SPECTRAL ASYMPTOTICS FOR THE THIRD ORDER OPERATOR WITH PERIODIC COEFFICIENTS

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Abstract. We consider the self-adjoint third order operator with 1-periodic coefficients on the real line. The spectrum of the operator is absolutely continuous and covers the real line. We determine the high energy asymptotics of the periodic, anti-periodic eigenvalues and of the branch points of the Lyapunov function. Furthermore, in the case of small coefficients we show that either whole spectrum has multiplicity one or the spectrum has multiplicity three except for a small spectral nonempty interval with multiplicity three. In the last case the asymptotics of this small interval is determined.

1. Introduction and main results

We consider the self-adjoint operator $H$ acting in $L^2(\mathbb{R})$ and given by

$$H = i\partial^3 + ip\partial + i\partial p + q$$

(1.1)

where the real 1-periodic coefficients $p, q$ belong to the space $L^1(T), T = \mathbb{R}/\mathbb{Z}$, equipped with the norm $\|f\| = \int_0^1 |f(s)|ds$. Without loss of generality we assume that

$$\int_0^1 q(t)dt = 0.$$  

(1.2)

A great number of papers is devoted to the inverse spectral theory for the Schrödinger operator with periodic potential: Dubrovin [D], Garnett–Trubowitz [GT], Its–Matveev [IM], Kargaev–Korotyaev [KK], Marchenko–Ostrovski [MO], Novikov [No] etc. Note that Korotyaev [K3] extended the results of [MO], [GT], [KK], for the case $-y'' + qy$ to the case of periodic distributions, i.e., $-y'' + q'y$ on $L^2(\mathbb{R})$, where periodic $q \in L^2_{loc}(\mathbb{R})$.

The results for the operator $H$ are used in the integration of the bad Boussinesq equation, given by

$$\ddot{\bar{p}} = \frac{1}{3} \partial^2 \left( \partial^2 p + 4p^2 \right), \quad \dot{p} = \partial q,$$

(1.3)

on the circle, see [McK] and references therein. Here $\dot{u}$ (or $\partial u$) means the derivatives of the function $u$ with respect to the time (or space) variable. It is equivalent to the Lax equation $\dot{H} = HK - KH$ where $K = -\partial^2 + \frac{4}{3}p$.

The inverse scattering theory for the self-adjoint third order operator $i\partial^3 + ip\partial + i\partial p + q$ with decreasing coefficients was developed in [DTT]. McKean [McK] obtained the numerous results in the inverse spectral theory for the non-self-adjoint operator $H_s = \partial^3 + p\partial + \partial p + q$ on the real line with smooth and sufficiently small $p$ and $q$. Results for non-self-adjoint operator $H_s$ was applied for integration of the good Boussinesq equation.

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It is known much less about the self-adjoint third order operators with periodic coefficients. Even the direct problem is not well developed.

Results about the spectrum of the higher order differential operators with smooth periodic coefficients are given in the book [DS], see also McGarvey’s paper [McG]. The case of the even order differential operators with non-smooth coefficients are given in [BK3].

The operator $H$ was considered by the authors in [BK4]. In order to describe our main results we recall needed results from [BK4]. We consider the differential equation

$$iy'''+ipy'+i(py')'+qy = \lambda y, \quad (t, \lambda) \in \mathbb{R} \times \mathbb{C}. \quad (1.4)$$

Introduce the $3 \times 3$ matrix-valued function $M(t, \lambda)$ by

$$M(t, \lambda) = \begin{pmatrix} \varphi_1 & \varphi_2 & \varphi_3 \\ \varphi'_1 & \varphi'_2 & \varphi'_3 \\ \varphi''_1 + p\varphi_1 & \varphi''_2 + p\varphi_2 & \varphi''_3 + p\varphi_3 \end{pmatrix} (t, \lambda), \quad (t, \lambda) \in \mathbb{R} \times \mathbb{C}, \quad (1.5)$$

where $\varphi_1, \varphi_2, \varphi_3$ are the fundamental solutions of equation (1.4) satisfying the conditions $M(0, \lambda) = \mathbb{1}_3, \quad \forall \lambda \in \mathbb{C}. \quad (1.6)$

Henceforth $\mathbb{1}_3$ is the $3 \times 3$ identity matrix. We define the modified monodromy matrix by $M(1, \lambda)$, below it is called shortly the monodromy matrix. The matrix-valued function $M(1, \cdot)$ is entire and its characteristic polynomial $D$ is given by

$$D(\tau, \lambda) = \det(M(1, \lambda) - \tau \mathbb{1}_3), \quad (\tau, \lambda) \in \mathbb{C}^2. \quad (1.7)$$

An eigenvalue of $M(1, \lambda)$ is called a multiplier, it is a zero of the algebraic equation $D(\cdot, \lambda) = 0$. If $\tau(\lambda)$ is a multiplier for some $\lambda \in \mathbb{C}$, then $\overline{\tau(\lambda)}^{-1}$ is also a multiplier. Each $M(1, \lambda), \lambda \in \mathbb{C}$, has exactly 3 (counting with multiplicities) multipliers $\tau_j(\lambda), j = 1, 2, 3,$ which satisfy

$$\tau_j(\lambda) = e^{i\omega_j^{-1}} (1 + O(|\lambda|^{-1})) \quad \text{as} \quad |\lambda| \to \infty. \quad (1.8)$$

Henceforth we put

$$z = \lambda^{\frac{1}{3}}, \quad \arg \lambda \in \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right], \quad \arg z \in \left(-\frac{\pi}{6}, \frac{\pi}{2}\right], \quad \omega = e^{i\frac{2\pi}{3}}.$$

The operator $H$ is self-adjoint on the domain

$$\text{Dom}(H) = \left\{ f \in L^2(\mathbb{R}) : i(f'' + pf)' + ipf' + qf \in L^2(\mathbb{R}), f'', (f'' + pf)' \in L^1_{\text{loc}}(\mathbb{R}) \right\}. \quad (1.9)$$

The spectrum of $H$ is absolutely continuous and satisfies

$$\sigma(H) = \left\{ \lambda \in \mathbb{R} : |\tau_j(\lambda)| = 1, \text{ some } j = 1, 2, 3 \right\} = \mathfrak{G}_1 \cup \mathfrak{G}_3 = \mathbb{R}, \quad \mathfrak{G}_2 = \emptyset,$$

where $\mathfrak{G}_j$ is the part of the spectrum of $H$ having the multiplicity $j = 1, 2, 3$. The spectrum of $H$ has multiplicity 1 at high energy. Note that the spectrum of the even order operator with periodic coefficients has multiplicity 2 at high energy, see [BK3]. Recall that the spectrum of the Hill operator $-\partial^2 + q$ on the real line is absolutely continuous and consists of spectral bands separated by gaps. In the case of the “generic” potential all gaps in the spectrum are open, see [MO], [K3]. In the case of third order operators with periodic coefficients there are no gaps in the spectrum for all $p, q$.

The coefficients of the polynomial $D(\cdot, \lambda)$ are entire functions of $\lambda$. Due to (1.8) there exist exactly three roots $\tau_1(\cdot), \tau_2(\cdot)$ and $\tau_3(\cdot)$, which constitute three distinct branches of some
analytic function \( \tau(\cdot) \) that has only algebraic singularities in \( \mathbb{C} \), see, e.g., Ch. 8 in [Fo]. Thus the function \( \tau(\cdot) \) is analytic on some 3-sheeted Riemann surface \( \mathcal{R} \). There are only a finite number of the algebraic singularities in any bounded domain. In order to describe these points we introduce the discriminant \( \rho(\lambda), \lambda \in \mathbb{C} \), of the polynomial \( D(\cdot, \lambda) \) by
\[
\rho = (\tau_1 - \tau_2)^2(\tau_1 - \tau_3)^2(\tau_2 - \tau_3)^2. \tag{1.10}
\]
The function \( \rho = \rho(\lambda) \) is entire and real on \( \mathbb{R} \). Due to McKean [McK] a zero of \( \rho \) is called a ramification and it is the branch point of the corresponding Riemann surface \( \mathcal{R} \). Ramification is a geometric term used for 'branching out', in the way that the square root function, for complex numbers, can be seen to have two branches differing in sign. We also use it from the opposite perspective (branches coming together) as when a covering map degenerates at a point of a space, with some collapsing together of the fibers of the mapping. In the case of the non-self-adjoint operator \( H \), the ramifications are invariant with respect to the Boussinesq flow and they can consider as the spectral data, see [McK].

If \( p = q = 0 \), then the function \( \rho \) and its zeros \( r_n^{0, \pm} \) have the form
\[
\rho_0 = 64 \sinh^2 \frac{\sqrt{3}z}{2} \sinh^2 \frac{\sqrt{3}\omega z}{2} \sinh^2 \frac{\sqrt{3}\omega^2 z}{2}, \quad r_n^{0, +} = r_n^{0, -} = i \left( \frac{2\pi n}{\sqrt{3}} \right)^3, \quad n \in \mathbb{Z}. \tag{1.11}
\]
We formulate our first results about the zeros \( r_n^{\pm}, n \in \mathbb{Z} \), of the function \( \rho \) (the ramifications).

**Theorem 1.1.**

i) The function \( \rho \) is entire, real on \( \mathbb{R} \), and satisfies
\[
\rho(\lambda) = |T(\lambda)|^4 - 8 \Re T^3(\lambda) + 18|T(\lambda)|^2 - 27, \quad \forall \ \lambda \in \mathbb{R}, \tag{1.12}
\]
where
\[
T(\cdot) = \text{Tr} M(1, \cdot). \tag{1.13}
\]

ii) Let \( \mathcal{G}_3 \) be the part of the spectrum of \( H \) having the multiplicity 3. Then
\[
\mathcal{G}_3 = \{ \lambda \in \mathbb{R} : |\tau_j(\lambda)| = 1, \ \forall \ j = 1, 2, 3 \} = \{ \lambda \in \mathbb{R} : \rho(\lambda) \leq 0 \}. \tag{1.14}
\]
Moreover, there exists only a finite number \( (\geq 0) \) of the bounded spectral bands with the spectrum of multiplicity 3.

iii) The ramifications \( r_n^{\pm} \) as \( n \rightarrow +\infty \) satisfy:
\[
r_n^{\pm} = r_n^{-\pm} = r_n^{0, \pm} - i \frac{4\pi n}{\sqrt{3}} \left( \hat{p}_n \mp |\hat{p}_n| + O(w_n) + O(n^{-1}) \right), \tag{1.15}
\]
for some sequence \( w_n, n \in \mathbb{N} \), such that \( \sum_{n \geq 1} \frac{|w_n|^2}{n} < \infty \).

**Remark.**

i) Identities (1.11) show that the endpoints of every spectral interval with the spectrum of multiplicity 3 are the ramifications.

ii) In this paper we analyze the spectrum at high energy using the multipliers. The functions \( \Delta_j = \frac{1}{2}(\tau_j + \tau_j^{-1}), j = 1, 2, 3 \), are the branches of the standard Lyapunov function \( \Delta(\lambda) \) analytic on a 3-sheeted Riemann surface. This function is more convenient for analysis of the spectrum at finite energy. Note that identity (1.14) implies:
\[
\mathcal{G}_3 = \{ \lambda \in \mathbb{R} : \Delta_j(\lambda) \in [-1, 1], \ \forall \ j = 1, 2, 3 \}. \tag{1.14}
\]

The graph of a typical Lyapunov function and the spectrum \( \mathcal{G}_3 \) are shown by Fig. 1.
Figure 1. The function $ρ$, the Lyapunov function and the spectrum $Σ_3$

Let $(λ_{2n})_{n ∈ Z}$ be the sequence of eigenvalues of equation (1.4) with the 1-periodic boundary condition $y(x + 1) = y(x), x ∈ R$. Let $(λ_{2n+1})_{n ∈ Z}$ be the sequence of eigenvalues of the equation (1.4) with the anti-periodic boundary condition $y(x + 1) = -y(x), x ∈ R$. They are labeling (counted with multiplicity) by

\[\begin{align*}
    &\ldots \leq λ_{-4} \leq λ_{-2} \leq λ_0 \leq λ_2 \leq λ_4 \leq \ldots, & \text{the periodic eigenvalues,} \\
    &\ldots \leq λ_{-3} \leq λ_{-1} \leq λ_1 \leq λ_3 \leq \ldots, & \text{the anti-periodic eigenvalues.}
\end{align*}\]  (1.16)

If $p = q = 0$, then the periodic and antiperiodic eigenvalues are given by $λ_n^0 = (πn)^3, n ∈ Z$.

**Theorem 1.2.** i) The periodic and antiperiodic eigenvalues satisfy

\[λ_n = (πn)^3 - 2\hat{p}_0πn + \frac{o(1)}{n} \quad \text{as} \quad n \rightarrow ±∞.\]  (1.17)

ii) The entire function $T = \text{Tr} M(1, ·)$ and the spectrum $Σ_3$ of the multiplicity three are recovered by the periodic spectrum plus one antiperiodic eigenvalue or by the antiperiodic spectrum plus one periodic eigenvalue.

**Remark.** i) Asymptotics (1.15) of the ramification is sharp, since it is determined in terms of Fourier coefficients of $p, q$. Unfortunately, the asymptotics of the periodic and antiperiodic eigenvalues in (1.17) is not sharp.
Knowing the function $T$ and using identities (1.12), (2.6) we recover the functions $\rho(\lambda), D(\tau, \lambda)$ and all multipliers.

We consider the operator $H_\varepsilon$ acting in $L^2(\mathbb{R})$ on the domain (1.9) and given by

$$H_\varepsilon = i\partial^3 + \varepsilon(ip\partial + i\partial p + q)$$  \hspace{1cm} (1.18)

where $\varepsilon \in \mathbb{R}$ is a small coupling constant. We have the following result.

