Classifying theory for simplicial parametrized groups

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May 5, 2014

Abstract

In this paper we describe a classifying theory for families of simplicial topological groups. If $B$ is a topological space and $G$ is a simplicial topological group, then we can consider the non-abelian cohomology $H(B, G)$ of $B$ with coefficients in $G$. If $G$ is a topological group, thought of as a constant simplicial group, then the set $H(B, G)$ is the set of isomorphism classes of principal $G$-bundles, or $G$-torsors, on $B$. For more general simplicial groups $G$, the set $H(B, G)$ parametrizes the set of equivalence classes of higher $G$-torsors. In this paper we consider a more general setting where $G$ is replaced by a simplicial group in the category of spaces over $B$. The main result of the paper is that under suitable conditions on $B$ and $G$ there is an isomorphism between $H(B, G)$ and the set of isomorphism classes of fiberwise principal bundles on $B$, with structure group given by the fiberwise geometric realization of $G$.

2010 Mathematics Subject Classification 18G55, 18F20, 55R35.

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*Supported by the Engineering and Physical Sciences Research Council [grant number EP/I010610/1]
1 Introduction

Let $G$ be a presheaf of groups on a topological space $B$. Then the Čech cohomology $\check{H}^1(B, G)$ of $B$ with coefficients in $G$ is traditionally defined in terms of $G$-valued cocycles $g_{ij} \in G(U_i \cap U_j)$ relative to some open cover $(U_i)_{i \in I}$ of $B$. The cocycle condition satisfied by the $g_{ij}$ is the equation

$$g_{ij}g_{jk} = g_{ik}$$

in $G(U_i \cap U_j \cap U_k)$. The set $\check{H}^1(B, G)$ of equivalence classes of such cocycles parametrizes isomorphism classes of $G$-torsors on $B$, i.e. presheaves on $B$ equipped with a principal action of $G$. The typical case is when $G$ is the sheaf of groups on $B$ represented by a topological group $G$, in which case $\check{H}^1(B, G)$ parametrizes the set of principal $G$-bundles on $B$. Slightly more generally, we could consider a parametrized situation where the group $G$ is replaced by a family of topological groups parametrized by the points of $B$. In other words, we consider $G$ as a group object in the category of spaces over $B$. Then $G$ determines a sheaf on $B$, which we also denote by $G$, whose sections are the local sections of the projection to $B$, and the set $\check{H}^1(B, G)$ parametrizes the set of isomorphism classes of fiberwise principal $G$-bundles on $B$ (see [11],[20]).

Now suppose that for any open set $U \subset B$, the group of sections $G(U)$ is replaced by a simplicial group of sections $G(U)$. In other words, suppose that $G$ is a group object in the category of simplicial presheaves on $B$. There is a generalization of the Čech cohomology set above to a cohomology set $\check{H}(B, G)$. Analogously there is a notion of ‘higher $G$-torsor’ on $B$ and under suitable hypotheses $\check{H}(B, G)$ parametrizes the set of isomorphism classes of these higher torsors. While higher torsors are implicitly the subject matter of this paper, we will have no need to consider them explicitly and we instead refer the interested reader to [4],[5],[6],[7],[26],[29],[45] among others for further details.

A cocycle in this generalized Čech cohomology is again defined relative to some open cover $U = (U_i)_{i \in I}$ of $B$, but now the data defining the cocycle becomes harder to describe. To begin with we have sections $g_{ij} \in G_0(U_i \cap U_j)$ of 0-simplices, but now the equation $g_{ij}g_{jk} = g_{ik}$ above will only be satisfied up to homotopy — thus there will be sections $g_{ijk} \in G_1(U_i \cap U_j \cap U_k)$ which satisfy the equations

$$d_0(g_{ijk}) = g_{ik}, \quad d_1(g_{ijk}) = g_{ij}g_{jk}$$

in $G_0(U_i \cap U_j \cap U_k)$. In turn these sections $g_{ijk}$ will satisfy a cocycle equation up to homotopy over quadruple intersections and so on.

It is much more illuminating to give this a more conceptual reformulation and think of the cocycle as a morphism of simplicial presheaves on $B$. Furthermore, working in the category $sPre(B)$ of simplicial presheaves on $B$ allows us to take advantage of the model category structure on $sPre(B)$ constructed in [22] so that we have powerful homotopy theoretic tools at our disposal.

Roughly speaking, this model structure gives rise to a non-abelian version of the derived category construction. If $X$ is a simplicial presheaf on $B$, then for any point $p \in B$ one can define a simplicial set $X_p$, called the stalk of $X$ at $p$. A map of simplicial presheaves $X \rightarrow Y$ is said to be a weak equivalence if it induces a weak homotopy equivalence $X_p \rightarrow Y_p$ on stalks for all points $p$ of $B$. This notion of weak equivalence is a generalization of the notion of quasi-isomorphism of complexes of sheaves: if $X$ and $Y$ are presheaves of simplicial abelian groups then a map $X \rightarrow Y$ is a weak equivalence precisely if it is a quasi-isomorphism of the corresponding complexes of sheaves.

In analogy with the derived category construction we can consider the homotopy category $Ho(sPre(B))$ with the same objects as $sPre(B)$ but in which these weak equivalences are now isomorphisms. The fact that these weak equivalences in $sPre(B)$ form part of the structure of a model category means that there is a simple description of the set of morphisms $[X,Y]$ in $Ho(sPre(B))$. It turns out that when $G$ is a presheaf of
groups on $B$ as above, then there is a bijective correspondence between the Čech cohomology set $\check{H}^1(B, G)$ and $[1, \mathbb{W}G]$, where 1 denotes the terminal simplicial presheaf on $B$ and $\mathbb{W}G$ is the classifying simplicial presheaf of the group $G$.

More generally, if $A$ is a sheaf of abelian groups on $B$, then the sheaf cohomology group $H^n(B, A)$ is isomorphic to $[1, K(A, n)]$, where $K(A, n)$ is a simplicial sheaf on $B$ whose stalk at a point $p \in B$ is the Eilenberg-Mac Lane space $K(A_p, n)$. In general there is a subtle difference between Čech cohomology and sheaf cohomology. This distinction forces us to work with so-called ‘hyper-Čech cohomology’. If $G$ is a presheaf of simplicial groups on $B$, then we will define $H(B, G) = [1, \mathbb{W}G]$, where $\mathbb{W}G$ denotes the classifying simplicial presheaf of the simplicial group $G$. Elements of $H(B, G)$ are represented by $G$-cocycles on $B$ in a sense that we will make precise in Section 4.3.

In this paper we want to study $H(B, G)$ in the following special case. Suppose that $B$ is a paracompact, Hausdorff space and that $G$ is a simplicial group in $\mathcal{K}/B$. Then $G$ determines a presheaf (in fact a sheaf) of simplicial groups on $B$, which we will also denote by $G$. The simplicial set of sections of $G$ over an open set $U \subset B$ has as its set of $n$-simplices the set of sections of the map $G \to B$ over $U$.

By taking the fiberwise geometric realization of the simplicial group $G$ in $\mathcal{K}/B$ we obtain a group $\vert G \vert$ in $\mathcal{K}/B$. We can therefore form the ordinary Čech cohomology $\check{H}^1(B, \vert G \vert)$ as described above. The main result of the paper is the following.

**Theorem 1.1.** Let $B$ be a paracompact, Hausdorff space and let $G$ be a simplicial group in $\mathcal{K}/B$. If $G_n$ is well sectioned for all $n \geq 0$ then there is an isomorphism of sets

$$H(B, G) = \check{H}^1(B, \vert G \vert).$$

In [40] it is proven that the set $\check{H}^1(B, \vert G \vert)$ is isomorphic to the set $[B, B[G]_\mathcal{K}/B]$ of homotopy classes of maps in $\mathcal{K}/B$ from $B$ into the classifying space $B[G]$ of the group object $\vert G \vert$. Therefore the isomorphism of Theorem 1.1 can be restated as an isomorphism of sets

$$H(B, G) = [B, B[G]_\mathcal{K}/B].$$  \hspace{1cm} (1)$$

Our proof of Theorem 1.1 shows more than what is stated: in the course of our proof we construct a universal $G$-cocycle on $B[G]$ with the property that every $G$-cocycle on $B$ is equivalent to one obtained from this universal cocycle via a map in $\mathcal{K}/B$ from $B$ to $B[G]$, unique up to homotopy of maps in $\mathcal{K}/B$.

In another direction, if $G$ is a simplicial abelian group in $\mathcal{K}/B$, then $\mathbb{W}G$ is also a simplicial abelian group in $\mathcal{K}/B$ and it follows that $H(B, G)$ has a natural structure as an abelian group. Likewise, the fiberwise geometric realization $\vert G \vert$ has the structure of an abelian group in $\mathcal{K}/B$ and hence $\check{H}^1(B, \vert G \vert)$ also has a natural structure of an abelian group. Another by-product of our proof of Theorem 1.1 is that, under the above hypotheses, the isomorphism (1) above is an isomorphism of abelian groups in this case (see Section 5.1). In particular, if $A$ is well pointed abelian group in $\mathcal{K}$, then (by the usual mechanism of the Dold-Kan correspondence and using the results of [8]) there is a simplicial abelian group $A[n]$ in $\mathcal{K}$ for any integer $n \geq 0$ with the property that $H(B, A[n])$ is isomorphic to the Čech cohomology group $H^{n+1}(B, A)$ (at least under the assumption that $B$ is paracompact), where $A$ denotes the sheaf of abelian groups on $B$ whose sections over an open set $U \subset B$ are the continuous maps from $U$ into $A$. In this case Theorem 1.1 can be interpreted as an isomorphism between $H^{n+1}(B, A)$ and $[B, B[A[n]]]$. In the special case that $A$ is discrete, we recover the result of Huber [20] giving an identification of $H^{n+1}(B, A)$ with $[B, K(A, n + 1)]$.

We need to comment on the relationship of this paper with previous works of other authors. We are not aware of any analogous studies in the parametrized setting. In the case where the simplicial group object $G$ is trivial, in the sense that it takes the form of a trivial bundle of groups $B \times G \to B$ for a simplicial topological group $G$, then the most general result related to ours that we are aware of is contained in the paper [11] of Graeme Segal. In this paper Segal proves (Proposition 4.3 of [11]) that if $B$ is a paracompact space and $A$ is a good simplicial space then there is a bijection between $[B, \vert A \vert]$ and the set of *concordance classes* of $A$-bundles on $B$. In [1] a related version of Segal’s theorem is proven, in which the set of homotopy classes $[B, \vert A \vert]$ is shown to be isomorphic to the set of concordance classes of $A$ valued cocycles on $B$, where $A$ is again a good simplicial space and $B$ is a space with the homotopy type of a CW complex. We caution
the reader that there is an\textit{a priori} difference between the concordance relation and the relation on simplicial maps given by simplicial homotopy. For example, it is not true in general that the set of isomorphism classes of principal groupoid bundles is in a bijective correspondence with the set of concordance classes of principal groupoid bundles (see for instance \cite{38}). It is of course well known that there is a bijection between the set of isomorphism classes and concordance classes of ordinary principal bundles with structure group a topological group. One outcome of our work is that the simplicial homotopy relation and the concordance relation agree when \( A = \mathbb{W}G \) for a simplicial topological group \( G \) (under the conditions on \( G \) described in Theorem 1.1).

In another direction, again in the case of the trivial bundle of groups \( G \times B \to B \), there is the work \cite{9} on crossed complexes. In this paper the authors show that there is a bijective correspondence between the set of homotopy classes of maps \([M, BC]\) for \( M \) a filtered space and \( C \) a crossed complex, and the set of equivalence classes of \( C \)-valued cocycles on \( M \). The equivalence relation considered in \cite{9} arises from the structure of \( M \) as a filtered space and it is not clear how this relates to the simplicial homotopy relation on simplicial maps.

Our proof of Theorem 1.1 is a substantial generalization of the methods of \cite{4}. We relate the non abelian cohomology set \( H(B, G) \) to the ordinary non abelian cohomology \( H^1(B, \pi) \) by constructing a simplicial group \( |\text{Dec} G| \) in \( \mathcal{X}/B \) together with homomorphisms \( \pi \leftarrow |\text{Dec} G| \to G \) which give us a means to compare \( G \) and \( |\text{Dec} G| \) through the intermediate group \( |\text{Dec} G| \). The simplicial group \( |\text{Dec} G| \) is the fiberwise geometric realization of the bisimplicial group \( \text{Dec} G \) in \( \mathcal{X}/B \) constructed from \( G \) using Illusie’s total décalage functor \cite{21}. The bisimplicial group \( \text{Dec} G \) also plays a prominent role in Porter’s work \cite{39} on \( n \)-types. We then prove that these maps induce isomorphisms \( H(B, |\text{Dec} G|) \cong H^1(B, \pi) \) and \( H(B, |\text{Dec} G|) \cong H(B, G) \).

In summary, the contents of the paper are as follows. In Section 2 we describe some background material on parametrized spaces from \cite{36}, some simplicial techniques based around the décalage comonad, and finally some material on generalized matching objects. In Section 3 we recall some background on the homotopy theory of simplicial sheaves and simplicial presheaves, and also study the notion of a locally fibrant simplicial object in a category equipped with a Grothendieck topology introduced in \cite{18}. We also study the \textquoteleft internal\textquoteright{} local simplicial homotopy theory corresponding to this notion of locally fibrant simplicial object. A key result in this section is Proposition 3.2 where we give a criterion to detect when certain homomorphisms of group objects are locally acyclic local fibrations. In Section 4 we prove some results about the internal local simplicial homotopy theory developed in the previous section, and give a precise definition of the non-abelian cohomology sets \( H(B, G) \). Finally in Section 5 we give our proof of Theorem 1.1.

