Akane Nakamura

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Tome 69, n° 2 (2019), p. 845-894.

<http://aif.centre-mersenne.org/item/AIF_2019__69_2_845_0>
AUTONOMOUS LIMIT OF THE 4-DIMENSIONAL PAINLEVÉ\textsubscript{\textfrac{1}{2}}-TYPE EQUATIONS AND DEGENERATION OF CURVES OF GENUS TWO

by Akane NAKAMURA (*)

Abstract. — In recent studies, 4-dimensional analogs of the Painlevé\textsubscript{\textfrac{1}{2}} equations were listed and there are 40 types. The aim of the present paper is to geometrically characterize these 40 Painlevé\textsubscript{\textfrac{1}{2}}-type equations. For this purpose, we study the autonomous limit of these equations and degeneration of their spectral curves. The spectral curves are 2-parameter families of genus two curves and their generic degeneration are one of the types classified by Namikawa and Ueno. Liu’s algorithm enables us to find the degeneration types of the spectral curves for our 40 types of integrable systems. This result is analogous to the following fact; the families of the spectral curves of the autonomous 2-dimensional Painlevé\textsubscript{\textfrac{1}{2}} equations $P_{I}$, $P_{II}$, $P_{IV}$, $P_{D_{8}}$, $P_{D_{7}}^{II}$, $P_{D_{6}}$, $P_{V}$ and $P_{V_{I}}$ define elliptic surfaces with the singular fiber at $H = \infty$ of the Dynkin types $E_{8}^{(1)}$, $E_{7}^{(1)}$, $E_{6}^{(1)}$, $D_{8}^{(1)}$, $D_{7}^{(1)}$, $D_{6}^{(1)}$, $D_{5}^{(1)}$ and $D_{4}^{(1)}$, respectively.

Keywords: integrable system, Painlevé\textsubscript{\textfrac{1}{2}}-type equations, isospectral limit, spectral curve, hyperelliptic curve, degeneration of curves.

2010 Mathematics Subject Classification: 34M55, 33E17, 14H70.

(*) I would like to express my utmost gratitude to Prof. Hiroshi Kawakami, Prof. Eric Rains, and Prof. Hidetaka Sakai. I am also grateful to Prof. Hayato Chiba, Prof. Yusuke Nakamura, Prof. Seiji Nishioka, Prof. Akishi Kato, Prof. Hiroyuki Kawamuko, Prof. Kinji Kimura, Prof. Kazuki Hiroe, Prof. Shinobu Hosono, Prof. Kenji Iohara, Prof. Takayuki Okuda, and Prof. Ralph Willox for valuable discussions and advices.
Résumé. — Dans des études récentes, des analogues en 4 dimensions des équations de Painlevé ont été perturbés et il existe 40 types. Le but du présent article est de caractériser géométriquement ces 40 équations de type Painlevé. A cet effet, nous étudions la limite autonome de ces équations et la dégénérescence de leurs courbes spectrales. Les courbes spectrales sont des familles paramétrées de courbes de genre deux et leur dégénérescence géométrique est d’un des types classiques par Namikawa et Ueno. L’algorithme de Liu nous permet de trouver le types de dégénérescence de courbes spectrales pour nos 40 types de systèmes intégrables. Ce résultat est analogue au fait suivant; les familles des courbes spectrales des équations de Painlevé autonomes bidimensionnelles $P_1, P_{II}, P_{III}, P_{IV}, P_{III}^{DS}, P_{III}^{D_7}, P_{III}^{D_6}, P_V$ et $P_{VI}$ finissent des surfaces elliptiques avec une fibre singulière $H = \infty$ des types Dynkin $E_8^{(1)}, E_7^{(1)}, E_6^{(1)}, D_8^{(1)}, D_7^{(1)}, D_6^{(1)}, D_5^{(1)}$ et $D_4^{(1)}$, respectivement.

1. Introduction

Les équations de Painlevé sont 8 types d’équations aux secondes ordres ordinaire des équations différentielles non linéaires avec la propriété de Painlevé qui n’ont pas de solutions élémentaires. Les équations de Painlevé ont des propriétés intéressantes; elles sont dérivées de la déformation isomonodromique de certaines équations linéaires, elles sont liées par le processus de dégénérescence, elles ont des symétries de Weyl affine, elles peuvent être dérivées des réductions des équations de soliton, et elles sont équivalentes aux systèmes Hamiltoniens non autonomes. De plus, leurs systèmes autonomes sont des systèmes intégrables solvables par des fonctions elliptiques. Nous appelons ainsi les systèmes autonomes des équations de Painlevé.

Les équations de Painlevé ont été proposées en tant que généralisation de l’une des propriétés des équations de Painlevé. Les deux directions principales de ces généralisations sont des analogues à des dimensions plus élevées et des analogues de différences. Nous traitons les analogues 4 dimensionnels de ce papier. Les huit types des équations de Painlevé sont appelés les équations au Painlevé bidimensionnelles $P_1, P_{II}, P_{III}, P_{IV}, P_{III}^{DS}, P_{III}^{D_7}, P_{III}^{D_6}, P_V$ et $P_{VI}$ finissent des surfaces elliptiques avec une fibre singulière $H = \infty$ des types Dynkin $E_8^{(1)}, E_7^{(1)}, E_6^{(1)}, D_8^{(1)}, D_7^{(1)}, D_6^{(1)}, D_5^{(1)}$ et $D_4^{(1)}$, respectivement.

(1) Une équation différentielle est dite d’avoir la propriété de Painlevé si sa solution générale n’a pas de singularités critiques qui dépendent des valeurs initiales.
equations\(^{(2)}\). As for the 4-dimensional case, classification and derivation of the Painlevi\(_4\)\(^{1/2}\)-type equations from isomonodromic deformation have been completed [21, 23, 22, 25, 51]. According to their result, there are 40 types of 4-dimensional Painlevi\(_4\)\(^{1/2}\)-type equations. Among these 40 types of equations, some of the equations coincide with equations already known from different contexts. Such well-known equations along with new equations are all organized in a unified way: isomonodromic deformation and degeneration.

There are problems to their classification. They say that there are “at most” 40 types of 4-dimensional Painlevi\(_4\)\(^{1/2}\)-type equations. There is no guarantee that these 40 equations are actually distinct. Some Hamiltonians of these equations look very similar to each other. Since the appearance of Hamiltonians or equations may change significantly by changes of variables, we can not classify equations by their appearances. Intrinsic or geometrical studies of these equations may be necessary to distinguish the Painlevi\(_4\)\(^{1/2}\)-type equations. The aim of this paper is to initiate geometrical studies of 4-dimensional Painlevi\(_4\)\(^{1/2}\)-type equations starting from these concrete equations.

Let us first review how the 2-dimensional Painlevi\(_2\)\(^{1/2}\) equations are geometrically classified. Okamoto initiated the studies of the space of initial conditions of the Painlevi\(_2\)\(^{1/2}\) equations [42]. He constructed the rational surfaces, whose points correspond to the germs of the meromorphic solutions of the Painlevi\(_2\)\(^{1/2}\) equations by resolving the singularities of the differential equations. Sakai extracted the key features of the spaces of initial conditions and classified what he calls the “generalized Halphen surfaces” with such features [49]. The classification of the 2-dimensional difference Painlevi\(_2\)\(^{1/2}\) equations correspond to the classification of generalized Halphen surfaces, and 8 types of such surfaces give the Painlevi\(_2\)\(^{1/2}\) differential equations. Such surfaces are distinguished by their anticanonical divisors. For the autonomous 2-dimensional Painlevi\(_2\)\(^{1/2}\) equations, the spaces of initial conditions are elliptic surfaces and their anticanonical divisors are one of the Kodaira types [50]. We can say that the 2-dimensional autonomous Painlevi\(_2\)\(^{1/2}\) equations are characterized or distinguished by the corresponding Kodaira types.

\(^{(2)}\) In this paper we use the term the Painlevi\(_4\)\(^{1/2}\)-type equations synonymously with isomonodromic deformation equations, despite the fact that Painlevi\(_4\)\(^{1/2}\) has never investigated isomonodromy. We prefer to use the term the Painlevi\(_4\)\(^{1/2}\)-type equations rather than the Schlesinger-type equations in order to express that our aim is to understand various higher-dimensional analogs of the Painlevi\(_4\)\(^{1/2}\) equations.
It is expected to carry out a similar study for the 4-dimensional Painlevé $\frac{1}{2}$-type equations. However, a straightforward generalization of the 2-dimensional cases seems to contain many difficulties. One of the reason is that the classification of 4-folds is much more difficult than that of surfaces. We want to avoid facing such difficulties by considering the spectral curves of the autonomous limit of these Painlevé $\frac{1}{2}$-type equations. While the geometry is made simple considerably in the autonomous limit, important characteristics of the original non-autonomous equations are retained.

Integrable systems are Hamiltonian systems on symplectic manifolds $(M^{2n}, \omega, H)$ with $n$ functions $f_1 = H, \ldots, f_n$ in involution $\{ f_i, f_j \} = 0$. The regular level sets of the momentum map $F = (f_1, \ldots, f_n): M \to \mathbb{C}^n$ are the Liouville tori. The image under $F$ of critical points are called the bifurcation diagram, and it is studied for characterizing integrable systems [6]. However, studying the bifurcation diagrams may become complicated for higher dimensional cases. When an integrable system is expressed in a Lax form, it is not difficult to determine the discriminant locus of spectral curves. We can often find correspondence between the bifurcation diagrams and the discriminant locus of spectral curves [5]. Therefore, we mainly study the degeneration of spectral curves in this paper.

Let $\{ H_i \}_{i=1, \ldots, g}$ be a set of functionally independent invariants of an integrable system $(M, \omega, H)$.

In the 2-dimensional case, the following holds.

**THEOREM** (cf. Theorem 4.1). — *Each autonomous 2-dimensional Painlevé $\frac{1}{2}$ equation defines an elliptic surface, whose general fibers are spectral curves of the system. The Kodaira types of the singular fibers at $H_J = \infty$ are listed as follows.*

| Hamiltonian | $H_{V_1}$ | $H_V$ | $H_{III(D_6)}$ | $H_{II(D_7)}$ | $H_{III(D_8)}$ | $H_{IV}$ | $H_{I_1}$ | $H_{II_1}$ | $H_{I_1}$ |
|-------------|------------|--------|----------------|----------------|----------------|---------|-----------|-----------|-----------|
| Kodaira type | $I_0^*$ | $I_1^*$ | $I_2^*$ | $I_3^*$ | $I_4^*$ | $IV^*$ | $III^*$ | $II^*$ |
| Dynkin type | $D_4^{(1)}$ | $D_5^{(1)}$ | $D_6^{(1)}$ | $D_7^{(1)}$ | $D_8^{(1)}$ | $E_6^{(1)}$ | $E_7^{(1)}$ | $E_8^{(1)}$ |

Table 1.1. The singular fiber at $H_J = \infty$ of spectral curve fibrations of the autonomous 2-dimensional Painlevé $\frac{1}{2}$ equations.

It is well-known that the Dynkin’s types in the above diagram appear in the configurations of vertical leaves of the Okamoto’s spaces of initial conditions [42].

For the autonomous 4-dimensional Painlevé $\frac{1}{2}$-type equations, general invariant sets $\bigcap_{i=1,2} H_i^{-1}(h_i)$ are the 2-dimensional Liouville tori. Such Liouville tori are the Jacobian varieties of the corresponding spectral curves.
Instead of studying the degenerations of 2-dimensional Liouville tori, we study the degenerations of the spectral curves of genus 2. As an analogy of the above theorem for the 2-dimensional Painlevé-I½ equations, we find the following:

**Main theorem** (cf. Theorem 4.5). — The spectral curves of the autonomous 4-dimensional Painlevé-I½-type equations have the following types of generic degenerations as in Table 4.6 in Section 4.

The table shows, for example, that generic degeneration of the spectral curves of the autonomous matrix Painlevé-I½ equations $H_{\text{Mat}}^{\text{IV}}, H_{\text{Mat}}^{\text{III}}, H_{\text{Mat}}^{\text{III(D6)}}, H_{\text{Mat}}^{\text{III(D7)}}$ and $H_{\text{Mat}}^{\text{III(D8)}}$ are $I_0 - I^*_0 - 1$, $I_0 - I^*_1 - 1$, $I_0 - I^*_2 - 1$, $I_0 - I^*_3 - 1$, $I_0 - I^*_4 - 1$ and $I_0 - I^*_5 - 1$ in Namikawa–Ueno’s notation. Those of $H_{\text{Mat}}^{\text{IV}}, H_{\text{Mat}}^{\text{III}}$ and $H_{\text{Mat}}^{\text{III}}$ are $I_0 - IV^* - 1$, $I_0 - III^* - 1$ and $I_0 - II^* - 1$. Therefore, generic degeneration of the spectral curves of the autonomous 4-dimensional matrix Painlevé-I½ equations have one additional elliptic curve to those counterpart of the 2-dimensional systems.

