Method for classifying multi-qubit states via the rank of coefficient matrix and its application to four-qubit states

Xiangrong Li¹, Dafa Li²,³

¹ Department of Mathematics, University of California, Irvine, CA 92697-3875, USA
² Department of mathematical sciences, Tsinghua University, Beijing 100084, CHINA
³ Center for Quantum Information Science and Technology, Tsinghua National Laboratory for information science and technology (TNList), Beijing 100084, CHINA

We construct coefficient matrices of size $2^n$ by $2^n - 1$ associated with pure $n$-qubit states and prove the invariance of the ranks of the coefficient matrices under stochastic local operations and classical communication (SLOCC). The ranks give rise to a simple way of partitioning pure $n$-qubit states into inequivalent families and distinguishing degenerate families from one another under SLOCC. Moreover, the classification scheme via the ranks of coefficient matrices can be combined with other schemes to build a more refined classification scheme. To exemplify we classify the nine families of four qubits introduced by Verstraete et al. [Phys. Rev. A 65, 052112 (2002)] further into inequivalent subfamilies via the ranks of coefficient matrices, and as a result, we find 28 genuinely entangled families and all the degenerate classes can be distinguished up to permutations of the four qubits. We also discuss the completeness of the classification of four qubits into nine families.

I. INTRODUCTION

Quantum entanglement plays a crucial role in quantum information theory, with applications to quantum teleportation, quantum cryptography, and quantum computation [1]. The equivalence under stochastic local operations and classical communication (SLOCC) induces a natural partition of quantum states. The central task of SLOCC classification is to classify quantum states according to a criterion that is invariant under SLOCC.

SLOCC entanglement classification has been the subject of intensive study during the last decade [2–21]. For three qubits, there are six SLOCC equivalence classes of which two are genuinely entangled classes: GHZ and W [2] and four degenerate classes can be distinguished by the local ranks (i.e., ranks of single-qubit reduced density matrices obtained by tracing out all but one qubit [2]). For four or more qubits, there are infinite SLOCC classes and it is highly desirable to partition the infinite classes into a finite number of families. The key lies in finding criteria to determine which family an arbitrary quantum state belongs to. In a pioneering work, Verstraete et al. [3] obtained nine SLOCC inequivalent families of four qubits using Lie group theory: $G_{abcd}$, $L_{abc2}$, $L_{ab2}$, $L_{ab3}$, $L_{a31}$, $L_{a0031}$, $L_{0031}$, $L_{0031}$, and $L_{00031}$. It is clear that, some families obtained by Verstraete et al. [3] contain an infinite number of SLOCC classes and some contain both degenerate classes and genuinely entangled classes. It is of great importance to find a more refined partition of four-qubit states such that the degenerate classes are distinguished from the genuinely entangled families. Many other efforts have been devoted to the SLOCC entanglement classification of four qubits [4–13]. More recently, a few attempts have been made toward the generalization to higher number of qubits, including odd $n$ qubits [14], even $n$ qubits [15], symmetric $n$ qubits [16–18], and general $n$ qubits [19, 20].

This paper is organized as follows. We first construct coefficient matrices of size $2^n$ by $2^n - 1$ associated to pure $n$-qubit states and prove the invariance of the ranks of coefficient matrices under SLOCC in Section II. In Section III, we present a recursive formula which allows us to easily calculate the ranks of coefficient matrices of $n$-qubit biseparable states. We next show that the degenerate families of general $n$ qubits are inequivalent to one another under SLOCC in Section IV. Section V is devoted to the classification of four qubits via the ranks of coefficient matrices. Section VI provides the discussion of the completeness of the nine families obtained by Verstraete et al. [3]. We finally conclude this paper in Section VII.

II. THE INVARIANCE OF THE RANKS OF COEFFICIENT MATRICES

Let $|\psi\rangle_{1\ldots n} = \sum_{i=0}^{2^n-1} a_i |i\rangle$ be an $n$-qubit pure state. We associate with the state $|\psi\rangle_{1\ldots n}$ a $2^n$ by the $2^n - 1$ coefficient matrix $C_{1\ldots, (\ell+1)\ldots n}(|\psi\rangle_{1\ldots n})$ whose entries are the coefficients $a_0, a_1, \ldots, a_{2^n-1}$ of the state $|\psi\rangle_{1\ldots n}$ arranged in ascending lexicographical
order. To illustrate, we list $C_{1\ldots\ell, (\ell+1)\ldots n}(|\psi\rangle_{1\ldots n})$ below as:

$$
\begin{pmatrix}
a_{0\ldots 0} & a_{0\ldots 1} & \cdots & a_{0\ldots n-\ell} & a_{0\ldots n-1} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \ddots & a_{0\ldots n-\ell} & a_{0\ldots n-1} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
a_{1\ldots 0} & a_{1\ldots 1} & \cdots & a_{1\ldots n-\ell} & a_{1\ldots n-1}
\end{pmatrix}
$$

In the binary form of the coefficient matrix in Eq. 1, bits 1 to $\ell$ and $\ell + 1$ to $n$ are referred to as the row bits and column bits, respectively. If $\ell = 0$, $C_{0\ldots n}(|\psi\rangle_{1\ldots n})$ reduces to the row vector $(a_0, \ldots, a_{2^n-1})$, and if $\ell = n$, $C_{1\ldots n}(|\psi\rangle_{1\ldots n})$ reduces to the column vector $(a_0, \ldots, a_{2^n-1})^T$.

Let $\{q_1, q_2, \ldots, q_n\}$ be a permutation of $\{1, 2, \ldots, n\}$. Let $C_{q_1, q_2, \ldots, q_n}(|\psi\rangle_{1\ldots n})$ be the $2^\ell \times 2^n - \ell$ coefficient matrix of the state $|\psi\rangle_{1\ldots n}$, which is constructed from the coefficient matrix $C_{1\ldots \ell, (\ell+1)\ldots n}$ in Eq. 1 by taking the corresponding permutation. Here $q_1, \ldots, q_{\ell}$ are the row bits and $q_{\ell+1}, \ldots, q_n$ are the column bits. Indeed, we only need to specify the row bits, as the column bits would simply be the rest of the bits. In the sequel, we will omit the subscripts $q_{\ell+1}, \ldots, q_n$ and simply write $C_{q_1, \ldots q_{\ell}}$, whenever the column bits are clear from the context.

It is known that two $n$-qubit pure states $|\psi\rangle_{1\ldots n}$ and $|\psi'\rangle_{1\ldots n}$ are equivalent to each other under SLOCC if and only if there are invertible operators $A_1, A_2, \ldots, A_n$ such that

$$
|\psi\rangle_{1\ldots n} = A_1 \otimes A_2 \otimes \cdots \otimes A_n |\psi'\rangle_{1\ldots n}.
$$

In terms of coefficient matrices, it can be verified that the following result holds: For any two SLOCC equivalent $n$-qubit pure states $|\psi\rangle_{1\ldots n}$ and $|\psi'\rangle_{1\ldots n}$, their coefficient matrices $C_{q_1, \ldots, q_{\ell}}$ satisfy the equation:

$$
C_{q_1, \ldots, q_{\ell}}(|\psi\rangle_{1\ldots n}) = (A_{q_1} \otimes \cdots \otimes A_{q_{\ell}})C_{q_1, \ldots, q_{\ell}}(|\psi\rangle_{1\ldots n})(A_{q_{\ell+1}} \otimes \cdots \otimes A_{q_n})^T,
$$

where $A_1, A_2, \ldots, A_n$ are the local operators in Eq. 2. Conversely, if there are local invertible operators $A_1, A_2, \ldots, A_n$ such that Eq. 3 holds true for some $C_{q_1, \ldots, q_{\ell}}$, then $|\psi\rangle_{1\ldots n}$ and $|\psi'\rangle_{1\ldots n}$ are equivalent under SLOCC.

