On universal realizability of spectra.

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Abstract

A list $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ of complex numbers is said to be realizable if it is the spectrum of an entrywise nonnegative matrix. The list $\Lambda$ is said to be universally realizable ($UR$) if it is the spectrum of a nonnegative matrix for each possible Jordan canonical form allowed by $\Lambda$. It is well known that an $n \times n$ nonnegative matrix $A$ is co-spectral to a nonnegative matrix $B$ with constant row sums. In this paper, we extend the co-spectrality between $A$ and $B$ to a similarity between $A$ and $B$, when the Perron eigenvalue is simple. We also show that if $\epsilon \geq 0$ and $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ is $UR$, then $\{\lambda_1 + \epsilon, \lambda_2, \ldots, \lambda_n\}$ is also $UR$. We give counter-examples for the cases: $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ is $UR$ implies $\{\lambda_1 + \epsilon, \lambda_2 - \epsilon, \lambda_3, \ldots, \lambda_n\}$ is $UR$, and $\Lambda_1, \Lambda_2$ are $UR$ implies $\Lambda_1 \cup \Lambda_2$ is $UR$.

Key words: nonnegative matrix, inverse eigenvalue problem, universal realizability.

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1 Introduction

Let $M_n$ denote the set of $n \times n$ real matrices and $M_{k,l}$ the set of $k \times l$ real matrices. Let $A \in M_n$ and let

$$J(A) = S^{-1}AS = \text{diag}(J_{n_1}(\lambda_1), J_{n_2}(\lambda_2), \ldots, J_{n_k}(\lambda_k))$$

be the Jordan canonical form of $A$ (hereafter JCF of $A$), where the $n_i \times n_i$ submatrices

$$J_{n_i}(\lambda_i) = \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & \cdots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix}, \quad i = 1, \ldots, k,$$

are called the Jordan blocks of $J(A)$. The elementary divisors of $A$ are the characteristic polynomials of $J_{n_i}(\lambda_i)$, $i = 1, \ldots, k$. The nonnegative inverse elementary divisors problem (hereafter NIEDP) is the problem of determining necessary and sufficient conditions for the existence of an $n \times n$ entrywise nonnegative matrix with prescribed elementary divisors \[3, 5, 10, 11, 13, 14, 15, 16\]. If there exists a nonnegative matrix with spectrum $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ for each possible Jordan canonical form allowed by $\Lambda$, we say that $\Lambda$ is universally realizable (UR). If $\Lambda$ is the spectrum of a nonnegative diagonalizable matrix, then $\Lambda$ is said to be diagonalizable realizable (DR).

The NIEDP is closely related to the nonnegative inverse eigenvalue problem (hereafter NIEP), which is the problem of characterizing all possible spectra of entrywise nonnegative matrices. If there is a nonnegative matrix $A$ with spectrum $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, we say that $\Lambda$ is realizable and that $A$ is a realizing matrix. Both problems, the NIEDP and the NIEP, remain unsolved. A complete solution for the NIEP is known only for $n \leq 4$.

Throughout this paper, the first written element of a list $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, i.e., $\lambda_1$, is the Perron eigenvalue of $\Lambda$, $\lambda_1 = \max\{|\lambda_i|, \lambda_i \in \Lambda\}$. If $\Lambda$ is the spectrum of a nonnegative matrix $A$, we write $\rho(A) = \lambda_1$ for the spectral radius of $A$.

In this paper, we ask whether certain properties of the NIEP, such as the three rules that characterize the $C$-realizability of lists (see \[2\]), extend or not to the NIEDP. In particular, we ask:

1) If $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ is UR, is $\{\lambda_1 + \epsilon, \lambda_2, \ldots, \lambda_n\}$ also UR for any $\epsilon > 0$?
2) If $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ is UR and $\lambda_2$ is real, is $\{\lambda_1 + \epsilon, \lambda_2 - \epsilon, \lambda_3, \ldots, \lambda_n\}$
also $\mathcal{UR}$ for any $\epsilon > 0$?

3) If the lists $\Lambda_1$ and $\Lambda_2$ are $\mathcal{UR}$, is $\Lambda_1 \cup \Lambda_2$ also $\mathcal{UR}$?

In [4], Cronin and Laffey examine the subtle difference between the symmetric nonnegative inverse eigenvalue problem (SNIEP), in which the realizing matrix is required to be symmetric, and the real diagonalizable nonnegative inverse eigenvalue problem (DRNIEP), in which the realizing matrix is diagonalizable. The authors in [4] give examples of lists of real numbers, which can be the spectrum of a nonnegative matrix, but not the spectrum of a diagonalizable nonnegative matrix.

The set of all $n \times n$ real matrices with constant row sums equal to $\alpha \in \mathbb{R}$ will be denoted by $\mathcal{CS}_\alpha$. It is clear that $e = [1, 1, \ldots, 1]^T$ is an eigenvector of any matrix $A \in \mathcal{CS}_\alpha$, corresponding to the eigenvalue $\alpha$. Denote by $e_k$ the vector with 1 in the $k^{th}$ position and zeros elsewhere. The importance of matrices with constant row sums is due to the well known fact that an $n \times n$ nonnegative matrix $A$ with spectrum $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, $\lambda_1$ being the Perron eigenvalue, is co-spectral to a nonnegative matrix $B \in \mathcal{CS}_1$. In this paper, we extend the co-spectrality between $A$ and $B$ to similarity between $A$ and $B$, when $\lambda_1$ is simple, and therefore $J(A) = J(B)$. In what follows, we use the following notations and results: we write $A \geq 0$ if $A$ is a nonnegative matrix, and $A > 0$ if $A$ is a positive matrix, that is, if all its entries are positive. We shall use the same notation for vectors.

**Theorem 1.1** [1, (2.7) Theorem p. 141] Let $A \in \{M = (m_{ij}) \in M_n : m_{ij} \leq 0, i \neq j\}$ be an irreducible matrix. Then each one of the following conditions is equivalent to the statement: “$A$ is a nonsingular $M$-matrix”.

i) $A^{-1}$ is positive.

ii) $Ax \geq 0$ and $Ax \neq 0$ for some $x$ positive.

**Theorem 1.2** [13] Let $q = [q_1, \ldots, q_n]^T$ be an arbitrary $n$-dimensional vector and $E_{11} \in M_n$ with 1 in the $(1, 1)$ position and zeros elsewhere. Let $A \in \mathcal{CS}_{\lambda_1}$ with JCF

$$J(A) = S^{-1}AS = \text{diag}(J_1(\lambda_1), J_{n_2}(\lambda_2), \ldots, J_{n_k}(\lambda_k)).$$

If $\lambda_1 + \sum_{i=1}^n q_i \neq \lambda_i$, $i = 2, \ldots, n$, then the matrix $A + eq^T$ has Jordan canonical form $J(A) + (\sum_{i=1}^n q_i)E_{11}$. In particular, if $\sum_{i=1}^n q_i = 0$, then $A$ and $A + eq^T$ are similar.

