Jump processes on the boundaries of random trees

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Abstract

In [7], Kigami showed that a transient random walk on a deterministic infinite tree \( T \) induces its trace process on the Martin boundary of \( T \). In this paper, we will deal with trace processes on Martin boundaries of random trees instead of deterministic ones, and prove short time log-asymptotic of on-diagonal heat kernel estimates and estimates of mean displacements.

1 Introduction

Consider an infinite tree \( T \) and a transient random walk \( \{Z_n\}_{n \geq 0} \) on \( T \). The transient random walk on \( T \) finally hits its Martin boundary, which is the collection of “infinities”. It is well-known that under suitable assumptions, the transient random walk on \( T \) induces a Hunt process (equivalently a Dirichlet form) on its Martin boundary \( M \) in the following way: let \((E, F)\) be the Dirichlet form associated with \( \{Z_n\}_{n \geq 0} \) and \( \text{HARM}_T \) be the hitting distribution (called the harmonic measure) of \( M \) started from a certain point in \( T \). By the theory of Martin boundaries, we have the map \( H \) which transforms functions on \( M \) into functions on \( T \) in such a way that for a given function \( f \) on \( M \), \( Hf \) is harmonic on \( T \) and has the boundary value \( f \) on \( M \). Then the induced form \((E_M, F_M)\) on the Martin boundary \( M \) is given by

\[
F_M := \{ f \in L^2(M, \text{HARM}_T) : Hf \in F \},
\]

\[
E_M(f, g) := E(Hf, Hg) \quad \text{for} \quad f, g \in F_M.
\]

Since \( Hf \) solves the Dirichlet problem at “infinity”, \((E_M, F_M)\) can be regarded as the trace of \((E, F)\) on \( M \). In [7], Kigami constructed a Hunt process \( \{X_t\}_{t \geq 0} \) on \( M \) associated with \((E_M, F_M)\) and obtained estimates of its heat kernel \( p_t(\cdot, \cdot) \) for a deterministic tree. In particular, detailed two sided heat kernel estimates are obtained when \( \text{HARM}_T \) has the volume doubling property with respect to the intrinsic metric \( D \) on \( M \), which will be defined in Definition 2.14. We refer [2] to history and related topics.

We now give a classical example to which the above construction of jump processes is analogous. Consider the reflected Brownian motion on the unit disc \( D := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\} \). Note that the corresponding Dirichlet form \((E, F)\) is given by

\[
E(u, v) := \int_D \left( \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} \right) dx dy,
\]

\[
F := \left\{ u \in L^2(D) : \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \in L^2(D) \right\}.
\]

Let \( H \) be an operator of taking the Poisson integral of a given function \( \varphi : \partial D \to \mathbb{R} \), which is defined as follows:

\[
H \varphi(re^{i\theta}) := \int_0^{2\pi} \frac{1 - r^2}{1 - 2r \cos(\theta - \theta') + r^2} \varphi(\theta') dv(\theta'),
\]

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where $\nu$ is the normalized uniform measure on $\partial \mathcal{D}$. Note that the probability measure $\nu$ coincides with the hitting distribution on $\partial \mathcal{D}$ of the Brownian motion starting at 0 due to its rotation invariance. Now we define a quadratic form $(E_{\partial \mathcal{D}}, F_{\partial \mathcal{D}})$ by

$$E_{\partial \mathcal{D}}(\varphi, \psi) := E(H\varphi, H\psi),$$

$$F_{\partial \mathcal{D}} := \{ \varphi \in L^2(\partial \mathcal{D}, \nu) ; H\varphi \in F \}.$$

It is well-known that $(E_{\partial \mathcal{D}}, F_{\partial \mathcal{D}})$ yields a regular Dirichlet form on $L^2(\partial \mathcal{D}, \nu)$, and it corresponds to the trace process of the reflecting Brownian motion on $\partial \mathcal{D}$. We remark that $E_{\partial \mathcal{D}}$ has the following explicit expression known as the Douglas integral:

$$E_{\partial \mathcal{D}}(\varphi, \psi) = \frac{\pi}{4} \int_0^{2\pi} \int_0^{2\pi} \frac{(\varphi(\theta) - \varphi(\theta'))(\psi(\theta) - \psi(\theta'))}{\sin^2(\frac{\theta - \theta'}{2})} d\nu(\theta) d\nu(\theta').$$

In the context of potential theory on Euclidean domains, the general analogue of the Douglas integral was obtained in [5], where the kernel $1/\sin^2(\theta/2)$ of $E_{\partial \mathcal{D}}$ is replaced by the Naim kernel which was introduced in [15]. Later, in the setting of the Martin boundary of reversible Markov chains on discrete graphs, Silverstein (17) studied a similar problem. See the references introduced above for details.

In this paper, we will consider random trees instead of deterministic ones, and we are going to study properties of processes on the Martin boundary induced by transient random walks on random trees. In particular, we are interested in random trees generated by branching processes. In [7], it is assumed to analyze properties of processes on the boundary that the harmonic measure satisfies the volume doubling property with respect to the intrinsic distance $D$. But for random trees, the volume doubling property of the harmonic measure does not hold in general. We overcome this difficulty by utilizing the ergodic theory on the space of trees developed in [12] and [13].

We now explain the framework more precisely. Consider a Galton-Watson branching process with offspring distribution $\{p_k\}_{k \geq 0}$. Starting from a single individual called the root, which is denoted by $o$, this process yields a random tree $T$, which is called a Galton-Watson tree with offspring distribution $\{p_k\}_{k \geq 0}$. In this paper, we assume that $p_0 = 0$ and $T$ is supercritical (namely $m := \sum_{k \geq 0} kp_k > 1$) to guarantee that $T$ is almost surely infinite. Under the above assumptions, the Galton-Watson tree $T$ can be regarded as the $T$-valued random variable, where $T := \{ T ; T \text{ is an infinite rooted tree} \}$, and we will denote the distribution of $T$ by $\mathbb{P}_{GW}$. The structure of $T$ and its electric network has been studied extensively for many years: see [11] for references and details. Given a rooted tree $T$, we consider a $\lambda$-biased random walk on $T$ under the probability measure $P_T^\lambda$. Precisely speaking, for $\lambda > 0$, we define a Markov chain $\{Z_n^\lambda\}_{n \geq 0}$ on the vertices of $T$ such that if $u \neq o$, $u$ has $k$ children $u_1, ..., u_k$ and the parent $\pi(u)$, then

$$P_T^\lambda(Z_{n+1}^\lambda = \pi(u) \mid Z_n^\lambda = u) = \frac{\lambda}{\lambda + k},$$

$$P_T^\lambda(Z_{n+1}^\lambda = u_i \mid Z_n^\lambda = u) = \frac{1}{\lambda + k}, \quad \text{for } 1 \leq i \leq k,$$

and if $u = o$, the random walk moves to its children equally likely. It is proved in [9] that $\{Z_n^\lambda\}_{n \geq 0}$ on the supercritical Galton-Watson tree $T$ is transient for almost every $T$, if and only if $0 < \lambda < m$. Thus for $0 < \lambda < m$, we have the harmonic measure $\text{HARM}^\lambda_T$, the induced Dirichlet form $(\mathcal{E}^\lambda, \mathcal{F}^\lambda)$ and the heat kernel $p_t^\lambda(\cdot, \cdot)$ associated with $(\mathcal{E}^\lambda, \mathcal{F}^\lambda) \mathbb{P}_{GW}$-a.s. In [12] and [13], Lyons, Pemantle and Peres showed that for $0 < \lambda < m$, $\beta_\lambda := \dim \text{HARM}^\lambda_T$ is a deterministic constant for almost every $T$, see Theorem 3.1.

We now state the results on short time log-asymptotic of on-diagonal heat kernel estimates and estimates of mean displacements. Note that $d(\cdot, \cdot)$ is the natural metric on $M$ defined in Definition 2.9.

**Theorem 1.1.** For $0 < \lambda < m$, the following holds $\mathbb{P}_{GW}$-a.s.

$$- \lim_{t \to 0} \frac{\log p_t^\lambda(\omega, \omega)}{\log t} = \frac{\beta_\lambda}{\beta_\lambda - \log \lambda}, \quad \text{HARM}^\lambda_T \text{ a.e.-}\omega.$$

**Theorem 1.2.** For $0 < \lambda < m$, the following holds $\mathbb{P}_{GW}$-a.s.

$$\lim_{t \to 0} \frac{\log E_{\omega}[d(\omega, X_t)^\gamma]}{\log t} = \left( \frac{\gamma}{\beta_\lambda - \log \lambda} \right) \wedge 1, \quad \text{HARM}^\lambda_T \text{ a.e.-}\omega.$$
Since the volume doubling property of the harmonic measure, which is assumed to analyze the heat kernel in \[7\], holds only when \( p_1 = 0 \) and \( \sup\{n : p_n > 0\} < \infty \), the heat kernel estimates proved in \[7\] cannot be applied for this problem in general. Note that the above results imply that the spectral dimension (resp. the walk dimension) is \( 2\beta_\lambda/(\beta_\lambda - \log \lambda) \) (resp. \( (\beta_\lambda - \log \lambda) \lor 1 \)).

This paper is organized as follows. In Section 2, we will introduce notation and results in \[7\]. In Section 3, we will introduce notation and results on Galton-Watson trees studied in \[12\], \[13\] and \[8\], and prove the asymptotic of the effective resistance along infinite rays, which will be important for the proof of the main results. We then prove the lower bound for the dimension of the harmonic measures, which is of independent interest. In Section 4, we will give the proofs of our main results.

