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Hybrid Structure Reliability Analysis Based on the Damped Newton Method

Hongwei Zheng, Guangwei Meng, Feng Li*, Tonghui Wei, Wei Luo, and Yaming Guo

Abstract: This paper presents a hybrid model reliability analysis method based on the damped Newton method with both random and interval variables to solve the hybrid structure reliability problem. The method combines an outer iterative solution and inner layer numerical calculation. In the outer iteration, the method seeks an optimized solution to the interval variable iterative by adding the boundary constraint condition based on the damped Newton optimization theory. In the inner layer solution, the method first reduces the dimension of the random variable through the dimension reduction method, then obtains the first four-order central moment of the function through the application of the Taylor expansion method, and finally calculates the reliability index of the structure according to the fourth-order moment calculation structure of the function. The results of a numerical example and an engineering ten-rod truss structure show that the proposed method can effectively solve the random-interval hybrid reliability problem and has better calculation accuracy than that of the two-layer iterative method.

Key words: random-interval variable hybrid model; reliability analysis; damped Newton optimization; single-layer iteration; fourth-order moment method

1 Background

Mechanical structural reliability design and analysis are affected by various uncertainties, such as the property parameters, load, and geometric dimensions of the materials. This uncertainty information can be processed by a probability model when the sample information of the parameters is sufficient. For variables with insufficient information, only the interval value is available as it cannot be described by an exact distribution function. Therefore, the reliability analysis of the probability-interval hybrid model must be conducted.

Many scholars have studied the reliability analysis method of hybrid models. Chowdhury et al.[1] discussed the reliability analysis of crack uniformity and a bimaterial structure with random and interval uncertainties. Zhang et al.[2] studied the structural reliability analysis of random variables and interval variables, defined them as three types of problems, and processed them with a support vector machine. Wang et al.[3] proposed a new formulation and numerical solution of Reliability-Based Design (RBD) optimization structures exhibiting random and uncertain-but-bound (interval and convex) mixed uncertainties. Shi and Li[4] analyzed the safety of the dynamic structure involving both input random variables and the interval variables. A new dynamic reliability analysis model is presented by constructing a second level limit state function. Chen et al.[5] presented a robust topology optimization method for random-interval uncertainty structures. Du et al.[6] made attempts to solve this problem by proposing an RBD method to deal with
uncertainty variables characterized by a hybrid of the probability distribution and interval. Xiao et al.\cite{7} proposed a Kriging-based effective subset simulation method for Hybrid Reliability Analysis under Random and Interval variables (HRA-RI). Elishakoff\cite{8} worked on the hybrid problem of the uncertainty probability model and convex model. Jiang et al.\cite{9-12} identified a variety of efficient hybrid reliability analysis methods based on a probability model and non-probability-interval model. Kang and Luo\cite{13} applied a single-layer optimization model to embed interval variables in the optimization problem to iteratively search the Most Probable failure Point (MPP). Xie et al.\cite{14} presented a single-loop method that searched for constrained MPP with interval-optimized Karush–Kuhn–Tucker. Subsequently, Xie et al.\cite{15} represented the function with a high-dimensional model and converted the model into a single-step reliability calculation model to solve the hybrid reliability problem. Zhang et al.\cite{16} proposed a new method for reliability analysis of random-interval hybrid structures based on the first-order reliability method.

Although there has been considerable research on reliability analysis methods for probabilistic and interval hybrid models, most require a two-layer iteration of the function to identify the most likely failure point. When the function is nonlinear and complex, the efficiency of the two-layer iteration is relatively low. Therefore, it is crucial to develop a new algorithm to reduce the layer of iteration to increase the computational efficiency.

