The relationship between loads and power of a rotor and an actuator disc

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Abstract. Most state of the art rotor design methods are based on the actuator disc theory developed about one century ago. The actuator disc is an axisymmetric permeable surface carrying a load that represents the load on a real rotor with a finite number of blades \( N \). However, the mathematics of the transition from a real rotor load to an axisymmetrically loaded disc is not yet presented in literature. By formulating an actuator disc equation of motion in which the Bernoulli constant \( H \) is expressed in kinematical terms, a comparison of the power conversion and load on the disc and rotor is possible. For both the converted power is expressed as a change of angular momentum times rotational speed. The limits for \( N \to \infty \) while the chord \( c \to 0 \), the rotational speed \( \Omega \to \infty \), the load \( F \) becoming uniform by \( \partial F / \partial r \to 0 \) and the thickness \( \varepsilon \to 0 \) confirm that the classical disc represents the rotor with an infinite number of blades. Furthermore, the expressions for the blade load are compared to the expressions in current design and analysis tools. The latter do not include the load on chord-wise vorticity. Including this is expected to give a better modelling of the tip and root flow.

1. Introduction

The actuator disc is the oldest and simplest representation of a rotor, as developed by the founders of propeller theory e.g. Froude [1], Lanchester [2], Joukowsky [3], Betz [4]. The load on the rotor is uniformly spread on a permeable disc with equal diameter, after which the momentum theory enables an easy performance prediction. The results of the momentum theory still are the basis for the design and load calculations of real rotors, having a finite number of blades. However, the mathematics of the transition from a real rotor to an axisymmetrically loaded disc is not yet presented in rotor aerodynamic literature like [5] - [10]. The formulation of this transition is hampered by an important difference between the disc and rotor. In the inertial coordinate system the rotor flow is unsteady while the disc flow is steady so the power conversion process is different.

In contrast to momentum theory, vortex theory allows for solutions of the wake flow generated by a finite number of blades. Betz [11] and Goldstein [12] have published such solutions based on some assumptions. Okulov & Sørensen [13] have discussed these methods and brought the Goldstein solution in agreement with the
actuator disc momentum theory using a new analytical solution to the wake vortex problem. By presenting solutions for several tip speed ratios and number of blades \( N = 1, 2, 3, 5, 10, 20, \infty \) quantitative information on the thrust and power as a function of \( N \) is provided. The method of Okulov & Sørensen [13] does not give a qualitative relation between the load and converted power of a rotor and a disc by formal limit transitions. This is the topic of the present paper.

The approach of this paper is to use the rotating coordinate system when possible, since both flows are steady in this system. The next § presents the formulation of the equation of motion in both coordinate systems. Thereafter §3 treats the power conversion for the disc and rotor. For both this leads to a change of the angular momentum. §4 gives equivalent expressions for the load on the disc and rotor blade, allowing a comparison in §5 and an answer to the question: is the actuator disc the result of a rotor subjected to the limit of the number of blades \( N \to \infty \) and the chord \( c \to 0 \) keeping \( Nc \) constant, then the rotational speed \( \Omega \to \infty \), the non-uniformity of the disc load \( \partial F/\partial r \to 0 \) and finally the thickness \( \epsilon \to 0 \), meanwhile keeping the converted power \( P \) constant?

2. The Euler equation in the inertial and rotating coordinate system

The inertial cylindrical coordinate system \((x, r, \varphi)\) of the disc and rotor is shown in Figures 1 and 2. The undisturbed parallel flow \( U_0 \) is aligned with the disc centerline. The force fields rotate with a constant angular speed \( e_x \Omega \) with \( e_x \) denoting the unit vector with an appropriate index. Besides the inertial coordinate system also the rotating system\((x, r, \varphi)_{\text{rot}}\) with angular velocity \( \Omega \) is used. The velocity \( v = e_x v_x + e_r v_r + e_\varphi v_\varphi \) and vorticity \( \omega = e_x \omega_x + e_r \omega_r + e_\varphi \omega_\varphi \) in both systems are related by

\[
\begin{align*}
v_{\text{rot}} &= v - e_\varphi \Omega r \\
\omega_{\text{rot}} &= \omega - 2e_x \Omega
\end{align*}
\]

