The gauge theory of dislocations: a uniformly moving screw dislocation

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In this paper, we present the equations of motion of a moving screw dislocation in the framework of the translation gauge theory of dislocations. In the gauge field theoretical formulation, a dislocation is a massive gauge field. We calculate the gauge field theoretical solutions of a uniformly moving screw dislocation. We give the subsonic and supersonic solutions. Thus, supersonic dislocations are not forbidden from the field theoretical point of view. We show that the elastic divergences at the dislocation core are removed. We also discuss the Mach cones produced by supersonic screw dislocations.

Keywords: dislocation dynamics; gauge theory of dislocations; supersonic motion; dislocation theory

1. Introduction

In the dynamics of dislocations, it is usually assumed that the screw dislocation possesses a Lorentz symmetry (Frank 1949; Hirth & Lothe 1968, Günther 1988) in contrast to the edge dislocation, which does not have a Lorentz-type symmetry because of two characteristic velocities, namely the velocities of transversal and longitudinal waves, entering the field equations. By means of such a Lorentz transformation, the screw dislocation can be transformed into a steady-state one, as in Maxwell’s theory of electromagnetic fields. The question arises if the ‘classical’ dynamics of dislocations reproduces the correct behaviour of dislocations. Usually, the conventional theories of dislocation dynamics have several drawbacks (e.g. inertial effects are missing, singularities, etc.). Unlike special relativity, where the speed of light is an upper limit, in elastodynamics the speed of sound is not a limit velocity. Shock waves can move faster than the velocity of sound and form Mach cones. Similar predictions that dislocations can move faster than the speed of sound have been given in the literature, based on the dynamics of dislocations (e.g. Eshelby 1956; Weertman 1967; Günther 1968, 1969; Callias & Markenscoff 1980; Weertman & Weertman 1980; Markenscoff & Huang 2008) and computer simulations (Gumbsch & Gao 1999; Koizumi et al. 2002; Li & Shi 2002; Tsuzuki et al. 2008). Supersonic dislocations have been recently observed in plasma crystals (Nosenko et al. 2007).

Another question is whether dislocations cause symmetric or asymmetric force stresses. In classical dislocation theories, the elastic distortion tensor is asymmetric, and the force stress tensor is assumed to be symmetric. Owing to
this asymmetry between the distortion and the stress tensors, Kröner (1976) has introduced a modulus of rotation for the antisymmetric part of the force stress tensor. In addition, Hehl & Kröner (1965) and Kröner (1968) argued that dislocations produce moment stresses such as response and antisymmetric force stresses. Pan & Chen (1994) calculated the antisymmetric stress corresponding to the body couple using linear and nonlinear continuum mechanics. Gao (1999) has developed an asymmetric theory of non-local elasticity based on an atomic lattice model and the three-dimensional quasi-continuum field theory. He pointed out that both strain and local rotation should be regarded as basic variables of geometric deformation, and he has also shown that the local rotation makes a very important contribution to the internal energy. For isotropic materials, the antisymmetric stress comes from the long-range property of interaction of atoms in the metal materials and non-uniform distribution of atomic forces in the more microscopic structures (Gao 1999).

A promising and straightforward candidate for an improved dynamical theory of dislocations is the so-called translational gauge theory of dislocations (Kadić & Edelen 1983, 1996; Edelen & Lagoudas 1988; Lazar & Anastassiadis 2008, 2009). In the gauge theory of defects, dislocations arise naturally as a consequence of broken translational symmetry, and their existence is not required to be postulated a priori. Moreover, such a theory uses the field theoretical framework that is well accepted in theoretical physics.

Recently, Sharma & Zhang (2006) applied the gauge theory of dislocations to a uniformly moving screw dislocation. They derived a gauge field theoretical solution to the problem of a uniformly moving screw dislocation. However, we will show that their solution is not the correct one for a supersonic screw dislocation. First of all, they used the constitutive relations given by Kadić & Edelen (1983) and Edelen & Lagoudas (1988), which are too simple. Even in Edelen’s version (Kadić & Edelen 1983; Edelen & Lagoudas 1988) of the gauge theory of dislocations, there are at least three characteristic velocities. For the anti-plane strain problem of a screw dislocation, Edelen’s theory possesses two characteristic velocities, namely the velocity of shear waves and a gauge theoretical characteristic velocity. This makes it impossible, without further simplifications and arguments, to use a Lorentz transformation, or at least it is not possible to transform a moving screw dislocation into a steady-state dislocation. Nevertheless, Sharma & Zhang (2006) have constructed a solution of a screw dislocation using the Lorentz transformation with respect to a gauge theoretical characteristic velocity. In addition, they have considered the gauge theoretical characteristic velocity as an upper limit velocity. Finally, they solved the equations of motion of a screw dislocation in a steady state. The stress field of their solution does not possess the correct far-field behaviour because their solution is only given in terms of the characteristic gauge velocity and not in terms of the shear speed of sound. No Mach cones appear in the supersonic regime. At best, the solution given by Sharma & Zhang (2006) is valid in the subsonic regime only if the characteristic gauge velocity is equal to the shear speed of sound.

