DIFFERENT VERSIONS OF THE NERVE THEOREM AND RAINBOW SIMPLICIES

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Abstract. Given a simplicial complex and a collection of subcomplexes covering it, the nerve theorem, a fundamental tool in topological combinatorics, guarantees a certain connectivity of the simplicial complex when connectivity conditions on the intersection of the subcomplexes are satisfied.

We show that it is possible to extend this theorem by replacing some of these connectivity conditions on the intersection of the subcomplexes by connectivity conditions on their union. While this is interesting for its own sake, we use this extension to generalize in various ways the Meshulam lemma, a powerful homological version of the Sperner lemma. We also prove a generalization of the Meshulam lemma that is somehow reminiscent of the polytopal generalization of the Sperner lemma by De Loera, Peterson, and Su. For this latter result, we use a different approach and we do not know whether there is a way to get it via a nerve theorem of some kind.

1. Introduction

The nerve theorem is a fundamental result in topological combinatorics. It has many applications, not only in combinatorics, but in category and homotopy theory and also in applied and computational topology. Roughly speaking, it relates the topological “complexity” of a simplicial complex to the topological “complexity” of the intersection complex of a “nice” cover of it. Stating the nerve theorem with conditions on intersection seems to be somehow dictated by its very nature. It might thus come as a surprise that a nerve theorem for unions also holds.

In this paper, we prove such a theorem. Actually, we prove a theorem that interpolates between a version of the nerve theorem with intersections and a version with unions. Given a simplicial complex $X$ and a finite collection of subcomplexes $\Gamma$, the nerve of $\Gamma$, denoted $N(\Gamma)$, is the simplicial complex with vertices the subcomplexes in $\Gamma$ and whose simplices are the subcollections of $\Gamma$ with a nonempty intersection.

Theorem 1. Consider a simplicial complex $X$ with a finite collection $\Gamma$ of subcomplexes such that $\bigcup \Gamma = X$. Let $k$ and $\ell$ be two integers such that $-1 \leq k \leq \ell < |\Gamma|$. Suppose that the following two conditions are satisfied:

1. $\tilde{H}_{k-|\sigma|}(\bigcap \sigma) = 0$ for every $\sigma \in N(\Gamma)$ of dimension at most $k$.
2. $\tilde{H}_{|\sigma|-2}(\bigcup \sigma) = 0$ for every $\sigma \in N(\Gamma)$ of dimension at least $k+1$ and at most $\ell$.

Then $\tilde{H}_\ell(N(\Gamma))$ is isomorphic to a subgroup of $\tilde{H}_\ell(X)$.

The case $k = \ell$ was obtained in [13] and is a generalization of a classical version of the homological nerve theorem (see [11, Theorem 6.1]). There are actually many “classical”

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versions, some of them with a connectivity condition in place of the acyclicity condition. It seems that the oldest reference to a nerve theorem with an acyclicity condition is due to Leray [9]. The case \( k = -1 \) is the nerve theorem for unions mentioned above.

The second purpose of this paper is to provide a generalization of a related result – Meshulam’s lemma [11, Proposition 1.6] and [10, Theorem 1.5] – which has several applications in combinatorics, such as the generalization of Edmonds’ intersection theorem by Aharoni and Berger [1]. Meshulam’s lemma is a Sperner-lemma type result, dealing with coloured simplicial complexes and colourful simplices, and in which the classical boundary condition of the Sperner lemma is replaced by an acyclicity condition. Its original proof relies on a certain version of the nerve theorem and we show that Theorem 1 can be used in the same vein to prove some variations of Meshulam’s lemma. We also prove – with a completely different approach – the following generalization of this lemma, which can be seen as a homological counterpart of the polytopal Sperner lemma by De Loera, Peterson, and Su [5], in a same way Meshulam’s lemma is a homological counterpart of the classical Sperner lemma. It can also be seen as a homological counterpart of Musin’s Sperner-type results for pseudomanifolds [15] and of Theorem 4.7 in the paper by Asada et al. [2]. We leave as an open question the existence of a proof based on a nerve theorem of some kind.

We recall that a pseudomanifold is a simplicial complex that is pure, non-branching (each ridge is contained in exactly two facets), and strongly connected (the dual is connected). A colourful simplex in a simplicial complex whose vertices are partitioned into subsets \( V_0, \ldots, V_m \) is a simplex with at most one vertex in each \( V_i \). Given a simplicial complex \( K \) and a subset \( U \) of its vertices, \( K[U] \) is the subcomplex induced by \( U \), i.e. the simplicial complex whose simplices are exactly the simplices of \( K \) whose vertices are all in \( U \).

**Theorem 2.** Consider a simplicial complex \( K \) whose vertices are partitioned into \( m \) subsets \( V_0, \ldots, V_m \) and let \( M \) be an orientable \( d \)-dimensional pseudomanifold with vertex set \( \{0, \ldots, m\} \). Suppose that \( \tilde{H}_{|\sigma|-2}(\bigcup_{\sigma \in M} V_i) = 0 \) for every \( \sigma \in M \). If \( \tilde{H}_d(K) = 0 \) as well, then the number of \((d + 1)\)-dimensional colourful simplices in \( K \) is at least \( m - d \).

In this paper, we use reduced homology with coefficients in an arbitrary field \( \mathbb{F} \). This assumption holds for all statements throughout the paper. In only one proof, integer coefficients will be used instead (in the proof of Lemma 7, in order to apply the Hurewicz theorem). At the end of our paper, a brief discussion on the statements that still hold when passing to integer coefficients will be provided. The simplicial complexes are all abstract and for a simplicial complex \( X \), we say that \( \tilde{H}_{-1}(X) = 0 \) if and only if \( X \) is nonempty.

Theorems 1 and 2 are respectively proved in Sections 3 and 5. Theorem 1 is proved by induction. The base case, which is the case \( k = -1 \) (“nerve theorem for unions”), is proved in Section 2. Applications of Theorem 1 are proposed in Section 4. Some of these applications are new generalizations of Meshulam’s lemma.

**2. A Nerve Theorem for Unions**

In this section, we prove the following nerve theorem for unions.