## 2 Background

### 2.1 Parametrized spaces

Let \( \mathcal{X} \) denote the category of \( k \)-spaces \cite{14} and let \( \mathcal{U} \) denote the subcategory of compactly generated spaces (i.e. weakly Hausdorff \( k \)-spaces). We will be interested in the category \( \mathcal{X}/B \) of spaces over \( B \), where \( B \) is an object of \( \mathcal{U} \). Recall from \cite{36} that \( \mathcal{X}/B \) is a\textit{topological bicomplete category}, in the sense that \( \mathcal{X}/B \) is enriched over \( \mathcal{X} \), the underlying category is complete and cocomplete, and that it is tensored and cotensored over \( \mathcal{X} \). For any space \( K \) and space \( X \) over \( B \) the tensor \( K \odot X \) is defined to be the space \( K \times X \) in \( \mathcal{X} \), considered as a space over \( B \) via the obvious map \( K \times X \to B \). Similarly, the cotensor \( X^K \) is defined to be the space \( \text{Map}_B(K, X) \) given by the pullback diagram

\[
\begin{array}{ccc}
\text{Map}_B(K, X) & \longrightarrow & \text{Map}(K, X) \\
\downarrow & & \downarrow \\
B & \longleftarrow & \text{Map}(K, B)
\end{array}
\]

in \( \mathcal{X} \), where the map \( B \to \text{Map}(K, B) \) is the adjoint of \( B \times K \to B \). Recall also (see \cite{36}) that \( \mathcal{X}/B \) is cartesian closed under the fiberwise cartesian product \( X \times_B Y \) and the fiberwise mapping space \( \text{Map}_B(X, Y) \) over \( B \). The definition of the fiberwise mapping space \( \text{Map}_B(X, Y) \) is rather subtle and we will not give it.
here, we instead refer the reader to Definition 1.37 of [36]. Let us note though that Map_\mathcal{B}(X, Y) is generally not weak Hausdorff even if X and Y are, which is one of the main reasons why May and Sigurdsson choose to work with the category \mathcal{X}/B rather than the category \mathcal{U}/B.

In [36] several model structures on \mathcal{X}/B are introduced. We shall be interested in the f-model structure (for fiberwise) with the weak equivalences, fibrations and cofibrations labelled accordingly. Thus a map \ f: X \rightarrow Y in \mathcal{X}/B is called an f-equivalence if it is a fiberwise homotopy equivalence. This needs the notion of homotopy over B, which is formulated in terms of X \times_B I. A map \ f: X \rightarrow Y in \mathcal{X}/B is called an f-fibration if it has the fiberwise covering homotopy property, i.e. if it has the right lifting property (RLP) with respect to all maps of the form i_0: Z \rightarrow Z \times_B I for all Z \in \mathcal{X}/B. A map \ f: X \rightarrow Y in \mathcal{X}/B is called an f-cofibration if it has the left lifting property (LLP) with respect to all f-acyclic f-fibrations. We have the following result from [36].

**Theorem 2.1** (May-Sigurdsson). \mathcal{X}/B has the structure of a proper, topological model category for which

- the weak equivalences are the f-equivalences,
- the fibrations are the f-fibrations,
- the cofibrations are the f-cofibrations.

Recall that a model category is said to be topological if it is a \mathcal{K}-model category in the sense of Definition 4.2.18 of [19], for the monoidal model structure on \mathcal{K} given by the classical Strom model structure [13] on \mathcal{K}.

In addition to the notion of f-cofibration, there is also the notion of an f-cofibration: this is a map \ f: X \rightarrow Y which satisfies the LLP with respect to all maps of the form p_0: Map_\mathcal{B}(I, Z) \rightarrow Z for some Z \in \mathcal{X}/B. Every f-cofibration g: X \rightarrow Y in \mathcal{X}/B is an f-cofibration. The converse is not true in general. However May and Sigurdsson prove (see Theorems 4.4.4 and 5.2.8 of [36]) that if \ f: X \rightarrow Y is a closed f-cofibration then g is an \bar{f}-cofibration.

Moreover, in analogy with the standard characterization of closed Hurewicz cofibrations in terms of NDR pairs, May and Sigurdsson give a criterion (see Lemma 5.2.4 of [36]) which detects when a closed inclusion i: A \rightarrow X in \mathcal{X}/B is an f-cofibration. Such an inclusion i: A \rightarrow X in \mathcal{X}/B is an f-cofibration if and only if (X, A) is a fiberwise NDR pair in the sense that there is a map u: X \rightarrow I for which A = u^{-1}(0) and a homotopy h: X \times_B I \rightarrow X over B such that h_0 = id, h_t|_A = id_A for all 0 \leq t \leq 1 and h_1(x) \in A whenever u(x) < 1.

The analogue of a pointed space in this parametrized context is the notion of an ex-space. An ex-space over B is a space X in \mathcal{X}/B together with a section of the map X \rightarrow B. The category of ex-spaces and maps between them is denoted by \mathcal{X}_B. An ex-space X is said to be well-sectioned if the section B \rightarrow X is an \bar{f}-cofibration.

### 2.2 Some simplicial techniques

Recall (see for example VII Section 5 of [31]) that the augmented simplex category \Delta_a is a monoidal category under the operation of ordinal sum denoted here by \sigma([m], [n]), with unit given by the empty set \([-1]\]. Moreover [0] is a monoid in \Delta_a, with multiplication given by the unique map [1] \rightarrow [0], which is universal in a certain precise sense (see VII Section 5 Proposition 1 of [31]). It follows that [0] determines a comonoid in \Delta_a^\text{op} and hence a comonad

\[\sigma(-, [0]): \Delta_a^\text{op} \rightarrow \Delta_a^\text{op}\]

on the opposite category of the simplex category \Delta. This comonad induces through composition a comonad on the category s\mathcal{C} = [\Delta_a^\text{op}, \mathcal{C}] of simplicial objects in \mathcal{C}, for any category \mathcal{C}. This comonad is denoted

\[\text{Dec}_0: s\mathcal{C} \rightarrow s\mathcal{C}\]
and if $X$ is an object of $s\mathcal{C}$, then $\text{Dec}_0 X$ is called the *décalage* of $X$. It is not hard to see that $\text{Dec}_0 X$ is an augmented simplicial object whose object of $n$-simplices is

$$(\text{Dec}_0 X)_n = X_{n+1},$$

whose face maps $d_i : (\text{Dec}_0 X)_n \to (\text{Dec}_0 X)_{n-1}$ are given by $d_i : X_{n+1} \to X_n$ for $i = 0, 1, \ldots, n$, and whose degeneracy maps $s_i : (\text{Dec}_0 X)_n \to (\text{Dec}_0 X)_{n+1}$ are given by $s_i : X_{n+1} \to X_{n+2}$ for $i = 0, 1, \ldots, n$. The augmentation $(\text{Dec}_0 X)_0 \to X_0$ is given by $d_0 : X_1 \to X_0$. We can picture $\text{Dec}_0 X$ as follows:

\[
\begin{array}{cccccc}
X_0 & \xrightarrow{d_0} & X_1 & \xrightarrow{d_1} & X_2 & \xrightarrow{d_2} & \cdots \\
 & s_0 & & s_0 & & s_0 & \\
& & & & & & \\
\end{array}
\]

Thus $\text{Dec}_0 X$ is obtained from $X$ by ‘stripping off’ the last face and degeneracy map at each level and re-indexing by shifting degrees up by one. If $\mathcal{C}$ is cocomplete then the left over degeneracy map at each level can be used to define a contraction of $\text{Dec}_0 X$ onto $X_0$. More precisely there is a simplicial homotopy $h : \text{Dec}_0 X \otimes \Delta^1 \to \text{Dec}_0 X$ which fits into a commutative diagram

\[
\begin{array}{ccc}
\text{Dec}_0 X & \xrightarrow{s_0d_0} & \text{Dec}_0 X \otimes \Delta^1 \\
\downarrow & \downarrow & \downarrow h \\
\text{Dec}_0 X & \xrightarrow{id} & \text{Dec}_0 X
\end{array}
\]

Here the tensor $\text{Dec}_0 X \otimes \Delta^1$ is defined with respect to the usual simplicial structure on $s\mathcal{C}$: thus if $A$ is an object of $s\mathcal{C}$ and $K$ is a simplicial set then $A \otimes K$ is the simplicial object of $\mathcal{C}$ whose object of $n$-simplices is

$$(A \otimes K)_n = \coprod_{k \in K_n} A_n,$$

see for example page 85 of [17]. As mentioned above, the functor $\text{Dec}_0$ is a comonad on $s\mathcal{C}$, whose counit is the simplicial map $\text{Dec}_0 X \to X$ given in degree $n$ by the last face map, i.e. the map

$$d_{n+1} : X_{n+1} \to X_n.$$ 

We will write $d_{\text{last}} : \text{Dec}_0 X \to X$ for this map. Observe that since the functor $\text{Dec}_0 : s\mathcal{C} \to s\mathcal{C}$ is obtained by restriction along the functor $\sigma(-, [0]) : \Delta^{op} \to \Delta^{op}$ it has both a left adjoint and a right adjoint — we give a description of the left adjoint in Corollary 2 below.

The comonad $\text{Dec}_0$ gives rise to a simplicial resolution of any object $X \in s\mathcal{C}$, in other words it gives rise to a bisimplicial object in $\mathcal{C}$ which, when thought of as a simplicial object in $s\mathcal{C}$, has as its object of $n$-simplices the simplicial object $\text{Dec}_n X := (\text{Dec}_0 X)^n$. This bisimplicial object is denoted by $\text{Dec} X$. Its object of $(m,n)$-bisimplices is $X_{m+n+1}$. Clearly this construction extends to define a functor

$$\text{Dec} : s\mathcal{C} \to ss\mathcal{C}.$$ (2)

This functor can also be described as restriction along the opposite functor of the ordinal sum map $\sigma : \Delta \times \Delta \to \Delta$. The bisimplicial object $\text{Dec} X$ was introduced by Illusie in [21] and is called the *total décalage* of $X$. For more details we refer to [11, 42].

When $\mathcal{C}$ has sufficiently many limits and colimits, the functor $\sigma^* = \text{Dec} : s\mathcal{C} \to ss\mathcal{C}$ has both a left adjoint $\sigma_!$ and a right adjoint $\sigma_*$. We will later need some elementary facts about the left adjoint of $\text{Dec}$. 

6
when \( C = \text{Set} \). In this case the left adjoint of \( \text{Dec} \) is closely related to the join operation on simplicial sets (see [16] [27] [30]). To see this first observe that ordinal sum induces a map \( \sigma_a^*: s\text{Set}_a \to ss\text{Set}_a \), where \( s\text{Set}_a = [\Delta_a^{op}, \text{Set}] \) denotes the category of augmented simplicial sets and \( ss\text{Set}_a = [\Delta_a^{op} \times \Delta_a^{op}, \text{Set}] \) denotes the category of biaugmented bisimplicial sets, and that we have a commutative diagram

\[
\begin{array}{ccc}
s\text{Set}_a & \xrightarrow{\sigma_a^*} & ss\text{Set}_a \\
i^* \downarrow & & \downarrow (i \times i)^*
\end{array}
\]

Here \( i^* \) and \( (i \times i)^* \) denote the functors induced by the inclusions \( i: \Delta \subset \Delta_a \) and \( i \times i: \Delta \times \Delta \subset \Delta_a \times \Delta_a \) respectively. Let \( i_! \) and \( (i \times i)_! \) denote the functors left adjoint to \( i^* \) and \( (i \times i)^* \) respectively. Thus if \( S \) is a simplicial set then \( i_!(S) \) is the augmented simplicial set with \( i_! S(-1) = \pi_0(S) \) and similarly for \( (i \times i)_! \). It follows that we have a natural isomorphism of functors

\[
i_! \sigma_i = (\sigma_a)_!(i \times i)_!
\]

and hence

\[
\sigma_i = i^* (\sigma_a)_!(i \times i)_!,
\]

where \( (\sigma_a)_! \) denotes the left adjoint of the functor \( \sigma_a^*: s\text{Set}_a \to ss\text{Set}_a \) induced by restriction along \( \sigma_a: \Delta_a \times \Delta_a \to \Delta_a \). Hence we have the following lemma.

**Lemma 2.1.** For any connected simplicial sets \( K \) and \( L \) we have the identity

\[
\sigma_i(K \square L) = K \star L,
\]

natural in \( K \) and \( L \), where \( K \star L \) denotes the join of \( K \) and \( L \).

Here \( K \square L \) denotes the box product of the simplicial sets \( K \) and \( L \): recall that the box product is the functor \( \square: s\text{Set} \times s\text{Set} \to ss\text{Set} \) which sends \( (K, L) \) to the bisimplicial set whose set of \( (m, n) \)-bisimplices is \( K_m \times L_n \). Thus \( \Delta^m \square \Delta^n \) is the classifying bisimplex \( \Delta^{m,n} \) for example. Note that \( \square \) extends canonically to a functor \( \square: s\text{Set}_a \times s\text{Set}_a \to ss\text{Set}_a \).

**Proof.** Observe that \( (i \times i)_!(K \square L) = i_!(K) \square i_!(L) \). Since \( K \) and \( L \) are connected we have \( (i \times i)_!(K \square L) = i_!(K) \square i_!(L) \), where \( i_! \) denotes the functor right adjoint to \( i^* \), so that \( i_!(S)([-1]) = 1 \) for any simplicial set \( S \). Therefore

\[
\sigma_i(K \square L) = i^* (\sigma_a)_!(i_!(K) \square i_!(L)),
\]

which is by definition equal to the join \( K \star L \) (see [27]). \( \square \)

As a corollary we have the following result.

**Corollary 2.1.** The functor \( C = \sigma((-) \square \Delta^0): s\text{Set} \to s\text{Set} \) is left adjoint to the functor \( \text{Dec}_0 \). In particular when \( X \) is a connected simplicial set we have

\[
C(X) = \sigma_i(X \square \Delta^0) = X \star \Delta^0.
\]

Thus when \( X \) is connected, \( C(X) \) is the cone construction on \( X \) (see Section III.5 of [17]).

**Proof.** Let \( X \) be a simplicial set. Then for any simplicial set \( Y \), there is a sequence of natural isomorphisms

\[
s\text{Set}(\sigma_i(X \square \Delta^0), Y) = ss\text{Set}(X \square \Delta^0, \text{Dec} Y) = s\text{Set}(X, (\text{Dec} Y)/\Delta^0),
\]

where we recall [28] that the functor \( -(\square \Delta^0): s\text{Set} \to ss\text{Set} \) is right adjoint to the functor \( -(\text{Dec})/\Delta^0: ss\text{Set} \to s\text{Set} \) which sends a bisimplicial set \( S \) to its first row. Hence there is an isomorphism

\[
s\text{Set}(\sigma_i(X \square \Delta^0), Y) = s\text{Set}(X, \text{Dec}_0 Y),
\]

natural in \( X \) and \( Y \), which proves the result. \( \square \)
2.3 Generalities on matching objects

Recall (see VII Proposition 1.21 of [17]) that if $X$ is a simplicial object in a complete category $\mathcal{C}$ and $K$ is a simplicial set, then we can define an object $M_K X$, a generalized matching object of $X$, by the formula

$$M_K X = \lim_{\Delta^n \to K} X_n.$$  

In other words $M_K X$ is the limit of the diagram $(\Delta/K)^{op} \to \Delta^{op} \to \mathcal{C}$ on the opposite of the simplex category $\Delta/K$ of $K$. This process is functorial in $X$ and so defines a functor $M_K: s\mathcal{C} \to \mathcal{C}$. It turns out that this functor is right adjoint to the functor $\mathcal{C} \to s\mathcal{C}$ which sends an object $A$ of $\mathcal{C}$ to the simplicial object $A \otimes K$. Hence the functor $M_K$ preserves limits, and therefore sends groups in $s\mathcal{C}$ to groups in $\mathcal{C}$.

