Rationality of inner twisted flags of type $A_n$

Saša Novaković

January 2020

Abstract. We prove that $\text{rcodim}(X) \geq 2$ if $X$ is a rational inner twisted flag of type $A_n$.

1. Introduction

A potential measure for rationality was introduced by Bernardara and Bolognesi [6] with the notion of categorical representability. We use the definition given in [1]. A $k$-linear triangulated category $T$ is said to be \textit{representable in dimension} $m$ if there is a semiorthogonal decomposition (see Section 3 for the definition) $T = \langle A_1, \ldots, A_n \rangle$ and for each $i = 1, \ldots, n$ there exists a smooth projective connected variety $Y_i$ with $\dim(Y_i) \leq m$, such that $A_i$ is equivalent to an admissible subcategory of $D^b(Y_i)$. We set

$$\text{rdim}(T) := \min\{m \mid T \text{ is representable in dimension } m\},$$

whenever such a finite $m$ exists. Let $X$ be a smooth projective $k$-variety. One says $X$ is \textit{representable in dimension} $m$ if $D^b(X)$ is representable in dimension $m$. We will use the following notations:

$$\text{rdim}(X) := \text{rdim}(D^b(X)), \quad \text{rcodim}(X) := \dim(X) - \text{rdim}(X).$$

Note that when the base field $k$ of a variety $X$ is not algebraically closed, the existence of $k$-rational points on $X$ and whether $X$ is $k$-rational is a major open question in arithmetic geometry. We recall the following question which was formulated in [7]:

**Question.** Let $X$ be a smooth projective variety over $k$ of dimension at least 2. Suppose $X$ is $k$-rational. Do we have $\text{rcodim}(X) \geq 2$?

There are several results suggesting that this question has a positive answer (see [1], [2], [4], [15], [19], [17] and references therein). In this context, we prove:

**Theorem 1.1.** Let $A$ be a central simple $k$-algebra of degree $n$ over a field of characteristic zero. We fix a sequence of integers $0 < n_1 < n_2 < \cdots < n_r < n$ and let $X = BS(n_1, \ldots, n_r, A)$ be a twisted flag of dimension at least two. If $X$ is rational over $k$, then $\text{rcodim}(X) \geq 2$.

Acknowledgement. I would like to thank Nikita Semenov for answering questions concerning twisted flag varieties. This research was conducted in the framework of the research training group GRK 2240: Algebro-geometric Methods in Algebra, Arithmetic and Topology, which is funded by the DFG.

2. Inner twisted flags of type $A_n$

Recall that a finite-dimensional $k$-algebra $A$ is called \textit{central simple} if it is an associative $k$-algebra that has no two-sided ideals other than 0 and $A$ and if its center equals $k$. If the algebra $A$ is a division algebra it is called \textit{central division algebra}. Note that $A$ is a central simple $k$-algebra if and only if there is a finite field extension $k \subset L$, such that $A \otimes_k L \simeq M_n(L)$. This is also equivalent to $A \otimes_k \bar{k} \simeq M_n(\bar{k})$. An extension $k \subset L$ such that $A \otimes_k L \simeq M_n(L)$ is called splitting field for $A$. The \textit{degree} of a central simple algebra $A$ is defined to be $\text{deg}(A) := \sqrt{\dim A}$. According to the \textit{Wedderburn Theorem}, for any central simple $k$-algebra $A$ there is an unique integer $n > 0$ and a division $k$-algebra $D$ such
The degree of the unique central division algebra $D$ that
setting

Note that the usual Brauer–Severi schemes are obtained from the generalized one by
in $(A)$. Two central simple algebras $A$ and $B$ are said to be Brauer-equivalent if there
are positive integers $r, s$ such that $M_r(A) \simeq M_s(B)$.

A Brauer–Severi variety of dimension $n$ can also be defined as a scheme $X$ of finite
type over $k$ such that $X \otimes_k L \simeq \mathbb{P}^n$ for a finite field extension $k \subset L$. A field extension $k \subset L$ for which $X \otimes_k L \simeq \mathbb{P}^n$ is called splitting field of $X$. Clearly, $k^+$ and $\tilde{k}$ are splitting
fields for any Brauer–Severi variety. In fact, every Brauer–Severi variety always splits over
a finite Galois extension. It follows from descent theory that $X$ is projective, integral and
smooth over $k$. For details we refer to [3] and [10].