**Theorem 3.** Consider a simplicial complex \( X \) with a finite collection \( \Gamma \) of subcomplexes such that \( \bigcup \Gamma = X \). Let \( \ell \) be an integer such that \(-1 \leq \ell < |\Gamma|\). Suppose that \( \tilde{H}_{|\sigma|-2}(\bigcup \sigma) = 0 \) for every \( \sigma \in N(\Gamma) \) of dimension at most \( \ell \). Then \( \tilde{H}_\ell(N(\Gamma)) \) is isomorphic to a subgroup of \( \tilde{H}_\ell(X) \).
It is the special case of Theorem 1 when $k = -1$ and it will be used in Section 3 to prove this latter theorem in its full generality.

If $\ell = -1$, the proof is easy: if $X$ is nonempty, then $N(\Gamma)$ is nonempty as well since $\Gamma$ covers $X$. Let us thus consider the case where $\ell \geq 0$. The general structure of the proof, in particular the use of a carrier argument, shares similarities with the proof Björner proposed for his generalization of the nerve theorem [3].

**Lemma 4.** There exists an augmentation-preserving chain map $f_\sharp: C(N(\Gamma)) \to C(sd X)$ such that for any $s$-dimensional simplex $\sigma \in N(\Gamma)$ with $0 \leq s \leq \ell$, the chain $f_\sharp(\sigma)$ is carried by $sd \bigcup \sigma$.

**Proof.** Given a 0-dimensional simplex $\{A\}$ of $N(\Gamma)$, the subcomplex is nonempty by definition (condition for $|\sigma| = 1$) and there exists thus a vertex $v_A$ in $A$. We define $f_\sharp(\{A\})$ to be $\{v_A\}$. Note that this latter is a vertex of $sd A$. Suppose now that $f_\sharp(z)$ has been defined for every chain $z \in C_i(N(\Gamma))$ for $i$ up to $s - 1 < \ell$ and satisfies $\partial f_\sharp(z) = f_\sharp(\partial z)$ (where we use the augmentation map if $z$ is a 0-chain). Suppose moreover that $f_\sharp(\sigma) \in C_i(sd \bigcup \sigma)$ for every $i$-dimensional simplex $\sigma \in N(\Gamma)$ for $i$ up to $s - 1$. Consider an $s$-dimensional simplex $\sigma$ of $N(\Gamma)$. The chain $f_\sharp(\partial \sigma)$ has been defined, it belongs to $C_{s-1}(sd \bigcup \sigma)$ and $\partial f_\sharp(\partial \sigma) = 0$. Since $s \leq \ell$, we have $\tilde{H}_{\dim \sigma - 1}(\bigcup \sigma) = 0$ and there exists a chain in $C_s(sd \bigcup \sigma)$ whose boundary is $f_\sharp(\partial \sigma)$. We define $f_\sharp(\sigma)$ to be this chain.

For any simplex $\tau$ of $X$, we set $\lambda(\tau) = \{A \in \Gamma: \tau \in A\}$.

**Lemma 5.** The map $\lambda$ is a simplicial map $sd X \to sd N(\Gamma)$.

**Proof.** Let $\tau$ be a simplex of $X$ and $\tau'$ any subset of $\tau$. We obviously have $\lambda(\tau) \subseteq \lambda(\tau')$. The map $\lambda$ reverses the order in the posets. \hfill $\square$

We are going to show that there is a chain homotopy between $\lambda_\sharp \circ f_\sharp$ and $sd_\sharp$. This will be done with the help of the acyclic carrier theorem, which we state here for sake of completeness.

An *acyclic carrier* from a simplicial complex $K$ to a simplicial complex $L$ is a function $\Psi$ that assigns to each simplex $\sigma$ in $K$ a subcomplex $\Psi(\sigma)$ of $L$ such that

- $\Psi(\sigma)$ is nonempty and acyclic
- If $\tau$ is a face of $\sigma$, then $\Psi(\tau) \subseteq \Psi(\sigma)$.

A chain map $\mu: C(K) \to C(L)$ is *carried* by $\Psi$ if for each simplex $\sigma$, the chain $\mu(\sigma)$ is carried by the subcomplex $\Psi(\sigma)$ of $L$.

**Theorem** (Acyclic carrier theorem – short version). Let $\Psi$ be an acyclic carrier from $K$ to $L$. If $\phi$ and $\psi$ are two augmentation-preserving chain maps from $C(K)$ to $C(L)$ that are carried by $\Psi$, then $\phi$ and $\psi$ are chain-homotopic.

The acyclic carrier theorem is more general and we refer to the book by Munkres [14, Theorem 13.3] for the statement in its full generality. By the way, we borrow from this book the definition of the acyclic carrier stated above. The proof of the acyclic carrier theorem given in that book is actually for homology with integer coefficients, but it is easy yet tedious to check that the very same proof works for coefficients in any field. We are not aware of a bibliographical reference where an acyclic carrier theorem for coefficients in any field is available.
In order to apply the acyclic carrier theorem, we define for each simplex \( \sigma \in N(\Gamma) \) the subcomplex
\[
\Phi(\sigma) = \triangle \{ \sigma' \in N(\Gamma) : \sigma' \cap \sigma \neq \emptyset \}.
\]

**Lemma 6.** The map \( \Phi \) is an acyclic carrier from \( N(\Gamma) \) to \( sd N(\Gamma) \).

**Proof.** Let \( \sigma \) be a simplex of \( N(\Gamma) \). The simplicial complex \( \Phi(\sigma) \) is nonempty, and for any subset \( \omega \) of \( \sigma \), we have \( \Phi(\omega) \subseteq \Phi(\sigma) \). The only thing that remains to be proved is thus the fact that \( \Phi(\sigma) \) is acyclic. Actually, we have more: it is contractible. To see it, consider the following two simplicial maps \( g, h : \Phi(\sigma) \to \Phi(\sigma) \) defined for any vertex \( \sigma' \in V(\Phi(\sigma)) \) by \( g(\sigma') = \sigma' \cap \sigma \) and by \( h(\sigma') = \sigma \) (constant map). Seeing \( \Phi(\sigma) \) as the poset \( \{ \{ \sigma' \in N(\Gamma) : \sigma' \cap \sigma \neq \emptyset \} \subseteq \} \), we have \( g \leq \text{id} \) and \( g \leq h \). By the order homotopy lemma [6, Lemma C.3], the maps \( h \) and \( id \) are homotopic, which means that the identity map is homotopic to the constant map. \( \square \)

