THE VERTEX ALGEBRAS $\mathcal{R}^{(p)}$ AND $\mathcal{V}^{(p)}$

DRAŽEN ADAMOVIĆ, THOMAS CREUTZIG, NAOKI GENRA AND JINWEI YANG

ABSTRACT. The vertex algebras $\mathcal{V}^{(p)}$ and $\mathcal{R}^{(p)}$ introduced in [2] are very interesting relatives of the famous triplet algebras of logarithmic CFT. The algebra $\mathcal{V}^{(p)}$ (respectively, $\mathcal{R}^{(p)}$) is a large extension of the simple affine vertex algebra $L_k(sl_2)$ (respectively, $L_k(sl_2)$ times a Heisenberg algebra), at level $k = -2 + 1/p$ for positive integer $p$. Particularly, the algebra $\mathcal{V}^{(2)}$ is the simple small superconformal vertex algebra with $c = -9$, and $\mathcal{R}^{(2)}$ is $L_{-3/2}(sl_3)$. In this paper, we derive structural results of these algebras and prove various conjectures coming from representation theory and physics.

We show that $SU(2)$ acts as automorphisms on $\mathcal{V}^{(p)}$ and we decompose $\mathcal{V}^{(p)}$ as an $L_k(sl_2)$-module and $\mathcal{R}^{(p)}$ as an $L_k(gl_2)$-module. The decomposition of $\mathcal{V}^{(p)}$ shows that $\mathcal{V}^{(p)}$ is the large level limit of a corner vertex algebra appearing in the context of $S$-duality. We also show that the quantum Hamiltonian reduction of $\mathcal{V}^{(p)}$ is the logarithmic doublet algebra $A^{(p)}$ introduced in [12], while the reduction of $\mathcal{R}^{(p)}$ yields the $B^{(p)}$-algebra of [39]. Conversely, we realize $\mathcal{V}^{(p)}$ and $\mathcal{R}^{(p)}$ from $A^{(p)}$ and $B^{(p)}$ via a procedure that deserves to be called inverse quantum Hamiltonian reduction. As a corollary, we obtain that the category $KL_k$ of ordinary $L_k(sl_2)$-modules at level $k = -2 + 1/p$ is a rigid vertex tensor category equivalent to a twist of the category Rep($SU(2)$). This finally completes rigid braided tensor category structures for $L_k(sl_2)$ at all complex levels $k$.

We also establish a uniqueness result of certain vertex operator algebra extensions and use this result to prove that both $\mathcal{R}^{(p)}$ and $B^{(p)}$ are certain non-principal $W$-algebras of type $A$ at boundary admissible levels. The same uniqueness result also shows that $\mathcal{R}^{(p)}$ and $B^{(p)}$ are the chiral algebras of Argyres-Douglas theories of type $(A_1, D_{2p})$ and $(A_1, A_{2p-3})$.

1. Introduction

The singlet [11][13][12], doublet [12], triplet [10][11][51][52][71], and $B^{(p)}$ [39][13] algebras are the best understood examples of vertex operator algebras with non semi-simple representation theory and they are of significant importance for logarithmic conformal field theory [38][39][13]. These algebras are large extensions of the Virasoro vertex operator algebra $L^{Vir}(c_1,p,0)$ at central charge $c_1,p = 1 - 6(p-1)^2/p$ for $p$ in $\mathbb{Z}_{\geq 2}$. The Virasoro algebra in turn is the quantum Hamiltonian reduction of the affine vertex operator algebra $L_k(sl_2)$ of $sl_2$ at level $k = -2 + 1/p$. In this work, we realize and study vertex operator algebras whose quantum Hamiltonian reductions are these well-known singlet, triplet and $B^{(p)}$-algebras. These algebras provide important sources of logarithmic conformal field theories and we will investigate their representation theory in future work. The importance of the present work is to resolve various open questions motivated from four dimensional physics, i.e. questions in Argyres-Douglas theories and in $S$-duality. Along the way, we discover a few additional interesting structure, which we shall describe in detail. First we introduce the definitions and the main properties of these important algebras.
1.1. The $\mathcal{V}^{(p)}$-algebra. The $\mathcal{V}^{(p)}$-algebra introduced in [2] is a certain abelian intertwining algebra that we shall study first. Let us briefly recall its definition (see (12) for the detail): The $\mathcal{V}^{(p)}$-algebra is a subalgebra of $M \otimes F_p^2$ where $M$ is the Weyl vertex operator algebra (also often called the $\beta\gamma$-system) and $F_p^2$ is the abelian intertwining algebra associated to the weight lattice of $\mathfrak{sl}_2$ rescaled by $\sqrt{p}$. It is characterized as the kernel of a screening operator $\tilde{Q}$ (see (4)):

$$\mathcal{V}^{(p)} = \ker M \otimes F_p^2 \tilde{Q}.$$ 

We think of $\mathcal{V}^{(p)}$ as an analogue of the doublet algebra $A^{(p)}$ introduced in [12]. The doublet algebra is an abelian intertwining algebra with $SL(2, \mathbb{C})$ acting as automorphisms [8] and it is a large extension of the Virasoro algebra at central charge $c_{1,p}$. Its even subalgebra is the famous triplet algebra. We elaborate various relations between $A^{(p)}$ and $\mathcal{V}^{(p)}$. Firstly, our Corollary 4 says that $\mathcal{V}^{(p)}$ is a subalgebra of $A^{(p)} \otimes \Pi(0)_{\frac{1}{2}}$, where $\Pi(0)_{\frac{1}{2}}$ is a certain extension along a rank one isotropic lattice of a rank two Heisenberg vertex operator algebra. $\mathcal{V}^{(p)}$ is then characterized as the kernel of another screening operator $S$ (see (22)):

$$\mathcal{V}^{(p)} = \ker A^{(p)} \otimes \Pi(0)_{\frac{1}{2}} S.$$ 

Set $k = -2 + \frac{1}{p}$ and $p \in \mathbb{Z}_{\geq 1}$. We denote by $L_k^{(p)}$ the simple highest-weight module of $L_k(\mathfrak{sl}_2)$ of highest-weight $nw$ with $\omega$ the fundamental weight of $\mathfrak{sl}_2$. We also use the short-hand notation $\rho_n = \rho_{nw}$ for the integrable $\mathfrak{sl}_2$-modules. One of our main aims was to prove

**Theorem 1.** (Theorem 5 and Corollary 5)

*The Lie algebra $\mathfrak{sl}_2$ acts on $\mathcal{V}^{(p)}$ as derivations and $SL(2, \mathbb{C})$ is a group of automorphisms. Moreover, $\mathcal{V}^{(p)}$ decomposes as an $\mathfrak{sl}_2 \otimes L_k(\mathfrak{sl}_2)$-module as

$$\mathcal{V}^{(p)} = \bigoplus_{n=0}^{\infty} \rho_n \otimes L_n^{(p)}.$$ 

This resolves the conjecture of [25] stated at the end of Section 1.2 of that work. There are then several small useful results that we establish about $\mathcal{V}^{(p)}$,

1. Corollary 6 tells us that $\mathcal{V}^{(p)}$ is strongly generated by $x = x(-1)1 \otimes 1$, $x \in \{e, f, h\}$ and the four vectors stated in (14).
2. Proposition 3 tells us that $\mathcal{V}^{(p)}$ is a simple abelian intertwining algebra.
3. Corollary 8 characterizes $\mathcal{V}^{(p)}$ as the subalgebra of $A^{(p)} \otimes \Pi(0)_{\frac{1}{2}}$ that is integrable with respect to the $\mathfrak{sl}_2$-action of the horizontal subalgebra of $L_k(\mathfrak{sl}_2)$,

$$\mathcal{V}^{(p)} = \left(A^{(p)} \otimes \Pi(0)_{\frac{1}{2}}\right)^{\text{int}}.$$ 

4. Let us also note that $\mathcal{V}^{(p)}$ has the following structure:
   a. If $p \equiv 2 \pmod{4}$, $\mathcal{V}^{(p)}$ is a $\frac{1}{2}\mathbb{Z}_{\geq 0}$-graded vertex operator superalgebra.
   b. If $p \equiv 0 \pmod{4}$, $\mathcal{V}^{(p)}$ is a $\mathbb{Z}_{\geq 0}$-graded vertex operator algebra.
   c. If $p \equiv 1, 3 \pmod{4}$, $\mathcal{V}^{(p)}$ is an abelian intertwining algebra.
1.2. The $\mathcal{R}^{(p)}$-algebra. The vertex algebra $\mathcal{R}^{(p)}$ appeared in [2] is motivated by the free-field realization of the affine vertex algebra $L_{-3/2}(\mathfrak{sl}_2)$ that is isomorphic to $\mathcal{R}^{(2)}$. These algebras are also studied in [25], where they are conjecturally identified as certain affine vertex algebras or vertex algebras for Argyres-Douglas theories.

The vertex algebra $\mathcal{R}^{(p)}$ is defined as a subalgebra of $\mathcal{V}^{(p)} \otimes F_{-\frac{p}{2}}$ where $F_{-\frac{p}{2}}$ is the abelian intertwining algebra associated to the weight lattice of $\mathfrak{sl}_2$ rescaled by $\sqrt{-p}$. It is generated under operator products by $L_k(\mathfrak{sl}_2) \otimes M(1)$ ($M(1)$ is the Heisenberg vertex operator algebra) together with four vectors stated in [15]. The $\mathcal{R}^{(p)}$-algebra is related to the $\mathcal{B}^{(p)}$-algebra of [59], which is characterized as

$$\mathcal{B}^{(p)} = \left( \mathcal{A}^{(p)} \otimes F_{-\frac{p}{2}} \right)^{U(1)}.$$ 

The $\mathcal{R}^{(p)}$-algebra is mainly studied in Section 4. Most properties are inherited from $\mathcal{V}^{(p)}$ and we list the main results as follows:

1. (Corollary 12)
$$\mathcal{R}^{(p)} = \left( \mathcal{V}^{(p)} \otimes F_{-\frac{p}{2}} \right)^{U(1)}$$

and especially $\mathcal{R}^{(p)}$ is simple.

2. As $L_k(\mathfrak{sl}_2) \otimes M(1)$-module
$$\mathcal{R}^{(p)} \cong \bigoplus_{\ell \in \mathbb{Z}} \bigoplus_{s=0}^{\infty} L^{(p)}_{\ell|+2s} \otimes M_{\ell+1,-\ell}$$

with $M_{\ell+1,-\ell}$ certain Fock modules.

3. $\mathcal{R}^{(p)} = \text{Ker}_{\mathcal{B}^{(p)} \otimes \Pi(0)^{1/2}} S$

4. (Corollaries 12 and 3) $\mathcal{R}^{(p)} = \left( \mathcal{B}^{(p)} \otimes \Pi(0)^{1/2} \right)^{\text{int}}$

1.3. Tensor categories related to $\text{KL}_k(\mathfrak{sl}_2)$. Let $V$ be a vertex operator algebra and $\mathcal{C}$ a category of $V$-modules. A crucial problem is whether $\mathcal{C}$ has a (rigid) vertex tensor category structure. Having such a tensor category facilitates proving structural results as e.g. an effective theory of vertex operator superalgebra extensions [58, 52, 53] and orbifolds [58, 59]. As will be explained in the next subsection, we are able to employ our vertex tensor category findings together with just mentioned theory to prove powerful uniqueness results of vertex operator algebra structures.

Let $\mathfrak{g}$ a simple Lie algebra, $k \in \mathbb{C}$ and $\text{KL}_k(\mathfrak{g})$ be the category of ordinary modules for the simple affine vertex operator algebra $L_k(\mathfrak{g})$ of $\mathfrak{g}$ at level $k$. A general aim is to establish rigid vertex tensor category structure on this category for all $\mathfrak{g}$ and $k$. Generically, that is if $k \notin \mathbb{Q}_{>-h_{\mathfrak{g}}}$, this has been achieved in seminal work by Kazhdan and Lusztig [61, 65]. For $k \in \mathbb{Z}_{>1}$ this follows from [56, 57], while for $k$ an admissible level the vertex tensor category structure has been proven to exist in [29] and rigidity in the simply-laced case in [26]. In the accompanying work [44], it is proven that semi-simplicity of $\text{KL}_k(\mathfrak{g})$ implies the existence of vertex tensor category. This result together with a main Theorem of [68] and our Theorem 1 implies that $\text{KL}_k(\mathfrak{sl}_2)$ for $k = -2 + \frac{1}{p}$ and $p$ in $\mathbb{Z}_{>1}$ is a rigid vertex tensor category and as such braided equivalent to a twist by some abelian 3-cocycle of $\text{Rep}(\text{SU}(2))$. Together with [61, 65, 56, 57, 29], this result completes the case $\mathfrak{sl}_2$ and thus

**Corollary 1.** For all $k \in \mathbb{C}$, the category of ordinary modules $\text{KL}_k(\mathfrak{sl}_2)$ is a rigid vertex tensor category.
1.5. Let $KL_k(sl_2)^{\text{even}}$ be the full tensor subcategory whose simple objects are the $L_{2n}^{(p)}$. We also prove that $KL_k(sl_2)^{\text{even}} \cong \text{Rep}(SO(3))$ as symmetric tensor categories.

A corollary of the vertex tensor category structure is that we have many simple currents as discussed in Subsection 6.3. For example, since $\mathcal{R}^{(p)}$ is realized as a $U(1)$-orbifold of some larger abelian intertwining algebra one gets that the $\mathcal{R}^{(p)}$-modules appearing in the decomposition are all simple currents due to results of [46].

