The Shi arrangement of the type $D_\ell$

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Abstract: In this paper, we give a basis for the derivation module of the cone over the Shi arrangement of the type $D_\ell$ explicitly.

Key words: Hyperplane arrangement; Shi arrangement; Free arrangement.

1. Introduction. Let $V$ be an $\ell$-dimensional vector space. An affine arrangement $A$ is a finite collection of affine hyperplanes in $V$. If every hyperplane $H \in A$ goes through the origin, then $A$ is called to be central. When $A$ is central, for each $H \in A$, choose $\alpha_H \in V^*$ with $\ker(\alpha_H) = H$. Let $S$ be the algebra of polynomial functions on $V$ and let $\text{Der}_S$ be the module of derivations

$$\text{Der}_S := \{\theta : S \to S \mid \theta(fg) = f\theta(g) + g\theta(f), f, g \in S, \theta \text{ is } \mathbb{R}\text{-linear}\}.$$ 

For a central arrangement $A$, recall

$$D(A) := \{\theta \in \text{Der}_S \mid \theta(\alpha_H) \in \alpha_H S \text{ for all } H \in A\}.$$ 

We say that $A$ is a free arrangement if $D(A)$ is a free $S$-module. The freeness was defined in $[15]$. The Factorization Theorem $[16]$ states that, for any free arrangement $A$, the characteristic polynomial of $A$ factors completely over the integers.

Let $E = \mathbb{R}^\ell$ be an $\ell$-dimensional Euclidean space with a coordinate system $x_1, \ldots, x_\ell$, and $\Phi$ be a crystallographic irreducible root system. Fix a positive root system $\Phi^+ \subset \Phi$. For each positive root $\alpha \in \Phi^+$ and $k \in \mathbb{Z}$, we define an affine hyperplane

$$H_{\alpha,k} := \{v \in V \mid \langle \alpha, v \rangle = k\}.$$ 

In $[10]$, J.-Y. Shi introduced the Shi arrangement

$$S(A_\ell) := \{H_{\alpha,k} \mid \alpha \in \Phi^+, 0 \leq k \leq 1\}$$

when the root system is of the type $A_\ell$. This definition was later extended to the generalized Shi arrangement (e.g., $[3]$)

Embed $E$ into $V = \mathbb{R}^{\ell+1}$ by adding a new coordinate $z$ such that $E$ is defined by the equation $z = 1$ in $V$. Then, as in $[7]$, we have the cone $\mathbf{cS}(\Phi)$ of $S(\Phi)$

$$\mathbf{cS}(\Phi) := \{cH_{\alpha,k} \mid \alpha \in \Phi^+, 0 \leq k \leq 1\} \cup \{\{z = 0\}\}.$$ 

In $[13]$, M. Yoshinaga proved that the cone $\mathbf{cS}(\Phi)$ is a free arrangement with exponents $(1, h, \ldots, h)$ ($h$ appears $\ell$ times), where $h$ is the Coxeter number of $\Phi$. (He actually verified the conjecture by P. Edelman and V. Reiner in $[3]$, which is far more general.) He proved the freeness without finding a basis.

In $[13]$, for the first time, the authors gave an explicit construction of a basis for $D(\mathbf{cS}(A_\ell))$. Then D. Suyama constructed bases for $D(\mathbf{cS}(B_\ell))$ and $D(\mathbf{cS}(C_\ell))$ in $[14]$. In this paper, we will give an explicit construction of a basis for $D(\mathbf{cS}(D_\ell))$. A defining polynomial of the cone over the Shi arrangement of the type $D_\ell$ is given by

$$Q := z \prod_{1 \leq s < t \leq \ell-1} \prod_{\epsilon = -1, 1} (x_s + c\epsilon x_t - z)(x_s + c\epsilon x_t).$$

Note that the number of hyperplanes in $\mathbf{cS}(D_\ell)$ is equal to $2(\ell - 1) + 1$. Our construction is similar to the construction in the case of the type $B_\ell$. The essential ingredients of the recipe are the Bernoulli polynomials and their relatives.

