EXPLICIT STRUCTURE OF THE FOKKER-PLANCK EQUATION WITH FLAT CONFINEMENT

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Abstract. We study the pointwise (in the space and time variables) behavior of the Fokker-Planck Equation with flat confinement. The solution has very clear description in the $xt$-plane, including large time behavior, initial layer and asymptotic behavior. Moreover, the structure of the solution highly depends on the potential function.

1. Introduction

1.1. The Models. The Fokker-Planck equations arise in many areas of sciences, including probability, statistical physics, plasma physics, gas and stellar dynamics. The term “Fokker-Planck” is widely used to represent various diffusion processes (Brownian motion).

In this paper, we study the kinetic Fokker-Planck equation with flat confinement in $\mathbb{R}^3$, it reads

$$
\begin{cases}
\partial_t F + v \cdot \nabla_x F = \nabla_v \cdot [\nabla_v F + (\nabla_v \Phi)F], & x, v \in \mathbb{R}^3, t > 0, \\
F(0, x, v) = F_0(x, v),
\end{cases}
$$

where $\gamma > 0$ and the potential $\Phi(v)$ takes the form

$$\Phi = \frac{1}{\gamma} \langle v \rangle^\gamma + \Phi_0,$$

for some constant $\Phi_0$. We define

$$M(v) = e^{-\Phi(v)},$$

with $\Phi_0 \in \mathbb{R}$ such that $M$ is a probability measure, it is easy to see that $M$ is a steady state to the Fokker-Planck equation (1). Thus it is nature to study the fluctuations of the Fokker-Planck equation (1) around $M(v)$, with the standard perturbation $f(t, x, v)$ to $M$ as

$$F = M + M^{1/2}f.$$

The Fokker-Planck equation for $f(t, x, v) = G^t f_0$ now takes the form

$$
\begin{cases}
\partial_t f + v \cdot \nabla_x f = \Delta_v f - \frac{1}{4} |v|^2 \langle v \rangle^{2\gamma-4} f + \left( \frac{3}{2} \langle v \rangle^{\gamma-2} + \frac{\gamma-2}{2} |v|^2 \langle v \rangle^{\gamma-4} \right) f = Lf, \\
f(0, x, v) = f_0(x, v),
\end{cases}
$$

here $G^t$ is the solution operator of the Fokker-Planck equation (2). It is obvious that $L$ is a non-positive self-adjoint operator on $L^2_v$. More precisely, its Dirichlet form is given by

$$\langle Lf, f \rangle_v = -\int \left| \nabla_v f + \frac{\nabla \Phi}{2} f \right|^2 dv = -\int \left| \nabla_v \left( \frac{f}{\sqrt{M}} \right) \right|^2 Mdv.$$
where $E_D = \sqrt{\mathcal{M}}$. Based on this property, we can introduce the Macro-Micro decomposition as follows: the macro projection $P_0$ is the orthogonal projection with respect to the $L_x^2$ inner product onto $\text{Ker}(L)$, and the micro projection $P_1 \equiv \text{Id} - P_0$.

1.2. Main theorem. Before the presentation of the main theorem, let us define some notations in this paper. We denote $\langle v \rangle^s = (1 + |v|^2)^{s/2}$, $s \in \mathbb{R}$. For the microscopic variable $v$, we denote

$$|f|_{L_v^2} = \left( \int_{\mathbb{R}^3} |f|^2 dv \right)^{1/2},$$

and the weighted norms $| \cdot |_{L_v^2(m)}$ and $| \cdot |_{L_v^2}$ by

$$|f|_{L_v^2(m)} = \left( \int_{\mathbb{R}^3} |f|^2 m dv \right)^{1/2}, \quad |f|_{L_v^2} = \left( \int_{\mathbb{R}^3} \langle v \rangle^{2\theta} |f|^2 dv \right)^{1/2}$$

respectively. The $L_v^2$ inner product in $\mathbb{R}^3$ will be denoted by $\langle \cdot, \cdot \rangle_v$. For the space variable $x$, we have similar notations. In fact, $L_x^2$ is the classical Hilbert space with norm

$$|f|_{L_x^2} = \left( \int_{\mathbb{R}^3} |f|^2 dx \right)^{1/2}.$$

We denote the sup norm as

$$|f|_{L_x^\infty} = \sup_{x \in \mathbb{R}^3} |f(x)|.$$

The standard vector product will be denoted by $(\cdot, \cdot)$. For the Fokker-Planck equation, the natural space in $v$ variable is adopted with the norm $| \cdot |_{L_v^2}$, which is defined by

$$|f|_{L_v^2} = |\langle v \rangle^\gamma f|_{L_v^2} + |\nabla_v f|_{L_v^2},$$

and the corresponding weighted spaces:

$$|f|_{L_v^2(m)} = |\langle v \rangle^\gamma f|_{L_v^2(m)} + |\nabla_v f|_{L_v^2(m)}, \quad |f|_{L_v^2,\omega} = |\langle v \rangle^\gamma f|_{L_v^2} + |\nabla_v f|_{L_v^2}.$$

Moreover, we define

$$\|f\|_{L_v^2} = \int_{\mathbb{R}^3} |f|_{L_v^2}^2 dx, \quad \|f\|_{L_v^2} = \int_{\mathbb{R}^3} |f|_{L_v^2}^2 dx,$$

and

$$\|f\|_{L_x^\infty L_v^2} = \sup_{x \in \mathbb{R}^3} |f|_{L_v^2}, \quad \|f\|_{L_x^1 L_v^2} = \int_{\mathbb{R}^3} |f|_{L_v^2} dx.$$

Finally, we define the high order Sobolev norm in $x$ variable: Let $k \in \mathbb{N}$,

$$\|f\|_{H^k_x L_v^2} = \sum_{|\alpha| \leq k} \|\partial^\alpha_x f\|_{L_v^2}.$$

The weighted spaces in $x$-$v$ norm can be defined by similar way.

The domain decomposition plays an essential role in our analysis, hence we define a cut-off function $\chi : \mathbb{R} \to \mathbb{R}$, which is a smooth nonincreasing function, $\chi(s) = 1$ for $s \leq 1$, $\chi(s) = 0$ for $s \geq 2$ and $0 \leq \chi \leq 1$. Moreover, we define $\chi_R(s) = \chi(s/R)$.

For simplicity of notations, hereafter, we abbreviate "$\leq C$ " to "$\lesssim$ ", where $C$ is a positive constant depending only upon fixed numbers.

Here are the precise description of our main results:

**Theorem 1.** Let $f$ be a solution of the Fokker-Planck equation (24) with initial data compact in the $x$-variable and bounded in $L_x^2$ (we need some algebraic weight for $0 < \gamma < 1$) space

$$f_0(x, v) \equiv 0 \text{ for } |x| \geq 1,$$

There exists a positive constant $M$ such that the following hold:

1. If $\gamma \geq 3/2$, there exist positive constants $C$ and $c$ such that the solution $f$ satisfies:

   (a) For $t \geq 1$ and $|x| \leq 2Mt$,

   $$|f|_{L_v^2} \lesssim \left[ e^{-Ct} e^{-\frac{|x|^2}{\gamma t}} + (1 + t)^{-3/2} e^{-\frac{|x|^2}{\gamma (1+t)}} + e^{-ct} \right] \|f_0\|_{L_x^\infty L_v^2}.$$
(b) For $\langle x \rangle \geq 2Mt$,
\[
|f(t,x)|_{L^2} \lesssim \left[ e^{-Ct} t^{-3/2} e^{-\frac{|x|^2}{1+t}} + t e^{-c(|x|+t)} \right] \|f_0\|_{L^\infty_x L^2_t}.
\]

(2) $1 \leq \gamma < 3/2$, for any given positive integer $N$, and any given sufficiently small $\alpha > 0$, there exist positive constants $C_N$, $C$, $c$ and $c_\alpha$ such that the solution $f$ satisfies
(a) For $t \geq 1$ and $\langle x \rangle \leq 2Mt$,
\[
|f|_{L^2_x} \lesssim \left[ e^{-Ct} t^{-3/2} e^{-\frac{|x|^2}{1+t}} + (1+t)^{-\gamma} (1 + \frac{|x|^2}{1+t})^{-N} + e^{-c(|x|+t)} \right] \|f_0\|_{L^\infty_x L^2_t}.
\]

(b) For $\langle x \rangle \geq 2Mt$,
\[
|f(t,x)|_{L^2_x} \lesssim \left[ e^{-Ct} t^{-9/2} e^{-\frac{|x|^2}{1+t}} + t(1+t) e^{-c_\alpha(|x|+t)} \right] \|f_0\|_{L^2}.
\]

(3) $0 < \gamma < 1$, for any given sufficiently small $\alpha > 0$ and any integer $j > 36$, there exist positive constants $C$, $c$ and $c_\alpha$ such that the solution $f$ satisfies
(a) For $t \geq 1$ and $\langle x \rangle \leq 2Mt$,
\[
|f(t,x)|_{L^2_x} \lesssim \left[ (1+t)^{-3/2} + t^3 \left( 1 + \frac{t}{j} \right) \right] \|f_0\|_{L^2((v)^{2(1-\gamma)s})}.
\]

(b) For $\langle x \rangle \geq 2Mt$,
\[
|f(t,x)|_{L^2_x} \lesssim \left[ t^{-9/2} e^{-\frac{|x|^2}{1+t}} + t(1+t) e^{-c_\alpha(|x|+t)} \right] \|f_0\|_{L^2((v)^{2(1-\gamma)s})}.
\]

1.3. Review of previous works and significant points of the paper. The study of the Fokker-Planck equation can be traced back to 1930’s. When the potential $\Phi = 0$, the equation \cite{1} is known as the Kolmogorov Fokker-Planck equation. In 1934 Kolmogorov \cite{2} derived the Green’s function for the whole space problem. The explicit formula surprisingly shows that the solution become smooth in $t, x, v$ variables when $t > 0$ immediately.

Later the regularization effect of it has been investigated further and been recovered by some more general and more robust methods. For example, it is known that the Fokker-Planck operator $-v \cdot \nabla_x + \Delta_v$ is a hypoelliptic operator. One can apply Hörmander’s commutator \cite{3} to the linear Fokker-Planck operator to obtain that diffusion in $v$ together with the transport term $v \cdot \nabla_x$ has a regularizing effect for solutions not only in $v$ but also in $t$ and $x$. It can also be obtained through the functional method, see \cite{4} \cite{5} \cite{6}. On the other hand, the Fokker-Planck operator is also known as a hypo coercive operator, which concerns the rate of convergence to equilibrium. Indeed, the trend to equilibria with a certain rate has been investigated in many papers (cf.\cite{1} \cite{2} \cite{3} \cite{4} \cite{5} \cite{6} \cite{7} \cite{8} \cite{9}) in the Maxwellian regime and in the whole space or in the periodic box.

Let us point out the most recent and important results constructed by Mouhot and Mischler \cite{10}. They developed an abstract method for deriving decay estimates on the semigroup associated to non-symmetric operators in Banach spaces, then applied this method to the kinetic Fokker-Planck equation in the torus with a flat confinement in the closed to equilibrium setting. They obtained spectral gap estimates for the associated semigroup for various norms, including Lebesgue norms, negative Sobolev norms, and the Monge-Kantorovich-Wasserstein distance $W_1$.

In this paper, we study the Fokker-Planck equation with flat confinement in the close to equilibrium setting. In the literature, this kind of problem basically focuses on the rate of convergence to equilibrium (see the reference listed above). In contrast, in this paper we supply a very explicit description of the solution in the sense of pointwise estimate. It turns out the structure of the solution highly depends on the potential function. Here are the significant points of the paper:

- The complete picture of the solution consists of three parts: the time-like region (large time behavior), the small time region (initial layer) and the space-like region (asymptotic behavior).
(1) In time-like region (the large time behavior of the solution), we have distinctly different descriptions according to potential functions. For $\gamma \geq 1$, thanks to the spectrum analysis [18] and our generalization, we have pointwise fluid structure in time-like region, which is much more precise than previous results. The leading term of wave propagation have been recognized. More specifically, for $\gamma \geq 3/2$ the leading term is diffusion wave with heat kernel type, while for $1 \leq \gamma < 3/2$ the diffusion wave is of algebraic type. By contrast, the spectral information is missing for $0 < \gamma < 1$, which leads to the unavailability of pointwise structure. Nevertheless, we can apply Kawashima’s argument [11] to get optimal decay rate in time.

(2) Within the initial layer, the effect of external force is not remarkable. Indeed, it is shown that the solution is basically dominated by the Kolmogorov Fokker-Planck equation ($\Phi = 0$) in this region.

(3) Concerning space-like region (asymptotic behavior), we have exponential decay for $\gamma \geq 3/2$ and sub-exponential decay for $0 < \gamma < 3/2$. The results are consistent with the wave behaviors inside time-like region for different range of $\gamma$ respectively. We believe this is the first result for asymptotic behavior of the Fokker-Planck equation with flat confinement.

- The regularization estimate plays a crucial role in this paper (see Lemma 8 and Lemma 10), which enables us to obtain the pointwise estimate without regularity assumption on the initial condition. In the literature, the regularization estimates for kinetic Fokker-Planck equation and Landau equation have been proved for various purposes, see for instance [7], [19], [22] (Appendix A.21.2) for Fokker-Planck case and [2] for Landau case. In this paper, we construct the regularization estimates in suitable weighted functions (precisely, suitable for space-like region), the calculation of the estimate is interesting and much more sophisticated than before. Moreover, this type of regularization estimate is itself new.

- We believe that our idea in this paper can be generalized to apply to Landau equation or Boltzmann equation without cutoff. In fact, these projects are works in progress.

To the best of our knowledge, in the literature the only one pointwise result of the kinetic type equation is Boltzmann equation for hard sphere and hard potential with cutoff (see [13], [17], and references therein). Let us point out the similarities and differences between Fokker-Planck equation with flat confinement and Boltzmann equation for hard sphere or hard potential with cutoff.

- The solutions of both in large time are dominated by fluid parts. For Fokker-Planck with $\gamma \geq 1$ and for Boltzmann with hard sphere or hard potential with cutoff, the fluid parts are characterized by diffusion waves. To extract them, both need the long wave-short wave decomposition. However the wave structures of them are different. For Boltzmann there are diffusion waves propagating with different speeds: one with the background speed of global Maxwellian while the other with the superposed speed of background speed and sound speed. In comparison, there is only one diffusion wave for Fokker-Planck. The fluid behavior can be seen formally from Chapman-Enskog expansion, which indicates that the macroscopic part (fluid part) of solution satisfies the viscous conservation laws system. For Boltzmann there are conservation laws of mass, momentum and energy, while Fokker-Planck only preserves the mass, this explains the difference of their wave structures. This picture could be valid even for general kinetic equation as well since physically the kinetic model can be approximated by fluid equation in the long run.

- Since the leading term of solution in large time are fluid parts and they essentially have finite propagation speed, the solution in space-like region should be insignificant. In fact it is shown that the asymptotic behaviors exponentially or sub-exponentially decay. This is similar to the solution of Boltzmann equation outside finite Mach number region.

- The regularization mechanism of Fokker-Planck is distinct from that of Boltzmann. In the former one, the regularity comes from the combined effect of ellipticity in the velocity variable $v$ and transport term (see Lemma 8 and Lemma 10). The initial singularity
is identified and subtracted by using the Green’s function of the Kolmogorov Fokker-Planck equation and it will be smeared out immediately due to regularization. While for Boltzmann equation, the initial singularity will be preserved (although decays in time very fast), thus one has to single out the singular kinetic wave, and the regularity of resultant remainder part comes from the compact part of the collision operator. (See the Mixture lemma in [13], [15] and [16]).

1.4. Method of proof and plan of the paper. The main idea of this paper is to combine the long wave-short wave decomposition, the wave-remainder decomposition, the weighted energy estimate and the regularization estimate. The long-short wave decomposition is based on Fourier transform. The long wave (when Fourier variable \(|\eta|\) small) contains the leading term in large time, i.e., the fluid wave. The short wave contains the possible initial singularity. We use the wave-remainder decomposition to extract the singularity. Here the regularization estimate is crucial to show the remaining term become more regular. Thanks to comparison principle and explicit fundamental solution of Kolmogorov-Fokker-Planck equation, we have estimate of the singularity. This provides us the structure of the solution inside some wave cone. To complete the picture in whole space, we apply weighted energy estimate to the remainder. The structure of the solution highly depends on the potentials. We explain the idea in more details as below.

