Several basic properties of complex numbers

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Abstract. Complex numbers play a fundamental role in multiple engineering fields. In this paper, we first review what the algebraic form of a complex number is and define several operations on complex numbers. At last, we introduce Blaschke factors and discuss inequalities of complex numbers.

1. Introduction
The idea of complex first is thought of by Jhon Wallis in his A Treatise of Algebra (Wallis, Jhon (1685)) in 1685 [1]. Later, Jean-Robert Argand comes up with the proof of the fundamental theorem of algebra in his pamphlet on complex numbers in 1806. Complex number area is gradually getting well-rounded expand by many mathematics in the begin of 20 century [2-10].

In this paper, we first introduce the algebraic form of complex numbers and then define addition, multiplication, and conjugation operations. At last, we present several beautiful inequalities of complex numbers.

2. Main Works

2.1 Algebraic form
We use the symbol \( \mathbb{C} \) as the set of all complex numbers. That is, \( \mathbb{C} = \{x + i \cdot y; x, y \in \mathbb{R}\} \), where \( i^2 = 1 \). Given a complex number \( x + i \cdot y \), we define \( x \) to be its real part, \( y \) to be an imaginary part, and \( \sqrt{x^2 + y^2} \) to be its absolute value(modulus). Given a complex number \( z \), we use the symbol \( \text{Im}(z) \) to represent its imaginary part and \( \text{Re}(z) \) its real part.

If \( \text{Im}(z) = 0 \) we say \( z \) is real, and if \( \text{Re}(z) = 0 \) we say \( z \) is imaginary.

2.2 Operation on \( \mathbb{C} \)
\( \mathbb{C} \) admits operations of

Addition: Given two complex number \( z_1 = x_1 + i \cdot y_1 \) and \( z_2 = x_2 + i \cdot y_2 \),
\( z_1 + z_2 = (x_1 + i \cdot y_1) + (x_2 + i \cdot y_2) = (x_1 + x_2) + i \cdot (y_1 + y_2) \)  \( (1) \)

Multiplication: Given two complex number \( z_1 = x_1 + i \cdot y_1 \) and \( z_2 = x_2 + i \cdot y_2 \),
\( z_1 \cdot z_2 = (x_1 + i \cdot y_1) \cdot (x_2 + i \cdot y_2) = (x_1 x_2 - y_1 y_2) + i \cdot (x_1 y_2 + x_2 y_1) \)  \( (2) \)

complex conjugation:
Addition and multiplication are simply the natural extension of the corresponding operations on \( \mathbb{R} \) along with the rule that \( i^2 = -1 \). Both additions and multiplication satisfy associativity and commutativity. In details, if \( z_1, z_2, z_3 \) are three complex numbers, the following hold:

\[
(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3),
\]

\[
(z_1 \cdot z_2) \cdot z_3 = z_1 \cdot (z_2 \cdot z_3),
\]

\[
z_1 + z_2 = z_2 + z_1,
\]

\[
z_1 \cdot z_2 = z_2 \cdot z_1.
\]

The identity for the operation of addition is 0, and the identity for multiplication is 1. Given any non-zero complex number \( z \), its inverse is \( \frac{1}{z} \) in the sense of multiplication and \( -z \) in the sense of addition. Also, these arithmetic operations satisfy the distribution law. In details, given any three complex numbers \( z_1, z_2, z_3 \), the following hold:

\[
z_1 \cdot (z_2 + z_3) = z_1 \cdot z_2 + z_1 \cdot z_3.
\]

One also has \( \overline{z + w} = \overline{z} + \overline{w}, \quad \overline{zw} = \overline{z} \overline{w} \).

