Dynamical signature of moire pattern in non-Hermitian ladder

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We study the dynamical behavior of a non-Hermitian moire superlattice system, which consists of two-coupled SSH chains with staggered imaginary on-site potentials. There are two main spatial regions, in which systems are in unbroken symmetric phases with fully real spectrum, appearing periodically along the ladder. We show that the two quantum phases are dimerized and tetramerized, which determine the distinct dynamical behaviors. Dirac probability can oscillate periodically, increase quadratically and increase exponentially, which correspond to the unbroken phase, exceptional point and the broken phase of the tetramerized region. In comparison, the Dirac probability can exhibit high-frequency oscillation in the dimerized region. These phenomena demonstrate the dynamical signature and provide insightful information of the moire pattern in the non-Hermitian regime.

I. INTRODUCTION

One of the unique features of a non-Hermitian system is the violation of conservation law of the Dirac probability, based on which, the complex potential is employed to describe open systems phenomenologically. Furthermore, unconventional propagation of light associated with the gain/loss has been demonstrated by engineering effective non-Hermitian Hamiltonians in optical systems. Around exceptional point (EP), many unique optical phenomena have been observed, ranging from loss-induced transparency, power oscillations violating left-right symmetry, low-power optical diodes to single-mode lasers. A fascinating phenomenon of non-Hermitian optical systems in the application aspect is the gain-induced detection, such as enhanced spontaneous emission, enhanced nano-particle sensing, as well as the amplified transmission in the optomechanical system. Both theoretical and experimental works not only give an insight into the dynamical property of the non-Hermitian Hamiltonian but also provide a platform to implement the novel optical phenomenon.

Recently there has been a growing interest in the influence of the moire pattern in physical systems. The moire pattern as a new way to apply periodic potentials in van der Waals heterostructures to tune electronic properties, has been extensively studied. Many interesting phenomena have been observed in the heterostructure materials with small twist angles and mismatched lattice constants. Moire patterns in condensed matter systems are produced by the difference in lattice constants or orientation of two 2D lattices when they are stacked into a two-layer structure. The aim of this paper is to demonstrate the phenomenon of the moire pattern in a non-Hermitian system via a dynamical process. A fascinating feature of a non-Hermitian system is the existence of exceptional points, at which two eigenstates coalesce. The dynamics of the system with parameters far away from, near and at the EP, exhibits extremely different behaviors. (i) When the system is far from or near EP but with a finite energy gap $\epsilon$, the dynamics is a periodic oscillation with associated Dirac probability oscillating in the period of time $2\pi/\epsilon$. (ii) When the system is at EP, the Dirac probability increases quadratically with time. (iii) When the system has complex levels, the Dirac probability increases exponentially with time. The rich variety of dynamical behaviors can show up periodically along the ladder.

In the paper, we study a modified non-Hermitian ladder system, which consists of two SSH chains with staggered imaginary potentials. The irregular structure arises from the slight difference of lattice constants between two legs. There are three types of approximate regular ladder structures, with different phases. These phenomena indicate that these three regions should have distinguishable dynamical behaviors, which are the signature of moire patterns. The reality of the spectrum is sensitive to the symmetry of the structure. The corresponding non-Hermiticity enhances the influence of the effect of moire patterns, that are apparent in the dynamics of the non-Hermitian system. We show that the dynamics is profoundly changed by slightly mismatched lattice constants associated with long period moire patterns.

This paper is organized as follows. In Section II, we present the model Hamiltonian and analyze the structure of the lattice. In Sections III and IV we investigate the quantum phase diagrams of two typical lattices based on the exact solutions, respectively. Section VI introduces the simple version of two types of lattice, which capture the main original dynamical behaviors. VII devotes to the numerical simulation of the model, revealing the dynamical signature of moire patterns. Finally, we give a summary and discussion in Section VIII.