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## 2 Preliminaries and Kigami’s results

In this section, we will introduce some notation and the results studied in \[7\].

### 2.1 Weighted graphs and associated random walks

**Definition 2.1.** (1) A pair \((V, C)\) is called a weighted graph (an electric network) if \( V \) is a countable set and \( C : V \times V \to [0, \infty) \) satisfies \( C(x, y) = C(y, x) \) for any \( x, y \in V \) and \( C(x, x) = 0 \) for any \( x \in V \). The points of \( V \) are called the vertices of the graph \((V, C)\). Two vertices \( x, y \in V \) are said to be adjacent if and only if \( C(x, y) > 0 \).

(2) A weighted graph \((V, C)\) is called connected if and only if for any \( x, y \in V \), there exists a sequence of vertices of \( V \) \( x = x_0, x_1, x_2, \cdots, x_n = y \) such that \( x_k \) and \( x_{k+1} \) are adjacent for \( 0 \leq k \leq n - 1 \).

(3) A weighted graph \((V, C)\) is called locally finite if and only if \( \#\{y \in V : y \text{ is adjacent to } x\} < +\infty \) for all \( x \in V \).

In this paper, we always assume that \( V \) is a countable set and the weighted graph \((V, C)\) is connected and locally finite. A weighted graph defines a reversible Markov chain on \( V \) in the following way.

**Definition 2.2.** Define \( C(x) := \sum_{y \in V} C(x, y) \), \( p(x, y) := \frac{C(x, y)}{C(x)} \). For \( n \geq 0 \), we define \( p^{(n)}(x, y) \) for \( x, y \in V \) inductively by \( p^{(0)}(x, y) := 1_{x}(y) \) and

\[
p^{(n+1)}(x, y) := \sum_{z \in V} p^{(n)}(x, z)p(z, y).
\]

Define \( G(x, y) := \sum_{n=0}^{\infty} p^{(n)}(x, y) \in [0, \infty] \). \( G(x, y) \) is called the Green function of \((V, C)\). A weighted graph \((V, C)\) is said to be transient if and only if \( G(x, y) < +\infty \) for any \( x, y \in V \).

Let \( (\{Z_n\}_{n \geq 0}, \{P_x\}_{x \in V}) \) be the random walk on \( V \) associated with \((V, C)\), that is \( P_x(Z_n = y) = p^{(n)}(x, y) \).

**Definition 2.3.** (1) The Laplacian \( \Delta : l(V) \to l(V) \) associated with \((V, C)\) is defined by \( \Delta u(x) := \sum_{y \in V} p(x, y)(u(y) - u(x)) \), for any \( u \in l(V) \). A function \( u \in l(V) \) is said to be harmonic on \( V \) with respect to \((V, C)\) if and only if \( \Delta u(x) = 0 \) for any \( x \in V \). Define

\[
\mathcal{H}(V, C) := \{ u \in l(V) : u \text{ is harmonic on } V \},
\]

\[
\mathcal{H}^\infty(V, C) := \{ u \in \mathcal{H}(V, C) : u \text{ is bounded} \}.
\]

(2) Define \( \mathcal{F} := \{ u \in l(V) : \sum_{x, y \in V} C(x, y)(u(x) - u(y))^2 < +\infty \} \). For any \( u, v \in \mathcal{F} \), define

\[
\mathcal{E}(u, v) := \frac{1}{2} \sum_{x, y \in V} C(x, y)(u(x) - u(y))(v(x) - v(y)).
\]
In the rest of the section, we introduce the notion and fundamental results of the Martin boundary of the transient weighted graph. See [13, Chapter 9] for references and details.

**Definition 2.4.** Assume \((V, C)\) is transient. The Martin kernel of \((V, C)\) is 
\[
K_z(x, y) := \frac{G(x, y)}{G(z, y)} \quad \text{for } x, y, z \in V.
\]

**Proposition 2.5.** [13, Theorem 9.18.] Assume \((V, C)\) is transient. Then there exists a unique minimal compactification \(\tilde{V}\) of \(V\) (up to homeomorphism) such that \(K_z\) extends to a continuous function from \(V \times \tilde{V}\) to \(\mathbb{R}\). \(\tilde{V}\) is independent of the choice of \(z\). Moreover, there exists a \(\tilde{V} \setminus V\)-valued random variable \(Z_\infty\) such that 
\[
P_x(\lim_{n \to \infty} Z_n = Z_\infty) = 1 \quad \text{for any } x \in V.
\]

**Definition 2.6.** Assume \((V, C)\) is transient. \(\tilde{V}\) is called a Martin compactification of \(V\). Define \(M(V, C) := \tilde{V} \setminus V\), which is called the Martin boundary of \((V, C)\). Define a probability measure \(\text{HARM}_{V, z}\) on \(M(V, C)\) by 
\[
\text{HARM}_{V, z}(B) := P_x(Z_\infty \in B) \quad \text{for any Borel set } B \subseteq M(V, C).
\]

The probability measure \(\text{HARM}_{V, z}\) is called the harmonic measure of \((V, C)\) starting from \(x\). The harmonic measure actually depends on the weight \(C\), but we will denote it by \(\text{HARM}_{V, z}\) for simplicity of notation when the choice of the weight is clear from the context.

The following theorem gives the representation of harmonic functions on \((V, C)\).

**Theorem 2.7.** [13, Theorem 9.37.] Assume that \((V, C)\) is transient.
1. \(K_z(\cdot, y) \in \mathcal{H}^\infty(V, C)\) for any \(z \in V\) and any \(y \in \tilde{V}\).
2. If \(g \in \mathcal{H}^\infty(V, C)\), then there exists \(f \in L^\infty(M(V, C), \text{HARM}_{V, z})\) such that 
\[
g(x) = \int_{M(V, C)} K_z(x, y)f(y)d\text{HARM}_{V, z}(y).
\]

Note that the function \(f\) does not depend on the choice of \(z\) since by connectedness of \((V, C)\), the harmonic measures \(\text{HARM}_{V, x}\) and \(\text{HARM}_{V, z}\) are mutually absolutely continuous for \(x, z \in V\), and 
\[
K_z(x, \cdot) = \frac{d\text{HARM}_{V, x}}{d\text{HARM}_{V, z}}(\cdot).
\]

### 2.2 Transient trees and their Martin boundaries

We now consider transient trees and their Martin boundaries.

**Definition 2.8.** A weighted graph \((T, C)\) is called a tree if and only if it is connected and does not have cycles. If \((T, C)\) is a tree, then for all \(x, y \in T\), there exists a unique shortest path between \(x\) and \(y\), which will be denoted by \(\pi_{xy}\).

\((T, C)\) is said to be *rooted* when it has the fixed reference point, which will be denoted by \(o\). In the rest of this paper, we always assume that \((T, C)\) is a rooted tree.

**Definition 2.9.** (1) An infinite path \((x_0, x_1, \ldots) \in T^\mathbb{N}\) is said to be an infinite ray from \(x \in T\) if and only if \(x_0 = x\) and \((x_0, x_1, \ldots, x_n)\) is the shortest path between \(x\) and \(x_n\) for all \(n \geq 1\).

(2) For \(x \in T\), define the height of \(x\), \(h(x)\) by the length of the shortest path between \(x\) and \(o\) and \(x\). Define \(T_k := \{x \in T : h(x) = k\} \quad \text{for } k \in \mathbb{N}\).

(3) For \(x \in T\), define \(N(x) := \{y \in T : y\) is adjacent to \(x\}\). For \(x \neq o\), the parent of \(x\), which will be denoted by \(\pi(x)\), is the unique element of \(N(x)\) which satisfies \(h(\pi(x)) = h(x) - 1\). We set \(S(x) := N(x) \setminus \{\pi(x)\}\).

(4) Define \(\Sigma := \Sigma(T, C)\) to be the collection of infinite rays from \(o\) and \(\tilde{T} := T \cup \Sigma(T, C)\). For \(\omega = (\omega_0, \omega_1, \cdots) \in \Sigma\), define \(\omega_n := \omega_n\) for \(n \geq 0\), \(N(\omega, \eta) := \max\{n \geq 0 : |\omega_n| = |\eta_n|\}\) and \(|\omega, \eta| := |\omega|_{N(\omega, \eta)} = |\eta|_{N(\omega, \eta)}\) for \(\omega, \eta \in \Sigma\).

(5) Let \(d_\infty(\omega, \eta) := e^{-N(\omega, \eta)}\) with the convention \(e^{-\infty} = 0\). Then \(d(\cdot, \cdot)\) defines an ultrametric on \(\tilde{T}\). Define 
\[
B_d(\omega, r) := \{\eta \in \Sigma : d(\omega, \eta) < r\}.
\]

The following theorem due to [3] is a fundamental result on the Martin boundary of a tree.
Theorem 2.10. \cite{Kigami97} Theorem 9.22.] Assume (T, C) is transient. Then the Martin compactification \( \mathcal{T} \) of T is always homeomorphic to (\( \bar{T}, d \)).

By the above theorem, we will identify the Martin boundary \( M(T, C) \) with \( \Sigma \), then \((\Sigma, d)\) is compact. In the rest of this article, we will always assume the following condition.

Assumption 2.11. \((T(x), C|_{T(x)})\) is transient for any \( x \in T \) where \( T(x) = \{ y \in T : x \in \overline{\omega y} \} \) and \( C|_{T(x)} \) is the restriction of \( C \) to \( T(x) \).

