ON THREEFOLDS WITH THE SMALLEST NONTRIVIAL MONODROMY GROUP

SERGE IL’OVSKI

Abstract. Using an adjunction-theoretic result due to A. J. Sommese together with a proposition from SGA7, we obtain a complete list of smooth threefolds for which the monodromy group acting on $H^2$ of its smooth hyperplane section is $\mathbb{Z}/2\mathbb{Z}$. The possibility of such a classification was announced by F. L. Zak in 1991.

1. Introduction

Suppose that $X \subset \mathbb{P}^N$ is a smooth projective variety of dimension $n$ (throughout the paper, we will assume that the base field is $\mathbb{C}$ and $X$ is not a linear subspace of $\mathbb{P}^N$) and that $Y \subset X$ is its smooth hyperplane section. As $Y$ varies in the family of smooth hyperplane sections of $X$, a monodromy action on $H^{n-1}(Y, \mathbb{Q})$ arises; its image in $\text{GL}(H^{n-1}(Y, \mathbb{Q}))$ will be called monodromy group of $X$.

If $\dim X$ is odd, this group is trivial if and only if the dual variety $X^* \subset (\mathbb{P}^N)^*$ is not a hypersurface (indeed, if $X^*$ is not a hypersurface, then $\pi_1((\mathbb{P}^N)^* \setminus X^*)$ is trivial, and if $X^*$ is a hypersurface and $\dim X$ is odd then the monodromy group contains reflections in non-zero vanishing cycles; see [11, 6.3.3]). It is known that, for a smooth threefold $X$, its dual $X^*$ is not a hypersurface if and only if $X$ is a $\mathbb{P}^2$-scroll over a curve (see [5, Theorem 3.2]). In this paper we obtain a complete description of the next natural class of threefolds, that is, of those with monodromy group $\mathbb{Z}/2\mathbb{Z}$.

Theorem 1.1. Suppose that $X \subset \mathbb{P}^N$ is a smooth projective threefold. Then the monodromy group of $X$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ if and only if $X$ is one of the following varieties:

1. the quadric $Q \subset \mathbb{P}^4$;
2. the Veronese variety $v_2(\mathbb{P}^3) \subset \mathbb{P}^9$ or its isomorphic projection;
3. the blowup of $\mathbb{P}^3$ at a point, embedded in $\mathbb{P}^8$ by the complete linear system $|2H - E|$, where $H$ is the preimage of a plane in $\mathbb{P}^3$ and $E$ is the exceptional divisor, or an isomorphic projection of this variety;
4. the Segre variety $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^7$.

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In his paper [18], F. L. Zak announced without proof that, for a smooth variety \( X \subset \mathbb{P}^N \) of odd dimension \( n = 2k + 1 \) the following three assertions are equivalent.

1. The monodromy group of \( X \) is \( \mathbb{Z}/2\mathbb{Z} \);
2. \( b_{2k}(Y) = b_{2k}(X) + 1 \), where \( Y \) is a smooth hyperplane section.
3. The dual variety \( X^* \subset (\mathbb{P}^N)^* \) is a normal hypersurface.

Thus, Theorem 1.1 yields a complete description of, say, smooth threefolds the dual of which is a normal hypersurface, too.

The equivalence (1) \( \Leftrightarrow \) (2) is easy (see the proof of Proposition 5.3 below); the proof of the equivalence (2) \( \Leftrightarrow \) (3) will be published in a separate paper, joint with Zak.

In our proof of Theorem 1.1 we heavily use the following two results: a proposition (due to Deligne) from SGA7 (see Proposition 3.1 below) and the main result of Sommese’s paper [15]. To wit, it turns out that the classification of smooth threefolds \( X \subset \mathbb{P}^N \) for which \( h^0(\omega_X(1)) = 0 \), which is contained in [15], is the same as the classification of smooth threefolds with finite monodromy group. Using Deligne’s result mentioned above, one can extract from Sommese’s list of such varieties those with \( \mathbb{Z}/2\mathbb{Z} \) monodromy group. Actually, we obtain a complete description of monodromy groups of varieties on Sommese’s list, with one exception.

In the above mentioned paper [18] it was announced that a complete classification of smooth threefolds \( X \subset \mathbb{P}^N \) with monodromy group \( \mathbb{Z}/2\mathbb{Z} \) could be obtained basing on a detailed study of the second fundamental form of \( X \). Be it for better or for worse, the results of the paper [15] (i.e., the adjunction theoretic methods) have lead to success sooner. Yet it would be interesting to find a more conceptual proof of Theorem 1.1, especially a proof that could be extended to higher (odd) dimensions.

Besides Theorem 1.1 we prove the following two results.

First, if \( E \) is a very ample vector bundle of rank 2 over a smooth surface \( S \), \( c_2(E) = r \), and if we vary a section of \( E \) in the space of sections with precisely \( r \) zeroes, then this variation induces a group of permutations of the zero locus of one such section. It turns out that this group is always the entire symmetric group \( S_r \) (Proposition 5.6).

Second, we show that if \( X \subset \mathbb{P}^N \) is a three-dimensional scroll over a smooth projective surface, then the curve that is its general one-dimensional linear section cannot be “too special” (Proposition 5.7).

After \( \mathbb{Z}/2\mathbb{Z} \), the next smallest monodromy group of an odd-dimensional variety is the symmetric group \( S_3 \) (it is the Weyl group of the \( A_2 \) root system; see Section 3.4). It seems possible to extract a list of threefolds with such monodromy from Sommese’s list as well; to that end, the main result of [14] may be of help. Conjecturally, the only smooth threefolds with monodromy group \( S_3 \) (“with the \( A_2 \) monodromy”) are the scroll \( \mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(3)) \) and the smooth hyperplane section of the Segre variety \( \mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8 \). The latter
threefold is also a scroll over $\mathbb{P}^2$, namely, the projectivisation of $\mathcal{T}_{\mathbb{P}^2}$ (or of $\Omega^1_{\mathbb{P}^2}$, depending on the conventions).

The paper is organized as follows. In Section 3 we briefly recall what we need from Picard–Lefschetz theory. In Section 4 we extract a characterization of threefolds with finite monodromy group from Sommese’s main result in [15]. In Sections 5.1–5.5 we describe vanishing root systems (see the definition in Section 3.4) for the varieties on Sommese’s list and extract from the list those with $\mathbb{Z}/2\mathbb{Z}$ monodromy.

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2. Notation and conventions

The base field will always be the field of complex numbers.

If $X \subset \mathbb{P}^N$ is a Zariski closed subset and $x \in X$, then $T_x X \subset \mathbb{P}^N$ is the embedded Zariski tangent space to $X$ at $x$.

If $E$ is a vector bundle aka locally free sheaf on $X$, then (unlike the EGA notation) closed points of the projectivisation $\mathbb{P}(E)$ are lines in the fibers of $E$ and closed points of the projectivization $\mathbb{P}^*(E)$ are hyperplanes in the fibers of $E$. In particular, $p_* \mathcal{O}_{\mathbb{P}^*(E)}|_X(1) = E$, where $p : \mathbb{P}^*(E) \to X$ is the canonical projection.

If $(\mathbb{P}^N)^*$ is the dual projective space to $\mathbb{P}^N$ and $\alpha \in (\mathbb{P}^N)^*$, then $H_\alpha \subset \mathbb{P}^N$ is the corresponding hyperplane. If $X \subset \mathbb{P}^N$ is a smooth projective variety, then its dual variety is $X^* = \{ \alpha \in (\mathbb{P}^N)^*: H_\alpha \text{ is not transversal to } X \}$.

If $\dim X = n$, $\alpha \in (\mathbb{P}^N)^\ast \setminus X^\ast$, $\dim X = n$, and $H_\alpha \cap X = Y$, then the variation of $\alpha$ in $(\mathbb{P}^N)^\ast \setminus X^\ast$ induces an action of the fundamental group $\pi_1((\mathbb{P}^N)^\ast \setminus X^\ast, \alpha)$ on $H^{n-1}(Y, \mathbb{Q})$ which is called the monodromy action; slightly abusing the language, we will say that the image of $\pi_1((\mathbb{P}^N)^\ast \setminus X^\ast, \alpha)$ in $\text{GL}(H^{n-1}(Y, \mathbb{Q}))$ is the monodromy group of $X$; as a subgroup of $\text{GL}_{b_{n-1}(Y)}(\mathbb{Q})$, the monodromy group is defined up to conjugation.

3. A survey of some results from Picard–Lefschetz theory

None of the assertions in this section claims to novelty: we just briefly recall what we need from Picard–Lefschetz theory. Almost all the proofs will be omitted. For non-trivial details we refer the reader to SGA7.2, especially Exposés XVII and XIX ([3] in the bibliography) and Lamotke [11].

Suppose that $X \subset \mathbb{P}^N$ is a smooth projective variety of odd dimension $n = 2k+1$ and $H = H_\alpha \subset \mathbb{P}^N$, where $\alpha \in (\mathbb{P}^N)^\ast \setminus X^\ast$, is a hyperplane transversal to $X$; put $Y = H \cap X$. 
3.1. Pairings. We will identify $H_0(Y, \mathbb{Q})$ with $\mathbb{Q}$, mapping a singular 0-chain $\sum c_i a_i$, $a_i \in Y$, to $\sum c_i \in \mathbb{Q}$. If $\xi \in H^m(Y, \mathbb{Q})$, $\sigma \in H_m(Y, \mathbb{Q})$, we put, keeping this identification in mind,

$$\langle \xi, \sigma \rangle = \xi \cap \sigma \in H_0(Y, \mathbb{Q}) = \mathbb{Q}.$$  

The $\mathbb{Q}$-valued pairing $\langle \cdot, \cdot \rangle$ is non-degenerate.

Define, on $H^{2k}(Y, \mathbb{Q})$, the non-degenerate bilinear form

$$\langle \xi, \eta \rangle = (-1)^k \langle \xi \cup \eta, [Y] \rangle,$$

where $[Y] \in H_{4k}(Y, \mathbb{Q})$ is the fundamental class. If $\dim X = 3$, then the pairing (2) is the negated intersection index.

