Convergence of Variational Iteration Method for Solving Singular Partial Differential Equations of Fractional Order

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We are concerned here with singular partial differential equations of fractional order (FSPDEs). The variational iteration method (VIM) is applied to obtain approximate solutions of this type of equations. Convergence analysis of the VIM is discussed. This analysis is used to estimate the maximum absolute truncated error of the series solution. A comparison between the results of VIM solutions and exact solution is given. The fractional derivatives are described in Caputo sense.

1. Introduction

In recent years, considerable attention has been devoted to the study of the fractional calculus and its numerous applications in many areas such as physics and engineering. The applications of fractional calculus used in many fields such as electrical networks, control theory of dynamical systems, probability and statistics, electrochemistry of corrosion, chemical physics, optics, and signal processing can be successfully modeled by linear or nonlinear FDEs [1–7]. Further, fractional partial differential equations appeared in many fields of engineering and science, including fractals theory, statistics, fluid flow, control theory, biology, chemistry, diffusion, probability, and potential theory [8, 9].

The singular partial differential equations of fractional order (FSPDEs), as generalizations of classical singular partial differential equations of integer order (SPDEs), are increasingly used to model problems in physics and engineering. Consequently, considerable attention has been given to the solution of singular partial differential equations of fractional order. Finding approximate or exact solutions of SPDEs is an important task. Except for a limited number of these equations, we have difficulty in finding their analytical solutions. Therefore, there have been attempts to find methods for obtaining approximate solutions. Several such techniques have drawn special attention, such as variational iteration method [10], homotopy analysis method [11], and homotopy iteration method [12].

The variational iteration method (VIM) was proposed by He [13–16] due to its flexibility and convergence and efficiently works with different types of linear and nonlinear partial differential equations of fractional order and gives approximate analytical solution for all these types of equations without linearization or discretization; many authors have been studying it; for example, see [17–21]. In this paper, we discuss the VIM for solving FSPDEs and obtain the convergence results of this method. The contribution of this work can be summarized in three points.

(1) Based on the sufficient condition that guarantees the existence of a unique solution to our problem (see Theorem 6) and using the series solution, convergence of VIM is discussed (see Theorem 7).

(2) Using point one, the maximum absolute truncated error of series solution of VIM is estimated (see Theorem 8).

(3) Some numerical examples are given.
Consider fractional singular partial differential equations with variable coefficients
\[
\frac{\partial^\alpha u}{\partial t^\alpha} + \mu(x) \frac{\partial^4 u}{\partial x^4} + \lambda(y) \frac{\partial^4 u}{\partial y^4} + h(z) \frac{\partial^4 u}{\partial z^4} = 0,
\]
where the variable coefficients subject to initial conditions
\[
ut(x, y, z, 0) = f_0(x, y, z),
\]
and boundary conditions
\[
\begin{align*}
&u(a, y, z, t) = g_0(y, z, t), & u(b, y, z, t) = g_1(y, z, t), \\
&u(x, a, z, t) = g_2(x, z, t), & u(x, b, z, t) = g_3(x, z, t), \\
&u(x, y, a, t) = g_4(x, z, t), & u(x, y, b, t) = g_5(x, z, t),
\end{align*}
\]
where \(\frac{\partial^\alpha}{\partial t^\alpha}\) is the fractional derivative in the Caputo sense, \(a < x, y, z < b\), and \(g_i\) and \(k_i\), \(i = 0, \ldots, 5\) are continuous. The \(\frac{\partial^4 u}{\partial x^4}, \frac{\partial^4 u}{\partial y^4}\), and \(\frac{\partial^4 u}{\partial z^4}\) are linear bounded operators; that is, it is possible to find numbers \(m_1, m_2, m_3 > 0\) such that \(\|\frac{\partial^4 u}{\partial x^4}\| \leq m_1\|u\|, \|\frac{\partial^4 u}{\partial y^4}\| \leq m_2\|u\|, \|\frac{\partial^4 u}{\partial z^4}\| \leq m_3\|u\|\). Equation (1) can be written as
\[
\begin{align*}
&D^\alpha_x u(x, y, z, t) \\
&= f(t, u(x, y, z, t), D^\nu_x u(x, y, z, t), D^\sigma_x u(x, y, z, t)),
\end{align*}
\]
where \(n_1 = n_2 = n_3 = 4\).

2. Preliminares

In this section, we give some basic definitions and properties of fractional calculus theory used in this paper.

