Quantum order from string-net condensations and origin of light and massless fermions

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Recently, it was pointed out that quantum orders and the associated projective symmetry groups can produce and protect massless gauge bosons and massless fermions in local bosonic models. In this paper, we demonstrate that a state with such kind of quantum orders can be viewed as a string-net condensed state. The emerging gauge bosons and fermions in local bosonic models can be regarded as a direct consequence of string-net condensation. The gauge bosons are fluctuations of large closed string-nets which are condensed in the ground state. The ends of open strings (or nodes of open string-nets) are the charged particles of the corresponding gauge field. For certain types of strings, the nodes of string-nets can even be fermions. According to the string-net picture, fermions always carry gauge charges. This suggests the existence of a new discrete gauge field that couples to neutrinos and neutrons. We also discuss how chiral symmetry that protects massless Dirac fermions can emerge from the projective symmetry of quantum order.

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I. INTRODUCTION

A. Fundamental questions about light and fermions

We have known light and fermions for many years. But we still cannot give a satisfactory answer to the following fundamental questions: What are light and fermions? Where light and fermions come from? Why light and fermions exist? At moment, the standard answers to the above fundamental questions appear to be “light is the particle described by a gauge field” and “fermions are the particles described by anti-commuting fields”. Here, we like to argue that there is another possible answer to the above questions: our vacuum is filled with string-like objects that form network of arbitrary sizes and those string-nets form a quantum condensed state. According to the string-net picture, the light (and other gauge bosons) is a vibration of the condensed string-nets and fermions are ends of strings (or nodes of string-nets). The string-net condensation provides a unified origin of light and fermions.[62]
Before discussing the above fundamental questions in more detail, we would like to clarify what do we mean by “light exists” and “fermions exist”. We know that there is a natural mass scale in physics – the Planck mass. Planck mass is so large that any observed particle have a mass at least factor $10^{16}$ smaller than the Planck mass. So all the observed particles can be treated as massless when compared with Planck mass. When we ask why some particles exist, we really ask why those particles are massless (or nearly massless when compared with Planck mass). So the real issue is to understand what makes certain excitations (such as light and fermions) massless. We have known that symmetry breaking is a way to get gapless bosonic excitations. We will see that string-net condensation is another way to get gapless excitations. However, string-net condensations can generate massless gauge bosons and massless fermions.

Second, we would like to clarify what do we mean by “origin of light and fermions”. We know that everything has to come from something. So when we ask “where light and fermions come from”, we have assumed that there are some things simpler and more fundamental than light and fermions. In the section II, we define local bosonic models which are simpler than models with gauge fields coupled to fermions. We will regard local bosonic models as more fundamental (the locality principle). We will show that light and fermions can emerge from a local bosonic model if the model contains a condensation of nets of string-like object in its ground state.

After the above two clarifications, we can state more precisely the meaning of “string-net condensation provides another possible answer to the fundamental questions about light and fermions”. When we say gauge bosons and fermions originate from string-net condensation, we really mean that (nearly) massless gauge bosons and fermions originate from string-net condensation in a local bosonic model.

B. Gapless phonon and symmetry breaking orders

Before considering the origin of massless photon and massless fermions, let us consider a simpler massless (or gapless) excitation – phonon. We can ask three similar questions about phonon: What is phonon? Where phonon comes from? Why phonon exists? We know that those are scientific questions and we know their answers. Phonon is a vibration of a crystal. Phonon comes from a spontaneous translation symmetry breaking. Phonon exists because the translation-symmetry-breaking phase actually exists in nature. In particular, the gaplessness of phonon is directly originated from and protected by the spontaneous translation symmetry breaking.[1, 2] Many other gapless excitations, such as spin wave, superfluid mode etc., also come from condensation of point-like objects that break certain symmetries.

It is quite interesting to see that our understanding of a gapless excitation - phonon - is rooted in our understanding of phases of matter. According to Landau’s theory,[3] phases of matter are different because they have different broken symmetries. The symmetry description of phases is very powerful. It allows us to classify all possible crystals. It also provides the origin for gapless phonons and many other gapless excitations. Until a few years ago, it was believed that the condensations of point-like objects, and the related symmetry breaking and order parameters, can describe all the orders (or phases) in nature.

C. The existence of light and fermions implies the existence of new orders

Knowing light as a massless excitation, one may wonder maybe light, just like phonon, is also a Nambu-Goldstone mode from a broken symmetry. However, experiments tell us that a $U(1)$ gauge boson, such as light, is really different from a Nambu-Goldstone mode in 3+1 dimensions. Therefore it is impossible to use Landau’s symmetry breaking theory and condensation of point-like objects to understand the origin and the masslessness of light. Also, Nambu-Goldstone modes are always bosonic; thus it is impossible to use symmetry breaking to understand the origin and the (nearly) masslessness of fermions. It seems that there does not exist any order that can give rise to massless light and massless fermions. Because of this, we put light and electron into a different category than phonon. We regarded them as elementary and introduced them by hand into our theory of nature.

However, if we believe light and electrons, just like phonon, exist for a reason, then such a reason must be a certain order in our vacuum that protect the masslessness of light and electron. (Here we have assumed that light and electron are not something that we place in an empty vacuum. Our vacuum is more like an “ocean” which is not empty. Light and electron are collective excitations that correspond to certain patterns of “water” motion.) Now the question is that what kind of order can give rise to light and electron, and protect their masslessness.

If we really believe in the equality between light, electron and phonon, then the very existence of light and fermions indicates that our understanding of states of matter is incomplete. We should deepen and expand our understanding of the states of matter. There should be new states of matter that contain new kind of orders. The new orders will produce light and electron, and protect their masslessness.

D. Topological order and quantum order

After the discovery of fractional quantum Hall (FQH) effect,[4, 5] it became clear that the Landau’s symmetry breaking theory cannot describe different FQH states, since those states all have the same symmetry. It was proposed that FQH states contain a new kind of order - topological order.[6] Topological order is new because it cannot be described by symmetry breaking, long range correlation, and local order parameters. Non of the usual tools that we used to characterize phases applies to topological order. Despite of this, topological order is not an empty concept. Topological order can be characterized by a new set of tools, such as the number of degenerate
Topological field theory

Topological orders

String–net condensation

Projective symmetry group

Phase transitions

Quantum system

Classical system

Gapped

Non–gapped

Quantum orders

Classical orders

FIG. 1: A classification of different orders in matter. (We view our vacuum as one kind of matter.)

ground states, quasiparticle statistics, and edge states. It was shown that the ground state degeneracy of a topological ordered state is a universal property since the degeneracy is robust against any perturbations. Such a topological degeneracy demonstrates the existence of topological order. It can also be used to perform fault tolerant quantum computations.

Recently, the concept of topological order was generalized to quantum order. Quantum order is used to describe new kinds of orders in gapless quantum states. One way to understand quantum order is to see how it fits into a general classification scheme of orders (see Fig. 1). First, different orders can be divided into two classes: symmetry breaking orders and non-symmetry breaking orders. The symmetry breaking orders can be described by a local order parameter and can be said to contain a condensation of point-like objects. All the symmetry breaking orders can be understood in terms of Landau’s symmetry breaking theory. The non-symmetry breaking orders cannot be described by symmetry breaking, neither by the related local order parameters and long range correlations. Thus they are a new kind of orders. If a quantum system (a state at zero temperature) contains a non-symmetry breaking order, then the system is said to contain a non-trivial quantum order. We see that a quantum order is simply a non-symmetry breaking order in a quantum system.

Quantum order can be further divided into many sub-classes. If a quantum state is gapped, then the corresponding quantum order will be called topological order. The low energy effective theory of a topological ordered state will be a topological field theory. The second class of quantum orders appear in Fermi liquids (or free fermion systems). The different quantum orders in Fermi liquids are classified by the Fermi surface topology.

E. The quantum orders from string-net condensations

In this paper, we will concentrate on the third class of quantum orders – the quantum orders from condensation of nets of strings, or simply, string-net condensation. This class of quantum orders shares some similarities with the symmetry breaking orders of “particle” condensation. We know that different symmetry break-

ing orders can be classified by symmetry groups. Using group theory, we can classify all the 230 crystal orders in three dimensions. The symmetry also produces and protects gapless Nambu-Goldstone bosons. Similarly, as we will see later in this paper, different string-net condensations (and the corresponding quantum orders) can be classified by a mathematical object called projective symmetry group (PSG). Using PSG, we can classify over 100 different 2D spin liquids that all have the same symmetry. Just like symmetry group, PSG can also produce and protect gapless excitations. However, unlike symmetry group, PSG produces and protects gapless bosons and gapless fermions. Because of this, we can say light and massless fermions can have a unified origin. They can come from string-net condensations.

We used to believe that to have light and fermions in our theory, we have to introduce by hand a fundamental $U(1)$ gauge field and anti-commuting fermion fields, since at that time we did not know any collective modes that behave like gauge bosons and fermions. Now, we know that gauge bosons and fermions appear commonly and naturally in quantum ordered states, as fluctuations of condensed string-nets and ends of open strings. This raises an issue: do light and fermions come from a fundamental $U(1)$ gauge field and anti-commuting fields as in the 123 standard model or do they come from a particular quantum order in our vacuum? Clearly it is more natural to assume light and fermions come from a quantum order in our vacuum. From the connection between string-net condensation, quantum order, and massless gauge/fermion excitations, it is very tempting to propose the following answers to the fundamental questions about light and (nearly) massless fermions:

**What are light and fermions?**

Light is a fluctuation of condensed string-nets of arbitrary sizes. Fermions are ends of open strings.

**Where light and (nearly) massless fermions come from?**

Light and the fermions come from the collective motions of nets of string-like objects that fill our vacuum.

**Why light and (nearly) massless fermions exist?**

Light and the fermions exist because our vacuum chooses to have a string-net condensation.

Had our vacuum chosen to have a “particle” condensation, there would be only Nambu-Goldstone bosons at low energies. Such a universe would be very boring. String-net condensation and the resulting light and (nearly) massless fermions provide a much more interesting universe, at least interesting enough to support intelligent life to study the origin of light and massless fermions.

The string-net picture of fermions explains why there is always an even number of fermions in our universe. The string-net picture for gauge bosons and fermions also has an experimental prediction: all fermions must carry certain gauge charges. At first sight, this prediction
appears to contradict with the known experimental fact that neutron carry no gauge charges. Thus one may think the string-net picture of gauge bosons and fermions has already been falsified by experiments. Here we would like to point out that the string-net picture of gauge bosons and fermions can still be correct if we assume the existence of a new discrete gauge field, such as a $Z_2$ gauge field, in our universe. In this case, neutrons and neutrinos carry a non-zero charge of the discrete gauge field. Therefore, the string-net picture of gauge bosons and fermions predict the existence of discrete gauge excitations (such as gauge flux lines) in our universe.

We would like to remark that, despite the similarity, the above string-net picture of gauge bosons and fermions is different from the picture of standard superstring theory. In standard superstring theory, closed strings correspond to gravitons, and open string correspond to gauge bosons. All the elementary particles correspond to different vibration modes of small strings in the superstring theory. Also, the fermions in the standard superstring theory come from the fermion fields on the world sheet. In our string-net picture, the vacuum is filled with large nets of strings. The massless gauge bosons correspond to the fluctuations of large closed string-nets (i.e., nets of closed strings) and fermions correspond to the ends of open strings in string nets. Anti-commuting fields are not needed to produce (nearly) massless fermions. Massless fermions appear as low energy collective modes in a purely bosonic system.

The string-net picture for gauge theories have a long history. The closed-string description of gauge fluctuations is intimately related to the Wilson loop in gauge theory.[17–19] The relation between dynamical gauge theory and a dynamical Wilson-loop theory was suggested in Ref. [20, 21]. Ref. [22] studied the Hamiltonian of a non-local model - lattice gauge theory. It was found that the lattice gauge theory contains a string-net structure and the gauge charges can be viewed as ends of strings. In Ref. [23, 24] various duality relations between lattice gauge theories and theories of extended objects were reviewed. In particular, some statistical lattice gauge models were found to be dual to certain statistical membrane models.[25] This duality relation is directly connected to the relation between gauge theory and closed-string-net theory[13] in quantum models.

Emerging fermions from local bosonic models also have a complicated history. The first examples of emerging fermions/anyons were the fractional quantum Hall states,[4, 5] where fermionic/anyonic excitations were obtained theoretically from interacting bosons in magnetic field.[26] In 1987, fermion fields and gauge fields were introduced to express the spin-1/2 Hamiltonian in the slave-boson approach.[27, 28] However, writing a bosonic Hamiltonian in terms of fermion fields does not imply the appearance of well defined fermionic quasiparticles. Emerging fermionic excitations can appear only in deconfined phases of the gauge field. Ref. [29–32] constructed several deconfined phases where the fermion fields do describe well defined quasiparticles. However, depending on the property of deconfined phases, those quasiparticles may carry fractional statistics (for the chiral spin states)[29, 30, 33] or Fermi statistics (for the $Z_2$ deconfined states).[31, 32]

Also in 1987, in a study of resonating-valence-bond (RVB) states, emerging fermions (the spinons) were proposed in a nearest neighbor dimer model on square lattice.[34–36] But, according to the deconfinement picture, the results in Ref. [34, 35] are valid only when the ground state of the dimer model is in the $Z_2$ deconfined phase. It appears that the dimer liquid on square lattice with only nearest neighbor dimers is not a deconfined state.[35, 36] and thus it is not clear if the nearest neighbor dimer model on square lattice[35] has the fermionic quasiparticles or not.[36] However, on triangular lattice, the dimer liquid is indeed a $Z_2$ deconfined state.[37] Therefore, the results in Ref. [34, 35] are valid for the triangular-lattice dimer model and fermionic quasiparticles do emerge in a dimer liquid on triangular lattice.

All the above models with emerging fermions are 2+1D models, where the emerging fermions can be understood from binding flux to a charged particle.[26] Recently, it was pointed out in Ref. [14] that the key to emerging fermions is a string structure. Fermions can generally appear as ends of open strings. The string picture allows a construction of a 3+1D local bosonic model that has emerging fermions.

Comparing with those previous results, the new features discussed in this paper are: (A) Massless gauge bosons and fermions can emerge from local bosonic models as a result of string-net condensation. (B) Massless fermions are protected by the string-net condensation (and the associated PSG). (C) String-net condensed states represent a new kind of phases which cannot be described Landau’s symmetry breaking theory. Different string-net condensed states are characterized by different PSG’s. (D) QED and QCD can emerge from a local bosonic model on cubic lattice. The effective QED and QCD has $4N$ families of leptons and quarks. Each family has one lepton and two flavors of quarks. The bottom line is that, within local bosonic models, massless fermions do not just emerge by themselves. Emerging massless fermions, emerging massless gauge bosons, string-net condensations, and PSG are intimately related. They are just different sides of same coin - quantum order.

According to the picture of quantum order, elementary particles (such as photon and electron) may not be elementary after all. They may be collective excitations of a local bosonic system below Planck scale. Since we cannot do experiments close to Planck scale, it is hard to determine if photon and electron are elementary particles or not. In this paper, we would like to show that the string-net picture of light and fermions is at least self consistent by studying some concrete local boson models which produce massless gauge bosons and massless fermions through string-net condensations. The local boson models studied here are just a few examples among a long list of local boson models[8, 28, 29, 31–33, 35, 37–47] that contain emerging fermions and gauge fields.

Here we would like to stress that the string-net picture for the actual gauge bosons and fermions in our universe is only a proposal at moment. Although string-
net condensation can produce and protect massless photons, gluons, quarks, and other charged leptons, we do not know at moment if string-net condensations can produce neutrinos which are chiral fermions, and the weak-interaction SU(2) gauge field which couples chirally to the quarks and the leptons. Also, we do not know if string-net condensation can produce an odd number of families of quarks and leptons. The QED and QCD produced by the known string-net condensations all contain an even number of families so far. The correctness of string-net condensation in our vacuum depend on resolving the above problems. Nature has four fascinating and somewhat strange properties: gauge bosons, Fermi statistics, chiral fermions, and gravity. The string-net condensation picture provides a natural explanation for the first two properties. Two more to go.

On the other hand, if we concern about a condensed matter problem: How to use bosons to make artificial light and artificial fermions, we simply let certain string-nets to condense.

