Best Proximity Coincidence Point Results for \((\alpha, D)\)-Proximal Generalized Geraghty Mappings in JS-Metric Spaces

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Received 7 September 2020; Revised 2 November 2020; Accepted 6 November 2020; Published 23 November 2020

Academic Editor: Huseyin Isik

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We introduce a type of Geraghty contractions in a JS-metric space \(X\), called \((\alpha, D)\)-proximal generalized Geraghty mappings. By using the triangular-\((\alpha, D)\)-proximal admissible property, we obtain the existence and uniqueness theorem of best proximity coincidence points for these mappings together with some corollaries and illustrative examples. As an application, we give a best proximity coincidence point result in \(X\) endowed with a binary relation.

1. Introduction and Preliminaries

Let \(T : A \rightarrow B\) be a map where \(A\) and \(B\) are two nonempty subsets of a metric space \(X\). It is known that if \(T\) is a non-self-map, the equation \(Tx = x\) does not always have a solution, and it clearly has no solution when \(A\) and \(B\) are disjoint. However, it is possible to determine an approximate solution \(x^*\) such that the error is \(d(x^*, Tx^*) = d(A, B)\). Such point \(x^*\) is called a best proximity point of \(T\). The best proximity point theorem was first studied in [1]. Then, there has been a wide range of research in this framework. Many researchers have studied and generalized the result in many aspects (for example, see [2–15]). For some recent articles regarding these points, see [16, 17] where Geraghty type mappings were studied and [18] where cyclic and noncyclic nonexpansive mappings were considered.

One of the well-known generalizations of the Banach contraction principle is the result given by Geraghty [19] which enriches the principle by considering the class of mappings \(\theta: [0,\infty) \rightarrow [0, 1)\) such that \(t_n \rightarrow 0\) when \(\theta(t_n) \rightarrow 1\). By including 1 in the ranges of those mappings \(\theta\), Ayari [20] provided a new result on the existence and uniqueness of the best proximity point for \(\alpha\)-proximal Geraghty mappings.

The concept of the best proximity coincidence point, which is an extension of a best proximity point problem, was mentioned in [21] (see also [22]) where some results of mappings in generalized metric spaces were presented. A point \(a\) is called a best proximity coincidence point of the pair \((g, T)\), where \(g\) is a self-map on \(A\), if \(d(ga, Ta) = d(A, B)\). Clearly, if \(g\) is the identity map, then each best proximity coincidence point of the pair \((g, T)\) is a best proximity point for \(T\).

A large number of results concerning these point problems in various metric spaces have been investigated since then. Hussain and the coauthors contributed several interesting results and generalizations in [23–25], including the recent article [26] where best proximity point results for Suzuki-Edelstein proximal contractions were studied. (See also, [27–31] for his work.)

Zhang and Su [32] weakened the \(P\)-property, called the weak \(P\)-property, and improved a best proximity point theorem for Geraghty nonself-contractions. In 2018, Komal et al. [33] obtained best proximity coincidence point
Theorem 7 (see [34]). A function \( D : X \times X \to \mathbb{R} \) is called a generalized metric on \( X \) if it satisfies the following conditions:

1. For any \( x, y, z \in X \),
   \[ \alpha D(x, y) + \beta D(x, z) + \gamma D(y, z) \leq \delta D(x, y) \]
2. \( D(x, x) = 0 \)
3. If \( D(x, y) = 0 \), then \( x = y \)
4. For any \( x, y, z \in X \),
   \[ D(x, y) + D(y, z) \leq D(x, z) + \theta D(y, x) \]

where \( \alpha, \beta, \gamma, \delta, \theta \) are non-negative real numbers.

### 2. Main Results

Throughout this article, let \( X = (X, D) \) be a \( JS \)-metric space, and let \( A \) and \( B \) be nonempty disjoint subsets of \( X \). Also, we require the following notations:

\[
D(A, B) = \inf \{ D(a, b) : a \in A, b \in B \},
\]
\[
A_0 = \{ a \in A : \text{there exists } b \in B \text{ such that } D(a, b) = D(A, B) \},
\]
\[
B_0 = \{ b \in B : \text{there exists } a \in A \text{ such that } D(a, b) = D(A, B) \}.
\]

Clearly, if one of \( A_0 \) and \( B_0 \) is nonempty, then so is the other.

Definition 8 (see [32]). Suppose that \( A_0 \) is nonempty. The pair \( (A, B) \) is said to have the weak \( P \)-property if and only if \( D(x_1, y_1) = D(x_2, y_2) = D(A, B) \) implies \( D(x_1, x_2) \leq D(y_1, y_2) \), where \( x_1, x_2 \in A_0 \) and \( y_1, y_2 \in B_0 \).

Definition 9. Let \( T : A \to B \) and \( S : A \to B \) be mappings. The pair \((S, T)\) is said to be triangular- \((a, D)\)-proximal admissible if the following conditions hold.

(i) If \( \alpha(Sx_1, Sx_2) \geq 1 \) and \( D(Su_1, Tx_1) = D(Su_2, Tx_2) = D(A, B) \), then \( \alpha(Su_1, Su_2) \geq 1 \) and \( D(Su_1, Su_2) < \infty \) for all \( x_1, x_2, u_1, u_2 \in A \).

