A SUBGROUP THEOREM FOR HOMOLOGICAL FILLING FUNCTIONS

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ABSTRACT. We use algebraic techniques to study homological filling functions of groups and their subgroups. If \( G \) is a group admitting a finite \((n+1)\)-dimensional \( K(G,1) \) and \( H \leq G \) is of type \( F_{n+1} \), then the \( n \)th-homological filling function of \( H \) is bounded above by that of \( G \). This contrast with known examples where such inequality does not hold under weaker conditions on the ambient group \( G \) or the subgroup \( H \). We include applications to hyperbolic groups and homotopical filling functions.

1. Introduction

The \( n \)th homological and homotopical filling functions of a space are generalized isoperimetric functions describing the minimal volume required to fill an \( n \)-cycle or \( n \)-sphere with an \( (n+1) \)-chain or \( (n+1) \)-ball. These functions have been widely studied in Riemannian Geometry and Geometric Group Theory; see for example [2, 5, 8, 9, 13]. In this paper, we study the relation between the \( n \)th homological filling functions of a finitely presented group and its subgroups. Our main result provides sufficient conditions for the \( n \)th-filling function of a subgroup to be bounded from above by the \( n \)th-filling function of the ambient group. The hypotheses of our theorem are in terms of finiteness properties of the ambient group and the subgroup. Our result contrasts with known examples illustrating that this relation does not hold under weaker conditions [4, 18, 16].

1.1. Statement of Main Result. A \( K(G,1) \) for a group \( G \) is a cell complex \( X \) with contractible universal cover \( \tilde{X} \) and fundamental group isomorphic to \( G \). If \( G \) admits a \( K(G,1) \) with finite \( n \)-skeleton, then \( G \) is said to be of type \( F_n \). Such finiteness properties are natural (topological) generalizations of being finitely generated (type \( F_1 \)) and finitely presented (type \( F_2 \)).

If \( X \) is a \( K(G,1) \) with finite \((n+1)\)-skeleton, then the \( n \)th-homological filling function of \( G \) is an optimal function \( \mathrm{FV}^{n+1}_G : \mathbb{N} \rightarrow \mathbb{N} \) such that \( \mathrm{FV}^{n+1}_G(k) \) bounds the minimal volume required to fill an \( n \)-cycle \( \gamma \) of \( \tilde{X} \) of volume at most \( k \), with an \((n+1)\)-chain \( \mu \) of \( \tilde{X} \) having boundary \( \partial(\mu) = \gamma \). See Section 3 for precise definitions.

It can be shown that the growth rate of \( \mathrm{FV}^{n+1}_G \) is independent of the choice of \( X \) up to an equivalence relation \( \sim \), hence \( \mathrm{FV}^{n+1}_G \) is an invariant of the group \( G \). The relation \( f \sim g \) between functions is defined as \( f \leq g \) and \( g \leq f \), where \( f \leq g \) means that there is \( C > 0 \) such that for all \( n \in \mathbb{N} \), \( f(n) \leq Cg(Cn+C) + Cn + C \). Our main result is a generalization of a result of Gersten [10] Thm C] to higher dimensions.

**Theorem 1.1.** Let \( n \geq 1 \). Let \( G \) be a group admitting a finite \((n+1)\)-dimensional \( K(G,1) \) and let \( H \leq G \) be a subgroup of type \( F_{n+1} \). Then

\[
\mathrm{FV}^{n+1}_H \leq \mathrm{FV}^{n+1}_G.
\]

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Some examples that contrast with Theorem 1.1 are the following. In [4], Noel Brady constructed a group $G$ admitting a finite 3–dimensional $K(G, 1)$ such that $FV^3_G$ is linear, and $G$ contains a subgroup $H \leq G$ of type $F_2$ with $FV^2_H$ at least quadratic. Another source of examples are the generalized Heisenberg groups $\mathcal{H}_{2n+1}$, for which Robert Young computed the homological filling invariants in [16, 18]. For instance, $\mathcal{H}_5$ admits a finite 5–dimensional $K(\mathcal{H}_5, 1)$ and has quadratic $FV^2_H$. On the other hand, $\mathcal{H}_5$ can be embedded in $\mathcal{H}_5$, admits a 3–dimensional $K(\mathcal{H}_5, 1)$, and has cubic $FV^2_H$. Likewise, $\mathcal{H}_5$ has quadratic $FV^3_H$ and can be embedded in $\mathcal{H}_7$ which has $FV^3_H$ polynomial of degree 3/2.

Theorem 1.1 also imposes constraints on certain well known constructions. For example, given a finitely generated group $H$ with decidable word problem in nondeterministic polynomial time, Birget, Ol’shanskii, Rips and Sapir produce an embedding of $H$ into a finitely presented group $G$ with polynomial Dehn function [3]. For this construction, Theorem 1.1 implies that if $H$ has a finite 2–dimensional $K(H, 1)$ and $FV^2_H$ is not bound by a polynomial function, then $G$ does not admit a finite 2–dimensional $K(G, 1)$. A particular example of such a group $H$ is the Baumslag-Solitar group $B(m, n)$ with $|m| \neq |n|$, for which the embedding constraint is known [10, Thm A].

We discuss some applications of Theorem 1.1 to hyperbolic groups and homotopical filling functions below. Recall that a group $G$ is hyperbolic if it has a linear Dehn function. In [11], Gersten proved the following:

**Theorem 1.2.** [11 Thm 4.6] Let $G$ be a hyperbolic group of cohomological dimension 2. Then every finitely presented subgroup $H \leq G$ is hyperbolic.

Gersten’s result does not hold in higher dimensions as Brady has exhibited a hyperbolic group $G$ of cohomological dimension 3 containing a non–hyperbolic finitely presented subgroup $H \leq G$ [4]. We can however, obtain a result similar to Theorem 1.2 by considering homotopical filling functions of higher dimensions. The $n$–th–homotopical filling function $\delta^n_G$ of a group $G$ is defined analogously to $FV^{n+1}_G$ but restricts to filling $n$–spheres with $(n + 1)$–balls inside the universal cover of $K(G, 1)$ with finite $(n + 1)$–skeleton. Roughly speaking, $\delta^n_G(k)$ bounds the minimum volume required to fill an $n$–sphere of volume at most $k$, with an $(n + 1)$–ball. Precise definitions of “volume” and $\delta^n_G$ can be found in [2, 5].

