Entanglement for all quantum states

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Abstract
It is shown that a state that is factorizable in the Hilbert space corresponding to some choice of degrees of freedom becomes entangled for a different choice of degrees of freedom. Therefore, entanglement is not a special case but is ubiquitous in quantum systems. Simple examples are calculated and a general proof is provided. The physical relevance of the change of tensor product structure is mentioned.

1. Introduction
Entanglement is one of the most remarkable features of quantum mechanics. Consider two exclusive properties of a quantum system, \(A_1\) and \(A_2\), corresponding to two different eigenvalues of some observable (for instance, spin up and spin down) and also another unrelated pair of exclusive properties, \(B_1\) and \(B_2\) (for instance, located here or there). Furthermore, imagine two possible states of the system: \(\psi_1\), corresponding to the simultaneous appearance of the properties \(A_1\) and \(B_1\) and the other state, \(\psi_2\), corresponding to the appearance of the properties \(A_2\) and \(B_2\). The superposition, \(\psi_1 + \psi_2\), is an entangled state of the system. In this state, none of the properties \(A_1, A_2, B_1, B_2\) are objective (in the sense that the state is not an eigenvector corresponding to any of these eigenvalues) but there are strong quantum correlations among them because the observation of one property, say \(A_1\), forces the appearance of \(B_1\) although they may be totally unrelated (like spin and location). In entangled states all sort of astonishing quantum effects appear, such as violations of Bell’s inequalities, the Einstein–Podolsky–Rosen (so-called) paradox, Schrödinger’s cat, nonlocality, contextuality, teleportation, quantum cryptography and computation, etc. The principle of superposition that generates the entanglement, contains perhaps the central essence of quantum mechanics and almost all pondering concerning the foundations of quantum mechanics involves entangled states.

The opposite to the entangled states are the factorizable states, for instance \(\psi_1\) or \(\psi_2\) above, where the properties are objective and the behaviour of the system is closer to classical expectations; for instance, the correlations found are understood as a direct consequence of the
preparation of the system. One may erroneously think that there are two classes of states for
the quantum system, entangled and factorizable, that correspond to qualitative difference in
the behaviour of the system, close to classical in one case and with strong quantum correlations
in the other. We will see in this work that this is indeed wrong because factorizable states
also exhibit entanglement with respect to other observables. In this sense, all states are
entangled; entanglement is not an exceptional feature of some states but is ubiquitous in
quantum mechanics.

The fact that factorizability and entanglement are not preserved in a change of the degrees
of freedom used to describe the system has been analysed by experts, specially those involved
in quantum computation research [1, 2], but this important feature of quantum mechanics is
ignored in textbooks, even advanced ones. In this work we present simple calculations that
emphasize this remarkable feature and thereby provide a didactic complement for a modern
quantum mechanics course. The calculations mentioned would be quite involved without the
application of the quantum covariance function that has also received very little consideration in
textbooks. In the following sections we will define entanglement and factorizability with rigour
and we will prove that every factorizable state becomes entangled in a different factorization
of the Hilbert space. For this, we will recall a useful tool provided by the quantum covariance
function and we will calculate several explicit examples that may be useful for teachers and
students of quantum mechanics at the advanced undergraduate and graduate level.

2. Entanglement in compound systems

The state of a compound quantum system, \( S = (S_1, S_2) \), belongs to a Hilbert space built
as the tensor product of spaces corresponding to each subsystem: \( \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \). This
decomposition, denoted as a tensor product structure (TPS), may correspond to two individual
physical subsystems like, for instance, one electron and one proton building a hydrogen atom,
or to different degrees of freedom or coordinates of one system. The degrees of freedom
are expected to be independent in the sense that the assignment of one value to one degree
of freedom is compatible with an arbitrary assignment of any value for the other one. For
quantum mechanics, this means that, in the Hilbert space, the two degrees of freedom \( A \) and
\( B \) will correspond to two observables whose operators act individually in each factor space;
that is, they are of the form \( A \otimes 1 \) and \( 1 \otimes B \) and therefore they commute. If we define bases
in each factor space, \( \{ \psi_k \} \in \mathcal{H}_1 \) and \( \{ \phi_r \} \in \mathcal{H}_2 \), the most general state in the Hilbert space is
given by an expansion in the basis \( \{ \psi_k \otimes \phi_r \} \) as

\[
\Psi = \sum_{k,r} C_{kr} \psi_k \otimes \phi_r. \tag{1}
\]

