Superfield formulation of N=4 supersymmetric Yang–Mills theory in extended superspace

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Action of 4 dimensional N=4 supersymmetric Yang–Mills theory is written by employing the superfields in N=4 superspace which were used to prove the equivalence of its constraint equations and equations of motion. Integral forms of the extended superspace are engaged to collect all of the superfields in one “master” superfield. The proposed N=4 supersymmetric Yang–Mills action in extended superspace is shown to acquire a simple form in terms of the master superfield.

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1 Introduction

Maximally supersymmetric gauge model in four dimensions that contains fields with spins at most one is N=4 supersymmetric Yang-Mills (SYM) theory \[1\]. This theory is distinguished for its finiteness and duality properties and studied extensively since last three decades (for some reviews see \[2, 3\]). In spite of these facts, a superfield formulation of N=4 SYM in extended superspace is still lacking. An off-shell formulation in terms of auxiliary fields of the N=4 SYM multiplet is still unknown (for a formulation with an infinite number of fields see \[4\]) A progress made in this direction was to accomplish the equivalences of the superfield constraint equations and the equations of motion for N=3 and N=4 SYM theories \[5, 6\]. In \[7\] a complete proof of this equivalence relation for N=3 SYM was given by introducing a suitable gauge choice which eliminates gauge freedom depending on Grassmann coordinates. Obviously, by superfields we mean fields written as functions of superspace variables. Indeed, this gauge choice permits one to find some recursion relations from the constraint equations to construct superfields order by order in Grassmann variables. Unlike the accustomed superfields, components of the superfields constructed in \[7, 8\] do not encompass any auxiliary field. Hence, they do not demand an off-shell supersymmetric formulation, but at the cost of considering superfields which do not possess the usual supersymmetry properties.

Superfields constructed in \[8\] were employed to construct physical states in Berkovits quantization of superparticles and superstrings in ten dimensions \[9\]. Also, the approach of \[7\] was applied to define deformed N=4 SYM equations \[10\]. Recently, in terms of these superfields an alternative superfield formulation of N=1 SYM without auxiliary fields in 4 dimensions was given \[11\].

We present a formulation of 4 dimensional N=4 SYM action in terms of the superfields of \[7\]. Moreover, we show that these fields can be written as integral forms and be collected in a “master” superfield such that the N=4 action can be expressed in a simple, compact form.

Though how to determine components of the superfields by recursion relations is known, actual calculation of components which are third or fourth order in Grassmann variables is a hard task. One of the other important issues to write an action in extended superspace is to define a measure which is invariant under the global $SU(4)$ group. We propose a measure which is suitable to achieve our goal. Acquainted with these we write an action in terms of superfields and prove that the action of N=4 SYM theory in terms of component fields results, after a lengthy calculation. For the coefficients appearing in the action there are more than one solution.

Engaging differentials of Grassmann variables the N=4 SYM superfields can be written as integral forms \[12\] (see also \[13\]) and be collected into a “master” superfield. Then, the N=4 SYM action can be written in a compact way. This action is apparently first order in space-time derivatives and there are two terms which are quadratic and cubic in master superfield. Indeed, all other powers of master superfield give vanishing contributions to the action. Also in this case,
there are some different solutions for the coefficients involved.

In the next section we recall formulation of 4 dimensional N=4 SYM theory in terms of component fields. In Section 3 after giving the definitions of superfields, we present their first two components in Grassmann variables of the extended superspace. The higher components are listed in Appendix. In Section 4 we present the general formulation in terms of superfields after a choice of measure. In Section 5 we give the definition of master superfield as a collection of integral forms. We demonstrate that the N=4 SYM theory action in extended superspace acquires a simple form. In the last section we discuss the results obtained and some open questions.

2 Component fields formulation

The N=4 Yang–Mills supermultiplet consists of one gauge field, eight Weyl fermions $\lambda^{a\alpha}$, $\bar{\lambda}^{\dot{a}\dot{\alpha}}$ and six scalars $\phi_{ij} = -\phi_{ji} = \frac{1}{2} \epsilon_{ijkl}\sigma^{kl}$. Spinor indices are $\alpha, \dot{\alpha} = 1, 2, i, j = 1, \ldots, 4$, denote indices of the global symmetry group $SU(4)$. In fact, $a_{a\dot{a}}$ is a singlet, $\lambda^{a\alpha}$ and $\bar{\lambda}^{\dot{a}\dot{\alpha}}$ are in the 4 and 4 representation and $\phi_{ij}$ are in the second rank, self dual 6 representation of $SU(4)$. All of the fields are in the adjoint representation of a non–abelian gauge group.

Hermitian conjugation which we attribute to the fields is

$$(a_{a\dot{a}})^\dagger = -a_{\dot{a}a}, (\lambda^{a\alpha})^\dagger = \bar{\lambda}^{\dot{a}\dot{\alpha}}, (\phi_{ij})^\dagger = \phi_{ji}. $$

N=4 extended SYM action in the component fields $a_{a\dot{a}}, \lambda^{a\alpha}, \phi_{ij}$ can be written as

$$I = \int d^4x \text{Tr} \left( \frac{1}{8} f_{\dot{a}\dot{b}} f^{\dot{a}\dot{b}} + \frac{1}{8} f_{\dot{a}\dot{b}} f^{\dot{a}\dot{b}} + \frac{1}{16} D_{a\dot{a}} \phi_{ij} D^{a\dot{a}} \phi_{ij} - \frac{i}{4} \lambda^{a\alpha} D_{a\dot{a}} \bar{\lambda}^{i\dot{a}} \bar{\phi}_{ij} \right),$$

where $D_{a\dot{a}} = \partial_{a\dot{a}} + [a_{a\dot{a}}, \cdot]$. $f_{\alpha\beta}$ and $f_{\dot{\alpha}\dot{\beta}}$ are self-dual and anti-self-dual field strengths defined as

$$f_{\alpha\beta} = -\frac{1}{2} \epsilon^{\dot{a}\dot{b}} \left( \partial_{\dot{a}\dot{b}} a_{a\dot{a}} - \partial_{a\dot{a}} a_{a\dot{a}} + [a_{a\dot{a}}, a_{\beta\dot{\beta}}] \right),$$

$$f_{\dot{\alpha}\dot{\beta}} = \frac{1}{2} \epsilon^{\dot{a}\dot{b}} \left( \partial_{\dot{a}\dot{b}} a_{\alpha\beta} - \partial_{\alpha\beta} a_{a\dot{a}} + [a_{a\dot{a}}, a_{\beta\dot{\beta}}] \right).$$