Definition 1. A real function \(f(x), x > 0\) is said to be in space \(C_\mu, \mu \in R\) if there exists a real number \(p > \mu\), such that \(f(x) = x^p f_1(x)\), where \(f_1(x) \in C(0, \infty)\), and it is said to be in the space \(C_\mu\) if \(f^m \in R_\mu, n \in N\).

Definition 2. The Riemann-Liouville fractional integral operator of order \(\alpha \geq 0\) of a function \(f \in C_\mu, \mu \geq -1\) is defined as
\[
\begin{align*}
J^\alpha f(x) &= \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) dt, \\
\alpha &> 0, \ t > 0.
\end{align*}
\]
In particular \(J^0 f(x) = f(x)\).

For \(\beta \geq 0\) and \(\gamma \geq -1\), some properties of the operator \(J^\alpha\) are
\[
\begin{align*}
(1) \quad &J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x), \\
(2) \quad &J^\alpha f(x) = f(x), \\
(3) \quad &J^\alpha x^\gamma = (\Gamma(\gamma+1)/\Gamma(\alpha+\gamma+1)) x^{\alpha+\gamma}.
\end{align*}
\]

Lemma 4. If \(m-1 < \alpha \leq m, m \in N, f \in C_\mu, \mu > -1\), then the following two properties hold:
\[
\begin{align*}
(1) \quad &D^\alpha f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f(t) dt, \\
&\quad m - 1 < \alpha \leq m.
\end{align*}
\]

Definition 3. The Caputo fractional derivative of \(f \in C_{m-1}^m, m \in N\) is defined as
\[
D^\alpha f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f(t) dt,
\]
where \(\frac{\partial^\alpha}{\partial t^\alpha}\) is the fractional derivative in the Caputo sense, \(a < x, y, z < b\), and \(g_i\) and \(k_i\), \(i = 0, \ldots, 5\) are continuous. The \(\frac{\partial^4 u}{\partial x^4}, \frac{\partial^4 u}{\partial y^4}\), and \(\frac{\partial^4 u}{\partial z^4}\) are linear bounded operators; that is, it is possible to find numbers \(m_1, m_2, m_3 > 0\) such that \(\|\frac{\partial^4 u}{\partial x^4}\| \leq m_1\|u\|, \|\frac{\partial^4 u}{\partial y^4}\| \leq m_2\|u\|, \|\frac{\partial^4 u}{\partial z^4}\| \leq m_3\|u\|\). Equation (1) can be written as
\[
\begin{align*}
&D^\alpha_x u(x, y, z, t) \\
&= f(t, u(x, y, z, t), D^\nu_x u(x, y, z, t), D^\sigma_x u(x, y, z, t)),
\end{align*}
\]
where \(n_1 = n_2 = n_3 = 4\).

3. Analysis of the Variational Iteration Method

To solve the fractional singular partial differential equations (4) by using the variational iteration method, with initial and
boundary conditions (2) and (3), where \(\|D^n u(t)\| = M\|u\|\), we construct the following correction functional:

\[
u_{n+1}(x, y, z, t) = u_n(x, y, z, t) + f^\alpha \left[ cD^n u(x, y, z, t) - f \left((x, y, z, t), u_n(x, y, z, t), \right.\right.
\left.\left. D^n_x u_n(x, y, z, t), D^n_y u_n(x, y, z, t), D^n_z u_n(x, y, z, t) \right) \right]
\]

or

\[
u_{n+1}(x, y, z, t) = u_n(x, y, z, t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \lambda(s) \left( cD^n u(x, y, z, s) - f \left(s, u_n(x, y, z, s), D^n_x u_n(x, y, z, s), D^n_y u_n(x, y, z, s) \right) \right) ds.
\]

We obtain the following iteration formula by substitution of (11) in (9)

\[
u_{n+1}(x, y, z, t) = u_n(x, y, z, t) + \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} \left( cD^n u(x, y, z, s) - f \left(s, u_n(x, y, z, s), D^n_x u_n(x, y, z, s), D^n_y u_n(x, y, z, s) \right) \right) ds.
\]

This yields the following iteration formula:

\[
u_{n+1}(x, y, z, t) = u_n(x, y, z, t) - \frac{(\alpha-1)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left( cD^n u(x, y, z, s) - f \left(s, u_n(x, y, z, s), D^n_x u_n(x, y, z, s), D^n_y u_n(x, y, z, s) \right) \right) ds.
\]

