Chapter 5
Path Tracking for Automated Driving: A Tutorial on Control System Formulations and Ongoing Research

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Nomenclature

The superscript “∗” is used to indicate the complex conjugate transpose.

\(a\): front semi-wheelbase
\(\bar{a}\): longitudinal distance between the center of gravity and the front end of the vehicle
\(a_x, a_{x,\text{max}}\): longitudinal acceleration, maximum longitudinal acceleration
\(a_y\): lateral acceleration
\(A, B, C, D, E\): generic state-space formulation matrices
\(A_r, B_r, C_r\): state-space matrices for path profile modeling
\(A_v, B_v, C_v, D_v\): state-space matrices for vehicle modeling
\(A', B'\): state-space matrices for modeling the tracking dynamics at the centers of percussion
\(b\): rear semi-wheelbase
\(\bar{b}\): longitudinal distance between the center of gravity and the rear end of the vehicle
\(b_\delta\): multiplicative factor of steering angle in the yaw acceleration error formulation used for backstepping control design
\(B_1, B_2, B_3\): matrices of the state-space single-track model formulation
$B$: box used in the formulation of the tube-based model predictive controller

$\bar{c}$: half of vehicle width

$c_{\text{COP},l}, c_{\text{COP},r}$: coefficients used in the definition of the sliding variables for the front and rear centers of percussion

$c$: constant gain for sliding mode controller

$C_f, C_r$: front and rear cornering stiffnesses

$C_f, \mu_{c_0}, C_r, \mu_{c_0}$: front and rear cornering stiffnesses for the nominal tire-road friction coefficient $\mu_{c_0} = 1$

$C_\beta, C_\psi, C_\Delta_\psi, C_k$: coefficients used for backstepping controller design

$C_1, C_2$: controller formulations 1 and 2

$d$: distance from the summit of the bend

$d_{\text{min},k,t}$: minimum distance between the vehicle and the obstacle points calculated at time $t$ and associated with the time $k$ within the tracking horizon

$d_{k,t,j}$: distance between the vehicle and the obstacle point $j$ calculated at time $t$ and associated with the time $k$ within the tracking horizon

$d_1, d_2$: denominators of controller formulations $C_1$ and $C_2$

$e, e_k$: error, discretised error

$D$: damping ratio

$D(s_L)$: denominator of the transfer function

$f_a$: function expressing the system dynamics

$f_{dy,t,\mu_{ek,t}}$: system model function for the coordinate $s_{k,t}$ and the tire road friction coefficient $\mu_{e_{k,t}}$, calculated at time $t$ and associated with the time $k$ within the tracking horizon

$F_{b,l}, F_{b,r}$: braking forces on the left-hand and right-hand sides of the vehicle

$F_{y,f}, F_{y,r}$: lateral forces at the front and rear axles

$F_{y,f}^{\text{FB}}, F_{y,r}^{\text{FB}}$: feedback contribution to the reference lateral forces on the front and rear axles

$F_{y,f}^{\text{FFW}}, F_{y,r}^{\text{FFW}}$: feedforward contribution to the lateral forces on the front and rear axles

$F_{y,f}^{\text{TOT}}$: sum of the feedforward and feedback contributions to the lateral forces on the front axle

$g$: gravity

$g_{x,\text{PP}}, g_{y,\text{PP}}, g_{x,S}, g_{y,S}$: longitudinal and lateral coordinates of the goal points according to the pure pursuit and Stanley path tracking methods

$g(\xi)$: nonlinear term within the model formulation for the robust tube-based controller design
\( G_c \): compensator of actuator dynamics
\( G_H, \tilde{M}_H, \tilde{N}_H \): matrices used for the coprime factorization of the nominal plant within the \( H_\infty \) controller design
\( G_{H\Delta}, \Delta_{M_H}, \Delta_{N_H} \): matrices used for the coprime factorization of the perturbed plant within the \( H_\infty \) controller design
\( G_{\text{vsl}} \): virtual sensor look-ahead filter
\( h \): function for obtaining the outputs starting from the inputs
\( h_p \): generic parameter considered in the parameter space approach
\( \mathcal{H}, \mathcal{F} \): polytopes used in the definitions of Pontryagin difference and Minkowski sum
\( H_C, H_P \): control horizon, prediction horizon
\( \mathcal{H} \): set of parameters in the parameters space approach
\( i \): imaginary unit
\( I_z \): yaw moment of inertia
\( J \): cost function for optimal control
\( J_{\text{obs}, k} \): cost function at time \( t \) and associated with the time \( k \) to the predicted distance between the vehicle and the obstacle
\( J_1, J_2 \): tracking performance criteria
\( k \): discretization step number or step time
\( k_c, k_{\text{int}} \): controller gain and integrator to keep the steady-state tracking error small
\( k_{ch} \): control gain of the chained-form controller used for calculating \( k_{1CC}, k_{2CC}, \) and \( k_{3CC} \)
\( k_d \): multiplicative factor of the state vector in the discontinuous control law
\( k_D \): derivative gain
\( k_{PP} \): gain used in the PIDD^2 controller
\( k_i \): integral gain
\( k_{LK} \): control gain in the feedback force contribution on the rear axle, \( F_{FB} \)
\( k_p \): proportional gain
\( k_{PP} \): tuning gain of the pure pursuit algorithm
\( k_S \): tuning gain of the Stanley method
\( k_w_1, k_w_2, k_{\psi} \): gains used in the optimal preview steering control law
\( k_{U} \): understeer gradient
\( k_{\phi} \): multiplicative factor of \( \dot{\psi} \)
\( k_{\Delta CG} \): multiplicative factor of \( \Delta y_{CG} \)
\( k_{\Delta v_d} \): multiplicative factor of \( \Delta y_{vd} \)
\( k_{\Delta \psi}, k_{\Delta \dot{\psi}} \): multiplicative factor of the heading error, multiplicative factor of the yaw rate error
multiplicative factors of the scalar errors $\Delta \psi_w, \Delta \dot{\psi}_w, \Delta y_w$ in the linear quadratic regulator with preview

gains of the preview controller, to be multiplied by the weighted values of the heading angle error, yaw rate error and lateral displacement error

$K_1, K_2, K_3, K_4$: gains of the chained controller

$K_{\text{LC}}$: linear quadric matrix gain of the limit cornering controller

$K_{\text{LQ}}$: matrix gain of the linear quadratic controller

$K_{\text{LQP}}$: matrix gain of the linear quadratic regulator with preview

$K_{\text{OBS}}$: collision weight

$l$: wheelbase

$l_d$: look-ahead distance

$L$: vehicle wheelbase

$L_{\text{Lipschitz}}$: Lipschitz constant

$L_{\text{Lyapunov}}$: Lyapunov function

$m$: vehicle mass

$M$: constant big enough to disregard obstacles that do not lie within the vehicle line of sight

$M_{\text{COP}}, M_{\text{COP},r}$: gains to provide robustness against the variation of cornering stiffness

$M_n$: tuning parameter of the sliding mode controller

$M_z$: yaw moment

$n$: counter

$n_1, n_2$: numerators of controller formulations $C_1$ and $C_2$

$N$: matrix of the Riccati equation for linear quadratic regulator design

$p$: characteristic polynomial

$p_{x_k,t_j}, p_{y_k,t_j}$: coordinates of the $j$-th point of the obstacle in the body frame, calculated at time $t$ and associated with the time $k$ within the tracking horizon

$p_{x_i,t_j}, p_{y_i,t_j}$: coordinates of the $j$-th point of the obstacle in the inertial frame at time $t$

$P_l$: Lyapunov matrix

$P_{t_j}$: $j$-th point of the obstacle in the inertial frame at time $t$
\( P_1, P_2: \) transfer functions calculated from a linear single-track model of the system, used in the feedforward contribution \( \delta_{\text{FFW},1} \)

\( \hat{q}: \) observer output

\( q_{\Delta y_1}, q_{\Delta y_2}, q_{\Delta \psi}, q_i: \) parameters of the filters of the frequency-shaped linear quadratic controller

\( q_1, q_2, q_3, q_4: \) extreme operating points for the \( \Gamma \)-stability controller

\( Q, R: \) weighting matrices of the cost function formulation

\( \text{Reach}_1(S, \mathcal{W}): \) one-step robust reachable set from a given set of states \((S)\)

\( r: \) input of the model reference system obtained by scaling the input of the desired trajectory generator

\( R_{\text{tr}}: \) trajectory radius

\( s, \dot{s}: \) trajectory coordinate and its time derivative

\( \mathcal{L}: \) Laplace operator

\( \mathcal{S}: \) matrix of the Riccati equation for linear quadratic regulator design

\( t: \) time

\( t_{\text{la}}: \) time corresponding to the look-ahead distance (at the current vehicle speed)

\( t_r: \) time delay due to the driver’s reaction

\( t_d: \) time delay

\( t_1, t_2: \) initial and final time values

\( T: \) matrix describing the dynamics of the system considering the center of percussion

\( T_{b,\text{lf}}, T_{b,\text{rf}}, T_{b,\text{lr}}, T_{b,\text{rr}}: \) braking torques of the left front, right front, left rear, and right rear wheels

\( T_i: \) time constant of the first order filter in the derivative term of the PID controller and delete the existing line

\( T_s: \) sampling time

\( u: \) input vector in the state-space formulation

\( \bar{\theta}: \) vector of the front and rear steering angles

\( u_k, \bar{u}_k, \hat{u}_k(e_k): \) control law, nominal controller, and state feedback control action used in the robust tube-based model predictive controller

\( u_1, u_2: \) control outputs of the chained controller

\( \mathcal{U}, \mathcal{U}: \) polyhedra used in the tube-based model predictive control constraints

\( v, \dot{v}_x, \dot{v}_y, \dot{v}_{\text{max}}: \) vehicle speed, longitudinal and lateral components of vehicle speed, maximum speed

\( v_{\text{ki}}: \) speed of the vehicle at time \( k \) predicted at time \( t \)

\( \dot{v}_{y,\text{path,CG}}: \) time derivative of the lateral velocity of the path
$v_0$: speed at which the four-wheel-steering controller changes the sign of the steering angle of the rear axle (transition from opposite signs to the same signs of the front and rear steering angles)

$V_1, V_2$: vehicle dynamics transfer functions

$w$: system disturbance

$\hat{w}$: element of $\hat{W}$

$W_d, W_\phi$: weighting function adopted in the backstepping steering control law

$W$: polyhedron used in the robust tube-based model predictive control constraints

$\tilde{W}$: Minkowski sum of the two polytopes $W$ and $B$

$x, h$: elements of the polytopes used in the definition of Pontryagin difference and Minkowski sum

$x_{\text{COP}_f}, x_{\text{COP}_r}$: coordinates of the front and rear centers of percussion

$x_p$: longitudinal position of a generic point $P$ in the vehicle reference system

$\dot{x}_{\text{ref}}$: reference longitudinal speed

$x_v, y_v$: vehicle positions in the tracking coordinates

$X, Y$: coordinates in the inertial reference system

$X_r, Y_r, \dot{X}_r, \dot{Y}_r$: positions and velocities of the rear wheel according to the inertial reference system

$y, y_k$: output and discrete output of the state-space formulation

$y_{ni}$: disturbance in the form of white noise in the linear quadratic regulator with preview

$\ddot{y}_{ld}$: lateral acceleration at the look-ahead distance $l_d$

$\ddot{y}_{\text{ref}}$: reference lateral acceleration

$Y_{\text{ref}}$: reference lateral position in the inertial frame

$z$: complex number used in the $z$-transform representation

$z_1, z_2, z_3, z_4$: augmented states for modelling the filter dynamics

$z_{\text{MPC}}, z_{\text{MPCref}}$: system outputs and references in the model predictive controller

$Z, Z_\infty$: subset of $\mathbb{Z}$, minimal robust positively invariant set

$\alpha$: half of the angular extension of the circular arc defined by the pure pursuit algorithm

$\alpha_r$: slip angle of the rear axle

$\alpha_{\text{ref,f}}$: reference slip angle of the front axle

$\alpha_{\text{FFW}}, \alpha_{\text{FFW}}^r$: reference values of the feedforward contributions to the front and rear slip angles

$\alpha_1, \alpha_2$: functions of the states, reference path and steering wheel input in the chained controller

$\alpha_{1,ST}, \alpha_{2,ST}$: tuning constants of the super-twisting controller
\( \beta, \beta_{SS} \): vehicle sideslip angle, steady-state vehicle sideslip angle

\( \beta_{x,f}, \beta_{y,f}, \beta_{r} \): normalized longitudinal force on the front axle, normalized lateral force on the front axle, normalized longitudinal force on the rear axle

\( \gamma \): sensitivity parameter for LQR design

\( \Gamma, \partial \Gamma \): desired region for locating the poles of the closed-loop system, with the corresponding boundary

\( \delta, \dot{\delta} \): steering angle and its time derivative. In the case of absence of any subscript, this notation refers to the front axle. In the case of a four-wheel-steering vehicle, the subscript ‘\( f \)’ is used to indicate the front steering angle and the subscript ‘\( r \)’ is used to indicate the rear steering angle. The additional subscript ‘\( ss \)’ is used to indicate steady-state conditions

\( \delta_{eq} \): equivalent steering angle in the super-twisting sliding mode formulation

\( \delta_{FB,LC,1}, \delta_{FB,LC,2}, \delta_{FB,LC,3} \): feedback steering angle contributions according to different formulations of the path tracking controllers for limit cornering

\( \delta_{FFW,LC} \): feedforward steering angle contribution according to the path tracking controller for limit cornering

\( \delta_{FFW,1}, \delta_{FFW,2}, \delta_{FFW,3} \): feedforward contributions to the reference steering angle according to different formulations

\( \delta_{\text{min}}, \delta_{\text{max}} \): minimum steering angle, maximum steering angle

\( \delta_{ST, \delta_{ST,1}, \delta_{ST,2}} \): steering angle contribution of the super-twisting controller, consisting of the contributions \( \delta_{ST,1} \) and \( \delta_{ST,2} \)

\( \dot{\gamma}_f \): steering rate contribution depending on the lateral deviation error at the front end of the vehicle

\( \dot{\gamma} \): steering rate contribution depending on vehicle yaw rate

\( \Delta a_y \): difference between the reference and the actual lateral acceleration

\( \Delta u \): control input variation

\( \Delta U_t \): optimization vector at time \( t \)

\( \Delta y, \Delta \dot{y}, \Delta \ddot{y} \): lateral position error and its first and second time derivatives. They can be calculated at the front axle (hence the subscript “\( f \)”), at the rear axle (hence the subscript “\( r \)”), at the vehicle center of gravity (hence the subscript “\( CG \)”), or at any other point along the longitudinal axis of the vehicle reference.
system (e.g., at the center of percussion or at the look-ahead distance)

$\Delta Y_{CG,SS}$: steady-state value of $\Delta Y_{CG}$

$\Delta Y_{rms}$, $\Delta Y_{max}$: root mean square error on vehicle lateral position, maximum error on the vehicle lateral position

$\Delta \delta_{min}$, $\Delta \delta_{max}$: minimum and maximum variation of the steering angle

$\Delta \psi$, $\Delta \dot{\psi}$, $\Delta \ddot{\psi}$: yaw angle (i.e., heading) error and its first and second time derivatives. They can be calculated with respect to the reference path at the front axle (hence the subscript “f”), at the rear axle (hence the subscript “r”), at the vehicle center of gravity (hence the subscript “CG”), or at any other point on the longitudinal axis of the vehicle reference system (e.g., at the centers of percussion)

$\Delta \psi_{CG,SS}$: steady-state value of the yaw angle error

$\Delta \psi_{rms}$, $\Delta \psi_{max}$: root mean square error on the yaw angle, maximum error on the yaw angle

$\Delta \psi_w$, $\Delta \dot{\psi}_w$, $\Delta y_w$: scalar values of the weighted average of the heading errors and lateral position error along the preview distance

$\varepsilon$: small number

$\mathcal{E}$: subset of $\Xi$ containing \{0\}

$\epsilon$, $\epsilon_{max}$: stability margin and maximum stability margin within the $H_\infty$ controller design

$\xi$, $\xi_{ref}$: state vector, reference state vector

$\xi_{augm.}$: augmented state vector for the frequency-shaped linear quadratic controller

$\xi_k$, $\xi_{\dot{k}}$: discrete state vector and predicted state vector

$\xi_{vk}$: discrete state vector used for describing the vehicle system in the tracking coordinates

$\Xi$, $\Xi^+$: polyhedra used in the tube-based model predictive control constraints

$\eta$: corrective coefficient of the rear axle cornering stiffness

$\kappa$, $\widehat{\kappa}$: path curvature and its estimated value. They can be calculated at the front axle (hence the subscript “f”), at the rear axle (hence the subscript “r”), or at any other point along the longitudinal axis of the vehicle reference system (e.g., the front and rear centers of percussion)

$\kappa^+$: derivative of the path curvature with respect to the trajectory coordinate

$\lambda$: sliding mode lateral position gain
5.1 Introduction

In automated driving system architectures (see the classification according to [1]), three layers can be typically defined [2]:

(i) The perception layer, aimed at detecting the conditions of the environment surrounding the vehicle, e.g., by identifying the appropriate lane and the presence of obstacles on the track;

(ii) The reference generation layer, providing the reference signals, e.g., in the form of the reference trajectory to be followed by the vehicle, based on the inputs from the perception layer;
(iii) The control layer, defining the commands required for ensuring the tracking performance of the reference trajectory. These commands are usually expressed in terms of reference steering angles (usually on the front axle only) and traction/braking torques.

