EXISTENCE OF GLOBAL SOLUTION OF THE CAUCHY PROBLEM FOR
THE RELATIVISTIC BOLTZMANN EQUATION WITH HARD
INTERACTIONS

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Abstract. By using the DiPerna and Lions technique for the nonrelativistic Boltzmann equation, it is shown that there exists a global mild solution to the Cauchy problem for the relativistic Boltzmann equation with the assumptions of the relativistic scattering cross section including some relativistic hard interactions and the initial data satisfying finite mass, "inertia", energy and entropy.

1. Introduction

This paper is concerned with the study of the Cauchy problem for the relativistic Boltzmann equation (or briefly, RBE) \[1\] of the form

\[
\frac{\partial f}{\partial t} + \frac{\bf p}{p_0} \frac{\partial f}{\partial \bf x} = Q(f, f),
\]

where \( f = f(t, \bf x, \bf p) \) is a distribution function of a one-particle relativistic gas without external forces, \( t \in (0, +\infty), \bf x \in \mathbb{R}^3, \bf p \in \mathbb{R}^3, p_0 = (1 + |\bf p|^2)^{1/2} \) and \( Q(f, f) \) is the relativistic collision operator whose structure is described below. By using a similar technique to one given in the nonrelativistic case, this paper is to deduce the existence of a global mild solution to RBE \[1\] given an initial condition \( f|_{t=0} = f_0(\bf x, \bf p) \) in \( \mathbb{R}^3 \times \mathbb{R}^3 \) which satisfies

\[
f_0 \geq 0, \int_{\mathbb{R}^3 \times \mathbb{R}^3} f_0(1 + |\bf x|^2 + p_0 + |\ln f_0|) d^3\bf x d^3\bf p < \infty. \tag{1.2}
\]

In \[1.2\], the second term of the integral can be regarded as a finite initial "inertia" of the relativistic system while the fourth one represents the Boltzmann entropy at an initial time. The two other terms of the integral in \[1.2\], from left to right, respectively, represent the mass and the energy in the relativistic system at the initial time. The finiteness of all the integrals states that the relativistic system has finite mass, "inertia", energy and entropy at the initial state.

If \( \varphi \in \mathcal{D}(\mathbb{R}^3) \), then \( Q(\varphi, \varphi) \) is a function of \( \bf p \) given by

\[
Q(\varphi, \varphi) = \frac{1}{p_0} \int_{\mathbb{R}^3} \frac{d^3\bf p_1}{p_{10}} \int_{S^2} d\Omega[\varphi(\bf p')\varphi(\bf p_1') - \varphi(\bf p)\varphi(\bf p_1)] B(g, \theta), \tag{1.3}
\]

where the different parts are explained below. In \[1.1\], \( Q(f, f) \) means \( Q(f(t, \bf x, \cdot), f(t, \bf x, \cdot)) \).

\( \bf p \) and \( \bf p_1 \) are dimensionless momenta of two colliding particles immediately before collision while \( \bf p' \) and \( \bf p_1' \) are, respectively, dimensionless momenta of the particles corresponding to \( \bf p \) and \( \bf p_1 \) immediately after collision; \( p_0 = (1 + |\bf p|^2)^{1/2} \) and \( p_{10} = (1 + |\bf p_1|^2)^{1/2} \) are, respectively, the dimensionless energy of the particles with the momenta \( \bf p \) and \( \bf p_1 \) while \( p'_0 = (1 + |\bf p'|^2)^{1/2} \) and \( p'_{10} = (1 + |\bf p_1'|^2)^{1/2} \) are, respectively, the dimensionless energy of the particles with the momenta \( \bf p' \) and \( \bf p_1' \). As is standard, \( \varphi_1 = \varphi(\bf p_1) \) is denoted by \( \varphi_1 \), etc., and primes are used to represent the results of collisions. \( \mathbb{R}^3 \) is a three-dimensional Euclidean space and \( S^2 \) a unit sphere.
sphere surface. \( B(g,\theta) \) is given by \( B(g,\theta) = gs^2/2(\sigma(g,\theta)/2 \), where \( \sigma(g,\theta) \) is a scattering cross section, \( s = |p_{10} + p_0|^2 - |p_1 + p|^2, \) \( g = \sqrt{|p_1 - p|^2 - |p_{10} - p_0|^2}/2, \) \( \theta \) is the scattering angle defined in \([0,\pi]\) by \( \cos \theta = 1 - 2[(p_0 - p_{10})(p_0 - p_{10}) - (p - p_1)(p - p_1)]/(4 - s) \). Obviously, \( s = 4 + 4g^2 \). \( \, \, \, \, d\Omega = \sin \theta d\theta d\psi, \, 0 \leq \theta \leq \pi, \, 0 \leq \psi \leq 2\pi. \)

The DiPerna and Lions techniques for the nonrelativistic Boltzmann equation were first applied by Dudyński and Ekiel-Jeżewska \([2]\) to prove global existence of solutions to the Cauchy problem for RBE with the assumptions of the relativistic scattering cross section excluding the relativistic hard interactions and the initial data satisfying finite mass, energy and entropy. Unlike in the nonrelativistic case, the causality of solutions to RBE is used by Dudyński and Ekiel-Jeżewska into their proof and so the relativistic initial data is not required to have finite “inertia”. Their results are correct but the relativistic scattering cross section does not include the cases about the relativistic hard interactions. After that, a different device was also given in \([3]\) to show global existence of solutions to the large-data Cauchy problem for RBE with some relativistic hard interactions; in this proof, the property of the causality of RBE is not used directly in solving the Cauchy problem but it is assumed that the initial data satisfies

\[
of_0 \geq 0, \int_{\mathbb{R}^3 \times \mathbb{R}^3} f_0(1 + p_0|x|^2 + p_0 + |\ln f_0|)d^3xd^3p < \infty, \tag{1.4}\]
i.e., finite mass, “inertia”, energy and entropy. The objective of this paper is to show that there exists a global mild solutions to the large-data Cauchy problem for RBE with some relativistic hard interactions under the condition of the initial data \( f_0 \) satisfying \((1.2)\), that is,

