RARE EVENTS AND POISSON POINT PROCESSES

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Abstract. The aim of the present work is to show that the results obtained earlier on the approximation of distributions of sums of independent terms by the accompanying compound Poisson laws may be interpreted as rather sharp quantitative estimates for the closeness between the sample containing independent observations of rare events and the Poisson point process which is obtained after a Poissonization of the initial sample.

The aim of the present work is to show that the results obtained earlier on the approximation of distributions of sums of independent terms by the accompanying compound Poisson infinitely divisible laws may be interpreted as rather sharp quantitative estimates for the closeness between the sample containing independent observations of rare events and the Poisson point process which is obtained after a Poissonization of the initial sample.

Let us first introduce some notation. Let \( F \) denote the set of probability distributions defined on the Borel \( \sigma \)-field of subsets of the Euclidean space \( \mathbb{R}^d \) and \( \mathcal{L}(\xi) \in F_d \) is the distribution of a \( d \)-dimensional random vector \( \xi \). For \( F \in F_1 \), the concentration function is defined by \( Q(F, b) = \sup_{x} F\left\{ [x, x + b]\right\}, \ b \geq 0 \). For \( F \in F_d \), we denote the corresponding distribution functions by \( F(x) = F\left\{ (-\infty, x_1] \times \cdots \times (-\infty, x_d]\right\}, \ x = (x_1, \ldots, x_d) \in \mathbb{R}^d \), and the uniform Kolmogorov distance by \( \rho(F, H) = \sup_x |F(x) - H(x)| \).

By \( c \) we denote absolute positive constants. Note that constants \( c \) can be different in different (or even in the same) formulas. For some positive quantities \( a \) and \( b \) writing \( a \asymp b \) means that \( a \leq c b \) and \( b \leq c a \).

Let \( X_1, X_2, \ldots, X_n \) be independent not identically distributed elements of a measurable space \((\mathfrak{X}, \mathcal{S})\) and \( f : \mathfrak{X} \to \mathbb{R}^d \) be a Borel mapping. Let \( F_i = \mathcal{L}(f(X_i)), \ i = 1, 2, \ldots, n \), be the distributions of \( f(X_i) \). Then the sum

\[
S = f(X_1) + f(X_2) + \cdots + f(X_n)
\]

has the distribution \( \prod_{i=1}^{n} F_i \) (products and powers of measures will be understood in the convolution sense: \( FH = F * H, \ H^m = H \* m, \ H^0 = E \) \( \overset{\text{def}}{=} E_0 \), where \( E_a \) is the distribution

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concentrated at a point \( a \in \mathbb{R}^d \). A natural approximating infinitely divisible distribution for \( \prod_{i=1}^n F_i \) is the accompanying compound Poisson distribution \( \prod_{i=1}^n e(F_i) \), where

\[
e(H) \overset{\text{def}}{=} e^{-1} \sum_{m=0}^{\infty} \frac{H^m}{m!}, \quad H \in \mathfrak{F}_d,
\]

and, more generally,

\[
e(\alpha H) \overset{\text{def}}{=} e^{-\alpha} \sum_{m=0}^{\infty} \frac{\alpha^m H^m}{m!}, \quad \alpha > 0.
\]

It is easy to see that \( \prod_{i=1}^n e(F_i) \) is the distribution of

\[
T = \sum_{i=1}^n \sum_{j=1}^{\nu_i} f(X_{i,j}),
\]

where \( X_{i,j} \) and \( \nu_i, i = 1, \ldots, n, j = 1, 2, \ldots \), are independent in aggregate random elements of the space \( \mathfrak{X} \) and random variables respectively with \( \mathcal{L}(X_{i,j}) = \mathcal{L}(X_i) \) and \( \mathcal{L}(\nu_i) = e(E_1) \).

