Abstract: Using sieves and elementary manipulations, we show that the signs of partial sums of
the Liouville function over divisors are in a strong sense equally distributed.

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1 Introduction

The Liouville function $\lambda(n)$ is defined by $\lambda(n) = (-1)^{\Omega(n)}$ where $\Omega(n)$ counts the total number of
factors in the prime decomposition of $n$. A natural problem is to study the partial sums $L(n, z) := \sum_{d|n, d<z} \lambda(n)$. These sums will be larger if divisors come clumped in groups with the same
parity of number of prime divisors, and they will be smaller otherwise. In [2], it is proved that the
quantities

$$\lim_{x \to \infty} x^{-1} \sum_{n<x} L(n, z)^2$$

exist for each $z$, and converge to a finite limit as $z$ tends to infinity. These quantities were
further studied in [1]. The purpose of this note is to show that the signs of $L(n, z)$ are randomly
distributed over $n$ and $z$, in the following sense:

Theorem 1.1. Let $(a_{n,x})_{n \geq 1}$ be a sequence of complex numbers depending on $x$ such that

$$\limsup_{x \to \infty} x^{-1} \sum_{n<x} |a_{n,x}|^2 < \infty.$$ 

If the limits

$$\lim_{z \to \infty} x^{-1} \sum_{n<x} a_{n,x} L(n, z)$$

exist for all $z$, then they tend to 0 as $z$ tends to infinity.

2 Details and proof

We begin by clarifying the condition on $a_{n,x}$ used in Theorem 1.1.
Proposition 1. The quantities \( \lim_{x \to \infty} x^{-1} \sum_{n<x} a_{n,x} L(n, z) \) exist for all \( z \) if and only if the quantities

\[
\lim_{x \to \infty} x^{-1} \sum_{n<x} a_{q,n,x} L(n, z)
\]

exist for all integers \( q \geq 1 \).

Proof. Manipulating, we find

\[
\lim_{x \to \infty} x^{-1} \sum_{n<x} a_n L(n, z) = \lim_{x \to \infty} x^{-1} \sum_{n<x} a_n \sum_{d|n, d<z} \lambda(d)
\]

\[
= \lim_{x \to \infty} x^{-1} \sum_{d<z} \sum_{n<x/d} a_{dn} \lambda(d).
\]

Thus, letting \( z \) vary, we find that

\[
\lim_{x \to \infty} x^{-1} \sum_{n<x/d} a_{dn} \lambda(d)
\]

must exist for all \( d \). Since \( \lambda(d) \) is nonzero, we arrive at the desired conclusion. \( \square \)

We now prove a technical lemma:

Lemma 2.1. Fix a sequence \( (a_{n,x})_{n \geq 0} \). Define

\[
f_x(q) = \frac{q}{x} \sum_{n<x/q} a_{q,n,x}, \quad g_x(q) = \sum_{d|q} \mu(q/d) f_x(d),
\]

and \( S_x(b/q) = x^{-1} \sum_{n<x} a_{n,x} e^{2\pi i (b/q)n} \). The following equality holds:

\[
g_x(q) = x^{-1} \sum_{b \mod q}^* S_x(b/q).
\]

The star indicates the summation is taken over residue classes in \( (\mathbb{Z}/q\mathbb{Z})^* \).

Proof. We obverse via sum manipulations that

\[
x^{-1} \sum_{d|q} \sum_{b \mod d}^* S_x(b/d) = x^{-1} \sum_{b \mod q} S_x(b/q)
\]

\[
= x^{-1} \sum_{n<x} a_n \left( \sum_{b \mod q} e^{2\pi i (b/q)n} \right)
\]

\[
= \frac{q}{x} \sum_{n<x/q} a_{q,n}
\]

The desired formula follows by Möbius inversion. \( \square \)
Corollary 2.1. Using the notation of Lemma 2.1, if \((a_{n,x})\) is such that \(g(q) = \lim_{x \to \infty} g_x(q)\) exists for all \(q\), then

\[
\sum_{n=1}^{\infty} \frac{|g(q)|^2}{\varphi(q)}
\]

converges whenever \(\limsup_{x \to \infty} x^{-1} \sum_{n<x} |a_{n,x}|^2\) converges

Proof. Using Cauchy–Schwarz on \(g(q)\), we find

\[
\sum_{q<Q} |g(q)|^2 \varphi(q) \leq \limsup_{x \to \infty} x^{-2} \sum_{q<Q} \sum_{b \equiv q \pmod{q}} |S_x(b/q)|^2
\]

The large sieve inequality states that the left hand side of this expression is bounded above by

\[
\limsup_{x \to \infty} \frac{x + Q}{x^2} \sum_{n<x} |a_{n,x}|^2,
\]

which is bounded uniformly as \(Q\) varies by our assumption on \((a_{n,x})\).