It immediately follows from Eq. 3 that the rank of any coefficient matrix of an $n$-qubit pure state is invariant under SLOCC. This leads to the following theorem.

**Theorem 1.** If two $n$-qubit pure states are SLOCC equivalent then their coefficient matrices $C_{q_1, \ldots, q_{\ell}}$ given above have the same rank.

Restated in the contrapositive the theorem reads: If two coefficient matrices $C_{q_1, \ldots, q_{\ell}}$ associated with two $n$-qubit pure states differ in their ranks, then the two states belong necessarily to different SLOCC classes.

Coefficient matrices constructed above turn out to be closely related to reduced density matrices. We let $\rho_{12\ldots n}(|\psi\rangle_{1\ldots n}) = |\psi\rangle_{1\ldots n} \langle \psi|_{1\ldots n}$ be the density matrix of an $n$-qubit pure state $|\psi\rangle_{1\ldots n}$, and we let $\rho_{q_1, \ldots, q_{\ell}}$ be the $\ell$-qubit reduced density matrix obtained from $\rho_{12\ldots n}$ by tracing out $n - \ell$ qubits. As has been previously noted for bipartite systems of dimensions $d \times d$, a reduced density matrix has a full rank factorization in terms of the corresponding coefficient matrix and its conjugate transpose.

This factorization also holds for $n$-qubit states:

$$
\rho_{q_1, \ldots, q_{\ell}}(|\psi\rangle_{1\ldots n}) = C_{q_1, \ldots, q_{\ell}}(|\psi\rangle_{1\ldots n})C_{q_1, \ldots, q_{\ell}}^\dagger(|\psi\rangle_{1\ldots n}),
$$

where $C^\dagger$ is the conjugate transpose of $C$. An important relationship between reduced density matrices and SLOCC polynomial invariants can be obtained by taking the determinants of both sides of Eq. 4 for even $n$ and for $\ell = n/2$, yielding:

$$
\det \rho_{q_1, \ldots, q_{n/2}}(|\psi\rangle_{1\ldots n}) = |\det C_{q_1, \ldots, q_{n/2}}(|\psi\rangle_{1\ldots n})|^2.
$$

Here $\det C_{q_1, \ldots, q_{n/2}}(|\psi\rangle_{1\ldots n})$ is a SLOCC polynomial invariant of degree $2^{n/2}$ for even $n$ qubits and its absolute value can be used as an entanglement measure. Thus we have the following:

**Theorem 2.** For even $n$-qubit pure states, the determinants of $n/2$-qubit reduced density matrices are the squares of the SLOCC polynomial invariants of degree $2^{n/2}$, with the absolute values of the latter quantifying $n/2$-qubit entanglement of the even $n$-qubit states after tracing out the other $n/2$ qubits.

As an example, when $n = 4$ we have $\det \rho_{12} = |L|^2$, $\det \rho_{13} = |M|^2$, and $\det \rho_{14} = |N|^2$, where $L$, $M$, and $N$ are polynomial invariants of degree 4 [23]. When $n = 6$, there are 10 three-qubit reduced density matrices and 10 polynomial invariants of degree 8: $D^6_{11}, \ldots, D^6_{10}$ [24]. For reduced density matrix $\rho_{123}$ and polynomial invariant $D^6_{16}$, we have $\det \rho_{123} = |D^6_{16}|^2$. Similar equations hold for other reduced density matrices and polynomial invariants with appropriate permutations of qubits.
Remark 1. (i) The determinants of reduced density matrices are invariant under SLOCC. (ii) It is worth noting that Eq. (5) holds for bipartite systems of dimensions \( d \times d \) as well \([22]\).

As a particular case of Eq. (1), when \( q_i = i \) we have \( C_{1} | \psi \rangle_{1\cdots n} = C_{1} | \psi \rangle_{1\cdots n} C_{1} | \psi \rangle_{1\cdots n} \).

By virtue of Eq. (4), the rank of the \( n - q \)-qubit reduced density matrices obtained by tracing out \( n - q \) qubits is invariant under SLOCC. (ii). It is verified that
\[
C_{q_1\cdots q_k}(|\varphi\rangle_{j_1\cdots j_k} \otimes |\varphi\rangle_{j_{k+1}\cdots j_n})
= C_{q_1'\cdots q_k'}(|\varphi\rangle_{j_1\cdots j_k} \otimes |\varphi\rangle_{j_{k+1}\cdots j_n}).
\]

In view of the fact that the rank of the Kronecker product of two matrices is the product of their ranks, we arrive at the following recursive formula for the ranks of coefficient matrices of an \( n \)-qubit biseparable state:
\[
\text{rank}(C_{q_1\cdots q_k}(|\varphi\rangle_{j_1\cdots j_k} \otimes |\varphi\rangle_{j_{k+1}\cdots j_n}))
= \text{rank}(C_{q_1'\cdots q_k'}(|\varphi\rangle_{j_1\cdots j_k}) \text{rank}(C_{q_1'\cdots q_k'}(|\varphi\rangle_{j_{k+1}\cdots j_n})).
\]

Corollary. The ranks of \( \ell \)-qubit reduced density matrices obtained by tracing out \( n - \ell \) qubits are invariant under SLOCC.

This is particularly true for the local ranks \([2]\). Note also that any complex matrix has a singular value decomposition, with the number of nonzero singular values equal to the rank of the matrix. This means that the number of nonzero singular values of any coefficient matrix of an \( n \)-qubit pure state is invariant under SLOCC.

III. A RECURSIVE FORMULA FOR THE RANKS OF \( N \)-QUBIT BISEPARABLE STATES

In principle, we can calculate the ranks of coefficient matrices for \( n \)-qubit biseparable pure states by direct calculations. However, in practice, this is rather cumbersome from the computational point of view, and as \( n \) becomes large, this might pose a serious problem. In order to avoid this difficulty, we propose a simple recursive formula for the ranks of \( n \)-qubit biseparable states.

