Universal exponential solution of the Yang-Baxter equation

Abstract

Exponential solutions of the Yang-Baxter equation give rise to generalized Schubert polynomials and corresponding symmetric functions. We provide several descriptions of the local stationary algebra defined by this equation. This allows to construct various exponential solutions of the YBE. The $B_n$ and $G_2$ cases are also treated.

1. Introduction

Let $K$ be a field of zero characteristic. Let $A = K[u_1, u_2, \ldots](x, y, \ldots)$ be the associative algebra of formal power series in commuting variables $x$, $y$, $\ldots$ with coefficients in a local stationary algebra $[V2]$ with generators $u_1$, $u_2$, $\ldots$. To be precise, we assume exactly the following: non-adjacent generators $u_i$ and $u_j$ commute whereas adjacent generators $u_i$ and $u_{i+1}$ are subject to certain relations (perhaps infinitely many) which are invariant in $i$.

It was shown in [FK1] (see also [FS]) that a theory of generalized Schubert polynomials and corresponding Stanley’s symmetric functions can be developed whenever one has a solution of the Yang-Baxter equation

$$h_i(x)h_{i+1}(x + y)h_i(y) = h_{i+1}(y)h_i(x + y)h_{i+1}(x)$$
given by \( h_i(x) = e^{xu_i} \). In other words, this theory requires the condition

\[ e^{xa} e^{(x+y)b} e^{ya} = e^{yb} e^{(x+y)a} e^{xb} \]

to be satisfied by any pair of adjacent generators \( a \) and \( b \).

In this paper, a “minimal” set of relations which would guarantee (1) is given. (In other words, we characterize the local stationary algebra \( A_0 \) defined by (1).) This enables us to construct exponential solutions of the YBE related to the following quotient algebras of \( A_0 \) (cf. [FS, FK1, R]):

(i) the nilCoxeter algebra of the symmetric group;
(ii) the degenerate Hecke algebra \( H_\infty(0) \);
(iii) the universal enveloping algebra of \( U_+(gl(n)) \);
(iv) the local Heisenberg algebra.

In Section 5, the relation between the YBE and [generalized] Verma identities is treated. Sections 4 and 6 contain some parallel results for the \( B_n \) and \( G_2 \) cases.

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2. Main results

The equation (1) is actually an infinite set of conditions on \( a \) and \( b \), given by taking coefficients of \( x^n y^m \) on both sides of (1). For example, equating the coefficients of \( x^2 y \) gives the first nontrivial condition

\[ b^2a + ab^2 + 2aba = ba^2 + a^2b + 2bab \ . \]

We are going to write a minimal list of relations that imply (1). This can be done in many different ways, as shown in Section 3. For the purposes of this section, however, it suffices to have the simplest of these characterizations. To state it, we need some notation.

Let \( C_0 = a + b \) and \( C_{m+1} = [a, C_m] \) for \( n = 0, 1, 2, \ldots \); here \([,]\) stands for the commutator: \([f, g] = fg - gf\). Thus

\[ C_1 = [a, b] \ , \ C_2 = [a, [a, b]] \ , \ C_3 = [a, [a, [a, b]]] \ , \ldots \]

**Theorem 1.** The condition (1) is satisfied if and only if for any odd \( m \) the expression \( C_m \) commutes with \( a + b \).

A proof of Theorem 1, along with other equivalent characterizations of the algebra defined by (1), is given in Section 3.

This theorem enables us to construct various examples of exponential solutions of the Yang-Baxter equation.

**Example 1.** [FK1, R] Hecke algebras. Suppose the generators \( u_i \) of a certain algebra satisfy the relations \( u_i^2 = \beta u_i \). In our previous notation, it means that

\[ u_i^2 = \beta a \ , \ u_i^2 = \beta b \ . \]
Straightforward computations then give

\[ C_0 = a + b, \]
\[ C_1 = [a, b], \]
\[ C_2 = [a, [a, b]] = \beta(ab + ba) - 2aba, \]
\[ C_3 = [a, [a, [a, b]]] = \beta^2[a, b] = \beta^2C_1, \]
\[ C_4 = \beta^2C_2, \ldots \]

Therefore, in order to satisfy (1), one has to have, in addition to (2), the commutation

\[ [C_0, C_1] = 0. \tag{3} \]

Using (2) and its consequence \([a + b, ab + ba] = 0\), one can see that (3) is equivalent to the Coxeter relation \(aba = bab\). In other words, an exponential solution of the Yang-Baxter equation can be obtained from an algebra defined by

\[ u_iu_j = u_ju_i, \quad |i - j| \geq 2; \]
\[ u_i^2 = \beta u_i; \]
\[ u_iu_{i+1}u_i = u_{i+1}u_iu_{i+1}. \tag{4} \]

The main special cases are \(\beta = 0\) and \(\beta = -1\).

