GABOR DUALITY THEORY FOR MORITA EQUIVALENT C*-ALGEBRAS

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Abstract. The duality principle for Gabor frames is one of the pillars of Gabor analysis. We establish a far-reaching generalization to Morita equivalent C*-algebras where the equivalence bimodule is a finitely generated projective Hilbert C*-module. These Hilbert C*-modules are equipped with some extra structure and are called Gabor bimodules. We formulate a duality principle for standard module frames for Gabor bimodules which reduces to the well-known Gabor duality principle for twisted group C*-algebras of a lattice in phase space. We lift all these results to the matrix algebra level and in the description of the module frames associated to a matrix Gabor bimodule we introduce (n, d)-matrix frames, which generalize superframes and multi-window frames. Density theorems for (n, d)-matrix frames are established, which extend the ones for multi-window and super Gabor frames. Our approach is based on the localization of a Hilbert C*-module with respect to a trace.

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1. Introduction

Hilbert C*-modules are well-studied objects in the theory of operator algebras and Rieffel made the crucial observation that they provide the correct framework for the extension of Morita equivalence of rings to C*-algebras. In his seminal work [27] he noted that the equivalence bimodules between two C*-algebras are bimodules where the left and right Hilbert C*-module structures are compatible and the respective C*-valued inner products satisfy an associativity condition. We are interested in the features of these equivalence bimodules from the perspective of frame theory. In [12] the notion of standard module frame was introduced for countably generated Hilbert C*-modules. Rieffel has already in [28] observed that finitely generated equivalence bimodules may be described in terms

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of finite standard module frames and used it in his study of Heisenberg modules, which is a class of projective Hilbert $C^*$-modules over twisted group $C^*$-algebras. In [13] the properties of standard module frames for Heisenberg modules have been studied from the perspective of duality theory, which was motivated by the observation in [21] that these module frames are closely related to Gabor frames for an associated Hilbert space.

Gabor frames have some additional features not shared by wavelets and shearlets that is due to the seminal contributions [9, 19, 29], where they developed the duality theory of Gabor frames.

**Theorem (Duality Theorem).** The Gabor system $\{e^{2\pi i \beta t} g(t - \alpha k)\}_{k,l \in \mathbb{Z}}$ generated by a function $g \in L^2(\mathbb{R})$ is a frame for $L^2(\mathbb{R})$ if and only if $\{e^{2\pi i \alpha t} g(t - k/\beta)\}_{k,l \in \mathbb{Z}}$ is a Riesz sequence for the closed span of $\{e^{2\pi i \alpha t} g(t - k/\beta)\}_{k,l \in \mathbb{Z}}$ in $L^2(\mathbb{R})$ Here $t \in \mathbb{R}$.

Due to its far-reaching implications there have been attempts to extend the duality principle to other classes of frames [11, 15, 3, 4], see [6, 8, 31, 32] for the theory of R-duals.

Motivated by the link between the duality theory of Gabor frames and the Morita equivalence of noncommutative tori [21, 18] we extend the duality theory of Gabor frames to module frames for equivalence bimodules between Morita equivalent $C^*$-algebras. The setup for our duality theory has its roots in [18] and is as follows: Let $A$ and $B$ be $C^*$-algebras where $B$ is assumed to have a unit and be equipped with a faithful finite trace $\text{tr}_B$. We define a *left Gabor bimodule* to be a quadruple

$$\{(A, B, E, \text{tr}_B)\}$$

where $E$ is a Morita equivalence $A$-$B$-bimodule. We show that module frames for Gabor bimodules admit a duality theorem and by localization with respect to a trace we are able to connect these module frame statements to results on frames in Hilbert spaces. Note that in [11] a different notion of localization of frames was introduced which constructs frames with additional regularity, which we also establish in our general setting.

The main application of our duality results is a concise treatment of Gabor frames for closed cocompact subgroups of locally compact abelian phase spaces. Our general approach to duality principles has led us to the introduction of $(n, d)$-matrix Gabor frames that is a joint generalization of multi-window superframes and Riesz bases and we prove that their Gabor dual systems are $(d, n)$-matrix Gabor frames.

Let $G$ be a second countable LCA group and let $\Lambda$ be a closed subgroup of $G \times \hat{G}$. For $g \in L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d)$, let

$$\mathcal{G}(g; \Lambda) := \{\pi(\lambda) g_{i,j} \mid \lambda \in \Lambda\}_{i \in \mathbb{Z}_n, j \in \mathbb{Z}_d}.$$ 

We say $\mathcal{G}(g; \Lambda)$ is an $(n, d)$-matrix Gabor frame for $L^2(G)$ if the collection of time-frequency shifts $\mathcal{G}(g; \Lambda)$ is a frame for $L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d)$. Equivalently, there exists $h \in L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d)$ such that for all $f \in L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d)$ we have

$$f_{r,s} = \sum_{k \in \mathbb{Z}_d} \sum_{l \in \mathbb{Z}_n} \int_{\Lambda} \langle f_{r,k}, \pi(\lambda) g_{l,k} \rangle_{L^2(G)} \pi(\lambda) h_{l,s} d\lambda,$$

for all $r \in \mathbb{Z}_n$ and $s \in \mathbb{Z}_d$. We develop the theory of these $(n, d)$-matrix Gabor frames and prove a duality theorem for this novel type of frames.

Let us summarize the content of this paper. In [Section 2] we collect some facts about $C^*$-algebras, Hilbert $C^*$-modules, the localization of Hilbert $C^*$-modules and finitely generated projective Hilbert $C^*$-modules. In [Section 3] we introduce Gabor bimodules, and study...
the case when these have a single generator in terms of module frames. We establish the analog of the duality theorem for Gabor frames for Gabor bimodules with one generator. In Section 3.2 we extend all of these results to the case of finitely many generators which leads one naturally to matrix-valued extensions of the statements and definitions in the preceding section. We also prove a density theorem for module frames. In the final section, we discuss applications to Gabor frames for closed subgroups of the time-frequency plane of locally compact abelian groups.

2. Preliminaries on C*-algebras and Hilbert C*-modules

We assume basic knowledge about Banach *-algebras, C*-algebras, and of Banach modules and Hilbert C*-modules. In this section we collect definitions and basic facts of concepts crucial for this paper, such as positivity in C*-algebras, Morita equivalence of C*-algebras, and localization of Hilbert C*-modules. For these topics we refer to [20], [25], and [24].

For a C*-algebra A and a ∈ A, we denote by σ_A(a) the spectrum of a in A. We will need the following important result.

**Proposition 2.1** ([24], Theorem 2.1.11). Let A be a unital C*-algebra and let B be a C*-subalgebra of A containing the unit of A. If b ∈ B, then σ_B(b) = σ_A(b). Equivalently, if b is invertible in A, then b^{-1} ∈ B.

**Definition 2.2.** Let A be a unital C*-algebra and let B ⊂ A be a Banach *-subalgebra of A with the same unit. We say that B is spectral invariant in A if whenever b ∈ B is invertible with b^{-1} ∈ A, we have b^{-1} ∈ B.

Now recall that a selfadjoint element a in a C*-algebra with σ_A(a) ⊂ [0, ∞) is called positive. We state a useful characterization of positivity.

**Proposition 2.3.** Let A be a C*-algebra. For a = a* ∈ A we have σ_A(a) ⊂ [0, ∞) if and only if a = b*b for some b ∈ A.

Denote by A^+ the set of positive elements in the C*-algebra A. The positive elements form a cone. In particular, if a ∈ A^+ then ka ∈ A^+ for all k ∈ [0, ∞), and if a_1, a_2 ∈ A^+ then a_1 + a_2 ∈ A^+. We also obtain a partial order on A^+ by a ≤ b if and only if b - a ∈ A^+. Note that not all elements of A^+ are comparable, but all elements are comparable to 1_A in the case A is unital.

Central to our results in Section 3 will be the localization of a Hilbert C*-module. For this we need positive linear functionals.

**Definition 2.4.** A positive linear functional on a C*-algebra A is a linear functional φ such that φ(A^+) ⊂ [0, ∞). If ∥φ∥ = 1 we say φ is a state.

**Remark 2.5.** If φ: A → C is a positive linear functional and A is unital, it is known that φ is a state if and only if φ(1_A) = 1.

We will denote the set of adjointable operators on the Hilbert A-module E by End_A(E), and the set of compact module operators by K(E). The following two results will be of great importance in our approach to duality theorems.

**Proposition 2.6** ([20], Proposition 1.1). Let A be a C*-algebra. If E is an inner product A-module and f, g ∈ E, then

\[ A(g, f) A(f, g) \leq \|A(f, f)\| A(g, g), \]
where \( \langle \cdot, \cdot \rangle \) is the \( A \)-valued inner product. Also, whenever \( c \geq 0 \) in \( A \), we have \( a^*ca \leq \|c\|a^*a \) for all \( a \in A \).

**Proposition 2.7** ([25], Corollary 2.22). Let \( A \) be a \( C^* \)-algebra. If \( E \) is a Hilbert \( A \)-module and \( T \in \text{End}_A(E) \), then for any \( f \in E \)

\[
\mathcal{A}\langle T f, T f \rangle \leq \|T\|^2_A \langle f, f \rangle
\]
as elements of the \( C^* \)-algebra \( A \), where \( \mathcal{A}\langle \cdot, \cdot \rangle \) is the \( A \)-valued inner product.

Suppose \( \phi \) is a positive linear functional on a \( C^* \)-algebra \( B \), and that \( E \) is a right Hilbert \( B \)-module. We define an inner product

\[
\langle \cdot, \cdot \rangle_B : E \times E \to \mathbb{C}
\]

\[
(f, g) \mapsto \phi(\langle g, f \rangle_B),
\]
where \( \langle \cdot, \cdot \rangle_B \) is the \( B \)-valued inner product. We may have to factor out the subspace

\[
N_\phi := \{ f \in E \mid \langle f, f \rangle_B = 0 \},
\]
and complete \( E/N_\phi \) with respect to \( \langle \cdot, \cdot \rangle_\phi \). This yields a Hilbert space which we will denote by \( H_E \). This is known as the localization of \( E \) in \( \phi \). There is a natural map \( \rho_\phi : E \to H_E \) which induces a map \( \rho_\phi : \text{End}_B(E) \to \mathbb{B}(H_E) \). We will focus entirely on the case in which \( \phi \) is a faithful positive linear functional, that is, when \( b \in B^+ \) and \( \phi(b) = 0 \) implies \( b = 0 \). In that case \( N_\phi = \{ 0 \} \) and we have the following useful result from [20] p. 57-58.

**Proposition 2.8.** Let \( A \) be a \( C^* \)-algebra equipped with a faithful positive linear functional \( \phi : A \to \mathbb{C} \), and let \( E \) be a left Hilbert \( A \)-module. Then the map \( \rho_\phi : \text{End}_A(E) \to \mathbb{B}(H_E) \) is an injective \(*\)-homomorphism.

The Hilbert \( C^* \)-modules of interest will be of a very particular form in that they will be \( A \)-\( B \)-equivalence bimodules for \( C^* \)-algebras \( A \) and \( B \). We will denote the \( A \)-valued inner product by \( \langle \cdot, \cdot \rangle \), and the \( B \)-valued inner product by \( \langle \cdot, \cdot \rangle\_B \).

**Definition 2.9.** Let \( A \) and \( B \) be \( C^* \)-algebras. A **Morita equivalence bimodule** between \( A \) and \( B \), or an **\( A \)-\( B \)-equivalence bimodule**, is a Hilbert \( C^* \)-module \( E \) satisfying the following conditions.

1. \( \langle E, E \rangle = A \) and \( \langle \cdot, \cdot \rangle\_B = B \), where \( \langle E, E \rangle = \text{span}_C \{ \langle f, g \rangle \mid f, g \in E \} \) and likewise for \( \langle \cdot, \cdot \rangle\_B \).
2. For all \( f, g \in E \), \( a \in A \) and \( b \in B \),

\[
\langle af, g \rangle = \langle f, a^*g \rangle_B \text{ and } \langle fb, g \rangle = \langle f, gb^* \rangle.
\]
3. For all \( f, g, h \in E \),

\[
\langle f, g \rangle h = \langle f, g \rangle_B h = f \langle g, h \rangle_B.
\]

Now let \( \mathcal{A} \subset A \) and \( \mathcal{B} \subset B \) be dense Banach \(*\)-subalgebras such that the enveloping \( C^* \)-algebra of \( \mathcal{A} \) is \( A \), and the enveloping \( C^* \)-algebra of \( \mathcal{B} \) is \( B \). Suppose further there is a dense \( A \)-\( B \)-inner product submodule \( \mathcal{E} \subset E \) such that the conditions above hold with \( A, B, \mathcal{E} \) instead of \( A, B, E \). In that case we say \( \mathcal{E} \) is an **\( A \)-\( B \)-pre-equivalence bimodule**.

We will make repeated use of the following fact in the sequel without mention.

**Proposition 2.10** ([25], Proposition 3.11). Let \( A \) and \( B \) be \( C^* \)-algebras and let \( E \) be an \( A \)-\( B \)-equivalence bimodule. Then \( \| \langle f, f \rangle \| = \| \langle f, f \rangle_B \| \) for all \( f \in E \).
It is a well-known result that if $E$ is an $A$-$B$-equivalence bimodule, then $B \cong \mathbb{K}_A(E)$ through the identification $\Theta_{f,g} \mapsto \langle f, g \rangle$. Here $\Theta_{f,g}$ is the compact module operator $\Theta_{f,g} : h \mapsto \langle h, f \rangle g$. We make particular note of the case when $E$ is a finitely generated Hilbert $A$-module.

**Proposition 2.11.** Let $E$ be an $A$-$B$-equivalence bimodule. Then $E$ is a finitely generated projective $A$-module if and only if $B$ is unital.

**Proof.** Suppose first $A$ is finitely generated and projective as a Hilbert $A$-module. As $E$ is finitely generated, any $A$-endomorphism on $E$ is determined by its action on a finite set of generators. Hence $\text{End}_A(E) = \mathbb{K}_A(E)$, and the former is unital. Since $B \cong \mathbb{K}_A(E)$, have that $B$ is unital.

Conversely, we assume that $B$ is unital. As $B \cong \mathbb{K}_A(E)$, and the latter is an ideal in $\text{End}_A(E)$, it follows that $\mathbb{K}_A(E) = \text{End}_A(E)$. As $B$ is unital and $\langle E, E \rangle$ is dense in $B$, we can find elements $f_1, \ldots, f_n, g_1, \ldots, g_n \in E$ such that $\sum_{i=1}^n \langle f_i, g_i \rangle = 1_B$. The maps $s : E \to A^n$

$h \mapsto (\langle h, f_i \rangle)_{i=1}^n,$

and $r : A^n \to E$

$(a_i)_{i=1}^n \mapsto \sum_{i=1}^n a_i g_i,$

are $A$-module maps satisfying

$$r \circ s(z) = \sum_{i=1}^n \langle h, f_i \rangle g_i = \sum_{i=1}^n h \langle f_i, g_i \rangle = h \sum_{i=1}^n \langle f_i, g_i \rangle = h \cdot 1_B = h$$

for all $h \in E$. It follows that $E$ is a finitely generated projective $A$-module. \hfill \Box

Note that the systems $\{f_1, \ldots, f_n\}$ and $\{g_1, \ldots, g_n\}$ are not necessarily $A$-linearly independent, but they still provide a reconstruction formula: $z = \sum_{i=1}^n \langle h, f_i \rangle g_i$. Motivated by spanning sets in finite-dimensional vector spaces, also called frames, we call the system $\{f_1, \ldots, f_n\}$ a module frame for $E$ and $\{g_1, \ldots, g_n\}$ is referred to as a dual module frame. The properties of module frames for equivalence bimodules are the main objective of this work.

The following two results concern properties of module frames consisting of a single element, though we do not formally introduce module frames until Definition 3.8. For our setup it will turn out that it is enough to consider module frames consisting of only one element, see Section 3. The results will come into play when we relate module frames and Gabor frames in Section 4.

