Noise representations of open system dynamics

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We analyze the conditions under which the dynamics of a quantum system open to a given environment can be simulated with an external noisy field that is a surrogate for the environmental degrees of freedom. We show that such a field is either a subjective or an objective surrogate; the former is capable of simulating the dynamics only for the specific system–environment arrangement, while the latter is an universal simulator for any system interacting with the given environment. Consequently, whether the objective surrogate field exists and what are its properties is determined exclusively by the environment. Thus, we are able to formulate the sufficient criterion for the environment to facilitate its surrogate, and we identify a number of environment types that satisfy it. Finally, we discuss in what sense the objective surrogate field representation can be considered classical and we explain its relation with the formation of system–environment entanglement, the back-action exerted by the system onto environment, and the Gaussian approximation to the system dynamics.

1 Introduction

In recent years, we have witnessed a tremendous pace of advancement in the field of quantum technology; presently, devices that utilize quantum effects to perform useful tasks in practical circumstances, seem to be an inevitable part of not so far future [1, 2]. Then, it is only natural that the focus of modern applied quantum theory shifts from idealized closed systems to more realistic open systems where the system (S) of interest—which can be a component of quantum device—undergoes evolution due to application of various control protocols while experiencing the decoherence caused by the contact with its uncontrolled environment (E). Indeed, even the system that has been prepared, and is handled, with the utmost care is extremely unlikely to be perfectly isolated from the environment. Moreover, unlike classical systems, even weak interactions with the environment can lead to decoherence effects that fundamentally alter the properties of quantum system [3, 4]. Therefore, the development of effective and accurate description of the dynamics of open quantum systems is of paramount importance.

The most straightforward ab initio approach utilizes the exact two-party Hamiltonian

\[ \hat{H}_{SE} = \hat{H}_S \otimes \hat{1} + \hat{V}_S \otimes \hat{V}_E + \hat{1} \otimes \hat{H}_E, \]

where \( \hat{V}_S, \hat{V}_E \) are the system and the environment sides of the coupling and \( \hat{H}_E, \hat{H}_S \) are the free Hamiltonians of the environment and the system (note that the latter incorporates any applied control scheme). However, in overwhelming majority of cases it is impossible
to solve it exactly, and hence, the success of such an approach relies heavily on approximation schemes (e.g., the quantum master equation method [5]) that are specific to a given system–environment arrangement. The drawback is that successful schemes and techniques developed for one arrangement only rarely can be reused for treating different arrangements, even when the only modified element is the control scheme applied to the system (e.g., see [6]).

An alternative to \textit{ab initio} approach, that we will focus on in this paper, is the \textit{noise representation}. Essentially, it is an attempt at assigning the involved parties with the distinct roles they play in the dynamics—the environment is viewed as the “influencer”, or the “driver”, and the system is the “influencee” or the “driven”. In formal terms, this idea is implemented by replacing the exact description \( \hat{H}_{SE} \) with an effective system-only Hamiltonian where the environmental operators have been superseded by an external field [7–16]

\[
\hat{H}_{\xi}(t) = \hat{H}_S + \xi(t)\hat{V}_S.
\]

We will refer to this field as the \textit{surrogate field}.

If the surrogate field \( \xi(t) \) comprises stochastic elements (as it is often the case), then \( \hat{H}_{\xi}(t) \) model has to be supplemented with an \textit{averaging procedure} where any quantity computed for given realization (or trajectory) of \( \xi \) must be averaged over all such realizations. Assuming that only the dynamics of system-only observables are of concern, then the averaging can be incorporated into the description by adopting the following definition of the density matrix of \( S \)

\[
\hat{\varrho}_S(t) = \hat{U}(t|\xi)\hat{\varrho}_S\hat{U}^\dagger(t|\xi),
\]

where \( \hat{\varrho}_S \) is the initial state, the unitary evolution operator conditioned by the field realization is given by a standard time-ordered exponential

\[
\hat{U}(t|\xi) = Te^{-i\int_0^t d\tau \hat{H}_{\xi}(\tau)} = Te^{-i\int_0^t d\tau [\hat{H}_S + \xi(\tau)\hat{V}_S]},
\]

and the symbol \( \langle \ldots \rangle \) indicates the average over field’s trajectories.

The conditions for applicability of noise representation are currently not well understood. Over the years, a few hypotheses have been posed in the literature [17–28], but the definite answer has proven to be elusive. The primary goal of this paper is to provide a possibly complete answer to this question and to quantify the conditions under which the system–environment arrangement facilitates a valid surrogate field representation. We consider the representation to be valid when \( \hat{H}_{\xi} \) model allows for high fidelity simulation of the actual dynamics governed by \( \hat{H}_{SE} \) of any system-only observable; formally, this criterion is expressed as

\[
\text{Tr}_E \left( e^{-it\hat{H}_{SE}} \hat{\varrho}_S \otimes \hat{\varrho}_E e^{it\hat{H}_{SE}} \right) \approx \hat{U}(t|\xi)\hat{\varrho}_S\hat{U}^\dagger(t|\xi),
\]

where \( \hat{\varrho}_E \) is the initial state of the environment and \( \text{Tr}_E(\ldots) \) is the partial trace over \( E \) subspace. In our investigations we will not be satisfied with “existence theorems” where one states that in a given circumstances the representation exists but does not specify the surrogate \( \xi(t) \)—we will not count as a proof of existence anything less than an explicit scheme for generating the surrogate field (e.g., a scheme that allows to instantiate trajectories of \( \xi(t) \) in a numerical simulation) that is given in the terms of dynamical laws of the environment. By choosing such an approach, we most likely relinquish our ability to give an exhaustive account for all possible scenarios where the system–environment interaction could be simulated with a surrogate. In our opinion it is a price worth paying because in return we will be able to deliver a more palpable results, possessing a degree of flexibility.
that allow to utilize them in a variety of tasks. An important example of such a task are tests of the aforementioned hypotheses regarding the applicability of noise representation.

In the most basic terms, the surrogate field representation exchanges an exact two-party Hamiltonian for an effective evolution generator for the single party $S$. Generally speaking, the details of the representation depend on all elements of the system–environment arrangement: the initial states $\hat{\rho}_S$ and $\hat{\rho}_E$, the free Hamiltonians $\hat{H}_S$ and $\hat{H}_E$, and, of course, both the system and the environment sides of the coupling $\hat{V}_S$ and $\hat{V}_E$. It seems quite obvious that whenever the original environment is swapped for some other physical system—in the sense that any number of elements among $\hat{H}_E$, $\hat{V}_E$ and $\hat{\rho}_E$ are swapped for different operators—the fundamental changes to the representation should also be expected (assuming it would even still exist); after all, the field $\xi(t)$ is supposed to be a surrogate for the environmental degrees of freedom. It is much less obvious what happens to the representation when it is the system side of the arrangement that is modified, instead. This brings about the question of objectivity, or more precisely, of inter-subjectivity of surrogate field representation: for a fixed environment, how does the surrogate field depend on the context defined by the system? By context we mean here the choice of all the elements constituting the system $S$, its initial state $\hat{\rho}_S$, the free Hamiltonian $\hat{H}_S$, and the coupling operator $\hat{V}_S$. Whenever any of those elements is modified, we will consider it a different context.

These questions are of particular relevance for quantum sensing. Specifically, consider an approach where a simple quantum system acts as a probe that is used to gather information about the environment $E$. Then, one attempts to utilize this information to predict the course of decoherence of a more complex systems open to $E$. When the surrogate field representation is valid in the context of the probe, the acquired information would include the characteristics of the surrogate. If the representation happens to be context-independent, then one would be able to apply this information to simulate the evolution of an arbitrary open quantum systems. Therefore, the issue of surrogate field’s objectivity is a vital one.

Motivated by quantum sensing aspects of open system dynamics, initially we will focus our attention on the case of qubit based impulse driven axial spectrometer (QIDAS), also known as dynamical-decoupling-based noise spectrometer [9, 10], as a choice for system $S$. Such spectrometer comprises a qubit probe that interacts with $E$ through purely longitudinal coupling

$$\hat{V}_S(t) = \frac{1}{2} f(t) \hat{\sigma}_z \quad \text{and} \quad \hat{H}_S(t) = \Delta \frac{f(t)}{2} \hat{\sigma}_z,$$

where $\hat{\sigma}_z = |+\rangle \langle + | - | - \rangle \langle - |$ is the $z$-component of Pauli operator. The time-domain filter function $f(t)$ [8, 29] encapsulates the effects of the dynamical control exerted on the qubit where short and intense pulses, that cause effectively instantaneous $\pi$ rotations of its Bloch vector, are applied in a periodic sequence. Such a sequence of spin flips establishes an effective narrow passband filter that decouples the qubit from the environment, except for a handful of frequencies that are commensurate with the period of the pulse sequence. Thus, the qubit probe is used to analyze the spectral contents of a signal it receives from the environment, and simultaneously, to record so obtained information in its decoherence rate. The recovery and reconstitution of this information is the core of the method [30, 31]. This choice of the reference context has two important advantages. (i) The longitudinal coupling and impulse control results in simple, but still non-trivial, qubit dynamics. This simplicity helps in avoiding the pitfall of being bogged down by the technical difficulties inevitably present in more complex systems. (ii) Swapping one pulse sequence for another modifies the coupling and the free Hamiltonian, therefore, QIDASes subjected to various
control sequences form a whole class of easily swappable contexts. This means that we have here a built-in test for inter-subjectivity—unless the qubit probe reports the same surrogate field, no matter the choice of pulse sequence, there cannot be any talk of its objectivity.

Note that traditionally the surrogate field is referred to in the literature as the classical noise [7–9, 29, 32–34]. It is then contrasted with the quantum noise [35–37] which, in most cases, is treated as a synonym for an ab initio approach. However, some authors [38–40] reserve this name for the specific arrangement of a two-level system coupled with a thermal reservoir of independent quantum harmonic oscillators—the so-called spin-boson model. Here, we have chosen to abandon the traditional nomenclature because of the risk that connotations of the adjective “classical” might be too suggestive, and that they could provoke one to draw some far fetched conclusions about the system–environment arrangement on the basis of the name alone. For example, one might expect that there is a link between the validity of “classical noise” representation and “classicality” of the environment (which is not necessarily the case, as demonstrated in Sec. 4), or that the “classical noise” representation is incompatible with the formation of system–environment entanglement because entanglement is a “non-classical” type of correlation (we challenge this sentiment in Sec. 5.3). On the other hand, the name “surrogate field” is not burdened by such a baggage, and it represents exactly what it advertises—the surrogate field is a surrogate for the environmental degrees of freedom. Nevertheless, some analogies between the surrogate field representation and classical theories are expected, and so, we explore this issue in Sec. 5.4.

Further note regarding the nomenclature, the completely positive trace-preserving dynamical map established by Eq. (3), belongs to the class of random unitary maps [17]. The “prototypical” scenario described with random unitary map, one that stems from the basic physical interpretation, occurs when the system dynamics are generated by an actual single-party Hamiltonian of form (2), and where any stochasticity of the external field (as well as the averaging procedure) depicts uncertainty or ignorance of the observer [17]. In the terms we used here, such a scenario is described in the following way. (i) There is no fundamental $\hat{H}_{SE}$ that is being replaced by $S$-only Hamiltonian $\hat{H}_\xi$, i.e., $\xi$ is not a surrogate for any environment but is a genuine external field, instead. (ii) Any single instance of the system’s evolution is given by unitary $\hat{U}(t|\xi)$ where the trajectory of $\xi$ is specified, and could be uncovered in principle, but is unknown to the observer. (iii) Because of this uncertainty, any expectation value predicted by the observer has to be averaged over all possibilities. Based on this example, some authors [17, 22] choose to classify evolution maps as classical when the map can be written in random unitary form, and as truly quantum or non-classical otherwise. The discussion on the relationship between surrogate field representation and the formation of system–environment entanglement presented in Sec. 5.3 provides an argument that such a classification scheme could be enriched with additional nuance. Surrogate field representations provide also a number of non trivial examples that should be useful for developing intuitions regarding the underlying physics of random unitary map theory.

The outline of the paper is as follows. In Sec. 2 we look for surrogate field representation that is inter-subjective within QIDAS context. Even though we successfully formulate the sufficient criterion for valid surrogate representation in this context, it does not end our investigation. We continue in Sec. 3, where we apply the lessons learned from the preceding analysis, to formulate the context-independent validity criterion for objective surrogate field representation—the main result of the paper. In Sec. 4 we describe three distinct examples of environment types that satisfy the validity criterion, and thus, facilitate their
own surrogate fields. Section 5 discusses a variety of topics related to objective surrogate representation and a general concept of external noise field simulators. Among others, in Sec. 5.2 and 5.3 we inspect the hypotheses concerning the influence of back-action exerted onto environment and the formation of system–environment entanglement on the validity of surrogate representation, and in Sec. 5.4 we argue in what sense the surrogate representation can be considered classical.

2 Contextual surrogate field representation

As we discussed previously, initially we focus on QIDAS as the context. Then, the system consists of a qubit with the free Hamiltonian $\hat{H}_S$ and the coupling $\hat{V}_S$ given by (6). With purely longitudinal coupling, only the off-diagonal matrix elements of the system state are affected and its density matrix is of the following form [9]

$$\hat{\rho}_S(t) = \begin{bmatrix} \rho_{++} & \rho_{+-} e^{-i\phi(t)} W_{E/\xi}(t|f) \\ \rho_{-+} & \rho_{--} \end{bmatrix},$$

(7)

Here, $\phi(t) = \Delta \int_0^t f(\tau) d\tau$, $\rho_{ss'} = \langle s|\hat{\rho}_S|s'\rangle = \langle s'|\hat{\rho}_S|s\rangle$ $(s, s' = \pm)$ are the matrix elements of the initial state which for simplicity we choose to be real numbers. The quantity $W_{E/\xi}(t|f)$ is called the coherence function and it is conditioned by the choice of pulse sequence applied to the qubit; the subscript $E/\xi$ indicates whether the evolution was governed by a full quantum-mechanical Hamiltonian $\hat{H}_SE$ or by a model $\hat{H}_\xi$. Therefore, in the QIDAS context, the basic criterion (5) is satisfied by stochastic model $\xi(t)$ when the coherence functions are matched $W_{\xi}(t|f) \approx W_{E}(t|f)$. On top of this minimal requirement, we will also demand that the surrogate field is inter-subjective among QIDASes controlled with arbitrary pulse sequences. Hence, the same model $\xi$ has to match quantum-mechanical result for all possible choices of filter $f(t)$.

