Adaptive Observer-based Synchronization of Nonlinear Nonpassifiable Systems

V.O. Nikiforov, A.L. Fradkov, and B.R. Andrievsky

Abstract—In this paper the relative degree limitation for adaptive observer-based synchronization schemes is overcome. The scheme is extended to nonpassifiable systems. Two synchronization methods are described and justified based on augmented error adaptive observer and high-order tuners. The solution is based on modern theory of nonlinear adaptive control, particularly on nonlinear observer structure and new classes of adaptation algorithms. Conditions of parametric convergence of the parameter estimation are established for the noiseless case. Robustness of the scheme to the bounded measurement error is established. The results are illustrated by example of application the proposed adaptive synchronization of chaotic Lorenz systems.

Index Terms—Adaptive observer, Chaotic behavior, Synchronization, High order tuner, Augmented error.

I. INTRODUCTION

SYNCHRONIZATION has found various applications during last decade. Particularly, a lot of interest has been attracted to the problem of information transmission by means of chaotic signals modulation, see [1], [2], [3], [4]. A number of results in this area are based on adaptive synchronization approach [5], [6], [7], [8]. New method of adaptive synchronization exploiting Lyapunov functions and passification was developed in [5], [8] and later extended to observer-based adaptive synchronization with application to telecommunication [9], [10]. The possibility of fast transmitting messages in noisy channel using new approach has been demonstrated [11], [12]. However, applicability of the method, proposed in [5], [8] is restricted by the passifiability of the plant (the master system). It implies the relative degree limitation: the relative degree of the plant model should be equal to zero or one. Such a limitation prevents from increasing security of communications by using master system with higher relative degree [13].

In this paper we overcome the relative degree limitation for adaptive observer-based synchronization schemes [14] and extend them to nonpassifiable systems, particularly to systems with relative degree greater than one. The solution is based on modern theory of nonlinear adaptive control [15], [16], [17], particularly on nonlinear observer structure and new classes of adaptation algorithms [18], [19].

In the literature the methods of adaptive synchronization suitable for adaptive systems with relative degree greater than one were proposed [13], [20]. However, the algorithm of [20] provides state feedback rather than output feedback and does not allow for presence of unknown parameters in the master system. The approach of [13] is based on canonical forms for linear adaptive observers (see, e.g. [21]). Design of adaptive observer for nonlinear systems in [13] requires case by case consideration and special tricks for each nonlinear system. For example, it is not clear how to apply results of [13] to Lorenz systems. Finally, the results of [13] do not allow to incorporate measurement errors.

Unlike [13], [20], in this paper a unified approach based on scheme of [10], [5], [8], [9] and new classes of adaptation algorithms of [15], [18], [19] are proposed. It allows to cope with bounded noise by means of robust modification of adaptation algorithm. Simulation results demonstrate better convergence and robustness properties the adaptive observer (slave system) compared with results of [10].

In Sections III and IV general method of adaptive observer design for higher relative degree systems is described and justified. To clarify the presentation of main idea we start with the simple disturbance-free case and introduce basic schemes of adaptive observers in Section III. In Sec. IV we modify basic schemes to provide robustness in the presence of external disturbances (measurement noise). The method is applied to Lorenz system in Section V where simulation results conforming the theoretical statements are presented. Numerical examples demonstrating application of the proposed scheme to signal transmission are given. Preliminary version of the results was announced in [22].

II. PROBLEM STATEMENT

Following [9] we assume that the plant (the master system) is described by the state-space equations of the form:

\[ \dot{x} = Ax + \varphi_0(y) + b\varphi^*(y)\theta, \quad y = c^T x, \]

\[ y_r = y + \xi, \]

where \( x \in \mathbb{R}^n \) is the inaccessible state vector, \( y \) is the plant output (transmitted signal), \( y_r \) is the measurable noisy signal, \( \xi \) is the additive channel noise (presented by a bounded function of time), \( \theta \in \mathbb{R}^m \) is the unknown vector of the plant parameters (possibly representing a message). It is assumed that the nonlinearities \( \varphi_0(y), \varphi(y) \), matrix \( A \) and vectors \( b, c \) are known.

We accept the following plant-model assumptions.

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Assumption 1: for any bounded initial condition $x(0)$ and any value of vector $\theta$ the state $x(t)$ is a bounded function of time.

