Tight Bounds for Sandpile Transience on the Two-Dimensional Grid up to Polylogarithmic Factors

David Durfee
Georgia Institute of Technology
ddurfee@gatech.edu

Matthew Fahrbach *
Georgia Institute of Technology
matthew.fahrbach@gatech.edu

Yu Gao
Shanghai Jiaotong University
xyz2606@sjtu.edu.cn

Tao Xiao
Shanghai Jiaotong University
xt_1992@sjtu.edu.cn

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Abstract

We use techniques from the theory of electrical networks to give tight bounds for the transience class of the Abelian sandpile model on the two-dimensional grid up to polylogarithmic factors. The Abelian sandpile model is a discrete process on graphs that is intimately related to the phenomenon of self-organized criticality. In this process vertices receive grains of sand, and once the number of grains exceeds their degree, they topple by sending grains to their neighbors. The transience class of a model is the maximum number of grains that can be added to the system before it necessarily reaches a recurrent state. Through a more refined and global analysis of electrical potentials and random walks, we give an upper bound of $O(n^4 \log^4 n)$ and a lower bound of $\Omega(n^4)$ for the transience class of the $n \times n$ grid. Our methods naturally extend to $n^d$-sized $d$-dimensional grids to give an upper bound of $O(n^{3d-2} \log^{d+2} n)$ and a lower bound of $\Omega(n^{3d-2})$.

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1 Introduction

The Abelian sandpile model is the canonical dynamical system used to study self-organized criticality. In their seminal paper, Bak, Tang, and Wiesenfeld [BTW87] proposed the idea of self-organized criticality to explain several ubiquitous patterns in nature typically viewed as complex phenomena, such as catastrophic events occurring without any triggering mechanism, the fractal behavior of mountain landscapes and coastal lines, and the presence of pink noise in electrical networks and the luminosity of stars. Since their discovery, self-organized criticality has been observed in an abundance of disparate scientific fields [Bak96, WPC+16], including condensed matter theory [WWAM06], economics [BPR15, SW94], epidemiology [SMM14], evolutionary biology [Phi14, SM96], high-energy astrophysics [Asc11, MTN94], materials science [RAM09], neuroscience [BdACA+16, LHG07], statistical physics [Dha06, Man91], seismology [SS89], and sociology [KG09]. A stochastic process is a self-organized critical system if it naturally evolves to highly imbalanced critical states where slight local disturbances can completely alter the current state. For example, when pouring grains of sand onto a table, the pile initially grows in a predictable way, but as it becomes steeper and more unstable, dropping a single grain can spontaneously cause an avalanche that affects the entire pile. Self-organized criticality differs from the critical point of a phase transition in statistical physics, because a self-organizing system does not rely on tuning an external parameter. Instead, it is insensitive to all parameters of the model and simply requires time to reach criticality, which is known as the transient period. Natural events empirically operate at a critical point between order and chaos, thus justifying our study of self-organized criticality.

Dhar [Dha90] developed the Abelian sandpile model on finite, directed graphs with a sink vertex to further understand self-organized criticality. The Abelian sandpile model, also known as a chip-firing game [BLS91], on a graph with a sink is defined as follows. In each iteration, a grain of sand is added to a non-sink vertex of the graph. While any non-sink vertex contains at least $\text{deg}(v)$ grains of sand, a grain is transferred from $v$ to each of its neighbors. This is known as a toppling. When no vertex can be toppled, the state is stable and the iteration ends. The sink absorbs grains and never topples, and the presence of a sink guarantees that every toppling procedure eventually stabilizes. An important property of the Abelian sandpile model is that the order in which vertices topple does not affect the stable state. Therefore, as the process evolves it produces a sequence of stables states. From the theory of Markov chains, we say that a state is recurrent if it can be revisited in such a sequence and transient otherwise.

In the self-organized critical state of the Abelian sandpile model on a graph with a sink, the transient states have zero probability and the recurrent states occur with equal probability [Dha90]. Therefore, the natural algorithmic question to ask about self-organized criticality for the Abelian sandpile model is: How long does it take for the process to reach a recurrent state? Starting with an empty configuration, if the vertex that receives the grain of sand is chosen uniformly at random in each step, Babai and Gorodezky [BG07] give a simple solution that is polynomial in the number of edges of the graph using a coupon collector argument. In the worst case, however, an adversary can choose where to place the grain of sand in each iteration. Babai and Gorodezky analyze the transience class of the model to understand its worst-case behavior, which is defined as the maximum number of grains that can be added to the empty configuration before the configuration necessarily becomes recurrent. An upper bound for the transience class of a model is an upper bound for the time needed to enter self-organized criticality.
1.1 Results

We give the first tight bound up to polylogarithmic factors of the transience class of the Abelian sandpile model on the \( n \times n \) grid with all boundary vertices connected to the sink. This model was first studied in depth by Dhar, Ruelle, Sen, and Verma [DRSV95], and it has since been the most extensively studied Abelian sandpile model due to its role in algebraic graph theory, theoretical computer science, and statistical physics. Babai and Gorodezky [BG07] initially established that the transience class for the \( n \times n \) grid is polynomially bounded. Although they give a liberal upper bound of \( O(n^{30}) \), their result was surprising because the grid contains subgraphs with exponentially large transience classes. Choure and Vishwanathan [CV12] improved the upper bound for the transience class of the grid to \( O(n^{7}) \) and gave a lower bound of \( \Omega(n^{3}) \) by viewing the graph as an electrical network and relating the Abelian sandpile model to random walks on the underlying graph. Moreover, they conjectured that the transience class of the grid is \( O(n^{4}) \).

We answer their conjecture nearly affirmatively with the following theorems.

**Theorem 1.1.** The transience class of the Abelian sandpile model on an \( n \times n \) grid is \( O(n^{4} \log^{4} n) \).

**Theorem 1.2.** The transience class of the Abelian sandpile model on an \( n \times n \) grid is \( \Omega(n^{4}) \).

Our results establish how fast the system evolves into its self-organized critical state in the adversarial case. Furthermore, these results reaffirm the empirical observations that natural processes exhibiting self-organized criticality quickly reach steady state. Our methods also naturally extends to higher-dimensional cases, giving the following result.

**Theorem 1.3.** The transience class of the Abelian sandpile model on an \( n^{d} \) \( d \)-dimensional grid is \( O(n^{3d-2} \log^{d+2} n) \) and \( \Omega(n^{3d-2}) \).

1.2 Techniques

Our approach is motivated by the method of Choure and Vishwanathan [CV12] for bounding the transience class of the Abelian sandpile model on graphs using electrical potential theory and the analysis of random walks. We first use weak duality instead of strong duality to give a simplified reduction for bounding voltages. Viewing the graph as an electrical network with a voltage source at some vertex and a grounded sink, we give more accurate voltage estimates by carefully considering the geometry of the grid. We use various lines of symmetry to compare escape probabilities of random walks with different initial position, resulting in a new technique for comparing vertex potentials between certain pairs of vertices. These geometric arguments can likely be generalized to other lattice-based graphs. As a result, we get empirically tight inequalities for the sum of all vertex potentials in the network and the voltage drop between opposite corners of the grid. Precise voltage bounds have direct consequences for modern maximum flow algorithms and interior point methods on highly-structured graphs.

For some of our voltage bounds, we interpret a vertex potential as an escape probability and decouple the two-dimensional random walk on the grid into independent one-dimensional random walks on a path graph. This decoupling allows us to apply classical results about simple symmetric random walks on \( \mathbb{Z} \) such as the reflection principle, which we extend as needed using conditional probability arguments. We noticed that some of the probability inequalities...
we developed are directly related to problems in enumerative combinatorics without closed-form solutions [ES77]. Moreover, by reducing from two-dimensional random walks to one-dimensional walks, we utilize standard probabilistic tools, including Stirling’s approximation, Chernoff bounds, and the negative binomial distribution. Because we consider many different kinds of events in our analysis, Section 5 is an extensive collection of probability inequalities for symmetric $t$-step random walks on $\mathbb{Z}$ with various boundary conditions.

Lastly, we leverage well-known results about effective resistances in the $n \times n$ grid when viewed as an electrical network. We follow Choure and Vishwanathan in using the potential reciprocity theorem to swap the voltage source and any other non-sink network, but we use this theorem repeatedly along with the fact that the effective resistance between any non-sink vertex and the sink is bounded between a constant and $O(\log n)$. This enables us to analyze tractable, one-dimensional random walk problems at the expense of polylogarithmic factors. When viewed through the lens of graph connectivity, such bounds for the transience class are surprising because paths and grids have low algebraic connectivity, yet global structural arguments can be made just using effective resistances. The fact that we can do this suggests that low maximum effective resistance captures a different but similar phenomenon to high conductance and high expansion for stochastic processes on graphs.

We outline the paper as follows. In Section 2, we review the Abelian sandpile model, random walks on graphs, and electrical networks. This includes the framework developed by Choure and Vishwanathan for bounding the transience class of the Abelian sandpile model on bounded-degree graphs. In Section 3, we prove the upper bound for the transience class in Theorem 1.1. In Section 4, we prove the lower bound for the transience class in Theorem 1.2. In Section 5, we present a collection of probability inequalities for simple symmetric random walks. In Section 6, we extend our methods to higher-dimensional grids and prove Theorem 1.3.

2 Background

2.1 Abelian Sandpile Model

Let $G = (V, E)$ be an undirected multigraph with vertex set $V$ and edge set $E$. Throughout this paper, all of the graphs we consider have a sink vertex denoted by $v_{\text{Sink}}$. The Abelian sandpile model is a dynamical system on a graph $G$ used to study the phenomenon of self-organized criticality, which we briefly describe in the beginning of Section 1. In the Abelian sandpile model, a configuration $\sigma$ on $G$ is a vector of nonnegative integers indexed by the non-sink vertices where $\sigma(v)$ denotes the number of grains of sand on vertex $v$. We say that a configuration is stable if $\sigma(v) < \deg(v)$ for all non-sink vertices and unstable otherwise. An unstable configuration $\sigma$ moves towards stabilization by selecting a vertex $v$ such that $\sigma(v) \geq \deg(v)$ and sending one grain of sand from $v$ to each of its neighboring vertices. This event is called a toppling of $v$, and it creates a new configuration $\sigma'$ such that $\sigma'(v) = \sigma(v) - \deg(v)$, $\sigma'(u) = \sigma(u) + 1$ for all vertices $u$ adjacent to $v$, and $\sigma'(u) = \sigma(u)$ for all remaining vertices. This procedure eventually reaches a stable state because $G$ has a sink. Moreover, the order in which vertices topple does not affect the final stable state. The initial configuration of the Abelian sandpile model is typically the zero vector, and in each iteration a grain of sand is either placed at a vertex chosen uniformly at random or at some predetermined vertex. The system evolves by stabilizing the configuration and then placing another grain of sand.
A configuration $\sigma$ is \textit{recurrent} if the process can eventually return to $\sigma$. Any state that is not recurrent is \textit{transient}. Note that once the system enters a recurrent state, it can never visit a transient state. Babai and Gorodezky [BG07] introduced the following idea to give an upper bound on the number of steps for the Abelian sandpile model to reach self-organized criticality.

**Definition 2.1.** The \textit{transience class} of the Abelian sandpile model of $G$ is the maximum number of grains that can be added to the empty configuration before the configuration necessarily becomes recurrent. We denote this quantity by $tcl(G)$.

![Figure 1: Configurations of the Abelian sandpile model on the 500×500 grid during its transience period after placing (a) $10^{10}$ (b) $2 \cdot 10^{10}$ (c) $4 \cdot 10^{10}$ (d) $8 \cdot 10^{10}$ grains of sand at (1,1).](image)

In the figure above, we illustrate the transient configurations in the \textit{transient period} of the Abelian sandpile model as it advances towards its critical state. We specifically show in this paper that by repeatedly placing grains of sand in the top-left corner, we maximize the length of the transience period up to a polylogarithmic factor.

In earlier related works, Björner, Lovász, and Shor [BLS91] studied a variant of this process without a sink and characterized when stabilization terminates. They also related the rate at which the system converges to the spectrum of the underlying graph. In the model we study, an observation by Dhar [Dha90] and Kirchoff’s theorem show that the stable recurrent states of the system are in bijection with the number of spanning trees of $G$. Additionally, these configurations are the representatives of the quotient group called the \textit{sandpile group}, a popular object in algebraic combinatorics [BS13]. Choure and Vishwanathan [CV12] show that if every vertex in a configuration has topped, then the configuration is recurrent, which we use to bound the transience class. For a comprehensive survey on the Abelian sandpile model, see [HLM+08]. Lastly, the Abelian sandpile model also has broad applications to statistical physics, including a direct relation to the $q$-state Potts model and Markov chain Monte Carlo algorithms for sampling random spanning trees [Dha90, JLP15, Wil10].

### 2.2 Random Walks on Graphs

A walk $w$ on a graph $G$ is a sequence of vertices $w^{(0)}, w^{(1)}, \ldots, w^{(t_{\text{max}})}$ such that each $w^{(t)}$ is a neighbor of $w^{(t-1)}$. We use a superscript because of the relation between steps in a walk and iterations of an evolving system. A random walk is a process that begins at a vertex $w^{(0)}$, and at each time step $t$ transitions from $w^{(t-1)}$ to $w^{(t)}$ such that $w^{(t)}$ is chosen uniformly at random.
from the neighbors of \(w(t-1)\). Note that this captures the effect of walking on a multigraph. We define \(|w| = t_{\text{max}}\) to be the length of the walk. The walks that we consider continue until reaching a set of sink vertices. It will be useful to define the following families of walks.

**Definition 2.2.** For any set of starting vertices \(S\) and terminating vertices \(T\), let

\[
W^G(S \rightarrow T) \overset{\text{def}}{=} \left\{ w : w^{(0)} \in S, w^{(i)} \notin T \cup \{v_{\text{Sink}}\} \text{ for } 0 \leq i \leq |w| - 1, \text{ and } w^{(|w|)} \in T \right\}
\]

be the set of finite walks from \(S\) to \(T\).

Note that with this definition, walks \(w\) of length 0 are permissible if \(w^{(0)} \in S \cap T\). Throughout the paper, it will be convenient to consider random walks from a single vertex \(u\) to another vertex \(v\) or pair of vertices \(\{v, v_{\text{Sink}}\}\). We denote these cases by

\[
W^G(u \rightarrow v) \overset{\text{def}}{=} W^G(\{u\} \rightarrow \{v\})
\]

and

\[
W^G(u \rightarrow \{v, v_{\text{Sink}}\}) \overset{\text{def}}{=} W^G(\{u\} \rightarrow \{v, v_{\text{Sink}}\}).
\]

Lastly, we consider the set of nonterminating walks in our analysis, so it will be useful to define

\[
W^G(S) \overset{\text{def}}{=} \left\{ w \in \prod_{i=0}^{\infty} V : w^{(0)} \in S \text{ and } w^{(i)} \neq v_{\text{Sink}} \text{ for any } i \geq 0 \right\}
\]

as the set of infinite walks from \(S\). An analogous definition follows when \(S = \{u\}\).

The focus of our study is the \(n \times n\) grid, denoted as SQUARE\(_n\). Its one-dimensional projection is the length \(n\) path, which we denote by PATH\(_n\). As with previous works [BG07, CV12], we do not follow the usual graph-theoretic convention of using \(n\) to denote vertex count. Instead, we use \(n\) for the side length of SQUARE\(_n\). We formally define PATH\(_n\) to be the undirected graph with the vertex set \(\{1, 2, \ldots, n\} \cup \{v_{\text{Sink}}\}\) and edges between \(i\) and \(i+1\) for every \(1 \leq i \leq n-1\), as well as two edges connecting \(v_{\text{Sink}}\) to 1 and \(n\). Note that by this definition, \(v_{\text{Sink}}\) can be viewed as the vertices 0 and \(n+1\). If we remove the sink, which can be thought of as letting \(v_{\text{Sink}} = \infty\), then the resulting graph is the line with vertices \(i \in \mathbb{Z}\) and edges between every pair \((i, i+1)\). We denote this graph by LINE. We use the indices \(i, j,\) and \(k\) to represent vertices in the one-dimensional case. In our analysis of random walks on LINE, which is critical to understanding the transience class of the Abelian sandpile model on SQUARE\(_n\), it is useful to record the minimum and maximum position of a \(t\)-step random walk.