This paper is organized as follows: In Section 2, we extend the co-spectrality between a nonnegative matrix $A$ and a nonnegative matrix $B
with constant row sums to a similarity between $A$ and $B$, when the Perron eigenvalue is simple. In Section 3, we show that if a list of complex numbers $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ is $UR$, then $\{\lambda_1 + \epsilon, \lambda_2, \ldots, \lambda_n\}$ is also $UR$ for any $\epsilon > 0$. We also consider the universal realizability of the Guo perturbation $\{\lambda_1 + \epsilon, \lambda_2 - \epsilon, \lambda_3, \ldots, \lambda_n\}$, and of the union of two universally realizable lists $\Lambda_1$ and $\Lambda_2$. In Section 4, we study the nonsymmetric realizability of lists of size 5 with trace zero and three negative elements.

2 Nonnegative matrices similar to nonnegative matrices with constant row sums

It is well known that if $A$ is an irreducible nonnegative matrix, then $A$ has a positive eigenvector associated to its Perron eigenvalue. In this section, we extend this result to reducible matrices under certain conditions. As a consequence, in both cases, $A$ is similar to a nonnegative matrix $B$ with constant row sums when the Perron eigenvalue is simple. In this way, we extend a result attributed to Johnson [7], about the co-spectrality between a nonnegative matrix $A$ and a nonnegative matrix $B \in CS_{\lambda_1}$.

**Lemma 2.1** Let $A \in M_n$ be a nonnegative matrix of the form

\[
A = \begin{bmatrix}
A_1 & 0 \\
A_3 & A_2
\end{bmatrix},
\]

with $A_1 \in CS_{\lambda_1}$, $A_3 \neq 0$, $A_2$ irreducible and $\lambda_1 = \rho(A) = \rho(A_1) > \rho(A_2)$. Then $A$ has a positive eigenvector associated to $\lambda_1$. Moreover, there exists a nonnegative matrix $B \in CS_{\lambda_1}$ similar to $A$.

**Proof.** Let $A_1 \in M_k$ and $A_2 \in M_{n-k}$. Let $x = \begin{bmatrix} e \\ y \end{bmatrix}$ with $e \in M_{k,1}$, $y \in M_{n-k,1}$. Then, for

\[
A_1 e \\
A_3 e + A_2 y
\]

we have $A_3 e = (\lambda_1 I - A_2) y$, where $\lambda_1 I - A_2$ is an irreducible nonsingular $M$-matrix. Then, from Theorem 1.1, $(\lambda_1 I - A_2)^{-1} > 0$. Therefore,

\[
y = (\lambda_1 I - A_2)^{-1}(A_3 e) > 0,
\]

(1)
and so $\mathbf{x}^T = [\mathbf{e}^T, \mathbf{y}^T] = [x_1, \ldots, x_n]$ is positive. Then, for $D = \text{diag} (x_1, \ldots, x_n)$, $B = D^{-1}AD$ is similar to $A$. Since

$$Be = D^{-1}ADe = \lambda_1 \mathbf{e},$$

then $B \in \mathcal{CS}_{\lambda_1}$.

\textbf{Remark 2.1} Note that the eigenvector $\mathbf{x}$ obtained in the proof of Lemma 2.1 is $\mathbf{x}^T = [\mathbf{e}^T, \mathbf{y}^T]$, where $\mathbf{e}$ has the number of rows $A_1$ and

$$\mathbf{y} = (\lambda_1 I - A_2)^{-1}(A_3 \mathbf{e}) = [y_1, \ldots, y_{n-k}]^T > 0.$$

Let $Y = \text{diag} (y_1, \ldots, y_{n-k})$, then a matrix $B \in \mathcal{CS}_{\lambda_1}$ similar to $A$ is of the form

$$B = \begin{bmatrix} A_1 & 0 \\ Y^{-1}A_3 & Y^{-1}A_2Y \end{bmatrix}.$$

Note that in Lemma 2.1 it is not necessary that the spectral radius of $A$ be simple, as shown in matrix

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix},$$

which has a positive eigenvector $[1, 1, 2]$ associated to the double eigenvalue $\lambda_1 = 2$.

Now, suppose that $A$ is a block diagonal matrix. Then, for this case, we have the following result:

\textbf{Lemma 2.2} Let $A \in M_n$ be a nonnegative matrix of the form

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix},$$

with $A_1 \in \mathcal{CS}_{\lambda_1}$, $A_2$ irreducible and $\lambda_1 = \rho (A) = \rho (A_1) > \rho (A_2)$. Then $A$ is similar to a nonnegative matrix $\tilde{A} = \begin{bmatrix} A_1 & 0 \\ A_3 & A_2 \end{bmatrix}$, with $A_3 \neq 0$. Moreover, there exists a nonnegative matrix $B \in \mathcal{CS}_{\lambda_1}$ similar to $A$. 

5
Proof. Let \( A_1 \in M_k \) and \( A_2 \in M_{n-k} \). We suppose, without loss of generality, that \( A_2 \in \mathcal{CS}_{\rho(A_2)} \). Define the nonsingular matrix

\[
S = \begin{bmatrix} I_k & 0 \\ -Z & I_{n-k} \end{bmatrix}, \quad \text{with} \quad S^{-1} = \begin{bmatrix} I_k & 0 \\ Z & I_{n-k} \end{bmatrix},
\]

where \( Z = ez^T \in M_{n-k,k} \), with \( z \) being an eigenvector of \( A_1^T \) associated to \( \lambda_1 \). Then

\[
\tilde{A} = S^{-1}AS = \begin{bmatrix} A_1 & 0 \\ ZA_1 - A_2Z & A_2 \end{bmatrix}.
\]

We show that \( A_3 = ZA_1 - A_2Z \) is a nonzero nonnegative matrix. The entry in position \((r, j)\) of the matrix \( A_3 \) is,

\[
e^T_r (ZA_1 - A_2Z)e_j = z^T \text{col}_j(A_1) - z^T \text{row}_r(A_2)e
\]

\[= \sum_{i=1}^{k} a_{ij}z_i - z_j \rho(A_2), \]

for all \( r = 1, \ldots, n-k \), \( j = 1, \ldots, k \). Therefore, \( ZA_1 - A_2Z \) has all its rows equal, which can be expressed as

\[
(A_1^T - \rho(A_2)I_k)z. \tag{2}
\]

Since \( A_1^T - \rho(A_2)I_k \) and \( A_1^T \) have the same eigenvectors, then from (2)

\[
(A_1^T - \rho(A_2)I_k)z = A_1^T Z - \rho(A_2)z
\]

\[= \lambda_1 z - \rho(A_2)z
\]

\[= (\lambda_1 - \rho(A_2))z \geq 0.
\]

Therefore \( A_3 = ZA_1 - A_2Z \) is a nonzero nonnegative matrix. Since \( A \) and \( \tilde{A} \) are similar with \( A_3 \) nonzero nonnegative, then from Lemma 2.1 there exists a nonnegative matrix \( B \in \mathcal{CS}_{\lambda_1} \) similar to \( A \).