(ii) If \( \alpha(x, z) \geq 1 \) and \( \alpha(z, y) \geq 1 \), then \( \alpha(x, y) \geq 1 \), for all \( x, y, z \in X \).
We consider the class of mappings $\Theta$ which is a slight generalization of the well-known class of $[0, 1)$-valued functions introduced by Geraghty [19]:

$$\Theta = \{ \theta : [0, \infty) \to [0, 1] : \theta(t_n) \to 1 \text{ implies } t_n \to 0 \}.$$  \hfill (4)

Now, we introduce a class of our contractions as follows.

**Definition 10.** Let $T : A \to B$ and $S : A \to A$ be mappings. Given that $\alpha : X \times X \to [0, \infty)$ is a function, the pair $(S, T)$ is said to be an $(\alpha, D)$-proximal generalized Geraghty mapping if the following conditions hold.

(i) $(S, T)$ is triangular-$(\alpha, D)$-proximal admissible.

(ii) There is $\theta \in \Theta$ such that for all $x, y, u, v \in A$, if $D(Su, Tx) = D(Sv, Ty) = D(A, B)$ and $\alpha(Sx, Sy) \geq 1$, then

$$\alpha(Sx, Sy)D(Tx, Ty) \leq \theta(M(x, y, u, v))M(x, y, u, v),$$ \hfill (5)

where $M(x, y, u, v) = \max \{ D(Sx, Sy), D(Sx, Su), D(Sy, Sv) \}$.

We first give a useful lemma.

**Lemma 11.** Let $\alpha : X \times X \to [0, \infty)$ be a function. Let $T : A \to B$ and $S : A \to A$ be two mappings such that $(S, T)$ is an $(\alpha, D)$-proximal generalized Geraghty mapping, and let $(A, B)$ have the weak $P$-property. If $\alpha(Sx, Sy) \geq 1$ for all $x, y \in B \{C(S, T)\}$, then $Sx = Sy$.

**Proof.** Let $x, y \in BC(S, T)$, we have that

$$D(Sx, Tx) = D(Sy, Ty) = D(A, B).$$ \hfill (6)

From the assumption, $\alpha(Sx, Sx) \geq 1, \alpha(Sy, Sy) \geq 1$, and $\alpha(Sx, Sy) \geq 1$. Since $\alpha(Sx, Sy) \geq 1$, $(S, T)$ is triangular-$(\alpha, D)$-proximal admissible and (6), we have that $D(Sx, Sy) < \infty$. Also, since $D(Sx, Tx) = D(Sx, Tx) = D(A, B), \alpha(Sx, Sx) \geq 1$ and $(S, T)$ is triangular-$(\alpha, D)$-proximal admissible, then $D(Sx, Sx) < \infty$. Similarly, we can show that $D(Sy, Sy) < \infty$.

Note that

$$M(x, x, x, x) = \max \{ D(Sx, Sx), D(Sx, Sx), D(Sx, Sx) \} = D(Sx, Sx) < \infty.$$ \hfill (7)

Since $(S, T)$ is an $(\alpha, D)$-proximal generalized Geraghty mapping, and $(A, B)$ has the weak $P$-property,

$$D(Sx, Sx) \leq \alpha(Sx, Sx)D(Sx, Sx) \leq \alpha(Sx, Sx)D(Tx, Tx) \leq \theta(D(Sx, Sx))D(Sx, Sx),$$ \hfill (8)

for some $\theta \in \Theta$. From the property of $\theta$, we can conclude that $D(Sx, Sx) = 0$. Similarly, we also have that $D(Sy, Sy) = 0$. Then,

$$M(x, y, x, y) = \max \{ D(Sx, Sy), D(Sx, Sx), D(Sy, Sy) \} = D(Sx, Sy) < \infty.$$ \hfill (9)

Since $\alpha(Sx, Sy) \geq 1$, we have that

$$D(Sx, Sy) \leq \alpha(Sx, Sy)D(Sx, Sy) \leq \alpha(Sx, Sy)D(Tx, Ty) \leq \theta(M(x, y, x, y))M(x, y, x, y)$$ \hfill (10)

for some $\theta \in \Theta$. Thus, $D(Sx, Sy) = 0$ which implies that $Sx = Sy$.

**Theorem 12.** Let $\emptyset \neq A_0 \subseteq S(A_0)$, and let $(S(A_0), D)$ be $D$-complete. Given that $\alpha : X \times X \to [0, \infty)$ is a function, and let $T : A \to B$ and $S : A \to A$ be mappings such that $(S, T)$ is an $(\alpha, D)$-proximal generalized Geraghty mapping. Suppose that the following conditions hold.

(i) $T(A_0) \subseteq B_0$ and the pair $(A, B)$ has the weak $P$-property.

(ii) There exist $x, y \in A_0$ such that $D(Sx, Ty) = D(A, B), \alpha(Sx, Sy) \geq 1$ and $D(Sy, Sy) < \infty$.

(iii) For $\{S_{x_n}\} \subset C(D, S(A_0), Sx^*)$ such that $\alpha(S_{x_n}, S_{x_{n+1}}) \geq 1$ for all $n \in \mathbb{N}$, there is a subsequence $\{S_{x_k}\}$ with $\alpha(S_{x_{k_n}}, Sx^*) \geq 1$ for all $k \in \mathbb{N}$.

Then, there exists $x^* \in A_0$ such that $D(Sx^*, Tx^*) = D(A, B)$. Moreover, if $\alpha(Sx^*, Sy^*) \geq 1$ for all $x^*, y^* \in BC(S, T)$ and $S$ is injective, then $(S, T)$ has a unique best proximity coincidence point.

