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GARCH(1,1) Model of the Financial Market with the Minkowski Metric

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Abstract: We solve a stylised fact on a long memory process of the volatility cluster phenomena by using the Minkowski metric for GARCH(1,1) (generalised autoregressive conditional heteroskedasticity) under the assumption that price and time cannot be separated. We provide a Yang-Mills equation in financial market and an anomaly on superspace of time series data as a consequence of the proof from the general relativity theory. We use an original idea in the Minkowski spacetime embedded in Kolmogorov space in time series data with the behaviour of traders. The result of this work is equivalent to the dark volatility or the hidden risk fear field induced by the interaction of the behaviour of the trader in the financial market panic when the market crashes.

Keywords: GARCH; Market Panic; Minkowski Metric; Time Series; Volatility; Yang-Mills.

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1 Introduction

Bachelier [1] was the pioneer of the efficient market hypothesis (EMH) [2] established on the assumption that the price obeys a random walk [3] in a spectacular market [4]. In economics, the time series data is a record in a Euclidean plane where price and time are independent of each other. In general relativity, we cannot separate price and time under the Euclidean space with the Hausdorff separation [5]. It is still a hidden loop space in time series data where price and time cannot be separated in analogy with the Minkowski spacetime. There exists an empirical analysis that is found on the stabilised fact on volatility cluster inducing long memory processes in the financial time series. Mandelbrot introduced a new model of the generalised autoregressive conditional heteroskedasticity (GARCH), the so-called fractional Brownian motion, and gave a result in FGARCH. The main problem in FGARCH is an infinite variance, and the stylised fact still happens in the form of the power-law distribution. With the development of the quantum field theory [6], the Chern-Simons current induced from the interaction of the dark energy in a living organism [7], loop gravity [8], and some new emerging disciplines such as econophysics [9], complex system [10], and behaviour economic theory [11], economists and econophysicists are gradually getting a deeper understanding of the market-hidden geometry induced from the interaction of buying and selling order-book submissions of traders and hidden behaviour [12] of traders in the financial market [13], which we empirically observed from the financial time series data by using the GARCH model [14]. Some experts in complex systems and fractal geometry [15] had been arguing that EMH cannot be used to explain some weird phenomena of financial time series, e.g. the volatility clustering phenomenon. Scientists have realised that the Euclidean plane is not sufficient to explain such a chaotic phenomenon without adding something more of the statistical theory of spinor field [16] in Wilson loop phase transition [17] to the plane. We can extend the Euclidean space to a non-Euclidean space under the superspace in the time series data with the supersymmetric property. In the Yang-Mills theory [18], the superspace is composed of a quantum foam and of the lattice gauge field with some extra properties of the Khovanov cohomology [19] and the Chern-Simons theory [20].

The problem comes from classical financial mathematics. We just want to perform fundamental investigations to find an adequate stochastic process matching the real financial time series data [21] without considering the market microstructure with the underlying orderbook [22] of the financial market scaling [23] and the hidden structure of the financial market as a D-brane [24] with extra dimensions [25]. Financial time series data at the moment of a financial market crash [26] are typical experimental results that we have for the complex systems [27]. In the analysis of an empirical work on the financial time series,
nonlinearities and nonstationarities [28] are shown with the stylised fact of a volatility cluster phenomenon in a long memory effect. Some statisticians and economists typically predicted macroeconomics of the time series and the financial time series by using statistical data analysis of autoregressive integrated moving average (ARIMA), ARCH, GARCH, and the Markov switching model using the assumption of the linearity and the stationary time series process which cannot be found in the most of the typical financial time series data. The main defect of the ordinary least square (OLS) [29] ARIMA and GARCH models is based on fitting the problem with the parameters of fitting or learning with a single stochastic process for infinite factors, which is governed by an infinite stochastic process influence on the future expectation price. When we add one point of the future price and fit the curve by using the data mining tool for the regression with GARCH(1,1), the coefficient of the equation that we use to describe the historical data will update and change the historical path so that it makes a non-realistic situation. In real life, we cannot change a historical event but can influence an expected future event. We call this problem a prior effect in the scaling behaviour of the time series data. There appears a prior effect on the wavelet transformation when we add one more point and do a wavelet transform again, so the result is not the same as the previous result. In the case of Fourier transform or a Fourier series of an approximate function by using a periodic function, we call this the Joseph effect. In the Hilbert-Huang transform, we call this the end effect; in econometrics, we can also notice this by an OLS method so that the beta changes when we add more points. This problem will also happen in the hidden Markov model or the Bayesian network because all families of the tool are based on the simple statistic and not on the superstatistics or the hyperstatistical theory in which is contained an infinite stochastic process for fitting. We call this effect in a general term a prior effect or a prior problem, the problem where you lie to people by just changing your historical record in the event of what you did. In the present time, physicists [30] define another type of the geometrical object on the financial market, which may live on the string [31] and the D2-brane world [32]. In order to build a new mathematical superstructure suitable for the financial market, we need to take into account also a psychology factor of traders in the model of the equation. The spinor field in the model cannot be the price of the direct buy, but it should be a covering space of the price arising from the interaction of the trader with different expectations on the supply and the demand of the stock.

It has been found that the state space of the financial market is considered a smooth manifold containing events as a fibre space endowed with the particular affine connection. Let the behaviour of the trader be a sequence of the state spaces of the Riemannian manifold, the so-called behaviour state space model, a cohomological sequence of agents and dual agents. The representation of the space trader is a so-called agent $A_t$ at time $t$, and the sequence of the expectation price of the trader is a so-called dual agent $A_t^*$, and the sequence is a homological algebra that induces a cohomological sequence as an evolution feedback of price evolution. Let us consider the path-connected component of the quotient group between the circle and the boundary defined by the feedback evolution between the shape of the endpoint of the time series of the predictor and the predictant after performing the transform into zero band the transformation of the empirical mode decomposition.

In this paper, we use a new approach to solve this problem by using differential geometry. We build a new GARCH(1,1) model based on [33, 34] in the Wilson loop [35] of the connection for the description of the behaviour of the interaction of the traders in the financial market. The gauge group action with the connection in the Chern-Simons current is a source of the spinor field appearing as a hidden risk fear field in the financial market.

The rest of the paper is organised as follows: In Section 2, we review GARCH(1,1) invented by Bollerslev in another framework of differential geometry. The allowed volatility should be minus in order to interpret the result of the risk field. In order to open a new door to solve the stylised fact, we use a Minkowski space instead of the single Kolmogorov system for the volatility cluster. Volatility is used to express an emotional trade by defining a fear factor in the stock market or a stress factor. We discuss on the stylised fact and give a new definition of the Grothendieck and De Rham cohomology for the financial market with the Minkowski metric for GARCH(1,1). In Section 3, we provide a proof of the volatility clustering phenomenon in the hyperbolic space with the Yang-Mills equation for the financial market. In Section 4, we summarise the results of the work and discuss future work.