Clearly, \( e(E_1) \) is the Poisson distribution with mean 1. Thus, the sum \( T \) is defined similarly to \( S \), but the initial sample \( X = (X_1, X_2, \ldots, X_n) \) is replaced by its Poissonized version \( Y = \{ X_{i,j} : i = 1, \ldots, n, j = 1, 2, \ldots, \nu_i \} \). Poissonization of the sample is known as one of the most powerful tools in studying empirical processes. The random set \( Y \) may be considered as a realization of the Poisson point process on the space \( \mathfrak{X} \) with intensity measure \( \sum_{i=1}^n \mathcal{L}(X_i) \). The important property of the Poisson point process is the space independence: for any pairwise disjoint sets \( A_1, \ldots, A_m \subset \mathfrak{X} \), the random sets \( Y \cap A_1, \ldots, Y \cap A_m \subset \mathfrak{X} \) are independent in aggregate. As a consequence, the Poisson point process \( Y \) allows for better approximation than the process \( X \). One can use the independence property since the theory of independent objects is much more elaborated.

In this paper, we consider the problem of approximation of the sample by the Poisson point process which is obtained after a Poissonization of the initial sample in the case where the sample is obtained by observation of rare events. The situation is the following. The set \( \mathfrak{X} \) is represented as the union of two disjoint measurable sets: \( \mathfrak{X} = \mathfrak{X}_1 \cup \mathfrak{X}_2 \), with \( \mathfrak{X}_1, \mathfrak{X}_2 \in \mathcal{S} \), \( \mathfrak{X}_1 \cap \mathfrak{X}_2 = \emptyset \). We say that the \( j \)-th rare event occurs if \( X_j \in \mathfrak{X}_2 \). Respectively, it does not occur if \( X_j \in \mathfrak{X}_1 \).

To formulate the results we need some additional notation.

Let \( f : \mathfrak{X} \to \mathbb{R}^d \) be a Borel mapping defined above and \( F_i = \mathcal{L}(f(X_i)), i = 1, 2, \ldots, n \). Then distributions \( F_i \in \mathfrak{F}_d \) can be represented as mixtures

\[
F_i = (1 - p_i) U_i + p_i V_i,
\]

where \( U_i, V_i \in \mathfrak{F}_d \) are conditional distributions of \( f(X_i) \) given \( X_i \in \mathfrak{X}_1 \) and \( X_i \in \mathfrak{X}_2 \) respectively, that is

\[
0 \leq p_i = \mathbb{P}\{ X_i \in \mathfrak{X}_2 \} = 1 - \mathbb{P}\{ X_i \in \mathfrak{X}_1 \} \leq 1.
\]

Below the \( V_i \) are arbitrary distributions. We deal with rare events while the quantity

\[
p = \max_{1 \leq i \leq n} p_i
\]
is small. In other words, this is the case if our rare events are sufficiently rare. Let

$$a_i = \int_{\mathbb{R}^d} x U_i \{dx\}, \quad i = 1, 2, \ldots, n,$$

(5)

and, for $d = 1$,

$$|a|^2_2 = \sum_{i=1}^{n} a_i^2, \quad |a|_\infty = \max_{1 \leq i \leq n} |a_i|,$$

$$\sigma_i^2 = (1 - p_i) \int_{-\infty}^{\infty} (x - a_i)^2 U_i \{dx\}, \quad B^2 = \sum_{i=1}^{n} \sigma_i^2,$$

(6)

Denote

$$H_1 = \prod_{i=1}^{n} F_i, \quad H_2 = \prod_{i=1}^{n} e(F_i), \quad H_3 = \prod_{i=1}^{n} E_{a_i} E(F_i E_{-a_i}).$$

(6)

Usually, a good approximation of the distribution $H_1$ is given by the distribution $H_3$. But while estimating the closeness of the sample $X$ to the Poisson point process $Y$, we are interested in the closeness of distributions $H_1$ and $H_2$. Thus, we need to estimate the distances between $H_1$ and $H_2$ using that $H_2$ is the distribution of

$$T = \sum_{i=1}^{n} \sum_{j=1}^{n} f(X_{i,j}),$$

where $f(X_{i,j})$ and $\nu_i, i = 1, \ldots, n, j = 1, 2, \ldots$, are random vectors and variables with $\mathcal{L}(f(X_{i,j})) = F_i, \mathcal{L}(\nu_i) = e(E_i)$, which are independent in aggregate. Moreover, $H_3 = \mathcal{L}(T - \Delta)$, with $\Delta = \sum_{i=1}^{n} a_i (\nu_i - 1)$.