This paper presents a solution to the probability-interval hybrid reliability with only a single iteration based on the damped Newton method and the fourth-order moment reliability index theory. First, we introduce the random-interval variable hybrid reliability model, then decompose the two-layer iteration into the inner-level numerical calculation and outer-layer single iterative to address the reliability problem. In the outer iteration, the damped Newton optimization theory is applied and the boundary constraint is added based on the damped Newton optimization theory, where condition for iterative termination is that the modulus gradient of the objective function at this point is less than the allowable error. A numerical calculation method is used to address the random variable in the inner layer. The calculation first reduces the random variable dimensions through the dimension reduction method, then obtains the first order central moment of the function through the application of Taylor expansion method, and finally calculates the reliability index of the structure according to the moment information of the function.

2 Hybrid Model with Random-Interval Variables

The structure contains both random and interval variables, and the function of the structure can be stated as

\[ Z = g(x, y^I) \] (1)

where \( x = (x_1, x_2, \ldots, x_n) \) are independent variables and \( y^I = (y_{i+1}^I, y_{i+2}^I, \ldots, y_p^I) \) are interval variables.

\[ y_i^I \in [y_i^L, y_i^R], \quad i = n + 1, n + 2, \ldots, p \] (2)

where \( y_i^L \) is the lower bound of the \( i \)-th interval variable and \( y_i^R \) is the upper bound of the \( i \)-th interval variable. By performing a normal transformation of the random variable \( x \),

\[ \Phi(U) = F_x(x) \] (3)

where \( \Phi() \) represents the standard normal distribution function. \( U \) represents the standard normal space variable and \( F_x() \) represents the probability distribution function. Thus, the random variable \( x \) is

\[ x = F_x^{-1}[\Phi(U)] \] (4)

Substituting Eq. (4) into Eq. (1), the function in the standard normal space is

\[ Z = g(F_x^{-1}(\Phi(U)), y^I) \] (5)

For the convenience of writing, let \( g(F_x^{-1}(\Phi(U)), y^I) \) be denoted as \( G(U, y^I) \). With a plane composed of two standard normal random variables \( U_1 \) and \( U_2 \), as an example, the function \( G(U, y^I) = 0 \) can be defined as a strip-shaped region in a standard normal space, as shown in Fig. 1.

The shaded part in Fig. 1 stands for the reliable area, and the area outside the shaded area is the failure area. \( \beta^L \) is the minimum reliability index and \( \beta^R \) is the

![Fig. 1 Standard normal space limit state area.](image-url)
maximum reliability index. The maximum and minimum failure probability can then be expressed as

$$P_f^\text{max} = P_r\{\min_y G(U, y^I) < 0\} = \Phi(-\beta^I)$$

$$P_f^\text{min} = P_r\{\max_y G(U, y^I) < 0\} = \Phi(-\beta^R)$$

In Eqs. (6) and (7), $P_f^\text{max}$ is the maximum failure probability and $P_f^\text{min}$ is the minimum failure probability. In practical engineering, the maximum failure probability of the structure is most valued by the designer. Therefore, in the subsequent analysis, the reliability of the hybrid structure is measured by the maximum failure probability $P_f^\text{max}$.

In Eq. (6), the hybrid reliability calculation problem can be transformed into the following optimization problem:

$$\beta^I = \min_U \|U\|,$$

s.t. \quad \min G(U, y^I) = 0 \tag{8}$$

This optimization can be transformed into the following two-layer nested optimization problem.

The outer layer is

$$\beta^I = \min_U \|U\|,$$

s.t. \quad G(U, y^I) = 0 \tag{9}$$

The inner layer is

$$G(U, y^*) = \min G(U, y^I),$$

s.t. \quad y^I_k \leq y^I \leq y^R \tag{10}$$

The optimization problems shown in Eqs. (9) and (10) can be solved by a two-layer iteration. The computational efficiency of the two-layer iteration is relatively low when the structural function is high-dimensional nonlinear.