This chapter focuses on the control layer and, in particular, the steering control for autonomous driving, also defined as path tracking control. The foundations of path tracking control for autonomous driving date back to well-known theoretical and experimental studies on robotic systems and driver modeling, detailed in several papers and textbooks (e.g., see the driver model descriptions in [3–9]). Moreover, automated driving experiments with different controllers have been conducted since the 1950s and 1960s, by using inductive cables or magnetic markers embedded in roadways to indicate the reference path [10, 11].

This contribution presents a survey of the main control techniques and formulations adopted to ensure that the automatically driven vehicles follow the reference trajectory, including analysis of extreme maneuvering conditions. The discussion will be based on a selection of different control structures, at increasing levels of complexity and performance. The focus will be on whether complex steering controllers are actually beneficial to autonomous driving. This is an important point, considering that Stanley and Sandstorm, the vehicles that obtained the first two places at the DARPA Grand Challenge (2004–2005), used very simple steering control laws based on kinematic vehicle models. In contrast to this, Boss, the autonomous vehicle winning the DARPA Urban Challenge (2007), was characterized by a far more advanced model predictive control strategy [12–15].

The main formulas for the different steering control structures will be concisely provided as a tutorial on the control system implementations, so that the reader can actually appreciate the characteristics of each formulation, and ultimately refer to the original papers in the case of specific interest. Also, the main simulation and experimental results obtained through the implementation of each control structure will be reported and critically analyzed.

The chapter is organized as follows:

- Section 5.2 presents path tracking methods based on simple geometric relationships, and a chained controller relying on a vehicle kinematic model, i.e., developed under the approximation of zero slip angles on the front and rear tires.
- The first part of Sect. 5.3 deals with conventional feedback controllers designed with a simplified dynamic model of the vehicle system, i.e., the well-known linear single-track vehicle model. The second half of Sect. 5.3 discusses relatively simple optimal control formulations, e.g., linear quadratic regulators, without and with feedforward contributions, and including the concept of preview in their most advanced declination. The layout of Sects. 5.2 and 5.3 mostly follows the guideline of a very relevant previous survey work [16], dating back to 2009, which critically assessed path tracking control methods through vehicle simulations with the software package CarSim.
- Section 5.4 discusses a couple of sliding mode formulations, one of them based on the important concept of center of percussion, and briefly mentions
other examples of path tracking controllers, e.g., based on $H_\infty$ control and backstepping control.

- Section 5.5 presents in detail the latest developments in the subject area, through a selection of examples of advanced controllers (i.e., path tracking controllers for autonomous racing and model predictive controllers) from recently published papers, including critical analysis of their specific benefits.
- Section 5.6 provides concluding remarks and ideas for future research on the subject.

5.2 Methods Based on Geometric and Kinematic Relationships

5.2.1 Pure Pursuit Method

The most basic path tracking method is represented by the pure pursuit formula, derived by geometrically calculating the curvature of a circular arc (describing an angle $2\alpha$ in a top view of the single-track model of the system, see Fig. 5.1) that connects the rear axle location to the goal point on the reference trajectory and by applying the well-known Ackerman steering formula, $1/(R_t \delta) = 1/L$. The goal point has coordinates $(g_{x,PP}, g_{y,PP})$ and is located at a look-ahead distance $l_d$ on the reference trajectory, measured from the rear axle. This brings a reference steering angle $\delta(t)$ equal to:

$$\delta(t) = \tan^{-1}\left( \frac{2L \sin \alpha}{l_d} \right) = \tan^{-1}\left( \frac{2L \sin \alpha}{k_{PP} v_s(t)} \right) \quad (5.1)$$

It can be shown that the resulting curvature is $\kappa = 1/R_{tr} = (2\Delta y_{la})/l_d^2$, i.e., this controller acts like a proportional controller, with gain $2/l_d^2$, on the error $\Delta y_{la}$, defined as the lateral distance between the $x$-axis of the vehicle reference system and the goal point $(g_{x,PP}, g_{y,PP})$ in Fig. 5.1. As shown in the right term of Eq. (5.1), the look-ahead distance $l_d$ is often scaled as a function of vehicle speed, $v_s(t)$, i.e., $l_d(t) = k_{PP} v_s(t)$. In general, low values of $l_d$ result in high precision tracking and low stability. As Eq. (5.1) is based on vehicle kinematics, it can generate significant tracking errors, caused by the absence of consideration of the vehicle sideslip.

5.2.2 Stanley Method

Another geometry-based path tracking method is the Stanley method, usually more suitable for medium-high speed driving conditions than the pure pursuit method and adopted by the Stanford University’s entry (called Stanley) to the DARPA Grand
Challenge. According to this method (see Fig. 5.2), the steering angle consists of: (i) a component equal to the heading error (i.e., the yaw angle error), \( \Delta \psi_f = \psi - \psi_{\text{path,}f} \), where \( \psi_{\text{path,}f} \) is measured at the goal point \((g_x, g_y, S)\) on the reference path; and (ii) a term based on the lateral distance error at the front axle, \( \Delta y_f \) (or any other point in the front part of the vehicle), ensuring that the intended trajectory intersects the target path at an approximated distance \( v_x(t)/k_S \) from the front axle, with \( k_S \) being the tuning parameter of the controller:

\[
\delta(t) = \Delta \psi_f(t) + \tan^{-1} \left( \frac{k_S \Delta y_f(t)}{v_x(t)} \right) 
\]  

(5.2)

Many other variants of geometric path tracking methods can be found in the literature. For example, Wit [18] proposes a vector pursuit path tracking method (for specific details refer directly to [18]).

### 5.2.3 Chained Controller Based on Vehicle Kinematics

Similarly to the previous controllers, the chained controller formulation in [17] (see also [19, 20]) is based on the single-track model of vehicle kinematics, i.e., considering zero slip angles for the front and rear tires. In particular, the hypotheses are that (see Fig. 5.2): (i) the rear wheels move along a direction with angle \( \psi \) (i.e., the yaw angle) with respect to the X-axis of the inertial reference system; (ii) the front wheels move along a direction with an angle \( \psi - \delta \) (where the signs are according to the conventions in [17]) with respect to the same X-axis; and (iii) a
simple kinematic relationship between yaw rate and steering wheel angle is valid, i.e., $\dot{\psi} = v \tan(\delta) / L$. Under (i)–(iii), the kinematic model of the vehicle in matrix form is:

$$
\begin{bmatrix}
\dot{X}_r \\
\dot{Y}_r \\
\dot{\psi} \\
\dot{\delta}
\end{bmatrix} =
\begin{bmatrix}
\cos(\psi) \\
\sin(\psi) \\
\frac{\tan(\delta)}{L} \\
0
\end{bmatrix} v
\begin{bmatrix}
0 \\
0 \\
0 \\
1
\end{bmatrix} \delta
$$ (5.3)

By expressing the kinematic model in path coordinates, with $s$ being the trajectory coordinate and $\kappa$ its curvature, the model formulation becomes:

$$
\begin{bmatrix}
\dot{s} \\
\Delta \dot{y}_r \\
\Delta \dot{\psi}_r \\
\dot{\delta}
\end{bmatrix} =
\begin{bmatrix}
\frac{\cos(\Delta \psi_1)}{1-\Delta \gamma_{y}(s)} \\
\frac{\sin(\Delta \psi_1)}{1-\Delta \gamma_{y}(s)} \\
\tan(\delta) - \frac{\kappa(s) \cos(\Delta \psi_1)}{1-\Delta \gamma_{y}(s)} \\
0
\end{bmatrix} v
\begin{bmatrix}
0 \\
0 \\
0 \\
1
\end{bmatrix} \delta
$$ (5.4)

with $\Delta \psi_r = \psi - \psi_{\text{path,r}}$. Through an appropriate transformation of the system coordinates (the general formulation of the coordinate transformation and the theory are reported in [17] and [19, 20]), it is possible to express this system into a typical
two-input chained form with the following structure:

\[
\begin{cases}
\dot{x}_1 = u_1 \\
\dot{x}_2 = \xi_3 u_1 \\
\dot{x}_3 = \xi_4 u_1 \\
\dot{x}_4 = u_2
\end{cases}
\] (5.5)

where the change of coordinates is:

\[
\begin{cases}
\dot{s} = 0 \\
\dot{s} = \Delta y_r \\
\dot{\xi}_3 = (1 - \Delta y_r \kappa(s)) \tan(\Delta \psi_r) \\
\dot{\xi}_4 = -\kappa'(s) \Delta y_r \tan(\Delta \psi_r) + \\
- \kappa(s) (1 - \Delta y_r \kappa(s)) \frac{1 + \sin^2(\Delta \psi_r)}{\cos^2(\Delta \psi_r)} + \\
\frac{(1 - \Delta y_r \kappa(s))^2 \tan(\delta)}{L \cos^4(\Delta \psi_r)}
\end{cases}
\] (5.6)

and the input transformation is:

\[
\begin{cases}
v = \frac{1 - \Delta y_r \kappa(s)}{\cos(\Delta \psi_r)} \dot{u}_1 \\
\dot{\delta} = \alpha_2 (u_2 - \alpha_1 u_1)
\end{cases}
\] (5.7)

with \( \alpha_1 = \frac{\partial \xi_4}{\partial s} + \frac{\partial \xi_4}{\partial \Delta y_r} (1 - \Delta y_r \kappa(s)) \tan(\Delta \psi_r) + \frac{\partial \xi_4}{\partial \Delta \psi} \left( \frac{\tan(\delta) \left( 1 - \Delta y_r \kappa(s) \right)}{L \cos(\Delta \psi)} - \kappa(s) \right) \)

and \( \alpha_2 = \frac{L \cos^3(\Delta \psi) \cos^2(\delta)}{(1 - \Delta y_r \kappa(s))^4} \). The controller design in chained systems is carried out in two phases. According to [16], “the first phase assumes that one control input is given, while the additional input is used to stabilize the remaining sub-vector of the system states. The second phase simply consists of specifying the first control input so as to guarantee convergence while maintaining stability.” In practical terms, if vehicle speed \( v \) is imposed, from Eq. (5.7) it is possible to calculate \( u_1 \), and then the steering input is a function of both \( u_1 \) and \( u_2 \), where \( u_2 \) depends on \( u_1 \).

In the proposed two-input chained structure for path tracking, the two controllers are designed so that for any piecewise-continuous, bounded, and strictly positive (or negative) \( u_1, u_2 \) is expressed as:

\[
u_2(\xi_2, \xi_3, \xi_4, t) = -k_{1_{CC}} u_1(t) - k_{2_{CC}} u_1(t) \xi_3 - k_{3_{CC}} \xi_4 \] (5.8)

De Luca et al. [17] suggest imposing \( k_{1_{CC}} = k_{ch}^3 \), \( k_{2_{CC}} = 3k_{ch}^2 \), \( k_{3_{CC}} = 3k_{ch} \), so that the control system tuning consists of selecting a single gain. The main (and possibly only) benefit of this convolute control formulation is that its straightforward extension allows the automated control of articulated vehicles, including vehicle systems with multiple trailers.
5.3 Methods Based on Conventional Feedback Controllers and Simplified Vehicle Dynamics Models

5.3.1 Simple Feedback Formulations

A significant body of literature adopts simple feedback control structures, such as proportional integral derivative (PID) controllers, to solve the steering control problem for autonomous vehicles. These feedback controllers are usually designed starting from simplified models of the system dynamics, accounting for the fact that the actual vehicle behavior is different from the one predicted by a geometric model, because of (i) the slip angles on the front and rear tires, generally different among the two axles, with the subsequent vehicle understeer gradient in steady-state conditions [21] and (ii) vehicle inertia in transient conditions, providing second order yaw dynamics, with variable equivalent stiffness and damping characteristics as functions of vehicle speed [22, 23].

On the one hand, the relatively weak link between the basic feedback formulations and the respective linearized dynamic models used for control system design, and also the significant level of approximation of these models, do not automatically guarantee improved performance for all driving conditions with respect to the algorithms based on the system geometry alone, presented in Sect. 5.2. On the other hand, despite the existence of multiple state-of-the-art contributions focused on advanced control techniques, e.g., based on model predictive control, relatively simple feedback controllers for path tracking can provide good performance for a variety of operating conditions, if carefully tuned. For example, the ARGO autonomous vehicle prototype developed by the team of Prof. Broggi of the University of Parma was characterized by a proportional (P) controller for steering control [24, 25]. Even in a recent paper [26] authored by the investigators of the European Union FP7 V-Charge project, the path tracking controller is based on a simple proportional controller, with an advanced algorithm for the reference path generation, using an optimization considering the current vehicle position with respect to equivalent electric field lines. The following paragraphs provide an overview of some relatively simple control structures designed through linear single-track vehicle models.

Already in the 1960s in Japan [10, 27], experiments were conducted on a path tracking controller based on a Proportional Derivative (PD) structure on the lateral displacement error between the reference path and the actual path of the vehicle, and a P controller on the yaw angle error. The two contributions were summed together to provide $\delta$. The yaw angle error contribution allowed an improvement in tracking performance with respect to the path deviation feedback contribution alone. This is further confirmed in a more recent paper by Nissan [28], in which a PD controller exclusively based on lateral position error is designed through pole placement, with clear issues in the results, caused by the effects of yaw rate and yaw angle, not addressed by the position error controller.
Fig. 5.3 Output feedback path tracking control system based on the lateral displacement error measured at the front bumper (adapted from [29])

More systematically, [29] critically examines the limitations of pure output feedback on the lateral vehicle displacement error measured at the front bumper, $\Delta y_{ld} = \frac{1}{s_L} (\hat{y}_{ld} - \hat{y}_{ref})$ (Fig. 5.3). This was the control practice (dictated by necessity) in the initial look-down automated driving system implementations based on devices installed on the road and not based on vision systems implemented on the vehicle, intrinsically allowing look-ahead path tracking control.

The analysis of [29] is based on the lateral acceleration and yaw rate frequency response characteristics for a steering wheel input, obtained from the equations of the linear single-track vehicle model, with the lateral acceleration being considered at a distance $l_d$ in front of the vehicle. In [29], $l_d$ is used to indicate both the installation position of the lateral displacement error sensor and a virtual look-ahead distance.

The transfer functions of vehicle response (i.e., $V_1(s_L) = \frac{\dot{\psi}(s_L)}{\delta(s_L)}$ and $V_2(s_L) = \frac{\ddot{y}_{ld}(s_L)}{\delta(s_L)}$) are characterized by the same second order denominator, i.e., $D(s_L) = I_L m v^2 s_L^2 + v \{ I_L (C_t + C_r) + m (C_\alpha a^2 + C_b b^2) \} s_L + m v^2 (C_r b - C_t a) + C_t C_r I_L^2$, with the corresponding damping coefficient being a decreasing function of vehicle speed. In [29], the effect of the tire-road friction coefficient is modeled by imposing $C_t = C_{t,\mu c_0} \mu_c$ and $C_r = C_{r,\mu c_0} \mu_c$, which, according to the authors of this chapter, can be the object of discussions. The lateral acceleration transfer function has a second order numerator, while the yaw rate transfer function has a first order numerator. In formulas:

$$V_1(s_L) = \frac{\dot{\psi}(s_L)}{\delta(s_L)} = \frac{C_t a m v^2 s_L + C_t C_r L v}{D(s_L)} \quad (5.9)$$

$$V_2(s_L) = \frac{\ddot{y}_{ld}(s_L)}{\delta(s_L)} = \frac{C_t v^2 (m a l_d + I_z) s_L^2 + C_t C_r L v (l_d + b) s_L + C_t C_r L v^2}{D(s_L)} \quad (5.10)$$
Equations (5.9) and (5.10) are the essential plant transfer functions to be considered for linear path tracking control design in the frequency domain. The inclusion of the transfer function of the specific steering actuator is also recommended by the authors of this chapter for any path tracking control design activity, consistently with the practice of many sources in the literature, some of them indicating a typical actuation bandwidth of 5–10 Hz.

Figure 5.4 shows the Bode plots of $V_2(s_L)$ for different vehicle speeds and tire-road friction conditions. In particular: (i) the natural mode of vehicle dynamics is negligible for low speed but significant for high speed; (ii) the frequency of the natural mode is almost independent of speed, but decreases as a function of the friction coefficient; (iii) the steady-state gain depends on both $v$ and $\mu_c$ (the latter is true especially at high speed); (iv) the high-frequency gain depends on $\mu_c$ but not on $v$; and (v) the contribution of yaw motion to lateral acceleration diminishes by the inverse of vehicle speed, in favor of the sideslip contribution.

According to [29], realistic control system specifications should be in terms of maximum lateral displacement error at any vehicle speed and robust behavior for $0.5 \leq \mu_c \leq 1$. The variations of $\mu_c$ can happen very quickly; therefore, the same controller must be capable of providing robustness for the indicated range of friction conditions. For very low friction values, i.e., for $\mu_c \leq 0.5$, it is expected that the longitudinal controller imposes much lower values of vehicle speed than in normal friction conditions. The important conclusion of the analysis in [29] is that for
relatively high vehicle speeds look-down controllers are not sufficient to meet the expected performance requirements.