**Theorem 1.3.** Let $p \in L^1(T)$ satisfy $\int_0^1 p(t)dt = 0$. Then there exist two functions $r^\pm(\varepsilon)$, real analytic in the disk $\{|\varepsilon| < c\} \subset \mathbb{C}$ for some $c > 0$, such that $r^\pm(0) = 0$ and they satisfy

$$r^+(\varepsilon) - r^-(\varepsilon) = 4h^2 \varepsilon^3 + O(\varepsilon^4) \quad \text{as} \quad \varepsilon \to 0,$$  \hspace{1cm} (1.19)

$$\mathcal{S}_3 = \begin{cases} (r^-(\varepsilon), r^+(\varepsilon)) & \text{or} \quad (r^+(\varepsilon), r^-(\varepsilon)) \\
\emptyset & \text{if} \quad h > 0 \\
\emptyset & \text{if} \quad h < 0 \end{cases}$$  \hspace{1cm} (1.20)

where

$$h = \frac{2}{3} \sum_{n \geq 1} \left( \frac{||\hat{p}_n||^2}{(2\pi n)^2} - \frac{3||\hat{q}_n||^2}{(2\pi n)^4} \right).$$  \hspace{1cm} (1.21)

**Remark.**

i) The functions $r^\pm(\varepsilon)$ are the (nearest to $\lambda = 0$) zeros of the function $\rho(\lambda, \varepsilon)$.

ii) Let $\varepsilon > 0$ be small enough. If $h > 0$, then $r^\pm(\varepsilon)$ are real and there is a band of the spectrum of the multiplicity three. If $h < 0$, then $r^\pm(\varepsilon)$ are non-real and there is no a band of the spectrum of the multiplicity three.

iii) The proof of the theorem is based on the analysis of the monodromy matrix as $\varepsilon \to 0$, and identities (1.12), (1.14). We determine the asymptotics of $r^\pm(\varepsilon)$ in the form $r^\pm(\varepsilon) = r(\varepsilon) \pm 2h^2 \varepsilon^3 + O(\varepsilon^4)$ as $\varepsilon \to 0$ for some function $r$. This gives the asymptotics (1.19). In the proof we use the some technique developed for fourth order operator with small $p, q$, see [BK1, BK2].

We describe now the results for vector differential equations and higher order differential equations. We begin with the vector case, where more deep results are obtained. The inverse problem for vector-valued Sturm-Liouville operators on the unit interval with Dirichlet boundary conditions, including characterization, was solved by Chelkak-Korotyaev [CK1, CK2]. We mention that uniqueness for inverse problems for systems on finite intervals was studied in [Ma]. The periodic case is more complicated and a lot of papers are devoted only to the direct problem for periodic systems: Carlson [Ca1, Ca2], Gelfand–Lidskii [GL], Gesztesy and coauthors [CL, CG], Korotyaev and coauthors [CK, BBK, K1, K2], etc. The discrete periodic systems were studied in [KKu1, KKu2]. We describe results for the first and second order operators with the periodic $N \times N$ matrix-valued potential from [CK, K1, K2], which are important for us. In fact the direct problem is consisted from two steps:

First step:

1. the Lyapunov function on some Riemann surface is constructed and described,
2. sharp asymptotics of periodic eigenvalues and ramifications of the Lyapunov function are determined,
3. multiplicity of the spectrum, endpoints of gaps are the periodic or antiperiodic eigenvalues or the ramifications of the Lyapunov function are determined.

The second step is more complicated:
(4) the conformal mapping with real part given by the integrated density of states and imaginary part given by the Lyapunov exponent is constructed and the main properties are obtained,
(5) trace formulas (similar to the case of the Hill operators) are determined,
(6) global estimates of gap lengths in terms of $L^2$-norm of potentials are obtained.

Spectral theory for higher order operators with decreasing coefficients is well developed, see [BDT] and the references therein. Numerous papers are devoted to the fourth order operators on bounded interval: [B], [CPS], [McL], [S] etc. The third order operator on the bounded interval was considered by Amour [A1], [A2].

Even ($\geq 4$) order operators with periodic coefficients considered in the papers: Badanin–Korotyaev [BK1] – [BK3], Papanicolaou [P1] [P2], Mikhailets–Molyboga [MM1] [MM2], Tkachenko [Tk], see also references therein. The spectral analysis of the higher ($\geq 3$) order operators with periodic coefficients is more difficult, than the analysis of the first and second order systems with periodic matrix-valued potentials. The main reason is that the monodromy matrix for the first and second order systems has asymptotics in terms of cos and sin bounded on the real line. The asymptotics of the monodromy matrix for higher order operators has additional components in terms of cosh and sinh unbounded on the real line.

The $2N$-order ($N \geq 2$) operator with periodic coefficients was considered in [BK3] (the case $N = 2$ see also in [BK1], [BK2]) and only the first step was done. The conformal mapping for the higher order operator, which is important for the spectral analysis, is not still constructed and there are no gap length estimates in terms of the norms of potentials.

The plan of the paper is as follows. In Sect. 2 we describe the basic properties of the monodromy matrix $M$. In Sect. 3 we consider the function $\rho$ and the Riemann surface of the multipliers at high energy and prove Theorem 1.1(i),ii). In Sect. 4 we determine asymptotics of the ramifications and prove Theorem 1.1(iii). In Sect. 5 we determine asymptotics of the periodic and antiperiodic spectrum and prove Theorem 1.2. In Sect. 6 we consider the case of the small coefficients and prove Theorem 1.3. The technical proof of the asymptotics of the trace of the monodromy matrix $M$ is placed in Appendix.

2. Monodromy matrix

Consider the unperturbed equation $iy''' = \lambda y$. It has the solutions $e^{iz\omega j-1}j$, and the multipliers have the form $e^{iz\omega j-1}j$, $j = 1, 2, 3$, here and below

$$\omega = e^{i\frac{2\pi}{3}}, \quad z = x + iy = \lambda^{\frac{1}{3}} \in S, \quad \arg \lambda \in \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right], \quad S = \{z \in \mathbb{C} : \arg z \in (-\frac{\pi}{6}, \frac{\pi}{2})\};$$

(2.1)

and on the boundary of $S$ we identify each point $z = re^{i\frac{\pi}{3}}, r > 0$, with the point $re^{-i\frac{\pi}{3}}$. The trace $\text{Tr} M_0$ of the unperturbed monodromy matrix $M_0$ is an entire function in $\lambda$ given by

$$T_0 = \text{Tr} M_0 = e^{iz} + e^{i\omega z} + e^{i\omega^2 z}. \quad (2.2)$$

Consider the perturbed equation (1.4). The matrix-valued function $M(t, \lambda)$, given by (1.5), satisfies

$$M' - P(\lambda)M = Q(t)M, \quad M(0, \lambda) = I_3, \quad \text{all } (t, \lambda) \in \mathbb{R} \times \mathbb{C}, \quad (2.3)$$
The matrix-valued function $M$ of the monodromy matrix for the even order operator [BK3]. Moreover, $\lambda$ for all $z$ in (2.3) is only bounded. It is a crucial point for our analysis.

Using this transformation we rewrite the problem (2.3) in the form

$$U = \begin{pmatrix} 0 & 0 & i \\ 0 & -i & 0 \\ i & 0 & 0 \end{pmatrix}$$

The matrix-valued function $M(1, \cdot)$ is entire and satisfies

$$\det M(1, \lambda) = 1, \quad M^*(1, \lambda)JM(1, \lambda) = J$$

where $J = \begin{pmatrix} 0 & 0 & i \\ 0 & -i & 0 \\ i & 0 & 0 \end{pmatrix}$ (2.5)

for all $\lambda \in \mathbb{C}$, see [BK4]. This identity is an odd-dimensional analog of the symplectic property of the monodromy matrix for the even order operator [BK3]. Moreover,

$$D(\tau, \lambda) = -\tau^3 + t^2T(\lambda) - \tau \overline{T}(\lambda) + 1, \quad \text{all } (\tau, \lambda) \in \mathbb{C}^2.$$ (2.6)

Introduce the simple transformation

$$M = \mathcal{U}^{-1}M\mathcal{U}, \quad \mathcal{U}^{-1}P\mathcal{U} = iz\Omega, \quad Q = \mathcal{U}^{-1}Q\mathcal{U} = \frac{1}{3iz}(pQ_1 + \frac{q}{z}Q_2)$$ (2.7)

where $\mathcal{U} = ZU$ and

$$\Omega = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & iz & 0 \\ 0 & 0 & (iz)^2 \end{pmatrix}, \quad U = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} = (U^*)^{-1},$$ (2.8)

$$Q_1 = \begin{pmatrix} -2 & \omega^2 & \omega \\ 1 & -2\omega^2 & \omega \\ \omega^2 & -2\omega & \omega^2 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ \omega & \omega^2 & \omega \end{pmatrix}.$$ (2.9)

Using this transformation we rewrite the problem (2.3) in the form

$$\mathcal{M} - iz\Omega\mathcal{M} = \mathcal{Q}(t, \lambda)\mathcal{M}, \quad \mathcal{M}(0, \lambda) = \mathbb{I}_3.$$ (2.10)

Identities (2.7), (2.9) show that the matrix $Q$ in the right hand side of equation (2.10) satisfies $Q(t, \lambda) = O((|p(t)| + |q(t)|)|z|^{-1})$ as $|\lambda| \to \infty$, while the corresponding coefficient $Q$ in (2.3) is only bounded. It is a crucial point for our analysis.

In [BK4] we proved that the monodromy matrix $\mathcal{M}$ and its trace $T(\lambda) = \text{Tr} \mathcal{M}(1, \lambda)$ satisfy

$$|\mathcal{M}(1, \lambda) - e^{iz\Omega}| \leq \frac{\kappa}{|z|} e^{z_0 + \kappa}, \quad \text{all } |\lambda| \geq 1,$$ (2.11)

$$|T(\lambda)| \leq 3e^{z_0 + \kappa}, \quad \text{all } \lambda \in \mathbb{C},$$

$$|T(\lambda) - T_0(\lambda)| \leq \frac{3\kappa}{|z|} e^{z_0 + \kappa}, \quad \text{all } |\lambda| \geq 1$$ (2.12)

where

$$\kappa = ||p|| + ||q||, \quad z_0 = \max_{j=0,1,2} \text{Re}(iz\omega^j) = \text{Re}(iz\omega^2) = \frac{y + \sqrt{3}x}{2}, \quad \frac{|z|}{2} \leq z_0 \leq |z|,$$ (2.13)

and $z = x + iy$. Henceforth a matrix $A$ has the norm given by

$$|A| = \max \{ \sqrt{E} \geq 0 : E \text{ is an eigenvalue of the matrix } A^*A \}.$$ (2.14)
Below we need sharper estimates of the monodromy matrix. Equation (2.10) and the standard arguments provide that the function \( M(t, \lambda) \) satisfies the integral equation
\[
M(t, \lambda) = e^{iz\Omega} + \int_0^t e^{iz(t-s)\Omega} Q(s, \lambda) M(s, \lambda) ds, \quad \text{all } (t, \lambda) \in \mathbb{R} \times (\mathbb{C} \setminus \{0\}),
\]
where the matrix \( \Omega \) is given by (2.8). The standard iterations yield
\[
M(t, \lambda) = \sum_{n=0}^\infty M_n(t, \lambda), \quad M_0(t, \lambda) = e^{iz\Omega} \tag{2.15}
\]
where
\[
M_n(t, \lambda) = \int_0^t e^{iz(t-s)\Omega} Q(s, \lambda) M_{n-1}(s, \lambda) ds, \quad n \in \mathbb{N}. \tag{2.16}
\]

**Lemma 2.1.** The series (2.15) converges absolutely and uniformly on any compact in \( \mathbb{C} \setminus \{0\} \). The matrix-valued function \( M(1, \cdot) \) is analytic in \( \mathbb{C} \setminus \{0\} \) and satisfies
\[
|M(1, \lambda)| \leq e^{\pi+\kappa}, \quad |M(1, \lambda) - \sum_{n=0}^{N-1} M_n(1, \lambda)| \leq \frac{e^{\pi+\kappa}}{|\lambda|^N}, \quad \text{all } N \in \mathbb{N}, \quad |\lambda| \geq 1. \tag{2.17}
\]

**Proof.** Identities (2.10) give
\[
M_n(t, \lambda) = \int_{0<t_1<\cdots<t_n<t_{n+1}=t} \prod_{k=1}^n \left( e^{iz(t_{k+1} - t_k)\Omega} Q(t_k) \right) M_0(t_1, \lambda) dt_1 dt_2 \cdots dt_n,
\]
the factors are ordering from right to left. Using estimates \( |e^{iz\Omega}| \leq e^{\pi t} \) we obtain
\[
|M_n(t, \lambda)| \leq \frac{e^{\pi t}}{n!} \left( \int_0^t |Q(s)| ds \right)^n, \quad \text{all } (n, t, \lambda) \in \mathbb{N} \times \mathbb{R}_+ \times (\mathbb{C} \setminus \{0\}). \tag{2.18}
\]
These estimates show that for each fixed \( t \in \mathbb{R}_+ \) the formal series (2.15) converges absolutely and uniformly on any compact in \( \mathbb{C} \setminus \{0\} \). Each term of this series is an analytic function in \( \mathbb{C} \setminus \{0\} \). Hence the sum is analytic in this domain. Summing the majorants and using the estimate \( \int_0^t |Q(s)| ds \leq \frac{\pi}{|\lambda|} \) we get (2.17). \( \blacksquare \)

In the following Lemma (proof in Appendix) we will determine asymptotics of the trace \( T \) of the monodromy matrix. Introduce the auxiliary function \( \phi \) which will be used in this asymptotics:
\[
\phi(t, \lambda) = \sum_{0 \leq k < j \leq 2} \omega^{2(k+j)} e^{iz} e^{i(\omega^k - \omega^j)zt}, \quad (t, \lambda) \in \mathbb{R} \times \mathbb{C}. \tag{2.19}
\]

For \( p \geq 1, \alpha \in \mathbb{R} \) we introduce the real spaces
\[
\ell^p = \left\{ f = (f_n)_{n \in \mathbb{Z}}, \quad \|f\|_p < \infty \right\}, \quad \|f\|_p^p = \sum_n |f_n|^p < \infty,
\]
\[
\ell^p_\alpha = \left\{ f = (f_n)_{n \in \mathbb{Z}}, \quad \sum_n (1 + |2\pi n|^{2\alpha}) |f_n|^p < \infty \right\}.
\]
We will write \( a_n = \ell^p_\alpha(n) \) iff the sequence \((a_n)_{n \in \mathbb{Z}} \in \ell^p_\alpha\).
Lemma 2.2. Let $p, q \in L^2(\mathbb{T})$. Then the function $T$ satisfies

$$T = \Phi_0 + \frac{\Phi_1}{z^2} + \frac{\tilde{\Phi}}{z^3}$$

(2.20)

where

$$\Phi_0 = \sum_{k=0}^{2} e^{iz\omega_k + \frac{2i\pi k}{2\pi + \tau}}, \quad \Phi_1(\lambda) = -\frac{1}{9} \int_0^1 \phi(s, \lambda) \eta(s) ds, \quad \eta(s) = \int_0^1 p(t)p(t-s) dt,$$

(2.21)

and the function $\tilde{\Phi}$ satisfies

$$\tilde{\Phi}(\lambda) = e^{\pi} \begin{cases} O(|\lambda|^{-1}) \text{ as } |\lambda| \to \infty, \quad \arg \lambda \in [-\frac{\pi}{4}, \frac{5\pi}{4}] \\
\ell_1(n) + O(n^{-1}) \text{ as } n \to +\infty, \quad \lambda = -i(\frac{4\pi n}{\sqrt{3}})^3(1 + O(n^{-2})) \end{cases}$$

(2.22)

uniformly in $\arg \lambda \in [-\frac{\pi}{4}, \frac{5\pi}{4}]$.

