Construction and classification of \( p \)-ring class fields modulo \( p \)-admissible conductors

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Abstract: Each \( p \)-ring class field \( K_f \) modulo a \( p \)-admissible conductor \( f \) over a quadratic base field \( K \) with \( p \)-ring class rank \( \delta_f \mod f \) is classified according to Galois cohomology and differential principal factorization type of all members of its associated heterogeneous multiplet \( M(K_f) = [(N_{c,i})_{1 \leq i \leq m(c)}]_{c|f} \) of dihedral fields \( N_{c,i} \) with various conductors \( c \mid f \) having \( p \)-multiplicities \( m(c) \) over \( K \) such that \( \sum_{c|f} m(c) = \frac{f^{f-1}}{p-1} \). The advanced viewpoint of classifying the entire collection \( M(K_f) \), instead of its individual members separately, admits considerably deeper insight into the class field theoretic structure of ring class fields. The actual construction of the multiplet \( M(K_f) \) is enabled by exploiting the routines for abelian extensions in the computational algebra system Magma.

Keywords: \( p \)-ring class fields; \( p \)-admissible conductors; Quadratic base fields; Non-Galois cubic fields; \( S_3 \)-fields; Dihedral fields; Multiplicity of discriminants; \( p \)-ring spaces; Heterogeneous multiplets; Galois cohomology; Differential principal factorizations; Capitulation of \( p \)-class groups; Statistics.

MSC: 11R37; 11R11; 11R16; 11R20; 11R27; 11R29; 11Y40.

1. Introduction

The aim of this article is to present an entirely new technique for the construction and classification of non-Galois fields \( L \) of odd prime degree \( p \) as subfields \( L < K_f \) of a \( p \)-ring class field \( K_f \) modulo a \( p \)-admissible conductor \( f \) over a quadratic base field \( K \). The innovative idea underlying this new method is the fact that, if the Galois closure \( N \) of such a field \( L \) is absolutely dihedral of degree \( 2p \) with automorphism group \( \text{Gal}(N/Q) \simeq D_p = \langle \sigma, \tau \mid \sigma^p = \tau^2 = 1, \tau \sigma = \sigma^{-1} \tau \rangle \), then \( N \) is relatively cyclic of degree \( p \) with group \( G = \text{Gal}(N/K) \simeq C_p = \langle \sigma \rangle \) over its unique quadratic subfield \( K = \text{Fix}(\sigma) \) and can be viewed as an abelian extension modulo some conductor \( f \) over \( K \) within the scope of class field theory [1–4].

The construction process for the fields \( L \) is implemented as a program script for the computational algebra system Magma [5–7] using the class field theoretic routines by Fieker [3], and the normal fields \( N/L \) are classified according to the cohomology \( \hat{H}^0(G, U_N) \) and \( \hat{H}^1(G, U_N) \) of their unit group \( U_N \) as a Galois module over \( G \) [8–10].

For \( p \geq 5 \), the results are completely new, whereas for \( p = 3 \), they admit an independent verification and a class field theoretic illumination of classical tables of cubic fields by Angell 1972 [11,12] and 1975 [13,14], Ennola and Turunen 1983 [15,16], Llorente and Quer 1988 [17], Fung and Williams 1990 [18,19], and Belabas 1997 [20]. However, in contrast to these well-known tables, where the focus was on the computation of fundamental systems of units and the structure of ideal class groups [11–16,18], or even only of generating polynomials and prime decompositions [17,20], our innovative database establishes an arrangement according to conductors with an increasing number of prime factors, pays attention to the phenomenon of multiplicities of discriminants [21–25], and constitutes the first classification into 9, respectively 3, differential principal factorization types of totally real, respectively simply real, cubic number fields [8–10,26,27]. This is a progressive new kind of structural information which has never been provided for algebraic number fields before, except for pure cubic fields [28–31] and pure quintic fields [8], but the present paper emphasizes the advanced viewpoint of classifying an entire ring class field \( K_f \) by its associated heterogeneous multiplet \( M(K_f) \) of dihedral fields with various conductors \( c \mid f \).
2. Heterogeneous multiplets of objects and invariants