In the meantime, Lazar & Anastassiadis (2008, 2009) have presented an improved translational gauge theoretical formulation of dislocations. They have used the correct isotropic constitutive relations in the gauge theory of dislocations that are incomplete and too simple in the books of Kadić & Edelen (1983) and
Edelen & Lagoudas (1988). In this paper, we derive the equations of motion of a screw dislocation and we will solve them for a uniformly moving screw dislocation using the theory of Lazar & Anastassiadis (2008). We will show that a dislocation is a massive field in the gauge theoretical formulation. We will calculate subsonic and supersonic solutions of a uniformly moving screw dislocation.

2. Gauge theory of dislocations

In dislocation dynamics, we have the following state quantities of dislocations\(^1\) (Kosevich 1979; Landau & Lifschitz 1989):

\[
T_{ijk} = \beta_{ik,j} - \beta_{ij,k} \quad \text{and} \quad I_{ij} = -v_{i,j} + \dot{\beta}_{ij},
\]

(2.1)

which are called the dislocation density tensor and the dislocation current tensor, respectively. These are the kinematic quantities of dislocations and are given in terms of the incompatible elastic distortion tensor \(\beta_{ij,k}\) and incompatible physical velocity of the material continuum \(v_i\). The dislocation density tensor describes the location and the shape of the dislocation core. The dislocation current tensor \(I_{ij}\) describes the movement of the dislocation core and contains rate terms \(v_{i,j}\) and \(\dot{\beta}_{ij}\). The dislocation density and the dislocation current tensors have to satisfy the translational Bianchi identities (square brackets indicate skew symmetrization)

\[
\epsilon_{jkl} T_{ijk,l} = 0 \quad \text{and} \quad \dot{T}_{ijk} + 2 I_{[ij,k]} = 0.
\]

(2.2)

The first equation means that dislocations do not have sources and the second one represents that the circulation of the dislocation current is proportional to the time derivative of the dislocation density. On the other hand, equations (2.2) are compatibility conditions, ensuring that \(T_{ijk}\) and \(I_{ij}\) can be given in terms of \(\beta_{ij}\) and \(v_i\), respectively, according to equations (2.1).

In the dynamical translational gauge theory of dislocations, the Lagrangian is of the bilinear form (linear theory)

\[
L = T - W = \frac{1}{2} p_i v_i + \frac{1}{2} D_{ij} I_{ij} - \frac{1}{2} \sigma_{ij} \beta_{ij} - \frac{1}{4} H_{ijk} T_{ijk}.
\]

(2.3)

Here, the canonical conjugate quantities (response quantities) are defined by

\[
p_i := \frac{\partial L}{\partial v_i}, \quad \sigma_{ij} := -\frac{\partial L}{\partial \beta_{ij}}, \quad D_{ij} := \frac{\partial L}{\partial I_{ij}} \quad \text{and} \quad H_{ijk} := -2 \frac{\partial L}{\partial T_{ijk}},
\]

(2.4)

where \(p_i\), \(\sigma_{ij}\), \(I_{ij}\) and \(H_{ijk}\) are the momentum vector, the force stress tensor, the dislocation momentum flux tensor and the pseudomoment stress tensor, respectively.

The Euler–Lagrange equations derived from the total Lagrangian \(\mathcal{L} = \mathcal{L}(v_i, \beta_{ij}, I_{ij}, T_{ijk})\) are given by

\[
E^v_i (\mathcal{L}) = \partial_i \frac{\partial \mathcal{L}}{\partial \dot{v}_i} + \partial_j \frac{\partial \mathcal{L}}{\partial v_{i,j}} - \frac{\partial \mathcal{L}}{\partial v_i} = 0
\]

(2.5)

\(^1\)We use the usual notation \(\beta_{i,j,k} := \partial_k \beta_{ij}\) and \(\hat{\beta}_{ij} := \partial_t \beta_{ij}\).
and

\[ E^\beta_{ij}(\mathcal{L}) = \partial_t \frac{\partial \mathcal{L}}{\partial \dot{\beta}_{ij}} + \partial_k \frac{\partial \mathcal{L}}{\partial \beta_{ij,k}} - \frac{\partial \mathcal{L}}{\partial \beta_{ij}} = 0. \] (2.6)