**Definition 2.3.** For any starting position \(i \in \mathbb{Z}\) and any walk \(w \in W^{\text{LINE}}(i)\), let the minimum position in the first \(t\) steps be

\[
\min_{0 \leq \hat{t} \leq t} w^{(\hat{t})}
\]

and the maximum position in the first \(t\) steps be

\[
\max_{0 \leq \hat{t} \leq t} w^{(\hat{t})}.
\]
The graph $\text{Square}_n$ is defined similarly. Its vertex set is $\{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\} \cup \{v_{\text{Sink}}\}$, and its edges connect any pair of vertices that differ in one coordinate. Vertices on the boundary have edges going to $v_{\text{Sink}}$ so that every non-sink vertex has degree 4. With this definition of $\text{Square}_n$, each corner vertex has two edges to $v_{\text{Sink}}$ and non-corner vertices on the boundary share one edge with $v_{\text{Sink}}$. Since all vertices correspond to pairs of coordinates, we use the vector notation $u = (u_1, u_2)$ to denote coordinates on the grid. An added benefit to this choice is that it easily extends to higher dimensions. Therefore, a random walk $w$ on $\text{Square}_n$ is naturally a $(t_{\text{max}} + 1) \times 2$ matrix. We can decouple such a walk $w$ into its horizontal and vertical components, using the notation $w_1$ to denote the change in position of the first coordinate and $w_2$ to denote the change in position of the second coordinate. In general, we use the notation $w_{\hat{d}}$ to index into one of the dimensions $1 \leq \hat{d} \leq d$ of a $d$-dimensional walk. We do not record duplicate positions when the walk takes a step in a dimension different than $\hat{d}$. Therefore, we have $|w| = |w_1| + |w_2| - 1$ when $d = 2$, because the initial vertex is present in both $w_1$ and $w_2$.

2.3 Electrical Networks

Vertex potentials are central to our analysis. They have close connections with electrical voltages and belong to the class of harmonic functions [DS84]. We analyze their relation to the transience class of general graphs. For any vertex non-sink vertex $u$, we can define the unique potential vector $\pi(u)$ such that $\pi(u)_u = 1$, $\pi(u)_{v_{\text{Sink}}} = 0$, and for all vertices $v \in V \setminus \{u, v_{\text{Sink}}\}$ we have

$$
\pi(u)_v = \frac{1}{\deg(v)} \sum_{x \sim v} \pi(u)_x,
$$

where the sum is over the neighbors of $v$. Thus, $\pi(u)_v$ denotes the potential at $v$ when the boundary conditions are set to 1 at $u$ and 0 at the sink. Since we analyze potential vectors in both $\text{Path}_n$ and $\text{Square}_n$, we use superscripts to denote the graph when context is unclear. Note that this notation differs from that in Choure and Vishwanathan [CV12], as they define this vector as $\pi_u$ and index the potential at $v$ by $\pi_u(v)$. We use our notation because it is consistent with indexing the entries of a vector by subscripts and because a potential vector is a function of the source vertex $u$.

Choure and Vishwanathan showed that we can give upper and lower bounds on the transience class using potentials, which we rephrase in the following theorem.

**Theorem 2.4** ([CV12, Theorem 3.1 and Lemma 3.4]). If $G$ is a graph such that the degree of every non-sink vertex is bounded by a constant, then

$$
tcl(G) = O \left( \max_{u,v \in V \setminus \{v_{\text{Sink}}\}} \left( \sum_{x \in V} \pi^G(u)_x \right) \pi^G(u)_v^{-1} \right).
$$

and

$$
tcl(G) = \Omega \left( \max_{u,v \in V \setminus \{v_{\text{Sink}}\}} \pi^G(u)_v^{-1} \right).
$$

We give a simplified version of this reduction using weak duality instead of strong duality in Appendix A. As all non-sink vertices have degree 4, we can apply Theorem 2.4 to $\text{Square}_n$.

The following combinatorial interpretation of potentials as random walks is fundamental to our investigation of the transience class of $\text{Square}_n$. 

6
Fact 2.5 ([DS84]). For any graph $G$ and non-sink vertex $u$, the potential $\pi(u)_v$ is the probability of a random walk starting at $v$ and reaching $u$ before $v_{Sink}$.

Lemma 2.6. Let $u$ be a non-sink vertex of $\text{SQUARE}_n$. For any vertex $v$, we have

$$\pi_{\text{SQUARE}_n}(u)_v = \sum_{w \in \mathcal{V}_{\text{SQUARE}_n}(v \rightarrow u)} 4^{-|w|}.$$ 

We defer the proof of the previous lemma to Appendix A.

A systematic treatment of the connection between random walks and electrical networks can be found in the monograph by Doyle and Snell [DS84] or the survey by Lovász [Lov93]. The following lemma is a key result for our investigation, which states that a voltage source and a measurement point can be swapped at the expense of a distortion in the potential equal to the ratio of the effective resistances between the sink and the two vertices. The effective resistance between a pair of vertices $u$ and $v$, denoted as $R_{\text{eff}}(u, v)$, can be formalized in several ways. In the electrical interpretation as discussed in [DS84], effective resistance can be viewed as the voltage needed to send one unit of current from $u$ to $v$ if every edge in $G$ is a unit resistor. We also give a linear algebraic definition of effective resistance in Appendix A.

Lemma 2.7 (Potential Reciprocity, [CV12, Lemma 2.1]). Let $G$ be any graph (not necessarily degree-bounded) with sink $v_{Sink}$. For any pair of vertices $u$ and $v$, we have

$$R_{\text{eff}}(v_{Sink}, u) \pi^G(v) = R_{\text{eff}}(v_{Sink}, v) \pi^G(u).$$

This statement is particularly powerful for $\text{SQUARE}_n$, because the effective resistance between any pair of vertices is bounded between a constant and $O(\log n)$. The following lemma makes use of a classical result that can be obtained using Thompson’s principle of the electrical flow [DS84].

Lemma 2.8. For any non-sink vertex $u$ in $\text{SQUARE}_n$, we have

$$\frac{1}{4} \leq R_{\text{eff}}^{\text{SQUARE}_n}(v_{Sink}, u) \leq 2 \log n + 1.$$ 

The previous result and Lemma 2.7 immediately imply the following lemma, which allows us to swap vertices when computing potentials. Both proofs are deferred to Appendix A.

Lemma 2.9. For any pair of non-sink vertices $u$ and $v$ in $\text{SQUARE}_n$, we have

$$\pi_{\text{SQUARE}_n}(u)_v \leq (8 \log n + 4) \pi_{\text{SQUARE}_n}(v)_u.$$ 

Voltages and flows on electrical networks are also central to many recent developments in algorithmic graph theory [CKM+11, Mad13]. Variants of these algorithms are closely related with random processes on graphs, and have been examined extensively recently [BMV12, Meh13, BBD+13, SV16a, SV16b, SV16c]. The convergence of many of these algorithms depend on the extremal voltage values of the electrical flow that they construct. As a result, we believe some of our techniques are relevant to the grid-based instantiations of these algorithms.
3 Upper Bound for the Transience Class of the Square Grid

In this section, we prove the upper bound stated in Theorem 1.1 for the transience class of the Abelian sandpile model on the square grid. Our proof follows the framework of Choure and Vishwanathan in that we use Theorem 2.4 to reduce the proof to bounding the following two quantities for any non-sink vertex \( u \in V(\text{SQUARE}_n) \):

1. We upper bound the sum of vertex potentials
   \[
   \sum_{v \in V} \pi^{\text{SQUARE}_n}(u)_v.
   \]

2. For all non-sink vertices \( v \), we lower lower the potential \( \pi^{\text{SQUARE}_n}(u)_v \).

By symmetry, we assume without loss of generality that \( u \) is in the top-left quadrant of \( \text{SQUARE}_n \) (i.e., \( 1 \leq u_1, u_2 \leq \lceil n/2 \rceil \)). The first key idea is to use reciprocity from Lemma 2.7 and effective resistance bounds from Lemma 2.8 to instead bound \( \pi^{\text{SQUARE}_n}(v)_u \) at the expense of \( O(\log n) \) factors. The second key idea is to interpret potentials as random walks using Fact 2.5 and decouple two-dimensional walks on \( \text{SQUARE}_n \) into separate horizontal and vertical one-dimensional walks on \( \text{PATH}_n \). Observe that a random step in \( \text{SQUARE}_n \) is equivalent to choosing the horizontal or vertical direction and then taking a random step in that direction. Using well-studied properties of one-dimensional random walks, we achieve nearly-tight bounds on the transience class \( tcl(\text{SQUARE}_n) \).

Finally, we note that there is a natural trade-off in the choice of the source vertex \( u \). Setting \( u \) near the boundary decreases vertex potentials, as a random walk has a greater chance of stepping off the boundary into \( v_{\text{Sink}} \). This improves the upper bound of the sum of vertex potentials, but it weakens the lower bound of the minimum vertex potential. For vertices \( u \) that are not near the boundary, the opposite is true. Therefore, we account for the choice of \( u \) in our bounds. We proceed with the following outline. In Section 3.1, we give an upper bound for \( \sum_v \pi^{\text{SQUARE}_n}(u)_v \). In Section 3.2, we give a lower bound for \( \pi^{\text{SQUARE}_n}(v)_u \). In Section 3.3, we combine these bounds to prove Theorem 1.1.

3.1 Upper Bound for the Sum of Vertex Potentials

**Lemma 3.1.** For any non-sink vertex \( u \) in \( \text{SQUARE}_n \), we have

\[
\sum_{v \in V} \pi^{\text{SQUARE}_n}(u)_v = O\left( u_1 u_2 \log^3 n \right).
\]

**Proof.** We use Fact 2.5 and Lemma 2.6 to interpret vertex potentials as random walks. We can omit \( v_{\text{Sink}} \), because any random walk starting there immediately terminates. By Lemma 2.9,

\[
\pi^{\text{SQUARE}_n}(u)_v = O\left( \pi^{\text{SQUARE}_n}(v)_u \log n \right),
\]

so we can apply the random walk interpretation to potentials starting at \( u \) instead of \( v \). Recall that \( \pi^{\text{SQUARE}_n}(v)_u \) is the probability that a random walk starting from \( u \) reaches \( v \) before the sink. Consider one such walk \( w \in \mathcal{W}^{\text{SQUARE}_n}(u \to v) \) and its decomposition into the two dimensions, \( w_1 \) and \( w_2 \). The probability of a walk from \( u \) reaching \( v \) is equal to the probability...
that two interleaved walks in PATH\(_n\) starting from \(u_1\) and \(u_2\) reach \(v_1\) and \(v_2\), respectively, at the same time before either hits the one-dimensional sink \(v_{\text{Sink}} = \{0, n + 1\}\).

If we remove the restriction that these walks hit \(v_1\) and \(v_2\) at the same time and only require that they visit \(v_1\) and \(v_2\) before hitting \(v_{\text{Sink}}\), respectively, then each of these less restricted walks \(w_d\) belongs to the class \(\mathcal{W}_{\text{PATH}}(u_d \rightarrow v_d)\). Viewing a walk \(w\) on SQUARE\(_n\) as infinite two-dimensional walk on \(\mathbb{Z}^2\) gives independence between the walks \(w_1\) and \(w_2\). Therefore, we can obtain the upper bound

\[
\pi_{\text{SQUARE}}(v)_u = \Pr_{w \sim \mathcal{W}_{\mathbb{Z}^2}}(w \text{ hits } v \text{ before leaving SQUARE}_n) \\
\leq \Pr_{w \sim \mathcal{W}_{\mathbb{Z}^2}}(w_1 \text{ hits } v_1 \text{ before } v_{\text{Sink}} \text{ and } w_2 \text{ hits } v_2 \text{ before } v_{\text{Sink}}) \\
= \Pr_{w \sim \mathcal{W}_{\mathbb{Z}^2}}(w_1 \text{ hits } v_1 \text{ before } v_{\text{Sink}}) \Pr_{w \sim \mathcal{W}_{\mathbb{Z}^2}}(w_2 \text{ hits } v_2 \text{ before } v_{\text{Sink}}) \\
= \pi_{\text{PATH}}(v_1)_{u_1} \pi_{\text{PATH}}(v_2)_{u_2}.
\]

Summing over all choices of \(v = (v_1, v_2)\) gives

\[
\sum_{v \in V(\text{SQUARE}_n)} \pi_{\text{SQUARE}}(v)_u \leq \left( \sum_{v_1 = 1}^{n} \pi_{\text{PATH}}(v_1)_{u_1} \right) \left( \sum_{v_2 = 1}^{n} \pi_{\text{PATH}}(v_2)_{u_2} \right).
\]

The potentials of vertices in PATH\(_n\) have the following closed-form solution, as shown in [DS84]:

\[
\pi_{\text{PATH}}(v_1)_{u_1} = \begin{cases} 
\frac{n+1-u_1}{n+1-v_1} & \text{if } v_1 \leq u_1, \\
\frac{u_1}{v_1} & \text{if } v_1 > u_1.
\end{cases}
\]

Splitting the sum at \(u_1\) and using the fact that potentials are escape probabilities, we have

\[
\sum_{v_1 = 1}^{n} \pi_{\text{PATH}}(v_1)_{u_1} \leq u_1 + \sum_{v_1 = u_1+1}^{n} \frac{u_1}{v_1} = O(u_1 \log n).
\]

We similarly obtain an upper bound of \(O(u_2 \log n)\) in the other dimension. These bounds along with the initial \(O(\log n)\) overhead from swapping all \(u\) and \(v\) gives the desired upper bound. \(\square\)

### 3.2 Lower Bound for the Minimum Vertex Potential

The more involved part of our paper proves a lower bound for the minimum vertex potential

\[
\min_{v \in V \setminus \{v_{\text{Sink}}\}} \pi_{\text{SQUARE}}(u)_v
\]

as a function of some fixed vertex \(u = (u_1, u_2)\). Recall that we assume without loss of generality that \(u\) is in the top-left quadrant of SQUARE\(_n\). We begin by proving that the minimum potential occurs in the corner farthest from \(u\):

\[
\min_{v \in V \setminus \{v_{\text{Sink}}\}} \pi_{\text{SQUARE}}(u)_v = \Omega\left( \pi_{\text{SQUARE}}((n,n))_u \right).
\]

Using Lemma 2.9 to swap \(u\) and \((n,n)\) at the expense of a \(\Omega(1/\log n)\) factor, we reduce the problem to giving a lower bound for \(\pi_{\text{SQUARE}}((n,n))_u\). We proceed by decomposing a random
walk \( w \in \mathcal{W}_{\text{Square}}^n (u \rightarrow \{(n, n), v_{\text{Sink}}\}) \) into the two one-dimensional walks \( w_1 \in \mathcal{W}_{\text{Path}}^n (u_1) \) and \( w_2 \in \mathcal{W}_{\text{Path}}^n (u_2) \). We interpret \( \pi_{\text{Square}}^n ((n, n))_u \) as the probability that the individual processes \( w_1 \) and \( w_2 \) visit \( n \) at the same time before either walk leaves the interval \([1, n]\). Walks on \text{Line} that meet at \( n \) before leaving the interval \([1, n]\) are equivalent to walks on \text{Path}_n that meet at \( n \) before terminating at \( v_{\text{Sink}} \). Therefore, we will use conditional probabilities to analyze walks on \text{Line} instead of walks on \text{Path}_n in order to leverage well-known facts about random walks on \text{Line}.