In time-like region (inside \(|x| \leq Mt\) for some \(M\)), the solution is dominated by fluid part, which is contained in the long wave part. In order to obtain its estimate, we devise different methods for \(\gamma \geq 1\) and \(0 < \gamma < 1\) respectively. For \(\gamma \geq 1\), taking advantage of spectrum information of the Fokker-Planck operator \([18]\) (In fact, the paper \([18]\) only studies the case \(\gamma = 2\) and we can extend it to the case \(\gamma \geq 1\)), the complex analytic or Fourier multiplier techniques can be applied to obtain pointwise structure of fluid part. However, for \(0 < \gamma < 1\), the spectrum information is missing due to the weak damping for large velocity. Instead, we use Kawashima’s argument \([11]\) to get the optimal decay only in time. It is shown that the \(L^2\) norm of the short wave exponentially decays in time for \(\gamma \geq 1\) essentially due to spectrum gap, while it decays only algebraically for \(0 < \gamma < 1\) if imposing certain velocity weight on initial data.

As mentioned before, we use wave-remainder decomposition to extract the possible initial singularity in short wave. This decomposition is based on a Picard-type iteration. The first several terms in the iteration contain the most singular part of the solution, and they are so-called wave part. In virtue of the fundamental function of Kolmogorov-Fokker-Planck equation and the comparison principle for the iteration equation, we have rather accurate pointwise estimate for wave part. By functional method, we prove the iteration equation has regularization effect, which enables us to show the remainder become more regular. Noticing the singularity will disappear after initial time, the regularizing estimate together with \(L^2\) decay of short wave yields the \(L^\infty\) decay of short wave. Combing with long wave, we finish the pointwise structure inside wave cone.

Note that to complete the structure outside wave cone (space-like region), we only need to estimate the remainder part since we already have explicit estimate for wave part. The weighted energy estimates come to play a big role. The weighted function are carefully chosen for different \(\gamma\). It is remarkable that sufficient understanding of the structure inside wave cone, which has been obtained previously, is absolutely needed in the estimate. And the regularization effect makes it possible to do the higher order weighted energy estimate. Then the desired pointwise estimate follows from Sobolev inequality.

The rest of this paper is organized as follows: We first prepare some important properties in section 2 for long-short wave decomposition, wave-remainder decomposition and regularizing estimates. Then we study the large time behavior in section 3. Finally, we study the initial layer and asymptotic behavior in section 4.

2. Preliminary

2.1. The operator \(L\). First, we introduce a new norm \(|\cdot|_{L^2_{\tilde{\sigma},\theta}}\), which is equivalent to the natural norm \(|\cdot|_{L^{\infty}_{\tilde{\sigma},\theta}}\).
Lemma 2. Let \( \theta \in \mathbb{R}, \gamma > 0 \). There exists \( c = c ( \theta, \gamma ) > 0 \), such that
\[
|g|^2_{L^2_{\sigma, \theta}} := \int \langle v \rangle^{2\theta} \left| \nabla g \right|^2 dv + \frac{1}{2} \int \langle v \rangle^{2\theta} \left| \theta \right|^2 \left| \nabla g \right|^2 dv \geq c |g|^2_{L^2_{\sigma, \theta}}.
\]
It means that \( |g|_{L^2_{\sigma, \theta}} \) and \( |g|_{L^2_{\sigma, \theta}} \) are equivalent.

Proof. We only need to consider the integrand \( \left| \langle v \rangle^{\theta} \langle v \rangle^{\gamma-1} g \right|^2 \) for \( v \) near the origin. Notice that for a smooth cut-off function \( 0 \leq \chi \leq 1 \) with \( \chi (v) = 1 \), for \( |v| \leq 1 \), \( \chi (v) = 0 \) for \( |v| \geq 2 \), Poincare’s inequality implies
\[
\left( \int |v|^2 \langle v \rangle^{2(\gamma-1)} |g|^2 dv \right)^{1/2} \leq \left( \int \chi(v)^{\theta+\gamma-1} g |g|^2 dv \right)^{1/2} \leq C \left| \nabla \chi \right| \langle v \rangle^{\theta+\gamma-1} g |g|_{L^2_{\sigma, \theta}} + C \chi \left| \nabla \left( \langle v \rangle^{\theta+\gamma-1} g \right) \right|_{L^2_{\sigma, \theta}}.
\]
Through this equivalent norm, we derive the coercivity of the operator \( L \) for all \( \gamma > 0 \), as below.

Lemma 3 (Coercivity). Let \( \theta \in \mathbb{R}, \gamma > 0 \). For any \( m > 1 \), there is \( 0 < C ( m ) < \infty \), such that
\[
\left| \left\langle \langle v \rangle^{2\theta} \frac{\Delta \phi}{2} g_1, g_2 \right\rangle \right| \leq \frac{C}{m} \left| g_1 \right|_{L^2_{\sigma, \theta}} \left| g_2 \right|_{L^2_{\sigma, \theta}} + C \left( \left( \int |v|^2 \langle v \rangle^{\theta} g_1^2 dv \right)^{1/2} \left( \int |v|^2 \langle v \rangle^{\theta} g_2^2 dv \right)^{1/2} \right). \tag{3}
\]
Moreover, there exists \( \nu_0 > 0 \) such that
\[
\langle -L g, g \rangle \geq \nu_0 \left| P_1 g \right|_{L^2_{\sigma}}^2. \tag{4}
\]
Proof. We first prove (3). We split
\[- \int \langle v \rangle^{2 \theta} \frac{\Delta_v \Phi}{2} g_1 g_2 dv = \int_{\{v \leq m\}} + \int_{\{v \geq m\}}.\]
It suffices to consider the second integral over \{\{v \geq m\}\} since
\[
\frac{\Delta_v \Phi}{2} = \frac{1 + (\gamma + 1)|v|^2}{1 + |v|^2} (v)^{-2} \leq \tilde{C}(v)^{-2}.
\]
Hence, it follows from the Cauchy-Schwartz inequality and Lemma 2 that
\[
\int_{\{v \geq m\}} \langle v \rangle^{2 \theta} \frac{\Delta_v \Phi}{2} g_1 g_2 dv \leq \frac{\tilde{C}}{m^\gamma} \int_{\{v \geq m\}} \langle v \rangle^{2 \theta} (v)^{(\gamma - 1)} |g_1 g_2| dv \leq \frac{C}{m^\gamma} |g_1|_{L^2_{\sigma, \theta}} |g_2|_{L^2_{\sigma, \theta}}.
\]
To prove (4), we use the contradiction argument. Assuming the contrary, there is a sequence of normalized functions \(g_n(v)\) with \(|g_n|_{L^2_{\sigma}} = 1\), which also satisfy
\[
\int_{\mathbb{R}^3} g_n \mathcal{M}^{1/2} dv = 0 \quad \text{and} \quad 0 \leq \langle -Lg_n, g_n \rangle_v \leq \frac{1}{n}.
\]
We denote the weak limit, with respect to the inner product \(\langle \cdot, \cdot \rangle_\sigma\), of \(g_n\) (up to a subsequence) by \(g_0\). Here
\[
\langle g, h \rangle_\sigma \stackrel{def}{=} \int \langle v \rangle^{2 \theta} \nabla g \cdot \nabla hdv + \int \langle v \rangle^{2 \theta} \frac{|v|^2}{2} (v)^{2\gamma - 4} gh dv.
\]
Hence,
\[
|g_0|_{L^2_{\sigma}} \leq 1.
\]
Notice that
\[(5) \quad \langle -Lg_n, g_n \rangle_v = \|g_n\|_{L^2_{\sigma}}^2 - \left\langle \frac{\Delta_v \Phi}{2} g_n, g_n \right\rangle_v.
\]
We claim that
\[
\lim_{n \to \infty} \left\langle \frac{\Delta_v \Phi}{2} g_n, g_n \right\rangle_v = \left\langle \frac{\Delta_v \Phi}{2} g_0, g_0 \right\rangle_v.
\]
In fact, for any given \(m > 0\),
\[
- \int_{\{v \leq m\}} \frac{\Delta_v \Phi}{2} |g_n|^2 dv \to - \int_{\{v \leq m\}} \frac{\Delta_v \Phi}{2} |g_0|^2 dv
\]
since \(\partial_v g_n\) are bounded in \(L^2\) \(\{v \leq m\}\) from \(|g_n|_{L^2_{\sigma}} = 1\) and Lemma 2. On the other hand, by (3) we obtain
\[
\left| - \int_{\{v \geq m\}} \frac{\Delta_v \Phi}{2} |g_n|^2 dv \right| \leq \frac{C}{m^\gamma} |g_n|_{L^2_{\sigma}}^2 \leq \frac{C}{m^\gamma}
\]
Letting \(n \to \infty\) after choosing \(m\) sufficiently large gives
\[
\left\langle \frac{\Delta_v \Phi}{2} g_n, g_n \right\rangle_v \to \left\langle \frac{\Delta_v \Phi}{2} g_0, g_0 \right\rangle_v.
\]
Combined with the fact that \(0 \leq \langle -Lg_n, g_n \rangle_v \leq 1/n\), letting \(n \to \infty\) in (5) yields
\[
0 \leq \langle -Lg_0, g_0 \rangle_v = \|g_0\|_{L^2_{\sigma}}^2 - \left\langle \frac{\Delta_v \Phi}{2} g_0, g_0 \right\rangle_v \leq 1 - \left\langle \frac{\Delta_v \Phi}{2} g_0, g_0 \right\rangle_v = \lim_{n \to \infty} \langle -Lg_n, g_n \rangle_v \leq \lim_{n \to \infty} \frac{1}{n} = 0.
\]
It implies that \(\langle Lg_0, g_0 \rangle_v = 0\) and \(|g_0|_{L^2_{\sigma}}^2 = 1\), so \(g_0 = a \mathcal{M}^{1/2}\) for some \(a\). On the other hand,
\[
\lim_{n \to \infty} \int_{\mathbb{R}^3} g_n \mathcal{M}^{1/2} dv = \int_{\mathbb{R}^3} g_0 \mathcal{M}^{1/2} dv = 0.
\]
We thus have \(a = 0\), which contradicts \(|g_0|_{L^2_{\sigma}} = 1\).

We remark that the results of Lemma 2 and Lemma 3 have been proved in 3 for Landau case. The following lemma which will be used in Proposition 13 is a consequence of Lemma 4.
Lemma 4. For $\gamma \geq 3/2$, there exist $C_1, C_2 > 0$ such that
\[
\langle |v|^{2\gamma - 2} g, g \rangle_{L^2} \leq C_1 \langle -Lg, g \rangle_{L^2} + C_2 |g|_{L^2}^2.
\]

Now, let us decompose the collision operator $L = -\Lambda + K$, where
\[
\Lambda = -L + \varpi \chi_R (|v|), \quad K = \varpi \chi_R (|v|),
\]
here $\varpi > 0$ and $R > 0$ are as large as desired.

Regarding the discussion in space-like region, the following weighted functions $\mu(x, v)$ will be taken into account:
\[
\mu(x, v) = \begin{cases} 1 \quad \text{or} \quad \exp \left( \langle x \rangle / D \right) & \text{if} \quad \gamma \geq 3/2, \\ 0 \quad \text{or} \quad \exp (\alpha c(x, v)) & \text{if} \quad 0 < \gamma < 3/2, \end{cases}
\]
for $D$ large, and
\[
\mu(x, v) = \begin{cases} 1 \quad \text{or} \quad \exp (\alpha c(x, v)) & \text{if} \quad 0 < \gamma < 3/2, \end{cases}
\]
where
\[
c(x, v) = 5 \left( \delta \langle x \rangle \right) \left( 1 - \chi \left( \delta \langle x \rangle \langle v \rangle^{-3} \right) \right),
\]
the positive constants $\delta$ and $\alpha$ being determined later.

Lemma 5. Assuming that $\gamma > 0$, we have the following properties of the operators $\Lambda$ and $K$.

(i) There exists $c > 0$ such that
\[
\int (\Lambda g) \mu dxdv \geq c \|g\|_{L^2(\mu)}^2.
\]

(ii) \[
\int (K g) \mu dxdv \leq \varpi \|g\|_{L^2(\mu)}^2.
\]

Proof. We only prove part (i) when $\mu(x, v) = e^{\alpha c(x, v)}$, since the other cases of part (i) and part (ii) are trivial. Notice that there is a constant $c_1 > 0$ such that
\[
\|\nabla_v g\|_{L^2(\mu)}^2 + \int \left[ \frac{|v|^2 \langle v \rangle^{2\gamma - 4}}{4} - \frac{3}{2} \langle v \rangle^{2\gamma - 2} + \frac{(\gamma - 2)}{2} |v|^2 \langle v \rangle^{2\gamma - 4} \right] \mu dxdv \geq c_1 \|g\|_{L^2(\mu)}^2
\]
whenever $\varpi, R > 0$ are sufficiently large. On the other hand, it follows from
\[
|\nabla_v c(x, v)| \leq C(\gamma) \langle v \rangle^{\gamma - 1} \left| \chi' \left( \delta \langle x \rangle \langle v \rangle^{-3} \right) \right|,
\]
that
\[
\left| \int \nabla_c g \cdot \nabla_v (\mu) g dxdv \right| \leq \alpha C(\gamma) \sup (|\chi'|) \int \langle v \rangle^{\gamma - 1} |g| \|\nabla_v g\| \mu dxdv \leq \frac{\alpha Q}{2} \|g\|_{L^2(\mu)}^2,
\]
where $Q = [C(\gamma) \sup (|\chi'|)]$. Therefore, we choose $\alpha > 0$ sufficiently small with $\alpha Q < c_1$ and thus deduce that
\[
\int (\Lambda g) \mu dxdv = \|\nabla_v g\|_{L^2(\mu)}^2 + \int \nabla_v g \cdot \nabla_v (\mu) g dxdv + \int \left[ \frac{|v|^2 \langle v \rangle^{2\gamma - 4}}{4} - \frac{3}{2} \langle v \rangle^{2\gamma - 2} + \frac{(\gamma - 2)}{2} |v|^2 \langle v \rangle^{2\gamma - 4} \right] \mu dxdv \geq \frac{c_1}{2} \|g\|_{L^2(\mu)}^2 = c \|g\|_{L^2(\mu)}^2,
\]
which completes the proof.

In preparation for studying the time-like region, we provide the spectrum $\text{Spec}(\eta)$, $\eta \in \mathbb{R}^3$, of the operator $L_\eta = -iv \cdot \eta + L$. We remark that the spectrum analysis has been done in [18] only for the case $\gamma = 2$, and we can extend it to the case $\gamma \geq 1$. 

Lemma 6 (Spectrum of $L_\eta$). Assuming that $\gamma \geq 1$, given $\delta > 0$
(i) there exists $\tau = \tau(\delta) > 0$ such that if $|\eta| > \delta$,
(7) $\text{Spec}(\eta) \subset \{ z \in \mathbb{C} : \text{Re}(z) < -\tau \}$.
(ii) If $|\eta| < \delta$,
(8) $\text{Spec}(\eta) \cap \{ z \in \mathbb{C} : \text{Re}(z) > -\tau \} = \{ \lambda(\eta) \}$,
where $\lambda(\eta)$ is the eigenvalue of $L_\eta$ which is real and smooth in $\eta$ only through $|\eta|^2$, i.e., $\lambda(\eta) = \mathcal{A}(|\eta|^2)$ for some real smooth function $\mathcal{A}$; the eigenfunction $\epsilon_D(\eta)$ is smooth in $\eta$ as well. In addition, they are analytic in $\eta$ if $\gamma \geq 3/2$. Their asymptotic expansions are given as below:
(9) $\lambda(\eta) = -a_\gamma |\eta|^2 + O(|\eta|^4)$,
and $\lambda(\eta) = E_D + i\eta \omega E_D$, $\omega = |\eta|/|\eta|$. Here $\{ \epsilon_D(\eta) \}$ can be normalized by $\langle \epsilon_D(-\eta), \epsilon_D(\eta) \rangle_v = 1$.
(iii) Moreover, the semigroup $e^{(-i\eta \cdot v + L)t}$ can be decomposed as
(10) $e^{(-i\eta \cdot v + L)t} f = e^{(-i\eta \cdot v + L)t} \Pi_\eta^{D+} f + \lambda(\eta) \langle \epsilon_D(-\eta), f \rangle_v \epsilon_D(\eta)$,
and there exist $a(\tau) > 0$, $\tau > 0$ such that $|e^{(-i\eta \cdot v + L)t} \Pi_\eta^{D+} f| \lesssim e^{-a(\tau)t}$ and $|\epsilon(\eta)| \leq e^{-\tau|\eta|^2t}$.

