A conjecture on equitable vertex arboricity of graphs*

Xin Zhang\textsuperscript{a,}\footnote{Email address: xzhang@xidian.edu.cn}, Jian-Liang Wu\textsuperscript{b,}\footnote{Email address: jlwu@sdu.edu.cn}

\textsuperscript{a}Department of Mathematics, Xidian University, Xi’an 710071, P. R. China
\textsuperscript{b}School of Mathematics, Shandong University, Jinan 250100, P. R. China

Abstract

Wu, Zhang and Li \cite{4} conjectured that the set of vertices of any simple graph $G$ can be equitably partitioned into $\lceil (\Delta(G) + 1)/2 \rceil$ subsets so that each of them induces a forest of $G$. In this note, we prove this conjecture for graphs $G$ with $\Delta(G) \geq |G|/2$.

Keywords: Equitable vertex arboricity; Relaxed coloring; Tree coloring; Maximum degree

1 Introduction

All graphs considered in this paper are finite, undirected, loopless and without multiple edges. For a graph $G$, we use $V(G)$, $E(G)$, $\delta(G)$ and $\Delta(G)$ to denote the vertex set, the edge set, the minimum degree and the maximum degree of $G$, respectively. By $a'(G)$ and $G^c$, we denote the largest size of a matching in the graph $G$ and the completement graph of $G$. For other basic undefined concepts we refer the reader to \cite{1}.

The vertex-arboricity $a(G)$ of a graph $G$ is the minimum number of subsets into which the vertex set $V(G)$ can be partitioned so that each subset induces a forest. This notation was first introduced by Chartrand, Kronk and Wall \cite{2} in 1968, who named it point-arboricity and proved that $a(G) \leq \lceil (\Delta(G) + 1)/2 \rceil$ for every graph $G$. Recently, Wu, Zhang and Li \cite{4} introduced the equitable version of vertex arboricity. If the set of vertices of a graph $G$ can be equitably partitioned into $k$ subsets (i.e. the size of each subset is either $\lceil |G|/k \rceil$ or $\lfloor |G|/k \rfloor$) such that each subset of vertices induce a forest of $G$, then we call that $G$ admits an equitable $k$-tree-coloring. The minimum integer $k$ such that $G$ has an equitable $k$-tree-coloring is the equitable vertex arboricity $a_{eq}(G)$ of $G$. As an extension of the result of Chartrand, Kronk and Wall on vertex arboricity, Wu, Zhang and

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Li [4] raised the following conjecture and they proved it for complete bipartite graphs, graphs with maximum average degree less than 3, and graphs with maximum average degree less than 10/3 and maximum degree at least 4.

**Conjecture 1.1.** \( a_{eq}(G) \leq \lceil \frac{\Delta(G)+1}{2} \rceil \) for every simple graph \( G \).

In this note, we establish this conjecture for graphs \( G \) with \( \Delta(G) \geq |G|/2 \).

## 2 Main results and the proofs

For convenience, we set \( \Gamma(G) = \lceil \frac{\Delta(G)+1}{2} \rceil \) throughout this section. To begin with, we introduce two useful lemmas of Chen, Lih and Wu.

**Lemma 2.1.** [3] If \( G \) is a disconnected graph, then \( \alpha'(G) \geq \delta(G) \).

**Lemma 2.2.** [3] If \( G \) is a connected graph such that \( |G| > 2\delta(G) \), then \( \alpha'(G) \geq \delta(G) \).

**Lemma 2.3.** If \( G \) is a connected graph with \( \delta(G) \geq 2 \), then \( G \) contains a cycle of length at least \( \delta(G) + 1 \).

**Proof.** Consider the longest path \( P = [v_0v_1 \ldots v_i] \) in \( G \). We see immediately that \( N(v_0) \subseteq V(P) \), because otherwise we would construct a longer path. Let \( v_i \) be a neighbor of \( v_0 \) so that \( i \) is maximum. Since \( \delta(G) \geq 2 \), \( C = [v_0v_1 \ldots v_i v_0] \) is a cycle of length \( i + 1 \geq \delta(G) + 1 \). \( \square \)

In what follows, we prove three independent theorems, which together imply Conjecture 1.1 for graphs \( G \) with \( \Delta(G) \geq |G|/2 \).