3 Probability-Interval Hybrid Reliability Analysis

To solve the reliability analysis problem in Eq. (6), a structural hybrid method based on the damped Newton optimization model is proposed. The algorithm flow is shown in Fig. 2. The inner layer probability analysis in Fig. 2 is nested in the outer interval optimization, and only a single iterative solution is performed. In this process, the random variable is analyzed in the inner layer. First, we reduce the dimension of the function, use the Taylor expansion method to solve the fourth-order central moment of the function, and obtain the reliability index through the fourth-order central moment. The interval variable is analyzed in the outer layer and

![Fig. 2 Schematic diagram of hybrid reliability analysis method.](image)

the improved damped Newton optimization method is applied for the iterative solution.

The reliability index in Eq. (6) can be expressed as a function of standard normal random variables and interval variables,

$$\beta = -\Phi^{-1}\{P_r[G(U, y^I) < 0]\}$$

Let

$$\beta = \text{Beta}(U, y^I) \tag{11}$$

where Beta denotes a function of the reliability index. Therefore, the minimum reliability index can be obtained by establishing the following optimization model:

$$\beta^I = \min \text{Beta}(U, y^I),$$

s.t. \quad y^I \in (y^L, y^R) \tag{13}$$

3.1 Interval analysis based on damped Newton optimization

In Newton’s optimization, the method approximates the objective function Beta() with an appropriate quadratic function in each iteration and constructs the search direction using the direction of the minimum point of the approximate quadratic function. The precise minimum point of the approximate quadratic function is then determined, which is used as the minimum point of the function Beta(). The damped Newton optimization method is an improved Newton method, which introduces the optimal step. The search factor is calculated and a one-dimensional search is performed in each iteration. The boundary constraints are added in this paper on the basis of the damped Newton method.

According to Eq. (13), the optimization objective function is Beta$(U, y^I)$, the constraint is $y^I \in (y^L, y^R)$, and the design variable is $y^I$. The design variable becomes $y^1_k$ at the $k$-th iteration of the interval variable $y^I$, and the random variable $U$ is treated as an invariant in the outer iteration.

First, Beta() performs the Taylor expansion at $y^1_k$ and the second-order expansion is used.

$$\text{Beta}(U, y^I) \approx \text{Beta}(U, y^I) = \text{Beta}(U, y^I) +$$

$$\Lambda(U, y^I)^T(y^I - y^I_k) + \frac{1}{2}(y^I - y^I_k)^T H(U, y^I_k)(y^I - y^I_k) \tag{14}$$
where \( \beta(\cdot) \) denotes the approximation function when the objective function \( \Delta \) turns into the second-order through Taylor expansion. \( \Delta \) represents the first derivative of the optimization objective function to the interval variable, and \( H \) denotes the second derivative of the optimization objective function to the interval variable.

Equation (15) is the gradient of the second-order approximation objective function \( \beta \),

\[
\nabla \beta(U, y_1^k) = \Delta(U, y_1^k) + H(U, y_1^k)(y_1^k - y_1^l) \quad (15)
\]

In order to obtain the minimum value of \( \beta(U, y_1^1) \), let \( \nabla \beta(U, y_1^k) = 0 \). Then,

\[
\Delta(U, y_1^1) + H(U, y_1^1)(y_1^k - y_1^l) = 0 \quad (16)
\]

Regarding Eq. (16) as the minimum point of the approximation of the \((k + 1)\)-th order of the function \( \beta(U, y_1^k) \), the following expression occurs:

\[
y_1^{k+1} = y_1^1 - H^{-1}(U, y_1^1)\Delta(U, y_1^1) \quad (17)
\]

Let

\[
p_k = -H^{-1}(U, y_1^1)\Delta(U, y_1^1) \quad (18)
\]

where \( p_k \) stands for the search direction. When Eq. (18) is substituted into Eq. (17),

\[
y_1^{k+1} = y_1^k + p_k \quad (19)
\]