**Proof of Theorem 3.** We first check that both \( \lambda_2 \circ f_2 \) and \( sd_2 \) are carried by \( \Phi \). Take \( \sigma \in N(\Gamma) \) of dimension \( s \). Any simplex of \( sd X \) in the support of \( f_2(\sigma) \) is of the form \( \{ \tau_0, \ldots, \tau_s \} \) with \( \tau_0 \subseteq \cdots \subseteq \tau_s \) and \( \tau_i \in U_{A \in \sigma} A \) for all \( i \in \{0, \ldots, s\} \). In particular, \( \lambda(\tau_i) \cap \sigma \neq \emptyset \) for all \( i \) and \( \{ \lambda(\tau_0), \ldots, \lambda(\tau_s) \} \) is a simplex of \( \triangle \{ \sigma' \in N(\Gamma) : \sigma' \cap \sigma \neq \emptyset \} \). It shows that \( \lambda_2 \circ f_2 \) is carried by \( \Phi \). Any simplex in the support of \( sd_2(\sigma) \) is of the form \( \{ \sigma_0, \ldots, \sigma_s \} \) with \( \sigma_0 \subseteq \cdots \subseteq \sigma_s = \sigma \) and \( \sigma_i \cap \sigma = \sigma_i \neq \emptyset \). Thus \( \{ \sigma_0, \ldots, \sigma_s \} \) is a simplex of \( \triangle \{ \sigma' \in N(\Gamma) : \sigma' \cap \sigma \neq \emptyset \} \) and \( sd_2 \) is carried by \( \Phi \).

Since both \( \lambda_2 \circ f_2 \) and \( sd_2 \) preserve augmentation (here, we use the fact that \( f_2 \) is augmentation-preserving – see Lemma 4), we can apply the acyclic carrier theorem given above and there is a chain homotopy between \( \lambda_2 \circ f_2 \) and \( sd_2 \). A direct application of the algebraic subdivision theorem [14, Theorem 17.2] shows that \( \lambda_2 \circ f_2 \) is an isomorphism \( \tilde{H}_i(N(\Gamma)) \to \tilde{H}_i(sd N(\Gamma)) \).

(Again, Munkres proves this theorem for the case of integer coefficients, but it can be checked that the same proof works when the coefficients are taken in any field; and again, we are missing a reference.) Hence \( f_2 \) is an injective homomorphism \( \tilde{H}_i(N(\Gamma)) \to \tilde{H}_i(sd X) \), which gives the conclusion. \( \square \)

## 3. The mixed nerve theorem

The purpose of this section is to prove Theorem 1. We will proceed by induction on \( k \). The base case is given by Theorem 3 which is the special case when \( k = -1 \) (only unions are considered), proved in Section 2. A crucial ingredient in the induction is a lemma that shows how to make homology of a simplicial complex vanishes up to some dimension \( d \), while keeping homology untouched beyond \( d \), by attaching simplices of dimension at most \( d \).

**Lemma 7.** Given a simplicial complex \( K \), there always exists a way to attach simplices of dimension at most \( d \) to \( K \) to get a simplicial complex \( K' \) such that

- \( \tilde{H}_i(K') = 0 \) for all \( i \leq d - 1 \).
- \( \tilde{H}_i(K') \cong \tilde{H}_i(K) \) for all \( i \geq d \).

**Proof.** The case \( d \leq 1 \) is easy. Assume that \( d \geq 2 \).

As explained by Hatcher [7], there exists a \((d - 2)\)-connected space \( K(d - 2) \) such that \( \tilde{H}_i(K(d - 2), \mathbb{Z}) \) and \( \tilde{H}_i(K, \mathbb{Z}) \) are isomorphic for \( i \geq d - 1 \). Such a space can be obtained by attaching simplices of dimension at most \( d \) to \( K \). This implies that there exist continuous
maps $f_1, \ldots, f_\beta: S^{d-1} \to \|K(d-2)\|$ such the $f_j \ast(u)$ form a basis of $\tilde{H}_{d-1}(K(d-2))$, where $u$ is the fundamental class of $\tilde{H}_{d-1}(S^{d-1})$. (This is immediate when $d = 2$ and a consequence of a combination of the Hurewicz theorem and the universal coefficient theorem when $d \geq 3$: in this latter case $\tilde{H}_{d-1}(K(d-2))$ is isomorphic to $\pi_{d-1}(K(d-2)) \otimes \mathbb{F}$, where $\mathbb{F}$ is our coefficient field.) Since $K(d-2)$ is $(d-2)$-connected, we have moreover $\tilde{H}_i(K(d-2)) = 0$ for $i \leq d-2$.

We attach to $K(d-2)$ a simplicial complex homeomorphic to a $d$-dimensional ball, with boundary being homotopic to $f_1(u)$ (this is possible through simplicial approximation). According to the Mayer-Vietoris exact sequence, we have

$$
\tilde{H}_i(K(d-2) \cup_{f_1} B^d) \cong \begin{cases} 
\tilde{H}_i(K(d-2)) & \text{if } i \neq d-1 \\
\tilde{H}_{d-1}(K(d-2)) / \langle f_1, \ast(u) \rangle & \text{if } i = d-1.
\end{cases}
$$

(Here, we use the fact that $\mathbb{F}$ is a field and that the homology groups are then vector spaces.) In other words, we are able to keep the same homology in all dimensions, except when $i = d-1$, where we strictly reduce the rank of the homology group. We repeat this construction for $j = 2, \ldots, \beta$. We get a simplicial complex $K'$ such that

- $\tilde{H}_i(K') = 0$ for all $i \leq d-1$
- $\tilde{H}_i(K') \cong \tilde{H}_i(K(d-2))$ for all $i \geq d$.

