Coalescence for supercritical Galton-Watson processes with immigration

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Abstract

In this paper, we consider Galton-Watson processes with immigration. Pick \( i (\geq 2) \) individuals randomly without replacement from the \( n \)-th generation and trace their lines of descent back in time till they coalesce into 1 individual in a certain generation, which we denote by \( X_{n,1}^n \) and is called the coalescence time. Firstly, we give the probability distribution of \( X_{n,1}^n \) in terms of the probability generating functions of both the offspring distribution and the immigration law. Then by studying the limit behaviors of various functionals of the Galton-Watson process with immigration, we find the limit distribution of \( X_{2,1}^n \) as \( n \to \infty \).

Keywords: Galton-Watson processes; immigration; coalescence time.

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1 Introduction

Recent years, coalescence becomes an interesting research object in the community of branching processes. Athreya \cite{1} studied the distribution of the coalescence time and its limit behavior of critical and subcritical Galton-Watson processes whereas the supercritical case was considered by the same author in \cite{2}. Lambert \cite{9} investigated the limit distribution of the coalescence time in the subcritical case for some more general settings, including both discrete and continuous time and state space. Furthermore, Lambert and Popovic \cite{10} defined a coalescent point process, for which the coalescence time of two successive individuals alive at the same time is defined as the first point mass in it, and study its limit. For coalescence in Bellman-Harris and multitype branching processes, we refer the reader to Athreya and Hong \cite{4} and Hong \cite{8}. We also note that Grosjean and Huillet \cite{7} studied a more general coalescence for Galton-Watson processes, that is, coalescence of \( i (\geq 2) \) individuals into \( j (< i) \) individuals. For more results and details of coalescence in branching processes, we refer the reader to Athreya \cite{3} and references therein for a survey.

In this paper, we consider Galton-Watson branching processes with immigration. To give the precise model, let \( M \) be a \( \mathbb{Z}^+ := \{ 0, 1, 2, \ldots \} \)-valued random variable with
probability generating function (p.g.f.) \( f(z) = E(z^M), z \in [0,1] \), which serves as the offspring distribution and let \( I \) be also a \( \mathbb{Z}^+ \)-valued random variable with p.g.f. \( g(z) = E(z^I), z \in [0,1] \), which plays the role as the immigration law. Set \( N_0 = 0 \) and for \( n \geq 0 \), define

\[
N_{n+1} = \sum_{i=1}^{N_n+I_n} \xi_{n,i},
\]

where \( I_n, n \geq 0 \) are mutually independent and all distributed as \( I \), for \( n \geq 0 \), \( I_n \) are independent of \( \{N_n, N_{n-1}, \ldots, N_0\} \), and \( \xi_{n,i}, n \geq 0, i \geq 1 \) are mutually independent and have the same distribution as \( M \). We mention that in (1) and the remainder of the paper, empty sum equals 0. We call the process \( \{N_n\}_{n \geq 0} \) a Galton-Watson process with immigration. Throughout the paper, for convenience, we always assume

\[
f(0) = g(0) = 0,
\]

so that no extinction happens and for \( n \geq 1 \), there are at least \( n \) individuals in generation \( n \).

Let \( T \) be a typical tree generated by the above branching process with immigration. For \( 2 \leq i \leq n \), pick \( i \) individuals from the \( n \)-th generation at random without replacement and trace their lines of descent back in time till the generation, marked as \( X_{i,1}^n \), at which those \( i \) individuals coalesce into 1 individual for the first time. We call the generation number \( X_{i,1}^n \) the coalescence time. We emphasize that due to the existence of immigration, those \( i \) individuals may have no common ancestor. In this case, we set \( X_{i,1}^n = \infty \). Our concern is to study the exact distribution of the coalescence time \( X_{i,1}^n \) and find its limit as \( n \to \infty \).

We remark that without (2), it is enough to consider the distribution of \( X_{i,1}^n \), conditioned on the event \( \{N_n \geq i\} \). For \( n \geq 0 \), denote by

\[
\varphi_{\overline{n}}(z) := E(z^{N_n}), z \in [0,1]
\]

the p.g.f. of \( N_n \). Throughout, for a function \( h(z) \) we denote by \( h^{(j)}(z) \) its \( j \)-th derivative and for \( n \geq 1 \), we set \( h_n(z) = h(h_{n-1}(z)) \) with \( h_0(z) = z \). In (3), a subscript \( \overline{n} \) is used to emphasize that \( \varphi_{\overline{n}}(z) \) is not the \( n \)-fold composition of a function \( \varphi(z) \). Now we state the main results.

