Some Results Related to Group Actions in Several Complex Variables

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Abstract

In this talk, we’ll present some recent results related to group actions in several complex variables. We’ll not aim at giving a complete survey about the topic but giving some our own results and related ones.

We’ll divide the results into two cases: compact and noncompact transformation groups. We emphasize some essential differences between the two cases. In the compact case, we’ll mention some results about schlichtness of envelopes of holomorphy and compactness of automorphism groups of some invariant domains. In the noncompact case, we’ll present our solution of the longstanding problem – the so-called extended future tube conjecture which asserts that the extended future tube is a domain of holomorphy. Invariant version of Cartan’s lemma about extension of holomorphic functions from the subvarities in the sense of group actions will be also mentioned.

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1. Fundamentals of several complex variables

About one century ago, Hartogs discovered that there exist some domains in several complex variables on which any holomorphic functions can be extended to larger domains, being different with one complex variable. This causes a basic concept – domain of holomorphy.

Definition. A domain of holomorphy in \( \mathbb{C}^n \) is a domain on which there exists a holomorphic function which can’t be extended holomorphically across any boundary points.

A domain in \( \mathbb{C}^n \) is called holomorphically convex, if given any infinite discrete point sequence \( z_k \) there exists a holomorphic function \( f \) s.t. \( f(z_k) \) is unbounded (or \( |f(x_n)| \to +\infty \)). Consequently, there exists a holomorphic function which tends to

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at the boundary. By Cartan-Thullen’s theorem, a domain in $\mathbb{C}^n$ is a domain of holomorphy if and only if the domain is Stein, i.e., holomorphically convex.

**Definition.** A function $\varphi$ with value in $[-\infty, +\infty)$ on the domain $D$ in $\mathbb{C}^n$ is called plurisubharmonic (p.s.h.) if (i) $\varphi$ is upper semicontinuous (i.e., $\{\varphi < c\}$ is open for each $c \in \mathbb{R}$, or equivalently $\lim_{z \to z_0} \varphi(z) \leq \varphi(z_0)$ for $z_0 \in D$); (ii) for each complex line $L := \{z_0 + t r : z_0 \in D\}, \varphi|_{L \cap D}$ is subharmonic w.r.t. one complex variable $t$.

An equivalent definition in the sense of distributions is that $i\partial \bar{\partial} \varphi$ is a positive current; in particular, when $\varphi$ is $C^2$, this means Levi form $\left(\frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j}\right) \geq 0$ everywhere. In other words, $dJd\varphi \geq 0$, where $J$ is the complex structure. (If $i\partial \bar{\partial} \varphi > 0$, then $\varphi$ is called strictly p.s.h.)

**Example.** For a bounded domain or a domain biholomorphic to a bounded domain, the Bergman kernel $K(z, \bar{z})$ is strictly p.s.h.

A pseudoconvex domain in $\mathbb{C}^n$ is a domain on which there exists a p.s.h. function which tends to $+\infty$ at the boundary. It’s easy to see that a holomorphically convex domain is pseudoconvex, since $|f|^2$ is plurisubharmonic function where $f$ is given in the consequence of the definition of a Stein domain.

A deep characterization of a domain of holomorphy is given by a solution to Levi problem which is the converse of the above statement.

**Fact.** A domain $D$ in $\mathbb{C}^n$ is a domain of holomorphy if and only if the domain is pseudoconvex.

A natural corresponding concept of the domain of holomorphy in the setting of complex manifolds (complex spaces) is the so-called Stein manifold (Stein space), which is defined as a holomorphically convex and holomorphically separable complex manifold (space). A complex manifold (or space with finite embedding dimension) is Stein if and only if it is a closed complex submanifold (or subvariety) in some $\mathbb{C}^n$, and if and only if there exists a strictly p.s.h. exhaustion function which is $\mathbb{R}$-valued (i.e., the value $-\infty$ is not allowed). A complex reductive Lie group, in particular a complex semisimple Lie group, is a Stein manifold.

We know that a domain of holomorphy or a Stein manifold are defined by special holomorphic functions which are usually hard to construct in several complex variables. However, a pseudoconvex domain is defined by a special p.s.h. function which is a real function and then relatively easy to construct. Construction of various holomorphic objects in several complex variables and complex geometry is a fundamental and difficult problem. An important philosophy here is reducing the construction of holomorphic functions to the construction of plurisubharmonic functions, because of the solution of Levi problem and Hörmander’s $L^2$ estimates for $\partial$ and other results.

### 2. Group actions in several complex variables

**Definition.** A group action of the group $G$ on a set $X$ is given by a mapping $\varphi : G \times X \to X$ satisfying the following: 1) $e \cdot x = x$, 2) $(ab) \cdot x = a \cdot (b \cdot x)$, where $e$ is the identity of the group, $a, b, \in G, x \in X$, $a \cdot x := \varphi(a, x)$. 