II. MODEL

In material science, moire patterns are usually produced by stacking two two-dimensional (2D) crystals into van der Waals heterostructures with an twist angle. As a non-Hermitian variant of the moire pattern, we take a simple example by stacking two one-dimensional chains.
We consider a two-leg ladder system with the Hamiltonian

$$H = H_1 + H_2 + H_{12},$$  

where $H_\lambda \ (\lambda = 1, 2)$ describes the independent SSH chain with staggered imaginary on-site potentials

$$H_\lambda = \sum_l (w |2l - 1, \lambda \rangle \langle 2l, \lambda | + v |2l, \lambda \rangle \langle 2l + 1, \lambda |)$$

$$+ \text{H.c.} + i\gamma \sum_l (-1)^l |l, \lambda \rangle \langle l, \lambda |,$$  

and $H_{12}$ is the inter-chain tunneling term

$$H_{12} = \sum_{l', l} \kappa_{ll'} |l, 1 \rangle \langle l', 2 | + \text{H.c.}.$$  

FIG. 1. (Color online) Schematic illustration of the modified non-Hermitian two-leg ladder system. It consists of two SSH chains with staggered imaginary potentials. The irregular structure arises from the slight difference of lattice constants between two legs. There are three types of approximate regular ladder structures (circled by the red, blue and green dotted lines, respectively), which appear periodically in a large scale. The inter-leg hopping rates are $\kappa, \kappa', \kappa_1,$ and $\kappa_2,$ in various regions, respectively. (b1), (c1) and (d1) are drafts of three typical structures in different regions. (b2), (c2) and (d2) are drafts of the reductions of the structures in (b1), (c1) and (d1), respectively, in the limit case $\kappa, \kappa' \gg w \gg v$ limit. (b2-d2) capture the main features of (b1-d1). Systems (b2) and (d2) have fully real spectra for small enough $\gamma,$ but distinguishable dynamical behaviors (see text). (c2) can also have full real spectrum when $\gamma$ is small enough, which can be seen from Fig. 3(b). These phenomena indicate that three regions should have distinguishable dynamical behaviors, which are the signature of moire pattern.

Here basis $\{|l_1, 1 \rangle, |l_2, 2 \rangle, l_\lambda \in [1, 2N_\lambda]\}$ is an orthonormal complete set, satisfying $\langle l, \lambda | l', \lambda' \rangle = \delta_{l,l'} \delta_{\lambda,\lambda'}. Unlike the usual case of a ladder, $N_1$ and $N_2$ are not identical in the present work. The inter-leg tunneling amplitude $\kappa_{ll'}$ depends on positions of $|l, 1 \rangle$ and $|l', 2 \rangle.$ The geometry of the modified two-leg ladder is illustrated in Fig. 1. There are three typical cases: (i) a site on a leg only couples to a single site with the same imaginary potential on another leg; (ii) a site on a leg only couples to a single site with the opposite imaginary potential on another leg; (iii) a site on a leg couples to two sites on another leg.

We consider the case that lattice constants of two chains are slightly different. In a certain region, the structure of the ladder can be regarded as uniform in a large scale. The local dynamics obeys the corresponding uniform Hamiltonian.
III. TETRAMERIZED PHASE

We consider the first typical regular ladder system which is illustrated in Fig. 2(a). The Hamiltonian has the form

\[ H_T = \sum_{l=1}^{N} \sum_{\lambda=1,2}^{2N} (w |2l-1, \lambda \rangle \langle 2l, \lambda | + v |2l, \lambda \rangle \langle 2l + 1, \lambda |) + \kappa \sum_{l=1}^{N} |l, 1 \rangle \langle l, 2 | + \text{H.c.} + i\gamma \sum_{l=1}^{2N} \sum_{\lambda=1,2} (-1)^{l} |l, \lambda \rangle \langle l, \lambda |, \]

where the boundary condition is \(|2N + 1, \lambda \rangle \equiv |1, \lambda \rangle\). As illustrated in Fig. 2(a), it satisfies the \(\mathcal{PT}\)-symmetry. Here, the time reversal operation \(T\) is such that \(TiT = -i\), while the effect of the parity is such that \(\mathcal{P} |l, \lambda \rangle = |2N - l + 1, \lambda \rangle\) for \(\lambda = 1, 2\). Applying operators \(\mathcal{P}\) and \(T\) on the Hamiltonian \(H_T\), one has \([T, H_T] \neq 0\) and \([\mathcal{P}, H_T] \neq 0\), but \([\mathcal{PT}, H_T] = 0\). According to the non-Hermitian quantum theory, such a Hamiltonian may have fully real spectrum within a certain parameter region. The boundary of the region is the critical point of quantum phase transition associated with the \(\mathcal{PT}\)-symmetry breaking. In the following, we will diagonalize this Hamiltonian and get the phase diagram.