The action (1) is invariant under the on-shell N=4 supersymmetry transformations:

$$\delta a_{a\dot{a}} = -\xi_{a\dot{a}} \bar{\lambda}^{i\dot{a}} - \bar{\xi}^{i\dot{a}} \lambda^{a\alpha}$$

3We always make the identification $x_{\alpha\dot{\alpha}} = \sigma^{\mu\nu} x_{\mu\nu}$, $x^{\alpha\dot{\alpha}} = \sigma^{\mu\nu} x_{\mu\nu}$, $\partial_{\alpha\dot{\alpha}} = \sigma^{\mu\nu} \partial_{\mu\nu}$.

4We use Wess-Bagger conventions [11] to raise and lower the spinor indices: $\theta_\alpha = \epsilon_{\alpha\beta} \theta_\beta$, $\epsilon_{\alpha\beta} \epsilon^{\gamma\delta} = \delta_{\gamma\delta}$. Note that, $f_{\alpha\beta} = (\sigma^{\mu\nu})_{\alpha}^{\dot{\gamma}} \epsilon_{\dot{\gamma}\dot{\beta}} f_{\mu\nu}$ and $f_{\dot{\alpha}\dot{\beta}} = \epsilon_{\dot{\alpha}\dot{\gamma}} (\bar{\sigma}^{\mu\nu})_{\dot{\gamma}}^{\mu\nu} f_{\mu\nu}$ where $f_{\mu\nu} = \partial_{\mu} a_{\nu} - \partial_{\nu} a_{\mu} + [a_{\mu}, a_{\nu}]$ as usual.
where \( \xi, \bar{\xi} \) are constant Weyl spinors.

### 3 Superfields in N=4 superspace

N=4 extended superspace is parametrized by the coordinates

\[(x^\mu, \theta_1^\alpha, \bar{\theta}_1^{\dot{\alpha}}).\]  

(5)

Translations in this extended superspace

\[x_{\alpha\dot{\alpha}} \to x_{\alpha\dot{\alpha}} + 2i(\zeta_1^\alpha \theta_1^\alpha + \zeta_1^{\dot{\alpha}} \bar{\theta}_1^{\dot{\alpha}}), \quad \theta_i^\alpha \to \theta_i^\alpha + \zeta_i^\alpha, \quad \bar{\theta}_i^{\dot{\alpha}} \to \bar{\theta}_i^{\dot{\alpha}} + \bar{\zeta}_i^{\dot{\alpha}}\]

are generated by \( T \equiv \zeta_i^\alpha Q_i^\alpha + \bar{\zeta}_i^{\dot{\alpha}} \bar{Q}_i^{\dot{\alpha}}, \) where \( \zeta_i^\alpha, \bar{\zeta}_i^{\dot{\alpha}} \) are Grassmann constants. The supercharges

\[Q_i^\alpha = \frac{\partial}{\partial \theta_i^\alpha} - i\bar{\theta}_i^{\dot{\alpha}} \partial_{\alpha\dot{\alpha}}, \quad \bar{Q}_i^{\dot{\alpha}} = -\frac{\partial}{\partial \bar{\theta}_i^{\dot{\alpha}}} + i\theta_i^\alpha \partial_{\alpha\dot{\alpha}},\]

(6)

satisfy the graded algebra

\[\{Q_i^\alpha, Q_j^{\dot{\alpha}}\} = 2i\delta_j^i \partial_{\alpha\dot{\alpha}}, \quad \{Q_i^\alpha, \bar{Q}_j^{\dot{\alpha}}\} = \{\partial_{\alpha\dot{\alpha}}, Q_i^\alpha\} = \{\partial_{\alpha\dot{\alpha}}, \bar{Q}_j^{\dot{\alpha}}\} = 0.\]

(7)

To construct supersymmetric actions in superspace it is convenient to be acquainted with the differential operators

\[D_\alpha = \frac{\partial}{\partial \theta_1^\alpha} + i\bar{\theta}_1^{\dot{\alpha}} \partial_{\alpha\dot{\alpha}}, \quad \bar{D}_{\dot{\alpha}} = -\frac{\partial}{\partial \bar{\theta}_1^{\dot{\alpha}}} - i\theta_1^\alpha \partial_{\alpha\dot{\alpha}},\]

(7)

that anticommute with the supercharges (6):

\[\{Q_i^\alpha, D_\alpha\} = \{\bar{Q}_i^{\dot{\alpha}}, \bar{D}_{\dot{\alpha}}\} = \{Q_i^\alpha, \bar{D}_{\dot{\alpha}}\} = \{\bar{Q}_i^{\dot{\alpha}}, D_\alpha\} = 0.\]

(8)

An off–shell N=4 SYM formulation is not available which would lead to construction of N=4 superfields living in the N=4 superspace (5) making use of accustomed methods. However, there exists another approach of introducing superfields whose components are constituted by the fields which are not auxiliary, in terms of the constraint equations for the superconnections \( A_{\alpha\dot{\alpha}}, \omega_{\alpha}^i, \) and \( \tilde{\omega}_{\alpha\dot{\alpha}} \) (7). The supercovariant derivatives in N=4 superspace (5),

\[\nabla_i^\alpha = D_i^\alpha + [\omega_i^\alpha,], \quad \nabla_{\alpha\dot{\alpha}} = \partial_{\alpha\dot{\alpha}} + [A_{\alpha\dot{\alpha}},].\]