**Theorem 1.1.** Let \( B(g,\theta) \) be the relativistic collision kernel of RBE \((1.1)\), defined above, and \( B_R \) a ball with a center at the origin and a radius \( R \), \( A(g) = \int_{\mathbb{S}^2}d\Omega B(g,\theta) \). Assume that

\[
B(g,\theta) \geq 0 \, \, \, \, \text{a.e. in} \, \, \, [0,\, +\infty) \times S^2, \, B(g,\theta) \in L^1_{\text{loc}}(\mathbb{R}^3 \times S^2), \tag{1.5}\]

\[
\frac{1}{p_0} \int_{\mathbb{R}^3} \frac{d^3p_1}{p_{10}} A(g) \to 0 \, \, \, \, \text{as} \, \, \, |p| \to +\infty, \, \, \, \, \text{for} \, \, \, \forall R(0,\, +\infty). \tag{1.6}\]

Then RBE \((1.7)\) has a mild or equivalently a renormalized solution \( f \) through initial data \( f_0 \) with \((1.2)\), satisfying the following properties

\[
f \in C([0,\, +\infty); \, L^1(\mathbb{R}^3 \times \mathbb{R}^3)), \tag{1.7}\]

\[
L(f) \in L^\infty([0,\, +\infty); \, L^1(\mathbb{R}^3 \times \mathbb{R}^3)) \, \, \, \, \text{for all} \, \, \, \forall R(0,\, +\infty), \tag{1.8}\]

\[
\frac{Q^+(f,f)}{1 + f} \in L^1([0,\, T]; L^1(\mathbb{R}^3 \times \mathbb{R}^3)) \, \, \, \, \text{for} \, \, \, \forall R(0,\, +\infty) \, \, \, \, \text{and} \, \, \, \forall T(0,\, +\infty), \tag{1.9}\]

\[
\sup_{0 \leq t \leq T} \int_{\mathbb{R}^3 \times \mathbb{R}^3} f(1 + |x|^2 + p_0 + |\ln f|)d^3xd^3p < C_T, \tag{1.10}\]

where \( C_T \) is a positive constant only dependent of \( f_0 \) and \( T \) for \( \forall T(0,\, +\infty) \).

Obviously, the assumption \((1.2)\) is weaker than that of \((1.1)\) and so this result is better than that obtained in \([3]\). Also, our proof is simpler than that given in \([3]\).

It is clear that the condition \((1.3)\) is equivalent to the following one:

\[
\sigma(g,\theta) \geq 0 \, \, \, \, \text{a.e. in} \, \, \, [0,\, +\infty) \times S^2, \, g(1 + g^2)^{1/2}\sigma(g,\theta) \in L^1_{\text{loc}}([0,\, +\infty) \times S^2), \tag{1.11}\]

which was first defined by Jiang \([3]\). The assumption \((1.4)\) was originally introduced by Jiang \([4,\, 5]\). Obviously, the relativistic assumptions \((1.5)\) and \((1.6)\) are similar to the following nonrelativistic ones adopted by DiPerna and Lions \([6]\):

\[
B(z,\omega) \geq 0 \, \, \, \, \text{a.e. in} \, \, \, \mathbb{R}^N \times S^{N-1}, \, B(z,\omega) \in L^1_{\text{loc}}(\mathbb{R}^N \times S^{N-1}), \tag{1.12}\]

\[
\frac{1}{1 + |\xi|^2} \int_{|z - \xi| \leq R} dzA(\xi) \to 0 \, \, \, \, \text{as} \, \, \, |\xi| \to +\infty, \, \, \, \, \text{for} \, \, \, \forall R(0,\, +\infty), \tag{1.13}\]
where \( B(z, \omega) \) is a function of \(|z|, |(z, \omega)|\) only, \( \tilde{A}(z) = \int_{S^N-1} d\omega B(z, \omega) \). It is also easy to see that the condition (1.6) includes some relativistic hard interactions defined as \( \int_{S^2} d\Omega B(g, \theta) \geq Cg^2 \), where \( C \) is a positive constant (see [7]). But it was assumed by Dudyński and Ekiel-Jeżewska (see [2]) that \( B(g, \theta) \) satisfies (1.5) and the following condition:

\[
\frac{1}{p_0} \int_{B_R} \frac{d^3 p_1}{p_{10}} A(g) \rightarrow 0 \quad \text{as} \ |p| \rightarrow +\infty, \quad \text{for} \ \forall R \in (0, +\infty),
\]

where \( B_R \) and \( A(g) \) are the same as (1.6); it has been claimed in [2] that their assumptions of \( B(g, \theta) \) exclude the relativistic hard interactions. It follows that (1.14) is more restrictive than (1.6).

Apart from those mentioned above, there are many other outstanding results relevant to the study of the Cauchy problem for RBE, e.g., the works of Andréasson, Bancel, Bichteler, Cercignani, Glassey, Kremer, Strauss and so on (see [8][9][10][11][12][13]). It is worth mentioning that the recent work of Glassey [14] not only introduces many relevant books and papers but also is a breakthrough. These are very helpful to our further studying this problem.

### 2. Conservation Laws and Entropy

As in the nonrelativistic case, the structure of the relativistic collision operator maintains not only the conversation of mass, momenta and energy in the relativistic system, but also the property that the entropy of the system does not decrease.