We note that such a system also has another symmetry under the exchange of two chains, i.e., \(|l, 1 \rangle \rightleftharpoons |l, 2 \rangle\). This symmetry ensures the conservation of bonding or antibonding state between two sites coupled by \(\kappa\). We refer the collective bonding or antibonding state as to dimerized phase. Taking the linear transformation

\[ |l, \sigma \rangle = \frac{1}{\sqrt{2}} \left( |l, 1 \rangle + \sigma |l, 2 \rangle \right), \]

with \(\sigma = \pm\), the Hamiltonian can be rewritten as

\[ H_T = H_+ + H_-, \]

\[ H_\sigma = \sum_{l=1}^{N} (w |2l-1, \sigma \rangle \langle 2l, \sigma | + v |2l, \sigma \rangle \langle 2l + 1, \sigma |) + \kappa \sum_{l=1}^{2N} |l, \sigma \rangle \langle l, \sigma | + \kappa \sigma \sum_{l=1}^{2N} |l, \sigma \rangle \langle l, \sigma |. \]

Sub-Hamiltonian \(H_\sigma\) satisfies \([H_+, H_-] = 0\), representing two independent non-Hermitian SSH chains but with opposite chemical potentials \(\pm \kappa\), which has been studied in the previous work\(^{31}\). It turns out that the spectrum \(\epsilon_k^\sigma\) for a single chain \(H_\sigma\) is

\[ \epsilon_k^\sigma = \sigma \kappa \pm \sqrt{(\epsilon_k^0)^2 - \gamma^2}, \]

which consists of two branches separated by an energy gap

\[ \Delta = \sqrt{(w - v)^2 - \gamma^2}. \]

Obviously, it displays a full real spectrum within the region of \((w - v)^2 \geq \gamma^2\). Beyond this region, the imaginary eigenvalue appears and the \(\mathcal{PT}\) symmetry of the corresponding eigenfunction is broken simultaneously according to the non-Hermitian quantum theory. The phase diagram is plotted in Fig. 3(a). We note that the dimerization along the legs still exists \((w \neq v)\), when the gap vanishes in such a non-Hermitian model. In the case of \(\kappa \gg w \gg \nu\), the combination of two types of dimerizations, inter- and intra-leg dimers, result in tetramers. We refer the collective tetramerized states as to tetramerized phase. The extremely tetramerized phase is characterized by the ground states of the system with \(\nu = 0\) (see Fig. 3(b)).

IV. DIMERIZED PHASE

In this section, we investigate another type of uniform ladders, which is illustrated in Fig. 2(c). The Hamiltonian reads
The eigenvalue of states of the system with extremely dimerized phase is characterized by the ground dimer. Then it is referred as to dimerized phase. The phase boundary condition is $|2N + 1, \lambda\rangle \equiv |1, \lambda\rangle$. As illustrated in Fig. 2(c), there is still a $\mathcal{PT}$-symmetry defined as before. According to the non-Hermitian quantum theory, such a Hamiltonian may have fully real spectrum within a certain parameter region. The boundary of the region is the critical point of quantum phase transition associated with $\mathcal{PT}$-symmetry breaking. In the following, we will diagonalize this Hamiltonian and investigate the phase diagram.

Taking the transformation

$$
|k, a\rangle = \frac{N}{\sqrt{2N}} \left( e^{-ik} |2l - 1, 1\rangle + |2l, 2\rangle \right),
|k, b\rangle = \frac{N}{\sqrt{2N}} \left( |2l, 1\rangle + e^{-ik} |2l - 1, 2\rangle \right),
$$

we have

$$H_D = \sum_k \langle k, a\rangle \langle k, b\rangle h_k \left( \frac{\langle k, a\rangle}{\langle k, b\rangle} \right),$$

where the kernel matrix is

$$h_k = \begin{pmatrix}
we^{-ik} + ve^{ik} + \kappa & i\gamma \\
we^{-ik} + ve^{ik} + \kappa & i\gamma
\end{pmatrix}. $$

The eigenvalue of $h_k$ is

$$\varepsilon_k = \pm \sqrt{(\varepsilon_k^0)^2 - \gamma^2},$$

where

$$\varepsilon_k^0 = |we^{-ik} + ve^{ik} + \kappa|,$$

is eigenvalues of $h_k^0 = h_k(\gamma = 0)$. It has been shown that $\varepsilon_k^0$ is always nonzero except at the lines

$$w = v, |2v/\kappa| > 1,$$

and

$$|(w + v)/\kappa| = 1.$$

The phase diagram is plotted in Fig. 3(c). Then $\varepsilon_k$ can be real within the whole $wv$ plane, once a appropriate $\gamma$ is taken. There is only one type of dimerization, interdimer. Then it is referred as to dimerized phase. The extremely dimerized phase is characterized by the ground states of the system with $\nu = w = 0$ (see Fig. 4(d2)).