Remark 2.2 Note that the matrix \( A_3 \) in the proof of Lemma 2.2 is

\[
A_3 = ez^TA_1 - A_2ez^T, \tag{3}
\]

with \( z \) being an eigenvector of \( A_1^T \) associated to \( \lambda_1 \). Then, from Lemma 2.1, \( \tilde{A} \) has a positive eigenvector \( x = [e^T, y^T] \) associated to \( \lambda_1 \), where

\[
y = (\lambda_1 I - A_2)^{-1}(A_3e) = [y_1, \ldots, y_{n-k}]^T, \quad \text{with} \ A_3 \ \text{as in} \ (3).
\]
Let $Y = \text{diag}\{y_1, \ldots, y_{n-k}\}$, then a matrix $B \in CS_{\lambda_1}$ similar to $\tilde{A}$ is of the form

$$B = \begin{bmatrix} A_1 & 0 \\ Y^{-1}A_3 & Y^{-1}A_2Y \end{bmatrix}.$$  

Next we prove the main result in this section. This result extends the co-spectrality between a nonnegative matrix $A$ and a nonnegative matrix $B \in CS_{\lambda_1}$, to a similarity between $A$ and $B$.

**Theorem 2.1** Let $A \in M_n$ be a nonnegative matrix with $\lambda_1 = \rho(A)$ simple. Then there exists a nonnegative matrix $B \in CS_{\lambda_1}$ similar to $A$.

**Proof.** If $A$ is irreducible, then $A$ has a positive eigenvector $x = [x_1, \ldots, x_n]^T$ associated to $\lambda_1$. Let $D = \text{diag} (x_1, \ldots, x_n)$. Then $B = D^{-1}AD \in CS_{\lambda_1}$ is nonnegative and similar to $A$.

If $A$ is reducible, then $A$ is permutationally similar to

$$\tilde{A} = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1,k-1} & A_{1k} \\ A_{21} & A_{22} & \cdots & \cdots & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ A_{k1} & \cdots & A_{k,k-1} & A_{kk} \\ 0 & \cdots & 0 & 0 & A_{k+1,k+1} \\ \vdots & \vdots & \vdots & \ddots & \ddots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & A_{k+r,k+r} \end{bmatrix}$$

with blocks $A_{ii}$ irreducible of order $n_i$, or zero of size $1 \times 1$, such that $\sum_{i=1}^{k+r} n_i = n$, and $[A_{i1} \ A_{i2} \ \cdots \ A_{i,i-1}]$ nonzero, $i = 2, \ldots, k$. We may assume, without loss of generality, that $\lambda_1$ is an eigenvalue of $A_{11} \in CS_{\lambda_1}$, and $A_{ii} \in CS_{\rho(A_{ii})}$, $i = 2, 3, \ldots, k + r$.

From Lemma 2.1, the submatrix

$$A_1 = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix},$$

in the left upper corner of $\tilde{A}$, is similar to a nonnegative matrix $B_1 \in CS_{\lambda_1}$, with $B_1 = D_1^{-1}A_1D_1$. 

7
We define $\tilde{D}_1 = \begin{bmatrix} D_1 & I_{n-(n_1+n_2)} \end{bmatrix}$. Then

$$\tilde{D}_1^{-1}\tilde{A}\tilde{D}_1 = \begin{bmatrix} B_1 & \ast & A_{33} & \vdots & \vdots & \ast & A_{kk} & \vdots & \vdots & \ast & A_{k+1,k+1} & \vdots & \vdots & \ast & \ast & \ast & \ast \end{bmatrix}.$$ 

Again, from Lemma 2.1, the left upper corner submatrix of $\tilde{D}_1^{-1}\tilde{A}\tilde{D}_1$,

$$A_2 = \begin{bmatrix} B_1 & 0 \\ \ast & A_{33} \end{bmatrix},$$

is similar to a nonnegative matrix $B_2 \in \mathcal{CS}_{\lambda_1}$, with $B_2 = D_2^{-1}A_2D_2$. Then we define $\tilde{D}_2 = \begin{bmatrix} D_2 & I_{n-(n_1+n_2+n_3)} \end{bmatrix}$ and we obtain

$$\tilde{D}_2^{-1}\tilde{D}_1^{-1}\tilde{A}\tilde{D}_1\tilde{D}_2 = \begin{bmatrix} B_2 & \ast & A_{44} & \vdots & \vdots & \ast & A_{kk} & \vdots & \vdots & \ast & A_{k+1,k+1} & \vdots & \vdots & \ast & \ast & \ast & \ast \end{bmatrix}.$$ 

Proceeding in a similar way, after $k-1$ steps, we obtain

$$\tilde{D}_{k-1}^{-1}\cdots\tilde{D}_1^{-1}\tilde{A}\tilde{D}_1\cdots\tilde{D}_{k-1} = \begin{bmatrix} B_{k-1} & A_{k+1,k+1} & \vdots & \vdots & \ast & \ast & \cdots & \cdots & \ast & \ast \end{bmatrix},$$

which is a block diagonal matrix, with $B_{k-1} \in \mathcal{CS}_{\lambda_1}$. Now, from Lemma 2.2, the submatrix

$$A'_k = \begin{bmatrix} B_{k-1} & A_{k+1,k+1} \\ A_{k+1,k+1} & A_{k+r,k+r} \end{bmatrix},$$
is similar to a nonnegative matrix $B_k' \in CS_{\lambda_1}$, $B_k' = D_k^{-1} S_k^{-1} A_k' S_k D_k$, where

$$S_k = \begin{bmatrix} I_{n_1+\cdots+n_k} & \varepsilon z_k^T I_{k+1,k+1} \\ -\varepsilon z_k & I_{k+1,k+1} \end{bmatrix},$$

with $z_k$ being an eigenvector of $B_k^{T-1}$ associated to $\lambda_1$.

We define $\tilde{D}_k = \begin{bmatrix} S_k D_k & I_{n-(n_1+\cdots+n_{k+1})} \end{bmatrix}$. Then,

$$\tilde{D}_k^{-1} \cdots \tilde{D}_1^{-1} \tilde{A} \tilde{D}_1 \cdots \tilde{D}_k = \begin{bmatrix} B_k' & A_{k+2,k+2} \\ & \ddots \\ & & A_{k+r,k+r} \end{bmatrix}.$$

Proceeding in a similar way, after $r-1$ steps, we obtain a nonnegative matrix $B \in CS_{\lambda_1}$ similar to $A$.

Remark 2.3 Note that the condition of simple Perron eigenvalue cannot be deleted from Theorem 2.1, as shown in matrix

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Observe also that this means that it is not always possible to work with matrices with constant row sums in the NIEDP, this fact does not apply to the NIEP.

