Pizza Sharing is PPA-hard

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Abstract

We study the computational complexity of finding a solution for the straight-cut and square-cut pizza sharing problems. We show that computing an $\varepsilon$-approximate solution is PPA-complete for both problems, while finding an exact solution for the square-cut problem is FIXP-hard and in BU. Our PPA-hardness results apply for any $\varepsilon < 1/5$, even when all mass distributions consist of non-overlapping axis-aligned rectangles or when they are point sets, and our FIXP-hardness result applies even when all mass distributions are unions of squares and right-angled triangles. We also prove that the decision variants of both approximate problems are NP-complete, while it is ETR-complete for the exact version of square-cut pizza sharing.

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1 Introduction

*Mass partition problems* ask to fairly divide measurable objects that are embedded into Euclidean space [RS20]. Perhaps the most popular mass partition problem is the *ham sandwich* problem, in which three masses are given in three-dimensional Euclidean space, and the goal is to find a single plane that cuts all three masses in half. Recently, there has been interest in *pizza sharing* problems, which are mass partition problems in the two-dimensional plane, and in this paper we study the computational complexity of such problems.

In the *straight-cut* pizza sharing problem, we are given $2n$ two-dimensional masses in the plane, and we are asked to find $n$ straight lines that simultaneously bisect all of the masses. See Figure 1a for an example. It has been shown that this problem always has a solution: the first result on the topic showed that solutions always exist when $n = 2$ [BPS19], and this was subsequently extended to show existence for all $n$ [HK20].

![Figure 1: Partitions of the plane to $R^+$ and $R^-$ (shaded and non-shaded areas respectively).](image)

Another related problem is the *square-cut* pizza sharing. In this problem, there are $n$ masses in the plane, and the task is to simultaneously bisect all masses using cuts, but the method of generating the cuts is different. Specifically, we seek a *square-cut*, which consists of a single path that is the union of horizontal and vertical line segments. See Figure 1b and Figure 1c for two examples of square-cuts. Intuitively, we can imagine that a pizza cutter is placed on the plane, and is then moved horizontally and vertically without being lifted in order to produce the cut. Note that the path is allowed to wrap around on the horizontal axis: if it exits the left or right boundary, then it re-appears on the opposite boundary. So the cut in Figure 1c is still considered to be a single square-cut.

It has been shown by [KRPS16] that, given $n$ masses, there always exists a *square-cut-path* (termed SC-path) which makes at most $n - 1$ turns and simultaneously bisects all of the masses. This holds even if the SC-path is required to be $y$-monotone, meaning that the path never moves downwards.

Two-dimensional fair division is usually called *land division* in the literature. Land division is a prominent topic of interest in the Economics and AI communities that studies ways of fairly allocating two-dimensional objects among $n$ agents [Cha05, SHNHA17, SNHA20, ESS21, AD15, IH09, Hüs11]. The first popular appearance of such problems in a mathematical description was done by [Ste48], and since then, the existence of allocations under various fairness criteria have been extensively studied, together with algorithms that achieve them. These problems find applications from division of resources on land itself, to the Law of the Sea [SS03], to redistricting [LRY09, LS14].

Consensus halving is a problem that asks us to split a one-dimensional resource into two parts such that $n$ agents have equal value in both parts. Here, we study the same fairness criterion for $n$ agents, but for a two-dimensional resource. One can see that when we have the same fairness criterion at hand for any $k$-dimensional resource, $k \geq 2$, we can always translate the problem into
its one-dimensional version, by integrating each agent’s measure to a single dimension. Then a solution can be given by applying consensus halving. However, the solutions we get by doing so, are not taking into account the dimensionality of the problem, and as a result they might produce very unnatural solutions to a high-dimensional problem. For example, in land division, applying consensus halving would produce two parts, each of which can possibly be a union of $\lceil n/2 \rceil$ disjoint land strips. Can we get better solutions by exploiting all the dimensions of the problem?

In this work we investigate different cutting methods of the two-dimensional objects, and in particular, two pizza sharing methods for which a solution is guaranteed. While based on intuition one might assume that exploiting the two dimensions would allow the complexity of finding a solution to be lower, our results show that this is not the case. We present polynomial time reductions from the one-dimensional problem to the two-dimensional problems showing that the latter are at least as hard as the former, i.e., PPA-hard. Apart from the hardness results themselves, we believe that our reductions are interesting from another aspect too. They show ways to efficiently turn a problem into one of higher dimension, a task that has no standardised methods to be achieved (even for non-efficient reductions), and whose inverse is trivially achievable.

**Computational complexity of fair division problems.** There has been much interest recently in the computational complexity of fair division problems. In particular, the complexity class PPA has risen to prominence, because it appears to naturally capture the complexity of solving these problems. For example, it has recently been shown by [FRG18, FRG19] that the consensus halving problem, the ham sandwich problem, and the well-known necklace splitting problem are all PPA-complete.

More generally, PPA captures all problems whose solution is verifiable in polynomial time and is guaranteed by the Borsuk-Ulam theorem. Finding an approximate solution to a Borsuk-Ulam function, or finding an exact solution to a linear Borsuk-Ulam function are both known to be PPA-complete problems [Pap94, DFMS21]. The existence of solutions to the ham sandwich problem, the necklace splitting problem, and indeed the square-cut pizza sharing problem can all be proved via the Borsuk-Ulam theorem\(^1\).

**Theorem 1** (Borsuk-Ulam). Let $f : S^d \to \mathbb{R}^d$ be a continuous function, where $S^d$ is a $d$-dimensional sphere. Then, there exists an $x \in S^d$ such that $f(x) = f(-x)$.

The other class of relevance here is the class BU, which consists of all problems that can be polynomial-time reduced to finding an exact solution to a Borsuk-Ulam function. This class was defined by [DFMS21] and is believed to be substantially harder than the class PPA, because it is possible to construct a Borsuk-Ulam function that only has irrational solutions. Due to this, it is not currently expected that BU will be contained in FNP, whereas the containment of PPA in FNP is immediate.

Unfortunately, it is not currently known whether BU has complete problems. However, in [DFMS21] it was shown that exact consensus halving is in BU and also FIXP-hard, implying that FIXP $\subseteq$ BU. FIXP, defined by Etessami and Yannakakis [EY10], is the class of problems that can be reduced to finding an exact fixed point of a Brouwer function. It is known by the aforementioned work, that FIXP contains the problem Square Root Sum, which has as input positive integers $a_1, \ldots, a_n$ and $k$, and asks whether $\sum_{i=1}^{n} \sqrt{a_i} \leq k$. The question of whether Square Root Sum is in NP has been open for more than 40 years ([GGJ76, Pap77, Tiw92]).

\(^1\)It has also been shown by [CS17] that the Borsuk-Ulam theorem is equivalent to the ham sandwich theorem which states that the volumes of any $n$ compact sets in $\mathbb{R}^n$ can always be simultaneously bisected by an $(n - 1)$-dimensional hyperplane.
Furthermore, since there exist Brouwer functions that only have irrational fixed points, it is likewise not expected that $\text{FIXP}$ will be contained in $\text{FNP}$.

**Our contribution.** We study the computational complexity of the straight-cut and square-cut pizza sharing problems, and we specifically study the cases where (i) all mass distributions are unions of weighted polygons, and (ii) we are given unweighted point sets. We show that it is $\text{PPA}$-complete to find approximate solutions of the approximate versions of all the problems, while their decision variants are $\text{NP}$-complete. We also show that the exact square-cut pizza sharing problem with mass distributions is $\text{FIXP}$-hard and in $\text{BU}$, while its decision variant is $\text{ETR}$-complete. All of our hardness results are summarized in Table 1 and Table 2.

These results represent, to the best of our knowledge, the first $\text{PPA}$-hardness results for problems arising from computational geometry. We also note that pizza sharing problems do not need a circuit as part of the input, which makes them in some sense more “natural” than problems that are specified by circuits. Other known “natural” $\text{PPA}$-hard problems are one-dimensional, such as consensus halving [FHSZ20] and necklace splitting [FRG19]. Here we show the first known $\text{PPA}$-hardness result for a “natural” two-dimensional problem. Let us also mention here that shortly after the appearance of our result, Schneider in [Sch21] proved that the discrete version of straight-cut pizza sharing where each mass is represented by unweighted points is $\text{PPA}$-complete, while its continuous version for a more general input representation is $\text{FIPX}$-hard.

Recently, [BHH21] made great improvement towards showing $\text{BU}$-hardness of exact consensus halving. They introduced a class named $\text{BBU}$ whose typical problem is an equivalent, alternative definition of the Borsuk-Ulam theorem, and they considered the strong approximation version of the classes $\text{BU}$ and $\text{BBU}$, named $\text{BU}_a$ and $\text{BBU}_a$, respectively. Some of the most notable results of the aforementioned paper is that $\text{BU}_a = \text{BBU}_a$ and that the strong approximation version of consensus halving is complete for $\text{BU}_a$. We believe that some of our reductions will be able to be translated into the framework of strong approximation and yield analogous $\text{BBU}_a$-hardness results for the strong approximation version of pizza sharing problems. However, we remark that $\text{BU}$-completeness of either exact consensus halving or any of the pizza sharing problems is yet to be proven.

For both the straight-cut and the square-cut pizza sharing problems, we show that it is $\text{PPA}$-complete to find an $\varepsilon$-approximate solution for any constant $\varepsilon \in (0, 1/5)$. This holds even when $n + n^{1-\delta}$ lines are permitted in a straight-cut pizza sharing instance with $2n$ mass distributions, and when $n - 1 + n^{1-\delta}$ turns of the square-cut path are permitted in a square-cut pizza sharing instance with $n$ mass distributions, for constant $\delta > 0$. Furthermore, the $\text{PPA}$-hardness holds even when each mass distribution is uniform over polynomially many axis-aligned rectangles, and there is no overlap between any two mass distributions. The inapproximability for such high values of $\varepsilon$ is possible due to a recent advancement in the inapproximability of consensus halving [DFHM22]. The $\text{PPA}$-inclusion holds even for inverse polynomial and inverse exponential $\varepsilon$, and for weighted polygons with holes (that is, the most general type of allowed input). Furthermore, we show that there exists an $\varepsilon > 0$ such that deciding whether an $\varepsilon$-approximate solution of straight-cut pizza sharing with at most $n - 1$ lines (resp. an $\varepsilon$-approximate solution of square-cut pizza sharing with at most $n - 2$ turns) exists or, is $\text{NP}$-complete. All of these results hold also for the discrete version of the problems.

We then turn our attention to the computational complexity of finding an *exact* solution to the square-cut problem. We show that the problem of finding an $\text{SC}$-path with at most $n - 1$ turns that exactly bisects $n$ masses lies in $\text{BU}$, and is $\text{FIPX}$-hard. This hardness result applies even if all mass distributions are unions of weighted axis-aligned squares and right-angled triangles. In order to prove containment in $\text{BU}$, we provide a simpler existence proof for a solution to
the square-cut pizza sharing problem that follows the lines of the original proof by [KRPS16], while to prove the FIXP-hardness, we reduce from the problem of finding an exact Consensus-Halving solution [DFMS21]. Regarding the decision version of the square-cut problem, we show that deciding whether there exists an exact solution with at most \( n - 2 \) turns is ETR-complete, where ETR consists of every decision problem that can be formulated in the existential theory of the reals (see Section 2 for its definition).

From a technical viewpoint, our PPA containment result for straight-cut pizza sharing is based on a reduction that transforms mass distributions to point sets in general position and then employs a recent result by [Sch21]. On the contrary, our containment results for square-cut pizza sharing are shown by directly reducing it to the BORSUK-ULAM problem. Our hardness results are obtained by reducing from the consensus halving problem, historically the first fair-division problem shown to be PPA-complete [FRG18]. It is worth mentioning that, if in the future, consensus halving with linear valuation densities (see definitions of \( k \)-block-triangle valuations in Section 2) is shown to be BU-complete, then our work implies BU-completeness of exact square-cut pizza sharing.

### Table 1: A summary of our hardness results for \( \varepsilon \)- STRAIGHT-PIZZA-SHARING.

| Hardness | \( \varepsilon \) | Lines | Pieces | Overlap | Theorem |
|----------|----------------|-------|--------|---------|---------|
| Point sets | | | | | |
| PPA      | \( 0.2 \) | \( n + n^{1-\delta} \) | - | - | 13 |
| NP       | \( c \)    | \( n - 1 \) | - | - | 15 |
| Mass distributions | | | | | |
| PPA      | \( 0.2 \) | \( n + n^{1-\delta} \) | \( \text{poly}(n) \) | 1 | 5 |
| NP       | \( c \)    | \( n - 1 \) | \( \text{poly}(n) \) | 1 | 6 |

Table 1: A summary of our hardness results for \( \varepsilon \)- STRAIGHT-PIZZA-SHARING. Here, \( c \) and \( \delta \) are absolute, positive constants. “Lines” refers to the number of cut-lines allowed in a solution. “Pieces” refers to the maximum number of distinct polygons that define every mass distribution. “Overlap” refers to the maximum number of different mass distributions allowed to contain any point of \([0, 1]^2\).

### Table 2: A summary of our hardness results for \( \varepsilon \)-SC-PIZZA-SHARING.

| Hardness | \( \varepsilon \) | Turns | Pieces | Overlap | Theorem |
|----------|----------------|-------|--------|---------|---------|
| Point sets | | | | | |
| PPA      | \( 0.2 \) | \( n - 1 + n^{1-\delta} \) | - | - | 14 |
| NP       | \( c \)    | \( n - 2 \) | - | - | 16 |
| Mass distributions | | | | | |
| PPA      | \( 0.2 \) | \( n - 1 + n^{1-\delta} \) | \( \text{poly}(n) \) | 1 | 8 |
| NP       | \( c \)    | \( n - 2 \) | \( \text{poly}(n) \) | 1 | 9 |
| FIXP     | 0           | \( n - 1 \) | 6 | 3 | 18 |
| ETR      | 0           | \( n - 2 \) | 6 | 3 | 19 |

Table 2: A summary of our hardness results for \( \varepsilon \)-SC-PIZZA-SHARING. Here, “turns” refers to the number of turns a solution (SC-path) is allowed to have. The definitions of \( c \), \( \delta \) and the semantics of “pieces”, and “overlap” are the same as those of Table 1.

### Further related work.

Since mass partitions lie in the intersection of topology, discrete geometry, and computer science there are several surveys on the topic; [BFHZ18, DLGMM19, Mat08, Živ17] focus on the topological point of view, while [AE+99, Ede12, KK03, Mat02] focus on computational aspects. Consensus halving [SS03] is the mass partition problem that received the majority of attention in Economics and Computation so far [DFH21, FRFGZ18,
Recently, Haviv [Hav22] showed PPA-completeness of finding fair independent sets on cycle graphs, having as a starting point the latter problem.

2 Preliminaries

**Mass distributions.** A mass distribution \( \mu \) on \([0, 1]^2\) is a measure on the plane such that all open subsets of \([0, 1]^2\) are measurable, \(0 < \mu(\{0, 1\}^2) < \infty\), and \(\mu(S) = 0\) for every subset of \([0, 1]^2\) with dimension lower than 2. A mass distribution \( \mu \) is finite-separable, or simply separable, if it can be decomposed into a finite set of non-overlapping areas \(a_1, a_2, \ldots, a_d\) such that \(\mu([0, 1]^2) = \sum_{j=1}^{d} \mu(a_j)\). In addition, a separable mass distribution \( \mu \) is piece-wise uniform, if for every \(j\) and every \(S \subseteq a_j\) it holds that \(\mu(a_j \cap S) = w_j \cdot \text{area}(a_j \cap S)\) for some weight \(w_j > 0\) independent of \(S\). When additionally \(w_j = w_k\) for all \(j, k \in [d]\) then the mass distribution is called uniform.

Finally, a mass distribution is normalised if \(\mu([0, 1]^2) = 1\). The support of mass distribution \(\mu\), denoted by \(\text{supp}(i)\), is the area \(A_i \subseteq [0, 1]^2\) which has the property that for every \(S \subseteq A_i\) with non-zero Lebesgue measure we have \(\mu_i(S) > 0\). Let \(N := \{I \subseteq [n] : \bigcap_{i \in I} \text{supp}(i) \neq \emptyset\}\). A set of mass distributions \(\mu_1, \ldots, \mu_n\), or colours, has overlap \(k\) if \(\max_{I \in N} |I| = k\). In order to easily present our results that concern additive approximations, throughout this work we consider all mass distributions to be normalised, which is without loss of generality.

**Set of straight-cuts.** A set of straight-cuts, or cut-lines, or simply lines defines subdivisions of the plane \(R\). Figure 1a shows an example of a set of straight-cuts. Each line creates two half-spaces, and arbitrarily assigns number “0” to one and “1” to the other. A subdivision of \(R\) is labeled “+” (and belongs to \(R^+\)) if its parity is odd (according to the labels given to the half-spaces) and “−” (and belongs to \(R^-\)) otherwise. Observe that by flipping the numbers of two half-spaces defined by a line, we flip all the subdivisions’ labels. Thus, there are only two possible labelings of the subdivisions.

**Square-cut-path.** A square-cut-path, denoted for brevity SC-path, is a non-crossing directed path that is formed only by horizontal and vertical line segments and in addition it is allowed to “wrap around” in the horizontal dimension. Figure 1b, Figure 1c show two examples of SC-paths. A turn of the path is where a horizontal segment meets with a vertical segment. An SC-path is \(y\)-monotone if all of its horizontal segments are monotone with respect to the \(y\) axis. Any SC-path partitions the plane into two regions, that we call \(R^+\) and \(R^-\).

**Pizza sharing.** A set of lines (resp. a SC-path) \(\varepsilon\)-bisects a mass distribution \(\mu\), if \(|\mu(R^+) - \mu(R^-)| \leq \varepsilon\). It simultaneously \(\varepsilon\)-bisects a set of mass distributions \(M\) if \(|\mu_i(R^+) - \mu_i(R^-)| \leq \varepsilon\) for every \(\mu_i \in M\).

**Definition 2.** For any \(n \geq 1\), the problem \(\varepsilon\)-Straight-Pizza-Sharing is defined as follows:

- **Input:** \(\varepsilon \geq 0\) and mass distributions \(\mu_1, \mu_2, \ldots, \mu_{2n}\) on \([0, 1]^2\).
- **Output:** A partition of \([0, 1]^2\) to \(R^+\) and \(R^-\) using at most \(n\) lines such that for each \(i \in [2n]\), it holds that \(|\mu_i(R^+) - \mu_i(R^-)| \leq \varepsilon\).
**Definition 3.** For any $n \geq 1$, the problem $\varepsilon$-SC-PIZZA-SHARING is defined as follows:

- **Input:** $\varepsilon \geq 0$ and mass distributions $\mu_1, \mu_2, \ldots, \mu_n$ on $[0, 1]^2$.
- **Output:** A partition of $[0, 1]^2$ to $R^+$ and $R^-$ using a SC-path with at most $n - 1$ turns such that for each $i \in [n]$, it holds that $|\mu_i(R^+) - \mu_i(R^-)| \leq \varepsilon$.

In [KRPS16], it was proved that $\varepsilon$-SC-PIZZA-SHARING always admits a solution for arbitrary continuous measures with respect to the Lebesgue measure, and for any $\varepsilon \geq 0$. In this work, we are interested in the computational aspect of the problem, hence we need to specify its input representation. For simplicity, we deal with measures determined by sets of polygons with holes, since even these simply-defined measures suffice to yield PPA-hardness. The precise input representation is described in Appendix A.

Although it is not hard to construct instances of $\varepsilon$-SC-PIZZA-SHARING where $n - 1$ turns are necessary for any SC-path in order to constitute a solution, there might be cases where a solution can be achieved with an SC-path with fewer turns. Hence, we also study the decision version of the problem, in which we ask whether we can find a solution with $k$ turns, where $k < n - 1$.

**Consensus halving.** The hardness results that we will show in this paper will be shown by a reduction from the consensus halving problem.

In the $\varepsilon$-CONSENSUS-HALVING problem, there is a set of $n$ agents with valuation functions $v_i$ over the interval $[0, 1]$, and the goal is to find a partition of the interval into subintervals labelled either “+” or “−”, using at most $n$ cuts. This partition should satisfy that for every agent $i$, the total value for the union of subintervals $\mathcal{I}^+$ labelled “+” and the total value for the union of subintervals $\mathcal{I}^−$ labelled “−” is the same up to $\varepsilon$, i.e., $|v_i(\mathcal{I}^+) - v_i(\mathcal{I}^-)| \leq \varepsilon$. We will consider the following types for a valuation function $v_i$; see Figure 2 for a visualization.

- **$k$-block.** $v_i$ can be decomposed into at most $k$ non-overlapping (but possibly adjacent) intervals $[a_{i1}^l, a_{i1}^r], \ldots, [a_{ik}^l, a_{ik}^r]$ where interval $[a_{ij}^l, a_{ij}^r]$ has density $c_{ij}$ and 0 otherwise. So, $v_i([a_{ij}^l, x]) = (x - a_{ij}^l) \cdot c_{ij}$ for every $x \in [a_{ij}^l, a_{ij}^r]$ and $v_i([0, 1]) = \sum_j v_i([a_{ij}^l, a_{ij}^r]) = 1$.

- **$k$-block uniform.** $v_i$ is $k$-block and the density of every interval is $c_i$.

- **$k$-block-triangle.** $v_i$ is the union of a $k$-block valuation function and an extra interval $[a_{i1}^l, a_{i1}^r]$, where $v_i([a_{i1}^l, x]) = (x - a_{i1}^l)^2$ for every $x \in [a_{i1}^l, a_{i1}^r]$ and $(a_{i1}^l, a_{i1}^r) \cap [a_{ij}^l, a_{ij}^r] = \emptyset$ for every $j \in [k]$.

![Diagram](image)

(a) 2-block valuation  
(b) 2-block uniform valuation  
(c) 1-block-triangle valuation

**Figure 2**

**Complexity classes.** $\varepsilon$-SC-PIZZA-SHARING is an example of a total problem, which is a problem that always has a solution. The complexity class TFNP (Total Function NP) defined in [MP91], contains all total problems whose solutions can be verified in polynomial time.
There are several well-known subclasses of TFNP that we will use in this paper. The class PPA, defined by Papadimitriou [Pap94], captures problems whose totality is guaranteed by the parity argument on undirected graphs. The complexity class PPAD $\subseteq$ PPA is the subclass of PPA containing all problems whose totality is guaranteed by the parity argument on directed graphs.