2.3 The jump process on the boundary of a deterministic tree

In what follows, we will write \( K(\cdot, \cdot) = K_\omega(\cdot, \cdot) \) and \( \text{HARM}_T = \text{HARM}_{T, \omega} \) when \((T, C)\) is a rooted tree.

Definition 2.12. Define a linear map \( H : L^1(\Sigma, \text{HARM}_T) \to l(T) \) by

\[
Hf(x) := \int_{\Sigma} K(x, y) f(y) d\text{HARM}_T(y)
\]

for any \( x \in T \) and \( f \in L^1(\Sigma, \text{HARM}_T) \). Moreover, define \( \mathcal{F}_\Sigma := \{ f \in L^1(\Sigma, \text{HARM}_T) : Hf \in \mathcal{F} \} \) and \( \mathcal{E}_\Sigma(f, g) := \mathcal{E}(Hf, Hg) \) for any \( f, g \in \mathcal{F}_\Sigma \).

In \cite{Kigami97}, Kigami studies various properties of the quadratic form \((\mathcal{E}_\Sigma, \mathcal{F}_\Sigma)\). In particular, the following result is established.

Theorem 2.13. \cite{Kigami97} Theorem 5.6.] \((\mathcal{E}_\Sigma, \mathcal{F}_\Sigma)\) is a regular Dirichlet form on \( L^2(\Sigma, \text{HARM}_T) \).

By the above theorem, there exists a stochastic process on \( \Sigma \) which corresponds to \((\mathcal{E}_\Sigma, \mathcal{F}_\Sigma)\). Before explaining the results on the properties of this process studied in \cite{Kigami97}, we introduce the intrinsic metric on the boundary \( \Sigma \), and conditions which tell us when the harmonic measure satisfies the volume doubling property with respect to the metric.

Definition 2.14. Define \( D(x) = \text{HARM}_T(\Sigma(x)) | R(x) \) for \( x \in T \), where \( \Sigma(x) = \{ \omega \in \Sigma : [\omega]_n = x \ for \ some \ n \geq 0 \} \) and \( R(x) \) is the effective resistance from \( x \) to \( \infty \) in \( T(x) \). Note that Assumption 2.11 guarantees that \( R(x) < +\infty \) and \( D(x) < +\infty \) for any \( x \in T \). For \( \omega \neq \eta \in \Sigma \), define \( D(\omega, \eta) = D([\omega, \eta]) \) and \( D(\omega, \omega) = 0 \) for any \( \omega \in \Sigma \).

Proposition 2.15. \cite{Kigami97} Proposition 6.4.]

(1) For any \( \omega \in \Sigma \), \( \{ D([\omega]_n) \}_{n \geq 0} \) is a strictly decreasing sequence. In particular, \( D(\cdot, \cdot) \) is an ultrametric on \( \Sigma \), i.e., for any \( \omega, \tau, \eta \in \Sigma \),

\[
\max\{ D(\omega, \eta), D(\eta, \tau) \} \geq D(\omega, \tau).
\]

(2) Define \( B_D(\omega, r) = \{ \eta \in \Sigma : D(\omega, \eta) < r \} \) for any \( \omega \in \Sigma \) and \( r > 0 \). Then \( B_D(\omega, r) = [\omega]_n \) if and only if \( D([\omega]_n) < r \leq D([\omega]_{n-1}) \).

The next result tells us when the harmonic measure \( \text{HARM}_T \) satisfies the volume doubling property with respect to \( D \) (i.e., there exists a constant \( c > 0 \) such that \( \text{HARM}_T(B_D(\omega, 2r)) \leq c \text{HARM}_T(B_D(\omega, r)) \) for any \( r > 0 \), and \( \omega \in \Sigma \)), which is a critical assumption for the heat kernel estimates in \cite{Kigami97}.

Theorem 2.16. \cite{Kigami97} Theorem 6.5, Proposition 6.6.] The harmonic measure \( \text{HARM}_T \) has the volume doubling property with respect to \( D \) if and only if the following conditions \((\text{EL})\) and \((\text{D})\) hold.

\((\text{EL}) : \) There exists \( c_1 \in (0, 1) \) such that \( c_1 \leq \text{HARM}_T(\Sigma(x))/\text{HARM}_T(\Sigma(\pi(x))) \) for any \( x \in T \setminus \{ \omega \} \).

\((\text{D}) : \) There exist \( m \geq 1 \) and \( \theta \in (0, 1) \) such that \( D([\omega]_{n+m}) \leq \theta D([\omega]_n) \) for any \( n \geq 0 \) and \( \omega \in \Sigma \).

In particular, \( \text{HARM}_T \) satisfies the volume doubling property with respect to \( D \) only if \( 2 \leq \#S(x) < \infty \) for any \( x \in T \).
In [7, Section 7], Kigami gives the following expression of the heat kernel associated with the regular Dirichlet form \((E_\Sigma, F_\Sigma)\) by using an eigenfunction expansion.

\[
\begin{align*}
p_t(\omega, \eta) &= \sum_{n \geq 0} \exp\left(-t/D([\omega]_{n+1})\right) - \exp\left(-t/D([\omega]_n)\right) \frac{1}{HARM_T(\Sigma([\omega]_n))} \mathbf{1}_{\Sigma([\omega]_n)}(\eta) \\
&= \begin{cases} 
1 + \sum_{n=0}^\infty \left( \frac{1}{HARM_T(\Sigma([\omega]_{n+1}))} - \frac{1}{HARM_T(\Sigma([\omega]_n))} \right) \exp(-t/D([\omega]_n)) & \text{if } \omega = \eta \\
\sum_{n=0}^N(\omega,\eta) \frac{1}{HARM_T(\Sigma([\omega]_n))} \exp(-t/D([\omega]_{n+1})) - \exp(-t/D([\omega]_n)) & \text{if } \omega \neq \eta,
\end{cases}
\end{align*}
\]  

(2.1)

with the convention \(1/D([\omega]-1) = 0\). If we allow \(\infty\) as a value, \(p_t(\omega, \eta)\) is well-defined on \((0, \infty) \times \Sigma^2\). Note that we have \(p_t(\omega, \eta) = p_t(\eta, \omega)\) and \(p_t(\omega, \omega) \geq p_t(\omega, \eta)\) for any \(\omega, \eta \in \Sigma\) from the above expression. In fact, the heat kernel \(p_t(\omega, \eta)\) which is given above is shown to be the transition density of the Hunt process associated with the regular Dirichlet form \((E_\Sigma, F_\Sigma)\) under suitable assumptions.

**Theorem 2.17.** [7, Proposition 7.2, Theorem 7.3.] Assume that \(\lim_{n \to \infty} D([\omega]_n) = 0\) for any \(\omega \in \Sigma\). Then,

\[
\int_\Sigma p_t(\omega, \eta) dHARM_T(\eta) = 1, \quad \text{and} \quad \int_\Sigma p_t(\omega, \xi) p_s(\xi, \eta) dHARM_T(\xi) = p_{t+s}(\omega, \eta),
\]

for any \(\omega, \eta \in \Sigma\) with \(\omega \neq \eta\) and any \(t, s > 0\). Moreover, there exists a Hunt process \((\{X_t\}_{t \geq 0}, \{P_\omega\}_{\omega \in \Sigma})\) on \(\Sigma\) whose transition density is \(p_t(\omega, \eta)\) i.e.

\[
E_\omega(u(X_t)) = \int_\Sigma p_t(\omega, \eta) u(\eta) dHARM_T(\eta),
\]

(2.2)

for any \(\omega \in \Sigma\) and any Borel measurable function \(u : \Sigma \to \mathbb{R}\), where \(E_\omega(\cdot)\) is the expectation with respect to \(P_\omega\).

**Remark 2.18.** Since it is shown in [7, Theorem 2.7] that \(D_x = G(x, o)/C(o)\), the assumption in the above theorem is equivalent to the symmetrized Green function vanishing at infinity.

By the above theorem, if \(\lim_{n \to \infty} D([\omega]_n) = 0\) for any \(\omega \in \Sigma\), then \(p_t u(\omega) = T_t u(\omega)\) for \(HARM_T\)-a.e. \(\omega \in \Sigma\), where \(\{T_t\}_{t \geq 0}\) is the strongly continuous semigroup on \(L^2(\Sigma, HARM_T)\) associated with the Dirichlet form \((E_\Sigma, F_\Sigma)\) on \(L^2(\Sigma, HARM_T)\).

We will introduce the heat kernel estimates given in [7, Proposition 7.5]. First, the following estimate is shown without any further assumptions.

**Proposition 2.19.** [7, Proposition 7.5.]

1. For any \(\omega \in \Sigma\), and any \(t > 0\),

\[
p_t(\omega, \omega) \geq \frac{1}{e} \cdot \frac{1}{HARM_T(B_D(\omega, t))}.
\]

2. If \(0 < t \leq D(\omega, \eta)\), then

\[
p_t(\omega, \eta) \leq \frac{t}{D(\omega, \eta) HARM_T(\Sigma([\omega], \eta)))}.
\]

In [7, Theorem 7.6.], the following two-sided estimates of \(p_t(\omega, \eta)\) and the estimates of mean displacement are proved under the assumption of the volume doubling property of \(HARM_T\). Note that under the volume doubling property of \(HARM_T\), we have \(\lim_{n \to \infty} D([\omega]_n) = 0\) for any \(\omega \in \Sigma\) by [7, Theorem]. In the following, if \(f\) and \(g\) are two functions defined on a set \(U\), \(f \asymp g\) means there exists \(C > 0\) such that \(C^{-1} f(x) \leq g(x) \leq C f(x)\) for all \(x \in U\).