3 Perturbation of universally realizable lists

Guo in 1997 \cite{Guo97} proved that increasing the Perron eigenvalue of a realizable list preserves the realizability. We extend this result to $UR$ lists.

Theorem 3.1 Let $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ be a list of complex numbers with $\lambda_1$ simple. If $\Lambda$ is $UR$, then $\Lambda_\epsilon = \{\lambda_1 + \epsilon, \lambda_2, \ldots, \lambda_n\}$ is also $UR$ for any $\epsilon > 0$.

Proof. Let $\epsilon > 0$ and

$$J_\epsilon = J_1(\lambda_1 + \epsilon) \bigoplus_{i=2}^{k} J_{n_i}(\lambda_i)$$
be a JCF allowed by $\Lambda_\epsilon$. The matrix

$$J = J_1(\lambda_1) \bigoplus_{i=2}^{k} J_n(\lambda_i)$$

is an allowed JCF by $\Lambda$. Because $\Lambda$ is UR, there exists a nonnegative matrix $A$ with spectrum $\Lambda$ and Jordan canonical form $J$. Besides, from Theorem 2.1 there exists a nonnegative matrix $B \in CS_{\lambda_1}$ with $J(B) = J$. Then, from Theorem 1.2, for $B$ and $q^T = [\frac{\epsilon}{n}, \ldots, \frac{\epsilon}{n}]$, we have that the matrix $A_\epsilon = B + \epsilon q^T$ is nonnegative with spectrum $\Lambda_\epsilon$ and JCF

$$J(A_\epsilon) = J(B) + \epsilon E_{11} = J + \epsilon E_{11} = J_1(\lambda_1 + \epsilon) \bigoplus_{i=2}^{k} J_n(\lambda_i).$$

Thus, $\Lambda_\epsilon$ is UR. ■

Guo in 1997 [6] also proved that increasing by $\epsilon$ a Perron eigenvalue and decreasing by $\epsilon$ another real eigenvalue of a realizable list preserves the realizability. Soto and Ccapa in 2008 [13] proved that a list of real numbers of Suleimanova type, that is, a list $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ with $\lambda_i \leq 0$ for $i = 2, \ldots, n$, and $\sum_{i=1}^{n} \lambda_1 \geq 0$, is UR. As a consequence, the perturbed list $\{\lambda_1 + \epsilon, \lambda_2 - \epsilon, \lambda_3, \ldots, \lambda_n\}$ with $\epsilon > 0$ is UR for nonnegative lists $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ and also for Suleimanova type lists $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. As we show below, this is not true for general lists $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. The construction of a counterexample is based on the study of UR lists of size 5 with trace zero and three negative elements. This construction has been motivated by the work of Cronin and Laffey [4]. They show that a realizable list is not necessarily diagonalizable realizable. In particular, they observe that the lists $\{3 + t, 3 - t, -2 + \epsilon, -2, -2 - \epsilon\}$ are realizable for small positive values of $\epsilon$ and values of $t$ close to 0.44, but they are symmetrically realizable only for $t \geq 1 - \epsilon$ [17, Theorem 3]. Note that these lists are diagonalizable realizable, since the eigenvalues are distinct. However, this is not a continuous property in $\epsilon$ as Cronin and Laffey show via the following result.

**Proposition 3.2** [4] Suppose $\{3 + t, 3 - t, -2, -2, -2\}$ is diagonalizable realizable, then $t \geq 1$.

Note that the list $\{3 + t, 3 - t, -2, -2, -2\}$ represents any list of size 5 with trace zero, simple Perron eigenvalue and three negative elements all
equal, i.e., lists of the form \\{\lambda_1, \lambda_2, \lambda_3, \lambda_3\} with \lambda_1 > \lambda_2 \geq 0 > \lambda_3 and \\
\lambda_1 + \lambda_2 + 3\lambda_3 = 0. This list can be scaled by \(-2/\lambda_3\) to \\
\begin{align*}
\left\{-\frac{2\lambda_1}{\lambda_3}, \frac{-2\lambda_2}{\lambda_3}, -2, -2, -2\right\}
\end{align*}

and taking \\
t = \frac{-2\lambda_1}{\lambda_3} - 3 = 3 + \frac{2\lambda_2}{\lambda_3} we have \\
\Lambda_{\pm t} = \{3 + t, 3 - t, -2, -2, -2\}, \quad 0 < t \leq 3.

Analogously:

- The list \(A^t\) = \(\{3 + t - t_0, 3 - t, -2 + t_0, -2, -2\}\), with \(0 < t_0 < \min\{1 + t, 2t\} < 2\) and \(0 < t \leq 3\), represents the lists \(\{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_4\}\) with \(\lambda_1 > \lambda_2 \geq 0 > \lambda_3 > \lambda_4\) and \(\lambda_1 + \lambda_2 + 3\lambda_3 + 2\lambda_4 = 0\) (scaling by \\
\(-2/\lambda_4\) and taking \(t_0 = 2 - \frac{2\lambda_1}{\lambda_4}\) and \\
t = \frac{-2\lambda_1}{\lambda_4} - 3 + t_0 = 3 + \frac{2\lambda_2}{\lambda_4}e).

- The list \(A^t\) = \(\{3 + t + t_0, 3 - t, -2, -2, -2, -2\}\), with \(t_0 > \max\{0, -2t\}\) and \\
\(-1 < t \leq 3\), represents the lists \(\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}\) with \(\lambda_1 > \lambda_2 \geq 0 > \lambda_3 > \lambda_4\) and \(\lambda_1 + \lambda_2 + 2\lambda_3 + \lambda_4 = 0\) (scaling by \\
\(-2/\lambda_4\) and \\
taking \\
t_0 = -2 + \frac{2\lambda_1}{\lambda_4}\) and \\
t = \frac{-2\lambda_1}{\lambda_4} - 3 - t_0 = 3 + \frac{2\lambda_2}{\lambda_4}e).

We need the following result due to Šmigoc:

**Lemma 3.1 [12, Lemma 5]** Suppose \(B\) is an \(m \times m\) matrix with Jordan canonical form \(J(B)\) that contains at least one \(1 \times 1\) Jordan block corresponding to the eigenvalue \(c\):

\[
J(B) = \begin{bmatrix}
    c & 0 \\
    0 & I(B)
\end{bmatrix}.
\]

Let \(u\) and \(v\), respectively, be left and right eigenvectors of \(B\) associated with the \(1 \times 1\) Jordan block in the above canonical form. Furthermore, we normalize vectors \(u\) and \(v\) so that \(u^Tv = 1\). Let \(J(A)\) be a Jordan canonical form for an \(n \times n\) matrix

\[
A = \begin{bmatrix}
    A_1 & a \\
    b^T & c
\end{bmatrix},
\]

where \(A_1\) is an \((n - 1) \times (n - 1)\) matrix and \(a\) and \(b\) are vectors in \(\mathbb{C}^{n-1}\). Then the matrix

\[
C = \begin{bmatrix}
    A_1 & au^T \\
    vb^T & B
\end{bmatrix}
\]

has Jordan canonical form

\[
J(C) = \begin{bmatrix}
    J(A) & 0 \\
    0 & I(B)
\end{bmatrix}.
\]
We consider the lists $\Lambda_t^{\pm 0} = \{3 + t - t_0, 3 - t, -2 + t_0, -2, -2\}$ and $\Lambda_t^{\prime 0} = \{3 + t + t_0, 3 - t, -2, -2, -2 - t_0\}$ that have a better behavior than $\Lambda_{\pm t}$ with respect to the Guo result applied to $UR$.

