Proca stars with nonminimal coupling to the Einstein tensor

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We investigate Proca star (PS) solutions, namely boson star (BS) type solutions for a complex vector field with mass and nonminimal coupling to the Einstein tensor. Irrespective of the existence of nonminimal coupling, PS solutions are mini-BS type, but the inclusion of it changes properties. For positive nonminimal coupling parameters, PS solutions do not exist for central amplitudes above some certain value due to the singular behavior of the evolution equations. For negative nonminimal coupling parameters, there is no such singular behavior but sufficiently enhanced numerical resolutions are requested for larger amplitudes. Irrespective of the sign of the nonzero nonminimal coupling parameter, PSs with the maximal Arnoult-Deser-Misner mass and Noether charge are gravitationally bound. Properties of PSs are very similar to those of BSs in scalar-tensor theories including healthy higher-derivative terms.

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I. INTRODUCTION

Motivated by the observed acceleration of the Universe, many modified gravity (MG) models have been proposed. These models have been tested in the various contexts of astrophysics and cosmology \cite{1, 2}. It is known that many MG models can be expressed in terms of the subclasses of the Horndeski theories, namely the most general scalar-tensor theories with the second-order equations of motion (EOMs) \cite{3–6}. More recently, the extension of the Horndeski theories to the vector-tensor theories has been explored in Refs. \cite{7–12}, which are called the generalized Proca theories, as they also correspond to nonlinear extensions of the massive vector field theory where the $U(1)$ gauge symmetry is explicitly broken.

Properties of compact objects will be very important to distinguish MG models in light of future gravitational wave (GW) observations. The simplest compact objects are black holes (BHs), which in the context of generalized Proca theories have been studied in Refs. \cite{14–19}. These solutions have exhibited nontrivial stealth features for the nontrivial Proca and electric charges. Neutron star (NS) solutions in a subclass of generalized Proca theories with nonminimal coupling to the Einstein tensor were studied in Ref. \cite{17}. In this paper, as a candidate of more exotic compact objects, we investigate boson star (BS) type solutions in the generalized complex Proca theory with mass and nonminimal coupling to the Einstein tensor, namely Proca stars (PSs).\textsuperscript{1}

BSs are gravitationally bound nontopological solutions constituted by bosonic particles. BS solutions have been firstly constructed for a complex scalar field with mass $\mu^2/|\phi|^2$ \cite{20–23}. Mass and radius of BSs are typically $M_B^2/\mu \sim 1/\mu$, respectively, where $M_B = \sqrt{\hbar c/G} \times 10^{19}\text{GeV}/c^2$ is the Planck mass.\textsuperscript{2} Thus, for $\mu \sim 1\text{GeV}$ close to that of protons and neutrons, the mass of BSs becomes $10^{13}\text{g}$, which is much smaller than the Chandrasekhar mass for their fermionic counterparts $M_{\text{Ch}} \sim M_B^3/\mu^2 \gtrsim M_\odot$, where $M_\odot = 1.99 \times 10^{33}\text{g}$ is the Solar mass. On the other hand, for $\mu \sim 10^{-10}\text{eV}$, mass and radius of BSs become $\sim M_\odot$ and $\sim 10\text{km}$, respectively, which may be targets for future GW observations as important as BHs and NSs. See Ref. \cite{24} and references therein for BS solutions for other scalar potentials.

BSs are characterized by several conserved charges. The first is the Arnoult-Deser-Misner (ADM) mass, which corresponds to the gravitational mass. The second is the Noether charge $Q$ associated with the global internal symmetry in the scalar field sector, where $\mu Q$ measures the rest mass energy of bosonic particles. Thus, the binding energy of a BS is given by $B := M - \mu Q$. For $B < 0$ a BS is said to be gravitationally bound, while for $B > 0$ the excess energy would be translated into kinetic energy of individual bosons. Radial perturbations of BS solutions have been studied in Refs. \cite{23, 25–27}. $Q$ and $M$ are generically the functions of the amplitude of the scalar field at the center $\phi_0$, i.e., $Q = Q(\phi_0)$ and $M = M(\phi_0)$. It has been shown that the critical solution dividing stable BSs from unstable ones satisfies $dQ/d\phi_0(\phi_{0,c}) = dM/d\phi_0(\phi_{0,c}) = 0$, where $\phi_{0,c}$ describes the critical central amplitude and solutions for $\phi_0 > \phi_{0,c}$ are unstable \cite{27, 28}. Because $B(\phi_{0,c}) = M(\phi_{0,c}) - \mu Q(\phi_{0,c}) < 0$, this critical solution possesses negative binding energy. In other words, $B < 0$ does not necessarily correspond to perturbative stability of BSs. Nevertheless, because stable BS solutions satisfy $B < 0$, we may use it as a necessary condition of stability.

In Ref. \cite{29}, BS solutions have been studied for a complex scalar field with nonminimal derivative coupling to the Einstein tensor $G^{\mu\nu} \partial_\mu \phi \partial_\nu \bar{\phi}$, where $\bar{\phi}$ is the complex conjugate of $\phi$, which is analogous to a subclass of the Horndeski theories for a real

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\textsuperscript{1} In this paper, we will distinguish “boson stars (BSs)” and “Proca stars (PSs)” for condensates of complex scalar and vector fields, respectively.

\textsuperscript{2} In the rest, we will work in the units of $c = \hbar = 1$. 

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will be devoted to giving a brief summary and conclusion.

Recently, BS type solutions for a complex vector field, namely PS solutions, have been studied in Refs. [33–35]. For a massive vector field $\mu^2 A^\mu \tilde{A}_\mu$, where $\tilde{A}_\mu$ is the complex conjugate of $A_\mu$, PSs are mini-BS types with masses $\sim M_p^2/\mu$. The properties of PSs are quite similar to those of BSs, and the critical solution dividing stable and unstable PS solutions corresponds to that with the maximal ADM mass and Noether charge $[33]$. In this paper, we will investigate PS solutions in the presence of nonminimal coupling to the Einstein tensor $G^{\mu \nu} A_\mu \tilde{A}_\nu$, which is analogous to a subclass of the generalized Proca theories studied in the context of BH physics $[14–18]$. We will find properties of PSs which are very similar to those of BSs in the complex scalar-tensor theories with healthy higher-derivative interactions.

