Optimization of congested traffic flow in systems with a localized periodic inhomogeneity

Elad Tomer\textsuperscript{1,*}, Leonid Safonov\textsuperscript{1,2}, Nilly Madar\textsuperscript{1}, and Shlomo Havlin\textsuperscript{1}

\textsuperscript{1} Minerva Center and Department of Physics, Bar–Ilan University, 52900 Ramat–Gan, Israel
\textsuperscript{2} Department of Applied Mathematics and Mechanics, Voronezh State University, 394693 Voronezh, Russia

(Received November 21, 2018)

We study traffic flow on roads with a localized periodic inhomogeneity such as traffic signals, using a stochastic car-following model. We find that in cases of congestion, traffic flow can be optimized by controlling the inhomogeneity’s frequency. By studying the wavelength dependence of the flux in stop-and-go traffic states, and exploring their stability, we are able to explain the optimization process. A general conclusion drawn from this study is, that the fundamental diagram of traffic (density flux relation) has to be generalized to include the influence of wavelength on the flux, for the stop-and-go traffic. Projecting the generalized fundamental diagram on the density-flux plane yields a 2D region, qualitatively similar to that found empirically [B. S. Kerner, Phys. Rev. Lett. 81, 3797 (1998)] in synchronized flow.

The theory of traffic flow has been a subject of comprehensive study for more than half a century [1–13] due to its theoretical and practical importance. Much attention has been devoted to characterizing the different states of congested traffic [14–17], including synchronized flow and stop-and-go traffic.

Optimization of systems with localized inhomogeneities such as traffic signals or entrance ramps was also extensively studied in the last decades [18–24]. Conventional traffic signals and ramp control theories are able to optimize such quantities as the total delay time of all drivers in the system, assuming a well defined flux-density relation known as the 'fundamental diagram' of traffic flow.

This basic concept of a fundamental diagram was ingrained in traffic flow theory for many decades. It was believed that the density-flux relation can be displayed as a single curve or as a combination of two isolated curves (see e.g. [13,17]). Recently, an experimental study of Kerner [7,8] shows that such a fundamental diagram does not exist. Instead, synchronized traffic displays a two-dimensional region in the density-flux plane. That is, for a given value of density, there exist a range of possible flux values. Mechanisms to optimize traffic flow by approaching the highest values of this range have not yet been suggested.

In this paper we investigate whether this new insight on the nature of traffic flow can be applied for optimizing the flux close to its maximal value for a given congested density. For this purpose, systems with periodic localized inhomogeneities are studied, using a recent car-following model with multiple stable and metastable states [18]. Two types of periodic inhomogeneities are considered:

(a) A signalized intersection, and
(b) an on-ramp with a signalized entrance.

Our study focus on cases of oversaturation, i.e. on cases where traffic is congested upstream to these inhomogeneities.

In the common traffic flow microscopic models such as in Refs. [10–14], the acceleration $a$ of a car depends on its headway $\Delta x$, velocity $v$, and velocity difference $\Delta v$ with the car ahead, i.e. $a = a(\Delta x, v, \Delta v)$. In particular, for the model in [14]

$$a = A \left(1 - \frac{\Delta x_0}{\Delta x}\right) - \frac{Z^2(-\Delta v)}{2(\Delta x - D)} - kZ(v - v_{\text{per}}) + \eta' ,$$

(1)

where $T$ is the safety time gap, $D$ and $\Delta x_0 = vT + D$ are the minimal and the optimal distance to the car ahead. $A$, $k$, and $v_{\text{per}}$ are constants, and the function $Z$ is defined as $Z(x) = (x + |x|)/2$. In the following numerical solutions of (1), the random term $\eta'$ is represented by choosing a random number in every iteration, uniformly distributed in the range $-0.5\eta \leq \eta' \leq 0.5\eta$. In the following we use the same choice of parameters as in [14] with $A = 3m/sec^2$, and a numerical time interval $\Delta t = 0.1\text{sec.}$

(a) Signalized intersection: In an oversaturated signalized intersection, the congested flow in each direction is affected only by the parameters of the traffic signals of this direction. Therefore we consider a single direction for simplicity (as e.g. in [15]). To study this case, a single traffic light is placed at position $L/2$ on a road with length $L$ and periodic boundary conditions. The flux $f$ is measured at the position of the traffic light.