Let $P: H^{2k}(Y, \mathbb{Q}) \to H_{2k}(Y, \mathbb{Q})$ be the Poincaré duality isomorphism, defined by the formula $P(\xi) = \xi \cap [Y]$ (cap-product with the fundamental class). The isomorphism $P$ transplants the pairing (2) to $H_{2k}(Y, \mathbb{Q})$, by the formula

$$\langle x, y \rangle = (P^{-1}(x), P^{-1}(y)).$$

The notation coincides with that in (2); this will not lead to a confusion. For any $x, y \in H_{2k}(Y, \mathbb{Q})$, one has

$$\langle x, y \rangle = \langle P^{-1}(x), y \rangle.$$

3.2. Spaces of vanishing cycles. The mapping $i^*: H^{2k}(X, \mathbb{Q}) \to H^{2k}(Y, \mathbb{Q})$, where $i: Y \hookrightarrow X$ is the embedding, is injective, and the pairing (2) is non-degenerate on $i^*H^{2k}(X, \mathbb{Q})$. Let $\text{Ev}(Y)$ be the orthogonal complement to $i^*H^{2k}(X, \mathbb{Q})$ in $H^{2k}(Y, \mathbb{Q})$ with respect to the bilinear form (2). One has $\dim \text{Ev}(Y) = b_{2k}(Y) - b_{2k}(X)$. If $X^*$ is a hypersurface, $\text{Ev}(Y) \neq 0$ and the bilinear form (2) is non-degenerate on $\text{Ev}(Y)$. The variation of $H_\alpha$ in the family of all transversal to $X$ hyperplanes induces an action of $\pi_1(\mathbb{P}^N \setminus X^*)$ on $H^{2k}(Y, \mathbb{Q})$ preserving the bilinear form and the decomposition $H^{2k}(Y, \mathbb{Q}) = i^*H^{2k}(X, \mathbb{Q}) \oplus \text{Ev}(Y)$; this action is trivial on $i^*H^{2k}(X, \mathbb{Q})$ and irreducible on $\text{Ev}(Y)$.

Put

$$\text{ev}(Y) = \text{Ker}(i_*: H_{2k}(Y, \mathbb{Q}) \to H_{2k}(X, \mathbb{Q})).$$

If $P: H^{2k}(Y, \mathbb{Q}) \to H_{2k}(Y, \mathbb{Q})$ is the Poincaré duality isomorphism, then $P(\text{Ev}(Y)) = \text{ev}(Y)$. The isomorphism $P: \text{Ev}(Y) \to \text{ev}(Y)$ transplants the action of the monodromy group on $\text{Ev}(Y)$ to $\text{ev}(Y)$.

3.3. Vanishing cycles and the monodromy group. Suppose now that $\ell \subset (\mathbb{P}^N)^*$ is a Lefschetz pencil with respect to $X$ (see [3] Exposé XVII, 2.2) and put $\ell \cap X^* = \{\alpha_1, \ldots, \alpha_r\}$. If $\delta_1, \ldots, \delta_r \in H_{n-1}(Y, \mathbb{Z})$ are vanishing cycles corresponding to $\alpha_1, \ldots, \alpha_r$, there exists an exact sequence

$$\mathbb{Z}^r \xrightarrow{A} H_{n-1}(Y, \mathbb{Z}) \to H_{n-1}(X, \mathbb{Z}) \to 0,$$
where \( A(n_1, \ldots, n_r) = \sum_{i=1}^{r} n_i \delta_i \) (see [3] Exposé XIX, Section 4.3]). For any \( i, 1 \leq i \leq r \), put
\[
\tau_i = A(\delta_i) \otimes 1 \in H_{2k}(Y, \mathbb{Z}) \otimes \mathbb{Q} = H_{2k}(Y, \mathbb{Q}).
\]
The elements \( \tau_i \) are also called vanishing cycles.

The vanishing cycles \( \tau_i \) have the following properties.

1. The subspace \( \text{ev}(Y) \subset H_{2k}(Y, \mathbb{Q}) \) is spanned by \( \tau_1, \ldots, \tau_r \).
2. For any \( i, j, 1 \leq i \leq j \leq r \), there exists an element \( g \) of the monodromy group such that \( g(\tau_i) = \tau_j \).
3. \( \langle \tau_i, \tau_i \rangle = 2 \) for all \( i \) (recall definitions (2) and (3)).
4. The monodromy group acting on \( \text{ev}(Y) \) is generated by “reflections in vanishing cycles”, that is, by the linear mappings of the form
\[
s_i : x \mapsto x - \langle x, \tau_i^\vee \rangle \tau_i, \quad 1 \leq i \leq r,
\]
where \( \lambda^\vee = P^{-1}(\lambda) \in \text{Ev}(Y) \) for any \( \lambda \in \text{ev}(Y) \).

### 3.4. Finite monodromy groups.
Suppose, in the above setting, that \( X^* \) is a hypersurface, so the monodromy group \( G \subset \text{GL}(\text{ev}(Y)) \) is nontrivial. According to Proposition 3.4 in Exposé XIX of [3], the group \( G \) is finite if and only if \( \text{ev}(Y) \subset H_{2k}(Y, \mathbb{Q}) \cap H^{k,k}(Y) \).

Suppose now that this is the case and put
\[
R = \{ g(\tau_i) \mid \text{ all } g \in G \text{ and all } i, 1 \leq i \leq r \}.
\]

**Proposition 3.1** (P. Deligne). The subset \( R \) of the space \( \text{ev}(Y) \) endowed with the bilinear form \( (2) \) is an irreducible root system of the type \( A, D \), or \( E \), the root lattice of this root system is spanned by \( \tau_1, \ldots, \tau_r \), and its Weyl group coincides with the monodromy group \( G \).

This assertion, or rather an equivalent statement about \( \text{Ev}(Y) \) and \( l \)-adic cohomology, is contained in [3] Exposé XIX, Proposition 3.3]. For the reader’s convenience we provide some details.

**Proof of Proposition 3.1.** To show that \( R \) is a root system we are to check the conditions (SR₁)–(SR₃) from Bourbaki [2] Chapitre VI, §1. Observe that \( R \) is finite since \( G \) is finite and that none of the elements of \( R \) is zero since none of the \( \tau_i \) is zero by virtue of assertion (3) from Section 3.3. This checks Condition (SR₁). If \( \tau = g(\tau_i) \in R \), where \( g \in G \) and \( 1 \leq i \leq r \), and if \( s_i \) is the reflection from (3), then the linear automorphism \( g^{-1}s_ig \in G \) is the reflection in \( \tau = g(\tau_i) \in R \), of the form \( x \mapsto x - (x, \tau^\vee)x \), so this reflection maps \( R \) into itself and Bourbaki’s Condition (SR₃) is also satisfied. Finally, to check Condition (SR₃) observe that all the \( \tau_i \) lie in the image of the natural mapping \( H_{2k}(Y, \mathbb{Z}) \to H_{2k}(Y, \mathbb{Q}) \). Since the action of \( \pi_1((\mathbb{P}^N)^* \setminus X^*) \) on \( H_{2k}(Y, \mathbb{Q}) \) lifts to \( H_{2k}(Y, \mathbb{Z}) \), this is true for all the reflections in elements of \( R \) as well. It is clear that if \( x', y' \in H_{2k}(Y, \mathbb{Z}) \) and \( x, y \) are their images in \( H_{2k}(Y, \mathbb{Q}) \), then \( (x, y^\vee) = (-1)^k x' \cap y' \cap [Y]_Z \in \mathbb{Z} \) (here, \([Y]_Z \in H_{4k}(Y, \mathbb{Z}) \) is the fundamental class and we naturally identified \( H_{4k}(Y, \mathbb{Z}) \) with \( \mathbb{Z} \)). Thus, Condition (SR₃) is satisfied and \( R \) is a root system.
The monodromy group $G$ coincides with the Weyl group of $R$ by virtue of assertion 2 from Section 3.3, the root system $R$ is irreducible since the action of $G$ on $ev(Y)$ is irreducible, and $R$ is simply laced since the lengths of all roots are equal by virtue of assertion 3 from Section 3.3. It remains to show that the root lattice is spanned by $\tau_1, \ldots, \tau_r$, i.e., that any $\tau \in R$ is an integer linear combination of $\tau_1, \ldots, \tau_r$. To that end, it suffices to show that each $s_i(\tau_j)$ is an integer linear combination of $\tau_1, \ldots, \tau_r$, and this assertion follows from Condition (SRIII), which we already checked. □

Remark 3.2. Observe that $\tau_1, \ldots, \tau_r$ need not be a base of the root system $R$.

Further on, the words, say, “The variety $X$ has $A_2$ monodromy” will mean that the monodromy group of $X$ is finite and the root system $R$ constructed above is of the indicated type; in particular, it implies that dimension of the space of vanishing cycles equals the rank of this root system and the monodromy group is isomorphic to its Weyl group. We will refer to the root system $R$ above as the vanishing root system of the variety $X$.

4. Sommese’s list

From now on, we assume that the odd-dimensional variety $X \subset \mathbb{P}^N$ with monodromy group $\mathbb{Z}/2\mathbb{Z}$ has dimension 3; we aim at the classification of such varieties. We begin with varieties having finite monodromy group.

We will need the following ad hoc terminology, which is a variant of the notion of “minimal reduction” from [15].

