Golden lattices.

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This paper is dedicated to Boris Venkov.

Abstract. Let \( \vartheta := \frac{-1 + \sqrt{5}}{2} \) be the golden ratio. A golden lattice is an even unimodular \( \mathbb{Z}[\vartheta] \)-lattice of which the Hilbert theta series is an extremal Hilbert modular form. We construct golden lattices from extremal even unimodular lattices and obtain families of dense modular lattices.

1. Introduction

1.1. Even unimodular \( \mathbb{R} \)-lattices. Let \( \mathbb{R} \) be the ring of integers in a real number field \( K \). Let \( \Lambda \) be a full \( \mathbb{R} \)-lattice in Euclidean \( n \)-space \( (K^n, Q) \), so \( \Lambda \) is a finitely generated \( \mathbb{R} \)-submodule of \( K^n \) that spans \( K^n \) over \( K \) and \( Q : K^n \to K \) is a totally positive definite quadratic form. The polar form of \( Q \) is the positive definite symmetric \( \mathbb{K} \)-bilinear form \( B \) defined by \( B(x, y) := Q(x + y) - Q(x) - Q(y) \). The lattice \( (\Lambda, Q) \) is called an even unimodular \( \mathbb{R} \)-lattice, if \( Q : \Lambda \to \mathbb{R} \) is an integral quadratic form such that

\[
\Lambda = \Lambda^\# := \{ x \in K^n \mid B(x, \lambda) \in \mathbb{R} \text{ for all } \lambda \in \Lambda \}.
\]

For small dimension and small fields such lattices have been classified in [8], [9], [10].

1.2. Trace lattices. Any \( \mathbb{R} \)-lattice \( (\Lambda, Q) \) and any totally positive \( \alpha \in K_+ \) gives rise to a positive definite \( \mathbb{Z} \)-lattice

\[
L_\alpha := (\Lambda, \tr_{K/\mathbb{Q}}(\alpha Q))
\]

of dimension \( n[K : \mathbb{Q}] \). \( L_\alpha \) will be called a trace lattice of \( (\Lambda, Q) \), since the quadratic form

\[
q := q_\alpha : L_\alpha \to \mathbb{Q}, x \mapsto \tr_{K/\mathbb{Q}}(\alpha Q(x))
\]

is obtained as a trace. In this way an \( \mathbb{R} \)-lattice defines a \( [K : \mathbb{Q}] \)-parametric family of positive definite \( \mathbb{Z} \)-lattices \( \{ L_\alpha \mid \alpha \in K_+ \} \) of which the most important invariants like minimum and determinant, and also the theta series may be read off from the corresponding invariants of the lattice \( (\Lambda, Q) \) and its Hilbert theta series. For instance the dual lattice

\[
L_\alpha^* := \{ x \in K^n \mid \tr_{K/\mathbb{Q}}(\alpha B(x, \lambda)) \in \mathbb{Z} \text{ for all } \lambda \in \Lambda \} = \alpha^{-1} R^* \Lambda^\#
\]

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where $R^* := \{ s \in K \mid \text{tr}_{K/Q}(sR) \subseteq \mathbb{Z} \}$ is the inverse different of $R$.

1.3. Extremal even unimodular lattices. In particular if $(\Lambda, Q)$ is an even unimodular $R$-lattice and $\alpha$ is a totally positive generator of $R^* = \alpha R$ then the trace lattice $L_\alpha$ is an even unimodular integral lattice of dimension $N = n[K : Q]$. It is well known that the minimum

$$\min(L, q) = \min\{ q(x) \mid 0 \neq x \in L \}$$

of an even unimodular $\mathbb{Z}$-lattice $L$ of dimension $N$ is bounded by $\min(L, q) \leq 1 + \left\lfloor \frac{N}{24} \right\rfloor$. Lattices that achieve equality are called extremal. If the dimension $N$ is a multiple of 24 then the extremal even unimodular lattices are densest known lattices. There are 4 such lattices known, the Leech lattice $\Lambda_{24}$ in dimension 24, three lattices $P_{48p}$, $P_{48q}$, and $P_{48n}$ of dimension 48 and one lattice $\Gamma_{72}$ of dimension 72.