Let \( K = \mathbb{Q}(\sqrt{d}) \) be a quadratic base field with positive or negative fundamental discriminant \( d = d_K \equiv 0, 1 \pmod{4} \), essentially squarefree except possibly for the 2-contribution \( \nu_2(d) \). Suppose that \( p \) is an odd prime number and \( f \geq 1 \) is a \( p \)-admissible conductor over \( K \) [21,25]. Then the \( p \)-ring class field \( K_{p,f} \mod f \) of \( K \) contains all cyclic relative extensions \( N/K \) with some conductor \( c \mid f \) which are absolutely dihedral with automorphism group \( \text{Gal}(N/Q) \simeq D_p \) over the rational number field \( Q \). The crucial concept underlying this entire paper is the collection of all these dihedral fields in a heterogeneous multiplet \( \mathbf{M}(K_{p,f}) = \{ (N_{c,i})_{1 \leq i \leq m_p(K,c)} \}_{c \mid f} \) according to the \( p \)-multiplicities \( m_p(K,c) \) [21,25], which satisfy the relation
\[
\sum_{c \mid f} m_p(K,c) = \frac{p^{\nu_f(p)} - 1}{p - 1}
\]
in terms of the \( p \)-ring class rank \( e_{p,f} \) modulo \( f \) of \( K \). Since our principal aim is the classification of \( p \)-ring class fields \( K_{p,f} \), it is essential to distinguish between a multiplet of \textit{objects} (expressing the multiplicity of the discriminants \( d_N \)) and a corresponding multiplet of \textit{invariants} (expressing the Galois cohomology of the unit groups \( U_N \) and differential principal factorizations of the fields \( N \)).

Definition 1. By the type of the \( p \)-ring class field \( K_{p,f} \) modulo \( f \) of \( K \) we understand the pair \((\text{Obj}(K_{p,f}), \text{Inv}(K_{p,f}))\) of heterogeneous multiplet
\[
\begin{align*}
\text{Obj}(K_{p,f}) &= \{ (N_{c,i})_{1 \leq i \leq m_p(K,c)} \}_{c \mid f} \\
\text{Inv}(K_{p,f}) &= \{ (\tau(N_{c,i}))_{1 \leq i \leq m_p(K,c)} \}_{c \mid f}
\end{align*}
\]
consisting of all absolutely dihedral fields \( N_{c,i} \) with conductors \( c \) dividing \( f \) as \textit{objects} and their differential principal factorization types (DPF types) \( \tau(N_{c,i}) \) as \textit{invariants} [8,9].

3. Homogeneous multiplets of unramified extensions

The unique situation where the heterogeneous multiplets degenerate to homogeneous multiplets occurs for \textit{unramified} relative extensions \( N/K \) with conductor \( f = 1 \) which has only itself as a divisor \( c \mid f \). In this unramified case, which implies positive \( p \)-class rank \( e_p = e_{p,1} \geq 1 \) of the quadratic base field \( K \), there occur \textit{at most two} possible differential principal factorization types.

Theorem 1. An \textit{unramified} cyclic extension \( N \) with odd prime degree \( p \) of \( K \) possesses the conductor \( f = 1 \) without any prime divisors. For a \textit{totally real} field \( N \), there are two cases:

1. If the \( p \)-class rank of \( K \) is \( e_p = 1 \), then \( N \) is of type \( \delta_1 \).
2. If the \( p \)-class rank of \( K \) is \( e_p \geq 2 \), then \textit{two types} \( \alpha_1 \) and \( \delta_1 \) are possible for \( N \).

If \( N \) is \textit{totally complex}, then \( N \) is of type \( \alpha_1 \), independently of the \( p \)-class rank of \( K \).

Proof. Since the conductor \( f = q_1 \cdots q_t \) is essentially the square free product of all prime numbers \( q_i \in \mathbb{P} \), whose overlying prime ideals \( q_i \in \mathbb{P}_K \) are ramified in \( N \), the following chain of equivalent statements is true: \( N/K \) is unramified \( \iff \) None of the prime ideals of \( K \) ramifies in \( N \) \( \iff \) The conductor \( f = 1 \) has no prime divisors, i.e., \( t = 0 \).

Now we use the fundamental equation in [9, Corollary 5.1] and the estimates in [9, Corollary 5.2] for the decision about possible types of principal factorizations. If \( f = 1 \), then there neither exist absolute principal factorizations in \( L/Q \), since \( 0 \leq A = \min(t,2) = 0 \), nor relative principal factorizations in \( N/K \), since \( 0 \leq R = \min(s,2) = 0 \), where \( s \leq t \) denotes the number of prime divisors \( q_i \) of \( f \) which split in \( K \). Consequently, the fundamental equation degenerates to \( U + 1 = C \) with \( 1 \leq U + 1 \leq 2 \), which implies \( 1 \leq C = \min(e_p,2) \), thus, only type \( \delta_1 \) with \( C = 1 \) is possible for \( e_p = 1 \), whereas type \( \alpha_1 \) with \( C = 2 \) can arise additionally for \( e_p \geq 2 \).

4. Conductors with a single prime divisor

For a \textit{regular prime} conductor \( f \), only two cases are possible.

