Knot points of double–covariant system of elliptic equations and preferred frames in general relativity

V Pelykh

Pidstryhach Institute Applied Problems in Mechanics and Mathematics Ukrainian National Academy of Sciences, 3B Naukova Str., Lviv, 79601, Ukraine
E-mail: pelykh@lms.lviv.ua

Abstract. The elliptic system of equations, which is general-covariant and locally $SU(2)$-covariant, is investigated. The new condition of the Dirichlet problem solvability and the condition of zeros absence for solutions are obtained for this system, which contains in particular case the Sen-Witten equation. On this basis it is proved the existence of the wide class of hypersurfaces, in all points of which there exists a correspondence between the Sen-Witten spinor field and three-frame, which generalizes the Nester orthoframe. The Nester special orthoframe also exists on a certain subclass containing not only the maximal hypersurfaces.

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1. Introduction

The necessity for investigation of submanifolds, on which the solutions of elliptic equations are equal to zero, is connected with a fact that the necessary and sufficient conditions for absence of such closed submanifolds of codimension one are simultaneously the necessary and sufficient conditions for uniqueness of the Dirichlet problem for these equations in the domain. Since the elliptic equations refer to the static solutions of the given hyperbolic field equations, the non-uniqueness of solution for the boundary value problem defines the non-stability of ”zero modes” of given field equations. Additionally there appears the necessity to study not only the closed submanifolds and not only of codimension one, but all another ones, on which zeros of solutions are located. This is related, in particular, to Sen-Witten equations (SWE), for which the question about zeros existence had been discussed during a long time \([1−4]\). The zeros absence for SWE solutions had been proved for the case when initial data set for Einstein equations on the maximal hypersurface is asymptotically flat, and the local mass condition is fulfilled \([5]\). Since on maximal hypersurface the system of equations splits into separate equations, the choice of Cauchy surface as maximal enabled us to use the known results of investigations of zeros for single equations and to ascertain that in each point of maximal hypersurface there exists two-to-one correspondence between Sen-Witten spinor and Nester special orthonormal frame (SOF).

The purpose of this paper is to develop a new approach for establishing the conditions of solvability and zeros absence for general from the physical point of view elliptic systems of equations. This will give the possibility to prove the existence of the wide class of hypersurfaces, in all points of which there exists the two-to-one correspondence between Sen-Witten spinor and a certain three-frame; we will name it Sen-Witten orthonormal frame (SWOF). In all points on such hypersurfaces there exist also the well defined lapses and shifts, associated by Ashtekar and Horowitz \([1]\) with Sen-Witten spinor. On a subclass of this class, including also the maximal hypersurfaces, we establish the existence of two-to-one correspondence between Sen-Witten spinor and Nester three-frame.

2. Preliminaries

We introduce first three definitions.

Definition 1. The knot point of the component of the solution is a point, in which the component is equal zero.

Definition 2. The knot point of the solution for the elliptical system of equations is a point, in which the solution is equal to zero.

From the general theory of elliptic differential equations is known that nontrivial solutions cannot vanish on an open subdomain, but they can turn to zero on subsets of lower dimensions \(k, k = 0, 1, ...n − 1,\) where \(n\) is dimension of the domain.

Definition 3. The knot submanifold of dimension \(s, s = 1, 2, ...n − 1,\) is a maximal
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Discrete set of knot points is 0-submanifold. We will show in Sec. 3 that for the system of differential equations, interesting for us, all knot subsets are formed by intersection of knot surfaces of the components of the solution.

The connection between the unique solvability for the boundary value problem in \( \mathbb{R}^n \) and absence of \((n - 1)\)-dimensional closed knot submanifolds was established by Picone [6, 7]. The existence of such connection follows from the next consideration: if the boundary value problem in a certain domain \( \Omega \) is uniquely solvable, then the boundary value problem is also uniquely solvable for any subdomain \( \Omega_1 \subset \Omega \). This excludes the possibility of existence of nontrivial solutions which turn to zero on the boundary of the arbitrary domain \( \Omega_1 \), i.e., excludes the possibility of existence of closed knot submanifolds of codimension one, and vice versa.

The known investigation of elliptical equations of general form does not allow to obtain the conditions for all knot points absence. For example, even in the case of the only single equation of general form it is proved the absence of zeros only of infinite order \([8, 9]\). That is why further we will examine only such general equations, which possess also the necessary physical properties, in particular, symmetry properties.