so only \( v_\varphi \) and \( \omega_z \) change in this coordinate transformation. The flow is assumed to be incompressible, homogeneous and inviscid, so the Euler equation in the inertial coordinate system holds

\[
\frac{\partial v}{\partial t} + (v \cdot \nabla) v = \frac{1}{\rho} (f - \nabla p)
\]

where \( p \) is the pressure, \( \rho \) the specific mass of the flow and \( f \) the force density. Usually \( f \) is assumed to be conservative, so is the gradient of a potential \( F \). In this case, \( f = \nabla F \) can be included in \( \nabla p \) such that \( f \) does not appear in the equation of motion. Subsequently the load on a lifting surface is determined by applying the appropriate boundary conditions at the body surface after which the pressure is determined. The resultant load is obtained by integration of the pressure on the surface. Here \( f \) is retained to represent the load exerted on the flow by the disc or rotor blade, which are
modelled as volumes carrying bound vorticity. Since the vorticity is bound, it is able to carry a force \( f \).

Rewriting (3) with the vector identity \((\mathbf{v} \cdot \nabla) \mathbf{v} = \nabla (\mathbf{v} \cdot \mathbf{v})/2 - \mathbf{v} \times \mathbf{\omega}\) yields

\[
f = \nabla H + \rho \frac{\partial \mathbf{v}}{\partial t} - \rho \mathbf{v} \times \mathbf{\omega}
\]

(4)

where \( H \) is the Bernoulli constant \( p + \frac{1}{2} \rho \mathbf{v} \cdot \mathbf{v} \). For flows with \( f = 0 \) and \( \mathbf{\omega} = 0 \) the velocity may be written as the gradient of the velocity potential \( \Phi \). This gives the unsteady Bernoulli equation

\[
\nabla \left( H + \rho \frac{\partial \Phi}{\partial t} \right) = 0
\]

(5)

Batchelor [14], eq. (3.5.20), gives the steady Euler equation in the rotating coordinate system, including the fictitious centrifugal force

\[
\mathbf{v}_{\text{rot}} \times 2 \Omega = \nabla H_{\text{rot}} - \rho \mathbf{v}_{\text{rot}} \times \mathbf{\omega}_{\text{rot}}
\]

(6)

With (2) this becomes

\[
f = \nabla \left( H_{\text{rot}} - \frac{\rho}{2} \left(\Omega r\right)^2 \right) - \rho \mathbf{v}_{\text{rot}} \times \mathbf{\omega}_{\text{rot}}
\]

(7)

or, with \( \nabla H_{\text{rot}} = \nabla \left( H - \rho v_r \Omega r \right) + \frac{\varphi}{r} \left(\Omega r\right)^2 \)

\[
f = \nabla \left( H - \rho v_r \Omega r \right) - \rho \mathbf{v}_{\text{rot}} \times \mathbf{\omega}_{\text{rot}}
\]

(8)

For flows that are steady with respect to the rotating reference system and have \( f = 0 \) and \( \mathbf{\omega} = 0 \), (7) and (8) show that

\[
\nabla \left( H_{\text{rot}} - \frac{\rho}{2} \left(\Omega r\right)^2 \right) = \nabla \left( H - \rho v_r \Omega r \right) = 0.
\]

(9)

This is equivalent to the Bernoulli equation (5) showing that for such flows

\[
\frac{\partial \Phi}{\partial t} = -v_\varphi \Omega r.
\]

(10)

The disc force and flow field are axisymmetric, so all azimuthal derivatives are zero. For later use the expression for the vorticity \( \mathbf{\omega} \) in the inertial coordinate system is given:

\[
\mathbf{\omega} = e_x \frac{\partial (rv_r)}{\partial r} - e_r \frac{\partial v_\varphi}{\partial x} + e_\varphi \left( \frac{\partial v_r}{\partial x} - \frac{\partial v_x}{\partial r} \right) \quad \text{if} \quad \frac{\partial}{\partial \varphi} = 0.
\]

(11)

\( \|^\) Batchelor’s equation is without \( f \). Note that he included the centrifugal term \( \frac{1}{2} \nabla \left(\Omega r\right)^2 \) in \( \nabla H \).
3. Power conversion

De Vries [5], app. B has derived (9) too, and discusses the power conversion in terms of $H$ and $v_\varphi \Omega r$. He does not present a result for the total converted power which is done here for the disc as well as rotor. The interial coordinate system is used, since the force field cannot perform work in the rotating system, as will be shown in the next §§.