**Theorem 1.** For \( n \geq i \geq 2, n > m \geq 0 \), we have

\[
P(X_{i,1}^n = \infty) = 1 - \sum_{i=1}^{n} \frac{1}{\Gamma(i)} \int_0^1 (1 - z)^{i-1} f_l^{(i)}(z) g'(f_l(z)) \prod_{s \neq l, s=1}^n g(f_s(z)) dz, \quad (4)
\]

and

\[
P(m \leq X_{i,1}^n < \infty) = \frac{1}{\Gamma(i)} \int_0^1 (1 - z)^{i-1} f_l^{(i)}(z) \varphi_{\overline{n}}(f_{n-m}(z)) \prod_{l=1}^{n-m} g(f_l(z)) dz
\]

\[
+ \sum_{k=1}^{n-m} \frac{1}{\Gamma(i)} \int_0^1 (1 - z)^{i-1} f_k^{(i)}(z) g'(f_k(z)) \prod_{l \neq k, l=1}^n g(f_l(z)) dz. \quad (5)
\]
Remark 1. (i) What we have seen from Theorem 1 is the distribution of the coalescence time $X_{i,1}^n$ of $i$ individuals into 1 individual of the Galton-Watson process with immigration. Basically speaking, for the proof of Theorem 1, we adopt a method similar to [7]. Of course, because the existence of the immigration, things become more complicated and some difference arises. For example, if those $i$ individuals are chosen from the descendants of different immigrants, then they will never coalesce into 1 individual. So special light should be shed on the case $X_{i,1}^n = \infty$. (ii) We have results for the coalescence time $X_{i,j}^n$, that is, the coalescence of $i$ individuals into $j$ ($i < j$) individuals of the Galton-Watson process with immigration. The proofs are more or less similar to those of $X_{i,1}^n$ but the notations are very heavy, so we omit that part in this paper.

With the distribution of $X_{i,1}^n$ in (4) and (5) in hands, one may ask whether $X_{i,1}^n$ converges to a certain distribution as $n \to \infty$. This is our next concern. We concentrate only on the limit distribution of $X_{2,1}^n$ since that of $X_{i,1}^n$ can be treated similarly. To begin with, we introduce some useful notations. For $m < n$, let

\[ N_{m,n}^{(l)} := \text{the number of progenies in the } n\text{-th generation born to the } l\text{-th individual in the } m\text{-th generation}; \]

\[ \overline{N}_{m,n}^{(l)} := \text{the number of progenies in the } n\text{-th generation born to the } l\text{-th immigrant at generation } m. \]

Lemma 1. Assume $f(0) = g(0) = 0$, $g'(1) < \infty$, $f''(1) < \infty$ and write $\mu = f'(1)$. Then there exist random variables $W$, $X$ and $V$, which satisfy $P$-a.s.,

\[ \frac{1}{\mu^n} N_{0,n}^{(1)} \to W, \quad \frac{1}{\mu^n} \sum_{k=0}^{n-1} \sum_{l=1}^{I_k} N_{k,n}^{(l)} \to X, \quad \frac{1}{\mu^{2n}} \sum_{k=0}^{n-1} \sum_{l=1}^{I_k} \left( \overline{N}_{k,n}^{(l)} \right)^2 \to V, \quad (6) \]

as $n \to \infty$. Furthermore, we have $P(V < X^2) = 1$.

In the next theorem, we study both the quenched and annealed limit laws of the coalescence time $X_{2,1}^n$.

Theorem 2. Assume $f(0) = g(0) = 0$, $g'(1) < \infty$, $f''(1) < \infty$. Then

(i) for almost all family tree $\mathbb{T}$, we have

\[ \lim_{n \to \infty} P(m \leq X_{2,1}^n < \infty | \mathbb{T}) = \frac{\sum_{l=1}^{N_m} W_{m,l}^2 + V_m}{\left( \sum_{l=1}^{N_m} W_{m,l} + X_m \right)^2}, \quad (7) \]

for $0 \leq m < \infty$, where for $m, l \geq 0$, $W_{m,l}$, $V_m$, and $X_m$ are independent copies of $W$, $V$ and $X$, which appear in Lemma 4, respectively;

(ii) for $0 \leq m < \infty$,

\[ \lim_{n \to \infty} P(m \leq X_{2,1}^n < \infty) = E \left( \frac{\sum_{l=1}^{N_m} W_{m,l}^2 + V_m}{\left( \sum_{l=1}^{N_m} W_{m,l} + X_m \right)^2} \right), \quad (8) \]
and especially,
\[
\lim_{n \to \infty} P(X_{2,1}^n < \infty) = E \left( \frac{V}{X^2} \right) < 1.
\] (9)