The paper is constructed as follows: In Sec. II, we will introduce the generalized complex Proca theory with mass and nonminimal coupling to the Einstein tensor and provide the covariant EOMs. In Sec. III, we will arrange EOMs to a set of the evolution equations in the static and spherically symmetric background to determine the structure of PSs. In Sec. IV, we will numerically construct PS solutions and discuss their properties. Finally, Sec. V will be devoted to giving a brief summary and conclusion.

II. THE GENERALIZED COMPLEX PROCA THEORY WITH MASS AND NONMINIMAL COUPLING TO THE EINSTEIN TENSOR

We consider the generalized complex Proca theory with mass and nonminimal coupling to the Einstein tensor

$$S = \int d^4x \sqrt{-g} \left[ \frac{1}{2\kappa^2} R - \frac{1}{4} F^{\mu \nu} \tilde{F}_{\mu \nu} - \frac{1}{2} (\mu^2 g^{\mu \nu} - \beta G^{\mu \nu}) A_\mu \tilde{A}_\nu \right],$$

where the Greek indices $(\mu, \nu, ...)$ run the four-dimensional spacetime, $g_{\mu \nu}$ is the metric tensor, $g^{\mu \nu} := (g_{\mu \nu})^{-1}$ is the inverse metric tensor, $g = \det(g_{\mu \nu})$ is the determinant of $g_{\mu \nu}$, and $R$ and $G_{\mu \nu}$ are the Ricci scalar curvature and the Einstein tensor associated with $g_{\mu \nu}$. $A_\mu$ is the complex vector field, $F_{\mu \nu} := \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength, quantities with a “bar” are the complex conjugates of those without a “bar”, $\kappa^2 := 8\pi G$, $G$ is the gravitational constant, $\mu^2$ is mass of the complex vector field, and $\beta$ measures nonminimal coupling to the Einstein tensor. In our units, the Planck mass $M_p$ is given by $M_p = 1/\sqrt{G}$.

We obtain EOMs by varying the action Eq. (1) with respect to $g_{\mu \nu}$ and $A_\mu$, respectively. Varying Eq. (1) with respect to $g_{\mu \nu}$, we obtain the gravitational EOM

$$0 = \varepsilon_{\mu \nu} := T^{(f)}_{\mu \nu} + \beta T^{(A)}_{\mu \nu} - \frac{1}{\kappa^2} G_{\mu \nu},$$

where

$$T^{(f)}_{\mu \nu} := \frac{1}{2} F^{\mu \rho} \tilde{F}_{\rho \nu} + \frac{1}{2} F^{\rho \mu} \tilde{F}_{\rho \nu} - \frac{1}{4} g_{\mu \nu} F^{\rho \sigma} \tilde{F}_{\rho \sigma} + \frac{\mu^2}{2} (A_\mu \tilde{A}_\nu + A_\nu \tilde{A}_\mu) - \frac{\mu^2}{2} g_{\mu \nu} A^\rho \tilde{A}_\rho$$

represents the energy-momentum tensor for the ordinary massive Proca field, and

$$T^{(A)}_{\mu \nu} := \frac{1}{2} \left( A^\rho \tilde{A}_\rho G_{\mu \nu} + \frac{1}{2} R A_\mu \tilde{A}_\nu + \frac{1}{2} R A_\nu \tilde{A}_\mu \right)$$

$$- \frac{1}{2} g_{\mu \nu} \left( \nabla^\rho A_\rho \nabla^\sigma \tilde{A}_\sigma - 2 \nabla^\rho A^\sigma \nabla_\rho \tilde{A}_\sigma + \nabla^\rho A^\sigma \nabla_\sigma \tilde{A}_\rho - A_\rho \nabla^\sigma \tilde{A}_\sigma - A_\sigma \nabla_\rho \tilde{A}_\rho + A_\rho \nabla_\sigma \nabla^\sigma \tilde{A}_\rho + \tilde{A}_\rho \nabla_\sigma \nabla^\sigma A_\rho \right)$$

$$- \left( \frac{1}{2} \nabla_\mu A^\rho \nabla_\rho \tilde{A}_\mu + \frac{1}{2} \nabla_\mu \tilde{A}_\mu \nabla_\rho A_\rho - \frac{1}{2} \nabla_\mu A^\rho \nabla_\rho \tilde{A}_\mu - \frac{1}{2} \nabla_\rho \tilde{A}_\rho \nabla_\mu A_\mu \right)$$

$$- \left( \frac{1}{2} \nabla_\rho A_\mu \nabla_\mu \tilde{A}_\rho + \frac{1}{2} \nabla_\rho \tilde{A}_\rho \nabla_\mu A_\mu - \frac{1}{2} \nabla_\mu A^\rho \nabla_\rho \tilde{A}_\mu - \frac{1}{2} A^\rho \nabla_\rho \nabla_\mu \tilde{A}_\mu \right)$$

$$+ \left( \frac{1}{2} A^\rho \nabla_\rho \tilde{A}_\mu + \frac{1}{2} \tilde{A}_\mu \nabla_\rho A^\rho - \frac{1}{2} A^\rho \nabla_\rho \tilde{A}_\mu - \frac{1}{2} \tilde{A}_\mu \nabla_\rho A^\rho \right)$$

$$+ \frac{1}{2} A_\mu \nabla_\rho \tilde{A}_\rho - A_\mu \nabla_\rho \tilde{A}_\rho - \tilde{A}_\mu \nabla_\rho A^\rho + \frac{1}{2} A_\mu \nabla_\rho \tilde{A}_\rho A^\rho + \frac{1}{2} \tilde{A}_\mu \nabla_\rho A^\rho$$