3. Ramifications

In this Section we will consider the function $\rho$ given by (1.11). Now we will prove a Counting Lemma for the ramifications. Introduce the contours

$$C_a(r) = \{ \lambda \in \mathbb{C} : |z - a| < r \}, \quad a \in \mathbb{C}, \quad r > 0,$$

and the domains

$$D_{\pm n} = \{ \lambda \in \mathbb{C} : |z - e^{i\pm \frac{2\pi n}{2\sqrt{3}}}| < \frac{\pi}{2\sqrt{3}} \}, \quad n \geq 0.$$  

(3.1)

Lemma 3.1. i) The function $\rho$ is entire, real on $\mathbb{R}$, and satisfies identity (1.12) and asymptotics

$$\rho = \rho_0(1 + O(|z|^{-1})) \text{ as } |\lambda| \to \infty, \quad \lambda \in \mathbb{C} \setminus \bigcup_{n \in \mathbb{Z}} D_n.$$  

(3.2)

ii) For each odd $N > 0$ large enough the function $\rho$ has (counting with multiplicities) $2N$ zeros on the domain $\{ |\lambda| < (\frac{2N}{\sqrt{3}})^3 \}$ and for each $|n| > \frac{N-1}{2}$ exactly two zeros in the domain $D_n$. There are no other zeros.

iii) The function $\rho$ has a finite even number $2m \geq 0$ of real zeros, counted with multiplicities.

Proof. i) The standard formula for the discriminant $d$ of the cubic polynomial $-\tau^3 + \alpha^2 - \beta r + 1$ gives $d = (ab)^2 - 4(a^3 + b^3) + 18ab - 27$ (see, e.g., [Co], Ch.7.5) which implies that $\rho$ is entire and real on $\mathbb{R}$ and satisfies (1.12).

We will show (3.2). Let $|\lambda| \to \infty$. Asymptotics (1.8) yields

$$\frac{\tau_1(\lambda) - \tau_3(\lambda)}{e^{iz} - e^{i\omega z}} = 1 + \frac{e^{i(1-\omega^2)z}O(z^{-1}) + O(z^{-1})}{e^{i(1-\omega^2)z} - 1}.$$  

(3.3)

We have

$$-\text{Re} i(1-\omega^2)z = \text{Im}(1-\omega^2)z = \frac{\sqrt{3}}{2}(x + \sqrt{3}y) \geq 0.$$  

Then $|e^{i(1-\omega^2)z}| \leq 1$ for all $\lambda \in \mathbb{C}$. Moreover, using the standard estimate $|\sin z| > \frac{1}{4}e^{\text{Im} z}$ as $|z - \pi n| \geq \frac{\pi}{4}$ for all $n \in \mathbb{Z}$ (see [PT], Lemma 2.1), we deduce that

$$|e^{i(1-\omega^2)z} - 1| = 2|e^{i(1-\omega^2)z}|\left|\sin \frac{(1-\omega^2)z}{2}\right| > \frac{1}{2}e^{\frac{1}{2}\text{Re} i(1-\omega^2)z}e^{\frac{1}{4}\text{Im}(1-\omega^2)z} = \frac{1}{2}$$  

-
as \( \lambda \in \mathbb{C} \setminus \cup_{n \geq 1} D_{-n} \). Then asymptotics (3.3) gives

\[
\tau_1(\lambda) - \tau_3(\lambda) = (e^{iz} - e^{i\omega z})(1 + O(z^{-1})) \quad \text{as} \quad \lambda \in \mathbb{C} \setminus \cup_{n \geq 1} D_{-n}.
\]

(3.4)

The similar arguments show that

\[
\tau_2(\lambda) - \tau_3(\lambda) = (e^{i\omega z} - e^{iz})(1 + O(z^{-1})) \quad \text{as} \quad \lambda \in \mathbb{C} \setminus \cup_{n \geq 1} D_{-n}.
\]

(3.5)

Furthermore, asymptotics (1.8) yields

\[
\frac{\tau_1(\lambda) - \tau_2(\lambda)}{e^{iz} - e^{i\omega z}} = 1 + \frac{e^{i(1-\omega)z}O(z^{-1}) + O(z^{-1})}{e^{(1-\omega)z} - 1} = 1 + \frac{e^{i(\omega-1)z}O(z^{-1}) + O(z^{-1})}{1 - e^{i(\omega-1)z}}.
\]

(3.6)

The relations

\[
\text{Re}(1 - \omega)z = -\text{Im}(1 - \omega)z = \frac{\sqrt{3}}{2}(x - \sqrt{3}y) \quad \left\{ \begin{array}{ll}
\geq 0, & \text{Re} \lambda \geq 0 \\
< 0, & \text{Re} \lambda < 0
\end{array} \right.
\]

give

\[
|e^{i(1-\omega)z}| \leq 1, \quad \text{all } \lambda : \text{Re} \lambda < 0; \quad |e^{i(\omega-1)z}| \leq 1, \quad \text{all } \lambda : \text{Re} \lambda \geq 0.
\]

Moreover, if \( \lambda \in \mathbb{C} \setminus \cup_{n \geq 1} D_{n} \), then

\[
|e^{i(1-\omega)z} - 1| = 2|e^{i(1-\omega)\frac{z}{2}}|\left|\sin\left(\frac{1 - \omega}{2}z\right)\right| > \frac{1}{2}e^{\frac{1}{2}|\text{Re}(1 - \omega)z|}e^{\frac{1}{2}|\text{Im}(1 - \omega)z|} = \frac{1}{2}
\]

as \( \text{Re} \lambda < 0 \), and similarly \( |e^{i(\omega-1)z} - 1| = \frac{1}{2} \) as \( \text{Re} \lambda \geq 0 \). Then asymptotics (3.6) gives

\[
\tau_1(\lambda) - \tau_2(\lambda) = (e^{iz} - e^{i\omega z})(1 + O(z^{-1})) \quad \text{as} \quad \lambda \in \mathbb{C} \setminus \cup_{n \geq 1} D_{n}.
\]

(3.7)

Substituting asymptotics (3.4), (3.5), (3.7) into (1.10) we obtain (3.2).

ii) Let \( N \geq 1 \) be odd and large enough and let \( N' > N \) be another odd. Let \( \lambda \) belong to the contours \( C_0(\frac{2N}{\sqrt{3}}), C_0(\frac{2N'}{\sqrt{3}}) \) and \( \partial D_n \) for all \( |n| > \frac{N-1}{2} \). Asymptotics (3.2) yields

\[
|\rho(\lambda) - \rho_0(\lambda)| = \frac{|\rho(\lambda)|}{\rho_0(\lambda)} = 1 = |\rho_0(\lambda)|O(|z|^{-1}) < |\rho_0(\lambda)|
\]

on all contours. Hence, by Rouché’s theorem, \( \rho(\cdot) \) has as many zeros, as \( \rho_0(\cdot) \) in each of the bounded domains and the remaining unbounded domain. Since \( \rho_0(\cdot) \) has exactly one zero of multiplicity two at each \( D_n, n \in \mathbb{Z} \), and since \( N' > N \) can be chosen arbitrarily large, the statement i) follows.

iii) Asymptotics (3.2) shows that \( \rho(\lambda) > 0 \) on the real line for large \( |\lambda| > 0 \). Then \( \rho \) has a finite number of real zeros. The function \( \rho \) has even number of zeros in the large disk due to the statement ii). The function \( \rho \) is real on \( \mathbb{R} \), hence it has both an even number of non-real zeros in this disc and an even number of real zeros. ■

Recall that the function \( \rho \) has a finite number \( m \) of real zeros. Using the results of Lemma 3.1 we index the zeros \( r_n^+, n \in \mathbb{Z} \), of \( \rho \) by:

a) labeling of real zeros:

\[
m \text{ even : } r_n^- \leq r_{n-1}^- \leq \ldots \leq r_{-m}^- \leq r_{-m+1}^- \leq \ldots \leq r_1^- \leq r_0^+,
\]

\[
m \text{ odd : } r_n^- \leq r_{n+1}^- \leq \ldots \leq r_{-m+1}^- \leq r_{-m}^- \leq \ldots \leq r_1^- \leq r_0^+,
\]
b) labeling of non-real zeros:

\[ m \text{ even : } 0 < \text{Im} r_+^m \leq \text{Im} r_+^{m+1} \leq \ldots, \]
\[ m \text{ odd : } 0 < \text{Im} r_+^{m-1} \leq \text{Im} r_+^{m+1} \leq \ldots, \]

and

\[ r_{-n}^± = r_n^±. \] (3.8)

**Proof of Theorem 1.1 i), ii).**

i) The statement is proved in Lemma 3.1 i).

ii) The first identity in (1.14) was proved in [BK4]. We will prove the second one. Let \( \tau_j = e^{ik_j}, j = 1, 2, 3, \) where \( \text{Re} k_j \in (-\pi, \pi]. \) We have \( k_1 + k_2 + k_3 = 0, \) since \( \tau_1 \tau_2 \tau_3 = 1 \). If \( k \neq \ell \) and \( \lambda \in \mathbb{C} \) is not a ramification, then \( k_j(\lambda) \neq k_i(\lambda). \) Identity (1.10) gives

\[ \rho = (e^{ik_1} - e^{ik_2})^2(e^{ik_1} - e^{ik_3})^2(e^{ik_2} - e^{ik_3})^2 = -64 \sin^2 \frac{k_1 - k_2}{2} \sin^2 \frac{k_1 - k_3}{2} \sin^2 \frac{k_2 - k_3}{2}. \] (3.9)

We have two cases. If \( \lambda \in \mathcal{G}_3, \) then the first identity in (1.14) implies that each \( k_j(\lambda) \in \mathbb{R}, j = 1, 2, 3, \) and (3.9) yields \( \rho(\lambda) \leq 0. \)

If \( \lambda \in \sigma(H) \setminus \mathcal{G}_3, \) then exactly one \( k_j, \) say \( k_1, \) is real and \( k_3 = \overline{k_2} \) are non-real, since if \( \lambda \in \mathbb{R} \) and \( \tau_2 = e^{ik_2} \) is a multiplier, then \( \tau_3 = \tau_2^{-1} = \overline{e^{ik_2}} \) is also a multiplier. Thus identity (3.9) implies \( \rho(\lambda) = 64 |\sin \frac{k_1 - k_2}{2}|^2 \sin^2 \text{Im} k_3 > 0, \) which yields the second identity in (1.14).

By Lemma 3.1, the function \( \rho \) has a finite even number of real zeros. Therefore, there exists only a finite number of the bounded spectral bands with the spectrum of multiplicity 3.

We will show in the following Lemma that the large multipliers in \( \mathbb{C}_+ \) specify by the dominating multiplier \( \tau_3 \) in the sense that these multipliers are zeros of the function

\[ \psi(\lambda) = \tau_3(\lambda) - (\tau_3(\lambda))^2, \] (3.10)

see (3.15). Moreover, here we determine the rough high energy asymptotics of the ramifications and the multipliers at the large ramifications.

**Lemma 3.2.** Let the branches of the multipliers be define by (1.8). Then

i) The multipliers satisfy

\[ \tau_1^{-1}(\lambda) = \tau_1(\lambda), \quad \tau_2^{-1}(\lambda) = \tau_2(\lambda) \] (3.11)

for all \( \lambda \in \Lambda_R \) and for some \( R > 0 \) large enough.

ii) Let \( \lambda = r_n^± \) and let \( n \to +\infty. \) Then

\[ \tau_1(\lambda) = \tau_2(\lambda) = (-1)^n e^{-\frac{2\pi n}{\sqrt{3}}}(1 + O(n^{-1})), \quad \tau_3(\lambda) = e^{\frac{2\pi n}{\sqrt{3}}}(1 + O(n^{-1})), \] (3.12)

\[ \tau_1(\lambda) = \tau_3(\lambda) = (-1)^n e^{\frac{2\pi n}{\sqrt{3}}}(1 + O(n^{-1})), \quad \tau_2(\lambda) = e^{-\frac{2\pi n}{\sqrt{3}}}(1 + O(n^{-1})). \] (3.13)

iii) The ramifications satisfy

\[ r_n^± = i\left(\frac{2\pi n}{\sqrt{3}}\right)^3(1 + O(n^{-2})) \quad \text{as} \quad n \to +\infty. \] (3.14)

iv) Let \( \lambda \in \mathbb{C}_+ \cap \Lambda_R \) for some \( R > 0 \) large enough, where the domains \( \Lambda_R = \{ \lambda \in \mathbb{C} : |\lambda| > R \}. \) Then

\[ \lambda \text{ is a ramification} \quad \iff \quad \psi(\lambda) = 0. \] (3.15)
Proof. i) Recall that \( \tau(\lambda) \) is a multiplier iff \( \tau^{-1}(\lambda) \) is a multiplier. Asymptotics (1.8) and identity \( \bar{\omega} = \omega^2 \) give

\[
\tau^{-1}(\lambda) = e^{iz}(1 + O(|z|^{-1})), \quad \tau_2^{-1}(\lambda) = e^{i\omega z}(1 + O(|z|^{-1})), \quad \tau_3^{-1}(\lambda) = e^{i\omega z}(1 + O(|z|^{-1}))
\]
as \( |\lambda| \to \infty \), which yields (3.11).

ii), iii) If \( \lambda = r_n^\pm \), then \( \tau_j(\lambda) = \tau_k(\lambda) \) for some \( j, k = 1, 2, 3 \). Let \( n \to +\infty \). By Lemma 3.1, \( z = (r_n^\pm)^{1/3} = e^{i\frac{\pi}{3}}(\frac{2\pi n + \delta_n}{\sqrt{3}}) \) where \( |\delta_n| < \frac{\pi}{2\sqrt{3}} \). Asymptotics (1.8) gives \( \tau_j(\lambda) = O(e^{-\frac{\pi n}{\sqrt{3}}}), j = 1, 2, \) and

\[
\tau_3(\lambda) = e^{\frac{2\pi n + \delta_n}{\sqrt{3}}}(1 + O(n^{-1})).
\]

These asymptotics show that \( \tau_3(\lambda) \neq \tau_j(\lambda) \) for \( j = 1, 2 \) and for all \( n \in \mathbb{N} \) large enough. Then \( \tau_1(\lambda) = \tau_2(\lambda) \), which implies the first identity in (3.12).