In Newton’s method, the search step is always assumed to be 1. However, in the damped Newton method, an optimal step factor is introduced to ensure the convergence speed of the algorithm. The optimal step factor is calculated and a one-dimensional search is conducted in each iteration. The specific method is as follows:

\[
\beta(U, y_1^k + \lambda_k \times p_k) = \min \beta(U, y_1^k + \lambda \times p_k), \quad \text{s.t. } \lambda > 0 \quad (20)
\]

The unary function is applied to address the optimization problem and identify the optimal step factor. The formula is calculated by the following expression:

\[
y_1^l = y_1^k + 1 = y_1^k + \lambda_k \times p_k \quad (21)
\]

An interval constraint check on the new iteration point is conducted. If \( y_1^{k+1} \notin [y_1^l, y_1^R] \), the iteration will continue until the convergence condition is met. If \( y_1^{k+1} \notin [y_1^l, y_1^R] \), the step factor is shortened, and let \( n = 1 \) in \( y_1^{k+1} = y_1^k + p_k \times \lambda_k / 2^n \). \( n \) is the number of boundary constraint checks. Then, the second interval constraint check is performed after we get the value of \( y_1^{k+1} \). If \( y_1^{k+1} \notin [y_1^l, y_1^R] \), let \( n = 2 \) and continue to calculate the new iteration point until the iteration point satisfies the boundary constraint condition. The iteration then continues with the new iteration point until the convergence condition, when the gradient modulus of the objective function at the iteration point is less than the set tolerance is met.

### 3.2 Probability analysis based on the dimension reduction method

#### 3.2.1 Dimension reduction method

In the \( k \)-th iteration of the interval variable, the interval variable \( y_1^k \) is constant and the structural function \( Z_k = G(U, y_1^k) \) can be expressed as the sum of the lower order function at the increasing level, the formula of dimension reduction method[17,18] is as follows:

\[
G(U, y_1^k) = G_0 + \sum_{i=1}^{n} G_i(U_i, y_1^k) + \cdots + \sum_{1 \leq l_1 < \cdots < l_n \leq n} G_{i_1, i_2, \ldots, i_n}(U_{i_1}, U_{i_2}, \ldots, U_{i_l}, y_1^k) + \cdots + G_{1, 2, \ldots, n}(U_1, U_2, \ldots, U_n, y_1^k) \quad (22)
\]

where \( G_0 \) is a constant term, \( G_i(U_i, y_1^k) \) represents the first-order expression of the structural function when the random variable \( U_i \) and the interval variable \( y_1^k \) work together, and \( G_{ij}(U_i, U_j, y_1^k) \) stands for the second-order expression of the structural function when the variables \( U_i, U_j \), and the interaction \( y_1^k \) work together. The items reflect the influence of all increasing variables to the structural function when the variables work together.

The Random Sample Dimension Reduction Method (RS-DRM) and central section dimension reduction method (called Cut-DRM) are commonly used in the dimension reduction. Reference [19] adapted an alternative model based on the original input-output relationship in the Cut-DRM method. In the Cut-DRM method, \( G(U, y_1^k) \) is approximately defined by the information of the center point \( c = (c_1, c_2, \ldots, c_n) \) in input space \( G(U, y_1^k) \). The expressions of the expanded items are in the following:

\[
\left\{ \begin{array}{l}
G_0 = G_0(U, y_1^k); \\
G_i(U_i, y_1^k) = G(c_1, \ldots, c_{i-1}, U_i, c_{i+1}, \ldots, c_n, y_1^k) - G_0; \\
G_{ij}(U_i, U_j, y_1^k) = G(c_1, \ldots, c_{i-1}, U_i, c_{i+1}, \ldots, c_{j-1}, U_j, c_{j+1}, \ldots, c_n, y_1^k) - G_i(U_i, y_1^k) - G_j(U_j, y_1^k) - G_0
\end{array} \right. \quad (23)
\]

For a sufficiently smooth structural function, the high-order term has less impact than the low-order term. Therefore, only the low-order term is retained to approximate the function. In moment calculation and reliability analysis, Ref. [20] indicated that the most advantageous value of the center point \( c \) is the mean of
each variable, which is expressed as \( \mu U \).