3 By “healthy,” we mean that there are no ghosty degrees of freedom associated with Ostrogradsky’s theorem $[36]$. 
represents the effective energy-momentum tensor for nonminimal coupling to the Einstein tensor. Similarly, varying Eq. (1) with respect to $A_\mu$ and $\bar{A}_\mu$, we obtain the EOM for the complex vector field

$$0 = F_\nu := \nabla^\mu F_{\mu\nu} - \left(\mu^2 \delta_\nu^\mu - \beta G^\nu_\mu\right) A_\mu,$$

and its complex conjugate $\bar{F}_\nu = 0$, respectively. The $U(1)$ gauge symmetry is explicitly broken by mass and nonminimal coupling. By taking the derivative of Eq. (5), we obtain

$$0 = \mathcal{G} := \mu^2 \nabla^\mu A_\mu - \beta G^{\mu\nu} \nabla_\mu A_\nu,$$

which generalizes the constraint relation for $\beta = 0$, $\nabla^\mu A_\mu = 0$. In the theory (1), there is still global $U(1)$ symmetry under $A_\mu \rightarrow e^{i\alpha} A_\mu$, where $\alpha$ is a constant. The associated Noether current is given by

$$j^\mu = \frac{i}{2} \left( \bar{F}^{\mu\nu} A_\nu - F^{\mu\nu} \bar{A}_\nu \right),$$

which satisfies the conservation law $\nabla_\mu j^\mu = 0$.

### III. PROCA STARS

#### A. Static and spherically-symmetric spacetime

In this section, we discuss PSs in the theory (1). We consider a static and spherically symmetric spacetime

$$g_{\mu\nu} dx^\mu dx^\nu = -\sigma(r)^2 \left(1 - \frac{2m(r)}{r}\right) dt^2 + \left(1 - \frac{2m(r)}{r}\right)^{-1} dr^2 + r^2 d\Omega_2^2,$$

where $\hat{t}$ and $r$ are the time and radial coordinates, $d\Omega_2^2$ is the metric of the unit two sphere, and $m(r)$ and $\sigma(r)$ depend only on the radial coordinate $r$. Correspondingly, we consider the ansatz for the vector field [33]

$$A_\mu dx^\mu = e^{-i\omega \hat{t}} \left(a_0(r) d\hat{t} + i a_1(r) dr\right),$$

where $a_0(r)$ and $a_1(r)$ depend only on $r$. We assume that the frequency $\omega$ is real, ensuring that the vector field neither grows nor decays. Equation (9) avoids the restrictions from Derrick’s theorem that forbid stable localized solutions to nonlinear wave equations [37]. The explicit time dependence $e^{-i\omega \hat{t}}$ does not induce the time dependence in the metric. In order to find $m(r)$, $\sigma(r)$, $a_0(r)$, and $a_1(r)$, we solve the evolution equation derived from EOMs (2), (5), and (6).

From our ansatz (8) and (9), the $(\hat{t}, \hat{t})$, $(r, r)$ and angular components of the gravitational EOM (2)

$$\mathcal{E}^{\hat{t}}_{\hat{t}} = 0, \quad \mathcal{E}^{r}_{r} = 0, \quad \mathcal{E}^{i}_{i} = 0,$$

are given by Eq. (A1) with Eq. (A2). Similarly, the $\hat{t}$ and $r$ components of the vector field EOM (5),

$$\mathcal{F}_{\hat{t}} = 0, \quad \mathcal{F}_{r} = 0,$$

are given by Eq. (A3) with Eq. (A4). Finally, Eq. (6) is given by Eq. (A5) with Eq. (A6). These equations are constrained by the identity

$$\nabla_\mu \mathcal{E}^{\mu}_{r} = -\frac{1}{2} g^{\hat{t}\hat{t}} \left( \mathcal{F}_{\hat{t}} \times \mathcal{F}_{\hat{t}} + \bar{\mathcal{F}}_{\hat{t}} \times \mathcal{F}_{\hat{t}} \right) + \frac{1}{2} \left( \mathcal{G}_{\bar{A}_r} + \bar{\mathcal{G}} A_r \right).$$

Combining them, we obtain a set of the evolution equations (B1a), (B1b), (B1c), and (B5) which determine the structure of PSs from the center $r = 0$ to spatial infinity $r = \infty$. 
B. Center

Solving EOMs near the center \( r = 0 \), we obtain the boundary conditions

\[
a_0(r) = f_0 + \frac{f_0}{6} \left( \mu^2 - \frac{\omega^2}{\sigma_0^2} + \frac{f_0^2 \kappa^2}{2 \sigma_0^2} \mu^2 - \frac{2 \omega^2}{\sigma_0^4} \right) \beta + \frac{f_0^2 \kappa^2}{3 \mu^2 \sigma_0^6} \left( 3 f_0^2 \mu^2 \kappa^2 \left( 3 \mu^2 \sigma_0^2 - 14 \omega^2 \right) + 4 \omega^2 \left( 9 \mu^2 \sigma_0^2 - 8 \omega^2 \right) \beta^2 + O (\beta^3) \right) r^2 + O (r^4),
\]

(13a)

\[
a_1(r) = \frac{f_0 \omega}{3 \sigma_0^2} \left( 1 + \frac{f_0^2 \kappa^2}{18 \mu^2 \sigma_0^2} (21 f_0^2 \mu^2 \kappa^2 - 12 \mu^2 \sigma_0^2 + 16 \omega^2) \beta^2 + O (\beta^3) \right) r + O (r^3),
\]

(13b)

\[
m(r) = \frac{f_0^2 \kappa^2}{12 \sigma_0^4} \left( \mu^2 + \frac{3 f_0^2 \mu^2 \kappa^2 + 4 \omega^2}{6 \sigma_0^2} \beta + \frac{3 f_0^2 \mu^2 \kappa^4 + 20 f_0^2 \kappa^2 \omega^2}{12 \sigma_0^4} \beta^2 + O (\beta^3) \right) r^3 + O (r^5),
\]

(13c)

\[
\sigma(r) = \sigma_0 + \frac{f_0 \kappa^2}{\sigma_0} \left( \mu^2 + \frac{3 f_0^2 \mu^2 \kappa^2 - 3 \mu^2 \sigma_0^2 + 4 \omega^2}{18 \sigma_0^2} \beta + \frac{f_0^2 \kappa^2}{48 \sigma_0^4} (5 f_0^2 \mu^2 \kappa^2 - 8 \mu^2 \sigma_0^2 + 20 \omega^2) \beta^2 + O(\beta^3) \right) r^2
\]

+ \( O (r^4) \).