**Corollary 1.3.** Let $G$ be a hyperbolic group of geometric dimension $n + 1$, where $n \geq 2$. Let $H \leq G$ be of type $F_{n+1}$. Then $\delta^n_H$ is linear.

Recall that the geometric dimension of a group $G$ is the minimum dimension among $K(G, 1)$’s. The Eilenberg–Ganea Theorem [6, 7] states that the cohomological and geometric dimensions of a group $G$ are equal for dimensions greater or equal than 3. This justifies our use of geometric dimension in the corollary above. In addition to Corollary 1.3 we have the following homotopical version of Theorem 1.1 for sufficiently large $n$.

**Corollary 1.4.** Let $n \geq 3$. Let $G$ be a group admitting a finite $(n + 1)$–dimensional $K(G, 1)$. Let $H \leq G$ be of type $F_{n+1}$. Then $\delta^n_H \leq \delta^n_G$.

Corollaries 1.3 and 1.4 follow from Theorem 1.1 and the following results:

**Theorem 1.5.** [11 pg. 1 and references therein] For $n \geq 3$, the $n$–th–homotopical and homological filling functions $\delta^n_G$ and $FV^{n+1}_G$ are equivalent. For $n = 2$, $\delta^2_G \leq FV^3_G$.

**Theorem 1.6.** [14] Let $G$ be a hyperbolic group. Then $FV^{n+1}_G$ is linear for all $n \geq 1$.

**Proof of Corollary 1.4.** Rips theorem implies that $G$ is of finite type [12], and then the Eilenberg–Ganea Theorem implies that $G$ admits a compact $(n+1)$–dimensional $K(G, 1)$ [6].
Theorems 1.1 and 1.6 imply that $FV^i_H$ is linear. It then follows from Theorem 1.5 that $\delta^n_H$ is linear.

**Proof of Corollary 1.4.** Apply Theorems 1.5 and 1.1.

**Remark 1.7.** Corollary 1.3 does not apply to Brady’s subgroup $H$ as it is not of type $F_3$. It is an open question whether or not the subgroup $H$ in Corollary 1.3 is in fact hyperbolic.

**Remark 1.8.** It is an open question whether or not the statement of Corollary 1.4 holds for $n = 1$ or 2. In general $\delta^1_G + FV^2_G$ and $\delta^2_G + FV^3_G$, so extra work is required. Examples of such groups are given in [1, 17].

1.2. **Outline of the Paper.** The rest of the paper is organized into three sections. Section 2 contains the definition of a filling norm on a finitely generated $\mathbb{Z}G$-module and lemmas required for the proof of Theorem 1.1. Section 3 contains algebraic and topological definitions for $FV^i_G$. Section 4 contains the proof of Theorem 1.1.

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2. FILLING NORMS ON $\mathbb{Z}G$-MODULES

In this section we define the notion of a filling-norm on a finitely generated $\mathbb{Z}G$-module. Most ideas in this section are based on the work of Gersten’s in [11]. The section contains four lemmas on which the proof of the main result of the paper relies on.

**Definition 2.1** (Norm on Abelian Groups). A norm on an Abelian group $A$ is a function $\| \cdot \| : A \rightarrow \mathbb{R}$ satisfying the following conditions:

- $\|a\| \geq 0$ with equality if and only if $a = e$,
- $\|a\| + \|a'\| \geq \|a + a'\|$,
- $\|na\| = |n| \cdot \|a\|$, where $n \in \mathbb{Z}$.

If $A$ is free Abelian with basis $X$, then $X$ induces an $\ell_1$-norm on $A$ given by $\left\| \sum_{x \in X} n_x x \right\|_1 = \sum_{x \in X} |n_x|$, where $n_x \in \mathbb{Z}$.

**Definition 2.2** (Linearly Equivalent Norms). Two norms $\| \cdot \|$ and $\| \cdot \|'$ on a $\mathbb{Z}$-module $M$ are linearly equivalent if there exists a fixed constant $C > 0$ such that

$$C^{-1}|m| \leq \|m\|' \leq C\|m\|$$

for all $m \in M$. This is an equivalence relation and the equivalence class of a norm $\| \cdot \|$ is called the linear equivalence class of $\| \cdot \|$.

**Definition 2.3** (Based Free $\mathbb{Z}G$-modules and Induced $\ell_1$-norms). Suppose $G$ is a group and $F$ is a free $\mathbb{Z}G$-module with $\mathbb{Z}G$-basis $\{a_1, \ldots, a_n\}$. Then $\{g \cdot a_i : g \in G, 1 \leq i \leq n\}$ is a free $\mathbb{Z}$-basis for $F$ as a (free) $\mathbb{Z}$-module. This free $\mathbb{Z}$-basis induces a $G$-equivariant $\ell_1$-norm $\| \cdot \|_1$ on $F$. We call a free $\mathbb{Z}G$-module based if it is understood to have a fixed basis, and we use this basis for the induced $\ell_1$-norm $\| \cdot \|_1$. 
Definition 2.4 (Filling Norms on \(\mathbb{Z}G\)-modules). Let \(\eta: F \to M\) be a surjective homomorphism of \(\mathbb{Z}G\)-modules and suppose that \(F\) is free, finitely generated, and based. The filling norm on \(M\) induced by \(\eta\) and the free \(\mathbb{Z}G\)-basis of \(F\) is defined as

\[
\|m\|_\eta = \min \{ \|x\|_1 : x \in F, \eta(x) = m \}.
\]

Observe that this norm is \(G\)-equivariant.

Remark 2.5 (Induced \(\ell_1\)-norms are Filling Norms). If \(F\) is a finitely generated based free \(\mathbb{Z}G\)-module, then the \(\ell_1\)-norm induced by a free \(\mathbb{Z}G\)-basis is a filling norm.

The following lemma is reminiscent of the fact that linear operators on finite dimensional normed spaces are bounded.