The initial approximation \(u_0\) can be chosen by the following manner which satisfies initial conditions:

\[
u_0 = \frac{1}{\gamma_0} t^\gamma = \gamma_0 + \gamma_1 t,
\]

where \(\gamma_0 = f_0(x, y, z), \gamma_1 = f_1(x, y, z)\).
We can obtain the following first-order approximation by substitution of (15) into (14)
\[ u_1(x, y, z, t) \]
\[ = u_0(x, y, z, t) - (\alpha - 1) J_t^\alpha f(t, u_0(x, y, z, t)) \]
\[ \times \left[ D_x^\alpha u_0(x, y, z, t) - f(t, u_0(x, y, z, t)) \right. \]
\[ \left. D_y^\alpha u_0(x, y, z, t) \right] . \]

Finally, by substituting the constant values of \( y_0 \) and \( y_1 \) into (16), we have the results as the first approximate solutions of (4) with (2) and (3).

### 3.1. Convergence Analysis

#### 3.1.1. Existence and Uniqueness Theorem
Define \( F : X \to X \) as a contentious mapping, and the function \( F(t, u_0, u_1, \ldots, u_{m-1}) \) exists with continuous and bounded derivatives, where \( X \) is the Banach space \( (C(J), \| \cdot \|) \), the space of all continuous functions on \( J \) with the norm
\[ \| u \| = \max_{t \in J} |u(t)| , \]
and satisfies Lipschitz condition with Lipschitz constant \( L \), such that
\[ |f(t, u_1(x, y, z, t), D_x^\alpha u_1(x, y, z, t)) - f(t, u_2(x, y, z, t), D_x^\alpha u_2(x, y, z, t))| \]
\[ \leq L \| u_1(x, y, z, t) - u_2(x, y, z, t) \| , \]
\[ \frac{|D_y^\alpha u_1(x, y, z, t) - D_y^\alpha u_2(x, y, z, t)|}{\alpha - n} \]
\[ \leq L \| u_1(x, y, z, t) - u_2(x, y, z, t) \| , \]
\[ \frac{|D_z^\alpha u_1(x, y, z, t) - D_z^\alpha u_2(x, y, z, t)|}{\alpha - n} \]
\[ \leq L \| u_1(x, y, z, t) - u_2(x, y, z, t) \| , \]
\[ 0 < L < 1, \ t \geq 0. \]  

**Theorem 6.** Let \( f \) satisfy the Lipschitz condition (18) then the problem (4) with (2) and (3) has unique solution \( u(x, t) \), whenever \( 0 < L < 1 \).

**Proof.** (1) The existence of the solution. From equation (4) we have
\[ u = f \left( t, \sum_{j=0}^{m-1} c_j t^j + f^a u, f^a D_x^a u, f^a D_y^a u, f^a D_z^a u \right) . \]  
\[ (19) \]

The mapping \( F : X \to X \) is defined as
\[ F(u) = f \left( t, \sum_{j=0}^{m-1} c_j t^j + f^a u, f^a D_x^a u, f^a D_y^a u, f^a D_z^a u \right) . \]  
\[ (20) \]
Let \( u, v \in X \), then
\[ |F(u) - F(v)| \]
\[ = \left| f \left( t, \sum_{j=0}^{m-1} c_j t^j + f^a u_i, f^a D_x^a u_i, f^a D_y^a u_i, f^a D_z^a u_i \right) - f \left( t, \sum_{j=0}^{m-1} c_j t^j + f^a v, f^a D_x^a v, f^a D_y^a v, f^a D_z^a v \right) \right| \]
\[ \leq L \sum_{i=0}^{3} |f^a D_x^a u_i - f^a D_y^a v| \]
\[ \leq L \sum_{i=0}^{3} \left| \frac{1}{\Gamma(\alpha - n)} \int_0^t (t-s)^{\alpha-1} \right| u - v \right| ds \]
\[ \leq L \sum_{i=0}^{3} \left| \frac{1}{\Gamma(\alpha - n)} \int_0^t (t-s)^{\alpha-1} \right| u - v \right| ds \]
\[ \leq \gamma \| u - v \| , \]
\[ (21) \]
where \( \gamma = \sum_{i=0}^{3} LMT(\Gamma(\alpha - n)) ) < 1 \), then we get
\[ \| F(u) - F(v) \| \leq \| u - v \| , \]
therefore the mapping \( F \) is contraction, and there exists unique solution \( u \in C(J) \) to problem (4).

