Abstract

One-instanton predictions are obtained from certain non-hyperelliptic Seiberg-Witten curves derived from M-theory for $N=2$ supersymmetric gauge theories. We consider $SU(N_1)\times SU(N_2)$ gauge theory with a hypermultiplet in the bifundamental representation together with hypermultiplets in the defining representations of $SU(N_1)$ and $SU(N_2)$. We also consider $SU(N)$ gauge theory with a hypermultiplet in the symmetric or antisymmetric representation, together with hypermultiplets in the defining representation. The systematic perturbation expansion about a hyperelliptic curve together with the judicious use of an involution map for the curve of the product groups provide the principal tools of the calculations.
1. Introduction

The Seiberg-Witten approach [1] to deriving the exact low-energy properties of $N=2$ supersymmetric gauge theories depends on the following data: a curve, which for many cases represents a Riemann surface, and a preferred meromorphic one-form, the Seiberg-Witten (SW) differential $\lambda$. When dealing with a Riemann surface, one calculates the renormalized order parameters of the theory, and their duals, from

$$2\pi i a_k = \oint_{A_k} \lambda \quad \text{and} \quad 2\pi i a_{D,k} = \oint_{B_k} \lambda,$$

respectively, where $A_k$ and $B_k$ are a canonical basis of homology cycles for the Riemann surface. Given (1), the prepotential $F$ is obtained by integrating

$$a_{D,k} = \frac{\partial F}{\partial a_k}.$$  

(2)

In terms of $N=1$ superfields, the Wilson effective Lagrangian, to lowest order in the momentum expansion, is

$$L = \frac{1}{4\pi} \text{Im} \left[ \int d^4\theta \frac{\partial F(A)}{\partial A^i} \bar{A}^i + \frac{1}{2} \int d^2\theta \frac{\partial^2 F(A)}{\partial A^i \partial A^j} W^i W^j \right],$$

(3)

where $A^i$ are $N=1$ chiral superfields. Holomorphy implies that the prepotential of the Coulomb phase has the form

$$F(A) = F_{cl.}(A) + F_{1\text{-loop}}(A) + \sum_{d=1}^{\infty} \Lambda^{[I(G)-I(R)]d} F_{d\text{-inst.}}(A),$$

(4)

where $I(G)$ is the Dynkin index of the adjoint representation of the gauge group, $I(R)$ is the sum of the Dynkin indices of the matter hypermultiplets, with the sum in (4) being over the instanton expansion, and $\Lambda$ is the quantum scale.

A wide class of SW problems can be solved by means of hyperelliptic curves [2], for which methods to extract the instanton expansion, as well as strong coupling information, have been rather well developed [3, 4, 5]. However, not all SW problems lead to
hyperelliptic curves. In particular, M-theory [6]–[8] and geometric engineering [9] often lead to Riemann surfaces that are not hyperelliptic, and to varieties that are not Riemann surfaces at all. Methods required to extract explicit predictions for the prepotentials associated to non-hyperelliptic curves are just beginning to be developed. In particular, in two previous papers [10, 11], we provided a construction for the instanton expansion for SU($N$) gauge theory, with matter in the antisymmetric or symmetric representations, with explicit results for the one-instanton contribution to the prepotential. (These two cases involve non-hyperelliptic cubic curves.) It is extremely important to continue the program of computing explicit predictions for the prepotentials from M-theory and geometric engineering, and then checking these against independent microscopic field theoretic calculations [12, 13]. Such successful comparisons should increase our confidence in the ability of string theory to provide us with field theoretic information, and in the power of M-theory and geometric engineering.

This paper extends our work to three more cases, all involving non-hyperelliptic curves. We will consider the instanton expansions for (1) SU($N_1$)×SU($N_2$) gauge theory with one multiplet in the bifundamental representation, together with $K_0$ and $K_3$ hypermultiplets in the defining representation of SU($N_1$) and SU($N_2$) respectively, (2) SU($N$) gauge theory with one matter hypermultiplet in the symmetric representation, together with $N_f$ hypermultiplets in the defining representation, and (3) SU($N$) gauge theory with one matter hypermultiplet in the antisymmetric representation, together with $N_f$ hypermultiplets in the defining representation.

