TORIC VARIETIES WITH AMPLE TANGENT BUNDLE

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Abstract. We give a simple combinatorial proof of the toric version of Mori’s theorem that the only $n$-dimensional smooth projective varieties with ample tangent bundle are the projective spaces $\mathbb{P}^n$.

1. Introduction

It is a well-known theorem that the only smooth projective varieties (over an algebraically closed field $k$) with ample tangent bundles are the projective spaces $\mathbb{P}^n_k$. This is first conjectured by Hartshorne [Har70, Problem 2.3] and later proved by Mori [Mor79] using the full force of his now-celebrated “bend and break” technique. Here we say that a vector bundle $E$ is ample (resp. nef) if the line bundle $O_{\mathbb{P}E}(1)$ on the projectivized bundle $\mathbb{P}E$ is ample (resp. nef).

In this paper, we consider a toric version of this theorem and show that it admits a simple combinatorial proof.

Theorem 1.1. Let $X$ be an $n$-dimensional smooth projective toric variety (over an algebraically closed field $k$) with ample tangent bundle $T_X$. Then $X$ is isomorphic to $\mathbb{P}^n_k$.

In the proof we consider the polytope $P \subseteq \mathbb{R}^n$ corresponding to $X$ (together with any ample divisor $D$). The key observation we make is that the ampleness of $T_X$ implies that the sum of any pair of two adjacent angles on a 2-dimensional face of $P$ is smaller than $\pi$. It follows that $P$ has to be an $n$-simplex, and hence $X$ is isomorphic to $\mathbb{P}^n$.

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2. Preliminaries

Here we list out some definitions and facts regarding toric varieties and toric vector bundles that we will use in this article. One may refer to [Ful93, CLS11] for more details about toric varieties, and [Pay08, DRJS18] for more details about toric vector bundles.

2.1. Toric varieties. We work throughout over an algebraically closed field $k$. By a toric variety, we mean an irreducible and normal algebraic variety $X$ containing a torus $T \cong (k^*)^n$ as a Zariski open subset such that the action of $T$ on itself (by multiplication) extends to an algebraic action of $T$ on $X$.

Let $M$ be the group of the characters of $T$, and $N$ the group of the 1-parameter subgroups of $T$. Both $M$ and $N$ are lattices of rank $n$ (equal to the dimension of $T$), i.e. isomorphic to $\mathbb{Z}^n$. They are dual to each other in the sense that there is a natural pairing of $M$ and $N$ denoted by $\langle \cdot, \cdot \rangle : M \times N \rightarrow \mathbb{Z}$.

Every toric variety $X$ is associated to a fan $\Sigma$ in $N_\mathbb{R} := N \otimes_{\mathbb{Z}} \mathbb{R} (\cong \mathbb{R}^n)$. A fan $\Sigma$ is said to be complete if it supports on the whole $N_\mathbb{R}$, and is said to be smooth if every cone in $\Sigma$ is generated by a subset of a $\mathbb{Z}$-basis of
N. A toric variety $X$ is complete if and only if its associated fan $\Sigma$ is complete, and $X$ is smooth if and only if $\Sigma$ is smooth.

There is an inclusion-reversing bijection between the cones $\sigma \in \Sigma$ and the $T$-orbits in $X$. Let $O_{\sigma} \subseteq X$ be the orbit corresponding to $\sigma$. The codimension of $O_{\sigma}$ in $X$ is equal to the dimension of $\sigma$. Each cone $\sigma \in \Sigma$ also corresponds to an open affine set $U_{\sigma} \in X$, which is equal to the union of all the orbits $O_{\tau}$ corresponding to cones $\tau$ contained in $\sigma$. Given a 1-dimensional cone $\rho \in \Sigma$, the closure of $O_{\rho}$ is a Weil divisor, denoted by $D_{\rho}$. The class group of $X$ is generated by the classes of the divisors $D_{\rho}$ corresponding to the 1-dimensional cones in $\Sigma$.

2.2. Polytopes and toric varieties. Let $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^n$. A lattice polytope $P$ in $M_{\mathbb{R}}$ is the convex hull of finitely many points in $M$. The dimension of $P$ is the dimension of the affine span of $P$. When $\dim P = \dim M_{\mathbb{R}}$, we say that $P$ is full dimensional.