**Theorem 2.4.** If \( \Delta(G) \geq \frac{2}{3}|G| - 1 \), then \( a_{eq}(G) \leq \Gamma(G) \).

**Proof.** If \( \Delta(G) = |G| - 1 \), then \( a_{eq}(G) \leq \Gamma(G) \) and this upper bound can be attained by the complete graphs, since we can arbitrarily partition \( V(G) \) into \( \Gamma(G) \) subsets so that each of them consists of one or two vertices, thus we assume \( \Delta(G) \leq |G| - 2 \). Since \( \Delta(G) + \delta(G') = |G| - 1 \) and \( \Delta(G) \geq \frac{2}{3}|G| - 1 \), \( |G'| \geq 3\delta(G') \) and \( \delta(G') \geq |G'| - 2\Gamma(G) \). By Lemmas 2.1 and 2.2, we have \( \alpha'(G') \geq \delta(G') \), so there exists a matching \( M = [x_1y_1, \ldots, x_\delta y_\delta] \) of size \( \delta := \delta(G') \) in \( G' \). Since \( |G'| \geq 3\delta(G') \), \( |V(G') \setminus V(M)| \geq \delta \), thus we can select \( \delta \) distinct vertices \( z_1, \ldots, z_\delta \) among \( V(G') \setminus V(M) \). Denote \( \beta = |G'| - 2\Gamma(G) \) and \( \mu = 3\Gamma(G) - |G'| \). Since \( |G| - 2 \geq \Delta(G) \geq \frac{2}{3}|G| - 1 \), \( \beta, \mu \geq 0 \). We now use \( \beta \) colors to color \( 3\beta \) vertices of \( G \) so that the \( i \)-th color class consists of the three vertices \( x_i, y_i \) and \( z_i \), and then use \( \mu \) colors to color the remaining \( 2\mu \) vertices of \( G \) so that each color class consists of two vertices. One can check that each color class of \( G \) induces a (linear) forest and the coloring of \( G \) is equitable. Therefore, \( a_{eq}(G) \leq \beta + \mu = \Gamma(G) \). \( \square \)
**Theorem 2.5.** If \( \frac{2}{3}|G| - 1 > \Delta(G) \geq \frac{2}{3}|G| - 2 \), then \( a_{eq}(G) \leq \Gamma(G) \).

**Proof.** If \( |G| \leq 3 \), then the result is trivial, so we assume \( |G| \geq 4 \). If \( |G| = 3k \), then \( \Delta(G) = 2k - 2 \) and \( \delta(G^c) = k + 1 \), since \( \frac{2}{3}|G| - 1 > \Delta(G) \geq \frac{2}{3}|G| - 2 \) and \( \Delta(G) + \delta(G^c) = |G| - 1 \). By Lemmas 2.1 and 2.2, we have \( \alpha'(G^c) \geq \delta(G^c) > k \). Let \( M_1 = \{x_{11}y_{11}, \ldots, x_{1k}y_{1k}\} \) be a matching of \( G^c \).

We now partition the vertices of \( G \) into \( k \) subsets so that the \( i \)-th subset consists of the vertices \( x_{i1}, y_{i1} \) and one another vertex different from the vertices in \( V(M_1) \). It is easy to check that this is an equitable partition so that each subset induces a (linear) forest, therefore, \( a_{eq}(G) \leq k = \Gamma(G) \).

If \( |G| = 3k + 2 \), then \( \Delta(G) = 2k - 1 \) and \( \delta(G^c) = k + 1 \). This also implies, by Lemma 2.1 and 2.2, that \( \alpha'(G^c) \geq \delta(G^c) > k \). Let \( M_2 = \{x_{21}y_{21}, \ldots, x_{2k}y_{2k}\} \) be a matching of \( G^c \). We now partition the vertices of \( G \) into \( k + 1 \) subsets so that the \( i \)-th subset with \( i \leq k \) consists of the vertices \( x_{2i}, y_{2i} \) and one another vertex different from the vertices in \( V(M_2) \) and the \( (k + 1) \)-th subset consists of two vertices in \( V(G) \setminus V(M_2) \). It is easy to check that this is an equitable partition so that each subset induces a (linear) forest, therefore, \( a_{eq}(G) \leq k = \Gamma(G) \).

By Lemmas 2.1 and 2.2, we have \( \alpha'(G^c) \geq \delta(G^c) \). Let \( M_3 = \{x_{31}y_{31}, \ldots, x_{3(k+1)}y_{3(k+1)}\} \) be a matching of \( G^c \). If \( x_{31} \) has a neighbor in \( G^c \) among \( \{x_{32}, y_{32}, \ldots, x_{3(k+1)}, y_{3(k+1)}\} \) (without loss of generality, assume that \( x_{31}x_{32} \in E(G^c) \) ), then we can partition the vertices of \( G \) into \( k \) subsets so that the first subset consists of the four vertices \( x_{31}, y_{31}, x_{32} \) and \( y_{32} \) and the \( i \)-th subset with \( 2 \leq i \leq k \) consists of the vertices \( x_{3(i+1)}, y_{3(i+1)} \) and one another vertex different from the vertices in \( V(M_2) \). One can check that this is an equitable partition so that each subset induces a (linear) forest, therefore, \( a_{eq}(G) \leq k = \Gamma(G) \).

**Theorem 2.6.** If \( \frac{2}{3}|G| - 2 \geq \Delta(G) \geq \frac{1}{2}|G| \), then \( a_{eq}(G) \leq \Gamma(G) \).

**Proof.** Since \( \Delta(G) + \delta(G^c) = |G| - 1 \) and \( \Delta(G) \geq \frac{1}{2}|G| \), \( |G^c| \geq 2\delta(G^c) + 2 \). We split our proof into two cases.