For convenience, let us first introduce the following notations:

\[
Q^+(\varphi, \varphi) = \frac{1}{p_0} \int_{R^3} \frac{d^3 p_1}{p_{10}} \int_{S^2} d\Omega \varphi(p_1') \varphi(p_1) B(g, \theta),
\]

\[
L(\varphi) = \frac{1}{p_0} \int_{R^3} \frac{d^3 p_1}{p_{10}} \int_{S^2} d\Omega \varphi(p_1) B(g, \theta),
\]

\[
Q^-(\varphi, \varphi) = \varphi(p) L(\varphi).
\]

Obviously,

\[
Q(\varphi, \varphi) = Q^+(\varphi, \varphi) - Q^-(\varphi, \varphi).
\]

Finally, it can be known that (1.3) has the following equivalent form

\[
Q(f, f) = \frac{1}{2} \int \int \int \frac{W(p, p_1; p', p_1')}{p_0 p_1 p_0' p_1'} [f' p_1' - f f_1] d^3 p_1 d^3 p' d^3 p_1',
\]

where

\[
W(p, p_1; p', p_1') = s \sigma(g, \theta) \delta(3)(p + p_1 - p' - p_1')[\delta(p_0 + p_{10} - p_0' - p_{10}')],
\]

which is called transition rate for RBE (see [1]). Note that energy and momenta of two colliding particles conserve before and after collision. It can be then shown from (2.6) that the transition rate for RBE satisfies

\[
W(p, p_1; p', p_1') = W(p', p_1'; p, p_1) = W(p_1, p; p_1', p') = W(p_1', p'; p_1, p).
\]

By using (1.3), (2.5) and (2.7), it can be easily proved that

\[
\int_{R^3} \psi(p) Q(\varphi, \varphi) d^3 p = \frac{1}{4} \int \int \int \frac{d^3 p_1}{p_0 p_1} \int_{S^2} d\Omega B(g, \theta)[\varphi' \varphi_1' - \varphi \varphi_1] 
\]

\[
\cdot [\psi(p) + \psi(p_1) - \psi(p') - \psi(p_1')]
\]

where \( Q^\pm(\varphi, \varphi) \psi(p) \in L^1(R^3) \) for any given \( \psi(p) \in L^\infty(R^3) \) and \( \varphi(p) \in L^1(R^3) \). It follows from (2.8) that \( \int_{R^3} \psi Q(f,f) d^3 p = 0 \) if \( f = f(t, x, p) \) is a distributional solution to RBE (1.1) such that \( \int_{R^3} \psi Q(f,f) d^3 p < +\infty \) for almost all \( t \) and \( x \) and \( \psi = b_0 + b p + c_0 p_0 \), where \( b_0 \in R, b \in R^3, c_0 \in R \). Furthermore, it is at least formally found that \( \int \int \int R^3 x d^3 p \) is
independent of $t$ for any distributional solution $f$ to RBE (1.1). This yields the conservation of mass, momentum and energy of the relativistic system.

Fortunately, the desired estimate of the “inertia” $\int \int_{\mathbb{R}^3 \times \mathbb{R}^3} f |x|^2 d^3x d^3p$ under the assumption of (1.2) can be also made successfully (see [15]). This estimate is different from that of the “inertia” $\int \int_{\mathbb{R}^3 \times \mathbb{R}^3} f p_0 |x|^2 d^3x d^3p$ defined by Jiang [3] under the assumption (1.4). To show this estimate, it requires the following identity

$$\frac{d}{dt} \int \int_{\mathbb{R}^3 \times \mathbb{R}^3} f |x|^2 d^3x d^3p = 2 \int \int_{\mathbb{R}^3 \times \mathbb{R}^3} f (xp/p_0) d^3x d^3p$$

(2.9)

derived by multiplying RBE (1.1) by $|x|^2$ and integrating by parts over $x$ and $p$. It follows from (2.9) that

$$\frac{d}{dt} \int \int_{\mathbb{R}^3 \times \mathbb{R}^3} f |x|^2 d^3x d^3p \leq \int \int_{\mathbb{R}^3 \times \mathbb{R}^3} f |x|^2 d^3x d^3p + \int \int_{\mathbb{R}^3 \times \mathbb{R}^3} f d^3x d^3p.$$  

(2.10)

This yields the following inequality

$$\sup_{0 \leq t \leq T} \int \int_{\mathbb{R}^3 \times \mathbb{R}^3} f |x|^2 d^3x d^3p \leq e^T \int \int_{\mathbb{R}^3 \times \mathbb{R}^3} f_0 (1 + |x|^2) d^3x d^3p$$

(2.11)

for any given $T > 0$ by multiplying the two sides of (2.10) by $e^{-t}$ and using the conservation of the mass of the relativistic system. The inequality given by (2.11) illustrates that the relativistic “inertia” of $f |x|^2$ over all the space and momentum variables in any given time interval is controlled by both mass and “inertia” at the initial state of the relativistic system.

Next, let us show the property that the entropy is always a nondecreasing function of $t$ in the relativistic system. To do this, we first deduce at least formally the following relativistic entropy identity

$$\frac{d}{dt} \int \int_{\mathbb{R}^3 \times \mathbb{R}^3} f \ln f d^3x d^3p + \frac{1}{4p_0} \int \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{d^3p_1}{p_{10}} \int_{S^2} d\Omega B(g, \theta)$$

$$\cdot \left[ f' f_1' - f f_1 \right] \ln \left( \frac{f' f_1'}{f f_1} \right) = 0$$

(2.12)

by multiplying RBE (1.1) by $1 + \ln f$, integrating over $x$ and $p$ and using (2.8). In general, $\int \int_{\mathbb{R}^3 \times \mathbb{R}^3} f \ln f d^3x d^3p$ is denoted by $H(t)$ and called H-function. Boltzmann’s entropy is usually defined by $-H(t)$. The second term in (2.12) is nonnegative and so $H(t)$ is a nonincreasing function of $t$. This means that the entropy of the relativistic system does not decrease. This property allows the desired estimate of the relativistic entropy to be derived from the Cauchy problem for RBE (1.1).

