DIRICHLET SPECTRUM AND GREEN FUNCTION

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Abstract. In the first part of this article we obtain an identity relating the radial spectrum of rotationally invariant geodesic balls and an isoperimetric quotient \( \sum 1/\lambda_i^{\text{rad}} = \int V(s)/S(s)ds \), Thms 2.5 & 2.7. We also obtain upper and lower estimates for the series \( \sum \lambda_i^{-2}(\Omega) \) where \( \Omega \) is an extrinsic ball of a proper minimal surface of \( \mathbb{R}^3 \), Thm 3.1. In the second part we show that the first eigenvalue of bounded domains is given by iteration of the Green operator and taking the limit, \( \lambda_1(\Omega) = \lim_{k \to \infty} \|G_k(f)\|_2/\|G_{k+1}(f)\|_2 \) for any function \( f > 0 \), Thm 4.1. In the third part we obtain explicitly the \( L^1(\Omega, \mu) \)-momentum spectrum of a bounded domain \( \Omega \) in terms of its Green operator, Thm 5.2. In particular, we obtain the first eigenvalue of a weighted bounded domain in terms of the \( L^1(\Omega, \mu) \)-momentum spectrum, extending the work of Hurtado-Markvorsen-Palmer on the first eigenvalue of rotationally invariant balls, [25].

1. Introduction

Let \( \Omega \subseteq M \) be a subset of a Riemannian manifold \( (M, ds^2) \). The Laplace operator \( \Delta = \text{div} \circ \text{grad} : C_0^\infty(\Omega) \to C_0^\infty(\Omega) \), acting on the space of smooth functions with compact support in \( \Omega \), is symmetric, negative-definite and densely defined in \( L^2(\Omega) \), however it is not self-adjoint. Considering the Sobolev spaces \( W^1_0(\Omega) \) and \( W^2_0(\Omega) \), the former being the closure of \( C_0^\infty(\Omega) \) with respect to the norm
\[
\|u\|_{W^1(\Omega)}^2 = \int_\Omega u^2 d\nu + \int_\Omega |\text{grad} u|^2 d\nu
\]
while the latter consists of those functions \( u \in W^1_0(\Omega) \) whose weak Laplacian \( \Delta u \) exists and belongs to \( L^2(\Omega) \), i.e.
\[
W^2_0(\Omega) = \{ u \in W^1_0(\Omega) : \Delta u \in L^2(\Omega) \}
\]
we have that \( \mathcal{L} = -\Delta|_{W^2_0}(\Omega) \) is a non-negative self-adjoint extension of \( -\Delta|_{C_0^\infty(\Omega)} \), see [22]. Let us recall that the spectrum of \( \mathcal{L} \), denoted by \( \sigma(\Omega) \), is the set of all \( \lambda \in [0, \infty) \) for which \( \mathcal{L} - \lambda I \) is not injective or the inverse operator \( (\mathcal{L} - \lambda I)^{-1} \) is unbounded, see [10]. The set of all \( \lambda \) for which \( (\mathcal{L} - \lambda I) \) is not injective, (eigenvalues) is called point spectrum \( \sigma_p(\Omega) \). Each eigenvalue \( \lambda \in \sigma_p(\Omega) \) is associated to a vector

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space \( V_\lambda = \{ u \in L^2(\Omega) : \triangle u + \lambda u = 0 \} \). The set of all isolated eigenvalues of finite multiplicity, i.e., those \( \lambda \in \sigma_p(\Omega) \) for which there exists \( \epsilon > 0 \) such that \((\lambda - \epsilon, \lambda + \epsilon) \cap \sigma(\Omega) = \{ \lambda \} \) and \( \dim(V_\lambda) < \infty \) is called the discrete spectrum and it is denoted by \( \sigma_d(\Omega) \). The complement of the discrete spectrum is the essential spectrum, \( \sigma_{ess}(\Omega) = \sigma(M) \setminus \sigma_d(\Omega) \). When \( \Omega \) is bounded with smooth boundary \( \partial \Omega \), (possibly empty), then the spectrum of \( L \) is discrete, i.e. a sequence of non-negative real numbers

\[
0 \leq \lambda_1(\Omega) < \lambda_2(\Omega) < \cdots \to \infty,
\]

where each eigenvalue \( \lambda_k(\Omega) \) is associated to a finite dimensional vector subspace \( V_k = \{ \phi : \triangle \phi + \lambda_k \phi = 0 \} \) such that \( L^2(\Omega) = \bigoplus_{k=1}^{\infty} V_k \), where \( \phi \in C^\infty(\Omega) \cap C^0(\overline{\Omega}) \) and \( \phi|\partial \Omega = 0 \). The eigenvalues \( \lambda_k \) are sometimes called “eigenvalues of \( \Omega \)”.

A classical and important problem in Riemannian geometry is the study of the relations between the spectrum \( \sigma(\Omega) \) and the geometry of \( \Omega \), see [7], [8]. In full generality, this is a difficult problem. A reasonable problem is, the study of the spectrum of rotationally invariant geodesic balls, i.e. balls with metrics invariant by rotations around the center. The spectrum \( \sigma(B(o, r)) \) of a rotationally invariant geodesic ball \( B(o, r) \) can be decomposed as a union of spectra \( \sigma^l(B(o, r)) \) of a family of operators \( L^l \) acting on smooth functions on \([0, r] \) indexed by the eigenvalues of the sphere \( \nu_l = l(l + n - 2), l = 0, 1, \ldots \), this is, \( \sigma(B(o, r)) = \cup_{l=0}^{\infty} \sigma^l(B(o, r)), \) see [13, p. 41], [15] Chapters 7 & 8. Our first results concern each spectrum \( \sigma^l(B(o, r)) \) separately. More precisely, Theorem [2.5] & [2.7] states that

\[
\sum_{\lambda \in \sigma^l(B(o, r))} \lambda^{-1} = \int_0^r V(s)/S(s) ds,
\]

where \( V(s) = \text{vol}(B(o, s)) \) and \( S(s) = \text{vol}(\partial B(o, s)) \). And if \( B(o, r) \subset \mathbb{R}^n \) then

\[
\sum_{\lambda \in \sigma^l(B(o, r))} \lambda^{-1} = c(l, n) \cdot \int_0^r V(s)/S(s) ds,
\]

\( 0 < c(n, l) < 1, \) for \( l = 1, 2 \ldots \) We also observe that a lemma due to Cheng-Li-Yau [14, Lemma 7] implies that if \( \dim(V_k) \geq 2 \) then the origin of \( B(o, r) \), belongs to the nodal set of every \( \phi \in V_k \), see Theorem [2.2]. To close this part of the article regarding the spectrum of rotationally invariant balls, i.e., Section [2], we construct examples of 4-dimensional non-rotationally invariant geodesic balls \( B(o, r) \) with the same spectrum \( \sigma^0(B(o, r)) = \sigma^0(B(o, r)) \) as the geodesic ball \( B(o, r) \) of the hyperbolic space \( \mathbb{H}^4(-1) \), see Example [2.3].

In Section 3 we consider proper isometric minimal immersions \( \varphi : M \to \mathbb{R}^n \), with \( \varphi(p) = o \) and extrinsic balls \( \Omega_r \) of radius \( r \), i.e. the connected component \( \Omega_r \subset \varphi^{-1}(B(o, r)) \) containing \( p \). The main result in this section, Theorem 3.1 is the following estimate

\[
A(m) \cdot \frac{r^m}{\text{vol}(\Omega_r)} r^4 \leq \sum_{\lambda \in \sigma(\Omega_r)} \lambda^{-2} \leq B(m) \cdot \left( \frac{\text{vol}(\Omega_r)}{r^m} \right)^{4/m} r^4,
\]

if \( m = \dim(M) = 2, 3. \)
In Section 4 we consider bounded subsets $\Omega$ with smooth boundary of weighted manifolds $(M, ds^2, \mu)$, $d\mu = \psi d\nu$, $\psi > 0$, weighted Laplace operator

$$\Delta_{\mu} = \frac{1}{\psi} \text{div}(\psi \text{grad})$$

and its Green operator $G: L^2(\Omega, \mu) \to L^2(\Omega, \mu)$. A particular case of the main result of this section is that if $f \in L^2(\Omega, \mu)$, $f > 0$ then

$$\lim_{k \to \infty} \|G^k(f)\|_{L^2} = \lambda_1(\Omega) \text{ and } \lim_{k \to \infty} \|G^k(f)\|_{L^2} \in \text{Ker}(\Delta_{\mu} + \lambda_1),$$

where $G^k = G \circ \cdots \circ G$. More details see Theorem 4.1. In Section 5 we consider the following hierarchy Dirichlet problem on a bounded open subset with smooth boundary $\Omega \subset M$,

$$\begin{cases}
\phi_0 &= 1 \text{ in } \Omega \\
\Delta \phi_k + k \phi_{k-1} &= 0 \text{ in } \Omega \\
\phi_k &= 0 \text{ on } \partial \Omega.
\end{cases}$$

The set $\{A_k\}_{k=1}^{\infty}$, $A_k = \int_0^R \phi_k d\mu$ is called the $L^1(\Omega, \mu)$-momentum spectrum. The momentum spectrum is intertwined with the spectrum of $\Omega$. In the main result of this section, Theorem 5.2, we solve the hierarchy Dirichlet problem and explicitly give the momentum spectrum the in terms of the Green operator. In Section 6 the main result, see Theorem 6.1, gives necessary and sufficient conditions for the expansion of the Green function in $L^2$-sense,

$$g(x, y) = \sum_{i=1}^{\infty} \frac{u_i(x) \cdot u_i(y)}{\lambda_i(\Omega)},$$

where $\{u_i\}_{i=1}^{\infty}$ is an orthonormal basis of $L^2(\Omega, \mu)$ formed by eigenfunctions. This result is the main tool to prove Theorems 5.1.4 and 5.1.5. In Section 7 we present the proofs of all results. Notation: in this article, the spectrum of $\Omega$ will be written either as a sequence of eigenvalues with repetition, according to their multiplicities,

$$\sigma(\Omega) = \{0 \leq \lambda_1(\Omega) < \lambda_2(\Omega) \leq \cdots\} \text{ or as a sequence of eigenvalues without repetition } \sigma(\Omega) = \{0 \leq \lambda_1(\Omega) < \lambda_2(\Omega) < \cdots\}.$$

2. Spectrum of rotationally invariant balls

A Riemannian $m$-manifold is a rotationally invariant $m$-manifold, also called model manifold, with radial sectional curvature $-G(r)$ along the geodesics issuing from the origin, where $G: \mathbb{R} \to \mathbb{R}$ is a smooth even function, is defined as the quotient space

$$M^n_h = [0, R_h) \times S^{m-1} / \sim$$

with $((t, \theta) \sim (s, \beta)) \iff t = s = 0$ and $\forall \theta, \beta \in S^{m-1}$ or $t = s$ and $\theta = \beta$, endowed with the metric $ds^2_h(t, \theta) = dt^2 + h^2(t) d\theta^2$ where $h: [0, \infty) \to \mathbb{R}$ is the unique solution of the Cauchy problem

$$\begin{cases}
h'' - Gh = 0, \\
h(0) = 0, h'(0) = 1,
\end{cases}$$

and $R_h$ is the largest positive real number such that $h_{|([0, R_h))} > 0$. If $R_h = \infty$, the manifold $M^n_h$ is geodesically complete. The geodesic ball $B(o, r)$ centered at the origin $o = \{0\} \times S^{m-1} / \sim$ with radius $r < R_h$, i.e. the set $[0, r) \times S^{m-1} / \sim$, is
rotationally invariant. The volume \( V(r) \) of the ball \( B(o, r) \) and the volume \( S(r) \) of the boundary \( \partial B(o, r) \) are given by
\[
V(r) = \omega_m \int_0^r h^{m-1}(s) \, ds \quad \text{and} \quad S(r) = \omega_m h^{m-1}(r),
\]
respectively, where \( \omega_m = \text{vol}(S^{m-1}) \). The Laplace operator on \( B(o, r) \), expressed in polar coordinates, is given by
\[
\Delta = \frac{\partial^2}{\partial t^2} + (n - 1) \frac{h'}{h} \frac{\partial}{\partial t} + \frac{1}{h^2} \Delta_\theta.
\]
To search for the Dirichlet eigenvalues \( \lambda \) of \( B(o, r) \) it is enough to seek smooth functions of the form \( u(t, \theta) = T(t)H(\theta) \) satisfying \( \Delta u + \lambda u = 0 \) in \( B(o, r) \) and \( u|\partial B(o, r) = 0 \), see \cite{13} p. 42]. This is equivalent to the following eigenvalue problems
\[
(2.2) \quad \begin{cases} 
\Delta_\theta H + \nu H = 0 & \text{in} \ S^{m-1}(1) \\
T'' + (m - 1) \frac{h'}{h} T' + (\lambda - \frac{\nu}{h^2}) T = 0 & \text{in} \ [0, r]
\end{cases}
\]
with initial conditions \( T'(0) = 0 \) if \( l = 0 \), \( T(t) \sim c \cdot t^l \) as \( t \to 0 \) when \( l = 1, 2, \ldots \) and \( T(r) = 0 \). Here \( T' = \partial T/\partial t \), \( T'' = \partial^2 T/\partial t^2 \).