**Proof.** From (ii), there exist $x_0, x_1 \in A_0$ such that

$$D(Sx_1, Tx_0) = D(A, B), \alpha(Sx_0, x_1) \geq 1, D(Sx_0, Sx_1) < \infty.$$ \hfill (11)

Since $T(A_0) \subseteq B_0, A_0 \subseteq S(A_0)$, and $(S, T)$ is triangular-$(\alpha, D)$-proximal admissible, there exists $x_2 \in A_0$ such that

$$D(Sx_2, Tx_1) = D(A, B), \alpha(Sx_1, x_2) \geq 1, D(Sx_1, Sx_2) < \infty.$$ \hfill (12)

Continuing in this way, we obtain a sequence $\{Sx_n\} \subseteq S(A_0)$ such that for all $n \in \mathbb{N}$,

$$D(Sx_n, Tx_{n-1}) = D(A, B) = D(Sx_{n+1}, Tx_n), \alpha(Sx_{n-1}, Sx_n) \geq 1, D(Sx_{n-1}, Sx_n) < \infty.$$ \hfill (13)

Using the weak $P$-property to (13), for $n$ and $n + 1$, we have that
$$D(Sx_n, Sx_{n+1}) \leq D(Tx_{n-1}, Tx_n)$$ \quad for all \(n \in \mathbb{N} \). \quad (14)

If there exists \(n_0 \in \mathbb{N} \cup \{0\} \) such that \(Sx_{n_0} = Sx_{n_0+1} \), then from (13),

$$D(Sx_{n_0+1}, Tx_{n_0}) = D(Sx_{n_0}, Tx_{n_0}) = D(A, B). \quad (15)$$

Now suppose that \(Sx_n \neq Sx_{n+1} \) for all \(n \in \mathbb{N} \cup \{0\} \). By the definition of \(D, D(Sx_n, Sx_{n+1}) \neq 0 \). We will first show that \(\lim_{n \to \infty} D(Sx_{n-1}, Sx_n) = 0 \). Let \(n \in \mathbb{N} \). Since \((S, T)\) is an \((a, D)\)-proximal generalized Geraghty mapping together with (13) and (14), we obtain that

$$D(Sx_n, Sx_{n+1}) \leq D(Tx_{n-1}, Tx_n)$$

\[
\leq aD(Sx_{n-1}, Sx_n) + \theta(M(x_{n-1}, x_n, x_{n+1}))M(x_{n-1}, x_n, x_{n+1})
\]

$$\leq M(x_{n-1}, x_n, x_{n+1}). \quad (16)$$

where

$$M(x_{n-1}, x_n, x_{n+1}) = \max \{D(Sx_{n-1}, Sx_n), D(Sx_n, Sx_{n+1})\}. \quad (17)$$

If \(M(x_{n-1}, x_n, x_{n+1}) = D(Sx_{n-1}, Sx_{n+1})\), then by (16),

$$D(Sx_n, Sx_{n+1}) \leq \theta(D(Sx_n, Sx_{n+1}))D(Sx_n, Sx_{n+1}) \leq D(Sx_{n-1}, Sx_n). \quad (18)$$

Since \(D(Sx_n, Sx_{n+1}) > 0 \) for all \(n \geq 0\),

$$1 \leq \theta(D(Sx_n, Sx_{n+1})) \leq 1, \quad (19)$$

and thus,

$$\lim_{n \to \infty} \theta(D(Sx_n, Sx_{n+1})) = 1. \quad (20)$$

By the definition of \(\theta\), \(\lim_{n \to \infty} D(Sx_n, Sx_{n+1}) = 0 \).

If \(M(x_{n-1}, x_n, x_{n+1}) = D(Sx_{n-1}, Sx_n)\), we again have that

$$D(Sx_n, Sx_{n+1}) \leq \theta(D(Sx_{n-1}, Sx_n))D(Sx_{n-1}, Sx_n) \leq D(Sx_{n-1}, Sx_n). \quad (21)$$

Since \(n\) is arbitrary, \(\{D(Sx_n, Sx_{n+1})\}\) is nonnegative and nonincreasing. Therefore, \(\{D(Sx_n, Sx_{n+1})\}\) converges to \(s \geq 0\). Suppose on the contrary that \(s > 0\). From (21),

$$\frac{D(Sx_n, Sx_{n+1})}{D(Sx_{n-1}, Sx_n)} \leq \theta(D(Sx_{n-1}, Sx_n)) \leq 1. \quad (22)$$

It follows that \(\lim_{n \to \infty} \theta(D(Sx_{n-1}, Sx_n)) = 1\). Since \(\theta \in \Theta\), we have that \(\lim_{n \to \infty} D(Sx_{n-1}, Sx_n) = 0\) which is a contradiction. Thus, \(s\) must be 0 and that

$$\lim_{n \to \infty} D(Sx_{n-1}, Sx_n) = 0. \quad (23)$$

Next, we shall show that \(\{Sx_n\}\) is a \((D,\text{-}\text{Cauchy})\) sequence. Suppose that this is not the case. Then, there exists \(\varepsilon > 0\) such that for any \(k \in \mathbb{N}\), there are subsequences \(\{Sx_{n_k}\}\) and \(\{Sx_{m_k}\}\) of \(\{Sx_n\}\) satisfying \(D(Sx_{n_k}, Sx_{m_k}) \geq \varepsilon\) for \(m_k \geq n_k \geq k\).