There exists a lot of results on the compound Poisson approximation of the distributions $\prod_{i=1}^{n} F_i$, see Arak and Zaitsev [2], Barbour and Chryssaphinou [3], Zaitsev [23], Roos [18], Čekanavičius [6] and the bibliography therein. However, most of these results require that distributions $F_i$ are appropriately centered. In the general case a better approximation can be obtained for the accompanying compound Poisson approximation of $H_1$ by $H_3$ using centering constants $a_i$.

In this paper, we consider estimates of classical distances between the distributions $H_1$ and $H_2$ with remainder terms having the additive summand of the form $c p$ for $d = 1$ or $c(d) p$ for $d \geq 1$. The most natural result of such a type is given in the following theorem.

**Theorem 1.** Let the conditions above be satisfied for $d \geq 1$ and $f(x) = 0$, for all $x \in \mathfrak{X}_1$. Then

$$\rho(H_1, H_2) \leq c(d) p.$$  

(7)

Theorem 1 is a direct consequence of a result of Zaitsev [21] which provides inequality (7) in the case, where

$$U_i = E, \quad i = 1, 2, \ldots, n,$$

(8)
and the \( V_i \) are arbitrary distributions. For \( d = 1 \), this statement was obtained earlier in [19]. This was an improvement of a result of Le Cam [14] who has proved one-dimensional assertion with \( p^{1/3} \) instead of \( p \).

Approximation of the sample by a Poisson point process under the conditions of Theorem 1 was discussed in Zaitsev [23], see also Hipp [10] and Roos [18].

Inequality (7) shows that different samples with rare events of uniformly small \( p_i \) are close one to another if they have the same (or close) measures \( \sum_{i=1}^{n} p_i V_i \). Indeed, in this case they have the same (or close) approximating Poisson point processes \( Y \). In the case of identical distributions \( V_i \), inequality (7) can be essentially sharpened and improved. Even the distance in variation \( d_{TV}(H_1, H_2) \) can be estimated by \( cp \) (Prokhorov [17] for the case of identical \( p_i \) and Le Cam [13] for the case of arbitrary \( p_i \)) and, moreover, by \( \sum_{i=1}^{n} p_i^2 / \max\{1, \sum_{i=1}^{n} p_i\} \) (Barbour and Hall [4]). Thus, for identical \( V_i \), the statement on the closeness of our processes can be essentially strengthened. The condition that the rare events are identically distributed is, however, seldomly satisfied. Rare events are similar to extreme incidents; each of them is unique and has his own individual distribution. Thus, inequality (7) can be useful, for example, in insurance theory to estimate the probabilities that the cumulative influence of risk factors \( f(X_j) \) will not exceed fixed critical values \( x_j \) (see, e.g., Hipp [10]).

In Theorems 2–9 below, the remainder terms of the form \( cp \) or \( c(d) p \) are related to Theorem 1 which is used in the proofs.

The importance of centering in the accompanying approximation was noted by Le Cam [14]. He has shown that accompanying laws with centering (see distribution \( H_3 \)) provide the rate of approximation of the form \( cn^{-1/3} \) for distributions of sums of \( n \) i.i.d. random variables with an absolute constant \( c \) without any assumptions on the distribution of summands. The same rate was obtained earlier by Kolmogorov [12] for the approximation by infinitely divisible distributions. Le Cam’s result shows that accompanying laws are sufficient to prove such a rate. Later Arak [1] obtained the optimal rate \( cn^{-2/3} \) for infinitely divisible approximation (see Arak and Zaitsev [2] for the history of the problem). Arak’s infinitely divisible approximation is much more complicated than the approximation by accompanying laws. Ibragimov and Presman [6] have shown that the rate \( cn^{-1/3} \) is optimal for the accompanying approximations (see Arak and Zaitsev [2], inequality (1.3), p. 181)).