A group action on a set can be restricted on various cases. When the set is a topological space and the group is a topological group, the action is continuous, then one gets a topological transformation group; when the space is a metric space, the transformation preserves the metric, then one gets a motion group; when the set is a differentiable manifold and the group is a Lie group, the action is differentiable, then one gets a Lie transformation group; when the set is a vector space, the transformation preserves the vector space structure, then one gets a linear transformation group; when the set is a complex space, the transformation is holomorphic, and the action is real analytic, then one gets a (real) holomorphic transformation group; when the set is a complex space, the group is a complex Lie group, and the action is algebraic, one gets an algebraic transformation group; when the set is a complex space, the group is a complex Lie group, and the action is holomorphic, then one gets a complex (holomorphic) transformation group.

In this talk, we’re mainly concerned with the last case. We consider a complex Lie group $G_\mathbb{C}$ with a real form $G_\mathbb{R}$ acting holomorphically on a complex manifold (also called holomorphic $G_\mathbb{C}$-manifold) and a $G_\mathbb{R}$-invariant domain. It’s known that a complex reductive Lie group has a unique maximal compact subgroup up to conjugate as its real form, but it also has many noncompact real forms.

A group action on a set can be regarded as a representation of the group on the whole group of transformations. An effective group action means the representation is faithful, so it corresponds to a (closed) subgroup of the whole transformation group.

Actually, many domains in several complex variables such as Hartogs, circular, Reinhardt and tube domains can be formulated in the setting of group actions.

**Examples.**

a) Hartogs and circular domains: consider the Hartogs action of $\mathbb{C}$ with the real form $S^1$ on $\mathbb{C}^n$: $\mathbb{C}^* \times \mathbb{C}^n \to \mathbb{C}^n$ given by $(t, (z_1, \ldots, z_n)) \to (tz_1, tz_2, \ldots, z_n)$, then Hartogs domain is $S^1$-invariant domain; consider the circular action of $\mathbb{C}^*$ with the real form $S^1$ on $\mathbb{C}^n$: $\mathbb{C}^* \times \mathbb{C}^n \to \mathbb{C}^n$ given by $(t, (z_1, \ldots, z_n)) \to (tz_1, tz_2, \ldots, z_n)$, then circular domain is $S^1$-invariant domain.

b) Reinhardt domains: consider the Reinhardt action of $(\mathbb{C}^*)^n$ on $\mathbb{C}^n$ given by

$((t_1, \ldots, t_n), (z_1, \ldots, z_n)) \to (t_1z_1, \ldots, t_nz_n)$,

then Reinhardt domain is $(S^1)^n$-invariant domain. One can similarly defines matrix Reinhardt domains.

c) tube domains: consider the action of $\mathbb{R}^n$ on $\mathbb{C}^n$ given by $(r, z) \to r + z$, then $\mathbb{R}^n$-invariant domain is tube domain.

d) future tube: let $M^4$ be the Minkowski space with the Lorentz metric:

$x \cdot y = x_0y_0 - x_1y_1 - x_2y_2 - x_3y_3$, where $x = (x_0, x_1, x_2, x_3), y = (y_0, y_1, y_2, y_3) \in R^4$;

let $V^+$ and $V^- = -V^+$ be the future and past light cones in $R^4$ respectively, i.e. $V^\pm = \{ y \in M : y^2 > 0, \pm y_0 > 0 \}$, the corresponding tube domains $\mathbb{C}^4 \equiv V^+ = T^+ = R^4 + iV^\pm$ in $\mathbb{C}^4$ are called future and past tubes; let $L$ be the Lorentz group, i.e. $L = O(1, 3)$. $L$ has four connected components, denote the identity component of $L$ by $L_+$, which is called the restricted Lorentz group, i.e. $L_+ = SO_+(1, 3)$; let $L(\mathbb{C})$ be the complex Lorentz group, i.e. $L = O(1, 3, \mathbb{C}) \cong O(4, \mathbb{C})$. $L(\mathbb{C})$ has two
connected components, denote the identity component of $L(\mathbb{C})$ by $L_+(\mathbb{C})$, called the proper complex Lorentz group which has the restricted Lorentz group as its real form. Considering the linear action of $L_+(\mathbb{C})$ on $\mathbb{C}^4$, the future (or past) tube is $L^+_1$-invariant.