Since \((S, T)\) is triangular-\((a, D)\)-proximal admissible, it is easy to see that

$$\alpha(Sx_n, Sx_m) \geq 1 \text{ and } D(Sx_n, Sx_m) < \infty \text{ when } m \geq n \text{ for all } m, n \in \mathbb{N}. \quad (24)$$

It follows from (13) and (24) that for any \(k \in \mathbb{N}\),

$$\alpha(Sx_{n_k-1}, Sx_{m_k-1}) \geq 1 \text{ and } D(Sx_{n_k-1}, Tx_{n_k-1}) = D(A, B) = D(Sx_{m_k}, Tx_{m_k-1}). \quad (25)$$

Since \((S, T)\) is an \((a, D)\)-proximal generalized Geraghty mapping and \((A, B)\) has the weak \(P\)-property, we obtain that

$$D(Sx_n, Sx_m) \leq D(Tx_{n-1}, Tx_{m-1})$$

\[
\leq \alpha(Sx_{n-1}, Sx_m) + \theta(M(x_{n-1}, x_n, x_{m+1}))M(x_{n-1}, x_n, x_{m+1})
\]

$$\leq M(x_{n-1}, x_n, x_{m+1}). \quad (26)$$

where

$$M(x_{n-1}, x_n, x_{m+1}) = \max \{D(Sx_{n-1}, Sx_m), D(Sx_m, Sx_m)\}. \quad (27)$$

If \(M(x_{n-1}, x_n, x_{m+1})\) is either \(D(Sx_{n-1}, Sx_m)\) or \(D(Sx_{m+1}, Sx_m)\), then, by (23), \(\lim_{n \to \infty} D(Sx_n, Sx_m) = 0\). This contradicts the assumption that \(\{Sx_n\}\) is not \(D\)-Cauchy. Thus, \(M(x_{n-1}, x_n, x_{m+1}) = D(Sx_{n-1}, Sx_{m+1})\).

As a consequence,

$$D(Sx_n, Sx_m) \leq \theta(D(Sx_{n-1}, Sx_{m-1}))D(Sx_{n-1}, Sx_{m-1}). \quad (28)$$

By repeating the same steps, it follows that

$$D(Sx_{n-i}, Sx_{m-i}) \leq \theta(D(Sx_{n-i-1}, Sx_{m-i-1}))D(Sx_{n-i-1}, Sx_{m-i-1}). \quad (29)$$

where \(i = 0, 1, 2, \ldots, n_k - 1\). Therefore,

$$D(Sx_{n_k}, Sx_{m_k}) \leq \prod_{i=0}^{n_k} \theta(D(Sx_{n-i}, Sx_{m-n_i}))D(Sx_0, Sx_{m_k-n_k}). \quad (30)$$
Let \( i_k \in \{1, 2, \cdots, n_k\} \) such that
\[
\theta(D(Sx_{n_k-i_k}, Sx_{m_k-i_k})) = \max \{\theta(D(Sx_{n_k-i}, Sx_{m_k-i})) : 1 \leq i \leq n_k\}.
\] (31)

Define
\[
\eta = \limsup_{k \to \infty} \{\theta(D(Sx_{n_k-i_k}, Sx_{m_k-i_k}))\}. \tag{32}
\]

If \( \eta < 1 \), then, \( \lim_{k \to \infty} D(Sx_{n_k}, Sx_{m_k}) = 0 \) which is impossible. Thus, \( \eta = 1 \). Without loss of generality, we may assume that \( \lim_{k \to \infty} \theta(D(Sx_{n_k-i}, Sx_{m_k+1-i})) = 1 \). By the definition of \( \theta \), \( \lim_{k \to \infty} D(Sx_{n_k-i}, Sx_{n_k+m-i}) = 0 \). Then, there exists \( k_0 \in \mathbb{N} \) such that
\[
D(Sx_{n_k-i}, Sx_{m_k+1-i}) < \frac{\varepsilon}{2}. \tag{33}
\]

Now,
\[
\epsilon \leq D(Sx_{n_k}, Sx_{m_k}) \leq \prod_{j=1}^{n_k} \theta(D(Sx_{n_k-j}, Sx_{m_k-j})) D(Sx_{n_k-j}, Sx_{m_k-j}) \leq \frac{\varepsilon}{2}, \tag{34}
\]
which is a contradiction. Therefore, \( \{Sx_n\} \) is a D-Cauchy sequence.

Since \( (S(A_0), D) \) is D-complete, there exists \( x^* \in A_0 \) such that
\[
\lim_{n \to \infty} D(Sx_n, x^*) = 0. \tag{35}
\]
Equivalently,
\[
\{Sx_n\} \in C(D, S(A_0), x^*). \tag{36}
\]

Since \( A_0 \subseteq S(A_0) \) and \( T(A_0) \subseteq B_0 \), it follows that there exists \( a \in A_0 \) such that
\[
D(Sa, Tx^*) = D(A, B). \tag{37}
\]

By (13) and (iii), there is a subsequence \( \{Sx_{n_k}\} \) of \( \{Sx_n\} \) such that \( a(Sx_{n_k}, x^*) \geq 1 \) for all \( k \in \mathbb{N} \). From (13), we have that
\[
D(Sx_{n_k}, Tx_{n_k}) = D(A, B) \quad \text{for all } k \in \mathbb{N}. \tag{38}
\]

By the weak \( P \)-property, (37) and (38), we obtain that \( D(Sx_{n_k+1}, Sa) \leq D(Tx_{n_k}, Tx^*) \).