We will prove (3.14). Let \( \lambda = r_n^\pm \) and let \( n \to +\infty \). Then

\[
iz = i(r_n^\pm)^{1/3} = ie^{i\frac{\pi}{3}}\left(\frac{2\pi n + \delta_n}{\sqrt{3}} + O(n^{-1})\right) = e^{i\frac{2\pi n}{\sqrt{3}} + O(n^{-1})}
\]

and asymptotics (1.8) yields

\[
\tau_1(\lambda) = e^{i(2\frac{2\pi n}{\sqrt{3}} + \delta_n)}(1 + O(n^{-1})), \quad \tau_2(\lambda) = e^{i(2\frac{2\pi n}{\sqrt{3}} + \delta_n)}(1 + O(n^{-1})).
\]

Substituting these asymptotics into the first identity in (3.12) and using \( \omega - \omega^2 = i\sqrt{3} \) we obtain \( e^{i\delta_n} = 1 + O(n^{-1}) \). Then \( \delta_n = O(n^{-1}) \), which yields (3.14).

Substituting \( \delta_n = O(n^{-1}) \) into (3.16), (3.17) we obtain asymptotics (3.12). These asymptotics together with (3.11) yield (3.13).

iv) Let \( \lambda \in \mathbb{C}^3 \cap \Lambda_R \) be a ramification. Identities (3.11) and (3.12) yield \( \tilde{\tau}_3^{-1}(\lambda) = \tau_2(\lambda) = \tau_1(\lambda) \). Using the identity

\[
\tau_1(\lambda) \tau_2(\lambda) \tau_3(\lambda) = 1
\]

we obtain \( \psi(\lambda) = 0 \). Conversely, let \( \psi(\lambda) = 0 \) for some \( \lambda \in \mathbb{C}^3 \cap \Lambda_R \). Then identity (3.11) implies \( \tau_3(\lambda) = \tau_2^{-1}(\lambda) \). Identity (3.13) yields \( \tau_1(\lambda) = \tau_2(\lambda) \), therefore \( \lambda \) is a ramification. □

Remark. 1) We describe the surface of the multipliers for the case \( p = q = 0 \). We have \( \tau^0(\lambda) = e^{i\frac{\lambda^3}{3}} \). In this case, then our surface is the 3-sheeted Riemann surface of the function \( \lambda^3 \). We have two parametrization of this surface.

First parametrization. We construct the standard (escator type) parametrization \( \tilde{R}^0 \) of the surface of the function \( \lambda^3 \) if we take 3 replicas \( \tilde{R}_{j}^0, j = 1, 2, 3 \), of the cut plane \( \mathbb{C} \setminus \mathbb{R}_- \) and join the edge of the cut on the sheet \( \tilde{R}_1 \) with the edge of the cut on the sheet \( \tilde{R}_2 \), the edge of the cut on the sheet \( \tilde{R}_1 \) with the edge of the cut on the sheet \( \tilde{R}_3 \), and the edge of the cut on the sheet \( \tilde{R}_3 \) with the edge of the cut on the sheet \( \tilde{R}_1 \), in the usual (crosswise) way, see Fig. (2 a).

Describe the function \( \tau^0(\lambda) \). Consider the sectors \( S_j = \omega^{j-1}S, j = 1, 2, 3 \), on the \( z \)-plane, where \( S \) is given by (2.1). In each of the sectors the function \( \lambda^\frac{1}{3} \), and then \( \tau^0(\lambda) \), is univalent. Moreover, for each \( j = 1, 2, 3 \), the function \( \tau^0(\lambda) \) satisfies: \( \tau^0(\lambda) = \tau_j^0(\lambda) = e^{i\omega^{j-1}z} \) in the sector \( \omega^{j-1}S \). The function \( \lambda = z^3 \) maps each of the sectors \( S_j, j = 1, 2, 3 \), onto the sheet \( \tilde{R}_j^0 \) of the surface \( \tilde{R}^0 \). For each \( j = 1, 2, 3 \), the function \( \tau^0(\lambda) \) satisfies: \( \tau^0(\lambda) = \tau_j^0(\lambda) \) on the sheet \( \tilde{R}_j^0 \).

Second parametrization. In order to consider below the perturbed case it will be convenient to use the other parametrization \( R^0 \) of the Riemann surface of the function \( \lambda^3 \), see Fig. (2 b).
Figure 2. The Riemann surface of the function $\lambda^\pm$, which coincides with the Riemann surface of the multipliers for the case $p = q = 0$, in a) the standard (escalator type) parametrization, b) the parametrization convenient for the considering of the perturbed case.

We take 3 replicas of the cut plane $\mathcal{R}^0_1 = \mathbb{C} \setminus i\mathbb{R}_+$, $\mathcal{R}^0_2 = \mathbb{C} \setminus i\mathbb{R}$ and $\mathcal{R}^0_3 = \mathbb{C} \setminus i\mathbb{R}_-$. We obtain the Riemann surface $\mathcal{R}^0$ by joining the edges of the cut $i\mathbb{R}_+$ on $\mathcal{R}^0_1$ and on $\mathcal{R}^0_2$ and the edges of the cut $i\mathbb{R}_-$ on $\mathcal{R}^0_2$ and on $\mathcal{R}^0_3$ in the usual (crosswise) way.

If we deform continuously the surface $\mathcal{R}^0$ so that the right half-plane $\text{Re}\lambda > 0$ of the sheet $\mathcal{R}^0_1$ and the right half-plane of the sheet $\mathcal{R}^0_2$ swap places, then we obtain the surface $\tilde{\mathcal{R}}^0$. The function $\tau^0(\lambda)$ satisfies:

a) $\tau^0(\lambda) = \tau^0_3(\lambda)$ on the sheet $\mathcal{R}^0_3$;

b) $\tau^0(\lambda) = \tau^0_1(\lambda)$ on the left half-plane $\text{Re}\lambda < 0$ of the sheet $\mathcal{R}^0_1$ and on the right half-plane $\text{Re}\lambda > 0$ of the sheet $\mathcal{R}^0_2$;

c) $\tau^0(\lambda) = \tau^0_2(\lambda)$ on the left half-plane of the sheet $\mathcal{R}^0_2$ and on the right half-plane of the sheet $\mathcal{R}^0_1$.

2) Consider the perturbed case. The surface $\mathcal{R}$ of the multipliers for large $|\lambda|$ is close to the surface $\mathcal{R}^0$, see Fig. [3]. Let $R = \left(\frac{N\pi}{\sqrt{3}}\right)^3$, where $N$ is given by Lemma [3.1 ii]). Then $r^\pm_n \in \mathcal{D}_n$ for each $|n| > \frac{N-1}{2}$. Describe the surface $\mathcal{R}$ for $|\lambda| > R$. We take 3 replicas $\mathcal{R}_1, \mathcal{R}_2$ and $\mathcal{R}_3$ of the cut plane:

$$\mathcal{R}_1 \cap \Lambda_R = \Lambda_R \setminus \cup_{n \in \mathbb{N}} \Gamma_n, \quad \mathcal{R}_2 \cap \Lambda_R = \Lambda_R \setminus \cup_{n \in \mathbb{Z}} \Gamma_n, \quad \mathcal{R}_3 \cap \Lambda_R = \Lambda_R \setminus \cup_{n \in \mathbb{N}} \Gamma_{-n},$$

where $\Lambda_R = \{ \lambda \in \mathbb{C} : |\lambda| > R \}$, $\Gamma_n = [r^+_n, r^-_n] \subset \Lambda_R$, $n \in \mathbb{Z}$, are straight lines. The surface $\mathcal{R}$ at large $|\lambda|$ is obtained by joining the edges of the cuts $\Gamma_n$ on $\mathcal{R}_1$ with the edges of the same cuts on $\mathcal{R}_2$, the edges of the cuts $\Gamma_{-n}$ on $\mathcal{R}_2$ with the edges of the same cuts on $\mathcal{R}_3$ in the crosswise way.
Figure 3. The Riemann surface $\mathcal{R}$ of the multipliers and the ramifications for the case of the coefficients $p_0 + p, q$, with small $p, q$ and constant $p_0$.

4. ASYMPTOTICS OF THE RAMIFICATIONS

Introduce the entire function $\xi$ by

$$\xi(\lambda) = 4T(\lambda) - \overline{T}(\lambda), \quad \lambda \in \mathbb{C}. \quad (4.1)$$

Lemma 4.1. Let $\lambda = r_n^\pm$. Then the function $\psi(\lambda) = \tau_3(\lambda) - \overline{\tau_3}(\lambda)$ satisfies

$$\psi(\lambda) = \left(\frac{1}{4} + O(n^{-1})\right)(\xi(\lambda) + O(e^{-\pi n \sqrt{3}})) \quad \text{as} \quad n \to +\infty. \quad (4.2)$$

Proof. Identity (2.6) provides

$$D(\tau(\lambda), \lambda) = -\tau^2(\lambda)(\tau(\lambda) - T(\lambda) + \tau^{-1}(\lambda)\overline{T}(\lambda) + \tau^{-2}(\lambda)), \quad (4.3)$$

$$\overline{D}(\tau(\overline{\lambda}), \overline{\lambda}) = -\tau(\overline{\lambda})(\tau(\overline{\lambda}) - T(\overline{\lambda})\overline{T}(\overline{\lambda}) + T(\lambda) - \overline{T}(\lambda)) \quad (4.4)$$

for all $\lambda \in \mathbb{C}$. Let $\lambda = r_n^\pm$ and let $n \to +\infty$. Identity (4.3) and asymptotics (3.12) imply

$$D(\tau_3(\lambda), \lambda) = -\tau_3^2(\lambda)(\tau_3(\lambda) - T(\lambda) + O(e^{-\pi n \sqrt{3}})).$$

The identity $D(\tau_3(\lambda), \lambda) = 0$ yields

$$\tau_3(\lambda) = T(\lambda) + O(e^{-\pi n \sqrt{3}}). \quad (4.5)$$
Identities (3.15), (4.1) and asymptotics (3.13), (4.5) give
\[ \overline{D}(\tau_3, \lambda) = -\tau_3(\lambda)(\tau_3(\lambda) - T(\lambda) + O(e^{-\frac{\pi}{\sqrt{\lambda}}})) \]
\[ = -\tau_3(\lambda)(2T(\lambda) - T(\lambda) + O(e^{-\frac{\pi}{\sqrt{\lambda}}})). \]  
(4.6)

Asymptotics (3.12), (3.13) and identity \( T = \tau_1 + \tau_2 + \tau_3 \) give
\[ T(\lambda) = e^{\frac{2\pi n}{3}}(1 + O(n^{-1})), \quad \overline{T}(\lambda) = 2(-1)^n e^{\frac{2\pi n}{3}}(1 + O(n^{-1})). \]  
(4.7)

Asymptotics (4.5), (4.8) and the identity \( \overline{D}(\tau_3, \lambda) = 0 \) yield
\[ \tau_3^2(\lambda) = \frac{4T^2(\lambda)}{\overline{T}(\lambda)} + O(e^{-\frac{\pi}{\sqrt{\lambda}}}). \]  
(4.8)

Asymptotics (4.5), (4.8) give
\[ \psi(\lambda) = \tau_3(\lambda) - \tau_3^2(\lambda) = T(\lambda) - \frac{4T^2(\lambda)}{\overline{T}(\lambda)} + O(e^{-\frac{\pi}{\sqrt{\lambda}}}). \]

Then asymptotics (4.7) yields (4.2). ■

Below we will often use the following simple result.

**Lemma 4.2.** Let \( f \in C([0, 1]) \) and let \( \alpha > 0, \beta \in \mathbb{R} \). Then
\[ \int_0^1 e^{(-\alpha+i\beta)xu} f(u) du = \frac{f(0) + o(1)}{\alpha - i\beta} x \]  as \( x \to +\infty. \]  
(4.9)

**Proof.** We have
\[ \int_0^1 e^{(-\alpha+i\beta)xu} f(u) du = \frac{(1 + e^{(-\alpha+i\beta)x})f(0)}{\alpha - i\beta} x + \int_0^1 e^{(-\alpha+i\beta)xu} k(u) du, \quad \text{all } x > 0, \]  
(4.10)

where \( k(u) = f(u) - f(0) \). Furthermore,
\[ A = \left| \int_0^1 e^{(-\alpha+i\beta)xu} k(u) du \right| \leq \int_0^1 e^{-\alpha xu} |k(u)| du + \int_0^1 e^{-\alpha xu} |k(u)| du \]
\[ \leq \max_{[0, \delta]} |k(u)| \int_0^\delta e^{-\alpha xu} du + \max_{[0, 1]} |k(u)| \int_\delta^1 e^{-\alpha xu} du \leq \frac{\max_{[0, \delta]} |k(u)| + e^{-\alpha \delta} \max_{[0, 1]} |k(u)|}{\alpha x} \]
for any \( \delta \in (0, 1) \). Let \( x \to +\infty \) and let \( \delta = \frac{\log x}{x} \). Then \( A = o(x^{-1}) \). Substituting this estimate into (4.10) we obtain (4.11).

The following Lemma gives the asymptotics of the function \( \xi \), given by (1.1), at the large ramifications in \( \mathbb{C}_+ \).