2 GARCH(1,1) and Stylised Facts

Let $\sigma_t(r_t)^2$ be the volatility of the stock return. Using this mathematical transformation, we induce a dark volatility
equivalent to dark matter in theoretical physics discovered by Wolfgang Pauli. The experiment that took place more than 20 years after the death of Pauli confirmed the neutrino in the balance of energy and momentum in his quantum field equation. Let $F$ be a fibre bundle. Let $g_{ij} \in G$ be a cycle and $g^{ij} \in G^*$ be a cocycle, a group action acting on the fibre bundle of manifold. We have $g_{ij} = \sqrt{g^{ij}} g_{ij}$. Let $x_t$ be a time series in the section of the Riemannian manifold, $x_t \in T_{x_0}X = p^{-1}(X)$. So, we have a differential 2-form of time series arising from the scalar production of the group transformation of the vector bundle, the so-called principal bundle of the time series, $\sigma_t^2 = <x_t, x_t> = g_{ij}||x_t||^2$ with $g_{ij} = 1$. The volume form is invariant with the respect to translation and rotation, and the volatility in the normal form will preserve the transformation. In this paper, we are interested in the case of dark volatility, $\sigma_t^2 < 0$, which happens when we change the Euclidean metric to the Minkowski metric $g_{ij} = -1$ that allows the volatility minus, $\sigma_t^2 < 0$, because

$$\sigma_t^2 = <x_t, x_t> = g_{ij}||x_t||^2 = -||x_t||^2. \quad (1)$$

In this case, we make the assumption that every risk from investment in the stock market induces a fear field or risk field, the so-called volatility network. We denote this field $C^1(\sigma_t^2), i = 1, 2, 3, \ldots, n$, for $n$-stock in the stock market. Let us denote a price in the stock market as $p_t(s_i)$, where $s_i$ is the code of stock $i$. We have an assumption that there may be a source of risk field coming from the trader willing to win a high profit from the stock market without their excess demand, but, on the other side, they use too much leverage. This situation will induce both the covariance and contravariance tensor field of the volatility of the price. Let us consider the Jacobian flow of the GARCH(1,1) equation. Recall equation (2) of GARCH(1,1).

Discrete-time volatility models were developed before high-frequency data became readily available, and they are typically applied to the daily or lower frequency returns. Most discrete-time models for the daily financial return $r_t$ satisfy the canonical product structure

$$r_t = \sigma_t \epsilon_t \quad (2)$$

returns with the multiplicative shock, $\epsilon_t$.

Here, the observed financial return $r_t$ is modelled as the product of an independent and identically distributed (i.i.d.) innovation $\epsilon_t$ and a positive scale factor $\sigma_t > 0$. One usually assumes that $\epsilon_t$ has a mean value zero and, for the standardisation, a unit variance. Specific models differ in their specification of the scale factors; an example is the stationary GARCH(1,1) recursion

$$\sigma_t^2 = \kappa + \alpha \sigma_{t-1}^2 + \beta \sigma_{t-1}^2 \quad (3)$$

where $\kappa, \beta, \alpha > 0$, and $\alpha + \beta < 1$.

The scale factors $\sigma_t$ (Jacobian in this model) are not observed (hidden), and one may use the daily close-to-close returns $r_n$ to estimate and evaluate the models for $\sigma_t$. Let us consider GARCH(1,1) for the series of returns $r_t = \log(r_t)$, which can be written as an additive shock

$$r_t' = \delta + \epsilon_t = \delta + \eta_t \sqrt{h_t},$$

and

$$h_t = a_0 + a_1 + \epsilon_{t-1}^2 + \beta h_{t-1}. \quad (4)$$

The main problem of GARCH(1,1) is that the return is not in the log scale but we have a constant Euclidean distance. In this paper, we change the metric to the hyperbolic distance in order to solve the volatility cluster phenomenon. We found a complete solution in the complex time scale with the conjugate solution in the minus time scale. We call the volatility in the inert hidden time scale a hidden fear field or dark volatility $\sigma_t^2(A_{\mu})$, with the hidden variable of a risk field $A_{\mu}$ as a connection in the gauge field theory.

The difference between volatility and dark volatility is the classes of different topological spaces as their mathematical objects.

The volatility $\sigma_t^2$ is based on the statistical theory under the normal distribution and usually treated as a random variable satisfying the Kolmogorov axiom of probability under the Hausdorff space with an outer spinor field. We define the dark volatility to be the general form of the normal distribution over the spinor field in the quotient superspace in the time series data under the $T_0$ separation axiom. The volatility $\sigma_t^2(A_{\mu})$ is a spinor field of the behaviour of the traders with a group action by the market cocycle $g$. An example of the dark volatility is a left chiral projection into the Euclidean plane with the left translation of the mean value only. It is just a volatility in the normal distribution if we take a group action as a left translation in the mean value. In the equilibrium of the statistical system, we have a normal distribution with the identity of the left translation of the gauge group appearing as a Gaussian random variable $\sigma_t^2 := 0$, the identity of the Lie algebras as a result of the left adjoint map as volatility in the Euclidean plane. If we extend the group action as a spinor field action to the time series in the form of a gauge potential field $X_{t} := gA_{\mu}$ translation in the parallel direction of the left and the right translation with an extra property of the spinor invariant, we will get a new definition of the volatility as the Jacobian flow in both left and right directions simultaneously in the superspace of the time series data.
In the Minkowski space of time series data, we get supply and demand as focus points with the reflection of price in the light cone as a mirror symmetry from the left to the right chiral state in the financial market with the supply and the demand side. Therefore, we define the volatility in GARCH(1,1) as a gauge group action in the Jacobian flow with \( g := g_{ij} = \sigma^2_t + h_t \). We substitute \( h_t \) with \( g_{ij}(t) \), so we have a Jacobian flow equation of the parallel transport GARCH(1,1) model

\[ g_{ij}(t) = a_0 + a_1 ||e_i||^2 + \beta g_{ij}(t - 1), \tag{5} \]

so we have an extension of the Euclidean plane to the non-Euclidean plane with the same as GARCH(1,1), the so-called Minkowski GARCH(1,1) in this context:

\[ a_1 ||e_i||^2 = g_{ij}(t) - a_0 - \beta g_{ij}(t - 1), \tag{6} \]

where \( g_{ik} \) is the inverse of the \( g^{ij} \) representation for the market risk cycle and risk cycle. In economics, most people assume \( ||e_i||^2 \sim N(\mu, \sigma^2) \) to be a random variable with random walk. In this paper, we assume that this term is an induced spinor field from the interaction of the behaviour of the trader with a risk aversion with no concept of the normal distribution. We use differential geometry and the gauge theory approach for the approximation of this term with the Laurent polynomial in the Chern-Simons theory in the Kolmogorov space in the time series data with \( x \in X, y \in Y, x_0 \in X^*, y^* \in Y^* \). We have

\[ ||e_i(A_\mu)||^2 = \langle \frac{x \wedge x^*}{C}, \frac{y \wedge y^*}{C} \rangle, \quad x^*(x \otimes y) = C, \]

where \( C \) is a constant value representing a memory effect in the volatility clustering phenomenon of the time series data and \( A_\mu \) is a connection.