**Theorem 2.20.** [7, Theorem 7.6, Corollary 7.9.] Suppose \(HARM_T\) has the volume doubling property with respect to \(D\). Then, the following results hold.

1. The heat kernel \(p_t(\omega, \eta)\) is continuous on \((0, \infty) \times \Sigma^2\). Define

\[
q_t(\omega, \eta) = \begin{cases} 
\frac{t}{D(\omega, \eta) HARM_T(\Sigma([\omega], \eta)))} & \text{if } 0 < t \leq D(\omega, \eta), \\
\frac{1}{HARM_T(B_D(\omega, t))} & \text{if } t > D(\omega, \eta).
\end{cases}
\]

(2.3)
Then,
\[ p_t(\omega, \eta) \asymp q_t(\omega, \eta) \text{ on } (0, \infty) \times \Sigma^2. \]
(2) For any \( \omega \in \Sigma \) and any \( t \in (0, 1] \),
\[ E_\omega[D(\omega, X_t)^\gamma] \asymp \begin{cases} \frac{t}{\gamma} & \text{if } \gamma > 1, \\ t(|\log t| + 1) & \text{if } \gamma = 1, \\ t^\gamma & \text{if } 0 < \gamma < 1. \end{cases} \]

3 Electric networks on Galton-Watson trees

3.1 The asymptotics of the effective resistance along infinite rays

In the previous section, we have presented results on the construction and properties of jump processes on the boundaries of deterministic trees studied in [4]. In this section, we will consider random trees instead of deterministic trees, in particular Galton-Watson trees. First we will introduce the preliminary results of electric networks and corresponding random walks on Galton-Watson trees which will be important for our study.

Let \( T \) be a rooted Galton-Watson tree with offspring distribution \( \{p_k\}_{k \geq 0} \). Note that
\[ \{\xi_n\}_{n \geq 0} = \{\#(v \in T : h(v) = n)\}_{n \geq 0} \]
is a Galton-Watson process with offspring distribution \( \{p_k\}_{k \geq 0} \). Throughout this paper, we will assume the following condition on \( \{p_k\}_{k \geq 0} \).
\[
\begin{align*}
p_0 &= 0, \\
m &= \sum_{k=0}^{\infty} kp_k > 1. 
\end{align*}
\]
Recall that under the above assumptions, \( T \) can be regarded as the \( T \)-valued random variable, where \( T := \{T : T \text{ is an infinite rooted tree}\} \), and we will denote its distribution by \( P_{GW} \). Moreover, \( T_v \) is an infinite tree for all \( v \in T \) with \( P_{GW} \)-probability 1, so the extinction event has \( P_{GW} \)-probability 0.

For an infinite tree \( T \) and \( \lambda > 0 \), we consider the \( \lambda \)-biased random walk \( \{Z_n^\lambda\}_{n \geq 0} \) on \( T \) under the probability measure \( P_n^\lambda \). Let the initial state \( Z_0 \) to be \( o \) unless otherwise stated. This is equivalent to considering the weighted graph \( (T, C^\lambda(T)) \), where the conductance of an edge connecting vertices at level \( n \) and \( n + 1 \) is \( \lambda^{-n} \). For \( 0 < \lambda < m \), it is proved in [9] that the \( \lambda \)-biased random walk on the Galton-Watson tree \( T \) is transient with \( P_{GW} \)-probability 1. Thus for \( 0 < \lambda < m \), we have the harmonic measure of \( \{Z_n^\lambda\}_{n \geq 0} \), which is a probability measure on \( \Sigma \). In what follows, when the \( \lambda \)-biased random walk on an infinite rooted tree \( T \) is transient, we will denote the harmonic measure of the random walk staring from a vertex \( x \) by \( HARM_{T,x}^\lambda \).

Again, we set \( HARM_T^\lambda = HARM_{T,o}^\lambda \). Note that by [7] Theorem 6.5, Proposition 6.6., \( HARM_T^\lambda \) has the volume doubling property with respect to the intrinsic metric \( D \) with \( P_{GW} \)-probability 1 only if \( p_1 = 0 \) and \( \sup\{n : p_n > 0\} < +\infty \). Since the purpose of the paper is to obtain estimates of the heat kernel without the volume doubling property of the harmonic measure, estimates of the harmonic measure and the effective resistance will be important. As for the control of the volume, the following results are shown in [12] Theorem 1.1 | [13] Theorem 5.1.

**Theorem 3.1.** For \( 0 < \lambda < m \), the following results hold.
(1) There exists a deterministic constant \( 0 < \beta_\lambda < \log m \) such that \( \beta_\lambda = \dim HARM_T^\lambda P_{GW} \)-a.s.
(2) Define
\[ \Sigma_1 := \{\omega \in \Sigma : -\lim_{n \to \infty} \frac{\log HARM_T^\lambda(B_d(\omega, e^{-n}))}{n} = -\lim_{n \to \infty} \frac{\log HARM_T^\lambda(S([\omega]_n))}{n} = \beta_\lambda\} \]
then \( HARM_T^\lambda(\Sigma_1) = 1 P_{GW} \)-a.s.

Next, we will investigate the behavior of the effective resistance. Before giving the statement, we need some preparations. In [12] [13], Markov chains on “the space of trees” are studied and, in particular, Markov chains associated with harmonic flows are utilized to study the behavior of harmonic measures of \( \lambda \)-biased random walks on Galton-Watson trees.
**Definition 3.2.** (1) For a tree $T \in \mathbb{T}$ and $v \in T$, define $R^\lambda(T, v) = R^\lambda(v)$ as the effective resistance of $(T(v), C^\lambda(T(v)))$ from the vertex $v$ to infinity. Define the effective conductance $EC^\lambda(T(v))$ by $EC^\lambda(T(v)) := (R^\lambda(v))^{-1}$.

(2) For a tree $T \in \mathbb{T}$, a nonnegative function $\theta$ on $T$ is called a flow on $T$ if for all $x \in T$, $\theta(x) = \sum_{y \tau(x)=x} \theta(y)$. Note that flows on $T$ are in one-to-one correspondence with positive finite Borel measures $\mu$ on $\Sigma(T)$ by $\theta(x) = \mu(\Sigma(x))$.

(3) Define $\mathbb{F} := \{\theta; \theta$ is a flow on $T$ for some $T \in \mathbb{T}\}$. A Borel map $\Theta : T \to \mathbb{F}$ is called a flow rule if for any $T \in \mathbb{T}$, $\Theta(T)$ is a flow on $T$, and for any $x \in T$ with $|x| = 1$ and $\Theta(T)(x) > 0$, we have $\Theta(T)(y)/\Theta(T)(x) = \Theta(T(x))(y)$ for $y \in T(x)$.

For a given flow rule $\Theta$, we can associate a Markov chain on $\mathbb{T}$ in the following way: for a flow rule $\Theta$, define transition probabilities $p_\theta$ by $p_\theta(T, T'(x)) := \Theta(T)(x)$ for $T \in \mathbb{T}$ and $x \in T$ with $|x| = 1$. A path of this Markov chain on $\mathbb{T}$ is naturally identified with an element of $T_{ray} := \{(T, \omega); T \in \mathbb{T}, \omega \in \Sigma(T)\}$. Define the shift operator $S$ on $T_{ray}$ by $S((T, \omega)) := (T(\omega_1), \tau(\omega))$, where $\tau$ is the shift operator on $\Sigma$. A measure $\mu$ on $\mathbb{T}$ is called $\Theta$-stationary if for any Borel subset $A \subseteq \mathbb{T}$,

$$\mu(A) = \int \sum_{T' \in A} p_\theta(T, T')d\mu(T') = \int \sum_{|x|=1, T(x) \in A} \Theta(T)(x)d\mu(T).$$

When the $\lambda$-biased random walk on $T \in \mathbb{T}$ is transient, its path converges almost surely to a random element of $\Sigma$, and its distribution is $\text{HARM}^\lambda$. Define $\text{HARM}^\lambda : \mathbb{T} \to \mathbb{F}$ by $\text{HARM}^\lambda(T)(x) := \text{HARM}^\lambda(\Sigma(x))$ for $x \in T$ with $|x| = 1$. It is obvious that this defines a flow rule. In the rest of this section, we will use the following results obtained by [8] and [16] independently, which give the explicit expression of the $\text{HARM}^\lambda$-stationary probability measure.

**Definition 3.3.** For a tree $T \in \mathbb{T}$ rooted at $\hat{o}$, define $\hat{T}$ as the tree obtained by drawing an extra edge between $\hat{o}$ and an extra adjacent vertex $\hat{\delta}$, which is the root of $\hat{T}$. Let

$$\alpha^\lambda(T) := P^\lambda_\hat{o}(Z_n^\lambda \neq \hat{\delta} \text{ for all } n \geq 1 \mid Z_0^\lambda = \hat{o}).$$

be the probability that the $\lambda$-biased random walk on $T$ starting at $\hat{o}$ never visits $\hat{\delta}$. $$\theta^\lambda(T) := \int \frac{\alpha^\lambda(T')EC^\lambda(T)}{\lambda - 1 + \alpha^\lambda(T') + EC^\lambda(T)}d\mathbb{P}_{GW}(T'),$$

$$h^\lambda := \int \frac{\theta^\lambda(T)d\mathbb{P}_{GW}(T)}{\hat{T}}.$$ Then, the measure $d\mu^\lambda_{\text{HARM}}(T) := (h^\lambda)^{-1}\cdot \theta^\lambda(T)d\mathbb{P}_{GW}(T)$ is the unique $\text{HARM}^\lambda$-stationary probability measure which is mutually absolutely continuous with respect to $\mathbb{P}_{GW}$. Moreover, the $\text{HARM}^\lambda$-Markov chain with initial distribution $\mu^\lambda_{\text{HARM}}$ is ergodic.