1.4. Golden lattices. This article considers the situation where $R = \mathbb{Z}[\vartheta]$ and $\vartheta = \frac{-1 + \sqrt{5}}{2}$ is the golden ratio. Then $K$ is a real quadratic number field of minimal possible discriminant 5, $R$ is a principal ideal domain and

$$R^* = \eta^{-1} R \text{ where } \eta = 3 + \vartheta = \frac{5 + \sqrt{5}}{2}$$

is a totally positive generator of the prime ideal of norm 5. The extremal even unimodular lattices $\Lambda_{24}$, $P_{48n}$ and $\Gamma_{72}$ may be obtained as trace lattices $L_{\eta^{-1}}$ of even unimodular $\mathbb{Z}[\vartheta]$-lattices. This structure allows to construct interesting families of dense modular lattices (Theorem 2.9).

2. Hilbert theta series of golden lattices

2.1. Symmetric Hilbert modular forms. Let $R := \mathbb{Z}[\vartheta]$ be the ring of integers in the real quadratic number field $K := \mathbb{Q}[\sqrt{5}]$. Then the Hilbert theta series (Definition 2.1) of an $n$-dimensional even unimodular $R$-lattice $(\Lambda, Q)$ is a Hilbert modular form of weight $n/2$ (see [5, Section 5.7]). If $(\Lambda, Q)$ is Galois invariant, then so is its theta series and hence this Hilbert modular form is symmetric. Hilbert modular forms for $R$ are holomorphic functions on the direct product $\mathbb{H}_K := \mathbb{H} \times \mathbb{H}$ of 2 copies of the upper half plane

$$\mathbb{H} := \{ z \in \mathbb{C} \mid \Im(z) > 0 \}.$$

If $\sigma_1, \sigma_2$ denote the two embeddings of $K$ into $\mathbb{R} \subseteq \mathbb{C}$ then $\text{SL}_2(R)$ acts on $\mathbb{H}_K$ by

$$(z_1, z_2) \begin{pmatrix} a & b \\ c & d \end{pmatrix} := \begin{pmatrix} \sigma_1(a) z_1 + \sigma_1(b) & \sigma_2(a) z_2 + \sigma_2(b) \\ \sigma_1(c) z_1 + \sigma_1(d) & \sigma_2(c) z_2 + \sigma_2(d) \end{pmatrix}$$

and the Galois automorphism just interchanges the two copies of $\mathbb{H}$. The ring of symmetric Hilbert modular forms for the group $\text{SL}_2(R)$ is a polynomial ring

$$\mathcal{H} := \mathbb{C}[A_2, B_6, C_{10}]$$

where the explicit generators of weight 2, 6, and 10 have been obtained in [6] and can be found in [5]. We denote by

$$\mathcal{H}_w := \{ f \in \mathcal{H} \mid \text{weight of } f = w \}$$

the space of symmetric Hilbert modular forms of weight $w$. 

Theorem 2.9.
Definition 2.1. Let $(\Lambda, Q)$ be an even $R$-lattice. Then the Hilbert theta series of $\Lambda$ is

$$\Theta(\Lambda, Q) := \sum_{\lambda \in \Lambda} \exp(2\pi i \text{tr}_{K/Q}(zQ(\lambda))) = 1 + \sum_{X \in R_+} A_X \exp(2\pi i \text{tr}_{K/Q}(zX))$$

where $A_X := |\{\lambda \in \Lambda \mid Q(\lambda) = X\}|$ and $\text{tr}_{K/Q}(zX) = z_1\sigma_1(X) + z_2\sigma_2(X)$ for $z = (z_1, z_2) \in \mathbb{H}_K$.