We add to \( \mathcal{L} \) a so-called null Lagrangian, \( \mathcal{L}_N = \sigma^0_{ij} \beta_{ij} - p^0_i v_i \), with the ‘background’ stress \( \sigma^0_{ij} \) and momentum \( p^0_i \) as external source fields. Written in terms of the canonical conjugate quantities (2.4), equations (2.5) and (2.6) then take the form

\[ D_{ij,j} + p_i = p^0_i \] (momentum balance of dislocations) (2.7)

and

\[ \dot{D}_{ij} + H_{ijk,k} + \sigma_{ij} = \sigma^0_{ij} \] (stress balance of dislocations). (2.8)

Equations (2.7) and (2.8) represent the dynamical equations for the balance of dislocations. Equation (2.7) is the momentum balance law of dislocations, where the physical momentum is the source of the dislocation momentum flux. Equation (2.8) represents the stress balance of dislocations. Thus, the force stress and the time derivative of the dislocation momentum flux are the sources of the pseudomoment stress. The conservation of linear momentum appears as an integrability condition for the balance of dislocation equations. This can be obtained by applying \( \partial_i \) in equation (2.7) and \( \partial_j \) in equation (2.8) and subtracting the latter from the former,

\[ \dot{p}_i - \sigma_{ij,j} = 0 \] (force balance of elasticity), (2.9)

where the time derivative of the physical momentum is the source of the force stress.

The linear constitutive relations for the momentum, the asymmetric force stress, the dislocation momentum flux tensor and the pseudomoment stress of an isotropic and centrosymmetric medium are (Lazar & Anastassiadis 2008)

\[ p_i = \rho v_i, \] (2.10)

\[ \sigma_{ij} = \lambda \delta_{ij} \beta_{kk} + \mu (\beta_{ij} + \beta_{ji}) + \gamma (\beta_{ij} - \beta_{ji}), \] (2.11)

\[ D_{ij} = d_1 \delta_{ij} I_{kk} + d_2 (I_{ij} + I_{ji}) + d_3 (I_{ij} - I_{ji}) \] (2.12)

and

\[ H_{ijk} = c_1 T_{ijk} + c_2 (T_{jki} + T_{kij}) + c_3 (\delta_{ij} T_{lkl} + \delta_{ik} T_{ijl}), \] (2.13)

where \( \rho \) is the mass density with nine material constants \( \mu, \lambda, \gamma, c_1, c_2, c_3, d_1, d_2 \) and \( d_3 \). Here, \( \mu \) and \( \lambda \) are the so-called Lamé coefficients, \( \gamma \) denotes the modulus of rotation, \( c_1-c_3 \) are higher-order stiffness parameters and \( d_1-d_3 \) are related to higher-order inertia due to dislocations.

The requirement of non-negativity of the energy (material stability), \( E = T + W \geq 0 \), leads to the conditions of positive semi-definiteness of the material constants. Thus, the material parameters have to fulfill the following
conditions (Lazar & Anastassiadis 2008):

\[
d_2 \geq 0, \quad d_3 \geq 0, \quad 3d_1 + 2d_2 \geq 0, \quad \mu \geq 0, \quad \gamma \geq 0, \quad 3\lambda + 2\mu \geq 0, \quad c_1 - c_2 \geq 0, \quad c_1 + 2c_2 \geq 0 \quad \text{and} \quad c_1 - c_2 + 2c_3 \geq 0.
\] (2.14)

(2.15)

(2.16)

Substituting the constitutive equations in the above systems (2.7) and (2.8), we obtain

\[
d_1(\dot{\beta}_{ij,i} - v_{j,ji}) + (d_2 + d_3)(\dot{\beta}_{ij,j} - v_{i,j}) + (d_2 - d_3)(\dot{\beta}_{ji,j} - v_{j,ji}) + p_i = p_i^0 \quad (2.17)
\]

and

\[
d_1\delta_{ij}(\ddot{\beta}_{ii} - \dot{v}_{k,k}) + (d_2 + d_3)(\ddot{\beta}_{ij} - \dot{v}_{i,j}) + (d_2 - d_3)(\ddot{\beta}_{ji} - \dot{v}_{j,i})
+ c_1(\beta_{ik,jk} - \beta_{ij,kk}) + c_2(\beta_{ji,ik} - \beta_{jk,ik} + \beta_{kj,ik} - \beta_{ki,jk})
+ c_3[\delta_{ij}(\beta_{lk,ik} - \beta_{ll,sk}) + (\beta_{lk,ji} - \beta_{lj,ki})] + \sigma_{ij} = \sigma_{ij}^0, \quad (2.18)
\]

which are a coupled system of partial differential equations for the field quantities \(v\) and \(\beta\).