2.1 Joint probability distributions

We begin by calculating the coherence function when the evolution is simulated with a stochastic Hamiltonian $\hat{H}_\xi$

$$W_{\xi}(t|f) = \frac{1}{\rho_{++}} e^{-\frac{1}{2} \sum_{i=1}^k \int_0^t f(t_i) \xi(t_i) dt_i} \rho_{++} e^{i \sum_{i=1}^k \int_0^t f(t_i) \xi(t_i) dt_i} = e^{-\frac{1}{2} \int_0^t f(t) \xi(t) dt},$$

(8)

and we expand the resultant exponential into power series

$$W_{\xi}(t|f) = \sum_{k=0}^{\infty} (-i)^k \int_0^t dt_1 \prod_{l=1}^{k-1} \int_0^{t_l} dt_{l+1} \left( \prod_{l=1}^k f(t_l) \right) M_{\xi}^{(k)}(t_1, \ldots, t_k),$$

(9)

in which, we have identified the $f(t)$-independent element that determines the evolution—it is the infinite set of moments of stochastic process $\xi(t)$

$$M_{\xi}^{(k)}(t) = M_{\xi}^{(k)}(t_1, \ldots, t_k) = \xi(t_1) \ldots \xi(t_k) = \left( \prod_{l=1}^k \sum_{\xi_l \in \Omega_{\xi}} \xi_l \right) P_{\xi}^{(k)}(\xi_1, t_1; \ldots; \xi_k, t_k).$$

(10)

More precisely, $M_{\xi}^{(k)}$ is actually a $k$th moment of joint probability distribution $P_{\xi}^{(k)}(\xi_1, t_1; \ldots; \xi_k, t_k)$ of process $\xi(t)$ having the value $\xi_k$ at the initial time $t_k$, followed by $\xi_{k-1}$ at
\( t_{k-1}, \ldots \), and terminating with \( \xi_1 \) at \( t_1 \) (assuming \( t_1 > t_2 > \ldots > t_k \)). The sums, or integrals in the case of continuous process, over unique values of \( \xi \) are restricted to the set \( \Omega_\xi \)—the spectrum of process \( \xi \). The family of probability distributions \( \{P^{(k)}_\xi\}_{k=1,\ldots,\infty} \) defines \( \xi(t) \) completely, and thus, functions \( P^{(k)}_\xi \) cannot be arbitrary as they have to satisfy the following two conditions [41]:

(i) Since each \( P^{(k)}_\xi \) is a probability distribution, it has to be non-negative for all values of its arguments within the spectrum of the process

\[
P^{(k)}_\xi(\xi_1, t_1; \ldots; \xi_k, t_k) \geq 0. \tag{11}
\]

(ii) The joint probabilities belonging to one family are related through Chapman-Kolmogorov consistency criterion

\[
\sum_{\xi_l \in \Omega_\xi} P^{(k)}_\xi(\xi_1, t_1; \ldots; \xi_{l-1}, t_{l-1}; \xi_l, t_l; \xi_{l+1}, t_{l+1}; \ldots; \xi_k, t_k)
= P^{(k-1)}_\xi(\xi_1, t_1; \ldots; \xi_{l-1}, t_{l-1}; \xi_{l+1}, t_{l+1}; \ldots; \xi_k, t_k), \tag{12}
\]

for \( t_1 > t_2 > \ldots > t_k \), and

\[
\sum_{\xi_l \in \Omega_\xi} P^{(1)}_\xi(\xi_1, t_1) = 1. \tag{13}
\]

Conversely, any set of functions that satisfy both of the above conditions defines some stochastic process. This fact will be the linchpin of our search for surrogate field representation.

### 2.2 Joint quasiprobability distributions

The next step is to transform the coherence function calculated for fully quantum-mechanical Hamiltonian into a form that will most directly compare with expressions (9) and (10) obtained previously for the stochastic model. We achieve this by switching to superoperator representation (i.e., the representation of operators that act in operator space),

\[
W_E(t|f) = \text{Tr}_E(\mathcal{T} e^{-i \int_0^t d\tau f(\tau)e^{i\tau [\hat{H}_E, \cdot]} \{\frac{1}{2} \hat{V}_E, \cdot \}) e^{-i [\hat{H}_E, \cdot]} \hat{\rho}_E). \tag{14}
\]

See Appendix A for detailed derivation of this result. The action of coupling superoperator associated with the anticommutator of \( \hat{V}_E \) is defined as

\[
\{\frac{1}{2} \hat{V}_E, \cdot \} \hat{A} = \{\frac{1}{2} \hat{V}_E, \hat{A} \} = \frac{1}{2} \left( \hat{V}_E \hat{A} + \hat{A} \hat{V}_E \right), \tag{15}
\]

and the evolution superoperator associated with the free environmental Hamiltonian

\[
e^{-it[\hat{H}_E, \cdot]} \hat{A} = (e^{-it\hat{H}_E} \cdot e^{it\hat{H}_E}) \hat{A} = e^{-it\hat{H}_E} \hat{A} e^{it\hat{H}_E}, \tag{16}
\]

satisfy the standard composition rules

\[
e^{-it[\hat{H}_E, \cdot]} e^{-i(t+t')[\hat{H}_E, \cdot]} = e^{-i(t+t') [\hat{H}_E, \cdot]}. \tag{17}
\]

We now expand the time-ordered exponential into a series

\[
W_E(t|f) = \sum_{k=0}^{\infty} (-i)^k \int_0^t dt_1 \prod_{l=1}^{k-1} \int_0^{t_l} dt_{l+1} \left( \prod_{l=1}^k f(t_l) \right) M^{(k)}_{\text{ctx}}(t_1, \ldots, t_k), \tag{18}
\]
which allows us to extract the quantum equivalent of stochastic moments

\[ M^{(k)}_{\text{ctx}}(t) = M^{(k)}_{\text{ctx}}(t_1, \ldots, t_k) = \text{Tr}_E \left( \{ \frac{1}{2} \hat{V}_E, \cdot \} e^{-i t_1 \hat{H}_E \cdot} \prod_{l=2}^{k} e^{i t_l \hat{H}_E \cdot} \{ \frac{1}{2} \hat{V}_E, \cdot \} e^{-i t_l \hat{H}_E \cdot} \right) \hat{\rho}_E, \]

(19)

where the symbol \( \prod_{l=2}^{k} \mathcal{A}(l) \) is to be understood as an ordered composition \( \mathcal{A}(2)\mathcal{A}(3) \ldots \mathcal{A}(k) \).

We added the subscript \( \text{ctx} \) to remind us of the current context. Although \( M^{(k)}_{\text{ctx}} \) play the same role as \( M_{\xi}^{(k)} \)—i.e., they are the \( f(t) \)-independent element that determines the evolution—there is no evident formal resemblance between them and their stochastic counterparts. The next steps, where we construct the spectral decomposition of coupling superoperator \( \{ \hat{V}_E/2, \cdot \} \), are meant to expose those hidden similarities.

Let \( \{ |n\rangle \}_n \) be the basis of eigenstates of the coupling operator

\[ \hat{V}_E |n\rangle = v_n |n\rangle. \]

(20)

Out of those eigenstates we construct the set of operators \( \{ |n\rangle \langle m| \}_{n,m} \); it consists of projectors \( |n\rangle \langle n| \) and coherences \( |n\rangle \langle m| \) \( (n \neq m) \). All of these operators are mutually orthonormal in respect to the trace inner product \( \text{Tr} \left( |n\rangle \langle m| \right| n' \rangle \langle m'| \rangle = \delta_{n,n'} \delta_{m,m'} \) and since the eigenstates \( |n\rangle \) span the \( E \) subspace, they span the space of operators acting in \( E \). In other words, the set of projectors and coherences \( \{ |n\rangle \langle m| \}_{n,m} \) forms an orthonormal basis. Moreover, they are also the eigenoperators of the coupling superoperator

\[ \{ \frac{1}{2} \hat{V}_E, \cdot \} |n\rangle \langle m| = \frac{1}{2} (v_n + v_m) |n\rangle \langle m|, \]

(21)

and since the superoperator is Hermitian (in respect to the trace inner product), they are its left eigenoperators as well. Consequently, we can perform the spectral decomposition of the superoperator that is analogous to the decomposition for Hermitian operators \( \hat{V}_E = \sum_n v_n |n\rangle \langle n| \),

\[ \{ \frac{1}{2} \hat{V}_E, \cdot \} = \sum_{\nu \in \Omega_{|n\rangle \langle m|}} \nu \sum_{\xi, \zeta \in \Omega_{|n\rangle \langle m|}} \sum_{\xi + \zeta = 2\nu} |n\rangle \langle m| \text{Tr} \langle m| \langle n| \cdot \rangle. \]

(22)

The first sum over \( \nu \), is restricted to the spectrum of the coupling superoperator \( \Omega_{|n\rangle \langle m|} \) that contains all unique values of form \((v_n + v_m)/2\). The symbols for the subsequent sums are to be read according to the following rules

\[ \sum_{\xi, \zeta \in \Omega_{|n\rangle \langle m|}} \sum_{\xi + \zeta = 2\nu} \delta(\xi + \zeta - 2\nu), \]

(23)

\[ \sum_{\nu \in \Omega_{|n\rangle \langle m|}} \sum_{\xi \in \Omega_{|n\rangle \langle m|}} \delta(\nu - \xi) \delta(v_m - \zeta), \]

(24)

etc. Here, \( \Omega_{|n\rangle \langle m|} \) is the set of unique eigenvalues of coupling operator \( \hat{V}_E \) and the indexes \( n, m \) label its eigenstates \( |n\rangle, |m\rangle \).

Finally, we define the symbol for the matrix element (in the basis of projectors and coherences) of evolution superoperator—the propagator

\[ T_{\xi}^{(nm)} |n'\rangle \langle m'| = \text{Tr}( |n\rangle \langle m| e^{-i t \hat{H}_E \cdot} |n'\rangle \langle m'| ). \]

(25)
With all these elements in hand, we now substitute the spectral decompositions of \( \{ \hat{V}_k / 2, \bullet \} \) into Eq. (19) which gives us the form we sought after

\[
M^{(k)}_{\text{ctx}}(t) = \sum_{\nu_1 \in \Omega_{n_1} | m_1} \left( \prod_{l=2}^{k} \sum_{\nu_l \in \Omega_{n_l} | m_l} \right) \left( \prod_{l=1}^{k} n_l \right) Q^{(k)}_{\text{ctx}}(\nu_1, \nu_1; \ldots; \nu_k, t_k),
\]

where the \textit{joint quasiprobability distributions} \( Q^{(k)}_{\text{ctx}} \) of (quantum) process \( \nu(t) \) are given by

\[
Q^{(k)}_{\text{ctx}}(\nu, t) = Q^{(k)}_{\text{ctx}}(\nu_1, t_1; \ldots; \nu_k, t_k)
= \sum_{n_1, t_2} \left( \prod_{l=2}^{k} \sum_{\xi_l, \zeta_l \in \Omega_{n_l} | m_l} \right) \left( \prod_{l=1}^{k} \sum_{\nu_l \in \Omega_{n_l} | m_l} \right) \times T_{t_1-t_2} (n_1 m_1 | n_2 m_2) \left( \prod_{l=2}^{k-1} T_{t_l-t_{l+1}} (n_l m_l | n_{l+1} m_{l+1}) \right) (n_k | e^{-it_k \hat{R}_E} \hat{Q}_E e^{it_k \hat{R}_E} | m_k).
\]

We are retaining the subscript \textit{ctx} to make sure we do not prematurely dismiss the potential context-dependence.

The key feature of Eq. (26) is the analogy between families of quasiprobabilites \( \{ Q^{(k)}_{\text{ctx}} \}_k \) and proper probabilities \( \{ P^{(k)}_\nu \}_k \) which goes beyond simplistic formal resemblance of formulas for quantum and stochastic moments. Indeed, using the identity

\[
\sum_{\nu_1 \in \Omega_{n_1} | m_1} \sum_{\xi_1, \zeta_1 \in \Omega_{n_1} | m_1} \sum_{\nu_2 \in \Omega_{n_2} | m_2} \sum_{\xi_2, \zeta_2 \in \Omega_{n_2} | m_2} = \sum_{\nu_1 \in \Omega_{n_1} | m_1} \sum_{\nu_2 \in \Omega_{n_2} | m_2} = \sum_{n_1, m_1},
\]

and the superoperator variant of the decomposition of identity \( \sum_n |n\rangle \langle n| = \mathbb{1} \),

\[
\sum_{n_1, m_1} |n_1\rangle \langle m_1| \text{Tr}(|m_1\rangle \langle n_1| \bullet) = \bullet,
\]

plus the composition rule for evolution superoperators (17), one can verify that, just like joint probabilities, the quasiprobabilites also satisfy the Chapman-Kolmogorov consistency criterion

\[
\sum_{\nu_1 \in \Omega_{n_1} | m_1} Q^{(k)}_{\text{ctx}}(\nu_1, t_1; \ldots; \nu_{l-1}, t_{l-1}; \nu_l; t_l; \nu_{l+1}, t_{l+1}; \ldots; \nu_k, t_k)
= Q^{(k-1)}_{\text{ctx}}(\nu_1, t_1; \ldots; \nu_{l-1}, t_{l-1}; \nu_{l+1}, t_{l+1}; \ldots; \nu_k, t_k).
\]

Despite that, \( \nu(t) \) cannot be interpreted as a stochastic process yet because quasiprobabilites do not necessary satisfy the condition of non-negativity. In other words, in general, it is possible that there are regions of argument values such that \( Q^{(k)}_{\text{ctx}}(\nu_1, t_1; \ldots; \nu_k, t_k) < 0 \) — this is the motivation behind our choice to name those functions the \textit{"quasiprobability distributions"}.

2.3 Structure of joint quasiprobability distributions

Since the consistency criterion (12) is automatically satisfied by joint quasiprobabilities, our objective should now be to explain what makes \( Q^{(k)}_{\text{ctx}} \) fail the non-negativity criterion
and to determine under what circumstances this can be amended so that quasiprobabilities are no longer “quasi”.