Remark 2. (i) We use a method similar to [2] to prove Theorem 2. But, we require \(f''(1) < \infty\) or equivalently, \(\sigma^2 := \text{var}(M) < \infty\), which is not required in [2] when studying the coalescence time of Galton-Watson processes without immigration. In our case, by Lemma 2 below, \(X_{2,1}^n \geq m\) if and only if those two individuals are chosen from progenies born to one of the \(N_m\) individuals in the \(m\)-th generation or from progenies born to the immigrants which immigrate into the system between the \(m\)-th and \((n-1)\)-th generations. Therefore, we need to show the convergence of
\[
\frac{1}{\mu^{2(n-m)}} \sum_{k=m}^{n-1} \sum_{t=1}^{I_{k}} \left( N_{k,n}^{(l)} \right)^2,
\]
which is a submartingale. To apply the convergence theorem of submartingale, we should require the finiteness of \(\sigma^2\). (ii) We see from (9) that \(\lim_{n \to \infty} P(X_{2,1}^n < \infty) < 1\). This happens naturally since for any \(n\), with positive probability, \(X_{2,1}^n = \infty\), that is, those two individuals chosen from progenies in generation \(n\) may never coalesce into 1 individual.

The remainder of the paper is organized as follows. In Section 2, we give two computable examples. Then in Section 3, we provide some lemmas which are useful to prove the main result. Also Lemma 1 is proved in this section. Finally, we devote Section 4 to finishing the proofs of Theorem 1 and Theorem 2.

2 Examples

In this section, we give two computable examples. For two numbers \(n \geq k \geq 1\), we denote by
\[
(n)_k = n(n-1) \cdots (n-k+1),
\]
the falling factorial of \(n\).

Example 1 \((l\)-nary tree with \(k\) immigrants in each generation\). Let \(g(z) = z^k, k \geq 1\) and \(f(z) = z^l, l \geq 2\). Then by some easy computation, it follows from (4) and (5) that for \(n > 0\),
\[
P(X_{i,1}^n = \infty) = 1 - \sum_{t=1}^{n} \sum_{s=0}^{i-1} \frac{(-1)^{i-s-1}(i^1-s^1)(l^1-t^1)k(1-l)}{(i-1)!(kl(1-l^n) - s(1-l))},
\]
\[
P(m \leq X_{i,1}^n < \infty) = \sum_{s=0}^{i-1} \frac{(-1)^{i-s-1}(i^1-s^1)(l^{n-m}-l^m)kl(1-l^m)}{(i-1)!(kl(1-l^n) - s(1-l))} + \sum_{t=1}^{n} \sum_{s=0}^{i-1} \frac{(-1)^{i-s-1}(i^1-s^1)(l^1-t^1)k(1-l)}{(i-1)!(kl(1-l^n) - s(1-l))}.
\]

Especially, setting \(i = 2\), then
\[
P(X_{2,1}^n = \infty) = 1 - \frac{k(1-l)(l^2(l^2-1)-l(l+1)((1-l^n))}{(1+l)(kl(1-l^n) - (1-l))(kl(1-l^n))},
\]
\[
P(m \leq X_{2,1}^n < \infty) = \sum_{s=0}^{i-1} \frac{(-1)^{i-s-1}(i^1-s^1)(l^{n-m}-l^m)kl(1-l^m)}{(i-1)!(kl(1-l^n) - s(1-l))} + \sum_{t=1}^{n} \sum_{s=0}^{i-1} \frac{(-1)^{i-s-1}(i^1-s^1)(l^1-t^1)k(1-l)}{(i-1)!(kl(1-l^n) - s(1-l))}.
\]
\[ P(m \leq X^n_{2,1} < \infty) = \frac{l^{n-m}(l^{n-m} - 1)(1 - l^n)}{(kl - kl^{n+1} - (1 - l))(1 - l^n)} \]
\[ + \frac{k(1 - l) \left(l^2(1 - l^{2(n-m)}) - l(l + 1)(1 - l^{n-m})\right)}{(1 + l)(kl(1 - l^n) - (1 - l))(kl(1 - l^n))}, \]
and consequently,

\[
\lim_{n \to \infty} P(X^n_{2,1} = \infty) = \frac{(1+l)k - (l - 1)}{(1+l)k},
\]

\[
\lim_{n \to \infty} P(m \leq X^n_{2,1} < \infty) = l^{-2m}(l - 1) \left( \frac{l^n - 1}{l} + \frac{1}{1+l} \right).
\]

We see from Example 1 that due to the existence of immigration, the formulae for an \(l\)-nary deterministic tree with \(k\) immigrants in each generation are already very complicated. In general, it is difficult to give explicit formulae if the involved offsprings and immigrations are random. In the next example, we consider a binary tree with certain random immigration.