There are various ways to have integrable systems in general. To identify equations from different origins is often not easy; equations may change their appearance by transformations. It is hoped that such intrinsic geometrical studies will be helpful for such identification problems. One possible approach using the degeneration of the Painlevé-I½ divisors(3) is proposed in [39]. This direction will be investigated further in a forthcoming paper.

In recent years, Rains and his collaborators [43, 46, 48] brought about crucial developments in the Painlevé-I½-type (difference) equations using noncommutative geometry. In their theory, the anticanonical divisors of rational surfaces, determining the Poisson structures, are one of the key ingredients. The relation between the anticanonical divisors in their work and the generic degeneration of spectral curves will be explained in future work.

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### Contents

The organization of this paper is as follows. In Section 2, after summarizing preliminaries, we review the classification of the 4-dimensional Painlevé-I½-type equations. In Section 3, we consider the autonomous limit of these 40 equations. In Section 4, we study the generic degeneration of

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(3) The Painlevé-I½ divisors introduced by Adler–Moerbeke[3, 4] compactify affine Liouville tori.
the spectral curves to characterize these integrable systems. In Appendix A.1, we list integrals of the autonomous 4-dimensional Painlevé$_i$$_{1/2}$-type equations. In Appendix A.3, we list the dual graph of the singular fibers appeared in our table.

2. Classification of 4-dimensional Painlevé$_i$$_{1/2}$-type equations

In this section, we review some of the recent progresses in classification of the 4-dimensional Painlevé$_i$$_{1/2}$-type equations, and introduce notation we use in this paper. The contents of this section is a summary of the other papers [21, 23, 22, 25, 51] and references therein.

The Painlevé$_i$$_{1/2}$ equations were found by Painlevé$_i$$_{1/2}$ through his classification of the second order algebraic differential equations with the “Painlevé$_i$$_{1/2}$ property”. However, a straightforward application of Painlevé$_i$$_{1/2}$’s classification method to higher-dimensional cases seems to face difficulties(4). Therefore other properties which characterize the Painlevé$_i$$_{1/2}$ equations become important for further generalization. The Painlevé$_i$$_{1/2}$ equations can be expressed as Hamilton systems [37, 42].

\[
\frac{dq}{dt} = \frac{\partial H}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q}.
\]

(4) Works of Chazy [7] and Cosgrove [8, 9] are famous in this direction.
We list the Hamiltonian functions for all the 2-dimensional Painlevé equations for later use.

\[ t(t - 1)H_{VI}(\alpha, \beta, \gamma, \epsilon; t; q, p) = q(q - 1)(q - t)p^2 + \{\epsilon q(q - 1) - (2\alpha + \beta + \gamma + \epsilon)q(q - t) + \gamma(q - 1)(q - t)\}p + \alpha(\alpha + \beta)(q - t), \]

\[ tH_{V}(\alpha, \beta, \gamma; t; q, p) = p(p + t)q(q - 1) + \beta pq + \gamma p - (\alpha + \gamma)tq, \]

\[ H_{IV}(\alpha, \beta, \gamma; t; q, p) = pq(p - q - t) + \beta p + \alpha q, \]

\[ tH_{III}(D_6)(\alpha, \beta; t; q, p) = p^2q^2 - (q^2 - \beta q - t)p - \alpha q, \]

\[ tH_{III}(D_7)(\alpha; t; q, p) = p^2q^2 + \alpha qp + tp + q, \]

\[ tH_{III}(D_8)(t; q, p) = p^2q^2 + qp - q - \frac{t}{q}, \]

\[ H_{II}(\alpha; t; q, p) = p^2 - (q^2 + t)p - \alpha q, \]

\[ H_{I}(t; q, p) = p^2 - q^3 - tq. \]

The Painlevé equations have another important aspect initiated by R. Fuchs [11]. Namely, they can be derived from (generalized) isomonodromic deformation of linear equations [19].

Furthermore, these eight types of the Painlevé equations are linked by processes called degenerations. In fact, \( H_{I}, \cdots, H_{V} \) can be derived from \( H_{VI} \) through degenerations.

\[ H_{VI} \rightarrow H_{V} \leftarrow H_{III}(D_6) \rightarrow H_{III}(D_7) \rightarrow H_{III}(D_8) \rightarrow H_{IV} \leftarrow H_{II} \rightarrow H_{I} \]

2.1. The classification of Fuchsian equations and isomonodromic deformation

If we fix the number of accessory parameters and identify the linear equations that transform into one another by Katz’s operations (addition and middle convolutions) [10, 20], we have only finite types of linear equations. We explain the notion of spectral type used in this paper. We follow the notion used in Oshima [44] for Fuchsian linear equations, Kawakami–Nakamura–Sakai [25] for unramified linear equations and Kawakami [21] for ramified equations. For the classification of linear equations, we need
to discern the types of linear equations. We review the local normal forms of linear equations, and introduce symbols to express such data. We study linear systems of first-order equations

\[ \frac{dY}{dx} = A(x)Y. \]

(2.1)

We first consider the Fuchsian case where

\[ A(x) = \sum_{i=1}^{n} \frac{A_i}{x-t_i}. \]

We assume that each matrix \( A_i \) is diagonalizable. The equation can be transformed into

\[ \frac{d\hat{Y}(x)}{dx} = \frac{T^{(i)}}{x-t_i} \hat{Y}(x), \quad T^{(i)}: \text{diagonal} \]

by a local transformation \( Y = P(x)\hat{Y} \). We express the multiplicity of the eigenvalues by a non-increasing sequence of numbers.

**Example 2.1.** — When \( T^{(i)} = \text{diag}(a, a, b, b, c) \), we write the multiplicity as 321.

Collecting such multiplicity data for all the singular points, the spectral type of the linear equation is defined as the \( n+1 \)-tuples of partitions of \( m \),

\[
\left( \begin{array}{c}
\underbrace{m_1^1 m_2^1 \ldots m_l^1}, & \underbrace{m_1^2 \ldots m_l^2}, & \ldots, & \underbrace{m_1^n \ldots m_l^n}, & m_1^\infty \ldots m_l^\infty,
\end{array} \right)
\]

\[
\left( \sum_{j=1}^{l_i} m_j^i = m \text{ for } 1 \leq i \leq n \text{ or } i = \infty \right),
\]

where \( m \) is the size of matrices.

**Theorem 2.2** (Kostov [32]). — Irreducible Fuchsian equations with two accessory parameters result in one of the four types by successive additions and middle convolutions:

- 11, 11, 11, 11
- 111, 111, 111 22, 1111, 1111 33, 222, 11111.

**Remark 2.3.** — Note that only the equation of the type 11, 11, 11, 11 has four singular points and the other three types have three singular points. The three of the singular points can be fixed at 0, 1, \( \infty \) by a Möbius transformation. Thus the three equations with only three singularities do not admit the continuous deformation of position of singularities. Among these 4 types, only linear equation of type 11, 11, 11, 11 admit isomonodromic deformation, and it gives the sixth Painlevé \( \frac{1}{2} \) equation \( H_{VI} \).
The Katz’s operations are important for studying Painlevi\(_{\frac{1}{2}}\)-type equations, because the following theorem holds.

**Theorem 2.4** (Haraoka–Filipuk [12]). — *Isomonodromic deformation equations are invariant under Katz’s operations.*

**Remark 2.5.** — Katz’s operations that do not change the type of the linear equation induce the corresponding Bi\(_{\frac{1}{2}}\)cklund transformations on the isomonodromic deformation equation. In fact, all of the \(D^{(1)}_4\)-type affine Weyl group symmetry that \(P_{VI}\) possesses can be derived from Katz’s operations and the Schlesinger transformations on the linear equations [38].

### 2.2. The classification of Fuchsian linear equations and isomonodromic deformation

The starting point of the classification of the 4-dimensional Painlevi\(_{\frac{1}{2}}\)-type equation is the following result.

**Theorem 2.6** (Oshima [44]). — *Irreducible Fuchsian equations with four accessory parameters result in one of the following 13 types by successive additions and middle convolutions\(^{(5)}\):*

- 11, 11, 11, 11, 11
- 21, 21, 111, 111  31, 22, 22, 1111  22, 22, 22, 211
- 211, 1111, 1111  221, 221, 11111  32, 11111, 11111
- 222, 222, 2211  33, 2211, 111111  44, 2222, 22211
- 44, 332, 11111111  55, 3331, 22222  66, 444, 2222211.

**Remark 2.7.** — Note that the equation of type 11, 11, 11, 11, 11 has five singular points, and the next three types have four singular points, and the rest nine types have three singular points. The equation of type 11, 11, 11, 11, 11 has two singularities to deform after fixing three of the singularities to 0, 1, \(\infty\). The next three types of equations with four singularities have one singularity to deform after fixing three of the singularities to 0, 1, \(\infty\). The nine equations with only three singularities do not admit continuous isomonodromic deformation.

The invariance of the isomonodromic deformation equation by Katz’ operations is guaranteed by Haraoka–Filipuk [12]. Sakai derived explicit Hamiltonians of the above four equations with four accessory parameters.

\(^{(5)}\)These operations are called the Katz’ operations [10, 20]
Theorem 2.8 (Sakai [51]). — There are four 4-dimensional Painlevï¿½ \textit{-type equations governed by Fuchsian equations.}

- The Garnier system in two variables \((11,11,11,11,11)\).
- The Fuji–Suzuki system \((21,21,111,111)\).
- The Sasano system \((31,22,22,1111)\).
- The Sixth Matrix Painlevï¿½ equation of size 2 \((22,22,22,211)\).

In this paper, we call these 4 equations derived from Fuchsian equation the “source equations”.

2.3. Degeneration scheme of 4-dimensional Painlevï¿½ \textit{-type equations}

Other 4-dimensional Painlevï¿½ \textit{-type equations we consider are derived from these source equations by degeneration process. The degenerations corresponding to unramified linear equations are treated in Kawakami–Nakamura–Sakai [25]. The degenerations corresponding to ramified linear equations are treated in Kawakami [21, 23, 22]. There are 40 types of the 4-dimensional Painlevï¿½ \textit{-type equations with 16 partial differential equations corresponding to the differential Garnier equations and 24 ordinary differential equations. Among 24 ordinary differential equations, 8 types are the matrix Painlevï¿½ \textit{-equations. According to [45], the 7 types with the source equation \(H\text{}_{\text{Ds}}\) correspond to the symmetric \(q\)-difference Garnier equations, and the 9 types with the source equation \(H\text{}_{\text{FS}}\) correspond to the nonsymmetric \(q\)-difference Garnier equations in Rains [46].

As shown in Diagram 2.2 [22], there are 4 series of degeneration diagram corresponding to 4 “source equations”. Explicit forms of the Hamiltonians and the Lax pairs can be found in [21, 23, 22, 25].

Remark 2.9. — The names of these Hamiltonians are temporal. As Sakai [49] labeled the 2-dimensional systems by the types of the anticanonical divisors of the compactified spaces of initial conditions, it may be natural to label the 4-dimensional systems from geometrical characterization.

Remark 2.10. — While Kawakami–Nakamura–Sakai [25] studied the degeneration from Fuchsian types, the theory of unramified non-Fuchsian linear equations has developed. Hiroe and Oshima classified all the unramified linear equations with 4 accessory parameters up to some transformations [14, Theorem 3.29]. Yamakawa proved that analogous theorem of Haraoka–Filpuk [12] holds for unramified non-Fuchsian equations [57].
Table 2.1. The list of the 4-dimensional Painlevé$_6^{1/2}$-type equations

By comparing the results, all the unramified equations with 4 accessory parameters come from the 4 Fuchsian source equations by degenerations, and the list of 4-dimensional Painlevé$_6^{1/2}$-type equations corresponding to the unramified linear equations is complete. To the author’s knowledge, it is still an open question whether linear equations of ramified type with 4
accessory parameters can be reduce to this list of 40 types. Rains’ work [47] might be conclusive in this direction.

Remark 2.11. — According to the classification of the Lax pairs by Rains [47], there are 40 types of 4-dimensional families which admit continuous deformations, comprising of 16 differential Garnier equations, 7 symmetric \( q \)-difference Garnier equations, 9 nonsymmetric \( q \)-Garnier equations and 8 matrix Painlevï¿½ equations. The numbers match with Kawakami’s list [22].

Remark 2.12. — Some of these 40 equations look similar to each other. For instance, H. Chiba pointed out that \( H_{\text{Gar},t_1}^{k+1} \) and \( \tilde{H}_{\text{Mat}}^{\text{II}} \) look almost the same after a symplectic transformations.\(^6\)

\[
H_{\text{Gar},t_1}^{k+1} = p_1^2 - (q_1^2 + t_1) p_1 + \kappa_1 q_1 + p_1 p_2 + p_2 q_2 (q_1 - q_2 + t_2) + \theta_0 q_2,
\]

\[
\tilde{H}_{\text{Mat}}^{\text{II}} = p_1^2 - \left( \frac{q_1^2}{4} + t \right) p_1 - \left( \theta_0 + \frac{\kappa_2}{2} \right) q_1 + p_1 p_2 + p_2 q_2 (q_1 - q_2) + \theta_0 q_2.
\]

One of the key motivations of the present paper is to geometrically distinguish such cases. We will show in Section 4 that the types of generic degeneration of spectral curves are different\(^7\).