Let \( p : \Omega(\wedge^2 T_x \mathbb{R} \otimes \mathbb{R}) \rightarrow \mathbb{R} \) with section matrix be a half volatility model, i.e. the matrix of error of the time series prediction \( e_t \) in the section of the supermanifold of the financial market, \( p^{-1}(\mathbb{R}) = [e_t] \).

It is independent of the coordinate system, so we can use the Jacobian flow of the coordinate transform across a stack of time scale going in both directions of future and past:

\[ de_{t+1} = \sqrt{g_{ij}(t)} d\epsilon_t, \quad de_{t-1} = \sqrt{g^{ij}(t)} d\epsilon_t. \tag{7} \]

So we have a Jacobian flow as the Minkowski GARCH(1,1) model

\[ ||e_i||^2 = a + b \frac{dg_{ij}(t)}{dt}, \tag{8} \]

where \( a, b \) are constants and \( g_{ij}(t) \) is a market cocycle. A parallel transport is a Wilson loop of connection \( W_{a,\beta}(A_\mu) \) over a principal fibre of financial supermanifold as arbitrage opportunity in the financial market cocycle \((a, \beta)\). It is obtained from the Jacobian flow of the market cocycle over the fibre space by

\[ A_\mu = \Gamma^i_{ij} = \frac{1}{2} g^{ij} (\partial [g_{il} + \partial [g_{lj} - \partial [g_{lj}]]. \tag{9} \]

The interpretation of \( A_\mu \) is an error of the expected price from the physiology of the time series data, considering the trading induced from the interaction of the risk aversion and the hidden risk fear field from the behaviour of the fundamentalist \( A_{\mu=1} = A_1 \). It is a chartlist or noise trader \( A_{\mu=2} \) and bias traders \( A_{\mu=1} \) in the financial market in the positive and negative forward looking to the future price of the market as a group operation of the gauge group of an expected market cocycle \( g \) to the gauge field \( A_\mu \).

Here, \( A_\mu := \Gamma^i_{ij} \) is a risk gauge potential induced from market cocycle \( g = g_{ik}(k) \) as a gauge group action

\[ [g, A_\mu] = \sigma^2 = A_\mu g - gA_\mu = Ad_\mu (\{A_\mu\}). \tag{10} \]

We have \( gA = Ag + ||e_i||^2 \) with the gauge group action \( A^\mu_\mu = g^{-1} A_\mu g + g^{-1} \partial g \). The Wilson loop of the behaviour trader in the connection \( A_\mu \) of the fibre space is defined by

\[ W_{a,\beta}(A^\mu) = W_{a,\beta}(A) - \frac{1}{24\pi^2} \int d^3 x e_{ijk} <g^{-1} \partial g gg^{-1} \partial g g^{-1} \partial g g>. \tag{11} \]

From the typical classical differential geometry, the equation above allows us to work over the connection of the fibre space as an extra dimension in the moduli space of the Killing vector field approach. We just use the tool for the computation of the expected state in our new definition of the spinor field in the time series data. The spinor field induces the coupling state between four types of tensor fields over the Killing equation above.

**Definition 1.** Let the support spinor be an arbitrage opportunity \( \Gamma^i_{ij} \) in the financial market over the moduli state space model in the time series data as a connection preserving the scalar product over the parallel translation along the fibre space of the physiology layer in the time series data.

We define a Chern-Simons current as an arbitrage opportunity density by

\[ j^\mu_A = \partial [F^{\mu\nu} + [A_\mu, F^{\mu\nu}]], \tag{12} \]

\[ \frac{1}{4} <F^{\mu\nu} F_{\mu\nu}> = \partial K^\mu, \tag{13} \]
where

\[ R^\mu = e^{\mu a} e^\nu \left\{ \frac{1}{2} A_a \partial _\beta A_\gamma + \frac{1}{2} A_a A_\beta A_\gamma \right\}, \]  \hspace{1cm} \text{(14)}

where a connection \( t^\mu _\nu = (A_a)^\mu _\nu \) and \( A \) is a Chern-Simons current in the financial market. An arbitrage or arbitron is a Chern-Simons anomaly, \( R^\mu \), or an anomaly current induced from the behaviour of the trader as a twist ghost field between two sides of the market. This field of the financial market induces a Ricci curvature in the financial market as an arbitrage opportunity for each connection in the coupling ghost field of the behaviour of the trader. This curvature blends the Euclidean plane of the space of the observation to twist with the mirror plane behind the Euclidean plane and wraps to each other as the D-brane and the anti-D-brane interaction of the superspace of the financial time series data.

\[ F^\mu _{\nu \rho} = R^\mu _{\nu \rho} (T) = e^\mu _{\nu} R^\nu _{\rho} e^\rho _{\mu}. \]  \hspace{1cm} \text{(15)}

**Definition 2.** The Ricci tensor in the time series data is a contraction of the curvature tensor of the risk defined by \( R_{ik}(\sigma^2) = R_{ij}^k(\sigma^2) \) with respect to the natural frame of the connection

\[ F^\mu _{\nu \rho} := R^\mu _{\nu \rho} = \partial_k \Gamma^j _{\mu} - \partial_j \Gamma^j _{\mu} + \Gamma^j _{\mu} \Gamma^m _{jk} - \Gamma^m _{jk} \Gamma^j _{\mu}, \]  \hspace{1cm} \text{(16)}

where \( Ad_g \{ A \} := [g, A] := g \times \Gamma^j _{\mu} := g \times A = \sigma^2 \).

Market equilibrium occurs when the Ricci curvature is zero, \( R_{ik} = 0 \) (arbitrage opportunity disappears and the physiology of the time series data contains no curvature).

The solution of the Patterson-Godazzi differential equation is a curvature of the superspace in the time series data.