Denote the $N$-point future tube by $\tau_N^{\pm} = \tau^\pm \times \cdots \times \tau^\pm$ $N$-times, let $L_+(\mathbb{C})$ act diagonally on $\mathbb{C}^{4N}$, i.e. for $z = (z^{(1)}, \ldots, z^{(N)}) \in \mathbb{C}^{4N}$, $\wedge = (\wedge z^{(1)}, \ldots, \wedge z^{(N)})$ where $\wedge \in L_+(\mathbb{C})$, then $\tau_N^{\pm}$ is $L^+_1$-invariant.

e) matrix Reinhardt domains: let $\mathbb{C}^n[m \times m] = \{(Z_1, \ldots, Z_n) : Z_j \in \mathbb{C}[m \times m]\}$ be the space of $n$-tuples of $m \times m$ matrices. A domain $D \subset \mathbb{C}^n[m \times m]$ is called matrix Reinhardt if it is invariant under the diagonal $U(m) \times U(m)$ action $(U, V)(Z_1, \ldots, Z_n) \mapsto (UZ_1V, \ldots, UZ_nV)$. These domains are the usual Reinhardt domains in the case $m = 1$. $\text{Diag}(D)$ is defined as the intersection of $D$ with the diagonal matrices $(Z_1, \ldots, Z_n) \in \mathbb{C}^n[m \times m]$.

**Slice theory**

When $G$ is a Lie transformation group properly acting on a smooth manifold $X$ (e.g. when $G$ is compact), one has a satisfactory slice theory about the structure of a neighborhood of an orbit. This theory was extended to the case of an affine reductive group action regularly on an affine variety by D. Luna ([20]) and the case of a complex reductive Lie group $G$ action holomorphically on a Stein space $X$ by Snow ([27]). In these cases, the structure of a neighborhood of a closed orbit is finely determined. We state the result for reduced Stein spaces. Let $G \cdot x$ be a closed orbit, then there exists a locally closed $G_e$-invariant Stein subspace $B$ containing $x$ s.t. the natural map from the homogeneous fiber bundle $G \times_{G_e} B$ over $G/G_e \cong G \cdot x$ is biholomorphic onto a $\pi$-saturated open Stein subset of $X$, where $\pi : X \to X/G$ is the categorical quotient (or GIT quotient) which exists as a Stein space. Here $B$ is called a slice at $x$. The slice $B$ is transversal to the closed orbit $G \cdot x$. When $X$ is regular at $x$, then $B$ can be chosen to be regular.

As a consequence of the slice theorem, one has a stratification of the categorical quotient $X/G$ at least when $X$ is a Stein manifold. The stratum with maximal dimension is Zariski open in $X/G$ and is contained in the regular part of $X/G$. This is called principal stratum. The inverse of the principal stratum under $\pi : X \to X/G$ consists of all $G$-closed orbits satisfying that they are of maximal dimension $k$ among the dimensions of all $G$-closed orbits and their corresponding isotropy groups are of minimum number of components. Such orbits are called principal closed orbits, and the corresponding isotropy groups are called principal. When $k = \dim G$, then $X$ is called having FPIG.

3. **Some results on compact holomorphic transformation groups**

The relationship between orbit connectedness, orbit convexity, and holomorphic convexity goes back to the beginning of this century, when several complex variables was born. Due to Hartogs, Reinhardt, H.Cartan and others, one already knew some classical relations between completeness, logarithmic convexity and holo-
morphical convexity for circular domains, Hartogs domains, and Reinhardt domains. The orbit connectedness and orbit convexity are defined in a general setting (for arbitrary compact connected Lie group), which correspond to completeness and logarithmic convexity when one restricts to the above domains.

There are some fundamental relationships between orbit connectedness and orbit convexity with holomorphically convexity and envelope of holomorphy for invariant domains.

**Definition.** Let $G_C$ be a connected complex Lie group, $G_R$ be a connected closed real form of $G_C$. Let $X$ be a holomorphic $G_C$-space, $D \subset X$ be a $G_R$-invariant set, we call $D$ orbit connected, if for $b_z : G_C \to X, g \mapsto g \cdot z, b_z^{-1}(D)$ is connected for each $z \in D$. When $(G_C, G_R)$ is a geodesic convex pair (i.e. the map $\text{Lie}(G_R) \times G_R \ni (v, g) \to \exp(iv)g \in G_C$ is a homeomorphism, cf. [3]), $D$ is called orbit convex if for each $z \in D$, and $v \in i\text{Lie}(G_R)$ s.t. $\exp(v) \in b_z^{-1}(D)$ it follows $\exp(tv) \in b_z^{-1}(D)$ for all $t \in [0, 1]$.

Roughly speaking, orbit connectedness means that $G_C x \cap D$ is connected for every $x \in D$.