Assumption 2: functions $\varphi_0(y), \varphi(y)$ are bounded for any bounded $y$.

The problem is to design an adaptive observer (a dynamical system) of the form
\begin{equation}
\dot{z} = F(z, yr), \quad \dot{\theta} = h(z, yr),
\end{equation}
such that
\begin{equation}
\lim_{t \to \infty} |\theta - \hat{\theta}| \leq \Delta, \text{ where } \Delta \text{ is some positive constant.}
\end{equation}

III. DESIGN OF ADAPTIVE OBSERVERS: DISTURBANCE-FREE CASE

In this section we assume that $\xi(t) \equiv 0$. Then the adaptive observer (the slave system) may have the following structure:
\begin{equation}
\dot{x} = A\dot{x} + \varphi_0(y) + b\varphi(y)^T \hat{\theta} + k(y - \hat{y}), \quad \dot{\hat{y}} = c^T \dot{x},
\end{equation}
\begin{equation}
\dot{\hat{\theta}} = F_0(\hat{\theta}, \hat{x}, y),
\end{equation}
where $\hat{x}$ and $\hat{y}$ are the estimates of $x$ and $y$, $\hat{\theta}$ is the vector of adjustable parameters (representing estimate of the parameter $\theta$), and $z = \text{col}(x, \hat{\theta})$.

Such a structure was proposed in [9] where convergence conditions were established for the case of plant (master system) with passifiable linear part (for bounded disturbances see [10]). Passifiability condition imposes strong restriction on the plant model: its relative degree $r$ should be equal to 1. At the same time it would be important to have extended solutions to this problem in the case $r > 1$.

Below we present two solutions to the posed problem under assumptions accepted. The first solution is based on the use of the augmented error (AE) concept [15, 23, 25], while the second one utilizes the idea of the high-order tuners (HOT) [15, 18, 19, 24].

To employ the augmented error concept we use the adaptive observer [4, 5] where vector $k$ is chosen so that $F = A - kc^T$ is Hurwitz. To derive the adaptation algorithm, we first obtain so-called error model. Differentiating estimation error $e = x - \hat{x}$ in view of (11) and (4), we obtain
\begin{equation}
\dot{e} = F \varepsilon + b \varphi(t)^T \hat{\theta}, \quad e = c^T \varepsilon
\end{equation}
where $\varphi(t) = \varphi(y(t))$, $\hat{\theta} = \theta - \bar{\theta}$ is the parameter error, while $e = y - \hat{y}$ is the output error accessible to measurements. The error model [5] can be rewritten in the form:
\begin{equation}
e = H(p) \left[ \varphi(t)^T \hat{\theta} \right],
\end{equation}
where $p = d/dt$ is the differential operator, and the transfer function $H(p) = c^T(pI - F)^{-1}b$ is asymptotically stable.

The adaptation algorithm can be chosen in the form [15, 25, 23]:
\begin{equation}
\dot{\hat{\theta}} = \gamma \omega(t)^T \hat{e},
\end{equation}
where $\gamma > 0$ is the adaptation gain, $\omega(t) = H(p)\varphi(t)$ is the filtered regressor, while the augmented error $\hat{e}$ is defined by the following equation:
\begin{equation}
\hat{e} = e + H(p) \left[ \varphi(t)^T \hat{\theta} \right] - \omega(t)^T \hat{\theta}.
\end{equation}

Introduce the following definition.

Definition: A vector-function $f : [0, \infty) \to \mathbb{R}^m$ is called persistently exciting (PE) on $[0, \infty)$, if it is measurable and bounded on $[0, \infty)$ and there exist $\alpha > 0, T > 0$ such that
\begin{equation}
\int_0^T f(s)f(s)^T ds \geq \alpha I
\end{equation}
for all $t \geq 0$.