2. **SU($N_1$)×SU($N_2$) gauge theory**

Consider the N=2 supersymmetric gauge theory based on the gauge group SU($N_1$)×SU($N_2$), with one massless hypermultiplet in the ($N_1$, $\bar{N}_2$) bifundamental representa-
tion, together with \( K_0 \) and \( K_3 \) massless hypermultiplets in the defining representation of SU(\( N_1 \)) and SU(\( N_2 \)) respectively. The chiral multiplets in the adjoint representation of SU(\( N_1 \)) or SU(\( N_2 \)) contain a complex scalar field \( \phi_1 \) or \( \phi_2 \). Along the flat directions of the potential, \([\phi_j, \bar{\phi}_j]\) vanishes \((j = 1, 2)\), and the symmetry is broken to U(1)^{\( N_1-1 \)} × U(1)^{\( N_2-1 \)}. The \(( N_1 - 1 ) + ( N_2 - 1 )\) dimensional moduli space is parametrized classically by \( e_i \) (1 \( \leq i \leq N_1 \)) and \( \hat{e}_i \) (1 \( \leq i \leq N_2 \)), which are the eigenvalues of \( \phi_1 \) and \( \phi_2 \) respectively, and satisfy the constraints \( \sum_{i=1}^{N_1} e_i = 0 \) and \( \sum_{i=1}^{N_2} \hat{e}_i = 0 \).

The curve for this theory, derived by Witten [7], and made more explicit in ref. [14] is

\[
P_0(x) t^3 - \frac{P_1(x)}{L_1^2} t^2 + \frac{P_2(x)}{L_1^2} t - \frac{L_2^2 P_3(x)}{L_1^2} = 0, \tag{5}
\]

where

\[
\begin{align*}
P_0(x) &= x^{K_0}, & P_1(x) &= \prod_{i=1}^{N_1} (x - e_i) \\
P_2(x) &= \prod_{i=1}^{N_2} (x - \hat{e}_i), & P_3(x) &= x^{K_3}, \\
L_1^2 &= \Lambda_1^{2N_1-N_2-K_0}, & L_2^2 &= \Lambda_2^{2N_2-N_1-K_3}, \tag{6}
\end{align*}
\]

with \( \Lambda_1 \) and \( \Lambda_2 \) the quantum scales of the two gauge groups. The requirement of asymptotic freedom, and restriction to the Coulomb phase, implies that \( \Lambda_1 \) and \( \Lambda_2 \) appear with positive powers in (6). The change of variables \( t = y/(P_0(x) L_1^2) \) gives the curve

\[
y^3 - P_1(x) y^2 + L_1^2 P_0(x) P_2(x) y - L_1^4 L_2^2 P_0(x) P_3(x) = 0, \tag{7}
\]

in the form which we will analyze. The involution map

\[
y \to \frac{L_1^2 L_2^2 P_0(x) P_3(x)}{y} \tag{8}
\]

interchanges the order of the gauge groups, i.e. SU(\( N_1 \))×SU(\( N_2 \)) → SU(\( N_2 \))×SU(\( N_1 \)); this will be important in what follows. The curve (7) corresponds to a three-fold branched
covering of the Riemann sphere, with sheets one and two connected by \( N_1 \) square-root branch-cuts centered about \( x = e_i \) (\( i = 1 \) to \( N_1 \)), and sheets two and three connected by \( N_2 \) square-root branch-cuts centered about \( x = \hat{e}_i \) (\( i = 1 \) to \( N_2 \)), which is a Riemann surface of genus \( N_1 + N_2 - 2 \).

