Bases of the Quantum Cluster Algebra of the Kronecker Quiver

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Abstract We construct bar-invariant $\mathbb{Z}[q^{\pm \frac{1}{2}}]$-bases of the quantum cluster algebra of Kronecker quiver which are quantum analogues of the canonical basis, semicanonical basis and dual semicanonical basis of the corresponding cluster algebra. As a byproduct, we prove positivity of the elements in these bases.

Keywords Quantum cluster algebra, $\mathbb{Z}[q^{\pm \frac{1}{2}}]$-basis, positivity

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1 Introduction

Cluster algebras were introduced by Fomin and Zelevinsky [1, 2] in order to study total positivity in algebraic groups and the specialization of canonical bases of quantum groups at $q = 1$. The study of $\mathbb{Z}$-bases of cluster algebras is important. There are many results involving the construction of $\mathbb{Z}$-bases of cluster algebras (for example, see [3, 4] for cluster algebras of rank 2, [5] for finite type, [6] for type $\tilde{A}$, [7] for $\tilde{D}$, [8] for $\tilde{A}_2^{(1)}$, [9] for affine type and [10] for $Q$ without oriented cycles). As a quantum analogue of cluster algebras, quantum cluster algebras were defined by Berenstein and Zelevinsky [11] in order to study canonical bases. A quantum cluster algebra is generated by a set of generators called cluster variables inside an ambient skew-field $\mathcal{F}$. Under the specialization $q = 1$, the quantum cluster algebras are exactly cluster algebras.

Naturally, one may hope to construct $\mathbb{Z}[q^{\pm \frac{1}{2}}]$-bases for quantum cluster algebras and further quantum analogues of bases of the corresponding cluster algebras. In this paper, we deal with the case of the quantum cluster algebra of Kronecker quiver and construct various bar-invariant $\mathbb{Z}[q^{\pm \frac{1}{2}}]$-bases by applying the $q$-deformation of the Caldero–Chapoton formula [12] defined by Rupel [13]. Under the specialization $q = 1$, these $\mathbb{Z}[q^{\pm \frac{1}{2}}]$-bases are exactly the canonical basis, semicanonical basis and dual semicanonical basis of the corresponding cluster algebra in the sense of [3, 4, 10], respectively. As a byproduct, we prove positivity of the elements in these bases.

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Recently, in [14], Lampe attached to certain element $w$ in Weyl group a subalgebra $U_q^+(w)$ of the positive part $U_q(w)$ of the universal enveloping algebra of a Kac-Moody Lie algebra of type $A_1^{(1)}$. Lampe also proved that $U_q^+(w)$ is a quantum cluster algebra in the sense of Berenstein–Zelevinsky and gave explicit formulae for the cluster variables. Note that the cluster variables are some elements of $q$-deformation of dual canonical basis elements of $U_q^+(w)$. However it is not clear whether cluster monomials belong to the dual canonical basis. Thus comparing these $\mathbb{Z}[q^{\pm1/2}]$-bases constructed in this note with the dual canonical basis of $U_q^+(w)$ becomes an interesting thing.

2 Preliminaries

2.1 Quantum Cluster Algebras

We begin with some of the terminology related to quantum cluster algebras. One can refer to [11] for more details. Let $L$ be a lattice of rank $m$ and $\Lambda : L \times L \to \mathbb{Z}$ a skew-symmetric bilinear form. We will need a formal variable $q$ and consider the ring of integer Laurent polynomials $\mathbb{Z}[q^{\pm1/2}]$. Define the based quantum torus associated with the pair $(L, \Lambda)$ to be the $\mathbb{Z}[q^{\pm1/2}]$-algebra $T$ with a distinguished $\mathbb{Z}[q^{\pm1/2}]$-basis $\{X^e : e \in L\}$ and the multiplication given by

$$X^e X^f = q^{\Lambda(e,f)/2} X^{e+f}.$$ 

It is easy to see that $T$ is associative and the basis elements satisfy the following relations:

$$X^e X^f = q^{\Lambda(e,f)} X^f X^e, \quad X^0 = 1, \quad (X^e)^{-1} = X^{-e}.$$ 

It is known that $T$ is an Ore domain, i.e., is contained in its skew-field of fractions $\mathcal{F}$. The quantum cluster algebra will be defined as a $\mathbb{Z}[q^{\pm1/2}]$-subalgebra of $\mathcal{F}$.

A toric frame in $\mathcal{F}$ is a map $M : \mathbb{Z}^m \to \mathcal{F} \setminus \{0\}$ of the form

$$M(c) = \varphi(X^{\eta(c)}),$$

where $\varphi$ is an automorphism of $\mathcal{F}$ and $\eta : \mathbb{Z}^m \to L$ is an isomorphism of lattices. By the definition, the elements $M(c)$ form a $\mathbb{Z}[q^{\pm1/2}]$-basis of the based quantum torus $T_M := \varphi(T)$ and satisfy the following relations:

$$M(c)M(d) = q^{\Lambda_M(c,d)/2} M(c + d), \quad M(c)M(d) = q^{\Lambda_M(c,d)} M(d)M(c), \quad M(0) = 1, \quad M(c)^{-1} = M(-c),$$

where $\Lambda_M$ is the skew-symmetric bilinear form on $\mathbb{Z}^m$ obtained from the lattice isomorphism $\eta$. Let $\Lambda_M$ also denote the skew-symmetric $m \times m$ matrix defined by $\lambda_{ij} = \Lambda_M(e_i, e_j)$, where $\{e_1, \ldots, e_m\}$ is the standard basis of $\mathbb{Z}^m$. Given a toric frame $M$, let $X_i = M(e_i)$. Then we have

$$T_M = \mathbb{Z}[q^{\pm1/2}]\{X_1^{\pm1}, \ldots, X_m^{\pm1} : X_i X_j = q^{\lambda_{ij}} X_j X_i\}.$$ 