To give a lower bound for the desired probability \( \pi_{\text{Square}}^n ((n, n))_u \), we show that a subset of \( \mathcal{W}_{\text{Square}}^n (u \rightarrow (n, n)) \) containing interleaved one-dimensional walks starting at \( u_1 \) and \( u_2 \) that first reach \( n \) in approximately the same number of steps has a sufficient amount of probability mass. We do so by noticing that the distributions of the number of steps for the walks to first reach \( n \) without leaving the interval \([1, n]\) are concentrated around \((n - u_1)^2\) and \((n - u_2)^2\), respectively. Consequently, we show that this distribution is approximately uniform in a \( \Theta(n^2) \) interval, with each \( t \)-step having probability \( \Omega(u_1/n^3) \) and \( \Omega(u_2/n^3) \). These intervals of walk lengths have \( \Theta(n^2) \) overlap, because \((n - u_1)\) and \((n - u_2)\) are within constant factors due to the assumption of \( 1 \leq u_1, u_2 \leq [n/2] \). Finally, we use Chernoff bounds to show that both walks have take approximately the same number of steps with constant probability. Combining these facts, we prove the desired lower bound \( \Omega(u_1 u_2/n^4) \).

### 3.2.1 Opposite Corner Minimizes Potential

**Lemma 3.2.** If \( u \) is a vertex in the top-left quadrant of \( \text{Square}_n \), then

\[
\pi_{\text{Square}}^n (u)_v \geq \frac{1}{16} \pi_{\text{Square}}^n (u)_{(n,n)},
\]

for any non-sink vertex \( v \in V(\text{Square}_n) \setminus \{v_{\text{Sink}}\} \).

**Proof.** We use Lemma 2.6 to decompose \( \pi_{\text{Square}}^n (u)_v \) as a sum of probabilities of walks, and then construct maps for all \( 1 \leq v_1, v_2 \leq n \) that show

\[
\pi_{\text{Square}}^n (u)_{(v_1,v_2)} \geq \max \left\{ \frac{1}{4} \pi_{\text{Square}}^n (u)_{(n,v_2)}, \frac{1}{4} \pi_{\text{Square}}^n (u)_{(v_1,n)} \right\}.
\]

We begin by considering the first dimension: \( \pi_{\text{Square}}^n (u)_{(v_1,v_2)} \geq \pi_{\text{Square}}^n (u)_{(n,v_2)}/4 \). Let \( \ell_{\text{hor}} \) be the horizontal line of reflection passing through \( \left(\lceil (v_1 + n)/2 \rceil, 1\right) \) and \( \left(\lceil (v_1 + n)/2 \rceil, n\right) \) in \( \mathbb{Z}^2 \), and let \( u^* \) be the reflection of \( u \) over \( \ell_{\text{hor}} \). Note that \( u^* \) may be outside of the \( n \times n \) grid.

Next, define the map

\[
f : \mathcal{W}_{\text{Square}}^n ((n, v_2) \rightarrow u) \rightarrow \mathcal{W}_{\text{Square}}^n ((v_1, v_2) \rightarrow u)
\]

as follows. For any walk \( w \in \mathcal{W}_{\text{Square}}^n ((n, v_2) \rightarrow u) \):

1. Start the walk \( f(w) \) at \( (v_1, v_2) \), and if \( n - v_1 \) is odd move to \( (v_1 + 1, v_2) \).
2. Perform \( w \) but make opposite vertical moves before the walk hits \( \ell_{\text{hor}} \), so that the partial walk is a reflection over \( \ell_{\text{hor}} \).
3. After hitting \( \ell_{\text{hor}} \) for the first time, continue performing \( w \), but now use the original vertical moves.
4. Terminate this walk when it first reaches \( u \).

Denote the preimage of a walk \( w' \in W^{\text{Square}_n}((v_1, v_2) \to u) \) under \( f \) to be

\[
f^{-1}(w') = \{ w \in W^{\text{Square}_n}((n, v_2) \to u) : f(w) = w' \}.
\]

We claim that for any \( w' \in W^{\text{Square}_n}((v_1, v_2) \to u) \), we have

\[
\frac{1}{4} \sum_{w \in f^{-1}(w')} 4^{-|w|} \leq 4^{-|w'|}.
\]

If \( f^{-1}(w') = \emptyset \) the claim is true, so assume \( f^{-1}(w') \neq \emptyset \). We analyze two cases. If \( w' \) hits \( \ell_{\text{hor}} \), then \( f^{-1}(w') \) contains exactly one walk \( w \) of length \( |w'| \) or \( |w'| - 1 \). If \( w' \) does not hit \( \ell_{\text{hor}} \), then

\[
f^{-1}(w') = \{ w \in W^{\text{Square}_n}((n, v_2) \to u) : w \text{ is a reflection of } w' \text{ over } \ell_{\text{hor}} \text{ before } w \text{ hits } u^* \}.
\]

It follows that any walk \( w \in f^{-1}(w') \) can be split into \( w = w_1w_2 \), where \( w_1 \) is the unique walk from \( (n, v_2) \) to \( u^* \) that is a reflection of \( w' \), and \( w_2 \) is a walk from \( u^* \) to \( u \) that avoids \( v_{\text{Sink}} \) and hits \( u \) exactly once upon termination. Clearly \( w_1 \) has length \(|w'| \) or \(|w'| - 1 \), and the set of admissible \( w_2 \) walks is \( W^{\text{Square}_n}(u^* \to u) \). Therefore,

\[
\frac{1}{4} \sum_{w \in f^{-1}(w')} 4^{-|w|} = 4^{-|w_1| - 1} \sum_{w_2 \in W^{\text{Square}_n}(u^* \to u)} 4^{-|w_2|} = 4^{-|w_1| - 1} \pi^{\text{Square}_n}(u^*) \leq 4^{-|w'|},
\]

since \( \pi^{\text{Square}_n}(u^*) \) is an escape probability. Summing over all \( w' \in W^{\text{Square}_n}((v_1, v_2) \to u) \), it follows from Lemma 2.6 and the previous inequality that

\[
\pi^{\text{Square}_n}(u)_{(v_1, v_2)} = \sum_{w' \in W^{\text{Square}_n}((v_1, v_2) \to u)} 4^{-|w'|} \geq \sum_{w' \in W^{\text{Square}_n}((v_1, v_2) \to u)} 4^{-|w|} \geq \frac{1}{4} \pi^{\text{Square}_n}(u)_{(n, v_2)},
\]

because every \( w \in W^{\text{Square}_n}((n, v_2) \to u) \) is the preimage of a \( w' \in W^{\text{Square}_n}((v_1, v_2) \to u) \).

Similarly, we can show that \( \pi^{\text{Square}_n}(u)_{(v_1, v_2)} \geq \pi^{\text{Square}_n}(u)_{(v_1, v_2)}/4 \) for all \( 1 \leq v_1 \leq n \) by reflecting walks over the vertical line from \((1, [(n + v_2)/2]) \) to \((n, [(n + v_2)/2]) \). Combining both inequalities proves the claim. \( \square \)

### 3.2.2 Lower Bound for the Opposite Corner Potential

As mentioned at the start of Section 3.2, by decomposing two-dimensional walks on \( \text{Square}_n \) into one-dimensional walks on \( \text{Line} \), our lower bound relies on showing that there is a \( \Theta(n^2) \) length interval such that each one-dimensional walk of a given length in that interval has probability \( \Omega(u_1/n^3) \) or \( \Omega(u_2/n^3) \), respectively, of remaining above \( 0 \) and reaching \( n \) for the first time at the end of the walk. For our purposes, lower bounds on this probability will suffice, and they follow from the following key property for one-dimensional walks, which we prove in Section 5.
Lemma 3.3. Let $n$ be any positive integer and $1 \leq i \leq \lfloor n/2 \rfloor$ be any starting position. For any constant $c > 4$ and any integer $t$ such that $n^2/c \leq t \leq n^2/4$ with $t \equiv n - i \pmod{2}$, a simple symmetric random walk random walk $w$ on $\mathbb{Z}$ satisfies

$$
\Pr_{w \sim \mathcal{W}^{\text{LINE}}(i)} \left[ w(t) = n, \max(w) = n, \text{and} \min(w) \geq 1 \right] \geq e^{-2-2c \cdot i} / n^3.
$$

Using this property with the following lemma, we give a lower bound for $\pi_{\text{SQUARE}}(\langle n, n \rangle)_u$, the probability that a walk starting from $u$ reaches $(n, n)$ before $v_{\text{Sink}}$. The following lemma is a consequence of a Chernoff bound. We defer its proof to Appendix B.

Lemma 3.4. For all $n \geq 10$, we have

$$
\min \left\{ \frac{1}{2n} \sum_{k = \left[ \frac{n}{2} \right]}^{\left[ \frac{3n}{4} \right]} \binom{n}{k}, \frac{1}{2n} \sum_{k = \left[ \frac{n}{4} \right]}^{\left[ \frac{3n}{4} \right]} \binom{n}{k} \right\} \geq \frac{2}{5}.
$$

Lemma 3.5. For all $n \geq 10$ and $u \in \text{SQUARE}_n$ such that $1 \leq u_1, u_2 \leq \lfloor n/2 \rfloor$, we have

$$
\pi_{\text{SQUARE}}(\langle n, n \rangle)_u \geq e^{-100 \frac{u_1 u_2}{n^4}}.
$$

Proof. We decouple any walk $w \in \mathcal{W}^{\text{SQUARE}}(u \rightarrow (n, n))$ into a horizontal walk $w_1 \in \mathcal{W}^{\text{LINE}}(u_1)$ and a vertical walk $w_2 \in \mathcal{W}^{\text{LINE}}(u_2)$. We view $\pi_{\text{SQUARE}}(\langle n, n \rangle)_u$ as the probability that the walks $w_1$ and $w_2$ visit $n$ at the same time before either leaves the interval $[1, n]$.

We can decompose any $t$-step walk on $\text{SQUARE}_n$ such that it takes $t_1$ steps in the horizontal direction and $t_2$ steps in the vertical direction. Considering instances where $w_1$ and $w_2$ visit $n$ exactly once, we obtain the following bound:

$$
\pi_{\text{SQUARE}}(\langle n, n \rangle)_u \geq \sum_{t_1, t_2 \geq 0} \frac{(t_1 + t_2)}{2t_1 + t_2} \Pr_{w_1 \sim \mathcal{W}^{\text{LINE}}(u_1)} \left[ w_1(t_1) = n, \max(w) = n - 1, \text{and} \min(w) \geq 1 \right] \\
\cdot \Pr_{w_2 \sim \mathcal{W}^{\text{LINE}}(u_2)} \left[ w_2(t_2) = n, \max(w) = n - 1, \text{and} \min(w) \geq 1 \right].
$$

By our choice of $u$ and $n$, this probability is equivalent to

$$
\sum_{t_1, t_2 \geq 5} \frac{(t_1 + t_2)}{2t_1 + t_2} \left( \frac{1}{2} \Pr_{w_1 \sim \mathcal{W}^{\text{LINE}}(u_1)} \left[ w_1(t_1 - 1) = n - 1, \max(w) = n - 1, \text{and} \min(w) \geq 1 \right] \right) \\
\cdot \left( \frac{1}{2} \Pr_{w_2 \sim \mathcal{W}^{\text{LINE}}(u_2)} \left[ w_2(t_2 - 1) = n - 1, \max(w) = n - 1, \text{and} \min(w) \geq 1 \right] \right).
$$

Letting $t = t_1 + t_2$, we further refine the set of two-dimensional walks so that $t \in [n^2/5, n^2/4]$ and $t_1, t_2 \in [t/4, 3t/4]$, while still capturing a sufficient amount of probability mass. Note that for valid walks, the parities of $t_1$ and $t_2$ satisfy $t_1 \equiv n - u_1 \pmod{2}$ and $t_2 \equiv n - u_2 \pmod{2}$.
Let $I$ be an indexing of all such pairs $(t_1, t_2)$. Then by the previous inequalities, Lemma 3.3, and Lemma 3.4, we have

$$
\pi^{\text{Square}_n}((n, n)) \geq \sum_{(t_1, t_2) \in I} \left( \frac{1}{2} e^{-2 - 2(20)} \frac{u_1}{n^3} \right) \left( \frac{1}{2} e^{-2 - 2(20)} \frac{u_2}{n^3} \right)
$$

$$
\geq e^{-84} \frac{u_1 u_2}{4n^6} \sum_{t \in \left[ \frac{n^2}{5}, \frac{n^2}{4} \right]} \frac{2}{5} \sum_{t \equiv u_1 + u_2 \pmod{2}}
$$

$$
\geq e^{-84} \frac{u_1 u_2}{4n^6} \cdot \frac{n^2}{50} \cdot \frac{2}{5}
$$

$$
\geq e^{-100} \frac{u_1 u_2}{n^4}.
$$

For the first inequality, we apply Lemma 3.3 because $t_1, t_2 \geq n^2 / 20$. For the second inequality, we group pairs $(t_1, t_2)$ by their sum $t = t_1 + t_2$ and use Lemma 3.4. Finally, the number of $t \in \left[ \frac{n^2}{5}, \frac{n^2}{4} \right]$ with either parity restriction is at least $\left\lfloor \frac{n^2}{40} \right\rfloor \geq \frac{n^2}{50}$, for $n \geq 10$.

3.3 Proof of Theorem 1.1

We can now combine the upper bound for the sum given by Lemma 3.1 and the lower bound in Section 3.2 to obtain the overall bound.

**Proof of Theorem 1.1.** For any $u = (u_1, u_2)$ in the top-left quadrant of $\text{Square}_n$, we have

$$
\max_{u, v \in V \setminus \{v_{\text{Sink}}\}} \left( \sum_{x \in V} \pi^{\text{Square}_n}(u) \right) \pi^{\text{Square}_n}(u)^{-1}
$$

$$
\leq \max_{u \in V \setminus \{v_{\text{Sink}}\}} \left( \sum_{x \in V} \pi^{\text{Square}_n}(u) \right) \pi^{\text{Square}_n}(u)^{-1} \frac{16}{\pi^{\text{Square}_n}((n, n))}
$$

(by Lemma 3.2)

$$
= \left( \sum_{x \in V} \pi^{\text{Square}_n}(u) \right) O(\log n)
$$

(by Lemma 2.9)

$$
= O(u_1 u_2 \log^3 n) O\left( \frac{n^4 \log n}{u_1 u_2} \right)
$$

(by Lemma 3.5 and Lemma 3.1)

$$
= O\left( n^4 \log^4 n \right).
$$

The result then follows from Theorem 2.4.

4 Lower Bound for the Transience Class of the Square Grid

In this section, we will show the lower bound for $tcl(\text{Square}_n)$ in a similar way to Section 3. Since the lower bound of Theorem 2.4 takes the maximum inverse vertex potential over all pairs of non-sink vertices $u$ and $v$, it will suffice to simply give an upper bound on $\pi^{\text{Square}_n}((n, n))(1, 1)$, which intuitively will give the smallest vertex potential. As in Section 3 where we gave a lower
bound on vertex potentials, we will decompose two-dimensional walks on SQUARE_n into one-dimensional walks on LINE and give upper bounds on the probability that a t step walk on LINE starting at 1, ends at n and does not leave the interval [1, n]. More specifically, our upper bound on $\pi^{\text{SQUARE}_n}((n, n))_{(1,1)}$ will follow from the following key lemma regarding walks on LINE, which we will prove in Section 5

**Lemma 4.1.** For $n \geq 20$

$$\Pr_{w \sim \mathcal{W}_{\text{LINE}}(1)} \left[ w(t) = n \text{ and } \max(w) = n \text{ and } \min(w) \geq 1 \right] \leq \min \left( \frac{e^{25}}{n^3}, 64 \left( \frac{1}{t} \right)^3 \right)$$

By decoupling in a nearly identical manner to the proof of Lemma 3.5, we can then apply this property to one-dimensional walks and achieve our desired upper bound.