If we fix a simplicial object $X$, then we obtain a functor $s\mathcal{S}et^{op} \to \mathcal{C}$ which sends a simplicial set $K$ to the generalized matching object $M_K X$. It turns out that this functor is also a right adjoint, and hence preserves limits.

We would like to compare the generalized matching objects $M^B_K X$ in $\mathcal{C}/B$ and $M_K X$ in $\mathcal{C}$ for $X$ an object of $\mathcal{C}/B$. To do this we first need to recall how limits in $\mathcal{C}/B$ are constructed. Recall from [30] that products in $\mathcal{C}/B$ are constructed as fiber products and that pullbacks are constructed by first forming the ordinary pullback in $\mathcal{C}$, and then equipping it with the canonical map to $B$. More generally, if $X: I \to \mathcal{C}/B$ is a diagram in $\mathcal{C}/B$ and $U(X): I \to \mathcal{C}$ denotes the underlying diagram in $\mathcal{C}$, then $\lim_{\xymatrix{\leftarrow \ar[r]_{\iota} & I}} X_i$ is defined by the following pullback diagram in $\mathcal{C}$:

$$\begin{array}{c}
\lim_{\xymatrix{\leftarrow \ar[r]_{\iota} & I}} X_i \\
\downarrow \downarrow \\
\lim_{\xymatrix{\leftarrow \ar[r]_{\iota} & I}} U(X_i) \\
\downarrow \downarrow \\
B \\
\downarrow \downarrow \\
\lim_{\xymatrix{\leftarrow \ar[r]_{\iota} & I}} B,
\end{array}$$

where $B$ is regarded as a constant diagram on $I$. It follows that if $X$ is a simplicial object of $\mathcal{C}/B$ then $M^B_K X$ fits into a pullback diagram

$$\begin{array}{c}
M^B_K X \\
\downarrow \\
M_K X \\
\downarrow \\
B \\
\downarrow \\
M_K B,
\end{array}$$

where we think of $B$ as a constant simplicial object in $\mathcal{C}$. We have the following well known lemma (see for instance [14] Lemma 4.2).

**Lemma 2.2.** Let $X$ be an object of $\mathcal{C}/B$. Then we have

$$M^B_{\Lambda^n_k} X = M_{\Lambda^n_k} X$$

for all $0 \leq k \leq n$ and all $n \geq 1$. Similarly we have

$$M^B_{\partial\Delta^n} X = M_{\partial\Delta^n} X$$

for all $n \geq 2$.

**Proof.** Since the diagram (3) is a pullback, it is enough to prove that the maps $B \to M_{\Lambda^n_k} B$ and $B \to M_{\partial\Delta^n} B$ are isomorphisms, where $B$ is regarded as a constant simplicial object in $\mathcal{C}$. Therefore, it is enough to prove that the simplex categories $\Delta \downarrow \Lambda^n_k$ and $\Delta \downarrow \partial\Delta^n$ are connected. This follows from the fact that the spaces $|\Lambda^n_k|$ and $|\partial\Delta^n|$ are connected, since $N(\Delta \downarrow \Lambda^n_k)$ and $N(\Delta \downarrow \partial\Delta^n)$ are weakly equivalent to $\Lambda^n_k$ and $\partial\Delta^n$ respectively.

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The notion of generalized matching object has an analog for bisimplicial objects in \( \mathcal{C} \). If \( K \) is a bisimplicial set and \( X \) is a bisimplicial object of \( \mathcal{C} \), then the generalized matching object \( M_K X \) is defined by an analogous limit formula in \( \mathcal{C} \) to that above:

\[
M_K X = \lim_{\Delta^m \times \Delta^n \to K} X_{m,n}.
\]

For a fixed bisimplicial set \( K \) this defines a functor \( M_K : \text{ss} \mathcal{C} \to \mathcal{C} \). Note that if \( K \) and \( L \) are simplicial sets and \( X \) is a bisimplicial object of \( \mathcal{C} \) then we have an isomorphism

\[
M_K \Box L X = M_K M_L X \tag{4}
\]

where \( M_L X \) denotes the matching object of the simplicial object \( X \) of \( \text{ss} \mathcal{C} \).

We have the following result.

**Lemma 2.3.** Let \( K \) be a bisimplicial set. The functor \( M_K : \text{ss} \mathcal{C} \to \mathcal{C} \) has a left adjoint \( (-) \otimes K : \mathcal{C} \to \text{ss} \mathcal{C} \).

**Proof.** \( (-) \otimes K : \mathcal{C} \to \text{ss} \mathcal{C} \) is the functor whose value on an object \( X \) of \( \mathcal{C} \) is the bisimplicial object \( (X \otimes K) \) whose object of \((m, n)\)-bisimplices is

\[
(X \otimes K)_{m,n} = \coprod_{k \in K_{m,n}} X.
\]

A map \( \alpha : ([m], [n]) \to ([p], [q]) \) induces a map \( \alpha^* : (X \otimes K)_{p,q} \to (X \otimes K)_{m,n} \) in the obvious fashion. The proof that this functor is left adjoint to \( M_K \) is entirely analogous to the proof of VII Proposition 1.21 of [17], to which we refer the reader for details. \( \Box \)

The following proposition describes a useful property of matching objects for bisimplicial objects that we shall put to use in Section 5.2.

**Proposition 2.1.** Let \( X \) be a simplicial object in \( \mathcal{C} \) and let \( K \) be a bisimplicial set. Then there is an isomorphism

\[
M_{\sigma, K} X \cong M_K \text{Dec} X,
\]

natural in \( X \) and \( K \), where \( \sigma : \text{ssSet} \to \text{sSet} \) denotes the left adjoint to \( \text{Dec} : \text{sSet} \to \text{ssSet} \). In particular when \( K \) and \( L \) are connected simplicial sets, then we have an isomorphism

\[
M_K \Box L \text{Dec} X \cong M_K \ast L X,
\]

natural in \( K \), \( L \) and \( X \), where \( K \ast L \) denotes the join of \( K \) and \( L \).

As an immediate corollary to this Proposition we have the following result.

**Corollary 2.2.** Let \( X \) be a simplicial object in \( \mathcal{C} \) and let \( K \) be a simplicial set. Then there is an isomorphism

\[
M_K \text{Dec}_0 X \cong M_{C K} X,
\]

natural in \( X \) and \( K \), where \( C K \) denotes the cone construction on \( K \).

**Proof.** In Proposition 2.1 take the bisimplicial set \( K \) to be \( K \Box \Delta^0 \). Then by Corollary 2.1 we have natural isomorphisms

\[
M_{\sigma(K \Box \Delta^0)} X \cong M_K \Box \Delta^0 \text{Dec} X \cong M_K M_{\Delta^0} \text{Dec} X \cong M_K \text{Dec}_0 X,
\]

where in the second isomorphism we have used (1). To complete the proof we recall that \( C K = \sigma(K \Box \Delta^0) \). \( \Box \)
Proof of Proposition 2.1. Let $X$ and $Y$ be objects of $\mathcal{C}$ and let $K$ be a bisimplicial set. By hypothesis $\mathcal{C}$ is cocomplete and so the functor $\text{Dec} = \sigma^*: s\mathcal{C} \to \text{ss}\mathcal{C}$ has a left adjoint $\sigma_1: \text{ss}\mathcal{C} \to s\mathcal{C}$ given as a left Kan extension, so that

$$(\sigma_1X)_n = \lim_{\sigma([p],[q]) \to [n]} X_{p,q}.$$ 

Therefore we can use Lemma 2.3 to establish the following series of natural isomorphisms.

$$\mathcal{C}(X, M_K \text{Dec } Y) \cong \text{ss}\mathcal{C}(X \otimes K, \text{Dec } Y)$$

$$\cong \text{ss}\mathcal{C}(\sigma_1(X \otimes K), Y)$$

$$\cong s\mathcal{C}(\sigma_1(X \otimes K), Y)$$

$$\cong \mathcal{C}(X, M_{\sigma_1K} Y)$$

where we have used the fact (proved shortly below) that there is a natural isomorphism $\sigma_1(- \otimes K) \cong (-) \otimes \sigma_1(K)$. It follows that $M_K \text{Dec } Y$ and $M_{\sigma_1K} Y$ have the same universal property in $\mathcal{C}$ and hence must be isomorphic.

To see that there is such a natural isomorphism we use the fact that we can interchange colimits to perform the following computation:

$$\sigma_1(X \otimes K)_n = \lim_{\sigma([p],[q]) \to [n]} (X \otimes K)_{p,q}$$

$$= \lim_{\sigma([p],[q]) \to [n]} \prod X_{K_{p,q}}$$

$$= \prod X_{\sigma_1K_n}$$

$$= (X \otimes \sigma_1K)_n.$$ 

These isomorphisms are natural in $n$ and hence the simplicial objects $\sigma_1(X \otimes K)$ and $X \otimes \sigma_1K$ are isomorphic. 

\[ \square \]

3 Internal local simplicial homotopy theory

In this section we study some aspects of local homotopy theory internal to a category $\mathcal{C}$ equipped with a Grothendieck pretopology. Our treatment follows closely the discussion in [18] and the classic [22].

3.1 Grothendieck topologies

Let $\mathcal{C}$ be a category with all limits and colimits and which comes equipped with a Grothendieck pretopology. Thus for every object $U$ of $\mathcal{C}$ there is assigned a class of families of morphisms $(U_i \to U)_{i \in I}$, called covering families which satisfy three axioms, namely: isomorphisms are covering families, covering families are stable under pullback and transitive under composition. For example, to say that covering families are stable under pullback means that if $(U_i \to U)_{i \in I}$ is a covering family, and $V \to U$ is a morphism in $\mathcal{C}$, then $(V \times_U U_i \to V)_{i \in I}$ is also a covering family. For more details we refer to Definition 2 in Section III.2 of [32].

We’ll often refer to a covering family $(U_i \to U)_{i \in I}$ as a cover of $U$. It is clear that if $\mathcal{C}$ is equipped with a Grothendieck pretopology, then for any object $B$ of $\mathcal{C}$, the slice category $\mathcal{C}/B$ inherits a natural Grothendieck pretopology: a family of morphisms $(U_i \to U)_{i \in I}$ in $\mathcal{C}/B$ is a covering family in $\mathcal{C}/B$ if and only if the underlying family of morphisms $(U_i \to U)_{i \in I}$ is a covering family in $\mathcal{C}$.

We will say that a map $f: X \to Y$ in $\mathcal{C}$ is a local epimorphism if there exists a covering family $(Y_i \to Y)_{i \in I}$ with the property that for each $i \in I$ there is a section of the induced map $Y_i \times_Y X \to Y_i$. Note that local epimorphisms are closed under retracts.
We will also need to talk about presheaves represented by objects of \( \mathcal{C} \). When \( \mathcal{C} \) is small we will sometimes write \( \mathcal{C} \) for the category \( [\mathcal{C}^{op}, \text{Set}] \) of set-valued presheaves on \( \mathcal{C} \). In general however, we will run into set theoretic difficulties in trying to consider the category of presheaves on \( \mathcal{C} \). To avoid these difficulties we will suppose that in addition to \( \mathcal{C} \), there exists a small subcategory \( \mathcal{C}_0 \subset \mathcal{C} \) such that \( \mathcal{C}_0 \) contains a terminal object and is equipped with a Grothendieck topology in such a way that if \( (U_i \to U)_{i \in I} \) is a covering family of \( U \) in \( \mathcal{C}_0 \), then \( (U_i \to U)_{i \in I} \) is also a covering family in \( \mathcal{C} \).

Then every object \( X \) of \( \mathcal{C} \) represents a presheaf \( \mathcal{X} \) on \( \mathcal{C}_0 \). If \( f: X \to Y \) is a map in \( \mathcal{C} \) then we write \( \hat{f}: \mathcal{X} \to \mathcal{Y} \) for the induced map of representable presheaves on \( \mathcal{C}_0 \). Note that if \( f: X \to Y \) is a local epimorphism in \( \mathcal{C} \), then \( \hat{f}: \mathcal{X} \to \mathcal{Y} \) is a local epimorphism of presheaves on \( \mathcal{C}_0 \) in the sense that for every section \( y \in \mathcal{Y}(U) \), there is a covering family \( (U_i \to U)_{i \in I} \) and elements \( x_i \in \mathcal{X}(U) \) such that \( \hat{f}(x_i) = y|_{U_i} \) for each \( i \in I \).

As our main example we will consider the case where \( \mathcal{C} = \mathcal{X} \), the category of k-spaces, where a covering family of a space \( U \) in \( \mathcal{X} \) is understood to mean as usual a collection of maps of the form \( (U_i \to U)_{i \in I} \), where \( (U_i)_{i \in I} \) forms an open cover of \( X \). For this definition to make sense, we need to know that every open subspace of a k-space is itself a k-space. However this follows from results of [44]: (with the notations of that paper) Theorem 5.1 of [44] shows that \( \mathcal{X} \) (called \( \mathcal{X}' \) in [44]) satisfies Axiom 1*, and therefore Proposition 2.4 of [44] shows that the relative topology on any open subset of a k-space coincides with the topology of its k-ification.

With this choice of \( \mathcal{C} \), we will be interested in the following small subcategory \( \mathcal{C}_0 \) of \( \mathcal{C} \). Let \( B \) be a paracompact space in \( \mathcal{X} \) and let \( \mathcal{C}_0 = \text{Open}(B) \) be the subcategory of \( \mathcal{X} \) generated by all open subsets of \( B \). It is clear that in this case \( \mathcal{C}_0 \) has a Grothendieck topology compatible with the open cover topology on \( \mathcal{C} \).