In some recent work, quantum orders and their connection to emerging gauge bosons and fermions were studied using PSG’s, without realizing their connection to string-net condensation.[9, 15, 46] In this paper, we will show that the quantum ordered states described by PSG’s are actually string-net condensed states. The gauge bosons and fermions produced and protected by the PSG’s have a very natural string-net interpretation.[13, 14] Quantum order, PSG, and string-net condensation are different parts of the same story. Here we will summarize and expand those previous work and try to present a coherent picture for quantum order, PSG, and string-net condensation, as well as the associated emerging gauge bosons and fermions.

F. Organization

Section III reviews the work in Ref. [14]. We will study an exactly soluble spin-1/2 model on square lattice[8, 46]. The model was solved using slave-boson approach.[46] This allowed us to identify the PSG that characterizes the non-trivial quantum order in the ground state.[46] Here, following Ref. [14], we will solve the model from string-net condensation point of view. Since the ground state of the model can be described by both string-net condensation and PSG, this allows us to demonstrate the direct connection between string-net condensation and PSG in section IV. The model is also one of the simplest models that demonstrates the connection between string-net condensation and emerging gauge field and fermions.[8, 14]

However, the above exact soluble model does not contain gapless gauge boson and gapless fermions. If we regard the lattice scale as the Planck scale, then gauge bosons and fermions do not “exist” in our model in the sense discussed in section I.A. In section V, we will discuss an exact soluble local bosonic model that contain massless Dirac fermions. In sections VII and VIII, we will discuss local bosonic models that give rise to massless electrons, quarks, gluons, and photons. Gauge bosons and fermions “exist” in those latter models.

II. LOCAL BOSONIC MODELS

In this paper, we will only consider local bosonic models. Local bosonic models are important since they are really local. We note that a fermionic model are in general non local since the fermion operators at different sites do not commute, even when the sites are well separated. Due to their intrinsic locality, local bosonic models are natural candidates for the fundamental theory of nature. In the following we will give a detailed definition of local bosonic models.

To define a physical system, we need to specify (A) a total Hilbert space, (B) a definition of a set of local physical operators, and (C) a Hamiltonian. With this understanding, a local bosonic model is defined to be a model that satisfies: (A) The total Hilbert space is a direct product of local Hilbert spaces of finite dimensions. (B) Local physical operators are local bosonic operators. By definition, local bosonic operators are operators acting within a local Hilbert space or finite products of those operators for nearby local Hilbert spaces. Those operators are called local bosonic operators since they all commute with each other when far apart. (C) The Hamiltonian is a sum of local physical operators.

A spin-1/2 system on a lattice is an example of local bosonic models. The local Hilbert space is two dimensional which contains $|\uparrow\rangle$ and $|\downarrow\rangle$ states. Local physical operators are $\sigma^a_i$, $\sigma^a_i\sigma^b_{i+\hat{x}}$, etc., where $\sigma^a$, $a=x,y,z$ are the Pauli matrices.

A free spinless fermion system (in 2 or higher dimensions) is not a local bosonic model despite it has the same total Hilbert space as the spin-1/2 system. This is because the fermion operators $c_i$ on different sites do not commute and are not local bosonic operators. More importantly, the fermion hoping Hamiltonian in 2 and higher dimensions cannot be written as a sum of local bosonic operators. (Note in higher dimensions, we cannot write all the hoping terms $c_i^\dagger c_j$ as product of local bosonic operators. However, due to the Jordan-Wigner transformation, a 1D fermion hoping $c_i^\dagger c_{i+1}$ can be written as a local bosonic operators. Hence, a 1D fermion system can be a local bosonic model if we exclude $c_i$ from our definition of local physical operators.)

The bosonic field theory without cut-off is not a local bosonic model. This is because the local Hilbert space does not have a finite dimension. A lattice gauge theory is not a local bosonic model. This is because its total Hilbert space cannot be a direct product of local Hilbert spaces.

Another example of local bosonic model is a quantum closed-string-net model. A quantum closed-string-net model on lattice can be defined in the following way. Let us consider only strings that cover nearest neighbor links. A closed-string configuration may have many closed strings with or without overlap. We will
call a closed-string configuration a closed string-net. For every closed string-net, we assign a quantum state. All those quantum states form a basis of the total Hilbert space of the closed-string-net model. Just like lattice gauge theory, the closed-string-net model is not a local bosonic model since the total Hilbert space cannot be a direct product of local Hilbert spaces. It turns out that closed-string-net models and lattice gauge model are closely related. In fact some closed-string-net models (or statistical membrane models) are equivalent to lattice gauge models.[22–25]

III. $\mathbb{Z}_2$ SPIN LIQUID AND STRING-NET CONDENSATION ON SQUARE LATTICE

A. Hamiltonians with closed-string-net condensation

Let us start consider an arbitrary spin-$1/2$ model on a square lattice. The first question that we want to ask is what kind of spin interaction can give rise to a low energy gauge theory. If we believe the connection between gauge theory and closed-string-net theory,[13, 22–25] then one way to obtain a low energy gauge theory is to design a spin interaction that allow strong fluctuations of large closed strings, but forbid other types of fluctuations (such as local spin flip, open string-net fluctuations, etc.). (Note that closed string-nets are nets of strings formed by intersecting/overlapping closed strings, while open string-nets are nets of strings containing at least an open string.) We hope the presence of strong fluctuations of large closed-strings will lead to condensation of closed strings of arbitrary sizes, which in turn gives rise to a low energy gauge theory.

Let us start with

$$H_J = -J \sum_{\text{even}} \sigma_i^x - J \sum_{\text{odd}} \sigma_i^y$$

(1)

where $i = (i_x, i_y)$ labels the lattice sites, $\sigma_i^{x,y,z}$ are the Pauli matrices, and $\sum_{\text{even}}$ (or $\sum_{\text{odd}}$) is a sum over even sites with $(-1)^i \equiv (-1)^{i_x+i_y} = 1$ (or over odd sites with $(-1)^i \equiv (-1)^{i_x+i_y} = -1$). The ground state of $H_J$, $|0\rangle$, has spins pointing to $x$-direction on even sites and to $y$-direction on odd sites (see Fig. 2). Such a state will be defined as a state with no string.

To create a string excitation, we first draw a string that connect nearest neighbor even plaquettes (see Fig. 2). We then flip the spins in the string. Such a string state is created by the following string creation operator (or simply, string operator):

$$W(C) = \prod_C \sigma_i^{z_i}$$

(2)

where the product $\prod_C$ is over all the sites on the string, $a_i = y$ if $i$ is even and $a_i = x$ if $i$ is odd. A generic string state has a form

$$|C_1 C_2 \ldots \rangle = W(C_1) W(C_2) \ldots |0\rangle$$

(3)

where $C_1$, $C_2$, ... are strings with no overlapping ends. Such a state will be called a string-net state and

$$W(C_{\text{net}}) = W(C_1) W(C_2) \ldots$$

will be called a string-net operator. The state $|C_1 C_2 \ldots \rangle$ is an open-string-net state if at least one of $C_i$ is an open string. The corresponding operator $W(C_{\text{net}})$ will be called an open-string-net operator. If all $C_i$ are closed loops, then $|C_1 C_2 \ldots \rangle$ is an closed-string-net state and $W(C_{\text{net}})$ an closed-string-net operator. The Hamiltonian has no string-net condensation since its ground state $|0\rangle$ contains no string-nets. To obtain a Hamiltonian with closed-string-net condensation, we need to first find a Hamiltonian whose ground state contains a lot of closed string-nets of arbitrary sizes and do not contain open string-nets.

Let us first write down a Hamiltonian such that closed strings cost no energy and any open strings cost a large energy. One such Hamiltonian has a form

$$H_U = -U \sum_{\text{even}} \hat{F}_i$$

$$\hat{F}_i = \sigma_i^x \sigma_{i+x}^y \sigma_{i+x+y}^x \sigma_{i+y}^y$$

(4)

We find the no-string state $|0\rangle$ is one of the ground state of $H_U$ (assuming $U > 0$) with energy $-UN_{\text{site}}$. All the closed-string-net states, such as $W(C_{\text{close}})|0\rangle$, are also ground state of $H_U$ since $[H_U, W(C_{\text{close}})] = 0$. An open-string state $W(C_{\text{open}})|0\rangle$ is also an eigenstate of $H_U$ but with energy $-UN_{\text{site}} + 2U$. We see that each end of open string cost an energy $U$. We also note that the energy of closed strings does not depend on the length of closed strings. Thus the closed strings in $H_U$ have no tension. We can introduce a string tension by adding the $H_J$ to our Hamiltonian. The string tension will be $2J$ per site (or per segment). We note that, any string-net state $|C_1 C_2 \ldots \rangle$ is an eigenstate of $H_U + H_J$. Thus, string-nets in the model described by $H_U + H_J$ do not fluctuate and hence cannot condense. To make string-nets to fluctuate, we need a g-term

$$H_g = g \sum_p U(C_p)$$

(5)

where $p$ labels the odd plaquettes and $C_p$ is the closed string around the plaquette $p$. In fact

$$H_g = -g \sum_{\text{odd}} \hat{F}_i$$

(6)
This way, we obtain the Hamiltonian of our spin-1/2 model

\[ H = H_U + H_J + H_g \]  

(7)

B. String condensation and low energy effective theory

When \( J = 0 \) in Eq. (7), the model is exactly soluble since \([\hat{F}_i, \hat{F}_j] = 0\).[8, 46] All the eigenstates of \( H_U + H_g \) can be obtained from the common eigenstates of \( \hat{F}_i \). Since \( \hat{F}_i^2 = 1 \), the eigenvalues of \( \hat{F}_i \) are simply \( \pm 1 \). Thus all the eigenstates of \( H_U + H_g \) are labeled by \( \pm 1 \) on each plaquette. (Note, this is not true for finite systems where the boundary condition introduce additional complications.[46]) The energies of those eigenstates are sum of eigenvalues of \( \hat{F}_i \) weighted by \( U \) and \( g \).

From the results of exact soluble model, we suggest a phase diagram of our model as sketched in Fig. 3. We will show that the phase diagram contains four different string-net condensed phases and one phase with no string condensation. All the phases have the same symmetry and are distinguished only by their different quantum orders.

Let us first discuss the phase with \( U, g > 0 \). We will assume \( J = 0 \) and \( U \gg g \). In this limit, all states containing open strings will have an energy of order \( U \). The low energy states contain only closed strings (or more generally closed string-nets) and satisfy

\[ \hat{F}_i|_{i=\text{even}} = 1 \]  

(8)

For infinite systems, the different low energy states are labeled by the eigenvalues of \( \hat{F}_i \) on odd plaquettes:

\[ \hat{F}_i|_{i=\text{odd}} = \pm 1 \]  

(9)

In particular, the ground state is given by

\[ \hat{F}_i|_{i=\text{odd}} = 1 \]  

(10)

All the closed-string-net operators \( W(C_{\text{net}}) \) commute with \( H_U + H_g \). Hence the ground state \( \Psi_0 \) of \( H_U + H_g \) satisfies

\[ \langle \Psi_0|W(C_{\text{net}})|\Psi_0 \rangle = 1. \]  

(11)

Thus the \( U, g > 0 \) ground state has a closed-string-net condensation. The low energy excitations above the ground state can be obtained by flipping \( \hat{F}_i \) from 1 to \(-1\) on some odd plaquettes.

If we view \( \hat{F}_i \) on odd plaquettes as the flux in \( Z_2 \) gauge theory, we find that the low energy sector of model is identical to a \( Z_2 \) lattice gauge theory, at least for infinite systems. This suggests that the low energy effective theory of our model is a \( Z_2 \) lattice gauge theory.

However, one may object this result by pointing out that the low energy sector of our model is also identical to an Ising model with one spin on each the odd plaquette. Thus the the low energy effective theory should be the Ising model. We would like to point out that although the low energy sector of our model is identical to an Ising model for infinite systems, the low energy sector of our model is different from an Ising model for finite systems. For example, on a finite even by even lattice with periodic boundary condition, the ground state of our model has a four-fold degeneracy.[8, 46] The Ising model does not have such a degeneracy. Also, our model contains an excitation that can be identified as \( Z_2 \) charge (see below). Therefore, the low energy effective theory of our model is a \( Z_2 \) lattice gauge theory instead of an Ising model. The \( \hat{F}_i = -1 \) excitations on odd plaquettes can be viewed as the \( Z_2 \) vortex excitations in the \( Z_2 \) lattice gauge theory.

C. Three types of strings and emerging fermions

What is the \( Z_2 \) charge excitations? We note that, in the closed-string-net condensed state, the action of the closed-string operator Eq. (2) on the ground state is trivial. This suggests that the action of the open-string operators on the ground state only depend on the ends of strings, since two open strings with the same ends only differ by a closed string. Therefore, an open-string operator create two particles at its ends when acting on the string condensed state. Since the strings in Eq. (2) only connect even plaquettes, the particle corresponding to the ends of the open strings always live on the even plaquettes. We will call such a string T1 string. Form
the commutation relation between \( \hat{F}_i \) and the open-string operators, we find that the open-string operators flip the sign of \( \hat{F}_i \) at its ends. Thus each particle created by the open-string operators has an energy \( 2U \). Now, let us consider the hopping of one such particle around four nearest neighbor even plaquettes (see Fig. 4). We see that the product of the the four hopping amplitudes is given by the eigenvalue of \( \hat{F}_i \) on the odd plaquette in the middle of the four even plaquettes.\(^8, 14\) This is exactly the relation between charge and flux. Thus if we identify \( \hat{F}_i \) on odd plaquettes as \( Z_2 \) flux, then the ends of strings on even plaquettes will correspond to the \( Z_2 \) charges. We note that, due to the closed-string condensation, the ends of open strings are not confined and have only short ranged interactions between them. Thus the \( Z_2 \) charges behave like quasiparticles with no string attached.

Just like the \( Z_2 \) charges, a pair \( Z_2 \) vortices is also created by an open string operator. Since the \( Z_2 \) vortices correspond to flipped \( \hat{F}_i \) on odd plaquettes, the open-string operator that create \( Z_2 \) vortices is also given by Eq. (2), except now the product is over a string that connect odd plaquettes. We will call such a string a T2 string. (The strings connecting even plaquettes were called T1 strings.)

We would like to point out that the reference state \((ie \hspace{1em} \text{the no string state}) \) for the T2 string is different from that of the T1 string. The no-T2-string state is given by \( |0\rangle \) with spin pointing in \( y \)-direction on even sites and \( x \)-direction on odd sites. Since the T1 and T2 strings have different reference state, we cannot have a dilute gas of the T1 strings and the T2 strings at the same time. One can easily check that the T2 string operator also commute with \( H_U + H_y \). Therefore, the ground state \( |\Psi_0\rangle \), in addition to the T1 closed-string condensation, also has a T2 closed-string condensation.

The hopping of a \( Z_2 \) vortex is induced by a short T2 open-string. Since the T2 open-strings operators all commute with each other, the \( Z_2 \) vortex behave like bosons. Similarly, the \( Z_2 \) charges also behave like bosons. However, T1 open-string operators and T2 open-string operators do not commute. As a result, the ends of T1 string and the ends of T2 string have nontrivial mutual statistics. As we have already shown that moving a \( Z_2 \) charge around a \( Z_2 \) vortex generate a phase \( \pi \), the \( Z_2 \) charges and the \( Z_2 \) vortices have a semionic mutual statistics.

The T3 strings are defined as bound states of T1 and T2 strings. The T3 string operator has a form \( W(C) = \prod_{n} \sigma_{i_n}^{l_m} \), where \( C \) is a string connecting the mid-points of the neighboring links (see Fig. 6), and \( i_n \) are sites on the string. \( l_m = z \) if the string does not turn at site \( i_m \) (see Fig. 6). \( l_m = x \) or \( y \) if the string makes a turn at site \( i_m \). \( l_m = x \) if the turn forms a upper-right or lower-left corner. \( l_m = y \) if the turn forms a lower-right or upper-left corner. (See Fig. 6.) The ground state also has a condensation of T3 closed-strings. The ends of T3 string, as bound states of the \( Z_2 \) charges and the \( Z_2 \) vortices, are fermions. The bound state is formed by a \( Z_2 \) charge and a \( Z_2 \) vortex on the two plaquettes on the two side of a link \((ie \hspace{1em} \hat{F}_i = -1 \) on the two sides of the link). Thus the fermions live on the links. It is interesting to see that string-net condensation in our model directly leads to \( Z_2 \) gauge structure and three new type of quasiparticles: \( Z_2 \) charge, \( Z_2 \) vortex, and fermions. Fermions, as ends of open T3 strings, emerge from our purely bosonic model.

