Hamiltonian Dynamics for the Kepler Problem in a Deformed Phase Space

Mahouton Norbert Hounkonnou and Mahougnon Justin Landalidji

International Chair of Mathematical Physics and Applications (ICMPA-UNESCO Chair)
University of Abomey-Calavi, 072 B.P. 50 Cotonou, Republic of Benin
E-mails: norbert.hounkonnou@cipma.uac.bj with copy to hounkonno@yahoo.fr
E-mails: landalidjijustin@yahoo.fr

Abstract

This work addresses the Hamiltonian dynamics of the Kepler problem in a deformed phase space, by considering the equatorial orbit. The recursion operators are constructed and used to compute the integrals of motion. The same investigation is performed with the introduction of the Laplace-Runge-Lenz vector. The existence of quasi-bi-Hamiltonian structures is also elucidated. Related properties are studied.

Keywords: Hamiltonian Dynamics, Kepler problem, deformed phase space, Laplace-Runge-Lenz vector, quasi-bi-Hamiltonian structure.

Mathematics Subject Classification (2010): 37C10 ; 37J35.

1 Introduction

In 1601, Kepler obtained a detailed set of observations of the motion of the planet Mars from the Danish astronomer Tycho Brahe [4]. From his analysis of these data, Kepler determined that the path of Mars is an ellipse, with the sun located at a focal point, and that the radius vector from the sun to the planet sweeps out equal areas in equal times. The direct problem was to determine the nature of the force required to maintain elliptical motion about a focal force center. This direct problem remained unsolved until after 1679, when Newton determined the functional dependence on distance of the force required to sustain such an elliptical path of Mars about the sun as a center of force located at a focal point of the ellipse.

Building on Newton’s description of the nature and universality of the gravitational force, scientists of the eighteenth century shifted their interest almost exclusively from direct to inverse problems. They used the combined gravitational forces of the sun and the other planets to predict and explain perturbations in the conic paths of planets and comets. That interest continued through the nineteenth and twentieth centuries, and today scientists still concentrate upon the inverse problem rather than the direct one.

In particular, in the last few decades there was a renewed interest in the Kepler problem as one of completely integrable Hamiltonian systems (IHS), the concept of which goes back to Liouville in 1897 [19] and Poincaré in 1899 [22]. Loosely speaking, IHS are dynamical systems admitting a Hamiltonian description, and possessing sufficiently many constants of motion. Many of these systems are Hamiltonian systems with respect to two compatible symplectic structures [20, 12, 33, 11] leading to a geometrical interpretation of the so-called recursion operator [18]. The theory of integrable Hamiltonian systems, based on the use of the Nijenhuis torsion, is a part of the geometry of a particular class of manifolds, called Poisson-Nijenhuis manifolds [21]. In 1992, Marmo and Vilasi [23] constructed a recursion operator for the Kepler dynamics, and obtained related constants of motion.

From the Magri works [20, 24], it is known that the eigenvalues of the recursion operator of bi-Hamiltonian systems form a set of pairwise Poisson-commuting invariants [6]. It is, however, worth noticing that two kinds of difficulties often arise, while investigating these systems: (i) Firstly, it is in general very difficult to give locally an explicit second Hamiltonian structure for a given integrable Hamiltonian system [25] even if it is theoretically always possible in the neighborhood of a regular point of the Hamiltonian [12]; (ii) Secondly, the global or semi-local existence of such structures implies very strong conditions which are rarely satisfied [8, 10].

In 1996, R. Brouzet et al. defined a weaker notion under the name of quasi-bi-Hamiltonian system (QBHS) which relaxes these two difficulties for two degrees of freedom. In 2000, G. Sparano et al constructed recursion
operator for the Kepler dynamics, in the non-commutative case using the so-called Delauney action-angle coordinates [28]. Further, in 2013, Hosokawa and Takeuchi [15] solved the same problem, but using the Runge-Lenz-Pauli vector, and got new constants of motion. A bi-Hamiltonian formulation for a Kepler problem was also studied with Delauney-type variables [14]. In 2016, J. F. Cariñena et al. [9] investigated some properties of the Kepler problem related to the existence of quasi-bi-Hamiltonian structures. In this work, we investigate the Kepler dynamics in a deformed phase space.

The paper is organized as follows. In Section 2, we present the considered deformed phase space. In Section 3, we define, in action-angle coordinates, the deformed Hamiltonian function, symplectic form and vector field describing the Kepler dynamics. In Section 4, we construct recursion operators, and compute the associated integrals of motion. In Section 5, we give an alternative Hamiltonian description for the dynamical systems and obtain associated recursion operators in a non resonant case. In Section 6, we study the existence of quasi-bi-Hamiltonian structure for the considered Kepler dynamics. In Section 7, we end with some concluding remarks.