2. The basis construction.

Proposition 2.1. For $(p, q) \in \mathbb{Z}_{\geq -1} \times \mathbb{Z}_{\geq 0}$, consider the following two conditions for a rational function $B_{p,q}(x)$:

1. $B_{p,q}(x+1) - B_{p,q}(x) = \frac{(x+1)^p + (-x)^p}{(x+1) - (-x)} - (x+1)^q(-x)^q$,
2. $B_{p,q}(-x) = -B_{p,q}(x)$.
Then such a rational function $B_{p,q}(x)$ uniquely exists. Moreover, the $B_{p,q}(x)$ is a polynomial unless $(p,q) = (-1,0)$ and $B_{-1,0}(x) = -(1/x)$.

Proof. Suppose $(p,q) \neq (-1,0)$. Since the right hand side of the first condition is a polynomial in $x$, there exists a polynomial $B_{p,q}(x)$ satisfying the first condition. Note that $B_{p,q}(x)$ is unique up to a constant term. Define a polynomial $F(x) = B_{p,q}(x) + B_{p,q}(-x)$. Since

$$B_{p,q}(x) - B_{p,q}(-x - 1) = (-x)^p - (x + 1)^p = (x + 1)^p - (-x)^p$$

$$= \frac{(x+1)^p - (-x)^p}{(x+1) - (-x)}(x+1)^q(-x)^q$$

we have $F(x+1) = F(x)$ for any $x$. Therefore $F(x)$ is a constant function. Then the polynomial $B_{p,q}(x) - (F(0)/2)$ is the unique solution satisfying the both conditions. Next we suppose $(p,q) = (-1,0)$. Then we compute

$$B_{-1,0}(x+1) - B_{-1,0}(x) = (x+1)^{-1} - (-x)^{-1} = -\frac{1}{x+1} + \frac{1}{x}.$$ 

Thus $B_{-1,0}(x) = -(1/x)$ is the unique solution satisfying the both conditions. \[\square\]

Definition 2.2. Define a rational function \( \mathcal{B}_{p,q}(x,z) \) in $x$ and $z$ by

$$\mathcal{B}_{p,q}(x,z) := z^{p+2q}B_{p,q}(x/z).$$

Then $\mathcal{B}_{p,q}(x,z)$ is a homogeneous polynomial of degree $p+2q$ except the two cases: $\mathcal{B}_{-1,0}(x,z) = -(1/x)$ and $\mathcal{B}_{0,q}(x,z) = 0$.

For a set $I := \{y_1, \ldots, y_m\}$ of variables, let

$$\sigma^n_I := \sigma_n(y_1, \ldots, y_m), \quad \tau^n_I := \sigma_n(y_1', \ldots, y_m'),$$

where $\sigma_n$ stands for the elementary symmetric function of degree $n$.

Definition 2.3. Define derivations

$$\varphi_j := (x_j - x_{j+1} - z) \sum_{i=1}^{\ell} \sum_{K_1 \cup K_2 \subseteq J, K_1 \cap K_2 = \emptyset} \left( \prod_{K_1} K_1 \right) \left( \prod_{K_2} K_2 \right) z^{\left| K_1 \right|} \frac{\partial}{\partial x_i}$$

for $j = 1, \ldots, \ell - 1$ and

$$\varphi_\ell := \sum_{i=1}^{\ell} \sum_{K_1 \cup K_2 \subseteq J, K_1 \cap K_2 = \emptyset} \left( \prod_{K_1} K_1 \right) \left( \prod_{K_2} K_2 \right) z^{\left| K_1 \right|} \frac{\partial}{\partial x_i} \theta_{\mathcal{E}}$$

for $j = \ell$, where

$$J := \{x_1, \ldots, x_{j-1}, J_1 := \{x_j, x_{j+1}\}, J_2 := \{x_{j+2}, \ldots, x_{\ell}\},$$

$$\prod_{K_1} K_1 := \prod_{x_i \in K_1} x_i, \quad p = 1, 2, \quad k_0 := |J \setminus (K_1 \cup K_2)| \geq 0, \quad k := (|J_1| - n_1) + 2(|J_2| - n_2) - 1 \geq -1.$$  