**Theorem 3.3**

i) Let $\Lambda_t^{\pm 0} = \{3 + t - t_0, 3 - t, -2 + t_0, -2, -2\}$ with $0 < t_0 < 2$ and $\frac{t_0}{2} < t \leq 3$. If $\Lambda_t^{\pm 0}$ is realizable, then it is $UR$.

ii) Let $\Lambda_t^{\prime 0} = \{3 + t + t_0, 3 - t, -2, -2, -2 - t_0\}$ with $t_0 > \max\{0, -2t\}$ and $t \leq 3$. If $\Lambda_t^{\prime 0}$ is realizable, then it is $UR$.

**Proof.**

i) Observe that the list $\Lambda_t^{\pm 0}$ has two possible JCF, since the only repeated eigenvalue is $-2$ with double multiplicity.

Under the realizability conditions in [9, 18], the realizing matrices for lists $\Lambda_t^{\pm 0}$ have the form

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ * & 0 & 1 & 0 & 0 \\ * & * & 0 & 1 & 0 \\ * & * & * & 0 & 1 \\ * & * & * & * & 0 \end{bmatrix},$$

then rank$(A + 2I) = 4$ and $A$ has a JCF with a Jordan block of size two $J_2(-2)$.

If $\Lambda_t^{\pm 0}$ is symmetrically realizable (see Spector conditions in [17, Theorem 3]), then $\Lambda_t^{\pm 0}$ is $DR$.

If $\Lambda_t^{\pm 0}$ is realizable but not symmetrically realizable, which means that $t < 1$ (see next section), we show that $\Lambda_t^{\pm 0}$ is $DR$ via the Šmigoc method given in Lemma 3.1.

Let $\Gamma_1 = \{3 + t - t_0, 3 - t, -2 + t_0, -2\}$ and $\Gamma_2 = \{\text{tr}(\Gamma_1), -2\} = \{2, -2\}$.

Note that these spectra are realizable because they satisfy the Perron and trace conditions. The matrix

$$B = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \approx J(B) = \begin{bmatrix} c = 2 & 0 \\ 0 & -2 \end{bmatrix}$$

realizes $\Gamma_2$. Let $u^T = [1/2, 1/2]$ and $v^T = [1, 1]$ be, respectively, left and right normalized eigenvectors of $B$.

We need to find a realization of $\Gamma_1$ with diagonal $(0, 0, 0, c = \text{tr}(\Gamma_1) = 2)$ and the only realization that we know with this diagonal is the one given in
[16, Theorem 14] which is of the form

\[
A = \begin{bmatrix}
0 & 1 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots \\
0 & 1 & 0 & 0 \\
d_1 & 0 & 1 & 0 \\
b & 0 & 0 & 1 \\
a & 0 & d_3 & 2
\end{bmatrix}.
\]

The characteristic polynomial of \( A \) is

\[
P_A(x) = x^4 - 2x^3 - (d_1 + d_3)x^2 + (2d_1 - b)x + 2b + d_1d_3 - a
\]

\[
= (x - (3 + t - t_0))(x - (3 - t))(x - (-2 + t_0))(x + 2)
\]

\[
= x^4 + k_1x^3 + k_2x^2 + k_3x + k_4
\]

with

\[
k_2 = -(t^2 - t_0t + t_0^2 - 5t_0 + 11),
\]

\[
k_3 = (t_0 - 4)t^2 + t_0(4 - t_0)t + t_0^2 - 5t_0 + 12,
\]

\[
k_4 = 2(t_0 - 2)(t - t_0 + 3)(t - 3).
\]

Identifying coefficients we have the system:

\[
d_1 + d_3 = -k_2, \quad 2d_1 - b = k_3, \quad 2b + d_1d_3 - a = k_4
\]

which allows us to obtain realizations of \( \Gamma_1 \), in function of \( d_1 \), of the form

\[
A(d_1) = \begin{bmatrix}
0 & 1 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots \\
0 & 1 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots \\
2d_1 - k_3 & 0 & 0 & 1 \\
-d_1^2 + (4 - k_2)d_1 - 2k_3 - k_4 & 0 & -k_2 - d_1 & 2
\end{bmatrix}
\]

that has JCF

\[
J(A(d_1)) = \begin{bmatrix}
3 + t - t_0 & 0 & 0 & 0 \\
0 & 3 - t & 0 & 0 \\
0 & 0 & -2 + t_0 & 0 \\
0 & 0 & 0 & -2
\end{bmatrix}.
\]

Now, by Lemma 3.1, the bonding of matrices \( A(d_1) \) and \( B \) leads to the matrix

\[
C(d_1) = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 1 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
2d_1 - k_3 & 0 & 0 & 1/2 & 1/2 \\
-d_1^2 + (4 - k_2)d_1 - 2k_3 - k_4 & 0 & -k_2 - d_1 & 0 & 2 \\
-d_1^2 + (4 - k_2)d_1 - 2k_3 - k_4 & 0 & -k_2 - d_1 & 2 & 0
\end{bmatrix}
\]
which realizes diagonally the list $\Lambda_t^{t_0}$.

Finally, $\Lambda_t^{t_0}$ is $\mathcal{UR}$.

ii) Analogously, under the realizability conditions in [9 18], the realizing matrices for lists $\Lambda_t^{t_0}$ have a JCF with a Jordan block of size two $J_2(-2)$.

If $\Lambda_t^{t_0}$ is symmetrically realizable, then $\Lambda_t^{t_0}$ is $\mathcal{DR}$.

If $\Lambda_t^{t_0}$ is realizable but not symmetrically realizable (for $t < 1$), we apply the Šmigoc method to the spectra

$$\Gamma_1' = \{3 + t + t_0, 3 - t, -2, -2 - t_0\}$$

and $\Gamma_2 = \{2, -2\}$

and, in the same way, we obtain the following $\mathcal{DR}$ realization of $\Lambda_t^{t_0}$

$$C(d_1) = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
2d_1 - k_3 & 0 & 0 & 1/2 & 1/2 \\
-d_1^2 + (4 - k_2)d_1 - 2k_3 - k_4 & 0 & -k_2 - d_1 & 0 & 2 \\
-d_1^2 + (4 - k_2)d_1 - 2k_3 - k_4 & 0 & -k_2 - d_1 & 2 & 0
\end{bmatrix}$$

for the system (4), with

$$k_2 = -(t^2 - t_0t + t_0^2 + 5t_0 + 11),$$

$$k_3 = (t_0 + 4)t^2 + t_0(4 + t_0)t - t_0^2 - 5t_0 - 12,$$

$$k_4 = 2(t_0 + 2)(t + t_0 + 3)(3 - t).$$

Hence, $\Lambda_t^{t_0}$ is $\mathcal{UR}$. ■

**Corollary 3.1** Let $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ be a $\mathcal{UR}$ list with $\lambda_2$ real. The list $\{\lambda_1 + \epsilon, \lambda_2 - \epsilon, \lambda_3, \ldots, \lambda_n\}$, for $\epsilon > 0$, is not necessarily $\mathcal{UR}$.