(8)\(\text{Note that, } (A_{\alpha\dot{\alpha}})^\dagger = -A_{\beta\dot{\alpha}} \text{ but } (\omega_i^\alpha)^\dagger = \tilde{\omega}_{\alpha\dot{\alpha}}\)
should satisfy the constraint equations

\[
\{ \nabla^i_\alpha, \nabla^j_\alpha \} = -2i\delta^j_i \nabla_{\alpha\dot{\alpha}}, \tag{11}
\]

\[
\{ \nabla^i_\alpha, \nabla^j_\beta \} = -2i\epsilon_{\alpha\beta}\Phi_{ij}, \quad \{ \nabla^{\dot{i}}_{\dot{\alpha}}, \nabla^{\dot{j}}_{\dot{\beta}} \} = 2i\epsilon^{\dot{\alpha}\dot{\beta}}\Phi_{ij}, \tag{12}
\]

\[
[\nabla^i_\alpha, \nabla^{\dot{i}}_{\dot{\alpha}}] = \epsilon_{\alpha\beta}\bar{\Lambda}^i_{\beta}, \quad [\nabla^{\dot{i}}_{\dot{\alpha}}, \nabla^{\dot{j}}_{\dot{\beta}}] = -\epsilon_{\alpha\beta}\Lambda^i_{\beta}. \tag{13}
\]

Here the upper–case letters indicate superfields whose first components are proportional to the component fields given by the lower–case letters. Let us also define the operator

\[
\mathcal{D} = \theta^a\nabla^i_a - \bar{\theta}^{a\dot{i}}\bar{\nabla}^{i\dot{a}}. \tag{14}
\]

One can show that Bianchi identities resulting from (11–13) lead to the recursion relations

\[
\mathcal{D}A_{a\dot{a}} = -\theta_{a\dot{a}}\bar{\Lambda}^i_a - \bar{\theta}^{i\dot{a}}\Lambda_a, \tag{15}
\]

\[
\mathcal{D}\Lambda_\alpha = 2i\theta^i_\alpha F_{\alpha\beta} - i(\Phi_{ik}, \Phi^{kj})\theta_{j\alpha} - 2i\bar{\theta}^{i\dot{a}}\nabla_{\alpha\dot{a}}\Phi_{ij}, \tag{16}
\]

\[
\mathcal{D}\Phi_{ij} = \theta^a_i\Lambda_{j\alpha} - \theta^a_j\Lambda_{i\alpha} + \epsilon_{ij\dot{k}}\bar{\theta}^{\dot{k}\dot{a}}\bar{\Lambda}^{i\dot{a}}. \tag{17}
\]

Being superconnections there are some redundant parts in \(\omega, \bar{\omega}\) which should be eliminated, obviously leaving the usual Yang–Mills gauge transformations intact. Adopting the gauge fixing condition

\[
\theta^a_i\omega^i_\alpha + \bar{\theta}^{i\dot{a}}\bar{\omega}^{i\dot{a}} = 0, \tag{18}
\]

that eliminates all the gauge transformations depending on the Grassmann coordinates \(\theta^a_i, \bar{\theta}^{i\dot{a}}\), which is similar to the Wess–Zumino condition, the recursion relations for the spinor superconnections can be derived from (14) and (17) as

\[
(1 + \mathcal{D})\omega^i_\alpha = 2i\bar{\theta}^{i\dot{a}}A_{a\dot{a}} - 2i\Phi_{ij}\theta^j_\alpha, \tag{19}
\]

\[
(1 + \mathcal{D})\bar{\omega}^{i\dot{a}} = 2i\theta^a_iA_{a\dot{a}} + 2i\Phi_{ij}\bar{\theta}^{j\dot{a}}. \tag{20}
\]

Note that after the gauge choice (18) the operator (14) turned to be the counting operator of the anticommuting coordinates \(\theta^a_i\) and \(\bar{\theta}^{i\dot{a}}\):

\[
\mathcal{D} = \theta^a_i\frac{\partial}{\partial \theta^a_i} + \bar{\theta}^{i\dot{a}}\frac{\partial}{\partial \bar{\theta}^{i\dot{a}}}. \]

Therefore, the superfields \(\omega, A, \Phi, \Lambda\) can be found from the recursion relations (15–20) order by order in \(\theta, \bar{\theta}\).

When one replaces the upper–case letters with the lower–case ones in (15–10), \(\mathcal{D}\) can be replaced with \(\delta\) which is the supersymmetry transformation (2)–(3) with the replacements \(\xi \to \theta, \xi \to \bar{\theta}\). If the above superfields are written order by order in \(\theta, \bar{\theta}\), as

\[
A_{a\dot{a}} = s_0A^{(0)}_{a\dot{a}} + s_1A^{(1)}_{a\dot{a}} + \cdots + s_nA^{(n)}_{a\dot{a}}, \tag{21}
\]

\[
\Phi_{ij} = e_0\Phi^{(0)}_{ij} + e_1\Phi^{(1)}_{ij} + \cdots + e_n\Phi^{(n)}_{ij}, \tag{22}
\]

\[
\Lambda_{i\alpha} = z_0\Lambda^{(0)}_{i\alpha} + z_1\Lambda^{(1)}_{i\alpha} + \cdots + z_n\Lambda^{(n)}_{i\alpha}. \tag{23}
\]
where $s_m, e_m, z_m; m = 0, 1 \cdots 16$, are real constants, the unique solution to any desired order can also be found as,

$$A^{(m)}_{\alpha \dot{\alpha}} = \delta A^{(m-1)}_{\alpha \dot{\alpha}}, \quad \Phi_{ij}^{(m)} = \delta \Phi_{ij}^{(m-1)}, \quad \Lambda^{(m)}_{\alpha} = \delta \Lambda^{(m-1)}_{\alpha} \quad (24)$$

by setting $s_0 = e_0 = z_0 = 1$ and

$$s_m = e_m, \quad m z_m = s_{m-1}, \quad m s_m = z_{m-1}; \quad m = 1, \cdots, 16. \quad (25)$$

Hence, to obtain the superfields $A, \Lambda, \Phi$ one can proceed in two equivalent ways: Make use of the recursion relations (15)–(16) directly or perform the transformations (24).