Theorem 2. Let \( K \) be a quadratic base field with \( p \)-class rank \( q = e_p \). Suppose \( f = q \) is a \textit{regular} \( p \)-admissible \textit{prime conductor} for \( K \). Then the heterogeneous multiplet \( \mathbf{M}(K_{p,f}) \) associated with the \( p \)-ring class field \( K_{p,f} \mod f \) of \( K \) consists of two homogeneous multiplets with multiplicities \( m_p(K,1) \) and \( m_p(K,q) \). In this order, and in dependence on the \( p \)-ring space \( V_p(q) \), these two multiplicities are given by
1. \( (1 + p + \ldots + p^{e-1}, \ p^e) \), if \( V_p(q) = V \) (free situation),
2. \( (1 + p + \ldots + p^{e-1}, 0) \), if \( V_p(q) < V \) (restrictive situation).

Proof. See [25, Theorem 3.2, p. 2215, and Theorem 3.3, p. 2217].

In the special case \( p = 3 \), there also exists the possibility of an irregular prime power conductor \( f = 3^2 \), provided the discriminant of the quadratic field satisfies the congruence \( d \equiv -3 \pmod{9} \).

**Theorem 3.** Assume that \( p = 3 \). Let \( K \) be a quadratic base field with 3-class rank \( q = q_3 \) and discriminant \( d \equiv -3 \pmod{9} \). Consider the irregular 3-admissible prime power conductor \( f = 3^2 \) for \( K \). Then the heterogeneous multiplet \( M(K_{p,f}) \) associated with the 3-ring class field \( K_{p,f} \) mod \( f \) of \( K \) consists of three homogeneous multiplets with multiplicities \( m_3(K, 1) \), \( m_3(K, 3) \) and \( m_3(K, 9) \). In this order, and in dependence on the 3-ring spaces \( V_3(3) \) and \( V_3(9) \), these three multiplicities are given by

1. \( (1 + 3 + \ldots + 3^{e-1}, \ 3^e, \ 3^{e+1}) \), if \( V_3(9) = V_3(3) = V \) (free situation),
2. \( (1 + 3 + \ldots + 3^{e-1}, \ 3^e, 0) \), if \( V_3(9) < V_3(3) = V \),
3. \( (1 + 3 + \ldots + 3^{e-1}, 0, \ 3^e) \), if \( V_3(9) = V_3(3) < V \),
4. \( (1 + 3 + \ldots + 3^{e-1}, 0, 0) \), if \( V_3(9) < V_3(3) < V \) (maximal restriction).

Proof. See [25, Theorem 3.4, p. 2217].

5. Conductors with two prime divisors

For regular conductors \( f \) divisible by two primes, more distinct situations may arise.

**Theorem 4.** Let \( K \) be a quadratic base field with \( p \)-class rank \( q = q_p \) and discriminant \( d \equiv -3 \pmod{9} \). Suppose \( f = q_1 \cdot q_2 \) is a regular \( p \)-admissible conductor for \( K \) with two prime divisors \( q_1 \) and \( q_2 \). Then the heterogeneous multiplet \( M(K_{p,f}) \) associated with the \( p \)-ring class field \( K_{p,f} \) mod \( f \) of \( K \) consists of four homogeneous multiplets with multiplicities \( m_p(K, 1) \), \( m_p(K, q_1) \), \( m_p(K, q_2) \) and \( m_p(K, f) \). In this order, and in dependence on the \( p \)-ring spaces \( V_p(q_1) \), \( V_p(q_2) \) and \( V_p(f) \), these four multiplicities are given by

1. \( (1 + p + \ldots + p^{e-1}, \ p^e, \ p^e, \ p^e (p-1)) \), if \( V_p(f) = V_p(q_1) = V_p(q_2) = V \) (free case),
2. \( (1 + p + \ldots + p^{e-1}, \ p^e, 0, 0) \), if \( V_p(f) = V_p(q_2) < V_p(q_1) = V \),
3. \( (1 + p + \ldots + p^{e-1}, 0, \ p^e, 0) \), if \( V_p(f) = V_p(q_1) < V_p(q_2) = V \),
4. \( (1 + p + \ldots + p^{e-1}, 0, 0, \ p^e) \), if \( V_p(f) = V_p(q_1) = V_p(q_2) < V \),
5. \( (1 + p + \ldots + p^{e-1}, 0, 0, 0) \), if \( V_p(f) < V_p(q_1) \neq V_p(q_2) < V \) (maximal restriction).