For each value \( \nu_l = l(l + m - 2) \), the set of all \( \lambda \) such that the equation
\[
(2.3) \quad T'' + (n - 1) \frac{h'}{h} T' + (\lambda - \frac{\nu}{h^2}) T = 0
\]
has a non-trivial solution satisfying the initial conditions consist of an increasing sequence of positive real numbers \( \{\lambda_{l, j}\}_{j=1}^\infty \).

Moreover, each \( \lambda_{l, i} \) determine a 1-dimensional space of solutions, say, generated by \( T_{l, i} \). The sequence \( \{\nu_l\}_{l=0}^\infty \) is called the \( \nu_l \)-spectrum, without repetitions, of \( B(o, r) \), denoted by \( \sigma(\lambda)(B(o, r)) \).

It is well known that the set of eigenvalues of the sphere \( S^{m-1} \) are given by \( \nu_l = l(l + m - 2) \), \( l = 0, 1, 2, \ldots \) and their multiplicity of each \( \nu_l \) is given by
\[
\delta(l, m) = \binom{m - 1 + l}{l} - \binom{m - 2 + l}{l - 1}.
\]
Thus, there exists an orthonormal basis formed by eigenfunctions \( H_{l,1}(\theta), \ldots, H_{l,\delta}(\theta) \) of the vector space \( V_{\nu_l} = \{ \phi : \Delta \phi + \nu \phi = 0 \} \). This implies that the set of functions
\[
\{ T_{l,1}(t)H_{l,1}(\theta), T_{l,1}(t)H_{l,2}(\theta), \ldots, T_{l,1}(t)H_{l,\delta}(\theta) \}
\]
is an orthonormal basis of the vector space \( \{ \psi : \Delta \psi + \lambda_{l, i} \psi = 0 \} \). Therefore, the multiplicity of each eigenvalue \( \lambda_{l, i} \) of the sequence \( \{\nu_l\}_{l=0}^\infty \) in the spectrum \( \sigma(B(o, r)) \) is \( \delta(l, m) \). Since all of the eigenvalues of \( B(o, r) \) are obtained in this procedure above, the spectrum of \( B(o, r) \), without repetitions, is the union of the \( \nu_l \)-spectrum \( \sigma(\nu_l)(B(o, r)) \), \( l = 0, 1, \ldots \)
\[
\sigma(B(o, r)) = \bigcup_{l=0}^{\infty} \sigma(l)(B(o, r)) = \{ \lambda_{l, i} \}_{l=0, j=1}^{\infty, \infty},
\]
each \( \lambda_{l, i} \) with multiplicity \( \delta(l, m) \).

The details of this discussion can be found in \cite{13} pp. 40-42]. Observe that the eigenvalues associated to \( \nu_0 = 0 \) are those whose eigenfunctions are radial functions.
$u(t, \theta) = c \cdot T(t)$. We call them radial eigenvalues. More generally we have the following definition.

**Definition 2.1.** Let $B(p, r) \subset M$ be a geodesic ball of radius $r > 0$ and center $p$, (not necessarily rotationally invariant). The radial spectrum $\sigma^{\text{rad}}(B(p, r))$ of $B(p, r)$ is formed by those eigenvalues $\lambda_k$ of $\mathcal{L} = -\Delta|_{W_2^2(B(p, r))}$ whose associated eigenspace $V_k$ contains a radial eigenfunction.

The radial spectrum of a general geodesic ball may be empty, however, if $B(p, r)$ is rotationally invariant, then its radial spectrum is $\sigma^{\text{rad}}(B(p, r)) = \sigma^{l=0}(B(p, r))$.

It should be remarked that there are non-rotationally invariant geodesic balls with non-empty radial spectrum, see Example 2.3. A criteria due to S.Y. Cheng, P. Li

$\sigma$ is rotationally invariant, then its radial spectrum is

$(\ref{2.5})$ 

This property allows to see the heat kernel as a probability distribution in the space of random paths on $M$. More precisely, for $x \in M$ and $U \subset M$, open subset, $\int _U p_t(x, y)dv(y)$ is the probability that a random path emanating from $x$ lies in $U$ at time $t$. Thus if we have strict inequality, $\int _M p_t(x, y)dv(y) < 1$, then there is a positive probability that a random path will reach infinity in finite time $t$. This motivates the following definition.

**Definition 2.2.** A Riemannian manifold $M$ is stochastically complete if for every $x \in M$ and $t > 0$ one has that

(2.5) 

$$\int _M p_t(x, y)dv(y) = 1,$$

where $p_t(x, y) \in C^\infty((0, \infty) \times M \times M)$ is the heat kernel of $M$. Otherwise it is called stochastically incomplete.

The spectrum of a stochastically incomplete model manifolds $\mathbb{M}^m_h$ is discrete, see [11] Example 6.12, say

$$\sigma(\mathbb{M}^m_h) = \{0 < \lambda_1(\mathbb{M}^m_h) < \lambda_2(\mathbb{M}^m_h) \leq \cdots\}$$

and let

$$\sigma(B(o, r)) = \{0 < \lambda_1(B(o, r)) < \lambda_2(B(o, r)) \leq \cdots\}$$
be the spectrum (with repetition) of $B(o, r) \subset M^n_h$. It is well known that the $k^{th}$-eigenvalue $\lambda_k(M^n_h)$ is obtained as the limit
\begin{equation}
(2.6) \quad \lambda_k(M^n_h) = \lim_{r \to \infty} \lambda_k(B(o, r))
\end{equation}
for $k = 1, 2, \ldots$. Moreover, (2.6) holds for any geodesically complete Riemannian manifold $M$ regards the radial spectrum of rotationally invariant balls of model manifolds. The

\begin{equation}
Corollary 2.6.
\end{equation}

If the model manifold $M$ is also obtained as a limit "harmonic series" of the eigenvalues, (without repetitions), \{\lambda_i\}_{i=1}^{\infty} = \sigma^l(B(o, r))$, for $l = 0, 1, \ldots$, also converges.

\begin{equation}
Theorem 2.4 ([1], [19], [20], [26]). A geodesically complete model manifold $M^n_h$ is stochastically incomplete if and only if
\end{equation}

$$\int_0^\infty \frac{V(t)}{S(t)} dt < \infty.$$  

Where $V(r) = \omega_m \int_0^r h^{n-1}(s) ds$ and $S(r) = \omega_m h^{m-1}(r)$.

2.2. “Harmonic series” of radial eigenvalues. Our main result in this section regards the radial spectrum of rotationally invariant balls of model manifolds. The radial spectrum of a stochastically incomplete model manifolds $M^n_h$ is deeply related with this geometric criteria as shows our first result, Theorem 2.5.

\begin{equation}
Theorem 2.5. Let $B(o, r)$ be of the model $M^n_h$, centred at the origin $o$ of radius $r$, with $\lambda_1(B(o, r)) > 0$. Let $\sigma^{rad}(B(o, r)) = \{\lambda^i_{rad}(B(o, r)) < \lambda^2_{rad}(B(o, r)) < \cdots\}$ be the radial spectrum of $B(o, r)$. Then
\end{equation}

\begin{equation}
(2.7) \quad \sum_{i=1}^{\infty} \frac{1}{\lambda^1_{rad}(B(o, r))} = \int_0^r \frac{V(s)}{S(s)} ds.
\end{equation}

If the model $M^n_h$ is stochastically incomplete then

\begin{equation}
(2.8) \quad \sum_{i=1}^{\infty} \frac{1}{\lambda^1_{rad}(M^n_h)} = \int_0^\infty \frac{V(s)}{S(s)} ds.
\end{equation}

\begin{equation}
Corollary 2.6. Let $B(o, 1) \subset M^n_h$ be a geodesic ball of radius $r = 1$ in the model manifold $M^n_h$, where $h(t) = \sinh(t), t, \sin(t)$, respectively. Then,
\end{equation}

\begin{equation}
\sum_{i=1}^{\infty} \frac{1}{\lambda^1_{rad}(B(o, 1))} = \log \left( \frac{(1 + e)^2}{4e} \right) \approx 0.240229 \quad \text{if } h(t) = \sinh(t)
\end{equation}

\begin{equation}
\sum_{i=1}^{\infty} \frac{1}{\lambda^1_{rad}(B(o, 1))} = 0.25 \quad \text{if } h(t) = t
\end{equation}

\begin{equation}
\sum_{i=1}^{\infty} \frac{1}{\lambda^1_{rad}(B(o, 1))} = \log \left( \frac{\sec(\frac{1}{2})}{2} \right) \approx 0.261168 \quad \text{if } h(t) = \sin(t).
\end{equation}

When the model manifold $M^n_h$ is the Euclidean space $\mathbb{R}^m$, we can show that the “harmonic series” of the eigenvalues, (without repetitions), $\{\lambda_i\}_{i=1}^{\infty} = \sigma^l(B(o, r))$, for $l = 0, 1, \ldots$, also converges.
Theorem 2.7. Let $B(o,r)$ be the geodesic ball of $\mathbb{R}^m$ with radius $r$ centred at the origin $o$. Let $\sigma^l(B(o,r)) = \{\lambda_{l,1}(B(o,r)) < \lambda_{l,2}(B(o,r)) < \cdots\}$ be the $\nu_l$-spectrum of $B(o,r)$ without repetition, $l = 0, 1, \ldots$. Then

$$\sum_{i=1}^{\infty} \frac{1}{\lambda_{l,i}(B(o,r))} = \left(\frac{1}{1 + 2\frac{l}{m}}\right) \int_0^r V(s) ds = \left(\frac{1}{1 + 2\frac{l}{m}}\right) \frac{r^2}{2m}$$

and

$$\sum_{i=1}^{\infty} \frac{1}{\lambda_{l,i}^2(B(o,r))} = \frac{r^4}{2(2l + m)^2(2 + 2l + m)}.$$  

If we let $\sigma(B(o,r)) = \{0 < \lambda_1 < \lambda_2 < \cdots\}$ be the spectrum of $B(o,r) \subset \mathbb{R}^m$, repeating the eigenvalues according to their multiplicities, in this case, $\sigma(B(o,r))$ is a union of $\delta(l,m)$-copies of $\sigma^l(B(o,r))$ for $l = 0, 1, \ldots$, then we can compute the whole sum

$$\sum_{k=1}^{\infty} \frac{1}{\lambda_k^2} = \sum_{l=0}^{\infty} \left(\sum_{i=1}^{\infty} \frac{\delta(l,m)}{\lambda_{l,i}^2}\right) = \sum_{l=0}^{\infty} \delta(l,m) \cdot \frac{r^4}{2(2l + m)^2(2 + 2l + m)}.$$  

In the particular case of dimension $m = 2$ or dimension $m = 3$,

$$\sum_{l=0}^{\infty} \delta(l,2) \cdot \frac{r^4}{2(2l + 2)^2(2 + 2l + 2)} = \frac{\pi^2 - 6}{96} \cdot r^4$$  

$$\sum_{l=0}^{\infty} \delta(l,3) \cdot \frac{r^4}{2(2l + 3)^2(5 + 2l)} = \frac{12 - \pi^2}{64} \cdot r^4.$$  

However, this series diverges for $m \geq 4$ in agreement with Theorem 6.1 equation (6.3). The convergence of the series is related to the fact that the Green function belongs to $L^2$ only in the case of $m = 1, 2, 3$. We should remark that the divergence in higher dimension is because of the multiplicity of the eigenvalues. In fact, if we consider the spectrum $\tilde{\sigma}(B(o,r)) = \cup_{l=1}^{\infty} \sigma^l(B(o,r)) = \{0 < \tilde{\lambda}_1 < \tilde{\lambda}_2 < \cdots\}$ without repetition then

$$\sum_{k=1}^{\infty} \frac{1}{\lambda_k^2} = \sum_{l=0}^{\infty} \left(\sum_{i=1}^{\infty} \frac{1}{\lambda_{l,i}^2}\right) = \sum_{l=0}^{\infty} \frac{r^4}{2(2l + m)^2(2 + 2l + m)} < \infty.$$  

Moreover, $\lim_{m \to \infty} \sum_{k=1}^{\infty} \frac{1}{\lambda_k^2} = 0$.