**Proof.** Let

$$\Lambda = \Lambda_t^{t_0} = \{3 + t - t_0, 3 - t, -2 + t_0, -2, -2\}$$

be a $\mathcal{UR}$ list as in Theorem 3.3 with $t < 1$ (see Lemma 4.1 for its existence).

Now, applying Wuwen perturbation with $\epsilon = t_0$, we obtain the list

$$\{3 + t, 3 - t, -2, -2, -2\}$$

which is not diagonally realizable by Proposition 3.2 and therefore it is not $\mathcal{UR}$. ■
It is easy to see that if $\Lambda$ and $\Gamma$ are lists of nonnegative real numbers, then $\Lambda \cup \Gamma$ is \textit{UR}. Let

$$\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \text{ and } \Gamma = \{\mu_1, \mu_2, \ldots, \mu_m\}$$

be lists of real numbers of Suleimanova type with trace zero and $\lambda_1 > \mu_1$, the Perron eigenvalues of $\Lambda$ and $\Gamma$ respectively. Then, from [3], $\Lambda \cup \Gamma$ is \textit{UR}. Now we show that this is not true for general lists.

**Lemma 3.2** Let $\Lambda = \{\lambda_1, \lambda_1, \lambda_2, \lambda_2\}$ be a list of real numbers with $\lambda_1 > 0 > \lambda_2 \geq -\lambda_1$ and $\lambda_1 + 2\lambda_2 < 0$. Then $\Lambda$ has no nonnegative realization with Jordan canonical form

$$J = \begin{bmatrix}
\lambda_1 & 0 & 0 & 0 \\
0 & \lambda_1 & 0 & 0 \\
0 & 0 & \lambda_2 & 1 \\
0 & 0 & 0 & \lambda_2
\end{bmatrix}.$$ 

**Proof.** Suppose there exists a nonnegative realization $A$ of $\Lambda$ with Jordan canonical form $J(A) = J$. As $\lambda_1 + 2\lambda_2 < 0$, then $\Lambda$ only admits reducible realizations and must be partitioned as $\{\lambda_1, \lambda_2\} \cup \{\lambda_1, \lambda_2\}$. So we assume, without loss of generality, that $A$ is of the form

$$A = \begin{bmatrix} B & 0 \\ C & D \end{bmatrix},$$

where $B$ and $D$ are irreducible matrices with spectrum $\{\lambda_1, \lambda_2\}$. Therefore, from the minimal polynomial of $B$ and $D$, we have

$$B^2 = (\lambda_1 + \lambda_2)B - \lambda_1\lambda_2 I \text{ and } D^2 = (\lambda_1 + \lambda_2)D - \lambda_1\lambda_2 I.$$ 

Since the minimal polynomial of $A$ is

$$x^3 + (-\lambda_1 - 2\lambda_2)x^2 + (2\lambda_1\lambda_2 + \lambda_2^2)x - \lambda_1\lambda_2^2,$$

then

$$A^3 + (-\lambda_1 - 2\lambda_2)A^2 + (2\lambda_1\lambda_2 + \lambda_2^2)A - \lambda_1\lambda_2^2 I = 0,$$

with

$$A^2 = \begin{bmatrix} B^2 & 0 \\ CB + DC & D^2 \end{bmatrix} = \begin{bmatrix} (\lambda_1 + \lambda_2)B - \lambda_1\lambda_2 I & 0 \\ CB + DC & (\lambda_1 + \lambda_2)D - \lambda_1\lambda_2 I \end{bmatrix},$$

15
and

\[ A^3 = AA^2 = \begin{bmatrix} (\lambda_1 + \lambda_2)B^2 - \lambda_1 \lambda_2 B & 0 \\ (\lambda_1 + \lambda_2)CB - \lambda_1 \lambda_2 C + DCB + D^2C & (\lambda_1 + \lambda_2)D^2 - \lambda_1 \lambda_2 D \end{bmatrix} \]

\[ = \begin{bmatrix} (\lambda_1^2 + \lambda_1 \lambda_2 + \lambda_2^2)B - (\lambda_1^2 \lambda_2 + \lambda_1 \lambda_2^2)I & 0 \\ (\lambda_1 + \lambda_2)(CB + DC) + DCB - 2\lambda_1 \lambda_2 C & (\lambda_1^2 + \lambda_1 \lambda_2 + \lambda_2^2)D - (\lambda_1^2 \lambda_2 + \lambda_1 \lambda_2^2)I \end{bmatrix}. \]

Therefore,

\[ A^3 - (\lambda_1 + 2\lambda_2)A^2 + (2\lambda_1 \lambda_2 + \lambda_2^2)A - \lambda_1 \lambda_2^2 I \]

\[ = \begin{bmatrix} (\lambda_1^2 + \lambda_1 \lambda_2 + \lambda_2^2)B - (\lambda_1^2 \lambda_2 + \lambda_1 \lambda_2^2)I & 0 \\ (\lambda_1 + \lambda_2)(CB + DC) + DCB - 2\lambda_1 \lambda_2 C & (\lambda_1^2 + \lambda_1 \lambda_2 + \lambda_2^2)D - (\lambda_1^2 \lambda_2 + \lambda_1 \lambda_2^2)I \end{bmatrix} \]

\[-(\lambda_1 + 2\lambda_2) \begin{bmatrix} (\lambda_1 + \lambda_2)B - \lambda_1 \lambda_2 I & 0 \\ CB + DC & (\lambda_1 + \lambda_2)D - \lambda_1 \lambda_2 I \end{bmatrix} + (2\lambda_1 \lambda_2 + \lambda_2^2) \begin{bmatrix} B & 0 \\ C & D \end{bmatrix} \]

\[-\lambda_1 \lambda_2^2 I = 0. \]

Now, by equalizing the block in position (2, 1) to zero, we have:

\[ (\lambda_1 + \lambda_2)(CB + DC) + DCB - 2\lambda_1 \lambda_2 C - (\lambda_1 + 2\lambda_2)(CB + DC) + (2\lambda_1 \lambda_2 + \lambda_2^2)C \]

\[ = -\lambda_2(CB + DC) + DCB + \lambda_2^2 C = 0. \]

