HILBERT-KUNZ MULTIPLICITY AND THE F-SIGNATURE

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Abstract. This paper is a much expanded version of two talks given in Ann Arbor in May of 2012 during the computational workshop on F-singularities. We survey some of the theory and results concerning the Hilbert-Kunz multiplicity and F-signature of positive characteristic local rings.

Dedicated to David Eisenbud, on the occasion of his 65th birthday.

1. Introduction

Throughout this paper \((R, \mathfrak{m}, k)\) will denote a Noetherian local ring of prime characteristic \(p\) with maximal ideal \(\mathfrak{m}\) and residue field \(k\). We let \(e\) be a varying non-negative integer, and let \(q = p^e\). By \(I^{[q]}\) we denote the ideal generated by \(x^q, x \in I\). If \(M\) is a finite \(R\)-module, \(M/I^{[q]}M\) has finite length. We will use \(\lambda(-)\) to denote the length of an \(R\)-module. We assume knowledge of basic ideas in commutative algebra, including the usual Hilbert-Samuel multiplicity, Cohen-Macaulay, regular, and Gorenstein rings.

The basic question this paper studies is how \(\lambda(M/I^{[q]}M)\) behaves as a function on \(q\), and how understanding this behavior leads to better understanding of the singularities of the ring \(R\). In a seminal paper which appeared in 1969, [Ku1], Ernst Kunz introduced the study of this function as a way to measure how close the ring \(R\) is to being regular.

The Frobenius homomorphism is the map \(F : R \rightarrow R\) given by \(F(r) = r^p\). We say that \(R\) is \(F\)-finite if \(R\) is a finitely generated module over itself via the Frobenius homomorphism. It

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is not difficult to prove that if \((R, \mathfrak{m}, k)\) is a complete local Noetherian ring of characteristic \(p\), or an affine ring over a field \(k\) of characteristic \(p\), then \(R\) is F-finite if and only if \([k^{1/p} : k]\) is finite. When \(R\) is reduced we can identify the Frobenius map with the inclusion of \(R\) into \(R^{1/p}\), the ring of \(p\)th roots of elements of \(R\). If \(M\) is an \(R\)-module, we will usually write \(M^{1/q}\) to denote what is more commonly denoted \(F^e_*(M)\), where \(q = p^e\), the module which is the same as \(M\) as abelian groups, but whose \(R\)-module structure is coming from restriction of scalars via \(e\)-iterates of the Frobenius map. This is an exact functor on the category of \(R\)-modules. Notice that \(F^e_*(R)\) can be naturally identified with \(R^{1/q}\).

If the residue field \(k\) of \(R\) is perfect then the lengths of the \(R\)-modules \(R^{1/q}/IR^{1/q}\) and \(R/I[q]\) are the same. If \(k\) is not perfect, but \(R\) is F-finite, then we can adjust by \([k^{1/q} : k]\). We define \(\alpha(R) := \log_p([k^{1/p} : k])\), so that we can write \([k^{1/q} : k] = q^{\alpha(R)}\). With this notation, \(\lambda_R(R^{1/q}/IR^{1/q}) = \lambda_R(R/I[q])q^{\alpha(R)}\).

More broadly, the two numbers we will study, namely the Hilbert-Kunz multiplicity and the F-signature, are characteristic \(p\) invariants which give information about the singularities of \(R\), and lead to many interesting issues concerning how to use characteristic \(p\) methods to study singularities. There are four basic facts about characteristic \(p\) which make things work. Those facts are first that \((r + s)^p = r^p + s^p\) for elements in a ring of characteristic \(p\) (i.e., the Frobenius is an endomorphism); second, that the map from \(R \rightarrow R^{1/p}\) is essentially the same map as that of \(R^{1/q} \rightarrow R^{1/q}\) when \(R\) is reduced and \(q = p^e\); third that \(\sum_{i} \frac{1}{p^i}\) converges (!); and lastly that the flatness of Frobenius characterizes regular rings. Virtually everything we prove comes down to these interrelated facts.

Throughout this paper, whenever possible we have tried to give new (or at least not published) approaches to basic material. This is not done for the sake of whimsy, but to provide extra methods which may be helpful. Thus, the approach we take to proving the existence of the Hilbert-Kunz multiplicity and the F-signature, while following the general lines of the proofs of Paul Monsky [Mo1] and Kevin Tucker [Tu] respectively, uses a lemma of Sankar Dutta [D] as a central point, which is not present in the usual proofs. When we present the proof of the existence of a second coefficient, we veer from the paper [HMM]
to present another proof, based on the growth of the length of certain Tor modules, due to Moira McDermott and this author. In proving the theorem relating tight closure to the Hilbert-Kunz multiplicity, we use a lemma of Ian Aberbach \[\text{Ab1}\] as a crucial point in the proof instead of presenting the original proof in \[\text{HH1}\]. We provide examples of Hilbert-Kunz multiplicities throughout the paper, but often do not give details of the calculation.

We describe the contents of this paper. In the second section we give some early results of Kunz on the relationship between regular local rings and the Hilbert-Kunz function. Kunz was ahead of his time in this regard, though characteristic \(p\) methods in commutative algebra were being used to study various homological conjectures at around the same time. In section three, we develop basic results and definitions needed to give our main existence theorems. Our main technical tool we use is a lemma of Dutta \[D\] which gives information about the nature of prime filtrations of \(R^{1/q}\). We prove that the Hilbert-Kunz multiplicity exists. Section four proves that for formally unmixed rings, the Hilbert-Kunz multiplicity is one if and only if \(R\) is regular. Here formally unmixed means that for all associated primes \(Q\) of the completion of a local ring \(R\), \(\dim \hat{R}/Q = \dim R\). Section five provides the relationship between tight closure and Hilbert-Kunz multiplicity. In section six we prove that the F-signature exists and do some examples. Section seven proves the existence of a second coefficient in the Hilbert-Kunz function for normal rings. The final section takes up lower bounds on the Hilbert-Kunz multiplicity, introducing the volume estimates due to Watanabe and Yoshida \[\text{WY2}, \text{WY4}\], as well as the method of root adjunction of Aberbach and Enescu \[\text{AE3}, \text{AE4}\] and recent improvements by Celikbas, Dao, Huneke, and Zhang \[\text{CDHZ}\]. We close with some results of Doug Hanes \[\text{Ha}\].

This survey does not present the considerable research dealing with the many remarkable and difficult calculations of Hilbert-Kunz multiplicity. For example, for work on plane cubics, see Pardue’s thesis \[\text{BC}\] and \[\text{Mo2}\]. For plane curves in general see \[\text{Tr2}\], and for general two-dimensional graded rings either \[\text{Br1}\] or \[\text{Tr1}\]. For binomial hypersurfaces, see \[\text{Co}\] or \[\text{U}\]. For flag varieties see \[\text{FT}\]. The Hilbert-Kunz multiplicity of Rees algebras was the theme of \[\text{ElY}\]. Many other important examples or work are in \[\text{Br1}-\text{Br3}, \text{Co}, \text{E}\],
We borrow freely from these papers for some of the examples presented in this paper. We do not cover many new developments and calculations of the F-signature, for example see [BST1]-[BST2] and for toric rings see [S] and more recently [VK]. See [EY] for further extensions of Hilbert-Kunz multiplicity, and [Vr] for additional work. We also do not discuss the very interesting work being done on limiting value of Hilbert-Kunz multiplicities as $p$ goes to infinity, For example, see [BLM], [GM], and [Tr3]. For an excellent survey of other numerical invariants of singularities defined via Frobenius and their relationship to birational algebraic geometry and the theory of test ideals, see [STu].

2. Early History

Ernst Kunz was a pioneer in this study, realizing that studying the colengths of Frobenius powers of $m$-primary ideals would be an interesting idea.

**Theorem 2.1.** ([Ku1, Theorem 2.1, Proposition 3.2, Theorem 3.3]) Let $(R, m, k)$ be a Noetherian local ring of dimension $d$ and prime characteristic $p > 0$. For every $e \geq 0$, and $q = p^e$, $\lambda(R/m^q) \geq q^d$. Moreover, equality holds for some $q$ if and only if $R$ is regular, in which case equality holds for all $q$. If $R$ is F-finite, then $R^{1/q}$ is a free module for some $q > 1$ if and only if $R$ is regular.

**Proof.** We may complete $R$, and assume the residue field is algebraically closed to prove the first statement. We may also go modulo a minimal prime of $R$ to assume that $R$ is a complete local domain; this change will only potentially decrease $\lambda(R/m^q)$. We claim that $R^{1/q}$ has rank $q^d$ as an $R$-module in this case. Choose a coefficient field $k$ and a minimal reduction $x_1, ..., x_d$ of the maximal ideal. Let $A$ be the complete subring $k[[x_1, ..., x_d]]$ which is isomorphic with a formal power series. Note that $A^{1/q} \cong k[[x_1^{1/q}, ..., x_d^{1/q}]]$, which is a free $A$-module of rank $q^d$, whose basis is given by by arbitrary monomials of the form $x_1^{a_1/q} \cdots x_d^{a_d/q}$ where $0 \leq a_i \leq q - 1$. Since the rank of $R$ over $A$ and the rank of $R^{1/q}$ over $A^{1/q}$ are the same, it follows that the rank of $R^{1/q}$ over $R$ is exactly $q^d$. (We note that if $R$ is an F-finite complete
domain but the residue field is not perfect, then essentially the same proof shows that the rank of $R^{1/q}$ is exactly $q^{d+\alpha(R)}$. Since $R^{1/q}$ is a finite $R$-module, $\mu_R(R^{1/q}) \geq q^d$, with equality if and only if $R^{1/q}$ is a free $R$-module. However, $\mu_R(R^{1/q}) = \lambda_R(R^{1/q}/mR^{1/q}) = \lambda_R(R/m^{[q]})$, which implies that $\lambda_R(R/m^{[q]}) \geq q^d$. Notice that equality occurs in this case if and only if $R^{1/q}$ is a free $R$-module.

If $R$ is regular, then since $m$ is generated by a regular sequence, it easily follows that $\lambda(R/m^{[q]}) = q^d$. The second statement also easily is seen when $R$ is regular and $R$ is $F$-finite; one can complete and use the Cohen Structure theorem to do the complete case, and then descend using standard facts. It is the converse of both statements that is the most interesting part of the theorem.

Suppose that equality holds for some $q$, i.e., $\lambda(R/m^{[q]}) = q^d$. We can complete the ring and extend the residue field to be algebraically closed without changing this equality, so without loss of generality, $R$ is $F$-finite and $\alpha(R) = 0$. Note that $\lambda(R/m^{[q^n]}) = q^{nd}$ for all $n \geq 1$, by a simple induction.

We claim that $R$ is a domain; for if $Q$ is a minimal prime of $R$ of maximal dimension, then we have that $q^{nd} = \lambda(R/m^{[q^n]}) \geq \lambda(R/m^{[q^n]} + Q) \geq q^{nd}$. Hence we have equality throughout. But then $\lambda(R/m^{[q^n]}) = \lambda(R/m^{[q^n]} + Q)$ forces $\lambda((m^{[q^n]} + Q)/m^{[q^n]}) = 0$, so that $Q \subseteq \cap_i m^{[q^n]} = 0$. From the first part of this theorem, we then obtain that for all $n \geq 1$, $R^{1/q^n}$ is a free $R$-module.

We next claim that $R$ is Cohen-Macaulay. Let $x_1, \ldots, x_d$ be a system of parameters generating an ideal $J$. Then $\lambda(R/J^{[q^n]}) = \lambda(R^{1/q^n}/JR^{1/q^n}) = \lambda(R/J)q^{dn}$, since $R^{1/q^n}$ is a free $R$-module of rank $q^{dn}$. By a formula of Lech [SH Theorem 11.2.10]: $\lim_{q \to \infty} \lambda(R/J^{[q^n]})/q^{dn} = e(J)$, the usual multiplicity of $J$. Hence the multiplicity of $J$ is the colength of $J$. Since $J$ is generated by a system of parameters, it follows that $R$ is Cohen-Macaulay. (See [BH, Theorem 4.6.10]).

Now choose a system of parameters as above, and fix $n$ such that $m^{[q^n]} \subseteq J$, where $J$ is the ideal generated by the parameters. Suppose that the projective dimension of $k$ is infinite. We compute $\text{Tor}_{d+1}(R/J, R/m^{[q^n]})$ in two ways. From the fact that $J$ is generated
by a regular sequence of length $d$, this Tor module is 0. On the other hand, we can take
the free resolution of $k$ and tensor with $R^{1/q^n}$ and obtain an $R^{1/q^n}$ minimal free resolution of
$R^{1/q^n}/mR^{1/q^n}$. Identifying $R^{1/q^n}$ with $R$, we see that a free resolution of $R/m^{[q^n]}$ is obtained
by applying the Frobenius to the maps in the free resolution of $k$, which has the effect of
raising all entries in matrices in the resolution (after fixing bases of the free modules) to the
$q^n$th powers. Now tensoring with $R/J$, we see the homology at the $(d + 1)$st stage is 0 if and
only if the projective dimension of $k$ is at most $d$, since the maps become 0 after tensoring
with $R/J$. It follows that $R$ is regular.

□

Exercise 2.2. If $(R, m, k)$ is F-finite, and $Q$ is a prime ideal, prove that $\alpha(R_Q) = \alpha(R)p^{\dim(R/Q)}$. (See [Kü2, Proposition 2.3].)

Exercise 2.3. Let $(R, m, k)$ be a regular local ring of dimension $d$ and prime characteristic
$p$, and let $I$ be an $m$-primary ideal. Prove that $\lambda(R/I^{[q^d]}) = q^d \lambda(R/I)$ so that in particular,
$e_{HK}(I) = \lambda(R/I)$.

3. Basics

We begin with some estimates on the growth of the Hilbert-Kunz function, and some
examples.

Lemma 3.1. Let $(R, m, k)$ be a Noetherian local ring of dimension $d$ and prime characteristic
$p > 0$. We let $e(I)$ denote the multiplicity of the ideal $I$. Let $I$ be an $m$-primary ideal. Then
$(q = p^e)$,

$$e(I)/d! \leq \lim \inf \lambda(R/I^{[q^d]})/q^d \leq \lim \sup \lambda(R/I^{[q^d]})/q^d \leq e(I)$$

Proof. We can make an extension of $R$ to assume that the residue field is infinite without
changing any of the relevant lengths. Let $J$ be a minimal reduction of $I$, so that $J$ is
generated by a system of parameters. There are containments, $J^q \subseteq I^q \subseteq I^q$ which gives inequalities on the lengths,

$$\lambda(R/J^q) \geq \lambda(R/I^q) \geq \lambda(R/I^q).$$

For large $q$, the right hand length is given by a polynomial in $q$ of degree $d$ with leading coefficient $e(I)/d!$. Dividing by $q^d$ gives one inequality. For the other, we use a formula of Lech [SH, Theorem 11.2.10]: $\lim_{q \to \infty} \lambda(R/J^q)/q^d = e(J)$. Since $J$ is a reduction of $I$, $e(J) = e(I)$. 

**Corollary 3.2.** Let $(R, m, k)$ be a Noetherian local ring of dimension 1 and prime characteristic $p > 0$. Let $I$ be an $m$-primary ideal. Then $e(I) = \lim_{q \to \infty} \lambda(R/I^q)/q^d$

**Proof.** Set $d = 1$ in the above formula. 

**Example 3.3.** Although the one-dimensional case may seem very transparent, as the usual multiplicity equals the Hilbert-Kunz multiplicity, the actual Hilbert function is by no means obvious. Here is one example from [Mo1]. Let $k$ be a field of characteristic $p$ congruent to 2 or 3 modulo 5. Set $R = k[[X, Y]]/(X^5 - Y^5)$. $R$ is a one-dimensional local ring with maximal ideal $m = (x, y)$, and the multiplicity of $R$ is 5. The difference $|\lambda(R/m^q) - 5q|$ is bounded by a constant. But it is not a constant in general. If we write the constant as $d_e$ where $q = p^e$, then when $e$ is even $d_e = -4$ while when $e$ is odd, $d_e = -6$. For one-dimensional complete local rings Monsky shows that the ‘constant’ term is a periodic function. See [Mo1] for details. See also [Kr] for work in the graded case.