If \( \sigma^2 \) is constant, we have an equal risk field in the market or the risk-free market \( \sigma^2 = 1 \), and we have the price persistence with the shock given by

\[ \frac{\partial p_t}{\partial \sigma^2} = p_{t-1} - p_t > 0, \]  \hspace{1cm} \text{(17)}

but in the case of dark volatility (as a hidden fear field) that induces an inertia in the price in the hidden direction (and also if \( \sigma^2 = -1 \))

\[ \frac{\partial p_t}{\partial \sigma^2} = p_{t-1} - p_t < 0. \]  \hspace{1cm} \text{(18)}

We conclude that \( \sigma^2 \in \mathbb{H} \) is called the dark volatility. We can monitor or measure on the border of the universe outside our galaxy only with the speed of light.

Let us define an underlying coordinate of the Jacobian flow. If \( g_{ij} = 1 \), we use an \( S^2 \) Riemann sphere as the GARCH(1,1) coordinate. When \(-1 < g_{ij} < 1\), we use a cone or hyperbolic coordinate, and, when \( g_{ij} = -1 \), we use the Minkowski space.

**2.1 Dark Volatility**

In this paper, we assume that the financial market is an asymmetric superspace \( X_t / Y_t \) with the left and right supersymmetry in the Yang-Mills theory. The left chiral supersymmetry is a source of the demand \( D \in X_t \), and the right chiral supersymmetry is a source of the supply in the hidden space \( S \in Y_t \). The duality in the general equilibrium with the invisible hand is a source of the hidden dual superspace with the hidden demand \( D^* \in X^* \) and the hidden supply \( S^* \in Y^* \). Most economists treat the price \( p_t \in \mathbb{R}^n \) as the time series data embedded in the Euclidean plane with an outer extra property of the spinor field. We use an alternative approach of the differential geometry. Our price is defined by an extension of the Euclidean plane to a quaternionic projective space \( \mathbb{H}P^1 \), and \( p_t(S, S^*, D, D^*) \) is this space with the extra property of the spinor field from the interaction of the behaviour of the trader from the supply and demand side as the connection \( A_\mu \) in the fibre space with the Hopf fibration. In econophysics, market depth is defined by a Taylor expansion of the price over an excess demand \( D \). We have an equivalent class of supply and demand when the price is a nonequilibrium of the matching by

\[ S^* \sim D \leftrightarrow S^* \alpha = D, S \sim D^* \leftrightarrow S\beta = D^*, \]  \hspace{1cm} \text{(19)}

where \( (\alpha, \beta) \) is an equivalent class market cocycle of the price matching.

We assume that the risk is composed of the observation risk and the hidden risk in the supersymmetric property of the left and the right chiral supersymmetry. The left chiral risk is the volatility in economics with the systematic risk \( \sigma^2 \) and the right chiral risk is the dark volatility, the market risk with the hidden risk fear field induced from the behaviour of the trader \( \sigma^2 (A_\mu = 1, 2, 3) \). Let price be a transition state of the coupling hidden state between the supply pair \( (S, S^*) \) and the demand pair \( (D, D^*) \).

Hence, we have two systems of a hyperbola with the focus point \( D = \beta S^*, D^* = a S \) in the hidden supply and the hidden demand and the centre in the demand and supply with pairs states twists space to each other with two Minkowski cones with the superdistribution \( pp(k) \) of the...
price momentum $k$. If we rotate the hyperbola light cone with $\theta = \frac{\pi}{2}$, we get two systems of equations:

\[
\text{Cone}(x, y, pp(k)) := \{ (x, y) | (x - D)^2 - (y - S)^2 = k \}
\]

\[
\text{Cone}(\lambda x, \lambda y, pp(k'))
\]

\[
: (\lambda x')^2 - (\lambda y')^2 = (\lambda^2 - 1)(x^2 - y^2) = k'.
\]

(20)

A Wilson loop of the time series data, denoted as $W_{a,b}(A_x)$, is a twistor in a complex projective space $\mathbb{CP}^1$. It turns one side of the Euclidean plane twist into another side of the plane with the help of modified M"obius map $z = \frac{1}{z}$.

We have $W_{a,b}(\frac{\lambda S}{\lambda D}) = \frac{D}{S}$. The equivalent class of the supply and demand is induced from the queuing process of $0$ to $\infty$ by using the modified M"obius map $z = \frac{1}{z}$. Each of the components is defined by gluing an equivalent class of the supply and demand $S \in [s], D \in [d]$ such that $[s] \sim [d]$ if and only if there exists $\lambda_i$ such that $x_i = \lambda_i y_i, \lambda = (\alpha, \beta)$.

The projective coordinate is a local coordinate on the Minkowski space, a Wilson loop of the unitary operator in quantum mechanics. Let $g_{ij}$ be the Jacobian matrix of the supply and demand, and we have a shallow to the hidden state of the transformation by using the Wigner ray transform

\[
g_{ij} \mapsto \lambda g_{ij}, \lambda \in \mathbb{U}(1) \simeq \text{Spin}(2),
\]

(21)

where $\lambda = \alpha_x, \alpha_y$ is the Pauli matrix in the $\text{Spin}(2)$ group.

**Definition 3.** Let $S$ be a supplied linear compact operator in Banach space and $D$ be a demand operator. A Wilson loop in the time series data at a general equilibrium point in the financial time series data is a point of price $x_i(S, D)$ such that there exists a ray of the unitary operator $\lambda \in SU(2) \simeq \text{Spin}(3)$,

\[
W_{a,b}(S, D) := \{ \lambda S, \lambda D \}, \lambda < S, D > := < S, D >.
\]

(22)

The determinant of the correlation matrix or a wedge product of the column of the correlation of all returns of stocks in the stock market can be $-1$ and induces a complex structure as a spin structure in the principal bundle of the time series data. We found that when $\text{det}(\text{Corr}(M))^2 = 1$, where $\text{Corr}(M), M \in SL(2, \mathbb{C})$ is a correlation matrix of the financial network $M$ with M"obius map. This space of the time series data is in equilibrium with two equilibrium points.

One is a real and explicit form, whereas the other is hidden in a complex structure like dark matter or invisible hand in the economics concept. If $\text{det}(\text{Corr}(M)) = 0$, the space of time series data is out of equilibrium and it can induce the market crash with a more herding behaviour of the noise trader. This result is related to the definition of the log return of an arbitrage opportunity in econophysics.

