Pythagorean quantization, action(s) and the arrow of time

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Abstract. Searching for the first well-documented attempts of introducing some kind of “quantization” into the description of nature inevitably leads to the ancient Greeks, in particular Plato and Pythagoras. The question of finding the so-called Pythagorean triples, i.e., right-angled triangles with integer length of all three sides, is, surprisingly, connected with complex nonlinear Riccati equations that occur in time-dependent quantum mechanics. The complex Riccati equation together with the usual Newtonian equation of the system, leads to a dynamical invariant with the dimension of an action. The relation between this invariant and a conserved “angular momentum” for the motion in the complex plane will be determined. The “Pythagorean quantization” shows similarities with the quantum Hall effect and leads to an interpretation of Sommerfeld’s fine structure constant that involves another quantum of action, the “least Coulombic action” \( e^2/c \). Since natural evolution is characterized by irreversibility and dissipation, the question of how these aspects can be incorporated into a quantum mechanical description arises. Two effective approaches that also both possess a dynamical invariant (like the one mentioned above) will be discussed. One uses an explicitly time-dependent (linear) Hamiltonian, whereas the other leads to a nonlinear Schrödinger equation with complex logarithmic nonlinearity. Both approaches can be transformed into each other via a non-unitary transformation that involves Schrödinger’s original definition of a (complex) action via the wave function.

1. Introduction
Looking for the first well-documented attempts of introducing some kind of “quantization” into the description of nature inevitably leads to the ancient Greeks, in particular, to Democritus, Plato and Pythagoras. The question of finding so-called Pythagorean triple, i.e. right(-angled) triangles where the length of all sides are integers, is surprisingly connected with complex nonlinear Riccati equations that occur in quantum mechanics. This Riccati equation, together with the Newtonian equation of motion for the respective system, leads to a dynamical invariant that has (almost) the dimension of an \( \text{action} \). This is, insofar, remarkable since the physical quantity that is actually quantized in nature is \( \text{action} \) – Planck’s constant (divided by \( 2\pi \)) \( \hbar \) is a quantum of action. The often-quoted quantization of energy is only a consequence of multiplying this quantum of action by a constant with the dimension of a frequency. Another physical quantity with the dimension of \( \text{action} \) is \( \text{angular momentum} \) which, in quantum mechanics, is also quantized in terms of \( \hbar \). But, like action, also angular momentum cannot be measured directly since it is a product of two measurable quantities (in this case, position and momentum).
In this context, it is interesting to state that even for systems where the Hamiltonian is no longer invariant in time (e.g., a harmonic oscillator (HO) with time-dependent frequency $\omega(t)$), or where not even a conventional Hamiltonian exists (e.g., for dissipative systems with (linear) velocity dependent friction forces), invariants still exist that have the dimension of action and are related to some kind of angular momentum.

In this paper, the following questions shall be essentially considered:

1. In the history of science and mathematics, what are the first known attempts at “quantization” and do they have any manifestation in modern (quantum) physics?
2. Apart from angular momentum, are there other quantities in physics that have the dimension of action and are they of any significant importance (e.g., as a constant of motion, etc.)?
3. Apart from $h$, are there other quanta of action in physics? What is their physical meaning and where do they occur?
4. In cases where no conserved (physical) Hamiltonian exists (e.g., in dissipative systems), are there conserved action-type quantities and what is their meaning?

2. Pythagorean quantization and its manifestation in modern physics

The idea of dividing our world into the smallest components that are not further divisible is well known from the philosophy of Democritus (ca. 460 - ca. 371 BC). This led to the word atom that is still used to describe the building blocks of our physical world that, for some time, seemed further indivisible.

A similar idea was formulated in a more abstract way by another equally famous Greek philosopher living around the same time. In his work Timaios, Plato (428/27 - 348/47 BC) gives his view of how the world is built up in terms of right triangles. Werner Heisenberg, who was equally fascinated with and puzzled by this text, summarizes this idea in his book “Der Teil und das Ganze” [1] in the general sense as follows: “Matter is made up of right triangles which, after being paired to form isosceles triangles or squares, are just joined together to build the regular bodies of stereometry: cube, tetrahedron, octahedron and icosahedrons. These four solids are then supposed to be the basic units of the four elements earth, fire, air and water.”

Even some 2000 years later this idea of Platonic polyhedral fascinated Johannes Kepler (1571-1630) so much that, in his quest for harmony in nature, he tried to explain the orbits of the planets in our solar system by fitting one polyhedron into another so that the radii of spheres enveloping this polyhedra would just represent the mean distance of the planets from the sun.

An aspect of quantization was brought into this picture by Titius of Wittenberg (1729-1796) and Johann Elert Bode (1747-1826) who proposed a series of numbers (integers!) that predicts the position of the planets (Titius–Bode law), showing similarities with Bohr’s model of the atom. (Remarkably, a new formulation of this law has been found by Reinsch [2] using a nonlinear formulation of a (formal) macroscopic Schrödinger equation (SE) to describe the solar system. This nonlinear formalism is equivalent to the one that will be presented in section 2.1 in the context of time-dependent quantum mechanics.)

Kepler tried to connect the geometry of the planetary orbits and movement of the planets with some kind of imaginary sounds – the music of the spheres. This takes us back to the ancient Greeks and the right triangles.

The Greek philosopher Pythagoras lived around 570-500 BC. In today’s society, even if (almost) nothing from our early mathematical education survived, most people can (hopefully) at least still recall the theorem named after him and might even be able to quote it as $a^2 + b^2 = c^2$ where $a$ and $b$ are the catheti and $c$ the hypotenuse of a right triangle. Pythagoras and his disciples were well-known for their dogma “everything is number”, with number meaning integer. They applied it to develop a musical scale (see Kepler’s music of the spheres) and also to the right triangle. So, the Pythagorean triple are three integers denoting the lengths of the three

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sides of a right-angled triangle, thus fulfilling Pythagoras’ theorem. The most common example is \((3, 4, 5)\) with \(9 + 16 = 25\). But, asked for a few more examples of the kind, even people accustomed to mathematics have difficulties finding any – although an infinite number of triples exist! Moreover, there is even a rather simple rule to finding these triples. This rule was probably already known in Mesopotamia around 2000 BC, but at least Diophantus of Alexandria (around 250 AC) knew it \(^1\).

That this rule is also somehow present in modern physics, particularly in quantum theory, will be discussed in the next sub-section.

2.1. Time-dependent quantum mechanics and dynamical invariants

In the following, it shall be shown how time-dependent quantum mechanics can be related to the “Pythagorean quantization” mentioned above. For this purpose, one-dimensional problems with exact analytic solutions in the form of Gaussian wave packets will be considered. This kind of solutions are obtained, e.g., for the free motion (potential cannot be fulfilled for \(n > d\) the form

\[ \Psi(x, t) = \text{N}(t) \exp \left\{ i \left[ y(t)x^2 + \frac{< p >}{\hbar} \tilde{x} + K(t) \right] \right\} \]

can be written as

\[ \Psi(x, t) = \Psi_{WP, L}(x, t) = \text{N}(t) \exp \left\{ i \left[ y(t)x^2 + \frac{< p >}{\hbar} \tilde{x} + K(t) \right] \right\} \]

with the shifted coordinate \(\tilde{x} = x - < x >\) and the mean value \(< x >\) of the classical trajectory, \(< p > = m\tilde{y}\) represents the classical momentum and the coefficient of the quadratic term in the exponent, \(y(t) = y_R(t) + iy_I(t)\), is a complex function of time. The (possibly) time-dependent normalization factor \(N(t)\) and the purely time-dependent function \(K(t)\) in the exponent are not relevant to the following discussion in this section.