Since \( a(Sx_{n_k}, x^*) \geq 1 \) and \( (S, T) \) is an \( (a, D) \)-proximal generalized Geraghty mapping,
\[
D(Sx_{n_k+1}, Sa) \leq D(Tx_{n_k}, Tx^*) \leq a(Sx_{n_k}, x^*) D(Tx_{n_k}, Tx^*) \leq \theta(M(x_{n_k}, x^*, x_{n_k+1}, a)) M(x_{n_k}, x^*, x_{n_k+1}, a) \leq M(x_{n_k}, x^*, x_{n_k+1}, a), \quad \text{for all } k \geq 1, \tag{39}
\]
where
\[
M(x, x^*, x_{n_k+1}, a) = \max \{D(Sx_n, x^*), D(Sx_n, x_{n_k+1}), D(x^*, Sa)\}. \tag{40}
\]

By (23) and (35), we immediately have that
\[
\lim_{k \to \infty} M(x_{n_k}, x^*, x_{n_k+1}, a) = D(x^*, Sa) \geq 0. \tag{41}
\]

If \( D(x^*, Sa) > 0 \), by letting \( k \to \infty \) in (39),
\[
1 \leq \liminf_{k \to \infty} \theta(M(x_{n_k}, x^*, x_{n_k+1}, a)) \leq 1. \tag{42}
\]
We subsequently have that
\[
\lim_{k \to \infty} \theta(M(x_{n_k}, x^*, x_{n_k+1}, a)) = 1. \tag{43}
\]

By the property of \( \theta \),
\[
\lim_{k \to \infty} M(x_{n_k}, x^*, x_{n_k+1}, a) = D(x^*, Sa) = 0, \tag{44}
\]
which is a contradiction. It follows that \( D(x^*, Sa) \) must be equal to 0, and thus \( x^* = Sa \). Therefore, from (37), there exists \( x^* \in A \) such that
\[
D(Sx^*, Tx^*) = D(A, B). \tag{45}
\]

Suppose further that \( x^*, y^* \in BC(S, T) \) and \( a(x^*, y^*) \geq 1 \). By Lemma 11, \( Sx^* = Sy^* \). Since \( S \) is injective, \( x^* = y^* \). The proof is now completed.

**Example 13.** Let \( X = [-3, 3] \) be equipped with the JS-metric \( D \) given by
\[
D(x, y) = \begin{cases} |x| + |y|, & x \neq 0 \text{ and } y \neq 0, \\ \frac{|x|^2}{2}, & y = 0, \\ \frac{|y|^2}{2}, & x = 0. \end{cases} \tag{46}
\]

Choose \( A = [-2, 0] \) and \( B = [0, 1] \). Let \( T : A \to B \) be a mapping defined by
\[
T(x) = -\frac{x}{3}, \quad \text{for all } x \in A, \tag{47}
\]
and let a mapping \( S : A \to A \) be defined by
\[
S(x) = \frac{x}{2}, \quad \text{for all } x \in A. \tag{48}
\]

It is not difficult to see that \( D(A, B) = 0 \) and \( (A, B) \) has the weak \( P \)-property. Next, define the map \( a : X \times X \to [0, \infty) \) by
\[
a(x, y) = \begin{cases} 1, & \text{if } x \neq 0 \text{ or } y = 0, \\ 0, & \text{otherwise}. \end{cases} \tag{49}
\]
for all \( x, y \in X \). Since \( A_0 = \{ 0 \} = B_0 \), then \( T(A_0) = \{ 0 \} \subseteq B_0 = \{ 0 \} \) and \( A_0 = \{ 0 \} \subseteq S(A_0) = \{ 0 \} \). Also, there is \( 0 \in A_0 \) satisfying
\[
D(S(0), T(0)) = D(0, 0) = 0 = D(A, B), \alpha(0, 0) \geq 1. \tag{50}
\]

We will first show that \( (S, T) \) is triangular-(\( \alpha, D \))-proximal admissible.
Let \( x_1, x_2, u_1, u_2 \in A \) such that \( \alpha(x_1, x_2) \geq 1 \) and
\[
D(Su_1, Tx_1) = D(Su_2, Tx_2) = D(A, B). \tag{51}
\]

Then, \( Sx_1 \neq 0 \) or \( Sx_2 = 0 \) and
\[
D\left( \frac{u_1}{2} - \frac{x_1}{3} \right) = D\left( \frac{u_2}{2} - \frac{x_2}{3} \right) = 0. \tag{52}
\]

Assume that \( \alpha(Su_1, Su_2) \neq 1 \), then \( u_1 / 2 = Su_1 = 0 \) and \( u_2 / 2 = Su_2 \neq 0 \).
Since \( \alpha(x_1, x_2) \geq 1 \), we consider the following two cases.

Case 1. If \( Sx_2 \neq 0 \), then \( Sx_1 \neq 0 \), and thus,
\[
\left| -\frac{x_1}{6} \right| = D\left( 0, -\frac{x_1}{3} \right) = 0. \tag{53}
\]

Then \( x_1 = 0 \). This implies that \( Sx_1 = 0 \) which is impossible.

