A NEW NECESSARY AND SUFFICIENT CONDITION FOR
THE RIEMANN HYPOTHESIS

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ABSTRACT. We give a new equivalent condition for the Riemann hypothesis consisting in an order condition for certain finite rational combinations of the values of \( \zeta(s) \) at even positive integers.

1. INTRODUCTION AND PRELIMINARIES

In this note we shall prove the following theorem:

Theorem 1.1. Let

\[
(1.1) \quad c_k := \sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{1}{\zeta(2j+2)},
\]

then the Riemann hypothesis is true if and only if

\[
(1.2) \quad c_k \ll k^{-\frac{3}{4}+\epsilon}, \quad (\forall \epsilon > 0).
\]

Remark 1.1. It will be seen below that unconditionally

\[
(1.3) \quad c_k \ll k^{-\frac{1}{2}}.
\]

Remark 1.2. It is quite obvious how one can trivially modify the proof of the theorem to obtain a more general result:

Theorem 1.2. A necessary and sufficient condition for \( \zeta(s) \neq 0 \) in the half-plane \( \Re(s) > 2(1-\alpha) \) is

\[
(1.4) \quad c_k \ll k^{-\alpha+\epsilon}, \quad (\forall \epsilon > 0).
\]

However we shall eschew such gratuitous generalizing at this stage.

Necessary and sufficient conditions for the Riemann hypothesis depending only on values of \( \zeta(s) \) at positive integers have been known for a long time,
e.g. those of M. Riesz \cite{5} and Hardy-Littlewood \cite{2}. M. Riesz’s criterion, for example, states that the Riemann hypothesis is true if and only if

$$
\sum_{k=1}^{\infty} \frac{(-1)^{k+1}x^k}{(k-1)!\zeta(2k)} = O(x^{1+\epsilon}), \quad (x \to +\infty).
$$

We believe our condition is new and it is definitely simpler, as it only involves finite rational combinations of the values $\zeta(2h)$, and seems well posed for numerical calculations. This work however did not originate as an attempt to simplify Riesz’s criterion. It arose rather as a consequence of our note \cite{1} on Maslanka’s expression of the Riemann zeta function \cite{3,4} in the form

$$(s - 1)\zeta(s) = \sum_{k=0}^{\infty} A_k P_k \left(\frac{s}{2}\right).$$

Here the $P_k(s)$ are the \textit{Pochhammer polynomials}

$$(1.5)\quad P_k(s) := \prod_{r=1}^{k} \left(1 - \frac{s}{r}\right),$$

which will appear prominently in the proof of Theorem \ref{thm:1}. Two elementary facts about them shall be needed: firstly

$$(1.6)\quad (-1)^k \left(\frac{n}{2} - 1\right) = P_k \left(\frac{s}{2}\right),$$

which is essentially a matter of notation, and secondly a standard estimate given here without proof:

\textbf{Lemma 1.1.} \textit{For every circle $|s| < r < \infty$ there is a positive constant $C_r$ such that}

$$(1.7)\quad |P_k(s)| \leq C_r k^{-\Re(s)}.$$  

\textbf{2. Sufficiency of the condition}

The sufficiency of the condition \ref{cond:1} follows from writing $(\zeta(s))^{-1}$ as a series of Pochhammer polynomials.

\textbf{Proposition 2.1} ( Sufficiency of the condition). \textit{If } $c_k \ll k^{-\frac{3}{4} + \frac{1}{4}\epsilon}$ \textit{for any } $\epsilon > 0$, \textit{then}

$$(2.1)\quad \frac{1}{\zeta(s)} = \sum_{k=0}^{\infty} c_k P_k \left(\frac{s}{2}\right), \quad (\Re(s) > \frac{1}{2}),$$

where the series converges uniformly on compact subsets of the half-plane.
Remark 2.1. Since it shall be shown that actually $c_k \ll k^{-\frac{1}{2}}$ it follows modifying trivially the above argument that the representation (2.1) for $(\zeta(s))^{-1}$ is unconditionally valid at least in the half-plane $\Re(s) > 1$.

We need a lemma before proving Proposition 2.1.

Lemma 2.1. Define

\begin{equation}
q_k := \sum_{n=1}^{\infty} \frac{1}{n^2} \left(1 - \frac{1}{n^2}\right)^k,
\end{equation}

then

\begin{equation}
q_k \ll k^{-\frac{1}{2}}.
\end{equation}

Proof. Let $B_1(x) = x - [x] - \frac{1}{2}$. By the Euler-MacLaurin formula we have for $k \geq 1$

\begin{equation}
q_k = \int_{1}^{\infty} \frac{1}{x^2} \left(1 - \frac{1}{x^2}\right)^k \, dx + \int_{1}^{\infty} \frac{d}{dx} \frac{1}{x^2} \left(1 - \frac{1}{x^2}\right)^k \, dx
\end{equation}

\begin{equation}
= \sqrt{\pi} \frac{\Gamma(k+1)}{\Gamma(k+\frac{3}{2})} + \int_{1}^{\infty} B_1(x) \frac{d}{dx} \left(1 - \frac{1}{x^2}\right)^k \, dx.
\end{equation}

Clearly

\begin{equation}
\frac{\Gamma(k+1)}{\Gamma(k+\frac{3}{2})} \ll k^{-\frac{1}{2}},
\end{equation}

and, letting $V(f(x))$ denote the total variation of $f(x)$ in $[1, \infty)$, we see that

\begin{equation}
\left| \int_{1}^{\infty} B_1(x) \frac{d}{dx} \left(1 - \frac{1}{x^2}\right)^k \, dx \right| \ll \int_{1}^{\infty} \left| \frac{d}{dx} \left(1 - \frac{1}{x^2}\right)^k \right| \, dx
\end{equation}

\begin{equation}
= V \left(1 - \frac{1}{x^2}\right)^k
\end{equation}

\begin{equation}
= 2 \max_{1 \leq x < \infty} \frac{1}{x^2} \left(1 - \frac{1}{x^2}\right)^k
\end{equation}

\begin{equation}
= \frac{2}{k+1} \left(1 - \frac{1}{k+1}\right)^k \ll k^{-1}.
\end{equation}