In summary, the univariate representation of the \( n \)-dimension function is the following expression:

\[
G(U, y^k) \approx \sum_{i=1}^{n} G(U_i, y^k_i) - (n-1)G(U_0, y^k) \tag{24}
\]

where \( G(U_i, y^k_i) = G(\mu U_i, \ldots, \mu U_{i-1}, U_i, \mu U_{i+1}, \ldots, \mu U_n, y^k) \) represents the one-dimensional function, and \( G(U_0, y^k) = G(\mu U_1, \ldots, \mu U_i, \ldots, \mu U_n, y^k) \) is a constant term.

### 3.2.2 Statistical moment of the structure function

The point estimation method is commonly used to solve the statistical moments of functions in the reliability analysis of engineering structures. However, only the first three moments of the random variables are used in the point estimation method, and the calculation accuracy is not as high as the Taylor series expansion even after the improvement by the fourth moment. Moreover, the decrease in the accuracy of the calculation results from many various factors, such as the mean value and coefficient of variation. Therefore, this paper obtains the first four-order central moment of the structural function through the Taylor expansion theory. The structural function after the dimension reduction is computed as

\[
Z_k^1 = G(U, y^k_1),
\]

\[
\mu Z_k^1 = E(Z_k^1) = G(\mu U, y^k_1) + \frac{1}{2} \sum_{i=1}^{n} A_{ik} \mu U_i \tag{25}
\]

\[
\mu Z_k^2 = E[(Z_k^1 - \mu Z_k^1)^2] = \sum_{i=1}^{n} (B_{ik}^1)^2 \mu U_i^2 + \frac{1}{2} \sum_{i=1}^{n} A_{ik}^1 B_{ik}^1 \mu U_i^3 + \frac{1}{2} \sum_{i=1}^{n} A_{ik}^1 \mu U_i^4 - 3(\mu U_i^2)^2 + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} (C_{ijk}^1)^2 \mu U_i^2 \mu U_j^2 \tag{26}
\]

\[
\mu Z_k^3 = E[(Z_k^1 - \mu Z_k^1)^3] = \sum_{i=1}^{n} (B_{ik}^1)^3 \mu U_i^3 + \frac{3}{2} \sum_{i=1}^{n} A_{ik}^1 (B_{ik}^1)^2 (\mu U_i^4 - 3(\mu U_i^2)^2) + \frac{3}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} B_{ik}^1 C_{ijk}^1 D_{ijk}^1 \mu U_i^2 \mu U_j^2 \tag{27}
\]

\[
\mu Z_k^4 = E[(Z_k^1 - \mu Z_k^1)^4] = \sum_{i=1}^{n} (B_{ik}^1)^4 (\mu U_i^4 - 3(\mu U_i^2)^2) + \frac{3}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} (B_{ik}^1)^2 (D_{ijk}^1)^2 \mu U_i^2 \mu U_j^2 \tag{28}
\]

where \( E[\cdot] \) is the expected operator, and \( \mu U_1, \mu U_2, \ldots, \mu U_n \) are the second-, third-, and fourth-order central moments of the random variable in sequence, respectively. Similarly, \( \mu Z_k^1, \mu Z_k^2, \mu Z_k^3, \) and \( \mu Z_k^4 \) are the central moments of the four orders of the structural function.