Since $(1, \eta^{-1})$ is a $\mathbb{Q}$-basis of $K$ the trace of $Q(\lambda)$ and $\eta^{-1}Q(\lambda)$ uniquely determine the value $Q(\lambda) \in K$. So $\Theta(\Lambda, Q)$ is determined by the $(q_0, q_1)$-expansion

$$\Theta(\Lambda, Q) := \sum_{\lambda \in \Lambda} q_0^{\text{tr}_{K/Q}(\eta^{-1}Q(\lambda))} q_1^{\text{tr}_{K/Q}(Q(\lambda))} \in \mathbb{C}[\![q_0, q_1]\!]$$

which is very convenient for computations. Replacing $q_1$ by 1 yields the usual theta series of the trace lattice $L_{\eta^{-1}}$ and substituting $q_0$ by 1 gives the theta series of $L_1$. We obtain $A_2(q_0, 1) = \Theta(E_8)$ the Eisenstein series of weight 4, $B_6(q_0, 1) = \Delta$, the cusp form of weight 12, and $C_{10}(q_0, 1) = 0$. In particular replacing $q_1$ by 1 yields a surjective ring homomorphism onto the ring of elliptic modular forms of weight divisible by 4 for the full modular group $SL_2(\mathbb{Z})$.

2.2. Extremal Hilbert modular forms.

Definition 2.2. Define a valuation on the field of fractions of $\mathbb{C}[\![q_0, q_1]\!]$ by

$$\nu : \mathbb{C}[\!(q_0, q_1)\!] \setminus \{0\} \rightarrow \mathbb{Z} \times \mathbb{Z}, \nu\left(\sum_{(i,j) = (s,t)}^{(\infty, \infty)} A_{(i,j)} q_0^s q_1^t\right) := \min\{(i, j) \mid A_{(i,j)} \neq 0\}$$

where the total ordering on $\mathbb{Z} \times \mathbb{Z}$ is lexicographic, so

$$(s, t) \leq (s', t')$$

if $s < s'$ or $s = s'$ and $t \leq t'$.

This gives rise to a valuation on the ring of Hilbert modular forms via the $(q_0, q_1)$-expansion. A symmetric Hilbert modular form $f \in \mathcal{H}_w$ is called an extremal Hilbert modular form of weight $w$, if

$$\nu(f - 1) \geq \nu(f' - 1) \text{ for all } f' \in \mathcal{H}_w.$$

One computes

$$\nu(A_2 - 1) = (1, 2), \ \nu(B_6) = (1, 2), \ \nu(C_{10}) = (2, 4), \ \nu(X_{12}) = (2, 5)$$

where $X_{12} = \frac{1}{4}(A_2C_{10} - B_6^2)$. The valuations of the first few extremal Hilbert modular forms are given in the table in Example 2.6 below.

2.3. Golden lattices.

Definition 2.3. An even unimodular $R$-lattice $(\Lambda, Q)$ is called a golden lattice, if its Hilbert theta series is an extremal symmetric Hilbert modular form.

Proposition 2.4. Let $(L, q)$ be an even unimodular lattice of dimension $N$ and $\vartheta \in \text{End}_{\mathbb{Z}}(L)$ be a symmetric endomorphism of $L$ with minimal polynomial $X^2 + X - 1$. Then $L$ is a $\mathbb{Z}[\vartheta]$-lattice $\Lambda$ and $L = L_{\eta^{-1}}$ for the even unimodular $\mathbb{Z}[\vartheta]$-lattice $(\Lambda, Q)$ with quadratic form

$$Q : \Lambda \rightarrow R, Q(\lambda) := \frac{1}{2}(q(\lambda) + q(\vartheta\lambda)) + \frac{1}{2}(q(\vartheta\lambda) - q(\lambda))\sqrt{5}.$$
Assume that there is some automorphism $\sigma \in \text{Aut}(L, q)$ such that $\sigma \vartheta = (-1 - \vartheta)\sigma$. Then for $N = 8, 16, 24, 32, 48, 56, 72$ the lattice $(\Lambda, Q)$ is a golden lattice, if and only if $(L, q)$ is an extremal even unimodular lattice.

**Proof.** Clearly any such endomorphism $\vartheta$ defines a $\mathbb{Z}[\vartheta]$-structure on the $\mathbb{Z}$-lattice $L$. Since $\vartheta$ is a symmetric endomorphism, the form $q$ is a trace form, $q(\lambda) = \text{tr}_{K/\mathbb{Q}}(\eta^{-1}Q(\lambda))$ for some $K$-valued quadratic form $Q$. If $Q(\lambda) = a + b\sqrt{5}$, then

$$q(\lambda) = \text{tr}_{K/\mathbb{Q}}(\eta^{-1}Q(\lambda)) = a - b$$

and hence

$$Q(\lambda) = \frac{1}{2}(q(\lambda) + q(\vartheta\lambda)) + \frac{1}{2}(q(\vartheta\lambda) - q(\lambda))\sqrt{5}.$$  

Clearly if $q(\lambda) \in \mathbb{Z}$ and $q(\vartheta\lambda) \in \mathbb{Z}$ then also $Q(\lambda) \in R$, therefore $(\Lambda, Q)$ is even. By Equation (1.1) the lattice $(\Lambda, Q)$ is unimodular.