(2) The uniqueness of the solution (see [23]).

\[ \square \]

#### 3.1.2. Proof of Convergence

**Theorem 7.** Suppose that \( X \) is Banach space and \( F : X \to X \) satisfies condition (18). Then, the sequence (14) converges to the solution of (4) with (2) and (3).

**Proof.** Defined \( (C(J), \| \cdot \|) \) is the Banach space, the space of all continuous functions on \( J \) with the norm
\[ \| u(x, y, z, t) \| = \max_{t \in J} |u(x, y, z, t)| . \]  
\[ (23) \]
We need to show that \( \{u_n\} \) is a Cauchy sequence in this Banach space:

\[
\|u_n - u_m\| = \max |u_n - u_m| = \max |u_{n-1} - \frac{(\alpha - 1)}{\Gamma(\alpha)} \times \int_0^t (t-s)^{\alpha-1} \times [D^\alpha u_{n-1}(x, y, z, s) - F(s, u_{n-1}(s), D_x^n u_{n-1}(s), \ldots) ds - u_{m-1} + \frac{(\alpha - 1)}{\Gamma(\alpha)} \times \int_0^t (t-s)^{\alpha-1} \times [D^\alpha u_{m-1} - F(s, u_{m-1}(s), D_x^n u_{m-1}, \ldots) ds - F(s, u_{m-1}, D_x^n u_{m-1}, D_y^n u_{m-1}, D_z^n u_{m-1}) ds] | u_{n-1} - u_{m-1} | ds |} \leq \max |u_{n-1} - u_{m-1}| - \frac{(\alpha - 1)}{\Gamma(\alpha)} \times \int_0^t (t-s)^{\alpha-1} |u_{n-1} - u_{m-1}| ds \leq \max |u_{n-1} - u_{m-1}| - \frac{(M + (m_1 + m_2 + m_3) RT)}{\Gamma(\alpha)} \times \int_0^t (t-s)^{\alpha-1} |u_{n-1} - u_{m-1}| ds \leq \max |u_{n-1} - u_{m-1}| - \frac{(M + (m_1 + m_2 + m_3) RT)}{\Gamma(\alpha-1)} \times \int_0^t (t-s)^{\alpha-1} |u_{n-1} - u_{m-1}| ds \]

where

\[
R = \max \{ (t-s)^{\alpha-1} \}_{0 \leq s \leq t, 0 \leq t \leq T},
\]

Finally, we have

\[
\|u_n - u_m\| \leq \left( 1 - \frac{(M + (m_1 + m_2 + m_3) RT)}{\Gamma(\alpha - 1)} \right) \times \|u_{n-1} - u_{m-1}\| \leq \gamma \|u_{n-1} - u_{m-1}\|,
\]

where \( M, R, \Gamma(\alpha) \) are constants and

\[
y = \left( 1 - \frac{(M + (m_1 + m_2 + m_3) RT)}{\Gamma(\alpha - 1)} \right).
\]

Let \( n = m + 1 \). Then

\[
\|u_{n+1} - u_m\| \leq \gamma \|u_{m-1} - u_{m-2}\| \leq \cdots \leq \gamma^n \|u_1 - u_0\|.
\]

From the triangle inequality, we have

\[
\|u_n - u_m\| \leq \|u_{n+1} - u_m\| + \|u_{n+2} - u_{n+1}\| + \cdots + \|u_{n+1} - u_0\| = \gamma^n \|u_1 - u_0\| + \gamma^{n+1} \|u_1 - u_0\| + \cdots + \gamma^{n-1} \|u_1 - u_0\| \leq \gamma^n [1 + \gamma + \gamma^2 + \cdots + \gamma^{n-1}] \|u_1 - u_0\| \leq \gamma^n \left( \frac{1 - \gamma^{n-m}}{1 - \gamma} \right) \|u_1 - u_0\|.
\]

Since \( 0 < \gamma < 1 \), so \( 1 - \gamma^{n-m} < 1 \), and then

\[
\|u_n - u_m\| \leq \gamma^n \|u_1 - u_0\|.
\]

But \( \|u_1 - u_0\| < \infty \); then \( \|u_n - u_m\| \to 0 \) as \( m \to \infty \). We conclude that \( u_n \) is a Cauchy sequence in \( C[J] \), so the sequence converges and the proof is complete. \( \Box \)

3.1.3. Error Analysis

**Theorem 8.** The maximum absolute error of the approximate solution \( u_m \) to problem (4)-(3) is estimated to be

\[
\max_{t \in J} |u_{\text{exact}} - u_n| \leq k,
\]