Theorem 1: The closed-loop system consisting of the master system (11), adjustable observer (4) and algorithm of adaptation (8) has the following properties:

i) for any initial conditions and any $\gamma > 0$ all the closed-loop signals are bounded and
\begin{equation}
\lim_{t \to \infty} (y(t) - \hat{y}(t)) = 0;
\end{equation}

ii) if the vector function $\varphi(t)$ satisfies PE condition and the transfer function $H(p)$ is minimum phase then in addition to (i)
\begin{equation}
\lim_{t \to \infty} (\theta - \hat{\theta}(t)) = 0.
\end{equation}

Proof: It is known that for the augmented error $\hat{e}$ one can write the following equivalent model [15, 25], neglecting exponentially vanishing term due to nonzero initial conditions:
\begin{equation}
\dot{\hat{e}} = \omega(t)^T \hat{\theta}.
\end{equation}

Then differentiating Lyapunov function $V(\hat{\theta}) = \frac{1}{2\gamma} \hat{\theta}^2$ along solutions of (8) and in view of (13), we obtain $\dot{V}(\hat{\theta}) = -\varepsilon^2$. The latter means boundedness of $\hat{\theta}$ and zeroing $\hat{e}(t)$ (since the right-hand sides of (11) and (8) are locally Lipschitz in $x$, $\hat{x}$ and $\hat{\theta}$ uniformly in $t$ [16]). Since $\omega(t)$ is bounded, from (8) we have that $\theta \to 0$ as $t \to \infty$. Therefore, from (13) we obtain that $e \to 0$ and, as a consequence, $e \to 0$. Part (i) is proved. Part (ii) can be straightforwardly proved with the use of standard arguments [25].

Now we present an alternative solution to the posed problem utilizing an idea of the high-order tuners. For this, we use the following adjustable observer:
\begin{equation}
\dot{x} = A\dot{x} + \varphi_0(y) + b\nu(y, \hat{\theta}) + k(y - \hat{y}), \quad \dot{\hat{y}} = c^T \dot{x}
\end{equation}
where the adjustable feedback $\nu(y, \hat{\theta})$ will be defined below. In this case the error model takes the view
\begin{equation}
\dot{e} = F \varepsilon + b \varphi(t)^T \theta - \nu, \quad e = c^T \varepsilon
\end{equation}
or
\begin{equation}
e = H(p) \left[ \varphi(t)^T \theta - \nu \right].
\end{equation}

Chose a transfer function $W(p)$ obeying the equation:
\begin{equation}
W(p) = (p + \lambda)H(p)
\end{equation}
where $\lambda$ is any positive constant. Then the model (15) can be rewritten in the form
\begin{equation}
e = \frac{1}{p + \lambda} \left[ \omega(t)^T \theta - W(p) \nu \right].
\end{equation}
where \( \varpi(t) = W(p)[\varphi(t)] \). Analysis of the model motivates the following choice of the adjustable feedback:

\[
\nu = W(p)^{-1}[\varpi(t)^\top \hat{\theta}]
\]

where \( \hat{\theta} \) is the vector of adjustable parameters. To realize the feedback we need to generate not only the adjustable parameters \( \hat{\theta} \), but also their high-order derivatives up to order \( r - 1 \), where \( r \) is the relative degree of the transfer function \( H(p) \). To overcome this problem we use the following algorithm of adaptation:

\[
\begin{align*}
\dot{\psi}_i &= \varpi \varepsilon, \\
\dot{\eta}_i &= (1 + \mu \varpi^\top \varpi)(\Gamma \eta_i + \hat{h} \psi_i), \\
\dot{\hat{\theta}}_i &= I^\top \eta_i
\end{align*}
\]

where \( i = 1, 2, \ldots, m, \mu > 0 \) is a design parameter, and \((l, \Gamma, h)\) is a minimal realization of the transfer function \( \alpha/(\alpha(p)) \) with a Hurwitz polynomial \( \alpha(p) \) of order \( r - 2 \), i.e., \( \alpha(0)/\alpha(p) = l^\top (\mu I - \Gamma^\top h) \).

Remark. If \( r \leq 2 \), one needs not to use additional filters. In this particular case the algorithm of adaptation takes the form \( \hat{\theta} = \varpi \varepsilon \).

Theorem 2: The closed-loop system consisting of the master system, observer and adjustable feedback has the following properties:

i) for any initial conditions and any \( \mu > \frac{3}{4\lambda}(l + |PF^{-1}h|)^2 \) where the positive definite matrix \( P \) obeys the equality \( F^\top P + PF = -2I \), all the closed-loop signals are bounded and regulation is achieved;

ii) if the vector function \( \varphi(t) \) satisfies PE condition and the transfer function \( H(p) \) is minimum phase then in addition to (i) asymptotic convergence is guaranteed.

Proof of the theorem is based on the standard arguments which can be found, for example, in [15], [18].