An easy computation shows that

$$M(c) = q^{\frac{1}{2} \sum_{i < j} c_i c_j \lambda_{ij}} X_1^{c_1} X_2^{c_2} \cdots X_m^{c_m} =: X^c, \quad c \in \mathbb{Z}^m.$$ 

Let $\Lambda$ be an $m \times m$ skew-symmetric matrix and let $\tilde{B}$ be an $m \times n$ matrix, $n \leq m$. We call the pair $(\Lambda, \tilde{B})$ compatible if $\tilde{B}^T \Lambda = (D|0)$ is an $n \times m$ matrix with $D = \text{diag}(d_1, \ldots, d_n)$,
where $d_i \in \mathbb{N}$ for $1 \leq i \leq n$. The pair $(M, \tilde{B})$ is called a quantum seed if the pair $(\Lambda_M, \tilde{B})$ is compatible. Define the $m \times m$ matrix $E = (e_{ij})$ by

$$e_{ij} = \begin{cases} 
\delta_{ij}, & \text{if } j \neq k; \\
-1, & \text{if } i = j = k; \\
\max(0, -b_{kk}), & \text{if } i \neq j = k.
\end{cases}$$

For $n, k \in \mathbb{Z}$, $k \geq 0$, denote $[n]_q = \frac{(q^n-q^{-n})-...-(q^{n-k+1}-q^{-n-k+1})}{(q^k-q^{-k})-...-(q-q^{-1})}$. Let $c = (c_1, \ldots, c_m) \in \mathbb{Z}^m$ with $c_k \geq 0$. Define the toric frame $M' : \mathbb{Z}^m \to \mathcal{F} \setminus \{0\}$ as follows:

$$M'(c) = \sum_{p=0}^{c_k} \left[ \begin{array}{c} c_k \\ p \end{array} \right] q^{pk/2} M(EC + pb^k), \quad M'(-c) = M'(-c)^{-1}, \quad (2.1)$$

where the vector $b^k \in \mathbb{Z}^m$ is the $k$-th column of $\tilde{B}$. Then the quantum seed $(M', \tilde{B}')$ is defined to be the mutation of $(M, \tilde{B})$ in the direction $k$. We say that two quantum seeds are mutation-equivalent if they can be obtained from each other by a sequence of mutations. Let $C = \{M'(e_i) : i \in [1, n]\}$ where $(M', \tilde{B}')$ is mutation-equivalent to $(M, \tilde{B})$. The elements of $C$ are called cluster variables. Let $P = \{M(e_i) : i \in [n+1, m]\}$, and the elements of $P$ be called coefficients. The quantum cluster algebra $A_q(\Lambda_M, \tilde{B})$ is the $\mathbb{Z}[q^{\pm 1/2}]$-subalgebra of $\mathcal{F}$ generated by $C \cup P$. We associated with $(M, \tilde{B})$ the $\mathbb{Z}$-linear bar-involution on $T_M$ by setting

$$\overline{q^{r/2}M(c)} = q^{-r/2}M(c), \quad r \in \mathbb{Z}, \quad c \in \mathbb{Z}^n.$$

It is easy to show that $\overline{XY} = Y \overline{X}$ for all $X, Y \in A_q(\Lambda_M, \tilde{B})$ and each element of $C \cup P$ is bar-invariant.

### 2.2 Kronecker Quiver

Given a compatible pair $(\Lambda, \tilde{B})$, we can associate a valued quiver (see [13, Section 2] for more details). Now we set $\Lambda = \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right)$ and $\tilde{B} = \left( \begin{array}{cc} 0 & 2 \\ 2 & 0 \end{array} \right)$. The quiver $Q$ associated with this pair is the Kronecker quiver:

$$1 \bullet \overbrace{\cdots} \bullet 2.$$

Let $k$ be a finite field with cardinality $|k| = q^2$. The category rep($kQ$) of finite-dimensional representations can be identified with the category of mod-$kQ$ of finite-dimensional modules over the path algebra $kQ$. It is well known (see [15]) that indecomposable $kQ$-module contains (up to isomorphism) three families: the indecomposable regular modules with dimension vector $(nd_p, nd_p)$ for $p \in \mathbb{P}_k^1$ of degree $d_p$ (in particular, denoted by $R_p(n)$ for $d_p = 1$), the preprojective modules with dimension vector $(n-1, n)$ (denoted by $M(n)$) and the preinjective modules with dimension vector $(n, n-1)$ (denoted by $N(n)$). Here $n \in \mathbb{N}$.

