$L^\infty$ ESTIMATES FOR THE BANACH-VALUED $\bar{\partial}$-PROBLEM IN A DISK

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Abstract. We study the differential equation $\frac{\partial G}{\partial \bar{z}} = g$ with an unbounded Banach-valued Bochner measurable function $g$ on the open unit disk $\mathbb{D} \subset \mathbb{C}$. We prove that under some conditions on the growth and essential support of $g$ such equation has a bounded solution given by a continuous linear operator. The obtained results are applicable to the Banach-valued corona problem for the algebra of bounded holomorphic functions on $\mathbb{D}$ with values in a complex commutative unital Banach algebra.

1. Formulation of Main Results

Let $K$ be a Lebesgue measurable subset of the open unit disk $\mathbb{D} \subset \mathbb{C}$ and $X$ be a complex Banach space. Two $X$-valued functions on $\mathbb{D}$ are equivalent if they coincide a.e. on $\mathbb{D}$. The complex Banach space $L^\infty(K,X)$ consists of equivalence classes of Bochner measurable essentially bounded functions $f : \mathbb{D} \rightarrow X$ equal 0 a.e. on $\mathbb{D} \setminus K$ equipped with norm $\|f\|_\infty := \text{ess sup}_{z \in K} \|f(z)\|_X$.

In the present paper we study the differential equation

$$\frac{\partial F}{\partial \bar{z}} = \frac{f(z)}{1 - |z|^2}, \quad |z| < 1, \quad f \in L^\infty(K,X).$$

Such equations play an essential role in the area of the Banach-valued corona problem for the algebra $H^\infty(\mathbb{D},A)$ of bounded holomorphic functions on $\mathbb{D}$ with values in a complex commutative unital Banach algebra $A$, see, e.g., [2], [3] and references therein. We prove that for a certain class of sets $K$ equation (1.1) has a weak solution $F \in L^\infty(\mathbb{D},X)$, i.e., such that for every $C^\infty$ function $\rho$ with compact support in $\mathbb{D}$

$$\iint_{\mathbb{D}} F(z) \cdot \frac{\partial \rho(z)}{\partial \bar{z}} dz \wedge d\bar{z} = - \iint_{\mathbb{D}} \frac{f(z)}{1 - |z|^2} \cdot \rho(z) dz \wedge d\bar{z},$$

given by a bounded linear operator $L_K^X : L^\infty(K,X) \rightarrow L^\infty(\mathbb{D},X)$. The operator $L_K^X$ is constructed explicitly that provides effective bounds of its norm in terms of some characteristics of $K$.

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1
To formulate our results, recall that a sequence \( \{z_n\} \subset \mathbb{D} \) is said to be interpolating for \( H^\infty \), the Banach space of bounded holomorphic functions on \( \mathbb{D} \), if every interpolation problem
\[
g(z_n) = a_n, \quad n \geq 1,
\]
with a bounded data \( \{a_n\} \subset \mathbb{C} \) has a solution \( g \in H^\infty \).

By the Banach open mapping theorem, there is a constant \( M \) such that problem (1.3) has a solution \( g \in H^\infty \) satisfying
\[
\|g\|_\infty \leq M \sup_n |a_n|.
\]
The smallest possible \( M \) is said to be the constant of interpolation of \( \{z_n\} \).

Clearly, every finite subset of \( \mathbb{D} \) is an interpolating sequence for \( H^\infty \). In general, by the Carleson theorem, see, e.g., [6, Ch. VII, Thm. 1.1], a sequence \( \zeta = \{z_n\} \subset \mathbb{D} \) is interpolating for \( H^\infty \) if and only if for some \( \delta > 0 \) the characteristic
\[
\delta(\zeta) := \inf_k \prod_{j, j \neq k} \rho(z_j, z_k) \geq \delta, \tag{1.4}
\]
where
\[
\rho(z, w) := \left| \frac{z - w}{1 - \overline{w}z} \right|, \quad z, w \in \mathbb{D},
\]
is the pseudohyperbolic metric on \( \mathbb{D} \).

In turn, the constant of interpolation \( M_\zeta \) of \( \zeta \) satisfies
\[
\frac{1}{\delta} \leq M_\zeta \leq \min \left\{ \frac{2e}{\delta} \log \frac{e}{\delta^2}, \left( \frac{1 + \sqrt{1 - \delta^2}}{\delta} \right)^2 \right\}. \tag{1.5}
\]
(The first term in the braces is an upper bound of the P. Jones linear interpolation operator [7, Thm. 6] obtained in [9] by elementary arguments, and the second one is due to Earl [5, Thm. 2].)

A subset \( S \) of a metric space \( (\mathcal{M}, d) \) is said to be \( \epsilon \)-separated if \( d(x, y) \geq \epsilon \) for all \( x, y \in S, x \neq y \). A maximal \( \epsilon \)-separated subset of \( \mathcal{M} \) is said to be an \( \epsilon \)-chain. Thus, if \( S \subset \mathcal{M} \) is an \( \epsilon \)-chain, then \( S \) is \( \epsilon \)-separated and for every \( z \in \mathcal{M} \setminus S \) there is \( x \in S \) such that \( d(z, x) < \epsilon \). Existence of \( \epsilon \)-chains follows from the Zorn lemma.

A subset \( K \subset \mathbb{D} \) is said to be quasi-interpolating, if an \( \epsilon \)-chain of \( K \), \( \epsilon \in (0, 1) \), with respect to \( \rho \) is an interpolating sequence for \( H^\infty \). (In fact, in this case every \( \epsilon \)-chain of \( K \), \( \epsilon \in (0, 1) \), with respect to \( \rho \) is an interpolating sequence for \( H^\infty \), this easily follows from [6 Ch. X, Cor. 1.6, Ch. VII, Lm. 5.3].)

An important example of a quasi-interpolating set is a pseudohyperbolic neighbourhood (see (1.7) for its definition) of a Carleson contour used in the proof of the corona theorem [4], [10].

\(^1\)Here \( \delta(\zeta) = 1 \) if \( \zeta \) consists of one element.
Given a complex Banach space $X$ and a subset $U \subset \mathbb{D}$ we denote by $C_\rho(U, X)$ the Banach space of bounded continuous functions $f : U \to X$ uniformly continuous with respect to $\rho$ equipped with norm $\|f\|_\infty := \sup_{z \in U} \|f(z)\|_X$.

We are ready to formulate the main result of the paper.

**Theorem 1.1.** Suppose a quasi-interpolating set $K \subset \mathbb{D}$ is Lebesgue measurable and $\zeta = \{z_j\}$ is an $\epsilon$-chain of $K$, $\epsilon \in (0, 1)$, with respect to $\rho$ such that $\delta(\zeta) \geq \delta > 0$. There is a bounded linear operator $L_K^X : L^\infty(K, X) \to C_\rho(\mathbb{D}, X)$ of norm

$$
\|L_K^X\| \leq \frac{ce}{1 - \epsilon} \max \left\{ 1, \frac{\log \frac{1}{\epsilon}}{(1 - \epsilon)^2} \right\}, \quad \epsilon_* := \max \left\{ \frac{1}{2}, \epsilon \right\},
$$

for a numerical constant $c < 5^2 \cdot 10^6$ such that for every $f \in L^\infty(K, X)$ the function $L_K^X f$ is a weak solution of equation (1.1).

The operator $L_K^X$ has the following properties:

(i) If $T : X \to Y$ is a bounded linear operator between complex Banach spaces, then

$$
TL_K^X = L_K^Y T,
$$

where $(Tf)(z) := T(f(z))$, $z \in \mathbb{D}$, $f : \mathbb{D} \to X$;

(ii) If $f \in L^\infty(K, X)$ has a compact essential range, then the range of $L_K^X f$ is relatively compact;

(iii) If $f \in L^\infty(X, K)$ is continuously differentiable on an open set $U \subset \mathbb{D}$, then $L_K^X f$ is continuously differentiable on $U$.

**Remark 1.2.** (1) The construction of the operator $L_K^X$ involves integration of elements of $L^\infty(K, X)$. This implies that if $\text{supp} \ K$ is the essential support of the characteristic function of $K$ and $K' := K \cap \text{supp} \ K$ (a Lebesgue measurable subset of full measure in $K$), then the correspondence $f \mapsto f|_{K'}$ determines an isometric isomorphism $I_K^{K'} : L^\infty(K, X) \to L^\infty(K', X)$ such that $L_K^X = L_K^{X'} \circ I_K^{K'}$. Thus without loss of generality we may tacitly assume hereafter that $K \subset \text{supp} \ K$.

(2) A nonquantitative version of Theorem 1.1 for a class of $C^\infty$ functions $f$ with relatively compact images was presented earlier in [2, Thm. 3.5].

Let us formulate another property of the operator $L_K^X$.

In the sequel, we use the following notation.

$\mathbb{D}_r(x) := \{z \in \mathbb{C} : |z - x| < r\}$, $\mathbb{D}_r := \mathbb{D}_r(0)$ and $D(x, r) := \{z \in \mathbb{C} : \rho(z, x) < r\}$.

For an interpolating sequence $\zeta \subset \mathbb{D}$ by $B_\zeta$ we denote the interpolating Blaschke product having simple zeros at points of $\zeta$.

For a subset $S \subset \mathbb{D}$ its open $\nu$-pseudohyperbolic neighbourhood, $\nu \in (0, 1)$, is given by the formula

$$
[S]_\nu := \left\{ y \in \mathbb{D} : \inf_{z \in S} \rho(y, z) < \nu \right\}.
$$

We denote by $\bar{S}$ the closure of $S$. 


Finally, $\mathcal{B}(K, X)$ stands for the Banach space of bounded linear operators from $L^\infty(K, X)$ in $H^\infty(D, X)$ equipped with the operator norm.

**Theorem 1.3.** Suppose $K$ is a quasi-interpolating set of positive Lebesgue measure and $\zeta$ is an $\epsilon$-chain of $K$, $\epsilon \in (0, 1)$, with respect to $\rho$ such that $\delta(\zeta) \geq \delta > 0$.

Given $\nu \in (0, 2 - \sqrt{3}]$ let

$$\epsilon_\nu := \frac{(2 - \sqrt{3})^3}{6} \cdot \nu.$$

There exist

- a natural number

$$k^*_\nu \leq \frac{c}{\nu^2(1 - \epsilon)} \cdot \max \left\{ 1, \frac{\log \frac{1}{\delta}}{(1 - \epsilon^*_\nu)^2} \right\}$$

for a numerical constant $c < 5^3 \cdot 10^5$;
- interpolating sequences $\zeta^i_\nu \subset K$ with $\delta(\zeta^i_\nu) > \frac{1}{2}$, $1 \leq i \leq k^*_\nu$, such that

$$K \subset \bigcup_{i=1}^{k^*_\nu} B_{\zeta^i_\nu}^{-1}(\mathbb{D}_{\epsilon_\nu}) \subset \bigcup_{i=1}^{k^*_\nu} B_{\zeta^i_\nu}^{-1}(\mathbb{D}_{6\epsilon_\nu}) \subset [K]_\nu;$$
- holomorphic functions $H^i_\nu \in H^\infty(\mathbb{C} \setminus \mathbb{D}_{\epsilon_\nu}, \mathcal{B}(K, X))$ vanishing at $\infty$ of norms

$$\|H^i_\nu\|_\infty \leq \frac{3}{5} \cdot \nu, \quad 1 \leq i \leq k^*_\nu;$$
- an operator $E^0_\nu \in \mathcal{B}(K, X)$ such that for every $f \in L^\infty(K, X)$, $z \in D \setminus [K]_\nu$

$$L^X_K(f)(z) = (E^0_\nu f)(z) + \sum_{i=1}^{k^*_\nu} (H^i_\nu(B_{\zeta^i_\nu}(z)))f(z).$$