**Example 2.** The nilCoxeter algebra of the symmetric group. This algebra (see [FS]) can be defined as the algebra spanned by permutations of \(S_n\), with the multiplication rule

\[ w \cdot v = \begin{cases} 
\text{usual product } wv & \text{if } l(w) + l(v) = l(wv) \\
0 & \text{otherwise}
\end{cases} \]

where \(l(w)\) is the length of a permutation \(w\) (the number of inversions). Another definition can be given in terms of generators \(u_i\) (adjacent transpositions) satisfying the relations (4) with \(\beta = 0\). This example leads to the ordinary Schubert polynomials of Lascoux and Schützenberger [L] (see also [M]), as shown in [FS] (cf. [FK1]).

**Example 3.** The degenerate Hecke algebra. Setting \(\beta = -1\) gives a Hecke algebra \(H_\infty(0)\). The theory of generalized Schubert polynomials leads in this case to the Grothendieck polynomials of Lascoux and Schützenberger [LS], as shown in [FK2].

**Example 4.** [FK1] Universal enveloping algebra of \(U_+(gl(n))\). This algebra can be defined as the local algebra with generators \(u_1, u_2, \ldots\) subject to Serre relations

\[ [u_i, [u_i, u_{i+1}]] = 0. \]

Redenoting \(a = u_i\) and \(b = u_{i+1}\), we have

\[ [a, [a, b]] = 0 \tag{5} \]

and

\[ [b, [b, a]] = 0. \tag{6} \]

This implies \(C_2 = [a, [a, b]] = 0\) and \([C_0, C_1] = [a + b, [a, b]] = 0\) which guarantees the conditions of Theorem 1.
Example 5. The local Heisenberg algebra. This is a local stationary algebra with generators \( u_1, u_2, \ldots \) satisfying

\[ [u_i, u_{i+1}] = \lambda_i \]

where \( \lambda_i \) are some constants. Since these relations imply both (5) and (6), this is a quotient algebra of the algebra of the previous example.

3. Proof of the main theorem

Define the sequence \( \{T_n\} \) of elements of our algebra \( A \) by

\[
T_0 = 1, \quad T_{n+1} = aT_n + T_nb
\]

Thus

\[
T_1 = a + b, \quad T_2 = a^2 + 2ab + b^2, \quad T_3 = a^3 + 3a^2b + 3ab^2 + b^3, \ldots
\]

Let \( C(x) \) and \( T(x) \) be the following generating functions for the sequences \( \{C_n\} \) and \( \{T_n\} \):

\[
C(x) = \sum_{n=0}^{\infty} \frac{C_n}{n!} x^n,
\]

\[
T(x) = \sum_{n=0}^{\infty} \frac{T_n}{n!} x^n.
\]

Lemma 1. \( C(x) = e^{xa}(a + b)e^{-xa} \), \( T(x) = e^{xa}e^{xb} \).

Proof. The defining recurrences for \( \{C_n(x)\} \) and \( \{T_n(x)\} \) can be rewritten, in terms of generating functions, as

\[
C'(x) = aC(x) - C(x)a
\]

and

\[
T'(x) = aT(x) + T(x)b.
\]

Together with the constant terms \( C(0) = a + b \) and \( T(0) = 1 \) these identities determine the functions \( C(x) \) and \( T(x) \) uniquely. It only remains to check that the functions \( C(x) = e^{xa}(a + b)e^{-xa} \) and \( T(x) = e^{xa}e^{xb} \) do satisfy these identities.

The following result is an extension of Theorem 1.