Proof. Let $L = -\Lambda + K$ with
$\Lambda_\eta f = (-\Lambda - i\eta \cdot v) f$, $L_\eta = (L - i\eta \cdot v) f$.
Here $f \in D(\Lambda_\eta) = \{ f \in L^2_v; \Lambda_\eta f \in L^2_v \}$ and $D(\Lambda_\eta) = D(L_\eta)$. Since $K$ is a bounded operator in $L^2_v$, $L_\eta$ is regarded as a bounded perturbation of $\Lambda_\eta$. We shall verify that such a decomposition satisfies the four hypotheses H1-H4 stated in [24]. Under the assumptions H1-H4, using semigroup theory and linear operator perturbation theory, Theorem 1.1 in [24] asserts that the spectrum of $L_\eta$ has the similar structure of the Boltzmann equation with cutoff hard potential. Since the null space of the linear Fokker-Planck operator is one-dimensional, for $|\eta|$ small enough, we only obtain one smooth eigenvalue of $L_\eta$ while there are five smooth eigenvalues for the Boltzmann equation with cutoff hard potentials. As to the verification of H1-H4, the proof is a slight modification of the paper [18] and hence we omit the details. The hypothesis H1 is worthy of being mentioned, for $\varpi$ sufficiently large, there exists a positive constant $c > 0$ such that
$\langle A f, f \rangle_v \geq c |f|_{L^2_v}^2 \geq c |f|_{L^2_v}^2$,
for all $\gamma \geq 1$, the last inequality holds since $|f|_{L^2_v}$ is stronger than $|f|_{L^2_v}$ as $\gamma \geq 1$. This is why we miss the spectrum structure for the case $0 < \gamma < 1$.
To prove (ii), we need to explore the symmetric properties of $\lambda(\eta)$ and $\epsilon_D(\eta)$. Here we follow the framework of section 7.3 in [17]. First we note there is a natural three dimensional orthogonal group $O(3)$-action on $L^2_v$: Let $a \in O(3)$, $f \in L^2_v$,
$$(a \circ f)(v) \equiv f(a^{-1}v).$$
Then it is easy to check the $O(3)$-action commutes with operators $L$, $P_0$ and $P_1$. Consider eigenvalue problem
(11) $L_\eta \epsilon_D(\eta) = (-iv \cdot \eta + L) \epsilon_D(\eta) = \lambda(\eta) \epsilon_D(\eta)$.
Apply $a \in O(3)$ to (11), by commutative properties and the fact that $a$ preserves $\mathbb{R}^3$ vector inner product,
$$(-iv \cdot (a\eta) + L)(a \circ \epsilon_D(\eta)) = \lambda(\eta)(a \circ \epsilon_D(\eta))(\eta).$$
Then $\lambda(\alpha\eta) = \lambda(\eta)$, $e_D(\alpha\eta) = a \circ e(\eta)$, which implies that $\lambda(\eta)$ is dependent only upon $|\eta|$. Now let $a \in O(3)$ be an orthogonal transformation that sends $\frac{\alpha}{|\eta|}$ to $(1, 0, 0)^T$. Thus the original eigenvalue problem (11) is reduced to

\begin{equation}
( -i\nu |\eta| + L ) e(|\eta|) = \lambda(|\eta|) e(|\eta|),
\end{equation}

with $\lambda(\eta) = \lambda(|\eta|)$, $e_D(\eta) = a^{-1} \circ e(|\eta|)$. We emphasize that in the new eigenvalue problem (12), the dependence on $\eta$ is only through $|\eta|$. Apply the Macro-Micro decomposition to (12) to yield

\begin{align}
(13a) & \quad -i|\eta|P_0v_1 (P_0e + P_1e) = \Lambda P_0e, \\
(13b) & \quad -i|\eta|P_1v_1P_0e - i|\eta|P_1v_1P_1e + LP_1e = \lambda P_1e.
\end{align}

Set $\lambda(|\eta|) = i|\eta|\zeta(|\eta|)$ We can solve $P_1e$ in terms of $P_0e$ from (13b),

\begin{equation}
P_1e = i|\eta|[L - i|\eta|P_1v_1 - i|\eta|\zeta(|\eta|)]^{-1}P_1v_1P_0e,
\end{equation}

then substitute this back to (13a) to get

\begin{equation}
\left( P_0v_1 + i|\eta|P_0[L - i|\eta|P_1v_1 - i|\eta|\zeta(|\eta|)]^{-1}P_1v_1 \right) P_0e = -\zeta P_0e.
\end{equation}

We notice that this is actually a finite dimensional eigenvalue problem. The solvability of it and asymptotic expansions of eigenvalue and eigenfunction for $|\eta| \ll 1$ are essentially due to implicit function theorem. The procedure is basically the same as the case $\gamma = 2$, we refer the readers to Theorem 3.2 in [18] for details. We obtain $\lambda(|\eta|)$ and $P_0e(|\eta|) = \beta(|\eta|)E_D$ with $\lambda$ and $\beta$ being smooth functions. Furthermore, $\lambda(|\eta|)$ and $\beta(|\eta|)$ are not merely smooth but analytic for $\gamma \geq 3/2$. To prove this, it suffices to check that the perturbation $i\nu f$ is $L$-bounded, i.e.,

\[ |v|^{2/2}_L \leq C_1|L|f^{2/2}_L + C_2|f|^{2/2}_L. \]

Then the Kato-Rellich theorem guarantees the operator $B(z) = -i\nu z + L$ is the analytic family of Type (A), see [10], which in turn implies the eigenvalue and eigenfunction associated with (12) are analytic in $|\eta|$, cf. [9]. Now, let us calculate $\langle \Lambda f, \Lambda f \rangle_v$ first. For simplicity of notation, let

\[ \psi(v) = \frac{1}{4} |v|^2 \langle v \rangle^{2\gamma-4} - \left( \frac{3}{2} \langle v \rangle^{\gamma-2} + \frac{\gamma - 2}{2} |v|^2 \langle v \rangle^{\gamma-4} \right) + \varpi \chi_R(|v|), \]

then

\[ \langle \Lambda f, \Lambda f \rangle_v = |\Delta v f|^{2}_L + |\psi(v)f|^{2}_v \]

\[ + 2 \langle \psi(v)\nabla_v f, \nabla_v f \rangle_v + 2 \langle f\nabla_v \psi(v), \nabla_v f \rangle_v. \]

By the Cauchy inequality, we have

\[ \langle f\nabla_v \psi(v), \nabla_v f \rangle_v \leq \langle \psi(v)\nabla_v f, \nabla_v f \rangle_v + \frac{1}{4} \left( \frac{f^2}{\psi(v)} \right) \langle \psi(v)\nabla_v \psi(v), \nabla_v \psi(v) \rangle_v. \]

Let us compare $(\nabla_v \psi(v), \nabla_v \psi(v))$ and $\psi^3(v)$. For $|v|$ large, we have

\[ \langle \nabla_v \psi(v), \nabla_v \psi(v) \rangle \sim |v|^{4\gamma-6} \]

and

\[ \psi^3(v) \sim |v|^{6\gamma-6}. \]

For $|v|$ small, one can choose $\varpi$ large enough such that

\[ \langle \nabla_v \psi(v), \nabla_v \psi(v) \rangle \ll \psi^3(v). \]

This means

\[ \langle \Lambda f, \Lambda f \rangle_v \geq |\Delta v f|^{2}_L + \frac{1}{2} \langle \psi(v)f|^{2}_v \geq \langle v \rangle^{2\gamma-2} f^{2}_L. \]

Hence if $\gamma \geq 3/2$,

\[ |v|^{2/2}_L \leq C \langle \Lambda f, \Lambda f \rangle_v = C \langle Lf - Kf, Lf - Kf \rangle_v \leq C_1|L|f^{2/2}_L + C_2|f|^{2/2}_L. \]
However, smooth (analytic) in $|\eta|$ does not ensure smooth (analytic) in $\eta$. Our goal is to show $\lambda(z)$ and $\beta(z)$ are in fact even functions in $z$. If so, due to a classical theorem of Whitney [23], we have

$$\lambda(|\eta|) = \mathcal{A}(|\eta|^2), \quad \beta(|\eta|) = \mathcal{B}(|\eta|^2),$$

for some smooth or analytic functions $\mathcal{A}$ and $\mathcal{B}$ provided $\lambda(|\eta|)$ and $\beta(|\eta|)$ are smooth or analytic respectively. To show they are even, let us define a map $\mathcal{R} : (v_1, v_2, v_3) \mapsto (-v_1, v_2, v_3)$, then obviously $\mathcal{R} \in O(3)$. We apply $\mathcal{R}$ to (12),

$$(\mathcal{R} \circ e)(|\eta|) = \lambda(|\eta|)(\mathcal{R} \circ e(|\eta|)),$$

which is an eigenvalue problem with $|\eta| \rightarrow -|\eta|$. This follows that the eigenpair $\{\lambda(|\eta|), \mathcal{R} \circ e(|\eta|)\}$ coincides with $\{(\lambda(-|\eta|), e(-|\eta|))\}$. Hence

$$(16) \quad \lambda(|\eta|) = \lambda(-|\eta|), \quad \mathcal{R} \circ e(|\eta|) = e(-|\eta|).$$

In addition, use $\mathcal{R} \circ \mathcal{P}_0 e(|\eta|) = \mathcal{P}_0 \mathcal{R} \circ e(|\eta|) = \mathcal{P}_0 e(-|\eta|)$ and $\mathcal{R} \circ \mathcal{E}_D = \mathcal{E}_D$ to find $\beta(|\eta|) = \beta(-|\eta|)$, namely $\beta$ is also an even function. We can show

$$(17) \quad \lambda(|\eta|) = \lambda(-|\eta|), \quad \mathcal{R} \circ e(|\eta|) = e(-|\eta|),$$

by taking complex conjugate of (12). This together with (16) show $\lambda(|\eta|)$ and $\beta(|\eta|)$ are real functions. By (14), we can construct $e(|\eta|)$ from $\mathcal{P}_0 e(|\eta|)$,

$$e(|\eta|) = \mathcal{P}_0 e(|\eta|) + \mathcal{P}_1 e(|\eta|) = \left(1+i|\eta| \right) \left[L - i|\eta| \mathcal{P}_1 v_1 - \lambda(|\eta|) \right]^{-1} \mathcal{P}_1 v_1 \beta(|\eta|) \mathcal{E}_D.$$

The eigenfunction $e_D(\eta)$ to original eigen-problem (11) can be recovered by applying $a^{-1}$,

$$e_D(\eta) = a^{-1} \circ e(|\eta|) = \left(1 + \left[L - \mathcal{P}_1 i\eta \cdot v - \mathcal{A}(|\eta|^2) \right]^{-1} \mathcal{P}_1 i\eta \cdot v \mathcal{B}(|\eta|^2) \mathcal{E}_D.$$

Therefore the proof is complete. \qed

Let $h$ be the solution of the equation

$$\begin{cases}
\partial_t h = \mathcal{L} h, \quad \text{where} \quad \mathcal{L} h = -v \cdot \nabla_x h - \Lambda h, \\
h(0, x, v) = h_0(x, v),
\end{cases}$$

(18)

In the next subsection, we will study some properties of the semigroup operator $e^{t \mathcal{L}}$.

2.2. The semigroup operator $e^{t \mathcal{L}}$.

Lemma 7. For any $k \in \mathbb{N} \cup \{0\}$,

(i) If $\gamma \geq 1$, there exists $C > 0$ such that

$$\|e^{t \mathcal{L}} h_0\|_{H^k_x L^2(\mu)} \lesssim C \|h_0\|_{H^k_x L^2(\mu)}.$$  

(ii) If $0 < \gamma < 1$, we have

$$\|e^{t \mathcal{L}} h_0\|_{H^k_x L^2(\mu)} \lesssim \|h_0\|_{H^k_x L^2(\mu)}.$$  

Proof. It suffices to show that there exists $c_0 > 0$ such that for any multiindex $\beta$,

$$\frac{d}{dt} \|\partial_x^\beta h\|_{L^2(\mu)}^2 \leq -c_0 \|\partial_x^\beta h\|_{L^2(\mu)}^2.$$  

Now, by Lemma[5] we have

$$\frac{1}{2} \frac{d}{dt} \|\partial_x^\beta h\|_{L^2(\mu)}^2 = -\int -v \cdot \nabla_x \left(\partial_x^\beta h\right) \partial_x^\beta h \, dx \, dv - \int \Lambda \left(\partial_x^\beta h\right) \partial_x^\beta h \, dx \, dv$$

$$\leq \int \frac{1}{2} \left(\partial_x^\beta h\right)^2 v \cdot \nabla_x h \, dx \, dv - c_0 \left\|\partial_x^\beta h\right\|_{L^2(\mu)}^2.$$  

If $\mu(x, v) \equiv 1$, (21) is obvious since $\int \frac{1}{2} \left(\partial_x^\beta h\right)^2 v \cdot \nabla_x h \, dx \, dv = 0.$
If \( \mu(x,v) = \exp((x)/D) \) and \( \gamma \geq 3/2 \), we choose \( D \) sufficiently large such that \( 1/D < \min\{c_0, 1\} \) and thus obtain
\[
\left| \int \frac{1}{2} (\partial_x^2 h)^2 v \cdot \nabla_x \mu dx dv \right| = \left| \int \frac{1}{2} (\partial_x^2 h)^2 v \cdot \frac{x}{D(x)} \mu dx dv \right| \\
\leq \frac{1}{2D} \int \langle v \rangle^{2\gamma-2} (\partial_x^2 h)^2 \mu dx dv \leq \frac{c_0}{2} \| \partial_x^2 h \|_{L^2(\mu)}^2.
\]
If \( \mu(x,v) = e^{\alpha c(x,v)} \), \( \gamma \in (0,3/2) \), since
\[
|\nabla_x c(x,v)| \leq \delta C \langle v \rangle^{2\gamma-3},
\]
for some constant \( C > 0 \) depending only upon \( \gamma \), we have
\[
\left| \int \frac{1}{2} (\partial_x^2 h)^2 v \cdot \nabla_x \mu dx dv \right| \leq \frac{\delta C \alpha}{2} \int \langle v \rangle^{2\gamma-2} (\partial_x^2 h)^2 \mu dx dv \leq \frac{c_0}{2} \| \partial_x^2 h \|_{L^2(\mu)}^2
\]
by choosing \( 0 < \delta, \alpha \ll 1 \) such that \( \delta C \alpha < \min\{c_0, 1\} \).

Grouping the above discussion, we obtain (21) and thus deduce that for \( \gamma > 0, k \in \mathbb{N} \cup \{0\} \),
\[
\|e^{t\mathcal{L}}h_0\|_{H^k_x L^2(\mu)} \leq \|h_0\|_{H^k_x L^2(\mu)}.
\]
Moreover, if \( \gamma \geq 1 \), then (21) becomes
\[
\frac{1}{2} \frac{d}{dt} \| \partial_x^2 h \|_{L^2(\mu)}^2 \leq -\frac{c_0}{2} \| \partial_x^2 h \|_{L^2(\mu)}^2 \leq -\frac{c_0}{2} \| \partial_x^2 h \|_{L^2(\mu)}^2,
\]
which leads to the exponential time decay of all \( x \)-derivatives of the solution \( e^{t\mathcal{L}}h_0 \) in the weighted \( L^2 \) norm.

The following is the regularization estimate of the Fokker-Planck equation in small time.