### 3.2 Local homotopy theory of simplicial presheaves

Suppose that \( \mathcal{C} \) is a small category. Recall that there are two model structures on the category of diagrams \( s\mathcal{C} = [\mathcal{C}^{op}, s\text{Set}] \). The weak equivalences for each of these model structures are the object-wise weak homotopy equivalences, i.e. the maps \( X \to Y \) in \( s\mathcal{C} \) such that \( X(C) \to Y(C) \) is a weak homotopy equivalence of simplicial sets for all \( C \in \mathcal{C} \). The injective model structure has as its cofibrations the monomorphisms, with the fibrations determined by the RLP with respect to the trivial cofibrations. The projective model structure has as its fibrations the object-wise Kan fibrations with the cofibrations determined by the LLP with respect to the trivial fibrations. To distinguish between the injective and projective model structures we will write \( s\text{Pre}(\mathcal{C}) \) for the injective model structure on the category of simplicial presheaves and, following [12], we will write \( U\mathcal{C} \) for the projective model structure (in [12] this is called the universal model structure).

For us, the projective model structure has a slight advantage over the injective model structure: the fibrant objects are easy to understand and there is a criterion to detect cofibrant objects which is relatively easy to check in practice. Namely, we have the following result from [12].

**Proposition 3.1** (Dugger [12]). A simplicial presheaf \( X \) is cofibrant in \( U\mathcal{C} \) if it is split, and is a degree-wise coproduct of representables.

Recall that a simplicial object \( X \) is said to be split if for all \( n \geq 0 \) there exist subobjects \( U_n \subset X_n \) such that the canonical map

\[
\prod_{\sigma: [n] \to [m]} U_\sigma \to X_n
\]

is an isomorphism, where \( U_\sigma \) denotes a copy of \( U_m \) and the map \( U_\sigma \to X_n \) is the composition \( U_m \to X_m \overset{\sigma^*}{\to} X_n \). In fact, Dugger shows in [12] that there is a very convenient cofibrant replacement functor in \( U\mathcal{C} \) which replaces any simplicial presheaf with a split one.

Now suppose that \( \mathcal{C} \) is a small category equipped with a Grothendieck topology. In the paper [22] the category \( s\mathcal{C} \) of simplicial presheaves on \( \mathcal{C} \) was equipped with the structure of a model category. Following [14] we will write \( s\text{Pre}(\mathcal{C})_L \) for this model category. The weak equivalences of this model structure are defined in terms of certain sheaves of simplicial homotopy groups as we now recall.
Let $X$ be a simplicial presheaf on $\mathcal{C}$. If $n \geq 1$ then for any object $C$ of $\mathcal{C}$ and any vertex $v \in X_0(C)$, Jardine defines sheaves $\pi_n(X_C, v)$ on $\mathcal{C}/C$ as the sheaves associated to the presheaves

$$(C' \to C) \mapsto \pi_n(X(C'), v).$$

Similarly the sheaf $\pi_0(X_C)$ on $\mathcal{C}/C$ is defined to be the sheaf associated to the presheaf

$$(C' \to C) \mapsto \pi_0(X(C')).$$

A map $f : X \to Y$ in $s\mathcal{C}$ is called a local weak equivalence if the induced maps $\pi_0(X_C) \to \pi_0(Y_C)$ and $\pi_n(X_C, v) \to \pi_n(Y_C, f(v))$ are isomorphisms of sheaves on $\mathcal{C}/C$ for all objects $C$ of $\mathcal{C}$ and all choices of vertices $v \in X_0(C)$. In [22] it is proven that the local weak equivalences are the weak equivalences for a model structure on $s\mathcal{C}$ whose cofibrations are the monomorphisms.

We will only be interested in a very special case of this general theory — namely when $\mathcal{C}$ is the category $\text{Open}(B)$ of open subsets of a paracompact space $B$ in $\mathcal{K}$, as described above. We will write $s\text{Pre}(B)$ for the category $s\mathcal{C}$ of simplicial presheaves on $\mathcal{C}$ and say that an object $X \in s\text{Pre}(B)$ is a simplicial presheaf on $B$ in this case. In this case the local weak equivalences are much easier to describe: a map $f : X \to Y$ in $s\text{Pre}(B)$ is a local weak equivalence if and only if the induced map on stalks $f_p : X_p \to Y_p$ is a weak homotopy equivalence for all points $p$ in $B$.

An important fact proven in [14] is that the Jardine model structure $s\text{Pre}(\mathcal{C})_{\mathcal{L}}$ is a localization of the injective model structure $s\text{Pre}(\mathcal{C})$ at a certain collection of maps. We recall some of the details here as it will be important for us later. Let $C$ be an object of $\mathcal{C}$ and write $C$ also for the corresponding representable presheaf $C$ on $\mathcal{C}$. Recall that a map $U \to C$ in $s\mathcal{C}$ is called a hypercover of $C$ if each $U_n$ is a coproduct of representables and the maps

$$U_n \to M^\partial_n U$$

are local epimorphisms of presheaves on $\mathcal{C}$. Here the matching objects are computed in the category $s\mathcal{C}$ as described above. A hypercover $U \to C$ is said to be split if the underlying simplicial presheaf $U$ is split. One of the main theorems of [14] is the following.

**Theorem 3.1 ([14]).** Let $S$ be the class of all hypercovers on $\mathcal{C}$. Then the localization $s\text{Pre}(\mathcal{C})/S$ exists and coincides with $s\text{Pre}(\mathcal{C})_{\mathcal{L}}$. Moreover the localization $U\mathcal{C}_{\mathcal{L}} = U\mathcal{C}/S$ exists and there is a Quillen equivalence

$$s\text{Pre}(\mathcal{C})_{\mathcal{L}} \simeq U\mathcal{C}_{\mathcal{L}}.$$

As pointed out in [14], the localized projective model structure $U\mathcal{C}_{\mathcal{L}}$ has an advantage over $s\text{Pre}(\mathcal{C})_{\mathcal{L}}$ in that the fibrant objects are easier to describe and there is a explicit formula for cofibrant replacement. In [14], a simplicial presheaf $X$ is said to satisfy descent for a hypercover $U \to C$ if the natural map

$$F(C) \to \text{holim}_n F(U_n)$$

is a weak homotopy equivalence of simplicial sets. In [14] the following characterization of the fibrant objects in $U\mathcal{C}_{\mathcal{L}}$ is obtained.

**Theorem 3.2 ([14]).** A simplicial presheaf $X$ is fibrant in $U\mathcal{C}_{\mathcal{L}}$ if and only if

1. $X$ is objectwise fibrant, i.e. $X(C)$ is a Kan complex for all objects $C$ of $\mathcal{C}$,
2. $X$ satisfies descent for all hypercovers.

In general it is difficult to compute the set of homotopy classes $[X, Y]$ in $U\mathcal{C}_{\mathcal{L}}$, since it is difficult to compute fibrant replacements in $U\mathcal{C}_{\mathcal{L}}$. The generalized Verdier hypercovering theorem [8, 14, 22] allows one to compute some invariants of $[X, Y]$, at least when $Y$ is locally fibrant in the following well known sense.
Suppose that \( i: K \to L \) is a map of simplicial sets. A map \( f: X \to Y \) in \( s\mathcal{E} \) is said to have local liftings relative to \( i: K \to L \) if for all objects \( C \) in \( \mathcal{E} \) and all diagrams in \( s\mathcal{E} \) of the form

\[
\begin{array}{ccc}
K \otimes C & \longrightarrow & X \\
\downarrow & & \downarrow f \\
L \otimes C & \longrightarrow & Y,
\end{array}
\]

there exists a covering sieve \( R \subset \mathcal{E}(\cdot, C) \) such that for every map \( \phi: U \to C \) in \( R \), the diagram obtained by restricting along \( \phi \) has a lift \( L \otimes U \to X \). A map \( f: X \to Y \) in \( s\mathcal{E} \) is said to be a local fibration if \( f \) has local liftings relative to \( A^k_0 \subset \Delta^0 \) for all \( 0 \leq k \leq n \) and all \( n \geq 1 \). An object \( X \) in \( s\mathcal{E} \) is said to be locally fibrant if the canonical map \( X \to 1 \) is a local fibration. It can be shown (see [13, 22]) that a map \( f: X \to Y \) in \( s\mathcal{E} \) is a hypercover if and only if \( f \) is a locally acyclic local fibration and \( U_n \) is a coproduct of representables for all \( n \geq 0 \).

When \( \mathcal{E} = \text{Open}(B) \), a map \( f: X \to Y \) in \( s\mathcal{E} \) is a local fibration if and only if the induced map on stalks \( f_p: X_p \to Y_p \) is a Kan fibration for all points \( p \) of \( M \). If \( f: X \to Y \) is an objectwise Kan fibration then \( f \) is a local fibration, but not conversely.

To state the hypercovering theorem we need some notation. For an object \( C \) of \( \mathcal{E} \), let \( HR(C) \) denote the full subcategory of \( s\mathcal{E} \) whose objects are the hypercovers of \( C \). We write \( \pi HR(C) \) for the category with the same objects as \( HR(C) \) but whose morphisms are simplicial homotopy classes of morphisms in \( HR(C) \). We have the following result.

**Theorem 3.3 (generalized Verdier hypercovering theorem [8, 14, 22]).** Let \( X \) be a locally fibrant simplicial presheaf on \( \mathcal{E} \) and let \( C \) be an object of \( \mathcal{E} \). Then there is an isomorphism

\[
[C, X] = \lim_{U \in \pi HR(C)} \pi(U, X)
\]

in \( \text{Ho}(U\mathcal{E}_C) \) where \( \pi(\cdot, \cdot) \) denotes simplicial homotopy classes of maps.

As pointed out in [14], the colimit above could just as well be taken over any full subcategory of \( \pi HR(C) \) whose objects belong to a dense set of hypercovers. In the case of main interest for us, namely when \( \mathcal{E} = \text{Open}(B) \), this means that the colimit above could be taken over the full subcategory of \( \pi HR(C) \) whose objects are the split hypercovers. Note that since \( U\mathcal{E}_C \) is a left Bousfield localization, every cofibrant object of \( U\mathcal{E} \) is automatically cofibrant in \( U\mathcal{E}_C \). In particular every split hypercover is cofibrant in \( U\mathcal{E}_C \) (Proposition 3.1). As a consequence we have the following simple observation.

**Lemma 3.1.** Let \( p: X \to Y \) be an objectwise Kan fibration of simplicial presheaves on \( B \) and suppose that \( Y \) is locally fibrant. Then the map

\[
[1, X] \to [1, Y]
\]

in \( \text{Ho}(U\mathcal{E}_C) \) is surjective, where \( 1 \) denotes the terminal object.

**Proof.** Since \( Y \) is locally fibrant and \( p \) is a local fibration, \( X \) is locally fibrant. Since \( p: X \to Y \) is a fibration in \( U\mathcal{E} \), for any split hypercover \( U \) of \( 1 \), the map

\[
\text{map}(U, X) \to \text{map}(U, Y)
\]

is a Kan fibration and hence is surjective on path components. Since filtered colimits preserve surjections it follows that the map

\[
\lim_{U \in \pi HR_s(1)} \pi_0\text{map}(U, X) \to \lim_{U \in \pi HR_s(1)} \pi_0\text{map}(U, Y)
\]

is surjective, where \( \pi HR_s(1) \) denotes the full subcategory of \( \pi HR(1) \) consisting of the split hypercovers. By the remarks above this map is isomorphic to the map \([1, X] \to [1, Y]\) in \( \text{Ho}(U\mathcal{E}_C) \).

\( \square \)
3.3 Internal local fibrations

Recall the following definition from [18].

**Definition 3.1 ([18])**. We say that a morphism \( f : X \to Y \) in \( s\mathcal{C} \) is a local fibration if for every \( n \geq 1 \) and every \( 0 \leq k \leq n \) the map
\[
X_n \to Y_n \times_{M_{\Delta^k} Y} M_{\Delta^n} X
\]
(5)
is a local epimorphism. We say that a simplicial object \( X \) in \( \mathcal{C} \) is locally fibrant if the canonical map \( X \to 1 \) to the terminal simplicial object is a local fibration: thus \( X \) is locally fibrant if for every \( n \geq 1 \) and every \( 0 \leq k \leq n \) the map
\[
X_n \to M_{\Delta^k} X
\]
is a local epimorphism. We say that a morphism \( f : X \to Y \) in \( s\mathcal{C} \) is a locally acyclic local fibration if for every \( n \geq 0 \) the map
\[
X_n \to Y_n \times_{M_{\Delta^n} Y} M_{\theta \Delta^n} X
\]
is a local epimorphism. Similarly we say that a simplicial object \( X \) of \( \mathcal{C} \) is locally acyclic if the map \( X \to 1 \) to the terminal simplicial object is an acyclic local fibration: thus \( X \) is locally acyclic if for every \( n \geq 0 \) the map
\[
X_n \to M_{\theta \Delta^n} X
\]
is a local epimorphism.

Clearly if \( f : X \to Y \) is a local fibration in \( s\mathcal{C} \) then the induced map \( \hat{f} : \hat{X} \to \hat{Y} \) on representable simplicial presheaves in \( s\text{Pre}(\mathcal{C}_0) \) is a local fibration. An analogous remark applies if \( f : X \to Y \) is an acyclic local fibration in \( s\mathcal{C} \).

**Lemma 3.2.** Let \( f : X \to Y \) be a morphism in \( s\mathcal{C}/B \). Then \( f \) is a local fibration in \( s\mathcal{C}/B \) if and only if the underlying map \( f : X \to Y \) is a local fibration in \( s\mathcal{C} \). In particular an object \( X \) in \( s\mathcal{C}/B \) is locally fibrant in \( s\mathcal{C} \) if and only if the underlying object \( X \) is locally fibrant in \( s\mathcal{C} \).

