The eigenvalue problem for the resonances of the infinite-dimensional Friedrichs model on the positive half line with Hilbert-Schmidt perturbations

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Abstract

A Gelfand triplet for the Hamiltonian $H$ of the infinite-dimensional Friedrichs model on the positive half line with Hilbert-Schmidt perturbations is constructed such that exactly the resonances (poles of the inverse of the Livšic-matrix) are eigenvalues of the extension $H^\times$ of $H$. The corresponding eigentantilinear forms are calculated explicitly. Using the wave matrices for the Abelian wave (Möller) operators the corresponding eigentantilinear forms for the unperturbed Hamiltonian $H_0$ turn out to be of pure Dirac type and can be characterized by their corresponding Gamov vector which is uniquely determined by restriction to the intersection of the Gelfand space for $H_0$ with $P_+\mathcal{H}_+^2$, where $\mathcal{H}_+^2$ is the Hardy space of the upper half plane. Simultaneously, this restriction yields a truncation of the unitary evolution $t \to e^{-itH_0}$ to the well-known decay semigroup for $t \geq 0$ of the Toeplitz type on $P_+\mathcal{H}_+^2$. That is, exactly those eigenvectors $\lambda \to k(\lambda - \zeta)^{-1}$, $k$ element of the multiplicity space $\mathcal{K}$, of the decay semigroup have an extension to an eigentantilinear form for $H_0$ hence for $H$ if $\zeta$ is a resonance and $k$ is from that subspace of $\mathcal{K}$ which is uniquely determined by its corresponding Dirac type antilinear form. Moreover, the scattering matrix which is meromorphic in the lower half plane has only simple poles there and the main part of its Laurent representation is a linear combination of Gamov vectors.

Keywords Resonances, Friedrichs model, scattering theory, Gamov vectors

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1 Introduction

Breit-Wigner formulas $E \to c((E - E_0)^2 + (\Gamma/2)^2)^{-1}$ describe bumps in quantum scattering cross sections, where $E_0$ is the resonance energy, $\Gamma/2$ the half width. These bumps are associated to unstable particles with finite lifetimes. If the scattering matrix is meromorphically continuable into the lower half plane, they can be connected with poles $E_0 - i(\Gamma/2)$ there. Then $c(E - E_0 - i(\gamma/2))^{-1}$ is called the Breit-Wigner amplitude (see e.g. Bohm [1]). These poles are called resonances. Their rigorous
mathematical description requires knowledge on the analytical properties of the scattering matrix in dependence of the Hamiltonian $H$, which is difficult to obtain, in general.

A first step in this direction is to investigate the resolvent of $H$, the meromorphic continuations of its matrix elements and their poles which are candidates for resonances. These poles are then associated with the eigenvalues of non-selfadjoint operators connected with $H$. That is, already in this approach the emphasis is to obtain certain spectral properties of these poles. The so-called Aigular-Balslev-Combes-Simon theory is representative for this efforts (see Aigular/Combes [2], Simon [3], see also Hislop/Sigal [4]).

Another approach is to derive states associated with resonances which represent directly their connection with decay, i.e. by satisfying the exponential decay law. These vectors are called *Gamov vectors* in the literature (see e.g. Bohm/Gadella [5]). The most simple method in this direction is to use the so-called decay semigroup for $t \geq 0$ of the Toeplitz type (see Eisenberg et al [6], Strauss [7]) as a truncation of the quantum evolution and to inspect its eigenvalue spectrum (which consists of all points of the lower half plane). For this approach the Hardy spaces play an essential role. However, the crucial problem is then the selection of the true Gamov vectors $\lambda \to k/(\lambda - \zeta)^{-1}$. This requires explicit knowledge of the scattering matrix, in particular for the calculation of the actual parameters $k \in \mathcal{K}$, the multiplicity space.

The present paper presents a direct approach in the sense to answer the question of the spectral properties of the resonances w.r.t. the Hamiltonian $H$ for an infinite-dimensional Friedrichs-like model (perturbation by Hiibert-Schmidt operators). In this model the resonances are characterized directly as the eigenvalues of an extension of $H$ by an appropriate Gelfand triplet. (Note that also for the ”first step” triplets are used, e.g. Banach space triplets by Agmon [8]). Moreover, from the eigenantilinear forms of the resonances, which act on the Gelfand space of the triplet, the associated Gamov vectors (eigenvectors of the Toeplitz type semigroup) are uniquely derived by their restriction to the dense ”Hardy space part” of the Gelfand space. Furthermore, the analytic structure of the scattering matrix in the lower half plane is characterized completely: there are only simple poles there (among them are the resonances) and the main part (in the Laurent sense) is a sum of a finite linear combination of Gamov vectors (associated to the resonances) and a second part, associated to poles which are not resonances (in this model).

The paper is related to [9]. There the finite-dimensional Friedrichs model on the whole real line is considered within the Schwartz space framework. In this case one has the full spectral analogy to the Lax-Phillips scattering theory. This fact suggested to combine Lax-Phillips ideas (mainly the Lax-Phillips semigroup) with the eigenvalue problem for resonances in general.
2 Preliminaries

2.1 Basic concepts of the model

2.1.1 Assumptions

The scattering model considered is a modified infinite-dimensional Friedrichs model on the positive half line where the perturbation is of the Hilbert-Schmidt type. Put

\[ \mathcal{H}_{0,+} := L^2(\mathbb{R}_+, \mathcal{K}, d\lambda), \quad \mathbb{R}_+ := (0, \infty), \]

where \( \mathcal{K} \) is a separable multiplicity Hilbert space, and

\[ \mathcal{H} := \mathcal{H}_{0,+} \oplus \mathcal{E}, \]

where \( \mathcal{E} \) is a separable Hilbert space, too. \( H_0 \) denotes the multiplication operator on \( \mathcal{H}_{0,+} \),

\[ H_0 f(\lambda) := \lambda f(\lambda), \quad f \in \mathcal{H}_{0,+}, \]

and \( A \) a selfadjoint compact operator on \( \mathcal{E} \) with only positive eigenvalues. Further let \( \Gamma \in L^2(\mathcal{H}) \) be a Hilbert-Schmidt operator on \( \mathcal{H} \) with \( \Gamma f = 0 \) for \( f \in \mathcal{H}_{0,+} \) and \( \Gamma \mathcal{E} \subseteq \mathcal{H}_{0,+} \), i.e.

\[ \Gamma e(\lambda) = M(\lambda)e, \quad e \in \mathcal{E}, \]

where \( M(\lambda) \) is necessarily a Hilbert-Schmidt operator of \( L^2(\mathcal{E} \to \mathcal{K}) \), well-defined a.e. on \( \mathbb{R}_+ \), satisfying \( \int_0^\infty \| M(\lambda) \|^2_2 d\lambda < \infty \) because of

\[ \| \Gamma \|^2_2 = \int_0^\infty \| M(\lambda) \|^2_2 d\lambda. \]

The perturbation is given by \( \Gamma + \Gamma^* \), i.e. we put

\[ H := (H_0 \oplus A) + \Gamma + \Gamma^*. \]

\( H \) is selfadjoint and \( \text{dom } H = \text{dom } H_0 \oplus \mathcal{E} \).

In the finite-dimensional case where \( \dim \mathcal{E} < \infty, \dim \mathcal{K} < \infty \) this includes the case of the usual (finite-dimensional) Friedrichs model, where \( \Gamma \) is a partial isometry with \( \Gamma^* \Gamma = P_\mathcal{E} \), the projection onto \( \mathcal{E} \), and \( \Gamma \Gamma^* < 1 - P_\mathcal{E} \), such that \( \int_0^\infty M(\lambda)^* M(\lambda) d\lambda = \mathbb{I}_\mathcal{E} \).

In the following also the case of "small perturbations" is of interest, where \( \Gamma \) is replaced by \( \epsilon \Gamma \), \( 0 < \epsilon \leq 1 \), \( \epsilon \) the so-called coupling constant.

For convenience in the following \( \Gamma \mathcal{E} \) is identified with \( \Gamma \) without confusion. On this model we impose the following assumptions:

(i) \( \mathcal{H}_{0,+} = \text{clo spa}\{E_0(\Delta)f, f \in \Gamma \mathcal{E}\} \) and \( \mathcal{H} = \text{clo spa}\{E(\Delta)e, e \in \mathcal{E}\} \), where \( E_0(\cdot), E(\cdot) \) denote the spectral measures of \( H_0, H \), respectively. This means: \( \Gamma \mathcal{E} \) is generating for \( \mathcal{H}_{0,+} \) w.r.t. \( H_0 \) and \( \mathcal{E} \) is generating for \( \mathcal{H} \) w.r.t. \( H \). (In the case \( \dim \mathcal{E} < \infty \) this implies \( \dim \mathcal{E} = \dim \mathcal{K} \).) If \( \dim \mathcal{E} = \infty \) then necessarily \( \dim \mathcal{K} = \infty \).