Finally, consider a more general case when the master system is described by the following equations:

\[
\begin{align*}
\dot{x} &= A(y)x + \varphi_0(y) + b \varphi(t)^\top \hat{\theta}, \\
y &= c^\top x.
\end{align*}
\]

We accept the following additional plant model assumptions.

Assumption 3: There exist a vector function \( k(y) \in \mathbb{R}^n \) and scalar function \( V(x) \) such that

\[
\begin{align*}
&c_1 |x|^2 \leq V(x) \leq c_2 |x|^2, \\
&\frac{\partial V}{\partial x}(x) (A(c^\top x) - k(c^\top x)c^\top)x \leq -c_3 |x|^2, \\
&\left| \frac{\partial V}{\partial x} \right| \leq c_4 |x|,
\end{align*}
\]

where \( c_i \) are some positive constants \( i = 1, 2, 3, 4 \).

Assumption 4: All entries of matrix \( A(y) \) are bounded for any bounded \( y \).

In other words, Assumption 3 means that the autonomous system

\[
\dot{x} = G(c^\top x)x,
\]

where \( G(c^\top x) = G(y) = A(y) - k(y)c^\top \), is exponentially stable.

It is worth noting that the design methods presented above are not applicable to the model. Therefore we introduce one more method presenting a special kind of the scheme with augmented error. For this we employ the following adjustable observer

\[
\begin{align*}
\dot{x} &= A(y)\hat{x} + \varphi_0(y) + b \varphi(t)^\top \hat{\theta} + k(y)(y - \hat{y}), \\
\dot{\hat{y}} &= c^\top \hat{x}
\end{align*}
\]

where the time-varying vector \( k(y) \) is chosen so that Assumption 3 is valid. In this case the error model takes the view:

\[
\begin{align*}
\dot{e} &= G(t) \varepsilon + b \varphi(t)^\top \hat{\theta}, \\
e &= c^\top \varepsilon,
\end{align*}
\]

where \( G(t) = G(c^\top x(t)) \).

Define the augmented error as follows:

\[
\dot{e} = e + c^\top \eta,
\]

where the auxiliary vector \( \eta \) is generated by the filters:

\[
\begin{align*}
\dot{\eta} &= G(t) \eta - \Omega \hat{\theta}, \\
\dot{\Omega} &= G(t) \Omega + b \varphi(t)^\top, \\
\Omega &\in \mathbb{R}^{n \times n}.
\end{align*}
\]

Then adaptation algorithm can be chosen in the form:

\[
\dot{\hat{\theta}} = \gamma \omega^\top e,
\]

where \( \omega = c^\top \Omega \).

Theorem 3: The closed-loop system consisting of the master system, adjustable observer, scheme of augmentation and algorithm of adaptation has the following properties:

i) for any initial conditions and any \( \gamma > 0 \) all the closed-loop signals are bounded and regulation is achieved;

ii) if, in addition, the vector function \( \varphi(t) \) satisfies PE condition then asymptotic convergence is guaranteed.

Proof: Differentiating the following auxiliary error \( \delta = e + \eta - \Omega \hat{\theta} \) in view of equations we obtain

\[
\begin{align*}
\dot{\delta} &= G\varepsilon + b \varphi^\top \hat{\theta} + G\eta - \Omega \dot{\hat{\theta}} - G\Omega \hat{\theta} - b \varphi^\top \hat{\theta} + \Omega \dot{\hat{\theta}} \\
&= G(\varepsilon + \eta - \Omega \hat{\theta}) = G\delta.
\end{align*}
\]

Then for the augmented error defined by equation we can write the following equivalent model: \( \dot{\varepsilon} = \omega^\top \theta + \delta_e \), where \( \delta_e = c^\top \varepsilon \) exponentially vanishes. Then using the same arguments as in the proof of Theorem we can show boundedness of all the closed-loop signals, regulation and convergence (under PE condition).