Proof. We use the terminology and notation in [25]. Generally, the \( p \)-ring class rank is given by \( q_{p,f} = q + t + w - \delta_p(f) \). Here, we have either \( t = 2 \), \( w = 0 \) or \( t = 1 \), \( w = 1 \), and thus \( q_{p,f} = q + 2 - \delta_p(f) \). Also, we know that generally \( m_p(K, 1) = \frac{p^{e-1}}{p-1} \). Since \( f = q_1 \cdot q_2 \) is \( p \)-admissible, \( q_1 \) and \( q_2 \) must also be \( p \)-admissible, both.

1. In the free case with defect \( \delta_p(f) = 0 \), we have \( V_p(f) = V_p(q_1) = V_p(q_2) = V \) and

\[
\frac{p^{e+2} - 1}{p-1} - \frac{p^e - 1}{p-1} = \frac{p^e (p^2 - 1)}{p-1} = \frac{p^e (p+1) = p^e + p^{e+1} (p-1)},
\]

which is exactly the desired partition

\[
\frac{p^{e+1} - 1}{p-1} - m_p(K, 1) = m_p(K, q_1) + m_p(K, q_2) + m_p(K, f).
\]

2. If \( q_1 \) is free and \( q_2 \), \( f \) are restrictive, then \( V_p(f) = V_p(q_2) < V_p(q_1) = V \) and the relation

\[
\frac{p^{e+1} - 1}{p-1} - \frac{p^e - 1}{p-1} = \frac{p^e (p-1)}{p-1} = p^e,
\]

must be interpreted as \( m_p(K, q_1) = p^e \) and \( m_p(K, q_2) = m_p(K, f) = 0 \).

3. This case arises by interchanging the roles of \( q_1 \) and \( q_2 \) in the previous case.

4. Additionally to (2) and (3), there is another case of defect \( \delta_p(f) = 1 \) where neither \( q_1 \) nor \( q_2 \) is free but their \( p \)-ring spaces coincide \( V_p(f) = V_p(q_1) = V_p(q_2) < V \). Then the formula in (2) has to be interpreted as \( m_p(K, q_1) = m_p(K, q_2) = 0 \) and \( m_p(K, f) = p^e \).
5. Finally, in the case of maximal restriction with defect $\delta_p(f) = 2$, which occurs for distinct $p$-ring spaces $V_p(f) < V_p(q_1) \neq V_p(q_2) < V$, there is no rank increment from $q$ to $q_{p,f}$, and thus $m_p(K, q_1) = m_p(K, q_2) = m_p(K, f) = 0$. 

6. Construction of $p$-ring class fields

This section describes how the classification of non-trivial $p$-ring class fields is prepared by their construction and rigorous count. The intended class field theoretic illumination of the structure of heterogeneous multiplets $M(K_{p,f}) = \{(N_{c,1}, \ldots, N_{c,m(c)}) \}_{c \in f}$ associated with $p$-ring class fields $K_{p,f}$ modulo $p$-admissible conductors $f$ over quadratic fields $K$ must pay primary attention to the $p$-class rank $q_p$ of the quadratic base fields $K = \mathbb{Q}(\sqrt{d})$, since $q_p$ enters the formula for the multiplicities $m(c)$. More precisely, since the existence of a torsion free fundamental unit $\epsilon > 1$ in real quadratic fields $K$ with $d > 0$, and the occurrence of the 3-torsion unit $\xi_3$ in the particular imaginary quadratic field $K$ with $d = -3$ in the case $p = 3$, exerts a crucial impact on the codimension of $p$-ring spaces $V_p(\epsilon)$, the invariant $q_p$ must rather be replaced by the $p$-Selmer rank $\sigma_p$ of $K$ which describes all $p$-virtual units of $K$, those which arise from non-trivial $p$-classes and the units in the usual sense:

$$\sigma_p = \begin{cases} q_p & \text{if } p > 5, d < 0 \text{ or } p = 3, d < -3, \\ q_p + 1 & \text{if } d > 0 \text{ or } p = 3, d = -3. \end{cases} \quad (2)$$

The secondary attention is devoted to various $p$-admissible conductors $f = q_1 \cdots q_t$ with an increasing number $t \geq 0$ of prime divisors, starting with unramified extensions having $t = 0$, $f = 1$, and continuing with ramified extensions, beginning with prime or prime power conductors having $t = 1$, $f = q_1$ with a prime $q_1 \in \mathcal{P}$ or the critical prime power $q_1 = p^2$.

7. Multiplets over imaginary quadratic fields for $p = 3$

The focus of this section and most of the further sections is on $p = 3$, where the components $N_{c,j}$ of multiplets are cyclic cubic extensions of quadratic base fields $K$. Here, we begin with imaginary base fields $K$ having the smallest possible 3-Selmer rank $\sigma_3 = q_3$. The behavior of the particular imaginary quadratic field $K$ with $d = -3$ where the extensions $N_{c,i}/K$ contain pure cubic fields is rather similar to real quadratic base fields $K$ with $\sigma_3 = q_3 + 1$, and thus the case $d = -3$ will be treated separately.