Definition 4.1. Let us say that a projective variety $Y \subset \mathbb{P}^n$ is $k$-subordinate to the variety $X \subset \mathbb{P}^{N+k}$ if there exists a sequence of smooth projective varieties $X_i \subset \mathbb{P}^{N+k-i}$, $X_0 = X$, $X_k = Y$, and birational morphisms

$$X = X_0 \xrightarrow{\varphi_0} X_1 \xrightarrow{\varphi_1} \ldots \xrightarrow{\varphi_{k-1}} X_k = Y$$

such that each $\varphi_i: X_i \dashrightarrow X_{i+1}$ is the projection from a point $a_i \in X_i$ and each $\varphi_i$ induces an isomorphism between $X_{i+1}$ and the blowup of $X_i$ at $a_i$. If $Y$ is $k$-subordinate to $X$ for some $k > 0$, we will just say that $Y$ is subordinate to $X$.

Proposition 4.2. Suppose that $X \subset \mathbb{P}^N$ is a smooth threefold such that $X^*$ is a hypersurface (equivalently, the monodromy group of $X$ is non-trivial). Then the monodromy group of $X$ is finite if and only if $X$ is one of the varieties listed in Table 1.

Remark 4.3. The categories in Table 1 are not disjoint.

We will refer to the list in Table 1 as Sommese’s list.

Proof. According to Proposition 6.1 of the paper [12], the monodromy group of $X$ is finite if and only if $H^0(X, \omega_X(1)) = 0$. Now Main Theorem of Sommese’s paper [15] contains a complete classification of pairs $(X, \mathcal{L})$, where $X$ is a smooth projective variety of dimension $n \geq 3$ and $\mathcal{L}$ is an
Table 1. Smooth threefolds \( X \subset \mathbb{P}^N \) with finite and non-trivial monodromy group

| No | Description of \( X \) |
|----|-------------------------|
| 1  | \( X \) is a scroll over a surface, that is, there exists a locally free sheaf \( \mathcal{E} \) of rank 2 over a smooth surface \( S \) such that \((X, \mathcal{O}_X(1)) \cong (\mathbb{P}^*(\mathcal{E}), \mathcal{O}_{\mathbb{P}^*(\mathcal{E})}|_S(1))\). |
| 2  | \( X \) is a pencil of quadrics, that is, there exists a morphism \( p: X \to C \), where \( C \) is a smooth curve, such that the fiber of \( p \) over a general point of \( C \) is a smooth quadric (i.e., a smooth surface of degree 2 in \( \mathbb{P}^N \)). |
| 3  | \( X \) is a Veronese pencil, that is, there exists a morphism \( p: X \to C \), where \( C \) is a smooth curve, such that, for a general point \( a \in C \), the fiber \( X_a = p^{-1}(a) \) is a smooth surface and \((X_a, \mathcal{O}_{X_a}(1)) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))\). |
| 4  | \( X \) is a Del Pezzo threefold, i.e., a Fano variety embedded by one half of the anticanonical class, that is, \( \omega_X \cong \mathcal{O}_X(-2) \). |
| 5  | \( X \) is a smooth quadric in \( \mathbb{P}^4 \). |
| 6  | \( X \) is the Veronese image \( v_2(Q) \subset \mathbb{P}^{13} \) or its isomorphic projection. |
| 7  | \( X \subset \mathbb{P}^N \) is subordinate to \( v_2(Q) \), that is, \( X \) is the blowup of the smooth three-dimensional quadric \( Q \) at \( k \geq 1 \) points, and \( \mathcal{O}_X(1) \cong \mathcal{O}_X(2\sigma^*H - E_1 - \cdots - E_k) \), where \( \sigma: X \to Q \) is the blowdown morphism, \( H \) is a hyperplane section of \( Q \), and \( E_1, \ldots, E_k \subset X \) are exceptional divisors. |

invertible sheaf on \( X \) that is ample and spanned by global sections and \( H^0(X, \omega_X \otimes \mathcal{L}^{\otimes n-2} = 0) \) (Sommese allows the variety \( X \) to have some mild singularities as well). The list of such pairs is contained in sections (0.2), (0.3), and (0.4) of [15].

Extracting smooth embedded threefolds with very ample \( \mathcal{L} \) from the list in Sommese’s Section (0.2) and excluding threefolds with trivial monodromy (i.e., those whose dual is not a hypersurface, i.e., \( \mathbb{P}^2 \)-scrolls over curves), one obtains smooth quadrics (item (5) in Table [1]).

Applying the same procedure to the list in [15, Section (0.3)], one obtains varieties from items (1), (2), and (4).

Now, translating Sommese’s result into the language of embedded projective varieties, it is easy to see that the pairs \((X, \mathcal{O}_X(1))\) from the list in [15, Section (0.4)], where \( \dim X = 3 \), \( X \) is smooth and embedded in \( \mathbb{P}^N \), are as follows. Either \((X, \mathcal{O}_X(1)) \cong (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2))\), or \((X, \mathcal{O}_X(1)) \cong (Q, \mathcal{O}_Q(2))\), where \( Q \subset \mathbb{P}^4 \) is a smooth quadric, or there exists a morphism \( p: X \to C \), where \( C \) is a curve, such that \( \omega_X^{\otimes 2}(3) \cong p^*\mathcal{L} \), where \( \mathcal{L} \) is a line bundle on \( C \), or, finally, \( X \) is subordinate to one of the above.
If \((X, \mathcal{O}_X(1)) \cong (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2))\), then \(X\) is a variety in item (4) of Sommese’s list (Table 1). If one projects such an \(X\) from a point \(a \in X\) to obtain a smooth variety \(X' \subset \mathbb{P}^{N-1}\) that is isomorphic to the blowup of \(X\) at \(a\), then \(X'\) is also a variety in item (4), and it is impossible to further extend the chain from Definition (4.1) since the variety \(X'\) is swept by lines, which will be blown down by the next projection.

If \((X, \mathcal{O}_X(1)) \cong (Q, \mathcal{O}_Q(2))\), where \(Q\) is the three-dimensional quadric, then \(X\) is \(v_2(Q) \subset \mathbb{P}^{13}\) or its isomorphic projection; this is item (6) in Sommese’s list.

If \(X\) is subordinate to a variety from item (6), then it is a variety from item (7) (observe that, in the chain of projections (5), each point \(a_i\) does not lie on exceptional divisors of the previous blowups since otherwise the projection would blow down the lines lying on this exceptional divisor and passing through \(a_i\)).

Finally, if there exists a morphism \(p: X \to C\) such that \(\omega_X^{\otimes 2}(3) \cong p^*\mathcal{L}\), where \(\mathcal{L}\) is an invertible sheaf on \(C\), then, denoting by \(F \subset X\) a general fiber of \(p\) and restricting to \(F\), one obtains the isomorphism \(\omega_F^{\otimes 2} \cong \mathcal{O}_F(-3)\), whence \(F \cong \mathbb{P}^2\) (since \(F\) is a Del Pezzo surface and the canonical class of \(F\) is divisible by 3) and \(\mathcal{O}_F(1) \cong \mathcal{O}_{\mathbb{P}^2}(2)\); thus, such an \(X\) belongs to item (3) in Table 1, so \(X\) is a Veronese pencil. Since a variety subordinate to a Veronese pencil is also a Veronese pencil, this completes the proof. \(\square\)

5. Inspection of Sommese’s list

In this section we will find vanishing root systems for varieties from Sommese’s list. The method for identification of root systems was used by Yu. I. Manin [13, Chapter 4], who attributes it to Deligne.

We will use the following

**Notation 5.1.** Suppose that \(L\) is a lattice in a finite dimensional linear space \(V\) over \(\mathbb{Q}\), and suppose that \(V\) is endowed with a symmetric and non-degenerate definite \(\mathbb{Q}\)-valued bilinear form \((\cdot, \cdot)\) such that \((x, y) \in \mathbb{Z}\) for any \(x, y \in L\). We will put

\[ L^\vee = \{ \lambda \in L \otimes \mathbb{Q} : (x, \lambda) \in \mathbb{Z} \text{ for all } x \in L \}. \]

**Remark 5.2.** If \(\langle e_1, \ldots, e_r \rangle\) is a \(\mathbb{Z}\)-basis of \(L\), then index of \(L\) in \(L^\vee\) equals \(\det \|(e_i, e_j)\|\). Hence, if \(L \subset L_1\) are two lattices such that \((x, y) \in \mathbb{Z}\) for any \(x, y \in L_1\), then

\[ [L^\vee : L] = [L_1^\vee : L_1] \cdot [L_1 : L]^2. \]

5.1. Scrolls over a surface. We begin with item (1) of Sommese’s list (Table 1).

Throughout this section, \(X\) is a three-dimensional scroll over a surface, that is, \(X = \mathbb{P}^1(\mathcal{E})\), where \(\mathcal{E}\) is a very ample bundle of rank 2 over a smooth projective surface \(S\) and \(\mathcal{O}_X(1) = \mathcal{O}_X|_S(1)\).

The answer to the question which scrolls over surfaces have \(A_1\) monodromy can be immediately read off from the book [1].
Proposition 5.3. In the above setting, $X$ has $A_1$ monodromy if and only if $(X, \mathcal{E}) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(2))$ or $(X, \mathcal{E}) \cong (Q, \mathcal{O}_Q(1) \oplus \mathcal{O}_Q(1))$, where $Q \subset \mathbb{P}^3$ is the smooth quadric.

Proof. Observe that for any smooth threefold $X$ with a smooth hyperplane section $Y$, the assertions “$X$ has $A_1$ monodromy” and “$b_2(Y) = b_2(X) + 1$” are equivalent. Indeed, the only reflection on a one-dimensional linear space is $x \mapsto -x$, so if $b_2(Y) = b_2(X) + 1$, then the monodromy group that acts on the one-dimensional $\text{Ev}(Y)$, being generated by reflections, is isomorphic to $\mathbb{Z}/2\mathbb{Z}$, whence $A_1$; the opposite implication is trivial.