**Lemma 2.12.** Let $A$ be any $C^*$-algebra, and let $E$ be a (left) Hilbert $A$-module. Suppose $T \in \text{End}_A(E)$ is such that there exist $C, D > 0$ such that

$$C \cdot \langle f, f \rangle \leq \langle T f, f \rangle \leq D \cdot \langle f, f \rangle, \quad (2.1)$$

for all $f \in E$. Then $T$ is invertible, and

$$\frac{1}{D} \cdot \langle f, f \rangle \leq \langle T^{-1} f, f \rangle \leq \frac{1}{C} \cdot \langle f, f \rangle,$$

for all $f \in E$. \hfill \Box
Proof. By (2.1), we see that $T$ is positive and invertible with $C\text{Id}_E \leq T \leq D\text{Id}_E$. Positivity is preserved when multiplying by positive commuting operators, so it follows that $CT^{-1} \leq T^{-1}T \leq DT^{-1}$, from which we get $\frac{1}{T} \text{Id}_E \leq T^{-1} \leq \frac{1}{T} \text{Id}_E$. □

Lemma 2.13. Let $A$ be any $C^*$-algebra, and let $E$ be a (left) Hilbert $A$-module. Let $T \in \text{End}_A(E)$ be such that there exist $C, D > 0$ such that

\[ C\ast\langle f, f \rangle \leq \langle T f, f \rangle \leq D\ast\langle f, f \rangle. \tag{2.2} \]

for all $f \in E$. Then the smallest possible value of $D$ is $\|T\|$, and the largest possible value for $C$ is $\|T^{-1}\|^{-1}$.

Proof. Since $T$ is positive we have $\|T\| = \sup_{\|f\|=1} \{\|\langle T f, f \rangle\|\}$. It follows that the smallest value for $D$ is $\|T\|$. By [Lemma 2.12] we see by the same argument that the minimal value for $\frac{1}{T}$ is $\|T^{-1}\|^{-1}$. Hence the largest value for $C$ is $\|T^{-1}\|^{-1}$. □

It is an interesting question when the property of being finitely generated projective passes to dense subalgebras and corresponding dense submodules.

Proposition 2.14. Let $E$ be an $A$-$B$-equivalence bimodule as in [Definition 2.9] with $B$ unital. Suppose there are dense Banach $*$-subalgebras $A \subset A$ and $B \subset B$, where $B$ is spectral invariant in $B$ and has the same unit as $B$. Suppose further that $E \subset E$ is an $A$-$B$-pre-equivalence bimodule. If $E$ is a finitely generated projective $A$-module, then $E$ is a finitely generated projective $A$-module.

Proof. The latter part of the proof of Proposition 2.11 can be adapted to this situation. The full proof can be found in [28 Proposition 3.7]. □

Since we aim to mimic the situation of Gabor analysis, which we will treat in [Section 4] the positive linear functional which we localize our Morita equivalence bimodule with respect to will have a particular form. In particular it will be a faithful trace. For unital Morita equivalent $C^*$-algebras $A$ and $B$ Rieffel showed in [26] that there is a bijection between non-normalized finite traces on $A$ and non-normalized finite traces on $B$ under which to a trace $\text{tr}_B$ on $B$ there is an associated trace $\text{tr}_A$ on $A$ satisfying

\[ \text{tr}_A(\ast\langle f, g \rangle) = \text{tr}_B((g, f)\ast) \tag{2.3} \]

for all $f, g \in E$. Here $E$ is the Morita equivalence bimodule. We will in the sequel almost always consider $A$ or $B$ unital, and so instead we will suppose the existence of a finite faithful trace on one $C^*$-algebra (the unital one) and induce a possibly unbounded trace on the other $C^*$-algebra. The following was proved in [2 Proposition 2.7] and ensures that this procedure works.

Proposition 2.15. Let $E$ be an $A$-$B$-equivalence bimodule, and suppose $\text{tr}_B$ is a faithful finite trace on $B$. Then the following hold:

i) There is a unique lower semi-continuous trace on $A$, denoted $\text{tr}_A$, for which

\[ \text{tr}_A(\ast\langle f, g \rangle) = \text{tr}_B((g, f)\ast) \tag{2.4} \]

for all $f, g \in E$. Moreover, $\text{tr}_A$ is faithful, and densely defined since it is finite on $\text{span}\{\ast\langle f, g \rangle : f, g \in E\}$. Setting

\[ \langle f, g \rangle_{\text{tr}_A} = \text{tr}_A(\ast\langle f, g \rangle), \quad \langle f, g \rangle_{\text{tr}_B} = \text{tr}_B((g, f)\ast), \tag{2.5} \]

for $f, g \in E$ defines inner products on $E$, with $\langle f, g \rangle_{\text{tr}_A} = \langle f, g \rangle_{\text{tr}_B}$ for all $f, g \in E$. Consequently, the Hilbert space obtained by completing $E$ in the norm $\|f\|' = \text{tr}_A(\ast\langle f, f \rangle)^{1/2}$ is just the localization of $E$ with respect to $\text{tr}_B$. \]
ii) If $E$ and $F$ are equivalence $A$-$B$-bimodules, then every adjointable $A$-linear operator $E \to F$ has a unique extension to a bounded linear operator $H_E \to H_F$. Furthermore, the map $\text{End}_A(E,F) \to \text{End}(H_E,H_F)$ given by sending $T$ to its unique extension is a norm-decreasing linear map of Banach spaces. Finally, if $E = F$, the map $\text{End}_A(E) \to \mathfrak{B}(H_E)$ is an isometric $*$-homomorphism of $C^*$-algebras.

If both $C^*$-algebras are unital then the induced trace is also a finite trace as in [26], see [2] p. 8.

**Convention 2.16.** We have the following as a standing assumption for the rest of the manuscript unless otherwise specified: Suppose we have a faithful trace $\text{tr}_B$ on a unital $C^*$-algebra $B$, and an $A$-$B$-equivalence bimodule $E$. If $A$ is a unital $C^*$-algebra, we pick the normalization of $\text{tr}_B$ such that $\text{tr}_A$ becomes a state. That is, we normalize the trace on the $C^*$-algebra on the left in the Morita equivalence.

### 3. Gabor Bimodules

#### 3.1. The single generator case.

Throughout this section we discuss properties of equivalence bimodules of the following type.

**Definition 3.1.** Let $A$ and $B$ be $C^*$-algebras where $B$ is assumed to have a unit and is equipped with a faithful finite trace $\text{tr}_B$. We define a left Gabor bimodule to be a quadruple

$$(A, B, E, \text{tr}_B)$$

where $E$ is an $A$-$B$-equivalence bimodule.

We define a right Gabor bimodule analogously, that is, it is a quadruple

$$(A, B, E, \text{tr}_A)$$

where $A$ is unital, $\text{tr}_A$ is a faithful finite trace on $A$, and $E$ is an $A$-$B$-equivalence bimodule.

**Remark 3.2.** By [Proposition 2.15] we may always induce a possibly unbounded trace $\text{tr}_A$ on $A$ given a left Gabor bimodule $(A, B, E, \text{tr}_B)$. Indeed, this will be of great importance in the sequel, and we will use $\text{tr}_A$ without mentioning that it is induced by $\text{tr}_B$. Likewise with $\text{tr}_B$ if we treat the case of right Gabor bimodules.

We are also interested in Gabor bimodules possessing some regularity, see Section 4.

**Definition 3.3.** A left Gabor bimodule with regularity is a septuple

$$(A, B, E, \text{tr}_B, A, B, \mathcal{E})$$

such that

1. $(A, B, E, \text{tr}_B)$ is a left Gabor bimodule.
2. $A \subset A$ and $B \subset B$ are dense Banach $*$-subalgebras.
3. $\mathcal{E} \subset E$ is an $A$-$B$-pre-equivalence bimodule.
4. $B$ is spectral invariant in $B$ with the same unit.

We define a right Gabor bimodule with regularity analogously.

The rest of this section will be devoted to exploring properties of Gabor bimodules, mostly left Gabor bimodules. In Section 4 we show that Gabor bimodules over twisted group $C^*$-algebras for LCA groups are Rieffel’s Heisenberg modules and provide a different approach to Gabor analysis. We start with some basic definitions from [12]. We restrict to a single generator in this section, and extend the results to finitely many generators in
Section 3.2. Indeed, we will see in that section that even the case of finitely many generators can be reduced to the case of a single generator of an associated Morita equivalence bimodule.

Definition 3.4. For \( g \in E \) we define the analysis operator by
\[
\Phi_g : E \to A \quad f \mapsto \langle f, g \rangle,
\]
and the synthesis operator:
\[
\Psi_g : A \to E \quad a \mapsto a \cdot g.
\]

An elementary computation shows that \( \Phi_g^* = \Psi_g \).

Remark 3.5. As \( E \) is an \( A\)-\( B\)-bimodule, we could just as well have defined the analysis operator and the synthesis operator with respect to the \( B\)-valued inner product. Indeed we will need this later, but it will then be indicated by writing \( \Phi_g^B \). Unless otherwise indicated the analysis operator and synthesis operator will be with respect to the left inner product module structure.

Definition 3.6. For \( g, h \in E \) we define the frame-like operator \( \Theta_{g,h} \) to be
\[
\Theta_{g,h} : E \to E \quad f \mapsto \langle f, g \rangle h.
\]
In other words, \( \Theta_{g,h} = \Psi_h \Phi_g = \Phi_h^* \Phi_g \). The frame operator of \( g \) is the operator
\[
\Theta_g := \Theta_{g,g} = \Phi_g^* \Phi_g : E \to E \quad f \mapsto \langle f, g \rangle g.
\]

Remark 3.7. The module frame operator \( \Theta_g \) is a positive operator since \( \Theta_g = \Phi_g^* \Phi_g \).

Definition 3.8. We say \( g \in E \) generates a (single) module frame for \( E \) if \( \Theta_g \) is an invertible operator \( E \to E \). Equivalently, there exist constants \( C, D > 0 \) such that
\[
C \langle f, f \rangle \leq \langle f, g \rangle \langle g, f \rangle \leq D \langle f, f \rangle,
\]
holds for all \( f \in E \).

Remark 3.9. When \( \{g\} \) is a module frame for \( E \), \( \Theta_g \) is a positive invertible operator on \( E \).

What follows will largely be a study of \( g \) and \( h \) in \( E \) such that \( \Theta_{g,h} \) is invertible, and how this relates to inner product inequalities for a localization \( H_E \) of \( E \). Our interest in this question is due to the fact that in certain cases module frames can be localized to obtain Hilbert space frames, see Section 4. We begin with a result which generalizes the Wexler-Raz biorthogonality condition for Gabor frames, which we also look at in Section 4.

Proposition 3.10 (Wexler-Raz for Gabor modules). Let \( g, h \in E \). Then \( f = \Theta_{g,h} f = \Theta_{h,g} f \) for all \( f \in E \) if and only if \( B \) is unital and \( \langle g, h \rangle = \langle h, g \rangle = 1_B \).

Proof. Suppose \( f = \Theta_{g,h} f = \Theta_{h,g} f \) for all \( f \in E \). Then \( E \) is a finitely generated projective \( A\)-module as it is generated by \( g \), and we can use \( g \) and \( h \) to make the maps \( r \\text{ and } s \) from the proof of Proposition 2.11. Hence \( B \cong \mathbb{K}_A(E) = \text{End}_A(E) \), and as \( \text{End}_A(E) \) is unital, we deduce \( B \) is unital. By Morita equivalence
\[
f = \Theta_{g,h} f = \langle f, g \rangle h = f \langle g, h \rangle = f \langle g, h \rangle.
\]
for all $f \in E$. Since $B$ acts faithfully on $E$ we deduce $\langle g, h \rangle_\ast = 1_B$. Then also

$$\langle h, g \rangle_\ast = \langle h, g \rangle_\ast^\ast = 1_B^\ast = 1_B.$$  

Conversely, suppose $B$ is unital and $\langle g, h \rangle_\ast = \langle h, g \rangle_\ast = 1_B$. Then

$$f = f1_B = f(g, h)_\ast = \ast(f, g)h = \Theta_{g, h}f,$$

and

$$f = f1_B = f(h, g)_\ast = \ast(f, h)g = \Theta_{h, g}f,$$

which finishes the proof. \hfill \Box

The following result showcases a duality between certain $A$-submodules of $E$ and $B$-submodules of $E$, which is very particular to our setting of Morita equivalence bimodules.

**Proposition 3.11.** For any $g, h \in E$ the following two statements are equivalent.

(i) $\ast(f, h)g = f$ for all $f \in \overline{Ag}$.

(ii) $f = g(h, f)_\ast$ for all $f \in \overline{gB}$.

**Proof.** Suppose first $\ast(f, h)g = f$ for all $f \in \overline{Ag}$. By Morita equivalence of $A$ and $B$,

$$f = f(h, g)_\ast$$

for all $f \in \overline{Ag}$, hence $\langle h, g \rangle_\ast$ fixes all elements in $\overline{Ag}$. In particular, since $A$ has an approximate unit, $g \in \overline{Ag}$, so $g(h, g)_\ast = g$. Now let $f' \in gB$. We then write $f' = gb$ for some $b \in B$, and so we deduce

$$g(h, f')_\ast = g(h, gb)_\ast = g(h, g)_\ast b = gb = f',$$

since $g(h, g)_\ast = g$ by the above. We extend the reconstruction formula to all of $\overline{gB}$ by continuity. The proof of the converse is completely analogous. \hfill \Box

In the special case where Proposition 3.11 (i) holds for all $f \in E$ we get another reconstruction formula. Note the (subtle) difference in the placement of $g$ and $h$ in the statement compared to statement (ii) in the preceding proposition.

**Proposition 3.12.** Let $g, h \in E$ be so that $\ast(f, h)g = f$ for all $f \in E$. Then

$$f = h(g, f)_\ast$$

for all $f \in \overline{hB}$.

**Proof.** Suppose that $\ast(f, h)g = f$ for all $f \in E$. Then $E$ is finitely generated and projective as an $A$-module as before, since it is singly generated by $g$, and we may use $g$ and $h$ to make the maps $r$ and $s$ from the proof of Proposition 2.11. Hence $B \cong \mathbb{K}_A(E) = \text{End}_A(E)$ and $B$ is unital. We may rewrite the equality to $f = f(h, g)_\ast$ for all $f \in E$, which implies $\langle h, g \rangle_\ast = 1_B$ as $B$ acts faithfully on $E$. But then

$$\langle g, h \rangle_\ast = \langle h, g \rangle_\ast^\ast = 1_B^\ast = 1_B$$

as well. Then if we let $f \in hB$ we may write $f = hb$ for some $b \in B$, and we get

$$h(g, f)_\ast = h(g, hb)_\ast = h(g, h)_\ast b = h1_Bb = hb = f.$$  

We extend the reconstruction formula to $\overline{hB}$ by continuity. \hfill \Box

Note that in the setting of Proposition 3.11 we may of course interchange $g$ and $h$. However the subspaces $\overline{Ah}$ and $\overline{Ag}$ do not need to coincide. We may however guarantee $\overline{Ag} = \overline{Ah}$ when $h$ has a special form.
Lemma 3.13. Let \( g \in E \) be such that \( \Theta_g|_{\overline{Ag}} \) is invertible as a map \( \overline{Ag} \to \overline{Ag} \). For \( h = \Theta_g|_{\overline{Ag}}^{-1} \) we have \( \overline{Ag} = \overline{Ah} \).

Proof. Let \( f \in \overline{Ag} \). As \( \Theta_g \), and hence also \( \Theta_g|_{\overline{Ag}} \), is an \( A \)-module operator, so is \( \Theta_g|_{\overline{Ag}}^{-1} \). Thus we get

\[
f = \Theta_g|_{\overline{Ag}}^{-1} \Theta_g f = \Theta_g|_{\overline{Ag}}^{-1}(\langle f, g \rangle g) = \langle f, g \rangle \Theta_g|_{\overline{Ag}}^{-1} g = \langle f, g \rangle h \in \overline{Ah}.
\]

Hence we have \( \overline{Ag} \subseteq \overline{Ah} \). Also \( g \in \overline{Ag} \) as \( A \) has a left approximate unit, and as \( \Theta_g \) is invertible as a map \( \overline{Ag} \to \overline{Ag} \) it follows that \( h = \Theta_g|_{\overline{Ag}}^{-1} g \in \overline{Ag} \). Hence \( \overline{Ag} = \overline{Ah} \). \( \square \)

In the remaining part of the article we focus mostly on the case where \( \Theta_{g,h} \) is invertible as a map \( E \to E \).

Definition 3.14. If \( g \in E \) is such that \( \Theta_g \) is invertible, then \( h = \Theta_g^{-1} g \) is called the canonical dual atom of \( g \).

Remark 3.15. Note that if \( g \) is such that \( \Theta_g : E \to E \) is invertible, then \( \overline{Ag} = E \). To see this, let \( f \in E \). Then

\[
f = \Theta_g \Theta_g^{-1} f = \Theta_g^{-1}(\Theta_g^{-1} f, g) g \in \overline{Ag}.
\]

Given \( g \in E \) such that \( \Theta_g : E \to E \) is invertible and with \( h = \Theta_g^{-1} g \) the canonical dual atom, one may ask what the canonical dual atom of \( h \) is. The following lemma tells us that it is exactly what one would expect from Hilbert space frame theory.