**Proof.** \( f : X \to Y \) is a local fibration in \( s\mathcal{C}/B \) if and only if
\[
X_n \to M_{\Delta^k}^B X \times_{M_{\Delta^n}^B Y} Y_n
\]
is a local epimorphism in \( \mathcal{C}/B \) for all \( 0 \leq k \leq n \) and all \( n \geq 1 \). Hence Lemma 2.2 implies that \( f : X \to Y \) is a local fibration in \( s\mathcal{C} \) if and only if
\[
X_n \to M_{\Delta^k} X \times_{M_{\Delta^n} Y} Y_n
\]
is a local epimorphism in \( \mathcal{C} \) for all \( 0 \leq k \leq n \) and all \( n \geq 1 \), i.e. \( f : X \to Y \) is a local fibration in \( s\mathcal{C} \). \( \square \)

**Lemma 3.3.** Let \( G \) be a group in \( s\mathcal{C}/B \). Then \( G \) is locally fibrant.

**Proof.** Lemma 2.2 shows that it is enough to check that the underlying map \( G \to B \) in \( s\mathcal{C} \) is locally fibrant. First let us observe that if \( X \) is a group object in \( s\text{Set}/Y \), where \( Y \) is a discrete simplicial set, then the map \( X \to Y \) is a Kan fibration. Returning to the case at hand, for any object \( A \) in \( \mathcal{C} \), if we set \( X = \mathcal{C}(A,G) \) and \( Y = \mathcal{C}(A,B) \), then \( X \) is a group object in \( s\text{Set}/Y \) whose group object of \( n \)-simplices is \( \mathcal{C}(A,G_n) \). Therefore by the observation just made,
\[
\mathcal{C}(A,G_n) \to M_{\Delta^k} \mathcal{C}(A,G)
\]
is surjective for all \( 0 \leq k \leq n \) and all \( n \geq 1 \). Since \( \mathcal{C} \) is complete we have
\[
M_{\Delta^k} \mathcal{C}(A,G) = \mathcal{C}(A,M_{\Delta^k} G).
\]
Therefore
\[
\mathcal{C}(A,G_n) \to \mathcal{C}(A,M_{\Delta^k} G)
\]
is surjective for all \( 0 \leq k \leq n \) and all \( n \geq 1 \). Taking \( A = M_{\Delta^k} G \) gives the result. \( \square \)
Corollary 3.1. Suppose that \( f : G \to H \) is a homomorphism of group objects in \( s\mathcal{E}/B \) where \( H \) is the constant simplicial object associated to a group object \( H \) in \( \mathcal{E}/B \). Then \( f : G \to H \) is a local fibration.

Proof. We need to check that the maps

\[
G_n \to M_{\Lambda_k^n} G \times_{M_{\Lambda_k^n} H} H_n
\]

are local epimorphisms in \( \mathcal{E} \) for all \( 0 \leq k \leq n \) and all \( n \geq 1 \). Since \( H \) is constant we have

\[
M_{\Lambda_k^n} H = H_n
\]

and so the result follows by Lemma 3.3.

Corollary 3.2. Suppose that \( f : G \to H \) is a homomorphism between group objects in \( s\mathcal{E}/B \) such that \( f_n : G_n \to H_n \) is a local epimorphism for all \( n \geq 0 \). Then \( f \) is a local fibration.

Proof. We leave the proof of this to the reader.

Lemma 3.4. Let \( f : X \to Y \) be a local fibration in \( \mathcal{E}/B \). Let \( K \subset L \) be a weak equivalence of finite simplicial sets. Then the map

\[
M_L X \to M_K X \times_{M_K Y} M_L Y
\]

is a local epimorphism in \( \mathcal{E}/B \).

Proof. The proof of this is just an obvious reformulation of the proof of the corresponding fact for simplicial sets. In more detail, observe that \( f : X \to Y \) has the local RLP with respect to any map of the form \( J \to J \cup_{\Lambda^n} \Delta^n \) for any simplicial set \( J \). Similarly observe that if \( f : X \to Y \) has the local RLP with respect to \( J \to J \) and \( K \to L \) is a retract of \( I \to J \), then \( f : X \to Y \) has the local RLP with respect to \( K \to L \). Finally, \( K \to L \) is a retract of a finite composition of pushouts of maps of the form \( \Lambda^n_k \to \Delta^n \).

3.4 A criterion for local acyclicity

Our first main technical result gives a criterion to detect when certain homomorphisms between group objects in \( s\mathcal{E}/B \) are locally acyclic local fibrations.

Proposition 3.2. Suppose that \( H \) is a group object in \( \mathcal{E}/B \), and that \( f : G \to H \) is a homomorphism between group objects in \( s\mathcal{E}/B \), where \( H \) is thought of as a constant simplicial object in \( \mathcal{E}/B \). Then \( f : G \to H \) is a locally acyclic local fibration in \( s\mathcal{E}/B \) for the induced topology on \( \mathcal{E}/B \) if the following conditions hold:

1. \( G_0 \to H \) is a local epimorphism in \( \mathcal{E}/B \),
2. \( G_1 \to G_0 \times_H G_0 \) is a local epimorphism in \( \mathcal{E}/B \),
3. the underlying map \( G_n \to M_{\partial \Delta^n} G \) is an effective epimorphism in \( \mathcal{E} \) for all \( n \geq 2 \).

Proof. We need to prove that

\[
G_n \to M_{\partial \Delta^n} G \times_{M_{\partial \Delta^n} H} H_n
\]

is a local epimorphism in \( \mathcal{E}/B \) for all \( n \geq 0 \). When \( n = 0 \) this reduces to the requirement that \( G_0 \to H \) is a local epimorphism. When \( n = 1 \) we have \( M_{\partial \Delta^1} G = G_0 \times B G_0 \) and \( M_{\partial \Delta^1} H = H \times_B H \), so we need to prove that

\[
G_1 \to (G_0 \times_B G_0) \times_{H \times_B H} H
\]

is a local epimorphism in \( \mathcal{E}/B \) — which reduces to the requirement that \( G_1 \to G_0 \times_H G_0 \) is a local epimorphism.

Therefore the proposition reduces to the claim that under the given hypotheses \( \partial_n : G_n \to M_{\partial \Delta^n} G \) is a local epimorphism in \( \mathcal{E} \) for all \( n \geq 2 \). We will show that in fact the given hypotheses imply that this map has a section.
Let \( \ker(\partial_n) \) denote the group object \( 1 \times_{\partial_0 \Delta^n} G \) in \( \mathscr{C}/B \). We will first construct a \( \ker(\partial_n) \) equivariant retraction \( r_n: G_n \to \ker(\partial_n) \) in \( \mathscr{C}/B \). To do this we will show that it suffices to prove an analogous statement for simplicial sets. By the Yoneda Lemma, it suffices to find a \( \mathscr{C}(-, \ker(\partial_n)) \) equivariant retraction

\[
r: \mathscr{C}(-, G_n) \to \mathscr{C}(-, \ker(\partial_n))
\]

of the map of representable presheaves

\[
\mathscr{C}(-, \ker(\partial_n)) \to \mathscr{C}(-, G_n)
\]

induced by the homomorphism of group objects \( \ker(\partial_n) \to G_n \). Let \( A \) be an object of \( \mathscr{C} \). Then \( \mathscr{C}(A, G) \) is a group object in \( sSet/K \), where \( K \) denotes the constant simplicial set \( \mathscr{C}(A, B) \). The set of \( n \)-simplices of \( \mathscr{C}(A, G) \) is the group \( \mathscr{C}(A, G_n) \) of group objects in \( sSet/K \). Since the functor \( \mathscr{C}(A, -) \) preserves limits, we have an isomorphism

\[
\mathscr{C}(A, M_{\partial \Delta^n} G) = M_{\partial \Delta^n} \mathscr{C}(A, G)
\]

in \( sSet/K \). Therefore the homomorphism \( \mathscr{C}(A, G_n) \to \mathscr{C}(A, M_{\partial \Delta^n} G) \) is isomorphic to the homomorphism \( \mathscr{C}(A, G)_n \to M_{\partial \Delta^n} \mathscr{C}(A, G) \) and hence \( \mathscr{C}(A, \ker(\partial_n)) \) is isomorphic to the kernel of the homomorphism \( \mathscr{C}(A, G)_n \to M_{\partial \Delta^n} \mathscr{C}(A, G) \) of group objects in \( sSet/K \). Hence it suffices to prove that for any group \( G \) in \( sSet/K \), there is a \( \ker(\partial_n) \) equivariant retraction \( r_n: G_n \to \ker(\partial_n) \) which is natural with respect to homomorphisms \( G \to G' \) of group objects in \( sSet/K \).

To prove this, first observe that if \( G \) is a group object in \( sSet/K \) then \( \ker(\partial_n) \) is the set of \( n \)-simplices \( g \in G_n \) such that \( d_i(g) = 1 \) for all \( 0 \leq i \leq n \). Define a map

\[
G_n \to \ker(\partial_n)
\]

\[
g \mapsto s_0d_0(g)^{-1}g.
\]

Note that this map is the identity on \( \ker(\partial_n) \) and is \( \ker(\partial_n) \) equivariant. Now let \( 0 \leq m < n \) and define a map

\[
\bigcap_{0 \leq i \leq m+1} \ker(d_i) \to \bigcap_{0 \leq i \leq m} \ker(d_i)
\]

by sending \( g \mapsto s_{m+1}d_{m+1}(g)^{-1}g \). To see that the image of this map does lie in the subgroup \( \bigcap_{0 \leq i \leq m+1} \ker(d_i) \), we note that if \( i < m + 1 \) then

\[
d_is_{m+1}d_{m+1}(g) = s_md_id_{m+1}(g) = s_md_m(g) = 1.
\]

Note that this map again restricts to the identity on \( \ker(\partial_n) \) and is \( \ker(\partial_n) \) equivariant. We have constructed a sequence of \( \ker(\partial_n) \) equivariant maps

\[
G_n \to \ker(\partial_n) \to \ker(\partial_n) \cap \ker(\partial_1) \to \cdots \to \ker(\partial_n)
\]

which restrict to the identity on \( \ker(\partial_n) \). Composing these maps gives the required retraction \( r_n \) which is clearly natural with respect to homomorphisms \( G \to G' \) of group objects in \( sSet/K \).

To complete the proof we need to show that the existence of the \( \ker(\partial_n) \) equivariant retraction \( r_n: G_n \to \ker(\partial_n) \) in \( \mathscr{C}/B \) implies the existence of a section of \( \partial_n: G_n \to M_{\partial \Delta^n} G \) in \( \mathscr{C}/B \). For this we have the following Lemma.

**Lemma 3.5.** Let \( \mathscr{D} \) be a category with finite limits and let \( \phi: G \to H \) be a homomorphism between group objects in \( \mathscr{D} \). Suppose that \( \ker(\phi) \) is a retract of \( G \). If \( \phi: G \to H \) is an effective epimorphism in \( \mathscr{D} \) then there is a section of \( \phi \).

**Proof.** As we will shortly observe in Section 4.1 below, there is a canonical isomorphism

\[
G \times_H G \cong G \times \ker(\phi).
\]
Given a retraction \( r: G \to \ker(\phi) \), define a map \( \hat{s}: G \to G \) by the composite

\[
G \xrightarrow{(1, r^{-1})} G \times \ker(\phi) \xrightarrow{1 \times i} G \times G \xrightarrow{m} G.
\]

Note that \( \phi \hat{s} = \phi \). Note also that \( \hat{s} \) is \( \ker(\phi) \) invariant in the sense that the diagram

\[
\begin{array}{ccc}
G \times \ker(\phi) & \xrightarrow{m} & G \\
p_1 \downarrow & & \downarrow \hat{s} \\
G & \xrightarrow{\hat{s}} & G
\end{array}
\]

in \( \mathcal{D} \) commutes. From the isomorphism \( \hat{s} \) above it follows that \( \hat{s} p_1 = \hat{s} p_2 \), where \( p_1 \) and \( p_2 \) denote the canonical projections in the coequalizer diagram

\[
\begin{array}{ccc}
G \times H & \xrightarrow{P_1} & G \\
p_2 \downarrow & & \downarrow \phi \\
G & \xrightarrow{\phi} & H
\end{array}
\]

Therefore there is a unique map \( s: H \to G \) such that \( s \phi = \hat{s} \). Hence \( \phi s \phi = \phi \) from which it follows that \( \phi s = \text{id} \), i.e. \( s \) is a section of \( \phi \).

To conclude the proof we observe that since pullbacks and colimits in \( \mathcal{C}/B \) are computed in \( \mathcal{C} \), if the underlying map \( G_n \to M_{\beta \Delta^n} G \) is an effective epimorphism in \( \mathcal{C} \) then the map \( G_n \to M_{\beta \Delta^n} G \) is an effective epimorphism in \( \mathcal{C}/B \).

4 Non-abelian cohomology

In this section we study some examples of the internal local homotopy theory of the previous section, before we recall the definition and some properties of non-abelian cohomology. Throughout this section \( \mathcal{C} \) denotes a category equipped with a Grothendieck pretopology, if we need to we will suppose that there exists a small subcategory \( \mathcal{C}_0 \) of \( \mathcal{C} \) satisfying the condition described in Section 3.1. We begin with a discussion of simplicial torsors.

4.1 Simplicial torsors

One source of examples of local Kan fibrations comes from torsors in \( s\mathcal{C} \). These are studied in detail in the monograph [15] of Duskin where they are used to give an interpretation of cohomology for monads. Recall that if \( G \) is a group in \( s\mathcal{C} \) and \( X \) is an object of \( s\mathcal{C} \), then a \( G \)-torsor in \( s\mathcal{C} \) over \( X \) consists of an object \( P \) of \( s\mathcal{C}/X \) equipped with an action \( P \times G \to P \) of the group object \( X \times G \) in \( s\mathcal{C}/X \) such that the diagram

\[
\begin{array}{ccc}
P \times G & \xrightarrow{P_1} & P \\
p_1 \downarrow & & \downarrow \ \\
P & \xrightarrow{X} & X
\end{array}
\]

is a pullback in \( s\mathcal{C} \) (in fact Duskin refers to such an object as a \( G \)-pseudo-torsor over \( X \), however we will use the more standard terminology above). Note that this definition can be expressed entirely in terms of finite limits in \( s\mathcal{C} \). Of course the notion of \( G \)-torsor over an object \( X \) makes sense in any category \( \mathcal{E} \) with finite limits for \( G \) a group in \( \mathcal{E} \) — in particular it follows that for \( P \) and \( G \) as above, \( P_n \) is a \( G_n \) torsor in \( \mathcal{C} \) over \( M_n \) for all \( n \geq 0 \).
A recurring example of a torsor in this paper is the following: if \( \phi: G \to H \) is a homomorphism of group objects in \( s\mathcal{E} \), then \( G \) is a \( \ker(\phi) \) torsor over \( H \) in \( s\mathcal{E} \). One way to see that the diagram (3) is a pullback in this case is to write it as the composite of the two pullback squares
\[
\begin{array}{ccc}
G \times \ker(\phi) & \to & G \\
\downarrow & & \downarrow \\
G & \to & G \times H \\
\end{array}
\]
where the left square is a pullback by definition of \( \ker(\phi) \) and where the horizontal maps in the right square are given by multiplication in \( G \) and the composite of \( \phi \times 1 \) and multiplication in \( H \) respectively.