Let \( \Omega \) be a bounded closed spherical-type domain in three-dimensional Riemannian space \( V^3 \), otherwise, (i) its boundary \( \partial \Omega \) in every point have a tangent plane; (ii) for every point \( P \) on the boundary there exists a sphere, which belongs to \( \Omega \), and the boundary of sphere includes the point \( P \).

In the domain \( \Omega \) let us consider the system of elliptic second order equations

\[
\frac{1}{\sqrt{-h}} \frac{\partial}{\partial x^\alpha} \left( \sqrt{-h} h^{\alpha\beta} \frac{\partial}{\partial x^\beta} u_A \right) + C_A^B u_B = 0, \tag{1}
\]

where \( h^{\alpha\beta} \) — components of the metric tensor in \( V^3 \); they are arbitrary real functions of independent real variables \( x^\alpha \), continuous in \( \Omega \) and the quadratic form \( h^{\alpha\beta} \xi_\alpha \xi_\beta \) is negative definite. The unknown functions \( u_A \) of independent variables \( x^\alpha \) are the elements of complex vector space \( \mathbb{C}^2 \), in which the skew symmetric tensor \( \varepsilon^{AB} \) is defined, and the group \( SU(2) \) acts. \( C_A^B \) is Hermitian \((1, 1)\) spinorial tensor.

The system of equation (1) is covariant under the arbitrary transformations of coordinates in \( V^3 \), and covariant under the local \( SU(2) \)-transformations of unknown functions in a local space isomorphic to the complexified tangent space in every point to \( V^3 \).

Picone had ascertained that at arbitrary coefficients of elliptic equations the boundary value problem is uniquely solvable, and the closed knot submanifolds of codimension one are absent, respectively, only in the domains with enough small intrinsic diameter.

The general conditions for the absence of closed knot surfaces for strong elliptic system (1) are ascertained by Theorem 1 [10].

† Maximal connected subset \( A \) is a nonempty connected subset such that the only connected subset containing \( A \) is \( A \).
**Theorem 1.** If in domain $\Omega$ there exist symmetrical quadratic functional second-order matrices $B_1, B_2, B_3$ of $C^1$ class, such that matrix

$$\sqrt{-h}C - \sum_{\alpha=1}^{3} \frac{\partial B_{\alpha}}{\partial x^\alpha} + B^T G^{-1} B$$

is positive definite, where $B = (B_1, B_2, B_3), G = \sqrt{-h} \text{diag}(\|h_{\alpha\beta}\|, \|h_{\alpha\beta}\|, \|h_{\alpha\beta}\|)$, then the solutions of system of equations [1] with matrix $C = \|C_A^B\|$ of $C^1$ class do not have the closed knot surfaces in domain $\Omega$.

The effective geometrical conditions of B-matrix existence and corresponding unique solvability of Dirichlet problem in dependence on the domain intrinsic diameter were obtained [10] for the Euclidean space. Such conditions are important, for example, in the theory of nuclear reactors. Since the conditions of knot manifolds absence for quantum fields equations are the point of our interest, we further will concentrate our attention on the conditions of knot points absence in the domains of arbitrary as well as infinite intrinsic diameter.

Evidently, if matrix $C$ is positive definite, then the conditions of Theorem 1 are fulfilled for $B \equiv 0$, and closed knot surfaces are absent in the domain with arbitrary intrinsic diameter. Simultaneously, the boundary value problem for the system of equations [1] is uniquely solvable.

Theorem 1 does not indicate the conditions at which knot points, lines as well as all knot surfaces for solutions of equations [1] are absent. We will obtain them in Sec. 3.

**3. Conditions for the absence of knot points**

In the case of a single selfadjoint elliptic equation in $V^3$ the knot submanifolds can be only the surfaces which divide the domain, but in the case of a system of equations the topology of knot submanifolds becomes more various: it can be also the lines and the points. We can take this fact into account and ascertain the conditions for the knot manifolds absence exploiting the double covariance of the system of equations [1] and using Zaremba-Giraud Lemma, generalized at first by Keldysh and Lavrentiev [11] and later by Oleynik [12].