(31)
where
\[
k = \left( \frac{y^m}{1 - y} \right) \| u_0 \|, \tag{32}\]
\[
\beta = \left( \frac{\alpha - 1}{\Gamma(\alpha)} \right). \tag{33}\]

**Proof.** From Theorem (9) and inequality (30) we have
\[
\| u_n - u_m \| \leq \left( \frac{y^m}{1 - y} \right) \| u_1 - u_0 \|, \tag{34}\]
as \( n \to \infty \); then \( u_n \to u_{\text{exact}} \) and
\[
\| u_1 - u_0 \| = \max_{t \in J} \left| - \frac{(\alpha - 1)}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} d\xi \right| \tag{35}\]
\[
= \left[ (M + (m_1 + m_2 + m_3 RT) \beta) \right] \| u_0 \|, \tag{36}\]
where \( \beta = \left| \left( (\alpha - 1)/\Gamma(\alpha) \right) \right| \), and thus, the maximum absolute error in the interval \( J \) is
\[
\| u_{\text{exact}} - u_n \| \leq \max_{t \in J} | u_{\text{exact}} - u_n | \leq k. \tag{37}\]

This completes the proof. \( \square \)

### 4. Numerical Examples

**Example 1.** Consider the following fourth-order fractional singular partial differential equation:
\[
\frac{\partial^\alpha u}{\partial t^\alpha} + \left( \frac{1}{x + \frac{x^4}{120}} \right) \frac{\partial^4 u}{\partial x^4} = 0, \tag{38}\]
\[
\frac{1}{2} < x < 1, \quad t > 0, \quad 1 < \alpha \leq 2. \tag{39}\]

With initial conditions
\[
u(x, 0) = 0, \quad \frac{1}{2} < x < 1, \tag{40}\]
\[
\frac{\partial u}{\partial t} (x, 0) = 1 + \frac{x^5}{120}, \quad 0 < x < 1 \tag{41}\]
and boundary conditions
\[
u \left( \frac{1}{2} t \right) = \left( 1 + \frac{(1/2)^5}{120} \right) \sin t, \tag{42}\]
\[
u (1, t) = \frac{121}{120} \sin t, \quad t > 0, \tag{43}\]
\[
\frac{\partial^2 u}{\partial x^2} \left( \frac{1}{2} t \right) = \frac{1}{6} \left( \frac{1}{6} \right)^3 \sin t, \tag{44}\]
\[
\frac{\partial^2 u}{\partial x^2} (1, t) = \frac{1}{6} \sin t, \quad t > 0, \tag{45}\]

the exact solution in special case \( \alpha = 2 \) is
\[
u (x, t) = \left( 1 + \frac{x^5}{120} \right) \sin t \tag{46}\]
and we solve the problem (36) by variational iteration method. According to variational iteration method, formula (14) for (36) can be expressed in the following form:
\[
u_{m+1} (x, t) = u_n (x, t) - (\alpha - 1) \int_0^t \left( F (s, u_0, D_x^s u_0, \ldots, D_x^{\alpha} u_0) \right) ds \tag{47}\]

\[
\int \left( \frac{\partial^\alpha u_0 (x, t)}{\partial t^\alpha} + \left( \frac{1}{x + \frac{x^4}{120}} \right) \frac{\partial^4 u_0}{\partial x^4} \right) dt \tag{48}\]

Suppose that an initial approximation has the following form which satisfies the initial conditions:
\[
u_0 (x, t) = \left( 1 + \frac{x^5}{120} \right) t. \tag{49}\]
Now by iteration formula (16), we obtain the following approximations:
\[
u_1 (x, t) = \nu_0 (x, t) - (\alpha - 1) \int_0^t \left( \frac{\partial^\alpha u_0 (x, t)}{\partial t^\alpha} + \left( \frac{1}{x + \frac{x^4}{120}} \right) \frac{\partial^4 u_0}{\partial x^4} \right) dt \tag{50}\]
\[
\int \left( \frac{\partial^\alpha u_1 (x, t)}{\partial t^\alpha} + \left( \frac{1}{x + \frac{x^4}{120}} \right) \frac{\partial^4 u_1}{\partial x^4} \right) dt \tag{51}\]

