Sums of powers of integers and generalized Stirling numbers of the second kind

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Abstract

By applying the Newton-Gregory expansion to the polynomial associated with the sum of powers of integers $S_k(n) = 1^k + 2^k + \cdots + n^k$, we derive a couple of infinite families of explicit formulas for $S_k(n)$. One of the families involves the $r$-Stirling numbers of the second kind $\{k\}_{j}^{r}$, $j = 0, 1, \ldots, k$, while the other involves their duals $\{k\}_{j}^{-r}$, with both families of formulas being indexed by the non-negative integer $r$. As a by-product, we obtain three additional formulas for $S_k(n)$ involving the numbers $\{k\}_{j}^{n+m}$, $\{k\}_{j}^{n-m}$ (where $m$ is any given non-negative integer), and $\{k\}_{j}^{k-j}$, respectively. Moreover, we provide a formula for the Bernoulli polynomials $B_k(x - 1)$ in terms of $\{k\}_{x}$ and the harmonic numbers.

1 Introduction

Following Broder [5, Equation 57] (see also Carlitz [6, Equation (3.2)]) we define the generalized (or weighted) Stirling numbers of the second kind by

$$R_{k,j}(x) = \sum_{i=0}^{k-j} \binom{k}{i} \binom{k-i}{j} x^i,$$

where $x$ stands for any arbitrary real or complex value, and where the $\{k\}_{j}$’s are the ordinary Stirling numbers of the second kind [3]. Note that $R_{k,j}(x)$ is a polynomial in $x$ of degree $k - j$ with leading coefficient $\binom{k}{j}$ and constant term $R_{k,j}(0) = \binom{k}{j}$. Furthermore, we have that $R_{k,j}(1) = \binom{k+1}{j+1}$. In general, when $x$ is the non-negative integer $r$, $R_{k,j}(r)$ becomes the $r$-Stirling number of the second kind $\{k\}_{j+r}$ [5]. A combinatorial interpretation of the polynomial $R_{k,j}(x)$ is given in [5, Theorem 27] (see also the definition provided by Bényi and Matsusaka in [1, Definition 2.13]).

For convenience and notational simplicity, in this paper we employ the notation $\{k\}_{j}^{r}$ to refer to Broder’s $r$-Stirling numbers of the second kind $\{k\}_{j+r}^{r}$. The former notation has been used recently by Ma and Wang in [14] (see also [1] and [15]). The numbers $\{k\}_{j}^{r}$ are then given by

$$\{k\}_{j}^{r} = \sum_{i=0}^{k-j} \binom{k}{i} \binom{k-i}{j} r^i,$$

integer $r \geq 0$.

Likewise, adopting the notation in [14], we define the counterpart or dual of $\{k\}_{j}^{r}$ for negative integer $r$ as

$$\{k\}_{j}^{-r} = \sum_{i=0}^{k-j} (-1)^i \binom{k}{i} \binom{k-i}{j} r^i,$$

integer $r \geq 0$. 

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Alternatively, \( \{k\}_{j} \) and \( \{k\}_{j}^{r} \) can be equivalently expressed in the form

\[
\{k\}_{j}^{r} = \frac{1}{j!} \sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} (i+r)^{k}, \quad \text{integer } r \geq 0,
\]

and

\[
\{k\}_{j}^{r} = \frac{1}{j!} \sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} (i-r)^{k}, \quad \text{integer } r \geq 0,
\]

respectively. Clearly, both \( \{k\}_{j}^{r} \) and \( \{k\}_{j}^{r} \) reduce to \( \{k\}_{j} \) when \( r = 0 \). It is to be noted that the numbers \( \{k\}_{j}^{r} \) were introduced and studied by Koutras under the name of non-central Stirling numbers of the second kind and denoted by \( S_r(k, j) \) (see [13, Equations (2.5) and (2.6)]).