Our goal of this section is to prove that $\lim_{q \to \infty} \lambda(R/I^q)/q^d$ always exists. We call it the Hilbert-Kunz multiplicity. The history of how Monsky came to prove its existence is interesting. One might think that he was inspired by the paper of Kunz, but in fact he did not know about it when he proved the existence. The situation was additionally complicated by the fact that Kunz had erroneously thought that the limit did not actually exist, and proposed a counterexample in his paper. This author asked Monsky how he came to think about it, and here is what he replied:
“Craig asked me how I was led into looking into Kunz’s papers on the characterization of regular local rings in characteristic $p$ (and defining and studying the Hilbert-Kunz multiplicity as a result). But that’s not the order in which things occurred.

At Brandeis I was on the thesis committee of Al Cuoco, who was working in Iwasawa theory. He studied the growth of the $p$-part of the ideal class group as one moves up the levels in a tower of number fields, where the Galois group is a product of 2 copies of the $p$-adic integers. I extended his results to a product of $s$ copies; this involved the study of modules over power series rings, with the base ring being the $p$-adics or $\mathbb{Z}/p\mathbb{Z}$. In particular I considered the following—let $M$ be a finitely generated module over the power series ring in $s$ variables over $\mathbb{Z}/p\mathbb{Z}$, and $J$ be the ideal generated by the $p^n$th powers of the variables. How does the length of $M/JM$ grow with $n$? I got an asymptotic formula for this growth, put it into a more general setting and wrote things up. In analogy with the Hilbert-Samuel terminology I intended to speak of the Hilbert-Frobenius function and the Hilbert-Frobenius multiplicity.

But when I showed my result to David Eisenbud he told me that it was wrong, and that Kunz had given examples in which there wasn’t an asymptotic formula. So I looked into Kunz’s papers, discovering that he had considered such questions before me. So it was only proper to call the function the Hilbert-Kunz function. And call the associated limiting value the Hilbert-Kunz multiplicity, even though Kunz had thought that it needn’t exist!”

To prove the existence of the Hilbert-Kunz multiplicity, we will consider modules as well as rings. We use a somewhat different treatment than the paper of Monsky [Mo1], organizing our approach through a lemma proved by Dutta [D], which is not only interesting in its own right, but has the additional benefit that we can directly apply it to show the existence of the F-signature as well. However, in the end, all the approaches use that the map from $R$ to $R^{1/p}$ is essentially the same as $R^{1/q}$ to $R^{1/wp}$, and that the sum of the reciprocals of the powers of $p$ converges.

**Lemma 3.4.** [D] see proof of Proposition, page 428] Let $(R, m, k)$ be a local Noetherian domain of dimension $d$ and prime characteristic $p$. Assume that $R$ is F-finite. Then there
exists a constant $C$ and a fixed finite set of nonzero primes, $\{Q_1, ..., Q_n\}$ such that for every $q = p^e$, the $R$-module $R^{1/q}$ has a prime filtration having at most $Cq^d$ copies of $R/Q_i$ for $i \geq 1$, and $q^{d+\alpha(R)}$ copies of $R$.

**Proof.** The proof we give, similar to Dutta’s proof, was shown to me by Karen Smith, and is essentially found in Appendix 2 of [Hu], proof of Exercise 10.4.

Use induction on $d$; the $d = 0$ case is trivial.

Fix a maximal rank free submodule $F$ of $R^{1/p}$. We know that the rank of $F$ is $p^d + \alpha(R)$. Let $T$ be the cokernel of the inclusion $F \subset R^{1/p}$. Fix a prime cyclic filtration of $T$, and extend it by $F$ to a filtration of $R^{1/p}$.

$$0 \subset F = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_t = R^{1/p}.$$  

Because $F$ is maximal rank, the prime cyclic factors $M_{i+1}/M_i = R/\mathfrak{A}_i$ all have dimension strictly less than the dimension of $R$. Let $C_i$ be the constant which (by induction) works for $R/\mathfrak{A}_i$, let $C$ be twice the sum of all the $C_i$, and let $\Omega$ be the collection of the (finite) sets of primes appearing in the filtrations of all the $(R/\mathfrak{A}_i)^{1/q}$, as well as the prime $(0)$. We claim that $\Omega$ and $C$ satisfy the conclusion of the problem.

By induction on $q$, we prove that $R^{1/q}$ has a prime filtration using primes from $\Omega$, with at most $\frac{C}{2}(1 + 1/p + ... + 1/q)q^{d+\alpha(R)}$ copies of each one. Assume this is true for $q$. Take $p^e = q$ roots of all the modules above. We have a prime cyclic filtration (except at zeroth spot, where it is obvious how to extend to one) of $R^{1/q}$ modules

$$0 \subset F^{1/q} = M_0^{1/q} \subset M_1^{1/q} \subset M_2^{1/q} \subset \cdots \subset M_t^{1/q} = R^{1/qp},$$

where each factor has the form $(R/\mathfrak{A}_i)^{1/q} = R^{1/q}/\mathfrak{A}_i^{1/q}$.

To make this into a prime cyclic filtration of $R$ modules, we simply refine each inclusion $M_i^{1/q} \subset M_{i+1}^{1/q}$ of $R$ modules by a prime cyclic filtration. This amounts to filtering $M_{i+1}^{1/q}/M_i^{1/q} = (R/\mathfrak{A}_i)^{1/q}$ by $R/\mathfrak{A}_i$ prime cyclic modules. By induction on $d$, this can be done with only primes from $\Omega$, and appearing with multiplicities at most $\leq C_id^{d-1+\alpha(R/\mathfrak{A}_i)} = ...$
\[ C_i q^{d-1 + \alpha(R)} \]. Thus the primes appearing in this prime cycle filtration of \( R^{1/q_p} / F^{1/q} \) all come from \( \Omega \), and each one appears at most \( (\sum_i C_i) q^{d-1 + \alpha(R)} \) times.

To refine the \( R \) submodule \( F^{1/q} \) into a prime filtration we deal with each of the free summands \( R^{1/q} \) separately. By induction there are only primes from \( \Omega \) appearing and the multiplicity of \( R/Q_i \) in \( F^{1/q} \) is no more than \( \text{rank } F \cdot (C_2(1 + 1/p + ... + 1/q)(q^{d+\alpha(R)}) + C_1 q^{d-1 + \alpha(R)} \leq C_2(1 + ... + 1/(qp))(qp)^d + \alpha(R) \leq C(qp)^d + \alpha(R). \)

\[ \square \]

**Lemma 3.5.** Let \((R, m, k)\) be a Noetherian local ring of dimension \( d \) and prime characteristic \( p > 0 \). Let \( M \) be a finitely generated \( R \)-module. There exists a constant \( C > 0 \) such that for all \( e \geq 0 \) and any \( m \)-primary ideal \( I \) of \( R \) with \( m^{[q]} \subseteq I \), where \( q = p^e \), we have that

\[ \lambda(R/I \otimes_R M) \leq C q^{\dim M}. \]

**Proof.** Set \( t = \mu(m) \). Since \( m^{[q]} \subseteq m^{[q]} \), we see that \( R/m^{[q]} \otimes_R M \) surjects onto \( R/I \otimes_R M \). Therefore \( \lambda(R/I \otimes_R M) \leq \lambda(R/(m^{[q]}) \otimes_R M) \). The Hilbert polynomial of \( M \) with respect to \( m^i \) has degree \( \dim(M) \). If the leading coefficient of this polynomial is \( c \), it is clear that any \( C >> c \) satisfies the desired bound. \[ \square \]

**Lemma 3.6.** Let \((R, m, k)\) be a local ring of dimension \( d \) and prime characteristic \( p \). If \( T \) is a finitely generated torsion \( R \)-module then there exists a constant \( D \) such that for all \( q = p^e \), and for all \( I \) containing \( m^{[q]} \), \( \lambda(\text{Tor}_1^R(R/I, T)) \leq D q^d \).

**Proof.** Choose a nonzerodivisor \( c \in R \) which annihilates \( T \), and consider an \( R/(c) = A \) presentation of \( T \):

\[ ... \to A^s \to A^r \to T \to 0. \]

Let \( N \) be the kernel of the surjection of \( A^r \) onto \( T \). Tensoring with \( R/I \), we obtain an exact sequence,

\[ \text{Tor}_1^R(A^r, R/I) \to \text{Tor}_1^R(T, R/I) \to N/IN \to (A/I)^r \to T/IT \to 0. \]
Since $N$ is torsion, Lemma 3.5 implies that the length of $N/IN$ is bounded above by $Eq^{d-1}$, for some fixed constant $E$ depending only on $N$. Thus it suffices to bound the length of $\text{Tor}_1^R(A^r, R/I)$. Notice that $r$ does not depend upon $q$ or $I$. Hence it suffices to bound the length of $\text{Tor}_1^R(A, R/I)$. From the exact sequence $0 \to R \to R \to A \to 0$, we obtain after tensoring with $R/I$ that $\text{Tor}_1^R(A, R/I) \cong (I : c)/I$. However, the length of $(I : c)/I$ is the same as the length of $R/(I, c)$, and by Lemma 3.5, this length is bounded by $Gq^{d-1}$ for some constant $G$ depending only on $A$. □

**Exercise 3.7.** Prove Lemma 3.6 with the modification that $\lambda(\text{Tor}_1^R(R/I, T)) \leq Dq^{\text{dim}(T)}$ (this is not so easy).

These lemmas have the following crucial consequence, which is a key point in the paper of Tucker [11u, Corollary 3.5]:

**Corollary 3.8.** Let $(R, \mathfrak{m}, k)$ be a Noetherian local domain of dimension $d$ and prime characteristic $p$. Assume that $R$ is $F$-finite. There exists a constant $C$ such that for all $q = p^e$ and all $q' = p^{e'}$ and for all ideals $I$ containing $\mathfrak{m}^g$,

$$|\lambda(R/I[q']) - (q')^{d+\alpha(R)}\lambda(R/I)| \leq C(q')^{d+\alpha(R)}q^{d-1}.$$  

**Proof.** Fix the constant $C$ and the primes $\{Q_1, ..., Q_n\}$ as in the statement of Lemma 3.4. Then for all $q'$ there is an exact sequence,

$$0 \to R(q')^{d+\alpha(R)} \to R^{1/q'} \to T \to 0,$$

where $T$ has a prime filtration by at most $C(q')^{d+\alpha(R)}$ copies of each $R/Q_i$. Tensoring with $R/I$, we see that the difference of lengths, $|\lambda(R/I[q']) - (q')^{d+\alpha(R)}\lambda(R/I)|$ is bounded by the sum of $\lambda(T/IT) + \lambda(\text{Tor}_1^R(T, R/I))$. This sum in turn is bounded by

$$\sum_{i=1}^n C(q')^{d+\alpha(R)}(\lambda(R/(Q_i, I)) + \lambda(\text{Tor}_1^R(R/Q_i, R/I))).$$

To prove the Corollary it suffices to prove that there is a constant $D$, not depending on $q$, $q'$, or $I$ such that $\lambda(R/(Q_i, I)) \leq Dq^{d-1}$ for each $i$, and $\lambda(\text{Tor}_1^R(R/Q_i, R/I)) \leq Dq^{d-1}$. The existence of such a constant $D$ follows from Lemmas 3.5 and 3.6 respectively. □
Remark 3.9. We can now easily prove that the Hilbert-Kunz multiplicity exists for the ring itself and arbitrary \( \mathfrak{m} \)-primary ideals \( I \) in the case \( R \) is an F-finite domain. To do the general case, however, requires a little more work which one needs in any case to deal with additivity properties of the Hilbert-Kunz multiplicity. However, it is worth seeing this easy case deduced from the corollary. We may assume that \( k \) is algebraically closed. Set \( c_q = \lambda(R/I^{[q]})/q^d \). Apply Corollary 3.8 with \( I \) replaced by \( I^{[q]} \). Divide by \((q'q)^d\). We obtain that for all \( q, q' \),

\[ |c_{qq'} - c_q| \leq \frac{C}{q}. \]

This inequality forces the set of \( c_q \) to be a Cauchy sequence, and hence they converge.

Lemma 3.10. Let \((R, \mathfrak{m}, k)\) be a Noetherian local reduced ring of dimension \( d \) and prime characteristic \( p > 0 \). Let \( P_1, \ldots, P_m \) be those minimal primes of \( R \) with \( \dim(R/P_i) = d \). If \( M \) and \( N \) are finitely generated \( R \)-modules such that \( M_{P_i} \cong N_{P_i} \) for each \( i \), then there exists a positive constant \( C \) such that for all \( e \geq 0 \) and for every ideal \( I \) of \( R \) with \( \mathfrak{m}^{[q]} \subseteq I \), where \( q = p^e \), we have \( |\lambda(R/I \otimes_R M) - \lambda(R/I \otimes_R N)| \leq Cq^{d-1} \).

Proof. Let \( W = R \setminus (\cup_i P_i) \), so that \( R_W \cong R_{P_1} \times \cdots \times R_{P_m} \), and we have that \( M_W \cong N_W \). Since \((\text{Hom}_R(M, N))_W \cong \text{Hom}_{R_W}(M_W, N_W)\), there is some \( \phi \in \text{Hom}_R(M, N) \) such that \( \phi_W \) is an isomorphism. Since \( \text{coker}(\phi) \) satisfies \( \text{coker}(\phi)_W = 0 \) and thus has dimension strictly smaller than \( d \), we can find a positive constant \( C \) such that for all \( e \geq 0 \) and for any ideal \( I \) of \( R \) which contains \( \mathfrak{m}^{[q]} \), we have that \( |\lambda(R/I \otimes_R R/ \text{coker}(\phi))| \leq Cq^{d-1} \). \( \square \)

We use some well-known notation in the next few results. Let \( f, g : \mathbb{N} \to \mathbb{R} \) be functions from the nonnegative integers to the real numbers. Recall that \( f(n) = O(g(n)) \) if there exists a positive constant \( C \) such that \( |f(n)| \leq Cg(n) \) for all \( n \gg 0 \), and we write \( f(n) = o(g(n)) \) if \( \lim_{n \to \infty} f(n)/g(n) = 0 \).

Proposition 3.11. Let \((R, \mathfrak{m}, k)\) be a Noetherian local ring of dimension \( d \) and prime characteristic \( p > 0 \). \( 0 \to N \to M \to K \to 0 \) be a short exact sequence of finitely generated
$R$-modules. Then,

$$\lambda(M/I^{[q]}M) = \lambda(N/I^{[q]}N) + \lambda(K/I^{[q]}K) + O(q^{d-1}).$$

**Proof.** First suppose that $R$ is reduced. Then $M$ and $N \oplus K$ have isomorphic localizations at each minimal prime of $R$, and the claim follows from Lemma 3.10.

If $R$ is not reduced, choose $q'$ such that $(\text{nilrad}(R))^{[q']} = 0$, and consider the same exact sequence as a sequence of $R^{q'}$-modules. This ring is reduced and applying the reduced case with the ideal $I^{[q']} \cap R^{q'}$ yields that

$$\lambda(M/I^{[qq']}M) = \lambda(N/I^{[qq']}N) + \lambda(K/I^{[qq']}K) + O(q^{d-1}).$$

Since $O(q^{d-1}) = O((qq')^{d-1})$, the Proposition is proved. \[\square\]

We are now able to prove the existence of the Hilbert-Kunz multiplicity:

**Theorem 3.12.** Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring of dimension $d$ and prime characteristic $p > 0$. Let $M$ be a finitely generated $R$-module, and let $I$ be an $\mathfrak{m}$-primary ideal. There is a real constant $\alpha = e_{HK}(I, M) \geq 1$ such that $\lambda(M/I^{[q]}M) = \alpha q^d + O(q^{d-1})$. If

$$0 \to N \to M \to K \to 0$$

is a short exact sequence of finitely generated $R$-modules, then

$$e_{HK}(I, M) = e_{HK}(I, K) + e_{HK}(I, N).$$

**Proof.** By making a faithfully flat extension there is no loss of generality in assuming that $R$ is a complete local ring with algebraically closed residue field. By taking a prime filtration of $M$ and using Proposition 3.11 it suffices to do the case in which $M = R/P$ for some prime $P$ of $R$. Thus there is no loss of generality in assuming that $R$ is an F-finite domain and $M = R$ in proving the first assertion. The second assertion follows immediately from the first assertion and Proposition 3.11.

