi - w_x, (\psi - w_x - \psi_t)) \]
\[ = -\frac{\partial}{\partial t} \left[ \|\psi - w_x\|^2 + G(\psi - w_x, \psi_t) \right]. \]

Therefore
\[ \frac{1}{2} \frac{d}{dt} \left[ \rho \|w_t\|^2 + G \|\psi - w_x\|^2 \right] - G(\psi - w_x, \psi_t) + a \|w_t\|^2 = 0, \quad t \geq 0. \quad (7) \]

Again multiplying the third equation of (1) by \( s_t \) and integrating on \( (0, 1) \) we obtain
\[ I_\rho (s_{tt}, s_t) + G(\psi - w_x, s_t) + \frac{4\gamma}{3} (s, s_t) - (s_{xx}, s_t) + \frac{4\beta}{3} \|s_t\|^2 = 0 \]
or
\[ \frac{1}{2} \frac{d}{dt} \left[ I_\rho \|s_t\|^2 + \frac{4\gamma}{3} \|s\|^2 + \|s_x\|^2 \right] + G(\psi - w_x, s_t) = -\frac{4\beta}{3} \|s_t\|^2, \quad t \geq 0. \quad (8) \]

Now adding (7) and (8) we obtain
\[ \frac{3}{2} \frac{d}{dt} \left[ I_\rho \|s_t\|^2 + \frac{4\gamma}{3} \|s\|^2 + \|s_x\|^2 \right] + \frac{1}{2} \frac{d}{dt} \left[ \rho \|w_t\|^2 + G \|\psi - w_x\|^2 \right] \]
\[ + 3G(\psi - w_x, s_t) + 4\beta \|s_t\|^2 - G(\psi - w_x, \psi_t) + a \|w_t\|^2 = 0 \]
or
\[ \frac{1}{2} \frac{d}{dt} \left[ I_\rho \|s_t\|^2 + \frac{4\gamma}{3} \|s\|^2 + \|s_x\|^2 + \rho \|w_t\|^2 + G \|\psi - w_x\|^2 \right] \]
\[ + G(\psi - w_x, 3s_t - \psi_t) + 4\beta \|s_t\|^2 + a \|w_t\|^2 = 0, \quad t \geq 0. \quad (9) \]

Next, we use the second equation in (1) and multiply it by \((3s - \psi)_t\) to find
\[ \frac{L_\rho}{2} \frac{d}{dt} \|3s_t - \psi_t\|^2 = (G(\psi - w_x) + (3s - \psi)_x) (3s - \psi)_t \]
\[ = (3s - \psi)_x (3s - \psi)_t + (3s - \psi)_t (3s - \psi)_xt \]
or
\[ \frac{1}{2} \frac{d}{dt} \left[ I_\rho \|3s_t - \psi_t\|^2 + \|3s - \psi\|^2 \right] = G(\psi - w_x, (3s - \psi)_t), \quad t \geq 0. \quad (10) \]

Finally, summing up (9) and (10) we get
\[ E'_1(t) = -a \|w_t\|^2 - 4\beta \|s_t\|^2, \quad t \geq 0. \]

Although from this lemma we see that the energy is uniformly bounded and decreasing, it is not clear how to prove exponential decay from this functional. Our objective is to modify it in a suitable manner so as to obtain an equivalent functional \( L \) which satisfies a differential inequality of the form
\[ \frac{d}{dt} L(t) \leq -\kappa L(t) \]
for some positive constant \( \kappa \). To this end, for \( \delta_i, i = 1, ..., 4 \) positive to be chosen, define
\[ L(t) = E_1(t) + \sum_{i=1}^{4} \delta_i H_i(t), \quad t \geq 0 \]
with
\[ H_1(t) = I_\rho (s_t, s) - \rho(w_t, W), \quad H_2(t) = I_\rho (3s_t - \psi_t, 3s - \psi) + \rho(w_t, Z), \]
\[ H_3(t) = -I_\rho (3s_t - \psi_t, \psi) + \rho(w_t, W), \quad H_4(t) = 4\rho(w_t, U) - I_\rho (\psi_t, \psi), \]
where \( W, Z \) and \( U \) are defined by
\[ W(x, t) = -\int_0^x s(r, t) dr, \quad (11) \]
Proof. Using the second equation of (1), we find for $t > 0$

\[ Z(x,t) = -\int_0^x (3s - \psi)(r,t)dr, \]  

(12)

and

\[ U(x,t) = -\int_0^x \psi(r,t)dr, \]  

(13)

respectively.

The Cauchy-Schwarz inequality and the Poincaré inequality allow us to prove easily the equivalence between $E_1(t)$ and $L(t)$. It remains to prove the differential inequality. To this end we proceed to prepare several lemmas which will be of great help in proving our result later.

Lemma 1. The derivative of $H_1(t)$ along solutions of (1)-(2) is estimated by

\[ H'_1(t) \leq (I_\rho + C_1 \beta_0 + \varepsilon_0) \|s_t\|^2 - (1 - \varepsilon_0) \|s_x\|^2 + \frac{4}{3} (\varepsilon_0 - \gamma) \|s\|^2 + \frac{a^2 + \beta^2}{4\varepsilon_0} \|w_t\|^2, \]

for $t > 0$ and some $\varepsilon_0 > 0$.

Proof. From the definition of $H_1(t)$, the first and the third equations in the system (1) we find

\[ H'_1(t) = I_\rho (s_{tt}, s) + I_\rho (s_t, s_t) - \rho(w_{tt}, W) - \rho(w_t, W_t) \]

\[ = I_\rho \|s_t\|^2 - G(\psi - w_x, s) - \frac{4\gamma}{3} \|s\|^2 - \|s_x\|^2 - \frac{4\beta}{3} (s_t, s) \]

\[ + G((\psi - w_x)_x, W) + a(w_t, W) - \rho(w_t, W_t), \]

and as the term $G((\psi - w_x)_x, W)$ cancels out with $-G(\psi - w_x, s)$ because $G((\psi - w_x)_x, W) = -G(\psi - w_x, W_x) = G(\psi - w_x, s)$, $t \geq 0$,

we infer that

\[ H'_1(t) \leq I_\rho \|s_t\|^2 - \|s_x\|^2 - \frac{4\gamma}{3} \|s\|^2 + \frac{4\beta}{3} \|s\|^2 + \frac{a^2 + \beta^2}{4\varepsilon_0} \|w_t\|^2 \]

\[ + \varepsilon_0 \left( \|W\|^2 + \|W_t\|^2 \right) + \frac{a^2 + \beta^2}{4\varepsilon_0} \|w_t\|^2, \]

for $t > 0$.

Now from the definition (11) it is clear that $\|W\|^2 \leq \|s\|^2$ and $\|W_t\|^2 \leq \|s_t\|^2$. Consequently, for $t > 0$

\[ H'_1(t) \leq \left( I_\rho + \varepsilon_0 + \frac{\beta^2}{3\varepsilon_0} \right) \|s_t\|^2 - (1 - \varepsilon_0) \|s_x\|^2 + \frac{4}{3} (\varepsilon_0 - \gamma) \|s\|^2 + \frac{a^2 + \beta^2}{4\varepsilon_0} \|w_t\|^2. \]

Lemma 2. The derivative of $H_2(t)$ along solutions of (1)-(2) satisfies for $t > 0$

\[ H'_2(t) \leq (I_\rho + \varepsilon_0) \|3s_t - \psi_t\|^2 + (\varepsilon_0 - 1) \|3s_x - \psi_x\|^2 + \frac{a^2 + \beta^2}{4\varepsilon_0} \|w_t\|^2, \]

where $\varepsilon_0 > 0$.

Proof. Using the second equation of (1), we find for $t \geq 0$

\[ I_\rho \frac{d}{dt} (3s_t - \psi_t, 3s - \psi) = I_\rho \|3s_t - \psi_t\|^2 + G(\psi - w_x, 3s - \psi) - \|3s_x - \psi_x\|^2. \]

Next, in view of

\[ \rho \frac{d}{dt} (w_t, Z) \]

\[ = \rho(w_{tt}, Z) + \rho(w_t, Z_t) \]
where \( \varepsilon_0 > 0 \), we infer that
\[
\begin{align*}
H_2'(t) &\leq I_p \|3s_t - \psi_t\|^2 - \|3s_x - \psi_x\|^2 \\
& + \varepsilon_0 \left( \|3s - \psi\|_x^2 + \|(3s - \psi)_t\|^2 \right) + \frac{a^2 + \rho^2}{4\varepsilon_0} \|w_t\|^2, \quad t \geq 0
\end{align*}
\]
or
\[
H_2'(t) \leq (I_p + \varepsilon_0) \|3s_t - \psi_t\|^2 + (\varepsilon_0 - 1) \|3s_x - \psi_x\|^2 + \frac{a^2 + \rho^2}{4\varepsilon_0} \|w_t\|^2, \quad t > 0.
\]

\[\Box\]

**Lemma 3.** We have the estimate
\[
H_3'(t) \leq (2\varepsilon_0 - G) \|\psi - w_x\|^2 + \frac{\varepsilon_0}{2} \|s_t\|^2 + 2\varepsilon_0 \|\psi\|^2 + (1 + \varepsilon_0) \|3s_t - \psi_t\|^2 + \frac{9I_F^2}{4\varepsilon_0} \|s_t\|^2 + \left( \frac{a^2 + \rho^2}{4\varepsilon_0} \right) \|w_t\|^2, \quad t > 0,
\]
for \( \varepsilon_0 > 0 \).

