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Upper bound estimate of incomplete Cochrane sum

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Abstract: By using the properties of Kloosterman sum and Dirichlet character, an optimal upper bound estimate of incomplete Cochrane sum is given.

Keywords: Dedekind sum, Cochrane sum, Kloosterman sum, Gauss sum

MSC: 11F20, 11L05, 11L07

1 Introduction

Let \( q \) be a positive integer, then for an arbitrary integer \( h \), the famous Dedekind sum \( S(h, q) \) is defined as

\[
S(h, q) = \sum_{a=1}^{q} \left( \left( \frac{a}{q} \right) \left( \frac{ha}{q} \right) \right),
\]

where

\[
\left( \frac{x}{y} \right) = \begin{cases} 
 x - [x] - \frac{1}{2}, & \text{if } x \text{ is not an integer}, \\
 0, & \text{if } x \text{ is an integer}.
\end{cases}
\]

In October 2000, during his visit to Xi'an, Todd Cochrane introduced a sum analogous to it as follows:

\[
C(h, q) = \sum_{a=1}^{q} \left( \left( \frac{a}{q} \right) \left( \frac{ha}{q} \right) \right),
\]

where \( \sum_{a=1}^{q} \) denotes the summation over all \( 1 \leq a \leq q \) such that \( (a, q) = 1 \), and \( \overline{a} \) is defined by \( a \overline{a} \equiv 1 \pmod{q} \).

Since then, various properties of \( C(h, q) \) are studied by many scholars. For example, Zhang and Yi [1] obtained the upper bound estimate

\[
|C(h, q)| \ll \sqrt{q}d(q) \ln^2 q,
\]

where \( d(q) \) is the classical divisor function. For the case \( q = p \), an odd prime, they also gave a sharp asymptotic formula

\[
\sum_{h=1}^{p-1} C^2(h, p) = \frac{5}{144} p^2 + O \left( p \exp \left( \frac{4 \ln p}{\ln \ln p} \right) \right).
\]
which proved that (1) is optimal. For arbitrary integer $q \geq 3$, Zhang [2] studied the mean square of the Cochrane sum $C(h, q)$, and obtained
\[
\sum_{h=1}^{q} C^2(h, q) = \frac{5}{144} \Phi(q) \prod_{p^2 \parallel q} \left( \frac{p+1}{p+2} \right)^2 + \frac{1}{p+1} + \frac{1}{p^3} + O\left( q \exp \left( \frac{4 \ln q}{\ln \ln q} \right) \right).
\]
where $\Phi(q)$ denotes the Euler function and $\prod_{p^2 \parallel q}$ the product over all prime divisors of $q$ with $p^2 \parallel q$ and $p^2 + 1 \parallel q$.

Later Lu and Yi [3] gave the mean square value of $C(h, q)$ over incomplete intervals. In fact, under the conditions that $q > 3$ is a square-free integer and $\alpha$ a real number with $\alpha \in (0, 1]$, they got
\[
\sum_{h=1}^{N} C^2(h, q) = \frac{5\delta}{144} \omega^3(q) \prod_{p \mid q} \frac{p^2 + 2p - 1}{p^2 + 1} + O\left( q^{2-\frac{1}{2}\delta} + \epsilon \right),
\]
where $\omega(q) = \sum_{d \mid q} 1$, $\epsilon$ is a sufficiently small positive constant and the $O$ constant depends only on $\epsilon$.

Other properties like the high-dimensional generalizations and hybrid mean values involving $C(h, q)$ can be found in references [4-8] and therein.

For arbitrary integers $m$ and $n$, Estermann [9] gave an upper bound estimate of the classical Kloosterman sum $S(m, n; q)$ as
\[
|S(m, n; q)| \leq (m, n, q) \frac{1}{2} q \frac{1}{2} d(q), \tag{2}
\]
where $S(m, n; q)$ is defined by
\[
S(m, n; q) = \sum_{a \mod q} e \left( \frac{ma + n\overline{a}}{q} \right),
\]
where $e(x) = e^{2\pi i x}$, and $(m, n, q)$ denotes the greatest common divisor of $m, n, q$. By completing method, one can derive immediately from (2) an upper bound estimate of the incomplete Kloosterman sum
\[
S(m, n; q, I) = \sum_{a \in I} e \left( \frac{ma + n\overline{a}}{q} \right)
\]
as the following:
\[
|S(m, n; q, I)| \ll q^{1+\epsilon} (m, q) \frac{1}{2}. \tag{3}
\]
where $I$ is an interval with length not exceeding $q$.