IV. ROBUST ADAPTIVE OBSERVERS: NOISY MEASUREMENTS

It is known that adaptation algorithms of pure integral action can loss stability in the presence of external disturbances or noise of measurements. In this section we present robustified modifications of above schemes which are applicable in the case of noisy measurements.
the form
\[ \Delta(t) = \varphi(y_r(t)) - \varphi(y_r(t)) + b(\varphi(y_r) - \varphi(y_r)) \nabla - k(y_r) \xi, \]
where \( \xi = c^T \hat{\alpha} \),

scheme of augmentation
\[ \hat{e} = y_r - \dot{\hat{y}} + H(p) \left[ \hat{\varphi}(t)^T \hat{\theta} - \omega(t)^T \hat{\theta} \right], \] (28)
where \( \hat{\varphi}(t) = \varphi(y_e(t)), \) \( \omega(t) = H(p)[\hat{\varphi}(t)] \), and robustified algorithm of adaptation
\[ \hat{\theta} = \gamma \hat{\omega}(t) \hat{e} - \alpha(\hat{\theta}) \hat{\theta}, \] (29)
where the function \( \alpha(\hat{\theta}) \) obeys the following relations
\[ \alpha(\hat{\theta}) = \begin{cases} 
0, & |\hat{\theta}| < \theta^* \\
\frac{|\hat{\theta}|}{\theta^*} - 1, & \theta^* \leq |\hat{\theta}| \leq 2\theta^* \\
1, & |\hat{\theta}| > 2\theta^* 
\end{cases}, \] (30)
with any positive constant \( \theta^* \).

**Theorem 4:** The closed-loop system consisting of the master system (1), (2), adjustable observer (27), scheme of augmentation (28) and algorithm of adaptation (29), (30) has the following properties:

i) For any initial conditions and any \( \gamma > 0, \theta^* > 0 \) all the closed-loop signals are bounded and the parameter error \( \hat{\theta} \) converges to the residual set
\[ D = \left\{ \theta : |\theta|^2 \leq \max \left\{ (|\theta| + 2\theta^*)^2, \gamma \| \xi + \xi_e \|_\infty^2 + |\theta|^2 \right\} \right\}, \] (31)
where the bounded variable \( \xi_e \) obeys the equations:
\[ \dot{\xi}_e = F \xi_e + \varphi_0(y) - \varphi_0(y_r) + b(\varphi(y) - \varphi(y_r))^T \theta - k \xi_e, \] (32)

ii) If \( \xi(t) \equiv 0 \) and \( \theta^* > |\theta| \) then, in addition to (i), regulation (11) is guaranteed.

iii) If, additionally, the vector-function \( \varphi(t) \) satisfies PE condition and the transfer function \( H(p) \) is minimum phase, then convergence (14) is achieved.

**Proof:** Differentiating estimation error \( \hat{e} = x - \hat{x} \) in view of equations (1), (2) and (27) after simple calculations we obtain:
\[ \dot{\hat{e}} = F \hat{e} + b \hat{\varphi}(t)^T \hat{\theta} + \Delta(t), \] (33)
where \( \Delta(t) = \varphi(y_r(t)) - \varphi(y_r(t)) + b(\varphi(y_r) - \varphi(y_r)) \nabla - k(y_r) \xi, \)

Then the augmented error \( \hat{e} \) defined by equation (28) takes the form
\[ \hat{e} = \hat{\omega}(t)^T \hat{\theta} + \xi_e + \xi, \] (34)
where the bounded variable \( \xi_e \) obeys the equations (12).

Choose the Lyapunov function \( V(\hat{\theta}) = \frac{1}{2} \hat{\theta}^T \hat{\theta} \). Its time derivative in view of (29) and (31) takes the form:
\[ \dot{V}(\hat{\theta}) = \hat{\theta}^T (\gamma \hat{\omega}^T \hat{\theta} - \gamma \hat{\omega}_e \hat{\theta}_e - \alpha \hat{\theta}) \]
\[ = \hat{\theta}^T (\gamma \hat{\omega}_e \hat{\theta}_e - \gamma \hat{\omega}_e \hat{\theta}_e + \hat{\theta}_e - \theta^* \hat{\theta}) \]
\[ \leq -\gamma |\hat{\omega}_e \hat{\theta}_e|^2 + 2 \gamma |\hat{\omega}_e \hat{\theta}_e| \| \xi_e + \xi \|_\infty + \alpha |\theta| \| \theta\|_\infty \]
\[ \leq -\frac{\gamma}{2} |\hat{\omega}_e \hat{\theta}_e|^2 - \frac{\gamma}{2} \| \xi_e + \xi \|_\infty^2 + \frac{1}{2} \| \theta \|_\infty^2 \]
\[ \leq -\frac{\gamma}{2} |\hat{\omega}_e \hat{\theta}_e|^2 - \frac{1}{2} \| \xi_e + \xi \|_\infty^2 + \frac{1}{2} |\theta|^2. \]
The latter inequality proves boundedness of all the closed-loop signals and validity of the estimate (31).