For $m \in \mathbb{Z} \setminus \{1, 2\}$, set

$$V(m) = \begin{cases} 
N(m-2), & \text{if } m \geq 3; \\
M(-m+1), & \text{if } m \leq 0.
\end{cases}$$

Now, let $\mathcal{T} = \mathbb{Z}[q^{\pm 1/2}]\langle X_1^{\pm 1}, X_2^{\pm 1} : X_1X_2 = qX_2X_1 \rangle$ and $\mathcal{F}$ be the skew field of fractions of $\mathcal{T}$ and thus the quantum cluster algebra of Kronecker quiver $A_q(\Lambda, \tilde{B})$ (denoted by $A_q(2, 2)$
in the following) is the \( \mathbb{Z}[q^{\pm 1/2}] \)-subalgebra of \( \mathcal{F} \) generated by the cluster variables \( X_k \), \( k \in \mathbb{Z} \), defined recursively by

\[
X_{m-1}X_{m+1} = qX_m^2 + 1. \tag{2.2}
\]

The quantum Laurent phenomenon (see [11]) implies that each \( X_k \) belongs to the subring of \( \mathcal{T} \) generated by \( q^{\pm 1/2} \), \( X_1^{\pm 1} \) and \( X_2^{\pm 1} \). The explicit Laurent expansion of each \( X_k \) in \( X_1, X_2 \) is given in [13, 14].

Let \( V \) be a representation of the Kronecker quiver with dimension vector \( \text{dim} \, V = (v_1, v_2) \). For \( e = (e_1, e_2) \in \mathbb{Z}_{\geq 0}^2 \), denote by \( Gr_e(V) \) the set of all subrepresentations \( M \) of \( V \) with \( \text{dim} \, M = e \). In [13], Rupel defines the element \( X_V \) of the quantum torus \( \mathcal{T} \) by

\[
X_V = \sum_e q^{-\frac{1}{2}d_e^V} |Gr_e(V)|X^{(-v_1+2v_2-2e_2,e_1-v_2)}, \tag{2.3}
\]

where \( d_e^V = 2e_1(v_1-e_1) - 2(2e_1-e_2)(v_2-e_2) \). This formula is called a \( q \)-deformation of the Caldero–Chapoton formula. Here and in the following, we simply write \( X^e \) instead of \( X^{(e)} \) for \( e \in \mathbb{Z}^2 \).

**Theorem 1** ([13]) For any \( m \in \mathbb{Z} \setminus \{1, 2\} \), the \( m \)-th cluster variable \( X_m \) of \( \mathcal{A}_q(2,2) \) equals \( X_{V(m)} \).

### 3 Bases of the Quantum Cluster Algebra \( \mathcal{A}_q(2,2) \)

In this section, we will construct various bar-invariant \( \mathbb{Z}[q^{\pm \frac{1}{2}}] \)-bases of quantum cluster algebra \( \mathcal{A}_q(2,2) \). Under the specialization \( q = 1 \), these bases are just bases of the cluster algebra of Kronecker quiver.

**Definition 1** For any \((r_1, r_2)\) and \((s_1, s_2)\) \( \in \mathbb{Z}^2 \), we write \((r_1, r_2) \leq (s_1, s_2)\) if \( r_i \leq s_i \) for \( 1 \leq i \leq 2 \). Moreover, if there exists some \( i \) such that \( r_i < s_i \), then we write \((r_1, r_2) < (s_1, s_2)\).

**Remark 1** By the definition of the \( q \)-deformation of the Caldero–Chapoton formula and the partial order in Definition 1, we obtain that the expansion of \( X_{V(m)} \) has a minimal non-zero term \( f(q^{\frac{1}{2}}, q^{-\frac{1}{2}})X^{-\text{dim} \, V(m)} \) where \( f(q^{\frac{1}{2}}, q^{-\frac{1}{2}}) \in \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}] \). In fact, \( f(q^{\frac{1}{2}}, q^{-\frac{1}{2}}) = 1 \) by the explicit expansion of \( X_{V(m)} \) given in [13, 14].

**Lemma 2** Let \( R_p(1) \) be the indecomposable regular module of degree 1 as above. Then

\[
X_{R_p(1)} = X^{(-1,1)} + X^{(1,-1)} + X^{(-1,-1)}.
\]

**Proof** Note that \( R_p(1) \) contains the three submodules: 0, \( M(1) \) and \( R_p(1) \). Thus the lemma immediately follows from the \( q \)-deformation of the Caldero–Chapoton formula.

By Lemma 2, the expression of \( X_{R_p(1)} \) is independent of the choice of \( p \in \mathbb{P}_k \) of degree 1. Hence, we set \( X_0 := X_{R_p(1)} \).

**Definition 2** (1) The \( n \)-th Chebyshev polynomials of the first kind is the polynomial \( F_n(x) \in \mathbb{Z}[x] \) defined by

\[
F_0(x) = 1, \ F_1(x) = x, \ F_2(x) = x^2 - 2, \ F_{n+1}(x) = F_n(x)F_1(x) - F_{n-1}(x) \quad \text{for} \ n \geq 2.
\]

(2) The \( n \)-th Chebyshev polynomials of the second kind is the polynomial \( S_n(x) \in \mathbb{Z}[x] \) defined by

\[
S_0(x) = 1, \ S_1(x) = x, \ S_2(x) = x^2 - 1, \ S_{n+1}(x) = S_n(x)S_1(x) - S_{n-1}(x) \quad \text{for} \ n \geq 2.
\]
It is obvious that $F_n(x) = S_n(x) - S_{n-2}(x)$. We denote $z = X_\delta$, $z_n = F_n(z)$, $s_n = S_n(z)$ for $n \geq 0$ and $z_n = s_n = 0$ for $n < 0$. Set

\[ B = \{ X_m^a X_{m+1}^b \mid m \in \mathbb{Z}, (a, b) \in \mathbb{Z}_2^2 \} \cup \{ z_n \mid n \in \mathbb{N} \}, \]
\[ S = \{ X_m^a X_{m+1}^b \mid m \in \mathbb{Z}, (a, b) \in \mathbb{Z}_2^2 \} \cup \{ s_n \mid n \in \mathbb{N} \}, \]
\[ D = \{ X_m^a X_{m+1}^b \mid m \in \mathbb{Z}, (a, b) \in \mathbb{Z}_2^2 \} \cup \{ z_n \mid n \in \mathbb{N} \}. \]