(ii) \( H \) has no real eigenvalues.

(iii) \( \int_0^\infty \lambda^2 \| M(\lambda) \|^2_2 d\lambda < \infty \). This is equivalent with \( \Gamma \mathcal{E} \subseteq \text{dom } H_0 \).
The essential parameter of the model is the Hilbert-Schmidt-value operator function $M(\cdot)$ on the positive half line. For this parameter we require the following further conditions of analytic continuability:

(iv) The operator function $M(\cdot)$ is analytically continuable into the complex plane,

\[ \mathbb{C} \ni z \rightarrow M(z) \in L_2(\mathcal{E} \rightarrow \mathcal{K}), \]

which is rational, with no simple poles and with no poles on the real line. That is, $M(\cdot)$ is holomorphic on the real line. The point $\infty$ is a holomorphic point of $M(\cdot)$ and $M(\infty) = 0$.

The set of all poles of $M(\cdot)$ is finite, the operator function $\mathbb{C} \ni z \rightarrow M(z)^* \in L_1(\mathcal{E})$ is rational, too. The poles of this function are complex-conjugated to the poles of $M(\cdot)$. The set of all these poles is denoted by $\mathcal{P}$, it is a finite set, symmetric w.r.t. complex conjugation.

(v) $\text{ima } M(z) =: \mathcal{A} \subset \mathcal{K}$ is a dense set in $\mathcal{K}$ for all $z \notin \mathcal{P}$ which is independent of $z$.

(vi) $\ker M(z) = \{0\}$ for all $z \notin \mathcal{P}$, i.e. $z \rightarrow M(z)^{-1}$ exists for $z \notin \mathcal{P}$.

(vii) $\mathbb{C} \ni z \rightarrow M(z)^{-1}$ is a holomorphic operator function of type A (in the sense of Kato [10]) on $\mathcal{A}$.

The conditions (v)-(vii) imply that $M(z) = M(z_0)C(z)$, where $z_0 \notin \mathcal{P}$ and $z \rightarrow C(z) \in L(\mathcal{E})$ is a rational operator function with poles in $\mathcal{P}$ such that $C(z)$ is bounded invertible for $z \notin \mathcal{P}$ and $z \rightarrow C(z)^{-1}$ is holomorphic at the points of $\mathcal{P}$, too.

A simple example of $M(\cdot)$ satisfying these conditions is given in the case $\mathcal{E} = \mathcal{K} = \mathbb{C}$ by $M(z) := (z + i)^{-2}$. Then $M(z)^{-1} = (z + i)^2$.

2.1.2 The operator function $\Phi$

We put

\[ \Phi(z) := \Gamma^* R_0(z) \Gamma = \int_0^\infty \frac{M(\lambda)^* M(\lambda)}{z - \lambda} d\lambda, \quad z \in \mathbb{C}_{>0}, \]

where $\mathbb{C}_{>0}$ is the complex plane with cut $[0, \infty)$, i.e. $\mathbb{C}_{>0} := \{z : z \in \mathbb{C} \setminus [0, \infty)\}$. $\Phi(\cdot)$ is holomorphic on $\mathbb{C}_{>0}$, $\Phi(z) \in L_1(\mathcal{E})$, $M(\lambda)^* M(\lambda) \in L_1(\mathcal{E})$ and

\[ \frac{\Gamma^* E_0(d\lambda) \Gamma}{d\lambda} = M(\lambda)^* M(\lambda), \quad \lambda > 0. \]

By straightforward residual calculation one obtains

\[ \Phi(z) = \log z \cdot M(z)^* M(z) - H(z), \quad z \in \mathbb{C}_{>0}, \quad (1) \]

where $\log(-1) = i\pi$ and $H(\cdot)$ denotes the (Laurent) main part of $\log z \cdot M(z)^* M(z)$ in $\mathbb{C}_{>0}$. That is, (1) confirms that $\Phi(\cdot)$ is holomorphic on $\mathbb{C}_{>0}$ and shows that $\Phi(\cdot)$ is meromorphic on the Riemannian surface of $z \rightarrow \log z$. For convenience we use the denotation $\Phi_\pm$ for $\Phi|_{\mathbb{C}_{\pm}}$ (where $\mathbb{C}_{\pm}$ are in the "first sheet" $\mathbb{C}_{>0}$). In the following
\( \Phi_{\pm} \) are mainly considered in \( \mathbb{C}_{<0} \), the complex plane with cut \(( -\infty, 0 ] \), i.e. \( \mathbb{C}_{<0} := \mathbb{C} \setminus ( -\infty, 0 ] \). Then

\[
\Phi_{-}(z) - \Phi_{+}(z) = 2\pi i M(\mathcal{P})^* M(z), \quad z \in \mathbb{C}_{<0}.
\]

The poles of \( \Phi_{+} \) are contained in \( \mathbb{C}_{-} \cap \mathcal{P} \), those of \( \Phi_{-} \) are contained in \( \mathbb{C}_{+} \cap \mathcal{P} \), i.e. there are only finitely many poles. One has

\[
\sup_{z \in \mathbb{C}_{<0}(R)} \| \Phi_{\pm}(z) \| < \infty, \quad \mathbb{C}_{<0}(R) := \mathbb{C}_{<0} \cap \{ z : |z| \geq R \},
\]

where \( R \) is sufficiently large.

### 2.1.3 Livšic-matrix and partial resolvent

The Livšic-matrix is defined by

\[
L_{\pm}(z) := z \mathbb{1}_{\mathcal{E}} - A - \Phi_{\pm}(z), \quad z \in \mathbb{C}_{<0}.
\]

The function \( z \to L_{\pm}(z) \in \mathcal{L}(\mathcal{E}) \) is meromorphic on \( \mathbb{C}_{<0} \), the poles are contained in \( \mathbb{C}_{\mp} \cap \mathcal{P} \).

The sandwiched resolvent \( R(z) := (z - H)^{-1} \) by the projection \( P_{\mathcal{E}} \) is called the partial resolvent. A straightforward calculation (see e.g. [11, pp. 136 f.]) gives

\[
P_{\mathcal{E}} R(z) P_{\mathcal{E}} \mathcal{E} = L_{\pm}(z)^{-1}, \quad z \in \mathbb{C}_{\pm}.
\]

(2) shows that \( L_{\pm}(\cdot)^{-1} \) is holomorphic on \( \mathbb{C}_{\pm} \). It is meromorphic on \( \mathbb{C}_{<0} \) because

\[
L_{\pm}(z) = z(\mathbb{1}_{\mathcal{E}} - z^{-1}(A + \Phi_{\pm}(z))) = z(\mathbb{1}_{\mathcal{E}} - K_{\pm}(z))
\]

where \( K_{\pm}(z) := z^{-1}(A + \Phi_{\pm}(z)) \) is compact for all \( z \in \mathbb{C}_{<0} \). One has \( L_{\pm}(z)^{-1} = z^{-1}(\mathbb{1}_{\mathcal{E}} - K_{\pm}(z))^{-1} \). Since \( z \to (\mathbb{1}_{\mathcal{E}} - K_{\pm}(z))^{-1} \) is holomorphic on \( \mathbb{C}_{\pm} \), it follows that \( L_{\pm}(z)^{-1} \) is meromorphic on \( \mathbb{C}_{<0} \), according to a well-known result due to Keldysch [12]. A straightforward estimation gives

\[
\sup_{z \in \mathbb{C}_{<0}(R)} \| L_{\pm}(z)^{-1} \| < \infty, \quad \mathbb{C}_{<0}(R) := \mathbb{C} \cap \{ z : |z| \geq R \},
\]

where \( R \) is sufficiently large. Therefore \( L_{\pm}(\cdot)^{-1} \) has at most finitely many poles which are contained in \( \mathbb{C}_{-} \). The set of these poles is denoted by \( \mathcal{R} \), they are called resonances. To simplify the treatment it is assumed that the sets of poles of \( L_{\pm}(\cdot) \) and \( L_{\pm}(\cdot)^{-1} \) are disjoint, \( \mathcal{R} \cap \mathcal{P} = \emptyset \).

Applying Theorem 1 of [11, p. 139] to the partial resolvent one obtains: \( \lambda \to L_{\pm}(\lambda)^{-1} \) is holomorphic on \( \mathbb{R}_{+} \) because of (ii). Moreover, for \( \lambda < 0 \) we have \( L_{\pm}(\lambda + i0) = L_{-}(\lambda - i0) \) and \( \lambda \to L_{\pm}(\lambda \pm i0)^{-1} \) are holomorphic there, again because of (ii). Further we have

**Lemma 1.** \( \zeta_{0} \in \mathbb{C}_{-} \) is a pole of \( L_{\pm}(\cdot)^{-1} \) iff \( \dim \ker L_{\pm}(\zeta_{0}) > 0 \).