Suppose that a biseparable \( n \)-qubit pure state \( |\psi\rangle_{1\cdots n} \) is of the form \( |\psi\rangle_{1\cdots n} = |\varphi\rangle_{j_1\cdots j_k} \otimes |\varphi\rangle_{j_{k+1}\cdots j_n} \) with \( |\varphi\rangle_{j_1\cdots j_k} \) being a \( k \)-qubit state and \( |\varphi\rangle_{j_{k+1}\cdots j_n} \) being an \((n-k)\)-qubit state. We let \( C_{q_1\cdots q_k}(|\psi\rangle_{1\cdots n}) \) be the coefficient matrix associated with the state \( |\psi\rangle_{1\cdots n} \). We let \( C_{q_1'\cdots q_k'}(|\varphi\rangle_{j_1\cdots j_k}) \) be the \( 2^s \) by \( 2^{k-s} \) coefficient matrix associated with the \( k \)-qubit state \( |\varphi\rangle_{j_1\cdots j_k} \). Here \( \{q_1',\ldots,q_s'\} = \{q_1,\ldots,q_k\} \cap \{j_1,\ldots,j_k\} \) are the row bits, and by convention, the rest \( k-s \) bits are the column bits. Moreover, we let \( C_{q_1'\cdots q_k'(|\varphi\rangle_{j_{k+1}\cdots j_n})} \) be the \( 2^t \) by \( 2^{n-k-t} \) coefficient matrix associated with the \((n-k)\)-qubit state \( |\varphi\rangle_{j_{k+1}\cdots j_n} \). Here \( \{q_1',\ldots,q_t'\} = \{q_1,\ldots,q_t\} \cap \{j_{k+1},\ldots,j_n\} \) are the row bits, and by convention, the rest \( n-k-t \) bits are the column bits. It can be verified that
\[
C_{q_1\cdots q_k}(|\varphi\rangle_{j_1\cdots j_k} \otimes |\varphi\rangle_{j_{k+1}\cdots j_n})
= C_{q_1'\cdots q_k'}(|\varphi\rangle_{j_1\cdots j_k} \otimes |\varphi\rangle_{j_{k+1}\cdots j_n}).
\]

The formula above allows us to calculate recursively the ranks of coefficient matrices of \( n \)-qubit biseparable states in terms of the ranks of coefficient matrices of \( k \)-qubit states and \((n-k)\)-qubit states.

To illustrate the use of the recursive formula, we start with the initial values \( \text{rank}(C_A(|\varphi\rangle_A)) = 1 \) and \( \text{rank}(C_B(|\varphi\rangle_B)) = 1 \). It is known that a two-qubit pure state can be either of the form \( A-B \) (separable) or the form \( AB \) (EPR). Using the recursive formula, we find \( \text{rank}(C_A(|\varphi\rangle_A)) = \text{rank}(C_A(|\varphi\rangle_A) \otimes \text{rank}(C_B(|\varphi\rangle_B)) = 1 \). On the other hand, a direct calculation shows that \( \text{rank}(C_A(|\varphi\rangle_A)) = 2 \). Using the results obtained above, we can find the ranks of coefficient matrices of three-qubit pure states. Consider, for example, \( \text{rank}(C_{AC}(|\varphi\rangle_B |\varphi\rangle_{AC})) \) for biseparable states being of the form \( B-AC \). Using the recursive formula, we have \( \text{rank}(C_{AC}(|\varphi\rangle_B |\varphi\rangle_{AC})) \) = \( \text{rank}(C_B(|\varphi\rangle_B) \otimes \text{rank}(C_{AC}(|\varphi\rangle_{AC})) = 2 \). In a similar fashion, we can fill in the rest of the entries in Table \( \text{III} \) except those in the last row which can be obtained by direct calculations. Proceeding in this way, we can construct Tables \( \text{II} \) and \( \text{III} \) for the ranks of coefficient matrices for four and five qubits.

Note that in Tables \( \text{II} \) and \( \text{III} \), the numbers are shown only for \( 2^n - 1 \) coefficient matrices. This is due to the fact that interchanging two row (resp. column) bits or exchanging the row and column bits of a coefficient matrix does not alter the rank of the matrix, since the former is equivalent to interchanging two rows (resp. columns) of the matrix and the latter is equivalent to transposing the matrix. Ignoring \( C_B \) and \( C_{1\cdots n} \) which always have rank 1, this amounts to totally \( 2^n - 1 \) potentially different coefficient matrices. For example, the ranks of \( C_{BA} \) and \( C_{BC} \) are not shown in Table \( \text{II} \) since \( C_{AB} \) and \( C_{BA} \) differ by the interchange of two rows, and \( C_{BC} \) is the transpose of \( C_{AD} \). As illustrated in Tables \( \text{II} \) and \( \text{III} \), the numbers are shown only for \( 2^n - 1 \) coefficient matrices.
the partitioning of the space of the pure states into inequivalent families under SLOCC (i.e., two states belong to the same family if and only if the ranks of coefficient matrices are all equal). In particular, degenerate families of three, four, and five qubits are inequivalent from one another under SLOCC.

### TABLE I. Ranks of coefficient matrices of three-qubit pure states.

| Families | Ranks of $C_A$ | $C_B$ | $C_C$ |
|----------|----------------|-------|-------|
| $A-B-C$  | 1              | 1     | 1     |
| $A-BC$   | 1              | 2     | 2     |
| $B-AC$   | 2              | 1     | 2     |
| $C-AB$   | 2              | 2     | 1     |
| $ABC$    | 2              | 2     | 2     |

### TABLE II. Ranks of coefficient matrices of four-qubit pure states.

| Families | Ranks of $C_A$ | $C_B$ | $C_C$ | $C_D$ | $C_{AB}$ | $C_{AC}$ | $C_{AD}$ |
|----------|----------------|-------|-------|-------|----------|----------|----------|
| $A-B-C-D$ | 1              | 1     | 1     | 1     | 1        | 1        | 1        |
| $A-B-CD$  | 1              | 1     | 2     | 2     | 1        | 2        | 2        |
| $A-C-BD$  | 1              | 2     | 1     | 2     | 2        | 1        | 2        |
| $A-D-BC$  | 1              | 2     | 2     | 1     | 2        | 2        | 1        |
| $B-C-AD$  | 2              | 1     | 1     | 2     | 2        | 2        | 1        |
| $B-D-AC$  | 2              | 1     | 2     | 1     | 2        | 2        | 1        |
| $C-D-AB$  | 2              | 2     | 1     | 1     | 2        | 2        | 2        |
| $A-BCD$   | 1              | 2     | 2     | 2     | 2        | 2        | 2        |
| $B-ACD$   | 2              | 1     | 2     | 2     | 2        | 2        | 2        |
| $C-ABD$   | 2              | 2     | 1     | 2     | 2        | 2        | 2        |
| $D-ABC$   | 2              | 2     | 2     | 1     | 2        | 2        | 2        |
| $AB-CD$   | 2              | 2     | 2     | 2     | 1        | 4        | 4        |
| $AC-BD$   | 2              | 2     | 2     | 2     | 4        | 1        | 4        |
| $AD-BC$   | 2              | 2     | 2     | 2     | 4        | 4        | 1        |
| $ABCD^a$  | 2              | 2     | 2     | 2     | ≥2       | ≥2       | ≥2       |

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\(^a\) $ABCD$ can be further partitioned under SLOCC in terms of the ranks of $C_{AB}$, $C_{AC}$ and $C_{AD}$.

Table III. Ranks of coefficient matrices of five-qubit pure states.

| Families | Ranks of $C_α$ | $C_β(β≠γ)$ |
|----------|----------------|-------------|
| $i-j-k-ℓ-m$ | 1\(^b\) | 1\(^c\) |
| $i-j-k-ℓm$ | 1, if $α = i, j, k$ | 1, if $β = i, j, k$ |
|           | 2, otherwise | or $β, γ = ℓ, m$ |
|           | 2, otherwise | or $β, γ = ℓ, m$ |
| $i-jk-ℓm$  | 1, if $α = i$ | 1, if $β = i, j$ |
|           | 2, otherwise | 2, otherwise |
| $i-jkℓm$   | 1, if $α = i$ | 2, if $β = i$ or $γ = i$ |
|           | 2, otherwise | 2, 3, or 4, otherwise |
| $ij-κℓm$   | 2\(^b\) | 1, if $β = i, j$ |
|           | 2, if $β = i, j, ℓ, m$ |
|           | 4, otherwise |
| $ijkℓm$    | 2\(^b\) | 2, 3, or 4\(^c\) |

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\(^a\) $\{i, j, k, ℓ, m\}$ is any permutation of $\{A, B, C, D, E\}$.