*email: tomer@alon.cc.biu.ac.il
Simulations are performed with different durations of the red, amber (yellow), and green lights ($\tau_r, \tau_y, \tau_g$ respectively), and of $\tau_-$ which is the total time in each signal period where the intersection is not in use by any of its approaches. Therefore $\tau_g = \tau - \tau_-$, $\tau_y = \tau + \tau_-$ where $\tau$ is the total signal period. $P_r = \tau_r/\tau$ is the relative duration of the red light.

Our goal here is to optimize the flux in a specific road without affecting the parameters of the whole complex system of roads. We therefore consider $\tau_-$, $\tau_y$ and especially $P_r$ as given constraints, and explore the relation between the flux $f$ and the signal period $\tau$. Typical examples of this relation are shown in Fig. 1.

Apart from the trivial flux oscillations explained below, Fig. 1 shows a monotonic increase of the flux as $\tau$ grows, for the deterministic model ($\eta = 0$). For the stochastic model ($\eta > 0$), however, an optimal signal period can be clearly seen. In this case, a crossover is observed with increasing $\tau$ from the deterministic $f(\tau)$ to decreased values of flux. The optimal value of $\tau$ approaches the crossover point for small $\eta$. 

\[
\begin{align*}
\text{Flux} & \quad [\text{veh/sec}] \\
\text{Density} & \quad [\text{veh/m}] \\
\text{Wavelength} & \quad [\text{veh}^{-1}] \\
\text{Density-Flux} & \quad \text{plane (thin curves in Fig. 2a)}
\end{align*}
\]

The projection of the surface in Fig. 2a on the density-flux plane (thin curves in Fig. 2a) provides a two-dimensional region in this plane, qualitatively similar to that found empirically by Kerner [7,8] for synchronized traffic. In our system this phenomena occurs since the flux of the periodic states depends not only on the density but also on the state’s wavelength $\lambda$ [27]. That is, for a given value of density, there exist many stable periodic states which have a range of wavelengths in the density-flux plane. Since the flux is also determined by the wavelength (see Fig 2b), the density-flux relation becomes multi-valued.

The common traffic flow and signal control theories do not predict such influence of $\tau$ on the flux itself (see e.g. [18–20,23]). The average flow in each direction of a signalized intersection $f(\tau)$ for our consideration is expected to be

\[
f = f_0 \frac{\tau_g + \tau_y}{\tau} = f_0 \left(1 - P_r - \frac{\tau_-}{\tau}\right), \tag{2}
\]

where $f_0$ does not depend on $\tau$ (see [19] and references within). For the signal period values $\tau \gg \tau_-$ that are displayed in Fig. 1, one would expect to find $f(\tau) \approx \text{const.}$

\[
\begin{align*}
\text{FIG. 1. Relation between signal period and flux for the values of acceleration noise amplitude } \eta & = 0, 0.5, 2, 5, 10 \text{ m/sec}^2 \text{ (top to bottom). Traffic lights parameters are } P_r = 1/3, \\
& \tau_y = \tau_- = 2 \text{ sec. The total number of cars in the system is } N = 400 \text{ and its length is } L = 10 \text{ km. The nine open circles correspond to the nine instances presented in Fig. 3.}
\end{align*}
\]
FIG. 2. (a) Density-wavelength-flux relation for the different states of the deterministic model, including stable periodic states (surface), stable and unstable homogeneous states (thick and dotted lines, respectively). Some curves with fixed wavelength \((\lambda^{-1} = 2/60, 3/60, \ldots, 12/60)\) were projected on the density-flux plane (thin curves). (b) A cross-section of (a) for a density \(\rho = 0.06 \text{veh/m}\), demonstrating the typical dependence of the flux on the wavelength. (c) Noise stability threshold amplitude \(\eta_h\), above which the states presented in (b) become unstable.