Lemma 3.16. Let \( g \in E \) be such that \( \Theta_g : E \to E \) is invertible, and let \( h = \Theta_g^{-1} g \). Then \( \Theta_h g = h \). Moreover, \( \Theta_h : E \to E \) is invertible with \( \Theta_h^{-1} = \Theta_g \), and the canonical dual atom of \( h \) is \( g \).

Proof. With \( h = \Theta_g^{-1} g \) the identity \( \Theta_h g = h \) is established as follows:

\[
\Theta_h g = \langle g, h \rangle h = \langle g, \Theta_g^{-1} g \rangle \Theta_g^{-1} g
= \Theta_g^{-1} \langle \Theta_g^{-1} g, g \rangle \Theta_g^{-1} g
= \Theta_g^{-1} g = h.
\]

We proceed to show that \( \Theta_h \) is invertible. By Lemma 3.13, \( \overline{Ag} = \overline{Ah} \), and we know \( \overline{Ag} = E \). Now let \( f \in Ag \), and write \( f = ag \) for some \( a \in A \). We then have

\[
\Theta_h \Theta_h f = \Theta_h (\langle f, h \rangle h) = \Theta_h (\langle ag, h \rangle h)
= a \Theta_h (\langle g, \Theta_g^{-1} g \rangle \Theta_g^{-1} g)
= a \Theta_h (\langle g, \Theta_g^{-1} g \rangle \Theta_g^{-1} g)
= a \Theta_g^{-1} (\langle g, g \rangle g) g
= ag.
\]

Likewise we have

\[
\Theta_h \Theta_g f = \Theta_h \Theta_g (ag) = \Theta_h (\langle g, ag \rangle)
= a \Theta_h (\langle g, g \rangle g) h
= a \Theta_h (\langle g, g \rangle g) h
= a \Theta_g^{-1} (\langle g, g \rangle g) g
= a \Theta_g^{-1} (\langle g, g \rangle g) g
= ag.
\]


\(Ag\) is dense in \(E\), and by extending the reconstruction formulas to all of \(E\) by continuity it follows that \(\Theta^{-1}_h = \Theta_g\). Then the canonical dual atom of \(h\) is

\[\Theta^{-1}_h h = \Theta_g h = \langle h, g \rangle g = \Theta_g \Theta^{-1}_g g = g,\]

which proves the result. \(\square\)

The following proposition tells us that \(g\) and \(\Theta^{-1}_g g\) then indeed have the desired properties as described in Proposition 3.11

**Proposition 3.17.** Let \(g \in E\) be such that \(\Theta_g\) is invertible and let \(h = \Theta^{-1}_g g\). Then

1. \(\langle f, g \rangle h = f\) for all \(f \in E = \overline{Ah} = \overline{Ag}\).
2. \(f' = h(g, f')\) for all \(f' \in \overline{hB}\).
3. \(\langle f, h \rangle g = f\) for all \(f \in E = \overline{Ah} = \overline{Ag}\).

**Proof.** Note first that \(E = \overline{Ag} = \overline{Ah}\) by Lemma 3.13. Now, let \(f \in Ah\) and write \(f = ah\) for some \(a \in A\). Then we have

\[\langle f, g \rangle h = \langle ah, g \rangle h = a \langle h, g \rangle h = a \langle \Theta^{-1}_g g, g \rangle h = a \langle g, \Theta^{-1}_g g \rangle h = a \langle g, h \rangle h = a \Theta_h g = ah = f,\]

where we have used \(\Theta_h g = h\), which holds by Lemma 3.16. We extend the reconstruction formula by continuity so it is valid for all \(f \in E\). By Proposition 3.11 this also implies \(f' = h(g, f')\) for all \(f' \in \overline{hB}\). Hence items (1) and (2) are true. To show (3), let \(f \in E\). Then

\[\langle f, h \rangle g = \langle f, \Theta^{-1}_g g \rangle g = \langle \Theta^{-1}_g f, g \rangle g = \Theta_g \Theta^{-1}_g f = f,\]

so item (3) is also true. \(\square\)

We may also prove the following additional reconstruction formula when \(h\) is the canonical dual atom. Note the (subtle) difference in where \(h\) and \(g\) are in the reconstruction formula compared to Proposition 3.11

**Proposition 3.18.** Let \(g \in E\) be such that \(\Theta_g\) is invertible, and let \(h = \Theta^{-1}_g g\). Then \(f = h(g, f)\) for all \(f \in \overline{gB}\) and \(f' = g(h, f')\) for all \(f' \in \overline{hB}\). As a consequence, \(\overline{gB} = \overline{hB}\).

**Proof.** Suppose \(f\) in \(\overline{gB}\). For \(g\) and \(h\) as stated, we have \(f = \langle f, h \rangle g\) for all \(f \in E\). By Proposition 3.11 we then have \(f = g(h, f)\) for all \(f \in \overline{gB}\). Then

\[f = g(h, f) = \langle g, h \rangle f = \langle g, \Theta^{-1}_g g \rangle f = \langle \Theta^{-1}_g g, g \rangle f = h(g, f)\]

The second statement follows from noting that our assumptions imply \(\overline{Ag} = \overline{Ah} = E\) by Lemma 3.13 and the fact that the canonical dual of \(h\) is \(g\) by Lemma 3.16. Then we may simply interchange \(g\) and \(h\) in the argument for the first assertion.

Lastly we prove \(\overline{gB} = \overline{hB}\). We know \(g \in \overline{gB}\) as \(B\) has an approximate unit, so

\[g = \Theta_g h = h(g, g) \in \overline{hB}\]

Likewise, \(h \in \overline{hB}\) and so

\[h = \Theta_h g = g(h, h) \in \overline{gB}\]

This finishes the proof. \(\square\)
There is a correspondence between projections in Morita equivalent $C^*$-algebras, see for example [28]. We formulate the following variant. Let $E$ be an $A$-$B$-equivalence bimodule, and let $B$ be unital. Then there is a way of constructing idempotents in $A$. This is the content of the following proposition.

**Proposition 3.19.** Let $E$ be an $A$-$B$-equivalence bimodule between a $C^*$-algebra $A$ and a unital $C^*$-algebra $B$. If $g, h \in E$ are such that $\langle g, h \rangle_\bullet = 1_B$, then $\bullet \langle g, h \rangle$ is an idempotent in $A$. In particular, the canonical dual atom $h = \Theta_g^{-1}g$ yields a projection $\bullet \langle g, h \rangle$ in $A$.

*Proof.* From $\langle g, h \rangle_\bullet = 1_B = 1_B^* = \langle h, g \rangle_\bullet$, we get

$$\bullet \langle g, h \rangle \bullet \langle g, h \rangle = \bullet (\bullet \langle g, h \rangle g, h) = \bullet (g(h, g) \bullet \langle g, h \rangle, h) = \bullet (g \cdot 1_B, h) = \bullet \langle g, h \rangle,$$

so $\bullet \langle g, h \rangle$ is an idempotent in $A$. If $h = \Theta_g^{-1}g$, we also have

$$\bullet \langle g, h \rangle = \bullet \langle g, \Theta_g^{-1}g \rangle = \bullet (\Theta_g^{-1}g, g) = \bullet \langle h, g \rangle = \bullet \langle g, h \rangle^\ast,$$

so $\bullet \langle g, h \rangle$ is a projection in $A$. \hfill \Box

One of the cornerstones of Gabor analysis is the duality principle, see for example [9, 19, 29]. One of the main intentions of this investigation is a reformulation of this duality principle in our module framework. To this end we introduce the following operator. For an element $g \in E$ we define the $B$-coefficient operator by

$$\Phi_g^B : E \to B$$

$$f \mapsto \langle g, f \rangle_\bullet.$$

Note that this operator is $B$-adjointable with adjoint

$$\Phi_g^B \ast b \mapsto g \cdot b.$$

We are now in the position to state and prove the module version of the duality principle.

**Proposition 3.20** (Module Duality Principle). Let $g \in E$. The following are equivalent.

1. $\Theta_g : E \to E$ is invertible.
2. $\Phi_g^B(\Phi_g^B)^\ast : B \to B$ is an isomorphism.

*Proof.* We show that both statements are equivalent to $\langle g, g \rangle_\bullet$ being invertible in $B$. Suppose $\Theta_g$ is invertible. Then $E$ is finitely generated and projective as an $A$-module, as we can make the maps $r$ and $s$ from the proof of Proposition 2.11 using $g$ and $\Theta_g^{-1}g$. Thus $B$ is unital. As

$$\Theta_g f = f \langle g, g \rangle_\bullet,$$

statement (1) is equivalent to $\langle g, g \rangle_\bullet$ being invertible in $B$. On the other hand,

$$\Phi_g^B(\Phi_g^B)^\ast b = \Phi_g^B(g \cdot b) = \langle g, g \cdot b \rangle_\bullet = \langle g, g \rangle_\bullet b.$$

Since $\Phi_g^B(\Phi_g^B)^\ast \in \text{End}_B(B)$ and $B$ is an ideal in $\text{End}_B(B)$, statement (2) implies that $B$ is unital and the statement is equivalent to $\langle g, g \rangle_\bullet$ being invertible in $B$. \hfill \Box

In Gabor analysis one is often concerned with the regularity of the atoms generating a Gabor frame, see Section 3. In case $g$ is so that $\Theta_g$ is invertible on all of $E$ with $g \in \mathcal{E}$, and $B \subset E$ is spectral invariant Banach $*$-subalgebra with the same unit as $B$, the canonical dual atom has the following important property.

**Proposition 3.21.** Let $E$ be an $A$-$B$-equivalence bimodule, with an $A$-$B$-pre-equivalence bimodule $\mathcal{E} \subset E$. Suppose $B \subset E$ is spectral invariant with the same unit. If $g \in \mathcal{E}$ is such that $\Theta_g : E \to E$ is invertible, then the canonical dual $\Theta_g^{-1}g$ is in $\mathcal{E}$ as well.
Proof. For \( f \in E \) we have
\[
\Theta_g f = \bullet (f, g) g = f (g, g) \bullet.
\]
We deduce that \( \langle g, g \rangle \bullet \) is invertible in \( B \) and
\[
\Theta_g^{-1} g = g (g, g)^{-1} \bullet.
\]
But as \( g \in \mathcal{E} \) we have \( \langle g, g \rangle \bullet \in \mathcal{B} \). By spectral invariance of \( \mathcal{B} \) in \( B \) it follows that \( \langle g, g \rangle^{-1} \bullet \in \mathcal{E} \). Then, since \( \mathcal{EB} \subset \mathcal{E} \), it follows that
\[
\Theta_g^{-1} g = g (g, g)^{-1} \bullet \in \mathcal{E},
\]
which is the desired assertion. \( \square \)

3.2. Extending to several generators. We extend the above theory to several generators. Indeed we will lift the \( A\text{-}B \)-equivalence bimodule \( E \) to an \( M_n(A)\text{-}M_d(B) \)-equivalence bimodule, for \( d, n \in \mathbb{N} \), and consider a type of module frame in this matrix setting. We will see in Section 4 that this generalizes \( n \)-multiwindow \( d \)-super Gabor frames of [18].

Note that we will index an \( n \times d \)-matrix by \( (i, j), i \in \mathbb{Z}_n, j \in \mathbb{Z}_d \), that is, we start indexing at 0. The reason for this is that in Section 4 we will need to incorporate the groups \( \mathbb{Z}_k, k \in \mathbb{N} \). Here \( \mathbb{Z}_k \) denotes the group \( \mathbb{Z}/(k\mathbb{Z}) \).

We will consider \( M_{n,d}(E) \) as an \( M_n(A)\text{-}M_d(B) \)-bimodule. Define an \( M_n(A) \)-valued inner product on \( M_{n,d}(E) \) by
\[
\bullet [-, -] : M_{n,d}(E) \times M_{n,d}(E) \rightarrow M_n(A)
\]
\[
(f, g) \mapsto \sum_{k \in \mathbb{Z}_d} \begin{pmatrix}
\langle f_{0,k}, 0, g_{0,k} \rangle & \langle f_{0,k}, 1, g_{1,k} \rangle & \cdots & \langle f_{0,k}, n-1, g_{n-1,k} \rangle \\
\langle f_{1,k}, 0, g_{0,k} \rangle & \langle f_{1,k}, 1, g_{1,k} \rangle & \cdots & \langle f_{1,k}, n-1, g_{n-1,k} \rangle \\
\vdots & \vdots & \ddots & \vdots \\
\langle f_{n-1,k}, 0, g_{0,k} \rangle & \langle f_{n-1,k}, 1, g_{1,k} \rangle & \cdots & \langle f_{n-1,k}, n-1, g_{n-1,k} \rangle
d\end{pmatrix}.
\]

The action of \( M_n(A) \) on \( M_{n,d}(E) \) is defined in the natural way, that is
\[
(a f)_{i,j} = \sum_{k \in \mathbb{Z}_n} a_{i,k} f_{k,j},
\]
for \( a \in M_n(A) \) and \( f \in M_{n,d}(E) \). Likewise we define an \( M_d(B) \)-valued inner product on \( M_{n,d}(E) \) in the following way
\[
[-,-] : M_{n,d}(E) \times M_{n,d}(E) \rightarrow M_d(B)
\]
\[
(f, g) \mapsto \sum_{k \in \mathbb{Z}_n} \begin{pmatrix}
\langle f_{k,0}, 0, g_{k,0} \rangle & \langle f_{k,0}, 0, g_{k,1} \rangle & \cdots & \langle f_{k,0}, 0, g_{k,d-1} \rangle \\
\langle f_{k,1}, 0, g_{k,0} \rangle & \langle f_{k,1}, 0, g_{k,1} \rangle & \cdots & \langle f_{k,1}, 0, g_{k,d-1} \rangle \\
\vdots & \vdots & \ddots & \vdots \\
\langle f_{k,d-1}, 0, g_{k,0} \rangle & \langle f_{k,d-1}, 0, g_{k,1} \rangle & \cdots & \langle f_{k,d-1}, 0, g_{k,d-1} \rangle
d\end{pmatrix}.
\]

The right action of \( M_d(B) \) on \( M_{n,d}(E) \) is defined by
\[
(f b)_{i,j} = \sum_{k \in \mathbb{Z}_d} f_{i,k} b_{k,j}
\]
for \( f \in M_{n,d}(E) \) and \( b \in M_d(B) \).

With this setup, \( M_{n,d}(E) \) becomes an \( M_n(A)\text{-}M_d(B) \)-equivalence bimodule. Indeed it is not hard to verify the three conditions of Definition 2.9. Verifying conditions ii) and iii) is a matter of verifying the statements in each matrix element using that \( E \) is an \( A\text{-}B \)-equivalence bimodule. Verifying condition i) is a matter of getting density in each matrix entry by choosing elements of \( M_{n,d}(E) \) in the correct way. Namely, if we want to get the
elements in place \((i, j)\) in \(M_n(A)\), then we may for example pair elements of \(M_{n,d}(E)\) with nonzero entry only in place \((i, k)\) with elements of \(M_{n,d}(E)\) with nonzero entry only in place \((j, k)\), for some \(k \in \mathbb{Z}_d\). The analogous procedure holds for \(M_d(B)\). Density then follows by \(\langle E, E \rangle = A\) and \(\langle E, E \rangle \bullet = B\). In particular, we have for \(f, g, h \in M_{n,d}(E)\) that

\[
\text{3.16} \quad \bullet[f, g]h = f[g, h] \bullet,
\]

and also

\[
\text{3.17} \quad M_n(A) = \mathbb{K}_{M_d(B)}(M_{n,d}(E)),
\]

\[
M_d(B) = \mathbb{K}_{M_n(A)}(M_{n,d}(E)).
\]

Also, since the new inner products are defined using the inner products \(\langle -, - \rangle\) and \(\langle -, - \rangle \bullet\), we see that in case we have Banach \(*\)-subalgebras \(A \subset A\) and \(B \subset B\), as well as an \(A\)-\(B\)-subbimodule \(E \subset E\) as above, we get

\[
\text{3.19} \quad \bullet[M_{n,d}(E), M_{n,d}(E)] \subset M_n(A), \quad [M_{n,d}(E), M_{n,d}(E)] \bullet \subset M_d(B),
\]

as well as

\[
\text{3.20} \quad M_n(A)M_{n,d}(E) \subset M_{n,d}(E), \quad M_{n,d}(E)M_d(B) \subset M_d(E).
\]

We wish to reduce the matrix algebra case to the Gabor bimodule case of Section 3.1 so we need to guarantee that spectral invariance of Banach \(*\)-subalgebras lifts to matrices. For convenience we include the following result.

**Lemma 3.22** \((30)\). If \(B\) is a spectral invariant Banach subalgebra of a Banach algebra \(B\), then \(M_m(B)\) is a spectral invariant Banach subalgebra of \(M_m(B)\), for all \(m \in \mathbb{N}\).