The parameters \(y(t)\) and \(\eta(t)\) have been introduced in the above definition of the Gaussian wave packet because the equations of motion for \(\left( \frac{\hbar}{m} y \right)\) and \(\eta(t)\) will be the important ones in what follows. Inserting the wave packet (2) into the SE (1) yields these two equations, where the one for \(\eta(t)\),

\[ \dot{\eta} + \omega^2 \eta = 0 , \]

(overdots denote derivatives with respect to time; the case \(V = 0\), in the following, is always obtained in the limit \(\omega \rightarrow 0\)), simply conveys that the maximum of the wave packet, located at \(x = < x >\), follows the classical trajectory. The equation for the quantity \(y(t)\) has the form a of a (complex) nonlinear (NL) Riccati equation and describes the time-dependence of the wave packet width that is related with the position uncertainty. This can be seen from the fact that \(y_I = \frac{1}{x \sqrt{2\pi}}\) with \(< \tilde{x}^2 > = < x^2 > - < x >^2\) being the mean square deviation of position. Defining the complex quantity \(\mathcal{C}(t) = \left( \frac{\hbar}{m} y \right) = \mathcal{R} + i\mathcal{I}\), the Riccati equation can be written in the form

\[ \frac{d}{dt} \mathcal{C} + \mathcal{C}^2 + \omega^2 = 0 . \]

This nonlinear complex equation will be the link to our “Pythagorean quantization” as well as to an action-type invariant.

\(^1\) Fermat claimed to have found an elegant proof that for \(a, b\) and \(c\) being integers, the relation \(a^n + b^n = c^n\) cannot be fulfilled for \(n > 2\) but the margin of Diophantus’ book was too small to write it down. This proof is still missing!
To show this, a new (real) variable $\alpha(t)$ is introduced via
\[ I = \left( \frac{2m \langle \hat{x}^2(t) \rangle}{\hbar} \right)^{1/2} \text{ (from which it follows that } \alpha = \left( \frac{2m \langle \hat{x}^2(t) \rangle}{\hbar} \right)^{1/2} \text{ ). Inserting this into the imaginary part of Eq. (4) allows one to determine } \mathcal{R}(t) \text{ as } \mathcal{R} = \frac{\dot{\alpha}}{\alpha}, \text{ which, when inserted into the real part of (4) together with the above definition of } I, \text{ finally turns the complex Riccati equation into the real nonlinear so-called Ermakov equation for } \alpha(t) \text{.}
\[ \ddot{\alpha} + \omega^2 \alpha = \frac{1}{\alpha^3}. \tag{5} \]

It had been shown by Ermakov [4] in 1880, i.e. 35 years before quantum mechanics was formulated by Schrödinger and Heisenberg, that from the pair of equations (3) and (5), coupled via $\omega^2$, by eliminating $\omega^2$ from the equations, a dynamical invariant, the Ermakov-invariant
\[ I_L = \frac{1}{2} \left[ (\eta \dot{\alpha} - \eta \dot{\alpha})^2 + \left( \frac{\eta}{\alpha} \right)^2 \right] = \text{ const}. \tag{6} \]
can be obtained (this invariant had been rediscovered by several authors, also in a quantum mechanical context; see, e.g., [5, 6, 7]).

Two remarkable properties of this invariant are i) it is also a constant of motion for $\omega = \omega(t)$, i.e., in the case where the corresponding Hamiltonian does not have this property and ii) apart from a missing constant factor proportional to mass $m$, it has the dimension of an action, not of an energy. This missing factor $m$ can be explained by the derivation of the invariant based on the mathematical properties of Eqs. (3) and (5). In a physical context, Eq. (3) would have to be replaced by the corresponding Newtonian equation of motion, i.e., by Eq. (3) multiplied by the missing factor $m$. Therefore, in the following we can assume that in a physical context the invariant has the dimension of an action. Furthermore, as will be shown below, an invariant of this type also exists for certain dissipative systems, i.e. systems for which a conventional Hamiltonian does not even exist.

In order to make the connection with the Pythagorean triple and also to make the corresponding physical interpretation more transparent, the complex Riccati equation (4) shall be linearized using the ansatz
\[ C = \mathcal{R} + i I = \frac{\dot{\lambda}}{\lambda}, \tag{7} \]
with complex $\lambda(t)$, leading to
\[ \ddot{\lambda} + \omega^2 \lambda = 0, \tag{8} \]
which has the form of the Newtonian equation (3) of the corresponding problem, but now for a complex variable.

First, a kind of geometric interpretation of the motion of $\lambda$ in the complex plane shall be given. Expressed in Cartesian coordinates, $\lambda$ can be written as $\lambda = u + i z$, or in polar coordinates as $\lambda = \alpha e^{i \varphi}$. Inserting the polar form into Eq. (7) leads to
\[ C = \frac{\dot{\alpha}}{\alpha} + i \varphi, \tag{9} \]
where the real part is already identical to $\mathcal{R}$, as defined above.

The quantity $\alpha$ defined above in $I$ as being proportional to the position uncertainty is identical to the absolute value of $\lambda$, if it can be shown that
\[ \varphi = \frac{1}{\alpha^2}. \tag{10} \]
This, however, can simply be proven by inserting real and imaginary parts of (9) into the imaginary part of the Riccati equation (4). Comparing relation (10), that can also be written in the form

\[ \dot{zu} - \dot{uz} = \alpha^2 \dot{\varphi} = 1, \]  

(11)

with the motion of a particle under the influence of a central force in two-dimensional physical space, it shows that this relation corresponds to the “conservation of angular momentum”, but here for the motion in the complex plane!

In the following, the case \( V = 0 \) shall be considered, turning the inhomogeneous Riccati equation (4) into the homogenous Bernoulli equation

\[ \frac{d}{dt} C + C^2 = 0 \]  

(12)

(in general, it is always possible to transform the inhomogeneous Riccati equation into a homogenous Bernoulli equation if one particular solution of the Riccati equation is known). From the definition of \( \mathcal{R} \) and \( \mathcal{I} \), it is obvious that

1) \( \mathcal{R} = \frac{\dot{\alpha}}{\alpha} \) is related to the (relative) change in time of the radius of the motion of \( \lambda \). In particular, for \( \frac{\dot{\alpha}}{\alpha} = k = \text{const.} \), the radius changes exponentially according to \( \alpha = \alpha_0 e^{kt} \), i.e., this radial change involves the number “\( e \)”.

2) \( \mathcal{I} = \frac{1}{\alpha^2} = \dot{\varphi} \) is related to the angular change of \( \lambda \). In particular, for \( \alpha = \text{const.} \), a constant angular velocity \( \dot{\varphi} = \omega_0 \) follows. The angular change involves the number “\( \pi \)”.

The (negative) temporal changes of real and imaginary parts of \( C \) are given by

\[ -\frac{d}{dt}\mathcal{R} = \mathcal{R}^2 - \mathcal{I}^2 = \Re\{C^2\} \]
\[ -\frac{d}{dt}\mathcal{I} = 2\mathcal{R}\mathcal{I} = \Im\{C^2\}. \]  

(13)

Now, in the complex plane, the absolute value of \( C^2 \), its real part and its imaginary part form a right triangle with the hypotenuse \( |C^2| = \mathcal{R}^2 + \mathcal{I}^2 \), i.e. the ”length” of \( C^2 \) and the two catheti \( \Re\{C^2\} = \mathcal{R}^2 - \mathcal{I}^2 \) and \( \Im\{C^2\} = 2\mathcal{R}\mathcal{I} \).

For integers \( \mathcal{R} \) and \( \mathcal{I} \) (with \( \mathcal{R} > \mathcal{I} \)), this, however, supplies us with the desired rule for the construction of the Pythagorean triple since all possible triples can be obtained if \( \mathcal{R} \) and \( \mathcal{I} \) take all possible integer values. In this case, the three sides of the right triangle will obviously always also be multiples of integers. For our dynamical system, this also entails that the changes of our (integer) quantities \( \mathcal{R} \) and \( \mathcal{I} \), \( \frac{d}{dt}\mathcal{R} \) and \( \frac{d}{dt}\mathcal{I} \), will also be expressed in terms of integers, i.e., a ”quantized” dynamics. This appears particularly interesting since complex Riccati equations and corresponding Ermakov systems occur in many different areas of physics, not only in time-dependent quantum mechanics, but also in time-independent QM, quantum optics, classical optics, Bose-Einstein condensates, systems in nonlinear dynamics, cosmological models and many more. So in all these systems, intrinsically, some kind of quantization is possible. (Relations to the complex quadratic family as it occurs in the theory of fractals are also obvious and will be discussed elsewhere.)

