Generation of the surfaces via quasi-rotation of higher order

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Abstract: In this study we report a change in the geometry of space during its quasi-rotation relative to an ellipse. Symmetric interrelation, generated by the quasi-rotation around the elliptic axis is considered. The possibility of an arbitrary point quasi-rotation in the $R^3$ space around the elliptic axis is confirmed and discussed. Further, we analyze the properties of quasi-rotation surfaces. In the considered examples, a circle within the quasi-rotation axis’s plane is adopted as a generatrix. The algorithm used to build 3D graphs is based on a mathematical description of the method of rotation of a point around a second-order axis curve. A symmetry interrelation with respect to the elliptic axis is considered. The generated images are analyzed to determine the structure of their self-intersection and self-touching. Flat intersections of the surfaces under study are generated. Geometric methods were used to determine the order of the surfaces. It was established that in the considered example that quasi-rotation of the circle that belongs to the quasi-rotational elliptic axis’ plane, a surface of the twentieth order is formed.

1. Introduction
One of the main problems of geometry is a description of the properties of surfaces and their modeling. Using a digital surface model, it is possible to physically manufacture its solid-state embodiment. The higher the order of the surface, the more efforts it takes to design and create its 3D model. Therefore, the search for simple methods for modeling of the complex surfaces is crucial in engineering and applied mathematics. In this work, we propose a method for the formation of cyclic surfaces that made it possible to simulate not only a torus or Dupin cyclide, but also a large number of new surfaces of higher orders.

The earlier reports [1, 2, 3] describe a method for rotating a point around a curvilinear axis of the second order, later termed “quasi-rotation”. By analogy with rotation, as a method of forming rotation surfaces, quasi-rotation can also serve as a tool for creating quasi-rotation surfaces. This report aims to study the properties of quasi-rotation surfaces in order to formulate the principles of the design with model predetermined properties.

2. Formulation of the problem
In order to study quasi-rotation surfaces, it is necessary to have an algorithm in place to create their digital models from the initial axis-forming pair. A tool to make such models can be created on the basis of computer mathematical modeling. The algorithm of the program should be based on a mathematical description of the quasi-rotation of a point around the ecliptic axis, which partly reported [2]. We aim to study the properties of the obtained surface models, and compare them with predicted ones, such as a multilayer structure and tangential self-intersection. We also aim to generate images of flat sections. This will assist in analyzing the surface structure. At last, we aim to create solid-state models of solids, molded by quasi-rotation surfaces using 3D printing. Such solids are expected to exhibit a variety of unique physical properties.
3. Theory
Since the “quasi-rotation” interrelations correspond to the respective “quasi-symmetries”, it is necessary to consider quasi-symmetry operation around the elliptic axis, and then proceed to quasi-rotation on its basis. Similarly to the case of symmetry rotation around a circular axis, \( AS_{ic \rightarrow QRT_{ie} \rightarrow QRT_{ie}} \), the symmetry rotation around elliptic axis \( AS_{ie \rightarrow QRT_{ie}} \), \( m = 1, 2, ..., \infty \) (\( A \)-axis, \( S \)-symmetry, \( c \)-circle, \( Q \)-quasi, \( R \)-rotation, \( T \)-transformation), the symmetry rotation around elliptic axis \( AS_{ie \rightarrow QRT_{ie}} \), \( m = 1, 2, ..., \infty \) (\( e \)-ellipse). The construction of the quasi-symmetry of point A relative to the elliptic axis \( ie \), is shown in figure 1.

![Figure 1](https://via.placeholder.com/150)

**Figure 1.** The point A symmetry relative to the elliptic axis \( i_e \).

Lines \( F_1A, F_2A \) intersect \( ie \) at four points \( S_1', S_1'', S_2', S_2'' \) \((F_1 \cap \widehat{ie} = S_1', S_1'' \cap \widehat{ie} = S_2', S_2'' \cap \widehat{ie} = S_2')\), through which we can draw four corresponding tangents to the ellipse \( ie \) of the lines \( i_1', i_1'' i_2', i_2'' \). Then four points \( A_1', A_1'', A_2', A_2'' \) correspond to any arbitrary point A).

\[
AS_{ie} (A) \rightarrow \begin{cases} \frac{AS_{ie} (A)}{AS_{ie}'' (A)} = A_1', A_1'', A_2', A_2'' \\ AS_{ie} (A) \end{cases}
\]