### 3.2.3 Reliability index by the moment method

After obtaining the statistical moment of the functional function, the failure probability interval is calculated based on the moment estimation method. According to the different degrees of complexity of the problems studied, the failure probability is calculated using the second-, third-, and fourth-order moments of the limit state function. Among them, the second- and fourth-order moment methods are relatively easy for implementation and the fourth-order moment method has better precision. Therefore, this paper combines the first four moments of the function to produce the expression of the failure probability as follows:

\[
\alpha g_k^1 = \mu Z_k^1 \tag{33}
\]

\[
\alpha g_k^2 = \sqrt{\mu Z_k^2} \tag{34}
\]

\[
\alpha g_k^3 = \left( \frac{\mu Z_k^3}{\alpha g_k^2} \right)^3 \tag{35}
\]

\[
\alpha g_k^4 = \left( \frac{\mu Z_k^4}{\alpha g_k^2} \right)^4 \tag{36}
\]

\[
\beta_k^1 = \begin{cases} 
\frac{3(\alpha g_k^4 - 1)\alpha g_k^1}{\alpha g_k^3 - \alpha g_k^3} + \alpha g_k^3 ((\alpha g_k^1)^2 - 1) & \alpha g_k^3 \neq 0 \\
\alpha g_k^1 & \alpha g_k^3 = 0
\end{cases} \tag{37}
\]

In Eq. (37), \( \beta_k^1 \) represents the reliability index of the structural function at the \( k \)-th iteration of the interval, \( \alpha g_k^1, \alpha g_k^2, \alpha g_k^3, \) and \( \alpha g_k^4 \) denote the mean, standard deviation, skewness coefficient, and kurtosis coefficient of the structural function, respectively.
The hybrid reliability analysis proposed in this paper is shown in the algorithm flow chart of Fig. 3.

1. Select the initial interval variable value $y_0$, determine the allowable error $\varepsilon$, and assume $k = 0$.
2. Apply Eqs. (33)–(36) to solve the statistical moments of the functional function.
3. Apply Eq. (37) to obtain the reliability index.
4. Calculate the gradient of the reliability index $\nabla(U, y^k)$ by using the interval variable $y^k$ as an independent variable.
5. Check the convergence. If $\|\nabla(U, y^k)\| \leq \varepsilon$, terminate the calculation, otherwise continue.
6. Construct a search direction according to Eq. (18) and conduct a one-dimensional search.
7. Update the point column according to Eq. (15).
8. Check the new point boundary following the method in Section 2.
9. Let $k = k + 1$, and turn to Eq. (21).

4 Examples of the Numerical Value

Two examples are given to verify the accuracy, validity, and stability of the method, in which the convergence precision is defined as $\varepsilon = 1 \times 10^{-6}$.

4.1 Example 1

Consider the following limit state function:

$$W = 3x_1^5 - 2x_2^3x_3y_1^2 + 1.2 \frac{x_1}{x_2x_3} y_2 - 60275$$  (38)

where $x_1$, $x_2$, and $x_3$ are random variables, and $y_1$ and $y_2$ are interval variables. The distribution type and distribution parameters of the variables are shown in Table 1.

Note: For random variables, Parameter 1 is the mean, Parameter 2 represents the standard deviation, and

![Flow chart of the hybrid reliability analysis based on the damped Newton method.](image-url)
Table 1 Statistical parameters of basic variables.

| Uncertain variable | Parameter 1 | Parameter 2 | Parameter 3 | Parameter 4 | Distribution type |
|--------------------|-------------|-------------|-------------|-------------|------------------|
| $x_1$              | 7.9200      | 0.3168      | 4.1977      | 3.1006×10$^{-2}$ | Lognormal         |
| $x_2$              | 1.3000      | 0.1040      | 0           | 3.4565×10$^{-4}$ | Normal distribution |
| $x_3$              | 0.9800      | 0.0196      | 8.6280×10$^{-6}$ | 7.9718×10$^{-7}$ | Extreme value I |
| $y_1$              | 2.8215      | 2.8785      | –           | –           | Interval variable |
| $y_2$              | 1.4850      | 1.5150      | –           | –           | Interval variable |

Parameters 3 and 4 denote the third- and fourth-order central moments of the random variable, respectively. For interval variables, Parameter 1 is the lower bound of the variable and Parameter 2 is the upper bound of the variable.