**Theorem 5.** Let $K$ be an imaginary quadratic field with fundamental discriminant $d < -3$ and trivial 3-class rank $q_3 = 0$. Assume that $f = q_1 \cdots q_t$ is a 3-admissible conductor with $t \geq 1$ regular prime or prime power divisors $q_i$ (that is, either $q_i \equiv \pm 1 \pmod{3}$ or $q_i = 3, d \equiv \pm 3 \pmod{9}$ or $q_i = 9, d \equiv \pm 1 \pmod{3}$ but not $q_i = 9, d \equiv -3 \pmod{9}$). Then the 3-ring class field $K_{3,f}$ modulo $f$ of $K$ contains a homogeneous multiplet $M(K_{3,f}) = (N_{f,1}, \ldots, N_{f,m})$ of dihedral fields with conductor $f$ and multiplicity $m = 2t - 1$ (singlet, doublet, quartet, octet, hexadecuplet, etc.).

**Proof.** All 3-ring spaces $V_3(q_t)$ coincide with 3-Selmer space $V.V_3$ [25, Theorem 3.2, p. 2215].

7.1. Classification of Angell’s 3169 simply real cubic fields

In order to demonstrate the powerful performance of our innovative techniques, we construct all 3-ring class fields $K_{3,f}$ which contain the normal closures $N$ of the simply real cubic fields $L$ in Angell’s table [11,12] as abelian extensions of the associated imaginary quadratic base fields $K < N$.

There arise four values of the multiplicity $m = 1, 2, 3, 4$, and accordingly simply real cubic fields are collected in singlets, doublets, triplets and quartets. Nillets with $m = 0$ complete the view.

The classification of the pure cubic fields, respectively non-pure simply real cubic fields, into differential principal factorization types was established in [28], respectively [9].

Although the types $\alpha$ and $\beta$ of pure cubic fields are similar to the types $\alpha_2$ and $\beta$ of non-pure simply real cubic fields, we do not mix the classifications, since firstly the existence of radicals among the principal factors distinguishes pure cubic fields from non-pure simply real cubic fields, and secondly, type $\gamma$ can only occur for the former, whereas type $\alpha_1$ is only possible for the latter.
Results

According to Table 1, the number of all non-pure simply real cubic fields \( L \) having discriminants \(-2 \cdot 10^4 < d_L < 0\) is given by 3134. Together with 35 pure cubic fields in Table 2, the total number is 3169, as announced correctly in [12].

Table 1. Cubic discriminants in the range \(-2 \cdot 10^4 < d_L = f^2 \cdot d < 0\)

| \( f \) | Condition | Total | Multiplicity | DPF |
|-------|-----------|-------|--------------|-----|
| \( q \) | \( \equiv -1 \pmod{3} \) | 454   | 0 1 2 3 4 a₁ a₂ β | 454 |
| 3     | \( d \equiv +3 \pmod{9} \) | 62    | 62           | 21  |
| 3     | \( d \equiv -3 \pmod{9} \) | 58    | 58           | 21  |
| 9     | \( d \equiv -3 \pmod{9} \) | 7     | 7            | 21  |
| 9     | \( d \equiv -1 \pmod{3} \) | 23    | 23           | 21  |
| 9     | \( d \equiv +1 \pmod{3} \) | 20    | 20           | 16  |
| \( q_{1q_2} \) | \( \equiv -1 \pmod{3} \) | 6     | 6            | 12  |
| 3q    | \( d \equiv +3 \pmod{9} \) | 7     | 7            | 14  |
| 3q    | \( d \equiv -3 \pmod{9} \) | 3     | 3            | 6   |
| 9q    | \( d \equiv -1 \pmod{3} \) | 3     | 3            | 6   |
| 9q    | \( d \equiv +1 \pmod{3} \) | 3     | 3            | 6   |
| 3ℓ    | \( d \equiv +3 \pmod{9} \) | 1     | 1            | 2   |
| \( q \ell \) | \( \equiv \mp1 \pmod{3} \) | 1     | 1            | 2   |
| 1     | \( \varrho_3 = 1 \) | 2143  | 2143         | 2143|

We emphasize the difference between the number of discriminants (without multiplicities)

\[
2824 + 24 + 58 + 22 = 2928,
\]

and the number of fields (including multiplicities in a weighted sum)

\[
1 \cdot 2824 + 2 \cdot 24 + 3 \cdot 58 + 4 \cdot 22 = 2824 + 48 + 174 + 88 = 3134,
\]

which can be confirmed by adding the contributions to the 3 DPF types \( a_1, a_2, \beta \)

\[
2344 + 65 + 725 = 3134.
\]