Since ends of T1 string are \( Z_2 \) charges, the T1 string can be viewed as strings of \( Z_2 \) “electric” flux. Similarly, the T2 string can be viewed as strings of \( Z_2 \) “magnetic” flux.

IV. CLASSIFICATION OF DIFFERENT STRING CONDENSATIONS BY PSG

A. Four classes of string-net condensations

As we have seen in last section that when \( U > 0 \), \( g > 0 \), and \( J = 0 \), the ground state of our model is given by

\[
\hat{F}_i |_{i = \text{even}} = 1, \quad \hat{F}_i |_{i = \text{odd}} = 1. \quad (12)
\]

We will call such a phase \( Z_2 \) phase to stress the low energy \( Z_2 \) gauge structure. In the \( Z_2 \) phase, the T1 string operator \( W_1(C_1) \) and the T2 string operator \( W_2(C_2) \) have the following expectation values

\[
\langle W_1(C_1) \rangle = 1, \quad \langle W_2(C_2) \rangle = 1 \quad (13)
\]

When \( U > 0 \), \( g < 0 \), and \( J = 0 \), the ground state is given by

\[
\hat{F}_i |_{i = \text{even}} = 1, \quad \hat{F}_i |_{i = \text{odd}} = -1. \quad (14)
\]

We see that there is \( \pi \)-flux through each odd plaquette. We will call such a phase \( Z_2 \)-flux phase. The T1 string operator and the T2 string operator have the following expectation values

\[
\langle W_1(C_1) \rangle = (-)^{N_{\text{odd}}}, \quad \langle W_2(C_2) \rangle = 1 \quad (15)
\]

where \( N_{\text{odd}} \) is the number of odd-plaquettes enclosed by the T1 string \( C_1 \).

When \( U < 0 \), \( g > 0 \), and \( J = 0 \), the ground state is

\[
\hat{F}_i |_{i = \text{even}} = -1, \quad \hat{F}_i |_{i = \text{odd}} = 1. \quad (16)
\]

The ground state has a \( Z_2 \) charge on each even plaquette. We will call such a phase \( \pi \)-flux charge phase. The T1 string operator and the T2 string operator have the following expectation values

\[
\langle W_1(C_1) \rangle = 1, \quad \langle W_2(C_2) \rangle = (-)^{N_{\text{even}}} \quad (17)
\]

where \( N_{\text{even}} \) is the number of even-plaquettes enclosed by the T2 string \( C_2 \). Note that the \( Z_2 \)-flux phase and the \( \pi \)-charge phase, different only by a lattice translation, are essentially the same phase.

When \( U < 0 \), \( g < 0 \), and \( J = 0 \), the ground state becomes

\[
\hat{F}_i |_{i = \text{even}} = -1, \quad \hat{F}_i |_{i = \text{odd}} = -1. \quad (18)
\]

There is a \( Z_2 \) charge on each even plaquette and \( \pi \)-flux through each odd plaquette. We will call such a phase \( Z_2 \)-flux-charge phase. The T1 string operator and the T2 string operator have the following expectation values

\[
\langle W_1(C_1) \rangle = (-)^{N_{\text{odd}}}, \quad \langle W_2(C_2) \rangle = (-)^{N_{\text{even}}} \quad (19)
\]
B. PSG and ends of condensed strings

From the different $\langle W_1(C_1) \rangle$ and $\langle W_2(C_2) \rangle$, we see that the above four phases have different string-net condensations. However, they all have the same symmetry. This raises an issue. Without symmetry breaking, how do we know the above four phases are really different phases? How do we know that it is impossible to change one string-net condensed state to another without a phase transition?

In the following, we will show that the different string-net condensations can be described by different PSG’s (just like different symmetry breaking orders can be described by different symmetry groups of ground states.) In Ref. [9, 10], different quantum orders were introduced via their different PSG’s. The connection between string-net condensation and PSG allows us to connect string-net condensation to the quantum order introduced in Ref. [9, 10]. In particular, the PSG’s are shown to be a universal property of a quantum phase, which can be changed only by phase transitions. Thus the different PSG’s for the different string-net condensed states indicate that those different string-net condensed states belong to different quantum phases.

When closed-string-nets condense, the ends of open strings behave like independent particles. Let us consider two particles states $|p_1, p_2\rangle$ described by the two ends of a T1 string. Note that the ends of the T1 strings, and hence the $Z_2$ charges, only live on the even plaquettes. Here $p_1$ and $p_2$ label the even plaquettes. For our model $H_{t1} + H_g$, $|p_1, p_2\rangle$ is an eigenstate and the $Z_2$ charges do not hop. Here we would like to add a term

$$H_t = t \sum_i (\sigma_i^x + \sigma_i^y) + t' \sum_i \sigma_i^z$$

(20)

to the Hamiltonian. The t-term $t \sum_i (\sigma_i^x + \sigma_i^y)$ makes the $Z_2$ charges to hop among the even plaquettes with a hopping amplitude of order $t$. The dynamics of the two $Z_2$ charges is described by the following effective Hamiltonian in the two-particle Hilbert space

$$H = H(p_1) + H(p_2)$$

(21)

where $H(p_1)$ describes the hopping of the first particle $p_1$ and $H(p_2)$ describes the hopping of the second particle $p_2$. Now we can define the PSG in a string-net condensed state. The PSG is nothing but the symmetry group of the hopping Hamiltonian $H(p)$.

Due to the translation symmetry of the underlying model $H_{t1} + H_g + H_t$, we may naively expect the hopping Hamiltonian of the $Z_2$ charge $H(p)$ also have a translation symmetry

$$H(p) = T_{xy}^\dagger H(p)T_{xy}, \quad T_{xy}|p\rangle = |p + x + y\rangle$$

$$H(p) = T_{x\bar{y}}^\dagger H(p)T_{x\bar{y}}, \quad T_{x\bar{y}}|p\rangle = |p + x - y\rangle$$

(22)

The above implies PSG = translation symmetry group. It turns out that Eq. (22) is too strong. The underlying spin model can have translation symmetry even when $H(p)$ does not satisfy Eq. (22). However, the possible symmetry groups of $H(p)$ (the PSG’s) are strongly constrained by the translation symmetry of the underlying spin model. In the follow, we will explain why the PSG can be different from the symmetry group of the physical spin model, and what conditions that the PSG must satisfy in order to be consistent with the translation symmetry of the spin model.

We note that a string always has two ends. Thus a physical state always has an even number of $Z_2$ charges. The actions of translation on a two-particle state are given by

$$T_{xy}^{(2)}|p_1, p_2\rangle = e^{i\theta_{xy}}|p_1 + x + y, p_2 + x + y\rangle$$

$$T_{x\bar{y}}^{(2)}|p_1, p_2\rangle = e^{i\theta_{x\bar{y}}}|p_1 + x - y, p_2 + x - y\rangle$$

(23)

The phases $e^{i\theta_{xy}}(p_1, p_2)$ and $e^{i\theta_{x\bar{y}}(p_1, p_2)}$ come from the ambiguity of the location of the string that connect $p_1$ and $p_2$. i.e the phases can be different if the string connecting the two $Z_2$ charges has different locations. $T_{xy}^{(2)}$ and $T_{x\bar{y}}^{(2)}$ satisfy the algebra of translations

$$T_{xy}^{(2)} T_{x\bar{y}}^{(2)} = T_{x\bar{y}}^{(2)} T_{xy}^{(2)}$$

(24)

$T_{xy}^{(2)}$ and $T_{x\bar{y}}^{(2)}$ are direct products of translation operators on the single-particle states. Thus, in some sense, the single-particle translations are square roots of two-particle translations.

The most general form of single-particle translations is given by $T_{xy} G_{xy}$ and $T_{x\bar{y}} G_{x\bar{y}}$ where the actions of operators $T_{xy} G_{xy}$ and $G_{xy} T_{xy}$ are defined as

$$T_{xy} |p\rangle = |p + x + y\rangle$$

$$T_{xy} |p\rangle = |p + x - y\rangle$$

$$G_{xy} |p\rangle = e^{i\phi_{xy}}|p\rangle$$

$$G_{x\bar{y}} |p\rangle = e^{i\phi_{x\bar{y}}}|p\rangle$$

(25)

In order for the direct product $T_{xy}^{(2)} = T_{xy} G_{xy} \otimes T_{xy} G_{xy}$ and $T_{x\bar{y}}^{(2)} = T_{x\bar{y}} G_{x\bar{y}} \otimes T_{x\bar{y}} G_{x\bar{y}}$ to reproduce the translation algebra Eq. (24), we only require $T_{xy} G_{xy}$ and $T_{x\bar{y}} G_{x\bar{y}}$ to satisfy

$$T_{xy} G_{xy} T_{xy} G_{xy} = T_{xy} G_{xy} T_{xy} G_{xy}$$

(26)

or

$$T_{xy} G_{xy} T_{xy} G_{xy} = -T_{xy} G_{xy} T_{xy} G_{xy}$$

(27)

The operators $T_{xy} G_{xy}$ and $T_{x\bar{y}} G_{x\bar{y}}$ generate a group. Such a group is the PSG introduced in Ref. [9]. The two different algebra Eq. (26) and Eq. (27) generate two different PSG’s, both are consistent with the translation group acting on the two-particle states. We will call the PSG generated by Eq. (26) $Z_{2A}$ PSG and the PSG generated by Eq. (27) $Z_{2B}$ PSG.

Let us give a more general definition of PSG. A PSG is a group. It is an extension of symmetry group (SG), i.e a PSG contains a normal subgroup (called invariant gauge group or IGG) such that

$$PSG/IGG = SG$$

(28)
For our case, the SG is the translation group $SG = \{1, T_x, T_y, T_{xy}, T_0, T_{0x}, T_{0y}, T_{0xy} \}$. For every element in SG, $\alpha(2) \in SG$, there are one or several elements in PSG, $a \in PSG$, such that $a \otimes \alpha = \alpha(2)$. The IGG in our PSG is formed by the transformations $G_0$ on the single-particle states that satisfy $G_0 \otimes G_0 = 1$. We find that IGG is generated by

$$G_0|p\rangle = -|p\rangle \quad \text{(29)}$$

and $T_{xy}G_{xy}$ and $T_{x0}G_{x0}$ generate the $Z2A$ and $Z2B$ PSG's.

Now we see that the underlying translation symmetry does not require the single-particle hopping Hamiltonian $H(p)$ to have a translation symmetry. It only require $H(p)$ to be invariant under the $Z2A$ PSG or the $Z2B$ PSG. When $H(p)$ is invariant under the $Z2A$ PSG, the hopping Hamiltonian has the usual translation symmetry. When $H(p)$ is invariant under the $Z2B$ PSG, the hopping Hamiltonian has a magnetic translation symmetry describing a hopping in a magnetic field with $\pi$-flux through each odd plaquette.

C. PSG's classify different string-net condensations

After understand the possible PSG’s for the hopping Hamiltonian of the ends of strings, now we are ready to calculate the actual PSG’s. Let us consider two ground states of our model $H_U + H_g + H_t$. One has $\tilde{F}_i |i=odd\rangle = 1$ (for $g > 0$) and the other has $\tilde{F}_i |i=odd\rangle = -1$ (for $g < 0$). Both ground states have the same translation symmetry in $x+y$ and $x-y$ directions. However, the corresponding single-particle hopping Hamiltonian $H(p)$ has different symmetries. For the $\tilde{F}_i |i=odd\rangle = 1$ state, there is no flux through odd plaquettes and $H(p)$ has the usual translation symmetry. It is invariant under the $Z2A$ PSG. While for the $\tilde{F}_i |i=odd\rangle = -1$ state, there is $\pi$-flux through odd plaquettes and $H(p)$ has a magnetic translation symmetry. Its PSG is the $Z2B$ PSG. Thus the $\tilde{F}_i |i=odd\rangle = 1$ state and the $\tilde{F}_i |i=odd\rangle = -1$ state have different orders despite they have the same symmetry. The different quantum orders in the two states can be characterized by their different PSG’s.

The above discussion also apply to the $Z2$ vortex and T2 strings. Thus the quantum orders in our model are described by a pair of PSG’s ($PSG_{charge}$, $PSG_{vortex}$), one for the $Z2$ charge and one for the $Z2$ vortex. The PSG pairs ($PSG_{charge}$, $PSG_{vortex}$) allows us to distinguish four different string-net condensed states of model $H = H_U + H_g + H_t$. (See Fig. 5.)

Now let us assume $U = g$ in our model:

$$H_U + H_g + H_t = H_t - V \sum_i \tilde{F}_i \quad \text{(30)}$$

The new physical spin model has a larger translation symmetry generated by $\Delta x = x$ and $\Delta i = y$ (see Fig. 5). Due to the enlarged symmetry group, the quantum orders in the new system should be characterized by a new PSG. In the following, we will calculate the new PSG.

![FIG. 5: The proposed phase diagram for the $H = H_U + H_g + H_t$ model. $t = t'$ is assumed to be positive. The four string-net condensed phases are characterized by a pair of PSG’s ($PSG_{charge}$, $PSG_{vortex}$). FM marks a ferromagnetic phase.](image-url)

The single-particle states are given by $|p\rangle$. When $p$ is even, $|p\rangle$ corresponds to a $Z2$ charge and when $p$ is odd, $|p\rangle$ corresponds to a $Z2$ vortex. We see that a translation by $x$ (or $y$) will change a $Z2$ charge to a $Z2$ vortex or a $Z2$ vortex to a $Z2$ charge. Therefore the effective single-particle hopping Hamiltonian $H(p)$ only contain hops between even plaquettes or odd plaquettes. The single-particle Hamiltonian $H(p)$ is invariant under the following two transformations $G_0$ and $G'_0$.

$$G_0|p\rangle = -|p\rangle, \quad G'_0|p\rangle = (-1)^p|p\rangle \quad \text{(31)}$$

We note that $G_0 \otimes G_0 = G'_0 \otimes G'_0 = 1$. Therefore both $G_0$ and $G'_0$ correspond to the identity element of the symmetry group of two-particle states. $(G_0, G'_0)$ generate the IGG of the new PSG. The new IGG is $Z2 \times Z2$.

The translations of single-particle states by $x$ and by $y$ are generated by $T_xG_x$ and $T_yG_y$. The translation by $x+y$ and by $x-y$ are given by

$$T_{xy}G_{xy} = T_yG_yT_xG_x$$
$$T_{xy}G_{xy} = (T_yG_y)^{-1}T_xG_x \quad \text{(32)}$$

Since $T_{xy}G_{xy}$ and $T_{x0}G_{x0}$ are the translations of the $Z2$ charge and the $Z2$ vortex discussed above, we find

$$(T_{xy}G_{xy})^{-1}T_{xy}G_{xy} = \eta \quad \text{(33)}$$

where $\eta = 1$ for the $(Z2A, Z2A)$ state with $\tilde{F}_i = 1$ and $\eta = -1$ for the $(Z2B, Z2B)$ state with $\tilde{F}_i = -1$. Also $T_xG_x$ and $T_yG_y$ must satisfy

$$(T_yG_y)^{-1}T_xG_x \in IGG \quad \text{(34)}$$

since on the two-particle states

$$(T_yG_y)^{-1}(T_xG_x)^{-1}T_yG_yT_xG_x = 1 \quad \text{(35)}$$

Therefore, $(T_yG_y)^{-1}(T_xG_x)^{-1}T_yG_yT_xG_x$ may take the following possible values $1, -1, (-1)^p, \text{ and } -(1)^p$. Only choices $\eta^p$ and $-\eta^p$ are consistent with Eq. (33) and we have

$$(T_yG_y)^{-1}(T_xG_x)^{-1}T_yG_yT_xG_x = \eta^p \quad \text{(36)}$$
We like to point out that the different choices of \( \eta' = \pm 1 \) do not lead to different PSG's. This is because if \( T_x G_x \) is a symmetry of the \( H(p) \), then \( T_x G_x(-) \) is also a symmetry of the \( H(p) \). However, the change \( G_x \rightarrow G_x(-) \) will change the sign of \( \eta' \). Thus \( \eta' = 1 \) and \( \eta' = -1 \) will lead to the same PSG. But the different signs of \( \eta \) will lead to different PSG's.