To prove the existence, we are now in the case of Remark 3.9 which finishes the proof. \[\square\]
We often suppress the $R$ in $e_{HK}(I, R)$ and just write $e_{HK}(I)$. When $I = \mathfrak{m}$, we set $e_{HK}(M) = e_{HK}(\mathfrak{m}, M)$, and refer to this value as the Hilbert-Kunz multiplicity of $M$.

**Example 3.13.** Unlike the usual multiplicity, the Hilbert-Kunz multiplicity is typically not an integer. The Hilbert-Kunz function can appear quite bizarre, at least to begin with. For example, let $R = \mathbb{Z}/5\mathbb{Z}[x_1, x_2, x_3, x_4]/(x_1^4 + \cdots + x_4^4)$, then with $I = (x_1, ..., x_4)$, $\lambda(R/I^{[5^e]}) = \frac{168}{61} (5^{3e}) - \frac{107}{61} (3^e)$ by [HaMo]. Note that $R$ is a 3-dimensional Gorenstein ring with isolated singularity.

Just as in the theory of usual multiplicity, it is now easy to prove some basic remarks on the behavior of the Hilbert-Kunz multiplicity. In particular, the following additivity theorem is highly useful.

**Theorem 3.14.** Let $(R, \mathfrak{m}, k)$ be a local Noetherian ring of dimension $d$ and prime characteristic $p$. let $I$ be an $\mathfrak{m}$-primary ideal, and let $M$ be a finitely generated $R$-module. Let $\Lambda$ be the set of minimal prime ideals $P$ of $R$ such that $\dim(R/P) = \dim(R)$. Then

$$e_{HK}(I, M) = \sum_{P \in \Lambda} e_{HK}(I, R/P)\lambda(M_P).$$

**Proof.** By Theorem 3.11, Hilbert-Kunz multiplicity is additive on short exact sequences. Fix a prime filtration of $M$, say

$$0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_n = M$$

where $M_{i+1}/M_i \cong R/P_i$ ($P_i$ a prime) for all $0 \leq i \leq n - 1$. As $e_{HK}(I, R/Q) = 0$ if $\dim(R/Q) < \dim(R)$, the additivity of multiplicity applied to this filtration shows that $e_{HK}(I, M)$ is a sum of the $e_{HK}(I, R/P)$ for $P \in \Lambda$, counted as many times as $R/P$ appears as some $M_{i+1}/M_i$. We can count this by localizing at $P$. In this case, we have a filtration of $M_P$, where all terms collapse except for those in which $(M_{i+1}/M_i)_P \cong (R/P)_P$, and the number of such copies is exactly the length of $M_P$. \qed
Corollary 3.15. Let \((R, m, k)\) be a local Noetherian domain of dimension \(d\) and prime characteristic \(p\). Let \(I\) be an \(m\)-primary ideal of \(R\), and \(M\) a finitely generated \(R\)-module. Then \(e_{HK}(I, M) = e_{HK}(I, R) \operatorname{rank}_R M\).

Proof. Recall that the rank of \(M\) is by definition the dimension of \(M \otimes_R K\) over \(K\), where \(K\) is the field of fractions of \(R\). We apply Lemma 3.10 with \(W = R \setminus 0\): if we set \(r = \operatorname{rank}_R M\), then \(W^{-1}M \cong K^r \cong W^{-1}R^r\), and the corollary follows. \(\square\)

Theorem 3.16. Let \((R, m, k)\) be a \(d\)-dimensional local Noetherian domain of prime characteristic \(p\), with field of fractions \(K\), and let \(I\) be an \(m\)-primary ideal. Let \(S\) be a module-finite extension domain of \(R\) with field of fractions \(L\). Then

\[
e_{HK}(I, R) = \sum_{Q \in \operatorname{Max}(S), \dim S_Q = d} \frac{e_{HK}(IS_Q, S_Q)[S/Q : k]}{[L : K]}.
\]

Proof. Since \(W^{-1}S \cong W^{-1}R^{[L:K]}\), we can apply Lemma 3.10 to conclude that \(e_{HK}(I, S) = e_{HK}(I, R)[L : K]\). On the other hand,

\[
e_{HK}(I, S) = \lim_{q \to \infty} \lambda_R(S/I^{[q]}S)/q^d.
\]

As every maximal ideal \(Q\) of \(S\) contains \(mS\), the Chinese Remainder Theorem implies that \(S/I^{[q]}S \cong \prod_{Q \in \operatorname{Max}(S)} S_Q/I^{[q]}S_Q\). In particular, \(\lambda_R(S/I^{[q]}S) = \sum_{Q \in \operatorname{Max}(S)} \lambda_R(S_Q/I^{[q]}S_Q) = \sum_{Q \in \operatorname{Max}(S)} \lambda_{S_Q}(S_Q/I^{[q]}S_Q)[S/Q : k]\). Therefore \(e_{hk}(I, S)\) equals

\[
\lim_{q \to \infty} \sum_{Q \in \operatorname{Max}(S)} \lambda_{S_Q}(S_Q/I^{[q]}S_Q)[S/Q : k]/q^d = \lim_{q \to \infty} \sum_{\dim S_Q = d} \lambda_{S_Q}(S_Q/I^{[q]}S_Q)[S/Q : k]/q^d.
\]

Hence

\[
e_{HK}(I, R) = \sum_{Q \in \operatorname{Max}(S), \dim S_Q = d} \frac{e_{HK}(IS_Q, S_Q)[S/Q : k]}{[L : K]}.
\]

\(\square\)

Example 3.17. Consider the Veronese subring \(R\) defined by

\[
R = k[[X_1^{i_1} \cdots X_d^{i_d} | i_1, \ldots, i_d \geq 0, \sum_{j=1}^d i_j = r]].
\]
Applying Theorem 3.16 to \( R \hookrightarrow S = k[[x, y]] \), we get

(3.1) \[ e_{HK}(R) = \frac{1}{r} \left( \frac{d + r - 1}{r} \right). \]

In particular, if \( d = 2, r = e(A) \), then \( e_{HK}(R) = \frac{e(R)+1}{2} \).

For other examples, consider the quotient singularities.

**Example 3.18.** See \[ WY1, \text{Theorem 5.4} \]. Let \( S \) be a regular local ring and suppose that \( G \) is a finite group of automorphisms of \( S \) with invariant ring \( R \) with maximal ideal \( \mathfrak{m} \). By Theorem 3.16 and Exercise 2.3, one sees that \( e_{HK}(R) = \frac{1}{|G|} \lambda(S/\mathfrak{m}S) \).

This formula is used, together with a lot more work, by Watanabe and Yoshida to give the following formulas for the Hilbert-Kunz multiplicities of the famous double points below:

Let \((R, \mathfrak{m}) = k[[x, y, z]]/(f)\) where \( f \) is one of the following:

| type | equation | char \( R \) | \( e_{HK}(R) \) |
|------|----------|-------------|-----------------|
| \(A_n\) | \( f = xy + z^{n+1} \) | \( p \geq 2 \) | \( 2 - 1/(n+1) \) \( (n \geq 1) \) |
| \(D_n\) | \( f = x^2 + yz^2 + y^{n-1} \) | \( p \geq 3 \) | \( 2 - 1/4(n-2) \) \( (n \geq 4) \) |
| \(E_6\) | \( f = x^2 + y^3 + z^4 \) | \( p \geq 5 \) | \( 2 - 1/24 \) |
| \(E_7\) | \( f = x^2 + y^3 + yz^3 \) | \( p \geq 5 \) | \( 2 - 1/48 \) |
| \(E_8\) | \( f = x^2 + y^3 + z^5 \) | \( p \geq 7 \) | \( 2 - 1/120 \) |

Each of these hypersurfaces is the invariant subring by a finite subgroup \( G \subseteq SL(2, k) \) which acts on the polynomial ring \( k[x, y] \). We have that \( e_{HK}(R) = 2 - 1/|G| \); see \[ WY1, \text{Theorem 5.1} \].

**Example 3.19.** Let \( S = k[x, y, z] \) where \( k \) is a field of characteristic at least five. Let \( h \in S \) be homogeneous of degree 3. Set \( R = S/(h) \), and let \( \mathfrak{m} = (x, y, z)R \). If \( h \) is smooth, then \( e_{hk}(\mathfrak{m}) = \frac{9}{4} \), while if \( h \) is a nodal or cuspidal cubic, \( e_{HK}(\mathfrak{m}) = \frac{7}{3} \). This has been done various ways. Pardue in his thesis did the nodal cubic; see also Buchweitz and Chen \[ BC \], Brenner \[ Br3 \], Monsky \[ Mo8 \], and Trivedi \[ Tr1 \], and in characteristic 2, \[ Mo2 \].

Here are a few more examples, which we leave as an exercise:
Exercise 3.20. We consider quadric hypersurfaces in $\mathbb{P}^3$. Let $k$ be a field of characteristic $p > 2$, and let $R$ be one of the following rings:

$$
\begin{align*}
&k[[X, Y, Z, W]]/(X^2), & \text{if } \text{rank}(q) = 1, \\
&k[[X, Y, Z, W]]/(X^2 - YZ), & \text{if } \text{rank}(q) = 2, \\
&k[[X, Y, Z, W]]/(XY - ZW), & \text{if } \text{rank}(q) = 3.
\end{align*}
$$

Prove that $e_{HK}(R) = 2$, $\frac{3}{2}$, or $\frac{4}{3}$, respectively.

For a long time it was thought that the Hilbert-Kunz multiplicity would always be a rational number. All the known examples were rational, e.g., for rings of finite Cohen-Macaulay type (see [Se]) or more generally $F$-finite type ([SVB], and [Y2]), for many computed hypersurfaces, for binomial hypersurfaces ([Co]), for graded normal rings of dimension two ([Br2],[Tr1]), and others. However, in recent years Monsky has given convincing evidence that this will not true, though as of the writing of this paper, there is only overwhelming evidence, but not a proof. One example given by Monsky is the following:

Example 3.21. Let $F$ be a finite field of characteristic 2 and $h = x^3 + y^3 + xyz \in F[[x, y, z]]$. Then Monsky conjectures, with a huge amount of evidence, that the Hilbert-Kunz multiplicity of the hypersurface $uv + h = 0$ is $\frac{4}{3} + \frac{5}{14\sqrt{7}}$. Even more, it appears that transcendental Hilbert-Kunz multiplicities exist. We refer to [Mo6] and [Mo7] for details.

4. Hilbert-Kunz Multiplicity Equal to One

We begin this section with an easy, but crucial estimate on the size of Hilbert-Kunz functions which was observed by Hanes [Ha].

Lemma 4.1. Let $(R, m, k)$ be a Noetherian local ring of dimension $d$ and prime characteristic $p_0$. Let $I \subseteq J$ be two ideals with $I$ $m$-primary (we allow $J = R$). Then $\lambda(R/I^{[d]}) \leq \lambda(J/I) \cdot \lambda(R/m^{[d]}) + \lambda(R/J^{[d]})$. 
Proof. Set $s = \lambda(J/I)$. Take a filtration of $I \subseteq J \subseteq R$

$$I = J_0 \subseteq J_1 \subseteq J_2 \subseteq \cdots \subseteq J_s = J \subseteq R$$

so that $\lambda(J_i/J_{i-1}) = 1$ i.e. $J_i/J_{i-1} \cong R/m$, $\forall i = 1, 2, \ldots, s$. That is to say $J_i = (J_{i-1}, x_i)$ for some $x_i \in J_i$ such that $J_{i-1} : x_i = m$.

For every $q = p^e$, there is a corresponding filtration of $I^q \subseteq J^q \subseteq R$

$$I^q = J_0^q \subseteq J_1^q \subseteq J_2^q \subseteq \cdots \subseteq J_s^q = J^q \subseteq R,$$

where $J_i^q/J_{i-1}^q \cong R/(J_i^q : x_i^q)$, which is a homomorphic image of $R/m^q$, for every $i = 1, 2, \ldots, s$. So $\lambda(J_i^q/J_{i-1}^q) \leq \lambda(R/m^q)$. Therefore $\lambda(R/I^q) \leq \lambda(J/I) \cdot \lambda(R/m^q) + \lambda(R/J^q)$. \hfill \Box

**Corollary 4.2.** Let $(R, m, k)$ be a Noetherian local ring of dimension $d$ and prime characteristic $p$. Let $I$ be a $m$-primary ideal of $R$. Then $\lambda(R/I^q) \leq \lambda(R/I) \cdot \lambda(R/m^q)$. If $I \subseteq J$ then $e_{HK}(I, R) \leq \lambda(J/I)e_{HK}(R) + e_{HK}(J, R)$.

**Proof.** To prove the first statement, we take $J = R$ and apply Lemma 4.1. For the second statement, the Corollary follows from Lemma 4.1 by dividing by $q^d$ and then taking the limits. \hfill \Box

Our goal is to prove that regularity is characterized by the Hilbert-Kunz multiplicity being one, if the ring is formally unmixed. This condition is necessary by the easy exercise below. Our treatment is taken directly from [HY].

**Exercise 4.3.** Let $R = k[[x, y, z]]/(xz, xy)$, where $k$ is a field of characteristic $p$. Prove that $e_{HK}(R) = 1$.

**Theorem 4.4.** Let $(R, m)$ be a Noetherian local ring of dimension $d$ and prime characteristic $p$. Let $J$ be an ideal such that $\dim R/J = 1$ and height $J = d - 1$. Assume that $x \in R$ is a non-zerodivisor in $R/J$, and set $I = (J, x)$. Assume that $R_P$ is regular for every minimal prime $P$ above $J$. Then $e_{HK}(I, R) \geq \lambda(R/I)$.
Proof. Using the properties of the usual multiplicity of parameter ideals, the associativity formula for the usual multiplicity, and we have

\[
e_{HK}(I, R) = \lim_{q \to \infty} \frac{1}{q^d} \cdot \lambda(R/I^{[q]}) = \lim_{q \to \infty} \frac{1}{q^d} \cdot \lambda(R/(J^{[q]}, x^q)) \]

\[
\geq \lim_{q \to \infty} \frac{1}{q^d} \cdot e(x^q; R/J^{[q]}) = \lim_{q \to \infty} \frac{q}{q^d} \cdot e(x; R/J^{[q]}) = \lim_{q \to \infty} \frac{1}{q^d} \cdot e(x; R/J^{[q]})
\]

\[
= \lim_{q \to \infty} \frac{1}{q^d-1} \cdot \sum_{P \in \text{min}(R/J)} e(x; R/P) \cdot \lambda_{R_P}(R_P/J_P^{[q]})
\]

\[
= \lim_{q \to \infty} \sum_{P \in \text{min}(R/J)} e(x; R/P) \cdot \lambda_{R_P}(R_P/J_P)
\]

\[
= \sum_{P \in \text{min}(R/J)} e(x; R/P) \cdot \lambda_{R_P}(R_P/J_P) = e(x; R/J) = \lambda(R/(J, x)) = \lambda(R/I).
\]

Observe that after we prove that \(e_{HK}(R) = 1\) implies the regularity of \(R\), then regularity forces \(e_{HK}(I) = \lambda(R/I)\) for all \(m\)-primary ideals \(I\), by using the work above.

A critical step in proving the main result of this section is in constructing an \(m\)-primary ideal \(I \subseteq m^{[p]}\) such that \(e_{HK}(I) \geq \lambda(R/I)\). This was proved by Watanabe and Yoshida [WY1, Theorem 1.5] but in a different way than is done here.