We rewrite the cubic curve (7) as

\[
y^3 + 2A(x) y^2 + B(x) y + \epsilon(x) = 0, \quad (9)
\]

where

\[
\epsilon(x) = -L_1^4 L_2^2 P_0^2(x) P_3(x) , \quad A(x) = -\frac{1}{2} P_1(x) , \quad B(x) = L_1^2 P_0(x) P_2(x). \quad (10)
\]

As in our previous work [10, 11], we will solve the prepotential for this problem by means of a systematic expansion in powers of \( \epsilon \), with the zeroth-order term being a hyperelliptic curve. The solutions to (9), correct to \( O(\epsilon) \), are

\[
y_1 = -A - r - \frac{\epsilon}{2r(A + r)} , \quad y_2 = -A + r + \frac{\epsilon}{2r(A - r)} , \quad y_3 = -\frac{\epsilon}{B}. \quad (11)
\]

where \( r \equiv \sqrt{A^2 - B} \). Notice that to this order, only sheets labelled by \( y_1 \) and \( y_2 \) are connected by branch-cuts, while \( y_3 \) is disconnected. (However, the involution map (8) will enable us to discuss the effects of the connection of sheets \( y_2 \) and \( y_3 \) by branch-cuts.)

The SW differential is

\[
\lambda = x \frac{dy}{y}, \quad (12)
\]

which takes a different value on each of the Riemann sheets. The perturbative expansion in \( \epsilon \), (11), induces a comparable expansion for the SW differential. For example, on sheet one

\[
\lambda_1 = (\lambda_1)_I + (\lambda_1)_{II} + ... \quad (13)
\]
where
\[(\lambda_1)_I = dx \left( \frac{A' - B'}{2B} + \frac{B'}{2B} \right), \quad (14)\]
is the usual expression for the SW differential for a hyperelliptic curve, while the \(O(\epsilon)\) correction is
\[(\lambda_1)_{II} = dx \partial_x \left( \frac{\epsilon}{2Br} + \frac{\epsilon r}{B^2} \right). \quad (15)\]

Equation (8) maps the sheets as follows: \(y_1 \leftrightarrow y_3\) and \(y_2 \leftrightarrow y_2\). Using \(y_3 = L_1^2L_2^2P_0P_3/y_1\), we may express the expansion for \(\lambda_3\) in terms of a comparable one for \(\lambda_1\), for which \(SU(N_1) \leftrightarrow SU(N_2)\), with the approximation (11) exhibiting the branch-cuts which connect sheets 2 and 3.

Given the SW differential to the required accuracy, we are able to compute the order parameters and dual order parameters to the comparable order in \(\epsilon\). Define a set of canonical homology cycles \(A_k\) and \(B_k\) for Riemann sheets \(y_1\) and \(y_2\), and cycles \(\hat{A}_k\) and \(\hat{B}_k\) for Riemann sheets \(y_2\) and \(y_3\). The cycle \(A_k\) is chosen to be a simple contour enclosing the slit centered about \(e_k\) (\(k = 1\) to \(N_1\)) on sheet 1, while \(\hat{A}_k\) (\(k = 1\) to \(N_2\)) similarly encloses the slit centered about \(\hat{e}_k\) on sheet 3. Then
\[2\pi i a_k = \oint_{A_k} \lambda_1 \quad \text{and} \quad 2\pi i \hat{a}_k = \oint_{\hat{A}_k} \lambda_3, \quad (16)\]
A calculation essentially identical to that of ref. [3] or of sec. 4 of ref. [10, 11] gives
\[a_k = e_k + \frac{1}{4}L_1^2 \frac{\partial S_k}{\partial x}(e_k) + \cdots \quad (k = 1 \text{ to } N_1), \]
\[\hat{a}_k = \hat{e}_k + \frac{1}{4}L_2^2 \frac{\partial \hat{S}_k}{\partial x}(\hat{e}_k) + \cdots \quad (k = 1 \text{ to } N_2). \quad (17)\]
where
\[S_k(x) = \frac{4x^{K_0} \prod_{i=1}^{N_2} (x - \hat{e}_i)}{\prod_{i \neq k}^{N_1} (x - e_i)^2} \quad (k = 1 \text{ to } N_1) \quad (18)\]
\[
\hat{S}_k(x) = \frac{4x^{K_3} \prod_{i=1}^{N_1} (x - e_i)}{\prod_{i \neq k} (x - \hat{e}_i)^2} \quad (k = 1 \text{ to } N_2). \tag{19}
\]