3. Equations of motion of a screw dislocation

We now proceed to derive the equations of motion for a moving screw dislocation. The symmetry of a screw dislocation leaves only the following non-vanishing components of the physical velocity vector and elastic distortion tensor: \(v_z\), \(\beta_{zx}\) and \(\beta_{zy}\). The equations of motion for a moving screw dislocation read

\[
(d_2 + d_3)(\ddot{\beta}_{zx,x} + \dot{\beta}_{zy,y} - \Delta v_z) + \rho v_z = \rho v_z^0, \quad (3.1)
\]

(3.2)

(3.3)

(3.4)

and

\[
(d_2 - d_3)(\ddot{\beta}_{zy} - \dot{v}_{z,y}) + c_1(\beta_{zy,xy} - \beta_{zx,yy}) + (\mu + \gamma)\beta_{zx} = (\mu + \gamma)\beta_{zx}^0, \quad (3.5)
\]

where \(\Delta = \partial_{xx} + \partial_{yy}\). In addition, the equilibrium condition is given by

\[
(\mu + \gamma)(\beta_{zx,x} + \beta_{zy,y}) = \rho \dot{v}_z. \quad (3.6)
\]

From the system of equations (3.2)–(3.5), we obtain

\[
\frac{c_1}{\mu + \gamma} = -\frac{c_2}{\mu - \gamma} \quad (3.7)
\]

(3.8)
We may introduce the following quantities:

\[
\ell_1^2 = \frac{c_1}{\mu + \gamma}, \quad (3.9)
\]

\[
L_1^2 = \frac{d_2 + d_3}{\rho}, \quad (3.10)
\]

and

\[
\tau_1^2 = \frac{d_2 + d_3}{\mu + \gamma}. \quad (3.11)
\]

Here, \(\ell_1\) and \(L_1\) are, respectively, the static and dynamic characteristic length scales and \(\tau_1\) is the characteristic time scale of the anti-plane strain problem. Moreover, \(\ell_1\) is related to dislocation stiffness and \(L_1\) is related to dislocation inertia.\(^2\) The velocity of the elastic shear waves is defined in terms of the dynamic length scale \(L_1\) and the time scale \(\tau_1\) as

\[
c_1^2 = \frac{L_1^2}{\tau_1^2} = \frac{\mu + \gamma}{\rho}. \quad (3.12)
\]

Owing to the presence of \(\gamma\), the velocity of the elastic shear waves is greater than that in a theory with symmetric force stresses. Moreover, the velocity of the shear waves has a similar form in micropolar elasticity (e.g. Nowacki 1986), where the force stress is also asymmetric. In a similar way, we may introduce the following transversal gauge theoretic velocity defined in terms of \(\ell_1\) and \(\tau_1\):

\[
a_1^2 = \frac{\ell_1^2}{\tau_1^2} = \frac{c_1}{d_2 + d_3} \quad (3.13)
\]

and therefore

\[
\frac{\ell_1^2}{L_1^2} = \frac{a_1^2}{c_1^2}. \quad (3.14)
\]

In the case \(\gamma = 0\), we recover symmetric force stresses and, from relations (3.7) and (3.8), \(c_1 = -c_2\) and \(d_3 = 0\). It can be seen that \(c_1 = -c_2\) gives the inequalities (2.16) with \(c_1 \geq 0\) and \(-c_1 \geq 0\). Thus, for non-negative \(c_1\) and \(c_1 \neq 0\), we cannot fulfill equations (2.16). This is the price we have to pay for symmetric force stresses of a screw dislocation in the dislocation gauge theory.

Because we are dealing with the physical state quantities \((v, \beta)\), no pseudo-Lorentz gauge (Kadić & Edelen 1983; Edelen & Lagoudas 1988) is used and allowed during the simplification of the equations of motion. Gauge conditions are allowed only for gauge potentials and not for physical state quantities. Of course, for anti-plane strain, the equilibrium condition (3.6), together with equation (3.12), looks like a ‘gauge’ condition, but it is not a gauge condition from the physical interpretation.

\(^2\)We want to mention that similar length scales (\(\ell_s\), static characteristic length; \(\ell_d\), dynamic characteristic length) have also been obtained by Gitman et al. (2005), Askes & Alfantis (2006) and Askes et al. (2007) in a dynamic gradient elasticity.
Applying the equilibrium condition (3.6), the equations of motion (3.1)–(3.5) can be written in the form

\[ \tau_1^2 \ddot{v}_z - L_1^2 \Delta v_z + v_z = v_0^z, \]  
(3.15)

\[ \tau_1^2 \ddot{\beta}_{zx} - \ell_1^2 \Delta \beta_{zx} - \tau_1^2 \left( 1 - \frac{\ell_1^2}{L_1^2} \right) \dot{v}_{z,x} + \beta_{zx} = \beta_{0}^{zx}, \]  
(3.16)

and

\[ \tau_1^2 \ddot{\beta}_{zy} - \ell_1^2 \Delta \beta_{zy} - \tau_1^2 \left( 1 - \frac{\ell_1^2}{L_1^2} \right) \dot{v}_{z,y} + \beta_{zy} = \beta_{0}^{zy}. \]  
(3.17)

These are the equations of motion of an arbitrary moving screw dislocation in the framework of dislocation gauge theory. Some important questions arise: whether the lengths \( \ell_1 \) and \( L_1 \) are independent or not and is there a physical reason to decouple the field equations (3.15)–(3.17)?