2 Deformed phase space and Kepler Hamiltonian

Let $\mathbb{R}^3_0 = \mathbb{R}^3 \setminus \{0,0,0\}$ be the configuration manifold $Q$, and $T^*Q = Q \times \mathbb{R}^3$ be the cotangent bundle with the local coordinates $(q,p)$. The cotangent bundle $T^*Q$ has a natural symplectic structure $\omega$ which, in local coordinates, is given by

$$\omega = \sum_{i=1}^{3} dq^i \wedge dp_i.$$  

Since $\omega$ is non-degenerate, it induces the map $\Lambda: T^*Q \rightarrow TQ$ defined by

$$\Lambda = \sum_{i=1}^{3} \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i},$$

where $TQ$ is the tangent bundle. The map $\Lambda$ is called the bivector field [34] and used to construct the Hamiltonian vector field $X_f$ of a Hamiltonian function $f$ by the relation

$$X_f = \Lambda df.$$  

(1)

The phase space deformation is here understood by replacing the usual product with the $\gamma-$star product, (also known as the Moyal product law) between two arbitrary functions of position and momentum [32, 22, 16] :

$$(f \ast_{\gamma} g)(q,p) = f(q_1,p_1) \exp \left( \frac{1}{2} \gamma^{ab} \frac{\partial}{\partial q^a} \frac{\partial}{\partial p^b} \right) g(q_2,p_2) \bigg|_{(q_1,p_1) = (q_2,p_2)} ,$$  

(2)

where

$$\gamma^{ab} = \begin{pmatrix} \Theta^{ij} & \delta^i_j \\ -\delta^i_j & 0 \end{pmatrix},$$  

(3)

$\Theta$ is an antisymmetric $n \times n$ matrix inducing the deformation in the coordinates. Without loss of generality, we restrict our study to the first two terms of the $\ast_{\gamma}$ deformed Poisson bracket expansion to obtain:

$$\{f,g\}_\gamma = \Theta^{ij} \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial q^j} + \left( \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i} \right) ,$$  

(4)

giving

$$\{q^i,q^j\}_\gamma = \Theta^{ij}, \quad \{q^i,p_j\}_\gamma = \delta^i_j , \quad \{p_i,p_j\}_\gamma = 0 .$$  

(5)

The Kepler Hamiltonian in $T^*Q$ takes the form:

$$H = \frac{p_i p^i}{2m} + V(r),$$  

(6)
yielding the Hamilton’s equations:
\[ \dot{q}^i := \{q^i, H\}_\gamma = \frac{p^i}{m} + \Theta^{ij} \frac{\partial V(r)}{\partial q^j}; \quad \dot{p}_i := \{p_i, H\}_\gamma = -\frac{\partial V(r)}{\partial q^i} \]  
(7)

and the following correction to the Newton second law [27]:
\[ m\ddot{q}^i = -\frac{q^i}{r^2} + m\varepsilon^{ijk}\dot{q}^j\Omega^k + m\varepsilon^{ijk}\dot{q}^i\dot{\Omega}^k, \]
(8)

where the deformation parameter \( \Theta^{ij} = \varepsilon^{ijk}\alpha^k \), and the angular velocity
\[ \Omega^i = \frac{k}{r^3}\alpha^i, i = 1, 2, 3. \]

Setting the deformation parameter \( \alpha^i = \delta^i_3 \alpha \) transforms \( H \) into
\[ H = \frac{m}{2} \left[ (\dot{q}^1 - q^2\Omega)^2 + (\dot{q}^2 + q^1\Omega)^2 + (\dot{q}^3)^2 \right] - \frac{k}{r}, \]
(9)

which is reduced to:
\[ H = \frac{p_r^2}{2m} + \frac{p_{\phi r}^2}{2mr^2} - \frac{k}{r}, \]
(10)
in spherical coordinates \((r, \varphi), \) and equatorial orbit corresponding to \( \varphi = \frac{\pi}{2} \), where \( p_r = m\dot{r} \) and \( p_{\phi r} = mr^2\dot{\varphi}_r \), with \( \dot{\varphi}_r = (\dot{\varphi} + \Omega) \) and \( \varphi_r = (\varphi + \Omega t) \in (0, 2\pi) \).

Equation (9) encodes the information on the phase space deformation through \( \Omega \), which depends on the deformation parameter \( \alpha \). However, it can evidently be interpreted as equivalent to the Hamiltonian for a charged particle in a homogeneous, independent of time, magnetic field along \( z \) axis, and the central Newtonian gravitational field in the usual commutative space.

Now considering the coordinate system \((r, \varphi_r, p_r, p_{\phi r})\), and using (1), we get the following Hamiltonian vector field:
\[ X_H = \frac{1}{m} \left[ p_r \frac{\partial}{\partial r} - \frac{1}{r^3} \left( -p_{\phi r}^2 + mkr \right) \frac{\partial}{\partial p_r} + \frac{p_{\phi r}}{mr^2} \frac{\partial}{\partial \phi_r} \right]. \]
(11)

### 3 Hamiltonian system in the action-angle coordinates

The Hamiltonian function (10) does not explicitly depend on the time. Then, setting \( V = W - Et \), it is possible to find a complete integral for the equation of motion by using the method of variable separation:
\[ W = W_r(r) + W_{\phi r}(\varphi_r). \]
(12)