Lemma 2.6 (\(\mathbb{Z}G\)-Morphisms between Free Modules are Bounded). \([11]\) Lemma 4.1] Let \(\varphi: F \to F'\) be a homomorphism between finitely generated, free, based \(\mathbb{Z}G\)-modules. Let \(\|\cdot\|_1\) and \(\|\cdot\|'_1\) denote the induced \(\ell_1\)-norms of \(F\) and \(F'\). Then there exists a constant \(C > 0\) such that for all \(x \in F\)

\[
\|\varphi(x)\|'_1 \leq C \cdot \|x\|_1.
\]

Proof. Let \(A = \{a_1, \ldots, a_n\}\) be the \(\mathbb{Z}G\)-basis of \(F\) inducing the norm \(\|\cdot\|_1\). Then \(\varphi\) is given by a finite \(n \times m\) matrix whose entries are elements of \(\mathbb{Z}G\). Observe that for any \(g \in G, x \in F\), we have \(\|x\|_1 = \|gx\|_1\). Define \(C = \max_{1 \leq i \leq n} \|\varphi(a_i)\|\) and let \(x \in F\) be arbitrary. Then

\[
\|\varphi(x)\|'_1 = \|\varphi(\lambda_1a_1 + \cdots + \lambda_na_n)\|'_1,
\]

where \(\lambda_i \in \mathbb{Z}G\)

\[
\leq \left\| \left( \sum_{g \in G} \lambda_{1,g}g \right) \varphi(a_1) \right\|'_1 + \cdots + \left\| \left( \sum_{g \in G} \lambda_{n,g}g \right) \varphi(a_n) \right\|'_1,
\]

where \(\lambda_j = \sum_{g \in G} \lambda_{j,g}g\) and \(\lambda_{j,g} \in \mathbb{Z}\)

\[
\leq \left( \sum_{g \in G} |\lambda_{1,g}| \right) \|\varphi(a_1)\|'_1 + \cdots + \left( \sum_{g \in G} |\lambda_{n,g}| \right) \|\varphi(a_n)\|'_1
\]

\[
\leq C \left( \sum_{j=1}^m \left( \sum_{g \in G} |\lambda_{j,g}| \right) \right) = C \|x\|_1. \quad \square
\]

Lemma 2.7 (\(\mathbb{Z}G\)-Morphisms with Projective Domain are Bounded). Let \(\varphi: P \to Q\) be a homomorphism between finitely generated \(\mathbb{Z}G\)-modules. Let \(\|\cdot\|_P\) and \(\|\cdot\|_Q\) denote filling norms on \(P\) and \(Q\) respectively. If \(P\) is projective then there exists a constant \(C > 0\) such that for all \(p \in P\)

\[
\|\varphi(p)\|_Q \leq C \cdot \|p\|_P.
\]

Proof. Consider the commutative diagram

\[
A \xrightarrow{\hat{\varphi}} B \\
\downarrow \varphi \downarrow \psi \downarrow \\
B \xrightarrow{\varphi} Q
\]

constructed as follows. Let \(A\) and \(B\) be finitely generated and based free \(\mathbb{Z}G\)-modules, and let \(A \to P\) and \(B \to Q\) be surjective morphisms inducing the filling norms \(\|\cdot\|_P\) and \(\|\cdot\|_Q\). Since \(P\) is projective and \(B \to Q\) is surjective, there is a lifting \(\psi: P \to B\) of \(\varphi\); then let \(\hat{\varphi}\).
be the composition $A \xrightarrow{\varphi} P \xrightarrow{\psi} B$. Let $C$ be the constant provided by Lemma 2.6 for $\varphi$. Let $p \in P$ and let $a \in A$ that maps to $p$. It follows that

$$\|\psi(p)\|_o \leq \|\varphi(p)\|_1 = \|\varphi(a)\|_1 \leq C|a|_1.$$  

Since the above inequality holds for any $a \in A$ with $\rho(a) = p$, it follows that

$$\|\varphi(p)\|_o \leq C \cdot \min_{\rho(a)=p} \|a\|_1 \leq C \cdot \|p\|_\rho.$$  

Proof. Consider a pair of surjective homomorphisms of $\mathbb{Z}G$–modules $\eta : F \rightarrow M$ and $\eta' : F' \rightarrow M$ such that $F$ and $F'$ are finitely generated, free, based modules inducing the filling norms $\| \cdot \|_\eta$ and $\| \cdot \|_{\eta'}$ on $M$. Since $\eta'$ is surjective, the universal property of $F$ provides a homomorphism $\varphi$ such that $\eta = \eta' \circ \varphi$. Let $m \in M$ be arbitrary and take $x \in F$ such that $\eta(x) = m$. Since $\eta' \circ \varphi(x) = m$, by Lemma 2.6 there exists $C > 0$ such that

$$\|m\|_{\eta'} = \min_{\eta'(x') = m} \|x'\|_1 \leq \|\varphi(x)\|_1' \leq C \cdot \|x\|_1.$$  

As this inequality holds for all $x \in F$ satisfying $\eta(x) = m$, we have

$$\|m\|_{\eta'} \leq C \cdot \min_{\eta'(x)=m} \|x\|_1 = C \cdot \|m\|_\eta.$$  

The other inequality proceeds in a similar manner. \hfill \square

Lemma 2.9 (Retraction Lemma). \cite{11} Prop. 4.4] Let $0 \rightarrow M \rightarrow N \rightarrow P \rightarrow 0$ be a short exact sequence of $\mathbb{Z}G$–modules where

1) $M$ is finitely generated and equipped with a filling-norm $\| \cdot \|_M$.  
2) $N$ is free, based, and equipped with the induced $\ell_1$-norm $\| \cdot \|_1$.  
3) $P$ is projective.

Then there exists a retraction $\rho : N \rightarrow M$ for the inclusion $\iota : M \rightarrow N$ and a fixed constant $C > 0$ such that $\|\rho(x)\|_M \leq C\|x\|_1$ for all $x \in N$.

Proof. Since $P$ is projective there is a retraction $\rho'$ for $\iota$. Since $M$ is finitely generated, $N$ is isomorphic to a product $I \oplus Q$ of free modules where $I$ is finitely generated and contains the image of $M$. Define $\rho : N \rightarrow M$ by $\rho|_I = \rho'|_I$ and $\rho|_Q = 0$. Then $\rho$ is a retraction for $\iota$ with support contained in $I$.