\( (G_0, G'_0) \) and \( (T_x G_x, T_y G_y) \) generate the new PSG. The single-particle Hamiltonian \( H(p) \) is invariant under such a PSG. \( \eta = 1 \) and \( \eta = -1 \) correspond to two different PSG's that characterize two different quantum orders. The ground state for \( V > 0 \) and \( |V| \gg t \) (see Eq. (30)) is described by the \( \eta = 1 \) PSG. The ground state for \( V < 0 \) and \( |V| \gg t \) is described by the \( \eta = -1 \) PSG. The two ground states have different quantum orders and different string-net condensations.

D. Different PSG's from the ends of different condensed strings

In this section we still assume \( U = g \) and consider only the translation invariant model Eq. (30). In the above we discussed the PSG for the ends of one type of condensed strings in different states. In this section, we will concentrate on only one ground state. We know that the ground state of our spin-1/2 model contain condensations of several type of strings. We like to calculate the the different PSG's for the different condensed strings.

The PSG's for the condensed T1 and T2 strings were obtained above. Here we will discuss the PSG for the T3 string. Since the ends of the T3 strings live on the links, the corresponding single-particle hopping Hamiltonian \( H_f(l) \) describes fermion hopping between links. Clearly, the symmetry group (the PSG) of \( H_f(l) \) can be different from that of \( H(p) \).

Let us consider fermion hopping around some small loops. The four hops of a fermion around a site \( \mathbf{i} \) (see Fig. 6) are generated by \( \sigma_i^x, \sigma_i^y, \sigma_i^z, \) and \( \eta_i \). The total amplitude of a fermion hopping around a site is \( \sigma_i^x \sigma_i^y \sigma_i^z = -1 \). The fermion hopping around a site always sees \( \pi \)-flux. The four hops of a fermion around a plaquette \( \mathbf{p} \) (see Fig. 6) are generated by \( \sigma_{i_0}^x, \sigma_{i_0+x}^x, \sigma_{i_0+x+y}^x, \) and \( \sigma_{i_0+y}^y \), where \( i_0 \) is the lower left corner of the plaquette \( \mathbf{p} \). The total amplitude of a fermion hopping around a plaquette is given by \( \sigma_{i_0+y}^y \sigma_{i_0+x+y}^x \sigma_{i_0+x}^x \sigma_{i_0}^x = \hat{F}_{i_0} \). When \( V > 0 \), the ground state has \( \hat{F}_{i_0} = 1 \). However, since site \( i_0 \) is next to the end of T3 string, we have \( \hat{F}_{i_0} = -1 \).

In this case, the fermion hopping around a plaquette sees \( \pi \)-flux. For \( V < 0 \) ground state, we find that fermion hopping around a plaquette sees no flux.

Let us define the fermion hopping \( l \rightarrow l + x \) as the combination of two hops \( l \rightarrow l + \frac{x}{2} - \frac{y}{2} \rightarrow l + x \) and the fermion hopping \( l \rightarrow l + y \) as the combination of \( l \rightarrow l + \frac{x}{2} + \frac{y}{2} \rightarrow l + y \) (see Fig. 6). Under such a definition, a fermion hopping around a square \( l \rightarrow l + x \rightarrow l + x + y \rightarrow l + y \rightarrow l \) correspond to a fermion hopping around a site and a fermion hopping around a plaquette discussed above (see Fig. 6). Therefore, the total amplitude for a fermion hopping around a square is given by the sign of \( V \): \( \text{sgn}(V) \). We find the translation symmetries \( (T_x G_x, T_y G_y) \) of the fermion hopping \( H_f(l) \) satisfies

\[
(\text{sgn}(V) = 0)^{-1} (T_x G_x)^{-1} T_y G_y T_x G_x = \text{sgn}(V)
\]

which is different from the translation algebra for \( H(p) \) Eq. (36). \( H_f(l) \) is also invariant under \( G_0' \):

\[
(G_0, T_x G_x, T_y G_y) \quad \text{generate the new PSG.}
\]

In Ref. [46], the spin-1/2 model Eq. (30) (with \( t = t' = 0 \)) was viewed as a hardcore boson model. The model was solved using slave-boson approach by splitting a boson into two fermions. Then it was shown the fermion hopping Hamiltonian for \( V > 0 \) and \( V < 0 \) states have different symmetries, or invariant under different PSG's. According to the arguments in Ref. [9], the different PSG's imply different quantum orders in the \( V > 0 \) and \( V < 0 \) ground state states. The PSG's obtained in Ref. [46] for the \( V > 0 \) and \( V < 0 \) phases agrees exactly with the fermion PSG’s that we obtained above. This example shows that the PSG’s introduced in Ref. [10, 46] are the symmetry groups of the hopping Hamiltonian of the ends of condensed strings. The PSG description and the string-net-condensation description of quantum orders are intimately related.

Here we would like to point out that the PSG’s introduced in Ref. [9, 16] are all fermion PSG’s. They are only one of many different kinds of PSG’s that can be used to characterize quantum orders. In general, a quantum ordered state may contain condensations of several types of strings. The ends of each type of condensed strings will have their own PSG.

V. MASSLESS FERMION AND PSG IN STRING-NET CONDENSED STATE

In Ref. [9, 16], it was pointed out that PSG can protect masslessness of the emerging fermions, just like symmetry can protect the masslessness of Nambu-Goldstone
bosons. In this section, we are going to study an exact soluble spin-$\frac{1}{2}$ model with string-net condensation and emerging massless fermions. Through this soluble model, we demonstrate how PSG that characterizes the string-net condensation can protect the masslessness of the fermions. The exact soluble model that we are going to study is motivated by Kitaev’s exact soluble spin-1/2 model on honeycomb lattice.[48]

A. Exact soluble spin-$\frac{1}{2}$ model

The exact soluble model is a local bosonic model on square lattice. To construct the model, we start with four Majorana fermions $\lambda_i^a$, $a = x, \bar{x}, y, \bar{y}$ and one complex fermion $\psi$. $\lambda_i^a$ satisfy

$$\{\lambda_i^a, \lambda_j^b\} = 2\delta_{ab}\delta_{ij}$$

(39)

We note that

$$\hat{U}_{i, i+x} = -i\lambda_i^x \lambda_{i+x}^x, \quad \hat{U}_{i, i+y} = -i\lambda_i^y \lambda_{i+y}^y, \quad \hat{U}_{ij} = \hat{U}_{ji}$$

(40)

form a commuting set of operators. Using such a commuting set of operators, we can construct the following exact soluble interacting fermion model

$$H = g \sum_i \hat{F}_i + t \sum_i (i\hat{U}_{i, i+x}\psi_i^{\dagger}\psi_{i+x} + i\hat{U}_{i, i+y}\psi_i^{\dagger}\psi_{i+y} + h.c.)$$

$$\hat{F}_i = \hat{U}_{i, i}\hat{U}_{i, i+\bar{i}}\hat{U}_{i, i}$$

(41)

where $i = i + x$, $i_2 = i + x + y$, $i_3 = i + y$, and $t$ is real. We will call $\hat{F}_i$ a $Z_2$ flux operator. To obtain the Hilbert space within which the Hamiltonian $H$ acts, we group $\lambda^x, \bar{x}, y, \bar{y}$ into two complex fermion operators

$$2\psi_{1, i} = \lambda_i^x + i\lambda_i^\bar{y}, \quad 2\psi_{2, i} = \lambda_i^\bar{x} + i\lambda_i^y$$

(42)

on each site. The complex fermion operators $\psi_{1, i}$ and $\psi_{2, i}$ generate an eight dimensional Hilbert space on each site.

Since $\hat{U}_{ij}$ commute with each other, we can find the common eigenstates of the $\hat{U}_{ij}$ operators: $|\{s_{ij}\}, n\rangle$, where $s_{ij}$ is the eigenvalue of $\hat{U}_{ij}$, and $n$ labels different degenerate common eigenstates. Since $(\hat{U}_{ij})^2 = 1$ and $\hat{U}_{ij} = \hat{U}_{ji}$, $s_{ij}$ satisfies $s_{ij} = \pm 1$ and $s_{ij} = s_{ji}$. Within the subspace with a fixed set of $s_{ij}$: $|\{s_{ij}\}, n\rangle = n = 1, 2, \ldots$, the Hamiltonian has a form

$$H = g \sum_i f_i + t \sum_i (is_{i, i+x}\psi_i^{\dagger}\psi_{i+x} + is_{i, i+y}\psi_i^{\dagger}\psi_{i+y} + h.c.)$$

$$f_i = s_{i, i}s_{i, i+\bar{i}}s_{i, i}s_{i, i}$$

(43)

which is a free fermion Hamiltonian. Thus we can find all the many-body eigenstates of $\psi_i$: $|\{s_{ij}\}, \Psi_n\rangle$ and their energies $E(\{s_{ij}\}, n)$ in each subspace. This way we solve the interacting fermion model exactly.

We note that the Hamiltonian $H$ can only change the fermion number on each site by an even number. Thus the $H$ acts within a subspace which has an even number of fermions on each site. We will call this subspace physical Hilbert space. The physical Hilbert space has only four states per site. When defined on the physical space, $H$ becomes a local bosonic system which actually describes a spin $\frac{1}{2} \times \frac{1}{2}$ system (with no spin rotation symmetry). We will call such a system spin-$\frac{1}{2}$ system. To obtain an expression of $H$ within the physical Hilbert space, we introduce two Majorana fermions $\eta_{1, i}$ and $\eta_{2, i}$ to represent $\psi_i$: $2\psi_i = \eta_{1, i} + i\eta_{2, i}$. We note that $\lambda^a\eta_i$, $a = x, \bar{x}, y, \bar{y}$, act within the four dimensional physical Hilbert space on each site, and thus are 4 by 4 matrices. Also $\{-i\lambda^x\eta_1, -i\lambda^y\eta_1\} = 2\delta_{ab}$, thus the four by 4 matrices $\lambda^a\eta_i$ satisfy the algebra of Dirac matrices. Therefore we can express $\lambda^a\eta_i$ in terms of Dirac matrices $\gamma^a$:

$$\lambda_i^a \eta_i = i\gamma_i^a$$

$$\gamma_i^x = \sigma^x \otimes \sigma^x$$

$$\gamma_i^y = \sigma^x \otimes \sigma^x$$

(44)

We can also define the $\gamma^5$

$$\gamma_i^5 \equiv \gamma_i^x \gamma_i^y \gamma_i^\bar{y} = -\sigma^0 \otimes \sigma^z = i\eta_i\eta_2$$

(45)

where we have used $1 - 2\psi_i^\dagger\psi_i = -i\eta_i\eta_2$ and $(-i\lambda^x\eta^5_1)(-i\lambda^y\eta^5_2)(-i\eta_1\eta_2) = 1$ for states with even numbers of fermions. With the above definition of $\gamma^a$ and $\gamma^5$, we find that

$$\lambda_i^a \eta_2 = \gamma^a \gamma^5$$

(46)

and

$$\lambda_i^a \psi_i = \frac{i}{2} (\gamma^a + \gamma^5 \gamma^5) = i\gamma_i^{-a}$$

(47)

We also have

$$\lambda_i^a \eta^5_i = \gamma^a \gamma^5 \gamma^5$$

(48)

The above relations allows us to write $H$ in terms of 4 by 4 Dirac matrices. For example

$$\hat{F}_i = -\gamma_i^x \psi_i^\dagger \psi_i^\dagger \bar{y}$$

(49)

and

$$\hat{U}_{i, i+x} \psi_i^{\dagger} \psi_{i+x} = -i\gamma_i^{x+y} \gamma_i^{\bar{y}+\bar{x}}$$

$$\hat{U}_{i, i+y} \psi_i^{\dagger} \psi_{i+y} = -i\gamma_i^y \gamma_i^{\bar{x}}$$

(50)

The physical states in the physical Hilbert space are invariant under local $Z_2$ gauge transformations generated by

$$G = \prod_i G_i^{n_i}$$

$$n_i = \psi_{1, i}^{\dagger}\psi_{1, i} + \psi_{2, i}^{\dagger}\psi_{2, i} + \psi_{i}^{\dagger}\psi_{i}$$

(51)

where $G_i$ is an arbitrary function with only two values $\pm 1$ and $n_i$ the number of fermions on site $i$. We note
that the $Z_2$ gauge transformations change $\psi_{1i} \rightarrow G_i \psi_{1i}$. The projection into the physical Hilbert space with even fermion per site makes our theory a $Z_2$ gauge theory.

Since the Hamiltonian $H$ in Eq. (41) is $Z_2$ gauge invariant: $[G, H] = 0$, eigenstate of $H$ within the physical Hilbert space can be obtained from $|\{s_{ij}\}, \Psi_n\rangle$ by projecting into the physical Hilbert space: $P|\{s_{ij}\}, \Psi_n\rangle$. The projected state $P|\{s_{ij}\}, \Psi_n\rangle$ (or the physical state), if non-zero, is an eigenstate of the spin-$\frac{1}{2}$ model with energy $E(\{s_{ij}\}, n)$. The $Z_2$ gauge invariance implies that

$$P|\{s_{ij}\}, \Psi_n\rangle = P|\{\tilde{s}_{ij}\}, \Psi_n\rangle$$

$$E(\{s_{ij}\}, n) = E(\{\tilde{s}_{ij}\}, n)$$

(52)

if $s_{ij}$ and $\tilde{s}_{ij}$ are $Z_2$ gauge equivalent

$$\tilde{s}_{ij} = G(i)s_{ij}G(j).$$

(53)

Let us count the states to show that the projected states $P|\{s_{ij}\}, \Psi_n\rangle$ generate all states in the physical Hilbert space. Let us consider a periodic lattice with $N_{\text{site}} = L_x L_y$ sites. First there are $2^{2N_{\text{site}}}$ choices of $s_{ij}$. We note that there are $2^{N_{\text{site}}}$ different $Z_2$ gauge transformations. But the constant gauge transformation $G(i) = -1$ does not change $s_{ij}$. Thus there are $2^{N_{\text{site}}}/2$ different $s_{ij}$'s in each $Z_2$ gauge equivalent class. Therefore, there are $2 \times 2^{N_{\text{site}}}$ different $Z_2$ gauge equivalent classes of $s_{ij}$'s. We also note that

$$\prod_{i} \psi_{1i} \psi_{2i} = (-1)^{L_x L_y} \prod_{i} (-i)^{\lambda^x_{1i} \lambda^x_{2i}} (-i)^{\lambda^y_{1i} \lambda^y_{2i}}$$

$$= (-1)^{L_x L_y + \sum_i (\psi_{1i} \psi_{1i} + \psi_{2i} \psi_{2i})}$$

(54)

Thus, among the $2 \times 2^{N_{\text{site}}}$ different classes of $s_{ij}$'s, $2^{N_{\text{site}}}$ of them satisfy $\prod_{i} \psi_{1i} \psi_{2i} = (-1)^{L_x L_y}$ and have even numbers of $\psi_{1i}$ and $\psi_{2i}$ fermions. The other $2^{N_{\text{site}}}$ of them satisfy $\prod_{i} \psi_{1i} \psi_{2i} = (-1)^{L_x L_y}$ and have odd numbers of $\psi_{1i}$ and $\psi_{2i}$ fermions. For each fixed $s_{ij}$, there are $2^{N_{\text{site}}}$ many-body states of the $\psi_{1i}$ fermions, ie in $|\{s_{ij}\}, \Psi_n\rangle$ runs from 1 to $2^{N_{\text{site}}}$. Among those $2^{N_{\text{site}}}$ many-body states, $2^{N_{\text{site}}}/2$ of them have even numbers of $\psi_{1i}$ fermions and $2^{N_{\text{site}}}/2$ of them have odd numbers of $\psi_{1i}$ fermions. In order for the projection $P|\{s_{ij}\}, \Psi_n\rangle$ to be non-zero, the total number of fermions must be even. A physical state has even numbers of $(\psi_{1i}, \psi_{2i})$ fermions and even numbers of $\psi_{1i}$ fermions, or it has odd numbers of $(\psi_{1i}, \psi_{2i})$ fermions and odd numbers of $\psi_{1i}$ fermions. Thus there are $2^{N_{\text{site}}} \times 2^{N_{\text{site}}}/2 + 2^{N_{\text{site}}} \times 2^{N_{\text{site}}}/2 = 4^{N_{\text{site}}}$ distinct physical states that can be produced by the projection. Thus the projection produces all the states in the physical Hilbert space.