As a consequence, [29] proposes:

(i) A feedforward steering contribution based on the reference path curvature;
(ii) Feedback control of vehicle absolute motion, i.e., control of yaw rate and lateral acceleration;
(iii) Modification of the system zeros in Eq. (5.10) by increasing the look-ahead distance $l_d$, which can be done even in look-down systems by having two position error sensors, installed at the front and rear bumpers.

With respect to (iii), Fig. 5.5 shows the increase of the damping ratio of the numerator of the transfer function in Eq. (5.10) as a function of $l_d$, for different velocities and $\mu_c = 0.5$. The zero pair in the numerator of Eq. (5.10) determines “the undershoot in Fig. 5.4 between 1 Hz and 2 Hz and the distribution of phase lag/lead in this frequency range. Conversely, prescribing a fixed maximum phase lag... yields look-ahead requirements for different speeds,” which is outlined in Fig. 5.6, i.e., the look-ahead distance should be an increasing function of vehicle speed. Section 5.5 will show that the most recent path tracking controllers, specifically implemented for high lateral acceleration conditions, actually include the combination of (i)–(iii).

Hsu and Tomizuka [30] extend the general analysis of [29] to include the specificities of vision-based control systems. In particular, the main conclusions are that (i) Look-ahead distance enhances stability but increases the error and decreases the closed-loop bandwidth; (ii) Higher vehicle speed increases the cross-over frequency and reduces stability; (iii) The time delays caused by the vision system decrease

![Fig. 5.5 Damping ratio of the numerator of the transfer function in Eq. (5.10) as a function of $l_d$, for different velocities and $\mu_c = 0.5$ (from [29])]
stability, through a reduction of the phase margin and equivalent effect of the right-hand-plane zeros appearing in the loop transfer function. Hsu and Tomizuka [30] propose a path tracking controller, consisting of a feedforward contribution of the form $\delta_{\text{FFW}, 1} = -P_1^{-1}(s_L) P_2 (s_L) e^{i \delta s_L} \kappa (s_L)$ (with $P_1 (s_L) = \Delta y_{CG}(s_L) / \delta (s_L)$ and $P_2 (s_L) = \Delta y_{CG}(s_L) / \kappa (s_L)$ being calculated from a linear single-track model of the system), and a PID feedback contribution on the lateral displacement error, receiving $\Delta y_{CG}$ as input, with $k_P = -0.01$, $k_I = 0$, $k_D = -0.0074$ and $T_i = 0.0001$.

Consistently with the conclusions of [29, 30], Tachibana et al. [31] propose a PD controller on the lateral position error at the look-ahead distance $l_d$, i.e., the position error at a single point in front of the vehicle is monitored, assuming that the heading angle of the vehicle does not change with distance (Fig. 5.7). The control law in discretized form is:

$$\delta_k = k_P (l_d, v) \Delta y_{\text{ld}, k} + k_D \left( \Delta y_{\text{ld}, k} - \Delta y_{\text{ld}, k-1} \right)$$  \hspace{1cm} (5.11)
Vehicle experiments showed that the control gains had to be varied as functions of vehicle speed and that at 50 km/h optimal look-ahead distances are in the range of 20 m, in the case of a curved trajectory, and 25 m, for a straight reference trajectory.

In [32], based on the Japanese Ministry of Construction Automated Highways System (AHS) project (1995–1996), the path tracking controller (Fig. 5.8) consists of a PID controller on $\Delta y_{CG}$, whose output is summed to that of a PI controller on $\Delta y_{ld}$, and to a feedforward contribution of the form:

$$\delta_{FFW,2} = \left(1 + k_U u^2\right) \frac{L}{R_{tr}}$$  \hspace{1cm} (5.12)

The feedforward contribution accounts for the steady-state vehicle understeer and brings a reduction of the maximum value of $\Delta y_{CG}$ from about 110 cm (with the feedback contribution only) to about 40 cm (with the feedback and feedforward contributions) during the case study tests. The comment of the authors of [32] is that the feedforward contribution allows achieving a good compromise between stability and tracking characteristics, without large gains of the feedback part. Interestingly, very recent papers by Prof. Gerdes of Stanford University reach the same conclusion, and the respective controllers significantly rely on nonlinear feedforward contributions (see Sect. 5.5). A similar combination of feedforward and feedback contributions was adopted for the anticollision system developed during the PRORETA research project, within a collaboration between Darmstadt University of Technology and Continental [33].

Marino et al. [34] propose a PID control architecture with two nested control blocks. The outer one calculates the reference yaw rate starting from the lateral position error, while the inner loop calculates the reference steering angle in order to track the reference yaw rate. According to the authors of [34], this architecture allows “to design standard PID controls in a multi-variable context.” Singular values analysis is used to assess the robustness of the controller with respect to the variation
of the main vehicle parameters. The simulation results show improved performance with respect to a model predictive driver model. The experiments on a Peugeot 307 prototype vehicle confirmed the adequate performance of the controller in normal driving conditions (i.e., for relatively low values of lateral acceleration).

Simple feedback formulations with constant gains can be designed to provide the required level of robustness for normal operating conditions. For example, [35] is an important study, focused on the design of a linear controller for an automated bus, with the main specifications being: (i) \(|\delta| \leq 40\) deg; (ii) \(|\dot{\delta}| \leq 23\) deg/s; (iii) lateral displacement error not exceeding 0.15 m in transient conditions and 0.02 m in steady state; (iv) lateral acceleration not exceeding 2 m/s² for passenger comfort with an ultimate limit of 4 m/s² for preventing vehicle rollover; and (v) a natural frequency of the lateral motion not exceeding 1.2 Hz.

The control system is designed through a linear single-track vehicle model, under the hypothesis of a lateral displacement sensor located on the front end of the vehicle, measuring the distance from the reference road path. The steering controller has the following formulation, where the benefit of the yaw rate-related contribution \(k_p \dot{\psi}\) is discussed in [35] through root-locus analysis:

\[
\delta = \dot{\delta}_v + \dot{\delta}_r = \dot{\delta}_v - k_p \dot{\psi}
\]  
(5.13)

The fixed controller design must be robust with respect to the variation of vehicle mass (very significant for a bus) and speed. In particular, the controller must provide stability for the four design points indicated in Fig. 5.9. The contribution \(\dot{\delta}_v(s_L)\) is designed from the frequency domain analysis. In the Laplace domain, \(\dot{\delta}_v(s_L)\) is defined as

\[
\dot{\delta}_v(s_L) = \omega_C^3 \frac{k_D s_L^2 + k_D s_L + k_P + k_I s_L}{(s_L^2 + 2D_s \omega_C s_L + \omega_C^2)(s_L + \omega_C)} \Delta y_i(s_L)
\]  
(5.14)

which, according to the authors of [35–37], is a PIDD² controller. This control structure was already recommended for high-speed path tracking by the authors of [29].

![Fig. 5.9 Vehicle operating domain (adapted from [35])](image-url)
A parameter space approach is used for the design of the compensator in Eq. (5.14). This method allows “to determine the set of parameters $\mathcal{H}$, for which the characteristic polynomial $p(s_L, h_p)$, $h_p \in \mathcal{H}$, is stable. The plant is robustly stable if the operating domain is entirely contained in the set of stable parameters.” For the specific problem, Hurwitz stability is not considered sufficient. A hyperbola in the $s_L$-plane is used to provide the required performance characteristics, i.e., the eigenvalues of the closed-loop system should lie in the region $\Gamma$ on the left of the boundary $\partial \Gamma$ defined as:

$$\partial \Gamma = \left\{ s_L = \sigma_L + i\omega_L \left| \left( \frac{\sigma_L}{\sigma_{L0}} \right)^2 - \left( \frac{\omega_L}{\omega_{L0}} \right)^2 = 1, \sigma_L \leq -\sigma_{L0} \right. \right\}$$  (5.15)

Values of $\sigma_{L0}$ equal to 0.12 and 0.35 are selected, respectively, for low and high velocities, with $\frac{\omega_{L0}}{\sigma_{L0}} = 5$. The $\Gamma$-stability boundaries for each of the four extreme operating points $(q_1, q_2, q_3, q_4)$ in Fig. 5.9 are calculated, resulting in four $\Gamma$-stable regions. The intersection of the stable regions is the set of controllers stabilizing the four considered plants (e.g., see Fig. 5.10).

The initial selection of the controller parameters $[k_{DD} \ k_D \ k_P \ k_I]$ was further refined in a simulation-based optimization procedure minimizing tracking performance criteria such as:

$$J_1 = \int_{t_1}^{t_2} \Delta y_t^2 dt, J_2 = \max_f |\Delta y_f|$$  (5.16)

A similar control design methodology, using the concept of $\Gamma$-stability, is presented in [37], this time based on the control of the errors of the front and tail lateral positions, instead of the front position and yaw rate. The option of a feedforward steering angle considering dynamic curvature preview was included in

Fig. 5.10 Set of $\Gamma$-stabilizing controllers for $\omega_C = 100$, $D_r = 0.5$, $k_I = 3$ and $k_P = 10$ (from [35])
the controller, which was validated with experimental tests in collaboration with the Californian PATH (Partners for Advanced Transportation Technology) center.

Another very relevant study [38], including experiments, assessed the performance of an automated Fiat Brava 1600 ELX. The control system design was based on the commonly used single-track vehicle model (the detailed formulation is provided in the following lines), together with the transfer function of the steering actuator.

Similarly to [35], the objective was to design fixed controllers (e.g., without any form of gain scheduling), capable of robustly stabilizing the plant for: (i) $v$ ranging between 60 km/h and 130 km/h; (ii) $m$ ranging between 1226 kg and 1626 kg; (iii) $I_z$ ranging between 1900 kgm$^2$ and 2520 kgm$^2$; and (iv) $C_f$ and $C_r$ ranging between 51 kN/rad and 69 kN/rad, and between 81.6 kN/rad and 110.4 kN/rad, respectively. The performance specifications included $|\Delta y_{CG}| \leq 0.2$ m, $|v_y(t)| \leq 1.5$ m/s, $|\Delta a_y| \leq 3.3$ m/s$^2$, and consideration of the steering actuator saturation.

Based on previous experimental analyses of human drivers [39] showing that the steering wheel action is applied “on the basis of the distance between the lane and the longitudinal axis of the car at a look-ahead point,” the controller in [38] uses feedback output on $\Delta y_{d} = \Delta y_{CG} + \Delta y_{path,d}$, with $l_d=11.5$ m. The controller is designed through classical loop-shaping techniques. In particular, two controllers, $C_1(z) = n_1(z)/d_1(z)$ and $C_2(z) = n_2(z)/d_2(z)$, were experimentally assessed, with numerical values of the controller parameters, in descending powers of $z$, given by:

$$ n_1 = [-7.844, 30.82, -47.37, 35.51, -13.24, 2.388, -0.2273] $$

$$ d_1 = [1, -4.92, 10.06, -10.96, 6.703, -2.181, 0.2949] $$

$$ n_2 = [-7.837, 29.03, -44.6, 33.43, -12.46, 2.2, -0.2138] $$

$$ d_2 = [1, -4.937, 10.13, -11.07, 6.794, -2.218, 0.3008] $$

The values of the gains are reported here for a quick implementation of the controller and reproduction of the results. $C_1(z)$ was designed with the purpose of providing good tracking performance, while $C_2(z)$ was specifically aimed at ride comfort. $C_1(z)$ and $C_2(z)$ were assessed along a curve with 1000 m radius followed by a straight section, at $v=100$ km/h. The tracking performance results are in Figs. 5.11 and 5.12, which show that $C_2(z)$ provokes higher amplitude and lower frequency oscillations of the lateral offset between the path and the vehicle center of gravity. Overall, the test drivers provided a better assessment for $C_2(z)$, which demonstrates the extreme importance of the human factor in the evaluation of path tracking algorithms.

Tan et al. [40] includes a wide set of experimental results obtained during public demonstrations at various sites and in very different operating conditions, ranging from docking to high speeds and relatively high lateral accelerations. The control system design is based on a simple but reliable controller, with the following form:

$$ \delta = -k_c(v)G_c(s_L) (k_{int} (s_L) \Delta y + l_d(v)G_{ld} (s_L) \Delta \psi) $$

(5.18)
where the measured inputs are $\Delta y$ and $\Delta \psi$. The tuning of the controller is carried out through a linear vehicle model, including roll dynamics. This peculiar (and interesting) choice is justified by the fact that a good match between the model and the experimental results is achievable, in the case of a comfort-oriented tuning of the vehicle suspension system, only through the inclusion of the roll dynamics and actuator dynamics in the model for control system design.

The formal design specifications are: (i) the maximization of the gain margin (with the requirement of being $> 2$) and phase margin (with a requirement of being $> 50$ deg) on the open-loop transfer function through the optimal selection
of the gain $k_c(v)$ and the preview distance $l_d(v)$ and (ii) to guarantee that the lateral displacement deviation of the closed-loop system shall not exceed a given threshold (reported in Fig. 5.13) for a 1 m/s$^2$ step input of the reference road acceleration.

In the control structure, $G_c(s_L)$ mainly compensates for the actuator dynamics and consists of a low-frequency integrator and high-frequency roll-off, “to reduce the effects of the steady-state tracking bias and the unwanted excitation of the high-frequency unmodeled actuator dynamics. $G_{ld}(s_L)$ is made of a high-frequency roll-off portion and a mid-frequency lead-lag filter to limit the look-ahead amplification and to provide extra look-ahead between 0.5 and 2 Hz.” In formulas:

$$G_c(s_L) = \frac{25\pi \left(s_L + 0.5\pi\right)}{(s_L + 0.02\pi) \left(s_L + 25\pi\right)} \quad (5.19)$$

$$G_{ld}(s_L) = \frac{20\pi \left(s_L + 0.4\pi\right)}{(s_L + 0.8\pi) \left(s_L + 10\pi\right)} \quad (5.20)$$

$k_{int}(s_L)$ is an integrator to keep the steady-state tracking error small. The main controller parameters and performance indicators based on the linear model of the system are reported in Fig. 5.13. Interestingly, $k_c(v)$ and $l_d(v)$ do not significantly change between 15 m/s and 30 m/s, while they are significantly different especially at low speed.

The comprehensive set of experimental results, collected on eight vehicle demonstrators including different number of passengers (from 0 to 4), shows: (i) 0.1 m maximum tracking error for highway driving up to 100 mph with standard deviation of less than 5 cm; (ii) 0.2 m maximum error in transient conditions for 0.5 g maneuvering; (iii) sharp curve tracking with less than 3 cm error; (iv) low speed precision docking with less than 1 cm error; and (v) smooth steering actions, i.e., the passengers feel no observable oscillations.

Based on these experimental proofs and also on the other references mentioned so far, the conclusion of this section is that in the absence of uncertainty and

![Fig. 5.13](image-url) Optimal controller parameters and performance indicators (from [40])
significant disturbances in the road scenario, simple, conventional, reliable, and
easily tunable control structures with appropriate gain scheduling are sufficient
for providing good performance in all operating conditions, and are actually
recommended by the authors of this chapter for vehicle implementation.

5.3.2 Linear Quadratic Regulators

5.3.2.1 Basic Linear Quadratic Formulation

This subsection focuses on the basic mathematical formulation of linear quadratic
regulator (LQR) control structures for path tracking. LQRs for path tracking are
based on the state-space formulation of the system dynamics. The main building
block is represented by the well-known single-track model of the vehicle [4, 21,
22], already used in the previous subsection, with a linear model of the tires,
parameterized by their cornering stiffness, and adopting the lateral slip velocity
of the vehicle center of gravity, $v_y$, and the yaw rate, $\psi$, as system states. This model
is suitable for describing vehicle dynamics at moderate lateral acceleration levels
with respect to the available tire-road friction conditions, and at small longitudinal
accelerations/decelerations (its equations are actually derived for the condition of
constant speed). In formulas:

$$
\begin{align*}
\begin{bmatrix}
\dot{v}_y \\
\dot{\psi}
\end{bmatrix} &= \begin{bmatrix}
-2\frac{(C_l+C_t)}{mv_x} & 2\frac{b(C_l-aC_t)}{mv_x} - v_x \\
2\frac{bC_l-aC_t}{I_v v_x} & -2\frac{(a^2C_l+b^2C_t)}{I_v v_x}
\end{bmatrix}
\begin{bmatrix}
v_y \\
\psi
\end{bmatrix} + \begin{bmatrix}
\frac{2C_l}{m}\frac{v_x}{2aC_l} \\
0
\end{bmatrix} \delta
\end{align*}
$$

(5.21)

The single-track vehicle model has to be expressed in the states relevant to the
implementation of the path tracking controller. The following control variables are
usually defined (minor variations are present in the different papers and reports):
(i) the lateral position error, $\Delta y_{CG}$, i.e., the length of the segment perpendicular
to the symmetry plane of the vehicle and connecting the center of gravity with the
corresponding point on the reference path (see Fig. 5.2, according to [16]).
In some of the papers, the distance is measured along a segment perpendicular to
the reference path, rather than perpendicular to the vehicle symmetry plane (i.e., a
reference system aligned with the road is adopted) and (ii) the heading angle error,
$\Delta \psi_{CG} = \psi - \psi_{path.CG}$ (indicated in Fig. 5.2). The controller formulation would
not significantly change if the errors were measured from any other point (different
from the center of gravity) located on the longitudinal axis of the vehicle reference
system or the road reference system (e.g., in front of the vehicle, in order to generate
a basic preview effect).