We may also compute the dual order parameters,

\[
2\pi i a_{D,k} = \oint_{B_k} \lambda_1 \quad \text{and} \quad 2\pi i \hat{a}_{D,k} = \oint_{\hat{B}_k} \lambda_3. \tag{20}
\]

where the \(B_k\) are curves going from \(x_1^-\) to \(x_k^-\) on the first sheet and from \(x_k^-\) to \(x_1^-\) on the second. (The cut centered about \(e_k\) goes from \(x_k^-\) to \(x_k^+\).) Analogously, \(\hat{B}_k\) are cycles which go from sheet 2 to 3. The branch cuts \(x_k^-\) are computed as in refs. [10] and [11]. A calculation along the lines of sec. 5 of ref. [10] gives, including the \(O(\epsilon)\) correction to \(\lambda_1\),

\[
2\pi i a_{D,k} = [2N_1 - N_2 - K_0 + 2 \log (-L_1)] a_k
- 2 \sum_{j \neq k} (a_k - a_j) \log (a_k - a_j) + \sum_{i=1}^{N_2} (a_k - \hat{a}_i) \log (a_k - \hat{a}_i) + K_0 a_k \log a_k
+ L_1^2 \left[ \frac{1}{2} \sum_{j=1}^{N_1} \frac{\partial S_j}{\partial x}(a_j) + \frac{1}{4} \frac{\partial S_k}{\partial x}(a_k) - \frac{1}{2} \sum_{i \neq k} \frac{S_i(a_i)}{a_k - a_i} \right]
+ L_2^2 \left[ \frac{1}{4} \sum_{j=1}^{N_1} \frac{\partial \hat{S}_j}{\partial x}(\hat{a}_j) + \frac{1}{4} \sum_{i=1}^{N_2} \frac{\hat{S}_i(\hat{a}_i)}{a_k - \hat{a}_i} \right] + \ldots \tag{21}
\]

Considerations analogous to those of Appendix D of ref. [10] give us the identities

\[
\sum_{j=1}^{N_1} \frac{\partial S_j}{\partial x}(e_j) = 0, \quad \sum_{j=1}^{N_2} \frac{\partial \hat{S}_j}{\partial x}(\hat{e}_j) = 0, \tag{22}
\]

implying \(\sum_{i=1}^{N_1} a_i = \sum_{i=1}^{N_1} e_i\) and \(\sum_{i=1}^{N_2} \hat{a}_i = \sum_{i=1}^{N_2} \hat{e}_i\) to the order that we are working.

Combining eqs. (21), (22), and identities analogous to (6.8) of ref. [10] gives

\[
2\pi i a_{D,k} = [2N_1 - N_2 - K_0 + 2 \log (-L_1)] a_k
- 2 \sum_{j \neq k} (a_k - a_j) \log (a_k - a_j) + \sum_{i=1}^{N_2} (a_k - \hat{a}_i) \log (a_k - \hat{a}_i) + K_0 a_k \log a_k
+ \frac{L_1^2}{4} \frac{\partial}{\partial a_k} \sum_{i=1}^{N_1} S_i(a_i) + \frac{L_2^2}{4} \frac{\partial}{\partial \hat{a}_k} \sum_{i=1}^{N_2} \hat{S}_i(\hat{a}_i) \quad (k = 1 \text{ to } N_1). \tag{23}
\]
Using the involution map (8) we obtain $2\pi i \hat{a}_{D,k}$ from (23) with the substitutions

$$a_i \longleftrightarrow \hat{a}_i, \quad L_1 \longleftrightarrow L_2, \quad K_0 \longleftrightarrow K_3 \quad S_i \longleftrightarrow \hat{S}_i.$$  \hspace{1cm} (24)