Case 2. If \( Sx_2 = 0 = x_2 / 2 \), then
\[
D\left( \frac{u_2}{2} - \frac{x_2}{3} \right) = D\left( 0, \frac{u_2}{2} \right) = \left| \frac{u_2}{4} \right| = 0. \tag{54}
\]

This implies that \( u_2 = 0 \) and \( Su_2 = 0 \) which is impossible. Thus, \( \alpha(Su_1, Su_2) \geq 1 \).

Next, assume that \( \alpha(x, z) \geq 1 \) and \( \alpha(z, y) \geq 1 \). Then, we can see that \( y = 0 \) if \( z = 0 \), and \( x \neq 0 \) if \( z \neq 0 \). Hence, \( x \neq 0 \) or \( y = 0 \), and thus, \( \alpha(x, y) \geq 1 \). This means that \( (S, T) \) is triangular-(\( \alpha, D \))-proximal admissible.

We note that there is a map \( \theta \in \Theta \) defined by \( \theta(t) = 2/3 \).

Now, for \( x, y \) satisfying \( \alpha(Sx, Sy) \geq 1 \), we have that \( Sx \neq 0 \) or \( Sy = 0 \). We consider the following two cases.

Case 1. If \( Sy = 0 \), then \( y = 0 \) and
\[
\alpha(Sx, Sy)D(Tx, Ty) = \alpha(Sx, 0)D(Tx, T(0)) = D\left( -\frac{x}{3}, 0 \right) = \left| -\frac{x_1}{6} \right| = \frac{2}{3} \left| \frac{x_1}{3} \right| = \frac{2}{3} \theta(M(x, y, u, v)).
\]

Case 2. If \( Sy \neq 0 \), then \( Sx \neq 0 \), and thus, \( x \neq 0 \) and \( y \neq 0 \). We obtain that
\[
\alpha(x, y)D(Tx, Ty) = D(Tx, Ty) = D\left( -\frac{x}{3}, -\frac{y}{3} \right) = \left| -\frac{x}{3} - \frac{y}{3} \right| \leq \frac{2}{3} |x| + \frac{2}{3} D(Sx, Sy) \leq \theta(M(x, y, u, v))M(x, y, u, v). \tag{55}
\]

Therefore, \( (S, T) \) is an \( (\alpha, D) \)-proximal generalized Geraghty mapping.
Finally, we will show that assumption (iii) in Theorem 12 holds. Let \( a \in A_0 \) and \( \{ Sx_n \} \in \mathcal{C}(D, S(A_0), S \alpha) \) such that \( \alpha(Sx_n, Sx_{n+1}) \geq 1 \) for all \( n \in \mathbb{N} \). Then,
\[
Sx_n \neq 0 \text{ or } Sx_{n+1} = 0 \text{ for each } n \in \mathbb{N}. \tag{57}
\]

If \( Sx_n \neq 0 \) for all \( n \in \mathbb{N} \), then \( \alpha(Sx_n, Sa) \geq 1 \) for all \( n \in \mathbb{N} \). Assume that there exists \( n_0 \in \mathbb{N} \) such that \( Sx_{n_0} = 0 \).
By (57), \( Sx_k = 0 \) for all \( k \geq n_0 \). Suppose that \( Sa \neq 0 \). Then,
\[
D(Sx_k, Sa) = D(0, a) = \frac{|a|}{2} \neq 0 \text{ for all } k \geq n_0. \tag{58}
\]

This contradicts with the fact that \( \{ Sx_n \} \in C(D, S(A_0), S \alpha) \). Thus, \( Sa = 0 \) and so \( \alpha(Sx_n, Sa) \geq 1 \). We also have that \( (S(A_0), D) \) is \( D \)-complete. Therefore, by Theorem 12, \( (S, T) \) has a best proximity coincidence point, which is 0.

**Example 14.** Let \( X = \mathbb{R}^2 \) be equipped with the IS-metric \( D \) given by
\[
D((x_1, y_1), (x_2, y_2)) = \begin{cases} |x_1 - x_2| + |y_1 - y_2|, & (x_1, y_1) \neq (0, 0), (y_1, y_2) \neq (0, 0), \\ |x_1 - x_2|, & (y_1, y_2) = (0, 0), \\ |y_1 - y_2|, & (x_1, x_2) = (0, 0). \end{cases} \tag{59}
\]

We consider the disjoint subsets A and B of X given by \( A = \{ (-1, y) ; 0 \leq y \leq 1 \} \) and \( B = \{ (1, y) ; 0 \leq y \leq 1 \} \). We can check that \( D(A, B) = 2 \) and the pair \( (A, B) \) has the weak P-property.

Let \( T : A \to B \) be a map defined by
\[
T(-1, y) = (1, \ln (1 + y)), \text{ for all } (-1, y) \in A. \tag{60}
\]

and let \( S : A \to A \) be a map defined by
\[
S(-1, y) = (-1, y), \text{ for all } (-1, y) \in A. \tag{61}
\]

Then, we consider a map \( \alpha : X \times X \to [0, \infty) \) given by
\[
\alpha((x_1, y_1), (x_2, y_2)) = \begin{cases} 1, & \text{if } x_1 \leq x_2, y_1 \geq y_2, \\ 0, & \text{otherwise}, \end{cases} \tag{62}
\]

for all \( x = (x_1, y_1), y = (x_2, y_2) \in X \).