This state \( \Psi \) is factorizable if there exist \( \Psi_1 \in \mathcal{H}_1 \) and \( \Psi_2 \in \mathcal{H}_2 \) such that
\( \Psi = \Psi_1 \otimes \Psi_2 \). Otherwise it is entangled. More precisely, we define entanglement by means of Schmidt
bi-orthogonal decomposition: for each state \( \Psi \) there exist two bases \( \{ \tilde{\psi}_k \} \in \mathcal{H}_1 \) and \( \{ \tilde{\phi}_r \} \in \mathcal{H}_2 \) such that

\[
\Psi = \sum_{k=1}^N \alpha_k \tilde{\psi}_k \otimes \tilde{\phi}_k, \tag{2}
\]

where \( N \leq \min\{D_1, D_2\} \) and where \( D_1 \) and \( D_2 \) are the dimensions of the Hilbert spaces.
Note that in this expansion we do not have a double sum as in the general expansion in
equation (1). The bases for the bi-orthogonal decomposition are not unique and, of course,
depend on the state \( \Psi \). If \( N = 1 \) the state is factorizable and if \( N \geq 2 \) the state is entangled.
Note that in the determination of whether a state is factorizable or entangled, the factorization of the Hilbert space (that is, the TPS) is crucial and this factorization depends on the choice of the observables corresponding to the degrees of freedom. From the mathematical point of view, every TPS is equivalent but from the physical point of view, the TPS are determined operationally by the measurements and operations that are accessible under given physical circumstances. For instance, if our composite system consists of two particles that are spatially separated, the most natural TPS is given by the tensor product of the single-particle Hilbert spaces associated with the individual particles. However, if in this same system, the overall motion is uninteresting and only the relative motion is relevant we may prefer a TPS corresponding, not to the position of the individual particles, but instead to the centre of mass and relative position.

A relevant question is whether some arbitrary state of a system, analysed with different choices of the degrees of freedom, that is, with different TPS, still maintains the property of being factorizable or not. In other words, is factorizability an objective property of the system or is it a feature of our description of the system. In order to approach this question we recall a useful tool provided by the quantum covariance function that relates a state $|\psi_1\rangle$ and two observables $A$ and $B$.

### 3. The quantum covariance function

Given two arbitrary Hermitian operators $A$ and $B$ and a normalized Hilbert space element $|\psi\rangle$, we define the quantum covariance function (QCF) by

$$Q(A, B, |\psi\rangle) = \langle\psi, AB|\psi\rangle - \langle\psi, A|\psi\rangle\langle\psi, B|\psi\rangle.$$  \hspace{1cm} (3)

This function has been studied in detail [3] and was used to provide an elegant proof of the uncertainty principle in the most general form given by Schrödinger [4]. Another interesting application of this function allows a complete study of manifest and concealed correlations in quantum systems [5]. For our purpose here, we note that when the observables correspond to two degrees of freedom, that is, they are of the form $A \otimes I$ and $I \otimes B$, and the state is factorizable, $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$, then the QCF vanishes. The proof is trivial: $Q(A \otimes I, I \otimes B, |\psi_1\rangle \otimes |\psi_2\rangle) = \langle|\psi_1\rangle \otimes |\psi_2\rangle, A \otimes B |\psi_1\rangle \otimes |\psi_2\rangle\rangle - \langle|\psi_1\rangle, A |\psi_1\rangle\rangle\langle|\psi_2\rangle, B |\psi_2\rangle\rangle = 0$. Note that $|\psi\rangle$ factorizable implies $Q(A, B, |\psi\rangle) = 0$ but the inverse is not true: there are cases with vanishing QCF but with entangled states as can be seen in [5]. In any case, if the QCF does not vanish, then we are sure that the state is not factorizable, that is, it is entangled.