Now we define an incomplete Cochrane sum as follows:
\[
C(h, q; \lambda) = \sum_{a=1}^{\lambda q} \left( \frac{a}{q} \right) \left( \frac{ha}{q} \right), \tag{4}
\]
where $\lambda \in (0, 1]$. By using the properties of Kloosterman sum and Dirichlet character, we shall prove the following:

**Theorem.** Let $q, h$ be integers with $q \geq 2$ and $(h, q) = 1$, $\lambda$ be a real number with $\lambda \in (0, 1]$. Then we have the upper bound estimate
\[
|C(h, q; \lambda)| \ll q^{1+\epsilon}.
\]

Taking $\lambda = 1$ in Theorem, we may immediately obtain

**Corollary.** Let $q, h$ be integers with $q \geq 2$ and $(h, q) = 1$. Then we have
\[
|C(h, q)| \ll q^{1+\epsilon},
\]
which is almost the same estimate as (1).
Some lemmas

To prove Theorem, we need the following several lemmas.

Lemma 1.1. Let \( q, h \) be integers with \( q \geq 2 \) and \((h,q) = 1\), \( \lambda \) be a real number with \( \lambda \in (0, 1] \). Then we have the identity

\[
C(h, q; \lambda) = -\frac{1}{2\pi^2 \varphi(q)} \sum_{\chi \mod q \atop \chi(-1) = -1} \chi(h) \left( \sum_{m=1}^{\infty} \frac{G(\chi, m; \lambda) - G(\chi, -m; \lambda)}{m} \right) \left( \sum_{n=1}^{\infty} \frac{G(\chi, n)}{n} \right).
\]

where \( \chi \) denotes a Dirichlet character modulo \( q \), \( G(\chi, m; \lambda) = \sum_{c=1}^{\lambda q} \chi(c) e \left( \frac{cm}{q} \right) \) denotes the partial Gauss sum corresponding to \( \chi \), and \( G(\chi, m) := G(\chi, m; 1) \).

Proof. From (4) and the orthogonality relation for characters modulo \( q \), we have

\[
C(h, q; \lambda) = \sum_{a \leq \lambda q \atop (a, q) = 1} \left( \frac{a}{q} \right) \left( \frac{\overline{ah}}{q} \right) = \frac{1}{\varphi(q)} \sum_{\chi \mod q \atop \chi(-1) = -1} \chi(a) \left( \sum_{b=1}^{q} \chi(b) \left( \frac{bh}{q} \right) \right).
\]

Note that

\[
((x)) = -\frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2\pi nx)}{n}, \quad \sin \chi = \frac{1}{2i}(e^{ix} - e^{-ix})
\]

and for arbitrary integer \( r \) with \((r, q) = 1\),

\[
G(\chi, rn) = \overline{\tau(r)} G(\chi, n)
\]

and

\[
\sum_{c=1}^{q} \chi(c) \left( \frac{cr}{q} \right) = 0 \quad \text{if} \quad \chi(-1) = 1.
\]

From these identities, we have

\[
C(h, q; \lambda) = \frac{1}{\pi^2 \varphi(q)} \sum_{\chi \mod q \atop \chi(-1) = -1} \left( \sum_{m=1}^{\infty} \frac{\chi(a) \sin \left( \frac{2\pi ma}{q} \right)}{m} \right) \left( \sum_{n=1}^{\infty} \frac{\chi(b) \sin \left( \frac{2\pi nbh}{q} \right)}{n} \right)
\]

\[
= -\frac{1}{4\pi^2 \varphi(q)} \sum_{\chi \mod q \atop \chi(-1) = -1} \left( \sum_{m=1}^{\infty} \frac{G(\chi, m; \lambda) - G(\chi, -m; \lambda)}{m} \right) \left( \sum_{n=1}^{\infty} \frac{G(\chi, n) - G(\chi, -n)}{n} \right)
\]