We conclude with the help of the universal coefficient theorem: the integral homology groups of $K(d-2)$ and $K$ are isomorphic for $i \geq d-1$, thus $\tilde{H}_i(K(d-2)) \cong \tilde{H}_i(K)$ for $i \geq d$. \qed

For the proof of Theorem 1, we also need the following technical lemma.

**Lemma 8.** Consider a simplicial complex $X$ and a nonempty finite collection $\sigma$ of subcomplexes such that $\tilde{H}_{|\sigma|-|\tau|-1}(\bigcap \tau) = 0$ for every nonempty subcollection $\tau \subseteq \sigma$. Then $\tilde{H}_{|\sigma|-2}(\bigcup \sigma) = 0$.

**Proof.** We start the proof with a preliminary remark that will be used several times in the proof: we have $\bigcap \sigma \neq \emptyset$ (obtained with $\tau = \sigma$).

We prove actually that we have $\tilde{H}_{|\sigma|-2}(\bigcup \sigma') = 0$ for every nonempty subcollection $\sigma' \subseteq \sigma$. We first prove this statement in the special case when $|\sigma'| = 1$. In this case, letting $\tau = \sigma'$, the condition of the lemma imposes that $\tilde{H}_{|\sigma|-2}(\bigcap \sigma') = 0$. Since $\sigma'$ has exactly one subcomplex, we have $\bigcap \sigma' = \bigcup \sigma'$, and the conclusion follows.

For the other cases, we proceed by induction on $m = |\sigma| + |\sigma'|$ and we start with the case $m = 2$. In this case $\sigma' = \sigma$ and it is a collection of exactly one subcomplex, which is a case we have already treated.

Consider now the case $m \geq 3$. The case $|\sigma'| = 1$ being known to be true, we assume that $|\sigma'| \geq 2$. We arbitrarily pick a subcomplex $A$ in $\sigma'$. We introduce the collection

$$
\omega = \{A \cap B : B \in \sigma \setminus \{A\}\}.
$$

(Note that $A \cap B \neq \emptyset$ in this formula, since $\bigcap \sigma \neq \emptyset$.) For every subcollection $\omega_0 \subseteq \omega$, there is a $\sigma'' \subseteq \sigma \setminus \{A\}$ such that $\omega_0 = \{A \cap B : B \in \sigma''\}$. It means that for every nonempty subcollection $\omega_0 \subseteq \omega$, we have $\tilde{H}_{|\sigma|-|\sigma''|+1-1}(\bigcap \omega_0) = 0$ by the condition of the lemma applied with $\tau = \sigma'' \cup \{A\}$, and thus $\tilde{H}_{|\sigma|-|\omega_0|-1}(\bigcap \omega_0) = 0$. The collection $\omega$ satisfies thus the condition of the lemma. The collection $\omega' = \{A \cap B : B \in \sigma' \setminus \{A\}\}$ is a nonempty
subcollection of $\omega$ and we have $|\omega| + |\omega'| = m - 2$. Hence the induction applies and we get $\widetilde{H}_{|\sigma|-2}(\bigcup \omega') = 0$, which means

$$\widetilde{H}_{|\sigma|-3}\left(A \cap \left(\bigcup_{B \in \sigma' \setminus \{A\}} B\right)\right) = 0.$$ 

By the Mayer-Vietoris exact sequence of the pair $\left(\bigcup_{B \in \sigma' \setminus \{A\}} B, A\right)$, we have that $\widetilde{H}_{|\sigma|-2}(\bigcup \sigma') = 0$, as required, because the condition of the lemma with $\tau = \{A\}$ imposes that $\widetilde{H}_{|\sigma|-2}(A) = 0$ and induction shows that $\widetilde{H}_{|\sigma|-2}\left(\bigcup_{B \in \sigma' \setminus \{A\}} B\right) = 0$. (Since $\bigcap \tau \neq \emptyset$, we have $A \cap \left(\bigcup_{B \in \sigma' \setminus \{A\}} B\right) \neq \emptyset$ and the Mayer-Vietoris exact sequence holds in the reduced case.) \hfill \square

**Proof of Theorem 1.** The proof is by induction on $k$. The integer $\ell$ is considered as fixed. If $k = -1$, then it follows directly from Theorem 3. Suppose our theorem is true for $k - 1$. We shall prove it for $k$.

Consider a simplicial complex $X$ with a finite collection $\Gamma$ of subcomplexes as in the statement of the theorem we want to prove.

**Claim.** For every nonnegative integer $j \leq k - 1$, there exists a simplicial complex $X(j)$, obtained by attaching to $X$ simplices of dimension at most $k - j - 1$, and a finite collection $\Gamma(j)$ of subcomplexes such that $\bigcup \Gamma(j) = X(j)$ and such that

a) $N(\Gamma) = N(\Gamma(j))$.

b) $\widetilde{H}_{k-|\tau|}\left(\bigcap \tau\right) = 0$ for every $\tau \in N(\Gamma(j))$ of dimension at most $k - 1$.

c) $\widetilde{H}_{|\tau|-2}\left(\bigcup \tau\right) = 0$ for every $\tau \in N(\Gamma(j))$ of dimension at least $k + 1$ and at most $\ell$.

d) $\widetilde{H}_{k-|\tau|-1}\left(\bigcap \tau\right) = 0$ for every $\tau \in N(\Gamma(j))$ of dimension at least $j$ and at most $k - 1$.

We prove the claim by decreasing induction on $j$, starting with the base case $j = k - 1$. This case is obviously true since in that case, we do not even have to add any simplex, and we set $\Gamma(k - 1) = \Gamma$.