For \(g \in M_{n,d}(E)\) we define as in Section 3.1 the analysis operator

\[
\text{3.21} \quad \Phi_g : M_{n,d}(E) \to M_n(A)
\]

\[
f \mapsto \bullet[f, g]
\]

which has as adjoint the operator

\[
\text{3.22} \quad \Phi^*_g : M_n(A) \to M_{n,d}(E)
\]

\[
a \mapsto ag.
\]

Using these we also define the frame-like operator \(\Phi^*_h \Phi_g =: \Theta_{g,h} : M_{n,d}(E) \to M_{n,d}(E)\) by

\[
\text{3.23} \quad \Theta_{g,h}f = \bullet[f, g]h, \quad \text{for } f \in M_{n,d}(E),
\]

and the frame operator \(\Phi^*_g \Phi_g =: \Theta_g : M_{n,d}(E) \to M_{n,d}(E)\) by

\[
\text{3.24} \quad \Theta_gf = \bullet[f, g]g, \quad \text{for } f \in M_{n,d}(E).
\]

As noted in Section 3.1, \(\Theta_g\) is a positive operator.

For simplicity, and since it is the case we will most often consider, suppose in the following that \(B\) is unital with a faithful finite trace. There is then an induced (possibly unbounded) trace on \(A\) as in Section 2. We may lift these traces to the matrix algebras. Indeed, there are traces on \(M_n(A)\) and \(M_d(B)\) satisfying

\[
\text{3.25} \quad \text{tr}_{M_n(A)}(\bullet[f, g]) = \text{tr}_{M_d(B)}([g, f] \bullet)
\]
for all \( f, g \in M_{n,d}(E) \). They are given by

\[
\begin{align*}
\text{tr}_{M_n(A)}(\bullet [f,g]) &= \frac{1}{n} \sum_{i \in \mathbb{Z}_n} \text{tr}_A(\bullet [f,g]_{i,i}), \\
\text{tr}_{M_d(B)}([f,g]_\bullet) &= \frac{1}{n} \sum_{i \in \mathbb{Z}_d} \text{tr}_B([f,g]_{i,i}).
\end{align*}
\]

The trace on \( M_d(B) \) extends to a finite trace on the whole algebra, but the same might not be true for the densely defined trace on \( M_n(A) \). It is however true if \( A \), and hence also \( M_n(A) \), is unital.

**Remark 3.23.** The normalization on the traces in (3.23) is so that if \( A \) is unital, then \( \text{tr}_{M_n(A)}(1_{M_n(A)}) = 1 \), that is, \( \text{tr}_{M_n(A)} \) is a faithful tracial state. In general \( \text{tr}_{M_d(B)} \) will not be a state, even if \( M_d(B) \) is unital.

The following lemma may be verified by elementary computations.

**Lemma 3.24.** Let \( B \) be a unital \( C^* \)-algebra. If \( \text{tr}_B \) is a faithful trace on \( B \), then the induced mapping \( \text{tr}_{M_m(B)} \) on the matrix algebra \( M_m(B) \), \( m \in \mathbb{N} \), given by

\[
\text{tr}_{M_m(B)}(b) = \sum_{i \in \mathbb{Z}_m} \text{tr}_B(b_{i,i}),
\]

for \( b \in M_m(B) \) is also a faithful trace.

We summarize the preceding discussion in the following proposition which allows us to study the \( M_n(A) \)-\( M_d(B) \)-equivalence bimodule \( M_{n,d}(E) \) by studying the \( A \)-\( B \)-bimodule \( E \).

**Proposition 3.25.** Let \( (A, B, E, \text{tr}_B) \) be a left Gabor bimodule. Then for all \( n, d \in \mathbb{N} \), the quadruple

\[
(M_n(A), M_d(B), M_{n,d}(E), \text{tr}_{M_d(B)}(B))
\]

with the above defined actions, inner products, and traces is also a left Gabor bimodule. Furthermore, if \( (A, B, E, \text{tr}_A, \text{tr}_B, A, B, E) \) is a left Gabor bimodule with regularity, then for all \( n, d \in \mathbb{N} \), the septuple

\[
(M_n(A), M_d(B), M_{n,d}(E), \text{tr}_{M_d(B)}(B), M_n(A), M_d(B), M_{n,d}(E)),
\]

with the above defined actions, inner products, and traces is also a left Gabor bimodule with regularity. The analogous statements hold for right Gabor bimodules.

Since our focus is on the description of frames in equivalence bimodules for Morita equivalent \( C^* \)-algebras, we want to do this now on the matrix algebra level and thus introduce an appropriate notion of module frames for the matrix-valued equivalence bimodules.

**Definition 3.26.** Let \( g = (g_{i,j})_{i \in \mathbb{Z}_n, j \in \mathbb{Z}_d} \in M_{n,d}(E) \). We say \( g \) generates a module \((n,d)\)-matrix frame for \( E \) with respect to \( A \) if there exists \( h = (h_{i,j})_{i \in \mathbb{Z}_n, j \in \mathbb{Z}_d} \in M_{n,d}(E) \) for which

\[
(f_{r,s})_{r \in \mathbb{Z}_n, s \in \mathbb{Z}_d} = \sum_{k \in \mathbb{Z}_d} \sum_{l \in \mathbb{Z}_n} \langle f_{r,k}, g_{l,k} \rangle h_{l,s},
\]

holds for all \( f = (f_{i,j})_{i \in \mathbb{Z}_n, j \in \mathbb{Z}_d} \in M_{n,d}(E), r \in \mathbb{Z}_n \), and \( s \in \mathbb{Z}_d \).
By definition of the above Hilbert $M_n(A)$-module structure on $M_{n,d}(E)$, we see that $g \in M_{n,d}(E)$ generates a module $(n,d)$-matrix frame for $E$ with respect to $A$ if and only if there is $h \in M_{n,d}(E)$ such that
\[(3.26) \quad f = \bullet[f, g]h\]
for all $f \in M_{n,d}(E)$. In other words, $g$ generates a module $(n,d)$-matrix frame for $E$ with respect to $A$ if and only if $g$ generates a single module frame for $M_{n,d}(E)$ with respect to $M_n(A)$. When $\bullet[3.26]$ is satisfied $M_{n,d}(E)$ is finitely generated projective as an $M_n(A)$-module, so as before it follows by Proposition 2.11 that $M_d(B)$ is unital. Then $B$ is also unital. By the identity
\[f = \bullet[f, g]h = f[g, h]\star,\]
we deduce that $\bullet[3.26]$ is satisfied if and only if $M_d(B)$ is unital and $[g, h]_\star = 1_{M_d(B)}$.

Remark 3.27. By the above discussion it follows that finding module $(n,d)$-matrix frames for the $A$-$B$-equivalence bimodule $E$ is the same as finding $g, h \in M_{n,d}(E)$ such that $[g, h]_\star = 1_{M_d(B)}$. That is, it is the same as finding single module frames for $M_{n,d}(E)$ as an $M_n(A)$-module. By Lemma 3.22 the corresponding statement is true of finding module frames with regularity. Hence all results of Section 3.1 can be carried over to the setup in this section.

Even though all results of Section 3.1 lift to the induced matrix algebra setup, we want to discuss explicitly two results relating the lifted traces. We show in Section 4 that these two results extend the density theorems of Gabor analysis to Gabor bimodules. Since we in Theorem 3.28 talk about left Gabor bimodules and in Theorem 3.29 talk about right Gabor bimodules, we will for the sake of avoiding confusion not have any convention on the normalization of traces in the two results.

**Theorem 3.28.** Let $(A, B, E, \text{tr}_B)$ be a left Gabor bimodule. If, in addition, $A$ is unital and $g \in M_{n,d}(E)$ is such that $\Theta_g : E \to E$ is invertible, then
\[(3.27) \quad d\text{tr}_B(1_B) \leq n\text{tr}_A(1_A).\]

**Proof.** The assumption that $\Theta_g$ is invertible implies $[g, g]_\star$ is invertible. Then
\[(3.28) \quad u = \Theta_g^{-1}g = g[g, g]^{-1}_\star\]
is the canonical dual frame for $M_{n,d}(E)$. We have $[g, u]_\star = [u, g]_\star = 1_{M_d(B)}$, and by Proposition 3.19 $[g, u]$ is a projection in $M_n(A)$. Using Lemma 3.21 and $\bullet[g, u] \leq 1_{M_n(A)}$ in $M_n(A)$, as well as (3.23), we get
\[
d\text{tr}_B(1_B) = n \cdot \frac{1}{n} \sum_{i=1}^{d} \text{tr}_B(1_B) = n \text{tr}_{M_d(B)}(1_{M_d(B)}) = n \text{tr}_{M_d(B)}([u, g]_\star)
= n \text{tr}_{M_n(A)}([g, u]) \leq n \text{tr}_{M_n(A)}(1_{M_n(A)}) = n \cdot \frac{1}{n} \sum_{i=1}^{n} \text{tr}_A(1_A) = n \text{tr}_A(1_A).
\]

**Theorem 3.29.** Let $(A, B, E, \text{tr}_A)$ be a right Gabor bimodule. If, in addition, $B$ is unital and $g \in M_{n,d}(E)$ is such that $\Phi_g \Phi_g^* : M_n(A) \to M_n(A)$ is an isomorphism, then
\[(3.29) \quad d\text{tr}_B(1_B) \geq n\text{tr}_A(1_A).\]
Proof. The assumptions imply $\bullet [g, g]^{-1} \in M_n(A)$, so it follows as in Section 3.1 that

$$1_{M_n(A)} = \bullet [g, g]^{-1} \bullet [g, g] = \bullet [\bullet [g, g]^{-1} g, g],$$

and $[\bullet [g, g]^{-1} g, g]_\bullet$ is a projection in $M_d(B)$ by Proposition 3.19. Since $B$ is unital, then, using Lemma 3.24 together with $[\bullet [g, g]^{-1} g, g]_\bullet \leq 1_{M_d(B)}$ in $M_d(B)$, as well as (3.23), we get

$$n \text{tr}_A(1_A) = n \cdot \frac{1}{n} \sum_{i=1}^n \text{tr}_A(1_A) = n \text{tr}_{M_n(A)}(1_{M_n(A)}) = n \text{tr}_{M_n(A)}(\bullet [\bullet [g, g]^{-1} g, g])$$

$$= n \text{tr}_{M_d(B)}([g, \bullet [g, g]^{-1} g]_\bullet) \leq n \text{tr}_{M_d(B)}(1_{M_d(B)})$$

$$= n \cdot \frac{1}{n} \sum_{i=1}^d \text{tr}_B(1_B) = d \text{tr}_B(1_B).$$

□

3.3. From a Gabor bimodule to its localization. In [21] the existence of multi-window Gabor frames for $L^2(\mathbb{R}^d)$ with windows in Feichtinger’s algebra was proved through considerations on a related Hilbert $C^*$-module. Furthermore, in [22] projections in noncommutative tori were constructed from Gabor frames with sufficiently regular windows. Thus being able to pass from an equivalence bimodule $E$ to a localization $H_E$ and back is quite important, and we dedicate this section to results on this procedure. We will interpret this in terms of standard Gabor analysis in Section 4 and we will explain how $L^2(G)$, for $G$ a second countable LCA group, relates to $H_E$ for specific modules $E$ which arise in the study of twisted group $C^*$-algebras.

We denote by $(-, -)_E$ the inner product on the localization of $E$ in $\text{tr}_A$. Concretely, we have $(f, g)_E = \text{tr}_A(\bullet (f, g))$.

Proposition 3.30. Let $(A, B, E, \text{tr}_B)$ be a left Gabor bimodule, and let $g \in E$. Then there exists an $h \in E$ such that we have $\bullet (f, g) h = f$ for all $f \in E$ if and only if there exist constants $C, D > 0$ such that

$$(3.30) \quad C(f, f)_E \leq (f(g, g) \bullet, f)_E \leq D(f, f)_E$$

for all $f \in H_E$. In other words, $g$ is a module frame for $E$ if and only if the inequalities in (3.30) are satisfied for some $C, D > 0$.

Proof. Suppose first that there is an $h \in E$ such that $\bullet (f, g) h = f$ for all $f \in E$. By Morita equivalence this implies

$$f = \bullet (f, g) h = f(g, h)_\bullet.$$
for all \( f \in E \). As before, this implies \( 1_B = \langle g, h \rangle \ast = \langle h, g \rangle \ast \). Since \( \text{tr}_B \) is a positive linear functional we obtain
\[
(f, f)_E = \text{tr}_A(\ast(f, f))
\]
\[
= \text{tr}_A(\ast(f \langle g, h \rangle \ast(h, g) \ast, f))
\]
\[
= \text{tr}_A(\ast(f \langle g, h \rangle \ast, f))
\]
\[
= \text{tr}_B(\langle f, f \rangle \ast \| (h, h) \| )
\]
\[
= \| (h, h) \| \text{tr}_B(\langle f, f \rangle \ast)
\]
\[
= \| (h, h) \| \text{tr}_A(\ast(f, f))
\]
\[
= \| (h, h) \| (f \langle g, g \rangle \ast, f)_E.
\]
for all \( f \in E \), where we have used Proposition 2.7 to deduce
\[
\langle g, \ast(h, h) \rangle \ast \leq \| (h, h) \| \langle g, g \rangle \ast.
\]
We then get the lower frame bound with \( C = \| (h, h) \| ^{-1} \), that is
\[
\frac{1}{\| (h, h) \| } (f, f)_E \leq (f \langle g, g \rangle \ast, f)_E
\]
for all \( f \in E \). By Proposition 2.8 all intermediate steps involve operators that extend to bounded operators on \( H_E \), so we may extend by continuity. We get the upper frame bound by use of Proposition 2.7 in the following manner
\[
(\ast(f \langle g, g \rangle \ast, f)_E = \text{tr}_A(\ast(f \langle g, g \rangle \ast, f))
\]
\[
= \text{tr}_A(\ast(f \langle g, g \rangle \ast 1/2, f \langle g, g \rangle \ast 1/2))
\]
\[
\leq \| (g, g) \| ^{1/2} \| \text{tr}_A(\ast(f, f))
\]
\[
= \| (g, g) \| \text{tr}_A(\ast(f, f))
\]
\[
= \| (g, g) \| (f \langle g, g \rangle \ast, f)_E.
\]
for all \( f \in E \). Once again all intermediate steps involve operators that extend to bounded operators on \( H_E \) by Proposition 2.8 so we may extend the result to all of \( H_E \). Thus we have shown that
\[
\frac{1}{\| (h, h) \| } (f, f)_E \leq (f \langle g, g \rangle \ast, f)_E \leq \| (g, g) \| (f, f)_E
\]
for all \( f \in H_E \).

Conversely, suppose there are \( C, D > 0 \) such that
\[
C(f, f)_E \leq (f \langle g, g \rangle \ast, f)_E \leq D(f, f)_E
\]
for all \( f \in H_E \). We wish to show that this implies there exist \( h \in E \) such that \( \ast(f, g)h = f \) for all \( f \in E \). The assumption implies that \( f \mapsto f \langle g, g \rangle \ast \) is a positive, invertible operator on \( H_E \). By Proposition 2.1 it follows that \( \langle g, g \rangle \ast \) is invertible in \( B \). Thus \( f \mapsto f \langle g, g \rangle \ast \) is a positive, invertible operator on \( E \) as well. Hence the operator
\[
\Theta_g : E \to E
\]
\[
f \mapsto \ast(f, g)g = f \langle g, g \rangle \ast
\]
is invertible with inverse
\[
\Theta_g^{-1} f = f \langle g, g \rangle \ast^{-1}.
\]
Define $h := \Theta^{-1}_g f$, and let $f \in E$ be arbitrary. Then we have
\[ \bullet(f, g)h = \bullet(f, g)\Theta^{-1}_g g = \Theta^{-1}_g (\bullet(f, g)g) = \Theta^{-1}_g \Theta_g f = f, \]
from which the result follows.

We are interested in module frames and module Riesz sequences, and their relationship to frames and Riesz sequences in Gabor analysis for LCA groups. To get results on Riesz sequences in Section 4 we need a module version of Riesz sequences which, when localized, yields the Riesz sequences we know from Gabor analysis. For this we let $A$ be unital with a faithful trace $\text{tr}_A$, and we need to localize $A$ as a Hilbert $A$-module in the trace $\text{tr}_A$. We let $(a_1, a_2)_A := \text{tr}_A(a_1 a_2^*)$. The completion of $A$ in this inner product will be denoted $H_A$, and the action of $A$ on $H_A$ is the continuous extension of the multiplication action (from the right) of $A$ on itself.