We will say that a \( G \)-torsor \( P \to X \) in \( s\mathcal{E} / B \) is locally trivial if each map \( P_n \to X_n \) is a local epimorphism in \( \mathcal{E} / B \). When \( \mathcal{E} = \mathcal{X} \) then the notion of locally trivial \( G \)-torsor for a group object in \( s\mathcal{X} / B \) reduces to the notion of simplicial principal bundle in \( \mathcal{X} / B \). Recall (see for example [40]) that a simplicial principal bundle in \( s\mathcal{X} / B \) consists of a map \( P \to X \) in \( s\mathcal{X} / B \) together with a simplicial map \( P \times G \to P \) such that each map \( P_n \to X_n \) has the structure of a principal \( G_n \)-bundle in \( \mathcal{X} / B \) for this action of \( G_n \).

We have the following result, directly analogous to the familiar result in the category of simplicial sets which states that every principal bundle in \( s\text{Set} \) is a Kan fibration.

**Lemma 4.1.** Let \( G \) be a group in \( s\mathcal{E} / B \) and let \( P \to X \) be a locally trivial \( G \)-torsor over \( X \) in \( s\mathcal{E} / B \). Then \( P \to X \) is a local Kan fibration in \( s\mathcal{E} / B \). Moreover, if each \( P_n \to X_n \) admits a global section, the maps
\[
P_n \to M_{\Lambda^n_k} \times_{M_{\Lambda^n_k} X} X_n
\]
admit global sections in \( \mathcal{E} / B \).

**Proof.** We need to show that the maps
\[
h^n_k: P_n \to M_{\Lambda^n_k} \times_{M_{\Lambda^n_k} X} X_n
\]
are local epimorphisms in \( \mathcal{E} / B \). Since the functor \( M_{\Lambda^n_k}(-) \) is a right adjoint, it sends torsors in \( s\mathcal{E} / B \) to torsors in \( \mathcal{E} / B \). In particular, \( M_{\Lambda^n_k} P \to M_{\Lambda^n_k} X \) is a \( M_{\Lambda^n_k} G \)-torsor in \( \mathcal{E} / B \). Hence \( M_{\Lambda^n_k} P \times_{M_{\Lambda^n_k} X} X_n \to X_n \) is a \( M_{\Lambda^n_k} G \)-torsor in \( \mathcal{E} / B \). For convenience of notation write \( Y_{n,k} = M_{\Lambda^n_k} \times_{M_{\Lambda^n_k} X} X_n \).

Let \( (X_{n,i})_{i \in I} \) be a covering family of \( X_n \) and let \( \sigma_i \) be a section of \( X_{n,i} \times_{X_n} P_n \to X_{n,i} \) for every \( i \in I \). For every \( i \in I \) we have a commutative diagram
\[
\begin{array}{ccc}
X_{n,i} \times_{X_n} P_n & \to & P_n \\
\downarrow & & \downarrow \\
X_{n,i} \times_{X_n} Y_{n,k} & \to & Y_{n,k} \\
\downarrow & & \downarrow \\
X_{n,i} & \to & X_n \\
\end{array}
\]
in which each square is a pullback. The section \( \sigma_i \) induces a map \( X_{n,i} \times_{X_n} Y_{n,k} \to P_n \) such that the projection to \( Y_{n,k} \) differs from the canonical projection \( X_{n,i} \times_{X_n} Y_{n,k} \to Y_{n,k} \) by a map \( X_{n,i} \times_{X_n} Y_{n,k} \to M_{\Lambda^n_k} G \). By Lemma [3] there exists a lift \( X_{n,i} \times_{X_n} Y_{n,k} \to G_n \) which can be used to rescale the induced map \( X_{n,i} \times_{X_n} Y_{n,k} \to P_n \) to obtain a map such that the diagram
\[
\begin{array}{ccc}
X_{n,i} \times_{X_n} P_n & \to & P_n \\
\downarrow & & \downarrow \\
X_{n,i} \times_{X_n} Y_{n,k} & \to & Y_{n,k} \\
\end{array}
\]
commutes for every \( i \in I \). These maps, together with the induced covering family \( (X_{n,i} \times_{X_n} Y_{n,k} \to Y_{n,k})_{i \in I} \), exhibit the maps \( h^n_k \) as local epimorphisms in \( \mathcal{E} / B \).
4.2 Twisted cartesian products and the $\mathbf{W}$ construction

Suppose that $P$ is a $G$-torsor over and $U: \mathcal{C} \to \mathcal{D}$ is a functor. Following [15], we say that $P$ is a $G$-torsor over $M$ (rel. $U: \mathcal{C} \to \mathcal{D}$) when $P \to M$ is equipped with a section $s: U(M) \to U(P)$. This notion allows for a neat definition of the notion of principal twisted cartesian product in the category $s\mathcal{C}$ of simplicial objects of $\mathcal{C}$, where $\mathcal{C}$ admits finite limits.

**Definition 4.1** (Duskin [15]). Let $G$ be a group object in $s\mathcal{C}$. A *principal twisted cartesian product* in $s\mathcal{C}$ with structure group $G$ is a $G$-torsor rel. $\text{Dec}_0: s\mathcal{C} \to s_0\mathcal{C}$, where $s_0\mathcal{C}$ denotes the category of augmented simplicial objects in $\mathcal{C}$ and coherent maps.

Thus a principal twisted cartesian product in $s\mathcal{C}$ with structure group consists of a $G$-torsor $P \to M$ in $s\mathcal{C}$ which is equipped with a *pseudo-cross section*, i.e. for every $n \geq 0$ there is a map $\sigma_n: M_n \to P_n$ such that $\sigma_n$ is a section of $P_n \to M_n$ for all $n \geq 0$, which satisfies $d_i\sigma_n = \sigma_{n-1}d_i$ for all $0 < i \leq n$ and $s_i\sigma_{n-1} = \sigma_n s_i$ for all $0 \leq i \leq n$, $n \geq 1$ (see Definition 18.5 of [34]).

As an immediate application of Lemma 4.1 we have the following result.

**Lemma 4.2.** Suppose that $P \to M$ is a principal twisted cartesian product in $s\mathcal{C}/B$ with structure group $G$. Then $\pi: P \to M$ is a local fibration in $s\mathcal{C}/B$.

A prime example of a twisted cartesian product is the universal $G$ torsor $WG \to \mathbf{W}G$. Recall (see [40]) that if $G$ is a group in $s\mathcal{C}$, then there is a canonical $G$-torsor

$$G//G \to G$$

in $s\text{Gpd}(\mathcal{C})$. Here $\text{Gpd}(\mathcal{C})$ denotes the category of groupoids in $\mathcal{C}$ and $G$ is regarded as a group in $s\text{Gpd}(\mathcal{C})$ whose underlying groupoid is discrete. Applying the nerve functor $N$ gives a canonical $G$-torsor

$$N(G//G) \to NG$$

in $s\mathcal{C}$, where $G$ is thought of as a constant simplicial group in $s\mathcal{C}$. Applying the total simplicial object functor $\sigma_*: s^s\mathcal{C} \to s\mathcal{C}$ right adjoint to $\text{Dec}: s\mathcal{C} \to s^s\mathcal{C}$ (see [11] [42]) we get a $G$-torsor

$$\sigma_*N(G//G) \to \sigma_*NG$$

in $s\mathcal{C}$. This $G$-torsor is denoted $WG \to \mathbf{W}G$. It can be shown (see [40]) that $WG = \text{Dec}_0\mathbf{W}G$ and that when $\mathcal{C} = \text{Set}$, we recover the usual description of the classifying complex $\mathbf{W}G$ of a simplicial group. Thus in this case $\mathbf{W}G$ is the simplicial set with exactly one 0-simplex and whose set of $n$-simplices, $n \geq 1$, is the set

$$(\mathbf{W}G)_n = G_{n-1} \times \cdots \times G_0$$

with face and degeneracy maps defined by the formulas

$$d_i(g_{n-1}, \ldots, g_0) = \begin{cases} (g_{n-2}, \ldots, g_0) & \text{if } i = 0, \\ (d_i(g_{n-1}), \ldots, d_1(g_{n-i+1}), g_{n-i-1}d_0(g_{n-i}), g_{n-i-2}, \ldots, g_0) & \text{if } 1 \leq i \leq n \end{cases}$$

and

$$s_i(g_{n-1}, \ldots, g_0) = \begin{cases} (1, g_{n-1}, \ldots, g_0) & \text{if } i = 0, \\ (s_i-1(g_{n-1}), \ldots, s_0(g_{n-i}), 1, g_{n-i-1}, \ldots, g_0) & \text{if } 1 \leq i \leq n. \end{cases}$$

respectively. When $G$ is a group in $s\mathcal{C}$ whose underlying simplicial object is constant, one can show that $\mathbf{W}G$ reduces to the usual classifying space $BG$ of $G$, thus $BG$ is the simplicial object in $\mathcal{C}$ whose object of $n$-simplices is the product $G \times \cdots \times G$ ($n$ factors) and whose face and degeneracy maps are defined using the product in $G$ and insertion of identities.

For the case of simplicial groups it is well known that $\mathbf{W}G$ is a fibrant simplicial set — see for instance Lemma 21.3 of [34]. In the next lemma we prove a more general statement.
Lemma 4.3. Let $G$ be group in $s\mathcal{C}/B$. Then $\overline{W}G$ is a locally fibrant simplicial object in $\mathcal{C}/B$.

Proof. We need to prove that $(\overline{W}G)_n \to M_{\Lambda^n_k} \overline{W}G$ is a local epimorphism for all $0 \leq k \leq n$ and all $n \geq 1$. Observe that if $k < n$ then

$$\Lambda^n_k = C\Lambda^{n-1}_k \cup \Lambda^{n-1}_k \Delta^{n-1}.$$

One way to see this is to write

$$\Lambda^{n-1}_k = \bigcup_{i=0 \atop i \neq k} d^i(\Delta^{n-2})$$

and recall that the functor $C(-) = (-) \star \Delta^0$ preserves unions (see [27]) so that

$$CA^{n-1}_k = \bigcup_{i=0 \atop i \neq k} d^i(\Delta^{n-2}) \star \Delta^0 = \bigcup_{i=0 \atop i \neq k} d^i(\Delta^{n-1}).$$

Therefore

$$CA^{n-1}_k \cup \Lambda^{n-1}_k \Delta^{n-1} = \bigcup_{i=0 \atop i \neq k} d^i(\Delta^{n-1}) \cup d^n(\Delta^{n-1}) = \bigcup_{i=0 \atop i \neq k} d^i(\Delta^{n-1}) = \Lambda^n_k.$$

Using this description of $\Lambda^n_k$ and Corollary 2.2 we have

$$M_{\Lambda^n_k} \overline{W}G = M_{CA^{n-1}_k \cup \Lambda^{n-1}_k \Delta^{n-1}} \overline{W}G$$

$$= M_{CA^{n-1}_k \overline{W}G} \times M_{\Lambda^{n-1}_k \overline{W}G} (\overline{W}G)_{n-1}$$

$$= M_{\Lambda^{n-1}_k W G} \times M_{\Lambda^{n-1}_k \overline{W}G} (\overline{W}G)_{n-1}.$$

Therefore the map $(\overline{W}G)_n \to M_{\Lambda^n_k} \overline{W}G$ is isomorphic to the map

$$(W G)_{n-1} \to M_{\Lambda^{n-1}_k W G} \times M_{\Lambda^{n-1}_k \overline{W}G} (\overline{W}G)_{n-1}$$

and so in this case the result follows from Lemma 4.1. For the case where $k = n$, note that we have

$$M_{\Lambda^n_n} \overline{W}G = M_{\Lambda^n_n} \overline{W}G^\circ,$$

where $G^\circ$ denotes the opposite simplicial object to $G$, so that $G^\circ$ is the composite of the functor $G: \Delta^{op} \to \mathcal{C}$ and the functor $\tau^{op}$, where $\tau: \Delta \to \Delta$ is the automorphism of $\Delta$ which reverses the order of each ordinal. Therefore the map $(\overline{W}G)_n \to M_{\Lambda^n_n} \overline{W}G$ is isomorphic to the map $(\overline{W}G^\circ)_n \to M_{\Lambda^n_n} \overline{W}G^\circ$ which we have just seen is a local epimorphism.

4.3 Non abelian cohomology

As mentioned in the introduction, the Čech cohomology $\check{H}^1(B, G)$ of a topological space $B$ with coefficients in a presheaf of groups $G$ on $B$ is traditionally thought of in terms of cocycles $g_{ij} \in G(U_i \cap U_j)$ relative to some open cover $U = (U_i)_{i \in I}$ of $B$. It is often more convenient however to package the data of such a cocycle $g_{ij}$ into a simplicial map

$$\check{C}(U) \to \overline{W}G$$
from the Čech resolution $\check{C}(U)$ of the open cover $U$ to the classifying simplicial presheaf $\overline{W}G$. The Čech resolution $\check{C}(U)$ is the simplicial presheaf on $B$ defined as follows. Let $rU_i$ denote the presheaf on $\text{Open}(B)$ represented by $U_i$, so that

$$rU_i(V) = \begin{cases} 1 & \text{if } V \subset U_i, \\ \emptyset & \text{otherwise} \end{cases}$$

for $V$ an open subset of $B$. Let $U$ also denote the coproduct $U = \coprod_{i \in I} rU_i$. Then $\check{C}(U)$ is the simplicial presheaf whose presheaf of $n$-simplices is

$$[n] \mapsto U \times U \times \cdots \times U \ (n + 1 \text{ factors}).$$

Thus the presheaf of 2-simplices of $\check{C}(U)$ is the coproduct $\coprod_{i,j \in I} r(U_i \cap U_j)$ and so on. It is not hard to see that a morphism of simplicial presheaves $\check{C}(U) \to \overline{W}G$ corresponds to the data of a family of sections $g_{ij} \in G(U_i \cap U_j)$ satisfying the cocycle equation $g_{ij}g_{jk} = g_{ik}$. We then have the following description of the Čech cohomology set $\check{H}^1(B, G)$ as the colimit over the set $\text{Cov}(B)$ of all covers $U$ of $B$

$$\check{H}^1(B, G) = \lim_{U \in \text{Cov}(B)} \pi(\check{C}(U), \overline{W}G)$$

of the sets $\pi(\check{C}(U), \overline{W}G)$ of simplicial homotopy classes of maps. The canonical map $\check{C}(U) \to 1$ to the terminal simplicial sheaf on $B$ is a hypercover and so we have a canonical map

$$\check{H}^1(B, G) \to \lim_{V \in \pi_0 \text{H}(B)} \pi(V, \overline{W}G).$$

It turns out that this map is an isomorphism. This fact was first observed in \cite{21} and (at this level of generality) rests on the fact that for any hypercover $V$, there is a bijection $\pi(V, \overline{W}G) \cong \pi(\text{cosk}_0 V, \overline{W}G)$. To prove this it is sufficient to check that the canonical map $V \to \text{cosk}_0 V$ is an isomorphism on fundamental groupoids, and by passing to stalks it is sufficient to check this for a contractible Kan complex, which is straightforward. In \cite{21} Jardine observes that one obtains an identification

$$\check{H}^1(B, G) \cong [1, \overline{W}G]$$

on applying the generalized Verdier hypercovering theorem (Theorem 3.3). In other words, Čech cohomology and so-called hyper-Čech cohomology coincide in this case. This identification opens the way to define higher order generalizations of the Čech cohomology $\check{H}^1(B, G)$. The following definition can be found in \cite{6} for the case when $G$ is 2-truncated.