The section of the tangent of the manifold is a Killing vector field of the manifold of the financial market. We introduce three types of the Killing vector field for the market potential field of the behaviour of the traders in the market:

\[
A_{+,1}(f) = \frac{\partial}{\partial f} : \text{SO}(2) \to \text{Spin}(2) : M \to T_x M
\]

(23)

\[
A_{+,2}(\sigma) = \frac{\partial}{\partial \sigma} : \text{SO}(2) \to \text{Spin}(2) : M \to T_x M
\]

(24)

\[
A_{+,3}(\omega) = \frac{\partial}{\partial \omega} : \text{SO}(2) \to \text{Spin}(2) : M \to T_x M
\]

(25)

where $+$ means optimistic behaviour of the trader, $-$ means the pessimistic trader, $f$ is a fundamentalist trader, $\sigma$ is a noise trader, and $\omega$ is a bias trader. $A_1$ is an agent field of the adult behaviour trader or the fundamentalist. $A_2$ is an agent field of the teenager behaviour trader or the herding behaviour or the noise trader. $A_3$ is an agent field of the child behaviour trader or the bias trader. The behaviour is spin up and down as the expected states in the market communication layer of the transactional analysis framework. The up-state is signified as the optimistic market expected state and pessimistic as the market crash state of an intuition state of the forward looking trader cut each other as in the transactional analysis framework of the market communication between the behaviour of the expected state and the behaviour of the market state.

The strategy of the fundamentalist (we denoted in short $f$) is the expected price in a period, where the price is in the minimum point $s_4$ and maximum point $s_2$ (crash and bubble price). Then the optimistic fundamentalist $f_+$ will buy at the minimum point $s_4$ of the price below the fundamental value, and the pessimistic fundamentalist $f_-$ will sell the product or short position at the maximum point of the price if the maximum point is over a fundamental value. The strategy of a chartist or noise trader, $\sigma_{\pm}$, is different from that of the fundamentalist. Let $\omega$ be a basic instinct bias behaviour of the trader.

We can use the Pauli matrix $\sigma(t)$ to define a strategy of all agents in the stock market as the basis of the quaternionic field span by the noise trader and the fundamentalist. Let $+1$ stay for buy at once and $-1$ for sell suddenly. Let $+1$ stay for being inert to buy and $-1$ for being inert to sell. The row of the Pauli matrix represents the position of the predictor, and the position in the column represents the predicted state.
**Definition 4.** For the noise trader, we use a finite state machine for the physiology of the time series to the accepted pattern defined by

\[
A_2 := \sigma(t) = \sigma_y = \begin{bmatrix}
    s_2^*(t) & s_4(t) \\
    s_2(t) & 0 & -i \\
    s_4^*(t) & i & 0
\end{bmatrix}
\]

where \(\sigma_y\) is a Pauli spin matrix.

For the fundamentalist trader, we use a finite state machine for the physiology of the time series to the accepted pattern, defined by

\[
A_1 := f_\pm = -W(\sigma_z) = \begin{bmatrix}
    s_2^* & s_4^* \\
    s_2(t) & 0 & -1 \\
    s_4(t) & 1 & 0
\end{bmatrix}
\]

Let \(\omega\) be a biased behaviour \(A_3\) of the market micropotential field. Since \([\sigma_y, \sigma_z] = 2i\sigma_x\),

\[
[f, -W^{-1}(\sigma)] = 2i\omega,
\]

where \(W\) is Wilson loop or knot state between the predictor and the predictant for the time series data, and \(W^{-1}\) is the inverse of the Wilson loop for the time series data (unknotted state between the predictor and the predictant).

We have an entanglement state inducing the strategy of the noise trader as herding behaviour explained by

\[
s_2 \rightarrow -i s_4^*, \quad s_4 \rightarrow i s_2^*
\]

with

\[
W_{a,\beta} \begin{bmatrix} s_2 \\ s_4 \end{bmatrix} = \begin{bmatrix} s_4^* \\ s_2 \end{bmatrix}.
\]

We use a Laurent series to decompose the risk field into the left excess demand \(D\) and the right excess demand \(D^*\) in the hidden space knot the Laurent polynomial with the coefficient in the Wilson loop of the behaviour of the trader over the knot of the market cocycle \((a, \beta), W_{a,\beta}(A_\mu)\) by

\[
\sigma^2_t = \int_{H^1(\mathbb{C}^{p_i} \times \mathbb{C}^{p_j})} \frac{\Pi W_{a,\beta}(A_\mu)}{D - D_0} dD + \int_{H^1(\mathbb{C}^{p_i} \times \mathbb{C}^{p_j})} \frac{\Pi W_{a,\beta}(A_\mu)}{D^* - D_0} dD^*
\]

where \((S, S')\) is a pair of the supply and the hidden demand and \((D, D')\) is a pair of the demand and a hidden demand field. The main problem of GARCH(1,1) is that return is not in a log scale, but we have a constant Euclidean distance. In this paper, we change the metric to the hyperbolic distance in order to solve the volatility of the cluster phenomenon. We found a complete solution in the complex time scale with a conjugate solution in the minus time scale. We interpret the results with the index cohesive force. We call the volatility in the inert hidden time scale the hidden fear field or the dark volatility.

We model the financial market in the unified theory of \(E_8 \times E_8\) (Fig. 1) under the mathematical structure of the Kolmogorov space underlying the time series data. Let \(M \in \mathbb{HP}^1/\text{Spin}(3)\) be a market with the supply and demand sides, and \(M^* : \rightarrow \mathbb{HP}^1/\text{Spin}(3) \rightarrow \mathbb{HP}^1\) be a dual market of the behaviour of the noise trader \(\sigma\) and fundamentalist \(f\). Then we prove the existence of the general equilibrium point in the market by using equivalent classes of the supply \([s]\) and the demand \([d]\) in the complex projective space \([s], [d] \in \mathbb{C}P^1\). The Kolmogorov space \(S' = \mathbb{HP}^1 \simeq E_8\) can use explicit states of the financial market and its dual \(\mathbb{HP}^1/\text{spin}(3) \simeq E_8\) uses dark states in the financial market. It is a ground field of the first knot cohomology group tensor with the first de Rahm cohomology group \(H_1(x_1; [G, G^*]; \mathbb{HP}^1/\text{spin}(3)) \otimes H^1(M; \mathbb{HP}^1/\text{spin}(3)).\) We induce free duality maps \(\xi : H_1(x_1; [G, G^*]; \mathbb{HP}^1/\text{spin}(3)) \rightarrow H^1(M; \mathbb{HP}^1/\text{spin}(3))\) as the market states \(E_8 \times E_8\) in the grand unified theory model of the financial market. For simplicity, we let a piece of the surface cut out from the Riemann sphere with a constant curvature \(g_{\Omega}\) defined as a parameter of a D-brane field basis with the same orientation as a definition of the supply and demand in the complex plane (flat Riemann surface). Now we blend the complex plane to the hyperbolic coordinate in eight hidden dimensions of \(\Psi = (\Psi_1, \ldots, \Psi_8) \in S',\) eight states in the explicit form.