Now if $Y \subset X$ is a general smooth hyperplane section, where $X$ is a $\mathbb{P}^1$-scroll over a surface $S$, then the hyperplane section $Y$ is isomorphic to $S$ with $c_2(\mathcal{E})$ points blown up, so $b_2(Y) = b_2(S) + c_2(\mathcal{E})$. Since $b_2(X) = b_2(S) + 1$, the variety $X$ has $A_1$ monodromy (equivalently, $b_2(Y) = b_2(X) + 1$) if and only if $c_2(\mathcal{E}) = 2$, and Theorem 11.4.5 from [1] does the job. \hfill \Box

Remark 5.4. One observes that if $(X, \mathcal{E}) \cong (Q, \mathcal{O}_Q(1) \oplus \mathcal{O}_Q(1))$ then $X$ is the Segre variety $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^7$ (variety $V_6$ in Table 2 below), and if $(X, \mathcal{E}) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(2))$ (variety $V_7$ in Table 2) then $X$ is isomorphic to the projection of the Veronese variety $v_2(\mathbb{P}^3) \subset \mathbb{P}^9$ from a point lying on $v_2(\mathbb{P}^3)$, or to an isomorphic projection of this projection.

For the sake of completeness we find the vanishing root system for other scrolls over surfaces as well.

Proposition 5.5. In the above setting, if $c_2(\mathcal{E}) = 1$, then $X^*$ is not a hypersurface in $(\mathbb{P}^N)^*$, and if $c_2(\mathcal{E}) = r > 1$, then the vanishing root system of $X$ is $A_{r-1}$.

Proof. Let $p : X \to S$ be the natural projection and let $Y \subset X$ be a general smooth hyperplane section; $Y$ is isomorphic to $S$ with $r$ points blown up. Let $\ell_1, \ldots, \ell_r \subset Y \subset X$ be the corresponding exceptional curves, which are lines on $Y$ and fibers of the projection $p$ on $X$. If $l_j \in H_2(Y, \mathbb{Z})$ is the class of $\ell_j$, then the images of all the $l_j$ in $H_2(X, \mathbb{Z})$ are equal, whence the image of $\text{Ker}(H_2(Y, \mathbb{Z}) \to H_2(X, \mathbb{Z}))$ in $H_2(Y, \mathbb{Q})$, with the pairing $(\cdot, \cdot)$ from Section 3.1 (i.e., with the negated intersection index), is isomorphic to the lattice

$$L = \left\{ \sum_{j=1}^r c_j l_j : c_j \in \mathbb{Z}, \sum_{j=1}^r c_j = 0 \right\},$$

where $(l_i, l_j) = \delta_{ij}$. If $r = 1$, this implies that $\text{Ev}(Y) = 0$, so $X^*$ is not a hypersurface. If $r > 1$, then rank $L = r - 1$, and a well-known computation (see for example [2, Planche I, (VIII)]) shows that $[L' : L] = r$. For a simply laced irreducible root system of rank $r - 1$, this is possible only if it is $A_{r-1}$.

\hfill \Box

We end Section 5.1 with two by-products of Proposition 5.5. The first one concerns the monodromy action on zero loci of sections of vector bundles on surfaces.
Suppose that $E$ is a very ample bundle of rank 2 on a smooth surface $S$, $c_2(E) = r$. Put $V = H^0(S, E)$ and let $D \subset V$ be the set of sections that are not transversal to the zero section. Any section in $V \setminus D$ has precisely $r$ zeroes. Fix a section $s_0 \in V \setminus D$, and let $Z(s_0) = \{p_1, \ldots, p_r\}$ be its zero locus. As sections of $E$ vary in $V \setminus D$, the fundamental group $\pi_1(V \setminus D, s_0)$ acts on $Z(s_0)$ by permutations. Let us say that the image of $\pi_1(V \setminus D, s_0)$ in the group of permutations of $p_1, \ldots, p_r$ is the monodromy group of sections of $E$.

**Proposition 5.6.** In the above setting, the monodromy group of sections of $E$ is the entire group of permutations of the set $Z(s_0)$.

*Proof.* Denoting the group in question by $\Gamma$, put $X = \mathbb{P}^*(E)$ and embed $X$ into $\mathbb{P}^N$ via the complete linear system $|\mathcal{O}_{X|S}(1)|$. Let $Y \subset X$ be the smooth hyperplane section of $X$ corresponding to the section $s_0 \in H^0(S, E) \cong H^0(X, \mathcal{O}_{X|S}(1))$; the surface $Y$ is isomorphic to the blowup of $S$ at the points of $Z(s_0)$, so the monodromy transformations acting on $H_2(Y, \mathbb{Q})$ are induced by the permutations from $\Gamma$. Thus, $\Gamma$ surjects onto the monodromy group of $X$. Proposition 5.5 implies that the monodromy group of $X$ is the symmetric group $S_r$, so $\Gamma$ surjects onto $S_r$. Since, on the other hand, $\Gamma$ is a subgroup of the permutation group $\text{Aut}(Z(s_0)) \cong S_r$, we conclude that $\Gamma = \text{Aut}(Z(s_0))$. \qed

Another by-product of Proposition 5.5 is an assertion to the effect that if $X \subset \mathbb{P}^n$ is a scroll over a surface then the curve that is its general linear section of codimension 2 cannot be “too special”.

If $\pi: C \to C_1$ is a finite morphism of (smooth, projective, and connected) algebraic curves and if $x_0 \in C_1 \setminus B$, where $B \subset C_1$ is the branch locus of $\pi$, then $\pi_1(C_1 \setminus B)$ acts on $\pi^{-1}(x_0)$; the induced subgroup in the group of permutations of $\pi^{-1}(x_0)$ will be called monodromy group of the fibers of $\pi$.

**Proposition 5.7.** Suppose that $X = \mathbb{P}^*(E) \subset \mathbb{P}^N$, where $E$ is a rank 2 very ample bundle on a smooth surface $S$, is embedded with $|\mathcal{O}_{X|S}(1)|$. If $c_1(E) = r$ and $C$ is a general one-dimensional linear section of $X$, then there exists a morphism $\pi: C \to \mathbb{P}^1$ such that $\deg \pi = r$ and the monodromy group of the fibers of $\pi$ is the entire symmetric group $S_r$.

*Remark 5.8.* The assertion that $C$ possesses a morphism of degree $r$ onto $\mathbb{P}^1$ is a particular case of Theorem 11.1.2(2) from [1]. It is only the second part of this proposition that claims to novelty.

*Proof.* By lines of the ruling we will mean the lines on $X$ that are fibers of the natural projection $X \to S$.

Suppose that $\ell \subset (\mathbb{P}^N)^*$ is a general Lefschetz pencil and $L \subset \mathbb{P}^N$, where $\dim L = N - 2$, is its axis. For a general $\ell$, the linear subspace $L$ does not contain any line of the ruling of $X$, so we may and will assume that this is the case. Putting $C = X \cap L$, define the mapping $\pi: C \to \ell$ as follows. For any $x \in C$, put $\pi(x) = \alpha \in \ell$, where the hyperplane $H_\alpha \subset \mathbb{P}^N$.
is the linear span of $L$ and the line of the ruling passing through $x$. (This is just a geometric description of the morphism $C' \to \mathbb{P}^1$ from the proof of [1, Theorem 11.1.2(2)].) If $\alpha \in \ell$ is such that $H_\alpha$ is transversal to $X$, then $H_\alpha \cap X$ is isomorphic to the blowup of $S$ at the points $s_1, \ldots, s_r$ that are the zeroes of the section $\sigma \in H^0(S, \mathcal{E}) = H^0(X, \mathcal{O}_X(1))$ defining the hyperplane $H_\alpha$, and $H_\alpha$ contains precisely $r$ lines of the ruling that are fibers over $s_1, \ldots, s_r$. If $\alpha \in \ell$ is such that $H_\alpha \cap X$ is singular, then $H_\alpha$ contains precisely $r - 1$ lines of the ruling, on one of which the unique singular point of $H_\alpha \cap X$ sits.

If we fix an $\alpha_0 \in \ell$ for which $H_{\alpha_0} \cap X$ is smooth, then it is clear from the proof of Proposition 5.5 that the action of the monodromy group on $\text{ev}(X \cap H_{\alpha_0})$ is induced by the monodromy permutations of lines of the ruling lying on $X \cap H_{\alpha_0}$, i.e., of the points of fiber $\pi^{-1}(\alpha)$. Thus, the latter group of permutations must be the entire $S_r$. □

5.2. Pencils of quadrics. Throughout this section $X \subset \mathbb{P}^N$ will be a pencil of quadrics in the sense of Proposition 4.2 (item (2) in Table 1). That is, $X \subset \mathbb{P}^N$ is a smooth projective threefold and $p: X \to C$ is a morphism onto a smooth curve $C$ such that its fibers are isomorphic to quadrics in $\mathbb{P}^3$. For each $t \in C$, put $X_t = p^{-1}(t)$.

It is well known that the subset $S \subset C$ of points such that fibers over them are not smooth is finite and fibers over the points of $S$ are quadratic cones of corank one.

We begin with a simple observation.

**Lemma 5.9.** If $X \subset \mathbb{P}^N$ is a pencil of quadrics and $Y \subset X$ is a smooth hyperplane section, then $\dim \text{ev}(Y) > 0$.

**Proof.** If $\text{ev}(Y) = 0$, then the dual variety $X^*$ is not a hypersurface, so $X$ is swept by planes (see [4, Theorem 5.1]). Since any morphism from $\mathbb{P}^2$ to a curve is constant, these planes should be contained in the fibers of the morphism $p: X \to C$, which is absurd. □

If $t_0 \in C \setminus S$, then $\pi_1(C \setminus S, t_0)$ acts by monodromy on $H_2(X_{t_0}, \mathbb{Q})$. Since this action preserves the intersection index and the class of the hyperplane section of the smooth quadric $X_{t_0}$, the monodromy transformation is either identity, or it swaps the classes of the lines $\ell, m \subset X_{t_0}$ of different rulings.

We will distinguish between two types of pencils of quadrics.

**Definition 5.10.** Let us say that a pencil of quadrics $p: X \to C$ is ordinary if this monodromy action on $H_2(X_t, \mathbb{Q})$ is nontrivial, and that it is extraordinary if this action is trivial.