**Proposition 2.2.1** Given a complex number \( z \),

\[
\text{Re}(z) = \frac{1}{2}(z + \overline{z}),
\]

\[
\text{Im}(z) = \frac{1}{2i}(z - \overline{z}),
\]

\[
|z| = \sqrt{z \overline{z}}.
\]

**Proof.** Assume \( z = x_1 + i \cdot y_1; \omega = x_2 + i \cdot y_2 \), where \( x_1, y_1 \) are real numbers.

For the first identity, we show as follows:

\[
(z + \omega) = x_1 + x_2 - i \cdot (y_1 + y_2) = (x_1 - i \cdot y_1) + (x_2 - i \cdot y_2) = (z + \overline{\omega}).
\]

For the second identity, firstly we notice that

\[
(z \cdot \omega) = (x_1 x_2 - y_1 y_2) - i \cdot (x_1 y_2 + x_2 y_1) = (z \cdot \overline{\omega}).
\]

For the third equality, we show as follows:

\[
z + \overline{z} = x_1 + i \cdot y_1 + x_1 - i \cdot y_1 = 2x_1
\]

\[
= 2\text{Re}(z).
\]

For the fourth equality, we show:

\[
z - \overline{z} = x_1 + iy_1 - x_1 + iy_1 = 2iy_1
\]

\[
= 2\text{Im}(z).
\]

For the last one, notice that \( z \cdot \overline{z} = (x_1 + i \cdot y_1) \cdot (x_1 - i \cdot y_1) = x_1^2 + y_1^2 \).

**Remark.** For all \( z, w \in \mathbb{C} \) is real if and only if \( \overline{z} = z \), and \( z \) is imaginary if and only if \( \overline{z} = -z \).

**Proposition 2.2.2** (Properties of Modulus)

For any complex number \( w_1, w_2 \), it holds that \( |w_1 \cdot w_2| = |w_1| \cdot |w_2| \).

**Proof.** If \( w_1 = x_1 + i \cdot y_1 \) and \( w_2 = x_2 + i \cdot y_2 \), then

\[
|w_1 \cdot w_2| = |(x_1 + i \cdot y_1)(x_2 + i \cdot y_2)| = \sqrt{(x_1 \cdot x_2 + y_1 \cdot y_2)^2 + (x_1 \cdot y_2 + x_2 \cdot y_1)^2}.
\]

And since \( |w_1| = \sqrt{x_1^2 + y_1^2}, \quad |w_2| = \sqrt{x_2^2 + y_2^2} \), we have
$|w_1| \cdot |w_2| = \sqrt{(x_1^2 + y_1^2) \cdot (x_2^2 + y_2^2)} = \sqrt{x_2^2 (x_1^2 + y_1^2) + y_2^2 (x_1^2 + y_1^2)} = \sqrt{x_2^2 \cdot x_1^2 + x_2^2 \cdot y_1^2 + y_2^2 \cdot x_1^2 + y_2^2 \cdot y_1^2} = \sqrt{(x_1 \cdot x_2 + y_1 \cdot y_2)^2 + (x_1 \cdot y_2 + x_2 \cdot y_1)^2} = |w_1 \cdot w_2|$.

2.3 Visualization of $\mathbb{C}$

Given a complex number $z = a + bi$ in the complex plane shown in Figure 2, we know that $a = r \cos \theta, b = r \sin \theta$. Thus, we have that $z = r \cos \theta + r \sin \theta$ and $r = |z| = \sqrt{a^2 + b^2}$.

Notice that $\cos(\theta + 2k\pi) = \cos(\theta), \sin(\theta + 2k\pi) = \sin(\theta)$ ($k \in \mathbb{Z}$). We say $\theta + 2k\pi$ ($k \in \mathbb{Z}$) are the arguments of $z$. If $\theta$ satisfies $-\pi < \theta \leq \pi$, we say $\theta$ is the principal argument angle, and we use the symbol $\text{arg}(z)$ to represent the principal argument angle.

The set $\mathbb{C}$ of all complex numbers should be visualized as a coordinate plane $\mathbb{R}^2$ where $z = x + iy$ is plotted as the coordinate pair $(x, y)$ since there exists a bijection between complex numbers and points on the plane $\mathbb{R}^2$. In this case, the x-axis is called the real axis, y-axis is called the imaginary axis, and the coordination place is called the complex plane. Recalling the distance formula for $\mathbb{R}^2$, we see that $|z|$ can represent the distance from the point $z$ to the origin in the complex plane. More generally, $|z - w|$ is the distance from the point $z$ to the point $w$. Note that the addition of complex numbers coincides with the addition in the vector space $\mathbb{R}^2$. The addition of two complex numbers can be visualized in the sense of the addition of two vectors.