The complexity class ETR consists of all decision problems that can be formulated in the existential theory of the reals [Mat14, Sch09]. It is known that NP $\subseteq$ ETR $\subseteq$ FNP [Can88], and it is generally believed that ETR is distinct from the other two classes. The class FETR (Function ETR) consists of all search problems whose decision version is in ETR. The class TFETR is the subclass of FETR which contains only problems that admit a solution (i.e. all the instances of their decision version are “yes” instances). Both FETR and TFETR were introduced in [DFMS21] as the natural analogues of FNP and TFNP in the real RAM model of computation. For a definition of the real RAM model we refer the reader to the detailed work of Erickson, van der Hoog, and Miltzow [EvM20].

In this paper, our focus regarding computational complexity will be on the following two classes contained in TFETR. As mentioned earlier, the class BU $\subseteq$ TFETR was introduced in [DFMS21] and captures problems whose totality is guaranteed by the Borsuk-Ulam theorem. The class FIXP $\subseteq$ BU was defined in [EY10] and captures problems whose totality is guaranteed by Brouwer’s fixed point theorem.

3 Hardness results

Here we show all hardness results regarding the exact and approximate versions of our pizza sharing problems for mass distributions, as well as for point sets.

3.1 Hardness of approximate STRAIGHT-PIZZA-SHARING

We start by proving that $\epsilon$-STRAIGHT-PIZZA-SHARING is PPA-hard for any $\epsilon < 1/5$, even for very simple mass distributions. We prove our result via a reduction from $\epsilon$-CONSENSUS-HALVING with $k$-block valuations, which for the special case of 3-block uniform valuations has been shown to be PPA-complete [DFHM22]. In addition, we explain how to combine the machinery of our reduction with that of [FRFGZ18] in order to get NP-hardness for $\epsilon$-STRAIGHT-PIZZA-SHARING, where $\epsilon > 0$ is a small constant.

The reduction. We reduce from CONSENSUS-HALVING with $2n$ agents, and for each agent we create a corresponding mass in STRAIGHT-PIZZA-SHARING. Firstly, we finely discretize the $[0, 1]$ interval into blocks and we place the blocks on $y = x^2$, where $x \geq 0$. So, the $[0, 1]$ interval corresponds to the part of the quadratic equation. This guarantees that every line can cut this “bent” interval at most twice and in addition the part of each mass that is in $R^+$ is almost the same as value of the corresponding agent for the piece of $[0, 1]$ labelled with “+”. Next we show how to construct an instance $I_P$ of $(\epsilon - \epsilon')$-STRAIGHT-PIZZA-SHARING with $2n$ mass distributions, for any $\epsilon' = \frac{1}{\text{poly}(n)} < \epsilon$, given an instance $I_{CH}$ of $\epsilon$-CONSENSUS-HALVING with $2n$ agents with $k$-block valuations.

Let $c_{\text{max}} := \max_{i \in [2n], m \in [k]} c_{im}$, where $c_{im}$ is the value density of agent $i$’s $m$-th block in $I_{CH}$, and observe that $c_{\text{max}} \geq 1$ since the total valuation of any agent over $[0, 1]$ is 1. In what follows, it will help us to think of the interval $[0, 1]$ in $I_{CH}$ as being discretized in increments of $d := \frac{1}{4n^2c_{\text{max}}}$. We refer to the subinterval $[(j - 1) \cdot d, j \cdot d]$ as the $j$-th $d$-block of interval $[0, 1]$ in $I_{CH}$, for $j \in [1/d]$. 

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Figure 3: Placing the big-tiles on the $y = x^2$ curve. The $j$-th, $(j+1)$-st and $(j+2)$-nd big-tiles are centered on the curve with great enough distance in the $x$-axis such that any straight line (red/dashed) can cut at most two big-tiles.

We now describe the instance $I_P$. For ease of presentation, the space of the instance is inflated to $\left[0, \left(\frac{6}{d^2}\right)^2 + 1\right]^2$, but by scaling the construction down, the results are attained. We consider two kinds of square tiles; $1/d$ large square tiles of size $1 \times 1$, each of which contains $2/n$ smaller square tiles of size $1/2n \times 1/2n$ on its diagonal. We will call the former type big-tile and denote it by $t_j$ and the latter one small-tile and denote it by $t_{ij}$ for some $i \in [2n], j \in [1/d]$.

The centers of consecutive big-tiles are positioned with $6/d$ distance apart in the $x$-axis. For $j \in [1/d]$ we define $s_j = \frac{6}{d} \cdot j$. For every agent $i \in [2n]$ we will create a uniform mass distribution $\mu_i$ that consists of at most $1/d$ many axis-aligned small-tiles. Each big-tile $t_j$, $j \in [1/d]$, is centered at $(s_j, s_j^2)$ while, in it, each small-tile $t_{ij}$, $i \in [2n]$, belonging to mass distribution $\mu_i$ has its bottom left corner at $(s_j + \frac{i-1}{2n}, s_j^2 + \frac{i-1}{2n})$. Each small-tile $t_{ij}$ contains total mass (belonging to $\mu_i$) of exactly the same total value $v_{ij}$ as that of agent $i$’s $j$-th $d$-block in $I_{CH}$. Observe that, by definition, the value $v_{ij}$ of the $j$-th $d$-block for agent $i$ is at most $d \cdot c_{\max} \leq \frac{1}{2n^2} \cdot 2n$, and therefore it fits inside the small tile of size $\frac{1}{2n} \times \frac{1}{2n}$. In particular, the mass inside $t_{ij}$ has width $\frac{1}{2n}$ and height $v_{ij} \cdot 2n$. Figure 3 and Figure 4 depict our construction.

The proof of the following lemma can be found in Appendix B.

**Lemma 4.** Fix a constant $\delta > 0$ and some $\varepsilon' \in \left[\frac{1}{m^2}, \varepsilon\right)$, where $r \geq 1$ and $\frac{1}{m^2} < \varepsilon < 1$. Let $L = \{l_1, \ldots, l_m\}$ be a set of lines, where $m \leq n + n^{1-\delta}$. If $L$ is a solution to $(\varepsilon - \varepsilon')$-STRAIGHT-PIZZA-SHARING instance $I_P$, then we can find in polynomial time a solution to $\varepsilon$-CONSSENSUS-HALVING instance $I_{CH}$ with at most $2(n + n^{1-\delta})$ cuts.
The $(j+1)$-st (left) and $(j+2)$-nd (right) $d$-block (right) of the Consensus-Halving instance.

The $(j+1)$-st (left) and the $(j+2)$-nd (right) big-tile of the Straight-Pizza-Sharing instance with their small-tiles. The small-tiles contain the mass distributions of the $(j+1)$-st and $(j+2)$-nd $d$-block, respectively.

Figure 4: The construction for a part of an instance with four mass distributions.

This, together with the fact that $\varepsilon$-Consensus-Halving is PPA-hard for any $\varepsilon < 1/5$, due to [DFHM22], implies the main theorem of this section.

**Theorem 5.** $\varepsilon$-Straight-Pizza-Sharing with $2n$ mass distributions is PPA-hard for any constant $\varepsilon < 1/5$, even when $n + n^{1-\delta}$ lines are allowed for any given constant $\delta > 0$, every mass distribution is uniform over polynomially many rectangles, and there is no overlap between any two mass distributions.

We will now shift our attention to studying the decision version of the problem, where we are asking to find a solution that uses at most $n - 1$ straight lines, and notice that there is no guarantee for such a solution. We employ the NP-hard instances of $\varepsilon$-Consensus-Halving for their constant $\varepsilon$ from [FRFGZ18], and reduce them according to the above reduction procedure to $(\varepsilon - \varepsilon')$-Straight-Pizza-Sharing instances for some $\varepsilon'$ that is inverse polynomial in the input size, e.g., $\varepsilon' = 1/n^2$. This gives the following.

**Theorem 6.** There exists a constant $\varepsilon > 0$ for which it is NP-hard to decide if an $\varepsilon$-Straight-Pizza-Sharing instance with $2n$ mass distributions admits a solution with at most $n - 1$ lines, even when every mass distribution is uniform over polynomially many rectangles, and there is no overlap between any two mass distributions.

### 3.2 Hardness of approximate SC-Pizza-Sharing

In this section we prove hardness results for $\varepsilon$-SC-Pizza-Sharing. We provide a reduction from $\varepsilon$-Consensus-Halving with $k$-block valuations, which was shown to be PPA-complete in [DFHM22] for any constant $\varepsilon < 1/5$ even for 3-block uniform valuations. Our construction shows that $\varepsilon$-SC-Pizza-Sharing remains PPA-hard even when there is no overlap between any two mass distributions. Notice that the case of non-overlapping mass distributions is the most simple type of an instance, since we can easily reduce it to one where an arbitrarily large number of masses overlap: make “dummy” exact copies of an existing mass distribution. Also, the machinery that we present, combined with the reduction by [FRFGZ18] implies NP-hardness for the decision version of $\varepsilon$-SC-Pizza-Sharing when $\varepsilon$ is a small constant.
The reduction. We reduce from a general $\varepsilon$-Consensus-Halving instance to an $\varepsilon$-SC-Pizza-Sharing instance, and the idea is to create a mass for each agent. For any $\varepsilon' = \frac{1}{\text{poly}(n)} < \varepsilon$, given an instance $I_{CH}$ of $\varepsilon$-Consensus-Halving with $n$ agents and $k$-block valuations we will show a polynomial time construction to an $(\varepsilon - \varepsilon')$-SC-Pizza-Sharing instance $I_{SC}$.

For our construction, we will use the same components as those in the proof of Lemma 4. In particular, let $c_{\text{max}} := \max_{i \in [n], m \in [k]} c_{im}$, where $c_{im}$ is the value density of agent $i$'s $m$-th block in $I_{CH}$ (and again note that $c_{\text{max}} \geq 1$ since the total valuation of the agent over $[0, 1]$ is 1). Similarly to the aforementioned proof, we will discretize the $[0, 1]$ interval of $I_{CH}$ in increments of $d := \frac{\max_{i}c_{im}}{n^2c_{\text{max}}}$. Also, let us restate that for any given $j \in [1/d]$, the subinterval $[(j - 1) \cdot d, j \cdot d]$ is called $j$-th $d$-block of interval $[0, 1]$ in $I_{CH}$.

To simplify the presentation, all the analysis of instance $I_{SC}$ will be for the scaled version of it, that is, in $[0, \frac{1}{n}]^2$. It is easy to verify, though, that by proper re-scaling we can put the instance in $[0, 1]^2$ as required. We will be using the same gadgets that were constructed for the proof of Lemma 4, namely the big-tiles, which contain small-tiles. In particular, we have $1/d$ square big-tiles of size $1 \times 1$, each of which contains $n$ square small-tiles of size $\frac{1}{n} \times \frac{1}{n}$ on its diagonal. For any $i \in [n], j \in [1/d]$, we denote them by $t_{ij}$ and $t_{ij}'$, respectively.

In this construction, however, the positioning of big-tiles will be different than that of the aforementioned proof. In particular, we will be placing them on the diagonal of $[0, \frac{1}{n}]^2$ as shown in Figure 4b. For every agent $i$ we will create a uniform mass distribution $\mu_i$ that consists of at most $1/d$ many axis-aligned small-tiles. For each $j \in [1/d]$, the bottom-left corner of big-tile $t_j$ is at $(j - 1, j - 1)$. In it, each small-tile $t_{ij}, i \in [n]$, belonging to mass distribution $\mu_i$ has its bottom left corner at $(j - 1 + \frac{i - 1}{n}, j - 1 + \frac{i - 1}{n})$.

Inside a given big-tile $t_j$, the total mass of each small-tile $t_{ij}$ contains total mass (belonging to $\mu_i$) of exactly the same total value $v_{ij}$ as that of agent $i$'s $j$-th $d$-block in $I_{CH}$. The shape of that mass is rectangular, and has width $1/n$ and height $v_{ij} \cdot n$. Using identical arguments to those of Lemma 4 we conclude that the total value of the $d$-block fits inside the small tile of size $\frac{1}{n} \times \frac{1}{n}$. Figure 4 depicts our construction.

We will now define how a solution to $I_{SC}$, i.e., an SC-path, is mapped back to a solution of $I_{CH}$, i.e., a set of cuts. This is identical to that of the aforementioned lemma’s translation. In particular, we consider again the big-tiles in sequential order and we add one cut at $j \cdot d$ whenever we find two big-tiles $t_j$ and $t_{j+2}$ that belong to different regions. Suppose that, following the aforementioned procedure, the next $I_{CH}$ cut falls at $j \cdot d'$ for some $d' > d$. If $t_j$ belongs to region “+” (resp. “−”) and $t_{j+2}$ belongs to “−” (resp. “+”), then the interval $[j \cdot d, j \cdot d']$ gets label “+” (resp. “−”), and vice versa. This translation obviously takes polynomial time.

What remains is to prove that the aforementioned translation of the aforementioned solution of a $(\varepsilon - \varepsilon')$-SC-Pizza-Sharing instance $I_{SC}$ into a solution of the $\varepsilon$-Consensus-Halving instance $I_{CH}$ is indeed correct. Notice that, if the solution of $I_{SC}$ has $r \in \mathbb{N}$ many turns on the SC-path, there can be at most $r + 1$ small-tiles that are intersected by it. For our reduction, let $r = n - 1 + n^{1 - \delta}$ for any constant $\delta > 0$. Consider sequentially the big-tiles $t_1, \ldots, t_{1/d}$, and without loss of generality, let $t_1', \ldots, t'_{r+1}$ be its subset, where $t_j'$ is the $j$-th big-tile that has an intersected small-tile. In the big-tile sequence of $t_1, \ldots, t_{1/d}$, the change of region can happen at most $n + n^{1 - \delta}$ times. Therefore, we have at most $n + n^{1 - \delta}$ cuts in $I_{CH}$ with the corresponding labels as defined previously. Each subinterval defined by two cuts in $I_{CH}$ follows the label of the corresponding big-tiles in $I_{SC}$. Now we are ready to prove the following lemma.

**Lemma 7.** Fix a constant $\delta > 0$, and some $\varepsilon' \in [\frac{\delta}{n}, \varepsilon)$, where $r \geq 1$ and $\frac{\delta}{n} < \varepsilon < 1$. Let an SC-path with at most $r = n - 1 + n^{1 - \delta}$ turns be a solution to $(\varepsilon - \varepsilon')$-SC-Pizza-Sharing instance $I_{SC}$. Then we can find in polynomial time a solution to $\varepsilon$-Consensus-Halving instance $I_{CH}$ with at most $n + n^{1 - \delta}$ cuts.
Proof. The above translation of the SC-path to $I_{CH}$ cuts indicates that each cut introduces a discrepancy between the “+” and “−” regions of $I_{CH}$, of value at most $c_{\text{max}} \cdot d$ for each valuation $v_i$, $i \in [n]$. This results in total discrepancy of $(n + n^{1-\delta}) \cdot c_{\text{max}} \cdot d$ for each agent $i$. We now denote by $I^+$ and $I^-$ the regions in $I_{CH}$ which correspond to the regions $R^+$ and $R^-$ of $I_{SC}$ respectively, induced by the SC-path after disregarding the intersected big-tiles. Then for the discrepancy in the valuation of agent $i$ in $I_{CH}$ we get

$$|v_i(I^+) - v_i(I^-)| \leq |\mu_i(R^+) - \mu_i(R^-)| + (n + n^{1-\delta}) \cdot c_{\text{max}} \cdot d$$

$$\leq (\varepsilon - \varepsilon') + (n + n^{1-\delta}) \cdot c_{\text{max}} \cdot d$$

$$< (\varepsilon - \varepsilon') + 2n \cdot c_{\text{max}} \cdot d$$

$$\leq \varepsilon,$$

where the last inequality is due to the fact that in our reduction we can pick $d \leq \frac{\varepsilon'}{2n \cdot c_{\text{max}}}$. This is always possible since, by assumption, $\varepsilon' \geq \frac{2}{n^r}$ for some $r \geq 1$, and additionally, according to our reduction, $d$ is required to be at most $\frac{1}{n^r \cdot c_{\text{max}}}$. $\square$

The above, together with the fact that $\varepsilon$-Consensus-Halving is PPA-hard for any $\varepsilon < 1/5$ ([DFHM22]) implies the main theorem of this section.

**Theorem 8.** $\varepsilon$-SC-Pizza-Sharing with $n$ mass distributions is PPA-hard for any constant $\varepsilon < 1/5$, even when $n - 1 + n^{1-\delta}$ turns are allowed in the SC-path for any given constant $\delta > 0$, every mass distribution is uniform over polynomially many rectangles, and there is no overlap between any two mass distributions.

Similarly to the case of the decision variant of $\varepsilon$-Straight-Pizza-Sharing (Theorem 6), we can get the following result by reducing from the NP-hard instances of $\varepsilon$-Consensus-Halving for constant $\varepsilon$.

**Theorem 9.** There exists a constant $\varepsilon > 0$ for which it is NP-hard to decide if an $\varepsilon$-SC-Pizza-Sharing instance with $n$ mass distributions admits a solution consisting of an SC-path with at most $n - 2$ turns, even when every mass distribution is uniform over polynomially many rectangles, and there is no overlap between any two mass distributions.

### 3.3 Hardness of discrete Straight-Pizza-Sharing and SC-Pizza-Sharing

In this section, we study the discrete versions of Straight-Pizza-Sharing and SC-Pizza-Sharing.

**Definition 10.** For any $n \geq 1$, the problem $\varepsilon$-Discrete-Straight-Pizza-Sharing is defined as follows:

- **Input:** $\varepsilon \geq 0$ and $2n$ point sets $P_1, P_2, \ldots, P_{2n}$ on $[0, 1]^2$.

- **Output:** One of the following.
  
  (a) Three points that can be intersected by the same line.
  
  (b) A partition of $[0, 1]^2$ to $R^+$ and $R^-$ using at most $n$ lines such that for each $i \in [2n]$ it holds that $|P_i \cap R^+| - |P_i \cap R^-| \leq \varepsilon \cdot |P_i|$.

A point that is intersected by a line does not belong to any of $R^+$, $R^-$.  

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Notice that the first kind of allowed output for both problems (Definition 10(a), Definition 11(a)) is a witness that the input points are not in general position or that their x- or y-coordinates are not unique, respectively, which can be checked in polynomial time. The second kind of output (Definition 10(b), Definition 11(b)) is the one that is interesting and can encode the hard instances studied here. In case the first kind of output does not exist, the other one is guaranteed to exist due to [Sch21] for \( \varepsilon \)-DISCRETE-straIGHT-Pizza-Sharing, while for \( \varepsilon \)-DISCRETE-SC-Pizza-Sharing its existence is guaranteed for every \( \varepsilon \in [0, 1] \) due to a reduction we present in Section 4.4 which shows containment in \( \text{PPA} \).

The definition of \( \varepsilon \)-DISCRETE-straIGHT-Pizza-Sharing is a slightly modified form of the one that appears in [Sch21], where it is referred to as \( \text{DiscretePizzaCutting} \). In our definition, we avoid having to “promise” an input of points that are in general position, by allowing as output a witness of an inappropriate input. Furthermore, the definition we present is more general, since it accommodates an approximation factor \( \varepsilon \); in particular, for \( \varepsilon < \min_i \{1/|P_i|\} \) we get the definition of the aforementioned paper. Similarly, we define \( \varepsilon \)-DISCRETE-SC-Pizza-Sharing, which to the best of our knowledge, has not been stated in previous work. Note that, if the input consists of points that are in general position, in an \( \varepsilon \)-DISCRETE-straIGHT-Pizza-Sharing solution only up to two points are allowed to be intersected by the same line, while in an \( \varepsilon \)-DISCRETE-SC-Pizza-Sharing solution only up to two points are allowed to be intersected by the same line segment.

Recall that in \( \varepsilon \)-SC-Pizza-Sharing we are given \( n \) mass distributions, while in \( \varepsilon \)-straIGHT-Pizza-Sharing we are given \( 2n \) mass distributions as input. In Appendix C, we describe a general construction that takes as an input \( q \in \{n, 2n\} \) mass distributions \( \mu_1, \ldots, \mu_q \) normalized on \([0, 1]^2\), represented by weighted polygons with holes (see Appendix A for the detailed description), and turns it into \( q \) sets of points \( P_1, \ldots, P_q \) on \([0, 1]^2\) whose union is in general position and furthermore, they have unique \( x \)- and \( y \)-coordinates. We prove that, given a set of \( m \leq 2n \) lines or an SC-path of at most \( m' \leq 2n - 1 \) turns, that partition \([0, 1]^2\) into \( R^+ \) and \( R^- \) such that \( |P_i \cap R^+| = |P_i \cap R^-| \leq (\varepsilon - \varepsilon') \cdot |P_i| \), the same lines and SC-path, respectively, separate the mass distributions such that \( |\mu_i(R^+) - \mu_i(R^-)| \leq \varepsilon \).

Let \( N \geq 2n \) be the input size of any of our two pizza sharing problems, and let the smallest area triangle in the mass distributions’ triangulation be \( \alpha \). For any \( \varepsilon' < \varepsilon \), where \( \varepsilon \) and \( \alpha \) are at least inverse polynomial in \( N \), the construction results to a polynomial time reduction from \( \varepsilon \)-straIGHT-Pizza-Sharing to \((\varepsilon - \varepsilon')\)-DISCRETE-straIGHT-Pizza-Sharing and from \( \varepsilon \)-SC-Pizza-Sharing to \((\varepsilon - \varepsilon')\)-DISCRETE-SC-Pizza-Sharing. The reduction can be performed in time polynomial in the input size and in \( 1/\alpha \). It consists of two parts: (i) First we “pixelate” the mass distributions finely enough so that they are represented by a sufficiently large number of pixels. This will ensure a high enough “resolution” of the pixelated distributions. (ii) The pixels will then be turned into points, which we have to perturb in order to guarantee they are
Lemma 12. Let $N$ be the input size of an approximate pizza sharing problem (either $\varepsilon$-Straight-Pizza-Sharing or $\varepsilon$-SC-Pizza-Sharing) whose triangulation has no triangle with area less than $\alpha > 0$. Also, let $\varepsilon' \in \left[\frac{6}{N^2}, \varepsilon\right]$, where $\varepsilon > 0$ is a fixed constant, and $N\varepsilon < \varepsilon < 1$. Then, the instance can be reduced in time $\text{poly}(N, 1/\alpha)$ to its approximate discrete version, that is, $(\varepsilon - \varepsilon')$-Discrete-Straight-Pizza-Sharing or $(\varepsilon - \varepsilon')$-Discrete-SC-Pizza-Sharing.