We can now prove the main theorem:

Proof of Theorem 1.1. We work with the notation of Lemma 2.1. To begin, we see via elementary sum manipulations

\[
\lim_{x \to \infty} x^{-1} \sum_{n<x} a_{n,x} L(n, z) = \lim_{x \to \infty} x^{-1} \sum_{n<x} a_{n,x} \sum_{d|n, d<z} \lambda(d)
\]

\[
= \sum_{d<z} \frac{\lambda(d) f(d)}{d},
\]

where \(f(d) = \lim_{x \to \infty} f_x(d)\). This limit exists for all \(d\) by Proposition 1. By Möbius inversion, we know that \(g(d) = \lim_{x \to \infty} g_x(d)\) must exist as well. Thus, we can manipulate our sum further as

\[
\sum_{d<z} \frac{\lambda(d) f(d)}{d} = \sum_{d<z} \frac{\lambda(d)}{d} \left( \sum_{q|d} g(q) \right)
\]

\[
= \sum_{q<z} \frac{\lambda(q) g(q)}{q} \left( \sum_{d<z/q} \frac{\lambda(d)}{d} \right).
\]

Note the key use of the fact that \(\lambda\) is completely multiplicative. Combining our work thus far, we get the following:

\[
\lim_{x \to \infty} x^{-1} \sum_{n<x} a_n L(n, z) = \sum_{q<z} \frac{\lambda(q) g(q)}{q} \left( \sum_{d<z/q} \frac{\lambda(d)}{d} \right)
\]
By the prime number theorem \[ \sum_{d<z/q} \frac{\lambda(d)}{d} \ll \frac{1}{\log^*(z/q)} \] where \( \log^*(z/q) = \max(1, \log(z/q)) \).

Hence,

\[
\left| \sum_{q<z} \frac{\lambda(q)g(q)}{q} \left( \sum_{d<z/q} \frac{\lambda(d)}{d} \right) \right| \ll \sum_{q<z} \frac{|g(q)|}{q \log^*(z/q)}.
\]

Fixing large \( T > 0 \), we find by Cauchy–Schwarz that

\[
\sum_{q<z} \frac{|g(q)|}{q \log^*(z/q)} \leq \sum_{q<z/T} \frac{|g(q)|}{\varphi(q)^{1/2}} \cdot \sum_{q<z} \frac{\varphi(q)}{q^2 \log^*(z/q)^2}^{1/2}
\]

\[
+ \sum_{z/T \leq q<z} \frac{|g(q)|^2}{\varphi(q)} \cdot \sum_{z/T \leq q<z} \frac{\varphi(q)}{q^2 \log^*(z/q)^2}^{1/2}
\]

We treat these summations one by one. By the conditions of the proposition we get that the first sum is bounded uniformly in terms of \( a_n \), and that the first sum on the second row is \( o_T(1) \).

For the second sum in the first row, we note that \( \log^*(z/g) \geq \log(T) \). The second sum in the bottom row is clearly \( O_T(1) \), and hence collecting we get that

\[
\limsup_{z \to \infty} \sum_{q<z} \frac{|g(q)|}{q \log^*(z/q)} \ll \limsup_{z \to \infty} \left| \sum_{q<z/T} \frac{1}{q \log^*(z/q)^2} \right|^{1/2}.
\]

Decomposing along intervals \( 2^{-(k+1)} \cdot z/T < q \leq 2^{-k} \cdot z/T \) it is clear that not only is the right hand side bounded but it tends to 0 as \( T \to \infty \). Hence, we conclude the result. \( \square \)

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References

[1] Régis de la Bretèche, François Dress, and Gérald Tenenbaum. Remarques sur une somme liée à la fonction de möbius. Mathematika, 66(2):416–421, 2020.

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