When examined as a diagram, $Q_{\text{ctx}}^{(k)}$ can be viewed as a superposition of a selection of \emph{propagator chains} (or simply \emph{chains}) where propagators $T_t(nm|n'm')$ play the role of “chain links” and the connections between consecutive links are established through eigenoperators $|n\rangle\langle m|$ that enforce the matching of indexes

\[
\text{a two-link segment of the chain} \quad \ldots T_{t'\rightarrow t}(nm|n'm')T_{t''\rightarrow t'}(n'm'|n''m'') \ldots ,
\]  

Each chain begins with a special link in a form of density matrix element and ends with a propagator that carries a disconnected projector

\[
T_{t_1-t_2}(n_1n_1|n_2m_2)T_{t_2-t_3}(n_2m_2|n_3m_3)\ldots \quad \ldots \quad T_{t_k-1-t_k}(n_{k-1}m_{k-1}|n_km_k) \langle n_k|e^{-it_k\hat{H}_E}\hat{\varrho}_Ee^{it_k\hat{H}_E}|m_k\rangle .
\]  

In general, propagators are complex functions

\[
T_t(nm|n'm') = \langle n|e^{-it\hat{H}_E}|n'\rangle \langle m|e^{-it\hat{H}_E}|m'\rangle^*,
\]  

and hence, the chains cannot be assigned with a definite sign (in particular, they are not necessary non-negative). However, among all chains constituting a given quasiprobability distribution we can distinguish a class composed entirely of propagators connected through projectors—the \emph{projector-connected chains}—such that each link is of the form

\[
T_t(nn|mm) = |\langle n|e^{-it\hat{H}_E}|m\rangle|^2 \geq 0,
\]  

including the initial link $\langle n_k|e^{-it_k\hat{H}_E}\hat{\varrho}_Ee^{it_k\hat{H}_E}|n_k\rangle \geq 0$. Consequently, the sum of all such chains is \emph{non-negative}

\[
P_{\text{ctx}}^{(k)}(\nu, t) \equiv \left( \prod_{l=1}^{k} \sum_{n_l \in n_l} \right) \left( \prod_{l=1}^{k-1} T_{t_l-t_{l+1}}(n_{l}m_{l}|n_{l+1}m_{l+1}) \right) \langle n_k|e^{-it_k\hat{H}_E}\hat{\varrho}_Ee^{it_k\hat{H}_E}|n_k\rangle \geq 0.
\]  

Although $P_{\text{ctx}}^{(k)}(\nu, t)$ formally resembles a composition of probabilities for alternative outcomes, it is not necessarily a proper probability distribution. For example, it does not have to be normalized as it is only a partial sum of all propagator chains that constitute the joint quasiprobability distribution $Q_{\text{ctx}}^{(k)}(\nu, t)$. The reminder $\Delta Q_{\text{ctx}}^{(k)}(\nu, t)$ that consists of all the chains with, at least, one connection through coherence $|n\rangle\langle m|$ ($n \neq m$), defined by the following decomposition

\[
Q_{\text{ctx}}^{(k)}(\nu, t) = P_{\text{ctx}}^{(k)}(\nu, t) + \Delta Q_{\text{ctx}}^{(k)}(\nu, t),
\]  

has to be also taken into account for the consistency criterion (that include normalization as a special case) to be satisfied. On the other hand, it is the contribution from those \emph{coherence-connected propagator chains} that hinders the compliance with the non-negativity criterion as $\Delta Q_{\text{ctx}}^{(k)}$, in contrast to $P_{\text{ctx}}^{(k)}$, cannot be guaranteed to have a definite sign. From
an oversimplified point of view, the “quantumness” of process $\nu(t)$ manifests itself through coherence-connected chains as their composition $\Delta Q^{(k)}_{ctx}$ is a proper quantum superposition where the amplitudes for alternative outcomes have the ability to interfere with each other.

To summarize, based on what we have presented so far, the sufficient criterion for validity of surrogate field representation in the context of QIDAS, or the contextual surrogate field representation for short, can be specified as follows:

**Criterion 1 (Contextual surrogate field representation)** For QIDAS–environment arrangement to facilitate the surrogate field representation $\nu(t)$ that is inter-subjective in respect with the choices of filter functions $f(t)$—the contextual surrogate field representation—it is sufficient that the members of the family of joint quasiprobability distributions $\{Q^{(k)}_{ctx}\}_k$ are non-negative functions.

For the joint quasiprobability distribution $Q^{(k)}_{ctx}(\nu, t)$ to be a non-negative function, it is sufficient that the coherence-connected propagator chains it consists of either

(a) vanish individually, resulting in $\Delta Q^{(k)}_{ctx}(\nu, t) \approx 0$, or

(b) interfere destructively so that $\Delta Q^{(k)}_{ctx}(\nu, t) \approx 0$, or

(c) interfere constructively so that $\Delta Q^{(k)}_{ctx}(\nu, t) \geq 0$.

Note that when the superpositions of all coherence-connected propagator chains are such that either option (a) or (b) is satisfied and $\Delta Q^{(k)}_{ctx}$ is negligible we get that $Q^{(k)}_{ctx}(\nu, t) \approx P^{(k)}_{ctx}(\nu, t) \geq 0$. Then, it follows that $\{P^{(k)}_{ctx}\}_K$ composed of the remaining projector-connected propagator chain combinations, becomes a proper stochastic process-defining family of joint probability distributions that satisfy the consistency criterion.

### 3 Objective surrogate field representation

Having discussed the validity of the surrogate field representation in the reference QIDAS context, we now turn to the wider question of the surrogate’s objectivity. By objectivity we mean here the inter-contextuality of the representation; the feature that would allow for the deployment of the same surrogate field model in any context.

Following the guidelines of Sec. 2, we begin by expanding the reduced density matrix of $S$ [see the general criterion (5)] into a series analogous to the quantum moment expansion of Eq. (18), but this time, in an unspecified context

$$\text{Tr}_E \left( e^{-i t \hat{H}_{SE}} \hat{\rho}_S \otimes \hat{\rho}_E e^{i t \hat{H}_{SE}} \right) = \sum_{k=0}^{\infty} (-i)^k \int_0^t dt_1 \left( \prod_{l=1}^{k-1} \int_0^{t_l} dt_{l+1} \right) \hat{M}^{(k)}_{SE}(t).$$ (37)

Here, $\hat{H}_{SE}$ is a generic two-party Hamiltonian (1), and the moment operators, that act in $S$ subspace only, read

$$\hat{M}^{(k)}_{SE}(t) = \sum_{\xi_1 \in \Omega(n)} \left( \prod_{l=2}^{k} \sum_{\xi_l \xi_{l+1} \in \Omega(n)} \right) \hat{m}^{(k)}_{S}(\xi, \xi, t) q_{e}^{(k)}(\xi, \xi, t),$$ (38)

where we have introduced the shorthand notation

$$q_{e}^{(k)}(\xi, \xi, t) = q_{e}^{(k)}(\xi_1, \xi_1, t_1; \xi_2, \xi_2, t_2; \ldots ; \xi_k, \xi_k, t_k).$$ (39)
Note that the values of the first two arguments are matched. The operator $\hat{m}_S^{(k)}(\xi, \zeta, t)$, that describes the impact of $\hat{H}_S$, $\hat{V}_S$ and $\hat{q}_S$ on the evolution (i.e., the explicitly context-dependent part of the expression), is defined in Appendix B but its specific form is of no concern to us at the moment. Instead, we focus our attention on the family of functions

$$q_E^{(k)}(\xi, \zeta, t) = \sum_{\xi_1=\nu_1}^{n_1} \left( \prod_{l=2}^{k} \sum_{\xi_l=\nu_l}^{n_l} \sum_{m_l=1}^{m_l} \right) \times T_{t_1-t_2}(n_1n_1|n_2m_2) \left( \prod_{l=2}^{k-1} T_{t_l-t_{l+1}}(n_lm_l|n_{l+1}m_{l+1}) \right) \langle n_k|e^{-it_k\hat{H}_E}\hat{q}_Ee^{it_k\hat{H}_E}|m_k \rangle,$$

(40)

that (i) automatically satisfy the consistency criterion

$$\sum_{\xi, \zeta \in \Omega(n)} q_E^{(k)}(\xi, \zeta, t_1; \ldots; \xi_l-1, \zeta_l-1, t_l-1; \xi_l, \zeta_l, t_l; \xi_{l+1}, \zeta_{l+1}, t_{l+1}; \ldots; \xi_k, \zeta_k, t_k) = q_E^{(k-1)}(\xi_1, \xi_1; \ldots; \xi_l-1, \zeta_l-1, t_l-1; \xi_l, \zeta_l, t_l; \xi_{l+1}, \zeta_{l+1}, t_{l+1}; \ldots; \xi_k, \zeta_k, t_k),$$

(41)

and (ii) due to the contribution from coherence-connected propagator chains can violate the non-negativity condition. Therefore, just like previously discussed $Q^{(k)}_{\text{ctx}}$’s, the members of $\{q_E^{(k)}\}_k$ can be interpreted as joint quasiprobabilities distributions but, in this case, of a two-component quantum process $(\xi(t), \zeta(t))$.

Analogously to the decomposition (36) of $Q^{(k)}_{\text{ctx}}$, we can split the superposition of propagator chains constituting $q_E^{(k)}$’s into two terms

$$q_E^{(k)}(\xi, \zeta, t) = \delta(\xi - \zeta) p_E^{(k)}(\xi, t) + \Delta q_E^{(k)}(\xi, \zeta, t),$$

(42)

where $p_E^{(k)}$ is the combination of all projector-connected chains, while $\Delta q_E^{(k)}$ is the superposition of chains with, at least, one connection through coherence. As it turns out, the first term $p_E^{(k)}$ is identical to the non-negative part of $Q^{(k)}_{\text{ctx}}$, which, of course, makes it non-negative as well

$$p_E^{(k)}(\xi_1, t_1; \ldots; \xi_k, t_k) = E^{(k)}_{\text{ctx}}(\xi_1, t_1; \ldots; \xi_k, t_k) = \left( \prod_{l=1}^{k} \sum_{\xi_l=\nu_l}^{n_l} \right) \left( \prod_{l=1}^{k-1} T_{t_l-t_{l+1}}(n_lm_l|n_{l+1}m_{l+1}) \right) \langle n_k|e^{-it_k\hat{H}_E}\hat{q}_Ee^{it_k\hat{H}_E}|m_k \rangle \geq 0,$$

(43)

The sign of the remainder $\Delta q_E^{(k)}$ is, in general, indeterminate, and it is also related to the respective part of $Q^{(k)}_{\text{ctx}}$

$$\Delta Q^{(k)}_{\text{ctx}}(\xi_1, t_1; \nu_2, t_2; \ldots; \nu_k, t_k) = \left( \prod_{l=2}^{k} \sum_{\xi_l, \zeta_l \in \Omega(n_l)} \right) \Delta q_E^{(k)}(\xi, \zeta, t).$$

(44)
Both relations can be verified by a direct comparison of definitions (27) and (40).

The relation (44) calls for some additional discussion. Firstly, we see that $\Delta q_{\text{ctx}}^{(k)}$ is, essentially, a superposition of $\Delta q_E^{(k)}$’s, that in turn are superpositions of coherence-connected chains. Secondly, by inspecting its definition (40), we recognize that the sole reason for superpositions within $\Delta q_E^{(k)}$ is the natural degeneracy of the coupling operator $\hat{V}_E$. Simultaneously, the extra summation in Eq. (44) arises due to accidental degeneracy specific to the QIDAS context. The particular form of $\hat{H}_S$ and $\hat{V}_S$ of this context allows for “reduction” of the general form of moment operator into much simpler contextual moments we have found in Sec. 2.2. Therefore, while $q_{\text{ctx}}^{(k)}$’s are contextual, $q_E^{(k)}$’s can be considered to be context-independent: quasiprobabilities $q_E^{(k)}$ are defined completely and exclusively by the environment side of $SE$ arrangement (hence, the index $E$). Interestingly, QIDAS is the only example we were able to find of a context that allows for this sort of moment operator reduction and the emergence of contextual quasiprobabilities. As far as we can tell, it cannot be ruled out that QIDAS context is unique in this way.

Suppose now that environmental dynamics are such that $\Delta q_E^{(k)}$’s are negligible and the quasiprobabilities become compliant with the non-negativity criterion, $q_E^{(k)}(\xi, \zeta, t) \approx \delta(\zeta - \xi) p_E^{(k)}(\xi, t) \geq 0$. As a result, we obtain the family of proper joint probability distributions $\{ p_E^{(k)} \}_k$ that defines stochastic process $\xi(t)$. However, we still have to show that this process is actually a surrogate field, i.e., that the evolution of reduced state of $S$ can be simulated with the model Hamiltonian $\hat{H}_S(t) = \hat{H}_S + \xi(t) \hat{V}_S$. In Appendix B we show that if $\Delta q_E^{(k)} = 0$ (note that $\Delta q_E^{(k)} > 0$ would not be enough), then the moment operators take on the following form

$$\hat{M}_E^{(k)}(t) = \bar{\xi}(t_1) \ldots \bar{\xi}(t_k) e^{-i t [\hat{H}_S, \bullet]} \left( \prod_{l=1}^{k} e^{i \tau_l [\hat{H}_S, \bullet]} \hat{V}_S, \bullet \right) e^{-i t [\hat{H}_S, \bullet]} \hat{S},$$

(45)

where the stochastic average is defined in a standard way

$$\bar{\xi}(t_1) \ldots \bar{\xi}(t_k) = \left( \prod_{l=1}^{k} \sum_{\xi_l \in \Omega_{(m)}} \right) \left( \prod_{l=1}^{k} \xi_l \right) p_E^{(k)}(\xi, t).$$

(46)

When this form of moment operator is substituted back into Eq. (37) we get

$$\text{Tr}_E \left( e^{-i t \hat{H}_S} \hat{S} \otimes \hat{E} e^{i t \hat{H}_S} \right) = e^{-i t [\hat{H}_S, \bullet]} \text{Tr} e^{-i \int_0^t d\tau e^{i \tau [\hat{H}_S, \bullet]} [\xi(\tau) \hat{V}_S, \bullet]} e^{-i t [\hat{H}_S, \bullet]} \hat{S}$$

$$= \text{Tr} e^{-i \int_0^t d\tau [\hat{H}_S + \xi(\tau) \hat{V}_S, \bullet]} \hat{S} = \hat{U}(t|\xi) \hat{S} \hat{U}^\dagger(t|\xi),$$

(47)

with the stochastic evolution operators

$$\hat{U}(t|\xi) = \text{Tr} \exp \left( -i \int_0^t \left[ \hat{H}_S + \xi(\tau) \hat{V}_S \right] d\tau \right),$$

(48)

which confirms that, indeed, $\xi(t)$ is a surrogate field. Also, since the context remained unspecified throughout, the validity of representation $\xi(t)$ is context-independent, i.e., it is the objective surrogate field that represents the coupling to the specific environment.