**Proof.** We infer from the first and third equations in (1) and the definition of \( H_3(t) \) that
\[
\begin{align*}
H_3'(t) &= -I_p (3s_{tt} - \psi_{tt}, \psi) - I_p (3s_t - \psi_t, \psi_t) + \rho (w_{tt}, w) + \rho \|w_t\|^2 \\
&= -G (\psi - w_x, \psi) - ((3s - \psi)_x, \psi) - I_p (3s_t - \psi_t, \psi_t) \\
& - G ((\psi - w_x)_x, w) - a(w_t, w) + \rho \|w_t\|^2 \\
&= -G \|\psi - w_x\|^2 + ((3s - \psi)_x, \psi_x) - I_p (3s_t - \psi_t, \psi_t) - a(w_t, w) + \rho \|w_t\|^2.
\end{align*}
\]

Observe that
\[
((3s - \psi)_x, \psi_x) = -(3s - \psi)_x, 3s_x - \psi_x) + ((3s - \psi)_x, 3s_x)
\]
\[
= -\|3s_x - \psi_x\|^2 + (3s_x - \psi_x, 3s_x)
\]
\[
\leq -\|3s_x - \psi_x\|^2 + \|3s_x - \psi_x\|^2 + \frac{9}{4} \|s_x\|^2 = \frac{9}{4} \|s_x\|^2,
\]
\[
-I_p (3s_t - \psi_t, \psi_t) = I_p (3s_t - \psi_t, 3s_t - \psi_t) - 3I_p (3s_t - \psi_t, s_t)
\]
\[
\leq (1 + \varepsilon_0) \|3s_t - \psi_t\|^2 + \frac{9I_F^2}{4\varepsilon_0} \|s_t\|^2,
\]
and
\[
a(w_t, w) \leq \varepsilon_0 \|w\|^2 + \frac{a^2}{4\varepsilon_0} \|w_t\|^2 \leq \varepsilon_0 \|w_x\|^2 + \frac{a^2}{4\varepsilon_0} \|w_t\|^2
\]
\[
\leq \varepsilon_0 \|w_x - \psi\|^2 + \frac{a^2}{4\varepsilon_0} \|w_t\|^2
\]
\[
\leq 2\varepsilon_0 \|w_x - \psi\|^2 + 2\varepsilon_0 \|\psi_x\|^2 + \frac{a^2}{4\varepsilon_0} \|w_t\|^2.
\]
for \( \varepsilon_0 > 0 \). Therefore

\[
H_3'(t) = -G \| \psi - w_x \|^2 + \frac{9}{4} \| s_x \|^2 + (1 + \varepsilon_0) \| 3s_t - \psi_t \|^2 + \frac{9I_2^2}{4\varepsilon_0} \| s_t \|^2 \\
+ 2\varepsilon_0 \| w_x - \psi \|^2 + 2\varepsilon_0 \| \psi_x \|^2 + \frac{\alpha^2}{4\varepsilon_0} \| w_t \|^2 + \rho \| w_t \|^2, \quad t > 0
\]
or

\[
H_3'(t) = (2\varepsilon_0 - G) \| \psi - w_x \|^2 + \frac{9}{4} \| s_x \|^2 + (1 + \varepsilon_0) \| 3s_t - \psi_t \|^2 \\
+ \frac{9I_2^2}{4\varepsilon_0} \| s_t \|^2 + 2\varepsilon_0 \| \psi_x \|^2 + \left( \frac{\alpha^2}{4\varepsilon_0} + \rho \right) \| w_t \|^2, \quad t > 0.
\]

Lemma 4. The functional \( H_4(t) \) satisfies

\[
H_4'(t) \leq 2 (\varepsilon_0 + 1) \| \psi_x \|^2 + (-I_\rho + \varepsilon_0) \| \psi_t \|^2 + 4\gamma^2 \| s \|^2 \\
+ \frac{4\delta^2}{\varepsilon_0} \| s_t \|^2 + \frac{4(a^2 + \rho^2)}{\varepsilon_0} \| w_t \|^2, \quad t > 0
\]

where \( \varepsilon_0 > 0 \).

Proof. Clearly

\[
H_4'(t) = -4G (\psi - w_x) - \psi_x + 4\rho (w_t, U_t) - I_\rho \| \psi_t \|^2 \\
+ (4G (\psi - w_x) - \psi_w + 4\gamma s + 4\beta s_t, \psi_t)
\]

\[
= -4G(\psi - w_x, \psi) - 4a(w_t, U_t) + 4\rho (w_t, U_t) - I_\rho \| \psi_t \|^2 + 4G(\psi - w_x, \psi) \\
+ \| \psi_x \|^2 + 4\gamma (s, \psi) + 4\beta (s_t, \psi)
\]

\[
= -4a(w_t, U_t) + 4\rho (w_t, U_t) - I_\rho \| \psi_t \|^2 + \| \psi_x \|^2 + 4\gamma (s, \psi) + 4\beta (s_t, \psi)
\]

where we have used (5) i.e.

\[
I_\rho \psi_t + 4G (\psi - w_x) - \psi_x + 4\gamma s + 4\beta s_t = 0, \quad t \geq 0
\]

obtained by summing three times the third equation with the second in the system (1).

The expression in (14) may be estimated as follows

\[
H_3'(t) \leq \varepsilon_0 \| \psi_x \|^2 + \frac{4a^2}{\varepsilon_0} \| w_t \|^2 + \varepsilon_0 \| \psi_t \|^2 + \frac{4\rho^2}{\varepsilon_0} \| w_t \|^2 - I_\rho \| \psi_t \|^2 + \| \psi_x \|^2 \\
+ 4\gamma^2 \| s \|^2 + \| \psi_x \|^2 + \varepsilon_0 \| \psi_x \|^2 + \frac{4\beta^2}{\varepsilon_0} \| s_t \|^2
\]
or

\[
H_3'(t) \leq 2 (\varepsilon_0 + 1) \| \psi_x \|^2 + (-I_\rho + \varepsilon_0) \| \psi_t \|^2 + 4\gamma^2 \| s \|^2 \\
+ \frac{4\delta^2}{\varepsilon_0} \| s_t \|^2 + \frac{4(a^2 + \rho^2)}{\varepsilon_0} \| w_t \|^2.
\]

That is

\[
H_3'(t) \leq 2 (\varepsilon_0 + 1) \| \psi_x \|^2 + (-I_\rho + \varepsilon_0) \| \psi_t \|^2 + 4\gamma^2 \| s \|^2 \\
+ \frac{4\delta^2}{\varepsilon_0} \| s_t \|^2 + \frac{4(a^2 + \rho^2)}{\varepsilon_0} \| w_t \|^2, \quad t > 0.
\]

\[\square\]
Using the previous lemmas we obtain the following proposition.

**Theorem 1.** For the energy $E_1$ defined above in (6), there exist two positive constants $K$ and $\kappa_0$ such that

$$E_1(t) \leq Ke^{-\kappa_0 t}, \ t \geq 0.$$  

**Proof.** In view of Lemmas 1-4, we see that

$$\begin{align*}
\sum_{i=1}^{4} \delta_i H_{i}'(t) &\leq \delta_1 \left( I_{\rho} + \varepsilon_0 + \frac{\beta^2}{3\varepsilon_0} \right) \| s_t \|^2 - \delta_1 (1 - \varepsilon_0) \| s_x \|^2 + \frac{4}{3} \delta_1 (\varepsilon_0 - \gamma) \| s \|^2 \\
+ \delta_1 \frac{a^2 + \rho^2}{4\varepsilon_0} \| u_t \|^2 + \delta_2 (I_{\rho} + \varepsilon_0) \| 3s_t - \psi_t \|^2 + \delta_4 (\varepsilon_0 - 1) \| 3s_x - \psi_x \|^2 \\
+ \delta_2 \frac{a^2 + \rho^2}{4\varepsilon_0} \| w_t \|^2 + \delta_3 (2\varepsilon_0 - G) \| \psi - w_x \|^2 + \frac{9}{4} \delta_3 \| s_x \|^2 + 2 \delta_3 \varepsilon_0 \| \psi_x \|^2 \\
+ \delta_3 (1 + \varepsilon_0) \| 3s_t - \psi_t \|^2 + \delta_4 \frac{4\beta^2}{4\varepsilon_0} \| s_t \|^2 + \delta_4 \frac{4\beta^2}{4\varepsilon_0} \| s_t \|^2 + 2 \delta_4 (\varepsilon_0 + 1) \| \psi_x \|^2 \\
+ \delta_4 (-I_{\rho} + \varepsilon_0) \| \psi_t \|^2 + 4\delta_4 \gamma^2 \| s \|^2 + 2 \delta_4 \frac{4\beta^2}{\varepsilon_0} \| s_t \|^2 + \frac{4}{3} \delta_4 \frac{4\beta^2}{\varepsilon_0} \| w_t \|^2 , \ t > 0
\end{align*}$$

or

$$\begin{align*}
\sum_{i=1}^{4} \delta_i H_{i}'(t) &\leq \left[ \delta_1 \frac{a^2 + \rho^2}{4\varepsilon_0} + \frac{a^2 + \rho^2}{4\varepsilon_0} + \delta_2 \frac{a^2 + \rho^2}{4\varepsilon_0} \right] \| u_t \|^2 \\
+ \delta_1 \left( I_{\rho} + \varepsilon_0 + \frac{\beta^2}{3\varepsilon_0} \right) \| s_t \|^2 - \delta_1 (1 - \varepsilon_0) \| s_x \|^2 + \frac{4}{3} \delta_1 (\varepsilon_0 - \gamma) \| s \|^2 \\
+ \delta_1 \frac{a^2 + \rho^2}{4\varepsilon_0} \| w_t \|^2 + \delta_2 (I_{\rho} + \varepsilon_0) \| 3s_t - \psi_t \|^2 + \delta_3 (2\varepsilon_0 - G) \| \psi - w_x \|^2 \\
+ \delta_3 (1 + \varepsilon_0) \| 3s_t - \psi_t \|^2 + \delta_3 (2\varepsilon_0 - G) \| \psi - w_x \|^2 \\
+ \delta_3 (1 + \varepsilon_0) \| 3s_t - \psi_t \|^2 + \delta_3 (2\varepsilon_0 - G) \| \psi - w_x \|^2 \\
+ \delta_3 (1 + \varepsilon_0) \| 3s_t - \psi_t \|^2 + \delta_3 (2\varepsilon_0 - G) \| \psi - w_x \|^2 \\
+ \delta_3 (1 + \varepsilon_0) \| 3s_t - \psi_t \|^2 + \delta_3 (2\varepsilon_0 - G) \| \psi - w_x \|^2 \\
+ \delta_3 (1 + \varepsilon_0) \| 3s_t - \psi_t \|^2 + \delta_3 (2\varepsilon_0 - G) \| \psi - w_x \|^2 \\
+ \delta_3 (1 + \varepsilon_0) \| 3s_t - \psi_t \|^2 + \delta_3 (2\varepsilon_0 - G) \| \psi - w_x \|^2 \\
+ \delta_3 (1 + \varepsilon_0) \| 3s_t - \psi_t \|^2 + \delta_3 (2\varepsilon_0 - G) \| \psi - w_x \|^2 , \ t > 0.
\end{align*}$$

