SYMPLECTIC STRUCTURES ON THE COTANGENT BUNDLES
OF OPEN 4-MANIFOLDS

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Abstract. We show that, for any two orientable smooth open 4-manifolds
$X_0, X_1$ which are homeomorphic, their cotangent bundles $T^*X_0, T^*X_1$ are
symplectomorphic with their canonical symplectic structure. In particular, for
any smooth manifold $R$ homeomorphic to $\mathbb{R}^4$, the standard Stein structure on
$T^*R$ is Stein homotopic to the standard Stein structure on $T^*\mathbb{R}^4 = \mathbb{R}^8$. We
use this to show that any exotic $\mathbb{R}^4$ embeds in the standard symplectic $\mathbb{R}^8$
as a Lagrangian submanifold. As a corollary, we show that $\mathbb{R}^8$ has uncount-
ablely many smoothly distinct foliations by Lagrangian $\mathbb{R}^4$s with their standard
smooth structure.

1. Introduction

We begin with some basics, which can be found in [10]. Throughout, assume that
all manifolds are orientable, smoothable, and come with a fixed smooth structure
unless otherwise indicated. Let $N$ be a smooth manifold of real dimension $n$. The
cotangent bundle of $N$, $T^*N$, carries a canonical 1-form $\lambda_0$ defined, in local
coordinates $x_1, \ldots, x_n, y_1 = dx_1, \ldots, y_n = dx_n$, by $\lambda_0 = \sum_{i=1}^n y_i dx_i$. The 1-form
$\lambda_0 \in \Omega^1(T^*N)$ is uniquely characterized by the property that $\sigma^* \lambda = \sigma$ for any
1-form $\sigma$ on $N$ thought of as a section of $T^*N$. Then, for any diffeomorphism
$\varphi : M \rightarrow N$, $\lambda_N = \varphi^* \lambda_0$ where $\varphi^*$ is the induced map $\varphi^* : T^*(T^*M) \rightarrow
T^*(T^*N)$.

From the canonical 1-form we obtain a canonical symplectic form $\omega_0 = -d\lambda_0$ on
$T^*N$. In local coordinates, $\omega_0$ has the form $\omega_0 = \sum_{i=1}^n dx_i \wedge dy_i$. As $\omega_0$ depends
only on $\lambda_0$, any two diffeomorphic manifolds have symplectomorphic cotangent
bundles. This fact is the basis of an idea of V. I. Arnol’d which states that the
smooth topology of a manifold should be reflected in the symplectic topology of its
cotangent bundle.

As a realization of this idea, M. Abouzaid showed in [1] that in dimension $n \equiv 1$
mod 4, an exotic $n$-sphere $S$ which does not bound a paralizable manifold does
not embed as a Lagrangian submanifold of $T^*S^n$ with the the standard symplectic
structure; hence the cotangent bundle of such an exotic sphere cannot be symplec-
tomorphic to $T^*S^n$. As $T^*S$ and $T^*S^n$ are diffeomorphic, the canonical symplectic
structure on $T^*S$ can be considered an exotic symplectic structure on $T^*S^n$.

The existence and classification of exotic symplectic structures on a given smooth
manifold is of independent interest. In [14] P. Seidel and I. Smith and in [2] M.
Abouzaid and P. Seidel construct exotic symplectic structures on $\mathbb{R}^{2n}$ for $n \geq 4$.
Also, in [11], M. McLean constructs exotic symplectic structures on $T^*S^n$ for $n \geq$
3. However, these constructions do not arise as cotangent bundles, nor are they symplectomorphic to them in any obvious way.