By considering the approximated system kinematics, it is $\Delta \dot{y}_{CG} = (\dot{v}_y + v_x \dot{\psi}) -
\dot{v}_{y,path.CG}(s) = \dot{v}_y + v_x (\dot{\psi} - \dot{\psi}_{path.CG}(s)) = \dot{v}_y + v_x \Delta \dot{\psi}_{CG}$ and
$\Delta \dot{y}_{CG} = v_y + v_x \Delta \dot{\psi}_{CG}$. Equation (5.21) can be transformed into the path coordinates, thus obtaining
the state-space formulation directly applicable to the path tracking LQR control system.
design:

\[
\begin{bmatrix}
\Delta \dot{y}_{CG} \\
\Delta \dot{\psi}_{CG} \\
\Delta \dot{\psi}_{CG}
\end{bmatrix} =
\begin{bmatrix}
0 & \frac{1}{m v_x} & 0 \\
0 & 0 & \frac{2 b C_l - a C_i}{I_z v_x} \\
0 & \frac{2 b C_l - a C_i}{I_z v_x} & -\frac{2 (a^2 C_l + b^2 C_i)}{I_z v_x}
\end{bmatrix}
\begin{bmatrix}
\Delta y_{CG} \\
\Delta \psi_{CG} \\
\Delta \psi_{CG}
\end{bmatrix} +
\begin{bmatrix}
0 \\
\frac{2 C_l}{m} \\
0
\end{bmatrix} \delta
\]

\[
\begin{bmatrix}
\dot{\psi}_{\text{path,CG}}(s) \\
\dot{\psi}_{\text{path,CG}}(s)
\end{bmatrix} =
\begin{bmatrix}
2 b C_l - a C_i \\
0 \\
-2 (a^2 C_l + b^2 C_i)
\end{bmatrix}
\begin{bmatrix}
\Delta \dot{y}_{CG} \\
\Delta \dot{\psi}_{CG} \\
\Delta \dot{\psi}_{CG}
\end{bmatrix}.
\]

which corresponds to the canonical form:

\[
\dot{\xi} = A \xi + B_1 \delta + B_2 \dot{\psi}_{\text{path,CG}}(s) + B_3 \ddot{\psi}_{\text{path,CG}}(s)
\] (5.23)

where \( \xi = [\Delta y_{CG}, \Delta \dot{y}_{CG}, \Delta \psi_{CG}, \Delta \dot{\psi}_{CG}]^T \), and the steering angle \( \delta \) is the control output. The term in Eqs. (5.22) and (5.23) including the reference yaw acceleration, \( \dot{\psi}_{\text{path,CG}}(s) \), is usually neglected in the literature. The term \( B_2 \dot{\psi}_{\text{path,CG}} \) represents a disturbance within the control system design.

The system is controllable as the controllability matrix \([B_1 \ AB_1 \ A^2 B_1 \ A^3 B_1]\) has full rank. The LQR formulation for path tracking is normally based on state feedback regulation, i.e., on the control of the lateral position and velocity errors at the center of gravity, and the control of the heading angle and heading rate errors, with all references set to 0. This means that \( \delta = -K_{LQ} \xi = -k_{1\text{LQ}} \Delta y_{CG} - k_{2\text{LQ}} \Delta \dot{y}_{CG} - k_{3\text{LQ}} \Delta \psi_{CG} - k_{4\text{LQ}} \Delta \dot{\psi}_{CG} \). The LQR controller minimizes the following quadratic cost function \( J \):

\[
J = \int_0^\infty \left( \xi^T Q \xi + u^T R u \right) dt
\] (5.24)

where the diagonal 4x4 weighting matrix \( Q \) is selected to define the relative importance of the tracking performance for the different states of the system, while the weighting factor \( R \) defines the relative importance of control effort and tracking performance. The gain \( K_{LQ} \) can be designed through the well-known algebraic Riccati equation [41]. In formulas:

\[
K_{LQ} = R^{-1} B_1^T S
\] (5.25)

\[
A^T S - SA - (SB_1^T + N) R^{-1} B_1^T S + Q = 0
\] (5.26)
thus bringing the following closed-loop formulation of the controlled system dynamics:

$$\dot{\xi} = (A - B_1 K_{LQ}) \xi + B_2 \tilde{\psi}_{\text{path,CG}}(s)$$  \hspace{1cm} (5.27)

The continuous form of the LQR path tracking controller was presented here. The continuous system in Eq. (5.27) can be easily subject to discretization for the design of a discrete LQR controller.

An LQR implementation was experimentally assessed by Nissan in [28] and compared with the performance provided by the PD controller on the lateral position error mentioned in Sect. 5.3.1. The tests (Fig. 5.14) were conducted at 80 km/h and consisted of a step change of 20 cm in the lateral coordinate of the target path. The study included a sensitivity analysis as a function of $\gamma = Q(1, 1)/R$, i.e., the ratio between the lateral deviation weighting and the control effort weighting in Eq. (5.24).

When commenting their experimental and simulation results, the authors of [28] state that “with PD control, large overshoot occurred when response was improved and the system became more susceptible to noise. This made it impossible to set the control constants at larger values.” In any case, the same authors observe that the model for LQR design exhibits “large modeling errors on the curved segments of
the test road, making it unable to track the path accurately on curves.” A possible solution to this limitation is to adopt a feedforward contribution based on road curvature, which will be discussed in the next subsection. The alternative proposed in [28] is a Kalman filter based on the vehicle motion equations, combined with a curvature approximation model. The output of the Kalman filter is the curvature estimation, $\hat{\kappa}$, which is used as a state variable in the augmented LQR scheme.

5.3.2.2 Linear Quadratic Regulator with Feedforward Contribution

The feedback LQR formulation of the controller can be enhanced through a feedforward contribution, aimed at canceling the steady-state lateral position error of the center of gravity, which is a problem especially in the case of curved paths. Hence, the control output and closed-loop system dynamics assume the following shape:

$$\delta = -K_{LQ}\dot{\xi} + \delta_{\text{FFW,3}}$$

$$\dot{\xi} = (A - B_1K_{LQ})\xi + B_1\delta_{\text{FFW,3}} + B_2\dot{\psi}_{\text{path,CG}}(s)$$

By manipulating Eq. (5.29), it is possible to obtain the analytical expression of the steady-state errors of the controlled system. By imposing $\Delta\psi_{\text{CG,SS}} = 0$ and under the hypothesis of a constant radius trajectory, it is:

$$\delta_{\text{FFW,3}} = \frac{L}{R_{\text{tr}}} + \left(\frac{mb}{2C_t} - \frac{ma}{2C_t}\right)\frac{a_y}{L} + k_{3LQ}\Delta\psi_{\text{CG,SS}} = \frac{L}{R_{\text{tr}}} + k_Ua_y + k_{3LQ}\Delta\psi_{\text{CG,SS}}$$

(5.30)

where the resulting steady-state value of the heading error becomes:

$$\Delta\psi_{\text{CG,SS}} = -\frac{b}{R_{\text{tr}}} + a\frac{mv_x^2}{2C_tL}$$

(5.31)

The important conclusion is that $\Delta\psi_{\text{CG,SS}}$ is not controllable through the feedforward contribution of the steering angle, if this is aimed at achieving $\Delta\psi_{\text{CG,SS}} = 0$.

From a practical viewpoint, as the yaw dynamics of the vehicle are strongly dependent on vehicle speed, i.e., yaw damping is a decreasing function of vehicle speed, a careful scheduling of the feedback controller gains is also required as a function of $v_x$, in order to provide consistent tracking performance at different vehicle speeds, as already discussed in Sect. 5.3.1.
5.3.2.3 **Linear Quadratic Regulator with Preview**

The LQR formulations in the previous sections can be significantly enhanced through a preview scheme, i.e., by augmenting the system to include the future profile of the reference path. To this purpose, the reference path is discretized, and a vector with its lateral coordinates is progressively updated at each time step. The vehicle system model in the tracking coordinates (Eq. 5.22) can be discretized as:

\[
\begin{align*}
\xi_{t+1} &= A_{t} \xi_t + B_{t} \delta_t \\
y_{t+1} &= C_{t} x_{t} + D_{t} \delta_t
\end{align*}
\]

(5.32)

The future reference path profile, represented by the vector \(y_{r_{t+1}}\), is modeled as:

\[
A_{r} = \begin{bmatrix}
0 & 0 & 0 & \cdots & 0 \\
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 1 & 0 \\
0 & 0 & \cdots & 0 & 0
\end{bmatrix}, \quad B_{r} = \begin{bmatrix}
0 \\
0 \\
0 \\
\vdots \\
0 \\
1
\end{bmatrix}
\]

(5.33)

The reference lateral position of the vehicle is located into a register that is progressively updated at each time step. \(y_{r_{t}}\) is considered as a disturbance in the form of a white noise.

The discretized model of the vehicle system, including the reference position coordinates, can be expressed as the combination of the models in Eqs. (5.32) and (5.33):

\[
\begin{align*}
\xi_{t+1} &= A_{t} \xi_t + B_{t} \delta_t + E_{t} y_{r_{t}} \\
y_{t} &= C_{t} \xi_t + D_{t} \delta_t
\end{align*}
\]

(5.34)

with augmented state vector \(\xi_t = [\xi_{t} \: y_{r_{t}}]^T\) (see [16] for the details regarding the formulation of the different matrices). The state feedback control law becomes \(u_t = -K_{LQp} \xi_t\), where the states related to the future values of the lateral coordinates of the road have an effect on the control action. The control gain matrix \(K_{LQp}\) can be calculated by using the well-known LQR discrete formulation and the respective Riccati equation.

It is interesting to observe that the path tracking control formulation deriving from Eq. (5.34), presented in [16], is very similar to the driver model with preview in [5] (originally not conceived for automated driving), based on the reference steering angle \(\delta = k_{\Delta \psi} \Delta \psi + k_{\Delta YCG} \Delta YCG + \sum_{k=1}^{n} \Delta y_k\), where the index \(k\) refers to points located in front of the vehicle. Actually, the formulation in Eq. (5.34) is identical to
the motorbike driver model in [7]. An extension of the linear quadratic formulation with preview could be based on the driver model in [8], which is an enhancement of the driver model in [5], with consideration of the future heading angle errors and their time derivatives, to give \( \delta = k_{\Delta \psi_w} \Delta \psi_w + k_{\Delta \dot{\psi}_w} \Delta \dot{\psi}_w + k_{\Delta y_w} \Delta y_w \). In this case, the errors \( \Delta \psi_w \), \( \Delta \dot{\psi}_w \), and \( \Delta y_w \) are scalar variables, calculated as the linear combination of the errors at different points \( k \).

5.3.2.4 Frequency-Shaped Linear Quadratic Control

A relevant contribution to the science of LQR path tracking control was provided by Peng and Tomizuka in the PATH framework in the 1990s [42–44]. Their frequency-shaped linear quadratic controller is based on the linear equations of the system, according to the formulation in Eqs. (5.21) and (5.22). The output is represented by the lateral deviation measured by a sensor located in the front part of the vehicle. The main advantages of the frequency-shaped LQR formulation with respect to conventional LQR control are: (i) the robustness on measurement noise at high frequencies and (ii) the possibility of including ride quality explicitly in the performance index (which is very important, as pointed out in some of the references including actual experimental tests and not only computer simulations).

The adopted performance indicator is:

\[
J = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ \Delta a_y^* (j\omega_L) \frac{q_{\Delta a_y}^2}{1 + \lambda_{\Delta a_y}^2 \omega_L^2} \Delta a_y (j\omega_L) + \Delta y_{CG}^* (j\omega_L) \frac{q_{\Delta \psi}^2}{1 + \lambda_{\Delta \psi}^2 \omega_L^2} \Delta \psi (j\omega_L) \\
\times \Delta y_{id}^* (j\omega_L) \frac{q_{\Delta y_{id}}^2}{(j\omega_L)^2} \Delta y_{id} (j\omega_L) + \delta^* (j\omega_L) R \delta (j\omega_L) \right] d\omega_L
\]

(5.35)

where \( \Delta a_y \) is the difference between the reference and the actual lateral acceleration. The coefficients \( q_{\Delta a_y} \) and \( \lambda_{\Delta a_y} \) are chosen to provide the expected ride quality, while the coefficients of the other three terms are selected to provide responsiveness to the road curvature and robustness with respect to the measurement noise. The disturbance term in the vehicle model, \( B_2 \dot{\psi}_{\text{path}} \) (i.e., the effect of the curvature, see Eq. (5.27)), is previewed with a preview time \( t_{ia} \). The problem is solved as a conventional linear quadratic controller after augmenting the system state variables with the states \( z_1, \ldots, z_4 \) corresponding to the four filters in Eq. (5.35), so that the augmented state vector is \( \xi_{\text{augm.}} = [\Delta y_{CG} \Delta \dot{y}_{CG} \Delta \psi \Delta \dot{\psi} z_1 z_2 z_3 z_4] \). The minimization of \( J \) requires the knowledge of the disturbance, i.e., \( \dot{\psi}_{\text{path}} \), from the current time to infinity. As this is not practically possible, an exponential decay of the curvature-related disturbance \( w(t) = 1/k(t) \) beyond the preview region is assumed. The resulting optimal preview steering
control law is expressed by:

$$
\delta = -k_\xi \xi_{augm.} + \int_{0}^{t_{d}} k_{w1} (l_{a}) w (t + \tau) \, d\tau + k_{w2} w (t + t_{d})
$$

(5.36)

where the first term is the state feedback controller, and the second and third terms are the preview control terms.

5.4 Other Control Structures for Path Tracking and Remarks

The controllers discussed in Sects. 5.2 and 5.3 are only an arbitrary selection of the very wide literature on the subject. This section presents an overview of the variety of less conventional control structures used for path tracking control.

5.4.1 Sliding Mode Controllers

Many papers (e.g., [35] and [45, 46]) present sliding mode controllers for path tracking. Utkin in [35] proposes a relatively simple first order sliding mode control structure (Fig. 5.15), resulting in a yaw control law of the form:

$$
\dot{\delta} = -M_\delta \text{sgn} (\sigma)
$$

(5.37)

where the sliding variable, $\sigma$, is given by $\sigma = c \Delta \psi + \Delta \dot{\psi}$, with $\Delta \dot{\psi} = \dot{\psi} - \dot{\psi}_{\text{ref}}$. As a consequence, the sliding variable is a function of $\psi$, $\dot{\psi}$, $\dot{\psi}_{\text{ref}}$ and $\dot{\psi}_{\text{ref}}$. The reference yaw rate is based on the lateral position error, having a time derivative $\Delta \dot{y}_{l_{a}} = v (\beta + \Delta \psi) + l_{a} \dot{\psi}$. According to traditional feedback linearization, $\dot{\psi}_{\text{ref}}$

Fig. 5.15 Nonlinear controller structure with observer (adapted from [35])
can be expressed as:

\[ \dot{\psi}_{\text{ref}} = -\frac{1}{l_d} \left[ v (\beta + \Delta \psi) + K \Delta y_{la} \right] \]  

\( (5.38) \)

with \( K \) determining the rate of decay of \( \Delta y_{la} \). In order to estimate the term \( \hat{q} = v (\beta + \Delta \psi) \), a dynamic observer (Observer 1 in Fig. 5.15) is implemented. As \( \Delta \dot{\psi} \) is not directly measured but is necessary for the computation of the sliding variable, the time derivative of \( \Delta \dot{\psi} \) is computed through a robust observer (Observer 2 in Fig. 5.15), as a simple differentiator would imply a significant risk of chattering. The comparison of the performance of this sliding mode formulation with the continuous controller discussed in Eqs. (5.13)–(5.16), carried out in [35], does not allow clear conclusions.

Particularly relevant, also considering the recent experimental developments at Stanford University to be discussed in Sect. 5.5, is the sliding mode path tracking controller developed in [46], for a four-wheel-steering vehicle, according to the strong tradition of Japanese vehicle engineering in four-wheel-steering systems. The controller is based on the important concept of centers of percussion of the front and rear axles. The centers of percussion (COPs) are located on the symmetry plane of the vehicle. Their longitudinal position with respect to the center of gravity is defined by (see also Fig. 5.16):

\[ |x_{\text{COP},f}| = \frac{I_z}{m b} \quad |x_{\text{COP},r}| = \frac{I_z}{m a} \]  

\( (5.39) \)

The four-wheel-steering vehicle layout of [46] allows the independent control of two variables, i.e., in the specific case the lateral position errors at the front and rear centers of percussion, respectively, \( \Delta y_{\text{COP},f} \) and \( \Delta y_{\text{COP},r} \).