\[
= -\frac{1}{4\pi^2 \varphi(q)} \sum_{\chi \mod q \atop \chi(-1) = -1} \overline{\chi(h)} \left( \sum_{m=1}^{\infty} \frac{G(\chi, m; \lambda) - G(\chi, -m; \lambda)}{m} \right) \left( \sum_{n=1}^{\infty} \frac{G(\chi, n) - G(\chi, -n)}{n} \right)
\]

\[
= -\frac{1}{2\pi^2 \varphi(q)} \sum_{\chi \mod q \atop \chi(-1) = -1} \overline{\chi(h)} \left( \sum_{m=1}^{\infty} \frac{G(\chi, m; \lambda) - G(\chi, -m; \lambda)}{m} \right) \left( \sum_{n=1}^{\infty} \frac{G(\chi, n)}{n} \right). \quad \square
\]

Lemma 1.2. Let \( q, h \) be integers with \( q \geq 2 \) and \((h,q) = 1\), \( \lambda \) be a real number with \( \lambda \in (0, 1] \). Then we have the estimates

\[
\sum_{\chi \mod q \atop \chi(-1) = -1} \overline{\chi(h)} \left( \sum_{m=1}^{\infty} \frac{G(\chi, m; \lambda)}{m} \right) \left( \sum_{n=1}^{\infty} \frac{G(\chi, n)}{n} \right) \ll \varphi(q)q^{1+\varepsilon}.
\]
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\[
\sum_{\chi \mod q \chi(1)=-1} \mathcal{Z}(h) \left( \sum_{m=1}^{\infty} \frac{G(\chi, m; \lambda)}{m} \right) \left( \sum_{n=1}^{\infty} \frac{G(\chi, n)}{n} \right) \ll \varphi(q)q^{\frac{1}{2}+\varepsilon}.
\]

**Proof.** For any non-principal Dirichlet character \( \chi \) modulo \( q \) and any parameter \( N \geq q \), applying Abel’s identity [10], we have

\[
\sum_{m=1}^{\infty} \frac{G(\chi, m; \lambda)}{m} = \sum_{1 \leq m \leq N} \frac{G(\chi, m; \lambda)}{m} + \int_{N}^{\infty} \frac{A(y, \chi)}{y^2} dy,
\]

\[
\sum_{n=1}^{\infty} \frac{G(\chi, n)}{n} = \sum_{1 \leq n \leq N} \frac{G(\chi, n)}{n} + \int_{N}^{\infty} \frac{B(y, \chi)}{y^2} dy,
\]

where \( A(y, \chi) = \sum_{N < m \leq y} G(\chi, m; \lambda) \), \( B(y, \chi) = \sum_{N < n \leq y} G(\chi, n) \).

So we have

\[
\sum_{\chi \mod q \chi(1)=-1} \mathcal{Z}(h) \left( \sum_{m=1}^{\infty} \frac{G(\chi, m; \lambda)}{m} \right) \left( \sum_{n=1}^{\infty} \frac{G(\chi, n)}{n} \right)
\]

\[
= \sum_{\chi \mod q \chi(1)=-1} \mathcal{Z}(h) \left( \sum_{1 \leq m \leq N} \frac{G(\chi, m; \lambda)}{m} \right) \left( \sum_{1 \leq n \leq N} \frac{G(\chi, n)}{n} \right)
\]

\[
+ \sum_{\chi \mod q \chi(1)=-1} \mathcal{Z}(h) \left( \sum_{1 \leq n \leq N} \frac{G(\chi, n)}{n} \right) \left( \int_{N}^{\infty} \frac{A(y, \chi)}{y^2} dy \right)
\]

\[
+ \sum_{\chi \mod q \chi(1)=-1} \mathcal{Z}(h) \left( \sum_{1 \leq m \leq N} \frac{G(\chi, m; \lambda)}{m} \right) \left( \int_{N}^{\infty} \frac{B(y, \chi)}{y^2} dy \right)
\]

\[
+ \sum_{\chi \mod q \chi(1)=-1} \mathcal{Z}(h) \left( \int_{N}^{\infty} \frac{A(y, \chi)}{y^2} dy \right) \left( \int_{N}^{\infty} \frac{B(y, \chi)}{y^2} dy \right)
\]

\[
:= M_1 + M_2 + M_3 + M_4.
\]