Let us introduce the matrix

$$R := \|R_A^B\| := \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}, \quad \alpha \bar{\alpha} + \beta \bar{\beta} = 1,$$

which is of the group $SU(2)$, and let its elements additionally satisfy the condition

$$C_0^1 \beta^2 + (C_0^0 - C_1^1) \alpha \beta - C_0^1 \alpha^2 = 0.$$

Therefore,

$$C_0^1 = R_0^A C_A^B R_1^B = 0,$$
and in accordance with Hermicity of matrix $C$ also $\overline{C_{0'}} = C_{1'}$. Then $C_0 := C_{0'}$ and $C_1 := C_{1'}$ are eigenvalues of matrix $C = ||C_A^B||$. This follows from a fact that for arbitrary matrix $R \in SU(2)$ the identity

$$-\varepsilon R \varepsilon \equiv R^{T^+}$$

is valid, where $\varepsilon = ||\varepsilon^{AB}||$. Therefore

$$C' = -\varepsilon R \varepsilon C R^{T^+} = R^{T^+} C R^T = \text{diag}(C_0, C_1).$$

Let us denote

$$\Delta := C_1 - C_0 = \left[ \left( C_1 - C_0 \right)^2 + 4|C_0^1|^2 \right]^{1/2},$$

and let us denote by $S$ a set of points in domain $\Omega$, in all points of which $C_0^1$ does not equal to zero, and let us denote by $T$ a set of points, in which $C_0^1$ is equal to zero. Then the elements of the matrix $R$, which transforms the matrix $C$ to diagonal form, satisfy on the set $S$ the conditions

$$\alpha = 1 + \frac{\Delta^2}{4|C_0^1|^2}, \quad \beta = \frac{\alpha \Delta}{2C_0^1}$$

and on the set $T$ the conditions

$$\alpha = 1, \quad \beta = 0.$$

Functions $u_0'$ and $u_1'$ on the set $S$ will be following:

$$u_0' = \tilde{\alpha} \left( u_0 + \frac{\Delta}{2C_0^1} u_1 \right), \quad u_1' = \alpha \left( -\frac{\Delta}{2C_0^1} u_0 + u_1 \right),$$

(2)

and on set $T$ they will be

$$u_0' = \bar{\alpha} u_0, \quad u_1' = \alpha u_1.$$  

(3)

Respectively, eigenvalue $C_0$ on $S$ is:

$$C_0 = \frac{4C_0^0|C_0^1|^4 + (4\Delta |C_0^1|^2 + C_1^1\Delta^2) (4|C_0^1|^2 + \Delta^2)}{4|C_0^1|^2 (4|C_0^1|^2 + \Delta^2)}$$

and coincides with $C_0^0$ on the set $T$.

**Lemma.** If real and imaginary parts of functions $u_A$ and of elements of matrix $C_A^B$ are functions of class $C^2$ in domain $\Omega$, then the real and imaginary parts of functions $u_A'$ defined by conditions (2)-(3) are also the functions of class $C^2$ in this domain.

**Proof.** Taking into account that it is always possible to choose $\text{Im} \alpha \in C^2(\Omega)$, from direct calculation we obtain that on a set $S$ there are exists first and second derivatives of real and imaginary parts of functions $u_A'$ and $\alpha$ with respect to arguments $(\Delta^2/4|C_0^1|^2)$ and $(\Delta/2|C_0^1|^2)$ and that

$$\lim_{P \to S \to Q \in T} \text{Re} \alpha^{(m)}(P) = \text{Re} \alpha^{(m)}(Q), \quad \lim_{P \to S \to Q \in T} \text{Im} \alpha^{(m)}(P) = \text{Im} \alpha^{(m)}(Q),$$

$$\lim_{P \to S \to Q \in T} \text{Re} u_A'^{(m)}(P) = \text{Re} u_A'^{(m)}(Q), \quad \lim_{P \to S \to Q \in T} \text{Im} u_A'^{(m)}(P) = \text{Im} u_A'^{(m)}(Q),$$
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where symbol \( f^{(m)} \) denotes arbitrary partial derivatives of order \( m = 0, 1, 2 \).

The following theorem is valid.

**Theorem 2.** Let:

a) real and imaginary parts of the elements of matrix \( C \) be of \( C^2 \) class in the domain \( \Omega \);

b) at least one eigenvalue of matrix \( C \), for definiteness \( C_0 \), is non-negative everywhere in \( \Omega \);

c) real or imaginary part of the function

\[
v := \begin{cases} 
(u_0 + \frac{\Delta}{2C_0^1} u_1) & |s \cap \partial \Omega| \\
u_0 & |T \cap \partial \Omega|
\end{cases}
\]

does not equal to zero in any point.