1.4. **Uniqueness of vertex operator algebra structure.** Given two vertex operator algebras $V$ and $W$ that share common properties, e.g. they have the same character or they have the same type of strong generators or they are isomorphic as modules for some common subalgebra. In such a case one usually would like to know if these two vertex operator algebras are actually isomorphic leading to the general question: Under which assumptions can we guarantee that two vertex operator algebras are isomorphic? For example a simple affine vertex operator algebra is uniquely specified by the Lie algebra structure on its weight one subspace together with the invariant bilinear form restricted to the weight one subspace. Similarly vertex operator algebras that are strongly generated by fields in weight one and $3/2$ are also uniquely specified by certain structure [21].

We shall apply the correspondence between the vertex operator algebra extensions and the commutative and associative algebra objects in the vertex tensor category [58]. We first use that the $U(1)$-orbifold of $A^{(1)}$ is nothing but the rank one Heisenberg vertex operator algebra to deduce that a certain object in (a completion of) $\text{Rep}(SO(3))$ can be given a unique simple commutative and associative $\mathcal{R}$-algebra $\mathcal{M}$. We first use that the $U(1)$-orbifold of $A^{(1)}$ is isomorphic as an $L_k(sl_2)$ module to $\mathcal{R}^{(p)}$ must even be isomorphic to $\mathcal{R}^{(p)}$ as a vertex operator algebra. A similar argument applies to $B^{(p)}$-algebras using the novel vertex tensor category results of the Virasoro algebra [49].

**Corollary 2.** (Corollaries 13 and 15) For $p$ in $Z_{\geq 1}$ and $k = -2 + \frac{1}{p}$, let $\mathcal{X}$ be a simple vertex operator algebra such that $\mathcal{X} \cong \mathcal{R}^{(p)}$ as an $L_k(sl_2) \otimes M(1)$-module. Then $\mathcal{X} \cong \mathcal{R}^{(p)}$ as vertex operator algebras.

Analogously if a simple vertex operator algebra $\mathcal{Y} \cong \mathcal{B}^{(p)}$ as an $L^{Vir}(c_1,p,0) \otimes M(1)$-module, then $\mathcal{Y} \cong \mathcal{B}^{(p)}$ as vertex operator algebras.

This conclusion solves the conjectures of [25] concerning $\mathcal{W}$-algebras at boundary admissible levels and chiral algebras for Argyres-Douglas theories. We now explain the $\mathcal{W}$-algebra connections and turn to the physics at the end of this introduction.

1.5. **$\mathcal{W}$-algebras and conformal embeddings.** Let $\mathfrak{g}$ be a simple Lie algebra, $f$ a nilpotent element in $\mathfrak{g}$ and $k$ a complex number. Then to this data one associates via quantum Hamiltonian reduction from the affine vertex operator algebra $V^k(\mathfrak{g})$ the universal $\mathcal{W}$-algebra of $\mathfrak{g}$ at level $k$ corresponding to $f$, denoted by $W^k(\mathfrak{g},f)$ [67]. Let $W_k(\mathfrak{g},f)$ denote the unique simple quotient of $W^k(\mathfrak{g},f)$. The level $k$ is admissible if $k = -h^\vee + \frac{\mathfrak{h}}{\mathfrak{h}}$ and $u, v$ positive, coprime integers with $u \geq h^\vee$ if $v$ is coprime to the lacity of $\mathfrak{g}$ and $u \geq h$ otherwise. Here $h^\vee$ and $h$ denote the dual Coxeter and Coxeter number of $\mathfrak{g}$. An interesting question that has been
The vertex algebras $\mathcal{R}(p)$ and $\mathcal{V}(p)$ studied much recently is to classify $\mathcal{W}(g, f)$ that are conformal extensions of affine vertex operator algebras and moreover to understand their decomposition in terms of modules of this affine vertex operator algebra. This has been particularly well understood if $f$ is trivial or minimal nilpotent [7, 5, 6]. In these cases one knows the operator product algebra and also has a powerful uniqueness theorem [67, 21].

Finding examples of conformal embeddings and branching rules for $\mathcal{W}$-algebras corresponding to other nilpotent elements is difficult. We successfully give explicit $\mathcal{W}$-algebra realizations of $\mathcal{R}(p)$ and $\mathcal{B}(p)$-algebras through character computations [25, 21] together with our uniqueness results. Furthermore, the $\mathcal{R}(p)$-case is a conformal embedding.

**Theorem 2.** (Theorems [10] and [12]) For $p \in \mathbb{Z}_{\geq 2}$

1. Let $\ell = -\frac{p^2 - 1}{p}$ and $f$ a nilpotent element in $\mathfrak{sl}_{p+1}$ corresponding to the partition $(p-1, 1, 1)$ of $p + 1$, then $\mathcal{W}_\ell(\mathfrak{sl}_{p+1}, f) \cong \mathcal{R}(p)$ as vertex operator algebras.

2. Let $\ell = -\frac{(p-1)^2}{p}$, then $\mathcal{W}_\ell(\mathfrak{sl}_{p-1}, f_{\text{sub}}) \cong \mathcal{B}(p)$ as vertex operator algebras.

The second statement solves the conjecture that $\mathcal{B}(p)$ is a simple quotient of an affine $\mathcal{W}$-algebra of type $A$ [39].

1.6. **Inverting Quantum Hamiltonian reduction.** The quantum Hamiltonian reduction realizes the Virasoro algebra at central charge $c_{1,p}$ as a certain cohomology of a complex associated to $L_k(\mathfrak{sl}_2)$ with $k = -\frac{2 + \frac{1}{p}}{p}$. The cohomology of the $L_k^{\mathfrak{sl}_2}$ are then corresponding Virasoro algebra modules and it is no problem to verify that $H^0_{\text{DS}}(\mathcal{V}(p)) \cong \mathcal{A}(p)$ and $H^0_{\text{DS}}(\mathcal{R}(p)) \cong \mathcal{B}(p)$ as Virasoro algebra modules. We work out the quantum Hamiltonian reduction of relaxed-highest weight modules (Proposition [7]) in order to prove that

**Theorem 3.** (Theorem [14]) As vertex operator algebras

$$H^0_{\text{DS}}(\mathcal{R}(p)) \cong \mathcal{B}(p),$$

and as abelian intertwining algebras

$$H^0_{\text{DS}}(\mathcal{V}(p)) \cong \mathcal{A}(p).$$

The above Theorem resolves Conjecture 5.11 of [25] and we will return to it when discussing the physics applications. Having this Theorem in mind one sees that the statements

$$\mathcal{R}(p) = \left(\mathcal{B}(p) \otimes \Pi(0)\right)^{\text{int}}, \quad \mathcal{V}(p) = \left(\mathcal{A}(p) \otimes \Pi(0)\right)^{\text{int}}$$

invert the quantum Hamiltonian reduction. This and also our other findings are interesting in the context of vertex algebras for $S$-duality and Argyres-Douglas theories as we will finally explain now.

1.7. **Vertex Algebras for $S$-duality.** Let $G$ be a compact Lie group with Lie algebra $\mathfrak{g}$ and let $\Psi$ be a complex number. One associates to this data so-called GL-twisted $\mathcal{N} = 4$ superconformal four-dimensional gauge theories. $G$ is the gauge group and $\Psi$ the coupling of the theory and GL indicates the connection to the geometric Langlands program [66] (see also [18, 50] for recent work on the connection between quantum geometric Langlands and $S$-duality). The physics motivation is
to generalize Montonen-Olive electro-magnetic duality and the overall picture is that there are dualities between gauge theories associated to $G$ and either $G$ itself or its Langlands dual $L^\ast G$ with coupling constants related by Möbius transformation $\Psi \mapsto \frac{a\Psi + b}{c\Psi + d}$ with $(a\ b\ c\ d)$ in GL$(2, Z)$. The inversion of $\Psi$ is referred to as $S$-duality.

Vertex algebras appear at the intersections of three-dimensional topological boundary conditions, while categories of modules are attached to the various boundary conditions. The type of vertex operator algebra depends on the type of boundary conditions, see [36, 55]. Most importantly, the conjecture is that the following object has the structure of a simple vertex operator superalgebra,

$$A^n[g, \Psi] = \bigoplus_{\lambda \in P^+_n} V^k(\lambda) \otimes V^\ell(\lambda)$$

where $k, \ell$ are related to the coupling $\Psi$ via $\Psi = k + h^\vee$ and

$$\frac{1}{k + h^\vee} + \frac{1}{\ell + h^\vee} = n \in \mathbb{Z}_{\geq 1}.$$ 

Here $P^+_n$ is a subset of dominant integral weights depending on $n$ such that it includes all dominant integral weights that lie in the root lattice $Q$, i.e. $Q \cap P^+ \subset P^+_n$ and the $V^k(\lambda)$ are Weyl modules, namely the $V^k(g)$-modules induced from the irreducible highest-weight representation $\rho_\lambda$ of $g$ of highest-weight $\lambda$. Other interesting vertex operator algebras appear by applying the quantum Hamiltonian reduction functor of either level $\ell$ or $k$ for some nilpotent element $f$ of $g$. The existence of these vertex operator algebras is mostly open, except for $g$ simply-laced and $f$ principal nilpotent [16] and $g = \mathfrak{sl}_2$ and $n = 1, 2$ [36, 55]. These latter cases are related to the exceptional Lie superalgebra $\mathfrak{osp}(2, 1, \alpha)$ and its minimal $\mathcal{W}$-superalgebra, the large $N = 4$ superconformal algebra. The algebras $A^n[g, \Psi]$ are expected to play an important role in quantum geometric Langlands, while its large $\Psi$-limit should relate to the classical geometric Langlands program. The expectation is that $A^n[g]$ is a deformable family of vertex operator algebras in the sense of [37] so that the limit $\Psi \to \infty$ exists. Moreover, the simple quotient of this limit should be a vertex operator algebra with $G$ as a subgroup of automorphisms and it should be of the form

$$A^n[g, \infty] = \bigoplus_{\lambda \in P^+_n} \rho_\lambda \otimes V^\ell(\lambda)$$

as $G \otimes V^\ell(g)$-module and $\ell = -h^\vee + \frac{1}{n}$. All these are conjectural, see Section 1.3.1 of [36] and our uniqueness result proves these large $\psi$-limit conjectures for $G = SU(2)$, i.e we have that [36] Conjecture 1.2 is true for $G = SU(2)$,

$$A^n[g, \infty] \cong \begin{cases} V^{(n)} & n \text{ even} \\ (V^{(n)})^2 & n \text{ odd.} \end{cases}$$

Quantum Hamiltonian reduction of $V^{(n)}$ gives $A^{(n)}$, which is the large $\Psi$-limit of another such corner vertex operator algebra.

1.8. Chiral Algebras for Argyres-Douglas theories. The second physics instance relevant to our work are chiral algebras of Argyres-Douglas theories. These are also four dimensional but $\mathcal{N} = 2$ supersymmetric gauge theories [17] associated to pairs of Dynkin diagrams $(X, Y)$ of simple Lie algebras. Vertex algebras appear
as chiral algebras of protected sectors of these gauge theories \cite{24} and in this instance the central charge, the rank of the Heisenberg subalgebra, affine subalgebras and their levels and the graded character of the chiral algebra can be determined from physics considerations, see e.g. \cite{22,43}. The clear question is then if indeed a vertex operator algebra with the desired properties exists and if it is uniquely determined by them. We also require that the chiral algebra is a simple vertex operator algebra.

Set $X = A_1$ and $Y$ either of type $A$ or $D$. The Schur index and central charge of $(A_1, A_{2n})$ Argyres-Douglas theories coincide with the character and the central charge of $L^{Vir}(c_2,2n+3,0)$ with $c_2,n = 1 - 6(2n + 1)^2 / (4n + 6)$ \cite{70} and there is no flavor symmetry meaning that there is no Heisenberg or affine subalgebra. In the case of $(A_1, D_{2n+1})$, the physics data determine the chiral algebra as the simple affine vertex operator algebra of $sl_2$ at level $k = -4n/(2n + 1)$. The uniqueness of these vertex operator algebras is obvious, i.e. any simple vertex operator algebra whose character and central charge coincides with the simple Virasoro vertex operator algebra or simple affine vertex operator algebra $L_k(sl_2)$ must be isomorphic to this vertex operator algebra. The cases of the chiral algebras of Argyres-Douglas theories of types $(A_1, D_{2p})$ and $(A_1, A_{2p-3})$ are much more complicated. In \cite{25}, the Schur-index was identified with the one of the $W$-algebra of our Theorem \ref{thm:main} in the $(A_1, D_{2p})$-case and with the character of the $B^{(p)}$-algebra in the type $(A_1, A_{2p-3})$-case. Our uniqueness Theorems identify the chiral algebras of these Argyres-Douglas theories, see Section \ref{sec:uniqueness}. That is, we have for $p \geq 2$:

1.9. Outlook. Higher rank analogues of the triplet algebras are introduced by Feigin and Tipunin \cite{53}. Not much is known about these algebras \cite{14,41} and they deserve further study. For example there are higher rank analogues of $B^{(p)}$ whose character coincides with a Schur index of a higher rank Argyres-Douglas theory \cite{27,23}. Our current aim building on this work is to solve more decomposition problems of conformal embeddings of $W$-algebras and to understand quantum Hamiltonian reduction on the category of relaxed-highest weight modules and their spectrally flown images better.