If \( \xi(t) \equiv 0 \) and \( \theta^* > |\theta| \), then the time derivative of the Lyapunov function \( \dot{V}(\hat{\theta}) = \frac{1}{2} \hat{\theta}^T \hat{\theta} \) obeys the following expressions
\[ \dot{V}(\hat{\theta}) = -\gamma |\hat{\omega}_e \hat{\theta}_e|^2 - \gamma |\hat{\omega}_e \hat{\theta}_e| \| \xi_e + \xi \|_\infty + \frac{1}{2} |\theta|^2. \]
The latter means validity of (11).

To extend the above results to the master system (24) with noisy output (24), we use the adjustable observer of the form
\[ \hat{x} = A(y_r) \hat{x} + \varphi_0(y_r) + b_{\hat{\gamma}}^T(y_r) \hat{\theta} + (k(y_r)(y_r - \hat{y}), (35) \]
\[ \hat{y} = c^T \hat{x}, \]
scheme of augmentation
\[ \hat{e} = y_r - \hat{y} + c^T \hat{\eta} \] (36)
\[ \hat{\eta} = \hat{G}(t) \hat{\eta} - \bar{\Omega} \hat{\theta}, \hat{\eta} \in \mathbb{R}^n, \] (37)
\[ \bar{\Omega} = \hat{G}(t) \bar{\Omega} + b_{\hat{\gamma}}^T(t), \bar{\Omega} \in \mathbb{R}^{n \times n} \] (38)
where \( \hat{G}(t) = A(y_r(t)) - k(y_r(t))c^T \), and robustified algorithm of adaptation
\[ \hat{\theta} = \gamma \hat{\omega}(t) \hat{e} - \alpha(\hat{\theta}) \hat{\theta}, \] (39)
where \( \hat{\omega}(t) = c^T \bar{\Omega}(t) \) and the function \( \alpha(\hat{\theta}) \) obeys relations (30).

**Theorem 5:** The closed-loop system consisting of the master system (24), (2), adjustable observer (35), scheme of augmentation (36), (38) and algorithm of adaptation (39), (30) has the following properties:

i) For any initial conditions and any \( \gamma > 0, \theta^* > 0 \) all the closed-loop signals are bounded and the parameter error \( \hat{\theta} \) converges to the residual set
\[ D = \left\{ \hat{\theta} : |\hat{\theta}|^2 \leq \max \left\{ (|\hat{\theta}| + 2\theta^*)^2, \gamma \| \xi + \xi_y \|_\infty^2 + |\theta|^2 \right\} \right\}, \] (40)
where the bounded variable \( \xi_y \) obeys the equations:
\[ \dot{\xi}_y = \hat{\bar{G}}(t) \xi_y + \varphi_0(y) - \varphi_0(y_r) + b(\varphi(y) - \varphi(y_r))^T \theta - k(y_r) \xi, \xi_y = c^T \xi_y; \]

ii) If \( \xi(t) \equiv 0 \) and \( \theta^* > |\theta| \) then, in addition to (i), regulation (11) is guaranteed.

iii) If, additionally, the vector-function \( \varphi(t) \) satisfies PE condition and the transfer function \( H(p) \) is minimum phase, then convergence (14) is achieved.
Proof: Differentiating estimation error \( e = x - \hat{x} \) in view of equations (21), (22) and (35) after simple calculations we obtain:

\[
\dot{\varepsilon} = \dot{G}(t)\varepsilon + b\dot{\varphi}(t)^T\dot{\theta} + \Delta(t),
\]

where \( \Delta(t) = (A(y) - A(\varphi_0))x + \varphi_0(y) - \varphi_0(\varphi_0) + b(\varphi(y) - \varphi(\varphi_0))\dot{\theta} - k(y_\varphi)\dot{\xi}. \) Then differentiating the auxiliary error \( \Xi_y = \varepsilon + \dot{\eta} - \Omega\dot{\theta} \) in view of equations (41), (42) and (43) we obtain:

\[
\dot{\Xi}_y = \dot{G}(t)\Xi_y + \varphi_0(y) - \varphi_0(\varphi_0) + b(\varphi(y) - \varphi(\varphi_0))\dot{\theta} - k(y_\varphi)\xi + (A(y) - A(\varphi_0))x, \\
\xi_y = c^T\Xi_y,
\]

Therefore, for the augmented error \( \bar{e} \) defined by equation (46) we can write \( \bar{e} = \bar{\omega}^T\dot{\theta} + \xi_\delta + \xi, \) where \( \xi_\delta = c^T\Xi_\delta. \) Finally, using the same approach as in the proof of Theorem 4 we show validity of the all parts of Theorem 5. 