The Galois invariance of $\Theta(\Lambda, Q)$ follows from the fact that $Q(\vartheta\sigma(\lambda)) = Q(\lambda)$ for all $\lambda \in \Lambda$.

To see this write $Q(\lambda) = a + b\sqrt{5}$ and $Q(\sigma(\lambda)) = a' + b'\sqrt{5}$, so

$$Q(\vartheta\sigma(\lambda)) = \vartheta^2Q(\sigma(\lambda)) = \frac{3a' - 5b'}{2} + \frac{3b' - a'}{2}\sqrt{5}.$$  

Then

$$\text{tr}_{K/\mathbb{Q}}(\eta^{-1}Q(\lambda)) = \text{tr}_{K/\mathbb{Q}}(\eta^{-1}Q(\sigma(\lambda))) \quad \text{yields} \quad a - b = a' - b'$$

$$\text{tr}_{K/\mathbb{Q}}(\eta^{-1}Q(\vartheta\lambda)) = \text{tr}_{K/\mathbb{Q}}(\eta^{-1}Q(\vartheta\sigma(\lambda))) \iff 2a - 4b = a' + b'$$

so in total $a' = \frac{3a - 5b}{2}, b' = \frac{a - 3b}{2}$ which gives

$$Q(\sigma(\lambda)) = \vartheta^2Q(\lambda)$$

for all $\lambda \in \Lambda$.

Therefore the mapping $\lambda \mapsto \vartheta\sigma(\lambda)$ gives a bijection between

$$\{\lambda \in \Lambda \mid Q(\lambda) = \alpha\} \text{ and } \{\lambda \in \Lambda \mid Q(\lambda) = \pi\}$$

so the Hilbert theta series of $(\Lambda, Q)$ is symmetric.

The last statement follows from explicit computations in the ring of Hilbert modular forms. These show that for weight 2, 4, 6, 8, 12, 14, and 18 the condition that $f(q_0, 1)$ be an extremal elliptic modular form and that $f(q_0, q_1)$ has non negative coefficients imply that $f$ is an extremal Hilbert modular form. The opposite direction follows from the table in Example 2.6.

**2.4. Associated modular lattices.** Generalising unimodular lattices Quebbemann [16] introduced the notion of $p$-modular lattices.

**Definition 2.5.** An even $\mathbb{Z}$-lattice $L$ in euclidean space is called $p$-modular, if there is a similarity $\sigma$ of norm $N$ (so $(\sigma(v), \sigma(w)) = p(v, w)$ for all $v, w \in L$) such that $\sigma(L^*) = L$.

If $p$ is one of the 6 primes for which $p + 1$ divides 24, then the theta series of $p$-modular lattices generate a polynomial ring with 2 generators from which one obtains a similar notion of extremality as for unimodular lattices: Let $k := \frac{24}{p+1}$. Then any $p$-modular lattice $L$ of dimension $N$ satisfies $\text{min}(L) \leq 1 + \lfloor \frac{N}{2k} \rfloor$. $p$-modular lattices achieving this bound are called extremal.