B. Physical properties of the spin-$\frac{1}{2}$ model

Let us define a closed-string operator to be

$$W(C_{\text{close}}) = \hat{U}_{i+1} \hat{U}_{i+2} \cdots \hat{U}_{i+n}$$

(55)

where $C_{\text{close}}$ is an closed oriented string $C_{\text{close}} = i_1 \rightarrow i_2 \cdots \rightarrow i_n \rightarrow i_1$ formed by nearest neighbor links. Since $C_{\text{close}}$ can intersect with itself, $C_{\text{close}}$ can also be viewed as a closed string-net. We will also call $W(C_{\text{close}})$ a closed-string-net operator.

The closed-string-net operators act within the physical Hilbert space and commute with the Hamiltonian Eq. (41). Thus there is a string-net condensation since $\langle W(C_{\text{close}}) \rangle = \pm 1$ in the ground state of Eq. (41). The above strings correspond to the T3 string discussed in section III C. Unlike the spin-$1/2$ model, we do not have condensed T1 and T2 closed strings in the spin-$1/2$ model.

We can also define open-string operators that act within the physical Hilbert space

$$W(C_{\text{open}}) = \lambda^x_{i} \hat{U}_{i+1} \hat{U}_{i+2} \cdots \hat{U}_{i+n} \lambda^y_{i}$$

$$W(C_{\text{open}}) = \psi_{1i} \hat{U}_{i+1} \hat{U}_{i+2} \cdots \hat{U}_{i+n} \psi_{1i}$$

(56)

where $C_{\text{open}}$ is an open oriented string $C_{\text{open}} = i_1 \rightarrow i_2 \cdots \rightarrow i_n$ formed by nearest neighbor links. $W(C)$ correspond to the open T3 string defined in section III C. Just like the spin-$1/2$ model Eq. (30), the ends of such strings correspond to gapped fermions (if $|g| \gg |t|$). The ends of $W$ strings only differ from the ends of $W$ strings by a local bosonic operator. Thus the ends of $W$ strings are also fermions.

To really prove the ends of $W$ strings are fermions, we need to show the hopping of the ends of $W$ strings satisfy the fermion hopping algebra introduced in Ref. [14]:

$$t_{ij} t_{kj} t_{ji} = - t_{ij} t_{kj} t_{ji},$$

$$[t_{ij}, t_{kl}] = 0, \text{ if } i, j, k, l \text{ are all different},$$

(57)

where $t_{ij}$ describes the hopping from site $i$ to site $j$. It was shown that the particles are fermions if their hopping satisfy the algebra Eq. (57). We note that the ends of the $W$ strings live on the sites. The labels $i, j, i$... in the above equation correspond to lattice sites $i, j$.... The hops between sites $i, j, k, l$ in Fig. 7 are given by

$$t_{i+a, i} = \psi_{i+a}^1 \hat{U}_{i+a} \psi_{i}, \quad a = \pm x, \pm y$$

(58)

Note that the hops between nearest neighbors are taken from the Hamiltonian Eq. (41). Since $U_{ij}$ commute with each other, the algebra of the above hopping operators is just that of fermion hopping operators. In particular, the above hopping operators satisfy the fermion hopping algebra Eq. (57). Hence, the ends of the $W$ strings are fermions.
For each fixed configuration $s_{ij}$, there are $2^{N_{site}}/2$ different states (with even or odd numbers of total $\psi$ fermions). Their energy are given by the fermion hopping Hamiltonian Eq. (43). Let $E_0(\{s_{ij}\})$ be the ground state energy of Eq. (43). The ground state and the ground state energy of our spin-1/2 model Eq. (61) is obtained by choosing a configuration $s_{ij}$ that minimize $E_0(\{s_{ij}\})$. We note that $E_0(\{s_{ij}\})$ is invariant under the $Z_2$ gauge transformation Eq. (53).

When $g \gg |\ell|$, the ground state of Eq. (41) has $\tilde{F}_i = -1$ which minimize the dominating $g \sum_i F_i$ term. The ground state configuration is given by

$$s_{i,i+x} = (-)^{s_i}, \quad s_{i,i+y} = 1.$$  \hfill (59)

The $\psi_i$ fermion hopping Hamiltonian Eq. (43) for the above configuration describes fermion hopping with $\pi$-flux per plaquette. The fermion spectrum has a form

$$E_k = \pm 2 \sqrt{t^2 \sin^2(k_x) + t^2 \sin^2(k_y)}.$$  \hfill (60)

The low energy excitations of such a hopping Hamiltonian are described by two two-component massless Dirac fermions in 2+1D. We see that the ends of the $\tilde{W}$ strings are massless Dirac fermions.

Our model also contain $Z_2$ gauge excitations. The $Z_2$ vortices are created by flipping $\tilde{F}_i = -1$ to $\tilde{F}_i = 1$ in some plaquettes. The $Z_2$ vortex behaves like a $\pi$-flux to the gapless fermions. Thus the gapless fermions carry a unit $Z_2$ charge. The low energy effective theory of our model is massless Dirac fermions coupled to a $Z_2$ gauge field.

C. Projective symmetry and massless fermions

We know that symmetry breaking can produce and protect gapless Nambu-Goldstone modes. In Ref. [9, 16], it was proposed that, in addition to symmetry breaking, quantum order can also produce and protect gapless excitations. The gapless excitations produced and protected by quantum order can be gapless gauge bosons and/or gapless fermions. In this paper we show that the quantum orders discussed in Ref. [9, 16] are due to string-net condensations. Therefore, more precisely it is string-net condensations that produce and protect gapless gauge bosons and/or gapless fermions. The string-net condensations and gapless excitations are connected in the following way. Let us consider a Hamiltonian that has a symmetry described by a symmetry group $SG$. We assume the ground state has a string-net condensation. Then, the hopping Hamiltonian for the ends of condensed string will be invariant under a larger group - the projective symmetry group $PSG$, as discussed in section IVB. $PSG$ is an extension of the symmetry group $SG$, ie $PSG$ contain a normal subgroup $IGG$ such that $PSG/IGG = SG$. The relation between $PSG$ and gapless gauge bosons is simple. Let $G$ be the maximum continuous subgroup of $IGG$. Then the gapless gauge bosons are described by a gauge theory with $G$ as the gauge group.[9, 15] Some times the ends of strings are fermions. However, the relation between gapless fermions and $PSG$ is more complicated. Through a case by case study of some $PSG$'s[9, 16], we find that certain $PSG$'s indeed guarantee the existence of gapless fermions.

In this section, we are going to study a large family of exact soluble local bosonic models which depends on many continuous parameters. The ground states of the local bosonic models have a string-net condensation and do not break any symmetry. We will show that the projective symmetry of the ends of condensed strings protects a massless fermion. As a result, our exact soluble model always has massless fermion excitations regardless the value of the continuous parameters (as long as they are within a certain range). This puts the results of Ref. [9, 16], which were based on mean-field theory, on a firmer ground.

The exact soluble local bosonic models are the spin-1/2 model

$$H_{\frac{1}{2}+\frac{1}{2}} = -g \sum_i \gamma_{i}^{xy} \bar{\psi}_{i+x} \gamma_{i}^{xy} \bar{\psi}_{i+x} + \gamma_{i}^{xy} \bar{\psi}_{i+y} \gamma_{i}^{xy} \bar{\psi}_{i+y} + h.c.)$$   \hfill (61)

where $\gamma_{ab}$ and $\gamma_{±,a}$ are given in Eq. (48) and Eq. (47). We will discuss a more general Hamiltonian later.

The Hamiltonian is not invariant under $x \rightarrow -x$ parity $P_x$. But it has a $x \rightarrow -x$ parity symmetry if $P_x$ is followed by a spin rotation $\gamma^x \leftrightarrow \gamma^x$. That is $\gamma_{P_x} P_x H(\gamma_{P_x} P_x)^{-1} = H$ with

$$\gamma_{P_x} = \gamma \gamma^x \gamma^x \frac{\sqrt{2}}{\gamma^x}$$  \hfill (62)

Similarly for $y \rightarrow -y$ parity $P_y$, we have $\gamma_{P_y} P_y H(\gamma_{P_y} P_y)^{-1} = H$ with

$$\gamma_{P_y} = \gamma \gamma^y \gamma^y \frac{\sqrt{2}}{\gamma^x}$$  \hfill (63)

In the fermion representation $\gamma_{P_x}$ and $\gamma_{P_y}$ generate the following transformations

$$\gamma_{P_x} : \lambda^x \leftrightarrow \lambda^x, \quad \psi^\dagger_{i} \psi_{i} \rightarrow \psi^\dagger_{i},$$

$$\gamma_{P_y} : \lambda^y \leftrightarrow \lambda^y, \quad \psi^\dagger_{i} \psi_{i} \rightarrow \psi^\dagger_{i}.$$  \hfill (64)

Now let us study how the symmetries $T_{x,y}$ and $\gamma_{P_x} P_{x,y}$ are realized in the hopping Hamiltonian Eq. (43) for the ends of condensed strings. As discussed in section IVB, the hopping Hamiltonian may not be invariant under the symmetry transformations $T_{x,y}$ and $\gamma_{P_x} P_{x,y}$ directly. The hopping Hamiltonian only has a projective symmetry generated by a symmetry transformation followed by a $Z_2$ gauge transformation $G(i)$. Since the $\pi$-flux configuration does not break any symmetries, we expect the hopping Hamiltonian for the $\pi$-flux configuration to be invariant under $G_T$, $G_T T_{x,y}$, $G_{P_x} \gamma_{P_x} P_x$, and $G_{P_y} \gamma_{P_y} P_y$, where $G_{x,y}$ and $G_{P_{x,y}}$ are the corresponding gauge transformations. The action of $T_{x,y}$...
and $\gamma_{P_{x,y}}$, $P_{x,y}$ on the $\psi$ fermion are given by

$$T_x : \psi_{(x,y)} \rightarrow \psi_{(x+1,y)},$$

$$T_y : \psi_{(x,y)} \rightarrow \psi_{(x,y+1)},$$

$$\gamma_{P_x} P_x : \psi_{(x,y)} \rightarrow \psi_{(-1,x,y)},$$

$$\gamma_{P_y} P_y : \psi_{(x,y)} \rightarrow \psi_{(x,-1,y)}.$$  \hfill (65)

For the $\pi$-flux configuration Eq. (59), we need to choose the following $G_{x,y}$ and $G_{P_{x,y}}$ in order for the combined transformation $G_{x,y} T_{x,y}$ and $G_{P_{x,y}} \gamma_{P_{x,y}} P_{x,y}$ to be the symmetries of the hopping Hamiltonian Eq. (43)

$$G_x = 1, \quad G_y = (-)^y,$$

$$G_{P_x} = (-)^x, \quad G_{P_y} = (-)^y.$$  \hfill (66)

The hopping Hamiltonian is also invariant under a global $Z_2$ gauge transformation

$$G_0 : \psi_i \rightarrow -\psi_i$$  \hfill (67)

The transformations $\{G_0, G_{x,y}, T_{x,y}, G_{P_{x,y}} \gamma_{P_{x,y}} P_{x,y}\}$ generate the PSG of the hopping Hamiltonian.

To show that the above PSG protects the masslessness of the fermions, we consider a more general Hamiltonian by adding

$$\delta H_{\frac{\pi}{2}} = \sum_{C_{ij}} \left( t(C_{ij}) \tilde{W}(C_{ij}) + h.c. \right)$$  \hfill (68)

to $H_{\frac{\pi}{2}}$, where $C_{ij}$ is an open string connecting site $i$ and site $j$ and $\tilde{W}(C_{ij})$ is given in Eq. (56). The new Hamiltonian is still exactly soluble. We will choose $t(C_{ij})$ such that the new Hamiltonian has the translation symmetries and the $P_{x,y}$ parity symmetries. In the following, we would like to show that the new Hamiltonian with those symmetries always has massless Dirac fermion excitations (assuming $t(C_{ij})$ is not too big comparing to $g$).

When $t(C_{ij})$ is not too large, the ground state is still described by the $\pi$-flux configuration. The new hopping Hamiltonian for $\pi$-flux configuration has a more general form

$$H = \sum_{(ij)} (\chi_{ij} \psi_i^\dagger \psi_j + h.c.)$$  \hfill (69)

The symmetry of the physical spin-$1 \frac{1}{2}$ Hamiltonian requires that the above hopping Hamiltonian to be invariant under the PSG discussed above. Such an invariance will guarantee the existence of massless fermions.

The invariance under $G_{x,y} T_x$ and $G_y T_y$ require that

$$\chi_{i,i+1} = (-)^y m_x \chi_m$$  \hfill (70)

In the momentum space,

$$\chi(k_1, k_2) = N_{\text{site}}^{-1} \sum_{ij} e^{-i(k_1 + i k_2) \cdot j} \chi_{ij}$$

$$= e_0(k_2) \delta_{k_1-k_2} + e_1(k_2) \delta_{k_1+k_2+Q_y}$$  \hfill (71)

where

$$e_0(k) = \sum_{m_x = \text{even}} e^{i k \cdot m} \chi_m,$$

$$e_1(k) = \sum_{m_x = \text{odd}} e^{i k \cdot m} \chi_m.$$  \hfill (72)

We note that $e_0(k)$ and $e_1(k)$ are periodic function in the Brillouin zone. They also satisfy

$$e_0(k) = e_0(k + Q_x), \quad e_1(k) = -e_1(k + Q_x).$$  \hfill (73)

where $Q_x = \pi x$ and $Q_y = \pi y$. In the momentum space, we can rewrite $H$ as

$$H = \sum_k \Psi^\dagger_k \Gamma(k) \Psi_k$$  \hfill (74)

where $\Psi^T_k = (\psi_k, \psi_{k+Q_y})$. The sum $\sum_k$ is over the reduced Brillouin zone: $-\pi < k_x < \pi$ and $-\pi/2 < k_y < \pi/2$. $\Gamma(k)$ has a form

$$\Gamma(k) = \begin{pmatrix} e_0(k) & e_1(k+Q_y) \\ e_1(k) & e_0(k+Q_x) \end{pmatrix}$$  \hfill (75)

Note that the transformation $\gamma_{P_{x,y}} : \psi \leftrightarrow \psi^\dagger$ changes

$$\sum \chi_{ij} \psi_i^\dagger \psi_j \rightarrow \sum \chi_{ij} \psi_i^\dagger \psi_j$$

with $\chi_{ij} = -\chi_{ji}$. Thus the invariance under $G_{P_{x,y}} \gamma_{P_{x,y}} P_{x,y}$ requires that

$$-\chi_{P_{x,y} P_{x,y}} = e^{-i m_x m_y} \chi \chi_m$$  \hfill (76)

In the momentum space, the above becomes

$$\epsilon_0(P_x k) = -e_0(k)$$

$$\epsilon_1(P_y k) = e_1(k + Q_y)$$  \hfill (77)

Similarly, the invariance under $G_{P_{x,y}} \gamma_{P_{x,y}} P_{x,y}$ requires that

$$-\chi_{P_{x,y} P_{x,y}} = e^{-i m_x m_y} \chi \chi_m$$  \hfill (78)

In the momentum space

$$\epsilon_0(P_x k) = -e_0(k + Q_y)$$

$$\epsilon_1(P_y k) = -e_1(k)$$  \hfill (79)

We see that the translation $T_{x,y}$ and $x \rightarrow -x$ parity $\gamma_{P_{x,y}} P_{x,y}$ symmetries of the spin-$1 \frac{1}{2}$ Hamiltonian require that $\epsilon_0(k) = -e_0(P_y k)$ and hence $\epsilon_0(k)|_{k_y = 0} = 0$. Similarly, the translation $T_{x,y}$ and $y \rightarrow -y$ parity $\gamma_{P_{x,y}} P_{x,y}$ symmetries require that $\epsilon_1(k)|_{k_y = 0} = 0$. Thus $T_{x,y}$ and $\gamma_{P_{x,y}} P_{x,y}$ symmetries require that $\Gamma(0) = 0$. Using Eq. (73), we find that $\Gamma(0) = 0$ implies that $\Gamma(Q_x y) = 0$. The spin-$1 \frac{1}{2}$ Hamiltonian Eq. (61) has (at least) two two-component massless Dirac fermions if it has two translation $T_{x,y}$ and two parity $\gamma_{P_{x,y}} P_{x,y}$ symmetries. We see that string-net condensation and the associated projective symmetry produce and protect massless Dirac fermions.
VI. MASSLESS FERMIONS AND STRING-NET CONDENSATION ON CUBIC LATTICE

The above calculation and the 2D model can be generalized to 3D cubic lattice. We introduce six Majorana fermions $\lambda^a_i$, where $a = x, \bar{x}, y, \bar{y}, z, \bar{z}$. One set of commuting operators on square lattice has a form

$$
\begin{align*}
\hat{U}_{i,i+x} &= -i\lambda^x_i\lambda^{\bar{x}}_{i+i+x} \\
\hat{U}_{i,i+y} &= -i\lambda^y_i\lambda^{\bar{y}}_{i+i+y} \\
\hat{U}_{i,i+z} &= -i\lambda^z_i\lambda^{\bar{z}}_{i+i+z}
\end{align*}
$$

$$\hat{U}^\dagger_{i,j} = \hat{U}_{j,i}. \quad (82)$$

Using $\hat{U}_{i,j}$ and a complex fermion $\psi_i$, we can construct an exact soluble interacting Hamiltonian on cubic lattice

$$
\begin{align*}
H^{\hat{U}_{i,j}} &= g\sum_p \hat{F}_p + \tau \sum_{i} \sum_{a=x,y,z} \left(i\hat{U}_{i,i+a}\psi_i^\dagger \psi_{i+a} + h.c.\right), \\
\hat{F}_p &= \hat{U}_{i_1,i_2}\hat{U}_{i_2,i_3}\hat{U}_{i_3,i_4}\hat{U}_{i_4,i_1}
\end{align*}
$$

where $\sum_p$ sum over all the square faces of the cubic lattice. $i_1, i_2, i_3, i_4$ label the four corners of the square $p$. The Hilbert space of system is generated by complex fermion operators $\psi_i$ and

$$
\begin{align*}
2\psi_{1,i} &= \lambda^x_i + i\lambda^{\bar{x}}_i \\
2\psi_{2,i} &= \lambda^y_i + i\lambda^{\bar{y}}_i \\
2\psi_{3,i} &= \lambda^z_i + i\lambda^{\bar{z}}_i
\end{align*}
$$

and there are 16 states per site.