One has known for a long time that the envelope of holomorphy of a domain in $\mathbb{C}^n$ (or more general a Riemann domain over $\mathbb{C}^n$) exists uniquely as a Riemann domain over $\mathbb{C}^n$. There is a difficult problem of univalence: When is the envelope of holomorphy of a domain in $\mathbb{C}^n$ itself a domain in $\mathbb{C}^n$? We have the following criteria for the univalence of the envelope of holomorphy for certain invariant domains:

**Theorem 1 ([36]).** Let $X$ be a Stein manifold, $K^C$ be a complex reductive Lie group holomorphically acting on $X$, where $K$ is a connected compact Lie group and $K^C$ be its universal complexification. Let $D \subset X$ be a $K$-invariant orbit connected domain. Then the envelope of holomorphy $E(D)$ of $D$ is schlicht and orbit convex if and only if the envelope of holomorphy $E(K^C \cdot D)$ of $K^C \cdot D$ is schlicht. Furthermore, in this case, $E(K^C \cdot D) = K^C \cdot E(D)$.

When $K = S^1$ and the action is circular (or $\alpha$-circular) and Hartogs, the corresponding concepts of orbit connectedness for such domains were introduced separately and the above results were obtained and stated separately by Casadio Tarabushi and Trapani in [1, 2].

When $K = (S^1)^n$ and the action is Reinhardt, the result is well known as a classical result about Reinhardt domain which asserts that any Reinhardt domain in $(\mathbb{C}^*)^n$ has schlicht envelope of holomorphy.

Some other results were also included in the above theorem. So our result can also be regarded as an extension of their results and the classical result on Reinhardt domains in a unified way.

In the proof, a theorem due to Harish-Chandra on the infinite dimensional representation of Lie groups plays an important role.

We also give some examples of orbit connected domains. Let $X = K^C / L^C$, the action of $K^C$ on $X$ be given by the left translations. When $L$ is connected or $(K, L)$ is a symmetric pair, then any $K$-invariant domain is orbit connected. The simplest example is Reinhardt domains in $(\mathbb{C}^*)^n$.

The origin of orbit connectedness could at least go back to [28].

**Example.** A theorem of V.Bargmann, D. Hall and A.S. Wightman (cf.
Wightman [32], Jost [12], Streater-Wightman [28]) asserts that $\tau_+^N$ is orbit connected.

We also consider the homogeneous embeddings of $K^C/L^C$. Let $X$ be a smooth homogeneous space embedding of $K^C/L^C$, $D \subset X$ be a $K$-domain. Assume that $L$ is connected or $(K, L)$ is a symmetric pair. Then $E(D)$ is schlicht and orbit convex. In particular, every matrix Reinhardt domain of holomorphy $D$ is orbit convex. Since an orbit convex matrix Reinhardt domain has a path connected $\text{Diag}(D)$, so a matrix Reinhardt domain of holomorphy has a connected $\text{Diag}(D)$.

Theorem 2([37]). Let $K$ be a connected compact Lie group, $L$ be a closed (not necessarily connected) subgroup of $K$. Let $K^C$ and $L^C$ be respectively universal complexification of $K$ and $L$. Suppose that $D$ is $K$-invariant relatively compact domain in $K^C/L^C$ (Here the action of $K^C$ is given by left translations). Then (i) $\text{Aut}(D)$ is a compact Lie group; (ii) Any proper holomorphic self-mapping of $D$ is biholomorphic if $K$ is semisimple or a direct product of a semisimple compact Lie group and a compact torus.

By a result of Matsushima, $K^C/L^C$ is a Stein manifold which is a holomorphic $K^C$-manifold w.r.t. left translation action.

The motivations of the present work are two-folds: the result (i) is to extend a main result of [4], where the same result was obtained by requiring a restrictive condition that $(K, L)$ is a symmetric pair, i.e., $K/L$ is a compact Riemannian symmetric space; the result (ii) is to extend a classical result which asserts that proper self mapping of the relatively compact Reinhardt domains in $(\mathbb{C}^*)^n$ is biholomorphic.

The proof is involved with many famous results such as Mostow decomposition theorem, H. Cartan’s theorem about compactness of automorphism groups, Andreotti-Frankel’s theorem on homology group of a Stein manifold, the holomorphic version of de Rham’s theorem on a Stein manifold, a result of Milnor’s about CW complex, a result from iteration theory, Poincaré duality theorem, degree theory for proper mappings, covering lifting existence theorem, and a result about compact semisimple Lie groups et al.

4. Extended future tube conjecture

Let’s keep the notation in Example d of the section 2. The set $\tau_+^N := \{\wedge z : z \in \tau_+^N, \wedge \in L^+_+(\mathbb{C})\}$ is called the extended future tube.