We will now summarize the above deliberations with formally stated sufficient criterion for validity of the objective surrogate field representation; this criterion can be considered as the main result of the paper.
and hence, the joint quasiprobabilities are turned into proper probability distributions. On the other hand, the projector-connected chains (and their combinations) are preserved, and hence, the superposition of coherence-connected propagator chains $\Delta q_E^{(k)}$ is negligible so that

$$q_E^{(k)}(\xi, \zeta, t) \approx \delta(\xi - \zeta) p_E^{(k)}(\xi, t).$$

Then, the stochastic process $\xi(t)$ is defined by the family of joint probability distributions $\{p_E^{(k)}\}_k$.

When the surrogate representation is valid, and the environmental Hamiltonian $\hat{H}_E$, the initial state $\hat{\rho}_E$, and the eigensystem of the coupling $\{|n\rangle, \Omega_n\}$ are known, then, in principle, the following algorithm allows to instantiate trajectories of surrogate field $\xi(t)$:

(i) Choose an arbitrary time grid $t_{\text{grd}} = (t_1, \ldots, t_k)$ ($t_1 > \ldots > t_k$). (ii) Calculate the joint probability distribution $p_E^{(k)}(\xi, t_{\text{grd}})$ according to (43) for all values of $\xi \in \Omega_n \times \ldots \times \Omega_n = \Omega_n^k$. Although straightforward, this is the most difficult and resource intensive step. (iii) Draw at random from previously obtained distribution the sequence $\xi_{\text{smp}} = (\xi_1, \ldots, \xi_k)$; such a sequence is a sample trajectory of the process spanned on grid $t_{\text{grd}}$. This concludes the procedure.

Once the probability distribution has been successfully calculated in the second step of the above procedure, the last step can be repeated any number of times at relatively low cost. The resultant collection of sample trajectories—provided the time grid is fine enough and the number of samples is sufficiently large—can be used to carry out the averaging procedure of any quantity. This includes not only the expectation values of system-only observables, but also quantities that characterize the process itself, like its moments.

### 4 Examples of environments that facilitate objective surrogate field

#### 4.1 Quasi-static coupling

Assume that the environmental free Hamiltonian and the coupling operator commute

$$[\hat{H}_E, \hat{V}_E] = 0. \quad (49)$$

Then, the eigenstates of the coupling operator $|n\rangle$ are, simultaneously, eigenstates of the Hamiltonian $\hat{H}_E|n\rangle = \epsilon_n |n\rangle$. It follows that, within each propagator, the evolution superoperator preserves the orthogonality between projectors $|n\rangle\langle n|$ and coherences $|m\rangle\langle m'|$ ($m \neq m'$)

$$T_i(nn|mm') = \text{Tr}(|n\rangle\langle n| e^{-it[\hat{H}_E, \hat{V}_E]}|m\rangle\langle m'|) = e^{-it(\epsilon_m - \epsilon_{m'})} \text{Tr}(|n\rangle\langle n| |m\rangle\langle m'|) = 0. \quad (50)$$

Therefore, all coherence-connected chains vanish because each one of those chains contains at least one instance of propagator linking a coherence and a projector [see Eq. (32)]. In such a case, any superposition of those chains, including $\Delta q_E^{(k)}$’s, vanish as well. On the other hand, the projector-connected chains (and their combinations) are preserved, and hence, the joint quasiprobabilities are turned into proper probability distributions $q_E^{(k)}(\xi, \zeta, t) = \delta(\xi - \zeta) p_E^{(k)}(\xi, t)$ that read

$$p_E^{(k)}(\xi, t) = \left(\prod_{l=2}^k \delta_{\xi_l, \xi_{l-1}}\right) \sum_{\xi_1 = \nu_n}^k |n\rangle\langle n| \hat{\rho}_E |n\rangle.$$  

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This leads to the following form of the reduced state of the system
\[
\operatorname{Tr}_E(e^{-t\hat{H}_{SE}} \hat{\rho}_S \otimes \hat{\rho}_E e^{t\hat{H}_{SE}}) = \sum_{n} \sum_{\xi=v_n} \langle n| \hat{\rho}_E|n\rangle e^{-it(H_S + \xi\hat{V}_S)} \hat{\rho}_S = e^{-it(H_S + \xi\hat{V}_S)}. \]

The resultant surrogate field \(\xi\) is of the quasi-static noise type—a stochastic process that is time-independent (essentially, a random variable). The process is governed by the probability distribution \(p(\xi) = \sum_{n} \langle n| \hat{\rho}_E|n\rangle\) given by the initial state of \(E\) and the spectrum of values that coincide with the spectrum of coupling operator \(\Omega_\xi = \Omega_{(n)}\).

### 4.2 Open environment

Suppose that the environmental degrees of freedom can be further separated into two subspaces: one that is in direct contact with the system (let us still label it as \(E\)), and the other part \((D)\) that is decoupled from the system but interacts with \(E\)
\[
\hat{H} = \hat{H}_S \otimes \hat{1} \otimes \hat{1} + \hat{V}_S \otimes \hat{V}_E \otimes \hat{1} + \hat{1} \otimes \hat{H}_E \otimes \hat{1} + \hat{1} \otimes \hat{V}_ED + \hat{1} \otimes \hat{1} \otimes \hat{H}_D. \tag{53}
\]
Essentially, \(D\) is an environment of \(E\) but not of \(S\).

In Appendix C we show that the joint quasiprobability distributions resultant from this form of environmental dynamics are given by an effective average over \(D\) degree of freedom (a partial trace over \(D\)):
\[
q^{(k)}_E(\xi, \zeta, t) = \sum_{n_{l_1}} \sum_{\xi_{l_1}=v_{n_{l_1}}} \left( \prod_{l=2}^{k} \sum_{\xi_{l_2}=v_{n_{l_2}}} \sum_{\zeta_{l_2}=v_{m_{l_2}}} \right) \langle n_1| \operatorname{Tr}_D \left( \prod_{l=2}^{k} \hat{U}_{ED}(t_{l-1} - t_l)|n_l\rangle\langle n_l| \right) \times \hat{U}_{ED}(t_k) \hat{\rho}_E \otimes \hat{\rho}_D \hat{U}_{ED}^\dagger(t_k) \left( \prod_{l=k}^{2} |m_l\rangle\langle m_l| \hat{U}_{ED}^\dagger(t_{l-1} - t_l) \right) |n_1\rangle, \tag{54}
\]

where the symbol \(\prod_{l=1}^{e_{l_i}} \hat{A}(t)\) applied to operators is to be read as an ordered composition \(\hat{A}(l_b)\hat{A}(l_b + 1)\ldots\hat{A}(l_c)\) (for \(l_b < l_c\)) or \(\hat{A}(l_b)\hat{A}(l_b - 1)\ldots\hat{A}(l_c)\) (for \(l_b > l_c\)), and the unitary evolution operator
\[
\hat{U}_{ED}(t) = e^{-it(\hat{H}_E + \hat{V}_E + \hat{H}_D)}, \tag{55}
\]
operates in \(ED\) subspace while the projectors \(|n_l\rangle\langle n_l|\) onto eigenstates of \(\hat{V}_E\) act only in \(E\) subspace.

Assume the initial state \(\hat{\rho}_D\) and the relation between coupling \(\hat{V}_{ED}\) and the free Hamiltonians are such that we can invoke the Born approximation
\[
\hat{U}_{ED}(t) \hat{A} \otimes \hat{\rho}_D \hat{U}_{ED}^\dagger(t) \approx \hat{A}(t) \otimes \hat{\rho}_D. \tag{56}
\]

In addition, in order to parametrize the undergoing dynamical process only in the terms of environment part that couples directly to the system, assume the Markov and secular approximations that specify the form of the dynamical map acting on \(E\)
\[
\hat{U}_{ED}(t - t') \hat{A} \otimes \hat{\rho}_D \hat{U}_{ED}^\dagger(t - t') \approx (\Lambda(t, t') \hat{A}) \otimes \hat{\rho}_D. \tag{57}
\]
Here, superoperators \(\Lambda(t, t')\) satisfy the following composition rule
\[
\Lambda(t, t')\Lambda(t', t'') = \Lambda(t, t''), \text{ for } t > t' > t'', \tag{58}
\]
and are generated by non-hermitian superoperator $L_E(\tau)$—the so-called Lindbladian—that acts in the $E$-operators subspace only

$$\Lambda(t, t') = T e^{\int_{t'}^t L_E(\tau) d\tau}, \text{ for } t > t'. \quad (59)$$

In the terms of open system theory, it is this full suite of approximations that lead to a quantum master equation for the evolution of reduced state of a system open to $D$

$$\frac{\partial}{\partial t} \hat{\rho}_E(t) = L_E(t) \hat{\rho}_E(t). \quad (60)$$

In our case, the secular Born–Markov approximation leads to quasiprobability distributions in a standard form of propagator chain superpositions (40) but with propagator links (25) modified according to

$$T_{t-t'}(nm|n'm') = \text{Tr}_E(|m\rangle \langle \Lambda(t, t')|n\rangle \langle m'|), \quad (61)$$

and the analogous modification to the initial link where $\langle n_k|e^{-it_k[H_E, \bullet]}\hat{\rho}_E|m_k\rangle$ is replaced with $\langle n_k|\Lambda(t_k, 0)\hat{\rho}_E|m_k\rangle$. Note that the composition rule (58) is crucial, as it is required for quasiprobability distributions to satisfy the consistency criterion.

The fact that dynamical map $\Lambda(t, t')$ is not unitary (Lindbladian is non-Hermitian in general), opens new possibilities for breaking the coherence-connected propagator chains. One way to achieve such an effect, is for the evolution superoperator to satisfy

$$\Lambda(t, t')|n\rangle\langle n| = \sum_m u_{m,n}(t, t')|m\rangle\langle m|, \quad (62a)$$

$$\Lambda(t, t')|n\rangle\langle n'| = \sum_{m\neq m'} w_{mm',nn'}(t, t')|m\rangle\langle m'| \quad (n \neq n'). \quad (62b)$$

That is, the superoperator maps projectors onto combination of projectors and coherences onto combination of coherences, thus, preserving their mutual orthogonality. When this is the case, then all coherence-connected chains constituting $\Delta q^{(k)}_E$ vanish because each one of them contains at least one instance of propagator of form $T_l(|n_m|n_{m+1}|m_{l+1}i) \propto \delta_{n_{l+1},m_{l+1}} = 0 \ (m_{l+1} \neq m_{l+1})$; note the similarity to quasi-static coupling case from Sec. 4.1.

Moreover, the remaining combinations of projector-connected chains $p^{(k)}_E$ are guaranteed to be non-negative because $\Lambda(t, t')$ is a trace-preserving and completely positive map so that $u_{m,n}(t, t') \geq 0$ and $\sum_m u_{m,n}(t, t') = 1$, for all $t > t'$ and $n, m$. Therefore, when environment dynamics have the property (62), $\{p^{(k)}_E\}_{k=1,\ldots,\infty}$ is a family of proper joint probability distributions and they define a surrogate field.

The following simple example showcases how this type of environmental dynamics supports an objective surrogate field representation. Let $E$ be a two-level system that is driven by time-independent Lindbladian $L_E = -(\gamma/2)[\hat{\sigma}_x, [\hat{\sigma}_x, \bullet]]$ and the coupling operator is $\hat{V}_E = \hat{\sigma}_z/2$. Then, the coupling has two eigenvalues $v_\pm = \pm 1/2$ corresponding to $|\pm\rangle$ eigenstates. It is a matter of straightforward algebra to verify that conditions (62) are satisfied here. The resultant probability distributions are given by

$$p^{(k)}_E(\xi, t) = \langle \text{sign}(\xi_k)|e^{iL_E t}\hat{\rho}_E|\text{sign}(\xi_k) \prod_{l=1}^{k-1} \left(1 + \text{sign}(\xi_l)\text{sign}(\xi_{l+1})e^{-2\gamma(t-l+1)}\right) \right. \quad (63)$$

with process spectrum $\Omega_\xi = \{+1/2, -1/2\}$. We recognize that this family of probability distributions describe a well known random telegraph noise [41]—a stochastic process that switches between two values, $\xi = \pm 1/2$ in this case, at the rate $\gamma$. 

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4.3 Environment of least action

In the last example, we will require that the spectrum of the coupling is dense so that the sums in (40) can be replaced with integration

$$\sum_{m_i} \sum_{m_j} \rightarrow \int_{\Gamma_{\xi_l}} \int_{\Gamma_{\xi_l}} dx_i \int_{\Gamma_{\xi_l}} dy_i,$$

(64)

where the intervals $\Gamma_{\xi_l}$ are the degenerate subspaces corresponding to eigenvalues $\xi_l$ and $\Gamma_\infty$ is the whole configuration space. Using this representation we rewrite (40) into a form that will be better suited for our current purposes

$$q_E^{(k)}(\xi, \zeta, t) = \int_{\Gamma_{\xi_l}} dx_i \int_{\Gamma_{\xi_l}} dy_i \delta(x_1 - y_1) \int_{\Gamma_{\xi_l}} dx_0 \int_{\Gamma_{\xi_l}} dy_0 \times K(x_1, x_0, t|\Gamma_{\xi_2} \ldots \Gamma_{\xi_k}) K(y_1, y_0, t|\Gamma_{\xi_2} \ldots \Gamma_{\xi_k})^* \langle x_0 | \hat{\rho}_E | y_0 \rangle,$$

(65)

where we have defined the Schrödinger chains

$$K(x_1, x_0, t|\Gamma_{\xi_2} \ldots \Gamma_{\xi_k}) = \left( \prod_{l=2}^{k} \int_{\Gamma_{\xi_l}} dx_l \right) \left( \prod_{l=1}^{k-1} \langle x_l | e^{-i(t_l-t_{l+1})\hat{H}_E(x_{l+1})} \right) \langle x_k | e^{-i t_k \hat{H}_E(x_0)} \rangle,$$

(66)

which are simply an alternative to the propagator chain description.