Batchelor [14], eq. (3.5.1), gives the expression for the work done by a distribution of volume forces in steady, viscous flow. For inviscid, unsteady flow this becomes the dot-product of the force density $f$ with the velocity vector $v$, so with (4)

$$f \cdot v = (v \cdot \nabla)H + \frac{\rho}{2} \frac{\partial \|v\|^2}{\partial t}. \quad (12)$$

The power produced or absorbed in an annulus $2\pi \varphi r dr$ must equal torque times rotational speed in that annulus. With $\bar{p}$ denoting the local power and $\bar{e}_p$ the azimuthally averaged power at position $r$ this gives

$$\bar{e}_p = \int f \cdot \bar{v} = \rho \Omega r \bar{f}_\varphi. \quad (13)$$

3.1. The actuator disc

For the steady axisymmetric disc flow (12) and (13) give

$$\rho \Omega r f_\varphi = (v \cdot \nabla)H. \quad (14)$$

The expression for $r f_\varphi$ is derived from the $\varphi$-component of (4)

$$r f_\varphi = -\rho r (v_\varphi \omega_r - v_r \omega_\varphi) = \rho (v \cdot \nabla) (rv_\varphi), \quad (15)$$

since $\omega_r$ as well as $\omega_\varphi$ are a function of $v_\varphi$ only, see (11). Substitution in (14) yields

$$(v \cdot \nabla)H = \rho (v \cdot \nabla) (\Omega rv_\varphi) \quad (16)$$

or

$$\bar{p} = \rho (v \cdot \nabla) (\Omega rv_\varphi) \quad (17)$$

This relation between the converted power and azimuthal velocity has been obtained by e.g. De Vries [5], app. C2, and Thwaites [15], p. 473. Assuming that $\|v\| \neq 0$ this power conversion equation becomes

$$\nabla H = \rho \nabla (\Omega rv_\varphi) = \rho \mathbf{e}_\varphi \Omega r \times \mathbf{\omega}. \quad (18)$$

where (11) is used to express the gradient term in vorticity. This equation is identical to (9), now allowing non-zero $f$ and $\mathbf{\omega}$ but for axisymmetric flows only.
3.2. The rotor

For the unsteady rotor flow the right hand side of (13) is evaluated with the \( \phi \)-component of (4). Since the flow is periodic \( \frac{\partial}{\partial t} v_\phi \) and \( \frac{\partial H}{\partial \phi} \) do not contribute to the azimuthal average of this component, so

\[
\overline{e_p} = \rho \int r (v \cdot \nabla) (\int rv_\phi) dr d\phi.
\]

which can be written as

\[
\overline{e_p} = \rho (v \cdot \nabla) (\int rv_\phi).
\]

Unlike the derivation of (15), the complete expression for \( \omega \) is used instead of the axisymmetric equation (11). The result is equivalent, since the periodicity of the flow field cancels the terms containing the azimuthal derivatives of \( v_\phi \). For the disc as well as the rotor the converted power is expressed in a change of \( \rho \Omega v_\phi \) meaning an increase of the angular momentum times the rotational speed. For the disc the local and azimuthally averaged value are identical, so for both devices the integrated power \( P \) is given by the same expression

\[
P = \rho 2\pi \int \int (v \cdot \nabla) (\int rv_\phi) r dr d\phi.
\]

4. Loads

4.1. The actuator disc

The combination of (8) and (16) gives the equation of motion for an axisymmetric actuator disc converting power

\[
f = -\rho v_{rot} \times \omega
\]
This equation, first presented by van Kuik [16], looks as the expression for the Kutta-Joukowsky force, but the subscript $\text{rot}$ distinguishes it from this. A physical interpretation of (22) is provided by looking at the force-free wake. In the inertial system the convection velocity of the vorticity is $v$, but since the vorticity is released by the rotating force field, the path and direction of the vorticity particles are determined by $v_{\text{rot}}$. The direction of the vorticity $\omega$ is the direction of $v_{\text{rot}}$ so the wake is force-free because $v_{\text{rot}} \times \omega = 0$. Since $f$ is perpendicular to the local velocity it does not perform work in the rotating system. Compared to the steady version of the Euler equation (4) the Bernoulli constant $H$ is absent since it is expressed in kinematical terms.