Then solution \( u_A \) of class \( C^2 \) for the system of equations (1) does not have any knot points in the domain \( \Omega \) of spherical type.

**Proof:** The system of equations (1) is covariant under the arbitrary transformations of coordinates and under the local transformations from the group \( SU(2) \) that allows to use them independently. Let us apply on the first step the \( SU(2) \) spinor transformation \( u_A \rightarrow R_{AB} u_B \), which transforms the matrix \( C \) to the diagonal form, and under which the equation (1) is covariant.

The eigenvalues of matrix \( C \) are real, therefore, the resulting system of equations (1) splits into a system of four independent equations for real and imaginary parts of spinor \( u_A \). Taking into account that \( u_A, C_0 \) and \( C_1 \) are scalars under transformations of coordinates, and \( C_0 \geq 0 \), we can apply the Zaremba-Giraud principle in the general form grounded by Oleynik [12] to every equation containing \( C_0 \). According to this principle, if in a certain point \( P_0 \) on the sphere the nonconstant function in the ball turns to zero, and everywhere in the ball \( \text{Re} \ u_{\varphi} < 0 \), then \( \langle d\text{Re} u_{\varphi}, l \rangle \big|_{P_0} < 0 \). Here \( l \) – arbitrary vector field, for which \( \langle n, l \rangle \big|_{P_0} > 0 \), and \( n \) is one-form of intrinsic normal to the sphere in the point \( P_0 \).

Let us show further that a set of the knot points for function \( \text{Re} u_{\varphi} \) does not contain the isolated points. Let us assume that such point exists, i.e. \( \text{Re} u_{\varphi} = 0 \), and in a certain neighborhood of the point \( P_0 \) the function has a constant sign. For definiteness let in this neighborhood be \( u_{\varphi} < 0 \). Let us consider a sphere, on which the point \( P \) lies and is so small that completely belongs to the mentioned neighborhood of the point \( P \). Then, using Zaremba-Giraud principle, we obtain \( \langle d\text{Re} u_{\varphi}, n \rangle \big|_{P_0} > 0 \), and therefore in any neighborhood of the point \( P_0 \), located outside the ball the function \( \text{Re} u_{\varphi} \) changes its sign, and that is why its zeros are not isolated. Therefore, they form the surfaces which divide \( \Omega \). Since \( C_0 \geq 0 \), then it follows from the maximum principle that the closed knot surfaces for the components of solution \( \text{Re} u_{\varphi} \) are absent. Analogous conclusion is true also for the component of solution \( \text{Im} u_{\varphi} \). This means that the only surfaces having common points with the boundary of domain \( \Omega \) can be the knot surfaces of real
or imaginary part of function $u_0'$. According to condition c), if, for definiteness,

$$\text{Re} \left( u_0 + \frac{\Delta}{2C_0} u_1 \right) |_{S \cap \partial \Omega} \neq 0, \quad \text{Re} u_0 |_{T \cap \partial \Omega} \neq 0,$$

then we can choose

$$\text{Re} \pi |_{S \cap \partial \Omega} \neq \left\{ \left( \text{Re} \left( u_0 + \frac{\Delta}{2C_0} u_1 \right) \right) \right\}^{-1} \text{Im} \pi \text{Im} \left( u_0 + \frac{\Delta}{2C_0} u_1 \right) |_{S \cap \partial \Omega},$$

$$\text{Re} \pi |_{T \cap \partial \Omega} \neq \left( \text{Re} u_0 \right)^{-1} \text{Im} \pi \text{Im} u_0 |_{T \cap \partial \Omega}$$

and obtain

$$\left[ \text{Re} \pi \text{Re} \left( u_0 + \frac{\Delta}{2C_0} u_1 \right) - \text{Im} \pi \text{Im} \left( u_0 + \frac{\Delta}{2C_0} u_1 \right) \right] |_{S \cap \partial \Omega}$$

$$\equiv \text{Re} u_0' |_{S \cap \partial \Omega} \neq 0,$$

$$\left( \text{Re} \pi \text{Re} u_0 - \text{Im} \pi \text{Im} u_0 \right) |_{T \cap \partial \Omega} \equiv \text{Re} u_0' |_{T \cap \partial \Omega} \neq 0.$$

Therefore knot surfaces as well as lines and points of the real (or imaginary) part are absent, and that is why any knot points of complete solution $u_A$ are also absent. The statement of the theorem is proved.