In this case, the Hamilton-Jacobi equation [3] is reduced to
\[ E = \frac{1}{2m} \left( \frac{\partial W}{\partial r} \right)^2 + \frac{1}{2mr^2} \left( \frac{\partial W}{\partial \phi_r} \right)^2 - \frac{k}{r}, \]
(13)

leading to the following set of equations:
\[ \begin{cases} \left( \frac{dW_{\phi r}(\varphi_r)}{d\varphi_r} \right)^2 = D_{\varphi r}^2 \\ -r^2 \left( \frac{dW_r(r)}{dr} \right)^2 + 2mr^2E + 2mrk = D_{\varphi r}^2, \end{cases} \]

where \( D_{\varphi r} \) is constant. In the compact case [34], characterized by \( E < 0 \), we can introduce the action variables [2] \( J_r \) and \( J_{\phi r} \) such that:
\[ \begin{align*} J_{\varphi r} &= \frac{1}{2\pi} \oint \frac{dW_{\phi r}(\varphi_r)}{d\varphi_r} d\varphi_r \\ J_r &= \frac{1}{2\pi} \oint \frac{dW_r(r)}{dr} dr. \end{align*} \]
Using the method of residue \[1, 34\], we get

\[ J_r = -p_{\varphi_\alpha} + \frac{mk}{\sqrt{-2mE}}, \quad D_{\varphi_\alpha} = p_{\varphi_\alpha}, \]

and the integrable system \[5\]:

\[
\begin{aligned}
\dot{J}_i &= 0 \\
\dot{\varphi}_i &= \frac{\partial H}{\partial J_i},
\end{aligned}
\Rightarrow
\begin{aligned}
\dot{J}_1 &= J_r; \quad J_2 = J_{\varphi_\alpha} \\
\dot{\varphi}_1 &= \frac{mk^2}{(J_1 + J_2)^2}; \quad \dot{\varphi}_2 = \frac{mk^2}{(J_1 + J_2)^2}, \quad \varphi^1(0) = \varphi^2(0) = 0.
\end{aligned}
\tag{14}
\]

**Proposition 1.** In action-angle coordinates \((J, \varphi)\), the Hamiltonian \(H\), the symplectic form \(\omega\), and the Hamiltonian vector field \(X_H\) are, respectively:

\[ H = E = -\frac{mk^2}{2(J_1 + J_2)^2}; \quad \omega = \sum_{h=1}^{2} dJ_h \wedge d\varphi^h, \tag{15} \]

and

\[ X_H = \{H, .\} := \frac{mk^2}{(J_1 + J_2)^2} \left( \frac{\partial}{\partial \varphi^1} + \frac{\partial}{\partial \varphi^2} \right), \tag{16} \]

where \(\{., .\}\) is the usual Poisson bracket.

### 4 Recursion operators

Let us define a 2-form \(\omega_1\) and a vector field \(\Delta\),

\[ \omega_1 := \sum_{h,k=1}^{2} S^h_k dJ_k \wedge d\varphi^h = \sum_{h=1}^{2} d\lambda_h \wedge d\varphi^h, \quad \Delta := \lambda_h \frac{\partial}{\partial J_h}, \tag{17} \]

where \(S = \begin{pmatrix} J_1 & J_2 \\ J_2 & J_1 \end{pmatrix}\), \(\begin{cases} \lambda_1 = \frac{1}{2}(J_1^2 + J_2^2) \\ \lambda_2 = J_2J_1 \end{cases}\) such that \(\omega_1\) is the Lie derivative of the symplectic form \(\omega\) in \(15\) with respect to the vector field \(\Delta\), i.e.:

\[ \mathcal{L}_\Delta \omega = \omega_1. \]

The vector field \(\Delta\) generates a sequence of finitely many (Abelian) symmetries according to the following scheme:

\[ X_{i+1} := [X_i, \Delta]_\mu = \frac{2}{\mu}(X_i(\Delta) - \Delta(X_i)), \]

where \(\mu = 3 - i, \quad i = 0, 1, 2\) and \(X_0 = X_H\) in \(16\). The \(X_i\)'s are given by

\[ X_0 = \frac{mk^2}{(J_1 + J_2)^3} \left( \frac{\partial}{\partial \varphi^1} + \frac{\partial}{\partial \varphi^2} \right), \quad X_1 = \frac{mk^2}{(J_1 + J_2)^3} \left( \frac{\partial}{\partial \varphi^1} + \frac{\partial}{\partial \varphi^2} \right), \tag{18} \]

\[ X_2 = \frac{mk^2}{(J_1 + J_2)} \left( \frac{\partial}{\partial \varphi^1} + \frac{\partial}{\partial \varphi^2} \right), \quad X_3 = mk^2 \left( \frac{\partial}{\partial \varphi^1} + \frac{\partial}{\partial \varphi^2} \right), \tag{19} \]

are:

(i) in involution, i.e.,

\[ [X_h, X_k]_\mu = 0, \quad h, k = 0, 1, 2, 3, \quad \mu = 1, 2, 3. \tag{20} \]

(ii) Hamiltonian vector fields, i.e., can be expressed as:

\[ X_i = \{H_i, .\} = \{H_{i+1}, .\}, \quad i = 0, 1, 2, \tag{21} \]
Proposition 2. The recursion operator for the Kepler dynamics in the action-angle coordinates $(J, \varphi)$ is given by:

$$T = \sum_{h,k} (S^{-1})^{h}_{k} \left( \frac{\partial}{\partial \varphi_{h}} \otimes dJ_{k} + \frac{\partial}{\partial J_{h}} \otimes d\varphi_{k} \right),$$

where

$$S^{-1} = \begin{pmatrix}
\frac{J_1}{(J_1 - J_2)(J_1 + J_2)} & -\frac{J_2}{(J_1 - J_2)(J_1 + J_2)} \\
\frac{J_1}{(J_1 - J_2)(J_1 + J_2)} & \frac{J_1 - J_2}{J_1}
\end{pmatrix},$$

and

$$H_0 = -\frac{mk^2}{2(J_1 + J_2)^2}, \quad H_1 = -\frac{mk^2}{(J_1 + J_2)}, \quad H_2 = mk^2 \ln(J_1 + J_2), \quad H_3 = mk^2(J_1 + J_2).$$