Let $P \subseteq M_{\mathbb{R}}$ be a full dimensional lattice polytope, and let $P_1, ..., P_m$ be the facets of $P$, i.e. codimension 1 faces of $P$. For each facet $P_k$, there exists a unique primitive lattice point $v_k \in N$ and a unique integer $c_k \in \mathbb{Z}$ such that $P_k = \{ u \in P | \langle u, v_k \rangle = -c_k \}$ and $\langle u, v_k \rangle \geq -c_k$ for all $u \in P$.

Define $\Sigma_P$ to be the complete fan whose 1-dimensional cones are exactly those generated by $v_k$. This fan $\Sigma_P$ is called the (inner) normal fan of $P$. The toric variety $X_{\Sigma_P}$ associated to $\Sigma_P$ is called the toric variety of $P$, and denoted by $X_P$. Denote by $D_k$ the divisor corresponding to the 1-dimensional cone generated by $v_k$. Then we may define a divisor on $X_P$ by $D_P := \sum_{k=1}^m c_k D_k$. Such a divisor $D_P$ is necessarily ample.

This process is reversible, and there is a 1-to-1 correspondence between full dimensional lattice polytopes $P \subseteq M_{\mathbb{R}}$ and a pair $(X, D)$ of a complete toric variety $X$ together with an ample $T$-invariant divisor $D$ on $X$.

2.3. Toric vector bundles. A vector bundle $\pi : E \to X$ over a toric variety $X = X_\Sigma$ is said to be toric (or equivariant) if there is a $T$-action on $E$ such that $t \circ \pi = \pi \circ t$ for all $t \in T$.

Given a cone $\sigma \in \Sigma$ and $u \in M$, define the line bundle $L_u|_{U_{\sigma}}$ over $U_{\sigma}$ to be the trivial line bundle $U_{\sigma}$ equipped with the $T$-action given by $t \cdot (x, z) := (t \cdot x, \chi^u(t) \cdot z)$. If $u, u' \in M$ satisfy $u - u' \in \sigma^\perp$, then $\chi^{u-u'}$ is a non-vanishing regular function on $U_{\sigma}$ which gives an isomorphism $L_u|_{U_{\sigma}} \cong L_{u'}|_{U_{\sigma}}$. In fact, the group of toric line bundles on $U_{\sigma}$ is isomorphic to $M_\sigma := M/(M \cap \sigma^\perp)$. Therefore, we also write $L_{[u]}|_{U_{\sigma}}$, where $[u] \in M_\sigma$ is the class of $u$.

Let $E \to X$ be a toric vector bundle of rank $r$. Its restriction to an invariant open affine set $U_{\sigma}$ splits into a direct sum of toric line bundles with trivial underlying line bundles [Pay98, Proposition 2.2]; i.e. we have $E|_{U_{\sigma}} \cong \bigoplus_{i=1}^r L_{[u_i]}|_{U_{\sigma}}$ for some $[u_i] \in M_\sigma$. Define the associated characters of $E$ on $\sigma$ to be the multiset $u_E(\sigma) \subseteq M_\sigma$ of size $r$ that contains the $[u_i]$ showing up in the splitting.

Example 2.1 (Associated characters of tangent bundles). Let $X = X_\Sigma$ be an $n$-dimensional smooth projective toric variety, and consider its tangent bundle $T_X$. Fix a maximal cone $\sigma \in \Sigma$. Since $X$ is smooth, the dual cone $\check{\sigma}$ of $\sigma$ is generated by some $u_1, ..., u_n \in M$ that form a $\mathbb{Z}$-basis of $M$. Denote by $x_1, ..., x_n \in \Gamma(U_{\sigma}, O_X)$ the coordinates on $U_{\sigma} \cong \mathbb{R}^n$ corresponding to $u_1, ..., u_n$. Then $\{ \frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n} \}$ is a local frame of $T_X$ on $U_{\sigma}$.

Each non-vanishing section $\frac{\partial}{\partial x_i} \in \Gamma(U_{\sigma}, T_X)$ naturally generates a toric line bundle on $U_{\sigma}$ isomorphic to $L_{u_i}|_{U_{\sigma}}$. Thus we have $T_X|_{U_{\sigma}} \cong \bigoplus_{i=1}^n L_{u_i}|_{U_{\sigma}}$, and hence the associated characters of $T_X$ on $\sigma$ are $u_{T_X}(\sigma) = \{ u_1, ..., u_n \}$. 