\(^b\) $α = i, j, k, ℓ, m$.

\(^c\) $β, γ = i, j, k, ℓ, m$.

### IV. DEGENERATE FAMILIES OF GENERAL N QUBITS ARE SLOCC INEQUIVARIANT TO ONE ANOTHER

The recursive formula above further gives rise to a criterion for biseparability of an $n$-qubit pure state. Indeed, we note that Eq. 7 holds particularly true for $\{q_1, \ldots, q_t\} = \{j_1, \ldots, j_k\}$. In this case, the coefficient matrices $C_{q_1 \ldots q_t}$ and $C_{q_i \ldots q_t}$ reduce to a column vector and a row vector respectively, and therefore both of them have rank 1. It follows that $\text{rank}(C_{q_1 \ldots q_t}(\langle ϕ_{q_1 \ldots q_t} | ψ_{1 \ldots n} \rangle)) = 1$. Conversely, if $\text{rank}(C_{q_1 \ldots q_t}(\langle ψ_{1 \ldots n} | ψ_{1 \ldots n} \rangle)) = 1$ for an $n$-qubit pure state $|ψ_{1 \ldots n}\rangle$, then $|ψ_{1 \ldots n}\rangle$ is biseparable, being of the form $|ψ_{1 \ldots n}\rangle = |ϕ_{q_1 \ldots q_t} \rangle ⊗ |ψ_{q_{t+1} \ldots q_n}\rangle$. This can be seen as follows. For simplicity, we assume $q_i = i$ with $i = 1, \ldots, n$. If $\text{rank}(C_{1 \ldots t}(\langle ψ_{1 \ldots n} | ψ_{1 \ldots n} \rangle)) = 1$, then all columns of $C_{1 \ldots t}$ are proportional to each other and each column can be written into the form $(a_0b_j, a_1b_j, \ldots, a_{2^t-1}b_j)^T$. Hence, $|ψ_{1 \ldots n}\rangle$...
can be written as $|\psi_{1\ldots n}\rangle = |\phi_{1\ldots \ell}\rangle \otimes |\varphi_{(\ell+1)\ldots n}\rangle$ with $|\phi_{1\ldots \ell}\rangle = \sum_{i=0}^{2^\ell-1} a_i |i\rangle_{1\ldots \ell}$ and $|\varphi_{(\ell+1)\ldots n}\rangle = \sum_{j=0}^{2^{n-\ell}-1} b_j |j\rangle_{(\ell+1)\ldots n}$. This leads to the following biseparability criterion for $n$-qubit pure states.

Biseparability criterion for $n$-qubit pure states.

For any coefficient matrix $C_{q_1\ldots q_\ell}$ associated with an $n$-qubit pure state $|\psi_{1\ldots n}\rangle$, $\text{rank}(C_{q_1\ldots q_\ell}(|\psi_{1\ldots n}\rangle)) = 1$ if and only if $|\psi\rangle$ is biseparable, being of the form $|\psi_{1\ldots n}\rangle = |\phi_{q_1\ldots q_\ell}\rangle \otimes |\varphi_{q_{\ell+1}\ldots q_n}\rangle$ (see also [21, 23]).

Invoking the fact that an $n$-qubit pure state is entangled if it is not full separable, we have the following criterion to identify $n$-qubit entangled (respectively, genuinely entangled) pure states: An $n$-qubit pure state is entangled (respectively, genuinely entangled) if and only if the rank of at least one of its coefficient matrices is (respectively, the ranks of its all coefficient matrices) greater than 1.

Note that all the above criteria can be rephrased in terms of the ranks of $\ell$-qubit reduced density matrices obtained by tracing out $n - \ell$ qubits or the number of nonzero singular values of coefficient matrices.

Theorem 1 together with the biseparability criterion above yield the following theorem.

Theorem 3. Degenerate families of general $n$ qubits are inequivalent to one another under SLOCC and they can be distinguished in terms of the ranks of coefficient matrices (or in terms of the ranks of the $\ell$-qubit reduced density matrices obtained by tracing out $n - \ell$ qubits).

The validity of Theorem 3 can be seen as follows. Given an $n$-qubit pure state, a partition $P$ of the $n$ particles is a collection of disjoint sets in such a way that the particles within any one set are entangled and any two particles from different sets are not entangled. Suppose $F_1$ and $F_2$ are two different degenerate families with partitions $P_1$ and $P_2$ respectively. Without loss of generality, we assume that there exists a set $S$ such that $S \in P_1$ and $S \notin P_2$. Then the states in $F_1$ can be written in the biseparable form $|\phi_{S}\rangle |\varphi_{\bar{S}}\rangle$, where $\bar{S}$ is the set of all particles except those in $S$. According to the biseparability criterion above, $\text{rank}(C_S) = 1$ for states in $F_1$. Since the states in $F_2$ cannot be written in the above biseparable form, $\text{rank}(C_S) > 1$ for states in $F_2$. In light of Theorem 1, the two degenerate families are inequivalent to each other under SLOCC.

In addition, we remark that degenerate families of general $n$ qubits can also be distinguished from one another under SLOCC in terms of the ranks of the $\ell$-qubit reduced density matrices obtained by tracing out $n - \ell$ qubits or the number of nonzero singular values of coefficient matrices.

V. SLOCC CLASSIFICATION OF FOUR QUBITS VIA THE RANKS OF COEFFICIENT MATRICES

Suppose that the states $|\psi\rangle$ and $|\psi'\rangle$ of four qubits are SLOCC equivalent to each other, then there are local invertible operators $A_1, A_2, A_3,$ and $A_4$ such that

$$|\psi'\rangle = A_1 \otimes A_2 \otimes A_3 \otimes A_4 |\psi\rangle.$$

For a four-qubit state $|\psi\rangle = \sum_{i=0}^{15} a_i |i\rangle$, we consider three coefficient matrices $C_{AB}, C_{AC},$ and $C_{AD}$ as follows:

$$C_{AB} = \begin{pmatrix}
a_0 & a_1 & a_2 & a_3 \\
a_4 & a_5 & a_6 & a_7 \\
a_8 & a_9 & a_{10} & a_{11} \\
a_{12} & a_{13} & a_{14} & a_{15}
\end{pmatrix},$$

$$C_{AC} = \begin{pmatrix}
a_0 & a_1 & a_4 & a_5 \\
a_2 & a_3 & a_6 & a_7 \\
a_8 & a_9 & a_{12} & a_{13} \\
a_{10} & a_{11} & a_{14} & a_{15}
\end{pmatrix},$$

$$C_{AD} = \begin{pmatrix}
a_0 & a_4 & a_2 & a_6 \\
a_1 & a_5 & a_3 & a_7 \\
a_8 & a_{12} & a_{10} & a_{14} \\
a_{13} & a_{13} & a_{11} & a_{15}
\end{pmatrix}.$$