**Example 2.** Let \(g(z) = \frac{1}{2}(z^2 + z)\) and \(f(z) = z^2\). Then some tedious computations from (1) and (5) yield that

\[
P(m \leq X^n_{i,1} < \infty) = \sum_{i=1}^{n-m} \sum_{j=0}^{2^{n-m-1}2^{i-1} - 1} \sum_{h=0}^{2^{n-m-1}2^{i-1} - 1} \sum_{k=0}^{(i-1)\left(1 - i - s(2^{n-m})\right)} \frac{(i-1)(-1)^{i-1-s(2^{n-m})}2^{-n+l+1}}{(i-1)!(A(n, m, s, l, j, k) + 2^{n-m} + l)}
\]

\[
+ \sum_{s=0}^{i-1} \sum_{l=1}^{m} \sum_{j=0}^{2^{n-m-1}2^{i-1} - 1} \sum_{h=0}^{2^{n-m-1}2^{i-1} - 1} \sum_{k=0}^{(i-1)\left(1 - i - s(2^{n-m})\right)} \frac{(i-1)(-1)^{i-1-s(2^{n-m})}2^{-n+l}}{(i-1)!(A(n, m, s, l, h, k) + 2^{n-m} + l)}
\]

\[
+ \sum_{k=1}^{n-m} \sum_{s=0}^{i-1} \sum_{j=0}^{2^{n-k-1}2^{i-1} - 1} \sum_{l=0}^{2^{n-k-1}2^{i-1} - 1} \sum_{h=0}^{2^{n-k-1}2^{i-1} - 1} \sum_{l=0}^{(i-1)(-1)^{i-1-s(2^{k})}2^{-n+1}} \frac{(i-1)(-1)^{i-1-s(2^{k})}2^{-n+1}}{(i-1)!B(n, s, l, j, k)} \]

\[
+ \sum_{k=1}^{n-m} \sum_{s=0}^{i-1} \sum_{j=0}^{2^{n-k-1}2^{i-1} - 1} \sum_{l=0}^{2^{n-k-1}2^{i-1} - 1} \sum_{h=0}^{2^{n-k-1}2^{i-1} - 1} \sum_{l=0}^{(i-1)(-1)^{i-1-s(2^{k})}2^{-n+1}} \frac{(i-1)(-1)^{i-1-s(2^{k})}2^{-n+1}}{(i-1)!B(n, s, l, j, k)} \]

\[(10)\]

where we write \(A(n, m, s, l, j, h, k) := 2^{n+1} - 2 - s + 2j + h2^{n-m+1} + k2^{n-m+l+1}\) and \(B(n, s, l, j, k) := 2^{n+1} + l2^{k+1} + 2j - 2 - s\) for simplicity.

We remark that though the formula in (10) looks very ugly, given \(n, m\) and \(i\), it is indeed computable.

### 3 Preliminary results

In this section, we present several lemmas which are useful to prove Theorem 1 and 2. To begin with, we study the quenched distribution of \(X^n_{i,1}\).
Lemma 2. We have

\[ P \left( m \leq X_{n,1}^n < \infty \right) = \sum_{i=1}^{n} \left( N_{i,m,n}^{(i)} \right) + \sum_{k=m}^{n-1} \sum_{l=1}^{I_k} \left( N_{k,n}^{(l)} \right), \tag{11} \]

and especially,

\[ P \left( X_{n,1}^n < \infty \right) = \sum_{k=0}^{n-1} \sum_{l=1}^{I_k} \left( \right). \tag{12} \]

Proof. Note that the event \( m \leq X_{n,1}^n < \infty \) occurs if and only if those \( i \) individuals are randomly chosen from progenies in the \( n \)-th generation born to one individual in the \( m \)-th generation or born to one of those immigrants which immigrate into the system between the \( m \)-th and \( (n-1) \)-th generation. Thus (11) follows. Since \( N_0 = 0 \), letting \( m = 0 \) in (11), we get (12). \( \square \)

To obtain the annealed distribution of \( X_{n,1}^n \), it is necessary to take expectation in both sides of (11) and (12). The following lemma supplies a tool to compute those expectations.