Each $x \in N$ has a unique decomposition $x = y + q$ where $y \in I$, $q \in Q$ such that $\rho(x) = \rho(y)$ and $\|y\|_1 \leq \|x\|_1$. Apply Lemma 2.7 to the restriction $\rho : I \rightarrow M$ to obtain $C > 0$ such that

$$\|\rho(x)\|_M = \|\rho(y)\|_M \leq C\|y\|_1 \leq C\|x\|_1.$$  

\hfill \square

3. Homological Filling Functions of Groups

In this section, given a group $G$ of type $FP_{n+1}$, where $n \geq 1$, we define the group invariant $FV^m_G$. In the first part of the section we provide an algebraic definition of $FV^m_G$ and prove that it is well defined. This algebraic approach, while naturally inspired in the topological approach, has not appeared previously in the literature and it provides a convenient algebraic framework suitable for some of the arguments in this paper. In the second part, we recall the topological approach to $FV^m_G$ and show that the topological and
algebraic approaches are equivalent for finitely presented groups of type $FP_{n+1}$. The final subsection discusses why $FV_{G}^{n+1}(k)$ is a finite number.

3.1. Algebraic Definition of $FV_{G}^{n+1}$.

Definition 3.1 (Linearly Equivalent Functions). Let $f$ and $g$ be functions from $\mathbb{N}$ to $\mathbb{N}$. Define $f \preceq g$ if there exists $C > 0$ such that for all $n \in \mathbb{N}$

$$f(n) \leq Cg(Cn + C) + C.$$ 

The functions $f$ and $g$ are linearly equivalent, $f \sim g$, if both $f \preceq g$ and $g \preceq f$ hold. This is an equivalence relation and the equivalence class containing a function $f$ is called the linear equivalence class of $f$.

Definition 3.2 ($FP_{n}$ group). A group $G$ is of type $FP_{n}$ if there is a resolution of $\mathbb{Z}G$–modules

$$P_n \xrightarrow{\partial_n} P_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_{i+1}} P_i \xrightarrow{\partial_{i}} P_{i-1} \xrightarrow{\partial_{i-1}} \cdots \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_{0}} \mathbb{Z} \rightarrow 0,$$

such that for each $i \in \{0, 1 \ldots, n\}$ the module $P_i$ is a finitely generated projective $\mathbb{Z}G$–module. In this case, such a resolution is called an $FP_{n}$–resolution.

Definition 3.3 (Algebraic definition of $FV_{G}^{n+1}$). Let $G$ be a group of type $FP_{n+1}$. The algebraic $n^{th}$–filling function is the (linear equivalence class of the) function

$$FV_{G}^{n+1} : \mathbb{N} \rightarrow \mathbb{N}$$

defined as follows. Let

$$P_{n+1} \xrightarrow{\partial_{n+1}} P_n \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_{i+1}} P_i \xrightarrow{\partial_{i}} P_{i-1} \xrightarrow{\partial_{i-1}} \cdots \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_{0}} \mathbb{Z} \rightarrow 0,$$

be a resolution of $\mathbb{Z}G$–modules for $G$ of type $FP_{n+1}$. Choose filling norms for $P_n$ and $P_{n+1}$, denoted by $\| \cdot \|_{P_n}$ and $\| \cdot \|_{P_{n+1}}$ respectively. Then

$$FV_{G}^{n+1}(k) = \max \left\{ \| \gamma \|_{\partial_{n+1}} : \gamma \in \ker(\partial_n), \| \gamma \|_{P_n} \leq k \right\},$$

where

$$\| \gamma \|_{\partial_{n+1}} = \min \{ \| \mu \|_{P_{n+1}} : \mu \in P_{n+1}, \partial_{n+1}(\mu) = \gamma \}.$$ 

Remark 3.4 (Finiteness of $FV_{G}^{n+1}$). It is not immediately clear that the maximum in Definition 3.3 is a finite number. In Section 3.5 we recall some results from the literature which, under the assumption $G$ is finitely presented, imply that $FV_{G}^{n+1}$ is a finite valued function for $n = 1$ and $n \geq 3$. The authors are not aware of a proof for the case $n = 2$.

For $n = 2$, all results in this paper regarding $FV_{G}^{3}$ hold under the following natural modifications. First, work with the standard extensions of addition, multiplication, and order, of the positive integers $\mathbb{N}$ to $\mathbb{N} \cup \{\infty\}$. Definition 3.3 is extended to functions $\mathbb{N} \rightarrow \mathbb{N} \cup \{\infty\}$, but we emphasize that the constant $C$ remains a finite positive integer. In Definition 3.3 the function $FV_{G}^{3}$ is defined as an $\mathbb{N} \rightarrow \mathbb{N} \cup \{\infty\}$ function. We observe that all arguments in this paper do not rely on finiteness of $FV_{G}^{n+1}(k)$.

Theorem 3.5 ($FV_{G}^{n+1}$ is a Well-defined Group Invariant). Let $G$ be a group of type $FP_{n+1}$. Then the algebraic $n^{th}$–filling function $FV_{G}^{n+1}$ of $G$ is well defined up to linear equivalence.

Proof. Let $(F_i, \partial_\ast)$ and $(P_i, \partial_\ast)$ be a pair of resolutions of $\mathbb{Z}G$–modules of type $FP_{n+1}$ with choices of filling-norms for their $n^{th}$ and $(n+1)^{th}$ modules denoted by $\| \cdot \|_{P_i}$ and $\| \cdot \|_{P_{n+1}}$, and $\| \cdot \|_{P_n}$ and $\| \cdot \|_{P_{n+1}}$, respectively. Let $FV_{F_i}$ and $FV_{P_i}$ be the induced functions according to Definition 3.3. By symmetry, it is enough to show that $FV_{F_i} \leq FV_{P_i}$.
It is well known that any two projective resolutions of a $\mathbb{Z}G$-module are chain homotopy equivalent, see for example [6, pg.24, Thm 7.5], and hence the resolutions $F_*$ and $P_*$ are chain homotopy equivalent. Therefore there exists chain maps filling-norms provided by Lemma 2.7. We claim that for every equivalence, see for example [6, pg.24, Thm 7.5], and hence the resolutions Fix $\parallel \cdot \parallel$ equivalence, Since defined as