The coefficient matrices above satisfy the following equations:

$$C_{AB}(|\psi'\rangle) = A_1 \otimes A_2 C_{AB}(|\psi\rangle)(A_3 \otimes A_4)^T, \quad (12)$$

$$C_{AC}(|\psi'\rangle) = A_1 \otimes A_3 C_{AC}(|\psi\rangle)(A_2 \otimes A_4)^T, \quad (13)$$

$$C_{AD}(|\psi'\rangle) = A_1 \otimes A_4 C_{AD}(|\psi\rangle)(A_3 \otimes A_2)^T. \quad (14)$$

It follows from Eqs. (12)-(14) that if two four-qubit states are SLOCC equivalent then their coefficient matrices $C_{AB}$ (and also $C_{AC}$ and $C_{AD}$) have the same rank. Conversely, if one of the coefficient matrices $C_{AB}, C_{AC},$ and $C_{AD}$ differ in the ranks, then the two four-qubit states are SLOCC inequivalent. Let family $F_{r_{AB}}^{C_{AB}}$ be the set of all four-qubit states with the same rank $r_{AB}$ of the coefficient matrix $C_{AB}$. Here $r_{AB}$ ranges over the values 1, 2, 3, and 4. Clearly, each one of the nine families introduced by Verstraete et al. [3] can be further divided into four SLOCC inequivalent subfamilies corresponding to the four possible values of $r_{AB}$. In a similar manner, we can define the families $F_{r_{AC}}^{C_{AC}}$.  


and $F_{r_{AB}}^{C_{AD}}$. One can obtain a more refined partition by further dividing the families $F_{r_{AB}}^{C_{AB}}$, $F_{r_{AC}}^{C_{AC}}$, and $F_{r_{AD}}^{C_{AD}}$ into subfamilies $F_{r_{AB}r_{AC}r_{AD}}^{C_{AB}C_{AC}C_{AD}} = F_{r_{AB}}^{C_{AB}} \cap F_{r_{AC}}^{C_{AC}} \cap F_{r_{AD}}^{C_{AD}}$. Clearly, the subfamilies $F_{r_{AB}r_{AC}r_{AD}}^{C_{AB}C_{AC}C_{AD}}$ and $F_{r_{AB}r_{AC}r_{AD}}^{C_{AD}r_{AC}C_{AD}}$ are SLOCC inequivalent when $r_{AB}r_{AC}r_{AD} \neq r_{AB}r_{AC}r_{AD}'$.

We now further partition the nine families introduced by Verstraete et al. \cite{3} into SLOCC inequivalent subfamilies via the rank of coefficient matrix. For convenience, we rewrite the families $G_{abcd}$ and $L_{abc2}$ as:

$$G_{abcd} = \alpha((0) + |15\rangle) + \beta(|3\rangle + |12\rangle) + \gamma(|5\rangle + |10\rangle) + \delta(|6\rangle + |9\rangle),$$  \hspace{1cm} (15)

$$L_{abc2} = \alpha'((0) + |15\rangle) + \beta'(|3\rangle + |12\rangle) + \gamma'(|5\rangle + |10\rangle) + \delta'(|6\rangle).$$  \hspace{1cm} (16)

In Table \ref{tab:VI} we show the subfamilies $F_{r_{AB}}^{C_{AB}}$, $F_{r_{AC}}^{C_{AC}}$, and $F_{r_{AD}}^{C_{AD}}$ of $G_{abcd}$. As illustrated in Table \ref{tab:VI}, $G_{abcd}$ can be further partitioned into nine genuinely entangled subfamilies and three biseparable subfamilies (marked with "*"), via $r_{AB}$, $r_{AC}$, and $r_{AD}$ (subfamilies not listed in the table are empty). For simplicity, the detailed descriptions of the subfamilies are not shown as they can be easily obtained by taking the intersections of the corresponding descriptions in Table \ref{tab:VI}. Tables \ref{tab:VI} and \ref{tab:VI} illustrate the partitions of the other eight families introduced by Verstraete et al. into inequivalent subfamilies. In total, we find 28 genuinely entangled subfamilies and all the degenerate classes can be distinguished up to permutations of the four qubits (i.e., $A-B-C-D$, $A-B-C-D$, $AB-CD$, $AB-CD$, $|0\rangle_A|W\rangle_{BCD}$, and $|0\rangle_A|GHZ\rangle_{BCD}$).

VI. DISCUSSION OF THE COMPLETENESS OF THE NINE FAMILIES OBTAINED BY VERSTRAEET ET AL.

The family $L_{ab3}$ in Ref. \cite{3} was defined as

$$L_{ab3} = a(|0000\rangle + |1111\rangle) + \frac{a + b}{2}(|0101\rangle + |1010\rangle) + \frac{a - b}{2}(|0110\rangle + |1001\rangle) + \frac{i}{\sqrt{2}}(|0001\rangle + |0010\rangle + |0111\rangle + |1011\rangle).$$  \hspace{1cm} (17)

In later work, Chiterental et al. \cite{3} obtained nine SLOCC inequivalent families of four qubits using invariant theory. Let $L'_{ab3}$ be defined by

$$L'_{ab3} = a(|0000\rangle + |1111\rangle) + \frac{a + b}{2}(|0101\rangle + |1010\rangle) + \frac{a - b}{2}(|0110\rangle + |1001\rangle) + \frac{i}{\sqrt{2}}(|0001\rangle + |0010\rangle + |0111\rangle + |1011\rangle).$$  \hspace{1cm} (18)

A. $L_{ab3}(a = 0)$ is SLOCC equivalent to $L'_{ab3}(a = 0)$

It is readily verified that the following equation holds between $L'_{ab3}(a = 0)$ and $L_{ab3}(a = 0)$:

$$L'_{ab3}(a = 0) = I \otimes I \otimes i\sigma_z \otimes i\sigma_z L_{ab3}(a = 0),$$  \hspace{1cm} (19)

where $I$ is the identity and $\sigma_z = \text{diag}\{1, -1\}$.

It follows from Eq. \cite{19} that $L_{ab3}(a = 0)$ and $L'_{ab3}(a = 0)$ are SLOCC equivalent. In particular, setting $b = 0$ yields that the states $\frac{1}{\sqrt{2}}(|0001\rangle + |0010\rangle - |0111\rangle - |1011\rangle)$ and $\frac{1}{\sqrt{2}}(|0001\rangle + |0010\rangle + |0111\rangle + |1011\rangle)$ are equivalent under SLOCC.

B. $L'_{ab3}(a \neq 0)$ [respectively, $L_{ab3}(a \neq 0)$] is SLOCC inequivalent to $L_{ab3}$ (respectively, $L'_{ab3}$)

We first show that the family $L'_{ab3}(a \neq 0)$ is SLOCC inequivalent to the family $L_{ab3}$. In Table \ref{tab:VII} we show the partition of $L'_{ab3}$ into SLOCC inequivalent subfamilies via $r_{AB}$, $r_{AC}$, and $r_{AD}$. Consulting Tables \ref{tab:VII} and \ref{tab:VII} and using the fact that the subfamilies with different ranks of coefficient matrices are SLOCC inequivalent to each other, it suffices to consider the following six cases.