The maximum failure probability of the function is calculated by the method proposed in this paper. In order to verify the method correctness, the Interval Monte Carlo Method (IMCM)\cite{21} is applied to calculate the failure probability of the function, the flow chat is shown in Fig. 4. IMCM divides the two interval variables into

Fig. 4 IMCM calculation flow chart.

Table 2 Failure probability calculation results.

| Method               | $P_{max}$ | Relative error (%) | Number of samples |
|----------------------|-----------|--------------------|-------------------|
| Interval Monte Carlo | 0.0023    | –                  | $10^6$            |
| Second iteration     | 0.0026    | 13.04              | 18                |
| Algorithm in this paper | 0.0024 | 4.35              | 10                |

$n$ intervals in the interval range for $n + 1$ interval points. According to Example 1, the interval points of the two interval variables are combined in pairs to obtain the $(n + 1)^2$ interval points. The Monte Carlo method is used to calculate the failure probability of each interval point, the $(n + 1)^2$ failure probabilities are obtained, and the maximum value is taken as the maximum failure probability. The following shows the calculation process of the IMCM of Example 1.

According to the calculation flow shown in Fig. 4, the greater the number $n$ of the interval variables is divided, the larger the amount of the calculation is required. In this paper, $n = 100$ and the number of samples of the random variables is $10^6$. The calculation results are shown in Table 2. As shown in Table 2, the relative error between the proposed method and IMCM is 4.35%. If the second iteration method is used, the relative error between the results and IMCM is 13.04%. Consequently, the accuracy is higher in the method adopted in this paper.

Figure 5 compares the proposed method and two-layer iterative convergence process. The abscissa is the number of iterations, and the ordinate is the reliability index. In this paper, the method tends to converge after five iterations, while it requires six iterations before convergence in the two-layer iterative method. In conclusion, this method has a faster convergence speed.

4.2 Example 2

A planar ten-bar truss structure is shown in Fig. 6. $E = 2.1 \times 10^{11}$ Pa denotes the elastic modulus, $\rho = 7.8 \times 10^3$ kg/m$^3$ denotes the material density, and $[\sigma] = 160$ MPa denotes the allowable stress. The length of each rod $L$, from Rod 1 to Rod 6, is 3.6 m and the area $A_{1-6}$ is 0.03 m$^2$. The area $A_{7-10}$ of Rods 7–10 is
Fig. 5 Convergence comparison of the proposed method with the two-layer iterative method.

Fig. 6 Ten-rod truss structure.

0.02 m². Planar constraints are applied at Nodes 5 and 6, a vertical downward load $P_1$ is applied at Node 4, and vertical load $P_2$ and horizontal load $P_3$ are applied at Node 2. Among them, $P_1$, $P_2$, and $P_3$ are random variables and $L$ are interval variables. The corresponding statistical parameters are listed in Table 3.

According to the theory of structural mechanics, $\Delta$ can be used to define the analytic relationship of the truss geometry, cross-sectional area parameters, material parameters of the bar, and external load. The expression is as follows:

$$\Delta = \left( \sum_{i=1}^{6} N_i^0 N_{i_i} / A_i + \sqrt{2} \sum_{i=7}^{10} N_i^0 N_{i_i} / A_i \right) L / E$$

(39)

where $N_i$ is expressed as

$$N_1 = P_2 - \frac{\sqrt{2}}{2} N_8$$

(40)

$$N_2 = -\frac{\sqrt{2}}{2} N_{10}$$

(41)

$$N_3 = -P_1 - 2P_2 + P_3 - \frac{\sqrt{2}}{2} N_8$$

(42)

$$N_4 = -P_2 + P_3 - \frac{\sqrt{2}}{2} N_{10}$$

(43)

$$N_5 = -P_2 - \frac{\sqrt{2}}{2} N_8 - \frac{\sqrt{2}}{2} N_{10}$$

(44)

$$N_6 = -\frac{\sqrt{2}}{2} N_{10}$$

(45)