**Remark 1.4.** Let $R_{K;\nu} : C_\rho(D, X) \to C_\rho(D \setminus [K]_\nu, X)$, $f \mapsto f|_{D \setminus [K]_\nu}$, be the restriction operator. Expanding holomorphic functions $H^i_\nu$ in the Laurent series:

$$H^i_\nu(w) = \sum_{j=1}^{\infty} E^i_{\nu,j} w^{-j}, \quad w \in \mathbb{C} \setminus \mathbb{D}_{\epsilon_\nu},$$

for some $E^i_{\nu,j} \in \mathcal{B}(K, X)$, $1 \leq i \leq k^*_\nu$, $j \in \mathbb{N}$, we obtain from (1.12):

$$R_{K;\nu} \circ L^X_K = R_{K;\nu} \circ E^0_\nu + \sum_{i=1}^{k^*_\nu} \sum_{j=1}^{\infty} R_{K;\nu} \circ (E^i_{\nu,j} \cdot B_{\zeta^i_\nu}^{-j}),$$

where all series converge uniformly on $D \setminus [K]_\nu$ due to (1.10).
Let $\mathcal{A} := \langle \mathcal{H}_\infty, \mathcal{H}_\infty^\ast \rangle$ be the Banach subalgebra of the algebra of bounded complex-valued continuous functions on $\mathbb{D}$ equipped with supremum norm generated by functions in $\mathcal{H}_\infty$ and their complex conjugate. For $X = \mathbb{C}$ we set $L_\infty(K) := L_\infty(K, \mathbb{C})$, $C_\rho(\mathbb{D}) := C_\rho(\mathbb{D}, \mathbb{C})$ and $L_K := L_K^\mathbb{C}$. Since $\mathcal{H}_\infty \subset C_\rho(\mathbb{D})$, $\mathcal{A}$ is a closed subalgebra of the Banach algebra $C_\rho(\mathbb{D})$.

As a corollary of Theorems 1.1 and 1.3 we obtain:

**Corollary 1.5.** $L_K$ is a bounded linear operator from $L_\infty(K)$ into $\mathcal{A}$.

In particular, for every $f \in L_\infty(K)$ the function $L_K f$ has radial limits a.e. on the boundary $S := \{e^{i\theta} \in \mathbb{C} : \theta \in \mathbb{R}\}$ of $\mathbb{D}$.

**Remark 1.6.** The algebra $\mathcal{A}$ is isometrically isomorphic via the Gelfand transform to the Banach algebra $C(M)$ of complex-valued continuous functions on the maximal ideal space $M$ of $\mathcal{H}_\infty$. Recall that $M$ is the weak∗ compact subset of the unit ball of the dual space $(\mathcal{H}_\infty)^*$ of nonzero complex homomorphisms of $\mathcal{H}_\infty$. The Gelfand transform $\hat{\cdot} : \mathcal{H}_\infty \to C(M)$, $\hat{\phi}(f) := \phi(f)$, $\phi \in M$, $f \in \mathcal{H}_\infty$, is an isometric monomorphism of Banach algebras. The correspondence $\mathbb{D} \ni z \mapsto \delta_z \in M$, where $\delta_z$ is the evaluation functional at $z \in \mathbb{D}$, embeds $\mathbb{D}$ as an open dense (due to the Carleson corona theorem [4]) subset of $M$.

The paper is organized as follows.

Section 2 contains some auxiliary results required for the proof of Theorem 1.1. Section 3 contains proofs of special cases of Theorems 1.1 and 1.3 for quasi-interpolating sets of small width. They are used in Section 4 to prove Theorems 1.1 and 1.3 for quasi-interpolating sets having chains of large characteristics. In turn, the latter results are used in Section 5 to prove Theorems 1.1 and 1.3 in the general case. Section 6 contains the proof of Corollary 1.5. In Section 7, we describe an application of Theorem 1.1 to the Banach-valued corona problem for the algebra $H_\infty(\mathbb{D}, \mathcal{A})$, where $\mathcal{A}$ is a complex commutative unital Banach algebra.

In a forthcoming paper, we present other applications of Theorems 1.1 and 1.3 to the theory of bounded Banach-valued holomorphic functions on $\mathbb{D}$.

## 2. Auxiliary Results

### 2.1. For $h \in L_\infty(\mathbb{D}, X)$, $0 < s \leq 1$, we define

\begin{equation}
(\mathcal{E}h)(z) = \frac{1}{2\pi i} \int_{\mathbb{D}_s} \frac{h(w)}{w - z} dw \wedge \bar{dw}, \quad z \in \mathbb{D}.
\end{equation}

Set

\begin{equation}
\omega(t) := t \log(\frac{2}{t}), \quad 0 < t \leq 2.
\end{equation}

Let $C^\omega(\mathbb{D}, X)$ be the Banach space of bounded continuous $X$-valued functions $f$ on $\mathbb{D}$ equipped with the norm

\begin{equation}
\|f\|_\omega := \max\{\|f\|_\infty, |f|_\omega\}, \quad |f|_\omega := \sup_{z_1, z_2 \in \mathbb{D}, z_1 \neq z_2} \frac{\|f(z_1) - f(z_2)\|_X}{\omega(|z_1 - z_2|)}.
\end{equation}
Lemma 2.1. $E$ is a bounded linear operator from $L^\infty(\mathbb{D}, X)$ into $C^\infty(\mathbb{D}, X)$ such that for every $h \in L^\infty(\mathbb{D}, X)$ the function $Eh$ is a weak solution of the equation

$$\frac{\partial H}{\partial z} = h(z), \quad z \in \mathbb{D},$$

and $Eh|_{\mathbb{D}\setminus\mathbb{D}_s}$ is a bounded $X$-valued holomorphic function.

Proof. Making the substitution $u := w - z$, then passing to polar coordinates $u = re^{i\phi}$ and using the Fubini theorem we write (2.1) as follows

$$(Eh)(z) = \frac{1}{2\pi i} \int_0^{2\pi} \left( \int_0^{1+s} h(re^{i\phi} + z)e^{-i\phi} dr \right) d\phi.$$ 

This, the triangle inequality for integrals and the fact that $h = 0$ a.e. on $\mathbb{C}\setminus\mathbb{D}_s$ imply

$$\|Eh\|_\infty \leq 2\pi \|h\|_\infty.$$ 

Next, for $z_1, z_2 \in \mathbb{D}$ and $r_0 := \rho(|z_1 - z_2|)$, $z_0 := \frac{z_1 + z_2}{2}$ we get

$$\|(Eh)(z_1) - (Eh)(z_2)\|_X \leq \frac{1}{2\pi} \left\| \int \int_{\mathbb{D}\setminus\mathbb{D}_s(r_0)} \left( \frac{h(w)}{w - z_1} - \frac{h(w)}{w - z_2} \right) dw \wedge d\bar{w} \right\|_X + \frac{1}{2\pi} \left\| \int \int_{\mathbb{D}_s(r_0)} \frac{(z_2 - z_1)h(w)}{(w - z_1)(w - z_2)} dw \wedge d\bar{w} \right\|_X + \frac{1}{2\pi} \left\| \int \int_{\mathbb{D}_s(r_0)} \left( \frac{z_2 - z_1}{2} \right) \frac{h(w)}{w - z_1} \frac{h(w)}{w - z_2} dw \wedge d\bar{w} \right\|_X =: I_1 + I_2 + I_3.$$ 

Integrals in $I_1$ and $I_2$ can be written in polar coordinates with respect to centers $z_1$ and $z_2$ similarly to (2.1). These and triangle inequalities for integrals lead to the following estimates

$$\max\{I_1, I_2\} \leq \left( \frac{|z_1 - z_2|}{2} + r_0 \right) \cdot \|h\|_\infty.$$ 

For the integral in $I_3$ we use polar coordinates with the center at $z_0$. Then using that

$$r_0^2 - \frac{|z_1 - z_2|^2}{4} = |z_1 - z_2|^2 \left( \log \left( \frac{8}{|z_1 - z_2|} \right)^2 - \frac{1}{4} \right) \geq \frac{3}{2} |z_1 - z_2|^2,$$
we obtain

\[ I_3 \leq \|h\|_\infty \cdot \frac{|z_2 - z_1|}{2\pi} \int_0^{2\pi} \left( \int_{r_0}^{r_2} \frac{1}{r^2 - \frac{|z_1 - z_2|^2}{4}} r\,dr \right) d\phi \]

\[ \leq \frac{1}{2} \|h\|_\infty \cdot |z_2 - z_1| \left( \log \left( 4 - \frac{|z_1 - z_2|^2}{4} \right) - \log \left( r_0^2 - \frac{|z_1 - z_2|^2}{4} \right) \right) \]

\[ \leq \frac{1}{2} \|h\|_\infty \cdot |z_2 - z_1| \log \left( \frac{8}{3\|z_2 - z_1\|^2} \right) \leq \|h\|_\infty \cdot (r_0 - \log 4 \cdot |z_2 - z_1|). \]

Combining this with (2.6) we get

(2.7) \[ \|(Eh)(z_1) - (Eh)(z_2)\|_X \leq 3r_0 \cdot \|h\|_\infty. \]

This and (2.5) complete the proof of the first statement of the lemma.

Next, the claim that \(Eh\) is a weak solution of equation (2.3) can be proved in the same way as a similar statement in [6, Ch. VIII.1] for \(X = \mathbb{C}\).

Finally, for \(z \in \overline{\mathbb{D}} \setminus \mathbb{D}_s\)

\[ (Eh)(z) = \frac{1}{2\pi i} \iint_{\mathbb{D}_s} \frac{h(w)}{z(w^2 - 1)} \, dw \wedge d\bar{w} = -\frac{1}{2\pi i} \sum_{i=1}^\infty \left( \iint_{\mathbb{D}_s} h(w)w^{i-1} \, dw \wedge d\bar{w} \right) z^{-i}. \]

The series converges uniformly on compact subsets of \(\mathbb{C} \setminus \overline{\mathbb{D}}_s\) to an \(X\)-valued holomorphic function. This proves the last statement of the lemma. \(\square\)

**Remark 2.2.** Note that properties (i)-(iii) of Theorem 1.1 are valid for the operator \(E\). Indeed, \(TE = ET\) for a bounded linear operator of complex Banach spaces \(T : X \to Y\) by the definition of the Bochner integral.

Next, if \(h \in L(\mathbb{D}, X)\) has a compact essential range \(R(h)\), then by the definition of a Bochner measurable essentially bounded function, \(h\) is a limit a.e. on \(\mathbb{D}\) of a sequence \(\{h_n\}_{n \in \mathbb{N}}\) of \(X\)-valued measurable simple functions with values in \(R(h)\). Then due to (2.4) every \((E(h_n))\) has range in \(\overline{\text{co}}_C(R(h))\), the closure of the convex complex hull of \(R(h)\), which is compact by the Mazur theorem. Since clearly \(\{(Eh_n)(z)\}_{n \in \mathbb{N}}\) converges to \((Eh)(z)\) for all \(z \in \mathbb{D}\), the range of \(Eh\) is contained in the compact set \(2 \cdot \overline{\text{co}}_C(R(h)) \subset X\), as required. Hereafter for \(S_1, S_2 \in X\), \(\lambda \in \mathbb{C}\)

\[ \lambda \cdot S_1 := \{\lambda x : x \in S_1\}, \quad S_1 + S_2 := \{x_1 + x_2 : x_1 \in S_1, i = 1, 2\}. \]

Finally, if \(h\) is continuously differentiable on an open set \(U\), then it easily follows from equation (2.4) that \(E(\phi h)\) is continuously differentiable on \(\mathbb{D}\) for every \(C^\infty\) function \(\phi\) with compact support in \(U\) equals 1 on an open subset \(V\) of \(U\). Also, since \(g_V := Eh |_V - E(\phi h) |_V\) is a weak solution of the equation \(\frac{\partial F}{\partial z} = 0\), it is a bounded \(X\)-valued holomorphic function on \(V\). Then \(Eh |_V = E(\phi h) |_V + g_V\) is continuously differentiable on \(V\). This gives the required statement.
2.2. We require the following result.