**Definition 4.2** (\cite{6}). Let $B$ be a topological space and let $G$ be a presheaf of simplicial groups on $B$. Then we define

$$H(B, G) := [1, \overline{W}G]$$

where the square brackets $[ , ]$ denotes the set of morphisms in the homotopy category $\text{Ho}(s\text{Pre}(B)_{\mathbb{C}})$.

We will call $H(B, G)$ the **hyper-Čech cohomology** of $B$ with coefficients in $G$. Elements of $H(B, G)$ have the following explicit description using Jardine’s notion of a *cocycle category* \cite{22, 23}. A $G$-cocycle on $B$ consists of a locally fibrant, locally acyclic simplicial object $V$ in $\sK/B$, together with a simplicial map $V \to \overline{W}G$. A morphism of $G$-cocycles from $p: V \to \overline{W}G$ to $q: W \to \overline{W}G$ consists of a map $f: V \to W$ in $\sK/B$ such that $qf = p$. There is an obvious notion of composition of morphisms of $G$-cocycles so that $G$-cocycles form the objects of a category $H(B, \overline{W}G)$. It can be shown (using Theorem 3 of \cite{22}) that there is an isomorphism $\pi_0 H(B, \overline{W}G) = H(B, G)$.

When $G$ is the presheaf of constant simplicial groups associated to a presheaf of groups on $B$, the cohomology set $H(B, G)$ reduces to the familiar Čech cohomology set $\check{H}^1(B, G)$. This set $\check{H}^1(B, G)$ is of course a shadow of a richer structure, namely $\check{H}^1(B, G)$ is the set of path components of the groupoid of $G$-torsors on $B$. Similarly, $H(B, G)$ is also a shadow of a richer structure, in this case $H(B, G)$ is the set of
path components of the ∞-groupoid of \( G \)-torsors on \( B \). This ∞-groupoid can be explicitly described as the Kan complex \( \hom(1, RWG) \), where \( RWG \) is a fibrant replacement (i.e. (hyper-)∞-stackification) for \( WG \), and where \( \hom(-,-) \) denotes the simplicial enrichment for \( sPre(B) \).

One can define Čech versions \( \check{H}(B,G) \) of the sets \( H(B,G) \) by setting

\[
\check{H}(B,G) = \lim_{U \in \text{Cov}(B)} \pi(\check{C}(U),WG).
\]

As above we still have a canonical map

\[
\check{H}(B,G) \to H(B,G)
\]

which is essentially the comparison map between Čech and sheaf cohomology. It is well known that this map need not be an isomorphism. However in the special case when \( B \) is paracompact and \( G \) is \( k \)-truncated for some \( k \geq 0 \), in the sense that the sheaves of homotopy groups \( \pi_i(G) \) are trivial for \( i \geq k \), then it can be shown (using Lemma 7.2.3.5 of [30] for instance) that this comparison map is an isomorphism.

An interesting case of this theory is the following. Suppose that \( G \) is a presheaf of groups on \( B \). Then \( G \) determines a group object \( AUT(G) \) in the category of presheaves of groupoids on \( B \) whose presheaf of objects is the presheaf Aut(\( G \)) on \( B \), so that Aut(\( G \))(\( U \)) = Aut(\( G(U) \)) for \( U \) an open subset of \( B \). The presheaf of morphisms is the presheaf

\[
\text{Aut}(G) \ltimes G
\]
on \( B \) for the obvious left action of \( \text{Aut}(G) \) on \( G \). \( \text{AUT}(G) \) is an example of a 2-group or categorical group [2]. Applying the nerve construction \( N \) to the presheaf of groupoids \( \text{AUT}(G) \) yields a group in \( sPre(M) \) that we will also denote by \( \text{Aut}(G) \). We will write \( H^1(B,\text{AUT}(G)) \) for the corresponding cohomology set.

The set \( H^1(B,\text{AUT}(G)) \) parametrizes the set of equivalence classes of \( G \)-gerbes on \( B \). A \( G \)-gerbe on \( B \) is a generalization of the notion of \( G \)-torsor on \( B \). For more details we refer to [7]. In this case the group \( N\text{AUT}(G) \) is \( 2 \)-truncated so that if \( B \) is paracompact we have an isomorphism \( \check{H}^1(B,\text{AUT}(G)) = H^1(B,\text{AUT}(G)) \).

When \( G \) is the presheaf of groups on \( B \) represented a group \( G \) in \( \mathcal{X} \), there is a slight variant of this construction in which the sheaf of groups \( \text{Aut}(G) \) is replaced by the sheaf represented by the group Aut(\( G \)) in \( \mathcal{X} \). In this case one obtains a group \( N\text{AUT}_0(G) \) in \( sSh(B) \), represented by a group in \( s\mathcal{X}/B \), and one defines as above the set \( H(B,\text{AUT}_0(G)) \). Note that there is a canonical homomorphism \( \text{AUT}_0(G) \to \text{AUT}(G) \) but the induced map \( H(B,\text{AUT}_0(G)) \to H(B,\text{AUT}(G)) \) need not be an isomorphism.

## 5 Proof of the main result

Let us recall the statement of the main theorem of this paper.

**Theorem 1.1.** Let \( B \) be a paracompact, Hausdorff space and let \( G \) be a group in \( s\mathcal{X}/B \). If \( G_n \) is well sectioned for all \( n \geq 0 \) then there is a canonical isomorphism

\[
H(B,G) = H(B,|G|).
\]

In this section we will give a proof of this theorem. We begin by describing a universal \( G \)-cocycle.

### 5.1 The universal \( G \)-cocycle

Let \( G \) be a simplicial group in \( \mathcal{X}/B \). If we apply Illusie’s total décalage functor [2] to \( G \) we obtain a biaugmented bisimplicial group Dec(\( G \)) in \( \mathcal{X}/B \). The bisimplicial group Dec(\( G \)) plays a prominent role in Porter’s work [30] on homotopy \( n \)-types. We will think of Dec(\( G \)) as the simplicial simplicial group in \( \mathcal{X}/B \).
whose simplicial group of $p$-simplices in $s\mathcal{K}'/B$ is $\text{Dec}_p(G)$ (i.e. the $p$-th row of $\text{Dec}(G)$). It is helpful to keep the following picture in mind:

\[
\begin{array}{cccc}
G_2 & d_0 & \text{Dec}_2G \\
\uparrow & & \uparrow \\
G_1 & d_0 & \text{Dec}_1G \\
\uparrow & & \uparrow \\
G_0 & d_0 & \text{Dec}_0G \\
\uparrow & & \uparrow \\
& & G
\end{array}
\]

We can take fiberwise geometric realizations in the horizontal direction to get the following diagram of simplicial groups in $\mathcal{K}'/B$:

\[
\begin{array}{cccc}
G_2 & d_0 & |\text{Dec}_2G| \\
\uparrow & & \uparrow \\
G_1 & d_0 & |\text{Dec}_1G| \\
\uparrow & & \uparrow \\
G_0 & d_0 & |\text{Dec}_0G| \\
\uparrow & & \uparrow \\
& & |d_{\text{last}}| \\
\uparrow & & \uparrow \\
& & |G|
\end{array}
\]

We can apply the functor $\overline{W}$ to this diagram and obtain the following diagram of simplicial spaces

\[
\overline{W}G \rightarrow \overline{W}|\text{Dec}(G)| \rightarrow \overline{W}|G|
\]

We can now explain the main idea of our proof of Theorem 1.1. The diagram (9) induces a diagram

\[
\begin{array}{cccc}
H(B, G) & \longrightarrow & H(B, |\text{Dec} G|) \\
\downarrow & & \downarrow \\
H(B, |G|)
\end{array}
\]

so that we can compare $H(B, G)$ and $H(B, |G|)$ via $H(B, |\text{Dec} G|)$. We will prove that each map in this diagram is an isomorphism with the hypotheses on $G$ and $B$. It is easy to check that if $G$ is an abelian group object in $s\mathcal{K}'/B$ then the maps in the diagram above are homomorphisms of abelian groups.
To prove that the maps in the above diagram are isomorphisms we will prove the following two propositions. After we had completed this paper, we discovered that a result related to, but less general than Proposition 5.1 below had been proven earlier by Moerdijk — see 3.7 of [37].

**Proposition 5.1.** Let $G$ be a well sectioned group in $s\mathcal{K}/B$. Then the map

\[ \mathcal{W}|\text{Dec}G| \to \mathcal{W}|G| \]

is a locally acyclic local fibration in $s\mathcal{K}/B$.

It follows that the induced map $\mathcal{W}|\text{Dec}G| \to \mathcal{W}|G|$ between representable simplicial presheaves on $B$ is a locally acyclic local fibration in $s\text{Pre}(B)$, and therefore in particular is a weak equivalence in the localized projective model structure on $s\text{Pre}(B)$. Therefore the map

\[ H(B,|\text{Dec}G|) \to H(B,|G|) \]

is an isomorphism. Theorem 1.1 then follows from the next proposition.

**Proposition 5.2.** Suppose that $B$ is paracompact and that $G$ is a well sectioned group in $s\mathcal{K}/B$. Then the induced map

\[ H(B,|\text{Dec}G|) \to H(B,G) \]

is an isomorphism.

Observe that the diagram (9) gives rise to a $G$-cocycle on the classifying space $B|G|$ using the canonical map $\hat{\mathcal{C}}(E|G|) \to \mathcal{W}|G|$. We call this $G$-cocycle the *universal $G$-cocycle* for the following reason: any $G$-cocycle over $B$ is equivalent to a $G$-cocycle obtained by pulling back the universal $G$-cocycle along a map $B \to B|G|$ unique up to homotopy. To see this observe that the isomorphism $\hat{H}^1(M,|G|) \cong H(M,G)$ is realized by the map which sends a $|G|$-cocycle on $B$ determined by a map $\hat{\mathcal{C}}(P) \to \mathcal{W}|G|$, where $P \to B$ is a principal $G$-bundle in $\mathcal{K}/B$, to the map $X \to \mathcal{W}G$, where $X$ is defined by the pullback diagram

\[
\begin{array}{ccc}
X & \longrightarrow & \mathcal{W}|\text{Dec}G| \\
\downarrow & & \downarrow \\
\hat{\mathcal{C}}(P) & \longrightarrow & \mathcal{W}|G|,
\end{array}
\]

and $X \to \mathcal{W}G$ is the canonical map induced by $\mathcal{W}|\text{Dec}G| \to \mathcal{W}G$. Since the map $\hat{\mathcal{C}}(P) \to \mathcal{W}|G|$ factors through $\hat{\mathcal{C}}(E|G|)$ via a classifying map $P \to E|G|$ for the bundle $P$, it follows that the cocycle $X \to \mathcal{W}G$ is the pullback of the universal $G$-cocycle under the map $B \to B|G|$. Theorem 1.1 implies that any $G$-cocycle on $B$ is equivalent to a cocycle of this form.

### 5.2 Proof of Proposition 5.1

To prove Proposition 5.1 it suffices to prove that $|\text{Dec}G| \to |G|$ is a locally acyclic local fibration, since $\mathcal{W}$ preserves surjective local fibrations and local weak equivalences (it suffices to check this on stalks, and it is well known that the analogous statements are true for simplicial sets). Therefore, in view of Proposition 3.2 it suffices to prove three things:

1. the map $|\text{Dec}_0G| \to |G|$ admits local sections,
2. the map $|\text{Dec}_1G| \to |\text{Dec}_0G| \times_{|G|} |\text{Dec}_0G|$ admits local sections,
3. the map $|\text{Dec}_nG| \to M_{\partial\Delta^n}|\text{Dec}G|$ is an effective epimorphism for all $n \geq 2$.

The main technical tool that we will use in proving these results is the following theorem from [40].
Theorem 5.1 (§3). Let $G$ be a group in $s\mathcal{K}/B$. Suppose that $P \to M$ is a locally trivial $G$-torsor in $s\mathcal{K}/B$, such that $P_n \to M_n$ is a numerable principal $G_n$ bundle in $\mathcal{K}/B$ for all $n \geq 0$. If $M$ is a proper simplicial object, then the induced map
\[ |P| \to |M| \]
on fiberwise geometric realizations is a locally trivial principal $|G|$ bundle in $\mathcal{K}/B$.

Recall that a simplicial object $X$ in $s\mathcal{K}/B$ is said to be proper if the inclusion $L_n X \subset X_{n+1}$ is an $f$-cofibration in $\mathcal{K}/B$ for all $n \geq 0$. In other words $X$ is proper if the inclusion $(X_{n+1}, sX_n)$ is a fiberwise NDR pair for all $n \geq 0$ where $sX_n = \cup_{i=0}^n s_i X_n$.

We need a recognition theorem to identify when a simplicial object in $\mathcal{K}/B$ is proper. It is sometimes easier to recognize when a simplicial object $X$ in $\mathcal{K}/B$ is good in the sense that $s_i: X_n \to X_{n+1}$ is a $f$-cofibration in $\mathcal{K}/B$ for all $0 \leq i \leq n$ and all $n \geq 0$. The following result from [40] (which is essentially folklore) provides a useful recognition criterion.

Proposition 5.3 (§4). If $X$ is a good simplicial object in $\mathcal{K}/B$, then $X$ is proper. If $G$ is a well sectioned group in $s\mathcal{K}/B$, then $G$ is good and hence proper.