V. CROSSOVER PHASE

In this section, we investigate the third type of uniform ladders, which is a crossover between two types of structures above and illustrated in Fig. 2(b). The Hamiltonian reads

$$H_C = \sum_{l=1}^N \sum_{\lambda=1,2} (w |2l - 1, \lambda\rangle \langle 2l, \lambda| + v |2l, \lambda\rangle \langle 2l + 1, \lambda|)$$

$$+ \kappa' \sum_{l=1}^{2N} (|l, 1\rangle \langle l, 2| + |l, 1\rangle \langle l + 1, 2|) + \text{H.c.}$$

Taking the Fourier transformation

$$|2l - 1, 1\rangle = \frac{1}{\sqrt{N}} \sum_k e^{ikl} |k, a\rangle$$

$$|2l, 1\rangle = \frac{1}{\sqrt{N}} \sum_k e^{ikl} |k, b\rangle$$

$$|2l - 1, 2\rangle = \frac{1}{\sqrt{N}} \sum_k e^{ikl} |k, c\rangle$$

$$|2l, 2\rangle = \frac{1}{\sqrt{N}} \sum_k e^{ikl} |k, d\rangle$$

we get

$$H_C = \sum_k \langle \psi_k \rangle h_k \langle \psi_k \rangle,$$

where the $4 \times 4$ matrix

$$h_k = \begin{pmatrix}
-i\gamma & \lambda_k & \kappa' & \kappa' \\
\lambda_k & i\gamma & \kappa' e^{-ik} & \kappa' \\
\kappa' & \kappa' e^{ik} & -i\gamma & \lambda_k \\
\kappa' & \lambda_k & i\gamma & -i\gamma
\end{pmatrix}, $$

with $\lambda_k = w + ve^{ik}$, and the vector $|\psi_k\rangle = (\langle k, a|, \langle k, b|, \langle k, c|, \langle k, d|)$. It is hard to get the simplified analytical expression of the eigen values of the matrix. However, we only concern the difference of the phase diagram of $H_C$ from that of the above two systems. First of all, when taking $\nu = w$, $H_C$ reduces to a uniform chain with staggered imaginary potentials. There is no energy gap in the spectrum of $H_C$ for $\gamma = 0$. Then any nonzero $\gamma$ can induce imaginary levels at $k = \pi$. The conclusion is true for all values of $\nu = w$, which is different from the case of dimer. Second, when taking $\lambda_k = w + ve^{ik} = \sigma k' (\sigma = \pm)$, i.e., $k = 0$ and $w + v = \sigma \kappa'$, matrix $h_k$ reduces to

$$h' = \kappa' \begin{pmatrix}
-i\gamma' & \sigma 1 & 1 & 1 \\
\sigma 1 & i\gamma' & 1 & 1 \\
1 & 1 & -i\gamma' & \sigma 1 \\
1 & 1 & \sigma 1 & i\gamma'
\end{pmatrix},$$

with $\gamma' = \gamma/\kappa'$. It is easy to check that two of four eigenvalues $\varepsilon_{\sigma}$ can be expressed as
The phase diagram is plotted in Fig. 3(b). This result is different from the case of tetramer. Beyond these regions, there always exist common parameters $(w, v, \gamma)$ to maintain the full real spectrum for three types of non-Hermitian systems. This fact ensures the probable existence of stable dynamical signature of moire pattern.

\[ \varepsilon_{\sigma} = -\sigma \gamma \pm i \gamma', \]

which indicate that any nonzero $\gamma$ can induce complex levels. This result is different from the case of tetramer. The phase diagram is plotted in Fig. 3(b).