**Example 2.6.** For the first few weights the extremal Hilbert modular forms $f$ turn out to be unique and start with non-negative integral coefficients. The valuation $\nu(f - 1) = (s, t)$ can be read off from the following table. Any golden
lattice Λ with theta series \( f \) defines an even unimodular \( \mathbb{Z} \)-lattice \( L_{\eta^{-1}} \) of minimum \( s \) and a 5-modular lattice \( L_1 \) of minimum \( t \). The Hilbert modular form \( f \) also gives us information about the minimal vectors \( \text{Min}(L, q) := \{ v \in L \mid q(v) = \min(L) \} \). The kissing number \( s_{\eta^{-1}} = |\text{Min}(L_{\eta^{-1}})| \) and \( s_1 = |\text{Min}(L_1)| \) of these two lattices can be read off from column 3 and 4 of the table. That all minimal vectors of \( L_1 \) are also minimal vectors of \( L_{\eta^{-1}} \) is indicated by a + in the last column. A – means that only half of the minimal vectors of \( L_1 \) are also contained in \( \text{Min}(L_{\eta^{-1}}) \).

| Weight | \( \nu(f - 1) \) | \( s_{\eta^{-1}} \) | \( s_1 \) | \( \subset \) |
|--------|-----------------|-----------------|-------|---|
| 2      | (1, 2)          | 240             | 120   | + |
| 4      | (1, 2)          | 480             | 240   | + |
| 6      | (2, 4)          | 196560          | 37800 | + |
| 8      | (2, 4)          | 146880          | 21600 | + |
| 10     | (2, 5)          | 39600           | 79200 | - |
| 12     | (3, 6)          | 52416000        | 26208000  | + |
| 14     | (3, 6)          | 155904000       | 537600 | + |
| 16     | (3, 7)          | 2611200        | 2611200 | - |
| 18     | (4, 8)          | 6218175600      | 75411000 | + |
| 20     | (4, 9)          | 1250172000      | 609840000 | - |
| 24     | (5, 10)         | 56586632880     | 1655821440 | + |
| 30     | (6, 13)         | 45792819072000  | 3217294080000 | - |

Note that [14] Proposition 3.3 applied to an even unimodular \( R \)-lattice \((\Lambda, Q)\) gives that

\[
\frac{5}{2} \min(L_{\eta^{-1}}) \geq \min(L_1) \geq 2 \min(L_{\eta^{-1}}).
\]

The argument is that \( \eta = 1 + \vartheta^2 \) and \( 5 \eta^{-1} = 1 + \vartheta^2 \) so for all \( \lambda \in \Lambda \)

\[
Q(\lambda) = (1 + \vartheta^2)\eta^{-1}Q(\lambda) = \eta^{-1}Q(\lambda) + \eta^{-1}Q(\vartheta \lambda) \quad \text{and} \quad 5 \eta^{-1}Q(\lambda) = (1 + \vartheta^2)Q(\lambda) = Q(\lambda) + Q(\vartheta \lambda)
\]

Immitating the proof of this proposition we find the following bound.

**Proposition 2.7.** Let \( f \) be a symmetric Hilbert modular form of weight \( w \), \( s := 1 + \left\lfloor \frac{w}{2} \right\rfloor \) and \( t := \left\lfloor \frac{w}{2} \right\rfloor \). Then \( \nu(f - 1) \leq (s, t) \). More precisely put \( \nu(f - 1) = (s', t') \). Then \( s' \leq s \) and \( \left\lfloor \frac{s'}{2} \right\rfloor \geq t' \geq 2s' \).

**Proof.** Let \( \nu(f - 1) = (s', t') \). Since \( f(q_0, 1) \) is an elliptic modular form of weight \( 2w \) we get that \( s' \leq s \). To obtain the other two inequalities write

\[
f = 1 + \sum_{X \in R^+} A_X q_0^{\text{tr}(\eta^{-1} X)} q_1^{\text{tr}(X)}.
\]

Then \( f \) is invariant under the transformation by \( \text{diag}(u, u^{-1}) \in \text{SL}_2(R) \) for any \( u \in R^* \). This shows that

\[
A_X = A_{\vartheta^2 X} = A_{\vartheta X}.
\]

To see that \( t' \geq 2s' \) choose some totally positive \( X = a + b \sqrt{5} \in R \) such that \( t' = 2a \) and \( A_X = A_{\vartheta X} \neq 0 \). Then

\[
a - b = \text{tr}(\eta^{-1} X) \geq s' \quad \text{and} \quad a + b = \text{tr}(\eta^{-1} \vartheta^2 X) \geq s'.
\]
and so \( t' = 2a = (a-b) + (a+b) \geq 2s' \). To see that \( t' \leq \frac{5t}{2} \) we let \( X := a+b\sqrt{5} \in R \) be a totally positive element with \( \text{tr}_{K/Q}(\eta^{-1}X) = a-b = s' \) and \( A_X \neq 0 \). Then
\[
2a = \text{tr}_{K/Q}(X) \geq t' \quad \text{and} \quad 3a - 5b = \text{tr}_{K/Q}(\vartheta^2X) \geq t'
\]
and hence \( 5s' = 5a - 5b \geq 2t' \).