Corollary 2.8. Let $B(o,r)$ be the geodesic ball of $\mathbb{R}^m$ with radius $r$ centred at the origin $o$. Let $\sigma^l(B(o,r)) = \{\lambda_{l,1}(B(o,r)) \leq \lambda_{l,2}(B(o,r)) \leq \cdots\}, l = 0, 1, \ldots$. Then

$$\lambda_{l,k}(B(o,r)) \geq \frac{k(m + 2l)}{m} \cdot \frac{\int_0^r V(s) ds}{r^2} = \frac{2k(m + 2l)}{r^2}.$$  

Corollary 2.8 should be compared with the estimates for the first eigenvalue obtained in [3, Thm.2.1] and [10, Thm.2.7]. We believe that Theorem 6.1 is valid for rotationally invariant geodesic balls of model manifolds. We formalize this in the following conjecture
Conjecture 2.9. There exists a constant $c = c(m, l) < 1$ such that for a geodesic ball $B(o, r) \subset \mathbb{M}_h^n$ the harmonic series converges
\[ \sum_{i=1}^{\infty} \frac{1}{\lambda_i(B(o, r))} = c \cdot \int_0^r \frac{V(s)}{S(s)} ds, \quad l = 0, 1, 2, \ldots \]

2.3. Example. Let $\{\partial/\partial x, \partial/\partial y, \partial/\partial z\}$ be a globally defined non-zero vector fields on $\mathbb{S}^3$ satisfying these conditions
\[ [\partial/\partial x, \partial/\partial y] = 2 \partial/\partial x, \quad [\partial/\partial y, \partial/\partial z] = 2 \partial/\partial y, \quad [\partial/\partial z, \partial/\partial x] = 2 \partial/\partial y \]
and let $dx, dy$ and $dz$ be their dual 1-forms. Consider, on $\Omega = [0, r] \times \mathbb{S}^3 / \sim$, where $(t, \theta) \sim (s, \beta) \iff t = s = 0$ or $t = s$ and $\theta = \beta$, the following metric
\[ ds^2 = dt^2 + a^2(t, \theta) dx^2 + b^2(t, \theta) dy^2 + c^2(t, \theta) dz^2, \]
where $a, b, c : [0, r] \times \mathbb{S}^3 \rightarrow \mathbb{R}$ are defined by
\[ a(t, \theta) = \sinh^2[t]/t \cdot f(t, \theta) \]
\[ b(t, \theta) = t \cdot f(t, \theta) \]
\[ c(t, \theta) = \sinh[t] \]
with $f : [0, r] \times \mathbb{S}^3 \rightarrow (0, \infty)$ given by $f(t, \theta) = 1 + t^k q(\theta), \quad k \geq 3$ and $q : \mathbb{S}^3 \rightarrow (0, \infty)$ is smooth. This metric is clearly smooth in $(0, r] \times \mathbb{S}^3 / \sim$. We need only check that it is smooth at the origin $\{0\} \times \mathbb{S}^3 \sim$. The coefficients near $t = 0$ and every $\theta \in \mathbb{S}^3$ fixed are given by $a(t, \theta) = t + t^3/3 + 2t^5/45 + O(t^6)$, $b(t, \theta) = t + t^3/6 + t^5/120 + O(t^6)$, $c(t, \theta) = t + O(t^6)$. This shows that $ds^2 \approx \text{can}_{\mathbb{H}^4}$ in the $C^2$ topology as $t \approx 0$, where $\text{can}_{\mathbb{H}^4} = dt^2 + \sinh^2[t] \left( dx^2 + dy^2 + dz^2 \right)$ is the canonical metric on the hyperbolic space $\mathbb{H}^4(-1)$. Let $\Omega = B(o, r)$ be the geodesic ball with center at the origin $o = \{0\} \times \mathbb{S}^3 / \sim$ and radius $r$ with respect to the metric $ds^2$. It is clear that $\Omega$ is not rotationally symmetric. The Laplace operator $\triangle_{ds^2}$ is given by
\[ \triangle_{ds^2}(t, \theta) = \frac{\partial^2}{\partial t^2} + 3 \coth(t) \frac{\partial}{\partial t} + \frac{t^2(1 + t^k q(\theta))^2}{\sinh^4(t)} \frac{\partial^2}{\partial x^2} + \frac{1}{t^2(1 + t^k q(\theta))^2} \frac{\partial^2}{\partial y^2} + \frac{1}{\sinh^2(t) \frac{\partial^2}{\partial z^2}} \]
\[ + \frac{2t^{2+k}(1 + t^k q(\theta))}{\sinh^2(t)} \frac{\partial q}{\partial x} \frac{\partial}{\partial x} - \frac{2t^{k-2}}{(1 + t^k q(\theta))^3} \frac{\partial q}{\partial y} \frac{\partial}{\partial y} \frac{\partial}{\partial y} \]
Let $\tilde{\Omega} = (\{0\} \times \mathbb{S}^3 / \sim, \text{can}_{\mathbb{H}^4})$ be the geodesic ball of radius $r$ centered at the origin in the Hyperbolic space $\mathbb{H}^4(-1)$. The Laplace operator $\triangle_{\text{can}_{\mathbb{H}^4}}$ on $\tilde{\Omega}$ is given by
\[ \triangle_{\text{can}_{\mathbb{H}^4}}(t, \theta) = \frac{\partial^2}{\partial t^2} + 3 \coth(t) \frac{\partial}{\partial t} + \frac{1}{\sinh^2(t)} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right). \]
Observe that $\triangle_{ds^2} = \triangle_{\text{can}_{\mathbb{H}^4}}$ on the set of the smooth radial functions $u(t, \theta) = u(t)$ with $u'(0) = 0$, in particular, $\Omega$ and $\tilde{\Omega}$ has the same radial eigenfunctions and radial eigenvalues.

Thus, $(\Omega, ds^2)$ is an example of a non-rotationally symmetric geodesic ball with the same radial spectrum of a rotationally symmetric ball $(\tilde{\Omega}, \text{can}_{\mathbb{H}^4})$. We should notice that the volume functions $t \rightarrow V(t)$ and $t \rightarrow S(t)$ are the same for both metrics, that can be checked directly, coherently with the identity (2.7). This
example is a variation of the example [9, Examp. 2] which, by its turn, was inspired by the example of G. Perelman in [33].

3. Spectrum of extrinsic balls of minimal submanifolds

Let \( \varphi : M \to \mathbb{R}^n \) be a proper and minimal immersion of a complete \( m \)-dimensional Riemannian manifold into \( \mathbb{R}^n \). Let \( \Omega_r = \varphi^{-1}(B(0, r)) \) be the extrinsic ball of radius \( r \). It was proved in [6] and [14] that the first Dirichlet eigenvalue of \( \Omega_r \) is bounded below as

\[
\lambda_1(\Omega_r) \geq \lambda_1(B(0, r)) = \frac{c_m^2}{r^2},
\]

where \( c_m \) is a constant depending only on the dimension of \( M \) and \( \lambda_1(B(0, r)) \) is the first Dirichlet eigenvalue of the ball of radius \( r \) in the Euclidean \( m \)-space \( \mathbb{R}^m \).

This inequality can be read as

\[
\frac{1}{\lambda_k^2(\Omega_r)} \leq C_m \cdot r^4,
\]

where \( C_m = 1/c_m^4 \). By Theorem 3.1 we have that \( \sum_{k=1}^{\infty} \frac{1}{\lambda_k^2(\Omega_r)} \) is finite in dimensions \( m = 2, 3 \). For a totally geodesic \( \mathbb{R}^m \subset \mathbb{R}^n \) with \( m = 2, 3 \) we have, see (2.11),

\[
\sum_{k=1}^{\infty} \frac{1}{\lambda_k^2(\Omega_r)} = C_m r^4
\]

where \( C_m \) is a constant depending only on the dimension. For a non-totally geodesic minimal submanifold we will give lower and upper bounds for this series in terms of the volume \( \text{vol}(\Omega_r) \), of the radius \( r \) and the dimension \( m = 2, 3 \). We have the following result.

**Theorem 3.1.** Let \( \varphi : M \to \mathbb{R}^n \) be a complete Riemannian \( m \)-manifold, properly and minimally immersed into the Euclidean \( n \)-space. Given \( o \in M \) and let \( \Omega_r \) be the extrinsic \( r \)-ball with center \( o \), namely,

\[
\Omega_r = \varphi^{-1}(B(\varphi(o), r))
\]

If \( m = 2, 3 \), then

\[
A_m \cdot \omega_m \cdot \left( \frac{r^m}{\text{vol}(\Omega_r)} \right) \cdot r^4 \leq \sum_{k=1}^{\infty} \frac{1}{\lambda_k^2(\Omega_r)} \leq B_m \cdot \zeta(4/m) \cdot \left( \frac{\text{vol}(\Omega_r)}{r^m} \right)^{4/m} \cdot r^4,
\]

where \( A_m = \frac{1}{4m^2} \left( 1 + \frac{m}{4+m} - \frac{2m}{4+m} \right) \) and \( B_m = \frac{c^{4/m}}{16\pi^2} \) are constants, \( \omega_m \) is the volume of the geodesic ball of radius 1 in \( \mathbb{R}^m \) and \( \zeta(4/m) = \sum_{k=1}^{\infty} \frac{1}{k^{4/m}} \).

If \( \varphi : M \to \mathbb{R}^n \) is a minimal \( m \)-submanifold with finite \( L^m \)-norm of the second fundamental form \( \alpha \),

\[
\int_M \|\alpha\|^m d\mu < \infty,
\]

then \( M \) has finite number of ends \( \{\text{End}_1, \cdots, \text{End}_k\} \), see [21, 27]. Moreover, the function \( r \to \frac{\text{vol}(\Omega_r)}{\omega_m r^m} \) is increasing, [31, 32] and

\[
\lim_{r \to \infty} \frac{\text{vol}(\Omega_r)}{\omega_m r^m} = \mathcal{E}
\]
where $E$ is related with the finite number of ends of $M$ in the following way:

$$
E = \begin{cases} 
\sum_{i=1}^{k} I_i & \text{if } m = 2 \\
 k & \text{if } m = 3,
\end{cases}
$$

where $I_i$ is the geometric index of the end $\text{End}_i$, see [2, 27, 34].

**Corollary 3.2.** Let $\varphi: M \to \mathbb{R}^n$ be a complete Riemannian $m$-manifold, properly and minimally immersed into the Euclidean $n$-space with finite $L^m$-norm of the second fundamental form $\alpha$,

$$
\int_M \|\alpha\|^m d\mu < \infty.
$$

If $m = 2, 3$ then

$$
(3.4) \quad \left(\frac{A_m}{E}\right) \cdot r^4 \leq \sum_{k=1}^{\infty} \frac{1}{\lambda_k^2(\Omega_r)} \leq B_m \cdot \zeta(4/m) \cdot (\omega_m E)^{4/m} \cdot r^4.
$$

4. **Weighted Green operator and the Dirichlet spectrum**

A weighted manifold is a triple $(M, ds^2, \mu)$ consisting of a Riemannian manifold $(M, ds^2)$ and a measure $\mu$ with positive density function $\psi \in C^\infty(M)$, i.e., $d\mu = \psi dv$, where $dv$ is the Riemannian density of $(M, ds^2)$. Let $\Omega \subset M$ be a relatively compact open subset with smooth boundary $\partial \Omega \neq \emptyset$ and consider the weighted Laplace operator $\Delta_\mu: C^\infty(\Omega) \to C^\infty(\Omega)$, acting on $C^\infty(\Omega)$, the space of smooth functions with compact support on $\Omega$, defined by

$$
\Delta_\mu = \frac{1}{\psi} \text{div}(\psi \cdot \text{grad}).
$$

The weighted Laplace operator is densely defined and symmetric with respect to $L^2(\Omega, \mu)$-inner product, but it is not self-adjoint. As in the classic case, we may consider the Sobolev spaces $W^1_0(\Omega, \mu)$ as the closure of $C^\infty(\Omega)$ with respect to the norm

$$
\|u\|^2_{W^1(\Omega, \mu)} := \int_{\Omega} u^2 d\mu + \int_{\Omega} |\text{grad} u|^2 d\mu
$$

and $W^2_0(\Omega, \mu)$, formed by those functions $u \in W^1_0(\Omega, \mu)$ whose weak Laplacian $\Delta_\mu u$ exists and belongs to $L^2(\Omega, \mu)$, i.e.

$$
W^2_0(\Omega, \mu) = \{ u \in W^1_0(\Omega, \mu): \Delta_\mu u \in L^2(\Omega, \mu) \}.
$$