Since the matrices involved in the last equality are nonnegative and \( \lambda_2 < 0 \), this is only possible if each addend is zero. In particular, \( C = 0 \). Then

\[ \dim(\ker(A - \lambda_2 I)) = 4 - \text{rank}(A - \lambda_2 I) \]

\[ = 4 - \text{rank} \begin{bmatrix} B - \lambda_2 I & 0 \\ 0 & D - \lambda_2 I \end{bmatrix} \]

\[ = 4 - (\text{rank}(B - \lambda_2 I) + \text{rank}(D - \lambda_2 I)) \]

\[ = 4 - (1 + 1) = 2. \]

However, from \( J(A) = J \) we have

\[ \dim(\ker(A - \lambda_2 I)) = 4 - \text{rank} \begin{bmatrix} \lambda_1 - \lambda_2 & 0 & 0 & 0 \\ 0 & \lambda_1 - \lambda_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = 1, \]

16
which contradicts the existence of a nonnegative realization $A$ with Jordan canonical form $J$. ■

As an example, consider $\Lambda = \{1, -1\}$. It is clear that $\Lambda$ is UR. However, from Lemma 3.2, the list $\Lambda \cup \Lambda = \{1, 1, -1, -1\}$ has no nonnegative realization with $JCF$

$$J = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$ 

Therefore, $\Lambda \cup \Lambda$ is not UR.

4 Lists of size 5 with trace zero and three negative elements

We are interested in the realizability of the lists with size 5 and trace zero

$$\Lambda_{\pm t} = \{3 + t, 3 - t, -2, -2, -2\},$$
$$\Lambda_{t}^0 = \{3 + t - t_0, 3 - t, -2 + t_0, -2, -2\},$$
$$\Lambda_{t}^{0'} = \{3 + t + t_0, 3 - t, -2, -2 - 2 - t_0\}$$

introduced in Section 3. It is well known that the list $\Lambda_{\pm t}$ is realizable if and only if $t \geq \sqrt{16 \sqrt{6} - 39} = 0.43799 \cdots$ (see [8]), and symmetrically realizable if and only if $t \geq 1$ (see [17]). Now, we study when the lists $\Lambda_{t}^0$ and $\Lambda_{t}^{0'}$ are realizable but not symmetrically realizable. We need the following result:

**Theorem 4.1** [18, Theorem 39 for $n = 5$ and $p = 2$] Let $P(x) = x^5 + k_2 x^3 + k_3 x^2 + k_4 x + k_5$. Then the following statements are equivalent:

i) $P(x)$ is the characteristic polynomial of a nonnegative matrix;

ii) the coefficients of $P(x)$ satisfy:

a) $k_2, k_3 \leq 0$;

b) $k_4 \leq \frac{k_2^2}{4}$;

$c) k_5 \leq \begin{cases} k_2 k_3 & \text{if } k_4 \leq 0, \\
 k_3 \left( \frac{k_2}{2} - \sqrt{\frac{k_2^2}{4} - k_4} \right) & \text{if } k_4 > 0. 
\end{cases}$
Lemma 4.1 1. \( \Lambda_t^{t_0} = \{3 + t - t_0, 3 - t, -2 + t_0, -2, -2\} \) with \( 0 < t_0 < 2t < 2 \) is realizable, but not symmetrically realizable, in the region
\[
t \geq \frac{t_0 + \sqrt{16\sqrt{6} - 4t_0(4 - t_0) - 3t_0^2 + 52t_0 - 156}}{2}.
\] (5)

2. \( \Lambda_t^{t_0} = \{3 + t + t_0, 3 - t, -2, -2 - t_0\} \) with \( 0 < t_0, t < 1 \) and \( t + t_0 < 1 \) is realizable, but not symmetrically realizable, in the region
\[
t \geq \frac{-t_0 + \sqrt{16\sqrt{6} + 4t_0(4 + t_0) - 3t_0^2 - 52t_0 - 156}}{2}.
\] (6)

Proof. 1. Note that \((t_0, t)\) varies in the interior of the triangle \(T\) with vertices \((0, 0), (0, 1)\) and \((2, 1)\). The hypothesis \(t < 1\) guarantees that \(\Lambda_t^{t_0}\) is not symmetrically realizable (see [17, Theorem 3]). Let us see that \(\Lambda_t^{t_0}\) is realizable using Theorem 4.1

The characteristic polynomial \(x^5 + k_2x^3 + k_3x^2 + k_4x + k_5\) of \(\Lambda_t^{t_0}\) is
\[
(x - (3 + t - t_0))(x - (3 - t))(x - (-2 + t_0))(x + 2)^2
\]
where
\[
k_2 = -t^2 + t_0t - t_0^2 + 5t_0 - 15
k_3 = -(6 - t_0)t^2 + t_0(6 - t_0)t - t_0^2 + 5t_0 - 10
k_4 = 4((t_0 - 3)t^2 + t_0(3 - t_0)t + 2t_0^2 - 10t_0 + 15)
k_5 = 4(t - 3)(t - t_0 + 3)(t_0 - 2).
\]

Clearly \(k_2\) is negative in the triangle \(T\) because \(k_2 < t_0t + 5t_0 - 15 < -3\). The derivative of \(k_3\) with respect to \(t\) is \(k_3' = -2(6-t_0)t + t_0(6-t_0)\), which is 0 in \(t = t_0/2\) and then the maximum value of \(k_3\) is \(k_3(t_0/2) = (2-t_0)(t_0^2-20)/4\) which is negative for \(0 < t_0 < 2\) and so \(k_3\) is also negative in \(T\).

The inequality \(k_4 \leq \frac{k_2^2}{4}\) holds if and only if \(k_2^2 - 4k_4\) is nonnegative. We have
\[
k_2^2 - 4k_4 = (t^2 - t_0t + 4(4 - t_0)\sqrt{6 - t_0 + t_0^2 - 13t_0 + 39})(t^2 - t_0t - 4(4 - t_0)\sqrt{6 - t_0 + t_0^2 - 13t_0 + 39})
\]
where the first factor is positive and the second is nonnegative in the triangle \(T\) if
\[
t \geq \frac{t_0 + \sqrt{16\sqrt{6} - 4t_0(4 - t_0) - 3t_0^2 + 52t_0 - 156}}{2}.
\]
The coefficient $k_4$ is positive in $T$ because

$$k_4 > 4\left(t_0^3/4 - 3 + (3 - t_0)t_0^2/2 + 2t_0^2 - 10t_0 + 15\right) = -t_0^3 + 14t_0^2 - 40t_0 + 48 > 0,$$

and $k_5 \leq k_3 \left(\frac{k_2}{2} - \sqrt{\frac{k_2^2}{4} - k_4}\right)$ in $T$ if the inequality (5) holds.

Therefore, by Theorem 4.1 we conclude that $\Lambda_{t_0}^{t_0}$ is realizable in the region (5).