Case 1. $L'_{ab3}(a = b \neq 0)$ is SLOCC inequivalent to $L_{ab3}(b = -3a \neq 0)$. 


be verified that if equivalent states, that is, they satisfy Eq. (2), then

\[ r_{xy} | \psi' \rangle = r_{xy} | \psi \rangle \quad \text{and} \quad r_{xy} | \psi \rangle, \text{either } D_{xy}(|\psi\rangle) \text{ and } D_{xy}(|\psi'\rangle) \text{ both vanish or neither vanishes.} \]

A direct calculation shows that

\[ D_{xy} = -\frac{1}{32} (a - b)^3 (a + b)^3 \quad (21) \]

for both \( L_{ab3} \) and \( L'_{ab3} \). The desired result then follows by noting that \( D_{xy} = 16a^6 \neq 0 \) for \( L_{ab3}(b = -3a \neq 0) \) whereas \( D_{xy} = 0 \) for \( L'_{ab3}(a = b \neq 0) \).

Case 2. \( L'_{ab3}(a = -b \neq 0) \) is SLOCC inequivalent to \( L_{ab3}(b = 3a \neq 0) \).

This case can be dealt with similarly as case 1 by noting that \( D_{xy} = 16a^6 \neq 0 \) for \( L_{ab3}(b = 3a \neq 0) \) whereas \( D_{xy} = 0 \) for \( L'_{ab3}(a = -b \neq 0) \).

Case 3. \( L'_{ab3}(b = -3a \neq 0) \) is SLOCC inequivalent to \( L_{ab3}(b = -3a \neq 0) \).

In this case, the semi-invariants defined in Ref. [7] turn out to be useful. More specifically, for any four-qubit state \( |\psi\rangle = \sum_{i=0}^{15} c_i |i\rangle \), the semi-invariants \( F_1 \) and \( F_2 \) are defined in Ref. [7] as

\[ F_1(\psi) = (c_0 c_7 - c_2 c_5 + c_1 c_6 - c_3 c_4)^2 - 4(c_2 c_4 - c_0 c_6)(c_3 c_5 - c_1 c_7), \quad (22) \]

\[ F_2(\psi) = (c_8 c_{15} - c_{11} c_{12} + c_9 c_{14} - c_{10} c_{13})^2 - 4(c_{11} c_{13} - c_9 c_{15})(c_{10} c_{12} - c_8 c_{14}). \quad (23) \]

Let \( |\phi\rangle \) be any four-qubit state SLOCC equivalent to \( L_{ab3} \) i.e., they satisfy Eq. (2). Let

\[ A_1 = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix}. \quad (24) \]
TABLE VII. SLOCC classifications of $L_{abc2}$ via $r_{AB}$, $r_{AC}$, and $r_{AD}$. The subfamilies marked with "*" are biseparable.

| $r_{AB}$ $r_{AC}$ $r_{AD}$ | Subfamily description |
|-----------------------------|-----------------------|
| 233                         | $\alpha' = \beta' = 0 \& \gamma' \neq 0$ |
| 244                         | $\alpha' = \pm \beta' \neq 0 \& \gamma' = 0$ |
| 332                         | $\alpha' = \gamma' = 0 \& \beta' \neq 0$ |
| 333                         | $\alpha' \neq 0 \& \beta' = \gamma' = 0$ |
| 333                         | $\alpha' = \pm \beta' = \pm \gamma' \neq 0$ |
| 344                         | $\gamma' = 0 \& \alpha' \beta' \neq 0 \& \alpha' \neq \pm \beta' \neq 0 \& \alpha' = \pm \beta' \neq 0 \& \alpha' \neq \pm \gamma'$ |
| 424                         | $\beta' = 0 \& \alpha' = \pm \gamma' \neq 0$ |
| 434                         | $\beta' = 0 \& \alpha' \gamma' \neq 0 \& \alpha' \neq \pm \gamma' \neq 0 \& \alpha' = \pm \gamma' \neq 0 \& \alpha' \neq \pm \beta'$ |
| 442                         | $\alpha' = 0 \& \beta' = \pm \gamma' \neq 0$ |
| 444                         | $\alpha' = 0 \& \beta' \neq \pm \gamma' \neq 0 \& \alpha' \neq \pm \gamma' \neq 0 \& \alpha' \neq \pm \beta'$ |
| 111*                        | $\alpha' = \beta' = \gamma' = 0$, (i.e., $A-B-C-D$) |

TABLE VII. SLOCC classifications of $L_{ab3}$, $L_{a2b2}$, $L_{a4}$, $L_{a2a3b1}$, $L_{0a53}$, $L_{07g1}$, and $L_{0a53a031}$ via $r_{AB}$, $r_{AC}$, and $r_{AD}$. The subfamilies marked with "*" are biseparable.

| Family $L_{ab2}$ $L_{a4}$ $L_{a2a3b1}$ $L_{0a53a031}$ | Subfamily description |
|-----------------------------|-----------------------|
| 333                         | $ab = 0 \& a \neq b$ |
| 424                         | $a = \pm b \neq 0$ |
| 434                         | $ab \neq 0 \& a \neq \pm b$ |
| 212*                        | $a = b = 0$ (i.e., $A-C-BD$) |
| 323                         | $L_{a4}(a = 0)$ |
| 434                         | $L_{a4}(a \neq 0)$ |
| 333                         | $L_{a2a3b1}(a \neq 0)$ |
| 222*                        | $a = 0$ (i.e., $|0\rangle_A|W\rangle_{BCD}$) |
| 333                         | $L_{0a53}$ |
| 444                         | $L_{0a53a031}$ |
| 222*                        | $|0\rangle_A|GHZ\rangle_{BCD}$ |

TABLE VIII. SLOCC classification of $L'_{ab3}$ via $r_{AB}$, $r_{AC}$, and $r_{AD}$.

| $r_{AB}$ $r_{AC}$ $r_{AD}$ | Subfamily description |
|-----------------------------|-----------------------|
| 222                         | $a = b = 0$ (i.e., $|W\rangle_{ABCD}$) |
| 344                         | $ab = 0 \& a \neq b$ |
| 424                         | $\emptyset$ |
| 444                         | $a = b \neq 0 \& b = -3a \neq 0$ |
| 444                         | $a = -b \neq 0 \& b = 3a \neq 0$ |
| 444                         | $ab \neq 0 \& b \neq \pm a \& b \neq \pm 3a$ |

A tedious but straightforward calculation yields

\[
F_1(\phi) = \frac{1}{2}(a^2 - b^2)\alpha_4^4 \left( \prod_{i=2}^{4} \det A_i \right)^2,
\]

\[
F_1(\phi) = \frac{1}{2}(a^2 - b^2)\alpha_3^4 \left( \prod_{i=2}^{4} \det A_i \right)^2.
\]

In view of Eqs. 25 and 26 and the fact that $A_1$ is invertible, it follows at once that if $|\phi\rangle$ is SLOCC equivalent to $L_{ab3}(a \neq \pm b)$, then the following equation holds:

\[
|F_1(\phi)| + |F_2(\phi)| \neq 0.
\]

Let $|\phi\rangle$ be any state SLOCC equivalent to $L'_{ab3}$ [i.e., they satisfy Eq. 2]. Again, a tedious but
straightforward calculation yields

\[
F_1(\varphi) = \frac{-1}{2\sqrt{2}} i \alpha_1 \left( -i \sqrt{2} (3a^2 + b^2) \alpha_1 + 8a(a^2 - b^2) \alpha_2 \right) \times \left[ \prod_{i=2}^{4} \det A_i \right]^2,
\]

\[
F_2(\varphi) = \frac{-1}{2\sqrt{2}} i \alpha_3 \left( -i \sqrt{2} (3a^2 + b^2) \alpha_3 + 8a(a^2 - b^2) \alpha_4 \right) \times \left[ \prod_{i=2}^{4} \det A_i \right]^2.
\]