Fig. 2c shows the noise stability threshold amplitude \(\eta_h\), above which the states presented in Fig. 2b become unstable. A comparison of the last two figures shows that states with flux values close to the minimal are the most stable in the presence of noise, while states with higher values of flux (close to the upper bound of the two dimensional region) are metastable. Therefore real-life stop-and-go traffic might be expected to show values of flux far below the optimum.

This new insight on the nature of the two dimensional region in the density-flux plane provides an explanation to our finding of a non-trivial relation between flux and signal period. For \(\eta = 0\), the monotonically increasing \(f(\tau)\) displayed in Fig. 1 corresponds to the left branch of Fig. 2b. High values of \(\tau\) stimulate transition to states with high values of \(\lambda\) and therefore with high values of flux, according to Fig. 2b. These metastable states survive since there is no noise \([28]\). The oscillations in flux observed for relatively small \(\tau\)'s can be easily explained \([29]\).

The crossover observed in Fig. 1 for the stochastic model \((\eta > 0\) curves) can be related to a crossing of the noise stability threshold \(\eta_h\). For values of \(\tau\) below the crossover point, the value of \(f(\tau)\) is close to that of the deterministic model, since \(\eta < \eta_h\). When \(\tau\) (and therefore \(\lambda\)) is increased, two trends are expected according to Figs 2b and 2c: an increase in the flux and a decrease in \(\eta_h\), until \(\eta \geq \eta_h\), where a crossover to values of flux lower than that of the deterministic case. Therefore the optimal \(\tau\) is usually close to the crossover point. Note that this crossover from deterministic to non-deterministic behavior with changing \(\tau\) occurs in spite of the fact that the noise amplitude is fixed, and is related to the \(\tau\) (and \(\lambda\)) dependence of \(\eta_h\).

We therefore see that the non-trivial flux-wavelength relation is the reason for the unexpected behavior of \(f(\tau)\). The deviations between the theoretical prediction of \([3]\) and the numerical measurements presented in Fig. 1 and the crossover observed in \(f(\tau)\) for \(\eta > 0\) are related to the relation between flux and wavelength discussed above and to differences in noise stability threshold of different periodic states.

To visualize the effect of the signal period and the acceleration noise amplitude on the flow, nine space-time diagrams are presented in Fig. 3a, below, at, and above the crossover. These nine diagrams correspond to the nine instances denoted by open circles in Fig. 1. Each dot in these space-time diagrams represents the position of a single car at a certain time. The dark regions therefore show the dense regions on the road. Looking at the diagrams in Fig. 3a, one can see the increase of the dominant wavelength with increasing \(\tau\). But unlike the deterministic case where the flow is periodic and the flux is increasing with \(\lambda\) and therefore with \(\tau\), in the stochastic model \((\eta > 0)\) small jams emerge in the low density regions when the noise or the signal period exceed certain thresholds. In such cases, other values of wavelength, smaller than that induced by the traffic light are effectively involved, resulting values of flux lower than that of the \(\eta = 0\) case (see Fig. 1).

For further support of this interpretation we evaluate the periodicity in the flow using single vehicle data collected at the intersection. Inspired by \([3]\) we calculate the auto-covariance \(ac_v(t)\) of the velocity function \(v(t')\) measured at the intersection,

\[
ac_v(t) = \frac{< v(t')v(t' + t) > - < v(t') > < v(t' + t) >}{< v(t')^2 > - < v(t') >^2},
\]

where a linear continuation of the discrete function \(v(t')\) is used. The brackets \(< \ldots >\) indicate averaging over time \(t'\). Displayed in Fig. 3b are the auto-covariance functions for the 9 instances of Fig. 3a, respectively. As can be seen from this figure, \(ac_v(t = \tau) = 1\) for all the \(\eta = 0\) instances, implying the flow for these cases is completely periodic, and that the time period is equal to the period of the traffic light. For \(\eta > 0\), \(ac_v(t = \tau)\) decreases as \(\eta\) or \(\tau\) are increased. This decrease is related to the appearance of small jams as can be seen in Fig. 3a. The decrease in \(ac_v(t = \tau)\) with \(\tau\) also implies that noise becomes more effective as \(\tau\) grows. A comparison of Fig. 3b to Fig. 1 shows that when \(ac_v(t = \tau) \approx 1\), the flux approaches the deterministic value.
To the position of the traffic light is at \( x = 5 \text{km} \). Gray lines in (b) correspond to \( \eta = 0 \), solid lines to \( \eta = 2 \), and dashed lines to \( \eta = 5 \).