In contrast, 235 is the number of formal cubic discriminants \( d_L = f^2 \cdot d_K \) with fundamental discriminants \( d_K \) of imaginary quadratic fields and 3-admissible conductors \( f \) for each \( K \), where the relevant multiplicity formula [25] yields the value zero. So the formal cubic discriminants belong to nillets, i.e., multiplets with multiplicity \( m_3(K,f) = 0 \). The total number of all (actual) cubic discriminants and formal cubic discriminants is the number of admissible cubic discriminants

\[
2928 + 235 = 3163.
\]

According to Theorem 5, Nillets can only arise for \( \varrho_3 \geq 1 \), but not for \( \varrho_3 = 0 \).
Table 2. Pure cubic discriminants in the range \(-2 \cdot 10^4 < d_L = -3 \cdot f^2 < 0\)

| \(f\) | Condition       | Total | 0 | 1 | 2 | 3 | 4 | \(\alpha\) | \(\beta\) | \(\gamma\) |
|-------|-----------------|-------|---|---|---|---|---|-------------|-------------|-------------|
| \(q\) | \(\equiv -1 (\text{mod } 3)\) | 11    | 3 | 5 | 2 | 2 | 1 |            |             |             |
| 9     | \(d = -3\)     | 1     | 1 | 1 |   |   |   |            |             |             |
| \(\ell\) | \(\equiv +1 (\text{mod } 3)\) | 10    | 3 | 5 | 2 | 2 | 1 |            |             |             |
| \(q_1q_2\) | \(\equiv -1 (\text{mod } 3)\) | 6     | 1 | 1 |   |   |   |            |             |             |
| \(9q\) | \(d = -3\)     | 5     | 1 | 4 |   |   |   |            |             |             |
| \(9\ell\) | \(d = -3\)     | 2     | 2 |   |   |   |   |            |             |             |
| \(q_1q_2\ell\) | \(\equiv +1 (\text{mod } 3)\) | 8     | 2 | 6 | 2 |   | 4 |            |             |             |
| \(3q_1q_2\) | \(d = -3\)     | 1     | 1 |   |   |   |   |            |             |             |
| \(3q\ell\) | \(d = -3\)     | 2     | 2 |   |   |   |   |            |             |             |
| **Summary** |               | 52    | 20| 29| 3 |   | 11| 20         | 4           |             |

According to Table 2, the number of pure cubic fields \(L\) with discriminant \(-2 \cdot 10^4 < d_L < 0\) is 35. Actually, triplets and quartets of pure cubic fields do not occur in this range.

There is a difference between the number of discriminants (without multiplicities)

\[
29 + 3 = 32,
\]

and the number of fields (including multiplicities in a weighted sum)

\[
1 \cdot 29 + 2 \cdot 3 = 29 + 6 = 35,
\]

which can be confirmed by adding the contributions to the 3 DPF types

\[
11 + 20 + 4 = 35.
\]

The total number of all (actual) cubic discriminants and formal cubic discriminants (of the 20 nilets) is the number of admissible pure cubic discriminants \(d_L = -3 \cdot f^2\),

\[
32 + 20 = 52.
\]

8. Multiplets over real quadratic fields for \(p = 3\)

We continue with real quadratic base fields \(K\) having elevated 3-Selmer rank \(\sigma_3 = q_3 + 1\), due to the existence of a torsion free fundamental unit \(\varepsilon > 1\).

8.1. Classification of Angell’s 4804 totally real cubic fields

In order to demonstrate our progressive perspective of classification of heterogeneous multiplets \(M(K_{3,f})\) into an enigmatic variety of differential principal factorization types, we construct all 3-ring class fields \(K_{3,f}\) which contain the normal closures \(N\) of the totally real cubic fields \(L\) in Angell’s table [13,14] as abelian extensions of the associated real quadratic base fields \(K < N\).

Again there arise four values of the multiplicity \(m = 1, 2, 3, 4\), and accordingly totally real cubic fields are collected in singlets, doublets, triplets and quartets. Formal nilets complete the view.

The classification into differential principal factorization types for non-cyclic totally real cubic fields was developed in [9,26,27].

