Black-hole event horizons — Teleology and Predictivity

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Abstract

General Relativity predicts the existence of black-holes. Access to the complete space-time manifold is required to describe the black-hole. This feature necessitates that black-hole dynamics is specified by future or teleological boundary condition. Here we demonstrate that the statistical mechanical description of black-holes, the raison d’être behind the existence of black-hole thermodynamics, requires teleological boundary condition. Within the fluid-gravity paradigm — Einstein’s equations when projected on space-time horizons resemble Navier-Stokes equation of a fluid — we show that the specific heat and the coefficient of bulk viscosity of the horizon-fluid are negative only if the teleological boundary condition is taken into account. We argue that in a quantum theory of gravity, the future boundary condition plays a crucial role. We briefly discuss the possible implications of this at late stages of black-hole evaporation.
Physical laws, since the time of Newton, are predictive. Given an initial condition, the physical laws predict the state of a system at a later time. The two biggest revolutions in Physics of the twentieth century — Relativity and Quantum Theory — are predictive. The solutions to the dynamical equations subject to a future boundary condition are viewed as the future influencing the present and is generally deemed unphysical.

General relativity, while still predictive, brought in some far-reaching changes. First major change, as compared to special relativity, is that the space-time is dynamic. Einstein equations can be solved subject to initial conditions, however, the predictions are physical only after solving the space-time metric [1, 2, 3, 4]. The second major change took place via the concept of event horizon [2, 3, 4, 5]. To define the event horizon, one requires the complete knowledge of the space-time manifold [2, 3, 5]. Thus, the dynamics of the event horizon cannot be predictive and the existence of the horizon requires imposing future boundary condition on the horizon [4, 5, 6].

The discovery of laws of black-hole thermodynamics, and the Hawking radiation [7, 8, 9, 4], allowed to identify, the surface gravity of a black-hole with temperature and the area of the event horizon with entropy. This also made clear that the event-horizons via black-hole entropy play a crucial role in identifying the microscopic degrees of freedom and hence, the fundamental theory of Gravity [10, 11].

Given that we do not yet have the complete description of quantum gravity, there are two possible ways to go about obtaining the microscopic description of the event horizon: First approach is to assume the microscopic structures (like in Loop gravity or String theory) and arrive at the entropy. Second approach — pursued in this essay — is to map the gravity equations near the horizon to a familiar system for which the microscopic degrees of freedom are known.

It has been shown three decades ago that the Einstein’s equation projected on to the event horizon is similar to Navier-Stokes equation [12, 13, 14, 15, 6]. Thus, if energy and entropy of the fluid — that are identical to that of the black-hole — can be obtained from the microscopic description of a fluid, it may be possible to associate microscopic degrees of freedom of the fluid to the horizon. The fluid contains two sets of parameters — susceptibilities and transport coefficients. While the first set of parameters correspond to changes in local variables; the other set involves fluxes of thermodynamic quantities [16, 17, 18]. These parameters can be determined via statistical mechanical description of the fluid fluctuations. In other words, treating the fluid not far from equilibrium, one can study the statistical mechanical fluctuations from the equilibrium and relate susceptibility/transport
Green-Kubo relations connect non-equilibrium processes to the thermal fluctuations in equilibrium via fluctuation-dissipation theorem \[16, 17, 18\]. Mathematically, the change in the expectation value of any operator is linearly related to the perturbing source, i.e.,

\[
\delta \langle O(t) \rangle = \int dt' \chi(t-t')\phi(t')
\]

where \( \chi \) is the response (or Green) function and \( \phi(t) \) is the source. Assuming that the operator is Hermitian and the source is real leads to the fact that response \( \chi(t-t') \) must also be real. The Fourier transform of \( \chi(t-t') \) can be written as:

\[
\chi(\omega) = \text{Re}\chi(\omega) + i\text{Im}\chi(\omega) \equiv \chi'(\omega) + i\chi''(\omega)
\]