**Theorem 4.5.** Let \((R, m, k)\) be a formally unmixed Noetherian local ring of dimension \(d\) and prime characteristic \(p\). Then \(e_{HK}(R) = 1\) if and only if \(R\) is regular.

Proof. We have already observed that if \(R\) is regular, then the Hilbert-Kunz multiplicity is one. We prove the converse. Since the Hilbert-Kunz multiplicity of \(R\) is the same as that of its completion, we may assume \(R\) is complete. The additivity formula for Hilbert-Kunz multiplicity Theorem 3.14 shows that \(e_{HK}(R) = \sum_{P} e_{HK}(R/P) \cdot \lambda(R_P)\) where the sum is over all minimal primes of maximal dimension. Since \(e_{HK}(R) = 1\), we deduce that \(R\) can
have only one minimal prime \( P \) and \( R_P \) has to be field, i.e. \( P_P = 0 \). Hence \( P = 0 \) since \( R \setminus P \) consists of non-zero divisors. Thus \( R \) is a domain.

It suffices to prove that \( \lambda(R/m^{[p]}) \leq p^d \) (where \( d = \dim(R) \)) as then Theorem 2.1 gives that \( R \) must be regular.

The singular locus of \( R \) is closed and not equal to \( \text{Spec}(R) \). It follows that we can choose a prime \( P \) such that \( \dim(R/P) = 1 \) and \( R_P \) is regular, which we leave as an exercise for the reader. Since the intersection of the symbolic powers of \( P \) is zero and \( R \) is complete, Chevalley’s lemma gives that some sufficiently large symbolic power of \( P \) lies inside \( m^{[p]} \). Call this symbolic power \( J \). Choose \( x \in m^{[p]} \) such that \( x \notin P \). The ideal \( I = (J, x) \) lies in \( m^{[p]} \) and satisfies the hypothesis of Theorem 4.4. Hence

\[
e_{HK}(I) \geq \lambda(R/I).
\]

On the other hand we have \( e_{HK}(I, R) \leq \lambda(m^{[p]}/I) \cdot e_{HK}(R) + e_{HK}(m^{[p]}, R) = \lambda(m^{[p]}/I) + e_{HK}(m^{[p]}, R) \leq \lambda(m^{[p]}/I) + \lambda(R/m^{[p]}), \) by Lemma 4.1 and Corollary 4.2.

That is to say

\[
\begin{align*}
\lambda(m^{[p]}/I) + \lambda(R/m^{[p]}) &= \lambda(R/I) \leq e_{HK}(I, R) \\
&\leq \lambda(m^{[p]}/I) + e_{HK}(m^{[p]}, R) \\
&\leq \lambda(m^{[p]}/I) + \lambda(R/m^{[p]}),
\end{align*}
\]

which forces \( \lambda(R/m^{[p]}) = e_{HK}(m^{[p]}, R) \). However,

\[
e_{HK}(m^{[p]}, R) = \lim_{q \to \infty} \frac{\lambda(R/m^{[pq]})}{q^d} = \lim_{p \to \infty} \frac{p^d \cdot \lambda(R/m^{[pq]})}{(pq)^d} = p^d \cdot e_{HK}(R) = p^d.
\]

Together the equalities imply that \( \lambda(R/m^{[p]}) = p^d \), which implies that \( R \) is regular by Theorem 2.1. □

The basic filtration lemmas, together with Kunz’s theorem already give a better result, provided the ring is Cohen-Macaulay. In fact, this is one of the more subtle and difficult points, to prove that Hilbert-Kunz multiplicity near one should imply that the ring is Cohen-Macaulay. A crucial step is provided by results of Goto and Nakamura, see [GN]. They prove...
the following beautiful generalization of the result of Serre which proves that the multiplicity of a parameter ideal is its colength if and only if the ring is Cohen-Macaulay.

**Theorem 4.6. [GN]** Let \((R, \mathfrak{m}, k)\) be an unmixed Noetherian local ring of prime characteristic \(p\) which is the homomorphic image of a Cohen-Macaulay local ring. Let \(J\) be an ideal generated by a system of parameters. If \(\lambda(R/J^*) = e(J)\), then \(R\) is \(F\)-rational (and therefore is Cohen-Macaulay).

The general philosophy is that the closer the Hilbert-Kunz multiplicity is to one, the better the singularities of the ring. The following proposition was proved by Blickle and Enescu, using results of Goto and Nakamura and Watanabe and Yoshida to first obtain that the ring is Cohen-Macaulay. We state the full result here, but only give the proof assuming Cohen-Macaulay.

**Proposition 4.7. [BE]** Let \((R, \mathfrak{m})\) be a Noetherian local ring of dimension \(d\) and prime characteristic \(p\). If \(R\) is not regular, then \(e_{HK}(R) > 1 + \frac{1}{p^d!}\).

**Proof.** We give the proof assuming that \(R\) is Cohen-Macaulay. Let \(e\) be the multiplicity of \(R\). We may assume the residue field is infinite. Fix a minimal reduction \(K\) of the maximal ideal. We apply Corollary 4.2 with \(I = K^{[p]}\) and \(J = \mathfrak{m}^{[p]}\). This gives that \(e p^d = e_{HK}(K^{[p]}) \leq \lambda(\mathfrak{m}^{[p]}/K^{[p]}) e_{HK}(R) + e_{HK}(\mathfrak{m}^{[p]}) = \lambda(\mathfrak{m}^{[p]}/K^{[p]}) e_{HK}(R) + p^d e_{HK}(R)\). By Theorem 2.1 \(\lambda(\mathfrak{m}^{[p]}/K^{[p]}) = e p^d - \lambda(R/\mathfrak{m}^{[p]}) \leq e p^d - (p^d + 1)\) because \(R\) is not regular. Putting these inequalities together and cancelling terms yields that \(e p^d \leq (e p^d - 1) e_{HK}(R)\) or \(1 + \frac{1}{e p^d - 1} \leq e_{HK}(R)\). Since \(e/d! \leq e_{HK}(R)\), if \(e > d!\), then \(1 + \frac{1}{d!} < e_{HK}(R)\), a stronger statement than what we claim. Otherwise, \(e p^d - 1 < p^d d!\), and the proposition follows. \(\square\)

The reader should ask themselves where the assumption that \(R\) is Cohen-Macaulay is used in the above proof.

The methods in this section also give a proof of a result of Kunz concerning the behavior of Hilbert-Kunz multiplicity under specialization. It is still an open problem whether or not Hilbert-Kunz multiplicity is upper semi-continuous. See, however, the interesting papers
of Shepherd-Barron \[\text{SB}\] (but be careful–Corollary 2 is not quite correct) and Enescu and Shimomoto \[\text{ES}\].

**Proposition 4.8.** \[\text{[Ku2, Cor. 3.8]}\] Let \((R, \mathfrak{m}, k)\) be a Noetherian local ring of dimension \(d\) and prime characteristic \(p\), and let \(P\) be a prime ideal of \(R\) such that \(\text{height}(P) + \text{dim}(R/P) = \text{dim}(R)\). Then \(e_{HK}(R_P) \leq e_{HK}(R)\). In fact, if \(t = \text{dim}(R/P)\), then \(q^t \cdot \lambda_{R_P}((R/P[q])_P) \leq \lambda(R/m^{[q]})\) for every \(q = p^e\).

**Proof.** By induction, it is enough to prove the case where \(\text{height}(P) = \text{dim}(R) - 1\). Notice it suffices to prove the second inequality.

Choose \(f \in \mathfrak{m} - P\). Then, using the properties of the usual multiplicity of parameter ideals, the associativity formula for the usual multiplicity, we have, for all \(q = p^e\),

\[
\begin{align*}
(4.4) & \quad \lambda(R/(P, f)^{[q]}) = \lambda(R/(P^{[q]}, f^q)) \\
(4.5) & \quad \geq e(f^q; R/P^{[q]}) \\
(4.6) & \quad = \lambda_{R_P}((R/P^{[q]})_P) \cdot e(f^q; R/P) \\
(4.7) & \quad = \lambda_{R_P}((R/P^{[q]})_P) \cdot q \cdot \lambda(R/(f, P)).
\end{align*}
\]

By Corollary 4.2, we know that \(\lambda(R/(f, P)) \cdot \lambda(R/m^{[q]}) \geq \lambda(R/(P, f)^{[q]})\). Hence \(\lambda(R/m^{[q]}) \geq q \cdot \lambda_{R_P}((R/P^{[q]})_P)\) for every \(q = p^e\).

\[\square\]

5. **Hilbert-Kunz Multiplicity and Tight Closure**

There is almost an exact parallel between the relationship of integral closure to the usual Hilbert-Samuel multiplicity, and the relationship between tight closure to Hilbert-Kunz multiplicity. Just as in the case of the Hilbert-Samuel multiplicity, this relationship is important both theoretically and necessary to fully understand multiplicity. We use a key result of Aberbach \[\text{[Ab1]}\] to make the proofs easier than the original proof in \[\text{[HH1]}\].
Let $R^o$ denote the complement of the union of all minimal primes of a ring $R$. The definition of tight closure for ideals is:

**Definition 5.1.** Let $R$ be a Noetherian ring of prime characteristic $p$. Let $I$ be an ideal of $R$. An element $x \in R$ is said to be in the tight closure of $I$ if there exists an element $c \in R^o$ such that for all large $q = p^e$, $cx^q \in I^{[q]}$.

There is also a definition of the tight closure of submodules of finitely generated $R$-modules, which we do not use in these notes. Of particular interest are rings in which every ideal is tightly closed.

**Definition 5.2.** A Noetherian ring in which every ideal is tightly closed is called weakly $F$-regular. A Noetherian ring $R$ such that $R_W$ is weakly $F$-regular for every multiplicative system $W$ is called $F$-regular.

We list a few of the main properties satisfied by tight closure.

**Proposition 5.3.** Let $R$ be a Noetherian ring of prime characteristic $p$ and let $I$ be an ideal.

1. $(I^*)^* = I^*$. If $I_1 \subseteq I_2 \subseteq R$, then $I_1^* \subseteq I_2^*$.
2. If $R$ is reduced or if $I$ has positive height, then $x \in R$ is in $I^*$ if and only if there exists $c \in R^o$ such that $cx^q \in I^{[q]}$ for all $q = p^e$.
3. An element $x \in R$ is in $I^*$ iff the image of $x$ in $R/P$ is in the tight closure of $(I+P)/P$ for every minimal prime $P$ of $R$.

**Proof.** Part (1) and (2) follow immediately from the definition.

We prove (3). One direction is clear: if $x \in I^*$, then this remains true modulo every minimal prime of $R$ since $c \in R^o$. Let $P_1, \ldots, P_n$ be the minimal primes of $R$. If $c_i^t \in R/P_i$ is nonzero we can always lift $c_i^t$ to an element $c_i \in R^o$ by using the Prime Avoidance theorem. Suppose that $c_i^t \in R/P_i$ is nonzero and such that $c_i^t x_i^q \in I_i^{[q]}$ for all large $q$, where $x_i$ (respectively $I_i$) represent the images of $x$ (respectively $I$) in $R/P_i$. Choose a lifting $c_i \in R^o$ of $c_i^t$. Then $c_i x_i^q \in I^{[q]} + P_i$ for every $i$. Choose elements $t_i$ in all the minimal primes
except \(P_i\). Set \(c = \sum_i c_i t_i\). It is easy to check that \(c \in R^p\). Choose \(q' \gg 0\) so that \(N^{[q']} = 0\), where \(N\) is the nilradical of \(R\). Then \(cx^q \in I^{[q]} + N\), and so \(c't^q x^{q'} \in I^{[qq']}\), which proves that \(x \in I^*\).

One direction of our main result of this section is quite easy from the definition:

**Proposition 5.4.** Let \((R, \mathfrak{m}, k)\) be a Noetherian local ring of dimension \(d\) and prime characteristic \(p\). Let \(I\) be an \(\mathfrak{m}\)-primary ideal, and suppose that \(I \subseteq J \subseteq I^*\). Then \(e_{HK}(I) = e_{HK}(J)\).

**Proof.** By assumption there is an element \(c \in R^p\) such that \(c\) annihilates the modules \(J^{[q]}/I^{[q]}\) for all large \(q = p^e\). These modules have a bounded number of generators, say \(t\), given by the number of generators of \(J\). In particular, \((R/(c, J^{[q]}))^t\) maps onto \(J^{[q]}/I^{[q]}\), so that the length is at most \(t \cdot \lambda(R/(c, J^{[q]}))\). However, the length of \(R/(c, J^{[q]})\) is at most \(O(q^{d-1})\) since the dimension of \(R/(c)\) is \(d - 1\). It follows that \(|\lambda(R/J^{[q]}) - \lambda(R/I^{[q]})| = O(q^{d-1})\), and so \(e_{HK}(I) = e_{HK}(J)\).

The main result of this section is the following:

**Theorem 5.5.** Let \((R, \mathfrak{m}, k)\) be a Noetherian local ring of dimension \(d\) and prime characteristic \(p\) which is formally unmixed. Let \(I \subseteq J\) be \(\mathfrak{m}\)-primary ideals. Then \(e_{HK}(I) = e_{HK}(J)\) if and only if \(J \subseteq I^*\).

**Proof.** One direction has already been done. To prove the other, we first observe that for \(\mathfrak{m}\)-primary ideals \(K\), \(e_{HK}(K) = e_{HK}(\widehat{K})\) and \((\widehat{K})^* = \widehat{K}^*\). We leave this latter equality as an exercise (see also [HH1 Proposition 4.14]). Hence we may assume that \(R\) is complete. Suppose that \(e_{HK}(I) = e_{HK}(J)\). We need to prove that \(J \subseteq I^*\). If not, there exists a minimal prime \(P\) of \(R\) such that the image of \(J\) in \(R/P\) is not in the tight closure of the image of \(I\) in \(R/P\), by Proposition 5.3. By the additivity formula for Hilbert-Kunz multiplicity, Proposition 3.14, as well as our assumption that \(R\) is formally unmixed, we must have that \(e_{HK}((I + P)/P) = e_{HK}((J + P)/P)\). Hence we may assume that \(R\) is a complete local domain.
Suppose by way of contradiction that $J$ is not in $I^*$. We may assume that $J = (x, I)$ for some $x \notin I^*$. We now use a result of Aberbach [Ab1]: since $x \notin I^*$, there exists a fixed integer $k$ such that for all $q = p^e$, $I^{[q]} : x^q \subseteq m^{\lfloor q/k \rfloor}$. But now for all large enough $q$, 

$$\lambda(R/I^{[q]}) - \lambda(R/(I^{[q]}, x^q)) = \lambda(R/m^{\lfloor q/k \rfloor}) \geq \delta q^d, \quad \text{where } \delta \text{ is any positive real strictly less than } \frac{e}{d!},$$

where $e$ is the multiplicity of $R$. This proves that $e_{HK}(I) \neq e_{HK}(J)$, a contradiction. □

With this tight closure characterization of the Hilbert-Kunz multiplicity, we can give an important estimate on it in the case the ring is Gorenstein, but not F-rational, meaning that systems of parameters are not tightly closed. This is due to Blickle and Enescu [BE].

**Proposition 5.6.** Let $(R, m, k)$ be a Noetherian local ring of dimension $d$ and prime characteristic $p$ which is Gorenstein but not F-rational. Set $e$ equal to the multiplicity of $R$. Then $e_{HK}(R) \geq 1 + \frac{1}{e-1}$.

**Proof.** We may assume that the residue field is infinite. Choose a minimal reduction of the maximal ideal and let $J$ be the ideal generated by that reduction. Observe that $\lambda(R/J) = e$.

Since $R$ is not F-rational, $J^* \neq J$. We use Lemma 3.4 to see that

$$e = e_{HK}(J) = e_{HK}(J^*) = \lambda(R/J^*)e_{HK}(R) \leq (e - 1)e_{HK}(R)$$

giving the result. □

If $e > d!$, then since $e_{HK}(R) \geq e/d!$, we see that $e_{HK}(R) \geq 1 + \frac{1}{d!}$. On the other hand, if $e \leq d!$, then $e - 1 < d!$, and Proposition 5.6 shows that in the Gorenstein but not F-rational case, we have the same estimate that $e_{HK}(R) \geq 1 + \frac{1}{d!}$.