The entropy can be controlled by the integral $\int \int_{\mathbb{R}^3 \times \mathbb{R}^3} f |\ln f| d^3x d^3p$ for any nonnegative solution $f$ to RBE (1.1) and so it is natural to make the considered estimate of the integral instead of the entropy. Note that

$$\int \int_{\mathbb{R}^3 \times \mathbb{R}^3} f |\ln f| d^3x d^3p = \int \int_{\mathbb{R}^3 \times \mathbb{R}^3} f \ln f d^3x d^3p + 2 \int \int_{|f| \leq 1} f |\ln f| d^3x d^3p$$

$$\leq \int \int_{\mathbb{R}^3 \times \mathbb{R}^3} f \ln f d^3x d^3p + 2 \int \int_{\mathbb{R}^3 \times \mathbb{R}^3} f |x|^2 d^3x d^3p$$

$$+ 2 \int \int_{f \leq \exp(-|x|^2/p_0)} f \ln(1/f) d^3x d^3p$$

$$\leq \int \int_{\mathbb{R}^3 \times \mathbb{R}^3} f \ln f d^3x d^3p + 2 \int \int_{\mathbb{R}^3 \times \mathbb{R}^3} f |x|^2 d^3x d^3p + C_1$$

(2.13)
where $C_1$ is some positive constant independent of $f$. By using (2.12) and (2.13), it can be then deduced that
\[
\sup_{0 \leq t \leq T} \left[ \int \int_{\mathbb{R}^3 \times \mathbb{R}^3} f \ln f \, d^3x d^3p \right] 
\leq \int \int_{\mathbb{R}^3 \times \mathbb{R}^3} f_0 [2e^T(|x|^2 + 1) + 2p_0 + |\ln f_0|] d^3x d^3p + C_1. 
\]

(2.14)

3. PROOF OF THEOREM 1.1

In this section we show our theorem given in Section 1 by use of the DiPerna and Lions global existence proof with only minor adjustments.

Let us first consider a similar approximation scheme to that given by DiPerna and Lions in the nonrelativistic case. Put $f_0^n = \min(f_0 1_{|x|^2 + |p|^2 \leq n}, n) + \frac{1}{n} e^{-((|x|^2 + p_0)}$ and $B_n(g, \theta) = g s^{+} \sigma_n(g, \theta)$ where $\sigma_n(g, \theta) = \sigma(g, \theta) 1_{\sigma(g, \theta) < n}(g, \theta) 1_{g > \frac{1}{4}}(g) 1_{\sin \theta > \frac{1}{4}}(\theta) 1_{p_0 + p_{10} \leq n}$ for $n = 1, 2, 3, \ldots$, and through this paper, all the functions denoted by 1 with subscript expressions, such as $1_{|x|^2 + |p|^2 \leq n}$, represent characteristic functions. It follows that
\[
\int \int_{\mathbb{R}^3 \times \mathbb{R}^3} d^3x d^3p |f_0 - f_0^n| (1 + |x|^2 + p_0) \to 0 \text{ as } n \to +\infty, 
\]

(3.1)
\[
\int \int_{\mathbb{R}^3 \times \mathbb{R}^3} d^3x d^3p f_0^n |\ln f_0^n| \leq C, 
\]

(3.2)
\[
\int_{B_R \times S^2} d^3p d\Omega |B_n(g, \theta) - B(g, \theta)| \to 0 
\]

(3.3) uniformly in $\{p_1 : |p_1| \leq k\}$ as $n \to +\infty$ for $\forall R, k \in (0, +\infty)$.

Then the collision kernel of RBE (1.1) is replaced by $B_n(g, \theta)$ to solve the following Cauchy problem
\[
\frac{\partial f^n}{\partial t} + \frac{p}{p_0} \frac{\partial f^n}{\partial x} = \tilde{Q}_n(f^n, f^n), f^n|_{t=0} = f^n_0. 
\]

(3.4)
Here and below, $\tilde{Q}_n$ is defined by $\tilde{Q}_n(\varphi, \varphi) = (1 + \frac{1}{n} \int_{\mathbb{R}^3} |\varphi| d^3p)^{-1} Q_n(\varphi, \varphi)$ and
\[
Q_n(\varphi, \varphi) = \frac{1}{p_0} \int_{\mathbb{R}^3} \int_{p_1} \int_{S^2} d\Omega [\varphi(p') \varphi(p') - \varphi(p) \varphi(p)] B_n(g, \theta). 
\]

(3.5)
It follows from (3.5) that for all $\varphi, \psi \in L^\infty((0, +\infty) \times \mathbb{R}^3 \times \mathbb{R}^3) \cap L^1(\mathbb{R}^3 \times \mathbb{R}^3)$),
\[
||\tilde{Q}_n(\varphi, \varphi)||_{L^\infty((0, +\infty) \times \mathbb{R}^3 \times \mathbb{R}^3)} \leq C_n ||\varphi||_{L^\infty((0, +\infty) \times \mathbb{R}^3 \times \mathbb{R}^3)}, 
\]

(3.6)
\[
||\tilde{Q}_n(\varphi, \varphi)||_{L^1(\mathbb{R}^3 \times \mathbb{R}^3)} \leq C_n ||\varphi||_{L^1(\mathbb{R}^3 \times \mathbb{R}^3)}, 
\]

(3.7)
\[
||\tilde{Q}_n(\varphi, \varphi) - \tilde{Q}_n(\psi, \psi)||_{L^1(\mathbb{R}^3 \times \mathbb{R}^3)} \leq C_n ||\varphi - \psi||_{L^1(\mathbb{R}^3 \times \mathbb{R}^3)}, 
\]

(3.8) here and below everywhere, $C_n$ is a nonnegative constant independent of $\varphi$ and $\psi$.