The extended future tube conjecture, which arose naturally from axiomatic quantum field theory at the end of 1950’s, asserts that the extended future tube $\tau_+^N$ is a domain of holomorphy for $N \geq 3$. This conjecture turns out to be very beautiful and natural. In their papers, Vladimirov and Sergeev said that the importance of the conjecture is also due to the fact that there are some assertions in QFT, such as the finite covariance theorem of Bogoliubov-Vladimirov, proved only assuming that this conjecture is true.

According to the axiomatic quantum field theory (cf. [12,13,28]), one may describe physical properties of a quantum system using the Wightman functions which correspond to holomorphic functions in $\tau_+^N$ invariant w.r.t. the diagonal action of $L^+_+$, This sort of functions have the following extension property.
BHW Theorem (due to Bargman, Hall, and Wightman 1957). An \( L_+^+ \)-invariant holomorphic function on \( \tau_N^+ \) can be extended to an \( L_+(\mathbb{C}) \)-invariant holomorphic function on \( \tau_N^+ \) (cf. \cite{[12,13,28]}).

In the proof, the orbit connectedness of \( \tau_N^+ \) play a key role. With this and Identity Theorem, one can easily define the invariant holomorphic extension.

So, a natural question arises, i.e., can these holomorphic functions be extended further? Or, is \( \tau_N' \) holomorphic convex w.r.t. \( L_+(\mathbb{C}) \)-invariant holomorphic function? After some argument, this is equivalent to ask if \( \tau_N' \) is a domain of holomorphy.

Streater’s theorem. A holomorphic function on the Dyson domain \( \tau_N^+ \cup \tau_N^- \cup J \) (where \( J := \tau_N' \cap M_{4N} \) is the set of Jost points which was proved to exist and characterized by R. Jost) can be extended to a holomorphic function on \( \tau_N' \) (cf. \cite{[12,28]}).

So, a natural question is to construct the envelope of holomorphy of the Dyson domain \( \tau_N^+ \cup \tau_N^- \cup J \) (This question is mentioned in the article “Quantum field theory” of the Russian’s great dictionary “Encyclopedia of Mathematics”). That the extended future tube conjecture holds is equivalent to that this envelope of holomorphy is exactly the extended future tube \( \tau_N' \).

The conjecture have been mentioned as an open problem in many classical (\cite{[12,28]}) and recent references (\cite{[11,21-24,28-31]}) and references therein. In \cite{[38,39]}, we found a route to solve the conjecture via Kiselman-Loeb’s minimum principle and Luna’s slice theory. Let’s recall the minimum principle.

**Minimum principle**

Let \( X \) be a complex manifold, \( G_{\mathbb{C}} \) a connected complex Lie group, \( G_{\mathbb{R}} \) a connected closed real form of \( G_{\mathbb{C}} \). Denote \( \psi : G_{\mathbb{C}} \to G_{\mathbb{C}}/G_{\mathbb{R}} \), and \( p : X \times G_{\mathbb{C}} \to X \) the natural projections.

\( G_{\mathbb{C}} \) acts on \( X \times G_{\mathbb{C}} \) on the right by:

\[
(X \times G_{\mathbb{C}}) \times G_{\mathbb{C}} \to X \times G_{\mathbb{C}}
\]

\[
((x,g),h) \mapsto (x,gh)
\]

Let \( \Omega \subset X \times G_{\mathbb{C}} \) be a right \( G_{\mathbb{R}} \)-invariant domain and have connected fibres of \( p \); and \( u \in C^\infty(\Omega) \) be a right \( G_{\mathbb{R}} \)-invariant function. \( u \) naturally induces a smooth function \( \hat{u}(x,\psi(g)) \) on \( \hat{\Omega} := (id_X,\psi)(\Omega) \).

Suppose that (1) \( u \) is p.s.h on \( \Omega \), (2) \( \forall x \in p(\Omega) \), \( u(x,\cdot) \) is strictly p.s.h. on \( \Omega_x = \Omega \cap p^{-1}(x) \), and (3) \( \hat{u}(x,\cdot) \) is exhaustive on \( \hat{\Omega}_x = \hat{\psi}(\Omega_x) \), then the minimum principle asserts that \( v(x) = \inf_{g \in \Omega_x} u(x,g) \) is \( C^\infty \) and p.s.h. on \( p(\Omega) \).

**Remark.** C.O. Kiselman in \cite{[14]} first obtained the minimum principle when \( X = \mathbb{C}^n, G_{\mathbb{C}} = \mathbb{C}^m, G_{\mathbb{R}} = Im\mathbb{C}^m \), J.J. Loeb in \cite{[18]} generalized Kiselman’s result to the present general case.

It’s easy to construct invariant p.s.h. functions w.r.t. compact Lie group via “averaging technique”. However, such a technique doesn’t hold again for non compact Lie group.