Turns out that the operator $L = -\Delta_\mu|_{W^2_0(\Omega, \mu)}$ is a self-adjoint, non-negative definite elliptic operator and its spectrum, denoted by $\sigma(\Omega, \mu)$, is a discrete increasing sequence of non-negative real numbers $\{\lambda_k\}_{k=1}^\infty \subset [0, \infty)$, (counted according to multiplicity), with $\lim_{k \to \infty} \lambda_k = \infty$, see [22, Thm.10.3]. The weighted Laplace operator $\Delta_\mu$ extends the classical Laplace operator $\triangle$, in the sense that if the density function $\psi \equiv 1$ then $\Delta_\mu = \triangle$. Let $(\Omega, \mu)$ be a weighted bounded open subset with smooth boundary $\partial \Omega \neq \emptyset$ of a Riemannian weighted manifold $(M, ds^2, \mu)$. The (weighted) Green operator $G^\Omega: L^2(\Omega, \mu) \to L^2(\Omega, \mu)$ is given by

$$
G^\Omega(f)(x) = \int_{\Omega} \int_{0}^{\infty} p^\Omega_t(x, y) f(y) d\mu(y) dt
$$

$$
= \int_{\Omega} g(x, y) f(y) dy,
$$

(4.1)
where \( p_t^\Omega(x, y) \) is the heat kernel of the operator \( \mathcal{L} = -\triangle_\mu |_{W_2^2(\Omega, \mu)} \)
and

\[
(4.2) \\
g_\Omega(x, y) = \int_0^\infty p_t^\Omega(x, y) dt
\]
is the Green function of \( \Omega \). The Green operator is a bounded self-adjoint operator
in \( L^2(\Omega, \mu) \) and it is the inverse of \( \mathcal{L} \), i.e., \( G^\Omega = \mathcal{L}^{-1} \). Thus for any \( f \in L^2(\Omega, \mu) \)
there is a unique solution \( u = G^\Omega(f) \in W_2^2(\Omega, \mu) \) to the equation \(-\triangle_\mu u = f\). If \( f \in C_0^\infty(\Omega) \)
then \( G^\Omega(f) \in C^\infty(\Omega) \) see details in [22 Thm. 13.4]. Applying \( G^\Omega \) to the equation
\[
\triangle_\mu u + \lambda_i(\Omega) u = 0
\]
we obtain that the \( i \)-th eigenvalue \( \lambda_i(\Omega) \) of \( \Omega \) is given by \( \lambda_i(\Omega) = u/G^\Omega(u) \). The difficulty
to know precisely the \( i \)-th eigenvalue applying the Green operator is that one needs
to know an \( i \)-th eigenfunction to start. For simplicity of notation, if no
ambiguity arises, we will suppress the reference to \( \Omega \) in the gadgets associated
to \( \Omega \). Thus, \( g \) and \( G \) will be, respectively, the Green function and the Green operator
of the operator \( \mathcal{L} = -\triangle_\mu |_{W_2^2(\Omega, \mu)} \). Our main result in this section is that
in order to obtain the first Dirichlet eigenvalue, assuming \( \lambda_1(\Omega) > 0 \), one picks a positive
function \( f \in L^2(\Omega, \mu) \) and compute the limit
\[
\lambda_1 = \lim_{k \to \infty} \frac{\|G^k(f)\|_2}{\|G^{k+1}(f)\|_2}
\]
where \( \|u\|_2 = \int_{\Omega} |u|^2 d\mu \) is the \( L^2(\Omega, \mu) \)-norm and \( G^k = G \circ \cdots \circ G \). More generally,
let \( f \in L^2(\Omega, \mu) \) and let \( \ell \) be the smallest positive integer such that
\[
\int_{\Omega} f(x) u_i(x) d\mu(x) = 0
\]
for \( i = 1, 2, \ldots, \ell - 1 \) and \( \int_{\Omega} f(x) u_\ell(x) d\mu(x) \neq 0 \), where \( \{u_i\} \) is an orthonormal
basis of \( L^2(\Omega, \mu) \) formed by eigenfunctions \( u_\ell \) with eigenvalues \( \lambda_\ell \). Then the \( \ell \)-th
eigenvalue is given by
\[
\lambda_\ell = \lim_{k \to \infty} \frac{\|G^k(f)\|_2}{\|G^{k+1}(f)\|_2}
\]

**Theorem 4.1.** Let \( (\Omega, ds^2, \mu) \) be a weighted bounded open subset of a weighted \( m \)-manifold \((M, ds^2, \mu)\), with smooth boundary \( \partial \Omega \neq \emptyset \). Let \( G \) be the Green operator
of \( \mathcal{L} \) and let \( \{u_i\} \) be an orthonormal basis of \( L^2(\Omega, \mu) \) formed with eigenfunctions
\( u_i \) with eigenvalues \( \lambda_i \). Then, for any \( f \in L^2(\Omega) \),
\[
(4.3) \\
\lim_{k \to \infty} \frac{\|G^k(f)\|_2}{\|G^{k+1}(f)\|_2} = \lambda_\ell,
\]
where \( \ell \) is the first positive integer such that,
\[
(4.4) \int_{\Omega} f(x) u_i(x) d\mu(x) = 0 \text{ for } i = 1, 2, \ldots, \ell - 1 \text{ and } \int_{\Omega} f(x) u_\ell(x) d\mu(x) \neq 0.
\]
Moreover,
\[
(4.5) \\
\lim_{k \to \infty} \frac{G^k(f)}{\|G^k(f)\|_2} = \phi_\ell \in \ker(\triangle_\mu + \lambda_\ell) \text{ in } L^2(\Omega, \mu).
\]
In particular, for any positive $f \in L^2(\Omega, \mu)$,

\[(4.6) \lim_{k \to \infty} \|G^k(f)\| = \lambda_1 \quad \text{and} \quad \lim_{k \to \infty} \|G^k(f)\| = u_1 \quad \text{in} \quad L^2(\Omega, \mu).\]

In addition, if $f \in L^2(\Omega, \mu)$ satisfying \[4.3\] and denoting by $f_1$ the function defined by

\[(4.7) \quad f_1 = f - \lambda_1 G(f),\]

then,

\[(4.8) \quad \lim_{k \to \infty} \|G^k(f_1)\| = \lambda_n\]

with

\[\lambda_n \geq \lambda_l.\]

And

\[(4.9) \quad \lim_{k \to \infty} \frac{G^k(f_1)}{G^k(f)} = \phi_n \in \ker(\Delta_\mu + \lambda_n).\]

Theorem \[4.1\] is effective if $\Omega = B(o, r)$ is a rotationally invariant geodesic ball of a model manifold $M_h^m$ and $\Delta_\mu = \Delta$. Indeed, let $C_{\text{rad}}(B(o, r))$ be the set of continuous radial functions $\phi: B(o, r) \to \mathbb{R}$, $\phi(t, \theta) = \phi(t)$. Define the operator $T: C_{\text{rad}}(B(o, r)) \to C_{\text{rad}}(B(o, r))$ by

\[(4.10) \quad T(\phi)(t, \theta) = \int_t^r \int_0^{h^{n-1}(s)} \phi(s) ds \, d\sigma.\]

Observe that $T$ is the Green operator for $L_0(f) = f'' + (m - 1) \frac{h'}{h} f' = \Delta_\mu$ for the measure $d\mu = \omega_m h^{m-1} d\nu$, hence Theorem \[4.1\] can be applied to obtain the following theorem.

**Theorem 4.2.** Let $B(o, r) \subset M_h^m$ be rotationally invariant geodesic ball of a model manifold with boundary $\partial B(o, r) \neq \emptyset$. For any $\phi \in C_{\text{rad}}(B(o, r))$ we have

\[(4.11) \quad \frac{\|T^k \phi\|_{L^2}}{\|T^{k+1} \phi\|_{L^2}} \to \lambda^\text{rad}_l(B(o, r)) \quad \text{and} \quad \frac{T^k \phi}{\|T^k \phi\|_{L^2}} \to \phi_l \quad \text{in} \quad L^2,\]

as $k \to \infty$. With $\phi_l \in \ker(\Delta + \lambda_l)$, $\lambda_l = \lambda_l(B(o, r))$ is $l^{th}$ radial eigenvalue, where $l$ is the first integer such that

\[(4.12) \quad \int_{B(o, r)} \phi \, u_i \, d\nu = 0 \quad \text{for} \quad i = 1, 2, \ldots, \ell - 1 \quad \text{and} \quad \int_{B(o, r)} \phi \, u_i \, d\nu \neq 0,\]

where $\{u_i\}$ is an orthonormal basis of $L^2([0, r], \mu)$ eigenfunctions of $L_0$. Thus, in particular, for any positive $\phi \in C_{\text{rad}}(B(o, r))$ we have

\[(4.13) \quad \frac{\|T^k \phi\|_{L^2}}{\|T^{k+1} \phi\|_{L^2}} \to \lambda_1(B(o, r)) \quad \text{and} \quad \frac{T^k \phi}{\|T^k \phi\|_{L^2}} \to \phi_1 \quad \text{in} \quad L^2,\]

as $k \to \infty$. 
To show how efficient Theorem 4.2 is, we let $B(o,r)$ be the geodesic ball centered at the origin $o$ and radius $r$ in rotationally symmetric manifold $M^n_h$. Consider the following two maps:

$$T^k : C^{rad}(B(o,r)) \rightarrow \mathbb{R}, \quad T^k(\phi) := \frac{\|T^k\phi\|_{L^2}}{\|T^{k+1}\phi\|_{L^2}}$$

$$T^\infty : C^{rad}(B(o,r)) \rightarrow \mathbb{R}, \quad T^\infty(\phi) := \lim_{k \rightarrow \infty} \frac{\|T^k\phi\|_{L^2}}{\|T^{k+1}\phi\|_{L^2}}$$

and the following family of functions arising from the $\phi_0 = 1$ \hspace{1cm} (4.14)

$$\{\phi_0 = 1, \phi_k = \phi_{k-1} - (T^\infty(\phi_{k-1})) T\phi_{k-1}\}.$$  

Applying Theorem 4.2 we can obtain a subset of the radial spectrum, namely, \hspace{1cm} (4.15)

$$\{T^\infty(\phi_k)\}_{k=0}^\infty \subset \sigma^{rad}(B(o,r)).$$

If $M^2_t = \mathbb{R}^2$, the radial spectrum of the $B(o,1)$ is given by $\sigma^{rad}(B(o,1)) = \{j^2_{0,k}\}_{k=1}^\infty$, where $j_{0,k}$ is the $k$th-zero of the Bessel function $J_0$. The following table shows the $T^j(\phi_i)$, $j = 1, 2, 3, 9$ and $i = 0, 1, 2$. Observe that $T^9(\phi_1)$ agrees with $j_{0,1}$ to the 5th-decimal place.

| $\phi_0 = 1$ | $\phi_1$ | $\phi_2$ |
|--------------|----------|----------|
| $T^1$        | 5.80381  | 31.8311  | 85.7823  |
| $T^2$        | 5.78388  | 30.6656  | 77.4423  |
| $T^3$        | 5.78321  | 30.5022  | 75.5737  |
| $T^9$        | 5.78319  | 30.4713  | 74.8874  |
| $\lambda$ $j_{0,1} \approx 5.78319$ | $j_{0,2} \approx 30.4713$ | $j_{0,3} \approx 74.8874$ |

**Remark 4.3.** This bootstrapping argument using the Green operator in Theorem 4.1 was discovered, and applied to a particular kind of functions, by S. Sato [35] to obtain explicit estimates for the first eigenvalue of spherical caps of $S^2(1)$. It was applied by Barroso-Bessa [4] to obtain estimates for the first eigenvalue of balls in rotationally symmetric manifolds.

5. $L^1(\Omega, \mu)$-MOMENTUM SPECTRUM

Let $(\Omega, ds^2, \mu)$ be a weighted bounded open subset of a Riemannian weighted Riemannian manifold with smooth boundary $\partial \Omega \neq \emptyset$. Consider a sequence of functions $\phi_k : \Omega \rightarrow \mathbb{R}, k = 0, 1, \ldots$ defined inductively as the sequence of solutions to the following hierarchy of boundary value problems on $\Omega$. Let $\phi_0 = 1$ in $\Omega$ and for $k \geq 1$

\hspace{1cm} (5.1)

$$\begin{align*}
\sum_{i=0}^k \Delta_{\mu} \phi_i + k \phi_{k-1} &= 0 \quad \text{in} \ \Omega \\
\phi_k &= 0 \quad \text{on} \ \partial \Omega.
\end{align*}$$

The solution $\phi_1(x)$ is the mean time for the first exit of $\Omega$ of a Brownian motion $t \rightarrow X_t$ in $\Omega$, with $X_0 = x$, see [29].

**Definition 5.1.** The $L^1(\Omega, \mu)$-momentum spectrum of $\Omega$ is the set $\{A_k(\Omega)\}_{k=1}^\infty$ of the $L^1(\Omega, \mu)$-norms of $\phi_k$,

$$A_k(\Omega) = \int_{\Omega} \phi_k d\mu.$$
The number $A_1(\Omega)$ is called the torsion rigidity of $\Omega$ because, when $\Omega \subset \mathbb{R}^2$, $A_1(\Omega)$ is the torque required for a unit angle of twist per unit length when twisting an elastic beam of uniform cross section $\Omega$, see [25], [29] and [30].