Note that $\varphi_j(z) = 0$ $(1 \leq j \leq \ell)$. In the rest of the paper, we will give a proof of the following theorem:

Theorem 2.4. The derivations $\varphi_1, \ldots, \varphi_\ell$, together with the Euler derivation

$$\theta_E := z \frac{\partial}{\partial z} + \sum_{i=1}^{\ell} x_i \frac{\partial}{\partial x_i},$$

form a basis for $D(\mathfrak{cS}(D_1))$.

Note that $\theta_E(x_i) = x_i$ $(1 \leq i \leq \ell)$ and $\theta_E(z) = z$.

Lemma 2.5. Let $1 \leq i \leq \ell$ and $1 \leq j \leq \ell$. Suppose $\varphi_j(x_i)$ is nonzero. Then $\varphi_j(x_i)$ is a homogeneous polynomial of degree $2(\ell - 1)$.  

Proof. Define

$$F_{ij} := (x_j - x_{j+1} - z) \left( \prod_{K_1} K_1 \right) \left( \prod_{K_2} K_2 \right) z^{\left| K_1 \right|} \sigma_{n_1}^{r_1} \tau_{n_2}^{r_2} \mathcal{B}_{k,k_0}(x_i, z)$$

for $1 \leq j \leq \ell - 1$, and

$$F_{i\ell} := \left( \prod_{K_1} K_1 \right) \left( \prod_{K_2} K_2 \right) z^{\left| K_1 \right|} x_\ell \mathcal{B}_{-1,0}(x_i, z)$$

when $K_1, K_2, n_1, n_2$ are fixed. Then $\varphi_j(x_i)$ is a linear combination of the $F_{ij}$’s over $\mathbb{R}$.

Note that $\mathcal{B}_{k,k_0}(x_i, z)$ is a polynomial unless $(k, k_0) = (-1,0)$.

Assume that $1 \leq j \leq \ell - 1$ and $(k, k_0) = (-1,0)$. Then $J = K_1 \cup K_2, n_1 = |J_1|, n_2 = |J_2|$, and $\mathcal{B}_{-1,0}(x_i, z) = -1/x_i$. Therefore each $F_{ij}$ is a polynomial. Thus $\varphi_j(x_i)$ is a nonzero polynomial and there exists a nonzero polynomial $F_{ij}$. Compute

$$\deg \varphi_j(x_i) = \deg F_{ij}$$

$$= 1 + |K_1| + 2|K_2| + |K_1| + n_1 + 2n_2.$$
and thus has

\[ f(n) = \sum (\prod_{i=1}^{k} a_i)^2 \cdot (\prod_{j=1}^{k} b_j)^2 \cdot (\prod_{k=1}^{n} c_k) \]

Therefore each \( F_{\ell} \) is a polynomial. Thus so is \( \varphi_t(x_i) \). Compute

\[
\deg \varphi_t(x_i) = \frac{1}{2} (|K_1| + 2|K_2| + |K_1| + 1 + \deg B_{-1,k_0}(x_i, z))
\]

where \( K_1, K_2, n_1, n_2 \) are fixed. Let \( \deg(x_i) \) denote the degree of \( f \) with respect to \( x_i \) when \( f \neq 0 \).

(1) Since, for every nonzero \( F_{ij} \), we obtain

\[
\deg(x_i) F_{ij} \leq 2 \quad (1 \leq p < i), \quad \deg(F_{ij}) = 2 \ell - 2.
\]

Hence we may conclude

\[
\in(F_{ij}) \leq x_i \cdots x_{i-1}^2 x_i^{2\ell-2i}
\]

and thus

\[
\in(\varphi_t(x_i)) \leq \max \{ \in(F_{ij}) \} \leq x_i \cdots x_{i-1}^2 x_i^{2\ell-2i}.
\]

(2) Suppose \( i < j \leq \ell \). Since \( x_i > x_j > z \), one has

\[
in(\sigma_{n_1}^{J_1} \tau_{n_2}^{J_2} B_{k_0}(x_i, z))
\]

\[
\leq x_i^{2\ell-2j+2k_0-1}
\]

when \( B_{k_0}(x_i, z) \) is nonzero. The equality holds if and only if \( n_1 = n_2 = 0 \).