Assume the environment is such that the least action principle approximation is applicable to the Feynman path integral representation of its Schrödinger propagators [44]

$$\langle x_e | e^{-i(t_e - t_0)\hat{H}_E} | x_b \rangle = \int_{x(t_0)=x_b}^{x(t_e)=x_e} \mathcal{D}x(t) e^{i S[x(t)]} \propto e^{i S_{cl}(x_e t_e; x_b t_b)},$$

(67)

where $S_{cl}(x_e t_e; x_b t_b)$ is the environment action [45] associated with the classical trajectory of coordinate $x$—i.e., the trajectory $x(t)$ that satisfy the corresponding Euler-Lagrange equation [45]—that begins at point $x_b$ at initial time $t_b$, and ends at point $x_e$ at time $t_e$. The approximation is justified using the stationary phase method: Since the action $S[x(t)]$ is large (e.g., like for massive macroscopic systems), the destructive interference between rapidly oscillating phase factors $\exp\{i S[x(t)]\}$ suppresses the integration over almost all trajectories, except for the immediate vicinity of the stationary point (or rather, the stationary trajectory) of action. The least action principle of classical mechanics asserts that the trajectory which satisfies the classical equation of motion is such a stationary point (and vice versa). Therefore, the only significant contribution to the integral comes from the neighborhood of $S_{cl}$, where the phase slows down and the interference is constructive.

First, we will consider one of the intermediate segments along the Schrödinger chain,

$$\int_{\Gamma_{\xi_l}} d x_l \langle x_1 \ldots x_{l-1} | e^{-i(t_{l-1} - t_l)\hat{H}_E} | x_l \rangle \langle x_l | e^{-i(t_l - t_{l+1})\hat{H}_E} | x_{l+1} \rangle \propto \int_{\Gamma_{\xi_l}} d x_l e^{i S_{cl}(x_{l-1} t_{l-1}; x_l t_l) + i S_{cl}(x_l t_l; x_{l+1} t_{l+1})},$$

(68)

where, for now, we will treat the time arguments $t_{l-1} > t_l > t_{l+1}$ and the end points $x_{l-1}, x_{l+1}$ as fixed values. Since the action is large, according to stationary phase method, the integral “stitching” the propagators will vanish due to destructive interference, unless the degenerate subspace $\Gamma_{\xi_l}$ contains a stationary point of the phase. Of course, to determine if $x_l$ is such a point we have to check the derivative of the phase,

$$\frac{\partial [S_{cl}(x_l t_l; x_{l+1} t_{l+1}) + S_{cl}(x_{l-1} t_{l-1}; x_l t_l)]}{\partial x_l} = p_e - p_b,$$

(69)
Here, we have utilized the theorem from classical theory that the derivative of the action in respect to the end/beginning point of the trajectory equals the momentum/minus momentum at the corresponding time \[46\], and so, \( p_0 \) is the momentum at the end of trajectory from \( x_{l+1} \) to \( x_l \), and \( p_b \) is the initial momentum at the beginning of trajectory from \( x_1 \) to \( x_{l-1} \). In general, \( p_b \neq p_e \) and \( x_1 \) is not a stationary point. Indeed, if we set the initial momentum at \( q_1 \) to \( p_e \), then the coordinate would propagate, in accordance with Euler-Lagrange equation, from \( x_1 \) to a certain point \( \tilde{x}_{l-1} \) that is different than the expected end point \( x_{l-1} \). In order to make the end point match the desired \( x_{l-1} \), the initial momentum has to be adjusted, which can be visualized as an application of impulse force that causes the discontinuity in momentum. However, there is one instance when such an intervention is not necessary: \( x_1 \) is the stationary point (i.e., \( p_e = p_b \)) when it happens to lie on the classical trajectory from \( x_{l+1} \) directly to \( x_{l-1} \).

We can now apply the above reasoning to the Schrödinger chain \( K(x_1, x_k, t|\Gamma_{\xi_2} \ldots \Gamma_{\xi_k}) \) as a whole. For given \( t \) and the end points of the trajectory \( x_1, x_0 \), the interference effects restrict the choice of \( \Gamma_{\xi_2}, \ldots, \Gamma_{\xi_k} \) to only one sequence where each interval overlaps with the classical trajectory from \( x_0 \) to \( x_1 \). Since each \( \Gamma_{\xi} \) corresponds to eigenvalue \( \xi_l \), the choice of arguments \( \xi \) for which \( \chi^{(k)}_E(\xi, \xi, t) \neq 0 \), is identically restricted. For the same reasons, but applied to the other Schrödinger chain \( K(y_1, y_0, t|\Gamma_{\xi_2} \ldots \Gamma_{\xi_k}) \), the same is true for \( \zeta \).

In order to turn \( \chi^{(k)}_E \)'s into \( p^{(k)}_E \)'s, and thus, obtain the valid surrogate field, the sequences of arguments \( \xi = (\xi_1, \ldots, \xi_k) \) and \( \zeta = (\zeta_1, \ldots, \zeta_k) \) have to be forced to match up exactly. The first elements of the sequences match up by default because the classical trajectories corresponding to each Schrödinger chain, end in the same point. If the beginning points \( x_0 \) and \( y_0 \) would be the same as well, then the classical trajectories would overlap and, as a result, the sequences would overlap too. The initial positions of each trajectory are determined by the initial state \( \hat{\rho}_E \). Therefore, when the least action approximation applies, the environment facilitates the objective surrogate field representation when its initial state satisfies

\[
\langle x_0|\hat{\rho}_E|y_0 \rangle = \delta(x_0 - y_0)\rho_E(x_0). \quad (70)
\]

Physically, this means that the environment should not be initialized in the Schrödinger’s cat type of state.

5 Discussion

5.1 Impostor field representations

The most common methods for spectral analysis with QIDAS are designed to operate under so-called Gaussian approximation \[8–10, 29, 30, 47, 48\]

\[
W_E(t|f) \approx e^{-\int_0^t dt_1 \int_0^{t_1} dt_2 f(t_1)f(t_2)M^{(2)}_{\text{class}}(t_1,t_2)}. \quad (71)
\]

For simplicity we assumed here the stationary state of the environment \( e^{-i\hat{H}_E \hat{\rho}_E}e^{i\hat{H}_E} = \hat{\rho}_E \) and a zero average value of the coupling operator \( M^{(1)}_{\text{class}}(t) = \text{Tr}(\hat{V}_E\hat{\rho}_E) = 0 \). Within this approximation, the dynamics of the qubit are determined by a single function—the second order autocorrelation \( C(t_1-t_2) = M^{(2)}_{\text{class}}(t_1,t_2) \).

It is a well established feature of Gaussian dynamics in QIDAS context that it can always be simulated with a stochastic model. The reason why it is so are the relatively lenient constraints imposed on the model: the simulator has to be a Gaussian process itself.
and its second order autocorrelation function (equivalently, its second moment, when the average is zero) has to match \( C(t) \). Although there is no known procedure for generating Gaussian stochastic process (in the sense of having the ability to instantiate its trajectories) with a given form of autocorrelation function, it is always possible to approximate it with arbitrary precision by taking a combination of simpler statistically independent processes. One example of such a model is constructed out of two-component random variables

\[
\xi_{\text{Gauss}}(t) = \sum_{\omega \in \Omega_s} \left[ x_\omega \cos(\omega t) - y_\omega \sin(\omega t) \right],
\]

(72)

where \( \Omega_s = \{ \omega : s_\omega = \int_{-\infty}^{\infty} e^{-i\omega t} C(t) dt \neq 0 \} \), and \( \{ x_\omega, y_\omega \}_\omega \) is a set of independent Gaussian random variables distributed according to

\[
p(\{ x_\omega, y_\omega \}_\omega) = \prod_{\omega \in \Omega_s} \frac{1}{s_\omega} \exp \left[ -\frac{\pi(x_\omega^2 + y_\omega^2)}{s_\omega} \right].
\]

(73)

In an appropriate limit \( \xi_{\text{Gauss}}(t) \) converges to the unique Gaussian process determined by the original autocorrelation function \( C(t) \). Of course, this means that only because the autocorrelation function of \( \xi_{\text{Gauss}}(t) \) approximates \( C(t) \), one should not ascribe any physical meaning to variables \( x_\omega, y_\omega \) used in its construction. The same is true for any other example of such a construct.

There is an interesting consequence to this property of Gaussian dynamics. Since the stochastic simulation is always possible in QIDAS context, it also means that a Gaussian model can be constructed even when \( \Delta q^{(k)}_E \neq 0 \) and the objective surrogate field representation is invalid. In Appendix D, we discuss an illustrative example of such a case. In order to distinguish this type of Gaussian model from the objective surrogate we adopt the following definition: Any stochastic model that is not explicitly constructed by the means of the family of joint probability distributions \( q^{(k)}_E(\xi, \zeta, t) \approx \delta(\xi - \zeta)p^{(k)}(\xi, t) \), but instead, is postulated or constructed in any other way under the constraint that its autocorrelation functions match certain form, will be referred to as an impostor field representation.

The concept of the impostor can be generalized beyond the context of QIDAS or Gaussian approximation. Suppose that the system–environment arrangement is such that the moment operators (38) have the form analogous to Eq. (45)

\[
\hat{M}^{(k)}_{SE}(t) = F^{(k)}(t)e^{-it[H \hat{S}, \cdot]} \left( \prod_{l=1}^{k} e^{it[H \hat{S}, \cdot][\hat{V}_S, \cdot]} e^{-it[H \hat{S}, \cdot]} \right) \hat{\rho}_S.
\]

(74)

Now, if one assumes that it is possible to identify functions \( F^{(k)}(t) \) with the moments of certain stochastic process, i.e., \( F^{(k)}(t) = M^{(k)}_{\xi}(t) = \xi(t_1) \cdots \xi(t_k) \) [see Eq. (10)], then, in this specific context, it is possible to construct the model Hamiltonian \( \hat{H}_\xi(t) = \hat{H}_S + \xi(t)\hat{V}_S \) that would simulate the dynamics of the system \( S \) [compare with Eq. (47)]. To tie this back to our previous discussion, note that, in the context of QIDAS, the moment operators can always be cast into form (74). If, on top of that, one also assumes Gaussian approximation, then \( F^{(k)}(t) \)’s factorize into specific combinations of \( F^{(2)}(t_1, t_2) \)’s (assuming that \( F^{(1)}(t_1) = 0 \)), e.g., \( F^{(4)}(t) = F^{(2)}(t_1, t_2)F^{(2)}(t_3, t_4) + F^{(2)}(t_1, t_3)F^{(2)}(t_2, t_4) + F^{(2)}(t_1, t_4)F^{(2)}(t_2, t_3) \). The moments of Gaussian process factorize in the same way with the second order autocorrelation function playing the role of \( F^{(2)} \). Note that this factorization is the unique property of Gaussian processes—there cannot exist a kind of “super-Gaussian” stochastic process, where all of its moments are expressed by a finite, but greater than two, number of independent autocorrelation functions [49]. Because of that, when the Gaussian approximation does not apply, the candidate for stochastic model \( \xi(t) \) has to be postulated

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or constructed in such a way so that each of the infinitely many moments \( M^{(k)}_\xi(t) \) equals the corresponding multivariate function \( F^{(k)}(t) \). Needless to say, in general, constructing a process that fits such criteria is an extremely difficult task, and, to make the matter even more daunting, there is no guarantee it can be accomplished given the arbitrary set of functions \( \{F^{(k)}\}_k \). When one succeeds anyway and finds \( \xi(t) \) that fits all \( F^{(k)} \)'s, by any other means than explicitly constructing the family \( \{q^{(k)}_E\}_k \), then this stochastic model is an example of an impostor field representation.

The way we proposed to generalize the notion of impostor representations provokes the following important question: Knowing that the full set of moments uniquely defines a stochastic process, and that the objective surrogate field representation results in moment operators of the form analogous to (74), is it true that the impostor that fits all the context specific \( F^{(k)} \)'s is automatically identical to the objective surrogate? The answer is negative, and to demonstrate why it is so, let us again consider the QIDAS context. In Sec. 2 we showed that, when the Gaussian approximation is not applied, the stochastic simulator takes on the form of the contextual surrogate field. However, the contextual surrogate does not imply the validity of the objective surrogate. Indeed, even if QIDAS–environment arrangement satisfy the criterion 1 of Sec. 2.3, the vanishing of \( \Delta Q^{(k)}_{\text{ctx}} \) (which is one of the requirements of the criterion) does not guarantee that \( \Delta q^{(k)}_E \)'s are also negligible [see relation (44)]. In fact, according to our definition, the contextual surrogate field should be classified as an impostor because the way it is constructed does not match exactly the blueprint for objective surrogate. Hence, even though we were unable to list any other example of contextual surrogate representation that for QIDAS, we can now say that an impostor that fits all context specific \( F^{(k)} \)'s without any additional arrangement-enabled approximations, would also be an example of a contextual type of surrogate field. This observation helps us to clearly define the fundamental difference between impostor representations and the objective surrogate field representation: Impostors are not inherently inter-contextual (inter-subjective) and could happen to be context-independent only by accident. In contrast, the objective surrogate representation is inter-contextual by design.

5.2 Surrogate field and back-action

A commonly entertained hypothesis (e.g., see Refs. [19, 21, 24, 25]) proposes that for the stochastic modeling of system–environment interaction to work, the coupling between \( S \) and \( E \) has to cause no back-action. The absence of back-action is understood here as the asymmetry between the system and the environment where \( E \) influences \( S \) but \( S \) does not influence \( E \).