### 2.2 Yang-Mills Equation for Financial Market

In this section, we explain the source of de Rahm cohomology of the financial market. The mathematical structure
in this section is related to the Chern-Simons theory of the so-called gravitational field in three forms of the connection \( F^\nabla \in \Omega^2(T,M \otimes T^*M) \) over three vector fields in the financial market: \( \Psi_i \in M \) a market state field, \( A_i \) an agent behaviour field, and \( [s_i] \) a field of physiology of financial time series data over Kolmogorov space. We simplify the Chern-Simons theory to a suitable form to easily understand this section and also to just simplify the prototype of the real and useful new cohomology theory for the financial market. Let \( A = (A_1, A_2, A_3) \) be a market micro potential field of the agent or the behaviour of the trader in the market. Let \( \Psi_i^{(S,D)}([s_i], [s_i]) \) be a field of the market 8 – states of the equivalent class of the physiology of the time series \([s_i], [s_i] \).

In the financial market microstructure, we define a new quantity in microeconomics induced from the interaction of the order submission from the supply and demand side of the orderbook, the so-called market microvector potential field or the connection in the Chern-Simons theory for the financial time series \( A(x, t; g_{ij}, \Psi_i^{(S,D)}, s_i(x_i)) \), where \( \Psi_i^{(S,D)} \) is a state function for the supply \( S \) and the demand \( D \) of the financial market. Let \( \Psi_i^{(S,D)}(F_i) \) be a scalar field induced from news of the market factor \( F_i, i = 1, 2, 3, 4 \). Let \( x_i(A, x^*, t^*) \) be a financial time series induced from field of market behaviour of the trader, where \( x^* \) and \( t^* \) are the hidden dimensions of time series data in the Kolmogorov space.

In the equilibrium state, the sequence is exact with \( \partial^2 = 0 \). We have demand in equilibrium state \( D = \partial^2 \) \( A = 0 \) as the property of short exact sequence in which we can induce an infinite cohomology sequence of sphere as a sheave sequence of financial time series data.

**Definition 5.** Let \( S \) be a supply potential field defined by the rate of change of the supply side of market potential fields of behaviour trader in the induced field of behaviour of agent \( A \) in the stock market.

\[
\vec{S} = -\frac{\partial A}{\partial [s_4]} \quad (31)
\]

\[
\vec{N}S = -i\frac{\partial A}{\partial [s_4]} \quad (32)
\]

\[
\vec{D} = \frac{\partial A}{\partial [s_2]} \quad (33)
\]

\[
\vec{N}D = +i\frac{\partial A}{\partial [s_2]} \quad (34)
\]

with

\[
A = (A_1(f), A_2(\sigma), A_3(\omega)). \quad (35)
\]

**Definition 6.** Let \( \vec{D} \) be a demand potential field (analogous to a magnetic field). It is defined by the rate of change of market potential fields in the induced field of \( \vec{S} \):

\[
\vec{D} = \nabla \times \vec{S}. \quad (36)
\]

**Definition 7.** Let \( F_{\mu\nu} \) be a stress tensor for the market microstructure with component \( A_\mu \) of the market potential field (analogous to connection one form in Chern-Simons theory) defined by the rank 2 antisymmetric tensor field:

\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad (37) \]

\[
\frac{dA}{d[s_2]} = S_\nu = F_{\mu\nu} \Psi^\nu, \quad \frac{dA}{d[s_4]} = D_\mu = \epsilon_{\mu\nu} F_{\nu\lambda} \Psi^\lambda. \quad (38)
\]
These three vector fields of the behaviour trader play the role of three forms in the tangent of complex manifold. It is an element of the section \( \Gamma \) of the manifold of market, \( A \in \Gamma (\wedge^3 T_s^* M \otimes \mathbb{H}) = \Omega^3(M) \).

Let us consider the three differential forms of behaviour of the trader as the market potential field \( A \):

\[
dA = \sum_{ijk=1,2,3} F_{ijk} dA_i \wedge dA_j \wedge dA_k
\]

with a market state \( \Psi^{(S, D)} \in C^1(M) \). A de Rahm cohomology for the financial market is defined by using the equivalent class over these differential forms. The first class is the so-called market cocycle \( Z^n(M) = \{ \omega^*: C^n(M) \to C^{n+1}(M) \} \) with the commutative diagram between the covariant and contravariant functor:

\[
\begin{array}{ccccccc}
\cdots & -\to & H^3(M) & -\to & H^2(M) & -\to & H^1(M) & -\to & H^0(M) & -\to & 0 \\
\downarrow \xi & & \downarrow \xi & & \downarrow \xi & & \downarrow \xi & & \downarrow \xi & & \\
\cdots & -\to & H_3(X_t) & -\to & H_2(X_t) & -\to & H_1(X_t) & -\to & H_0(X_t) & -\to & 0 \\
\downarrow \nu & & \downarrow \nu & & \downarrow \nu & & \downarrow \nu & & \downarrow \nu & & \\
\cdots & -\to & H^3(x_t: [G, G^*]) & -\to & H^3(x_t: [G, G^*]) & -\to & H^1(x_t: [G, G^*]) & -\to & H^0(x_t: [G, G^*]) & -\to & 0
\end{array}
\]

It is a kernel of the co-differential map between supply and demand to the market potential field:

\[
\text{Ker}(d: \Omega^3(M) \to \Omega^3(M)), d: [s_i](S, D)
\]

\[
\mapsto A(A_1, A_2, A_3), d[s_i](S, D) = 0 = A_
\]

where \([s_i](S, D)\) is an equivalent class of physiology of time series data. The boundary map of the cochain of market \( \Omega^2(M) \) is a second equivalent class used for the modulo state:

\[
\text{Im}(d: \Omega^1(M) \to \Omega^2(M)), d: \Psi^{(S, D)} \mapsto [s_i](S, D).
\]

**Definition 8.** The de Rahm cohomology for financial time series is an equivalent class of second cochain of the market

\[
H^3_{\text{BR}}(M) = \text{Ker}(d: \Omega^2(M) \to \Omega^3(M))/\text{Im}(d: \Omega^1(M) \to \Omega^2(M)).
\]

The meaning of the newly defined mathematical object is its use for measuring a market equilibrium in the algebraic topology approach.

The section of manifold induces a connection of the differential form. The connection is typically the gravitational field in physics. In finance, the connection is used for measuring the arbitrage opportunity. The connection allows us to use the Peterson-Codazzi equations of the Killing form of parallel transport of the geodesic curve as the hidden equilibrium equation for financial market over the Riemannian surface of the market.