The relationship between the vehicle state variables and the time derivatives of the lateral position errors at the centers of percussion is:

\[
\begin{align*}
\Delta \dot{y}_{\text{COP},f} & = - \left[ v (\beta + \Delta \psi_{\text{COP},f}) + |x_{\text{COP},f}| \dot{\psi} \right] \\
\Delta \dot{y}_{\text{COP},r} & = - \left[ v (\beta + \Delta \psi_{\text{COP},r}) - |x_{\text{COP},r}| \dot{\psi} \right]
\end{align*}
\]  

\( (5.40) \)

This means that the state vector \( \xi = \begin{bmatrix} \beta & \dot{\psi} \end{bmatrix}^T \) of a linear single-track vehicle model similar to that reported in Eq. (5.21) can be expressed as a function of \( \Delta \dot{y}_{\text{COP}} = \begin{bmatrix} \Delta \dot{y}_{\text{COP},f} & \Delta \dot{y}_{\text{COP},r} \end{bmatrix}^T \) and \( \Delta \psi_{\text{COP}} = \begin{bmatrix} \Delta \psi_{\text{COP},f} & \Delta \psi_{\text{COP},r} \end{bmatrix}^T \):

\[ \xi = T \Delta \dot{y}_{\text{COP}} + v T \Delta \psi_{\text{COP}} \]  

\( (5.41) \)

The substitution of Eq. (5.41) into the single-track vehicle model equations leads to the following tracking position dynamics at the centers of percussion:

\[ \Delta \ddot{y}_{\text{COP}} = A' \Delta \dot{y}_{\text{COP}} + B' u + v A' \Delta \psi_{\text{COP}} + v \dot{\psi}_{\text{path,COP}} + h' w \]  

\( (5.42) \)
where \( \dot{\psi}_{\text{path,COP}} = \begin{bmatrix} \dot{\psi}_{\text{path,COP,f}} \\ \dot{\psi}_{\text{path,COP,r}} \end{bmatrix} \) is the vector formulation of the time derivatives of the target path heading angles at the front and rear centers of percussion.

With a proper selection of the feedback controller configuration and by imposing that the two control points are the centers of percussion of the vehicle, Eq. (5.42) simplifies into the form:

\[
\Delta \gamma_{\text{COP}} = B' \ddot{u} + v \dot{\psi}_{\text{path,COP}} + \dot{h}'w \tag{5.43}
\]

where:

\[
B' = \begin{bmatrix} -\frac{2Gl}{mb} & 0 \\ 0 & -\frac{2Gl}{ma} \end{bmatrix} \tag{5.44}
\]

The diagonal matrix \( B' \) very importantly indicates that the position tracking problems at the front and rear centers of percussion are decoupled. Hence, each center of percussion path deviation can be independently controlled by the front and rear steering angles. As a consequence, the control laws for the front and rear axles can be separately designed, as single-input single-output controllers. This means that the lateral displacement dynamics of the front center of percussion are independent from the lateral force of the rear tires, while the lateral displacement
dynamics of the rear center of percussion are independent from the lateral force of the front tires. A different selection of the longitudinal position of the control points would imply the design of a multivariable controller.

In the specific case of [46], the resulting sliding mode control laws have the following shape:

$$\begin{align*}
\delta_t &= \beta + \frac{a}{v} \dot{\psi} + \frac{c_{\text{COP},t}}{k_{\text{COP},t}} \Delta \dot{y}_{\text{COP},t} + M_{\text{COP},t} \text{sgn} (\sigma_t) \\
\delta_r &= \beta - \frac{b}{v} \dot{\psi} + \frac{c_{\text{COP},r}}{k_{\text{COP},r}} \Delta \dot{y}_{\text{COP},r} + M_{\text{COP},r} \text{sgn} (\sigma_r)
\end{align*}$$

(5.45)

with the sliding variables \(\sigma_f\) and \(\sigma_r\) defined as \(\sigma_i = c_{\text{COP},i} \Delta y_{\text{COP},i} + \Delta \dot{y}_{\text{COP},i}\), with \(i = f, r\). The formulation in Eq. (5.45) requires an estimation of sideslip angle, while \(\dot{\psi}\) and \(\Delta \dot{y}_{\text{COP},i}\) can be easily measured or estimated. The sliding mode formulation of the specific paper is designed, through the appropriate definition of the gains, \(M_{\text{COP},f}\) and \(M_{\text{COP},r}\), to provide robustness against the variations of cornering stiffness and target path radius, and cross-wind disturbances. In order to prevent chattering, the following approximation of the discontinuous part of the control law is used:

$$\text{sgn} (\sigma_i) \approx \sin \left( \tan^{-1} \left( \frac{\sigma_i}{\mu_i} \right) \right), \mu_i > 0 \ (i = f, r)$$

(5.46)

The resulting steady-state values of the front and rear steering angles, \(\delta_{f,\text{SS}}\) and \(\delta_{r,\text{SS}}\), are given by:

$$\begin{align*}
\delta_{f,\text{SS}} &= \pm \frac{1}{R_t} \left( a - \frac{a-b}{2mab} I_z + \frac{mbv^2}{2C_f L} \right) \\
\delta_{r,\text{SS}} &= \pm \frac{1}{R_t} \left( -b - \frac{a-b}{2mab} I_z + \frac{mav^2}{2C_r L} \right)
\end{align*}$$

(5.47)

In practice, the sign of \(\delta_{r,\text{SS}}\) changes at the vehicle speed \(v_0\) (about 55 km/h for the case study vehicle of [46]), i.e., at low speeds the rear wheels steer in counterphase with respect to the front wheels, while the opposite happens for \(v > v_0\).

$$v_0 = \sqrt{\frac{2C_f I_z}{ma} \left( b + \frac{a-b}{2mab} I_z \right)}$$

(5.48)

The four-wheel-steering controller based on the COP concept was validated through CarSim simulations of a cross-wind disturbance situation. Figure 5.17 reports a sample of the simulation results, showing that the maximum deviations at the front and rear control points of the four-wheel-steering vehicle are not influenced by vehicle velocity, while the rear path deviation of the two-wheel-steering vehicle used as a term of comparison increases at exponential rate.

Overall, the main benefit of sliding mode control is the low-complexity of the resulting control law (see Eqs. (5.37) and (5.45)), however: (1) “it needs knowledge about the bounds of the disturbances and uncertainties in advance” [47]; (2) “it is not robust outside the sliding surface” [47]; and (3) it can present chattering.
In order to alleviate (3), Tagne et al. [48] adopted and experimentally validated a super-twisting algorithm of the form $\delta = \delta_{ST} + \delta_{eq}$, where $\delta_{ST}$ is the super-twisting contribution:

$$
\delta_{ST} = \delta_{ST,1} + \delta_{ST,2}, \quad \begin{cases} 
\delta_{ST,1} = -\alpha_{ST} |\sigma|^{0.5} \text{sign}(\sigma) \\
\delta_{ST,2} = -\alpha_{ST} \text{sign}(\sigma)
\end{cases} \tag{5.49}
$$

with $\sigma = \Delta \dot{y}_{CG} + \lambda \Delta y_{CG}$. The equivalent term corresponding to $\dot{\sigma} = 0$ is calculated starting from the equations of the single-track model of the system and is given by:

$$
\delta_{eq} = \frac{C_f + C_r}{C_f} \beta + \frac{a C_f - b C_r}{C_f v_x} \dot{\psi} + \frac{m v_x^2}{C_f} \kappa - \frac{m \lambda}{C_f} \Delta \dot{y}_{CG} \tag{5.50}
$$

[49, 50] are other useful references in the area of sliding mode control applied to automated driving. The recent paper [47] presents and experimentally validates, including comparison with the super-twisting algorithm of Eqs. (5.49) and (5.50), a path tracking controller based on the theory of immersion and invariance, where the target dynamics for the system are selected during the control design phase, similarly to sliding mode control. The main benefits with respect to sliding mode control are that: (i) “the manifold does not necessarily have to be reached”; (ii) it allows more flexibility in the selection of the target dynamics; and (iii) “it avoids the use of a discontinuous term in the control law.”

In the area of control structures with discontinuous control action, [51] experimentally demonstrates a nonlinear controller (in practice a sliding mode
controller including saturation functions, even if it is not explicitly called in this way in the original paper), based on the lateral offset at a look-ahead point (estimated by a Kalman filter, together with its time derivative) and the yaw angle error. [52] presents a discontinuous control law of the form

\[ \delta = - (|k_d \xi| + |r|) \text{sign} (b^T P_L (\xi - \xi_{\text{ref}})) \]

generated through model-reference control and a Lyapunov approach, with simulation results showing significant chattering, unacceptable for a real vehicle implementation.

### 5.4.2 Other Control Structures

Many other path tracking controllers were implemented in the literature, covering most of the control structure options.

For example, O’Brien et al. [53] discuss an \( H_\infty \) controller based on the theory of loop-shaping [54, 55], with the purpose of providing robustness with respect to the variation of the plant parameters. Given a plant with a co-prime factorization \( G_H = M_H^{-1} N_H \) [54, 55], the perturbed plant is expressed as \( G_{H\Delta} = (\tilde{M}_H + \Delta M_H)^{-1} (\tilde{N}_H + \Delta N_H) \), where \( \| \Delta M_H \Delta N_H \|_\infty < \epsilon \) formulates the fact that the actual plant can be different from the nominal plant. The purpose of the \( H_\infty \) optimization is to find a controller stabilizing the system and maximizing the value of \( \epsilon \). The value of \( \epsilon_{\text{max}} \) represents a measure of the stability margin (i.e., the so-called robust stability margin) for the nominal system to perturbations in the co-prime factorization of the plant. The paper assesses the performance of the \( H_\infty \) controller through simulations with a basic 3-degree-of-freedom vehicle model, including robustness analysis with respect to varying speeds, icy roads, and wind gusts. As the controller is based on two inputs (i.e., the lateral displacement error and yaw angle error) and produces one output, a singular value decomposition analysis [55] would allow discussing its functional controllability. A more recent second example of \( H_\infty \) controller implementation and experimental demonstration on a prototype vehicle is included in [56].

Shin and Joo [57] present a backstepping controller design (see [58] for the theory of backstepping), based on a linear single-track vehicle model, in this case with the following states: (i) sideslip angle; (ii) yaw rate; (iii) the difference between the actual and reference yaw angles, \( \Delta \psi = \psi - \psi_{\text{ref}} \); and (iv) the vehicle offset from the center of the lane, \( \Delta y_{\text{ld}} \). In particular, in the model for control system design, the equation of \( \Delta \dot{y}_{\text{ld}} \) is \( \Delta \dot{y}_{\text{ld}} = v (\beta + \Delta \psi) + l_d \dot{\psi} \). Hence, the reference yaw rate is defined as

\[ \dot{\psi}_{\text{ref}} = - \frac{v (\beta + \Delta \psi) + K_d \Delta y_{\text{ld}}}{l_d} \]

(5.51)

By defining \( \Delta \dot{\psi} = \dot{\psi} - \dot{\psi}_{\text{ref}} \) and substituting into the equation of \( \Delta \dot{y}_{\text{ld}} \), the lateral displacement error dynamics are described by \( \Delta \dot{y}_{\text{ld}} = -K_d \Delta y_{\text{ld}} + l_d \Delta \dot{\psi} \). After differentiation of \( \Delta \dot{\psi} \) with respect to time and re-substitution into the single-track
model equations, the yaw rate error dynamics are given by Eq. (5.52), based on the expressions of the coefficients $C_\beta$, $C_\dot{\psi}$, $C_{\Delta \psi}$, $C_\kappa$, and $b_\delta$.

$$\Delta \ddot{\psi} = C_\beta \dot{\psi} + C_\ddot{\psi} + C_{\Delta \psi} \Delta \psi - C_\kappa \kappa + b_\delta \delta$$

$$C_\beta = \frac{-2C_\beta + 2C_\kappa}{m} \frac{I_d}{l_d} + \frac{C_{\Delta \psi} - 2C \alpha}{r \alpha}$$

$$C_\ddot{\psi} = \frac{v(-1 + \frac{2C_\beta - 2C_\kappa}{m \omega}) + v + Kd - \frac{2C \beta + 2C \alpha}{r \alpha}}{C_{\Delta \psi}}$$

$$C_{\Delta \psi} = \frac{k_d \psi}{l_d} \quad C_\kappa = \frac{v^2}{l_d} \quad b_\delta = \frac{2C_\alpha}{l_d} + \frac{2C_\kappa}{l_d \alpha}$$

(5.52)

The steering control law is then chosen as:

$$\delta = \frac{1}{b_\delta} \left[ - \left( C_\beta \dot{\psi} + C_\ddot{\psi} + C_{\Delta \psi} \Delta \psi - C_\kappa \kappa \right) - \frac{W_d l_d}{W_\psi} \Delta y_{la} - k_\psi \dot{\psi} \right]$$

(5.53)

which brings the following yaw rate error dynamics: $\Delta \ddot{\psi} = -k_\psi \dot{\psi} - \frac{W_d l_d}{W_\psi} \Delta y_{la}$.

The term $k_\psi \dot{\psi}$ in Eq. (5.53) is used to decouple the lateral displacement and yaw rate error dynamics. The stability of the controller can be demonstrated through the Lyapunov function $L_{Lyapunov} = \frac{1}{2} \left( W_d \Delta y_{la}^2 + W_\psi \Delta \dot{\psi}^2 \right)$.

The controller was assessed in relatively low lateral acceleration conditions, through experimental tests with a vehicle prototype (a Hyundai sedan) on a proving ground consisting of a straight section with a length of about 1.2 km and a curved section with a radius of 260 m. Interestingly, the experimental analysis included the comparison of the lateral offset of the vehicle trajectory with respect to the reference one for the cases of human driving and autonomous driving. In general, the proposed controller “has the properties of deviating outwards in the lane during curve entry and inwards during curve exit, similar to the human driver. This can be reduced by adjusting the look-ahead distance to vary proportionally to the vehicle speed and by tuning the feedforward gain related to the curvature” [57].

Many papers (e.g., [59–62]) present fuzzy logic-based controllers for vehicle path tracking control. Given the general caution with respect to fuzzy control still present in industry (e.g., because of the lack of formal proof of stability), this chapter will not report the detailed descriptions of the available fuzzy implementations. The only note is that Naranjo et al. [62] include experimental results (on a Citroen Berlingo) with a fuzzy controller based on a real-time kinematic differential global positioning system, used as the main sensor for vehicle positioning.

5.4.3 Remarks

Unluckily, apart from [16], assessing geometric controllers and different LQR formulations on a CarSim vehicle model (see Table 5.1 summarizing the main
Table 5.1  Comparison of a selection of the path tracking controller formulations discussed so far (adapted from [16])

| Tracking method         | Robustness to disturbances | Path requirements                      | Cutting Corners                        | Overshooting                      | Steady state error             | Suitable applications                        |
|-------------------------|----------------------------|----------------------------------------|----------------------------------------|-----------------------------------|--------------------------------|----------------------------------------------|
| Pure pursuit            | Good                       | None within reason                     | Significant as speed increases         | Moderate as speed increases       | Significant as speed increases | Slow driving and/or on discontinuous paths  |
| Stanley                 | Fair                       | Continuous curvature                   | No                                     | Moderate as speed increases       | Significant as speed increases | Smooth highway driving and/or parking maneuvers |
| Kinematic (chained controller) | Poor                  | Continuous through second derivative of curvature | No                                     | Moderate as speed increases, significant during rapidly changing curvature | Significant as speed increases | Smooth parking maneuvers                       |
| LQR with feedforward    | Poor                       | Continuous curvature                   | No                                     | Significant during rapidly/changing curvature | Minimal until much higher speeds | Smooth high speed urban driving at speed       |
| LQR with preview        | Fair                       | None within reason                     | Moderate in rapidly changing curvature and/or speed | Moderate in rapidly changing curvature and/or speed | Minimal until much higher speeds | Highway driving at relatively constant speed |
conclusions of that study), there is limited available literature objectively comparing the performance of the different path tracking controllers discussed so far in this chapter.

[63] is an exception, presenting a simulation-based comparison of four controllers, i.e., a self-tuning regulator (for the details, see [64]), an $H_\infty$ controller, a fuzzy logic controller, and a P controller (of the form $\delta = k_{\Delta \psi} \Delta \psi + k_{\Delta v_d} \Delta y_d$). The assessment includes consideration of the effects of curvature, wind, and variations of vehicle speed and tire-road friction coefficients, along the simulated test track circuit at Satory – Versailles, France. The model used for the comparison is a simple linear single-track vehicle model, with a limited level of realism. The comparison shows that the self-tuning controller provides the best performance, followed by the $H_\infty$ controller and the fuzzy logic controller, which are approximately at the same level (even if the authors of [63] mention that fuzzy control is generally less reliable than a conventional controller), and finally by the proportional controller. However, this important analysis would require further development and level of detail.

5.5 Recent Advances in Path Tracking Control

The conclusion of the comparative study of different path tracking controllers in [16] (see Table 5.1), dating back to 2009, was that the expected evolution of the science of path tracking control would have been in the directions of (a) controllers combining different structures and formulations depending on the operating condition of the vehicle, in order to provide consistently reliable automated driving and (b) model predictive controllers, for autonomous driving even in extreme conditions, for example, at high lateral accelerations.

Based on the literature discussed so far, it is evident that there are already extensive experimental demonstrations of gain scheduled controllers capable of simultaneously providing the required vehicle tracking response for a wide range of speeds and lateral accelerations, and very precise maneuverability in docking conditions. In particular, [40] explicitly mentions that with the specific experimentally validated path tracking controller, based on linear control theory and implemented with realistic vehicle actuators, there is no need for multiple control structures, in order to achieve consistently reliable and comfortable path tracking behavior. On the other hand, the very recent paper [65] still includes kinematic model-based controllers in the analysis and considers them useful in low-speed conditions. Nevertheless, given results such as those in [40], the authors of this chapter do not consider the development of heterogeneous control structures to be a priority or major obstacle for the development of the automated driving agenda.
Two main trends can be observed in the recent research in the subject area of path tracking control:

(i) Development of path tracking controllers characterized by the capability of controlling the vehicle at its cornering limit, for example, even at lateral accelerations of 9.5 m/s² (e.g., for automated car racing).