The minimum of an extremal \( \text{5-modular lattice} \) is \( 1 + \left\lfloor \frac{N}{8} \right\rfloor \) an for large \( N \) this is strictly bigger than \( \frac{5}{2}(1 + \left\lfloor \frac{N}{24} \right\rfloor) \) which yields:

**Corollary 2.8.** Let \( L \) be a golden lattice of dimension \( N \). Then \( L_1 \) is an extremal \( \text{5-modular lattice} \) if and only if \( N = 8 \) or \( N = 24 \).

Any golden lattice defines a family of modular lattices:

**Theorem 2.9.** Let \((\Lambda, Q)\) a golden lattice of dimension \( n \) and \((s, t) := \nu(\Theta(\Lambda, Q) - 1)\). For \( a \in \mathbb{N}_0 \) the trace lattice \( L_{1+an} \) is an \( (a^2 + 5a + 5) \)-modular lattice of minimum \( \geq t + as \).

**Proof.** Recall that \( L_{1+an} = (\Lambda, \text{tr}_{K/Q}(1 + an^{-1})Q) \). Since \( \text{tr}_{K/Q}(Q) \) and \( \text{tr}_{K/Q}(Q) \) are positive definite and take integral values on \( \Lambda \), the lattice \( L_{1+an} \) is even and positive definite for any \( a \in \mathbb{Z}_{\geq 0} \). Clearly \( \min(L_{1+an}) \geq \min(L_1) + a \min(L_{n^{-1}}) = t + as \). Now \((\Lambda, Q)\) is unimodular, so Equation (1.1) yields that the \( \mathbb{Z} \)-dual of \( L_{1+an} \) is
\[
L_{1+an}^* = \eta^{-1}(1 + an^{-1})^{-1} \Lambda^\# = \eta^{-1}(1 + an^{-1})^{-1} L_{1+an}.
\]

The element \( \eta(1 + an^{-1}) \in K \) hence defines a similarity between \( L_{1+an}^* \) and \( L_{1+an} \) or norm \( N(\eta + a) = a^2 + 5a + 5 \).

### 3. Examples

All even unimodular \( \mathbb{Z}[\vartheta] \)-lattices are classified in dimension 4, 8, and 12 \([7], [8]\). In each of these dimensions there is a unique golden lattice. For the other dimensions 16 to 36 we inspect automorphisms of some known extremal even unimodular lattice to find a golden lattice with the method from Proposition 2.4. The tensor symbol \( \otimes \) denotes the Kronecker product of matrix groups, which is group theoretically the central product.

**3.1. Dimension 4.** Already Maass \([12]\) has shown that there is a unique golden lattice, \( F_4 \), of dimension 4. It can be constructed from the maximal order \( \mathcal{M} \) in the definite quaternion algebra with center \( K \) that is only ramified at the two infinite places. Its \( R \)-automorphism group is
\[
\text{Aut}_R(F_4) \cong (\text{SL}_2(5) \otimes \text{SL}_2(5)) : 2.
\]

**3.2. Dimension 8.** Maass also showed that the only 8-dimensional even unimodular \( R \)-lattice is the orthogonal sum \( F_4 \perp F_4 \).

**3.3. Dimension 12.** The golden lattices of dimension 12 are exactly the \( \mathbb{Z}[\vartheta] \)-structures of the unique extremal even unimodular \( \mathbb{Z} \)-lattice of dimension 24, the Leech lattice. \([8]\) shows that there is a unique such golden lattice \( \Lambda \). Its automorphism group is
\[
\text{Aut}_R(\Lambda) \cong 2.J_2 \otimes \text{SL}_2(5).
\]
3.4. Dimension 16. By [7] Table (1.2)] the mass of all even unimodular \( R \)-lattices of rank 16 is \( > 10^6 \), so a complete classification seems to be out of reach. Here it would be desirable to have a mass formula for the lattices without roots in analogy to the classical case of even unimodular \( \mathbb{Z} \)-lattices [11].