The second approximation takes the following form:
\[
u_2 (x, t) = \nu_1 (x, t) - (\alpha - 1) \int_0^t \left( \frac{\partial^\alpha u_1 (x, t)}{\partial t^\alpha} + \left( \frac{1}{x + \frac{x^4}{120}} \right) \frac{\partial^4 u_1}{\partial x^4} \right) dt \tag{52}\]
\[
\int \left( \frac{\partial^\alpha u_2 (x, t)}{\partial t^\alpha} + \left( \frac{1}{x + \frac{x^4}{120}} \right) \frac{\partial^4 u_2}{\partial x^4} \right) dt \tag{53}\]
\begin{equation}
\begin{aligned}
u_3(x,t) &= \nu_2(x,t) - (\alpha - 1) J_t^\alpha \\
&\quad \times \left( \frac{\partial^4 \nu_2(x,t)}{\partial t^4} + \left( 1 + \frac{x^4}{120} \right) \frac{\partial^4 \nu_1}{\partial x^4} \right) \\
&= \left( 1 + \frac{x^5}{120} \right) \\
&\quad \times \left( t - (\alpha - 1)^3 \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + (\alpha - 1)^2 \\
&\quad \times \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} - (\alpha - 1)^3 \frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)} \right)
\end{aligned}
\end{equation}

\text{(43)}

Table 1 shows the absolute error of VIM solution of example (36) (when $\alpha = 1.999$, $x = 0.1$, and $n = 2$), while Table 2 shows the maximum absolute truncated error of VIM solution (using Theorem 8) at different values of $n$ (when $t = 2$).

\textbf{Example 2.} Consider the following fourth-order fractional singular partial differential equation:

\begin{equation}
\begin{aligned}
&\frac{\partial^\alpha \nu}{\partial t^\alpha} + (\frac{x}{\sin x} - 1) \frac{\partial^4 \nu}{\partial x^4} = 0, \\
&0 < x < 1, t > 0, 1 < \alpha \leq 2.
\end{aligned}
\end{equation}

With initial conditions

\begin{equation}
\begin{aligned}
u(x,0) &= x - \sin x, \quad 0 < x < 1 \\
\frac{\partial \nu}{\partial t}(x,0) &= -(x - \sin x), \quad 0 < x < 1
\end{aligned}
\end{equation}

and boundary conditions

\begin{equation}
\begin{aligned}
u(0,t) &= 0, \quad \nu(1,t) = e^{-t} (1 - \sin 1), \quad t > 0, \\
\frac{\partial^2 \nu}{\partial x^2}(0,t) &= 0, \quad \frac{\partial^2 \nu}{\partial x^2}(1,t) = e^{-t} \sin 1, \quad t > 1,
\end{aligned}
\end{equation}

the exact solution in special case $\alpha = 2$ is

\begin{equation}
\begin{aligned}
u(x,t) &= (x - \sin x) e^{-t}.
\end{aligned}
\end{equation}

According to variational iteration method, formula (14) for (44) can be expressed in the following form:

\begin{equation}
\begin{aligned}
u_{k+1}(x,t) &= \nu_k(x,t) - (\alpha - 1) J_t^\alpha \\
&\quad \times \left( \frac{\partial^\alpha \nu_k(x,t)}{\partial t^\alpha} + \left( \frac{x}{\sin x} - 1 \right) \frac{\partial^4 \nu_k}{\partial x^4} \right) \\
&= \nu_1(x,t) - (\alpha - 1) J_t^\alpha \\
&\quad \times \left( \frac{\partial^\alpha \nu_1(x,t)}{\partial t^\alpha} + \left( \frac{x}{\sin x} - 1 \right) \frac{\partial^4 \nu_1}{\partial x^4} \right) \\
&= \nu_2(x,t)
\end{aligned}
\end{equation}

\text{(48)}

\begin{table}[h]
\centering
\caption{Absolute error.}
\begin{tabular}{|c|c|}
\hline
$t$ & Error of VIM ($n = 2$) \\
\hline
0.2 & $1.9635 \times 10^{-7}$ \\
0.4 & $9.44308 \times 10^{-6}$ \\
0.6 & $5.15266 \times 10^{-5}$ \\
0.8 & $1.7613 \times 10^{-4}$ \\
1 & $4.98115 \times 10^{-4}$ \\
1.2 & 0.0012731 \\
1.4 & 0.00301987 \\
1.6 & 0.00699301 \\
1.8 & 0.0139188 \\
2 & 0.02729 \\
\hline
\end{tabular}
\end{table}

\begin{table}[h]
\centering
\caption{Maximum absolute error.}
\begin{tabular}{|c|c|}
\hline
$n$ & Maximum error VIM \\
\hline
2 & 0.0272901 \\
3 & 0.00186871 \\
4 & 0.00328421 \\
\hline
\end{tabular}
\end{table}