This hypothesis can be motivated by the following intuitive reasoning. When there is no back-action, it stands to reason that \( E \) evolves as if \( S \) did not exist, and hence, the environment can always be assigned with a definite state \( \hat{\varrho}_E(t) \) as the dynamical equation of its motion is decoupled from the system. Moreover, if the state of one of the parties is definite at all times, then the state of the total system can only be separable

\[
\hat{\varrho}_{SE}(t) \sim \hat{\varrho}_S(t\{\hat{\varrho}_E(\tau) : 0 < \tau < t\}) \otimes \hat{\varrho}_E(t).
\]

where the evolution of the system state is, in general, dependent on the history of the environment. [Compare with Born approximation (56) of Sec. 4.2.] When this is the case, it seems reasonable to anticipate that, from the point of view of the system, \( E \) would act as a source of external (i.e., independent of \( S \)) field that drives its evolution. On the other hand, if the system evolves as if driven by an external field, it seems self evident that it
would be a contradiction if $S$ was able to influence the field’s source. In what follows, we will investigate if this line of argument holds up.

To our best knowledge, there is no universally accepted standard criterion for verifying the absence of back-action that we could make use of. Therefore, in order to analyze the above proposition and its justification, we will adopt a criterion that is the simplest and agrees with the most basic intuition: The back-action will be considered absent (or, at least, negligible) when the expectation value of any $E$-only observable is unchanged in comparison to the value obtained in the case when there is no system–environment coupling. Formally, this criterion is expressed as

$$\text{Tr}_S \left( e^{-it[H_S + \hat{\mathcal{H}}_E + \hat{V}_E \otimes \hat{V}_E]} \hat{\rho}_S \otimes \hat{\rho}_E \right) \approx \text{Tr}_S \left( e^{-it[H_S + \hat{\mathcal{H}}_E] \hat{\rho}_S \otimes \hat{\rho}_E} \right),$$  \hspace{1cm} (76)$$

where $\text{Tr}_S(\bullet)$ indicates the partial trace over system degrees of freedom. With the use of this criterion, and the following counter examples, we will now show that the intuitive understanding of the role of back-action presented above is faulty and that there is no causal link between the lack of back-action and the validity of surrogate field representation.

First, we select QIDAS for $S$ and an arbitrary $\hat{H}_E, \hat{V}_E$. Then, when the interaction is present, with some algebra, we can express the reduced density matrix of the environment in the terms of propagator chains

$$\text{Tr}_S \left( e^{-i[t\hat{H}_E + \frac{1}{2} \hat{\mathcal{H}}_E \otimes \hat{V}_E \int_0^t f(\tau) d\tau \cdot \bullet] \hat{\rho}_S \otimes \hat{\rho}_E} \right) = \sum_{s=\pm} \langle s | \hat{\rho}_S | s \rangle e^{-i[t\hat{H}_E + \frac{1}{2} s \int_0^t f(\tau) \hat{V}_E \cdot \bullet] \hat{\rho}_E}$$

$$\sum_{k=0}^{\infty} \left( -i \frac{s}{2} \right)^k \left( \prod_{l=1}^{k} \sum_{n_l \neq m_l} \int_0^t dt_1 \left( \prod_{l=1}^{k-1} \int_0^{t_l} dt_{l+1} \right) \right) \langle \prod_{l=1}^{k} (v_{n_l} - v_{m_l}) \rangle \langle n_k | e^{-it_k \hat{H}_E \otimes \hat{\rho}_E} e^{it_k \hat{H}_E} | m_k \rangle \times e^{-i(t-t_1)\hat{\mathcal{H}}_E \cdot \bullet} | n_1 \rangle \langle m_1 |,$$  \hspace{1cm} (77)$$

Note that the links in the chains are connected only through coherences (i.e., the index pairs in each sum cannot match up). With this in mind, we take the initial state $\hat{\rho}_E \propto \mathbb{1}$. Then, in the above expression, only $k = 0$ term survives because the initial link of each chain vanishes as $\langle n_k | e^{-it_k \hat{H}_E \mathbb{1} \otimes \hat{\rho}_E} e^{it_k \hat{H}_E} | m_k \rangle = 0$, which leaves us with

$$\text{Tr}_S \left( e^{-i[t\hat{H}_E + \frac{1}{2} \hat{\mathcal{H}}_E \otimes \hat{V}_E \int_0^t f(\tau) d\tau \cdot \bullet] \hat{\rho}_S \otimes \hat{\rho}_E} \right) = \sum_{s=\pm} \langle s | \hat{\rho}_S | s \rangle e^{-i[t\hat{\mathcal{H}}_E] \hat{\rho}_E}$$

$$= \text{Tr}_S \left( e^{-i[t\hat{\mathcal{H}}_E] \hat{\rho}_S \otimes \hat{\rho}_E} \right) \text{Tr}_S \left( e^{-i[H_S + \hat{\mathcal{H}}_E] \hat{\rho}_S \otimes \hat{\rho}_E} \right),$$  \hspace{1cm} (78)$$

i.e., according to criterion (76), the back-action disappears. On the other hand, restricting the form of the initial state is not sufficient to ensure that each $\Delta_H^{(k)} \approx 0$ (or that $Q^{(k)}_{\text{cx}}$‘s are non-negative). Therefore, even though there is no back-action, the surrogate field representation is not guaranteed to be valid.

With another counter example, we can also disprove the reciprocal assertion that a valid surrogate field implies the lack of back-action. In this case, we keep the same choice for $S$ as before, but we specify $E$ to be of the quasi-static coupling type discussed in Sec. 4.1. When $[\hat{H}_E, \hat{V}_E] = 0$, the expressions for the reduced density matrix with, and
without, system–environment coupling simplify as follows:

\[
\text{Tr}_S (e^{-it([\hat{H}_E + \frac{1}{2} \hat{V}_E \otimes \hat{\rho}_E] f(t) d\tau, \hat{\rho}_S \otimes \hat{\rho}_E)) = \sum_{n,m} |n\rangle \langle m| e^{-it(\epsilon_n - \epsilon_m)} \langle m| \hat{\rho}_E|n\rangle} \sum_{s=\pm} \langle s| \hat{\rho}_S|s\rangle e^{-\frac{i}{2}s(v_n - v_m)} \int_0^t f(\tau) d\tau,
\]

(79)

\[
\text{Tr}_S (e^{-it[\hat{H}_E, \hat{\rho}_S \otimes \hat{\rho}_E]) = \sum_{n,m} |n\rangle \langle m| e^{-it(\epsilon_n - \epsilon_m)} \langle m| \hat{\rho}_E|n\rangle}.
\]

(80)

where \( \hat{H}_E|n\rangle = \epsilon_n|n\rangle \). On the one hand, we have demonstrated in Sec. 4.1 that quasi-static coupling facilitates valid surrogate field representation. On the other hand, by comparing Eqs. (79) and (80) we can see that, in those same circumstances, the qubit can still influence the environment. Hence, it is possible that a valid surrogate field representation exists, while \( S \) exerts the back-action onto \( E \).

The above examples demonstrate that, contrary to the “common sense” intuition, there is no causal link between the lack of back-action and surrogate field representation. We believe the reason for this counter-intuitive disconnect can be explained with another intuitive picture. As we argued previously, no back-action means that the state of \( E \) remains definite and is independent of \( S \), hence, it is a statement about the environment as a whole. On the other hand, the surrogate field representation deemphasizes the role of the state of the environment \( \hat{\rho}_E(t) \), and instead, places the focus on the coupling \( \hat{V}_E(t) \) and its dynamics: when the valid surrogate exists, one could say that it is the “state” of the coupling operator which remains definite (or that it can be assigned with a definite “value”), and so it can be as well superseded with an external field. As it turns out, the way the state of the environment evolves is not necessarily the decisive factor in determining the “state” of the coupling.

5.3 Surrogate field and system–environment entanglement

Another often considered hypothesis states that the absence of system–environment entanglement is a necessary condition for valid stochastic modeling of the system dynamics [25, 28]. It seems that this supposition is partially motivated by a view that the causes for decoherence can be segregated into two distinct categories [3]: the “smearing” of coherence due to formation of entanglement with the environment, or the decay of coherence due to ensemble average of stochastic evolution, i.e., coupling with external noise. (Although, it has been demonstrated that such categorization is not completely watertight [50–55].) The hypothesis might also ring true to some because it agrees with a common-sense intuition: since stochastic models are commonly described as “classical” and the entanglement is “non-classical”, they cannot coexist. However, we have to point out that such a reasoning is flawed; it is an example of deceptively easy to commit association fallacy where one asserts that certain qualities are shared by a collection of things because they are label with similar or related descriptors. In this case, the culprit is, of course, an association due to ambiguous notion of classicality.

In what follows, we will demonstrate that the hypothesis is incorrect, and that there is no good reason to establish a causal link between entanglement and stochastic modeling with the surrogate field representation. The recently discovered criterion for the absence of entanglement between QIDAS and its environment [54] will be instrumental in our task, as it allows us to transparently incorporate into its structure our own criteria for surrogate field validity. The no-entanglement criterion reads: the QIDAS is not entangled with its
environment at time $t$ if and only if

$$e^{-i(t\hat{H}_E + \frac{1}{2}\hat{V}_E \int_0^t f(\tau) d\tau, \bullet)}_{\hat{\varrho}E} = e^{-i(t\hat{H}_E - \frac{1}{2}\hat{V}_E \int_0^t f(\tau) d\tau, \bullet)}_{\hat{\varrho}E}. \tag{81}$$

First, set the initial state of $E$ to $\hat{\varrho}E \propto \hat{1}$, then Eq. (81) is trivially satisfied. On the other hand, specifying the initial state is not sufficient for ensuring non-negativity of quasiprobability distributions. Therefore, even though there is no entanglement with the environment, the surrogate field might not be valid.

Second, choose $E$ to be of the quasi-static coupling type (i.e., $[\hat{H}_E, \hat{V}_E] = 0$), then the surrogate field representation is guaranteed to be valid, but the criterion (81) is not necessarily satisfied because

$$e^{-i(t\hat{H}_E \pm \frac{1}{2}\hat{V}_E \int_0^t f(\tau) d\tau, \bullet)}_{\hat{\varrho}E} = \sum_{n,m} |n\rangle \langle m| e^{-i(\epsilon_n - \epsilon_m)t \pm \frac{1}{2}(v_n - v_m) \int_0^t f(\tau) d\tau} \langle n|_{\hat{\varrho}E}|m\rangle. \tag{82}$$

Hence, even when the surrogate field representation is valid, the system can still become entangled with its environment.

As it was the case in the previous section, also here, the causal link has to be dismissed. The reasons for the disconnect are essentially the same as before: the entanglement is a statement about the state of system–environment complex, while the surrogate field representation is concerned only with substituting for the coupling.

### 5.4 What is classical about surrogate field?

In classical theory, a particle is considered an element of objective reality—it is assumed that it unconditionally exists in some definite state at all times. In the formalism of the theory, the state of the particle is equated to continuous single-valued trajectory $r(t)$ representing the position of its center of mass as a function of time. If the system is composed of multiple particles labeled with index $i$, the description is extended by simply including a trajectory $r_i(t)$ for each constituent so that each one of them is an element of objective reality.

Note that the unconditional existence assumption implies that the state of classical particle is inter-subjective. Indeed, since the position and the momentum are definite at all times, then all observers will report the same result when they measure them at the given moment in time. This points to the first analogy between classical theory and the surrogate field representation. When we know that any system coupled to the environment that facilitates its objective surrogate will experience the same field, and that the experience of such systems is the only possible record about the surrogate, then it makes no practical difference if we choose to presume that the surrogate exists even if no one is “looking”. Therefore, we can say that the objective surrogate field can be considered an element of objective reality.

Although the very fact of the classical particle’s existence—formally represented by uninterrupted generation of its trajectory—does not rely on any other agent, these “other agents” can intervene and cause the particle’s trajectory to be modified. In the formalism of the theory, the modifications due to particles’ interactions are governed by an appropriate set of coupled equations of motion for all trajectories. However, it is impossible to store an unambiguous record about the form of equations of motion in any of those modified trajectories. Or in other words, the same set of trajectories could result from whole plethora of different sets of equations. In particular, it is always possible to replace equations that couple many trajectories through interaction potentials with a set of decoupled equations where each particle experiences an external force field. Equivalently, one can describe
the dynamics of these particles in terms of constraint motion—the method that allows to “conceal” most of (or even all) such force fields by switching to properly chosen set of generalized coordinates. Hence, one can always describe a multi-particle system in terms of independent particles, each riding on an elaborately constructed track that leads it over trajectory that is identical to one generated in the presence of interactions. The model of epicycles in Ptolemaic system of astronomy is an example of such an approach.

The concept of external force fields and the method of constraint motion, naturally supported by classical theory, are, in general, not compatible with the formalism of quantum mechanics. However, the cases when the surrogate field (objective or contextual), or even the impostor fields, are valid, represent exceptions when a multi-party quantum system allows this kind of semi-classical description. It is the second reason why surrogate field representation can be considered classical.

5.5 Is it possible to observe surrogate’s trajectories?

The theory of surrogate field representation is constructed under the assumption that its predictions of measurement results are given in the terms of quantities averaged over surrogate’s trajectories. This means that the scope of the theory is too narrow to provide a meaningful answer to the question whether the trajectories can be observed or measured. Therefore, if one wishes to tackle this question, an expanded theory has to be considered. Note that we have faced a similar issue in Sec. 5.3, where, in order to analyze the system–environment entanglement, we had to use the criterion taken from outside the surrogate field theory. Even though the criterion did not utilize the averaging procedure at any point, it would not be possible to arrive at satisfactory solution without the input from the surrogate’s theory on the properties of system–environment arrangement.