**Remark 2**

(1) It is easy to check that

\[ X^{(r,s)} X^{(t,s)} = X^{(r+s,t+s)} \]

for any $r, s, t \in \mathbb{Z}$, thus the expansions of $z_n, s_n$ and $z^n$ have a minimal non-zero term $f(q^\pm\frac{1}{2}, q^{-\frac{1}{2}})X^{-(n,n)}$ according to the partial order in Definition 1.

(2) The elements $c$ associated with these minimal non-zero terms $f(q^\pm\frac{1}{2}, q^{-\frac{1}{2}})X^c$ in the expansion of the elements in the set $B$ are different from each other. Indeed, it is easy to compute

\[ c = \begin{cases} 
(b, -a), & \text{if } X^c = X_0^a X_1^b; \\
(a, b), & \text{if } X^c = X_1^a X_2^b; \\
(-b, a), & \text{if } X^c = X_2^a X_3^b; \\
(a \cdot \dim V(m) + b \cdot \dim V(m + 1), & \text{if } X^c = X_m^a X_{m+1}^b \text{ for } m \neq 0, 1 \text{ or } 2; \\
(n, n), & \text{if } X^c = z_n. 
\end{cases} \]

We note that there is at most one exceptional module in each dimension vector.

Now we define a ring homomorphism of the quantum cluster algebra $A_q(2, 2)$:

\[ \sigma_1 : A_q(2, 2) \rightarrow A_q(2, 2), \]

which sends $X_m$ to $X_{m+1}$ and $q^{\pm\frac{1}{2}}$ to $q^{\pm\frac{1}{2}}$. It is obviously an automorphism which preserves the defining relations. The following lemma is easy but important.

**Lemma 3**  \[ \sigma_1(X_\delta) = X_\delta. \]

**Proof** By Theorem 1 and the definition of the $q$-deformation of the Caldero–Chapoton formula, we have

\[ X_0 = X_{V(0)} = X^{(2,-1)} + X^{(0,-1)}, \]
\[ X_3 = X_{V(3)} = X^{(-1,2)} + X^{(-1,0)}, \]
\[ X_{-1} = X_{V(-1)} = X^{(3,-2)} + X^{(-1,-2)} + (q + q^{-1})X^{(1,-2)} + X^{(-1,0)}. \]

Following these identities and Lemma 2, one easily confirms the relations

\[ X_\delta = q^\pm\frac{1}{2}(X_0 X_3 - q X_1 X_2) = q^\pm\frac{1}{2}(X_{-1} X_2 - q X_0 X_1). \]

Thus \[ \sigma_1(X_\delta) = \sigma_1(q^\pm\frac{1}{2}(X_{-1} X_2 - q X_0 X_1)) = q^\pm\frac{1}{2}(X_0 X_3 - q X_1 X_2) = X_\delta. \]

**Lemma 4** For any $n \in \mathbb{Z}$, we have $X_n X_\delta = q^{-\frac{1}{2}} X_{n-1} + q^{\frac{1}{2}} X_{n+1}$.

**Proof** By an easy computation, we have

\[ X_0 = X^{(2,-1)} + X^{(0,-1)}, \]
\[ X_{-1} = X^{(3,-2)} + X^{(-1,-2)} + (q + q^{-1})X^{(1,-2)} + X^{(-1,0)}. \]

Then by Lemma 2, it is easy to prove $X_0 X_\delta = q^{-\frac{1}{2}} X_{-1} + q^{\frac{1}{2}} X_1$. Thus we can finish the proof by Lemma 3 and applying the automorphism $\sigma_1$. □
Lemma 4. We assume that (2) holds for polynomials.

Proof. The proof of (1) follows from the inductive relations in the definition of Chebyshev polynomials. This proves (2). Now we prove (3). If \( m > n \geq 1 \), we have

\[
z_n z_m = z_{m+n} + z_{m-n}, \quad z_n z_m = z_{2n} + 2.
\]

(2) For \( m \geq 1 \) and \( n \in \mathbb{Z} \), we have

\[
X_n z_m = q^{\frac{m}{2}} X_{n+m} + q^{-\frac{m}{2}} X_{n-m}.
\]

(3) For \( m \geq 0 \) and \( n \in \mathbb{Z} \), we have

\[
X_n X_{n+2m} = q^m X_{n+m}^2 + \sum_{l=0}^{m-1} q^{-m+2l+1} \sum_{k=l+1}^m z_{2(m-k)},
\]

\[
X_n X_{n+2m+1} = q^m X_{n+m} X_{n+m+1} + \sum_{l=0}^{m-1} q^{-m+2l+\frac{1}{2}} \sum_{k=l+1}^m z_{2(m-k)+1}.
\]