Remark 2.13. — Some of the equations in the list, such as the Noumi–Yamada systems, had been known from different context. See [25] for the references for other derivations.

3. Autonomous limit of Painlevï¿½ type equations

In the previous section, we saw that there are 40 types of 4-dimensional Painlevï¿½\(1/2\)-type equations. In this section, we consider the autonomous limit of these 40 equations by taking the isospectral limit of the isomonodromic deformation equations. Using the Lax pair, we obtain two functionally independent invariants for each system. Therefore, the autonomous limit of 4-dimensional Painlevï¿½\(1/2\)-type equations are integrable in Liouville’s sense.

\(^6\) The following transformation to the \( H_{\text{Mat}}^{\text{II}} \) in [25] yields \( \tilde{H}_{\text{Mat}}^{\text{II}} \): \( p_1 \to 2p_1, q_1 \to \frac{q_1}{2}, p_2 \to -q_2, q_2 \to p_2, \kappa_1 \to -\theta_0 - \kappa_2 \).

\(^7\) It is hoped to prove a theorem stating “If the types of generic degeneration of the spectral curves are different, the corresponding space of initial conditions are not isomorphic, or, do not admit biregular map from one another.”
3.1. Integrable system and Lax pair representation

Let us recall the definition of integrability in Liouville’s sense. A Hamiltonian system is a triple \((M, \omega, H)\), where \((M, \omega)\) is a symplectic manifold and \(H\) is a Hamiltonian function on \(M\); \(\iota_{X_H}\omega = dH\). A function \(f\) Poisson commutes with the Hamiltonian \(H\), that is \(\{f, H\} = 0\), if and only if \(f\) is constant along integrable curve of the Hamiltonian vector field \(X_H\). Such a function \(f\) is called a conserved quantity or first integral. A Hamiltonian system is (completely) integrable in Liouville’s sense if it possesses \(n := \frac{1}{2} \dim M\) independent integrals of motion, \(f_1, f_2, \ldots, f_n\), which are pairwise in involution with respect to the Poisson bracket; \(\{f_i, f_j\} = 0\) for all \(i, j\). This definition of integrability is motivated by Liouville’s theorem. Let \((M, \omega, H)\) be a real integrable system of dimension \(2n\) with integral of motion \(f_1, \ldots, f_n\), and let \(c \in \mathbb{R}^n\) be a regular value of \(f = (f_1, \ldots, f_n)\). Liouville’s theorem states that any compact component of the level set \(f^{-1}(c)\) is a torus. The complex Liouville theorem is also known [4].

Many integrable systems are known to have Lax pair expressions:

\[
\frac{dA(x)}{dt} + [A(x), B(x)] = 0, \tag{3.1}
\]

where \(A(x)\) and \(B(x)\) are \(m\) by \(m\) matrices and \(x\) is a spectral parameter. From this differential equation, \(\text{tr} (A(x)^k)\) are conserved quantities of the system:

\[
\frac{d}{dt} \text{tr} (A(x)^k) = \text{tr} (k [B(x), A(x)] A(x)^{k-1}) = 0.
\]

Therefore, the eigenvalues of \(A(x)\) are all conserved quantities since the coefficients of the characteristic polynomial are expressible in terms of these traces. In fact, the Lax pair is equivalent to the following isospectral problem:

\[
\begin{cases}
A(x) = Y A_0(x) Y^{-1}, \\
\frac{dY}{dt} = B(x)Y,
\end{cases}
\]

where \(A_0(x)\) is a matrix satisfying \(\frac{dA_0(x)}{dt} = 0\). The curve defined by the characteristic polynomial is called the spectral curve:

\[
\det (yI_m - A(x)) = 0.
\]
3.2. Isomonodromic deformation to isospectral deformation

The isomonodromic problems have the following forms:

\[
\begin{align*}
\frac{\partial Y}{\partial x} &= A(x,t)Y, \\
\frac{\partial Y}{\partial t} &= B(x,t)Y,
\end{align*}
\]

and the deformation equation is expressed as

\[
\frac{\partial A(x,t)}{\partial t} - \frac{\partial B(x,t)}{\partial x} + [A(x,t), B(x,t)] = 0.
\] (3.2)

We find the similarities in isospectral and isomonodromic problems; the only difference is the existence of the term \(\frac{\partial B}{\partial x}\) in isomonodromic deformation equation. In fact, we can consider isospectral problems as the special limit of isomonodromic problem with a parameter \(\delta\). We restate the isomonodromic problem as follows\(^{(8)}\):

\[
\begin{align*}
\delta \frac{\partial Y}{\partial x} &= A(x,\tilde{t})Y, \\
\frac{\partial Y}{\partial \tilde{t}} &= B(x,\tilde{t})Y,
\end{align*}
\]

where \(\tilde{t}\) is a variable which satisfies \(\frac{d\tilde{t}}{dt} = \delta\). The integrability condition \(\frac{d^2 Y}{dx \partial t} = \frac{d^2 Y}{dt \partial x}\) is equivalent to the following:

\[
\frac{\partial A(x,\tilde{t})}{\partial \tilde{t}} - \delta \frac{\partial B(x,\tilde{t})}{\partial x} + [A(x,\tilde{t}), B(x,\tilde{t})] = 0.
\] (3.3)

The case when \(\delta = 1\) is the usual one\(^{(9)}\). When \(\delta = 0\), the term \(\delta \frac{\partial B}{\partial x}\) drops off from the deformation equation and we have a Lax pair in a narrow sense\(^{(10)}\). The deformation equation 3.2 with \(\delta\) is solved by a Hamiltonian \(H(\delta)\). When \(\delta = 1\), the Hamiltonian \(H(1)\) is equal to the original Hamiltonian of the isomonodromic problem. Therefore, \(H(\delta)\) is a slight modification of the Hamiltonian. When \(\delta = 0\), \(H(0)\) is a conserved quantity of the system.

\(^{(8)}\) Here \(\delta\) plays the role of \(\lambda\) in Deligne’s lambda connections [52]. In some literature such as Levin–Olshanetsky–Zotov [34], it is customary to use \(\kappa\) instead of \(\delta\).

\(^{(9)}\) Adams–Harnad–Hurtubise [2] and Adams–Harnad–Previo [1] studied finite dimensional integrable systems by embedding them into rational coadjoint orbits of loop algebras. Harnad [13] further generalized their theory as applicable to the isomonodromic systems. Such nonautonomous isomonodromic systems are obtained by identifying the time flows of the integrable system with parameters determining the moment map.

\(^{(10)}\) In other words, we mean a Lax pair in the sense of integrable systems. The isomonodromic problems are often called as Lax pairs, but they do not give first integrals.
Remark 3.1. — Isomonodromic equations are flows on moduli space of connections [18]. Isospectral limit correspond to $\lambda \to 0$ limit of moduli of $\lambda$-connections [52] to moduli of Higgs bundles. The Painlevé half-type equations become the Hitchin systems [15] at the limit.

Taking the isospectral limit of 8 types of 2-dimensional Painlevé equations, we can state the following classically-known result.

Proposition 3.2. — As the autonomous limits of 2-dimensional Painlevé equations, we obtain 8 types of integrable systems with a first integral for each system (11).

Proof. — We take the second Painlevé half equation as an example to demonstrate a proof. Proofs of the other equations are similar.

\[
\begin{align*}
\delta \frac{\partial Y}{\partial x} &= A(x, \tilde{t})Y, & A(x, \tilde{t}) &= \left( A_{\infty}^{(-3)}(\tilde{t})x^2 + A_{\infty}^{(-2)}(\tilde{t})x + A_{\infty}^{(-1)}(\tilde{t}) \right), \\
\frac{\partial Y}{\partial t} &= B(x, \tilde{t})Y, & B(x, \tilde{t}) &= \left( A_{\infty}^{(-3)}(\tilde{t})x + B_1(\tilde{t}) \right), \\
\end{align*}
\]

where

\[
\begin{align*}
\hat{A}_{\infty}^{(-3)}(\tilde{t}) &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, & \hat{A}_{\infty}^{(-2)}(\tilde{t}) &= \begin{pmatrix} 0 & 1 \\ -p + q^2 + \tilde{t} & 0 \end{pmatrix}, \\
\hat{A}_{\infty}^{(-1)}(\tilde{t}) &= \begin{pmatrix} -p + q^2 + \tilde{t} & q \\ (p - q^2 - \tilde{t})q - \kappa_2 & p - q^2 \end{pmatrix}, & \hat{B}_1(\tilde{t}) &= \begin{pmatrix} q \\ p - q^2 - \tilde{t} \end{pmatrix}, \\
A_{\infty}^{(-i)} &= U^{-1} \hat{A}_{\infty}^{(-i)}U \quad \text{for } i = 1, 2, 3, & B_1 &= U^{-1} \hat{B}_1 U, & U &= \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}.
\end{align*}
\]

The deformation equation (3.3) is equivalent to the following differential equations.

\[
\frac{dq}{dt} = 2p - q^2 - \tilde{t}, \quad \frac{dp}{dt} = 2pq + \delta - \kappa_1, \quad \frac{du}{dt} = 0.
\]

The first two equations are equivalent to the Hamiltonian system

\[
\frac{dq}{dt} = \frac{\partial H_{11}(\delta)}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H_{11}(\delta)}{\partial q},
\]

with the Hamiltonian $H_{11}(\delta) := p^2 - (q^2 + \tilde{t})p + (\kappa_1 - \delta)q$. When $\delta = 1$, it is the usual Hamiltonian of $H_{11}$. Moreover, when $\delta \neq 0$, we can normalize to the $\delta = 1$ case. Taking the limit $\delta \to 0$, we obtain an autonomous system with a Hamiltonian $H_{11}(0) = p^2 - (q^2 + \tilde{t})p + \kappa_1 q$. Since it is an autonomous system, the Hamiltonian is a first integral. The dimension of

\[\text{(11)}\]

These first integrals are the autonomous Hamiltonians. They are rational in the phase variables $q, p$. 
the phase space is two, the number of first integrals is half of the dimension. Therefore, the autonomous second Painlevé $\frac{1}{2}$ equation is integrable in Liouville’s sense. The Lax pair\(^{(12)}\) and the spectral curve of the autonomous second Painlevé $\frac{1}{2}$ equation are

$$\frac{dA(x)}{dt} + [A(x), B(x)] = 0,$$

$$\det(yI - A(x)) = y^2 - (x^2 + \tilde{t})y - \kappa_1 x - H_{\Pi}(0) = 0.$$ \(\square\)

**Remark 3.3.** — For the 2-dimensional cases, parameters of the Painlevé $\frac{1}{2}$ equations can be thought as roots of affine root systems, and $\delta$ corresponds to the null root [49, 50].

### 3.3. The autonomous limit of the 4-dimensional Painlevé $\frac{1}{2}$-type equations

We can also consider such autonomous limit for higher dimensional Painlevé $\frac{1}{2}$-type equations. From the coefficients of the spectral curves, we obtain first integrals.

**Theorem 3.4.** — As the autonomous limits of 4-dimensional Painlevé $\frac{1}{2}$-type equations, we obtain 40 types of integrable systems with two functionally independent first integrals for each system\(^{(13)}\).

**Proof.** — One of the simplest 4-dimensional Painlevé $\frac{1}{2}$-type equation is the first matrix Painlevé $\frac{1}{2}$ equation [21]. The linear equation is given by

$$\frac{dA(x)}{dt} + [A(x), B(x)] = 0,$$

$$A(x) = (A_0 x^2 + A_1 x + A_2), \quad B(x) = A_0 x + B_1.$$ 

where

$A_0 = \begin{pmatrix} O_2 & I_2 \\ O_2 & O_2 \end{pmatrix}, \quad A_1 = \begin{pmatrix} O_2 & Q \\ I_2 & O_2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -P & Q^2 + \tilde{t} I_2 \\ -Q & P \end{pmatrix},$

$B_1 = \begin{pmatrix} O_2 & 2Q \\ I_2 & O_2 \end{pmatrix}, \quad O_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$

$Q = \begin{pmatrix} q_1 & u \\ -q_2/u & q_1 \end{pmatrix}, \quad P = \begin{pmatrix} p_{1/2} & -p_{2u} \\ (p_{2q_2} - \kappa_2)/u & p_{1/2} \end{pmatrix}.$

\(^{(12)}\) We rewrite $A(x) = A(x, \tilde{t})$ and $B(x) = B(x, \tilde{t}).$

\(^{(13)}\) These invariants are rational in the phase variables $q_1, p_1, q_2, p_2.$
The spectral curve is defined by the characteristic polynomial of the matrix \( A(x) \):

\[
\det (yI_4 - A(x)) = y^4 - (2x^3 + 2\hat{t}x + h)y^2 + x^6 + 2\hat{t}x^4 + hx^3 + \hat{t}^2x^2 + (\hat{t}h - \kappa_2^2)x + g.
\]

The explicit forms of \( h \) and \( g \) are

\[
h := H^\text{Mat}_1 = \text{tr} \left( P^2 - Q^3 - \hat{t}Q \right) = -2p_2(p_2q_2 - \kappa_2) + \frac{p_1^2}{2} - 2q_1\hat{t} - 2q_1(q_1^2 - q_2) + 4q_1q_2,
\]

\[
g := G^\text{Mat}_1 = q_2(p_1p_2 + 3q_1^2 - q_2 + \hat{t})^2 - \kappa_2p_1(p_1p_2 + 3q_1^2 - q_2 + \hat{t}) - 2\kappa_2^2q_1.
\]

Since \( h \) and \( g \) are coefficient of the spectral curve, they are invariants of the autonomous system. We can also check that \( h \) and \( g \) are conserved by direct computation:

\[
\dot{h} = X_h h = \{h, h\} = 0, \quad \dot{g} = X_h g = \{g, h\} = 0,
\]

where \( X_h \) is the Hamiltonian vector field associated to the Hamiltonian \( h \).