**Lemma 8** (Regularization estimate). For \( \gamma > 0 \) and \( 0 < t \leq 1 \), we have
\[
\int |\nabla_x e^{t\mathcal{L}}h_0|^2 \mu dx dv = O(t^{-1}) \int |h_0|^2 \mu dx dv
\]
and
\[
\int |\nabla_x e^{t\mathcal{L}}h_0|^2 \mu dx dv = O(t^{-3}) \int |h_0|^2 \mu dx dv.
\]

**Proof.** Recall that \( \Lambda = -L + K \) with
\[
\Lambda f = -\triangle f + \left[ \frac{1}{4} |\nabla \Phi|^2 - \frac{1}{2} \Delta \Phi \right] f + \varpi \chi_R(|v|) f, \quad K = \varpi \chi_R(|v|) f.
\]
Here we choose \( R > 0 \) and \( \varpi > 0 \) sufficiently large such that
\[
\begin{cases}
\frac{1}{4} |\nabla \Phi|^2 - \frac{1}{2} \Delta \Phi \leq \frac{1}{3} \langle v \rangle^{2\gamma-2}, & |\nabla v \left( \frac{1}{4} |\nabla \Phi|^2 - \frac{1}{2} \Delta \Phi \right) | \lesssim \langle v \rangle^{2\gamma-3} \quad \text{for} \ |v| > 2R, \\
\frac{1}{4} |\nabla \Phi|^2 + \frac{1}{2} \Delta \Phi, & |\nabla v \left( \frac{1}{4} |\nabla \Phi|^2 - \frac{1}{2} \Delta \Phi \right) | < \frac{\varpi}{2} \quad \text{for} \ |v| \leq 2R,
\end{cases}
\]
Now, define the energy functional
\[
\mathcal{F}(t, h) := A \|h\|_{L^2(\mu)}^2 + at \|\nabla \Phi \|_{L^2(\mu)}^2 + 2ct^2 \langle \nabla_x h, \nabla \Phi \rangle_{L^2(\mu)} + bt^3 \|\nabla_x h\|_{L^2(\mu)}^2,
\]
with \( a, b, c > 0 \) and \( c < \sqrt{ab} \) (positive definite) and \( A \) sufficiently large. We shall show that \( d\mathcal{F}/dt \leq 0, \ t \in (0,1) \), via choosing suitable positive constants \( A, a, b \) and \( c \).

In (21), it has been shown that
\[
\frac{d}{dt} \|h\|_{L^2(\mu)}^2 \leq -c_0 \|h\|_{L^2(\mu)}^2,
\]
and
\[
\frac{d}{dt} \|\partial_x h\|_{L^2(\mu)}^2 \leq -c_0 \|\partial_x h\|_{L^2(\mu)}^2.
\]
Next, we show that
\begin{equation}
\frac{1}{2} \frac{d}{dt} \| \partial_v h \|^2_{L^2(\mu)} \leq - \int \partial_x h \partial_v h \mu dx dv - \frac{c_0}{2} \| \partial_v h \|^2_{L^2(\mu)} + C_\varepsilon \| h \|^2_{L^2(\mu)} + \varepsilon \| \partial_v h \|^2_{L^2(\mu)},
\end{equation}
where $\varepsilon > 0$ is arbitrarily small and $C_\varepsilon = O(1/\varepsilon)$. Compute
\begin{equation}
\frac{1}{2} \frac{d}{dt} \| \partial_v h \|^2_{L^2(\mu)} = - \int \partial_x h \partial_v h \mu dx dv + \int \left[ \frac{v}{2} (\partial_v h)^2 \right] \cdot \nabla x \mu dx dv
\end{equation}
\begin{equation}
- \int (\Lambda \partial_v h) \partial_v h \mu dx dv - \int [\partial_v, \Lambda] h \partial_v h \mu dx dv,
\end{equation}
where
\begin{equation}
[\partial_v, \Lambda] h = \partial_v \left[ \frac{1}{4} |\nabla_v \Phi|^2 - \frac{1}{2} \Delta_v \Phi \right] h + \varpi \partial_v (\chi R) h.
\end{equation}
In the proof of Lemma 5, one can see that
\begin{equation}
|\nabla x \mu| \leq \min\{c_0, 1\} \cdot \langle v \rangle^{2\gamma - 3} \mu \quad \text{and} \quad |\nabla x \mu| \leq \min\{c_0, 1\} \mu,
\end{equation}
so
\begin{equation}
\left| \int \left[ \frac{v}{2} (\partial_v h)^2 \right] \cdot \nabla x \mu dx dv \right| \leq \frac{c_0}{2} \int \langle v \rangle^{2\gamma - 2} (\partial_v h)^2 \mu dx dv \leq \frac{c_0}{2} \| \partial_v h \|^2_{L^2(\mu)}.
\end{equation}
Furthermore, by (22) we obtain
\begin{equation}
\left| \int [\partial_v, \Lambda] h \partial_v h \mu dx dv \right| \leq C \int \langle v \rangle^{2\gamma - 2} |h \partial_v h| \mu dx dv + \frac{c_0}{2} \int \chi R \| h \partial_v h \| \mu dx dv + \frac{c_0}{2} \int \chi \mu dx dv \leq C' \int \langle v \rangle^{2\gamma - 2} |h \partial_v h| \mu dx dv \leq C_\varepsilon \| h \|^2_{L^2(\mu)} + \varepsilon \| \partial_v h \|^2_{L^2(\mu)},
\end{equation}
where $\varepsilon > 0$ is arbitrary small and $C_\varepsilon = O(1/\varepsilon)$. It turns out that
\begin{equation}
\frac{1}{2} \frac{d}{dt} \| \partial_v h \|^2_{L^2(\mu)} \leq - \int \partial_x h \partial_v h \mu dx dv - \frac{c_0}{2} \| \partial_v h \|^2_{L^2(\mu)} + C_\varepsilon \| h \|^2_{L^2(\mu)} + \varepsilon \| \partial_v h \|^2_{L^2(\mu)}.
\end{equation}
Finally, direct computation gives
\begin{equation}
\frac{d}{dt} \int \partial_x h \partial_v h \mu dx dv
\end{equation}
\begin{equation}
= - \int |\partial_x h|^2 \mu dx dv - 2 \int \nabla_v (\partial_x h) \cdot \nabla_v (\partial_v h) \mu dx dv - 2 \int \left( \frac{|\nabla_v \Phi|^2}{4} - \frac{\Delta_v \Phi}{2} + \varpi \chi R \right) \partial_x h \partial_v h \mu dx dv
\end{equation}
\begin{equation}
+ \int (v \cdot \nabla x \mu) \partial_v h \partial_x h \mu dx dv + \frac{1}{2} \int \partial_v \left( \frac{|\nabla_v \Phi|^2}{4} - \frac{\Delta_v \Phi}{2} + \varpi \chi R \right) h^2 \partial_x h \mu dx dv
\end{equation}
\begin{equation}
- \int \nabla_v (\partial_x h) \cdot \nabla_v (\mu) \partial_v h dx dv - \int \nabla_v (\partial_v h) \cdot \nabla_v (\mu) \partial_x h dx dv.
\end{equation}
From (16), it follows that
\begin{equation}
\left| \int \nabla_v (\partial_x h) \cdot \nabla_v (\mu) \partial_v h dx dv + \int \nabla_v (\partial_v h) \cdot \nabla_v (\mu) \partial_x h dx dv \right|
\end{equation}
\begin{equation}
\leq aC(\gamma) \sup \langle |\chi'| \rangle \int \langle v \rangle^{\gamma - 1} (|\nabla_v (\partial_x h)||\partial_v h| + |\nabla_v (\partial_v h)||\partial_x h|) \mu dx dv
\end{equation}
\begin{equation}
= aQ \int \left( \langle v \rangle^{\gamma - 1} |\nabla_v (\partial_x h)||\partial_v h| + \langle v \rangle^{\gamma - 1} |\nabla_v (\partial_v h)||\partial_x h| \right) \mu dx dv, \quad Q = C(\gamma) \sup \langle |\chi'| \rangle.
\end{equation}
Note that this inequality is valid in the cases $\mu (x, v) = 1$ and $\mu (x, v) = \exp (\langle x \rangle / D)$ as well, since $\nabla_v \mu = 0$ in both cases. Therefore,
\begin{equation}
\frac{d}{dt} \int \partial_x h \partial_v h \mu dx dv
\end{equation}
\begin{equation}
\leq - \int |\partial_x h|^2 \mu dx dv - 2 \int \nabla_v (\partial_x h) \cdot \nabla_v (\partial_v h) \mu dx dv - 2 \int \left( \frac{|\nabla_v \Phi|^2}{4} - \frac{\Delta_v \Phi}{2} + \varpi \chi R \right) \partial_x h \partial_v h \mu dx dv.
\[
+ \int (v \cdot \nabla_x \mu) \partial_{x_i} h \partial_{x_i} h dx dv + \frac{1}{2} \int \partial_{x_i} \left( \left| \nabla_v \Phi \right|^2 \frac{\nabla_v \Phi \cdot \partial_{x_i} h}{2} + \frac{\triangle_v \Phi}{2} + \varpi \chi_R \right) h^2 \partial_{x_i} \mu dx dv \\
+ \alpha Q \int \left( (v)^{-1} \left| \nabla_v (\partial_x h) \right| \left| \partial_{x_i} h \right| + (v) \gamma^{-1} \left| \nabla_v (\partial_v, h) \right| \left| \partial_{x_i} h \right| \right) \mu dx dv.
\]

Collecting terms gives
\[
\frac{d}{dt} F(t, \mu) \\
\leq -c_0 A \| h \|_{L^2_\mu}^2 + a \| \nabla_v h \|_{L^2_\mu}^2 + 4ct \left\langle \nabla_x h, \nabla_v h \right\rangle_{L^2_\mu} + 3b \| \nabla_x h \|_{L^2_\mu}^2 \\
+ 2at \left[ -3 \sum_{i=1}^3 \int \partial_{x_i} h \partial_{x_i} v h dx dv - \frac{c_0}{2} \| \nabla_v h \|_{L^2_\mu}^2 + 3C_\varepsilon \| h \|_{L^2_\mu}^2 + \varepsilon \| \nabla_v h \|_{L^2_\mu}^2 \right] \\
+ 2ct^2 \left[ -3 \sum_{i=1}^3 \int \partial_{x_i} h \partial_{x_i} v h dx dv - \sum_{i=1}^3 \int (v \cdot \nabla_x \mu) \partial_{x_i} h \partial_{x_i} h dx dv \\
- 2\sum_{i=1}^3 \int \left( \frac{\left| \nabla_v \Phi \right|^2}{4} - \frac{\triangle_v \Phi}{2} + \varpi \chi_R \right) \partial_{x_i} h \partial_{x_i} h dx dv - \sum_{i=1}^3 \int (v \cdot \nabla_x \mu) \partial_{x_i} h \partial_{x_i} h dx dv \\
+ \frac{1}{2} \sum_{i=1}^3 \int \partial_{x_i} h \partial_{x_i} \left( \frac{\left| \nabla_v \Phi \right|^2}{4} - \frac{\triangle_v \Phi}{2} + \varpi \chi_R \right) h^2 \partial_{x_i} \mu dx dv \\
+ \alpha Q \sum_{i=1}^3 \int \left( (v)^{-1} \left| \nabla_v (\partial_x h) \right| \left| \partial_{x_i} h \right| + (v) \gamma^{-1} \left| \nabla_v (\partial_v, h) \right| \left| \partial_{x_i} h \right| \right) \mu dx dv \\
- b c_0 t \| \nabla_x h \|_{L^2_\mu}^2 .
\]

Since (using (22) and (26))
\[
\left| -3 \sum_{i=1}^3 \int \left( \frac{\left| \nabla_v \Phi \right|^2}{4} - \frac{\triangle_v \Phi}{2} + \varpi \chi_R \right) \partial_{x_i} h \partial_{x_i} v h dx dv + 3 \int \partial_{x_i} (v \cdot \nabla_x \mu) \partial_{x_i} h \partial_{x_i} h dx dv \right| \\
\leq 3 \sum_{i=1}^3 \int (v)^{2\gamma-2} \left| \partial_{x_i} h \partial_{x_i} h \right| \mu dx dv + 3\varpi \int \chi_R |\nabla_x h \cdot \nabla_v h| \mu dx dv \\
\leq 3 \left[ \frac{b c_0}{8c} t \| (v)^{-1} (\partial_x h) \|_{L^2_\mu}^2 + \frac{8c}{bc_0 t} \| (v) \gamma^{-1} (\partial_v, h) \|_{L^2_\mu}^2 \right] + 3\varpi \int \chi_R |\nabla_x h \cdot \nabla_v h| \mu dx dv \\
= \frac{b c_0}{8c} t \| \nabla_x h \|_{L^2_\mu}^2 + \frac{8c}{bc_0 t} \| \nabla_v h \|_{L^2_\mu}^2 + 3\varpi \int \chi_R |\nabla_x h \cdot \nabla_v h| \mu dx dv,
\]

and (using the Cauchy-Schwartz inequality)
\[
\left| 2 \sum_{i=1}^3 \int \nabla_v (\partial_x h) \cdot \nabla_v (\partial_v, h) \mu dx dv \right| \leq \sum_{i=1}^3 \left[ \frac{b c_0}{8c} t \| \nabla_v (\partial_x h) \|_{L^2_\mu}^2 + \frac{8c}{bc_0 t} \| \nabla_v (\partial_v, h) \|_{L^2_\mu}^2 \right] \\
\leq \frac{b c_0}{8c} t \| \nabla_x h \|_{L^2_\mu}^2 + \frac{8c}{bc_0 t} \| \nabla_v h \|_{L^2_\mu}^2 ,
\]

and (using (22) and (26))
\[
\left| \frac{1}{2} \sum_{i=1}^3 \int \partial_{x_i} \left( \frac{\left| \nabla_v \Phi \right|^2}{4} - \frac{\triangle_v \Phi}{2} + \varpi \chi_R \right) h^2 \partial_{x_i} \mu dx dv \right| \\
\leq \frac{C}{2} \int (v)^{2\gamma-3} h^2 \mu dx dv + \frac{\varpi}{4} \int (v)^{2\gamma-3} \chi_R h^2 \mu dx dv + \frac{\varpi}{2R} \int (v)^{2\gamma-3} |\chi_R| h^2 \mu dx dv \\
\leq M' \int (v)^{2\gamma-2} h^2 \mu dx dv \leq M' \| h \|_{L^2_\mu}^2 .
\]
where $M’ > 0$ is dependent only upon $R$ and $\varepsilon$, and
\[
\left| \sum_{i=1}^{3} \int \left( (v)^{\gamma-1} |\nabla_v (\partial_x h)| |\partial_v h| + (v)^{\gamma-1} |\nabla_v (\partial_x h)| |\partial_x h| \right) \mu dx dv \right|
\leq \sum_{i=1}^{3} \left[ \frac{2c}{bc_0 t} \int \langle v \rangle^{2\gamma-2} |\partial_v h| \mu dx dv + \frac{bc_0 t}{8c} \int |\nabla_v (\partial_x h)|^2 \mu dx dv \right]
\leq \frac{2c}{bc_0 t} |\nabla \phi h|^2_{L^2(\mu)} + \frac{bc_0}{8c} t \|\nabla_x h\|^2_{L^2(\mu)},}
we deduce that
\[
\frac{d}{dt} F(t, h_t) \leq \|h\|^2_{L^2(\mu)} \left[ -a_0 A + a + 6atC_c + (4c + 2a + 6c\varepsilon t) C_c + 2cM’t^2 \right]
+ \|\nabla_x h\|^2_{L^2(\mu)} \left( -2c + 3b + (4c + 2a + 6c\varepsilon t) \varepsilon \right) t^2
+ \|\nabla_x h\|^2_{L^2(\mu)} \left( -\frac{1}{2} bc_0 + \frac{\alpha bc_0}{4} \right) t^3
+ \|\nabla_x h\|^2_{L^2(\mu)} \left( -\alpha c_0 + 2a \varepsilon + \frac{32c^2}{bc_0} + \frac{4c^2 \alpha}{bc_0} \right) t.
\]
Set $a = \varepsilon$, $4b = c = \varepsilon^{3/2}$. Choosing $A$ sufficiently large, $\varepsilon > 0$ sufficiently small, we obtain
\[
\frac{d}{dt} F(t, h_t) \leq 0, \quad t \in (0, 1),
\]
which implies that
\[
F(t, h_t) \leq F(0, h_0) = A \|h_0\|^2_{L^2}, \quad t \in [0, 1].
\]
This completes the proof. \(\square\)

Before the end of this section, we will introduce the wave-remainder decomposition, which is the key decomposition in our paper. To this end, we design a Picard-type iteration, treating $Kf$ as a source term. The zero order approximation of the Fokker-Planck equation (2) is
\[
\left\{ \begin{array}{l}
\partial_t h^{(0)} + v \cdot \nabla_x h^{(0)} + \Lambda h^{(0)} = 0, \\
h^{(0)}(0, x, v) = f_0(x, v).
\end{array} \right.
\]
Thus the difference $f - h^{(0)}$ satisfies
\[
\left\{ \begin{array}{l}
\partial_t (f - h^{(0)}) + v \cdot \nabla_x (f - h^{(0)}) + \Lambda (f - h^{(0)}) = K(f - h^{(0)}) + Kh^{(0)}, \\
(f - h^{(0)})(0, x, v) = 0.
\end{array} \right.
\]
Therefore the first order approximation $h^{(1)}$ can be defined by
\[
\left\{ \begin{array}{l}
\partial_t h^{(1)} + v \cdot \nabla_x h^{(1)} + \Lambda h^{(1)} = Kh^{(0)}, \\
h^{(1)}(0, x, v) = 0.
\end{array} \right.
\]
In general, we can define the $j^{th}$ order approximation $h^{(j)}$, $j \geq 1$, as
\[
\left\{ \begin{array}{l}
\partial_t h^{(j)} + v \cdot \nabla_x h^{(j)} + \Lambda h^{(j)} = Kh^{(j-1)}, \\
h^{(j)}(0, x, v) = 0.
\end{array} \right.
\]
The wave part and the remainder part can be defined as follows:
\[
W^{(3)} = \sum_{j=0}^{3} h^{(j)}, \quad R^{(3)} = f - W^{(3)}.
\]
Note that $\mathcal{R}^{(3)}$ solves the equation
\begin{align}
\begin{cases}
\partial_t \mathcal{R}^{(3)} + v \cdot \nabla_x \mathcal{R}^{(3)} = L \mathcal{R}^{(3)} + Kh^{(3)}, \\
\mathcal{R}^{(3)}(0, x, v) = 0.
\end{cases}
\end{align}
(31)

Here are some basic estimates of the wave part.