(ii) Progressive increase of the level of sophistication of the implemented control structures, with particular focus on model predictive control, now extensively implemented in simulation and preliminarily demonstrated at the experimental level (which confirms the conclusion of [16]).

The following subsections describe examples of recently published path tracking controllers, with concise critical analyses and discussions.

### 5.5.1 Advanced Feedforward and Feedback Controllers for Limit Cornering

Important recent contributions in the area of path tracking control, mainly developed at Stanford University, are aimed at achieving high path tracking performance at the cornering limit of the vehicle, e.g., at lateral acceleration levels up to 9.5 m/s² in high friction conditions (which represents the cornering limit for a typical passenger car), or for extreme combined cornering and braking/traction [66–70]. Particular focus is on the development of feedforward steering formulations, allowing a relaxation of the specifications of the feedback part of the controller and a reduction of the issues related to the effect of measurement disturbances.

For example, Kritayakirana and Gerdes [67] present and experimentally validate a feedforward/feedback steering controller for a two-wheel-steering vehicle, based on the path tracking control of the front center of percussion (COP), already defined in Eq. (5.39). By considering $\Delta \hat{\psi}_{CG} = \dot{\psi} - \hat{\psi}_{path,CG} = \dot{\psi} - \kappa \dot{s}$ and the yaw moment balance equation of a single-track vehicle model, the time derivative of $\Delta \hat{\psi}_{CG}$ (i.e., the yaw acceleration error) is:

$$\Delta \ddot{\psi}_{CG} = \ddot{\psi} - \kappa \ddot{s} - \kappa \dot{s} = \frac{aF_{y,f} - bF_{y,r}}{I_x} - \kappa \ddot{s} - \kappa \dot{s}$$

(5.54)

The lateral position error at a generic point $P$ along the $x$-axis of the vehicle reference system is given by: $\Delta y_P = \Delta y_{CG} + x_P \sin \Delta \psi_{CG}$ and its acceleration $\Delta \ddot{y}_P$ is approximated with:

$$\Delta \ddot{y}_P = \frac{F_{y,f} + F_{y,r}}{m} - v_x \kappa \dot{s} + x_P aF_{y,f} - bF_{y,r}}{I_x} - x_P (\kappa \ddot{s} + \kappa \dot{s})$$

(5.55)

The issue is that the steering actuator can command the front tire force, but it does not have any direct control over the rear tire force, which, thus, represents a disturbance in Eq. (5.55). However, by imposing $x_P = x_{COP,f}$ and hence that $\frac{F_{y,f}}{m} +$
\[ x_{COP,f} \frac{-bF_{y,f}}{I_z} = 0 \]

Eq. (5.55) can be simplified into:

\[
\Delta \ddot{y}_{COP,f} = \frac{F_{y,f} + F_{y,r}}{m} - v_x \kappa \dot{s} + x_{COP,f} \frac{aF_{y,f} - bF_{y,r}}{I_z} - x_{COP,f} \left( \kappa \ddot{s} + \dot{k} \dot{s} \right)
\]

\[
= \frac{LF_{y,f}}{b} - v_x \kappa \dot{s} - x_{COP,f} \left( \kappa \ddot{s} + \dot{k} \dot{s} \right)
\]

(5.56)

The rear tire force is thus eliminated from the equation of the lateral acceleration error at the control point, which now depends only on \( F_{y,f} \), directly controllable through the steering input. The conclusion, similar to the outcome of the analysis in [46] referred to a four-wheel-steering vehicle, is that at the front COP the effect of the rear tire forces on the lateral position error dynamics can be neglected.

The feedforward contribution of the steering controller in [67] has the purpose of eliminating the dynamics of the lateral acceleration error, i.e., it can be obtained by imposing \( \Delta \ddot{y}_{COP,f} = 0 \) in Eq. (5.56). Hence, the feedforward lateral force on the front axle, \( F_{FFW,y,f} \), is given by:

\[
F_{FFW,y,f} = \frac{mb}{L} \left( v_x \kappa \dot{s} + x_{COP,f} \left( \kappa \ddot{s} + \dot{k} \dot{s} \right) \right)
\]

(5.57)

Eq. (5.57) allows ideal tracking performance, independently from the rear lateral tire force contribution, provided that the terms related to the reference path can be accurately estimated. By substituting Eq. (5.57) into the yaw moment error equation and introducing the feedback part of the control force on the front axle, \( F_{FB,y,f} \), such that the total control force is \( F_{TOT,y,f} = F_{FFW,y,f} + F_{FB,y,f} \), the system dynamics become:

\[
\left\{ \begin{array}{c}
\Delta \ddot{\psi}_{CG} = \frac{a}{L} F_{FB,y,f} + \frac{v_x}{L} \kappa \dot{s} - \frac{b}{L} F_{y,r} - \frac{b}{L} \left( \kappa \ddot{s} + \dot{k} \dot{s} \right) \\
\Delta \ddot{y}_{COP,f} = \frac{L}{b} \frac{F_{FB,y,f}}{m}
\end{array} \right.
\]

(5.58)

Through the terms \( \frac{v_x}{L} \kappa \dot{s} - \frac{b}{L} \left( \kappa \ddot{s} + \dot{k} \dot{s} \right) \), Eq. (5.58) shows that the disturbance caused by the curvature cannot be eliminated from the yaw tracking equation, unless an independent actuator controlling the rear tire force is adopted (as already proposed in [46]).

The objective of the feedback part of the controller in [67, 68] is to provide path tracking and yaw stability even when the rear tires are saturated, while the scenario in which the front tires are saturated is not considered. Note that passenger cars are usually characterized by an understeering behavior for vehicle safety, i.e., the absolute value of the front slip angles is normally larger than the absolute value of the rear slip angles, and the front lateral forces saturate first.

During the design of the feedback controller, the nonlinear behavior of the rear tires is considered through the model proposed in [69], according to which \( F_{y,r} = -2 \eta_r C_r \alpha_r \), with \( \eta_r \) (0 \( \leq \eta_r \leq 1 \)) being a monotonically decreasing function of the absolute value of the rear slip angle. By substituting the control variables...
into the expression of $\alpha_r \approx \frac{v_2 - \dot{\psi}}{v}$, it is $\alpha_r \approx \frac{\Delta \gamma_{COP,f} - (b + \psi_{COP,f}) \Delta \psi_{CG} - v_1 \Delta \psi_{CG} - bx_2}{v_1}$. By including this formulation into the expression for $F_{y,f}$ and then into the equations of the single-track vehicle model in path coordinates (see also Eq. (5.22)), the state-space formulation of the system is obtained, including the effect of the feedforward controller.

The feedback contribution of the front lateral force is then expressed as a full-state feedback controller:

$$F_{y,f}^{FB} = -k_{1LC} \dot{x} = -k_{1LC} \Delta \gamma_{COP,f} - k_{2LC} \Delta \dot{\gamma}_{COP,f} - k_{3LC} \Delta \psi_{CG} - k_{4LC} \Delta \dot{\psi}_{CG} \quad (5.59)$$

By substituting Eq. (5.59) into the state-space formulation, the closed-loop system equations can be used for control system design:

$$\begin{bmatrix} \Delta \dot{\gamma}_{CG} \\ \Delta \ddot{\gamma}_{CG} \\ \Delta \dot{\psi}_{CG} \\ \Delta \ddot{\psi}_{CG} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{L} & 0 & 0 \\ -\frac{k_{1LC}}{L} \frac{L}{b m} & -\frac{k_{2LC}}{L} \frac{L}{b m} & -\frac{k_{3LC}}{L} \frac{L}{b m} & -\frac{k_{4LC}}{L} \frac{L}{b m} \\ -\frac{k_{1LC}}{I_z} & \frac{k_{2LC}}{I_z} & 0 & 0 \\ \frac{2b_n C_r}{I_z v_x} & -\frac{k_{3LC}}{I_z} & 2b_n C_r & \frac{2(b + \psi_{COP}) b_n C_r}{I_z v_x} \end{bmatrix} \begin{bmatrix} \Delta \gamma_{CG} \\ \Delta \dot{\gamma}_{CG} \\ \Delta \psi_{CG} \\ \Delta \dot{\psi}_{CG} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{v_2 k_3 \dot{x}}{L} - \frac{b}{L} (k_3 \dot{x} + \hat{k} \dot{x}) - \frac{2b^3 b_n C_r \dot{x}}{I_z v_x} \end{bmatrix} \quad (5.60)$$

In the specific controller of [67], it is $k_{2LC} = 0$. In particular, $F_{y,f}^{FB}$ of Eq. (5.59) is manipulated to become $F_{y,f}^{FB} = -k_{LK} (\Delta \gamma_{COP,f} + (l_d - a) \Delta \psi_{CG}) - k_{\hat{\psi}} \Delta \dot{\psi}_{CG}$, where clearly the control point of the feedback part of the controller is not located at the front center of percussion any more. During the control system implementation, the following values are adopted: $k_{LK} = 4000$ N/m, $l_d = 20$ m and $k_{\Delta \dot{\psi}} = 9500$ Ns/.rad. The stability of the control system at the cornering limit is demonstrated through Lyapunov method applied to Eq. (5.60), without considering the disturbance from the curvature (which does not affect stability). Detailed tuning criteria of the feedback control gains are reported in [71].

Starting from the previous formulations of the reference front lateral force, $F_{y,f}^{TOT}$, the Fiala tire model for pure cornering conditions is used to obtain the reference slip angle on the front axle, $\alpha_{ref,f}$. Then, based on the measured vehicle yaw rate and estimated sideslip angle, the reference steering angle $\delta$ for the front axle is calculated, based on the kinematic relationship $\delta \approx \beta + \frac{aw}{v_2} - \alpha_{ref,f}$. In practice, these steps can introduce significant errors in the process, in the absence of very accurate state estimation.

A more recent paper of the same research group [70] presents a further development of the controller discussed so far, as the experimental tests show a suboptimal tracking performance of the controller based on Eqs. (5.57) and (5.59) above 7 m/s$^2$ of lateral acceleration. The authors suggest considering a simplification of
the feedforward contribution in Eq. (5.57), by imposing \( x_{COP_f} (\kappa\dot{s} + \ddot{k}) = 0 \) and \( \dot{s} = v_x \), in order to reduce vehicle response oscillations and increase damping. As a consequence, the feedforward values of the lateral force at the front and rear axles simply become:

\[
\begin{align*}
F_{y,f}^{FFW} &= \frac{mb}{L} v_x^2 k \\
F_{y,r}^{FFW} &= \frac{ma}{L} v_x^2 k
\end{align*}
\] (5.61)

which are the steady-state values of lateral force in cornering at the reference lateral acceleration \( v_x^2 k \). From \( F_{y,f}^{FFW} \) and \( F_{y,r}^{FFW} \), it is possible to calculate \( \alpha_f^{FFW} \) and \( \alpha_r^{FFW} \), which are the corresponding slip angles. The new expression of the feedforward steering angle is:

\[ \delta_{FFW,LC} = L\kappa - \alpha_f^{FFW} + \alpha_r^{FFW} \] (5.62)

which is the well-known equation of vehicle response in steady-state cornering conditions. The main challenge is to actually implement the controller and state estimators computing the correct values of \( \alpha_f^{FFW} \) and \( \alpha_r^{FFW} \).

In [70], the feedback control law of Eq. (5.59) is simplified into the proportional controller:

\[ \delta_{FB,LC,1} = -k_p (\Delta y_{CG} + l_d \Delta \psi_{CG}) \] (5.63)

The overall control law is, thus, given by \( \delta = \delta_{FFW,LC} + \delta_{FB,LC,1} \). The steady-state response of the system as a function of \( v \) is reported in Fig. 5.18, at lateral accelerations of 3 m/s² and 7 m/s², where for the first case the calculation is based on a linear vehicle model and for the latter it is based on a nonlinear model.

Since the controller is aimed at eliminating a weighted sum of \( \Delta y_{CG} \) and \( \Delta \psi_{CG} \), the feedback part of the controller actually tries to reduce \( \Delta y_{d,i} \), while the steady-state values of \( \Delta y_{CG} \) and \( \Delta \psi_{CG} \) are nonzero. From a physical viewpoint, this is the situation corresponding to Fig. 5.19a. An important observation from Fig. 5.18 is that the steady-state value of path deviation is close to zero for vehicle speeds of 17 m/s and 20 m/s, depending on the considered vehicle model. This means that at these speeds the velocity vector at the center of gravity is tangent to the path and \( \Delta \psi_{CG} \approx -\beta \) (see Fig. 5.19b).

A new form of look-ahead feedback control law is suggested, with the aim of constantly keeping the vehicle in the operating condition of Fig. 5.19b:

\[ \delta_{FB,LC,2} = -k_p (\Delta y_{CG} + l_d (\Delta \psi_{CG} + \beta)) \] (5.64)

With this control law, the steady-state error \( \Delta y_{CG} \) is significantly reduced (i.e., it is ideally zero when including the feedforward contribution according to Eq. (5.62)); however, the system is now characterized by reduced stability margins, with respect to the feedback control law of Eq. (5.63). This is evident through
Fig. 5.18  Steady-state path tracking error $\Delta y_{CG}$, sideslip angle $\beta$, and heading deviation $\Delta \psi_{CG}$ as a function of vehicle speed (from [70])

Fig. 5.19  (a) Steady-state cornering when the vehicle has lateral position error at the center of gravity but no look-ahead error; (b) Steady-state cornering with zero lateral deviation at the center of gravity, which requires the velocity vector to be tangent to the path, i.e., $\Delta \psi_{CG} = -\beta$ (from [70])
the root-locus analysis of Fig. 5.20, comparing the systems governed by the same feedforward control law (Eq. 5.62), and the feedback laws corresponding to Eqs. (5.63) and (5.64). For example, at 25 m/s the closed-loop steering response with Eq. (5.63) is well damped, with a damping ratio of 0.9, while in the same conditions the damping ratio is only 0.2 with the feedback control of Eq. (5.64).

In order to prevent the stability issues of path tracking enforced through feedback, Kapanja and Gerdes [70] finally propose to eliminate the sideslip-related feedback contribution from Eq. (5.64). Nevertheless, the zero displacement error condition (corresponding to $\Delta \psi_{CG} = -\beta$) is incorporated into the feedback contribution by using $\beta_{SS}$ (i.e., the expected steady-state value of sideslip angle) instead of the estimated $\beta$. Hence, Eq. (5.64) becomes:

$$
\delta_{\text{FB,LC,3}} = -k_P \left( \Delta y_{CG} + l_d \left( \Delta \psi_{CG} + \beta_{SS} \right) \right) = -k_P \left( \Delta y_{CG} + l_d \left( \Delta \psi_{CG} + \alpha_{FFW} + b \right) \right)
$$

(5.65)
Actually, the sideslip contribution of Eq. (5.65), i.e., $-k_p l_d (\alpha_{FFW} + bk)$, is now a feedforward term.

Vehicle experiments with an autonomous Audi TTS were executed at the Thunderhill Raceway Park, in order to compare the performance of (i) the controller corresponding to Eqs. (5.62) and (5.65) “With Sideslip FFW” in the legend; and (ii) the controller corresponding to Eqs. (5.62) and (5.63), “Original Controller” in the legend (from [70]).

The experimental results, reported in Figs. 5.21 and 5.22, show that controller (i) implies significantly improved path tracking. In any case, the whole approach needs further developments, as the feedforward contribution is sensitive to system uncertainty (e.g., tire-road friction conditions), which is much more important in the case of a vehicle operating on a real road rather than on a race track.
**5.5.2 Model Predictive Control**

With respect to the path tracking formulations discussed in the previous sections, model predictive control [72–74] brings the following benefits:

- Inclusion of constraints on inputs and states;
- Systematic approach to the control problem, with the possibility of considering multiple actuators and models at different levels of complexity within the same control design framework;
- Enhanced tracking performance at medium-high lateral accelerations and during emergency conditions, depending on the complexity level of the selected model for control system design.

Extensive literature provides simulation and experimental results of model predictive control applications for path tracking. For example, one of the first attempts is presented in [75], with a path tracking model predictive controller based on a single-track vehicle model. This adopts a nonlinear tire model with constant
cornering stiffness and lateral force saturation at the value corresponding to the estimated tire-road friction coefficient.