**Proposition 3.31.** Let $(A, B, E, \text{tr}_A)$ be a right Gabor bimodule, and let $g \in E$. Then $\Phi_g \Phi_g^* : A \to A$ is an isomorphism if and only if there exist $C, D > 0$ such that for all $a \in A$ it holds that
\[ C(a, a) \leq (ag, ag)_E \leq D(a, a)_A. \]

**Proof.** First suppose $\Phi_g \Phi_g^* : A \to A$ is an isomorphism. Then, since by Proposition 2.6
\[ \bullet(\langle ag, ag \rangle) = a \langle g, g \rangle a^* \leq \| \langle g, g \rangle \| a a^*, \]
we may deduce
\[ (ag, ag)_A = \text{tr}_A(\langle ag, ag \rangle) \leq \| \langle g, g \rangle \| \text{tr}_A(a a^*) = \| \langle g, g \rangle \|(a, a)_A. \]
Hence in (3.31) we may set $D = \| \langle g, g \rangle \|$. Since $\Phi_g \Phi_g^* : A \to A$ is an isomorphism and $\Phi_g \Phi_g^* a = a \langle g, g \rangle$, it follows that there is $\langle g, g \rangle^{-1} \in A$. Then
\[ (a, a)_A = \text{tr}_A(a a^*)_A = \text{tr}_A(a \langle g, g \rangle^{1/2} \langle g, g \rangle^{-1/2} a^*) \leq \| \langle g, g \rangle^{-1} \| \text{tr}_A(a \langle g, g \rangle a^*) = \| \langle g, g \rangle^{-1} \| \langle ag, ag \rangle), \]
which implies that we may set $C = \| \langle g, g \rangle^{-1} \|^{-1}$ in (3.31). All intermediate steps extend to $H_A$ by Proposition 2.8.

Suppose now that (3.31) is satisfied. The lower inequality in (3.31) tells us that for all $a \in A$,
\[ (a(\langle g, g \rangle - C), a)_A = \text{tr}_A(a(\langle g, g \rangle - C)a^*) \]
\[ = \text{tr}_A(a(\langle g, g \rangle a^*) - C \text{tr}_A(aa^*)) \]
\[ = \text{tr}_A(\langle ag, ag \rangle) - C \text{tr}_A(aa^*) \]
\[ = (ag, ag)_E - C(a, a)_A \geq 0. \]
Note that we need the upper inequality of (3.31) to extend all intermediate steps to $H_A$ via Proposition 2.8. It follows that $\langle g, g \rangle$ is a positive invertible operator on $H_A \supset A$. By Proposition 2.1 it follows that $\langle g, g \rangle$ is invertible in $A$. Then, since
\[ \Phi_g \Phi_g^* a = a \langle g, g \rangle, \]
it follows that $\Phi_g \Phi_g^* : A \to A$ is an isomorphism. \qed
Both Proposition 3.30 and Proposition 3.31 were proved for Gabor bimodules, so by Remark 3.27 the results lift to the corresponding matrix setting of Section 3.2.

Remark 3.32. Note that in the proofs of the two preceding results the upper bounds in (3.30) and (3.31) were both satisfied with $D = \| \langle g, g \rangle \|$. We will see in Section 4 that in the Gabor analysis setting, this means that all atoms coming from the Hilbert $C^*$-module are Bessel vectors for the localized frame system.

Remark 3.33. The two preceding results actually have shorter proofs using Proposition 2.1 in a more direct way, but these proofs would not give us values for $C$ and $D$ in (3.30) and (3.31), only the existence. The values of the constants are of interest on their own, see Section 4.

For use in Section 4, we introduce the following notion.

**Definition 3.34.** Let $(A, B, E, \text{tr}_A)$ be a right Gabor bimodule, and let $g \in E$. If $\Phi_g \Phi_g^* : A \to A$ is an isomorphism, we say $g$ generates a module Riesz sequence for $E$ with respect to $A$. If $h \in M_{n,d}(E)$ generates a module Riesz sequence for $M_{n,d}(E)$ with respect to $M_n(A)$, we will also say that $h$ generates a module $(n, d)$-matrix Riesz sequence for $E$ with respect to $A$.

4. Applications to Gabor analysis

In this section we show how the above results reproduce some of the core results of Gabor analysis for LCA groups. We will see how some of the cornerstones of Gabor analysis on LCA groups are trivial consequences of the above framework. Of particular interest is the reproduction of some of the main results of [18] on $n$-multiwindow $d$-super Gabor frames with windows in the Feichtinger algebra. Indeed we will show the corresponding results for localized module $(n, d)$-matrix frames, which generalize $n$-multiwindow $d$-super Gabor frames.

To present the results we will need to explain how time frequency analysis on LCA groups relates to Morita equivalence of twisted group $C^*$-algebras. In the interest of brevity, we refer the reader to [18] for a more in-depth treatment of time frequency analysis and its relation to twisted group $C^*$-algebras, and to [16] for a survey on the Feichtinger algebra.

Throughout this section, we fix a second countable LCA group $G$ and let $\hat{G}$ be its dual group. We fix a Haar measure $\mu_G$ on $G$ and normalize the Haar measure $\mu_{\hat{G}}$ on $\hat{G}$ such that the Plancherel theorem holds. By $\Lambda$ we denote a closed subgroup of the time-frequency plane $G \times \hat{G}$. The induced topologies and group multiplications on $\Lambda$ and $(G \times \hat{G})/\Lambda$ turn them into LCA groups as well, and we may equip them with their respective Haar measures. Having fixed the Haar measures on $G$, $\hat{G}$, and $\Lambda$, we will assume $(G \times \hat{G})/\Lambda$ is equipped with the unique Haar measure such that Weil’s formula holds, that is, such that for all $f \in L^1(G \times \hat{G})$ we have

$$\int_{G \times \hat{G}} f(\xi)d\mu_{G \times \hat{G}} = \int_{(G \times \hat{G})/\Lambda} \int_{\Lambda} f(\xi + \lambda)d\mu_\Lambda(\lambda)d\mu_{(G \times \hat{G})/\Lambda}(\hat{\xi}), \quad \hat{\xi} = \xi + \Lambda.$$  

In this setting we can define the size of $\Lambda$ by

$$s(\Lambda) := \int_{(G \times \hat{G})/\Lambda} 1d\mu_{(G \times \hat{G})/\Lambda}.$$  

Note that $s(\Lambda)$ is finite if and only if $\Lambda$ is cocompact in $G \times \hat{G}$. 
For any \( x \in G \) and \( \omega \in \hat{G} \) we define the translation operator (or time shift) \( T_x \) by
\[
T_x f(t) = f(t - x), \quad t \in G,
\]
and the modulation operator (or frequency shift) \( E_\omega \) by
\[
E_\omega f(t) = \omega(t)f(t), \quad t \in G.
\]
These operators are unitary on \( L^2(G) \), and satisfy the commutation relation
\[
(4.1) \quad E_\omega T_x = \omega(x)T_x E_\omega.
\]
For any \( \xi = (x, \omega) \in G \times \hat{G} \) we may then define the time-frequency shift operator
\[
(4.2) \quad \pi(\xi) = \pi(x, \omega) = E_\omega T_x.
\]
We define the 2-cocycle
\[
(4.3) \quad c : (G \times \hat{G}) \times (G \times \hat{G}) \rightarrow \mathbb{T}
\]
for \( \xi_1 = (x_1, \omega_1), \xi_2 = (x_2, \omega_2) \in G \times \hat{G} \). Note then that
\[
(4.4) \quad \pi(\xi_1)\pi(\xi_2) = c(\xi_1, \xi_2)\pi(\xi_1 + \xi_2),
\]
and
\[
(4.5) \quad \pi(\xi_1)\pi(\xi_2) = c(\xi_1, \xi_2)\pi(\xi_1, \xi_2)\pi(\xi_2)\pi(\xi_1) = c_s(\xi_1, \xi_2)\pi(\xi_1),
\]
where we have introduced the symplectic cocycle \( c_s \) by
\[
(4.6) \quad c_s : (G \times \hat{G}) \times (G \times \hat{G}) \rightarrow \mathbb{T}
\]
for all \( \xi \in G \times \hat{G} \).

Using the symplectic cocycle \( c_s \) we define for a closed subgroup \( \Lambda \subset G \times \hat{G} \) the adjoint subgroup \( \Lambda^\circ \) by
\[
(4.8) \quad \Lambda^\circ := \{ \xi \in G \times \hat{G} \mid c_s(\xi, \lambda) = 1 \quad \forall \lambda \in \Lambda \}.
\]
Then \( (\Lambda^\circ)^\circ = \Lambda \) and \( \Lambda^\circ \cong (G \times \hat{G})/\Lambda \), see for example [17]. Note that \( \Lambda \) is cocompact if and only if \( \Lambda^\circ \) is discrete. With these identifications we put on \( \Lambda^\circ \) the Haar measure such that the Plancherel theorem holds with respect to \( \Lambda^\circ \) and \( (G \times \hat{G})/\Lambda \).

We define the \textit{short time Fourier transform} with respect to \( g \in L^2(G) \) as the operator
\[
(4.9) \quad V_g : L^2(G) \rightarrow L^2(G \times \hat{G})
\]
\[
V_g f(\xi) = \langle f, \pi(\xi)g \rangle,
\]
for \( \xi \in G \times \hat{G} \). The \textit{Feichtinger algebra} \( S_0(G) \) is then defined by
\[
(4.10) \quad S_0(G) := \{ f \in L^2(G) \mid V_g f \in L^1(G \times \hat{G}) \}.
\]
A norm on \( S_0(G) \) is given by
\[
(4.11) \quad \|f\|_{S_0(G)} := \|V_g f\|_{L^1(G \times \hat{G})}, \quad \text{for some } g \in S_0(G) \setminus \{0\}.
\]
It is a nontrivial fact that all elements of \( S_0(G) \setminus \{0\} \) determine equivalent norms on \( S_0(G) \).

In case \( G \) is discrete it is known that \( S_0(G) = \ell^1(G) \) with equivalent norms. Furthermore, \( S_0(G) \) consists of continuous functions and is dense in both \( L^1(G) \) and \( L^2(G) \).
For two functions $F_1, F_2$ over $\Lambda \subset G \times \hat{G}$ we define the twisted convolution by
\begin{equation}
F_1 \ast F_2(\lambda) := \int_{\Lambda} F_1(\lambda') F_2(\lambda - \lambda') c(\lambda', \lambda - \lambda') \, d\lambda',
\end{equation}
and the twisted involution $F^*_1(\lambda) := c(\lambda, \lambda) F_1(-\lambda)$. In [18] it was shown that $S_0(\Lambda)$ is a Banach $\ast$-D-algebra for some $D > 0$ when equipped with twisted convolution, and indeed it is possible to choose an equivalent norm on $S_0(G)$ such that it becomes a Banach $\ast$-algebra. We denote the resulting Banach $\ast$-algebra by $S_0(\Lambda, c)$. Using this we may then define two Banach $\ast$-algebras
\begin{align}
\mathcal{A} &:= \{ a \in \mathbb{B}(L^2(G)) \mid a = \int_{\Lambda} a(\lambda) \pi(\lambda) \, d\lambda, a \in S_0(\Lambda) \}, \\
\mathcal{B} &:= \{ b \in \mathbb{B}(L^2(G)) \mid b = \int_{\Lambda^0} b(\lambda^0) \pi(\lambda^0)^* \, d\lambda^0, b \in S_0(\Lambda^0) \}.
\end{align}
Note that $\mathcal{A} \cong S_0(\Lambda, c)$ and $\mathcal{B} \cong S_0(\Lambda^0, \overline{c})$ via the natural maps. We will use these identifications without mention in the sequel.

The following was proved in [18].

**Lemma 4.1.** Suppose $\Lambda \subset G \times \hat{G}$ is a closed subgroup. Then $\xi \mapsto \pi(\xi)$ is a faithful unitary $c$-projective representation of $\Lambda$. As a result, the integrated representation is a non-degenerate $\ast$-representation of $S_0(\Lambda, c)$.

We may then obtain the minimal universal enveloping algebra $C^*_\ast(\Lambda, c)$ of $S_0(\Lambda, c)$ through the integrated representation of $S_0(\Lambda, c)$ on $L^2(G)$, that is, the representation
\begin{equation}
a \cdot f = \int_{\Lambda} a(\lambda) \pi(\lambda) f d\lambda,
\end{equation}
for $a \in S_0(\Lambda, c)$ and $f \in L^2(G)$. As $\Lambda$ is abelian, hence amenable, the minimal and maximal enveloping algebras coincide, so we write $C^*(\Lambda, c)$ for the universal enveloping algebra of $S_0(\Lambda, c)$. We do the same for $S_0(\Lambda^0, \overline{c})$, and denote its universal enveloping $C^*$-algebra by $C^*(\Lambda^0, \overline{c})$.

**Theorem 4.2 ([28]).** The twisted group $C^*$-algebras $C^*(\Lambda, c)$ and $C^*(\Lambda^0, \overline{c})$ are Morita equivalent.

Indeed, $S_0(G)$ becomes a pre-equivalence bimodule between $\mathcal{A}$ and $\mathcal{B}$ as in [Definition 2.9] when equipped with the inner products
\begin{equation}
\ast\langle f, g \rangle := \int_{\Lambda^0} \langle f, \pi(\lambda^0) g \rangle \pi(\lambda^0) d\lambda^0,
\end{equation}
and
\begin{equation}
\langle f, g \rangle \ast := \int_{\Lambda^0} \langle g, \pi(\lambda^0)^* f \rangle \pi(\lambda^0)^* d\lambda^0,
\end{equation}
and the actions
\begin{equation}
a \cdot f = \int_{\Lambda^0} a(\lambda) \pi(\lambda) f d\lambda
\end{equation}
and
\begin{equation}
f \cdot b = \int_{\Lambda^0} b(\lambda^0) \pi(\lambda^0)^* f d\lambda^0,
\end{equation}
with $a \in \mathcal{A}$, $b \in \mathcal{B}$, and $f, g \in S_0(G)$. That these are well-defined was noted in Section 3 of [18]. In the remainder of the section we denote by $A$ the $C^*$-completion of $\mathcal{A}$, $B$ the
The $C^*$-completion of $B$, $\mathcal{E} = S_0(G)$, and by $E$ the Hilbert $C^*$-module completion of $\mathcal{E}$. Hilbert $C^*$-modules $E$ as in this setting are called Heisenberg modules.

Remark 4.3. The fact that we get the same twisted group $C^*$-algebras by using $S_0(\Lambda, c)$ as we get when using the more traditional approach with $L^1(\Lambda, c)$ was noted in [2].

Since $S_0$-functions are continuous, there are also well-defined canonical faithful traces on $A$ and $B$ given by

$$\text{tr}_A : A \to \mathbb{C} \quad a \mapsto a(0),$$

(4.19)

and

$$\text{tr}_B : B \to \mathbb{C} \quad b \mapsto b(0).$$

(4.20)

In general, these traces do not extend to $A$ and $B$, but we will nonetheless denote them by $\text{tr}_A$ and $\text{tr}_B$. These are indeed related as in (2.3), which can be seen from

$$\text{tr}_A(\langle f, g \rangle) = \langle f, g \rangle(0) = \langle f, g \rangle_{L^2(G)} = \langle g, f \rangle_{\mathcal{E}}(0) = \text{tr}_B(\langle g, f \rangle_{\mathcal{E}}).$$

(4.21)

In our discussion the following two results are crucial. The first follows immediately by [23], and the second is a consequence of [14].

Lemma 4.4. $C^*(\Lambda, c)$ is unital if and only if $\Lambda$ is discrete.

Proposition 4.5. For a discrete subgroup $\Lambda$ in $G \times \hat{G}$ the involutive Banach algebra $S_0(\Lambda, c)$ is spectral invariant in $C^*(\Lambda, c)$.

Remark 4.6. Although the traces $\text{tr}_A$ and $\text{tr}_B$ do not in general extend to the algebras $A$ and $B$, we can guarantee they extend in one case. Namely, $\text{tr}_A$ extends to all of $A$ if $A$ is unital, which is equivalent to $\Lambda$ being discrete. The same is of course true for $B$ and $\text{tr}_B$, with the discreteness condition on $\Lambda^0$. This is due to the fact that the trace given by evaluation in the identity extends to twisted group $C^*$-algebras when the underlying group is discrete [5, p. 951].

The case of $\Lambda$ or $\Lambda^0$ being discrete is the case we will almost exclusively restrict to after Proposition 4.8.

The following is now an immediate consequence.