Consider a screw dislocation moving in the \( x \)-direction. If we want to construct a solution that is consistent with the classical solution, we have to fulfill the condition \( I_{zx}^0 = -v_0^z + \dot{\beta}_0^{zx} = 0 \), because only the classical dislocation density \( T_{zx}^0 \) and dislocation current \( I_{zy}^0 \) are non-zero. If we use the field equations (3.15) and (3.16), we obtain

\[ -\left[ \frac{\ell_1^2}{c_T^2} \partial_{tt} - L_1^2 \Delta + 1 \right] v_{z,x} + \left[ \frac{L_1^2}{c_T^2} \partial_{tt} - \ell_1^2 \Delta + 1 \right] \dot{\beta}_{zx} = 0. \]  
(3.18)

To guarantee that equation (3.18) is fulfilled, we choose

\[ v_{z,x} = \dot{\beta}_{zx} \]  
(3.19a)

and

\[ L_1 = \ell_1. \]  
(3.19b)

Equation (3.19a) ensures that \( I_{zx} = 0 \) and equation (3.19b), with equation (3.14), gives the relation \( a_T = c_T \), which means that only one characteristic velocity \( c_T \) survives for a screw dislocation in the gauge theory of dislocations. Thus, for a physically consistent solution, we obtain the uncoupled Klein–Gordon equations

\[ \left[ 1 + \ell_1^2 \square_T \right] v_z = v_0^z, \]  
(3.20)

\[ \left[ 1 + \ell_1^2 \square_T \right] \beta_{zx} = \beta_{0}^{zx} \]  
(3.21)

and

\[ \left[ 1 + \ell_1^2 \square_T \right] \beta_{zy} = \beta_{0}^{zy}, \]  
(3.22)

with the following \((1+2)\)-dimensional d’Alembert operator (wave operator)

\[ \square_T = \frac{1}{c_T^2} \partial_{tt} - \Delta. \]  
(3.23)

Moreover, the uncoupled systems (3.20)–(3.22) with only one characteristic velocity possess a Lorentz symmetry. However, \( c_T \) is not a limit velocity, unlike the speed of light in special relativity. In field theories, the Klein–Gordon equations describe massive fields (e.g. Rubakov 2002). Thus, a dislocation is a massive
gauge field. From the condition $L_1 = \ell_1$, we find, for the inertia term of a screw dislocation,

$$d_2 + d_3 = \frac{c_1}{c_T} = \rho \ell_1^2,$$

(3.24)

which is given in terms of the characteristic length scale $\ell_1$. Under these assumptions, the dynamical dislocation gauge theory of a screw dislocation possesses only one internal length scale $\ell_1$.

If we multiply equations (3.20)–(3.22) by $\Box_T$ and use the ‘classical’ result (Günther 1968, 1969), we obtain

$$\left[1 + \ell_1^2 \Box_T\right] \Box_T v_z = I_{zy}^0,$$  

(3.25)

$$\left[1 + \ell_1^2 \Box_T\right] \Box_T \beta_{xz} = T_{xzy}^0,$$  

(3.26)

and

$$\left[1 + \ell_1^2 \Box_T\right] \Box_T \beta_{zy} = -T_{zyx}^0 + \frac{1}{c_T^2} I_{zy}^0$$  

(3.27)

as a set of fourth-order partial differential equations. As source terms, only the classical dislocation density $T_{zy}^0$ and dislocation current $I_{zy}^0$ are acting. Equations (3.25)–(3.27) have the two-dimensional form of the Bopp–Podolsky equations (Bopp 1940; Podolsky 1942; see also Iwanenko & Sokolow 1953) in generalized electrodynamics, introduced by Bopp and Podolsky to avoid singularities in electrodynamics.

In general, the velocity $V$ of a screw dislocation might be subsonic or supersonic relative to the material space. A subsonic velocity lies in the range $0 < V < c_T$ and for supersonic screw dislocations, the velocity reads $c_T < V$.