At this point, only one further aspect shall be mentioned. Considering the time-dependent SE in momentum space [8], similar Gaussian wave packet solutions can be obtained where the time-dependence of the momentum uncertainty is again determined by a complex Riccati equation, the complex variable now being just the inverse of the one in position space, i.e., \( C^{-1} = \frac{\lambda}{\lambda'} \), fulfilling

\[ \frac{d}{dt}(C^{-1}) - \omega^2 (C^{-1})^2 - 1 = 0. \]  

(14)
In particular, for $V = 0$ (s. Bernoulli equation in position space), one obtains

$$C^{-1} = t - i \alpha_0^2$$

(15)

with $\alpha_0^2 = \frac{\hbar}{2m} < \tilde{x}^2 > (t = 0) = \text{const.}$, i.e., a kind of complex time-variable, where the imaginary part is related to the initial position uncertainty. In this context, it should be mentioned that the possibility of complexification of physical quantities (also in the context of dissipative systems or $\mathcal{PT}$-symmetry) has recently gained growing interest (some examples can also be found in other contributions to these proceedings; see also [9, 10, 11, 12, 13]).

A quantum phenomenon where the real part of $\mathcal{C}$, i.e. $\mathcal{R} = \frac{\alpha}{\dot{\alpha}}$, and its possible quantization play a role will be presented in section 3.

2.2. On the physical interpretation of the Ermakov invariant

It has been shown in section 2.1. that the position uncertainty can be expressed in terms of $\alpha(t)$. It is straightforward to show (see, e.g., [14]) that, similarly, the momentum uncertainty $< \tilde{p}^2 > = < p^2 > - < p >^2$ can also be expressed in terms of $\alpha$, $\dot{\alpha}$ and $\dot{\varphi}$.

This finally allows one to write down the quantum mechanical contribution to the energy of the wave packet (2) (for the HO), which is identical to the ground state energy $\tilde{E} = \frac{\hbar}{2} \omega$, in the form [15]

$$\tilde{E} = \frac{< \tilde{p}^2 >}{2m} + \frac{m}{2} \omega^2 < \tilde{x}^2 > = \frac{\hbar}{4} \left\{ \dot{\alpha}^2 + \alpha^2 \dot{\varphi}^2 + \omega^2 \alpha^2 \right\} .$$

(16)

In search of a Hamiltonian formalism for the uncertainties similar to the classical Hamiltonian formalism for position and momentum, one can try using the difference $\mathcal{L} = < \tilde{x}^2 > - \frac{m}{2} \omega^2 < \tilde{x}^2 >$, expressed in terms of $\alpha$, $\dot{\alpha}$ and $\dot{\varphi}$ and as a kind of Lagrangian to determine the corresponding canonical momenta, leading to

$$\frac{\partial \mathcal{L}}{\partial \dot{\alpha}} = p_\alpha = \frac{\hbar}{2} \dot{\alpha} \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = p_\varphi = \frac{\hbar}{2} \alpha^2 \dot{\varphi} = \frac{\hbar}{2} \dot{\varphi} .$$

(17)

So, the ground state energy $\tilde{E}$ can finally be written as a Hamiltonian in the form

$$\mathcal{H}(\alpha, p_\alpha, \varphi, p_\varphi) = \frac{p_\alpha^2}{\hbar} + \frac{p_\varphi^2}{\hbar \alpha^2} + \frac{\hbar}{4} \omega^2 \alpha^2$$

(18)

which yields the correct Hamiltonian equation of motion equivalent to the Ermakov equation (5) for $\alpha(t)$.

It shall only be mentioned here (for details see [16]) that the time-dependent Green function (also called Feynman kernel) for the time-propagation of the initial wave packet corresponding to (2) can be given in terms of $u$ and $z$, the real and imaginary parts of $\lambda(t)$. Comparing the resulting time-dependent wave packet with the form used in section 2.1. shows that $u$ and $z$ can be expressed in terms of $\eta$, $\dot{\eta}$, $\alpha$ and $\dot{\alpha}$. In particular,

$$z = \frac{m}{\alpha_0 p_0} \eta(t) = \alpha \sin \varphi$$

(19)

can be verified.

Rewriting the Ermakov invariant in terms of $z$ and $\dot{z}$, instead of $\eta$ and $\dot{\eta}$, yields

$$I_L = \frac{1}{2} \left( \frac{\alpha_0 p_0}{m} \right)^2 \left[ \left( \dot{z} \alpha - z \dot{\alpha} \right)^2 + \left( \frac{z}{\alpha} \right)^2 \right] = \text{const.} .$$

(20)
Since \((\frac{\dot{z}}{\alpha})^2 = \sin^2 \varphi\), it is necessary for \(I_L\) to be constant that \((\dot{z} - \dot{\alpha})^2 = (\frac{u}{\alpha})^2 = \cos^2 \varphi\).

Therefore, it follows that

\[
u = \dot{z}\alpha^2 - z\dot{\alpha} = (\frac{m}{\alpha_0 p_0}) \alpha^2 \left(\eta - \frac{\dot{\alpha}}{\alpha} \eta\right) \tag{21}\]

and

\[
I_L = \frac{1}{2} \left(\frac{p_0 \alpha_0}{m}\right)^2, \tag{22}\]

i.e., the Ermakov invariant corresponds (apart from the constant factor \(\frac{1}{m}\)) to the initial energy (for \(\eta_0 = 0\)) times the imaginary part of the “complex time” \((\alpha^2)\).

As we have seen before, there are two conserved quantities in the Ermakov systems; one is the Ermakov invariant \((6)\), the other is given by the relation \((11)\), \(\dot{z}u - \dot{u}z = 1\), connected with the conservation of angular momentum in the complex plane.

Expressing now \(u\), \(z\), \(\dot{u}\) and \(\dot{z}\) according to Eqs. \((19)\) and \((21)\) in terms of \(\eta\), \(\dot{\eta}\), \(\alpha\) and \(\dot{\alpha}\) and using the Eqs. \((3)\) and \((5)\) for \(\eta(t)\) and \(\alpha(t)\), relation \((11)\) can be rewritten in the form

\[
\dot{z}u - \dot{u}z = \left[\frac{m}{\alpha_0 p_0}\right]^2 \left[\dot{\eta}^2 \alpha^2 - 2\dot{\eta}\dot{\alpha} + \eta^2 \left(\dot{\alpha}^2 + \frac{1}{\alpha^2}\right)\right]
= 2 \left(\frac{m}{\alpha_0 p_0}\right)^2 I_L, \tag{23}\]

i.e., the action-type invariant \(I_L\) is (up to a constant factor) identical to the conserved “angular momentum” for the motion of \(\lambda(t)\) in the complex plane.

### 3. Currents, (fractional) Quantum Hall Effect and more action(s)

The probability density \(\rho = \Psi^*\Psi\) corresponding to the solution of the time-dependent SE \((1)\) obeys the continuity equation

\[
\frac{\partial}{\partial t} \varrho + \frac{\partial}{\partial x} (\varrho \, v_-) = 0 \tag{24}\]

with the probability current density \(j = \varrho v_-\) and the velocity field \(v_- = \frac{\hbar}{2mi} \left(\frac{\partial}{\partial \Psi} - \frac{\partial}{\partial \Psi^*}\right)\).

For Gaussian wave packets, this can be written as

\[
v_- = \dot{\eta} + \frac{\dot{\alpha}}{\alpha} \, \dot{\bar{x}} \tag{25}\]

with the classical contribution \(\dot{\eta} = <v>\).