**Remark 5.11.** If $S \neq \emptyset$, then the pencil is ordinary. Indeed, if $t$ travels along a small circle around a point in $S$, then the classes of the lines of two rulings $\ell, m \subset X_t$ are interchanged.

Let $Y \subset X$ be a smooth hyperplane section, and let $\pi: Y \to C$ be the restriction of $p$ to $Y$. It is clear that singular fibers of $\pi$ are pairs of different
intersecting lines. Let \( T = \{ t_1, \ldots, t_m \} \subset C \) be the set of points such that fibers of \( \pi \) over them are singular. Observe that \( T \neq \emptyset \): a simple dimension count shows that no hyperplane in \( \mathbb{P}^N \) can be transversal to all the fibers of \( p: X \to C \).

For each \( t_j \in T \), choose once and for all a line \( \ell_j \subset \pi^{-1}(t_j) \). Each \( \ell_j \) is a \((-1)\)-curve on \( Y \); blowing down all the \( \ell_j \), one obtains a smooth surface \( Y' \) which is a \( \mathbb{P}^1 \)-bundle over \( C \). Let \( \sigma: Y \to Y' \) be the blowdown morphism.

A standard argument using the fact that the field of rational functions on a curve is a \( C_1 \) field shows that the projection \( \pi': Y' \to C \) has a section, so there exists a divisor \( D \subset Y' \) having intersection index 1 with all the fibers of the projection \( \pi' \). Let \( e \in H_2(Y, \mathbb{Z}) \) be the class of the divisor \( \sigma^*D \), and let \( f \in H_2(Y, \mathbb{Z}) \) be the class of the divisor that is \( \sigma^* \) of a fiber of the projection \( \pi' \). It is clear that \( H_2(Y, \mathbb{Z}) \) is a free abelian group with basis \( \langle e, f, l_1, \ldots, l_m \rangle \), where \( l_j, 1 \leq j \leq m \), are the classes of \( \ell_j \). The class \( e \) has intersection index one with \( f \) and zero with each \( l_j \).

**Proposition 5.12.** Suppose that \( X \subset \mathbb{P}^N \) is an ordinary pencil of quadrics over a curve \( C \) and \( Y \subset X \) is a smooth hyperplane section; let \( m \) be the number of degenerate fibers of the induced morphism \( \pi: Y \to C \). Then \( \dim \text{ev}(Y) = m \); the vanishing root system of \( X \) is \( D_m \) if \( m \geq 4 \), it is \( A_3 \) if \( m = 3 \), and the cases \( m = 1, 2 \) are impossible.

**Proof.** Let \( i: Y \hookrightarrow X \) be the embedding and let \( i_*: H_2(Y, \mathbb{Z}) \to H_2(X, \mathbb{Z}) \) be the induced homomorphism. Suppose that

\[
(6) \quad ae + bf + \sum_{j=1}^{m} c_j l_j \in \text{Ker} \ i_*. 
\]

Intersecting \( i_* \) of the left-hand side of (3) with the fiber of \( p \), one obtains \( a = 0 \). Since the pencil \( X \) is ordinary, all the \( i_*(l_j) \) are equal to the same element \( l \in H_2(X, \mathbb{Z}) \), and \( i_*(f) = 2l \). Thus, \( (6) \) is equivalent to

\[
(7) \quad c_1 + \cdots + c_m + 2b = 0, 
\]

so the root lattice of the vanishing root system of \( X \) is

\[
L = \{ bf + \sum_{j=1}^{m} c_j l_j : c_1 + \cdots + c_m = -2b \}. 
\]

If \( \langle \cdot, \cdot \rangle \) is the pairing on \( H_2(Y, \mathbb{Q}) \) defined as in (3), Section 5.11 then in \( H_2(Y, \mathbb{Q}) \) one has \( (f, l_j) = (f, f) = 0 \) for all \( j \) and \( (l_i, l_j) = \delta_{ij} \). Putting \( \tilde{l}_i = l_i - \frac{1}{2} f \), one has \( (\tilde{l}_i, \tilde{l}_j) = \delta_{ij} \) and

\[
(8) \quad L = \left\{ \sum_{j=1}^{m} c_j \tilde{l}_j : c_j \in \mathbb{Z}, \sum_{j=1}^{m} c_j \equiv 0 \pmod{2} \right\}, \text{ where } (\tilde{l}_i, \tilde{l}_j) = \delta_{ij}. 
\]

It follows from Lemma 5.34 that \( m > 0 \).

A well-known computation (see for example Planche IV, (VIII)) shows that the order of the group \( L^\vee / L \) equals 4. So, the vanishing root system is
a simply laced irreducible root system of rank \( m \) and the index of the root lattice in the weight lattice is 4. For \( m \geq 4 \), such a system is \( D_m \); for \( m = 3 \), such a system is \( A_3 \); for \( m = 1 \) or 2, such a system does not exist. \( \square \)

**Proposition 5.13.** Suppose that \( X \subset \mathbb{P}^N \) is an extraordinary pencil of quadrics over a curve \( C \) and \( Y \subset X \) is a smooth hyperplane section; let \( m \) be the number of degenerate fibers of the induced morphism \( Y \to C \). Then \( \dim \text{ev}(Y) = m - 1 \) and the vanishing root system of \( X \) is \( A_{m-1} \).

**Proof.** Since the pencil \( X \) is extraordinary, all its fibers are smooth (see Remark 5.11), and if \( Q \) is the generic fiber (in the scheme-theoretic sense) of the morphism \( p \), then both families of lines on the quadric \( Q \) are defined over \( C(C) \).

Hence, there exist two 2-dimensional families of lines \( \mathcal{U} \) and \( \mathcal{V} \) on \( X \) such that, for each \( t \), the family of lines from \( \mathcal{U} \) (resp. \( \mathcal{V} \)) lying on \( X_t \) is the family of all lines of one of the rulings of the quadric \( X_t \).

Since the base of the family of the lines of each ruling of a smooth two-dimensional quadric is isomorphic to \( \mathbb{P}^1 \), the same argument by which we showed that the morphism \( \pi': Y' \to C \) has a section shows that there exists a surface \( F \subset X \) such that, for each \( t \in C \), \( F \cap p^{-1}(t) \) is a line from the family \( \mathcal{U} \). Now it is clear that if a line \( \ell \subset X \) belongs to the family \( \mathcal{U} \), then \( \ell \cdot F = 0 \). Hence, if \( a \in H_2(X, \mathbb{Z}) \) (resp. \( v \in H_2(X, \mathbb{Z}) \)) stands for the class of any line from the family \( \mathcal{U} \) (resp. \( \mathcal{V} \)), then the elements \( u \) and \( v \) are linearly independent in \( H_2(X, \mathbb{Z}) \).

Using notation from the beginning of this section we may, without loss of generality, assume that all the lines \( \ell_j \) we have chosen on the surface \( Y \) belong to the family \( \mathcal{U} \). Suppose now that (6) holds, where \( i : Y \hookrightarrow X \) is the embedding. Take \( i_* \) of the left-hand side; intersecting the resulting homology class with the class of a fiber of \( X_t \) for some \( t \in C \), one obtains a \( = 0 \). Since, in \( H_2(X, \mathbb{Z}) \), \( i_*(f) = u + v \), one infers that

\[
b(u + v) + (c_1 + \cdots + c_m)u = 0;
\]

since \( u \) and \( v \) are linearly independent in \( H_2(X, \mathbb{Z}) \), this implies that \( b = 0 \) and \( c_1 + \cdots + c_m = 0 \). Thus, the root lattice of the vanishing root system is isomorphic to the lattice

\[
L = \left\{ \sum_{j=1}^{m} c_j l_j : c_j \in \mathbb{Z}, \ c_1 + \cdots + c_m = 0 \right\}, \quad \text{where } (l_i, l_j) = \delta_{ij}
\]

(Lemma 5.9 implies that \( m > 1 \)). In the proof of Proposition 5.5 we have seen that this implies that the root system in question is \( A_{m-1} \). \( \square \)

Now we can find out which pencils of quadrics have vanishing root system \( A_1 \).
Suppose that $14$ SERGE LVOVSKI
quadrics. Then its vanishing root system is $A_p$ and will assume that $P$ and (9), one sees that $\deg E$ coordinates that agree with the decomposition can be represented as a 4 $H_d$ $\phi$ Taking into account that $\deg E$ $R$ the only such varieties containing a one-dimensional family of $\sigma$ $\deg E$ degenerate fibers of the induced pencil $E$ $2$ deg $\phi$ $\sigma$ and that the discriminant of $\sigma$ $\deg E$ $E$ $\sigma$ and putting together equations (10) one has $2$ deg $E'$ and putting together equations (10) and (9) one sees that $Y \subset \mathbb{P}^*(E')$ is the zero locus of a section $\sigma' \in H^0(\text{Sym}^2(E')) = H^0(E)$ and that the discriminant of $\sigma'$ is a section of $(\det E') \otimes L^{\otimes 3}$, since the vanishing root system is $A_1$, Proposition 5.13 implies that there are precisely 2 degenerate fibers of the induced pencil $Y \to C$, so the degree of the invertible sheaf $(\det E') \otimes L^{\otimes 3}$ equals 2, whence

\begin{equation}
2 \deg E' + 3 \deg L = 2.
\end{equation}

Taking into account that $\deg E' = \deg E$ and putting together equations (10) and (9) one sees that $\deg E = 4$ and $\deg L = -2$.

Put $Z = \bigcup_{i \in C} (X_i) \subset \mathbb{P}^N$. The natural homomorphism $H^0(\mathcal{O}_{\mathbb{P}^N}(1)) \to H^0(\mathcal{O}_{\mathbb{P}^N}(1)) = H^0(\mathcal{E})$ induces a morphism $\varphi: \mathbb{P}^*(E) \to \mathbb{P}^N$ such that $\varphi^*\mathcal{O}_Z(1) = \mathcal{O}_{\mathbb{P}^*(E)}(1)$ and $\varphi(\mathbb{P}^*(E)) = Z$. Since $Z$ contains a one-dimensional family of $\mathbb{P}^3$'s, it is not a quadric; since $\deg(\varphi|_{\mathbb{P}^*(E)}) \cdot \deg Z = \deg \det E = 4$, the morphism $\varphi|_{\mathbb{P}^*(E)}$ is birational onto $Z$ and $\deg Z = 4$.