**Lemma 4.2.**

$$\pi^{\text{SQUARE}_n}((n, n))_{(1,1)} \leq 2 \max_t \left( \Pr_{w \sim \mathcal{W}_{\text{LINE}}(1)} \left[ w(t) = n \text{ and } \max(w) = n \text{ and } \min(w) \geq 1 \right] \right) \cdot \sum_{t \geq 0} \Pr_{w \sim \mathcal{W}_{\text{LINE}}(1)} \left[ w(t) = n \text{ and } \max(w) = n \text{ and } \min(w) \geq 1 \right]$$

**Proof.** Analogously to our lower bound proof of $\pi^{\text{SQUARE}_n}((n, n))_{(1,1)}$, we decouple any walk $w \in \mathcal{W}_{\text{SQUARE}_n}((1,1) \rightarrow (n, n))$ into a horizontal walk $w_1 \in \mathcal{W}_{\text{LINE}}(1)$ and a vertical walk $w_2 \in \mathcal{W}_{\text{LINE}}(1)$. We view $\pi^{\text{SQUARE}_n}((n, n))_{(1,1)}$ as the probability that the walks $w_1$ and $w_2$ visit $n$ at the same time before either leaves the interval [1, n].

Similarly, we can decompose any t-step walk on SQUARE_n such that it takes $t_1$ steps in the horizontal direction and $t_2$ steps in the vertical direction. Now however, we only require that $w_1$ and $w_2$ both visit $n$ on the final step, so both walks could have previously visited $n$, and possibly at the same time. This then gives the upper bound:

$$\pi^{\text{SQUARE}_n}((n, n))_{(1,1)} \leq \sum_{t_1, t_2 \geq 0} \frac{(t_1 + t_2)}{2t_1 + t_2} \Pr_{w_1 \sim \mathcal{W}_{\text{LINE}}(1)} \left[ w_1^{(t_1)} = n, \max(w) = n, \text{ and } \min(w) \geq 1 \right] \cdot \Pr_{w_2 \sim \mathcal{W}_{\text{LINE}}(1)} \left[ w_2^{(t_2)} = n, \max(w) = n, \text{ and } \min(w) \geq 1 \right].$$

By separating the summations we are given

$$\pi^{\text{SQUARE}_n}((n, n))_{(1,1)} \leq \sum_{t_1 \geq 0} \Pr_{w_1 \sim \mathcal{W}_{\text{LINE}}(1)} \left[ w_1^{(t_1)} = n, \max(w) = n, \text{ and } \min(w) \geq 1 \right] \cdot \sum_{t_2 \geq 0} \frac{(t_1 + t_2)}{2t_1 + t_2} \Pr_{w_2 \sim \mathcal{W}_{\text{LINE}}(1)} \left[ w_2^{(t_2)} = n, \max(w) = n, \text{ and } \min(w) \geq 1 \right].$$

The inner summation can be upper bounded by using Fact 4.3.
\[
\sum_{t_2 \geq 0} \left( \frac{(t_1 + t_2)}{2^{t_1 + t_2}} \right) \Pr_{w_2 \sim \mathcal{W}_{\text{LINE}}(1)} \left[ w_2^{(t_2)} = n, \max(w) = n, \text{and} \min(w) \geq 1 \right] \leq \\
2 \max_{t_2} \left( \Pr_{w_2 \sim \mathcal{W}_{\text{LINE}}(1)} \left[ w_2^{(t_2)} = n, \max(w) = n, \text{and} \min(w) \geq 1 \right] \right)
\]

We can then bring this term in front of the summation, which concludes our proof.

**Fact 4.3.** For any nonnegative integer \( t_1 \), we have

\[
\sum_{t_2 \geq 0} \left( \frac{(t_1 + t_2)}{2^{t_1 + t_2}} \right) \frac{1}{2^{t_1 + t_2}} = 2.
\]

**Proof.** This follows directly from the negative binomial distribution. Observe that

\[
\sum_{t_2 \geq 0} \left( \frac{(t_1 + t_2)}{2^{t_1 + t_2}} \right) \frac{1}{2^{t_1 + t_2}} = 2 \sum_{t_2 \geq 0} \left( \frac{(t_1 + 1) - 1 + t_2}{2^{t_1 + 1}} \right) \frac{1}{2^{t_2}} = 2,
\]

as desired.

The upper bound on the maximum term in the RHS of Lemma 4.2 follows immediately from Lemma 4.1, and we will now upper bound the summation in the RHS of Lemma 4.2 which will be a simple consequence of Lemma 4.1.

**Lemma 4.4.**

\[
\sum_{t \geq 0} \Pr_{w \sim \mathcal{W}_{\text{LINE}}(1)} \left[ w^{(t)} = n \text{ and} \max(w) = n \text{ and} \min(w) \geq 1 \right] \leq \frac{e^{26}}{n}
\]

**Proof.** We first break the summation into

\[
\sum_{t \geq 0} \Pr_{w \sim \mathcal{W}_{\text{LINE}}(1)} \left[ w^{(t)} = n \text{ and} \max(w) = n \text{ and} \min(w) \geq 1 \right] = \\
\sum_{n^2 \geq t \geq 0} \Pr_{w \sim \mathcal{W}_{\text{LINE}}(1)} \left[ w^{(t)} = n \text{ and} \max(w) = n \text{ and} \min(w) \geq 1 \right] + \sum_{t > n^2} \Pr_{w \sim \mathcal{W}_{\text{LINE}}(1)} \left[ w^{(t)} = n \text{ and} \max(w) = n \text{ and} \min(w) \geq 1 \right]
\]

and then upper bound both terms by \( O\left(\frac{1}{n} \right) \). The upper bound of the first term follows immediately from Lemma 4.1 and the fact that we are summing over \( n^2 + 1 \) terms.

\[
\sum_{n^2 \geq t \geq 0} \Pr_{w \sim \mathcal{W}_{\text{LINE}}(1)} \left[ w^{(t)} = n \text{ and} \max(w) = n \text{ and} \min(w) \geq 1 \right] \leq \frac{e^{25}}{n}
\]
To bound the second summation, we again use Lemma 4.1 that

\[ \Pr_{w \sim W_{\text{Line}}(1)} \left[ w(t) = n \text{ and } \max(w) \leq t \geq t \right] \leq 64 \left( \frac{n}{t} \right)^3 \]

The fact that \(64 \left( \frac{n}{t} \right)^3\) is a decreasing function in \(t\) implies

\[ 64 \left( \frac{n}{t} \right)^3 \leq \int_{t-1}^{t} 64 \left( \frac{n}{t} \right)^3 dt \]

which gives

\[ \sum_{t>n^2} \Pr_{w \sim W_{\text{Line}}(1)} \left[ w(t) = n \text{ and } \max(w) = n \text{ and } \min(w) \geq 1 \right] \leq \int_{n^2}^{\infty} 64 \left( \frac{n}{t} \right)^3 dt = \frac{32}{n} \]

4.1 Proof of Theorem 1.2

We can now easily combine the lemmas in this section along with bounds relating vertex potential to the lower bound of the transience class to obtain our overall bound.

**Proof of Theorem 1.2.** It follows immediately from Lemmas 4.2, 4.1, and 4.4 that

\[ \pi_{\text{Square}}((n,n))((1,1)) \leq 2e^{25} \cdot \frac{e^{26}}{n^3} \leq \frac{e^{100}}{n^4} \]

Therefore, \( \pi_{\text{Square}}((n,n))^{-1} = \Omega(n^4) \) and the result then follows from Theorem 2.4. \(\square\)

5 Properties of Simple Symmetric Random Walks

Our proofs for upper and lower bounding the sandpile transience class on the grid heavily utilized decoupling 2-D walks into two independent 1-D walks because we claimed that these walks were easier to work with. This claim was immediately apparent for working with vertex potentials in 1-D walks that helped prove the upper bound on the sum of vertex potentials.

But, the two essential lemmas regarding 1-D walks for proving the lower and upper bound of the minimum vertex potential were assumed. Consequently, this section examines the probability

\[ \Pr_{w \sim W_{\text{Line}}(i)} \left[ w(t) = n \text{ and } \max(w) = n \text{ and } \min(w) \geq 1 \right] \]

and prove lower and upper bounds required for Lemma 3.3 and Lemma 4.1 by extending some of the previously known properties of random walks in 1-D. The key ideas in these proofs are:

1. The position of a walk in one dimension follows the binomial distribution.

2. Known explicit expressions for the maximum/minimum point of a walk given a certain number of steps.
3. Strong bounds on binomial coefficients via Stirling’s approximation.

The properties we need will not immediately follow from previously known facts because we require both a minimum and maximum. Section 5.1 will give proofs of the known explicit expressions for the maximum/minimum point of a walk along with several other useful facts that fall out of this proof. Section 5.2 will apply Stirling’s bound to give lower bounds on certain binomial coefficients. Section 5.3 and 5.4 will prove a several necessary lower bound lemmas, and Lemma 3.3 will be proven at the end of Section 5.4. Section 5.5 will then give the necessary upper bound lemmas and a proof of Lemma 4.1

Lower Bound

For lower bounding this probability, we first split our desired probability into the product of two probabilities due to the definition of conditional probability, and separately prove lower bounds of each.

1. In Lemma 5.6, we show that for $t \in \Theta(n^2)$, the probability that a 1-D walk starting at $1 \leq i \leq \lceil n \rceil$,

   \[
   \Pr_{w \sim \mathcal{W}_{\text{LINE}}(i)} \left[ \min_{\leq t}(w) \geq 1 \right] \geq \Omega \left( \frac{i}{n} \right).
   \]

2. In Lemma 5.8 and Lemma 5.7, we bound the probability of such a walk, under the conditions of $1 \leq i \leq \lfloor n/2 \rfloor$ and $t \in \Theta(n^2)$ reaches $n$ at step $t$ without ever going above $n$, while conditioned on not going below 1:

   \[
   \Pr_{w \sim \mathcal{W}_{\text{LINE}}(i)} \left[ w(t) = n \text{ and } \max_{\leq t}(w) = n \text{ and } \min_{\leq t}(w) \geq 1 \right] \geq \Omega \left( \frac{1}{n^2} \right).
   \]

Lemma 3.3 immediately follows multiplying these two bounds together. This division allows us to separate proving a minimum and maximum, and in turn simplifies applying known bounds on binomial distributions.

Specifically, Lemma 5.6 is an immediate consequence of explicit expressions for the minimum point of a walk and bounds on binomial coefficients, both of which will be given rigorous treatment in Section 5.1. These proofs will also output a known explicit expression for the probability of the walk reaching $n$ at step $t$, while only staying to its left. All that remains then is to add in the condition of the walk not going to the left of 1. To this end, note that 1 is in the opposite direction of $n$, with respect to the starting position $i$. So we will formally show that the probability of reaching $n$ without going above $n$ only improves if the walk cannot move too far in the wrong direction, but only for $t \leq (n - i + 1)^2$, giving the reason we need to upper bound $t$ by $n^2/4$, which completes the proof.

Upper Bound

For upper bounding this probability, our desired lemma only concerns walks that start at $i = 1$, which will be critical for our proof, and the key idea will then be to split the walk in half and consider the probability that the necessary conditions are satisfied in the first $t/2$ steps and in
the second \( t/2 \) steps. The midpoint of the walk at \( t/2 \) steps can be any point in the interval \([1, n]\), so we must sum over all these possible midpoints. Removing the upper and lower bound conditions, respectively, will then give the upper bound in Lemma 5.9.

\[
\Pr_{w \sim \mathcal{W}^{\text{line}}(1)} \left[ w^{(t)} = n \text{ and } \max_{\leq t} (w) = n \text{ and } \min_{\leq t} (w) \geq 1 \right] \leq \\
\sum_{i=1}^{n} \Pr_{w \sim \mathcal{W}^{\text{line}}(1)} \left[ w^{(\lfloor \frac{t}{2} \rfloor)} = i \text{ and } \min_{\leq \lfloor \frac{t}{2} \rfloor} (w) \geq 1 \right] \Pr_{w \sim \mathcal{W}^{\text{line}}(i)} \left[ w^{(\lceil \frac{t}{2} \rceil)} = n \text{ and } \max_{\leq \lfloor \frac{t}{2} \rfloor} (w) = n \right]
\]

Due to the first \( t/2 \) step walk starting at 1 and the second \( t/2 \) step walk ending at \( n \), the condition \( \min_{\leq t} (w) \geq 1 \) for the first walk and \( \max_{\leq \lceil \frac{t}{2} \rceil} (w) = n \) for the second walk, will be the more difficult property for each respective walk to satisfy. We can then apply known facts proven in Section 5.1 to obtain expressions for each term within the summation, and the remainder of the upper bound analysis will then focus on upper bounding those expressions, that contain binomial coefficients, with simpler expressions.

### 5.1 Maximum Position of a Walk

As mentioned above, our proofs will mostly leverage folklore facts regarding the maximum / minimum position of a walk and corresponding bounds for the expressions of these probabilities. This section will first give the statement and proof for the maximum / minimum position, then will show how Stirling’s approximation immediately gives strong bounds on certain binomial coefficients.

Note that if we are only concerned with a single end point, we can fix the starting location at 0 by shifting things accordingly. In these cases, the following bounds are well known in combinatorics.

**Fact 5.1 ([RB79]).** For any nonnegative integers \( t \) and \( n \), we have

\[
\Pr_{w \sim \mathcal{W}^{\text{line}}(0)} \left[ \max_{\leq t} (w) = n \right] = \begin{cases} 
\Pr_{w \sim \mathcal{W}^{\text{line}}(0)} \left[ w^{(t)} = n \right] = \left( \frac{t+n}{2} \right) \frac{1}{2^t} & \text{if } t + n \equiv 0 \pmod{2}, \\
\Pr_{w \sim \mathcal{W}^{\text{line}}(0)} \left[ w^{(t)} = n + 1 \right] = \left( \frac{t+n+1}{2} \right) \frac{1}{2^t} & \text{if } t + n \equiv 1 \pmod{2}.
\end{cases}
\]

**Proof.** For any \( k \leq n \), consider a walk \( w \in \mathcal{W}^{\text{line}}(0) \) that satisfies \( w^{(t)} = k \) and \( \max_{\leq t} (w) \geq n \). Let \( t^* \) be the first time that \( w^{(t^*)} = n \), and construct the walk \( m \) ending at \( 2n - k \) such that

\[
m(i) = \begin{cases} 
w(i) & \text{if } 0 \leq i \leq t^*, \\
2n - w(i) & \text{if } t^* < i \leq t.
\end{cases}
\]

This reflection map is a bijection, so it follows for \( k \leq n \) that

\[
\Pr_{w \sim \mathcal{W}^{\text{line}}(0)} \left[ w^{(t)} = k \text{ and } \max_{\leq t} (w) \geq n \right] = \Pr_{w \sim \mathcal{W}^{\text{line}}(0)} \left[ w^{(t)} = 2n - k \right].
\]
Subtracting the probability of the maximum position being at least $n + 1$ gives
\[
\Pr[\omega \sim W^{\text{LXN}}(0) \mid w(t) = k \text{ and } \max(w) = n] = \Pr[\omega \sim W^{\text{LXN}}(0) \mid w(t) = 2n - k] - \Pr[\omega \sim W^{\text{LXN}}(0) \mid w(t) = 2(n + 1) - k].
\]
Summing over all $k \leq n$, we have
\[
\Pr[\omega \sim W^{\text{LXN}}(0) \mid \max(w) = n] = \Pr[\omega \sim W^{\text{LXN}}(0) \mid w(t) = 2n - k] + \Pr[\omega \sim W^{\text{LXN}}(0) \mid w(t) = n + 1].
\]
Considering the parity of $t$ and $n$ completes the proof.