Let $C$ denote the maximum of the constants for the maps $g_{n+1}$, $h_n$, and $f_n$ and the chosen filling-norms provided by Lemma 2.7. We claim that for every $k \in \mathbb{N}$,

$$FV_{F_*}^{n+1}(k) \leq C \cdot FV_{P_*}^{n+1}(Ck + C) + Ck + C.$$  

Fix $k$. Let $\alpha \in \ker(\partial_n)$ be such that $\|\alpha\|_{F_*} \leq k$. Choose $\beta \in P_{n+1}$ such that $\partial_{n+1}(\beta) = f_n(\alpha)$ and $\|f_n(\alpha)\|_{P_{n+1}} = \|\beta\|_{P_{n+1}}$. By commutativity of the chain maps and the chain homotopy equivalence,

$$\partial_{n+1} \circ h_n(\alpha) + h_{n-1} \circ \partial_n(\alpha) = g_n \circ f_n(\alpha) - \alpha = g_n \circ \partial_{n+1}(\beta) - \alpha = \partial_{n+1} \circ g_{n+1}(\beta) - \alpha.$$  

Since $\alpha \in \ker(\partial_n)$, we have that $h_{n-1} \circ \partial_n(\alpha) = 0$. Rearranging the above equation, we obtain

$$\alpha = \partial_{n+1} \circ g_{n+1}(\beta) - \partial_{n+1} \circ h_n(\alpha) = \partial_{n+1} (g_{n+1}(\beta) - h_n(\alpha)).$$  

Hence $g_{n+1}(\beta) - h_n(\alpha)$ has boundary $\alpha$. Observe that

$$\|\alpha\|_{\partial_{n+1}} \leq \|g_{n+1}(\beta) - h_n(\alpha)\|_{F_{n+1}} = \|g_{n+1}(\beta)\|_{F_{n+1}} + \|h_n(\alpha)\|_{F_{n+1}} \leq \|\beta\|_{P_{n+1}} + C \cdot \|\alpha\|_{F_n} \leq C \cdot FV_{P_*}^{n+1}(\|f_n(\alpha)\|_{P_{n+1}}) + C \cdot \|\alpha\|_{F_n} \leq C \cdot FV_{P_*}^{n+1}(C\|\alpha\|_{F_n}) + C \cdot \|\alpha\|_{F_n} \leq C \cdot FV_{P_*}^{n+1}(Ck + C) + Ck + C \leq C \cdot FV_{F_*}^{n+1}(Ck + C) + Ck + C$$  

since $\|\alpha\|_{F_*} \leq k$.

Since $\alpha$ was arbitrary, $FV_{F_*}^{n+1}(k) \leq C \cdot FV_{P_*}^{n+1}(Ck + C) + Ck + C$ for all $k \in \mathbb{N}$. This shows that $FV_{F_*}^{n+1} \leq FV_{P_*}^{n+1}$ completing the proof.

3.2. Topological Definition of $FV_{G}^{n+1}$. For a cell complex $X$, the cellular chain group $C_i(X)$ is a free Abelian group with basis the collection of all $i$-cells of $X$. This natural basis induces an $\ell_1$-norm on $C_i(X)$ that we denote by $\| \cdot \|_1$. Recall that a complex $X$ is $n$-connected if its first $n$-homotopy groups are trivial.

**Definition 3.6 (F_n group).** A group $G$ is of type $F_n$ if there is a $K(G, 1)$-complex with a finite $n$-skeleton, i.e., with only finitely many cells in dimensions $\leq n$.

**Definition 3.7 (Topological Definition of $FV_{G}^{n+1}$).** [17] Let $G$ be a group acting properly, cocompactly by cellular automorphisms on an $n$-connected cell complex $X$. The topological $n^{th}$-filling function of $G$ is the (linear equivalence class of the) function $FV_{G}^{n+1} : \mathbb{N} \rightarrow \mathbb{N}$ defined as

$$FV_{G}^{n+1}(k) = \max \{ \|y\|_0 : y \in Z_n(X), \|y\|_1 \leq k \},$$  

where

$$\|y\|_0 = \min \{ \|\mu\|_1 : \mu \in C_{n+1}(X), \partial(\mu) = y \}.$$
Young provides a geometric proof that the topological $n^{th}$-filling function $FV^n_G$ is well defined as an invariant of the group [17].

**Theorem 3.8.** [17] Lemma 1] Let $G$ be a group admitting a proper and cocompact action by cellular automorphisms on an $n$–connected cell complex. Then the topological $n^{th}$–filling invariant $FV^n_G$ of $G$ is well defined up to linear equivalence.

Even in the topological definition, it is not trivial that $FV^n_G$ is a finite valued function and Remark 3.4 also applies in this case. For the rest of the section, we show that the topological and algebraic approaches to $FV^n_G$ are equivalent for finitely presented groups of type $FP_{n+1}$.

**Proposition 3.9.** Let $n \geq 1$ and let $G$ be a group of type $F_{n+1}$. Then $G$ is of type $FP_{n+1}$ and the algebraic and topological $n^{th}$–filling functions of $G$ are linearly equivalent.

**Proof.** Let $X$ be a $K(G, 1)$ with finite $(n+1)$-skeleton. The $G$-action on the universal cover $\tilde{X}$ of $X$ induces the structure of a $\mathbb{Z}G$-module to the group of cellular chains $C_i(\tilde{X})$ and each boundary map $\partial_i$ is a morphism of $\mathbb{Z}G$-modules. Since the $G$-action on $\tilde{X}$ is cellular and free, each $C_i(\tilde{X})$ is a free $\mathbb{Z}G$-module with basis any collection of representatives of the $G$-orbits of $i$-cells. Since the action is cocompact on the $(n+1)$–skeleton, each $C_i(\tilde{X})$ is a finitely generated free $\mathbb{Z}G$-module for $i \in \{0, 1, \ldots, n+1\}$. Since $\tilde{X}$ is a contractible space, all its homology groups are trivial and therefore we have a resolution of $\mathbb{Z}G$-modules

$$\cdots \longrightarrow C_{n+1}(X) \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \to \mathbb{Z} \to 0,$$

of type $FP_{n+1}$. Under our assumptions, the induced topological $n^{th}$–filling function of $G$ is a particular instance of an algebraic $n^{th}$–filling function of $G$. The conclusion then follows from Theorems 3.8 and 3.9.

**Proposition 3.10.** [6] pg 205, proof of Thm. 7.1] Let $G$ be finitely presented and of type $FP_n$ where $n \geq 2$. Then $G$ is of type $F_{n-1}$.