The prepotential, which satisfies

$$a_{D,k} = \frac{\partial F}{\partial a_k} \quad (k = 1 \text{ to } N_1), \quad \hat{a}_{D,k} = \frac{\partial F}{\partial \hat{a}_k} \quad (k = 1 \text{ to } N_2),$$  \hspace{1cm} (25)

is

$$\mathcal{F}_{\text{1-loop}} = \frac{i}{8\pi} \left[ \sum_{i,j=1}^{N_1} (a_i - a_j)^2 \log (a_i - a_j)^2 + \sum_{\alpha,\beta=1}^{N_2} (\hat{a}_\alpha - \hat{a}_\beta)^2 \log (\hat{a}_\alpha - \hat{a}_\beta)^2 ight. - \sum_{i=1}^{N_1} \sum_{\alpha=1}^{N_2} (a_i - \hat{a}_\alpha)^2 \log (a_i - \hat{a}_\alpha)^2 - K_0 \sum_{i=1}^{N_1} a_i^2 \log a_i^2 - K_3 \sum_{\alpha=1}^{N_2} \hat{a}_\alpha^2 \log \hat{a}_\alpha^2 \bigg],$$  \hspace{1cm} (26)

and

$$2\pi i \mathcal{F}_{\text{1-inst}} = \frac{1}{4} L_1^2 \sum_{i=1}^{N_1} S_i (a_i) + \frac{1}{4} L_2^2 \sum_{i=1}^{N_2} \hat{S}_i (\hat{a}_i),$$  \hspace{1cm} (27)

where

$$S_k(a_k) = \frac{4a_k^{K_0} \prod_{i=1}^{N_2} (a_k - \hat{a}_i)}{\prod_{i \neq k}^{N_1} (a_k - a_i)^2} \quad (k = 1 \text{ to } N_1),$$

$$\hat{S}_k(\hat{a}_k) = \frac{4\hat{a}_k^{K_3} \prod_{i=1}^{N_1} (\hat{a}_k - a_i)}{\prod_{i \neq k}^{N_2} (\hat{a}_k - \hat{a}_i)^2} \quad (k = 1 \text{ to } N_2).$$  \hspace{1cm} (28)

Since $S_k(a_k)$ and $\hat{S}_k(\hat{a}_k)$ depend on both $a_i$ and $\hat{a}_i$, eq. (27) is not just the naive sum of instanton contributions from each subgroup.

The one-loop prepotential (26) agrees with the perturbation theory result. Further, we have one check available from the work of D’Hoker and Phong (see the last paper in ref. [4]), who consider various decoupling limits for $N=2$ SU($N$) gauge theory with a massive hypermultiplet in the adjoint representation. In their eq. (6.17) ff. they give the one-instanton correction for SU($N$)$\times$SU($N$) theory with a bifundamental representation,
$K_0 = K_3 = 0$, and a single quantum-scale. Our result (27-28) agrees with theirs, up to an overall constant for the quantum scales.

3. SU($N$) with a symmetric tensor flavor and $N_f$ fundamentals

In this section, we derive the one-instanton contribution to the prepotential for the $N=2$ SU($N$) gauge theory with one matter hypermultiplet in the rank two symmetric tensor representation (with mass $m$) together with $N_f$ matter hypermultiplets in the defining representation (with masses $m_k$, $k = 1, \cdots, N_f$). Asymptotic freedom restricts $N_f$ to be less than $N-2$. (In previous work [11], we obtained the one-instanton prediction for $N_f = 0$.) The Seiberg-Witten curve for this theory, constructed by Landsteiner, Lopez, and Lowe [15], is

$$y^3 + f(x) y^2 + L^2 x^2 j(x)f(-x)y + L^6 x^6 j^2(x)j(-x) = 0,$$

where

$$f(x) = \prod_{i=1}^{N}(x-e_i), \quad j(x) = \prod_{j=1}^{N_f}(x-f_j), \quad \text{and} \quad L^2 = \Lambda^{N-2-N_f}.$$  