4. Uniformly moving screw dislocation

We now study a screw dislocation moving with a uniform velocity $V$ in the $x$-direction. Here, $V$ denotes the dislocation velocity in the material space. The dislocation moves relative to the material space, which plays the role of an ‘aether’. If a screw dislocation is moving with the velocity $V$, then $v_z$ is the physical velocity of the material space, where the dislocation lives, in order to transport the dislocation core to another position in the material space. Let $x'$ be the coordinate in the direction of motion in the moving coordinate system. For the uniformly moving screw dislocation, we use the transform

$$x' = x - Vt,$$  

(4.1)

with

$$\partial_t = -V \partial_{x'},$$  

(4.2)
and we obtain the equations
\[
\begin{align*}
&\left[ 1 - \ell^2_1 \left( (1 - M^2_T) \partial_{x'x'} + \partial_{yy} \right) \right] v_z = v^0_z, \quad (4.3) \\
&\left[ 1 - \ell^2_1 \left( (1 - M^2_T) \partial_{x'x'} + \partial_{yy} \right) \right] \beta_{zx} = \beta^0_{zx}, \quad (4.4) \\
&\left[ 1 - \ell^2_1 \left( (1 - M^2_T) \partial_{x'x'} + \partial_{yy} \right) \right] \beta_{zy} = \beta^0_{zy}, \quad (4.5) \\
&\left[ 1 - \ell^2_1 \left( (1 - M^2_T) \partial_{x'x'} + \partial_{yy} \right) \right] T_{zxy} = T^0_{zxy} \quad (4.6)
\end{align*}
\]
and
\[
\left[ 1 - \ell^2_1 \left( (1 - M^2_T) \partial_{x'x'} + \partial_{yy} \right) \right] I_{zy} = I^0_{zy}, \quad (4.7)
\]
with the Mach number of a moving screw dislocation relative to \( c_T \)
\[
M_T = \frac{V}{c_T}. \quad (4.8)
\]
Equations (4.3)–(4.7) are two-dimensional modified inhomogeneous Helmholtz equations (subsonic case). In the supersonic case, they change to one-dimensional inhomogeneous Klein–Gordon equations (see below).

(a) Subsonic case

We now study a screw dislocation moving with a uniform velocity \( V < c_T \) in the \( x \)-direction \((M_T < 1)\). The ‘background’ solution reads (e.g. Günther 1968)
\[
\begin{align*}
v^0_z &= V \partial_y F^0, \quad \beta^0_{zx} = -\partial_y F^0, \quad \beta^0_{zy} = \beta^2_T \partial_x F^0, \\
T^0_{zxy} &= b \delta(x - Vt) \delta(y) \quad \text{and} \quad I^0_{zy} = -b V \delta(x - Vt) \delta(y),
\end{align*}
\]
with
\[
F^0 = \frac{b}{2\pi} \frac{1}{\beta_T} \ln r_T, \quad (4.10)
\]
\[
r_T = \sqrt{(x - Vt)^2 + \beta^2_T y^2} \quad \text{and} \quad \beta_T = \sqrt{1 - M^2_T}. \quad (4.11)
\]
In the subsonic region, the field equations (4.3)–(4.7) are
\[
\begin{align*}
&\left[ 1 - \ell^2_1 (\beta^2_T \partial_{x'x'} + \partial_{yy}) \right] v_z = v^0_z, \quad (4.12) \\
&\left[ 1 - \ell^2_1 (\beta^2_T \partial_{x'x'} + \partial_{yy}) \right] \beta_{zx} = \beta^0_{zx}, \quad (4.13) \\
&\left[ 1 - \ell^2_1 (\beta^2_T \partial_{x'x'} + \partial_{yy}) \right] \beta_{zy} = \beta^0_{zy}, \quad (4.14) \\
&\left[ 1 - \ell^2_1 (\beta^2_T \partial_{x'x'} + \partial_{yy}) \right] T_{zxy} = T^0_{zxy} \quad (4.15)
\end{align*}
\]
and
\[
\left[ 1 - \ell^2_1 (\beta^2_T \partial_{x'x'} + \partial_{yy}) \right] I_{zy} = I^0_{zy}, \quad (4.16)
\]
Equations (4.12)–(4.16) can easily be solved with the inhomogeneous parts (4.9) using Fourier transforms or other techniques. The solutions for the dislocation
density and the dislocation flux of a screw dislocation are given by
\begin{equation}
T_{xy} = \frac{b}{2\pi} \frac{1}{\ell^2} \beta \frac{K_0 \left( \frac{r_T}{\ell_1} \right)}{\beta_T}
\end{equation}
and
\begin{equation}
I_{zy} = -\frac{b}{2\pi} \frac{V}{\ell^2} \beta \frac{K_0 \left( \frac{r_T}{\ell_1} \right)}{\beta_T}
\end{equation}