Theorem 2. The following statements are equivalent:

(i) \( e^{xa}e^{(x+y)b}e^{ya} = e^{yb}e^{(x+y)a}e^{xb} \);

(ii) \( [T(x), T(y)] = 0 \);

(iii) \( [C(x), C(y)] = 0 \);

(iv) \( [T(x), a + b] = 0 \);

(v) \( [C(x), a + b] = 0 \).
(vi) \[ [T_n, T_m] = 0 \] for all \( n \) and \( m \);

(vii) \[ [C_n, C_m] = 0 \] for all \( n \) and \( m \);

(viii) \[ [T_n, a + b] = 0 \] for all even \( n \);

(ix) \[ [C_m, a + b] = 0 \] for all odd \( m \);

(x) \[ [T_n, T_{n+1}] = 0 \] for all \( n \);

(xi) \[ [C_n, C_{n+1}] = 0 \] for all \( n \);

(xii) \[ [a, [a, C_n] = [b, [b, C_n]] \] for all even \( n \);

(xiii) \[ [a, [a, \ldots, [a, b] \ldots]] = [b, [b, \ldots, [b, a] \ldots]] \] for all even \( m \).

**Proof.**

(i) \( \iff \) (ii) Restate (i) as \( [e^{xa}e^{xb}, e^{yb}e^{ya}] = 0 \) and note that \( e^{xa}e^{xb} \) commutes with \( e^{yb}e^{ya} \) iff it commutes with \((e^{yb}e^{ya})^{-1} = e^{-ya}e^{-yb}\) which means the same as commuting with \( e^{ya}e^{yb} \).

(iii) \( \iff \) (v) Assume (v). Then

\[
e^{xa}(a + b)e^{-xa}e^{ya}(a + b)e^{-ya} = e^{ya}(a + b)e^{(x-y)a}(a + b)e^{-ya} = e^{ya}(a + b)e^{(x-y)a}e^{-ya} = e^{ya}(a + b)e^{-ya}e^{xa}(a + b)e^{-xa}.
\]

The implication (iii) \( \implies \) (v) is trivial since \( a + b = C(0) \).

(ii) \( \iff \) (vi) Follows from Lemma.

(iii) \( \iff \) (vii) Follows from Lemma.

(ii) \( \iff \) (iii) The implication (ii) \( \implies \) (iii) follows from the identity

\[
(7) \quad C(x) = T'(x)(T(x))^{-1}
\]

which is a formal consequence of Lemma:

\[
T'(x)(T(x))^{-1} = (e^{xa}ae^{xb} + e^{xa}be^{xb})(e^{-xb}e^{-xa}) = e^{xa}(a + b)e^{-xa} = C(x).
\]

To prove the converse, rewrite (7) as

\[
T'(x) = C(x)T(x);
\]

this means that \( T_{n+1} \) is a certain noncommutative polynomial in \( C_0, \ldots, C_n, T_1, \ldots, T_n \); also \( T_1 = C_0 \). Therefore \( T_{n+1} \) is a polynomial in \( C_0, \ldots, C_n \); and if \( C_i \)'s commute, then so do \( T_i \)'s.

(ii) \( \iff \) (iv) Since \( a + b = T(0) \), (ii) \( \implies \) (iv) is trivial. Assume (iv); then (7) implies (v) implies (iii) implies (ii).
The part (vii) $\iff$ (xi) is trivial; the part (xi) $\implies$ (vii) follows from the identity

$$
[C_j, C_i] = [[a, C_{j-1}], C_i] = -[[C_{j-1}, C_i], a] - [[C_i, a], C_{j-1}]
$$

by induction on $j - i = 1, 2, 3, \ldots$

This is proved analogously to (vii) $\iff$ (xi); use

$$
[T_j, T_i] = [[a, T_{j-1}] + T_{j-1}T_1, T_i]
= [[a, T_{j-1}], T_i] + [T_{j-1}T_1, T_i]
= -[[T_{j-1}, T_i], a] - [[T_i, a], T_{j-1}] + [T_{j-1}T_1, T_i]
$$

instead of (8).

Again, (vii) $\implies$ (ix) is obvious. Assume (ix). We will prove $[C_j, C_i] = 0$ by induction on $i + j$. If $i + j$ is even (and, say, $i < j$), then (8) and the induction assumption give $[C_j, C_i] = -[C_{j-1}, C_{i+1}]$; then repeat this argument until the indices coincide. If $i + j$ is odd (now let us take $i > j$), then repeatedly apply (8) and the induction hypothesis to get

$$
[C_j, C_i] = -[C_{j-1}, C_{i+1}] = [C_{j-2}, C_{i+2}] = \cdots = \pm[C_0, C_{i+j}] = 0;
$$

we used (ix) in the last step.

