On the Fractional Mean Value

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Abstract

This work, dealt with the classical mean value theorem and took advantage of it in the fractional calculus. The concept of a fractional critical point is introduced. Some sufficient conditions for the existence of a critical point is studied and an illustrative example relevant to the concept of the time dilation effect is given. The present paper also includes, some connections between convexity (and monotonicity) with fractional derivative in the Riemann-Liouville sense.

Keywords: Fractional derivative, Fractional critical point, Fractional mean value, convexity.

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1 Introduction

Fractional calculus which dates back to more than 300 years ago, has became one of the thriving areas that is supported by the recent seminal papers together with some special phenomenological viewpoints. Evidently, there are many dynamic mechanisms in the universe (such as anomalous diffusion and fractal dynamics) that are much more better to be described via the fractional calculus than the classical one (see e.g. [5, 9, 11]).

In the present work, the concept of the mean value is nothing more than one is given in the classical calculus. However, the prefix fractional, is to emboss its concept for the next purposes which will be given in sections 2-4. Also it has been several works in the fractional context that are devoted to this concept. The generalized mean value theorem was first given
in [10] and this is about a generalized Taylor formula in a fractional case. In [4], the similar mean value theorem is obtained but it is more likely to the classical case. This theorem in both mentioned forms have been used in many papers. In [5], the Cauchy type mean value is obtained. Moreover, the mean value is studied in a more abstract and operational form. Another generalized mean value theorem is proved in [1] with slight changes, compared to the one is obtained in [10].

The accurate relation between convexity and increase of the differential in local sense, depends on the usual order on the real line. Connecting the convexity to the differential (in non-local sense) in the same manner, needs bringing an order on the domains of integrations, since we are dealing with the integro-differential operators. This is done in section 4. Moreover, a regularization is introduced, that upon which, the distribution of the fractional mean values have became uniform. This is because we make use of the fractional mean value in Theorem 4.5.

This paper is organized as follows: In section 2, some preliminaries on the concept of the mean value is introduced and an approximation of it as a root of a polynomial is given. In section 2, the concept of the fractional critical point as the root of the fractional derivative is introduced, and sufficient conditions for its existence are studied. Also, an interpretation of the fractional critical point is given based on the time dilation effect (according to [7]). Connections between fractional derivative and convexity (as mentioned above) and monotonicity are studied in section 3. A fundamental condition for this connection is introduced. This condition, defines some order on the fractional derivatives with respect to the orders on the intervals that are being involved in the Riemann-Liouville integro-differential operator.

2 Fractional mean value

**Definition 2.1.** [2] Let \([a, b]\) be an interval and \(f\) be an integrable function. The left Riemann-Liouville fractional integral and derivative of order \(\alpha \in (0, 1)\), (provided they exist) are defined by

\[
I^\alpha_{a+} f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)}{(x-t)^{1-\alpha}} dt \quad \forall x \in (a, b], \tag{2.1}
\]

\[
D^\alpha_{a+} f(x) := \frac{d}{dx} I^{1-\alpha}_{a+} f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x \frac{f(t)}{(x-t)^{\alpha}} dt \quad \forall x \in (a, b]. \tag{2.2}
\]
where $\Gamma(\alpha)$ is the Gamma function. The space $I_+^\alpha(L^p)$ for $1 \leq p \leq \infty$, is defined by
\[
I_+^\alpha(L^p) := \left\{ f \mid f = I_+^\alpha \phi, \ \phi \in L^p(a, b) \right\}.
\] (2.3)
The left Caputo fractional derivative of a function $f$ (provided it exists) is defined by
\[
C^\alpha_a f(x) := \frac{1}{\Gamma(1-\alpha)} \int_a^x \frac{f'(t)}{(x-t)^\alpha} dt \quad \forall x \in (a, b].
\] (2.4)
The set of absolutely continuous functions, that is denoted by $AC[a, b]$, is the set of all functions $f$, which has representation of the form (2):
\[
f(x) = c + I_+^\alpha \phi, \ c \in \mathbb{R}, \ \phi \in L(a, b).
\] (2.5)

**Proposition 2.2.** [2]

a) If $f \in L^p(a, b)$, $(1 \leq p < +\infty)$, then
\[
D_+^\alpha I_+^\alpha f(x) = f(x).
\] (2.6)
b) If $f \in L^1(a, b)$ and $f_{1-\alpha} = I_+^{1-\alpha} f \in AC[a, b]$, then
\[
I_+^\alpha D_+^\alpha f(x) = f(x) - \frac{f_{1-\alpha}(a)}{\Gamma(\alpha)} (x-a)^{\alpha-1},
\] (2.7)
which holds almost everywhere on $[a, b]$.
c) If $f \in I_+^\alpha(L^p)$, $(1 \leq p \leq \infty)$ then
\[
I_+^\alpha D_+^\alpha f(x) = f(x).
\] (2.8)