Consider the case $j \leq k - 2$. We start with $X(j + 1)$ and $\Gamma(j + 1)$, which we know to exist. Consider a $\sigma$ in $N(\Gamma(j + 1))$ of dimension exactly $j$ such that $\widetilde{H}_{k-j-2}\left(\bigcap \sigma\right) \neq 0$. If such a simplex does not exist, we are done. By Lemma 7, we can attach a collection $\mathcal{C}$ of simplices of dimension at most $k - j - 1$ to $\bigcap \sigma$ so that

$$\widetilde{H}_{k-j-2}\left(\bigcap \sigma'\right) = 0 \quad \text{and} \quad \widetilde{H}_{k-j-1}\left(\bigcap \sigma'\right) = 0,$$

where $\sigma' = \{A \cup C : A \in \sigma\}$. We get the right-hand side equality as a consequence of $\widetilde{H}_{k-j-1}\left(\bigcap \sigma'\right) = \widetilde{H}_{k-j-1}\left(\bigcap \sigma\right)$. Define

$$X' = X(j + 1) \cup \mathcal{C} \quad \text{and} \quad \Gamma' = (\Gamma(j + 1) \setminus \sigma) \cup \sigma'.$$

Property a) is automatically satisfied for $\Gamma'$: if $\tau \setminus \sigma \neq \emptyset$, then the corresponding simplex $\tau'$ in $\Gamma'$ is such that $\bigcap \tau = \bigcap \tau'$; if $\tau \subseteq \sigma$, then both $\bigcap \tau$ and $\bigcap \tau'$ are nonempty.

Consider a simplex $\tau' \in N(\Gamma')$ of dimension at most $k - 1$. Denote by $\tau$ the corresponding simplex in $\Gamma(j + 1)$. If $|\tau'| \leq j$, property b) is satisfied since we have added simplices of dimension at most $k - j - 1$. If $|\tau'| \geq j + 1$, either $\tau = \sigma$, in which case $\widetilde{H}_{k-|\tau|}\left(\bigcap \tau'\right) = 0$, or $\tau \neq \sigma$, in which case as above $\bigcap \tau = \bigcap \tau'$. In both cases, property b) is satisfied.
Property \( c \) is satisfied because of the dimension of the attached simplices.

We repeat this operation as many times as necessary to satisfy property \( d \). Note that when we attach \( \mathcal{C} \), we do not alter the satisfaction of property \( d \) for the simplices \( \tau \) that already satisfy it: as above, when \( \tau \neq \sigma \), we have \( \bigcap \tau = \bigcap \tau' \).

At the end of the process, we get a simplicial complex \( X(j) \) and a finite collection \( \Gamma(j) \) of subcomplexes covering it, so that properties \( a \), \( b \), \( c \), and \( d \) are simultaneously satisfied.

The claim is proved.

We can now finish the proof of Theorem 1. Apply the claim for \( j = 0 \). It ensures the existence of a simplicial complex \( X(0) \) and a collection \( \Gamma(0) \) of subcomplexes covering it, with the properties \( a)–d) \) satisfied. We want to apply Theorem 1 for \( k - 1 \). Consider a \( \sigma \in \text{N}(\Gamma(0)) \) of dimension at most \( k - 1 \). We have \( \tilde{H}_{k-|\sigma|-1}(\bigcap \sigma) = 0 \) because of property \( d \).

Together with \( c \), it implies that condition \( (1) \) is satisfied. Consider now a \( \sigma \in \text{N}(\Gamma(0)) \) of dimension exactly \( k \). Every strict subset \( \tau \) of \( \sigma \) is such that \( \tilde{H}_{k-|\tau|}(\bigcap \tau) = 0 \) because of \( b \), and \( \tilde{H}_{-1}(\bigcap \sigma) = 0 \) because \( \bigcap \sigma \neq \emptyset \). Lemma 8 implies then that \( \tilde{H}_{-1}(\bigcup \sigma) = 0 \). Hence condition \( (2) \) is satisfied.

The simplicial complex \( X(0) \) and the collection \( \Gamma(0) \) satisfy the induction hypothesis for \( k - 1 \) and therefore \( \tilde{H}_\ell(\text{N}(\Gamma(0))) = \tilde{H}_\ell(\text{N}(\Gamma)) \) is isomorphic to a subgroup of \( \tilde{H}_\ell(X(0)) \). Since \( X(0) \) has been obtained from \( X \) by attaching simplices of dimension at most \( k - 1 < \ell \), we have \( \tilde{H}_\ell(X(0)) = \tilde{H}_\ell(X) \), and the conclusion follows.

\[ \Box \]

4. Applications of the mixed nerve theorem

4.1. Homological Helly-type results. We start our applications with the following Helly-type result.

**Theorem 9.** Consider a simplicial complex \( X \) with a finite collection \( \Gamma \) of subcomplexes. Let \( k \) be an integer such that \(-1 \leq k \leq |\Gamma| - 2\). Suppose that the following two conditions are satisfied for every subcollection \( \Gamma' \subseteq \Gamma \):

1. \( \tilde{H}_{k-|\Gamma'|}(\bigcap \Gamma') = 0 \) whenever \( 0 \leq |\Gamma'| \leq k + 1 \).
2. \( \tilde{H}_{|\Gamma'|-2}(\bigcup \Gamma') = 0 \) whenever \( k + 2 \leq |\Gamma'| \leq |\Gamma| \).

Then \( \bigcap \Gamma \neq \emptyset \).

**Proof.** Suppose for a contradiction that there exists a nonempty \( \Gamma' \subseteq \Gamma \) such that \( \bigcap \Gamma' = \emptyset \). Choose such a \( \Gamma' \) of minimal cardinality. Because of condition \( (1) \), we have \( |\Gamma'| \geq k + 2 \). Since \( \tilde{H}_{|\Gamma'|-2}(\bigcup \Gamma') = 0 \) by condition \( (2) \), Theorem 1 for \( X' = \bigcup \Gamma' \) and \( \ell = |\Gamma'|-2 \) implies that \( \tilde{H}_\ell(\text{N}(\Gamma')) = 0 \). By definition of \( \Gamma' \), the \( \ell \)-skeleton of \( \text{N}(\Gamma') \) is the boundary of the \((\ell+1)\)-dimensional simplex and is thus homeomorphic to \( S^\ell \). The fact that the \( \ell \)-th homology group of \( \text{N}(\Gamma') \) vanishes implies that there is at least one \((\ell+1)\)-dimensional simplex in \( \text{N}(\Gamma') \), i.e. that we have \( \bigcap \Gamma' \neq \emptyset \), which is a contradiction. \( \Box \)

Note that when \( X \) is embedded in \( \mathbb{R}^d \), the above theorem gives rise to the following “topological Helly theorem” [13] (see also [8]).