(1)

Below are the formulas for finding the coordinates of the image A through the coordinates of the inverse image (preimage) for the interrelation \( AS_{ie} \) for any arbitrary point \( A(x_A, y_A) \).

\[
y_A' = 2 \left( \frac{a^4b^2y_S - a^4b^2x_Sy_Sx_A + b^4x_A^2y_A}{b^4x_S^2 + a^4y_S^2} \right) - y_A \tag{2}
\]

\[
x_A' = 2 \left( \frac{a^2}{x_S} - \frac{a^2y_S(a^4b^2y_S - a^4b^2x_Sy_Sx_A + b^4x_A^2y_A)}{b^2x_S(b^4x_A^2 + a^4y_S^2)} \right) - x_A \tag{3}
\]

The coordinates \( x_S, y_S \) determine the position of the point S and are the solution of the equations:

\[
\left( \frac{b^2(x_A - c)}{a^2y_A^2} \right) + \left( \frac{2cb^2(x_A - c)}{a^2y_A} \right)y_S + b^2c^2/a^2 - b^2 = 0 \tag{4}
\]

\[
x_S = y_S(x_A - c) + c \tag{5}
\]

In equations (2), (3), (4), (5), the variables \( a \) and \( b \) are the major and minor semiaxes of the ellipse \( ie \). The variable \( c \) corresponds to the \( x \) coordinate of the focus of the ellipse \( ie \) and is calculated by the formula:

\[
c = \pm \sqrt{a^2 - b^2}
\]
The derivation of equations (2), (3), (4), (5) is reported in [2]. Thus, using the derived equations, we can interrelate and draw the required number of image points using the corresponding number of inverse image (preimage) points of an arbitrary line \( y = f(x) \). Quasi-symmetry is a type of symmetry rotation around a circular axis, such as inversion. Inversion and other quadratic transformations are described elsewhere [4, 5].

The result of the quasi-rotation of point \( A \) around the elliptic \( i_e \) is four circles \( k_1', k_1'', k_2', k_2'' \). The centers of these circles belong to the plane of the axis \( i_e \). The planes of circles \( k_1', k_1'', k_2', k_2'' \) are perpendicular to the plane of the axis \( i_e \). Point \( A \) and its image belong to the corresponding circle \( k_i \). As a result, four circles are brought into the interrelation with each respected point of an arbitrary line \( y = f(x) \) by the QRT\(_{ie}\) apparatus:

\[
QRT_{ie} (A) \rightarrow \begin{cases} 
RT_{i_1'}(A) \\
RT_{i_1''}(A) \\
RT_{i_2'}(A) \\
RT_{i_2''}(A)
\end{cases} = k_1', k_1'', k_2', k_2''
\] (6)

For an arbitrary point \( A \in \mathbb{R}^3 \), there is a “quasi-rotation” interrelation transformation apparatus - QRT\(_{ie}\), which includes: plane of rotation \( \alpha \), \( (A \in \alpha, \alpha \perp i') \), rotation centers \( O_1', O_1'', O_2', O_2'' \), \( (O = \alpha \cap i') \) and the constant values of variable \( r_1' = |O_1'A|, r_1'' = |O_1''A|, r_2' = |O_2'A|, r_2'' = |O_2''A| \).

Any arbitrary ellipse \( (i_e) \) in 3D space (\( \mathbb{R}^3 \)) defines a bundle of incident straight circular cones \( \{\Delta_c\} \).

Figure 2 shows two cones with vertices \( F_1' \) and \( F_2' \). These cones are the result of a quasi-rotation of the plane \( \Delta_c \) by an arbitrary angle. In quasi-rotation, the vertices of these cones follow a hyperbolic trajectory. The conic generation by a corresponding interrelated second conic, located in a plane perpendicular to it, is described elsewhere [6]. When turning by angle equal to 180°, the cone with the vertex \( F_1' \) transitions into the cone with the vertex \( F_2' \), the straight line \( F_1S' \) will transition into the straight line \( F_2S' \). In quasi-rotation relative to the ellipse, the straight line \( F_1S' \) describes a straight circular cone incident to the hyperbola \( j \). In the case of quasi-rotation around the hyperbolic axis \( j \), the plane \( \Delta \) will transition into the cone with the vertex \( S \), incident to the hyperbole \( j \). On any cone \( \Delta_c \), the interrelation QRT\(_{ie}\) induces a symmetry transformation with respect to the axis \( i_e \). As a result, the QRT\(_{ie}\) interrelation splits into a bundle of cones \( \{\Delta_c\} \) by \( \infty \) transformations "axial symmetry" - AS\(_{ie}\).