**Proposition 2.3.** Assume that $f$ is differentiable on $[a, b]$ with $f(a) = 0$, then

(a) [2],
\[
D_+^\alpha f(x) = C^\alpha D_+^\alpha f(x) = I_+^{1-\alpha} f'(x).
\] (2.9)

(b) [6],
\[
\lim_{\alpha \to 0^+} I_+^\alpha f(x) = f(x).
\] (2.10)

**Proposition 2.4.** [8]

(a) If $f \in AC[a, b]$, then $f_{1-\alpha} \in AC[a, b]$.

(b) Abel’s integral equation is solvable in $L^1(a, b)$ if and only if $f_{1-\alpha} \in AC[a, b]$ and we have
\[
f_{1-\alpha}(x) = \frac{1}{\Gamma(2-\alpha)} \left\{ f(a)(x-a)^{1-\alpha} + \int_a^x f'(t)(x-t)^{1-\alpha} dt \right\}.
\] (2.11)
Assumptions 2.5. We assume that $f \in C^1[a, b]$ with $f(a) = 0$ and $D_{a+}^\alpha f \in C[a, b]$.

In this paper, all statements except Proposition 4.6, presume Assumption 2.5. Also Proposition 2.4, together with Assumption 2.5, yield the existence of $D_{a+}^\alpha f(x)$ for every $\alpha \in (0, 1)$, since $C^1[a, b] \subset AC[a, b]$.

Now we begin with this known lemma in the first semester calculus: Let $\phi \in L^1(a, b)$, then
\[ \exists \xi \in (a, b), \quad f(\xi) \cdot \int_a^b \phi(t) dt = \int_a^b \phi(t) f(t) dt. \] (2.12)

Letting $\phi_x(t) = \frac{(x-t)^{1-\alpha}}{\Gamma(1-\alpha)}$ we have
\[ \exists \xi \in (a, x), \quad f(\xi) \cdot \frac{(x-a)^{1-\alpha}}{\Gamma(2-\alpha)} = \frac{1}{\Gamma(1-\alpha)} \int_a^x \frac{f(t)}{(x-t)^\alpha} dt = I_{a+}^{1-\alpha} f(x). \] (2.13)

We name $\xi$ the fractional mean value. Since $\xi$ is not unique, we define
\[ \xi(x, \alpha) := \sup \left\{ \xi \in (a, x) \mid f(\xi) = \Gamma(2-\alpha) \left( I_{a+}^{1-\alpha} f(x) \right) (x-a)^{\alpha-1} \right\}. \] (2.14)

In general, all possible $\xi$’s are belonging to the set $\Lambda(\alpha, f, x)$ which is defined by
\[ \Lambda(\alpha, f, x) := f^{-1} \left\{ \Gamma(2-\alpha) \left( I_{a+}^{1-\alpha} f(x) \right) (x-a)^{\alpha-1} \right\}. \] (2.15)

The set $\Lambda$ is closed, since $f$ is continuous. Let $g(x) := \Gamma(2-\alpha) \left( I_{a+}^{1-\alpha} f(x) \right) (x-a)^{\alpha-1}$. If $f$ is monotone on $(a, b)$, then for $x \in (a, b)$, there exists a unique $\xi(x, \alpha) \in (a, x)$ for which $f(\xi(x, \alpha)) = g(x)$. In other words, $\forall x \in (a, b), g(x) \in Im(f)$.

Proposition 2.6. Suppose $f$ is monotone on $(a, b)$. Then $\xi(x, \alpha) \in C^1(a, b)$.

The following lemma gives some details about the mean value function $\xi$, with respect to the variable $x$, but under the certain condition.