Propositions 3.9 and 3.10 imply the following statement.

**Corollary 3.11.** Let $G$ be a finitely presented group of type $FP_{n+1}$. Then the topological and algebraic definitions of $FV^n_G$ are equivalent.

3.3. **Finiteness of $FV^n_G(k)$**, Let $G$ be a finitely presented group of type $FP_{n+1}$, or equivalently assume that $G$ is of type $F_{n+1}$; see Proposition 3.10. We will sketch why $FV^n_G$ is a finite valued function for $n = 1$ and $n \geq 3$.

3.3.1. **Case $n = 1$**. Finiteness of $FV^1_G$ follows from that of the Dehn function $\delta_G$. We summarize the argument from Gersten’s article [9] Prop 2.4. Let $X$ be a $K(G, 1)$ with finite 2–skeleton and let $z \in Z_1(\tilde{X})$ be a 1–cycle with $\|z\|_1 \leq k$. Then $z = z_1 + \ldots + z_m$ for some $m \leq k$ where each $z_i$ is the 1–cycle induced by a simple edge circuit $\gamma_i$ in $\tilde{X}$ and

$$\sum_{i=1}^m \ell(\gamma_i) = \|z\|_1.$$  

Then

$$\|z\|_2 \leq \sum_{i=1}^m \text{Area}(\gamma_i) \leq \sum_{i=1}^m \delta_G(\ell(\gamma_i)) \leq k \cdot \delta_G(k) < \infty.$$
3.3.2. Case n ≥ 3. A group $G$ of type $F_{n+1}$ has a well defined invariant called the $n^{th}$-homotopical filling function $\delta^n_G : \mathbb{N} \rightarrow \mathbb{N}$. There are multiple approaches to define $\delta^n_G$, we sketch the approach found in [1, 5]. Roughly speaking, if $X$ is a $K(G, 1)$ with finite $(n+1)$-skeleton, then $\delta^n_G(k)$ measures the number of $(n+1)$-balls required to fill a sphere $S^n \rightarrow \tilde{X}$ comprised of at most $k$ $n$-balls. Here the maps $f : S^n \rightarrow \tilde{X}$ and fillings $\tilde{f} : D^{n+1} \rightarrow \tilde{X}$ are required to be in a particular class of maps called admissible maps. This allows one to define the volumes, $\text{vol}(f)$ and $\text{vol}(\tilde{f})$, as the number of $n$-balls and $(n+1)$-balls of $S^n$ and $D^{n+1}$ respectively, mapping homeomorphically to open cells of $\tilde{X}$. The filling volume of $f$ is given by

$$F\text{Vol}(f) = \sup \{ \text{vol}(\tilde{f}) \mid \tilde{f} : D^{n+1} \rightarrow \tilde{X}, \tilde{f}|_{\partial D^{n+1}} = f \}$$

and $\delta^n_G$ by

$$\delta^n_G(k) = \max \{ F\text{Vol}(f) \mid f : S^n \rightarrow \tilde{X}, \text{vol}(f) \leq k \}.$$  

Alonso et al. use higher homotopy groups as $\pi_1(X)$–modules to provide a more algebraic approach to $\delta^n_G$, in particular they show that $\delta^n_G$ is a finite valued function [2, Corollary 1]. It is observed in [5, Remark 2.4(2)] that Alonso’s approach and the approach described above are equivalent.

The finiteness of $F\text{Vol}^{n+1}_G$ then follows from the inequality

$$F\text{Vol}^{n+1}_G \leq \delta^n_G,$$

which holds for all $n \geq 3$. We outline the argument for this inequality described in the introduction of [1]. Let $X$ be a $K(G, 1)$ with finite $(n+1)$-skeleton and let $\gamma \in Z_n(\tilde{X})$ with $\|\gamma\| \leq k$. Using the Hurewicz Theorem, one can show (see [13, 15]) that $\gamma$ is the image of the fundamental class of an $n$–sphere for a map $f : S^n \rightarrow \tilde{X}$ such that $\text{vol}(f) = ||\gamma||_1$. If $\tilde{f} : D^{n+1} \rightarrow \tilde{X}$ is an extension of $f$ to the $(n+1)$–ball $D^{n+1}$, then the image of the fundamental class of $D^{n+1}$ is an $(n+1)$–chain $\mu$ with $\partial(\mu) = \gamma$ and $\text{vol}(\tilde{f}) \geq ||\mu||_1$. Therefore the filling volume

$$F\text{Vol}(f) = \sup \{ \text{vol}(\tilde{f}) \mid \tilde{f} : D^{n+1} \rightarrow \tilde{X}, \tilde{f}|_{\partial D^{n+1}} = f \}$$

is greater than or equal to the filling norm $||\gamma||_{b,n}$. It follows that $F\text{Vol}^{n+1}_G(k) \leq \delta^n_G(k)$

4. Main Result

As we will be working with cell complexes, all relevant computations in this section are understood to occur within cellular chain complexes.

Definition 4.1 (Stably finitely generated free). A $\mathbb{Z}G$-module $P$ is stably finitely generated free if there exists finitely generated free $\mathbb{Z}G$ modules $F$ and $F'$ such that $P \oplus F' \cong F$.

Lemma 4.2 (The Eilenberg Trick). [6, pg.207] Let $G = \pi_1(X, x_0)$, where $X$ is a cell complex. Then $X$ is a subcomplex of a complex $Y$ such that the inclusion $X \hookrightarrow Y$ is a homotopy equivalence, and the cellular $n$-cycles of the universal covers $\tilde{Y}$ and $\tilde{X}$ satisfy $Z_n(\tilde{Y}) \cong Z_n(\tilde{X}) \oplus \mathbb{Z}G$ as $\mathbb{Z}G$-modules.