(13d)

With Eq. (13), we numerically integrate Eqs. (B1a), (B1b), (B1c), and (B5) toward the spatial infinity \( r = \infty \). From Eq. (13), for \( f_0 = 0 \) we obtain \( \sigma(r) = \sigma_0 \) and \( m(r) = a_0(r) = a_1(r) = 0 \), namely the Minkowski solution.

C. Spatial infinity

For a correct eigenvalue of \( \dot{\omega} \), we find the asymptotically flat solution where \( m(r) \) and \( \sigma(r) \) exponentially approach constant values, \( m_\infty > 0 \) and \( \sigma_\infty > 0 \), respectively, while \( a_0(r) \) and \( a_1(r) \) exponentially approach zero as \( e^{-\sqrt{\mu^2 - \omega^2/\sigma_\infty^2} r} \). Thus, in the large \( r \) limit the metric exponentially approaches the Schwarzschild form

\[
ds^2 \to -\sigma_\infty^2 \left( 1 - \frac{2 m_\infty}{r} \right) dt^2 + \left( 1 - \frac{2 m_\infty}{r} \right)^{-1} dr^2 + r^2 d\Omega_2^2,
\]

(14)

where the proper time measured at \( r = \infty \) is given by \( t = \sigma_\infty \hat{t} \), and correspondingly the proper frequency \( \omega \) is given by

\[
\omega := \frac{\dot{\omega}}{\sigma_\infty}.
\]

(15)

The exponential fall-off condition \( e^{-\sqrt{\mu^2 - \omega^2} r} \) requires \( \omega < \mu \). We numerically confirmed that in the limit \( f_0 \to 0 \), namely in the limit of the Minkowski solution, \( \omega \to \mu \), leaving no \( r \) dependence in the large \( r \) limit.

D. ADM mass, Noether charge, effective radius, and compactness

Having numerical solutions, we then evaluate the conserved quantities which characterize PSs. The first is associated with the time translational symmetry and corresponds to the ADM mass

\[
M := \frac{m_\infty}{G} = M_p^2 m_\infty.
\]

(16)

The second is the Noether charge associated with the global \( U(1) \) symmetry, which is given by integrating \( j^j \) in Eq. (7) over a constant time hypersurface

\[
Q = \int \Sigma \ d^3 x \sqrt{-g} j^j = 4 \pi \int_0^\infty \ dr \ r^2 a_1(r) \left( \dot{\omega} a_1(r) - \dot{a}_1(r) \right). \]

(17)

As for a BS, a PS is said to be gravitationally bound if

\[
B := M - \mu Q < 0.
\]

(18)

As for BSs mentioned in Sec. I, for PSs we use Eq. (18) as a necessary condition for stability.
We then define the effective radius \( R := \frac{1}{Q} \int_\Sigma d^3 x \sqrt{-g} (r \dot{r})^2 \) and the (effective) compactness

\[
C := \frac{GM}{R} = \frac{M}{R^2} = \frac{m_\infty}{R}.
\]

In contrast to the cases of BHs and NSs, for PSs \( r = R \) is not the surface of \( A_\mu = 0 \). Nevertheless, due to the exponential falloff property of \( A_\mu \), \( R \) represents the characteristic length scale of energy localization. The parameter \( C \) is very important to distinguish various compact objects in different theories in light of future GW observations.

E. Parameters

For numerical analyses, we may fix some parameters to unity. First, by rescaling \( \hat{\omega} \rightarrow \mu \hat{\omega}, \quad r \rightarrow r \mu, \quad m(r) \rightarrow \mu m(r), \quad \sigma(r) \rightarrow \sigma(r), \quad a_0(r) \rightarrow \kappa a_0(r), \quad a_1(r) \rightarrow \kappa a_1(r), \) we can rewrite Eqs. (B1a), (B1b), (B1c), and (B5) into equations without \( \mu \) and \( \kappa \). Hence, we may work by setting \( \mu = \kappa = 1 \). Moreover, as \( \sigma_0 \) corresponds to the time rescaling at the center \( r = 0 \), we may also set \( \sigma_0 = 1 \) for convenience. In general, then \( \sigma_\infty \neq 1 \) and quantities measured at the spatial infinity can be obtained by performing the rescalings discussed in Sec. III.C. Thus, the only remaining parameters are \( f_0 \) and \( \beta \). For a fixed value of \( \beta \), by varying \( f_0 \) we numerically integrate them and evaluate the above quantities. In Fig. 1, by setting \( \mu = \kappa = \sigma_0 = 1 \), \( m(r), \sigma(r), a_0(r), \) and \( a_1(r) \) are shown as functions of \( r \) for \( \beta = 0.2 \) and \( f_0 = 1.0 \), which are very similar to the case of \( \beta = 0 \) [33, 34].