Note. If the conditions a) and b) of the Theorem be fulfilled, and the matrix $C$ be non-negative definite in domain $\Omega$, then both eigenvalues are non-negative, and, therefore, the boundary value problem for the system of equations $\Pi$ is uniquely solvable in arbitrary bounded domain, as it follows from the classical maximum principle. Otherwise the solution in finite domain exists only in the case when its intrinsic diameter does not overcome a certain value.

4. The conditions of knot points absence for the solutions of Sen-Witten equation

After Witten’s positive energy proof the attempts of development of tensor method for proof were performed along two lines. The attempts of the tensor interpretation for Sen-Witten spinor field belong to the first line. In particular, Ashtekar and Horowitz [1] used Sen-Witten spinor field for determination of a class of preferred lapses $T := \lambda$ and shifts $T^a := -\sqrt{2} i \lambda^{(A} \lambda^{B)}$. Dimakis and Müller-Hoissen [2, 3] had defined a preferred class of orthonormal frame fields in which spinor field take a certain standard form. Frauendiener [13] had noticed a correspondence between Sen-Witten spinor field and a triad. But, as it was shown by Dimakis and Müller-Hoissen, frame fields cannot exist in the knot points of the spinor field.

Among the works of the second line the most developed is Nester’s method which is grounded on the new gauge conditions for the special orthonormal frame (SOF):

$$* q := \varepsilon^{abc} \omega_{abc} = 0, \quad \tilde{q}_b := \omega^a_{ba} = F_b,$$  \hspace{1cm} (4)
where $\omega_{abc}$ are the connection one forms coefficients and $F$ is arbitrary everywhere on $\Sigma_t$ defined exact one-form.

An essential part of Witten’s proof of nonnegativity for ADM mass is application of Sen-Witten equation (SWE)

$$D^B_{\ C} \beta^C = 0$$

(5)

with appropriate asymptotic conditions on the spacelike hypersurface $\Sigma$ in four-dimensional Riemannian manifold $M = \Sigma \times \mathbb{R}$ with each $\Sigma_t = \Sigma \times \{t\}$ spacelike.

Initial data set $(\Sigma_t, h_{\mu\nu}, K_{\pi\rho})$ satisfies the constraints, and is asymptotically flat. An action of operator $D_{AB}$ on spinor fields is

$$D_{AB} \lambda_C = D_{AB} \lambda_C + \frac{\sqrt{2}}{2} K_{ABC} D_{D} \lambda_D,$$

where $D_{AB}$ — spinorial form of the derivative operator $D_\alpha$ compatible with the metric $h_{\mu\nu}$ on $C^\infty$ hypersurface $\Sigma_t$, $K_{ABCD}$ — spinorial tensor of extrinsic curvature of hypersurface $\Sigma_t$.

The existence and uniqueness theorem for solution of equation (5) in corresponding Hilbert space with some asymptotic conditions was proved by Reula [14] (see also [1]).

Let us ascertain the conditions of zeros absence for these solutions on $\Sigma_t$ using the results of Sec. 2. From equation (5), taking into account the equation of Hamiltonian constraint on $\Sigma_t$, in Gauss normal coordinates we obtain [5]:

$$D^A B D^B C \lambda^C = \frac{1}{2\sqrt{\hbar}} \frac{\partial}{\partial x^\alpha} \left( \sqrt{\hbar} h^{\alpha\beta} \frac{\partial}{\partial x^\beta} \lambda_A \right) - \frac{\sqrt{2}}{2} K D_{AB} \lambda^B -$$

$$- \frac{\sqrt{2}}{4} \lambda^B D_{AB} K + \frac{1}{4} K^2 \lambda_A + \frac{1}{8} K_{\alpha\beta} K^{\alpha\beta} \lambda_A + \frac{1}{4} \mu \lambda_A = 0.$$  

(6)

Therefore, the system of equations (6) is a system of the form (1); if it does not have the knot points, the SWE also does not have them.