There are several known extremal even unimodular lattices in dimension 32 which have a fairly big automorphism group. In particular there are two golden lattices \( \Lambda_1 \) and \( \Lambda_2 \) that have are \( \mathcal{M} \)-lattices for \( \mathcal{M} \) as in [3.1 (see [3] and [14] Table 2] for the automorphism group). The automorphism groups \( G_i = \text{Aut}_{\mathbb{Z}[\vartheta]}(\Lambda_i) \) are

\[
G_1 \cong (\otimes^4 \text{SL}_2(5)) : S_4, \quad G_2 \cong \text{SL}_2(5) \otimes 2^{1+6}_-, O^-_6(2).
\]

3.5. Dimension 20. No golden lattice of dimension 20 is known. It is an interesting problem to construct such a golden lattice or to prove its non-existence, since this is the smallest dimension for which \( \min(L_1) > 2 \min(L_{n-1}) \).

From the extremal even unimodular lattice \( L \) in [2] with automorphism group \(( U_5(2) \times 2^{1+4}.\text{Alt}_5).2 \) and an automorphism \( z \in L \) of order 5 with irreducible minimal polynomial one obtains a Galois invariant \( \mathbb{Z}[z + z^{-1}]-lattice (\Lambda, Q) \) for which the 5-modular trace lattice \( L_1 \) has 19800 minimal vectors of norm 4 and no vectors of norm 5.

3.6. Dimension 24. The extremal even unimodular lattice \( P_{48n} \) constructed in [13] Theorem 5.3] has an obvious structure as a \( \mathbb{Z}[\vartheta]-lattice \) with automorphism group \( \text{SL}_2(13) \otimes \text{SL}_2(5) \). This provides one example of a golden lattice of \( \mathbb{Z}[\vartheta]- \)dimension 24.

3.7. Dimension 32. No golden lattice of dimension 32 is known. There is one extremal even unimodular lattice \( L \) of dimension 64 constructed in [13] p. 496] of which the extremality is proven in [14] Proposition 4.2]. The subgroup \( G := \text{SL}_2(17) \otimes \text{SL}_2(5) \) of the automorphism group of \( L \) has endomorphism ring \( \mathbb{Z}[(1 + \sqrt{17})/2, \vartheta] \) and with Proposition [2.4] the structure over \( \mathbb{Z}[\vartheta] \) yields a lattice \((\Lambda, Q) \) whose Hilbert theta series is symmetric, \( L = L_{n-1} \) is extremal, but the minimum of the 5-modular lattice \( L_1 \) is only 6 (and not 7, as required for a golden lattice).

3.8. Dimension 36. In [15] an extremal even unimodular lattice \( \Gamma_{72} \) of dimension 72 is constructed. Any Galois invariant \( \mathbb{Z}[\vartheta]-structure \) on \( \Gamma_{72} \) will give rise to a golden lattice of rank 36. The lattice \( \Gamma_{72} \) can be obtained has a Hermitian tensor product \( 4 \)

\[
\Gamma_{72} = P \otimes_{\mathbb{Z}[(1 + \sqrt{-3})/2]} P
\]

where \( P \) is the \( \mathbb{Z}[(1 + \sqrt{-3})/2]-structure \) of the Leech lattice with automorphism group \( \text{SL}_2(25) \). The group \( \text{SL}_2(25) \leq \text{GL}_{24}(\mathbb{Z}) \) contains an element \( \zeta \) of order 5 with irreducible minimal polynomial. Put \( \vartheta := \zeta + \zeta^{-1} \). Then \( P \) is a \( \mathbb{Z}[\vartheta, \frac{1 + \sqrt{-3}}{2}]-lattice \) with automorphism group \( U := (C_5 \times C_5) : C_4 \). This yields a \( \mathbb{Z}[\vartheta]-structure \) on the Hermitian tensor product \( \Gamma_{72} \) which defines a golden lattice of rank 36 whose automorphism group contains \( U \times P \text{SL}_2(7) \).

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