In terms of the arbitrary scale factors $l$ and $b$ which are real constants, let us define the zeroth order components as

$$A^{(0)}_{\alpha \dot{\alpha}} = a_{\alpha \dot{\alpha}}, \quad \Lambda^{(0)}_{\alpha} = l \lambda_{\alpha}, \quad \Phi^{(0)}_{ij} = b \phi_{ij}. \quad (26)$$

The first order components of the superfields $A, \Phi, \Lambda$ can be derived from these as

$$A^{(1)}_{\alpha \dot{\alpha}} = -l \theta^{\alpha} \bar{\chi}^{\dot{i}}_{\dot{\alpha}} - l \bar{\theta}^{\dot{i}}_{\dot{\alpha}} \chi_{\alpha}, \quad (27)$$

$$\Lambda^{(1)}_{\alpha} = 2i \theta^{\beta} f^{\alpha \beta \dot{\gamma}} \theta^{\dot{\gamma}} \phi_{j\alpha} - 2ib \bar{\theta}^{\dot{i} \alpha} D \phi_{ij}, \quad (28)$$

$$\Phi^{(1)}_{ij} = l \theta^{\alpha} \lambda_{j\alpha} - l \theta^{\dot{\alpha} \dot{\beta}} \chi_{\alpha} + l \epsilon_{ijkl} \bar{\theta}^{\dot{i} \alpha} \bar{\chi}^{\dot{\gamma} \alpha}. \quad (29)$$

On the other hand, the spinor superconnection $\omega$ can be separated into two parts:

$$\omega^{\alpha \dot{\alpha}} = v^{\alpha \dot{\alpha}} + u^{\alpha \dot{\alpha}},$$

such that the gauge condition (18) takes the form

$$\theta^{\alpha} v_{\alpha}^{\dot{i}} + \bar{\theta}^{\dot{i} \alpha} u_{\dot{i} \alpha} = 0, \quad \theta^{\alpha} u_{\alpha}^{\dot{i}} + \bar{\theta}^{\dot{i} \alpha} v_{\dot{i} \alpha} = 0. \quad (30)$$

There are no zeroth order components, their first and the second order components can be calculated from the recursion relations (19)–(20) and (26)–(28) as

$$v_{\alpha}^{(1)i} = i \bar{\theta}^{\dot{i} \alpha} a_{\alpha \dot{\alpha}}, \quad v_{\alpha}^{(2)i} = \frac{2il}{3} \bar{\theta}^{\dot{i} \alpha} (\theta_{k\alpha} \bar{\chi}^{k}_{\dot{\alpha}} + \bar{\theta}^{k}_{\dot{\alpha}} \chi_{k\alpha}), \quad (29)$$

$$u_{\alpha}^{(1)i} = -ib \theta_{j\alpha} \bar{\phi}^{\dot{j}}_{\dot{\alpha}}, \quad u_{\alpha}^{(2)i} = \frac{2il}{3} \theta_{j\alpha} (\bar{\theta}^{\dot{i} \alpha} \bar{\chi}^{j}_{\dot{\alpha}} - \bar{\theta}^{j \alpha} \chi^{\dot{i} \alpha}). \quad (30)$$

Here we presented the first two components of the superfields. The higher order terms are listed in Appendix.

### 4 N=4 SYM action in terms of superfields

We wish to find an action in terms of the superfields $\omega, A, \Lambda, \Phi$ and the derivatives $\partial_{\alpha \dot{\alpha}}$, such that after performing integrals over the Grassmann variables $\theta, \bar{\theta}$
it attains the action in terms of component fields (1). Inspecting components of the superfields $\omega, A, \Lambda, \Phi$ one observes that if we do not restrict the integration over $\theta, \bar{\theta}$ but integrate over the whole superspace, the desired result cannot be achieved.

We propose the action, in terms of the constant parameters $k_1, \ldots, k_6$,

$$S = ik_1 < \tilde{\omega}_{i\alpha} \tilde{\partial}^{i\alpha} \omega_\alpha > + ik_2 < \tilde{\omega}_{i\alpha} [A^{i\alpha}, \omega_\alpha] > + ik_3 < \omega^{i\alpha} A_{i\alpha} - \tilde{\omega}_{i\alpha} \Lambda^{i\alpha} > + ik_4 < A_{i\alpha\bar{\alpha}} A^{i\alpha\bar{\alpha}} > + ik_5 < A_{ij} \{ \omega^{i\alpha}, \omega_j^{\bar{\alpha}} \} + \Phi^{ij} \{ \tilde{\omega}_{i\alpha}, \tilde{\omega}_j^{\bar{\alpha}} \} > + k_6 < \Phi^{ij} \Phi^{ij} >,$$

(31)

where we defined, by the normalization constant $\mathcal{N} = 1/3200$,

$$< O > \equiv \mathcal{N} \left( \int d^4 x \ d\theta_\alpha^i \ d\bar{\theta}^\alpha_i \ d\tilde{\theta}_\alpha^i \ d\tilde{\bar{\theta}}^\alpha_i \ Tr O \right)_{\theta=\bar{\theta}=0}.$$  

(32)

Thus, the only non–vanishing $\theta, \bar{\theta}$ contribution to the integral is

$$< \theta^a \tilde{\theta}^\alpha \tilde{\theta}^{k\alpha} \tilde{\theta}^{\bar{\beta}\bar{\alpha}} K(x) >= \left( \frac{\mathcal{N}}{8} e^{a\beta} e^{\bar{a} \bar{\beta}} (\delta^a_i \delta^\beta_j + \delta^\beta_i \delta^a_j) \right) \int d^4 x \ Tr K(x),$$

(33)

for any function $K(x)$. With this choice of measure (32), due to mass dimensions and R-charges of the superfields (Table 1), (31) is the most general action one can write up to total derivatives.

| $A_{i\bar{\alpha}}$ | $\lambda_i$ | $\Phi_{ij}$ | $\omega^i$ | $\theta_i$ |
|-------------------|-------------|------------|-----------|-----------|
| $d$               | 1           | 3/2        | 1         | 1/2       |
| $R$               | 0           | -1         | -2        | 1         | -1        |

Table 1: Dimensions $d$, and R-weights.