Results

According to Table 3, the number of non-cyclic totally real cubic fields \(L\) with discriminant \(0 < d_L < 10^5\) is 4753, in perfect accordance with the results by Llorente and Oneto [32,33], who discovered the omission of ten fields in the table by Angell [13,14]. Together with 51 cyclic cubic fields in Table 4, the total number is 4804 (not 4794, as announced erroneously in [14]).
Again we emphasize the difference between the number of discriminants (without multiplicities)

\[ 4652 + 9 + 21 + 5 = 4687, \]

and the number of fields (including multiplicities in a weighted sum)

\[ 1 \cdot 4652 + 2 \cdot 9 + 3 \cdot 21 + 4 \cdot 5 = 4652 + 18 + 63 + 20 = 4753, \]

which can be confirmed by adding the contributions to the 7 DPF types \((a_2, a_3) \text{ do not occur})

\[ 16 + 10 + 76 + 106 + 3349 + 79 + 1117 = 4753. \]

### Table 3. Cubic discriminants in the range \(0 < d_L = f^2 \cdot d < 10^5\)

| \(f\) | Condition | Multiplicity | Differential Principal Factorization |
|------|-----------|--------------|--------------------------------------|
| \(q\) | \(d \equiv -1 \pmod{3}\) | 3025 | 2219 | 806 |
| \(3\) | \(d \equiv +3 \pmod{9}\) | 396 | 287 | 109 |
| \(3\) | \(d \equiv -3 \pmod{9}\) | 389 | 284 | 105 |
| \(9\) | \(d \equiv -3 \pmod{9}\) | 48 | 9 | 38 |
| \(9\) | \(d \equiv -1 \pmod{3}\) | 136 | 102 | 34 |
| \(9\) | \(d \equiv +1 \pmod{3}\) | 127 | 96 | 31 |
| \(l\) | \(d \equiv +1 \pmod{3}\) | 402 | 316 | 86 |
| \(q_1q_2\) | \(d \equiv -1 \pmod{3}\) | 70 | 30 | 38 | 2 |
| \(3q\) | \(d \equiv +3 \pmod{9}\) | 46 | 23 | 23 |
| \(3q\) | \(d \equiv -3 \pmod{9}\) | 45 | 19 | 25 | 1 |
| \(9q\) | \(d \equiv -3 \pmod{9}\) | 5 | 4 | 1 |
| \(9q\) | \(d \equiv -1 \pmod{3}\) | 14 | 6 | 8 |
| \(9q\) | \(d \equiv +1 \pmod{3}\) | 15 | 5 | 10 |
| \(9\ell\) | \(d \equiv -1 \pmod{3}\) | 1 | 1 |
| \(3\ell\) | \(d \equiv +3 \pmod{9}\) | 6 | 1 | 5 |
| \(3\ell\) | \(d \equiv -3 \pmod{9}\) | 5 | 2 | 3 |
| \(q\ell\) | \(\equiv +1 \pmod{3}\) | 43 | 13 | 29 | 1 |
| \(3q_1q_2\) | \(d \equiv +3 \pmod{9}\) | 2 | 3 |
| \(1\) | \(\varepsilon_3 = 1\) | 3300 | 3300 |
| \(q\) | \(d \equiv -1 \pmod{3}\) | 275 | 261 | 14 | 4 | 36 |
| \(3\) | \(d \equiv -3 \pmod{9}\) | 35 | 34 | 1 |
| \(l\) | \(\equiv +1 \pmod{3}\) | 28 | 25 | 3 | 3 |
| \(3q\) | \(d \equiv -3 \pmod{9}\) | 2 | 1 |
| \(1\) | \(\varepsilon_3 = 2\) | 5 | 5 | 16 | 10 | 76 | 106 | 3349 | 79 | 1117 |

In contrast, \(3733\) is the number of formal cubic discriminants \(d_L = f^2 \cdot d_K\) with fundamental discriminants \(d_K\) of real quadratic fields and 3-admissible conductors \(f\) for each \(K\), where the relevant multiplicity formula \([25]\) yields the value zero. So the formal cubic discriminants belong to nilets, i.e., multiplets with multiplicity \(m_3(K, f) = 0\). The total number of all (actual) cubic discriminants and formal cubic discriminants is the number of admissible cubic discriminants

\[ 4687 + 3733 = 8420. \]

### Table 4. Cyclic cubic discriminants in the range \(0 < d_L = f^2 < 10^5\)

| \(f\) | Condition | \(M\) | \(\text{DPF}\) |
|------|-----------|------|---------------|
| \(9\) | \(d = 1\) | 1 | 1 |
| \(l\) | \(\equiv +1 \pmod{3}\) | 30 | 30 |
| \(9l\) | \(d = 1\) | 4 | 8 |
| \(l_1l_2\) | \(\equiv +1 \pmod{3}\) | 6 | 12 |
| Summary | | 31 | 10 | 51 |
According to Table 4, the number of cyclic cubic fields $L$ with discriminant $0 < d_L < 10^5$ is 51, with 31 arising from singlets having conductors $f$ with a single prime divisor, and 20 from doublets having two prime divisors of the conductor $f$. (M denotes the multiplicity.)