For \(\dot{\alpha} = 0\), i.e. constant wave packet width, there is no quantum mechanical contribution to the current. However, for \(\dot{\alpha} \neq 0\), typical quantum effects like tunnelling or Quantum Hall Effect (QHE) may occur. \(^2\)

In the QHE, the current \(j\) (here only one-dimension shall be considered)

\[
j = \varrho \, v_- \propto \sigma \, E \tag{26}\]

\(^2\) The contribution \(\frac{\dot{\alpha}}{\alpha} \, \dot{\bar{x}}\) to the velocity should not be mistaken for the so-called “osmotic velocity” \(u = \frac{\hbar}{2mi} \, \frac{\partial}{\partial \bar{x}}\) applied in some alternative interpretations of quantum mechanics. In the case of our wave packet, the osmotic velocity would be \(u = -\frac{1}{\alpha^2} \dot{x} = -\dot{\phi} \dot{x}\) and therefore not be related to the radial change of \(\lambda\), \(\frac{\dot{\alpha}}{\alpha}\), but to the angular change \(\dot{\phi}\).
is proportional to the electric field strength $E$ and the conductivity $\sigma$ with

$$\sigma = n \, c \, \left(\frac{e^2}{c\hbar}\right) = n \, c \, \alpha_{\text{Som}}$$

(27)

where $n = \text{integer}$, $c = \text{velocity of light}$ and $\alpha_{\text{Som}} = \left(\frac{e^2}{c\hbar}\right) \approx 1/137$ is Sommerfeld’s fine structure constant (not to be confused with $\alpha(t)$ as defined in section 2!).

For fixed position of the system’s centre of mass, $\dot{\eta} = 0$, the conductivity or current displays the following proportionality:

$$j \propto n \, \left(\frac{e^2}{c\hbar}\right) \propto \frac{\dot{\alpha}}{\alpha}$$

(28)

therefore changes of $j$ by integers $n$ would correspond to integer changes of $\frac{\dot{\alpha}}{\alpha}$, a possibility discussed in section 2.1.

Apart from the QHE, there is also a fractional QHE observed where

$$j \propto \frac{n}{k} \, \left(\frac{e^2}{c\hbar}\right)$$

(29)

with $n, k$ both being integers. There are different interpretations of this effect in the literature and not all possible combinations of $n$ and $k$ (being odd, even or both) have been observed. To my knowledge, none of the existing theories is able to explain all observed combinations of $n$ and $k$ or even predict yet unobserved ones. Therefore, there might still be room for some unknown in the theory of the fractional QHE.

One interpretation that had been given by the author and colleagues [17] some time ago assumes that the fractional QHE is a manifestation of a phenomenon where two effects are involved, each being quantized separately. Since it was already stated in the introduction that action is the quantity that is usually quantized in nature, we assumed that in the fractional QHE one of the effects is quantized in terms of $\hbar$, while the other is in terms of $\frac{e^2}{c}$, so the ratio of both is just Sommerfeld’s fine structure constant $\alpha_{\text{Som}} = \frac{e^2}{c\hbar}$.

Certainly the action $h$ is well known in quantum mechanics; however, the action $\frac{e^2}{c}$ was also already mentioned by Einstein [18], Schrödinger [19] and Eddington [20], assuming its importance but not knowing its physical relevance. Our interpretation of the fractional QHE is, therefore, that this is the first observed physical effect that shows the relevance of a second quantum of action $\frac{e^2}{c}$.

In [17], we called $\frac{e^2}{c}$ the “quantum of electrostatic action”. Why this name? For two electrons interacting via Coulomb’s law, the electrostatic action involved is

$$\text{action} = \text{energy} \times \text{time} = \frac{e^2}{r} \times t$$

(30)

Since the distance between the electrons, $r$, and their interaction time, $t$, can be related via the interaction velocity $v = r/t$, relation (30) can be written as

$$\text{action} = \frac{e^2}{v}$$

(31)

This action becomes a minimum when the velocity attains its maximum value $v = c = \text{velocity of light}$. So, the least Coulombic action is $\frac{e^2}{c}$.

But then the question arises, since nature is usually very economical, as expressed by the various extremal principles, e.g., in optics, mechanics, etc., why does it allow the luxury of two
elementary quanta of action? Of course, no definite answer shall be given at this point; but the following remarks, based on the aforementioned shall be allowed. Planck’s constant (divided by $2\pi$), $\hbar$, is well known particularly in quantum mechanics of oscillating systems, where, e.g., the energy is quantized in terms of $\hbar \omega = \hbar \dot{\varphi}$, i.e., the frequency is related to some kind of angular velocity and thus to some angular aspect. On the other hand, it has just been shown that $\frac{e^2}{c}$ is somehow related with a physical situation when a radial distance plays the important role.

As radius and circumference of a circle cannot be expressed in terms of the same units ("quanta") since they are related via the irrational number $\pi$, maybe the “radial” action and the “angular” action also cannot be expressed in terms of the same fundamental units.

As seen in our discussion of integer value for $\frac{\dot{\alpha}}{\alpha}$ and $\varphi$ related to $\left(\frac{2\hbar}{m\gamma}\right) = \frac{\lambda}{\chi}$, also there, the radial aspect is for integer values of $\frac{\dot{\alpha}}{\alpha}$ connected with $e = 2.718...$, whereas the angular aspect $\varphi = \frac{1}{2\pi}$ is connected with $\pi = 3.141...$. So, maybe $\frac{e^2}{c}$ and $\hbar$ just offer the possibility to quantize in a (complex ?) two-dimensional (phase ?) space radial changes as well as angular ones.

4. Irreversibility (the “arrow of time”) and dissipation

In classical Hamiltonian mechanics, the time-evolution of a dynamical system can be described by canonical transformations that leave the algebraic structure based on the Poisson brackets unchanged and (usually) conserve the energy of the system. (For systems with explicitly time-dependent Hamiltonians, e.g. HOs with time-dependent frequency $\omega(t)$, the Hamiltonian is no longer a constant of motion, but the energy is still not lost due to a dissipative force.)

In quantum mechanics, the time-evolution of a corresponding quantum system can be described by unitary transformations which, essentially, represent a pure rotation of the wave function or state vector in an abstract Hilbert space with conserved norm, i.e., length of the vector. 3

Considering natural evolution processes as observed in the macroscopic world around us, growth processes (in two or more dimensions) are usually also connected with some kind of radial changes. So the shell of a nautilus or the horns of a ram not only undergo angular changes when growing, but also radial ones leading to forms like (often logarithmic) spirals.

So, not only rotation but, also, radial expansion (or contraction) are essential elements of these kinds of irreversible evolutions thus offering also the possibility of defining something like a direction of time (e.g., larger radius = later in time). Another characteristic feature of macroscopic evolution processes is the loss of energy by transfer from the dynamical system to the environment due to dissipation, transforming (mechanical) energy into heat.

Irreversibility and dissipation are usually not elements contained in the description of a single system in terms of the conventional Hamiltonian formalism. In the following, it shall be shown how these essential elements of nature might be incorporated into this formalism (classical and quantum mechanical) and what price must be paid for this. It should also be mentioned here briefly that irreversibility and dissipation often occur simultaneously and are often also not independent of each other (see, e.g., fluctuation-dissipative theorems, etc.). However, this coupling is not necessarily always the case. There are, for instance, irreversible processes possible without any dissipation of energy and we will mention such cases subsequently.

4.1. Irreversible aspects in conventional quantum mechanics

One example of an irreversible behaviour of a quantum system is well known from the textbooks but not usually recognized as such.

3 Due to the probabilistic interpretation of the wave function this normalizability condition has to be fulfilled, but still irreversible effects can occur that would cause a change of the length of the state vector if they would not be compensated by some kind of “renormalization”; for an example in conventional quantum mechanics, see section 4.1.
Considering the time-evolution of a dynamical system, by "reversible" we usually mean that we cannot distinguish between the motion of the system going forward or backward in time. This is reflected in the corresponding equation of motion where, under change of $t$ into $-t$ (and consequently also changing the signs of velocities or momenta), the form of these equations does not change. This also applies to quantum mechanical equations where one must take into account that time-reversal also requires the change $i = \sqrt{-1}$ into $-i$ in these equations (which, in position space, automatically changes $p$ into $-p$).