Because of the choice of measure (32), which is manifestly $SU(4)$ invariant, components of the superfields at most up to the fourth order in $\theta, \tilde{\theta}$ are required. Carrying out integrals over the variables $\theta, \tilde{\theta}$ in (31) is a very lengthy calculation although it is straightforward. Nevertheless, using the identity

$$[\phi_{ij}, \phi^{kl}] [\phi_{kl}, \phi^{ij}] = \frac{1}{2} [\phi^{ij}, \phi^{kl}] [\phi_{ij}, \phi_{kl}],$$

(34)

and performing the integrals over $\theta, \tilde{\theta}$, we conclude that to get the action (1) from (31) the coefficients $k_1, \ldots, k_6$, should satisfy the equations

$$12k_2 - 3k_1 - 4k_3 + 2k_4 = 0,$$

(35)

$$-3k_1 + 10k_3 - 8k_4 = \frac{3}{20\mathcal{N}},$$

(36)

$$k_2 - 2k_5 = 0,$$

(37)

$$k_4 - 2k_6 = -\frac{1}{10\mathcal{N}^2},$$

(38)
\[ 3k_1 - 10k_3 + 16k_6 = \frac{3}{10Nb^2}, \quad (39) \]
\[ -k_2 + k_3 - k_4 + 2k_5 + 2k_6 = -\frac{3}{20Nbl^2}, \quad (40) \]
\[ -3k_2 - 7k_3 - 3k_4 + 18k_5 + 16k_6 = \frac{3}{16Nbl^2}. \quad (41) \]

Although these equations possess some different solutions, by fixing
\[ k_6 = -\frac{3k_4}{2} \quad (42) \]
one obtains the solution
\[ k_1 = -8(104 + b(282 + b(16 + 63b))), \quad (43) \]
\[ k_2 = \frac{1}{10}(-2815 - 4b(1974 + b(115 + 441b))), \quad (44) \]
\[ k_3 = -\frac{12}{5}(105 + b(282 + b(20 + 63b))), \quad (45) \]
\[ k_4 = -3(21 + 4b^2), \quad (46) \]
\[ k_5 = \frac{k_2}{2}, \quad (47) \]

with the scale factors
\[ b = \sqrt{-2 + \sqrt{26}/2}, \quad l = 4\sqrt{(5 - 4b^2)/39}, \quad (48) \]
whose signs can be taken diversely, i.e. \( b \to \pm b, \ l \to \pm l \) are also solutions.

**5 A formalism by integral forms**

To acquire an understanding of underlying geometrical aspects of the formulation given in the previous section, we would like to write superfields as integral forms \[12, 13\]. Let us introduce the differentials \( d\theta, d\bar{\theta} \) whose (wedge) products are commutative\[7\]:
\[
\begin{align*}
    d\theta_i^\alpha \wedge d\theta_j^\beta &= d\theta_j^\beta \wedge d\theta_i^\alpha, \\
    d\bar{\theta}^{i\dot{\alpha}} \wedge d\bar{\theta}^{j\dot{\beta}} &= d\bar{\theta}^{j\dot{\beta}} \wedge d\bar{\theta}^{i\dot{\alpha}}, \\
    d\theta_i^\alpha \wedge d\bar{\theta}^{j\dot{\alpha}} &= d\bar{\theta}^{j\dot{\alpha}} \wedge d\theta_i^\alpha.
\end{align*}
\]

Obviously, to each superfield one can associate an integral form and write the action \[51\] in terms of these forms. This would not give a new insight. However, we can collect differential forms possessing different degrees in a “master” superfield as

\[7\]Here, we write the wedge product symbol \( \wedge \) explicitly to avoid the notational confusion.
\[ \Omega = c_1(u^i + v^i) \partial \theta^a_i + ic_2(2A_{\alpha \alpha} \partial \theta^a_i \wedge \partial \bar{\theta}^{\alpha a} + 3\Phi^{ij} \epsilon_{\alpha \beta} \partial \theta^a_i \wedge \partial \bar{\theta}^{\beta a}) \\
-2ic_3(\Lambda_{\alpha} \epsilon_{\alpha \beta} \partial \theta^a_i \wedge \partial \bar{\theta}^{\alpha a} \wedge \partial \bar{\theta}^{\beta b} + 4\Lambda_{\alpha} \epsilon_{\alpha \beta} \partial \bar{\theta}^{\alpha a} \wedge \partial \theta^b_i \wedge \partial \bar{\theta}^{\beta i}) \\
+2c_4(2F_{\alpha \beta \epsilon \alpha \beta} \epsilon_{\alpha \beta} \partial \theta^a_i \wedge \partial \bar{\theta}^{\beta i} \wedge \partial \bar{\theta}^{\alpha i} \wedge \partial \bar{\theta}^{\beta i} + 8F_{\alpha \beta} \epsilon_{\alpha \beta} \partial \theta^a_i \wedge \partial \bar{\theta}^{\alpha i} \wedge \partial \bar{\theta}^{\beta i} \wedge \partial \theta^b_i \wedge \partial \bar{\theta}^{\beta b} \\
+3[\Phi_{ik}, \Phi_{kj}] \epsilon_{\alpha \beta} \epsilon_{\alpha \beta} \partial \theta^a_i \wedge \partial \theta^b_i \wedge \partial \bar{\theta}^{\alpha a} \wedge \partial \bar{\theta}^{\beta b}). \] (49)