When \( n \neq 4 \), smoothing theory tells us that there is a unique smooth structure on the topological manifold \( \mathbb{R}^n \) up to homeomorphism. (See [15] for \( n > 4 \), [12] for \( n = 3 \).) So among the symplectic structures on \( T^*\mathbb{R}^n \cong \mathbb{R}^{2n} \), there is a fixed one which corresponds to the canonical structure on \( T^*\mathbb{R}^n \). However in dimension \( n = 4 \), the topological manifold \( \mathbb{R}^4 \) admits uncountably infinitely many inequivalent smoothings. For each smooth manifold \( R \), homeomorphic to \( \mathbb{R}^4 \), \( T^*R \) and \( T^*\mathbb{R}^4 \) are diffeomorphic.

It is then natural to ask: Are any of the exotic symplectic structures on \( \mathbb{R}^8 \) symplectomorphic to the canonical structure on \( T^*R \) for some smooth \( R \) homeomorphic to \( \mathbb{R}^4 \)? We answer a stronger question in the negative.

**Theorem 1.** Let \( X_0, X_1 \) be smooth, open, homeomorphic 4-manifolds. If \( \pi_1(X_i) \neq 0 \) we assume that there is an \( s \)-cobordism between \( X_0 \) and \( X_1 \). Then each of \( T^*X_0 \) and \( T^*X_1 \) have a fixed Stein structure up to Stein homotopy and the Stein structures on \( T^*X_0 \) and \( T^*X_1 \) are Stein homotopic. Therefore, \( T^*X_0 \) and \( T^*X_1 \) are symplectomorphic with their canonical symplectic structures.

We reserve the definition of a Stein manifold until the next section. A special case of Theorem 1 is the following:

**Corollary 2.** Let \( R \) be a smooth 4-manifold homeomorphic to \( \mathbb{R}^4 \). Then the Stein structure on \( T^*R \) is Stein homotopic to the standard Stein structure on \( \mathbb{R}^8 \); hence \( T^*R \) and \( \mathbb{R}^8 \) are symplectomorphic.

A Lagrangian submanifold \( L \) of a symplectic manifold \( (V, \omega) \) is a submanifold of maximal dimension where \( \omega|_L \equiv 0 \). In a cotangent bundle, graphs of closed 1-forms are Lagrangian.

**Corollary 3.** Let \( R \) be any exotic \( \mathbb{R}^4 \). Then \( R \) embeds in the standard symplectic \( (\mathbb{R}^8, \omega) \) as a Lagrangian submanifold.

**Proof.** The zero section of \( T^*R \) is a Lagrangian copy of \( R \) which sits inside \( \mathbb{R}^8 \) as its image under the symplectomorphism. \( \square \)

Currently, there are no known smooth manifolds which

1. are homeomorphic to \( \mathbb{R}^4 \),
2. have finite handlebody decompositions, and
3. which are known to be not diffeomorphic to the standard \( \mathbb{R}^4 \).

If the smooth 4-dimensional Poincaré conjecture is false, such an object exists and arises by puncturing an exotic 4-sphere. Potential examples arise via Gluck twists [7] or from proposed counterexamples to the Andrews-Curtis conjecture. Recall that the Andrews-Curtis conjecture states that balanced presentations of the trivial group can be trivialized using the Andrews-Curtis moves; a collection of moves on group presentations related to elementary Morse moves of 1 and 2 handles. (See Remark 5.1.11 of [8] for examples.) Since no such finite handlebody is currently

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1Recall that an \( s \)-cobordism is an \( h \)-cobordism with vanishing Whitehead torsion. In dimension 4, Freedman proved that in the case of “good” fundamental group, such an \( s \)-cobordism has a topological product structure. See Theorem 7.1A of [6]. We will only require the existence of such an \( s \)-cobordism, not a product structure on it, so we can remove the qualification on \( \pi_1 \).
known to be exotic, all known examples involve highly complicated behavior at infinity.

This complicated behavior at infinity obstructs the usual definitions of Lagrangian Floer homology for non-compact Lagrangians. We then ask the question: Can a Lagrangian Floer homology be defined for Lagrangians such as we have describe? If so, can any of the above Lagrangian exotic $\mathbb{R}^4$’s be distinguished by their Floer homologies?