**Lemma 9** (Estimate of the wave part). *If we assume the initial condition $f_0$ has compact support in $x$, then for all $0 \leq k \leq 3$ and $t > 0$,

(i) if $\gamma \geq 1$, there exists $C > 0$ such that

$$|h^{(k)}|_{L^2_t L^2_x} \lesssim e^{-Ct} t^{-(9-2k)/2} e^{-\frac{|v|^2}{4t^2}} \|f_0\|_{L^1_t L^2_x}, \quad \|h^{(k)}\|_{L^2_x(\mu)} \lesssim t^k e^{-Ct} \|f_0\|_{L^2(\mu)},$$

(ii) if $0 < \gamma < 1$, we have

$$|h^{(k)}|_{L^2_t L^2_x} \lesssim t^{-(9-2k)/2} e^{-\frac{|v|^2}{4t^2}} \|f_0\|_{L^1_t L^2_x}, \quad \|h^{(k)}\|_{L^2_x(\mu)} \lesssim t^k \|f_0\|_{L^2(\mu)}.$$  

*Proof.* To prove this lemma, we need to recall the fundamental solution of Kolmogorov-Fokker-Planck equation [12]: Let $g$ solve
\begin{align}
\begin{cases}
\partial_t g + v \cdot \nabla_x g = \Delta_v g, \\
g(0, x, v) = g_0(x, v).
\end{cases}
\end{align}
(32)

Then
\begin{align}
g(t, x, v) = \int_{\mathbb{R}^6} G_{FP}(t, x, v; 0, y, u)g_0(y, u) dy du,
\end{align}
(33)

where the Green function $G_{FP}$ is
\begin{align}
G_{FP}(t, x, v; \tau, y, u) = \frac{1}{(t-\tau)^6} \exp \left( - \frac{3((x-y)-(t-\tau)/2)((v+u))^2}{(t-\tau)^3} - \frac{|v-u|^2}{4(t-\tau)} \right).
\end{align}
(34)

Moreover,
\begin{align}
G_{FP}(t, x, v; t', x', v') = \int_{\mathbb{R}^6} G_{FP}(t, x, v; \tau, y, u)G_{FP}(\tau, y, u; t', x', v') dy du.
\end{align}
(35)

If $g_0$ has compact support in the $x$ variable, by (33) and (34), we have
\begin{align}
|g|_{L^2_x} \lesssim t^{-9/2} e^{-\frac{|v|^2}{4t^2}} \|g_0\|_{L^1_t L^2_x}.
\end{align}

Note that $h^{(0)}$ satisfies the Kolmogorov-Fokker-Planck equation with damping term, by comparison principle, we have that for $\gamma \geq 1$,
\begin{align}
|h^{(0)}| \lesssim e^{-Ct} \int_{\mathbb{R}^6} G_{FP}(t, x, v; 0, y, u)|f_0(y, u)| dy du,
\end{align}

hence
\begin{align}
|h^{(0)}|_{L^2_t L^2_x} \lesssim e^{-Ct} t^{-9/2} e^{-\frac{|v|^2}{4t^2}} \|f_0\|_{L^1_t L^2_x}.
\end{align}

By (35) and Duhamel’s formula, we have
\begin{align}
|h^{(k)}|_{L^2_t L^2_x} \lesssim e^{-Ct} t^{-(9-2k)/2} e^{-\frac{|v|^2}{4t^2}} \|f_0\|_{L^1_t L^2_x}.
\end{align}

The case for $0 < \gamma < 1$ is similar and we omit the detail. This completes the $L^2_t$ estimate. The $L^2_x(\mu)$ estimate of $h^{(k)}$ can be constructed by Lemma 7 and Duhamel’s formula easily and we omit the proof. \qed

We need the regularization estimate of $h^{(3)}$ since it appears in the source term of the remainder part.
Lemma 10 (x-derivatives estimate of $h^{(3)}$). Assuming that $\gamma > 0$, $j = 1, 2$, then

(i) for $0 < t \leq 1$, we have

$$\| \nabla_x^j h^{(3)} \|_{L^2(\mu)} \lesssim t^{3-\frac{j}{2}}\| f_0 \|_{L^2(\mu)}.$$ 

(ii) for $t > 1$, we have that if $\gamma \geq 1$,

$$\| \nabla_x^j h^{(3)} \|_{L^2(\mu)} \lesssim \| f_0 \|_{L^2(\mu)},$$

and if $0 < \gamma < 1$,

$$\| \nabla_x^j h^{(3)} \|_{L^2(\mu)} \lesssim t^3\| f_0 \|_{L^2(\mu)}.$$ 

Proof. We divide our proof into several steps:

Step 1: First derivative of $h^{(j)}$, $0 \leq j \leq 3$ in small time. We want to show that for $0 < t \leq 1$,

$$\| \nabla_x h^{(j)} \|_{L^2(\mu)} \lesssim t^{(-3+2j)/2}\| f_0 \|_{L^2(\mu)}.$$

The estimate of $h^{(0)}$ is immediately from Lemma 8. Note that

$$h^{(1)} = \int_0^t e^{(t-s)\mathcal{L}} Ke^{s\mathcal{L}} f_0 ds,$$

hence

$$\nabla_x h^{(1)} = \int_0^t \frac{(t-s)+s}{t} \nabla_x e^{(t-s)\mathcal{L}} Ke^{s\mathcal{L}} f_0 ds,$$

we then have (by Lemma 8)

$$\left\| \nabla_x h^{(1)} \right\|_{L^2(\mu)} \lesssim \int_0^t t^{-1} \left[ (t-s)^{-1/2} + s^{-1/2} \right] ds \| f_0 \|_{L^2(\mu)}$$

$$\lesssim t^{-1/2}\| f_0 \|_{L^2(\mu)}.$$ 

Similarly, note that

$$h^{(2)} = \int_0^t \int_0^{s_1} e^{(t-s_1)\mathcal{L}} Ke^{(s_1-s_2)\mathcal{L}} Ke^{s_2\mathcal{L}} f_0 ds_2 ds_1,$$

hence

$$\nabla_x h^{(2)} = \int_0^t \int_0^{s_1} \frac{s_1-s_2}{s_1} \nabla_x e^{(t-s_1)\mathcal{L}} Ke^{(s_1-s_2)\mathcal{L}} Ke^{s_2\mathcal{L}} f_0 ds_2 ds_1,$$

we then have

$$\left\| \nabla_x h^{(2)} \right\|_{L^2(\mu)} \lesssim \int_0^t \int_0^{s_1} s_1^{-1} \left[ (s_1-s_2)^{-1/2} + s_2^{-1/2} \right] ds_2 ds_1 \| f_0 \|_{L^2(\mu)}$$

$$\lesssim t^{1/2}\| f_0 \|_{L^2(\mu)}.$$ 

The estimate of $h^{(3)}$ is similar and hence we omit the details.

Step 2: Second derivatives of $h^{(j)}$, $0 \leq j \leq 3$ in small time. We want to show that for any $0 < t \leq 1$,

$$\| \nabla_x^2 h^{(3)} \|_{L^2(\mu)} \lesssim C_j t^{-3+j}\| f_0 \|_{L^2(\mu)}.$$

We only give the estimates of $h^{(0)}$ and $h^{(1)}$. The case $h^{(2)}$ and $h^{(3)}$ are similar. Let us estimate $h^{(0)}$. For any $0 < t_0 \leq 1$ and $t_0/2 < t \leq t_0$, we have

$$\nabla_x h^{(0)}(t) = e^{(t-t_0)\mathcal{L}} \nabla_x h^{(0)}(t_0/2),$$

hence (by Lemma 8)

(36)  $$\left\| \nabla_x^2 h^{(0)}(t) \right\|_{L^2(\mu)} \lesssim \left( t - t_0/2 \right)^{-3/2} \left( t_0/2 \right)^{-3/2}\| f_0 \|_{L^2(\mu)}.$$ 

If we take $t = t_0$, we have

(37)  $$\left\| \nabla_x^2 h^{(0)}(t_0) \right\|_{L^2(\mu)} \lesssim t_0^{-3}\| f_0 \|_{L^2(\mu)}.$$
Note that $t_0 \in (0, 1)$ is arbitrary. This completes the estimate of $h^{(0)}$. For $h^{(1)}$, let $0 < t_1 \leq 1$ and $t_1/2 < t \leq t_1$, then
\[
\nabla_x h^{(1)}(t) = e^{(t-t_1/2)c} \nabla_x h^{(1)}(t_1/2) + \int_{t_1/2}^t e^{(t-s)c} K \nabla_x h^{(0)}(s) ds,
\]
hence (by Lemma 8 and (37))
\[
\left\| \nabla_x h^{(1)}(t) \right\|_{L^2(\mu)} \lesssim (t - t_1/2)^{-3/2} (t_1/2)^{-1/2} \left\| f_0 \right\|_{L^2(\mu)} + \int_{t_1/2}^t s^{-3} \left\| f_0 \right\|_{L^2(\mu)} ds.
\]
Now, take $t = t_1$, we get
\[
\left\| \nabla_x h^{(1)}(t_1) \right\|_{L^2(\mu)} \lesssim t_1^{-2} \left\| f_0 \right\|_{L^2(\mu)}.
\]
This completes the estimate of $h^{(1)}$.

Next, we shall prove the large time behavior for $\gamma \geq 1$; the case $0 < \gamma < 1$ is similar and we omit the details.

Step 3: First derivative of $h^{(j)}$, $0 \leq j \leq 3$, in large time ($\gamma \geq 1$). We want to show that for $t > 1$,
\[
\left\| \nabla_x h^{(j)} \right\|_{L^2(\mu)} \leq C \left\| f_0 \right\|_{L^2(\mu)}.
\]
For $h^{(0)}$, by Lemma 7, we have
\[
(38) \quad \left\| \nabla_x h^{(0)}(t) \right\|_{L^2(\mu)} \leq \left\| \nabla_x h^{(0)}(1) \right\|_{L^2(\mu)} \lesssim \left\| f_0 \right\|_{L^2(\mu)}.
\]
For $h^{(1)}$, we have
\[
h^{(1)}(t) = e^{(t-1)c} h^{(1)}(1) + \int_1^t e^{(t-s)c} K h^{(0)}(s) ds,
\]
and hence (using Lemma 7 and (35))
\[
\left\| \nabla_x h^{(1)}(t) \right\|_{L^2(\mu)} \leq e^{-C(t-1)} \left\| \nabla_x h^{(1)}(1) \right\|_{L^2(\mu)} + \int_1^t e^{-C(t-s)} \left\| \nabla_x h^{(0)} \right\|_{L^2(\mu)} (s) ds
\]
\[
\lesssim \left[ e^{-C(t-1)} + 1 \right] \left\| f_0 \right\|_{L^2(\mu)}.
\]
The estimate of $\nabla_x h^{(2)}$ and $\nabla_x h^{(3)}$ are similar and hence we omit the details.

Step 4: Second derivatives of $h^{(j)}$, $0 \leq j \leq 3$ in large time ($\gamma \geq 1$). We can show that for $t > 1$,
\[
\left\| \nabla_x^2 h^{(j)} \right\|_{L^2(\mu)} \leq C \left\| f_0 \right\|_{L^2(\mu)}.
\]
This estimate is similar to step 3 and hence we omit the details. \hfill \Box

3. Behavior for time like region

In this section, we will see the large time behavior of the solution.

3.1. The case $\gamma \geq 1$: Fluid structure. By the Fourier transform (in the $x$ variable), the solution of the Fokker-Planck equation can be represented as
\[
f(t, x, v) = \int_{\mathbb{R}^3} e^{i p x + (-i v \cdot \eta + L)t} \hat{f}_0(\eta, v) d\eta.
\]
We can decompose the solution $f$ into the long wave part $f_L$ and the short wave part $f_S$ given respectively by
\[
f_L = \int_{|\eta| < \delta} e^{i p x + (-i v \cdot \eta + L)t} \hat{f}_0(\eta, v) d\eta,
\]
\[
f_S = \int_{|\eta| > \delta} e^{i p x + (-i v \cdot \eta + L)t} \hat{f}_0(\eta, v) d\eta.
\]
The following short wave analysis relies on spectral analysis (Lemma 8).
Proposition 11 (Short wave $f_S$). Assuming that $\gamma \geq 1$, let $f_0 \in L^2$, then
\[ \|f_S\|_{L^2} \leq e^{-\alpha \gamma t} \|f_0\|_{L^2}. \]

In order to study the long wave part $f_L$ for $\gamma \geq 1$, we need to decompose the long wave part as the fluid part and non-fluid part, i.e., $f_L = f_{L,0} + f_{L,\perp}$, where
\[ f_{L,0} = \int_{|\eta|<\delta} e^{\lambda(n) t} e^{ix \cdot (\eta D(-\eta) \cdot \hat{f}_0)} e^{D(\eta)} d\eta, \]
\[ f_{L,\perp} = \int_{|\eta|<\delta} e^{ix \cdot \eta} e^{(i\eta \cdot \eta + L) t} \Pi_{\eta} \hat{f}_0 d\eta. \]

Using lemma 6, we have the exponential decay of the non-fluid long wave part.