Now we estimate \( M_1, M_2, M_3, M_4 \) respectively. First note that

\[
G(\chi, n)G(\chi, m; \lambda) = \sum_{a=1}^{q} \chi(a)e\left(\frac{n\bar{a}}{q}\right) \sum_{b=1}^{\lambda \bar{q}} \chi(b)e\left(\frac{mb}{q}\right) = \sum_{b \leq \lambda \bar{q}} \sum_{a=1}^{q} \chi(a)e\left(\frac{n\bar{a}b + mb}{q}\right).
\]

By the orthogonality relation for characters modulo \( q \) as the following:

\[
\sum_{\chi \mod q \chi(1)=-1} \chi(a) = \begin{cases} 
\frac{1}{2} \varphi(q), & \text{if } a \equiv 1 \pmod{q}, \\
\frac{-1}{2} \varphi(q), & \text{if } a \equiv -1 \pmod{q}, \\
0, & \text{otherwise},
\end{cases}
\]

we have

\[
M_1 = \sum_{\chi \mod q \chi(1)=-1} \mathcal{Z}(h) \left( \sum_{1 \leq m \leq N} \frac{G(\chi, m; \lambda)}{m} \right) \left( \sum_{1 \leq n \leq N} \frac{G(\chi, n)}{n} \right)
\]
\[
\begin{align*}
&= \sum_{1 \leq m \leq N} \sum_{1 \leq n \leq N} \frac{1}{m} \sum_{a=1}^{q} \sum_{b \leq \lambda q} e \left( \frac{na\overline{h} + mb}{q} \right) \sum_{\chi \mod q, \chi(-1)=-1} \chi(a) \overline{\chi}(h)
&= \frac{1}{2} \varphi(q) \sum_{1 \leq m \leq N} \sum_{1 \leq n \leq N} \frac{1}{mn} \sum_{b \leq \lambda q} e \left( \frac{mb + nh\overline{h}}{q} \right) \\
&= \frac{1}{2} \varphi(q) \sum_{1 \leq m \leq N} \sum_{1 \leq n \leq N} \frac{1}{mn} \sum_{b \leq \lambda q} e \left( \frac{mb - nh\overline{h}}{q} \right) \\
&\ll \varphi(q) \sum_{1 \leq m \leq N} \sum_{1 \leq n \leq N} \frac{1}{mn} \left| \sum_{b \leq \lambda q} e \left( \pm nh\overline{h} + mb \right) \right| \\
&\ll \varphi(q)q^{1/2+\varepsilon} \sum_{1 \leq m \leq N} \sum_{1 \leq n \leq N} \frac{1}{mn} (m, q)^{1/2} \\
&= \varphi(q)q^{1/2+\varepsilon} \sum_{u | q} \sum_{1 \leq m \leq N/u} \sum_{1 \leq n \leq N} \frac{u^{1/2}}{mun} \\
&\ll \varphi(q)q^{1/2+\varepsilon} \ln^2 N,
\end{align*}
\]
where we have used the upper bound of (3).

Then from the estimates for trigonometric sum and Gauss sum we can also get
\[
|A(y, \chi)| = \left| \sum_{b \leq \lambda q} \chi(b) \sum_{N < m \leq y} e \left( \frac{mb}{q} \right) \right| \\
\ll \sum_{b \leq \lambda q} \left| \sin \frac{\pi bq}{q} \right| \ll \sum_{b \leq \lambda q} \frac{q}{b} \\
\ll q \ln(\lambda q).
\]

and
\[
\sum_{\chi \mod q, \chi(-1)=-1} |G(\chi, n)| = \sum_{\chi \mod q, \chi(-1)=-1} |G(\chi, 1)| \ll q^{1/2} \varphi(q).
\]

Thus we have the estimate
\[
M_2 \ll \left| \sum_{\chi \mod q, \chi(-1)=-1} \overline{\chi}(h) \left( \sum_{1 \leq n \leq N} \frac{G(\chi, n)}{n} \right) \left( \int_{N}^{\infty} \frac{A(y, \chi)}{y^2} \, dy \right) \right| \\
\ll \sum_{1 \leq n \leq N} \frac{1}{n} \sum_{\chi \mod q, \chi(-1)=-1} |G(\chi, n)| \int_{N}^{\infty} \frac{q \ln(\lambda q)}{y^2} \, dy \\
\ll q^{1/2} \ln(\lambda q) \cdot \varphi(q) \ln N.
\]

