Transmission and reflection coefficients for a scalar field inside a charged black hole

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Using the late-time expansion we calculate the leading-order coefficients describing the evolution of a massless scalar field inside a Reissner-Nordstrom black hole. These coefficients may be interpreted as the reflection and transmission coefficients for scalar-field modes propagating from the event horizon to the Cauchy horizon. Our results agree with those obtained previously by Gursel et al by a different method.
In a recent paper [1], we analysed the evolution of a massless scalar field inside a Reissner-Nordstrom black hole. The analysis was based on a new method, the *late-time expansion*, which is essentially an expansion in the small parameter $1/t$. This method takes advantage of the relatively simple form of the ingoing radiation tails, which are known to obey an inverse-power law. [2] Assuming initial data $\psi \cong v^{-n}$ at the event horizon (EH), the asymptotic behavior near the Cauchy horizon (CH) was found in Ref. [1] to be

$$\psi = \sum_{j=0}^{\infty} \left[ a_j u^{-n-j} + b_j v^{-n-j} \right] + O(\delta r),$$

where $\delta r \equiv r - r_-$, $u$ and $v$ are the double-null Eddington-Finkelstein-like coordinates, and $a_j$ and $b_j$ are constants (the notation here is the same as that of Ref. [1], except for a few changes specified below). Of special importance are the coefficients $a_0$ and $b_0$, which dominate the evolution near the CH at late time (i.e. $v \to \infty$, $u \to -\infty$). These parameters may be interpreted (in a somewhat vague sense) as the reflection and transmission coefficients, respectively, for low-frequency scalar-field modes propagating from the EH to the CH. The explicit value of these coefficients (like all other coefficients $a_j$ and $b_j$) was not calculated in Ref. [1]. In this paper we shall use the late-time expansion to calculate $a_0$ and $b_0$. The problem of a massless scalar field inside a charged black hole was previously analysed [3, 4] by a different method (Fourier integration in the complex plain), and the transmission and reflection coefficients were calculated in Ref. [4]. Our goal here is to compare the results of the late-time expansion to those obtained in Ref. [4] by the other method.

We shall use here the same notation as in Ref. [1], except for the following changes:

(i) We denote here the "tortoise" coordinate by $r^*$ (not $x$);

(ii) $\kappa_-$, the inner horizon’s surface gravity, is defined here in the standard way, $\kappa_- = (r_+ - r_-)/(2r_-^2)$;

(iii) We denote the total scalar field by $\Psi$ (we reserve the symbol $\psi$ for the spherical-harmonics modes of the field). $\Psi$ satisfies the Klein-Gordon (KG) equation, $\Psi_{;\alpha}^{\alpha} = 0$. 

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We first decompose $\Psi$ into spherical harmonics:

$$
\Psi(r, \theta, \varphi, t) = \sum_{lm} Y_{lm}(\theta, \varphi) \psi_{lm}(r, t).
$$

For brevity, we shall omit the indices $l,m$, and denote $\psi_{lm}$ by $\psi$. We then use the late-time expansion:

$$
\psi = \sum_{k=0}^{\infty} \psi_k(r)t^{-n-k}.
$$

(2)

Here, like in Ref. [1], when studying the asymptotic behavior at the CH, we restrict attention to the leading-order of $\psi$ (and $\psi_k$) in $\delta r \equiv r - r_-$. At this order, the functions $\psi_k$ are polynomials in $r^*$ [1], of order $k$:

$$
\psi_k \approx \sum_{i=0}^{k} c^{ki} r^{*i}.
$$

(3)

From the analysis in Ref. [1] it is obvious that the leading-order term in Eq. (1), $a_0u^{-n} + b_0v^{-n}$, is equal to $t^{-n} F^1(w)$ (i.e. the term $j = 1$ in Eq. (29) there [7]), namely,

$$
a_0u^{-n} + b_0v^{-n} = t^{-n} \left[ c^{0,0} + c^{1,1}(r^*/t) + c^{2,2}(r^*/t)^2 + \ldots \right].
$$

(4)