To a central simple $k$-algebra $A$ one can also associate twisted forms of Grassmannians.

We also recall the basics of generalized Brauer–Severi schemes (see [13]). Let $X$ be a
noetherian $k$-scheme and $A$ a sheaf of Azumaya algebras of rank $n^2$ over $X$ (see [11], [12]
for details on Azumaya algebras). For an integer $1 \leq n_1 < n$ the generalized Brauer–Severi scheme $p : BS(n_1, A) \rightarrow X$ is defined as the scheme representing the functor
$F : \text{Sch}/X \rightarrow \text{Sets}$, where $(\psi : Y \rightarrow X)$ is mapped to the set of left ideals $I_1 \subset \cdots \subset I_r$ of $\psi^*A$ such that $I_r/A \otimes I_s$ is locally free of rank $n(n-n_1)$. By definition, there is an étale covering $U \rightarrow X$ and a locally free sheaf $E$ of rank $n$ with the following trivializing diagram:

$$
\begin{array}{ccc}
\text{Grass}(n_1, E) & \xrightarrow{\pi} & BS(n_1, A) \\
q \downarrow & & \downarrow p \\
U & \xrightarrow{\eta} & X
\end{array}
$$

In the same way one defines the twisted relative flag $BS(n_1, \ldots, n_r, A)$ as the scheme
representing the functor $F : \text{Sch}/X \rightarrow \text{Sets}$, where $(\psi : Y \rightarrow X)$ is mapped to the set of left ideals $J_1 \subset \cdots \subset J_r$ of $\psi^*A$ such that $J_r/A \otimes J_s$ is locally free of rank $n(n-n_1)$. As
for the generalized Brauer–Severi schemes, there is an étale covering $U \rightarrow X$ and a locally
free sheaf $E$ of rank $n$ with diagram

$$
\begin{array}{ccc}
\text{Flag}_U(n_1, \ldots, n_r, E) & \xrightarrow{\pi} & BS(n_1, \ldots, n_r, A) \\
q \downarrow & & \downarrow p \\
U & \xrightarrow{\eta} & X
\end{array}
$$

Note that the usual Brauer–Severi schemes are obtained from the generalized one by
setting $n_1 = 1$. In this case one has a well known one-to-one correspondence between
sheaves of Azumaya algebras of rank $n^2$ on $X$ and Brauer–Severi schemes of relative
dimension $n-1$ via $H^1(X, \text{PGL}_{n^2})$ (see [11]). Note that if the base scheme $X$ is a point a
sheaf of Azumaya algebras on $X$ is a central simple $k$-algebra and the generalized Brauer–
Severi schemes are the generalized Brauer–Severi varieties from above. Consider a twisted
flag $X = SB(n_1, \ldots, n_r, A) \rightarrow \text{Spec}(k)$. Such an $X$ is an inner form of a partial flag variety.
That is, there is a cartesian square of the form

$$
\begin{array}{ccc}
\text{Grass}_L(n_1, \ldots, n_r, V) & \xrightarrow{\pi} & BS(n_1, \ldots, n_r, A) \\
q \downarrow & & \downarrow p \\
\text{Spec}(L) & \xrightarrow{\pi} & \text{Spec}(k)
\end{array}
$$
where \( L/k \) is a Galois extension and the 1-cocycle \( \text{Gal}(L/k) \to \text{Aut}(\text{Grass}_{L}(n_1, \ldots, n_r, V)) \) factors through \( \text{PGL}(V) \).

3. **Semiorthogonal decompositions**

Let \( D \) be a triangulated category and \( C \) a triangulated subcategory. The subcategory \( C \) is called **thick** if it is closed under isomorphisms and direct summands. For a subset \( A \) of objects of \( D \) we denote by \( \langle A \rangle \) the smallest full thick subcategory of \( D \) containing the elements of \( A \). For a smooth projective variety \( X \) over \( k \), we denote by \( D^b(X) \) the bounded derived category of coherent sheaves on \( X \). Moreover, if \( B \) is an associated \( k \)-algebra, we write \( D^b(B) \) for the bounded derived category of finitely generated left \( B \)-modules.