According to the classification of varieties of degree 4 from the paper [16], the only such varieties containing a one-dimensional family of $\mathbb{P}^3$'s are the Segre variety $\mathbb{P}^1 \times \mathbb{P}^3 \subset \mathbb{P}^7$ and its regular projections. Thus, $C \cong \mathbb{P}^1$ and $E \cong \bigoplus_{i=1}^4 \mathcal{O}(d_i)$, where $0 \leq d_1 \leq d_2 \leq d_3 \leq d_4$ and $d_1 + d_2 + d_3 + d_4 = 4$.

Since $C \cong \mathbb{P}^1$ and $\deg L = -2$, one has $L \cong \mathcal{O}_{\mathbb{P}^1}(-2)$.

The quadratic form defining $X \subset \mathbb{P}^*(E)$, which is a section of $\text{Sym}^2(E) \otimes L$, can be represented as a $4 \times 4$-matrix $[a_{ij}]_{1 \leq i, j \leq 4}$, where $a_{ij} \in H^0(\mathcal{O}_{\mathbb{P}^1}(d_i + d_j - 2))$. Let us show that the case $d_1 = 0$ is impossible. Indeed, if this is the case, then $a_{11}$ is identically zero, so in each fiber of the bundle $\mathbb{P}^*(E)$ the point with homogeneous coordinates $(1 : 0 : 0 : 0)$ lies in $X$ (we use homogeneous coordinates that agree with the decomposition $E = \bigoplus \mathcal{O}_{\mathbb{P}^1}(d_i)$). On the other hand, since $d_1 = 0$, the mapping $\varphi: \mathbb{P}(E) \to \mathbb{P}^N$ maps the points with coordinates $(1 : 0 : 0 : 0)$ in all the fibers of $\mathbb{P}^*(E)$ to one and the same point.
of $\mathbb{P}^N$. Thus, there exists a point contained in all the fibers of the pencil $p: X \to \mathbb{P}^1$, which is absurd.

We have proved that $d_1 \neq 0$, whence $d_1 = d_2 = d_3 = d_4 = 1$. Thus, $Z$ is the Segre variety $\mathbb{P}^1 \times \mathbb{P}^3 \subset \mathbb{P}^7$, and the matrix $\|a_{ij}\|$ consists of constants. This proves that $X = \mathbb{P}^1 \times Q \subset \mathbb{P}^1 \times \mathbb{P}^3$, where $Q \subset \mathbb{P}^3$ is a smooth quadric, and the proposition follows. \hfill \Box

5.3. Veronese pencils. In this section we account for item (3) in Sommese’s list (Table 1). We will see that none of such varieties has monodromy group of the type $A_1$.

Throughout this section $X \subset \mathbb{P}^N$ will be a smooth projective threefold such that there exists a morphism $p: X \to C$ onto a smooth curve $C$; for $t \in C$, we put $X_t = p^{-1}(t)$. We assume that for a general $t \in C$ one has $(X_t, \mathcal{O}_{X_t}(1)) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$.

Observe that if, in this setting, $Y \subset X$ is a smooth hyperplane section, then $\dim \text{ev}(Y) \neq 0$; the proof is the same as that of Lemma 5.9.

Let $S = \{u_1, \ldots, u_p\} \subset C$ be the set of points such that fibers of $p$ over them are not irreducible and reduced surfaces (of course, $S$ may be empty); each fiber $X_{u_j}$ is the union of $n_j$ irreducible components. We begin with simple lemmas.

**Lemma 5.15.** Suppose that $F_1, \ldots, F_n \subset \mathbb{P}^N$ are irreducible projective surfaces satisfying the following conditions:

(1) for any two different $F_i, F_j$, the set-theoretic intersection $F_i \cap F_j$ is isomorphic to $\mathbb{P}^1$ or empty;

(2) the graph of which the vertices are $F_1, \ldots, F_n$ and such that the vertices $F_i$ and $F_j$ are joined by an edge if and only if $F_i \cap F_j \neq \emptyset$ (i.e., the incidence graph of $F_1, \ldots, F_n$), is a tree.

(3) $H^3(F_j, \mathbb{Q}) = 0$ for each $j$.

Then $H^3(F_1 \cup \cdots \cup F_n, \mathbb{Q}) = 0$.

**Proof.** Condition (2) implies that the surfaces $F_1, \ldots, F_n$ may be ordered so that, for each $m$, $F_m \cap F_{m-1} \neq \emptyset$ and $F_m \cap F_j = \emptyset$ for $j < m$. Now we prove, by induction on $m$, that $H^3(F_1 \cup \cdots \cup F_m, \mathbb{Q}) = 0$ for all $m$. In the induction step one uses the Mayer–Vietoris sequence for $(F_1 \cup \cdots \cup F_{m-1}) \cup F_m$, taking into account that $C = (F_1 \cup \cdots \cup F_{m-1}) \cap F_m$ is isomorphic to $\mathbb{P}^1$ and the mapping

$$H^2(F_1 \cup \cdots \cup F_{m-1}, \mathbb{Q}) \oplus H^2(F_m, \mathbb{Q}) \to H^2(C, \mathbb{Q})$$

is an epimorphism. \hfill \Box

**Lemma 5.16.** Suppose that $X \subset \mathbb{P}^N$ is a smooth three-dimensional projective variety which is not a linear subspace of $\mathbb{P}^N$.

(a) If $X$ contains two planes $L_1$ and $L_2$, then $L_1 \cap L_2 = \emptyset$.

(b) If $X$ contains a plane $L$ and an irreducible quadric $Q$, then either $L \cap Q = \emptyset$ or $L \cap Q$ is (set-theoretically) a line.
Proof. The proofs of both assertions being similar, we will prove (b). If, by way of contradiction, $C = L \cap Q$ is not empty and not a line, then $C$ is an irreducible conic or the union of two distinct lines. Hence, if $H$ is the linear span of $Q \cup L$, then $\dim H = 3$ and $H = T_x(Q \cup L)$ for any $x \in Q \cap L$. Since $T_x(Q \cup L) \subset T_x X$ and $\dim T_x X = 3$, we conclude that $T_x X = H$ for all $x \in C$, so the Gauss map of $X$ is not finite. This contradicts Zak’s theorem on tangencies [19, Chapter I, Corollary 2.8]. □

Lemma 5.17. For each $u \in S$ one has $H^3(X_u, \mathbb{Q}) = 0$.

Proof. Since $C$ is a smooth curve, both $X_u$ and its hyperplane section are connected. Since $\dim X = 3$, the intersection of any two components of $X_u$ is purely one-dimensional. Now we consider two cases.

(a) The divisor $p^*u$ has no multiple components.

If $H$ is a general hyperplane (which does not contain $X_u$), then the scheme $H \cap X_u$ is reduced and connected, whence $h^0(\mathcal{O}_{X_u}) = 1$. Since the Hilbert polynomial of $X_u$ is the same as that of $v_2(\mathbb{P}^2)$, the arithmetic genus of $H \cap X_u$ equals zero.

It follows from the above that all the components of $H \cap X_u$ are isomorphic to $\mathbb{P}^1$ and the incidence graph of the components of $H \cap X_u$ is a tree; in particular, any two components of $H \cap X_u$ are either disjoint or intersect at precisely one point. Hence, if $F_1$ and $F_2$ are two components of $X_u$, then either $F_1 \cap F_2 = \emptyset$ or $F_1 \cap F_2$ is a line. Besides, the incidence graph of the components of $X_u$ is a tree and any component of $X_u$ is a surface of degree $\leq 3$ whose general hyperplane sections are isomorphic to $\mathbb{P}^1$. It follows from the latter assertion that $H^3(F, \mathbb{Q}) = 0$ whenever $F$ is a component of $X_u$. Thus, $X_u$ satisfies the hypothesis of Lemma 5.15 whence $H^3(p^{-1}(u), \mathbb{Q}) = 0$.

(b) The divisor $p^*u$ has a multiple component. Since $\deg p^*u = 4$ and the set $X_u = p^{-1}(u)$ is connected, Lemma 5.16 implies that $X_u$ is either a plane, or an irreducible quadric, or the union of an irreducible quadric and a plane intersecting in a line. In each of these cases the vanishing $H^3(X_u, \mathbb{Q}) = 0$ is obvious. □

Lemma 5.18.

$$b_2(X) = 2 + \sum_{j=1}^r (n_j - 1)$$

Proof. Since $b_2(X) = b_4(X)$ by Poincaré duality, it suffices to find $b_4(X)$.

Put $X_0 = p^{-1}(C \setminus S)$, $D = p^{-1}(S)$; taking into account Lemma 5.17 one has the exact sequence

$$(11) \quad 0 = H^3(D, \mathbb{Q}) \to H^4_c(X_0, \mathbb{Q}) \to H^4(X, \mathbb{Q}) \to H^4(D, \mathbb{Q})$$

$$\to H^5_c(X_0, \mathbb{Q}) \to H^5(X, \mathbb{Q}) \to H^5(D, \mathbb{Q}) = 0.$$  

Observe that if $D$ is a projective variety having $n$ irreducible components $D_1, \ldots, D_n$ of equal dimension $d$, then $H^{2d}(D, \mathbb{Q}) \cong \mathbb{Q}^d$; hence, $b_4(D) = n_1 + \cdots + n_r$.  