Proposition 3. In the coordinate system $(\xi, \phi)$, the Hamiltonian function $H'$, the symplectic form $\omega'$, the Hamiltonian vector field $X'_{H'}$, and the recursion operator $T'$ are, respectively:

$$H' = \frac{1}{m\alpha}(\xi_1 - \xi_2); \quad \omega' = \sum_{h=1}^{2} d\xi_{h} \wedge d\phi_{h}; \quad X'_{H'} = \frac{1}{m\alpha} \left( \frac{\partial}{\partial \phi_{1}} - \frac{\partial}{\partial \phi_{2}} \right),$$

$$T' = \sum_{i=1}^{2} R_{i} \left( \frac{\partial}{\partial \xi_{i}} \otimes d\xi_{i} + \frac{\partial}{\partial \phi_{i}} \otimes d\phi_{i} \right),$$

where $R = \begin{pmatrix} \xi_1 & 0 \\ 0 & \xi_2 \end{pmatrix}$. 
Two interesting cases deserve investigation:

I) Introduce the Laplace-Runge-Lenz (LRL) vector $A$ given by

$$A = p \times L - mkq/r, \quad (31)$$

where $p$ is the momentum vector, $q$ is the position vector of the particle of mass $m$, and $L$ is the angle momentum vector, $L = q \times p$. We obtain:

$$L_1 = 0; \quad L_2 = 0; \quad L_3 = mr^2 \dot{\varphi}_\alpha = p_{\varphi_\alpha}. \quad (32)$$

$$A_1 = C \sin \beta + D \cos \beta; \quad A_2 = C \cos \beta - D \sin \beta; \quad A_3 = 0, \quad (33)$$

$$\{A_1, H\} := \left(\frac{\partial A_1}{\partial r} \frac{\partial H}{\partial p_r} - \frac{\partial A_1}{\partial p_r} \frac{\partial H}{\partial r}\right) + \left(\frac{\partial A_1}{\partial \varphi_\alpha} \frac{\partial H}{\partial p_{\varphi_\alpha}} - \frac{\partial A_1}{\partial p_{\varphi_\alpha}} \frac{\partial H}{\partial \varphi_\alpha}\right)$$

$$= \frac{3k\alpha p_r}{mr^4}(D \sin \beta - B \cos \beta), \quad (34)$$

$$\{A_2, H\} = \frac{3k\alpha p_r}{mr^4}(B \sin \beta - D \cos \beta), \quad (35)$$

where

$$C = -p_r p_{\varphi_\alpha} \cos \varphi_\alpha + \frac{p^2}{r} \sin \varphi_\alpha - mk \sin \varphi_\alpha$$

and

$$D = p_r p_{\varphi_\alpha} \sin \varphi_\alpha + \frac{p^2}{r} \cos \varphi_\alpha - mk \cos \varphi_\alpha, \quad \text{and} \quad \beta = \Omega t.$$

**Remark 1.** We have:

(i) The $A_i, i = 1, 2, 3,$ commute with the Hamiltonian $H$ in $\{10\}$, i.e., \{A_i, H\} = 0, if

$$\beta = \frac{\pi}{4}; \quad \frac{p_r p_{\varphi_\alpha}}{p^2} = - \cot(\beta + \varphi_\alpha), \quad (\beta + \varphi_\alpha) \in (0, \pi). \quad (36)$$

(ii) $\{A_1, A_2\} = (-2mH + \frac{3k\alpha p_r}{r^4})p_{\varphi_\alpha}, \quad \{A_1, L_3\} = A_2, \quad \{A_2, L_3\} = A_1.$

(iii) Setting $L_3 = A_3$ and $p^2_{\varphi_\alpha} = r^2[2mk - r(p_r(3\Omega - rp_r))] \equiv A^2_3$, then, the $A_i$'s generate an $su(2)$ Lie algebra, i.e., $\{A_i, A_j\} = \varepsilon_{ijl}A_l$.

II) Consider a scaled Runge-Lenz-Pauli vector $\Gamma$, defined on the domain $\{(q, p) \in \mathcal{T}^*(\mathbb{R}^3 \setminus \{0, 0, 0\}) | H(q, p) < 0\}$ by

$$\Gamma = \frac{1}{\sqrt{-2mH}}A, \quad (37)$$

where $H$ is the Hamiltonian function given in $\{10\}$. The components $\Gamma_i$ are:

$$\Gamma_1 = \frac{1}{\sqrt{-2mH}}(C \sin \beta + D \cos \beta); \quad \Gamma_2 = \frac{1}{\sqrt{-2mH}}(C \cos \beta - D \sin \beta); \quad \Gamma_3 = 0, \quad (38)$$

with

$$|\Gamma|^2 = -\frac{mk^2}{2H} + L_3^2. \quad (39)$$

The quantities $H$, $|\Gamma|^2$, and $L_3$ are in involution, i.e.,

$$\{|\Gamma|^2, L_3\} = 0, \quad \{|\Gamma|^2, H\} = 0, \quad \{L_3, H\} = 0.$$
Putting $\pi_1 = |\Gamma|^2$ and $\pi_2 = p_{\pi, \alpha}$, the equations of motion in the $(\pi, \chi)$ system become:

$$
\begin{align*}
\dot{\pi}_1 &= 0, \\
\chi^i &= \frac{\partial H''}{\partial \pi_i},
\end{align*}
$$

$$
H'' = \frac{mk^2}{2(\pi_2^2 - \pi_1)},
$$

$$
\Rightarrow \left\{ \begin{array}{l}
\pi_i = \text{cst.}, \ i = 1, 2, \\
\chi^i = \frac{\partial H''}{\partial \xi^i} + \chi^i(0), \ \chi^i(0) = 0.
\end{array} \right.
$$