A physical Hilbert space is defined as a subspace with even numbers of fermions per site. The physical Hilbert space has 8 states per site. When restricted in the physical Hilbert space, $H^{\hat{U}_{i,j}}$ defines our spin-$\frac{1}{2}$$\frac{1}{2}$ system, which is a local bosonic system.

When $g \gg |t|$, our spin-$\frac{1}{2}$$\frac{1}{2}$ model has two four-component massless Dirac fermions as its low lying excitations. The model also has a $Z_2$ gauge excitations and the massless Dirac fermions carry unit $Z_2$ gauge charge. Again, the model has a string-net condensation in its ground state. Both the $Z_2$ gauge excitation and the massless fermion are produced and protected by the string-net condensation and the associated PSG.

VII. ARTIFICIAL LIGHT AND ARTIFICIAL MASSLESS ELECTRON ON CUBIC LATTICE

In this section, we are going to combine the above 3D model and the rotor model discussed in Ref. [44] and Ref. [13] to obtain a quasi-exact soluble local bosonic model that contains massless Dirac fermions coupled to massless $U(1)$ gauge bosons.

A. 3D rotor model and artificial light

A rotor is described by an angular variable $\theta$. The angular momentum of $\theta$, denoted as $S^z$, is quantized as integers. The 3D rotor model under consideration has one rotor on every link of a cubic lattice. We use $ij$ to label the nearest neighbor links. $ij$ and $ji$ label the same links. For convenience, we will define $\hat{\theta}_{ij} = -\hat{\theta}_{ji}$ and $S^z_{ij} = -S^z_{ji}$. The 3D rotor Hamiltonian has a form

$$
H_{rotor} = U \sum_i \left(\sum_a (S^z_{i,i+a})^2 + \frac{1}{2} \sum_{i,a} (S^z_{i,i+a})^2 \right) + g_1 \sum_p \cos(\hat{\theta}_{i_1,i_2} + \hat{\theta}_{i_2,i_3} + \hat{\theta}_{i_3,i_4} + \hat{\theta}_{i_4,i_1}) \quad (85)
$$

Here $i = (i_x, i_y, i_z)$ label the sites of the cubic lattice, and $a = \pm x, \pm y, \pm z$. The $\sum_p$ sum over all the square faces of the cubic lattice. $i_1, i_2, i_3,$ and $i_4$ label the four corners of the square $p$. When $J = g_1 = 0$ and $U > 0$, the state with all $S^z_{ij} = 0$ is a ground state. Such a state will be regarded as a state with no strings. We can create a string or a string-net from the no-string state using the following string (or string-net) operator

$$
W_{U(1)}(C) = \prod_C e^{i\hat{\theta}_{ij}} \quad (86)
$$

where $C$ is a string (or a string-net) formed by the nearest neighbor links, and $\prod_C$ is a product over all the nearest neighbor links $ij$ on the string (or string-net). Since the closed-string-net operator $W_{U(1)}(C_{close})$ commute with $H_{rotor}$ when $J = g_1 = 0$, $W_{U(1)}(C_{close})$ generate a large set of degenerate ground states. The degenerate ground states are described by closed string-nets.

There is another way to generate the degenerate ground states. We note that all the degenerate ground states satisfy $\sum_a S^z_{i,i+a} = 0$. Let $|\{\theta_{ij}\}\rangle$ be the common eigenstate of $\hat{\theta}_{ij}$: $\hat{\theta}_{ij}|\{\theta_{ij}\}\rangle = \theta_{ij}|\{\theta_{ij}\}\rangle$. Then the projection into $\sum_a S^z_{i,i+a} = 0$ subspace: $\mathcal{P}|\{\theta_{ij}\}\rangle$ give us a degenerate ground state. We note that

$$
e^{-\sum_i \phi_i \sum_a S^z_{i,i+a}} \quad (87)
$$

generate a $U(1)$ gauge transformation $|\{\theta_{ij}\}\rangle \rightarrow |\{\tilde{\theta}_{ij}\}\rangle$, where

$$\tilde{\theta}_{ij} = \theta_{ij} + \phi_i - \phi_j \quad (88)$$

Thus two $U(1)$ gauge equivalent configurations $\theta_{ij}$ and $\tilde{\theta}_{ij}$ give rise to the same projected state

$$\mathcal{P}|\{\theta_{ij}\}\rangle = \mathcal{P}|\{\tilde{\theta}_{ij}\}\rangle \quad (89)$$

We find that the degenerate ground states are described by $U(1)$ gauge equivalent classes of $\theta_{ij}$. The degenerate ground states also have a $U(1)$ gauge structure.

When $J = 0$ but $g_1 \neq 0$, the degeneracy in the ground states are lifted. One can show that, in this case, $\mathcal{P}|\{\theta_{ij}\}\rangle$ is an energy eigenstate with an energy $g_1 \sum_p \cos(\theta_{i_1,i_2} + \theta_{i_2,i_3} + \theta_{i_3,i_4} + \theta_{i_4,i_1})$. Clearly two $U(1)$ gauge equivalent configurations $\theta_{ij}$ and $\tilde{\theta}_{ij}$ have the same energy. The non-zero $g_1$ makes the closed string-nets to fluctuate and vanishing $J$ means that the strings in the
string-nets have no tension. Thus \( J = 0 \) ground state has strong fluctuations of large closed string-nets, and the ground state has a closed-string-net condensation.\cite{13}

When \( J \neq 0 \), \( \mathcal{P}(\{\theta_{ij}\}) \) is no longer an eigenstate. The fluctuations of \( \theta_{ij} \) describe a dynamical \( U(1) \) gauge theory with \( \theta_{ij} \) as the gauge potential.\cite{15,44}

**B. (Quasi-)exact soluble QED on cubic lattice**

To obtain massless Dirac fermions and \( U(1) \) gauge bosons from a local bosonic model, we mix the spin-\(\frac{1}{2}\) model and the rotor model to get

\[
H_{\text{QED}} = U \sum_i \left( \psi_i^\dagger \psi_i + \sum_a S_{i,i+a}^z \right)^2 + \frac{J}{2} \sum_{i,a} (S_{i,i+a}^z)^2 \\
+ g_1 \sum_p \cos(\Phi_p) + g_2 \sum_p \bar{F}_p \\
+ t \sum_{i,a} \sum_{a=x,y,z} \left( e^{i\theta_{ij}} U_{i,i+a} \psi_i^\dagger \psi_i + h.c. \right)
\]

where \( \Phi_p = \theta_{ij} + \theta_{ij} + \theta_{ij} + \theta_{ij} \). If we restrict ourselves within the physical Hilbert space with even numbers of fermions per site, the above model is a local bosonic model.

Let us first set \( J = 0 \). In this case, the above model can be solved exactly. First let us also set \( U = 0 \). In this case \( \theta_{ij} \) and \( \bar{U}_{ij} \) commute with \( H_{\text{QED}} \) and commute with each other. Let \( \{\{\theta_{ij}, s_{ij}\}, n\} \) be the common eigenstates of \( \theta_{ij} \) and \( \bar{U}_{ij} \), where \( n = 1, 2, ..., 2^{N_{\text{site}}} \) labels different degenerate common eigenstates. Within the subspace expanded by \( \{\{\theta_{ij}, s_{ij}\}, n\} \), \( n = 1, 2, ..., 2^{N_{\text{site}}} \), the \( H_{\text{QED}} \) reduces to

\[
H_{\text{hop}} = g_1 \sum_p \cos(\Phi_p) + g_2 \sum_p \bar{F}_p \\
+ t \sum_{i,a} \sum_{a=x,y,z} \left( e^{i\theta_{ij}} s_{i,i+a} \psi_i^\dagger \psi_i + h.c. \right)
\]

which is a free fermion hopping model. Let \( \{\{\theta_{ij}, s_{ij}\}, \Psi_n\} \) be the many-fermion eigenstate of the above fermion hopping model and let \( E(\{\theta_{ij}, s_{ij}\}, n) \) be its energy. Then \( \{\{\theta_{ij}, s_{ij}\}, \Psi_n\} \) is also an eigenstate of \( H_{\text{QED}} \mid J = 0, U = 0 \) with energy \( E(\{\theta_{ij}, s_{ij}\}, n) \).

We note that

\[
\hat{N}_i = \psi_i^\dagger \psi_i + \sum_a S_{i,i+a}^z
\]

commute with each other and commute with \( H_{\text{QED}} \). Thus the eigenstates of \( H_{\text{QED}} \mid J = 0, U = 0 \) can be obtained from the eigenstates of \( H_{\text{QED}} \mid J = 0, U = 0 \) by projecting into the subspace with \( \hat{N}_i = N_i \):

\[
\mathcal{P}(\{N_i\}) \{\theta_{ij}, s_{ij}\}, \Psi_n\}
\]

The above state is an eigenstate of \( H_{\text{QED}} \mid J = 0 \) with an energy

\[
U \sum_i N_i + E(\{\theta_{ij}, s_{ij}\}, n).
\]

Eq. (93) and Eq. (94) are our exact solution of \( H_{\text{QED}} \mid J = 0 \). (We have implicitly assumed that \( \mathcal{P}(\{N_i\}) \) also perform the projection into the physical Hilbert space of even numbers of fermions per site.)

When \( U \) is positive and large, the low energy excitations only appear in the sector \( N_i = 0 \). Those low energy eigenstates are given by \( \mathcal{P}(\{\theta_{ij}, s_{ij}\}, \Psi_n) \) where \( \mathcal{P} \) is the projection into the \( N_i = 0 \) subspace and the even-fermion subspace. Their energy is \( E(\{\theta_{ij}, s_{ij}\}, n) \).

Let us further assume that \(-g_1 \gg |t| \) and \( g \gg |t| \). In this limit, the ground state have \( f_p = -1 \) and \( \Phi_p = 0 \). We can choose

\[
\theta_{i,i+a} = 0, \quad a = x, y, z \\
s_{i,i+x} = 1, \\
s_{i,i+y} = (-t)^x, \\
s_{i,i+z} = (-t)^{x+y}
\]

to describe such a configuration. For such a configuration, Eq. (91) describes a staggered fermion Hamiltonian.\cite{24,49,50} The ground state wave function \( \mathcal{P}(\{\theta_{ij}, s_{ij}\}, \Psi_0) \) is an eigenstate of the \( U(1) \) closed-string-net operator \( W_{U(1)}(C_{\text{close}}) \) with eigenvalue 1. It is also a eigenstate of the \( Z_2 \) closed-string-net operator \( W(C_{\text{close}}) \) with eigenvalue \((-1)^{N_p}\) where \( N_p \) is the number of the square plaquettes enclosed by \( C_{\text{close}} \). We see that there is a condensation of closed \( U(1) \) and \( Z_2 \) string-nets in the \( J = 0 \) ground state. In such a string-net condensed state, there are gapless fermionic excitations, which are described by fermion-hopping in \( \pi \)-flux phase.

In the momentum space, the fermion hopping Hamiltonian Eq. (91) for the \( \pi \)-flux configuration has a form

\[
H_{\text{hop}} = \sum_k \Psi_{a,k}^T \Gamma(k) \Psi_{a,k} + \text{Const.}
\]

where

\[
\Psi_{a,k}^T = (\psi_{a,k}, \psi_{a,k+Q_x}, \psi_{a,k+Q_y}, \psi_{a,k+Q_z+Q_y}), \\
\Gamma(k) = 2t(\sin(k_x) \Gamma_1 + \sin(k_y) \Gamma_2 + \sin(k_z) \Gamma_3)
\]

and \( \Gamma_1 = \tau^3 \otimes \tau^0, \Gamma_2 = \tau^1 \otimes \tau^3, \) and \( \Gamma_3 = \tau^1 \otimes \tau^1 \). Here \( \tau^{1,2,3} \) are the Pauli matrices and \( \tau^0 \) is the 2 by 2 identity matrix. The momentum summation \( \sum_k \) over a range \( k_x \in (-\pi/2, \pi/2), k_y \in (-\pi/2, \pi/2), \) and \( k_z \in (-\pi, \pi) \). Since \( \{\Gamma_i, \Gamma_j\} = 2\delta_{ij} \), \( i, j = 1, 2, 3 \), we find the fermions have a dispersion

\[
E(k) = \pm 2t \sqrt{\sin^2(k_x) + \sin^2(k_y) + \sin^2(k_z)}
\]

We see that the dispersion has two nodes at \( k = 0 \) and \( k = (0,0,\pi) \). Thus, Eq. (91) will give rise to 2 massless four-component Dirac fermions in the continuum limit.

After including the \( U(1) \) gauge fluctuations described by \( \theta_{ij} \) and the \( Z_2 \) gauge fluctuations described by \( s_{ij} \), the massless Dirac fermions interact with the \( U(1) \) and the \( Z_2 \) gauge fields as fermions with unit charge. Therefore the total low energy effective theory of our model is a QED with 2 families of Dirac fermions of unit charge (plus an extra \( Z_2 \) gauge filed). We will call those fermions...
artificial electrons. The continuum effective theory has a form
\[
\mathcal{L} = \bar{\psi}_1 D_{\mu} \psi_1^0 \psi_1 + v_f \bar{\psi}_1 D_1 \gamma^1 \psi_1 + \frac{C}{J_0} E^2 - l_0 g_1 B^2 + ... \tag{98}
\]
where \( l_0 \) is the lattice constant, \( I = 1, 2, D_0 = \partial_t + i a_0, \) \( D_i = \partial_i + i a_i \| i = 1, 2, 3, \) \( v_f = 2 l_0 v, \) \( \gamma^0 \| \mu = 0, 1, 2, 3 \) are \( 4 \times 4 \) Dirac matrices, and \( \bar{\psi} = \psi^0. \)

We like to pointed out the constant \( C \) is Eq. \( \text{(98)} \) is of order 1. Thus the coefficient of the \( E^2 \) term \( \frac{C}{l_0} \rightarrow \infty \) when \( J = 0. \) For a finite \( J, \) the \( U(1) \) gauge field will have a non-trivial dynamics. We also like to point out that, without fine tuning, the speed of artificial light, \( c_a \sim \frac{1}{\sqrt{J_0 g_1}}, \) and the speed of artificial electrons, \( v_f, \) do not have to be the same in our model. Thus the Lorentz symmetry is not guaranteed.