Proposition 2.7. Suppose that $h : [a, b] \to [a, b]$ is differentiable. Assume that there exists some $x_0 \in (a, b)$ at which $h$ has a local extremum and also $h(x_0) = \xi(x_0)$, where $\xi(x) \in \Lambda(\alpha, f, x)$. Then there exists some $x \in (a, b)$, such that
\[ D_{a+}^\alpha f(x) = D \left( f(h(x)) \cdot \frac{(x-a)^{1-\alpha}}{\Gamma(2-\alpha)} \right). \] (2.16)
Proof. In contrary assume that for every $x \in (a, b)$, we have:

$$D_{a^+}^\alpha f(x) \neq D\left(f(h(x)), \frac{(x-a)^{1-\alpha}}{\Gamma(2-\alpha)}\right).$$  \hfill (2.17)

Letting

$$F(x, h) = I_{a^+}^{1-\alpha} f(x) - f(h(x)), \frac{(x-a)^{1-\alpha}}{\Gamma(2-\alpha)},$$

we have $F(x_0, h(x_0)) = 0$ and $\frac{\partial F}{\partial x}(x_0, h(x_0)) \neq 0$. By Implicit Function theorem, $x$ will be a function of $h$ in the vicinity of $x_0$ and this is impossible since $h'(x_0) = 0$. This proves the assertion of the Lemma. □

Regarding to Proposition 2.7, when a given function $h \in C^1([a, b], [a, b])$, satisfies (2.17) everywhere, one conclude that $h$ is monotone, or $h(x) \neq \xi(x)$ holds everywhere.

Proposition 2.8. Let $f \in C^{n+1}[a, b]$. Then $\xi(\alpha)$ can be estimated by finding the roots of the following polynomial

$$\forall \quad \delta < b - a,
\sum_{j=1}^{n} f^{(j)}(a) \left\{ \frac{\delta^{1-\alpha}}{\Gamma(2-\alpha)j!} x^j - \frac{\delta^{j+1-\alpha}}{\Gamma(j+2-\alpha)} \right\}
- \left[ I_{a^+}^{n+2-\alpha} f^{(n+1)} \right](a + \delta) + o\left(\frac{1}{n!}\right) = 0,$$

where $x = \xi - a$.

Proof. First observe that:

$$f(\xi) \cdot \frac{\delta^{1-\alpha}}{\Gamma(2-\alpha)} = \left[ I_{a^+}^{1-\alpha} f \right](a + \delta) = \sum_{j=0}^{n} f^{(j)} \cdot \frac{\delta^{j+1-\alpha}}{\Gamma(j+2-\alpha)} + \left[ I_{a^+}^{n+2-\alpha} f^{(n+1)} \right](a + \delta). \hfill (2.19)$$

On the other hand, taking into account the expansion of $f(\xi)$ about the point $a$ that is:

$$f(\xi) = \sum_{j=0}^{n} f^{(j)}(a) \cdot \frac{(\xi - a)^j}{j!} + o\left(\frac{1}{n!}\right),$$

and equalizing with $f(\xi)$ in (2.19), gives the result. □
3 Fractional critical point

In this section, the roots of the fractional derivative of a function \( f \), will be studied. This may be thought of as a parallel study to the critical points in the ordinary calculus. For the similarity, we name the root of \( D_a^\alpha f \), the fractional critical point of \( f \).

To find the fractional critical point of a function \( f \) of order \( 1-\alpha \), it is necessary and sufficient to find the nodes, those are intersections of two curves \( (x-a)^\alpha \) and \( f(\xi(x)) \). This can be stated in a rigorous way as follows:

**Lemma 3.1.** Suppose \( f \) is monotone, then

\[
 f(\xi(x)) = (x-a)^\alpha, \quad \xi(x) \in \Lambda(\alpha, f, x)
\]

if and only if \( D_a^{1-\alpha} f(x) = 0 \).

**Proof.** Let \( g(x) = f(-\eta(x)) \), where \( \eta(x) = -\xi(x) \). By Proposition 2.6, \( \eta \) is differentiable. The point \( x_0 \in (a, b) \) satisfies the equation \( g(x_0) = (x_0-a)^\alpha \), if and only if it satisfies the following equation

\[
 (x-a)g'(x) - \alpha g(x) = 0.
\]

Multiplying (3.2) by \( \frac{(x-a)^{\alpha-1}}{\Gamma(1+\alpha)} \) we get:

\[
 0 = \frac{1}{\Gamma(1+\alpha)} \left( -(x_0-a)^\alpha f'(-\eta(x_0))\eta'(x_0) - \alpha(x_0-a)^{\alpha-1} f(-\eta(x_0)) \right)
\]

\[
 = -\frac{d}{dx} \left( \frac{(x-a)^\alpha}{\Gamma(1+\alpha)} f(\xi(x)) \right) \bigg|_{x=x_0}
\]

\[
 = -\frac{d}{dx} I_a^{\alpha} f(x) \bigg|_{x=x_0}
\]

\[
 = -D_a^{1-\alpha} f(x_0). \quad \Box
\]

**Theorem 3.2.** Assume that \( f(x) = 0 \) for some \( x \in (a, b) \), then there exists some \( \xi \in (a, x] \) for which \( D_a^{\alpha} f(\xi) = 0 \).