**Remark 5.7.** It is worth noting that the relationship between the Hilbert-Kunz multiplicity of ideals and the tight closure was an important idea in the construction by Brenner and Monsky [BM] of a counterexample to the localization problem in tight closure theory.
The work of Hochster and Roberts on the Cohen-Macaulayness of rings of invariants [HoR] focused attention on the splitting properties of the map from $R$ to $R^{1/p}$. If $R$ is F-finite, then this map splits as a homomorphism of $R$-modules if and only if $R$ is F-pure, i.e. the Frobenius homomorphism is a pure map. Thus the idea of splitting copies of $R$ out of $R^{1/p}$ clearly had something to say about the singularities of $R$. This idea was further explored during the development of tight closure, with the concept of strong F-regularity. In [SVB], Smith and Van den Bergh studied the asymptotic behavior of summands of $R^{1/q}$ for rings of finite F-representation type which are strongly F-regular. Yao [Y1] later removed the assumption of strong F-regularity from their work. For free summands, in [HL], the idea of the F-signature was introduced as a way to asymptotically key track of the number such summands of $R^{1/q}$ as $q$ varies. As it turns out, almost the exact same ideas were introduced at the same time by Watanabe and Yoshida [WY5] in their study of minimal Hilbert-Kunz multiplicity. The F-signature provides delicate information about the singularities of $R$, as we shall see. One immediate problem was to show that a limit exists in this asymptotic construction. When $R$ is Gorenstein, this was done in [HL], and we reproduce that argument here since it is not difficult and has the additional benefit of expressing the F-signature as a difference of the Hilbert-Kunz multiplicities of two ideals. The case when $R$ is not Gorenstein proved to be considerably harder. After many partial results (see [Ab2], [Y2], for example) Kevin Tucker recently proved the limit always exists. We give a modified version of his proof here.

We first set up the basic ideas. Let $(R, \mathfrak{m}, k)$ be a $d$-dimensional reduced Noetherian local ring with prime characteristic $p$ and residue field $k$. We assume that $R$ is F-finite. By $a_q$ we denote the largest rank of a free $R$-module appearing in a direct sum decomposition of $R^{1/q}$, where as usual $q = p^e$. We write $R^{1/q} \cong R^{a_q} \oplus M_q$ as an $R$-module, where $M_q$ has no free direct summands. The number $a_q$ is called the $e$-th Frobenius splitting number of $R$.

**Definition 6.1.** The F-signature of $R$, denoted $s(R)$, is $s(R) = \lim_{q \to \infty} \frac{a_q}{q^{d+\alpha(R)}}$, the limit taken as $q$ goes to infinity, provided the limit exists.
We first prove that the limit exists in the Gorenstein case, partly due to the ease of the proof, and partly due to the fact that it gives a precise value for the F-signature in terms of Hilbert-Kunz multiplicities. This theorem is found in [HL].

**Theorem 6.2.** Let $(R, m, k)$ be a Noetherian local reduced Gorenstein ring of dimension $d$ and prime characteristic $p$. Then \( \lim_{q \to \infty} \frac{a_q}{q^d m^d} \) exists and is equal to the difference between the Hilbert-Kunz multiplicity of the ideal $I$ generated by a system of parameters, and the Hilbert-Kunz multiplicity of the ideal $I : m$.

**Proof.** Let $I = (x_1, ..., x_d)$ be generated by a system of parameters. We claim that the difference $\lambda(M/IM) - \lambda(M/(I : m)M)$ is zero for all maximal Cohen-Macaulay modules $M$ without a free summand. We state this as a separate lemma.

**Lemma 6.3.** Let $(R, m)$ be a Gorenstein local ring and let $M$ be a maximal Cohen-Macaulay $R$-module without a free summand. Let $I$ be an ideal generated by a system of parameters for $R$, and let $\Delta \in R$ be a representative for the socle of $R/I$. Then $\Delta M \subseteq IM$.

**Proof.** Choose generators \( \{m_1, \ldots, m_n\} \) for $M$ and define a homomorphism $R \to M^n$ by $1 \mapsto (m_1, \ldots, m_n)$. Let $N$ be the cokernel, so that we have an exact sequence

\[
0 \to R \to M^n \to N \to 0.
\]

Since $M$ has no free summands, this exact sequence is nonsplit. This implies, since $R$ is Gorenstein, that $N$ is not Cohen-Macaulay. When we kill $I$, therefore, there is a nonzero $\text{Tor}$:

\[
0 \to \text{Tor}_1^R(N, R/I) \to \overline{R} \to \overline{M}^n \to \overline{N} \to 0.
\]

Since the map $\overline{R} \to \overline{M}^n$ has a nonzero kernel, we must have $\overline{\Delta} \mapsto 0$. Since the elements $m_1, \ldots, m_n$ generate $M$, this says precisely that $\Delta M \subseteq IM$. \(\square\)

Returning to the proof of Theorem 6.2, we write $R^{1/q} = R^{aq} \oplus M_q$, where $M_q$ is a maximal Cohen-Macaulay module without free summands. Applying Lemma 6.3, we then see that
\[ q^{\alpha(R)}(\lambda(R/I^{[q]}) - \lambda(R/(I, \Delta)^{[q]})) = a_q \] and therefore

\[ e_{HK}(I, R) - e_{HK}((I, \Delta), R) = s(R). \]

\[ \square \]

**Remark 6.4.** The proof above shows that the F-signature of a Gorenstein local ring is 0 if and only if for some (or equivalently for all) ideals \( I \) generated by a system of parameters, \( e_{HK}(I) = e_{HK}(I : m) \). As we have seen, this equality holds if and only if \( I \) and \( I : m \) have the same tight closure, which is true if and only if \( I \) is not tightly closed, since every ideal properly containing \( I \) must contain \( I : m \). Thus the F-signature is positive in this case if and only if \( R \) is F-rational (and then is strongly F-regular, as \( R \) is Gorenstein.) Aberbach and Leuschke [AL] proved in general that the F-signature is positive if and only if \( R \) is strongly F-regular. In fact the ideas of the proof above extend to prove something a little less than strong F-regularity, namely, that [HL, Theorem 11] if the lim sup of \( a_q/q^d \) is positive, then \( R \) must be weakly F-regular, and in particular is Cohen-Macaulay and integrally closed. Thus, if \( R \) is not weakly F-regular, \( s(R) \) exists and is 0. We prove this important fact next. For graded rings, it is known that strong and weak F-regularity are equivalent [LS].

**Remark 6.5.** Watanabe and Yoshida [WY5] systematically studied minimal possible difference between the Hilbert-Kunz multiplicity of two \( m \)-primary ideals. They go further, and introduced the notion of minimal relative Hilbert-Kunz multiplicity \( mHK(R) \). By their definition, \( mHK(R) = \lim \inf \lambda_R(R/\text{ann}_R Az^{\alpha}) \), where \( z \) is a generator of the socle of the injective hull \( E_R(k) \). They prove that \( mHK(R) \leq e_{HK}(I) - e_{HK}(I') \) for \( m \)-primary ideals \( \subset I' \) with \( \lambda_R(I'/I) = 1 \). If \( R \) is Gorenstein, they prove the minimal relative Hilbert-Kunz multiplicity is in fact \( e_{HK}(J) - e_{HK}(J : m) \) for any parameter ideal \( J \) of \( R \). As an example, we quote one of their theorems: Let \( k \) be a field of characteristic \( p > 0 \), and let \( R = k[x_1, \ldots, x_d]^G \) be the invariant subring by a finite group \( G \) of \( GL(d, k) \) with \( (p, |G|) = 1 \). Also, assume that \( G \) contains no pseudo-reflections. Then the minimal relative Hilbert-Kunz multiplicity is \( 1/|G| \).
Lemma 6.6. Assume that \((R, \mathfrak{m})\) is a reduced \(F\)-finite local ring containing a field of prime characteristic \(p\) and let \(d = \dim R\). We adopt the notation from the beginning of this section. If \(s(R) > 0\), then \(R\) is weakly \(F\)-regular.

Proof. Assume that \(s(R) > 0\), but \(R\) is not weakly \(F\)-regular, that is, not all ideals of \(R\) are tightly closed. By [HH, Theorem 6.1] \(R\) has a test element, and then [HH1, Proposition 6.1] shows that the tight closure of an arbitrary ideal in \(R\) is the intersection of \(\mathfrak{m}\)-primary tightly closed ideals. Since \(R\) is not weakly \(F\)-regular, there exists an \(\mathfrak{m}\)-primary ideal \(I\) with \(I \neq I^*\). Choose an element \(\Delta\) of \(I: \mathfrak{m}\) which is not in \(I^*\).

\[
q^{\alpha(R)}(\lambda(R/I^{[q]}) - \lambda(R/(I, \Delta)^{[q]})) = \lambda(R^{1/q}/IR^{1/q}) - \lambda(R^{1/q}/(I, \Delta)R^{1/q}) \geq a_q
\]

Dividing by \(q^{d+\alpha(R)}\) and taking the limit gives on the left-hand side a difference of Hilbert-Kunz multiplicities,

\[
e_{HK}(I) - e_{HK}((I, \Delta)) \geq s(R).
\]

But by Theorem [5.5] this difference is zero, showing that \(s(R) = 0\). \(\square\)

The beautiful idea of Tucker’s proof that the \(F\)-signature exists in general is to represent it as a limit of certain normalized Hilbert-Kunz multiplicities, which are decreasing. To capture this, we first discuss some general facts about free summands of modules.

Discussion 6.7. Let \((R, \mathfrak{m})\) be a Noetherian local reduced ring, and let \(M\) be a torsion-free \(R\)-module. We can always write \(M = N \oplus F\), where \(F\) is free and \(N\) has no free summands. We define a submodule \(M_{nf}\) of \(M\) to be \(N + \mathfrak{m}F\). On the face of it, this submodule depends on the choice of \(N\). However we can also describe this submodule by the following:

\[
\{ x \in M \mid \phi(x) \in \mathfrak{m} \forall \phi \in \text{Hom}_R(M, R) \}.
\]

To see that these are the same, simply note that clearly \(M_{nf}\) is inside the above submodule (note it is a submodule!), and conversely, if \(x\) is in the submodule, then \(x \in M_{nf}\); otherwise we can write \(x = n + y\), where \(y\) is a minimal generator of \(F\), and where \(n \in N\). The
submodule $R_y$ of $M$ clearly splits off as a free summand, so there is a $\phi : M \to R$ such that $\phi(y) = 1$. Then $\phi(x) = 1 + \phi(n) \notin m$, a contradiction. Note that $M/M_{nf}$ is a vector space of dimension equal to the rank of $F$.

**Definition 6.8.** Let $(R, m, k)$ be a reduced local Noetherian ring of prime characteristic $p$. For $q = p^e$, we let $I_q := (R^{1/q})_{nf}^{[q]}$, an ideal in $R$.

This ideal was considered in work of Yongwei Yao [Y1] as well as Florian Enescu and Ian Aberbach [AE2]. Observe that Tucker defines it as follows, which from the discussion above is equivalent to our definition:

$$I_q = \{ r \in R | \phi(r^{1/q}) \in m \forall \phi \in \text{Hom}_R(R^{1/q}, R) \}.$$ 

We group some basic remarks about these ideals in the following proposition:

**Proposition 6.9.** Let $(R, m, k)$ be a reduced local Noetherian ring of prime characteristic $p$. Then $m^{[q]} \subseteq I_q$ for all $q = p^e$. Furthermore $I_q^{[q']} \subseteq I_{qq'}$ for all $q = p^e$ and $q' = p^{e'}$. If the residue field is perfect, $\lambda(R/I_q) = a_q$.

**Proof.** Since $mR^{1/q} \subseteq (R^{1/q})_{nf}$, it is immediate from the definition that $m^{[q]} \subseteq I_q$. To prove the second statement, let $r \in I_q$, so that $r^{1/q} \in (R^{1/q})_{nf}$. Then $(r^{q'})^{1/(qq')} = r^{1/q} \in R^{1/(qq')}$ is clearly $I_{qq'}$ by the second description of these ideals, since if $\phi : R^{1/(qq')} \to R$ was such that $\phi(r^{1/q}) \notin m$, restricting $\phi$ to $R^{1/q}$ would give the contradiction that $r \notin I_q$. The last statement of the proposition follows since $\lambda(R/I_q) = \lambda(R^{1/q}/I_q^{1/q}R^{1/q}) = \lambda(R^{1/q}/(R^{1/q})_{nf}) = a_q$. \qed

We are ready to prove Tucker’s theorem:

**Theorem 6.10.** [Tm. Theorem 4.9] Let $(R, m, k)$ be a Noetherian local ring of dimension $d$ and prime characteristic $p$. Assume that $R$ is $F$-finite. Then $s(R) = \lim_{q} \frac{a_q}{q^{d+\alpha(R)}}$ exists.

**Proof.** We can complete $R$ and extend the residue field to assume that $R$ is complete with perfect residue field. By Lemma 6.6 if $R$ is not weakly $F$-regular, then $s(R) = 0$. Hence we
may assume that \( R \) is weakly F-regular, and is in particular a Cohen-Macaulay domain. We use Corollary 3.8. We have that there is a constant \( C \) such that for all \( q, q' \),

\[
|\lambda(R/I_q^{[q']}) - (q')^d \lambda(R/I_q)| \leq C(q')^d q^{d-1}.
\]

Dividing by \((q')^d\) we obtain that

\[
|\lambda(R/I_q^{[q']})/(q')^d - \lambda(R/I_q)| \leq C q^{d-1}.
\]

Taking the limit as \( q' \) goes to infinity, we see that

\[
|e_{HK}(I_q) - a_q| \leq C q^{d-1}.
\]

Dividing by \( q^d \) shows that the F-signature exists if and only if the limit of \( e_{HK}(I_q)/q^d \) exists. This follows by noting that \( I_q^{[p]} \subseteq I_{qp} \) for all \( q \), so that \( e_{HK}(I_{qp}) \leq e_{HK}(I_q^{[p]}) = p^d e_{HK}(I_q) \), so that dividing through by \( qp \) shows that the sequence \( \{e_{HK}(I_q)/q^d\} \) is decreasing, and thus has a limit, necessarily equal to \( s(R) \).

\( \square \)

**Example 6.11.** We return to Example 3.18 where the Hilbert-Kunz multiplicity of simple quotient singularities were given. Let \((R, \mathfrak{m})\) be a two-dimensional complete Cohen-Macaulay ring. Assume that \( R \) is F-finite and is Gorenstein and F-rational. Then \( R \) is a double point and is isomorphic to \( k[[x, y, z]]/(f) \), where \( f \) is one of the following:

| type | equation | char \( R \) | \( s(R) \) |
|------|----------|-------------|------------|
| \( A_n \) | \( f = xy + z^{n+1} \) | \( p \geq 2 \) | \( 1/(n + 1) \) (\( n \geq 1 \)) |
| \( D_n \) | \( f = x^2 + yz^2 + y^{n-1} \) | \( p \geq 3 \) | \( 1/4(n - 2) \) (\( n \geq 4 \)) |
| \( E_6 \) | \( f = x^2 + y^3 + z^4 \) | \( p \geq 5 \) | \( 1/24 \) |
| \( E_7 \) | \( f = x^2 + y^3 + yz^3 \) | \( p \geq 5 \) | \( 1/48 \) |
| \( E_8 \) | \( f = x^2 + y^3 + z^5 \) | \( p \geq 7 \) | \( 1/120 \) |

As in Example 3.18 in each of these examples a minimal reduction \( J \) of the maximal ideal \( \mathfrak{m} \) has the property that \( \mathfrak{m}/J \) is a vector space of dimension 1. Hence \( e_{HK}(J) - e_{HK}(R) = s(R) \) by Theorem 6.2. Since \( J \) is generated by a regular sequence and is a reduction of \( \mathfrak{m} \), \( e_{HK}(J) = e(J) = e(\mathfrak{m}) = 2 \). On the other hand, Example 3.18 gives the Hilbert-Kunz
multiplicity for each of these examples, and in each case it is $2 - 1/|G|$, where each ring is the invariant ring of a finite group $G$ acting on a power series ring, giving our statement. Notice that the F-signature is exactly $1/|G|$. The same reasoning applies to Example 3.21 to show that if the Hilbert-Kunz multiplicity is irrational in this example, as expected, then so is the F-signature in the same example.