Proof. If \( \zeta_{0} \) is a pole of \( L_{\pm}(\cdot)^{-1} \) then one has \( 0 \in \text{spec} L_{\pm}(\zeta_{0}) = \text{spec} \{ \zeta_{0}(\mathbb{1}_{\mathcal{E}} - K_{\pm}(\zeta_{0})) \} \) and this means 0 is an eigenvalue of \( \mathbb{1}_{\mathcal{E}} - K_{\pm}(\zeta_{0}) \), i.e. also an eigenvalue of \( L_{\pm}(\zeta_{0}) \), hence \( \ker L_{\pm}(\zeta_{0}) \supset \{ 0 \} \). The converse is obvious. \( \Box \)
The sandwiched spectral measure $E(\cdot)$ of $H$ satisfies
\[
P_{E}E(d\lambda)P_{E} = \frac{1}{2\pi i} \left( (L_{-}(\lambda)^{-1} - L_{+}(\lambda)^{-1}) = L_{\pm}(\lambda)^{-1}M(\lambda)^{\ast}M(\lambda)L_{\mp}(\lambda)^{-1} \right), \quad \lambda > 0.
\]

It vanishes for $\lambda < 0$. Therefore \(\text{spec } H = [0, \infty)\) and it follows that $H$ is pure absolutely continuous.

### 2.2 Spectral representations by spectral integrals

The spectral integral
\[
\mathcal{H}_{0,+} \ni x := \int_{0}^{\infty} E_{0}(d\lambda) \Gamma f(\lambda), \quad \mathbb{R} \ni \lambda \to f(\lambda) \in \mathcal{E},
\]
exists iff \(\int_{0}^{\infty} \| M(\lambda)f(\lambda) \|_{K}^{2} d\lambda < \infty\). According to assumption (i) the linear manifold of all such spectral integrals is dense in $\mathcal{H}_{0,+}$. It defines a spectral representation of $\mathcal{H}_{0,+}$ w.r.t. $H_{0}$ which is given by the isometric isomorphism between $\mathcal{H}_{0,+}$ and $L^{2}(\mathbb{R}_{+}, \mathcal{E}_{\lambda}, d\lambda)$, where the Hilbert space $\mathcal{E}_{\lambda}$ is the completion of $\mathcal{E}$ w.r.t. the scalar product $\langle e_{1}, e_{2} \rangle_{\lambda} := (M(\lambda)e_{1}, M(\lambda)e_{2})_{K}$. Note that $\Gamma \mathcal{E}$ is a spectral manifold w.r.t. $E_{0}(\cdot)$. If $\dim \mathcal{E} < \infty$ then $\mathcal{E}_{\lambda} = \mathcal{E}$ for all $\lambda$. (See e.g. [13, p. 90 f.] for details.)

The same procedure for $\mathcal{H}$ and $H$ leads to a distinguished spectral representation defined by the spectral integral
\[
\mathcal{H} \ni y := \int_{0}^{\infty} E(d\lambda) g(\lambda), \quad \mathbb{R} \ni \lambda \to g(\lambda) \in \mathcal{E},
\]
where (5) exists iff
\[
\int_{0}^{\infty} \| M(\lambda)L_{\pm}(\lambda)^{-1}g(\lambda) \|_{K}^{2} d\lambda < \infty.
\]
That is, for these dense sets of spectral integrals we call $f(\cdot)$ the $E_{0}$-representer of $x$, $x(\lambda) = M(\lambda)f(\lambda)$, and $g(\cdot)$ the $E$-representer of $y$. Note that if $y \in \text{dom } H$ then $\lambda \to \lambda g(\lambda)$ is the $E$-representer of $Hy$.

These two spectral representations we call the natural spectral representations of $\mathcal{H}, H_{0}$ w.r.t. $H, H_{0}$, respectively.

### 2.3 Wave operators, wave matrices and scattering matrix

The strong wave operators are defined by
\[
W_{\pm} := \text{s-lim}_{t \to \pm \infty} e^{itH}e^{-itH_{0}} P_{ac}^{0},
\]
where $P_{ac}^{0}$ denotes the projection onto the absolutely continuous subspace of $H_{0}$. If $H$ is pure absolutely continuous and the condition of asymptotic completeness is satisfied then
\[
W_{\pm}^{*} \text{ } = \text{s-lim}_{t \to \pm \infty} e^{itH_{0}}e^{-itH}.
\]
If the wave operators (6) exist, they can be rewritten as so-called Abelian limits
\[
W_{+} = \Omega_{+} := \text{s-lim}_{\epsilon \to +0} \int_{0}^{\infty} e^{-\epsilon t} e^{itH}e^{-itH_{0}} P_{ac}^{0} dt,
\]
(7)
\[ W_- = \Omega_- := \text{s-lim}_{\epsilon \to 0} \int_0^\infty ee^{-it} e^{-itH} e^{itH_0} P_{ac}^0 dt. \] (8)

Note that the existence of the Abelian limits \( \Omega_\pm \) does not imply the existence of the wave operators (6). The time integrals in (7) and (8) can be transformed into spectral integrals (see [14, p. 361]):

\[ \int_0^\infty ee^{-it} e^{itH} e^{-itH_0} P_{ac}^0 dt = \int_0^\infty E(d\lambda)(\mathbb{1}_H - VR_0(\lambda + i\epsilon)) P_{ac}^0, \] (9)

\[ \int_0^\infty ee^{-it} e^{-itH} e^{itH_0} P_{ac}^0 dt = \int_0^\infty E(d\lambda)(\mathbb{1}_H - VR_0(\lambda - i\epsilon)) P_{ac}^0, \] (10)

where \( V := H - H_0 = \Gamma + \Gamma^*. \)

**Lemma 2.** The limits

\( \Omega_\pm := \text{s-lim}_{\epsilon \to 0} \int_0^\infty E(d\lambda)(\mathbb{1}_H - VR_0(\lambda \pm i\epsilon)) P_{ac}^0, \) (11)

\( \tilde{\Omega}_\pm := \text{s-lim}_{\epsilon \to 0} \int_0^\infty E_0(d\lambda)(\mathbb{1}_H + VR(\lambda \pm i\epsilon)) \) (12)

exist, \( \Omega_\pm \) is isometric from \( \mathcal{H}_{0,+} \) onto \( \mathcal{H} \) and \( \tilde{\Omega}_\pm = \Omega^*_\pm. \) That is, the Abelian limits of the wave operators exist and they are isometric from \( \mathcal{H}_{0,+} \) onto \( \mathcal{H}. \)

Proof. The existence of (12) follows by straightforward calculation. The result is

\[ \tilde{\Omega}_\pm y = \int_0^\infty E_0(d\lambda) \Gamma L_\pm(\lambda)^{-1} g(\lambda), \] (13)

where \( y := \int_0^\infty E(d\lambda) g(\lambda), \) i.e. \( g(\cdot) \) is the \( \mathcal{L}-\)resrenter of \( y. \) (13) says that the \( E_0\)-resrenter of \( \tilde{\Omega}_\pm y \) is given by \( \lambda \to L_\pm(\lambda)^{-1} g(\lambda). \) One calculates easily that the "multiplication operator" \( \lambda \to L_\pm(\lambda)^{-1} \) acts, w.r.t. the natural spectral representations of \( \mathcal{H}, \mathcal{H}_0, \) isometrically from \( \mathcal{H} \) into \( \mathcal{H}_0, \) because

\[ \left\| \int_0^\infty E(d\lambda) g(\lambda) \right\|^2_{\mathcal{H}} = \left\| \int_0^\infty E_0(d\lambda) \Gamma L_\pm(\lambda)^{-1} g(\lambda) \right\|^2_{\mathcal{H}_{0,+}}. \]

and the spectral integrals in \( \mathcal{H} \) are dense, i.e.

\[ \| \tilde{\Omega}_\pm y \| = \| y \|, \ y \in \mathcal{H} \]

follows. This means \( \tilde{\Omega}_\pm \) is isometric on \( \mathcal{H}. \) To show that the image of \( \tilde{\Omega}_\pm \) is \( \mathcal{H}_{0,+} \) it is sufficient to show that the dense set of all spectral integrals belongs to the image: choose an \( E_0\)-resrenter \( \lambda \to f(\lambda) \in \mathcal{E} \) of a vector \( x \in \mathcal{H}_{0,+}. \) Then \( \lambda \to L_\pm(\lambda)^{-1} f(\lambda) \in \mathcal{E} \) is an \( \mathcal{L}\)-resrenter of a vector \( y \in \mathcal{H} \) because

\[ \left\| \int_0^\infty E(d\lambda) L_\pm(\lambda)^{-1} f(\lambda) \right\|^2 = \int_0^\infty \| M(\lambda)f(\lambda) \|^2 d\lambda < \infty, \]

and one obtains \( \tilde{\Omega}_\pm y = x, \) hence \( \tilde{\Omega}_\pm \mathcal{H} = \mathcal{H}_{0,+} \) follows or \( \tilde{\Omega}^*_\pm \tilde{\Omega}_\pm = \mathbb{1}_\mathcal{H}, \tilde{\Omega}_\pm \tilde{\Omega}_\pm^* = \mathbb{1}_{\mathcal{H}_{0,+}}. \)