**Lemma 4.3.** Let \( p, q \in L^2(\mathbb{T}) \). Let \( \lambda = r_n^\pm = z^3, \zeta = i\omega^2 z \), and let \( n \to +\infty \). Then the function \( \xi \), given by (4.1), satisfies
\[ \xi(\lambda) = 4e^{\frac{2\pi n}{3} \zeta} \left[ \sin^2 \left( \frac{\sqrt{3}\zeta}{2} + \frac{\hat{p}_n}{2\pi n} + O(n^{-3}) \right) - \frac{|\hat{p}_n|^2}{12(\pi n)^2} + \ell_1^1(n) + O(n^{-4}) \right]. \]  
(4.11)
Proof. Asymptotics (2.20) yields
\[ \xi = \xi_0 + \frac{\xi_1}{z^\frac{1}{2}} + \frac{\xi_2}{z^\frac{3}{2}} \] (4.12)
where
\[ \xi_0(\lambda) = 4\Phi_0(\lambda) - \overline{\Phi_0(\lambda)}, \quad \xi_1(\lambda) = 4\Phi_1(\lambda) - 2\overline{\Phi_0(\lambda)\Phi_1(\lambda)}, \] (4.13)
\[ \xi_2(\lambda) = 4\Phi(\lambda) - 2\overline{\Phi_0(\lambda)\Phi(\lambda)} + O(e^{-2n\nu_1}). \] (4.14)
Asymptotics (3.14) implies \( \zeta = \frac{2\pi n}{\sqrt{3}} + O(n^{-1}) \). Substituting this asymptotics into identity (2.21) we get
\[ \Phi_0(\lambda) = e^{\zeta - \frac{2\pi n}{\sqrt{3}}} + O(e^{-\frac{2\pi n}{\sqrt{3}}}), \quad \overline{\Phi_0(\lambda)} = 2e^{\frac{2\pi n}{\sqrt{3}}} \cos\left(\frac{\sqrt{3}\xi}{2} + \frac{\hat{\nu}_0}{\sqrt{3}\xi}\right) + O(e^{-\frac{2\pi n}{\sqrt{3}}}). \] (4.15)
Substituting (2.22), (4.15) into (4.14) we obtain
\[ \Phi_0(\lambda) = e^{\zeta_0(t^1(n) + O(n^{-1}))} \] (4.16)
Asymptotics (4.13) give
\[ \xi_0(\lambda) = 4e^{\zeta - \frac{2\pi n}{\sqrt{3}}} \sin^2\left(\frac{\sqrt{3}\xi}{2} + \frac{\hat{\nu}_0}{\sqrt{3}\xi}\right) + O(e^{-\frac{2\pi n}{\sqrt{3}}}). \] (4.17)
Assume that
\[ \xi_1(\lambda) = \frac{4}{9} \omega^2 e^\xi \left| \nu_n \right|^2 + \ell^1_2(n) + O(n^{-2}). \] (4.18)
Substituting asymptotics (4.17), (4.18) into (4.12) we obtain
\[ \xi(\lambda) = 4e^{\zeta - \frac{2\pi n}{\sqrt{3}}} \sin^2\left(\frac{\sqrt{3}\xi}{2} + \frac{\hat{\nu}_0}{\sqrt{3}\xi}\right) - \frac{3}{9} \left| \nu_n \right|^2 + \ell^1_2(n) + O(n^{-4}), \] which yields (4.11).
We will prove (4.18). Substituting identity (2.21) into (4.13) we obtain
\[ \xi_1(\lambda) = -\frac{2}{9} \int_0^1 \alpha(u, \lambda)\eta(u)du \] (4.19)
where
\[ \eta(u) = \int_0^1 p(t)p(t-u)dt, \quad \alpha(u, \lambda) = 2\phi(u, \lambda) - \overline{\Phi_0(\lambda)}\overline{\phi(u, \lambda)}, \] (4.20)
\( \phi \) is given by (2.19). Identities (2.19) imply
\[ \phi(t, \lambda) = e^\xi e^{(\omega^2 - 1)\xi t} + \omega e^{\xi e^{(\omega^1 - 1)\xi t}} + \omega^2 e^{\omega^2 \xi e^{(\omega - \omega^2)\xi t}}, \quad \text{all } (t, \lambda) \in \mathbb{R} \times \mathbb{C}. \]
Using \( \omega - \omega^2 = i\sqrt{3} \) we have \( e^{\omega^2 \xi e^{(\omega - \omega^2)\xi t}} = O(e^{-\frac{2\pi n}{\sqrt{3}}}) \) and then
\[ \phi(t, \lambda) = e^\xi \left( e^{(\omega^2 - 1)\xi t} + \omega e^{(\omega^1 - 1)\xi t} + O(e^{-\sqrt{3}\pi n}) \right) \] (4.21)
uniformly on \( t \in [0, 1] \). Moreover,
\[ \overline{\phi(t, \lambda)} = e^{-\omega^2 \xi e^{(\omega^2 - 1)\xi t}} + \omega e^{-\omega^1 e^{(\omega - \omega^1)\xi t}} + \omega^2 e^{-\omega^2 \xi e^{(\omega^2 - \omega)\xi t}}, \quad \text{all } (t, \lambda) \in \mathbb{R} \times \mathbb{C}. \] (4.22)
Asymptotics (4.13) implies
\[ \overline{\Phi_0(\lambda)} = 2(-1)^n e^{\frac{2\pi n}{\sqrt{3}}} \left( 1 + O(n^{-1}) \right). \] (4.23)
Relations (4.22), (4.23) yield
\[ \Phi_0(\lambda) \phi(t, \lambda) = 2(-1)^n e^\xi (e^{i\frac{2}{3} \omega^2 \xi t} + \omega e^{-i\frac{2}{3} \omega^2 \xi (1-t)} + \omega^2 e^{i\frac{2}{3} \omega^2 \xi (1-t)} (1 + O(n^{-1})) \]
\[ = 2e^\xi (e^{(\omega^2-1) t} + \omega e^{(\omega^2-1) (1-t)} + \omega^2 e^{(\omega^2-1) (1-t)} (1 + O(n^{-1})) \] (4.24)
uniformly on \( t \in [0, 1] \). Substituting identities (4.21), (4.24) into (4.20) we obtain
\[ \alpha(t, \lambda) = 2e^\xi \left( \omega e^{(\omega^2-1) t} - \omega e^{(\omega^2-1) (1-t)} - \omega^2 e^{-(\omega^2-1) (1-t)} (1 + O(n^{-1})) \right) \] (4.25)
uniformly on \( t \in [0, 1] \). Using the identity \( \eta(1-t) = \eta(t) \) we obtain
\[ \int_0^1 e^{(\omega-1) \xi t} \eta(t) dt = \int_0^1 e^{(\omega-1) (1-t) \xi \eta(t) dt. \] (4.26)
Substituting (4.25) into (4.19) and using (4.26) we obtain
\[ \xi_1(\lambda) = \frac{4}{9} \omega^2 e^\xi \int_0^1 \left( e^{(\omega^2-1) t} (1 + O(n^{-1})) + e^{(\omega^2-1) (1-t) (O(n^{-1})) + e^{(\omega^2-1) (1-t) O(n^{-1})} \right) \eta(t) dt \]
\[ = \frac{4}{9} \omega^2 e^\xi \int_0^1 \left( e^{-2(\omega^2-1) t} (1 + O(n^{-1})) + e^{-(\omega^2-1) (1-t) O(n^{-1})} + e^{-(\omega^2-1) (1-t) O(n^{-1})} \right) \eta(t) dt. \]
Asymptotics (4.9) and identity \( \int_0^1 e^{-2(\omega^2-1) n t} \eta(t) dt = |\hat{p}_n|^2 \) give (4.18).

Proof of Theorem 1.1 iii). Identities \( r_{-n} = r_{+n}^\pm \) are proved in (3.8). Let \( \lambda = r_n^\pm \) and let \( n \to +\infty \). Then (3.14) implies \( \lambda = r_n^\pm \) and \( \delta_n = O(n^{-1}) \). Relations (3.15), (4.2) give \( \xi(\lambda) = O(e^{-\frac{\pi}{\sqrt{n}}}) \). Substituting (4.11) into this asymptotics we get
\[ \sin^2 \left( \frac{\sqrt{3} \delta_n}{2} + \frac{\hat{p}_0}{2 \pi n} + O(n^{-3}) \right) = \frac{1}{12 (\pi n)^2} \left( |\hat{p}_n|^2 + \ell_1^2 (n) + O(n^{-1}) \right). \] (4.27)
Recall the estimate \(|(w^2 + \varepsilon^2)^{\frac{1}{2}} - w| \leq |\varepsilon| \) for all \( w, \varepsilon \in \mathbb{C} \) (see, e.g., [CKP], Ch.4.5). Substituting
\[ w = |\hat{p}_n|, \quad \varepsilon = (\ell_1^2 (n) + O(n^{-2}))^{\frac{1}{2}} = \ell_1^n (n) + O(n^{-1}) \]
into the last estimate we obtain
\[ \left( |\hat{p}_n|^2 + \ell_1^2 (n) + O(n^{-2}) \right)^{\frac{1}{2}} = |\hat{p}_n| + \ell_1^n (n) + O(n^{-1}). \]
Asymptotics (1.27) gives
\[ \delta_n = -\frac{\hat{p}_0}{\sqrt{3} \pi n} \pm \frac{1}{3 \pi n} \left( |\hat{p}_n| + \ell_1^n (n) + O(n^{-1}) \right), \]
which yields asymptotics (1.15). Theorem 1.1 is proved.
5. The periodic spectrum

In this Section we consider the periodic and antiperiodic eigenvalues labeling by (1.16). These eigenvalues are zeros of the entire functions \( D(\pm 1, \cdot) \), where \( D \) is given by (1.7). Identities (2.12) give
\[
D(1, \lambda) = 2i \text{Im} T(\lambda), \quad D(-1, \lambda) = 2 + 2 \text{Re} T(\lambda), \quad \forall \lambda \in \mathbb{R}. \tag{5.1}
\]
If \( p = q = 0 \), then the functions \( D(\pm 1, \lambda) \) have the form
\[
D_0(1, \lambda) = -8i \sin \frac{z}{2} \sin \frac{\omega z^2}{2}, \quad D_0(-1, \lambda) = 8 \cos \frac{z}{2} \cos \frac{\omega z^2}{2}. \tag{5.2}
\]
Now we will prove a Counting Lemma for the periodic and antiperiodic eigenvalues. Introduce the domains
\[
K_n = \left\{ \lambda \in \mathbb{C} : |z - \pi n| < \frac{\pi}{2} \right\}, \quad K_{-n} = \left\{ \lambda \in \mathbb{C} : |\omega z + \pi n| < \frac{\pi}{2} \right\}, \quad n \in \mathbb{N}.
\]

**Lemma 5.1.** i) The functions \( D(\pm 1, \lambda) \) as \(|\lambda| \to \infty \) satisfy
\[
D(1, \lambda) = D_0(1, \lambda)(1 + O(|z|^{-1})), \quad \lambda \in \mathbb{C} \setminus \bigcup_{n \in \mathbb{Z}} K_{2n}, \tag{5.3}
\]
\[
D(-1, \lambda) = D_0(-1, \lambda)(1 + O(|z|^{-1})), \quad \lambda \in \mathbb{C} \setminus \bigcup_{n \in \mathbb{Z}} K_{2n+1}. \tag{5.4}
\]
ii) For each odd \( N > n_0 \) for some \( n_0 \geq 1 \) the function \( D(1, \cdot) \) has exactly \( N \) zeros, counted with multiplicity, in the disk \( \{ \lambda : |\lambda| < (\pi N)^3 \} \) and for each even \( n : |n| > N \) exactly one simple zero in the domain \( K_n \). There are no other zeros.
iii) For each even \( N > n_0 \) for some \( n_0 \geq 1 \) the function \( D(-1, \cdot) \) has exactly \( N \) zeros, counted with multiplicity, in the disk \( \{ \lambda : |\lambda| < (\pi N)^3 \} \) and for each odd \( n : |n| > N \), exactly one simple zero in the domain \( K_n \). There are no other zeros.

**Proof.** We consider only the function \( D(1, \cdot) \). The proof for \( D(-1, \cdot) \) is similar.

i) Identities (5.1) and estimates (2.12) imply
\[
|D(1, \lambda) - D_0(1, \lambda)| \leq 2|T(\lambda) - T_0(\lambda)| \leq \frac{6k}{|\lambda|} e^{-\alpha \lambda + \kappa}, \quad \forall |\lambda| \geq 1. \tag{5.5}
\]
Substituting the estimate \(|\sin w| > \frac{1}{4} e^{0.5|\text{Im } w|} \) as \(|w - \pi n| > \frac{\pi}{4} \) for all \( n \in \mathbb{Z} \) into (5.2) and using the relations
\[
|\text{Im } z| + |\text{Im } \omega z| + |\text{Im } \omega z^2| = |y| + \frac{|\sqrt{3}x - y|}{2} + \frac{\sqrt{3}x + y}{2} \geq \sqrt{3}x + y = 2z_0
\]
we obtain
\[
|D_0(1, \lambda)| = 8 \left| \sin \frac{z}{2} \right| \left| \sin \frac{\omega z^2}{2} \right| > \frac{1}{8} e^{\frac{1}{2}(|\text{Im } z| + |\text{Im } \omega z| + |\text{Im } \omega z^2|)} \geq e^{\frac{\pi}{8}} \tag{5.6}
\]
for all \( \lambda \in \mathbb{C} \setminus \bigcup_{n \in \mathbb{Z}} K_{2n} \). Estimates (5.5) and (5.6) yield (5.3).

ii), iii) Let \( N \geq 1 \) be odd and large enough and let \( N' > N \) be another odd. Let \( \lambda \) belong to the contours \( C_0(\pi N), C_0(\pi N'), C_{\pi n}(\frac{3}{2}) \), \( |n| > N, n \) is even. Asymptotics (5.3) yield
\[
|D(1, \lambda) - D_0(1, \lambda)| = |D_0(1, \lambda)| \frac{|D(1, \lambda)|}{D_0(1, \lambda)} - 1 = |D_0(1, \lambda)| \frac{O(1)}{|\lambda|} < |D_0(1, \lambda)|
\]
on all contours. Hence, by Rouché’s theorem, \( D(1, \cdot) \) has as many zeros, as \( D_0(1, \cdot) \) in each of the bounded domains and the remaining unbounded domain. Since \( D_0(1, \cdot) \) has exactly one
Moreover, we will show that the function $T\lambda$ which gives the last asymptotics in (5.7). This asymptotics we obtain are zeros of the function $\tau$. Let $D$ statement for $\lambda$. Let $\lambda$. Proof. Then (1.8) gives $\tau\lambda = O(e^{\sqrt{T3\pi}\pi n})$. Then $|\tau\lambda| \neq 1$ and identity (3.11) implies $\tau_2(\lambda) = \tau_3^{-1}(\lambda)$. Then $(-1)^n = \tau_1(\lambda) = \frac{1}{\tau_2(\lambda)\tau_3(\lambda)} = \frac{\tau_3(\lambda)}{\tau_3(\lambda)}$, which yields the first identities in (5.7).