7. A Second Coefficient

In this section we take up a more careful study of the Hilbert-Kunz function, showing that a second coefficient exists in great generality. This was proved in [HMM], and further improved in [HoY]. The approach we give in this paper is a bit different than those appearing elsewhere, following an alternate proof developed by Moira McDermott and myself, but not previously published. The proof in [HMM] relies on the theory of divisors associated to modules. The approach here rests on growth of Tor modules. In some ways this method is less transparent than that in [HMM], but this author believes it has considerable value nonetheless. We are aiming to prove:

**Theorem 7.1.** Let $(R, \mathfrak{m}, k)$ be an excellent, local, normal ring of characteristic $p$ with a perfect residue field and $\dim R = d$. Let $I$ be an $\mathfrak{m}$-primary ideal. Then $\lambda(M/I^{[q]}M) = \alpha q^d + \beta q^{d-1} + O(q^{d-2})$ for some $\alpha$ and $\beta$ in $\mathbb{R}$.

In [HoY] the condition that $R$ be normal is weakened to just assuming that $R$ satisfies Serre’s condition $R_1$.

One could hope that this theorem could be generalized to prove that there exists a constant $\gamma$ such that $\lambda(M/I^{[q]}M) = \alpha q^d + \beta q^{d-1} + \gamma q^{d-2} + O(q^{d-3})$ whenever $R$ is non-singular in codimension two. However, this cannot be true. For instance, see Example 3.13.

We first discuss the growth of Tor modules, expanding on what we did in earlier sections.

**Lemma 7.2.** Let $(R, \mathfrak{m}, k)$ be a local ring of characteristic $p$. If $T$ is a finitely generated torsion $R$-module with $\dim T = \ell$, then $\lambda(\text{Tor}_1(T, R/I^{[q]})) \leq O(q^\ell)$. 
Proof. Set \( d = \dim R \). Choose a system of parameters \((x_1, \ldots, x_d) \subseteq I\). We induct on \( \lambda(I/(x_1, \ldots, x_d)) \). If \( \lambda(I/(x_1, \ldots, x_d)) > 0 \), then there exists \( J \subset I \) with \( \lambda(I/J) = 1 \) so that we may write \( I = (J, u) \) with \( J : u = \mathfrak{m} \). For every \( q = p^n \) there is an exact sequence

\[
0 \to R/J^{[q]} : u^q \to R/J^{[q]} \to R/I^{[q]} \to 0.
\]

Tensor with \( T \) and look at the following portion of the long exact sequence:

\[
\cdots \to \Tor_1(R/J^{[q]}, T) \to \Tor_1(R/I^{[q]}, T) \to \Tor_0(R/J^{[q]} : u^q, T) \to \cdots.
\]

We have \( \lambda(\Tor_1(R/J^{[q]}, T)) \leq O(q^{d-2}) \) by induction. Also, since \( J : u = \mathfrak{m} \), we have \( \mathfrak{m}^{[q]} \subseteq J^{[q]} : u^q \) and \( \lambda(\Tor_0(R/J^{[q]} : u^q, T)) \leq \lambda(\Tor_0(R/\mathfrak{m}^{[q]}, T)) \). But \( \lambda(\Tor_0(R/\mathfrak{m}^{[q]}, T)) \) is the Hilbert-Kunz function for \( T \), so \( \lambda(\Tor_0(R/\mathfrak{m}^{[q]}, T)) = O(q^{\dim T}) \) and \( \dim T \leq d - 2 \).

We have reduced to the case where \( \lambda(I/(x_1, \ldots, x_d)) = 0 \). We need a theorem which is implicitly in Roberts [Ro] and explicitly given as Theorem 6.2 in [HH]:

**Theorem 7.3.** Let \((R, \mathfrak{m})\) be a local ring of characteristic \( p \) and let \( G_* \) be a finite complex

\[
0 \to G_n \to \cdots \to G_0 \to 0
\]

of length \( n \) such that each \( G_i \) is a finitely generated free module and suppose that each \( H_i(G_*) \) has finite length. Suppose that \( M \) is a finitely generated \( R \)-module. Let \( d = \dim M \). Then there is a constant \( C > 0 \) such that \( \ell(H_{n-t}(M \otimes_R F^e(G_*))) \leq C q^{m(d,t)} \) for all \( t \geq 0 \) and all \( e \geq 0 \), where \( q = p^e \).

Consider \( K_*((x); R) \), the Koszul complex on \((x_1, \ldots, x_d)\). Let \( H_*((x); R) \) denote the homology of the Koszul complex. We apply the above theorem to conclude that there exists a constant \( C > 0 \) such that \( \lambda(H_{d-t}(T \otimes F^e(K_*))) \leq C q^{\min(t, \ell)} \) for all \( t \) and for all \( e \). Hence \( \lambda(H_i(T \otimes F^e(K_*))) \leq O(q^t) \) for all \( i \). In general, \( H_1(T \otimes F^e(K_*))) \) maps onto \( \Tor_1(T, R/I^{[q]}) \), which gives the stated result. \( \square \)

Next we study the growth of \( \Tor_2 \).
Lemma 7.4. Let \((R, \mathfrak{m}, k)\) be a Noetherian local ring of dimension \(d\) satisfying Serre’s condition \(S_2\), and having prime characteristic \(p\). Let \(T\) be an \(R\)-module with \(\dim T \leq d - 2\). Then \(\lambda(\text{Tor}_2(T, R/I^{[q]})) = O(q^{d-2})\).

Proof. Pick a regular sequence \(x, y\) contained in the annihilator of \(T\). There is an exact sequence

\[
0 \to T' \to (R/(x, y))^n \to T \to 0
\]

Note \(\dim T' = d - 2\). Next tensor with \(R/I^{[q]}\) and consider the following portion of the long exact sequence:

\[
\cdots \to \text{Tor}_2(R/(x, y), R/I^{[q]}) \oplus^n \to \text{Tor}_2(T, R/I^{[q]}) \to \text{Tor}_1(T', R/I^{[q]}) \to \cdots.
\]

Since \(x, y\) is regular sequence, we know \(\sum_{i=0}^{2} \lambda(\text{Tor}_i(R/(x, y), R/I^{[q]})) = 0\). Also, \(\lambda(\text{Tor}_1(R/(x, y), R/I^{[q]})) = O(q^{d-2})\) by Lemma 7.2. Then \(\lambda(\text{Tor}_2(R/(x, y), R/I^{[q]})) = O(q^{d-2})\) as well. We also know that \(\lambda(\text{Tor}_1(T', R/I^{[q]})) = O(q^{d-2})\) by Lemma 7.2. From the long exact sequence above, we conclude that \(\lambda(\text{Tor}_2(T, R/I^{[q]})) = O(q^{d-2})\). \(\square\)

The main surprise is the next lemma, which shows that for the first \(\text{Tor}\), modules which are torsion-free have slower growth than those which are torsion!

Lemma 7.5. Let \((R, \mathfrak{m}, k)\) be a normal local ring of dimension \(d\) and prime characteristic \(p\). Let \(M\) be a torsion-free \(R\)-module. Then \(\lambda(\text{Tor}_1(M, R/I^{[q]})) = O(q^{d-2})\).

Proof. Consider the following exact sequence where \(M^* = \text{Hom}_R(M, R)\):

\[
0 \to M \xrightarrow{\theta} M^{**} \to T \to 0.
\]

Note that \(\theta\) is an isomorphism in codimension one and consequently \(T\) is a torsion-module with \(\dim T \leq d - 2\). We obtain the following long exact sequence:

\[
\cdots \to \text{Tor}_2(T, R/I^{[q]}) \to \text{Tor}_1(M, R/I^{[q]}) \to \text{Tor}_1(M^{**}, R/I^{[q]}) \to \text{Tor}_1(T, R/I^{[q]}) \to \cdots.
\]
From this we conclude that

\[ |\lambda(\text{Tor}_1(M, R/I[\mathcal{q}])) - \lambda(\text{Tor}_1(M^{**}, R/I[\mathcal{q}]))) \leq \lambda(\text{Tor}_2(T, R/I[\mathcal{q}]))) + \lambda(\text{Tor}_1(T, R/I[\mathcal{q}]))) \]

\[ = O(q^{d-2}). \]

The last inequality follows from Lemma 7.2 and Lemma 7.4. So we may replace \( M \) by \( M^{**} \) and assume that \( M \) has depth 2. Therefore, \( M \) is \( S_2 \) and \( M_P \) is free for all height one primes \( P \).

We can choose a regular sequence \( x, y \) such that they kill all \( \text{Tor}_R^i(M, R/I[\mathcal{q}]) \) for \( i \geq 1 \). This can be done in many ways. For example, we leave as an exercise that there exists a sequence, \( x, y \), which is a regular sequence on \( R \) and on \( M \) such that multiplication by \( x \) on \( M \) factors through a free module \( F = R^r \) and multiplication by \( y \) on \( M \) also factors through \( F \). These multiplications then induce homotopies which can be used to prove our claim.

We let \( \cdots \rightarrow F_2 \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0 \) be the start of a minimal free resolution of \( M \). We tensor with \( R/I[\mathcal{q}] \), and write ‘ for images after tensoring. Let \( Z_q \) be the kernel of the induced map from \( F'_1 \) to \( F'_0 \), and \( B_q \) be the image of the induced map from \( F'_2 \) to \( F'_1 \).

Thus, \( \text{Tor}_1(M, R/I[\mathcal{q}]) = Z_q/B_q \). Consider the short exact sequence,

\[ 0 \rightarrow \text{Tor}_1(M, R/I[\mathcal{q}]) \rightarrow F'_1/B_q \rightarrow N/I[\mathcal{q}]N \rightarrow 0, \]

where \( N \) is the kernel of the map from \( F_0 \) onto \( M \). We tensor with \( R/(x, y) \) and use that both \( x \) and \( y \) annihilate \( \text{Tor}_1(M, R/I[\mathcal{q}]) \) to see that the length of this Tor is at most \( \lambda(\text{Tor}_1(R/(x, y), N/I[\mathcal{q}]N)) + \lambda(F'_1/(B_q + (x, y)F'_1)) \leq \lambda(\text{Tor}_1(R/(x, y), N/I[\mathcal{q}]N)) + \lambda(R/((x, y) + I[\mathcal{q}]))) \cdot \text{rank}(F_1). \)

If we had the term \( R/I[\mathcal{q}] \) in the first Tor module instead of \( N/I[\mathcal{q}]N \), we could apply Theorem 7.2 directly to see the sum is \( O(q^{d-2}) \). We leave it to the reader to show that this change does not affect the order of growth. \( \square \)

We record the following two corollaries to Lemma 7.5.

**Corollary 7.6.** Let \((R, \mathfrak{m}, k)\) be a local, normal ring of characteristic \( p \) with \( \dim R = d \). Let \( M \) be a finitely generated \( R \)-module. Then for all \( i \geq 2 \), \( \lambda(\text{Tor}_i(M, R/I[\mathcal{q}])) = O(q^{d-2}). \)
Proof. Consider the exact sequence $0 \to \Omega^1(M) \to F \to M \to 0$ where $F$ is free. Hence $\lambda(\text{Tor}_i(M, R/I[q])) \cong \lambda(\text{Tor}_{i-1}(\Omega^1(M), R/I[q]))$. It follows that to prove the lemma, we need only consider the case $i = 2$, and in this case since $\Omega^1(M)$ is torsion free, the Lemma above implies that $\lambda(\text{Tor}_1(\Omega^1(M), R/I[q])) = O(q^{d-2})$, giving that $\lambda(\text{Tor}_2(M, R/I[q])) = O(q^{d-2})$. \hfill \qed

The next corollary shows that $\lambda(\text{Tor}_1(-, R/I[q]))$ is additive on short exact sequences of torsion modules, up to $O(q^{d-2})$.

**Corollary 7.7.** If $T_1$, $T_2$ and $T_3$ are torsion $R$-modules, and $0 \to T_1 \to T_2 \to T_3 \to 0$ is exact, then $|\sum_{i=1}^{3} (-1)^{i+1} \lambda(\text{Tor}_1(T_i, R/I[q]))| = O(q^{d-2})$.

**Proof.** After tensoring the exact sequence with $R/I[q]$ we obtain the following long exact sequence:

$$
\cdots \to \text{Tor}_2(T_3, R/I[q]) \to \text{Tor}_1(T_1, R/I[q]) \to \text{Tor}_1(T_2, R/I[q]) \to \text{Tor}_1(T_3, R/I[q]) \\
\to \text{Tor}_0(T_1, R/I[q]) \to \text{Tor}_0(T_2, R/I[q]) \to \text{Tor}_0(T_3, R/I[q]) \to 0
$$

We examine the cokernel at one spot in the previous sequence. Consider

$$
\to \text{Tor}_2(T_3, R/I[q]) \to \text{Tor}_1(T_1, R/I[q]) \to \text{Tor}_1(T_2, R/I[q]) \to \text{Tor}_1(T_3, R/I[q]) \to C \to 0.
$$

We know that $\lambda(\text{Tor}_2(T_3, R/I[q])) = O(q^{d-2})$ by Corollary 7.6. It is therefore enough to show that $\lambda(C) = O(q^{d-2})$. We also have the exact sequence

$$
0 \to C \to \text{Tor}_0(T_1, R/I[q]) \to \text{Tor}_0(T_2, R/I[q]) \to \text{Tor}_0(T_3, R/I[q]) \to 0.
$$

Since the $T_i$ are torsion modules, $\dim T_i \leq d - 1$, and there are constants $c_i \geq 0$ such that $\lambda(\text{Tor}_0(T_i, R/I[q])) = c_i q^{d-1} + O(q^{d-2})$ so that

$$
\lambda(C) = c_1 q^{d-1} - c_2 q^{d-1} + c_3 q^{d-1} + O(q^{d-2}).
$$

But since the Hilbert-Kunz multiplicity is additive on short exact sequences, $c_2 = c_1 + c_3$, and hence $\lambda(C) = O(q^{d-2})$. \hfill \qed
Theorem 7.8. Let \((R, \mathfrak{m}, k)\) be an excellent, local, normal ring of characteristic \(p\) with perfect residue field and with \(\dim R = d\). Let \(T\) be a torsion \(R\)-module. Then there exists \(\gamma(T) \in \mathbb{R}\) such that 
\[
\lambda(\text{Tor}_1(T, R/I^{[q]})) = \gamma(T)q^{d-1} + O(q^{d-2}).
\]

Proof. We may complete \(R\) and henceforth assume \(R\) is complete. Hence \(R\) is F-finite.