W.r.t. the resrenters of \( x, y \) the operator \( \tilde{\Omega}_\pm^* \) is given by the transformation of \( \lambda \to f(\lambda) \) to \( \lambda \to L_\pm(\lambda)^{-1} f(\lambda) \) or

\[ \tilde{\Omega}_\pm^* x = \int_0^\infty E(d\lambda) L_\pm(\lambda)^{-1} f(\lambda). \]
However, this can be rewritten into
\[\tilde{\Omega}_\pm^* x = \int_0^\infty E(d\lambda)(1 - V R_0(\lambda \pm i0))x, \quad x \in \mathcal{H}_{0,+},\]
i.e. (11) exists and \(\Omega_\pm = \tilde{\Omega}_\pm^*\).  \(\square\)

The proof of Lemma 2 yields

**COROLLARY 1.** The (isometric) Abelian wave operators \(\Omega_\pm, \Omega_\pm^*\) are given by the formulas
\[\Omega_\pm \left(\int_0^\infty E_0(d\lambda)\Gamma f(\lambda)\right) = \int_0^\infty E(d\lambda)L_\pm(\lambda)f(\lambda),\]
\[\Omega_\pm^* \left(\int_0^\infty E(d\lambda)g(\lambda)\right) = \int_0^\infty E_0(d\lambda)\Gamma L_\pm(\lambda)^{-1}g(\lambda),\]
i.e. if \(\lambda \to f(\lambda)\) is the \(E_0\)-representer of \(x \in \mathcal{H}_{0,+}\) then the \(E\)-representer of \(\Omega_\pm x \in \mathcal{H}\) is given by \(\lambda \to L_\pm(\lambda)f(\lambda)\). Conversely, if \(\lambda \to g(\lambda)\) is the \(E\)-representer of \(y \in \mathcal{H}\) then the \(E_0\)-representer of \(\Omega_\pm^* y \in \mathcal{H}_{0,+}\) is given by \(\lambda \to L_\pm(\lambda)^{-1}g(\lambda)\).

For example, the \(E_0\)-representer of the vector \(\Omega_\pm^* e, e \in \mathcal{E}\), is given by
\[\lambda \to M(\lambda)L_+(\lambda)^{-1}e,\]the \(E_0\)-representer of \(\Omega_\pm^* \Gamma e\) is
\[\lambda \to M(\lambda)L_+(\lambda)^{-1}(\lambda - A)e,\]
because \((\lambda - A)e\) is the \(E\)-representer of \(\Gamma e\):
\[\int_0^\infty E(d\lambda)(\lambda - A)e = \int_0^\infty E(d\lambda)e - \int_0^\infty E(d\lambda)Ae = He - Ae = \Gamma e,\]
note that \(e \in \text{dom } H\), because \(He = Ae + \Gamma e\).

Corollary 1 means that the Abelian wave operators act by application of the Livšic-matrix resp. its inverse on the corresponding representers. In general, operator functions with these properties are called wave matrices of \(\Omega_\pm, \Omega_\pm^*\). Note that wave matrices are well-defined only if the spectral representations are fixed.

**COROLLARY 2.** The wave matrices of \(\Omega_\pm, \Omega_\pm^*\) w.r.t. the natural spectral representations of \(\mathcal{H}_{0,+} \mathcal{H}\) are given by \(\Omega_\pm(\lambda) = L_\pm(\lambda), \Omega_\pm(\lambda)^* = L_\pm(\lambda)^{-1}, \lambda \in \mathbb{R}_+\).

Note that Lemma 2 implies that \(H\) and \(H_0\) are unitarily equivalent.

**REMARK 1.** Since \(\Omega_\pm\) is isometric, (7) and (8) can be improved. In this case even
\[\lim_{\epsilon \to 0^+} \int_0^\infty \epsilon e^{-\epsilon t}\|e^{-iH_0}x - \Omega_+ x\|^2 dt = 0, \quad x \in \mathcal{H}_{0,+},\]
\[\lim_{\epsilon \to 0^+} \int_0^\infty \epsilon e^{-\epsilon t}\|e^{-iH_0}x - \Omega_- x\|^2 dt = 0, \quad x \in \mathcal{H}_{0,+},\]
are true. The conditions (16), (17) are strongly related to the existence of the strong limits (6), i.e. to the existence of the strong wave operators (for details see [13, p.
101 f.]). If $\Gamma$ is even trace class, $\Gamma \in \mathcal{L}_1(\mathcal{H})$, then the strong limits (6) exist. In particular, in the finite-dimensional case $\dim \mathcal{E} < \infty$ the wave operators $W_{\pm}$ exist.

The scattering operator $S$ is defined by $S := W^*_+ W_- = \Omega^* \Omega$, it is unitary on $\mathcal{H}_{0,+}$ and can be represented by its scattering matrix w.r.t. a given spectral representation of $\mathcal{H}_{0,+}$.

**LEMMA 3.** W.r.t. the natural spectral representation of $\mathcal{H}_{0,+}$ the corresponding scattering matrix $S_\mathcal{E}$ is given by

$$S_\mathcal{E}(\lambda) = L_+(\lambda)^{-1}L_-(\lambda) = L_+(\lambda)^{-1}L_+(\lambda)^*,$$

i.e. if $x \in \mathcal{H}_{0,+}$ and $f(\cdot)$ is its $E_0$-representer then $\lambda \to S_\mathcal{E}(\lambda)f(\lambda)$ is the $E_0$-representer of $Sx$. $S_\mathcal{E}(\cdot)$ is meromorphically continuable into $\mathbb{C}_{<0}$, its poles are contained in $\mathcal{R} \cup (\mathcal{P} \cap \mathbb{C}_+)$. 

The scattering matrix $S_K(\cdot)$ w.r.t. the original spectral representation of $\mathcal{H}_{0,+}$ given by the $K$-valued functions $\lambda \to x(\lambda) \in K$ for $x \in \mathcal{H}_{0,+}$ satisfies

$$S_K(\lambda)M(\lambda)f(\lambda) = M(\lambda)S_\mathcal{E}(\lambda)f(\lambda)$$

which is satisfied by

$$S_K(\lambda) := \mathbb{1}_K - 2\pi i M(\lambda)L_+(\lambda)^{-1}M(\lambda)^*.$$  \hspace{1cm} (18)

$S_K(\cdot)$ is meromorphically continuable into $\mathbb{C}_{<0}$, its poles are contained in $\mathcal{R} \cup \mathcal{P}$.

In the case of "small perturbations" where $\Gamma$ is replaced by $\epsilon \Gamma$, i.e. $M(\cdot)$ is replaced by $\epsilon M(\cdot)$, there is an essential difference between the resonances (poles of $L_+(-\cdot)^{-1}$) and the poles of $L_-(\cdot)$ (which are in $\mathbb{C}_+$) because the poles of $z \to \epsilon M(z)$ and $z \to \epsilon M(\bar{z})^*$ are independent of $\epsilon$, whereas the poles of $L_+(-\cdot)^{-1}$ depend on $\epsilon$. This implies that also these poles of $S_K(\cdot)$, contained in $\mathcal{R}$, depend on $\epsilon$, those in $\mathcal{P}$ are independent of $\epsilon$. The most interesting resonances are those whose trajectories for $\epsilon \to 0$ run into the embedded eigenvalues, i.e. the eigenvalues of $A$.

### 3 The Gelfand triplet

A Gelfand triplet is given by the Gelfand space and its topology.

The **Gelfand space** $\mathcal{G} \subset \mathcal{H}_{0,+}$ is defined to be the manifold of all $s \in \mathcal{H}_{0,+}$,

$$s(\lambda) = M(\lambda)L_+(\lambda)^{-1}g(\lambda), \quad g(\lambda) \in \mathcal{E},$$

($g(\cdot)$ is the $E$-representer of $g := \Omega_+ s$), such that $g(\cdot)$ is holomorphic on $\mathbb{R}_+$, meromorphic on $\mathbb{C}_{<0}$ with poles at most in $\mathcal{P}$, and $\lambda \to \lambda g(\lambda)$ is also an $E$-representer. The **Gelfand topology** in $\mathcal{G}$ is defined by the collection of norms

$$\mathcal{G} \ni s \to [s]_K := \|s\|_{\mathcal{H}_{0,+}} + \sup_{z \in K} \|g(z)\|_{\mathcal{E}}, \quad K \subset \mathbb{C}_{<0} \setminus \mathcal{P}, \quad K \text{ compact}.$$  

Obviously $\mathcal{G}$ is dense in $\mathcal{H}_{0,+}$ w.r.t. the Hilbert topology and $\mathcal{G}$ defines a Gelfand triplet

$$\mathcal{G} \subset \mathcal{H}_{0,+} \subset \mathcal{G}^\times,$$
where $\mathcal{G}^\times$ denotes the set of all continuous antilinear forms w.r.t. the Gelfand topology. Note that
\[ \Omega_+^e \mathcal{E} \oplus \Omega_+^e \Gamma \mathcal{E} \subset \mathcal{G}. \]
This follows from (14) and (15). Note further that $\Omega_+^e \Gamma \mathcal{E} \subset \text{dom } H_0$ because of (iii).