2.4. Positivity of toric vector bundles. Let $X = X_\Sigma$ be a complete toric variety. By an invariant curve on $X$, we mean a complete irreducible 1-dimensional subvariety that is invariant under the $T$-action. Via the cone-orbit correspondence, there is a one-to-one correspondence between the invariant curves and the codimension-1 cones; every invariant curve is the closure of an 1-dimensional orbit, which corresponds to a codimension-1 cone in $\Sigma$. For each codimension-1 cone $\tau \in \Sigma$, denote the corresponding invariant curve by $C_\tau$.

The positivity of toric vector bundles can be checked on invariant curves according to the following result in [HMP10].

**Theorem 2.2.** [HMP10 Theorem 2.1] A toric vector bundle on a complete toric variety is ample (resp. nef) if and only if its restriction to every invariant curve is ample (resp. nef).

Note that every invariant curve is a $\mathbb{P}^1$. By Birkhoff-Grothendieck theorem, every vector bundle on $\mathbb{P}^1$ splits into a direct sum of line bundles. Hence, the positivity of vector bundles on $\mathbb{P}^1$ is well understood, namely $\bigoplus_{i=1}^r O_{\mathbb{P}^1}(a_i)$ is ample (resp. nef) if and only if every $a_i$ is positive (resp. non-negative). It is common to call the $r$-tuple (or multiset) $(a_i)_{i=1}^r$ the splitting type of the vector bundle.

Fix a codimension-1 cone $\tau$, and let $\sigma, \sigma'$ be the two maximal cones containing $\tau$. Given $u, u' \in M$ satisfying $u - u' \in \tau^\perp$, define a toric line bundle $L_{u,u'}$ on $U_\sigma \cup U_{\sigma'}$ by gluing the toric line bundles $L_u|_{U_\sigma}$ and $L_{u'}|_{U_{\sigma'}}$, with the transition function $\chi_{u'-u}$.

Since the invariant curve $C_\tau$ is contained in $U_\sigma \cup U_{\sigma'}$, we may restrict $L_{u,u'}$ to get a toric line bundle $L_{u,u'}|_{C_\tau}$ on $C_\tau$.

**Proposition 2.3.** [HMP10 Corollary 5.5 and 5.10] Let $X$ be a complete toric variety. Any toric vector bundle $E|_{C_\tau}$ on the invariant curve $C_\tau$ splits equivariantly as a sum of line bundles

$$E|_{C_\tau} = \bigoplus_{i=1}^r L_{u_i,u_i'}|_{C_\tau}. $$

The splitting is unique up to reordering.

Combining this with the following lemma that computes the underlying line bundle of $L_{u,u'}|_{C_\tau}$, one gets the splitting type of $E|_{C_\tau}$.

**Lemma 2.4.** [HMP10 Example 5.1] Let $u_0$ be the generator of $M \cap \tau^\perp \cong \mathbb{Z}$ that is positive on $\sigma$, and let $m$ be the integer such that $u - u' = mu_0$. Then, the underlying line bundle of $L_{u,u'}|_{C_\tau}$ is isomorphic to $O_{\mathbb{P}^1}(m)$.

3. Restricting $T_X$ to invariant curves

Let $X = X_\Sigma$ be a smooth complete toric variety of dimension $n$. In this section, we consider the restrictions of the tangent bundle $T_X$ to the invariant curves. The goal is to get the splitting types in terms of the combinatorial data of the fan $\Sigma$ of $X$.

Fix an $(n-1)$-dimensional cone $\tau \in \Sigma$. Let $\sigma, \sigma' \in \Sigma(n)$ be the two maximal cones containing $\tau$. Let $v_1, ..., v_{n-1}, v_n, v'_1 \in N$ be primitive vectors such that $\tau$ is generated by $\{v_1, ..., v_{n-1}\}$, $\sigma$ is generated by $\{v_1, ..., v_{n-1}, v_n\}$, and $\sigma'$ is generated by $\{v_1, ..., v_{n-1}, v'_n\}$. There are unique $u_i, u'_i \in M$ $(i = 1, ..., n)$ such that $\langle u_i, v_i \rangle = \langle u'_i, v'_i \rangle = 1$ for all $i$ and $\langle u_i, v_j \rangle = \langle u'_i, v'_j \rangle = 0$ for all $i \neq j$, where we define $v'_i = v_i$ for $i = 1, ..., n-1$. The dual cones $\check{\sigma}$ and $\check{\sigma}'$ are generated by $\{u_1, ..., u_n\}$ and $\{u'_1, ..., u'_n\}$, respectively.