**Proposition 3.4.** For $0 < \lambda < m$, define $\theta^\lambda : \mathbb{T} \to \mathbb{R}$ and $h^\lambda \in \mathbb{R}_{\geq 0}$ by

Then, the measure $d\mu^\lambda_{\text{HARM}}(T) := (h^\lambda)^{-1}\cdot \theta^\lambda(T)d\mathbb{P}_{GW}(T)$ is the unique $\text{HARM}^\lambda$-stationary probability measure which is mutually absolutely continuous with respect to $\mathbb{P}_{GW}$. Moreover, the $\text{HARM}^\lambda$-Markov chain with initial distribution $\mu^\lambda_{\text{HARM}}$ is ergodic.

**Proof.** The explicit formula for $\mu^\lambda_{\text{HARM}}$ is given in [8] Lemma 5] and [16] Theorem 4.1. The ergodicity of the $\text{HARM}^\lambda$-Markov chain is proved in [12] [13].

The following result will be important for the proof of our main result.

**Proposition 3.5.** Under the assumption [8,16], for $0 < \lambda < m$, the following holds $\mathbb{P}_{GW}$-a.s.

$$\lim_{n \to \infty} \frac{1}{n} \log R^\lambda(|\omega|_n) = \log \lambda, \text{ HARM}^\lambda_\mathbb{T}$-a.e. \omega. \quad (3.2)$$

In other words, if define

$$\Sigma_2 := \{\omega \in \Sigma; \lim_{n \to \infty} \frac{1}{n} \log R^\lambda(|\omega|_n) = \log \lambda\},$$

then $\text{HARM}^\lambda_\mathbb{T}(\Sigma_2) = 1 \mathbb{P}_{GW}$-a.s.
Before giving the proof of Proposition 5.3, we prove the following moment estimates of the effective resistance, which are of independent interest.

Lemma 3.6. (1) For $0 < \lambda < 1$, $R^\lambda(x) \leq \frac{1}{\lambda^x}$. In particular, for $x \in \mathcal{T}_k$ we have $R^\lambda(x) \leq \lambda^k \cdot \frac{1}{\lambda^k}$ $\mathbb{P}_{\text{GW}}$-a.s.

(2) For $\lambda = 1$, there exists a constant $b > 0$ such that $\mathbb{P}_{\text{GW}}(R(o) > n) \leq e^{-bn}$ for any $n \in \mathbb{N}$.

(3) For $1 < \lambda < m$, if $p_1 > 0$, there exists a constant $c > 0$ such that $\mathbb{P}_{\text{GW}}(R^\lambda(o) > n) \leq cn^{\log p_1 / \log \lambda}$ for any $n \in \mathbb{N}$. If $p_1 = 0$, for any $\alpha > 0$ there exists a constant $c_\alpha > 0$ such that $\mathbb{P}_{\text{GW}}(R^\lambda(o) > n) \leq c_\alpha n^{-\alpha}$ for any $n \in \mathbb{N}$.

(4) For $0 < \lambda < m$, if $\sum_{k \geq 1} k^\alpha p_k < \infty$ for $\alpha \geq 1$, we have $\mathbb{E}_{\text{GW}}[(R^\lambda(o))^{-\alpha}] < \infty$.

Proof. The first claim immediately follows from Rayleigh's monotonicity principle. We will prove the rest. Remark that the following argument heavily relies on that in [14, Section 4]. First, we assume $p_1 > 0$. For $T \in \mathcal{T}$, it is easy to show that

$$\alpha^\lambda(T) = \frac{\sum_{v \in S(o)} \alpha^\lambda(T(v))}{\lambda + \sum_{v \in S(o)} \alpha^\lambda(T(v))} = \frac{1}{1 + \lambda R^\lambda(o)}.$$  \(\text{(3.3)}\)

Define

$$F_\lambda(s) := \mathbb{P}_{\text{GW}}(\alpha^\lambda(T) \leq s) = \mathbb{P}_{\text{GW}}\left(R^\lambda(o) \geq \frac{1 - s}{s \lambda}\right),$$

and denote the $k$-th fold of $F_\lambda$ by $F_\lambda^k$. Then by combining the branching property of Galton-Watson trees and \(\text{(3.3)}\), we obtain

$$F_\lambda(s) = \begin{cases} \sum_l p_l F_\lambda^l \left(\frac{\lambda}{1 - \lambda s}\right) & \text{if } s \in (0, 1), \\ 0 & \text{if } s \leq 0, \\ 1 & \text{if } s \geq 1. \end{cases}$$

By this expression, we get

$$\mathbb{P}_{\text{GW}}(R^\lambda(o) \geq n) = F_\lambda \left(\frac{1}{1 + \lambda n}\right) = \sum_l p_l F_\lambda^l \left(\frac{1}{n}\right) \leq \sum_l p_l F_\lambda \left(\frac{1}{n}\right) = \sum_l p_l \mathbb{P}_{\text{GW}}\left(R^\lambda(o) \geq \frac{n - 1}{\lambda}\right)^l.$$  

So if we define $G(x) := \sum_l p_l x^l$, we have

$$\mathbb{P}_{\text{GW}}(R^\lambda(o) \geq n) \leq G \left(\mathbb{P}_{\text{GW}}\left(R^\lambda(o) \geq \frac{n - 1}{\lambda}\right)\right) \leq G \circ G \left(\mathbb{P}_{\text{GW}}\left(R^\lambda(o) \geq \frac{n - \lambda - 1}{\lambda^2}\right)\right) \leq ....$$

Since $R^\lambda(o) < \infty$ a.s. for $0 < \lambda < m$, we have $\mathbb{P}_{\text{GW}}(R^\lambda(o) \geq n) < 1/2$ for $N$ large enough. Hence we obtain that $\mathbb{P}_{\text{GW}}(R^\lambda(o) \geq n) \leq G^{(l)}(1/2)$ for $l \in \mathbb{N}$ satisfying $\frac{n - \sum_{i=1}^{l-1} \lambda^i}{\lambda^l} > N$. Note that by [1] Chapter 1, Section 11, there exists a positive function $Q(s)$ on $[0, 1]$, such that $\lim_{n \to \infty} p_1^{-n} G^{(n)}(s) = Q(s)$ for any $s \in [0, 1]$. This implies that when $\lambda = 1$ there exists a constant $b > 0$ such that $\mathbb{P}_{\text{GW}}(R(o) > n) \leq e^{-bn}$, and when $1 < \lambda < m$ there exists a constant $c > 0$ such that $\mathbb{P}_{\text{GW}}(R^\lambda(o) > n) \leq cn^{\log p_1 / \log \lambda}$. These estimates imply the second claim, and the third claim in the case of $p_1 > 0$. When $1 < \lambda < m$ and $p_1 = 0$, for $\varepsilon > 0$ we define another offspring distribution $\{q_k\}_{k \geq 0}$ by $q_0 := 0, q_1 := \varepsilon, q_L := p_L - \varepsilon$ and $q_n := p_n$ for $n > L$, where $L := \inf\{k : p_k > 0\}$. If we take $\varepsilon > 0$ sufficiently small such that $\varepsilon < p_L$ and $\lambda < \sum_{k \geq 1} kq_k$, it is easy to see that we can define the Galton-Watson tree with offspring distribution $\{q_k\}_{k \geq 0}$ on the probability space $(\mathcal{T}, \mathbb{P}_{\text{GW}})$ in such a way that $R^\lambda(o)$ is stochastically dominated by the effective resistance of $\lambda$-biased random walk on the Galton-Watson tree with offspring distribution $\{q_k\}_{k \geq 0}$. Hence, we get the third claim. The fourth claim is immediate from the equality

$$\frac{1}{R^\lambda(o)} = \sum_{v \in S(o)} \frac{1}{1 + R^\lambda(v)} \leq \#S(o),$$