Therefore, the coefficients $a_0$ and $b_0$ depend only on the parameters $c^{k,k}$. We shall now show that $a_0$ and $b_0$ can be determined directly from $c^{0,0}$ and $c^{1,1}$. For simplicity, let us define

$$
\hat{u} \equiv -u, \quad \hat{a}_0 \equiv (-1)^n a_0,
$$

such that $a_0u^{-n} = \hat{a}_0\hat{u}^{-n}$. Recalling that $\hat{u} = t - r^*$, $v = t + r^*$, we may write

$$
a_0u^{-n} + b_0v^{-n} = t^{-n} \left[ \hat{a}_0(1 - r^*/t)^{-n} + b_0(1 + r^*/t)^{-n} \right]
$$

$$
= t^{-n} \left[ (\hat{a}_0 + b_0) + n(\hat{a}_0 - b_0)(r^*/t) + \frac{n(n + 1)(\hat{a}_0 + b_0)}{2} (r^*/t)^2 + \ldots \right].
$$

(5)

Comparing Eqs. (4) and (5), we find

$$
c^{0,0} = \hat{a}_0 + b_0, \quad c^{1,1} = n(\hat{a}_0 - b_0).
$$

(6)

Therefore, all we need to do is to calculate the coefficients $c^{0,0}$ and $c^{1,1}$. This, in turn, requires the solution of the differential equation for $\psi_{k=0}$ and $\psi_{k=1}$. Recall that for that purpose it
is not sufficient to consider the leading order of these functions in $\delta r$: Since the parameters $c^{0,0}$ and $c^{1,1}$ will be determined by matching the functions $\psi_0$ and $\psi_1$ to the initial data at the EH, we shall need the exact form of these functions in the entire range $r_- < r < r_+$. The two functions $\psi_0$ and $\psi_1$ satisfy the same "static" (i.e. t-independent) KG equation,

$$f \psi_{k,rr} + (f_r + 2f/r) \psi_{k,r} - \frac{l(l + 1)}{r^2} \psi_k = 0 \quad (k = 0, 1),$$

where $f \equiv 1 - 2M/r + e^2/r^2$. The general solution of this equation is

$$\psi_k = aQ_l(x) + bP_l(x),$$

where $a$ and $b$ are arbitrary constants, $P_l$ and $Q_l$ are the Legendre functions of the first and second kinds, respectively, and

$$x \equiv \frac{2r - r_+ - r_-}{r_+ - r_-}.$$  

Thus, the most general expressions for $\psi_0$ and $\psi_1$ are

$$\psi_0 = a^0 Q_l(x) + b^0 P_l(x)$$

and

$$\psi_1 = a^1 Q_l(x) + b^1 P_l(x). \quad (7)$$

The parameters $a^0$, $b^0$, $a^1$ and $b^1$ are to be determined from the initial conditions at the EH. The event and inner horizons correspond to $x = 1$ and $x = -1$, respectively. At both points, $P_l(x)$ is regular and $Q_l(x)$ diverges logarithmically. The asymptotic form of the two Legendre functions at the horizons is given by

$$P_l(x = 1) = 1 \quad , \quad Q_l(x) = -\frac{1}{2}\ln(1-x) + \text{regular term} \quad (x \to 1) \quad (8)$$

and

$$P_l(x = -1) = (-1)^l \quad , \quad Q_l(x) = \frac{(-1)^l}{2}\ln(1+x) + \text{regular term} \quad (x \to -1). \quad (9)$$

As was shown in Ref. [1], regularity at the EH implies $a^0 = 0$, so
\[ \psi_0 = b^0 P_l(x) . \] (10)

Comparing Eqs. (3), (8) and (10), we find

\[ c^{0,0} = (-1)^l b^0 . \] (11)

Also, since both \( r^* \) and \( Q_l(x) \) diverge logarithmically at the CH, \( c^{1,1} \) must be proportional to \( a^1 \). Since at the CH \( r - r_- \propto e^{-2\kappa_r r^*} \) and

\[ 1 + x = \frac{2}{r_+ - r_-} (r - r_-) , \]

we have

\[ \ln(1 + x) = -2\kappa_r r^* + \text{regular term} \quad (x \to -1) . \] (12)