Lemma 3. Suppose that \( Y_1, ..., Y_n, Z \) are mutually independent \( \mathbb{Z}^+ \)-valued random variables, \( Y_1, ..., Y_n \) share the same p.g.f. \( \phi(z) \), and \( Z \) has p.g.f. \( h(z) \). Then, for \( 1 \leq j < i, i_1 + i_2 + ... + i_j = i \) and \( i_1, i_2, ..., i_j \geq 1 \),

\[ E \left( \frac{(Y_1)_{i_1}(Y_2)_{i_2} \cdots (Y_j)_{i_j}}{(\sum_{l=1}^{n} Y_l + Z)_i} \right) = \frac{1}{\Gamma(i)} \int_0^1 (1 - z)^{i-1} \phi(z)^{n-j} \prod_{l=1}^{j} \phi^{(i_l)}(z)h(z)dz. \tag{13} \]

Proof. Setting \( \sum_{l=j+1}^{n} Y_l = Y \), we have

\[ E \left( \frac{(Y_1)_{i_1}(Y_2)_{i_2} \cdots (Y_j)_{i_j}}{(\sum_{l=1}^{n} Y_l + Z)_i} \right) = \sum_{m, m_1, ..., m_j, x=0}^{\infty} P(Y = m)P(Y_1 = m_1) \cdots P(Y_j = m_j)P(Z = x) \times \frac{(m_1)_{i_1}(m_2)_{i_2} \cdots (m_j)_{i_j}}{(m + m_1 + ... + m_j + x)_i}. \tag{13} \]

On the other hand, owing to the independence of \( Y_1, ..., Y_n, Z \), some careful computation yields that

\[ \frac{1}{\Gamma(i)} \int_0^1 (1 - z)^{i-1} \phi(z)^{n-j} \prod_{l=1}^{j} \phi^{(i_l)}(z)h(z)dz \]

\[ = \sum_{m, m_1, ..., m_j, x=0}^{\infty} P(Y = m)P(Y_1 = m_1) \cdots P(Y_j = m_j)P(Z = x) \times \frac{(m_1)_{i_1}(m_2)_{i_2} \cdots (m_j)_{i_j}}{(m + m_1 + ... + m_j + x)_i} \frac{1}{\Gamma(i)} \int_0^1 (1 - z)^{i-1} z^{m_1 + m_2 + ... + m_j + x} dz. \]
The convergence of $X$ is all $f > 0$ that $P(\text{no extinction happens to the branching process } N_{0,n}, n \geq 1 \text{ so that } P(0 < W < \infty) = 1$).

By [5], we have $P$-a.s., $W_n \to W$ as $n \to \infty$ for some random variable $W$. Since we assume $f(0) = 0$, no extinction happens to the branching process $N_{0,n}, n \geq 1$ so that $P(0 < W < \infty) = 1$.

The convergence of $X_n$ and $V_n$ can be shown by the convergence theorem of submartingale. We remark that the convergence of $X_n$ is a direct consequence of [6]. We give its proof here for convenience. Some direct computation yields that
\[
E^2 = \frac{\lambda}{\mu^2} \left( \sigma^2 \left( \frac{\mu(\mu^2-1)}{\mu^2-1} - \frac{\mu^n-1}{\mu-1} \right) + \frac{\mu^2(\mu^2-1)}{\mu^2-1} \right).
\]

Thus there exists some constant $C > 0$, such that for all $n \geq 1$,
\[
E(X_n) < C, \ E(V_n) < C.
\]  

Proof of Lemma 1

Write $\lambda := g'(1), \sigma^2 := f''(1) + f'(1) - f'(1)^2 \equiv \text{Var}(M)$ and for $n \geq 1$ set
\[
W_n := \frac{N_{0,n}^{(1)}}{\mu^n}, \ X_n := \frac{1}{\mu^n} \sum_{k=0}^{n-1} I_k \sum_{l=1}^{N_{k,n}^{(l)}} V_n := \frac{1}{\mu^n} \sum_{k=0}^{n-1} \sum_{l=1}^{I_k} \left( N_{k,n}^{(l)} \right)^2.
\]

By [5], we have $P$-a.s., $W_n \to W$ as $n \to \infty$ for some random variable $W$. Since we assume $f(0) = 0$, no extinction happens to the branching process $N_{0,n}, n \geq 1$ so that $P(0 < W < \infty) = 1$.