On the other hand, for non-negative integer \( k \), let \( S_k(n) \) denote the sum of \( k \)-th powers of the first \( n \) positive integers

\[
S_k(n) = 1^k + 2^k + \cdots + n^k,
\]

with \( S_k(0) = 0 \) for all \( k \). As is well known, \( S_k(n) \) can be expressed in terms of the Stirling numbers of the second kind as (see, e.g., [18])

\[
S_k(n) = -\delta_{k,0} + \sum_{j=0}^{k} j! \binom{n+1}{j+1} \{k\}_{j},
\]

where \( \delta_{k,0} \) is the Kronecker delta, which ensures that \( S_0(n) = n \). Additionally, \( S_k(n) \) admits the following variant of (3):

\[
S_k(n) = \sum_{j=1}^{k+1} (j-1)! \binom{n}{j} \{k+1\}_{j} = \sum_{j=0}^{k} j! \binom{n}{j+1} \{k+1\}_{j+1},
\]

(see, e.g., [20, Equation (9)], [7], [11, Corollary 2], and [8, Theorem 5]). The first expression in (4) can be readily obtained from the exponential generating function [4, Equation (11)]

\[
\sum_{n=1}^{\infty} \left(1^k + 2^k + \cdots + n^k \right) \frac{x^n}{n!} = e^x \sum_{j=1}^{k+1} \frac{1}{j} \binom{k+1}{j} x^j.
\]

Of course, (3) and (4) are equivalent formulas. Indeed, it is a simple exercise to convert (3) into (4), and vice versa, by means of the recursion \( \{k\}_{j} = j\{k-1\}_{j} \) and \( \{k-1\}_{j-1} \) and the well-known combinatorial identity \( \binom{n}{j+1} + \binom{n}{j} = \binom{n+1}{j+1} \).

Incidentally, it is worthwhile to mention that, in his 1928 Monthly article [10], Ginsburg wrote down explicitly the first few instances of (4) for \( k = 2, 3, 4, 5 \) in terms of the binomial coefficients
\( \binom{n}{j+1} \), where \( j = 0, 1, \ldots, k \), namely

\[
S_2(n) = \binom{n}{1} + 3 \binom{n}{2} + 2 \binom{n}{3},
\]

\[
S_3(n) = \binom{n}{1} + 7 \binom{n}{2} + 12 \binom{n}{3} + 6 \binom{n}{4},
\]

\[
S_4(n) = \binom{n}{1} + 15 \binom{n}{2} + 50 \binom{n}{3} + 60 \binom{n}{4} + 24 \binom{n}{5},
\]

\[
S_5(n) = \binom{n}{1} + 31 \binom{n}{2} + 180 \binom{n}{3} + 390 \binom{n}{4} + 360 \binom{n}{5} + 120 \binom{n}{6}.
\]

As noted by Ginsburg, the above formulas appeared on page 88 of the book by Schwatt, *Introduction to Operations with Series* (Philadelphia, The Press of the University of Pennsylvania, 1924).

In this paper, we obtain a unifying formula for \( S_k(n) \) giving (3) and (4) as particular cases. Indeed, we derive a couple of infinite families of explicit formulas for \( S_k(n) \), one of them involving the numbers \( \{k\}_{r} \) and the other the numbers \( \{k\}_{j-r} \), with \( j = 0, 1, \ldots, k \). Specifically, we establish the following theorem which constitutes the main result of this paper.

**Theorem 1.** Let \( k \) and \( n \) be any non-negative integers and let \( \{k\}_{r} \) and \( \{k\}_{j-r} \) be the numbers defined in (1) and (2), respectively, where \( r \) stands for any arbitrary but fixed non-negative integer. Then

\[
S_k(n) = \sum_{j=0}^{k} j! \left[ \binom{n+1-r}{j+1} + (-1)^j \binom{r+j-1}{j+1} \right] \{k\}_{r},
\]

(5)

and

\[
S_k(n) = \sum_{j=0}^{k} j! \left[ \binom{n+1+r}{j+1} - \binom{r+1}{j+1} \right] \{k\}_{j-r}.
\]

(6)

Before we prove Theorem 1 in the next section, a few observations are in order.