Here, the symbol $K_n$ stands for the modified Bessel function of second kind (McDonald function) and of order $n$. In figure 1, the dislocation density is plotted for different speeds that are subsonic with respect to $c_T$. The field of the dislocation density suffers a contraction when the value of its velocity approaches the velocity $c_T$. The curve of $T_{xy}$ is a circle at $V = 0$ and is a generalized ellipse at any other velocity $V < c_T$. The field is dilated in the directions orthogonal to the direction of motion and contracted along the line of motion. The solution for the elastic velocity is given by
\begin{equation}
v_z = \frac{b}{2\pi} \frac{\beta y}{r_T^2} \left[ 1 - \frac{r_T}{\ell_1} K_1 \left( \frac{r_T}{\ell_1 \beta_T} \right) \right].
\end{equation}

In figures 2 and 3, the physical velocity of a screw dislocation is plotted for subsonic velocities. It again shows a Fitzgerald contraction and it does not have a singularity. The physical velocity possesses maximum values depending on the dislocation velocity near the dislocation line. The solution of the elastic distortion reads
\begin{equation}
\beta_{zx} = -\frac{b}{2\pi} \frac{\beta y}{r_T^2} \left[ 1 - \frac{r_T}{\ell_1} K_1 \left( \frac{r_T}{\ell_1 \beta_T} \right) \right]
\end{equation}
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Figure 2. Contour plot of the elastic velocity $v_z$ of a screw dislocation moving with subsonic speed: (a) $V = 0.1c_T$ and (b) $V = 0.9c_T$.

Figure 3. Three-dimensional plots of the physical velocity $v_z$ in units of $bV/2\pi \ell_1^2 \beta_T$: (a) $V = 0.1c_T$ and (b) $V = 0.9c_T$.

and

$$\beta_{zy} = \frac{b}{2\pi} \frac{\beta_T (x - Vt)}{r_T^2} \left[ 1 - \frac{r_T}{\ell_1 \beta_T} K_1 \left( \frac{r_T}{\ell_1 \beta_T} \right) \right]. \quad (4.21)$$

All the elastic fields are non-singular in the gauge theoretic framework.

On the other hand, solutions (4.20) and (4.21) are of the same form as the expression given by Sharma & Zhang (2006) if $c_T = a_T$ and $\gamma = 0$. 

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Figure 4. Contour plot of the dislocation density $T_{xxy}$ of a screw dislocation moving with supersonic speed $V = \sqrt{2}c_T$.

(b) Supersonic case

We now consider the supersonic case, $V > c_T$ (supersonic case, $M_T > 1$). Therefore, the term $\beta^2_{T} = -\gamma^2_{T}$ alters, where

$$\gamma_{T} = \sqrt{M^2_{T} - 1}.$$  \hspace{1cm} (4.22)

Then, the field equation for the dislocation density changes to

$$\left[1 + \ell^2_{1}(\gamma^2_{T}\partial_{x'x'} - \partial_{yy})\right]T_{xxy} = T^0_{xxy},$$ \hspace{1cm} (4.23)

It is an inhomogeneous one-dimensional Klein–Gordon equation. The dislocation density of a supersonic screw dislocation has the form of the corresponding Green function. Therefore, it is given by (e.g. Iwanenko & Sokolow 1953)

$$T_{xxy} = \frac{b}{2\ell^2_{1}\gamma_{T}}J_0\left(\frac{\sqrt{(Vt - x)^2 - \gamma^2_{T}y^2}}{\ell_1\gamma_{T}}\right)H((Vt - x) - \gamma_{T}|y|).$$ \hspace{1cm} (4.24)

Here, $J_n$ denotes the Bessel function of first kind and of order $n$, and $H$ is the Heaviside step function. It is non-zero just for $Vt - x > \gamma_{T}|y|$. It builds a shear-wave Mach cone with the angle $\sin \theta_{T} = c_T / V$. The dislocation density has a maximum of $T_{xxy} = b/2\gamma_{T}\ell^2_{1}$ on the Mach cone. Inside the Mach cone, it oscillates with decreasing amplitude. For the dislocation current density, we obtain $I_{xy} = -V T_{xxy}$ with equation (4.24). The visualization of the Mach cone of the dislocation density $T_{xxy}$ produced by the screw dislocation in the supersonic regime is plotted in figure 4.
The equations of the elastic fields, altering their character from elliptic to hyperbolic, are

\[
[1 + \ell_1^2(\gamma_T^2 \partial_{x'x'} - \partial_{yy})] v_z = v_z^0, \quad (4.25)
\]

\[
[1 + \ell_1^2(\gamma_T^2 \partial_{x'x'} - \partial_{yy})] \beta_{zx} = \beta_{zx}^0, \quad (4.26)
\]

and

\[
[1 + \ell_1^2(\gamma_T^2 \partial_{x'x'} - \partial_{yy})] \beta_{zy} = \beta_{zy}^0, \quad (4.27)
\]