Proof. Let $x_0$ be a $0$-cell of $X$, and glue an $n$-cell $D^n$ to $(X, x_0)$ by mapping its boundary to $x_0$. The resulting space is the wedge sum of $X$ and an $n$-sphere $S^n$. To obtain $Y$, attach an $(n+1)$-cell $D^{n+1}$ by the attaching map that identifies $\partial D^{n+1}$ with the $n$-sphere $S^n$. Then $Z_n(\tilde{Y}) \cong Z_n(\tilde{X}) \oplus \mathbb{Z}G$ where the $\mathbb{Z}G$ factor is generated by a lifting of the $n$-cell $D^n$ to $\tilde{Y}$. It is clear that $X \hookrightarrow Y$ is a homotopy equivalence. \qed
Let $G$ be a group admitting a finite proof is based on Gersten’s proof of [11, Thm 4.6] and is adjusted for higher dimensions: after subdivisions, there exists a cellular map

$$0 \rightarrow P_n \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_0 \rightarrow M \rightarrow 0$$

and

$$0 \rightarrow P'_n \rightarrow P'_{n-1} \rightarrow \cdots \rightarrow P'_0 \rightarrow M \rightarrow 0$$

be exact sequences of $R$–modules with $P_i$ and $P'_i$ projective for $i \leq n - 1$. Then

$$P_0 \oplus P'_1 \oplus P_2 \oplus P'_3 \oplus \cdots \cong P'_0 \oplus P_1 \oplus P'_2 \oplus P_3 \oplus \cdots$$

We are now ready to prove our main result which is a generalization of [10, Thm C]. The proof is based on Gersten’s proof of [11] Thm 4.6] and is adjusted for higher dimensions:

**Theorem 4.4.** Let $G$ be a group admitting a finite $(n + 1)$-dimensional $K(G, 1)$ and let $H \leq G$ be a subgroup of type $F_{n+1}$. Then $FV^G_{n+1} \leq FV^G_{n+1}$.

**Proof.** Let $W$ be a finite $(n + 1)$-dimensional $K(G, 1)$. Let $X$ be the $(n + 1)$-skeleton of a $K(H, 1)$. Since $H$ is of type $F_{n+1}$, we may assume that $X$ is a finite cell complex. Then, after subdivisions, there exists a cellular map $f : X \rightarrow W$ inducing the inclusion $H \hookrightarrow G$ at the level of fundamental groups. Let $M_f$ be the mapping cylinder of $f$ and consider the exact sequences of $\mathbb{Z}G$-modules

(4.1) \hspace{1cm} 0 \rightarrow \mathbb{Z}_n(\tilde{M}_f) \rightarrow C_n(\tilde{M}_f) \rightarrow \cdots \rightarrow C_0(\tilde{M}_f) \rightarrow \mathbb{Z} \rightarrow 0

and

(4.2) \hspace{1cm} 0 \rightarrow C_{n+1}(\tilde{W}) \rightarrow C_n(\tilde{W}) \rightarrow \cdots \rightarrow C_0(\tilde{W}) \rightarrow \mathbb{Z} \rightarrow 0,

where $\tilde{W}$ and $\tilde{M}_f$ denote the universal covers of $W$ and $M_f$ respectively.

Applying Schanuel’s lemma to the above sequences shows that the $\mathbb{Z}G$–module $\mathbb{Z}_n(\tilde{M}_f)$ is stably finitely generated free. Let $Y$ be the space obtained by attaching a finite number of $(n + 1)$–balls to the base point of $M_f$ as in Lemma 4.2 such that $\mathbb{Z}_n(\tilde{Y})$ is finitely generated and free as a $\mathbb{Z}G$-module.

From here on, we are only concerned with the inclusion map $X \rightarrow Y$ realizing the inclusion $H \rightarrow G$ at the level of fundamental groups with the property that $\mathbb{Z}_n(\tilde{Y})$ is finitely generated and free as a $\mathbb{Z}G$-module. From the first two properties, it follows that any lifting $\tilde{X} \rightarrow \tilde{Y}$ is an embedding and hence, without loss of generality, we can assume that $\tilde{X} \subseteq \tilde{Y}$. Since the ring $\mathbb{Z}G$ is free as a $\mathbb{Z}H$–module, it follows that $C_i(\tilde{Y})$ is a free as a $\mathbb{Z}H$-module. Consequently, the $\mathbb{Z}H$–module $C_i(\tilde{X})$ is a free factor of $C_i(\tilde{Y})$. Hence the quotient $C_i(\tilde{Y})/C_i(\tilde{X})$ is a free $\mathbb{Z}H$–module.

Restricting our attention to $n$-skeleta, the following short exact sequence of chain complexes of $\mathbb{Z}H$-modules arises

(4.3) \hspace{1cm} 0 \rightarrow C_*^a(\tilde{X}^n) \rightarrow C_*^a(\tilde{Y}^n) \rightarrow C_*^a(\tilde{Y}^n, \tilde{X}^n) \rightarrow 0,

where $C_*^a(\tilde{Y}^n, \tilde{X}^n)$ is the free quotient complex $C_*^a(\tilde{Y}^n)/C_*^a(\tilde{X}^n)$. Consider the induced long exact homology sequence

(4.4) \hspace{1cm} 0 \rightarrow \tilde{H}_n(\tilde{X}^n) \rightarrow \tilde{H}_n(\tilde{Y}^n) \rightarrow \tilde{H}_n(\tilde{Y}^n, \tilde{X}^n) \rightarrow \tilde{H}_{n-1}(\tilde{X}^n) \rightarrow \cdots .

Since $X$ is the $(n + 1)$-skeleton of an $K(H, 1)$, the homology group $\tilde{H}_{n-1}(\tilde{X}^n)$ is trivial. Now the exact sequence (4.4) can be truncated, obtaining the short exact sequence

(4.5) \hspace{1cm} 0 \rightarrow \mathbb{Z}_n(\tilde{X}) \rightarrow \mathbb{Z}_n(\tilde{Y}) \rightarrow \mathbb{Z}_n(\tilde{Y}, \tilde{X}) \rightarrow 0,
where $i$ is induced by the inclusion $\tilde{X} \subseteq \tilde{Y}$. We claim that the short exact sequence (4.5) satisfies the three hypothesis of Lemma 2.9.

First, since $X$ is a finite cell complex, $C_{n+1}(\tilde{X})$ is finitely generated as a $\mathbb{Z}H$-module. Therefore $Z_n(\tilde{X})$ is also finitely generated as a $\mathbb{Z}H$-module.

Second, the construction of $Y$ guarantees that $Z_n(\tilde{Y})$ is a free $\mathbb{Z}G$-module, hence $Z_n(\tilde{Y})$ is a free $\mathbb{Z}H$-module.