Using the inequalities

$$\| 3s_t - \psi_t \|^2 \leq (1 + \varepsilon_0) \| \psi_t \|^2 + 9 \left( 1 + \frac{1}{\varepsilon_0} \right) \| s_t \|^2$$

and

$$\| \psi_x \|^2 \leq \frac{1}{4} \| 3s_x - \psi_x \|^2 + 9 \| s_x \|^2$$

we obtain

$$\begin{align*}
\sum_{i=1}^{4} \delta_i H_{i}'(t) &\leq \left[ \delta_1 \frac{a^2 + \rho^2}{4\varepsilon_0} + \frac{a^2 + \rho^2}{4\varepsilon_0} + \delta_2 \frac{a^2 + \rho^2}{4\varepsilon_0} \right] \| u_t \|^2 \\
+ \left\{ \delta_1 \left( I_{\rho} + \varepsilon_0 + \frac{\beta^2}{3\varepsilon_0} \right) \| s_t \|^2 + \frac{4}{3} \delta_1 (\varepsilon_0 - \gamma) \| s \|^2 \right\} \\
\times \| s_t \|^2 + \left\{ \frac{9}{4} \delta_3 \| s_t \|^2 + \frac{4}{3} \delta_3 \| s_t \|^2 \right\} \| s_x \|^2 \\
+ \left\{ \frac{1}{4} \delta_2 (I_{\rho} + \varepsilon_0) + \delta_3 (1 + \varepsilon_0) \right\} \| s_x \|^2 \\
+ \left\{ \frac{4}{3} \delta_4 \frac{4\beta^2}{\varepsilon_0} \| s_t \|^2 + \frac{4}{3} \delta_4 \frac{4\beta^2}{\varepsilon_0} \| s_t \|^2 \right\} \| w_t \|^2 , \ t > 0.
\end{align*}$$
Therefore, in view of Proposition 1, we can write

\[
E'_i(t) + \sum_{i=1}^{4} \delta_i H'_i(t) \leq -a \|w_i\|^2 - 4\beta \|s_i\|^2 + 4 \left[ \frac{1}{3} \delta_1 (\varepsilon_0 - \gamma) + \delta_4 \gamma^2 \right] \|s\|^2
\]

\[
+ \left\{ \delta_1 \left( I_{\rho} + \varepsilon_0 + \frac{\beta^2}{3\varepsilon_0} \right) + \delta_3 \frac{9H}{4\varepsilon_0} + \delta_4 \frac{a^2}{4\varepsilon_0} + 9 \left( 1 + \frac{1}{\varepsilon_0} \right) \left[ \delta_2 (I_{\rho} + \varepsilon_0) + \delta_3 (1 + \varepsilon_0) \right] \right\} \|s_i\|^2
\]

\[
+ \left\{ \frac{3}{2} \delta_3 - \delta_1 (1 - \varepsilon_0) + 9 \left[ \delta_3 \varepsilon_0 + 2\delta_4 (\varepsilon_0 + 1) \right] \|s_x\|^2
\]

\[
+ \left\{ \frac{3}{2} \delta_3 - \delta_1 (1 - \varepsilon_0) + 9 \left[ \delta_3 \varepsilon_0 + 2\delta_4 (\varepsilon_0 + 1) \right] \|s_x\|^2
\]

\[
+ \left\{ \frac{3}{2} \delta_3 - \delta_1 (1 - \varepsilon_0) + 9 \left[ \delta_3 \varepsilon_0 + 2\delta_4 (\varepsilon_0 + 1) \right] \|s_x\|^2
\]

\[
+ \left\{ \frac{3}{2} \delta_3 - \delta_1 (1 - \varepsilon_0) + 9 \left[ \delta_3 \varepsilon_0 + 2\delta_4 (\varepsilon_0 + 1) \right] \|s_x\|^2
\]

\[
+ \left\{ \frac{3}{2} \delta_3 - \delta_1 (1 - \varepsilon_0) + 9 \left[ \delta_3 \varepsilon_0 + 2\delta_4 (\varepsilon_0 + 1) \right] \|s_x\|^2
\]

\[
+ \left\{ \frac{3}{2} \delta_3 - \delta_1 (1 - \varepsilon_0) + 9 \left[ \delta_3 \varepsilon_0 + 2\delta_4 (\varepsilon_0 + 1) \right] \|s_x\|^2
\]

or, if \( L(t) = E_1(t) + \sum_{i=1}^{4} \delta_i H_i(t) \) (as defined above) then

\[
L'(t) \leq \left\{ -a + \left[ \frac{3}{2} \delta_3 - \delta_1 (1 - \varepsilon_0) + 9 \left[ \delta_3 \varepsilon_0 + 2\delta_4 (\varepsilon_0 + 1) \right] \|s_x\|^2
\]

\[
+ \left\{ \frac{3}{2} \delta_3 - \delta_1 (1 - \varepsilon_0) + 9 \left[ \delta_3 \varepsilon_0 + 2\delta_4 (\varepsilon_0 + 1) \right] \|s_x\|^2
\]

\[
+ \left\{ \frac{3}{2} \delta_3 - \delta_1 (1 - \varepsilon_0) + 9 \left[ \delta_3 \varepsilon_0 + 2\delta_4 (\varepsilon_0 + 1) \right] \|s_x\|^2
\]

To recover all the terms in \( E_1(t) \) (with negative coefficients) in the right hand side of this last relation we add and subtract \( \mu \|3s_t - \psi_t\|^2 \) (with a 'small' coefficient \( \mu \) to be determined) in this right hand side (notice that we could have done this at an earlier stage). Using the inequality

\[
\|3s_t - \psi_t\|^2 \leq (1 + \varepsilon_0) \|\psi_t\|^2 + 9 \left( 1 + \frac{1}{\varepsilon_0} \right) \|s_t\|^2
\]

say with \( \varepsilon_0 = 1 \), we see that the insertion of \( \mu \|3s_t - \psi_t\|^2 \), with positive \( \mu \), will give rise to the terms \( \|\psi_t\|^2 \) and \( \|s_t\|^2 \) with small coefficients. These coefficients will be easily absorbed by the corresponding ones in the last relation. Therefore

\[
L'(t) \leq \left\{ -a + \left[ \frac{3}{2} \delta_3 - \delta_1 (1 - \varepsilon_0) + 9 \left[ \delta_3 \varepsilon_0 + 2\delta_4 (\varepsilon_0 + 1) \right] \|s_x\|^2
\]

\[
+ \left\{ \frac{3}{2} \delta_3 - \delta_1 (1 - \varepsilon_0) + 9 \left[ \delta_3 \varepsilon_0 + 2\delta_4 (\varepsilon_0 + 1) \right] \|s_x\|^2
\]

\[
+ \left\{ \frac{3}{2} \delta_3 - \delta_1 (1 - \varepsilon_0) + 9 \left[ \delta_3 \varepsilon_0 + 2\delta_4 (\varepsilon_0 + 1) \right] \|s_x\|^2
\]

\[
+ \left\{ \frac{3}{2} \delta_3 - \delta_1 (1 - \varepsilon_0) + 9 \left[ \delta_3 \varepsilon_0 + 2\delta_4 (\varepsilon_0 + 1) \right] \|s_x\|^2
\]

\[
+ \left\{ \frac{3}{2} \delta_3 - \delta_1 (1 - \varepsilon_0) + 9 \left[ \delta_3 \varepsilon_0 + 2\delta_4 (\varepsilon_0 + 1) \right] \|s_x\|^2
\]

\[
+ \left\{ \frac{3}{2} \delta_3 - \delta_1 (1 - \varepsilon_0) + 9 \left[ \delta_3 \varepsilon_0 + 2\delta_4 (\varepsilon_0 + 1) \right] \|s_x\|^2
\]

\[
+ \left\{ \frac{3}{2} \delta_3 - \delta_1 (1 - \varepsilon_0) + 9 \left[ \delta_3 \varepsilon_0 + 2\delta_4 (\varepsilon_0 + 1) \right] \|s_x\|^2
\]

\[
+ \left\{ \frac{3}{2} \delta_3 - \delta_1 (1 - \varepsilon_0) + 9 \left[ \delta_3 \varepsilon_0 + 2\delta_4 (\varepsilon_0 + 1) \right] \|s_x\|^2
\]

\[
+ \left\{ \frac{3}{2} \delta_3 - \delta_1 (1 - \varepsilon_0) + 9 \left[ \delta_3 \varepsilon_0 + 2\delta_4 (\varepsilon_0 + 1) \right] \|s_x\|^2
\]

\[
+ \left\{ \frac{3}{2} \delta_3 - \delta_1 (1 - \varepsilon_0) + 9 \left[ \delta_3 \varepsilon_0 + 2\delta_4 (\varepsilon_0 + 1) \right] \|s_x\|^2
\]

\[
+ \left\{ \frac{3}{2} \delta_3 - \delta_1 (1 - \varepsilon_0) + 9 \left[ \delta_3 \varepsilon_0 + 2\delta_4 (\varepsilon_0 + 1) \right] \|s_x\|^2
\]

\[
+ \left\{ \frac{3}{2} \delta_3 - \delta_1 (1 - \varepsilon_0) + 9 \left[ \delta_3 \varepsilon_0 + 2\delta_4 (\varepsilon_0 + 1) \right] \|s_x\|^2
\]

\[
+ \left\{ \frac{3}{2} \delta_3 - \delta_1 (1 - \varepsilon_0) + 9 \left[ \delta_3 \varepsilon_0 + 2\delta_4 (\varepsilon_0 + 1) \right] \|s_x\|^2
\]

\[
+ \left\{ \frac{3}{2} \delta_3 - \delta_1 (1 - \varepsilon_0) + 9 \left[ \delta_3 \varepsilon_0 + 2\delta_4 (\varepsilon_0 + 1) \right] \|s_x\|^2
\]