The Poisson bracket in the above equations is defined by

\[
\{F, G\} := \sum_{i=1}^{2} \left( \frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} - \frac{\partial G}{\partial q_i} \frac{\partial F}{\partial p_i} \right).
\]

Since

\[
\text{rank} \begin{pmatrix}
\frac{\partial h}{\partial q_1} & \frac{\partial h}{\partial q_1} & \frac{\partial h}{\partial q_1} & \frac{\partial h}{\partial q_1} \\
\frac{\partial h}{\partial p_1} & \frac{\partial h}{\partial p_1} & \frac{\partial h}{\partial p_1} & \frac{\partial h}{\partial p_1} \\
\frac{\partial h}{\partial q_2} & \frac{\partial h}{\partial q_2} & \frac{\partial h}{\partial q_2} & \frac{\partial h}{\partial q_2} \\
\frac{\partial h}{\partial p_2} & \frac{\partial h}{\partial p_2} & \frac{\partial h}{\partial p_2} & \frac{\partial h}{\partial p_2}
\end{pmatrix} = 2
\]

for the general value of \((q_1, p_1, q_2, p_2)\), we have two functionally independent invariants of the system. Thus the autonomous Hamiltonian system with the Hamiltonian \( H^\text{Mat}_1 \) is integrable in Liouville’s sense.

From similar direct computations, we obtain the desired results for all the rest of 4-dimensional Painlev\(\text{\ddash}1/2\)-type equations. We list functions in involution for the ramified types in Appendix A.1. These spectral curves and conserved quantities can be calculated from the data in the papers [21, 23, 22, 25, 51]. The only troublesome part is to find appropriate modifications of the Hamiltonians in the presence of \( \delta \). The other parts are straightforward. \(\square\)
Remark 3.5. — It is an interesting problem to study the invariant surfaces defined by \( H^{-1}(c_1) \cap G^{-1}(c_2) \subset \mathbb{C}^4 \) for \( c_1, c_2 \in \mathbb{C} \), where \( H \) and \( G \) are functionally independent invariants of the system. As in the case of other integrable systems [4], these Liouville tori can be completed into Abelian surfaces by adjoining the Painlevé \( \frac{1}{2} \) divisors [39].

4. Degeneration of spectral curves

This section is the main part of this paper.

We study generic degeneration of spectral curves of the autonomous 4-dimensional Painlevé \( \frac{1}{2} \)-type equations, aiming to characterize these systems.

4.1. Genus 1 fibration and Tate’s algorithm

Before discussing the genus 2 cases, corresponding to the autonomous limit of 4-dimensional Painlevé \( \frac{1}{2} \)-type equations, we discuss the genus 1 cases, corresponding to the autonomous 2-dimensional Painlevé \( \frac{1}{2} \) equations.

Let us recall some of the basics we need. We can construct the Kodaira–Néron model of an elliptic curve \( E \) over \( \mathbb{A}^1 = \text{Spec}(\mathbb{C}[h]) \). The possible types of singular fibers of elliptic surfaces were classified by Kodaira [30, 31]. Tate’s algorithm provides a way to determine the Kodaira type of singular fibers without actually resolving the singularities [55].

We consider an elliptic curve \( E \) over \( \mathbb{A}^1 \) with a section in the Weierstrass form:

\[
y^2 = x^3 + a(h)x + b(h), \quad a(h), b(h) \in \mathbb{C}[h].
\]

We may assume that for \( a(h) \) and \( b(h) \), the polynomials \( l(h) \) such that \( l(h)^4|a(h), l(h)^6|b(h) \) are only constants. Otherwise, we may divide both sides of the equations by \( l(h)^6 \) and replace \( x, y \) by \( x/l(h)^2, y/l(h)^3 \) if necessary. Let \( X_1 \) be the affine surface defined by the equation (4.1):

\[
X_1 = \{(x, y, h) \in \mathbb{A}^2 \times \mathbb{A}^1 \mid y^2 = x^3 + a(h)x + b(h)\}.
\]

A general fiber of the projection \( \varphi_1 : X_1 \to \mathbb{A}^1 \) is an affine part of an elliptic curve. Let \( n \) be the minimal positive integer satisfying \( \deg a(h) \leq 4n \) and \( \deg b(h) \leq 6n \). Dividing equation (4.1) by \( h^{6n} \) and replacing \( \tilde{x} = x/h^{2n} \), \( \tilde{y} = y/h^{3n} \), \( \tilde{H} = 1/h \), we obtain the “\( \infty \)-model”:

\[
\tilde{y}^2 = \tilde{x}^3 + \tilde{a}(\tilde{H})\tilde{x} + \tilde{b}(\tilde{H}), \quad \tilde{a}(\tilde{H}), \tilde{b}(\tilde{H}) \in \mathbb{C}[\tilde{H}]
\]
where \( \tilde{a}(\tilde{H}) = a(h)/h^{4n}, \quad \tilde{b}(\tilde{H}) = b(h)/h^{6n} \) are polynomials in \( h \). Let \( X_1 \) be the affine surface defined by equation (4.2). Let \( \overline{X_1} \) and \( \overline{X_2} \) be the projectivized surfaces in \( \mathbb{P}^2 \times \mathbb{A}^1 \): \( \overline{X_1} \subset \mathbb{P}^2 \times \mathbb{A}^1, \overline{X_2} : X_1 \to \mathbb{A}^1 \). We glue \( \overline{X_1} \) and \( \overline{X_2} \) by identifying \((x, y, h)\) and \((\tilde{x}, \tilde{y}, \tilde{H})\) by the equations above. Let us denote the surface obtained this way by \( W \). We call \( W \) the Weierstrass model of the elliptic curve (4.1). The surface \( W \) has a morphism \( \phi : W \to \mathbb{P}^1 = \mathbb{A}^1 \cup \mathbb{A}^1 \). After the minimal resolution of the singular points of \( W \), we obtain a nonsingular surface \( X \). This nonsingular projective surface \( X \) together with the fibration \( \phi : X \to \mathbb{P}^1 \) is called the Kodaira–N\( \ddot{\text{e}} \)ron model of the elliptic curve \( E \) over \( \mathbb{A}^1 \).

The singular fibers of an elliptic surface are classified by Kodaira. The Kodaira type of the elliptic surface \( X \) can be computed from the equation (4.1) using Tate’s algorithm. From the Weierstrass form equation (4.1), we can associate two quantities: \( \Delta := 4a^3 + 27b^2, \quad j := 4a^3/\Delta \). Here, \( \Delta \) is the discriminant of the cubic \( x^3 + a(h)x + b(h) \) and \( j \) is the \( j \)-invariant. The Kodaira types of the singular fibers are determined as in Table 4.1 by the order of \( \Delta \) and \( j \) which we denote \( \text{ord}_v(\Delta), \text{ord}_v(j) \).

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
\text{Kodaira} & \text{Dynkin} & \text{ord}(\Delta) & \text{ord}(j) & \text{Kodaira} & \text{Dynkin} & \text{ord}(\Delta) & \text{ord}(j) \\
\hline
I_0 & - & 0 & \geq 0 & I_0^* & D_4^{(1)} & 6 & \geq 0 \\
I_m & A_{m-1}^{(1)} & m & -m & I_m^* & D_{4+m}^{(1)} & 6 + m & -m \\
II & - & 2 & \geq 0 & IV^* & E_6^{(1)} & 8 & \geq 0 \\
III & A_1^{(1)} & 3 & \geq 0 & III^* & E_7^{(1)} & 9 & \geq 0 \\
IV & A_2^{(1)} & 4 & \geq 0 & II^* & E_8^{(1)} & 10 & \geq 0 \\
\hline
\end{array}
\]

Table 4.1. Tate’s algorithm and Kodaira types

4.1.1. Elliptic surface associated with the spectral curves

We introduce the main subject of this article, fibration of spectral curves associated with integrable Lax equations. Let us consider a \( 2n \)-dimensional integrable system with a Lax pair. The spectral curve is parametrized by \( n \) functionally independent first integrals \( H_1, \ldots, H_n \).

**Theorem 4.1.** — Each elliptic surface whose general fiber is a spectral curve of the autonomous 2-dimensional Painlev\( \ddot{\text{e}} \)\( \ddot{\text{n}} \)-type equation has the following singular fiber at \( H = \infty \).

**Proof.** — First, let us consider the first Painlev\( \ddot{\text{e}} \)\( \ddot{\text{n}} \)-type equation

\[
\frac{d^2 q}{dt^2} = 6q^2 + t.
\]
Table 4.2. The singular fiber at $H = \infty$ of spectral curve fibrations of the autonomous 2-dimensional Painlevi$_{\frac{1}{2}}$ equations.

The first Painlevi$_{\frac{1}{2}}$ equation has a Lax form
\[
\frac{\partial A}{\partial t} - \delta \frac{\partial B}{\partial x} + [A, B] = 0,
\]
\[
A(x) = \left( \begin{array}{c} -p \\ x - q \\ p \end{array} \right), \quad B(x) = \left( \begin{array}{c} 0 \\ x + 2q \\ 0 \end{array} \right).
\]

The spectral curve associated with its autonomous equation is defined by
\[
\det (yH_2 - A(x)) = 0.
\]
This is equivalent to the following elliptic curve
\[
y^2 = x^3 + \tilde{t}x + H_1, \quad H_1 = p^2 - q^3 + \tilde{t}q.
\]

Let us write $H_1$ as $h$ for short. We consider this curve as an elliptic curve $E$ over an affine line $A^1$.

By changing variables as $\tilde{H} = h^{-1}$, $\tilde{x} = h^{-2}x$, $\tilde{y} = h^{-3}y$, obtain the “$\infty$-model”;
\[
(4.3) \quad \tilde{y}^2 = \tilde{x}^3 + \tilde{t}\tilde{H}^4 \tilde{x} + \tilde{H}^5.
\]

Thus we get the Weierstrass model $\varphi: W \to \mathbb{P}^1$. The Kodaira–Neron model $\phi: X \to \mathbb{P}^1$ of elliptic curve $E$ is obtained from $W$ by the minimal desingularization. The Kodaira-type of singular fiber at $h = \infty$ can be computed using the equation (4.3). The discriminant of the cubic $\tilde{x}^3 + \tilde{t}\tilde{H}^4 \tilde{x} + \tilde{H}^5$ and the $j$-invariant are
\[
\Delta = 4 \left( i\tilde{H}^4 \right)^3 + 27 \left( \tilde{H}^5 \right)^2 = \tilde{H}^{10} \left( 27 + 4i^3 \tilde{H}^2 \right),
\]
\[
j = \frac{4}{\Delta} \left( i\tilde{H}^4 \right)^3 = \frac{4i^3 \tilde{H}^{12}}{\tilde{H}^{10}(27 + 2i^3 \tilde{H}^2)} = \tilde{H}^2 \frac{4i^3}{27 + 4i^3 \tilde{H}^2}.
\]

Therefore, the order of zero of $\Delta$ and $j$ at $\tilde{H} = 0$ is $\text{ord}_\infty(\Delta) = 10$, $\text{ord}_\infty(j) = 2$. Using Tate’s algorithm, we find that the elliptic surface $X \to \mathbb{P}^1$ has the singular fiber of type $\Pi^*$. In Dynkin’s notation, this fiber is of type $E^*_8$. Let us express the other two zeros of the discriminant $\Delta$ by $h_+$ and $h_-$. Since $h_+$ and $h_-$ are both simple zeros of $\Delta$, the Kodaira type of
the fibers at $h_+$ and $h_-$ are $I_1$, from Tate’s algorithm.

The dual graph of the singular fiber of the Kodaira type $II^*$ (Dynkin type $E_8^{(1)}$). The numbers in circles denote the multiplicities of components in the reducible fibers.

\begin{figure}
\centering
\includegraphics[width=0.5\linewidth]{dual_graph}
\caption{The elliptic surface associated to the spectral curves of the autonomous $P_1$.}
\end{figure}

The spectral curves associated to other autonomous 2-dimensional Painlevé equations are also curves of genus one for the general values of the Hamiltonians. It is well known that curves with genus one can always be transformed into the Weierstrass normal form. With the aid of computer programs, we can transform the spectral curves of autonomous 2-dimensional Painlevé equations into the Weierstrass normal form\textsuperscript{(14)}. Once the spectral curves are in the Weierstrass form, we can construct the elliptic surfaces. We apply Tate’s algorithm to find the Kodaira types of the singular fibers of these elliptic surfaces.