The Gelfand space $\mathcal{G}$ can be transferred into $\mathcal{H}$ by the wave operator $\Omega_e^+$:
\[ \mathcal{D} := \Omega_e^+ \mathcal{G}. \]
The topology of $\mathcal{D}$ is defined by the injection of the topology of $\mathcal{G}$. Then $\mathcal{D}$ is a Gelfand space in $\mathcal{H}$ and
\[ \mathcal{D} \subset \mathcal{H} \subset \mathcal{D}^\times \]
is the corresponding Gelfand triplet. $\mathcal{D}$ satisfies
\[ E \oplus \Gamma E \subset \mathcal{D}, \quad \mathcal{D} \subset \text{dom } H, \quad HD \subset D. \]

Then, defining $\Phi$ by $\Phi := \Omega_e^+ \mathcal{D} \subset \mathcal{D}$ one obtains
\[ \mathcal{D} = \Phi \oplus \mathcal{E}, \quad \mathcal{D}^\times = \Phi^\times \times \mathcal{E} \] (cartesian product).

Further one has $\Phi \subseteq \text{dom } H_0$ because of $\Phi \subseteq \text{dom } H = \text{dom } H_0 \oplus \mathcal{E}$, thus $H_0 \Phi \subseteq \Phi$ follows.

For $\mathcal{D} \ni d = \phi + e$, $\phi \in \Phi$, $e \in \mathcal{E}$ and $d^\times = \{\Phi^\times, e^\times\} \in \Phi^\times \times \mathcal{E}$ one obtains
\[ \langle d \mid d^\times \rangle = \langle \phi \mid \phi^\times \rangle + (e,e^\times)_{\mathcal{E}}. \quad (19) \]

4 The eigenvalue problem for $H$ w.r.t. the Gelfand triplet $\mathcal{D} \subset \mathcal{H} \subset \mathcal{D}^\times$

4.1 The boundary condition and the solution

The Gelfand triplet $\mathcal{D} \subset \mathcal{H} \subset \mathcal{D}^\times$ yields a unique extension $H^\times$ on $\mathcal{D}^\times$ given by
\[ \langle d \mid H^\times d^\times \rangle := \langle Hd \mid d^\times \rangle, \quad d \in \mathcal{D}, \quad d^\times \in \mathcal{D}^\times. \quad (20) \]
The eigenvalue equation for eigenvalues $\zeta_0 \in \mathbb{C} \setminus \mathcal{P}$ of $H^\times$ reads then
\[ H^\times d^\times_0 = \zeta_0 d^\times_0, \quad d^\times_0 := \{\phi_0^\times(\zeta_0, e_0), e_0 \in \mathcal{E}, \zeta_0 \in \mathbb{C} \setminus \mathcal{P}, \phi_0^\times \in \Phi^\times\}. \quad (21) \]

For the part $\phi_0^\times$ of a solution we impose a

**Boundary condition:** $\phi_0^\times$ is required to be the analytic continuation into $\mathbb{C} \setminus \mathcal{P}$ of a holomorphic vector antilinear form $\phi_0^\times(z, e_0)$ on $\mathbb{C}_+$ such that the $\Phi$-part of the eigenvalue equation is an identity on $\mathbb{C}_+$.

The solution of this eigenvalue problem is given by

**THEOREM 1.** The point $\zeta_0 \in \mathbb{C} \setminus \mathcal{P}$ is an eigenvalue of $H^\times$ with eigenantilinear form $d^\times_0 := \{\phi_0^\times(\zeta_0, e_0), e_0 \}$ iff $\zeta_0$ is a resonance, $\zeta_0 \in \mathcal{R}$, and $e_0$ satisfies $L_+(\zeta_0) e_0 = 0$, i.e. $e_0 \in \ker L_+(\zeta_0)$. That is, the (generalized) eigenspace of $\zeta_0$ has the dimension $\dim \ker L_+(\zeta_0)$, the geometric multiplicity of the eigenvalue $0$ of $L_+(\zeta_0)$. 


Proof. According to (19) and (20) the eigenvalue equation (21) means
\[
\left\langle H \phi \mid d_0^\times \right\rangle = \left\langle \zeta_0 d \mid d_0^\times \right\rangle \quad d \in \mathcal{D},
\]
This is equivalent with
\[
(A_0 e - \zeta_0 e, e_0) + \langle Ge \mid \phi_0^\times \rangle = \langle \zeta_0 \phi - H_0 \phi \mid \phi_0^\times \rangle - \langle \Gamma^* \phi, e_0 \rangle,
\]
where \( d = \phi + e, \phi \in \Phi, e \in \mathcal{E} \). Since \( e \) and \( \phi \) vary independently we obtain two equations:
\[
\langle (\zeta_0 - A) e, e_0 \rangle = \langle Ge \mid \phi_0^\times \rangle, \quad e \in \mathcal{E},
\]
and
\[
\langle (\zeta_0 - H_0) \phi \mid \phi_0^\times \rangle = \langle \Gamma^* \phi, e_0 \rangle, \quad \phi \in \Phi.
\]
\( \phi_0^\times \) depends on \( \zeta_0 \), the possible eigenvalue (and on \( e_0 \)). According to the boundary condition for \( \phi_0^\times \) this antilinear form is required to be the analytic continuation of a holomorphic vector antilinear form \( \mathbb{C}_+ \ni z \to \phi_0^\times(z) \) such that the equation (23) is valid also on \( \mathbb{C}_+ \):
\[
((\mathcal{Z} - H_0) \phi \mid \phi_0^\times(z))_{\mathcal{H}_{0,+}} = \langle \Gamma^* \phi, e_0 \rangle, \quad z \in \mathbb{C}_+, \phi \in \Phi,
\]
or
\[
(\phi, (z - H_0) \phi_0^\times(z))_{\mathcal{H}_{0,+}} = \langle \phi, \Gamma e_0 \rangle_{\mathcal{H}_{0,+}}, \quad z \in \mathbb{C}_+, \phi \in \Phi.
\]
This means \( (z - H_0) \phi_0^\times(z) = \Gamma e_0 \) or
\[
\phi_0^\times(z, e_0) = (z - H_0)^{-1} \Gamma e_0, \quad z \in \mathbb{C}_+.
\]
That is, with \( \phi := \mathcal{P}_\mathcal{E}^+ \Omega_+ s, \ s \in \mathcal{G} \), one obtains
\[
\langle \phi \mid \phi_0^\times(z, e_0) \rangle = \langle \mathcal{P}_\mathcal{E}^+ \Omega_+ s, R_0(z) \Gamma e_0 \rangle_{\mathcal{H}_{0,+}} = \langle \Omega_+ s, R_0(z) \Gamma e_0 \rangle_{\mathcal{H}_{0,+}}, \quad z \in \mathbb{C}_+.
\]
Now one has to check that this antilinear form has an analytic continuation into \( \mathbb{C}_- \) across \( \mathbb{R}_+ \) which is meromorphic on \( \mathbb{C}_{<0} \). First we use the following identity which can be obtained by a straightforward calculation:
\[
R_0(z) \Gamma e + e = R(z) L_+(z) e, \quad e \in \mathcal{E}, z \in \mathbb{C}_+.
\]
Then we get
\[
(\Omega_+ s, R_0(z) \Gamma e_0) = \langle \Omega_+ s, R(z) L_+(z) e_0 \rangle - \langle \Omega_+ s, e_0 \rangle.
\]
(\( \Omega_+ s, e_0 \)) is a constant term. We put \( g := \Omega_+ s \in \mathcal{D} \) and obtain
\[
(g, R(z) L_+(z) e_0) = \langle R(\mathcal{Z}) g, L_+(z) e_0 \rangle
\]
\[
= \left( \int_0^\infty \frac{1}{\mathcal{Z} - \mu} E(\mu) g(\mu), L_+(z) e_0 \right)
\]
\[
= \left( \int_0^\infty \frac{1}{\mathcal{Z} - \mu} P_\mathcal{E}^i E(\mu) P_\mathcal{E}^i d\mu g(\mu), L_+(z) e_0 \right)
\]
\[
= \left( \int_0^\infty \frac{1}{\mathcal{Z} - \mu} L_-(\mu)^{-1} M(\mu)L_+(\mu)^{-1} g(\mu) d\mu, L_+(z) e_0 \right)
\]
\[
= \langle \Psi_-(\mathcal{Z}), L_+(z) e_0 \rangle,
\]
(25)
where
\[
\Psi_{\pm}(z) := \int_0^\infty \frac{1}{z - \mu} L_-(\mu)^{-1} M(\mu)^* M(\mu) L_+ \mu^{-1} g(\mu) d\mu, \quad z \in \mathbb{C}_{\pm},
\]
\(\Psi_{\pm}(\cdot)\) is meromorphically continuable into \(\mathbb{C}_{<0}\) across \(\mathbb{R}_+\), where
\[
\Psi_-(z) - \Psi_+(z) = 2\pi i L_-(z)^{-1} M(z)^* M(z) L_+ (z)^{-1} g(z), \quad z \in \mathbb{C}_{<0},
\]
and for \(z \in \mathbb{C}_-\) one obtains
\[
\Psi_-(\bar{z}) = \Psi_+(\bar{z}) + 2\pi i L_-(\bar{z})^{-1} M(z)^* M(z) L_+ (\bar{z})^{-1} g(\bar{z})
\]
and
\[
(\Psi_-(\bar{z}), L_+(z)e_0) = (\Psi_+(\bar{z}), L_+(z)e_0) + 2\pi i (M(\bar{z}) L_+ (z)^{-1} g(\bar{z}), M(z)e_0)
\]
\[
= (\Psi_+(\bar{z}), L_+(z)e_0) + 2\pi i (L_+(\bar{z})^{-1} g(\bar{z}), M(z)^* M(z)e_0).
\]
Inspection of (26) proves the assertion. Poles are necessarily in \(\mathcal{P}\).