1.10. Organization of this work. We start in section \ref{sec:introduction} by defining the $\mathcal{V}^{(p)}$ and $\mathcal{R}^{(p)}$-algebras along with stating a few basic properties. The next two sections are then devoted to establish most of the structural results about $\mathcal{V}^{(p)}$ and $\mathcal{R}^{(p)}$ that we mentioned in the introduction. The case $p = 1$ is different (and much simpler) than the general case and is discussed in section \ref{sec:level1}. Our structural results are then used in section \ref{sec:tensor} to determine tensor category structure and to use this to prove the uniqueness of vertex operator algebra structure on our algebras. As a consequence we identify $\mathcal{R}^{(p)}$ and $\mathcal{B}^{(p)}$ with $W$-algebras. The next section then uses these uniqueness results to identify $\mathcal{R}^{(p)}$ and $\mathcal{B}^{(p)}$ with chiral algebras of Argyres-Douglas theories. In section \ref{sec:qham} we study properties of the quantum Hamiltonian reduction functor from $L_k(sl_2)$ to the Virasoro algebra. Especially we give a procedure that goes back from Virasoro modules to $L_k(sl_2)$-modules. This is used to show that $\mathcal{V}^{(p)}$ and $\mathcal{R}^{(p)}$ are related to $\mathcal{A}^{(p)}$ and $\mathcal{B}^{(p)}$ via quantum Hamiltonian reduction.
2. Introduction of the Relevant Algebras

We recall [2] and [4] and realize the algebras that we are interested in side larger free field times lattice vertex algebras. For this let \( p \in \mathbb{Z}_{\geq 2} \) and let \( N(p) \) be the following lattice

\[
N(p) = \mathbb{Z} \alpha + \mathbb{Z} \beta + \mathbb{Z} \delta
\]

with the \( \mathbb{Q} \)-valued bilinear form \( \langle \cdot, \cdot \rangle \) such that

\[
\langle \alpha, \alpha \rangle = 1, \quad \langle \beta, \beta \rangle = -1 \quad \text{and} \quad \langle \delta, \delta \rangle = \frac{2}{p}
\]

and all other products of basis vectors are zero. Let \( V_{N(p)} \) be the associated abelian intertwining algebra and set \( k = -2 + \frac{1}{p} \). One defines the three elements

\[
e = e^{\alpha + \beta},
\]

\[
h = -2\beta(-1) + \delta(-1),
\]

\[
f = \left( (k + 1)(\alpha(-1)^2 - \alpha(-2)) - \alpha(-1)\delta(-1) + (k + 2)\alpha(-1)\beta(-1) \right) e^{-\alpha - \beta}.
\]

Then the components of the fields

\[
Y(x, z) = \sum_{n \in \mathbb{Z}} x(n)z^{-n-1}, \quad x \in \{e, f, h\}
\]

satisfy the commutations relations for the affine Lie algebra \( \widehat{sl}_2 \) of level \( k \). Moreover, the subalgebra of \( V_{L(p)} \) generated by the set \( \{e, f, h\} \) is isomorphic to the simple vertex operator algebra \( L_k(sl_2) \).

The screening operators that we need are

\[
Q = \text{Res}_z Y(e^{\alpha + \beta - p\delta}, z), \quad \tilde{Q} = \text{Res}_z Y(e^{-\frac{1}{2}(\alpha + \beta) + \delta}, z).
\]

They commute with the \( \widehat{sl}_2 \)-action. It is now useful to introduce some additional elements following [4]. Let

\[
\gamma := \alpha + \beta - \frac{1}{k+2}\delta = \alpha + \beta - p\delta, \\
\mu := -\beta + \frac{1}{2}\delta, \\
\nu := \frac{k}{2}\alpha - \frac{k+2}{2}\beta + \frac{1}{2}\delta = \alpha - \frac{1}{2p}(\alpha + \beta) + \frac{1}{2}\delta.
\]

Then

\[
\langle \gamma, \gamma \rangle = \frac{2}{k+2} = 2p, \quad \langle \mu, \mu \rangle = -\langle \nu, \nu \rangle = \frac{k}{2},
\]

and all other products are zero. The screening charges then take the form

\[
Q = \text{Res}_z Y(e^{\gamma}, z), \quad \tilde{Q} = \text{Res}_z Y(e^{-\frac{1}{2}\gamma}, z).
\]

For our calculation, it is useful to notice that

\[
\alpha = \nu + \frac{k+2}{2}\gamma, \quad \beta = -\frac{k+2}{2}\gamma + \frac{2}{k}\mu - \frac{k+2}{k}\nu, \\
\delta = -(k + 2)\gamma + \frac{2(k+2)}{k}\mu - \frac{2(k+2)}{k}\nu.
\]

Let

\[
e = \frac{2}{k}(\mu - \nu), \quad d = \mu + \nu.
\]

Then
Let $M$ be the subalgebra of $V_{N(p)}$ generated by
\[ a = e^{\alpha + \beta}, \quad a^* = -\alpha(-1)e^{-\alpha - \beta}. \]
Then $M$ is isomorphic to the Weyl vertex algebra. The Weyl vertex algebra is often also called the $\beta\gamma$-vertex operator algebra and in physics the symplectic boson algebra.

We have the following abelian intertwining algebra
\[ V_{(p)} = \ker_{M \otimes F_\frac{p}{2}} \tilde{Q}. \]
Moreover, $L_k(\mathfrak{sl}_2)$ can be realized as a subalgebra of $M \otimes M_{\delta}(1) \subset M \otimes F_{\frac{p}{2}}$, namely
\[ e(z) = a(z), \]
\[ h(z) = -2 : a^*(z)a(z) : + \delta(z), \]
\[ f(z) = - : a^*(z)^2a(z) : + k\partial_z a^*(z) + a^*(z)\delta(z). \]
Since $\tilde{Q}$ commutes with the action of $\hat{\mathfrak{sl}}_2$ we have that
\[ L_k(\mathfrak{sl}_2) \subset \mathcal{V}_{(p)}. \]
Moreover, one can show that $Q$ acts as a derivation on $\mathcal{V}_{(p)}$. Note that if $p$ is even, then $\mathcal{V}_{(p)}$ is a vertex superalgebra, while for odd $p$ it is not a vertex (super)algebra but only an abelian intertwining algebra.

Recall that the screening operators are the zero-modes $Q = e_0^\gamma$ and $\tilde{Q} = e_0^{-\frac{p}{2}}$.

We remark that
\[ [Q, \tilde{Q}] = (e_0^\gamma e^{-\frac{p}{2}})0 = \gamma(-1)e^{-\frac{p-1}{p}\gamma})0 = \frac{p}{p-1}(\partial e^{-\frac{p-1}{p}\gamma})0 = 0. \]
There are thus four important vectors that are obviously in $\mathcal{V}_{(p)}$, namely
\[ \tilde{\tau}^+_p = e^{\frac{p}{2}\delta}, \]
\[ \tau^+_p = e^{\frac{p}{2}\delta}, \]
\[ \tau^-_p = f(0)e^{\frac{p}{2}\delta}, \]
\[ \tilde{\tau}^-_p = -f(0)Qe^{\frac{p}{2}\delta}. \]

Now we introduce the second algebra we are interested in, which we call the $\mathcal{R}_{(p)}$-algebra. Let $\varphi$ satisfy $\langle \varphi, \varphi \rangle = -\frac{2}{p}$ and let $\Lambda_{\varphi}^{(p)} = \frac{p}{2}Z$ and $F_{\varphi} = V_{\Lambda_{\varphi}^{(p)}}$. 

\[ \frac{p}{2}\delta = -\frac{\gamma}{2} + \frac{c}{2}. \]
The vertex algebra $\mathcal{R}^{(p)}$ is defined to be the subalgebra of $\mathcal{V}^{(p)} \otimes F_{-\frac{p}{2}}$ generated by $x = x(-1) \otimes 1, x \in \{ e, f, h \}, 1 \otimes \varphi(-1) \otimes 1$ and

$$e_{\alpha_1, p} := \frac{1}{\sqrt{2}} e^{p}_{(p)} \otimes e^{\frac{p}{2}} = \frac{1}{\sqrt{2}} e^{\frac{p}{2}}(\delta + \varphi),$$

$$f_{\alpha_1, p} := \frac{1}{\sqrt{2}} e^{p}_{(p)} \otimes e^{-\frac{p}{2}} = -\frac{1}{\sqrt{2}} f(0) Q e^{\frac{p}{2}}(\delta + \varphi),$$

$$e_{\alpha_2, p} := \frac{1}{\sqrt{2}} e^{p}_{(p)} \otimes e^{-\frac{p}{2}} = \frac{1}{\sqrt{2}} f(0) e^{\frac{p}{2}}(\delta + \varphi),$$

$$f_{\alpha_2, p} := \frac{1}{\sqrt{2}} e^{p}_{(p)} \otimes e^{\frac{p}{2}} = \frac{1}{\sqrt{2}} f(0) e^{\frac{p}{2}}(\delta + \varphi).$$

(15)

The Heisenberg vertex algebra generated by $\varphi$ is denoted by $M_{\varphi}(1)$ and in general, $\mathcal{R}^{(p)}$ is an extension of

$$L_k(\mathfrak{sl}_2) \otimes M_{\varphi}(1)$$

by the four fields of conformal weight $p/2$ in (15). Set $M_{\delta, \varphi}(1) = M_{\delta}(1) \otimes M_{\varphi}(1)$ and let

$$\Phi(0) = M_{\delta, \varphi}(1) \otimes \mathbb{Z}\left[ \frac{P}{2} (\delta + \varphi) \right] \subset F_{\frac{p}{2}} \otimes F_{-\frac{p}{2}}.$$

Then $\Phi(0)$ contains a rank one isotropic lattice vertex operator algebra. In general we have that

$$\mathcal{R}^{(p)} \subset (M \otimes \Phi(0))^{\text{int}},$$

where $(M \otimes \Phi(0))^{\text{int}}$ is the maximal $\mathfrak{sl}_2$–integrable submodule of $M \otimes \Phi(0)$. The cases $p = 2, 3$ have been studied:

**Theorem 4.**

1. $\mathcal{R}^{(2)} \cong L_{-3/2}(\mathfrak{sl}_3)$.
2. $\mathcal{R}^{(3)} \cong W_{-8/3}(\mathfrak{sl}_4, f_\theta)$ with $f_\theta$ minimal nilpotent.
3. For $p = 2, 3$ we have $\mathcal{R}^{(p)} = \text{Ker}_{M \otimes \Phi(0)} \tilde{Q} = (M \otimes \Phi(0))^{\text{int}}$.

### 3. The $\mathcal{V}^{(p)}$-algebra

We first study $\mathcal{V}^{(p)}$ and $\mathcal{R}^{(p)}$ will inherit many properties from $\mathcal{V}^{(p)}$.

3.1. **From the doublet $\mathcal{A}^{(p)}$ to $\mathcal{V}^{(p)}$.** We will now realize $\mathcal{R}^{(p)}$ and $\mathcal{V}^{(p)}$ by lifting certain well-known extensions called doublet algebras $\mathcal{A}^{(p)}$ of the Virasoro vertex operator algebra, following [4]. The various useful lattice vectors have been recorded in [5]–[8]. For $\zeta \in \mathbb{C}$, we denote the universal Virasoro vertex operator algebra of central charge $\zeta$ by $V^{\text{Vir}}(\zeta, 0)$. For any two co-prime positive integers $p, q$ we set

$$c_{q, p} := 1 - \frac{(p - q)^2}{pq}.$$

Let $\omega$ be the conformal vector in $V^{\text{Vir}}(c_{1, p}, 0)$. Define

$$\Pi(0) := M_{c, d}(1) \otimes \mathbb{C}[\mathbb{Z} c] \quad \text{and} \quad \Pi(0)^{\frac{1}{2}} = M_{c, d}(1) \otimes \mathbb{C}\left[ \frac{\mathbb{Z} c}{2} \right].$$

(17)

Since $\langle c, c \rangle = 0$ these are vertex operator algebras that contain a rank one isotropic lattice vertex operator algebra as their subalgebra. There is an injective homomorphism of vertex algebras

$$\Phi : V^k(\mathfrak{sl}_2) \rightarrow V^{\text{Vir}}(c_{1, p}, 0) \otimes \Pi(0)$$
such that
\[ e \mapsto e^c, \]
\[ h \mapsto 2\mu(-1), \]
\[ f \mapsto [(k + 2)\omega - \nu(-1)^2 - (k + 1)\nu(-2)] e^{-c}. \]

The famous doublet \( \mathcal{A}(p) \) and triplet \( \mathcal{W}(p) \) algebras are realized as \([12]\):
\[
\mathcal{A}(p) = \text{Ker} \mathcal{V}_z \tilde{Q},
\]
\[
\mathcal{W}(p) = \text{Ker} \mathcal{V}_z \tilde{Q}.
\]

The triplet \( \mathcal{W}(p) \) are vertex operator algebras, while the doublets are vertex super-algebras for even \( p \) and otherwise only abelian intertwining algebras. Recall that \( \mathcal{A}(p) \) is generated by the doublet \( a^{\pm} \) together with the Virasoro element \( \omega \). These are explicitely
\[
a^+ = Qa^-, \quad a^- = e^{-\frac{r}{2}} \quad \text{and} \quad \omega = \frac{1}{4p} \gamma(-1)^2 + \frac{p-1}{2p} \gamma(-2).
\]

The abelian intertwining algebras \( \mathcal{A}(p) \) and \( \Pi(0)^{\frac{1}{2}} \) both have a \( \mathbb{Z}_2 \)-action with invariant subalgebras \( \mathcal{W}(p) \) and \( \Pi(0) \). So that they decompose as \( \mathcal{W}(p) \) and \( \Pi(0) \)-modules
\[
\mathcal{A}(p) = \mathcal{A}_0^{(p)} \oplus \mathcal{A}_1^{(p)} \quad \text{and} \quad \Pi(0)^{\frac{1}{2}} = \Pi(0)_0^{\frac{1}{2}} \oplus \Pi(0)_1^{\frac{1}{2}},
\]
with \( \mathcal{A}_0^{(p)} = \mathcal{W}(p) \) and \( \Pi(0)_0^{\frac{1}{2}} = \Pi(0) \). The diagonal \( \mathbb{Z}_2 \)-orbifold is thus
\[
\left( \mathcal{A}(p) \otimes \Pi(0)^{\frac{1}{2}} \right)^{\mathbb{Z}_2} \cong \mathcal{W}(p) \otimes \Pi(0)_0^{\frac{1}{2}} \oplus \mathcal{A}_1^{(p)} \otimes \Pi(0)_1^{\frac{1}{2}}.
\]

We have that \( \mathcal{W}(p) \subset \left( \mathcal{A}(p) \otimes \Pi(0)^{\frac{1}{2}} \right)^{\mathbb{Z}_2} \) and the expressions for the generators are
\[
\tau_{(p)}^+ = a^- e^{\frac{r}{2}},
\]
\[
\tau_{(p)}^- = a^+ e^{\frac{r}{2}},
\]
\[
\tau_{(p)} = f(0)a^- e^{\frac{r}{2}}
\]
\[
\tau_{(p)}^- = -f(0)a^+ e^{\frac{r}{2}}.
\]

Note that \( \mathcal{W}(p) \) can also be realized as a subalgebra of \( M \otimes F_{\frac{1}{2}} \), we need to consider screening operator
\[
S = e_0^\alpha = \text{Res}_z Y(e^\alpha, z),
\]
since \( M = \ker \Pi_0 S \) \([54]\).