Given Theorem 5 and Theorem 8, and since their instances are constructed such that $\alpha$ is an at least inverse polynomial function of the input size, the above lemma implies the following hardness results.

Theorem 13. $\varepsilon$-Discrete-Straight-Pizza-Sharing with $2n$ point sets is PPA-hard for any constant $\varepsilon < 1/5$, even when $n + n^{1-\delta}$ lines are allowed for any given constant $\delta > 0$.

Theorem 14. $\varepsilon$-Discrete-SC-Pizza-Sharing with $n$ point sets is PPA-hard for any constant $\varepsilon < 1/5$, even when $n - 1 + n^{1-\delta}$ turns are allowed in the SC-path for any given constant $\delta > 0$.

We note that PPA-hardness for $\varepsilon$-Discrete-Straight-Pizza-Sharing was so far known only for any $\varepsilon \in [0, \min_i \{1/|P_i|\}]$ (which is equivalent to $\varepsilon = 0$), due to [Sch21]. Since, in the aforementioned paper’s constructions, $\min_i \{1/|P_i|\} \in O(1/\text{poly}(N))$, our result strengthens the hardness of the problem significantly.

Lemma 12 is general enough that allows us to derive NP-hardness results for the decision variants of the two discrete versions of the pizza sharing problems. If we ask for a solution with at most $n - 1$ straight lines or $n - 2$ turns in $(\varepsilon - \varepsilon')$-Discrete-Straight-Pizza-Sharing and $(\varepsilon - \varepsilon')$-Discrete-SC-Pizza-Sharing, respectively, then we can easily reduce to them from the instances of Theorem 6 and Theorem 9, picking $\varepsilon'$ to be some inverse polynomial function of $N$, e.g., $\varepsilon' = 1/N$. In particular, we get the following.

Theorem 15. There exists a constant $\varepsilon > 0$ for which it is NP-hard to decide whether a solution of $\varepsilon$-Discrete-Straight-Pizza-Sharing with $2n$ point sets and at most $n - 1$ lines exists.

Theorem 16. There exists a constant $\varepsilon > 0$ for which it is NP-hard to decide whether a solution of $\varepsilon$-Discrete-SC-Pizza-Sharing with $n$ point sets and an SC-path with at most $n - 2$ turns exists.

3.4 Hardness of exact SC-Pizza-Sharing

In this section, we show hardness results for exact SC-Pizza-Sharing. We prove that solving SC-Pizza-Sharing is FXP-hard and that deciding whether there exists a solution for SC-Pizza-Sharing with fewer than $n - 1$ turns is ETR-hard.

As mentioned earlier, computing an exact solution of a FXP-hard problem may require computing an irrational number. To showcase this for SC-Pizza-Sharing, consider the following simple instance. Let us have a single mass distribution in the shape of a right-angled triangle, whose corners are on $(0, 1), (1, 1)$, and $(1, 0)$. It is normalised, i.e., its total mass is 1, therefore its weight is 2. An exact solution of SC-Pizza-Sharing is either a horizontal or a vertical straight line (with 0 turns), which cuts the triangle such that each half-space has half of the mass, that is, $1/2$. One can easily check that the solution is either the horizontal line $y = \sqrt{2}/2$, or the vertical line $x = \sqrt{2}/2$.

We provide a main reduction from (exact) Consensus-Halving to (exact) SC-Pizza-Sharing, whose gadgets are then employed to show the ETR-hardness of the decision version.
This time, as a starting point, we will use the instances of Consensus-Halving produced in [DFMS21], which we will denote by \( I_{\text{CH}}^{\text{DFMS}} \). When clear from context, by \( I_{\text{CH}}^{\text{DFMS}} \) we will denote a particular instance of the aforementioned family. We note that here the input consists of sets of points, i.e., we describe polygons by their vertices. For a detailed description of the input representation, see Appendix A. In [DFMS21], the \( \text{FIXP} \)-hard family of instances we reduce from has as input \( n \) arithmetic circuits capturing the cumulative valuation of \( n \) agents on \([0,1]\), which were piece-wise polynomials of maximum degree 2. However, since their (density) valuation functions are piece-wise linear, the input of SC-Pizza-Sharing suffices to consist of only rectangles and triangles, and no other shape. Therefore, there is no need for extra translation of the input of Consensus-Halving to the input of SC-Pizza-Sharing.

The reduction. Here we show the main reduction, which will conclude with the proof that finding an exact solution to SC-Pizza-Sharing is \( \text{FIXP} \)-hard, and then will be used to show that its decision version is \( \text{ETR} \)-hard. We reduce from a Consensus-Halving instance \( I_{\text{CH}}^{\text{DFMS}} \) with \( n \) agents and \( k \)-block-triangle valuations to an SC-Pizza-Sharing instance \( I_{\text{SC}} \) with \( n \) mass distributions. When requesting a solution in \( I_{\text{SC}} \) with at most \( n-1 \) turns in the SC-path, then we get \( \text{FIXP} \)-hardness, while when requesting to decide if \( I_{\text{SC}} \) is solvable with \( n-2 \) turns, then we get \( \text{ETR} \)-hardness. Both of these results are due to [DFMS21], and hold even for 6-block-triangle valuations.

As in our previous proofs, for ease of presentation, the space of the instance is inflated to \([0,2n(k+1)]^2\), and by scaling the construction down to \([0,1]^2\) the correctness is attained. The key difference between this reduction and the previous reductions on the approximate versions is that the starting point of the reduction, i.e., instance \( I_{\text{CH}}^{\text{DFMS}} \), besides rectangular, also contains triangular-shaped valuations for agents. More specifically, all of the following hold:

1. the valuation function of every agent is 4-block-triangle, or 6-block;
2. every triangle has height 2 and belongs to exactly one interval of interest of the form \([a,a+1]\);
3. for every agent \( i \in [n] \) there exists an interval \([a_i,b_i]\) that contains more than half of their total valuation, and in addition, for every \( i' \neq i \) we have \((a_i,b_i) \cap (a_{i'},b_{i'}) = \emptyset\).

Also, in this reduction, the resulting SC-Pizza-Sharing instances will contain weighted mass distributions (see definition in Section 2).

The first step is to partition \([0,1]\) of \( I_{\text{CH}} \) into subintervals that are defined by points of interest. We say that a point \( x \in [0,1] \) is a point of interest if it coincides with the beginning or the end of a valuation block of an agent; formally, \( x \) is a point of interest if \( x \in \{a_{ij}, a_{ij}'\} \) for some agent \( i \in [n] \) and some of her valuation blocks \( j \in [k] \) (for this notation see the Consensus-Halving preliminaries in Section 2). These points conceptually split \([0,1]\) into intervals of interest, since in between any pair of consecutive points of interest, all agents have a non-changing valuation density. Let \( 0 \leq x_1 \leq x_2 \leq \ldots \leq x_m \leq 1 \) denote the points of interest, and each interval of interest \([x_j,x_{j+1}]\), \( j \in [m] \) is called the \( j \)-th subinterval. Observe that \( m \leq 2 \cdot n \cdot k \).

For each subinterval \( j \in [m] \) of \( I_{\text{CH}} \), we will construct a tile \( t_j \) of size \( 1 \times 1 \). Let the total value of agent \( i \) in the \( j \)-th interval of \( I_{\text{CH}} \) be \( v_{ij} \). If for agent \( i \) the valuation in the \( j \)-th interval has constant density (i.e., the total valuation has a rectangular shape), then we create a square mass that belongs to \( \mu_i \), of size \( 1 \times 1 \) inside \( t_j \), with weight \( w_{ij} = v_{ij} \). If her valuation has linear density (i.e., the total valuation has a triangular shape), then we create a mass of right-triangular shape that belongs to \( \mu_i \), with two of its sides being of length 1 and touching the top and right sides of \( t_j \), while its right angle touches the top-right corner of \( t_j \), and its weight is \( w_{ij} = 2 \).
(a) An instance of Consensus-Halving with three agents. Here, there are nine points of interest.

(b) The corresponding instance of SC-Pizza-Sharing; observe that all three different mass distributions overlap in \(t_2\) and the green and blue distributions overlap in \(t_3\).

Figure 5

Observe that, for any given \(i \in [n]\), the total value of a subinterval in \(I_{CH}\) is equal to its total mass in the corresponding tile. Finally, we place the tiles sequentially in a diagonal manner, i.e., each tile \(t_j\) is axis-aligned and placed with its bottom-left corner at point \((j, j)\). See Figure 5 and Figure 6 for a depiction.

(a) Part of the Consensus-Halving instance with two agents and the corresponding regions of interest.

(b) The corresponding part in the SC-Pizza-Sharing instance.

Figure 6

Now we need to show how an exact solution of \(I_{SC}\), that is, an SC-path with \(n-1\) many turns is mapped back to a solution of \(I_{CH}\) with \(n\) cuts. Consider the first tile \(t_j\) that is intersected by the SC-path, resulting in a part of it belonging to \(R^+\) and the rest of it to \(R^-\). Let \(a_j^+ := \mu_i(t_j \cap R^+)\), and by our definition above, it is implied that \(\mu_i(t_j \cap R^-) = v_{ij} - a_j^+\). We then place a cut in the \(j\)-th subinterval of \(I_{CH}\) at point \(x_j' = x_j + (x_{j+1} - x_j) \cdot \frac{a_j^+}{v_{ij}}\), and label the interval \([0, x_j']\) with “+”, and the interval \([x_j', x_{j+1}]\) with “−”. Next, consider the second tile \(t_j'\) that is intersected by the SC-path, resulting in a part of it belonging to \(R^+\) and the rest of it to \(R^-\). For this tile, we will place a cut in the \(j'\)-th subinterval of \(I_{CH}\) at point \(x_{j'}' = x_{j'} + (x_{j'+1} - x_{j'}) \cdot \frac{v_{ij'} - a_{j'}^+}{v_{ij'}}\), and label the interval \([x_{j'+1}, x_{j'}']\) with “−”, and the interval \([x_{j'}, x_{j'+1}]\) with “+”. We continue in this fashion with the translation of the SC-path into cuts and labels of \(I_{CH}\), where, the \(r\)-th in
order intersected tile will be of the kind \( t_j \) if \( r \) is odd, and of the kind \( t_j' \) if \( r \) is even. This will result in a set of cuts that define regions with alternating signs.

It remains to prove that this is a solution to \( I_{CH} \). To do this, we will use the following crucial observation.

**Claim 17.** In any solution of \( I_{SC} \) created by \( I_{DFMS}^{CH} \), there can be no turn inside a tile.

**Proof.** The truth of the statement can become apparent if one considers Item 3 of the above facts on \( I_{DFMS}^{CH} \), together with the fact that the endpoints of each \([a_i, b_i] \) are points of interest. It is implied then, that in any of the \( I_{DFMS}^{CH} \) solutions, there needs to be at least one cut in each interval \([a_i, b_i] \) for each \( i \in [n] \). And since we are allowed to draw at most \( n \) cuts, there will be a single cut in each of those intervals. Also, due to the fact that \( a_i, b_i \) are points of interest, each cut in \([a_i, b_i] \) belongs to a different subinterval, and therefore, there will be exactly \( n \) cuts in \( n \) distinct subintervals. Focusing now on our \( I_{SC} \) construction, the SC-path with \( n - 1 \) turns consists of a total of \( n \) horizontal and vertical line segments. If any of those does not intersect any tile, then this SC-path will correspond to a solution of at most \( n - 1 \) cuts in \( I_{CH} \), which cannot be a solution. Therefore, every line segment intersects some tile.

Notice that, due to the diagonal placement of tiles, any SC-path that is a solution has to be \( x \)-monotone and \( y \)-monotone, i.e., to have a “staircase” form. Also, this diagonal placement of tiles dictates that, if there was a turn of the SC inside a tile, then two line segments are used to intersect it instead of one. This means that at most \( n - 1 \) tiles will be intersected, and therefore this translates to a set of at most \( n - 1 \) cuts in \( I_{CH} \), which cannot be a solution. \( \square \)

Put differently, having a turn inside a tile would be a “waste”, and in our instances, all turns are needed in order for a solution to exist.

Let us now focus on a tile \( t_r \). We consider two cases that can appear in \( I_{DFMS}^{CH} \):

- **For all \( i \in [n] \), \( v_i \) is nonnegative constant in the \( r \)-th subinterval.** Let \( t_r \) be a tile as the aforementioned \( t_j \). Now notice that the “+” part of the \( j \)-th subinterval equals \( v_{ij} \cdot \frac{x_j - x_{ij}}{x_{j+1} - x_j} = a_j^+ = \mu_i(t_j \cap R^+) \), and the “−” part equals \( v_{ij} \cdot \frac{x_{j+1} - x_j'}{x_{j+1} - x_j} = v_{ij} - a_j^+ = \mu_i(t_j \cap R^-) \). If on the other hand, \( t_r \) is a tile as the aforementioned \( t_j' \), it is again easy to see that, for every \( i \in [n] \), the total “+” and “−” parts of \( t_j' \) have mass equal to the corresponding “+” and “−” parts of the \( j \)-th subinterval.

- **There is an \( i \in [n] \) such that \( v_i \) is linear in the \( r \)-th subinterval.** Due to the previous case, for all \( i' \neq i \), the total “+” and “−” parts of the tile have mass equal to the corresponding “+” and “−” parts of the corresponding subinterval in \( I_{CH} \). For \( i \), due to the structure of \( I_{DFMS}^{CH} \), the slope of \( v_i \) is 2, and the length of the subinterval is 1, therefore we have \( v_{ij} = 1 \). Let \( t_r \) be a tile as the aforementioned \( t_j \), and without loss of generality, let the SC-path intersect it horizontally at \( j + s \) (y-coordinate), and its bottom part is labelled “+”, while the top part is labelled “−”. According to the previous case, a cut should be placed at point \( x_j' = x_j + s \) in the \( j \)-th subinterval, and its part to the left should be labelled “+” while the one to its right should be labelled “−”. Now notice that for agent \( i \) the “+” part of the \( j \)-th subinterval equals \( \frac{s \cdot a_j^+}{2} = s^2 \), while her “+” mass in \( t_j \) is \( \frac{s^2}{2} \cdot w_{ij} = s^2 \), since \( w_{ij} = 2 \). Similarly, both the “−” parts of \( I_{CH} \) and \( I_{SC} \) are equal in the subinterval and the tile, respectively. Finally, if, \( t_r \) is a tile as the aforementioned \( t_j' \), it is again easy to see that the total “+” and “−” parts of \( t_j' \) have mass equal to the corresponding “+” and “−” parts of the \( j \)-th subinterval.

The above analysis shows that, for any \( i \in [n] \), in each individual tile the total “+” mass is equal to the total “+” value of the corresponding subinterval. Therefore, given a solution to
where the total mass of $R^+$ will equal that of $R^-$ for every $i \in [n]$, the induced cuts on $I_{CH}$ constitute a solution. Finally, due to the $\mathsf{FIXP}$-hardness of exact Consensus-Halving shown in [DFMS21], we get the following.

**Theorem 18.** SC-Pizza-Sharing is $\mathsf{FIXP}$-hard even when every mass distribution is uniform, consists of at most six pieces that can be unit-squares or right-angled triangles, and have overlap at most 3.

We can also show that deciding whether there exists an exact SC-Pizza-Sharing solution with $n-2$ turns is ETR-hard. To show this, we will use a result of [DFMS21], where it was shown that deciding whether there exists an exact Consensus-Halving solution with $n$ agents and $n-1$ cuts is ETR-hard. We give a reduction from this version of Consensus-Halving to the decision problem for SC-Pizza-Sharing. The reduction uses the same ideas that we presented for the $\mathsf{FIXP}$-hardness reduction. The full details are in Appendix D, where the following theorem is shown.

**Theorem 19.** It is ETR-hard to decide if an exact SC-Pizza-Sharing instance admits a solution with a SC-path with at most $n-2$ turns, even when every mass distribution consists of at most six pieces that can be unit-squares or right-angled triangles, and have overlap at most 3.

## 4 Containment results

In this section, we present containment results for the exact and approximate versions of Straight-Pizza-Sharing and SC-Pizza-Sharing that we study in this paper. Our containment result for the former problem is possible by reducing it to its discrete version, which was recently shown by [Sch21] to be in PPA. For SC-Pizza-Sharing, our results revolve around a proof that solutions exist, which utilizes the Borsuk-Ulam theorem. A proof of this kind was already presented in [KRPS16], but we present a new one which is conceptually simpler, as it does not use any involved topological techniques. An advantage of our proof is that it can be made algorithmic, and so it can be used to show that SC-Pizza-Sharing is contained in BU. Then, by making simple modifications to the BU containment proof, we show that the other containment results hold. Then, we study the corresponding decision variants of the problems, and acquire containment in NP for approximate and discrete versions of the problems and ETR containment of exact SC-Pizza-Sharing.

### 4.1 Containment of approximate Straight-Pizza-Sharing

Here we prove that $\varepsilon$-Straight-Pizza-Sharing with $2n$ mass distributions is in PPA for any $\varepsilon \in \Omega(1/poly(N))$ and $\alpha \in \Omega(1/poly(N))$, where $N$ is the input size and $\alpha$ is the smallest area among the triangles of the triangulated mass distributions of $\varepsilon$-Straight-Pizza-Sharing. This answers a big open question left from [DFM22], where PPA containment was elusive. We also show that deciding whether a solution with at most $n-1$ straight lines exists is in NP. Both those results are derived by reducing the problem to its discrete version, where instead of mass distributions, the input consists of points, and the goal is to bisect (up to one point) each of the $2n$ point sets using at most $n$ straight lines. The latter was recently shown by [Sch21] to be in PPA.

**PPA containment.** We will employ Lemma 12 in a straightforward way. In particular, given an $\varepsilon$-Straight-Pizza-Sharing instance for some $\varepsilon \in \Omega(1/poly(N))$, we will pick an $\varepsilon' < \varepsilon$ as prescribed in the aforementioned lemma, and reduce our problem to Discrete-Straight-Pizza-Sharing in time $\text{poly}(N, 1/\alpha)$. 

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Theorem 20. \( \varepsilon\)-\textsc{Straight-Pizza-Sharing} with \(2n\) weighted mass distributions with holes is in \text{PPA} for any \(\varepsilon \in \Omega(1/\text{poly}(N))\) and \(\alpha \in \Omega(1/\text{poly}(N))\), where \(N\) is the input size and \(\alpha\) is the smallest area among the triangles of the triangulated mass distributions.

**NP containment.** Observe that Lemma 12 shows how to turn a set of \(2n\) mass distributions (weighted polygons with holes) into a set of (unweighted) points, which, if cut by at most \(2n\) straight lines, will result to an approximate cut of the mass distributions relaxed by an extra additive \(\varepsilon' \in \Omega(1/N^c)\) for any \(c > 0\). When the number of straight lines is at most \(n - 1\), this gives straightforwardly a reduction to 0-\textsc{Discrete-Straight-Pizza-Sharing}, since we can check how many of the polynomially many points of the latter instance are in \(R^+\) and \(R^-\).

Theorem 21. Deciding whether \(\varepsilon\)-\textsc{Straight-Pizza-Sharing} with \(2n\) weighted mass distributions with holes has a solution with at most \(n-1\) straight lines is in \text{NP} for any \(\varepsilon \in \Omega(1/\text{poly}(N))\) and \(\alpha \in \Omega(1/\text{poly}(N))\), where \(N\) is the input size and \(\alpha\) is the smallest area among the triangles of the triangulated mass distributions.

### 4.2 Containment of exact \textsc{SC-Pizza-Sharing}

**Existence of a \textsc{SC-Pizza-Sharing} solution.** We begin by proving that a solution to exact \textsc{SC-Pizza-Sharing} always exists. This proof holds for arbitrary mass distributions, but for our algorithmic results, we will only consider the case where the mass distributions are unions of polygons with holes. Our proof is based on the proof of Karasev, Roldán-Pensado, and Soberón [KRRPS16], but they use more involved techniques from topology, which we would like to avoid since our goal is to implement the result algorithmically.

Let \(S^n\) denote the \(L_1\) sphere in \((n + 1)\)-dimensions. The Borsuk-Ulam theorem states that if \(f : S^n \to \mathbb{R}^n\) is a continuous function, then there exists a point \(\vec{P} \in S^n\) such that \(f(\vec{P}) = f(-\vec{P})\). We will show how to encode SC-paths as points in \(S^n\), and then we will build a function \(f\) that ensures that \(f(\vec{P}) = f(-\vec{P})\) only when the SC-path corresponding to \(\vec{P}\) is a solution to the \textsc{SC-Pizza-Sharing} problem.

Figure 7 (see Appendix E) gives an overview for our SC-path embedding. The embedding considers only \(y\)-monotone SC-paths. The path itself is determined by the points \(x_1, x_2, \ldots, y_1, y_2, \ldots\), which define the points at which the path turns. The path begins on the boundary on the line \(y_1\), and then moves to \(x_1\), at which points it turns and moves upwards to \(y_2\), and it then turns to move to \(x_2\), and so on.

But these points alone do not fully specify the path, since the decision of whether to move to the right or to the left when moving from \(x_i\) to \(x_{i+1}\) has not been specified. To do this, we also include signs that are affixed to each horizontal strip. We will call the two sides of the cut side A (non-shaded) and side B (shaded). The \((i+1)\)-st strip \([y_i, y_{i+1}]\) is split into two by the vertical line segment on \(x = x_i\) that starts from point \((x_i, y_i)\) and ends at \((x_{i}, y_{i+1})\), with one part being on side A, and the other being on side B. If the strip is assigned the sign \(+\) then the area on the left of the strip is on side A of the cut, while if the strip is assigned the sign \(\text{—}\) then the areas on the right of the strip is on side A of the cut. Once the sides of the cut have been decided, there is then a unique way to move through the points in order to separate sides A and B.