with the inhomogeneous parts (A4)–(A6). The corresponding solutions read

\[
v_z = \frac{bV}{2\ell_1} \frac{y}{\sqrt{(Vt-x)^2 - \gamma_T^2 y^2}} J_1 \left( \frac{\sqrt{(Vt-x)^2 - \gamma_T^2 y^2}}{\ell_1 \gamma_T} \right) H\left( (Vt - x) - \gamma_T |y| \right),
\]

\[
\beta_{zx} = -\frac{b}{2\ell_1} \frac{y}{\sqrt{(Vt-x)^2 - \gamma_T^2 y^2}} J_1 \left( \frac{\sqrt{(Vt-x)^2 - \gamma_T^2 y^2}}{\ell_1 \gamma_T} \right) H\left( (Vt - x) - \gamma_T |y| \right)
\]

and

\[
\beta_{zy} = -\frac{b}{2\ell_1} \frac{(Vt-x)}{\sqrt{(Vt-x)^2 - \gamma_T^2 y^2}} J_1 \left( \frac{\sqrt{(Vt-x)^2 - \gamma_T^2 y^2}}{\ell_1 \gamma_T} \right) H\left( (Vt - x) - \gamma_T |y| \right).
\]

It can be seen that equation (4.28) is non-zero just for \( Vt - x > \gamma_T |y| \). In addition, it builds a shear-wave Mach cone with the angle \( \sin \theta_T = c_T / V \). It is important to note that the classical singularity as a Dirac delta function of supersonic dislocations on the Mach cone does not appear in our gauge theoretic result. The elastic distortions and the velocity have a maximum value on the Mach cone; \( v_z = bVy/4L_1^2 \gamma_T \), \( \beta_{zx} = -by/4\ell_1^2 \gamma_T \), \( v_z \) and \( \beta_{zx} \) are also zero at \( y = 0 \). Inside the Mach cone, they oscillate with decreasing amplitude. The visualization of the Mach cone of the physical velocity \( v_z \) produced by the screw dislocation in the supersonic regime is plotted in figure 5.

Let us mention that the solution of Sharma & Zhang (2006) does not possess the correct behaviour in the supersonic region. It does not show Mach cones, which are predicted in computer simulations (Gumbsch & Gao 1999; Koizumi et al. 2002; Li & Shi 2002; Tsuzuki et al. 2008) and found experimentally (Nosenko et al. 2007).
5. Conclusion

In this paper, we have investigated a moving screw dislocation in the gauge field theory of dislocations. We derived the equations of motion of an arbitrary moving screw dislocation in this field-theoretic framework. First, a coupled system of inhomogeneous Klein–Gordon equations is obtained in the dynamical case with two characteristic velocities. We have found one dynamical characteristic length scale $L_1$, one static characteristic length scale $\ell_1$ and one characteristic time scale $\tau_1$. Later, we have decoupled the field equations, following physical arguments, to construct a consistent solution. Owing to these arguments, we found $L_1 = \ell_1$ and $a_T = c_T$. The elastic fields $v_z$, $\beta_{zx}$ and $\beta_{zy}$ and the fields of the dislocation core $T_{zy}$ and $I_{zy}$ have the characteristic speed $c_T$. For a uniformly moving screw dislocation, we have given analytical solutions for the subsonic and supersonic cases. For the supersonic case, we found one Mach cone for the velocity $c_T$.

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Appendix A. Supersonic screw dislocation in elasticity

In elasticity, a supersonic screw dislocation moves with the velocity $V > c_T$. In symmetric elasticity, the speed of shear waves reads $c_T^2 = \mu/\rho$ and in asymmetric elasticity, it is given by $c_T^2 = (\mu + \gamma)/\rho$. In asymmetric elasticity, the antisymmetric part of the stress tensor produces body couples. The field equations for the elastic velocity and the elastic distortions read (Günther 1968)

$$\Box_T v_z^0 = -b V \partial_y \delta(x - Vt) \delta(y),$$

(A1)

$$\Box_T \beta_{zx}^0 = b \partial_y \delta(x - Vt) \delta(y)$$

(A2)
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\[ \Box_T \beta^0_{zy} = b \gamma^2_T \delta(x - Vt) \delta(y). \] (A3)

These are inhomogeneous wave equations describing massless fields. The supersonic solutions are

\[ v^0_z = \frac{b V}{2 |y|} \delta(Vt - x - \gamma_T |y|), \] (A4)

\[ \rho^0_{xz} = -\frac{b y}{2 |y|} \delta(Vt - x - \gamma_T |y|) \] (A5)

and

\[ \rho^0_{zy} = -\frac{b \gamma_T}{2} \delta(Vt - x - \gamma_T |y|), \] (A6)

where \( \delta \) denotes the Dirac delta function. Thus, the classical solutions are shock waves produced by a supersonic screw dislocation. Both in front of and behind the shock front, the material is undeformed and at rest.

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