Remark 1. Note that the screening operator \( S \) can be obtained as
\[
S = \text{Res}_z Y(e^{\gamma/2p + \nu}, z) = \text{Res}_z Y(v_{2,1} \otimes e^\nu, z),
\]
where \( v_{2,1} = e^{\gamma/2p} \) is a singular vector for the Virasoro algebra with conformal weight \( \frac{3}{2}k + 1 \). This screening operator has also appeared in \([4]\).

Proposition 1. \( \mathcal{W}(p) = \ker \left( \mathcal{A}(p) \otimes \Pi(0)^{\frac{1}{2}} \right)^{\mathbb{Z}_2} \).
Proof. Combining $\mathcal{A}^{(p)} = \text{Ker}_{\mathbb{Z}/2} \tilde{\mathcal{Q}}$, $M = \text{Ker}_{\Pi(0)} S$ and $\mathcal{V}^{(p)} = \text{Ker}_{F_{\mathbb{Z}}} \tilde{\mathcal{Q}}$ together with computing that $\Pi(0) \otimes F_{\mathbb{Z}}$ is the diagonal $\mathbb{Z}_2$-orbifold of $V_{\mathbb{Z}/2} \otimes \Pi(0)_{\frac{1}{2}}$ we immediately get the claim
\[
\mathcal{V}^{(p)} = \text{Ker}_{M \otimes F_{\mathbb{Z}}} \tilde{\mathcal{Q}}
\]
\[
= \text{Ker}_{\Pi(0) \otimes F_{\mathbb{Z}}} \tilde{\mathcal{Q}} \bigcap \text{Ker}_{\Pi(0) \otimes F_{\mathbb{Z}}} S
\]
\[
= \text{Ker} \left( V_{\mathbb{Z}/2} \otimes \Pi(0)_{\frac{1}{2}} \right)_{\frac{1}{2}} \tilde{\mathcal{Q}} \bigcap \text{Ker} \left( V_{\mathbb{Z}/2} \otimes \Pi(0)_{\frac{1}{2}} \right)_{\frac{1}{2}} S
\]
\[
= \text{Ker} \left( \mathcal{A}^{(p)} \otimes \Pi(0)_{\frac{1}{2}} \right)_{\frac{1}{2}} S.
\]

\[\square\]

Let $L^{\text{Vir}}(c, h)$ denote the irreducible lowest-weight module of the Virasoro algebra at central charge $c$ of lowest-weight $h$. Now we take the following decomposition of $\mathcal{A}^{(p)}$ which follows from e.g. [10, Thm 1.1 and 1.2]
\[
\mathcal{A}^{(p)} = \bigoplus_{n=0}^{\infty} \rho_n \otimes L^{\text{Vir}}(c_1, p, h_{1, n+1}),
\]
where $\rho_n$ is $n+1$–dimensional irreducible representation of $sl_2$ and $h_{1, n+1} = \frac{(1 - (n+1)p^2 - (p-1)^2)}{4p}$. Note that the additional $sl_2$-action is defined by $e = \frac{Q}{p}$ and certain action of $f$. Let
\[
v_{1, n, j} := Q^j e^{-\frac{n-1}{2} \gamma} \in \mathcal{A}^{(p)}, \quad j = 0, \ldots, n-1.
\]
Then $v_{1, n, j}$ is the highest weight vector in $L^{\text{Vir}}(c_1, p, h_{1, n})$ and $\bigoplus_{j=0}^{n-1} \mathbb{C} v_{1, n, j}$ is isomorphic to $\rho_{n-1}$ with respect to the additional $sl_2$-action, in which $v_{1, n, j}$ is the weight vector of the weight $2j - n + 1$. We denote by $L^{(p)}_s$ the irreducible highest-weight representation of $sl_2$ whose top level is $\rho_s$ and on which the central element acts by multiplication by the level $k = -2 + \frac{1}{p}$. Common notations are
\[
L^{(p)}_s = L_{\Lambda_1}, \quad (k + 2 - s)\Lambda_0 + s\Lambda_1 = L_k(s\omega_1), \quad (s \in \mathbb{Z}_{\geq 0}),
\]
with $\Lambda_0, \Lambda_1$ the affine fundamental weights and $\omega_1$ the fundamental weight of $sl_2$.

**Proposition 2.** [4 Proposition 6.1]. For every $s \in \mathbb{Z}_{\geq 0}$ and $j = 0, \ldots, s$, we have
\[
\tilde{\varphi}_{s, j} : L^{(p)}_s \rightarrow L_{k}(s\omega_{1})_{(v_{1, s+1, j} \otimes e^{\frac{1}{2} c})} \subset L^{\text{Vir}}(c_1, p, h_{1, s+1}) \otimes \Pi(0)_{\frac{1}{2}}.
\]

We present an important Lemma on uniqueness of singular vector in $L^{\text{Vir}}(c_1, p, h_{1, s}) \otimes \Pi(0)_{\frac{1}{2}}$. A more general version will be studied in [4].

**Lemma 1.** Assume that $w$ is a singular vector for $\mathfrak{sl}_2$ in $L^{\text{Vir}}(c_1, p, h_{1, s}) \otimes \Pi(0)_{\frac{1}{2}}$ with dominant integral weight. Then, $w = v_{1, s, j} \otimes e^{\frac{1}{2} c}$. Proof: Let $\mathfrak{b}$ be the Borel subalgebra of $\mathfrak{sl}_2$ generated by $e(n) = e_n^c$ and $h(n) = 2\mu(n)$. We consider $L^{\text{Vir}}(c_1, p, h_{1, s}) \otimes \Pi(0)_{\frac{1}{2}}$ as a module for $\mathfrak{b}$. 
Assume that $W \subset L^{\text{Vir}}(c_{1,p}, h_{1,s}) \otimes \Pi(0)^{1\over 2}$ is any non-zero $L_k(\mathfrak{sl}_2)$-submodule which is integrable with respect to $\mathfrak{sl}_2$. Using action of $\hat{h}$, we see that $W$ contains vector $u_1 \otimes e^{\frac{\gamma - c}{2}}$ for certain $u_1 \in L^{\text{Vir}}(c_{1,p}, h_{1,s})$ and $m \in \mathbb{Z}$. By using the action of the Sugawara Virasoro element $L_{\text{Sug}}(n)$ we easily get that

$$v_{1,s,j} \otimes e^{\frac{m - 1}{2}c} \in W.$$  

Direct calculation then shows that $v_{1,s,j} \otimes e^{\frac{m - 1}{2}c}$ is singular if and only if $m = s$. \hfill $\square$

**Corollary 3.** \(\text{Ker} L^{\text{Vir}}(c_{1,p}, h_{1,n+1}) \otimes \Pi(0)^{1\over 2} S \cong L_n^{(p)}.\)

**Proof.** Since $v_{1,n+1,0} \otimes e^{\frac{\gamma - c}{2}} = e^{-\frac{\gamma}{\sqrt{2p+\nu}}}$ and $S = e^{\gamma/2p+\nu}$, the relation $(\gamma/2p + \nu, \gamma - c) = 0$ implies that $S(v_{1,n+1,0} \otimes e^{\frac{\gamma - c}{2}}) = 0$. Since $[S, Q] = 0$, we have that $S(v_{1,n+1,0} \otimes e^{\frac{\gamma - c}{2}}) = Q^1 S(v_{1,n+1,0} \otimes e^{\frac{\gamma - c}{2}}) = 0$. \hfill $\square$

Recall that $A^{(p)} = A_0^{(p)} \oplus A_1^{(p)}$ and $\Pi(0)^{1\over 2} = \Pi(0)^{1\over 2}_0 \oplus \Pi(0)^{1\over 2}_1$, so that

$$A^{(p)} \otimes \Pi(0)^{1\over 2} = \left(A^{(p)} \otimes \Pi(0)^{1\over 2}_0\right)^{\mathbb{Z}_2} \oplus A_1^{(p)} \otimes \Pi(0) \oplus \mathcal{V}^{(p)} \otimes \Pi(0)^{1\over 2}_1.$$  

We inspect that

$$\text{Ker} A^{(p)} \otimes \Pi(0)^{1\over 2} S = \text{Ker} \mathcal{V}^{(p)} \otimes \Pi(0)^{1\over 2}_1 S = 0,$$

so that Proposition 4 improves to

**Corollary 4.** \(\mathcal{V}^{(p)} = \text{Ker} A^{(p)} \otimes \Pi(0)^{1\over 2} S.\)

### 3.2. The $\mathfrak{sl}_2$-action.

**Theorem 5.** $\mathcal{V}^{(p)}$ decomposes as a $\mathfrak{sl}_2 \otimes L_k(\mathfrak{sl}_2)$-module as

$$\mathcal{V}^{(p)} = \bigoplus_{n=0}^{\infty} \rho_n \otimes L_n^{(p)}.$$

**Proof.** The decomposition of $A^{(p)}$ \eqref{eq:23} yields

$$A^{(p)} \otimes \Pi(0)^{1\over 2} = \bigoplus_{n=0}^{\infty} \rho_n \otimes \left(L^{\text{Vir}}(c_{1,p}, h_{1,n+1}) \otimes \Pi(0)^{1\over 2}\right),$$

so that

$$\mathcal{V}^{(p)} = \text{Ker} A^{(p)} \otimes \Pi(0)^{1\over 2} S \cong \bigoplus_{n=0}^{\infty} \rho_n \otimes L_n^{(p)}.$$ \hfill $\square$

Strictly speaking the action of $\mathfrak{sl}_2$ is obtained in \cite{8} for triplet. But it extends easily for doublet:

**Remark 2.** In \cite{8} it was proven that the Lie algebra $\mathfrak{sl}_2$ acts on $\mathcal{V}^{(p)}$ by derivations. Let us recall the main steps in the proof.

- We use decomposition of $W^{(p)}$ as a Vir $\otimes \mathfrak{sl}_2$-module from \cite{10}.
- We use the fact that screening operator $Q$ is a derivation and we put $e = Q$. 
• We construct an automorphism of order two $\Psi$ of singlet algebra which extends to an automorphism of $\mathcal{W}^{(p)}$.
• The action of operator $f$ is given by $f = -\Psi^{-1}Q\Psi$.

But the same proof with minor modifications implies that $\mathfrak{sl}_2$ acts on $\mathcal{A}^{(p)}$ by derivations. So:

• Using [12], the doublet algebra $\mathcal{A}^{(p)}$ is a Vir$\otimes\mathfrak{sl}_2$–module and it decomposes as

$$\mathcal{A}^{(p)} = \bigoplus_{n=0}^{\infty} \rho_n \otimes L^{\text{Vir}}(c_{1,p}, h_{1,n+1}).$$

• The screening operator $Q$ is a derivation of $\mathcal{A}^{(p)}$ and we can put $e = Q$.
• The automorphism $\Psi$ of $\mathcal{W}^{(p)}$, easily extends to an automorphism of order two of $\mathcal{A}^{(p)}$ such that $\Psi(a^+) = a^-$.
• The operator $f$ can be again defined as $f = -\Psi^{-1}Q\Psi$.

**Theorem 6.** [8] The Lie algebra $\mathfrak{sl}_2$ acts on $\mathcal{A}^{(p)}$ by derivations and

$$(\mathcal{A}^{(p)})^{\mathfrak{sl}_2} = L^{\text{Vir}}(c_{1,p}, 0).$$

This immediately gives an $\mathfrak{sl}_2$-action of $\mathcal{A}^{(p)} \otimes \Pi(0)^{\frac{1}{2}}$ and from the above Theorem we see that this action restricts to an action on the $\mathcal{V}^{(p)}$-subalgebra, and so

**Corollary 5.** The Lie algebra $\mathfrak{sl}_2$ acts on $\mathcal{V}^{(p)}$ as derivations.

Essentially the same argument as the one of Proposition 1.3 of [10] gives now

**Corollary 6.** $\mathcal{V}^{(p)}$ is strongly generated by $x = x(-1)1 \otimes 1$, $x \in \{e, f, h\}$ and the four vectors stated in [14].