(28)

When \(a(a^2 - b^2) \neq 0\), consider the operator

\[
A_i^a = \left( \begin{array}{cc}
\alpha_1 & i \sqrt{2} (3a^2 + b^2) \alpha_1 \\
0 & \alpha_4
\end{array} \right),
\]

(30)

where \(\alpha_1 \alpha_4 \neq 0\). Clearly, \(A_i^a\) is invertible. In view of Eqs. (28)-(30), it follows that there exists a state \(|\varphi^*\rangle\) equivalent to \(|\varphi^\prime\rangle\) (\(a(a^2 - b^2) \neq 0\)) under local invertible operators \(A_1^a, A_2, A_3, \) and \(A_4\), such that

\[
|F_1(\varphi^*)| + |F_2(\varphi^*)| = 0.
\]

(31)

From Eqs. (27) and (31), \(|\varphi^*\rangle\) is SLOCC inequivalent to the state \(L_{ab}^\prime (a \neq \pm b)\). Therefore, \(L_{ab}^\prime (a(a^2 - b^2) \neq 0)\) is SLOCC inequivalent to \(L_{ab} (a \neq \pm b)\). In particular, \(L_{ab}^\prime (b = -3a \neq 0)\) is SLOCC inequivalent to \(L_{ab} (b = -3a \neq 0)\).

Case 4. \(L_{ab}^\prime (b = 3a \neq 0)\) is SLOCC inequivalent to \(L_{ab} (b = 3a \neq 0)\).

This case can be treated analogously to case 3.

Case 5. \(L_{ab}^\prime (a \neq 0 \& b = 0)\) is SLOCC inequivalent to \(L_{ab} (ab = 0 \& a \neq b)\).

In Ref. [10], we proved that \(L_{ab} (a = 0 \& b \neq 0)\) and \(L_{ab} (a \neq 0 \& b = 0)\) are SLOCC inequivalent. A proof analogous to that of Ref. [10] shows that \(L_{ab}^\prime (a = 0 \& b \neq 0)\) and \(L_{ab}^\prime (a \neq 0 \& b = 0)\) are SLOCC inequivalent. Using the fact that \(L_{ab} (a = 0 \& b \neq 0)\) is SLOCC equivalent to \(L_{ab}^\prime (a = 0 \& b \neq 0)\) [see Eq. (19)] yields that \(L_{ab}^\prime (a \neq 0 \& b = 0)\) is SLOCC inequivalent to \(L_{ab} (a = 0 \& b \neq 0)\). Furthermore, an argument analogous to case 3 shows that \(L_{ab}^\prime (a \neq 0 \& b = 0)\) is equivalent to \(L_{ab} (a \neq 0 \& b = 0)\).

Indeed, we can further conclude that \(L_{ab} (a = 0)\) and \(L_{ab} (a \neq 0)\) are SLOCC inequivalent and \(L_{ab}^\prime (a = 0)\) and \(L_{ab}^\prime (a \neq 0)\) are SLOCC inequivalent.

Case 6. \(L_{ab}^\prime (ab \neq 0 \& a \neq \pm b \& b \neq \pm 3a)\) is SLOCC inequivalent to \(L_{ab} (ab \neq 0 \& a \neq \pm b \& b \neq \pm 3a)\).

This case can be treated analogously to case 3.

As a consequence, \(L_{ab}^\prime (a \neq 0)\) is SLOCC inequivalent to \(L_{ab} (a \neq 0)\). An analogous argument shows that \(L_{ab} (a \neq 0)\) is SLOCC inequivalent to \(L_{ab}^\prime (a \neq 0)\).

C. The relation between \(L_{ab}^\prime\) and \(L_{ab}\) under permutations

Let \(|\gamma\rangle\) be the state of the subfamily \(L_{ab}^\prime (a \neq 0 \& b = 0)\), \(|\eta\rangle\) be the state of the subfamily \(L_{ab}^\prime (b = 3a \neq 0)\), \(|\vartheta\rangle\) be the state of the subfamily \(L_{ab} (b = -3a \neq 0)\), and \(|\nu\rangle\) be the state of the subfamily \(L_{ab} (ab \neq 0 \& a \neq \pm b \& b \neq \pm 3a)\). We argue that the above four subfamilies are SLOCC inequivalent to \(L_{ab}\) under any permutation of qubits. This can be seen as follows. Let \((i, j)\) be the transposition of qubits \(i\) and \(j\). A tedious calculation shows that the permutations giving rise to different \(|\gamma\rangle, |\eta\rangle, |\vartheta\rangle, |\nu\rangle\) are \(\kappa_1 = I, \kappa_2 = (1, 3), \kappa_3 = (1, 4), \kappa_4 = (1, 2)(1, 3), \kappa_5 = (1, 2)(1, 4), \) \(\kappa_6 = (1, 4)(1, 2)(1, 3)\). Similarly, the permutations giving rise to different \(|\eta\rangle, |\vartheta\rangle, |\nu\rangle\) are \(\pi_1 = I, \pi_2 = (1, 2), \pi_3 = (1, 3), \pi_4 = (1, 4), \pi_5 = (1, 3)(1, 2), \pi_6 = (1, 4)(1, 2), \pi_7 = (1, 2)(1, 3), \pi_8 = (1, 2)(1, 4), \pi_9 = (1, 2)(1, 3)(1, 2), \pi_{10} = (1, 2)(1, 4)(1, 2), \pi_{11} = (1, 4)(1, 2)(1, 3), \) \(\pi_{12} = (1, 4)(1, 2)(1, 3)(1, 2)\). The result that \(\kappa_1 |\gamma\rangle (i = 1, \ldots, 6), \pi_1 |\eta\rangle, \pi_2 |\vartheta\rangle,\) and \(\pi_3 |\nu\rangle (j = 1, \ldots, 12)\) are all SLOCC inequivalent to \(L_{ab}\) then follows by calculating the ranks \(r_{AB}, r_{AC},\) and \(r_{AD}\) of \(\kappa_1 |\gamma\rangle, \pi_1 |\eta\rangle, \pi_2 |\vartheta\rangle,\) and \(\pi_3 |\nu\rangle,\) using an argument analogous to that of case 3 in the previous section.

Remark 2. By using Tables VII and VIII one can verify that \((1, 4)L_{ab}^\prime (a = b \neq 0)\) is SLOCC equivalent to \(L_{ab} (a = 0 \& b \neq 0)\) under the invertible local operator \(\sigma_x \otimes \sigma_z \otimes iI \otimes \sigma_y,\) and \((1, 3)L_{ab}^\prime (a = -b \neq 0)\) is SLOCC equivalent to \(L_{ab} (a = 0 \& b \neq 0)\) under the invertible local operator \(\sigma_x \otimes \sigma_z \otimes \sigma_y \otimes iI.\)

D. \(L_{ab} (a \neq 0)\) is SLOCC inequivalent to the other eight families by Verstraete et al.

Here we show that \(L_{ab}^\prime (a \neq 0)\) is not only SLOCC inequivalent to \(L_{ab}\) but also SLOCC inequivalent to the other eight families by Verstraete et al. For simplicity, we only show that \(L_{ab}^\prime (a = -b \neq 0)\)
is SLOCC inequivalent to the other eight families obtained by Verstraete et al. From Table VII, $r_{\text{ABrACTAD}} = 443$ for $\mathcal{S}_{\text{abc}}(a = -b \neq 0)$. Consulting Tables V, VI, and VII and using the fact that the subfamilies with different ranks of coefficient matrices are SLOCC inequivalent to each other, it suffices to show that $L_{\text{abc}}(a = -b \neq 0)$ is SLOCC inequivalent to the subfamilies with $r_{\text{ABrACTAD}} = 443$ of $G_{\text{abcd}}$ and $L_{\text{abc}}$.