Before proceeding, we briefly comment on the massless case \( \mu = 0 \). In this case, by rescaling

\[
r \rightarrow \hat{\omega} r, \quad m \rightarrow \hat{\omega} m, \quad a_0(r) \rightarrow \kappa a_0(r), \quad a_1(r) \rightarrow \kappa a_1(r),
\]

the dependence on \( \hat{\omega} \) is completely eliminated from EOMs. Thus, the problem does not reduce to the eigenvalue one. As \( \beta \) is dimensionless, for \( \mu = 0 \) there is no physical scale which characterizes PSs. Thus, there is no PS solution only by nonminimal coupling.
IV. NUMERICAL SOLUTIONS

For each of \( \beta = 0.2, 0.1, 0, -0.1, \) and \(-0.2, \) for different values of \( f_0 \) we numerically integrate the equations (B1a), (B1b), (B1c) and (B5) with the boundary conditions (13), and find \( \omega \) that reproduces the asymptotic behaviors discussed in Sec. III C. We then evaluate \( M, Q, R, \) and \( \mathcal{C} \) discussed in Sec. III D.

A. Frequency

In Fig. 2, for \( \beta = 0.2, 0.1, 0, -0.1, \) and \(-0.2, \) \( \omega/\mu \) is shown as a function of \( f_0. \) In all cases, for \( f_0 \to 0, \omega/\mu \to 1 \) which reproduces the Minkowski solution.

For \( \beta = 0, \) there are several branches of PSs for a single value of \( \omega. \) As \( f_0 \) increases from zero, \( \omega/\mu \) decreases from 1 and reaches the minimal value 0.814 for \( f_0 \approx 0.199 M_p. \) Then, \( \omega/\mu \) increases and reaches the local maximal value 0.906 for \( f_0 \approx 2.39 M_p. \) As \( f_0 \) increases further, \( \omega/\mu \) gradually approaches 0.891. For nonzero values of \( \beta, \) irrespective of its sign, numerical PS solutions cease to exist for \( f_0 \) above some certain value. However, the reasons are different for \( \beta > 0 \) and \( \beta < 0. \)

For \( \beta > 0, \) there are several branches of PSs for a single value of \( \omega. \) However, for \( f_0 \) above some certain value depending on \( \beta, \) \( \bar{C}_1 \) in Eq. (B5) always vanishes at a very small radius, and no PS solutions exist. In our examples, no PS solutions are found for \( f_0 \gtrsim 0.349 M_p \) for \( \beta = 0.1 \) and for \( f_0 \gtrsim 0.231 M_p \) for \( \beta = 0.2. \) Before reaching these values of \( f_0, \omega/\mu \) takes the minimal values 0.827 for \( f_0 \approx 0.160 M_p (\beta = 0.1) \) and 0.836 for \( f_0 \approx 0.140 M_p (\beta = 0.2). \) Thus, for larger nonminimal coupling parameters, PS solutions cease to exist for smaller amplitudes.

For \( \beta < 0, \) \( \omega/\mu \) always decreases. Although \( \bar{C}_1 \) in Eq. (B5) does not vanish at any radius, it becomes extremely small for larger values of \( f_0, \) making numerical integrations unstable unless sufficiently high resolutions are taken. Thus, although we will show PS solutions only for smaller values of \( f_0 \) where numerical integration can be performed stably, in principle it does not forbid the existence of PS solutions for larger values of \( f_0. \) In our analysis, numerical integration could not performed stably for \( f_0 \lesssim 0.628 M_p \) for \( \beta = -0.1 \) and for \( f_0 \lesssim 0.409 M_p \) for \( \beta = -0.2. \)

B. ADM mass and Noether charge

In Figs. 3-5, for \( \beta = 0, 0.2, \) and \(-0.2, \) \( M \) and \( \mu Q \) are shown as functions of \( \omega/\mu. \) For all cases, \( f_0 = 0 \) gives the Minkowski solution \( M = Q = 0 \) with \( \omega = \mu. \) Moreover, even in the presence of the nonminimal coupling to the Einstein tensor PS solutions obtained in our analysis are mini-BS type, where \( M \) and \( \mu Q \) are typically \( \sim M_p^2/\mu. \)
The case of $\beta = 0$ is shown in Fig. 3. As $f_0$ increases from zero, $M$ and $\mu Q$ increase from zero while $\omega/\mu$ decreases from 1. $M$ and $Q$ then take their maximal values, $M_{\text{max}} \approx 1.06 M_p^2/\mu$ and $\mu Q_{\text{max}} \approx 1.09 M_p^2/\mu$ for $\omega/\mu \approx 0.870$, which satisfies Eq. (18) and agrees with Ref. [33]. Then $\omega/\mu$ still decreases, while $M$ and $\mu Q$ start to decrease. After $\omega/\mu$ reaches its minimal value 0.814, it starts to increase while $M$ and $\mu Q$ still decrease until reaching their local minimum values, $M_{\text{min}} \approx 0.496 M_p^2/\mu$ and $\mu Q_{\text{min}} \approx 0.412 M_p^2/\mu$, respectively. In this region, Eq. (18) is not satisfied. As $f_0$ increases further, $M$ and $\mu Q$ eventually converge to $M \approx 0.563 M_p^2/\mu$ and $\mu Q \approx 0.486 M_p^2/\mu$, respectively.

The case of $\beta = 0.2$ is shown in Fig. 4. As $f_0$ increases from zero, $M$ and $\mu Q$ increase from zero while $\omega/\mu$ decreases from 1. $M$ and $\mu Q$ then take their maximal values, $M_{\text{max}} \approx 1.05 M_p^2/\mu$ and $\mu Q_{\text{max}} \approx 1.07 M_p^2/\mu$, for $\omega/\mu \approx 0.872$, which satisfies Eq. (18). Then as $\omega/\mu$ still decreases, $M$ and $\mu Q$ start to decrease. After $\omega/\mu$ reaches its minimal value 0.836, $\omega/\mu$ starts to increase while $M$ and $\mu Q$ still decrease. In this region, Eq. (18) is not satisfied. As we discussed in Sec. IV A, for $\beta = 0.2$, we
can obtain PS solutions up to $f_0 = 0.231M_p$, for which $M \approx 0.768M_p^2/\mu$ and $\mu Q \approx 0.744M_p^2/\mu$.