Note that $\mathcal{V}^{(p)}$ has the following structure:

• If $p \equiv 2 \pmod{4}$, $\mathcal{V}^{(p)}$ is a $\frac{1}{2}\mathbb{Z}_{\geq 0}$-graded vertex operator superalgebra.
• If $p \equiv 0 \pmod{4}$, $\mathcal{V}^{(p)}$ is a $\frac{1}{2}\mathbb{Z}_{\geq 0}$-graded vertex operator algebra.
• If $p \equiv 1, 3 \pmod{4}$, $\mathcal{V}^{(p)}$ is an abelian intertwining algebra.

We also have the following vertex subalgebra:

$$(\mathcal{V}^{(p)})^{\mathbb{Z}_2} = \text{Ker}_{\mathcal{W}^{(p)} \otimes \Pi(0)^{\frac{1}{2}}} S \cong \bigoplus_{n=0}^{\infty} \rho_{2n} \otimes \mathcal{L}^{(p)}_{2n}.$$  

In all cases, Since $(\mathcal{V}^{(p)})^{\mathbb{Z}_2}$ is a subalgebra of the vertex operator algebra $\mathcal{W}^{(p)} \otimes \Pi(0)^{\frac{1}{2}}$, we have that $(\mathcal{V}^{(p)})^{\mathbb{Z}_2}$ has the structure of a vertex operator algebra.

Moreover, since $\text{Aut}(\mathcal{W}^{(p)}) = \text{PSL}(2, \mathbb{C})$ (cf. [8]), we get:

**Corollary 7.** The group $G = \text{PSL}(2, \mathbb{C})$ acts on $(\mathcal{V}^{(p)})^{\mathbb{Z}_2}$ as its automorphism group, and we have the following decomposition of $(\mathcal{V}^{(p)})^{\mathbb{Z}_2}$ as $G \otimes L_k(\mathfrak{sl}_2)$–module:

$$(\mathcal{V}^{(p)})^{\mathbb{Z}_2} = \bigoplus_{n=0}^{\infty} \rho_{2n} \otimes \mathcal{L}^{(p)}_{2n}.$$
3.3. Simplicity of $\mathcal{V}^{(p)}$.

**Proposition 3.** $\mathcal{V}^{(p)}$ is a simple abelian intertwining algebra.

**Proof.** Assume that $0 \neq I \subseteq \mathcal{V}^{(p)}$ is a non-trivial ideal in $\mathcal{V}^{(p)}$. Then $I$ is a $L_k(\mathfrak{sl}_2)$-module in $KL_k$ for $k = -2 + 1/p$, and therefore it must contain a non-trivial singular vector with dominant weight. Using Lemma 1 we get that

$$v_{1,n_0+1,j} \otimes e^{\frac{n_0}{2}c} \in I$$

for certain $n_0 \in \mathbb{Z}_{\geq 0}$, $0 \leq j \leq n_0$. We can take $n_0$ to be minimal with this property. Assume that $n_0 > 0$. Since $A^{(p)}$ is simple and generated by $a^\pm$, there is $m \in \frac{1}{2}\mathbb{Z}$, such that

$$a^+_m v_{1,n_0+1,j} = Cv_{1,n_0,j'} \quad \text{or} \quad a^-_m v_{1,n_0+1,j} = Cv_{1,n_0,j'}$$

for certain $C \neq 0$ and $0 \leq j' \leq n_0 - 1$. Now applying the action of the four generators of $\mathcal{V}^{(p)}$ on $v_{1,n_0+1,j} \otimes e^{\frac{n_0}{2}c}$ we get

$$v_{1,n_0,j'} \otimes e^{\frac{n_0-1}{2}c} \in I.$$ 

This is in contradiction with minimality of $n_0$. Therefore $n_0 = 0$, and $1 \in I$. So $I = \mathcal{V}^{(p)}$. This proves the simplicity of $\mathcal{V}^{(p)}$. $\square$

4. The $\mathcal{R}^{(p)}$ Vertex Algebra

The three abelian intertwining algebras $\mathcal{V}^{(p)}, A^{(p)}$ and $F_{-\frac{p}{2}}$ have a natural $U(1)$-action. In the first two cases it is just obtained by exponentiating the action of the Cartan subalgebra of $\mathfrak{sl}_2$, while in the last one it is obtained by exponentiating $\varphi(0)$. This action gives the decompositions in terms of $(\mathcal{V}^{(p)} U(1) \otimes U(1), A^{(p)} U(1) \otimes U(1))$ and $M_{\varphi}(1) \otimes U(1)$ modules

$$\mathcal{V}^{(p)} \cong \bigoplus_{\ell \in \mathbb{Z}} L^{(p)}_{\ell} \otimes \mathbb{C}_\ell,$$

$$A^{(p)} \cong \bigoplus_{\ell \in \mathbb{Z}} A^{(p)}_{\ell} \otimes \mathbb{C}_\ell,$$

$$F_{-\frac{p}{2}} \cong \bigoplus_{\ell \in \mathbb{Z}} M_{\varphi}(1, \ell) \otimes \mathbb{C}_\ell,$$

with

$$\mathcal{V}^{(p)} \cong \bigoplus_{s=0}^{\infty} L^{(p)}_{|\ell|+2s},$$

$$A^{(p)} \cong \bigoplus_{s=0}^{\infty} L^{Vir}(c_{1,p}, h_{1,|\ell|+2s+1}),$$

$$M_{\varphi}(1, \ell) \cong M_{\varphi}(1) e^{\ell \frac{p}{2}}.$$
The $B(p)$-algebra of \([39]\) is the diagonal $U(1)$-orbifold of $A(p) \otimes F_{-\frac{p}{2}}$

$$B(p) = \left( A(p) \otimes F_{-\frac{p}{2}} \right)^{U(1)} = \bigoplus_{\ell \in \mathbb{Z}} (A(p)_{\ell} \otimes M_{\varphi}(1, -\ell))$$

(27)

Similarly the diagonal $U(1)$-orbifold of $V(p) \otimes F_{-\frac{p}{2}}$ is

$$\tilde{R}(p) = \left( V(p) \otimes F_{-\frac{p}{2}} \right)^{U(1)} = \bigoplus_{\ell \in \mathbb{Z}} (V(p)_{\ell} \otimes M_{\varphi}(1, -\ell))$$

(28)

The vectors (15) are in $\tilde{R}(p)$ and hence $R(p) \subset \tilde{R}(p)$. We will prove in a moment that they actually coincide.

Recall \([39]\) that the $B(p)$ algebra is generated by $a^+ \otimes e^{\varphi/2}$, $a^- \otimes e^{-\varphi/2}$, $\varphi(-1)$ and $\omega$. The $U(1)$-invariant part of the isomorphism of Corollary 4 says that $\tilde{R}(p) = \ker_{B(p) \otimes \Pi(0)_{\frac{p}{2}}} S$.

### 4.1. Identification of the quotient.

As before, we assume that $k = -2 + \frac{1}{p}$ for $p$ a positive integer and we denote by $L_s(p)$ the simple highest-weight module of $L_k(s\mathfrak{sl}_2)$ of highest-weight $s\omega_1$. Let $s$ be a non-negative integer, then from Lemma 1 we have that

$$\left( L_{\text{Vir}}(c_k, h_{1,s+1}) \otimes \Pi(0) \right)^{\text{int}} \cong \begin{cases} L_s(p) & \text{if } s \text{ is even} \\ 0 & \text{if } s \text{ is odd} \end{cases}$$

(29)

$$\left( L_{\text{Vir}}^\mathbb{Z}(c_k, h_{1,s+1}) \otimes \Pi(0) \right)^{\text{int}} \cong \begin{cases} 0 & \text{if } s \text{ is even} \\ L_s(p) & \text{if } s \text{ is odd} \end{cases}$$

One can also define twisted modules, e.g. let $w_s$ be a twisted highest-weight vector satisfying

$$e(m-1)w_s = 0, \quad f(m+2)w_s = 0, \quad h(m)w_s = \delta_{m,0}(s - 2)w_s \quad \text{for } m \geq 0.$$

Then $w_s$ generates a module that we denote by $\rho_1(L_s(p)_{-\frac{p}{2}})$.

**Lemma 2.** Let $k = -2 + \frac{1}{p}$ with $p$ a positive integer and $s$ a non-negative integer, then

1. For $s$ even

$$0 \to L_s(p) \to L_{\text{Vir}}(c_k, h_{1,s+1}) \otimes \Pi(0) \to \rho_1(L_s(p)_{-\frac{p}{2}}) \to 0$$
(2) For $s$ odd
\[ 0 \to L_s^{(p)} \to L_{\text{Vir}}^{(p)}(c_k, h_{1,s+1}) \otimes \Pi(0)^{\frac{\delta}{2}} \to \rho_1 \left( L_s^{(p)} \right) \to 0 \]

Proof. Consider the vector
\[ w_s = v_{1,s+1} \otimes e^{\frac{c}{2} + c}. \]
Let us prove that $w_s$ is cyclic. Let $W = L_k(\mathfrak{sl}_2).w_s$. Assume that $s$ is even (the case $s$ is odd is completely analogous). By applying operators $e(-1)$ and $f(1)$ we get
\[ v_{1,s+1} \otimes e^{\frac{c}{2} + mc} \in W \quad (\forall m \in \mathbb{Z}). \]
Using the action of $\hat{b}$, we get that
\[ v_{1,s+1} \otimes z \in W \quad (\forall z \in \Pi(0)). \]
Then applying action of $L_{\text{sug}}(n)$ we obtain
\[ L_{\text{Vir}}^{(p)}(c_k, h_{1,s+1}) \otimes \Pi(0)^{\frac{\delta}{2}} = W. \]
Next we notice that $e(-1)w_s = v_{1,s+1} \otimes e^{\frac{c}{2}}$ which is singular in $L_{\text{Vir}}^{(p)}(c_k, h_{1,s+1}) \otimes \Pi(0)^{\frac{\delta}{2}}$. So in the quotient module $W_s$, the vector
\[ \overline{w}_s = w_s + L_s^{(p)} \]
satisfies twisted highest weight condition:
\[ e(m-1)\overline{w}_s = 0, \quad f(m+2)\overline{w}_s = 0, \quad h(m)\overline{w}_s = \delta_{m,0}(s-2)\overline{w}_s \quad \text{for} \quad m \geq 0. \]
Applying again the action of $\hat{b}$ we see that $\overline{w}_s$ is unique singular vector in $W_s$. We conclude that the quotient module is isomorphic to the module $\rho_1 \left( L_s^{(p)} \right)$. $\square$

As a consequence, we get another nice characterization of our algebras:

**Corollary 8.** The algebras satisfy
\[ \mathcal{V}^{(p)} = \left( A^{(p)} \otimes \Pi(0)^{\frac{\delta}{2}} \right)^{\text{int}} \quad \text{and} \quad \overline{\mathcal{R}}^{(p)} = \left( B^{(p)} \otimes \Pi(0)^{\frac{\delta}{2}} \right)^{\text{int}}. \]

5. THE CASE $p = 1$

The case $p = 1$ can be realized very similarly to the case $p \geq 1$. The only difference is that one doesn’t need the screening operators $Q$ and $\overline{Q}$. Consider the algebras $A^{(1)} = F_2$ and $\mathcal{V}^{(1)}$, $A^{(1)}$ is an abelian intertwining algebra that is a simple current extension of $L_1(\mathfrak{sl}_2)$:
\[ A^{(1)} = L_1(\mathfrak{sl}_2) \oplus L_1. \]
Thus $A^{(1)}$ has the natural $\mathfrak{sl}_2 \times \text{Vir}$ action so that
\[ A^{(1)} = \bigoplus_{n=0}^{\infty} \rho_n \otimes L_{\text{Vir}} \left( 1, \frac{n^2}{4} \right). \]
By Lemmas 1 and 2 we have the following realization of irreducible \( \widehat{\mathfrak{sl}_2} \)-module \( L_n^{(1)} = L_{-1}(n \omega_1) \), which has the highest weight \(- (1 + n) \Lambda_0 + n \Lambda_1\):

\[
L_n^{(1)} = L_{-1}(n \omega_1) = \left( L^{\text{Vir}}(1, \frac{n^2}{4}) \otimes \Pi(0) \right)^{\text{int}}.
\]

Define

\[
\mathcal{Y}^{(1)} = \left( A^{(1)} \otimes \Pi(0) \right)^{\text{int}}
\]

and apply the same arguments as for \( \mathcal{V}^{(p)} \) to give

**Proposition 4.** \( \mathcal{Y}^{(1)} \) is an abelian intertwining algebra, \( \mathfrak{sl}_2 \) acts on \( \mathcal{Y}^{(1)} \) by derivations, and we have the following decomposition of \( \mathcal{Y}^{(1)} \) as \( \mathfrak{sl}_2 \otimes L_{-1}(\mathfrak{sl}_2) \)-module:

\[
\mathcal{Y}^{(1)} = \bigoplus_{n=0}^{\infty} \rho_n \otimes L_n^{(1)}.
\]

The algebra \( \mathcal{Y}^{(1)} \) has a natural \( \mathbb{Z}_2 \)-gradation:

\[
\mathcal{Y}^{(1)} = \mathcal{Y}^{(1)}_0 \oplus \mathcal{Y}^{(1)}_1
\]

such that \( \mathcal{Y}^{(1)}_0 \) is a simple vertex operator algebra:

\[
\mathcal{Y}^{(1)}_0 = \left( \mathcal{Y}^{(1)} \right)^{\mathbb{Z}_2} = \bigoplus_{n=0}^{\infty} \rho_{2n} \otimes L^{(1)}_{2n},
\]

and \( \mathcal{Y}^{(1)}_1 \) is an irreducible \( \mathcal{Y}^{(1)} \)-module

\[
\mathcal{Y}^{(1)}_1 = \bigoplus_{n=0}^{\infty} \rho_{2n+1} \otimes L^{(1)}_{2n+1}.
\]

The abelian intertwining algebra \( \mathcal{Y}^{(1)} \) is also a building block for a realization of \( L_1(\mathfrak{psl}(2|2)) \).