Suppose that an initial approximation has the following form which satisfies the initial condition:

\begin{equation}
\begin{aligned}
u_0(x,t) &= (x - \sin x) - (x - \sin x) t.
\end{aligned}
\end{equation}

Now by iteration formula (48), we obtain the first approximation

\begin{equation}
\begin{aligned}
u_1(x,t) &= \nu_0(x,t) - (\alpha - 1) J_t^\alpha \\
&\quad \times \left( \frac{\partial^\alpha \nu_0(x,t)}{\partial t^\alpha} + \left( \frac{x}{\sin x} - 1 \right) \frac{\partial^4 \nu_0}{\partial x^4} \right) \\
&= (x - \sin x) - (x - \sin x) t \\
&\quad + (\alpha - 1)(x - \sin x) \frac{t^\alpha}{\Gamma(\alpha + 1)} - (\alpha - 1) \\
&\quad \times (x - \sin x) \frac{t^{\alpha+1}}{\Gamma(\alpha + 2)}
\end{aligned}
\end{equation}

\text{(50)}

and second approximation

\begin{equation}
\begin{aligned}
u_2(x,t) &= \nu_1(x,t) - (\alpha - 1) J_t^\alpha \\
&\quad \times \left( \frac{\partial^\alpha \nu_1(x,t)}{\partial t^\alpha} + \left( \frac{x}{\sin x} - 1 \right) \frac{\partial^4 \nu_1}{\partial x^4} \right) \\
&= (x - \sin x) - (x - \sin x) t \\
&\quad - (\alpha - 3)(\alpha - 1)(x - \sin x) \frac{t^\alpha}{\Gamma(\alpha + 1)}
\end{aligned}
\end{equation}
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\[ u_0 (x, t) = (x - \sin x) - (x - \sin x) t \]

\[ u_1 (x, t) = u_0 (x, t) - (\alpha - 1) t \]

\[ \times \left( \frac{\partial^\alpha}{\partial t^\alpha} + \left( \frac{x}{\sin x} - 1 \right) \frac{\partial^4 u_0}{\partial x^4} \right) \]

\[ = (x - \sin x) - (x - \sin x) t \]

\[ + (\alpha - 1) (x - \sin x) \frac{r^\alpha}{\Gamma(\alpha + 1)} \]

\[ - (\alpha - 1) (x - \sin x) \frac{r^{\alpha+1}}{\Gamma(\alpha + 2)} \]

\[ \times \left( \frac{\partial^\alpha}{\partial t^\alpha} + \left( \frac{x}{\sin x} - 1 \right) \frac{\partial^4 u_1}{\partial x^4} \right) \]

\[ = (x - \sin x) - (x - \sin x) t \]

\[ - (\alpha - 3) (\alpha - 1) (x - \sin x) \frac{r^\alpha}{\Gamma(\alpha + 1)} \]

\[ + (\alpha - 3) (\alpha - 1) (x - \sin x) \frac{r^{\alpha+1}}{\Gamma(\alpha + 2)} \]

\[ u_2 (x, t) = u_1 (x, t) - (\alpha - 1) t \]

\[ \times \left( \frac{\partial^\alpha}{\partial t^\alpha} + \left( \frac{x}{\sin x} - 1 \right) \frac{\partial^4 u_1}{\partial x^4} \right) \]

\[ = (x - \sin x) - (x - \sin x) t \]

\[ - (\alpha - 3) (\alpha - 1) (x - \sin x) \frac{r^\alpha}{\Gamma(\alpha + 1)} \]

\[ + (\alpha - 3) (\alpha - 1) (x - \sin x) \frac{r^{\alpha+1}}{\Gamma(\alpha + 2)} \]

Table 3 shows the absolute error of VIM solution of example (37) (when \( \alpha = 1.5, x = 0.1, \) and \( n = 2 \)), while Table 4 shows the maximum absolute truncated error of VIM solution (using Theorem 8) at different values of \( n \) (when \( t = 2 \)).