So, an SC-path with \(n - 1\) turns, can be represented by \(n\) variables and \([n/2]\) signs. We then embed these into the sphere \(S^n\) in the following way. We give an informal description here, and the full definition will be given in the proof of Theorem 22. We encode the \(y\) values as variables \(z_i \in [-1, 1]\) where \(y_{i+1} = y_i + |z_i|\), while the \(x\) values are encoded as values in \([-1, 1]\), where \(|x_i|\) defines the \(i\)th value on the \(x\) axis. We use the signs of the \(z_i\) variables to define the signs for.
the strips, and note that at least one of the \( z_i \)'s is non-zero since the sum of the strips' lengths should equal 1. Finally, we then shrink all variables so that their sum lies in \([-1, 1]\), and we embed them as a point in \( S^n \) using one extra dimension as a slack variable in case the sum of the absolute values of the variables does not equal 1.

The key property is that, if a point \( \bar{P} \in S^n \) represents an SC-path, then the point \(-\bar{P}\) represents exactly the same SC-path but with all signs flipped, and thus sides A and B will be exchanged. Therefore, we can write down a function \( f \), where \( f(\bar{P}) \) outputs the amount of mass of the \( i \)-th mass distribution that lies on the A side of the cut, and therefore any point \( \bar{P} \) satisfying \( f(\bar{P}) = f(-\bar{P}) \) must cut all mass distributions exactly in half.

So, we get the following theorem that is proved formally in Appendix E.

**Theorem 22** (originally by [KRPS16]). Let \( n \) be a positive integer. For any \( n \) mass distributions in \( \mathbb{R}^2 \), there is a path formed by only horizontal and vertical segments with at most \( n - 1 \) turns that splits \( \mathbb{R}^2 \) into two sets of equal size in each measure. Moreover, the path is y-monotone.

**BU-containment.** The next step is to turn this existence result into a proof that SC-Pizza-Sharing is contained in BU. We begin by recapping the definition of BU given in [DFMS21]. An arithmetic circuit is a circuit that operates on real numbers, and uses gates from the set \{\( c, +, -, \times, \max, \min \}\), where a \( c \)-gate outputs the constant \( c \), a \( \times c \)-gate multiplies the input by a constant \( c \), and all other gates behave according to their standard definitions. The class BU contains every problem that can be reduced in polynomial time to Borsuk-Ulam.

**Definition 23** (Borsuk-Ulam).
- **Input:** A continuous function \( f : \mathbb{R}^{d+1} \to \mathbb{R}^d \) presented as an arithmetic circuit.
- **Task:** Find an \( x \in S^d \) such that \( f(x) = f(-x) \).

While Theorem 22 utilizes the Borsuk-Ulam theorem, it does not directly show containment in BU, because it does not construct an arithmetic circuit. We now show how this can be done if the mass distributions are unions of polygons with holes.

The key issue is how to determine how much of a polygon lies on a particular side of the cut. To do this, we first triangulate all polygons, as shown in Figure 8a (see Appendix F.1) to obtain mass distributions that are unions of triangles. We then break each individual triangle down into a sum of axis-aligned right-angled triangles. This is shown in Figure 8b, where the triangle \( \triangle ABC \) is represented as the sum of \( \triangle XYB + \triangle XBZ - \triangle AYB - \triangle XAC - \triangle CBZ \).

We then explicitly build an arithmetic circuit that, given the point \( x \in S^{n+1} \) used in Theorem 3, and a specific axis-aligned right-angled triangle, can output the amount of mass of that triangle that lies on side A of the cut. Then the function \( f \) used in Theorem 3 can be built simply by summing over the right-angled triangles that arise from decomposing each of the polygons. So we get the following theorem, which is proved formally in Appendix F.

**Theorem 24.** Exact SC-Pizza-Sharing for weighted polygons with holes is in BU.

**ETR containment.** In the following theorem we show that the problem of deciding whether there exists an exact solution of SC-Pizza-Sharing with \( n \) mass distributions and \( k \in \mathbb{N} \) turns in the SC-path is in ETR. Consequently, this implies that the respective search problem lies in FETR, and as it is apparent from the BU containment result of this paper, when \( k \geq n - 1 \) the problem is in BU (\( \subseteq \text{TFETR} \subseteq \text{FETR} \)). To show this we use the proof of Theorem 24. The detailed proof can be found in Appendix G.
Theorem 25. Deciding whether there exists an SC-path with $k$ turns that is an exact solution for SC-Pizza-Sharing with $n$ mass distributions is in ETR.

4.3 Containment of approximate SC-Pizza-Sharing

Here we show that the problem of finding a solution to $\varepsilon$-SC-Pizza-Sharing is in PPA even for exponentially small $\varepsilon$, while deciding whether there exists a solution (an SC-path) with at most $k \in \mathbb{N}$ turns is in NP. Notice that, from Lemma 12, we directly get containment in PPA and in NP (similarly to Theorem 21) of the aforementioned problems, respectively, for any $\varepsilon$ and $\alpha$ (the smallest area in a triangle of the triangulated mass distributions) that are $\Omega(1/poly(N))$, where $N$ is the input size. That is possible by reducing $\varepsilon$-SC-Pizza-Sharing to $(\varepsilon - \varepsilon')$-Discrete-SC-Pizza-Sharing for some appropriate $\varepsilon' < \varepsilon$, and by employing our PPA containment result of the latter problem which appears later on in Theorem 30. However, by reducing our problems to $\varepsilon$-Borsuk-Ulam and its decision version we get containment in PPA and in NP, respectively, even for exponentially small $\varepsilon$.

**PPA containment.** The following theorem shows PPA containment of $\varepsilon$-SC-Pizza-Sharing via a reduction to the $\varepsilon$-Borsuk-Ulam problem which is in PPA [DFMS21].

**Definition 26 ($\varepsilon$-Borsuk-Ulam).**

- **Input:** A continuous function $f : \mathbb{R}^{d+1} \to \mathbb{R}^d$ presented as an arithmetic circuit, along with constants $\varepsilon, \lambda > 0$.
- **Task:** Find one of the following.
  1. Two points $x, y \in S^d$ such that $\|f(x) - f(y)\|_\infty > \lambda \cdot \|x - y\|_\infty$.
  2. A point $x \in S^d$ such that $\|f(x) - f(-x)\|_\infty \leq \varepsilon$.

If the first task is accomplished, then we have found witnesses $x, y \in S^d$ that function $f$ is not $\lambda$-Lipschitz continuous in the $L_\infty$-norm as required. But if the second task is accomplished then we have an approximate solution to the Borsuk-Ulam problem. To prove the theorem, we utilize Theorem 24 and the Lipschitzness of the function we have constructed in its proof. The detailed proof can be found in Appendix H.

**Theorem 27.** $\varepsilon$-SC-Pizza-Sharing for weighted polygons with holes is in PPA.

**NP containment.** Finally, we show that deciding whether there exists a solution for $\varepsilon$-SC-Pizza-Sharing with $k$ turns is in NP, for any $k \in \mathbb{N}$. We prove this via a reduction to $\varepsilon$-Borsuk-Ulam problem when $\varepsilon$-SC-Pizza-Sharing is parameterized by the number of turns the SC-path must have. Our proof combines ideas and results from the proofs of Theorem 25, Theorem 27, and Theorem 24. The detailed proof can be found in Appendix I.

**Theorem 28.** Deciding whether there exists an SC-path with $k$ turns that is a solution of $\varepsilon$-SC-Pizza-Sharing with $n$ mass distributions is in NP.

4.4 Containment of discrete SC-Pizza-Sharing

It has already been shown in [Sch21] that $\varepsilon$-Discrete-Straight-Pizza-Sharing is in PPA even for $\varepsilon = 0$. We complete the picture regarding inclusion of discrete pizza sharing problems, by showing that $\varepsilon$-Discrete-SC-Pizza-Sharing is also in PPA for $\varepsilon = 0$, and therefore, for
every \( \varepsilon \in [0, 1] \). We will reduce \( \varepsilon \)-DISCRETE-SC-PIZZA-SHARING to \( \varepsilon' \)-SC-PIZZA-SHARING for \( \varepsilon = 0 \) and \( \varepsilon' = 1/2N \), where \( N \) is the input size. Finally, as one would expect from the discrete version, its decision variant is in \( \text{NP} \) since its candidate solutions are verifiable in polynomial time.

**PPA containment.** Consider an instance of \( \varepsilon \)-DISCRETE-SC-PIZZA-SHARING with point sets \( P_1, \ldots, P_n \), and denote \( P := P_1 \cup \cdots \cup P_n \). We intend to turn each point into a mass of non-zero area. To do so, we need to first scale down the landscape of the points, to create some excess free space as a “frame” around them. It suffices to scale down by an order of 3 and center it in the middle of \([0, 1]^2\), that is, to map each point \((x, y)\) to \(\left(\frac{1}{3} + \frac{x}{3}, \frac{1}{3} + \frac{y}{3}\right)\). Now all our points are in \([1/3, 2/3]^2\). For convenience, for each \(i \in [n]\), we will be still denoting by \(P_i\) the new set of points after scaling and centering.

Now, we check whether for any pair \(i \neq j\) we have \(P_{i,j} := P_i \cap P_j \neq \emptyset\), which means that two points of two point sets have identical positions. Consider all \(P_{i,j} \neq \emptyset\) and let their union be \(P'\). Now consider all points that do not belong in \(P'\), that is \(R := P \setminus P'\). We want to find the minimum positive difference in the \(x\)- and \(y\)-coordinates between any pair of points in \(R\). Let a point of \(R\) be denoted \(p_t = (x_t, y_t)\), and let us denote

\[
x_{\min} := \min_{p_a, p_b \in R, x_a \neq x_b} |x_a - x_b|, \quad \text{and by} \quad y_{\min} := \min_{p_a, p_b \in R, y_a \neq y_b} |y_a - y_b|,
\]

and finally, \(d := \min\{x_{\min}, y_{\min}\}\).

We now turn each point of \(P_i\) into an axis-aligned square of size \(\frac{d}{3} \times \frac{d}{3} \), with its bottom-left corner having the point’s coordinates. Notice that the total area of the squares is \(|P_i| \cdot \frac{d^2}{9}\), therefore, by setting the weight of each square to \(\frac{9}{|P_i|d^2}\) we have the full description of a mass distribution \(\mu_i\). Notice that, since \(d \leq 1/3\), all of the mass distributions are in \([1/3 - 1/3 \cdot 3, 2/3 + 1/3 \cdot 3] = [2/9, 7/9]^2\).

**Lemma 29.** Any SC-path that is a solution to the resulting \(\varepsilon\)-SC-PIZZA-SHARING instance for \(\varepsilon = 1/2N\), can be turned into a solution of \(\varepsilon'\)-DISCRETE-SC-PIZZA-SHARING for \(\varepsilon' = 0\) in polynomial time.

**Proof.** If a horizontal (resp. vertical) line segment of the SC-path solution intersects two squares (of any two mass distributions), this means that their corresponding points in DISCRETE-SC-PIZZA-SHARING had the same \(y\)- (resp. \(x\)-) coordinate. To see this, without loss of generality, suppose that the two squares are intersected by the same horizontal line segment, and that their corresponding points do not have the same \(y\)-coordinate. Then, the distance between their bottom-left corners is positive but no greater than \(d/3\), which implies that \(d \leq d/3\) (by definition of \(d\)), a contradiction. A symmetric argument holds for two squares that are intersected by the same vertical line segment. In both the above cases, we return as a solution to 0-DISCRETE-SC-PIZZA-SHARING the two corresponding points of the squares, which are of the kind of “Output (a)” in Definition 11.

Let \(P_{\max} := \max_{i \in [n]} |P_i|\). Suppose \(|P_i|\) is odd for every \(i \in [n]\). Then, since we are asking for an \(1/2N\)-SC-PIZZA-SHARING solution, its SC-path cannot be non-intersecting with any of the squares, otherwise \(|\mu_i(R^+) - \mu_i(R^-)| \geq 1/|P_i| \geq 1/P_{\max} > 1/2P_{\max} \geq 1/2N\), a contradiction. Therefore, at least one square of \(P_i\) is intersected, and this holds for every \(i \in [n]\). If no line segment of the SC-path intersects two squares, we conclude that the SC-path with \(n - 1\) turns and \(n\) line segments will intersect at most \(n\) squares. But since, as discussed above, at least one square of each mass distribution has to be intersected by SC, we get that SC intersects exactly one square of each mass distribution. Each side, \(R^+\) and \(R^-\) of the SC-path includes at least \(\lfloor |P_i|/2 \rfloor\) entire squares for every \(i \in [n]\), and therefore, their bottom-left corners, i.e., the
corresponding points of Discrete-SC-Pizza-Sharing. This is a solution to the 0-Discrete-SC-Pizza-Sharing instance.

Now suppose $|P_i|$ is even for some $i \in [n]$. We can remove an arbitrary square from all mass distributions that come from point sets with even cardinality, and perform the aforementioned reduction to 1/2N-SC-Pizza-Sharing. Then, let us call “i-th segment” the one that intersects one square of $P_i$, called i-th square, and let it be a vertical segment, without loss of generality. By placing back the removed square, its bottom-left corner will be: (i) either on opposite sides with that of the i-th square, (ii) or on the same side (notice that due to the allowed discrepancy $1/2N < 1/2P_{\text{max}}$, the i-th segment cannot fall on the bottom-left corner of the i-th square). Then, in case (i) each side contains exactly $|P_i|/2$ bottom-left corners of squares (i.e., points). By scaling up the positions of SC-path’s segments (recall that we have scaled down), this is a solution to the 0-Discrete-SC-Pizza-Sharing instance. In case (ii), suppose without loss of generality that both the inserted square and the bottom-left corner of the i-th square are on the left side of the i-th segment. We modify the SC-path by shifting the i-th segment to the left such that it is now located $d/3$ to the left of i-th square’s bottom-left corner. Notice that this position is to the right of the inserted square since there is at least $2d/3$ distance between the two squares. We do the same for every $i \in [n]$ has even number of points/squares. Then, each side of the SC-path for every $i \in [n]$ has exactly $|P_i|/2$ bottom-left corners of squares which represent the initial points. After scaling up the positions of the SC-path’s segments, this is a solution to 0-Discrete-SC-Pizza-Sharing.

Finally, it is clear that the aforementioned operations can be performed in poly($N$) time. □

By the PPA containment of 1/2N-SC-Pizza-Sharing (see Theorem 27), we get the following.

**Theorem 30.** $\varepsilon$-Discrete-SC-Pizza-Sharing is in PPA for any $\varepsilon \in [0, 1]$.

**NP containment.** It is also easy to see that, by checking whether each of the points of each $P_i$ is in $R^+$ or $R^-$ as defined by a candidate SC-path solution, we can decide in polynomial time if indeed it is a solution or not to 0-Discrete-SC-Pizza-Sharing (since the points are polynomially many in $N$, by definition).

**Theorem 31.** Deciding whether there exists an SC-path solution to $\varepsilon$-Discrete-SC-Pizza-Sharing is in NP for any $\varepsilon \in [0, 1]$.

## 5 Conclusions

For $\varepsilon$-Straight-Pizza-Sharing we have shown that finding a solution is PPA-complete for any $\varepsilon \in [1/N^c, 1/5)$, where $N$ is the input size and $c > 0$ is any constant. This result holds for both its continuous (even when the input contains only axis-aligned squares) and its discrete version. We have also shown that the same result holds for $\varepsilon$-SC-Pizza-Sharing, where the PPA containment holds even for inverse exponential $\varepsilon$. One open question that remains is “Can we prove containment in PPA of $\varepsilon$-Straight-Pizza-Sharing for inverse exponential $\varepsilon$?”. For the decision variant of both these problem, we show that there exists a small constant $\varepsilon$ such that they are NP-complete. For both these problems and their search/decision variants, a big open question is “What is the largest constant $\varepsilon$ for which the problem remains PPA-hard and NP-hard, respectively?”. The most interesting open question is “are there any good algorithms that guarantee a solution in polynomial time for some constant $\varepsilon \in [1/5, 1)$, even when slightly more lines (resp. turns in an SC-path) are allowed?".
We have also shown that exact SC-Pizza-Sharing is FixP-hard and in BU. The interesting question that needs to be settled is “What is the complexity of the problem; is it complete for one of those classes or is it complete for some other class?”. Schnider in [Sch21] showed that exact Straight-Pizza-Sharing is FixP-hard for a more general type of input than ours. So, a natural question is “When the input consists of weighted polygons with holes, is the problem FixP-hard and inside BU, and if yes, is it complete for any of the two classes?”. For a strong approximation version of Consensus-Halving, [BHH21] showed that the problem is BU_a-complete. We conjecture that the same holds for the two pizza sharing problems studied here.

Another problem that remains open is the complexity of \( \varepsilon \)-Straight-Pizza-Sharing and \( \varepsilon \)-SC-Pizza-Sharing when every mass distribution consists of a constant number of non-overlapping rectangles. And finally, what is the complexity of the modified problem when instead of two parts we ask to fairly split the plane into \( d \geq 3 \) equal parts?

A Input representation

While the mathematical proof of Theorem 22 holds for general measures which are absolutely continuous with respect to the Lebesgue measure, for the computational problems Straight-Pizza-Sharing and SC-Pizza-Sharing we need a standardized way to describe the input, and therefore restrict to particular classes of measures. We consider the class of mass distributions that are defined by weighted polygons with holes. This class consists of mass distributions with the property that are succinctly represented in the input of a Turing machine.

We will use the standard representation of 2-d polygons in computational geometry problems, that is, a directed chain of points. Consider a polygon that is defined by \( k \) points \( p_i = (x_i, y_i) \), where \( x_i, y_i \in [0, 1] \cap \mathbb{Q} \), for \( i \in [k] \), which form a directed chain \( C = (p_1, \ldots, p_k) \). This chain represents a closed boundary defined by the line segments \((p_i, p_{i+1})\) for \( i \in [k-1] \) and a final one \((p_k, p_1)\). Since we consider polygons with holes, we need a way to distinguish between the polygons that define a boundary whose interior has strictly positive weight and polygons that define the boundary of the holes (whose interior has zero weight). We will call the former solid and the latter hollow polygon. To distinguish between the two, we define a solid polygon to be represented by directed line segments with counterclockwise orientation, while a hollow polygon to be represented similarly but with clockwise orientation. Furthermore, each solid polygon \( C_s \), its weight \( w \) and its \( r \geq 0 \) holes \( C_{h_1}, C_{h_2}, \ldots, C_{h_r} \) in the interior, are grouped together in the input to indicate that all these directed chains of points represent a single polygon \((w, C_s, C_{h_1}, \ldots, C_{h_r})\).

B Proof of Lemma 4

Observe that the number of big-tiles intersected by a line in the set \( \mathcal{L} \) is bounded by the number of times this line can cut \( y = x^2 \). Any straight line can cut \( y = x^2 \) at most twice, and the same holds when instead of points we have big-tiles on \( y = x^2 \) if we take care of the sparsity of the big-tiles \( s_j = \frac{6}{d} \cdot j \).

Recall that we present the instance as if it was \( \left[ 0, \left(\frac{6}{d^2} \right)^2 + 1 \right]^2 \) instead of \([0,1]^2\). But furthermore, let us do the analysis of the proof in the artificial square \([0, \frac{1}{d^2} + 1]^2\) and show how from there we go to \( \left[ 0, \left(\frac{6}{d^2} \right)^2 + 1 \right]^2 \). In particular, in the artificial square the barycenters of consecutive big-tiles have distance 1 on the \( x \)-axis, and their size is \( \frac{6}{d^2} \times \frac{d}{6} \). We will prove that
for any big-tile of size \(2r \times 2r\) with \(r \leq d/12\) there is no straight line that intersects more than two big-tiles.

For \(j \in \{1, 2, \ldots, \frac{1}{3} - 2\}\) let us consider the \(x\)-coordinates of interest \(j, j + 1, j + 2\). Consider also their corresponding points on \(y = x^2\), i.e., \(j^2, (j + 1)^2, (j + 2)^2\). For every \(j \in [1/d]\), each point \((j, j^2)\) is the barycenter of a \(2r \times 2r\) big-tile. Let us focus on the line \(\ell^j\) that passes from the top left corner of the \((j + 1)\)-st big-tile and the bottom right corner of the \((j + 2)\)-nd big-tile (see Figure 3). We also consider a direction for the line, which is the one pointing North-East, and therefore, we name the two half-planes left and right half-plane accordingly. If the \(j'\)-th tile for every \(j' \in [j]\) is to the left of \(\ell^j\) and none of them touches it, then we say that the line is nice.

Now let us consider the line \(\ell^j\) defined by two points \((x_{j+1}, y_{j+1})\) and \((x_{j+2}, y_{j+2})\). The shortest distance of a point \((x, y)\) from \(\ell^j\) is

\[
D(j, r) = \frac{2 - 6jr - 7r - r^2}{\sqrt{(1 + 2r)^2 + (2j + 3 - r)^2}},
\]

where if this value is positive then \((x, y)\) is strictly to the left of \(\ell^j\), otherwise it is to the right of \(\ell^j\). In particular, we have \((x_{j+1}, y_{j+1}) = (j + 1 - r, (j + 1)^2 + r)\) and \((x_{j+2}, y_{j+2}) = (j + 2 + r, (j + 2)^2 - r)\).

**Claim 32.** For \(j \in [\frac{1}{3} - 2]\) the point \((x, y) = (j, j^2)\) is strictly to the left of \(\ell^j\).

**Proof.** For \((x, y) = (j, j^2)\) the shortest distance is

\[
D(j, r) = \frac{2 - 6jr - 7r - r^2}{\sqrt{(1 + 2r)^2 + (2j + 3 - r)^2}}.
\]

We will now show that \(D(j, r) > r\sqrt{2}\), and therefore it does not intersect with the big-tile, since the greatest distance from a point of a big-tile to its barycenter is \(r\sqrt{2}\). By the choice of \(r \leq d/12\) we can see that \(6jr + 7r + r^2 \leq 1\) and \(2j + 3 - r > 1 + 2r\), for every \(j \in [\frac{1}{3} - 2]\). Therefore, we have

\[
D(j, r) > \frac{1}{\sqrt{2} \cdot (2j + 3 - r)^2} = \frac{1}{\sqrt{2} \cdot (2j + 3 - r)},
\]

which is at least \(r\sqrt{2}\). Therefore, the \(j\)-th big-tile is entirely to the left of line \(\ell^j\).