16 SERGE VOVSKI
Moreover, if we denote the genus of the curve \( C \) by \( g \), then \( b_2(X) = 2g \). Indeed, by Poincaré duality it suffices to check that \( b_1(X) = 2g \); now observe that, since fibers of \( p \) are swept by rational curves, any morphism from \( X \) to an Abelian variety factors through \( p \), so the Albanese variety of \( X \) is isomorphic to that of \( C \).

Let us find \( H^2_c(X_0, \mathbb{Q}) \) and \( H^5_c(X_0, \mathbb{Q}) \). To that end, let \( p_0 : X_0 \to C \) be the restriction of \( p \) to \( X_0 \). Pick a point \( u \in C \setminus S \) and put \( F = X_u \). Since \( F \) is the Veronese surface or its isomorphic projection, the ring \( H^*(F, \mathbb{Q}) \) is generated by the class of hyperplane section, whence \( \pi_1(C \setminus S, u) \) acts trivially on \( H^*(F, \mathbb{Q}) \). Hence, the Leray spectral sequence of \( p_0 \) has the form

\[
E_2^{pq} = H^p_c(C \setminus S, H^q(F, \mathbb{Q})) \Rightarrow H^{p+q}_c(X_0, \mathbb{Q}).
\]

Since \( F \cong \mathbb{P}^2 \) and this spectral sequence degenerates at \( E_2 \) by Griffiths’ theorem [8, Proposition 3.1], one finds that \( H^1_c(X_0, \mathbb{Q}) \cong \mathbb{Q} \) and \( H^5_c(X_0, \mathbb{Q}) \cong \mathbb{Q}^{2g+r-1} \).

Plugging these cohomology groups of \( X_0 \) and \( X \) in the exact sequence (11), one obtains the result. \( \square \)

**Proposition 5.19.** Suppose that \( X \) is a Veronese pencil, \( Y \subset X \) is a smooth hyperplane section, and \( m = \dim \text{ev}(Y) \). Then the vanishing root system of \( X \) is \( A_3 \) for \( m = 3 \), it is \( D_m \) for \( m \geq 4 \), \( m \neq 8 \), it is \( D_8 \) or \( E_8 \) for \( m = 8 \), and the case \( m \leq 2 \) is impossible.

**Proof.** Let \( i : Y \hookrightarrow X \) be the embedding of a general hyperplane section \( Y \), and put \( \pi = p \circ i \).

If \( t \in C \setminus S \), then \( \pi^{-1}(t) \) is either a smooth rational curve of degree 4 (i.e., a transversal hyperplane section of a smooth rational surface of degree 4) or the union of two different irreducible conics that intersect transversally at one point. If \( T = \{t_1, \ldots, t_m\} \subset C \) is the set of points with the latter property, put \( \pi^{-1}(t_j) = E_j \cup F_j \), where \( E_j \) and \( F_j \) are the above-mentioned conics.

If \( t = u_j \in S \), then \( \pi^{-1}(t) \) is a curve with \( n_j \) components; each of these components is a smooth rational curve and their intersection graph is a tree.

It is clear that the self-intersection index of each \( E_j \) and \( F_j \) in \( Y \) equals \(-1\). Blowing down all the \( E_j \)'s one obtains a morphism \( \sigma : Y \to Y' \), where \( Y' \) is a smooth surface possessing a morphism \( \pi' : Y' \to C \) such that \( \pi'^{-1}(t) \) is a smooth rational curve for \( t \notin S \) and \( \pi^{-1}(t) \) is a connected reducible curve of arithmetic genus zero with \( n_j \) rational components if \( t = u_j \in S \). Arguing as in the proof of Proposition 5.18, one sees that

\[
(12) \quad b_2(Y) = 2 + \sum_{j=1}^r (n_j - 1) + m,
\]

where \( m \) is the cardinality of \( T \subset C \). Now it follows from (12) and Lemma 5.18 that

\[
(13) \quad \text{rank} \ker(H_2(Y, \mathbb{Z}) \xrightarrow{i_*} H_2(X, \mathbb{Z})) = m.
\]
Let \( f \in H_2(Y, \mathbb{Z}) \) be the class of \( \sigma^*F' \), where \( F' \subset Y' \) is a smooth fiber of \( \pi' \), let \( l_j \in H_2(Y, \mathbb{Z}) \) be the class of \( E_j, \; 1 \leq j \leq m \). Put \( L_1 = \text{Ker}(H_2(Y, \mathbb{Z}) \to H_2(X, \mathbb{Z})) \) and

\[
L = \{ bf + \sum_{j=1}^{m} c_j l_j : c_1 + \ldots + c_m = -2b \}.
\]

Observe that, if for \( H \) the index of the root lattice in the weight lattice is \( 4 \), then \( E \)ither \( \text{eq} \)uals \( 4 \) or \( 1 \). If, for a simply laced irreducible root system \( R \) the index of the root lattice in the weight lattice is \( 4 \), then \( \text{eq} \)uals \( 4 \) and \( \text{rank} L = m; \) \cite{13} implies that the lattice \( L \) has finite index in \( L_1 \).

The same observation as in the proof of Proposition \( 5.12 \) shows that \( [L'] : L \) equals \( 4 \) or \( 1 \). If, for a simply laced irreducible root system \( R \) of rank \( m \), the index of the root lattice in the weight lattice is \( 4 \), then either \( m = 3 \) and \( R = A_3 \) or \( m \geq 4 \) and \( R = D_m \); if this index is \( 1 \), then \( m = 8 \) and \( R = E_8 \). This completes the proof. \( \square \)

### 5.4. Del Pezzo threefolds

In this section we account for item (4) in Table \( 1 \). The complete list of such varieties is well known (see \cite{9, 10, 6, 7}), and the Betti numbers of these varieties are well known, too, which already allows one to extract from the list those having 1-dimensional space of vanishing cycles. For the sake of completeness we compute the vanishing root systems as well.

**Proposition 5.20.** If \( X \subset \mathbb{P}^N \) is an embedded smooth Del Pezzo threefold, then its vanishing root system is as indicated in Table \( 2 \).

**Remark 5.21.** As Manin indicated in \cite{13} Section 23.13, the monodromy group for the smooth cubic \( V_3 \subset \mathbb{P}^4 \) was first computed by Todd in \cite{17}.

**Proof of Proposition 5.20.** It is obvious that the vanishing root system is \( A_1 \) for the variety \( V_6 \) in Table \( 2 \). The results about vanishing root systems of \( V_6' \) and \( V_7 \) follow from Proposition \( 5.5 \) and the result about the vanishing root system of \( V_6 \) follows from Proposition \( 5.14 \). It remains to find this root system for \( V_3, V_4, \) and \( V_5 \).

If \( 3 \leq m \leq 5 \), then it follows from the Lefschetz hyperplane theorem that \( H_2(V_m, \mathbb{Z}) \cong \mathbb{Z}, \) that the class of any line \( \ell \subset V_m \) is a generator if \( H_2(V_m, \mathbb{Z}) \), and that if \( C \subset V_m \) is a curve, then \( [C] = \text{deg} \; C \cdot [\ell] \in H_2(V_m, \mathbb{Z}) \) (brackets mean “the homology class”).

If \( Y \subset V_m \) is a smooth hyperplane section, then \( Y \) is isomorphic to \( \mathbb{P}^2 \) blown up at \( 9 - m \) points; let \( \sigma : Y \to \mathbb{P}^2 \) be the corresponding blowdown. If \( \ell_1, \ldots, \ell_{9-m} \) are the exceptional curves of \( \sigma \), which are lines on \( Y \), let \( l_i \in H_2(Y, \mathbb{Z}) \) be the class of \( \ell_i \). If \( H \subset \mathbb{P}^2 \) is a line and \( h \in H_2(Y, \mathbb{Z}) \) is the class of \( \sigma^*H \), then \( H_2(Y, \mathbb{Z}) \) is the free abelian group with basis \( (h, l_1, \ldots, l_{9-m}) \).

If the line \( H \subset \mathbb{P}^2 \) does not pass through the \( 9 - m \) points that are blown up, then \( \sigma^{-1}(H) \subset Y \subset \mathbb{P}^N \) is a twisted cubic. Hence, if \( i : Y \to X \) is
Table 2. Smooth embedded Del Pezzo threefolds

| Variety | Description | Root system |
|---------|-------------|-------------|
| $V_3 \subset \mathbb{P}^4$ | Cubic | $E_6$ |
| $V_4 \subset \mathbb{P}^5$ | Complete intersection of two quadrics | $D_5$ |
| $V_5 \subset \mathbb{P}^6$ | Codimension 3 linear section of the Grassmannian $G(2,5) \subset \mathbb{P}^9$ in the Plücker embedding | $A_4$ |
| $V_6 \subset \mathbb{P}^7$ | Segre variety $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ | $A_1$ |
| $V_6' \subset \mathbb{P}^7$ | $\mathbb{P}^*(T_{\mathbb{P}^2})$, where $T_{\mathbb{P}^2}$ is the tangent bundle, embedded by the complete linear system $|O_{V_6|\mathbb{P}^2}(1)|$ | $A_2$ |
| $V_7 \subset \mathbb{P}^8$ | $\mathbb{P}^*(O_{\mathbb{P}^2}(1) \oplus O_{\mathbb{P}^2}(2))$ embedded by the complete linear system $|O_{V_7|\mathbb{P}^2}(1)|$, or an isomorphic projection of this variety | $A_1$ |
| $V_8 \subset \mathbb{P}^9$ | Veronese variety $v_2(\mathbb{P}^3)$ or an isomorphic projection of this variety | $A_1$ |

The embedding, one has $i_*[h] = 3\iota_*[\tilde{l}_j] \in H_2(V_m, \mathbb{Z})$ for $1 \leq j \leq 9 - m$. Therefore,

$$\text{Ker}(i_* : H_2(Y, \mathbb{Z}) \to H_2(V_m, \mathbb{Z})) = \{ah + c_1\tilde{l}_1 + \ldots + c_r\tilde{l}_r : c_1 + \ldots + c_r + 3a = 0 \},$$

where $r = 9 - m$. Put $\tilde{l}_i = l_i - \frac{1}{3}h$. Then the root lattice $L \subset H_2(Y, Q)$ of the vanishing root system $R$, with the pairing $(\cdot, \cdot)$ on $H_2(Y, Q)$ from Section 3.1, is isomorphic to

$$L = \left\{c_1\tilde{l}_1 + \ldots + c_r\tilde{l}_r : c_j \in \mathbb{Z}, \sum c_j \equiv 0 \pmod{3} \right\}.$$ 