**Fact 5.2.** For any integers $n \geq 0$ and $k \leq n$, we have
\[
\Pr[\omega \sim W^{\text{LXN}}(0) \mid w(t) = k \text{ and } \max(w) \geq n] = \Pr[\omega \sim W^{\text{LXN}}(0) \mid w(t) = 2n - k].
\]

**Fact 5.3.** Let $t$ and $n$ be nonnegative integers. For any integer $k \leq n$, we have
\[
\Pr[\omega \sim W^{\text{LXN}}(0) \mid w(t) = k \text{ and } \max(w) = n] = \Pr[\omega \sim W^{\text{LXN}}(0) \mid w(t) = 2n - k] - \Pr[\omega \sim W^{\text{LXN}}(0) \mid w(t) = 2(n + 1) - k] = \begin{cases} 
\left(\frac{t}{2}\right)^{2t} - \left(\frac{t+2n-k}{2}\right)^{2t} & \text{if } t + k \equiv 0 \pmod{2}, \\
0 & \text{if } t + k \equiv 1 \pmod{2}.
\end{cases}
\]
Proof. The first equality follows from the proof of Fact 5.1. Using Fact 5.1 and analyzing the parity of the walks gives
\[
\left(\frac{t}{2}\right)^{2t} - \left(\frac{t+2n-k}{2}\right)^{2t} = \left(\frac{t}{2}\right)^{2t} - \left(\frac{t-2n+k}{2}\right)^{2t} - \left(\frac{t+2n-k}{2}\right)^{2t} = \left(\frac{t}{2}\right)^{2t} - \left(\frac{t-2n+k}{2}\right)^{2t} - \left(\frac{t+2n-k}{2}\right)^{2t} = \left(\frac{t}{2}\right)^{2t} \left(\frac{4n-2k+2}{t+2n-k+2}\right),
\]
as desired.

## 5.2 Lower Bound for Binomial Coefficients
Ultimately, our goal is to give strong lower bounds on closely related probabilities to the ones above. To do so, we need to use various bounds on binomial coefficients that are consequences of Stirling’s approximation.

**Fact 5.4** (Stirling’s Approximation). For any positive integer $n$, we have
\[
\sqrt{2\pi n!} \leq \frac{n!}{\sqrt{n} \left(\frac{n}{e}\right)^n} \leq e.
\]
An immediate consequence of this is a concentration bound on binomial coefficients.

**Fact 5.5.** Let $c, n \in \mathbb{R}_{>0}$ such that $c\sqrt{n} < n$. For any $k \in [(n - c\sqrt{n})/2, (n + c\sqrt{n})/2]$, we have

\[
\binom{n}{k} \geq e^{-1 - c^2} \frac{2^n}{\sqrt{n}}.
\]

**Proof.** We directly substitute Stirling’s approximation to the definition of binomial coefficients:

\[
\binom{n}{k} = \frac{n!}{(n-k)! (n-k)!} \geq \frac{\sqrt{2\pi n} (\frac{n}{e})^n}{e^{n/2}} \cdot \frac{2^n}{\sqrt{n}} \geq \frac{2\sqrt{2\pi} \cdot \frac{1}{e^2} \cdot 2^n}{\sqrt{n}} \geq e^{-1 - c^2} \frac{2^n}{\sqrt{n}},
\]

as desired. \(\square\)

### 5.3 Lower Bound for the Minimum Position of a Walk

We now bound the probability of the minimum position of a walk in $W_{\text{LINE}}(i)$ being at least 1 after $t$ steps.

**Lemma 5.6.** For any positive integer $n$, initial position $1 \leq i \leq \lceil n/2 \rceil$, and constant $c > 4$, if we have $t \in [n^2/c, n^2/4]$, then

\[
\Pr_{w \sim W_{\text{LINE}}(i)} \left[ \min_{\leq t} (w) \geq 1 \right] \geq e^{-1 - c} \frac{i}{n}.
\]

**Proof.** First observe that

\[
\Pr_{w \sim W_{\text{LINE}}(i)} \left[ \min_{\leq t} (w) \geq 1 \right] = \sum_{k=1}^{i} \Pr_{w \sim W_{\text{LINE}}(i)} \left[ \min_{\leq t} (w) = k \right].
\]

By symmetry, this sum is equal to

\[
\sum_{k=0}^{i-1} \Pr_{w \sim W_{\text{LINE}}(0)} \left[ \max_{\leq t} (w) = k \right].
\]

For each $0 \leq k \leq i - 1$, Fact 5.1 implies that

\[
\Pr_{w \sim W_{\text{LINE}}(0)} \left[ \max_{\leq t} (w) = k \right] \in \left\{ \left( \frac{t}{2^t} \right) \frac{1}{2^t}, \left( \frac{t}{t+k+1} \right) \frac{1}{2^t} \right\}.
\]
By assumption
\[ k \leq k + 1 \leq i \leq n \leq \sqrt{ct}, \]
so applying Fact 5.5 gives
\[
\min\left\{ \left( \frac{t}{2^i} \right) \frac{1}{2t}, \left( \frac{t}{2^i} \right) \frac{1}{2t} \right\} \geq \left( \frac{t}{t+\sqrt{ct}} \right) \frac{1}{2t} \\
\geq e^{-1-c} \frac{1}{\sqrt{t}} \\
\geq e^{-1-c} \frac{1}{n},
\]
because \( t \leq n^2/4 \). Summing over \( 0 \leq k \leq i - 1 \) gives the desired bound.

5.4 Lower Bound for the Final and Maximum Position of a Walk

Similarly, we can use binomial coefficient approximations to bound the probability of a \( t \)-step walk terminating at \( n \) while never moving to a position greater than \( n \).

**Lemma 5.7.** For any initial position \( 1 \leq i \leq \lceil n/2 \rceil \) and any time \( \max\{n, n^2/c\} \leq t \leq n^2/4 \) with \( t \equiv n - i \pmod{2} \), we have

\[
\Pr_{w \sim \mathcal{W}_{\text{line}}(i)} \left[ \max_{\leq t} (w) = n \text{ and } w(t) = n \right] \geq e^{-1-c} \frac{1}{n^2}. 
\]

**Proof.** First, by symmetry we will rewrite the probability as

\[
\Pr_{w \sim \mathcal{W}_{\text{line}}(0)} \left[ \max_{\leq t} (w) = n - i \text{ and } w(t) = n - i \right].
\]

Fact 5.3 gives that this probability equals to

\[
\frac{1}{2^t} \left( \frac{t}{2^{n-i}} \right) \frac{2(n-i+1)}{t+n-i+2}.
\]

We can then separately bound the last two terms according to the assumptions on \( t \) and \( i \). Setting \( i = 0 \) minimizes \( \left( \frac{t}{2^{n-i}} \right) \) for all \( i \geq 0 \). Setting \( i = \lceil n/2 \rceil \) in the numerator, \( i = 0 \) in the denominator, and \( t = n^2/4 \) minimizes \( \frac{2(n-i+1)}{t+n+2} \).

\[
\geq \frac{1}{2^t} \left( \frac{t}{2^{n+2}} \right) \frac{2(\lceil n/2 \rceil + 1)}{n^2/4 + n + 2} \geq \frac{1}{2^t} \left( \frac{t}{2^{(n+2)}} \right) \frac{n}{n^2/4}
\]

We can then again apply Fact 5.5 using the observation that \( n \leq \sqrt{c} \cdot \sqrt{t} \) to get:

\[
\geq \frac{1}{2^t} \left( \frac{t}{\sqrt{c} \cdot \sqrt{t}} \right) \frac{n}{n^2} \geq e^{-1-c} \frac{1}{n^2}.
\]

\[ \square \]
It remains to condition upon the minimum of a walk. This hinges upon the following statement about moving in the wrong direction only decreasing the probability a walk starting at some \(1 \leq i \leq \lfloor n/2 \rfloor\) ending at \(n\) without ever going past \(n\).

**Lemma 5.8.** For any \(1 \leq i \leq \lfloor n/2 \rfloor\), at any step \(t \leq n^2/4\) with \(t \equiv n - i \pmod{2}\), we have

\[
\Pr_{w \sim \mathcal{W}_{\text{LINE}}(i)} \left[ w^{(t)} = n \text{ and } \max(w) = n \right] 
\geq \Pr_{w \sim \mathcal{W}_{\text{LINE}}(i)} \left[ w^{(t)} = n \text{ and } \max(w) = n \mid \min(w) < 1 \right].
\]

**Proof.** If we condition on \(\min_{\leq t}(w) < 1\), then consider the first time the walk hits 0 and suppose this happens at step \(\hat{t}\). Note that this means \(i \equiv \hat{t} \pmod{2}\) and in turn \(n \equiv t - \hat{t} \pmod{2}\).

The probability of having \(\max_{\leq t}(w) = w^{(t)} = n\) via the walk in steps \(\hat{t} + 1 \ldots t\) is then at most

\[
\Pr_{w \sim \mathcal{W}_{\text{LINE}}(0)} \left[ w^{(t-\hat{t})} = \max(w) = n \right].
\]

Note that we have inequality since it’s possible that we already have \(\max_{\leq t}(w) > n\).

Therefore, it suffices to show for any \(n\) and any \(1 \leq \hat{t} \leq t\) we have:

\[
\Pr_{w \sim \mathcal{W}_{\text{LINE}}(0)} \left[ w^{(t-\hat{t})} = \max(w) = n \right] \leq \Pr_{w \sim \mathcal{W}_{\text{LINE}}(i)} \left[ w^{(t)} = \max(w) = n \right].
\]

There are two variables that are shifted from one side of the inequality to the other, the starting position of the walk and the number of steps. In order to then prove the inequality, we will show that both, taking more steps and starting further to the right, will only improve the probability of ending at \(n\) and not going above \(n\).

We begin by showing that taking more steps will only improve this probability.

\[
\Pr_{w \sim \mathcal{W}_{\text{LINE}}(0)} \left[ w^{(t-\hat{t})} = \max(w) = n \right] \leq \max \left\{ \Pr_{w \sim \mathcal{W}_{\text{LINE}}(0)} \left[ w^{(t-1)} = \max(w) = n \right], \Pr_{w \sim \mathcal{W}_{\text{LINE}}(0)} \left[ w^{(t)} = \max(w) = n \right] \right\}.
\]

There is no guarantee that \(t \equiv n \pmod{2}\), so we need to consider \(t\) or \(t - 1\) steps depending on parity. We are guaranteed that \(t - 1 \geq t - \hat{t}\) because \(\hat{t} \geq 1\), so without loss of generality, we will assume that \(t \equiv \hat{t} \pmod{2}\) and show

\[
\Pr_{w \sim \mathcal{W}_{\text{LINE}}(0)} \left[ w^{(t-\hat{t})} = \max(w) = n \right] \leq \Pr_{w \sim \mathcal{W}_{\text{LINE}}(0)} \left[ w^{(t)} = \max(w) = n \right]
\]

where the proof will be equivalent if \(t - 1 \equiv \hat{t} \pmod{2}\).

The explicit formula of these probability from Fact 5.3 gives:

\[
\Pr_{w \sim \mathcal{W}_{\text{LINE}}(0)} \left[ \max(w) = w^{(t)} = n \right] = \frac{1}{2^t} \left( \frac{t}{\lfloor t/2 \rfloor} \right) \frac{2n + 2}{\hat{t} + n + 2}.
\]
Substitute \( t \) by \( t - 2 \) in the above equation and compare the right hand sides. This gives

\[
\Pr_{w \sim W^{\text{line}}(0)} \left[ w^{(t-2)} = \max_{\leq t-2}(w) = n \right] \leq \Pr_{w \sim W^{\text{line}}(0)} \left[ w^{(t)} = \max_{\leq t}(w) = n \right]
\]

when

\[
\frac{1}{2^{t-2}} \left( \frac{t - 2}{\left( \frac{t}{2} \right)^2 - 1} \right) \frac{2n + 2}{t + n} \leq \frac{1}{2^t} \left( \frac{t + n}{\frac{t}{2}} \right) \frac{2n + 2}{t + n + 2}
\]

\[
\frac{1}{(t - 2)! \left( \frac{t - 2}{2} - 1 \right)! \frac{1}{t + n}} \leq \frac{1}{4 \left( \frac{t + n}{2} \right)! \left( \frac{t - n}{2} \right)! \frac{1}{t + n + 2}} \leq \frac{t(t - 1)}{t + n} \leq \frac{3t}{(t + n)(t - n)(t + n + 2)} \leq \frac{n^2}{2n}
\]

which is true by assumption. Inductively applying it for \( t - 2 \) then proves the inequality.

To complete our proof, it now suffices to show

\[
\max \left\{ \Pr_{w \sim W^{\text{line}}(0)} \left[ w^{(t-1)} = \max_{\leq t-1}(w) = n \right], \Pr_{w \sim W^{\text{line}}(0)} \left[ w^{(t)} = \max_{\leq t}(w) = n \right] \right\} \leq \Pr_{w \sim W^{\text{line}}(i)} \left[ w^{(t)} = \max_{\leq t}(w) = n \right].
\]

which will be proven similarly. First rewrite the RHS with the fact that

\[
\Pr_{w \sim W^{\text{line}}(i)} \left[ w^{(t)} = \max_{\leq t}(w) = n \right] = \Pr_{w \sim W^{\text{line}}(0)} \left[ w^{(t)} = \max_{\leq t}(w) = n - i \right]
\]

and initially assume that \( t \equiv n \pmod{2} \), which implies \( n \equiv n - i \pmod{2} \). Again, using the explicit formula from Fact 5.3 and substituting \( n \) by \( n - 2 \) gives

\[
\Pr_{w \sim W^{\text{line}}(0)} \left[ w^{(t)} = \max_{\leq t}(w) = n \right] \leq \Pr_{w \sim W^{\text{line}}(0)} \left[ w^{(t)} = \max_{\leq t}(w) = n - 2 \right]
\]

when \( t + 2 \leq n^2 \), which true by assumption, and can be inductively applied until \( n = (n - i + 2) \) because \( (n - i + 2) \geq \left\lceil n/2 \right\rceil + 1 \). Unfortunately, we cannot entirely apply the same proof when \( t - 1 \equiv n \pmod{2} \) because this implies \( n \not\equiv n - i \pmod{2} \). Applying the same proof as for \( t \equiv n \pmod{2} \) we can obtain

\[
\Pr_{w \sim W^{\text{line}}(0)} \left[ w^{(t-1)} = \max_{\leq t-1}(w) = n \right] \leq \Pr_{w \sim W^{\text{line}}(0)} \left[ w^{(t-1)} = \max_{\leq t-1}(w) = n - i + 1 \right]
\]

because \( (t - 1) + 2 \leq (n - i + 3)^2 \).

Therefore, we can conclude the proof by showing

\[
\Pr_{w \sim W^{\text{line}}(0)} \left[ w^{(t-1)} = \max_{\leq t-1}(w) = n - i + 1 \right] \leq \Pr_{w \sim W^{\text{line}}(0)} \left[ w^{(t)} = \max_{\leq t}(w) = n - i \right]
\]
This is then true when
\[ n - i \leq \frac{t}{t - (n - i)(n - i + 1)} \]
which holds if \( n - i \geq 0 \).

An immediate corollary of this lemma is that if we condition on the walk not going to the left of 1, it only becomes more likely to reach \( n \) without going above \( n \). This then proves the main result of this section.