Le Cam [14] considered not only i.i.d. summands, he obtained bounds for the accompanying compound Poisson approximation for distributions of sums of independent non-identically distributed random variables—with and without centering.

In Example 1 of Götze and Zaitsev [8], the case where \( F_1 = \cdots = F_n = F, \int |x|^3 F(dx) < \infty, a = \int x F(dx) \neq 0 \) and \( \sigma^2 = \int (x-a)^2 F(dx) > 0 \) was considered. Using the Berry–Esseen bound in the CLT, it was shown that \( \rho(H_1, H_2) \geq c \min\{1, a^2/\sigma^2\} \), for sufficiently large \( n \geq n_0 \). In Example 2 of Götze and Zaitsev [8], it was shown that in the degenerate case, for example, if \( F = E_1 \), we have \( H_1 = E_n \) and \( H_2 = e(nE_1) \), the Poisson law with parameter \( n \). Clearly, \( \rho(E_n, e(nE_1)) \geq c \).
Thus, the approximation without centering is not always successful. To ensure the validity of such an approximation, one needs additional assumptions such as zero mean, symmetry or a large atom at zero, see Zaitsev [23] and the bibliography therein.

Note, however, that these examples are rarely appearing in our scheme of rare events. It is much more plausible that the values of the function $f(x)$ are much larger for $x \in X_2$ than for $x \in X_1$. Moreover, the tails of distributions $V_i$ usually will be heavier than those of $U_i$. It is well-known that the concentration functions of convolutions of distributions with heavy tails decrease faster than those of distributions with light tails. Thus, we may expect that the concentration functions of distributions $H_1, H_2, H_3$ admit good bounds for the remainder terms in Theorems 3–6 below.

Le Cam [14] considered the rate of approximation by accompanying laws without centering proving the following result (see Le Cam [15], Theorem 2, p. 431).

**Theorem 2.** Let the above conditions be satisfied for $d = 1$ and, for some $\tau > 0$,

$$U_i\{[−\tau, \tau]\} = 1, \quad i = 1, 2, \ldots, n.$$  \hspace{1cm} (9)

Then

$$\rho(H_1, H_2) \leq c \left( p^{1/3} + \left( (1 + \tau^{-2}|a_2^2|)D^{-2}(\tau) \right)^{1/3} \right),$$  \hspace{1cm} (2)

where

$$D^2(\tau) = \sum_{i=1}^{n} \int_{-\infty}^{\infty} \min\{1, x^2\tau^{-2}\}F_i(dx).$$

Applying Theorem 2 in the case $a_i = 0$, $i = 1, 2, \ldots, n$, we obtain another result of Le Cam [14] which contains the rate of accompanying approximation with centering, see Le Cam ([15], Theorem 1, p. 429):

**Corollary 1.** Let the conditions of Theorem 2 be satisfied with $a = 0$. Then

$$\rho(H_1, H_2) \leq c \left( p^{1/3} + D^{-2/3}(\tau) \right).$$  \hspace{1cm} (3)

For the proof it suffices to notice that the distributions $F_iE_{-a_i}$ satisfy the conditions of Theorem 1 with $a_i = 0$ and with replacing $\tau$ by $2\tau$.

The statement of Corollary 1 was slightly improved by Ibragimov and Presman [9]. Zaitsev [20, 22] proved Theorem 3 below which sharpened and generalized the statement of Corollary 1. To formulate this theorem we need some additional notation.

Following Katz [11] and Petrov [16], we introduce the class $G$ of real valued functions $g(\cdot)$ on $\mathbb{R}$ satisfying the conditions

a) $g(\cdot)$ is a non-negative even function, which is strictly positive for $x \neq 0$ and does not decrease for $x \geq 0$.

b) the function $x/g(x)$ is non-decreasing for $x > 0$.