Existence and uniqueness of distributional solutions to the Cauchy problem given by (3.4) can be below established. The first step to show this is to construct a set $\mathcal{F}$ such that each $\varphi \equiv \varphi(t, x, p)$ belongs to $\mathcal{F}$ if and only if $\varphi$ is a measurable function defined in $[0, +\infty) \times \mathbb{R}^3 \times \mathbb{R}^3$ and satisfies
\[
(i) \quad \sup_{(t, x, p) \in [0, +\infty) \times \mathbb{R}^3 \times \mathbb{R}^3} e^{-C_n t} |\varphi(t, x, p)| \leq ||f_0^n||_{L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)}, 
\]
\[
(ii) \quad \sup_{0 \leq t < +\infty} e^{-C_n t} \int \int_{\mathbb{R}^3 \times \mathbb{R}^3} |\varphi| d^3x d^3p \leq ||f_0^n||_{L^1(\mathbb{R}^3 \times \mathbb{R}^3)}. 
\]
The second is to define a mapping on $\mathcal{F}$ as follows: for each $\varphi \in \mathcal{F}$, \begin{equation}
abla_{j}(\varphi)(t, x, p) = f_{0}^{n}(x - \frac{t}{p_{0}}, p) + \int_{0}^{t} \nabla_{n}(\varphi, \varphi)(t, x, (\sigma - t)\frac{p}{p_{0}}, p) \, d\sigma. \tag{3.9} \end{equation}
Then, by \cite{3.6}, \cite{3.7} and \cite{3.8}, it can be proved that $J_{n}(\mathcal{F}) \subseteq \mathcal{F}$ and that $J_{n}$ is contractive, i.e., for every $g, h \in \mathcal{F}$,
\begin{equation}
\sup_{0 \leq t < +\infty} e^{-2C_{n}t} \int_{\mathbb{R}^{3} \times \mathbb{R}^{3}} |J_{n}(g)(t, x, p) - J_{n}(h)(t, x, p)| d^{3}x d^{3}p \leq \frac{1}{2} \sup_{0 \leq t < +\infty} e^{-2C_{n}t} \int_{\mathbb{R}^{3} \times \mathbb{R}^{3}} |g(t, x, p) - h(t, x, p)| d^{3}x d^{3}p. \tag{3.10} \end{equation}
Therefore there exists a unique element $f^{n} = J_{n}(f^{n}) \in \mathcal{F}$ such that for almost every $(x, p) \in \mathbb{R}^{3} \times \mathbb{R}^{3}$,
\begin{equation}
(f^{n})^{\#}(t, x, p) = J_{n}(f^{n})^{\#}(t, x, p) \quad \text{for} \quad \forall t \in [0, +\infty), \tag{3.11} \end{equation}
here and below, $h^{\#}$ denotes for any measurable function $h$ on $(0, +\infty) \times \mathbb{R}^{3} \times \mathbb{R}^{3}$, the following restriction to characteristics: $h^{\#}(t, x, p) = h(t, x + t\frac{p}{p_{0}}, p)$. It can be also easily proved that \( J_{n}(f^{n}, f^{n}) \in L^{1}_{loc}(\mathbb{R}^{3} \times \mathbb{R}^{3}) \) and hence $f^{n}$ is a distributional solution to \( \mathbf{3.4} \).
It is below shown that $f^{n}(t, x, p) \geq 0$ for almost every $(t, x, p) \in [0, +\infty) \times \mathbb{R}^{3} \times \mathbb{R}^{3}$. To show this, another mapping $J_{n}^{+}$ is first defined as follows: $J_{n}^{+}(f) = \max(0, J_{n}(f))$ for $\forall f \in \mathcal{F}^{+} = \{ f : f \in \mathcal{F} \text{ and } f \text{ a.e. } \geq 0 \}$. $\mathcal{F}^{+}$ is a subset of $\mathcal{F}$. Similarly, it can be easily shown that the mapping $J_{n}^{+}$ maps $\mathcal{F}^{+}$ into itself and is uniformly contractive with an inequality similar to \( \mathbf{3.10} \), i.e., for every $g, h \in \mathcal{F}^{+}$,
\begin{equation}
\sup_{0 \leq t < +\infty} e^{-2C_{n}t} \int_{\mathbb{R}^{3} \times \mathbb{R}^{3}} |J_{n}^{+}(g)(t, x, p) - J_{n}^{+}(h)(t, x, p)| d^{3}x d^{3}p \leq \frac{1}{2} \sup_{0 \leq t < +\infty} e^{-2C_{n}t} \int_{\mathbb{R}^{3} \times \mathbb{R}^{3}} |g(t, x, p) - h(t, x, p)| d^{3}x d^{3}p. \tag{3.12} \end{equation}
Then there exists a unique element $\tilde{f} \in \mathcal{F}^{+}$ such that $J_{n}^{+}(\tilde{f}) = \tilde{f}$ for almost every $(t, x, \xi) \in [0, +\infty) \times \mathbb{R}^{3} \times \mathbb{R}^{3}$. Thus, if $\bar{f} = J_{n}(\tilde{f})$ for almost every $(t, x, \xi) \in [0, +\infty) \times \mathbb{R}^{3} \times \mathbb{R}^{3}$, $\tilde{f}$ is a distributional solution to $\mathbf{3.4}$ through $f_{0}^{n}$ in the time interval $[0, +\infty)$, and $\tilde{f} = f^{n}$. It will be below shown that $J_{n}(\tilde{f})^{\#} = \tilde{f}$, or equivalently, $J_{n}(\tilde{f})^{\#} = \tilde{f}^{\#}$. In fact, by \( \mathbf{3.9} \), it is known that $J_{n}(\tilde{f})^{\#}$ is absolutely continuous with respect to $t \in [0, +\infty)$ for almost every $(x, \xi) \in \mathbb{R}^{3} \times \mathbb{R}^{3}$. It may be assumed without loss of generality that $J_{n}(\tilde{f})^{\#}(t, x, \xi) = \tilde{f}$ is continuous for all $(t, x, \xi)$. To prove that $J_{n}(\tilde{f})^{\#} = \tilde{f}$, it suffices to prove that $J_{n}(\tilde{f})^{\#} \geq 0$. Assume that $J_{n}(\tilde{f})^{\#}(t_{0}, x_{0}, \xi_{0}) < 0$ for some point $(t_{0}, x_{0}, \xi_{0}) \in [0, +\infty) \times \mathbb{R}^{3} \times \mathbb{R}^{3}$. Note that $J_{n}(\tilde{f})^{\#}(t, x_{0}, \xi_{0}) = f_{0}^{n}(x_{0}, \xi_{0}) \geq 0$. Then two values $t^{*}$ and $t_{1}$ can be found in $[0, t_{0})$ such that $J_{n}(\tilde{f})^{\#}(t^{*}, x_{0}, \xi_{0}) = 0$ and $J_{n}(\tilde{f})^{\#}(t, x_{0}, \xi_{0}) < 0$ for all $t \in (t^{*}, t_{1}]$, so that $\tilde{f}(t, x_{0}, \xi_{0}) = 0$ for all $t \in [t^{*}, t_{1}]$. By \( \mathbf{3.9} \), it can be known that $0 > J_{n}(\tilde{f})^{\#}(t, x_{0}, \xi_{0}) \geq J_{n}(\tilde{f})^{\#}(t^{*}, x_{0}, \xi_{0})$ for all $t \in (t^{*}, t_{1}]$. This is a contradiction with $J_{n}(\tilde{f})^{\#}(t^{*}, x_{0}, \xi_{0}) = 0$. Therefore $f^{n} = \tilde{f} = J_{n}(\tilde{f})^{\#} \geq 0$ for almost every $(x, \xi) \in [0, +\infty) \times \mathbb{R}^{3} \times \mathbb{R}^{3}$.
The properties of $\{ f^{n}(t, x, p) \}_{n=1}^{\infty}$ are discussed below. Obviously, $f^{n}$ satisfies
\begin{equation}
0 \leq f^{n} \in L^{\infty} \cap L^{1}(\mathbb{R}^{3} \times \mathbb{R}^{3}) \quad (\forall T < +\infty). \tag{3.13} \end{equation}
Furthermore, by using \( \mathbf{3.11} \), it can be proved that
\begin{equation}
f^{n}(t, x, p) \in C([0, +\infty); L^{1}(\mathbb{R}^{3} \times \mathbb{R}^{3})). \tag{3.14} \end{equation}
For convenience, $\tilde{Q}_{n}^{+}, \tilde{L}_{n}$ and $\tilde{Q}_{n}$ are, respectively, denoted by
\begin{equation}
\tilde{Q}_{n}^{+}(\varphi, \varphi) = (1 + \frac{1}{n} \int_{\mathbb{R}^{3}} |\varphi| d^{3}p)^{-1} \frac{1}{p_{0}} \int_{\mathbb{R}^{3}} \frac{d^{3}p_{1}}{p_{10}} \int_{S^{2}} d\Omega \varphi(p') \varphi(p') B_{n}(g, \theta), \tag{3.15} \end{equation}
\[ \tilde{L}_n(\varphi) = (1 + \frac{1}{n} \int_{\mathbb{R}^3} |\varphi|d^3\mathbf{p})^{-1} \frac{1}{p_0} \int_{\mathbb{R}^3} \frac{d^3\mathbf{p}_1}{p_{10}} \int_{S^2} d\Omega \varphi(\mathbf{p}_1) B_n(g, \theta). \]  