(40)

The relationships between $(J, \varphi)$ and $(\pi, \chi)$ are deduced as:

$$
J_1 = -\pi_1 + \sqrt{\pi_1 - \pi_2^2}; \ J_2 = \pi_2; \ \chi^1 = \frac{1}{(J_1 + J_2)} \varphi^1; \ \chi^2 = -\frac{J_2}{(J_1 + J_2)} \varphi^2.
$$

(41)

Finally, we get:

**Proposition 4.** In the coordinate system $(\pi, \chi)$, the Hamiltonian function $H''$, the symplectic form $\omega''$, the Hamiltonian vector field $X''_{H''}$, and the recursion operator $T''$ are given as follows:

$$
H'' = \frac{mk^2}{2(\pi_2^2 - \pi_1)}; \ \omega'' = \sum_{h=1}^2 d\pi_h \wedge d\chi^h; \ X''_{H''} = \frac{mk^2}{2(\pi_2^2 - \pi_1)^2} \left( \frac{\partial}{\partial \chi^1} - 2\pi_2 \frac{\partial}{\partial \chi^2} \right)
$$

$$
T'' = \sum_{i=1}^2 F_i \left( \frac{\partial}{\partial \pi_i} \otimes d\pi_i + \frac{\partial}{\partial \chi^i} \otimes d\chi^i \right), \ \text{where} \ F = \begin{pmatrix} \pi_1 & 0 \\ 0 & \pi_2 \end{pmatrix}.
$$

(42)

(43)

5 Alternative Hamiltonian description

Let

$$
\Upsilon = J_1 X_1 + J_2 X_2
$$

(44)

be a dynamical system on the manifold $T^*Q$, with $X_1$ and $X_2$ obtained in (18) and (19). The relation (44) can be rewritten as:

$$
\Upsilon = \nu_a X^a + \nu_e X^e,
$$

(45)

where

$$
\nu_a = -2H_a, \ \nu_e = H_e; \ H_a = J_1 H_0, \ H_e = J_2 H_1; \ X^a = \frac{\partial}{\partial \varphi^a}, \ X^e = \frac{\partial}{\partial \varphi^e},
$$

$$
\frac{\partial}{\partial \Phi^a} = \left( \frac{\partial}{\partial \varphi^1} + \frac{\partial}{\partial \varphi^2} \right), \ \frac{\partial}{\partial \Phi^e} = -\left( \frac{\partial}{\partial \varphi^1} + \frac{\partial}{\partial \varphi^2} \right).
$$

(46)

(47)

The vector fields $X^a, X^e$ and the $C^\infty-$functions $H_a, H_e$ satisfy the following properties:

$$
[X^i, X^j] = 0; \ \mathcal{L}_{X^i} H_i = 0, \ i, j \in \{a, e\}.
$$

(48)

Let $\mathcal{N}$ be an open dense submanifold of $T^*Q$ on which $\Upsilon$ is explicitly integrable such that:

$$
X^a \wedge X^e \neq 0; \ dH_a \wedge dH_e \neq 0.
$$

(49)

Now, considering the coordinate system $(H, \Phi)$ with $\Phi^i, i \in \{a, e\}$, which are closed differential 1-forms, the equations of motion of $\Upsilon$ are given by

$$
\dot{\Phi}^a = -2H_a; \ \dot{\Phi}^e = H_e; \ \dot{H}_a = 0; \ \dot{H}_e = 0,
$$

(50)

with the functions $H_a$ and $H_e$ obeying the condition (49). We can construct a closed 2-form, for $i \in \{a, e\}$,

$$
\omega = \sum_i df^i(H_i) \wedge d\Phi^i,
$$

(51)

which is non degenerate as long as $df^a \wedge df^e \neq 0$, and

$$
\iota_X \omega = -df^i; \ \iota_\Phi \omega = -\sum_i \nu_i df^i; \ \sum_i d\nu_i \wedge df^i = 0.
$$

(52)
Notice that (52) is a necessary condition for \( \iota_{\nu} \hat{\omega} \) to be exact, i.e., it is closed. Since \( d\nu_a \wedge d\nu_e \neq 0 \), the solutions of (52) are given by linear functions (17)

\[
f^i = \sum_j L^{ij} \nu_j, \quad i, j \in \{a, e\}, \quad \text{where} \quad L = \begin{pmatrix} -1/2 & 0 \\ 0 & 1 \end{pmatrix}.
\]  

(53)

Then, we get:

\[
f^a = -\frac{1}{2} \nu_a; \quad f^e = \nu_e.
\]  

(54)

From (46) and (54), we can rewrite (51) in the new coordinate system \((\nu, \Phi)\) as:

\[
\tilde{\omega} = \sum_i df^i(\nu_i) \wedge d\Phi^i, \quad i \in \{a, e\},
\]  