**Proof.** By (2.11), \( f_{1-\alpha}(a) = 0 \). Also by Proposition 2.2 (b), we have

\[
 \exists \xi \in \Lambda(\alpha, D_a^{\alpha} f, x), \ D_a^{\alpha} f(\xi(x)) \frac{(x-a)^\alpha}{\Gamma(\alpha)} = I_a^{\alpha} D_a^{\alpha} f(x) = f(x) = 0,
\]

6
Remark 3.3: According to the interpretation of the fractional derivative, that is introduced in [7], let $v(t)$ be the velocity measured by an individual, that is trapped in a gravitational field, $V$ be the velocity measured by an independent observer, and $\tau$ and $t$ be the time in the presence of the gravitational field and the cosmic time respectively, then we have:

$$V(t) = D_{a+}^{\alpha} v(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_{a}^{t} \frac{v(\tau)}{(t-\tau)\alpha} d\tau.$$  \hspace{1cm} (3.5)

If $v(t)$ vanishes at specific time $t$, then Theorem 3.2 states that, the independent observer has measured zero velocity, sooner at some time $\xi(t) \leq t$.

Lemma 3.4. For $x \in [a, b]$, we have

$$\begin{cases} \lim_{\alpha \to 0^+} D_{a+}^{\alpha} f(x) = f(x), \\ \lim_{\alpha \to 1^-} D_{a+}^{\alpha} f(x) = f'(x). \end{cases}$$ \hspace{1cm} (3.6)

Proof. By assumption we have (Proposition 2.3(a))

$$D_{a+}^{\alpha} f(x) = C D_{a+}^{\alpha} f(x) = I_{a+}^{p} f'(x)$$ \hspace{1cm} (3.7)

where $p = 1 - \alpha$. Now by Proposition 2.3(b)

$$\lim_{\alpha \to 1^-} D_{a+}^{\alpha} f(x) = \lim_{p \to 0^+} I_{a+}^{p} f'(x) = f'(x).$$ \hspace{1cm} (3.8)

Again by $D_{a+}^{\alpha} f(x) = I_{a+}^{1-\alpha} f'(x)$ we have

$$\left| I_{a+}^{1-\alpha} f'(x) - I_{a+}^{1-\alpha} f'(x) \right| \leq M \left| \int_{a}^{x} \left( \frac{(x-t)^{-\alpha}}{\Gamma(1-\alpha)} - 1 \right) dt \right| = M \frac{(x-a)^{1-\alpha}}{\Gamma(2-\alpha)} - (x-a).$$ \hspace{1cm} (3.9)

where $M = \max_{t \in [a,b]} |f'(t)|$. Taking the limit $\alpha \to 0^+$, gives the result. \hfill \Box

Let $X_{\epsilon,\alpha} := \left\{ r \in B(x_0, \epsilon) \mid D_{a+}^{\alpha} f(r) = 0 \right\}$ and $r(\alpha) := \sup X_{\epsilon,\alpha}$ (provided $X_{\epsilon,\alpha} \neq \emptyset$), where $\epsilon \in \mathbb{R}^+$, $\alpha \in [0,1)$ and $x_0 \in [a,b]$. We have the following statement:

Theorem 3.5. Suppose $x_0$ and $x_1$ are the unique maximum (minimum) and root of $f$ in $(a,b]$ respectively, then

$$\begin{cases} \lim_{\alpha \to 0^+} r(\alpha) = x_1, \\ \lim_{\alpha \to 1^-} r(\alpha) = x_0. \end{cases}$$ \hspace{1cm} (3.10)
Proof. Without loss of generality, assume the $x_0$ is a maximum. For $t \in (a, b) \setminus \{x_0\}$ we have

$$\lim_{\alpha \to 1^-} D^\alpha_{a+} f(t) = \begin{cases} f'(t) > 0 & t \in (a, x_0), \\ f'(t) < 0 & t \in (x_0, b). \end{cases} \quad (3.11)$$