(3.16)

Obviously,

\[ \tilde{Q}_n(\varphi, \varphi) = \varphi(\mathbf{p}) \tilde{L}_n(\varphi), \]

(3.17)

Similarly, \eqref{eq:3.5} is equivalent to the form

\[ Q_n(\varphi, \varphi) = \frac{1}{2} \int \int \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3} \frac{W_n(\mathbf{p}, \mathbf{p}_1; \mathbf{p}', \mathbf{p}_1')}{p_0 p_{10} p_{0}'} [\varphi' \varphi_1' - \varphi \varphi_1] d^3\mathbf{p}_1 d^3\mathbf{p}_1' d^3\mathbf{p}' d^3\mathbf{p}_1'. \]

(3.19)

with

\[ W_n(\mathbf{p}, \mathbf{p}_1; \mathbf{p}', \mathbf{p}_1') = s \sigma_n(\mathbf{g}, \mathbf{p}) \delta(\mathbf{p} + \mathbf{p}_1 - \mathbf{p}' - \mathbf{p}_1') \delta(p_0 + p_{10} - p_0' - p_{10}'). \]

(3.20)

If \( \tilde{Q}_n^\pm(\varphi, \varphi)\psi(\mathbf{p}) \in L^1(\mathbb{R}^3) \) for every given \( \psi(\mathbf{p}) \in L^\infty(\mathbb{R}^3) \) and \( \varphi(\mathbf{p}) \in L^1(\mathbb{R}^3) \), then it can be deduced by using \eqref{eq:3.20} that

\[ \int_{\mathbb{R}^3} \psi(\mathbf{p}) \tilde{Q}_n(\varphi, \varphi) d^3\mathbf{p} = \frac{1}{4} (1 + \int_{\mathbb{R}^3} \varphi d^3\mathbf{p})^{-1} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{S^2} \frac{d^3\mathbf{p}_1 d^3\mathbf{p}_1' d\Omega}{p_{10} p_0} B_n(g, \theta) \cdot [\varphi' \varphi_1' - \varphi \varphi_1] [\psi(\mathbf{p}) + \psi(\mathbf{p}_1) - \psi(\mathbf{p}') - \psi(\mathbf{p}_1')]. \]

(3.21)

By \eqref{eq:3.15}, \eqref{eq:3.16} and \eqref{eq:3.17}, it is obvious to see that

\[ \tilde{Q}_n(\mathbf{f}, f) \in L^1((0, +\infty) \times \mathbb{R}^3 \times \mathbb{R}^3). \]

(3.22)

Then, by \eqref{eq:2.11}, \eqref{eq:2.12}, \eqref{eq:2.21} and Gronwall’s inequality, it can be found that

\[ \sup_{0 \leq t \leq T} \int_{\mathbb{R}^3} f^n(1 + x^2 + p_0 + \ln f^n) d^3\mathbf{x} d^3\mathbf{p} \leq C_T. \]

(3.23)

It also follows by \eqref{eq:2.12} that

\[ \frac{1}{4} \int_0^{+\infty} \int_{\mathbb{R}^3} \left\{(1 + \int_{\mathbb{R}^3} f^n d^3\mathbf{p})^{-1} \int_{\mathbb{R}^3} \int_{S^2} \frac{d^3\mathbf{p}_1 d^3\mathbf{p}_1' d\Omega}{p_{10} p_0} B_n(g, \theta) \cdot (f^n f_1' - f_1 f^n) \ln \left( \frac{f^n f_1'}{f_1 f^n} \right) \right\} d\sigma d^3\mathbf{x} \leq C_T. \]