We would like to remark that, for finite \( J, \) the \( U(1) \) closed-string operators no longer condense. A necessary (but not sufficient) condition for closed strings to condense is that the ground state expectation value of the closed-string operator satisfy the perimeter law
\[
\langle W_{U(1)}(C_{\text{close}}) \rangle = A e^{-L_C/\xi} \tag{99}
\]
where \( L_C \) is the length of the closed string and \( A, \xi \) are constants for large closed strings. We note that the closed-string operators are the Wilson loop operator of the \( U(1) \) gauge field. If the \( 3+1D \) \( U(1) \) gauge theory is in the Coulomb phase where the artificial light is gapless, it was found that\[18\]
\[
\langle W(C_{\text{close}}) \rangle = A(C) e^{-L_C/\xi} \tag{100}
\]
where \( A(C) \) depend on the shape of the closed string \( C_{\text{close}} \) even in the large-string limit. Thus the closed strings in our model do not exactly condense. The \( U(1) \) Coulomb phase is, in some sense, similar to the algebraic-long-range order phase of \( 1+1D \) interacting boson model where the bosons do not exactly condense but the boson operator has an algebraic-long-range correlation.

### C. Emerging chiral symmetry from PSG

Eq. \( \text{(98)} \) describes the low energy dynamics of the ends of open strings (the fermion \( \psi \)) and the “condensed” closed string-nets (the \( U(1) \) gauge field). The fermions and gauge boson are massless and interact with each other. Here we would like to address an important question: after integrating out high energy fermions and gauge fluctuations, do fermions and gauge boson remain to be massless? In general, interaction between massless excitations will generate a mass term for them, unless the masslessness is protected by symmetry or some other things. We know that due to the \( U(1) \) gauge invariance, the radiative corrections cannot generate counter term that break the \( U(1) \) gauge invariance. Thus radiative corrections cannot generate mass for the \( U(1) \) gauge boson. For the fermions, if the theory has a chiral symmetry \( \psi_1 \rightarrow e^{i \gamma^5} \psi_1, \) \( \gamma^5 = \gamma^0 \gamma^1 \gamma^2 \gamma^3, \) then the radiative corrections cannot generate counter terms that break the chiral symmetry and thus cannot generate mass for fermions. Although the low energy effective theory Eq. \( \text{(98)} \) appears to have the chiral symmetry, in fact it does not. This is because that Eq. \( \text{(98)} \) is derived from a lattice model. It contains many other higher order terms summarized by the \( ... \) in Eq. \( \text{(98)} \). Those higher order terms do not have the chiral symmetry. To see this, we note that the action of \( \gamma^5 \) on \( \Psi_{a,k} \) is realized by a \( 4 \times 4 \) matrix \( \gamma^5 \propto \Gamma_1 \Gamma_2 \Gamma_3 \propto \tau^3 \otimes \tau^2. \) We also note that the periodic boundary conditions of \( \Psi_{a,k} \) in the reduced Brillouin zone are given by
\[
\Psi_{a,k+Q_x} = \tau^1 \otimes \tau^0 \Psi_{a,k}, \quad \Psi_{a,k+Q_y} = \tau^0 \otimes \tau^1 \Psi_{a,k}, \tag{101}
\]
We find that the action of \( \gamma^5 \) is incompatible with the periodic boundary conditions since \( \gamma^5 \) does not commute with \( \tau^1 \otimes \tau^0 \) and \( \tau^0 \otimes \tau^1. \) Therefore the chiral symmetry generated by \( \gamma^5 \) cannot be realized on lattice. Due to the lack of chiral symmetry, it appears that the radiative corrections can generate a mass term
\[
\delta \mathcal{L} = \bar{\psi}_{1,a} m \psi_{1,a} \tag{102}
\]
which is allowed by the symmetry.

The lack of chiral symmetry on lattice makes it very difficult to study massless fermions/quarks in lattice gauge theory. In last a few years, this problem was solved using the Ginsparg-Wilson relation.[51–54] In the following, we would like to show that there is another way to solve the massless-fermion/chiral-symmetry problem. We will show that our model has an emerging chiral symmetry that appear only at low energies. The low energy chiral symmetry comes from the non-trivial quantum order and the associated PSG in the string-net condensed ground state.[9, 15, 16] The Dirac operator in our model satisfies a linear relation
\[
WD\Psi = D, \quad W \in \text{PSG} \tag{103}
\]
in contrast to the non-linear Ginsparg-Wilson relation
\[
D\gamma^5 + \gamma^5 D = a D\gamma^5 D \tag{104}
\]
Because of the low energy chiral symmetry, all the 2 families of Dirac fermions remain massless even after we include the radiative corrections from the interaction with the \( U(1) \) gauge bosons.

To see how the string-net condensation and the related PSG protect the massless fermions, we follow closely the discussion in section VC. The Hamiltonian Eq. \( \text{(90)} \) is a mixture of the rotor model and the spin-\( \frac{1}{2} \)\( \frac{1}{2} \) model. The symmetry properties of the rotor part is simple. Here, we will concentrate on the spin-\( \frac{1}{2} \)\( \frac{1}{2} \)\( \frac{1}{2} \) part. Eq. \( \text{(90)} \) is not invariant under the six parity transformations: \( P_{x,y,z} \) and \( P_{x,y,z} \) that generate \( x \leftrightarrow -x, y \leftrightarrow -y, z \leftrightarrow -z, \) \( x \leftrightarrow y, y \leftrightarrow z, \) and \( z \leftrightarrow x. \) But it is invariant under parity \( P_{x,y,z} \) and \( P_{x,y,z} \) followed by spin rotations \( \gamma_{P_{x,y,z}} \) and \( \gamma_{P_{x,y,z}} \) respectively. In the fermion representation \( \gamma_{P_{x,y,z}} \) and \( \gamma_{P_{x,y,z}} \) generate the following transforma-
tions
\begin{align*}
\gamma_{P_x} : & \lambda^x_i \leftrightarrow \lambda^x_{\bar{i}}, \quad \psi_i \leftrightarrow \psi^*_i, \\
\gamma_{P_y} : & \lambda^y_i \leftrightarrow \lambda^y_{\bar{i}}, \quad \psi_i \leftrightarrow \psi^*_i, \\
\gamma_{P_z} : & \lambda^z_i \leftrightarrow \lambda^z_{\bar{i}}, \quad \psi_i \leftrightarrow \psi^*_i, \\
\gamma_{P_{xy}} : & \lambda^x_i \leftrightarrow \lambda^y_{\bar{i}}, \quad \lambda^y_i \leftrightarrow \lambda^x_{\bar{i}}, \\
\gamma_{P_{xz}} : & \lambda^x_i \leftrightarrow \lambda^z_{\bar{i}}, \quad \lambda^z_i \leftrightarrow \lambda^x_{\bar{i}}, \\
\gamma_{P_{yz}} : & \lambda^y_i \leftrightarrow \lambda^z_{\bar{i}}, \quad \lambda^z_i \leftrightarrow \lambda^y_{\bar{i}}, \\
\end{align*}
(105)

The symmetries \( T_{x,y}, \gamma_{P_{x,y,z}} \), and \( \gamma_{P_{x,y,z},z} \) are realized in the hopping Hamiltonian Eq. (91) through PSG. The hopping Hamiltonian is invariant only under the symmetry transformations followed by proper \( Z_2 \) gauge transformations \( G(i) \).

For the \( \pi \)-flux configuration \( \delta G \) of the spin-1/2 sector and the zero-flux configuration \( \theta G \) of the rotor sector do not break any symmetries, we expect the hopping Hamiltonian Eq. (91) to be invariant under \( G_{x,y,z}T_{x,y,z}, G_{P_{x,y,z}}\gamma_{P_{x,y,z}}F_{x,y,z} \) and \( G_{P_{x,y,z},z}\gamma_{P_{x,y,z},z}P_{x,y,z,z} \).

The action of \( T_{x,y} \) and \( \gamma_{P_{x,y,z}}, P_{x,y},z \) on the \( \psi \) fermion are standard coordinate transformations. The action of \( \gamma_{P_{x,y,z}}, P_{x,y,z} \) on the \( \psi \) fermion are given by
\begin{align*}
\gamma_{P_x} P_x : & \psi_{i_x,i_y,i_z} \leftrightarrow \psi^*_{(-i_x,i_y,i_z)}, \\
\gamma_{P_y} P_y : & \psi_{i_x,i_y,i_z} \leftrightarrow \psi^*_{(i_x,-i_y,i_z)}, \\
\gamma_{P_z} P_y : & \psi_{i_x,i_y,i_z} \leftrightarrow \psi^*_{(i_x,i_y,-i_z)}. \\
\end{align*}
(106)

For the \( \pi \)-flux configuration Eq. (95), we need to choose the following \( G_{x,y,z}, G_{P_{x,y,z}}, \) and \( G_{P_{x,y,z},z} \), in order for the combined transformation \( G_{x,y,z}T_{x,y,z}, G_{P_{x,y,z}}\gamma_{P_{x,y,z}}F_{x,y,z} \), and \( G_{P_{x,y,z},z}\gamma_{P_{x,y,z},z}P_{x,y,z,z} \) to be the symmetries of the hopping Hamiltonian Eq. (91)
\begin{align*}
G_{x} = & (-)^{i_y+i_x}, \quad G_{y} = (-)^{i_z}, \quad G_{z} = 1 \\
G_{P_x} = & (-)^{i_z}, \quad G_{P_y} = (-)^{i_y}, \quad G_{P_z} = (-)^{i_x}, \\
G_{P_{xy}} = & (-)^{i_x+i_y}, \quad G_{P_{xz}} = (-)^{i_y+i_z}, \quad G_{P_{yz}} = (-)^{i_x+i_z}. \\
\end{align*}
(107)

The hopping Hamiltonian is also invariant under a global \( Z_2 \) gauge transformation
\begin{equation}
G_0 : \quad \psi_i \rightarrow -\psi_i \\
(108)
\end{equation}

The transformations \( \{ G_{x,y,z}T_{x,y,z}, G_{P_{x,y,z}}\gamma_{P_{x,y,z}}P_{x,y,z}, \) \( G_{P_{x,y,z},z}\gamma_{P_{x,y,z},z}P_{x,y,z,z}, G_0 \} \) generate a PSG (a part of the full PSG) of the hopping Hamiltonian.

To study the robustness of massless fermions, we consider a more general Hamiltonian by adding
\begin{equation}
\delta H = \sum_{C_{ij}} \left( t(C_{ij}) \tilde{W}_{U(1)}(C_{ij}) + h.c. \right) \\
(109)
\end{equation}
to \( H_{QED} \), where \( C_{ij} \) is an open string connecting site \( i \) and site \( j \) and \( \tilde{W}_{U(1)}(C_{ij}) \) is an open-string operator
\begin{align}
\tilde{W}_{U(1)}(C_{open}) = & \psi_i \gamma_1^{i_1} \gamma_2^{i_2} \cdots \psi_{i_n} \gamma_{m-1}^{i_{m-1}} \psi_{i_n} \\
(110)
\end{align}
The new Hamiltonian is still exactly soluble, when \( j = 0 \).

We will choose \( t(C_{ij}) \) such that the new Hamiltonian has the translation symmetries and the \( P_{x,y,z} \) parity symmetries. We find that the resulting projective symmetry imposes enough constraint on the hopping Hamiltonian for the ends of condensed strings such that the Hamiltonian always has massless Dirac fermions (assuming \( t(C_{ij}) \) is not too big comparing to \( g \) and \( g_1 \)). Despite the PSG transformations \( G_{P_{x,y,z},z} \gamma_{P_{x,y,z},z}P_{x,y,z,z} \), \( P_{x,y,z,z} \) are not needed for the existence of the massless fermions, we will still include them in the following discussion.

For small \( t(C_{ij}) \), the ground state is still described by the \( \pi \)-flux configuration. The new hopping Hamiltonian for \( \pi \)-flux configuration has a more general form
\begin{equation}
H = \sum_{(i,j)} (\chi_{ij} \psi_i^\dagger \psi_j + h.c.) \\
(111)
\end{equation}
The symmetry of the generalized \( H_{QED} \) requires that the above hopping Hamiltonian to be invariant under the PSG generated by \( \{ G_0, G_{x,y,z}T_{x,y,z}, G_{P_{x,y,z}}\gamma_{P_{x,y,z}}F_{x,y,z}, \)
\( G_{P_{x,y,z},z}\gamma_{P_{x,y,z},z}P_{x,y,z,z} \).

The invariance under \( G_{x,y,z}T_{x,y,z} \) require that
\begin{equation}
\chi_{i,i+m} = (-)^{m_x} (-)^{m_y} \chi_{m} \\
(112)
\end{equation}
In the momentum space,
\begin{align*}
\chi(k_1,k_2) = & \sum_{ij} e^{-ik_1 \cdot i + i k_2 \cdot j} \chi_{ij} \\
= & \sum_{\alpha,\beta=0,1} \epsilon_{\alpha \beta}(k) \delta_{k_1-k_2 + \alpha Q_x + \beta Q_y} \\
& \sum_{m_y+m_z=even, m_z=even} e^{-ik \cdot m} \chi_m. \\
(113)
\end{align*}
where
\begin{align*}
\epsilon_{00}(k) = & \sum_{m_y+m_z=even, m_z=even} e^{-ik \cdot m} \chi_m, \\
\epsilon_{10}(k) = & \sum_{m_y+m_z=odd, m_z=even} e^{-ik \cdot m} \chi_m, \\
\epsilon_{01}(k) = & \sum_{m_y+m_z=even, m_z=odd} e^{-ik \cdot m} \chi_m, \\
\epsilon_{11}(k) = & \sum_{m_y+m_z=odd, m_z=odd} e^{-ik \cdot m} \chi_m. \\
(114)
\end{align*}
We note that \( \epsilon_{\alpha \beta}(k) \) are periodic function in the lattice Brillouin zone \( -\pi < k_{x,y,z} < \pi \). They also satisfy
\begin{equation}
\epsilon_{\alpha \beta}(k) = (-)^\alpha \epsilon_{\alpha \beta}(k+Q_y+Q_z), \quad \epsilon_{\alpha \beta}(k) = (-)^\beta \epsilon_{\alpha \beta}(k+Q_x). \\
(115)
\end{equation}
The \( \Gamma(k) \) in Eq. (96) now has a form
\begin{align*}
\Gamma(k) = & \left( \epsilon_{00}(k) \epsilon_{10}(k+Q_y+Q_z), \epsilon_{01}(k+Q_y), \epsilon_{11}(k+Q_x+Q_y), \epsilon_{10}(k+Q_y), \epsilon_{00}(k+Q_y+Q_z) \right) \\
(116)
\end{align*}
\begin{align*}
& \left( \epsilon_{10}(k) \epsilon_{00}(k+Q_y+Q_z), \epsilon_{11}(k+Q_y), \epsilon_{01}(k+Q_y), \epsilon_{10}(k+Q_y), \epsilon_{00}(k+Q_y+Q_z) \right) \\
& \left( \epsilon_{01}(k) \epsilon_{10}(k+Q_y+Q_z), \epsilon_{11}(k+Q_y), \epsilon_{00}(k+Q_y+Q_z) \right) \\
& \left( \epsilon_{11}(k) \epsilon_{10}(k+Q_y+Q_z), \epsilon_{00}(k+Q_y+Q_z) \right). \\
\end{align*}
Just as discussed in section V C, the invariance under \( G_{P_{x,y,z}} \) requires that
\begin{equation}
-\chi_{P_x, i} = G_{P_x}(i) \chi_{ij} G_{P_x}(j) \\
(117)
\end{equation}
or
\[
\chi_{\pm m} = -(-)^{m_x \gamma m} \chi m
\] (118)

In the momentum space, the above becomes
\[
\epsilon_{00}(-P, k) = -\epsilon_{00}(k + Q),
\epsilon_{10}(-P, k) = -\epsilon_{10}(k),
\epsilon_{01}(-P, k) = -\epsilon_{01}(k + Q),
\epsilon_{11}(-P, k) = -\epsilon_{11}(k).
\] (119)

where \( Q = \pi z \). Similarly, the invariance under \( G_y \gamma_y P_y \) requires that
\[
\chi_{-y m} = -(-)^{m_y m} \chi m
\] (120)

In the momentum space
\[
\epsilon_{00}(-P, k) = -\epsilon_{00}(k),
\epsilon_{10}(-P, k) = -\epsilon_{10}(k + Q),
\epsilon_{01}(-P, k) = -\epsilon_{01}(k),
\epsilon_{11}(-P, k) = -\epsilon_{11}(k + Q).
\] (121)

The invariance under \( G_x \gamma_x P_x \) requires that
\[
\chi_{-x m} = -(-)^{m_x m} \chi m
\] (122)