(15)
In (16) we neglect \( \varepsilon_0 (\varepsilon_0 < G/2) \) and \( \mu \) at this stage

\[
\begin{align*}
\delta_1 &< 1, \\
\delta_4 &< \frac{1}{4} \delta_1, \\
\delta_2 I_\rho + \delta_3 &< I_\rho \delta_4, \\
\frac{1}{2} \delta_4 &< \delta_2.
\end{align*}
\] (17)

As there are only smallness conditions on \( \delta_3 \), we focus on

\[
\begin{align*}
18 \delta_4 &< \delta_1, \\
\delta_4 &< \frac{1}{4} \delta_1, \\
\delta_2 &< \delta_4, \\
\frac{1}{4} \delta_4 &< \delta_2.
\end{align*}
\] (18)

It suffices then to pick first \( \delta_4 = \frac{3}{2} \delta_2 \) and then \( \delta_1 > 3 \max \{6, \gamma\} \delta_4 \), for instance \( \delta_1 = 4 \max \{6, \gamma\} \delta_4 = 6 \max \{6, \gamma\} \delta_2 \). Second, we select \( \delta_3 \) fulfilling (17). Next, we choose \( \varepsilon_0, \mu \) and \( \delta_2 \) so that all the coefficients in (16) hold. The inequality \( L(t) \leq -\kappa L(t), t \geq t_0 > 0 \) implies the exponential decay of \( L(t) \). This property is shared by \( E_1(t) \) through the equivalence. It is easy to show it on \([0, t_0] \) as well. \( \square \)

3. Viscoelastic case (Problem (2)). Let the (modified) energy associated to the problem

\[
\begin{align*}
\rho w_{tt} + G(\psi - w_x)_x + G \int_0^t h(t-r)w_{xt}(r)dr &= 0, \\
I_\rho (3s_{tt} - \psi_{tt}) - G(\psi - w_x) - (3s - \psi)_{xx} + \int_0^t h(t-r)(3s - \psi)_{xx}(r)dr &= 0, \\
I_\rho s_{tt} + G(\psi - w_x) + \frac{1}{2} \gamma s - s_{xx} + \frac{1}{4} \beta s_t &= 0,
\end{align*}
\]

be defined, for \( t \geq 0 \), by

\[
E_2(t) = E_1(t) + \frac{1}{2} \left[ G(h \square w_x)(t) + (h \square (3s - \psi)_x)(t) - \left( \int_0^t h(s)ds \right) \left( G \|w_x\|^2 + \|(3s - \psi)_x\|^2 \right) \right]
\]

where

\[
(h \square u)(t) = \int_0^1 \int_0^t h(t-r)|u(t) - u(r)|^2 dr dx, \quad t \geq 0.
\]

**Proposition 2.** The energy \( E_2(t) \) given by (19) satisfies for \( t > 0 \)

\[
E'_2(t) = -4 \beta \|s_t\|^2 + \frac{G}{2} (h' \square w_x)(t) + \frac{1}{2} (h' \square (3s - \psi)_x)(t) - \frac{G}{2} h(t) \|w_x\|^2 - \frac{1}{2} h(t) \|(3s - \psi)_x\|^2, \quad t > 0.
\]

**Proof.** Multiplying the first equation of (3) by \( w_t \) and integrating over \((0, 1)\) we obtain

\[
\frac{\rho}{2} \frac{d}{dt} \|w_t\|^2 + G((\psi - w_x)_x, w_t) + G \left( \int_0^t h(t-r)w_{xt}(r)dr, w_t \right) = 0, \quad t > 0
\]

or

\[
\frac{\rho}{2} \frac{d}{dt} \|w_t\|^2 - G(\psi - w_x, w_{xt}) - G \left( \int_0^t h(t-r)w_x(r)dr, w_{tx} \right) = 0, \quad t > 0
\]
and by
\[
\left( \int_0^t h(t-r)w_x(r)dr, w_{tx} \right) = \frac{1}{2}(h'\square w_x)(t) - \frac{1}{2} \frac{d}{dt}(h\square w_x)(t)
\]
\[
+ \frac{1}{2} \frac{d}{dt} \left\{ \left( \int_0^t h(s)ds \right) \int_0^1 |w_x|^2 \, dx \right\} - \frac{1}{2} h(t) \int_0^1 |w_x|^2 \, dx, \ t \geq 0
\]
we get
\[
\frac{1}{2} \frac{d}{dt} \left\{ \rho \|w_t\|^2 + G(h\square w_x)(t) - G \left( \int_0^t h(s)ds \right) \|w_x\|^2 \right\} - G(\psi - w_x, w_{xt}) = \frac{G}{2}(h'\square w_x)(t) - \frac{G}{2} h(t) \|w_x\|^2, \ t > 0
\]
and note that
\[
G(\psi - w_x, w_{xt}) = -G(\psi - w_x, (\psi - w_x - \psi_t))
\]
\[
= -\frac{G}{2} \frac{d}{dt} \|\psi - w_x\|^2 + G(\psi - w_x, \psi_t).
\]

Again multiplying the third equation of (3) by \(s_t\) and integrating on \((0,1)\) we obtain
\[
I_p(s_{tt}, s_t) + G(\psi - w_x, s_t) + \frac{4\gamma}{3} (s, s_t) - (s_{xx}, s_t) + \frac{4\beta}{3} \|s_t\|^2 = 0, \ t > 0
\]
or
\[
\frac{1}{2} \frac{d}{dt} \left[ I_p \|s_t\|^2 + \frac{4\gamma}{3} \|s\|^2 + \|s_x\|^2 \right] + G(\psi - w_x, s_t) = -\frac{4\beta}{3} \|s_t\|^2, \ t > 0.
\]
Now adding (20) and (22) taking into account (21), we obtain
\[
\frac{3}{2} \frac{d}{dt} \left[ I_p \|s_t\|^2 + \frac{4\gamma}{3} \|s\|^2 + \|s_x\|^2 \right]
\]
\[
+ \frac{1}{2} \frac{d}{dt} \left[ \rho \|w_t\|^2 + G \|\psi - w_x\|^2 + G(h\square w_x)(t) - G \left( \int_0^t h(s)ds \right) \|w_x\|^2 \right]
\]
\[
+ 3G(\psi - w_x, s_t) + 4\beta \|s_t\|^2 - G(\psi - w_x, \psi_t)
\]
\[
- \frac{G}{2} (h'\square w_x)(t) + \frac{G}{2} h(t) \|w_x\|^2 = 0
\]
or
\[
\frac{1}{2} \frac{d}{dt} \left[ 3I_p \|s_t\|^2 + 4\gamma \|s\|^2 + 3 \|s_x\|^2 + \rho \|w_t\|^2 + G \|\psi - w_x\|^2 \right]
\]
\[
+ G(h\square w_x)(t) - G \left( \int_0^t h(s)ds \right) \|w_x\|^2 \right) + G(\psi - w_x, 3s_t - \psi_t)
\]
\[
+ 4\beta \|s_t\|^2 - \frac{G}{2} (h'\square w_x)(t) + \frac{G}{2} h(t) \|w_x\|^2 = 0, \ t > 0.
\]
Next we use the second equation in (3) to get
\[
\frac{1}{2} I_p \frac{d}{dt} \|3s_t - \psi_t\|^2 = (G(\psi - w_x) + (3s - \psi)_{xx}, (3s - \psi)_t)
\]
\[- \left( \int_0^t h(t-r) (3s - \psi)_{xx}(r)dr, (3s - \psi)_t \right)
\]
\[
= G(\psi - w_x, (3s - \psi)_t) - ((3s - \psi)_x, (3s - \psi)_{xt})
\]
\[
+ \left( \int_0^t h(t-r) (3s - \psi)_x(r)dr, (3s - \psi)_{tx} \right).
\]
The identity
\[
\left( \int_0^t h(t-r) (3s-\psi)_x(r) dr, (3s-\psi)_x \right) = \frac{1}{2} (h' \square (3s-\psi)_x) (t)
\]
implies
\[
\frac{1}{2} \frac{d}{dt} \left[ \int \|3s_t - \psi_t\|^2 + \|(3s - \psi)_x\|^2 + (h \square (3s - \psi)_x) (t)
- \left( \int_0^t h(s) ds \right) \|(3s - \psi)_x\|^2 \right] = G (\psi - w_x, (3s - \psi)_x) + \frac{1}{2} (h' \square (3s - \psi)_x) (t) - \frac{1}{2} h(t) \|(3s - \psi)_x\|^2.
\]
Gathering the previous relations (23) and (24), we are lead to
\[
E_2' (t) = -4 \beta \|s_t\|^2 + \frac{G}{2} (h' \square w_x) (t) + \frac{1}{2} (h' \square (3s - \psi)_x) (t)
- \frac{G}{2} h(t) \|w_x\|^2 - \frac{1}{2} h(t) \|(3s - \psi)_x\|^2, \quad t > 0.
\]

Again we find a decreasing energy functional. However the nonnegativity of \(E_2 (t)\) is not clear. This difficulty does not arise in the usual viscoelastic problems as there will be always a term of the form \(G \|w_x\|^2\) which will take of the negative term \(G \left( \int_0^t h(s) ds \right) \|w_x\|^2\). We are tempted to use the expression \(G ((\psi - w_x)_x, w_t)\) in the proof of Proposition 2 to get the derivative of the term \(G \|w_x\|^2\) but then we will be faced with an even more complicated situation which is the appearance of the term \(G (\psi_x, w_t)\). The argument below shows that this functional is indeed nonnegative under a certain relationship between \(h\) and \(G\). The user will have a certain freedom in the manner of imposing this condition.