**Claim 33.** Let \(j \in \{4, \ldots, \frac{1}{3} - 2\}\) and \(j' \in [j]\). The shortest distance of point \((x, y) = (j', (j')^2)\) from \(\ell^j\) is decreasing with respect to \(j'\).

**Proof.** Let us denote by \(D(j, j', r)\) the distance of the statement. We have

\[
D(j, j', r) = \frac{x_{j+1}(y_{j+2} - y_{j+1}) - y_{j+1}(x_{j+2} - x_{j+1}) - j'(2j + 3 + s) + (j')^2(1 + 2r)}{\sqrt{(x_{j+2} - x_{j+1})^2 + (y_{j+2} - y_{j+1})^2}}
\]

\[
= \frac{x_{j+1}(y_{j+2} - y_{j+1}) - y_{j+1}(x_{j+2} - x_{j+1}) - j'(2j + 3 + s) - j'[(2j + 3 + s) - j'(1 + 2r)]}{\sqrt{(x_{j+2} - x_{j+1})^2 + (y_{j+2} - y_{j+1})^2}}.
\]

According to the definition of \(\ell^j\), \(x_{j+1}, y_{j+1}, x_{j+2}\) and \(y_{j+2}\) depend on \(j\) and \(r\). Keeping the aforementioned two parameters fixed, let us focus on the dependence of \(D(j, j', r)\) on \(j'\). Observe that its denominator is positive, and that only the rightmost term of the numerator depends on \(j'\). Since \((2j + 3 + s) - j'(1 + 2r) \geq 0\) for every \(j' \leq j\) when \(j \geq 4\), we deduce that \(D(j, j', r)\) decreases with \(j'\) for the required domain of \(j\).
From the latter two claims, we conclude the following.

**Corollary 34.** For \( j \in \{4, \ldots, \frac{1}{2} - 2\} \) the line \( \ell^j \) is nice.

**Claim 35.** For \( j \in \{2, 3\} \) the line \( \ell^j \) is nice.

**Proof.** For these values of \( j \), although we know from Claim 32 that the entire \( j \)-th tile is strictly to the left of \( \ell^j \), i.e., that point \((j, j^2)\) is further than \( r\sqrt{2} \) away from \( \ell^j \), we do not know whether this is true also for points \((j', (j')^2)\) for \( j' \in [j - 1] \).

For \( j = 2 \) and \( j' = 1 \), from Equation (1) for \((x, y) = (j', (j')^2)\) we get

\[
D(2, 1, r) = \frac{6 - 26r - r^2}{\sqrt{(1 + 2r)^2 + (7 - r)^2}}.
\]

By the choice of \( r = d/12 \), the numerator is strictly greater than 2, and \( 1 + 2r < 7 - r \). Therefore, \( D(2, 1, r) > \frac{2}{(7 - r)\sqrt{2}} \). Similarly to the proof of Claim 32, we require \( D(2, 1, r) > r\sqrt{2} \), which is true since \( \frac{2}{(7 - r)\sqrt{2}} \geq r\sqrt{2} \).

For \( j = 3 \) and \( j' = 2 \), from Equation (1) for \((x, y) = (j', (j')^2)\) we get

\[
D(3, 2, r) = \frac{6 - 36r - r^2}{\sqrt{(1 + 2r)^2 + (9 - r)^2}}.
\]

Similarly to the previous case, the numerator is strictly greater than 2, and \( 1 + 2r < 9 - r \). Therefore, \( D(3, 2, r) > \frac{2}{(9 - r)\sqrt{2}} \). We require \( D(3, 2, r) > r\sqrt{2} \), which is true since \( \frac{2}{(9 - r)\sqrt{2}} \geq r\sqrt{2} \).

For \( j = 3 \) and \( j' = 1 \), from Equation (1) for \((x, y) = (j', (j')^2)\) we get

\[
D(3, 1, r) = \frac{12 - 43r - r^2}{\sqrt{(1 + 2r)^2 + (9 - r)^2}}.
\]

The numerator is strictly greater than 5, and \( 9 - r > 1 + 2r \). Therefore, \( D(3, 1, r) > \frac{5}{(9 - r)\sqrt{2}} \). We require \( D(3, 1, r) > r\sqrt{2} \), which is true since \( \frac{5}{(9 - r)\sqrt{2}} \geq r\sqrt{2} \). \( \Box \)

**Claim 36.** If for some \( j \in \left[\frac{1}{2} - 2\right] \) the line \( \ell^j \) is nice, then any line that intersects the \((j + 1)\)-st and \((j + 2)\)-nd tiles is also nice.

**Proof.** Consider an arbitrary line \( \ell^k \) that intersects both the \((j + 1)\)-st and \((j + 2)\)-nd tiles. Let us denote by \( L(\ell) \) the left half-plane defined by a directed line \( \ell \), and let us denote by \( A^{j+1} \) the half-plane for which \( y \leq (j + 1)^2 - r \), i.e. the bottom half-plane defined by the lower boundary of the \((j + 1)\)-th big-tile. We can observe that the area \( L(\ell^j) \cap A^{j+1} \), always includes \( L(\ell^j) \cap A^{j+1} \). Therefore, if \( \ell^j \) is nice, in other words \( L(\ell^j) \cap A^{j+1} \) includes all of the big-tiles \( t_{j'} \) for \( j' \in [j] \), then so does \( L(\ell^k) \cap A^{j+1} \), which means that \( \ell^k \) is also nice. \( \Box \)

So far we have shown that when a line intersects two consecutive big-tiles, it cannot intersect any other big-tile below them. We will now show that when a line intersects two non-consecutive big-tiles then it also cannot intersect any other big-tile below the lowest of the two. For some \( j \in \left[\frac{1}{2} - k\right] \) consider the \((j + 1)\)-st and the \((j + k)\)-th tile, where \( k \geq 3 \). Also, consider the line \( \ell^j_k \) that intersects them, and the line \( \ell^{j+k-2} \) that touches the \((j + k - 1)\)-st and the \((j + k)\)-th big-tiles (under the standard definition of \( \ell^j \), where \( j \leftarrow j + k - 2 \)). Let us borrow the notation from the proof of Claim 36, and observe that \( L(\ell^j_k) \cap A^{j+1} \) includes \( L(\ell^{j+k-2}) \cap A^{j+1} \). Therefore, if \( \ell^{j+k-2} \) is nice, then \( L(\ell^{j+k-2}) \cap A^{j+1} \) includes all of the big-tiles \( t_{j'} \) for \( j' \in [j] \), and so does \( L(\ell^j_k) \cap A^{j+1} \), or in other words \( \ell^j_k \) is also nice.
Corollary 37. Any line that intersects two big-tiles cannot intersect any big-tile located below the lowest of the two.

Now we are ready to prove the main claim.

Claim 38. Any line can intersect at most two big-tiles.

Proof. From Corollary 34, Claim 35, Claim 36 and the above paragraph, we have deduced Corollary 37. To prove the current claim it remains to prove that any line that intersects two big-tiles cannot intersect any big-tile between the two or above the highest of the two. However, this is easy to show by just shift in perspective. In particular, for the sake of contradiction, consider three big-tiles namely the $j$-th, $(j + k)$-th and $(j + k')$-th with $1 \leq k < k'$ that are intersected by a line. But then, by considering the latter two big-tiles and applying to them Corollary 37 we see that our assumption about the line intersecting the $j$-th big-tile is contradicting the corollary. This completes the proof. □

The instance we have created is in square $[0, \frac{1}{d^2} + 1]^2$. It is now easy to re-normalize it in $\left[0, \left(\frac{6}{d^2}\right)^2 + 1\right]^2$ by inflating it such that the big-tiles have size $1 \times 1$ instead of $2r \times 2r$. This results in the barycenters of consecutive big-tiles now having distance $1/(2r) = 6/d$ instead of 1 in the $x$-axis, since we picked $r = d/12$. Therefore, the barycenters of the big-tiles for $j \in \lfloor 1/d \rfloor$ have coordinates $(s_j, s_j^2)$, where $s_j = \frac{6}{2} \cdot j$. Similarly, by further scaling the instance we can put it in $[0, 1]^2$.

We conclude that each cut can intersect at most two big-tiles, thus our available cuts will intersect at most $2(n + n^{1-\delta})$ big-tiles. Let us disregard the big-tiles that are intersected by the set of lines $\mathcal{L}$. By doing so, each cut introduces at most $2 \cdot c_{\text{max}} \cdot d$ discrepancy for each valuation $v_i$, $i \in \{2n\}$, resulting to $2(n + n^{1-\delta}) \cdot c_{\text{max}} \cdot d$ discrepancy in a total.

Now, we are ready to define the cuts for the Consensus-Halving instance $I_{\text{CH}}$. We consider the big-tiles in sequential order and we add one cut at $j \cdot d$ whenever we find two big-tiles $t_j$ and $t_{j+2}$ that belong to different regions, i.e. “+”, “−”, and vice versa. This change of region can happen at most $2(n + n^{1-\delta})$ times as argued earlier. Hence, we have at most $2(n + n^{1-\delta})$ cuts in the instance $I_{\text{CH}}$. The labels of the pieces for $I_{\text{CH}}$ follow the labels of the big-tiles of the instance $I_P$ (excluding those who contain intersected small-tiles, which are arbitrarily given the same label as that of the next big-tile).

Let us denote by $\mathcal{I}^+$ and $\mathcal{I}^-$ the regions in $I_{\text{CH}}$ corresponding (according to the above mapping) to the regions $R^+$ and $R^-$ of $I_P$ respectively, induced by $\mathcal{L}$ after disregarding the intersected big-tiles. Then for the discrepancy in the valuation of agent $i$ in $I_{\text{CH}}$ we get

$$|v_i(\mathcal{I}^+) - v_i(\mathcal{I}^-)| \leq |\mu_i(R^+) - \mu_i(R^-)| + 2(n + n^{1-\delta}) \cdot c_{\text{max}} \cdot d$$

$$\leq (\varepsilon - \varepsilon') + 2(n + n^{1-\delta}) \cdot c_{\text{max}} \cdot d$$

$$< (\varepsilon - \varepsilon') + 4n \cdot c_{\text{max}} \cdot d$$

$$\leq \varepsilon,$$

where the last inequality comes from the fact that in our reduction we can pick $d \leq \frac{\varepsilon'}{4n \cdot c_{\text{max}}}$. Recall that this is always possible since $\varepsilon' \geq 1/n'$ for some $r \geq 1$, and additionally, according to our reduction, $d$ is required to be at most $\frac{1}{4n^2 \cdot c_{\text{max}}}$. This completes the proof.

### C Proof of Lemma 12

**Pixelation.** We will start with the task of pixelating the mass distributions. Consider the input of an $\varepsilon$-SC-Pizza-Sharing or an $\varepsilon$-Straight-Pizza-Sharing instance, that is, $q \in$
\(\{n, 2n\}\) mass distributions, respectively, on \([0, 1]^2\) consisting of weighted polygons with holes (see Appendix A for details on the input representation). Let the instance’s input size be \(N \geq 2n\) (by definition). As a first step, we perform a pixelation procedure: each polygon will be turned into a union of smaller squares that will have approximately the same total area as the polygon.

As shown in Appendix F.1, it is possible to decompose a polygon into a union of disjoint non-obtuse triangles. Each of those triangles’ area is rational since it can be computed by adding and subtracting the areas of five right-angled triangles with rational coordinates, which additionally, are axis-aligned. Furthermore, the cardinality of the non-obtuse triangles is a polynomial function in the input size of the polygon’s description, that is, the coordinates of the points that define its corners and the value that defines its weight. Therefore, the exact area of any mass distribution can be computed in polynomial time.

As we have discussed earlier, in order for our approximation parameter \(\varepsilon \in [0, 1]\) to make sense, we consider normalized mass distributions, meaning that \(\mu_i([0, 1]^2) = 1\) for all \(i \in [q]\). Notice that it can be the case that some mass distributions have total area constant (in which case their weight is constant), while some others might have total area exponentially small (and therefore exponentially large weight). Therefore, in our analysis, we make sure that the “resolution” we provide to any polygon \(F\) is relative to its actual area \(\text{area}(F)\) rather than its measure \(\mu_i(F)\).

Consider some polygon \(F\) of mass distribution \(\mu_i\) for \(i \in [q]\) and let it be triangulated into non-obtuse triangles, all with non-zero Lebesgue measure (strictly positive area). We will focus on one of \(F\)’s non-obtuse triangles, \(\overline{ABC}\) (see Figure 8b) with area \(S := \text{area}(\overline{ABC}) > 0\) and perimeter \(T > 0\). Notice that since \(\overline{ABC}\) is in \([0, 1]^2\), we have \(S \leq 1\) and \(T \leq 3 \cdot \sqrt{2} < 5\).

Suppose that among all triangles of all \(\mu_i\)’s, the minimum area triangle has area \(\alpha\). The first step is to pixelate \(\overline{ABC}\). Let our pixels have size \(t \times t\) for \(t = \alpha/15N^{1+c}\), where \(c > 0\) is any fixed constant, and consider an axis-aligned square grid of pixels in our space, \([0, 1]^2\). We create a pixel for \(\overline{ABC}\) if and only if the pixel’s intersection with \(\overline{ABC}\) has non-zero Lebesgue measure. Then, the pixelated version of the triangle, denoted \(\overline{ABC}_p\), has area \(S + S'\), where \(S'\) is the excess area induced by the pixels intersected by the three sides of the triangle. By definition, \(S'\) is at least 0, and at most the area of pixels that intersect the three sides of \(\overline{ABC}\). Therefore, by denoting the number of such pixels for each side by \(n_{AB}, n_{BC}, n_{CA}\) and referring to Figure 8b, we have \(n_{AB} \leq \lceil \frac{AY}{t} \rceil + 1 + \lceil \frac{YB}{t} \rceil + 1 \leq \frac{AX}{t} + \frac{YB}{t} + 4\), and similarly for \(n_{BC}, n_{CA}\). This gives

\[
S' \leq t^2 \cdot \left( \frac{AY}{t} + \frac{YB}{t} + \frac{BZ}{t} + \frac{ZC}{t} + \frac{CX}{t} + \frac{CA}{t} + 12 \right) \\
\leq t \cdot \left( (AY + YB) + (BZ + ZC) + (CX + CA) + 12t \right) \\
\leq t \cdot (2 \cdot AB + 2 \cdot BC + 2 \cdot CA + 12t) \\
= t \cdot (2 \cdot T + 12t) \\
< 3 \cdot T \cdot t \\
\leq \frac{\alpha}{N^{1+c}} \quad \text{(since } T < 5\text{),}
\]

where the last strict inequality comes from the fact that \(t < \frac{T}{12}\). To see this, we have to first notice that \(t = \frac{\alpha}{15N^{1+c}} \leq \frac{S}{15} < \frac{S}{27}\). Then we also have to use the known formula that connects \(S\) and \(T\), namely,

\[
S = \sqrt{\frac{T}{2} \left( \frac{T}{2} - AB \right) \left( \frac{T}{2} - BC \right) \left( \frac{T}{2} - CA \right)} < \sqrt{\left( \frac{T}{2} \right)^4} = \frac{T^2}{4},
\]

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Proof. It suffices to show that 

\[
\frac{S'}{S} < \frac{\alpha}{S \cdot N^{1+c}},
\]

which gives the following.

**Claim 39.** The pixelation of \(\overline{ABC}\) results in \(\overline{ABC_p}\), where area \(\overline{ABC_p}\) < \((1 + \frac{\alpha}{S \cdot N^{1+c}}) \cdot area(\overline{ABC})\).

Consider a straight line \(\ell\) that cuts \(\overline{ABC_p}\), splitting it into two shapes \(L_p\) and \(R_p\) with areas area\((L_p)\) and area\((R_p)\), respectively. Let the corresponding two shapes that \(\ell\) creates when intersecting \(\overline{ABC}\) be \(L\) and \(R\), with areas area\((L)\) and area\((R)\), respectively. Also, let \(M\) be the set of pixels of \(\overline{ABC_p}\) that are intersected by \(\ell\), and \(M_L, M_R\) be a partition of \(M\). When clear from the context, we will slightly abuse the notation by denoting \(D\) the set of points in the union of squares on \([0,1]^2\) corresponding to pixel set \(D\).

**Claim 40.** The total area of \(M\) is at most \(\alpha/5N^{1+c}\).

**Proof.** We start from the easy observation that the length of \(\ell \cap \overline{ABC_p}\) is at most \(\sqrt{2} < 2\) since we are in \([0,1]^2\). Therefore, the number of pixels that \(\ell\) intersects is at most \(\left\lfloor \frac{\ell}{t} \right\rfloor > \frac{\ell}{t} + 1 < \frac{\ell}{t}\) (since \(t < 1\)), resulting to their area being at most \(t^2 \cdot \frac{3}{t} = \frac{\alpha}{5N^{1+c}}\). \(\square\)

**Claim 41.** For any disjoint \(M_L, M_R\) with \(M_L \cup M_R = M\), we have \(|(\text{area}(L) - \text{area}(R)) - (\text{area}(L_p \cup M_L) - \text{area}(R_p \cup M_R))| \leq 2\alpha/N^{1+c}\).

**Proof.** It suffices to show that \(0 \leq \text{area}(L_p \cup M_L) - \text{area}(L) \leq 2\alpha/N^{1+c}\). The first inequality is easy to see, since \(L \subseteq L_p \cup M_L\), which implies \(\text{area}(L) \leq \text{area}(L_p \cup M_L)\). For the second inequality, we have

\[
\text{area}(L) > \frac{\text{area}(L_p)}{1 + \frac{\alpha}{SN^{1+c}}} \quad \text{(by Claim 39)}
\]

\[
= \text{area}(L_p) \cdot \left(1 - \frac{\alpha}{SN^{1+c}}\right)
\]

\[
\geq \text{area}(L_p \cup M_L) - \text{area}(M) \cdot \left(1 - \frac{\alpha}{SN^{1+c}}\right)
\]

\[
\geq \text{area}(L_p \cup M_L) - \frac{\alpha}{5N^{1+c}} \cdot \left(\text{area}(L_p \cup M_L) - \frac{\alpha}{5N^{1+c}}\right) \quad \text{(by Claim 40)}
\]

\[
\geq \text{area}(L_p \cup M_L) - \frac{\alpha}{5N^{1+c}} \cdot (S - \frac{\alpha}{5N^{1+c}})
\]

\[
\geq \text{area}(L_p \cup M_L) - \frac{2\alpha}{N^{1+c}}.
\]

Similarly, \(0 \leq \text{area}(R_p \cup M_R) - \text{area}(R) \leq 2\alpha/N^{1+c}\), or equivalently, \(-2\alpha/N^{1+c} \leq -(\text{area}(R_p \cup M_R) - \text{area}(R)) \leq 0\). Therefore,

\[-2\alpha/N^{1+c} \leq (\text{area}(L_p \cup M_L) - \text{area}(L)) - (\text{area}(R_p \cup M_R) - \text{area}(R)) \leq 2\alpha/N^{1+c}\]

\(\square\)
Recall that, in an $\varepsilon$-Straight-Pizza-Sharing solution, at most $2n$ lines can intersect $\overline{ABC}$ and $\overline{ABC}_p$ (even though the standardized version of the problem requires at most $n$ straight lines, as we showed in Theorem 5, PPA-hardness holds even for at most $n + n^{1-\delta}$ lines for any constant $\delta > 0$). Similarly for an $\varepsilon$-SC-Pizza-Sharing solution, since its SC-path comprises of at most $n - 1$ turns, i.e., $n$ straight line segments (and again, by Theorem 8 PPA-hardness holds even for at most $n + n^{1-\delta}$ line segments for any constant $\delta > 0$) By inductively applying Claim 40 and Claim 41 $N$ times (recall that $N \geq 2n$), we get the following.

**Lemma 42.** Let at most $2n$ straight lines intersect $\overline{ABC}_p$, and the side of each pixel be $\alpha/15N^{1+c}$ for any $c > 0$. Also, let $M$ be the set of its pixels that are intersected by the lines, and $M_L, M_R$ be an arbitrary partition of $M$. Then, $\text{area}(M) \leq \alpha/5N^c$, and furthermore, $\left| \left( \text{area}(L) - \text{area}(R) \right) - (\text{area}(L_p \cup M_L) - \text{area}(R_p \cup M_R)) \right| \leq 2\alpha/N^c$, for any $c > 0$.

**Turning pixels into points in general position.** So far, for $\varepsilon$-SC-Pizza-Sharing and $\varepsilon$-Straight-Pizza-Sharing, with $q \in \{n, 2n\}$ mass distributions, respectively, we have described how to turn each distribution $\mu_i, i \in \{q\}$ into a pixelated version of it. We will now turn each of those pixels into a set of points. Let $\mu_i$ consist of $b$ many polygons, and recall that each polygon has its own weight $w_{i,j} \geq 0$, $j \in [b]$. Let $T^{i,j} \in F_i$ be a non-obtuse triangle belonging to the $j$-th polygon of $\mu_i$, and $F_i$ be the set of these triangles composing $\mu_i$. Suppose a pixel contains non-zero Lebesgue measure of triangles $\{T^{i,k}\}_{k \in D}$ for some $D \subseteq [b]$, and let us denote $w_{\text{max}}^i := \max_{k \in D} w_{i,k}$. Observe that, due to our assumption that the mass distributions are normalised, we have $1 = \sum_{T^{i,j} \in F_i} w_{i,j} \cdot \text{area}(T^{i,j}) \geq \sum_{T^{i,j} \in F_i} w_{i,j} \cdot \alpha$, therefore,

$$w_{\text{max}}^i \leq \sum_{T^{i,j} \in F_i} w_{i,j} \leq 1/\alpha.$$  

We will place $\left[ w_{\text{max}}^i \cdot N^c \right]$ points at the pixel’s bottom-left corner, that is, all having the same position. Each of the pixels of mass distribution $i$ has no weight, as desired, and they form a set $P_i$. Recall that each of the $P_i$’s we created contains at most $2N^c/t^2\alpha = 1800N^{2+3c}/a^3$ points, i.e., polynomially many in the instance’s description size and $1/\alpha$. That is because its pixels can be at most $\left[ 1/\varepsilon \right] \cdot \left[ 1/\varepsilon \right] \leq 4/t^2 = 900N^{2+2c}/a^2$, with each pixel containing at most $\left[ w_{\text{max}}^i \cdot N^c \right] \leq \frac{N^c}{\alpha^2} + 1 \leq \frac{2N^c}{\alpha^2}$ points.