4.2. The rotor

Figure 2 shows a rotor blade having a cross-section $C$ at which the bound vorticity is distributed. Only the load on the radial bound vorticity is shown. The load on a blade is derived in the rotating coordinate system by integration of (7) on $C$:

$$L = -\rho \int_{C} v_{\text{rot}} \times \omega dC + \int_{C} \nabla \left( H_{\text{rot}} - \frac{\rho}{2} (\Omega r)^2 \right) dC$$

(23)

Although (23) gives the sectional load, $L$ may have a span-wise or radial component. Evaluation of the terms at the right hand side of (23) leads to the following remarks:

- When the radial component of the $\nabla$-term is ignored, the integral is two-dimensional, and is converted by Green’s theorem to $\oint (H_{\text{rot}} - \frac{\rho}{2} (\Omega r)^2) n ds$ where $S$ is a contour enclosing $C$ and $n$ the outward unit vector at $S$ in the plane of $C$. Since $f$ and $\omega$ are zero at $S$, (9) applies and the integral gives 0. The sectional load becomes

$$L = -\rho \int_{C} v_{\text{rot}} \times \omega dC$$

(24)

- If the integrated blade load is considered instead of the sectional load, $\nabla \left( H_{\text{rot}} - \frac{\rho}{2} (\Omega r)^2 \right)$ is integrated on a volume $V$ enclosing the blade. Again the volume integral is converted to the surface integral $\iint (H_{\text{rot}} - \frac{\rho}{2} (\Omega r)^2) n dA$, where $A$ is the three-dimensional surface of $V$, and $n$ the outward unit vector at $A$. Since $f$ and $\omega$ are zero at $A$, (9) is valid so $\iint (H_{\text{rot}} - \frac{\rho}{2} (\Omega r)^2) n dA = 0$. Although the gradient term has a local radial contribution, it does not contribute to the integrated blade load. This integrated load is obtained by span-wise integration of (24) including the radial component (see the Appendix). Since $L \perp v_{\text{rot}}$ the force field does not convert power in the rotating system, just as $f$ in the disc equation (22).

- Figure 2 shows the blade bound vorticity as well as the free root- and tip vorticity. The bound vorticity is mainly radial. Near the root and tip the bound vorticity changes into the root- and tip vortex which implies axial and azimuthal bound
vorticity. In case the blade has no twist and pitch angle, the blade bound $\omega_z$ is 0. For blade geometries including twist and pitch $\omega_z$ will still be small so may be neglected. This is not necessarily the same for the chord-wise bound vorticity $\omega_\phi$ as will be discussed below.

- When only the radial bound vorticity is taken into account

$$\int_C \mathbf{v}_{rot} \times \mathbf{e}_r \omega_r dC = \mathbf{\tilde{v}}_{rot} \times \mathbf{\Gamma}$$

with $\mathbf{\Gamma} = \mathbf{e}_r \int_C \omega_r dC$ and $\mathbf{\tilde{v}}_{rot}$ the effective or average velocity. This is the expression used in rotor aerodynamic textbooks like Stepniewsky & Keys [6] and Leishman [7]. Blade Element Momentum (BEM) design methods as presented by Hansen [8], Sørensen [9] and wind turbine textbooks like Burton, Sharpe, et al. [10] express the inviscid load as $L_{blade} = C_l \frac{1}{2} \rho \mathbf{v}_{rel}^2 c$ where $c$ is the chord, $C_l$ the lift coefficient and $\mathbf{v}_{rel}$ the velocity perceived by the blade, so identical to $\mathbf{v}_{rot}$. This expression is fully equivalent to $-\rho \mathbf{\tilde{b}}_{rot} \times \mathbf{\Gamma}$.