**Lemma 3.3.** Let \( n \) be any positive integer and \( 1 \leq i \leq \lceil n/2 \rceil \) be any starting position. For any constant \( c > 4 \) and any integer \( t \) such that \( n^2/c \leq t \leq n^2/4 \) with \( t \equiv n - i \pmod{2} \), a simple symmetric random walk \( w \) on \( \mathbb{Z} \) satisfies

\[
\Pr_{w \sim W_{\text{Line}(i)}} \left[ w(t) = n, \max_{\leq t}(w) = n, \min_{\leq t}(w) \geq 1 \right] \geq e^{-2-2c} \frac{i}{n^3}.
\]

**Proof.** (Of Lemma 3.3) Consider any starting position \( 1 \leq i \leq \lceil n/2 \rceil \) and any time \( n^2/c \leq t \leq n^2/4 \) with \( t \equiv n - i \pmod{2} \). By the definition of conditional probability we have:

\[
\Pr_{w \sim W_{\text{Line}(i)}} \left[ w(t) = n, \max_{\leq t}(w) = n, \min_{\leq t}(w) \geq 1 \right] = \Pr_{w \sim W_{\text{Line}(i)}} \left[ w(t) = n \right] \cdot \Pr_{w \sim W_{\text{Line}(i)}} \left[ \min_{\leq t}(w) \geq 1 \right].
\]

Lemma 5.6 gives that the second term is at least \( \exp(-1 - c) i / n \). Taking the probability under \( \min_{\leq t}(w) \geq 1 \) (i.e. the complementary event of \( \min_{\leq t}(w) < 1 \)) in Lemma 5.8 then allows us to upper bound the first term using Lemma 5.7:

\[
\Pr_{w \sim W_{\text{Line}(i)}} \left[ w(t) = n, \max_{\leq t}(w) = n, \min_{\leq t}(w) \geq 1 \right] \geq \Pr_{w \sim W_{\text{Line}(i)}} \left[ w(t) = n \right] \geq e^{-1-c} \frac{1}{n^2}.
\]

Putting these together then gives:

\[
\Pr_{w \sim W_{\text{Line}(i)}} \left[ w(t) = n, \max_{\leq t}(w) = n, \min_{\leq t}(w) \geq 1 \right] \geq \left( e^{-1-c} \frac{i}{n} \right) \cdot \left( e^{-1-c} \frac{1}{n^2} \right) = e^{-2-2c} \frac{i}{n^3}.
\]

\( \square \)
5.5 Upper Bound for the Final, Maximum, and Minimum position of a Walk

We begin by splitting every $t$ step walk in half, and instead consider the probability of each walk satisfying the given conditions. In order to give upper bounds of these probabilities, we will relax the requirements, allowing us to more easily relate the probabilities to previously known facts regarding 1-D walks that we proved in Section 5.1. Furthermore, by splitting the walk in half, we now have to look at all possible midpoints in $[1,n]$ that the walk could be at.

**Lemma 5.9.**

\[
\Pr_{w \sim W_{\text{LINE}}(1)} \left[ w(t) = n \text{ and } \max(w) = n \text{ and } \min(w) \geq 1 \right] \leq \sum_{i=1}^{n} \Pr_{w \sim W_{\text{LINE}}(1)} \left[ w(\lfloor t/2 \rfloor) = i \text{ and } \min(w) \geq 1 \right] \cdot \Pr_{w \sim W_{\text{LINE}}(i)} \left[ w(\lceil t/2 \rceil) = n \text{ and } \max(w) = n \right]
\]

**Proof.** By subdividing the walk roughly in half, we consider all possible positions of any walk after half of its steps that still satisfies the maximum and minimum conditions. The second half of the walk must end at $n$, which implies the maximum position of the walk must be at least $n$, so the first half of the walk only needs to ensure that its walks does not go above $n$. Accordingly, we can write

\[
\Pr_{w \sim W_{\text{LINE}}(1)} \left[ w(t) = n \text{ and } \max(w) = n \text{ and } \min(w) \geq 1 \right] = \sum_{i=1}^{n} \Pr_{w \sim W_{\text{LINE}}(1)} \left[ w(\lfloor t/2 \rfloor) = i \text{ and } \max(w) \leq \lfloor t/2 \rfloor \text{ and } \min(w) \geq \lfloor t/2 \rfloor \right] \cdot \Pr_{w \sim W_{\text{LINE}}(i)} \left[ w(\lceil t/2 \rceil) = n \text{ and } \max(w) = n \text{ and } \min(w) \geq \lfloor t/2 \rfloor \right]
\]

Removing some of the conditions that the walks must satisfy can only increase the probability, so our upper bound follows.

\[\square\]

From Fact 5.3 we can obtain explicit expressions for each inner term of the summation, which we then simplify into a strong bound on the summation in the following lemma.

**Lemma 5.10.**

\[
\Pr_{w \sim W_{\text{LINE}}(1)} \left[ w(t) = n \text{ and } \max(w) = n \text{ and } \min(w) \geq 1 \right] \leq \sum_{i=1}^{n} \left( \frac{16i(n - i + 1)}{t^2} \right) \left( \frac{\lfloor t/2 \rfloor}{\lfloor t/2 \rfloor + 1} \right) \frac{1}{\lfloor t/2 \rfloor} \left( \frac{\lceil t/2 \rceil}{\lceil t/2 \rceil + (n-i+1) - 1} \right) \frac{1}{\lceil t/2 \rceil}
\]

**Proof.** Apply the upper bound from Lemma 5.9 and more closely examine each inner term in the summation. By the symmetry of walks, there must be an equivalent number of $\lfloor t/2 \rfloor$ step
walks with endpoints 1 and \( i \) that never walk below 1 vs those that never walk above \( i \). Thus

\[
\Pr_{w \sim \mathcal{W}^{\text{LINE}}(1)} \left[ \min(w) \geq 1 \text{ and } w(\left\lfloor \frac{i}{2} \right\rfloor) = i \right] = \Pr_{w \sim \mathcal{W}^{\text{LINE}}(1)} \left[ \max(w) \leq i \text{ and } w(\left\lfloor \frac{i}{2} \right\rfloor) = i \right]
\]

Shifting the start of the walk to 0 allows us to apply Fact 5.3 because \( \max_{\leq \left\lfloor \frac{i}{2} \right\rfloor} w \leq i \) is equivalent to \( \max_{\leq \left\lfloor \frac{i}{2} \right\rfloor} w = i \) if the walk must end at \( i \). Therefore

\[
\Pr_{w \sim \mathcal{W}^{\text{LINE}}(1)} \left[ \min(w) \geq 1 \text{ and } w(\left\lfloor \frac{i}{2} \right\rfloor) = i \right] = \left( \frac{2i}{\left\lfloor \frac{i}{2} \right\rfloor + i + 1} \right) \left( \frac{\left\lfloor \frac{i}{2} \right\rfloor}{\left\lfloor \frac{i}{2} \right\rfloor + i + 1} \right) \frac{1}{2^\left\lfloor \frac{i}{2} \right\rfloor}
\]

when the parity is correct and 0 otherwise, so this will work as an upper bound. Similarly, by shifting the start to 0 and applying Fact 5.3 we get

\[
\Pr_{w \sim \mathcal{W}^{\text{LINE}}(i)} \left[ w(\left\lfloor \frac{i}{2} \right\rfloor) = n \text{ and } \max(w) = n \right] = \left( \frac{2(n - i + 1)}{\left\lfloor \frac{i}{2} \right\rfloor + (n - i + 1) + 1} \right) \left( \frac{\left\lfloor \frac{i}{2} \right\rfloor}{\left\lfloor \frac{i}{2} \right\rfloor + (n - i + 1) + 1} \right) \frac{1}{2^\left\lfloor \frac{i}{2} \right\rfloor}
\]

Applying Lemma 5.9, we know have expressions for the term inside the summation

\[
\Pr_{w \sim \mathcal{W}^{\text{LINE}}(1)} \left[ w(\left\lfloor \frac{i}{2} \right\rfloor) = i \right] \cdot \Pr_{w \sim \mathcal{W}^{\text{LINE}}(i)} \left[ w(\left\lfloor \frac{i}{2} \right\rfloor) = n \text{ and } \max(w) = n \right]
\]

\[
= \left( \frac{2i}{\left\lfloor \frac{i}{2} \right\rfloor + i + 1} \right) \left( \frac{\left\lfloor \frac{i}{2} \right\rfloor}{\left\lfloor \frac{i}{2} \right\rfloor + i + 1} \right) \cdot \left( \frac{2(n - i + 1)}{\left\lfloor \frac{i}{2} \right\rfloor + (n - i + 1) + 1} \right) \left( \frac{\left\lfloor \frac{i}{2} \right\rfloor}{\left\lfloor \frac{i}{2} \right\rfloor + (n - i + 1) + 1} \right) \frac{1}{2^\left\lfloor \frac{i}{2} \right\rfloor}
\]

\[
\leq \left( \frac{16i(n - i + 1)}{t^2} \right) \left( \frac{\left\lfloor \frac{i}{2} \right\rfloor}{\left\lfloor \frac{i}{2} \right\rfloor + i + 1} \right) \cdot \left( \frac{\left\lfloor \frac{i}{2} \right\rfloor}{\left\lfloor \frac{i}{2} \right\rfloor + (n - i + 1) + 1} \right) \frac{1}{2^\left\lfloor \frac{i}{2} \right\rfloor}
\]

which we upper bound by the fact that \( (\left\lfloor \frac{i}{2} \right\rfloor + i + 1)(\left\lfloor \frac{i}{2} \right\rfloor + (n - i + 1) + 1) \geq \frac{t^2}{4} \), finishing the proof.

Now that we have an explicit upper bound for the probability we are examining, we now only need to prove upper bounds on this expression with binomial coefficients that will be done in the next two lemmas. The following lemma will then give a upper bound the inner expression from Lemma 5.10 by simply bounding the binomial coefficients by the central coefficient and the known upper bounds from Stirling’s bounds.

**Lemma 5.11.** For any \( i \in [1, n] \)

\[
\left( \frac{16i(n - i + 1)}{t^2} \right) \left( \frac{\left\lfloor \frac{i}{2} \right\rfloor}{\left\lfloor \frac{i}{2} \right\rfloor + i + 1} \right) \cdot \left( \frac{\left\lfloor \frac{i}{2} \right\rfloor}{\left\lfloor \frac{i}{2} \right\rfloor + (n - i + 1) + 1} \right) \frac{1}{2^\left\lfloor \frac{i}{2} \right\rfloor} \leq 64 \frac{n^2}{t^3}
\]

**Proof.** Given that \( i \in [1, n] \), we can crudely upper bound \( i(n - i + 1) \) by \( n^2 \). Additionally, we can will use Stirling’s bound on the central binomial coefficient to upper bound our binomial coefficients.
\begin{align*}
\left( \frac{\left\lceil \frac{t}{2} \right\rceil + (n-i+1) - 1}{2} \right) & \leq 2 \left\lceil \frac{t}{2} \right\rceil \frac{1}{\sqrt{\left\lceil \frac{t}{2} \right\rceil}} \\
\text{and} \quad \left( \frac{\left\lfloor \frac{t}{2} \right\rfloor}{\left\lfloor \frac{t}{2} \right\rfloor + i - 1} \right) & \leq 2 \left\lfloor \frac{t}{2} \right\rfloor \frac{1}{\sqrt{\left\lfloor \frac{t}{2} \right\rfloor}}.
\end{align*}

The exponential terms will cancel and

\[ \frac{1}{\sqrt{\left\lceil \frac{t}{2} \right\rceil}} \frac{1}{\sqrt{\left\lfloor \frac{t}{2} \right\rfloor}} \leq 4 \frac{1}{t} \]

giving our desired bound \( \Box \)

The upper bound in Lemma 5.11 will not be sufficient for \( t \) that are asymptotically less than \( n^2 \), so for these \( t \) we will need to give a more detailed analysis. As a result, we will more carefully examine the binomial coefficients, which will be significantly smaller than the central coefficient for small \( t \). Consequently, the exponential term will not be sufficiently canceled by the binomial coefficient for \( t \) that are asymptotically smaller than \( n^2 \). More specifically, we can show that the function of \( t \) on the RHS of Lemma 5.10 is increasing in \( t \) up until approximately \( n^2/20 \). In the following lemma we will consider even length walks for simplicity, and the proof for odd length walks follows equivalently.

\textbf{Lemma 5.12.} \textit{Given} \( n \geq 20 \) \textit{and integer} \( 1 \leq i \leq n \) \textit{then for all} \( t \leq \frac{n^2}{40} \)

\[ \frac{16i(n-i+1)}{(2t)^2} \frac{1}{2^t} \left( \frac{t}{t+1} \right) \left( \frac{t}{t+(n-i+1)-1} \right) \leq \frac{16i(n-i+1)}{(2t+4)^2} \frac{1}{2^{t+1}} \left( \frac{t+2}{t+2+(n-i+1)-1} \right) \]

\textit{where we consider walks of length} \( 2t \) \textit{and} \( 2t+4 \) \textit{to ensure that} \( \frac{2t}{t} \) \textit{and} \( \frac{2t+4}{t} \) \textit{have the same parity}

\textbf{Proof.} Cancellation of like terms implies that our desired inequality is equivalent to

\[ \frac{1}{t^2} \left( \frac{t}{t+i-1} \right) \left( \frac{t}{t+(n-i+1)-1} \right) \leq \frac{1}{(t+2)^2} \frac{1}{2^{t+2}} \left( \frac{t+2}{t+2+(n-i+1)-1} \right) \]

Further examination of the binomial coefficients shows that

\[ \left( \frac{t}{t+i-1} \right) \left( \frac{t+2}{t+1} \right) = \left( \frac{t+2}{t+2+i-1} \right) \]

and

\[ \left( \frac{t}{t+(n-i+1)-1} \right) \left( \frac{t+2}{t+2+(n-i)+1} \right) = \left( \frac{t+2}{t+2+(n-i+1)-1} \right) \]

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Cancellation of the binomial coefficients according to these identities gives that our desired inequality is then equivalent to

\[
\frac{1}{t^2} \leq \frac{1}{(t+2)^2} \frac{(t+2)(t+1)}{16 \left( \frac{t+3-i}{2} \right) \left( \frac{t+3+(n-i)}{2} \right) \left( \frac{t+2+(n-i)}{2} \right)}
\]

Further cancellation of like terms and bringing the denominator on each side into the numerator on the other side implies that our desired inequality is equivalent to

\[
(t + 1 + i)(t + 3 - i)(t + 2 + (n - i))(t + 2 - (n - i)) \leq t^2(t + 1)^2
\]

It is straightforward to see that \((t + 1 + i)(t + 3 - i)\) is maximized when \(i = 1\) and decreases as \(i\) moves away from 1 in either direction. Similarly, \((t + 2 + (n - i))(t + 2 - (n - i))\) is maximized when \(n - i = 0\) and decreases as \(n - i\) moves away from 0 in either direction. Furthermore, it must be true that either \(i \geq n/2\) or \(n - i \geq n/2\), so we can upper bound the LHS of our inequality by plugging in \(n/2\) for \(i\) or \(n - i\) and set the other to the value that maximizes the product. Hence,

\[
(t + 1 + i)(t + 3 - i)(t + 2 + (n - i))(t + 2 - (n - i)) \leq (t + 2)^2 \left( t + 3 + \frac{n}{2} \right) \left( t + 3 - \frac{n}{2} \right)
\]

In order to prove our desired inequality it now suffices to show that

\[
(t + 2)^2 \left( t + 3 + \frac{n}{2} \right) \left( t + 3 - \frac{n}{2} \right) \leq t^2(t + 1)^2
\]

which is equivalent to

\[
\left( t + 3 + \frac{n}{2} \right) \left( t + 3 - \frac{n}{2} \right) \leq t^2 \left( 1 - \frac{1}{t + 2} \right)^2
\]

Expanding both sides of the inequality gives

\[
t^2 + 6t + 9 - \frac{n^2}{4} \leq t^2 - \frac{2t^2}{t + 2} + \left( \frac{t}{t + 2} \right)^2
\]

Cancellation and rearranging then reduces to

\[
6t + 9 + \frac{2t^2}{t + 2} - \left( \frac{t}{t + 2} \right)^2 \leq \frac{n^2}{4}
\]

Given that \(\frac{2t^2}{t+2} \leq 2t\), in order to prove our desired inequality, it therefore suffices to show that

\[
8t + 9 \leq \frac{n^2}{4}
\]

which is true when \(t \leq \frac{n^2}{32}\) if \(n \geq 20\)

\[\square\]
With the strong upper bounds on the expression on the RHS ofLemma 5.10 we can now prove our main upper bound result of this section.