**Theorem 3** (Zaitsev [22]). Let the conditions of Theorem 2 be satisfied without condition (9). Let $g \in G$. Denote

$$\beta_i = \beta_i(g) = (1 - p_i) \int_{-\infty}^{\infty} (x - a_i)^2 g(x - a_i) U_i \{dx\} < \infty,$$
\[ \beta = \beta(g) = \sum_{i=1}^{n} \beta_i(g). \]

Define \( \lambda = \lambda(g) = \min \{ B, \frac{\beta(g)}{E(g)} \} \), if \( B^2 > 0 \), and \( \lambda = 0 \) otherwise. Let \( B^2 > 0 \). Then

\[ \rho(H_1, H_3) \leq c \left( p + \min \left\{ Q(H_1, \lambda), Q(H_3, \lambda) \right\} \right) \leq p + \frac{\lambda}{B} Q\left( \prod_{i=1}^{n} e(p_i V_i E_{-a_i}), B \right). \quad (10) \]

If \( B^2 = 0 \), then \( \rho(H_1, H_3) \leq cp. \)

It can be shown that, under condition (9), \( Q(H_3, \lambda) \leq c(p + D^{-1}(\tau)) \), see [8]. Therefore, Theorem 3 is sharper with respect to order than Corollary 1. The main result of Zaitsev [20] is a particular case of Theorem 3 with \( g(x) \equiv |x| \). The optimality of the statement of Theorem 3 was analyzed in [8, Examples 3 and 4].

Theorem 3 is the main tool in the proof of the main results of the paper [8], Theorems 4–7, which improve Theorem 2, see a discussion in [8]. Theorems 4–6 provide sharper bounds.

**Theorem 4.** Let the conditions of Theorem 3 be satisfied. Then

\[ \rho(H_1, H_2) \leq c \left( p + P\{ |\Delta| > \gamma \} \right) + \min_{1 \leq k \leq 3} Q(H_k, \gamma), \]

for any \( \gamma \geq \lambda \), where \( \Delta = \sum_{i=1}^{n} a_i (\nu_i - 1) \) and \( \nu_i \) are i.i.d. Poisson with mean 1.

A particular case of Theorem 4 occurring when \( a_i = 0, i = 1, 2, \ldots, n \), implies inequality (10). Theorems 5–7 below are simple consecutive consequences of Theorem 4.

**Theorem 5.** Let the conditions of Theorem 3 be satisfied. Then, for any \( \varkappa \geq \lambda \),

\[ \rho(H_1, H_2) \leq c \left( p + (1 + |a|_2 \varkappa^{-1} \sqrt{\delta} + |a|_\infty \varkappa^{-1} \delta) \right) q, \]

where \( q = \min_{1 \leq k \leq 3} Q(H_k, \varkappa) \) and \( \delta = \log \left( 1 + \varkappa q^{-1} |a|_2^{-1} \right). \)

**Theorem 6.** Let the conditions of Theorem 2 be satisfied. Then

\[ \rho(H_1, H_2) \leq c \left( p + (1 + |a|_2 \tau^{-1} \sqrt{s} + |a|_\infty \tau^{-1} s) Q \right), \]

where \( Q = \min_{1 \leq k \leq 3} Q(H_k, \tau) \) and \( s = \log \left( 1 + \tau Q^{-1} |a|_2^{-1} \right). \)

**Theorem 7.** Let the conditions of Theorem 2 be satisfied. Then

\[ \rho(H_1, H_2) \leq c \left( p + (1 + |a|_2 \tau^{-1} \sqrt{r} + |a|_\infty \tau^{-1} r) D^{-1}(\tau) \right), \]

where \( r = \log \left( 1 + \tau D(\tau) |a|_2^{-1} \right). \)

Theorem 7 is better with respect to order than Theorem 2 (clearly, \( |a|_\infty \leq |a|_2 \)). Theorems 4–6 provide sharper bounds.