**Observation.** Let \( G \) be a real Lie group which acts on \( \mathbb{C}^n \) linearly. Let \( D \) be a Bergman hyperbolic domain which is \( G \)-invariant. Then the Bergman kernel \( K_D(z,\overline{w}) \) satisfies \( K_D(z,\overline{\zeta}) = K_D(g \cdot z,\overline{g^{-1} \zeta}) \) for \( g \in G \), when \( G \) is compact; when \( G \) is semisimple, we have \( K_D(z,\overline{w}) = K_D(g \cdot z,\overline{g \cdot w}) \).
Brief proof is as follows. Since $G$ linearly act on $\mathbb{C}^n$, one has a representation $G \to GL(n, \mathbb{C})$; if $G$ is semisimple, then the image of $G$ must be in $SL(n, \mathbb{C})$; if $G$ is compact, the image of $G$ is in $U(n)$. Using the transformation formula for the Bergman kernels and noting that the determinant of the Jacobian of the map $z \to g \cdot z$ is 1 for semisimple case, and is in $S^1$ for compact case, then we can get the result.

We consider the following question: Let $X$ be a Stein manifold, $G_C$ be a connected complex reductive Lie group acting on $X$ s.t. the action is holomorphic, $G_R$ a connected real form of $G_C$. Let $D \subset X$ be a $G_R$-invariant orbit connected Stein domain, is $G_C \cdot D$ also Stein?

When $G_R$ is compact, the answer is positive (cf. [22]). This is a special case of Theorem 1 in the section 3.

The extended future tube conjecture is a special case of the question, where $X = \mathbb{C}^{4N}, G_C = L_+(\mathbb{C}), G_R = L^1_+, D = \tau_1^+, G_C \cdot D = \tau_1^+$

Consider $X \times G_C \xrightarrow{\rho} X, \rho(x,g) = g^{-1} \cdot x$. Suppose that there is a suitable $G_R$-invariant s.p.s.h. function $\varphi$ on $D$. We have a p.s.h. function $u(x,g) = \varphi(g^{-1} \cdot x)$ on $\Omega = \rho^{-1}(D)$. Define $\psi(x) = \inf_{g \in \Omega_x} u(x,g)$ for $x \in p(\Omega)$, where $p : X \times G_C \to X$ is given by $p(x,g) = x$, and $\Omega_x := \{g \in G_C : (X,g) \in \Omega\}$.

In order to prove $\psi(x)$ is p.s.h. on $p(\Omega) = G_C \cdot D$, we can use the minimum principle due to Kiselman-Loeb.

Observation. $\Omega_x$ is connected if and only if $D$ is orbit connected.

In order to use the minimum principle, we still need to check two assumptions: (i) $u(x,\cdot)$ is s.p.s.h. on $\Omega_x$; (ii) $\tilde{u}(x,\cdot)$ is exhaustion on $\Omega_x$, where $\tilde{u}(x,\psi(g))$ is defined on $\Omega = (id,\psi)(\Omega) \subset X \times G_C/G_R$ and is induced by $u, \psi : G_C \to G_C/G_R, \tilde{\Omega}_x = \psi(\Omega_x)$. Usually speaking, assumption (i) fails on the whole $\Omega$. However, when $X$ has FPIG, then the assumption (i) is fulfilled on a Zariski open subset of $\Omega$. Let $X' := \{x \in X : G_C x \text{ is closed, } (G_C)_x \text{ is principal and finite}\}$, then, by the slice theory, $A = X \backslash A'$ is a $G_C$-invariant analytic subset of $X$. Let $D' = D \cap X', \Omega' := \rho^{-1}(D')$, then the assumption (i) is satisfied on $\Omega'$. If the assumption (ii) is also satisfied on $\Omega'$, then we can use the minimum principle on $\Omega'$ and get that $\psi(x)$ is p.s.h. on $p(\Omega') = G_C \cdot D \backslash A$ since $\psi(x)$ is upper semicontinuous on $G_C \cdot D$, by the extension theorem for p.s.h. functions, $\psi(x)$ can be extended to a p.s.h. function on $G_C \cdot D$.

If we can prove that the extended p.s.h. function is weak exhaustion, then $G_C \cdot D$ is Stein.

As a consequence of our observations, we deduce that the general question is true for pseudoconvex pair $(G_C, G_R)$ (i.e., there exists a $G_R$-invariant p.s.h. function on $G_C$ which is exhaustion on $G_C/G_R$(cf.[17])), which include the case when $G_R$ is compact and nilpotent(cf.[17]). However it's pity that $(L_+(\mathbb{C}), L^1_+)$ is not a pseudoconvex pair.