Proof. The proof of (1) follows from the inductive relations in the definition of Chebyshev polynomials. As for (2), we make induction on \( m \). If \( m = 1 \), the equation in (2) is a direct corollary of Lemma 4. We assume that (2) holds for \( m \leq k \). For \( m = k + 1 \), we have

\[
X_n z_{k+1} = X_n (z_k z_1 - z_{k-1})
\]

\[
= (q^{\frac{1}{2}} X_{n+k} + q^{-\frac{1}{2}} X_{n-k}) z_1 - (q^{-\frac{k}{2}} X_{n+k-1} + q^{-\frac{k-1}{2}} X_{n-k+1})
\]

\[
= q^{\frac{1}{2}} X_{n+k} z_1 + q^{-\frac{1}{2}} X_{n-k} z_1 - (q^{-\frac{k}{2}} X_{n+k-1} + q^{-\frac{k-1}{2}} X_{n-k+1})
\]

\[
= q^{\frac{1}{2}} (q^{\frac{1}{2}} X_{n+k+1} + q^{-\frac{1}{2}} X_{n-k+1}) + q^{-\frac{1}{2}} (q^{\frac{1}{2}} X_{n-k+1} + q^{-\frac{1}{2}} X_{n-k-1})
\]

\[
- (q^{-\frac{k}{2}} X_{n+k-1} + q^{-\frac{k-1}{2}} X_{n-k-1})
\]

\[
= q^{\frac{k+1}{2}} X_{n+k+1} + q^{-\frac{k-1}{2}} X_{n-k-1}.
\]

This proves (2). Now we prove (3). If \( m = 0 \), it is obvious. If \( m = 1 \), by the recurrence relations (2.2), we have \( X_n X_{n+2} = q X_{n+1}^2 + 1 \). We have proved that the equation \( X_0 X_3 = q X_1 X_2 + q^{-\frac{1}{2}} z \) (see (3.1)) holds, thus by Lemma 3, we have \( X_n X_{n+3} = q X_{n+2} X_{n+2} + q^{-\frac{1}{2}} z \). Now we assume that equations in (3) hold for \( m \leq k \). For \( m = k + 1 \), by Lemma 4, we have

\[
X_n X_{n+2k+2} = q^{-\frac{1}{2}} X_n (X_{n+2k+1} z_1 - q^{\frac{1}{2}} X_{n+2k}).
\]

Following the inductive assumption, it is equal to

\[
q^{-\frac{k+1}{2}} \left( q^k X_{n+k} X_{n+k+1} + \sum_{l=0}^{k-1} q^{-k+2l+\frac{1}{2}} \sum_{i=l+1}^k z_{2(k-i)+1} \right) z_1
\]

\[
- q^{-1} \left( q^k X_{n+k}^2 + \sum_{l=0}^{k-1} q^{-k+2l+1} \sum_{i=l+1}^k z_{2(k-i)} \right).
\]
Using Lemma 4 again and (1) of this proposition, it is
\begin{align*}
q^{k-\frac{1}{2}}X_{n+k}(q^{\frac{1}{2}}X_{n+k+2} + q^{-\frac{1}{2}}X_{n+k}) + \sum_{l=0}^{k-1} q^{-k+2l} \sum_{i=l+1}^{k} z_{2(k-i)+1}z_1 \\
- q^{-1}\left(q^{k}X_{n+k}^2 + \sum_{l=0}^{k-1} q^{-k+2l+1} \sum_{i=l+1}^{k} z_{2(k-i)}\right)
= q^{k+1}X_{n+k+1}^2 + \sum_{l=0}^{k} q^{-k+2l} \sum_{i=l+1}^{k+1} z_{2(k-i)+1}.
\end{align*}

Similarly, by Lemma 4, we have
\[X_nX_{n+2k+3} = q^{-\frac{1}{2}}X_n(X_{n+2k+2}z_1 - q^{-\frac{1}{2}}X_{n+2k+1}).\]

Using the equation (*) and a similar proof, we obtain
\[X_nX_{n+2k+3} = q^{k+1}X_{n+k+1}X_{n+k+2} + \sum_{l=0}^{k} q^{-k+2l-\frac{1}{2}} \sum_{i=l+1}^{k+1} z_{2(k-i)+1+1}.\]

**Remark 4** By Lemma 5 and properties of bar-invariant, we can easily obtain similar results for \(z_mX_n, X_{n+2m}X_n, X_{n+2m}X_n\).

We similarly define the quantized version of the definition of positivity in [3].

**Definition 3** A nonzero element \(x \in A_q(2,2)\) is positive if for every \(m \in \mathbb{Z}\), all the coefficients in the expansion of \(x\) as a Laurent polynomial in \(\{x_m, x_{m+1}\}\) belong to \(\mathbb{N}[q^{\pm \frac{1}{2}}]\).

**Corollary 7** Every element in \(B, S\) and \(D\) is a positive element of quantum cluster algebra \(A_q(2,2)\).

**Proof** By Lemma 2 and the fact that \(z_n(x) = s_n(x) - s_{n-2}(x)\), we only need to prove every element in \(B\) is positive. By the definition of \(\sigma_1\) and Remark 3, it is enough to prove the positivity in \(\{x_1, x_2\}\). We prove it by induction. For convenience, we write down the following equations according to Proposition 6 and Remark 4:

For \(m \geq 1\):
\[X_1z_m = q^{m}X_{1+m} + q^{-\frac{m}{2}}X_{1-m}.\]  
(3.2)