By Example 2.1, the associated characters of $T_X$ on $\sigma$ and $\sigma'$ are given by

$$u_{T_X}(\sigma) = \{u_1, ..., u_n\}, \quad u_{T_X}(\check{\sigma}') = \{u'_1, ..., u'_n\}. $$

Following Section 2.4, let \( C_\tau \) be the invariant curve corresponding to \( \tau \). The splitting of \( T_X|_{C_\tau} \) as in Proposition 2.3 is easy to get by the following fact.

**Lemma 3.1.** We have \( u_i - u'_i \in \tau^\perp \) for all \( i = 1, \ldots, n \), and \( u_i - u'_j \notin \tau^\perp \) for all \( i \neq j \).

**Proof.** The first part follows from \( \langle u_i - u'_i, v_j \rangle = 0 \) for all \( i' = 1, \ldots, n - 1 \), and the second part follows from \( \langle u_i - u'_j, v_i \rangle = -(u_i - u'_j, v_j) = 1 \), where at least one of \( i, j \) is not \( n \). \( \square \)

**Definition 3.2.** Define \( a_i \in \mathbb{Z} (i = 1, \ldots, n) \) to be the integers satisfying \( u_i = u'_i + a_i u_n \). Such integers exist since \( u_n \) is a primitive generator of \( \tau^\perp \cap M \cong \mathbb{Z} \). Note that \( u'_n = -u_n \) so that \( a_n = 2 \).

**Proposition 3.3.** On the invariant curve \( C_\tau \), the restriction \( T_X|_{C_\tau} \) of the tangent bundle (as a toric vector bundle) splits into the following direct sum of toric line bundles

\[
T_X|_{C_\tau} \cong \bigoplus_{i=1}^{n} \mathcal{L}_{u_i, u'_i}|_{C_\tau}.
\]

In particular, we have the following splitting of \( T_X|_{C_\tau} \) as a vector bundle

\[
T_X|_{C_\tau} \cong \bigoplus_{i=1}^{n} \mathcal{O}_{\mathbb{P}^1}(a_i).
\]

**Proof.** By Proposition 2.3, \( T_X|_{C_\tau} \) splits into a direct sum of toric line bundles of the form \( \mathcal{L}_{u,w}|_{C_\tau} \). This gives a bijection \( \iota : \mathbb{C}(\sigma) \to \mathbb{C}(\sigma') \) mapping \( u \) to \( u' \) whenever \( \mathcal{L}_{u,w}|_{C_\tau} \) shows up in the splitting. Note that \( u_i - \iota(u_i) \in \tau^\perp \) by the definition of \( \mathcal{L}_{u,w} \). Then Lemma 3.1 implies that we must have \( \iota(u_i) = u'_i \) for all \( i \), hence the splitting in the first part. The second part follows directly from the first part together with Lemma 2.4. \( \square \)

**Remark 3.4.** The integers \( a_i \) are the same as the integers \( b_i \) that show up in the “wall relation”

\[
b_1 v_1 + \cdots + b_{n-1} v_{n-1} + v_n + v'_n = 0,
\]

mentioned in [Sch18] and [DRJS18]. Indeed we have \( b_i = -\langle u_i, v'_n \rangle = a_i \) for all \( i \).

**Example 3.5.** For each of the following toric surfaces \( X \), we fix a 1-dimensional cone \( \tau \) in its fan (as shown in Figure 3.6) and compute the splitting type of \( T_X|_{C_\tau} \).

1. \( X = \mathbb{P}^2 \). The dual cones of the maximal cones containing \( \tau \) are given by \( \sigma = \text{Cone}\{(-1,0),(-1,1)\} \) and \( \sigma = \text{Cone}\{(0,-1),(1,-1)\} \). Therefore we get \( T_X|_{C_\tau} \cong \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(2) \). In fact, the restrictions of \( T_X \) to the other two invariant curves have the same splitting type, so \( T_X \) is ample by Proposition 2.2.

2. \( X = \mathbb{P}^1 \times \mathbb{P}^1 \). The dual cones of the maximal cones containing \( \tau \) are given by \( \sigma = \text{Cone}\{(-1,0),(0,1)\} \) and \( \sigma = \text{Cone}\{(-1,0),(0,-1)\} \). Therefore we get \( T_X|_{C_\tau} \cong \mathcal{O}_{\mathbb{P}^1}(0) \oplus \mathcal{O}_{\mathbb{P}^1}(2) \). In fact, the restrictions of \( T_X \) to the other three invariant curves have the same splitting type, so \( T_X \) is nef (but not ample) by Proposition 2.2.