The imaginary part of \( \chi(\omega) \) can be written as:

\[
\chi''(\omega) = -\frac{i}{2} \int_{-\infty}^{\infty} dt e^{i\omega t} [\chi(t) - \chi(-t)]
\]

\( \chi''(\omega) \) represents the part of the response function that is not invariant under time reversal. Evaluation of \( \chi''(\omega) \) provides three key information about the system. First, it provides information about the dissipative processes. Second, it tells us whether the processes are causal or anti-causal \[19, 20\]. Lastly, from Eq. (1), transport coefficients of normal fluids can be obtained once we know \( \chi(\omega) \) in the hydrodynamic limit \( \omega \to 0 \) \[16, 17, 18\]:

\[
C_{\text{Transport}} \propto \langle O(t,r)O(0,r) \rangle, \propto \chi''(\omega = 0)
\]

where, \( \langle O(t,r)O(0,r) \rangle \) is the auto-correlation function of \( O \). Evaluation of the auto-correlation function again requires the causal (anti-causal) response of the horizon properties of the fluid.

But, what does this digression have to do with black-holes? As we show in this essay, a similar analysis is possible for the horizon-fluid and the issue of imposing the future boundary condition comes to the fore. It also leads to a natural statistical mechanical explanation of some of the puzzling aspects of black-hole thermodynamics. A synopsis of the research programme advocated in this essay is provided in Fig. 1.

As mentioned above, Einstein equations projected on to the event horizon of an asymptotically flat black-hole lead to:

\[
\frac{D\theta}{dt} + \theta^2 = -\frac{\kappa}{8\pi} \theta - \frac{1}{16\pi} \theta^2 + \frac{1}{8\pi} \sigma_{AB}\sigma^{AB} + T_{\mu\nu}\xi^\mu \xi^\nu
\]
The above equation is similar to the energy conservation equation of a viscous fluid:

\[ \frac{\partial \mathcal{E}}{\partial t} + \mathbf{v} \cdot \nabla \mathcal{E} = -\nabla \cdot [(\mathcal{E} + P)\mathbf{v} - \mathbf{v} \cdot \sigma'] - \sigma'_{AB} \frac{\partial v_A}{\partial x_B}, \quad (6) \]

where \( \sigma' \) is the dissipative stress tensor for the fluid given by, \( \sigma'_{AB} = [v(A,B)]_{\text{trace free}} + \zeta \delta_{AB} \theta \), \( \theta = \nabla \cdot \mathbf{v} \), and \( \mathbf{v} \) is the velocity of the fluid. \( \zeta(\eta) \) is the coefficient of the bulk (shear) viscosity of the fluid.

Identifying \( T = \frac{\kappa}{8\pi} \), \( \zeta = -\frac{1}{16\pi} \), \( \eta = \frac{1}{16\pi} \), energy density of the fluid, \( \mathcal{E} = \theta \) and the momentum density of the fluid, \( \Pi_A = -\frac{1}{8\pi} n^a \xi_a^\alpha A \) and \( P = \frac{T}{4} \), equation (5) is similar to the energy conservation equation of the fluid (6) i.e.

\[ \frac{D \mathcal{E}}{dt} + \theta \mathcal{E} = -\frac{\kappa}{8\pi} \theta + \zeta \theta^2 + 2 \eta \sigma_{AB} \sigma^{AB} + T_{\mu\nu} \xi^\mu \xi^\nu \quad (7) \]