For \(m \geq 0\):
\[X_{1}X_{1+2m} = q^{m}X_{1+m}^2 + q^{-m+1} \sum_{k=1}^{m} z_{2(m-k)} + q^{-m+3} \sum_{k=2}^{m} z_{2(m-k)}
+ \cdots + q^{-m-3} \sum_{k=m-1}^{m} z_{2(m-k)} + q^{m-1};\]  
(3.3)
\[X_{1}X_{2+2m} = q^{m}X_{1+m}X_{2+m} + q^{-m+\frac{3}{2}} \sum_{k=1}^{m} z_{2(m-k)+1} + q^{-m+\frac{5}{2}} \sum_{k=2}^{m} z_{2(m-k)+1}
+ \cdots + q^{-m-\frac{1}{2}} \sum_{k=m-1}^{m} z_{2(m-k)+1} + q^{m-\frac{3}{2}}z_1;\]  
(3.4)
\[X_{1}X_{1-2m} = q^{-m}X_{1-m}^2 + q^{m-1} \sum_{k=1}^{m} z_{2(m-k)} + q^{-m-3} \sum_{k=2}^{m} z_{2(m-k)}
+ \cdots + q^{-m-3} \sum_{k=m-1}^{m} z_{2(m-k)} + q^{m-3}z_1.\]
\[ + \cdots + q^{-m+3} \sum_{k=m-1}^{m} z_{2(m-k)} + q^{-m+1}; \]

\[ X_1 X_{-2m} = q^{-m-1} X_{-m} X_1 + q^{m-\frac{1}{2}} \sum_{k=1}^{m} z_{2(m-k)+1} + q^{m-\frac{3}{2}} \sum_{k=2}^{m} z_{2(m-k)+1} + \cdots + q^{-m+\frac{7}{2}} \sum_{k=m-1}^{m} z_{2(m-k)+1} + q^{-m+\frac{7}{2}} z_1. \]

It is easy to check that \( \{X_{-2}, X_{-1}, X_0, X_1, X_2, z_1, z_2\} \) are positive elements in \( \{x_1, x_2\} \). Now assume that \( \{X_{-2m}, X_{-2m+1}, \ldots, X_{2m-1}, X_{2m}, z_1, \ldots, z_{2m}\} \) are positive elements in \( \{x_1, x_2\} \). Then by (3.2), (3.3) and (3.4), we know that \( X_{2m+1}, X_{2m+2} \) and \( X_{-1-2m} \) are positive. Thus we obtain that \( z_{2m+1} \) is positive by (3.1). Therefore by (3.5), we have that \( X_{-2m-2} \) is positive. Again by (3.2), we know that \( X_{2m+3} \) is positive. Thus we get \( z_{2m+2} \) is positive by (3.1) again. Throughout the above discussions, we obtain \( \{X_{-2m-2}, X_{-2m-1}, \ldots, X_{2m+1}, X_{2m+2}, z_1, \ldots, z_{2m+2}\} \) are positive elements in \( \{x_1, x_2\} \). The proof is finished.

\[ \square \]

**Remark 5** In fact, according to [13, 14, 16] the positivity in the cluster variables is obvious, then applying the equation \( X_{-1} z_m = q^{\frac{3}{2}} X_{1} + q^{-\frac{3}{2}} X_{-1} \), we can deduce the positivity in the elements \( z_m \) for any \( m \in \mathbb{N} \). Here, we give an alternative proof without needing the explicit expansions of cluster variables.

**Theorem 8** The sets \( B, S \) and \( D \) are \( \mathbb{Z}[q^{\pm \frac{1}{2}}] \)-bases of the quantum cluster algebra \( A_q(2, 2) \).

**Proof** Note that if \( B \) is a \( \mathbb{Z}[q^{\pm \frac{1}{2}}] \)-basis of the quantum cluster algebra \( A_q(2, 2) \), then \( S \) and \( D \) are naturally \( \mathbb{Z}[q^{\pm \frac{1}{2}}] \)-bases of quantum cluster algebra \( A_q(2, 2) \) because there are unipotent matrix transformations between \( \{z_n | n \in \mathbb{N}\} \), \( \{s_n | n \in \mathbb{N}\} \) and \( \{z^n | n \in \mathbb{N}\} \). In the following, we will focus on the set \( B \) and prove it is a \( \mathbb{Z}[q^{\pm \frac{1}{2}}] \)-basis of the quantum cluster algebra \( A_q(2, 2) \).

By Proposition 6, we obtain that any element of the quantum cluster algebra \( A_q(2, 2) \) can be a \( \mathbb{Z}[q^{\pm \frac{1}{2}}] \)-combination of the elements in the set \( B \). Thus we only need to prove the elements in \( B \) are \( \mathbb{Z}[q^{\pm \frac{1}{2}}] \)-independent.

By Remark 2, we know that the elements \( c \) associated with these minimal non-zero terms \( f(q^{\frac{1}{2}}, q^{-\frac{1}{2}})X^c \) in the expansion of the elements in the set \( B \) are different from each other. Now we suppose that a finite \( \mathbb{Z}[q^{\pm \frac{1}{2}}] \)-combination of the elements in the set \( B \) is equal to 0. Let \( S \subset \mathbb{Z}^2 \) be the set of all \( \alpha \) such that the corresponding element occurs with a non-zero coefficient in this \( \mathbb{Z}[q^{\pm \frac{1}{2}}] \)-combination. If \( S \) is non-empty, pick a minimal element \( \alpha \in S \), by Remarks 1 and 2, we know that \( X^\alpha \) does not occur in the expansion of any other element in the above equation which gives a contradiction. This completes the proof of the theorem.

\[ \square \]

Set

\[ B' = \{q^{-\frac{m}{2}} X_a X_{m+1} | m \in \mathbb{Z}, (a, b) \in \mathbb{Z}_{\geq 0}^2 \cup \{z_n | n \in \mathbb{N}\}, \]
\[ S' = \{q^{-\frac{m}{2}} X_a X_{m+1} | m \in \mathbb{Z}, (a, b) \in \mathbb{Z}_{\geq 0}^2 \cup \{s_n | n \in \mathbb{N}\}, \]
\[ D' = \{q^{-\frac{m}{2}} X_a X_{m+1} | m \in \mathbb{Z}, (a, b) \in \mathbb{Z}_{\geq 0}^2 \cup \{z^n | n \in \mathbb{N}\}. \]

Then we can obtain the following corollary.