Same argument as in (vii) $\iff$ (ix), with (8) replaced by (9).

Use an identity

$$
[a, [a, C_n]] - [b, [b, C_n]] = (a - b)[C_0, C_n] + [C_n, C_0](a - b) + [C_n, C_1]
$$

(one only needs to know that $C_0 = a + b$ and $C_1 = [a, b]$ to check this) to prove the $\implies$ part. The same identity, together with (8) and induction on $i + j$, proves the $\iff$ part.

Induction on $n$ and $m$ for the implications $\iff$ and $\implies$, respectively.

This completes the proof of Theorem 2. \qed

Theorem 1 is just the statement (i) $\iff$ (ix).

Comments. 1. The above proofs of (ii) $\iff$ (iv) and (iii) $\iff$ (v) could in fact be omitted; e.g., we gave an independent proof of of (ix) $\iff$ (vii) whereas obviously (v) $\implies$ (ix) and (vii) $\iff$ (iii).

2. In view of condition (vii) of Theorem 2, it is natural to suggest that the graded associative algebra $A_0$ defined by (1) (or by any of (i)-(xiii)) is isomorphic to the algebra formally generated by pairwise commuting elements $C_0, C_1, C_2, \ldots$.
and an element \( a \) satisfying the conditions \([a, C_i] = C_{i+1}\); the ranks are defined by \( r(k(a)) = 1, r(k(C_i)) = i + 1\).

The last statement has been recently proved by Nantel Bergeron [B]. It follows immediately that the Hilbert series of \( A_0 \) is

\[
(1 - t)^{-2}(1 - t^2)^{-1}(1 - t^3)^{-1}(1 - t^4)^{-1} \cdots,
\]

and a linear basis is given by \( \{a^iC_0^jC_1^{j_1} \cdots \} \). N.Bergeron also constructed a linear basis of words in \( A_0 \).

4. The \( B_n \) case

Many of the previous results can be extended to the \( B_n \) case where the analogue of the Yang-Baxter equation is

\[
h_2(x - y)h_1(x)h_2(x + y)h_1(y) = h_1(y)h_2(x + y)h_1(x)h_2(x - y)
\]

(for another adjacent pairs the \( A_n \)-YBE is kept). If we are interested in exponential solutions \( h_1(x) = e^{xa}, h_2(x) = e^{xb} \), then the YBE becomes

\[
[ e^{xb} e^{xa} e^{-xb}, e^{yb} e^{ya} e^{yb}] = 0.
\]

It can be shown [FK3] that the theory of [generalized] Schubert/Grothendieck/Stanley polynomials can be constructed whenever one has a local stationary algebra whose first two generators satisfy (10) and any other adjacent generators satisfy (1). We will show how solutions of (10) can be constructed using the approach of Sections 2-3.

Example 6. Let the generators \( a \) and \( b \) satisfy the relations

\[
[b, [b, [b, a]]] = 0,
\]

\[
[a, [b, [b, a]]] = 0,
\]

and

\[
[a, [a, [b, a]]] = 0.
\]

Note that \([a, [b, [b, a]]] = -[b, [a, [a, b]]]\), so (12) can be restated as

\[
[b, [a, [a, b]]] = 0.
\]

Theorem 3. (11)-(13) imply (10).

Proof. We start with introducing formal power series

\[
R(x) = e^{xb} e^{xa} e^{xb}
\]

and

\[
L(x) = R'(x) (R(x))^{-1}
\]
Similarly to the implication (iii) \(\iff\) (ii) of Theorem 2, \(R(x)\)’s form a commuting family (cf. (10)) provided so do \(L(x)\)’s. We are going to prove now that (11)-(13) imply \(\{L(x)\}\) is a commuting family.