**Lemma 2.3** ([6] Ch. X, Lm. 1.4, Ch. VII, Lm. 5.3). Let \( B_\zeta \) be the interpolating Blaschke product with zeros \( \zeta = \{ z_n \} \) such that

\[
\delta(\zeta) = \inf_n (1 - |z_n|^2)|B'_\zeta(z_n)| \geq \delta > 0.
\]

Suppose \( \lambda \in (0, 1) \) and \( r := r(\lambda) \in (0, 1) \) satisfy

\[
\frac{2\lambda}{1 + \lambda^2} < \delta \quad \text{and} \quad r = \frac{\delta - \lambda}{1 - \lambda \delta}. 
\]

Then

(i) \( B_\zeta^{-1}(\mathbb{D}_r) = \{ z \in \mathbb{C} : |B_\zeta(z)| < r \} \) is the union of pairwise disjoint domains \( V_{\zeta n}(\ni z_n) \) such that \( V_{\zeta n} \subset D(z_n, \lambda) \);

(ii) \( B_\zeta \) maps every \( V_{\zeta n} \) biholomorphically onto \( \mathbb{D}_r \);

(iii) Every sequence \( \omega = \{ w_n \} \) with \( w_n \in D(z_n, \lambda) \) for all \( n \), is interpolating for \( H^\infty \)

and

\[
\delta(\omega) \geq \frac{\delta - 2\lambda}{1 + \lambda}. 
\]

By \( b_{\zeta n} : \mathbb{D}_r \to V_{\zeta n} \) we denote the (holomorphic) inverse of \( B_\zeta|_{V_{\zeta n}} \). Since \( B_\zeta(z_n) = 0 \) and \( \| B_\zeta \|_\infty = 1 \), by the Schwarz-Pick theorem \( D(z_n, r) \subset b_{\zeta n}(\mathbb{D}_r) \subset V_{\zeta n}(\ni D(z_n, \lambda)) \).

Let \( M_\zeta \) be the constant of interpolation of \( \zeta \) and \( N \) be the set of indices of \( \{ z_n \} \).

**Proposition 2.4** ([2] Prop. 3.7). There exist functions \( f_j \in H^\infty(\mathbb{D} \times \mathbb{D} \ni M_\zeta) \), \( j \in N \), such that

\[
f_j(b_{\zeta j}(w), w) = 1, \quad f_j(b_{\zeta k}(w), w) = 0, \quad k \neq j,
\]

\[
\sum_{j \in N} |f_j(z, w)| \leq 2M_\zeta, \quad (z, w) \in \mathbb{D} \times \mathbb{D} \ni M_\zeta.
\]

**Proof.** Due to [6] Ch. VII, Th. 2.1 there exist functions \( g_j \in H^\infty, j \in N \), such that

\[
g_j(z_j) = 1, \quad g_j(z_k) = 0, \quad k \neq j, \quad \text{and} \quad \sum_{j \in N} |g_j(z)| \leq M_\zeta, \quad z \in \mathbb{D}.
\]

Consider a bounded linear operator \( L : \ell^\infty(N) \to H^\infty \) of norm \( \| L \| = M_\zeta \)

\[
L(a)(z) := \sum_{j \in N} a_j g_j(z), \quad a = \{ a_j \}_{j \in N}, \quad z \in \mathbb{D}.
\]

Let \( R(w) : H^\infty \to \ell^\infty(N) \) be the restriction operator to \( b(w) = \{ b_j(w) \}_{j \in N}, w \in \mathbb{D}_r \). Then

\[
(R(w) \circ L)(a) := \left\{ \sum_{j \in N} a_j g_j(b_k(w)) \right\}_{k \in N}.
\]
Lemma 2.6. Let \( \Gamma \) be bounded linear operators of norm \( \leq 1 \). Then the linear operator \( \hat{L}(w) := L \circ P(w)^{-1} \), \( w \in \mathbb{D}_r \). This and the Cauchy estimates for derivatives of bounded holomorphic functions imply that \( P(w) := R(w) \circ L : \ell^\infty(N) \to \ell^\infty(N) \), \( w \in \mathbb{D}_r \), is a family of bounded linear operators of norms \( \leq M_\zeta \) holomorphically depending on \( w \) and such that \( P(0) = \text{id} \). The Cauchy estimates yield \( \| \frac{dP}{dw}(w) \| \leq M_\zeta \frac{M_\zeta}{r-|w|} \). Thus for \( |w| \leq \frac{r}{2M_\zeta} \) we have

\[
\| P(w) \| - 1 := \| P(w) - P(0) \| \leq |w| \frac{M_\zeta}{r-|w|} \leq \frac{1}{2}
\]

In particular, the operator \( P(w) \) is invertible and \( \| P(w)^{-1} \| \leq 2 \).

Set

\[
\hat{L}(w) := L \circ P(w)^{-1}, \quad w \in \mathbb{D}_r.
\]

Then the linear operator \( \hat{L}(w) : \ell^\infty(N) \to \ell^\infty \) is continuous, holomorphically depends on \( w \in \mathbb{D}_r \) and \( \| \hat{L}(w) \| \leq 2M_\zeta \). Moreover, \( R(w) \circ \hat{L}(w) = \text{id} \).

Finally, define

\[
(2.10) \quad f_j(., w) := \hat{L}(w)(\delta_j), \quad j \in N;
\]

here \( \delta_j = \{ \delta_{ij} \}_{i \in N} \in \ell^\infty(N) \), \( \delta_{ij} = 1 \) if \( i = j \) and 0 otherwise. Clearly, functions \( f_j \in \ell^\infty(D \times \mathbb{D}_r) \), \( j \in N \), satisfy (2.8), (2.9).

Let us consider the open annulus

\[
(2.11) \quad A := \mathbb{D}_\frac{r}{2M_\zeta} \setminus \mathbb{D}_\frac{r}{M_\zeta}
\]

and set \( A_\zeta := B^{-1}_\zeta(A) \subset \mathbb{D} \). By the definition, see Lemma 2.3

\[
A_\zeta := \bigcup_n A_{\zeta n}, \quad \text{where} \quad A_{\zeta n} := V_{\zeta n} \cap A_\zeta.
\]

Moreover, \( B_\zeta \) maps every \( A_{\zeta n} \) biholomorphically onto \( A \). We also set

\[
S_{\zeta 1} := B_\zeta^{-1}(\mathbb{D} \setminus \mathbb{D}_r), \quad S_{\zeta 2} := B_\zeta^{-1}(\mathbb{D}_r).
\]

Let \( X \) be a complex Banach space. In what follows, for a complex manifold \( U \) we denote by \( H^\infty(U, X) \) the Banach space of bounded \( X \)-valued holomorphic functions \( f \) on \( U \) equipped with norm \( \| f \|_\infty := \sup_{z \in U} \| f(z) \|_X \).

**Theorem 2.5.** There exist bounded linear operators \( T_i : H^\infty(A_\zeta, X) \to H^\infty(S_{\zeta i}, X) \), \( i = 1, 2 \), of norms \( \| T_1 \| \leq 6M_\zeta, \| T_2 \| \leq 4M_\zeta \) such that

\[
(2.12) \quad (T_1 f)|_{A_\zeta} + (T_2 f)|_{A_\zeta} = f \quad \text{for all} \quad f \in H^\infty(A_\zeta, X).
\]

**Proof.** Let \( \Gamma_{B_\zeta}(A) := \{(z, w) \in A \times A : w = B_\zeta(z)\} \) be the graph of \( B_\zeta|_{A_\zeta} \), and let \( R_A : H^\infty(D \times A, X) \to H^\infty(\Gamma_{B_\zeta}(A), X), f \mapsto f|_{\Gamma_{B_\zeta}(A)} \), be the restriction operator.

**Lemma 2.6.** There exists a bounded linear operator \( S^X_A : H^\infty(A \times N, X) \to H^\infty(D \times A, X) \) of norm \( \leq 2M_\zeta \) such that

\[
(R_A \circ S^X_A)(g)(b_j(w), w) = g(w, j) \quad \text{for all} \quad (w, j) \in A \times N, \quad g \in H^\infty(A \times N, X).
\]
Proof. We define
\[ S^X_A(g)(z, w) := \sum_{j \in \mathbb{N}} f_j(z, w)g(w, j) \]
with \( f_j \) as in Proposition 2.4. Then the required result follows from (2.8) and (2.9). \( \square \)

Next, consider an isometric isomorphism \( I : H^\infty(A, X) \to H^\infty(A \times N, X) \),
\[ (If)(w, j) := (f \circ b_{\xi j})(w), \quad (w, j) \in A \times N, \quad f \in H^\infty(A, X). \]
Then the linear operator \( S^X_A \circ I : H^\infty(A, X) \to H^\infty(D \times A, X) \) of norm \( \leq 2M_\xi \) is given by
\[ ((S^X_A \circ I)f)(z, w) = \sum_{j \in \mathbb{N}} f_j(z, w)(f \circ b_{\xi j})(w), \quad (z, w) \in D \times A, \quad f \in H^\infty(A, X). \]
Let
\[ ((S^X_A \circ I)f)(z, w) = \sum_{n=-\infty}^{-1} a_n(z)w^n + \sum_{n=0}^{\infty} a_n(z)w^n =: g_1(z, w) + g_2(z, w), \]
(2.16)
\[ a_n(z) := \frac{1}{2\pi i} \oint_{|\xi| = \frac{r}{\pi \xi}} \frac{((S^X_A \circ I)f)(z, \xi)}{\xi^{n+1}} d\xi, \quad n \in \mathbb{Z}, \ z \in D, \]
be the Laurent series expansion of \( (S^X_A \circ I)f \) with respect to \( w \in A \).

By the Cauchy formula for a sufficiently small \( \varepsilon > 0 \) we get
\[ ((S^X_A \circ I)f)(z, w) \]
(2.17)
\[ = \frac{1}{2\pi i} \oint_{|\xi| = \frac{r}{\pi \xi}} \frac{((S^X_A \circ I)f)(z, \xi)}{w - \xi} d\xi + \frac{1}{2\pi i} \oint_{|\xi| = \frac{r}{\pi \xi} - \varepsilon} \frac{((S^X_A \circ I)f)(z, \xi)}{\xi - w} d\xi \]
\[ = g_1(z, w) + g_2(z, w), \quad z \in D, \quad \frac{r}{\pi \xi} + \varepsilon \leq |w| \leq \frac{r}{\pi \xi} - \varepsilon. \]
Formula (2.17) shows that \( g_2 \) extends to a function \( \tilde{g}_2 \in H^\infty(D \times D, X) \) of norm
\[ \|\tilde{g}_2\|_{\infty} \leq \|(S^X_A \circ I)f\|_{\infty} + \lim_{\varepsilon \to 0} \frac{\|(S^X_A \circ I)f\|_{\infty}}{\frac{r}{3M_\xi} - \frac{r}{6M_\xi} - 2\varepsilon} \cdot \left( \frac{r}{6M_\xi} + \varepsilon \right) \leq 4M_\xi \|f\|_{\infty}. \]
(2.18)
Similarly, \( g_1 \) extends to a function \( \tilde{g}_1 \in H^\infty(D \times C \setminus \overline{D}, X) \) having zero at \( \infty \) of norm
\[ \|\tilde{g}_1\|_{\infty} \leq \|(S^X_A \circ I)f\|_{\infty} + \|\tilde{g}_2\|_{\infty} \leq 6M_\xi \|f\|_{\infty}. \]
(2.19)
Finally, we set
\[ T_1f(z) := \tilde{g}_1(z, B_\xi(z)), \quad z \in S_{\xi 1}, \quad \text{and} \quad T_2f(z) := \tilde{g}_2(z, B_\xi(z)), \quad z \in S_{\xi 2}. \]
(2.20)
Then $T_i : H^\infty(A_\zeta, X) \to H^\infty(S_{\zeta i}, X), i = 1, 2$, are bounded linear operators of norms $\|T_1\| \leq 6M_\zeta, \|T_2\| \leq 4M_\zeta$ and due to Lemma 2.6 and (2.15), (2.16) one obtains for $z \in A_\zeta, j \in N$,

$$(T_1f)(z) + (T_2f)(z) = ((S \circ I)f)(z, B_\zeta(z)) = (I f)(B_\zeta(z), j) = (f \circ b_\zeta j)(B_\zeta(z)) = f(z).$$

This completes the proof of the theorem. □

**Remark 2.7.** Equation (2.13) shows that

$$T_1f = \sum_{n=-\infty}^{-1} a_n B_\zeta^n, \quad T_2f = \sum_{n=0}^{\infty} a_n B_\zeta^n, \quad \text{where}$$

(2.21)

$$a_n(z) := \frac{1}{2\pi i} \oint_{|\xi|=r \zeta} \frac{((S_A^X \circ I)f)(z, \xi)}{\xi^{n+1}} d\xi, \quad n \in \mathbb{Z}, \ z \in \mathbb{D}.$$  

Here all $a_n \in H^\infty(\mathbb{D}, X)$ and the series for $T_1f$ and $T_2f$ converge uniformly on subsets $B_\zeta^{-1}(U)$, where $U$ are compact subsets of $\mathbb{D} \setminus \mathbb{D}(r \zeta)$ and $\mathbb{D}(r \zeta)$, respectively.