**VI. DYNAMICAL SIGNATURES**

In this section, we investigate the dynamics of the regular ladder systems in the limit case $\kappa, \kappa' \gg w \gg v$. We start to reduce the original lattices by decoupling between tetramers in $H_T$ by taking $v = 0$ and dimers in $H_D$ by taking $v = w = 0$. We employ Hamiltonians $H_T$ and $H_D$ to describe two clusters, which are schematically illustrated in Fig. 1(b2) and (d2). In the following, we study the two sub-Hamiltonians.

**A. Tetramer cluster**

The 4-site Hamiltonian $h_T$ reads

\[ h_T = w (|1, 1\rangle \langle 2, 1| + |1, 2\rangle \langle 2, 2|) \]

\[ + \kappa (|1, 1\rangle \langle 1, 2| + |2, 1\rangle \langle 2, 2|) \] + H.c.

\[ + i \gamma \sum_{l=1}^{2} \sum_{\lambda=1,2} (-1)^{l} |l, \lambda\rangle \langle l, \lambda|, \]

\[ (25) \]

or the matrix form

\[ h_T = \begin{pmatrix}
- \gamma & \kappa & w & 0 \\
\kappa & - \gamma & 0 & w \\
w & 0 & i \gamma & \kappa \\
0 & w & \kappa & i \gamma
\end{pmatrix}, \]

\[ (26) \]

based on the basis set \{\[1, 1\], \[1, 2\], \[2, 1\], \[2, 2\]\}. The eigenvectors can be obtained explicitly as

\[ \begin{pmatrix}
|\chi_{++}\rangle \\
|\chi_{+-}\rangle \\
|\chi_{-+}\rangle \\
|\chi_{--}\rangle
\end{pmatrix} = \begin{pmatrix}
\Lambda^* & \Lambda^* & w & w \\
- \Lambda & - \Lambda & w & w \\
- \Lambda^* & - \Lambda^* & - w & w \\
\Lambda & - \Lambda & - w & w
\end{pmatrix} \begin{pmatrix}
|1, 1\rangle \\
|1, 2\rangle \\
|2, 1\rangle \\
|2, 2\rangle
\end{pmatrix}, \]

\[ (27) \]

with corresponding eigenvalues

\[ \varepsilon_{\sigma \rho} = \sigma \kappa + \rho \overline{\nu}, (\sigma, \rho = \pm), \]

\[ (28) \]

where $\Lambda = \overline{\nu} + i \gamma$ and $\overline{\nu} = \sqrt{w^2 - \gamma^2}$. The dynamics of the system depends on eigen levels: For real $\overline{\nu}$, the average Dirac probability is conserved, while varies exponentially for imaginary $\overline{\nu}$. The fascinating behavior occurs at $\overline{\nu} = 0$, i.e., $w = \gamma$, which is the EP of the system. We can see that

\[ |\chi_{++}\rangle = |\chi_{+-}\rangle, |\chi_{-+}\rangle = |\chi_{--}\rangle, \]

\[ (29) \]

from the expression of $\chi_{\sigma \rho}$ when taking $\gamma = \gamma_c = w$. Two pairs of eigenvectors coalesce to a single pair. It has been shown that the Dirac probability increases quadratically with time $\overline{\nu} t$.

To characterize the dynamics, we consider the time evolution of the initial state

\[ |\psi(0)\rangle = \frac{1}{2} \left( |1, 1\rangle + |1, 2\rangle + |2, 1\rangle + |2, 2\rangle \right). \]

\[ (30) \]

(i) For the real $\overline{\nu}$, the evolved state at instant $t$ is

\[ |\psi(t)\rangle = \frac{e^{- \frac{i \kappa t}{2}}}{2} \left( \left( \cos \left( \frac{\overline{\nu} t}{2} \right) - i \frac{w - i \gamma}{\overline{\nu}} \sin \left( \frac{\overline{\nu} t}{2} \right) \right) |1, 1\rangle \\
+ |1, 2\rangle \right) + \left( \left( \cos \left( \frac{\overline{\nu} t}{2} \right) + i \frac{w + i \gamma}{\overline{\nu}} \sin \left( \frac{\overline{\nu} t}{2} \right) \right) |2, 1\rangle \\
+ |2, 2\rangle \right). \]

\[ (31) \]
The Dirac probability is
\[
P^R_T(t) = ||\psi_T(t)||^2 = \left(\frac{w}{|\gamma|}\right)^2 - \left(\frac{\gamma}{|w|}\right)^2 \cos(2\pi|\overline{w}|t),
\]
which is a periodic function of time with the period \(\tau_T = \frac{\pi}{|\overline{w}|}\).