Proposition 4.7. Let $\Lambda$ be cocompact, which implies $\Lambda^0$ is discrete. Then under the above conditions on $A, B, E, \text{tr}_B$ the quadruple $(A, B, E, \text{tr}_B)$ is a left Gabor bimodule. In addition, the septuple $(A, B, E, \text{tr}_B, A, B, E)$ is a left Gabor bimodule with regularity.

If $\Lambda^0$ is cocompact and thus $\Lambda$ is discrete, then we obtain a right Gabor bimodule with regularity analogously.

We may then reprove Theorem 3.9 of [18] in this framework.

Proposition 4.8. Let $\Lambda \subset G \times \hat{G}$ be a closed subgroup. Then $E$ is a finitely generated projective $A$-module if and only if $\Lambda \subset G \times \hat{G}$ is a cocompact subgroup. Also, $\mathcal{E}$ is a finitely generated projective $A$-module if and only if $\Lambda \subset G \times \hat{G}$ is cocompact.

Proof. $E$ is finitely generated and projective over $A$ if and only if $K_A(E) = \text{End}_A(E)$. As $E$ is an $A$-$B$-equivalence bimodule, this is equivalent to $B$ being unital by Proposition 2.11. $B$ is unital if and only if $\Lambda^0$ is discrete by Lemma 4.4, so equivalently

$$\hat{\Lambda}^0 \cong (G \times \hat{G})/\Lambda$$

(4.22)
is compact, that is, $\Lambda$ is cocompact in $G \times \hat{G}$.

Now, if $\Lambda$ is cocompact, then $\mathcal{B}$ is unital, so we are in the situation of Proposition 2.14 by Proposition 4.5. Hence by the first part of this proposition it follows that $\mathcal{E}$ is a finitely generated projective $\mathcal{A}$-module.

Conversely, suppose $\mathcal{E}$ is a finitely generated projective $\mathcal{A}$-module. Then $\mathcal{E} \cong A^0 p$ isometrically for some $n \in \mathbb{N}$ and some $p \in M_n(\mathcal{A})$. Passing to the completions we obtain $E \cong A^0 p$, so $E$ is a finitely generated projective $\mathcal{A}$-module. By the first part of this proposition it follows that $\Lambda$ is cocompact. \hfill $\Box$

Remark 4.9. Proposition 4.8 shows that we can only have finite module frames for $E$ as an $\mathcal{A}$-module if $\Lambda$ is cocompact in $G \times \hat{G}$. Since we wish to study the relationship between finite module frames and Gabor frames this is the case we care most about in the sequel.

To get results on Gabor frames for $L^2(G)$ with windows in $E$ from the above setup, we will need to localize certain subsets of the $C^*$-algebras $\mathcal{A}$ and $\mathcal{B}$, as well as the Morita equivalence bimodule $E$, just as explained in Section 2. For simplicity, let $\Lambda$ be cocompact in $G \times \hat{G}$ from now on, unless otherwise specified. Then $\Lambda^o$ is discrete and $\text{tr}_B$ is defined on all of $\mathcal{B}$. The localization of $\mathcal{B}$ in $\text{tr}_B$ is induced by the inner product $(-, -)_B$ given by

$$(b_1, b_2) = \text{tr}_B(b_1^* b_2).$$

Since $\mathcal{B}$ is dense in $\mathcal{B}$ and $\text{tr}_B$ is continuous, it follows that their localizations in $\text{tr}_B$ are the same. For $b_1, b_2 \in \mathcal{B}$ we then have

$$(b_1, b_2)_B = \text{tr}_B(b_1^* b_2)
= \text{tr}_B(\sum_{\lambda^o \in \Lambda^o} b_1(\lambda^o)\pi(\lambda^o)^* \sum_{\xi \in \Lambda^o} b_2(\xi)\pi(\xi))
= \text{tr}_B(\sum_{\lambda^o \in \Lambda^o} \sum_{\xi \in \Lambda^o} b_1(\lambda^o)b_2(\xi)c(\lambda^o, \lambda^o)\pi(\lambda^o)\pi(\xi)(-\lambda^o)\pi(\xi))
= \text{tr}_B(\sum_{\lambda^o \in \Lambda^o} \sum_{\xi \in \Lambda^o} b_1(\lambda^o)b_2(\xi)c(\lambda^o, \lambda^o)c(-\lambda^o, \xi)c(-\lambda^o + \xi)\pi(\lambda^o))
= \sum_{\xi \in \Lambda^o} b_1(\xi)b_2(\xi)c(\xi, \xi)c(-\xi, \xi)
= \langle b_1, b_2 \rangle_{\mathcal{E}(\Lambda^o)}.$$

As $\mathcal{B} = S_0(\Lambda^o, \sigma) = \ell^1(\Lambda^o, \sigma)$ is dense in $\ell^2(\Lambda^o)$, we may identify the localization $H_B$ of $\mathcal{B}$ with $\ell^2(\Lambda^o)$. By [2, Proposition 3.2] we also obtain that the localization of $E$ in $\text{tr}_B$ is $L^2(G)$.

Note that this is the same as the localization of $E$ in $\text{tr}_A$ by construction, and that there is an action of $A$ on $L^2(G)$ by extending the action of $A$ on $E$.

It is slightly more tricky to localize subsets of $A$. Indeed, it is not in general possible as the trace might not be defined everywhere. However, even if $A$ is not unital we may localize the algebraic ideal $\mathcal{E}(E, E) \subset A$ in the trace $\text{tr}_A$. Indeed, by [2, Theorem 3.5], elements of $E$ are such that whenever $g \in E$ and $f \in L^2(G)$, then $\langle (f, \pi(\lambda)g) \rangle_{\lambda \in \Lambda} \in L^2(\Lambda)$. This is the property of being a Bessel vector, which we will discuss in more detail below. Hence
for any \( f, g \in E \), we may identify \( \langle f, g \rangle \in A \) with \( ((f, \pi(\lambda)g))_{\lambda \in \Lambda} \) in \( L^2(\Lambda) \) by doing the analogous procedure with \( \text{tr}_A \) as for \( \text{tr}_B \) above.

We may do the same for the matrix algebras and matrix modules considered in Section 3.2. Note that \( \bullet [M_{n,d}(E), M_{n,d}(E)] = M_{n,d}(\bullet (E, E)) \). Adapting the setting of twisted group \( C^* \)-algebras and Heisenberg modules above to the matrix algebra setting of Section 3.2 we see that we obtain the following identifications

\[
H_{M_d(B)} = \ell^2(\Lambda^0 \times \mathbb{Z}_d \times \mathbb{Z}_d)
\]

(4.23)

\[
H_{M_n(\bullet (E, E))} = L^2(\Lambda \times \mathbb{Z}_n \times \mathbb{Z}_n)
\]

(4.24)

\[
H_{M_{n,d}(E)} = L^2(\mathbb{G} \times \mathbb{Z}_n \times \mathbb{Z}_d).
\]

Remark 4.10. Should \( \Lambda^0 \) be cocompact and therefore \( \Lambda \) discrete, we do the obvious changes. Also if both \( \Lambda \) and \( \Lambda^0 \) are discrete, that is, they are both lattices, then we may localize all of \( M_n(A) \) and all of \( M_d(B) \).

We can finally treat the analogs of Gabor frames in our framework. In what follows we will consider a novel type of Gabor frames. To ease notation we will for \( f \in L^2(\mathbb{G} \times \mathbb{Z}_n \times \mathbb{Z}_d) \) write \( f_{i,j} \) instead of \( f(\cdot, i, j) \), and the same for elements of \( L^2(\Lambda \times \mathbb{Z}_n \times \mathbb{Z}_n) \) and \( L^2(\Lambda^0 \times \mathbb{Z}_d \times \mathbb{Z}_d) \).

Definition 4.11. Let \( \Lambda \) be a closed subgroup of \( \mathbb{G} \). For \( g \in L^2(\mathbb{G} \times \mathbb{Z}_n \times \mathbb{Z}_d) \) we define the coefficient operator \( C_g \) by

\[
C_g : L^2(\mathbb{G} \times \mathbb{Z}_n \times \mathbb{Z}_d) \rightarrow L^2(\Lambda \times \mathbb{Z}_n \times \mathbb{Z}_n)
\]

(4.25)

\[
C_g(f) = \{ \sum_{m \in \mathbb{Z}_d} \langle f_{k,m}, \pi(\lambda)g_{l,m}\rangle \lambda \in \Lambda, k, l \in \mathbb{Z}_n \}
\]

and the synthesis operator \( D_g \) by

\[
D_g : L^2(\Lambda \times \mathbb{Z}_n \times \mathbb{Z}_n) \rightarrow L^2(\mathbb{G} \times \mathbb{Z}_n \times \mathbb{Z}_d)
\]

(4.26)

\[
D_g a = \{ \sum_{m \in \mathbb{Z}_d} \int_{\Lambda} a_{k,m}(\lambda)\pi(\lambda)g_{l,m} d\lambda \}_{k \in \mathbb{Z}_n, l \in \mathbb{Z}_d}.
\]

Furthermore, we define the frame-like operator \( S_{g,h} = D_h C_g \), and for brevity we write \( S_g \) for \( D_g C_g \). We say \( S_g \) is the frame operator associated to \( g \).

We say \( g \) generates an \((n, d)\)-matrix Gabor frame for \( L^2(\mathbb{G}) \) with respect to \( \Lambda \) if \( S_g : L^2(\mathbb{G} \times \mathbb{Z}_n \times \mathbb{Z}_d) \rightarrow L^2(\mathbb{G} \times \mathbb{Z}_n \times \mathbb{Z}_d) \) is an isomorphism. Equivalently, the collection of time-frequency shifts

(4.27)

\[
G(g; \Lambda) := \{ \pi(\lambda)g_{i,j} \mid \lambda \in \Lambda \}_{i \in \mathbb{Z}_n, j \in \mathbb{Z}_d}
\]

is a frame for \( L^2(\mathbb{G} \times \mathbb{Z}_n \times \mathbb{Z}_d) \). We then say that \( G(g; \Lambda) \) is an \((n, d)\)-matrix Gabor frame for \( L^2(\mathbb{G}) \). Equivalently, there exists \( h \in L^2(\mathbb{G} \times \mathbb{Z}_n \times \mathbb{Z}_d) \) such that for all \( f \in L^2(\mathbb{G} \times \mathbb{Z}_n \times \mathbb{Z}_d) \) we have

(4.27)

\[
f_{r,s} = \sum_{k \in \mathbb{Z}_d} \sum_{l \in \mathbb{Z}_n} \int_{\Lambda} (f_{r,k}, \pi(\lambda)g_{l,k})\pi(\lambda)h_{l,s} d\lambda,
\]

for all \( r \in \mathbb{Z}_n \) and \( s \in \mathbb{Z}_d \). When \( g \) and \( h \) satisfy (4.27) we say \( G(g; \Lambda) \) and \( G(h; \Lambda) \) are a dual pair of \((n, d)\)-matrix Gabor frames. If \( \Lambda \) is implicit, we may also say \( h \) is a dual \((n, d)\)-matrix Gabor atom for \( g \), or just a dual atom of \( g \).

Remark 4.12. The equivalence of the definitions of \((n, d)\)-matrix Gabor frames given in Definition 4.11 follows by [7] Lemma 6.3.2 and Proposition 4.16 below.
Remark 4.13. When $\mathcal{G}(g; \Lambda)$ is an $(n, d)$-matrix Gabor frame for $L^2(G)$, there is always a dual $(n, d)$-matrix Gabor atom for $g$, namely $h = S_g^{-1}g$. This is known as the canonical dual of $g$.

Remark 4.14. One can verify that $C_g = D_g^*$. Thus $S_g$ is always a positive operator between Hilbert spaces, just as for the module frame operator in Section 3.

For general $g \in L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d)$ the operator $C_g$ will not be bounded. Elements $g$ such that $C_g$ is bounded are of interest on their own.

Definition 4.15. If $g \in L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d)$ is so that $C_g : L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d) \to L^2(\Lambda \times \mathbb{Z}_n \times \mathbb{Z}_d)$ is a bounded operator we say $g$ is an $(n, d)$-matrix Gabor Bessel vector for $L^2(G)$ with respect to $\Lambda$, or that $\mathcal{G}(g; \Lambda)$ is an $(n, d)$-matrix Gabor Bessel system for $L^2(G)$. Equivalently, there is $D > 0$ such that for all $f \in L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d)$ we have

$$\langle f, f \rangle \leq D \langle C_g f, C_g f \rangle,$$

which may also be written as

$$\sum_{i \in \mathbb{Z}_n} \sum_{j \in \mathbb{Z}_d} \int_G |f_{i,j}(\xi)|^2 d\xi \leq D \sum_{k,l \in \mathbb{Z}_n} \int_{\Lambda} |\sum_{m \in \mathbb{Z}_d} (f_{k,m}, \pi(\lambda)g_{l,m})|^2 d\lambda.$$

The smallest $D > 0$ such that the condition of (4.28) holds is called the optimal Bessel bound of $\mathcal{G}(g; \Lambda)$, or just the optimal Bessel bound of $g$ if the set $\Lambda$ is clear from the context.

The Gabor frames of Definition 4.11 seemingly generalize the $n$-multiwindow $d$-super Gabor frames of [13]. Indeed, we obtain $n$-multiwindow $d$-super Gabor frames if we only require reconstruction of $f \in L^2(G \times \mathbb{Z}_d)$ and we identify $L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d) \subset L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d)$ by embedding along a single element of $\mathbb{Z}_n$. Hence [14] generalizes both multiwindow Gabor frames and super Gabor frames as well, setting $d = 1$ or $n = 1$, respectively. However, we will in Proposition 4.16 show that any $n$-multiwindow $d$-super Gabor frame for $L^2(G)$ with respect to $\Lambda$ is an $(n, d)$-matrix Gabor frame for $L^2(G)$ with respect to $\Lambda$. However, we continue to call them by separate names, since, as mentioned above, they are used for reconstruction in different Hilbert spaces.

The following proposition was noted in the $(n, 1)$-matrix case in [2] Theorem 3.11, and its proof in the $(n, d)$-matrix Gabor case goes through the same except with more bookkeeping.

Proposition 4.16. Let $\Lambda \subset G \times \hat{G}$ be closed and cocompact. For every $g \in M_{n,d}(E)$, $C_g : L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d) \to L^2(\Lambda \times \mathbb{Z}_n \times \mathbb{Z}_n)$ is a bounded operator. In other words, every $g \in M_{n,d}(E)$ is a Bessel vector.

For ease of notation, the localization map in in $M_n(A)$ will be denoted by $\rho_{M_n(A)}$, even though we might not be able to localize all of $M_n(A)$. With the above definitions, the following calculation is justified for $f, g \in M_{n,d}(E) \subset L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d)$ by Proposition 4.16

$$\rho_{M_{n,d}(E)}(f) = \rho_{M_n(A)}(\Phi_g(f)) = \rho_{M_{n,d}(E)}(\{ \sum_{m \in \mathbb{Z}_d} \int_{\Lambda} \{ f_{k,m}, \pi(\lambda)g_{l,m} \} d\lambda \})$$

$$= \rho_{M_{n,d}(E)}(\{ \sum_{m \in \mathbb{Z}_d} \{ f_{k,m}, \pi(\lambda)g_{l,m} \} \} \lambda \in \Lambda, k, l \in \mathbb{Z}_n) = C_g \rho_{M_{n,d}(E)}(f).$$

Hence we obtain the following result.
Lemma 4.17. Let \( \Lambda \subset G \times \hat{G} \) be closed and cocompact. For every \( g \in M_{n,d}(E) \), the module coefficient operator \( \Phi_g \) localizes to give the coefficient operator \( C_g \). Equivalently, the diagram
\[
\begin{array}{ccc}
M_{n,d}(E) & \xrightarrow{\Phi_g} & M_n(A) \\
\rho_{M_{n,d}(E)} & & \rho_{M_n(A)} \\
L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d) & \xrightarrow{C_g} & L^2(\Lambda \times \mathbb{Z}_n \times \mathbb{Z}_n)
\end{array}
\]
commutes for all \( g \in M_{n,d}(E) \).

Likewise one may obtain \( C_g^* \rho_{M_n(A)} = \rho_{M_{n,d}(E)} \Phi_g^* : M_n(\bullet(E,E)) \rightarrow L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d) \) for all \( g \in M_{n,d}(E) \). Note that the domain might be larger, but we cannot guarantee this unless \( A \) is unital, that is, when \( \Lambda \) is discrete.