Taking a motion picture, e.g., of a classical HO, invariance under time-reversal implies that if a spectator watching the film closes the eyes in the same moment the film is stopped and ran backwards, on opening the eyes he/she will not be able to tell if the film is running forward or backward. The same would apply if one would show a film of the motion of the Gaussian wave packet (with constant width) that is usually given in the textbooks as solution of the SE for the HO.

However, if one would try the same with the very basic example of the Gaussian wave packet solution of the SE for the free motion, $V = 0$, the situation would change drastically. Due to the spreading of the wave packet width, it would immediately be possible to distinguish between the film running forward or backward in time; broadening width means forward, shrinking width means backward in time.

But how is this distinction possible although we know that the SE is invariant under time reversal? Obviously, the reason for this discrepancy is somehow connected with the temporal behaviour of the wave packet width, the typical quantum mechanical aspect of the system (as previously mentioned in section 3). Indeed, it is straightforward to show that for wave packets with time-dependent width, i.e., $\alpha(t) \neq \text{cont.}$, two imaginary contributions result when the kinetic energy operator is applied to the corresponding wave packet, $i\{ -\frac{\hbar}{2} \dot{\alpha} \alpha \frac{1}{m} \left( \dot{\eta} + \frac{\alpha}{\dot{\alpha}} \dot{\eta} \right) \dot{\eta} \}$. Shouldn’t these imaginary contributions cause problems with the physical interpretation of the kinetic energy? Obviously not since the mean value of the sum of these two terms vanishes.

How about the unitarity of the time-evolution and the normalizability of the wave packet? The second of the two terms definitely would cause an exponential decay of the wave packet, but, the first term, which is just the negative mean value of the second, compensates for this. This first term, however, is essentially the time-dependent normalization factor of the wave packet. A wave packet with normalization "constant" would decay; this decay, however, is "renormalized" by the purely time-dependent imaginary term $-i\frac{\hbar}{2} \frac{\dot{\alpha}}{\alpha}$.

Another consequence of the irreversible spreading of the wave packet should also briefly be mentioned. Adding a real term, like a potential, to the SE does not change the corresponding continuity equation (24) for the probability density $\rho = \Psi^* \Psi$. However, any additional imaginary term in the SE will also cause an additional term in the continuity equation (and vice versa, as will be shown below). In particular, the second of the above-mentioned terms occurs in the velocity field $v_- = \dot{\eta} + \frac{\alpha}{\dot{\alpha}} \dot{\eta} \dot{\alpha}$ and contains the contribution connected with the quantum aspects of the current (i.e., tunnelling, QHE, etc., see above). In contrast with the corresponding classical situation, for the quantum system now $\frac{\partial}{\partial x} v_- = \frac{\dot{\alpha}}{\alpha} \neq 0$ is valid for time-dependent wave packet width, just representing the first of the two above-mentioned imaginary terms. For classical Hamiltonian systems $\frac{\partial}{\partial x} v_- \text{ always vanishes!}$ (For further details, see also [21]).

As this rather trivial textbook example already shows, imaginary contributions to the Hamiltonian can still lead to physically acceptable results, providing their mean value vanishes, thus avoiding imaginary contributions to the energy of the system and guaranteeing normalizability of the corresponding wave function. They introduce an element of irreversibility without changing the energy of the system, particularly without dissipation.
4.2. Conventional treatment of dissipation in quantum mechanics

Let us now, however, just turn to this other aspect that is usually not part of the conventional formalism, namely, dissipation. In an attempt to describe dissipative systems within the conventional formalism of classical Hamiltonian mechanics or quantum mechanics, one usually applies the so-called system-plus-reservoir approach, i.e., system of interest and interacting environment together are considered a closed Hamiltonian system. With the help of additional assumptions and approximations (projection techniques, averaging procedures, Markovian limits, etc.), one finally tries to obtain irreversible equations of motion for the relevant subsystem that also properly describe the energy loss, e.g., due to the action of some velocity dependent friction force.

A conventional way of treating dissipative quantum systems uses the (reduced) density matrix/operator approach. In this context, frequently-used models are i) the (linear) coupling of the system to a bath of many HOs (often attributed to Caldeira and Leggett [22, 23]) which, however, can violate the positivity requirement of the density operator [24]; ii) and the application of the so-called Linblad generators [25] (proposed also by Kossakowski et al. [26] at the same time), whose specific form guarantees positivity but the particular choice of the operators occurring therein is usually “guided by intuition” [27] since no obvious physical guideline exists.

From a practical viewpoint, both approaches have the disadvantage that, in numerical calculations, the computational effort scales in the most favourable case at least with $N^2$ if $N$ is the size of the relevant Hilbert space. Furthermore, for a realistic description, usually a large number of environmental degrees of freedom must be considered, drastically increasing the computational effort and severely limiting the systems that can be treated. In contrast, methods based on pure state wave packets have the chance of scaling in the most favourable case linearly with $N$.

Therefore, in the following, modified SEs for “one-particle” wave functions/wave packets shall be considered for the description of dissipative quantum systems. There also seems to be no conceptual objection against the use of pure states for the description of the system of interest since, due to the interaction with the environment, the system itself does not decay, it only changes its dynamical properties like momentum and energy. Therefore, a description of the system alone in terms of a wave functions/wave packets should be acceptable. The degrees of freedom of the environment will not enter explicitly into our descriptions; only the effect of the environment on the system of interest will be taken into account by the modification of the SE.

4.3. Effective modified Schrödinger equations for the description of dissipative systems

Early attempts at finding classical Lagrangians/Hamiltonians that provide irreversible equations of motion including linear velocity dependent dissipative friction forces and allowing for usual canonical quantization to obtain the corresponding quantum mechanical versions are due to Caldirola [28] and Kanai [29]. They considered the Lagrangian

$$\hat{L}_{CK} = \frac{m}{2} \dot{x}^2 - V(x) e^{\gamma t}$$

yielding, via the Euler-Lagrange equation, the equation of motion

$$m \ddot{x} + m \gamma \dot{x} + \frac{\partial}{\partial x} V(x) = 0$$

including the desired damping force, with friction constant $\gamma$. In the usual way, the canonical momentum $\hat{p}$ is obtained as

$$\frac{\partial}{\partial \dot{x}} \hat{L}_{CK} = m \dot{x} e^{\gamma t} = p e^{\gamma t} = \hat{p}.$$
Note that this canonical momentum $\hat{p}$ is different from the physical kinetic momentum $p = m \dot{x}$ and the transition from the physical variables $(x, p)$ to the canonical variables $(\hat{x} = x, \hat{p} = e^{\gamma t} p)$ is given by a non-canonical transformation. It is straightforward to obtain the corresponding Hamiltonian $\hat{H}_{CK}(x, \hat{p})$,

$$\hat{H}_{CK} = e^{-\gamma t} \frac{\hat{p}^2}{2m} + e^{\gamma t} V(x) \quad (35)$$

which also supplies the correct equations of motion including the friction force.

Canonical quantization, i.e., replacing the canonical momentum $\hat{p}$ by the operator $\hat{p}_{op} = \frac{\hbar}{i} \frac{\partial}{\partial x}$ and applying the resulting (linear, but explicitly time-dependent) Hamiltonian $\hat{H}_{CK, op}$ to the canonical WF $\hat{\Psi}_{CK}(x, t)$, leads to the modified SE

$$i\hbar \frac{\partial}{\partial t} \hat{\Psi}_{CK}(x, t) = \hat{H}_{CK, op} \hat{\Psi}_{CK}(x, t) = \{ e^{-\gamma t} \left( -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \right) + e^{\gamma t} V(x) \} \hat{\Psi}_{CK}(x, t). \quad (36)$$

Also for this SE in the cases considered in this work, i.e., $V = 0$ and HO, it can be shown that exact analytic solutions in the form of Gaussian wave packets exist. The maximum follows the trajectory, determined now by Eq. (33), whereas the wave packets width can again be expressed in terms of a quantity $\alpha(t)$ as $\alpha_{CK} = \left( \frac{2m \langle \hat{x}^2(t) \rangle_C}{\hbar} \right)^{1/2}$, obeying the Ermakov-type equation

$$\ddot{\alpha}_{CK} + \gamma \dot{\alpha}_{CK} + \omega^2 \alpha_{CK} = e^{-2\gamma t} \frac{\alpha^3_{CK}}{\alpha^3_{CK}}. \quad (37)$$