Observe that if $\Omega = B(o, r)$ is a rotationally invariant geodesic ball then the solutions $\phi_k(x) = \phi_k(|x|)$ of (5.1) are radial functions since $\phi_1$ is radial. In [25], A. Hurtado, S. Markvorsen and V. Palmer, showed that the first eigenvalue of a rotationally invariant ball can be given in terms of the momentum spectrum. They proved that

$$
\lambda_1(B(o, r)) = \lim_{k \to \infty} \frac{k\phi_{k-1}(0)}{\phi_k(0)} = \lim_{k \to \infty} \frac{kA_{k-1}(B(o, r))}{A_k(B(o, r))}
$$

and $\phi_\infty(r) = \lim_{k \to \infty} \phi_k(r)/\phi_k(0)$ is a radial first eigenfunction. In our next result we solve explicitly the problem (5.1) in terms of the $\Delta_\mu$ Green operator $G$. It links the momentum spectrum with the Dirichlet spectrum of domains and in case of rotationally invariant geodesic balls and it recovers Hurtado-Markvorsen-Palmer’s result.

**Theorem 5.2.** Let $(\Omega, ds^2, \mu)$ be a bounded weighted open subset of a Riemannian weighted Riemannian manifold $(M, ds^2, \mu)$ with smooth boundary $\partial \Omega \neq \emptyset$. Let $\phi_k$ be the sequence of functions given by the boundary problem (5.1) and $G$ the $\Delta_\mu$ Green operator of $\Omega$. Then

$$
\phi_k(x) = k!G^k(1).
$$

Hence the exit moment spectrum is related to the Dirichlet spectrum as follows

$$
\phi_k(x) = k! \sum_{i=1}^\infty \frac{a_i u_i(x)}{\lambda_i^k}
$$

$$
A_k = k! \sum_{i=1}^\infty \frac{a_i^2}{\lambda_i^k}
$$

where $a_i = \int_\Omega u_i(x) d\mu(x)$ and $\{u_i\}$ is an orthonormal basis in $L^2(\Omega, \mu)$ formed by eigenfunctions of $L = -\Delta_\mu |W|^{2}(\Omega, \mu)$.

**Corollary 5.3.** Let $(\Omega, ds^2, \mu)$ be weighted bounded open subset of a Riemannian weighted Riemannian manifold manifold $(M, ds^2, \mu)$ with smooth boundary $\partial \Omega \neq \emptyset$. Then,

$$
\lambda_1 = \lim_{k \to \infty} \frac{kA_{k-1}}{A_k} \quad \text{and} \quad \lambda_2 \leq \lim_{k \to \infty} \frac{(k-1)A_{k-2} - \lambda_1 A_{k-1}}{kA_{k-1} - \lambda_1 A_k}
$$

Hence,

$$
\frac{\lambda_2}{\lambda_1} \leq \lim_{k \to \infty} \frac{(k-1)A_{k-2} - \lambda_1}{kA_{k-1} - \lambda_1} < \infty.
$$

6. $L^2$-expansion of the Green function $g(x, y)$

Let $(\Omega, ds^2, \mu)$ be a relatively compact open subset of a Riemannian weighted manifold $(M, ds^2, \mu)$ and let $\sigma(\Omega, \mu) = \{\lambda_k\}_{k=1}^\infty \subset [0, \infty)$ be the spectrum of $L = -\Delta_\mu |W|^{2}(\Omega, \mu)$. repeated accordingly to their multiplicity. Let $\{u_k\}_{k=1}^\infty$ be an orthonormal basis in $L^2(\Omega, \mu)$ such that each function $u_k$ is an eigenfunction.
of $L$ with eigenvalue $\lambda_k$. In this basis, the heat kernel $p_t^\Omega(x,y)$ of the operator $L$ admits an expansion

\begin{equation}
\label{eq:heat_kernel}
P_t^\Omega(x,y) = \sum_{k=1}^\infty e^{-\lambda_k t} u_k(x)u_k(y).
\end{equation}

This series converges absolutely and uniformly in the domain $t \geq \epsilon$, $x,y \in \Omega$ for any $\epsilon > 0$ as well as in the topology of $C^\infty(\mathbb{R}_+ \times \Omega \times \Omega)$, see the details in [22, Thm. 10.13]. If we could integrate the heat kernel expansion (6.1) in the variable $t$ we would obtain that

\begin{equation}
\label{eq:green_function_expansion}
g_t^\Omega(x,y) = \sum_{k=1}^\infty \frac{u_k(x) \cdot u_k(y)}{\lambda_k(\Omega)}.
\end{equation}

It is known that identity (6.2) holds in the sense of distributions, see [22, exercise 13.14] and we may ask whether this identity holds in stronger topology, for instance in $L^1(\Omega \times \Omega, \mu)$ or $L^2(\Omega \times \Omega, \mu)$? It is shown in [22, Exercise 13.7] that $g_t^\Omega(x,y) \in L^1(\Omega \times \Omega, \mu)$, however, $g_t^\Omega(x,\cdot)$ does not need to belong to $L^2(\Omega, \mu)$. Indeed, let $\Omega = B_{\mathbb{R}^4}(1)$ be the geodesic ball of the 4-dimensional Euclidean space $\mathbb{R}^4$, with its canonical metric, of radius $r = 1$ and center at the origin $0$. The Green function of $\Omega$ is given by

\[ g_0^\Omega(0,y) = \frac{1}{2\omega_4} \left( \frac{1}{|y|^2} - 1 \right), \]

which obviously does not belong to $L^2(B_{\mathbb{R}^4}(1))$. Here $\omega_4$ is the volume of the unit 3-sphere in $\mathbb{R}^4$. On the other hand, if $\Omega = B_{\mathbb{R}}(1)$ is the geodesic ball in the real line $\mathbb{R}$ then the Green function is $g_t^\Omega(x, y) = (|x-y| - x \cdot y + 1)/2$ which clear is in $L^2(\Omega \times \Omega)$. We start showing that a necessary and sufficient condition to $g_t^\Omega \in L^2(\Omega \times \Omega, \mu)$, for any measure $\mu$, is that $\dim(M) = 1, 2, 3$ and in these dimensions identity (6.2) holds in $L^2(\Omega \times \Omega, \mu)$. Precisely, we have the following result.

**Theorem 6.1.** Let $(M, ds^2, \mu)$ be a weighted Riemannian manifold and let $\Omega \subset M$ be a bounded open subset of $M$ with smooth boundary $\partial \Omega \neq \emptyset$. Then the Green function $g_t^\Omega \in L^2(\Omega \times \Omega, \mu)$ if and only if $\dim(M) = 1, 2, 3$. Furthermore,

\begin{equation}
\label{eq:green_function}
g_t^\Omega(x, y) = \sum_{k=1}^\infty \frac{u_k(x) \cdot u_k(y)}{\lambda_k(\Omega)},
\end{equation}

where the series converges in $L^2(\Omega \times \Omega, \mu)$, and $\{u_k\}_{k=1}^\infty$ is an orthonormal basis formed by eigenfunctions associated to the eigenvalues $\{\lambda_k(\Omega)\}_{k=1}^\infty$ of $\Omega$. Moreover

\begin{equation}
\label{eq:norm}
\|g_t^\Omega\|_{L^2(\Omega \times \Omega)} = \sum_{k=1}^\infty \frac{1}{\lambda_k^2(\Omega)}.
\end{equation}

If $n = 1$ then

\begin{equation}
\label{eq:average}
\int_\Omega g_t^\Omega(x, x) d\mu(x) = \sum_{k=1}^\infty \frac{1}{\lambda_k(\Omega)}.
\end{equation}

It should be remarked that this result above is a direct consequence of Weyl’s asymptotic formula for the eigenvalues $\lambda_i(\Omega)$. The first consequence of Theorem
regards a result due to Gruter and Widman, see [24, Thm. 1.1]. They proved that for any \( y \in \Omega \subset \mathbb{R}^n, n \geq 3 \), the function \( x \to g^\Omega(x,y) \in L^*_{\frac{n-2}{2}}(\Omega) \), where \( L^*_p(\Omega) \) is defined by
\[
L^*_p(\Omega) := \{ f : \Omega \to \mathbb{R} \cup \{ \infty \}, \text{measurable and } \| f \|_{L^*_p(\Omega)} < \infty \}
\]
with
\[
\| f \|_{L^*_p(\Omega)} := \sup_{t > 0} t \cdot \nu(\{ x \in \Omega : |f(x)| > t \})^{1/p}.
\]
It is easy to see that \( L^p(\Omega) \subset L^*_p(\Omega) \), thus coupling Theorem 6.1 with Gruter and Widman’s result we obtain a more precise statement.

**Corollary 6.2.** Let \( \Omega \subset \mathbb{R}^n, n \geq 3 \) be a bounded open subset with smooth boundary. Let \( g^\Omega \) its Green function (associated to the Laplacian \( \triangle \)).

- If \( n = 3 \) then \( x \to g^\Omega(x,y) \in L^2(\Omega) \cap L^*_3(\Omega), y \in \Omega \).
- If \( n \geq 4 \) then \( x \to g^\Omega(x,y) \in L^*_{\frac{n-2}{2}}(\Omega) \setminus L^2(\Omega), y \in \Omega \).

**Remark 6.3.** In general, \( L^*_3(\Omega) \setminus L^2(\Omega) \neq \emptyset \) and \( L^2(\Omega) \setminus L^*_3(\Omega) \neq \emptyset \). Indeed, letting \( \Omega = B_r(o) \subset \mathbb{R}^3 \) and \( f_\alpha(x) = 1/|x|^\alpha \) we have that \( f_\alpha \in L^*_3(\Omega) \setminus L^2(\Omega) \) if \( \alpha > 1 \) and \( f_\alpha \in L^2(\Omega) \setminus L^*_3(\Omega) \) if \( 0 < \alpha < 1/2 \).

### 7. Proofs of the results

The structure of this section is the following. First, we will prove Theorem 6.1 and then Theorem 3.1. Then we will prove Theorems 2.5 and 2.7. Finally we present the proofs of Theorems 4.1 & 5.2.

#### 7.0.1. Proof of Theorem 6.1

Let \( (\Omega, ds^2, \mu) \) be a bounded open subset of a weighted \( m \)-manifold \((M, ds^2, \mu)\) with smooth boundary \( \partial \Omega \neq \emptyset \). Let \( \sigma(\Omega) = \{ \lambda_k(\Omega) \}_{k=1}^\infty \) be set of eigenvalues of \( \mathcal{L} = -\Delta_\mu W^2_{2,\Omega}(\mu,\mu) \), repeated according to multiplicity. One has that the Weyl’s asymptotic formula holds for the eigenvalues \( \lambda_k \),

\[
(7.1) \quad \lambda_k(\Omega) \approx c_m \cdot \left( \frac{k}{\mu(\Omega)} \right)^{2/m} \quad \text{as } k \to \infty,
\]
where \( c_m > 0 \) is the same constant as in \( \mathbb{R}^m \), depending only on the dimension \( m = \dim(M) \), see [24] p.7. Thus
\[
\sum_{i=1}^\infty \frac{1}{\lambda_k^2(\Omega)} < \infty \Leftrightarrow m = 1, 2, 3
\]
and
\[
\sum_{i=1}^\infty \frac{1}{\lambda_k^2(\Omega)} < \infty \Leftrightarrow m = 1.
\]
Let \( \{ g_k : \Omega \times \Omega \to \mathbb{R} \} \subset L^2(\Omega \times \Omega, \mu) \) be a sequence of functions defined by
\[
g_k(x,y) = \sum_{i=1}^k \frac{u_i(x)u_i(y)}{\lambda_i(\Omega)}.
\]
Let $k_2 > k_1$ and compute
\[
\|g_{k_2} - g_{k_1}\|_{L^2(\Omega \times \Omega, \mu)}^2 = \sum_{i=k_1+1}^{k_2} \frac{1}{\lambda_i^2(\Omega)}.
\]
The sequence $\{g_k\}$ is a Cauchy sequence in $L^2(\Omega \times \Omega, \mu)$ iff $\sum_{i=1}^{\infty} \frac{1}{\lambda_i^2(\Omega)} < \infty$. Then, 
\[
g_{\infty}(x, y) = \sum_{i=1}^{\infty} \frac{u_i(x)u_i(y)}{\lambda_i(\Omega)} \in L^2(\Omega \times \Omega, \mu).
\]

On the other hand, it is known that the Green function $g^\Omega$ satisfies the functional identity
\[
g^\Omega(x, y) = \sum_{i=1}^{\infty} \frac{u_i(x)u_i(y)}{\lambda_i(\Omega)}
\]
in the sense of distributions, see [22, p. 348]. Therefore, 
\[
g^\Omega = g_{\infty} \in L^2(\Omega \times \Omega, \mu)
\]
if and only if $\dim(M) = 1, 2, 3$, where 
\[
g_{\infty}(x, y) = \sum_{i=1}^{\infty} \frac{u_i(x)u_i(y)}{\lambda_i(\Omega)}.
\]

7.0.2. Proof of Theorem Let $\varphi: M \to N$ be a properly immersed $m$-submanifold $M$ into of a Riemannian manifold $N$. Let $t_N(x) = \text{dist}_N(p, x)$ be the distance in $N$ from a fixed point $p = \varphi(q) \in N$. Let $\Omega_r = \varphi_1^{-1}(B_N(p, r))$ be an extrinsic ball that contains $q$, where $B_N(p, r)$ is the geodesic ball of $N$ with center at $p$ and radius $r < \min\{\text{inj}(p), \pi/\sqrt{k}\}$, $k = \sup K_N$ and where $\pi/\sqrt{k} = \infty$ if $k \leq 0$.