Proposition 12 (Non fluid long wave $f_{L,\perp}$). Assuming that $\gamma \geq 1$ and $s > 0$, let $f_0 \in L^2$, we have
\[ \|f_{L,\perp}\|_{H^s L^2} \leq e^{-\alpha \gamma t} \|f_0\|_{L^2}. \]

For the fluid part, we have the following structure:

Proposition 13 (Fluid long wave $f_{L,0}$). For $\gamma \geq 3/2$ and any given $M > 1$ there exists $C > 0$ such that for $|x| \leq Mt$,
\[ |f_{L,0}(x, t)|_{L^2} \leq C \left[ (1 + t)^{-3/2} e^{-\frac{|x|^2}{C(t+1)}} + e^{-t/C} \right] \|f_0\|_{L^2}^2. \]

On the other hand, for $1 \leq \gamma < 3/2$ and any given positive integer $N$, there exists positive constant $C$ (depending on $N$) such that
\[ |f_{L,0}(x, t)|_{L^2} \leq C \left[ (1 + t)^{-3/2} \left( 1 + \frac{|x|^2}{1 + t} \right)^{-N} + e^{-t/C} \right] \|f_0\|_{L^2}^2. \]

Proof. Before the proof of this proposition, we need the following two lemmas:

Lemma 14 (Lemma 7.11, [17]). Suppose that $g(\eta, t)$ is analytic in $\eta$ for $|\eta| < \delta \ll 1$ and satisfies
\[ g(\eta, t) = O(e^{-A|\eta|^2 t + O(|\eta|^4 t)}), \]
for some $A > 0$. Then in the region of $|x| < (2\Re + 1)t$, $\Re$ is any given positive constant, there exists a constant $C$ such that the following inequality holds:
\[ \left| \int_{|\eta|<\delta} e^{ix \cdot \eta} g(\eta, t) d\eta \right| \leq C \left[ (1 + t)^{-\frac{\delta}{2}} e^{-\frac{|x|^2}{1 + t}} + e^{-t/C} \right]. \]

Lemma 15 (Reformulation of Lemma 2.2, [14]). Let $x, \eta \in \mathbb{R}^n$. Suppose $g(\eta, t)$ has compact support in the variable $\eta$, and there exists a constant $b > 0$, such that $g(\eta, t)$ satisfies
\[ \left| D^\alpha_\eta g(\eta, t) \right| \leq C(|\eta|^{(|\alpha| + k - |\beta|) +} + |\eta|^{(|\alpha| + k + |\beta|/2)}(1 + t)|\eta|^2)^m e^{-b|\eta|^2 t} \]
for any $k, m \in \mathbb{N}$ and any multi-indices $\alpha, \beta$ with $|\beta| \leq 2N$, then there exists a positive constant $C$ such that
\[ \left| \int_{|\eta|<\delta} e^{ix \cdot \eta} g(\eta, t) d\eta \right| \leq C_N \left[ (1 + t)^{-\frac{n + |\alpha| + k}{2}} B_N(|x|, t) + e^{-t/C_N} \right], \]
where $N$ is any fixed integer, $(e) = \max(0, e)$ and
\[ B_N(|x|, t) = \left( 1 + \frac{|x|^2}{1 + t} \right)^{-N}. \]
Now, let us go back to the proof of this proposition. Noticing that
\[ |f_{L,0}(x,t)|_{L^2_x} \leq \left| \int_{|\eta|<\delta} e^{\lambda(\eta)t} e^{i\eta \cdot x} \langle e_D(-\eta), \hat{f}_0 \rangle_v d\eta \right|, \]
let
\[ g(\eta,t) = e^{\lambda(\eta)t} \langle e_D(-\eta), \hat{f}_0 \rangle_v. \]
When \( \gamma \geq 3/2 \), the eigenvalue \( \lambda(\eta) \) and eigenvector \( e_D(\eta) \) are analytic in \( \eta \), by the asymptotic expansion of \( \lambda(\eta) \) in (3), we have
\[ |g(\eta,t)| \leq e^{-a_3|\eta|^2 t + O(|\eta|^4)t} \| f_0 \|_{L^1_x L^2_x}, \]
and hence one can apply Lemma 14 to conclude that
\[ |f_{L,0}(x,t)|_{L^2_x} \leq \left[ (1 + t)^{-3/2} e^{-\frac{|x|^2}{2} C t} + e^{-ct} \right] \| f_0 \|_{L^1_x L^2_x}. \]
As for \( 1 \leq \gamma < 3/2 \), the eigenvalue and eigenvector are only smooth in \( \eta \). It is easy to see that
\[ |\hat{D}_\eta^2 g(\eta,t)| \leq (1 + t)|\beta|/2 e^{-a_3|\eta|^2 t/2} \| f_0 \|_{L^1_x L^2_x}. \]
We apply Lemma 15 and let \( n = 3, |\alpha| = k = 0 \) to conclude that
\[ |f_{L,0}(x,t)|_{L^2_x} \leq \left[ (1 + t)^{-3/2} \left( 1 + \frac{|x|^2}{1 + t} \right)^{-N} + e^{t/C} \right] \| f_0 \|_{L^1_x L^2_x}. \]
\( \square \)

We define the fluid part as \( f_F = f_{L,0} \) and the nonfluid part as \( f_\ast = f - f_F \). By fluid-nonfluid decomposition and wave-remainder decomposition, we have
\[ f = f_F + f_\ast = W^{(3)} + R^{(3)}, \]
one can define the kinetic part as \( f_K = W^{(3)} \) and the tail part as \( f_R = R^{(3)} - f_F \).
From Lemma 13 and Lemma 9, the estimate of the fluid part \( f_F \) and the kinetic part \( f_K \) are completed. We only need to study the tail part \( f_R \). Note that for \( t > 1 \), \( R^{(3)} \) has the following estimate (by Lemma 10)
\[ \| \nabla_x^2 R^{(3)} \|_{L^2_x} \leq \int_0^t \left\| \mathbb{G}^{t,s} K \mathbb{N}_2^2 h^{(3)}(s) \right\|_{L^2_x} ds \leq Ct \| f_0 \|_{L^2_x}. \]  
From (41), (42), (46), it is easy to check that
\[ \| f_R \|_{L^2_x} = \| f_\ast - W^{(3)} \|_{L^2_x} \leq e^{-Ct} \| f_0 \|_{L^2_x}, \]
\[ \| \nabla_x^2 f_R \|_{L^2_x} = \| \nabla_x^2 (R^{(3)} - f_F) \|_{L^2_x} \leq Ct \| f_0 \|_{L^2_x}, \]
this shows that
\[ |f_R|_{L^2_x} \leq e^{-Ct} \| f_0 \|_{L^2_x}. \]
In conclusion, we have that for time like region, if \( \gamma \geq 3/2 \),
\[ |f|_{L^2_x} \leq \left[ (1 + t)^{-3/2} e^{-\frac{|x|^2}{2(1+t)}} + e^{-ct} + e^{-Ct e^{-\frac{|x|^2}{c^2}}} \right] \| f_0 \|_{L^\infty_x L^2_x}, \]
and if \( 1 \leq \gamma < 3/2 \), any given \( N > 0 \),
\[ |f|_{L^2_x} \leq \left[ (1 + t)^{-3/2} \left( 1 + \frac{|x|^2}{1 + t} \right)^{-N} + e^{-c(|x|+t)} + e^{-Ct e^{-\frac{|x|^2}{c^2}}} \right] \| f_0 \|_{L^\infty_x L^2_x}. \]
3.2. The case $0 < \gamma < 1$: Large time behavior. In virtue of the loss of the spectral analysis for $0 < \gamma < 1$, we will instead use the method of the Fourier transform to deal with the time decay of the solution $f$ to equation (2) in this case. The main idea is to construct the desired weighted time-frequency Lyapunov functional to capture the total energy dissipation rate. In the proof we have to take great care to estimate the microscopic and macrosopic parts for $|\eta| \leq 1$ and $|\eta| > 1$ respectively. Consider (2), taking the Fourier transform in $x$ yields

\begin{equation}
\partial_t \hat{f} + i v \cdot \eta \hat{f} = L \hat{f}.
\end{equation}

Let us calculate the $L^2$ estimate first.

**Proposition 16** ($L^2$ estimate). Let $f$ be the solution to equation (2). Then there exists time-frequency functional $\mathcal{E}(t, \eta)$ such that

\begin{equation}
\mathcal{E}(t, \eta) \approx \left| \hat{f}(t, \eta) \right|^2_{L^2},
\end{equation}

where for any $t > 0$ and $\eta \in \mathbb{R}^3$, we have

\begin{equation}
\partial_t \mathcal{E}(t, \eta) + \sigma \tilde{\rho}(\eta) \left| (v)^{\ell} \hat{f}(t, \eta) \right|^2_{L^2} \leq 0.
\end{equation}

Here the notation $\tilde{\rho}(\eta) = \min\{1, |\eta|^2\}$.

**Proof.** We multiply equation (47) with $\hat{f}(t, \eta, v)$ and integrate over $v$ to obtain

\[
\frac{1}{2} \frac{d}{dt} \left| \hat{f}(t, \eta) \right|^2_{L^2} - \text{Re} \left( L \hat{f}, \hat{f} \right) = 0.
\]

From the coercivity in Lemma 3, it follows that

\begin{equation}
\frac{1}{2} \frac{d}{dt} \left| \hat{f}(t, \eta) \right|^2_{L^2} + \nu_0 \left| P_1 \hat{f} \right|^2_{L^2} \leq 0.
\end{equation}

Now, we need the estimate of $P_0 \hat{f}$. In the sequel, we will apply Strain’s argument to estimate the macroscopic dissipation, in the spirit of Kawashima’s work on dissipation of the hyperbolic-parabolic system. Let $a = \langle \mathcal{M}^{1/2}, f \rangle_v$ and $b = (b_1, b_2, b_3)$ with $b_i = \langle v, \mathcal{M}^{1/2}, f \rangle_v = \langle v_i, \mathcal{M}^{1/2}, P_1 f \rangle_v$. Then $P_0 f = a \mathcal{M}^{1/2}$ and from (2), $a$ and $b$ satisfy the fluid-type system

\begin{equation}
\begin{cases}
\partial_t a + \nabla_x \cdot b = 0 \\
\partial_t b + \alpha \nabla_x a + \nabla_x \cdot \Gamma (P_1 f) = -\int (\mathcal{M}^{1/2} \nabla_v \Phi) P_1 f dv,
\end{cases}
\end{equation}

where

\[
\alpha = \frac{1}{3} \int |v|^2 \mathcal{M} dv > 0,
\]

and $\Gamma = (\Gamma_{ij})_{3 \times 3}$ is the moment function defined by

\[
\Gamma_{ij}(g) = \left( (v_i v_j - 1) \mathcal{M}^{1/2}, g \right)_v, \quad 1 \leq i, j \leq 3.
\]

Note by the definition of $P_0$ that $\Gamma (P_1 f) = \int (v \otimes v) \mathcal{M}^{1/2} P_1 f dv$. Taking the Fourier transform in $x$ of (51), we have

\[
|\eta|^2 |\hat{a}|^2 = \langle i \eta \hat{a}, i \eta \hat{a} \rangle = \frac{1}{\alpha} \left( \langle i \eta \hat{a}, -\partial_t \hat{b} - i \Gamma (P_1 f) \eta - \int (\mathcal{M}^{1/2} \nabla_v \Phi) P_1 \hat{f} dv \rangle \right)
\]

\[
= \frac{1}{\alpha} \left[ -\langle i \eta \hat{a}, \hat{b} \rangle_t + \langle \eta \cdot \hat{b} \rangle^2 - \langle i \eta \hat{a}, i \Gamma (P_1 f) \eta \rangle - \langle i \eta \hat{a}, \int (\mathcal{M}^{1/2} \nabla_v \Phi) P_1 \hat{f} dv \rangle \right].
\]

Invoking on the rapid decay of $\mathcal{M}^{1/2}$ and using the Cauchy-Schwartz inequality, we have

\[
\left| \int (\mathcal{M}^{1/2} \nabla_v \Phi) P_1 \hat{f} dv \right|^2 \lesssim \mathcal{M}^{1/2} v \langle v \rangle^{-1} \left| \int (v) \gamma^{-1} P_1 \hat{f} \right|^2_{L^2} \lesssim 3 \alpha \left| P_1 \hat{f} \right|_{L^2}^2,
\]
and
\[ \left| \left( i\eta \hat{\mu}, i\Gamma (P_1 \hat{f}) \eta \right) \right| \leq \epsilon |\eta|^2 |\hat{\alpha}|^2 + C_\epsilon |\eta|^2 \left| P_1 \hat{f} \right|_{L^2_{\gamma-1}}, \]
for any small \( \epsilon > 0 \). Therefore, we can conclude

\[ \partial_t \text{Re} \left( \frac{i\eta \hat{\mu}, \hat{b}}{1 + |\eta|^2} \right) + \frac{\sigma |\eta|^2}{1 + |\eta|^2} |\hat{\alpha}|^2 \leq C \left| P_1 \hat{f} \right|_{L^2_{\gamma-1}}^2, \]
for some \( \sigma > 0 \). Now, we define
\[ \mathcal{E} (t, \eta) = \left| \hat{f} (t, \eta) \right|_{L^2_{\gamma}} + \kappa_3 \text{Re} \left( \frac{i\eta \hat{\mu}, \hat{b}}{1 + |\eta|^2} \right), \]
for a constant \( \kappa_3 > 0 \) to be determined later. One can fix \( \kappa_3 > 0 \) small enough such that
\( \mathcal{E} (t, \eta) \approx \left| \hat{f} (t, \eta) \right|_{L^2_{\gamma}} \). Furthermore, according Lemma 3 and (52), we choose \( \kappa_3 > 0 \) sufficiently small such that

\[ \partial_t \mathcal{E} (t, \eta) + \sigma \left| P_1 \hat{f} \right|_{L^2_{\gamma-1}}^2 + \frac{2\sigma |\eta|^2}{1 + |\eta|^2} |\hat{\alpha}|^2 \leq 0, \]
for some \( \sigma > 0 \). In conclusion, we now have

\[ \partial_t \mathcal{E} (t, \eta) + \sigma \hat{\rho} (\eta) \left| \hat{f} (t, \eta) \right|_{L^2_{\gamma-1}}^2 \leq 0. \]
Here the notation \( \hat{\rho} (\eta) = \min \{ 1, |\eta|^2 \} \).

Since \( \gamma - 1 < 0 \), it is insufficient to gain the time decay of the total energy of the solution \( f \). Therefore, in order to capture the total energy dissipation rate, we need to make further energy estimates on the microscopic part \( P_1 f \) and the macroscopic part \( P_0 f \).

**Proposition 17** (Weighted \( L^2 \) estimate). Fix \( \ell \geq 0 \). Let \( f \) be the solution to equation (2). Then there exists a weighted time-frequency functional \( \mathcal{E}_\ell (t, \eta) \) such that

\[ \mathcal{E}_\ell (t, \eta) \approx \left| \langle v \rangle^\ell \hat{f} (t, \eta) \right|_{L^2_{\gamma}}^2, \]
where for any \( t > 0 \) and \( \eta \in \mathbb{R}^3 \) we have

\[ \partial_t \mathcal{E}_\ell (t, \eta) + \sigma \hat{\rho} (\eta) \left| \langle v \rangle^\ell \hat{f} (t, \eta) \right|_{L^2_{\gamma-1}}^2 \leq 0. \]

**Proof.** Firstly, we shall prove the following Lyapunov inequality with a velocity weight \( \ell \in \mathbb{R} \):

\[ \frac{d}{dt} \left| \langle v \rangle^\ell P_1 \hat{f} (t, \eta) \right|_{L^2_{\gamma}}^2 + \sigma \left| \langle v \rangle^\ell P_1 \hat{f} (t, \eta) \right|_{L^2_{\gamma-1}}^2 \leq C_\sigma |\eta|^2 \left| \hat{f} \right|_{L^2_{\gamma-1}}^2 + C_\ell \left| P_1 \hat{f} \right|_{L^2_{\gamma}(B_{2\kappa})}^2, \]
where the constants \( C_\ell > 0 \) and \( R > 0 \) are depending only upon \( \ell \). We split the solution \( f \) into \( f = P_0 f + P_1 f \), and then apply \( P_1 \) to equation (14):

\[ \partial_t P_1 \hat{f} + iv \cdot \eta P_1 \hat{f} - LP_1 \hat{f} = -P_1 \left( iv \cdot \eta P_0 \hat{f} \right) + P_0 \left( iv \cdot \eta P_1 \hat{f} \right). \]
Multiply the above equation by \( \langle v \rangle^{2\ell} P_1 \hat{f} (\ell \neq 0) \) and integrate in \( v \) to obtain