If we put $\lambda_1 = 3\tilde{l}_1$ and $\lambda_j = \tilde{l}_i - \frac{1}{3}h$ for $2 \leq j \leq r$, then $(\lambda_1, \ldots, \lambda_r)$ is a $\mathbb{Z}$-basis of the lattice $L$. Since $(l_i, l_j) = \delta_{ij}$, $(h, l_j) = 0$ and $(h, h) = -1$ (recall that the pairing $(\cdot, \cdot)$ from Section 3.1 is the negated intersection index), one has

$$\langle \lambda_j, \lambda_k \rangle = \begin{cases} 8, & j = k = 1, \\ 3, & j = 1, \ k > 1, \\ 2, & j = k > 1, \\ 1, & j > 1, \ k > 1, \ j \neq k. \end{cases}$$
An easy induction shows that
\[
\begin{vmatrix}
    a & b & b & \ldots & b \\
    b & c & d & \ldots & d \\
    b & d & c & \ldots & d \\
    \cdots & \cdots & \cdots & \cdots & \cdots \\
    b & d & \cdots & \ldots & d & c
\end{vmatrix}
\]
\[
(14) \quad \det (c - d)^{r-2}(a(c + (r - 2)d) - (r - 1)b^2),
\]
where the matrix in the left-hand side is \((r \times r)\). Substituting \((a, b, c, d) = (8, 3, 2, 1)\) in (14), one obtains \([L^\nu : L] = \det ||(\lambda_j, \lambda_k)||_{1 \leq j, k \leq m} = 9 - r = m\). (Actually, we need only the cases \(r = 9 - m\), where \(m = 3, 4, 5\), and the three corresponding determinants can be evaluated by hand.) It remains to observe that if \(R\) is a simply laced irreducible root system of rank \(9 - m\), \(3 \leq m \leq 5\), for which the index of the root lattice in the weight lattice equals \(m\), then \(R = E_6\) for \(m = 3\), \(R = D_5\) for \(m = 4\), and \(R = A_4\) for \(m = 5\).

**Remark 5.22.** In Chapter IV of his book [13], Manin associated a root system of rank \(r = 9 - m\) to any Del Pezzo surface of degree \(m \leq 6\). In this remark we compare these results with Proposition 5.20.

If \(X \subset \mathbb{P}^N\) is a Del Pezzo threefold and \(Y \subset X\) is its smooth hyperplane section, then \(Y\) is a Del Pezzo surface. Since \(h^1(\mathcal{O}_Y) = h^2(\mathcal{O}_Y) = 0\), one can identify \(\text{Pic}(Y)\) with \(H^2(Y, \mathbb{Z})\), and one can identify \(H^2(Y, \mathbb{Z})\) with \(H_2(Y, \mathbb{Z})\) via Poincaré duality. Since \(\text{Ev}(Y)\) is the orthogonal complement to \(i^*H^2(X, \mathbb{Q})\), where \(i: Y \hookrightarrow X\) is the embedding, and since \(i^*\mathcal{O}_X(1) = \omega_Y^{-1}\), we see that, if one uses the identification above, all the elements of the vanishing root system \(R \subset \text{ev}(Y)\) are identified with elements \(l \in \text{Pic}(Y) \cong H^2(Y, \mathbb{Z})\) such that \((l, \omega_Y) = 0\) and \((l, l) = -2\) (here, we mean by \((\cdot, \cdot)\) the standard intersection index on \(\text{Pic}(Y)\), without negation). Thus, the vanishing root system is contained in Manin’s root system \(R_r\). Observe that if \(3 \leq m \leq 5\), then the vanishing root system equals Manin’s \(R_r\). Indeed, Theorem 25.4 of [13] asserts that \(R_r = E_6\) if \(r = 6\), \(D_5\) if \(r = 5\), and \(A_4\) if \(r = 4\). As we observed above, vanishing root system \(R\) is contained in \(R_r\), and Proposition 5.20 implies that the number of roots in \(R\) and \(R_r\) is the same. Thus, \(R\) coincides with Manin’s \(R_r = R_{9-m}\) if \(3 \leq m \leq 5\).

For both Del Pezzo threefolds \(X\) of degree 6, the vanishing root system differs from Manin’s root system \(A_1 \times A_2\). The reason is that in this case \(i^*H^2(X, \mathbb{Q})\), to which all the roots of the vanishing root system must be orthogonal, is not spanned by the class of \(\omega_Y\), so the space of vanishing cycles is smaller than the orthogonal complement to the class of \(\omega_Y\).

### 5.5. Odds and ends

Going through Sommese’s list, it remains to consider the varieties in categories (5), (6), and (7) (Table 1).

For the smooth quadric in \(\mathbb{P}^4\) (item (5)) it is obvious that the vanishing root system is \(A_1\). Now we account for items (6) and (7).

**Proposition 5.23.** If \(X \subset \mathbb{P}^N\) is a variety in the category (6) or (7) of Sommese’s list (Table 1), then the vanishing root system of \(X\) is \(D_5\).
Proof. Suppose first that \((X, \mathcal{O}_X(1)) \cong (Q, \mathcal{O}_Q(2))\), where \(Q\) is the smooth three-dimensional quadric. Let \(Y \subset X\) be a smooth hyperplane section. Then \((Y, \mathcal{O}_Y(1)) \cong (F, \mathcal{O}_F(-2K_F))\), where \(F\) is a Del Pezzo surface of degree 4; we identify \(Y\) with \(F\).

Let \(\sigma : F \to \mathbb{P}^2\) be a standard blowdown of five exceptional curves, let \(C_1, \ldots, C_5 \subset F\) be these curves (observe that they are conics on \(Y \subset X\)), and let \(l_i, 1 \leq i \leq 5\), be the class of \(C_i\) in \(H_2(F, \mathbb{Z})\). If \(L \subset \mathbb{P}^2\) is a line and \(h \in H_2(F, \mathbb{Z})\) is the class of \(\sigma^*L\), then \(H_2(Y, \mathbb{Z}) = H_2(F, \mathbb{Z})\) is the free abelian group with the basis \((h, l_1, \ldots, l_5)\).

Since \(X\) is isomorphic to the three-dimensional quadric, \(H_2(X, \mathbb{Z}) \cong \mathbb{Z}\) and this group is generated by the class of any conic \(C \subset X\); if \(C' \subset X\) is an arbitrary curve, then \(\sigma^*C' = (\deg C'/2)[C] \in H_2(X, \mathbb{Z})\) (brackets mean “the homology class”). In particular, if \(i_* : H_2(Y, \mathbb{Z}) \to H_2(X, \mathbb{Z})\) is the canonical surjection, then \(i_*(h) = 3[C]\). Hence,

\[
\ker i_* = \{ah + cl_1 + \ldots + c_5l_5 : c_1 + \ldots + c_5 + 3a = 0\},
\]

where \((l_i, l_j) = \delta_{ij}, (l_i, h) = 0, \text{ and } (h, h) = -1\). In the proof of Proposition 5.20 we showed that this implies that the vanishing root system is \(D_5\).

Now suppose that a variety \(X'\) belongs to the category (7) in Table 1; then \(X'\) is isomorphic to the blowup of \(Q\) at \(k\) points \(a_1, \ldots, a_k\). If \(Y' \subset X'\) is a smooth hyperplane section, then \(Y'\) is isomorphic to the blowup of \(Y\) at \(a_1, \ldots, a_k\), where \(Y \subset X\) is a smooth hyperplane section of \(X\) containing \(a_1, \ldots, a_k\). Let \(\sigma : X' \to X, \tau : Y' \to Y\) be the corresponding blowdown morphisms.

If \(E_j = \sigma^{-1}(a_j) \subset X', 1 \leq j \leq k\), are the exceptional divisors and \(\ell_j = \tau^{-1}(a_j) \subset Y', \) then

\[
H_2(X', \mathbb{Z}) \cong H_2(X, \mathbb{Z}) \oplus \mathbb{Z}[\ell_1] \oplus \cdots \oplus \mathbb{Z}[\ell_k],
\]

\[
H_2(Y', \mathbb{Z}) \cong H_2(Y, \mathbb{Z}) \oplus \mathbb{Z}[\ell'_1] \oplus \cdots \oplus \mathbb{Z}[\ell'_k],
\]

where we denoted by \([\ell_j]\) and \([\ell'_j]\) the class of \(\ell_j\) in \(H_2(X', \mathbb{Z})\) and in \(H_2(Y', \mathbb{Z})\).

Since \(\sigma_*([\ell_j]) = 0, \tau_*([\ell'_j]) = 0\) and \(i'_*([\ell'_j]) = [\ell_j]\) for all \(j\), where \(i' : Y' \to X'\) is the embedding, it is clear from the diagram

\[
\begin{array}{c}
Y' \xrightarrow{i'} X' \\
\downarrow\tau \quad \downarrow\sigma \\
Y \xrightarrow{i} X
\end{array}
\]

that

\[
\ker(i'_* : H_2(Y', \mathbb{Z}) \to H_2(X', \mathbb{Z})) \cong \ker(i_* : H_2(Y, \mathbb{Z}) \to H_2(X, \mathbb{Z}))
\]

and that this isomorphism respects the from \((\cdot, \cdot)\). Thus, the vanishing root system of \(X'\) is isomorphic to that of \(X\). \(\square\)

Putting together Propositions 4.2, 5.3, 5.14, 5.19, 5.20 and 5.23 one obtains a proof of Theorem 1.1.
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