Let \( A_c = F^L \in \Omega^3(T_s M \otimes T_s^* M), \nabla: \Omega^0(M) \to \Omega^1(M) \).

Let \( \Psi_i \in M: \mathbb{H}P^1/\text{Spin}(3) \to S^*, [s_i] \in \text{Physio}(x_i) := G : CP^1 = S^3/S^1 \to S^3 = \mathbb{H}, G \) is a Lie algebra of the predictor, and \( G^* \) is a predictant. \( \Psi^*_i \) is dual basis of the market state \( \Psi_i \). We have the Peterson-Codazzi equations of arbitrage opportunity as Jacobian flow \( g_{ij} \), given by

\[
g(\nabla \Psi[s^*], [s_i]) + g(\Psi, \nabla [s_i] \Psi^*) = 0.
\]
opportunity is the change of market state with respect to the expectation field of physiology \([s^*]\).

### 2.3 Grothendieck Cohomology for Financial Time Series

Let the Minkowski space of light cone in time series data be composed of two cones embedded into Euclidean plane in pointed space of time series as the equilibrium node. The upper cone is for the superspace of supply \(X_t\), known as the Kolmogorov space in time series data. The down cone is the superspace of the demand side of the market, \(Y_t\). We assume that a unit cell of the non-Euclidean plane of time series data is composed of two sheets with left- and right-hand supersymmetry. It is a superposition to each other in opposite directions with left-hand and right-hand supersymmetry of the hidden direction of time \(dr\) and reversed direction of time scale \(dt^2\) in which observation cannot be noticed from outside the system. The quantisation of the hidden state of behaviour of trader in the superspace of time series data is an unoriented supermanifold with a ghost field and an anti-ghost field in the side of D-brane and anti-D-brane sheet of normal distribution in the Minkowski space in time series data. We define a new lattice theory with ghost pairs state of two pairs of supply and demand ghost field \((\Psi_L, \Psi_R)\).

**Definition 9.** Let \(\Psi_R(D_t)\) be a market right state of the demand side over behaviour of trader \([A_\mu]\). Let \(\Psi_L(S_t)\) be a market right state of the supply side over the behaviour of trader \([A_\mu]\). Let \([A_\mu]\) be a gauge field of connection of behaviour of traders in the financial market. Let \([A_\mu^*_G]\) be a dual space of gauge field of connection of a genetic code of the host cell defined by

\[
\Phi(A_\mu)^*_G = \sum_{i=1}^{n} (\mu_i, A_\mu)|\Psi_R(D_t) > = \sum_{i=1}^{n} \sigma_i^{x^2} |\Psi_R(x^*) >
\]

(46)

and

\[
\Phi(A_\mu)^*_L = \sum_{i=1}^{n} (\mu_i, A_\mu)|\Psi_L(S_t) > = \sum_{i=1}^{n} \sigma_i^{x^2} |\Psi_L(y^*) >
\]

(47)

if there exists a market transition state shift as dark volatility \(\sigma'_i\) with gauge group of evolution behaviour of trader \(G\) such that

\[
[\Psi_R(x^*)^+, \Psi_L(y^*)^-] = \frac{\partial p_i(S_t, D_t)}{\theta(s'_i(t))}
\]

(48)

with \(\sigma'_i^{x^2} \in \mathcal{A}^G \otimes \mathcal{A}^*_G\).

The coupling of interaction of supernormal distribution induced from the behaviour of trader as pairs of ghost field and anti-ghost field can be classified by using the link operator in knot theory, with link state \(<L>_{+}, <L>_{-}\), and \(<L>_{-}\). A unoriented superdistribution is in knot state with left and right link operator associated with left and right supersymmetry in the chiral ghost field of supply and demand of the market superstate. We work in the level of Kolmogorov space. Let \(X_t\) be the Kolmogorov space of supply gauge field with \(S_t \in X_t\) and \(Y_t\) be the Kolmogorov space of demand gauge field with \(D_t \in Y_t\). Let a momentum space of orbital of quantum price structure in the supermanifold be coordinated by \((k_x, k_y) \in X_t / Y_t / L\).

We can choose \(X_t / Y_t := \mathbb{R}^p\), with twistor break a chiral supersymmetry from left to right in the mirror space by

\[
O^{p,q}_{X_t / Y_t} \rightarrow O^{p,q}_{Y_t / X_t}
\]

(49)

and

\[
O^{p,q}_{Y_t / X_t} \rightarrow O^{p,q}_{X_t / Y_t}
\]

(50)

with the skein relation over knot of Laurent polynomial \(q = e^{\frac{2\pi i}{n}}\)

\[- q^\frac{z}{2} <L>_{+} + (q^\frac{z}{2} - q^{-\frac{z}{2}})<L>_{0} + q^{-\frac{z}{2}}<L>_{-} = 0 \quad (51)

The dual behaviour trader pair is \((k'_x, k'_y) \in Y_t / X_t\). We have a short, exact, sequence-induced infinite sequence of Grothendieck cohomology for financial time series data:

\[
0 \rightarrow O^{p,q}_{\mathbb{Z}/2} \rightarrow O^{p,q}_{X_t} \rightarrow O^{p,q}_{Y_t} \rightarrow O^{p,q}_{X_t / Y_t} \rightarrow O^{p,q}_{X_t} \rightarrow O^{p,q}_{Y_t} \rightarrow \cdots \rightarrow H^1(O^{p,q}_{X_t / Y_t}) \rightarrow H^1(O^{p,q}_{X_t}) \rightarrow H^1(O^{p,q}_{Y_t}) \rightarrow \cdots
\]

\[
H^1(O^{p,q}_{X_t / Y_t}) \rightarrow H^1(O^{p,q}_{X_t}) \rightarrow H^1(O^{p,q}_{Y_t}) \rightarrow \cdots
\]

\[
H^2(O^{p,q}_{X_t / Y_t}) \rightarrow H^2(O^{p,q}_{X_t}) \rightarrow H^2(O^{p,q}_{Y_t}) \rightarrow \cdots
\]

\[
H^2(O^{p,q}_{X_t / Y_t}) \rightarrow H^2(O^{p,q}_{X_t}) \rightarrow H^2(O^{p,q}_{Y_t}) \rightarrow \cdots
\]

\[
H^3(O^{p,q}_{X_t / Y_t}) \rightarrow H^3(O^{p,q}_{X_t}) \rightarrow H^3(O^{p,q}_{Y_t}) \rightarrow \cdots
\]

\[
H^3(O^{p,q}_{X_t / Y_t}) \rightarrow H^3(O^{p,q}_{X_t}) \rightarrow H^3(O^{p,q}_{Y_t}) \rightarrow \cdots
\]