Estimates (2.12) yield

$$T(\lambda) = T_0(\lambda)(1 + O(z^{-1})) = (e^{iz} + e^{i\omega z} + e^{i\omega^2 z})(1 + O(z^{-1})) = O(e^{\sqrt{T3\pi}\pi n}).$$

Substituting (5.9) and (5.10) into (2.6) we obtain

$$D(\tau_3(\lambda), \lambda) = \tau_3^2(\lambda)(-\tau_3(\lambda) + T(\lambda) + O(1)).$$

The identity $D(\tau_3(\lambda), \lambda) = 0$ and (5.10) imply

$$\tau_3(\lambda) = T(\lambda) + O(1) = T(\lambda)(1 + O(e^{-\sqrt{T3\pi}\pi n})).$$

which gives the last asymptotics in (5.7).

Asymptotics (1.8) implies $(-1)^n = \tau_1(\lambda) = e^{iz}(1 + O(n^{-1}))$. Substituting $z = \pi n + \delta_n$ into this asymptotics we obtain $e^{i\delta_n} = 1 + O(n^{-1})$. Then $\delta_n = O(n^{-1})$ and $z = \pi n + O(n^{-1})$ which yields (5.8). ■

**Proof of Theorem 1.2 i).** Let $\lambda = \lambda_n, n \to +\infty$. Asymptotics (5.8) shows that $z = \pi n + \delta_n \in \mathbb{R}, \delta_n = O(n^{-1})$. Asymptotics (2.20) yields

$$T(\lambda) = e^{iz\omega^2} + \frac{2\delta_n}{i\omega\omega^2}(1 + O(e^{-\sqrt{T3\pi}\pi n}))) + \frac{\Phi_1(\lambda)}{z^2} + O(e^{-\omega^2}),$$

(5.11)

Identity (2.19) gives

$$\phi(t, \lambda) = e^{iz\omega^2}(e^{i(\omega - \omega^2) z t} + \omega^{-1}e^{i(\omega - \omega^2) z t} + \omega^2 e^{i(1 - \omega)(1 + t) z t}) = e^{iz\omega^2}(e^{-\sqrt{T3\pi} t + \omega e^{-\sqrt{T3\pi} e^{-\sqrt{T3\pi}} z t} + O(e^{-\sqrt{T3\pi} n}))$$

uniformly on $t \in [0, 1]$. Substituting this asymptotics into (2.21) and using (4.9) we obtain

$$\Phi_1(\lambda) = -\frac{e^{iz\omega^2}}{9} \left( \int_0^1 (e^{-\sqrt{T3}\omega} + \omega e^{-\sqrt{T3} e^{-\sqrt{T3}}\omega} \eta(u)) du + O(e^{-\sqrt{T3\pi} n}) \right) = e^{iz\omega^2} o(n^{-1}).$$
Substituting the last asymptotics into (5.11) we get
\[ T(\lambda) = e^{i\omega z + \frac{2i\delta_0}{3(\pi n + \delta_n)}} (1 + o(n^{-3})) \],
\[ \overline{T}(\lambda) = e^{-i\omega z - \frac{2i\delta_0}{3(\pi n + \delta_n)}} (1 + o(n^{-3})). \]
Substituting these asymptotics into (5.7) we obtain
\[ e^{-i\delta_n - \frac{2i\delta_0}{3(\pi n + \delta_n)}} = 1 + o(n^{-3}). \]
Then
\[ \delta_n = -\frac{2\rho_0}{3(\pi n + \delta_n)} + o(n^{-3}) = -\frac{2\rho_0}{3\pi n} - \frac{4\rho_0^2}{9(\pi n)^3} + o(n^{-3}), \]
which implies (1.17) for \( n \to +\infty \). The asymptotics for \( n \to -\infty \) can be obtained by substituting \(-t\) instead of \( t \) in equation (1.4).

We need the following Hadamard factorizations of the functions \( D(\pm 1, \cdot) \).

**Lemma 5.3.** The functions \( D(\pm 1, \lambda) \) satisfy
\[ D(1, \lambda) = i(\lambda_0 - \lambda) \prod_{n \neq 0} \frac{\lambda_{2n} - \lambda}{\lambda^0_{2n}}, \] (5.12)
\[ D(-1, \lambda) = 8 \prod_{n \neq 0} \frac{\lambda_{2n-1} - \lambda}{\lambda^0_{2n-1}}, \] (5.13)
oun uniformly on any bounded subset of \( \mathbb{C} \), where \( \lambda^0_n = (\pi n)^3, n \in \mathbb{Z} \).

**Proof.** We consider the functions \( D(1, \cdot) \). The proof for \( D(-1, \cdot) \) is similar. Asymptotics (1.17) show that the infinite product in (5.12) converges uniformly on any bounded subset of \( \mathbb{C} \) to the entire function of \( \lambda \), whose zeros are precisely \( \lambda_{2n}, n \in \mathbb{Z} \). Identity (5.2) yields
\[ D_0(1, \lambda) = -i\lambda \prod_{n \neq 0} \frac{\lambda^0_{2n} - \lambda}{\lambda^0_{2n}}. \] (5.14)

Asymptotics (5.3) and identity (5.2) show that the entire function \( D(1, \lambda) \) has order 1. Moreover, its zeros have asymptotics (1.17). Then
\[ D(1, \lambda) = ie^{A\lambda + B}(\lambda_0 - \lambda) \prod_{n \neq 0} \frac{\lambda_{2n} - \lambda}{\lambda^0_{2n}}, \text{ for some } A, B \in \mathbb{C}. \] (5.15)

Identities (5.14), (5.15) yield
\[ \log \frac{D(1, \lambda)}{D_0(1, \lambda)} = A\lambda + B + \log \frac{\lambda - \lambda_0}{\lambda} + \sum_{n \neq 0} \log \frac{\lambda_{2n} - \lambda}{\lambda^0_{2n} - \lambda}. \] (5.16)

Let \( \lambda \to +i\infty \). Asymptotics (5.3) implies \( \log \frac{D(1, \lambda)}{D_0(1, \lambda)} = O(|\lambda|^{-1}) \). Moreover, we have
\[ \sum_{n \neq 0} \log \frac{\lambda_{2n} - \lambda}{\lambda^0_{2n} - \lambda} = \sum_{n \neq 0} \log \left(1 + \frac{\lambda_{2n} - \lambda^0_{2n}}{\lambda^0_{2n} - \lambda} \right) = \sum_{n \neq 0} \log \left(1 + \frac{O(n)}{\lambda^0_{2n} - \lambda} \right) = o(1). \]

Identity (5.16) gives \( A = B = 0 \). Identity (5.14) gives (5.12).

Now we will prove the results about the recovering of the function \( \rho \) and the spectrum \( \sigma(H) \) by the periodic (or antiperiodic) spectrum.

**Proof of Theorem 1.2 ii.** Let \{\( \lambda_n, n \text{ even} \}\) be the periodic spectrum and \( \lambda_n \) be the antiperiodic eigenvalue. Identities (5.12), (5.1) give the function \( D(1, \cdot) \) and \( \text{Im} T(\lambda) = -\frac{1}{2} D(1, \lambda) \).
Then we reconstruct \( \text{Re} T(\lambda) \) up to some constant. This constant, and then the function \( T \), can be determined from the identity \( \text{Re} T(\lambda) = \frac{1}{2}D(-1, \lambda) - 1 = -1 \). Identity (1.12) provides the discriminant \( \rho \). Identity (1.14) gives the spectrum \( \mathcal{S}_3 \) of the multiplicity three.

Using the similar calculations we recover the function \( T \) and the spectrum of the multiplicity three by the antiperiodic spectrum and one periodic eigenvalue. \( \blacksquare \)

6. Small coefficients

We prove Theorem 1.3. Here we use some methods developed for fourth order operators with the small 1-periodic coefficients \([BK1], [BK2]\). We consider the equation

\[
y'''' + \varepsilon(y'' + iy') + qy = \lambda y, \quad \lambda \in \mathbb{C}.
\]

(6.1)

In this case the matrix-valued function \( M(t, \lambda) = M(t, \lambda, \varepsilon), (t, \lambda, \varepsilon) \in \mathbb{R} \times \mathbb{C} \times \mathbb{R} \), given by (1.5) is a solution of equation

\[
M' - P(\lambda)M = \varepsilon Q(t)M, \quad M(0, \lambda, \varepsilon) = I_3
\]

(6.2)

where the \( 3 \times 3 \) matrices \( P \) and \( Q \) are given by (2.4).

Consider the case \( \varepsilon = 0 \). The matrix-valued function \( M_0(t, \lambda) = M(t, \lambda, 0) \) has the form \( M_0(t, \lambda, z) = e^{\lambda t}P \). Each function \( M_0(t, \cdot), t \in \mathbb{R}, \) is entire. Eigenvalues of the matrix \( M_0 \) have the form \( e^{izt}, e^{i\omega z t}, e^{i \omega^2 t} \), since eigenvalues of the matrix \( P \) are given by \( iz, i\omega z, i\omega^2 z \). Estimates \( |e^{iz t}| \leq e^{z |t|} \) imply

\[
|M_0(t, \lambda)| \leq e^{z_0 |t|}, \quad \text{all} \ (t, \lambda) \in \mathbb{R} \times \mathbb{C},
\]

(6.3)

where \( z_0 \) is given by (2.13) and a matrix norm is given by (2.14).

Consider the case \( \varepsilon \neq 0 \). The solution \( M(t, \lambda, \varepsilon) \) of problem (6.2) satisfies the integral equation

\[
M(t, \lambda, \varepsilon) = M_0(t, \lambda) + \varepsilon \int_0^t M_0(t - s, \lambda)Q(s)M(s, \lambda, \varepsilon)ds.
\]

(6.4)

The standard iterations in (6.4) lead to the standard series

\[
M(t, \lambda, \varepsilon) = \sum_{n \geq 0} \varepsilon^n M_n(t, \lambda), \quad M_n(t, \lambda) = \int_0^t M_0(t - s, \lambda)Q(s)M_{n-1}(s, \lambda)ds, \quad n \geq 1.
\]

(6.5)

We need the uniform estimates of the monodromy matrix \( M(1, \lambda, \varepsilon) \).

**Lemma 6.1.** For each \( t \in \mathbb{R} \) the series (6.5) converges absolutely and uniformly on any bounded subset of \( \mathbb{C}^2 \). Each matrix-valued function \( M(t, \cdot, \cdot), t \in [0, 1] \) is entire in \( (\lambda, \varepsilon) \in \mathbb{C}^2 \) and satisfies:

\[
|M(1, \lambda, \varepsilon)| \leq e^{z_0 + \varepsilon}, \quad |M(1, \lambda, \varepsilon) - \sum_{n=0}^{N-1} M_n(1, \lambda, \varepsilon)| \leq |\varepsilon|^N N e^{z_0 + |\varepsilon|^N}
\]

(6.6)

for all \( (N, \lambda, \varepsilon) \in \mathbb{N} \times \mathbb{C}^2 \) where \( \varkappa = ||p|| + ||q|| \).

**Proof.** Consider the case \( t \geq 0 \). The proof for \( t < 0 \) is similar. Identity (6.5) gives

\[
M_n(t, \lambda) = \int_{0<t_1<\ldots<t_n<t_{n+1}=t} \prod_{k=1}^{n} (M_0(t_{k+1} - t_k, \lambda)Q(t_k)) M_0(t_1, \lambda)dt_1dt_2\ldots dt_n,
\]

(6.7)
the factors are ordering from right to left. Substituting estimates (6.3) into identities (6.7) we obtain
\[ |M_n(t, \lambda)| \leq \frac{e^{3t}}{n!} \left( \int_0^t |Q(s)| ds \right)^n, \quad \text{all } (n, t, \lambda) \in \mathbb{N} \times \mathbb{R}_+ \times \mathbb{C}. \] (6.8)

These estimates show that for each fixed \( t \geq 0 \) the formal series (6.5) converges absolutely and uniformly on any bounded subset of \( \mathbb{C}^2 \). Each term of this series is an entire function of \((\lambda, \varepsilon)\). Hence the sum is an entire function. Summing the majorants and using the estimate \( \int_0^1 |Q(s)| ds \leq \kappa \) we obtain (6.6). \( \blacksquare \)

We introduce the discriminant \( \rho(\lambda, \varepsilon), (\lambda, \varepsilon) \in \mathbb{C}^2 \), of the polynomial \( \det(M(\lambda, \varepsilon) - \tau \mathbb{I}_3) \) by identity (1.10).

**Lemma 6.2.** i) The function \( \rho(\lambda, \varepsilon) \) is entire in \( \mathbb{C}^2 \) and satisfies:
\[ \rho(\lambda, \varepsilon) = \rho_0(\lambda) \left( 1 + O(\varepsilon) \right) \quad \text{as} \quad \varepsilon \to 0, \] (6.9)
uniformly in \( \lambda \) on any bounded subset of \( D = \mathbb{C} \setminus \bigcup_{n \in \mathbb{Z}} D_n, D_n \) are given by (3.7).

ii) Let \( |\varepsilon| < c \) for some \( c > 0 \) small enough. Then the function \( \rho(\cdot, \varepsilon) \) has exactly two zeros, counted with multiplicities, in each domain \( D_n, n \in \mathbb{Z} \). There are no other zeros. In particular, the function \( \rho(\cdot, \varepsilon) \) has no any real zeros in the domain \(|\lambda| \geq 1\).