By Corollary 7.7, it is enough to prove the result for \(T = R/Q\) where \(Q\) is a height one prime of \(R\). If \(\dim T \leq d - 2\), we know that 
\[
\lambda(\text{Tor}_1(T, R/I^{[q]})) = O(q^{d-2})
\]
by Lemma 7.2 and \(\lambda(\text{Tor}_2(T, R/I^{[q]})) \leq O(q^{d-2})\) by Lemma 7.4. Let \(Q\) be a height one prime of \(R\) and consider the following exact sequence:
\[
0 \to (R/Q)^{p^{d-1}} \to (R/Q)^{1/p} \to T \to 0.
\]

Tensor with \(R/I^{[q]}\) and look at the following portion of the corresponding long exact sequence:
\[
\to \text{Tor}_2(T, R/I^{[q]}) \to \text{Tor}_1(R/Q, R/I^{[q]})^{p^{d-1}} \to \text{Tor}_1((R/Q)^{1/p}, R/I^{[q]}) \to \text{Tor}_1(T, R/I^{[q]}) \to .
\]

From this we see that
\[
|p^{d-1}\lambda(\text{Tor}_1(R/Q, R/I^{[q]})) - \lambda(\text{Tor}_1((R/Q)^{1/p}, R/I^{[q]}))| = O(q^{d-2}),
\]

Next consider the exact sequence \(0 \to Q^{1/p} \to R^{1/p} \to (R/Q)^{1/p} \to 0\). First note that 
\[
\lambda(\text{Tor}_1(R^{1/p}, R/I^{[q]})) = O(q^{d-2})\]
by Lemma 7.5. From the usual long exact sequence on Tor we observe that
\[
\lambda(\text{Tor}_1((R/Q)^{1/p}, R/I^{[q]})) \leq \lambda(\text{Tor}_0(Q^{1/p}, R/I^{[q]})) - \lambda(\text{Tor}_0(R^{1/p}, R/I^{[q]}))
\]
\[
+ \lambda(\text{Tor}_0((R/Q)^{1/p}, R/I^{[q]})) + O(q^{d-2})
\]
\[
\leq \lambda(\text{Tor}_0(Q, R/I^{[pq]})) - \lambda(\text{Tor}_0(R, R/I^{[pq]}))
\]
\[
+ \lambda(\text{Tor}_0(Q, R/I^{[pq]})) + O(q^{d-2})
\]
Now consider the sequence $0 \to Q \to R \to R/Q \to 0$. After tensoring with $R/I^{[pq]}$, it is clear from the usual long exact sequence that

$$\lambda(\text{Tor}_1(R/Q, R/I^{[pq]})) = \lambda(\text{Tor}_0(Q, R/I^{[pq]})) - \lambda(\text{Tor}_0(R, R/I^{[pq]})) + \lambda(\text{Tor}_0(R/Q, R/I^{[pq]})).$$

Combining this with the previous inequality shows that

$$\lambda(\text{Tor}_1((R/Q)^{1/p}, R/I^{[q]})) \leq \lambda(\text{Tor}_1(R/Q, R/I^{[pq]})) + O(q^{d-2}).$$

Combining (7.1) and the previous inequality yields

$$p^{d-1}\lambda(\text{Tor}_1(R/Q, R/I^{[q]})) - \lambda(\text{Tor}_1(R/Q, R/I^{[pq]})) \leq O(q^{d-2}).$$

Recall that $q = p^e$. Define $\delta_q = \lambda(\text{Tor}_1(R/Q, R/I^{[q]}))/q^{d-1}$. We claim that $\{\delta_q\}$ is a Cauchy sequence. We use the previous inequality to observe that

$$\delta_{pq} - \delta_q = \lambda(\text{Tor}_1(R/Q, R/I^{[pq]}))/(pq)^{d-1} - p^{d-1}\lambda(\text{Tor}_1(R/Q, R/I^{[q]}))/p^{d-1}q^{d-1}$$

$$= O(1/q)$$

The sequence $\{\delta_q\}$ converges to some $\gamma(R/Q) \in \mathbb{R}$. A simple argument shows further that $|\delta_q - \gamma(R/Q)| = O(q^{-1})$. Hence $\lambda(\text{Tor}_1(R/Q, R/I^{[q]})) = \gamma(R/Q)q^{d-1} + O(q^{d-2}).$  

**Proposition 7.9.** Let $(R, m, k)$ be an excellent, local, normal ring of characteristic $p$ with $\dim R = d$. Let $M$ be a torsion-free $R$-module of rank $r$. Then there exists $\gamma(M) \in \mathbb{R}$ such that

$$\lambda(\text{Tor}_0(M, R/I^{[q]})) - r\lambda(\text{Tor}_0(R, R/I^{[q]})) = \gamma(M)q^{d-1} + O(q^{d-2}).$$

**Proof.** We may complete $R$ and henceforth assume $R$ is complete. Since $M$ is torsion-free of rank $r$ as an $R$-module, we can choose an embedding $R^r \to M$ such that the cokernel $T$ is a torsion module over $R$, and so $\dim T \leq d - 1$. We have the following exact sequence:

$$0 \to R^r \to M \to T \to 0.$$ Tensor with $R/I^{[q]}$ and consider the usual long exact sequence:

$$0 \to \text{Tor}_1(M, R/I^{[q]}) \to \text{Tor}_1(T, R/I^{[q]}) \to \text{Tor}_0(R, R/I^{[q]}\oplus r$$

$$\to \text{Tor}_0(M, R/I^{[q]}) \to \text{Tor}_0(T, R/I^{[q]} \to 0.$$
We know that $\lambda(\text{Tor}_1(M, R/I[q])) = O(q^{d-2})$ by Lemma 7.3 and
\[
\lambda(\text{Tor}_1(T, R/I[q])) = \gamma(T)q^{d-1} + O(q^{d-2})
\]
by Theorem 7.8. Also, $\lambda(\text{Tor}_0(T, R/I[q]))$ is the Hilbert-Kunz function for $T$ and therefore there is a constant $C \geq 0$ such that $\lambda(\text{Tor}_0(T, R/I[q])) = Cq^{d-1} + O(q^{d-2})$. Thus,
\[
\lambda(\text{Tor}_0(M, R/I[q])) - r\lambda(\text{Tor}_0(R, R/I[q])) = \gamma(M)q^{d-1} + O(q^{d-2})
\]
for some $\gamma(M) \in \mathbb{R}$. □

Corollary 7.10. Let $R$ be an excellent, local, normal ring of characteristic $p$ with perfect residue field and $\dim R = d$. Then there exists $\gamma = \gamma(R^{1/p}) \in \mathbb{R}$ such that $\lambda(\text{Tor}_0(R, R/I[q])) - p^d\lambda(\text{Tor}_0(R, R/I[q])) = \gamma q^{d-1} + O(q^{d-2})$.

Proof. We complete $R$ and assume it is complete. Then $R^{1/p}$ is a finitely generated $R$-module of rank $p^d$. Thus,
\[
\lambda(\text{Tor}_0(R^{1/p}, R/I[q])) - p^d\lambda(\text{Tor}_0(R, R/I[q])) = \gamma q^{d-1} + O(q^{d-2})
\]
for some $\gamma \in \mathbb{R}$ by Proposition 7.9. As $\lambda(\text{Tor}_0(R^{1/p}, R/I[q])) = \lambda(\text{Tor}_0(R, R/I[q]))$, we have
\[
\lambda(\text{Tor}_0(R, R/I[q])) - p^d\lambda(\text{Tor}_0(R, R/I[q])) = \gamma(R^{1/p})q^{d-1} + O(q^{d-2}).
\]
□

The next two theorems are the main content in [HMM]. As mentioned earlier, the approach in this paper is through divisors attached to modules, rather than the growth of the length of Tor modules. See [K2] for further analysis of the second coefficient.

Theorem 7.11. Let $(R, m, k)$ be an excellent, local, normal ring of dimension $d$ and prime characteristic $p$ with a perfect residue field. Then there exists $\beta(R) \in \mathbb{R}$ such that $\lambda(R/I[q]) = e_{HK}(R)q^d + \beta(R)q^{d-1} + O(q^{d-2})$. 
Proof. We may complete \( R \) and henceforth assume \( R \) is complete. Define \( \epsilon_q := \lambda(R/I^{[q]}) - (\gamma(R^{1/p})/(p^{d-1} - p^d))q^{d-1} \). Recall that \( q = p^e \). We claim that \( \{\epsilon_q/q^d\} \) is a Cauchy sequence. Corollary 7.10 shows that \( \epsilon_{pq} - p^d\epsilon_q = O(q^{d-2}) \). Hence \( |\epsilon_{pq}/(pq)^d - \epsilon_q/q^d| = O(q^{-2}) \). The sequence \( \{\epsilon_q/q^d\} \) converges to some \( \alpha(R) \in \mathbb{R} \). Another simple geometric series argument shows that \( |\epsilon_q/q^d - \alpha(R)| = O(q^{-2}) \) and so \( \epsilon_q = \alpha(R)q^d + O(q^{d-2}) \). In other words, \( \lambda(R/I^{[q]}) = \alpha(R)q^d + \beta(R)q^{d-1} + O(q^{d-2}) \) where \( \beta(R) = \gamma(R^{1/p})/(p^{d-1} - p^d) \). Clearly \( \alpha(R) = \epsilon_{HK}(R) \) is forced. \[ \square \]

**Theorem 7.12.** Let \((R, m, k)\) be an excellent, local, normal ring of dimension \( d \) and prime characteristic \( p \) with a perfect residue field. Let \( M \) be finitely generated \( R \)-module. Then there exists \( \beta(M) \in \mathbb{R} \) such that \( \lambda(M/I^{[q]}M) = \epsilon_{HK}(M)q^d + \beta(M)q^{d-1} + O(q^{d-2}) \).

**Proof.** We may complete \( R \) and henceforth assume \( R \) is complete. Suppose \( M \) is a torsion-free \( R \)-module of rank \( r \). We know that \( \lambda(\text{Tor}_0(M, R/I^{[q]})) - r\lambda(\text{Tor}_0(R, R/I^{[q]})) = \gamma(M)q^{d-1} + O(q^{d-2}) \) for some \( \gamma(M) \in \mathbb{R} \) by Proposition 7.1. By Theorem 7.11 we know that \( \lambda(R/I^{[q]}) = \alpha(R)q^d + \beta(R)q^{d-1} + O(q^{d-2}) \). Combining these two results gives:

\[
\lambda(\text{Tor}_0(M, R/I^{[q]})) - r(\alpha(R)q^d + \beta(R)q^{d-1} + O(q^{d-2})) = \gamma(M)q^{d-1} + O(q^{d-2})
\]

\[
\lambda(\text{Tor}_0(M, R/I^{[q]})) = r\alpha(R)q^d + (r\beta(R) + \gamma(M))q^{d-1} + O(q^{d-2})
\]

If \( M \) is not torsion-free, then we have the following exact sequence where \( \overline{M} \) is torsion-free:

\[
0 \rightarrow T \rightarrow M \rightarrow \overline{M} \rightarrow 0.
\]

Tensor with \( R/I^{[q]} \) and consider the usual long exact sequence

\[
\cdots \rightarrow \text{Tor}_1(\overline{M}, R/I^{[q]}) \rightarrow T/I^{[q]}T \rightarrow M/I^{[q]}M \rightarrow \overline{M}/I^{[q]}\overline{M} \rightarrow 0.
\]

We know \( \lambda(\text{Tor}_1(\overline{M}, R/I^{[q]})) = O(q^{d-2}) \) by Lemma 7.3. Also, \( \lambda(T/I^{[q]}T) = \epsilon_{HK}(T)q^{\dim T} + O(q^{\dim T-1}) \) and \( \dim T \leq d - 1 \). Hence the result for \( M \) follows from the result for \( \overline{M} \). \[ \square \]
8. Estimates on Hilbert-Kunz Multiplicity

In this section we discuss estimates of the Hilbert-Kunz multiplicity. A key motivating idea in this process was introduced in the paper of Blickle and Enescu [BE] which proved that for rings which are not regular, the Hilbert-Kunz multiplicity is bounded away from 1 uniformly. This is the content of Proposition 4.7, which gives the lower bound of $1 + \frac{1}{p^d!}$ for formally unmixed non-regular rings. However, it was felt that the presence of the characteristic $p$ in the formula bounding the Hilbert-Kunz multiplicity away from 1 should not be necessary. Watanabe and Yoshida [WY4] made this explicit with the following conjecture:

**Conjecture 8.1.** Let $d \geq 1$ be an integer and $p > 2$ a prime number. Put $R_{p,d} := \mathbb{F}_p[[x_0, x_1, \ldots, x_d]]/(x_0^2 + \cdots + x_d^2)$. Let $(R, \mathfrak{m}, k)$ be a $d$-dimensional unmixed local ring with $k = \mathbb{F}_p$, an algebraic closure of the field with $p$-elements. Then the following statements hold.

1. If $R$ is not regular, then $e_{HK}(R) \geq e_{HK}(R_{p,d}) \geq 1 + a_d$, where $a_d$ is the $d$th coefficient of the power series expansion of $\sec(x) + \tan(x)$ around $0$.

2. If $e_{HK}(R) = e_{HK}(R_{p,d})$, then the $\mathfrak{m}$-adic completion $\hat{R}$ of $R$ is isomorphic to $R_{p,d}$ as local rings.

There are several methods which have been used to estimate the Hilbert-Kunz multiplicity. Perhaps the most effective method is due to Watanabe and Yoshida, the method of estimation by computing volumes. Closely related ideas were also introduced by Hanes [Ha]. We illustrate this method in the simplest case where $R$ is a Cohen-Macaulay local ring of dimension 2. Higher dimensional cases are of course more difficult, but the basic volume estimates are similar. The point is to estimate $l_A(m^{[q]}/J^{[q]})$ (where $J$ is a minimal reduction of $m$) using volumes in $\mathbb{R}^d$. In a later paper, Watanabe and Yoshida use the methods, somewhat refined, to study higher dimension. In [WY4], they prove their conjecture up to dimension four. Aberbach and Enescu [AE4] have extended this by verifying the first part of the conjecture up to dimension six. Dimension seven is open as of the time this article was written.
We need the following lemma to prove Theorem 8.3. Just as in [WY1], it is convenient to adopt the following notation: if $t$ is a real number, then $I^t := I^{[t]}$.

**Lemma 8.2.** Let $(R, m, k)$ be an unmixed local ring of dimension $R = 2$, of prime characteristic $p$, and infinite residue field. Let $J$ be a parameter ideal of $R$. Let $1 \leq s < 2$. Then we have the following limits:

$$\lim_{q \to \infty} \frac{\lambda(R/J^{sq})}{q^2} = e(J)\frac{s^2}{2}, \quad \lim_{q \to \infty} \lambda\left(\frac{J^{sq} + (J^*)[q]}{J[q]}\right) = e(J) \cdot \frac{(2 - s)^2}{2}$$

**Proof.** We leave these for the reader as an exercise. The first follows from the usual Hilbert-Samuel multiplicity, while the second can be immediately reduced to the case in which $R$ is a power series ring and the parameters are regular parameters. In this case the second limit can be thought of as computing a certain volume. We will describe the $d$-dimensional case after proving the theorem.

**Theorem 8.3.** [WY1 Corollary] Let $(R, m, k)$ be a two-dimensional Cohen–Macaulay local ring of prime characteristic $p$. Put $e = e(R)$, the multiplicity of $R$. Then the following statements hold:

1. $e_{HK}(R) \geq \frac{e+1}{2}$.
2. Suppose that $k = \overline{k}$. Then $e_{HK}(R) = \frac{e+1}{2}$ holds if and only if the associated graded ring $gr_m(R)$ is isomorphic to the Veronese subring $k[X,Y]^{(e)}$.

**Proof.** We will only prove the first statement. We claim that

$$e_{HK}(R) \geq \frac{r + 2}{2r + 2} e,$$

where $e$ is the multiplicity of $R$, and $r$ is the minimal number of generators of $m/J^*$. The theorem follows easily from this inequality, since the fact that $e \geq r - 1$ implies that $\frac{e+1}{2} \leq \frac{r+2}{2r+2} e$.

To prove the above claim, we let $s$ be a real number, $1 \leq s < 2$. We may assume that the residue field is infinite, and we then choose a minimal reduction $J$ of the maximal ideal.
Note that $\lambda(m^q/(J^*[q]) = e q^2 - e_{HK}(R)q + O(q)$, by the tight closure characterization of the Hilbert-Kunz multiplicity, Theorem 5.5 and Theorem 3.12.

We have the following:

$$\lambda(m^q/(J^*[q]) \leq \lambda((m^q + m^{sq})/(J^*[q] + m^{sq})) + \lambda(((J^*[q] + m^{sq})/(J^*[q] + J^{sq}) + \lambda(((J^*[q] + J^{sq})/J^q)).$$

The middle term in this sum is negligible, since $J$ is a reduction of $m$, so that there is a fixed power of $m$ annihilating these modules, and the number of generators of a power of $m$ grows as $O(q)$. Hence the entire term in $O(q)$.