**Remark 1.** It is easily seen that both (5) and (6) reduce to (3) when \( r = 0 \). Furthermore, (5) reduces to (4) when \( r = 1 \). Moreover, setting \( r = n \) in (5) leads to

\[
S_k(n) = n^{k+1} + \sum_{j=1}^{k} (-1)^j j! \left( \binom{n+j-1}{j+1} \right) \{k\}_n.
\]

(7)

Similarly, setting \( r = n+1 \) in (5) yields

\[
S_k(n) = \sum_{j=0}^{k} (-1)^j j! \left( \binom{n+j}{j+1} \right) \{k\}_{n+1}.
\]

(8)

retrieving the result obtained in [12, Equation (4.8)].
Remark 2. It should be stressed that both (5) and (6) hold irrespective of the value taken by the non-negative integer parameter $r$. This means that, actually, the right-hand side of (5) and (6) provides us with an infinite supply of explicit formulas for $S_k(n)$, one for each value of $r$. For example, for $r = 2$, and noting that $S_k(1) = 1$ for all $k$, we have from (5)

$$S_k(n) = 1 + \sum_{j=0}^{k} j! \binom{n-1}{j+1} \binom{k}{j},$$

where

$$\binom{k}{j}_2 = \frac{1}{j!} \sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} (i+2)^k.$$

Analogously, for $r = 2$, we have from (6)

$$S_k(n) = -\delta_{k,0} + (-1)^{k+1} (1 + 2^k) + \sum_{j=0}^{k} j! \binom{n+3}{j+1} \binom{k}{j},$$

where

$$\binom{k}{j}_{-2} = \frac{1}{j!} \sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} (i-2)^k.$$

Remark 3. In the last section, we obtain a more general formula for $S_k(n)$ involving the numbers $\{k\}_{n+m}$ and $\{k\}_{n-m}$, where $m$ is any given non-negative integer (see equations (19) and (20)). Furthermore, we provide another formula for $S_k(n)$ involving the numbers $\{k\}_{k-j}$ (see equation (21)), as well as a formula for the Bernoulli polynomials $B_k(x-1)$ in terms of $\{k\}_x$ and the harmonic numbers (see equation (22)).

2 Proof of Theorem 1

The proof of Theorem 1 is based on the following lemma.

Lemma 1. For $x$ a real or complex variable, let $S_k(x)$ denote the unique interpolating polynomial in $x$ of degree $k + 1$ such that $S_k(x) = 1^k + 2^k + \cdots + x^k$ whenever $x$ is a positive integer (with $S_k(0) = 0$). Then

$$S_k(x) = S_k(a - 1) + \sum_{j=0}^{k} j! \binom{x+1-a}{j+1} R_{k,j}(a),$$

where $a$ is a parameter taking any arbitrary but fixed real or complex value.

Proof. As is well known (see, e.g., [9, Equation (15)], $S_k(x)$ can be expressed in terms of the Bernoulli polynomials $B_k(x)$ as follows

$$S_k(x) = \frac{1}{k+1} [B_{k+1}(x+1) - B_{k+1}(1)].$$

(10)
Recall further that the forward difference operator $\Delta$ acting on the function $f(x)$ is defined by

$$\Delta f(x) = f(x + 1) - f(x).$$

Then, the following elementary result

$$\Delta S_k(x) = (x + 1)^k$$

follows immediately from (10) and the difference equation $\Delta B_{k+1}(x) = (k+1)x^k$ [9, Equation (12)].