(3.24)

In \eqref{eq:3.23} and \eqref{eq:3.24} \( C_T \) is a positive constant only dependent of \( f_0 \) and \( T \) except of \( n \). By using necessary and sufficient conditions of a weakly compact set in the \( L^1 \)-space (see \cite{16}, Page 347, IV.13.54), it can be deduced from \eqref{eq:3.23} that \( \{f^n\}_{n=1} \) is weakly compact in \( L^1((0, T) \times \mathbb{R}^3 \times \mathbb{R}^3) \) \((\forall T < +\infty) \) and so it may be assumed without loss of generality that \( f^n \) converges weakly in \( L^1((0, T) \times \mathbb{R}^3 \times \mathbb{R}^3) \) to \( f \in C_T = L^1_{loc}((0, +\infty) \times \mathbb{R}^3 \times \mathbb{R}^3) \) as \( n \to +\infty \) for all \( T \in (0, +\infty) \). Obviously, \( f \geq 0 \) and \( f|_{t=0} = f_0(\mathbf{x}, \mathbf{p}) \) for almost every \( (\mathbf{x}, \mathbf{p}) \in \mathbb{R}^3 \times \mathbb{R}^3 \). It is also known that \eqref{eq:3.23} yields \eqref{eq:1.10}.

It can be also claimed that for \( \forall T, R \in (0, +\infty) \), \( \{\tilde{Q}_n^\pm (f^n, f^n)/(1 + f^n)\}_{n=1} \) are weakly compact subsets of \( L^1((0, T) \times \mathbb{R}^3 \times B_R) \).

Once this claim is proven, it follows that \( f \) is a global mild solution to RBE \eqref{eq:1.1} satisfying \eqref{eq:1.7}, \eqref{eq:1.8}, \eqref{eq:1.9} and \eqref{eq:1.10}, by analyzing step by step the subsolutions and supersolutions of RBE \eqref{eq:1.1} with a similar device to that given by DiPerna and Lions \cite{6}. Our analysis not only allows for the relations among three different types of solutions to RBE \eqref{eq:1.1} (see \cite{3}) but also requires the momentum-averaged compactness of the transport operator of RBE \eqref{eq:1.1} (see \cite{17} or \cite{18}).

It remains to show this claim. In the case of \( \tilde{Q}_n^- \), to prove this, it suffices to prove that \( \tilde{L}_n(f^n) \) belongs to some weakly compact subset of \( L^1((0, T) \times \mathbb{R}^3 \times B_R) \). Denote \( \tilde{L}_{nk} \) \((k = 1, 2, 3, \cdots)\) by

\[ \tilde{L}_{nk}(f^n) = (1 + \frac{1}{n} \int_{\mathbb{R}^3} |f^n|d^3\mathbf{p})^{-1} \frac{1}{p_0} \int_{\mathbb{R}^3} \frac{d^3\mathbf{p}_1}{p_{10}} \int_{S^2} d\Omega f^n(\mathbf{p}_1) B_n(g, \theta)|_{|\mathbf{p}_1| \leq k}. \]

(3.25)
Then put \( \Psi(y) = y(\ln y)^+ \) and define \( \tilde{b}_k := \tilde{b}_k(t, x, p) \) by

\[
\tilde{b}_k = (1 + \frac{1}{n} \int_{\mathbb{R}^3} |f^n| d^3p)^{-1} \frac{1}{p_0} \int_{\mathbb{R}^3} \frac{d^3p_1}{p_{10}} \int_{S^2} d\Omega B_n(g, \theta) \mathbf{1}_{|p_1| \leq k}.
\]  

(3.26)

Thus

\[
\Psi(\tilde{L}_{nk}(f^n)) \leq \tilde{b}_k \Psi(\frac{\tilde{L}_{nk}(f^n)}{b_k}) + (\ln \tilde{b}_k) \tilde{L}_{nk}(f^n).
\]

(3.27)

Take \( \alpha_n = |f^n|_{L^\infty} + 1 \) and \( v = \tilde{b}_k^{-1} \tilde{L}_{nk}(f^n) \). Since \( \Psi(y) \) is convex, there exists a positive constant \( \beta_n \) which depends only on \( t, x, p \) and \( n \), such that, for \( 0 \leq u < v < w < \alpha_n \),

\[
0 \leq \frac{\Psi(u) - \Psi(v)}{u - v} \leq \beta_n \leq \frac{\Psi(v) - \Psi(w)}{v - w},
\]

which gives

\[
\Psi(u) \geq \Psi(v) + \beta_n(u - v) \quad (\forall u \in [0, \alpha_n]),
\]

in particular,

\[
\Psi(f^n) \geq \Psi(v) + \beta_n(f^n - v).
\]

It follows that

\[
\Psi(\frac{\tilde{L}_{nk}(f^n)}{b_k}) \leq \frac{1}{\tilde{b}_k} \int \int_{\mathbb{R}^3 \times S^2} \frac{B_n(g, \theta)}{p_0 p_{10}} \Psi(\mathbf{1}_{|p_1| \leq k} d^3p_1 d\Omega).
\]

(3.28)

By (3.27) and (3.28), it can be concluded that

\[
\int \int_{\mathbb{R}^3 \times B_R} \Psi(\tilde{L}_{nk}(f^n)) d^3x d^3p
\]

\[
\leq \int \int_{\mathbb{R}^3 \times B_R} \left\{ \int \int_{\mathbb{R}^3 \times S^2} \frac{B_n(g, \theta)}{p_0 p_{10}} \Psi(\mathbf{1}_{|p_1| \leq k} d^3p_1 d\Omega) \right\} d^3x d^3p
\]

\[
+ \int \int_{\mathbb{R}^3 \times B_R} (\ln \tilde{b}_k) \tilde{L}_{nk}(f^n) d^3x d^3p.
\]

(3.29)