To visualize multiplication, we first recall that every point in $\mathbb{R}^2$ can be described by polar coordinates and translate this for $\mathbb{C}$. In detail, for any non-zero complex number $z$, we have

$|\frac{z}{|z|}| = |\frac{z}{|z|}| = |\frac{1}{|z|}| = 1$; thus, there exists $\Theta \in \mathbb{R}$ so that $\frac{z}{|z|} = \cos \Theta + i \sin \Theta$. By using the Euler formula, $e^{i\Theta} = \cos \Theta + i \sin \Theta$, we have $z = |z| \cos \Theta + i |z| \sin \Theta$, so that $z = |z|e^{i\Theta}$. By invoking trigonometric identities, we can check that $e^{i\Theta_1} \cdot e^{i\Theta_2} = e^{i(\Theta_1 + \Theta_2)}$.

Thus, for a complex number $z$ in $\mathbb{C}$, we have

$z \cdot w = |z| \cdot e^{i \text{arg}(z)} \cdot |w| \cdot e^{i \text{arg}(w)} = |zw| \cdot e^{i(\text{arg}(z) + \text{arg}(w))}$

So, multiplication of a complex number $z$ with a complex number $w$ is composed of two steps: scaling $z$ by $|w|$ and rotating counterclockwise by $\text{arg}(w)$.

2.4 Inequality

Since the absolute value of a complex number corresponds to distance on the complex plane, we can obtain the triangle inequality $|z + w| \leq |z| + |w|$.

Also, we can obtain the reverse triangle inequality $||z| - |w|| \leq |z - w|$.

Indeed from the triangle inequality, we have $|z| = |z - w + w| \leq |z - w| + |w|$. Thus,
\[ |z| - |w| \leq |z - w|, \]
\[ |w| = |w - z + z| \leq |w - z| + |z|. \]

We also can obtain by direct computation that \(|\text{Re}(z)| \leq |z|\) and \(|\text{Im}(z)| \leq |z|\).

**Definition 2.4.1** \(\{z_n\}\) is a Cauchy Sequence if \(\forall \varepsilon > 0 \exists N \in \mathbb{N}\) so that \(\forall m, n \geq N\) are has
\[
|z_n - z_m| < \varepsilon.
\]

In other words, all terms in a Cauchy sequence are eventually as close as we would like. A Convergent sequence is a Cauchy sequence, since the terms are eventually close to the limit and close with each other. Next, we show its converse is true.

**Theorem 2.4.2** [10] All Cauchy sequence converge in \(\mathbb{C}\).

Proof let \((Z_n)_{n \in \mathbb{C}}\) be a Cauchy Sequence. Notice that \(|\text{Re}(z)|, |\text{Im}(z)| \leq |z|\). We can know that \((\text{Re}(z_n))_{n \in \mathbb{N}}, (\text{Im}(z_n))_{n \in \mathbb{N}} \in \mathbb{R}\) are Cauchy Sequence. Because \(\mathbb{R}\) is complete, the real and imaginary parts converge to some \(x, y \in \mathbb{R}\), respectively.

Define \(w := x + iy\)
\[
\lim_{n \to \infty} |Z_n - w| = \lim_{n \to \infty} ((\text{Re}(z_n - x)^2 + (\text{Im}(z_n) - y)^2)^{1/2} = 0
\]
Thus \((Z_n)_{n \in \mathbb{C}}\) converges to \(w\).

**Theorem 2.4.3** [10] Let \(z, w\) be two complex numbers such that \(\bar{z}w \neq 1\). Then, \(|\frac{w-z}{1-\bar{w}z}| < 1\) if \(|z| < 1\) and \(|w| < 1\).

Proof. Notice that \(|w - z|^2 = (w - z)(\bar{w} - \bar{z}) = w\bar{w} - w\bar{z} - z\bar{w} + z^2 = |w|^2 - w\bar{z} - z\bar{w} + |z|^2\), and \(|1 - \bar{w}z|^2 = (1 - \bar{w}z)(1 - w\bar{z}) = (1 - \bar{w}z)(1 - wz)\)
\[
= 1 - w\bar{z} - \bar{w}z + \bar{w}z + |w|^2 |z|^2.
\]
We have \(|1 - \bar{w}z|^2 - |w - z|^2 = 1 + |w|^2|z|^2 - |w|^2 - |z|^2\)
\[
= |w|^2(|z|^2 - 1) + 1 - |z|^2\]
\[
= (|w|^2 - 1)(|z|^2 - 1) > 0.
\]
Thus, \(|1 - \bar{w}z|^2 > |w - z|^2\). It follows that \(|\frac{w-z}{1-\bar{w}z}| < 1\).