**Proposition 3.** The functional \(E_2 (t)\) is nonnegative (and actually equivalent to the same functional without the term in \(\|w_x\|^2\)) provided that
\[
(a) \quad \frac{G + \sqrt{G^2 + 4G}}{2} < \xi < \frac{3G}{1 + 3G}, \quad \text{or}
\]
\[
(b) \quad 1 > \xi > \max \left\{ \frac{3G}{1 + 3G}, \frac{G + \sqrt{G^2 + 4G}}{2}, \frac{2G + \sqrt{4G(1 + 4G)}}{2(1 + 3G)} \right\}.
\]

**Proof.** We have for \(\eta_1, \eta_2 > 0\)
\[
\|w_x\|^2 \leq (1 + \eta_1) \|\psi - w_x\|^2 + \left( 1 + \frac{1}{\eta_1} \right) \|\psi_x\|^2
\]
\[
\leq (1 + \eta_1) \|\psi - w_x\|^2 + \left( 1 + \frac{1}{\eta_1} \right) \left[ (1 + \eta_2) \|\psi_x - 3s_x\|^2 + 9 \left( 1 + \frac{1}{\eta_2} \right) \|s_x\|^2 \right].
\]
Our goal is to control \(-G \left( \int_0^t h(s) ds \right) \|w_x\|^2\) by the expression \(G \|\psi - w_x\|^2 + I_p \|(3s - \psi)_x\|^2 + 3 \|s_x\|^2\) which is already in \(E_2(t)\). Clearly we will need

\[
\begin{aligned}
(1 - \xi)(1 + \eta_1) &\leq 1, \\
G(1 - \xi) \left(1 + \frac{1}{\eta_1}\right)(1 + \eta_2) &\leq \xi, \\
9G(1 - \xi) \left(1 + \frac{1}{\eta_0}\right)(1 + \frac{1}{\eta_2}) &\leq 3.
\end{aligned}
\]

To fix ideas let us pick \(\eta_1 = \frac{\xi}{1-\xi}\) to fulfill the first relation. For the other two relations to hold we will need

\[
\frac{3G(1 - \xi)}{\xi - 3G(1 - \xi)} \leq \eta_2 \leq \frac{\xi^2 - G(1 - \xi)}{G(1 - \xi)}.
\]

This condition is valid in case

(a) \(\frac{G + \sqrt{G^2 + 4G}}{2 < \xi < \frac{3G}{1+3G}}\) (which is possible if \(G > 1/6\)), or

(b) \(\xi > \max \left\{ \frac{3G}{1+3G}, \frac{2G + \sqrt{4G(1+4G)}}{2(1+3G)} \right\} = \frac{2G + \sqrt{4G(1+4G)}}{2(1+3G)}\) (note that the term in the right side is smaller than one).

Clearly \(E'_2(t) \leq 0\) (that is \(E_2(t)\) is nonincreasing as \(E_1(t)\)) but we have lost the term \(-\|w_t\|^2\) in \(E_2'(t)\) (which was in \(E'_1(t)\)) we need to recover it by introducing the functional

\[
\tilde{H}_5(t) = -\rho \left( w_t, \int_0^t h(t-r) [w(t) - w(r)] \, dr \right), \quad t \geq 0.
\]

Let \(\varepsilon_1 > 0\) and let \(t \geq t_0 > 0\) so that \(\int_0^t h(r) dr \geq \int_0^{t_0} h(r) dr = h_0 > 0\).

**Lemma 5.** The derivative of \(\tilde{H}_5(t)\) along solutions of the system (3)-(4) is estimated as follows

\[
\tilde{H}_5'(t) \leq \rho \left( \varepsilon_1 - h_0 \right) \|w_t\|^2 + G (1 - \xi) \left(1 + \frac{1}{|\varepsilon_1|}\right)(h \Box w_x)(t) + \varepsilon_1 G (1 - \xi)^2 \|w_x\|^2 + \varepsilon_1 G \|\psi - w_x\|^2 + \frac{\rho h(0)}{|\varepsilon_1|} (|h'| \Box w_x)(t), \quad t \geq t_0 > 0.
\]

**Proof.** It is clear that

\[
\tilde{H}_5'(t) = -\rho \left( w_t, \int_0^t h(t-r) [w(t) - w(r)] \, dr \right) - \rho \left( w_t, \int_0^t h(t-r) [w(t) - w(r)] \, dr \right) - \rho \left( w_t, \int_0^t h(r) \, dr \right), \quad t > 0
\]

and in virtue of the first equation in (3) we see that

\[
\tilde{H}_5'(t) = G (\psi - w_x) + G \int_0^t h(t-r) w_x(r) dr, \int_0^t h(t-r) [w(t) - w(r)] \, dr
\]

\[
-\rho \left( w_t, \int_0^t h(t-r) [w(t) - w(r)] \, dr \right) - \rho \left( \int_0^t h(r) \, dr \right) \|w_t\|^2, \quad t > 0.
\]

Therefore

\[
\tilde{H}_5(t) = -G (\psi - w_x + \int_0^t h(t-r) w_x(r) dr, \int_0^t h(t-r) [w_x(t) - w_x(r)] dr
\]

\[
-\rho \left( w_t, \int_0^t h(t-r) [w(t) - w(r)] \, dr \right) - \rho \left( \int_0^t h(r) \, dr \right) \|w_t\|^2, \quad t > 0.
\]
Note that

\[-G \left( \int_0^t h(t-r)w_x(r)dr, \int_0^t h(t-r) [w_x(t) - w_x(r)] dr \right) - \left( \int_0^t h(t-r)w_x(t) - w_x(r) dr, \int_0^t h(t-r) [w_x(t) - w_x(r)] dr \right) \]

\[= G \left( \int_0^t h(t-r)w_x(t) - w_x(r) dr, \int_0^t h(t-r) [w_x(t) - w_x(r)] dr \right) - \left( \int_0^t h(t-r)w_x(t) - w_x(r) dr, \int_0^t h(t-r) [w_x(t) - w_x(r)] dr \right) \]

\[\leq G (1 - \xi) \left( h \Box w_x \right) (t) + \varepsilon_1 G \left( 1 - \xi \right)^2 \| w_x \|^2 + \frac{G(1 - \xi)}{4\varepsilon_1} \left( h \Box w_x \right) (t) \]

\[\leq G (1 - \xi) \left( 1 + \frac{1}{2\varepsilon_1} \right) \left( h \Box w_x \right) (t) + \varepsilon_1 G \left( 1 - \xi \right)^2 \| w_x \|^2 , \]

\[\leq \varepsilon_1 G \| \psi - w_x \|^2 + \frac{G(1 - \xi)}{4\varepsilon_1} \left( h \Box w_x \right) (t) \]

and

\[-\rho \left( w_t, \int_0^t h(t-r) [w(t) - w(r)] dr \right) \leq \varepsilon_1 \rho \| w_t \|^2 + \frac{\rho(0)}{4\varepsilon_1} \left( h' \Box w_x \right) (t). \]

Hence

\[\hat{H}_1(t) \leq G (1 - \xi) \left( 1 + \frac{1}{2\varepsilon_1} \right) \left( h \Box w_x \right) (t) + \varepsilon_1 G \left( 1 - \xi \right)^2 \| w_x \|^2 + \varepsilon_1 G \| \psi - w_x \|^2 + \rho \left( \varepsilon_1 - \int_0^t h(r)dr \right) \| w_t \|^2 + \frac{\rho(0)}{4\varepsilon_1} \left( h' \Box w_x \right) (t). \]

For \( t \geq t_0 > 0 \) we have \( \int_0^t h(r)dr \geq h_0 > 0 \) and thus

\[\hat{H}_1(t) \leq \rho \left( \varepsilon_1 - h_0 \right) \| w_t \|^2 + G (1 - \xi) \left( 1 + \frac{1}{2\varepsilon_1} \right) \left( h \Box w_x \right) (t) + \varepsilon_1 G \left( 1 - \xi \right)^2 \| w_x \|^2 + \varepsilon_1 G \| \psi - w_x \|^2 + \frac{\rho(0)}{4\varepsilon_1} \left( h' \Box w_x \right) (t). \]

We modify \( H_1(t) \) by

\[\tilde{H}_1(t) := I_\rho \left( 1 - \int_0^t h(r)dr \right) (s_t, s) - \rho(w_t, W), \quad t \geq 0. \]

Then, Lemma 1 becomes

**Lemma 6.** We have for the derivative of \( \tilde{H}_1(t) \) along solutions of (3)-(4)

\[\tilde{H}_1(t) \leq \left( I_\rho + \frac{\beta^2}{4\varepsilon_1} + \frac{\beta^2}{4\varepsilon_1} \right) \| s_t \|^2 + \left[ \frac{3}{2} (\varepsilon_1 - \gamma \xi) + \varepsilon_2 + \varepsilon_1 \right] \| s \|^2 + \left[ -\xi + \frac{9G^2(1-\xi)^2}{2\varepsilon_2} \right] \| s \|^2 + \frac{G^2(1-\xi)^2}{2\varepsilon_2} \| \psi - 3s \|^2 + \frac{G^2(1-\xi)}{4\varepsilon_1} \left( h \Box w_x \right) (t) + \varepsilon_1 \| w_t \|^2 \]

for \( t > 0 \) and \( \varepsilon_1, \varepsilon_2 > 0. \)