Remark 4.2. — For the special values of the parameters and the constant $\tilde{t}$ of the equations, the order of zeros or poles of $\Delta$ and $j$ changes so that the Kodaira type changes. This corresponds to the situations when the equations have the special solutions. In this paper, we concentrate on the situations when the parameters are generic.

\footnote{Magma, Sage and Maple serve this purpose. Magma even calculates Kodaira types from the equations.}
4.1.2. The Liouville tori and elliptic surface

For the autonomous 2-dimensional Painlevé$_i^{1/2}$-type equations, the Liouville tori are elliptic curves. Therefore, elliptic surfaces are naturally associated to these integrable systems as Hamiltonian fibrations. We think of the time variable $\tilde{t}$ as a constant.

**Theorem 4.3.** — Each elliptic surface associated to the autonomous 2-dimensional Painlevé$_i^{1/2}$ equation as the Hamiltonian fibration has the following singular fiber at $h = \infty$.

| Hamiltonian | $H_{VI}$ | $H_V$ | $H_{III(D_6)}$ | $H_{III(D_7)}$ | $H_{III(D_8)}$ | $H_{IV}$ | $H_{II}$ | $H_I$ |
|------------|---------|-------|----------------|----------------|----------------|--------|---------|-------|
| Kodaira type | $I_0^*$ | $I_1^*$ | $I_2^*$ | $I_3^*$ | $I_4^*$ | $IV^*$ | $III^*$ | $II^*$ |
| Dynkin type | $D_4^{(1)}$ | $D_5^{(1)}$ | $D_6^{(1)}$ | $D_7^{(1)}$ | $D_8^{(1)}$ | $E_6^{(1)}$ | $E_7^{(1)}$ | $E_8^{(1)}$ |

Proof. — Let us first consider the easiest case: the first Painlevé$_i^{1/2}$ equation. We write $h = H_1$ for short. The Hamiltonian of the first Painlevé$_i^{1/2}$ equation is $h = p^2 - (q^3 + \tilde{t}q)$. We view it as an elliptic curve over $\mathbb{A}^1$:

\[
\{(q, p, h) \in \mathbb{A}^2_{(q,p)} \times \mathbb{A}^1_h \mid p^2 = q^3 + \tilde{t}q + h\} \rightarrow \mathbb{A}^1_h.
\]

As in the case of the spectral curve fibration, we can construct the Kodaira–Néron model of the elliptic surface from the equation. Replacing $\tilde{q} = q/h^2$, $\tilde{p} = p/h^3$, $\tilde{H} = 1/h$, we obtain the $\infty$-model:

\[
\bar{p}^2 = \tilde{q}^3 + \tilde{t}\tilde{H}^4 \tilde{q} + \tilde{H}^5.
\]

After compactification and the minimal desingularization, we obtain a regular elliptic surface whose general fiber at $h$ is the elliptic curve defined by $p^2 = q^3 + \tilde{t}q + h$. The discriminant and the $j$-invariant are:

\[
\Delta = 4 \left(\tilde{t}\tilde{H}^4\right)^3 + 27 \left(\tilde{H}^5\right)^2 = \tilde{H}^{10}(27 + 4\tilde{t}^3 \tilde{H}^2),
\]

\[
j = \frac{4(\tilde{t}\tilde{H}^4)^3}{4\tilde{H}^{10}(27 + 4\tilde{t}^3 \tilde{H}^2)} = \tilde{H}^2 \frac{\tilde{t}^3}{27 + 4\tilde{t}^3 \tilde{H}^2}.
\]

Thus the order of zero of $\Delta$ and $j$ at $\tilde{H} = 0$ are $\text{ord}_\infty(\Delta) = 10$, $\text{ord}_\infty(j) = 2$. It follows from Tate’s algorithm that the singular fiber at $h = \infty$ of the autonomous $P_1$-Hamiltonian fibration is of type $II^*$, or $E_8^{(1)}$ in the Dynkin’s notation.

We use computer programs to transform the other Hamiltonians into the Weierstrass normal form. The rest of the proof is similar to the case of $H_1$. □
The agreement of the singular fibers at \( h = \infty \) of the spectral curve fibrations and the Liouville torus fibrations is not a coincidence. Liouville tori are related to the Jacobian varieties of the spectral curves, and taking the Jacobians are isomorphism in genus 1 cases by Abel’s theorem. It might be natural to study the fibration of the Liouville tori, but we have to deal with families of 2-dimensional Abelian varieties to study the autonomous 4-dimensional Painlevé-type equations. Therefore studying the degeneration of the Liouville tori become harder compared to the cases of the 2-dimensional Painlevé equations. On the other hand, we only need to deal with genus 2 curves to study spectral curve fibrations of the 4-dimensional autonomous Painlevé type equations. Thus, studying the degeneration of spectral curves is the main object of this paper.

Using blowing-up process, Okamoto resolved the singularities of 2-dimensional Painlevé\( \frac{1}{2} \) differential equations and constructed the “spaces of initial conditions” [42]. While he deals with singularities of the systems of differential equations, we deal with spectral curves or Hamiltonians themselves for autonomous cases.

A space of initial conditions can be characterized by a pair \((X, D)\) of a rational surface \(X\) and the anti-canonical divisor \(D\) of \(X\). Each irreducible component of \(D\) is a rational curve and, in the case of the Painlevé\( \frac{1}{2} \) equations, is called as a vertical leaf [49]. The intersection diagram of \(D\) is given by that of the certain root lattice listed above.

Remark 4.4. — The spaces of initial conditions are also considered for several cases of 4-dimensional Garnier systems and Noumi–Yamada systems [28, 29, 53, 54].

If we restrict our attention to the autonomous cases, the geometrical studies are much simpler. The autonomous 2-dimensional Painlevé\( \frac{1}{2} \) equations constructed from the spaces of initial conditions were studied by Sakai [50].

4.2. Degeneration of genus 2 curves and Liu’s algorithm

We apply a similar method as in the previous subsection to the 40 types of the autonomous 4-dimensional Painlevé\( \frac{1}{2} \)-type equations. While genus of spectral curves of the autonomous 2-dimensional Painlevé\( \frac{1}{2} \) equations is 1, genus of the autonomous 4-dimensional Painlevé\( \frac{1}{2} \)-type equations is 2. As we use Tate’s algorithm to determine the fibers of the Ni\( \frac{1}{2} \)ron models for the genus 1 curves, we use Liu’s algorithm for the genus 2 curves. While
the spectral curves of the autonomous 2-dimensional Painlevé\(_1\) equations are 1-parameter families parameterized by the Hamiltonian, those of the 4-dimensional Painlevé\(_2\)-type equations are 2-parameters families parameterized by two first integrals.

The spectral curve is now 2-parameter family of genus two curves

\[ F(w, x, y, h_1, h_2) = 0, \]

where \( h_1 \) and \( h_2 \) are the functionally independent conserved quantities of the system and \( F(w, x, y, h_1, h_2) \in \mathbb{C}[w, x, y, h_1, h_2]. \) Let us consider a generic line in Spec(\( \mathbb{C}[h_1, h_2] \))

\[ ah_1 + bh_2 = c, \]

where \( a, b, c \in \mathbb{C} \) are generic. Upon replacing \( \tilde{a} = -a/b, \tilde{b} = c/b, \)

\[ h_2 = \tilde{a}h_1 + \tilde{b} \]

we obtain a one-parameter family of spectral curves

\[ \tilde{F}(w, x, y, h_1) := F(w, x, y, h_1, \tilde{a}h_1 + \tilde{b}) = 0. \]

We study degeneration of this one-parameter family of spectral curves assuming \( \tilde{a} \) and \( \tilde{b} \) are generic.

4.2.1. Liu’s algorithm

We summarize studies on the degeneration of genus two curves. The numerical classification of the fibers in pencils of genus 2 curves are given by Ogg [41] and Iitaka [17]. Namikawa and Ueno [40] completed the geometrical classification of such fibers (and added a few missing types in [41] and [17]). There are 120 types in Namikawa–Ueno’s classification, while there are only 10 types in Kodaira’s classification of the fibers in pencils of genus 1 curves\(^{(15)}\). Liu gave an algorithm similar to Tate’s algorithm for genus 2 curves [35, 36]. Using Liu’s algorithm, we can determine the Namikawa–Ueno type of singular fibers from explicit equations of pencils of genus 2 hyperelliptic curves in the Weierstrass form.

|                | genus of spectral curve | types of singular fibers in pencils | algorithm     |
|----------------|-------------------------|------------------------------------|---------------|
| 2-dim. Painlevé\(_1\) | 1                      | Kodaira                            | Tate’s algorithm |
| 4-dim. Painlevé\(_2\) | 2                      | Namikawa–Ueno                      | Liu’s algorithm |

\(^{(15)}\)Kodaira type \( I_n (n \geq 1) \) and \( I^*_n (n \geq 1) \) are counted as 1 type, respectively.
4.3. Generic degeneration of the spectral curves of the autonomous 4-dimensional Painlevï¿½-type equations

Let us state the main theorem of this paper.

**Theorem 4.5.** — The spectral curves of the autonomous 4-dimensional Painlevï¿½-type equations have the following types of generic degenerations as in Table 4.6.

**Remark 4.6.** — Let us explain the notations used in Table 4.6. The Hamiltonians are the Hamiltonians of the 4-dimensional Painlevï¿½-type equations. The explicit forms of the non-autonomous counterparts can be found in [21, 23, 22, 24, 26, 27, 51]. The spectral types indicate the type of corresponding linear equations. Such notations are explained in Section 2.1 and Appendix A.2. “Namikawa–Ueno type” means the types of degeneration of genus 2 curves in Namikawa–Ueno [40]. When the fiber contains components expressible by the Kodaira-type, we also write its Dynkin’s name in the column noted “Dynkin”. The column named “stable” tells us the type of the stable model [35]. The symbol (n) means the cyclic group with n elements. We also write Ogg’s type written in “On pencils of curves of genus two” [41]. Ogg uses the notation “Kod” to express Kodaira-type and do not distinguish them, while Namikawa and Ueno does. Ogg’s type might be helpful to see the rough classification. For example, all 8 types of matrix Painlevï¿½ equations have the same Ogg’s type 14. The column “monodromy” means 5 monodromy types in Namikawa–Ueno [40]. Elliptic types are those with finite degrees of monodromy, while parabolic types have infinite degrees. Elliptic[1] are those with stable model “I” in Liu’s notation. We abbreviate as “ell[1]”. We summarize such correspondences in Table 4.3. The column named “page” indicate the page number of Namikawa–Ueno’s paper where some data of the corresponding type can be found.

**Proof of Theorem 4.5.** — Let us take Gar, the most degenerated Garnier system, to demonstrate our computation. The Lax pair is given by

\[
\frac{dA(x)}{dt_i} + [A(x), B_i(x)] = 0, \quad i = 1, 2
\]

(16) When n = 0, (n) is the trivial group.
| Type          | Order of monodromy | stable type (Liu’s notation [35]) |
|--------------|--------------------|----------------------------------|
| elliptic[1]  | finite             | I                                |
| elliptic[2]  | finite             | V                                |
| parabolic[3] | infinite           | II, VI                           |
| parabolic[4] | infinite           | III, VII                         |
| parabolic[5] | infinite           | IV                               |

Table 4.3. Namikawa–Ueno’s elliptic and parabolic types and Liu’s stable model types.

| Hamiltonian | the Hamiltonian of the Painlevé\(_{1/2}\)-type equation [21, 23, 22, 25, 51] |
|------------|---------------------------------------------------------------------------------|
| spectral type | the spectral type of the corresponding linear equation [21, 25, 44], Appendix A.2 |
| monodromy | 5 types of monodromy (elliptic[1],[2], Parabolic[3],[4],[5]) as in Namikawa–Ueno [40] |
| Namikawa–Ueno | the type of fiber in the minimal model following the notation in Namikawa–Ueno [36, 40] |
| Dynkin | Dynkin type (when the fiber contains Kodaira-type component) |
| stable | the type of stable model of the fiber [35] |
| \(\Phi\) | the group of connected components of the \(\text{Ni}_1\cdot\text{Hon}\) model of the Jacobian \(J(C)\) [36] |
| Ogg | the type of fiber in the minimal model following the notation in Ogg [41] |
| page | the page number of the fiber in Namikawa–Ueno’s paper [40] |

Table 4.4.