Construction of this master superfield is twofold: Firstly, each component is chosen to possess mass dimension equal to its form degree, e.g., the first component is a one form and it has mass dimension one. Secondly, once the first order components are chosen as one form, the second order components are related to the first ones up to the constant \(c_2\), by the recursion relations (19) replacing in the right hand side explicit \( \theta, \bar{\theta} \) with the differentials \( \partial \theta, \partial \bar{\theta} \). The third order ones are obtained from the second order components utilizing the recursion relations (15), (17), up to the constant \(c_3\). Similarly, the fourth order terms are derived from the recursion relation (16) of the third order components, up to the constant \(c_4\).

To write an action we also need the hermitian conjugate of \(\Omega\):

\[ \Omega^\dagger = -c_1(\bar{u}^i + \bar{v}^i) \partial \bar{\theta}^{\alpha a} + ic_2(2A_{\alpha \alpha} \partial \bar{\theta}^{\alpha a} \wedge \partial \bar{\theta}^{\beta a} - 3\Phi^{ij} \epsilon_{\alpha \beta} \partial \bar{\theta}^{\alpha a} \wedge \partial \bar{\theta}^{\beta a}) \] (50)

\[ -2ic_3(4\Lambda_{\alpha} \epsilon_{\alpha \beta} \partial \bar{\theta}^{\alpha a} \wedge \partial \bar{\theta}^{\beta a} \wedge \partial \bar{\theta}^{\alpha a} \wedge \partial \bar{\theta}^{\beta a} + 4\Lambda_{\alpha} \epsilon_{\alpha \beta} \partial \bar{\theta}^{\alpha a} \wedge \partial \bar{\theta}^{\beta a} \wedge \partial \bar{\theta}^{\beta a} \wedge \partial \bar{\theta}^{\beta a} \\
+2c_4(8F_{\alpha \beta \epsilon \alpha \beta} \epsilon_{\alpha \beta} \partial \bar{\theta}^{\alpha a} \wedge \partial \bar{\theta}^{\beta a} \wedge \partial \bar{\theta}^{\alpha a} \wedge \partial \bar{\theta}^{\beta a} + 2F_{\alpha \beta} \epsilon_{\alpha \beta} \partial \bar{\theta}^{\alpha a} \wedge \partial \bar{\theta}^{\beta a} \wedge \partial \bar{\theta}^{\alpha a} \wedge \partial \bar{\theta}^{\beta a} \\
+3[\Phi_{ik}, \Phi_{kj}] \epsilon_{\alpha \beta} \epsilon_{\alpha \beta} \partial \bar{\theta}^{\alpha a} \wedge \partial \bar{\theta}^{\beta a} \wedge \partial \bar{\theta}^{\alpha a} \wedge \partial \bar{\theta}^{\beta a}). \] (51)

Let us introduce the operator

\[ d = i\partial_{\alpha a} \partial \theta^a_i \wedge \partial \bar{\theta}^{\alpha a} \]

which corresponds to derivatives \( \partial/\partial x^\mu \). In terms of the constants \(m_1, m_2, m_3\), we propose the action, suppressing superspace integrals and trace over the gauge group,

\[ I = m_1 \Omega^\dagger \wedge d \wedge \Omega + m_2 \Omega \wedge \Omega + m_3 \left( \Omega \wedge \Omega^\dagger \wedge \Omega + \Omega^\dagger \wedge \Omega \wedge \Omega^\dagger \right) \] (52)

and the \(SU(4)\) invariant 4-form

\[ d\theta^a_i \wedge \partial \theta^a_j \wedge \partial \bar{\theta}^{\alpha a} \wedge \partial \bar{\theta}^{\beta b} = \epsilon^{\alpha \beta} \epsilon^{\alpha \beta} (\delta^a_i \delta^b_j - \delta^b_j \delta^a_i) d\theta^a_i d\theta^b_j d\theta^a_i d\theta^b_j. \] (53)

All other powers of the superfields \(\Omega, \Omega^\dagger\) give vanishing contributions due to the choice of the measure (52) - (53). Comparing the coefficients of (52) with (31) one can show that they are related as

\[ k_1 = 3m_1 c_1^2, \quad k_2 = -12m_3 c_1^2 c_2, \quad k_3 = -48m_2 c_1 c_3, \]
\[ k_4 = -48m_2 c_2^2, \quad k_5 = -6m_3 c_2^2 c_2, \quad k_6 = 72m_2 c_2^2. \] (54)
$c_4$ does not play any role. Note that in this case the condition (42), namely

$$k_0 = -\frac{3k_4}{2}$$

is dictated spontaneously. Replacing $k_1, \ldots, k_6$ in (31) with the values given in (54), one obtains the equations which $c_1, \ldots, m_3$ coefficients should satisfy, so that (52) reproduces the action (1). There exist several solutions to these equations. By setting $c_1 = c_2 = 1$ one can show that there exists a solution such that

\begin{align*}
    c_3 &= 4 + \frac{12}{5} b (4 + b^2), \\
    m_1 &= -\frac{8}{3} (104 + b (282 + b (16 + 63 b))), \\
    m_2 &= \frac{1}{16} (21 + 4 b^2), \\
    m_3 &= \frac{1}{120} (2815 + 4 b (1974 + b (115 + 441 b)))
\end{align*}

(55) (56) (57) (58)

where $b$ and $l$ are given with (48) as before.