Equation (33) together with (37) leads, also in the dissipative case, to an exact Ermakov invariant $[30]$ of the form

$$I_{CK} = \frac{1}{2} \left[ e^{2\gamma t} \left( \eta \alpha_{CK} - \eta \dot{\alpha}_{CK} \right)^2 + \left( \frac{\eta}{\alpha_{CK}} \right)^2 \right] = \text{const.} \quad (38)$$

However, apparently a problem arises when calculating the uncertainty product of position and physical momentum according to

$$U_{CK,phys} = \langle \hat{X}^2 \rangle_C < \hat{P}^2 >_C = \langle \hat{X}^2 \rangle_C < \hat{P}^2 >_C \rightarrow e^{-2\gamma t} \quad (39)$$

where the brackets $< ... >_C$ denote mean values and the subscript $CK$ indicates that they are calculated with the functions $\Psi_{CK}(x, t)$ that are solutions of Eq. (36). In relation (39) $p = \hat{p} e^{-\gamma t}$ has been applied to replace the operator for the physical momentum $p$ by the operator for the canonical momentum $\hat{p}$ where the uncertainty product of position and canonical momentum $\hat{p}$ always retains the minimum contribution $\hbar^2/4$. But relation (39) would lead to arbitrarily small values for $U_{CK,phys}$ (smaller than $\hbar^2/4$) and let it even finally vanish for increasing time $t$.

Therefore, this approach was criticised by several authors $[31, 32, 33]$ and considered inappropriate for the quantum mechanical description of dissipative systems. Nevertheless, others still applied this approach or tried to justify or eliminate $[34, 35]$ the problematic behaviour expressed by (39). The situation became even more puzzling after Yu and Sun $[36, 37]$ showed that the CK-Hamiltonian operator can be derived from the conventional system-plus-reservoir approach. A solution to this paradoxical situation will be given below (see also [38]).

Other approaches try to avoid the problems arising from non-canonical transformations in classical mechanics and start already on the quantum mechanical level, trying to modify the
Hamiltonian operator by adding effective dissipative terms. The guiding line for the choice of the friction terms is Ehrenfest’s theorem in the sense that the classical equation of motion including the damping term should, on an average, be valid, i.e.

\[
\frac{d}{dt} \langle p \rangle + \gamma \langle p \rangle + \langle \frac{\partial}{\partial x} V \rangle = 0
\]  

(40)

which corresponds to the classical equation (33).

Assuming the modified Hamiltonian to be of the form

\[
H_{NL} = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x) + W = H_L + W ,
\]

(41)

applying Heisenberg’s equation of motion and comparing it with Eq. (40) leads to

\[
\langle \frac{\partial}{\partial x} W \rangle = \gamma \langle p \rangle
\]

(42)

for the definition of \( W \). Obviously, this definition is not without ambiguity as it only contains the derivative of \( W \) and, moreover, only the mean value of this quantity. Therefore, a variety of different modified Hamiltonians of this kind exist in the literature, usually leading to nonlinear (NL) SEs, at least due to the occurrence of mean values \( \langle ... \rangle = \int_{-\infty}^{+\infty} \Psi^* ... \Psi dx \) in the definition of \( W \).

The most frequently quoted and applied one is Kostin’s NLSE \([39]\) with the friction term

\[
W_K = \frac{\gamma \hbar}{2i} \left( \ln \frac{\Psi_{NL}}{\Psi_{NL}^*} - \langle \ln \frac{\Psi_{NL}}{\Psi_{NL}^*} \rangle \right).
\]

(43)

However, this approach also has several shortcomings. It allows all the solutions of the undamped HO (which are real functions with \( \Psi = \Psi^* \)) to also be solutions of the problem including damping. Additionally, the solutions of the damped HO contain the undamped frequency \( \omega \) instead of the reduced frequency \( \Omega = (\omega^2 - \frac{\gamma^2}{4})^{1/2} \). These problems are related to a third one, namely, that \( W_K \) is a real quantity and could thus be considered an additional potential, therefore not contributing to the continuity equation for the density \( \varrho \).

Thus the inconsistency occurs in that the equation of motion for the mean values contains the time-symmetry-breaking friction term \( -\gamma \langle p \rangle \), whereas the equation for the corresponding density \( \varrho \) is still the reversible continuity equation, which is unphysical since the system under consideration should display irreversible dynamics on all levels of description.

In order to avoid the conflicting aspects connected with the above-mentioned “derivation” of the “friction potentials” and to guarantee an irreversible time-evolution for the density \( \varrho \) with corresponding (non-hermitian) imaginary contributions in the respective modified SE, starting point of our approach is the addition of a time-symmetry-breaking diffusion term to the continuity equation, thus turning it into an irreversible Fokker–Planck-type equation (FPE), (more precisely, a Smoluchowski equation),

\[
\frac{\partial}{\partial t} \varrho + \frac{\partial}{\partial x} \left( \varrho \, \nu_\text{--} \right) - D \frac{\partial^2}{\partial x^2} \varrho = 0.
\]

(44)

To avoid problems with particle/antiparticle mixing etc., this real FPE was supposed to be separable into a pair of c.c. equations containing only \( \Psi \) or \( \Psi^* \), respectively (using an ansatz by Madelung \([40]\) and Mrowka \([41]\)). Due to the diffusion term, separability in general is not possible but can be achieved if an additional separation condition,

\[
-D \frac{\partial^2}{\partial x^2} \frac{\varrho}{\varrho} = \gamma \langle \ln \varrho - \langle \ln \varrho \rangle \rangle
\]

(45)
with \( D = \) diffusion coefficient (possibly time-dependent), is fulfilled.

This leads to a NLSE with \textit{complex logarithmic} nonlinearity [42, 43, 44],

\[
i \hbar \frac{\partial}{\partial t} \Psi_{NL}(x,t) = \left\{ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x) \right\} \Psi_{NL}(x,t) + \gamma \hbar \left( \ln \Psi_{NL} - \langle \ln \Psi_{NL} \rangle \right) \Psi_{NL}(x,t) + \{H_L + \tilde{W}\} \Psi_{NL}(x,t) = H_{NL} \Psi_{NL}(x,t).
\]

Considering the complex nonlinearity \( \tilde{W} \) in terms of real and imaginary parts,

\[
\tilde{W} = \tilde{W}_R + i\tilde{W}_I = \frac{\gamma \hbar}{2i} \left( \ln \frac{\Psi_{NL}}{\Psi_{NL}^*} - \langle \ln \frac{\Psi_{NL}}{\Psi_{NL}^*} \rangle \right) + \frac{\gamma \hbar}{2i} \left( \ln \rho_{NL} - \langle \ln \rho_{NL} \rangle \right)
\]

it is obvious that the real part is identical to Kostins term and thus provides the dissipative friction term in the equation of motion for the mean values. On the other hand, the imaginary part, originating from the diffusion term in the FPE, introduces irreversibility \(^4\) but, due to the property of vanishing mean value, causes no problems with imaginary energy contributions and also guarantees normalizability of the WF. In addition, the imaginary part also solves Kostin’s other problems since the solutions of the undamped oscillator are no longer admitted and the oscillation frequency of the damped motion is now the correct reduced frequency \( \Omega = \left( \omega^2 - \frac{\gamma^2}{\omega} \right)^{1/2} \).

Although the mean value of the energy is still given by the mean values of kinetic and potential energies (as the mean value of \( \tilde{W} \) vanishes), this energy is no longer constant since it is calculated with the help of \( \Psi_{NL}(x,t) \), the solution of the NLSE. In particular, the classical part of the energy decays exactly as expected from the classical equation of motion proportional to kinetic energy \( T_d \), i.e., \( \frac{d}{dt} E_d = -2\gamma T_d \). The quantum mechanical contribution can display rather unexpected behaviour; for details see, e.g., [47].