\[ \frac{1}{2} \frac{d}{dt} \left| \langle v \rangle^\ell P_1 \hat{f} (t, \eta) \right|_{L^2_{\gamma}}^2 - \text{Re} \left\langle \langle v \rangle^{2\ell} LP_1 \hat{f}, P_1 \hat{f} \right\rangle_v = \Gamma, \]
where
\[ \Gamma = -\text{Re} \left\langle P_1 \left( iv \cdot \eta P_0 \hat{f} \right), \langle v \rangle^{2\ell} P_1 \hat{f} \right\rangle + \text{Re} \left\langle P_0 \left( iv \cdot \eta P_1 \hat{f} \right), \langle v \rangle^{2\ell} P_1 \hat{f} \right\rangle. \]
Owing to the rapid decay of $\mathcal{M}^{1/2}$ we obtain
\[
|\Gamma| \leq \epsilon \left( \langle v \rangle^j P_1 \hat{f}(t, \eta) \right)_{L^2_{-1}}^2 + C_0 |\eta|^2 \left( \langle \langle v \rangle^{2-j} P_1 \hat{f} \rangle_{L^2_2}^2 + |P_0 \hat{f}|_{L^2_2}^2 \right),
\]
which holds for any small $\epsilon > 0$ and any large $j > 0$. Given $c_0 \in (0, \frac{1}{4})$, rewrite $L = -\Lambda + K$, $K = \varpi \chi_R (|v|)$, where $R > 0$ and $\varpi > 0$ are chosen sufficiently large such that
\[
\frac{|v|^2}{4} (\gamma - 2) - \frac{3}{2} (\gamma - 2) |v|^2 \langle v \rangle^2 \geq \varpi \chi_R (|v|) \geq c_0 \langle v \rangle^{2\gamma-2}
\]
and
\[
\frac{2 |\ell|}{\left[ 1 + (2R)^2 \right]^{\gamma/2}} < \min \{1, c_0\}.
\]
Hence, we have
\[
- \text{Re} \left( \langle v \rangle^{2\ell} L P_1 \hat{f}, P_1 \hat{f} \right)_v = \text{Re} \int \langle v \rangle^{2\ell} \left[ (A - K) P_1 \hat{f} \right] P_1 \hat{f} dv
\]
\[
\geq \int \langle v \rangle^{2\ell} \left| \nabla_v P_1 \hat{f} \right|^2 + \text{Re} \int 2 \ell \langle v \rangle^{2\ell-2} \left( v \cdot \nabla_v P_1 \hat{f} \right) P_1 \hat{f} dv + c_0 \int \langle v \rangle^{2\ell+2\gamma-2} \left| P_1 \hat{f} \right|^2 dv - C' \left| P_1 \hat{f} \right|_{L^2_2(B_{2R})}^2.
\]
Since
\[
\left| \text{Re} \int 2 \ell \langle v \rangle^{2\ell-2} \left( v \cdot \nabla_v P_1 \hat{f} \right) P_1 \hat{f} dv \right|
\]
\[
\leq \int 2 |\ell| \langle v \rangle^{2\ell-1} \left| \nabla_v P_1 \hat{f} \right| \left| P_1 \hat{f} \right| dv = \int 2 |\ell| \langle v \rangle^{\ell+\gamma-1} \left| \nabla_v P_1 \hat{f} \right| \left| P_1 \hat{f} \right| dv
\]
\[
\leq \frac{1}{2} \int \langle v \rangle^{2\ell} \left| \nabla_v P_1 \hat{f} \right|^2 dv + \frac{c_0}{2} \int \langle v \rangle^{2\ell+2\gamma-2} \left| P_1 \hat{f} \right|^2 dv + C'' (R) |\ell| \int |v| \langle v \rangle^{2\ell} \left| \nabla_v P_1 \hat{f} \right| \left| P_1 \hat{f} \right| dv
\]
\[
\leq \int \langle v \rangle^{2\ell} \left| \nabla_v P_1 \hat{f} \right|^2 dv + \frac{c_0}{2} \int \langle v \rangle^{2\ell+2\gamma-2} \left| P_1 \hat{f} \right|^2 dv + \frac{C'' (R) |\ell|^2}{2} \int |v| \langle v \rangle^{2\ell} \left| \nabla_v P_1 \hat{f} \right| \left| P_1 \hat{f} \right| dv,
\]
we deduce that
\[
- \text{Re} \left( \langle v \rangle^{2\ell} L P_1 \hat{f}, P_1 \hat{f} \right)_v \geq \frac{c_0}{2} \int \langle v \rangle^{2\ell+2\gamma-2} \left| P_1 \hat{f} \right|^2 dv - C_\ell \left| P_1 \hat{f} \right|_{L^2_2(B_{2R})}^2,
\]
where $C_\ell > 0$. Consequently,
\[
\left. \frac{d}{dt} \left| \langle v \rangle^{\ell} P_1 \hat{f}(t, \eta) \right|_{L^2_2}^2 + \sigma \left| \langle v \rangle^{\ell} P_1 \hat{f}(t, \eta) \right|_{L^2_{-1}}^2 \leq C_\eta |\eta|^2 \left| \hat{f} \right|_{L^2_{-1}}^2 + C_\ell \left| \hat{f} \right|_{L^2_2(B_{2R})}^2,
\]
for some constant $\sigma > 0$. In particular, when $\ell = 0$, following the procedure as above and using Lemma 3 yields
\[
\left(\begin{array}{c}
\frac{d}{dt} \left| P_1 \hat{f}(t, \eta) \right|_{L^2_2}^2 + \sigma \left| P_1 \hat{f}(t, \eta) \right|_{L^2_{-1}}^2 \\
\end{array}\right) \leq \left| \eta \right|^2 \left| P_0 \hat{f} \right|_{L^2_2}^2.
\]
In addition, if we multiply (47) with $\langle v \rangle^{2\ell} \hat{f}(t, \eta, v)$, integrate in $v$ and use the same procedure as above, we also obtain
\[
\left(\begin{array}{c}
\frac{1}{2} \frac{d}{dt} \left| \langle v \rangle^{\ell} \hat{f}(t, \eta) \right|_{L^2_2}^2 + \sigma \left| \langle v \rangle^{\ell} \hat{f}(t, \eta) \right|_{L^2_{-1}}^2 \\
\end{array}\right) \leq C_\ell \left| \hat{f} \right|_{L^2_2(B_{2R})}^2.
\]
To do the weighted estimate, we introduce a new energy splitting as follows:
\[
\mathcal{E}_\ell (t, \eta) := \mathcal{E}_\ell^0 (t, \eta) + \mathcal{E}_\ell^1 (t, \eta),
\]
where
\[
\mathcal{E}_\ell^0 (t, \eta) = 1_{\left| \eta \right| \leq 1} \left( \mathcal{E}_\ell (t, \eta) + \kappa_4 \left| \langle v \rangle^{\ell} P_1 \hat{f}(t, \eta) \right|_{L^2_2}^2 \right),
\]
\[ \mathcal{E}_t^1 (t, \eta) = 1_{|\eta| > 1} \left( \mathcal{E}_t (t, \eta) + \kappa_5 \left| \langle v \rangle^\ell \hat{f} (t, \eta) \right|^2_{L^2_t} \right), \]

the constants \( \kappa_4, \kappa_5 > 0 \) will be chosen small enough. Notice further that \( |\tilde{a}|^2 \gtrsim \left| P_0 \hat{f} \right|^2_{L^2_t} \) for all \( s \in \mathbb{R} \), and so \( \mathcal{E}_t (t, \eta) \approx \left| \langle v \rangle^\ell \hat{f} (t, \eta) \right|^2_{L^2_t} \) for \( \ell \geq 0 \).

For \( \mathcal{E}_t^0 (t, \eta) \), we combine (54) and (57) for \( |\eta| > 1 \) to obtain

\[ \partial_t \mathcal{E}_t^0 (t, \eta) + \sigma \left| \langle v \rangle^\ell \hat{f} (t, \eta) \right|^2_{L^2_{t-1}} 1_{|\eta| > 1} \leq 0, \]

for \( \kappa_4 > 0 \) small enough, since \( |\eta|^2 / \left( 1 + |\eta|^2 \right) \geq \frac{1}{2} \).

For \( \mathcal{E}_t^1 (t, \eta) \), since \( |\eta|^2 / \left( 1 + |\eta|^2 \right) \geq \frac{|\eta|^2}{2} \), for \( |\eta| \leq 1 \) and \( |\tilde{a}|^2 \gtrsim \left| P_0 \hat{f} \right|^2_{L^2_t} \) for all \( \ell \geq 0 \), combining (54) and (59) for \( |\eta| \leq 1 \) to obtain

\[ \partial_t \mathcal{E}_t^1 (t, \eta) + \sigma |\eta|^2 \left| \langle v \rangle^\ell \hat{f} (t, \eta) \right|^2_{L^2_{t-1}} 1_{|\eta| \leq 1} \leq 0, \]

for \( \kappa_5 > 0 \) small enough.

Now, it is enough to prove the estimate in time-like region. We apply the Hölder inequality to obtain that for \( j > 0 \),

\[ \mathcal{E} (t, \eta) \lesssim \left| \hat{f} (t, \eta) \right|^2_{L^2_t} = \int \langle v \rangle^{-2(1-\gamma)j} \left| \hat{f} (t, \eta) \right|^2_{L^2_t} \, dv \leq \left( \int \langle v \rangle^{-2(1-\gamma)j} \left| \hat{f} (t, \eta) \right|^2 \, dv \right)^{j/(j+1)} \left( \int \langle v \rangle^{2(1-\gamma)j} \left| \hat{f} (t, \eta) \right|^2 \, dv \right)^{1/(j+1)} \lesssim \left| \hat{f} (t, \eta) \right|^2_{L^2_{t-1}} \mathcal{E}^{1/j} (1-\gamma)j (t, \eta). \]

We thus conclude that

\[ \mathcal{E}^{(j+1)/j} (t, \eta) \lesssim \left| \hat{f} (t, \eta) \right|^2_{L^2_{t-1}} \mathcal{E}^{1/j} (1-\gamma)j (t, \eta) \lesssim \left| \hat{f} (t, \eta) \right|^2_{L^2_{t-1}} \mathcal{E}^{1/j} (1-\gamma)j (0, \eta). \]

Now we can rewrite (54), for any \( \eta \in \mathbb{R}^3 \), as

\[ \partial_t \mathcal{E} (t, \eta) + \sigma \tilde{\rho} (\eta) \mathcal{E}^{(j+1)/j} (t, \eta) \mathcal{E}^{-1/j} (1-\gamma)j (0, \eta) \leq 0. \]

Integrating this over time, we obtain

\[ j \mathcal{E}^{-1/j} (0, \eta) - j \mathcal{E}^{-1/j} (t, \eta) \lesssim -t \tilde{\rho} (\eta) \mathcal{E}^{-1/j} (1-\gamma)j (0, \eta). \]

As a consequence, for any \( j > 0 \), uniformly in \( \eta \in \mathbb{R}^3 \), we get

\[ \mathcal{E} (t, \eta) \lesssim \mathcal{E} (1-\gamma)j (0, \eta) \left( \frac{t \tilde{\rho} (\eta)}{j} + 1 \right)^{-j}, \]

since \( \mathcal{E} (0, \eta) \lesssim \mathcal{E} (1-\gamma)j (0, \eta) \).

Now for \( t \geq 1 \), we integrate over \( \eta \) and split into \( |\eta| \leq 1 \) and \( |\eta| > 1 \) to achieve

\[ \int_{|\eta| > 1} \mathcal{E} (t, \eta) \, d\eta \lesssim \left( \frac{t}{j} + 1 \right)^{-j} \int_{|\eta| > 1} \mathcal{E} (1-\gamma)j (0, \eta) \, d\eta \lesssim t^{-j} \| \tilde{f}_0 \|^2_{L^2_{t-1}}. \]

Alternatively, when \( |\eta| \leq 1 \) and \( k \in \mathbb{N} \), since

\[ \int_{|\eta| \leq 1} |\eta|^{2k} \left( \frac{t |\eta|^2}{j} + 1 \right)^{-j} \, d\eta \lesssim t^{-j - k} \]

for all \( t \geq 1 \) if \( j > \frac{3}{2} + k \),
we obtain
\[
\int_{|\eta| \leq 1} |\eta|^{2k} E(t, \eta) \, d\eta \lesssim \int_{|\eta| \leq 1} |\eta|^{2k} \left( \frac{t|\eta|^2}{j} + 1 \right)^{-j} E(1-\gamma) (0, \eta) \, d\eta
\]
Note that our initial data has exponential weight, so \( j \) can be as large as desired. Recall that the long wave part \( f_L \) and the short wave part \( f_S \) of the solution \( f \) are given respectively by
\[
f_L = \int_{|\eta| < 1} e^{i\eta \cdot x + (-1/2 + L) t} f_0(\eta, v) \, d\eta, \tag{64}
\]
\[
f_S = \int_{|\eta| > 1} e^{i\eta \cdot x - (1/2 + L) t} f_0(\eta, v) \, d\eta.
\]
Consequently, for short wave part, by (62), we have
\[
\|f_S\|_{L^2} \lesssim \left( 1 + \frac{t}{j} \right)^{-\frac{1}{2}} \|f_0\|_{L^2}^{(1-\gamma)j},
\]
and for long wave part, (63) and Sobolev inequality implies
\[
\|f_L\|_{L^\infty L^2_x} \lesssim \|f_L\|_{H^2_t L^2_x}^{3/4} \|f_L\|_{L^2_x}^{1/4} \lesssim t^{3/2} \|f_0\|_{L^1_x L^2_x}^{(1-\gamma)j}.
\]
Finally, note that \( f_S = f - f_L \), by regularization estimate (Lemma [10]), we have
\[
\|\nabla_x^2 f_S\|_{L^2} \lesssim t^4 \|f_0\|_{L^2}.
\]
hence
\[
\|f_S\|_{L^\infty L^2_x} \lesssim \|f_S\|_{H^2_t L^2_x}^{3/4} \|f_S\|_{L^2_x}^{1/4} \lesssim t^3 \left( 1 + \frac{t}{j} \right)^{-\frac{1}{2}} \|f_0\|_{L^2}^{(1-\gamma)j}.
\]
This completes the proof in time-like region.

4. Initial layer and behavior for space like region

4.1. The case \( \gamma \geq 3/2 \): Exponential decay.

**Proposition 18** (Weighted energy estimate for \( R^{(3)}, \gamma \geq 3/2 \)). Consider the weight
\[
w(x, t) = e^{\frac{x - Mt}{D^M}}, \quad \mu(x) = e^{\frac{x}{D}},
\]
where \( D \) and \( M \) are large numbers to be chosen later. Then we have
\[
\|w R^{(3)}\|_{L^2} \lesssim \|f_0\|_{L^2(\mu)}.
\]
**Proof.** Let \( u = w R^{(3)} \), then \( u \) solves the equation
\[
\partial_t u + v \cdot \nabla_x u + \frac{1}{2D} \left( M - \frac{x \cdot v}{\langle x \rangle} \right) u = Lu + K w h^{(3)},
\]
The energy estimate gives
\[
\frac{1}{2} \frac{d}{dt} \|u\|_{L^2}^2 + \frac{1}{2D} \int \left( M - \frac{x \cdot v}{\langle x \rangle} \right) u^2 \, dx \, dv
\]
\[
+ \int u(-Lu) \, dx \, dv = \int u K w h^{(3)} \, dx \, dv,
\]
here the good part is
\[
\int \frac{M}{2D} u^2 + u(-Lu) \, dx \, dv.
\]
Using Lemma [4] and noting \( 2(\gamma - 2) \geq 1 \) when \( \gamma \geq 3/2 \), we have
\[
\left| \frac{1}{2D} \int \frac{x \cdot v}{\langle x \rangle} u^2 \, dx \, dv \right| \leq \frac{1}{2D} \int C_2 u^2 + C_1 u(-Lu) \, dx \, dv,
\]
which means
\[
\frac{1}{2} \frac{d}{dt} \|u\|_{L^2}^2 + \left( \frac{M}{2D} - \frac{C_2}{2D} \right) \int u^2 dxdv \\
+ \left( 1 - \frac{C_1}{2D} \right) \int u(-Lu) dxdv \leq \int \left| uKw^{(3)} \right| dxdv.
\]
Choosing \( D \) and \( M \) large enough yields
\[
\frac{d}{dt} \|u\|_{L^2} \lesssim \|wh^{(3)}\|_{L^2},
\]
and hence it follows
\[
\|u\|_{L^2} \lesssim \int_0^t \|h^{(3)}\|_{L^2} ds \lesssim \|f_0\|_{L^2}.
\]
This completes the \( L^2 \) weighted energy estimate. For \( x \)-derivative estimate, we only need to control the commutator term:
\[
\int \frac{|\nabla u|}{D(x)} |u\partial_x u| dxdv \leq \frac{1}{2D} \left( C_2 \|u\|_{H^1 L^2}^2 + C_1 \int u(-Lu) + \partial_x u (-L\partial_x u) dxdv \right),
\]

hence by Lemma 10
\[
\|u\|_{H^1 L^2} \lesssim \int_0^t \|wh^{(3)}\|_{H^1 L^2} ds \lesssim \int_0^t \|h^{(3)}\|_{H^1 L^2} ds \lesssim \|f_0\|_{L^2}.
\]
The \( H^1 L^2 \) estimate is similar and hence we omit the details. This means that for \( \gamma \geq 3/2 \)
\[
|R^{(3)}|_{L^2} \lesssim t e^{-C'(x) + t} \|f_0\|_{L^2}.
\]

In conclusion, if \( f_0 \) has compact support in the \( x \)-variable, then for \( x \geq 2Mt \), we have
\[
\|f\|_{L^2} \lesssim \left( te^{-C'(x) + t} + e^{-C't^{-9/2}e^{-\frac{|x|^2}{4}}} \right) \|f_0\|_{L^\infty L^2}.
\]