Finally, the case of $\beta = -0.2$ is shown in Fig. 5. As $f_0$ increases from zero, $M$ and $\mu Q$ increase from zero while $\omega/\mu$ decreases from 1. $M$ and $\mu Q$ then take their maximal values, $M_{\text{max}} \approx 1.07M_p^2/\mu$ and $\mu Q_{\text{max}} \approx 1.10M_p^2/\mu$ for $\omega/\mu \approx 0.868$, which satisfies Eq. (18). As $f_0$ increases further, $M$ and $\mu Q$ decrease while $\omega$ still decreases. As we discussed in Sec. IV A, for $\beta = -0.2$, for $f_0$ above $0.409M_p$, we could not find numerical PS solutions due to the problem discussed in Sec. IV A. For $f_0 = 0.409M_p$, $M \approx 0.727M_p^2/\mu$ and $\mu Q \approx 0.667M_p^2/\mu$. For $\omega/\mu \lesssim 0.792$, Eq. (18) is not satisfied.

In Fig. 6, $M$ (the left panel) and $\mu Q$ (the right panel) are shown for all $\beta = 0.2, -0.1, -0.1$, and $-0.2$. $M$ and $\mu Q$ almost coincide for larger $\omega/\mu$, i.e., smaller $f_0$. As $M$ and $\mu Q$ approach their maximal values, $\beta$ dependence becomes more evident, and for negative (positive) values of $\beta$, both $M$ and $\mu Q$ take larger (smaller) values than those for $\beta = 0$ for the same $\omega$. Moreover, for both positive and negative values of $\beta$, spiraling features observed for $\beta = 0$ eventually disappear. Such behaviors are very similar to those observed for BSs in the EGB theory [30, 31], the EdGB theory [32] and the complex scalar-tensor theory with nonminimal derivative coupling to the Einstein tensor [29]. Thus, they may be generic for healthy higher-derivative theories.

FIG. 5. For $\beta = -0.2$, $M$ and $\mu Q$ are shown as functions of $\omega/\mu$. The solid red and blue dashed curves correspond to $M$ and $\mu Q$, respectively. Here, $M$ and $\mu Q$ are shown in $M_p^2/\mu$.

FIG. 6. $M$ (the left panel) and $\mu Q$ (the right panel) are shown as functions of $\omega/\mu$. In each panel, the solid red, thick red, black dotted-dashed, thick blue dashed, and blue dashed curves correspond to $\beta = 0.2, 0.1, 0, -0.1$, and $-0.2$, respectively. Here, $M$ and $\mu Q$ are shown in $M_p^2/\mu$. 
C. Mass-radius relation

In Fig. 7, for $\beta = 0.2, 0.1, 0, -0.1,$ and $-0.2$, $M$ is shown as a function of $\mu R$ defined in Eq. (19). In all cases, for larger $f_0$, $R$ becomes smaller, which means that energy is localized more efficiently around the center. For $\mu R \gtrsim 6.0$, no clear $\beta$ dependence is observed, while for $\mu R \lesssim 6.0$, $\beta$ dependence becomes more evident and a larger (smaller) value of $M$ is observed for $\beta < 0$ ($\beta > 0$). For $\beta = 0$, $M$ is a multivalued function of $R$, while for $\beta \neq 0$, it is a single-valued function of $R$. Very similar behaviors are observed also for $\mu Q$ as a function of $\mu R$. As a reference, we also show the case of $R = 2m_\infty$, although $R$ has nothing to do with the Schwarzschild radius of a BH.

![Graph](image)

FIG. 7. $M$ is shown as a function of $\mu R$. The solid red, thick red, thick black dotted-dashed, thick blue dashed, and blue dashed curves correspond to $\beta = 0.2, 0.1, 0, -0.1,$ and $-0.2$, respectively, and the black dotted line represents $R = 2m_\infty$. Here, $M$ is shown in $M_\odot^2/\mu$.

In Figs. 8-10, for $\beta = 0, 0.2$, and $-0.2$, $M$ and $\mu Q$ are shown as functions of $\mu R$, respectively. For all $\beta = 0, 0.2,$ and $-0.2$, for $\mu R \gtrsim 3.5$, Eq. (18) is satisfied, while for $\mu R \lesssim 3.5$, it is not satisfied.

D. Compactness

In Fig. 11, for $\beta = 0.2, 0,$ and $-0.2$, $C$ as defined in Eq. (20) is shown as the function of $\mu R$. For $\mu R \gtrsim 6.0$, $C$ becomes smaller and no clear $\beta$ dependence is observed, while for $\mu R \lesssim 6.0$, $\beta$ dependence becomes more evident and a larger (smaller) value of $C$ is observed for $\beta < 0$ ($\beta > 0$). For $\beta = 0$, $C$ is a multivalued function of $R$, while for $\beta \neq 0$, it is a single-valued function of $R$.

For $\beta = 0$, as $f_0$ increases, $R$ decreases, and $C$ increases and takes the maximal value 0.280 for $\mu R \approx 2.50$. After reaching the maximal value, $C$ starts to decrease, while $\mu R$ still decreases and reaches the minimal value 2.12. As $f_0$ further increases, $C$ eventually converges to 0.213.

For $\beta = 0.2$, as $f_0$ increases while $\mu R$ decreases, $C$ increases, takes the maximal value 0.243 for $\mu R \approx 3.44$, and then decreases until $\mu R$ reaches the minimal value 2.21.

For $\beta = -0.2$, as $f_0$ increases while $\mu R$ decreases $C$ monotonically increases and takes the maximal value 0.350 when $\mu R$ reaches the minimal value 2.08 due to the technical problem discussed in Sec. IV A. There may be PS solutions for $\mu R \lesssim 2.08$, for which $C$ may be larger values than 0.350. However, the condition (18) is not satisfied for such solutions.