Thus, \( QRT_{ie} \rightarrow \{AS_{ie}\}^\infty \). Based on the nonlinear interrelation QRT\(_{ie}\), as well as on the basis of the RT\(_{i}\) transformation, it becomes possible to create mathematical models of surfaces. The particular examples of using nonlinear interrelation as a basis for creating surfaces are described elsewhere [7].
4. Results
By using the law of $AS_{\nu}(A)$, interrelation, graphically presented in figure 1, let us construct the image of the line via $AS_{\nu}$ apparatus. Figure 3 shows the quasi-symmetry of the line parallel $a$ to the axis $x$ via rotation around the elliptic symmetry axis $i_e$. The curves $a_1', a_1'', a_2', a_2''$ are Bezier curves, constructed by the $AS_{\nu}$ interrelation of the certain number of points belonging to the preimage line $a$. The resulting image is a type of conchoid [8, 9].

To determine the order of quasi-symmetry interrelation, we utilize the graphical method. The maximum number of points of the image $AS_{\nu}(a)$, belonging on one straight line $l$ is eight, i.e., the order of conformity is eighth. The interrelation and their varieties are known from the source [10].

![Figure 3. Quasi-symmetry interrelation of line $a$ via elliptic axis $i_e$.](image)

Based on formulas (1) and (6), as well as equations (2), (3), (4), (5), within the framework of these studies, an algorithm was compiled that can be implemented in any environment of computer mathematical modeling.

![Figure 4. The quasi-rotation surface in three projections, and its corresponding axis-forming pair.](image)
The program creates the required number of circles corresponding to the points of the generator according to the $QRT_i$ device. The construction of a 3D surface model using an array of points was written in [11]. Sources [12, 13] describe approaches to modeling surfaces. Also, by means of the created algorithm, it is possible to generate the flat sections of surfaces. Such sections are a set of points satisfying $QRT_i(a)$.

An example of a quasi-rotation surface model in three projections is shown in figure 4. These images allow us to appreciate the high complexity of the structure displayed on their surface. However, it is obvious that the surface has two mutually perpendicular planes of symmetry. One of them is the plane of the axis $i$, the second passes through the foci $F_1$ and $F_2$ of the axis $i$. All four layers tangentially intersect each other along the circle $a$, since $a$ belongs to them all. Because the points of the generatrix $a$ belong to the straight line with the major semiaxis of the ellipse $i$, as a result of $QRT_i(a)$ there are not four, but two circles, since $a_1''=a_2''$, $a_1'''=a_2'$ → $k_1'=k_2''$, $k_1''=k_3'$. Therefore, the tangential self-intersection of the layers is observed not only along the initial generatrix of the circle $a$, but also around two more circles $k_1'=k_2''$, $k_1''=k_3'$. Pairwise tangential intersection of the layers is visible in the top view (figure 4). In the front view (figure 5), the curve of self-intersection of layers is highlighted in white. The points of this curve correspond to the hyperbola equation. The left branch of the hyperbola is the result of self-intersection of the layer $a_2'$, and the right branch is $a_1'$. This hyperbola is the axis $j$ (figure 2).

**Figure 5.** Frontal projection of a quasi-rotation surface model.

In more detail, the complex structure of the surface of quasi-rotation is revealed by its flat sections. Figure 6 shows the axial section of the surface shown in figure 5. This image is identical to the symmetry of the circle $a$ with respect to the ellipse $i$. The graphical method for determining the order of the curve shows that the maximum number of points $AS_i(a)$ lying on one straight line $l$ is twelve. The resulting points are numbered in figure 6 by numbers.