Then we shall estimate \(M_3\). Since
\[
\sum_{\chi \mod q, \chi(-1)=-1} |G(\chi, m; \lambda)| \leq q^{1/2} (q) \left( \sum_{\chi \mod q, \chi(-1)=-1} |G(\chi, m; \lambda)|^2 \right)^{1/2} \\
= q^{1/2} (q) \left( \sum_{a \leq \lambda q} \sum_{b \leq \lambda q} e \left( \frac{ma - nb}{q} \right) \sum_{\chi \mod q, \chi(-1)=-1} \chi(a) \overline{\chi}(b) \right)^{1/2}
\]

\[ \ll \phi^{1/2}(q) \left( \sum_{a \leq \lambda q} 1 \cdot \phi(q) \right)^{1/2} \]

and

\[ \ll \phi(q)(\lambda q)^{1/2}, \]

Hence

\[ |B(y, \chi)| \ll q \ln q. \]

At last, we have

\[ M_4 = \sum_{\chi \mod q \atop \chi(-1) = -1} \overline{\chi}(h) \left( \int_{-\infty}^{\infty} A(y, \chi) \frac{dy}{y^2} \right) \left( \int_{-\infty}^{\infty} B(y, \chi) \frac{dy}{y^2} \right) \]

\[ \ll \sum_{1 \leq m \leq N} \frac{1}{m} \sum_{\chi \mod q \atop \chi(-1) = -1} |G(\chi, m; \lambda)| \int_{-\infty}^{\infty} q \ln q \frac{dy}{y^2} \]

\[ \ll \frac{\lambda^{1/2}q^2\phi(q) \ln q \cdot \ln N}{N}. \]

Taking \( N = q^2 \), combining the estimates of \( M_1, M_2, M_3, M_4 \), we can get

\[ \sum_{\chi \mod q \atop \chi(-1) = -1} \overline{\chi}(h) \left( \sum_{n=1}^{\infty} \frac{G(\chi, n; \lambda)}{n} \right) \left( \sum_{n=1}^{\infty} \frac{G(\chi, n)}{n} \right) \ll \phi(q)q^{1/2+e}. \]

Similarly, we can also get

\[ \sum_{\chi \mod q \atop \chi(-1) = -1} \overline{\chi}(h) \left( \sum_{n=1}^{\infty} \frac{G(\chi, -n; \lambda)}{n} \right) \left( \sum_{n=1}^{\infty} \frac{G(\chi, n)}{n} \right) \ll \phi(q)q^{1/2+e}. \]

2 Proof of Theorem

In this section, we shall complete the proof of Theorem. For arbitrary integer \( h \) with \( (h, q) = 1 \), applying Lemma 1.1 and Lemma 1.2 we immediately have

\[ |C(h, q; \lambda)| = \frac{1}{2\pi \phi(q)} \left| \sum_{\chi \mod q \atop \chi(-1) = -1} \overline{\chi}(h) \left( \sum_{n=1}^{\infty} \frac{G(\chi, n; \lambda) - G(\chi, -n; \lambda)}{n} \right) \left( \sum_{m=1}^{\infty} \frac{G(\chi, m)}{m} \right) \right| \]
\[
\leq \frac{1}{2\pi^2 \varphi(q)} \sum_{\chi \mod q \atop \chi(-1) = -1} \chi(h) \left( \sum_{n=1}^{\infty} \frac{G(\chi, n; \lambda)}{n} \right) \left( \sum_{m=1}^{\infty} \frac{G(\chi, m)}{m} \right) + \frac{1}{2\pi^2 \varphi(q)} \sum_{\chi \mod q \atop \chi(-1) = -1} \chi(h) \left( \sum_{n=1}^{\infty} \frac{G(-\chi, -n; \lambda)}{n} \right) \left( \sum_{m=1}^{\infty} \frac{G(\chi, m)}{m} \right) \ll q^{1/2} + \varepsilon.
\]

This completes the proof of Theorem.

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