E. Speculation about stability

Although in the given subclass of the generalized complex Proca theory the explicit analysis of stability will be quite involved, in this subsection we will give the speculation about stability in terms of the insensitivity of the critical central amplitude of the
FIG. 8. $M$ and $\mu Q$ are shown as functions of $\mu R$, for $\beta = 0$. The solid red and blue dashed curves correspond to $M$ and $\mu Q$, respectively, and the black dotted line represents $R = 2m_\infty$. Here, $M$ and $\mu Q$ are shown in $M_p^2/\mu$.

FIG. 9. $M$ and $\mu Q$ are shown as functions of $\mu R$, for $\beta = 0.2$. The solid red and blue dashed curves correspond to $M$ and $\mu Q$, respectively, and the black dotted line represents $R = 2m_\infty$. Here, $M$ and $\mu Q$ are shown in $M_p^2/\mu$.

vector field $f_{0,c}$, for which PSs take the maximal values of the ADM mass $M$ and the Noether charge $Q$, to the choice of the nonminimal coupling parameter $\beta$.

We have confirmed that the value of the critical central amplitude, which in our analysis is found to be $f_{0,c} \approx 0.0399M_p$, is insensitive to the choice of $\beta$, although the absolute maximal values of $M$ and $\mu Q$ both depend on $\beta$. As we have mentioned in Sec. 1, in the case of $\beta = 0$ the PS solution with the maximal values of $M$ and $\mu Q$ corresponds to the critical solution which divides stable and unstable PS solutions \cite{33}, as in the case of BSs in the scalar-tensor theory \cite{27,28}. Thus, the insensitivity of the value of $f_{0,c}$ to the choice of $\beta$ indicates that even in the presence of nonminimal coupling $\beta \neq 0$ the PS solution with the maximal values of $M$ and $\mu Q$ obtained for $f_{0,c} \approx 0.0399M_p$ would also correspond to the critical PS solution which divides stable and unstable PS ones, and PS solutions for $f_0 < f_{0,c} \approx 0.0399M_p$ would be stable, irrespective of the choice of $\beta$. The explicit confirmation of this speculation is definitively important, but will be left for future studies.
V. CONCLUSIONS

In this paper, we have investigated boson star type solutions in the generalized complex Proca theory with mass and nonminimal coupling to the Einstein tensor, namely Proca star solutions. We have numerically constructed PS solutions for nonzero values of the nonminimal coupling parameter, and found that the inclusion of it changes properties of PSs.

For positive nonminimal coupling parameters, PS solutions did not exist for central amplitudes above some certain value, as the evolution equations to determine the structure of PSs became singular at very small radii. For negative nonminimal coupling parameters, although there was no such singular behavior, sufficiently enhanced numerical resolutions were requested for larger amplitudes. Thus, for larger absolute values of the nonminimal coupling parameter, spiraling features observed for the vanishing nonminimal coupling eventually disappeared. Moreover, for negative (positive) nonminimal coupling parameters, for the same
frequency PSs had larger (smaller) ADM mass and Noether charge than those for the vanishing nonminimal coupling parameter. Irrespective of the sign of the nonzero nonminimal coupling parameter, PSs with the maximal values of the ADM mass and the Noether charge were always gravitationally bound.

Such behaviors of PSs were similar to those observed for BSs in the EGB theory [30, 31], EdGB theory [32], and complex scalar-tensor theory with nonminimal derivative coupling to the Einstein tensor [29]. Thus, they would be generic for gravitational theories including healthy higher-derivative terms. Similarities between the scalar-tensor and generalized Proca theories with nonminimal coupling to the Einstein tensor were pointed out in the context of BH physics in Refs. [14, 15, 17, 18].

Finally, we have given the speculation on stability of PSs in the given subclass of the generalized complex theory. We have confirmed that the critical central amplitude of the vector field which provides the PS solutions with the maximal values of the ADM mass and the Noether charge is insensitive to the choice of the nonminimal coupling parameter. Since, in the case without nonminimal coupling, the PS solution with the maximal values of ADM mass and Noether charge corresponds to the critical PS solution dividing stable and unstable PS solutions [33], combined with the confirmed insensitivity to the choice of the nonminimal coupling parameter, we have speculated that even in the presence of nonzero nonminimal coupling PS solutions with the maximal values of ADM mass and Noether charge correspond to the critical PS solutions dividing stable and unstable PS ones. The explicit confirmation of this speculation will be left for the future studies.

PS solutions obtained in this paper were mini-BS type. A definitively interesting question is whether more massive PS solutions can be constructed in the presence of self-couplings and other nonminimal couplings to curvatures. We will leave these issues for our future studies.

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Appendix A: Components of EOMs

The \((\hat{t}, \hat{t}), (r, r),\) and angular components of the gravitational EOM (10) are given by

\begin{align}
\mathcal{E}_{\hat{t}\hat{t}} &= -\frac{1}{2r^4\kappa^2(r - 2m(r))\sigma(r)^2}U_{\hat{t}\hat{t}}(r), \\
\mathcal{E}_{r r} &= \frac{1}{2r^3\kappa^2(r - 2m(r))\sigma(r)^3}U_{r r}(r), \\
\mathcal{E}_{i i} &= \frac{1}{r^4\kappa^2(r - 2m(r))^2\sigma(r)^4}U_{i i}(r),
\end{align}
respectively.

The \( \ell \) and \( r \) components of the vector field EOM (11) are given by

\[
\mathcal{F}_\ell = \frac{e^{-i\omega t}}{r^2\sigma(r)} V_\ell(r), \tag{A3a}
\]
\[
\mathcal{F}_r = -\frac{ie^{-i\omega t}}{r^2(r-2m(r))\sigma(r)^2} V_r(r), \tag{A3b}
\]

where

\[
V_\ell(r) := -a_0(r)\sigma(r)\left(\mu^2 r^2 + 2\beta m'(r)\right) + (r - 2m(r))(\beta a_1(r) - 2r\omega a_0'(r) + r\omega a_0''(r)) + \sigma(r)(2a_0'(r) - r\omega a_1'(r) + ra_0''(r)), \tag{A4a}
\]
\[
V_r(r) := r^3\omega a_0'(r) - a_1(r)\left(r^3\omega^2 - (r - 2m(r))\sigma(r)^2(\mu^2 r^2 + 2\beta m'(r)) + 2\beta(r - 2m(r))^2\sigma(r)\sigma'(r)\right), \tag{A4b}
\]

respectively.