In the case of the extended future tube conjecture, we proved that the assumption (ii) in the minimum principle is satisfied and the constructed function is weak exhaustion. These are the main technical difficulties. We overcome them and finished our proof via a consideration of the matrix form of the conjecture and explicit calculations based on Hua's work and matrix techniques ([9,19]).
Theorem [38,39]. The extended future tube conjecture is true.

A.G. Sergeev posed an interesting idea to attack the mentioned question. He assumed an invariant version of Cartan’s lemma: if \( A \subset D \) is a \( G_R \)-invariant analytic subset, \( f \in \mathcal{O}(A)^{G_R} \), then there exists an \( F \in \mathcal{O}(D)^{G_R} \) s.t. \( F|_A = f \). If this is the case, we can prove that \( \pi(D) \) is Stein in \( X//G_C \). In order to prove it, it’s sufficient to prove \( \pi(D) \) is holomorphically convex. Let \( \{y_n\} \subset \pi(D) \) be an arbitrary discrete set. Then \( \{\pi^{-1}(y_n)\} \cap D \) is a \( G_R \)-invariant analytic subset in \( D \). By the assumption, then there exists a \( G_R \)-invariant holomorphic function \( F \) on \( D \) s.t. \( F|_{\pi^{-1}(y_n)} = n \). Since \( \mathcal{O}(\pi(D)) \cong \mathcal{O}(D)^{G_R} \), then we get a holomorphic function on \( \pi(D) \) which is unbounded on \( \{y_n\} \). This means that \( \pi(D) \) is holomorphically convex, and then \( \pi^{-1}(\pi(D)) \) is also Stein. When \( \pi^{-1}(\pi(D)) = G_C \cdot D \), i.e., \( G_C \cdot D \) is \( \pi \)-saturated, then \( G_C \cdot D \) is Stein.

It seems to be hard to prove directly the invariant version of Cartan’s lemma for a noncompact Lie group \( G_R \), although it’s trivially the case for a compact Lie group. Actually, we have the following:

Proposition ([41]). Suppose, furthermore, \( G_C \cdot D \) is \( \pi \)-saturated. Then the invariant version of Cartan’s lemma holds if and only if \( G_C \cdot D \) is Stein.

However, we recently observed that it should be possible to directly give an answer to the above question based on \( L^2 \)-methods and group actions.