Next, we will show that \( (S, T) \) is triangular-(\( \alpha, D \))-proximal admissible. Let \( x, y, u, v \in A \) such that \( x = (-1, x), y = (-1, y), u = (-1, u), \) and \( v = (-1, v) \) satisfying \( \alpha(Sx, Sy) \geq 1 \) and
\[ D(Su, Tx) = D(Sv, Ty) = D(A, B). \] 

Consequently, \( \bar{x} \geq \bar{y} \) and

\[ D((-1, \bar{u}), (-1, \ln (1 + \bar{x}))) = D((-1, \bar{v}), (-1, \ln (1 + \bar{y}))) = 2. \]

(64)

It follows that \( \bar{u} = \ln (1 + \bar{x}) \) and \( \bar{v} = \ln (1 + \bar{y}) \). Since \( \bar{x} \geq \bar{y} \), then \( \bar{u} \geq \bar{v} \), and thus, \( \alpha(\bar{u}, \bar{v}) \geq 1 \).

Assume that \( \alpha(x, y) \geq 1 \) and \( \alpha(y, u) \geq 1 \). Then, we can see that \( \bar{x} \geq \bar{y} \) and \( \bar{y} \geq \bar{u} \). Therefore, \( \bar{x} \geq \bar{u} \), and thus, \( \alpha(x, u) \geq 1 \).

This means that \( (S, T) \) is triangular-(\( \alpha, D \))-proximal admissible.

We choose a map \( \theta \in \Theta \) which is defined by

\[ \theta(t) = \begin{cases} 1, & t = 0, \\ \frac{\ln (1 + t)}{t}, & t > 0. \end{cases} \]

Let \( x, y, u, v \in A \) such that \( x = (1, \bar{x}), y = (1, \bar{y}), u = (1, \bar{u}) \), and \( v = (1, \bar{v}) \) satisfying \( \alpha(\bar{x}, \bar{y}) \geq 1 \). If \( x = y \), then we are done. Suppose that \( x \neq y \). It follows that \( D(x, y) > 0 \), and so, \( M(x, y, u, v) > 0 \). Thus,

\[ \alpha(\bar{x}, \bar{y})D(Tx, Ty) = D(Tx, Ty) \\
= D((1, \ln (1 + \bar{x})), (1, \ln (1 + \bar{y}))) \\
= | \ln (1 + \bar{x}) - \ln (1 + \bar{y}) | \\
\geq \ln |1 + \frac{\bar{x} - \bar{y}}{\bar{y}}| \\
= \ln (1 + M(x, y, u, v)) \\
= \ln (1 + M(x, y, u, v))M(x, y, u, v). \]

(66)

Therefore, \( (S, T) \) is an \( (\alpha, D) \)-proximal generalized Geraghty mapping.

Since \( A_0 = A = \{ (1, y) : 0 \leq y \leq 1 \} \) and \( B_0 = B = \{ (1, y) : 0 \leq y \leq 1 \} \),

\[ T(A_0) = \{ (1, y) : 0 \leq y \leq \ln 2 \} \subseteq B_0, \]

\[ A_0 = \{ (1, 0) \} \subseteq S(A_0) = A_0. \]

(67)

Also, \( (S(A_0), D) \) is \( D \)-complete, and there is \( (-1, 0) \in A_0 \) satisfying

\[ D(S(-1, 0), T(-1, 0)) = D((-1, 0), (1, 0)) = 2 = D(A, B), \]

\[ \alpha(S(-1, 0), T(-1, 0)) = \alpha((-1, 0), (1, 0)) \geq 1. \]

(68)

We have left to that show assumption (iii) in Theorem 12 holds. Let \( a = (-1, \bar{a}) \in A_0 \) and \( \{ Sx_n \} \in C(D, S(A_0), S\bar{a}) \) such that \( \alpha(\bar{x}, \bar{y}), \bar{x}n+1 = \alpha((-1, y), (-1, y_{n+1}) \geq 1 \) for all \( n \in \mathbb{N} \).

Then, \( y_n \geq y_{n+1} \) for all \( n \). Since \( S(A_0) = A_0 = \{ (-1, y) : 0 \leq y \leq 1 \} \) and \( \{ y_n \} \) is nonincreasing which \( \{ Sx_n \} \in C(D, S(A_0), S\bar{a}) = C(D, A_0, \alpha) \). It follows that \( y_n \geq \bar{a} \) for all \( n \in \mathbb{N} \). Then, \( \alpha(\bar{x}, \bar{y}) \geq 1 \) for all \( n \in \mathbb{N} \). Therefore, by Theorem 12, \( (S, T) \) has a best proximity coincidence point.

Next, we present a corollary of our result. The following definition is required.

Definition 15. Let \( T : A \to B \) and \( S : A \to A \) be mappings. Let \( \alpha : X \times X \to [0, \infty) \) be a function. Then, the pair \( (S, T) \) is said to be an \( (\alpha, D) \)-proximal mapping if the following conditions hold.

(i) The pair \( (S, T) \) is triangular-(\( \alpha, D \))-proximal admissible.

(ii) There exists \( k \in (0, 1] \) such that for all \( x, y, u, v \in X \), if \( D(Su, Tx) = D(Sv, Ty) = D(A, B) \) and \( \alpha(Sx, Sy) \geq 1 \), then

\[ D(Tx, Ty) \leq kD(x, y). \]

(69)

By putting \( \theta(t) = k \), where \( k \in (0, 1] \) in Theorem 12, we have the following result.