where

\[
A(x) = A_0x^3 + A_1x^2 + A_2x + A_3,
\]

\[
B_1(x) = A_0x^2 + A_1x + B_{10} = \frac{A(x)}{x} + C_1 - \frac{A_3}{x},
\]

\[
B_2(x) = -A_0x + B_{20},
\]
for
\[ A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & p_1 \\ 1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} q_2 & p_1^2 + p_2 + 2\tilde{t}_1 \\ -p_1 & -q_2 \end{pmatrix}, \]
\[ A_3 = \begin{pmatrix} q_1 - p_1q_2 & p_1^3 + 2p_1p_2 - q_2^2 + \tilde{t}_1p_1 - \tilde{t}_2 \\ -p_2 + \tilde{t}_1 & -q_1 + p_1q_2 \end{pmatrix}, \]
\[ B_{10} = \begin{pmatrix} q_2 & p_1^2 + 2p_2 + \tilde{t}_1 \\ -p_1 & -q_2 \end{pmatrix}, \quad B_{20} = \begin{pmatrix} 0 & -2p_2 \\ -1 & 0 \end{pmatrix}, \quad C_1 = \begin{pmatrix} 0 & p_2 - \tilde{t}_1 \\ 0 & 0 \end{pmatrix}. \]
The characteristic polynomial
\[ \det(yI_2 - A(x)) = 0 \]
is expressed as
\[ y^2 = x^5 + 3\tilde{t}_2x^3 - \tilde{t}_1x^2 + (2\tilde{t}_2^2 - h_1)x + h_2 - \tilde{t}_1\tilde{t}_2, \]
where \( h_1 = H_{\text{Gar},\tilde{t}_1}^9, \quad h_2 = H_{\text{Gar},\tilde{t}_2}^9 \). Note that it is already in the Weierstrass form. We consider the degeneration along a line \( h_2 = ah_1 + b \), where \( a \) and \( b \) are generic constants.
\[ y^2 = x^5 + 3\tilde{t}_2x^3 - \tilde{t}_1x^2 + (2\tilde{t}_2^2 - h_1)x + ah_1 + b - \tilde{t}_1\tilde{t}_2, \]
In order to see the degeneration at \( h_1 = \infty \), we introduce \( \tilde{x} = x/h_1, \quad \tilde{y} = y/h_3, \quad \tilde{H} = 1/h_1. \)
\[ \tilde{y}^2 = \tilde{H}x^5 + 3\tilde{t}_2\tilde{H}^3x^3 - \tilde{t}_1\tilde{H}^4x^2 - \tilde{H}^4x + 2\tilde{t}_2\tilde{H}^5x + \tilde{H}^5(a + b\tilde{H} - \tilde{t}_1\tilde{t}_2\tilde{H}). \]
The Igusa invariants of the quintic can be calculated as follows.
\[ J_2 = -5\tilde{H}^5 + \frac{67}{4}\tilde{t}_2\tilde{H}^6, \quad J_4 = \frac{15}{8}\tilde{H}^{10} + O(\tilde{H}^{11}), \quad J_6 = \frac{5}{16}\tilde{H}^{15} + O(\tilde{H}^{16}), \]
\[ J_8 = -\frac{325}{256}\tilde{H}^{20} + O(\tilde{H}^{21}), \quad J_{10} = -\frac{1}{16}\tilde{H}^{25} + O(\tilde{H}^{26}). \]
Since \( 5 \cdot \text{ord}_\infty J_{2i} - \text{ord}_\infty J_{10} = 0 \) for \( i \leq 5 \), the stable model has smooth fiber (type(I) in Liu’s notation of stable curves) at \( \tilde{H} = 0 \). With further computation, we find that generic degeneration is of type \( \text{VII}^* \) in Namikawa–Ueno’s notation. This type is type 22 in Ogg’s notation [41].
\[ \text{VII}^* : H_{\text{Gar},\tilde{t}_1}^2 \]

The numbers in circles denote the multiplicities of components in the reducible fibers. All curves are \((-2)\)-curves except the one expressed as “B”, which is a \((-3)\)-curve.
We can transform the spectral curves of the other autonomous 4-dimensional Painlevé\(^{1/2}\) equations into the Weierstrass normal form\(^{(17)}\). We restrict the 2-parameter families of genus 2 curves to a generic line on the base and apply Liu’s algorithm to find the generic degeneration. □

**Remark 4.7.** — In this paper, we proposed a possible clue to characterize the integrable systems studying the degeneration of spectral curves. The Namikawa–Ueno types and the monodromy matrices of the generic singular fibers are given. Our plan in the future work is to understand the phase spaces as the relative compactified Jacobian of the spectral curve fibration by studying the discriminant locus of the base space and combine with the results on monodromies of the singular fibers.

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
\text{Hamiltonian} & \text{monodromy} & \text{Namikawa–Ueno} & \Phi & \text{Ogg} & \text{page} \\
\text{spectral type} & \text{stable} & \text{Dynkin} & & & \\
\hline
H_{\text{VI}}^{\text{Mat}} & \text{ell}[2] & I_0 - I_0^* - 1 & (2)^2 & 14 & p. 159 \\
22,22,22,2211 & V & I_0 - D_4 - 1 & & & \\
\hline
H_{\text{V}}^{\text{Mat}} & \text{par}[3] & I_0 - I_1^* - 1 & (4) & 14 & p. 170 \\
(2)(11),22,22 & VI & I_0 - D_5 - 1 & & & \\
\hline
H_{\text{III(Da)}}^{\text{Mat}} & \text{par}[3] & I_0 - I_2^* - 1 & (2)^2 & 14 & p. 170 \\
(2)(2),(2)(11) & VI & I_0 - D_6 - 1 & & & \\
\hline
H_{\text{III(Dr)}}^{\text{Mat}} & \text{par}[3] & I_0 - I_3^* - 1 & (4) & 14 & p. 170 \\
(2)(2),(11)_2 & VI & I_0 - D_7 - 1 & & & \\
\hline
H_{\text{III(Da)}}^{\text{Mat}} & \text{par}[3] & I_0 - I_4^* - 1 & (2)^2 & 14 & p. 170 \\
(2)_2,(11)_2 & VI & I_0 - D_8 - 1 & & & \\
\hline
H_{\text{IV}}^{\text{Mat}} & \text{ell}[2] & I_0 - IV^* - 1 & (3) & 14 & p. 160 \\
((2))((11)),22 & V & I_0 - E_6 - 1 & & & \\
\hline
H_{\text{II}}^{\text{Mat}} & \text{ell}[2] & I_0 - II^* - 1 & (2) & 14 & p. 162 \\
((2))(((11))) & V & I_0 - E_7 - 1 & & & \\
\hline
H_{\text{I}}^{\text{Mat}} & \text{ell}[2] & I_0 - I^* - 1 & 0 & 14 & p. 160 \\
(((11)))_2 & V & I_0 - E_8 - 1 & & & \\
\hline
\end{array}
\]

Table 4.5. The generic degeneration of the spectral curves of the autonomous 4-dimensional matrix Painlevé\(^{1/2}\)-type equations

\(^{(17)}\)We used Maple’s command “Weierstrassform” implemented by van Hoeij.
| Hamiltonian spectral type | monodromy stable | Namikawa–Ueno | Dynkin | $\Phi$ | Ogg | page |
|---------------------------|------------------|---------------|--------|--------|------|------|
| $H_{\text{Gar}}^{1+1+1+1+1} \cdot 11,11,11,11,11$ | ell{1} II | $I_0^{--0}$ |       | (2)$^4$ | 33   | p. 155 |
| $H_{\text{Gar}}^{2+1+1+1+1} (1)(1),11,11,11,11$ | par{3} II | $I_1^{--0}$ |       | (4) $\times (2)^2$ | 33   | p. 171 |
| $H_{\text{Gar}}^{3/2+2+1+1} (2)(1),(1)(1)(1)$ | par{3} II | $I_2^{--0}$ |       | (2)$^4$ | 33   | p. 171 |
| $H_{\text{Gar}}^{2+2+1} (1)(1),(1)(1),11$ | par{4} III | $I_3^{--0}$ |       | (4)$^2$ | 33   | p. 180 |
| $H_{\text{Gar}}^{3/2+2+1} (1),2,(1)(1),11$ | par{4} III | $I_4^{--0}$ |       | (4) $\times (2)^2$ | 33   | p. 180 |
| $H_{\text{Gar}}^{3/2+3/2+1} (1),2,(1),11$ | par{4} III | $I_5^{--0}$ |       | (2)$^4$ | 33   | p. 180 |
| $H_{\text{Gar}}^{3+1+1} (1)(1)(1),11,11$ | ell{2} V | $I_6^{--0}$ |       | (6) $\times (2)$ | 29a  | p. 161 |
| $H_{\text{Gar}}^{4/2+1+1} (((1)(1)),11,11)$ | ell{2} V | $I_7^{--0}$ |       | (2)$^3$ | 29a  | p. 162 |
| $H_{\text{Gar}}^{4+1} (((1)(1)),11,11)$ | par{3} II | $I_8^{--0}$ |       | (8) | 23   | p. 178 |
| $H_{\text{Gar}}^{2+2} (((1)(1)),(1)(1))_2,11$ | par{3} II | $I_9^{--0}$ |       | (4) | 25   | p. 176 |
| $H_{\text{Gar}}^{3+2} (((1)(1)),(1)(1))_2,11$ | par{3} VI | $I_{10}^{--0}$ |       | (12) | 29a  | p. 175 |
| $H_{\text{Gar}}^{4+3} (((1)(1)),(1)(1))_2,11$ | par{3} VI | $I_{11}^{--0}$ |       | (4) $\times (2)$ | 29a  | p. 177 |
| $H_{\text{Gar}}^{5+2} (((1)(1)),(1)(1))_2,11$ | par{3} VI | $I_{12}^{--0}$ |       | (6) $\times (2)$ | 29a  | p. 175 |
| $H_{\text{Gar}}^{5+3/2} (((1)(1)),(1)(1))_2,11$ | par{3} VI | $I_{13}^{--0}$ |       | (2)$^3$ | 29a  | p. 177 |
| $H_{\text{Gar}}^{7/2} (((1)(1)),11,11)$ | ell{1} I | IX $-- 3$ |       | (5) | 21   | p. 157 |
| $H_{\text{Gar}}^{9/2} (((1)(1)),11,11)$ | ell{1} I | VII$^*$ |       | (2) | 22   | p. 156 |

*Table 4.6.* The generic degeneration of the spectral curves of the autonomous 4-dimensional Garnier equations
| Hamiltonian spectral type | monodromy stable | Namikawa–Ueno Dynkin | Φ | Ogg | page |
|---------------------------|-----------------|----------------------|---|-----|------|
| $H_{FS}^{2+2}$ 21,21,11111,1111 | par[5] II | $\Pi_{3-0}$ | (12) | 41 | p. 171 |
| $H_{FS}^{4+2}$ (11)(1),21,1111 | par[5] VI | $\Pi_{3-0}$ | (13) | 41 | p. 183 |
| $H_{FS}^{4+2}$ (1)2, 21, 1111 | par[5] IV | $\Pi_{3-0}$ | (14) | 41 | p. 183 |
| $H_{Suz}^{3+2}$ (11)(1), (1)21 | par[5] IV | $\Pi_{3-0}$ | (15) | 41 | p. 183 |
| $H_{KFS}^{3+2}$ (1)3, (1)21 | par[5] IV | $\Pi_{3-0}$ | (16) | 41 | p. 183 |
| $H_{KFS}^{3+2}$ (1)3, (1)21 | par[5] IV | $\Pi_{3-0}$ | (17) | 41 | p. 183 |
| $H_{KFS}^{3+2}$ (1)3, (1)21 | par[5] IV | $\Pi_{3-0}$ | (18) | 41 | p. 183 |
| $H_{NY}^{4+2}$ (2)(1),11111,111 | par[4] III | $\Pi_{3-0}$ | (6) $\times$ (2) | 41a | p. 182 |
| $H_{NY}^{4+2}$ (11)(1),111 | par[3] II | $IV^* - \Pi_{3}$ | (10) | 41b | p. 175 |
| $H_{SS}^{3+2}$ 31,22,22,22,1111 | par[3] VI | $I_2 - I_0 - 0$ | (2)$^3$ | 2 | p. 171 |
| $H_{SS}^{3+2}$ (111)(1),22,22 | par[4] VII | $I_2 - I_1^* - 0$ | (4) $\times$ (2) | 2 | p. 180 |
| $H_{SS}^{3+2}$ (2)(2), (111)(1) | par[4] VII | $I_2 - I_2^* - 0$ | (2)$^4$ | 2 | p. 180 |
| $H_{SS}^{3+2}$ (1)3, (2)(2) | par[4] VII | $I_2 - I_3^* - 0$ | (4) $\times$ (2) | 2 | p. 180 |
| $H_{SS}^{3+2}$ (1)3, (2)(2) | par[4] VII | $I_2 - I_4^* - 0$ | (2)$^3$ | 2 | p. 180 |
| $H_{SS}^{3+2}$ (1)4, (2)(2) | par[4] VII | $I_2 - I_6^* - 0$ | (4) $\times$ (2) | 2 | p. 180 |
| $H_{SS}^{3+2}$ (1)4, (2)(2) | par[4] VII | $I_2 - I_6^* - 0$ | (2)$^3$ | 2 | p. 180 |

Table 4.7. The generic degeneration of the spectral curves of the autonomous 4-dimensional Fuji–Suzuki and Sasano equations
Appendix

A.1. Conserved quantities

The autonomous 4-dimensional Painlevé½-type equations have two functionally independent first integrals. In this subsection, we list these first integrals for the ramified equations\(^{(18)}\). One of the reason is that the other first integrals than the Hamiltonians have long expressions. Writing them for “less-degenerated” systems take huge spaces. But they are easily computable from data in the previous paper [25]. We only give first integrals for autonomous version of equations in Kawakami [21, 23, 22]. We list Hamiltonians \(H's\) with \(\delta^{(19)}\) and the other invariants \(G's\).