We find from Eqs. (3,7,9,12)

\[ c^{1,1} = (-1)^{l+1} \kappa_- a^1 . \] (13)

We shall now calculate \( b^0 \) and \( a^1 \) from the initial data at the EH. Presumably, we have there \( \psi = v^{-n} \), namely,

\[ \psi = t^{-n} [1 + r^*/t]^{-n} = t^{-n} [1 - n(r^*/t) + (...) (r^*/t)^2 + ...] . \] (14)

It is obvious from Eq. (2) that the term proportional to \( (r^*/t)^k \) in the brackets at the right-hand side will come from \( \psi_k \). Considering first the contribution from \( \psi_{k=0} \), we find from Eqs. (8,10,12)

\[ b^0 = 1 . \] (15)

Consider next the contribution from \( k = 1 \). At the EH, too, \( Q_l(x) \) is proportional to \( r^* \) as both diverge logarithmically. In analogy with the above treatment of the CH, we now have

\[ \ln(1 - x) = 2\kappa_+ r^* + \text{regular term} \quad (x \to 1) . \]
Equations (7) and (8) then yield

\[ \psi_1 = -\kappa_+ a^1 r^* + \text{regular term} \quad (x \to 1), \]

so Eq. (14) implies

\[ a^1 = n/\kappa_+. \quad (16) \]

Returning to the asymptotic behavior at the CH, we find [cf. Eqs. (11, 13, 15, 16)]

\[ c^{0,0} = (-1)^l, \quad c^{1,1} = (-1)^{l+1} n\kappa_-/\kappa_+, \]

and therefore

\[ \hat{a}_0 = (-1)^l \frac{\kappa_+ - \kappa_-}{2\kappa_+}, \quad b_0 = (-1)^l \frac{\kappa_+ + \kappa_-}{2\kappa_+} \quad (17) \]

[cf. Eq. (3)]. Substituting \( \kappa_\pm = \frac{r_+ - r_-}{2r_\pm} \), and recalling that \( \hat{a}_0 \equiv (-1)^n a_0 \), we finally obtain the desired expression for the reflection and transmission coefficients:

\[ a_0 = (-1)^{l+n} \frac{r_-^2 - r_+^2}{2r_-^2}, \quad b_0 = (-1)^l \frac{r_-^2 + r_+^2}{2r_-^2} \quad (18) \]

This result is the same as the one obtained by Gursel et al [4] by a different method (Fourier integration in the complex plain). [9]

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[6] Equation (24) of Ref. [1] describes a polynomial of order $k + 1$. However, as was shown in section VII there, regularity at the EH implies $c^{0,1} = 0$. Since $c^{1,2}$ vanishes too, it follows from the recursion relation for $\psi_k$ [Eqs. (13,22) there] that $c^{k,k+1}$ vanishes for all $k$.

[7] Note that regularity at the EH implies $F^0(w) = 0$, because $c^{k,k+1} = 0$ [6], so the leading-order term in Eq. (29) of Ref. [1] is $j=1$.

[8] This is essentially the "$k = 0$ solution" in Ref. [4] [cf. Eq. (6) there], except that $\psi$ there is $r$ times our $\psi$.

[9] See Eqs. (10,11,15) in [4], and note the following differences in terminology: (i) The coefficient related to transmission is denoted $b_0$ here and $A(0)$ in Ref. [4], whereas the one related to reflection is denoted $a_0$ here and $B(0)$ in Ref. [4]; (ii) The field denoted there by $\psi$ is the KG field multiplied by $r$ (whereas here $\Psi$ is the KG field itself); (iii) In Ref. [4] it is presumed that $n \equiv 2l + 2$ (which is the case if the mode has a nonvanishing static initial multipole moment [2]), so $n$ is always even. In our analysis $n$ may be any integer. (Recall that for a mode with a vanishing static initial multipole moment, one expects $n = 2l + 3$ [2].)