We prove that

$$\lambda((m^q + m^{sq})/(J^*[q] + m^{sq})) = r \cdot \lambda(R/J^{(s-1)q}) + O(q).$$

By our assumption, we can write as $m = J^* + Ru_1 + \cdots + Ru_r$. Since $J^{(s-1)q}u_i^q \subseteq m^{sq} \subseteq m^{sq} + (J^*[q]$, we have

$$\lambda\left(\frac{m^q + m^{sq}}{(J^*[q] + m^{sq})}\right) \leq \sum_{i=1}^r \lambda\left(R/(J^*[q] + m^{sq}) : u_i^q\right) \leq r \cdot \lambda(R/J^{(s-1)q}).$$

Also, we have $\lambda((J^*[q] + J^q) = O(q^{d-1})$ by Theorem 5.5. Hence

$$\lambda(m^q/(J^*[q]) \leq r \cdot \lambda(R/J^{(s-1)q}) + \lambda\left(\frac{(J^*[q] + J^{sq})}{J^q}\right) + O(q).$$

Dividing by $q^2$ and letting $q$ go to infinity, it follows from Lemma 8.2 that

$$e_{HK}(J) - e_{HK}(m) \leq r \cdot e \cdot \frac{(s - 1)^2}{2} + e \cdot \frac{(2 - s)^2}{2}.$$ 

Setting $s = \frac{r+2}{r+1}$ proves the claim and finishes the proof of the theorem.

The more general situation is as follows. We take the next discussion directly from [WY4].

For any positive real number $s$, we put

$$v_s := \text{Vol}\left\{(x_1, \ldots, x_d) \in [0,1]^d \left| \sum_{i=1}^d x_i \leq s \right.\right\}, \quad v'_s := 1 - v_s,$$
where $\text{Vol}(W)$ denotes the volume of $W \subseteq \mathbb{R}^d$. With this notation, a key theorem in the work of Watanabe and Yoshida is the following:

**Theorem 8.4.** Let $(R, \mathfrak{m}, k)$ be an unmixed local ring of characteristic $p > 0$. Put $d = \dim R \geq 1$. Let $J$ be a minimal reduction of $\mathfrak{m}$, and let $r$ be an integer with $r \geq \mu_R(\mathfrak{m}/J^*)$, where $J^*$ denotes the tight closure of $J$. Also, let $s \geq 1$ be a rational number. Then we have

\[
e_{HK}(R) \geq e(R) \left\{ v_s - r \cdot \frac{(s-1)^d}{d!} \right\}.
\]

This has been extended in [AE4].

**Example 8.5.** [cf. [BC, WY1]] Let $(R, \mathfrak{m}, k)$ be a hypersurface local ring of characteristic $p > 0$ with $d = \dim R \geq 1$. Then

\[
e_{HK}(R) \geq \beta_{d+1} \cdot e(R),
\]

where $\beta_{d+1}$ is given by the formula:

\[
\text{Vol} \left\{ x \in [0,1]^d \left| \frac{d-1}{2} \leq \sum x_i \leq \frac{d+1}{2} \right. \right\} = 1 - v_{d-1} - v'_{d+1}.
\]

The first few values of $\beta_{d+1}$, beginning at $d = 0$ are the following: $1, 1, \frac{3}{4}, \frac{2}{3}, \frac{115}{192}$, and for $d = 5, \frac{11}{20}$.

**Exercise 8.6.** ([WY1] Theorem (2.15)) Let $(R, \mathfrak{m}, k)$ be a local ring of characteristic $p > 0$. Let $G = gr_{\mathfrak{m}}(R)$ the associated graded ring of $R$ with respect $\mathfrak{m}$ as above. Then $e_{HK}(R) \leq e_{HK}(G_{2\mathfrak{m}}) \leq e(R)$. Give an example to show that equality does not necessarily hold. (In fact, it seldom holds.)

Our final bounds rest on another technique, due to Aberbach and Enescu, as refined by Celikbas, Dao, Huneke, and Zhang, which allow one to give a uniform lower bound on the Hilbert-Kunz functions of non-regular rings. The basic idea of Aberbach and Enescu is to adjoint roots of elements in some fixed minimal reduction of the maximal ideal. In a bounded number of steps of such adjunctions, one reaches a ring which is not F-rational. In this case as we have seen, there are good lower bounds for the Hilbert-Kunz multiplicity. This reduces
the problem to understanding the relationship between Hilbert-Kunz multiplicity of a ring and the ring adjoined some root. At this point the estimates in [CDHZ] are helpful. The first uniform bound was given in [AE3]:

**Theorem 8.7** (Aberbach-Enescu). Let \((R, m, k)\) be an unmixed ring of dimension \(d \geq 2\) and prime characteristic \(p\). If \(R\) is not regular, then

\[
e_{HK}(R) \geq 1 + \frac{1}{d(d! (d-1)+1)^d}.
\]

This bound was improved in the paper [CDHZ] as we describe below. The essential new idea is in the following proposition:

**Proposition 8.8.** Let \(R\) be a local Noetherian domain, and let \(I = (J, u)\) where \(J\) is an integrally closed \(m\)-primary ideal of \(R\) and \(u \in \text{Soc}(J)\). If \(M\) is a finitely generated torsion-free \(R\)-module, then

\[
\ell(IM/JM) \geq \text{rank}(M).
\]

**Proof.** Set \(N = (JM :_M u)\). Since \(M/N \cong (J, u)M/JM\) and \(mM \subseteq N\), we can write \(M = N + N'\) with \(\mu(N') = \ell\left(\frac{IM}{JM}\right)\). Thus it suffices to prove \(\mu(N') \geq \text{rank}(M)\). Since \(u(M/N') \subseteq J(M/N')\), it follows from the determinantal trick [SH 2.1.8] that there is an element \(r = u^n + j_1 \cdot u^{n-1} + \cdots + j_n\) with \(j_i \in J^i\) for all \(i\) such that \(rM \subseteq N'\). Observe that \(r \neq 0\) since \(J\) is integrally closed and \(u \notin J\). Since \(M_r = N'_r\), this implies that \(\mu(N') \geq \text{rank}(N') = \text{rank}(M)\). \(\square\)

Given two ideals \(I\) and \(J\) with \(J \subseteq I\), \(\overline{\ell}(I/J)\) will denote the longest chain of integrally closed ideals between \(J\) and \(I\).

**Corollary 8.9.** Let \(R\) be a Noetherian local domain. Let \(J\) be an integrally closed \(m\)-primary ideal of \(R\) and let \(I\) be an ideal containing \(J\). If \(M\) is a finitely generated torsion-free \(R\)-module, then

\[
\ell(IM/JM) \geq \overline{\ell}(I/J) \cdot \text{rank}(M).
\]
Proof. Set \( n = \ell(I/J) \). Then there is a chain of ideals

\[ J = K_0 \subset K_1 \subset \ldots \subset K_{n-1} \subset K_n = I \]

with \( K_i = K_i \) for all \( i \). Then

\[
\ell(IM/JM) \geq \sum_{j=0}^{n} \ell(K_{j+1}M/K_jM) \geq \sum_{j=0}^{n} \ell((K_j, u_j)M/K_jM)
\]

for some \( u_j \in K_{j+1} \cap Soc(K_j) \). Thus the result follows from Proposition 8.8. \( \square \)

One of the important ideas in proving that Hilbert-Kunz multiplicity equal to one implies regularity was showing an inequality \( e_{HK}(I) \geq \lambda(R/I) \) for a suitable \( m \)-primary ideal \( I \). Recall that must have equality if \( R \) is regular. This idea was developed in [WY1, 2.17], where the following questions were raised:

Let \( R \) be a Cohen-Macaulay local ring of characteristic \( p > 0 \). Then for any \( m \)-primary ideal \( I \), do we have (1) \( e_{HK}(I) \geq \ell(R/I) \)? (2) If \( \text{pd}_R(R/I) < \infty \), is \( e_{HK}(I) = \ell(R/I) \)?

The answer to both questions turns out to be negative; for example, see the paper of Kurano [K1]. The next exercise shows that (1) is true for many \( m \)-primary ideals [CDHZ]:

**Exercise 8.10.** Assume \( R \) is an excellent normal ring with an algebraically closed residue field. If \( I \) is an integrally closed \( m \)-primary ideal of \( R \), then

\[
e_{HK}(I) \geq \ell(R/I) + e_{HK}(R) - 1.
\]

If \( I \) is an \( m \)-primary ideal such that there is an integrally closed ideal \( K \subset I \) with \( \ell(I/K) = 1 \), then

\[
e_{HK}(I) \geq \ell(R/I).
\]

(Hint: Use [Wa, 2.1] and Corollary 8.9)

We turn to better uniform lower bounds for the Hilbert-Kunz multiplicity. An important point is the following, which we leave as an exercise (see [CDHZ]):
Exercise 8.11. Assume $R$ is Cohen-Macaulay and normal, and let $x \in m - m^2$ be part of a minimal reduction of $m$. Let $S = R[y]$ with $y^n = x$. Then $mS + (y^i)$ is integrally closed for any nonnegative integer $i$.

Corollary 8.12. Assume that $(R, m, k)$ is a Cohen-Macaulay normal local ring of prime characteristic $p$ with infinite residue field. Let $x \in m - m^2$ be part of a minimal reduction of $m$ and let $S = R[y]$ with $y^n = x$. Then

$$e_{HK}(R) - 1 \geq \frac{e_{HK}(S) - 1}{n}.$$ 

Proof. It follows from Proposition 8.11 and Corollary 8.9 that

$$e_{HK}(mS) \geq \ell(S/mS) + e_{HK}(S) - 1$$

Note that $S/mS \cong k[y]/(y^n)$. So $\ell(S/mS) = n$. Moreover, $e_{HK}(mS) = n \cdot e_{HK}(R)$ by Theorem 3.16. Therefore,

$$n \cdot e_{HK}(R) \geq n + e_{HK}(S) - 1$$

and hence the result follows. \qed

We can now give a rough lower bound on the Hilbert-Kunz multiplicity of non-regular local ring, which depends only upon the dimension of the ring.

Theorem 8.13. Let $(R, m, k)$ be a formally unmixed Noetherian local ring of prime characteristic $p$, multiplicity $e > 1$, and dimension $d$. Then $e_{HK}(R) \geq 1 + \frac{1}{d!}$. 

Proof. If $e_{HK}(R) \geq 1 + 1/d!$, there is nothing to prove. Hence we may assume that $e_{HK}(R) < 1 + 1/d!$, and then $R$ is $F$-regular and Gorenstein by [AE3, 3.6] (see Proposition 5.6 as well). Thus we may assume that $R$ is $F$-rational and Gorenstein.

Let $(x) = (x_1, \cdots, x_d)$ be a minimal reduction of $m$. Consider the set of overrings $S = R[x_1^{1/n}, \ldots, x_i^{1/n}] = R_{i,n}$ which are not $F$-rational. Choose $n$ and $i$ such that we attain $\min \{n^i : R_{i,n} \text{ is not } F\text{-rational}\}$. Set $S = R_{i,n}$. Then by Proposition 5.6 applied to $x_1^{1/n}, \ldots, x_i^{1/n}, x_{i+1}, \ldots, x_d$,

$$e_{HK}(S) \geq \frac{e(S)}{e(S) - 1}.$$
However, since $S/(x_1^{1/n}, \ldots, x_i^{1/n}, x_{i+1}, \ldots, x_d) \cong R/(x)$, we have $e(S) = e$. Therefore, $e_{HK}(S) \geq 1 + \frac{1}{e-1}$.

Let $R_0 = R$, and for each $i \geq j \geq 1$, let $R_j = R_{j-1}[x_j^{1/n}]$, then by Corollary 8.12,

$$e_{HK}(R_j) - 1 \geq \frac{e_{HK}(R_{j-1}) - 1}{n}.$$ 

Since $e - 1 < d!$, it remains to prove that

$$\min \{ n^i : R_{e,n} \text{ is not F-regular} \} \leq d^d.$$ 

To do this we note that it suffices to prove that $R[x_1^{1/d}, \ldots, x_d^{1/d}]$ is not F-regular. Set $y_i = x_i^{1/d}$. Then a socle representative of $S/(x)$ is $u \cdot y_1^{d-1} \ldots y_d^{d-1}$, where $u$ generates the socle of $(xR)$. Let $v$ be any discrete valuation centered on the maximal ideal of $S$. Then we claim that

$$v(u \cdot y_1^{d-1} \ldots y_d^{d-1}) \geq dv(m).$$ 

Since $v(u) \geq v(m)$, this is clear.

It follows that $u \cdot y_1^{d-1} \ldots y_d^{d-1} \in (mS)^d$. By the tight closure Briançon-Skoda theorem [HH1, Section 5] this implies that $(x_1, \ldots, x_d)S$ is not tightly closed, which gives the desired conclusion. \hfill \Box

Another approach, closely related to the volume methods of Watanabe and Yoshida, was given by Douglas Hanes in [Ha]. We close this survey with some of his results. See in particular [Ha, Theorem 2.4] and [Ha, Corollary 2.8].

**Theorem 8.14.** Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring of prime characteristic $p$, and dimension $d \geq 2$. Let $I$ be an $\mathfrak{m}$-primary ideal, and set $t = \mu(I)$. Then,

$$e_{HK}(I) \geq \frac{e(I)}{d!} \cdot \frac{t}{(t^{1/(d-1)} - 1)^{d-1}}.$$ 

**Proof.** We note that $I^q \subseteq I^q$ for all $q = p^r$, and $\mu(I^q) \leq t$ for all $q$. Hence, for all $q = p^e$ and any $s \in \mathbb{N}$, $\lambda(I^q + I^{q+s})/I^{q+s}) \leq t \cdot \lambda(R/I^s)$. Therefore, for all $q = p^e$ and any $s \in \mathbb{N}$, we see that

$$\lambda(R/I^q) \geq \lambda(R/(I^q + I^{q+s})) \geq \lambda(R/I^{q+s}) - t \cdot \lambda(R/I^s).$$
Just as in the work of Watanabe and Yoshida, the key point is to choose $s$ carefully. Set $s = q\alpha$. We obtain that
\[
\left(\frac{e(I)}{d!}\right)[(q + q\alpha)^d - t(q\alpha)^d] \leq \lambda(R/I^{[q]}) + O(q^{d-1}).
\]
Ignoring the $O(q^{d-1})$ term and computing the maximal value of the function on the left-hand side of this equation, we obtain that a maximum is achieved when
\[
\alpha = \frac{1}{(t/\alpha - 1)^d - 1}.
\]
The best lower bound for $e_{HK}(I)$ is obtained by setting $s = \left\lfloor \frac{q}{(t/\alpha - 1)^d - 1} \right\rfloor$. Note that $s > 0$, since $t \geq d \geq 2$. We may write $s = q(\alpha - \epsilon)$ where $\epsilon < 1/q$. Applying the equations above with this value of $s$ gives us that
\[
\lambda(R/I^{[q]}) \geq \left(\frac{e(I)}{d!}\right)q^d[(1 + \alpha - \epsilon)^d - t(\alpha - \epsilon)^d] + O(q^{d-1}).
\]
Dividing through by $q^d$, and letting $q$ go to infinity (and $\epsilon$ toward 0), we obtain the estimate
\[
e_{HK}(I) \geq \left(\frac{e(I)}{d!}\right)[(1 + \alpha)^d - t(\alpha)^d],
\]
from which the theorem follows.

**Corollary 8.15.** Let $(R, m, k)$ be a d-dimensional hypersurface ring of prime characteristic $p$, where $d \geq 3$. Then $e_{HK}(R) \geq e(R)2^{d-1}/d!$.

**Proof.** Apply the previous theorem. Notice that the function $F(t) = \frac{t}{(t/\alpha - 1)^d - 1}$ is decreasing, and $F(2^{d-1}) = 2^{d-1}$. As long as $\mu(m) \leq 2^{d-1}$ we can then apply the theorem. Since $\mu(m) \leq d + 1$ and $d \geq 3$, the inequality holds.

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