Since for every $\epsilon \in \mathbb{R}^+$, $f'$ changes sign over $B(x_0, \epsilon)$, for $\alpha$ close enough to 1, $D^\alpha_{a+} f$ changes sign too and therefore, $X_{\epsilon, \alpha}$ is nonempty. Let $r = \limsup_{\alpha \to 1^-} r(\alpha)$. If $r \neq x_0$, then by continuity of $f'$ and $\lim_{\alpha \to 1^-} D^\alpha_{a+} f(r(\alpha)) = f'(r(\alpha))$, one obtain

$$0 = \limsup_{\alpha \to 1^-} D^\alpha_{a+} f(r(\alpha)) = f'(r(\alpha)) \neq 0. \quad (3.12)$$

Similarly we infer that, $\liminf_{\alpha \to 1^-} r(\alpha) = x_0$. The same result can be obtained in the case $\alpha \to 0^+$. □

4 Convexity

In classical derivative, convexity of a differentiable function is equivalent to that function has a nondecreasing derivative. In this section, our aim is to give a similar argument. But first we need an order to give a meaning to the concept of increase or decrease of the fractional derivative.

Regarding to [7], upon which, the fractional derivative, interpreted as the real velocity (with respect to cosmic time) is calculated for an event that happens in the presence of some strong gravitational field (with time dilation effect), we propose the following:

With respect to the non-locality of the fractional derivative that involves integrating over a time interval namely $[a, x]$, it seems to be reasonable if we define an order on the fractional derivative, in such a manner that we fix a constant length for two different time domains that the integrations are taken over them, that is:

**Definition 4.1.** A real valued function $f$ is called $\delta$-increasing, if for every $x_0, y_0, x, y \in \mathbb{R}$ and $\delta > 0$ with $x_0 < x < y_0 < y$ and $x - x_0 = y - y_0 = \delta$ we have

$$D^\alpha_{x_0} f(x - x_0) \leq D^\alpha_{y_0} f(y - y_0).$$

Next step is to set a discipline for the mean value corresponding to the fractional integration of order $1 - \alpha$:  

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8
Definition 4.2. We say that a continuous function $f$ has the property ($P$), if for every $x_0, y_0, x, y \in \mathbb{R}$ and $\delta > 0$ with $x_0 < x < y_0 < y$ and $x - x_0 = y - y_0 = \delta$, we have

$$\xi_x - \xi_y = y_0 - x_0,$$

where $\xi_{x_0+\delta} \in \Lambda(\alpha, f \circ (x - x_0), x_0 + \delta)$ and $\xi_{y_0+\delta} \in \Lambda(\alpha, f \circ (y - y_0), y_0 + \delta)$.

The property ($P$) is about the independence of the mean value $\xi$ to the initial point $x_0$ or $y_0$, when we are dealing with the fractional differentiation that involves integrating and the integration is taken over two different time intervals of the same length, $[x_0, x_0 + \delta]$ and $[y_0, y_0 + \delta]$.

Example 4.3. Let $f(x) = x^\beta$ ($\beta \in \mathbb{R}$), then $f$ is $\delta$-increasing since

$$D^{\alpha}_{x_0^+} f(x-x_0) = \frac{\Gamma(1 + \beta)}{\Gamma(\beta - \alpha + 1)} (x-x_0)^{\beta-\alpha} = \frac{\Gamma(1 + \beta)}{\Gamma(\beta - \alpha + 1)} \delta^{\beta-\alpha}. \quad (4.1)$$

Also we have

$$I^{1-\alpha}_{x_0^+} f(x-x_0)^\beta = \frac{\Gamma(\beta)}{\Gamma(\beta - \alpha + 1)} (x-x_0)^{\beta-\alpha+1} = \frac{(x-x_0)^{1-\alpha}}{\Gamma(2-\alpha)} (\xi(x) - x_0)^\beta, \quad (4.2)$$

and we obtain

$$\frac{\xi(x) - x_0}{x-x_0} = \left\{ \frac{\Gamma(2 - \alpha)\Gamma(\beta)}{\Gamma(\beta - \alpha + 1)} \right\}^{\frac{1}{\beta}}. \quad (4.3)$$

For $x_0 < x < y_0 < y$ with $x - x_0 = y - y_0 = \delta$, we have $\xi(x) - x_0 = \xi(y) - y_0$ if and only if $\left\{ \frac{\Gamma(2 - \alpha)\Gamma(\beta)}{\Gamma(\beta - \alpha + 1)} \right\}^{\frac{1}{\beta}}$ is a positive real number and it is equivalent to $f$ satisfies the property ($P$).