**Proposition 5.** [26, Remark 9.11] As vertex operator algebras \( L_1(\mathfrak{psl}(2|2)) \cong \mathcal{Y}^{(1)}_0 \otimes L_1(\mathfrak{sl}_2) \bigoplus \mathcal{Y}^{(1)}_1 \otimes L_1(\omega_1). \)

The algebra \( \mathcal{R}^{(1)} \) is defined to be

\[
\mathcal{R}^{(1)} = \left( \mathcal{Y}^{(1)} \otimes F_{-\frac{1}{2}} \right)^{U(1)}
\]

and it can be identified with \( M \otimes M \), see e.g. Proposition 5.1 of [25].

**Proposition 6.** As vertex operator algebras

\[
\mathcal{R}^{(1)} \cong M \otimes M.
\]
6. Tensor category of $L_k(\mathfrak{sl}_2)$

6.1. Tensor category of orbifold vertex operator algebra. We recall the following results on the tensor category theory of orbifold vertex operator algebras from [68]. Let $A$ be an abelian group and $(F, \Omega)$ be a normalized abelian 3-cocycle on $A$ with values in $\mathbb{C}^\times$, let $V$ be a simple abelian intertwining algebra ([47], [48]), a kind of generalized vertex operator algebra graded by $A$ with the usual associativity and commutativity properties of the vertex operator algebra modified by the abelian 3-cocycle $(F, \Omega)$. Let $G$ be a compact Lie group of continuous automorphisms of $V$ containing $\hat{A}$, let $\text{Rep}_{A,F,\Omega}(G)$ be the modified tensor category of $\text{Rep}(G)$ by the 3-cocycle $(F, \Omega)$.

Theorem 7 ([68]).

(1) As a $G \otimes V^G$-module, $V$ is semisimple with the decomposition:

$$V = \bigoplus_{\chi \in \hat{G}} M_{\chi} \otimes V_{\chi},$$

where the sum runs over all finite-dimensional irreducible characters of $G$, $M_{\chi}$ is the finite dimensional irreducible $G$-module corresponding to $\chi$, and the $V_{\chi}$ are nonzero, distinct, irreducible $V^G$-modules.

(2) Let $C_V$ be the category of $V^G$-modules generated by the $V_{\chi}$. If $V^G$ has a braided tensor category of modules that contains all $V_{\chi}$, then there is a braided tensor equivalence $C_V \cong \text{Rep}_{A,F,\Omega}(G)$.

6.2. Rigidity of $KL_k(\mathfrak{sl}_2)$. Note that the doublet vertex algebra $A^{(p)}$ is the kernel of the screening operator $\tilde{Q}$ on the abelian intertwining algebra $V_{\frac{1}{2}+n}$, and the vertex algebra $V^{(p)}$ is the kernel of the screening operator $S$ on $A^{(p)} \otimes \Pi_\frac{1}{2}$. They inherit the structure of the abelian intertwining algebra from $V_{\frac{1}{2}+n}$, i.e. we have:

Lemma 3. The vertex algebra $V^{(p)}$ has an abelian intertwining algebra structure with an $SL(2, \mathbb{C})$-action. Moreover, as an $SL(2, \mathbb{C}) \otimes L_k(\mathfrak{sl}_2)$-module,

$$V^{(p)} \cong \bigoplus_{n=0}^{\infty} \rho_n \otimes L_n^{(p)}.$$  

Corollary 9. Let $k = -2 + \frac{1}{p}$ for $p \in \mathbb{Z}_{\geq 1}$. Then $KL_k \cong \text{Rep}_{A,F,\Omega}(SU(2))$ as braided tensor categories for some abelian 3-cocycle $(F, \Omega)$ and in particular $KL_k$ is rigid.

Proof. From [59], the category $KL_k$ of ordinary $L_k(\mathfrak{sl}_2)$-modules is semisimple with simple objects $L_n$ for $n \in \mathbb{Z}_{\geq 0}$. Furthermore, it was shown in [44] that $KL_k$ has a braided tensor category structure. Thus by Theorem 7 that $KL_k \cong \text{Rep}_{A,F,\Omega}(SU(2))$ as braided tensor categories for some abelian 3-cocycle $(F, \Omega)$ and in particular $KL_k$ is rigid. 

Let $KL_k^{\text{even}}$ be the subcategory of $KL_k$ whose simple objects are the $L_{2n}$ with $n \in \mathbb{Z}_{\geq 0}$. Then the $SL(2, \mathbb{C})$-action on $V^{(p)}$ induces a $PSL(2, \mathbb{C})$-action on the
The orbifold \((\mathcal{V}(p))_{\mathbb{Z}_2}\) and the orbifold decomposes accordingly
\[
(\mathcal{V}(p))_{\mathbb{Z}_2} \cong \bigoplus_{n=0}^{\infty} \rho_{2n} \otimes L_{2n}^{(p)}.
\]

The orbifold \((\mathcal{V}(p))_{\mathbb{Z}_2}\) is a vertex operator algebra and hence there is no need for any abelian cocycle, i.e.

**Corollary 10.** \(\text{KL}_{k}^{\text{even}} \cong \text{Rep}(SO(3))\) as symmetric tensor categories.

**Remark 3.** Consider the case of \(A(1)\), then \((A^{(1)})^{SU(2)}\) is \(L_{\text{Vir}}(1, 0)\) and its representation category is braided equivalent to \(\text{Rep}_{A,F,\Omega}(SU(2))\) for some abelian 3-cocycle \((F, \Omega)\) [68, Example 4.11]. The \(\mathbb{Z}_2\) orbifold of \(A(1)\) is just \(L_1(\mathfrak{sl}_2)\) and it decomposes as
\[
L_1(\mathfrak{sl}_2) \cong \bigoplus_{n=0}^{\infty} \rho_{2n} \otimes L_{1, n}^{\text{Vir}}(1, n^2).
\]

We can apply Theorem 7, since \(\mathcal{V}(p)\) and \(\mathcal{V}(p) \otimes F_{-\frac{p}{2}}\) are simple vertex operator algebras by Proposition 3 and since \(\text{KL}_{k}\) as well as Fock modules of the Heisenberg vertex operator algebra have tensor category structure ([44] and [31, Theorem 2.3]) and hence also their extensions have this property [32]. Especially we have the following fusion rules

**Corollary 11.** The modules \(V^{(p)}_{\ell}\) are simple currents for the vertex operator algebra \(\mathcal{V}^{(p)} U(1)\) and the \(\tilde{R}^{(p)}_{\ell}\) are simple currents for \(\tilde{R}^{(p)}\). Especially the fusion rules
\[
V^{(p)}_{\ell} \boxtimes V^{(p)}_{\ell'} \cong V^{(p)}_{\ell + \ell'}, \quad \text{and} \quad \tilde{R}^{(p)}_{\ell} \boxtimes \tilde{R}^{(p)}_{\ell'} \cong \tilde{R}^{(p)}_{\ell + \ell'}
\]
hold.
We can thus use the theory of infinite order simple current extensions [30] and have that \( \tilde{\mathcal{R}}(p) \) is an infinite order simple current extension of \( \mathcal{V}(p)^{U(1)} \otimes M_\varphi(1) \). Moreover it is generated by the fields corresponding to the generating simple currents, that is the fields in \( \mathcal{V}(p)^{U(1)} \otimes M_\varphi(1) \) and \( \mathcal{V}(p)^{-1} \otimes M_\varphi(1) \) together with the ones of \( \mathcal{V}(p)^{U(1)} \otimes M_\varphi(1) \). But this is exactly how we introduced \( \mathcal{R}(p) \), so that we can conclude that

**Corollary 12.** The vertex operator algebras

\[ \tilde{\mathcal{R}}(p) = \mathcal{R}(p). \]

**6.4. Uniqueness of \( \mathcal{V}_0(p) \) and \( \mathcal{R}(p) \).** Consider the case \( p = 1 \), so that \( (\mathcal{A}(1))^{U(1)} = \mathcal{A}_0^{(1)} = M(1) \) is nothing but the simple rank one Heisenberg vertex operator algebra. The latter is strongly generated by the weight one field and this means that there is a unique simple vertex operator algebra structure up to isomorphism. Recall that there is a one-to-one correspondence between commutative and associative algebras in a vertex tensor category \( \mathcal{C} \) of a vertex operator algebra \( \mathcal{V} \) and vertex operator algebra extensions of \( \mathcal{V} \) in \( \mathcal{C} [55] \). Moreover the vertex tensor category of modules of the extended vertex operator algebra that lie in \( \mathcal{C} \) is braided equivalent to the category of local modules for the algebra object that lie in \( \mathcal{C} [32] \). Especially the extended vertex operator algebra is simple if and only if the corresponding algebra object is simple as a module for itself.

The braided equivalence \( KL_k^{\text{even}} \cong \text{Rep}(SO(3)) \) together with the unique simple vertex operator algebra structure on \( \mathcal{A}_0^{(1)} \) and the fact that the category of Virasoro modules whose simple objects are the \( L_{\text{Vir}}(1,n^2) \) is equivalent to \( \text{Rep}(SO(3)) \) as symmetric tensor categories (by Remark [3]) imply that

**Theorem 8.**

1. **Up to isomorphism, there is a unique commutative and associative algebra structure on the object**

\[ \mathcal{A} = \bigoplus_{s=0}^{\infty} \rho_{2s} \]

in \( \text{Rep}((SO(3))) \) such that \( \mathcal{A} \) is simple as a module for itself.

2. For every \( p \) in \( \mathbb{Z}_{\geq 1} \) and up to isomorphism there is a unique simple vertex operator algebra structure on

\[ \bigoplus_{s=0}^{\infty} \mathcal{L}_2(p) \]

**Corollary 13.** For \( p \) in \( \mathbb{Z}_{\geq 1} \) and \( k = -2 + \frac{1}{p} \), let \( \mathcal{X} \) be a simple vertex operator algebra, such that

\[ \mathcal{X} \cong \bigoplus_{n \in \mathbb{Z}} \mathcal{V}_n^{(p)} \otimes M_\varphi(1,-n) \]

as \( L_k(\mathfrak{sl}_2) \otimes M(1) \)-module. Then \( \mathcal{X} \cong \mathcal{R}(p). \)

**Proof.** The Heisenberg coset of \( \mathcal{X} \) is

\[ \text{Com}(M_\varphi(1), \mathcal{X}) \cong \mathcal{V}_0^{(p)}. \]
This is an isomorphism of vertex operator algebras by Theorem 8. Simple current extensions to $\frac{1}{2}\mathbb{Z}$-graded vertex operator (super)algebras are unique up to isomorphism by combining [28, Proposition 2.15] with [30, Remark 3.11]. Hence any two simple current extensions of $Y^{(p)}_L \otimes M(1)$ of the form $\bigoplus_{n \in \mathbb{Z}} Y_n^{(p)} \otimes M_p(1, -n)$ must be isomorphic as vertex operator algebras.

There is a similar uniqueness result for $B^{(p)}$ and $A_0^{(p)}$-algebras. For this we however need to assume the existence of vertex tensor category structure on the Virasoro modules appearing in $B^{(p)}$. This existence result is work in progress [45].

**Theorem 9.** [15] For any complex number $c$, the category of $C_1$-cofinite modules for $L^{\text{Vir}}(c, 0)$ has a vertex tensor category structure.

Let $\mathcal{O}_{1,p}$ be the category of $L^{\text{Vir}}(c_1, 0)$-modules whose objects are direct sums of the simple objects $L^{\text{Vir}}(c_1, h_{1,n})$ and $\mathcal{O}^{\text{even}}_{1,p}$ be the subcategory of $\mathcal{O}_{1,p}$ whose objects are direct sums of the simple objects $L^{\text{Vir}}(c_1, h_{1,2n+1})$. By Theorem 9 these two categories have braided tensor category structure. Since

$$\mathcal{A}^{(p)} \cong \bigoplus_{n=0}^{\infty} \rho_n \otimes L^{\text{Vir}}(c_1, h_{1,n+1})$$

as $SU(2) \otimes L^{\text{Vir}}(c_1, 0)$-module, we have in analogy to Corollaries 9 and 10:

**Corollary 14.** For $p$ in $\mathbb{Z}_{\geq 1}$, the category $\mathcal{O}_{1,p} \cong \text{Rep}_{A,F,\Omega}(SU(2))$ as braided tensor categories for some abelian 3-cocycle $(F, \Omega)$ and in particular $\mathcal{O}_{1,p}$ is rigid. Moreover, $\mathcal{O}^{\text{even}}_{1,p} \cong \text{Rep}(SO(3))$ as symmetric tensor categories.