Since

\[
L(x) = R'(x)(R(x))^{-1} = R'(x)R(-x) = (be^{xb}e^{xa}e^{xb} + e^{xb}ae^{xa}e^{xb} + e^{xb}e^{xa}be^{xb})e^{-xb}e^{-xa}e^{-xb} = b + e^{xb}ae^{-xb} + e^{xb}e^{xa}be^{-xa}e^{-xb},
\]

then

\[
L'(x) = e^{xb}[b,a]e^{-xb} + [b,e^{xb}e^{xa}be^{-xa}e^{-xb}] + e^{xb}e^{xa}[a,b]e^{-xa}e^{-xb}.
\]

Further straightforward calculations make a repeated use of the identity

\[
e^{xf}g e^{-xf} = g + x[f,g] + x^2[f,[f,g]]/2 + x^3[f,[f,[f,g]]]/6 + \ldots,
\]

along with (11)-(13):

\[
L'(x) = e^{xb}[b,a]e^{-xb} + [b,e^{xb}(b + x[a,b] + x^2[a,[a,b]]/2)e^{-xb}]
\]

\[
= x[b,e^{xb}([a,b] + x[a,[a,b]]/2)e^{-xb}] + xe^{xb}[a,[a,b]]e^{-xb}
\]

\[
= x[b,[a,b] + x[a,[a,b]] + x[a,[a,b]]/2 + x[a,[a,b]]]
\]

\[
= x[a + b,[a,b]].
\]

To prove that \(L(x)\) is a commuting family, we only need now

\[
[L(0), L'(x)] = 0
\]

which reduces to

\[
[a + 2b, [a + b, [a, b]]] = 0.
\]

The latter follows instantly from (11), (12), (13), and (12’). This completes the proof of the theorem. \(\square\)

It should be noted that, in fact, (14) is the first (i.e., the lowest-degree) condition on \(a\) and \(b\) in the “\(B_n\)-universal” algebra \(B_0\) defined by (10). There are no conditions in degree 5. We hope to give an exact list of defining relations for \(B_0\) in another publication.

\textbf{Example 7. The nilCoxeter algebra of the hyperoctahedral group.} (Compare to Example 2.) Assume \(a^2 = b^2 = 0\). Then (11) and (13) are guaranteed, and (12) reduces to \(abab - baba = 0\). We conclude that (11)-(13) are satisfied in the algebra defined by

\[
u_iu_j = u_ju_i, |i - j| \geq 2; \\
u_i^2 = 0; \\
u_iu_{i+1}u_i = u_{i+1}u_iu_{i+1}, i \geq 2; \\
u_1u_2u_1u_2 = u_2u_1u_2u_1
\]

which is the nilCoxeter algebra of the hyperoctahedral group (the definition is similar to the \(A_n\) case).
Example 8. Universal enveloping algebra of $U_+(so(2n+1))$. (Compare to Example 4.) This algebra can be defined as the local algebra with generators $u_1, u_2, \ldots$ subject to Serre relations

$$u_i u_j = u_j u_i, \quad |i-j| \geq 2;$$

$$[u_i, [u_i, u_{i+1}]] = 0, \quad i \geq 2;$$

$$[u_i, [u_{i+1}, u_{i+1}]] = 0, \quad i \geq 2;$$

(15)

(16) $[b, [b, a]] = 0$

(17) $[a, [a, [a, b]]] = 0$

where $a = u_1, b = u_2$. To show that (10) is satisfied in this case, just note that (16)-(17) immediately imply (11)-(13).

Example 9. Universal enveloping algebra of $U_+(sp(2n))$. This algebra is defined by (15) together with

$$[a, [a, b]] = 0;$$

(18) $[b, [b, [b, a]]] = 0.$

The condition (10) is satisfied, analogously to Example 8.

5. Verma relations

Let $\{u_i\}$ be the generators of the universal enveloping algebra of Example 4 (one could also use Example 5 instead). Let $\{t_i\}$ be arbitrary constants. Define

$$e_i = \ln(1+t_i u_i);$$

(19)

then, surprisingly, the $e_i$'s provide an exponential solution of the YBE. In other words,

$$e^x \ln(1+ta) e^{(x+y) \ln(1+sb)} e^{y \ln(1+ta)} = e^{y \ln(1+sb)} e^{(x+y) \ln(1+ta)} e^x \ln(1+sb)$$

(20)

where $a = u_i, b = u_{i+1}, t = t_i, s = t_{i+1}$. (The locality condition obviously holds.) It is not trivial at all that the last identity follows from (5) and (6). We will prove it by showing that it is true when $x = n$ and $y = m$ are arbitrary nonnegative integers, in which case it converts into the following generalized Verma identity (cf. [V1]).