Observe, however, that in the discrete version of the pizza sharing instance we created, the points of the $q \in \{n, 2n\}$ point sets lie on vertices of a square grid with edge length $t = \alpha/15N^{1+c}$. These points are not in general position, and therefore, not all solutions of that instance (a set of three points intersected by the same straight line) can be translated back to a solution of the original corresponding continuous version of the instance. That is why we need to turn this instance into one where the points in $P_1 \cup \cdots \cup P_q$ are in general position.

First, we have to slightly shift the points inside each $P_i$ which have identical positions. Notice that, by Equation (3), at most $\left[ w_{\text{max}}^i \cdot N^c \right] \leq 2N^c/\alpha$ such points have identical positions. Let these points be $p_1, \ldots, p_m$ for $m \leq 2N^c/\alpha$ such points have identical positions. Let these points be $p_1, \ldots, p_m$ for $m \leq 2N^c/\alpha$. We shift vertically each point $p_j = (x_j, y_j), j \in [m]$ to \( (x_j, y_j + (j - 1) \cdot \frac{N^c}{30N^{2+2c}}) \). It is easy to see that none of these points have the same position. Observe also, that none of those points acquired the same position as another one from $P_i$, since the closest point of $P_i$ to any of the points $p_1, \ldots, p_m$ for $m \leq 2N^c/\alpha$ has distance (in the $y$-axis) at least \( (y_j + \frac{N^c}{30N^{2+2c}}) - (y_j + m \cdot \frac{N^c}{30N^{2+2c}}) \geq \frac{N^c}{15N^{2+c}} (1 - \frac{1}{N^{2+c}}) > 0 \).

Then, we have to take care of some points of $P_i, P_j$ for $i \neq j$, which might have identical positions. To prevent such a case, we create a small gap between the position of any two point sets’ points, by shifting their grid in the $x$-direction. Formally, for each $i \in [2n]$, each point of $P_i$ with coordinates $(x_i, y_i)$ now acquires coordinates \( (x_i + (i - 1) \cdot \frac{\alpha^2}{30N^{2+2c}}, y_i) \). It is immediate
that two points that used to have identical positions now have different positions; it is also straightforward that we have preserved the uniqueness of position among the points inside each $P_i$. What remains is to check whether a shifted point of $P_i$ has the same position as a point of $P_j$.

There are two cases: if they used to have the same $y$-coordinate, then their minimum distance in the $x$-coordinate is at least 
\[
\left( x_i + \frac{\alpha}{15N+\pi} \right) - \left( x_i + (2n-1) \cdot \frac{\alpha^2}{30N^{2.5+2N}} \right) > \frac{\alpha}{15N+\pi} \left( 1 - \frac{\alpha^2}{N^{2N}} \right) > 0;
\]
if they used to have the same $x$-coordinate, then they must have had different $y$-coordinates, which remained the case after the shifting. From the shifting we performed, we get the following.

**Claim 43.** The points of $P_1 \cup \cdots \cup P_q$ can lie only on vertices of a square grid in $[0,1]^2$ with edge length $t' = \frac{\alpha^2}{30N^{2.5+2N}}$.

Now that each of the points of our instance has a unique position, we need to make sure that they are in general position, meaning that no triplet of them can be intersected by the same straight line. We will do this by further distorting the position of the points in a way, such that, even though the points will have a tiny distance from their original place (and therefore seem to remain on a square grid), they will be in general position.

To make the presentation of the analysis easier, instead of our fine grid (of Claim 43), we will show the proof of correctness of our construction for the inflated square grid $\{0, \ldots, k\} \times \{0, \ldots, k\}$ with edge length 1. Observe, that in such a grid, the only lines that intersect at least two points have a slope of the form $S = \frac{Y_2-Y_1}{X_2-X_1}$, where $(X_1,Y_1), (X_2,Y_2)$ is any pair of points. We will call those lines of interest. Since $X_1,Y_1,X_2,Y_2 \in \{0,\ldots,k\}$, we deduce that the slopes of interest are actually of the form $S = \frac{\Delta Y}{\Delta X}$, where $\Delta Y \in \{-k,\ldots,0,\ldots,k\}$, and $\Delta X \in \{-k,\ldots,-1,1,\ldots,k\}$, or it is $\infty$.

Let a line of interest $\ell$ intersect a point $(X_1,Y_1)$. If its slope is $\infty$, then the line intersects all points $(X_1,Y')$ for $Y' \in \{0,\ldots,k\}$. If its slope is the irreducible fraction $S = A/B < \infty$, we define 
\[
L := \max\{-\left\lfloor \frac{X_1}{B} \right\rfloor, -\left\lfloor \frac{Y_1}{A} \right\rfloor\} \quad \text{and} \quad \Upsilon := \min\left\{\left\lfloor \frac{k-X_1}{B} \right\rfloor, \left\lfloor \frac{k-Y_1}{A} \right\rfloor\right\}.
\]

The points intersected by $\ell$ are of the form $(X,Y)$ with $X = X_1 + j \cdot B$, $Y = X_1 + j \cdot A$, for $j \in \{L,L+1,\ldots,U\}$. Notice that the set of possible lines of interest $I$ is finite with $|I| \leq \binom{k+1}{2}$, and for each $\ell \in I$, define $R_\ell$ as the set of points that are intersected by $\ell$. Finally, we define the collection $R := \{R_\ell | \ell \in I\}$.

Consider now replacing each point in the grid, with a disk of radius $d > 0$ centered at its corresponding point’s position. Let us identify each disk by its center, that is, its initial point on the grid. For convenience, we will be calling the grid of points point-grid and the grid of disks disk-grid. Consider now a set $R_d$ of at least two disks that can be intersected by the same straight line, called distorted line of interest, and let $R'$ be the collection of all such sets (and again, $R'$ is a finite collection since $k$ is finite).

**Claim 44.** If $d \leq 1/42k^4$, for every set $R_d \in R'$ there is a set $R_\ell \in R$ such that $R_d \subseteq R_\ell$.

**Proof.** We will show that the fact that the points were replaced by disks, makes any initial line of interest $\ell$ able of changing its slope, but only slightly, so that the only disks it can intersect are still the ones in $R_\ell$. Put simply, the claim’s statement means that a line of interest $\ell$, on the point-grid, after replacing the points with disks, cannot deviate enough to intersect a disk that is not in $R_\ell$. To show this, we will compare the maximum increase (due to the disks) among all slopes of interest, with the minimum gap between any two slopes of interest in the disk-grid.

Consider a line of interest $\ell$ in the point-grid, with corresponding set $R_\ell$, and without loss of generality, set its angle to 0 radians. After replacing the points with disks, we want to give an upper bound on the range of angles that a straight line has when it is required to intersect at least two disks of $R_\ell$. The maximum angle range is achieved by lines that intersect two disks whose centers are the closest possible, namely in distance 1. For this angle range, called
\( \theta \in (0, \pi) \), we know that \( \sin \theta = \frac{d}{2d} = 2d \). This gives \( \tan \theta = \frac{\sin(\theta/2)}{\cos(\theta/2)} = -\frac{\sin(\theta/2)}{\sqrt{1-\sin^2(\theta/2)}} = \frac{2d}{\sqrt{1-4d^2}} \leq 2d \). Since \( d \leq 1/42k^4 \leq 1/6 \), therefore, \( |\tan \theta| = \frac{2d}{\sqrt{1-4d^2}} \leq \frac{6d}{1-9d^2} < 7d \).

Now we need to find the minimum positive gap between the slopes of any two lines of interest in the disk-grid. As discussed earlier, each of the lines of interest of the point-grid has slope either \( \infty \) or \( \frac{\Delta Y}{\Delta X} \) for \( \Delta Y \in \{-k, \ldots, 0, \ldots, k\} \) and \( \Delta X \in \{-k, \ldots, -1, 1, \ldots, k\} \). The former is symmetric with the case where \( \Delta Y = 0 \), so it suffices to consider the latter case. Let us consider a line of interest \( \ell \) with slope \( S \), and define similarly another line of interest \( \ell' \) with slope \( S' = \frac{\Delta Y'}{\Delta X'} \). Now consider these lines’ distorted versions on the disk-grid, and let their angles be \( \varphi \) and \( \varphi' \), respectively. After replacing the points with disks, let \( \psi > 0 \) be the smallest (in absolute value) angle between any two distorted lines of interest corresponding to \( \ell \) and \( \ell' \). Consider the following two cases:

(a) \( S = S' \): Then we have \( |\tan \psi| \geq \frac{|Y + 1 - d - (Y + d)|}{k + d - (0 - d)} \geq \frac{2d}{k + 2d} \geq 7d > |\tan \theta| \), where the last weak inequality holds for any \( d \leq \frac{1}{42k^4} \).

(b) \( S \neq S' \): Without loss of generality, let \( \varphi, \varphi' \in [0, 2\pi) \), and let \( \rho := \min\{|\varphi - \varphi'|, |\varphi - \varphi' - \pi|\} \). Then \( \psi \geq \rho - 2 \cdot \theta \), therefore, \( |\tan \psi| \geq |\tan (\rho - 2\theta)| = \frac{|\tan \rho - \tan 2\theta|}{1 + \tan \rho \cdot \tan 2\theta} \). Furthermore, we have that \( \tan \rho \leq \frac{1}{k} - 0 = \frac{1}{k} \), and that \( \tan 2\theta = \frac{2\tan \theta}{1 - \tan^2 \theta} \leq \frac{4d}{1 - 4d^2} \leq 4d \leq \frac{41}{12k^4} < 1 \). This implies that \( |\tan \psi| \geq \frac{|\tan \rho - \tan 2\theta|}{1 + \frac{1}{k}} \geq \frac{|\tan \rho - \tan 2\theta|}{2} \geq \frac{|\tan \rho|}{2} \). Also, for finite \( S := \tan \varphi, S' := \tan \varphi' \), we know that \( |\tan \varphi| \leq \frac{k}{2}, |\tan \varphi'| \leq \frac{k}{2} \), therefore \( |1 + \tan \varphi \cdot \tan \varphi'| \leq 1 + k^2 \). We also have that \( |S - S'| = |\frac{\Delta Y}{\Delta X} - \frac{\Delta Y'}{\Delta X'}| = |\frac{\Delta Y \cdot \Delta X' - \Delta Y' \cdot \Delta X}{\Delta X \cdot \Delta X'}| \geq \frac{1}{k^2} \), since \( \Delta Y \cdot \Delta X' - \Delta Y' \cdot \Delta X \) is a positive integer, and \( |\Delta X \cdot \Delta X'| \leq k \cdot k \). By definition of \( \rho \), we have \( |\tan \rho| = \frac{|\tan \varphi - \tan \varphi'|}{1 + \tan \varphi \cdot \tan \varphi'} \geq \frac{|S - S'|}{1 + k^2} \geq \frac{1}{2k^4} \).

and therefore, \( |\tan \psi| \geq \frac{1}{4k^4} \geq 7d > |\tan \theta| \), where the last weak inequality holds for any \( d \leq \frac{1}{42k^4} \).

For both of the above cases, we have shown that the maximum difference \( \theta \) in the angle of distorted lines of interest, as compared to their non-distorted version, is strictly smaller than the minimum angle \( \psi \) between any two distorted lines of interest that intersect two pairs of disks, where each pair defines a different line of interest.
Lemma 45. The skewed points are in general position.

Proof. First, observe that \( \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2} = \delta_i \sqrt{1 + r^2} \leq \frac{d\sqrt{2}}{2} < d \), therefore \((x_i, y_i)\) belongs to the disk with center \((X_i, Y_i)\). For the sake of contradiction, suppose that there exist three skewed points that can be intersected by a common line. As discussed above, the original points of those skewed points must be on the same line of interest. Let the three skewed points that can be intersected by a common line be \((x_0, y_0)\), \((x_1, y_1)\), \((x_2, y_2)\), with respective original points \((X_0, Y_0)\), \((X_1, Y_1)\), \((X_2, Y_2)\). We know that the latter points are all on the same line of interest \(\ell\). The slope of \(\ell\) can be either \(\infty\) or \(S \leq \frac{k}{t} = k\).

We define the auxiliary quantities \(W_1 := Y_1 - Y_0\), \(W_2 := Y_2 - Y_0\), \(Z_1 := X_1 - X_0\), \(Z_2 := X_2 - X_0\), \(D_1 := Z_1^2 + 2X_0Z_1 + W_1^2 + 2Y_0W_1\), \(D_2 := Z_2^2 + 2X_0Z_2 + W_2^2 + 2Y_0W_2\). Let us analyse the case where \(S\) is finite. Then \(S = \frac{W_1}{Z_1} = \frac{W_2}{Z_2} = \frac{Y_2 - Y_1}{X_2 - X_1}\). We have

\[
\begin{align*}
\frac{y_1 - y_0}{x_1 - x_0} = W_1 + r \cdot \left[(X_1^2 - X_0^2) + (Y_1^2 - Y_0^2)\right] & = \frac{W_1 + r \cdot D_1}{Z_1 + D_1} = \frac{S + r \cdot D_1}{1 + D_1/Z_1}, \quad (4) \\
\frac{y_2 - y_0}{x_2 - x_0} = W_2 + r \cdot \left[(X_2^2 - X_0^2) + (Y_2^2 - Y_0^2)\right] & = \frac{W_2 + r \cdot D_2}{Z_2 + D_2} = \frac{S + r \cdot D_2}{1 + D_2/Z_2}, \quad (5)
\end{align*}
\]

where the last equalities hold, since \(S < \infty\), and therefore, \(Z_1 \neq 0\), \(Z_2 \neq 0\). We have assumed that the three skewed points are intersected by a common line, which implies that \(\frac{y_2 - y_0}{x_2 - x_0} = \frac{y_1 - y_0}{x_1 - x_0}\).

From Equation (4), Equation (5), we get

\[
\frac{S + r \cdot D_1/Z_1}{1 + D_1/Z_1} = \frac{S + r \cdot D_2/Z_2}{1 + D_2/Z_2},
\]

which, after simplification, gives \((S - r) \cdot \frac{D_1}{Z_1} = (S - r) \cdot \frac{D_2}{Z_2}\), or equivalently, \(\frac{D_1}{Z_1} = \frac{D_2}{Z_2}\), since \(r = 1/2k\), and the smallest positive slope is \(S = 1/k\). The latter equality implies

\[
X_2 - X_1 + (Y_2 - Y_1) \cdot S = 0, \quad \text{or equivalently,} \quad 1 + S^2 = 0,
\]

a contradiction.

Now consider the case where the slope of \(\ell\) is \(\infty\). We have \(Z_1 = Z_2 = 0\), therefore, \(D_1 = W_1^2 + 2Y_0W_1\), \(D_2 = W_2^2 + 2Y_0W_2\), and similarly to the above analysis, we have

\[
\begin{align*}
\frac{y_1 - y_0}{x_1 - x_0} = W_1 + r \cdot \left[(X_1^2 - X_0^2) + (Y_1^2 - Y_0^2)\right] & = \frac{W_1 + r \cdot (W_1^2 + 2Y_0W_1)}{W_1^2 + 2Y_0W_1}, \quad (6) \\
\frac{y_2 - y_0}{x_2 - x_0} = W_2 + r \cdot \left[(X_2^2 - X_0^2) + (Y_2^2 - Y_0^2)\right] & = \frac{W_2 + r \cdot (W_2^2 + 2Y_0W_2)}{W_2^2 + 2Y_0W_2}, \quad (7)
\end{align*}
\]

We have assumed that the three skewed points are intersected by a common line, therefore, by equating the left-hand sides of Equation (6) and Equation (7), after simplification, we get \(W_1 = W_2\), or equivalently, \(Y_1 = Y_2\). Since also \(X_1 = X_2\), the points \((X_1, Y_1)\), \((X_2, Y_2)\) are identical, a contradiction. \(\Box\)

Recall now that Claim 44 and Lemma 45 were proven for a general (original) point-grid \(\{0, \ldots, k\} \times \{0, \ldots, k\}\), for any \(k \geq 1\). To use them in the context of our reduction, we need to set the proper values for \(k\) and then re-scale the entire grid so that it is inside \([0, 1]^2\). From Claim 43, we need \(k = \lceil \frac{\rho}{\Delta} \rceil = \lceil \frac{30\sqrt{2} + 2\gamma^2}{\alpha} \rceil\). Now we have a very large grid with edge length 1, so in order to be in \([0, 1]^2\) as required, consider the grid’s edge length to be \(t' = \frac{\rho^2}{30\sqrt{2} + 2\gamma^2}\). Finally, to ensure that all our auxiliary results go through, we have to also scale down \(r\) by further multiplying it with \(t'\).

We are now ready to prove Lemma 12.
Proof. Consider the input of an \(\varepsilon\text{-SC-Pizza-Sharing}\) or an \(\varepsilon\text{-Straight-Pizza-Sharing}\) instance, meaning, the description of \(q \in \{n, 2n\}\) sets, respectively, of weighted polygons with holes on \([0, 1]^2\). By definition, its size is \(N \in \Omega(n)\). In time \(\text{poly}(N)\), we perform a triangulation of each polygon into non-obtuse triangles, therefore, in total we require \(\text{poly}(N)\) time for this task. Then, we perform the “pixelation” procedure, which requires, for each of \(q\) mass distributions, checking whether each of \([225N^2+2\varepsilon/\alpha^2]\) pixels has a non-empty intersection with a triangle. This task can be performed in \(\text{poly}(N, 1/\alpha)\) time, since \(c > 0\) is a fixed constant. Next, the points of each point set \(P_i\) created from the respective pixels are shifted positively by an exponentially small value in their \(y\)- and \(x\)-coordinate, so that there is no overlap between any two points. This takes again \(\text{poly}(N, 1/\alpha)\) time. Finally, these points are skewed so that they are in general position, by adding very small values in their \(x\)- and \(y\)-coordinates, which takes once again \(\text{poly}(N, 1/\alpha)\) time. Also, we remark that the skewing procedure, results in points such that no two of them have the same \(x\)- or \(y\)-coordinate, thus Output (a) of Definition 11 cannot be produced. Finally, notice that, even though we have ensured that each of the created points’ position before skewing is on a vertex of an exponentially large grid, the number of points we have is \(\text{poly}(N, 1/\alpha)\).

We claim that the lines \(\ell_1, \ldots, \ell_m\) for some \(m \leq 2n\) that are a solution to \(\varepsilon\text{-Discrete-Straight-Pizza-Sharing}\) or the line segments of the SC-path solution of \(\varepsilon\text{-Discrete-SC-Pizza-Sharing}\) (recall that in Theorem 5 and Theorem 8 we allowed almost \(2n\) lines and line segments, respectively), are also a solution to \((\varepsilon - \varepsilon')\text{-Straight-Pizza-Sharing}\) and \((\varepsilon - \varepsilon')\text{-SC-Pizza-Sharing}\), respectively, for any \(\varepsilon, \varepsilon'\) with \(6/N^c \leq \varepsilon' < \varepsilon \leq 1\), where \(c > 0\) is any constant. What remains is to show the correctness of this statement. Notice that, after “pixelation”, we placed a set of at most \(2N^c/\alpha\) points at the bottom-left corner of the corresponding pixel. Then, we “shifted” each point up by at most distance \(2N^c/\alpha \cdot \frac{\alpha}{30N^{2+2\varepsilon/2N}} \leq \frac{\alpha}{15N^{2+2\varepsilon/2N}} \leq \frac{\alpha}{30N^{t+1+c}}\), and to the right by at most distance \(2n \cdot \frac{\alpha}{30N^{2+2\varepsilon/2N}} \leq N \cdot \frac{\alpha}{30N^{2+2\varepsilon/2N}} \leq \frac{\alpha}{60N^{1+c}}\). Finally, in the “skewing” process, we further moved each point to the right and up, but no more than distance \(c^2d \leq \frac{\alpha}{\sqrt{\alpha}^4} \leq \frac{\alpha}{\sqrt{12}} = \frac{\alpha^2}{12N^2} < \frac{\alpha}{60N^{t+1+c}}\). Therefore, each point’s total extra movement from the bottom-left of the pixel was to the right and up to a distance strictly less than \(\frac{\alpha}{30N^{t+1+c}} + \frac{\alpha}{60N^{1+c}} + \frac{\alpha}{60N^{1+c}} = \frac{\alpha}{15N^{1+c}} = t\), which means that the point remained in the pixel.

Consider now a line (resp. a line segment) \(\ell\) that is part of a solution of the \((\varepsilon - \varepsilon')\text{-Discrete-Straight-Pizza-Sharing}\) (resp. \((\varepsilon - \varepsilon')\text{-Discrete-SC-Pizza-Sharing}\)) instance, and intersects the corresponding non-obtuse triangle from mass distribution \(i \in [q]\), where \(q = 2n\) or \(q = n\), respectively. The triangle has been pixelated, and its corresponding points belonging to \(P_i\) have been created. As we discussed above, each point is inside its corresponding pixel. The line that cuts through the triangle can be thought of as intersecting a set \(M\) of the pixels of the triangle’s pixelated version. The points that correspond to the pixels of \(L_p\) are clearly in the \(L\)-part of the cut; the points that correspond to the pixels of \(R_p\) are clearly in the \(R\)-part of the cut. No matter what part of the cut the points corresponding to \(M\) join, Claim 41 and Lemma 42 apply. In the aforementioned results, notice that \(L_p \cup M_L\) and \(R_p \cup M_R\) correspond to regions that are defined by whole pixels, meaning that they are the union of pixel regions. Therefore, their respective areas are of the form \(k_L \cdot t^2\) and \(k_R \cdot t^2\), where \(k_L, k_R \in \mathbb{N}\) represent the number of pixels on each part of the cut. By construction, if our triangle at hand belongs to \(\mu_i\) and has weight \(w_{i,j}\), then each of the \(k_L\) pixels contains at least \([w_{i,j} \cdot N^c]\) points, and similarly for \(k_R\).