A symplectic manifold $(V, \omega)$ is called exact if $\omega = d\alpha$ for some 1-form $\alpha$. (In the case of the canonical structure on a cotangent bundle $\alpha = -\lambda_0$.) A Lagrangian submanifold $L$ of an exact symplectic manifold is exact if $\alpha|_L$ is exact. Note that $\alpha|_L$ is closed on $L$ as $d\alpha|_L = \omega|_L = 0$. When $L$ has $H^1(L; \mathbb{R}) = 0$, every closed form is exact.

The usual version of the nearby Lagrangian conjecture states that if a closed manifold $L$ is an exact Lagrangian in $T^*N$ (with $N$ compact) then $L$ is Hamiltonian isotopic to the zero section of $T^*N$. We have then shown that the corresponding non-compact version (when $N$ is 4-dimensional and open) is false – without some sort of control at infinity – since such a Hamiltonian isotopy would give a diffeomorphism between any two smooth structures on $N$.

Let $(X_0, J_0)$ and $(X_1, J_1)$ be two smooth manifolds with smooth foliations. We will call these two foliations smoothly equivalent if there exists a diffeomorphism $\phi : X_0 \to X_1$ such that $\phi_*(J_0) = J_1$. Suppose that a foliated manifold $(X, \mathcal{F})$ admits a smooth global slice by which we will mean a smooth manifold $Y$ and smooth embedding $Y \to X$ which intersects every leaf of $\mathcal{F}$ once transversely. It is straightforward to show that any two global slices of $\mathcal{F}$ are diffeomorphic; hence the leaf space of $\mathcal{F}$ is naturally identified with $Y$ with its smooth structure.

**Corollary 4.** The standard symplectic $\mathbb{R}^8$ admits uncountably infinitely many smoothly distinct foliations by Lagrangian $\mathbb{R}^4$’s with the standard smooth structure.

**Proof.** For each exotic $\mathbb{R}^4$, $R$, $T^*R$ has a codimension 4 foliation by the fibers of the projection $T^*R \to R$, each a Lagrangian $\mathbb{R}^4$ with its standard smooth structure. Since the leaf space $\mathcal{L}$ is naturally identified with $R$, $\mathcal{L}$ is a smooth manifold and its smooth type is an invariant of the foliation. The result follows by the uncountability of smooth structures on $\mathbb{R}^4$ together with Corollary 2. \qed

### 2. Stein Manifolds

A smooth manifold $V$ of real dimension $2n$, equipped with an almost complex structure $J$ is said to be Stein if $J$ is integrable and $V$ admits a proper holomorphic embedding into $\mathbb{C}^N$ for some $N$. By [9, 3, 13], a complex manifold $(V, J)$ is Stein if and only if it admits a smooth function $\phi : V \to \mathbb{R}$ which is

1. proper and bounded below (exhausting) and
2. $J$-convex in the sense that $-dd^c\phi(v, Jv) > 0$ for all $v$. Here $d^c\phi$ denotes $d\phi \circ J$.

We call the triple $(V, J, \phi)$ a Stein structure on $V$. Note that $-dd^c\phi$ is a symplectic form on $V$ compatible with $J$. In fact, the existence of a Stein structure only requires a weaker condition, due to the following theorem of Eliashberg:

**Theorem 5** (Eliashberg [5]). A smooth manifold $V^{2n}$ with $2n > 4$ admits a Stein structure if and only if it admits an almost complex structure $J$ and an exhausting...
Morse function $\phi$ with critical points of index $\leq n$. More precisely, $J$ is homotopic through almost complex structures to a complex structure $J'$ such that $\phi$ is $J'$ convex.