Lemma 2. Conditions (5)-(6) imply

$$e^x (1+ta)^n (1+sb)^{n+m} (1+ta)^m = (1+sb)^m (1+tb)^{n+m} (1+sb)^n$$

(21)
Let us first make clear why (21) implies (20). The equality (20) can be rewritten as
\[
\sum_{i, j, k, l \leq i + j} P_{ijkl}(a, b) x^i y^j t^k s^l = 0
\]
where \( P_{ijkl}(a, b) \) are some non-commutative polynomials in \( a \) and \( b \). We need \( P_{ijkl}(a, b) = 0 \) for any \( i, j, k, \) and \( l \). Let us fix \( k \) and \( l \). Then

(22) \[
\sum_{i, j} P_{ijkl}(a, b) x^i y^j = 0
\]
is a finite identity about \( a \) and \( b \); the finiteness is ensured by the condition \( i + j \leq k + l \). The condition (21) means that (22) is true for any nonnegative integers \( x = n \) and \( y = m \). Since the matrix \{\( \{n^i m^j\} \) whose rows are indexed by pairs \((i, j)\) and columns by pairs \((n, m)\) has maximal rank, it follows that all \( P_{ijkl}(a, b) \) are 0.

Now it remains to prove Lemma 2. First note that the elements \( a' = 1 + ta \) and \( b' = 1 + sb \) satisfy the same relations (5) and (6) as \( a \) and \( b \) do; then it suffices to show the ordinary Verma identity

(23) \[
a^n b^{n+m} a^m = b^m a^{n+m} b^n .
\]

Lemma 3 (cf. [S]). Assume that, in some associative algebra, \( ab - ba = \lambda \) where \( \lambda \) commutes with both \( a \) and \( b \). Then, for any nonnegative integer \( n \),

\[
a^n b^n = (ba + \lambda)(ba + 2\lambda) \cdots (ba + n\lambda)
\]
and

\[
b^n a^n = ba(ba - \lambda) \cdots (ba - (n-1)\lambda) .
\]

This lemma immediately implies that \( a^n b^n \) commutes with \( b^m a^m \) which is exactly (23).

In fact, the following generalization of (23) is true.

Lemma 4 [F]. Under assumptions of Lemma 3, any two “balanced” words in \( a \) and \( b \) (that is, words containing as many \( a \)’s as \( b \)’s) commute. □

Conjecture. The solution (19) is a universal exponential solution of the YBE. In other words, the associative algebra generated by the elements \( e_i \) is isomorphic to the local stationary algebra \( A_0 \) defined by the condition (1).

The above results can be generalized to the \( B_n \) case. The \( B_n \)-Verma relations are

\[
a^n b^{2n+m} a^{n+m} b^m = b^m a^{n+m} b^{2n+m} a^n
\]
where, as before, one can make substitutions \( a \leftarrow 1 + ta \) and \( b \leftarrow 1 + sb \). Consequently, the elements \( e_i = ln(1 + t_i u_i) \) give an exponential solution of the \( B_n \)-YBE.

6. The \( G_2 \) case

Let us denote

\[
\mathcal{P}(G_2) = \{ a, b \mid a^2 = \beta a, \ b^2 = \beta b, \ abab = bababa \},
\]
\[
\mathcal{U}(G_2) = \{ a, b \mid [a, [a, [a, [a, b]]]] = 0, \ [b, [b, a]] = 0 \}.
\]
The following results can be verified.

**Exponential solutions of the Yang-Baxter equation.** Let us define \( h_1(x) := \exp(xa), \) \( h_2(x) := \exp(xb), \) where \( a \) and \( b \) are either the generators of the algebra \( \mathcal{P}(G_2) \) or those of \( U_+(G_2). \)

Then

\[
h_1(x)h_2(3x+y)h_1(2x+y)h_2(3x+2y)h_1(x+y)h_2(y) = h_2(y)h_1(x+y)h_2(3x+y)h_1(2x+y)h_2(3x+2y)h_1(x) .
\]

**Generalized Verma relations.** In the algebra \( U_+(G_2), \)

\[
a^n b^{3n+m} a^{2n+m} b^{3n+2m} a^{n+m} b^m = b^m a^{n+m} b^{3n+2m} a^{2n+m} b^{3n+m} a^n .
\]

The same formula is true with \( a \) and \( b \) replaced by \( 1 + ta \) and \( 1 + sb, \) respectively.

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