Let $E_r(x)$ be the mean time of the first exit from $\Omega_r$ for a Brownian motion particle starting at $x \in \Omega_r$. A fundamental observation of Dynkin [22, vol.II, p.51] states that the function $E_r$ satisfies the Poisson equation
\[
\begin{align*}
\Delta E_r &= -1 \text{ in } \Omega_r \\
E_r &= 0 \text{ on } \partial \Omega_r
\end{align*}
\]
If $g^{\Omega_r}(x, y) = g(x, y)$ is the Green function of $\Omega_r$, with Dirichlet boundary data, then 
\[
E_r(x) = \int_{\Omega_r} g(x, y) d\nu(y).
\]

Applying Cauchy-Schwarz and assuming that $m = 2, 3$ we obtain
\[
\int_{\Omega_r} E_r^2(x) d\nu(x) \leq \text{vol}(\Omega_r) \cdot \int_{\Omega_r} \left( \int_{\Omega_r} g(x, y) d\nu(y) \right)^2 d\nu(x)
\]
\[
\leq \text{vol}(\Omega_r) \cdot \int_{\Omega_r} \int_{\Omega_r} g^2(x, y) d\nu(y) d\nu(x)
\]
\[
= \text{vol}(\Omega_r) \sum_{k=1}^{\infty} \frac{1}{\lambda_k(\Omega_r)^2}.
\]
Assume that $N = \mathbb{R}^n$, $p = o \in \mathbb{R}^n$. Let $\overline{E}_r : B^m(0, r) \to \mathbb{R}$ be the mean exit time of the first exit from the geodesic ball $B^m(o, r) \subset \mathbb{R}^m$. It is known that $\overline{E}_r$ is radial, i.e. $\overline{E}_r(y) = \overline{E}_r(t(y))$, $t(y) = |y-o|$, $y \in \mathbb{R}^m$. Denote by $E_r$ the transplant of $\overline{E}_r$ to $B^m(o, r)$, i.e., the function $E_r : B^m(o, r) \subset \mathbb{R}^n \to \mathbb{R}$ defined by $E_r(z) = \overline{E}_r(t(z))$. Consider the restriction of $\overline{E}_r(z)$ to the immersion $\varphi(M)$, i.e. $x \in \Omega_r \to \overline{E}_r(\varphi(x))$. In [28], Steen Markvorsen proved that if the immersion $\varphi : M \to \mathbb{R}^{n+1}$ is a minimal hypersurface then $E_r(x) = \overline{E}_r(\varphi(x)) = \overline{E}_r(t(\varphi(x)))$. Solving problem (7.2), we have

\begin{equation}
\overline{E}_r(s) = \int_{s}^{r} \frac{\text{vol}(B^m(o, \zeta))}{\text{vol}(\partial B^m(o, \zeta))} d\zeta = \frac{1}{2m} (r^2 - s^2),
\end{equation}

therefore, by inequality (7.3), we have

$$\text{vol}(\Omega_r) \sum_{k=1}^{\infty} \frac{1}{\lambda_k(\Omega_r)^2} \geq \frac{1}{4m^2} \int_{\Omega_r} (r^4 + t^4(x) - 2r^2t^2(x)) \, d\nu(x),$$

where we identified $t(\varphi(x)) = t(x)$. Applying co-area formula for the extrinsic distance function $t : M \to \mathbb{R}$, we obtain

$$\int_{\Omega_r} (r^4 + t^4(x) - 2r^2t^2(x)) \, d\nu(x) = \int_{0}^{r} \left( \int_{\partial \Omega_s} \frac{r^4 + t^4(x) - 2r^2t^2(x)}{|\nabla t|} \, dA \right) \, ds = \int_{0}^{r} \left( r^4 + s^4 - 2r^2s^2 \right) \left( \int_{\partial \Omega_s} \frac{1}{|\nabla t|} \, dA \right) \, ds \geq \int_{0}^{r} \left( r^4 + s^4 - 2r^2s^2 \right) \text{vol}(\partial \Omega_s) \, ds.
$$

On the other hand, we have that $\text{vol}(\partial \Omega_s) \geq m \omega_m s^{m-1}$, see [32]. Then

\begin{equation}
\int_{\Omega_r} (r^4 + t^4(x) - 2r^2t^2(x)) \, dV(x) \geq m \omega_m \int_{0}^{r} (r^4 + s^4 - 2r^2s^2) s^{m-1} \, ds = m \omega_m r^{4+m} \left( \frac{1}{m} + \frac{1}{4+m} - \frac{2}{2+m} \right).
\end{equation}

In order to simplify the notation let us denote by $A_m : = 1 + \frac{m}{4+m} - \frac{2m}{2+2m}$. Hence,

\begin{equation}
\text{vol}(\Omega_r) \cdot \sum_{k=1}^{\infty} \frac{1}{\lambda_k(\Omega_r)^2} \geq \frac{r^4}{4m^2} A_m \omega_m r^m.
\end{equation}

That proves the lower bound in (7.3). To prove the upper bound recall that Cheng, Li and Yau proved in [13] that

\begin{equation}
\lambda_k(\Omega_r) \geq 4\pi \left( \frac{k}{e} \right)^{2/m} \frac{1}{\text{vol}(\Omega_r)^{2/m}}.
\end{equation}

Therefore,

\begin{equation}
\sum_{k=1}^{\infty} \frac{1}{\lambda_k(\Omega_r)^2} \leq \frac{e^{4/m}}{16\pi^2} \text{vol}(\Omega_r)^{4/m} \sum_{k=1}^{\infty} \frac{1}{k^{4/m}} = \frac{e^{4/m}}{16\pi^2} \text{vol}(\Omega_r)^{4/m} \zeta(4/m).
\end{equation}
Observe that \( \zeta(2) = \pi^2/6 \). Putting together inequalities (7.6) and (7.8) we obtain

\[
A_m \cdot \omega_m \cdot \left( \frac{r^m}{\text{vol}(\Omega_r)} \right) \cdot r^4 \leq \sum_{k=1}^{\infty} \frac{1}{\lambda_k^2(\Omega_r)} \leq B_m \cdot \zeta(4/m) \cdot \left( \frac{\text{vol}(\Omega_r)}{r^{m}} \right)^{4/m} \cdot r^4.
\]

In order to obtain inequality (3.4) we have by the monotonicity formula, see [31, 32], that the function

\[
r \to \frac{\text{vol}(\Omega_r)}{\omega_m r^m}
\]

is an increasing function. Moreover by the classical results of Jorge-Meeks in [27], see also [2, 34],

\[
\lim_{r \to \infty} \frac{\text{vol}(\Omega_r)}{\omega_m r^m} = \mathcal{E}.
\]

Therefore

\[
\text{vol}(\Omega_r) \leq \omega_m \mathcal{E} r^m
\]

and the theorem follows taking in consideration that

\[
\mathcal{E} = \left\{ \sum_{i=1}^{k} I_i \right\} \text{ if } m = 2 \quad \text{and} \quad \left\{ \sum k \right\} \text{ if } m = 3,
\]

where \( I_i \) is the geometric index of the end \( E_i \), see details in [27].

#### 7.0.3. Proof of Theorem 2.5

The metric on the geodesic ball \( B(o, r) \subset M^n_r \) is expressed, in polar coordinates, as

\[
ds^2 = dt^2 + h^2(t) d\theta^2.
\]

The Laplacian \( \triangle \) of this metric is given by

\[
\triangle (t, \theta) = \frac{\partial^2}{\partial t^2} + (n-1) \frac{h'}{h}(t) \frac{\partial}{\partial t} + \frac{1}{h^2(t)} \triangle_{\theta}
\]

and \( \triangle_{\theta} \) is the Laplacian on \( S^{n-1} \). Observe that the radial eigenvalues of \( B(o, r) \) are the eigenvalues of the operator \( L_0 \) in the following eigenvalue problem.

(7.9)

\[
\begin{cases}
L_0 u + \lambda u = 0 \\
u'(0) = 0 \\
u(r) = 0.
\end{cases}
\]

In order to study this eigenvalue problem, define the following space of functions

(7.10)

\[\Lambda := \left\{ u \in W^2([0, r], \mu) : \lim_{t \to 0^+} u'(t) = 0 \text{ and } u(0) = 0 \right\}\]

and the density in \([0, r]\) given by \( d\mu(t) = \omega_n h^{n-1}(t) dt \). Observe that \( f \in L^2([0, r], \mu) \) if and only if \( t \circ f \in L^2(B(o, r)) \). Define the bilinear form \( \mathcal{E}_{bf} \) acting on Lipschitz functions

\[
\mathcal{E}_{bf}(f, g) = \int_0^r f'(s)g'(s) ds
\]

and let \( \mathcal{F} \) be the closure of \( \Lambda \) in \( L^2([0, r], \mu) \) with respect to the norm

(7.11)

\[\|f\|_{\mathcal{F}}^2 = \|f\|_{L^2([0, r], \mu)}^2 + \mathcal{E}_{bf}(f, f)\]

The bilinear form \( \mathcal{E}_{bf} \) acting on \( \mathcal{F} \), in the distributional sense, is a Dirichlet form, i.e. it has the following properties.

(1) **Positivity:** \( \mathcal{E}_{bf}(f) = \mathcal{E}_{bf}(f, f) \geq 0 \) for any \( f \in \mathcal{F} \).
(2) **Closedness**: the space $\mathcal{F}$ is a Hilbert space with respect to the following product

$$ (f, g) = (f, g) + \mathcal{E}_{bf}(f, g). $$

(3) **The Markov property**: if $f \in \mathcal{F}$ then the function

$$ g := \min\{1, \max\{f, 0\}\} $$

also belongs to $\mathcal{F}$ and $\mathcal{E}_{bf}(f) \leq \mathcal{E}_{bf}(g)$. Here we used the shorthand notation $\mathcal{E}_{bf}(f) := \mathcal{E}_{bf}(f, f)$.

Any Dirichlet form $\mathcal{E}_{bf}$ has a generator $L$ which is a non-positive definite self-adjoint operator on $L^2([0, r], \mu)$ with domains $\mathcal{D} = \mathcal{D}(L) \subset \mathcal{F}$ such that

$$\mathcal{E}_{bf}(f, g) = (L f, g)$$

for $f \in \mathcal{D}$ and $g \in \mathcal{F}$ where $\mathcal{D}$ is dense in $\mathcal{F}$, see details in [23, Sec.2.2].

**Proposition 7.1.** The operator $L|_{\mathcal{D}}$ is an extension of $L_0|_{\Lambda}$, this is,

$$ (7.13) \quad L f = L_0 f, \text{ for any } f \in \Lambda. $$

Proof: For any $f, g \in \Lambda$

$$ (L f, g) = -\mathcal{E}_{bf}(f, g) $$

$$ = -\omega_n \int_0^r f'(t)g'(t)h^{n-1}(t)dt $$

$$ = -\omega_n \int_0^r \left[ \frac{d}{dt} (f'(t)g(t)h^{n-1}(t)) - g(t) \frac{d}{dt} (f'(t)h^{n-1}(t)) \right] dt $$

$$ = -\omega_n \left[ f'(t)g(t)h^{n-1}(t) \right]_0^r + \int_0^r g(t)L_0f(t)d\mu(t) $$

$$ = (L_0 f, g). $$

The generator $L$ determines the heat semigroup $P_t = e^{-Lt}$, $t \geq 0$ which posses a heat kernel $p(t, x, y)$ and Green function $g(x, y) = \int_0^\infty p(t, x, y)dt$. Observe that $L|_{\mathcal{D}}$ is a self-adjoint extension of $L_0|_{\Lambda}$. Thus, the solution of eigenvalue problem (7.9) is an infinite sequence of eigenvalues $0 < \lambda_1^{rad} < \lambda_2^{rad} < \cdots$, (the radial spectrum of $B(o, r)$). Moreover, $L_0 = \Delta_{\mu}$, then we have by Theorem 6.1 that

$$\sum_{i=1}^{\infty} \frac{1}{\lambda_i^{rad}(B(o, r))} = \int_0^r g(x, x)dx. $$

We need to determine the Green function $g(x, y)$ for the operator $L$.

**Proposition 7.2.** The Green function $g(x, y)$ for the operator $L$ is given by

$$\sum_{i=1}^{\infty} \frac{1}{\lambda_i^{rad}(B(o, r))} = \int_0^r g(x, x)dx. $$

We need to determine the Green function $g(x, y)$ for the operator $L$.