Lemma 4.18. Let \( \Lambda \subset G \times \hat{G} \) be closed and cocompact. For every \( g \in M_{n,d}(E) \), the module synthesis operator \( \Phi_g^* \) localizes to the Gabor synthesis operator \( C_g^* \). Equivalently, the diagram
\[
\begin{array}{ccc}
M_n(\bullet(E,E)) & \xrightarrow{\Phi_g} & M_{n,d}(E) \\
\rho_{M_n(A)} & & \rho_{M_{n,d}(E)} \\
L^2(\Lambda \times \mathbb{Z}_n \times \mathbb{Z}_n) & \xrightarrow{C_g^*} & L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d)
\end{array}
\]
commutes for every \( g \in M_{n,d}(E) \).

Combining [Lemma 4.17] and [Lemma 4.18] we then obtain

Proposition 4.19. Let \( \Lambda \subset G \times \hat{G} \) be closed and cocompact. For all \( g,h \in M_{n,d}(E) \), \( S_{g,h} \rho_{M_{n,d}(E)} = \rho_{M_{n,d}(E)} \Theta_{g,h} \), meaning the module frame-like operator \( \Theta_{g,h} \) localizes to the frame-like operator \( S_{g,h} \). Equivalently, the diagram
\[
\begin{array}{ccc}
M_{n,d}(E) & \xrightarrow{\Theta_{g,h}} & M_{n,d}(E) \\
\rho_{M_{n,d}(E)} & & \rho_{M_{n,d}(E)} \\
L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d) & \xrightarrow{S_{g,h}} & L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d)
\end{array}
\]
commutes for all \( g \in M_{n,d}(E) \).

As \( \rho_{M_{n,d}(E)} : M_{n,d}(E) \rightarrow \rho_{M_{n,d}(E)}(M_{n,d}(E)) \) is a linear bijection intertwining both the \( A \)-actions and the \( B \)-actions, we see by [Proposition 4.19] that for \( g \in M_{n,d}(E) \), \( \Theta_g \) is invertible if and only if \( S_g |_{\rho_{M_{n,d}(E)}(M_{n,d}(E))} \) is invertible. But we also have the following result.

Lemma 4.20. Let \( \Lambda \subset G \times \hat{G} \) be closed and cocompact, and let \( g \in M_{n,d}(E) \). Then
\[
S_g |_{\rho_{M_{n,d}(E)}(M_{n,d}(E))} : \rho_{M_{n,d}(E)}(M_{n,d}(E)) \rightarrow \rho_{M_{n,d}(E)}(M_{n,d}(E))
\]
is invertible if and only if
\[
S_g : L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d) \rightarrow L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d)
\]
is invertible.

**Proof.** Suppose first $S_g|_{\rho_{M_{n,d}(E)}(M_{n,d}(E))} : \rho_{M_{n,d}(E)}(M_{n,d}(E)) \to \rho_{M_{n,d}(E)}(M_{n,d}(E))$ is invertible. Since any $g \in M_{n,d}(E)$ is a Bessel vector by [Proposition 4.16], we may extend the operator by continuity to obtain that $S_g : L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d) \to L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d)$ is invertible as well.

Conversely, suppose $S_g : L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d) \to L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d)$ is invertible. Since $S_g$ is the continuous extension of $\Theta_g$, it then follows by [Proposition 2.8] and [Proposition 2.1] that $\Theta_g$ is invertible, which implies $S_g|_{\rho_{M_{n,d}(E)}(M_{n,d}(E))}$ is invertible.

**Remark 4.21.** From now on we will identify $M_{n,d}(E)$ and its image in the localization, and we will do this without mention.

Combining [Proposition 4.19] and [Lemma 4.20] we obtain the following important result.

**Proposition 4.22.** Let $\Lambda \subset G \times \hat{G}$ be closed and cocompact. For $g \in M_{n,d}(E)$ we have that $\Theta_g$ is invertible if and only if $S_g$ is invertible. In other words, $g$ generates a module $(n,d)$-matrix frame for $E$ with respect to $\Lambda$ if and only if $S_g$ is invertible. Since any $g \in M_{n,d}(E)$ generates an $(n,d)$-matrix Gabor Riesz sequence and relate them to [Definition 3.34].

**Corollary 4.23.** Let $\Lambda \subset G \times \hat{G}$ be closed and cocompact, and let $g, h \in M_{n,d}(E)$. Then $g$ and $h$ generate dual $(n,d)$-matrix Gabor frames for $L^2(G)$ with respect to $\Lambda$ if and only if $[g,h]_\bullet$ extends to the identity operator on $L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d)$.

**Proof.** Suppose first $g,h \in M_{n,d}(E)$ generate dual $(n,d)$-matrix Gabor frames for $L^2(G)$ with respect to $\Lambda$. Then we know that for all $f \in M_{n,d}(E)$ we have

$$f = \bullet[f,g]h = [g,h]_\bullet,$$

from which we as before deduce that $[g,h]_\bullet = 1_{M_{d}(E)}$. This extends by continuity to the identity operator on all of $L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d)$.

Conversely, if $[g,h]_\bullet$ extends to the identity operator on $L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d)$, then $[g,h]_\bullet$ acts as the identity on $M_{n,d}(E)$. For any $f \in M_{n,d}(E)$ we then have

$$f = [g,h]_\bullet = \bullet[f,g]h,$$

hence (4.27) holds for all $f \in M_{n,d}(E)$. But this extends to $L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d)$ by continuity, which implies that $g$ and $h$ generate dual $(n,d)$-matrix Gabor frames.

We wish to establish a duality principle for $(n,d)$-matrix Gabor frames. For this we also need to treat $(n,d)$-matrix Gabor Riesz sequences and relate them to [Definition 3.34].

**Definition 4.24.** Let $g \in L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d)$. We say $g$ generates an $(n,d)$-matrix Gabor Riesz sequence for $L^2(G)$ with respect to $\Lambda$, or that $G(g;\Lambda)$ is an $(n,d)$-matrix Gabor Riesz sequence for $L^2(G)$, if

$$(4.30) \quad C_gC^*_g : L^2(\Lambda \times \mathbb{Z}_n \times \mathbb{Z}_n) \to L^2(\Lambda \times \mathbb{Z}_n \times \mathbb{Z}_n)$$

is an isomorphism. Equivalently, there exists $h \in L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d)$ such that for all $a \in L^2(\Lambda \times \mathbb{Z}_n \times \mathbb{Z}_n)$ we have

$$(4.31) \quad a_{r,s}(\mu) = \sum_{i,j} \sum_r \left( \int_{\Lambda} a_{r,j}(\lambda)\pi(\lambda)g_{j,i}d\lambda, \pi(\mu)h_{s,i} \right)$$

for all $r,s \in \mathbb{Z}_n$ and all $\mu \in \Lambda$. If (4.31) is satisfied we will say $h$ generates a dual $(n,d)$-matrix Gabor Riesz sequence of $g$.  

Remark 4.25. Note that the equivalence of the definitions of \((n,d)\)-matrix Gabor Riesz sequences in Definition 4.24 follows by [17, Theorem 3.6.6] and Proposition 4.16.

Remark 4.26. \((4.31)\) can be seen to be equivalent to \(C_g C^*_h = C_h C^*_g = \text{Id}_{L^2(\Lambda \times \mathbb{Z}_n \times \mathbb{Z}_n)}\).

Before treating localization of module matrix Riesz sequences and how they relate to matrix Gabor Riesz sequences, we do a necessary but justified simplification. Recall that existence of finite module matrix Riesz sequences for \(M_{n,d}(E)\) with respect to \(M_n(A)\) requires \(A\) to be unital by Proposition 3.20. In the following we therefore let \(\Lambda\) be discrete, but not necessarily cocompact. Hence in the following, \(A\) is unital with a faithful trace, but \(B\) might not have that property. By [17, p. 251] we know that \(\mathcal{G}(g; \Lambda)\) is a Bessel system with Bessel bound \(D\) if and only if \(\mathcal{G}(g; \Lambda^0)\) is a Bessel system with Bessel bound \(D\). Applying Proposition 4.10 we immediately get the following from Lemma 4.17 and Remark 4.25.

Proposition 4.27. Let \(\Lambda \subset G \times \hat{G}\) be discrete. For all \(g, h \in M_{n,d}(E)\) we have \((C_h C^*_g) \circ \rho_{M_n(A)} = \rho_{M_n(A)} \circ (\Phi_h \Phi^*_g)\). Equivalently, the diagram

\[
\begin{array}{ccc}
M_n(A) & \xrightarrow{\Phi_h \Phi^*_g} & M_n(A) \\
\rho_{M_n(A)} & & \rho_{M_n(A)} \\
L^2(\Lambda \times \mathbb{Z}_n \times \mathbb{Z}_n) & \xrightarrow{C_h C^*_g} & L^2(\Lambda \times \mathbb{Z}_n \times \mathbb{Z}_n)
\end{array}
\]

commutes.

As \(\rho_{M_n(A)} : M_n(A) \rightarrow \rho_{M_n(A)}(M_n(A))\) is a linear bijection respecting the actions of \(A\), we see by Proposition 4.27 that for \(g \in M_{n,d}(E)\), \(\Phi_g \Phi^*_g\) is an isomorphism if and only if \((C_g C^*_g)|_{\rho_{M_n(A)}(M_n(A))}\) is an isomorphism. In analogy with Lemma 4.20 we have the following result.

Lemma 4.28. Let \(\Lambda \subset G \times \hat{G}\) be discrete. For \(g \in M_{n,d}(E)\) we have that

\[
(C_g C^*_g)|_{\rho_{M_n(A)}(M_n(A))} : \rho_{M_n(A)}(M_n(A)) \rightarrow \rho_{M_n(A)}(M_n(A))
\]

is invertible if and only if

\[
C_g C^*_g : L^2(\Lambda \times \mathbb{Z}_n \times \mathbb{Z}_n) \rightarrow L^2(\Lambda \times \mathbb{Z}_n \times \mathbb{Z}_n)
\]

is invertible.

Proof. Suppose first that \((C_g C^*_g)|_{\rho_{M_n(A)}(M_n(A))} : \rho_{M_n(A)}(M_n(A)) \rightarrow \rho_{M_n(A)}(M_n(A))\) is invertible. Since any \(g \in M_{n,d}(E)\) is a Bessel vector by Proposition 4.16 we may extend the operator by continuity to obtain that \(C_g C^*_g : L^2(\Lambda \times \mathbb{Z}_n \times \mathbb{Z}_n) \rightarrow L^2(\Lambda \times \mathbb{Z}_n \times \mathbb{Z}_n)\) is invertible as well.

Conversely, suppose \(C_g C^*_g : L^2(\Lambda \times \mathbb{Z}_n \times \mathbb{Z}_n) \rightarrow L^2(\Lambda \times \mathbb{Z}_n \times \mathbb{Z}_n)\) is invertible. Since \(C_g C^*_g\) is the continuous extension of \(\Phi_g \Phi^*_g\), it then follows by Proposition 2.8 and Proposition 2.1 that \(\Phi_g \Phi^*_g\) is invertible as well, which implies \((C_g C^*_g)|_{\rho_{M_n(A)}(M_n(A))} : \rho_{M_n(A)}(M_n(A)) \rightarrow \rho_{M_n(A)}(M_n(A))\) is invertible. \(\square\)

Remark 4.29. From now on we will identify \(M_n(\bullet(E, E))\) (and potentially a larger domain) and its localization. The same goes for \(M_q(B)\).

Now the following is an immediate consequence.
Proposition 4.30. Let $\Lambda \subset G \times \hat{G}$ be discrete. For $g \in M_{n,d}(E)$ we have that $\Phi_g \Phi_g^*$ is invertible if and only if $C_g C_g^*$ is invertible. In other words, $g$ generates a module $(n,d)$-matrix Riesz sequence for $E$ with respect to $A$ if and only if $\mathcal{G}(g;\Lambda)$ is an $(n,d)$-matrix Gabor Riesz sequence for $L^2(G)$.

By the proof of Lemma 4.28 we then have the following statement.

Corollary 4.31. Let $\Lambda \subset G \times \hat{G}$ be discrete. Suppose $g,h \in M_{n,d}(E)$. Then $g$ and $h$ generate dual $(n,d)$-matrix Gabor Riesz sequences for $L^2(G)$ with respect to $\Lambda$ and if and only if $\mathcal{M}[g,h]$ extends to the identity operator on $L^2(\Lambda \times \mathbb{Z}_n \times \mathbb{Z}_n)$.

Proof. Suppose first that $g$ and $h$ generate dual $(n,d)$-matrix Gabor Riesz sequences for $L^2(G)$ with respect to $\Lambda$. Then for all $a \in M_n(A)$ we have

$$(a_{r,s}) = \left\{ \sum_{i \in \mathbb{Z}_n} \sum_{j \in \mathbb{Z}_n} \left( \int_{\Lambda} a_{r,j}(\lambda) \pi(\lambda) g_{j,i} d\lambda, \pi(\mu) h_{s,i} \right) \right\}_{\mu \in \Lambda, r,s \in \mathbb{Z}_n},$$

which is equivalent to $a = a \mathcal{M}[g,h]$ for all $a \in M_n(A)$. But the first expression extends by continuity to $L^2(\Lambda \times \mathbb{Z}_n \times \mathbb{Z}_n)$, so $\mathcal{M}[g,h]$ extends to the identity on $L^2(\Lambda \times \mathbb{Z}_n \times \mathbb{Z}_n)$.

Conversely, suppose $\mathcal{M}[g,h]$ extends to the identity on $L^2(\Lambda \times \mathbb{Z}_n \times \mathbb{Z}_n)$. Once again, for all $a \in M_n(A)$ we then have

$$(a_{r,s}) = \left\{ \sum_{i \in \mathbb{Z}_n} \sum_{j \in \mathbb{Z}_n} \left( \int_{\Lambda} a_{r,j}(\lambda) \pi(\lambda) g_{j,i} d\lambda, \pi(\mu) h_{s,i} \right) \right\}_{\mu \in \Lambda, r,s \in \mathbb{Z}_n},$$

which again extends to $L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_n)$. Hence $g$ and $h$ are dual $(n,d)$-matrix Gabor Riesz sequences for $L^2(G)$ with respect to $\Lambda$. \hfill $\square$

Note how the above results guarantee that when $\Lambda \subset G \times \hat{G}$ is closed and cocompact and $g \in M_{n,d}(E)$ is such that $\mathcal{G}(g;\Lambda)$ is an $(n,d)$-matrix Gabor frame for $L^2(G)$, the canonical dual frame $S_g^{-1} g \in M_{n,d}(E)$. Indeed,

$$S_g^{-1} g = \Theta_g^{-1} g = [g,g]^{-1} \in M_{n,d}(E).$$

Likewise, for Riesz sequences there is the notion of canonical biorthogonal atom, see for example [2, p. 160]. Restricting to $\Lambda$ discrete, it is given by $(S_g^B)^{-1} g$, where $S_g^B$ is the frame operator with respect to the right hand side, that is, with respect to $\Lambda^o$. We see that for all $f \in M_{n,d}(E)$

$$S_g^B f = (\Phi_g^B)^* \Phi_g^B f = (\Phi_g^B)^*[g,f] = g[f,g] = \mathcal{M}[g,g] f.$$

Thus it follows that

$$(S_g^B)^{-1} g = (\Theta_g^B)^{-1} g = [g,g]^{-1} \in M_{n,d}(E).$$

Hence for both matrix Gabor frames and matrix Gabor Riesz sequences with generating atom in $M_{n,d}(E)$, the canonically associated dual atoms are also in $M_{n,d}(E)$.

We have the following result which shows that in the cases we are interested in, if the generating atom is regular, the canonical dual atom has the same regularity.

Proposition 4.32. Let $g \in M_{n,d}(E)$.

i) If $\mathcal{G}(g;\Lambda)$ is an $(n,d)$-matrix Gabor frame for $L^2(G)$ and $\Lambda$ is closed and cocompact in $G \times \hat{G}$, then the canonical dual atom is in $M_{n,d}(E)$.

ii) If $\mathcal{G}(g;\Lambda)$ is an $(n,d)$-matrix Gabor Riesz sequence for $L^2(G)$ and $\Lambda$ is discrete, then the canonical biorthogonal atom is also in $M_{n,d}(E)$. 

Proof. For the proof of i), note that the assumption that \( \Lambda \) is cocompact implies that \( \Lambda^o \) is discrete, so by Lemma 4.4 we get that \( M_d(B) \) is unital. Also \( M_d(B) \) is a \( C^* \)-subalgebra of \( \mathbb{B}(H_{M_n,d}) \) by Proposition 2.8. That \( G(g; \Lambda) \) is an \((n, d)\)-matrix Gabor frame for \( L^2(G) \) then means that (3.30) is satisfied for our current setting. We deduce, as in the proof of Proposition 3.30 that \( g, g \) is invertible in \( M_d(B) \). Since \( g, g \) is spectral invariant in \( M_d(B) \) by Proposition 4.5 and Lemma 3.22 it follows that the canonical dual atom is \( g[g, g]^{-1} \in M_n,d(\mathcal{E}) \).