**Proof.** From the definition of \( \tilde{H}_1(t) \), the first and the third equations in the system we find

\[\tilde{H}_1(t) = I_\rho \left( 1 - \int_0^t h(r)dr \right) (s_t, s) + I_\rho \left( 1 - \int_0^t h(r)dr \right) (s_t, s) - I_\rho h(t) (s_t, s) - \rho(w_t, W) - \rho(w_t, W_t) \]

\[= I_\rho \left( 1 - \int_0^t h(r)dr \right) \| s_t \|^2 - G \left( 1 - \int_0^t h(r)dr \right) (\psi - w_t, s) \]

\[-\frac{4\varepsilon_1}{3} \left( 1 - \int_0^t h(r)dr \right) \| s \|^2 - \left( 1 - \int_0^t h(r)dr \right) \| s_x \|^2 - \frac{4\varepsilon_1}{3} \left( 1 - \int_0^t h(r)dr \right) (s_t, s) \]

\[+ G((\psi - w_x)_x, W) + G \left( \int_0^t h(t-r)w_x(r)dr, W \right) - \rho(w_t, W_t), \quad t > 0 \]
and as
\[ G((\psi - w_x)_x, W) = -G(\psi - w_x, W_x) = G(\psi - w_x, s) \]
we infer that
\[
\tilde{H}'_1(t) \leq I_p \| s_t \|^2 - \frac{2\xi}{3} \| s \|^2 - \xi \| s_x \|^2 + \frac{4}{3} \varepsilon_1 \| s \|^2 + \frac{\beta^2}{4\xi_1} \| s_t \|^2
\]
\[ + G \left( \int_0^t h(r)dr \right) (\psi - w_x, s) - G \left( \int_0^t h(t - r) [w_x(t) - w_x(r)] dr, s \right)
\]
\[ + G \left( \int_0^t h(r)dr \right) (w_x, s) + \varepsilon_1 \| w_t \|^2 + \frac{\beta^2}{4\xi_1} \| s_t \|^2 \]
or
\[
\tilde{H}'_1(t) \leq \left( I_p + \frac{\beta^2}{4\xi_1} + \frac{\beta^2}{4\xi_1} \right) \| s_t \|^2 + \frac{4}{3} \left( \varepsilon_1 - \gamma \xi \right) \| s \|^2 - \xi \| s_x \|^2
\]
\[ + G \left( \int_0^t h(r)dr \right) (\psi, s) - G \left( \int_0^t h(t - r) [w_x(t) - w_x(r)] dr, s \right) + \varepsilon_1 \| w_t \|^2. \]

It is easy to see that
\[ G \left( \int_0^t h(t - r) [w_x(t) - w_x(r)] dr, s \right) \leq \varepsilon_1 \| s \|^2 + \frac{G^2(1 - \xi)}{4\varepsilon_1} (h\Box w_x)(t), \]
and
\[ G \left( \int_0^t h(r)dr \right) (\psi, s) \leq \varepsilon_2 \| s \|^2 + \frac{G^2(1 - \xi)^2}{2\varepsilon_2} \| \psi \|^2\]
\[ \leq \varepsilon_2 \| s \|^2 + \frac{G^2(1 - \xi)^2}{2\varepsilon_2} \left[ \| \psi_x - 3s_x \|^2 + 9 \| s_x \|^2 \right]. \]
Therefore, for \( t > 0 \)
\[
\tilde{H}'_1(t) \leq \left( I_p + \frac{\beta^2}{4\xi_1} + \frac{\beta^2}{4\xi_1} \right) \| s_t \|^2 + \frac{4}{3} \left( \varepsilon_1 - \gamma \xi \right) \| s \|^2 - \xi \| s_x \|^2 + \varepsilon_2 \| s \|^2
\]
\[ + \frac{G^2(1 - \xi)^2}{2\varepsilon_2} \| \psi_x - 3s_x \|^2 + \varepsilon_1 \| s \|^2 + \frac{G^2(1 - \xi)^2}{4\varepsilon_1} (h\Box w_x)(t) + \varepsilon_1 \| w_t \|^2 \]
or
\[
\tilde{H}'_1(t) \leq \left( I_p + \frac{\beta^2}{4\xi_1} + \frac{\beta^2}{4\xi_1} \right) \| s_t \|^2 + \frac{4}{3} \left( \varepsilon_1 - \gamma \xi \right) \| s \|^2 + \frac{G^2(1 - \xi)^2}{2\varepsilon_2} \| \psi_x - 3s_x \|^2
\]
\[ + \frac{G^2(1 - \xi)^2}{2\varepsilon_2} \| \psi_x - 3s_x \|^2 + \frac{G^2(1 - \xi)^2}{4\varepsilon_1} (h\Box w_x)(t) + \varepsilon_1 \| w_t \|^2. \]

Let
\[
\tilde{H}_4(t) = -\left( (3s - \psi)_t, \int_0^t h(t - r) [(3s - \psi)(t) - (3s - \psi)(r)] dr \right), \ t \geq t_0 > 0. \]

**Lemma 7.** The derivative of \( \tilde{H}_4(t) \) along solutions of (3)-(4) is estimated as follows, for \( \varepsilon_0 > 0 \)
\[
\tilde{H}'_4(t) \leq \varepsilon_0 \| \psi - w_x \|^2 + \varepsilon_0 \left( 1 - \int_0^t h(r)dr \right) \| (3s - \psi)_x \| \]
\[ + I_p (\varepsilon_0 - h_0) \| (3s - \psi)_t \|^2 + \frac{I_p h_0}{4\varepsilon_0} (h\Box (3s - \psi)_x)(t)
\]
\[ + (1 - \xi) \left[ 1 + \frac{G^2}{4\varepsilon_0} \left( 2 - \int_0^t h(r)dr \right) \right] (h\Box (3s - \psi)_x)(t), \ t \geq t_0 > 0. \]
Proof. We have
\[
\dot{H}_4(t) = -I_\rho \left((3s - \psi)_{tt}, \int_0^t h(t-r) \left[(3s - \psi)(t) - (3s - \psi)(r)\right] dr \right) \\
- I_\rho \left(3s - \psi\right)_t, \int_0^t h'(t-r) \left[(3s - \psi)(t) - (3s - \psi)(r)\right] dr \\
- I_\rho \left(\int_0^t h(r) dr\right) \|3s_t - \psi_t\|^2, \; t \geq 0.
\]
(25)

The first term in the right hand side of (25) is equal to
\[
- I_\rho \left(3s - \psi\right)_{tt}, \int_0^t h(t-r) \left[(3s - \psi)(t) - (3s - \psi)(r)\right] dr \\
= -G(\psi - w_x, \int_0^t h(t-r) \left[(3s - \psi)(t) - (3s - \psi)(r)\right] dr) \\
+ \left(\int_0^t h(t-r) (3s - \psi)_x (t) - (3s - \psi)(r) dr\right)^2 \\
- \left(\int_0^t h(r) dr\right) \left(3s - \psi\right)_x (t) \left(3s - \psi\right)_x (r) dr
\]
or
\[
- I_\rho \left(3s - \psi\right)_{tt}, \int_0^t h(t-r) \left[(3s - \psi)(t) - (3s - \psi)(r)\right] dr \\
\geq \varepsilon_0 \|\psi - w_x\|^2 + \frac{\rho^2 (1 - \xi)}{4 \varepsilon_0} \left(h \Box (3s - \psi)_x (t) + (1 - \xi) (h \Box (3s - \psi)_x (t) + (1 - \xi) (h \Box (3s - \psi)_x (t))
\]
and may be estimated as follows
\[
- I_\rho \left(3s - \psi\right)_{tt}, \int_0^t h(t-r) \left[(3s - \psi)(t) - (3s - \psi)(r)\right] dr \\
\leq \varepsilon_0 \|\psi - w_x\|^2 + \frac{\rho^2 (1 - \xi)}{4 \varepsilon_0} \left(h \Box (3s - \psi)_x (t) + (1 - \xi) (h \Box (3s - \psi)_x (t))
\]
\[
+ \varepsilon_0 \left(1 - \int_0^t h(r) dr\right) \|3s - \psi\|^2
\]
Therefore
\[
\dot{H}_4(t) \leq \varepsilon_0 \|\psi - w_x\|^2 + \varepsilon_0 \left(1 - \int_0^t h(r) dr\right) \|3s - \psi\|^2 \\
+ I_\rho \left(\varepsilon_0 - h_{t0}\right) \|3s - \psi\|^2 + \frac{\rho h(t_0)}{4 \varepsilon_0} \left(|h'| \Box (3s - \psi)_x (t)
\]
\[
+ (1 - \xi) \left[1 + \frac{\rho^2 (1 - \xi)}{4 \varepsilon_0} \left(2 - \int_0^t h(r) dr\right)\right] (h \Box (3s - \psi)_x (t), \; t \geq t_0 > 0.
\]
Lemma 8. For the functional $\tilde{H}_3(t) = \rho(w_t, w)$, we have the estimate

$$\tilde{H}_3'(t) = -G \left( \frac{\xi}{2} - \varepsilon_0 \right) ||w_x||^2 + \frac{G(1-\xi)}{4\varepsilon_0} (h \square w_x)(t) + \rho ||w_t||^2 + \frac{G}{2\varepsilon} \left[ (1 + \eta) ||(3s - \psi)_x||^2 + 9 \left( 1 + \frac{1}{\eta} \right) ||s_x||^2 \right], \ t \geq 0$$

for $\varepsilon_0, \eta > 0$.