Now we know that the antilinear form \(\phi^\times_0(z, e_0)\) satisfies the equation (24) for \(z \in \mathbb{C}_+\). Therefore it satisfies the equation (23) for all \(z \in \mathbb{C}_{<0} \setminus \mathcal{P}\) and it is holomorphic there. For this reason we consider the second equation (22) first on \(\mathbb{C}_+\). Then it reads
\[
((\bar{z} - A)e, e_0) = (\Gamma e, \phi_0^\times(z, e_0)) = (\Gamma e, R_0(z)\Gamma e_0) = (e, \Gamma^* R_0(z)\Gamma e_0).
\]
Using \((z - A)e_0 - \Gamma^* R_0(z)\Gamma e_0 = L_+(z)e_0\) we have
\[
(e, (z - A)e_0) - (\Gamma e, \phi_0^\times(z, e_0)) = (e, L_+(z)e_0), \quad e \in \mathcal{E} \quad z \in \mathbb{C}_+
\]
and the equation (22) reads simply \((e, L_+(z)e_0) = 0\) for all \(e \in \mathcal{E}\) which obviously has no solution in \(\mathbb{C}_+ \cup \mathbb{R}_+\). But by analytic continuation the identity (27) is true also in \(\mathbb{C}_-\). That is, equation (22) is equivalent with
\[
L_+(\zeta_0)e_0 = 0, \quad \zeta_0 \in \mathbb{C}_- \setminus \mathcal{P},
\]
This means: equation (22) has a solution \(\zeta_0 \in \mathbb{C}_- \setminus \mathcal{P}\) with the corresponding parameter \(e_0 \in \mathcal{E}\) iff equation (28) is satisfied. Conversely, if \(\zeta_0 \in \mathbb{C}_- \setminus \mathcal{P}\) and \(e_0 \in \mathcal{E}\) satisfy equation (28) then \(\zeta_0\) is an eigenvalue of \(H^\times\) and \(d_0^\times := \{\phi^\times_0(\zeta_0, e_0), e_0\}\) is a corresponding eigenantilinear form. The dimension of the eigenspace of \(\zeta_0\) is then \(\dim \ker L_+(\zeta_0)\). \(\square\)

### 4.2 Characterization of the eigenantilinear forms by the Gelfand triplet \(\mathcal{G} \subset H_{0,+} \subset \mathcal{G}^\times\)

The solutions \(d_0^\times(\zeta, e_0)\) of the eigenvalue problem for \(H^\times\) refer to the Gelfand triplet \(\mathcal{D} \subset H \subset \mathcal{D}^\times\) which is given by the transfer \(\mathcal{D} = \Omega_+ \mathcal{G}\) from the Gelfand triplet of the Hilbert space \(H_{0,+}\). The "back transformation"
\[
s_0^\times(\zeta_0, e_0) := (\Omega_+^*)^\times d_0^\times(\zeta_0, e_0), \quad s_0^\times \in \mathcal{G}^\times
\]
of the eigensolution $d_0^\kappa$ to the Gelfand triplet $\mathcal{G} \subset \mathcal{H}_{0,+} \subset \mathcal{G}^\times$ has a surprising property.

**THEOREM 2.** The eigenantilinear form $s_0^\kappa$ of $H_0^\kappa$ w.r.t. the Gelfand triplet $\mathcal{G} \subset \mathcal{H}_{0,+} \subset \mathcal{G}^\times$, associated to $d_0^\kappa$ by (29) is of pure Dirac type w.r.t. the point $\zeta_0 \in \mathbb{C}_+$:

$$\langle s \mid s_0^\kappa(\zeta_0, e_0) \rangle = 2\pi i(s(\zeta_0), k_0)e_0,$$

where $k_0 := M(\zeta_0)e_0$.

Proof. For $z \in \mathbb{C}_+$ one has

$$\langle s \mid s_0^\kappa(z, e_0) \rangle = \langle \Omega_+s \mid d_0^\kappa(z, e_0) \rangle = \langle P_\xi \Omega_+s \mid \phi_0^\kappa(z, e_0) \rangle + \langle P_\xi \Omega_+s, e_0 \rangle,$$

i.e.

$$\langle s \mid s_0^\kappa(z, e_0) \rangle = (\Omega_+s, R(z)L_+(z)e_0).$$

For $z \in \mathbb{C}_-$ one has, according to (25) and (26),

$$\langle s \mid s_0^\kappa(z, e_0) \rangle = (\Psi_-(\Omega), L_+(z)e_0) + 2\pi i(M(\zeta)L_+(\zeta)^{-1}g(\zeta), M(z)e_0).$$

If $z = \zeta_0$ is a resonance then

$$\langle s \mid s_0^\kappa(\zeta_0, e_0) \rangle = 2\pi i(s(\zeta_0), k_0), \quad k_0 := M(\zeta_0)e_0$$

and this is the assertion. □

### 4.3 The associated Gamov vectors

According to Theorem 2, the back transformed eigenantilinear forms $s_0^\kappa(\zeta_0, e_0)$ are of pure Dirac type. This property is crucial for the association of Gamov vectors which are uniquely determined by $s_0^\kappa(\zeta_0, e_0)$.

In the literature there are several approaches to associate "Gamov vectors" to resonances. In one of them Gamov vectors are considered to be special eigenvectors of a truncated evolution $t \to T_+(t), t \geq 0$, on the Hilbert space $P_+\mathcal{H}_2^\kappa$, where $T_+(t) := P_+Q_+e^{-it\tilde{H}_0}P_-^\kappa$ and $\tilde{H}_0$ denotes the extension of $H_0$ to the multiplication operator on $\mathcal{H}_0 := L^2(\mathbb{R}, K, d\lambda)$, $P_+$ the projection of $\mathcal{H}_0$ onto $\mathcal{H}_{0,+}$, $Q_+$ the projection of $\mathcal{H}_0$ onto the Hardy space $\mathcal{H}_2^\kappa$ (see e.g. Eisenberg et al [6], Strauss [7], see also [15]).

The truncated evolution is a strongly continuous contraction semigroup on $P_+\mathcal{H}_2^\kappa$ of the Toeplitz type (see Strauss [7]). As it is well-known, each point $\zeta \in \mathbb{C}_-$ is an eigenvalue of the generator of this semigroup and the corresponding eigenspace is given by $\{P_+f : f \in \mathcal{H}_2^\kappa, f(\lambda) := k(\lambda - \zeta)^{-1}, k \in K\}$, i.e. the dimension of the eigenspace of $\zeta$ coincides with $\dim K$.

A crucial question is whether Gamov vectors of this type can be uniquely associated to eigenantilinear forms of $H_0^\kappa$. A first answer is that one has to select the poles of $L_+(\cdot)^{-1}$ resp. of $S_K(\cdot)$. However, it remains the question which values of $k \in K$ have to be chosen. To solve this problem we consider an appropriate dense subset of $\mathcal{G}$ by means of the Hardy space $\mathcal{H}_2^\kappa$. 