For the proof of ii), note that the assumption that \( \Lambda \) is discrete implies \( M_n(A) \) is unital. Also, \( M_n(A) \) is a \( C^* \)-subalgebra of \( \mathbb{B}(H_{M_n(A)}) \) by Proposition 2.8. That \( G(g; \Lambda) \) determines an \((n, d)\)-matrix Gabor Riesz sequence for \( L^2(G) \) then means that (3.31) is satisfied for our current setting. The middle term of (3.31) can be written \((a \cdot [g, g], a)\), so \( [g, g] \) extends to a positive, invertible operator on \( H_{M_n(A)} \). We deduce as in the proof of Proposition 3.31 that \( [g, g] \) is invertible in \( M_n(A) \). Then \( g \in M_{n,d}(\mathcal{E}) \) we have \( [g, g] \in M_n(A) \), and by Proposition 4.5 and Lemma 3.22 \( M_n(A) \) is spectral invariant in \( M_n(A) \). It follows that the canonical dual atom \( h := [g, g]^{-1} \) is in \( M_n.d(\mathcal{E}) \).

\( \square \)

Remark 4.33. In the special case \( n = d = 1 \) Proposition 4.32 gives a proof of the fact that the canonical dual atom of a Gabor frame vector in Feichtinger’s algebra \( S_0(G) \) is also in Feichtinger’s algebra whenever \( \Lambda \) is cocompact.

When applying the module setup of Section 3 to Gabor analysis, we take as a pre-equivalence bimodule \( \mathcal{E} = S_0(G \times \mathbb{Z}_n \times \mathbb{Z}_d) \), which is a proper subspace of \( L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d) \) unless \( G \) is a finite group. Even the Hilbert \( C^* \)-module completion \( E \) is properly contained in \( L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d) \) for most choices of \( \Lambda \), see [2] Example 3.8. As such, we cannot hope to treat general atoms in \( L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d) \) by applying just this method. But indeed the module reformulation is made exactly to guarantee some regularity of the atoms generating frames.

From Definition 4.11 we see that \((n, d)\)-matrix Gabor frames generalize \( n \)-multiwindow \( d \)-super Gabor frames considered in [18]. However, we now make clear how they fit into the module framework. As mentioned earlier, we obtain \( n \)-multiwindow \( d \)-super Gabor frames if we only require reconstruction of \( f \in L^2(G \times \mathbb{Z}_d) \) and we identify \( L^2(G \times \mathbb{Z}_d) \) by embedding it along a single element in \( \mathbb{Z}_n \). The module reformulation of this is that \( g, h \in M_{n,d}(\mathcal{E}) \) are dual \( n \)-multiwindow \( d \)-super Gabor frames if for all \( f \in M_{n,d}(\mathcal{E}) \) supported only one row we have

\[
(4.34) \quad f = \cdot f, g[h = f[g, h] \cdot .
\]

Likewise, it is clear that the \((n, d)\)-matrix Gabor Riesz sequences of Definition 4.24 generalize the \( n \)-multiwindow \( d \)-super Gabor Riesz sequences also considered in [18]. Indeed, we obtain \( n \)-multiwindow \( d \)-super Gabor Riesz sequences if we only require reconstruction of \( a \in L^2(\Lambda \times \mathbb{Z}_n) \) and we identify \( L^2(\Lambda \times \mathbb{Z}_n) \subset L^2(\Lambda \times \mathbb{Z}_n \times \mathbb{Z}_d) \) by embedding it along a single element in the middle copy of \( \mathbb{Z}_d \). The module reformulation of this is that \( g, h \in M_{n,d}(\mathcal{E}) \) are dual \( n \)-multiwindow \( d \)-super Gabor Riesz sequences if for all \( a \in M_n(A) \) supported only one row we have

\[
(4.35) \quad a = \cdot [ag, h] = a \cdot [g, h].
\]

We proceed to prove that all \( n \)-multiwindow \( d \)-super Gabor frames for \( L^2(G) \) with respect to \( \Lambda \) are \((n, d)\)-matrix Gabor frames for \( L^2(G) \) with respect to \( \Lambda \), as well as the analogous statement for Riesz sequences. The converse statement is true as well.
Proposition 4.34. Let $g$ be in $M_{n,d}(E)$.

i) If $\mathcal{G}(g; \Lambda)$ is an $n$-multiwindow $d$-super Gabor frame for $L^2(G)$ with a dual window $h \in M_{n,d}(E)$, then $\mathcal{G}(g; \Lambda)$ is an $(n,d)$-matrix Gabor frame for $L^2(G)$ with dual window $h$.

ii) If $\mathcal{G}(g; \Lambda)$ is an $n$-multiwindow $d$-super Gabor Riesz sequence for $L^2(G)$ with a dual Gabor Riesz sequence $\mathcal{G}(h; \Lambda)$ with $h \in M_{n,d}(E)$, then $\mathcal{G}(g; \Lambda)$ is an $(n,d)$-matrix Gabor Riesz sequence for $L^2(G)$ with dual Gabor Riesz sequence $\mathcal{G}(h; \Lambda)$.

Proof. If $\mathcal{G}(g; \Lambda)$ is an $n$-multiwindow $d$-super Gabor frame for $L^2(G)$ with respect to $\Lambda$ with a dual window $h \in M_{n,d}(E)$, we can, as noted above, reconstruct any $f \in M_{n,d}(E)$ supported on a single row. In other words,

$$f = f[g, h]_*$$

for all $f \in M_{n,d}(E)$ supported on a single row. Given arbitrary $f' \in M_{n,d}(E)$ we may then just write $f'$ as a sum of $n$ matrices $f'_i$, $i = 0, \ldots, n - 1$, with only one nonzero row, namely the $k$th row of $f'_i$ is given by

$$(f_{i,0}, \ldots, f_{i,d-1}) \delta_{ik}$$

for $k \in \mathbb{Z}_n$, and we can reconstruct each of these rows. Hence we can reconstruct arbitrary elements of $M_{n,d}(E)$. This passes to the localization $L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d)$, and thus finishes the proof of (i).

The proof of (ii) is completely analogous, writing elements $a \in M_n(A)$ as a sum of matrices with only one nonzero row and then using that we can reconstruct such matrices. This will also pass to the localization $L^2(\Lambda \times \mathbb{Z}_n \times \mathbb{Z}_n)$. □

Given a closed and cocompact subgroup $\Lambda$, we may ask if there are restrictions on $n, d \in \mathbb{N}$ for there to possibly exist $(n,d)$-matrix Gabor frames for $L^2(G)$ with respect to $\Lambda$. Conversely, if we fix $n$ and $d$, we may ask if there are restrictions on the size of the subgroup $\Lambda$ for there to possibly exist $(n,d)$-matrix Gabor frames for $L^2(G)$ with respect to $\Lambda$. When $\Lambda$ is a lattice, we have the following proposition.

Proposition 4.35. Let $\Lambda \subset G \times \hat{G}$ be a lattice. If there is $g \in M_{n,d}(E)$ such that $\mathcal{G}(g; \Lambda)$ is an $(n,d)$-matrix Gabor frame for $L^2(G)$, then

$$s(\Lambda) \leq \frac{n}{d}.$$  

Proof. Since $\Lambda$ is discrete and cocompact, both $A$ and $B$ are unital. We also know by Proposition 4.32 that the canonical dual of $g$ is in $M_{n,d}(E)$. Hence we are in the setting of Theorem 3.28. Since module $(n,d)$-matrix frames localize to $(n,d)$-matrix Gabor frames for the localization, and we have $\text{tr}_A(1_A) = 1$, and $\text{tr}_B(1_B) = s(\Lambda)$ (since the identity on $B$ is $s(\Lambda) \delta_0$, where $\delta_0$ is the indicator function in the group identity, see for example [28]), the result is immediate by Theorem 3.28. □

Likewise, given a lattice $\Lambda$, we may ask if there is a relationship between the size of $\Lambda$ and the integers $n$ and $d$ such that there can possibly exist $(n,d)$-matrix Gabor Riesz sequences for $L^2(G)$ with respect to $\Lambda$. This is the content of the following proposition.

Proposition 4.36. Let $\Lambda \subset G \times \hat{G}$ be a lattice. If $g \in M_{n,d}(E)$ is such that $\mathcal{G}(g; \Lambda)$ is an $(n,d)$-matrix Gabor Riesz sequence for $L^2(G)$, then

$$s(\Lambda) \geq \frac{n}{d}.$$
Proof. As before we know by the conditions on $\Lambda$ that both $A$ and $B$ are unital, and by Proposition 4.32 the canonical dual of $g$ is in $M_{n,d}(E)$. Thus we are in the setting of Theorem 3.29. Since module $(n,d)$-matrix Riesz sequences localize to $(n,d)$-matrix Gabor Riesz sequences for the localization, and $\text{tr}_A(1_A) = 1$ and $\text{tr}_B(1_B) = s(\Lambda)$ (once again since the identity on $B$ is $s(\Lambda)\delta_0$), the result is immediate by Theorem 3.29. \hfill $\square$

Remark 4.37. The two preceding propositions contain statements known as density theorems in Gabor analysis. This is due to the fact that they give conditions on the density of a lattice for there to possibly exist Gabor frames.

Now let $\Lambda \subset G \times \hat{G}$ be cocompact again. In this framework two of the cornerstones of Gabor analysis, namely the Wexler-Raz biorthogonality relations and the duality principle for Gabor frames, are then quite easy to prove for $(n,d)$-matrix Gabor frames for $L^2(G)$ with respect to $\Lambda$ with atoms in $M_{n,d}(E)$.

**Proposition 4.38** (Wexler-Raz Biorthogonality Relations). Let $\Lambda \subset G \times \hat{G}$ be a closed and cocompact subgroup, and let $g,h \in M_{n,d}(E)$. Then the following are equivalent:

i) $\mathcal{G}(g;\Lambda)$ and $\mathcal{G}(h;\Lambda)$ are dual $(n,d)$-matrix Gabor frames for $L^2(G)$.

ii) $\langle g, \pi(\Lambda^\circ) h \rangle_{L^2(G)} = s(\Lambda) \cdot \delta_0 \Lambda^\circ$.

Proof. As $\Lambda$ is cocompact we know $\Lambda^\circ$ is discrete, so $M_d(B)$ is unital. Knowing this, we can see that both the above statements are equivalent to the statement $[g,h]_\bullet = [h,g]_\bullet = 1_{M_d(B)}$. \hfill $\square$

**Theorem 4.39** (Duality principle). Let $\Lambda \subset G \times \hat{G}$ be a closed and cocompact subgroup, and let $g \in M_{n,d}(E)$. Then the following are equivalent.

i) $\mathcal{G}(g;\Lambda)$ is an $(n,d)$-matrix Gabor frame for $L^2(G)$.

ii) $\mathcal{G}(g;\Lambda^\circ)$ is a $(d,n)$-matrix Gabor Riesz sequence for $L^2(G)$.

Proof. Statement i) can be seen to be equivalent to $[g,g]_\bullet$ being invertible by Proposition 4.22. But statement ii) is also equivalent to $[g,g]_\bullet$ being invertible by Proposition 4.30 since we consider $\mathcal{G}(g;\Lambda^\circ)$, that is, we work over $M_d(B)$. This finishes the proof. \hfill $\square$

For completeness we also include the following result related to the duality principle. This is a strengthening of the corresponding result in [18].

**Proposition 4.40.** Let $\Lambda \subset G \times \hat{G}$ be closed and cocompact, and let $g,h \in M_{n,d}(E)$ be such that $[g,h]_\bullet$ extends to the identity operator on $L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d)$. Then $[g,h]_\bullet$ is an idempotent operator from $L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d)$ onto $\overline{\text{span}}\{\sum_{i \in \mathbb{Z}_n} \sum_{j \in \mathbb{Z}_d} \pi(\Lambda^\circ) g_{i,j})\}$.

Proof. Since $[g,h]_\bullet$ extends to the identity operator, we have $[g,h]_\bullet = [h,g]_\bullet = 1_{M_d(B)}$. That $[g,h]_\bullet$ is an idempotent then follows by Proposition 3.19. By Proposition 3.12 we get that $[g,h]_\bullet$ is an idempotent from $M_{n,d}(E)$ onto $gM_d(B)$. But this passes to the localization, and the localization of $gM_d(B)$ is

$$\overline{\text{span}}\{\sum_{i \in \mathbb{Z}_n} \sum_{j \in \mathbb{Z}_d} \pi(\Lambda^\circ) g_{i,j})\} \subset L^2(G \times \mathbb{Z}_n \times \mathbb{Z}_d).$$

Lastly in this section, we prove that whenever $\Lambda$ is cocompact, there is a close relationship between the module frame bounds and the Gabor frame bounds in the localization.
Proposition 4.41. Let $\Lambda \subset G \times \hat{G}$ be a closed and cocompact subgroup. Then $g \in M_{n,d}(E)$ generates a module $(n,d)$-matrix frame for $E$ as an $A$-module with lower frame bound $C$ and upper frame bound $D$ if and only if $G(g;\Lambda)$ is an $(n,d)$-matrix Gabor frame for $L^2(G)$ with lower frame bound $C$ and upper frame bound $D$.

Proof. By Lemma 2.13 it suffices to prove that the optimal frame bounds are equal for both the module frame and the Gabor frame. We know that the localization of a module $(n,d)$-matrix frame for $E$ as an $A$-module becomes an $(n,d)$-matrix Gabor frame for $L^2(G)$ with respect to $\Lambda$. Since $\Lambda$ is cocompact, we also know that if $g \in M_{n,d}(E)$ is such that $G(g;\Lambda)$ is an $(n,d)$-matrix Gabor frame for $L^2(G)$, then the canonical dual $S_g^{-1}g \in M_{n,d}(E)$ also. By Proposition 4.22 we have $\rho(\Theta_g) = S_g$. From standard Hilbert space frame theory we know that the optimal upper frame bound for $S_g$ is $\|S_g\|$, and the optimal lower frame bound for $S_g$ is $\|S_g^{-1}\|^{-1}$, see for example Section 5.1 of [13]. We know by Proposition 2.8 that $\|\Theta_g\| = \|\rho(\Theta_g)\| = \|S_g\|$ and $\|\Theta_g^{-1}\| = \|\rho(\Theta_g^{-1})\| = \|S_g^{-1}\|$. The result then follows by Lemma 2.13. □

Remark 4.42. A straightforward calculation will show that $\|\Theta_g\| = \|\overset{\cdot \cdot}{g}\|$. Indeed, as $\Theta_g x = \overset{\cdot \cdot}{g} x$ and $\overset{\cdot \cdot}{g}$ is positive, we see that

$$\|\Theta_g\| = \sup_{\|f\|=1} \{\|\overset{\cdot \cdot}{g} f\|\},$$

and it follows immediately that

$$\|\Theta_g\| \leq \|\overset{\cdot \cdot}{g}\| = \|\overset{\cdot \cdot}{g}\|.$$

Inserting $f = \|\overset{\cdot \cdot}{g}\|^{-1/2} g$ we obtain the equality. Note that this is the same upper bound we obtained in Proposition 3.30.

Corollary 4.43. Let $g \in M_{n,d}(E)$ and let $\Lambda \subset G \times \hat{G}$ be a closed and cocompact subgroup. Then $g$ has the same Bessel bound both as an $(n,d)$-matrix Gabor atom for $L^2(G)$ with respect to $\Lambda$ and as a $(d,n)$-matrix Gabor atom for $L^2(G)$ with respect to $\Lambda^\circ$.

Proof. It suffices to prove that the optimal Bessel bounds agree. Let $D_\Lambda$ be the optimal Bessel bound for $G(g;\Lambda)$, and let $D_{\Lambda^\circ}$ be the optimal Bessel bound for $G(g;\Lambda^\circ)$. By Proposition 4.41 and Remark 4.42 it follows that $D_\Lambda = \|S_g\| = \|\overset{\cdot \cdot}{g}\|$. But the analogous argument works with $\Lambda^\circ$ instead of $\Lambda$, since the important part for the setup with localization as done in this paper is that $\Lambda$ or $\Lambda^\circ$ is cocompact. Indeed, this is really a consequence of Proposition 2.8. Hence we may just as well apply the frame operator $S_g^B$, which is given by the continuous extension of multiplication by $\overset{\cdot \cdot}{g}$ on the left. That is, for all $f \in M_{n,d}(E)$ we have

$$S_g^B f = \overset{\cdot \cdot}{g} f.$$

An analogous argument to the one in Remark 4.42 will show that $D_{\Lambda^\circ} = \|\overset{\cdot \cdot}{g}\|$, and hence $D_\Lambda = D_{\Lambda^\circ}$. □

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