The $e_i$ parametrize the classical moduli space, and the $f_k$ are related to the masses via $f_k = \frac{1}{2}m - m_k$. (We independently derived the curve for this theory for $m = m_k = 0$ using R-symmetry, and checking against M-theory.) The curve (29) has the involution

$$y \rightarrow \frac{L^4 x^4 j(x)j(-x)}{y}, \quad x \rightarrow -x.$$  

We begin by calculating the prepotential when the symmetric hypermultiplet is massless ($m = 0$). First, we define the residue function

$$S_k(x) = \frac{4(-1)^N x^2 \prod_{j=1}^{N_f}(x-f_j) \prod_{i=1}^{N}(x+e_i)}{\prod_{i \neq k}(x-e_i)^2}.$$  

8
A calculation along the lines of refs. [3, 10, 11] gives the renormalized order parameters

\[ a_k = e_k + \frac{L^2}{4} \frac{\partial S_k}{\partial x} (e_k) + \ldots, \quad (k = 1 \text{ to } N). \]  

(33)

The dual order parameters can then be computed in terms of \( a_k \) as

\[ 2\pi i a_{D,k} = [N - N_f - 2 + 2 \log L + (N + 2) \log (-1)] a_k \]

\[ -2 \sum_{j \neq k} (a_k - a_j) \log (a_k - a_j) + \sum_{i=1}^{N} (a_k + a_i) \log (a_k + a_i) \]

\[ + 2 a_k \log a_k + \sum_{i=1}^{N_f} (a_k - f_i) \log (a_k - f_i) + \frac{L^2}{4} \frac{\partial a_k}{\partial a_k} \sum_{i=1}^{N} S_i(a_i). \]  

(34)

This enables us to integrate (2), and obtain the instanton expansion in (4), accurate to one-instanton. We find that \( \mathcal{F}_{1\text{-loop}} \) agrees with perturbation theory, and

\[ 2\pi i \mathcal{F}_{1\text{-inst}} = \frac{1}{4} \sum_{i=1}^{N} S_i(a_i), \]  

(35)

where

\[ S_k(a_k) = \frac{4(-1)^N a_k^2 \prod_{j=1}^{N_f} (a_k - f_j) \prod_{i=1}^{N} (a_k + a_i)}{\prod_{i \neq k} (a_k - a_i)^2}. \]  

(36)

Equation (36) reduces to our previous result [11] for \( N_f = 0 \).

For a symmetric hypermultiplet with mass \( m \), one shifts \( a_k \to a_k + \frac{1}{2}m \) in \( \mathcal{F}(a) \). Thus, eq. (35) remains valid, but with

\[ S_k(a_k) = \frac{4(-1)^N (a_k + m/2)^2 \prod_{j=1}^{N_f} (a_k + m_j) \prod_{i=1}^{N} (a_k + a_i + m)}{\prod_{i \neq k} (a_k - a_i)^2}. \]  

(37)

Equation (37) has all the required double-scaling limits as \( m \) or \( m_k \to \infty \).

4. SU(\( \text{N} \)) with an antisymmetric tensor flavor and \( N_f \) fundamentals

In this section, we derive the one-instanton contribution to the prepotential for the N=2 SU(\( \text{N} \)) gauge theory with one matter hypermultiplet in the rank two antisymmetric
tensor representation (with mass $m$) together with $N_f$ matter hypermultiplets in the defining representation (with masses $m_k$, $k = 1, \cdots, N_f$). Asymptotic freedom restricts $N_f$ to be less than $N + 2$. (The one-instanton prediction for $N_f = 0$ was presented in ref. [10].)

The Seiberg-Witten curve for this theory was derived by Landsteiner, Lopez, and Lowe [15]. Through a redefinition of $y$, their curve can be written as

$$y^3 + [x^2 f(x) + xB + 3A]y^2 + L^2 j(x)[x^2 f(-x) - xB + 3A]y + L^6 j^2(x)j(-x) = 0, \quad (38)$$

where $f(x)$ and $j(x)$ are defined in (30), and

$$L^2 = \Lambda^{N+2-N_f}, \quad A = L^2 \prod_{j=1}^{N_f} (-f_j), \quad B = L^2 \sum_{j=1}^{N_f} \prod_{l \neq j} (-f_l). \quad (39)$$

The curve (38) has the involution

$$y \rightarrow \frac{L^4 j(x) j(-x)}{y}, \quad x \rightarrow -x. \quad (40)$$

When $N_f = 0$, eq. (39) implies $A = L^2$, $B = 0$, and the curve (38) reduces to that given in [8].