**Definition 3.1.** Let \( A \) be a division algebra over \( k \), not necessarily central. An object \( \mathcal{E}^* \in D^b(X) \) is called **\( A \)-exceptional** if \( \text{End}(\mathcal{E}^*) = A \) and \( \text{Hom}(\mathcal{E}^*, \mathcal{E}^*[r]) = 0 \) for \( r \neq 0 \). By **generalized exceptional object**, we mean \( A \)-exceptional for some division algebra \( A \) over \( k \). If \( A = k \), the object \( \mathcal{E}^* \) is called **exceptional**.

**Definition 3.2.** A totally ordered set \( \{ \mathcal{E}_{\mathbf{1}}, \ldots, \mathcal{E}_{\mathbf{n}} \} \) of generalized exceptional objects on \( X \) is called an **generalized exceptional collection** if \( \text{Hom}(\mathcal{E}_{\mathbf{i}}, \mathcal{E}_{\mathbf{j}}[r]) = 0 \) for all integers \( r \) whenever \( i > j \). A generalized exceptional collection is **full** if \( \langle \{ \mathcal{E}_{\mathbf{1}}, \ldots, \mathcal{E}_{\mathbf{n}} \} \rangle = D^b(X) \) and **strong** if \( \text{Hom}(\mathcal{E}_{\mathbf{i}}, \mathcal{E}_{\mathbf{j}}[r]) = 0 \) whenever \( r \neq 0 \). If the set \( \{ \mathcal{E}_{\mathbf{1}}, \ldots, \mathcal{E}_{\mathbf{n}} \} \) consists of exceptional objects it is called **exceptional collection**.

Recall that a full thick triangulated subcategory \( D \) of \( D^b(X) \) is called **admissible** if the inclusion \( D \hookrightarrow D^b(X) \) has a left and right adjoint functor.

**Definition 3.3.** Let \( X \) be a smooth projective variety over \( k \). A sequence \( D_1, \ldots, D_n \) of full admissible triangulated subcategories of \( D^b(X) \) is called **semiorthogonal** if \( D_i \subset D_j = \{ \mathcal{F}^* \in D^b(X) \mid \text{Hom}(\mathcal{F}^*, \mathcal{F}^*) = 0, \forall \mathcal{G}^* \in D_i \} \) for \( i > j \). Such a sequence defines a **semiorthogonal decomposition** of \( D^b(X) \) if the smallest thick full subcategory containing all \( D_i \) equals \( D^b(X) \).

For a semiorthogonal decomposition we write \( D^b(X) = \langle D_1, \ldots, D_n \rangle \).

**Remark 3.4.** Let \( \mathcal{E}_{\mathbf{1}}, \ldots, \mathcal{E}_{\mathbf{n}} \) be a full generalized exceptional collection on \( X \). It is easy to verify that by setting \( D_i = \langle \mathcal{E}_{\mathbf{i}} \rangle \) one gets a semiorthogonal decomposition \( D^b(X) = \langle D_1, \ldots, D_n \rangle \).

4. **Proof of Theorem 1.1**

**Lemma 4.1.** Let \( X \) be the inner twisted flag from Theorem 1.1. Then \( D^b(X) \) admits a semiorthogonal decomposition with components being equivalent to \( D^b(A^{\oplus j}) \) for suitable positive integers \( j \geq 0 \).

**Proof.** On a relative flag \( \text{Grass}(n_1, \ldots, n_r, \mathcal{E}) \) one has tautological subbundles sitting in the following sequence

\[
0 \longrightarrow R_1 \longrightarrow \cdots \longrightarrow R_r \longrightarrow q^* \mathcal{E} \longrightarrow T_1 \longrightarrow \cdots \longrightarrow T_r \longrightarrow 0.
\]