which follows from the parallel and serial laws of basic electric network theory. \(\square\)
**Proof of Proposition 3.5.** We shall verify below that the function \( f^\lambda : T_{\text{ray}} \to \mathbb{R} \) defined by
\[
f^\lambda((T, \omega)) := \log \frac{R^\lambda([\omega]_1)}{R^\lambda(o)},
\]
is integrable with respect to \( \mu^\lambda_{\text{HARM}} \times \text{HARM}^\lambda_T \). Then Proposition 3.4 and Birkhoff’s ergodic theorem imply that for \( \mu^\lambda_{\text{HARM}} \times \text{HARM}^\lambda_T \)-almost every \((T, \omega)\),
\[
\lim_{n \to \infty} \frac{1}{n} \log R^\lambda([\omega]_n) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^n S^k f^\lambda(T, \omega) = \int_{T_{\text{ray}}} \log \frac{R^\lambda([\xi]_1)}{R^\lambda(o)} \, d\mu^\lambda_{\text{HARM}} \times \text{HARM}^\lambda_T(T, \xi).
\]
By stationarity of \( \mu^\lambda_{\text{HARM}} \), we get
\[
\int_{T_{\text{ray}}} \log \frac{R^\lambda([\xi]_1)}{R^\lambda(o)} \, d\mu^\lambda_{\text{HARM}} \times \text{HARM}^\lambda_T(T, \xi) = \log \lambda,
\]
which yields the claim of Proposition 3.5, since \( \mathbb{P}_{GW} \ll \mu^\lambda_{\text{HARM}} \) by Proposition 3.4. The integrability of \( f^\lambda \) with respect to \( \mu^\lambda_{\text{HARM}} \times \text{HARM}^\lambda_T \) can be proved as follows: since
\[
\int_{T_{\text{ray}}} \left| \log \frac{R^\lambda([\xi]_1)}{R^\lambda(o)} \right| \, d\mu^\lambda_{\text{HARM}} \times \text{HARM}^\lambda_T(T, \xi) \leq \int_{T_{\text{ray}}} \left( |\log R^\lambda([\xi]_1)| + |\log R^\lambda(o)| \right) \, d\mu^\lambda_{\text{HARM}} \times \text{HARM}^\lambda_T(T, \xi),
\]
\[
\leq |\log \lambda| + 2 \int_{T_{\text{ray}}} |\log R^\lambda(o)| \, d\mu^\lambda_{\text{HARM}} \times \text{HARM}^\lambda_T(T, \xi),
\]
it suffices to prove the integrability of \( |\log R^\lambda(o)| \) with respect to \( \mu^\lambda_{\text{HARM}} \) because the random variable \( \log R^\lambda(o) \) does not depend on \( \xi \). It is shown in the proof of [3, Lemma 5] that \( 0 < \theta^\lambda(x) < 1 \) for any \( x > 0 \), hence we only need to prove \( \log R^\lambda(o) \in L^1(T, \mathbb{P}_{GW}) \). Since
\[
\mathbb{P}_{GW}(\log R^\lambda(o) \geq n) = \mathbb{P}_{GW}(R^\lambda(o) \geq e^n) + \mathbb{P}_{GW}(R^\lambda(o) \leq e^{-n}),
\]
we get the desired result by Lemma 3.6. □

As a corollary of Proposition 3.5, we obtain the following lower bound of \( \beta_{\lambda} \).

**Corollary 3.7.** For \( 0 < \lambda < m \), we have \( \beta_{\lambda} > 0 \lor \log \lambda \).

**Proof.** It is proved in [12, 13] that for \( 0 < \lambda < m \),
\[
\beta_{\lambda} = \int_{T_{\text{ray}}} \log \frac{1}{\text{HARM}^\lambda_T([\xi]_1)} \, d\mu^\lambda_{\text{HARM}} \times \text{HARM}^\lambda_T(T, \xi).
\]
By using [7, Theorem 3.8], we have for \( 0 < \lambda < m \),
\[
\beta_{\lambda} - \log \lambda = \int_{T_{\text{ray}}} \log \left( \frac{1 + R^\lambda([\xi]_1)}{R^\lambda(o)} \cdot \frac{R^\lambda(o)}{R^\lambda([\xi]_1)} \right) \, d\mu^\lambda_{\text{HARM}} \times \text{HARM}^\lambda_T(T, \xi) > 0,
\]
and we get the conclusion. □

**Remark 3.8.** Corollary 3.7 and the fact that \( \beta_{\lambda} < \log m \) imply that \( \lim_{\lambda \to m} \beta_{\lambda} = \log m \), where \( \log m \) is the Hausdorff dimension of the boundary \((\Sigma, d) \mathbb{P}_{GW} \)-a.s. This is a part of the results stated in [3, Theorem 1].
4 Proof of the main theorems

In this section, we construct jump processes on the boundaries of Galton-Watson trees and give the proof of the short time log-asymptotic of the on-diagonal heat kernel and the estimates of mean displacements. The first result is on the construction of the jump process on the boundary of the Galton-Watson tree $\mathcal{T}$. For $0 < \lambda < m$, let $D^\lambda$ be the intrinsic metric of $(\mathcal{T}, C^\lambda)$ on $\Sigma$, $(\mathcal{E}_\Sigma^\lambda, \mathcal{F}_\Sigma^\lambda)$ be the Dirichlet form on $L^2(\Sigma, H\text{ARM}_T^\lambda)$ which corresponds to the $\lambda$-biased random walk on $\mathcal{T}$ and $p^\lambda_t(\omega, \eta)$ be the heat kernel associated with the Dirichlet form $(\mathcal{E}_\Sigma^\lambda, \mathcal{F}_\Sigma^\lambda)$. Define $\Sigma_* := \Sigma_1 \cap \Sigma_2$. Note that $H\text{ARM}_T^\lambda(\Sigma_*) = 1 \mathbb{P}_{GW}$-a.s.

**Theorem 4.1.** For $0 < \lambda < m$, the following results hold $\mathbb{P}_{GW}$-a.s.

$$\int_{\Sigma} p^\lambda_t(\omega, \tau) dH\text{ARM}_T^\lambda(\tau) = 1, \quad \text{and} \quad \int_{\Sigma} p^\lambda_t(\omega, \xi)p^\lambda_s(\xi, \eta) dH\text{ARM}_T^\lambda(\xi) = p^\lambda_{t+s}(\omega, \eta),$$

for any $\omega, \eta \in \Sigma_*$ with $\omega \neq \eta$ and any $t, s > 0$. Moreover, there exists a Hunt process $\{(X^\lambda_t)_{t \geq 0}, \{P^\lambda_\omega\}_{\omega \in \Sigma}\}$ on $\Sigma_*$ whose transition density is $p^\lambda_t(\omega, \eta) \mathbb{P}_{GW}$-a.s.

**Proof.** By Theorem 3.1, Proposition 3.5 and Corollary 3.7 for $0 < \lambda < m$, we have $\lim_{n \to \infty} D^\lambda(\lfloor \omega \rfloor_n) = 0$ for $\omega \in \Sigma_* \mathbb{P}_{GW}$-a.s. By using (2.1), a routine calculation yields the first statement. In order to prove the second statement, it suffices to show that for $0 < \lambda < m$, $p^\lambda_t(C(\Sigma_*)) \subseteq C(\Sigma_*)$, and $\|p^\lambda_t u - u\|_\infty \to 0$ as $t \to 0$ for any $u \in C(\Sigma_*) \mathbb{P}_{GW}$-a.s., where $C(\Sigma_*)$ is the collection of continuous functions on $\Sigma_*$. It can be proved by a similar argument to that in [7, Theorem 7.3, Lemma 7.4]. \hfill $\Box$

We now prove Theorem 3.1. In the rest of this paper, we write $D^\lambda$, $H\text{ARM}_T^\lambda$, $R^\lambda v$ and $p^\lambda_t$ as $D$, $H\text{ARM}$, $R(v)$ and $p_t$ respectively.

**Proof of Theorem 3.1.** It is sufficient to prove the claim for any $\omega \in \Sigma_*$. We will write $D_n = D(\lfloor \omega \rfloor_n)$, $R_n = R(\lfloor \omega \rfloor_n)$ and $H_n = H\text{ARM}(\Sigma(\lfloor \omega \rfloor_n))$. Let $l = l(\omega, t)$ be the unique integer which satisfies $D_l < t \leq D_{l+1}$. Then we have $\lim_{t \to +0} l(\omega, t) = \infty$ for all $\omega \in \Sigma_*$. Note that by Proposition 2.19 we have the following lower bound.

$$p_t(\omega, \omega) \geq \frac{1}{e \cdot H_l} \cdot \frac{1}{H_{D_l}(\omega, t)} = \frac{1}{e} \cdot \frac{1}{H_t}.$$ 

So we will prove the upper bound.

$$1 + \sum_{n=0}^{l-1} \left( \frac{1}{H_{n+1}} - \frac{1}{H_n} \right) \exp(-t/D_n) \leq 1 + \sum_{n=0}^{l-1} \left( \frac{1}{H_{n+1}} - \frac{1}{H_n} \right) = \frac{1}{H_l} = \frac{1}{H\text{ARM}(\Sigma(\lfloor \omega \rfloor_0))} = \frac{1}{H\text{ARM}(B_d(\omega, e^{-l})).}$$

On the other hand,

$$\sum_{n=l}^{\infty} \left( \frac{1}{H_{n+1}} - \frac{1}{H_n} \right) \exp(-t/D_n) \leq \sum_{n=l}^{\infty} \frac{1}{H_{n+1}} \exp(-t/D_n) \leq \frac{1}{H_l} \sum_{n=l}^{\infty} \frac{H_l}{H_{n+1}} \exp\left(\frac{D_l}{D_n}\right).$$

By Theorem 3.1, we have the following control of the volume. For all $\epsilon > 0$, there exists a random integer $L = L(\omega, \epsilon)$ such that for all $n \geq L$,

$$\exp\{-\beta_\lambda + \epsilon\} n \leq H_n \leq \exp\{-\beta_\lambda - \epsilon\} n.$$ 