Moreover, if the range $\text{im}(f)$ of $f$ is a relatively compact subset of $X$, then due to (2.15), $\text{im}((S_A^X \circ I)f) \subset 2M_\zeta \cdot \overline{c_{\mathbb{C}}}(\text{im}(f))$. In turn, due to (2.17) arguing as in (2.18), (2.19) we obtain

$$\text{im}(\tilde{g}_2) \subset \text{im}((S_A^X \circ I)f) + \frac{\rho}{3M_\zeta} \cdot \overline{c_{\mathbb{C}}}(\text{im}((S_A^X \circ I)f)) \subset 2 \cdot \overline{c_{\mathbb{C}}}(\text{im}((S_A^X \circ I)f)),$$

$$\text{im}(\tilde{g}_1) \subset \text{im}((S_A^X \circ I)f) + \text{im}(\tilde{g}_2) \subset 3 \cdot \overline{c_{\mathbb{C}}}(\text{im}((S_A^X \circ I)f)).$$

Hence, $\text{im}(T_i f) \subset 6 \cdot \overline{c_{\mathbb{C}}}(\text{im}((S_A^X \circ I)f), i = 1, 2$. These implications and the classical Mazur theorem show that $\text{im}(T_i f) \subset X, i = 1, 2$, are relatively compact.

Finally, note that $L_1T_i = T_iL_i, i = 1, 2$, for a bounded linear operator of complex Banach spaces $L : X \to Y$ by (2.13) and the uniqueness of Laurent series expansions.

### 3. Theorems 1.1 and 1.3 for Sets of Small Width

We retain notation of Lemma 2.3. In this section we prove Theorems 1.1 and 1.3 (see Theorem 3.3 and Corollary 3.4) for a Lebesgue measurable set $K \subseteq \overline{B_{\zeta}^{-1}(\mathbb{D}(r \zeta))}$, where $\zeta = \{z_n\}_{n \in N}$ is an $\varepsilon$-chain with respect to $\rho$ such that $\delta(\zeta) \geq \delta > 0$. In this case,

(3.1)

$$K = \bigcup_n K_n, \quad \text{where} \quad K_n := V_{\zeta n} \cap K, \ n \in N,$$

and, moreover, since $V_{\zeta n} \subset D(z_n, \lambda)$ (see Lemma 2.3), $b_{\zeta n}$ maps $\mathbb{D}(r)$ into $D(z_n, \lambda)$ for all $n \in N$. Then applying to $b_{\zeta n}$ the Schwarz-Pick theorem we obtain

(3.2)

$$D(z_n, \frac{\lambda}{6M_\zeta}) \subset b_{\zeta n}(\mathbb{D}(\frac{\lambda}{6M_\zeta})) \subset D(z_n, \frac{\lambda}{6M_\zeta}).$$
Therefore, since \( K_n \subset b_{\zeta_n}(\mathbb{D}, \overline{E}_\rho) \) and \( \frac{1}{b_{\zeta_n}} < \delta \) \( \frac{1}{b_{\zeta_n}} < \delta(\zeta) \) while \( \rho(z_i, z_j) > \delta(\zeta) \) for all \( i \neq j \) (if \( N \) contains at least two elements),

\[
K_n \subset \begin{cases} 
D(z_n, \epsilon) & \text{if } \epsilon \leq \frac{r}{6M_{\zeta}}, \\
D(z_n, \frac{\lambda}{6M_{\zeta}}) & \text{if } \epsilon > \frac{r}{6M_{\zeta}},
\end{cases} \quad n \in N.
\]

We set

\[
c(\epsilon) := \begin{cases} \epsilon & \text{if } \epsilon \leq \frac{r}{6M_{\zeta}}, \\
\frac{\lambda}{6M_{\zeta}} & \text{if } \epsilon > \frac{r}{6M_{\zeta}}.
\end{cases}
\]

Recall that \( C_\rho(S, X), S \subset \mathbb{D}, \) stands for the Banach space of bounded continuous functions \( f : S \to X \) uniformly continuous with respect to \( \rho \) equipped with norm \( \|f\|_\infty = \sup_{z \in S} \|f(z)\|_X \).

First, we prove the following result.

**Lemma 3.1.** There is a bounded linear operator \( E^X_K : L^\infty(K, X) \to C_\rho(B_{\zeta}^{-1}(\mathbb{D}_r), X) \) of norm

\[
\|E^X_K\| \leq \frac{2c(\epsilon)}{1 - c(\epsilon)^2}
\]

such that for every \( f \in L^\infty(K, X) \) the \( X \)-valued function \( E^X_K f \) is a weak solution of the equation

\[
\frac{\partial F}{\partial \bar{z}} = \frac{f(z)}{1 - |z|^2}, \quad z \in B_{\zeta}^{-1}(\mathbb{D}_r).
\]

**Proof.** Recall that \( \zeta = \{z_n\}_{n \in N} \) and \( D(z_n, r) \subset V_{\zeta_n} \subset D(z_n, \lambda) \), see Lemma 2.3. For \( f \in L^\infty(K, X) \), we set \( f_n := f|_{V_{\zeta_n}} \in L^\infty(K_n, X) \). Let \( g_n(w) := \frac{w + z_n}{1 + z_n w}, w \in \mathbb{D}, \) be the Möbius transformation of \( \mathbb{D} \) which maps \( \mathbb{D}_\lambda \) biholomorphically onto \( D(z_n, \lambda) \). Making the substitution \( z = g_n(w) \) we obtain

\[
\frac{f_n(z)}{1 - |z|^2} dz = \frac{(f_n \circ g_n)(w)}{1 - |g_n(w)|^2} g_n(w) d\bar{w} = \frac{1 + z_n w}{1 + z_n w} \cdot \frac{(f_n \circ g_n)(w)}{1 - |w|^2} d\bar{w} =: \tilde{f}_n(w) d\bar{w}.
\]

Here \( \tilde{f}_n \in L^\infty(K_n, X), K_n := g_n^{-1}(K_n) \subset \mathbb{D}_{\epsilon}(\mathbb{D}_\lambda) \) and

\[
\|\tilde{f}_n\|_\infty \leq \frac{\|f_n\|_\infty}{1 - c(\epsilon)^2}.
\]

Applying to \( \tilde{f}_n \) the linear operator \( E \) of Lemma 2.4 we obtain that \( E\tilde{f}_n \) is a weak solution of the equation \( \frac{\partial F}{\partial \bar{w}} = \tilde{f}_n(w), w \in \mathbb{D}, \) such that, see (2.3), (2.7),

\[
\|E\tilde{f}_n\|_\infty \leq 2c(\epsilon)\|\tilde{f}_n\|_\infty \leq \frac{2c(\epsilon)\|f_n\|_\infty}{1 - c(\epsilon)^2} \quad \text{and} \quad |E\tilde{f}_n|_\omega \leq 3\|\tilde{f}_n\|_\infty \leq \frac{3\|f_n\|_\infty}{1 - c(\epsilon)^2}.
\]

We define

\[
(E^X_K f)(z) := (E\tilde{f}_n)(g_n^{-1}(z)), \quad z \in V_{\zeta_n}, \quad n \in N.
\]

Then due to (3.6), \( E^X_K f \) is a weak solution of equation (3.5).
Let $K$ be an isomorphic isometry of the metric space $(\mathbb{D}, \rho)$, the latter implies (recall that $B_\zeta^{-1}(\mathbb{D}_r) = \bigcap_{n \in \mathbb{N}} V_{\zeta_n}$)

\begin{equation}
\sup_{z \in B_\zeta^{-1}(\mathbb{D}_r)} |(E_K^X f)(z)| \leq \frac{2c(\epsilon)\|f_n\|_\infty}{1 - c(\epsilon)^2},
\end{equation}

and given $\epsilon > 0$ there is $\delta_\epsilon \in (0, 1)$ such that for all $z_1, z_2 \in V_{\zeta_n}, n \in \mathbb{N}$, satisfying $\rho(z_1, z_2) < \delta_\epsilon$,

\begin{equation}
|(E_K^X f)(z_1) - (E_K^X f)(z_2)| < \epsilon.
\end{equation}

Moreover, according to Lemma 2.3 (see also the proof of [6, Ch. X, Lm. 1.4]) we have for $N$ containing at least two elements, $n_1, n_2 \in N$, $n_1 \neq n_2$,

\begin{equation}
\text{dist}_\rho(V_{\zeta_{n_1}}, V_{\zeta_{n_2}}) = \inf_{y_i \in V_{\zeta_{n_i}}, i = 1, 2} \rho(y_1, y_2) \geq \delta - \frac{2\lambda}{1 + \lambda^2} > 0.
\end{equation}

Then for $\tilde{\delta}_\epsilon := \min\{\delta_\epsilon, \delta - \frac{2\lambda}{1 + \lambda^2}\}$ we get from (3.9)–(3.11) for all $z_1, z_2 \in B_\zeta^{-1}(\mathbb{D}_r)$, $\rho(z_1, z_2) < \tilde{\delta}_\epsilon$,

\begin{equation}
|(E_K^X f)(z_1) - (E_K^X f)(z_2)| < \epsilon.
\end{equation}

This and (3.9) imply that $E_K^X f \in C_{\rho}(B_\zeta^{-1}(\mathbb{D}_r), X)$ and $\|E_K^X f\| \leq \frac{2c(\epsilon)}{1 - c(\epsilon)^2}$, as required. \qed

**Remark 3.2.** Since analogs of properties (i)–(iii) of the operator of Theorem 1.1 are valid for the operator $E$, see Remark 2.2, such properties are valid for the operator $E_K^X$ as well.

Suppose $K \subset B_\zeta^{-1}(\mathbb{D}, r_{M_\zeta})$ is a Lebesgue measurable quasi-interpolating set having an $\epsilon$-chain $\zeta = \{z_j\}_{j \in \mathbb{N}}$ with respect to $\rho$ of the characteristic $\delta(\zeta) \geq \delta > 0$.