The main difference between (24) and the sectional load in the current rotor design tools is that the latter does not include the impact of $\omega_\phi$. This chord-wise component of the bound vorticity is reflected in a difference between the radial velocity on the suction and pressure side of the blade. It is known to give additional lift at the tip of low aspect ratio wings. Experiments on model rotors of hovering helicopters have shown an increased normal load for $0.97 < r/R < 1$, see Gray et al. [17]. Recently Akay et al. [18] and Micallef et al. [19], [20] reported detailed experimental and numerical investigations on wind turbine model rotors (the 2m Ø TU-Delft rotors and the 4.5m Ø 'Mexico-project' rotor in open wind tunnel test sections of 3 respectively 9.5m). The radial
flow changes from inboard below the radial position of maximum chord to outboard for larger radii, thereby releasing the root vortex [18]. For the Mexico-project rotor [20] and TU-Delft rotor [21] the tip vortex trajectory after leaving the tip goes inboard, after which the wake expansion moves the vortex to a larger radius. The explanation is that the tip vortex trajectory is determined by two counteracting velocity inductions. For the tip vortex of any lifting surface the self-induced velocity at and near the surface is inboard, so for the rotor towards a smaller radius. The wake induced velocity (the wake expansion) moves the vortex to a larger radius. Apparently the tendency of the tip vortex to first move inboard is stronger than the wake expansion. The inboard motion is strongly connected to the radial velocities at the blade surface, so to the occurrence of chord-wise vorticity $\omega_{\varphi}$. The phenomenon of root- and tip-lift $-e_r v_r \omega_{\varphi}$ and the associated radial component $e_r v_r \omega_{\varphi}$ is not modelled in most of the current blade design tools. The vorticity-based code of Micallef et al. includes all vorticity components, so enables an analysis of the impact of $\omega_{\varphi}$. In [19] the components of the sectional force coefficients $L/(\tfrac{1}{2} \rho \| v_{\text{rot}}^2 \|)$ have been calculated showing a radial coefficient having the same order of magnitude as the azimuthal coefficient for $0.4 \leq r/R \leq 1$. Furthermore the axial coefficient with and without the radial velocities have been calculated, showing an appreciable difference in the root region and a smaller difference near the tip. Both findings confirm the qualitative analysis of van Kuik [16].

Including $\omega_{\varphi}$ in blade design tools will provide a better modelling of the root and tip flow, with local changes in the blade loads. However, small changes in the root and tip flow may have an impact on the wake vorticity position and by that on the induced velocities in the rotor plane.

Equation (24) is expressed in all components of the vorticity. In the Appendix it is shown that (24), integrated along the radius, is equivalent to the integration of the pressure on the blade surface, resulting in an axial, azimuthal and radial load.

5. The transition from rotor to disc

A comparison between both force fields and the limit transition from the rotor blade to the disc can be done based on the disc expression (22) and the equivalent rotor expression (23).

First $\Omega$ is kept constant. When the limit $N \to \infty$, $c \to 0$ with $Nc = \text{constant}$ is applied, the vorticity is spread uniformly over the disc azimuth angle while the solidity of the rotor/disc remains constant. When the strength of the centerline vortex $\Gamma = \int r v_{\varphi} d\varphi$ is kept constant, the total bound vorticity of the rotor remains constant for $N \to \infty$, see Figures 1 and 2. Furthermore the power $P$ is kept constant which gives, by (21), $(\mathbf{v} \cdot \nabla) (\Omega \Gamma) = \text{constant}$ for $N \to \infty$. Since $\Omega \Gamma$ is invariant, this implies that $\frac{1}{\Omega} (\mathbf{v} \cdot \nabla) = \text{constant}$ for $N \to \infty$. In other words, the azimuthally averaged flow field will not change when $N \to \infty$, $c \to 0$. 
With (9) the load for a rotor with \( N \) blades is
\[
\mathbf{L}_{N \text{ blades}} = -N \rho \oint C \mathbf{v}_{\text{rot}} \times \mathbf{\omega} dC + N \oint C \nabla (H - \rho v_{\varphi} \Omega r) dC.
\] (26)

The limit transition gives
\[
\lim_{N \rightarrow \infty} \mathbf{L}_{N \text{ blades}} = -2 \pi r \rho \left[ \int C \mathbf{v}_{\text{rot}} \times \mathbf{\omega} dx + \int C \nabla \left( \frac{1}{\rho} v_{\varphi} - v_{\varphi} \Omega r \right) dx \right].
\] (27)