Proof. We infer from the first equation and the definition of $\tilde{H}_3(t)$ that

$$\tilde{H}_3'(t) = \rho (w_t, w) + \rho ||w_t||^2$$

$$= -G ((\psi - w_x)_x, w) - G \left( \int_0^t h(t-r)w_{xx}(r)dr, w \right) + \rho ||w_t||^2$$

$$= G (\psi - w_x, w_x) - G \left( \int_0^t h(t-r)w_{xx}(r)dr, w \right) + \rho ||w_t||^2$$

$$= -G ||w_x||^2 + G (\psi, w_x) + G \left( \int_0^t h(t-r)w_x(r)dr, w_x \right) + \rho ||w_t||^2, \ t \geq 0.$$}

The identity

$$\left( \int_0^t h(t-r)w_x(r)dr, w_x \right)$$

$$= - \left( \int_0^t h(t-r)[w_x(t) - w_x(r)] dr, w_x \right) + \left( \int_0^t h(r)dr \right) ||w_x||^2, \ t \geq 0$$

implies that

$$\tilde{H}_3'(t) = -G \xi ||w_x||^2 - G (\psi_x, w) - G \left( \int_0^t h(t-r)[w_x(t) - w_x(r)] dr, w_x \right) + \rho ||w_t||^2, \ t > 0.$$}

As

$$G (\psi_x, w) \leq \frac{\xi}{2} ||w_x||^2 + \frac{1}{2\varepsilon} ||\psi_x||^2$$

$$\leq \frac{\xi}{2} ||w_x||^2 + \frac{1}{2\varepsilon} \left[ (1 + \eta) ||(3s - \psi)_x||^2 + 9 \left( 1 + \frac{1}{\eta} \right) ||s_x||^2 \right], \ \eta > 0$$

we get

$$\tilde{H}_3'(t) = -G \left( \frac{\xi}{2} - \varepsilon_0 \right) ||w_x||^2 + \frac{G(1-\xi)}{4\varepsilon_0} (h \square w_x)(t) + \rho ||w_t||^2 + \frac{G}{2\varepsilon} \left[ (1 + \eta) ||(3s - \psi)_x||^2 + 9 \left( 1 + \frac{1}{\eta} \right) ||s_x||^2 \right], \ t > 0.$$
Proof. Using the second equation of (3), we find
\[
I_{\rho} \frac{d}{dt} (3s_t - \psi_t, 3s - \psi) = I_{\rho} \|3s_t - \psi_t\|^2 + G(\psi - w_x, 3s - \psi)
- \left( \int_0^t h(t - r) (3s - \psi)_x(r) dr, 3s - \psi \right) - \|3s_x - \psi_x\|^2
= I_{\rho} \|3s_t - \psi_t\|^2 + G(\psi - w_x, 3s - \psi) - \left( 1 - \int_0^t h(r) dr \right) \|3s_x - \psi_x\|^2
- \left( \int_0^t h(t - r) [(3s - \psi)_x(t) - (3s - \psi)_x(r)] dr, (3s - \psi)_x \right), \ t > 0.
\]
Then
\[
I_{\rho} \frac{d}{dt} (3s_t - \psi_t, 3s - \psi) \leq I_{\rho} \|3s_t - \psi_t\|^2 + G(\psi - w_x, 3s - \psi)
- \left( 1 - \int_0^t h(r) dr \right) \|3s_x - \psi_x\|^2 + \varepsilon_0 \|3s_x - \psi_x\|^2
+ \frac{1}{4\varepsilon_0} \left( \int_0^t h(r) dr \right) (h \Box (3s - \psi)_x(t)). \tag{26}
\]
On the other hand
\[
\rho \frac{d}{dt} (w_t, Z) = \rho(w_{tt}, Z) + \rho(w_t, Z_t)
= -G((\psi - w_x)_x, Z) - G \left( \int_0^t h(t - r) w_{xx}(r) dr, Z \right) + \rho(w_t, Z_t)
= G(\psi - w_x, Z_x) - G \left( \int_0^t h(t - r) w_{xx}(r) dr, Z \right) + \rho(w_t, Z_t)
= -G(\psi - w_x, 3s - \psi) - G \left( \int_0^t h(t - r) [w_x(t) - w_x(r)] dr, Z_x \right)
+ G \left( \int_0^t h(r) dr \right) (w_x, Z_x) + \rho(w_t, Z_t)
\leq -G(\psi - w_x, 3s - \psi) + (\varepsilon_3 + \varepsilon_0) G \|3s - \psi\|_x \|w_x\|^2 + \frac{\rho}{4\varepsilon_3} \left( \int_0^t h(r) dr \right) \|w_x\|^2
+ \frac{\rho}{4\varepsilon_3} \left( \int_0^t h(r) dr \right) (h \Box w_x(t)) + \varepsilon_4 \|3s - \psi\|_x \|w_t\|^2 + \frac{\rho^2}{4\varepsilon_3} \|w_t\|^2, \ t \geq 0 \tag{27}
\]
where \(\varepsilon_0, \varepsilon_3, \varepsilon_4 > 0\). Therefore (26) and (27) lead to
\[
\dot{H}_2(t) \leq \left[ (\varepsilon_3 + \varepsilon_0) G - \left( 1 - \int_0^t h(r) dr \right) + \varepsilon_0 \right] \|3s_x - \psi_x\|^2
+ (I_{\rho} + \varepsilon_4) \|3s_t - \psi_t\|^2 + \frac{1}{4\varepsilon_0} \left( \int_0^t h(r) dr \right) (h \Box (3s - \psi)_x(t))
\]
\[
+ \frac{\rho}{4\varepsilon_3} \left( \int_0^t h(r) dr \right) (h \Box w_x(t)) + \frac{\rho}{4\varepsilon_3} \left( \int_0^t h(r) dr \right) \|w_x\|^2 + \frac{\rho^2}{4\varepsilon_3} \|w_t\|^2
\]
or
\[
\dot{H}_2(t) \leq \left[ (\varepsilon_3 + \varepsilon_0) G - \xi + \varepsilon_0 \right] \|3s_x - \psi_x\|^2 + (I_{\rho} + \varepsilon_4) \|3s_t - \psi_t\|^2
+ \frac{1}{4\varepsilon_0} \left( \int_0^t h(r) dr \right) (h \Box (3s - \psi)_x(t)) + \frac{\rho}{4\varepsilon_3} \left( \int_0^t h(r) dr \right) (h \Box w_x(t))
\]
\[
+ \frac{\rho}{4\varepsilon_3} \left( \int_0^t h(r) dr \right) \|w_x\|^2 + \frac{\rho^2}{4\varepsilon_3} \|w_t\|^2, \ t > 0.
\]
\[\square\]

**Theorem 2.** The energy of system (3)-(4) goes to zero exponentially as time goes to infinity provided that the \(L^1\)-norm of \(h\) is small enough (see (32), (33) and (b))
in Proposition 3). That is there exist two positive constant $A$ and $b$ such that

$$E_2(t) \leq Ae^{-bt}, \quad t \geq 0.$$  

**Proof.** Taking into account all the previous lemmas, as well as Proposition 2, we obtain for

$$L(t) = E_2(t) + \sum_{i=1}^{5} \delta_i \dot{H}_i(t)$$

$$\dot{L}(t) \leq -4\beta \|s_i\|^2 + \frac{\xi}{2}(h' \square w_x)(t) + \frac{1}{2}(h' \square (3s - \psi)_x)(t) + \delta_1 \left( I_p + \frac{\beta^2}{4\varepsilon_1} + \frac{\varepsilon^2}{4\varepsilon_1} \right) \|s_i\|^2 + \delta_1 \left[ \frac{\xi}{2} \varepsilon_1 - \frac{1}{2} \gamma \varepsilon_2 \right] \|s\|^2$$

$$+ \delta_1 \left[ -\xi + \frac{9G^2(\xi)}{4\varepsilon_1} \right] \|s_x\|^2 + \delta_1 \frac{G^2(\xi)}{4\varepsilon_1} \|\psi - 3s_x\|^2 + \delta_1 \frac{G^2(\xi)}{4\varepsilon_1} (h \square w_x)(t)$$

$$+ \delta_1 \xi_1 \|w_t\|^2 + \delta_2 (\varepsilon_3 + \varepsilon_0) G - \xi + \varepsilon_0 \|3s_x - \psi_x\|^2 + \delta_2 (I_p + \varepsilon_4) \|3s_t - \psi_t\|^2$$

$$+ \frac{\xi}{2\varepsilon_1} \int_0^t h(r) dr \left( h \square (3s - \psi)_x)(t) + \frac{\xi}{2\varepsilon_1} \int_0^t h(r) dr \right) (h \square w_x)(t) + \frac{\xi^2}{4\varepsilon_1} \|w_t\|^2$$

$$+ \frac{\xi}{2\varepsilon_1} \left( \int_0^t h(r) dr \right) \|w_x\|^2 - \delta_3 G \left( \frac{\xi}{2} - \varepsilon_0 \right) \|w_x\|^2 + \delta_3 G \left( \frac{\xi}{2} - \varepsilon_0 \right) \|w_x\|^2 + \delta_3 G \left( \frac{\xi}{2} - \varepsilon_0 \right) \|w_x\|^2 + \delta_3 G \left( \frac{\xi}{2} - \varepsilon_0 \right) \|w_x\|^2$$

or

$$\dot{L}(t) \leq \left[ -4\beta + \frac{\xi}{2\varepsilon_1} \left( I_p + \frac{\beta^2}{4\varepsilon_1} + \frac{\varepsilon^2}{4\varepsilon_1} \right) \right] \|s_i\|^2 + \delta_1 \left[ \frac{\xi}{2} \varepsilon_1 - \frac{1}{2} \gamma \varepsilon_2 \right] \|s\|^2$$

$$+ \left\{ \delta_3 \rho + \delta_1 \xi_1 + \delta_2 \frac{\beta^2}{4\varepsilon_1} + \delta_2 \rho (\xi_1 - \varepsilon_0) \right\} \|w_t\|^2 + \delta_4 \xi_0 + \delta_5 \xi_1 \|\psi - w_t\|^2$$

$$+ \left[ \frac{\xi}{2\varepsilon_1} \int_0^t h(r) dr \right] \|w_x\|^2 - \delta_3 G \left( \frac{\xi}{2} - \varepsilon_0 \right) \|w_x\|^2 + \delta_3 G \left( \frac{\xi}{2} - \varepsilon_0 \right) \|w_x\|^2$$

$$+ \left( \frac{\xi}{2\varepsilon_1} \int_0^t h(r) dr \right) \|w_x\|^2 - \delta_3 G \left( \frac{\xi}{2} - \varepsilon_0 \right) \|w_x\|^2 + \delta_3 G \left( \frac{\xi}{2} - \varepsilon_0 \right) \|w_x\|^2$$