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LEMMA 4. The inclusions
\[ \mathcal{G} \cap P_+ \mathcal{H}_+^2 \subset P_+ \mathcal{H}_+^2 \subset \mathcal{H}_{0,+} \]
are dense inclusions w.r.t. the Hilbert topology of \( \mathcal{H}_{0,+} \).

Proof. The density of the inclusion \( P_+ \mathcal{H}_+^2 \subset \mathcal{H}_{0,+} \) is a standard result. To prove the density of the first inclusion we consider all Schwartz functions \( \mathbb{R}_+ \ni \lambda \to v(\lambda) \in \mathcal{E} \) with compact support. Then the functions \( \lambda \to M(\lambda)v(\lambda) \) are \( \mathcal{K} \)-valued Schwartz functions with compact support in \( \mathbb{R}_+ \). The linear span of all these functions is denoted by \( \mathcal{U} \subset \mathcal{H}_{0,+} \). \( \mathcal{U} \) is dense in \( \mathcal{H}_{0,+} \). Note that the functions of \( \mathcal{U} \) can be considered also as functions on \( \mathbb{R} \), i.e. as functions from \( \mathcal{H}_0 \) and that the functions \( \lambda \to (M(\lambda)v(\lambda))' \) belong to \( \mathcal{H}_0 \), too. Then
\[ (F^{-1}u(z)) := \int_0^\infty e^{iz\lambda}u(\lambda)d\lambda, \quad z \in \mathbb{C}, \ u \in \mathcal{U}, \]
is holomorphic on \( \mathbb{C} \) and one has \( F^{-1}u \in \mathcal{H}_+^2 \) and \( (F^{-1}u)(z) \in \mathcal{A} \). Then \( w := P_+F^{-1}u \in P_+\mathcal{H}_+^2 \) and \( w \in \text{dom} \mathcal{H}_0 \). Moreover, \( P_+F^{-1}\mathcal{U} \) is dense in \( P_+\mathcal{H}_+^2 \). Put
\[ s(\lambda) := M(\lambda)^{-1}w(\lambda), \quad \lambda \in \mathbb{R}_+. \]
Then \( s(z) := M(z)^{-1}w(z) \in \mathcal{E} \) is holomorphic on \( \mathbb{C}_{<0} \). Now we define \( g(\cdot) \) by
\[ g(z) := L_+(z)s(z) \in \mathcal{E}, \quad z \in \mathbb{C}_{<0} \setminus \mathcal{P}. \]
Then \( z \to g(z) \) is holomorphic on \( \mathbb{C}_{<0} \setminus \mathcal{P} \) and
\[ w(z) = M(z)L_+(z)^{-1}g(z). \]
This means that \( g \) is the \( \mathcal{E} \)-representor of \( w \). Therefore one obtains \( w \in \mathcal{G} \) because also \( \lambda \to \lambda g(\lambda) \) is an \( \mathcal{E} \)-representor. Hence \( P_+F^{-1}\mathcal{U} \subset \mathcal{G} \cap P_+\mathcal{H}_+^2 \) follows, i.e. a fortiori also \( \mathcal{G} \cap P_+\mathcal{H}_+^2 \) is dense in \( P_+\mathcal{H}_+^2 \). \( \square \)

Now we introduce in \( \mathcal{G} \cap P_+\mathcal{H}_+^2 \) a third topology by
\[ P_+\mathcal{H}_+^2 \ni f \to \langle f \rangle := \|P_+^{-1}f\|_{\mathcal{H}_0}. \]
Note that the projection \( P_+ \) restricted to \( \mathcal{H}_+^2 \) with image \( P_+\mathcal{H}_+^2 \) is a bijection (see e.g. [13]). Moreover, \( P_+\mathcal{H}_+^2 \) is a Hilbert space w.r.t. the norm \( \langle \cdot \rangle \).

LEMMA 5. The inclusion \( \mathcal{G} \cap P_+\mathcal{H}_+^2 \subset P_+\mathcal{H}_+^2 \) is also dense w.r.t. the Hilbert norm \( \langle \cdot \rangle \) of \( P_+\mathcal{H}_+^2 \).

Proof. It is obvious from the proof of Lemma 4 because \( P_+^{-1}(\mathcal{G} \cap P_+\mathcal{H}_+^2) \supset F^{-1}\mathcal{U} \) which is dense in \( \mathcal{H}_+^2 \) w.r.t. its Hilbert norm. \( \square \)

The next step to obtain associated Gamov vectors for the eigenantilinear forms \( s_0^\varphi(\zeta_0,e_0) \) is to restrict them from \( \mathcal{G} \) to \( \mathcal{G} \cap P_+\mathcal{H}_+^2 \).

COROLLARY 3. The restricted eigenantilinear form \( s_0^\varphi(\zeta_0,e_0)|_{\mathcal{G} \cap P_+\mathcal{H}_+^2} \)
\[ \mathcal{G} \cap P_+\mathcal{H}_+^2 \ni s \to 2\pi i(s(\zeta_0),k_0)_K, \quad k_0 = M(\zeta_0)e_0, \]

\[ \varphi = \sum_{\lambda \in \mathcal{K}} e^{i\lambda} \mathcal{K}_{\varphi}(\lambda). \]

\[ \langle f \rangle := \|P_+^{-1}f\|_{\mathcal{H}_0}. \]
is even continuous w.r.t. the Hilbert topology $⟨⋅⟩$ of $P_+\mathcal{H}_+^2$, i.e. it can be continuously extended to $\text{clo}_{⟨⋅⟩}(\mathcal{G} \cap P_+\mathcal{H}_+^2) = P_+\mathcal{H}_+^2$. That is, $s_0^*(ζ_0, e_0)\upharpoonright P_+\mathcal{H}_+^2$ is realized by the $P_+\mathcal{H}_+^2$-vector

$$R_+ \ni λ \mapsto \frac{k_0}{ζ_0 - λ}$$

via the relation

$$2\pi i(s(ζ_0), k_0)_K = \int_{-∞}^{∞} \left(s(λ), \frac{k_0}{ζ_0 - λ}\right)_K dλ,$$  \hspace{1cm} (30)

where in (30) the (unique) extensions of $s(⋅)$ and $λ \mapsto k_0(ζ_0 - λ)^{-1}$ onto the whole real line have to be used.

Proof. It follows immediately from the Paley-Wiener theorem. □

Corollary 3 means: the restriction to $P_+\mathcal{H}_+^2$ of the eignantilinear form $s_0^*(ζ_0, e_0)$, associated to the resonance $ζ_0$ and to the parameter vector $e_0 \in \ker L_+(ζ_0)$, which is the back transform $S_0^*(ζ_0, e_0) = (Ω_+^*)^x d_0^*(ζ_0, e_0)$ of $d_0^*(ζ_0, e_0)$ to the Hilbert space $\mathcal{H}_{0,+}$ resp. to the corresponding Gelfand triplet, yields the associated Gamov vector $λ \mapsto k_0(ζ_0 - λ)^{-1}$ where $k_0 = M(ζ_0)e_0$. Conversely, exactly those eigenvectors $λ \mapsto k(ζ_0 - λ)^{-1}$ of the Toeplitz type semigroup $T_+(⋅)$ have an extension (or "continuation") to an eignantilinear form of the extended Hamiltonian $H_0^x$ resp. to the extended Hamiltonian $H^x$ if $ζ = ζ_0$ is a resonance and $k = k_0 = M(ζ_0)$ with $e_0 \in \ker L_+(ζ_0)$. That is, Corollary 3 solves the problem of the selection of the "true" Gamov vectors.

Obviously, the parameter spaces $M(ζ_0)\ker L_+(ζ_0)$ resp. $\ker L_+(ζ_0)$ can be expressed by the scattering matrix $S_ξ(⋅)$ at $ζ_0$. Moreover, if $ζ_0$ is a simple pole of $S_ξ(⋅)$ then the parameter space can be calculated using the Laurent expansion of $S_ξ(⋅)$ at $ζ_0$. Note that $S_ξ(z)^{-1} = L_-(z)^{-1}L_+(z)$ is holomorphic in $C_- \setminus P$, hence at $ζ_0$.

**PROPOSITION 1.** One has $\ker L_+(ζ_0) = \ker S_ξ(ζ_0)^{-1}$. If $ζ_0$ is a simple pole of $S_ξ$ then

$$\ker L_+(ζ_0) = \text{ima}\{\text{Res}_{z=ζ_0} S_ξ(z)\}.$$  \hspace{1cm} (31)

Proof. An easy calculation gives

$$\ker L_+(ζ_0) = \text{ima} L_{-1} = \text{ima}(L_{-1}L_-(ζ_0)),$$

where $L_{-1} = \text{Res}_{z=ζ_0} L_+(z)^{-1}$. This gives (31). Note that $L_-(ζ_0)^{-1}$ exists and is bounded. □

**REMARK 3.** $ζ_0$ is a simple pole of $L_+(⋅)^{-1}$, i.e. of $S_ξ(⋅)$, iff $ζ_0$ is a simple pole of $S_ξ(⋅)$ because $S_{-1} \neq 0$ iff $M(ζ_0)S_{-1}M(ζ_0)^* \neq 0$, where $S_{-1}$ denotes the residuum of $ζ_0$ as a pole of $L_+(⋅)^{-1}$. This is obvious from (18) because $M(ζ_0)S_{-1}M(ζ_0)^* = 0$ implies, according to (vi), $S_{-1}M(ζ_0)^* = 0$ or $M(ζ_0)S_{-1}^* = 0$, hence $S_{-1} = 0 = S_{-1}$ follows. The converse is trivial.