Lemma 5.1. Let $G$ be a well sectioned group in $s\mathcal{K}/B$. Then the fiberwise geometric realization
\[ |d_{last}|: |Dec_0 G| \to |G| \]
is the projection in a numerable, locally trivial principal $|\ker(d_{last})|$ bundle on $|G|$. In particular $|d_{last}|$ admits local sections.

Proof. The short exact sequence of groups $1 \to \ker(d_{last}) \to Dec_0 G \to G \to 1$ in $s\mathcal{K}/B$ makes it clear that $d_{last}: Dec_0 G \to G$ has the structure of a ker($d_{last}$) torsor in $\mathcal{K}/B$: the action of ker($d_{last}$) on Dec$_0 G$ is the obvious one and the diagram

\[
\begin{array}{ccc}
Dec_0 G \times \ker(d_{last}) & \longrightarrow & Dec_0 G \\
\downarrow & & \downarrow \\
Dec_0 G & \longrightarrow & G
\end{array}
\]

is clearly a pullback in $s\mathcal{K}/B$. We need to check that each $(Dec_0 G)_n \to G_n$ is a numerable ker($d_{n+1}$) bundle in $\mathcal{K}/B$. This is clear however, since the map $d_{n+1}: (Dec_0 G)_n \to G_n$ has a section, and hence is a trivial bundle. We now wish to apply Theorem 5.1 to conclude that $|Dec_0 G| \to |G|$ is a locally trivial principal bundle in $\mathcal{K}/B$, and so we need to know that $G$ is proper. This follows from Proposition 5.3. $G$ is well sectioned, hence good, and hence proper. \qed

We now turn our attention to item 2. We have the following lemma.

Lemma 5.2. Let $G$ be a well sectioned group in $s\mathcal{K}/B$. Then the fiberwise geometric realization
\[ |Dec_1 G| \to |Dec_0 G| \times_{|G|} |Dec_0 G| \]
is the projection in a locally trivial principal bundle in $\mathcal{K}/B$, and hence admits local sections.

Proof. We need to verify the hypotheses from Theorem 5.1. We first check that each map
\[(Dec_1 G)_n \to (Dec_0 G)_n \times_{G_n} (Dec_0 G)_n \quad (10)\]
is the projection in a numerable fiberwise principal bundle. To do this we will show that each of these maps has a section. The map (10) is the map
\[ G_{n+2} \to G_{n+1} \times_{G_n} G_{n+1} \]
\[ g \mapsto (d_{n+1}(g), d_{n+2}(g)). \]
Suppose that \((x, y) \in G_{n+1} \times_{G_n} G_{n+1}\) so that \(d_{n+1}(x) = d_{n+1}(y)\). Then it is easy to check that
\[
\sigma(x, y) = s_n(xy^{-1})s_{n+1}(y)
\]
satisfies \(d_{n+1}(\sigma(x, y)) = x\) and \(d_{n+2}(\sigma(x, y)) = y\). Hence (10) is the projection in a trivial fiberwise principal bundle.

To satisfy the hypotheses for Theorem 5.1 we need to check that \(\text{Dec}_G \times_G \text{Dec}_G\) is a proper simplicial object. Observe that there is an isomorphism \(\text{Dec}_G \times_G \text{Dec}_G \cong \text{Dec}_G \times_B \ker(d_{\text{last}})\). It suffices to prove that \(\text{Dec}_G \times_B \ker(d_{\text{last}})\) is well sectioned. Clearly \(\text{Dec}_G\) is well sectioned; since \(\ker(d_{\text{last}})\) is a retract of \(\text{Dec}_G\) we see that \(\ker(d_{\text{last}})\) is also well sectioned. Therefore the result follows from the fact that if \(A \to X\) and \(A' \to X'\) are \(\bar{f}\)-cofibrations in \(\mathcal{K}/B\) then \(A \times_B A' \to X \times_B X'\) is also an \(\bar{f}\)-cofibration (see [10]).

**Lemma 5.3.** The map \(|\text{Dec}_n G| \to M_{\Delta^n} \cdot |\text{Dec} G|\) is an effective epimorphism for all \(n \geq 2\).

**Proof.** Observe that since fiberwise geometric realization preserves finite limits, it suffices to prove that
\[
|\text{Dec}_n G| \to |M_{\Delta^n} \cdot \text{Dec} G|.
\]
is an effective epimorphism in \(\mathcal{K}/B\). Since fiberwise geometric realization preserves colimits, and colimits in \(s\mathcal{K}/B\) are computed pointwise, it suffices to prove that
\[
(\text{Dec}_n G)_m \to (M_{\Delta^n} \cdot \text{Dec} G)_m
\]
is an effective epimorphism in \(\mathcal{K}/B\) for all \(m \geq 0\) and all \(n \geq 2\). Since colimits in \(\mathcal{K}/B\) are computed in \(\mathcal{K}\) and then equipped with the canonical map to \(B\), it is sufficient to prove that (11) is an effective epimorphism in \(\mathcal{K}\) for all \(m \geq 0\) and all \(n \geq 2\). Since colimits in \(\mathcal{K}\) are computed in \(\text{Top}\) it is sufficient to prove that the diagram
\[
(\text{Dec}_n G)_m \times (M_{\Delta^n} \cdot \text{Dec} G)_m \to (\text{Dec}_n G)_m \Rightarrow (\text{Dec}_n G)_m \to (M_{\Delta^n} \cdot \text{Dec} G)_m
\]
is a coequalizer in \(\text{Top}\) for all \(m \geq 0\) and all \(n \geq 2\). Therefore it is sufficient to prove that (11) is a surjective quotient map for all \(m \geq 0\) and all \(n \geq 2\).

Clearly
\[
(M_{\Delta^n} \cdot \text{Dec} G)_m = M_{\Delta^n} M_{\Delta^n} \cdot \text{Dec} G = M_{\Delta^n} \cdot \Delta G,
\]
using [4]. Therefore, Proposition 2.1 shows that the map (11) is isomorphic to the map
\[
G_{n+m+1} \to M_{\Delta^n \cdot \Delta G},
\]
where \(\Delta^m \star \partial \Delta^n\) denotes the join of the simplicial sets \(\Delta^m\) and \(\partial \Delta^n\). The join \(\Delta^m \star \partial \Delta^n\) is computed in [27] to be equal to
\[
\bigcup_{j=0}^{n} \partial_{j+m+1} \Delta^{n+m+1}.
\]
It is easy to see that the geometric realization of \(\Delta^m \star \partial \Delta^n\) contracts onto its final vertex and therefore the inclusion \(\Delta^m \star \partial \Delta^n \to \Delta^{n+m+1}\) is an acyclic cofibration of simplicial groups. Therefore the map \(G_{n+m+1} \to M_{\Delta^n \cdot \Delta G}\) admits local sections by Lemma 5.4 and hence is a surjective quotient map.

### 5.3 Proof of Proposition 5.2

We now consider the map \(\overline{W} |\text{Dec} G| \to \overline{W} G\) from (9). This is induced by the homomorphism of groups \(|\text{Dec} G| \to G\) in \(s\mathcal{K}/B\) which is in turn induced by the homomorphism of simplicial groups \(d_{\text{first}} : \text{Dec} G \to G\) in \(s\mathcal{K}/B\) which in degree \(n\) is given by the augmentation homomorphism
\[
d_{\text{first}} : \text{Dec}_n G \to G_n.
\]
Since this augmentation homomorphism can be re-written as the augmentation homomorphism
\[ \text{Dec}_0(\text{Dec}_{n-1}G) \to (\text{Dec}_{n-1}G)_0 \]
it is enough to study the homomorphism \( d_0 : \text{Dec}_0G \to G_0 \) for any group \( G \). For this we have the following lemma.

**Lemma 5.4.** Let \( G \) be a well sectioned group in \( s\mathcal{K}/B \). Then the following are true.

1. \( \text{Dec}_0G \) is a split augmented group in \( s\mathcal{K}/B \) with augmentation \( d_0 : \text{Dec}_0G \to G_0 \) and splitting \( s_0 : G_0 \to G_1 \).

2. The group \( \ker(d_0) \) is well sectioned in \( s\mathcal{K}/B \).

3. The short exact sequence of groups in \( \mathcal{K}/B \)
\[ 1 \to |\ker(d_0)| \to |\text{Dec}_0G| \to G_0 \to 1 \]

obtained by applying the fiberwise geometric realization functor \(|\cdot|\) is split exact and \(|\ker(d_0)|\) is fiberwise contractible in \( \mathcal{K}/B \).

**Proof.** The first statement of the lemma is clear. For the second statement, observe that \( \ker(d_0) \) is a retract of \( G \), and hence is well sectioned since \( \mathcal{f}-\text{cofibrations are closed under retracts.} \)
Finally, for the last statement, we prove that \(|\ker(d_0)|\) is fiberwise contractible in \( \mathcal{K}/B \) (the split exactness is clear from 1). To accomplish this we will prove that the simplicial object \( \ker(d_0) \) in \( \mathcal{K}/B \) has extra degeneracies. In fact, because of the simplicial identity \( d_0s_{n+1} = s_ns_0 \) for \( i < j \), the left over degeneracy map \( s_{n+1} : (\text{Dec}_0G)_n \to (\text{Dec}_0G)_{n+1} \) restricts to define a homomorphism
\[ s_{n+1} : (\ker(d_0))_n \to (\ker(d_0))_{n+1} \]
for all \( n \geq 0 \). One can therefore define a simplicial homotopy \( h : \ker(d_0) \otimes \Delta^1 \to \ker(d_0) \) which fits into a commutative diagram

\[
\begin{array}{ccc}
\ker(d_0) & \xrightarrow{1} & \ker(d_0) \\
\downarrow & & \downarrow \\
\ker(d_0) \otimes \Delta^1 & \xrightarrow{h} & \ker(d_0) \\
\downarrow & & \downarrow \text{id} \\
\ker(d_0) & & \ker(d_0)
\end{array}
\]

Applying the fiberwise geometric realization functor \(|\cdot|\), and using the fact that \(|\cdot| : s\mathcal{K}/B \to \mathcal{K}/B \) is compatible with the geometric realization functor \(|\cdot| : s\text{Set} \to \mathcal{K}\) in the sense that there is an isomorphism \( |X \otimes K| \cong |X| \otimes |K| \) for all objects \( X \) of \( s\mathcal{K}/B \) and simplicial sets \( K \), we see that \( h \) induces a fiberwise contraction of \( \ker(d_0) \) onto \( B \).

Write \( p \) for the homomorphism \( \text{Dec}G \to G \). From the lemma we see that \( p_n : \text{Dec}_nG \to G_n \) is split for all \( n \geq 0 \) and hence the induced homomorphism \( |\text{Dec}_nG| \to G_n \) is also split. Therefore Corollary 3.2 implies that \(|\text{Dec}G| \to G \) is a local fibration in \( s\mathcal{K}/B \). The following lemma is an immediate consequence of this, using the well known fact that \( \overline{\text{W}} : sGp \to s\text{Set} \) sends surjective Kan fibrations to Kan fibrations.

**Lemma 5.5.** The induced map \( p : \overline{\text{W}}[\text{Dec}G] \to \overline{\text{W}}G \) is a local fibration of simplicial presheaves on \( B \).

Since \( p \) is a local fibration with fiber \( \overline{\text{W}} \ker(p) \) we have (see [8]) an exact sequence of sets
\[ [1, \overline{\text{W}}|\ker(p)|] \to [1, \overline{\text{W}}|\text{Dec}G|] \to [1, \overline{\text{W}}G] \tag{12} \]

Recall from Lemma 5.4 above that for every \( n \geq 0 \) the group \(|\ker(p_n)|\) in \( s\mathcal{K}/B \) is fiberwise contractible and well sectioned. Therefore, from the exactness of the sequence (12) we see that to prove the map \( \overline{\text{W}}[\text{Dec}G] \to \overline{\text{W}}G \) induces an isomorphism \([1, \overline{\text{W}}|\text{Dec}G|] \to [1, \overline{\text{W}}G]\) it is enough to prove the following lemmas.
Lemma 5.6. Let $G$ be a group in $s\mathcal{K}/B$. Then the map
\[ [1, \underline{W}|\text{Dec } G|] \to [1, \underline{W} G]. \]
in $\text{Ho}(s\text{Pre}(B))$ induced by the map $\underline{W}|\text{Dec } G| \to \underline{W} G$ in $s\mathcal{K}/B$ is surjective.

Lemma 5.7. Let $G$ be a well sectioned group in $s\mathcal{K}/B$ such that $G_n$ is fiberwise contractible for all $n \geq 0$. Suppose that $B$ is paracompact and Hausdorff. Then $[1, \underline{W} G]$ is trivial.

Proof of Lemma 5.6. We have seen above that the map $|\text{Dec } G| \to G$ is degree-wise a split homomorphism of groups. It follows that the corresponding homomorphism of groups $|\text{Dec } G| \to G$ in $s\text{Pre}(B)$ is object-wise surjective, in the sense that for each open set $U \subset B$, the homomorphism of simplicial groups $|\text{Dec } G|(U) \to G(U)$ is surjective. Therefore $\underline{W}|\text{Dec } G| \to \underline{W} G$ is a fibration in the projective model structure on $s\text{Pre}(B)$ and the result follows from Lemma 3.1.

Proof of Lemma 5.7. From Proposition 5.1 above we see that for any well sectioned simplicial group $G$ in $\mathcal{K}/B$ there is an isomorphism
\[ [1, \underline{W}|\text{Dec } G|] \cong [1, \underline{W} G] \]
and we have just seen that there is a surjection
\[ [1, \underline{W}|\text{Dec } G|] \to [1, \underline{W} G]. \] (13)

Consider the set $[1, \underline{W} G]$. Earlier we have recalled how this is shown in [24] to be isomorphic to the Čech cohomology set $H^1(B, |G|)$. Since $G_n$ is fiberwise contractible for all $n \geq 0$, $G$ is good and the constant simplicial object $B$ is good, it follows that $|G|$ is fiberwise contractible. Hence every principal $|G|$ bundle on the paracompact space $B$ is trivial and so $[1, \underline{W} G]$ is trivial. Therefore $[1, \underline{W}|\text{Dec } G|]$ is trivial. Since the map (13) is surjective it follows that $[1, \underline{W} G]$ is trivial as required.

Acknowledgements

I foremost would like to thank Urs Schreiber for his encouragement, and for many helpful email discussions as well as for his comments on earlier drafts of these notes. I would also like to thank Tom Leinster and Thomas Nikolaus for useful conversations. Finally, I would like to thank John Baez for introducing me to simplicial homotopy theory.

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