The convergence of $X_n$ and $V_n$ can be shown by the convergence theorem of submartingale. We remark that the convergence of $X_n$ is a direct consequence of [6]. We give its proof here for convenience. Some direct computation yields that
\[
E(X_n) = \frac{\mu(\mu^n-1)}{\mu^n(\mu-1)},
\]
\[
E(V_n) = \frac{\lambda}{\mu^2} \left( \sigma^2 \left( \frac{\mu(\mu^2-1)}{\mu^2-1} - \frac{\mu^n-1}{\mu-1} \right) + \frac{\mu^2(\mu^2-1)}{\mu^2-1} \right).
\]

Thus there exists some constant $C > 0$, such that for all $n \geq 1$,
\[
E(X_n) < C, \ E(V_n) < C.
\]  

Notice that by Markov property,
\[
E(V_{n+1}|V_n, V_{n-1}, ..., V_1) = E(V_{n+1}|V_n)
\]
\[
= \frac{1}{\mu^{2n+2}} \left[ \sum_{k=0}^{n-1} \left( N_{k,n+1}^{(l)} \right)^2 + \sum_{l=1}^{I_n} \left( N_{n,n+1}^{(l)} \right)^2 \left| V_n \right) \right]
\]
\[
= \frac{1}{\mu^{2n+2}} \left[ \sum_{k=0}^{n-1} \left( N_{k,n+1}^{(l)} \right)^2 + \sum_{l=1}^{I_n} \left( N_{n,n+1}^{(l)} \right)^2 \left| V_n \right) \right],
\]  

where $\xi_{n,k,l,j}, n, k, l, j \geq 1$ are mutually independent random variables whose p.g.f. are all $f(z)$. Since $E \left( \sum_{l=1}^{I_n} \left( N_{n,n+1}^{(l)} \right)^2 \right) = \lambda(\sigma^2 + \mu^2)$ and
\[
E \left( \sum_{k=0}^{n-1} \sum_{l=1}^{I_k} \left( \sum_{j=1}^{\xi_{n,k,l,j}} \right)^2 \right) \left| V_n \right)
\]
\[
\begin{align*}
&= \sum_{k=0}^{n-1} I_k \left( \sum_{l=1}^{n-1} E(\xi_{n,k,l,j}^2) + \sum_{i,j=1, i \neq j}^{n-1} E(\xi_{n,k,l,i})E(\xi_{n,k,l,j}) \right) \\
&= \sum_{k=0}^{n-1} I_k \left( \left( \frac{\mathcal{N}_{k,n}^{(0)}}{\mathcal{N}_{k,n}} \right)^2 \mu^2 + \frac{\mathcal{N}_{k,n}^{(1)}}{\mathcal{N}_{k,n}} \sigma^2 \right),
\end{align*}
\]

from (15), we get

\[
E(V_{n+1}|V_n, V_{n-1}, ..., V_1) = \frac{1}{\mu^{2n+2}} \left( \lambda(\sigma^2 + \mu^2) + \sum_{k=0}^{n-1} I_k \left( \left( \frac{\mathcal{N}_{k,n}^{(0)}}{\mathcal{N}_{k,n}} \right)^2 \mu^2 + \frac{\mathcal{N}_{k,n}^{(1)}}{\mathcal{N}_{k,n}} \sigma^2 \right) \right)
\]

\[
> \frac{1}{\mu^{2n}} \sum_{k=0}^{n-1} I_k \left( \frac{\mathcal{N}_{k,n}^{(0)}}{\mathcal{N}_{k,n}} \right)^2 = V_n.
\]

A similar argument also yields that \( E(X_{n+1}|X_n, ..., X_1) > X_n \). Consequently, taking (15) into consideration, we conclude that both \( V_n, n \geq 1 \) and \( X_n, n \geq 1 \) are submartingales with bounded means. Thus, applying the convergence theorem of submartingale, we have \( P \)-a.s.,

\[
\lim_{n \to \infty} V_n = V, \quad \lim_{n \to \infty} X_n = X
\]

for some random variables \( V \) and \( X \) with \( E(V) < C \) and \( E(X) < C \). Clearly, we have for all \( n \geq 1 \) \( X_n > \frac{N_{0,n}^{(1)}}{\mu^n} \) and \( V_n > \frac{N_{0,n}^{(1)}}{\mu^n} \). Therefore, we have \( P \)-a.s., \( 0 < V, X < \infty \). We thus finish the proof of (6). Finally, by (6), we have \( P \)-a.s.,

\[
X^2 - V = \lim_{n \to \infty} (X_n^2 - V_n) \geq \lim_{n \to \infty} \frac{N_{0,n}^{(1)}}{\mu^n} \frac{N_{0,n}^{(2)}}{\mu^n} = W^2 > 0.
\]

Lemma 1 is proved. \( \square \)