Falcone et al. [76] discuss and compare three model predictive control formulations. The first one (here called Controller A) is based on a nonlinear single-track vehicle model. This considers constant vertical load on the front and rear axles and uses Pacejka magic formula [77], under the hypothesis of zero longitudinal slip ratio (i.e., pure cornering conditions). The model is expressed in the form:

$$\begin{align*}
\dot{\xi}_{k+1} &= f_{\delta(k),\mu(k)}^d (\xi_k, \Delta u_k) \\
u_k &= u_{k-1} + \Delta u_k
\end{align*}$$

(5.66a)

(5.66b)

The system output is $z_{\text{MPC}_k} = [\psi_k \ y_k]^T$. The cost function to be minimized is:

$$J(\xi_t, \Delta U_t) = \sum_{n=1}^{H_p} \|z_{\text{MPC}_k+n} - z_{\text{MPC ref},k+n}\|_Q^2 + \sum_{n=0}^{H_e-1} \|\Delta u_{k+n}\|_R^2$$

(5.67)

The first contribution, $\sum_{n=1}^{H_p} \|z_{\text{MPC}_k+n} - z_{\text{MPC ref},k+n}\|_Q^2$, relates to the tracking performance of the system ($z_{\text{MPC ref}}$ is the vector of the reference signals), while the second contribution, $\sum_{n=0}^{H_e-1} \|\Delta u_{k+n}\|_R^2$, considers the control effort. Similarly to the case of the linear quadratic regulator, the parameters of $Q$ and $R$ can be tuned to define the performance of the model predictive controller, i.e., the variables that need to be tracked with higher precision, and the relative weight between tracking performance and control effort. At each time step, the following finite horizon optimal control problem is solved:

$$\begin{align*}
\arg \min_{\Delta U} & \quad J(\xi_t, \Delta U_t) \\
\text{s.t.} & \quad \dot{\xi}_{k+1,t} = f_{\delta(k,t),\mu(k,t)}^d (\xi_{k,t}, \Delta u_{k,t}) \\
& \quad z_{\text{MPC},k,t} = h(\xi_{k,t}) \\
& \quad \mu_{c,k,t} = \mu_{c,t}, \quad s_{k,t} = s_{t,t}, \quad k = t, \ldots, t + H_p \\
& \quad \delta_{\text{min}} \leq u_{t,t} \leq \delta_{\text{max}} \\
& \quad \Delta \delta_{\text{min}} \leq \Delta u_{k,t} \leq \Delta \delta_{\text{max}}
\end{align*}$$

(5.68a)

(5.68b)

(5.68c)

(5.68d)

(5.68e)

(5.68f)
The optimization vector at time $t$ is $\Delta U_t = [\Delta u_{k,t} \ldots \Delta u_{t+H_C-1,t}]^T$. $H_P$ and $H_C$ denote the prediction and control horizons, respectively. The solution of problem (5.68) implies a nonlinear optimization, with a very significant computational burden. The optimization of Controller A is solved through the commercial NPSOL software package [78].

Falcone et al. [76] present only experiments at low vehicle speed with the controller based on the nonlinear model in the form of Eq. (5.66a) and the optimization problem (5.68), i.e., Controller A. In fact, as speed increases, larger prediction and control horizons are required “in order to stabilize the vehicle along the path.” This implies more evaluations of the objective function and increased size of the optimization problem, which becomes unpractical. As a consequence, Falcone et al. [76] also discuss an alternative formulation, Controller B, based on the linearization of the system at each time step, around the current operating point. This procedure significantly decreases the computational complexity of the optimization problem, even if additional calculations are required for system linearization at each time step. In the case of Controller B, the model output vector is $z_{MPC_k} = \begin{bmatrix} \psi_k & \dot{\psi}_k & y_k \end{bmatrix}^T$. In Controller B, tire slip angle variation is an additional output that is constrained (through a soft constraint and a slack variable) but not tracked.

Controllers A and B were assessed in double lane change tests through simulations and experiments. Controller parameters have a significant effect on the control system performance; therefore, the main parameter values used in [76] are reported for completeness:

- Controller A: $T_s = 0.05$ s, $H_P = 7$, $H_C = 3$, $\delta_{\text{min}} = -10$ deg, $\delta_{\text{max}} = 10$ deg, $\Delta \delta_{\text{min}} = -1.5$ deg, $\Delta \delta_{\text{max}} = 1.5$ deg, $\mu_c = 0.3$, $Q = \begin{bmatrix} 500 & 0 \\ 0 & 75 \end{bmatrix}$, $R = 150$;
- Controller B: $T_s = 0.05$ s, $H_P = 25$, $H_C = 10$, $\delta_{\text{min}} = -10$ deg, $\delta_{\text{max}} = 10$ deg, $\Delta \delta_{\text{min}} = -0.85$ deg, $\Delta \delta_{\text{max}} = 0.85$ deg, $\mu_c = 0.3$, $Q = \begin{bmatrix} 200 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{bmatrix}$, $R = 5 \times 10^4$.

Finally, Falcone et al. [76] include a simplified version of Controller B, here called Controller C, with $H_C = 1$, allowing a further reduction of the computational load for implementation on actual automotive control hardware (in this case, the set of required calculations at each time step can be predicted a priori).
Table 5.2 Maximum computation times for Controller A and Controller B during lane change tests at different vehicle speeds (from [76])

| v [m/s] | Controller A [s] | Controller B [s] |
|---------|-----------------|-----------------|
| 10      | 0.15 (H_D = 7, H_C = 2) | 0.03 (H_D = 7, H_C = 3) |
| 15      | 0.35 (H_D = 10, H_C = 4) | 0.03 (H_D = 10, H_C = 4) |
| 17      | 1.3 (H_D = 10, H_C = 7)  | 0.03 (H_D = 10, H_C = 10) |

Fig. 5.23 Experimental results with Controller B at 21.5 m/s entry speed. From the top: lateral position, yaw angle, and yaw rate (from [76])

Table 5.2 compares the maximum computation times of Controller A and Controller B during a lane change maneuver in low friction conditions. Clearly, with the control hardware available in [76], it is not possible to run Controller A for vehicle speeds larger than 10 m/s. Figures 5.23 and 5.24 show vehicle performance with Controller B during the double lane change test executed at 21.5 m/s. The control action is characterized by significant chattering. The authors of [76] state that this aspect is not critical, as the vehicle cornering dynamics act as a filter, and, therefore, the vehicle passengers do not perceive the oscillations.

Tables 5.3 and 5.4 report the main tracking performance indicators during the tests, for different vehicle speeds, respectively, for Controller B and Controller C. According to [76], “Controller C performs slightly worse than Controller B;” nevertheless, “it is able to stabilize the vehicle at high speed.”

In [79], the same research group presents a model predictive controller (here called Controller D) based on a four-wheel vehicle model including wheel and tire dynamics. As a consequence, the state vector of the model for control system design
Table 5.3 Performance indicators (i.e., tracking errors) for Controller B for double lane change tests at different initial speeds (from [76])

| v [m/s] | μ_c | Δψ_{rms} [deg] | ΔY_{rms} [m] | Δψ_{max} [deg] | ΔY_{max} [m] |
|---------|-----|----------------|--------------|----------------|-------------|
| 10      | 0.3 | 0.53           | 1.28 \times 10^{-2} | 8.21 | 0.8 |
| 15      | 0.3 | 1.172          | 4.64 \times 10^{-2} | 14.71 | 2.51 |
| 19      | 0.3 | 1.23           | 7.51 \times 10^{-2} | 16.38 | 3.10 |
| 21.5    | 0.25| 1.81           | 1.11 \times 10^{-1} | 19.02 | 2.97 |

Table 5.4 Performance indicators (i.e., tracking errors for Controller C for double lane change tests at different initial speeds (from [76])

| v [m/s] | μ_c | Δψ_{rms} [deg] | ΔY_{rms} [m] | Δψ_{max} [deg] | ΔY_{max} [m] |
|---------|-----|----------------|--------------|----------------|-------------|
| 10      | 0.2 | 9.52 \times 10^{-1} | 5.77 \times 10^{-2} | 13.12 | 3.28 |
| 17      | 0.25| 8.28 \times 10^{-1} | 2.90 \times 10^{-2} | 12.26 | 1.81 |
| 21      | 0.2 | 1.037          | 7.66 \times 10^{-2} | 12.49 | 3.20 |

is \( \xi = \begin{bmatrix} \dot{y} & v_x & \dot{y} & Y & X & \omega_{lf} & \omega_{lf} & \omega_{rf} \end{bmatrix}^T \). The tires are modeled through the magic formula, this time including consideration of the interaction between longitudinal and lateral forces. However, the vehicle model considers a constant vertical load on each tire, which is a substantial limitation, as the load transfers induced by longitudinal and lateral accelerations are an important cause of nonlinearity and variation of the understeer characteristic.
The main benefit is that Controller D allows systematic and concurrent control of the steering angle and the individual friction brake torques (and potentially the drivetrain torque as well). This feature is an important point for actual vehicles including stability control systems based on independent caliper pressure control. Therefore, the control output vector for Controller D is \( u = [\delta T_{b,lf} \, T_{b,rf} \, T_{b,lr} \, T_{b,rr}]^T \). As the controller was tested for a lane change maneuver only, the traction torque is not considered in \( u \) within [79].

Controller D provides better performance in extreme conditions than a controller (here called Controller E) based on a nonlinear single-track vehicle model, in which the control output is represented by \( u = [\delta M_z]^T \) [79]. In Controller E, heuristics are adopted (within a low-level controller) to calculate the individual friction brake torques required to generate the reference yaw moment, \( M_z \), output by the model predictive controller. The simulation results show that lane change tests can be executed at a higher initial vehicle speed with Controller D than with Controller E. However, the authors admit that the duration of the simulation runs with Controller D was about 15 min, and, therefore, they did not have the time to fine tune the parameters of Controller D. This justifies the simulation results for Controller D (e.g., see Fig. 5.25), showing significant vibrations of the control action, which requires further investigations in the opinion of the authors of this chapter. Falcone et al. [79] also include experimental results with Controller F, which is based on the linearized model used for Controller D (i.e., the model with four vehicle corners), where the linearizations are carried out online, around the current operating point of the vehicle. The resulting controller can, thus, run online with a fixed step size of 50 ms on the control hardware available in [79].

Yin et al. [80] propose a model predictive controller, Controller G, for an autonomous electric vehicle with individually controlled drivetrains, with a formulation very similar to that of Controller F of [79]. The controller is based on a linearized vehicle model including the four vehicle corners, where the reference steering angle and the four slip ratios are the control outputs. This means that a low-level controller is used to calculate the individual wheel torques required to achieve the reference longitudinal slips. In practice, this is very difficult to implement.

Fig. 5.25  Performance of Controller D at an entry speed of 14 m/s (from [79])
because of the approximations in the slip ratio estimation during normal driving. Slip ratio estimation is much easier in extreme conditions, i.e., when the absolute values of the slip ratios are larger and the conventional traction control and antilock braking systems are usually activated. The benefit of Controller G is that it allows the control of the traction torque during autonomous driving, without the requirement of two separate controllers for steering and longitudinal tracking.

Similarly to [79], Attia et al. [81] present a nonlinear model predictive controller, Controller H, based on a model coupling the longitudinal and lateral vehicle dynamics. The model for control synthesis is a discretized single-track vehicle model, including the degrees of freedom corresponding to the longitudinal and lateral displacements of the center of gravity, the vehicle yaw motion, and the equivalent front and rear wheel dynamics. Therefore, the state vector is 
\[ \xi = [v_x \; v_y \; \psi \; \omega_r \; X \; Y]^T. \]

The main model nonlinearity is represented by the Burckhardt tire model. The model predictive controller is responsible for the steering angle demand only. The longitudinal vehicle dynamics are exclusively used for considering the interaction between longitudinal and lateral tire forces; in fact, the wheel torque demand is controlled by an independent controller based on Lyapunov approach. The paper does not report the details of the numerical aspects related to the algorithm implementation and online optimization, apart that the considered sample time is \( T_S = 10 \text{ ms}. \)

Controller H includes some consideration of the interactions between longitudinal and lateral control. In fact, an excessive level of vehicle speed reference can originate problems in terms of lateral dynamics, as “no active lateral stabilization is considered in the control design.” In this respect, the reference vehicle speed of the longitudinal controller can be saturated based on the expected road curvature and the estimated tire-road friction coefficient, according to the following formulations proposed in [82, 83]:

\[
v_{\text{max}} = \sqrt{\frac{g \mu_c}{\kappa}} \]

\[
v_{\text{max}} = \sqrt{\frac{g}{\kappa} \left( \frac{\phi_t + \mu_c}{1 - \phi_t \mu_c} \right)} \] (5.70)

Also, according to the US National Highway Traffic Safety Administration (NHTSA) [83], the longitudinal acceleration to bring vehicle speed to the maximum value specified in Eq. (5.70) should be limited to:

\[
a_{x,\text{max}} = \frac{v^2 - v_{\text{max}}^2}{2 (d - l_t v)} \] (5.71)

Criteria (5.69)–(5.71) are easily applicable for reference speed generation; however, safety-critical conditions could happen, for example, caused by an erroneous friction coefficient estimation, thus determining higher reference speed profile than
the one compatible with the actual friction limits. In these situations, the stability control system of the vehicle is expected to intervene and overrule the inputs of the autonomous driving controller, as it happens in normal humanly driven passenger cars. Nevertheless, the authors of [81] report a couple of sideslip-related stability criteria (from [84, 85]), mentioned as relevant to automated driving, without clearly specifying how to organically include them within their automated steering controller:

\[
\left| \frac{1}{24} \dot{\beta} + \frac{4}{24} \beta \right| \leq 1
\]

(5.72)

\[
\beta \leq 10 \text{ deg} - 7 \text{ deg} \frac{v^2}{(40 \text{ m/s})^2}
\]

(5.73)

The performance of Controller H was assessed through simulations with a nonlinear vehicle dynamics model developed by the same authors, along a highway exit scenario. The results (Fig. 5.26) show a lateral position error not exceeding 6 cm and a heading angle error not exceeding 4 deg.

Fig. 5.26 Combined longitudinal and lateral control test with Controller H: (a) real-world road; (b) reference and vehicle trajectories; (c) tracking errors; and (d) reference speed tracking (simulation results from [81])
The previous contributions discuss model predictive controllers without any specific feature aimed at providing system robustness. Nevertheless, the presented model predictive controllers guarantee enhanced tracking performance in nominal conditions, i.e., when the model used by the controller provides a good fit with the actual plant. To the purpose of conjugating excellent tracking performance in nominal conditions and robustness, recent contributions are focused on robust model predictive control [65]. For example, Gao et al. [86] discuss a robust tube-based model predictive controller (Controller I, for the theory refer to [87–89]), conceived with the specific purpose of relatively low computational load.

The controller is based on a system model formulation of the form:

$$\xi_{k+1} = A\xi_k + g(\xi_k) + Bu_k + w_k$$  \hspace{1cm} (5.74)

with $\xi_k \in \mathcal{Z}, u_k \in \mathcal{U}, w_k \in \mathcal{W}$. Equation (5.74) is derived starting from a single-track vehicle model, including the longitudinal force balance equation of the system. The path tracking controller is based on the lateral displacement error at the front center of percussion to eliminate the rear lateral tire forces from the lateral displacement error equation. The state vector is $\xi = [\dot{y}_{\text{COP}}, f_{\text{COP}}, \dot{\psi}, \Delta \psi, \Delta y_{\text{CG}}, s]^T$, and the input vector (i.e., the output of the controller) is $u = [\beta_{x,f}, \beta_{y,f}, \beta_{r}]^T$. $\beta_{x,f}$ and $\beta_{y,f}$ are the normalized longitudinal and lateral forces on the front wheels, which are controlled through the drivetrain/friction brakes and the steering system, respectively. $\beta_r$ is the normalized longitudinal force on the rear axle. A linear model is used for the lateral force of the rear axle. Essentially, the model in Eq. (5.74) includes a linear term, $A\xi_k + Bu_k$, a small (under the hypotheses of the discussion in [86]) nonlinear term, $g(\xi_k)$, and a disturbance term, $w_k$.

The control action consists of two contributions: (i) a nominal control input for the nominal system, i.e., the system in Eq. (5.74) under the hypothesis of zero disturbance and (ii) a state feedback controller acting on the error, $e_k = \xi_k - \xi_{\text{nom}}$, between the actual state of the system and the predicted state of the nominal system. The nominal system is defined as the system with the nominal control input and zero disturbance sequence. As a consequence, the control law has the following shape:

$$u_k = \bar{u}_k + \hat{u}(e_k)$$  \hspace{1cm} (5.75)

where $\bar{u}_k$ is the nominal controller and $\hat{u}(e_k)$ is the state feedback control action. In the specific case, the stabilizing feedback contribution is based on a linear quadratic regulator: $\hat{u}(e_k) = K_{LQ}(\xi_k - \bar{\xi}_k)$. In practice, in [86], the linear part of the dynamics is separated into two contributions, the first one including the longitudinal dynamics and the second one including the lateral dynamics.

In general, the error dynamics are given by

$$e_{k+1} = Ae_k + Bu_k + g(\xi_k) + g(\bar{\xi}_k) + w_k$$  \hspace{1cm} (5.76)
Yu et al. [87] demonstrated that if $\mathcal{Z}$ is a robust positively invariant set of the error system in Eq. (5.76) with control law $\hat{u}$ and if $\xi_k \in \left\{ \overline{\xi}_k \right\} \ominus \mathcal{Z}$, then $
abla_{k+i} \in \left\{ \overline{\xi}_{k+i} \right\} \ominus \mathcal{Z}$ for all $i > 0$ and all admissible disturbance sequences $w_{k+i} \in \mathcal{W}$ (see Appendix for the definitions of invariant set and Minkowski sum, $\ominus$). This means that if the system states start close to the nominal state, then the control law in Eq. (5.75) will keep the system trajectory within the robust positively invariant set $\mathcal{Z}$ centered at the predicted nominal states. This statement also suggests that “if a feasible solution can be found for the nominal system subject to the tightened constraints $\overline{\mathcal{Z}} = \mathcal{Z} \ominus \mathcal{Z}$ and $\overline{\mathcal{U}} = \mathcal{U} \ominus \hat{u} (\mathcal{Z})$, then the control law” in Eq. (5.75) “will ensure constraint satisfaction for the controlled uncertain system” (see Appendix for the definition of Pontryagin difference, $\ominus$).