2. Now $(t_0, t)$ varies in the interior of the triangle $R$ with vertices $(0, 0), (0, 1)$ and $(1, 0)$. Again, the hypothesis $t < 1$ implies no symmetric realization of $\Lambda_{t_0}^{t_0}$ (see [17, Theorem 3]).

The characteristic polynomial $x^5 + k_1x^4 + k_2x^3 + k_3x^2 + k_4x + k_5$ of $\Lambda_{t_0}^{t_0}$ is

$$(x - (3 + t + t_0))(x - (3 - t))(x + 2)(x - (-2 - t_0))$$

where

$$k_2 = -(t^2 + t_0t + t_0^2 + 5t_0 + 15)$$
$$k_3 = -((t_0 + 6)t^2 + t_0(t_0 + 6)t + t_0^2 + 5t_0 + 10)$$
$$k_4 = -4((t_0 + 3)t^2 + t_0(t_0 + t_0)t - 2t_0^2 - 10t_0 - 15)$$
$$k_5 = 4(3 - t)(t + t_0 + 3)(t_0 + 2).$$

Clearly $k_2$ and $k_3$ are negative in the triangle $R$. For $k_4 \leq \frac{k_2^2}{4}$ we have

$$k_2^2 - 4k_4 = (t^2 + t_0t + 4(4 + t_0)\sqrt{6 + t_0 + t_0^2 + 13t_0 + 39})(t^2 + t_0t - 4(4 + t_0)\sqrt{6 + t_0 + t_0^2 + 13t_0 + 39})$$

where the first factor is positive and the second is nonnegative in the triangle $R$ if

$$t \geq \frac{-t_0 + \sqrt{16\sqrt{6 + t_0(4 + t_0) - 3t_0^2 - 52t_0 - 156}}}{2}.$$

The coefficient $k_4$ is positive in $T$ because

$$k_4 > -4((t_0 + 3)(1 - t_0)^2 + t_0(t_0 + 3)(1 - t_0) - 2t_0^2 - 10t_0 - 15) = 12(t_0^2 + 4t_0 + 4) > 0,$$

and $k_5 \leq k_3 \left(\frac{k_2}{2} - \sqrt{\frac{k_2^2}{4} - k_4}\right)$ in $R$ if the inequality (6) holds.

Therefore, by Theorem 4.1 we conclude that $\Lambda_{t_0}^{t_0}$ is realizable in the region (6).

Figure 1 and Figure 2 show graphically the regions of realizability (the grey regions) of $\Lambda_{t_0}^{t_0}$ and $\Lambda_{t_0}^{t_0}$ respectively, described in the previous lemma.
In the following example we give a diagonalizable nonsymmetric realization of the lists $\Lambda_{t_0}$ and $\Lambda'_{t_0}$ for particular values of $(t_0, t)$ in the corresponding regions.

Example 4.1 Let us consider the list $\Lambda_{t_0}$ for $t_0 = 1$. By Lemma 4.1, the list $\Lambda_1 = \{2 + t, 3 - t, -1, -2, -2\}$ is realizable for $t \geq \frac{1}{2} (1 + \sqrt{48} - 107) = 0.7877 \ldots$. Let us consider $t_0 = 0.8$ and realize diagonalizably the list $\Lambda_{0.8} = \{2.8, 2.2, -1, -2, -2\}$. The characteristic polynomial of the list $\Gamma_1 = \{2.8, 2.2, -1, -2\}$ is $x^4 - 2x^3 - \frac{171}{25}x^2 + \frac{212}{25}x + \frac{308}{25}$. Following the proof of Theorem 3.3 we obtain $d_3 = \frac{171}{25} - d_1$, $b = 2d_1 - \frac{212}{25}$, $a = -d_2^2 + \frac{271}{25}d_1 - \frac{732}{25}$.

The entries $d_3$ and $b$ are nonnegative for $\frac{106}{25} \leq d_1 \leq \frac{171}{25}$. The entry $a$ is nonnegative for $d_1 \in [\frac{271 - \sqrt{241}}{50}, \frac{271 + \sqrt{241}}{50}] = [5.10951 \ldots, 5.73048 \ldots]$. Then the rank of $a$ is between 0 and its maximum value attained in $d_1 = \frac{271}{50}$, i.e., $a \in [0, 0.094]$. If we take $d_1 = 5.5$ we obtain the matrices

$$A(5.5) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 5.5 & 0 & 1 & 0 \\ 2.52 & 0 & 0 & 1 \\ 0.09 & 2.58 & 2 \end{bmatrix}$$

and

$$C(5.5) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 5.5 & 0 & 1 & 0 \\ 2.52 & 0 & 0.5 & 0.5 \\ 0.09 & 2.58 & 0 & 2 \\ 0.09 & 2.58 & 2 & 0 \end{bmatrix}$$

that realize $\Gamma_1$ and $\Lambda_{0.8}$ respectively.

Finally, we consider the list $\Lambda'_{0.5} = \{3.5 + t, 3 - t, -2, -2, -2.5\}$ that, by Lemma 4.1, is realizable for $t \geq \frac{-1 + \sqrt{144\sqrt{26} - 731}}{4} = 0.2013 \ldots$. Let us consider

\[t = \frac{t_0}{2}\]
$t = 0.3$ and realize diagonalizably the list $\Lambda^{0.3}_{0.5} = \{3.8, 2.7, -2, -2, -2.5\}$. The characteristic polynomial of the list $\Gamma_1' = \{3.8, 2.7, -2, -2.5\}$ is $x^4 - 2x^3 - \frac{1399}{100} x^2 + \frac{1367}{100} x + \frac{513}{10}$. From the proof of Theorem 3.3 we obtain

$$d_3 = \frac{1399}{100} - d_1, \quad b = 2d_1 - \frac{1367}{100}, \quad a = -d_1^2 + \frac{1799}{100} d_1 - \frac{1966}{25}.$$ 

The entries $d_3$ and $b$ are nonnegative for $\frac{1367}{200} \leq d_1 \leq \frac{1399}{100}$. The entry $a$ is nonnegative for $d_1 \in \left[\frac{1799-9\sqrt{1121}}{200}, \frac{1799+9\sqrt{1121}}{200}\right] = [7.483 \cdots, 10.501 \cdots]$. Then the rank of $a$ is between 0 and its maximum value attained in $d_1 = \frac{1799}{200}$, i.e., $a \in [0, 2.270025]$. If we take $d_1 = 9$ we obtain the matrices

$$A(9) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 9 & 0 & 1 & 0 \\ 4.33 & 0 & 0 & 1 \\ 2.27 & 4.99 & 2 \end{bmatrix} \quad \text{and} \quad C(9) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 9 & 0 & 1 & 0 \\ 4.33 & 0 & 0 & 0.5 \\ 2.27 & 0 & 4.99 & 0.5 \\ 2.27 & 0 & 4.99 & 2 \end{bmatrix}$$

that realize $\Gamma_1'$ and $\Lambda^{0.3}_{0.5}$ respectively.

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