### 4. Transformation entanglement

Let us consider a quantum system with two subsystems $S = (S_A, S_B)$ that may correspond to two degrees of freedom $A$ and $B$. The state of the system belongs then to the Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ and the two degrees of freedom are represented by operators $A \otimes I$ and $I \otimes B$. Let us consider a factorizable, non-entangled, state $|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$ with $|\psi_A\rangle$ and $|\psi_B\rangle$ arbitrary states (not necessarily eigenvectors of $A$ and $B$) in the spaces $\mathcal{H}_A$ and $\mathcal{H}_B$. Then there exists a transformation of the degrees of freedom $F = F(A, B)$ and $G = G(A, B)$ that suggests a different factorization or TPS, $\mathcal{H} = \mathcal{H}_F \otimes \mathcal{H}_G$, where the state is no longer factorizable: $|\psi\rangle \neq |\psi_F\rangle \otimes |\psi_G\rangle$ with $|\psi_F\rangle \in \mathcal{H}_F$ and $|\psi_G\rangle \in \mathcal{H}_G$. The state becomes entangled in these new degrees of freedom; the factorizability of states is not invariant under a different factorization of the Hilbert space.

We will next clarify this with two simple examples and we will later give a general proof.
4.1. System with two coordinates

Let us consider a very simple system characterized by two space coordinates \( X_1 \) and \( X_2 \). This may correspond to the position of two free particles moving in a line or to one particle moving in a plane (the equivalence of \( n \) free particles moving in \( \mathbb{R} \) with one particle moving in \( \mathbb{R}^n \) is an interesting fact that has been used to relate nonseparability with contextuality \([6]\)). The state of the system is then an element of \( \mathcal{H}_1 \otimes \mathcal{H}_2 \) and the two coordinates correspond to the operators \( X_1 = X \otimes I + I \otimes X \) and \( X_2 = I \otimes X \). If we use the basis \( \{ \phi_x \} \in \mathcal{H}_1 \) and \( \{ \psi_x \} \in \mathcal{H}_2 \) corresponding to the eigenvectors of the position operator in both spaces, then the most general factorizable state is given by

\[
\Psi = \left( \int dx_1 \ f(x_1) \phi_{x_1} \right) \otimes \left( \int dx_2 \ g(x_2) \psi_{x_2} \right) = \int dx_1 \int dx_2 \ f(x_1) g(x_2) \phi_{x_1} \otimes \psi_{x_2}, \tag{4}
\]

where \( f(x) \) and \( g(x) \) are two properly normalized arbitrary functions. (For more rigour, we should mention that the spaces \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) are rigged Hilbert spaces that contain not only the normalizable square integrable functions but also the improper eigenvectors \( \{ \phi_x \} \) and \( \{ \psi_x \} \). More details on this can be found in advanced quantum mechanics books \([7]\)).

Instead of \( X_1 \) and \( X_2 \) we can now consider another pair of degrees of freedom given by

\[
A = X_1 + X_2 = X \otimes I + I \otimes X \tag{5}
\]

\[
B = X_1 - X_2 = X \otimes I - I \otimes X. \tag{6}
\]

Physically, these new coordinates are related to the centre of mass and relative distance, in the case of two particles in a line, or to a rotation and reflection of the coordinate axis, in the case of one particle moving in a plane. It is a trivial change of variables but, as we will see, with significant consequences for the treatment of the quantum system. Let \( X_{a,b} \) in \( \mathcal{H}_1 \otimes \mathcal{H}_2 \) denote the eigenvectors of \( A \) and \( B \) corresponding to the eigenvalues \( a \) and \( b \). That is, \( A X_{a,b} = a X_{a,b} \) and \( B X_{a,b} = b X_{a,b} \). One can easily check that \( X_{a,b} = \psi_{\frac{1}{2}(a+b)} \otimes \phi_{\frac{1}{2}(a-b)} \). Consider now the degree of freedom \( A \) alone, isolated from the other degree of freedom \( B \). To this degree of freedom we can associate a Hilbert space \( \mathcal{H}_A \) spanned by the eigenvectors of \( A \), \( \{ \eta_a \} \). Similarly, the eigenvectors \( \{ \xi_b \} \) of \( B \), considered independently, generate another Hilbert space \( \mathcal{H}_B \). The tensor product of these spaces, \( \mathcal{H}_A \otimes \mathcal{H}_B \), provide a different factorization of the Hilbert space of the compound system, spanned by the basis \( \{ \eta_a \otimes \xi_b \} \). The two bases \( \{ \eta_a \otimes \xi_b \} \) and \( \{ \psi_{x_1} \otimes \phi_{x_2} \} \) are related by