We point out that cyclic cubic fields are rather contained in ray class fields over $\mathbb{Q}$ than in ring class fields over real quadratic base fields. The single possible DPF type $\zeta$ has nothing to do with the 9 DPF types $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \gamma, \delta_1, \delta_2, \epsilon$ of non-abelian totally real cubic fields in [9].

9. Conclusion and outlook

In this paper, we have classified all multiplets $\text{Obj}(K_{3,f})$ of non-pure simply real cubic fields $L$ (more precisely of their normal closures $N$) according to the associated multiplets of invariants, namely the differential principal factorization types, $\text{Inv}(K_{3,f})$, where $K_{3,f}$ denotes the 3-ring class field modulo a 3-admissible conductor $f$ of the imaginary quadratic subfield $K < N$: (Recall that $\text{Obj}(K_{3,f}) = (N_{f,L})_{1 \leq i \leq m}$ and $\text{Inv}(K_{3,f}) = (\pi(N_{f,L}))_{1 \leq i \leq m}$, here homogeneously.)

- 2824 singlets of type either $(\alpha_1)$ or $(\alpha_2)$ or $(\beta)$, according to Table 1;
- 24 doublets of exclusive type $(\beta, \beta)$ (without 3 pure cubic doublets);
- 58 triplets with the following distribution of types:
  - 7 triplets of type $(\beta, \beta, \beta)$ for $f = 9$ singular, $e_3 = 0$,
  - 51 triplets sharing common 3-class rank $e_3 = 1$ of $K$ [34, Table 1, pp. 118–121], namely
    - 34 triplets of type $(\alpha_1, \alpha_1, \alpha_1)$ for $f = q, \ell, 3$,
    - 3 triplets of type $(\alpha_1, \alpha_1, \beta)$ for $f = \ell, 9$ split,
    - 5 triplets of type $(\alpha_1, \beta, \beta)$ for $f = q, 3$, and
    - 9 triplets of type $(\beta, \beta, \beta)$ for $f = q, 3, 3q, q\ell, q_9$ finer than [34] since $a_2$ does not occur;
- 22 quartets of exclusive type $(\alpha_1, \alpha_1, \alpha_1, \alpha_1)$ (see [35] for details concerning the capitulation).

Similarly, we have classified all multiplets $\text{Obj}(K_{3,f})$ of non-cyclic totally real cubic fields $L$ (more precisely of their normal closures $N$) according to the associated multiplets of invariants, namely the differential principal factorization types, $\text{Inv}(K_{3,f})$, where $K_{3,f}$ denotes the 3-ring class field modulo a 3-admissible conductor $f$ of the real quadratic subfield $K < N$:

- 4652 singlets of type either $(\beta_2)$ or $(\gamma)$ or $(\delta_1)$ or $(\delta_2)$ or $(\epsilon)$, according to Table 3;
- 9 doublets, 4 of type $(\gamma, \gamma)$ for $f = 9q, 3q_1q_2$ and 5 of type $(\epsilon, \epsilon)$ for $f = 3q, 9q, q_1q_2, q\ell$;
- 21 triplets with the following distribution of types:
  - 1 triplet of type $(\epsilon, \epsilon, \epsilon)$ for $f = 9$ singular, $e_3 = 0$,
  - 1 triplet of type $(\gamma, \gamma, \gamma)$ for $f = 9q$ singular, $e_3 = 0$, and
  - 19 triplets sharing common 3-class rank $e_3 = 1$ of $K$ (with considerable refinement of Schmittlein’s coarse distinction of only two alternatives [34, Table 2, pp. 122–123]), namely
    - 13 triplets of type $(\delta_1, \delta_1, \delta_1)$ for $f = 3, q,
    - 1 triplet of type $(\beta_1, \beta_1, \beta_1)$ for $f = 3q$,
    - 2 triplets of type $(\beta_1, \beta_1, \epsilon)$ for $f = q$ (conspicuously with symbol “−” in [34]), and
    - 3 triplets of type $(\beta_1, \beta_1, \beta_1)$ for $f = \ell$;
- 5 quartets, 1 of type $(\alpha_1, \alpha_1, \alpha_1, \alpha_1)$ and 4 of type $(\alpha_1, \alpha_1, \alpha_1, \delta_1)$ (more details in [36,37]).

In the same manner, we shall refine more extensive tables by Fung and Williams [18], Ennola and Turunen [15,16], Llorente and Quer [17] in the new year 2021.

Moreover, we shall provide extensive evidence of the truth of Scholz’ conjecture, which we have proved for $p = 3$ in [9], also for $p = 5$ and $p = 7$, and probably for any odd prime $p$.

Data Availability: Implementations of our innovative algorithms in Magma [5–7] may be requested via email.

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