Suppose we have turned the \(\varepsilon\text{-Straight-Pizza-Sharing}\) (resp. \(\varepsilon\text{-SC-Pizza-Sharing}\)) to \((\varepsilon - \varepsilon')\text{-Straight-Pizza-Sharing}\) (resp. \((\varepsilon - \varepsilon')\text{-SC-Pizza-Sharing}\)) as described above, so that all points are in general position. A solution of the latter problems
always exists due to [Sch21] and Lemma 29. We are given a solution of any of the latter two problems, that is, a set of lines (resp. line segments) \( \ell_1, \ldots, \ell_m \) for \( m \leq 2n \), that partition \([0, 1]^2\) to \( R^+ \) and \( R^- \) and for every \( i \in [q], \) where \( q \in \{2n, n\} \), we have \( ||P_i \cap R^+ | - | P_i \cap R^- || \leq (\varepsilon - \varepsilon') \cdot |P_i| \).

Recall that \( T^{i,j} \in F_i \) is a non-obtuse triangle which belongs to the \( j \)-th polygon of \( \mu_i \), and \( F_i \) is the set of such triangles that compose \( \mu_i \). By \( T_p^{i,j} \) we denote the pixelated version of \( T^{i,j} \), while \( M_p^{i,j} \) are \( T_p^{i,j} \)'s respective parts of the pixels intersected by lines, that join the \( R^+ \) and the \( R^- \) sides, respectively. To bound \( |P_i| \) we will use the fact that \( \sum_{T^{i,j} \in F_i} w_{i,j} \cdot \text{area}(T^{i,j}) = 1 \), or equivalently, \( \sum_{T^{i,j} \in F_i} w_{i,j} \cdot N^c \cdot \text{area}(T^{i,j}) = N^c \). We have

\[
\sum_{T^{i,j} \in F_i} w_{i,j} \cdot N^c \cdot \text{area}(T^{i,j}) + \sum_{T^{i,j} \in F_i} w_{i,j} \cdot N^c \cdot \frac{\alpha}{N^{1+c}} \cdot \text{area}(T^{i,j}) \leq N^c + w_{i,j} \cdot N^c \cdot \frac{\alpha}{N^{1+c}},
\]

and since \( \sum_{T^{i,j} \in F_i} \text{area}(T^{i,j}) \leq 1 \), we get

\[
\sum_{T^{i,j} \in F_i} [w_{i,j} \cdot N^c] \cdot \text{area}(T^{i,j}) + \sum_{T^{i,j} \in F_i} \left[ w_{i,j} \cdot N^c \right] \cdot \frac{\alpha}{N^{1+c}} \cdot \text{area}(T^{i,j}) \leq N^c + 1 + \left( w_{i,j} \cdot N^c + 1 \right) \cdot \frac{\alpha}{N^{1+c}}.
\]

Now observe that, after pixelation, only the pixels at the boundary of each triangle \( T^{i,j} \) can correspond to \( \left[ w_{i,j} \cdot N^c \right] \) points instead of \( \left[ w_{i,j} \cdot N^c \right] \). Therefore, using the notation of Equation (2), where \( S := \text{area}(T^{i,j}) \), only at most a fraction \( S'/S \) can correspond to \( \left[ w_{i,j} \cdot N^c \right] \) points. So,

\[
|P_i| \leq \sum_{T^{i,j} \in F_i} \left[ w_{i,j} \cdot N^c \right] \cdot \frac{\text{area}(T^{i,j})}{t^2} + \sum_{T^{i,j} \in F_i} \left[ w_{i,j} \cdot N^c \right] \cdot \frac{\alpha}{N^{1+c}} \cdot \frac{\text{area}(T^{i,j})}{t^2}
\]

\[
\leq \frac{1}{t^2} \cdot \left( N^c + 1 + \left( w_{i,j} \cdot N^c + 1 \right) \cdot \frac{\alpha}{N^{1+c}} \right)
\]

\[
\leq \frac{1}{t^2} \cdot \left( N^c + 1 + \left( \frac{N^c}{\alpha} + 1 \right) \cdot \frac{\alpha}{N^{1+c}} \right)
\]

\[
\leq \frac{1}{t^2} \cdot \left( N^c + 1 + \frac{2}{N} \right)
\]

\[
\leq \frac{1}{t^2} \cdot (N^c + 3),
\]

where the second to last inequality comes from the fact that \( 1 \leq N^c/\alpha \).
Putting everything together, we have

\[
|\mu_i(R^+) - \mu_i(R^-)| = \left| \sum_{T^{i,j} \in F_i} w_{i,j} \cdot \text{area}(T^{i,j} \cap R^+) - \sum_{T^{i,j} \in F_i} w_{i,j} \cdot \text{area}(T^{i,j} \cap R^-) \right|
\]

\[
= \left| \sum_{T^{i,j} \in F_i} w_{i,j} \cdot (\text{area}(T^{i,j} \cap R^+) - \text{area}(T^{i,j} \cap R^-)) \right|
\]

\[
\leq \left| \sum_{T^{i,j} \in F_i} w_{i,j} \cdot (\text{area}(T^{i,j} \cap M^{i,j}_+) - \text{area}(T^{i,j} \cap M^{i,j}_-)) \right| + \frac{2\alpha}{N^c} \sum_{T^{i,j}_p \in F_i} w_{i,j}
\]

\[
\leq t^2 \cdot \frac{1}{N^c} \cdot \sum_{T^{i,j}_p \in F_i} |P_{i,j} \cap R^+| - \frac{1}{N^c} \cdot \sum_{T^{i,j}_p \in F_i} |P_{i,j} \cap R^-| + \frac{t^2}{\alpha} + \frac{2}{N^c}
\]

\[
\leq t^2 \cdot \frac{1}{N^c} \cdot (\varepsilon - \varepsilon')|P_i| + \frac{t^2}{\alpha} + \frac{2}{N^c}
\]

\[
\leq (\varepsilon - \varepsilon') \cdot \frac{1}{N^c} \cdot (N^c + 3) + \frac{\alpha}{225N^{2+2\varepsilon}} + \frac{2}{N^c} \quad \text{(by Equation (8))}
\]

\[
\leq (\varepsilon - \varepsilon') + \frac{3}{N^c} + \frac{\alpha}{225N^{2+2\varepsilon}} + \frac{2}{N^c}
\]

\[
\leq (\varepsilon - \varepsilon') + \frac{6}{N^c}
\]

\[
\leq \varepsilon,
\]

where the first inequality is acquired by the reverse triangle inequality in combination with Lemma 42, and the second inequality is due to the fact that the number of points in a pixel of $T^{i,j}_p$ is $\left\lceil \frac{w_{i,j}}{N^c} \right\rceil$, where $w_{i,j} \geq w_{i,j}$, by definition. \qed

### D Proof of Theorem 19

Before we prove the theorem, let us give a brief sketch of the ETR-hardness proof of the aforementioned Consensus-Halving decision version of [DFMS21]. We are given an instance of the following problem which was shown to be ETR-complete (Lemma 15 of the aforementioned paper).

**Definition 46 (Feasible\(_{(0,1)}\)).** Let $p(x_1, \ldots, x_m)$ be a polynomial. We ask whether there exists a point $(x_1, \ldots, x_m) \in [0,1]^m$ that satisfies $p(x_1, \ldots, x_m) = 0$.

Given the polynomial $p$, we first normalize it so that the sum of the absolute values of its terms is in $[0,1]$ (thus not inserting more roots), resulting to a polynomial $q$. Then, we separate the terms that have positive coefficients from those that have negative coefficients, thus creating two positive polynomials $q_1, q_2$ such that $q = q_1 - q_2$. Therefore, $p(\overline{x}) = 0$ for some $\overline{x} \in [0,1]^m$ if
and only if \( q_1(\vec{x}) = q_2(\vec{x}) \). We then represent \( q_1, q_2 \) in a circuit form with gates that implement the operations \{c, +, \times c, \times \} (where \( c \in [0, 1] \cap \mathbb{Q} \) is a constant input, and \( \times c \) is multiplication by constant). By the scaling we know that \( q_1, q_2 \in [0, 1] \), and in addition, the computation of the circuit using the aforementioned operations can be simulated by a Consensus-Halving instance with \( n - 1 \) agents, where \( n - 1 \in \text{poly}(\#\text{gates}) \); the argument of \( p \) becomes a set of “input” cuts and according to the circuit implementation by Consensus-Halving, two output cuts encode \( q_1 \) and \( q_2 \). Finally, checking whether \( q_1 = q_2 \) is true is done by an additional \( n \)-th agent that can only be satisfied (have her total valuation split in half) if and only if \( q_1(\vec{x}) = q_2(\vec{x}) \) for some \( \vec{x} \in [0, 1]^m \). In other words, the Consensus-Halving instance has a solution if and only if there are “input” cuts \( \vec{x} = (x_1, \ldots, x_m) \in [0, 1]^m \) that force the rest of the cuts (according to the circuit implementation) that encode the values \( v_{m+1}, \ldots, v_{n-1} \) at the output of each of the circuit’s gates such that they also cut the \((n)-\)th agent’s valuation in half without the need for an additional cut.

We are now ready to prove the theorem.

Proof. We will use exactly the same technique up to the point where we have a Consensus-Halving instance that checks whether \( q_1 = q_2 \). Then, we use the gadgets described in the FIXP-hardness reduction of Section 3.4 that reduce the valuation functions of \( n \) agents in Consensus-Halving into mass distributions of a SC-Pizza-Sharing instance with \( n \) colours. According to Claim 17 in any solution of the resulting SC-Pizza-Sharing instance, the SC-path does not have turns inside any unit-square. This means that each horizontal/vertical segment of the SC-path that cuts a unit square in SC-Pizza-Sharing has a 1-1 correspondence to a cut of a Consensus-Halving solution, thus a Consensus-Halving solution that uses \( n - 1 \) cuts would correspond to a SC-path with \( n - 1 \) line segments, i.e., \( n - 2 \) turns. Therefore, if and only if there is a SC-path that solves SC-Pizza-Sharing with \( n \) colours and \( n - 2 \) turns, there is a \((n-1)\)-cut that solves Consensus-Halving with \( n \) agents. Equivalently, there is a \( \vec{x} \in [0, 1]^m \) such that \( p(\vec{x}) = 0 \), making the Feasible\([0,1]\) instance satisfiable.

\[ \square \]

E Proof of Theorem 22

Consider the exact SC-Pizza-Sharing problem, whose definition is the same as Definition 3 for \( \varepsilon = 0 \). We are given \( n \) measures in \( \mathbb{R}^2 \). For ease of presentation, we consider the measures to be normalized in \([0, 1]^2\) instead of \( \mathbb{R}^2 \).

We shall first consider the case where \( n \) is even, i.e., \( n = 2m \) for some \( m \in \mathbb{N} \setminus \{0\} \). The proof for \( n = 2m - 1 \) is similar and is mentioned at the end of this case. Let us first split the unit-square into \( m + 1 \leq n \) stripes using \( m \) horizontal cuts \( 0 \leq y_1 \leq y_2 \leq \cdots \leq y_m \leq 1 \), and denote also by \( y_0 = 0 \) and \( y_{m+1} = 1 \). Then, let us cut with directed (pointing upwards) vertical lines \( x_1, x_2, \ldots, x_m \in [0, 1] \) each of the stripes, except for the bottom one, where a cut \( x_i \) starts from the horizontal segment \( y = y_i \) and ends up touching \( y = y_{i+1} \). As stated earlier, the bottom stripe \([0, y_1]\) is not cut by any vertical cut (see Figure 7 for an example).

We also define the variables \( z_1, z_2, \ldots, z_{m+1} \) where \( |z_i| = y_i - y_{i-1} \), \( i \in [m+1] \). The sign of \( z_i \), \( i \in [m+1] \) indicates the sign of the leftmost part of slice \([y_{i-1}, y_i]\) that \( x_{i-1} \) defines, and we set the whole slice \([0, y_1]\) to have the sign of \( z_1 \). Clearly, \( (z_1, z_2, \ldots, z_{m+1}) \in S^m \), since \( \sum_{i=1}^{m+1} |z_i| = 1 \), and \( (x_1, \ldots, x_m) \in [0, 1]^m \), by definition. A feasible solution of SC-Pizza-Sharing is then the vector \( (z_1, \ldots, z_{m+1}, x_1, \ldots, x_m) \); this defines a directed path that consists of horizontal and vertical line segments with at most \( 2m - 1 \) turns. We can recover this path using Algorithm 1.

Note that in the algorithm we implicitly suggest to navigate using the following trick: if we are at a horizontal part of the path, say \( y = y' \), and we hit the boundary \( x = 0 \) or \( x = 1 \) before we
reach a vertical cut, then we wrap around in the horizontal dimension and continue from point 
\((1, y')\) or \((0, y')\) respectively.

We will now define a Borsuk-Ulam function, namely a continuous function 
\(f : S^d \mapsto \mathbb{R}^d\), for a suitable dimension \(d > 0\) to be determined later. It will turn out that \(d = 2m = n\), but having it undetermined for as long as we can makes the proof transparent enough to help in the understanding of the cases where \(d \neq n\) (see Theorem 25, Theorem 28 and their proofs). Let the variables \(Z_1, \ldots, Z_m, R, X_1, \ldots, X_m\) be in the \((2m)\)-sphere \(S^{2m}\) under the \(L_1\) norm, i.e., \(\sum_{i=1}^{m} |Z_i| + |R| + \sum_{i=1}^{m} |X_i| = 2m\). Notice that now instead of a \(Z_{m+1}\) variable we have \(R\). A vector in \(S^{2m}\) maps back to a feasible solution \((z_1, \ldots, z_{m+1}, x_1, \ldots, x_m\)) of SC-PIZZA-SHARING in the following way:

\[
\forall i \in [m], \quad |z_i| = \min \left\{ \sum_{j=1}^{i} |Z_j|, 1 \right\} - \sum_{j=1}^{i-1} |z_j| = \begin{cases} |Z_i|, & \text{if } \sum_{j=1}^{i} |Z_j| < 1, \\ 1 - \sum_{j=1}^{i-1} |z_j|, & \text{otherwise,} \end{cases}
\]

\(|z_{m+1}| = 1 - \sum_{i=1}^{m} |z_i|,\)

and \(\text{sign}(z_i) = \text{sign}(Z_i), \forall i \in [m],\)

\(\text{sign}(z_{m+1}) = \text{sign}(R).\)

Also,

\(x_i = \min\{|X_i|, 1\}, \forall i \in [m].\)
Finally, we highlight that in the map-back process, \( R \) is only used to indicate the sign of variable \( z_{m+1} \), while its absolute value has no other purpose than to serve as a \textit{remainder}: it ensures that \((Z_1, \ldots, Z_m, R, X_1, \ldots, X_m) \in S^{2m}\), i.e., it is \(|R| = 2m - \sum_{i=1}^{m} |Z_i| + \sum_{i=1}^{m} |X_i|\).

For any given point \( \vec{P} = (Z_1, \ldots, Z_m, R, X_1, \ldots, X_m) \in S^{2m} \), the Borsuk-Ulam function is defined to be the total “+” (positive) measure on \([0,1]^2\) induced by \( \vec{P} \), and we denote it by \( \mu(R^+; \vec{P}) \), that is, \( f(\vec{P}) = \mu(R^+; \vec{P}) \). The total positive measure is a continuous function of the variables: \( f \) is a continuous function of \((z_1, \ldots, z_{m+1}, x_1, \ldots, x_m)\), by the interpretation of the variables on \([0,1]^2\); and the mapping we defined from \( \vec{P} \) to \((z_1, \ldots, z_{m+1}, x_1, \ldots, x_m)\) is also continuous. To see the latter, note that for fixed signs of the variables of \( f \), \( \vec{P} \) is a map to \( R \times \mathbb{R} \), and hence the mapping is continuous.

By the Borsuk-Ulam theorem, there exist two antipodal points \( \vec{P}^*, -\vec{P}^* \in S^{2m} \) such that \( f(\vec{P}^*) = f(-\vec{P}^*) \). Notice that \( f(-\vec{P}) = \mu(R^-; \vec{P}) \), since by flipping the signs of the variables of \( \vec{P} \), we consider the “−” (negative) measure of \([0,1]^2\) induced by \( \vec{P} \). Therefore, when \( f(\vec{P}^*) = f(-\vec{P}^*) \) we will have \( \mu(R^+; \vec{P}^*) = \mu(R^-; \vec{P}^*) \), that is, in each of the \( 2m \) measures, the total positive measure equals the negative one. By mapping back \( \vec{P}^* \in S^{2m} \) to the corresponding point \( \vec{P} = (z_1, \ldots, z_{m+1}, x_1, \ldots, x_m) \in S^m \times [0,1]^m \), we get a solution to SC-PIZZA-SHARING. The total number of turns of the directed path is \( 2m - 1 = n - 1 \).

The proof is similar when \( n = 2m - 1 \). The horizontal cuts are again \( 0 \leq y_1 \leq y_2 \leq \cdots \leq y_m \leq 1 \), but the vertical cuts are \( x_1, x_2, \ldots, x_{m-1} \in [0,1] \), meaning that the top slice is not vertically cut. Also, it is easy to see that one could consider the path to be again \( \mu \)-monotone but in the opposite direction, meaning that there is no line segment pointing upwards.

Remark 47. \textit{Note that a symmetric proof exists, where the slices are vertical instead of horizontal, and the cuts within the slices are horizontal instead of vertical. The analysis is similar to the one we give here, and it guarantees the existence of an SC-path which is allowed to wrap around in the vertical dimension, it bisects all \( n \) measures and is \( x \)-monotone with no line segment heading left (or right).}

\section{F Proof of Theorem 24}

Consider an arbitrary instance of exact SC-PIZZA-SHARING, i.e., the one of Definition 3, where we additionally restrict the mass distributions to be weighted polygons (with holes). For ease of presentation, we will give distinct colours to the mass distributions \( \mu_i \), \( i \in [n] \) and call them \textit{colours} 1, 2, \ldots, \( n \). We are given a set of polygons with holes in the input form described in Appendix A. We will first do a preprocessing of the input: (a) normalization in \([0,1]^2\), and (b) triangulation. The former is needed in order to simulate the space of the mathematical existence proof of Theorem 22, while the latter allows us to construct the Borsuk-Ulam function of the aforementioned proof via circuits. For the computation of the Borsuk-Ulam function described in Appendix E we need a way of computing the “positive” measure of a polygon, as dictated by a given feasible solution \( \vec{P} \). To simplify this computation, we will triangulate further the polygon to end up with only non-obtuse triangles, which can be decomposed into axis-aligned right-angled triangles (Appendix F.1).

For task (a) we first check among all polygons of all colours, what the smallest and greatest coordinates of their vertices are, which defines a rectangle that includes all our polygons. Then, we find the square with smallest sides within which the aforementioned rectangle can be
Algorithm 1 Mapping labelled cuts to SC-paths

Input: A vector \((z_1, \ldots, z_{m+1}, x_1, \ldots, x_m)\).

Output: A feasible solution of SC-Pizza-Sharing (i.e., a SC-path).

1: Find the set \(T = \{t_1 \ldots, t_r\} \subseteq \{1, 2, \ldots, m + 1\}\) of all indices of \(z_1, \ldots, z_{m+1}\), where \(t_1 < \cdots < t_r\) and for each \(\ell \in [r]\) it holds that \(z_{t_\ell} \neq 0\).
2: if \(z_{t_1} \cdot z_{t_2} < 0\) then
3: starting point of the path is \((0, y_{t_1})\)
4: end if
5: if \(z_{t_1} \cdot z_{t_2} \geq 0\) then
6: starting point of the path is \((1, y_{t_1})\)
7: end if
8: \(i \leftarrow 2\)
9: \(i' \leftarrow 3\)
10: while \(i' \leq r\) do
11: if \(z_{t_i} \cdot z_{t_i'} \cdot (x_{t_i'-1} - x_{t_i-1}) > 0\) then
12: Go right (\(\rightarrow\))
13: end if
14: if \(z_{t_i} \cdot z_{t_i'} \cdot (x_{t_i'-1} - x_{t_i-1}) < 0\) then
15: Go left (\(\leftarrow\))
16: end if
17: if \(z_{t_i} \cdot z_{t_i'} > 0\) and \(x_{t_i'-1} - x_{t_i-1} = 0\) then
18: Go up (\(\uparrow\))
19: end if
20: if \(z_{t_i} \cdot z_{t_i'} < 0\) and \(x_{t_i'-1} - x_{t_i-1} = 0\) then
21: Go left (\(\leftarrow\))
22: end if
23: \(i \leftarrow i + 1\)
24: \(i' \leftarrow i' + 1\)
25: end while

inscribed. Finally, by shifting the rectangle so that its bottom left vertex is at \((0, 0)\), and then scaling it down so that each side is of length 1, we get a normalized input where each polygon is in \([0, 1]^2\) and their relative positions and areas are preserved. Task (b) is easily taken care of via already known algorithms (e.g., [GJPT78, AAP86, Meh84, GM91]) that triangulate polygons with holes in \(O(n \log n)\) time (which is optimal) without inserting additional vertices.

F.1 Computing areas of polygons via axis-aligned right-angled triangle decomposition

Here we first show how an arbitrary triangle can be decomposed into right-angled triangles whose right angle is additionally axis-aligned. Then, by using only the allowed gates of BU, we present a way to construct a Borsuk-Ulam function. We show that any solution of BORSUK-ULAM whose input is the aforementioned function can be mapped back in polynomial time to an exact SC-Pizza-Sharing solution.

By the definition of the Borsuk-Ulam function of Section 4, it is apparent that we need to be able to compute parts of the area of a polygon, depending on where square-cuts fall. The function we provide can compute parts of the area of axis-aligned right-angled triangles. For that reason, after a standard triangulation (e.g., using the technique of [AAP86]) we further
preprocess it and decompose each obtuse triangle into two right-angled triangles, by adding an extra line segment.