Spinorial tensor

$$C^A_B := \frac{\sqrt{2}}{4} D^A B K + \frac{1}{4} \varepsilon^A_B \left( 2 K^2 + \frac{1}{2} K_{\pi\rho} K^{\pi\rho} + \mu \right)$$

is Hermitian because $(D^A_B K)^+ = (\varepsilon^{BC} D^A_{AC} K)^+ = (\varepsilon^{BC})^+ (D^A_{AC} K)^+ = -(D^A_{AC} K) \varepsilon^{CB} = (D^A_B K)$.

So, the SWE solutions of class $C^2$ do not have the knot points in a bounded closed domain $\Omega$ of spherical type on $\Sigma_t$, if for spinorial tensor $C^A_B$ in this domain and for the boundary values of the solution the conditions of Theorem 2 are fulfilled.

Let us consider further a sequence $\Omega_n$ of increasing domains of spherical type covering $\Sigma_t$. If in every domain the conditions of Theorem 2 are fulfilled, then all solutions of class $C^2$ do not have the knot points in $\Omega_n$. According to Reula, on $\Sigma_t$ there exists the SWE solution of $\lambda^C = \lambda^C_\infty + \beta^C$ form, where $\lambda^C_\infty$ is asymptotically covariant constant spinor field on $\Sigma_t$, $\beta^C$ is an element of Hilbert space $\mathcal{H}$, which is the Cauchy completion of $C^\infty$ spinor fields under the norm

$$|| \beta^E ||^2_{\mathcal{H}} = \int_{\Sigma_t} (D^A_{B \beta^B} \beta^C) (D^A_{AC} \beta^C) dV.$$
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Solution $\lambda^C$ belongs properly to $C^\infty$ class. From the asymptotical flatness condition it follows that $(\Delta^2/(4|C_0|^2))$, as well as real and imaginary parts of functions $(\Delta/2C_0)$ and $(\Delta/2C_0^\ast)$ vanish asymptotically. Therefore, condition c) from Theorem 2 asymptotically take a form: $\text{Re}\lambda_\infty^0 \neq 0$ or $\text{Im}\lambda_\infty^0 \neq 0$. In such a way we obtain the following theorem:

**Theorem 3.** Let:

a) initial data set be asymptotically flat;

b) everywhere on $\Sigma_t$ the matrix of spinorial tensor \( \mathbb{I} \) have at least one non-negative eigenvalue, for definiteness $C_0$;

c) $\text{Re}\lambda_\infty^0$ or $\text{Im}\lambda_\infty^0$ asymptotically nowhere equal zero.

Then the asymptotically constant nontrivial solution $\lambda^C$ to SWE does not have the knot points on $\Sigma_t$.

The conditions of Theorem 3 are fully admissible from the physical point of view.

5. Towards Sen-Witten equation, special orthonormal frame and preferred time variables

Usually the question about existence of system of coordinates or orthonormal basis, which satisfy certain gauge conditions, is reduced to the question about existence of solution for nonlinear system of differential equations and often can be solved only at some additional limitations and assumptions [15].

The existence theorem for Sen-Witten (linear) equation and the Theorem 3 about their zeros (Sec. 4) on surfaces, which can be not maximal, allow us to prove the existence of a certain class of orthonormal three-frames in all points of these hypersurfaces which satisfies gauge conditions.

$$
\varepsilon^{abc}\omega_{abc} \equiv *q = 0, \quad \omega^a_{1a} \equiv -\tilde{q}_1 = F_1, \quad \omega^a_{2a} = -\tilde{q}_2 = F_2,
\omega^a_{3a} = -\tilde{q}_3 = K + F_3,
$$

and generalizes Nester’s SOF. Such three-frame we will name as Sen-Witten orthonormal frame (SWOF).

**Theorem 4.** Let the conditions of Theorem 3 be fulfilled. Then everywhere on $\Sigma_t$ there exists a two-to-one correspondence between Sen-Witten spinor and Sen-Witten orthonormal frame.