Associated to every symplectic manifold $(V, \omega)$ there is a contractible space of almost complex structures $J$ which are compatible with $\omega$ in the sense that $g(v, w) = \omega(v, Jw)$ is a Riemannian metric and $\omega(Jv, Jw) = \omega(v, w)$. When $(V, \omega) = (T^*N, \omega_0)$, we can construct a contractible subspace of these structures explicitly in terms of Riemannian metrics on $N$. That is, pick a Riemannian metric $g_N$ on $N$ and define $J_g$ in local coordinates $x_1, \ldots, x_n, y_1, \ldots, y_n$ (for $T^*N$) by

$$J_g \left( \frac{\partial}{\partial x_i} \right) = \sum_{j=1}^n (g_N)_{ij} \frac{\partial}{\partial y_j} \quad \text{and} \quad J_g \left( \frac{\partial}{\partial y_i} \right) = \sum_{j=1}^n -(g_N)^{ij} \frac{\partial}{\partial x_j}$$

Then $J_g$ is compatible with $\omega_0$ and, on the zero section $Z$, $g|_Z \equiv g_N$. This almost complex structure is not, in general, integrable.

Now, pick an exhausting Morse function $f : N \to \mathbb{R}$. Let $\phi : T^*N \to \mathbb{R}$ be defined by $\phi(x, y) = f(x) + \frac{1}{2} |y|^2$. Then $\phi$ is again an exhausting Morse function, now for $T^*N$, whose critical points occur along the zero section and have an index-preserving bijection with those of $f$. As long as the dimension of $N$ is $n > 2$, the conditions of Theorem 5 are satisfied and $T^*N$ admits a Stein structure.

An exhausting Morse function on $V^{2n}$ is called subcritical if it has only critical points of index $< n$. A Stein structure $(\phi, J)$ is called subcritical if $\phi$ is subcritical. In the subcritical case, Y. Eliashberg and K. Cieliebak showed that Stein structures are unique up to homotopy:

**Theorem 6** (Eliashberg, Cieliebak [4]). Let $n > 3$ and let $(\phi_0, J_0)$, $(\phi_1, J_1)$ be two subcritical Stein structures on $V^{2n}$. If $J_0$ and $J_1$ are homotopic as almost complex structures, then $(\phi_0, J_0)$ and $(\phi_1, J_1)$ are homotopic as Stein structures.

Here a Stein homotopy consists of a concatenation of “simple Morse homotopies” i.e. sequences of Morse birth-deaths and handle slides. In this case, critical points of the $\phi_i$ do not escape to infinity and Moser’s trick\(^2\) applies to give us a 1-parameter family of diffeomorphisms taking one Stein structure to the other. See [4] for details. Consequently, the underlying symplectic manifolds for the Stein structures $(\phi_0, J_0)$ and $(\phi_1, J_1)$ are symplectomorphic.

Before we begin with the proof of Theorem 1. We show the following:

**Lemma 7.** If two 4-manifolds $X_0, X_1$ are homeomorphic, then their cotangent bundles are diffeomorphic as 8-manifolds. If $\pi_1(X_i) \neq 0$, we assume that the $X_i$ are s-cobordant.

*Proof.* Let $W$ be an $h$-cobordism between $X_0$ and $X_1$. If $\pi_1(X_i) \neq 0$, then assume that the Whitehead torsion of $W$ vanishes so that $W$ is an s-cobordism. There is a rank 4 real vector bundle $T$ on $W$ which restricts to $T^*X_0$ and $T^*X_1$ on $\partial W$. The bundle $T$ can be obtained by pulling back $T^*X_0$ via the homotopy equivalence $W \to X_0$. Then we see that $T|_{X_1}$ is isomorphic to $T^*X_1$ by noting that it has the requisite characteristic classes.

The unit sphere bundle $S(T)$ is then an s-cobordism between 7-manifolds – the unit sphere bundles of $T^*X_i$ – and, by the s-cobordism theorem, a product.

Now, taking the unit disc bundle $D(T)$, we get an s-cobordism of 8-manifolds with boundary. As we have seen, it is a product on the boundary. As this is again

\(^2\)See Section 3.2 of [10] for the compact case.
a product by the $s$-cobordism theorem, the diffeomorphism of the $T^*X_i$ follows by restricting to the interior.