4 Proofs of main results

4.1 Proof of Theorem 1

To begin with, we show (5). By Lemma 2, we have

\[
P \left( m \leq X_{i,1}^n \leq \infty \right) = E \left( \sum_{l=1}^{N_m} \left( \frac{N_{m,n}^{(l)}}{(N_n)_i} \right) + \sum_{t=m}^{n-1} \sum_{l=1}^{I_t} \left( \frac{N_{t,n}^{(l)}}{(N_n)_i} \right) \right)
\]

\[
= E \left( \sum_{l=1}^{N_m} \left( \frac{N_{m,n}^{(l)}}{(N_n)_i} \right) \right) + \sum_{t=m}^{n-1} E \left( \sum_{l=1}^{I_t} \left( \frac{N_{t,n}^{(l)}}{(N_n)_i} \right) \right).
\]
\[ L = \sum_{t=m}^{n-1} L_t. \] (17)

To compute \( L \), noticing that \( N_n = \sum_{l=1}^{N_m^{(l)} N_m^{(l)_m}} + \sum_{k=m}^{n-1} I_k \sum_{l=1}^{N_k^{(l)}} \), thus,

\[
L = E \left( \frac{\sum_{l=1}^{N_m^{(l)} N_m^{(l)_m}}}{(N_n)_i} \right)
= \sum_{n_m, i_m, \ldots, i_{n-1}=0}^{\infty} P(N_m = n_m) \prod_{k=m}^{n-1} P(I_k = i_k)
\times \sum_{l=1}^{N_m^{(l)}} E \left( \frac{(N_m^{(l)} i_l)}{(N_m^{(l)} i_l + \sum_{k=m}^{n-1} I_k \sum_{l=1}^{N_k^{(l)}} i_l)} \right). \tag{18}
\]

In order to apply Lemma 3, we set \( Z = \sum_{k=m}^{n-1} \sum_{l=1}^{N_k^{(l)}} N_{k,n}^{(l)} \) and \( Y_l = N_{m,n}^{(l)}, 1 \leq l \leq n_m. \) Then \( Z, Y_1, \ldots, Y_{n_m} \) are mutually independent. By some classical argument of Galton-Watson processes, \( Y_1, \ldots, Y_{n_m} \) share the common p.g.f. \( f_{n-m}(z), z \in [0,1]. \) From the assumption, it is clear that \( N_{k,n}^{(l)}, m \leq k \leq n-1, 1 \leq l \leq i_k \) are mutually independent and for \( m \leq k \leq n-1, 1 \leq l \leq i_k \), the p.g.f. of \( N_{k,n}^{(l)} \) equals \( f_{n-k}(z), z \in [0,1]. \) Thus, the p.g.f. of \( Z \) equals \( \prod_{k=m}^{n-1} f_{n-k}^{(i_k)}(z), z \in [0,1]. \) Therefore, an application of Lemma 3 yields that

\[
E \left( \frac{(N_m^{(l)} i_l)}{(N_m^{(l)} i_l + \sum_{k=m}^{n-1} I_k \sum_{l=1}^{N_k^{(l)}} i_l)} \right)
= \frac{1}{\Gamma(i)} \int_0^1 (1-z)^{i-1} f_{n-m}^{(i)}(z) f_{n-m}^{(i)}(z) \prod_{k=m}^{n-1} f_{n-k}^{(i_k)}(z) dz. \tag{19}
\]

Substituting (19) into (18), we get that

\[
L = \sum_{n_m, i_m, \ldots, i_{n-1}=0}^{\infty} P(N_m = n_m) \prod_{k=m}^{n-1} P(I_k = i_k)
\times \sum_{l=1}^{N_m^{(l)}} \int_0^1 (1-z)^{i-1} f_{n-m}^{(i)}(z) f_{n-m}^{(i)}(z) \prod_{k=m}^{n-1} f_{n-k}^{(i_k)}(z) dz
\]

\[
= \frac{1}{\Gamma(i)} \int_0^1 (1-z)^{i-1} f_{n-m}^{(i)}(z) f_{n-m}^{(i)}(z) \prod_{k=m}^{n-1} g(f_{n-k}(z)) dz
\]

\[
= \frac{1}{\Gamma(i)} \int_0^1 (1-z)^{i-1} f_{n-m}^{(i)}(z) f_{n-m}^{(i)}(z) \prod_{k=1}^{n-m} g(f_k(z)) dz. \tag{20}
\]

\( L_t \) can be computed similarly. Indeed,

\[
L_t = E \left( \frac{\sum_{l=1}^{l_t} N_{t,n}^{(l)} i_k}{(N_n)_i} \right) = E \left( \frac{\sum_{l=1}^{l_t} N_{t,n}^{(l)} i_k}{(N_n)_i} \right).
\]
Thus, it follows from (23) and (25) that

$$L_t = \frac{1}{\Gamma(i)} \int_0^1 (1 - z)^{i-1} f_{n-t}^{(i)}(z) g_{n-t}^{(i)}(z) \prod_{l \neq n-t, l=1}^n g(f_l(z)) dz.$$  \hspace{1cm} (26)
Consequently, taking (17), (20) and (26) together, we finish the proof of (5).