V. Example: Signal Transmission via Adaptive Synchronization of the Lorenz Systems

A. Design of the adaptive observer

Let us consider, for example, application of the proposed method to adaptive synchronization of the Lorenz systems, exhibiting chaotic behavior.

Let the master system be Lorenz system, described by the following equations [1], [28]:

\[
\begin{cases}
\dot{x}_1 = \sigma x_2 - \sigma x_1, \\
\dot{x}_2 = -x_2 - x_1 x_3 + \theta x_1, \\
\dot{x}_3 = -\beta x_3 + x_1 x_2.
\end{cases}
\]

(42)

Constant parameters \( \beta, \sigma \) are assumed to be known; parameter \( \theta \) is varying depending on the information signal and its value has to be reconstructed by the observer. It is also assumed that the component \( x_1 \) is taken as a transmitted signal, i.e. \( y \equiv x_1. \)

Evidently, the system (42) is a special case of (21) with the following components:

\[
A(y) = \begin{bmatrix} -\sigma & \sigma & 0 \\ 0 & -1 & -y \\ 0 & y & -\beta \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},
\]

\[
\varphi_0(y) = 0_{3,1}, \quad \varphi(y) = y, \quad c^T = [1, 0, 0].
\]

(43)

It is clear, that Assumption 4 is valid. To apply Theorem 3, accordingly with Assumption 3, one has to find a vector-function \( k(y) \in \mathbb{R}^3 \) so as the system \( \dot{x} = (A(y) - k(y)c^T)x, \) \( y = c^Tx \) be asymptotically stable. Let us pick up \( k(y) \equiv k \equiv [0, -\sigma, 0]^T. \) Then the matrix-function \( G(y) = A(y) - k(y)c^T \) is sum of a diagonal and a skew-symmetric matrices:

\[
G(y) = \begin{bmatrix} -\sigma & \sigma & 0 \\ -\sigma & -1 & -y \\ 0 & y & -\beta \end{bmatrix}.
\]

(44)

It can be easily shown that this choice provides fulfillment of the Assumption 3. Indeed, let us consider the system \( \dot{x} = G(c^T x)x, \) with the matrix \( G(y) \) given by (44), and introduce the Lyapunov function \( V(x) = 0.5x^Tx. \) Differentiating \( V(x(t)) \) on \( t \) one obtains:

\[
\dot{V}(x) = 0.5x^T(G(c^T x) + G(c^T x))x \\
= x^T \begin{bmatrix} -\sigma & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -\beta \end{bmatrix} x \\
= -\sigma x_1^2 - x_2^2 - \beta x_3^2.
\]

Then the exponential stability of the system \( \dot{x} = G(c^T x)x \) immediately follows. Therefore, Assumptions 3,4 are valid and Theorem 5 can be applied. The adjustable observer for Lorenz-based master system (42) has a form (22), where the matrices are given in (43). For estimation the unknown plant parameter \( \theta \) (and, thereby, for recovering the message signal), the tuning algorithm (24)–(26) with \( n = 3 \) has to be implemented in the observer. In the case of significant magnitude of the channel noise the robustified adaptation law (27), (28) can be used.

Some numerical examples of implementation of the proposed method for adaptive synchronization of Lorenz systems are given below.

B. Numerical example: square waveform recovering via adaptive synchronization of Lorenz systems

Let us use the algorithm (22), (24)–(26) for recovering parameter \( \theta \) of the master system (42). It was assumed above that \( \theta \) is an unknown constant. In practice this is a varying information signal, \( \theta = \theta(t), \) and applicability of the proposed method depends on the rate of tuning of the observer parameter \( \dot{\theta}(t). \) This rate can be found by means of the numerical examinations.