**Theorem 3.3.** There is a bounded linear operator $L_K^X : L^\infty(K, X) \to C_{\rho}(\mathbb{D}, X)$ satisfying conditions (i)–(iii) of Theorem 1.1 of norm

$$\|L_K^X\| \leq \frac{12c(\epsilon)M_\zeta}{1 - c(\epsilon)^2}$$

such that for every $f \in L^\infty(K, X)$ the function $L_K^X f$ is a weak solution of equation (1.1).

**Proof.** Let $R_A : C_{\rho}(B_\zeta^{-1}(\mathbb{D}_r), X) \to C_{\rho}(A_\zeta, X), f \mapsto f|_{A_\zeta}, A_\zeta = B_\zeta^{-1}(A)$, see (2.11), be the restriction operator. The composite operator $R_A \circ E_K^X$ maps $L^\infty(A_\zeta, X)$ (because $K \subset B_\zeta^{-1}(\mathbb{D}, r_{M_\zeta})$) and therefore Theorem 2.5 can be applied.

We set

\begin{equation}
(L_K^X f)(z) := \begin{cases}
((E_K^X - T_2 \circ R_{A_\zeta} \circ E_K^X)f)(z) & \text{if} \quad z \in S_{\zeta_2} = B_\zeta^{-1}(\mathbb{D}, r_{M_\zeta}) \\
(T_1 \circ R_{A_\zeta} \circ E_K^X)f)(z) & \text{if} \quad z \in S_{\zeta_1} = B_\zeta^{-1}(\mathbb{D} \setminus \bar{\mathbb{D}}, r_{M_\zeta}).
\end{cases}
\end{equation}
Since $\mathbb{D} = S_{\zeta_1} \cup S_{\zeta_2}$, $A_\zeta = S_{\zeta_1} \cap S_{\zeta_2}$ and since due to Theorem 2.5
\[ ((E_K^X - T_2 \circ R_{A_\zeta} \circ E_K^X)f)(z) = ((T_1 \circ R_{A_\zeta} \circ E_K^X)f)(z), \quad z \in A_\zeta, \]
the operator $L_K^X$ is well defined. Since functions $(T_1 \circ R_{A_\zeta} \circ E_K^X)f$ are holomorphic, the function $L_K^Xf$ is a weak solution of (1.1) by Lemma 3.1. Moreover,
\[
\|L_K^Xf\|_\infty \leq \max\{\|E_K^X - T_2 \circ R_{A_\zeta} \circ E_K^X\|_\infty, \|T_1 \circ R_{A_\zeta} \circ E_K^X\|_\infty \}
\leq \max\{1 + \|T_2\| \cdot \|R_{A_\zeta}\|, \|T_1\| \cdot \|R_{A_\zeta}\|\} \cdot \|E_K^X\| \cdot \|f\|_\infty \leq 6M_\zeta \cdot \frac{2c(\varepsilon)}{1 - c(\varepsilon)^2} \cdot \|f\|_\infty.
\]
Hence, $\|L_K^Xf\| \leq \frac{12c(\varepsilon)M_\zeta}{1 - c(\varepsilon)^2}$, as required.

Next, let us prove that $L_K^Xf \in C_\rho(\mathbb{D}, X)$.

To this end, consider the open cover
\[
\mathbb{D} = B_\zeta^{-1}(\mathbb{D}, \frac{r}{2M_\zeta}) \cup B_\zeta^{-1}(\mathbb{D} \setminus \frac{r}{2M_\zeta}) =: W_1 \cup W_2.
\]

Since $E_K^Xf$ is holomorphic on $B_\zeta^{-1}(\mathbb{D}) \setminus B_\zeta^{-1}(\mathbb{D} \setminus \frac{r}{6M_\zeta}) \supset A_\zeta$, formula (3.12) shows that $(T_2 \circ R_{A_\zeta} \circ E_K^X)f$ extends to a bounded $X$-valued holomorphic function, say $g$, on $B_\zeta^{-1}(\mathbb{D})$. We define a holomorphic function $G \in H^\infty(\mathbb{D}_r \times N, X)$ by the formula (cf. (2.14))
\[ G(w, j) := (g \circ b_{\zeta j})(w), \quad (w, j) \in \mathbb{D}_r \times N. \]

Expanding $G$ in the Maclaurin series in $w \in \mathbb{D}_r$ and then substituting $w = B_\zeta(z)$ we obtain (as $b_{\zeta j}(B_\zeta(z)) = z, z \in V_{\zeta j}, j \in N$)
\[ (3.13) \quad g(z) = \sum_{i=1}^{\infty} c_i(z) B_\zeta^i(z), \quad z \in B_\zeta^{-1}(\mathbb{D}_r), \]
where $c_i \in H^\infty(B_\zeta^{-1}(\mathbb{D}_r), X)$ are locally constant functions and the series converges uniformly on subsets $B_\zeta^{-1}(\mathbb{D}_t), t < r$, and, in particular, on $W_1$. Due to inequality (3.11), all $c_i \in C_\rho(B_\zeta^{-1}(\mathbb{D}_r), X)$. Also, $B_\zeta \in C_\rho(\mathbb{D}, \mathbb{C})$. Hence, uniform convergence of series (3.13) on $W_1$ implies that $g|_{W_1} \in C_\rho(W_1, X)$. Thus, since $E_K^Xf \in C_\rho(B_\zeta^{-1}(\mathbb{D}_r), X)$ (see Lemma 3.1), $L_K^Xf|_{W_1} = E_K^Xf|_{W_1} + g|_{W_1} \in C_\rho(W_1, X)$ as well.

Further, equation (2.21) shows that $L_K^Xf|_{S_{\zeta 1}} := (T_1 \circ R_{A_\zeta} \circ E_K^X)f|_{S_{\zeta 1}}$ can be expanded in a series $\sum_{n=1}^{\infty} d_n B_\zeta^{-n}$ for some $d_n \in H^\infty(\mathbb{D}, X)$ converging uniformly on subsets $B_\zeta^{-1}(\mathbb{D} \setminus \frac{r}{t_\varepsilon})$, $t \in (\frac{r}{6M_\zeta}, 1)$, and, in particular, on $W_2$. Since, all $d_n \in C_\rho(\mathbb{D}, X)$ and $\frac{1}{B_\zeta} \in C_\rho(W_2, \mathbb{C})$, uniform convergence of the series implies that $L_K^Xf|_{W_2} = (T_1 \circ R_{A_\zeta} \circ E_K^X)f|_{W_2} \in C_\rho(W_2, X)$.

Now, every open disk $\mathbb{D}_{\frac{r}{12M_\zeta}}(w) \subset \mathbb{D}$ is a subset of either $\mathbb{D}_{\frac{r}{2M_\zeta}}$ or $\mathbb{D} \setminus \frac{r}{12M_\zeta}$. Moreover, if $z \in \mathbb{D}$ is such that $w = B_\zeta(z)$, then $B_\zeta(D(z, \frac{r}{12M_\zeta})) \subset \mathbb{D}_{\frac{r}{12M_\zeta}}(w)$ (due to the Schwarz–Pick theorem). Hence, every open pseudohyperbolic disk $D(z, \frac{r}{12M_\zeta})$ is a subset of either $W_1$ or
$W_2$ Since $L^X_K f$ is uniformly continuous with respect to $\rho$ on $W_i$, $i = 1, 2$, the latter implies that $L^X_K f \in C_\rho(\mathbb{D}, X)$, as stated.

Finally, statements (i)–(iii) of Theorem 1.1 follow from Remarks 2.2, 2.7 and 3.2. The proof of the theorem is complete. 

The next result describes the composition of the operator $L^X_K$ of Theorem 3.3 with the restriction operator to $B^{-1}_\zeta(\mathbb{D} \setminus \bar{\mathbb{D}}} \frac{r}{\|M\zeta\|})$ (see Theorem 1.3). Recall that $\mathcal{B}(K, X)$ stands for the Banach space of bounded linear operators from $L^\infty(K, X)$ in $H^\infty(D, X)$ equipped with the operator norm.

**Corollary 3.4.** There exists a bounded holomorphic function $H : \mathbb{C} \setminus \bar{\mathbb{D}}} \frac{r}{\|M\zeta\|} \to \mathcal{B}(K, X)$ vanishing at $\infty$ of norm

$$\|H\|_\infty \leq \frac{12c(\epsilon)M\zeta}{1 - c(\epsilon)^2}$$

such that

$$L^X_K (f)(z) = (H(B\zeta(z))f)(z), \quad z \in B^{-1}_\zeta(\mathbb{D} \setminus \bar{\mathbb{D}}} \frac{r}{\|M\zeta\|}), \quad f \in L^\infty(K, X).$$

**Proof.** By the definition, see (2.15), (3.12), $S^X_A \circ I \circ R_A \circ E^X_K$ is a bounded linear operator from $L^\infty(K, X)$ in $H^\infty(D \times \mathbb{C} \setminus \bar{\mathbb{D}}} \frac{r}{\|M\zeta\|}, X) \approx H^\infty(C \setminus \bar{\mathbb{D}}} \frac{r}{\|M\zeta\|}, H^\infty)$. We define for $f \in L^\infty(K, X), \quad w \in \mathbb{C} \setminus \bar{\mathbb{D}}} \frac{r}{\|M\zeta\|}, \quad z \in \mathbb{D},$

$$((H(w)f)(z) := \lim_{\epsilon \to 0^+} \frac{1}{2\pi i} \oint_{|\xi| = \frac{r}{\|M\zeta\|} + \epsilon} \frac{(S^X_A \circ I \circ R_A \circ E^X_K)f(z, \xi)}{w - \xi} d\xi.$$ 

Then using (2.19) and Lemma 3.1 we obtain

$$\sup_{w \in \mathbb{C} \setminus \bar{\mathbb{D}}} \frac{r}{\|M\zeta\|} \left( \sup_{z \in \mathbb{D}} |(H(w)f)(z)| \right) \leq \frac{12c(\epsilon)M\zeta}{1 - c(\epsilon)^2} \cdot \|f\|_\infty.$$ 

Equations (3.16), (3.17) show that $H \in H^\infty(C \setminus \bar{\mathbb{D}}} \frac{r}{\|M\zeta\|}, \mathcal{B}^X_K)$, vanishes at $\infty$ and has norm $\|H\|_\infty \leq \frac{12c(\epsilon)M\zeta}{1 - c(\epsilon)^2}$. Now, equation (3.15) follows from (2.20) and (3.12). 

4. **Theorems 1.1 and 1.3 for Sets with Chains of Large Characteristic**

4.1. Let $\zeta = \{z_j\}_{j \in \mathbb{N}} \subset K$ be an $\epsilon$-chain of a Lebesgue measurable set $K \subset \mathbb{D}$ with respect to $\rho$. In this and the next sections we prove Theorems 1.1 and 1.3 under the assumption

$$\delta(\zeta) \geq 1 - \frac{(1 - \sqrt{\epsilon_\ast})^2}{8};$$

here $\epsilon_\ast := \max\{\frac{1}{2}, \epsilon\}$.

Specifically, in this setting, first we prove
Theorem 4.1. There is a bounded linear operator $L^X_K : L^\infty(K, X) \to C_\rho(\mathbb{D}, X)$ satisfying conditions (i)–(iii) of Theorem 1.1 of norm
\[
\|L^X_K\| \leq c \cdot \frac{\epsilon}{1 - \epsilon}
\]
for some $c < 389423$ such that for every $f \in L^\infty(K, X)$ the function $L^X_K f$ is a weak solution of equation (1.1).

We require the following result.