It can be known from (1.5) that \( 0 \leq \tilde{b}_k \leq b_{Rk} \) for all \( (t, x, p) \in (0, +\infty) \times \mathbb{R}^3 \times B_R \) and

\[
\int \int_{\mathbb{R}^3 \times S^2} \frac{B_n(g, \theta)}{p_0 p_{10}} \mathbf{1}_{|p_1| \leq k} d^3p d\Omega
\]

\[
\leq \int \int_{\mathbb{R}^3 \times S^2} \frac{B(g, \theta)}{p_0 p_{10}} \mathbf{1}_{|p_1| \leq k} d^3p d\Omega \leq b_{Rk},
\]

(3.30)

where \( b_{Rk} \) is a positive constant which depends only on \( R \) and \( k \), thus it can be shown from (3.25) and (3.29) that

\[
\int \int_{\mathbb{R}^3 \times B_R} \Psi(\tilde{L}_{nk}(f^n)) d^3x d^3p
\]

\[
\leq b_{Rk} \int \int_{\mathbb{R}^3 \times \mathbb{R}^3} \Psi(f^n) d^3x d^3p + b_{Rk} |\ln b_{Rk}| \int \int_{\mathbb{R}^3 \times \mathbb{R}^3} f^n d^3x d^3p.
\]

Hence, by (3.23), \( \{\Psi(\tilde{L}_{nk}(f^n))\}_{n=1}^\infty \) is bounded in \( L^\infty((0, T); L^1(\mathbb{R}^3 \times B_R)) \) for every given \( k \). It can be also known from (3.23), (3.25) and (3.30) that

\[
\sup_{n \geq 1} \int \int_{\mathbb{R}^3 \times B_R} \tilde{L}_{nk}(f^n)(1 + |x|^2 + p_0) d^3x d^3p < C_{Rk}
\]

(3.31)

where \( C_{Rk} \) is a positive constant which only depends on \( R \) and \( k \). It follows that \( \tilde{L}_{nk}(f^n) \) belongs to a weakly compact subset of \( L^1((0, T) \times \mathbb{R}^3 \times B_R) \) for any given \( R \) and \( k \) in \( (0, +\infty) \).

On the other hand, it can be shown that as \( k \to +\infty \),

\[
\sup_{t \in [0, T]} \sup_{n \geq 1} |\tilde{L}_{nk}(f^n) - \tilde{L}_n(f^n)|_{L^1(\mathbb{R}^3 \times B_R)} \to 0.
\]

(3.32)
Indeed, it can be known from (1.6) that, for every small $\sigma > 0$, there exists $k_0 > 0$, such that, as $|p_1| > k_0$,
\[
\int \int_{B_R \times S^2} \frac{B(g, \theta)}{p_0 p_{10}} d^3 p d\Omega < \sigma p_{10}.
\] (3.33)

By using (3.25) and (3.33), it can be shown that, as $k > k_0$,
\[
\sup_{t \in [0, T]} \sup_{n \geq 1} \int \int_{\mathbb{R}^3 \times \mathbb{R}^3} \left\{ \left[ \int \int_{B_R \times S^2} \frac{B(g, \theta)}{p_0 p_{10}} d^3 p d\Omega \right] \cdot 1_{|p_1| > k} f_n^1 \right\} d^3 x d^3 p
\leq \sigma \cdot \sup_{t \in [0, T]} \sup_{n \geq 1} \int \int_{\mathbb{R}^3 \times \mathbb{R}^3} p_{10} f_n^1 d^3 x d^3 p.
\]

This leads easily to (3.32) by using (3.23).

Therefore, by (3.31) and (3.32), it can be easily deduced that $\{ \tilde{L}_n(f_n) \}$ is weakly compact in $L^1((0, T) \times \mathbb{R}^3 \times B_R)$.

In the case of $\tilde{Q}_n^+$, the following inequality is used:
\[
\tilde{Q}_n^+(f_n, f_n) \leq K \tilde{Q}_n^+(f_n, f_n) + \frac{\tilde{e}_n}{\ln K}
\]
for every $K > 1$, where $\tilde{e}_n$ is defined by
\[
\tilde{e}_n = \int \int \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times S^2} \frac{d^3 p_1 d^3 p d\Omega}{p_0 p_{10}} B_n(g, \theta)(f_n f_n - f_n f_{n})^\prime \ln \left( \frac{f_n f_n^\prime}{f_n^\prime} \right),
\]
and thus, by (3.24) and the above result in the case of $\tilde{Q}_n^-$, it can be shown that $\frac{\tilde{Q}_n^+(f_n, f_n)}{1 + f_n f_{n}^\prime}$
belongs to some weakly compact subset of $L^1((0, T) \times \mathbb{R}^3 \times B_R)$ for any given $R$ and $T$ in $(0, +\infty)$.

The proof of this claim is finished and the theorem holds.

**Remark 1.** The proof of the existence and uniqueness of global solution to the Cauchy problem (3.4) is simpler than that given in [3].

**Remark 2.** By using the device of Dudyński and Ekiel-Jeżewska [2], the causality of RBE (1.1) (see [19] [20]) can be directly applied into the further proof of the existence of global solution to the Cauchy problem for RBE (1.1) in some relativistic hard interaction cases with the finite initial physically natural bounds excluding the finite initial “inertia”, i.e., with the initial data $f_0(x, p)$ satisfying
\[
f_0 \geq 0, \int \int_{\mathbb{R}^3 \times \mathbb{R}^3} f_0(1 + p_0 + \ln f_0) d^3 x d^3 p < \infty.
\] (3.34)

Exactly speaking, under the assumptions of (1.5), (1.6) and (3.34), RBE (1.1) has a mild or equivalently a renormalized solution $f$ through initial data $f_0$, satisfying (1.7), (1.8), (1.9) and
\[
\sup_{t \geq 0} \int \int_{\mathbb{R}^3 \times \mathbb{R}^3} f(1 + p_0 + \ln f) d^3 x d^3 p < +\infty.
\] (3.35)

The first author of this paper will give another paper to show a detail proof of this result.

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