Finally, we mention a result on strong approximation of sums of independent random vectors by infinitely divisible distributions. Theorem 8 below is a consequence of the main
result of Zaitsev [21], Lemma A of Berkes and Philipp [5], and the Strassen–Dudley theorem (see Dudley [7]). The norms in the space $\mathbb{R}^d$ are denoted $|| \cdot ||_2$ and $|| \cdot ||_\infty$.

**Theorem 8.** Let conditions (2)–(6) be satisfied for $d \geq 1$, and let, for some $\tau \geq 0$,

$$U_i \{ \{ x \in \mathbb{R}^d : ||x||_2 \leq \tau \} \} = 1, \quad i = 1, 2, \ldots, n,$$

and the $V_i \in \mathfrak{F}_d$ are arbitrary distributions. Then, for each fixed $\lambda > 0$, the random vectors $S, T$ and $\Delta$ can be constructed on the same probability space so that

$$\mathcal{L}(S) = H_1, \quad \mathcal{L}(T) = H_2, \quad \mathcal{L}(T - \Delta) = H_3,$$

where $\Delta = \sum_{i=1}^n (\nu_i - 1) a_i$ and $\nu_i$ are i.i.d. Poisson with mean 1, and

$$P \{ ||S - T + \Delta||_2 > \lambda \} \leq c(d) \left( p + \exp \left( - \frac{\lambda}{c(d) \tau} \right) \right) + \sum_{i=1}^n p_i^2. \quad (11)$$

Moreover, for all $x \in \mathbb{R}^d$ and $\lambda > 0$,

$$H_1(x) \leq H_3(x + \lambda \cdot 1) + c(d) \left( p + \exp \left( - \frac{\lambda}{c(d) \tau} \right) \right), \quad (12)$$

$$H_3(x) \leq H_1(x + \lambda \cdot 1) + c(d) \left( p + \exp \left( - \frac{\lambda}{c(d) \tau} \right) \right), \quad (13)$$

where $1 = (1, 1, \ldots, 1) \in \mathbb{R}^d$ and $c(d)$ depends only on $d$. If the distributions $V_i$ are identical, then the term $\sum_{i=1}^n p_i^2$ in (11) can be dropped.

Theorem 8 implies the following assertions about the closeness of distributions $H_1$ and $H_2$.

**Theorem 9.** Let the conditions of Theorem 8 be satisfied. Then

$$P \{ ||S - T||_2 > 2 \lambda \} \leq c(d) \left( p + \exp \left( - \frac{\lambda}{c(d) \tau} \right) \right) + \sum_{i=1}^n p_i^2 + P \{ ||\Delta||_2 \geq \lambda \}. \quad (14)$$

If the distributions $V_i$ are identical, then the term $\sum_{i=1}^n p_i^2$ in (14) can be removed. Moreover, for all $x \in \mathbb{R}^d$ and $\lambda > 0$,

$$H_1(x) \leq H_2(x + 2 \lambda \cdot 1) + c(d) \left( p + \exp \left( - \frac{\lambda}{c(d) \tau} \right) \right) + P \{ ||\Delta||_\infty \geq \lambda \}, \quad (15)$$

$$H_2(x) \leq H_1(x + 2 \lambda \cdot 1) + c(d) \left( p + \exp \left( - \frac{\lambda}{c(d) \tau} \right) \right) + P \{ ||\Delta||_\infty \geq \lambda \}. \quad (16)$$

Choosing an appropriate $\lambda$, we see that inequalities (14) and (15)–(16) imply bounds for the Prokhorov and Lévy distances respectively.

Using Bernstein's inequality, we proved in [8] that, for $d = 1$,

$$P \{ |\Delta| \geq \gamma \} \leq 2 \max \{ e^{-\gamma^2/4|a|^2}, e^{-\gamma/4|a|_\infty} \}.$$  

(17)
for any $\gamma \geq 0$. This inequality can be used for the estimation of the right-hand sides of inequalities (14)–(16). For $d > 1$, we can apply inequality (17) coordinatewise.

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