To show that $L_{\text{abc}}'(a = -b \neq 0)$ is SLOCC inequivalent to the subfamily with $r_{\text{ABrACTAD}} = 443$ of $G_{\text{abcd}}$, we use the degree 6 polynomial invariant $D_{\text{xy}}$ given in Eq. (20). It is readily seen from Eq. (21) that $D_{\text{xy}} = 0$ for $L_{\text{abc}}'(a = -b \neq 0)$. A simple calculation shows that

$$D_{\text{xy}} = (\alpha \beta - \gamma \delta)(\alpha \beta + \gamma \delta)(\alpha^2 + \beta^2 - \gamma^2 - \delta^2)$$

(32)

for $G_{\text{abcd}}$ [as defined in Eq. (15)]. It is readily seen from Eq. (32) that $D_{\text{xy}} \neq 0$ for the subfamily with $r_{\text{ABrACTAD}} = 443$ of $G_{\text{abcd}}$ and then the desired result follows.

Next we show that $L_{\text{abc}}'(a = -b \neq 0)$ is SLOCC inequivalent to the subfamily with $r_{\text{ABrACTAD}} = 443$ of $L_{\text{abc}}$ [as defined in Eq. (16)]. A calculation shows that

$$D_{\text{xy}} = (\alpha' \beta')^2(\alpha'^2 - \gamma'^2 + \beta'^2)$$

(33)

for $L_{\text{abc}}$. From Table VI we distinguish the following two cases.

Case 1. $\alpha' \neq 0$ & $\beta' = \pm \gamma' \neq 0$ & $\alpha' \neq \pm \beta'$.

In this case $D_{\text{xy}} \neq 0$ and then the desired result follows.

Case 2. $\alpha' = 0$ & $\beta' \neq \pm \gamma' \neq \beta' \gamma' \neq 0$.

In this case $D_{\text{xy}} = 0$. We can resort to the semi-invariants given in Eqs. (22) and (28). Let $|\varphi\rangle$ be any state SLOCC equivalent to $L_{\text{abc}}'(a = -b \neq 0)$ with $A_1$ given by Eq. (24). A tedious but straightforward calculation yields

$$F_1(\varphi) = -2a^2\alpha_3^4 \left(\prod_{i=2}^{4} \text{det} A_i\right)^2,$$

(34)

$$F_2(\varphi) = -2a^2\alpha_3^4 \left(\prod_{i=2}^{4} \text{det} A_i\right)^2.$$  

(35)

In view of Eqs. (31) and (32) and the fact that $A_1$ is invertible, it follows at once that if $|\varphi\rangle$ is SLOCC equivalent to $L_{\text{abc}}'(a = -b \neq 0)$, then the following equation holds:

$$|F_1(\varphi)| + |F_2(\varphi)| \neq 0.$$  

(36)

The desired result then follows by noting that $F_1 = F_2 = 0$ for $L_{\text{abc}}$ with $\alpha' = 0$ & $\beta' \neq \pm \gamma' \gamma'$. As a consequence, $L_{\text{abc}}'(a = -b \neq 0)$ is SLOCC inequivalent to the nine families obtained by Verstraete et al. [3].

The discussion suggests that the partition in Ref. [3] is incomplete. For completeness, one may add the family $L_{\text{abc}}$ to the family $L_{\text{abc}}$ in Ref. [3]. An analogous argument shows that the partition in Ref. [6] is incomplete as well, and for completeness, one may add the family $L_{\text{abc}}$ to the family 6 in Ref. [6].

**VII. CONCLUSION**

We have recast the necessary and sufficient condition for two $n$-qubit states to be equivalent under SLOCC into an equivalent form in terms of the coefficient matrices associated with the states. As a direct consequence of the new necessary and sufficient condition, we have showed that the rank of the coefficient matrix as well as the rank of the $\ell$-qubit reduced density matrix is invariant under SLOCC.

We have also presented a recursive formula for the calculation of the rank of coefficient matrix of an $n$-qubit biseparable state. The recursive formula further gives rise to a biseparability criterion in terms of the rank of coefficient matrix to determine if an arbitrary $n$-qubit pure state is biseparable. The invariance of the rank of coefficient matrix together with the biseparability criterion reveals that all the degenerate families of general $n$ qubits are inequivalent under SLOCC.

We have then classified four-qubit states under SLOCC via the ranks of coefficient matrices and the nine families introduced by Verstraete et al. were further partitioned into inequivalent subfamilies. In particular, we have found 28 genuinely entangled families and all the degenerate classes can be distinguished up to permutations of the four qubits. We have performed a detailed study of the relation between the family $L_{\text{abc}}$ and the family $L_{\text{abc}}'$ with corrections to the signs of the last two terms in the formula of $L_{\text{abc}}$ via the ranks of coefficient matrices. By using a degree 6 polynomial invariant and two semi-invariants of four qubits, we have found that $L_{\text{abc}}'(a = 0)$ is SLOCC equivalent to $L_{\text{abc}}'(a = 0)$ whereas $L_{\text{abc}}'(a = 0)$ is SLOCC inequivalent to $L_{\text{abc}}'(a = 0)$. We have also demonstrated that $L_{\text{abc}}'(a = 0 & b = 0)$, $L_{\text{abc}}'(b = \pm 3a \neq 0)$, and $L_{\text{abc}}'(ab \neq 0 & a = \pm b & b \neq \pm 3a)$ are SLOCC inequivalent to $L_{\text{abc}}$ under any permutation of qubits.
whereas $L_{ab}'(a = \pm b \neq 0)$ are SLOCC equivalent to $L_{ab}(a = 0 \& b \neq 0)$ under some permutations. This suggests that the partition of four-qubit states into the nine families by Verstraete et al. is incomplete, and for completeness, one may simply add the family $L_{ab}'$ to the family $L_{ab}$. 

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APPENDIX

Following [25], $D_{xy}$ can be constructed as

$$D_{xy} = \begin{vmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{vmatrix},$$

where the entries of $D_{xy}$ are given by:

$$d_{11} = a_0a_3 - a_1a_2,$$
$$d_{12} = a_0a_7 - a_1a_6 - a_2a_5 + a_3a_4,$$
$$d_{13} = a_4a_7 - a_5a_6,$$
$$d_{21} = a_0a_{11} - a_1a_{10} - a_2a_9 + a_3a_8,$$
$$d_{22} = a_0a_{15} - a_1a_{14} - a_2a_{13} + a_3a_{12} + a_4a_{11} - a_5a_{10} - a_6a_9 + a_7a_8,$$
$$d_{23} = a_4a_{15} - a_5a_{14} - a_6a_{13} + a_7a_{12},$$
$$d_{31} = a_8a_{11} - a_9a_{10},$$
$$d_{32} = a_8a_{15} - a_9a_{14} - a_{10}a_{13} + a_{11}a_{12},$$
$$d_{33} = a_{12}a_{15} - a_{13}a_{14}.$$

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