(55)

leading to the following form:

\[
\tilde{\omega} = -\frac{1}{2} d\nu_a \wedge d\Phi_a + d\nu_e \wedge d\Phi_e.
\]  

(56)

The corresponding Hamiltonian description for \( \Upsilon \) is given with the following quadratic Hamiltonian function

\[
\tilde{H} = -\frac{1}{4} \nu_a^2 + \frac{1}{2} \nu_e^2.
\]  

(57)

In addition, from (34) other symplectic structures of the form (55) can be constructed, in which any \( f_i \) depending only on the corresponding frequency \( \nu_i, i \in \{a, e\} \), will be admissible as long as \( \tilde{\omega}_b, b \in \{1, ..., n\} \), is non-degenerate, i.e., as long as \( df^a \wedge df^e \neq 0 \). From above, putting:

\[
f^a = \nu_a; \quad f^e = \nu_e \quad \text{and} \quad f^a = \nu_a^2; \quad f^e = \nu_e^2
\]  

(58)

we obtain, respectively:

\[
\tilde{\omega}_1 = d\nu_a \wedge d\Phi_a + d\nu_e \wedge d\Phi_e \quad \text{and} \quad \tilde{\omega}_2 = 2\nu_a d\nu_a \wedge d\Phi_a + 2\nu_e d\nu_e \wedge d\Phi_e.
\]  

(59)

Then, the \((1,1)\)-tensor field \( T = \tilde{\omega}_2 \circ \tilde{\omega}_1^{-1} \) is constructed, taking the form

\[
T = T_1 + T_2,
\]  

(60)

where

\[
T_1 = 2\nu_a \left( \frac{\partial}{\partial \nu_a} \otimes d\nu_a + \frac{\partial}{\partial \Phi_a} \otimes d\Phi_a \right) \quad \text{and} \quad T_2 = 2\nu_e \left( \frac{\partial}{\partial \nu_e} \otimes d\nu_e + \frac{\partial}{\partial \Phi_e} \otimes d\Phi_e \right).
\]  

(61)

Finally, basing on (29) and (30), \( T_1 \) and \( T_2 \) are recursion operators for the dynamical system \( \Upsilon \). Hence, \( T \) is also a recursion operator for the dynamical system \( \Upsilon \) as a sum of two recursion operators.

### 6 Quasi-bi-Hamiltonian structures

Basing on (8) and (9), in this part, we investigate the recursion operators for quasi-bi-Hamiltonian structures.

**Definition 1.** A Hamiltonian vector field \( Y \) on a symplectic manifold \((M, \omega)\) is called quasi-bi-Hamiltonian if there exist another symplectic structure \( \omega_1 \), and a nowhere-vanishing function \( g \), such that \( gY \) is a Hamiltonian vector field with respect to \( \omega_1 \), i.e.,

\[
\iota_{\nu} \omega_0 = -dH_0; \quad \iota_{\nu} \omega_1 = \iota_{\nu}(g\omega_1) = -dH_1,
\]  

(62)

where \( H_0 \) and \( H_1 \) are integrals of motion for the Hamiltonian vector field \( Y \). \( g\omega_1 \) is not closed in general.
A consequence of this definition is that the pair \((\omega_0, \omega_1)\) determines a \((1, 1)\)-tensor field \(T\) defined as 
\[ T := \hat{\omega}_0^{-1} \circ \hat{\omega}_1, \]
that is, \(\omega_0(Y, X) = \omega_1(TY, X)\), where \(X, Y\) are two Hamiltonian vector fields, and \(\hat{\omega} := \iota_Y \omega\). In the action-angle coordinates \((J, \varphi)\), the decomposition of the symplectic form 
\[ \omega' = \omega'_1 + \omega'_2, \]

\[
\omega'_1 = dJ_1 \wedge d\varphi^1 - \left( \frac{2(J_1 + J_2)^3}{m^2k^2\alpha} + 1 \right) dJ_2 \wedge d\varphi^2
\]

\[
\omega'_2 = -dJ_1 \wedge d\varphi^2 + \left( \frac{2(J_1 + J_2)^3}{m^2k^2\alpha} + 1 \right) dJ_2 \wedge d\varphi^1
\]

shows that:

(i) \(\omega'_1\) and \(\omega'_2\) are not closed, i.e., \(d\omega'_1 \neq 0, d\omega'_2 \neq 0\), where \(d\) is the exterior derivative. So, \(\omega'_1\) and \(\omega'_2\) are not symplectic.

(ii) \(\iota_{x_H} \omega'_1 = -dh'_1, \ i_{x_H} \omega'_2 = -dh'_2\), where \(h'_1 = -h'_2 = \frac{2J_2}{m\alpha}\).

(iii) The functions \(h'_1\) and \(h'_2\) are first integrals of \(X_H\), i.e., \(X_H(h'_1) = X_H(h'_2) = 0\).