References

[1] E. Casadio Tarabushi, S. Trapani: Envelopes of holomorphy of Hartogs and circular domains. Pacific J. Math. 149 (1991), no. 2, 231–249.
[2] E. Casadio Tarabushi, S. Trapani: Construction of envelopes of holomorphy for some classes of special domains. J. Geom. Anal. 4 (1994), no. 1, 1–21.
[3] G. Coeuré, J.J. Loeb: Univalence de certaines enveloppes d’holomorphie. (French) C. R. Acad. Sci. Paris Sér. Math. 302 (1986), no. 2, 59–61.
[4] G. Fels, L. Geati: Invariant domains in complex symmetric spaces, J.rein und angew. Math. 454 (1994), 97-118.
[5] H. Grauert: Selected papers, With commentary by Y. T. Siu et al. Springer-Verlag, 1994.
[6] H. Grauert, R. Remmert: Coherent analytic sheaves, Springer-Verlag, Berlin Heidelberg, 1984.
[7] H. Grauert, R. Remmert: Theory of Stein spaces, Grundl. 236, Springer-Verlag, 1979.
[8] L. Hörmander: Introduction to complex analytic in several variables, third revised ed., North-Holland Mathematical Library, Vol.7, North-Halland, Amsterdam, 1991.
[9] L.-K. Hua: Harmonic analysis of functions of several complex variables in the classical domains, (in Chinese) Science Press, Beijing, 1958; English translation, Amer.Math.Soc., Providence, RI, 1963.
[10] M. Jarnicki, P. Pflug: Extension of holomorphic functions. de Gruyter Expositions in Mathematics, 34. Walter de Gruyter Co., Berlin, 2000.
[11] M. Jarnicki, P. Pflug: On the extended tube conjecture. *Manuscripta Math.*, 89 (1996), no. 4, 461–470.
[12] R. Jost: *The general theory of quantized fields*. Amer. Math. Soc., Providence, R.I., 1965.
[13] David Kazhdan: Introduction to QFT, in *Quantum fields and strings: a course for mathematicians*. Vol. 1, 2, pp. 377–418 American Mathematical Society, Providence, RI; Institute for Advanced Study (IAS), Princeton, NJ, 1999.
[14] C.O. Kiselman: The partial Legendre transformation for plurisubharmonic functions. *Invent. Math.* 49, 137–148 (1978).
[15] C.O. Kiselman: Plurisubharmonic functions and potential theory in several complex variables, in *“Developments of Mathematics 1950-2000”*, ed. by J.-P. Pier, pp. 655–714, Birkhäuser-verlag, 2000.
[16] M. Lassalle: Séries de Laurent des fonctions holomorphes dans la complexification d’un espace symétrique compact. (French) *Ann. Sci. école Norm. Sup.* (4) 11 (1978), no. 2, 167–210.
[17] J.J. Loeb: Pseudo-convexité des ouverts invariants et convexité géodésique dans certains espaces symétriques. *Sém. Lelong-Skoda, Lect. Notes in Math.* 1198, 172–190.
[18] J.J. Loeb: Action d’une forme réelle sur un groupe de Lie complexe. *Ann. Inst. Fourier*, fasc. 4, t. 35 (1985).
[19] Qikeng Lu: *Classical manifolds and classical domains* (in Chinese), Shanghai Scientific and Technical Press, 1963.
[20] D. Luna: Slices étales. *Bull. Soc. Math. France*. Mem 33, 81–105 (1973).
[21] A.G. Sergeev: Around the extended future tube conjecture, *Lect. Notes in Math.*, v. 1574, Springer-Verlag, 1994.
[22] A. G. Sergeev, P. Heinzner: The extended matrix disk is a domain of holomorphy. *Math. USSR Izvestija*, Vol.38 (1992), no. 3.
[23] A. G. Sergeev, V.S. Vladimirov: Complex analysis in the future tube, in *Encyclopedia of Math. Sci.*, Vol. 8 (Several Complex Variables, II), Springer-Verlag, 1994.
[24] A.G. Sergeev, X.Y. Zhou: On invariant domains of holomorphy (in Russian). *Proc. of Steklov Math. Institute*, Tom 203, 159–172, 1994.
[25] A.G. Sergeev, X.Y. Zhou: Extended future tube conjecture. *Proc. of Steklov Math. Institute*, Tom 228, 32–51, 2000.
[26] Y.-T. Siu: Pseudoconvexity and the problem of Levi. *Bull. Amer. Math. Soc.* 84 (1978), no. 4, 481–512.
[27] D. M. Snow: Reductive Group Actions on Stein Spaces. *Math. Ann.* 259, 79–97 (1982).
[28] R. F. Streater, A.S. Wightman: *PCT, Spin and statistics, and all that*. Benjamin, Reading, Mass, 1964.
[29] V. S. Vladimirov: Analytic functions of several complex variables and quantum field theory. *Proc. of the Steklov Inst. of Math*. 1978, Issue 1, 69–81.
[30] V. S. Vladimirov: Several complex variables in mathematical physics. *Sém. Lelong-Skoda, Lecture Notes in Math.*, Vol. 919, 1982, 358–386.
Some Results Related to Group Actions in Several Complex Variables

[31] V. S. Vladimirov, V. V. Zharinov: Analytic methods in mathematical physics. 
Proc. of the Steklov Inst. of Math. 1988, Issue 2, 117–137.

[32] A. S. Wightman: Quantum field theory and analytic function of several complex variables, 
J. Indian Math. Soc. (N.S.) 24(1960/1961), 625–677.

[33] B. I. Zav’yalov, V.B.Trushin: On the extended n-point tube, Teoret. Mat. Fiz. 
27, 1(1976), 3–15.

[34] X.Y. Zhou: On matrix Reinhardt domains. Math. Ann. 287, 35–46(1990).

[35] X.Y. Zhou: On orbit convexity of certain torus invariant domain of holomorphy. 
Dokl. AN SSSR, T.322, N.2, 1992, 262–267.

[36] X.Y. Zhou: On orbit connectedness, orbit convexity, and envelopes of holomorphy. Izvestiya Ross. Akad. Nauk, Series Math. T.58, N.2, 1994, 196–205.

[37] X.Y. Zhou: On invariant domains in certain homogeneous spaces. Ann. L’Inst. 
Fourier, T.47, N.4, 1997, 1101–1115.

[38] X.Y. Zhou: A proof of the extended future tube conjecture(in Russian). Izvestiya Ross. Akad. Nauk, Series Math. T.62, N.1, 1998, 211–224.

[39] X.Y. Zhou: The extended future tube is a domain of holomorphy. Math. 
Research Letters 5, 185–190(1998).

[40] X.Y. Zhou: Quotients, invariant version of Cartan’s lemma, and the minimum 
principle. Proc. of first ICCM., 335–343, Amer. Math. Soc. and International 
Press, 2001.

[41] X.Y. Zhou: Invariant version of Cartan’s lemma and complexification of 
invariant domains (in Russian). Dokl. Ross. Akad. Nauk, vol.366, no.5, 1999, 
608–612.