**Proof:**

Really, let all conditions of Theorem 3 be fulfilled on $\Sigma_t$. Then SWE solution $\lambda_A$ does not have the knot points anywhere on $\Sigma_t$. This allows to prove on such $\Sigma_t$ the Sommers [16] assumption that spatial null one-form $L = -\lambda_A\lambda_B$ on $\Sigma_t$ is nonzero, and allows to turn everywhere on $\Sigma_t$ to the ”squared” SWE represented in the form:

$$
\langle \bar{L}, D \otimes L \rangle - \mathcal{K}L + 3!i*(n \land D \land L) = 0,
$$

where $\langle \bar{L}, D \otimes L \rangle$ is one–form with components $\bar{L}_\nu D_\mu L^\nu$, $\bar{L} = |L|$ $|L|^{-1} * (L \land \mathcal{T})$ is nonzero spatial one–form, and $n$ is one form of unit normal to $\Sigma_t$. 
The bilinear form
\[
\frac{1}{\sqrt{2}} n^A \lambda_A \bar{\lambda}_A = \lambda A \lambda^{A\dagger} \equiv \lambda,
\]
where \( n \) is one-form of a unit normal to \( \Sigma_t \), is Hermitian positive definite one, and the solution \( \lambda_A \) does not have the knot points on \( \Sigma_t \). Consequently, we can further introduce real nowhere degenerated orthonormal 4-coframe \( \theta^m \) as
\[
\theta^0 \equiv n = N dt, \quad \theta^1 = \frac{\sqrt{2}}{2\lambda} (L + \bar{L}), \quad \theta^2 = \frac{\sqrt{2}}{2\lambda t} (L - \bar{L}), \quad \theta^3 = \bar{L}
\]
and represent immediately \((\text{9})\) in the form
\[
- \langle \theta^1, D \otimes \theta^3 \rangle - \mathcal{K} \theta^1 + 3! \left[ n \wedge (D + F) \wedge \theta^2 \right] = 0, \quad (11)
\]
\[
\langle \theta^2, D \otimes \theta^3 \rangle + \mathcal{K} \theta^3 + 3! \left[ n \wedge (D + F) \wedge \theta^1 \right] = 0, \quad (12)
\]
where \( F = D \ln \lambda \). The system of equations \((\text{11})-(\text{12})\) includes only four independent equations, and they are equations \((\text{8})\) for the connection one-forms coefficients. From this it follows that if on \( \Sigma_t \) the conditions of Theorem 3 and SWE are fulfilled, then on \( \Sigma_t \) there exists three-frame \( \theta^a \) defined by \((\text{10})\) in which conditions \((\text{8})\) are fulfilled.

Inversely, if on \( \Sigma_t \) in some three-frame \( \theta^a \) the conditions of Theorem 3 and conditions \((\text{8})\) are fulfilled, then it follows from condition of Theorem 3 that these one-forms have a form \( \theta^a = \theta^a_\infty + \phi^a \), where \( \theta^a_\infty \) tend asymptotically to the covariant constant forms and \( \phi^a \) belongs to \( \mathcal{H} \). We can turn from four-frame \( \theta^m \equiv \{n, \theta^a\} \) to one-forms \( \theta^0, L, \bar{L} \), assuming \( \lambda_A|_{\Sigma_t} \neq 0 \). After this we obtain equation \((\text{9})\) and further \((\text{5})\)† for spinor field \( \lambda^A \), which, as we have demonstrated previously, indeed does not have the knot points on selected hypersurface \( \Sigma_t \) and which together with asymptotical conditions defines up to the sign the spinor field \( \lambda^A \). Mentioned in conditions of the Theorem correspondence between Sen-Witten spinor field and Nester’s SOF is defined by relationship \((\text{10})\).

We have proved \((\text{5})\) that if initial set \((\Sigma_t, h_{\mu\nu}, K_{\pi\rho})\) on maximal hypersurface‡ \( \Sigma_t \) is asymptotically flat and satisfies the dominant energy condition, then everywhere on \( \Sigma_t \) from existence Sen-Witten spinor field follows existence Nester’s three-frame and conversely. Theorem 3 allows to strengthen significantly this result taking away the assumption that \( \Sigma_t \) is maximal. Indeed, if all conditions of Theorem 3 are fulfilled on \( \Sigma_t \), and additionally the one-form \( \mathcal{K} \bar{L} \) is globally exact, we can perform in conditions the identification \( F \equiv d \ln \lambda + \mathcal{K} \theta^3 \) and obtain the Nester’s gauge \((\text{11})\), or to perform the inverse transition — from Nester’s gauge to SWE. Therefore, if on \( \Sigma_t \) the conditions of Theorem 3 are fulfilled, then SWE and Nester’s gauge are equivalent if and only if the one-form \( \mathcal{K} \lambda^{A\dagger} \lambda^B \) is exact. In this case the correspondence between Sen-Witten spinor and Nester’s SOF is also ascertained by relationship \((\text{10})\).