**Proof.** i) Identity (1.12) and Lemma 6.1 show that the function \( \rho(\lambda, \varepsilon) \) is entire. Estimates (6.6) give \( M(1, \lambda, \varepsilon) = M_0(1, \lambda) + O(\varepsilon) \) as \( \varepsilon \to 0 \) uniformly in \( \lambda \) on any compact in \( \mathbb{C} \). Let \( \lambda \in D \). Then all eigenvalues \( e^{iz}, e^{i\omega z}, e^{i\omega^2 z} \) of the matrix \( M_0(1, \lambda) \) are simple and the standard matrix perturbation theory (see, e.g., [HJ], Corollary 6.3.4) gives that the eigenvalues \( \tau_j(\lambda, \varepsilon), j = 1, 2, 3, \) of the matrix \( M(\lambda, \varepsilon) \) satisfy
\[ \tau_j(\lambda, \varepsilon) = e^{i\omega_{j-1} z} + O(\varepsilon) = e^{i\omega_{j-1} z} \left( 1 + O(\varepsilon) \right) \quad \text{as} \quad \varepsilon \to 0 \] uniformly in \( \lambda \) on any bounded subset of \( D \). Substituting these asymptotics into (1.10) and using the identity
\[ \rho_0 = (e^{iz} - e^{i\omega z})^2(e^{iz} - e^{i\omega^2 z})^2(e^{i\omega z} - e^{i\omega^2 z})^2 \] we obtain (6.9).

ii) Using asymptotics (6.9) and repeating the arguments from the proof of Lemma 3.1 ii) we obtain the statement. \( \blacksquare \)

Introduce the entire functions \( T_n(\lambda) = \text{Tr} M_n(1, \lambda), n \geq 0 \). Estimates (6.6) imply
\[ T(\lambda, \varepsilon) = \text{Tr} M(1, \lambda, \varepsilon) = T_0(\lambda) + \varepsilon T_1(\lambda) + \varepsilon^2 T_2(\lambda) + \varepsilon^3 T_3(\lambda) + O(\varepsilon^4) \quad \text{as} \quad \varepsilon \to 0 \] (6.10)
uniformly in \( \lambda \) on any compact in \( \mathbb{C} \). Below we will use the following relations.

**Lemma 6.3.** Let \( p \in L^1(\mathbb{T}) \) satisfy \( \int_0^1 p(t) dt = 0 \). Then the functions \( T_1 \) and \( T_2 \) satisfy
\[ T_1 = 0, \] (6.11)
\[ \text{Re} T_2(\lambda) = -3h(1 + O(\lambda)) \quad \text{as} \quad \lambda \to 0 \] (6.12)
where \( h \) is given by (1.24).

**Proof.** Identity (6.5) implies
\[ T_1 = \text{Tr} M_1(1, \cdot) = \text{Tr} \int_0^1 e^{(1-t)P} Q(t) e^{tP} ds = \text{Tr} e^P \int_0^1 Q(t) dt = 0, \]
which yields (6.11). Moreover,
\[ T_2 = \text{Tr} M_2(1, \cdot) = \text{Tr} \int_0^1 dt \int_0^t e^{(1-t+s)P} Q(t) e^{(t-s)P} Q(s) ds. \] (6.13)

We rewrite identities (2.4) for the matrices \( P, Q \) in the form
\[ P = P_0 - i\lambda P_1, \quad Q = -pP_0 + iqP_1, \quad P_0 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad P_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \]

Using the identities
\[ e^{tP_0} = I_3 + tP_0 + \frac{t^2}{2} P_0^2 = \begin{pmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}, \]
we obtain
\[ e^{tP(\lambda)} Q(s) = e^{tP_0} Q(s)(I_3 + O(\lambda)) = (-p(s)K_1(t) + iq(s)K_2(t))(I_3 + O(\lambda)) \]
as \( \lambda \to 0 \) uniformly on \( (t, s) \in [0, 1]^2 \) where
\[ K_1 = e^{tP_0} P_0^{*} = \begin{pmatrix} t & \frac{t^2}{2} & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad K_2 = e^{tP_0} P_1 = \begin{pmatrix} \frac{t^2}{2} & 0 & 0 \\ t & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \]

Using the identities
\[ \text{Tr} K_1(1-u)K_1(u) = 2(1-u)u + \frac{(1-u)^2}{2} + \frac{u^2}{2} = \frac{1}{2} + u(1-u), \]
\[ \text{Tr} K_2(1-u)K_2(u) = \frac{(1-u)^2u^2}{4}, \]
we obtain
\[ \text{Re} \text{Tr} e^{(1-t+s)P} Q(t) e^{(t-s)P} Q(s) = J(t, s)(1 + O(\lambda)) \]
as \( \lambda \to 0 \) uniformly on \( (t, s) \in [0, 1]^2 \) where
\[ J(t, s) = p(t)p(s)\left( \frac{1}{2} + u(1-u) \right) - q(t)q(s)\frac{(1-u)^2u^2}{4}, \quad u = t - s. \]

Substituting this asymptotics into (6.13) we obtain
\[ \text{Re} T_2(\lambda) = \int_0^1 dt \int_0^t J(t, s) ds(1 + O(\lambda)) = \frac{1}{2} \int_0^1 dt \int_{t-1}^t J(t, s) ds(1 + O(\lambda)) \]
as \( \lambda \to 0 \), since \( J(t, s) = J(s, t-1) \). Using the simple identity
\[ \int_0^1 dt \int_{t-1}^t g(t-s)f(t)f(s) ds = \sum_{k \in \mathbb{Z}} |\hat{f}_k|^2 \hat{g}_k \]
for all \( f \in L^1(0, 1), g, g' \in L^2(0, 1), g(0) = g(1) = 0 \), and identities
\[ \int_0^1 u(1-u)e^{-i2\pi n u} du = -\frac{1}{2(\pi n)^2}, \quad \int_0^1 u^2(1-u)^2 e^{-i2\pi n u} du = -\frac{3}{2(\pi n)^4} \]
for all \( n \neq 0 \), we obtain (6.12).
Now we will determine the asymptotics of the first spectral interval of multiplicity 3 for the small coefficients.

**Proof of Theorem 1.3.** Identities (2.2) imply

\[ T_0(\lambda) = e^{i\zeta} + e^{i\omega z} + e^{i\omega^2 z} = 3 - \frac{i\lambda}{2} - \frac{\lambda^2}{240} + O(\lambda^3) \quad \text{as} \quad \lambda \to 0. \]  

(6.14)

Recall that the functions \( T(\lambda, \varepsilon), \rho(\lambda, \varepsilon) \) are entire in \((\lambda, \varepsilon) \in \mathbb{C}^2\). Let the entire functions \( a(\lambda, \varepsilon), b(\lambda, \varepsilon) \) and the numbers \( b_j, j = 2, 3 \), be given by

\[
T(\lambda, \varepsilon) = 3 + a(\lambda, \varepsilon) + ib(\lambda, \varepsilon), \quad b(\lambda, \varepsilon) = \text{Im} T(\lambda, \varepsilon), \quad \text{all} \quad (\lambda, \varepsilon) \in \mathbb{R}^2, \quad b_j = \text{Im} T_j(0).
\]

Substituting relations (6.11), (6.12), (6.14) into (6.10) we obtain

\[
T(\lambda, \varepsilon) = 3 - \frac{i\lambda}{2} - \frac{\lambda^2}{240} - (3h - ib_2)\varepsilon^2 + \varepsilon^3T_3(0) + O(\lambda^3) + O(\lambda\varepsilon^2) + O(\varepsilon^4)
\]

(6.15)
as \((\lambda, \varepsilon) \to (0, 0)\). Asymptotics (6.15) gives

\[
a(\lambda, \varepsilon) = -3h\varepsilon^2 + O(\lambda^3) + O(\lambda\varepsilon^2) + O(\varepsilon^3), \quad b(\lambda, \varepsilon) = \mu(\lambda, \varepsilon) + O(\lambda^3) + O(\lambda\varepsilon^2) + O(\varepsilon^4)
\]

(6.16)
as \((\lambda, \varepsilon) \to (0, 0)\) where

\[
\mu = -\frac{\lambda}{2} + \frac{r}{2}, \quad r(\varepsilon) = 2b_2\varepsilon^2 + 2b_3\varepsilon^3.
\]

(6.17)

Identity (1.12) gives

\[
\rho = a^3(a + 4) + b^2(108 + 2(a + 18)a + b^2).
\]

(6.18)

Substituting asymptotics (6.16) into (6.18) we obtain

\[
\rho(\lambda, \varepsilon) = f(\mu, \varepsilon) = 108(\mu^2 - h^3\varepsilon^6 + O(\mu^4) + O(\mu^2\varepsilon^2) + O(\mu^4) + O(\varepsilon^7))
\]

as \((\mu, \varepsilon) \to (0, 0)\) where

\[
\lambda = r - 2\mu
\]

(6.19)

and \(f(\mu, \varepsilon)\) is entire in \((\mu, \varepsilon)\). Introduce the new variable \(u\) by \(u = \mu\varepsilon^3\). Then

\[
f(u\varepsilon^3, \varepsilon) = 108\varepsilon^6E(u, \varepsilon), \quad E(u, \varepsilon) = u^2 - h^3 + O(\varepsilon)
\]

(6.20)
as \(\varepsilon \to 0\) uniformly on any compact in \(\mathbb{C}\), the function \(E(u, \varepsilon)\) is entire in \((u, \varepsilon)\).

Consider the equation \(E(u, \varepsilon) = 0\). Let \(h \neq 0\). Using \(\frac{\partial}{\partial u}E(\pm h^\frac{3}{2}, 0) \neq 0\) and the Implicit Function Theorem we deduce that there exist two real analytic functions \(u(\varepsilon)\) in the disk \(|\varepsilon| < c\) for some \(c > 0\) such that each \(u(\varepsilon)\) is a zero of the function \(E(\cdot, \varepsilon)\). Asymptotics (6.20) yields

\[
u(\varepsilon) = \pm h^\frac{3}{2} + O(\varepsilon) \quad \text{as} \quad \varepsilon \to 0.
\]

Substituting \(\mu = u(\varepsilon)\varepsilon^3\) into the identity (6.19) we deduce that there exist two real analytic functions \(r(\varepsilon)\) in the disk \(|\varepsilon| < c\) such that

\[
r(\varepsilon) = r(\varepsilon) \pm 2h^\frac{3}{2}\varepsilon^3 + O(\varepsilon^4) \quad \text{as} \quad \varepsilon \to 0
\]

which yields (1.19).

Consider \(h > 0\). Let \(\varepsilon > 0\) be small enough. Then \(r^{-}(\varepsilon) = \pm r(\varepsilon)\) is \(\rho(r(\varepsilon), \varepsilon) = f(0, \varepsilon) < 0\). Then \(\rho(\cdot, \varepsilon) < 0\) on the whole interval \((r^{-}(\varepsilon), r^{+}(\varepsilon))\) and, by Lemma (6.2 iii), \(\rho(\cdot, \varepsilon) > 0\) out of this interval. Then, due to (1.14), the spectrum of \(H_\varepsilon\) in this interval has multiplicity 3 and the other spectrum has multiplicity 1. The proof for the case \(\varepsilon < 0\) is similar.
If \( h < 0 \), then, by Lemma 4.2, the function \( \rho \) has no any real zeros and \( \rho(\cdot, \varepsilon) > 0 \) on the whole real axis. Hence all the spectrum has multiplicity 1. ■

7. Appendix

In this Section we will prove Lemma 2.2. Introduce the functions \( e_{jk}(t, \lambda) \) by

\[
e_{jk}(t, \lambda) = e^{i\omega t} e^{i(\omega t - \omega^2 t)z}, \quad \text{all} \quad (t, \lambda) \in \mathbb{R} \times \mathbb{C}, \quad j, k = 1, 2, 3.
\]

Lemma 7.1. Let \( f, g \in L^2(\mathbb{T}) \). Then all functions \( \alpha_{jk}, 1 \leq j < k \leq 3 \), given by

\[
\alpha_{jk}(\lambda) = \int_0^1 e_{jk}(t, \lambda) \int_0^1 f(u)g(u - t)dudt,
\]

satisfy asymptotics

\[
\alpha_{jk}(\lambda) = e^{\alpha_0} \begin{cases} 
O(|z|^{-1}) & \text{as} \ |\lambda| \to \infty, \ 
\arg \lambda \in \left[-\frac{\pi}{4}, \frac{5\pi}{4}\right] \\
\ell_1(n) + O(n^{-1}) & \text{as} \ n \to +\infty, \ \lambda = -i\left(\frac{2n\pi}{\sqrt{3}}\right)^3 \left(1 + O(n^{-2})\right)
\end{cases}
\]

uniformly in \( \arg \lambda \in \left[-\frac{\pi}{4}, \frac{5\pi}{4}\right] \).

Proof. Let \( \lambda = -i\frac{2n\pi}{\sqrt{3}}(1 + O(n^{-2})) \) as \( n \to +\infty \). Then \( z = i\frac{2n\pi}{\sqrt{3}} + O(n^{-1}) \) or \( z = e^{-i\frac{2n\pi}{\sqrt{3}}} + O(n^{-1}) \). Consider the first case, the proof for the second one is similar. Using the identities

\[
e^{-i\omega^2 z} e_{12}(t, \lambda) = e^{i(\omega - \omega^2)z} e^{i(1 - \omega) t} = e^{-\sqrt{3}/2 (2 - (1 + \sqrt{3})t)z}, \\
e^{-i\omega^2 z} e_{13}(t, \lambda) = e^{i(1 - \omega) z} = e^{-\sqrt{3}/2 z}, \\
e^{-i\omega^2 z} e_{23}(t, \lambda) = e^{-i(\omega - \omega^2) t} = e^{-\sqrt{3} z t}
\]

for all \( (t, \lambda) \in \mathbb{R} \times \mathbb{C} \), we obtain

\[
e^{-i\omega^2 z} e_{12}(t, \lambda) = e^{-(\sqrt{3} - 1)\pi n t} (1 + O(n^{-1})) , \quad e^{-i\omega^2 z} e_{13}(t, \lambda) = e^{-i(\sqrt{3} + 1)\pi n t} (1 + O(n^{-1})) , \\
e^{-i\omega^2 z} e_{23}(t, \lambda) = e^{-i2\pi n t} (1 + O(n^{-1})) \quad \text{as} \ n \to +\infty \quad \text{uniformly on} \ t \in [0, 1].
\]

Substituting this asymptotics into identity (7.2) and using (4.9) we obtain that the function \( a_{jk} \) satisfies asymptotics (7.3) for \( \lambda = -i\left(\frac{2n\pi}{\sqrt{3}}\right)^3 (1 + O(n^{-2})) \).