Next we proceed to prove (4). Letting $m = 0$ in (5), we get

$$P(X_{i,1}^n < \infty) = \sum_{l=1}^{n} \frac{1}{\Gamma(i)} \int_{0}^{1} (1 - z)^{i-1} f_l^{(i)}(z) g'(f_l(z)) \prod_{s \neq l, s=1}^{n} g(f_s(z)) dz.$$  

Then (4) is proved and so is Theorem 1. \hfill \Box

4.2 Proof of Theorem 2

Fix $0 \leq m < \infty$. Then it follows from Lemma 2 that

$$P(m \leq X_{2,1}^n < \infty | \mathbb{T}) = \frac{\sum_{l=1}^{N_m} N_{m,n}^{(l)} (N_{m,n}^{(l)} - 1) + \sum_{k=m}^{n-1} \sum_{l=1}^{l_k} N_{k,n}^{(l)} (N_{k,n}^{(l)} - 1)}{N_{k,n} - 1}$$

$$= \frac{\sum_{l=1}^{N_m} N_{m,n}^{(l)} (N_{m,n}^{(l)} - 1) + \sum_{k=m}^{n-1} \sum_{l=1}^{l_k} N_{k,n}^{(l)} (N_{k,n}^{(l)} - 1)}{(\sum_{l=1}^{N_m} N_{m,n}^{(l)} + \sum_{k=m}^{n-1} \sum_{l=1}^{l_k} N_{k,n}^{(l)}) (\sum_{l=1}^{N_m} N_{m,n}^{(l)} + \sum_{k=m}^{n-1} \sum_{l=1}^{l_k} N_{k,n}^{(l)} - 1)}$$

$$= \Delta_1 + \Delta_2,$$

where

$$\Delta_1 := \frac{\sum_{l=1}^{N_m} N_{m,n}^{(l)} (N_{m,n}^{(l)} - 1)}{(\sum_{l=1}^{N_m} N_{m,n}^{(l)} + \sum_{k=m}^{n-1} \sum_{l=1}^{l_k} N_{k,n}^{(l)}) (\sum_{l=1}^{N_m} N_{m,n}^{(l)} + \sum_{k=m}^{n-1} \sum_{l=1}^{l_k} N_{k,n}^{(l)} - 1)}$$

$$\Delta_2 := \frac{\sum_{k=m}^{n-1} \sum_{l=1}^{l_k} N_{k,n}^{(l)} (N_{k,n}^{(l)} - 1)}{(\sum_{l=1}^{N_m} N_{m,n}^{(l)} + \sum_{k=m}^{n-1} \sum_{l=1}^{l_k} N_{k,n}^{(l)}) (\sum_{l=1}^{N_m} N_{m,n}^{(l)} + \sum_{k=m}^{n-1} \sum_{l=1}^{l_k} N_{k,n}^{(l)} - 1)}.$$

Applying Lemma 1 we get that P-a.s.,

$$\Delta_1 \rightarrow \frac{\sum_{l=1}^{N_m} W_{m,l}^2}{(\sum_{l=1}^{N_m} W_{m,l} + X_m)^2} \text{ as } n \rightarrow \infty,$$

$$= \frac{\sum_{l=1}^{N_m} W_{m,l}^2}{(\sum_{l=1}^{N_m} W_{m,l} + X_m)^2} \text{ as } n \rightarrow \infty,$$

(28)

and

$$\Delta_2 = \frac{V_m}{(\sum_{l=1}^{N_m} W_{m,l} + X_m)^2} \text{ as } n \rightarrow \infty,$$

where for $m, l \geq 0, W_{m,l}, V_m$ and $X_m$ are independent copies of $W, V$ and $X$, which appear in Lemma 1 respectively.
Substituting (28) and (29) into (27), we obtain (7). We thus finish the proof of Part (i) of Theorem 2.

Now we proceed to prove the second part. Taking expectation in both sides of (7) and using the bounded convergence theorem, we get (8). Finally, putting especially \( m = 0 \) in (8), we have

\[
\lim_{n \to \infty} P(0 \leq X_{2,1}^n < \infty) = E\left(\frac{V}{X^2}\right).
\]

But by Lemma 1, we have \( P(0 < V < X^2) = 1 \), implying \( E(V/X^2) < 1 \). Consequently, Part (ii) of Theorem 2 is proved. \( \square \)

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