3. Let \( X \) be the Hirzebruch surface \( \mathbb{F}_1 \), which is isomorphic to \( \mathbb{P}^2 \) blown up at one point. The dual cones of the maximal cones containing \( \tau \) are given by \( \sigma = \text{Cone}\{(-1,0),(0,1)\} \) and \( \sigma = \text{Cone}\{(-1,1),(0,-1)\} \). Therefore we get \( T_X|_{C_\tau} \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(2) \), and hence \( T_X \) is not nef by Proposition 2.2.
4. POLYTOPES AND AMPLENESS OF THE TANGENT BUNDLE

Let $X = X_{\Sigma}$, $T_X$, $\tau$, $\sigma$, $\sigma'$, $u_i$, $u'_i$, $a_i$ be as in the previous section.

Fix an ample $T$-invariant divisor $D$, and let $P = P(X, D)$ be the corresponding polytope. Note that $X$ and $\Sigma$ are simplicial as they are smooth; in particular, every maximal cone in $\Sigma$ has exactly $n$ faces of dimension $(n-1)$, and every $(n-1)$-dimensional cone has exactly $(n-1)$ faces of dimension $(n-2)$. This implies that there are exactly $n$ edges emanating from every vertex of $P$ and that every edge of $P$ is contained in exactly $(n-1)$ faces of dimension 2.

Let $p_\sigma \in P$ be the vertex corresponding to the maximal cone $\sigma$. Denote by $P - p_\sigma$ the translation of $P$ by $-p_\sigma$. The cone generated by $P - p_\sigma$ is given by $\{u \in M_\mathbb{R} | \langle u, u_j \rangle \geq 0 \text{ for all } j = 1, ..., n\}$, which is exactly the dual cone $\sigma^\vee$ of $\sigma$. Thus, the $n$ edges of $P$ emanating from $p_\sigma$ are parallel to $u_1, ..., u_n$. Similarly, the $n$ edges emanating from the vertex $p_{\sigma'}$ corresponding to $\sigma'$ are parallel to $u'_1, ..., u'_n$.

Recall that the $u_i$ and $u'_i$ satisfy $u'_i = u_i - a_i u_n$ for all $i = 1, ..., n-1$ and $u'_n = -u_n$. Since $\sigma$ and $\sigma'$ contain the $(n-1)$-dimensional cone $\tau$ as a common face, the convex hull of $p_\sigma, p_{\sigma'}$ of $p_\sigma$ and $p_{\sigma'}$ is an edge of $P$; it corresponds to $\tau$ and is parallel to $u_n$ and $u'_n$. Fix a $j \in \{1, ..., n-1\}$. Consider the points $p_\sigma + u_j, p_{\sigma'} + u'_j \in M$. The point $p_\sigma + u_j$ is on an edge emanating from $p_\sigma$, and $p_{\sigma'} + u'_j$ is on an edge emanating from $p_{\sigma'}$. In addition, $p_\sigma + u_j - (p_{\sigma'} + u'_j) = (p_\sigma - p_{\sigma'}) + m_j u_n$, $p_\sigma + u_j, p_{\sigma'} + u'_j$ is parallel to $p_\sigma, p_{\sigma'}$. Thus, the four points $p_\sigma, p_{\sigma'}, p_\sigma + u_j, p_{\sigma'} + u'_j$ are contained in a common 2-dimensional face $A_i \subseteq P$. In fact, $A_i$ is the 2-dimensional face of $P$ corresponding to the $(n-2)$-dimensional cone $\tau \cap (u_i)^\perp = \tau \cap (u'_i)^\perp$.

Denote the angles at $p_\sigma$ and $p_{\sigma'}$ on $A_j$ by $\theta(p_\sigma, A_j)$ and $\theta(p_{\sigma'}, A_j)$, respectively. Their sum is related to the integer $a_j$ in the following way.

**Proposition 4.1.** The sum $\theta(p_\sigma, A_j) + \theta(p_{\sigma'}, A_j)$ is smaller than $\pi$ if and only if $a_j > 0$, equal to $\pi$ if and only if $a_j = 0$, and greater than $\pi$ if and only if $a_j < 0$.