Inventing an expanded theory that would define the measurement scheme for trajectories lies beyond the scope of this paper. Instead, we would like to engage in preliminary speculations about conditions that should be met by a potential solution: (i) The surrogate field represents a given environment defined by the triplet of operators \( \hat{H}_E, \hat{\varrho}_E, \) and \( \hat{V}_E, \) hence, if one wishes to observe certain aspects of the surrogate, it is necessary that this environment is participating in the proposed measurement scheme. (ii) The result of the proposed measurement scheme should be in the form of time sequence \( r_t = (r_{t1}, r_{t2}, \ldots, r_{tk}), \) with \( t_1 > \ldots > t_k. \) (iii) Assuming that the environment represented by the surrogate of interest satisfies criterion 2 (i.e., that \( \Delta q_E^{(k)} = 0 \) for all \( k \)), the probability of measuring a given sequence \( p(r_t) \) should be related to the surrogate’s joint probability distributions:

\[
p(r_t) \approx \left( \prod_{l=1}^{k} \sum_{\xi_l: h(\xi_l) = r_{tl}} \right) p_E^{(k)}(\xi, t), \quad (83)
\]

Here, the real function \( h \) defines the measurement resolution, e.g., if the resolution is perfect \( h(\xi) = \xi, \) then \( p(r_t) \approx p_E^{(k)}(r_{t1}, t_1; \ldots; r_{tk}, t_k) \) and the scheme would result in a direct measurement of a sample trajectory spanned on a time grid \( t. \)

5.6 Markovian surrogate field

According to the theory of stochastic processes, the process has the Markov property (or the process is markovian) when, given the time sequence \( t_1 > t_2 > \ldots > t_k, \) the probability for having the value \( \xi_1 \) at time \( t_1 \) conditioned by the past steps

\[
P^{(k)}_\xi(\xi_1, t_1| \xi_2, t_2; \ldots; \xi_k, t_k) = P^{(k)}_\xi(\xi_1, t_1; \xi_2, t_2; \ldots; \xi_k, t_k)/P^{(k-1)}_\xi(\xi_2, t_2; \ldots; \xi_k, t_k), \quad (84)
\]
depends only on the previously taken step [41],

\[ P_{\xi}^{(k)}(\xi_1, t_1 | \xi_2, t_2; \ldots; \xi_k, t_k) = P_{\xi}^{(2)}(\xi_1, t_1 | \xi_2, t_2). \]  

(85)

The conditional probability \( P_{\xi}^{(2)}(\xi_1, t_1 | \xi_2, t_2) \), often called the transition matrix, is the fundamental building block of Markovian process: it follows from (84) that any joint probability distribution can be expressed with these transition matrices and the initial distribution,

\[ P_{\xi}^{(k)}(\xi, t) = P_{\xi}^{(1)}(\xi_k, t_k) \prod_{l=1}^{k-1} P_{\xi}^{(2)}(\xi_l, t_l | \xi_{l+1}, t_{l+1}). \]  

(86)

When the criterion (86) is applied to surrogate field representations, described by probability distributions \( p_{E}^{(k)} \), we can easily verify that the surrogate has the Markov property when the coupling \( \hat{V}_E \) is not degenerate [see the example of open environment (63) discussed in Sec. 4.2], or when the sums over degenerate subspaces can be neglected (see the environment of least action discussed in Sec 4.3), so that

\[ p_{E}^{(k)}(v_{n_1}, t_1; \ldots; v_{n_k}, t_k) = \langle n_k | e^{-i\hat{t}_k \hat{H}_E} \hat{\phi}_E e^{i\hat{t}_k \hat{H}_E} | n_k \rangle \prod_{l=1}^{k-1} T_{t_l-t_{l+1}}(n_l n_l | n_{l+1} n_{l+1}), \]  

(87)

In such a case, the propagator is identified with the transition matrix, and the initial distribution corresponds to the matrix element of the environment state.

The major advantage of having the Markov property is the algorithmic procedure (86) for construction of joint probability distributions out of a few relatively simple elements. This may be considered as a main reason why Markovian processes play such a vital role in the theory of stochastic processes. Indeed, it is prohibitively difficult to analyze a general non-Markovian process because there is no concise way to describe their defining family of joint probabilities [56]. However, the significance of this aspect of Markovianity is negligible for surrogate fields; after all, all surrogates, Markovian or not, are by design characterized with a procedurally constructed probability distributions (see Sec. 3).

The question is then, if this feature of the stochastic process theory is so inconsequential for surrogate representations, then why are we even bringing it up? Historically, the adjective “Markovian” has been freely (and often recklessly) used in a variety of contexts. In our opinion, its apparent adequateness was (and still is) explained by a commonly shared belief that “Markovian” is synonymous with “memory-less”. Regrettably, this belief is mostly unfounded; one should realize that, originally, the “memory-less” descriptor was used as an attempt at intuitive, but not exactly exhaustive, summary of the Markov property. To give an example of such a “liberal” use of word Markov and Markovian, let us consider the Markov approximation that we have invoked in Sec. 4.2. Without going into much detail, the approximation is usually justified by stating that the second order autocorrelation of environment (subsystem \( D \) in our case) is a short-ranged function of time, i.e., it rapidly decays to zero as its time argument crosses a small threshold value. If we adopted this explanation to the domain of stochastic processes, then it would become a statement about the properties of the transition matrix \( P_{\xi}^{(2)}(\xi_1, t_1 | \xi_2, t_2) \), and as such, it would not influence whether the process has or has not the Markov property. However, given that the interpretation of autocorrelation as a measure of memory is rather uncontroversial, then its short range would indicate a short memory, or the “lack of memory”. When one believes that “Markovian” is interchangeable with “memory-less”, then the choice of the name makes perfect sense. The point of this digression, and this subsection as a whole, is to sensitiz...
the Reader to, essentially, a semantic issues with Markovian and memory-less descriptors. The fact that there are several theories with a well defined element that share the name Markov (e.g., stochastic processes, open systems, quantum Markovianity [57–61]), even though they have next to nothing to do with each other, makes the adjective “Markovian” a likely culprit in provoking association fallacies and causing a lot of confusion.

6 Conclusion

We have formulated the sufficient criterion for the dynamics of an open quantum system to be simulated using the external field that is a surrogate for the environmental degrees of freedom—the surrogate field representation. To achieve this, we have developed the approach in which the influence of the environment is wholly described by the family of joint quasiprobabilities \( \{q^{(k)}_{E}\}_k \), with each of its members constructed out of simple basic elements. This language has proven to be flexible enough to allow us not only to carry out a comprehensive analysis of microscopic origins of so-called classical noise approximations and random unitary dynamical maps, but also to explore some of the most interesting accompanying issues. Two important examples of such issues were the previously hypothesized incompatibility of surrogate representation with the formation of system–environment entanglement, and the causal relation between the absence of system’s back-action and the existence of valid surrogate representation; we have disproved both propositions.

We have concluded that it is impossible to point to one reason for the validity of the surrogate field representation (like e.g., the absence of back-action). Instead, whether the simulation with surrogate field is valid is determined by the relationship between the dynamical laws governing the environment (the free Hamiltonian \( \hat{H}_E \) and the initial state \( \hat{\rho}_E \)) and operator \( \hat{V}_E \) that couples it to the system. The examples of environment types that facilitate their surrogate fields presented here illustrate this point by showing a variety of ways to satisfy the validity criterion.

We have addressed the issue of subjectivity and inter-subjectivity of the surrogate field representation. Even though the question of the objectivity of external field simulator is an important one—both from practical and purely theoretical point of view—previous studies on classical noise or random unitary maps were unable to engage with it in satisfactory capacity. We had taken this particular shortcoming into consideration, and we had set fixing this specific blind spot as one of the main design goals of our approach. The resultant quasiprobability formulation leads to the system state decomposition (38) where the contributions from the system and the environment are clearly separated. This separation is crucial; it allows for the influence exerted by the environment—e.g., whether this influence can be represented with the surrogate field—to be considered independently of the influenced system. Thus, the quasiprobability formulation was an ideal tool for finding the answer to the question of surrogate’s objectivity; one can hope that it will also open new avenues for the development of the quantum open systems theory.

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A Superoperator representation of coherence function

The system side free Hamiltonian and coupling in QIDAS context are given by Eq. (6). In this form, the impulse control is already embedded into system-side operators. Let us take a step back, and write

$$
\dot{H}_{SE} = \frac{\Delta}{2} \hat{\sigma}_z + \frac{1}{2} \hat{\sigma}_z \otimes \hat{V}_E + \hat{H}_E,
$$

and define the total system–environment Hamiltonian as a sum of $\dot{H}_{SE}$ and the time-dependent control Hamiltonian $\dot{H}_{\text{imp}}(t)$. The control is designed in such a way that it intervenes at selected time point with a short and intense impulse of external field that causes an instantaneous $\pi$ rotation of qubit’s Bloch vector. Aside those specific timings \(\{\tau_1, \tau_2, \ldots, \tau_n\} \ (0 < \tau_1 < \tau_2 \ldots < \tau_n < t)\) the control Hamiltonian is turned off.

The application of the control impulse is modeled with an unitary transformation

$$
\dot{U}_{\text{imp}} = e^{-i \frac{\pi}{2} \hat{\sigma}_x} = -i \hat{\sigma}_x
$$

and the evolution in between control pulses is given by

$$
\dot{U}_t = e^{-i \tau H_{SE}} = e^{-i \tau (\frac{\Delta}{2} \hat{\sigma}_z + \frac{1}{2} \hat{\sigma}_z \otimes \hat{V}_E + \hat{H}_E)}.
$$

With this parameterization, the evolution operator for the total system reads

$$
\dot{U}_{\text{QIDAS}}(t) = \dot{U}_{-\tau_n} \left( \prod_{l=1}^{1} \dot{U}_{\text{imp}} \dot{U}_{\tau_{l-1}} \right) \dot{U}_{\text{imp}} \dot{U}_{\tau_1},
$$

where the symbol $\prod_{l=1}^{1} \dot{A}(l)$ is to be understood as an ordered composition $\dot{A}(n)\dot{A}(n - 1)\ldots\dot{A}(1)$.

Now, we switch to the superoperator picture

$$
\dot{U}_{\text{QIDAS}}(t) \hat{\sigma}_S \otimes \hat{\sigma}_E \dot{U}_{\text{QIDAS}}^\dagger(t) = \dot{U}_{\text{QIDAS}}(t) \hat{\sigma}_S \otimes \hat{\sigma}_E
$$

where the evolution superoperator is defined as

$$
\dot{U}_{\text{QIDAS}}(t) = \dot{U}_{-\tau_n} \left( \prod_{l=1}^{1} \dot{U}_{\text{imp}} \dot{U}_{\tau_{l-1}} \right) \dot{U}_{\text{imp}} \dot{U}_{\tau_1},
$$

(the symbol $\prod_{l=1}^{1} \dot{A}(l)$ is to be read as $\dot{A}(n)\dot{A}(n - 1)\ldots\dot{A}(1)$) with the impulse and inter-pulse evolution superoperators given by

$$
\dot{U}_{\text{imp}} \dot{A} = (\hat{\sigma}_x \cdot \hat{\sigma}_x) \dot{A} = \hat{\sigma}_x \dot{A} \hat{\sigma}_x,
$$

$$
\dot{U}_t \dot{A} = e^{-i \tau [\hat{H}_{\text{SE}}, \dot{\cdot}]} \dot{A} = (e^{-i \tau \hat{H}_{\text{SE}}} \cdot e^{i \tau \hat{H}_{\text{SE}}}) \dot{A} = e^{-i \tau \hat{H}_{\text{SE}}} \dot{A} e^{i \tau \hat{H}_{\text{SE}}}.
$$

Let us define an orthonormal basis in the subspace of qubit operators $\{|+\rangle\langle+|, |-\rangle\langle-|, |+\rangle\langle-|, |-\rangle\langle+|\}$. This basis can be use, e.g., to decompose the initial state of the system, $\hat{\sigma}_S = \sum_{s,s'} \hat{\sigma}_{ss'} |s\rangle\langle s'|$ with $\hat{\sigma}_{ss'} = \operatorname{Tr}(|s\rangle\langle s'| \hat{\sigma}_S)$. The most important property of those basis operators, from our point of view, is the fact that they are the eigenoperators of the generator of inter-pulse evolution,

$$
[\hat{H}_{\text{SE}}, \cdot]|{\pm\rangle\langle\pm}| \dot{A} = |{\pm\rangle\langle\pm}| \left( [\hat{H}_E \pm \frac{1}{2} \hat{V}_E, \cdot] \right) \dot{A},
$$

$$
[\hat{H}_{\text{SE}}, \cdot]|{\mp\rangle\langle\mp}| \dot{A} = |{\mp\rangle\langle\mp}| \left( [\pm \Delta \cdot + [\hat{H}_E, \cdot] \pm \frac{1}{2} \hat{V}_E, \cdot \right) \dot{A}.
$$
In addition, they also behave well under the action of the impulse superoperator

\[
U_{\text{imp}} |\pm\rangle \otimes \hat{A} = |\mp\rangle \otimes \hat{A},
\]

\[
U_{\text{imp}} |\mp\rangle \otimes \hat{A} = |\mp\rangle \otimes \hat{A}.
\]