We note that the above equations are valid for an asymptotically flat space-time with a single horizon. In this essay, our focus is to provide an understanding of the negativity of the bulk viscosity and show that the same can help to understand the negative specific heat of asymptotically flat black-hole space-times [21] [22] (See Fig. 1).
To obtain statistical mechanical understanding of the transport coefficients of the horizon-fluid, we need to know how an external influence or source produces a response in the horizon-fluid [23, 24, 25, 26]. For simplicity, we focus on the horizon-fluid corresponding to Schwarzschild black-hole [27], however, the analysis can be extended for any asymptotically flat black-holes in General Relativity [28]. The change in the area of the horizon-fluid is given by the dynamics of the volume expansion coefficient, $\theta^H$ of a congruence of null geodesics on the horizon. It is governed by the Raychaudhury equation, also the energy conservation equation for the horizon-fluid [26]. It can be cast in the form of a linear response, by retaining only the volume change of the fluid, given by,

$$
- \frac{d\theta^H}{dt} + g_H \theta^H \simeq 8\pi I^H,
$$

where, $g_H = 2\pi T$ and $I^H$ is the source which in this case is the energy flux. From the above equation, one sees that $\theta^H$ increases exponentially with time. Demanding that the horizon exists in the future necessitates to impose future boundary condition on $\theta_H$ [6]; imposing initial boundary condition on $\theta_H$ implies $\theta_H$ increases exponentially and destroys the black-hole. Since, $\theta_H$ is assumed to be small in deriving Eq. (8), initial boundary condition can not be imposed. Thus the teleological boundary condition can be viewed as a condition for the stability of the black-hole event horizon or the condition that small external influences do not drive the fluid far from equilibrium [26].

From Eq. (4), the transport coefficients for the horizon-fluid can be rewritten as:

$$
C_{\text{Transport}} = f(T, A) \langle O(t, r)O(0, r) \rangle,
$$

where, $f(T, A)$ is a well-behaved function of $A$ and $T$, and $O(t) \equiv \delta A(t)$. $\delta A$ is the change in the black-hole area and is proportional to $\theta_H$. From general properties of any fluid (including horizon-fluid) and using (8), it can be argued that $\lim_{\epsilon \to 0} \delta A(t) \sim \delta A_0(t) \exp[i(\omega - i\epsilon)t]$, i.e. fluctuations in the area undergo damped oscillations in the hydrodynamic limit.

For normal fluids in the presence of an external influence, the response is causal. However, as mentioned above, the response of an horizon-fluid to any external influence is anti-causal [6]. Mathematically, this implies: $\delta A_{\text{causal}}(t) = \delta A(t) \Theta(t)$ and $\delta A_{\text{anti-causal}}(t) = \delta A(t) \Theta(-t)$. Hence, $C_{\text{anti-causal}} = -C_{\text{causal}}$. Thus, it can be shown that the bulk viscosity ($\zeta$) of the horizon-fluid is negative [27, 28]. A similar argument shows the specific heat of the horizon-fluid is negative [29]. It is important to stress that apart from the conventional techniques from the theory of fluctuations [16, 17, 18], the key physics input is
the *future or teleological* boundary condition in order to evaluate the statistical correlation of the change in the horizon area ($\delta A$) (See Fig. 1).

The analysis presented until now is semi-classical. We provide arguments that the teleological boundary condition plays a key role in the quantum theory. Assuming that the underlying theory for these fluctuations is a quantum theory, it is possible to define an operator corresponding to this variable [30, 31]. As long as the theory is in the space-time continuum, our analysis clearly demonstrates the need to impose *teleological boundary condition*. This is in stark contrast to all other physical theories.

While our analysis is for black-hole event horizons, it is possible to extend the analysis to isolated horizons [32]. For such horizons, the response is causal in nature, hence the coefficient of bulk viscosity is positive. However, for the second law to be satisfied [3, 4], we need to impose future boundary condition. This condition makes sure that the black-hole horizon is stable at late times and is the famous Cosmic Censorship Conjecture [4]. This seems to suggest that the future boundary condition is integral to black-hole physics. One crucial question that remains is whether future boundary condition is required at the late stages of black-hole evaporation. Our ignorance about the physics in this phase prevents us from saying anything concrete [33, 34]. However, if the generalized second law remains valid, the horizon evolution might be governed by future boundary condition.

Ultimately the fundamental theory of Quantum Gravity might turn out to be very different from a theory based on a spacetime continuum [35]. It is interesting whether the future boundary condition plays a fundamental role there.

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