**Proof.** Suppose $a_j < 0$. Consider the quadrilateral with vertices $p_\sigma, p_{\sigma'}, p_{\sigma'} + u'_j, p_\sigma + u_j$. It is a trapezoid with the edges $p_\sigma + u_j, p_{\sigma'} + u'_j$ and $p_\sigma, p_{\sigma'}$ parallel to each other. Since

$$((p_{\sigma'} + u'_j) - (p_{\sigma'} + u_j)) - (p_{\sigma'} - p_\sigma) = -a_j u_1,$$

the edge $p_{\sigma'} + u'_j$ is longer than $p_\sigma, p_{\sigma'}$, implying $\theta(p_\sigma, A_j) + \theta(p_{\sigma'}, A_j) > \pi$.

The other two cases are similar.

**Remark 4.2.** Although the angles $\theta(p_\sigma, A_j), \theta(p_{\sigma'}, A_j)$ themselves are not invariant under a change of bases of $M$, whether their sum is smaller than, equal to, or greater than $\pi$ is.

\[\begin{align*}
(1) \ X = \mathbb{P}^2 \\
(2) \ X = \mathbb{P}^1 \times \mathbb{P}^1 \\
(3) \ X = \mathbb{F}_1
\end{align*}\]

\textbf{Figure 3.6. Fans of toric surfaces}
Example 4.3. In Figure 4.4 are polytopes $P(X, -K_X)$ corresponding to the toric surfaces $X$ in Example 3.5 together with their anticanonical line bundles $-K_X$. The cones $\tau, \sigma, \sigma'$ are the same as in Example 3.5.

1. $X = \mathbb{P}^2$. Recall $T_X|_{C_\tau} \cong \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(2)$ so that $a_1 = 1 > 0$. Here we see $\theta(p_\sigma, P) + \theta(p_{\sigma'}, P) < \pi$.
2. $X = \mathbb{P}^1 \times \mathbb{P}^1$. Recall $T_X|_{C_\tau} \cong \mathcal{O}_{\mathbb{P}^1}(0) \oplus \mathcal{O}_{\mathbb{P}^1}(2)$ so that $a_1 = 0$. Here we see $\theta(p_\sigma, P) + \theta(p_{\sigma'}, P) = \pi$.
3. $X = \mathbb{F}_1$. Recall $T_X|_{C_\tau} \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(2)$ so that $a_1 = -1 < 0$. Here we see $\theta(p_\sigma, P) + \theta(p_{\sigma'}, P) > \pi$.

![Polytopes $P(X, -K_X)$ of toric surfaces](image)

Proof of Theorem 1.1. As promised, we will show that the polytope $P$ corresponding to $X$ (together with any ample $T$-invariant divisor $D$) is an $n$-simplex.

Let $A$ be a 2-dimensional face of $P$. Let $m$ be the number of vertices of $A$, and let $p_1, \ldots, p_m$ be the vertices of $A$, ordered so that $p_k$ is adjacent to $p_{k+1}$ for all $k = 1, \ldots, m$ (where $p_{m+1} := p_1$). Since $T_X$ is ample, its restriction to every invariant curve is ample. Then, by Proposition 3.3 and Proposition 4.1, $\theta(p_k, A) + \theta(p_{k+1}, A) < \pi$ for all $k$. This implies

$$m\pi > \sum_{k=1}^{m} (\theta(p_k, A) + \theta(p_{k+1}, A)) = 2 \sum_{k=1}^{m} \theta(p_k, A) = 2(m-2)\pi.$$  

We get $m < 4$, implying $A$ is a triangle. The same is true for all 2-dimensional faces of $P$.

Now, we start with a vertex $q_0$ of $P$. Recall that every vertex of $P$ is adjacent to exactly $n$ vertices since $X$ is smooth and hence simplicial. Let $q_1, \ldots, q_n$ be the $n$ points adjacent to $q_0$. Given $1 < j \leq n$, let $A_j$ be the 2-dimensional face containing the edges $q_0q_j$ and $q_0q_j$. Since $A_j$ is in fact a triangle, $q_1$ is also adjacent to $q_j$. Thus $q_1$ is adjacent to $q_0, q_2, \ldots, q_n$. Similarly, every $p_j$ is adjacent to exactly $p_0, \ldots, \hat{p}_j, \ldots, p_n$. Consequently, $p_0, p_1, \ldots, p_n$ are the only vertices of $P$, and hence $P$ is the $n$-simplex with vertices $p_0, p_1, \ldots, p_n$. □

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