Now the flow has become axisymmetric, (18) is valid showing that the second integral is zero. The local load \( \mathbf{F}_{\text{disc}} \) is
\[
\mathbf{F}_{\text{disc}} = \lim_{N \rightarrow \infty} \frac{1}{2 \pi r} \mathbf{L}_{N \text{ blades}} = -\rho \int C \mathbf{v}_{\text{rot}} \times \mathbf{\omega} dx.
\] (28)

Since \( \mathbf{F}_{\text{disc}} = \int C \mathbf{f}_{\text{disc}} dx \) the local force density becomes
\[
\mathbf{f}_{\text{disc}} = -\rho \mathbf{v}_{\text{rot}} \times \mathbf{\omega}.
\] (29)

The latter is identical to (22). The equivalency of (22) and (29) relates the disc load and bound vorticity to the blade load and bound vorticity.

So far the rotor blade and the disc still have a non-uniform load distribution \( \mathbf{f} \), a finite \( \Omega \) and finite thickness \( \epsilon \). Sørensen and van Kuik [22] have analysed the limit \( \Omega \rightarrow 0 \); here the limit \( \Omega \rightarrow \infty \) is taken. As the power \( P \) remains constant, the torque \( Q \) disappears and the load becomes normal. For \( \Omega \rightarrow \infty \) (14) and (15) yield
\[
rf_{\varphi} \propto \Omega^{-1} \quad \text{and} \quad rv_{\varphi} \propto \Omega^{-1} \quad \text{if} \quad \lim_{\Omega \rightarrow \infty} \mathbf{f} \cdot \mathbf{v} = \text{constant}.
\] (30)

This implies that \( f_{\varphi} \) and \( v_{\varphi} \) vanish but not \( v_{\varphi} \Omega r \). Similarly \( \omega_x \) and \( \omega_r \) vanish according to (11). Evaluation of (28) yields
\[
\lim_{\Omega \rightarrow \infty} \mathbf{F}_{\text{disc}} = \mathbf{e}_x \rho \int C \left( v_{\varphi} \omega_{\varphi} - \frac{\partial (v_{\varphi} \Omega r)}{\partial x} \right) dx - \mathbf{e}_r \rho \int C \left( v_{\varphi} \omega_{\varphi} + \frac{\partial (v_{\varphi} \Omega r)}{\partial r} \right) dx.
\] (31)

By (18) \( \nabla H = \rho \nabla (v_{\varphi} \Omega r) \) so (31) becomes
\[
\lim_{\Omega \rightarrow \infty} \mathbf{F}_{\text{disc}} = -\mathbf{e}_x \Delta H - \mathbf{e}_r \int \frac{\partial H}{\partial r} dx + \rho \int C \mathbf{v} \times \mathbf{e}_{\varphi} \omega_{\varphi} dx.
\] (32)

For a radially uniform distribution \( \partial H/\partial r = 0 \) and \( \omega_{\varphi} = 0 \) except at the disc edge, so
\[
\lim_{\Omega \rightarrow \infty} \mathbf{F}_{\text{disc}} = -\mathbf{e}_x \Delta H
\] (33)

The last limit is \( \epsilon \rightarrow 0 \). For an infinitely thin disc with \( v_{\varphi} = 0 \), \( e_x \Delta H = e_x \Delta p \) so finally
\[
\lim_{\Omega \rightarrow \infty} \frac{1}{2 \pi r} \mathbf{L}_{N \text{ blades}} = \lim_{\Omega \rightarrow \infty} \lim_{\epsilon \rightarrow 0} \mathbf{F}_{\text{disc}} = -\mathbf{e}_x \Delta p.
\] (34)

which represents the classical actuator disc.
6. Conclusions

The power converted by the actuator disc as well as the rotor shows itself as a change of
the angular momentum times rotational speed. The expressions for the rotor and disc
are identical.

By expressing the Bernoulli constant $H$ in kinematical terms, a simple actuator
disc equation of motion has been derived: $f = -\rho \mathbf{v}_{\text{rot}} \times \omega$. This equation allows a
comparison of loads and bound vorticity of a rotor blade with those of the disc.