On the other hand, the Newton-Gregory expansion of the function $f(x)$ is given by (see, e.g., [17, Equation (A.9), p. 230])

$$f(x) = \sum_{j=0}^{\infty} \binom{x-a}{j} \Delta^j f(a),$$

where, for any integer $j \geq 1$, the $j$-th order difference operator $\Delta^j$ is defined by $\Delta^j f(x) = \Delta(\Delta^{j-1} f(x)) = \Delta^{j-1}(\Delta f(x))$ and $\Delta^0 f(x) = f(x)$, and where $\Delta^j f(a) = \Delta^j f(x)|_{x=a}$. Hence, applying the Newton-Gregory expansion to the power sum polynomial $S_k(x)$ and using (11) yields

$$S_k(x) = S_k(a) + \sum_{j=0}^{k} \binom{x-a}{j+1} \Delta^j(a+1)^k,$$

(12)

where the terms in the summation with index $j$ greater than $k$ have been omitted because $\Delta^j(x+1)^k = 0$ for all $j \geq k + 1$ [17, Equation (6.16), p. 68].

The connection between (12) and the generalized Stirling numbers $R_{k,j}(x)$ stems from the fact that (see, e.g., [5, Theorem 29] and [6, Equation (3.8)])

$$R_{k,j}(x) = \frac{1}{j!} \Delta^j x^k.$$

(13)

Therefore, combining (12) and (13), and making $a \to a - 1$, we get (9).

When $x$ and $a$ are the non-negative integers $n$ and $r$, respectively, (9) becomes

$$S_k(n) = S_k(r-1) + \sum_{j=0}^{k} j! \binom{n+1-r}{j+1} \binom{k}{j}_r,$$

(14)

with $S_k(-1) = 0$ for all $k \geq 1$, and $S_0(-1) = -1$. Now, letting $n = 0$ in (14) and using the relation

$$\binom{-x}{k} = (-1)^k \binom{x+k-1}{k}$$

(15)

gives

$$S_k(r-1) = \sum_{j=0}^{k} (-1)^j j! \binom{r+j-1}{j+1} \binom{k}{j}_r.$$  

(16)

Then, substituting (16) in (14), we get (5).

Moreover, making the transformation $r \to -r$ in (14) and invoking the symmetry property of the power sum polynomial (see, e.g., [16, Theorem 10])

$$S_k(-r - 1) = -\delta_{k,0} + (-1)^{k+1} S_k(r),$$

with $\delta_{k,0} = 1$ for $k = 0$ and $\delta_{k,0} = 0$ otherwise, we get

$$S_k(-r - 1) = -\delta_{k,0} + (-1)^{k+1} S_k(r),$$

which is (5).
it follows that
\[
S_k(n) = -\delta_{k,0} - (-1)^k S_k(r) + \sum_{j=0}^{k} j! \binom{n + 1 + r}{j + 1} \left\{ \binom{k}{j} \right\}_{-r},
\] (17)

from which, upon setting \( n = 0 \), we further obtain
\[
(-1)^k S_k(r) = -\delta_{k,0} + \sum_{j=0}^{k} j! \binom{r + 1}{j + 1} \left\{ \binom{k}{j} \right\}_{-r}.
\] (18)

Finally, substituting (18) in (17), we get (6).

We conclude this section with the following two remarks.

**Remark 4.** By renaming \( r \) as \( n \) in (18) we find that
\[
S_k(n) = (-1)^k \left\{ -\delta_{k,0} + \sum_{j=0}^{k} j! \binom{n + 1 + r}{j + 1} \left\{ \binom{k}{j} \right\}_{-r} \right\},
\]

which may be compared with (3).

**Remark 5.** It is to be noted that the equation (14) above is equivalent to the equation appearing in [2, Corollary 2.2] in which \( d = 1 \) and \( a = r \).