There are 5 ramified cases from the degeneration of \(A_5\) Fuji–Suzuki system\(^{(20)}\).

\[
H_{\text{FS}}^{A_3} = H_{\text{III}}(D_6)(-\theta_2^\infty, \delta + \theta_1^0 + \theta_1; \tilde{t}; q_1, p_1) + H_{\text{III}}(D_6)(\theta_2^0 - \theta_1^0, \theta_2^0 - \theta_1^0 - \theta_1^1; \tilde{t}; q_2, p_2) - \frac{1}{\tilde{t}}p_1p_2(q_1q_2 + \tilde{t}),
\]

\[
G_{\text{FS}}^{A_3} = \theta_2^0\tilde{t}(H_{\text{III}}(D_6)(-\theta_2^\infty, \theta_1^0 + \theta_1^1; \tilde{t}; q_1, p_1) - p_1p_2)
\]

\[
+ (q_1q_2 - \tilde{t})(\theta_2^\infty p_2 - (\theta_2^0 - \theta_2^0) p_1
\]

\[
+ p_1p_2 (\theta_1 + \theta_1^0 - \theta_1^0 + (p_1 - 1) q_1 - (p_2 - 1) q_2)),
\]

\[
H_{\text{KFS}}^{\frac{5}{2}} = H_{\text{III}}(D_7)(-\theta_1^0, \tilde{t}; q_1, p_1) + H_{\text{III}}(D_7)(\theta_2^0 - \theta_1^0, \tilde{t}; q_2, p_2)
\]

\[
+ \frac{1}{\tilde{t}}(p_2q_1(p_1(q_1 + q_2) + \theta_2^\infty) - q_1),
\]

\[
G_{\text{KFS}}^{\frac{5}{2}} = (q_2 - q_1)((\theta_2^0 - \theta_1^0) p_1p_2q_1 - \theta_2^\infty p_2^2q_1 + p_1p_2^2q_1q_2 + p_1^2p_2q_1^2 + p_1q_1
\]

\[
+ p_1p_2\tilde{t}) + \theta_2^0(\tilde{t}H_{\text{III}}(D_7)(-\theta_1^0; \tilde{t}; q_1, p_1) - q_1 + p_1p_2q_1^2(\theta_2^0 - \theta_2^\infty),
\]

\[
H_{\text{KFS}}^{\frac{7}{2} + \frac{3}{2}} = H_{\text{III}}(D_7)(\theta_1^0; \tilde{t}; q_1, p_1) + H_{\text{III}}(D_7)(\delta - \theta_1^\infty; \tilde{t}; q_2, p_2)
\]

\[
- \frac{1}{\tilde{t}}p_1q_1p_2q_2 - \left(p_2^2 q_1 + p_1 + p_2\right),
\]

\[
G_{\text{KFS}}^{\frac{7}{2} + \frac{3}{2}} = \frac{1}{q_1}(p_2(\theta_1^\infty + p_1q_1 - p_2q_2)(\tilde{t} - p_1q_1^2q_2) + (p_1 - p_2) q_2q_1^2 - \tilde{t}),
\]

\(^{(18)}\) We do not write first integrals of the Garnier equations here, since the first integrals are just autonomous limit of two Hamiltonians. Such Hamiltonians are listed by Kimura [27], Kawamuko [26] and Kawakami [22].

\(^{(19)}\) Hamiltonians for the case \(\delta = 0\) is the first integral.

\(^{(20)}\) Although we obtain the Garnier equations of ramified types from certain degenerations of \(A_5\)-type Fuji–Suzuki system, we exclude the Garnier systems.
There are also 4 systems that are ramified derived from $D_6$-Sasano system.

\[ H_{KFS}^{3 + \frac{3}{2}} = H_{III}(D_7)(\theta_1^\infty - \theta_2^\infty; \tilde{t}; q_1, p_1) + H_{III}(D_7)(\delta - \theta_1^\infty; \tilde{t}; q_2, p_2) - \frac{1}{t}p_1q_1p_2q_2 - (p_1p_2 + p_1 + p_2), \]

\[ C_{KFS}^{3 + \frac{3}{2}} = (p_1p_2q_1 - p_2^2q_2 + \theta_1^\infty p_2 - 1) \left( -p_1 (q_1q_2 - \tilde{t}) + \theta_2^\infty q_2 \right) - p_2 (q_1q_2 - \tilde{t}), \]

\[ H_{KFS}^{4 + \frac{4}{3}} = \frac{1}{t} \left( p_2^2q_1^2 + \delta q_1p_1 - q_1 - \frac{\tilde{t}}{q_1} \right) + H_{III}(D_8)(\tilde{t}; q_2, p_2) + \frac{1}{t} \left( -p_1q_1p_2q_2 + \frac{q_1q_2}{t} + q_1 + q_2 \right), \]

\[ C_{KFS}^{4 + \frac{4}{3}} = \frac{1}{tq_1q_2} \left( (p_1q_1 - p_2q_2) \left( p_1p_2q_2^2q_1^2\tilde{t} + p_1q_2q_2^2\tilde{t} + q_2^2q_1^2 \right) + \tilde{t}^2 (p_1q_1^2 - p_2q_2^2 - q_2^2) \right). \]

There are also 4 systems that are ramified derived from $D_6$-Sasano system.
We have three ramified systems of matrix Painlevé equations. In this subsection, we explain the notation to express spectral types for ramified equations, which is introduced by Kawakami [21]. The Fuchsian case is explained in 2.1.

A.2. Local data of linear equations

For the classification of linear equations, we need to discern the types of linear equations. In this subsection, we explain the notation to express spectral types for ramified equations, which is introduced by Kawakami [21]. The Fuchsian case is explained in 2.1.

\[ G_{KSS}^{3+2} = -3\theta_0^2\theta_1^\infty p_2 q_2 - \theta_0(\theta_1^\infty)^2 p_2 q_2 + 3\theta_0^2\theta_1^\infty q_2^2 + \theta_0^\infty p_2 q_1 + 2\theta_0^2\theta_1^\infty p_1 q_2 - 2\theta_0^3 p_2 q_2 + 2\theta_0^2 p_2^2 q_2^2 + 2\theta_0^\infty p_2 q_1 + 3\theta_0^2 p_1 q_2 q_2 - \theta_0^\infty p_1 q_2 q_2 - 3\theta_0^2 p_1 q_2 q_1 q_2^2 - \theta_0^\infty p_1 q_2^2 q_2.

\[ H_{KSS}^{5+2} = H_{III}(D_8)(\bar{t}; q_1, p_1) + H_{III}(D_7)(-\theta_0, \bar{t}; q_2, p_2) + 2p_2 q_1 - (\theta_0 + 1 - \delta)p_1 q_1 + 1 - p_2, \]

\[ G_{KSS}^{3+2} = \frac{1}{q_1^2} \left( p_2^2 q_1^2 (2\bar{t} - \theta_0 q_1 q_2) + p_1^2 q_2^2 q_1^2 - \theta_0 q_2 q_1 \bar{t} - q_2^2 q_3^2 + \bar{t}^2 \right) - p_2 q_2^2 (\theta_0 p_1^2 q_2 q_1 + \theta_0 p_1 (\bar{t} - \theta_0 q_1 q_2) - \theta_0 q_2 q_1 + \bar{t}) + q_2^3 q_2 (p_1 q_1 - 1) + q_1^2 p_1 (\bar{t} - \theta_0 q_1 q_2), \]

\[ H_{KSS}^{3+2} = \frac{1}{t} \left( p_1^2 q_1^2 + \theta_1 q_1 p_1 - q_1 - \bar{t}/q_1 \right) + H_{III}(D_8)(\bar{t}; q_1, p_2) - 2\frac{q_1^2 q_2}{\bar{t}^2} + q_1 + q_2 \frac{\bar{t}}{q_1 q_2} - \frac{1}{4q_1^2} \left( 2p_2 q_2 + 1 \right) \left( \bar{t} \left( 2p_2 q_2 + 1 \right) \left( \bar{t} - q_1^2 \theta_0 q_1 \right) + 4p_1 q_2 q_1^2 \right). \]

We have three ramified systems of matrix Painlevé equations. \( G_{III(D_7)}^{\text{Mat}} \) and \( G_{III(D_8)}^{\text{Mat}} \) are too long and \( H_1^{\text{Mat}} \) is already written in the main part of this paper. So we skip writing first integrals of autonomous matrix Painlevé equations.
Now we explain the way to obtain the formal canonical form around each non-Fuchsian singular point. We also explain that these canonical forms can be expressed by the refining sequences of partition. Let us assume that the coefficient matrix $A(x)$ of the equation has a singularity at the origin, and that $A(x)$ is expanded in the Laurent series as follows:

\[
\frac{dY}{dx} = \left( \frac{A^0}{x^{r+1}} + \frac{A^1}{x^r} + \cdots \right) Y.
\]

Here, $A^j$ ($j = 0, 1, \ldots$) are $m \times m$ matrices. We assume that $A^0$ is diagonalizable.

With an appropriate choice of the gauge matrix, we can assume that $A^0$ is diagonal and that its eigenvalues are $t_0^0, \ldots, t_m^0$. When $r = 0$, then the origin is a regular singular point. Let us assume that $r > 0$. If $t_i^0 \neq t_j^0$ ($1 \leq i \leq l$, $l + 1 \leq j \leq m$), then a gauge transformation by a formal power series $Y = P(x)Z$ ($P(x) = I + P_1x + P_2x^2 + \cdots$) leads to the following form:

\[
\frac{dZ}{dx} = \left( \frac{B^0}{x^{r+1}} + \frac{B^1}{x^r} + \cdots \right) Z.
\]

Here, we can transform $B^i$ into the following form:

\[
B^i = \begin{pmatrix} B_{11}^i & O \\ O & B_{22}^i \end{pmatrix}, \quad B_{11}^i \in M_l(\mathbb{C}), \quad B_{22}^i \in M_{m-l}(\mathbb{C}).
\]

With successive application of this process, the equation (A.4) is formally decomposed to direct sum of equations whose leading terms have only one eigenvalues respectively. When the leading term of the block is diagonalizable, that is when it is scalar matrix, then this part can be canceled by a gauge transformation by a scalar function, so that the equation is reduced to a equation with smaller $r$.

Remark A.8. — When $A^0$ is not diagonalizable, in order to decompose the system into equations of smaller sizes, we need to take an appropriate covering $x = \xi^k$. In that case, the transformation matrix $P(x)$ is a Puiseux series in $x$. The equations with this property are called ramified. When we do not need to take coverings, that is when $k = 1$, we say that the equations are unramified.

A.2.1. Unramified non-Fuchsian case

When the equation (A.4) is unramified, it can be transformed into the following form:

\[
\frac{dY}{dx} = \left( \frac{T_0}{x^{r+1}} + \frac{T_1}{x^r} + \cdots + \frac{T_r}{x} + \cdots \right) Y.
\]
We can assume that $T_j$’s are diagonal matrices and that $T_0 = A_0$. Furthermore, we can eliminate the regular terms by an appropriate diagonal matrix with formal power series components. Thus, the equation (A.4) can be transformed into the following form by gauge transformation of a formal power series:

\[
\frac{dY}{dx} = \left( \frac{T_0}{x^{r+1}} + \frac{T_1}{x^r} + \cdots + \frac{T_r}{x^1} \right) Y.
\]

If we write the diagonal components of $T_i$ as $t_{ij}$ ($j = 1, \ldots, m$), then a canonical form around the origin can be described by the following data:

\[
\begin{array}{cccc}
  x = 0 & t_{0j} & t_{1j} & \cdots & t_{rj} \\
  : & : & : & & :
\end{array}
\]

We write down such formal canonical forms for each singular point, and put them together. This kind of table is called the Riemann scheme of the linear equation. As we can see from the procedure to obtain the canonical form, the leftmost column splits into several groups as equivalence class of values. In the second column from the left, these groups splits further, and so on, we get a nested columns.