\[
\eta_a \otimes \xi_b = \psi_{\frac{1}{2}(a+b)} \otimes \phi_{\frac{1}{2}(a-b)}, \tag{7}
\]

\[
\psi_{x_1} \otimes \phi_{x_2} = \eta_{x_1+x_2} \otimes \xi_{x_1-x_2}. \tag{8}
\]

Note that this change of basis is trivial, it amounts only to a relabelling or reordering of the basis elements. This is expected because the new and old degrees of freedom commute and therefore they share the same basis. Anyway, if we perform the variable change \( x_1 + x_2 = a \), \( x_1 - x_2 = b \), \( dx_1 \ dx_2 = \frac{1}{2} \ d a \ d b \) in the factorizable state in equation (4) we get the same state in the new basis given as

\[
\Psi = \frac{1}{2} \int da \ db \ f \left( \frac{1}{2}(a + b) \right) g \left( \frac{1}{2}(a - b) \right) \eta_a \otimes \xi_b. \tag{9}
\]

One can, of course, find many examples of factorizable states that remain factorizable after the change of variables. For instance, the Gaussian \( f(x_1) g(x_2) = \exp \left[ -x_1^2 - x_2^2 \right] \) becomes \( \exp[-a^2 - b^2]/2 \) that is also factorizable, or also a plane wave \( \exp[\pm ik_1 x_1 + k_2 x_2] \) remains factorizable. However, not every factorizable state remains so and, in general,
\[
f \left( \frac{1}{2}(a + b) \right) g \left( \frac{1}{2}(a - b) \right) \neq F(a)G(b). \]
As an example for nonfactorizability we can take, for instance, a double Gaussian for \( f(x_1) = \exp(-(x_1 - d)^2) + \exp(-(x_1 + d)^2) \) and one single Gaussian for \( g(x_2) = \exp \left( -x_2^2 \right) \).

A more elegant proof that the variable change destroys the factorizability is obtained using the QCF. Clearly, with the factorizable state in equation (4), that is with \( \Psi = \Psi_1 \otimes \Psi_2 \), the QCF vanishes:

\[
Q(X_1, X_2, \Psi) = \langle \Psi, X \otimes X \Psi \rangle - \langle \Psi, X \otimes I \Psi \rangle \langle I \otimes X \Psi \rangle = 0.
\]

However, for the same state, a similar calculation gives

\[
Q(A, B, \Psi) = \langle \Psi_1, X^2 \Psi_1 \rangle - \langle \Psi_2, X^2 \Psi_2 \rangle - \langle \Psi_1, X \Psi_1 \rangle^2 + \langle \Psi_2, X \Psi_2 \rangle^2
\]

and replace it on the right-hand side of equation (10) obtaining

\[
\psi_k \pm \}

\[
\Delta^2_{X_1} - \Delta^2_{X_2} \neq 0
\]

then we reach a contradiction. In order to see this, let us choose, among all possible \( f(x_1) \), one that has a zero in \( x_0 \), that is, \( f(x_0) = 0 \). Therefore, in the \((x_1, x_2)\) plane, the product \( f(x_1)g(x_2) \) vanishes along a straight line perpendicular to the \( x_1 \) axis. This line is mapped in the \((a, b)\) plane into a curve given by \( x_1(a, b) = x_0 \). We can solve this equation for \( a \), that is, \( a = a_0(b) \) and replace it on the right-hand side of equation (13) obtaining \( F(a_0(b))G(b) = 0 \) for all \( b \). Now, since \( \forall b, G(b) \neq 0 \) by definition, we must have \( F(a_0(b)) = 0 \) for all \( b \), in contradiction with the assumption that \( F \) does not vanish everywhere.