For the cases considered in this paper \((V = 0 \text{ and } \text{HO})\) also Gaussian wave packets solutions of the form (46) exist for the log NLSE. Inserting this wave packet into the log NLSE (46) yields a pair of equations of motion for the maximum,

\[
\ddot{\eta} + \gamma \dot{\eta} + \omega^2 \eta = 0,
\]

and for the width, expressed in terms of the complex variable \( \mathcal{C} = \left( \frac{2\hbar}{m} \right) = \mathcal{R} + i \mathcal{I} \) as the complex Riccati equation

\[
\frac{d}{dt} \mathcal{C} + \gamma \mathcal{C} + \mathcal{C}^2 + \omega^2 = 0.
\]

Using the same definition for \( \mathcal{I} \) as in the conservative case, i.e., \( \mathcal{I} = \frac{1}{\alpha_{NL}} \), the real part \( \mathcal{R} \), obtained by inserting \( \mathcal{I} \) into the imaginary part of Eq. (49), is modified according to

\[
\mathcal{R} = \frac{\dot{\alpha}_{NL}}{\alpha_{NL}} - \frac{\gamma}{2}.
\]

\(^4\) The formal similarity between our nonlinear imaginary term and an approach by Beretta introducing irreversibility and thermodynamic aspects into quantum mechanics without necessarily dissipative effects should be pointed out. See also Ref.[45-46] and Berettas contribution to these proceedings.
Inserting $I$ and $R$ into the real part of (49) yields, again, an Ermakov equation for $\alpha(t)$ and thus for the wave packet width,

$$\ddot{\alpha}_{NL} + \left(\omega^2 - \frac{\gamma^2}{4}\right) \alpha_{NL} = \frac{1}{\alpha_{NL}^3}. \tag{51}$$

Also in the dissipative case where no conserved physical Hamiltonian exists that corresponds to the actual energy of the system, the pair of equations of motion for the wave packet maximum (corresponding to the classical trajectory) and the wave packet width (proportional to the position uncertainty) still provides an exact dynamical invariant of Ermakov type [48],

$$I_{NL} = \frac{1}{2} e^{\gamma t} \left[ \left( \dot{\eta} - \frac{\gamma}{2} \alpha_{NL} \right) \eta \right]^2 + \left( \frac{\eta}{\alpha_{NL}} \right)^2 = \text{const}. \tag{52}$$

Again, the complex nonlinear Riccati equation (49) can be linearized using the ansatz

$$C = \left( \frac{2\hbar}{m} y \right) = \frac{\dot{\lambda}}{\lambda} = \frac{\dot{\lambda}}{\lambda} - \frac{\gamma}{2} = \left( \frac{\dot{\alpha}}{\alpha} - \frac{\gamma}{2} \right) + i \varphi \tag{53}$$

with $\dot{\lambda} = \lambda e^{-\gamma t/2} = \alpha_{NL} e^{-\gamma t/2 + i\varphi}$, i.e., the radius of $\dot{\lambda}$ is exponentially shrinking, which is also expressed by the equation of motion for $\dot{\lambda}$,

$$\ddot{\lambda} + \gamma \dot{\lambda} + \omega^2 \lambda = 0, \tag{54}$$

which is just a complex Newtonian equation including the friction term.

Compared with the conservative HO, where $\lambda(t)$ moved on a circle with constant radius $\alpha = \alpha_0 = 1/\sqrt{\omega_0}$, for the damped HO, $\dot{\lambda}(t)$ spirals towards the origin with the radius decreasing exponentially. (Again, as in the conservative case, it can be shown [49] that now the imaginary part of $\dot{\lambda}(t)$ is directly proportional to the classical trajectory, i.e., $\dot{\lambda} = \frac{m}{\alpha_0 \omega_0} \eta(t)$.)

Expressed in terms of $\lambda$ (i.e., compared to $\dot{\lambda}$, in a system expanding exponentially), the linearized Riccati equation (49) takes the form

$$\ddot{\lambda} + \left(\omega^2 - \frac{\gamma^2}{4}\right) \lambda = 0, \tag{55}$$

i.e., like for an undamped HO with shifted frequency $\Omega = \left(\omega^2 - \frac{\gamma^2}{4}\right)^{1/2}$.

4.4. Nonunitary connection between the CK-Hamiltonian and the log NLSE

It can be shown that both approaches, the linear explicitly time-dependent one by CK and the NLSE with complex logarithmic term, are equivalent descriptions of the same physical situation. Only i) the CK-approach keeps the canonical Hamiltonian structure and the linearity of the quantized version, but looses the direct physical interpretation of the occurring quantities like operators and WFs; whereas ii) the log NLSE keeps the direct physical interpretation of all these quantities, but looses properties like the superposition principle (thus, solving actually en passant the problem of Schrödinger’s cat for a non-isolated system).

The price one must pay in order to link these two approaches is that of a non-unitary transformation, based on the definition of action according to Schrödinger, used in his original
derivation of the SE. In his first communication [50] Schrödinger starts from the Hamilton–Jacobi equation

$$\frac{\partial}{\partial t} S_C + H = 0$$

(56)

with the action function $S_C$ and the momentum $p_C = \frac{\hbar}{i} \frac{\partial}{\partial x} S_C$. He introduces the wave function $\Psi(x,t)$ through

$$S_C = \left(\frac{\hbar}{i}\right) \ln \Psi$$

(57)

(where in his original paper he omitted the imaginary unit $i = \sqrt{-1}$, which he corrected later on after he was forced to accept that $\Psi$ and thus also $S_C$ can be complex functions of space and time (therefore the subscript “c”). Via a variational ansatz Schrödinger finally arrived at the (linear) SE.

One can now reverse Schrödinger’s procedure, starting with the log NLSE and, using the definition (57) of $S_C$, arriving at

$$\left(\frac{\partial}{\partial t} + \gamma\right) S_C + H = -\gamma < S_C > .$$

(58)

This is, of course, as little rigorous as Schrödinger’s first attempt was; however, it follows his way of thinking of how to connect the classical Hamilton–Jacobi theory with a wave (mechanical) equation. The term $-\gamma < S_C >$ is necessary mainly for normalization purposes, as mentioned above, and shall be neglected for the moment. Multiplying the remaining lhs of Eq. (58) by $e^{\gamma t}$ and using the definitions

$$\hat{S}_C = e^{\gamma t} S_C \quad \text{and} \quad \hat{H} = e^{\gamma t} H$$

(59)

finally yields the transformed Hamilton–Jacobi equation

$$\frac{\partial}{\partial t} \hat{S}_C + \hat{H} = 0 .$$

(60)

From the definition of the action, it follows that the WF $\hat{\Psi}$ in the transformed (canonical) system is connected with the WF $\Psi_{NL}$ in the physical system via the non-unitary transformation

$$\ln \hat{\Psi} = e^{\gamma t} \ln \Psi_{NL} .$$

(61)

Consequently, the complex momenta in the two systems are connected via

$$\hat{p} = \frac{\hbar}{i} \frac{\partial}{\partial x} \ln \hat{\Psi} = e^{\gamma t} \frac{\hbar}{i} \frac{\partial}{\partial x} \ln \Psi_{NL} = e^{\gamma t} p_C ,$$

(62)

which is equivalent to the connection between canonical and kinetic momentum in the CK theory. The non-canonical connection between the classical variables $(x,p)$ and $(\hat{x},\hat{p})$ corresponds to the non-unitary transformation between $\Psi_{NL}$ and $\Psi$.