In particular, we check the obtuseness of a triangle $\triangle ABC$ by computing the squared lengths of its sides $AB^2, BC^2, AC^2$ (each is rational; a sum of squares of rationals), taking the largest one, w.l.o.g. $AC^2$ and then checking whether $AB^2 + BC^2 < AC^2$. If the inequality is not true then $\triangle ABC$ is non-obtuse and we proceed. Otherwise, we add the line segment $BD$ that starts from $B$ and ends at $D$ on side $AC$, where $\angle BDC = \angle ADB = 90^\circ$\textsuperscript{3}. The coordinates $(x_D, y_D)$ of $D$ are rationals since they are the solution of the following two equations: (a) one that dictates that $D$ is on $AC$: 

$$
\frac{y_D - y_A}{x_D - x_A} = \frac{y_A - y_C}{x_A - x_C},
$$

and (b) one that captures the fact that $BD$ and $AC$ are perpendicular:

$$
\frac{y_B - y_D}{x_B - x_D} \cdot \frac{y_A - y_C}{x_A - x_C} = -1.
$$

In fact,

$$
x_D = \frac{\text{Num}}{\text{Den}}, \quad \text{and} \quad y_D = \frac{y_A - y_C}{x_A - x_C} (x_D - x_A) + y_A,
$$

where \text{Num} = (y_A - y_D)[(y_B - y_A)(x_A - x_C) + (y_A - y_C)x_C] + (y_A - y_C)(x_A - x_C)^2,

and \text{Den} = (x_A - x_C)(x_A - x_C + y_A - y_C).

At this point the triangulation of each polygon consists of non-obtuse triangles. The following proposition makes the computation of areas of these triangles’ parts possible via a decomposition into axis-aligned right-angled triangles.

**Proposition 48.** The area of any non-obtuse triangle can be computed by the areas of five axis-aligned right-angled triangles.

**Proof.** A proof by picture is presented in Figure 8b, where we draw a segment from the top-left corner to the bottom-right one, and $\overrightarrow{XYB} + \overrightarrow{XZB} - \overrightarrow{AYB} - \overrightarrow{AXC} - \overrightarrow{CBZ}$.

![Figure 8](image)

(a) The triangulation with only non-obtuse triangles. After the standard triangulation, extra line segments (in red colour) are added to ensure non-obtuseness.

(b) A non-obtuse triangle $\triangle ABC$ and its decomposition into axis-aligned right-angled triangles: $\overrightarrow{XYB} + \overrightarrow{XZB} - \overrightarrow{AYB} - \overrightarrow{AXC} - \overrightarrow{CBZ}$.

\textsuperscript{3}We will denote by $\triangle ABC$ a triangle with vertices $A, B, C$ and, when clear from context, we will also use the same notation to indicate the area of the triangle. Two intersecting line segments $AB, BC$ define two angles, denoted $\angle ABC$ and $\angle CBA$. The order of the vertices implies a direction of the segments, i.e., in the former angle we have $AB, BC$ and in the latter we have $CB, BA$. We consider the direction of the segments and define the angle to be the intersection of the left halfspaces of the segments. Therefore $\angle ABC = 360^\circ - \angle CBA$. This order will not matter if clear from context (e.g., in triangles).
The proof is immediate if we show that every non-obtuse triangle $\triangle ABC$ can be tightly inscribed inside a rectangle, meaning that all of its vertices touch the rectangle’s perimeter. In particular, the rectangle has coordinates

$$(\min\{x_A, x_B, x_C\}, \min\{y_A, y_B, y_C\}), \quad (\max\{x_A, x_B, x_C\}, \min\{y_A, y_B, y_C\}),$$

$$(\max\{x_A, x_B, x_C\}, \max\{y_A, y_B, y_C\}), \quad \text{and} \quad (\min\{x_A, x_B, x_C\}, \max\{y_A, y_B, y_C\}).$$

First, we argue that one of $\triangle ABC$’s vertices is on a corner of the rectangle. W.l.o.g. suppose $\arg\min\{x_A, x_B, x_C\} = A$. If $\arg\min\{y_A, y_B, y_C\} = A$ or $\arg\max\{y_A, y_B, y_C\} = A$ then $A$ is the bottom-left or top-left corner of the rectangle, respectively. Otherwise, suppose w.l.o.g. that $\arg\min\{y_A, y_B, y_C\} = B$. If $\arg\max\{x_A, x_B, x_C\} = B$ then $B$ is the bottom-right corner of the rectangle. Otherwise, $\arg\max\{x_A, x_B, x_C\} = C$. Then, if $\arg\max\{y_A, y_B, y_C\} = C$, $C$ is the top-right corner of the rectangle. Otherwise, $\arg\max\{y_A, y_B, y_C\} = A$ and $A$ is the top-left corner of the rectangle. We conclude that a vertex of the triangle, w.l.o.g. $A$, is on a corner of the rectangle.

Now we need to show that $B$ and $C$ lie on the perimeter of the rectangle. By the definition of the aforementioned rectangle’s vertices, it cannot be that both $C$ and $D$ are not on the perimeter; if, for example, $A$ is the top-right corner then $\arg\min\{y_A, y_B, y_C\} \in \{B, C\}$, therefore vertex $\arg\min\{y_A, y_B, y_C\}$ touches the lower side of the rectangle, and similarly if $A$ is any of the other corners. Now, for the sake of contradiction, suppose that the remaining vertex, w.l.o.g. $C$ does not touch the perimeter of the rectangle. Then, by definition of the rectangle’s vertices, $B$ is on another corner. If $A$ and $B$ are on the same side of the rectangle, then it is clear by the definition of the rectangle’s coordinates that $C$ must be on the perimeter of it. Otherwise $A$ and $B$ are diagonal corners of the rectangle. Then, $AB$ is the largest side of the triangle and if $C$ is not on the boundary, it holds that $AC^2 + BC^2 < AB^2$, meaning that $\triangle ABC$ is obtuse, a contradiction. Therefore, any non-obtuse triangle can be tightly inscribed in a rectangle. $\square$

### F.2 Constructing the Borsuk-Ulam function

Here we show how to construct the Borsuk-Ulam function $f : S^n \to \mathbb{R}^n$ given $n$ sets of weighted polygons. We will focus on an arbitrary colour $i \in [n]$ and present the coordinate $f_i$.

Consider the $\tau \geq 1$ weighted polygons of the $i$-th colour, and let us focus on a particular polygon $t \in [\tau]$. We have triangulated the polygon into $m_t$ non-obtuse triangles. Consider one such triangle $j \in [m_t]$ and the virtual triangles $T^j_1, T^j_2, T^j_3, T^j_4, T^j_5$, which are the corresponding five axis-aligned right-angled triangles described in the proof of Proposition 48, also see Figure 8b). W.l.o.g. we consider $T^j_1$ and $T^j_2$ to be the positively contributing triangles and the rest to be the negatively contributing triangles. For each of them we will be computing the positive measure that the cut $\overrightarrow{p}$ of SC-PIZZA-SHARING defines (see Appendix E). By the axis-aligned right-angled triangle decomposition described in the proof of Proposition 48, it suffices to show how to compute parts of areas of such a triangle, for all of its four possible orientations: $Q_I, Q_{III}, Q_{III}, Q_{IV}$, where $Q_o$ is the orientation when, by shifting the triangle so that the vertex of the right angle is on $(0,0)$, the whole triangle is in the $o$-th quadrant.

First, we test the orientation of our triangle. Observe that for $T^j_1$ and $T^j_2$ we know that the orientation is $Q_I$ and $Q_{III}$, respectively (see $XYB$ and $XZB$ in Figure 8b). For $T^j_3, T^j_4, T^j_5$, we check the coordinates and identify what kind of triangles they are.

For a fixed colour $i \in [n]$, for each possible category $Q_o, o \in \{I, III, IIII, IV\}$ we show how to compute the term that an axis-aligned right-angled triangle $T^j_o = \overrightarrow{ABC}$ (as in Figure 8b), $j \in [m], r \in [5]$ contributes to the Borsuk-Ulam function $f(\overrightarrow{p})_i$. Let us denote by $g_{j,r}(x)$ the linear function of the line segment of $\overrightarrow{ABC}$’s hypotenuse (with respect to the horizontal axis $x$),
and note that it is easy to derive from the segment’s end-points. We also denote by \((x_{j,r}, y_{j,r})\) and \((x'_{j,r}, y'_{j,r})\) the bottom and top points that define the hypotenuse of \(ABC\). The vertex of the right angle is defined by these four coordinates depending on \(Q_o\).

For any given point (feasible solution) \(p\) that defines a SC-path of SC-Pizza-Sharing (see Appendix E), recall that there are \([n/2]\) slices \(0 = y_0 \leq y_1 \leq \cdots \leq y_{[n/2]} \leq y_{[n/2]+1} = 1\) that determine \([n/2] + 1\) slices \([z_1, \ldots, z_{[n/2]+1}]\) that partition \([0, 1]^2\). One can see that, given the \(z_s’s, s \in \{[n/2]+1\}\) from \(p\), the \(y_s’s\) can be computed by the recursive expression \(y_s = |z_s - y_{s-1}|.\)

We first define the following auxiliary functions \(A'_j(z_s), B'_j(-z_s)\) for each \(Q_o\) and \(r \in [5]:\)

\(Q_1:\)

\[
A'_j(z_s) := \frac{1}{2} \cdot \left[ \min\{\max\{x_{j,r} - x_{s-1}, 0\}, g_{j,r}^{-1}(y_{s-1}) - x'_{j,r}\} + \\
\min\{\max\{g_{j,r}^{-1}(y_{s-1} + \max\{-z_s, 0\}) - x_{s-1}, 0\}, g_{j,r}^{-1}(y_{s-1} + \max\{-z_s, 0\}) - x'_{j,r}\}\right].
\]

\[
B'_j(-z_s) := \frac{1}{2} \cdot \left[ (x'_{j,r} - x_{j,r}) + \max\{x'_{j,r} - g_{j,r}^{-1}(y_{s-1} + \max\{-z_s, 0\}), 0\}\right] - \\
\min\{y'_{j,r} - y_{j,r}, \max\{y_{s-1} - y_{j,r}, 0\} + \max\{-z_s, 0\}\}.
\]

\(Q_2:\)

\[
A'_j(z_s) := \frac{1}{2} \cdot \left[ \min\{\max\{x_{j,r} - x_{s-1}, 0\}, x'_{j,r} - g_{j,r}^{-1}(y_{s-1})\} + \\
\min\{\max\{x_{s-1} - g_{j,r}^{-1}(y_{s-1} + \max\{z_s, 0\}), 0\}, x'_{j,r} - g_{j,r}^{-1}(y_{s-1} + \max\{z_s, 0\})\}\right].
\]

\[
B'_j(-z_s) := \frac{1}{2} \cdot \left[ (x'_{j,r} - x_{j,r}) + \max\{x'_{j,r} - g_{j,r}^{-1}(y_{s-1} + \max\{-z_s, 0\}), 0\}\right] - \\
\min\{y'_{j,r} - y_{j,r}, \max\{y_{s-1} - y_{j,r}, 0\} + \max\{-z_s, 0\}\}.
\]

\(Q_3:\)

\[
A'_j(z_s) := \frac{1}{2} \cdot \left[ \min\{\max\{x_{j,r} - x_{j,r}, 0\}, x_{j,r} - g_{j,r}^{-1}(1 - y_{s-1})\} + \\
\min\{\max\{x_{s-1} - g_{j,r}^{-1}((1 - y_{s-1}) + \max\{z_s, 0\}), 0\}, x_{j,r} - g_{j,r}^{-1}((1 - y_{s-1}) + \max\{z_s, 0\})\}\right].
\]

\[
B'_j(-z_s) := \frac{1}{2} \cdot \left[ (x'_{j,r} - x_{j,r}) + \max\{x_{j,r} - g_{j,r}^{-1}(y_{s-1} + \max\{-z_s, 0\}), 0\}\right] - \\
\min\{y'_{j,r} - y_{j,r}, \max\{y_{j,r} - y_{s-1}, 0\} + \max\{-z_s, 0\}\}.
\]

\(Q_4:\)

\[
A'_j(z_s) := \frac{1}{2} \cdot \left[ \min\{\max\{x_{j,r} - x_{j,r}, 0\}, g_{j,r}^{-1}(1 - y_{s-1}) - x_{j,r}\} + \\
\min\{\max\{g_{j,r}^{-1}((1 - y_{s-1}) - x_{s-1} + \max\{-z_s, 0\}), 0\}, g_{j,r}^{-1}((1 - y_{s-1}) - x_{j,r} + \max\{-z_s, 0\})\}\right].
\]

\[
B'_j(-z_s) := \frac{1}{2} \cdot \left[ (x'_{j,r} - x_{j,r}) + \max\{g_{j,r}^{-1}(y_{s-1} + \max\{z_s, 0\}) - x_{j,r}, 0\}\right] - \\
\min\{y'_{j,r} - y_{j,r}, \max\{y_{j,r} - y_{s-1}, 0\} + \max\{z_s, 0\}\}.
\]
For an example of case $Q_{II}$ see Figure 9.

Then, we get the total area that all slices up to $z_s$ (i.e., $[0, y_s]$) define together with $x_s$ from

$$D_j^s(s, z_s) := A_j^s(z_s) + (B_j^s(-z_s) - A_j^s(-z_s)),$$

which has the property that if $z_s > 0$ (resp. $z_s < 0$), then the computed area considers the thickness $|z_s|$ only in the part of the slice that is left (resp. right) to $x_s$. For $z_s = 0$, $|z_s|$ has no thickness, its measure is 0, and consistently vanishes from the above functions.

Then, to isolate the part that only slice $z_s$ contributes to the positive measure, for $r \in [5]$ (see definition of virtual triangles $T_j^r$ above), we define

$$d_j^r(s) := D_j^r(s, z_s) - D_j^r(s, 0).$$

Consequently, the positive measure that an (unweighted) non-obtuse triangle $j$ contributes to the Borsuk-Ulam function according to the SC-path $\overrightarrow{p}$ is

$$q_j := \sum_{s=1}^{[n/2]+1} \left( d_j^1(s) + d_j^2(s) - d_j^3(s) - d_j^4(s) - d_j^5(s) \right).$$

Finally, given that colour $i \in [n]$ has $\tau$ many weighted polygons, each of weight $w_t$, $t \in [\tau]$, which has been decomposed into $m_t$ many non-obtuse triangles, $i$’s positive measure (i.e., the $i$-th coordinate of the Borsuk-Ulam function) is

$$f(\overrightarrow{p})_i = \sum_{t=1}^{\tau} w_t \sum_{j=1}^{m_t} q_j.$$

G Proof of Theorem 25

The proof is by showing that the aforementioned problem can be formulated as an ETR problem. We will use the machinery of the BU containment proof (Theorem 24) to build the piece-wise polynomial function $f$. In the proof of the aforementioned theorem we construct the function $f : S^n \rightarrow R^n$ so that it computes the "positive" part of measure $i \in [n]$ in its $i$-th coordinate, and
we require (at most) \( n - 1 \) turns in the SC-path. It is easy to modify this construction for any given number \( k \) of turns: by the proof of Theorem 22, what we need to do is to cut \( \lceil (k + 1)/2 \rceil + 1 \) slices (i.e., place \( \lceil (k + 1)/2 \rceil \) horizontal cuts) in \([0,1]^2\) and another \( k + 1 - \lceil (k + 1)/2 \rceil \) vertical cuts on them starting with the second slice from the bottom. Then, by the same mapping as in the aforementioned proof, we have a function \( f : S^{k+1} \to R^n \) for which, if \( f(\vec{P}^*) = f(-\vec{P}^*) \) for some \( \vec{P}^* \) we have a solution to exact SC-PIZZA-SHARING with \( k \) turns in its SC-path. One can see that the SC-PIZZA-SHARING problems of Theorem 22 and Theorem 24 were special cases for \( k \geq n - 1 \), for which the aforementioned equality is guaranteed by the Borsuk-Ulam theorem.

The requirement \( f(\vec{P}) = f(-\vec{P}) \) can be implemented as an ETR formula, that is, it can be written in the form: \( \exists \vec{P} \in \mathbb{R}^m \cdot \Phi \), where \( \Phi \) is a Boolean formula using connectives \( \{\land, \lor, \neg\} \) over polynomials with domain \( \mathbb{R}^m \) for some \( m \in \mathbb{N} \) compared with the operators \( \{<, \leq, =, \geq, >\} \). Recall from Appendix F.2 that function \( f \) is constructed using the operations \( \{c, +, -, \times c, \times, \max, \min\} \), since also the \( | \cdot | \) operation can be implemented as: \( |t| := \max\{t,0\} + \max\{-t,0\} \). In \( f \), we only need to replace max\{\( y, z \)\} and min\{\( y, z \)\} occurrence with a new variable \( g_{\text{max}} \) and \( g_{\text{min}} \) respectively and include the following constraints \( C_{\text{max}}, C_{\text{min}} \) in formula \( \Phi \):

\[
C_{\text{max}} = ( (g_{\text{max}} = y) \land (y \geq z) ) \lor ( (g_{\text{max}} = z) \land (z > y) ),
\]

\[
C_{\text{min}} = ( (g_{\text{min}} = y) \land (y \leq z) ) \lor ( (g_{\text{min}} = z) \land (z < y) ).
\]

Let us denote by \( \vec{g} \in \mathbb{R}^{m'} \) all the variables needed to replace the \( m' \in \mathbb{N} \) occurrences of max and min operations in \( f \). Also, denote by \( C \) the conjunction of all the aforementioned \( C_{\text{max}}, C_{\text{min}} \) constraints. Then our ETR formula is:

\[
\exists \vec{P}, \vec{g} \in \mathbb{R}^{m+m'}. \left( \sum_{j=1}^{k+2} |P_j| = k + 1 \right) \land C \land \left( \bigwedge_{i=1}^{n} f(\vec{P})_i = f(-\vec{P})_i \right).
\]

This completes the proof.

H Proof of Theorem 27

The proof is straight-forward with the help of the proof of Theorem 24. Given the \( \varepsilon \)-SC-PIZZA-SHARING instance with \( n \) mass distributions, we just have to construct the Borsuk-Ulam function \( f \) with a circuit as in the aforementioned proof (a task implementable in polynomial time). The function, as the proof shows, is continuous piece-wise polynomial, and therefore it is \( \lambda \)-Lipschitz continuous for \( \lambda = \max_{j=1}^{n+1} \left\{ \sup_{x} \left\| \frac{\partial f(x)}{\partial z_j} \right\|_{\infty} \right\} \) (and note that points where \( f_i(x), i \in [n] \) is non-differentiable do not matter for Lipschitzness). By the definition of \( f \) (proof of Theorem 24) one can see that \( \lambda \) is constant: the polynomial pieces of \( f \) are of degree at most 2, and the partial derivative of each \( f_i \) is defined by the rational points given in the input that define the polygons. By formulating it as an \( \varepsilon \)-BORSUK-Ulam instance, any solution of that instance corresponds to a solution of \( \varepsilon \)-SC-PIZZA-SHARING, since \( f_i(x) \) and \( f_i(-x) \) capture the measures \( \mu_i(\mathbb{R}^+) \) and \( \mu_i(\mathbb{R}^-) \), respectively in the definition of \( \varepsilon \)-SC-PIZZA-SHARING.

I Proof of Theorem 28

The problem stated in the theorem is reduced to the respective version of \( \varepsilon \)-BORSUK-Ulam parameterized by \( k \) in the same way as it was done in the proof of Theorem 25. Recall, \( k \) is the exact number of turns the SC-path should have. Suppose now we are given an instance as
described in the statement of Theorem 28. Using exactly the same arguments as in the proof of Theorem 25, we can construct the Borsuk-Ulam function $f : S^{k+1} \to \mathbb{R}^n$ in polynomial time.

What remains to be shown in order to prove inclusion in $NP$ is that if there is a solution to the $\varepsilon$-SC-PIZZA-SHARING instance with $k$ turns in the SC-path, then there exists a rational solution with bounded bit length.

As shown in the proof of Theorem 27, the function $f$ we use (proof of Theorem 24) is continuous, piece-wise polynomial, and therefore $\lambda$-Lipschitz continuous, where $\lambda = \max_{j=1}^{n+1} \left\{ \sup_{x} \left\| \frac{\partial f(x)}{\partial x_j} \right\|_{\infty} \right\}$ (again, note that $\lambda$ is constant, and points where $f_i(x), i \in [n]$ is non-differentiable do not matter for Lipschitzness). Suppose that a point $x \in S^{k+1}$ satisfies $\|f(x) - f(-x)\|_{\infty} \leq \varepsilon$ for a given $\varepsilon > 0$, and therefore it is a solution to the decision version of $\varepsilon$-BORSUK-ULAM when parameterized by the number of turns $k$. Then, the following inequalities hold for any other point $y \in S^{k+1}$:

$$\|f(x) - f(-x)\|_{\infty} \leq \varepsilon,$$

$$\|f(x) - f(y)\|_{\infty} \leq \lambda \cdot \|x - y\|_{\infty},$$

$$\|f(-x) - f(-y)\|_{\infty} \leq \lambda \cdot \|x - y\|_{\infty}.$$

By the triangle inequality we get

$$\|f(y) - f(-y)\|_{\infty} \leq 2 \cdot \lambda \cdot \|x - y\|_{\infty} + \varepsilon,$$

for any $y \in S^{k+1}$. \hspace{1cm} (9)

Consider now a number $M$ that is upper-bounded by an inverse-polynomial of the input size. Then, for any $y$ such that $\|x - y\|_{\infty} \leq \frac{M}{\lambda}$ we have from Equation (9) that

$$\|f(y) - f(-y)\|_{\infty} \leq M + \varepsilon,$$

meaning that $y$ is also a solution to $\varepsilon'$-BORSUK-ULAM parameterized by $k$, for $\varepsilon' = M + \varepsilon$. Therefore, $y$ is a solution to an instance of the same problem when additively relaxed by at most an inverse polynomial (in the input size) quantity.

By the above, we conclude that there are always polynomial-size solutions to the problem (and thus to the initial $\varepsilon$-SC-PIZZA-SHARING problem) given that there are exact solutions to it. Therefore, a candidate solution can be verified in polynomial time.

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