**Figure 6.** Intersection of the surface shown in Figure 5 by the plane of the axis $i$. 

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5
The applied computational algorithm allows you to set any value of the minor and major semiaxis of the ellipse \( i \). Thus, indicating equal values for these constants in the initial data of the algorithm, we obtain the correspondence of the quasi-rotation relative to the circle \( QRT_{ic(a)} \). Figure 7 shows an example of a model of the surface of a quasi-rotation in three projections. This surface is bilayered. Its layers tangentially intersect each other along the initial generatrix of the circle \( a \). Below in figure 8 (a) is a section of this surface by the plane of the axis \( i_c \). This image is identical to the symmetry of the circle \( a \) relative to the circle \( i_c \). The graphical method for determining the order of the curve shows that the maximum number of points \( AS_{ic(a)} \) lying on one straight line \( l \) is six. In figure 8 (b) is a cross section of the surface under study with a plane at \( 0.1r_a \) spaced apart from the plane of the axis \( i_c \). When determining the order of a given curve graphically, we see that the number of points of the curve under study lying on the straight line \( l \) is ten.

**Figure 7.** The quasi-rotation surface in three projections, and its corresponding rotation axis-generatrix pair.

**Figure 8.** The cross-section of the surface shown in figure 5 by the plane of the axis \( i_c \) (a) and the plane by \( 0.1r_{ax} \) spaced from the plane of the axis \( i_c \) (b).
5. Discussion
The described analysis of quasi-rotation surface models gives an idea about the quasi-rotation surface order. The resulting data are listed in table 1. To fill in the table’s cells containing information about the order of the line image, generated by the symmetric interrelation operation with respect to the circular axis, the data [1] was systematized, accordingly:

Table 1. Relationship between values of orders of geometric object in quasi-symmetry and quasi-rotation.

| Axis   | Order of the symmetry interrelation | Generatrix (Preimage) | Preimage order | Symmetric image order | Quasi-rotation surface order |
|--------|-------------------------------------|-----------------------|----------------|-----------------------|-----------------------------|
| Circle | 4                                   | Line                  | 1              | 4                     | 6                           |
|        |                                     | Circle                | 2              | 6                     | 10                          |
| Ellipse| 8                                   | Line                  | 1              | 8                     | 12                          |
|        |                                     | Circle                | 2              | 12                    | 20                          |

Figure 8 (b) shows an intersection of a surface with a plane of 0.1 $r_a$ spaced from the plane of the axis $l$. Evidently, the original circle $a$ in the plane of the intersection transitions into two closed lines that intersects the straight line $l$ at four points. Notably, the appearance of the other lines of intersection of the surface with the plane does not visually change and still gives 6 points of intersection with the straight line $l$. Therefore, the order of the resulting surface is ten. It follows that in the case shown in figure 6, when the section plane is shifted by a small amount, the initial circle will transition into four closed lines. These lines intersect the straight line $l$ at eight points. Thus, the order of the resulting surface is equal to twenty. The same reasoning applies to the case shown in figure 3. During calculating of the order of the quasi-rotation surface for the line around an ellipse axis, four must be added to the order of the image of the quasi-symmetry of the line. The given examples of quasi-rotation surfaces are selected to detect quasi-rotation surfaces of the maximum possible order. The QRTiе apparatus leads to the formation of a bilayered torus [1], if the center of the generatrix circle coincides with the center of the axis. The order of such a surface is eight. The QRTiе apparatus leads to the formation of a two-layered Dupin cyclide, as a combination of two layers of a four-layer surface. This is possible if the center of the generatrix of the circle coincides with one of the foci of the elliptic axis. This conclusion follows from the study of the geometric properties of Dupin cyclides described elsewhere [14]. In this case, the order of the four-layered surface is sixteen.

6. Conclusions
In this work, an algorithm is developed and tested that allows to numerically simulate the surfaces generated by quasi-rotation of an arbitrary line lying in the plane of the axis. The original data for the calculation are the values that determine the shape of the generatrix and the axis, as well as their relative position. This algorithm can be implemented in any software environment of digital mathematical modelling. Thus, it was possible to generate 3D models of quasi-rotation surfaces. Based on these models, one can create models of solids within one or more layers of the generated surface, as well as their compartments. Such models can be used to create parts using materials processing machines with software control. Solid-state models of quasi-rotation surfaces will allow us to study their optical, mechanical, and other properties. Studying the properties of such solids will help to predict their application areas. Higher-order surfaces are used in construction and mechanical engineering [15], for designing of the favorable conditions for catalytic processes in chemical technology [16], and in other spheres of human activity [17].
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