**Corollary 9** The sets \( B', S' \) and \( D' \) are bar-invariant \( \mathbb{Z}[q^{\pm \frac{1}{2}}] \)-bases of the quantum cluster algebra \( A_q(2, 2) \).
4 A Representation-theoretic Interpretation of the Element $s_n$

Recall we denote by $R_p(n)$ the indecomposable regular modules with dimension vector $(n, n)$ for $n \geq 1$ and some $p \in \mathbb{P}_1$ of degree 1. In this section, we will prove that $s_n$ is equal to $X_{n\delta}$ for every $n \in \mathbb{N}$. The following proposition shows the Laurent expansion of $X_m$ in $A_q(2, 2)$.

**Proposition 10** ([13, 14]) For every $n \geq 0$, we have

\[
X_{-n} = X^{(n+2, -n-1)} + \sum_{p+r \leq n} \left[\begin{array}{c} n-r \\ p \end{array}\right]_q \left[\begin{array}{c} n+1-p \\ r \end{array}\right]_q X^{(2r-n, 2p-n-1)}, \tag{4.1}
\]

\[
X_{n+3} = X^{(-n-1, n+2)} + \sum_{p+r \leq n} \left[\begin{array}{c} n-r \\ p \end{array}\right]_q \left[\begin{array}{c} n+1-p \\ r \end{array}\right]_q X^{(2p-n-1, 2r-n)}. \tag{4.2}
\]

**Lemma 11** For every $n \in \mathbb{N}$, we have $s_n = q^{n+1}X_1X_{n+3} - q^{n+2}X_2X_{n+2}$.

**Proof** It is easy to check that $s_1 = q^{\frac{1}{2}}X_1X_4 - q^{\frac{3}{2}}X_2X_3$. Assume that it holds for $n \leq k$, then

\[
s_{k+1} = s ks_1 - s_{k-1}
= (q^{k-\frac{1}{2}}X_1X_{k+3} - q^{k+\frac{1}{2}}X_2X_{k+2})s_1 - (q^{k-\frac{1}{2}}X_1X_{k+2} - q^{k+\frac{1}{2}}X_2X_{k+1})
= q^{k+\frac{1}{2}}X_1(q^{\frac{3}{2}}X_{k+4} + q^{-\frac{1}{2}}X_{k+2}) - q^{k+\frac{1}{2}}X_2(q^{\frac{1}{2}}X_{k+3} + q^{-\frac{1}{2}}X_{k+1})
- (q^{k-\frac{1}{2}}X_1X_{k+2} - q^{k+\frac{1}{2}}X_2X_{k+1})
= q^{k+\frac{1}{2}}X_1X_{k+4} - q^{k+\frac{1}{2}}X_2X_{k+3}. \tag{4.3}
\]

Denote the quantum binomial coefficients \( \binom{n}{r}_q = \frac{(q^n-1)(q^{n-1}-1)\cdots (q^{n-r+1}-1)}{(q^r-1)(q^{r-1})\cdots (q-1)} \) and take \( \binom{n}{0}_q = 1 \) for any $n \in \mathbb{Z}$, \( \binom{n}{l}_q = 0 \) for any $l < 0$, and \( \binom{n}{l}_q = 0 \) for any $0 \leq n < l$. Then we have the following theorem proved in [16]:

**Theorem 12** ([16, Theorem 4.6]) Let $e = (a, b)$ for $(a, b) \in \mathbb{Z}_2^2$. Then for $n \geq 1$,

\[
|Gr_e(R_p(n))| = \binom{n-a}{n-b}_q \binom{b}{a}_q. \tag{4.4}
\]

**Proposition 13** $X_{R_p(n)} = s_n$ for any $n \in \mathbb{N}$.

**Proof** By the $q$-deformation of the Caldero–Chapoton formula and Theorem 12, we have

\[
X_{R_p(n)} = \sum_{(a,b)} q^{-\frac{1}{2}d_{R_p(n)}^{R_p(n)}} |Gr_{(a,b)}(R_p(n))| X^{(n-2b, 2a-n)}
= \sum_{(a,b)} q^{(a-b)(n-b)} \binom{n-a}{n-b}_q q^{(a-b)\binom{b}{a}_q} X^{(n-2b, 2a-n)}
= \sum_{(a,b)} \binom{n-a}{n-b}_q \binom{b}{a}_q X^{(n-2b, 2a-n)}
= \sum_{p+r \leq n} \left[\begin{array}{c} n-r \\ p \end{array}\right]_q \left[\begin{array}{c} n-p \\ r \end{array}\right]_q X^{(2p-n, 2r-n)}. \tag{4.5}
\]