## 3 Concluding remarks

Let us observe that, by letting \( r = n + m \) in (5), where \( m \) is any given non-negative integer, and using (15), we obtain
\[
S_k(n) = \sum_{j=0}^{k} (-1)^j j! \left[ \binom{n + m + j - 1}{j + 1} - \binom{m + j - 1}{j + 1} \right] \left\{ \binom{k}{j} \right\}_{n+m}.
\] (19)

Of course, (19) reduces to (7) and (8) when \( m = 0 \) and \( m = 1 \), respectively. Similarly, putting \( r = n - m \) in (5), where \( m \) is any given non-negative integer, we obtain
\[
S_k(n) = \sum_{j=0}^{k} j! \left[ \binom{m + 1}{j + 1} + (-1)^j \binom{n + j - m - 1}{j + 1} \right] \left\{ \binom{k}{j} \right\}_{n-m}.
\] (20)

Note that, when \( m = n \), (20) reduces to (3).

On the other hand, applying the formula \( \left\{ \binom{k}{j} \right\}_{-r} = (-1)^{k-j} \left\{ \binom{k}{j} \right\}_{r-j} \) (see [14, Equation (2.4)]) and taking \( r = k \) in (6) yields
\[
S_k(n) = \sum_{j=0}^{k} (-1)^{k-j} j! \left[ \binom{n + k + 1}{j + 1} - \binom{k + 1}{j + 1} \right] \left\{ \binom{k}{j} \right\}_{k-j}.
\] (21)

Incidently, for \( n = 1 \), (21) gives us the identity
\[
\sum_{j=0}^{k} (-1)^{k-j} j! \left( \frac{k+1}{j} \right) \left\{ \binom{k}{j} \right\}_{k-j} = 1.
\]
Moreover, from (10) and (16), we obtain the following formula for the Bernoulli polynomials evaluated at the non-negative integer \( r \)

\[
B_{k+1}(r) = B_{k+1}(1) + (k + 1) \sum_{j=0}^{k} (-1)^j j! \binom{r + j - 1}{j + 1} \binom{k}{j}, \quad \text{integer } r \geq 0.
\]

Likewise, making \( r \to -r \) in the preceding equation and using (15) gives the following formula for the Bernoulli polynomials evaluated at the negative integer \(-r\)

\[
B_{k+1}(-r) = B_{k+1}(1) - (k + 1) \sum_{j=0}^{k} j! \binom{r + 1}{j + 1} \binom{k}{j} , \quad \text{integer } r \geq 0.
\]

One can naturally extend the above formula for \( B_{k+1}(r) \) to apply to any real or complex variable \( x \) as follows

\[
B_{k+1}(x) = B_{k+1}(1) + (k + 1) \sum_{j=0}^{k} j! \binom{x + j - 1}{j + 1} \binom{k}{j} , \quad \text{where, using the notation in [1], } \{ \binom{k}{j}_x \} \text{ refers to the Stirling polynomial of the second kind } R_{k,j}(x), \text{ namely}
\]

\[
\{ \binom{k}{j}_x \} = \frac{1}{j!} \sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} (i + x)^k.
\]

Finally, we point out that \( B_k(x - 1) \) can be expressed in the form

\[
B_k(x - 1) = \sum_{j=0}^{k} (-1)^j j! H_{j+1} \{ \binom{k}{j}_x \}, \quad \text{(22)}
\]

where \( H_j = 1 + \frac{1}{2} + \cdots + \frac{1}{j} \) is the \( j \)-th harmonic number. In particular, setting \( x = 1 \) in (22) yields the following known formula for the Bernoulli numbers (see, e.g., [19, Equation (5.9)])

\[
B_k = \sum_{j=0}^{k} (-1)^j j! H_{j+1} \{ \binom{k+1}{j+1} \}.
\]

Furthermore, from (10) and (22), we arrive at the following formula for the sum of powers of integers

\[
S_{k-1}(n) = \frac{1}{k} \sum_{j=0}^{k} (-1)^j j! H_{j+1} \left( \binom{k}{j} - \binom{k}{j} \right),
\]

which holds for any integers \( n \geq 0 \) and \( k \geq 1 \), with \( S_{k-1}(0) = 0 \) for all \( k \geq 1 \).

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