(ii) For the imaginary \(\overline{w}\), the evolved state at instant \(t\) is
\[
|\psi_T(t)\rangle = \frac{e^{-i\lambda t}}{2} \left\{ \cos(|\overline{w}|t) - i\frac{w}{|\gamma|} \sin(|\overline{w}|t) \right\} (|1, 1\rangle + |1, 2\rangle)
+ \left\{ \cos(|\overline{w}|t) - i\frac{w}{|\gamma|} \sin(|\overline{w}|t) \right\} (|2, 1\rangle + |2, 2\rangle).
\]
The Dirac probability is
\[
P^I_T(t) = ||\psi_T(t)||^2 = \left(\frac{\gamma}{|w|}\right)^2 \cosh(2|\overline{w}|t) - \left(\frac{w}{|\gamma|}\right)^2,
\]
which is an exponential function of time with the characteristic time constant \(\Gamma = 1/2|\overline{w}|\).

(iii) At the EP with zero \(\overline{w}\), Jordan blocks appear in the matrix \(h_T\). According to the appendix, the evolved state at instant \(t\) is
\[
|\psi_T(t)\rangle = \frac{e^{-i\lambda t}}{2} \left\{ [1 - t\gamma(1 + i)] (|1, 1\rangle + |1, 2\rangle)
+ [1 + t\gamma(1 - i)] (|2, 1\rangle + |2, 2\rangle) \right\}.
\]
The Dirac probability is
\[
P^{\text{EP}}_T(t) = ||\psi_T(t)||^2 = 1 + 2\gamma^2 t^2,
\]
which increases quadratically with time, as the form \(t^2\). Plots of three typical dynamical behaviors are presented in Fig. 4.

B. Dimer cluster

The 2-site Hamiltonian \(h_D\) reads
\[
h_D = \kappa |1, 1\rangle \langle 1, 2| + \text{H.c.}
+ i\gamma (|1, 1\rangle \langle 1, 1| - |1, 2\rangle \langle 1, 2|).
\]
The eigenvectors can be obtained explicitly as
\[
\begin{align*}
|\chi_+\rangle &= \left(\begin{array}{c}
i\gamma + \varepsilon_+ \\
i\gamma + \varepsilon_- \end{array}\right) |1, 1\rangle, \\
|\chi_-\rangle &= \left(\begin{array}{c}
i\gamma + \varepsilon_+ \\
i\gamma + \varepsilon_- \end{array}\right) |1, 2\rangle,
\end{align*}
\]
with corresponding eigenvalues
\[
\varepsilon_\sigma = \sigma\sqrt{\kappa^2 - \gamma^2}, (\sigma = \pm).
\]
In parallel, we consider the time evolution of the initial state
\[
|\varphi_D(0)\rangle = \frac{1}{\sqrt{2}} (|1, 1\rangle + |1, 2\rangle),
\]
which is a part of \(|\psi(0)\rangle\). The evolved state is
\[
|\varphi_D(t)\rangle = \frac{1}{\sqrt{2}} \left[ \cos(\varepsilon_D t) - i\frac{\kappa + i\gamma}{\varepsilon_D} \sin(\varepsilon_D t) \right] |1, 1\rangle
+ \left[ \cos(\varepsilon_D t) - i\frac{\kappa - i\gamma}{\varepsilon_D} \sin(\varepsilon_D t) \right] |1, 2\rangle.
\]
The Dirac probability is
\[ P_D(t) = |\langle \varphi_D(t) \rangle|^2 \]
\[ = \left( \frac{\kappa}{\varepsilon_D} \right)^2 \left( \left( \frac{\gamma}{\varepsilon_D} \right)^2 \cos(2\varepsilon_D t) \right), \tag{42} \]
which is a periodic function of time with the period \( T_D = \pi/\varepsilon_D \). In Fig. 5, we plots the dynamical behaviors in comparison with that of tetramer. The dynamics of such two systems are distinct. In the parameter region \( \kappa \gg w, \gamma \gg \nu \), for real \( \tau_T \), we have \( \tau_T \gg \tau_D \). And the tetramerized phase supports amplification of the probability.