Suppose that \( F_{ij} \) is nonzero. For \( 1 \leq i < j \leq \ell - 1 \), we have

\[
in(F_{ij}) = x_i \cdots x_{i-1} + x_i \cdots x_{i-1} x_i^{2\ell-2i}
\]

Thus

\[
in(\varphi_t(x_i)) = x_i \cdots x_{i-1} x_i^{2\ell-2i}.
\]

For \( 1 \leq i < j = \ell \),

\[
in(F_{ij}) = x_i \cdots x_{i-1} x_i^{2\ell-2i}.
\]

This proves (2).

Now we only need to prove (3). Let \( i < j \leq \ell \) in (3). Then the equality

\[
in(F_{ij}) = x_i \cdots x_{i-1} x_i^{2\ell-2i}
\]

holds if and only if

\[
K_{1} = \emptyset, \quad K_{2} = J, \quad n_{1} = n_{2} = k_{0} = 0, \quad k = 2\ell - 2i - 1
\]

because the leading term of \( B_{2\ell-2i-1,0}(x_i, z) \) is equal to

\[
x^{2\ell-2i-1}
\]

Next let \( i = \ell \) in (3). Then the equality

\[
in(F_{\ell i}) = x_i \cdots x_{i-1} x_i^{2\ell-2i}
\]

holds if and only if

\[
K_{1} = \emptyset, \quad K_{2} = J = \{ x_{1}, \ldots, x_{\ell-1} \}, \quad k_{0} = 0.
\]

Therefore, for \( 1 \leq i \leq \ell \),

\[
in(\varphi_t(x_i)) = x_i \cdots x_{i-1} x_i^{2\ell-2i}.
\]
Corollary 2.7. (1) \[ \text{in}(\det [\varphi_j(x_i)]) = \ell \prod_{i=1}^{\ell} \text{in}(\varphi_i(x_i)) = \prod_{i=1}^{\ell-1} x_i^{4(\ell-1)}. \]

(2) Moreover, the leading term of \( \det [\varphi_j(x_i)] \) is equal to
\[ \frac{1}{(2\ell - 3)!} \prod_{i=1}^{\ell-1} x_i^{4(\ell-1)}. \]

(3) In particular, \( \det [\varphi_j(x_i)] \) does not vanish.

Next, we will prove \( \varphi_j \in D(S(D_\ell)) \) for \( 1 \leq j \leq \ell \). We denote \( S(D_\ell) \) simply by \( S_\ell \) from now on. Before the proof, we need the following two lemmas:

Lemma 2.8. Fix \( 1 \leq j \leq \ell - 1 \) and \( \epsilon \in \{ -1, 1 \} \). Then
(1) \[ \prod_{x_i \in J} (x_i - x_s)(x_i - \epsilon x_t) = \sum_{K_1 \cup K_2 \subseteq J \atop K_1 \cap K_2 = \emptyset} (\prod K_1) \times (\prod K_2)^2 \frac{(-x_s + \epsilon x_t)^{|K_1|}}{|K_1|!} (\epsilon x_s x_t)^{|K_2|}. \]

(2) \[ \sum_{0 \leq n_1 \leq |J_1|, 0 \leq n_2 \leq |J_2|} (-1)^{|J_1| + |J_2| - n_1 - n_2 \sigma_{n_1} J_1 \times J_2} (\epsilon x_s)^{k+1} \]
\[ = \prod_{x_i \in J_1} (x_i - x_s) \prod_{x_i \in J_2} (x_i^2 - x_s^2). \]

Proof. (1) is easy because the left hand side is equal to
\[ \prod_{x_i \in J} (x_i^2 - (x_s + \epsilon x_t)x_i + \epsilon x_s x_t). \]