**Proposition 5.** The Hamiltonian vector field \(X_H\) is quasi-bi-Hamiltonian with respect to the two 2-forms \((\omega, \omega'_1)\). Idem for \((\omega, \omega'_2)\). The weaker \(\omega'_1\) recursion operators are given by:

\[
\hat{T}'_1 := \omega^{-1} \circ \omega'_1 = \frac{\partial}{\partial J_1} \circ dJ_1 + \frac{\partial}{\partial \varphi^1} \circ d\varphi^1 - (2K + 1) \left( \frac{\partial}{\partial J_2} \circ dJ_2 + \frac{\partial}{\partial \varphi^2} \circ d\varphi^2 \right),
\]

and

\[
\hat{T}'_2 := \omega^{-1} \circ \omega'_2 = (2K + 1) \left( \frac{\partial}{\partial J_1} \circ dJ_1 + \frac{\partial}{\partial \varphi^1} \circ d\varphi^1 \right) - \frac{\partial}{\partial J_2} \circ dJ_2 - \frac{\partial}{\partial \varphi^2} \circ dJ_1,
\]

where

\[ K = \frac{(J_1 + J_2)^3}{m^2k^2\alpha} \]

and the 2-vector field \(\omega^{-1} = \frac{\partial}{\partial J_1} \wedge \frac{\partial}{\partial \varphi^1}\).

Similarly, \(\omega''\) can be re-expressed as the sum of two 2-forms as follows:

\[
\omega'' = \omega''_1 + \omega''_2,
\]

where

\[
\omega''_1 = 2dJ_1 \wedge d\varphi^1 - \left( \frac{(J_1 + J_2)(2J_2 + 1)}{(J_1 + J_2)^2} \right) dJ_2 \wedge d\varphi^2
\]

\[
\omega''_2 = -J_2dJ_1 \wedge d\varphi^2 - \frac{2(1 + J_1)\varphi^1}{(J_1 + J_2)} dJ_1 \wedge dJ_2 + \left( \frac{2J_2 + 1}{(J_1 + J_2)} \right) dJ_2 \wedge d\varphi^1
\]

\[
+ \left( \frac{2}{(J_1 + J_2)} + \frac{2J_2 + 1}{(J_1 + J_2)^2} \right) (J_2 - 1)\varphi^1 dJ_2 \wedge dJ_1.
\]

As above:

(iv) \(\omega''_1\) and \(\omega''_2\) are not symplectic, i.e., \(d\omega''_1 \neq 0, d\omega''_2 \neq 0\).

(v) \(\iota_{x_H} \omega''_1 = -dh''_1, \ i_{x_H} \omega''_2 = -dh''_2\), where

\[
h''_1 = \frac{k^2m(3J_2(8J_2 - 6J_1 + 3) + J_1(2J_1 + 5))}{6(J_1 + J_2)^3},
\]

\[
h''_2 = \frac{k^2m(J_2(-J_2 - 3J_1 + 12) + 8J_1)}{6(J_1 + J_2)^3}.
\]
(vi) $h''_1$ and $h''_2$ are also first integrals of $X_H$, i.e., $X_H(h''_1) = X_H(h''_2) = 0$.

**Proposition 6.** The Hamiltonian vector field $X_H$ is quasi-bi-Hamiltonian with respect to the two 2-forms $(\omega, \omega''_1)$. Idem for $(\omega, \omega''_2)$. The weaker $\omega''_i$ recursion operators $\tilde{T}''_1$ and $\tilde{T}''_2$ are:

\[
\tilde{T}''_1 := \omega^{-1} \circ \omega''_1 = 2\left(\frac{\partial}{\partial \varphi^1} \otimes d\varphi^1 + \frac{\partial}{\partial J_1} \otimes dJ_1\right) - J_2\left(2 + \frac{\tilde{V}}{V^2}\right)\left(\frac{\partial}{\partial J_2} \otimes dJ_2 + \frac{\partial}{\partial \varphi^2} \otimes d\varphi^2\right),
\]

\[
\tilde{T}''_2 := \omega^{-1} \circ \omega''_2 = \left(2 + \frac{\tilde{V}}{V}\right)\left(\frac{\partial}{\partial J_1} \otimes dJ_2 + \frac{\partial}{\partial \varphi^2} \otimes d\varphi^1\right) - J_2\left(\frac{\partial}{\partial \varphi^1} \otimes d\varphi^2 + J_2\frac{\partial}{\partial J_2} \otimes dJ_1\right)
- \left(2 + \frac{\tilde{V}}{V^2}(J_2 - 1)\right)\varphi^1\left(\frac{\partial}{\partial \varphi^1} \otimes dJ_2 - \frac{\partial}{\partial \varphi^2} \otimes dJ_1\right),
\]

where $\tilde{V} = 2J_2 + 1$, $V = J_1 + J_2$.

**7 Concluding remarks**

In this paper, we have constructed recursion operators for the Kepler dynamics in a deformed phase space by considering the equatorial orbit, computed the associated integrals of motion, and proved the existence of quasi-bi-Hamiltonian structures for the Kepler dynamics.

**Acknowledgments**

This work is supported by TWAS Research Grant RGA No.17-542 RG/MATHS/AF/AC_G-FR3240300147. The ICMPA-UNESCO Chair is in partnership with the Association pour la Promotion Scientifique de l’Afrique (APSA), France, and Daniel Iagolnitzer Foundation (DIF), France, supporting the development of mathematical physics in Africa.

**References**

[1] M. J. Ablowitz, A. S. Fokas, *Complex Variables: Introduction and applications* 2nd Edition, Cambridge University press, 2003.

[2] R. Abraham and J. E. Marsden, *Foundation of Mechanics (2nd edition)*, Benjamin/Cummings, Reading, Mass. 1978.

[3] V. I. Arnold, *Mathematical Methods of Classical Mechanics*, 2nd Edition, Springer-Verlag, 1978.