**Corollary 10.** Consider a simplicial complex \( X \) embedded in \( \mathbb{R}^d \) with a finite collection \( \Gamma \) of subcomplexes. Suppose that we have \( \tilde{H}_{d-|\Gamma'|}(\bigcap \Gamma') = 0 \) for every subcollection \( \Gamma' \subseteq \Gamma \) of cardinality at most \( d + 1 \). Then \( \bigcap \Gamma \neq \emptyset \).
4.2. Homological Sperner-type results. All results of this subsection deal with a simplicial complex $K$ whose vertices are partitioned into subsets $V_0, \ldots, V_m$ as colours and ensure the existence of a rainbow simplex under some homological condition. Formally, a *rainbow simplex* in such a simplicial complex is a simplex that has exactly one vertex in each $V_i$. We show how the nerve theorems introduced in the present paper can be used to get results of this type. We were however not able to prove Theorem 2 within this framework and a proof with a completely different approach is given in Section 5.

For $S \subseteq \{0, \ldots, m\}$, we denote by $K_S$ the subcomplex of $K$ induced by the vertices in $\bigcup_{i \in S} V_i$. The following theorem is a generalization of the Meshulam lemma cited in the introduction.

**Theorem 11.** Consider a simplicial complex $K$ whose vertices are partitioned into $m + 1$ subsets $V_0, \ldots, V_m$. Suppose that $\tilde{H}_{|S|-2}(K_S) = 0$ for every nonempty $S \subseteq \{0, \ldots, m\}$. Then there exists at least one rainbow simplex in $K$.

Meshulam’s lemma is the same statement with the stronger requirement that $K_S$ is $(|S| - 2)$-acyclic instead of $\tilde{H}_{|S|-2}(K_S) = 0$. Theorem 11 has been recently introduced by the second author [12]. We present here a new proof showing that it is a consequence of our “union” version of the nerve theorem (Theorem 3). Right after this proof, we will present a generalization of Theorem 11, with yet another proof.

We need a preliminary lemma. For an integer $i \in \{0, \ldots, m\}$, we define $A_i$ to be subcomplex of $sd K$ induced by the vertices $\tau \in V(sd K)$ such that $\tau \cap V_i \neq \emptyset$.

**Lemma 12.** For any nonempty $S \subseteq \{0, \ldots, m\}$, the simplicial complexes $\bigcup_{i \in S} A_i$ and $K_S$ have same homology groups.

**Proof.** For $\tau \in V(\bigcup_{i \in S} A_i)$, we define $\lambda(\tau)$ to be $\tau \setminus \bigcup_{i \in S} V_i$. It induces a simplicial map $\bigcup_{i \in S} A_i \to sd K_S$. Now, consider the simplicial inclusion map $j : sd K_S \to \bigcup_{i \in S} A_i$. We have clearly $j \circ \lambda = \text{id}_{sd K_S}$. We also have $j \circ \lambda \leq \text{id}_{\bigcup_{i \in S} A_i}$ (with the order-preserving map point of view; see the proof of Lemma 6). The order homotopy lemma [6, Lemma C.3] implies then that $j$ and $\lambda$ are homotopy inverse. \hfill $\square$

**Proof of Theorem 11.** We prove by induction on $|S|$ that $\bigcap_{i \in S} A_i \neq \emptyset$ for any nonempty $S \subseteq \{0, \ldots, m\}$. Since $\tilde{H}_{|T|-2}(K_S) = 0$ for any singleton $S \subseteq \{0, \ldots, m\}$, every $A_i$ is nonempty. It proves that the above statement is correct for $|S| = 1$. Consider now a set $S \subseteq \{0, \ldots, m\}$ of cardinality $s \geq 2$. We denote by $N_S$ the nerve of $\{A_i : i \in S\}$. By induction, $N_S$ contains the boundary of the $(s - 1)$-dimensional simplex (with vertex set $\{A_i : i \in S\}$). According to Lemma 12, we have $\tilde{H}_{|T|-2}(\bigcup_{i \in T} A_i) = 0$ for every nonempty subset $T$ of $S$. Theorem 3 with $X = \bigcup_{i \in S} A_i$, $\Gamma = \{A_i : i \in S\}$, and $\ell = s - 2$ implies then that $\tilde{H}_{s-2}(N_S) = 0$ and in particular that there is at least one $(s - 1)$-dimensional simplex in $N_S$: this latter simplicial complex is thus exactly the $(s - 1)$-dimensional simplex and $\bigcap_{i \in S} A_i \neq \emptyset$.

To conclude, note that any vertex of $\bigcap_{i=0}^m A_i$ is a simplex of $K$ intersecting every $V_i$. \hfill $\square$

Theorem 11 tells us that the responsibility for the existence of a rainbow simplex is due to the homology of the subcomplexes $K_S$. Next we shall see that this responsibility can be shared by other subcomplexes. For $S \subseteq \{0, \ldots, m\}$, we denote by $\tilde{K}_S$ the subcomplex of $K$ consisting of those simplices $\sigma$ of $K$ for which the subset of colours assigned to $\sigma$ does not contain $S$. 

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Theorem 13. Consider a simplicial complex $K$ whose vertices are partitioned into $m + 1$ subsets $V_0, \ldots, V_m$ and such that $\tilde{H}_{m-1}(K) = 0$. Let $k$ be an integer such that $-1 \leq k \leq m - 1$. Suppose that for every $S \subseteq \{0, \ldots, m\}$:

1. $\tilde{H}_{k-|S|}(K_{\{0, \ldots, m\}\setminus S}) = 0$ whenever $1 \leq |S| \leq k + 1$.
2. $\tilde{H}_{|S|-2}(\tilde{K}_S) = 0$ whenever $k + 2 \leq |S| \leq m$.

Then $K$ contains a rainbow simplex.

Proof. Let $I = \{0, \ldots, m\}$. For $i \in I$, we define $B_i$ to be $K_{I \setminus \{i\}}$. Note that

$$\bigcap_{i \in S} B_i = K_{I \setminus S} \quad \text{and} \quad \bigcup_{i \in S} B_i = \tilde{K}_S.$$ 

Note also that $K_I = K$, $\tilde{K}_{\{i\}} = B_i$, and $\tilde{K}_I$ is $K$ minus the rainbow simplices.