**b) On-ramp with signalized entrance:** Next, we study another example of a localized inhomogeneity, caused by an on-ramp. To make this inhomogeneity periodic, we introduce traffic signals at the downstream end of the ramp, and study its effect on the flux in the main road. We focus on cases where the average incoming flux \( f_{in} \) is high enough to induce congestion on the main road (see [5]), but still low enough to avoid congestion on the secondary road. Similarly to [17], we introduce also an off-ramp with equal flux of outgoing vehicles \( f_{out} = f_{in} \) in a large distance from the on-ramp, so that the total number of cars in the system is conserved. The entrance and the exit of cars from the ramps are performed in an adiabatic manner as in [16]. The incoming vehicles are allowed to enter the main road during the green light period \( \tau_g \), and are stopped during the red light period \( \tau_r \). Here the signal period is \( \tau = \tau_g + \tau_r \), and the relative duration of the green light is \( P_g = \frac{\tau_g}{\tau} \). But unlike the signalized intersection where \( P_r \) was predetermined, here \( P_g \) is one of the optimization parameters, in addition to \( \tau \). The range of possible values for this parameter is \( f_{in}/f_{max} \leq P_g \leq 1 \) where \( f_{max} \) is the maximal possible value of the incoming flux. This lower bound of \( P_g \) is considered to avoid congestion on the secondary road, since cars approach the queue near the traffic light with rate \( f_{in} \), and this queue is discharged with rate \( f_{max} \) during green light. The upper bound \( P_g = 1 \) is related to an unsignalized on-ramp.

Typical relations between flux and signal period are shown in Fig. 4a, for different values of \( P_g \) and for \( \eta = 2m/s^2 \). The lower curve corresponds to an unsignalized on-ramp \( (P_g = 1) \), so it is found that the introduction of a traffic light increases the flux on the main road. Since the lower bound of \( P_g \) ensures that the average influx from the on-ramp remain the same as in the unsignalized on-ramp, we can say that the increases in the flux on the main road is obtained without causing congestion on the secondary road.

As can be seen from Fig. 4a, for each value of \( P_g \) there usually exist a single maximum in the flux. The short relative durations of green light \( P_g \approx f_{in}/f_{max} \) usually (but not always) yields a higher flux. A comparison of the maximal flux at the optimal signal parameters to the flux without a traffic light is presented in Fig 4b, for different values of noise. The relative increase in the flux due to the introduction of a traffic light varies from 1.0% for \( \eta = 10m/s^2 \), through 10.0% for \( \eta = 2m/s^2 \) up to 13.9% for \( \eta = 0 \).

![Auto-covariance](image)

**FIG. 3.** (a) Space-time diagrams and (b) auto-covariance functions of systems with single traffic light with parameters as the nine instances denoted with open circles in Fig. 1. The functions of systems with single traffic light with parameter \( s \).

![Flux vs. Signal Period](image)

**FIG. 4.** (a) Relation between on-ramp signal period and flux on the main road for \( P_g = 0.3, 0.5, 0.6, 0.8, 1.0 \) (top to bottom) and noise amplitude \( \eta = 2m/s^2 \). The total number of cars in the system is \( N = 300 \), its length is \( L = 10 \text{km} \), and the flux is locally measured on the main road 100m upstream to the on-ramp. Here \( f_{in} = 0.1 \) and \( f_{max} = 0.333 \). (b) A comparison between the optimal flux (upper curve) and the flux without the presence of a traffic light (lower curve), as a function of \( \eta \).