By Theorem 8 there is a unique simple vertex operator algebra structure on $\bigoplus_{n=0}^{\infty} L^{\text{Vir}}(c_1, h_{1,2n+1})$ and so any simple vertex operator algebra that is isomorphic to the singlet algebra $A_0^{(p)}$ as a module for the Virasoro algebra is isomorphic to the singlet algebra $A_0^{(p)}$ as a vertex operator algebra. The same argument as for Corollary 13 applies and we also get uniqueness of $B^{(p)}$-algebras:

**Corollary 15.** Let $Y$ be a simple vertex operator algebra that is isomorphic to $B^{(p)}$ as a module for the Virasoro algebra times the Heisenberg algebra. Then $Y \cong B^{(p)}$.

We remark that $h_{1,n} > h_{1,m}$ for $n < m$ and hence it follows inductively that any module $M$ for $L^{\text{Vir}}(c_1, 0)$ whose character (graded by conformal weight) coincides with $\bigoplus_{n=0}^{\infty} L^{\text{Vir}}(c_1, h_{1,2n+1})$ is actually isomorphic as a Virasoro module to $\bigoplus_{n=0}^{\infty} L^{\text{Vir}}(c_1, h_{1,2n+1})$. This in turn means that above corollary can be improved to:

**Corollary 16.** Let $Y$ be a simple vertex operator algebra that is a module for $L^{\text{Vir}}(c_1, 0)$ times a rank one Heisenberg algebra, such that the character graded by conformal weight and Heisenberg weight of $Y$ coincides with the one of $B^{(p)}$. Then $Y \cong B^{(p)}$.

As a final remark in this section, let us note that a similar uniqueness Theorem applies for the triplet algebras $W^{(p)}$. For this let $Z$ be a simple vertex operator
algebra of central charge $c_{1,p}$. Assume that $U(1)$ is a subgroup of the automorphism group of $Z$ so that

$$Z \cong \bigoplus_{n \in \mathbb{Z}} C_n \otimes Z_n$$

as $U(1) \otimes Z^{U(1)}$ module. Then the $Z_n$ are all simple $Z^{U(1)}$ modules by [46] and the $Z_n$ are simple currents by [31, Thm. 3.1] (since vertex tensor category assumption is satisfied by [45] together with [32]). We assume that the character of $Z$, graded by $U(1)$ weight and also by conformal weight coincides with the corresponding graded character of $\mathcal{W}^{(p)}$. Then as just remarked this means that the two algebras are already isomorphic as $U(1) \otimes \text{Vir}(c_{1,p},0)$-modules. Especially our uniqueness Theorem thus implies that $Z^{U(1)} = Z_0$ is isomorphic to the singlet vertex operator algebra $A^{(p)}_0$ as a vertex operator algebra. Uniqueness of simple current extensions then implies that $Z \cong \mathcal{W}^{(p)}$ as vertex operator algebras. We summarize:

**Corollary 17.** Let $Z$ be a simple vertex operator algebra of central charge $c_{1,p}$ with $U(1)$ as subgroup of automorphism such that the character, graded by $U(1)$ weight and conformal weight, coincides with the graded character of $\mathcal{W}^{(p)}$. Then $Z \cong \mathcal{W}^{(p)}$ as vertex operator algebras.

7. $\mathcal{W}$-algebras and chiral algebras of Argyres-Douglas theories

Let $g = \mathfrak{sl}_n$ and let $f$ be a nilpotent element corresponding to the partition $(n-2, 1, 1)$ of $n$. This means that there exists an $\mathfrak{sl}_2$-triple $e, h, f$ in $\mathfrak{sl}_n$ such that the standard representation of $\mathfrak{sl}_n$ decomposes as $\rho_{n-2} \oplus \rho_1 \oplus \rho_1$ under this $\mathfrak{sl}_2$-action. We denote the corresponding simple $\mathcal{W}$-algebra at level $\ell$ by $\mathcal{W}_\ell(\mathfrak{sl}_n, f)$. It contains an affine subalgebra of $\mathfrak{gl}_2$ at level $k = \ell + n - 3$ as subalgebra. We set $p = n - 1$ and $\ell = -\frac{p^2 - 1}{p}$ so that $k = \ell + p - 2 = -2 + \frac{1}{p}$. Note that $\ell + n = \frac{p^2 - 1}{p}$ is a boundary admissible level. The main result of [25, Theorem 5.7] tells us that for this choice $\mathcal{W}_\ell(\mathfrak{sl}_{p+1}, f) \cong \mathcal{R}^{(p)}$ as $L_k(\mathfrak{sl}_2) \otimes M(1)$-modules. By Corollary 13 we have that

**Theorem 10.** Let $\ell = -\frac{p^2 - 1}{p}$. Then $\mathcal{W}_\ell(\mathfrak{sl}_{p+1}, f) \cong \mathcal{R}^{(p)}$ as vertex operator algebras.

Argyres-Douglas theories are four dimensional $\mathcal{N} = 2$ superconformal field theories. They have associated chiral algebras, that are actually vertex operator algebras [24]. The theories are characterized by pairs of Dynkin diagrams and we are interested in the case of $(A_1, D_{2p})$, see [22, 33]. Usually not much is known about these chiral algebras. However in this case, the Schur-index, that is the character of the chiral algebra is known and it agrees with the character of $\mathcal{W}_\ell(\mathfrak{sl}_{p+1}, f)$ by [25, Theorem 5.7]. Moreover the flavour symmetries of these algebras are known and in this case the translation is that the chiral algebra is an extension of $L_k(\mathfrak{sl}_2) \otimes M(1)$ with $k = -2 + \frac{1}{p}$. The question of simplicity has not appeared in the physics literature yet, but it is a natural requirement that chiral algebras of these gauge theories will most often be simple. Let us summarize

**Remark 4.** The chiral algebra $\mathcal{X}$ of $(A_1, D_{2p})$ Argyres-Douglas theories has the properties
(1) $X$ is a simple vertex operator algebra,
(2) $X$ is an extension of $L_k(\mathfrak{sl}_2) \otimes M(1)$ with $k = -2 + \frac{1}{p}$.
(3) The character of $X$ coincides with the one of $R^{(p)}$.

**Theorem 11.** $R^{(p)}$ is the chiral algebra of $(A_1, D_{2p})$ Argyres-Douglas theories.

Analogous results hold for the $B^{(p)}$-algebra. Theorem 27 of [15] says that the character of $B^{(p)}$ and of the simple subregular $W$-algebra of $\mathfrak{sl}_{p-1}$, $W_{\ell}(\mathfrak{sl}_{p-1}, f_{\text{sub}})$, at level $\ell = 1 - p + \frac{1}{p}$ coincide. The central charges of the Virasoro subalgebras also coincide so that Corollary 16 applies, i.e.

**Theorem 12.** Let $\ell = -\frac{(p-1)^2}{p}$. Then $W_{\ell}(\mathfrak{sl}_{p-1}, f_{\text{sub}}) \cong B^{(p)}$ as vertex operator algebras.

The chiral algebra of $(A_1, A_{2p-3})$ Argyres-Douglas theories is a vertex operator algebra whose character and central charge coincide with the corresponding data of the $B^{(p)}$-algebra [13, 22, 25]. Again it is natural to require it to be a simple vertex operator algebra.

**Remark 5.** The chiral algebra $\mathcal{Y}$ of $(A_1, A_{2p-3})$ Argyres-Douglas theories has the properties

(1) $\mathcal{Y}$ is a simple vertex operator algebra,
(2) $\mathcal{Y}$ is an extension of $L^{\text{Vir}}(c, t, 0) \otimes M(1)$.
(3) The character of $\mathcal{Y}$ coincides with the one of $B^{(p)}$.

The uniqueness result (Corollary 16) tells us that

**Theorem 13.** $B^{(p)}$ is the chiral algebra of $(A_1, A_{2p-3})$ Argyres-Douglas theories.

8. Quantum Hamilton Reduction

The aim of this section is to prove that $B^{(p)}$ is as a vertex operator algebra the quantum Hamiltonian reduction of $R^{(p)}$ and also that $A^{(p)}$ is the quantum Hamiltonian reduction of $\mathcal{Y}^{(p)}$. For this we need to understand the reduction of modules of $\Pi(0)$ first.

8.1. Reduction of $\Pi(0)$-modules. Recall that $\Pi(0) = M(1) \otimes \mathbb{C}[Z_c]$, and we have modules $\Pi_{\ell}(\lambda) = \Pi(0, e^{\ell \frac{c}{p} + \lambda c})$. We set $\Pi(\lambda) := \Pi_0(\lambda)$.

In this section we shall introduce the Drinfeld-Sokolov reduction cohomology for $V(n)$-modules and apply for modules which we constructed above, where $\mathfrak{n} = \mathbb{C} e \subset \mathfrak{sl}_2$ and $V(n)$ is the vertex subalgebra of $V_k(\mathfrak{sl}_2)$ generated by $e(z)$. Let $\mathcal{F}$ be the fermionic vertex superalgebra generated by the fields

$$\Psi^+(z) = \sum_{n \in \mathbb{Z}} \Psi^+ (n) z^{-n-1}, \quad \Psi^- (z) = \sum_{n \in \mathbb{Z}} \Psi^- (n) z^{-n}$$

such that the components of the fields $\Psi^\pm (z)$ satisfy the anti-commutation relation for the Clifford algebra

$$\{ \Psi^\pm (n), \Psi^\pm (m) \} = 0, \quad \{ \Psi^\pm (n), \Psi^\mp (m) \} = \delta_{n+m, 0} \quad (n, m \in \mathbb{Z}).$$

The conformal vector

$$\omega_{\text{fer}} = \Psi^- (-1) \Psi^+ (-1) 1$$
defines on \( F \) the structure of a vertex operator superalgebra with central charge \( c = -2 \). The fermionic vertex superalgebra \( F \) has the charge decomposition by charge of \( \Psi^\pm(n) \). We have \( F = \bigoplus_{i \in \mathbb{Z}} F^i \). By using the boson-fermion correspondence \( F \) can be realized as lattice vertex superalgebra

\[
F \xrightarrow{\sim} V_{\mathbb{Z},\phi} = M_\phi(1) \otimes \mathbb{C}[\mathbb{Z}\phi], \quad \Psi^\pm \mapsto e^{\pm\phi}, \quad \langle \phi, \phi \rangle = 1,
\]

and conformal vector is given by

\[
\omega_\text{fer} = \frac{1}{2}(\phi(-1)^2 - \phi(-2))1.
\]

Given a \( V(n) \)-module \( M \), set the complex \( C(M) = M \otimes F \) and the differential \( d_{DS(0)} \), where

\[
d_{DS} = (e + 1) \otimes e^\phi \in V(n) \otimes F.
\]

Then \( C(M) = \bigoplus_{i \in \mathbb{Z}} C^i(M) \), where \( C^i(M) = M \otimes F^i \). Since \( d_{DS(0)} : C^i(M) \subset C^{i+1}(M) \) and \( d_{DS(0)}^2 = 0 \), the pair \( (C(M), d_{DS(0)}) \) forms the cochain complex. The Drinfeld-Sokolov reduction \( H^*_\text{DS}(M) \) for \( M \) is defined by

\[
H^*_\text{DS}(M) = H^*(C(M), d_{DS(0)}).
\]

Since \( d_{DS(0)} \) is a vertex operator of \( d_{DS} \), if \( M \) is a vertex superalgebra with a map \( V(n) \to V \) of vertex superalgebras, \( H_{DS}(V) \) inherits a vertex superalgebra structure from that of \( C(V) \). Moreover, if \( M \) is a \( V \)-module, \( H_{DS}(M) \) is a \( H_{DS}(V) \)-module.

Consider now the Drinfeld-Sokolov reductions for the vertex algebra \( \Pi(0) \) with a vertex algebra homomorphism

\[
V(n) \ni e \mapsto e^c \in \Pi(0).
\]

and its irreducible modules \( \Pi(\lambda), \Pi_r(\lambda) \).

**Proposition 7.**

1. \( H_{DS}(\Pi(0)) = \delta_{1,0} C1 \),
2. \( H_{DS}(\Pi(\lambda)) = \delta_{1,0} C e^{\lambda(\alpha + \beta)} \),
3. \( H_{DS}(\Pi_r(\lambda)) = 0 \) if \( r \neq 0 \).

**Proof.** Let

\[
\omega_{C(\Pi(0))} = \frac{1}{2}(-1)^d(-1) + \omega_\text{fer}.
\]

Then \( \omega_{C(\Pi(0))} \) is the conformal vector of \( C(\Pi(0)) \) with the central charge 0. The conformal weights of \( c, d, e^c, e^{-c}, e^\phi, e^{-\phi} \) are 1, 1, 0, 0, 1, 0 respectively. Denote by \( L^\Pi(0) \) the gradation operator with respect to \( \omega_{C(\Pi(0))} \), and by \( C(\Pi(0))_l \) the homogeneous subspace of \( C(\Pi(0)) \) with the conformal weight \( l \in \mathbb{Z} \). Then \( C(\Pi(0)) = \bigoplus_{l \in \mathbb{Z}} C(\Pi(0))_l \). Now, we have

\[
\omega_{C(\Pi(0))} = d_{DS(0)} \cdot \frac{1}{2} (-d(-1)\phi(-1) + \phi(-1)^2 - \phi(-2)) e^{-c-\phi} \in \text{Im} d_{DS(0)}.
\]

Since \( \text{Im} d_{DS(0)} \) is an ideal of \( \text{Ker} d_{DS(0)} \), if \( v \) is a vector in \( \text{Ker} d_{DS(0)} \cap C(\Pi(0))_l \) with \( l \neq 0 \), we have

\[
v = \frac{1}{l} L^\Pi(0)v \in \text{Im} d_{DS(0)}.
\]
Thus, $H_{DS}(\Pi(0)) = H(C(\Pi(0)), d_{DS(0)})$. Using the facts that
\[ C(\Pi(0))_0 = \text{Span}\{e^{ic}, e^{ic-\phi} \mid i \in \mathbb{Z}\} \]
and that
\[ d_{DS(0)} \cdot e^{ic-\phi} = e^{ic} + e^{(i+1)c} \neq 0, \quad d_{DS(0)} \cdot e^{ic} = 0, \]
it follows that
\[ \text{Ker} d_{DS(0)} \cap C(\Pi(0))_0 = \text{Span}\{e^{ic} \mid i \in \mathbb{Z}\}, \]
\[ \text{Im} d_{DS(0)} \cap C(\Pi(0))_0 = \text{Span}\{e^{ic} + e^{(i+1)c} \mid i \in \mathbb{Z}\}. \]
Hence
\[ H_{DS}(\Pi(0)) = H_{DS}^0(\Pi(0)) = \frac{\text{Ker} d_{DS(0)} \cap C(\Pi(0))_0}{\text{Im} d_{DS(0)} \cap C(\Pi(0))_0} = \mathbb{C}1. \]

Similarly we get $H_{DS}(\Pi(\lambda)) = \mathbb{C}e^{(\alpha+\beta)}. This proves assertion (1) and (2).