Lemma 4.2. Assume that $\{z_1, \ldots, z_k\}$ is an $L$-chain with respect to $\rho$ in the open pseudo-hyperbolic disk $D(z, R) \subset \mathbb{D}, 0 < R < 1$. Then
\[
\frac{R^2}{1 - R^2} \cdot \frac{1 - L^2}{L^2} \leq k \leq \frac{(2R + L)^2}{1 - R^2} \cdot \frac{1}{L^2}.
\]

Proof. By the definition of an $L$-chain, the pseudo-hyperbolic disks $D(z_i, \frac{L}{2})$, $1 \leq i \leq k$, are pairwise disjoint so that
\[
\bigcup_{i=1}^k D(z_i, \frac{L}{2}) \subset D(z, R'),
\]
where $R' = R + \frac{L}{1 - \frac{L}{2}}$ (for the triangle inequality for $\rho$, see, e.g., [6 Lm. 1.4]).

Similarly,
\[
D(z, R) \subset \bigcup_{i=1}^k D(z_i, L).
\]

Next, the 2-form $\omega := \frac{i}{2} \frac{dz \wedge d\overline{z}}{1 - |z|^2} = \frac{dx \wedge dy}{(1 - (x^2 + y^2))^2}$ is invariant with respect to the action of the group of Möbius transformations of $\mathbb{D}$ and, hence, using the polar coordinates $z = re^{i\phi} \in \mathbb{D}$ we get for every $s \in (0, 1),$
\[
\int_{D(z, s)} \omega = \int_{D(z, s)} \frac{dx \wedge dy}{1 - (r^2)^2} = \frac{2\pi}{1} \frac{1}{1 - r^2} \int_{0}^{s} r \ dr = \frac{\pi s^2}{1 - s^2}.
\]

Using this formula, we obtain from (4.3), (4.4)
\[
\frac{k \pi \left(\frac{L}{2}\right)^2}{1 - \left(\frac{L}{2}\right)^2} \leq \frac{\pi (R')^2}{1 - (R')^2} = \frac{\pi (2R + L)^2}{4 - L^2(1 - R^2)} \quad \text{and} \quad \frac{\pi R^2}{1 - R^2} \leq k \frac{\pi L^2}{1 - L^2}.
\]

These imply inequalities (4.2). \qed

Proof of Theorem 4.1 We choose in Lemma 2.3
\[
\delta := 1 - \frac{(1 - \sqrt{\epsilon_\ast})^2}{8}, \quad \lambda := \sqrt{\epsilon_\ast} - \frac{1 - \epsilon_\ast}{4}.
\]

Then
\[
\frac{2\lambda}{1 + \lambda^2} = 1 - \frac{(1 - \lambda)^2}{1 + \lambda^2} < 1 - \frac{(1 - \sqrt{\epsilon_\ast})^2(1 + \frac{1 + \sqrt{\epsilon_\ast}}{4})^2}{1 + \epsilon_\ast} < 1 - \frac{25(1 - \sqrt{\epsilon_\ast})^2}{32} < \delta,
\]
as required.

Further, we have

**Lemma 4.3.**

\[ r := \frac{\delta - \lambda}{1 - \lambda \delta} \lambda > \varepsilon. \]

**Proof.** By the definition,

\[
1 - \lambda \delta = 1 - \left(1 - \frac{(1 - \sqrt{\varepsilon})^2}{8}\right) \cdot \left(\sqrt{\varepsilon} - \frac{1 - \varepsilon}{4}\right)
\]

\[
\leq (1 - \sqrt{\varepsilon}) + \frac{1 - \sqrt{\varepsilon}}{4} \left(1 + \sqrt{\varepsilon} + \frac{(1 - \sqrt{\varepsilon}) \sqrt{\varepsilon}}{2}\right) = (1 - \sqrt{\varepsilon}) \left(1 + \frac{2 + 3\sqrt{\varepsilon} - \varepsilon}{8}\right)
\]

\[
< \frac{3(1 - \sqrt{\varepsilon})}{2}.
\]

Also,

\[
\delta - \lambda = (1 - \sqrt{\varepsilon}) + \frac{1 - \sqrt{\varepsilon}}{4} \left(1 + \sqrt{\varepsilon} - \frac{1 - \sqrt{\varepsilon}}{2}\right) = (1 - \sqrt{\varepsilon}) \left(\frac{9 + 3\sqrt{\varepsilon}}{8}\right).
\]

Hence,

\[
r - \varepsilon = \frac{\delta - \lambda}{1 - \lambda \delta} \lambda - \varepsilon > \frac{2}{3(1 - \sqrt{\varepsilon})} \cdot \frac{(9 + 3\sqrt{\varepsilon})(1 - \sqrt{\varepsilon})}{8} \cdot \left(\sqrt{\varepsilon} - \frac{1 - \varepsilon}{4}\right) - \varepsilon
\]

\[
= \frac{3 + \sqrt{\varepsilon}}{4} \left(\sqrt{\varepsilon} - \frac{1 - \varepsilon}{4}\right) - \varepsilon = \frac{1 - \sqrt{\varepsilon}}{4} \left(3\sqrt{\varepsilon} - \frac{3 + \sqrt{\varepsilon}}{4}(1 + \sqrt{\varepsilon})\right)
\]

\[
= \frac{1 - \sqrt{\varepsilon}}{16} (8\sqrt{\varepsilon} - \varepsilon - 3) \geq \frac{1 - \sqrt{\varepsilon}}{16} \left(8 \frac{1}{2} - \frac{1}{2} - 3\right) > 0
\]

because the function \( t \mapsto (-t^2 + 8t - 3) \) is increasing for \( t \in \left[\frac{1}{\sqrt{2}}, 1\right) \).

Next, we prove the following result.

**Lemma 4.4.**

\[ \delta_m := \frac{\delta - \frac{2\lambda}{1 + \lambda}}{1 - \frac{2\lambda}{1 + \lambda}} > \frac{1}{2}. \]

**Proof.** Using the implication

\[ r = \frac{\delta - \lambda}{1 - \lambda \delta} \lambda \implies \delta = \frac{r}{1 + r} \]

we obtain by Lemma 4.3

\[
\delta_m = \frac{\delta - \frac{2\lambda}{1 + \lambda}}{1 - \frac{2\lambda}{1 + \lambda}} = \frac{\frac{r}{1 + r} - \lambda}{1 - \frac{2\lambda}{1 + \lambda}} > \frac{\frac{r}{1 + r} \left(1 + \frac{1 - \varepsilon}{\sqrt{\varepsilon}}\right) - \left(\sqrt{\varepsilon} - \frac{1 - \varepsilon}{4}\right)}{1 - \varepsilon} = \frac{1}{2}.
\]
From the previous estimates as a corollary of Lemma 2.3 we obtain:

**Lemma 4.5.** Let $B_\zeta$ be the interpolating Blaschke product with zeros $\zeta = \{z_n\}_{n \in \mathbb{N}}$. Then for $\lambda$, $\delta$ and $r$ as in (4.5), (4.6)

(i) $B_\zeta^{-1}(D_\epsilon) = \{z \in \mathbb{C} : |B_\zeta(z)| < r\}$ is the union of pairwise disjoint domains $V_{\zeta_n}(\exists z_n)$ such that

\[ B_\zeta^{-1}(D_\epsilon) \subset V_{\zeta_n} \subset D(z_n, \lambda); \]

(ii) $B_\zeta$ maps every $V_{\zeta_n}$ biholomorphically onto $\mathbb{D}_r$;

(iii) Every sequence $\omega = \{w_n\}_{n \in \mathbb{N}}$ with $w_n \in D(z_n, \lambda)$ for all $n$, is interpolating for $H^\infty$ and

\[ \delta(\omega) \geq \delta_m > \frac{1}{2}. \]

We set for $0 < \nu \leq 2 - \sqrt{3}$,

\[ \epsilon_\nu := \frac{(2 - \sqrt{3})^3 \cdot \nu}{6} \]

and define a sequence $\zeta_\nu \subset K$ as follows:

\[ \zeta_\nu := \begin{cases} 
\zeta & \text{if } \epsilon \leq \epsilon_\nu \\
\text{an } \epsilon_\nu - \text{chain of } K & \text{if } \epsilon > \epsilon_\nu.
\end{cases} \]

Then, according to Lemma 4.2 for every $z \in \zeta$ the (nonempty) set $\zeta_\nu \cap D(z, \epsilon)$ is finite of cardinality

\[ k_\zeta = 1 \text{ if } \epsilon \leq \epsilon_\nu \quad \text{and} \quad k_\zeta \leq \frac{(2\epsilon_\nu)^2}{\epsilon_\nu^2(1 - \epsilon^2)} =: k_\nu \quad \text{if } \epsilon > \epsilon_\nu. \]

Let

\[ k^*_\nu := \max_{z \in \zeta} k_\zeta. \]

Equation (4.10) easily implies that $\zeta_\nu$ can be presented as the disjoint union of subsequences $\zeta^i_\nu$, $1 \leq i \leq k^*_\nu$, such that for every $z \in \zeta$ the set $\zeta^i_\nu \cap D(z, \epsilon)$ is either empty or consists of a single element.

Given $1 \leq i \leq k^*_\nu$, let $\zeta_i$ be the subset of $\zeta$ such that for every $z \in \zeta_i$ the set $\zeta^i_\nu \cap D(z, \epsilon)$ consists of a single element. Since $\delta(\zeta_i) \geq \delta(\zeta) \geq \delta$, Lemma 4.5(iii) implies that

\[ \delta(\zeta^i_\nu) > \frac{1}{2} \text{ for all } 1 \leq i \leq k^*_\nu. \]

Next, we require

**Lemma 4.6.** Given $1 \leq i \leq k^*_\nu$ there exists a $\lambda \in (0, \nu)$ in Lemma 2.3 such that for the corresponding $r := r(\lambda)$,

\[ \frac{r}{6M_{\zeta_\nu}} = \epsilon_\nu. \]
Proof. By the definition,

\begin{equation}
\tag{4.14}
r(\lambda) := \frac{\delta(\zeta^i_{\nu}) - \lambda}{1 - \lambda \delta(\zeta^i_{\nu})} \cdot \lambda, \quad \text{where} \quad \frac{2\lambda}{1 + \lambda^2} < \delta(\zeta^i_{\nu}).
\end{equation}

The latter inequality implies that

\[ \lambda < \frac{1}{\delta(\zeta^i_{\nu})} - \sqrt{\left(\frac{1}{\delta(\zeta^i_{\nu})}\right)^2 - 1} =: \lambda^i_{\nu}. \]

Using (4.12) we obtain

\[ \lambda^i_{\nu} > \frac{1}{2} - \sqrt{\left(\frac{1}{2}\right)^2 - 1} = 2 - \sqrt{3} \geq \nu. \]

This implies that \( r \) is a continuous function in \( \lambda \in [0, \nu] \).

Also, \( r(0) = 0 \) and

\begin{equation}
\tag{4.15}
r(\nu) = \frac{\delta(\zeta^i_{\nu}) - \nu}{1 - \nu \delta(\zeta^i_{\nu})} \cdot \nu > \frac{1}{2} - \frac{(2 - \sqrt{3})}{1 - (2 - \sqrt{3}) \cdot \frac{1}{2}} \cdot \nu = (2 - \sqrt{3}) \cdot \nu.
\end{equation}

Moreover, according to (1.5)

\begin{equation}
\tag{4.16}
M_{\zeta^i_{\nu}} \leq \left(\frac{1 + \sqrt{1 - (\delta(\zeta^i_{\nu}))^2}}{\delta(\zeta^i_{\nu})}\right)^2 < \left(\frac{1 + \sqrt{1 - (1/2)^2}}{1/2}\right)^2 = (2 + \sqrt{3})^2.
\end{equation}

Hence,

\begin{equation}
\tag{4.17}
\frac{r(\nu)}{6M_{\zeta^i_{\nu}}} > \frac{(2 - \sqrt{3}) \cdot \nu}{6(2 + \sqrt{3})^2} = \frac{(2 - \sqrt{3})^3 \cdot \nu}{6} = \epsilon_{\nu}.
\end{equation}

Thus by the intermediate value theorem applied to the continuous function \( \frac{r(\lambda)}{6M_{\zeta^i_{\nu}}}, \lambda \in [0, \nu] \), we obtain that there is some \( \lambda \in (0, \nu) \) such that for the corresponding \( r := r(\lambda) \),

\[ \frac{r}{6M_{\zeta^i_{\nu}}} = \epsilon_{\nu}, \]

as required. \( \square \)

We apply Theorem 3.3 and Corollary 3.4 with the \( r \) of Lemma 4.6. Then we obtain for

\begin{equation}
\tag{4.18}
K^i_{\nu} := K \cap \bigcup_{z \in \zeta^i_{\nu}} D(z, \epsilon_{\nu}) \subset B^{-1}_{\zeta^i_{\nu}}(\epsilon_{\nu})
\end{equation}

(here the implication in the brackets is due to the Schwarz-Pick theorem):
There is a bounded linear operator $L_{K^i}^X : L^\infty(K^i, X) \to C_\rho(\mathbb{D}, X)$ satisfying conditions (i)–(iii) of Theorem 1.1 of norm (4.19)

$$\|L_{K^i}^X\| \leq \frac{12 \epsilon \nu M_{\zeta^i}}{1 - \epsilon^2 \nu^2}$$

such that for every $f \in L^\infty(K^i, X)$ the function $L_{K^i}^X f$ is a weak solution of equation (1.1).