Hence, (2.4), (2.5) and (2.6) achieve (2.3). □

Proof of Proposition 2.1. First note that
\begin{equation}
(2.7) \quad c_k = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{1}{\zeta(2j+2)}
\end{equation}

\begin{align*}
&= \sum_{j=0}^{k} (-1)^j \binom{k}{j} \sum_{n=1}^{\infty} \frac{\mu(n)}{n^{2j+2}} \\
&= \sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} \sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{1}{n^{2j}} \\
&= \sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} \left(1 - \frac{1}{n^2}\right)^k.
\end{align*}

Starting now with $\Re(s) > 1$ we have

\begin{align*}
\frac{1}{\zeta(s)} &= \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} \left(\frac{1}{n^2}\right)^{\frac{s}{2}-1} \\
&= \sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} \left(1 - \left(1 - \frac{1}{n^2}\right)\right)^{\frac{s}{2}-1} \\
&= \sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} \sum_{k=0}^{\infty} (-1)^k \binom{\frac{s}{2}-1}{k} \left(1 - \frac{1}{n^2}\right)^k \\
&= \sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} \sum_{k=0}^{\infty} P_k \left(\frac{s}{2}\right) \left(1 - \frac{1}{n^2}\right)^k.
\end{align*}

These summations can be interchanged because calling

\begin{equation}
S = \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{1}{n^2} \left|P_k \left(\frac{s}{2}\right)\right| \left(1 - \frac{1}{n^2}\right)^k,
\end{equation}

we see from Lemma 1.1 and Lemma 2.2 that

\begin{align*}
S &= \sum_{k=0}^{\infty} \left|P_k \left(\frac{s}{2}\right)\right| \sum_{n=1}^{\infty} \frac{1}{n^2} \left(1 - \frac{1}{n^2}\right)^k \\
&= \sum_{k=0}^{\infty} \left|P_k \left(\frac{s}{2}\right)\right| q_k \ll \sum_{k=1}^{\infty} k^{-\frac{\Re(s)}{2} - \frac{1}{2}} < \infty.
\end{align*}

Thus we proceed to interchange summations in (2.8), taking into account (2.7), to obtain unconditionally for $\Re(s) > 1$,
\[
\frac{1}{\zeta(s)} = \sum_{k=0}^{\infty} P_k\left(\frac{s}{2}\right) \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \left(1 - \frac{1}{n^2}\right)^k \\
= \sum_{k=0}^{\infty} c_k P_k\left(\frac{s}{2}\right). 
\]
(2.9)

But Lemma 1.1 together with the hypothesis \(c_k \ll k^{\frac{3}{2}+\frac{1}{2}\varepsilon}\) implies that the above series converges uniformly on compacts of the half-plane \(\Re(s) > \frac{1}{2} + \varepsilon\). This means that the series extends \((\zeta(s))^{-1}\) analytically to the half-plane \(\Re(s) > \frac{1}{2}\). □

3. Necessity of the condition

Proof of the necessity of the condition. Assume now that the Riemann hypothesis is true. If as usual we write

\[ M(x) := \sum_{n \leq x} \mu(n), \]

we then have

\[ M(x) \ll x^{\frac{1}{2}+2\varepsilon}, \quad (\forall \varepsilon > 0). \]

We can transform the second expression for \(c_k\) in (2.7) summing it by parts to obtain

\[
c_k = \int_1^{\infty} M(x) \frac{d}{dx} \left(\frac{1}{x^2} \left(1 - \frac{1}{x^2}\right)^k\right) dx \\
= 2 \int_0^1 M\left(\frac{1}{x}\right) (1 - x^2)^k ((k+2)x^3 - x) dx.
\]

Therefore

\[ |c_k| \leq 2(k+2) \int_0^1 \left| M\left(\frac{1}{x}\right) \right| x^3(1 - x^2)^k dx + \int_0^1 \left| M\left(\frac{1}{x}\right) \right| x(1 - x^2)^k dx, \]

but (on the Riemann hypothesis)

\[ M\left(\frac{1}{x}\right) \ll x^{-\frac{3}{2}-2\varepsilon}, \quad (x \downarrow 0), \]

so that

\[ c_k \ll k \int_0^1 x^{\frac{5}{2}-2\varepsilon}(1 - x^2)^k dx + \int_0^1 x^{\frac{3}{2}-2\varepsilon}(1 - x^2)^k dx. \]

On the other hand, for \(\Re(\lambda) > -1\) a classical beta integral result is
\[
\int_0^1 x^\lambda (1 - x^2)^k dx = \Gamma \left( \frac{\lambda + 1}{2} \right) \frac{\Gamma(k + 1)}{\Gamma(k + \frac{1}{2}(\lambda + 3))} \ll k^{-\frac{\lambda}{2} - \frac{1}{2}},
\]
so that (3.1) becomes
\[
c_k \ll k^{-\frac{3}{4} + \epsilon}.
\]

4. RESULTS OF SOME CALCULATIONS

A test for the first \(c_k\) up to \(k = 1000\) shows a very pleasant smooth curve which, on the meager strength of so limited a calculation, would seem to indicate that

\[
c_k k^{3/4} \log^2 k
\]
tends to a finite limit in a very regular way.

References

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