Surprisingly it turns out that $S_ξ(⋅)$ has only simple poles in $C_-$. This is pointed out in the next section.
5 The function-theoretic characterization of the scattering matrix $S_K(\cdot)$ on the lower half plane

According to (18) the scattering matrix $z \to S_K(z)$ is meromorphic on $\mathbb{C}_{<0}$. In $\mathbb{C}_-$ there are poles at the resonances (points of $\mathcal{R}$), there are no other poles there because of $\mathcal{R} \cap \mathcal{P} = \emptyset$. Possible poles in $\mathbb{C}_+$ are at the points of $\mathcal{P}$. Since $S_K(\cdot)$ is unitary on the positive half line one has

$$S_K(z)^{-1} = S_K(\overline{z})^*, \quad z \in \mathbb{C}_{<0}. \quad (32)$$

On the upper border of $\mathbb{C}_{<0}$ (the negative half line) $S_K(\cdot)$ is holomorphic, i.e. $\lambda \to S_K(\lambda + i0)$ is holomorphic for $\lambda < 0$ because $L_{+}(\lambda + i0)^{-1} = P_{\xi}(\lambda - H)^{-1}P_{\xi}^* \mathcal{E}$ for $\lambda < 0$ and $(\lambda - H)^{-1}$ is holomorphic there. Further we have

$$L_{+}(\lambda - i0) = L_{-}(\lambda - i0) + 2\pi i M(\lambda)^* M(\lambda), \quad \lambda < 0. \quad (33)$$

That is, $L_{+}(\lambda - i0)$ is holomorphic for $\lambda < 0$, hence $L_{+}(\lambda - i0)^{-1}$ remains meromorphic for $\lambda < 0$ and there is no pole on the negative half line.

In the contrary case, if $-\lambda_0$, $\lambda_0 > 0$, is a pole then, according to Lemma 1, we have $\ker L_{+}(-\lambda_0 - i0) \supset \{0\}$, i.e. there is $e_0 \in \mathcal{E}$ such that

$$(L_{-}(-\lambda_0 - i0) + 2\pi i M(-\lambda_0)^* M(-\lambda_0)) e_0 = 0$$

or

$$(-\lambda_0 - A + \int_{0}^{\infty} \frac{M(\mu)^* M(\mu)}{\lambda_0 + \mu} d\mu + 2\pi i M(-\lambda_0)^* M(-\lambda_0)) e_0 = 0$$

hence

$$(e_0, (\lambda_0 + A) e_0) = \left( e_0, \int_{0}^{\infty} \frac{M(\mu)^* M(\mu)}{\lambda_0 + \mu} d\mu e_0 \right) + 2\pi i ||M(-\lambda_0) e_0||^2. \quad (34)$$

This implies $M(-\lambda_0) e_0 = 0$ and $e_0 = 0$.

Therefore, $S_K(\cdot)$ is holomorphic also on the lower border of $\mathbb{C}_{<0}$, i.e. $\lambda \to S_K(\lambda - i0)$ is holomorphic for $\lambda < 0$. Then from (32)

$$S_K(\lambda \pm i0)^{-1} = S_K(\lambda \mp i0)^*$$

follows, i.e. $S_K(\lambda \pm i0)$ is bounded invertible for $\lambda < 0$, but not necessarily unitary. Moreover, from (3) and (iv) we have

$$\sup_{z \in \mathbb{C}_{<0}(R)} ||S_K(z)||_K < \infty, \quad \mathbb{C}_{<0}(R) := \mathbb{C}_{<0} \cap \{z : |z| \geq R\}, \quad (35)$$

where $R$ is sufficiently large. Now it turns out that these three conditions

- $z \to S_K(z)$ is meromorphic on $\mathbb{C}_{<0}$, holomorphic on $\mathbb{R}_+$,
- there exist the norm limits $\lim_{\epsilon \to 0} S_K(\lambda \pm i\epsilon) =: S_K(\lambda \pm i0)$ for $\lambda < 0$, holomorphic on $\mathbb{R}_- := (-\infty, 0)$,
- $S_K(\cdot)$ is bounded at infinity on $\mathbb{C}_{<0}$,
are sufficient to describe the function-theoretic behaviour of $S_K(\cdot)$ on the lower half plane (note that these conditions already imply that there are at most finitely many poles in $\mathbb{C}_-$).

First we extend the scattering operator $S$, originally a unitary operator on $\mathcal{H}_{0,+}$, to a bonded operator on $\mathcal{H}_0$ using the boundary values $S_K(\lambda \pm i0)$ for $\lambda < 0$. We define the bounded operator

$$\mathcal{H}_0 \ni f \to S_{\pm}f \in \mathcal{H}_0$$

by

$$(S_{\pm}f)(\lambda) := \begin{cases} S_K(\lambda)f(\lambda), & \lambda > 0, \\ S_K(\lambda \pm i0)f(\lambda), & \lambda < 0. \end{cases}$$

Then it turns out that for vectors $g \in \mathcal{H}_2^-$ the projection of $S_-g$ onto the Hardy space $\mathcal{H}_2^+$ has a very simple form:

**THEOREM 3.** The relation

$$(Q_+S_-g)(z) = \sum_{\zeta \in R \cup P_-} S_{-1\zeta}g(\zeta), \quad z \in \mathbb{C}_+, \; g \in \mathcal{H}_2^-,$$  

(33)

is valid, where $S_{-1\zeta}$ denotes the residuum of $S_K(\cdot)$ at the pole $\zeta$ and $P_- := P \cap \mathbb{C}_-$.

**Proof.** It is given in [15], see also Gadella [16] for related calculations. \[\square\]

Since the right hand side of (33) is also well-defined on $\mathbb{C}_-$ and rational on $\mathbb{C}$ we have

**COROLLARY 4.** The poles of the scattering matrix $S_K(\cdot)$ in $\mathbb{C}_-$ are necessarily simple, the Laurent representation of $S_K(\cdot)$ in $\mathbb{C}_-$ is given by

$$S_K(z) = \sum_{\zeta \in R \cup P_-} \frac{S_{-1\zeta}}{z - \zeta} + H_K(z), \quad z \in \mathbb{C}_-, \quad (34)$$

where $z \to H_K(z)$ is holomorphic on $\mathbb{C}_-$.

**Proof.** The projection of $S_-g$ onto $\mathcal{H}_2^-$ is given by

$$(Q_-S_-g)(z) = -\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{S_-(\lambda)g(\lambda)}{\lambda - z} \, d\lambda, \quad z \in \mathbb{C}_-.$$

The right hand side of (33) is a rational function on $\mathbb{C}$, i.e. $(Q_+S_-g)(\cdot)$ has an analytic continuation into $\mathbb{C}_-$ given by this expression. On the other hand $Q_+S_-g + Q_-S_-g = S_-g$. This means: the function $\mathbb{R}_+ \ni \lambda \to S_K(\lambda)g(\lambda)$ has an analytic continuation into $\mathbb{C}_-$ given by

$$(S_-g)(z) = S_K(z)g(z) = \sum_{\zeta \in R \cup P_-} \frac{S_{-1\zeta}g(\zeta)}{z - \zeta} - \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{S_-(\lambda)g(\lambda)}{\lambda - z} \, d\lambda, \quad z \in \mathbb{C}_-.$$

This is true for all $g \in \mathcal{H}_2^-$. Putting, for example, $g(z) := k(z - i)^{-1}, \; k \in \mathcal{K}$ then one obtains (34), where the first term is the "main part" and

$$H_K(z) = \sum_{\zeta \in R \cup P_-} \frac{S_{-1\zeta}}{z - \zeta} - \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{S_-(\lambda)}{(\lambda - i)(\lambda - z)} \, d\lambda$$
is the holomorphic part of $S_K(\cdot)$ in $\mathbb{C}_-$. □

Note that, according to Proposition 1, the terms

$$z \to \frac{S_{-1,\zeta} g}{z - \zeta_0}, \quad g \in \mathcal{K}, \, \zeta_0 \in \mathcal{R}$$

for resonances are "true" Gamov vectors because for simple poles $\ker L_+(\zeta_0) = \text{ima} \text{Res}_{z=\zeta_0} S_{\zeta}(z) = \text{ima} S_{-1,\zeta_0}$.

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