The transformed canonical Hamiltonian, the corresponding SE and the transformed WF are identical to those of the CK approach. This, however, explains the apparent violation of the uncertainty principle by the CK approach. The mean values of position and momentum uncertainty are first calculated with the operators corresponding to the canonical variables, $\hat{x}_{op} = x$ and $\hat{p}_{op} = \frac{\hbar}{i} \frac{\partial}{\partial x}$, using the canonical WF $\Psi_{CK}(x,t)$. The apparent violation occurs when only the canonical momentum operator $\hat{p}_{op}$ is changed to the physical momentum operator $p_{op}$ via $p_{op} = \hat{p}_{op} e^{-\gamma t}$, but then the mean values are still calculated, applying the canonical WF $\Psi_{CK}(x,t)$, instead of the physical WF $\Psi_{NL}(x,t)$, since, according to Eqs. (61) and (62),
a change of the momentum is also connected with a change of the definition of the action and thus a corresponding (non-unitary) change of the WF. Unfortunately, this is not so obvious at first sight since in the sets of physical and canonical variables, the position variable remains unchanged under the non-canonical transformation so both, the physical WF $\Psi_{NL}(x, t)$ and the canonical WF $\Psi_{CK}(x, t)$, depend on the same variables, however having a different physical meaning (and different analytical forms as functions of $x$ and $t$), in particular when mean values are calculated with them. Taking this into account, the violation of the uncertainty principle vanishes. That means, to be consistent, WFs and operators must be transformed simultaneously.

This particularly means that the canonical position uncertainty $<\hat{x}^2>_{CK}$ and the physical position uncertainty $<\hat{x}^2>_{NL}$ for the dissipative case, and hence $\alpha_{CK}$ and $\alpha_{NL}$, are not identical but, using the non-unitary transformation (61), are related via

$$\alpha_{CK} = e^{-\gamma t/2} \alpha_{NL}.$$  

Inserting this into Eq. (37) for $\alpha_{CK}$, turns it exactly into Eq.(51) for $\alpha_{NL}$ and, hence, the Ermakov invariant $I_{CK}$ is identical to $I_{NL}$ in Eq. (52),

$$I_{CK} = I_{NL}. \quad (64)$$

In an earlier work [51], it had been shown that the dissipative equation of motion (33) can also be obtained via the canonical formalism if a different, but also non-canonical, transformation, is involved; in this case leading (at least for the damped HO) to a canonical Hamiltonian that is a constant of motion. This transformation uses an expanding coordinate system according to

$$Q = e^{\gamma t/2} \hat{x}, \quad P = e^{\gamma t/2} \hat{p} + \frac{m}{2} \gamma e^{\gamma t/2} \hat{x}, \quad (65)$$

leading to the Hamiltonian

$$\hat{H}_{exp} = \frac{1}{2m} P^2 + \frac{m}{2} \left( \omega^2 - \frac{\gamma^2}{4} \right) Q^2 = \text{const.} \quad (66)$$

with the equation of motion

$$\ddot{Q} + \left( \omega^2 - \frac{\gamma^2}{4} \right) Q = 0 \quad (67)$$

which, after back transformation to the physical variables, yields Eq. (33). It should be mentioned that the variables $(Q, P)$ of this expanding system and the variables $(\hat{x}, \hat{p})$ of the CK approach are connected via a canonical transformation,

$$Q = e^{\gamma t/2} \hat{x}, \quad P = e^{-\gamma t/2} \hat{p} + \frac{m}{2} \gamma e^{\gamma t/2} \hat{x}. \quad (68)$$

Keeping in mind that the maximum of the wave packet solving the NLSE (46) obeys Eq. (48), which is identical to Eq. (33) for the classical trajectory, and applying the transformation (65), then Eq. (48) for $\eta(t)$ can be written in the form of Eq. (67). Except now, this equation and Eq. (51) for $\alpha_{NL}$ allow for rewriting the Ermakov invariant (52) in the form

$$I_{dissip} = \frac{1}{2} \left[ \left( \frac{Q}{\alpha_{NL}} - Q \right) \dot{\alpha}_{NL} \right]^2 + \left( \frac{Q}{\alpha_{NL}} \right)^2 = \text{const.}, \quad (69)$$

i.e., exactly in a form as in the conservative case (see. Eq. (6)).

It means, however, that this invariant is independent of $\omega$ and $\gamma$, i.e., it exists also for time-dependent parameters $\omega = \omega(t)$ and $\gamma = \gamma(t)$, in cases where definitely no conserved physical Hamiltonian function exists.
5. Conclusions
In the first part of this paper, it has been shown that the rule for finding the so-called Pythagorean triple, a rule that only involves integers and can therefore be considered as a kind of “quantization condition”, is also found in different contexts in modern physics. In particular, all systems that can be described by complex nonlinear Riccati equations exhibit this property, e.g., time-dependent and time-independent [52] SEs. As an example, we considered the time-dependent SE and found that the formal similarity between the above-mentioned rule and the time-evolution of the quantum uncertainties, related to probability currents, showed the possibility for a quantization of the dynamical quantum properties (like they are already experimentally observed), e.g., in the QHE.

Linearization of the Riccati equation allowed for a geometric interpretation of the quantized dynamical properties in terms of angular and radial changes. It has also been shown that particularly the radial changes are related to an element of irreversibility (already in conventional quantum mechanics,) connected to imaginary contributions to the SE and thus additional terms (compared to the classical situation) to the continuity equation. Radial changes of a similar kind were also found in connection with dissipation, an aspect not usually directly included in conventional quantum mechanics.

The complex Riccati equation, together with the classical Newtonian equation of the system, allowed for the definition of a constant of motion, also in cases where the classical Hamiltonian of the system did not have this property. This Ermakov invariant has (apart from a constant factor $\frac{1}{m}$) the dimension of an action and can be identified with the conserved angular momentum for the motion in a complex plane. In our example, the value of this angular momentum is $\hbar^2$, usually attributed to the non-classical quantity “spin”.

Quantization of currents connected to some quantized radial changes had already drawn our attention to the QHE. However, there is also a second element of quantization observable in the fractional QHE. The interpretation that this fraction might originate from two effects where each of them is quantized separately but only one in terms of $\hbar$, the other in terms of $\frac{e^2}{c}$, another quantum of action, the so-called quantum of least Coulombic action. The ratio of these two quanta actually yields Sommerfeld’s fine structure constant. As reason for the occurrence of a second quantum of action in nature, it was speculated that both might be incommensurable in a sense like radius and circumference of a circle. One action, probably $\hbar$, should then quantize phenomena where angular aspects are relevant, whereas the other action, $\frac{e^2}{c}$, should quantize radial properties. These of course are, so far, only assumptions and further studies are necessary to clarify this; however, work in this direction is in progress (see also the contribution of Reinisch in these proceedings).

In the second part, systems were considered where no conserved physical Hamiltonian exists, in particular dissipative systems with linear velocity dependent friction forces. It had been shown that different approaches to incorporate these effects into the formalism of classical Hamiltonian mechanics as well as into quantum mechanics exist and what problems can rise in this context. An approach in the form of a NLSE with complex logarithmic nonlinearity was found that did not exhibit the problematic aspects of some other attempts, but combined their positive aspects. The imaginary part introduced an element of irreversibility but still kept real energy values and normalizable WFs while the real part introduced the irreversible friction term into the equation of motion for the classical aspect (mean value of position) and caused dissipation of the energy. Both terms however are, in this case, not independent of each other due to the coupling of amplitude and phase of the WF. Also in this case, an exact invariant of Ermakov-type with the same properties as in the conservative case could be found for our log NLSE and be compared with a corresponding approach by Caldirola and Kanai. An equivalence of these two approaches can be established but involves non-canonical transformations in the classical case and non-unitary transformations in the quantum mechanical case. This non-unitary transformation of
the corresponding WFs is mainly a transformation of the (complex) action function, introduced by Schrödinger, when the description in terms of physical quantities obeying a NLSE shall be changed into a description in terms of canonical quantities obeying a linear SE. Again, in this case, the action (now complex) is the crucial quantity. Depending on the problem under investigation, one can choose the more convenient approach for the major calculations and still obtain the information contained in the other one via a non-unitary transformation. For an extension to a many-particle theory the linear CK-approach seems at first sight more appealing, but to obtain physically meaningful results, the necessary non-unitary transformation might in this case be more complicated. However, for a many-particle system where the corresponding wave function can be written as a product of one-particle wave functions, the logarithm allows for a convenient separation of the product into a sum, but terms that cannot be treated with a simple product ansatz might cause more problems. Work in this direction is in progress.

Although some answers to the questions posed in the introduction could be found, there are still sufficient unsolved problems, leaving plenty of room for further work and action!

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