(2) The left hand side is equal to
\[ \sum_{0 \leq n_1 \leq |J_1|} (-\epsilon x_s)^{|J_1| - n_1 \sigma_{n_1} J_1 \times J_2} \sum_{0 \leq n_2 \leq |J_2|} (-x_s^2)^{|J_2| - n_2 \sigma_{n_2} J_2 \times J_2} \]
which is equal to the right hand side. \[ \square \]

Lemma 2.9. (1) The polynomial \[ x_s \overline{B}_{k, 0}(x, z) - x_s \overline{B}_{k, 0}(x_t, z) \]
is divisible by \( x_s^2 - x_t^2 \).

(2) For \( \epsilon \in \{ -1, 1 \} \), the polynomial
\[ (x_s - \epsilon x_t) x_s x_t \left\{ \overline{B}_{k, 0}(x, z) + \epsilon \overline{B}_{k, 0}(x_t, z) \right\} \]
\[ - (x_s + \epsilon x_t)(x_s x_t)^{k_0} \left[ x_s x_t^{k+1} - x_s (x_t)^{k+1} \right] \]
is divisible by \( x_s + \epsilon x_t - z \).

Proof. (1) follows from the fact that \( -B_{k, 0}(x, z) = \overline{B}_{k, 0}(-x, z) \) in Proposition 2.1.

(2) follows from the following congruence relation of polynomials modulo the ideal \( (x_s + \epsilon x_t - z) \):
\[ (x_s - \epsilon x_t) x_s x_t \left\{ B_{k, 0}(x, z) + \epsilon \overline{B}_{k, 0}(x_t, z) \right\} \]
\[ = (x_s - \epsilon x_t) x_s x_t \left( x_s + \epsilon x_t \right)^{k+2} - B_{k, 0} \left( \frac{x_s}{x_t} \right) \]
\[ = (x_s - \epsilon x_t) x_s x_t \left( x_s + \epsilon x_t \right)^{k+2k_0} \]
\[ \left[ \frac{\overline{B}_{k, 0}(x, z)}{x_s + \epsilon x_t} \right] = \left[ \frac{\overline{B}_{k, 0}(x_t, z)}{x_s + \epsilon x_t} \right] \]
\[ = (x_s - \epsilon x_t)(x_s x_t)^{k_0} \left[ x_s x_t^{k+1} - x_s (x_t)^{k+1} \right]. \]

Proposition 2.10. Every \( \varphi_j \) lies in \( D(S_\ell) \).

Proof. For \( 1 \leq j \leq \ell - 1, 1 \leq s < t \leq \ell, \) and \( \epsilon \in \{ -1, 1 \} \), by Lemma 2.9 and Lemma 2.8, we have the following congruence relation of polynomials modulo the ideal \( (x_s + \epsilon x_t - z) \):
\[ (x_s - \epsilon x_t) x_s x_t [\varphi_j(x_s + \epsilon x_t - z)] \]
\[ = (x_s - \epsilon x_t) x_s x_t \left[ (x_s + \epsilon x_t)^{k+1} \right] \]
\[ \times \left[ \prod_{K_1, K_2} (-1)^{|K_1| + |K_2|} \sum_{0 \leq n_1 \leq |J_1|, 0 \leq n_2 \leq |J_2|} \sigma_{n_1} J_1 \times J_2 \]
\[ \times (x_s - \epsilon x_t)(x_s x_t)/\overline{B}_{k, 0}(x, z) + \epsilon \overline{B}_{k, 0}(x_t, z) \]
\[ \equiv (x_s - \epsilon x_t) x_s x_t \left[ (x_s + \epsilon x_t)^{k+1} \right] \]
\[ \left[ \prod_{K_1, K_2} (-1)^{|K_1| + |K_2|} \sum_{0 \leq n_1 \leq |J_1|, 0 \leq n_2 \leq |J_2|} \sigma_{n_1} J_1 \times J_2 \]
\[ \times \left[ \prod_{n_1, n_2} (x_s - \epsilon x_t) x_s x_t^{k+1} - x_s (x_t)^{k+1} \right] \]
\[ = (x_s - \epsilon x_t) x_s x_t \left[ (x_s + \epsilon x_t) \sum_{x_i \in J_1} (x_i - x_s)(x_i - \epsilon x_t) \right] \]
\[ \times (-1)^{|J_1|} \prod_{x_i \in J_1} (x_i - x_s) \prod_{x_i \in J_2} (x_i^2 - x_s^2) \]
\[-x_s \prod_{i \in J_1} (x_i - cx_t) \prod_{i \in J_2} (x_i^2 - x_i^2) \quad (\dagger).\]