**Lemma 4.1.** For \( n \geq 20 \)

\[
\Pr_{w \sim W^{\text{line}}(1)} \left[ w^{(t)} = n \text{ and } \max_{\leq t}(w) = n \text{ and } \min_{\leq t}(w) \geq 1 \right] \leq \min \left( \frac{e^{25}}{n^3}, 64 \left( \frac{n}{t} \right)^3 \right)
\]

**Proof.** Applying Lemma 5.10 and 5.11 we have

\[
\Pr_{w \sim W^{\text{line}}(1)} \left[ w^{(t)} = n \text{ and } \max_{\leq t}(w) = n \text{ and } \min_{\leq t}(w) \geq 1 \right] \leq \sum_{i=1}^{n} 64 \left( \frac{n^2}{t^3} \right)
\]

which immediately gives an upper bound of \( 64 \left( \frac{n}{t} \right)^3 \). Similarly, Lemma 5.10 and 5.12 imply that for all \( t \leq \frac{n^2}{40} \)

\[
\Pr_{w \sim W^{\text{line}}(1)} \left[ w^{(t)} = n \text{ and } \max_{\leq t}(w) = n \text{ and } \min_{\leq t}(w) \geq 1 \right] \leq \sum_{i=1}^{n} 64 \frac{n^2}{T^3}
\]

where \( T = \frac{n^2}{40} \). We then plug in the upper bound of this expression from Lemma 5.11 to obtain for all \( t \leq \frac{n^2}{40} \)

\[
\Pr_{w \sim W^{\text{line}}(1)} \left[ w^{(t)} = n \text{ and } \max_{\leq t}(w) = n \text{ and } \min_{\leq t}(w) \geq 1 \right] \leq \sum_{i=1}^{n} 64 \frac{n^2}{T^3}
\]

Plugging back in \( \frac{n^2}{40} \) for \( T \) and summing over 1 to \( n \) gives

\[
\Pr_{w \sim W^{\text{line}}(1)} \left[ w^{(t)} = n \text{ and } \max_{\leq t}(w) = n \text{ and } \min_{\leq t}(w) \geq 1 \right] \leq \sum_{i=1}^{n} 64 \frac{n^2}{T^3} \leq \frac{e^{25}}{n^3}
\]

for all \( t \leq \frac{n^2}{40} \). Further using the fact that \( 64 \left( \frac{n}{t} \right)^3 \) is a decreasing function in \( t \), so \( 64 \left( \frac{n}{t} \right)^3 \leq \frac{e^{25}}{n^3} \) for all \( t \geq \frac{n^2}{40} \), we have proven our desired lower bound.

### 6 Extension to Higher Dimensions

In this section we show how to naturally extend our upper and lower bounds to grids of dimension \( d \), where \( d \) is a constant.

**Theorem 1.3.** The transience class of the Abelian sandpile model on an \( n^d \) \( d \)-dimensional grid is \( O(n^{3d-2} \log^{d+2} n) \) and \( \Omega(n^{3d-2}) \).
Here we denote the $d$-dimensional grid or hyper-cube as $d\text{Cube}_n$, which is defined analogously to $\text{Square}_n$, and is the vertex set \{1, 2, ..., $n$\}$^d \cup \{v_{\text{Sink}}\}$ where its edges connect any pair of vertices that differ in one coordinate. Additionally, vertices on the boundary will have the requisite number of edges such that every non-sink vertex has degree $2d$. We denote all coordinates with vector notation $\mathbf{u} = (u_1, u_2, \ldots, u_d)$, and can similarly decouple a walk $w$ on $d\text{Cube}_n$ into each of it’s one dimensional walks $w_1, w_2, \ldots, w_d$. As with our analysis of $\text{Square}_n$, each step of a random walk in $d\text{Cube}_n$ can be equivalently considered as choosing a random direction with probability $1/d$, then choosing a corresponding step in the one-dimensional walk with probability $1/2$.

Our bounds for the two-dimensional grid heavily relied on decoupling into interleaving one-dimensional walks, then applying bounds on these walks shown in Section 5. Extending these bounds to $d$-dimensional hypercubes will follow comparably and only require simple extensions of our necessary lemmas for the two-dimensional grids. Accordingly, in order to prove the desired upper and lower bounds, we will reference the necessary lemmas from previous sections and show how minor modifications will give analogous lemmas for the $d$-dimensional grid. The upper bound proof required several key lemmas, and will be more involved, while extending the lower bound will only require one simple addition to our proof in Section 4.

**Upper Bound**

As Theorem 2.4 from [CV12] only relies on non-sink vertices being constant degree, our assumption that $d$ is constant and definition that all non-sink vertices have degree $2d$, allows us to apply the upper bound on the transience class in the same manner as the one-dimensional analysis. Besides utilizing the properties of one-dimensional walks, specifically Lemma 3.3 proven in Section 5, the proof of our upper bound then relied upon four key lemmas.

- **Lemma 2.9**: The source and sink of vertex potentials could be flipped while only losing a log $n$ approximation factor.

- **Lemma 3.1**: An upper bound on the sum of vertex potentials, by decomposing into 1-dimensional vertex potentials.

- **Lemma 3.2**: The fact that, up to constant factors, for any vertex, the opposite corner minimizes vertex potential.

- **Lemma 3.5**: A lower bound on the vertex potential $\pi^{\text{Square}_n}(n, n)_{\mathbf{u}}$ for any $\mathbf{u}$ such that $1 \leq u_1, u_2 \leq \lceil n/2 \rceil$.

We will describe how to extend each of these lemmas to constant dimensions, which for the most part will follow immediately from our decoupling walks into one-dimensional walks.

**Lemma 6.1.** For any pair of non-sink vertices $\mathbf{u}$ and $\mathbf{v}$ in $d\text{Cube}_n$, we have

$$\pi^{d\text{Cube}_n}(\mathbf{u})_{\mathbf{v}} \leq (8 \log n + 4) \pi^{d\text{Cube}_n}(\mathbf{v})_{\mathbf{u}}.$$ 

*Proof.* This is a direct consequence of Rayleigh’s monotonicity theorem. We fix an underlying $n \times n$ grid in the hyper-cube with corners at the source and sink, and set the rest of the resistors to infinity. Then the upper bound for the $n \times n$ grid is an upper bound for the hyper-cube. \hfill \square
Our analogous lemma to that of Lemma 3.1 will follow immediately from the analogous application of Lemma 6.1 and the decoupling of walks into one-dimension.

**Lemma 6.2.** For any non-sink vertex \( u \) in \( d\text{CUBE}_n \), we have

\[
\sum_{u \in V} \pi^{d\text{CUBE}_n}(u)_v = O \left( \log n \prod_{i=1}^{d} u_i \log n \right).
\]

**Proof.** We follow identically to the proof of Lemma 3.1. First take the reciprocal by our extension of Lemma 2.9:

\[
\pi^{d\text{CUBE}_n}(u)_v = O \left( \pi^{d\text{CUBE}_n}(v)_u \log n \right).
\]

Extending to higher dimensions,

\[
\pi^{d\text{Cube}_n}(v)_u = \text{Pr}_{w \sim W^{d\text{Cube}_n}(u)} \left[ w \text{ hits } v \text{ before leaving } d\text{Cube}_n \right]
\]

\[
\leq \text{Pr}_{w \sim W^{d\text{Cube}_n}(u)} \left[ \bigcap_{i=1}^{d} w_i \text{ hits } v_i \text{ before } v_{\text{Sink}} \right]
\]

\[
= \prod_{i=1}^{d} \text{Pr}_{w \sim W^{d\text{Cube}_n}(u)} \left[ w_i \text{ hits } v_i \text{ before } v_{\text{Sink}} \right]
\]

\[
= \prod_{i=1}^{d} \pi^{\text{Path}_n}(v_i)_{u_i}.
\]

Similarly, summing over all choices of \( v \) gives

\[
\sum_{v \in V(d\text{Cube}_n)} \pi^{d\text{Cube}_n}(v)_u \leq \prod_{i=1}^{d} \left( \sum_{v_i=1}^{n} \pi^{\text{Path}_n}(v_i)_{u_i} \right).
\]

The rest of our proof then follows from exactly as that in Lemma 3.1.

We can also easily extend our proof of Lemma 3.2 to higher dimensions due to the argument dealing with each dimension separately.

**Lemma 6.3.** If \( u \) is a vertex of \( d\text{Cube}_n \) such that \( 1 \leq u_i \leq \lceil n/2 \rceil \) for all \( 1 \leq i \leq d \), then

\[
\pi^{d\text{Cube}_n}(u)_v \geq \left( \frac{1}{2d} \right)^d \pi^{d\text{Cube}_n}(u)_{(n,n,...,n)},
\]

for any non-sink vertex \( v \in V(d\text{Cube}_n) \setminus \{v_{\text{Sink}}\} \).

**Proof.** Directly extend the proof of Lemma 3.2 by reflecting walks across a \((d-1)\)-dimensional hyperplane.

Finally, we will need an analogous Lemma 3.5, where the key idea was to considers walks of length \( \Theta(n^2) \) and show that there is a constant fraction such that both dimensions have taken \( \Theta(n^2) \) steps, which then allows for the application of Lemma 3.3 over each of these possible walks. To extend this analysis, we will essentially union bound Lemma 3.4 over \( d \) dimensions and have that walks of \( \Theta(n^2) \) steps will take \( \Theta(n^2) \) steps in each direction with probability at least \( 2^{-O(d)} \), which is still constant by our assumption that \( d \) is constant.
Lemma 6.4. For all \( n \geq 10 \) and \( u \in d\text{CUBE}_n \) such that \( 1 \leq u_i \leq \lfloor n/2 \rfloor \) for all \( i \), we have
\[
\pi^{d\text{CUBE}_n}\left( (n^d) \right)_u \geq \Omega\left( \frac{\prod_{i=1}^d u_i}{n^{3d-2}} \right).
\]

Proof. Equivalently to the proof of Lemma 3.5, we will decouple walks \( w \in \mathcal{W}^{d\text{CUBE}_n}(u \rightarrow (n^d)) \) into one-dimensional walks \( w_i \in \mathcal{W}^{\text{LINE}}(u_i) \), and view \( \pi^{\text{SQUARE}_n}\left( (n^d) \right)_u \) as the probability that each walk \( w_i \) visits \( n \) at the same time before any leaves the interval \([1,n]\). If each walk has taken \( t_1, t_2, \ldots, t_d \) steps, respectively, then the total number of possible interleavings of these walks is the multinomial
\[
\left( \begin{array}{c} t_1 + t_2 + \cdots + t_d \\ t_1, t_2, \ldots, t_d \end{array} \right).
\]
As with the proof of Lemma 3.5 we obtain the lower bound
\[
\pi^{d\text{CUBE}_n}\left( (n^d) \right)_u \geq \sum_{t_1, t_2, \ldots, t_d \geq 0} \left( \begin{array}{c} t_1 + t_2 + \cdots + t_d \\ t_1, t_2, \ldots, t_d \end{array} \right) \frac{1}{d^{t_1+t_2+\cdots+t_d}} \prod_{i=1}^d \left( \frac{1}{2} \Pr_{w_i \sim \mathcal{W}^{\text{LINE}}(u_i)} \left[ w_i(t_i-1) = n - 1, \max_{\leq t_i-1} (w) = n - 1, \text{and } \min_{\leq t_i-1} (w) \geq 1 \right] \right)
\]
In order to apply Lemma 3.3 to each walk, we need each \( t_i \) to be within the interval \([n^2/c, n^2/4]\), where here we will let \( c = 16d \) be a large constant with respect to the dimension. Similar to the proof of Lemma 3.5, we will consider all walks of length \( n^2/8 \leq t \leq n^2/4 \) where \( t = t_1 + t_2 + \cdots + t_d \), and show that a constant fraction of these walks satisfy all \( t_i \geq n^2/c \) and \( t_i \) having the correct parity. Note that we can ignore the parity conditions by simply lower bounding the probability of all having correct parity by \( 4^{-d} \). It then remains to show that all will satisfy \( t_i \geq n^2/c \) with constant probability.

We will do so with very crude bounds with respect to each dimension. Specifically, we consider the probability that \( t_1 \geq n^2/c \) and the remaining dimensions follow symmetrically. Having each dimension take at least \( n^2/c \) steps are not independent events, so we will consider the probability that \( t_1 \geq n^2/c \) when conditioning upon \( t_2, t_3, \ldots, t_d \geq n^2/c \) (which will only decrease the probability \( t_1 \geq n^2/c \)). This is equivalent to fixing \( n^2/c \) steps in each of those directions and randomly choosing all remaining steps with probability \( 1/d \) for each direction. The remaining number of steps is then at least \( dn^2/c \) by our assumption that \( t \geq n^2/8 \) and \( c = 16d \). As a result, the expected number of steps in the first dimension is at least \( n^2/c \), which implies \( t_1 \geq n^2/c \) with probability at least \( 1/2 \). Multiplying this probability over all dimensions gives that each \( t_i \geq n^2/c \) with probability at least \( 2^{-d} \).

As a result, we have \( O(n^2) \) possible values of \( t \) for which we can decompose into appropriate 1-dimensional walks, occurring with constant probability, and apply Lemma 3.3 to each decomposition. The lower bound then follow from summing \( \Omega(\prod_{i=1}^d u_i/n^3) \) over the \( O(n^2) \) possible walks \( \square \)

The upper bound for higher dimensions then follows from applying all of these lemmas, which have been extended to higher dimensions, in the same manner of our proof for Theorem 1.1:
For any \( u = (u_1, u_2, \ldots, u_d) \) in the top-left orthant of \( d\text{Cube}_n \), we have

\[
\max_{u, v \in V \setminus \{v_{\text{Sink}}\}} \left( \sum_{x \in V} \pi_{d\text{Cube}_n}(u)_x \right) \pi_{d\text{Cube}_n}(u)^{-1} \leq \max_{u \in V \setminus \{v_{\text{Sink}}\}} \left( \sum_{x \in V} \pi_{d\text{Cube}_n}(u)_x \right) \left( \frac{(2d)^d}{\pi_{d\text{Cube}_n}(u)(n^d)} \right) \quad \text{(by Lemma 6.3)}
\]

\[
= \left( \sum_{x \in V} \pi_{d\text{Cube}_n}(u)_x \right) \frac{O(\log n)}{\pi_{d\text{Cube}_n}((n^d))} \quad \text{(by Lemma 6.1)}
\]

\[
= O \left( \log n \prod_{i=1}^{d} u_i \log n \right) O \left( \frac{n^{3d-2} \log n}{\prod_{i=1}^{d} u_i} \right) \quad \text{(by Lemma 6.4 and Lemma 6.2)}
\]

\[
= O \left( n^{3d-2} \log^{d+2} n \right).
\]

The result \( tcl(d\text{Cube}_n) = O(n^{3d-2} \log^{d+2} n) \) then follows from Theorem 2.4.