6 Discussions

We presented a superfield formulation of N=4 SYM theory in 4 dimensions. Superfields which we deal with do not possess auxiliary fields, in contrast to the standard superfields which one engages to formulate off-shell supersymmetric theories. Thus, techniques to carry out calculations like taking variations or performing path integrals of their functionals with respect to these superfields are obscure at the moment. Hence, we also do not know how to imply supersymmetry invariance of the action (31) at the level of superfields. In spite of all these facts, being able to introduce integral forms to write the action (31) in terms of the master superfield $\Omega$ (52) is very promising. Getting a better knowledge of the geometrical aspects of the master field (49) can shed some light on the use of our formalism. One of the tools to deepen the understanding how to operate with these superfields is to study the analogous formulations of the N=1 SYM in 4 and 10 dimensions. Although, the former is available the latter case is still missing.

Possessing a superfield formulation of N=4 SYM, even though without auxiliary fields, should also be helpful to study deformations of it in terms of Moyal brackets: In spite of the fact that deformed equations of motion of N=4 SYM were worked out [10] an underlying action is still missing.

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A  Higher components of superfields

Here we list components of the superfields which are not given in Section 3. Because of the choice of measure (32)–(33) some of the terms of components evidently give vanishing contribution to the action (51). Below, “…” indicates these terms which are not needed for our calculation.

From (26) by making use of the recursion relations (15)–(16) or performing the supersymmetry transformations (24), the components of the superfield $A_{\alpha\dot{\alpha}}$ can be obtained as

$$A^{(2)}_{\alpha\dot{\alpha}} = -i \left( \theta_{\alpha a} \bar{\theta}^{\dot{\beta}} f_{a\dot{\beta}} + \bar{\theta}^{\dot{\alpha}} \theta_i f_{\alpha\beta} - b^2 \theta_{\alpha a} \theta^i [\phi^k, \phi_{kj}] \right) + \ldots, \quad (59)$$

$$A^{(3)}_{\alpha\dot{\alpha}} = -i \frac{\bar{\theta}^{\dot{\alpha}}}{6} \theta_i \left( \theta_{k\beta} D_{\alpha\beta} \lambda^{k\dot{\alpha}} + \theta_{k\alpha} D_{\beta\alpha} \lambda^{k\dot{\beta}} + \bar{\theta}^{\dot{\beta}} D_{\alpha\beta} \lambda_{\dot{\alpha} \dot{\beta}} + \bar{\theta}^{\dot{\alpha}} D_{\beta\alpha} \lambda_{\dot{\beta} \dot{\alpha}} \right) + \frac{\bar{\theta}^{\dot{\alpha}}}{6} \theta_i \left( \bar{\theta}^i D_{\beta\alpha} \lambda^{\dot{\beta}} + \bar{\theta}^i D_{\alpha\beta} \lambda^{\dot{\alpha}} \right) + \theta_{\alpha a} \lambda_{\dot{\alpha} \dot{\beta}} \left( \theta_{kj} \lambda^{k\dot{\alpha}} + \theta_{kj} \lambda^{k\dot{\beta}} \right) + \theta_{\alpha a} \lambda^{k\dot{\alpha}} \left( \theta_{kj} \lambda_{\dot{\beta} \dot{\alpha}} + \theta_{kj} \lambda_{\dot{\beta} \dot{\alpha}} \right) + \ldots. \quad (60)$$

$$A^{(4)}_{\alpha\dot{\alpha}} = -i \frac{\bar{\theta}^{\dot{\alpha}}}{24} \theta_i \left( \theta_{k\beta} \left( \bar{\theta}^i \lambda^{k\dot{\alpha}} + \theta^i \lambda^{k\dot{\beta}} \right) + \theta_{k\alpha} \lambda_{\dot{\alpha} \dot{\beta}} \right) \lambda_{\dot{\beta} \dot{\alpha}} + \frac{\theta_i}{2} \left( -2 \theta_{\alpha a} \lambda_{\dot{\alpha} \dot{\beta}} \lambda_{\dot{\beta} \dot{\alpha}} \right) + \theta_{\alpha a} \lambda_{\dot{\alpha} \dot{\beta}} \left( -2 \theta_{\alpha a} \lambda_{\dot{\beta} \dot{\alpha}} \right) + \theta_{\alpha a} \lambda^{k\dot{\alpha}} \left( -2 \theta_{\alpha a} \lambda_{\dot{\beta} \dot{\alpha}} \right) + \ldots. \quad (61)$$

To derive components of the superfield $A_{\alpha a}$ one departs from (27) and uses the recursion relations (15)–(16) or performs the supersymmetry transformations (24):

$$\Lambda^{(2)}_{\alpha a} = i \frac{\theta_i}{2} \theta^i \left( \theta_{k\beta} D_{\alpha\beta} \lambda^{k\dot{\alpha}} + \theta_{k\alpha} D_{\beta\alpha} \lambda^{k\dot{\beta}} + 3 \theta^k \theta_a \lambda_{\dot{\alpha} \dot{\beta}} \lambda_{\dot{\beta} \dot{\alpha}} \right) + \bar{\theta}^{\dot{\alpha}} \theta_i \left( \theta_{k\beta} D_{\alpha\beta} \lambda_{\dot{\alpha} \dot{\beta}} + \theta_{k\alpha} D_{\beta\alpha} \lambda_{\dot{\beta} \dot{\alpha}} \right) \lambda_{\dot{\beta} \dot{\alpha}} + \theta_{\alpha a} \lambda_{\dot{\alpha} \dot{\beta}} \left( \theta_{kj} \lambda^{k\dot{\alpha}} + \theta_{kj} \lambda^{k\dot{\beta}} \right) + \theta_{\alpha a} \lambda^{k\dot{\alpha}} \left( \theta_{kj} \lambda_{\dot{\beta} \dot{\alpha}} + \theta_{kj} \lambda_{\dot{\beta} \dot{\alpha}} \right) + \ldots. \quad (62)$$
$$+2ib^2\theta_{\alpha}\theta_k^\alpha(\epsilon_{iklm}\bar{\theta}_k^\alpha[D_{\gamma\alpha}\phi^{mn},\phi^{kj}] + \bar{\theta}_l^\alpha[D_{\gamma\alpha}\phi^{nk},\phi_{ik}])$$