[4] J. B. Brackenridge, *The Key to Newton’s Dynamics*, University of California Press, 1995.

[5] O. I. Bogoyavlenski, *Theory of tensor invariants of integrable Hamiltonian systems: I. Incompatible Poisson structures*, Comm. Math. Phys. **180** (1996), 529–586.

[6] R. Brouzet, *systèmes bihamiltoniens et complète intégrabilité en dimension 4*, CRAS **311** (Série 1), 895–8, 1990.

[7] R. Brouzet, *Géométrie des systèmes bihamiltoniens en dimension 4*, Thèse de l’Université Montpellier 2, 1991.

[8] R. Brouzet, R. Caboz, J. Rabenivo, V. Ravoson, *Two degrees of freedom quasi-bi-Hamiltonian systems*, J. Phys. A: Math. Gen. **29** (1996), 2069–2076.
[9] J. F. Carinena, M. F. Rana, *Quasi-bi-Hamiltonian structures of the 2-Dimensional Kepler problem*, SIGMA 12 (2016), 010.

[10] L. Fernandes, *Completely integrable bihamiltonian systems*, J. Dynam. Differential Equations 6 1994, 53–69.

[11] S. De Filippo, G. Marmo, M. Salerno and G. Vilasi, *A New Characterization of Completely Integrable Systems*, Nuovo Cimento B 83 (1984), 97–112.

[12] I. M. Gelfand, I. Ya. Dorfman, *The Schouten Bracket and Hamiltonian Operators*, Funct. Anal. Appl. 14 (1980), 71–74.

[13] H. Goldstein, *Prehistory of the "Runge-Lenz" vector*, Amer. J. Phys 43, 737 (1975).

[14] Y. A. Grigoryev, Tsiganov A. V., *On bi-Hamiltonian formulation of the perturbed Kepler problem*, J. Phys. A: Math. Theor. 48 (2015), 175206, 7 pages.

[15] K. Hosokawa, T. Takeuchi, *A Construction for the Concrete Example of a Recursion Operator*, JP J. Geom. Topol. 14 (2013), 99–183.

[16] N. Khosravi, S. Jalalzadeh, H. R. Sepangi, *Non-commutative multi-dimensional cosmology*, JHEP 01 (2006) 134, 1–10.

[17] G. Landi, G. Marmo and G. Vilasi, *Recursion Operators: Meaning and Existence for Completely Integrable Systems*, J. Math. Phys. 35 (1994), 808–815.

[18] P. D. Lax *Integrals of Nonlinear Equations of Evolution and Solitary Waves*. Comm. Pure Appl. Math. 21 (1968), 467–490.

[19] R. Liouville, *Sur le mouvement d’un corps solide pesant suspendu par l’un de ses points*. Acta Math. 20 (1897), 239–284.

[20] F. Magri, *A simple model of the integrable Hamiltonian equation*, J. Math. Phys. 19 (1978), 1156–62.

[21] F. Magri and C. Morosi, *A geometrical characterization of integrable Hamiltonian systems through the theory of Poisson-Nijenhuis manifolds*, Publ. Dep. Math. Univ. Milan, 1984.

[22] B. Malekolkalami, K. Atazadeh and B. Vakili, *Late time acceleration in a non-commutative model of modified cosmology*, Phys. Lett. B 739 (2014), 400–404.

[23] G. Marmo and G. Vilasi, *When Do Recursion Operators Generate New Conservation Laws?* Phys. Lett. B 277 (1992), 137–140.

[24] H. Poincaré, *Sur les quadratures mécaniques*, Acta Math. 13, 1 (1899).

[25] V. Ravoson, *(ρ,s)-structure bihamiltonienne, séparabilité, paires de Lax et intégrabilité*, Thèse de Doctorat de Math. Appliquées Univ. Pau et Pays de l’Adour, 1992.

[26] J. M. Romero, J. A. Santiago, J. D. Vergara, *Newton’s second law in a non-commutative space*, Phys. Lett. A 310 (2003), 9–12.

[27] J. M. Romero, J. D. Vergara, *The Kepler problem and deformation*. Mod. Phys. Lett. A 18 (2003), 1673–1680.

[28] G. Sparano, G. Vilasi, *Noncommutative Integrability and Recursion Operators*, J. Geom. Phys. 36 (2000), 270–284.

[29] T. Takeuchi, *On the construction of recursion operators for some geodesic flows*, Departement of Mathematics, Tokyo University of Science, 2015.
[30] T. Takeuchi, *A Construction of a Recursion Operator for Some Solutions of Einstein Field Equations*, Proceedings of the Fifteenth International Conference on Geometry, Integrability and Quantization 15 (2014), 249–258.

[31] T. Takeuchi, *On the Construction of Recursion Operators for the Kerr-Newman and FLRW Metrics*, J. Geom. Phys. 37 (2015), 85–96.

[32] B. Vakili, P. Pedram and S. Jalalzadeh, *Late time acceleration in a deformed phase space model of dilaton cosmology*, Phys. Lett. B 687 (2010), 119–123.

[33] G. Vilasi, *On the Hamiltonian Structures of the Korteweg-de Vries and Sine-Gordon Theories*, Phys. Lett. B 94, 195?198 (1980).

[34] G. Vilasi, *Hamiltonian Dynamics*, 1st Edition, World Scientific Publishing Co., Inc., River Edge, NJ, 2001.