When the matter hypermultiplets are massless ($m = 0$, $m_k = 0$), the form of the curve simplifies. For $N_f = 1$, eq. (39) yields $A = 0$, $B = L^2$. For $N_f \geq 2$, eq. (39) yields $A = B = 0$, and the curve simplifies to

$$y^3 + x^2 f(x) y^2 + L^2 x^{N_f+2} f(-x)y + L^6 (-1)^{N_f} x^{3N_f} = 0. \quad (41)$$

We independently derived the curve (41).

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6Our definitions of $A$ and $B$ differ from ref. [15] by the signs in front of the $f_j$. These changes are necessary for the one-instanton contribution to the prepotential to make sense (no logarithmic dependence), and to agree with known results in overlapping cases (see below).
First, we calculate the prepotential for a massless antisymmetric flavor \((m = 0)\). We introduce the residue functions

\[
S_k(x) = \frac{4(-1)^N \prod_{j=1}^{N_f} (x - f_j) \prod_{i=1}^N (x + e_i)}{x^2 \prod_{i \neq k} (x - e_i)^2}
\]

\[
S_0(x) = \frac{4(-1)^N \prod_{j=1}^{N_f} (x - f_j) \prod_{i=1}^N (x + e_i)}{\prod_{i=1}^N (x - e_i)^2}
\]

\[
L^2 R_k(x) = \frac{3A + Bx}{x^2 \prod_{i \neq k} (x - e_i)}
\]

where, as before, the \(e_i\) parametrize the classical moduli space, and \(f_k = \frac{1}{2} m - m_k\). The renormalized order parameters are calculated to be

\[
a_k = e_k + L^2 \left( \frac{1}{4} \frac{\partial S_k}{\partial x}(e_k) - R_k(e_k) \right) + \ldots, \quad (k = 1 \text{ to } N).
\]

(45)

Following the strategy of refs. [10, 11], one obtains the dual order parameters

\[
2\pi i \ a_{D,k} = [N - N_f + 2 + 2 \log L + (N + 2) \log (-1)] a_k - 2 \sum_{j \neq k} (a_k - a_j) \log (a_k - a_j) + \sum_{i=1}^N (a_k + a_i) \log (a_k + a_i) - 2 a_k \log a_k
\]

\[
+ \sum_{i=1}^{N_f} (a_k - f_i) \log (a_k - f_i) + L^2 \frac{\partial}{\partial a_k} \left( \frac{1}{4} \sum_{i=1}^N S_i(a_i) - \frac{1}{2} S_0(0) \right).
\]

(46)

Integrating this expression, we find that the one-loop prepotential \(F_{1-loop}\) agrees with the perturbation theory, and that the one-instanton contribution to the prepotential is given by

\[
2\pi i F_{1-inst.} = \frac{1}{4} \sum_{k=1}^N S_k(a_k) - \frac{1}{2} S_0(0),
\]

(47)

where

\[
S_k(a_k) = \frac{4(-1)^N \prod_{j=1}^{N_f} (a_k - f_j) \prod_{i=1}^N (a_k + a_i)}{a_k^2 \prod_{i \neq k} (a_k - a_i)^2}.
\]
\[ S_0(0) = \frac{4(-1)^N \prod_{j=1}^{N_f} (-f_j)}{\prod_{i=1}^N a_k}. \]  

Equations (47-48) reduce to our previous result [10] for \( N_f = 0 \).