Remark 4.4. Under the monotonicity assumption of $f$ with property ($P$) and denoting $\xi(x,a)$ the mean value defined by (2.14), we have

$$\frac{\partial}{\partial a} \left\{ \Gamma(2 - \alpha)\left( I^{1-\alpha}_{a^+} f \right)(x).\left( x-a \right)^{\alpha-1} \right\} = \frac{\partial}{\partial a} f(\xi(x,a)). \quad (4.4)$$

Indeed because of the independence of $\xi(x) - a$ to the initial point $a$, there must be a function $h(x)$ such that $\xi(x) = a + h(x)$. So we have

$$h(x) = f^{-1}\left\{ \Gamma(2 - \alpha)\left( I^{1-\alpha}_{a^+} f \right)(x).\left( x-a \right)^{\alpha-1} \right\} - a.$$
Now differentiating with respect to \( a \) and utilizing \((f^{-1})' = \frac{1}{f^2}\), gives (4.1).

**Theorem 4.5.** Let \( f' \) satisfies the property \((P)\). Then \( f \) is convex if and only if it is \( \delta \)-increasing.

**Proof.** For \( x_0, y_0 \in [a, b] \) and positive \( \delta \) with \( x_0 < x < y_0 < y \) and \( x - x_0 = y - y_0 = \delta \), we have

\[
D_{x_0}^\alpha f(x - x_0) - D_{y_0}^\alpha f(y - y_0) = I_{x_0}^{1-\alpha} f'(x - x_0) - I_{y_0}^{1-\alpha} f'(y - y_0)
\]

\[
= \frac{\delta^{1-\alpha}}{\Gamma(2-\alpha)} \left( f'(\xi(x)) - f'(\xi(y)) \right). \tag{4.5}
\]

The fractional derivative is increasing if and only if \( f'(\xi(x)) \leq f'(\xi(y)) \) and by continuity of \( f' \), we infer that \( f' \) is increasing and therefore \( f \) is convex. \( \square \)

For a positive \( \tau \), let \( \Delta f(x) := f(x + \tau) - f(x) \). If for a constant \( \tau \), we have \( \Delta f(x) \geq 0 \), then we say that \( f \) is increasing up to the positive constant \( \tau \). Next Lemma is about the relationship between the monotonicity of a function and its fractional derivative.

**Proposition 4.6.** Let \( f \in C[0, b] \) and \( f(x) = 0 \) for \( x \in \mathbb{R}\setminus[0, b] \). Let \( D_{a+}^\alpha f \) exists and for \( 0 < \tau < b \), we have \( D_{a+}^\alpha f(x) \leq D_{a+}^\alpha f(x + \tau) \). Then \( f \) is increasing (up to \( \tau \)) if \( \Delta f(0) \geq 0 \).

**Proof.** The difference \( \Delta D_{a+}^\alpha f(x) \) (that is a function of \( x \) and \( \tau \)) is positive. Taking the Laplace transform we obtain

\[
L \Delta D_{a+}^\alpha f(x) = s^\alpha \left( Lf(x + \tau) - Lf(x) \right) - \Delta f(0). \tag{4.6}
\]

Then taking the inverse gives

\[
\Delta f(x) = \frac{1}{\Gamma(\alpha)} \left( x^{\alpha-1} \Delta f(0) + x^{\alpha-1} \ast \Delta D_{a+}^\alpha f(x) \right)
\]

\[
= \frac{x^{\alpha-1}}{\Gamma(\alpha)} \Delta f(0) + I_a^\alpha \Delta D_{a+}^\alpha f(x) \geq 0, \tag{4.7}
\]

where asterisk means convolution. \( \square \)

**Remark 4.7.**

It is easy to verify that, if \( D_{a+}^\alpha f(x) \leq D_{a+}^\alpha g(x) \) and \( f(0) = g(0) \), then \( f(x) \leq g(x) \).
Let $f$ be periodic with period $\tau$. Then (4.7) gives $I_\alpha^a, \Delta D_\alpha^a f(x) = 0$ and therefore

$$D_\alpha^a f(x + \tau) = D_\alpha^a f(x).$$

In other words, periodic functions have periodic fractional derivatives with the same period.

**Conclusion** In this paper, the fractional mean value is introduced and its approximation as the roots of a polynomial is obtained. Fractional critical point is introduced, some sufficient conditions for its existence are studied and an interpretation of such critical point is given with respect to the time dilation effect in a gravitational field. Moreover, Convexity and monotonicity and their possible connections to increase of a fractional derivative in the Riemann-Liouville sense are discussed.

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