VII. MOIRE PATTERN

In the original system, the inter-leg hopping rates are \( \kappa, \kappa', \kappa_1 \) and \( \kappa_2 \), in various regions, respectively. Three types of regular lattices emerge periodically along the legs. Based on the above analysis, it is expected that the dynamical behaviors in two cases are still distinct for nonzero \( w \) and \( \nu \). To demonstrate this point, we perform a numerical simulation for the dynamics of the original Hamiltonian. We take the initial state being distributed on each site with the equal probability
\[ |\psi(0)\rangle = \frac{1}{\sqrt{2N}} \sum_{l} (|l, 1\rangle + |l, 2\rangle). \tag{43} \]

The evolved state is
\[ |\psi(t)\rangle = e^{-iHt} |\psi(0)\rangle, \tag{44} \]
based on which the average value of all physical quantities can be obtained. Our primary interest here is the influence of the moire pattern on the dynamics of the system, or the mapping between the periodic structure and the time-dependent Dirac probability distribution
\[ P(j, t) = \sum_{\lambda=1,2} |\langle j, \lambda | \psi(t) \rangle|^2. \tag{45} \]

The inter-chain hopping rate is taken as the form
\[ \kappa_{ij} = \kappa_0 e^{-\alpha^2|x_i - y_j|^2} = \kappa_0 e^{-\alpha^2|i - (1 - \Delta)j|^2}, \tag{46} \]
where \( x_i = i, y_j = (1 - \Delta)j \) denote the dimensionless lattice site coordinates in chain 1 and 2, respectively. Here \( \alpha \) controls the range of inter-chain tunneling amplitude \( \kappa_{ij} \). We take \( \alpha = 2 \), which results in \( \kappa = \kappa_0, \kappa' = 0.37\kappa_0 \) and ensures \( \kappa_{ij} \approx 0 \) except the rates \( \kappa, \kappa_1, \kappa_2, \) and \( \kappa' \). The numerical simulation is performed by exact diagonalization for a finite system with several typical imaginary potentials \( \gamma > \gamma_c, \gamma \approx \gamma_c \) and \( \gamma < \gamma_c \). The probability distributions \( P(j, t) \) are plotted in Fig. 5, which show that Moire patterns are apparent for each cases. It is worth mentioning briefly that the moire pattern above discussed can be implemented through the two dimensional array of evanescently coupled optical waveguides with alternating regions of optical gain and absorption.\[ \square \]

VIII. SUMMARY

We have shown that a super periodicity in the coordinate space (along the ladder) is imposed on two coupled SSH chains if there exists a slight difference of lattice constants between two legs, in spite of a staggered imaginary potential. There are two main gapped phases in each period, which have full real spectra for appropriate system parameters. We have characterized such phases within unbroken \( PT \)-symmetry regions in terms of the dimerization and tetramerization. Analytical analysis and numerical simulation for the dynamics of two phases have shown that imaginary potentials can enhance their distinguishability extremely. Hence, the dynamics of the whole system is profoundly changed by slightly mismatched lattice constants associated with long period moire patterns. Our result for this concrete system proves insightful information about the moire pattern in the non-Hermitian regime. In contrast to a Hermitian system, the amplification of Dirac probability and parameter sensitivity of a non-Hermitian system in the vicinity of EP demonstrate the remarkable dynamic signature of moire patterns. The generalization to higher dimensions is straight forward.