$$N_7 = \sqrt{2}(P_1 + P_2) + N_8$$

(46)

$$N_8 = \frac{a_{22} b_1 - a_{12} b_2}{a_{11} a_{22} - a_{12} a_{21}}$$

(47)

$$N_9 = \sqrt{2} P_2 + N_{10}$$

(48)

$$N_{10} = \frac{a_{11} b_2 - a_{21} b_1}{a_{11} a_{22} - a_{12} a_{21}}$$

(49)

where

$$a_{11} = \left( \frac{1}{A_1} + \frac{1}{A_3} + \frac{1}{A_5} + \frac{2\sqrt{2}}{A_7} + \frac{2\sqrt{2}}{A_8} \right) \frac{L}{2E}$$

(50)

$$a_{22} = \left( \frac{1}{A_2} + \frac{1}{A_4} + \frac{1}{A_5} + \frac{1}{A_6} + \frac{2\sqrt{2}}{A_9} + \frac{2\sqrt{2}}{A_{10}} \right) \frac{L}{2E}$$

(51)

Table 3 Statistical eigenvalues of ten-rod truss variables.

| Uncertain variable | Mean (kN) | Coefficient of variation | Skew coefficient | Kurtosis coefficient | Distribution type |
|-------------------|-----------|--------------------------|------------------|---------------------|------------------|
| $P_1$             | 750       | 0.10                     | 0.3014           | 3.1631              | Lognormal        |
| $P_2$             | 950       | 0.12                     | 0.3625           | 3.2351              | Lognormal        |
| $P_3$             | 950       | 0.11                     | 0                 | 3.0000              | Normal distribution |
| $L$               | 3.55      | 3.65                     | –                | –                   | Interval variable |


\[ a_{12} = a_{21} = \frac{L}{2A_5E} \]  \hspace{1cm} (52)

\[ b_1 = \left( \frac{P_2 - 2P_2 + P_1 - P_3}{A_3} \right) \frac{\sqrt{2}}{A_7} \left( \frac{\sqrt{2}L}{2E} \right) \]  \hspace{1cm} (53)

\[ b_2 = \left( \frac{\sqrt{2}(P_3 - P_2)}{A_4} - \frac{\sqrt{2}P_2}{A_5} - \frac{4P_2}{A_9} \right) \frac{L}{2E} \]  \hspace{1cm} (54)

\[ P_1 = 1, \quad P_2 = 0, \quad \text{and} \quad N_i^0 \text{ indicates the axial force of the rod. According to the above derivation, the structural function expression of the ten-rod truss structure is} \]

\[ g = 0.008 - \left( \sum_{i=1}^{6} \frac{N_i^0 N_i}{A_i} + \sqrt{2} \sum_{i=7}^{10} \frac{N_i^0 N_i}{A_i} \right) \frac{L}{E} \]  \hspace{1cm} (55)

Matlab software is used to solve the algorithm flow of this paper and the calculation results are shown in Table 4. The maximum failure probability calculated by the algorithm is 0.0145. In order to verify the correctness of the algorithm, the IMCM results were used as the reference value. When the number of random variables is 10⁶, the IMCM calculation is 0.0149. The relative error between the proposed algorithm and IMCM is 2.68%, which proves that the proposed algorithm has better computational accuracy.

5 Conclusion

This paper proposed a hybrid reliability method based on the improved damped Newton optimization, addressing the problem of reliability analysis of the probability-interval hybrid structure. First, the damped Newton optimization method was improved and the boundary conditions were added. The method then combines the improved damped Newton optimization method, the dimension reduction method, and the fourth-order moment method of random variables before it conducts the numerical calculation of the failure probability of the function. The computational efficiency is improved as the original two iterations were reduced to one. Examples of numerical calculation and engineering show that the calculation results of this method are close to those of the IMCM when applied to solving the probability-interval hybrid structure reliability problem. Therefore, this method has important engineering significance for the reliability design of engineering structures.

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