**Lower Bound**

Extending our lower bound proof to \( d \) dimensions will be simpler because by the same decoupling argument of \( d \)-dimensional walks to 1-dimensional walks, we can easily obtain an extension of the lower bound in Lemma 4.2 for \( \pi_{d\text{Cube}_n}(n^d)_{(1,1,\ldots,1)} \) as

\[
\pi_{d\text{Cube}_n}(n^d)_{(1,1,\ldots,1)} \leq d \left( \max_t \left( \Pr_{w \sim \text{VLine}(1)} \left[ w^{(t)} = n \text{ and } \max_{\leq t} (w) = n \text{ and } \min_{\leq t} (w) \geq 1 \right] \right) \right)^{d-1} \cdot \sum_{t \geq 0} \Pr_{w \sim \text{VLine}(1)} \left[ w^{(t)} = n \text{ and } \max_{\leq t} (w) = n \text{ and } \min_{\leq t} (w) \geq 1 \right]
\]

This can follow the same proof of Lemma 4.2 except instead apply the following extension of Fact 4.3

**Fact 6.5.** For any nonnegative integer \( t_1 \), we have

\[
\sum_{t_2, \ldots, t_d \geq 0} \left( t_1 + t_2 + \cdots + t_d \right) \frac{1}{t_1 t_2 \cdots t_d} \frac{1}{n^{t_1 + t_2 + \cdots + t_d}} = d.
\]

**Proof.** Generalize the proof of Fact 4.3 using the negative multinomial distribution. \( \square \)

We are then able to apply Lemma 4.1 and 4.4 to get

\[
\pi_{d\text{Cube}_n}(n^d)_{(1,1,\ldots,1)} \leq O \left( \left( \frac{1}{n^d} \right)^{d-1} \frac{1}{n} \right) = O(n^{-3d+2})
\]

Theorem 2.4 then yields that \( tcl(d\text{Cube}_n) = \Omega(n^{3d-2}) \).
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A Deferred Proofs for Section 2

A.1 Reduction to Vertex Potential

We prove the vertex potential upper and lower bounds on $tcl(G)$ given in Theorem 2.4. First we will prove the upper bound given in Theorem 2.4, then we will prove the lower bound given in Theorem 2.4.

Proof of upper bound in Theorem 2.4. We will set up a linear program fractional relaxation. Let the number of pebbled added at vertex $u$ be $x_u$, the degree of vertex $u$ be $d_u$ and the number of topples at vertex $u$ be $z_u$. We can bound the maximum number of (fractional) pebbles added before vertex $v$ topples by:

$$\max \sum_u x_u$$

$$\left( \sum_{w \sim v, w \neq v, \text{Sink}} z_w \right) + x_v \leq d_v - 1$$

$$\forall u \neq v, v \left( \sum_{w \sim u, w \neq v, \text{Sink}} z_w \right) - d_u z_u + x_u \leq d_u - 1$$

$$z \geq 0$$

We can relate these constraints to vertex potentials by rescaling and summing these inequalities by $\pi(G)(v)_u$ in a manner akin to weak duality. Let $\hat{v} \neq v_{\text{Sink}}$ be the minimizer of $\pi(G)(v)_u$. Then adding the constraints together with coefficients

$$\frac{\pi(G)(v)_u}{\pi(G)(v)_{\hat{v}}}$$

gives:

$$\frac{1}{\pi(G)(v)_{\hat{v}}} \left( \pi(G)(v)_v \left( x_v + \sum_{w \sim v, w \neq \text{Sink}} z_w \right) + \sum_{u \neq v, \text{Sink}} \pi(G)(v)_u \left( -d_u z_u + x_u + \sum_{w \sim u, w \neq v, \text{Sink}} z_w \right) \right)$$

$$\leq \sum_{u \neq v_{\text{Sink}}} \frac{\pi(G)(v)_u}{\pi(G)(v)_{\hat{v}}} (d_u - 1)$$
Combining the terms involving $z_u$ on the LHS gives:

$$\sum_{u \neq v, v_{\text{Sink}}} -d_u \pi^{(G)}(v)_u z_u + \sum_{u \neq v, v_{\text{Sink}}, w \sim u} \pi^{(G)}(v)_u z_w$$

$$= \sum_{u \neq v, v_{\text{Sink}}} z_u \left(-d_u \pi^{(G)}(v)_u + \sum_{w \sim u, w \neq v_{\text{Sink}}} \pi^{(G)}(v)_w\right) = 0.$$ 

So the only terms remaining are terms involving $x_u$ on the LHS, leaving:

$$\sum_{u \neq v_{\text{Sink}}} \pi^{(G)}(v)_u x_u \leq \sum_{u \neq v_{\text{Sink}}} \pi^{(G)}(v)_u (d_u - 1).$$

The overall bound then follows from $\pi^{(G)}(v)_u \geq \pi^{(G)}(v)_{\overline{v}}$ and the degrees being constant.

Proof of lower bound in Theorem 2.4. Similarly, we can also obtain a lower bound on $\text{tcl}(S)$ by considering only adding pebbles to some vertex $s$ and lower bounding the number of topples that happens at $s$ before some vertex $v$ receives a topple via the linear program:

$$\min z_s$$

$$\forall u \neq s, v_{\text{Sink}} \sum_{w \sim u, w \neq v, v_{\text{Sink}}} z_w \geq d_u z_u$$

$$z_v \geq 1$$

In this linear program, however, we do not allow $z_v$ to topple and just let it accumulate pebbles. The linear program’s inequalities

$$\sum_{w \sim v, w \neq v_{\text{Sink}}} z_w \geq d_v z_v$$

and $z_v \geq 1$ then require that $v$ accumulates enough pebbles such that it has more than it’s degree.

Here given a set of potentials $\pi^{(G)}(v)$, consider adding together these inequalities for all the vertices $u$ with coefficients

$$\frac{\pi^{(G)}(v)_u}{\pi^{(G)}(v)_s}$$

gives:

$$\sum_{u \neq s, v_{\text{Sink}}} \pi^{(G)}(v)_u \sum_{w \sim u, w \neq v, v_{\text{Sink}}} z_w \geq \sum_{u \neq s, v_{\text{Sink}}} \pi^{(G)}(v)_u d_u z_u$$

Rearranging terms on LHS gives

$$\sum_{u \neq v, v_{\text{Sink}}} z_u \sum_{w \sim u, w \neq v_{\text{Sink}}} \frac{\pi^{(G)}(v)_w}{\pi^{(G)}(v)_s} \geq \sum_{u \neq s, v_{\text{Sink}}} \frac{\pi^{(G)}(v)_u}{\pi^{(G)}(v)_s} d_u z_u \quad (1)$$
Then using the fact that
\[ \sum_{w \sim u, w \neq v \text{Sink}} \frac{\pi^{(G)}(v)_w}{\pi^{(G)}(v)_s} = \frac{\pi^{(G)}(v)_u}{\pi^{(G)}(v)_s} d_u \]
this reduces to the inequality below
\[ d_s z_s \geq \frac{\pi^{(G)}(v)_u}{\pi^{(G)}(v)_s} d_v z_v. \]
Since all degrees are constant, the condition of \( z_v \geq 1 \) and \( \pi^{(G)}(v)_v = 1 \) then implies
\[ z_s \geq \frac{1}{\deg \max \cdot \pi^{(G)}(v)_s}. \]

\[ \square \]

A.2 Definition of Effective Resistance

We define the *Laplacian matrix* of a simple graph \( G \) to be
\[ L \overset{\text{def}}{=} D - A, \]
where \( D \) is the degree matrix and \( A \) is the adjacency matrix of \( G \). An entry of \( L \) is given by
\[ L_{uv} = \begin{cases} \deg(u) & \text{if } u = v, \\ -1 & \text{if } u \sim v, \\ 0 & \text{otherwise}. \end{cases} \]

Let \( \chi_{uv} \) denote the indicator vector that is \(-1\) at \( u \) and \( 1 \) at \( v \). The effective resistance between a pair of vertices \( u \) and \( v \), is given by
\[ R_{\text{eff}}^{(G)}(u, v) = \chi_{vu}^\top L^\dagger \chi_{uv}, \]
where \( L^\dagger \) is the pseudoinverse of \( L \). For more dedicated expositions, see [ESVM+11] and [DS84].

A.3 Proof of Lemma 2.6

We show the relation between voltage potentials and the probability of a walk escaping at the source instead of the sink.

**Proof of Lemma 2.6.** By definition, we have
\[ \pi^{\text{Square}_n}(u)_v = \frac{\sum_{w \in V^{\text{Square}_n}(v \to u)} 4^{-|w|}}{\sum_{w \in V^{\text{Square}_n}(v \to \{u, v\text{Sink}\})} 4^{-|w|}}. \]

For any \( v \in V(\text{Square}_n) \), let
\[ f(v) = \sum_{w \in V^{\text{Square}_n}(v \to \{u, v\text{Sink}\})} 4^{-|w|} \]
be the normalizing constant for \( \pi^{\text{Square}_n}(u)_v \). It follows that \( f(u) = 1 \) and \( f(v_{\text{Sink}}) = 1 \), because the only such walk for each has length 0. For all other \( v \in V(\text{Square}_n) \setminus \{u, v_{\text{Sink}}\} \),

\[
f(v) = \frac{1}{4} \sum_{x \sim v} f(x).
\]

Therefore, \( f(v) \) is a harmonic function with constant boundary values, so \( f(v) = 1 \) for all vertices \( v \in V(\text{Square}_n) \).

A.4 Proof of Lemma 2.8

We verify that the effective resistance between \( v_{\text{Sink}} \) and any internal vertex is \( \Omega(1) \) and \( O(\log n) \) using a triangle inequality for effective resistances and the fact that the effective resistance between opposite corners in an \( n \times n \) resistor network is \( \Theta(\log n) \).

**Proposition A.1** ([LPW09, Proposition 9.16]). Let \( G \) be an \( n \times n \) network of unit resistors. If \( u \) and \( v \) are vertices at opposite corner vertices, then

\[
\frac{\log(n - 1)}{2} \leq R_{\text{eff}}^{G}(u, v) \leq 2 \log n.
\]

**Proof of Lemma 2.8.** We first prove the lower bound

\[
\frac{1}{4} \leq R_{\text{eff}}^{\text{Square}_n}(v_{\text{Sink}}, u).
\]

The effective resistance between \( v_{\text{Sink}} \) and \( u \) is the reciprocal of the total current flowing into the circuit when \( \pi^{\text{Square}_n}(u)_{u} = 1 \) and \( \pi^{\text{Square}_n}(u)_{v_{\text{Sink}}} = 0 \). Since \( \pi^{\text{Square}_n}(u) \) is a harmonic function, we have \( \pi^{\text{Square}_n}(u)_v \geq 0 \) for all \( v \in V(\text{Square}_n) \). Moreover, \( \deg(u) = 4 \), so

\[
R_{\text{eff}}^{\text{Square}_n}(v_{\text{Sink}}, u) = \left( \sum_{v \sim u} \pi^{\text{Square}_n}(u)_u - \pi^{\text{Square}_n}(u)_v \right)^{-1} \geq \frac{1}{4}.
\]

For the upper bound, we use Rayleigh’s monotonicity law, Proposition A.1, and the triangle inequality for effective resistances to show that

\[
R_{\text{eff}}^{\text{Square}_n}(v_{\text{Sink}}, u) \leq 2 \log n + 1,
\]

for \( n \) sufficiently large. Rayleigh’s monotonicity law [DS84] states that if the resistances of a circuit are increased, the effective resistance between any two points can only increase. The following triangle inequality for effective resistances is given in [Tet91]:

\[
R_{\text{eff}}^{G}(u, v) \leq R_{\text{eff}}^{G}(u, x) + R_{\text{eff}}^{G}(x, v).
\]

Define \( H \) to be the subgraph of \( \text{Square}_n \) obtained by deleting \( v_{\text{Sink}} \) and all edges incident to \( v_{\text{Sink}} \). Let \( m \) be the largest positive integer such that \( u_1 + i \leq n \) and \( u_2 + j \leq n \) for all \( 0 \leq i, j < m \), and let \( H(u) \) be the subgraph of \( H \) induced by the vertex set

\[
\{(u_1 + i, u_2 + j) : 0 \leq i, j < m\}.
\]

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We can view \( H(\mathbf{u}) \) as the largest square resistor network in \( H \) such that \( \mathbf{u} \) is the top-left vertex. Let \( \mathbf{v} = [\mathbf{u}_1 + m - 1, \mathbf{u}_2 + m - 1] \) be the bottom-right vertex in \( H(\mathbf{u}) \). Using infinite resistors to remove every edge in \( E(\text{SQUARE}_n) \setminus E(H(\mathbf{u})) \), we have

\[
\mathcal{R}_{\text{eff}}^{\text{SQUARE}_n}(\mathbf{v}, \mathbf{u}) \leq \mathcal{R}_{\text{eff}}^{H(\mathbf{u})}(\mathbf{v}, \mathbf{u})
\]

by Rayleigh’s monotonicity law. Proposition A.1 implies that

\[
\mathcal{R}_{\text{eff}}^{H(\mathbf{u})}(\mathbf{v}, \mathbf{u}) \leq 2 \log n
\]

since \( m \leq n \). The vertex \( \mathbf{v} \) is incident to \( v_{\text{Sink}} \) in \( \text{SQUARE}_n \), so Rayleigh’s monotonicity law gives

\[
\mathcal{R}_{\text{eff}}^{\text{SQUARE}_n}(v_{\text{Sink}}, \mathbf{v}) \leq 1.
\]

By the triangle inequality for effective resistances, we have

\[
\mathcal{R}_{\text{eff}}^{\text{SQUARE}_n}(v_{\text{Sink}}, \mathbf{u}) \leq \mathcal{R}_{\text{eff}}^{\text{SQUARE}_n}(v_{\text{Sink}}, \mathbf{v}) + \mathcal{R}_{\text{eff}}^{\text{SQUARE}_n}(\mathbf{v}, \mathbf{u}) \leq 2 \log n + 1,
\]

which completes the proof.

B Deferred Proofs for Section 3

B.1 Proof of Lemma 3.4

For \( n \geq 10 \), we want to give a constant lower bound for

\[
\frac{1}{2^n} \sum_{\begin{subarray}{c} k = \left[ \frac{n}{4} \right] \\ k \text{ odd} \end{subarray}} \binom{n}{k} \quad \text{and} \quad \frac{1}{2^n} \sum_{\begin{subarray}{c} k = \left[ \frac{n}{4} \right] \\ k \text{ even} \end{subarray}} \binom{n}{k}.
\]

The proof we give uses the recursive definition of binomial coefficients and a Chernoff bound for symmetric random variables.

Proof of Lemma 3.4. First observe that for \( n \geq 10 \), we have

\[
\frac{1}{2^n} \sum_{\begin{subarray}{c} k = \left[ \frac{n}{4} \right] \\ k \text{ odd} \end{subarray}} \binom{n}{k} \geq \frac{1}{2^n} \sum_{k \in \left( \frac{n-1}{4}, \frac{3(n-1)}{4} \right)} \binom{n-1}{k}
\]

and

\[
\frac{1}{2^n} \sum_{\begin{subarray}{c} k = \left[ \frac{n}{4} \right] \\ k \text{ even} \end{subarray}} \binom{n}{k} \geq \frac{1}{2^n} \sum_{k \in \left( \frac{n-1}{4}, \frac{3(n-1)}{4} \right)} \binom{n-1}{k}.
\]

To see this, use the parity restriction and expand the summands as

\[
\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.
\]
Let $X_1, X_2, \ldots, X_{n-1}$ be independent Bernoulli random variables such that $\Pr[X_i = 0] = 1/2$ and $\Pr[X_i = 1] = 1/2$. Let $S_{n-1} = X_1 + X_2 + \cdots + X_{n-1}$ and $\mu = E[S_{n-1}] = (n-1)/2$. Using a Chernoff bound, we have

\[
\frac{1}{2^n} \sum_{k \in \binom{n-1}{\frac{3(n-1)}{4}}} \left( \begin{array}{c} n-1 \\ k \end{array} \right) = \frac{1}{2} \left( 1 - \Pr \left[ |S_{n-1} - \mu| \geq \frac{1}{2} \mu \right] \right) \\
\geq \frac{1}{2} - e^{-(n-1)/24} \\
\geq \frac{2}{5},
\]

for $n \geq 60$. Checking the remaining cases numerically when $10 \leq n < 60$ proves the claim. \(\square\)

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