Starting from a rotor with a finite number of blades $N$ having a finite thickness $c$ and a finite rotational speed $\Omega$, the limits $N \to \infty$, $c \to 0$ transform the expressions of
the blade load to the expressions of the axisymmetric disc load. The limit $\Omega \to \infty$ gives
d the disc with a normal load, so without torque and swirl. Finally the limits $\partial F/\partial r \to 0$
and $\epsilon \to 0$ return the classical actuator disc with a uniform pressure jump across the
disc.

The expressions for the sectional blade load as used in many rotor design and
analysis methods do not include the span-wise gradient of $H_{\text{rot}}$, not the load on chord-
wise vorticity. Recent detailed measurements of the flow field near the blade root and
tip indicate that this vorticity has a significant impact on the local flow. Including it will
provide better modelling of the root and tip flow, enabling a more accurate calculation
of the local induced velocities.

Appendix A. The blade load expressed as pressure distribution

In (24) $L$ is expressed in kinematical terms. However, physically it is pressure at
a surface that creates a resultant load, so (24) has to have an equivalent pressure
formulation. Integrated along the span on a volume $V$ enclosing the blade gives the
resultant force $\mathbf{R}$

$$\mathbf{R} = -\rho \iiint_V \mathbf{v}_{\text{rot}} \times \mathbf{\omega} dV = -\rho \iiint_V \mathbf{v}_{\text{rot}} \times \mathbf{\omega}_{\text{rot}} dV + 2\rho \iiint_V \mathbf{v}_{\text{rot}} \times \mathbf{e}_z \Omega dV$$  \hspace{1cm} (35)

The distribution of $\omega$ at $V$ is equivalent to the concentration of $\omega$ in an infinitely thin
vortex sheet $\gamma = \int \omega dn$ at the surface $A$ of $V$, with $\mathbf{n}$ normal to $A$ and $\mathbf{v}_{\text{rot}} = 0$
inside $V$. Consequently the last integral does not contribute to $\mathbf{R}$. Since $\gamma = \mathbf{e}_n \times (\mathbf{v}_{\text{rot,inside}} - \mathbf{v}_{\text{rot,outside}})$ and the velocity at the sheet $\mathbf{v}_{\text{rot}} = \frac{1}{2}(\mathbf{v}_{\text{rot,inside}} + \mathbf{v}_{\text{rot,outside}})$
the first integral becomes

$$-\rho \iiint_V \mathbf{v}_{\text{rot}} \times \mathbf{\omega}_{\text{rot}} dV = -\rho \int_A \bar{\mathbf{v}}_{\text{rot}} \times \mathbf{\gamma} dA = -\rho \int_A \mathbf{e}_n \| \mathbf{v}_{\text{rot,outside}} \|^2 dA.  \hspace{1cm} (36)$$

$\mathbf{v}_{\text{rot,outside}}$ is determined at streamlines tangent to the blade surface / vortex sheet, where
$\omega = 0$ and $f = 0$. By (9) it follows that $H_{\text{rot,outside}} = \frac{c}{2}(\Omega r)^2 + c$ where $c$ is a constant,
so $\frac{c}{2} \| \mathbf{v}_{\text{rot,outside}} \|^2 = c - (p - p_0) + \frac{c}{2}(\Omega r)^2$. The contribution of $c + p_0 + \frac{c}{2}(\Omega r)^2$ vanishes
in the closed contour integral. Herewith (35) becomes
\[ \mathbf{R} = -\rho \int \int \int \mathbf{v}_{\text{rot}} \times \mathbf{\omega} dV = \int \int p \mathbf{e}_n dA \] (37)
The product \( \mathbf{e}_x \cdot \mathbf{e}_n dA \) is equal to \( dA_x \) where \( A_x \) is the projection of the blade surface in axial direction. \( \mathbf{R} \) can be decomposed in the axial, radial or span-wise and azimuthal components:
\[ \mathbf{R} = \mathbf{e}_x \int_A p dA_x + \mathbf{e}_r \int_A p dA_r + \mathbf{e}_\varphi \int_A p dA_\varphi \] (38)
The axial and azimuthal components of \( \mathbf{R} \) contribute to the rotor thrust and torque. The radial component does not give a resultant rotor load.

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