From the example seen, it becomes clear that every change to new degrees of freedom that mixes the old ones destroys the factorization of the state. The only possibility to preserve the factorization is when the degrees of freedom are not mixed, that is, the transformations are of the type \( A = A(X_1) \) and \( B = B(X_2) \). Therefore, the unitary transformations in the Hilbert space \( \mathcal{H}_1 \otimes \mathcal{H}_2 \) that leave the TPS invariant are of the type \( U_A \otimes U_B \), for instance, the time evolution of internal and external degrees of freedom [2]. A generalization of this to TPS involving any number of factors is evident.

### 4.2. System with two spins

In this example, let us consider a system of two particles with spin 1/2. As degrees of freedom to characterize the system we can take, as is usually done, the \( z \) component of both spins \( S_1 \) and \( S_2 \). Their corresponding two-dimensional Hilbert spaces \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) are spanned by the two bases \( \{ \psi_{k,l} \} \) and \( \{ \phi_{k,l} \} \). The four-dimensional Hilbert space \( \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \) for the system has a basis \( \{ \psi_{k,l} = \psi_k \otimes \phi_r, k, r = +, - \} \) and the most general factorizable state is given by

\[
\psi = \psi_1 \otimes \psi_2 = \sum_k \alpha_k \psi_k \otimes \sum_r \beta_r \phi_r = \sum_{k,r} \alpha_k \beta_r \psi_{k,l}. \]
We will see that when we factorize the Hilbert space corresponding to other degrees of freedom this state becomes entangled.

As different degrees of freedom we can take, for instance, the square of two orthogonal components of total spin:

\[ S_z^2 = (S_z \otimes I + I \otimes S_z)^2 = \frac{\hbar^2}{2} I \otimes I + 2 S_z \otimes S_z \]  
(15)

\[ S_x^2 = (S_x \otimes I + I \otimes S_x)^2 = \frac{\hbar^2}{2} I \otimes I + 2 S_z \otimes S_z \]  
(16)

These two operators commute \( [S_z^2, S_x^2] = 0 \) and their eigenvalues are 0 and \( \hbar^2 \) corresponding to the degeneracy eigenvectors \{\chi_{s,t}, s, t = 0, 1\}. That is,

\[ S_z^2 \chi_{s,t} = s \hbar^2 \chi_{s,t} \]
\[ S_x^2 \chi_{s,t} = t \hbar^2 \chi_{s,t} \]
(17)

We have then two different bases \{\psi_{k,r}, k, r = +, -\} and \{\chi_{s,t}, s, t = 0, 1\}. In this example, the new and old degrees of freedom do not commute and they do not share the basis as was the case in the previous example. The unitary transformation relating both bases is

\[
\begin{pmatrix}
\chi_{1,1} \\
\chi_{1,0} \\
\chi_{0,1} \\
\chi_{0,0}
\end{pmatrix} = \frac{1}{\sqrt{2}}
\begin{pmatrix}
1 & 1 & 0 & 0 \\
1 & -1 & 0 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 1 & -1
\end{pmatrix}
\begin{pmatrix}
\psi_{+,+} \\
\psi_{-,-} \\
\psi_{+,+} \\
\psi_{-,+}
\end{pmatrix}.
\]  
(18)

Now, if we take the most general factorizable state in the TPS related with the basis \{\psi_{k,r}\}, given in equation (14), and we make the change of basis with the unitary transformation, then one can prove that the resulting expression is, in general, not factorizable in the TPS related to \{\chi_{s,t}\}, that is, it cannot always be written in the form

\[ \Psi = \sum_{s,t} \delta_{s,t} \eta_{s,t} \chi_{s,t}. \]  
(19)

The change of basis destroys the factorizability. Instead of this explicit long calculation it is simpler to use the QCF to show that the state is entangled in the Hilbert space factorization corresponding to the new variables. Doing this we obtain

\[ Q(S_z^2, S_x^2, \Psi) = -\hbar^2 \langle \psi_1, S_z \psi_1 | \psi_2, S_z \psi_2 \rangle - 4 \langle \psi_1, S_z \psi_1 | \psi_2, S_z \psi_2 \rangle \langle \psi_1, S_z \psi_1 | \psi_2, S_z \psi_2 \rangle - 4 \langle \psi_1, S_z \psi_1 | \psi_2, S_z \psi_2 \rangle \langle \psi_1, S_z \psi_1 | \psi_2, S_z \psi_2 \rangle \neq 0 \]  
(20)

for arbitrary \( \psi_1 \) and \( \psi_2 \).