**Theorem 2.4.4** Let \(n\) be an arbitrary positive integer and \(a_j (j = 1,2,\cdots,n)\) be complex numbers.
If these complex number satisfy that \(\prod_{j \in I}(1 + a_j - 1) \leq \frac{1}{2}\), where \(I\) is an arbitrary subset of \(1,2,\cdots,n\). Then \(\sum_{j=1}^{n} |a_j| < 3\).

Proof. Assume that \(1 + a_j = r_j e^{i \theta_j}, r_j \geq 0, |\theta_j| \leq \pi, j = 1,2,\cdots\).

Thus, \(|\prod_{j \in I} r_j \cdot e^{i \sum_{j \in I} \theta_j}| \leq \frac{1}{2}\).

First we claim that for real number \(r, \theta\), where \(r > 0, |\theta| \leq \pi, |re^{i \theta} - 1| \leq \frac{1}{2}\), then \(\frac{1}{2} \leq r \leq \frac{3}{2}, |\theta| \leq \frac{\pi}{6}, |re^{i \theta} - 1| \leq |r - 1| + |\theta|\). This claim is easy to be proved by invoking geometric properties of complex numbers. It suffices to show \(|re^{i \theta} - 1| \leq |r - 1| + |\theta|\).

\[
|re^{i \theta} - 1| = |r(c \cos \theta + i \sin \theta) - 1|
\]
\[
= |(r - 1)(c \cos \theta + i \sin \theta) + [(c \cos \theta - 1) + i \sin \theta]|\]
\[
\leq |r - 1| + \sqrt{(c \cos \theta - 1)^2 + \sin^2 \theta} = |r - 1| + \sqrt{2(1 - c \cos \theta)}
\]
\[
= |r - 1| + 2 |\sin \frac{\theta}{2}| \leq |r - 1| + |\theta|.
\]

By induction, it is easy to obtain that
\[
\frac{1}{2} \leq \prod_{j \in I} r_j \leq \frac{3}{2}, |\sum_{j \in I} \theta_j| \leq \frac{\pi}{6}.
\]

It follows that \(|a_i| = |r_j e^{i \theta_j} - 1| \leq |r_j - 1| + |\theta_j|\).
Thus \(\sum_{j=1}^{n} |a_j| \leq \sum_{j=1}^{n} |r_j - 1| + \sum_{j=1}^{n} |\theta_j|\).
\[
\sum_{j \in I} |\theta_j - 1| = \sum_{j \in I} (r_j - 1) - \prod_{j \in I} (1 + (\eta - 1)) - 1 \leq \frac{3}{2} - 1 = \frac{1}{2}.
\]
\[ \sum_{r_j < 1} |r_j - 1| = \sum_{r_j < 1} (1 - r_j) \leq \prod_{r_j > 1} [1 - (1 - r_j)] - 1 \leq 2 - 1 = 1, \]
\[ \sum_{\theta_j \geq 0} |\theta_j| + \sum_{\theta_j < 0} |\theta_j| = \sum_{\theta_j \geq 0} \theta_j - \sum_{\theta_j < 0} \theta_j \leq \frac{\pi}{3}. \]

To conclude, \( \sum_{j=1}^n |a_j| \ll \frac{1}{2} + 1 + \frac{\pi}{3}. \)

### 3. Conclusion

In this paper, we mainly discuss the arithmetic and geometric structures of complex numbers. First, we introduce the algebraic form of complex numbers and then define addition, multiplication, and conjugation operations. We also discuss inequalities of complex numbers. At last, we give a beatiful inequality of complex numbers. In the future, we will report more on applications of Blaschke factors. Blaschke factors provide a powerful tool for the manipulation of the zeros of a holomorphic function analogously to factors for complex polynomials.

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