For $t > 0$ sufficiently small, we have $l \geq L$, and

$$\sum_{n=l}^{\infty} \frac{H_l}{H_{n+1}} \exp\left(\frac{D_l}{D_n}\right) \leq C \sum_{n=l}^{\infty} \exp\{(\beta_\lambda + \epsilon)(n-l) + 2\epsilon l\} \cdot \exp\left[-C' \frac{R_l}{R_n} \exp\{(\beta_\lambda - \epsilon)(n-l) - 2\epsilon l\}\right] \leq C \exp(2\epsilon l) \sum_{k=0}^{\infty} \exp\{(\beta_\lambda + \epsilon)k\} \exp\left[-C' \frac{R_l}{R_l+k} \exp\{(\beta_\lambda - \epsilon)k\}\right].$$
where $C$ and $C'$ are constants which do not depend on $t$. By Proposition 3.5 for all $\varepsilon > 0$, there exists $K = K(\omega, \varepsilon)$ such that for all $n \geq K$,

$$\exp\{(\log \lambda - \varepsilon)n\} \leq R_n \leq \exp\{(\log \lambda + \varepsilon)n\}.$$  

Hence, for $0 < \lambda < m$, $\varepsilon > 0$, $l \geq N \lor K$ and $t > 0$ sufficiently small,

$$\sum_{n=l}^{\infty} \frac{H_l}{H_{n+1}} \exp\left(-\frac{D_l}{D_n}\right) \leq C \exp(2\varepsilon l) \sum_{k=0}^{\infty} \exp\{\alpha \lambda + \varepsilon k\} \cdot \exp\left[-\frac{C'}{\exp(2\varepsilon l)} \exp\{\log \lambda - \varepsilon l\} \exp\{\beta \lambda - \varepsilon k\}\right] \exp\{\alpha \lambda + \varepsilon k\}$$

$$= \sum_{n=l}^{\infty} \frac{H_l}{H_{n+1}} \exp\left(-\frac{D_l}{D_n}\right) \leq C \exp(2\varepsilon l) \sum_{k=0}^{\infty} \exp\{\alpha \lambda + \varepsilon k\} \exp\left[-\frac{C'}{\exp(4\varepsilon l)} \exp\{\alpha \lambda - \log \lambda - \varepsilon k\}\right] \exp\{\alpha \lambda + \varepsilon k\},$$

where $\alpha = \exp(\beta \lambda - \log \lambda - \varepsilon) > 1$ and $Q := \frac{\alpha + \varepsilon - \beta \lambda - \log \lambda - \varepsilon}{\log \lambda - \varepsilon} + 1$. If we define $f(x) = x^{Q+1} \exp(-\gamma x)$ for $\gamma > 0$, one can easily check that $f(x) \leq f(Q+1) = \frac{(Q+1)^{Q+1} \exp(-Q+1)}{\gamma^{Q+1}}$. Thus,

$$\sum_{n=l}^{\infty} \frac{H_l}{H_{n+1}} \exp\left(-\frac{D_l}{D_n}\right) \leq C \exp(2\varepsilon l) \sum_{k=0}^{\infty} \alpha^{-k} C' \exp\{4(Q+1)\varepsilon l\} \leq C'' \exp\{4Q + 6\varepsilon l\}.$$  

By combining the above estimates, we have

$$\log p_t(\omega, \omega) \leq \log \frac{1}{H_t} + \log \left(1 + C'' \exp\{4Q + 6\varepsilon l\}\right).$$  

(4.1)

By Theorem 3.1. again, this implies

$$\lim_{t \to 0} \frac{\log p_t(\omega, \omega)}{l(\omega, t)} = \beta_\lambda, \quad \text{for all } \omega \in \Sigma_\ast, \quad \P_{GW}-a.s.$$  

So the proof will be finished if we prove the following.

$$\lim_{t \to 0} \frac{l(\omega, t)}{\log t} = \frac{1}{\beta_\lambda - \log \lambda}, \quad \text{for all } \omega \in \Sigma_\ast, \quad \P_{GW}-a.s.$$  

(4.2)

We now prove (4.2). By the definition of $l(\omega, t)$, Theorem 3.1 and Proposition 3.5 for all $\varepsilon > 0$ and $\omega \in \Sigma_\ast$, we have the following inequality for $t > 0$ sufficiently small $\P_{GW}$-a.s.

$$\exp\{-(\beta \lambda + \varepsilon)l\} \exp\{\log \lambda - \varepsilon\} \leq H_t R_t = D_t < t \leq D_{t-1} = H_{t-1} R_{t-1} \leq \exp\{-(\beta \lambda - \varepsilon)l\} \exp\{\log \lambda + \varepsilon\}.$$  

(4.3)

So, we have

$$-(\beta \lambda - \log \lambda + 2\varepsilon)l < \log t \leq -(\beta \lambda - \log \lambda - 2\varepsilon)l,$$

and (4.2) is proved. \hfill \sqr

In order to prove Theorem 1.2, we first show the following proposition corresponding to Theorem 2.20.

**Proposition 4.2.** For $0 < \lambda < m$, the following holds $\P_{GW}$-a.s.

$$\lim_{t \to 0} \frac{\log E_\omega[D(\omega, X_t)^\gamma]}{\log t} = \gamma \land 1, \quad \text{HARM}_T^\lambda \text{ a.e.}. \omega.$$  

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Define $I_0 := \{ \eta \in \Sigma : D(\omega, \eta) \leq t \}$, $I_n := \{ \eta \in \Sigma : 2^{n-1} t < D(\omega, \eta) \leq 2^n t \}$, for $n \geq 1$. From the expression of the heat kernel, it follows that $p_t(\omega, \eta) \geq p_t(\omega, \eta)$ for all $\omega, \eta \in \Sigma$. So by using the on-diagonal upper bound of the heat kernel, for $t > 0$ sufficiently small, we have the following for all $\omega \in \Sigma$.

$$\int_{I_0} p_t(\omega, \eta) D(\omega, \eta)^\gamma \text{dHARM}(\eta) \leq p_t(\omega, \omega) \cdot t^\gamma \cdot \text{HARM}(I_0) \leq t^\gamma (1 + C \exp((4Q + 6)\varepsilon t)).$$

Note that by Proposition 2.19 we have the off-diagonal upper bound for the heat kernel:

$$p_t(\omega, \eta) \leq \frac{t}{D(\omega, \eta) \text{HARM}(BD(\omega, D(\omega, \eta)))}.$$

Thus for $n \geq 1$,

$$\int_{I_n} p_t(\omega, \eta) D(\omega, \eta)^\gamma \text{dHARM}(\eta) \leq t \int_{I_n} \frac{D(\omega, \eta)^{\gamma-1}}{\text{HARM}(BD(\omega, D(\omega, \eta)))} \text{dHARM}(\eta) \leq t(2^n t)^{\gamma-1} \left( \frac{\text{HARM}(BD(\omega, 2^n t))}{\text{HARM}(BD(\omega, 2^n t))} - 1 \right).$$

By the proof of Theorem 4.1, there exists $t_0 = t_0(\omega, \varepsilon) > 0$ such that $t^{\kappa_\lambda + \varepsilon} \leq \text{HARM}(BD(\omega, t)) \leq t^{\kappa_\lambda - \varepsilon}$ for all $0 < t \leq t_0$ and for all $\omega \in \Sigma_\bullet$, where $\kappa_\lambda = \frac{\delta_2}{\mu(\lambda \log \lambda)}$. So we have,

$$\frac{\text{HARM}(BD(\omega, 2t))}{\text{HARM}(BD(\omega, t))} \leq \frac{(2t)^{\kappa_\lambda - \varepsilon}}{t^{\kappa_\lambda + \varepsilon}} \leq Ct^{-2\varepsilon}, \quad \text{for all } 0 < t \leq t_0, \text{ and for all } \omega \in \Sigma_\bullet,$$

where $C$ is a constant which does not depend on $t$. Define $M := \max\{ n : 2^n t \leq t_0 \}$ and $I := \bigcup_{n \geq M+1} I_n$. Then we have the following results for $\varepsilon > 0$ and $t > 0$ sufficiently small.

$$\sum_{n=1}^M \int_{I_n} p_t(\omega, \eta) D(\omega, \eta)^\gamma \text{dHARM}(\eta) \leq t \sum_{n=1}^M (2^n t)^{\gamma-1} \{ C(2^n t)^{-2\varepsilon} - 1 \} \leq \begin{cases} Ct^{\gamma - 2\varepsilon} & 0 < \gamma < 1 \\ Ct^{1 - 2\varepsilon} & \gamma = 1 \\ Ct & \gamma > 1, \end{cases}$$

and

$$\int_{I_n} p_t(\omega, \eta) D(\omega, \eta)^\gamma \text{dHARM}(\eta) \leq t \int_{I_n} \frac{D(\omega, \eta)^{\gamma-1}}{\text{HARM}(BD(\omega, D(\omega, \eta)))} \text{dHARM}(\eta) \leq t R(\alpha)^{\gamma-1} \cdot \frac{\text{HARM}(I)}{\text{HARM}(BD(\omega, t_0))} \leq t R(\alpha)^{\gamma-1} \cdot \frac{1 - \text{HARM}(BD(\omega, \frac{t_0}{2}))}{\text{HARM}(BD(\omega, t_0))}.$$ 

Since

$$E_\omega[D(\omega, X_t)^\gamma] = \sum_{n=0}^M \int_{I_n} p_t(\omega, \eta) D(\omega, \eta)^\gamma \text{dHARM}(\eta) + \int_Z p_t(\omega, \eta) D(\omega, \eta)^\gamma \text{dHARM}(\eta),$$

by combining the above estimates, we obtain

$$\liminf_{t \to 0} \frac{\log E_\omega[D(\omega, X_t)^\gamma]}{\log t} \geq \gamma \land 1, \quad \text{for all } \omega \in \Sigma_\bullet.$$

**Upper bound.** In order to establish the upper bound, we have to obtain a lower bound of $E_\omega[D(\omega, X_t)^\gamma]$. First, we will prove the following.