Corollary 16. Let \( A_0 \subseteq S(A_0) \) and \( (S(A_0), D) \) be \( D \)-complete. Given that \( \alpha : X \times X \to [0, \infty) \) is a function, and let \( T : A \to B \) and \( S : A \to A \) be mappings such that \( (S, T) \) is an \( (\alpha, D) \)-proximal mapping. Suppose that the following conditions hold.

(i) \( T(A_0) \subseteq B_0 \) and the pair \( (A, B) \) has the weak \( P \)-property.

(ii) There exist \( x, y \in A_0 \) such that \( D(Sx, Ty) = D(A, B) \) and \( \alpha(Sx, Sy) \geq 1 \) and \( D(Sy, Sx) < \infty \).

(iii) For \( \{ Sx_n \} \subseteq C(D, S(A_0)), Sx^* \), if \( \alpha(Sx_n, Sx_{n+1}) \geq 1 \) for all \( n \in \mathbb{N} \), then there is a subsequence \( \{ Sx_n^*_k \} \) with \( \alpha(Sx_n^*_k, Sx^*) \geq 1 \) for all \( k \in \mathbb{N} \).

Then, there exists \( x^* \in A \) such that \( D(Sx^*, Tx^*) = D(A, B) \).

Moreover, if \( \alpha(Sx^*, Sx^*) \geq 1 \) for all \( x^*, y^* \in BC(S, T) \), then \( (S, T) \) has a unique best proximity coincidence point.

3. Consequence

We will apply our result on the best proximity coincidence point on a \( JS \)-metric space endowed with a binary relation \( R \).

Let \( T : A \to B \) and \( S : A \to A \) be mappings. The pair \( (S, T) \) is said to be \( (R, D) \)-proximally comparative if \( SxRSy \) and \( D(Su_1, Tx) = D(Su_2, Ty) = D(A, B) \Rightarrow Su_1RSu_2 \) and \( D(Su_1, Su_2) < \infty \) for all \( x, y, u_1, u_2 \in A \).

Definition 17. Let \( T : A \to B \) and \( S : A \to A \) be mappings. The pair \( (S, T) \) is said to be an \( (R, D) \)-proximally comparative generalized Geraghty mapping if the following hold.

(i) The pair \( (S, T) \) is \( (R, D) \)-proximally comparative.
(2) There exist $\theta \in \Theta$ such that for all $x, y, u, v \in A$, if $D(Su, Tx) = D(Sy, Ty) = D(A, B)$ and $SxRSy$, then

$$D(Tx, Ty) \leq \theta(M(x, y, u, v))M(x, y, u, v),$$

where $M(x, y, u, v) = \max \{D(Sx, Sy), D(Sx, Su), D(Sy, Sv)\}$.

**Corollary 18.** Let $X$ be endowed with a symmetric, transitive binary relation $R$. Let $T : A \rightarrow B$ and $S : A \rightarrow A$ be mappings such that $\emptyset \neq A_0 \subseteq S(A_0)$ and $(S(A_0), D)$ be $D$-complete. If $(S, T)$ is an $(R, D)$-proximally comparative generalized Geraghty mapping and the following conditions hold:

(i) $T(A_0) \subseteq B_0$ and the pair $(A, B)$ has the weak $P$-property;

(ii) there exist $x, y \in A_0$ such that $D(Sx, Ty) = D(A, B)$ and $SyRSx$ and $D(Sy, Sx) < \infty$;

(iii) for $(Sx_n) \in C(D, S(A_0), Sx^*)$, if $Sx_nR$ for all $n \in \mathbb{N}$, then there is a subsequence $(Sx_{n_k})$ with $Sx_{n_k}R$ for all $k \in \mathbb{N}$.

then there exists $x^* \in A_0$ such that $D(Sx^*, Tx^*) = D(A, B)$. Moreover, if $Sx^*RSy^*$ for all $x^*, y^* \in B(C(S, T)$ and $S$ is injective, then $(S, T)$ has a unique best proximity coincidence point.

**Proof.** Define

$$\alpha(x, y) = \begin{cases} 1, & \text{if } xRy, \\ 0, & \text{otherwise,} \end{cases}$$

for all $x, y \in X$. We can see that the hypotheses of Theorem 12 hold which imply that there is $x^* \in A$ such that $D(Sx^*, Tx^*) = D(A, B)$. Let $x^*, y^* \in B(C(S, T)$. Then, $Sx^*RSy^*$ which implies that $\alpha(Sx^*, Sy^*) \geq 1$. Again, by Theorem 12, $x^* = y^*$.

**4. Conclusion and Open Questions**

We have introduced new classes of Geraghty's type mappings called $(\alpha, D)$-proximal generalized Geraghty mappings. Then, we investigated some conditions for this type of mappings to have a best proximity coincidence point in JS-metric spaces using the weak P-property. The question is whether one can extend Theorem 12 to the framework of common best proximity point in a JS-metric space $X$. Can we also extend the result when $X$ is other generalized metric spaces?

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors have no conflict of interests regarding the publication of this paper.

**Acknowledgments**

This research was partially supported by Chiang Mai University and by the Centre of Excellence in Mathematics, CHE Thailand.

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