To verify the third hypothesis, restrict attention to $n$-skeleta. Consider $Y^{(n)}$ and observe that

\begin{equation}
0 \to Z_n(\tilde{Y}^{(n)}) \to C_n(\tilde{Y}^{(n)}) \to \cdots \to C_0(\tilde{Y}^{(n)}) \to \mathbb{Z} \to 0
\end{equation}

is an exact sequence of $\mathbb{Z}G$-modules, and hence it is also an exact sequence of $\mathbb{Z}H$-modules. Analogously, considering $X^{(n)}$, the sequence of $\mathbb{Z}H$-modules

\begin{equation}
0 \to Z_n(\tilde{X}^{(n)}) \to C_n(\tilde{X}^{(n)}) \to \cdots \to C_0(\tilde{X}^{(n)}) \to \mathbb{Z} \to 0
\end{equation}

is exact. Exactness of (4.6) and (4.7), in addition with the assumption that $\tilde{X}$ is a subcomplex of $\tilde{Y}$, implies that the sequence of $\mathbb{Z}H$-modules

\begin{equation}
0 \to Z_n(\tilde{Y}^{(n)}, \tilde{X}^{(n)}) \to C_n(\tilde{Y}^{(n)}, \tilde{X}^{(n)}) \to \cdots \to C_0(\tilde{Y}^{(n)}, \tilde{X}^{(n)}) \to \mathbb{Z} \to 0
\end{equation}

is also exact. Applying Schanuel's Lemma to the exact sequences (4.6) and (4.8), together with $Z_n(\tilde{Y}^{(n)})$ being a free $\mathbb{Z}H$-module, we have that $Z_n(\tilde{Y}, \tilde{X})$ is a projective $\mathbb{Z}H$-module.

Thus we have shown that the short exact sequence (4.5) satisfies the three hypothesis of Lemma 2.9. Before invoking this lemma and concluding the proof, we set up notation for the norms required to specify representatives of $FV_{\mathbb{Z}G}^{n+1}$ and $FV_{\mathbb{Z}H}^{n+1}$.

Let $|| \cdot ||_1$ denote the $\ell_1$-norm on $C(i)(\tilde{Y})$ induced by the basis consisting on all $i$-cells of $\tilde{Y}$. Let $|| \cdot ||_{Z_0(\tilde{Y})}$ denote the $\ell_1$-norm on $Z_0(\tilde{Y})$ induced by a free $\mathbb{Z}G$-basis; by definition this is also filling norm on $Z_0(\tilde{Y})$. Then (a representative of) $FV_{\mathbb{Z}G}^{n+1}$ is given by

\begin{equation}
FV_{\mathbb{Z}G}^{n+1}(k) = \max \left\{ ||\gamma||_{Z_0(\tilde{Y})} : \gamma \in Z_1(\tilde{Y}), ||\gamma||_1 \leq k \right\}.
\end{equation}

Since $C_{n+1}(\tilde{X}) \subseteq C_{n+1}(\tilde{Y})$ is a free factor, the $\ell_1$-norm on $C_{n+1}(\tilde{X})$ induced by the $(n+1)$-cells of $\tilde{X}$ equals the restriction of $|| \cdot ||_1$ to $C_{n+1}(\tilde{X})$. Let $|| \cdot ||_{Z_0(\tilde{X})}$ denote the filling-norm on $Z_0(\tilde{X})$ as a $\mathbb{Z}H$-module induced by the boundary map $C_{n+1}(\tilde{X}) \xrightarrow{\partial_{n+1}} Z_n(\tilde{X})$. Then

\begin{equation}
FV_{\mathbb{Z}H}^{n+1}(k) = \max \left\{ ||\gamma||_{Z_0(\tilde{X})} : \gamma \in Z_1(\tilde{X}), ||\gamma||_1 \leq k \right\}.
\end{equation}

By Lemma 2.9 applied to the short exact sequence (4.5), there exists a constant $C_1 > 0$ and a morphism of $\mathbb{Z}H$-modules $\rho : Z_n(\tilde{Y}) \to Z_n(\tilde{X})$ such that

\begin{equation}
||\rho(\alpha)||_{Z_0(\tilde{X})} \leq C_1 \cdot ||\alpha||_{Z_0(\tilde{Y})},
\end{equation}

for every $\alpha \in Z_n(\tilde{Y})$, and $\rho \circ i$ is the identity on $Z_n(\tilde{X})$.

Let $k \in \mathbb{N}$ and let $\gamma \in Z_n(\tilde{X})$ such that $||\gamma||_1 \leq k$. Then (4.11) implies that

\begin{equation}
||\gamma||_{Z_0(\tilde{X})} = ||\rho \circ (\gamma)||_{Z_0(\tilde{Y})} \leq C \cdot ||(\gamma)||_{Z_0(\tilde{Y})} \leq C \cdot FV_{\mathbb{Z}G}^{n+1}(k).
\end{equation}

Since $\gamma$ was arbitrary, we have $FV_{\mathbb{Z}H}^{n+1}(k) \leq C \cdot FV_{\mathbb{Z}G}^{n+1}(k)$. \hfill $\square$

**Remark 4.5.** Recall that the cohomological dimension of a group $G$, $cd(G)$, is the smallest natural number $n$ for which $G$ admits a projective resolution of $\mathbb{Z}G$-modules for $\mathbb{Z}$ of length $n$. The Eilenberg–Ganea Theorem states that the geometric and cohomological dimensions of a group are equal for all dimensions greater or equal to 3 \cite{7}. In the context of
Theorem 4.4, if the ambient group \( G \) is of dimension \( n + 1 \), where \( n \geq 2 \), it cannot happen that \( \mathbb{Z}_m(\tilde{M}_f) \) is stably finitely generated free as a \( \mathbb{Z}_G \)-module for \( m < n \) since

\[
0 \to \mathbb{Z}_m(\tilde{M}_f) \to C_m(\tilde{M}_f) \to \cdots \to C_0(\tilde{M}_f) \to \mathbb{Z} \to 0
\]

would be a projective resolution of \( \mathbb{Z}_G \)-modules of length \( m + 1 \) – contradicting the fact that \( \text{cd}(G) = n + 1 \). Thus the proof of Theorem 4.4 does not generalize to \( m \)-th homological filling functions for \( m < n \). As illustrated in the introduction, it generally is not true for such groups that \( F^{m+1}_H \leq F^{m+1}_G \).

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