$$\limsup_{t \to 0} \frac{\log E_\omega[D(\omega, X_t)^\gamma]}{\log t} \leq 1, \quad \text{for all } \omega \in \Sigma_\bullet.$$

Proof. Analogously to Theorem 4.1, it is sufficient to prove the claim for any $\omega \in \Sigma_\bullet$.
Define $F_{\omega, \eta}(t) := p_t(\omega, \eta)$ for $0 < t \leq D(\omega, \eta)$ and $\omega, \eta \in \Sigma$. Then by Proposition 2.19, we have

\[
F_{\omega, \eta}(t) \geq F'_{\omega, \eta}(D(\omega, \eta))t, \quad \text{for } 0 < t \leq D(\omega, \eta),
\]

\[
F'_{\omega, \eta}(t) = \sum_{n=0}^{N(\omega, \eta)} \frac{(D(\mathbf{t}_n))^1} \left( \exp(-t/D(\mathbf{t}_n)) - (D(\mathbf{t}_n-1))^{-1} \exp(-t/D(\mathbf{t}_n-1)) \right)
\]

Hence, for all $\omega, \eta \in \Sigma$

\[
F'_{\omega, \eta}(D(\omega, \eta)) \geq (D(o))^{-1}e^{-1} \geq e^{-1}R(o),
\]

therefore, $p_t(\omega, \eta) \geq e^{-1}R(o)t$, which implies the desired result. Now, the proof will be finished once we prove the following.

\[
\limsup_{t \to 0} \frac{\log E_{\omega}[D(\omega, X_t^\gamma)]}{\log t} \leq \gamma \quad \text{for all } \omega \in \Sigma, \quad (4.4)
\]

In order to prove (4.3), we first show the following near diagonal lower bound for the heat kernel. For $\varepsilon > 0$ sufficiently small and for all $\omega \in \Sigma$, there exist $C_1 = C_1(\omega, \varepsilon) > 0$, $t_1 = t_1(\omega, \varepsilon) \in (0, t_0)$, $\delta' = \delta'(\omega, \varepsilon) > 0$ $\delta'' = \delta''(\omega, \varepsilon) > 0$ with $\delta'' > \delta'$ and $\delta', \delta'' \to 0$ as $\varepsilon \to 0$ such that

\[
p_t(\omega, \eta) \geq \frac{C_1}{t^{\delta' - \varepsilon}}, \quad \text{for all } 0 < t \leq t_1 \text{ and all } \eta \in B_D(\omega, t^{1+\delta'}) \setminus B_D(\omega, t^{1+\delta''}). \quad (4.5)
\]

For $\omega, \eta \in \Sigma$, and $N(\omega, \eta)$ in Definition 2.9, we have

\[
p_t(\omega, \eta) = \sum_{n=0}^{N(\omega, \eta)} \frac{\exp(-t/D(\mathbf{t}_n-1)) - \exp(-t/D(\mathbf{t}_n))}{\text{HARM}(\Sigma(\mathbf{t}_n))/n)}
\]

\[
= p_t(\omega, \omega) - \sum_{n=N(\omega, \eta) + 1}^{\infty} \frac{\exp(-t/D(\mathbf{t}_n-1)) - \exp(-t/D(\mathbf{t}_n))}{\text{HARM}(\Sigma(\mathbf{t}_n))/n)}
\]

\[
\geq \frac{C}{\text{HARM}(B_D(\omega, t))} - \sum_{n=N(\omega, \eta) + 1}^{\infty} \frac{\exp(-t/D(\mathbf{t}_n-1))}{\text{HARM}(\Sigma(\mathbf{t}_n))/n)}
\]

If $N = N(\omega, \eta) \geq L(\omega, \varepsilon) \lor K(\omega, \varepsilon)$, where $L(\omega, \varepsilon)$ and $K(\omega, \varepsilon)$ are given in the proof of Theorem 1.1, then

\[
\sum_{n=N+1}^{\infty} \frac{\exp(-t/D(\mathbf{t}_n-1))}{\text{HARM}(\Sigma(\mathbf{t}_n))/n)} \leq C \sum_{n=N+1}^{\infty} \exp\{\beta \lambda + \varepsilon\} \exp\{-\lambda \log x - \varepsilon n\} \exp\{\beta \lambda - \varepsilon n\}
\]

\[
\leq C \sum_{n=N+1}^{\infty} \exp\{\beta \lambda + \varepsilon\} \exp\{-C' t \log \lambda - 2\varepsilon n\} = C \sum_{n=N+1}^{\infty} \alpha^\delta \exp(-C' t \alpha^n)
\]

\[
\leq C \int_{\alpha}^{\infty} x^P \exp(-C' tx)dx,
\]

where $\alpha = \exp(\beta \lambda - 2\varepsilon)$, $\delta = \frac{\beta \lambda + \varepsilon}{\beta \lambda - \log \lambda - 2\varepsilon}$ and $P := \lceil \delta \rceil + 1$. It is easy to check that

\[
\int x^P \exp(-C' tx)dx = \exp(-C' tx) \sum_{i=0}^{P} (-1)^{P-i} \frac{P!}{\alpha^{(C' t)^{P-i+1}} x^i}.
\]
Therefore, we have
\[ \sum_{n=N+1}^{\infty} \frac{\exp(-t/D([\omega]_{n-1}))}{\text{HARM}(\Sigma([\omega]_n))} \leq C' \alpha^{-1} \sum_{i=0}^{P} t^{\alpha^{-N}i}. \]

When \( t^{1+\theta} \leq D(\omega, \eta) \leq t^{1+\theta'}, \) for \( 0 < \theta' < \theta, \) and \( N_\alpha = L(\omega, \varepsilon) \geq L(\omega, \varepsilon) \lor K(\omega, \varepsilon), \) we have \( t^{-\theta'} \leq \alpha^N t \leq t^{-\theta'}. \) Hence, when \( t^{1+\theta} \leq D(\omega, \eta) \leq t^{1+\theta'} \) for \( 0 < \theta' < \theta, \) and \( t > 0 \) sufficiently small,
\[ \sum_{n=N+1}^{\infty} \frac{\exp(-t/D([\omega]_{n-1}))}{\text{HARM}(\Sigma([\omega]_n))} \leq C' \exp(-C't^{-\theta'}) t^{-P-1} \sum_{i=0}^{P} (t^{\frac{i+\theta}{\alpha}})^i \leq C' \exp(-C't^{-\theta'}) t^{\frac{P(1+\theta)}{\alpha}}. \]

Since \( x^k \exp(-C't^x) \leq \frac{k^x \exp(-k)}{(C')^x} \) for any \( x > 0 \) and \( k \in \mathbb{N}, \) we have completed the proof of \( \text{Proposition 4.2}. \) Next, define \( B := \{ \eta \in \Sigma : t^{1+d} \leq D(\omega, \eta) \leq t^{1+d'} \}, \) where \( \delta'' > d > d' > \delta > 0. \) Then according to (4.4), we have
\[ E_\omega[D(\omega, X_t)^\gamma] \geq \int_B \mu(\omega, \eta) D(\omega, \eta)^\gamma d\text{HARM}(\eta) \geq \frac{1}{t^{\kappa_\lambda - \varepsilon}} \cdot t^{(1+d)\gamma} \cdot \text{HARM}(B) \geq t^{\kappa_\lambda - \kappa_\lambda + \gamma d' + \varepsilon} \left\{ t^{(1+d')\kappa_\lambda} - t^{(1+d')\kappa_\lambda - \varepsilon} \right\}. \]

By taking \( \varepsilon, \) \( d \) and \( d' \) sufficiently small such that \( d\kappa_\lambda - d\varepsilon > d'\kappa_\lambda + d\varepsilon + 2\varepsilon, \) we get the desired result. \( \square \)

As a corollary of Proposition \( \text{Proposition 4.2}, \) we can easily show Theorem \( \text{Theorem 1.2}. \)

**Proof of Theorem \( \text{Theorem 1.2}. \)** For \( \omega \in \Sigma, \) \( \eta \in \Sigma, \) \( d(\omega, \eta) = \exp(-N(\omega, \eta)), \) and for \( N(\omega, \eta) \geq L(\omega, \varepsilon) \lor K(\omega, \varepsilon), \) by (4.2) in the proof of Theorem \( \text{Theorem 1.1}, \) we have
\[ \exp\{-(\beta_\lambda - \log \lambda + 2\varepsilon)N(\omega, \eta)\} \leq D(\omega, \eta) = D_N \leq \exp\{-(\beta_\lambda - \log \lambda - 2\varepsilon)N(\omega, \eta)\}. \]
Hence for \( d(\omega, \eta) \leq \exp\{-L(\omega, \eta) \lor K(\omega, \eta)\}, \) we have
\[ d(\omega, \eta)^{\beta_\lambda - \log \lambda + 2\varepsilon} \leq D(\omega, \eta) \leq d(\omega, \eta)^{\beta_\lambda - \log \lambda - 2\varepsilon}. \]
This implies that there exists \( C_2 = C_2(\omega, \varepsilon) > 0 \) such that
\[ C_2^{-1} D(\omega, \eta)^{\frac{1}{\beta_\lambda - \log \lambda + 2\varepsilon}} \leq d(\omega, \eta) \leq C_2 D(\omega, \eta)^{\frac{1}{\beta_\lambda - \log \lambda - 2\varepsilon}}. \]
This estimate and Proposition \( \text{Proposition 4.2}, \) yield the claim. \( \square \)

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