IX. APPENDIX

In this appendix we present the derivation of time evolution of the initial state in Eq. (30) under the system \( h_{\gamma_c} \) at the EP. It can be reduced to the problem of \( 2 \times 2 \) matrix
\[ M = \begin{pmatrix} \kappa - i\gamma & w \\ w & \kappa + i\gamma \end{pmatrix}, \tag{47} \]
where \( w = \gamma \). We have
\[ M = \begin{pmatrix} \kappa - i\gamma & \gamma \\ \gamma & \kappa + i\gamma \end{pmatrix}, \tag{48} \]
which is in the Jordan block form
\[ M = \mathcal{V} \mathcal{H} \mathcal{V}^{-1} = \begin{pmatrix} -i\gamma & 1 \\ \gamma & 0 \end{pmatrix} \begin{pmatrix} \kappa & 0 \\ 0 & \kappa + i\gamma \end{pmatrix} \begin{pmatrix} 0 & 1/\gamma \\ 1 & i \end{pmatrix}, \]
with
\[ \mathcal{H} = \begin{pmatrix} \kappa & 1 \\ 0 & \kappa \end{pmatrix}, \mathcal{V} = \begin{pmatrix} -i\gamma & 1 \\ \gamma & 0 \end{pmatrix}. \]
FIG. 5. (Color online) Propagation of the initial state being distributed on each site with equal probability. The system parameters are (a) $\kappa = 1$, $w = 0.5$, $v = 0.1$, $\gamma = 0.395$, $\Delta = 1/301$, (b) $\kappa = 0.6$, $w = 0.5$, $v = 0.1$, $\gamma = 0.395$, $\Delta = 1/301$, (c) $\kappa = 1$, $w = 0.5$, $v = 0.1$, $\gamma = 0.645$, $\Delta = 1/101$, and (d) $\kappa = 1$, $w = 0.5$, $v = 0.1$, $\gamma = 0.655$, $\Delta = 1/101$ respectively. The dashed red and black lines in each panel denote the two distinct dynamical behaviors which are determined by the tetramerized and dimerized substructures. The panels (a) and (b) show that probability $P(j,t)$ varied periodically with different period as time $t$ increases. In panels (c)-(d), one can see that the Dirac probability increases in the tetramerized phase while it exhibits periodical oscillation in the dimerized phase. Note that $P(t)$ in the tetramerized phase increases quadratically with time $t$ in panel (c), which is associated with the EP as given in Eq. (36). In comparision, $P(t)$ increases exponentially in the broken phase of the tetramerized region, which is in agreement with the Eq. (34). These four panels show apparently that the slight mismatched lattice constants lead to the long period moire patterns.
The evolved state $|\Psi(t)\rangle$ obeys the Schrodinger equation
\[ i \frac{d}{dt} |\Psi(t)\rangle = M |\Psi(t)\rangle, \tag{49} \]
which leads to
\[ i \frac{d}{dt} |\Psi(t)\rangle = V h V^{-1} |\Psi(t)\rangle, \tag{50} \]
or
\[ i \frac{d}{dt} V^{-1} |\Psi(t)\rangle = h V^{-1} |\Psi(t)\rangle. \tag{51} \]
Setting
\[ |\psi\rangle = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = V^{-1} |\Psi(t)\rangle, \]
we have
\[ i \frac{d}{dt} |\psi\rangle = h |\psi\rangle, \tag{52} \]
or the explicit form
\[ \begin{cases} i \frac{d\psi_1}{dt} = \kappa \psi_1 + \psi_2 \\ i \frac{d\psi_2}{dt} = \kappa \psi_2 \end{cases}. \tag{53} \]
Then the solution is
\[ \begin{cases} \psi_2 = c_2 e^{-i\kappa t} \\ i \frac{d\psi_1}{dt} = \kappa \psi_1 + c_2 e^{-i\kappa t} \\ \psi_1 = c_1 e^{-i\kappa t} - itc_2 e^{-i\kappa t} \end{cases}, \tag{54} \]
where $c_1$ and $c_2$ are constants determined by initial state.
Finally we have
\[ |\Psi(t)\rangle = V |\psi(t)\rangle = V \left( c_1 e^{-i\kappa t} - itc_2 e^{-i\kappa t} \right). \tag{55} \]
For initial state
\[ |\Psi(0)\rangle = \begin{pmatrix} a \\ b \end{pmatrix}, \tag{56} \]
we have
\[ |\Psi(t)\rangle = e^{-i\kappa t} \left( \frac{1 - t\gamma}{1 + t\gamma} a - it\gamma b \right) \left( \frac{1 + t\gamma}{1 - t\gamma} b - it\gamma a \right). \tag{57} \]
Taking $a = b = 1/\sqrt{2}$, we obtain
\[ |\Psi(t)\rangle = \frac{e^{-i\kappa t}}{\sqrt{2}} \left( \begin{pmatrix} 1 - t\gamma (1 + i) \\ 1 + t\gamma (1 - i) \end{pmatrix} \right). \tag{58} \]
It shows that the dynamics at EP is peculiar, which is linearly time dependent.

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