We describe such nesting structure by refining sequences of partitions of $m$ and call it the spectral type of the singular point. We line up such spectral types of each singular point, and separate them by commas. We call it the spectral type of the equation. In such a case,

\[
\exp \left( -\frac{T_0}{rx^r} + \cdots - \frac{T_{r-1}}{x} \right) x^{Tr}
\]

is the fundamental solution matrix for the formal canonical form (A.5). The degree $r$ of the polynomial is called the Poincaré rank of the singular point. When the singular point is of regular type, the Poincaré rank is 0. When the equation is ramified, this part is a polynomial in $x^{-1/k}$, and the Poincaré rank is non-integer rational number. If the thing we want to express is just the Poincaré ranks at each singularity, we attach Poincaré rank plus 1 to each singularity, and line them up, and separate them by + signs. When the equation is unramified, Poincaré rank plus 1 is as same as numbers of columns appeared in refining sequences of partitions at each singularities.
Example A.9. — For instance, let us consider the following normal form:
\[
\frac{d\hat{Y}(x)}{dx} = \begin{cases} 
\frac{1}{x^3} \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & b \end{pmatrix} + \frac{1}{x^2} \begin{pmatrix} c & 0 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & d & 0 \\ 0 & 0 & 0 & e \end{pmatrix} + \frac{1}{x} \begin{pmatrix} f & 0 & 0 & 0 \\ 0 & g & 0 & 0 \\ 0 & 0 & h & 0 \\ 0 & 0 & 0 & i \end{pmatrix} \end{cases} \hat{Y}.
\]

We align the diagonal entries as \(\{a \ a \ b \ b\} \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \). We express the degeneracy of the eigenvalues by \(\{22\ \ 211\ \ 1111\ \}\). Each row expresses a partition of the matrix size \(m\). The partitions in the lower rows are a refinement of a partition in the upper rows.

In order to express the degeneracy of the eigenvalues briefly, we use parentheses. Firstly, write the finest partition of \(m\) in the lowest row, which expresses the degeneracy of the eigenvalue of \(T^{(i)}\). Secondly, put the numbers that are grouped together in the second lowest partition in parentheses. We continue this process until the highest row.

Example A.10. — The local data of the example above can be expressed concisely using the parentheses.

\[
\begin{array}{ccc}
\begin{pmatrix} a & c & f \\ a & c & g \\ b & d & h \\ b & e & i \end{pmatrix} & \rightarrow & \begin{pmatrix} a & a & b & b \\ c & c & d & e \\ f & g & h & i \end{pmatrix} \\
22 & 211 & 1111 \\
((11)||(1)(1)) & (11)(1)(1) & 1111
\end{array}
\]

The way to restore the degeneracy of eigenvalues from the symbol \(((11)||(1)(1))\) is as follows.
- Add the numbers in the outermost parenthesis \(\rightarrow 22\)
- Add the numbers in the inner parenthesis \(\rightarrow 211\)
- Write the numbers in the innermost parentheses \(\rightarrow 1111\)

We express the types of linear equations by aligning such data for each singular point.

A.2.2. Ramified case

We have the following formal normal form at each singularities (Hukuhara [16], Levelt [33] and Turrittin [56]). Let us assume that \(x = 0\) is
an irregular singular point. Then, there exist a positive integer \( q \), rational numbers with the common denominator \( q \) such that \( r_0 < r_1 < \cdots < r_{n-1} < r_n = -1 \), diagonal matrices \( T_0, \ldots, T_n \), a transformation \( z = F(x^{1/q}) \) in the class of formal series in \( x^{1/q} \), such that the transformed system have the following form;

\[
\frac{dz}{dx} = (T_0x^{r_0} + \cdots + T_{n-1}x^{r_{n-1}} + T_nx^{-1})z.
\]

Let us assume that the diagonal matrix \( T_k \) has \( t_{ki} \) for \( i = 1, \ldots, m \) as diagonal components. We express the local data by the following table:

\[
\begin{array}{cccccc}
\hline
x = 0 & \left( \begin{array}{c}
\frac{n}{q}
\end{array} \right) \\
\hline
t_0^0 & t_1^1 & \cdots & t_n^1 \\
\vdots & \vdots & \ddots & \vdots \\
t_m^0 & t_{m1} & \cdots & t_m^n.
\end{array}
\]

Let us introduce a way to express such local data compactly with examples. Let us assume the following normal form;

\[
\frac{dz}{dx} = \left\{ c_0\Omega x^{-\frac{8}{3}} + c_1\Omega^2 x^{-\frac{7}{3}} + c_2I_3x^{-2} + c_3\Omega x^{-\frac{5}{3}} + c_4\Omega^2 x^{-\frac{4}{3}} + c_5I_3x^{-1} \right\} z,
\]

where \( \Omega = \text{diag}(1, \omega, \omega^2) \) and \( \omega \) is a primitive third root of unity. This normal form can be expressed as

\[
\begin{array}{cccccc}
\hline
x = 0 & \left( \begin{array}{c}
\frac{5}{3}
\end{array} \right) \\
\hline
c_0 & c_1 & c_2 & c_3 & c_4 & c_5, \\
c_0\omega & c_1\omega & c_2 & c_3\omega & c_4\omega^2 & c_5 \\
c_0\omega^2 & c_1\omega & c_2 & c_3\omega^2 & c_4\omega & c_5.
\end{array}
\]

Two systems in the lower rows \( \frac{dz_2}{dx} = (c_0\omega x^{-\frac{8}{3}} + \ldots)z_2 \) and \( \frac{dz_3}{dx} = (c_0\omega x^{-\frac{8}{3}} + \ldots)z_3 \) can be obtained by the first row \( \frac{dz_1}{dx} = (c_0x^{-\frac{8}{3}} + \ldots)z_1 \) upon replacement \( x^{\frac{1}{3}} \mapsto \omega x^{\frac{1}{3}} \mapsto \omega^2 x^{\frac{1}{3}} \). Since we have 3 copies of the first equation, we express the local data as \( (((((1))))))_3 \).

Example A.11. — We show some other examples.

\[
\begin{array}{cccccc}
\hline
x = 0 & \left( \begin{array}{c}
\frac{1}{2}
\end{array} \right) & x = 0 & \left( \begin{array}{c}
\frac{1}{2}
\end{array} \right) \\
\alpha & \beta & \alpha & \beta \\
-\alpha & \beta & \rightarrow (1)_{211}, & \alpha\omega & \beta & \rightarrow (1)_{31}, \\
0 & \gamma & \alpha\omega & \beta \\
0 & \epsilon & 0 & \gamma.
\end{array}
\]

(21) Kawakami [21] devised such notation.
A.3. The dual graphs of the singular fibers

We list the dual graphs of singular fibers appeared in the table. The numbers in circles indicate multiplicities of the components. We adopt, as in Ogg [41], the following symbol for component $\Gamma$ of singular fibers (Table A.8). $K_X$ is the canonical divisor of surface $X$. The matrices next to or below the dual graphs are the monodromy [40].

| Symbol | Genus | $\Gamma^2$ | $\Gamma \cdot K_X$ |
|--------|-------|------------|-------------------|
| A      | 1     | -1         | 1                 |
| B      | 0     | -3         | 1                 |
| C      | 1     | -2         | 2                 |
| D      | 0     | -4         | 2                 |
| none   | 0     | -2         | 0                 |

Table A.8. Components of singular fibers

$I_{0-0-0}^* : H_{Gar}^{1+1+1+1+1+1}$

\[
I_{0-0-0}^* : H_{Gar}^{2+1+1+1}
\]

\[
\begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 \\
\end{pmatrix}
\]

\[
I_{1-0-0}^* : H_{Gar}^{2+1+1+1}
\]

\[
\begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & -1 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 \\
\end{pmatrix}
\]
\[ I^*_1 - 0 - 0 : H_1^{2+1+1} \]

\[ I^*_2 - 0 - 0 : H_2^{3+1+1} \]

\[ I^*_0 - IV^* - (-1) : H_0^{3+1+1} \]
$I_0^* - III^* - (-1): H_{Gar}^{5+1+1}$

\[
\begin{pmatrix}
0 & 0 & -1 & 0 \\
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}
\]

$III^* - II_0^* - (-1): H_{Gar}^{4+1}$

\[
\begin{pmatrix}
0 & 0 & -1 & 0 \\
0 & -1 & -1 & -1 \\
1 & 0 & 0 & 1 \\
0 & 0 & 0 & -1
\end{pmatrix}
\]

$II^* - II_0^* - (-1): H_{Gar}^{7+1}$

\[
\begin{pmatrix}
0 & 0 & -1 & 0 \\
0 & -1 & 1 & 0 \\
1 & 0 & 1 & -1 \\
0 & 0 & 0 & -1
\end{pmatrix}
\]

$IV^* - I_1^* - (-1): H_{Gar}^{3+2}$

\[
\begin{pmatrix}
-1 & 0 & -1 & 0 \\
0 & -1 & 0 & -1 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}
\]
IV* − I*₂ − (−1): $H^{3+\frac{3}{2}}_{\text{Gar}}$

III* − I*₁ − (−1): $H^{\frac{5}{2}+2}_{\text{Gar}}$

III* − I*₂ − (−1): $H^{\frac{5}{2}+\frac{3}{2}}_{\text{Gar}}$

II* − II*₀: $H^{\frac{7}{2}+1}_{\text{Gar}}$
IX $- 3$: $H^5_{\text{Gar}}$

\[
\begin{pmatrix}
0 & -1 & -1 & 0 \\
-1 & 0 & 0 & -1 \\
1 & -1 & -1 & 0 \\
0 & 1 & 0 & 0 \\
\end{pmatrix}
\]

VII*: $H^2_{\text{Gar}}$

\[
\begin{pmatrix}
0 & -1 & -1 & 0 \\
-1 & 1 & 0 & -1 \\
1 & -1 & -1 & 0 \\
0 & 1 & 0 & 0 \\
\end{pmatrix}
\]

para[3], II$_3$–0: $H^{A_2}_{\text{FS}}$

\[
\begin{pmatrix}
-1 & 0 & 0 & 0 \\
-1 & 1 & 0 & 3 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}
\]

para[5], II$_3$–1: $H^{A_4}_{\text{FS}}$

\[
\begin{pmatrix}
-1 & 0 & -1 & 0 \\
1 & 1 & 1 & 3 \\
0 & 0 & -1 & 1 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}
\]
para[5], II₃₋₂: $H^{3_3}_{FS}$

para[5], II₃₋₃: $H^{3_3+2}_{Suz}$

para[5], II₃₋₄: $H^{3_3+3}_{KFS}$

para[5], II₃₋₅: $H^{3_3+4}_{KFS}$

para[5], II₃₋₆: $H^{3_3+5}_{KFS}$
para[4], \( \Pi_{3-1}: H_{NY}^{A_5} \)

\[
\begin{pmatrix}
-1 & 0 & -1 & -1 \\
0 & 1 & 1 & 3 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

\( \Pi_{4} - \Pi_{3}: H_{NY}^{A_4} \)

\[
\begin{pmatrix}
-1 & 0 & -1 & -1 \\
-1 & 1 & 0 & 3 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

\( I_{2} - I_{0}^*: H_{SS}^{D_6} \)

\[
\begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 2 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

\( I_{2} - I_{1}^*: H_{SS}^{D_5} \)

\[
\begin{pmatrix}
-1 & 0 & -1 & 0 \\
0 & 1 & 0 & 2 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

\( I_{2} - I_{2}^*: H_{SS}^{D_4} \)

\[
\begin{pmatrix}
-1 & 0 & -2 & 0 \\
0 & 1 & 0 & 2 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]
\( I_2 - I_3^* - 0: H_{Ks}^{3+2} \)

\[
\begin{pmatrix}
-1 & 0 & -3 & 0 \\
0 & 1 & 0 & 2 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

\( I_2 - I_4^* - 0: H_{Ks}^{4+2} \)

\[
\begin{pmatrix}
-1 & 0 & -4 & 0 \\
0 & 1 & 0 & 2 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

\( I_2 - I_5^* - 0: H_{Ks}^{5+2} \)

\[
\begin{pmatrix}
-1 & 0 & -5 & 0 \\
0 & 1 & 0 & 2 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

\( I_2 - I_6^* - 0: H_{Ks}^{3+\frac{5}{3}} \)

\[
\begin{pmatrix}
-1 & 0 & -6 & 0 \\
0 & 1 & 0 & 2 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]
$I_0 - I_0^* - 1: H_{VI}^{\text{Mat}}$

$\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}$

$I_0 - I_1^* - 1: H_{V}^{\text{Mat}}$

$\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & -1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}$

$I_0 - I_2^* - 1: H_{\text{III}(D_6)}^{\text{Mat}}$

$\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & -2 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}$

$I_0 - I_3^* - 1: H_{\text{III}(D_7)}^{\text{Mat}}$

$\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & -3 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}$

$I_0 - I_4^* - 1: H_{\text{III}(D_8)}^{\text{Mat}}$

$\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & -4 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}$
I₀ − IV* − 1: $H_{IV}^{\text{Mat}}$

\[
\begin{pmatrix}
-1 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

I₀ − III* − 1: $H_{II}^{\text{Mat}}$

\[
\begin{pmatrix}
0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

I₀ − II* − 1: $H_{I}^{\text{Mat}}$

\[
\begin{pmatrix}
0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

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Manuscrit reçu le 14 novembre 2016,
révisé le 12 août 2017,
accepté le 14 novembre 2017.

Akane NAKAMURA
Dept. of Mathematics,
Josai University,
1-1 Keyakidai,
Sakado,
Saitama 350-0295 (Japan)
a-naka@josai.ac.jp