1. M. J. Lighthill, and G. B. Whitham, Proc. R. Soc. London Ser. A 229 (1955) 281.
2. K. Nagel, Phys. Rev. E 53, 4655 (1996).
3. D. E. Wolf, Physica A 263, 438 (1999).
4. D. Chowdhury, L. Santen, and A. Schadschneider, Phys. Rep. 329, 199 (2000).
5. D. Helbing, cond-mat/0012229.
[6] B. S. Kerner, H. Rehborn, Phys. Rev. Lett. 79, 4030 (1997).
[7] B. S. Kerner, Physics World 12 (8), 25 (1999).
[8] L. Neubert, L. Santen, A. Schadschneider, and M. Schreckenberg, Phys. Rev. E 60, 6480 (1999).
[9] M. Treiber, A. Hennecke, and D. Helbing, Phys. Rev. E 62, 1805 (2000).
[10] R. Herman, R. W. Rothery in Proceedings of the 2ns International Symposium on the Theory of Road Traffic Flow, London, 1963, edited by J. Almond, (Organization for Economic Cooperation and Development, Paris), 1 (1965).
[11] K. Nagel, M. Schreckenberg, J. Phys. I (France) 2, 2221 (1992).
[12] O. Biham, A. A. Middleton, and D. Levine, Phys. Rev. A. 46, R6124 (1992).
[13] M. Bando, K. Hasebe, A. Nakayama, A. Shibata and Y. Sugiyama, Phys. Rev. E 51, 1035 (1995).
[14] N. Mitarai and H. Nakanishi, Phys. Rev. Lett. 85, 1766 (2000).
[15] E. Tomer, L. Safonov, and S. Havlin, Phys. Rev. Lett. 84, 382 (2000).
[16] H. Y. Lee, H.-W. Lee, D. Kim, Phys. Rev. Lett. 81, 1130 (1998).
[17] A. D. May, Traffic Flow Fundamentals, Prentice Hall, Englewood Cliffs, N.J., 1990.
[18] D. Gazis and R. B. Potts in: Proceedings of the 2ns International Symposium on the Theory of Road Traffic Flow, London, 1963, edited by J. Almond, (Organization for Economic Cooperation and Development, Paris), 221 (1965).
[19] W. B. Cronje, Transport. Res. Rec. 905, 80 (1983).
[20] N. H. Gartner, C. Stamatiadis, and P. J. Tarnoff, Transport. Res. Rec. 1494, 98 (1995).
[21] H. Zhang, S. G. Ritchie, and W. W. Recker, Transport. Res. C 4, 51 (1996).
[22] T. H. Chang and J. T. Lin, Transport. Res. B 34, 471 (2000).
[23] D. Chowdhury and A. Schadschneider, Phys. Rev. E 59, R1311 (1999).
[24] When the light is changed from green to amber, all simulated drivers upstream to the intersection estimate the intersection crossing time $t_n$ by linear extrapolation of their position. The first car that begins to stop, $s$, is the first car that will not be able to cross before the light will change to red (i.e. $t_s = \min\{t_n | t_n > \tau_y\}$). This car is stopped by setting $\Delta x$ in Eq. (1) to be the distance to the traffic light position. The consecutive cars follow $s$ and stop according to (1). Due to this procedure cars might still pass the intersection during time $\tau_y$.
[25] Interactive simulations of the deterministic model with and without traffic lights can be found at http://ory.ph.biu.ac.il/2000/traffic/
[26] The wavelength is defined as the average distance (in units of cars) between two nearest dense regions in stop-and-go traffic.
[27] Similar increase of the flux with decreasing $\tau$ for relatively low values of $\tau$ is suppressed in Fig. 1 due to relatively large values of $\tau - /\tau$ in (2).
[28] These oscillations are caused by the discretization of the flowing media (vehicles) and by the fact that all drivers are identical. Therefore, the magnitude of these oscillations is highest for $\eta = 0$, for which the crossing times in each cycle are the same. It is therefore clear that the magnitude of oscillations decay linearly with $\tau$. 