Using these remarks, it is easy to check that conditions (1s) and (2s) imply that $X = K$ and $\Gamma = \{B_i : i \in I\}$ satisfy conditions (1h) and (2h) of Theorem 9, except for $|\Gamma'| = |\Gamma| = m + 1$. Since $\bigcap_{i \in I} B_i = \emptyset$, we have thus $\tilde{H}_{m-1}(\bigcup_{i \in I} B_i) \neq 0$. Note that $\bigcup_{i \in I} B_i = \tilde{K}_I$, hence $\tilde{K}_I$ is different from $K$ because by hypothesis $\tilde{H}_{m-1}(K) = 0$. Consequently there must be a rainbow simplex in $K$. This completes the proof of our theorem.

Note that Theorem 13 for $k = m - 1$ is exactly Theorem 11. For the next variant of Theorem 11 we need the following definition. Let $K$ be a simplicial complex whose vertex set is partitioned into colour sets $V_0, \ldots, V_m$. We say that a vertex $v \in V_i$ is isolated on its colour if $v$ is an isolated point in $K_{\{i\}}$.

Theorem 14. Let $K$ be a simplicial complex whose vertex set is partitioned into colour sets $V_0, \ldots, V_m$ and suppose the vertex $v \in V(K)$ is isolated on its colour. If $\tilde{H}_{|S|-2}(K_S) = 0$ for every nonempty $S \subseteq \{0, \ldots, m\}$, then there exists a rainbow simplex $\sigma$ in $K$ containing the vertex $v$.

Proof. Consider the following subcomplex of $K$, called the link of $v$:

$$\text{lk}(v, K) = \{\sigma \in K : v \notin \sigma \text{ and } (\{v\} \cup \sigma) \in K\}.$$ 

Suppose without loss of generality $v \in K_{\{0\}}$. Since $v$ is isolated on its color, then the vertices of $\text{lk}(v, K)$ are partitioned into $m$ color classes $V'_1, \ldots, V'_m$ with $V'_i \subseteq V_i$. For $S \subseteq \{1, \ldots, m\}$, we denote by $\text{lk}(v, K)_S$ the subcomplex of $\text{lk}(v, K)$ induced by $\bigcup_{i \in S} V'_i$. Note that we have $\text{lk}(v, K)_S = \text{lk}(v, K') \cap K_S$. We wish to prove that for every nonempty subset $S \subseteq \{1, \ldots, m\}$ we have $\tilde{H}_{|S|-2}(\text{lk}(v, K)_S) = 0$, which will imply, by Theorem 2, that there is a rainbow simplex that contains $v$.

Let us consider the Mayer-Vietoris exact sequence of the pair $(K_S, v \ast \text{lk}(v, K)_S)$:

$$\cdots \to \tilde{H}_{|S|-1}(K_S) \oplus \tilde{H}_{|S|-1}(v \ast \text{lk}(v, K)_S) \to \tilde{H}_{|S|-1}(K') \to \tilde{H}_{|S|-2}(\text{lk}(v, K)_S) \to 0 \to \cdots$$

where $K_S \cap (v \ast \text{lk}(v, K)_S) = \text{lk}(v, K)_S$ and $K' = K_S \cup (v \ast \text{lk}(v, K)_S)$. Consequently, we have $\tilde{H}_{|S|-2}(\text{lk}(v, K)_S) = 0$, provided the homomorphism $\tilde{H}_{|S|-1}(K_S) \to \tilde{H}_{|S|-1}(K')$ induced by the inclusion is an epimorphism.

For that purpose let us consider $K''$ be the subcomplex of $K$ induced by the vertices in $\bigcup_{i \in S} V'_i \cup (V_0 \setminus \{v\})$. Note that $K'$ is the subcomplex of $K$ induced by the vertices in
\[ \bigcup_{i \in S} V_i \cup \{v\}. \] Since \( v \) is isolated of \( K_{\{0\}} \), we have that \( K'' \cup K' = K_{S \cup \{0\}} \) and \( K'' \cap K' = K_S \). The Mayer-Vietoris exact sequence of the pair \((K'', K')\) is
\[ \cdots \rightarrow \tilde{H}_{|S|-1}(K_S) \rightarrow \tilde{H}_{|S|-1}(K') \oplus \tilde{H}_{|S|-1}(K'') \rightarrow \tilde{H}_{|S|-1}(K_{S \cup \{0\}}) = 0 \rightarrow \cdots \]
which implies that the homomorphism \( \tilde{H}_{|S|-1}(K_S) \rightarrow \tilde{H}_{|S|-1}(K') \) induced by the inclusion is an epimorphism as we wished.

**Corollary 15.** Let \( K \) be a simplicial complex whose vertex set is partitioned into colour sets \( V_0, \ldots, V_m \) and suppose \( K_{\{i\}} \) is 0-dimensional for \( i \in \{0, \ldots, m\} \). If \( \tilde{H}_{|S|-2}(K_S) = 0 \) for every nonempty \( S \subseteq \{0, \ldots, m\} \), then every simplex is contained in a rainbow simplex.

**Proof.** The proof is by induction on \( n \), the dimension of \( \sigma \). If \( n = 0 \), then the corollary follows from Theorem 14. Suppose the corollary is true for \( n - 1 \), we shall prove it for \( n \). Suppose \( \sigma = \{v_0, \ldots, v_n\} \). Then, by the proof of Theorem 14, \( \text{lk}(v_0, K) \) satisfies the hypothesis of the corollary. By induction \( \{v_1, \ldots, v_n\} \) is contained in a rainbow simplex \( \{v_1, \ldots, v_n, \ldots, v_m\} \) of \( \text{lk}(v_0, K) \). Consequently \( \{v_0, \ldots, v_m\} \) is a rainbow simplex of \( K \) containing \( \sigma \).