$$+ \left( \frac{\xi}{2\varepsilon_1} \int_0^t h(r) dr \right) \|w_x\|^2 - \delta_3 G \left( \frac{\xi}{2} - \varepsilon_0 \right) \|w_x\|^2 + \delta_3 G \left( \frac{\xi}{2} - \varepsilon_0 \right) \|w_x\|^2$$

$$+ \left( \frac{\xi}{2\varepsilon_1} \int_0^t h(r) dr \right) \|w_x\|^2 - \delta_3 G \left( \frac{\xi}{2} - \varepsilon_0 \right) \|w_x\|^2 + \delta_3 G \left( \frac{\xi}{2} - \varepsilon_0 \right) \|w_x\|^2$$

$$+ \left( \frac{\xi}{2\varepsilon_1} \int_0^t h(r) dr \right) \|w_x\|^2 - \delta_3 G \left( \frac{\xi}{2} - \varepsilon_0 \right) \|w_x\|^2 + \delta_3 G \left( \frac{\xi}{2} - \varepsilon_0 \right) \|w_x\|^2$$

$$+ \left( \frac{\xi}{2\varepsilon_1} \int_0^t h(r) dr \right) \|w_x\|^2 - \delta_3 G \left( \frac{\xi}{2} - \varepsilon_0 \right) \|w_x\|^2 + \delta_3 G \left( \frac{\xi}{2} - \varepsilon_0 \right) \|w_x\|^2$$

$$+ \left( \frac{\xi}{2\varepsilon_1} \int_0^t h(r) dr \right) \|w_x\|^2 - \delta_3 G \left( \frac{\xi}{2} - \varepsilon_0 \right) \|w_x\|^2 + \delta_3 G \left( \frac{\xi}{2} - \varepsilon_0 \right) \|w_x\|^2$$

$$+ \left( \frac{\xi}{2\varepsilon_1} \int_0^t h(r) dr \right) \|w_x\|^2 - \delta_3 G \left( \frac{\xi}{2} - \varepsilon_0 \right) \|w_x\|^2 + \delta_3 G \left( \frac{\xi}{2} - \varepsilon_0 \right) \|w_x\|^2$$

$$+ \left( \frac{\xi}{2\varepsilon_1} \int_0^t h(r) dr \right) \|w_x\|^2 - \delta_3 G \left( \frac{\xi}{2} - \varepsilon_0 \right) \|w_x\|^2 + \delta_3 G \left( \frac{\xi}{2} - \varepsilon_0 \right) \|w_x\|^2$$

Note that

$$\|\psi - w_x\|^2 \leq 2 \|\psi_x\|^2 + 2 \|w_x\|^2$$

$$\leq 2 \left(1 + \eta \right) \|\psi_x\|^2 + 2 \|w_x\|^2.$$
Therefore
\[ L'(t) \leq \left[ -4 \beta + \delta_1 \left( I_{\rho} + \frac{\delta_2^2}{\delta_4^2} + \frac{\varepsilon_1^2}{2\gamma} \right) \right] \|s_t\|^2 + \delta_1 \left( \frac{\varepsilon_1}{3} \varepsilon_1 - \frac{\varepsilon_2^2}{3} + \varepsilon_2 \right) \|s\|^2 + \left\{ \begin{array}{l}
\delta_1 \left( \frac{\varepsilon_1}{3} + \delta_2^2 + \frac{\delta_3}{\delta_4} \right) \|\varepsilon_t\|^2 + \delta_3 \delta_{\rho} (\varepsilon_1 - \eta_0) \|s_t - \psi_t\|^2 \\
\delta_2 (I_{\rho} + \varepsilon_t) + \delta_4 I_{\rho} (\varepsilon_t - \eta_0) \|s_t - \psi_t\|^2 \\
\left[ \delta_1 \left( \frac{\varepsilon_1}{3} + \frac{9G^2(1-\xi)^2}{2\gamma} \right) + \frac{9G^2}{2\gamma} \left( 1 + \frac{1}{\eta} \right) + 18 \left( 1 + \frac{1}{\eta} \right) \delta_4 \varepsilon_0 + \delta_5 \varepsilon_1 G \right] \|s_t\|^2 \\
\delta_2 G^2(1-\xi) + \delta_5 \varepsilon_1 G (1 - \xi) - \delta_5 \varepsilon_1 (1 - \xi)^2 + 2 \delta_4 \varepsilon_0 + \delta_5 \varepsilon_1 G \right] \|w_t\|^2 \\
\delta_2 \left( \frac{G^2(1-\xi)}{\varepsilon_t} \right) + \delta_4 (1 - \xi) \left[ 1 + \frac{G^2}{4\varepsilon_t} (2 - \eta_0) \right] + \delta_5 \varepsilon_0 \delta_4 (3 - \eta_0 + \eta_0) + \delta_5 \varepsilon_0 (3 - \eta_0 + \eta_0) \\
\right. \\
\times (h \Box w_t)(t) + \frac{\delta_2 G^2(1-\xi)}{\varepsilon_t} + \delta_4 (1 - \xi) \left[ 1 + \frac{G^2}{4\varepsilon_t} (2 - \eta_0) \right] + \frac{\delta_5 \varepsilon_0 \delta_4 (3 - \eta_0 + \eta_0)}{4\varepsilon_t} - \frac{\delta_1}{2} \\
\times (h \Box (3\varepsilon - \psi_x))(t), \ t \geq t_0 > 0. \\
\right) \]

We need all the coefficients in (28) to be negative. Ignoring for the moment \( \varepsilon_1, \varepsilon_0 \), the first term and the last two terms in the right hand side of (28), we impose

\[
\begin{align*}
\varepsilon_2 &< \frac{4\gamma}{9}, \\
\delta_3 \frac{\varepsilon_2}{\varepsilon_4} + \delta_3 \rho &< \delta_5 h_0 \rho, \\
\delta_2 \left( I_{\rho} + \varepsilon_4 \right) &< \delta_4 I_{\rho} h_0, \\
\frac{9G^2}{2\gamma} \frac{\delta_3}{\delta_4} \left( 1 + \frac{1}{\eta} \right) &< \delta_1 \left[ \frac{\varepsilon_1}{3} - \frac{9G^2(1-\xi)^2}{2\gamma} \right], \\
\delta_2 \frac{G^2(1-\xi)^2}{\varepsilon_t^2} + \frac{\delta_2 G}{2\gamma} (1 + \eta) &< \delta_2 \left( \frac{\varepsilon_1}{3} - \frac{9G^2(1-\xi)^2}{2\gamma} \right), \\
\delta_2 \left( \frac{G^2(1-\xi)}{\varepsilon_t} \right) &< \delta_2 \left( \frac{\varepsilon_1}{3} - \frac{9G^2(1-\xi)^2}{2\gamma} \right).
\end{align*}
\]

Taking \( \varepsilon_2 = \gamma, \varepsilon_3 = \frac{\xi}{G^2} \), \( \delta_3 \) and \( \delta_5 \) large enough allows us to concentrate on the last three relations in (29). Note that

\[
\frac{\xi}{G^2} - \frac{9G^2(1-\xi)}{2\varepsilon_t} = \xi - \frac{9G^2(1-\xi)^2}{2\gamma} \geq \frac{\xi}{2}
\]

if

\[
1 + \frac{2\gamma}{9G^2} - \sqrt{\left( 1 + \frac{2\gamma}{9G^2} \right)^2 - 1} < \xi < 1.
\]

We are left with

\[
\begin{align*}
\frac{9G^2}{2\gamma} \frac{\delta_3}{\delta_4} \left( 1 + \frac{1}{\eta} \right) &< \frac{\delta_1}{2}, \\
\delta_1 \frac{G^2(1-\xi)^2}{\varepsilon_t^2} + \frac{G^2}{\varepsilon_t} \delta_3 (1 + \eta) &< \xi \delta_2, \\
\delta_2 \left( \frac{G^2(1-\xi)}{\varepsilon_t} \right) &< \frac{\varepsilon_1}{3} \delta_3.
\end{align*}
\]

Take \( \delta_2 = \frac{\varepsilon_2^2}{2G(1-\xi)} \delta_3 \) and \( \eta = 1 \)

\[
\begin{align*}
\delta_3 &< \frac{\delta_1}{2}, \\
\delta_2 \frac{G^2(1-\xi)^2}{\varepsilon_t^2} + \frac{G^2}{\varepsilon_t} \delta_3 &< \frac{\varepsilon_1^3}{2G(1-\xi)} \delta_3.
\end{align*}
\]

If \( \frac{2G}{\xi} < \frac{\varepsilon_2^2}{2G(1-\xi)} \) i.e.

\[
4G^2 (1 - \xi)^2 < \xi^4
\]

and

\[
18G^3 (1 - \xi)^2 < \xi^2(\frac{\varepsilon_2^3}{2G(1-\xi)} - \frac{2G}{\xi})
\]
which is valid when $\xi$ is close to 1 that is $1 - \xi$ is small enough, we see that (31) is possible. We select, say $\delta_1$ in terms of $\delta_3$. Then we consider (29) and choose $\delta_4$ and $\delta_5$ in terms of $\delta_3$ (any value for $\varepsilon_4$ would do, for instance $\varepsilon_4 = 1$). Finally, we turn to (28) and select $\varepsilon_1$, $\varepsilon_0$ and $\delta_3$ so that the ignored terms have negative coefficients. The exponential decay of the functional $\tilde{L}$ and thereafter (by equivalence) of the energy $E_2$ is derived on $[t_0, +\infty)$. On the interval $[0, t_0]$ if follows from the continuity.

Remark. 1- It is apparent from the argument that the conditions on $\xi$ may be improved. We have not managed any effort in this regards. Of course removing these conditions would be nice.

2- Our main goal is the handling of the interfacial slip and the stabilization of the system. We have assumed the standard conditions on the relaxation function and did not try to enlarge this class. We refer the reader to [6,13-15,19,26-31] for this issue.

3- The relaxation function $h$ may be different in the first two equations of (3).

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