The constraint relation (6) is given by

\[
\mathcal{G} = \frac{ie^{-i\omega t}}{r^4(r - 2m(r))\sigma(r)^2} W(r), \tag{A5}
\]
where

$$W(r) := r^3 \omega_a(r)(\mu^2 r^2 + 2 \beta m'(r))$$

$$+ (r - 2m(r))\sigma(r) \left[ r(r - 2m(r))\sigma'(r) - 2 \beta (r - 2m(r))\sigma'(r) \right]$$

$$+ a_1(r) \left\{ -2\sigma(r) \left( \mu^2 r^2 m'(r) - 2 \beta m'(r) + r(\mu^2 r + 2 \beta m''(r)) \right) + m(r) \left( -2 \beta m'(r) + r(\mu^2 r + 2 \beta m''(r)) \right) \right\}.$$  \[(A6)\]

These equations are constrained by Eq. (12).

**Appendix B: Evolution equations**

Combining $E_{\tau}, E_{\chi} = 0,$ and $F_{\xi} = 0$ in Eqs. (A1) and (A3), we obtain the evolution equations for $m(r), \sigma(r),$ and $a_0'(r)$ as

$$m'(r) = -\frac{\kappa^2}{r(r^2 + 3\beta \kappa^2 a_0(r))^2 - (2r + 3\beta \kappa^2 a_1(r)(r - 2m(r)))(r - 2m(r))}\sigma(r)^2$$

$$\times \left\{ \mu^2 r^5 a_0(r)^2 + (r - 2m(r)) \left[ a_1(r) \left( r^4 \omega_a(r)^2 + (r - 2m(r)) \right) \right] \right\},$$  \[(B1a)\]

$$\sigma'(r) = -\frac{\kappa^2 \sigma(r)}{r(r - 2m(r)) \left[ r^2 + 3\beta \kappa^2 a_0(r)^2 - (2r + 3\beta \kappa^2 a_1(r)(r - 2m(r)))(r - 2m(r))\sigma(r)^2 \right] \sigma(r)^2}$$

$$\times \left\{ a_0(r)^2 \left( \mu^2 r^5 + 2r^2 \beta m(r) \right) - 2r^2 \beta a_0(r)(r - 2m(r))a_0'(r)$$

$$+ a_1(r) \left( r - 2m(r) \right)^2 \sigma(r)^2 \left[ a_1(r) \left( \mu^2 r^3 + 2 \beta m(r) \right) + 2r \beta (r - 2m(r))a_1'(r) \right] \right\},$$  \[(B1b)\]

$$a_0''(r) = \frac{1}{r(r - 2m(r))} \times \left\{ \mu^2 r^5 a_0(r)^2 + 2r^2 \omega a_1(r) - 4 \omega a_1(r)m(r) - 2r a_0'(r) + 4m(r)a_0'(r)$$

$$+ \frac{2r^2 \beta \kappa^2 a_0(r)^2 - 2(2r + 3\beta \kappa^2 a_1(r)(r - 2m(r)))(r - 2m(r)) \sigma(r)^2}{r^2 \kappa^2 \sigma(r)^2}$$

$$\times \left[ a_0(r)^2 \left( \mu^2 r^4 + 4r \beta m(r) \right) - 4r^2 \beta a_0(r)(r - 2m(r))a_0'(r)$$

$$- (r - 2m(r)) \left[ a_1(r) \left( r^4 \omega_a(r)^2 + (r - 2m(r)) \sigma(r)^2 \right) - 2r^3 \omega a_1(r)a_0(r) + r^3 a_0'(r) \right]$$

$$+ r^2 \omega a_1'(r) - 2r \omega m(r)a_1'(r)$$

$$+ \frac{\kappa^2 (2 - \beta a_0(r) + r \omega a_1(r) - 2a_0'(r))}{2r^2 + 3\beta \omega a_1(r)(r - 2m(r)) + 4\beta m(r) \sigma(r)^2} \right\},$$  \[(B1c)\]

Then, $E_{\chi} = 0$ in Eq. (A1) and $G = 0$ (A5) contain the combination

$$m''(r) - (r - 2m(r)) \frac{\sigma''(r)}{\sigma(r)}.$$  \[(B2)\]

By combining them, we obtain

$$a_0''(r) = C \left[ a_0(r), a_1(r), a_0'(r), a_1'(r), m(r), \sigma(r), m'(r), \sigma'(r) \right],$$  \[(B3)\]

where $C$ is the nonlinear combination of the given variables, which is too involved to be shown explicitly. Then substituting Eqs. (B1) into Eq. (B3), we can replace $m'(r), \sigma'(r),$ and $a_0'(r)$ in Eq. (B3) with the lower derivative terms, and obtain the equation

$$\tilde{C}_0 \left[ a_0(r), a_1(r), a_0'(r), m(r), \sigma(r) \right] + \tilde{C}_1 \left[ a_0(r), a_1(r), a_0'(r), m(r), \sigma(r) \right] a_1'(r) = 0,$$  \[(B4)\]
where $\tilde{C}_i$ ($i = 0, 1$) are the nonlinear combinations of the given variables, which are too involved to be shown explicitly. Hence, we can solve Eq. (B4) in terms of $a'_i(r)$ as

$$a'_i(r) = \frac{\tilde{C}_0 [a_0(r), a_1(r), a'_0(r), m(r), \sigma(r)]}{\tilde{C}_1 [a_0(r), a_1(r), a'_0(r), m(r), \sigma(r)]} \quad (B5)$$

BS solutions do not exist if $\tilde{C}_1 [a_0(r), a_1(r), a'_0(r), m(r), \sigma(r)]$ vanishes at some radius.

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