In our case \( \mathcal{E} \) is a \( k \)-vector space of dimension \( n \) denoted by \( V \). Recall that \( X = \text{BS}(n_1, \ldots, n_r, A) \) is the inner twisted flag obtained from \( \text{Grass}(n_1, \ldots, n_r, V) \) by descent. Denote by \( \alpha(1), \ldots, \alpha(r) \) partitions of the forms

\[
(\alpha_1, \ldots, \alpha_{i_1}), (\alpha_1, \ldots, \alpha_{i_2}), \ldots, (\alpha_1, \ldots, \alpha_{i_r}),
\]

with \( 0 \leq \alpha_i \leq l_2 - l_1, \ldots, 0 \leq \alpha_i \leq l_m - l_{m-1}, 0 \leq \alpha_i \leq n - l_m \). Let \( |\alpha| = \sum_{i=1}^r |\alpha(i)| \), where \( |\alpha(j)| = \sum_{i=1}^{\alpha(j)} \alpha_i \) for \( 1 \leq j \leq r \). Then the sheaf \( (S^{\alpha(1)}R_1 \otimes \cdots \otimes S^{\alpha(r)}R_r)^{\oplus |\alpha|} \) descents to a sheaf \( \mathcal{J}_\alpha \) on \( X \) (see [BJ], Section 4). As in the case of generalized Brauer–Severi varieties, one can show that \( \text{End}(\mathcal{J}_\alpha) \) is Brauer-equivalent to \( A^{\oplus |\alpha|} \). Note that \( D^b(\text{End}(\mathcal{J}_\alpha)) \) is equivalent to \( D^b(A^{\oplus |\alpha|}) \) by Morita equivalence. Now it is proved in [4] that the subcategories \( \langle \mathcal{J}_\alpha \rangle \simeq D^b(\text{End}(\mathcal{J}_\alpha)) \) form a semiorthogonal decomposition. \( \square \)
Proposition 4.2. Let $X$ be the inner twisted flag from above. Then $\text{rdim}(X) \leq \text{ind}(A) - 1$.

Proof. According to Lemma 4.1, $D^b(X)$ admits a semiorthogonal decomposition with components being equivalent to $D^b(A^{\oplus j})$ for suitable positive integers $j \geq 0$. Now let $A = M_\lambda(D)$ according to the Wedderburn theorem. Since any of the derived categories $D^b(A^{\oplus j})$ can be embedded into $D^b(Y)$, where $Y$ is the Brauer–Severi variety corresponding to $D$ (see [5]), the assertion follows from the fact that $\text{deg}(D) = \text{ind}(A)$ and $\text{dim}(Y) = \text{ind}(A) - 1$. \hfill $\square$

Lemma 4.3. Let $\alpha \geq 1$ be an integer. Then $\alpha x^2 - x - 1 \geq 0$ for all $x \geq 2$.

Proof. The function $\alpha x^2 - x - 1$ is strictly monotonically increasing for $x > 1/2\alpha$ and has its positive root at $(1/2\alpha) \cdot (1 + \sqrt{1 + 4\alpha})$ which is $< 2$ for all $\alpha \geq 1$. \hfill $\square$

Proof. (of Theorem 1.1)
Recall the well known fact that $X(k) \neq \emptyset$ iff $\text{ind}(A) \mid d$, where $d = \gcd(n, n_1, \ldots, n_r)$ denotes the greatest common divisor. So if $X$ is $k$-rational, it has a $k$-rational point and therefore $\text{ind}(A) \mid d$. Now let $n = b \cdot \text{ind}(A)$ and $n_i = b_i \cdot \text{ind}(A)$ for $i = 1, \ldots, r$. Then $b, b_1, \ldots, b_r$ are positive integers satisfying $0 < b_1 < b_2 < \cdots < b_r < b$. This implies that $a \geq 1$ for

$$a = b_1(b - b_1) + \sum_{i=2}^r (b_i - b_{i-1})(b - b_i).$$

Assume $\text{ind}(A) > 1$ and let $x = \text{ind}(A)$. With $\alpha = a$, Lemma 4.3 implies

$$a \cdot \text{ind}(A)^2 - \text{ind}(A) - 1 \geq 0.$$

It is easy to see that

$$a \cdot \text{ind}(A)^2 = n_1(n - n_1) + \sum_{i=2}^r (n_i - n_{i-1})(n - n_i).$$

This implies

$$n_1(n - n_1) + \sum_{i=2}^r (n_i - n_{i-1})(n - n_i) \geq (\text{ind}(A) - 1) + 2.$$