Assuming for simplicity that \( \varrho_{++} = \varrho_{--} \) and that the number of impulses \( n \) is even, let us apply the total evolution superoperator to each of those basis operators

\[
U_{\text{QIDAS}}(t) |\pm\rangle \otimes \hat{A} = e^{i\Delta \int_0^t f(\tau) d\tau} \otimes \left( e^{-i(t-\tau_0)\hat{H}_{\text{imp}} \pm \frac{1}{2} \hat{V}_{\text{E}} \cdot \hat{\sigma}} \left( \prod_{l=1}^{n-1} e^{-i(\tau_l-\tau_{l-1})\hat{H}_{\text{imp}} \pm \frac{1}{2} \hat{V}_{\text{E}} \cdot \hat{\sigma}} \right) e^{-i\tau_l \hat{H}_{\text{E}} \pm \frac{1}{2} \hat{V}_{\text{E}} \cdot \hat{\sigma}} \right) \hat{A},
\]

\[
U_{\text{QIDAS}}(t) |\mp\rangle \otimes \hat{A} = e^{i\Delta \int_0^t f(\tau) d\tau} |\mp\rangle \otimes \left( e^{-i(t-\tau_0)\hat{H}_{\text{imp}} \pm \frac{1}{2} \hat{V}_{\text{E}} \cdot \hat{\sigma}} \left( \prod_{l=1}^{n-1} e^{-i(\tau_l-\tau_{l-1})\hat{H}_{\text{imp}} \pm \frac{1}{2} \hat{V}_{\text{E}} \cdot \hat{\sigma}} \right) e^{-i\tau_l \hat{H}_{\text{E}} \pm \frac{1}{2} \hat{V}_{\text{E}} \cdot \hat{\sigma}} \right) \hat{A},
\]

where \( T \) indicates time-ordering of superoperators, and we have introduced the time-domain filter function

\[
f(\tau) = \Theta(t-\tau)\Theta(\tau-\tau_0) + \sum_{l=1}^{n-1} (-1)^l \Theta(\tau_{l+1} - \tau)\Theta(\tau - \tau_l) + \Theta(\tau_1 - \tau)\Theta(\tau),
\]

here, \( \Theta(s) \) is the Heaviside step function. Finally, we are ready to calculate the coherence function (14):

\[
W_E(t|f) = \frac{1}{\varrho_{++}} e^{i\Delta \int_0^t f(\tau) d\tau} \text{Tr}_E \left( |\mp\rangle \langle +| \otimes U_{\text{QIDAS}}(t) \left( \sum_{s,s' = \pm} \varrho_{ss'} |s\rangle \langle s'| \otimes \varrho_{E} \right) \right)
\]

\[
= \frac{\varrho_{--}}{\varrho_{++}} \text{Tr}_E \left( |\mp\rangle \langle +| \otimes \left( e^{-i t \hat{H}_{\text{E}}} \left( e^{-i \int_0^t f(\tau) d\tau \hat{e}^r \hat{H}_{\text{E}} \cdot \hat{\sigma}} \right) \hat{\sigma} \right) \otimes \varrho_{E} \right)
\]

\[
= \text{Tr}_E \left( e^{-i t \hat{H}_{\text{E}}} \left( e^{-i \int_0^t f(\tau) d\tau \hat{e}^r \hat{H}_{\text{E}} \cdot \hat{\sigma}} \right) \otimes \varrho_{E} \right)
\]

\[
= \text{Tr}_E \left( \left( e^{-i \int_0^t f(\tau) d\tau \hat{e}^r \hat{H}_{\text{E}} \cdot \hat{\sigma}} \right) \otimes \varrho_{E} \right)
\]

If the number of impulses \( n \) was odd, then we would obtain \( W_E(t|-f) \), which amounts to simple redefinition of the filter function \( f'(t) = -f(t) \).
B Moment operator

The explicit form of the reduced state of the system, that is discussed in Sec. 3, reads as follows

\[
\text{Tr}_E(e^{-it\tilde{H}_{SE}} \hat{\rho}_S \otimes \hat{\rho}_E e^{it\tilde{H}_{SE}}) = \text{Tr}_E(e^{-it[\hat{H}_S + \hat{H}_E, \hat{\rho}_S \otimes \hat{\rho}_E]} \hat{\rho}_S \otimes \hat{\rho}_E)
\]

\[
= \text{Tr}_E(e^{-it[\hat{H}_S + \hat{H}_E, \hat{\rho}_S \otimes \hat{\rho}_E]} e^{-it[\hat{H}_S + \hat{H}_E, \hat{\rho}_S \otimes \hat{\rho}_E]} \hat{\rho}_S \otimes \hat{\rho}_E)
\]

\[
= \sum_{k=0}^{\infty} (-i)^k \int_0^t dt_1 \sum_{l=1}^{k-1} \int_0^{t_l} dt_{l+1} \ldots \int_0^{t_{k-1}} dt_k \text{Tr}_E \left[ \prod_{l=1}^{k} e^{it_l[\hat{H}_S + \hat{H}_E, \hat{\rho}_S \otimes \hat{\rho}_E]} \hat{V}_{SE} \hat{V}_{E} \cdot \hat{\rho}_S \otimes \hat{\rho}_E \right].
\]

where symbol \( \prod_{l=1}^{k} \mathcal{A}(l) \) is to be understood as an ordered composition \( \mathcal{A}(1)\mathcal{A}(2) \ldots \mathcal{A}(k) \). Given the eigenstates and the corresponding eigenvalues of system-side coupling \( \hat{V}_S[d] = s_d[d] \) we identify the orthonormal basis of the eigenoperators of superoperator associated with the commutator of the interaction

\[
[\hat{V}_S \otimes \hat{V}_E, \cdot] = \sum_{\alpha, \beta} \sum_{\xi, \zeta} (\alpha \xi - \beta \zeta) \sum_{d, b, m} \sum_{n,m} \langle d | b \otimes | n \rangle \langle m | \text{Tr}(|b| \langle d | \otimes | m \rangle \langle n |). \]

Substituting the decomposition into Eq. (104) leads directly to Eqs. (37) and (38), with the context-dependent operator given by

\[
\hat{m}^{(k)}_S(\xi, \zeta, \mathbf{t}) = \left( \prod_{l=1}^{k} \sum_{\alpha, \beta l \in \Omega_{1d}, \xi, \zeta \in \Omega_m} \left( \prod_{l=1}^{k} \sum_{d_l, b_l} \sum_{\alpha_l = s_{d_l}, \beta_l = s_{b_l}} (\alpha_l - \beta_l \xi_l \zeta_l) \right) \left( \prod_{l=2}^{k} (\alpha_l \xi_l - \beta_l \zeta_l) \right) \right)
\]

\[
\times e^{-i(t-t_1)\hat{H}_S, \cdot} |d_1| \langle b_1 | \left( \prod_{l=1}^{k-1} S_{t_l-t_{l+1}}(d_l b_l | d_{l+1} b_{l+1}) \right) \langle d_k | e^{-i t_k \hat{H}_S, \cdot} \hat{\rho}_S \otimes \hat{\rho}_E | b_k \rangle,
\]

where the system propagators are defined as

\[
S_{t_l}(d b | d' b') = \text{Tr}_S(|b| \langle d | e^{-i t_l \hat{H}_S, \cdot} \hat{\rho}_S \otimes \hat{\rho}_E | b' \rangle).
\]

When the criterion 2 is satisfied, then

\[
\hat{m}^{(k)}_S(\xi, \zeta, \mathbf{t}) q^{(k)}_E(\xi, \zeta, \mathbf{t}) \approx \delta(\xi - \zeta) \hat{m}^{(k)}_S(\xi, \xi, \mathbf{t}) p^{(k)}(\xi, \mathbf{t}).
\]

Once \( \xi = \zeta \) is set in Eq. (107), each factor that entwines eigenvalues of \( \hat{V}_S \) and \( \hat{V}_E \), simplifies to \( (\alpha \xi_l - \beta \zeta_l) = \xi_l (\alpha_l - \beta_l) \), and as a result, the moment operator factorizes into context-dependent operator and context-independent real number. The latter is identified with the
moment of the surrogate field \( \xi(t) \). In the context-dependent part, we look for the spectral decomposition of superoperator associated with the commutator of \( \hat{V}_S \)

\[
\sum_{\alpha, \beta \in \Omega_{\alpha}} (\alpha - \beta) \sum_{d : \alpha = s_d} \sum_{b : \beta = s_b} |d\rangle \langle b| \text{Tr}(|d\rangle \langle b| \bullet \) = \[\hat{V}_S \bullet \),
\]

by noting that

\[
e^{-it[H_S, \bullet]}[\hat{V}_S \bullet]e^{-it[H_S, \bullet]}|d'\rangle \langle b'| = \sum_{\alpha, \beta \in \Omega_{\alpha}} (\alpha - \beta) \sum_{d : \alpha = s_d} \sum_{b : \beta = s_b} e^{-it[H_S, \bullet]}|d\rangle \text{S}_t(db|d'b'),
\]

which leads us to the form (45).

C Joint quasiprobability distributions for open environment

Consider an orthonormal basis in \( ED \) subspace \( \{|n; i\rangle \}_{n,i} \), where \(|n; i\rangle = |n\rangle \otimes |i\rangle \), \( \{|i\rangle \}_i \) is an arbitrary basis in subspace \( D \), and \(|n\rangle \) are the eigenstates of \( \hat{V}_E \). Using this basis and the Schrödinger representation of propagators [see Eqs. (33) and (55)]

\[
T_i(ni, mj|m'n'i', m'j') = \langle ni; \hat{U}_{ED}(t)|m'n'i'; \hat{U}_{ED}(t)'m'j'|j\rangle
\]

we will now rewrite the general definition (40) of \( q_E^{(k)} (\xi, \zeta, t) \)

\[
q_E^{(k)} (\xi, \zeta, t)
\]

\[
= \sum_{n_1; \xi_1 = v_1}^{n_k; \xi_k = v_k} \left( \prod_{l=2}^{k} \sum_{n_l; \xi_l = v_l}^{n_l; \xi_l = v_l} \sum_{i_1}^{n_1} \sum_{i_2...i_k} \left( \prod_{l=1}^{k-1} (n_l; i_l|\hat{U}_{ED}(t_l-t_{l+1})|n_{l+1}; i_{l+1}) \right) \right.
\]

\[
\times \langle n_k; i_k|\hat{U}_{ED}(t_k)\hat{V}_E \otimes \hat{V}_D \hat{U}_{ED}(t_k)'|m_k; j_k\rangle \left( \prod_{l=1}^{k-1} (m_l; j_l|\hat{U}_{ED}(t_l-t_{l+1})|m_{l+1}; j_{l+1}) \right)^* + \text{c.c.}
\]

\[
= \sum_{n_1; \xi_1 = v_1}^{n_k; \xi_k = v_k} \left( \prod_{l=2}^{k} \sum_{n_l; \xi_l = v_l}^{n_l; \xi_l = v_l} \sum_{i_1}^{n_1} \left( \prod_{l=2}^{k} \hat{U}_{ED}(t_{l-1} - t_l)|n_l| \sum_{i_l} (|i_l\rangle \langle i_l|) \right) \right)
\]

\[
\times \hat{U}_{ED}(t_k)\hat{V}_E \otimes \hat{V}_D \hat{U}_{ED}(t_k)' \left[ \prod_{l=k}^{k} \left( \sum_{j_l} (|j_l\rangle \langle j_l|) \right) \right] |n_k| \langle m_k| \hat{U}_{ED}^+(t_{k-1} - t_k) |n_1; i_1\rangle \right),
\]

where the symbol \( \prod_{l=1}^{k} \hat{A}(l) \) is to be understood as an ordered composition: \( \hat{A}(l_b) \hat{A}(l_b+1) \ldots \hat{A}(l_e) \) for \( l_b < l_e \), or \( \hat{A}(l_e) \hat{A}(l_b-1) \ldots \hat{A}(l_b) \) for \( l_b > l_e \). Since the sums over \( i_l \) and \( j_l \) are not constraint in any way, we get \( \sum_{i_1} \langle n_1; i_1| \hat{V}_D (\bullet) |n_1; i_1\rangle = \langle n_1| \text{Tr}_D (\bullet) |n_1\rangle \) and \( \sum_{i_l} (|i_l\rangle \langle i_l|) = \hat{1} \), which leads to Eq. (54).

D Impostor example: the case of bias-induced phase shift

Consider the context where a standard QIDAS arrangement is swapped with a setup where the system-side coupling is modified in the following way

\[
\hat{V}_S = \frac{1}{2} f(t) \hat{\sigma}_z \rightarrow \hat{V}_\lambda(t) = \frac{\lambda}{2} \hat{1} + \hat{V}_S(t).
\]
For example, for \( \lambda = 1 \) one gets \( \hat{V}_\lambda(0) = |+\rangle\langle+| \) and the subsequent pulses switch the coupling between the eigenstate projectors \( |\pm\rangle\langle\pm| \). The identity operator in the modified coupling—the bias—is not affected in any way by the dynamical control exerted on the qubit. The initial state \( \hat{\rho}_S \) and the free Hamiltonian \( \hat{H}_S \) remain the same as in the original context.

Because of the structure of the coupling operator \( \hat{V}_\lambda \), whatever the form of the stochastic model, the results of the simulation are the same with and without the bias \([35, 62]\). On the other hand, the non zero bias does affect the course of fully quantum-mechanical dynamics. In particular, within Gaussian approximation, the off-diagonal element of qubit’s density matrix reads

\[
\rho_{+-}(t) = \langle +|\hat{\rho}_S(t)|-\rangle = \langle +|\text{Tr}_E(e^{-i\int_0^t d\tau[H_S(\tau)+\hat{V}_S(\tau)\otimes \hat{V}_E]})\hat{\rho}_S \otimes \hat{\rho}_E|\rangle 
\approx \rho_{+-} e^{-i\phi(t)} e^{-\int_0^t dt_1 \int_0^{t_1} dt_2 f(t_1)f(t_2) F(t_1,t_2)},
\]

(115)

Instead of autocorrelation function \( C(t) \), that would be expected in cases of stochastic simulation or unbiased coupling, we get

\[
F(t_1,t_2) = \sum_{\nu_1 \in \Omega_{\nu_1}} \sum_{\nu_2 \in \Omega_{\nu_2}} \nu_1 \nu_2 \Omega^{(2)}_{\text{ctx}}(\nu, t) + i\lambda \sum_{\xi_1,\xi_2,\xi_3,\xi_4} \xi_1 f(t_2) [\xi_2 - \zeta_2] \text{Im}\{\Delta q^{(2)}_E(\xi, \zeta, t)\}
\]

\[
\equiv C(t_1-t_2) + i\lambda K(t_1,t_2|f),
\]

(116)

where the autocorrelation function is accompanied by an additional biased-induced phase shift \( \Delta \phi(t) = \lambda \int_0^t dt_1 \int_0^{t_1} dt_2 K(t_1,t_2|f) \) \([35, 62]\). This demonstrates that the impostor field \( \xi_{\text{Gauss}}(t) \) is not inter-contextual: even though it produces a Gaussian simulations in QIDAS context, it is incapable of taking into account the phase shift appearing in the context of biased coupling.

When the objective surrogate field representation is valid, the resultant simulations are not disrupted by the bias. Indeed, since \( K(t_1,t_2|f) \) is spanned by superpositions of coherence-connected propagator chains \( \Delta q^{(2)}_E(\xi, \zeta, t) \), the bias-induced phase shift is automatically negligible when the environment facilitates its surrogate field. This can be interpreted in a different way: if one observes a non zero bias-induced phase shift, then it indicates that the objective surrogate field representation is invalid, but even if this is the case, the impostor \( \xi_{\text{Gauss}}(t) \) would still be a valid representation in the unbiased context.

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