**Example 3.** Consider the following singular two-dimensional partial differential equation of fractional order:

\[ \frac{\partial^\alpha}{\partial t^\alpha} + 2 \left( \frac{1}{x^2} + \frac{x^4}{6!} \right) \frac{\partial^4 u}{\partial x^4} + 2 \left( \frac{1}{y^2} + \frac{y^4}{6!} \right) \frac{\partial^4 u}{\partial y^4} = 0, \]

\[ 0 < x, \ y < 1, \ t > 0, \ 1 < \alpha \leq 2. \]

With initial conditions

\[ u (x, y, 0) = 0, \ 0 < x < 1 \]

\[ \frac{\partial u}{\partial t} (x, y, 0) = 2 + \frac{x^6}{6!} + \frac{y^6}{6!}, \ 0 < x < 1 \]

and boundary conditions

\[ u (0.5, y, t) = \left( 2 + \frac{0.5^6}{6!} + \frac{y^6}{6!} \right) \sin t, \]

\[ u (1, y, t) = \left( 2 + \frac{1^6}{6!} + \frac{y^6}{6!} \right) \sin t, \ t > 0, \]
\[ \frac{\partial^2 u}{\partial x^2} (0.5, y, t) = \frac{(0.5)^4}{6!} \sin t, \]
\[ \frac{\partial^2 u}{\partial y^2} (1, 0.5, t) = \frac{1}{6!} \sin t, \quad t > 1, \]
\[ \frac{\partial^2 u}{\partial y^2} (x, 0.5, t) = \frac{(0.5)^4}{6!} \sin t, \]
\[ \frac{\partial^2 u}{\partial y^2} (x, 1, t) = \frac{1}{6!} \sin t, \quad t > 1, \]

the exact solution in special case \( \alpha = 2 \) is
\[ u(x, y, t) = \left( 2 + \frac{x^6}{6!} + \frac{y^6}{6!} \right) \sin t. \]

According to variational iteration method, formula (14) for (52) can be expressed in the following form:
\[ u_{k+1}(x, t) = u_k(x, t) - (\alpha - 1) f_t^\alpha \]
\[ \times \left( \frac{\partial^4 u_k(x, t)}{\partial t^4} + 2 \left( \frac{x^4}{6!} \right) \frac{\partial^4 u_k(x, t)}{\partial x^4} \right) \]
\[ + 2 \left( \frac{1}{y^4} \right) \frac{\partial^4 u_k}{\partial y^4}. \]

Suppose that an initial approximation has the following form which satisfies the initial conditions:
\[ u_0(x, t) = \left( 2 + \frac{x^6}{6!} + \frac{y^6}{6!} \right) t. \]

Now by iteration formula (56), we obtain the following approximations:
\[ u_1(x, t) = u_0(x, t) - (\alpha - 1) f_t^\alpha \]
\[ \times \left( \frac{\partial^4 u_0(x, t)}{\partial t^4} + 2 \left( \frac{x^4}{6!} \right) \frac{\partial^4 u_0(x, t)}{\partial x^4} \right) \]
\[ + 2 \left( \frac{1}{y^4} \right) \frac{\partial^4 u_0}{\partial y^4} \]
\[ = \left( 2 + \frac{x^6}{6!} + \frac{y^6}{6!} \right) t - 2 (\alpha - 1) \left( 2 + \frac{x^6}{6!} + \frac{y^6}{6!} \right) \frac{t^{\alpha+1}}{\Gamma(\alpha + 2)}. \]

The second approximation takes the following form:
\[ u_2(x, t) = u_1(x, t) - (\alpha - 1) f_t^\alpha \]
\[ \times \left( \frac{\partial^4 u_1(x, t)}{\partial t^4} + 2 \left( \frac{x^4}{6!} \right) \frac{\partial^4 u_1(x, t)}{\partial x^4} \right) \]
\[ + 2 \left( \frac{1}{y^4} \right) \frac{\partial^4 u_1}{\partial y^4} \]
\[ = \left( 2 + \frac{x^6}{6!} + \frac{y^6}{6!} \right) t - 2 (\alpha - 1) \left( 2 + \frac{x^6}{6!} + \frac{y^6}{6!} \right) \frac{t^{2\alpha+1}}{\Gamma(2\alpha + 1)} \]
\[ + (\alpha - 1)^2 \left( 2 + \frac{x^6}{6!} + \frac{y^6}{6!} \right) \frac{t^{3\alpha+1}}{\Gamma(3\alpha + 2)}. \]

Table 5 shows the absolute error of VIM solution of example (38) (when \( \alpha = 1.999 \), \( x = y = 0.1 \), and \( n = 2 \)), while Table 6 shows the maximum absolute truncated error of VIM solution (using Theorem 8, resp.) at different values of \( n \) (when \( t = 2 \)).
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