Next we notice that $(c-1) \in \text{Ker} d_{DS(0)}$, and since it has conformal weight 1, it is in $\text{Im} d_{DS(0)}$. Since $\text{Im} d_{DS(0)}$ is an ideal in $\text{Ker} d_{DS(0)}$, we conclude that for each $w \in \Pi_r(\lambda)$:
\[ w = \frac{1}{r} c(0) w \in \text{Im} d_{DS(0)}. \]
This proves assertion (3). \qed

Let $c \in \mathbb{C}, U_1$ any $V^{\text{Vir}}(c, 0)$-module and $U^2$ is any $L^{\text{Vir}}(c, 0)$-module. As above, we will identify $U_i \otimes \Pi(0)$ with a $V(\mathfrak{n})$-module only acting on the second factor $\Pi(0)$. Using Proposition 7, the following is clear:

Corollary 18.

(1) Assume that $k \in \mathbb{C} \setminus \{-2\}$ and that $U_1$ is any $V^{\text{Vir}}(c_k, 0)$-module. Then
\[ H^i_{DS}(V^{\text{Vir}}(c, 0) \otimes \Pi(0)) \cong \delta_{i,0} V^{\text{Vir}}(c, 0) \]
and
\[ H^i_{DS}(U_1 \otimes \Pi(\lambda)) \cong \delta_{i,0} U_1 \]
as $V^{\text{Vir}}(c_k, 0)$-modules.

(2) Assume that $V$ is any vertex operator algebra extension of $V^{\text{Vir}}(c_k, 0)$. Then as $V^{\text{Vir}}(c_k, 0)$-modules
\[ H^i_{DS}(V \otimes \Pi(0)) = \delta_{i,0} V. \]

8.2. Quantum Hamilton reduction of $V^{(p)}$ and $R^{(p)}$. In this section we will prove

Theorem 14. As vertex operator algebras,
\[ H^0_{DS}(R^{(p)}) = B^{(p)}, \]
and as abelian intertwining algebras,
\[ H^0_{DS}(V^{(p)}) = A^{(p)}. \]
First we need to recall some known statements

**Lemma 4.** As vertex operator algebras,

\[ H_0^{DS}(V^k(s\mathfrak{sl}_2)) \cong V^{Vir}(c_{1,p}, 0). \]

As \( V^{Vir}(c_{1,p}, 0) \)-modules

\[ H_0^{DS}(L_{s}^{(p)}) \cong L^{Vir}(c_{1,p}, h_{1,s+1}); \quad H_i^{DS}(L_{s}^{(p)}) = 0, \quad i \neq 0 \]

and

\[ H_0^{DS}(R^{(p)}) \cong B^{(p)}, \quad H_0^{DS}(V^{(p)}) \cong A^{(p)}, \quad H_i^{DS}(R^{(p)}) = H_i^{DS}(V^{(p)}) = 0, \quad i \neq 0. \]

**Proof.** The first statement follows from [19], the second one follows from Theorem 6.7.1 and Theorem 6.7.4 in [19], and the last one follows from the decomposition of \( R^{(p)}, B^{(p)}, V^{(p)} \) and \( A^{(p)} \) as Virasoro algebra \( V^{Vir}(c_{1,p}, 0) \)-modules. See also Proposition 5.10 of [25]. □

By Proposition [1] we have the short exact sequence of \( V^{(p)} \)-modules:

\[ 0 \to V^{(p)} \xrightarrow{\varphi} (A^{(p)} \otimes \Pi(0))_{Z_2} \xrightarrow{S} \text{Im}(S) \to 0. \]

(32)

Then the cohomology functor \( H^{DS}(?) \) yields the long exact sequence of \( H_0^{DS}(V^{(p)} \)-modules from the exact sequence (32). From Lemma 4 as Virasoro algebra \( V^{Vir}(c_{1,p}, 0) \)-modules

\[ H_i^{DS}(V^{(p)}) \cong \delta_{i,0} A^{(p)}. \]

Also, it follows from Corollary [18] and both of equations (20) and \( \Pi(0)_{\frac{1}{2}} = \Pi(\frac{1}{2}) \) that as Virasoro algebra \( V^{Vir}(c_{1,p}, 0) \)-modules

\[ H_i^{DS}(A^{(p)} \otimes \Pi(0))_{\frac{1}{2}} \cong \delta_{i,0} A^{(p)}. \]

Thus, the long exact sequence induced from (32) gives rise to the exact sequence of \( V^{Vir}(c_{1,p}, 0) \)-modules

\[ 0 \to H_0^{-1}(\text{Im}(S)) \to H_0^{DS}(V^{(p)}) \xrightarrow{\phi} H_0^{DS}(A^{(p)} \otimes \Pi(0))_{\frac{1}{2}} \to H_0^{DS}(\text{Im}(S)) \to 0, \]

and the vanishing results

\[ H_i^{DS}(\text{Im}(S)) = 0, \quad i \neq 0, -1. \]

\[ (34) \]

\( \phi \) is a homomorphism \( \phi : A^{(p)} \to A^{(p)} \) of abelian intertwining algebras and it actually is an isomorphism:

**Lemma 5.** \( \phi : A^{(p)} \to A^{(p)} \) is an isomorphism of abelian intertwining algebras.

**Proof.** As \( \tilde{\phi} \) in (32) is a homomorphism of abelian intertwining algebras, so is \( \phi \). Thus, it is enough to show that \( \phi \) is an isomorphism of \( L^{Vir}(c_{1,p}, 0) \)-modules. Notice that \( \Pi \left( \frac{s}{2} \right) \) for \( s \in \mathbb{Z} \) is isomorphic to \( \Pi(0) = \Pi(0)_{\frac{1}{2}} \) (resp. \( \Pi \left( \frac{1}{2} \right) = \Pi(0)_{\frac{1}{2}} \))
as a $\Pi(0)$-module if $s$ is even (resp. odd). First, the injective map $\tilde{\phi}_{s,j} : L_s^{(p)} \hookrightarrow L_{\text{Vir}}(c_1,p; h_{1,s+1}) \otimes \Pi(\frac{s}{2})$ given in Proposition 2 yields an exact sequence

\begin{equation}
0 \to L_s^{(p)} \xrightarrow{\delta_{s,j}} L_{\text{Vir}}(c_1,p; h_{1,s+1}) \otimes \Pi(\frac{s}{2}) \xrightarrow{S_s} \text{Im } S_s \to 0,
\end{equation}

where $\text{Im } S_s = (L_{\text{Vir}}(c_1,p; h_{1,s+1}) \otimes \Pi(\frac{s}{2})) / L_s^{(p)}$ and $S_s$ is the canonical projection. Consider the map

$$\phi_{s,j} : H_{DS}^0(L_s^{(p)}) \to L_{\text{Vir}}(c_1,p; h_{1,s+1})$$

induced from $\tilde{\phi}_{s,j}$ through the Drinfeld-Sokolov reduction functor $H_{DS}(? \frac{s}{2})$. By Proposition 4 we have

$$H_{DS}^i(L_{\text{Vir}}(c_1,p; h_{1,s+1}) \otimes \Pi(\frac{s}{2})) = L_{\text{Vir}}(c_1,p; h_{1,s+1}) \otimes H_{DS}^i(\Pi(\frac{s}{2})) = \delta_{s,0} L_{\text{Vir}}(c_1,p; h_{1,s+1}) = H_{DS}^0(L_s^{(p)}).$$

Recall that $v_{1,s+1,j} \otimes e^{\tilde{z}_c}$ is the image of the highest weight vector of $L_s^{(p)}$ by $\tilde{\phi}_{s,j}$. Since

$$d_{DS(0)}(v_{1,s+1,j} \otimes e^{\tilde{z}_c}) = 0$$

for the differential $d_{DS(0)}$ of the Drinfeld-Sokolov reduction for $L_{\text{Vir}}(c_1,p; h_{1,s+1}) \otimes \Pi(\frac{s}{2})$, the vector $v_{1,s+1,j} \otimes e^{\tilde{z}_c}$ is in $\text{Ker } d_{DS(0)}$ with degree 0. Moreover, the vector $v_{1,s+1,j} \otimes e^{\tilde{z}_c}$ is non-zero in $H_{DS}^0(L_{\text{Vir}}(c_1,p; h_{1,s+1}) \otimes \Pi(\frac{s}{2}))$ by the proof of Proposition 4. Thus, the map $\phi_{s,j}$ is a non-trivial homomorphism of $L_{\text{Vir}}(c_1,p; 0)$-modules, and so is an isomorphism. Now, we have the decompositions

$$\mathcal{Y}^{(p)} = \bigoplus_{s=0}^{\infty} \rho_s \otimes L_s^{(p)}, \quad \mathcal{A}^{(p)} \otimes \Pi(\frac{s}{2}) = \bigoplus_{s=0}^{\infty} \rho_s \otimes L_{\text{Vir}}(c_1,p; h_{1,s+1}) \otimes \Pi(\frac{s}{2})$$

as $\mathfrak{sl}_2 \otimes L_{\mathfrak{sl}_2}$-modules. Set the projections

$$p_s : \mathcal{Y}^{(p)} \to \rho_s \otimes L_s^{(p)},$$

$$q_s : \mathcal{A}^{(p)} \otimes \Pi(\frac{s}{2}) \to \rho_s \otimes L_{\text{Vir}}(c_1,p; h_{1,s+1}) \otimes \Pi(\frac{s}{2}),$$

$$r_{s,j} : \rho_s \to \rho_{s,j},$$

where $\rho_{s,j}$ is the weight space of $\rho_s$ with the weight $2j - s$ for $j = 0, \ldots, s$, and let

$$p_{s,j} = (r_{s,j} \otimes \text{id}_{L_s^{(p)}}) \circ p_s,$$

$$q_{s,j} = (r_{s,j} \otimes \text{id}_{L_{\text{Vir}}(c_1,p; h_{1,s+1}) \otimes \Pi(\frac{s}{2})}) \circ q_s.$$
Hence, the exact sequence $\text{(32)}$ consists of the direct sum of the exact sequence $\text{(35)}$ for all $s, j$. We conclude that

$$\bar{\phi} = \bigoplus_{s=0}^{\infty} \bigoplus_{j=0}^{s} \bar{\phi}_{s,j},$$

which implies that

$$\phi = \bigoplus_{s=0}^{\infty} \bigoplus_{j=0}^{s} \phi_{s,j}.$$  

As each $\phi_{s,j}$ is an isomorphism of $L^{Vir}(c_1, p, 0)$-modules, so is $\phi$. This completes the proof.  

As a consequence of Lemma 5, the exact sequence $\text{(33)}$ and the formulae $\text{(34)}$, we have

$$H_{DS}(\text{Im}(S)) = 0.$$  

Again from the long exact sequence $\text{(33)}$, the vertex operator algebra homomorphism $\phi$ is an isomorphism. We proved Theorem 14 for the algebra $V(p)$, the proof for $R(p)$ is similar.

### 8.3. Reduction of $L_1(\mathfrak{psl}(2|2))$

Recall that

$$L_1(\mathfrak{psl}(2|2)) = V_0^{(1)} \otimes L_1(\mathfrak{sl}_2) \bigoplus V_1^{(1)} \otimes L_1(\omega_1).$$

Since

$$H_{DS}(V_0^{(1)}) = A_0^{(1)} = L_1(\mathfrak{sl}_2), \quad H_{DS}(V_1^{(1)}) = A_1^{(1)} = L_1(\omega_1),$$

we get

$$H_{DS}(L_1(\mathfrak{psl}(2|2))) \cong L_1(\mathfrak{sl}_2)^{\otimes 2} \oplus L_1(\omega_1)^{\otimes 2} \cong F_1^{\otimes 2},$$

where $F_1$ is the Clifford vertex algebra (bc system) of central charge $c = 1$.

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D.A.: Department of Mathematics, Faculty of Science, University of Zagreb, Bijenička 30, 10 000 Zagreb, Croatia; adamovic@math.hr
T.C.: Department of Mathematical and Statistical Sciences, University of Alberta, 632 CAB, Edmonton, Alberta, Canada T6G 2G1; creutzig@ualberta.ca

N.G.: Department of Mathematical and Statistical Sciences, University of Alberta, 632 CAB, Edmonton, Alberta, Canada T6G 2G1; genra@ualberta.ca

J.Y.: Department of Mathematical and Statistical Sciences, University of Alberta, 632 CAB, Edmonton, Alberta, Canada T6G 2G1; jinwei2@ualberta.ca