**Proposition 7.2.** The Green function $g(x, y)$ for the operator $L$ is given by

$$ (7.16) \quad g(x, y) = \int_x^r \frac{1}{\omega_n h^{n-1}(t)} \theta_y(t)dt, $$

where

$$ (7.17) \quad \theta_y(t) = \begin{cases} 1 \quad \text{if} \quad t \geq y \\ 0 \quad \text{if} \quad t < y. \end{cases} $$

Moreover,

$$ (7.18) \quad G(f)(x) = \int_0^r g(x, y)f(y)d\mu(y) = T(f)(x). $$

Here $T$ is the operator defined in (4.10).
Proof: We need to prove that \( g(r, y) = 0 \), \( \lim_{x \to 0^+} \frac{\partial}{\partial x} g(x, y) = 0 \), \( \mathcal{L}_x(x, y) = 0 \) for \( x \neq y \) and \( \mathcal{L}_g(x, \cdot) = -\delta_x \). The two first properties are straightforward from equation (7.16). When \( x \neq y \) we have

\[
(7.19) \quad g(x, y) = \begin{cases} 
\int_x^r \frac{1}{\omega_n h^{n-1}(t)} \, dt & \text{if } x \geq y \\
\int_y^r \frac{1}{\omega_n h^{n-1}(t)} \, dt & \text{if } x \leq y.
\end{cases}
\]

Then,

\[
(7.20) \quad \mathcal{L}_x g(x, y) = \begin{cases} 
L_0 \left( \int_x^r \frac{1}{\omega_n h^{n-1}(t)} \, dt \right) & \text{if } x \geq y \\
0 & \text{if } y \leq x.
\end{cases}
\]

The last requirement \( \mathcal{L}_g(x, \cdot) = -\delta_x \) is proven now. We have to show that if \( f \in \mathcal{F} \) then

\[
(7.21) \quad -\int_0^r \mathcal{L}_x g(x, y) f(y) \, d\mu(y) = -\mathcal{L}_x \int_0^r g(x, y) f(y) \, d\mu(y) = -\mathcal{L}_x G(f)(x) = f(x).
\]

On the other hand,

\[
G(f)(x) = \int_0^r g(x, y) f(y) \, d\mu(y) = \int_0^r \left( \int_x^r \frac{\theta_y(t)}{h^{n-1}(t)} \, dt \right) f(y) h^{n-1}(y) \, dy = \int_x^r \frac{1}{h^{n-1}(t)} \left( \int_0^t \theta_y(t) f(y) h^{n-1}(y) \, dy + \int_t^r \theta_y(t) f(y) h^{n-1}(y) \, dy \right) \, dt = \int_x^r \frac{1}{h^{n-1}(t)} \left( \int_0^t f(y) h^{n-1}(y) \, dy \right) \, dt = T(f)(x).
\]

Hence we have to prove that \( \mathcal{L} \circ T = -\text{id} \). Since \( T: \Lambda \to \Lambda \), for any \( u \in \Lambda \)

\[
(7.22) \quad \mathcal{L} \circ T(u) = D^2_h \circ T(u) = \frac{\partial^2 T(u)}{\partial t^2} + (n - 1) \frac{h'(t)}{h(t)} \frac{\partial T(u)}{\partial t}.
\]

However,

\[
\frac{\partial^2 T(u)}{\partial t^2}(t) = (n - 1) \frac{h'(t)}{h(t)} \frac{1}{h^{n-1}} \int_0^t h^{n-1}(s) u(s) \, ds - u(t)
\]

\[
(n - 1) \frac{h'(t)}{h(t)} \cdot \frac{\partial T(u)}{\partial t}(t) = -(n - 1) \frac{h'(t)}{h(t)} \frac{1}{h^{n-1}(t)} \int_0^t h^{n-1}(s) u(s) \, ds.
\]
Proposition 7.3. The Green function with density $d\mu$ We need to find the Green function discrete, say $\sigma$ have from 7.15 that

$$T = \frac{1}{\lambda_i(B(o, r))} = \int_0^r g(x, x) d\mu(x)$$

$$= \int_0^r \left( \int_x^{r} \frac{dt}{h^{n-1}(t)} \right) h^{n-1}(x) dx$$

$$= \int_0^r \frac{\int_0^{x} h^{n-1}(t) dt}{h^{n-1}(x)} dx$$

$$= \int_0^r \frac{V(s)}{S(s)} ds,$$

(7.23)

This proves identity (2.7). If $M^n_h$ is stochastically incomplete, then its spectrum is discrete, say $\sigma(M^n_h) = \{\lambda_1(M^n_h) < \lambda_2(M^n_h) \leq \cdots \}$. Taking the limits in (7.24) we obtain

$$\lim_{r \to \infty} \sum_{i=1}^{\infty} \frac{1}{\lambda_i^{(r)}(B(o, r))} = \int_0^\infty \frac{V(s)}{S(s)} ds < \infty.$$ 

To prove identity (2.8) we recall that $\lambda_i^{rad}(M^n_h) = \lim_{r \to \infty} \lambda_i^{rad}(B(o, r))$. This proves that

$$\sum_{i=1}^{\infty} \frac{1}{\lambda_i^{rad}(M^n_h)} = \int_0^\infty \frac{V(s)}{S(s)} ds < \infty,$$

(7.25)

7.0.4. Proof of Theorem 2.7. The proof of Theorem 2.7 is closed to the proof of Theorem 2.5. The spectrum of $B(o, r)$, without repetitions, is the union of the $\nu_i$-spectrums $\sigma^{i}(B(o, r))$, $l = 0, 1, \ldots$

$$\sigma(B(o, r)) = \bigcup_{i=0}^{\infty} \sigma^{i}(B(o, r)) = \{\lambda_{l,j}\}_{l=0,j=1}^{\infty},$$

each $\lambda_{l,j}$ with multiplicity $\delta(l, m)$. The eigenvalues of the $l$-spectrum $\sigma^{l}(B(o, r))$, $l \geq 1$, are the eigenvalues of the the operator

$$L_l(T(t)) = T''(t) + (n-1) \frac{1}{l} T'(t) - \frac{\nu_l}{l^2} T(t),$$

$$\nu_l = l(l + m - 2),$$
in the following Dirichlet eigenvalue problem on $[0, r]$.

$$T'' + (m-1) \frac{1}{l} T' + (\lambda - \frac{\nu_l}{l^2}) T = 0 \quad \text{in} \quad [0, r]$$

(7.24)

with initial conditions $T(t) \sim c \cdot t^l$ as $t \to 0$ when $l = 1, 2 \ldots$ and $T(r) = 0$. The procedure to show that the operator $L_l$ has a self-adjoint extension $L_l$ and a Green function $g_l$ is similar to the procedure in the proof of Theorem 2.5. By Theorem 6.1 we have that

$$\sum_{i=1}^{\infty} \frac{1}{\lambda_{l,i}(B(o, r))} = \int_0^r g_l(x, x) dx.$$ 

We need to find the Green function $g_l$.

**Proposition 7.3.** The Green function $g_l(x, y)$ for the $L_l$ operator on $M = [0, r]$ with density $d\mu(x) = \omega_i x^{n-1} dx$ boundary conditions

$$u'(0) = u(r) = 0,$$

with $u(x) = g(x, y)$ for any $y \in (0, r)$
is given by

\[
(7.26) \quad g(x, y) = \begin{cases} 
\frac{x y^\alpha}{\beta \omega_n y^\gamma - 1} \left( \frac{1}{y^\beta} - \frac{1}{r^\beta} \right) & \text{if } 0 \leq x < y \\
\frac{x y^\alpha}{\beta \omega_n y^\gamma - 1} \left( \frac{1}{x^\beta} - \frac{1}{r^\beta} \right) & \text{if } y \leq x \leq r 
\end{cases}
\]

with \( \alpha = l + n - 1 \) and \( \beta = 2l + n - 2 \).

**Proof.** Observe first of all that

\[
u'(0) = \frac{\partial}{\partial x} g(x, y) \bigg|_{x=0} = 0, \quad u(r) = g(r, y) = 0, \quad \forall y \in [0, r].
\]

On the other hand

\[
L_l(u'(x)) = u''(x) + \frac{l(l + n - 2)}{x^2} u(x) = 0
\]

because

\[
L_l(x^\alpha) = L_l(x^{l-\beta}) = 0.
\]

For any function \( f: [0, r] \to \mathbb{R} \), we have then

\[
G_l(f)(x) = \int_0^r g_l(x, y) f(y) d\mu(y)
\]

\[
= \int_0^x g_l(x, y) f(y) d\mu(y) + \int_x^r g_l(x, y) f(y) d\mu(y)
\]

\[
= \frac{x^l}{\beta} \left( \frac{1}{x^\beta} - \frac{1}{r^\beta} \right) \int_0^x y^\alpha f(y) dy + \frac{x^l}{\beta} \int_x^r y^\alpha \left( \frac{1}{y^\beta} - \frac{1}{r^\beta} \right) f(y) dy
\]

\[
= \frac{x^{l-\beta}}{\beta} \int_0^x y^\alpha f(y) dy + \frac{x^l}{\beta} \int_x^r y^{\alpha-\beta} f(y) dy - \frac{x^l}{\beta r^\beta} \int_0^r y^\alpha f(y) dy.
\]

and

\[
L_l(G_l(f)(x)) = L_l \left( \frac{x^{l-\beta}}{\beta} \int_0^x y^\alpha f(y) dy \right) + L_l \left( \frac{x^l}{\beta} \int_x^r y^{\alpha-\beta} f(y) dy \right) - L_l \left( \frac{x^l}{\beta r^\beta} \right) \int_0^r y^\alpha f(y) dy
\]

\[
= L_l \left( \frac{x^{l-\beta}}{\beta} \right) \int_0^x y^\alpha f(y) dy + L_l \left( \frac{x^l}{\beta} \right) \int_x^r y^{\alpha-\beta} f(y) dy - x^{l+\alpha-\beta-1} f(x)
\]

\[
= -f(x),
\]

where we have applied \( l + \alpha - \beta - 1 = 0 \). Therefore we conclude that,

\[
L_l \circ G_l(f)(x) = -f(x),
\]
and \( g_i \) is a Green function for our problem. Now
\[
\left( \sum_{i=1}^{\infty} \frac{1}{\lambda_{l,i}} \right) = \int_0^r g_i(x,x) d\mu(x) = \int_0^r \frac{x^{l+\alpha}}{m} \left( \frac{1}{x^\beta} - \frac{1}{r^\beta} \right) dx
\]
\[
= \frac{r^{l+\alpha-\beta+1}}{(l+\alpha+1)(l+\alpha-\beta+1)}
\]
\[
= \frac{r^2}{2(2l+n)} = \left( \frac{1}{1+\frac{2}{n}} \right) \frac{r^2}{2n}
\]
\[
= \left( \frac{1}{1+\frac{2}{n}} \right) \max_{x \in B_{y^n}(r)} E_r(x).
\]
This proves (2.9).

To prove (2.10) we proceed as follows.
\[
\sum_{i=1}^{\infty} \frac{1}{(\lambda_{l,i})^2} = \int_0^r \int_0^r g_i(x,y) g_i(y,x) d\mu(y) d\mu(x)
\]
\[
= \int_0^r \int_0^r \omega_n^2 \frac{x^{n-1} y^{n-1} g_i(x,y)}{r^{n-1}} g_i(y,x) dxdy
\]
\[
= \int_0^r \omega_n^2 \frac{x^{n-1}}{r^{n-1}} \left( \int_0^x g_i(x,y) y^{n-1} dy + \int_x^r g(x,y) g_i(y,x) dy \right) dx
\]
\[
= \frac{1}{\beta^2} \int_0^r \left( \int_0^x x^{l+\alpha+y^{l+\alpha}} \left( \frac{1}{x^\beta} - \frac{1}{r^\beta} \right)^2 dy + \int_x^r x^{l+\alpha+y^{l+\alpha}} \left( \frac{1}{y^\beta} - \frac{1}{r^\beta} \right)^2 dy \right) dx
\]
\[
= \frac{r^{2(l+\alpha-\beta+1)}}{(\alpha+1+\beta+1)(\alpha-\beta+1)}
\]
\[
= \frac{r^4}{2(2l+n)^2(2l+n)}
\]
\[
\square
\]

7.0.5. Proof of Theorem 4.1 Let \((\Omega, ds^2, \mu)\) be a weighted bounded open subset, with smooth boundary \(\partial\Omega \neq \emptyset\), of a Riemannian weighted \(m\)-manifold \((M, ds^2, \mu)\). The Green operator \(G: L^2(\Omega, \mu) \to L^2(\Omega, \mu)\) is given by
\[
G(f)(x) = \int_0^\infty \int_{\Omega} p_t(x,y) f(y) d\mu(y) dt
\]
(7.27)
where \(p_t(x,y)\) is the heat kernel of the operator \(\mathcal{L} = -\triangle_{\mu} |_{W^2(\Omega, \mu)}\) and \(g(x,y)\) is the Green function of \(\Omega\). Let \(\{u_1, u_2, u_3, \ldots\}\) be a \(L^2(\Omega, \mu)\)-orthonormal basis of \(L^2(\Omega, \mu)\) formed by eigenfunctions \(u_i\) with eigenvalue \(\lambda_i(\Omega) \in \sigma(\Omega)\). The proof Theorem 4.1 is divided in few simple propositions.