Let us rewrite the master/slave systems equations as follows:

**Master system:**

\[
\begin{cases}
\dot{x}_1 = \sigma \dot{x}_2 - \sigma x_1, \\
\dot{x}_2 = -\dot{x}_2 - x_1 x_3 + \theta x_1, \\
\dot{x}_3 = -\beta x_3 + x_1 x_2, \\
y(t) = x_1(t).
\end{cases}
\]

(45)

where \( r \) is some known constant factor, \( \vartheta(t) \) is a varying parameter (in the case of communication via chaotic signal modulation, \( \vartheta(t) \) can be treated as an information signal). It is fulfilled that \( \theta = r(1 + \vartheta(t)). \)

**Adjustable observer:**

\[
\begin{cases}
\dot{\hat{x}}_1 = \sigma \hat{x}_2 - \sigma \hat{x}_1, \\
\dot{\hat{x}}_2 = -\hat{x}_2 - y_r(t)\hat{x}_3 + \sigma e(t) \\
\dot{\hat{x}}_3 = -\beta \hat{x}_3 + y_r(t)\hat{x}_2, \\
e(t) = y_r(t) - \hat{x}_1(t).
\end{cases}
\]

(46)

where \( e(t) \) can be referred to as an observation error, \( y_r(t) \) is a measurable signal (in the case of communication systems, \( y_r \) is referred to as a received signal). For the noiseless case it is valid that \( y_r(t) \equiv y(t). \)
Augmented error filters:
\[
\begin{align*}
\dot{\Omega}_1 &= \sigma \Omega_2 - \sigma \Omega_1, \\
\dot{\Omega}_2 &= -\sigma \Omega_1 - \Omega_2 + y_r(t) \Omega_3, \\
\dot{\Omega}_3 &= -\beta \Omega_3 + y_r(t) \Omega_2, \\
\omega(t) &= \Omega_1(t), \\
\dot{\eta}_1 &= \sigma \eta_2 - \sigma \eta_1 - \Omega_1(t) \dot{\vartheta}(t), \\
\dot{\eta}_2 &= -\sigma \eta_1 - \eta_2 + y_r(t) \eta_3 - \Omega_2(t) \dot{\vartheta}(t), \\
\dot{\eta}_3 &= -\beta \eta_3 + y_r(t) \eta_2 - \Omega_3(t) \dot{\vartheta}(t),
\end{align*}
\]
(47)

Adaptation algorithm:
\[
\begin{align*}
\dot{e}(t) &= e(t) + c^T \eta(t), \\
\dot{\vartheta} &= \gamma \omega \dot{e}, \\
\vartheta(0) &= \vartheta_0,
\end{align*}
\]
(49)
where parameter $\gamma > 0$ is an adaptation gain.

The following numerical values of the master/slave systems parameters were chosen:
\[
\sigma = 10, \; \beta = 8/3, \; r = 97; \; \gamma = 0.45.
\]

In our experiments, the “square wave” process $\vartheta(t)$ had been taken. In Fig. 1 a the time history of the measured signal $y_r(t)$ is shown. In the Fig. 1 b the time histories of the original square waveform $\vartheta(t)$ and the waveform, recovered by means of the adaptive observer (46)–(49) are shown.

Figure 2 shows the corresponding time histories of the second and third components of the state estimation errors $\varepsilon_i(t) = x_i(t), \; i = 1, 2$. Similar results for analogous information signal are shown in Figs. 3, 4. The simulation results demonstrate high adaptation rate of the proposed algorithm. It can be seen that the transient time for state estimates is the same as the one for parameter estimates.

VI. CONCLUSIONS

In this paper an unified approach for nonlinear adaptive synchronization is proposed based on scheme of [5], [8], [9], [10] and new class of adaptation algorithms of [15], [18], [19]. It allows one to use chaotic signals generated by non-passifiable nonlinear systems, particularly, by systems with
relative degree greater than one which potentially increases security of communications. Two versions of synchronization scheme are proposed based on augmented error adaptive observer and high-order tuners. Conditions of the estimate convergence are established for the noiseless case (Theorems 1–3). Robustness of the scheme to the bounded measurement noise is established (Theorems 4–5). The proposed approach allows one to cope with bounded noise by means of robust modification of adaptation algorithm.

Theoretical results are illustrated by simulation example by signal transmission based on adaptive synchronization of Lorenz systems. That example demonstrates high of parameter identification rate. The proposed algorithms may be applied to transmission of both binary (digital) and analogous signals in communication systems.

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