There exists a holomorphic function $h^\nu_{\zeta^i} \in H^\infty(\mathbb{C} \setminus \mathbb{D}_\epsilon, X_{K^i})$ vanishing at $\infty$ of norm (4.20)

$$\|h^\nu_{\zeta^i}\|_\infty \leq \frac{12 \epsilon \nu M_{\zeta^i}}{1 - \epsilon^2 \nu^2}$$

such that

$$(L_{K^i}^X f)(z) = (h^\nu_{\zeta^i}(B_{\zeta^i}(z)) f)(z), \quad z \in B_{\zeta^i}^{-1}(\mathbb{D} \setminus \mathbb{D}_\epsilon), \quad f \in L^\infty(K^i, X).$$

Further, we use that

$$K = \bigcup_{0 \leq i \leq k^*_\nu} K^i_{\nu}, \quad \text{where} \quad K^0_{\nu} := \emptyset.$$

Let $\chi^\nu_j \in L^\infty(\mathbb{D})$ be the equivalence class of the characteristic function of the set $S^j_{\nu} := K^j_{\nu} \setminus \bigcup_{i=1}^{j-1} K^i_{\nu}, \quad 1 \leq j \leq k^*_\nu$ (here each $S^j_{\nu} \neq \emptyset$ by the definition of a pseudohyperbolic chain of $K$). Sets $S^j_{\nu}$ are Lebesgue measurable, mutually disjoint and cover $K$; hence, $\sum_{j=1}^{k^*_\nu} \chi^\nu_j = 1.$

Using the bounded linear projections

$$M_{\chi^\nu_j} : L^\infty(K, X) \to L^\infty(K^j_{\nu}, X), \quad M_{\chi^\nu_j} f := \chi^\nu_j \cdot f, \quad 1 \leq j \leq k^*_\nu,$$

and operators $L_{K^j_{\nu}}^X : L^\infty(K^j_{\nu}, X) \to C_\rho(\mathbb{D}, X), \quad 1 \leq j \leq k^*_\nu,$ of statement (*) we define

$$(4.22) \quad L_{K^j_{\nu}}^X := \sum_{j=1}^{k^*_\nu} L_{K^j_{\nu}}^X \circ M_{\chi^\nu_j}.$$

By the definition, $L_{K^j_{\nu}}^X : L^\infty(K, X) \to C_\rho(\mathbb{D}, X)$ is a bounded linear operator satisfying the statement of Theorem 4.1. To get the required upper bound of the operator norm we define

$$(4.23) \quad L_K^X := L_{K^\tilde{\nu}}^X, \quad \tilde{\nu} := 2 - \sqrt{3}.$$
Then using equations (4.8)–(4.10) and (4.16) we obtain
\[
\|L^X_{K_i}\| \leq \sum_{j=1}^{k^*_\nu} \|L^X_{K_i} \circ M_{\nu}^j\| \\
\leq \begin{cases} \\
\frac{12\epsilon M_{\nu}}{1 - \epsilon^2} \leq \frac{12\epsilon(2 + \sqrt{3})^2}{1 - \epsilon^2} \leq \frac{167\epsilon}{1 - \epsilon} & \text{if } \epsilon \leq \epsilon_{\nu} \\
 \frac{2(2 - \sqrt{3})^2}{1 - \epsilon_{\nu}^2} \leq \frac{(2 + \epsilon_{\nu})^2(2 - \sqrt{3})^2}{\epsilon_{\nu}^2(1 - \epsilon_{\nu}^2)} \cdot \frac{\epsilon}{1 - \epsilon} < \frac{389423\epsilon}{1 - \epsilon} & \text{if } \epsilon > \epsilon_{\nu}.
\end{cases}
\]
This completes the proof of the theorem. \qed

4.2. In this section, we prove Theorem 1.3 under assumption (4.1). We retain notation of the previous section, see (4.8), (4.11), (4.12), (4.20).

**Lemma 4.7.** The following hold:
\[
\begin{align*}
(4.24) & \quad k^*_\nu \leq \frac{c}{\nu^2(1 - \epsilon)} \text{ for some } c < 194712; \\
(4.25) & \quad K \subset \bigcup_{i=1}^{k^*_\nu} B^{-1}_{\nu}(\mathbb{D}_{\epsilon_{\nu}}) \subset \bigcup_{i=1}^{k^*_\nu} B^{-1}_{\nu}(\mathbb{D}_{6\epsilon_{\nu}}) \subset [K]_{\nu}; \\
(4.26) & \quad \|h^i_{\nu}\|_\infty \leq \frac{3}{5} \cdot \nu, \quad 1 \leq i \leq k^*_\nu.
\end{align*}
\]

**Proof.** By the definition, see (4.8), (4.10),
\[
k^*_\nu \leq \frac{(2\epsilon + \epsilon_{\nu})^2}{\epsilon_{\nu}^2(1 - \epsilon^2)} = \frac{(6(2 + \sqrt{3})^3(2\epsilon + \epsilon_{\nu})^2}{\nu^2(1 - \epsilon)(1 + \epsilon)} \leq \frac{(6(2 + \sqrt{3})^3(2 + \frac{(2 - \sqrt{3})^2}{6})^2}{\nu^2(1 - \epsilon)^2 \cdot 2} \leq \frac{194712}{\nu^2(1 - \epsilon)},
\]
as stated.

Further, according to the Schwarz-Pick theorem and Lemma 2.3(i) with \(\lambda\) and \(r\) as in Lemma 4.6, given \(1 \leq i \leq k^*_\nu\) we have
\[
K^i_{\nu} \subset [\nu]_{\nu_{\nu}} \subset B_{\nu_{\nu}}^{-1}(\mathbb{D}_{\epsilon_{\nu}}) \subset B_{\nu_{\nu}}^{-1}(\mathbb{D}_{6\epsilon_{\nu}}) \subset [\nu]_{\nu_{\nu}} \subset [K]_{\nu}.
\]
This and (4.18) imply (4.25):
\[
K := \bigcup_{i=1}^{k^*_\nu} K^i_{\nu} \subset \bigcup_{i=1}^{k^*_\nu} B_{\nu_{\nu}}^{-1}(\mathbb{D}_{\epsilon_{\nu}}) \subset \bigcup_{i=1}^{k^*_\nu} B_{\nu_{\nu}}^{-1}(\mathbb{D}_{6\epsilon_{\nu}}) \subset [K]_{\nu}.
\]

Finally, due to (4.8), (4.16), (4.20),
\[
\|h^i_{\nu}\|_\infty \leq \frac{12\epsilon_{\nu} M_{\nu}}{1 - \epsilon_{\nu}^2} \leq \frac{2 \cdot (2 - \sqrt{3}) \cdot \nu}{1 - \frac{(2 - \sqrt{3})^2}{36}} \leq \frac{3}{5} \cdot \nu.
\]

The proof of the lemma is complete. \qed
Let us define holomorphic functions \( H_i^\nu \in H^\infty(\mathbb{C} \setminus \overline{D}_\epsilon^\nu, \mathcal{B}_K^X) \) vanishing at \( \infty \) by the formulas
\[
H_i^\nu(w) := h_i^\nu(w) \circ M_{\chi_j}, \quad w \in \mathbb{C} \setminus \overline{D}_\epsilon^\nu, \quad 1 \leq i \leq k^*_\nu.
\]
Then (4.26) implies that
\[
\|H_i^\nu\|_\infty \leq \frac{3}{5} \cdot \nu, \quad 1 \leq i \leq k^*_\nu.
\]
Now, the required version of Theorem 1.3 reads as follows:

**Theorem 4.8.** Under assumption (4.1) and in notation of Lemma 4.7 there exists an operator \( E_0^\nu \in \mathcal{B}(K, X) \) such that for every \( f \in L^\infty(K, X), \ z \in \mathbb{D} \setminus K_\nu^\nu \)
\[
(L_K^X f)(z) = (E_0^\nu f)(z) + \sum_{i=1}^{k^*_\nu} (H_i^\nu(B_{\zeta_i}(z)) f)(z).
\]
Here \( k^*_\nu, B_{\zeta_i} \) and \( H_i^\nu \) satisfy (4.24), (4.25), (4.27), (4.28).

**Proof of Theorem 4.8.** We define, see (4.22), (4.23),
\[
E_0^\nu := L_K^X - L_{K,\nu}^X.
\]
Then \( E_0^\nu \) is a bounded linear operator from \( L^\infty(K, X) \) in \( H^\infty(\mathbb{D}, X) \).

Next, due to (4.25),
\[
\mathbb{D} \setminus K_\nu^\nu \subset \bigcap_{i=1}^{k^*_\nu} B_{\zeta_i}^{-1}(\mathbb{D} \setminus D_{\epsilon^\nu}) \subset \bigcap_{i=1}^{k^*_\nu} B_{\zeta_i}^{-1}(\mathbb{D} \setminus \mathbb{D}_{\epsilon^\nu}).
\]
Hence, all functions \( h_i^\nu \circ B_{\zeta_i} \) are defined on \( \mathbb{D} \setminus K_\nu^\nu \). Therefore statement (***) and formulas (4.24), (4.27) imply that
\[
(L_K^X f)(z) = (E_0^\nu f)(z) + \sum_{i=1}^{k^*_\nu} ((h_i^\nu(B_{\zeta_i}(z)) \circ M_{\chi_j}) f)(z), \quad z \in \mathbb{D} \setminus K_\nu^\nu, \quad f \in L^\infty(K, X).
\]
This completes the proof of the theorem. \( \square \)

5. Proofs of Theorems 1.1 and 1.3

In this section, we prove Theorems 1.1 and 1.3 in the general case.

**Proof of Theorem 1.1.** We use the following result.

**Lemma 5.1.** ([6] Ch. X, Cor. 1.6). Every interpolating sequence \( \zeta \) containing at least two elements can be presented as the disjoint union of interpolating sequences \( \zeta_1 \) and \( \zeta_2 \) such that
\[
\delta(\zeta_j) \geq \sqrt{\delta(\zeta)}, \quad j = 1, 2.
\]
For $\delta$ and $\epsilon$ as in Theorem 1.1 let $l \in \mathbb{N}$ be such that
\[
\log \delta \log (1 - (1 - \sqrt{\epsilon}^2) \leq 2^l < \frac{2 \log \delta}{\log (1 - (1 - \sqrt{\epsilon})^2)}
\]
if $\delta \leq 1 - (1 - \sqrt{\epsilon})^2$ and $l = 0$ otherwise.