4.3. General proof

The simple example concerning two spatial coordinates can be generalized to provide a general proof. Let us consider a quantum system with two subsystems \( S = (S_A, S_B) \) described in the Hilbert space \( \mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \) corresponding to the TPS associated with the two degrees of freedom \( A \otimes I \) and \( I \otimes B \). Let us assume a factorizable, non-entangled, normalized state \( \Psi = \Psi_A \otimes \Psi_B \) where \( \Psi_A \) and \( \Psi_B \) are two arbitrary normalized states in the factor spaces \( \mathcal{H}_A \) and \( \mathcal{H}_B \). Then there exists a transformation of the degrees of freedom, \( F = A \otimes I + I \otimes B \) and \( G = A \otimes I - I \otimes B \), whose TPS has a different factorization, \( \mathcal{H} = \mathcal{H}_F \otimes \mathcal{H}_G \), where the state is no longer factorizable. In order to prove this we show that the QCF \( Q(F, G, \Psi) \neq 0 \) and therefore the state \( \Psi \) is entangled in the Hilbert space factorization \( \mathcal{H} = \mathcal{H}_F \otimes \mathcal{H}_G \).
Entanglement for all quantum states

corresponding to the degrees of freedom $F$ and $G$. Note that $FG = (A \otimes I + I \otimes B)(A \otimes I - I \otimes B) = A^2 \otimes I - I \otimes B^2$ and then we have

$$Q(F, G, \Psi) = \langle \Psi_A \otimes \Psi_B, (A^2 \otimes I - I \otimes B^2)\Psi_A \otimes \Psi_B \rangle - \langle \Psi_A \otimes \Psi_B, (A \otimes I + I \otimes B)\Psi_A \otimes \Psi_B \rangle$$
$$\times \langle \Psi_A \otimes \Psi_B, (A \otimes I - I \otimes B)\Psi_A \otimes \Psi_B \rangle$$
$$= \langle \Psi_A, A^2\Psi_A \rangle - \langle \Psi_B, B^2\Psi_B \rangle - \langle \Psi_A, A\Psi_A \rangle^2 + \langle \Psi_B, B\Psi_B \rangle^2$$
$$= \Delta_A^2 - \Delta_B^2 \neq 0 \quad (21)$$

because the indeterminacies of $A$ and $B$ in the arbitrary states $\Psi_A$ and $\Psi_B$ are, in general, different.

5. Conclusions

We have seen that the factorizability of a state is a property that is not invariant under a change of the degrees of freedom that we use in order to describe the system. This proof is made simple by the use of the QCF whose non-vanishing is a criterion for entanglement.

The fact that the appearance of entanglement depends on the choice of degrees of freedom can find an interesting application in the 'disentanglement’ of a state. One can, sometimes, transform an entangled state into a factorizable one by a judicious choice of the degrees of freedom. In some sense this is the inverse problem to the one presented in section 4. One example of this is provided in [8] where the entangled state of the compound system of one proton and one electron with Coulomb interaction becomes factorizable when we use centre of mass and relative position coordinates instead of the individual spatial coordinates of the proton and electron. In this case, the two-particle system consisting of one proton and one electron in an entangled state is described in a simpler, factorizable, state of two fictitious noninteracting particles: one with the total mass of the hydrogen atom, moving freely in space, and another particle with the effective mass, moving in a fixed Coulomb potential.

Perhaps the most important manifestation of quantum correlations, that is, those that cannot be explained in terms of some classical interaction, involves violations of Bell’s inequalities. Furthermore, it has been shown [9] that in every nonfactorizable or entangled state there are observables that violate Bell’s inequalities. In this work, we have seen that for any system in a factorizable state, we can find different degrees of freedom that suggest a different factorization of the Hilbert space where the same state becomes entangled. As a consequence of this we can conclude that in every state, even for those factorizable, we can find pairs of observables that will violate Bell’s inequalities. This violation of the classical behaviour is then not exceptional but is ubiquitous in quantum systems.

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