Case 1. When \(x_s \in J_1, (\dagger) = 0.\)
Case 2. When \(x_s \in J_2 \) and \(x_t \in J_2, (\dagger) = 0.\)
Case 3. When \(x_s \in J_1 \) and \(x_t \in J_2, (\dagger) = 0.\)
Case 4. When \(x_s \in J_1, x_t \in J_1 \) and \(\epsilon = 1, (\dagger) = 0.\)
Case 5. If \(x_s \in J_1, x_t \in J_1 \) and \(\epsilon = -1, \) then \(s = j < t = j + 1.\) So \((\dagger)\) is divisible by \(x_s + cx_t - z.\)

We also have the following congruence relation of polynomials modulo the ideal \((x_s + cx_t - z):\)

\[(x_s - cx_t)x_s x_t [\varphi(x_s + cx_t - z)] = \sum_{K_1 \cup K_2 \subseteq J} (\prod_{K_1}) (\prod_{K_2})^2 (-z)^{|K_1|}(-x_t)\]
\[(x_s - cx_t)x_s x_t (B_{-1,k_0}(x,s) + \epsilon B_{-1,k_0}(x,t), z)] \equiv (x_s + cx_t)(-x_t)(x_t - x_s)\]
\[\sum_{K_1 \subseteq J} (\prod_{K_1}) (\prod_{K_2})^2 [-(x_s + cx_t)] (\prod_{K_1}(x_s x_t))^{k_0}\]
\[= (x_s^2 - x_t^2) x_t \prod_{i \in J} (x_i - x_s)(x_i - x_t) \quad (\dagger\dagger).\]

Since \(s < t \leq \ell,\) we have \(x_s \in J = \{x_1, \ldots, x_{t-1}\}.\) Thus \((\dagger\dagger) = 0.\) Therefore \(\varphi_j(x_s + cx_t - z)\) is divisible by \(x_s + cx_t - z\) for \(1 \leq j \leq \ell, 1 \leq s < t \leq \ell.\)

For \(1 \leq j \leq \ell,\)
\[\varphi_j(x_s^2 - x_t^2) = 2x_s \varphi_j(x_s) - 2x_t \varphi_j(x_t)\]

is divisible either by \(x_s B_{k,k_0}(x_s,z) - x_t B_{k,k_0}(x_t,z)\) or by \(x_s B_{-1,k_0}(x_s,z) - x_t B_{-1,k_0}(x_t,z),\) we have
\[\varphi_j(x_s^2 - x_t^2) \equiv 0 \mod (x_s^2 - x_t^2)\]
by Lemma 2.9 (1). This implies \(\varphi_j \in D(S_\ell).\)

Applying Saito’s lemma [9, 7] Theorem 4.19, we complete our proof of Theorem 2.3 thanks to Lemma 2.5 Corollary 2.7 (3) and Proposition 2.10. Theorem 2.4 implies that \(\det[\varphi_j(x_i)]\) is a nonzero multiple of \((Q/z).\) By Corollary 2.7 (2) one obtains Corollary 2.11.

\[\det[\varphi_j(x_i)] = \frac{1}{(2\ell - 3)!} \prod_{1 \leq s \leq \ell} \prod_{\ell \in \{-1,1\}} (x_s + cx_t - z)(x_s + cx_t).\]

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