Now, since

$$\text{dim}(X) = n_1(n - n_1) + \sum_{i=2}^r (n_i - n_{i-1})(n - n_i)$$

and since $\text{rdim}(X) \leq \text{ind}(A) - 1$ according to Proposition 4.2, we conclude

$$\text{rcodim}(X) = \text{dim}(X) - \text{rdim}(X) \geq 2.$$

It remains to consider the case $\text{ind}(A) = 1$. In this case the inner twisted flag $X$ is split, i.e. is isomorphic to $\text{Flag}_{k}(n_1, \ldots, n_r, V)$, where $V$ is a $n$-dimensional $k$-vector space. From Kapranov [13], we know that $X$ admits a full exceptional collection. But then $\text{rdim}(X) = 0$ according to [1], Proposition 6.1.6 and hence $\text{rcodim}(X) \geq 2$. \hfill $\square$

Remark 4.4. If the inner twisted flag $X$ from Theorem 1.1 is of dimension one, it is actually a Brauer–Severi curve. In this case [13] shows that rationality of $X$ is equivalent to $\text{rdim}(X) = 0$. Hence $\text{rcodim}(X) = 1$. This is one reason why the question in the introduction assumes $\text{dim}(X) \geq 2$. 


References

[1] A. Auel and M. Bernardara: Cycles, derived categories, and rationality, in Surveys on Recent Developments in Algebraic Geometry, Proceedings of Symposia in Pure Mathematics 95 (2017), 199-266.
[2] A. Auel and M. Bernardara: Semiorthogonal decompositions and birational geometry of del Pezzo surfaces over arbitrary fields. Proc. London Math. Soc. 117 (2018) 1-64.
[3] M. Artin: Brauer-Severi varieties. Brauer groups in ring theory and algebraic geometry, Lecture Notes in Math. 917, Notes by A. Verschoren, Berlin, New York: Springer-Verlag (1982), 194-210
[4] S. Baek: Semiorthogonal decompositions for twisted Grassmannians. arXiv:1205.1175v1 [math.AG] (2012).
[5] M. Bernardara: A semiorthogonal decomposition for Brauer–Severi schemes. Math. Nachr. 282 (2009), 1406-1413.
[6] M. Bernardara and M. Bolognesi: Categorical representability and intermediate Jacobians of Fano threefolds. EMS Ser. Congr. Rep., Eur. Math. Soc. (2013), 1-10.
[7] M. Bernardara: Semiorthogonal decompositions and noncommutative motives in algebraic geometry. Habilitation thesis, available at https://www.math.univ-toulouse.fr/~mbernard/ (2016).
[8] M. Bernardara: Categorical dimension of birational automorphisms and filtrations of Cremona maps. arXiv:1701.06833v3 (2017).
[9] A. Blanchet: Function fields of generalized Brauer–Severi varieties. Comm. Algebra. Vol. 19 (1991), 97-118.
[10] P. Gille and T. Szamuely: Central simple algebras and Galois cohomology. Cambridge Studies in advanced Mathematics. 101. Cambridge University Press. (2006)
[11] A. Grothendieck: Le group de Brauer I: Algebras d Azumaya et interpretations diverses, Seminaire Bourbaki. No. 290 (1964).
[12] A. Grothendieck: Le group de Brauer II: Theorie cohomoligique, Seminaire Bourbaki. No. 297 (1965).
[13] M.M. Kapranov: On the derived categories of coherent sheaves on some homogeneous spaces. Invent. Math. Vol. 92 (1988), 479-508.
[14] M. Levine, V. Srinivas and J. Weyman: K-Theory of Twisted Grassmannians. K-Theory Vol. 3 (1989), 99-121.
[15] S. Novaković: Non-existence of exceptional collections on twisted flags and categorical representability via noncommutative motives. arXiv:1607.01033v1 [math.AG] (2016).
[16] S. Novaković: On non-existence of full exceptional collections on some relative flags. arXiv:1607.04833v4 [math.AG] (2016).
[17] S. Novaković: Rational points on symmetric powers and categorical representability. arXiv:1701.02173v1 [math.AG] (2017).

MATHEMATISCHES INSTITUT, HEINRICH–HEINE–UNIVERSITÄT 40225 DÜSSELDORF, GERMANY
E-mail adress: novakovic@math.uni-duesseldorf.de