4.2. The case \( 0 < \gamma < 3/2 \): Subexponential decay. If \( 0 < \gamma < 3/2 \), we consider the weight
\[
w(t, x, v) = e^{\frac{\rho(t, x, v)}{2}}, \quad \mu(x, v) = e^{\alpha c(x, v)},
\]
where
\[
\rho(t, x, v) = 5 \left( \delta \langle x \rangle - M \right)^{\frac{2\gamma}{3}} \left( 1 - \chi \left( \delta \langle x \rangle - M \right) \langle v \rangle^{\gamma - 3} \right)
+ \left[ 1 - \chi \left( \delta \langle x \rangle - M \right) \langle v \rangle^{\gamma - 3} \right] \delta \langle x \rangle - M \langle v \rangle^{2\gamma - 3} + 3 \langle v \rangle \chi \left( \delta \langle x \rangle - M \right) \langle v \rangle^{\gamma - 3},
\]
and
\[
c(x, v) = 5 \left( \delta \langle x \rangle \right)^{\frac{2\gamma}{3}} \left( 1 - \chi \left( \delta \langle x \rangle \right)^{\gamma - 3} \right).
\]
Here \( M \) is a large positive constant, \( \delta, \alpha \) are small positive constants, all of them need to be chosen later. For simplicity, we define
\[
H_+ = \{ (x, v) : |\delta \langle x \rangle - M \rangle | \geq 2 \langle v \rangle^{3-\gamma} \},
H_0 = \{ (x, v) : \langle v \rangle^{3-\gamma} < |\delta \langle x \rangle - M \rangle | < 2 \langle v \rangle^{3-\gamma} \},
\]
and
\[
H_- = \{ (x, v) : |\delta \langle x \rangle - M \rangle | \leq \langle v \rangle^{3-\gamma} \}.
\]

Proposition 19 (Weighted energy estimate for \( R^{(3)} \), \( 0 < \gamma < 3/2 \)). Consider the weight
\[
w(t, x, v) = e^{\frac{\rho(t, x, v)}{2}}, \quad \mu(x, v) = e^{\alpha c(x, v)},
\]
then we have that for \( 1 \leq \gamma < 3/2 \)
\[
\|wR^{(3)}\|_{H^2 L^2} \lesssim t(1 + t) \|f_0\|_{L^2}.
\]
and for $0 < \gamma < 1$, 
$$||w{\mathcal R}^{(3)}||_{H^2_2 L^2_x} \lesssim t(1 + t^4)||f_0||_{L^2((v)^{2N})}.$$ 

Proof. Let $u = w{\mathcal R}^{(3)} = e^{\frac{t^2}{4}}{\mathcal R}^{(3)}$. Note that $u$ solves the equation 
$$\partial_t u + v \cdot \nabla_x u - \frac{\alpha}{2}(\partial_t \rho + v \cdot \nabla_x \rho)u - e^{\frac{t^2}{4}}L(e^{-\frac{t^2}{4}}u) = Kwh^{(3)},$$ 

the energy estimate gives 
$$\frac{d}{dt} \int_{R^3} \langle u, u \rangle_v dx - \int_{R^3} \langle u, Kwh^{(3)} \rangle_v dx$$ 
$$= \int_{R^3} \frac{\alpha}{2} \langle u, (\partial_t \rho + v \cdot \nabla_x \rho)u \rangle_v dx + \int_{R^3} \langle u, e^{\frac{t^2}{4}}L(e^{-\frac{t^2}{4}}u) \rangle_v dx.$$ 

It is easy to see that 
$$\langle u, e^{\frac{t^2}{4}}L(e^{-\frac{t^2}{4}}u) \rangle_v = \langle u, Lu \rangle_v + \frac{\alpha^2}{4} \langle u^2, |\nabla_v \rho|^2 \rangle_v,$$ 

direct calculation gives 
$$\nabla_v \rho = \left[(\gamma - 3)(1 - 2\chi)\delta(\langle x \rangle - Mt) \langle v \rangle^{2\gamma - 3} + 3(\gamma - 3) \langle v \rangle^\gamma - 5(\gamma - 3) (\delta(\langle x \rangle - Mt))^{\frac{2\gamma - 3}{\gamma}} \right]$$ 
$$\times \left[\delta(\langle x \rangle - Mt) \langle v \rangle^{\gamma - 4} \right] \frac{v}{\langle v \rangle} \chi'$$ 
$$+ \left[(2\gamma - 3)\delta(\langle x \rangle - Mt) \langle v \rangle^{2\gamma - 4} \right] \frac{v}{\langle v \rangle} (1 - \chi) + 3\gamma \langle v \rangle^{\gamma - 1} \frac{v}{\langle v \rangle} \chi.$$ 

This implies 
$$||\nabla_v \rho|| \lesssim \langle v \rangle^{\gamma - 1} \text{ on } H_0 \cup H_-, $$ 

direct calculation gives 
$$\alpha^2 \left| \int_{R^3} \langle u^2 \nabla_v \rho, \nabla_v \rho \rangle_v dx \right| \lesssim \alpha^2 \int_{R^3} |P_1 u|^2_{L^2_x} dx + \alpha^2 \int_{H_0 \cup H_-} \langle v \rangle^{2\gamma - 2} |P_0 u|^2 dv dx.$$ 

One can easily check that 
$$\partial_t \rho = -\delta M \langle v \rangle^{2\gamma - 3} \left(\frac{5\gamma}{3 - \gamma} \left[\delta(\langle x \rangle - Mt) \langle v \rangle^{\gamma - 3} \right]^{\frac{2\gamma - 3}{\gamma}} - (1 - \chi) + \chi(1 - \chi) \right)$$ 
$$+ \delta M \left[5 \left[\delta(\langle x \rangle - Mt) \langle v \rangle^{\gamma - 3} \right]^{\frac{2\gamma - 3}{\gamma}} - (1 - 2\chi) \left[\delta(\langle x \rangle - Mt) \langle v \rangle^{\gamma - 3} \right] - 3 \right] \langle v \rangle^{2\gamma - 3} \chi' \leq 0,$$ 

the constants 5 and 3 are chosen artificially such that the quantity in the latter bracket is non-negative on $H_0$ and, 
$$\nabla_x \rho = \delta (\nabla_x \langle x \rangle) \langle v \rangle^{2\gamma - 3} \left(\frac{5\gamma}{3 - \gamma} \left[\delta(\langle x \rangle - Mt) \langle v \rangle^{\gamma - 3} \right]^{\frac{2\gamma - 3}{\gamma}} - (1 - \chi) + \chi(1 - \chi) \right)$$ 
$$- \delta (\nabla_x \langle x \rangle) \left(5 \left[\delta(\langle x \rangle - Mt) \langle v \rangle^{\gamma - 3} \right]^{\frac{2\gamma - 3}{\gamma}} - (1 - 2\chi) \left[\delta(\langle x \rangle - Mt) \langle v \rangle^{\gamma - 3} \right] + 3 \right) \langle v \rangle^{2\gamma - 3} \chi',$$ 

hence 
$$\partial_t \rho = v \cdot \nabla_x \rho = 0 \text{ on } H_-, $$ 

for $H_0$, we get 
$$|\partial_t \rho| \lesssim \delta M \langle v \rangle^{2\gamma - 3} \text{ and } |v \cdot \nabla_x \rho| \lesssim \delta \langle v \rangle^{2\gamma - 2}.$$ 

Finally, on $H_+$, we have 
$$\partial_t \rho = -\frac{5\delta M \gamma}{3 - \gamma} [\delta(\langle x \rangle - Mt)]^{2\gamma - 3},$$ 
$$v \cdot \nabla_x \rho = \frac{5\delta \gamma \nu x}{3 - \gamma} [\delta(\langle x \rangle - Mt)]^{2\gamma - 1}. $$
Direct calculation shows

\[
\alpha \int_{\mathbb{R}^3} \langle u, \nabla_x \rho \mu \rangle_v dx \lesssim \alpha \delta \int_{\mathbb{R}^3} |\langle v \rangle^{\gamma-1} P_1 u|_L^2 dx
\]

\[
+ \alpha \delta \int_{H_+^3} [\delta(\langle x \rangle - M t)]^{2/3 - 2} |P_0 u|^2 dv dx + \alpha \delta \int_{H_0} |P_0 u|^2 dv dx
\]

and

\[
\alpha \int_{\mathbb{R}^3} \langle u, \partial_t \rho \mu \rangle_v dx \leq \alpha \delta M \int_{\mathbb{R}^3} |\langle v \rangle^{\gamma-2} P_1 u|_L^2 dx
\]

\[
- \alpha \delta M \int_{H_+^3} [\delta(\langle x \rangle - M t)]^{2/3} |P_0 u|^2 dv dx + \alpha \delta M \int_{H_0} |P_0 u|^2 dv dx.
\]

In conclusion, we get (here we ignore some fixed positive constants)

\[
\frac{d}{dt} \int_{\mathbb{R}^3} \langle u, u \rangle_v dx - \int_{\mathbb{R}^3} \langle u, K \rho \mu (3) \rangle_v dx
\]

\[
\leq -(C - \alpha^2 - \alpha \delta - \alpha \delta M) \int_{\mathbb{R}^3} |P_1 u|_L^2 dx
\]

\[
- \alpha (\delta M - \delta) \int_{H_+^3} [\delta(\langle x \rangle - M t)]^{2/3} |P_0 u|^2 dv dx
\]

\[
+ \alpha (\alpha + \delta + \delta M) \int_{H_0} |P_0 u|^2 dv dx
\]

\[
+ \alpha^2 \int_{H_+^3} |P_0 u|^2 dv dx.
\]

If we choose \(\delta, \alpha\) small and \(M\) large enough, we have

\[
\frac{d}{dt} ||u||^2_{L^2} \lesssim ||u||_{L^2} ||K \rho \mu (3)||_{L^2} + ||u||_{L^2} ||\mathcal{R}(3)||_{L^2} \lesssim ||u||_{L^2} \left( ||K(3)||_{L^2 \langle K^2 \delta \rangle} + ||\mathcal{R}(3)||_{L^2} \right),
\]

due to the fact that \(\partial_t \rho \leq 0\). For the \(x\)-derivative estimate, we only need to control the commutator terms:

\[
\alpha \int_{\mathbb{R}^3} \langle \partial_{x_i} u, \partial_{x_i} (\partial_t \rho + v \cdot \nabla_x \rho) u \rangle_v dx,
\]

(65)

\[
\int_{\mathbb{R}^3} \langle \partial_{x_i} u, \partial_{x_i} \left( \left( e^{\frac{\alpha \delta}{2}} \Delta_v e^{-\frac{\alpha \delta}{2}} \right) u \right) \rangle_v dx,
\]

(66)

and

\[
\alpha \int_{\mathbb{R}^3} \langle \partial_{x_i} u, \partial_{x_i} (\nabla_v \rho \cdot \nabla_v u) \rangle_v dx.
\]

(67)

It is obvious that the decay of \(\partial_{x_i} (\partial_t \rho + v \cdot \nabla_x \rho)\) is faster than \(\langle \partial_t \rho + v \cdot \nabla_x \rho \rangle\), hence the first term (65) is easy to control. For the terms (66) and (67), direct calculation gives

\[
\int_{\mathbb{R}^3} \langle \partial_{x_i} u, \partial_{x_i} \left( \left( e^{\frac{\alpha \delta}{2}} \Delta_v e^{-\frac{\alpha \delta}{2}} \right) u \right) \rangle_v dx
\]

\[
= \frac{\alpha^2}{2} \int_{\mathbb{R}^3} \langle \partial_{x_i} \nabla_v \rho \cdot \nabla_v u, \partial_{x_i} u \rangle_v dx + \frac{\alpha^2}{2} \int_{\mathbb{R}^3} |(\nabla_v \rho) \partial_{x_i} u|^2_{L^2} dx
\]

\[
+ \alpha \int_{\mathbb{R}^3} \langle \partial_{x_i} \nabla_v \rho \cdot \nabla_v u, \partial_{x_i} u \rangle_v dx + \alpha \int_{\mathbb{R}^3} \langle \partial_{x_i} \nabla_v \rho \cdot \nabla_v \partial_{x_i} u, u \rangle_v dx
\]

\[
+ \alpha \int_{\mathbb{R}^3} \langle \nabla_v \rho \cdot \nabla_v \partial_{x_i} u, \partial_{x_i} u \rangle_v dx,
\]

and

\[
\int_{\mathbb{R}^3} \langle \partial_{x_i} u, \partial_{x_i} (\nabla_v \rho \cdot \nabla_v u) \rangle_v dx = \int_{\mathbb{R}^3} \langle \partial_{x_i} u, \partial_{x_i} \nabla_v \rho \cdot \nabla_v u \rangle_v dx + \int_{\mathbb{R}^3} \langle \partial_{x_i} u, \nabla_v \rho \cdot \nabla_v \partial_{x_i} u \rangle_v dx
\]
Similar to the first term, the decay of $\partial_x \nabla_v \rho$ is faster than $\nabla_v \rho$, hence it is also easy to control those two terms (66) and (67). The second derivative estimate is similar and hence we omit the details. We then have

$$\frac{d}{dt} \|u\|_{H^2_x L^2_v}^2 \lesssim \|u\|_{H^2_x L^2_v} \|K w h^{(3)}\|_{H^2_x L^2_v} + \|u\|_{H^2_x L^2_v} \|R^{(3)}\|_{H^2_x L^2_v}$$

i.e.,

$$\frac{d}{dt} \|u\|_{H^2_x L^2_v} \lesssim \|h^{(3)}\|_{H^2_x L^2_v(K^2 w^2)} + \|R^{(3)}\|_{H^2_x L^2_v}.$$  

Note that

$K^2 w^2 \leq \omega^2 \exp \left\{ \alpha \left[ 8\sqrt{2} R^\gamma + 5 \left( \delta \langle x \rangle \right)^{\frac{\gamma}{\gamma-3}} \left( 1 - \chi \left( \delta \langle x \rangle \gamma^{-3} \right) \right) \right\} \lesssim \mu,$

this means

$$\frac{d}{dt} \|u\|_{H^2_x L^2_v} \lesssim \|h^{(3)}\|_{H^2_x L^2_v(\mu)} + \|R^{(3)}\|_{H^2_x L^2_v}.$$  

Hence, it follows that (by Lemma 10) for $1 \leq \gamma < 3/2$,  

$$\|u\|_{H^2_x L^2_v} \lesssim t(1 + t^k) \left( \|f_0\|_{L^2(\mu)} + \|f_0\|_{L^2} \right),$$  

and for $0 < \gamma < 1$,  

$$\|u\|_{H^2_x L^2_v} \lesssim t(1 + t^k) \left( \|f_0\|_{L^2(\mu)} + \|f_0\|_{L^2(\mu)} \right),$$  

here $\langle v \rangle^{2(1-\gamma)}$ is the weighted function defined in time like region. Note that

$$\|f_0\|_{L^2(\mu)} \lesssim \|f_0\|_{L^2}.$$  

This completes the proof of the proposition.  

Observe that for $\langle x \rangle > 2Mt$,

$$\rho(t, x, v) > (\delta(\langle x \rangle - Mt))^{\frac{\gamma}{\gamma-3}}.$$  

and

$$\langle x \rangle - Mt > \frac{\langle x \rangle}{3} + \frac{Mt}{3}.$$  

It follows from the Sobolev inequality that there exist positive constants $C$ and $c_\alpha$ such that for $1 \leq \gamma < 3/2$,  

$$|R^{(3)}(t, x)|_{L^2_v} \leq t(1 + t^k)Ce^{-c_\alpha(\langle x \rangle + t)} \|f_0\|_{L^2},$$  

and for $0 < \gamma < 1$,  

$$|R^{(3)}(t, x)|_{L^2_v} \leq t(1 + t^k)Ce^{-c_\alpha(\langle x \rangle + t)} \|f_0\|_{L^2(\mu)} \langle v \rangle^{2(1-\gamma)}.$$  

here $\alpha > 0$ can be chosen as small as we want. In conclusion, if $f_0$ has compact support in the $x$-variable, then for $\langle x \rangle \geq 2Mt$, we have for $1 \leq \gamma < 3/2$,  

$$|f|_{L^2_v} \lesssim \left( t(1 + t^k)e^{-c_\alpha(\langle x \rangle + t)} + e^{-C t}e^{-9/2 \langle u \rangle^2} \right) \|f_0\|_{L^2},$$  

and for $0 < \gamma < 1$,  

$$|f|_{L^2_v} \lesssim \left( t(1 + t^k)e^{-c_\alpha(\langle x \rangle + t)} + t^{-9/2}e^{-\langle u \rangle^2} \right) \|f_0\|_{L^2(\mu)} \langle v \rangle^{2(1-\gamma)}.$$  

This completes the estimate of the initial layer and space-like region.
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