On the other hand, by Proposition 10, we have

\[
q^{\frac{1}{2}}X_1X_{n+3} = q^{\frac{1}{2}}X_1X^{(-n-1, n+2)} + q^{\frac{1}{2}}X_1 \sum_{p+r \leq n} \left[\begin{array}{c} n-r \\ p \end{array}\right]_q \left[\begin{array}{c} n+1-p \\ r \end{array}\right]_q X^{(2p-n-1, 2r-n)}
= q^{n+1}X^{(-n, n+2)} + \sum_{p+r \leq n} \left[\begin{array}{c} n-r \\ p \end{array}\right]_q \left[\begin{array}{c} n+1-p \\ r \end{array}\right]_q q^r X^{(2p-n, 2r-n)}. \tag{4.6}
\]
And
\[ q^{\frac{n+1}{2}}X_2X_{n+2} = q^{\frac{n+1}{2}}X_2X^{(-n,n+1)} + q^{\frac{n+1}{2}}X_2 \sum_{p+r \leq n-1} \left[ \begin{array}{c} n-1-r \\ p \end{array} \right] q^{\left[ \begin{array}{c} n-p \\ r \end{array} \right]} X^{(2p-n,2r+1-n)} \]
\[ = q^{n+1}X^{(-n,n+2)} + \sum_{p+r \leq n-1} \left[ \begin{array}{c} n-1-r \\ p \end{array} \right] q^{\left[ \begin{array}{c} n-p \\ r \end{array} \right]} q^{n-p+1}X^{(2p-n,2r-2-n)} \]
\[ = q^{n+1}X^{(-n,n+2)} + \sum_{p+r \leq n} \left[ \begin{array}{c} n-r \\ p \end{array} \right] q^{\left[ \begin{array}{c} n-p \\ r-1 \end{array} \right]} q^{n-p+1}X^{(2p-n,2r-n)}. \]

Note that it is easy to confirm the following identity:
\[ \left[ \begin{array}{c} n+1-p \\ r \end{array} \right] q^r = \left[ \begin{array}{c} n-p \\ r \end{array} \right] + \left[ \begin{array}{c} n-p \\ r-1 \end{array} \right] q^{n-p+1}. \]

Then by Lemma 11, we have
\[ s_n = q^{\frac{n}{2}}X_1X_{n+3} - q^{\frac{n}{2}}X_2X_{n+2} = \sum_{p+r \leq n} \left[ \begin{array}{c} n-r \\ p \end{array} \right] q^{\left[ \begin{array}{c} n-p \\ r \end{array} \right]} X^{(2p-n,2r-n)} = X_{R_n}. \]

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**References**

[1] Fomin, S., Zelevinsky, A.: Cluster algebras. I. Foundations. *J. Amer. Math. Soc.*, 15(2), 497–529 (2002)

[2] Fomin, S., Zelevinsky, A.: Cluster algebras. II. Finite type classification. *Invent. Math.*, 154(1), 63–121 (2003)

[3] Sherman, P., Zelevinsky, A.: Positivity and canonical bases in rank 2 cluster algebras of finite and affine types. *Mosc. Math. J.*, 4(4), 947–974 (2004)

[4] Caldero, P., Zelevinsky, A.: Laurent expansions in cluster algebras via quiver representations. *Mosc. Math. J.*, 6(3), 411–429 (2006)

[5] Caldero, P., Keller, B.: From triangulated categories to cluster algebras. *Invent. Math.*, 172(1), 169–211 (2008)

[6] Dupont, G.: Generic variables in acyclic cluster algebras. *J. Pure Appl. Alg.*, 215, 628–641 (2011)

[7] Ding, M., Xu, F.: A Z-basis of cluster algebras for \( \tilde{D}_4 \). *Algebra Colloq.*, accepted

[8] Cerulli Irelli, G.: Cluster algebras of type \( A_m^{(1)} \). *Algebra Represent Theor.*, doi: 10.1007/s10468-011-9317-5

[9] Ding, M., Xiao, J., Xu, F.: Integral bases of cluster algebras and representations of tame quivers. *Algebra Represent Theor.*, doi: 10.1007/s10468-011-9317-z

[10] Geiss, C., Leclerc, B., Schröer, J.: Generic bases for cluster algebras and the Chamber Ansatz. *J. Amer. Math. Soc.*, 25, 21–76 (2012)

[11] Berenstein, A., Zelevinsky, A.: Quantum cluster algebras. *Adv. Math.*, 195, 405–455 (2005)

[12] Caldero, P., Chapoton, F.: Cluster algebras as Hall algebras of quiver representations. *Comment. Math. Helv.*, 81(3), 595–616 (2006)

[13] Rupel, D.: On a quantum analogue of the Caldero–Chapoton formula. *Int. Math. Res. Not.*, doi: 10.1093/imrn/rnq192 (2010)

[14] Lampe, P.: A quantum cluster algebra of Kronecker type and the dual canonical basis. *Int. Math. Res. Not.*, doi: 10.1093/imrn/rnq162 (2010)

[15] Diab, V., Ringel, C. M.: Indecomposable representations of graphs and algebras. *Mem. Amer. Math. Soc.*, 173, (1976)

[16] Szanto, C.: On the cardinalities of Kronecker quiver Grassmannians. *Math. Z.*, doi: 10.1007/s00209-010-0762-x (2010)

[17] Ding, M., Xiao, J., Xu, F.: Realizing enveloping algebras via varieties of modules. *Acta Mathematica Sinica, English Series*, 26(1), 29–48 (2010)