In general, the controller and invariant set pair are very difficult to calculate, unless the nonlinear term in Eq. (5.76) is small, i.e., $\|g (\xi_1) - g (\xi_2)\|_2 \leq L_{\text{Lipschitz}} \|\xi_1 - \xi_2\|_2$. For this case, [86] provides an algorithm for the computation of the minimal positively invariant set $\mathcal{Z}$ (see Appendix for the definition) associated with the linear quadratic regulator gain $K_{\text{LQ}}$ applied to the system defined by $A$ and $B$. In the actual calculations of [86], as the system was split into its longitudinal and lateral dynamics contributions, the respective invariant sets were calculated separately.

From a practical viewpoint, the robust model predictive control system design procedure reduces to the following relatively simple steps:

(i) Formulation of the model equations according to the structure of Eq. (5.76).
(ii) Off-line computation of the linear quadratic regulator gain matrix $K_{\text{LQ}}$ (for an infinite horizon) based on the linear part of the system model.
(iii) Off-line estimation of the bounded disturbance $w$. This can be carried out by comparing the one-step prediction of the model in Eq. (5.76), discretized at 50 ms in [86], with the measured vehicle states. For example, in [86] a simple bound (i.e., $[0.2 \ 0.2 \ 0.2 \ 0.005 \ 0.05 \ 0.05]$) is used “to conservatively estimate the disturbance bound” on the system state prediction.
(iv) Off-line computation of $\mathcal{Z}$ through the algorithm in [86], starting from $\mathcal{W}$.
(v) Online computation of the nominal control action through conventional nonlinear model predictive control methods (e.g., see Eq. (5.68)), e.g., by solving the optimization with the NPSOL tool.
(vi) Online computation of $u_k = \overline{u}_k + \hat{u} (e_k) = \overline{u}_k + K_{\text{LQ}} e_k$. 

$$ e_{k+1} = A e_k + B \hat{u} (e_k) + \tilde{w}_k $$

(5.77)

with $\tilde{w}_k \in \mathcal{W} = \mathcal{W} \oplus B (\mathcal{E})$, where

$$ B (\mathcal{E}) = \left\{ x \in \mathbb{R}^n | \| x \|_\infty \leq L_{\text{Lipschitz}} (\mathcal{Z}) \max_{e \in \mathcal{E}} \| e \|_2 \right\} $$

(5.78)
The robust formulation in [86] is simply a linear quadratic regulator coupled with an implicit model predictive controller, with an additional algorithm to calculate the invariant set. The calculation of $\mathcal{Z}$ is beneficial to know and consider the expected boundaries of the states, i.e., the projection of the bounds of the robust invariant set, Proj($\mathcal{Z}$), while the controller is running (hence the concept of tube-based model predictive control).

Gao et al. [86] report simulation results of obstacle avoidance maneuvers, including introduction of random bounded disturbances with uniform distribution into the simulation model, and comparison of the performance of the robust controller, $u_k = \bar{u}_k + \hat{u}(e_k)$, with the performance of the nominal controller, $\bar{u}_k$ (the difference is evident in Figs. 5.27 and 5.28). Also, Gao et al. [86] include experimental results, such as multiple obstacle avoidance tests carried out on a surface with a tire-road friction coefficient $\mu_c=0.1$, while the controller is set for $\mu_c=0.3$ (Figs. 5.29 and 5.30).

In addition to the tube-based model predictive controller, Carvalho et al. [65] also suggest time-varying stochastic model predictive control for dealing with system

![Fig. 5.27](image1) Simulation results: trajectory of the nominal model predictive controller under a random external disturbance (from [86])

![Fig. 5.28](image2) Simulation results: trajectory of the robust model predictive controller under a random external disturbance (from [86])
uncertainty. The theory and an example of application to the automated driving problem are provided in [90, 91].

Another interesting example of comparison of control structures for path tracking based on model predictive control in uncertain conditions is included in [92], dealing with the problem of obstacle avoidance on a slippery road. The paper supposes that the reference generation layer (see the introduction of this chapter) outputs the reference trajectory, but does not correct it to avoid an obstacle located on the desired path. Therefore, the reference trajectory has to be modified by the control layer, i.e., it has to be replanned to take the obstacle into account. Figures 5.31 and 5.32 show
the two control layer architectures that are contrasted in [92]:

(i) An architecture (Controller J) consisting of a single-level model predictive controller, based on a four-wheel vehicle model with a nonlinear tire model. The desired trajectory and the obstacle position are fed to the control algorithm. The controller directly calculates the reference steering angle and the reference longitudinal forces for the two sides of the vehicle, $F_{b,l}$ and $F_{b,r}$, in order to avoid the obstacle. These two forces are converted into individual braking torques by the “Braking logic” (see Fig. 5.31);

(ii) An architecture (Controller K) consisting of a two-level model predictive controller. The top-layer (the “High-level path replanner obstacle avoiding” in Fig. 5.32) is based on a simple point-mass model, with the purpose of replanning the reference trajectory starting from the detected position of the obstacle. The lower layer (the “Low-level path follower”), based on the same
four-wheel vehicle model as the one used in controller architecture (i), receives the replanned reference trajectory and calculates the same outputs as the single-layer controller.

Controller J and the top layer of Controller K use cost functions with similar structure to the one in Eq. \((5.67)\), with a term referring to the tracking performance and a term referring to the control effort, but they also include an additional term given by:

\[
\sum_{k=t}^{t+H_p-1} J_{\text{obs},t} = \sum_{k=t}^{t+H_p-1} \frac{K_{\text{obs}} v_{k,t}}{d_{\text{min},j} + \varepsilon} \tag{5.79}
\]

\(J_{\text{obs},t}\) in Eq. \((5.79)\) is the cost at time \(k\) associated to the predicted distance between the vehicle and the obstacle, under the assumption that obstacle position is known for a collection of discretized points \(P_{t,j}\). In particular, \(d_{\text{min},j} = \min_j d_{k,t,j}\). \(d_{k,t,j}\) is defined as:

\[
d_{k,t,j} = \begin{cases} 
    p_{x,t,j} - \bar{a} & \text{if } p_{y,t,j} \in [-\bar{c}, \bar{c}] \text{ and } p_{x,t,j} > \bar{a} \\
    0 & \text{if } p_{y,t,j} \in [-\bar{c}, \bar{c}] \text{ and } p_{x,t,j} \in [-\bar{b}, \bar{a}] \\
    M & \text{otherwise} \end{cases} \tag{5.80}
\]

where \(d_{k,t,j} = 0\) indicates the occurrence of a collision (see also Fig. 5.33).

\[\begin{array}{c}
\text{Fig. 5.33} \text{ Obstacle avoidance scenario and main geometrical parameters used in the model predictive control formulation (adapted from [92])}
\end{array}\]
The simulation and experimental results demonstrate that the case study obstacle avoidance maneuver can be executed at higher values of vehicle speed with the two-level model predictive controller (Controller K), while the single-level architecture (Controller J) tends to cause stability problems. In general, when the vehicle deviates too far from the reference trajectory, the system becomes uncontrollable. In these conditions, “the vehicle state is outside the region of attraction of the equilibrium trajectory associated to the desired reference.” In the single-layer architecture, this phenomenon happens quite often, as it is induced by tire saturation. This situation does not occur with the simple point-mass model of Controller K, “since the path replanner always replans a path starting from the current state of the vehicle, and therefore ensures that the low-level reference is close to current state. This explains why the performance of the two-level approach is better than the one-level approach.”

Also, the computational performance of the two-level approach is much better than that of the single-level approach. Hence, the important conclusion is that for effective automated driving through model predictive control, the separation between the reference generation layer, which should replan the trajectory considering the presence of obstacles, and the control layer (see the introduction of this chapter) is not only convenient for control system simplification but also beneficial to control system performance.

Table 5.5 is included as a summary of the main characteristics of the model predictive controllers discussed in this section, with an overview of the adopted models, the possible control action (lateral control or combined longitudinal and lateral control), the involved complexity, and the form of validation presented in the literature, i.e., through simulations and/or experiments.

All the algorithms discussed in this section are based on implicit model predictive control formulations, i.e., the optimizations are run online, which implies a significant computational load for the vehicle control unit. Also, implicit model predictive control does not permit formal analysis of performance, suboptimality and stability [93]. An option that should be assessed in future research is explicit model predictive control, which has already been successfully implemented, including experiments, for concurrent yaw moment and active steering control in [94]. The validation of robust and explicit model predictive control formulations could represent the next step of path tracking control research.
Table 5.5 Comparison of the selected model predictive controllers for path tracking

| Controller | Vehicle and tire model | Control action | Controller complexity | Validation |
|------------|------------------------|----------------|-----------------------|------------|
| A [76]     | Single-track vehicle model with Pacejka tire model (pure cornering) with constant vertical load | Steering control only | Nonlinear model and high computational load | Experiments for vehicle speeds lower than 10 m/s due to computational constraints |
| B [76]     | Single-track vehicle model with Pacejka tire model with constant vertical load | Steering control only | Linearized model for computationally efficient optimization | Experimental lane change tests executed at various speeds |
| C [76]     | Single-track vehicle model with Pacejka tire model with constant vertical load | Steering control only | Further simplified version of Controller B, with $H_C = 1$ | Experimental lane change tests executed at various speeds; performance slightly worse than for Controller B |
| D [79]     | Four-wheel vehicle model with Pacejka tire model (combined slip) with constant vertical load | Longitudinal control, steering control, and direct yaw moment control | Nonlinear optimization with high computational load | Simulations of lane change tests executed at 50 km/h and 70 km/h with good tracking performance |
| E [79]     | Single-track vehicle model with Pacejka tire model (combined slip) with constant vertical load | Steering control and direct yaw moment control | Nonlinear (relatively) low complexity model for online optimization | Simulation of lane change tests executed at 50 km/h; controller not able to stabilize the vehicle at higher entry speeds |
| F [79]     | Four-wheel vehicle model with Pacejka tire model (combined slip) with constant vertical load | Steering control and direct yaw moment control | Linearized model based on Controller D for computationally efficient online optimization | Experimental lane change tests at 50 km/h |
| G [80]     | Four-wheel vehicle model and Pacejka tire model (combined slip) with constant vertical load | Longitudinal control, steering control, and direct yaw moment control | Linearized model for computationally efficient optimization | Simulations of a single lane change test |
### Table 5.5 (continued)

| Controller | Vehicle and tire model | Control action | Controller complexity | Validation |
|------------|------------------------|----------------|-----------------------|------------|
| H [81]     | Single-track vehicle model with Burckhardt tire model with constant vertical load | Only steering control by the model predictive controller; independent Lyapunov approach for longitudinal control | Nonlinear model predictive controller; no details provided regarding the online optimization | Simulations of a highway exit scenario |
| I [86]     | Single-track vehicle model with simplified nonlinear tire model with constant vertical load | Longitudinal control, steering control and direct yaw moment control | Nonlinear model with significant simplifications for computationally efficient online optimization providing system robustness | Experimental obstacle avoidance tests at high speed on a snow track |
| J [92]     | Four-wheel vehicle model with Pacejka tire model (combined slip) with constant vertical load | Longitudinal control, steering control and direct yaw moment control | Single-level nonlinear model predictive controller with significant computational load | Experimental lane change tests in icy road conditions up to 40 km/h |
| K [92]     | Simple point-mass model for the high level controller; the same as Controller J in terms of vehicle model for the low level controller | Longitudinal control, steering control, and direct yaw moment control | Two-level hierarchical nonlinear model predictive controller with significant computational load reduction with respect to Controller J | Experimental lane change tests in icy road conditions up to 55 km/h |

### 5.6 Concluding Remarks

This chapter provided an overview of control structures for path tracking in autonomous vehicles, ranging from basic kinematic controllers to robust model predictive controllers. The presented analysis and results bring the following conclusions:

- Without disturbances (e.g., caused by side wind and the banking of the road), and uncertainties (e.g., caused by the vision systems), the path tracking performance of simple control structures is adequate and was already experimentally demonstrated in research programs in the 1990s [95]. For an assigned vehicle
speed, fixed parameter controllers can deal with the range of axle cornering stiffnesses, vehicle masses, and tire-road friction coefficients typical of a real vehicle operation. Apparently, fixed parameter controllers can be effective even in the case of buses and heavy goods vehicles, characterized by significant mass variations during their operation.

- Look-down controllers show significant performance limitations at medium-high speeds. Effective feedback control design for path tracking must be based on the combination of lateral displacement and heading angle control, or lateral displacement evaluated at a look-ahead distance.
- Gain scheduling of the controller parameters, including the preview distance, is recommended as a function of vehicle speed.
- The separation among the reference trajectory level and the control layer is not only convenient for simplifying the automated driving control system design but also allows better system response from the viewpoint of vehicle stability, conjugated with reduced computational effort.
- In order to reduce the effect of disturbances and relax the tuning of the feedback part of the path tracking controller, properly designed feedforward contributions are essential. The main limitation of existing feedforward controllers for path tracking is their reliance on very advanced state estimators, which have to provide smooth outputs. Machine learning techniques, adaptive control and sensor fusion could significantly help with this challenging task.
- An interesting concept is represented by the center of percussion. A path tracking controller based on the position error at the front center of percussion eliminates the effect of the rear axle cornering forces on the lateral position error dynamics, which significantly facilitates the design of a feedback controller for a two-wheel-steering vehicle. A four-wheel-steering path tracking controller based on the lateral position errors at the front and rear centers of percussion simplifies into the design of two decoupled single-input single-output controllers.
- Some of the available experimental studies show that from the vehicle passengers’ perspective, vehicle comfort (determined by the frequency and amplitude of the oscillations induced by the control action) is more important than the excellence of the tracking performance. Unluckily, only very limited data are available with respect to the comfort behavior of the most recent and performing path tracking controllers, including analysis of the subjective feedback of the occupants.
- Despite many path tracking formulations have been assessed through experimental tests, an objective comparative assessment of the performance of different control structures for the same vehicle and set of state estimators is missing in the literature. On the one hand, the authors presenting the most advanced control formulations tend to show their benefits without going through a comparison with fine-tuned simple controllers. On the other hand, the authors presenting relatively simple control formulations tend to highlight their performance even at medium-high lateral acceleration levels. Also, the subjective performance of the different path tracking formulations, i.e., in terms of oscillation of the control action and the subsequent vehicle response, should be carefully assessed through
experimental tests, in order to draw clear conclusions on the required level of control system sophistication.

- **Current research developments** are in the area of automated driving for limit conditions, i.e., at extreme lateral and longitudinal accelerations, and in the area of robust model predictive control, with the main aim of systematically dealing with system uncertainty.

- **Future research activities** should also systematically cover the interaction between automated steering and direct yaw moment control, which currently represents the main actuation technique for stabilizing the vehicle in extreme transient conditions. Direct yaw moment control can be sporadically actuated through the friction brakes within stability control systems [96] or, in the case of vehicles with torque-vectoring differentials or multiple electric drivetrains, can be continuously actuated during normal vehicle operation [97–100]. Especially in the latter case, further investigations of automated driving during extreme cornering are required, including more detailed and comprehensive experimental demonstrations.

**Appendix: Definitions of Invariant Sets, Minkowski Sum ⊕, and Pontryagin Difference ⊖**

The following definitions are provided:

(a) **Reachable set** for systems with external inputs. Consider a system \( \xi_{k+1} = f(\xi_k, u_k) + w_k \), with \( \xi_k \in \Xi \), \( u_k \in U \), \( w_k \in W \). The one-step robust reachable set from a given set of states \( S \) is \( \text{Reach}_f(S, W) \triangleq \{ \xi \in \mathbb{R}^n | \xi_0 \in S, \exists u \in U, \exists w \in W : \xi = f(\xi_0, u, w) \} \);

(b) **Robust positively invariant set**. A set \( Z \subseteq \Xi \) is said to be a robust positively invariant set for the autonomous system \( \xi_{k+1} = f_0(\xi_k) + w_k \), with \( \xi_k \in \Xi \) and \( w_k \in W \), if \( \xi_0 \in Z \Rightarrow \xi_k \in Z, \forall w_k \in W, \forall k \geq 1 \in \mathbb{N}^+ \);

(c) **Minimal robust positively invariant set**. The set \( Z_\infty \subseteq \Xi \) is the minimal robust positively invariant set for the defined autonomous system, if \( Z_\infty \) is a robust positively invariant set and \( Z_\infty \) is contained in every closed robust positively invariant set in \( \Xi \) (see [89] for the details);

(d) **Minkowski sum**. The Minkowski sum of two polytopes, \( \varphi \) and \( \mathcal{H} \), is the polytope \( \varphi \oplus \mathcal{H} := \{ x + h \in \mathbb{R}^n | x \in \varphi, h \in \mathcal{H} \} \);

(e) **Pontryagin difference**. The Pontryagin difference of two polytopes, \( \varphi \) and \( \mathcal{H} \), is the polytope \( \varphi \ominus \mathcal{H} := \{ x \in \mathbb{R}^n | x + h \in \varphi, \forall h \in \mathcal{H} \} \).
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