Identities (7.4) yield

\[
|e^{-i\omega^2 z} e_{12}(t, \lambda)| = e^{-\sqrt{3}(x(1-t) + \xi t)}, \quad |e^{-i\omega^2 z} e_{13}(t, \lambda)| = e^{-\sqrt{3}t}, \quad |e^{-i\omega^2 z} e_{23}(t, \lambda)| = e^{-\sqrt{3} z t}
\]

for all \( (t, \lambda) \in [0, 1] \times \mathbb{C} \), where \( \xi = \frac{x^2 + y\sqrt{3}}{2} \). Substituting these estimates into identity (7.2) we obtain

\[
|e^{-i\omega^2 z} \alpha_{jk}(\lambda)| \leq \max_{t \in [0, 1]} \int_0^1 f(u)g(u - t)du \int_0^1 (e^{-\sqrt{3}(x(1-t) + \xi t)} + e^{-\sqrt{3} t} + e^{-\sqrt{3} z t}) dt
\]

for all \( \lambda \in \mathbb{C} \). Let arg \( \lambda \in \left[-\frac{\pi}{4}, \frac{5\pi}{4}\right] \). Then arg \( z \in \left[-\frac{\pi}{2}, \frac{5\pi}{12}\right] \) and max \( \{x, \xi\} \geq |z| \sin \frac{\pi}{12} \). Estimates (7.5) yield (7.3). ■

Lemma 7.2. Let \( p, q \in L^2(\mathbb{T}) \). Then the functions \( T_k = \text{Tr} \mathcal{M}_k(1, \cdot), k \in \mathbb{N}, \) satisfy

\[
T_1 = i\frac{2\rho_0 \theta_2}{3z}, \quad T_2 = -\frac{2\rho_0^2 \theta_1}{9z^2} + \frac{\Phi_1}{z^2} + \frac{T_2}{z^3}, \quad T_3 = -i\frac{4\rho_0^3 \theta_0}{81z^3} + \frac{T_3}{z^3}
\]

(7.6)
where $\Phi_1$ is given by (2.21) and
\[
\theta_m(\lambda) = \sum_{j=0}^{2} \omega^{mj} e^{i\omega^j z}, \quad m = 0, 1, 2, \quad \lambda \in \mathbb{C},
\]
the functions $\tilde{T}_2, \tilde{T}_3$ satisfy (7.3), and
\[
T_k(\lambda) = e^{z_0}O(|z|^{-1}) \quad \text{as} \quad |\lambda| \to \infty, \quad \text{all} \quad k \geq 4. \tag{7.7}
\]
**Proof.** Estimates (2.17) give (7.7). Identities (2.7), (2.9) and $\int_0^1 q(t)dt = 0$ give
\[
\int_0^1 Q_{jj}(s, \lambda)ds = \frac{2}{3z} \theta_0 \omega^{2(j-1)}. \tag{7.8}
\]
Then identities
\[
T_1(\lambda) = \text{Tr} \int_0^1 e^{iz(1-s)\Omega} Q(s, \lambda)e^{izs\Omega} ds = \text{Tr} e^{iz\Omega} \int_0^1 Q(s, \lambda)ds = \sum_{j=1}^{3} e^{iz\omega^j - 1} \int_0^1 Q_{jj}(s, \lambda)ds,
\]
yield the first identity in (7.6). We will prove the second one. We have
\[
T_2(\lambda) = \text{Tr} \int_0^1 \int_0^t e^{iz(1-t)\Omega} Q(t, \lambda)e^{iz(t-s)\Omega} Q(s, \lambda)e^{izs\Omega} dsdt = \sum_{j,k=1}^{3} a_{jk}(\lambda) \tag{7.9}
\]
where
\[
a_{jk}(\lambda) = \int_0^1 \int_0^t e_{jk}(t-s, \lambda)v_{jk}(t, s, \lambda)dsdt, \quad v_{jk}(t, s, \lambda) = Q_{jk}(t, \lambda)Q_{kj}(s, \lambda), \tag{7.10}
\]
and $e_{jk}$ has the form (7.11). Identity (7.8) give
\[
a_{jj}(\lambda) = e^{iz\omega^j - 1} \left( \int_0^1 Q_{jj}(t, \lambda)dt \right)^2 = -\frac{2\theta_0^2 \omega^{j-1} e^{iz\omega^j - 1}}{9z^2}, \quad \text{all} \quad j = 1, 2, 3.
\]
Substituting these identities into (7.9) we obtain
\[
T_2(\lambda) = -\frac{2\theta_0^2}{9z^2} \theta_1(\lambda) + \sum_{j,k=1}^{3} a_{jk}(\lambda). \tag{7.11}
\]
Identity (7.1) yields $e_{jk}(t, \lambda) = e_{jk}(1-t, \lambda)$ for all $j, k = 1, 2, 3, (t, \lambda) \in \mathbb{R} \times \mathbb{C}$. Then (7.10) gives
\[
a_{kj}(\lambda) = \int_0^1 du \int_0^u e_{jk}(1-u + s, \lambda)v_{jk}(s, u, \lambda)ds = \int_0^1 dt \int_{t-1}^t e_{jk}(t-u, \lambda)v_{jk}(t, u, \lambda)du,
\]
which yields
\[
a_{jk}(\lambda) + a_{kj}(\lambda) = \int_0^1 dt \int_{t-1}^t e_{jk}(t-s, \lambda)v_{jk}(t, s, \lambda)ds = \int_0^1 e_{jk}(u, \lambda) \int_0^1 v_{jk}(t, t-u, \lambda)dtdu
\]
for all $j, k = 1, 2, 3, \lambda \in \mathbb{C}$. Identities (2.7), (2.9) give
\[
\sum_{j,k=1}^{3} a_{jk}(\lambda) = \sum_{1 \leq j < k \leq 3} \int_0^1 du e_{jk}(u, \lambda) \int_0^1 v_{jk}(t, t-u, \lambda)dt = \Phi_1(\lambda) \frac{1}{z^2} + \overline{\Phi}_1(\lambda) \frac{1}{z^3} + O\left(\frac{e^{z_0}}{|z|^4}\right) \tag{7.12}
\]
as $|\lambda| \to \infty$ where

$$\tilde{\Phi}_1(\lambda) = -\frac{1}{9} \sum_{1 \leq j < k \leq 3} \int_0^1 e_{jk}(u, \lambda) \int_0^1 (p(t)q(t-u) + p(t-u)q(t)) dt du.$$

By Lemma 7.1 the function $\tilde{\Phi}_1$ satisfies (7.3). Substituting (7.12) into (7.11) we obtain the second identity in (7.6) and asymptotics (7.3) for $\tilde{T}_2$.

We will prove identity (7.6) for $T$. We have

$$T_3(\lambda) = \text{Tr} \int_0^1 \int_0^t \int_0^s e^{iz(1-t)\Omega} Q(t, \lambda) e^{iz(t-s)\Omega} Q(s, \lambda) e^{iz(s-u)\Omega} Q(u, \lambda) e^{izu\Omega} dt du ds dt$$

$$= \sum_{j, k, \ell = 1}^3 \int_0^1 \int_0^t \int_0^s e^{iz(\omega_j^{-1}(1-u-t) + \omega_k^{-1}(t-s) + \omega_\ell^{-1}(s-u))} Q_jk(t, \lambda) Q_\ell(s, \lambda) Q_\ell(u, \lambda) dt du ds dt.$$ 

Identity (7.8) and identity (2.7) for $Q$ give

$$T_3(\lambda) = -\frac{4i \rho_0^3}{81 \lambda} - \frac{i}{27 \lambda} (-2 B_1(\lambda) + B_2(\lambda)) + O\left(\frac{e^{z_0}}{|z|^4}\right)$$

(7.13) as $|\lambda| \to \infty$ where

$$B_1 = \sum_{j, k, \ell = 1}^3 \omega^{j+2k} \beta_{jk}, \quad \beta_{jk} = b_{jjk} + b_{jkj} + b_{kjj}, \quad B_2 = \sum_{j, k, \ell = 1}^3 b_{jk\ell},$$

(7.14)

$$b_{jk\ell} = \int_0^1 \int_0^t \int_0^s e^{iz(\omega_j^{-1}(1-u-t) + \omega_k^{-1}(t-s) + \omega_\ell^{-1}(s-u))} f(t, s, u) dt du ds dt, \quad f(t, s, u) = p(t)p(s)p(u),$$

(7.15)

$\gamma = \gamma(j, k) \in \{1, 2, 3\}, \gamma \neq j, \gamma \neq k$. Assume that the functions $B_1, B_2$ satisfy asymptotics (7.3). Then identity (7.13) gives the third identity in (7.6) and asymptotics (7.3) for $T_3$.

We will prove that the functions $B_1, B_2$ satisfy asymptotics (7.3). Consider the function $B_1$. We have

$$b_{jjk}(\lambda) = \int_0^1 \int_0^t \int_0^s e_{jk}(s-u, \lambda) f(t, s, u) dt du ds dt = \int_0^1 \int_0^t \int_0^s e_{jk}(t-s, \lambda) f(t, s, u) dt du ds dt,$$

$$b_{jkj}(\lambda) = \int_0^1 \int_0^t \int_0^s e_{jk}(t-s, \lambda) f(t, s, u) dt du ds dt,$$

$$b_{kjj}(\lambda) = \int_0^1 \int_0^t \int_0^s e_{jk}(1-t+u, \lambda) f(t, s, u) dt du ds dt = \int_0^1 \int_0^t \int_0^s e_{jk}(t-s, \lambda) f(t, s, u) dt du ds dt.$$

Substituting these identities into (7.14) we obtain

$$\beta_{jk}(\lambda) = \int_0^1 \int_0^t \int_0^s e_{jk}(t-s, \lambda) f(t, s, u) dt du ds dt = \int_0^1 \int_0^t \int_0^s e_{jk}(v, \lambda) \int_0^t p(t-v)p(t)(\tilde{p}(t) - p(t-v)) dt dv$$

where $\tilde{p}(t) = \int_0^t p(u) du$. By Lemma 7.1, the functions $\beta_{jk}$, and then the function $B_1$, satisfy asymptotics (7.3).
Consider the function $B_2$. Identities (7.15) yield
\[
b_{jk\gamma}(\lambda) = \int_0^1 dt \int_0^t \bar{e}_{jk}(t, s, \lambda) h(t, s) ds,
\]
(7.16)
\[
b_{kj\gamma}(\lambda) = \int_0^1 dt \int_0^t ds \bar{e}_{jk}(t, s, \lambda) \int_s^t p(u - s)p(u - t)p(u) du
\]
for all $\lambda \in \mathbb{C}, j, 1 \leq j < k \leq 3$ where
\[
h(t, s) = \int_t^1 p(u - t + s)p(u - t) p(u) du,
\]
and
\[
\bar{e}_{jk}(t, s, \lambda) = e_{jk}(t, \lambda) e^{i(\omega^{-1} - \omega^k - 1)zs},
\]
e_{jk} are given by (7.11). Consider the function $b_{123}$, the estimates for the other functions $b_{jk\gamma}, j, k = 1, 2, 3, j \neq k$, are similar. We have
\[
e^{-i\omega^2 z^2}e_{12}(t, s, \lambda) = e^{i(\omega - \omega^2)z} e^{iz(t-s-\omega t + \omega^2 s)} = e^{-\frac{\omega}{\sqrt{3}}(2-t-s-i\sqrt{3}(t-s))},
\]
(7.17)
which gives $|e^{-i\omega^2 z^2}e_{12}(t, s, \lambda)| \leq e^{-\sqrt{3}(1-t)}$ for all $(t, s, \lambda) \in [0, 1]^2 \times \mathbb{C}, s \leq t$. Substituting this estimate into (7.16) and using the continuity of the function $h(t, s)$ in $s$ we obtain
\[
|e^{-i\omega^2 z^2}b_{123}(\lambda)| \leq \int_0^1 dt \int_0^t e^{-\sqrt{3}(1-t)}|h(t, s)| ds \leq \int_0^1 \max_{t \in [0, t]} |h(t, s)| \int_t^0 e^{-\sqrt{3}(1-t)} ds dt = O\left(\frac{1}{|z|}\right)
\]
as $|\lambda| \to \infty$ uniformly in $\arg \lambda \in [-\frac{\pi}{4}, \frac{5\pi}{4}]$. Thus the function $b_{123}$ satisfy (7.3) for $\arg \lambda \in [-\frac{\pi}{4}, \frac{5\pi}{4}]$.

Let $\lambda = -i(\frac{2\sqrt{n}}{\sqrt{3}})^3(1 + O(n^{-2}))$ as $n \to +\infty$. Then $z = i\frac{2\sqrt{n}}{\sqrt{3}} + O(n^{-1})$ or $z = e^{-i\frac{2\sqrt{n}}{\sqrt{3}}} + O(n^{-1})$.

Consider the first case, the proof for the second one is similar. Identities (7.17) give
\[
e^{-i\omega^2 z^2}e_{12}(t, s, \lambda) = e^{-\sqrt{3}n(t-s)}(e^{i\pi n(t+s)} + O(n^{-1})).
\]
Substituting this asymptotics into (7.16) and using (1.9) we obtain
\[
|e^{-i\omega^2 z^2}b_{123}(\lambda)| \leq \int_0^1 dt \int_0^t e^{-\sqrt{3}n(t-s)}|h(t, s)| ds (1 + O(n^{-1}))
\]
\[
= \int_0^1 du e^{-\sqrt{3}n u} \int_u^t |h(t, t - u)| dt (1 + O(n^{-1})) = O(n^{-1}),
\]
which shows that the functions $b_{123}$ satisfy asymptotics (7.3).

The similar arguments show that all functions $b_{jk\gamma}$, and then the function $B_2$, satisfy asymptotics (7.3), which proves Lemma. ■

Proof of Lemma 2.2 Substituting (2.2), (7.6), (7.7) into the identity $T = \sum_{n \geq 0} T_n$ we obtain (2.20). ■

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