Then
\[
\delta_l := \delta^{\frac{1}{2^l}} \geq 1 - (1 - \sqrt{\epsilon})^2.
\]

If $\zeta \subset K$, $\delta(\zeta) \geq \delta$, is an $\epsilon$-chain of $K$ with respect to $\rho$ of Theorem 1.1, then due to Lemma 5.1 we can present $\zeta$ as the disjoint union of $s \leq 2^l$ subsets $\zeta_j$, $1 \leq j \leq s$, such that $\delta(\zeta_j) \geq \delta_l$ for all $j$. Let
\[
\zeta_j := \frac{K \cap \bigcup_{z \in \zeta_j} D(z, \epsilon)}{\chi_j}, \quad 1 \leq j \leq s.
\]

Then $\zeta_j$ is an $\epsilon$-chain of $K_j$ and
\[
K = \bigcup_{j=0}^{s} K_j, \quad K_0 := \emptyset.
\]

Let $\chi_j \in L^\infty(\mathbb{D})$ be the equivalence class of the characteristic function of the set $R_j := K_j \setminus \bigcup_{i=1}^{j-1} K_i$, $1 \leq j \leq s$ (here each $R_j \neq \emptyset$ by the definition of an $\epsilon$-chain). Since sets $R_j$ are Lebesgue measurable, mutually disjoint and cover $K$,
\[
\sum_{j=1}^{s} \chi_j = 1.
\]

Applying Theorem 4.1 to $K_j$ we define
\[
L^X_K := \sum_{j=1}^{s} L^X_{K_j} \circ M_{\chi_j},
\]
where
\[
M_{\chi_j} : L^\infty(K, X) \to L^\infty(K_j, X), \quad M_{\chi_j} f := \chi_j \cdot f, \quad 1 \leq j \leq s,
\]
is a bounded linear projection.

By the definition, $L^X_K : L^\infty(K, X) \to C_\rho(\mathbb{D}, X)$ is a linear operator satisfying the statement of Theorem 1.1 of norm, see Theorem 4.1 and (5.1),
\[
\|L^X_K\| \leq \sum_{j=1}^{s} \|L^X_{K_j}\| \leq 389423 \epsilon \cdot \max \left\{ 1, \frac{2 \log \frac{1}{\delta}}{\log \frac{1 - (1 - \sqrt{\epsilon})^2}{8}} \right\} \leq 389423 \epsilon \cdot \max \left\{ 1, \frac{64 \log \frac{1}{\delta}}{1 - \epsilon} \right\}
\]
\[
\leq 24923072 \epsilon \cdot \max \left\{ 1, \frac{\log \frac{1}{\delta}}{1 - \epsilon} \right\}.\]
We used here the inequalities \( \log(1 + t) \leq t \) for \( t \geq 0 \) and \( \max\{1, ab\} \leq a \cdot \max\{1, b\} \) for \( a \geq 1, b \geq 0 \).

The proof of the theorem is complete. \( \square \)

**Proof of Theorem 1.3.** The proof goes in the same way as that of Theorem 4.8. Specifically, if \( H^i_{\nu,j}, B^i_{\nu,j}, 1 \leq i \leq k^s_{\nu,j} \), are the corresponding objects of Theorem 4.8 for \( K_j \), \( 1 \leq j \leq s \), then the holomorphic functions of Theorem 1.3 are defined as follows, see (5.4),

\[
H^j_{\nu}(w) := H^i_{\nu,j}(w) \circ M_{\chi_j} \quad w \in \mathbb{C} \setminus \mathbb{D}_{\epsilon_\nu}, \quad 1 \leq i \leq k^s_{\nu,j}, \quad 1 \leq j \leq s.
\]

By the definition, see (4.21), (5.1), the number \( k^s_{\nu} \) of such functions is

\[
\sum_{j=1}^{s} k^s_{\nu,j} \leq \frac{194712}{\nu^2(1 - \epsilon)} \cdot \max \left\{ 1, \frac{2 \log \frac{1}{\nu}}{\log \frac{1}{(1 - \epsilon) - \nu^2(1 - \epsilon)}} \right\} \leq \frac{1246158}{\nu^2(1 - \epsilon)} \cdot \max \left\{ 1, \frac{\log \frac{1}{\nu}}{(1 - \epsilon)^2} \right\}.
\]

Next, if \( E^0_{\nu,j} \in \mathcal{B}_K^X \) are operators (4.30) for \( K = K_j, 1 \leq j \leq s \), we define

\[
E^0_{\nu} := \sum_{j=1}^{s} E^0_{\nu,j} \circ M_{\chi_j} \in \mathcal{B}_K^X.
\]

This, (5.4) and Theorem 4.8 imply (1.12), i.e., for every \( f \in L^\infty(K, X), z \in \mathbb{D} \setminus [K]_{\nu}, \)

\[
(L^X_K f)(z) = (E^0_{\nu} f)(z) + \sum_{j=1}^{s} \sum_{i=1}^{k^s_{\nu,j}} (H^i_{\nu,j}(B^i_{\nu,j}(z)) f)(z).
\]

The proof of Theorem 1.3 is complete. \( \square \)

6. **Proof of Corollary 1.5**

Let \( \mathcal{M} \) be the maximal ideal space of \( H^\infty \) equipped with the Gelfand topology, the induced weak* topology on \( \mathcal{M} \) (see Remark 1.7). Then \( \mathcal{M} = \mathcal{M}_a \cup \mathcal{M}_a \), where \( \mathcal{M}_a \) and \( \mathcal{M}_a \) are sets of nontrivial (maximal analytic disks) and one-pointed Gleason parts for \( H^\infty \).

The set \( \mathcal{M}_a \) is open and dense in \( \mathcal{M} \) and is the union of closures in \( \mathcal{M} \) of all interpolating sequences for \( H^\infty \), see, e.g., \cite{6} Ch. X]. Using the definition of a Gleason part one easily shows that a subset of \( \mathbb{D} \) is quasi-interpolating if and only if its closure in \( \mathcal{M} \) lies in \( \mathcal{M}_a \).

In what follows, we naturally identify \( \mathbb{D} \) with an open dense subset of \( \mathcal{M}_a \) formed by evaluation homomorphisms at points of \( \mathbb{D} \).

Let \( K \) be a Lebesgue measurable quasi-interpolating set. Given \( \nu \in (0, 1) \) consider the \( \nu \)-hyperbolic neighbourhood \( [K]_\nu \) of \( K \), see (1.7). Clearly, \( [K]_\nu \) is a quasi-interpolating subset of \( \mathbb{D} \) as well; thus, the closure \( \overline{\text{cl}}([K]_\nu) \) of \( [K]_\nu \) in \( \mathcal{M} \) is a compact subset of \( \mathcal{M}_a \).

Let \( g \in \text{range}(L_K) \subset C_\nu(\mathbb{D}) \), see Theorem 1.3. Then due to \cite{11} Thm. 1.2 (c) (see also \cite{8} Thm. 2.1), \( g \) admits a continuous extension \( G \) to \( \mathcal{M}_a \).

Next, due to Theorem 1.3 (see (1.13)), the restriction \( g' := g|_U, U := \mathbb{D} \setminus [K]_\nu \), is the limit of a uniformly convergent on \( U \) sequence of meromorphic functions \( \{f_n\}_{n \in \mathbb{N}} \) such that all \( f_n, h_n \in H^\infty \) and all \( \frac{1}{|h_n|} \) are bounded from above on \( U \). In particular, functions
are continuously extended via the Gelfand transform \( \hat{\cdot} \) (see Remark 1.6) to the open set \( M \setminus \text{cl}([K]_\nu) \subset M \). By the Carleson corona theorem, \( U \) is an open dense subset of \( M \setminus \text{cl}([K]_\nu) \); hence, the extended sequence \( \{ \frac{\hat{g}_n}{\hat{a}_n} \}_{n \in \mathbb{N}} \) converges uniformly on \( M \setminus \text{cl}([K]_\nu) \) to a continuous function \( G' \) which extends \( g' \). Since \( \text{cl}([K]_\nu) \subset M_a \), the union of domains of \( G \) and \( G' \) is

\[
M_a \cup (M \setminus \text{cl}([K]_\nu)) = M.
\]

In turn, by the Carleson corona theorem, \( U \) is dense in the open set \( M_a \setminus \text{cl}([K]_\nu) \), the intersection of domains of \( G \) and \( G' \). Hence, \( G \) and \( G' \) coincide on \( M_a \setminus \text{cl}([K]_\nu) \) (as they are continuous extensions of the same function) and the formula

\[
\hat{G} := \begin{cases} 
G & \text{on } M_a \\
G' & \text{on } M \setminus \text{cl}([K]_\nu)
\end{cases}
\]

determines a continuous function on \( \hat{M} \) which extends \( g \).

Applying the Stone-Weierstrass theorem, we obtain that \( C(\hat{M}) \) coincides with the algebra \( \mathcal{A} := \{ f \in C(\hat{M}) : f \in A \} \). This implies that \( g = \hat{G}|_D \in \mathcal{A} \), as stated.

This completes the proof of the corollary.

7. Banach-valued Corona Problem for \( H^\infty(\mathbb{D}, A) \)

In this section we describe an application of the nonquantitative version of Theorem 1.1 to the Banach-valued corona problem for \( H^\infty \) presented in [3].

Let \( A \) be a uniform algebra defined on the maximal ideal space \( \mathcal{M}(A) \) and \( H^\infty(\mathbb{D}, A) \) be the Banach algebra of bounded \( A \)-valued holomorphic functions on \( \mathbb{D} \). There is a continuous embedding \( \iota \) of \( \mathbb{D} \times \mathcal{M}(A) \) into the maximal ideal space \( \mathcal{M}(H^\infty(\mathbb{D}, A)) \) taking \((z, x) \in \mathbb{D} \times \mathcal{M}(A)\) to the evaluation homomorphism \( f \mapsto (f(z))(x), f \in H^\infty(\mathbb{D}, A) \). The complement of the closure of \( \iota(X) \) in \( \mathcal{M}(A) \) is called the corona. The corona problem asks whether the corona is empty. The problem can be equivalently reformulated as follows, see, e.g., [3] Ch. V, Thm. 1.8:

A collection \( f_1, \ldots, f_n \in H^\infty(\mathbb{D}, A) \) satisfies the corona condition if

\[
1 \geq \max_{1 \leq j \leq n} |(f_j(z))(x)| \geq \delta > 0 \quad \text{for all } (z, x) \in \mathbb{D} \times \mathcal{M}(A).
\]

The corona problem being solvable (i.e., the corona is empty) means that for all \( n \in \mathbb{N} \) and \( f_1, \ldots, f_n \) satisfying the corona condition, the Bezout equation

\[
f_1 g_1 + \cdots + f_n g_n = 1
\]

has a solution \( g_1, \ldots, g_n \in H^\infty(\mathbb{D}, A) \).

The next result established in [3] asserts that the Bezout equation is solvable if and only if it is locally solvable.

**Theorem 7.1** ([3] Thm. 6.1). Suppose \( f_1, \ldots, f_n \in H^\infty(\mathbb{D}, A) \) satisfy (7.1). Equation (7.2) is solvable if and only if there exist a finite open cover \((U_j)_{1 \leq j \leq m}\) of \( \mathcal{M} \) and holomorphic functions \( g_{ij} \in H^\infty(U_j \cap \mathbb{D}, A), 1 \leq i \leq n, 1 \leq j \leq m, \) such that

\[
f_1|_{U_j \cap \mathbb{D}} \cdot g_{1j} + \cdots + f_n|_{U_j \cap \mathbb{D}} \cdot g_{nj} = 1 \quad \text{for all } j.
\]
The proof of the theorem follows the lines of the proof of [2, Thm. 1.11], where instead of [2, Thm. 3.5] one uses Theorem 1.1.

In a forthcoming paper we apply Theorem 1.1 to the corona problem for the algebra of bounded holomorphic functions on a polydisk.

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