Ashtekar and Horowitz \((\text{1})\) have accentuated on the necessity of zeros investigations for SWE solutions introducing the vector interpretation of Sen-Witten’s spinor which

† The equivalence of the SWE \((\text{5})\) and of the equation \((\text{6})\) is proved by Reula \((\text{14})\).
‡ Maximal surfaces are spacelike submanifolds of a Lorentzian manifold which locally maximize the induced area functional.
defines a preferred lapse and shift. Evidently, the fulfilling of the Theorem 3 conditions ensures the existence of corresponding lapses and shifts well defined everywhere on $\Sigma_t$. And also the preferred class of orthonormal four-frame fields introduced by Dimakis and Müller-Hoissen exists in all points of $\Sigma_t$ under fulfilling of the Theorem 3 conditions.

6. Conclusions

The presence of zeros in the solutions of elliptic equations is rather ordinary than exceptional case, therefore, it is necessary to prove the absence of zeros for concrete cases.

The represented investigation demonstrates the possibility for obtaining the condition of the knot manifolds absence for enough general system of elliptic second order equations owing to its double covariance.

The application of this result to SWE allows to prove the equivalence of SWE and gauge conditions, and, respectively, the existence of an everywhere well defined two-to-one correspondence between Sen-Witten spinor field and the SWOF, which is the Nester SOF in the particular case, when one of the one-forms $K^a$ is exact. Therefore, the indicated correspondence exists not only on the unique — maximal — hypersurface, but on the whole set of asymptotically flat hypersurfaces.

Ashtekar and Horowitz had shown that the Reula results hold even if the energy condition is mildly violated. Also the conclusion about existence of special three- and four-frames, as well as preferred lapses and shifts, is stable under the violation of the energy condition, because, as it is seen from , there exist the hypersurfaces, on which this condition of knot points absence is fulfilled at violation of the energy condition.

References

[1] Ashtekar A and Horowitz G T 1984 Phase space of general relativity revisited: A canonical choice of time and simplification of the Hamiltonian J. Math. Phys. 25 1473–1480
[2] Dimakis A Müller-Hoissen F 1990 Spinor fields and the positive energy theorem Class. Quantum Grav. 7 283–295
[3] ——1989 On a gauge condition for orthonormal three–frames Phys. Lett. A 112 73–74
[4] Nester J 1991 Special orthonormal frames end energy localization CQG 8 L19–23
[5] Pelykh V 2000 Equivalence of the spinor and tensor methods in the positive energy problem J. Math. Phys. 41 5550–5556
[6] Picone M 1911 Una teorema sulle soluzioni delle equazioni lineari ellitiche autoaggiunte alle derivate parziali del secondo ordine Rend. Acc. Sci. Lincei 20 331–338
[7] ——1913 Teorema di unicita nei problemi del valori al contorno per le equazioni ellitiche e paraboliche Ib. 22 2p sem. 275–282
[8] Aronszajn N 1957 An unique continuation theorem for solutions of elliptic partial differential equations or inequalities of second order J. Math. pur. et appl. XXVI 235–249
[9] Cordes H O 1956 Über die eindeutige Bestimmheit der Lösungen elliptischer Differentialgleichungen durch Anfangsverhalten Nachricht. Akad. Wiss. Göttingen, Math.–phys. Klasse 11 239–258
[10] Bobyk O I, Bodnarchuk P I, Ptashnyk B Y and Skorobohat’ko V Y 1972 Elements of qualitative theory of differential equations with partial derivatives (Kyiv: Naukova Dumka) p 216
Knot points of double-covariant system of elliptic equations

[11] Keldysh M V and Lavrentiev M A 1937 On uniqueness of Neumann problem *Dokl. Akad. Nauk SSSR* **16**, 151–152 (1937)

[12] Oleynik O A 1952 About the properties of the solutions of some boundary values problem for elliptic type equations *Math. Sbornik* **30** 695–702

[13] Frauendiener J 1991 Triads and the Witten equations *Class. Quantum Grav.* **8** 1881–1887

[14] Reula O 1982 Existence theorem for solutions of Witten’s equation and nonnegativity of total mass *J. Math. Phys.* **23** 810–814

[15] Nester J 1989 Gauge condition for orthonormal three-frames *J. Math. Phys.* **30** 624–626

[16] Sommers P 1980 Space spinors *J. Math. Phys.* **21**, 2567–2571