In the momentum space
\[
\epsilon_{00}(-P, k) = -\epsilon_{00}(k),
\epsilon_{10}(-P, k) = -\epsilon_{10}(k + Q),
\epsilon_{01}(-P, k) = -\epsilon_{01}(k),
\epsilon_{11}(-P, k) = -\epsilon_{11}(k + Q).
\] (123)

The invariance under \( G_y \gamma_y P_y \) requires that
\[
\chi_{x y i, y z j} = G_{y x} (i) \chi_{y z} G_{y x} (j)
\] (124)

or
\[
\chi_{x y m} = (-)^{m_x m} \chi m
\] (125)

In the momentum space
\[
\epsilon_{00}(P, k) = -\epsilon_{00}(k),
\epsilon_{10}(P, k) = -\epsilon_{10}(k + Q),
\epsilon_{01}(P, k) = -\epsilon_{01}(k),
\epsilon_{11}(P, k) = -\epsilon_{11}(k).
\] (126)

The invariance under \( G_x \gamma_x P_x \) requires that
\[
\chi_{x y m} = (-)^{m_y m} \chi m
\] (127)

or
\[
\epsilon_{00}(P, k) = -\epsilon_{00}(k),
\epsilon_{10}(P, k) = -\epsilon_{10}(k),
\epsilon_{01}(P, k) = -\epsilon_{01}(k),
\epsilon_{11}(P, k) = -\epsilon_{11}(k).
\] (128)

The invariance under \( G_{xy} \gamma_{xy} P_{xy} \) requires that
\[
\chi_{xy m} = (-)^{m_x m} \chi m
\] (129)

or
\[
\epsilon_{00}(P_{xy}, k) = -\epsilon_{00}(k),
\epsilon_{10}(P_{xy}, k) = -\epsilon_{10}(k + Q),
\epsilon_{01}(P_{xy}, k) = -\epsilon_{01}(k),
\epsilon_{11}(P_{xy}, k) = -\epsilon_{11}(k + Q).
\] (130)

We see that Eq. (119) require that \( \epsilon_{10}(k) |_{k_y = k_y = 0} = 0 \) and \( \epsilon_{11}(k) |_{k_y = k_y = 0} = 0 \). Eq. (123) require that \( \epsilon_{00}(k) |_{k_y = k_y = 0} = 0 \) and \( \epsilon_{01}(k) |_{k_y = k_y = 0} = 0 \). Thus the \( \epsilon_{a, b}(0) = 0 \). When combined with Eq. (115), Eq. (119), and Eq. (123), we find
\[
\epsilon_{a, b}(\alpha\beta_Q, x + \alpha\gamma_Q, + \alpha\gamma_Q) = 0, \quad \alpha_x, \alpha_y, \alpha_z = 0, 1
\] (131)

Therefore \( \Gamma(k) = 0 \) when \( k = 0, Q \). The two translation \( T_{xy} \) and the three parity \( \gamma_{y, z} P_{xy, z} \) symmetries of \( H_{QED} \) guarantee the existence of at least 4 four-component massless Dirac fermions. Or more precisely, no symmetric local perturbations in the local bosonic model \( H_{QED} \) can generate mass terms for the two massless Dirac fermions in the unperturbed Hamiltonian.

Since the mass term in the continuum effective field theory is not allowed by the underlying lattice PSG, we say that our model has an emerging chiral symmetry. The masslessness of the Dirac fermion is protected by the quantum order and the associated PSG.

**VIII. QED AND QCD FROM A BOSONIC MODEL ON CUBIC LATTICE**

In this section, we are going to generalize the results in Ref. [55] and Ref. [15] and use a bosonic model on cubic lattice to generate QED and QCD with \( 2N_f \) families of massless quarks and leptons. To describe the local Hilbert space on site \( i \) in our bosonic model, it is convenient to introduce fermions \( \lambda^a_i \) and \( \psi_i^{\mu a} \), where \( a = 1, \ldots, N_f, n = 1, \ldots, 2N_f \) and \( \alpha = 1, 2, 3 \). \( \lambda^a_i \) is in the fundamental representation of a \( SU(N_f) \) group. \( \psi_i^{\mu} \) is in the fundamental representation of a \( SU(3) \) color group and a \( SU(2N_f) \) group. The Hilbert space of fermions is bigger than the Hilbert space of our boson model. Only a physical subspace of the fermions Hilbert space becomes the Hilbert space of our boson model. The physical states on each site is formed by color singlet states that satisfy
\[
\left( \lambda_i^a \lambda_j^b \delta^{ab} + \psi_i^{\mu a} \psi_j^{\mu b} - \delta^{ab} i^3 N_f \right) \Phi_{\text{phys}} = 0
\] (132)

where \( N_f \) is assumed to be even. Once restricted within the physical Hilbert space, the fermion model becomes our local bosonic model.

In the fermion representation, the local physical oper-
Let us assume, for the time being, $J_3 = 0$. In terms of fermions, the above Hamiltonian can be rewritten as

$$H = -\frac{J_1}{N_f} \sum_{(ij)} \psi_i^a \psi_i^b \psi_j^a \psi_j^b - J_2 \sum_{(ij)} \lambda^a_{ij} \lambda_{ij}^a + \text{Const.}$$

Using path integral, we can rewrite the above model as

$$Z = \int D(\chi) D(\psi) D(a_0) D(u) D(\chi) e^{i \int dt L}$$

$$L = \psi_i^{n\dagger} i[\partial_t + i a_0(i)] \psi_i^n - \sum_{(ij)} (\psi_i^{n\dagger} u_{ij} \psi_j^n + h.c.)$$

$$+ \lambda_i^{a\dagger} i[\partial_t + i \text{Tr} a_0(i)] \lambda_i^a - \sum_{(ij)} (\lambda_i^{a\dagger} \chi_{ij} \lambda_i^a + h.c.)$$

$$- \frac{N_f}{J_1} \sum_{(ij)} \text{Tr}(u_{ij} u_{ij}^\dagger) - \frac{N_f}{J_2} \sum_{(ij)} \chi_{ij} \chi_{ij}^\dagger$$

$$- \frac{C N_f}{J_1} \sum_{(ij)} [\chi_{ij} \text{det}(u_{ij}) + h.c.]$$

where $\psi_i^a = (\psi_i^{n1}, \psi_i^{n2}, \psi_i^{n3})$, and $a_0(i)$ and $u_{ij}$ are $3 \times 3$ complex matrices that satisfy

$$u_{ij}^\dagger = u_{ji}, \quad a_0(i) = a_0^\dagger(i)$$

When $J_3 \neq 0$, the Lagrangian may contain terms that mix $\chi_{ij}$ and $u_{ij}$:

$$L = \psi_i^{n\dagger} i[\partial_t + i a_0(i)] \psi_i^n - \sum_{(ij)} (\psi_i^{n\dagger} u_{ij} \psi_j^n + h.c.)$$

$$+ \lambda_i^{a\dagger} i[\partial_t + i \text{Tr} a_0(i)] \lambda_i^a - \sum_{(ij)} (\lambda_i^{a\dagger} \chi_{ij} \lambda_i^a + h.c.)$$

$$- \frac{N_f}{J_1} \sum_{(ij)} \text{Tr}(u_{ij} u_{ij}^\dagger) - \frac{N_f}{J_2} \sum_{(ij)} \chi_{ij} \chi_{ij}^\dagger$$

$$- \frac{C N_f}{J_1} \sum_{(ij)} [\chi_{ij} \text{det}(u_{ij}) + h.c.]$$

where $C$ is a $O(1)$ constant. We note that the above Lagrangian describes a $U(1) \times SU(3)$ lattice gauge theory coupled to fermions.

The field $a_0(i)$ in the Lagrangian is introduced to enforce the constraint

$$\psi_i^{n\dagger} \psi_i^a - \psi_i^a \psi_i^{n\dagger} + \lambda_i^{a\dagger} \lambda_i^a \delta^{a\beta} - \lambda_i^{a\dagger} \delta^{a\beta} = 0$$
to make the \( \pi \)-flux ansatz to have the minimal energy.) Despite the \( i \) dependence, the above ansatz actually describe translation, rotation, parity, and charge conjugation symmetric states. This is because the symmetry transformed ansatz, although not equal to the original ansatz, is gauge equivalent to the original ansatz.

The mean-field Hamiltonian for fermions has a form

\[
H = \sum_{\langle ij \rangle} (\psi^\dagger_i u_{ij} \psi_j^a + \lambda^a_{ij} \chi_{ij} \lambda^a_j + h.c.)
\] (143)

The fermion dispersion has two nodes at \( k = 0 \) and \( k = (0, 0, \pi) \). Thus there are \( 2N_f \times 7 \) massless four-component Dirac fermions in the continuum limit. They correspond to quarks and leptons of \( 2N_f \) different families. Each family contains six quarks (two flavors times three colors) that carry \( SU(3) \) colors and charge 1/3 for the \( U(1) \) gauge field, and one lepton that carry no \( SU(3) \) colors and charge 1 for the \( U(1) \) gauge field.

Including the collective fluctuations of the ansatz, the \( U(1) \times SU(3) = U(3) \) fermion theory has a form

\[
L = \sum_i \bar{\psi}_{\gamma}^i i(\partial_i + i a_0(i)) \psi^n_j + \sum_{ij} \bar{\psi}_{\gamma}^i u_{ij} e^{ia_0} \psi^n_j \\
+ \sum_i \lambda^a_{ij} i(\partial_i + i \text{Tr} a_0(i)) \lambda^a_j + \sum_{ij} \lambda^a_{ij} \chi_{ij} \det(e^{ia_0}) \lambda^a_j
\]

(144)

where \( a_{ij} \) are \( 3 \times 3 \) hermitian matrices, describing \( U(1) \) and \( SU(3) \) gauge fields. In the continuum limit, the above becomes

\[
\mathcal{L} = \bar{\psi}_{I,n} D_0 \gamma_0 \psi_{I,n} + v_f \bar{\psi}_{I,n} D_\gamma \psi_{I,n} \\
+ \bar{\lambda}_{I,a} D_0' \gamma_0 \lambda_{I,a} + v_f \bar{\lambda}_{I,a} D_\gamma' \lambda_{I,a}
\]

with \( v_f \sim l_0 J_1, \quad v_f' \sim l_0 J_2, \quad D_\mu = \partial_\mu + i a_\mu, \quad D_\gamma' = \partial_\mu + i \text{Tr} a_\mu, \quad I = 1, 2, \) and \( \gamma^a \) are \( 4 \times 4 \) Dirac matrices.\[24, 49, 50] \( \lambda_{I,a} \) and \( \psi_{I,n} \) are Dirac fermion fields. \( \psi_{I,n} \) form a fundamental representation of color \( SU(3) \).

If we integrate out \( a_0 \) and \( a_{ij} \) in Eq. (144) first, we will recover the bosonic model Eq. (134). If we integrating out the high energy fermions first, the \( U(1) \times SU(3) \) gauge field \( a_{ij} \) will acquire a dynamics. We obtain the following low energy effective theory in continuum limit

\[
\mathcal{L} = \bar{\psi}_{I,n} D_0 \gamma_0 \psi_{I,n} + v_f \bar{\psi}_{I,n} D_\gamma \psi_{I,n} \\
+ \bar{\lambda}_{I,a} D_0' \gamma_0 \lambda_{I,a} + v_f \bar{\lambda}_{I,a} D_\gamma' \lambda_{I,a} \\
+ \frac{1}{\alpha_S} [\text{Tr} F_{0} F^{0} + c_a^2 \text{Tr} F_{ij} F^{ij}] + ...
\]

(146)

where the velocity of the \( U(3) \) gauge bosons is \( c_a \sim l_0 J_{1,2} \), and \( \ldots \) represents higher derivative terms and the coupling constant \( \alpha_S \) is of order \( 1/N_f \).

In the large \( N_f \) limit, fluctuations of the gauge fields are weak. The model Eq. (146) describes a \( U(1) \times SU(3) \) gauge theory coupled weakly to \( 2N_f \) families of massless fermions. Therefore, our bosonic model can generate massless artificial quarks and artificial leptons that couple to artificial light and artificial gluons. As discussed in Ref. [15], the PSG of the ansatz Eq. (142) protects the masslessness of the artificial quarks and the artificial leptons. Our model has an emerging chiral symmetry.

\section{IX. CONCLUSION}

In this paper, we have studied a new class of ordered states - string-net condensed states - in local bosonic models. The new kind of orders does not break any symmetry and cannot be described by Landau’s symmetry breaking theory. We show that different string-net condensations can be characterized (and hopefully, classified) by the projective symmetry in the hopping Hamiltonian for ends of condensed strings. Similar to symmetry breaking states (or “particle” condensed states), string-net condensed states can also produce and protect gapless excitations. However, unlike symmetry breaking states which can only produce and protect gapless scalar bosons (or Nambu-Goldstone modes), string-net condensed states can produce and protect gapless gauge bosons and gapless fermions. It is amazing to see that gapless fermions can even appear in local bosonic models.

Motivated by the above results, we propose the following locality principle: The fundamental theory for our universe is a local bosonic model. Using several local bosonic models as examples, we try to argue that the locality principle is not obviously wrong, if we assume that there is a string-net condensation in our vacuum. The string-net condensation can naturally produce and protect massless photons (as well as gluons) and (nearly) massless electrons/quarks. However, to really prove the string-net condensation in our vacuum, we need to show that string-net condensation can generate chiral fermions. Also, the above locality principle has not been taken quantum gravity into account. It may need to be generalized to include quantum gravity. In any case, we can say that we have a plausible understanding where light and fermions come from. The existence of light and fermions is no longer mysterious once we realize that they can come from local bosonic models via string-net condensations.

The string-net condensation and the associated PSG also provide a new solution to the chiral symmetry and the fermion mass problem in lattice QED and lattice QCD. We show that the symmetry of a lattice bosonic model leads to PSG of the hopping Hamiltonian for the ends of condensed strings. If the ends of condensed strings are fermions, then PSG can some times protect the masslessness of the fermions, even though the chiral symmetry in the continuum limit cannot be generalized to the lattice. Thus PSG can lead to an emerging chiral symmetry that protect massless Dirac fermions.

In this paper, we have been stressing that string-net condensation and the associated PSG can protect the masslessness of fermions. However, most fermions in nature do have masses, although very small comparing to the Planck mass. One may wonder where those small masses come from. Here we would like to point out that the PSG argument for masslessness only works for radiative corrections. In other words, the fermions protected by string-net condensation and PSG cannot gain any mass from additive radiative corrections caused by high energy fluctuations. However, if the model has infrared divergence, then infrared divergence can give the
would-be-massless fermions some masses. The acquired masses should have the scale of the infrared divergence. The 3+1D QED model studied in this paper do not have any infrared divergence. Thus, the artificial electrons in the model are exactly massless. But in the bosonic model discussed in section VIII, the $SU(3)$ gauge coupling $\alpha_S$ runs as

$$\frac{d\alpha_S^{-1}}{d\ln(M^2)} = \frac{11 - (2/3)(2N_f)}{4\pi}$$

(147)

where $M$ is the cut-off scale. Thus when $N_f \leq 8$, $\alpha_S$ has a logarithmic infrared divergence. In general, for models with $U(1)$ and $SU(3)$ gauge interactions and a right content of fermions, the $SU(3)$ gauge interactions can have a weak logarithmic infrared divergence in 3+1D.\[57, 58\] This weak divergence could generate mass of order $e^{-C/\alpha_S(M_P)}M_P$, where $M_P$ is the Planck mass or the GUT scale (the cut-off scale of the lattice theory), $C = O(1)$ and $\alpha_S(M_P)$ is the dimensionless gauge coupling constant at the Planck scale. An $C/\alpha_S(M_P) \sim 40$ can produce a desired separation between the Planck mass/GUT scale and the masses of observed fermions. It is interesting to see that, in order to use string-net condensation picture to explain the origin of gauge bosons and nearly massless fermions, it is important to have a four dimensional space-time. When space-time has five or more dimensions, the gauge-fermion interaction do not have any infrared divergence. In this case, if a string-net condensation produces massless fermions, those fermions will remain to be massless down to zero energy. In 2+1D, the gauge interaction between massless fermions is so strong that one cannot have fermionic quasiparticles at low energies.\[59–61\] It is amazing to see that 3+1 is the only space-time dimension that the gauge bosons and fermions produced by string-net condensation have weak enough interaction so that they can be identified at low energies and, at same time, have strong enough interaction to have a rich non-trivial structure at low energies.

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Here, by “string-net condensation” we mean condensation of nets of string-like objects of arbitrary sizes.