$$-2ib\bar{\theta}^\beta\theta_{\alpha}\theta_k^\beta(\theta_{m\beta}D_{\alpha\alpha}[\phi_{ik},\phi^{km}] + 2\bar{\theta}^{k\beta}D_{\alpha\alpha}D_{\beta\beta}\phi_{ik})$$

$$+4ib\bar{\theta}^\beta\theta_{\alpha}\theta_k^\beta[f_{\alpha\beta},\phi_{ij}]$$

$$+ib^3\theta_{\alpha}\bar{\theta}_k^\alpha(\epsilon_{iklm}\bar{\theta}_k^\alpha[\phi^{np},\phi_{pm}],\phi^{kj}) + \bar{\theta}_l^\alpha[\phi^{nj},\phi_{nm},\phi_{ik}]$$

$$+ib^3\bar{\theta}_j^\alpha\bar{\theta}_k^\alpha(\theta_{mn}[\phi^{nk},\phi_{nm}],\phi_{ij}) + 4\theta_{ka}[\phi^{km},\phi_{nm},\phi_{ij}], + \cdots. \quad (63)$$

Similarly components of the superfield \(\Phi^{ij}\) are calculated from (28) in terms of the recursion relations (15–16) or the supersymmetry transformations (24) as

\[
\Phi^{(2)ij} = -\frac{i}{2} \left( 2b\bar{\theta}_j^\alpha D_{\alpha\alpha}\phi^{jk} - be^{ijkl}\bar{\theta}_k^\alpha\theta_{\alpha}\phi_{lm} - b^2\bar{\theta}_j^\alpha\theta_{\alpha}[\phi^{kj},\phi_{km}] \right) + \frac{i}{2} \left( i \leftrightarrow j \right) + \cdots \quad (64)
\]

\[
\Phi^{(3)ij} = \frac{i}{6} \left( 2\bar{\theta}_j^\alpha\bar{\theta}_k^\alpha D_{\alpha\alpha}\lambda^j_\lambda - \lambda e^{ijkl}\phi^{\gamma\alpha}\theta_{\alpha}\phi_{lm}\lambda_{\beta} + be^{ijkl}\bar{\theta}_k^\alpha\theta_{\alpha}\phi^{\gamma\alpha}[\lambda_{\alpha},\phi_{km}] \right) + \frac{ib^3\bar{\theta}_j^\alpha\theta_{\alpha}\theta_k^\beta[\phi_{ij},\phi_{km}] + ib^2\bar{\theta}_j^\alpha\theta_{\alpha}[\phi^{nj},\phi_{nm},\phi_{ij}], + \cdots. \quad (65)\right.
\]

\[
\Phi^{(4)ij} = \frac{1}{24} \left( e^{ijkl}\phi^{\gamma\alpha}\theta_{\alpha}\phi_{lm}[\phi_{nr},\phi^{pr}],\phi_{km}] + 2ib^2\theta_{\alpha}[\phi_{mj},\phi_{nr}],\phi_{km}] \right) - 4ib\bar{\theta}^\beta\bar{\theta}_j^\alpha\theta_{\alpha}[\phi_{jk},\phi_{mn}] - \frac{i}{6} \left( i \leftrightarrow j \right) + \cdots. \quad (66)\right.
\]

To find the third and fourth order components in \(\theta, \bar{\theta}\) of the spinor superconnections \(\omega^{\alpha}_{\beta} \equiv \psi^\alpha_{\beta} + u_{\alpha}^{\beta}\) one takes (29–30) and operates with the recursion relations (15–16):

\[
\psi^{(3)\alpha}_{\beta} = \frac{1}{2} \bar{\theta}_j^\alpha \theta_k^\beta f_{\alpha\beta} + b^2\bar{\theta}_j^\alpha\theta_{\alpha}[\phi^{jk},\phi_{km}], + \cdots. \quad (67)
\]

\[
u^{(3)\alpha}_{\beta} = -\frac{b^2}{4} \theta_{\alpha}\bar{\theta}_k^\alpha[\phi^{jk},\phi_{km}] - \bar{\theta}_j^\alpha[\phi^{jk},\phi_{km}] - 2b\bar{\theta}^\beta\theta_{\alpha}\theta_{\beta}^\beta\phi^{jk} + \cdots. \quad (68)
\]

\[
u^{(4)\alpha}_{\beta} = -\frac{1}{15} b\bar{\theta}^\beta\theta_{\alpha}\phi^{jk}[\lambda_{\alpha},\lambda_{\beta}] + \bar{\theta}_j^\alpha\theta_{\alpha}[\phi^{nk},\phi_{nm},\phi_{ij}], + \cdots. \quad (69)\]

12
\[ u^{(4)\alpha}_i = -\frac{l}{15} \theta_{i\alpha} \bar{\theta}_m \left( b \theta^{i\alpha} [\phi^{jk}, \lambda_{k\beta}] - \theta^\beta_m (b \bar{\theta}^{j\dot{\alpha}} [\phi^{ik}, \lambda_{k\beta}] - 2 \bar{\theta}^{j\dot{\alpha}} D_{\beta} \gamma^{\dot{\alpha}} + 2 \bar{\theta}^{j\dot{\alpha}} D_{\beta} \gamma^{\dot{\alpha}}) \right. \\
-3b \theta^k \bar{\theta}^{i\dot{\alpha}} [\phi^{ik}, \lambda_{n\beta}] - b e^{iklm} \theta^l \bar{\theta}^{j\dot{\alpha}} [\lambda_{n\beta}, \phi_{km}] \left) + \cdots. \right(70) \]

The higher components in \( \theta, \bar{\theta} \) which are not listed here do not play any role in our calculations.

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