5. A POLYTOPAL GENERALIZATION OF MESHULAM’S LEMMA

This section is devoted to a proof of Theorem 2. Note that Theorem 11 is the special case where \( d = m - 1 \) and \( M \) is the boundary of the \( m \)-dimensional simplex with vertex set \( \{0, \ldots, m\} \).

The following counting lemma will be used in the proof of that theorem. The **supporting complex** of a chain is the simplicial complex whose simplices are all simplices in the support of the chain as well as their faces.

**Lemma 16.** Let \( L \) be a simplicial complex and let \( c \in C_s(L) \) be such that the supporting complex of \( \partial c \) is a pseudomanifold with \( n \) vertices. Then the support of \( c \) is of cardinality at least \( n - s \).

**Proof.** Consider the graph \( G(c) \) whose vertices are the \( s \)-dimensional simplices in the support of \( c \) and whose edges connect two simplices having a common facet. For a connected component \( K \) of \( G(c) \), we denote by \( c_K \) the chain obtained from \( c \) by keeping only the \( s \)-dimensional simplices corresponding to vertices in \( K \). Since \( \partial c \) is a pseudomanifold, it is strongly connected (see the definition in Section 1) and only one connected component \( K_0 \) is such that \( \partial c_{K_0} \) is nonzero. The supporting complex of \( c_{K_0} \) has at least \( n \) vertices.

We prove now that any chain \( c' \) in \( C_s(L) \), such that \( G(c') \) is connected and whose supporting complex has at least \( n \) vertices, has a support of cardinality at least \( n - s \). This implies then directly the desired result. The proof works by induction on the cardinality \( k \) of the support of \( c' \). If \( k = 1 \), the statement is obviously true: \( s + 1 - s = 1 \). Suppose that \( k > 1 \). In a connected graph with at least one edge, there is at least one vertex whose removal does not disconnect the graph. We can thus remove a simplex from the support of \( c' \) and obtain a new chain \( c'' \) such that \( G(c'') \) is still connected. Note that the removed simplex has a facet in common with a simplex in the support of \( c' \). It means that at most one vertex has been removed from the supporting simplex of \( c' \). By induction, we have \( k - 1 \geq n - 1 - s \), and thus \( k \geq n - s \), as required.

The proof of Theorem 2 we propose uses a technique presented in the recent survey by De Loera et al. \[4, Proposition 2.5\] for proving Meshulam’s lemma.
Proof of Theorem 2. First, we prove the existence of a map \( f_z : \mathcal{C}(\mathcal{M}) \rightarrow K \) that is augmentation preserving and such that for every \( \sigma \in \mathcal{M} \) the support of \( f_z(\sigma) \) is contained in \( K[\bigcup_{i \in \sigma} V_i] \). We proceed by induction by \( k \) and prove that the statement is true for \(|\sigma| \leq k \). When \( k = 0 \), we define \( f_z(i) \) to be any vertex in \( V_i \) (which exists because \( \tilde{H}_{k-1}(V_i) = 0 \)). Suppose now that the statement is true up to \( k - 1 \). For a simplex \( \sigma \) such that \(|\sigma| = k \), we have \( \partial f_z(\partial \sigma) = 0 \) (we apply the induction hypothesis: it is chain map). Since \( \tilde{H}_{k-2}(K[\bigcup_{i \in \sigma} V_i]) = 0 \), there exists an element \( f_z(\sigma) \) in \( C_{k-1}(K[\bigcup_{i \in \sigma} V_i]) \) such that \( \partial f_z(\sigma) = f_z(\sigma) \).

Second, define \( \lambda : V(K) \rightarrow \{0, \ldots, m\} \) by \( \lambda(v) = i \) for \( v \in V_i \). It induces a simplicial map \( \lambda : K \rightarrow \Delta \), where \( \Delta \) is the \( m \)-dimensional simplex with \( \{0, \ldots, m\} \) as vertex set, and considered as a simplicial complex. Note that \( \lambda \) applied on an \( m \)-dimensional simplex is nonzero if and only if that simplex is rainbow. We claim that \( (\lambda \circ f_z)(\sigma) = \sigma \) for any oriented simplex \( \sigma \) of \( \mathcal{C}(\mathcal{M}) \) (note that \( \mathcal{M} \) is a subcomplex of \( \Delta \)) and we will prove it by induction. Since \( f_z \) is augmentation-preserving, this is obviously true when \( \sigma \) is 0-dimensional. Take now any oriented simplex \( \sigma \in \mathcal{M} \). Since the support of \( f_z(\sigma) \) is contained in \( K[\bigcup_{i \in \sigma} V_i] \), the chain \( (\lambda \circ f_z)(\sigma) \) is of the form \( x\sigma \) for some \( x \in \mathbb{F} \). By induction, we have

\[
\partial(\lambda \circ f_z)(\sigma) = (\lambda \circ f_z)(\partial \sigma) = \partial \sigma.
\]

Thus \( x\partial \sigma = \partial \sigma \), which means that \( x = 1 \).

Third, consider the chain \( z \in C_d(\mathcal{M}) \) equal to the sum of all \( d \)-dimensional oriented simplices of \( \mathcal{M} \) (with unitary coefficients) so that \( \partial z = 0 \). This is possible because \( \mathcal{M} \) is orientable. Now, consider the chain \( c' \in C_{d+1}(K) \) defined by \( \partial c' = f_z(z) \). Such a \( c' \) exists because of the condition \( \tilde{H}_d(K) = 0 \). We have \( \lambda \circ f_z(c') = z \) since \( \lambda \circ f_z \) is the inclusion chain map. According to Lemma 16 with \( n = m + 1 \) and \( s = d + 1 \), there are at least \( m - d \) simplices in \( c = \lambda(c') \), which means that there exist at least that number of rainbow simplices in \( K \).

6. Complementary comments

- While Theorem 3 and Theorem 2 hold for homology with integer coefficients – the same proof works in that case – it could be that this is not true for Theorem 1. Indeed, this latter theorem relies on Lemma 7, whose proof crucially uses the fact that the coefficients are in a field.
- Theorem 2 holds also when \( \mathcal{M} \) is a non-orientable pseudomanifold provided that we work with coefficients in \( \mathbb{Z}_2 \).

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