The result for an antisymmetric hypermultiplet with mass \( m \) is obtained by shifting \( a_i \rightarrow a_i + \frac{1}{2}m \) in \( \mathcal{F}(a) \). Thus, eq. (47) remains valid, but with

\[
S_k(a_k) = \frac{4(-1)^N \prod_{j=1}^{N_f} (a_k + m_j) \prod_{i=1}^N (a_k + a_i + m)}{(a_k + \frac{1}{2}m)^2 \prod_{i \neq k} (a_k - a_i)^2},
\]

\[
S_0(0) = \frac{4(-1)^N \prod_{j=1}^{N_f} (m_j - \frac{1}{2}m)}{\prod_{i=1}^N (a_k + \frac{1}{2}m)}. \tag{49}
\]

Equation (47) may be compared with previously available results for SU(2) and SU(3) [3]. The SU(2) theory with one antisymmetric hypermultiplet of mass \( m \) and \( N_f \) hypermultiplets in the defining representation with masses \( m_k \) is equivalent to SU(2) with \( N_f \) hypermultiplets in the defining representation with masses \( m_k \). For \( N_c = 2 \), we have checked that one-instanton prepotential (47) is equal to that given in eq. (4.33b) of [3] for \( N_f = 1 \) (with the change of scale \( L^2 = \frac{1}{16} \tilde{\Lambda}_{\text{DKP}}^2 \)), and differs by a constant (namely, \( -2 \) and \( m - 2 \sum_{k=1}^3 m_k \) respectively) for \( N_f = 2 \) and 3 (using \( a_1 + a_2 = 0 \)). Also, SU(3) with one antisymmetric hypermultiplet of mass \( m \) and \( N_f \) fundamental hypermultiplets with masses \( m_k \) is equivalent to SU(3) with \( N_f + 1 \) defining hypermultiplets with masses \( m'_k \). For \( N_c = 3 \), we found that eq. (47) is equal to eq. (4.33b) of [3] for \( N_f = 1 \) and 2 (with the change of scale \( L^2 = \frac{1}{16} \tilde{\Lambda}_{\text{DKP}}^2 \) and using \( a_1 + a_2 + a_3 = 0 \), and differs by a constant (namely, \( -2 \) and \( m - 2 \sum_{k=1}^4 m_k \) respectively) for \( N_f = 3 \) and 4, provided that \( m'_k = m_k \) for \( k = 1, \cdots, N_f \) and \( m'_{N_f+1} = -m \).
5. Concluding remarks

In this paper, we have derived the one-instanton contribution to the prepotential for N=2 supersymmetric SU(N) gauge theory with a matter hypermultiplet in the symmetric or antisymmetric representation, together with \( N_f \) hypermultiplets in the defining representation from the cubic non-hyperelliptic curves obtained from M-theory. We also studied the comparable expansion for SU\((N_1) \times SU(N_2)\) with one hypermultiplet in the bifundamental representation, and \( K_0 \) and \( K_3 \) massless hypermultiplet in the defining representations of SU\((N_1)\) and SU\((N_2)\) respectively, using the non-hyperelliptic curve derived from M-theory by Witten [7]. This latter calculation makes essential use of the involution map (8) in order to obtain the contributions of the complete set of periods to the prepotential. Thus, the techniques of section two of this paper generalizes the methods described in refs. [10] and [11].

What is striking about the one-instanton contribution to the prepotential \( F_{1\text{-inst.}} \) obtained from our work and that of DKP [3, 4] is its remarkable universality of form when expressed in terms of the renormalized order parameters \( a_k \). In all cases one can express it as

\[
2\pi i F_{1\text{-inst.}} = \frac{1}{4} \sum_{k=1}^{N} S_k(a_k) - \frac{1}{2} S_0(0),
\]

(50)

where the particular group and representation content appear only in the form of the residue functions \( S_k(a_k) \) and \( S_0(0) \), and the latter depend only on the leading order (in \( L \)) coefficients of the hyperelliptic approximation. \( S_0(0) \) vanishes if there is no second order pole at the origin.) This universality of form is badly hidden when \( F_{1\text{-inst.}} \) is expressed in terms of SU\((N)\)-invariant moduli. It is our opinion that the universality of form of \( F_{1\text{-inst.}} \) when expressed in terms of the renormalized order parameters has not been adequately
explained (see, however, ref. [5] for progress on this issue).

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