Proposition 7.4. If \(\lambda_1(\Omega) > 0\) then the Green operator \(G\) satisfies
\[
G(f)(x) = \sum_{i=1}^{\infty} \frac{f^i u_i(x)}{\lambda_i(\Omega)} \quad \text{for any } f \in L^2(\Omega, \mu),
\]
(7.28)
where \( f^i = \int_\Omega f(x)u_i(x)\,d\mu(x) \). Moreover,

\[
G^k(f) = \sum_{i=1}^{\infty} \frac{f^i u_i(x)}{\lambda_i^k(\Omega)},
\]

where \( G^k = G \circ \cdots \circ G \).

**Proof.** Let \( \{u_i\}_{i=1}^{\infty} \) be a complete orthonormal basis of \( L^2(\Omega) \) of eigenfunctions. For any \( f \in L^2(\Omega, \mu) \) we have

\[
f(x) = \sum_{i=1}^{\infty} f^i u_i(x) \quad \text{in} \quad L^2 - \text{sense}.
\]

This means that \( \|f - \sum_{i=1}^{k} f^i u_i\|_{L^2} \to 0 \) as \( k \to \infty \). Set

\[
f_k(x) = \sum_{i=k+1}^{\infty} f^i u_i(x).
\]

Since \( \sum_{i=k+1}^{\infty} (f^i)^2 < \infty \) we have then \( f_k \in L^2(\Omega, \mu) \). Thus

\[
U^k = f - f_k = \sum_{i=1}^{k} f^i u_i(x) \in L^2(\Omega, \mu).
\]

Therefore,

\[
\|f - \sum_{i=1}^{k} f^i u_i(x)\|_{L^2} = \|f - U^k\| = \|f_k\|_{L^2} \to 0 \quad \text{as} \quad k \to \infty.
\]

For any \( k \in \mathbb{N} \),

\[
G(f)(x) = \int_\Omega g(x, y)f(y)\,d\mu(y)
= \int_\Omega g(x, y)U^k(y)\,d\mu(y) + \int_\Omega g(x, y)f_k(y)\,d\mu(y)
= \int_\Omega g(x, y) \sum_{i=1}^{k} f^i u_i(y)\,d\mu(y) + \int_\Omega g(x, y)f_k(y)\,d\mu(y)
= \sum_{i=1}^{k} f^i \int_\Omega g(x, y)u_i(y)\,d\mu(y) + G(f_k)(x)
= \sum_{i=1}^{k} \frac{f^i u_i(x)}{\lambda_i(\Omega)} + G(f_k)(x).
\]

Therefore, from (7.31) and [22 Exercise 13.6] we have

\[
\|G(f) - \sum_{i=1}^{k} \frac{f^i u_i}{\lambda_i(\Omega)}\|_{L^2} = \|G(f_k)\|_{L^2} \leq \frac{\|f_k\|_{L^2}}{\lambda_1(\Omega)} \to 0 \quad \text{as} \quad k \to \infty.
\]
This proves (7.28). We used that \( \int_\Omega g(x,y)u_i(y)d\mu(y) = G(u_i)(x) = u_i(x)/\lambda_i(\Omega) \).

We will prove (7.29) by induction. Assume that

\[
G^k(f)(x) = \sum_{i=1}^{\infty} \frac{f^i u_i(x)}{\lambda^i(\Omega)} \in L^2(\Omega).
\]

Then, since \((G^k(f))^i = \frac{f^i}{\lambda^i(\Omega)}\), we have

\[
G^{k+1}(f)(x) = G(G^k(f))(x) = \sum_{i=1}^{\infty} \frac{(G^k(f))^i u_i(x)}{\lambda_i(\Omega)} = \sum_{i=1}^{\infty} \frac{f^i u_i(x)}{\lambda_i^{k+1}(\Omega)}.
\]

\[\Box\]

**Proposition 7.5.** Let \( \Omega \subset M \) be a bounded open set with smooth boundary \( \partial \Omega \) such that \( \lambda_1(\Omega) > 0 \). Then, for any \( f \in L^2(\Omega) \),

\[
\|G^k(f)\|^2 = \sum_{i=1}^{\infty} \frac{(f^i)^2}{\lambda_i^{2k}(\Omega)}.
\]

Moreover,

\[
\lim_{k \to \infty} \frac{\|G^k(f)\|}{\|G^{k+1}(f)\|} = \lambda_l(\Omega),
\]

\[
\lim_{k \to \infty} \frac{G^k(f)}{\|G^k(f)\|} \to \phi_l \in \text{Ker}(\Delta_\mu + \lambda_l) \text{ in } L^2,
\]

where \( l \) is the first integer such that,

\[
\int_\Omega f(y)u_l(y)d\mu(y) \neq 0.
\]

**Proof.** Identity (7.34) follows from equation (7.29) and Passerval’s identity.

Using (7.29) we have

\[
\frac{\|G^k(f)\|^2}{\|G^{k+1}(f)\|^2} = \frac{\sum_{i=1}^{\infty} \frac{(f^i)^2}{\lambda_i^{2k}(\Omega)}}{\sum_{i=1}^{\infty} \frac{(f^i)^2}{\lambda_i^{2k+2}(\Omega)}} = \lambda_l^2(\Omega) \frac{\sum_{i=1}^{\infty} (f^i)^2 \left(\frac{\lambda_l(\Omega)}{\lambda_i(\Omega)}\right)^{2k}}{\sum_{i=1}^{\infty} (f^i)^2 \left(\frac{\lambda_l(\Omega)}{\lambda_i(\Omega)}\right)^{2k+2}}.
\]
Since $\frac{\lambda_i(\Omega)}{\lambda_l(\Omega)} < 1$ for any $i > l$, we have

\begin{equation}
\lim_{k \to \infty} \left( \frac{\lambda_l(\Omega)}{\lambda_i(\Omega)} \right)^{2k+2} = \frac{\lambda_l(\Omega)}{\lambda_i(\Omega)}^{2k} = \delta_{il}.
\end{equation}

Then,

\begin{equation}
\lim_{k \to \infty} \frac{\|G_k(f)\|^2}{\|G_{k+1}(f)\|^2} = \lambda_l^2(\Omega).
\end{equation}

This proves identity (4.3) of Theorem 4.1. By the identities (7.28) and (7.34) we have that

\begin{equation}
\int_{\Omega} G_k(f) u_i d\mu = \frac{1}{\lambda_i^k(\Omega)} \sum_{j=l}^{\infty} (f^j)^2 \lambda_{j}^{2k}(\Omega)
\end{equation}

Since $\lambda_j(\Omega) < \lambda_l(\Omega)$, we have that

\begin{equation}
\lim_{k \to \infty} \frac{G_k(f)}{\|G_k(f)\|} u_i d\mu = \int_{\Omega} \frac{G_k(f)}{\|G_k(f)\|} u_i d\mu = \delta_{il}.
\end{equation}

This shows that $\phi_l = \lim_{k \to \infty} \frac{G_k(f)}{\|G_k(f)\|} \in \text{Ker}(\Delta_l + \lambda_l)$ and proves identity (4.5) of Theorem 4.1.

**Corollary 7.6.** Under the assumptions of the above proposition and letting $f_1$ be the function

\begin{equation}
f_1 := f - \lambda_l(\Omega)G(f),
\end{equation}

we have

\begin{equation}
\lim_{k \to \infty} \frac{\|G_k(f_1)\|}{\|G_{k+1}(f_1)\|} = \lambda_n(\Omega)
\end{equation}

\begin{equation}
\lim_{k \to \infty} \frac{G_k(f_1)}{\|G_k(f_1)\|} \to \phi_n \in \text{Ker}(\Delta_n + \lambda_n) \text{ in } L^2.
\end{equation}

with,

$\lambda_n(\Omega) > \lambda_l(\Omega)$.

**Proof.** We only have to apply the above proposition taking into account that by equality (7.28)

\begin{equation}
f - \lambda_l(\Omega)G(f) = \sum_{i=l}^{\infty} f^i u_i - \sum_{i=l}^{\infty} f^i u_i \frac{\lambda_i(\Omega)}{\lambda_l(\Omega)}
\end{equation}

\begin{equation}
= \sum_{i=l+1}^{\infty} f^i u_i \left(1 - \frac{\lambda_i(\Omega)}{\lambda_l(\Omega)}\right).
\end{equation}

□
In order to show the uniqueness, note that first of all that
\[
\phi_k \text{ unique.}
\]
Theorem 5.2 states that this problem admits a unique family of solutions \(\{\phi_k\}_{k=1}^{\infty}\),

\[
\int_{B_{d_0}(r)} f(x)u_1(x)d\mu(x) \neq 0
\]
positive or negative) function \(f\). Hence, using Proposition \(\text{7.6}\) we have

**Corollary 7.7.** Let \(\Omega \subset M\) be a bounded open set with smooth boundary \(\partial \Omega \neq \emptyset\). Then, for any positive (or negative) \(f \in L^2(\Omega, \mu)\),

\[
\lim_{k \to \infty} \frac{\|G^k(f)\|}{\|G^{k+1}(f)\|} = \lambda_1(\Omega),
\]

(7.46)

\[
\lim_{k \to \infty} \frac{G^k(f)}{G^{k+1}(f)} = \phi_1 \in \text{Ker}(\Delta_\mu + \lambda_1) \text{ in } L^2.
\]

7.0.6. **Proof of Theorem 5.2.** Let \(\Omega\) be an open relatively compact domain with smooth boundary \(\partial \Omega \neq \emptyset\) of a Riemannian manifold \(M\). Consider the following hierarchy Dirichlet problem

\[
\begin{align*}
\Delta \phi_k + k \phi_{k-1} &= 0 \quad \text{on } \Omega \\
\phi_k|_{\partial \Omega} &= 0.
\end{align*}
\]

(7.47)

Theorem 5.2 states that this problem admits a unique family of solutions \(\{\phi_k\}_{k=1}^{\infty}\), given by

\[
\phi_k(x) = k! G^k(1)(x).
\]

In order to show the uniqueness, note that first of all that \(\phi_1\) is unique. Otherwise we would have two different functions \(\phi_1^1\) and \(\phi_1^2\) such that

\[
\begin{align*}
\Delta \phi_1^1 &= \Delta \phi_1^2 = -1 \\
\phi_1^1|_{\partial \Omega} &= \phi_1^2|_{\partial \Omega} = 0.
\end{align*}
\]

(7.49)

Hence \(\phi_1^1 - \phi_1^2\) would be an harmonic function in \(\Omega\) with \(\phi_1^1 - \phi_1^2 = 0\) in the boundary \(\partial \Omega\), then by the minimum principle \(\phi_1^1 = \phi_1^2\), a contradiction. Assume that in the family of solutions \(\{\phi_k\}_{k=1}^{\infty}\) the first \(j\) functions \(\phi_1, \phi_2, \ldots, \phi_j\) are unique but there exists two different \(\phi_{j+1}^1\) and \(\phi_{j+1}^2\) solutions in the \(j+1\)-th slot. Then \(\Delta(\phi_{j+1}^1 - \phi_{j+1}^2) = 0\) and \(\phi_{j+1}^1 - \phi_{j+1}^2 = 0\) on \(\partial \Omega\). Then \(\phi_{j+1}^1 = \phi_{j+1}^2\) Now, functions defined by \(\phi_k(x) = k G(\phi_{k-1})(x), k = 1, \ldots\) are the solutions of the problem (7.47). Since the solution to the problem (7.47) is unique we only have to check that

\[
\Delta (k G(\phi_{k-1})) = -k \phi_{k-1}.
\]

(7.50)

But that is straightforward because the Green operator is the inverse of \(-\Delta\). To prove (7.48) let us use the induction method. Observe that \(\phi_1 = G(1)(x)\). Suppose that equation (7.48) is true and let us compute \(\phi_{k+1}(x)\).

\[
\phi_{k+1}(x) = (k+1)G(\phi_k)(x) = (k+1)G(k! G^k(1)(x))
\]

(7.51)

\[
= (k+1)k! G(G^k(1))(x) = (k+1)! G^{k+1}(1)(x).
\]

This proves (7.2) in Theorem 5.2. On the other hand, by (7.29),

\[
G^k(1)(x) = \sum_{i=1}^{\infty} \frac{a_i u_i(x)}{\lambda^k_i(\Omega)},
\]
where \( a_i = \int_\Omega u_i(y)d\mu(y) \). Thus

\[
\phi_k(x) = k! G^k(1) = k! \sum_{i=1}^{\infty} \frac{a_i u_i(x)}{\lambda_i^k(\Omega)}.
\]

Moreover, the \( L^1(\Omega,\mu) \)-momentum spectrum of \( \Omega \) is readily obtained by

\[
A_k(\Omega) = \int_\Omega \phi_k d\mu = k! \sum_{i=1}^{\infty} \frac{a_i^2}{\lambda_i^k(\Omega)}.
\]

This proves Theorem 5.2. The proof of Corollary 5.3 is straightforward.

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