**Case 1:** \( G^c \) is connected.

Since \( |G^c| \geq 2\delta(G^c) + 2 \), there exists a path \( P = \{x_0, x_1, \ldots, x_{2\delta}\} \) of length \( 2\delta := 2\delta(G^c) \) in \( G^c \) (see [1, Exercise 4.2.9]). Denote \( \beta = |G| - 3\Gamma(G) \) and \( \mu = 4\Gamma(G) - |G| \). Since \( \frac{2}{3}|G| - 2 \geq \Delta(G) \geq \frac{1}{2}|G| \), \( \beta, \mu \geq 1 \). Since \( 2\Gamma(G) > \Delta(G) = |G| - \delta(G^c) - 1 \), \( \delta(G^c) \geq |G| - 2\Gamma(G) = 2\beta + \mu \).
Thus, the vertex sets $V_i = \{x_{4i-4}, x_{4i-3}, x_{4i-2}, x_{4i-1}\}$ with $1 \leq i \leq \beta$ and $U_i = \{x_{4\beta+2i-2}, x_{4\beta+2i-1}\}$ with $1 \leq i \leq \mu$ are well defined. Note that $V(P) \supseteq \bigcup_{i=1}^{\beta} V_i \cup \bigcup_{i=1}^{\mu} U_i$. Since $|G| - 4\beta - 3\mu = \mu$, $|G - \bigcup_{i=1}^{\beta} V_i - \bigcup_{i=1}^{\mu} U_i| = \mu$. Let $V(G) \setminus \left( \bigcup_{i=1}^{\beta} V_i \cup \bigcup_{i=1}^{\mu} U_i \right) = \{y_1, \ldots, y_\mu\}$ and let $W_i = U_i \cup \{y_i\}$ with $1 \leq i \leq \mu$. We now partition the vertices of $G$ into $\beta + \mu$ subsets $V_1, \ldots, V_\beta, W_1, \ldots, W_\mu$. One can check that this is an equitable partition so that each subset induces a (linear) forest, therefore, $a_{eq}(G) \leq \beta + \mu = \Gamma(G)$.

**Case 2:** $G^c$ is disconnected.

Let $G_1, \ldots, G_t$ be the components of $G^c$ with $t \geq 2$. Since $\Delta(G) + \delta(G^c) = |G| - 1$ and $\Delta(G) < \frac{2}{3}|G| - 2$, $\min\{\delta(G_1), \ldots, \delta(G_t)\} \geq \delta(G^c) \geq 2$. This implies, by Lemma 2.3, that $G_i$ contains a cycle $C_i = \{x_i^0, x_i^1, \ldots, x_i^{l(C_i)}\}$ of length $l(C_i) + 1 \geq \delta(G_i) + 1$ for each $1 \leq i \leq t$. Let $V_j = \{x_{4j-4}, x_{4j-3}, x_{4j-2}, x_{4j-1}\}$ with $1 \leq i \leq t$ and $1 \leq j \leq n_i$, in which $4n_i - 1 \leq l(C_i)$ and $n_1 + \ldots + n_t = \beta$. Note that $V_j$ is well defined by Claim 1.

**Claim 1.** $\sum_{i=1}^{t} \left[ \frac{\delta(G^c)+1-4n_i}{2} \right] \geq \mu$.

**Proof.** Otherwise, $\delta(G^c) \leq \frac{1}{2}\delta(G^c) \leq \frac{1}{2} \sum_{i=1}^{t} \delta(G_i) \leq \sum_{i=1}^{t} \left[ \frac{\delta(G^c)+1}{2} \right] < 2\beta + \mu = |G| - 2\Gamma(G) \leq |G| - \Delta(G) - 1$, contradicting to $\Delta(G) + \delta(G^c) = |G| - 1$. \hfill $\square$

We conclude, by Claim 1, that there exists a matching $M$ of size at least $\mu$ in $G^c - \bigcup_{i=1}^{t} \bigcup_{j=1}^{n_i} V_j$. Therefore, we can partition the vertices of $G$ into $\beta + \mu$ subsets so that the $i$-th subset with $1 \leq i \leq \mu$ consists of a pair of vertices matched under $M$ and one vertex in $V(G) \setminus \left( V(M) \cup \bigcup_{i=1}^{t} \bigcup_{j=1}^{n_i} V_j \right)$ and the last $\beta$ subsets are $V_1, \ldots, V_{n_1}, \ldots, V_1, \ldots, V_{n_\mu}$. One can check that this is an equitable partition so that each subset induces a (linear) forest, therefore, $a_{eq}(G) \leq \beta + \mu = \Gamma(G)$. \hfill $\square$

From the proofs of the above three theorems, we can immediately deduce the following conclusions.

**Conclusion 2.7.** If $G$ is a simple graph with $\Delta(G) \geq \frac{1}{2}|G|$, then $|V(G)|$ can be equitably partitioned into $\Gamma(G)$ subsets so that each of them induces a linear forest of $G$, i.e., the equitable linear vertex arboricity of $G$ is at most $\Gamma(G)$, and the upper bound $\Gamma(G)$ is sharp.

**Conclusion 2.8.** An equitable $\Gamma(G)$-tree-coloring of any simple graph $G$ can be constructed in linear time.

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