Proof of Theorem 1. If $X_0$ and $X_1$ are homeomorphic, then $T^*X_0$ and $T^*X_1$ are diffeomorphic by Lemma 7. Choose some representative $V$ of this diffeomorphism type and particular diffeomorphisms $\varphi_i : V \to T^*X_i$. The canonical 1-forms, symplectic forms and choices of almost complex structures for the cotangent bundles pull back to $\lambda_i$, $\omega_i$ and $J_i$ on $V$.

First, we show that $J_0$ and $J_1$ are homotopic as almost complex structures. Write $\mathcal{I}(n)$ for the space $GL^+(2n, \mathbb{R})/GL(n, \mathbb{C})$ which classifies almost complex structures on $\mathbb{R}^{2n}$. The space $\mathcal{I}(n)$ is homotopy equivalent to $SO(2n)/U(n)$ and, when $n = 4$, we can compute several of the homotopy groups:

$$\pi_0 \mathcal{I}(4) = 0, \quad \pi_1 \mathcal{I}(4) = 0, \quad \pi_2 \mathcal{I}(4) = \mathbb{Z},$$
$$\pi_3 \mathcal{I}(4) = 0, \quad \pi_4 \mathcal{I}(4) = 0, \quad \pi_5 \mathcal{I}(4) = 0,$$
$$\pi_6 \mathcal{I}(4) = \mathbb{Z},$$

The obstructions to a homotopy between $J_0$ and $J_1$ lie in $H^i(V, V^{(i-1)}; \pi_i(\mathcal{I}(4))$ where $V^{(i-1)}$ is the $i-1$-skeleton of $V$. As $H^i(V; \mathbb{Z}) = 0$ for $i \geq 4$, the only non-trivial group in this list is $H^2(V, V^{(1)}; \pi_2(\mathcal{I}(4))) \cong H^2(V, V^{(1)}; \mathbb{Z}) \cong H^2(X_i, X_i^{(1)})$. However, $J_0$ and $J_1$ are homotopic over the 2-skeleton so this obstruction vanishes. (Clearly they both have the same first Chern class.) Therefore, $J_0$ and $J_1$ are homotopic as almost complex structures.

As we saw above, for each Morse function $f_i$ on $X_i$, we obtain Morse functions $\phi_i$ on $T^*X_i$ whose critical points (and, once a metric is chosen, flow lines along the zero section) can be identified with those of $f_i$. Note that the $\phi_i$ will be proper and bounded below when the $f_i$ are.

In order to apply Theorem 6, we will need to show that the $X_i$ admit Morse functions without index 4-critical points. To construct such a Morse function, begin with a Morse function which is bounded below and whose critical values are discrete, with a single critical point per critical value. With such a choice, $X_i$ is identified with a composition of elementary cobordisms.

Let $x_4$ be an index 4 critical point. Then the unstable manifold of $x_4$ must include trajectories to at most finitely many index 3 critical points. Further, if $x_3$ is an index 3 critical point, its stable manifold must meet at most 2 index 4 critical points. (Since the stable manifold is 1 dimensional.) As $H_4(X_i; \mathbb{Z}) = 0$, the boundary map $\partial : C_4(X) \to C_3(X)$ is injective; so, after possibly performing some handle-slides, we can cancel any index 4 critical point with an index 3 critical point. Therefore, each of the $X_i$ admit a Morse function $f_i$ without index 4-critical points.

Let $(\phi_i, J_i)$ be the Stein structures on $V$ constructed using Theorem 5 where $J_i$ is a complex structure homotopic to the almost complex structure $J_i'$ constructed from the metric on $X_i$. As each of the $\phi_i$ are subcritical, we then apply Theorem 6 to see that the $(\phi_i, J_i)$ are Stein homotopic.

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