(52)

Let a behaviour of traders be a ghost pairs \(\Phi^\pm := (\Psi^R, \Psi^L)\) with the left supersymmetry

\[
\Psi_L := \Psi_{L_{-d^1}} \rightarrow \Psi_{L_{-d^2}} \rightarrow \cdots \rightarrow \Psi_{L_{-d^n}} \rightarrow 0
\]

(53)

where \(d^i_l\) is an adjoint map of the market chain of cocyclics:

\[
g^l_i : O^{p,q}_{0 \rightarrow \mathbb{Z}/2 \rightarrow X_t \rightarrow Y_t \rightarrow X_t / Y_t \rightarrow \cdots}
\]

\[
\rightarrow O^{p,q}_{X_t / Y_t \rightarrow Y_t^* \rightarrow X_t^* \rightarrow \cdots}
\]

(54)
Let a supply side of the market as a right supersymmetry be
\[
\Psi_R := \Psi_R^{x^R} = s_t^{x^R} \cdots s_1^{x^R} \cdots s_{x^R}^R
\]  
(55)
\[
g_i^R : \ominus \mathbb{P} \{ \cdots \rightarrow X_t \rightarrow X_t/Y_t \rightarrow 0 \}
\]
\[
\leftarrow \mathbb{P} \{ \cdots \rightarrow Y_t/Y_t \rightarrow X_t \rightarrow 0 \}
\]  
(56)

with
\[
Ad_{\phi} \Psi^R = \{ \Psi^L, \Psi^R \} = 0.
\]  
(57)

The inversion property of CPT produces parity inversion as a ghost field and an anti-ghost field pairs. Let the ghost field pair be \( \Phi_{\Psi_1}^-(A_\mu) \) with arbitrage opportunity as a connection \( A_\mu \) and the anti-ghost field pair be \( \Phi_{\Psi_1}^+(A_\mu) \) with partition map
\[
p(\Phi_{\Psi_1}^-(A_\mu)) + p(\Phi_{\Psi_1}^+(A_\mu)) = -1,
\]  
(58)

where the parity map separates the hidden supersymmetry of the left right in the supply demand equilibrium node represented as the market general equilibrium point between supply and demand gauge field in two cones of price surfaces by \( p : \{ \Phi_{\Psi_1}^-(A_\mu), \Phi_{\Psi_1}^+(A_\mu) \} \rightarrow \mathbb{Z}_2 = \{ -1, 1 \} \) with \( p(\Phi_{\Psi_1}^-(A_\mu)) = 1 - \Phi_{\Psi_1}^+(A_\mu) \) and \( p(\Phi_{\Psi_1}^+(A_\mu)) = 1 - \Phi_{\Psi_1}^-(A_\mu) \) as the time reversal parity operator in the moduli state space model of the financial market.

Let us define the quantum price for the time reversal wave function as \( \Psi(x) = \Psi(x)^t = e^{-ik_0 x^t} \). We have the superspace in time series data is used to measure (see Fig. 2 for details) the dual coupling behaviour field derived from Laurent series over four hidden momentum spaces, \( X_t, Y_t, X_t^*, Y_t^* \) with both positive and negative coefficients of the knot polynomial representing optimistic and pessimistic fundamentalist and noise trader behaviour field. The knot behaviour induces an arbitrage opportunity as a connection over the Wilson loop representing the market phase transition with \( (a, b) \) market cocycle in \( \mathcal{W}_{\beta,\alpha}(A_\mu) \).

Let a momentum coordinate of price quantum be in functional coordinate over the path integral of the Wilson loop along the loop space of the ghost field behaviour pairs. We have
\[
k_x = \frac{i}{\hbar} \int_0^{\psi_0} \frac{\Pi W_{\alpha,\beta}(A_\mu)}{x^t - x_0^t} dx^t,
\]  
(59)
\[
k_y = \frac{i}{\hbar} \int_0^{\psi_0} \frac{\Pi W_{\alpha,\beta}(A_\mu)}{y^t - y_0^t} dy^t.
\]  
(60)

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{Figure2}
\caption{The superspace in time series data with moduli stack [s]. The cone inside the superspace is a Minkowski space with light cone for time series data. We have spinor field in time series data as support spinor machine in this model.}
\end{figure}
where \( p^*(k') \) is a superdistribution of the price quantisation, and \( x', y' \) are the hidden demand and hidden supply, \( 'D', 'S' \). We define Galois connection to map between two dual Minkowski cones as the market equilibrium path by \( \{(x^*, y^*) \} \subset \text{Cone}(x', y'; pp^*(k')) \) with

\[
\text{Cone}(x, y; pp(k)) \xrightarrow{f} \text{Cone}(x', y'; pp^*(k')) \]

\[
\text{Cone}(x^*, y^*; pp^*(k'))
\]

(65)

\[
\{(x^*, y^*)\} = g(f(\{(x^*, y^*)\})), \{(x^*, y^*)\} = f(g(\{(x^*, y^*)\})).
\]

(66)

In a supernormal distribution, we have to take into account spin as the coupling field of behaviour of trader appearing as the Wilson loop of Pauli matrix \( f_+ := A_{\mu=1} := -W_{\mu}(\sigma_2) \) as behaviour of the fundamentalist. The behaviour of the chatist is defined by the Pauli matrix \( \sigma_+ := A_{\mu=2} := \sigma_y \). The behaviour of bias traders is an adjoint representation of fundamentalist and the inverse of the Wilson loop of chatist \( A_3 := \frac{1}{2}[f, -W^{-1}(\sigma)] \). The optimistic behaviour is denoted by \( A_{\mu} \) and the pessimistic one is denoted by \( A^\mu = (A_{\mu})^* \). Hence

\[
pp^*(k') := A_{\mu} \Phi^+_\varphi k + A^\alpha \Phi^-_{\varphi k} \tag{67}
\]

with \( ||pp(k')||^2 = 1 \) and \( ||A_{\mu} \Phi^+_\varphi k||^2 = 1, ||A^\alpha \Phi^-_{\varphi k}||^2 = 1. \) \( A_{\mu} \) is a Pauli matrix representing the spinor field of behaviour of the trader. The spinor invariant property of price quantum \( \varphi_t \) in these normalisation formulae is a source of stationary process in the superspace of time series data extended volatility to dark volatility by \( ||\varphi_t||^2 = 1 \) to \( ||\sigma_1 \varphi_t||^2 = 1 \).

Let \( S^2 \) be financial market as a Riemann sphere of price momentum space of price quantum without singularity, glued up from dual side of supply and demand in market with two disjoint cones with two equilibrium nodes. The market equilibrium node is composed of pairs of opposite spinor field of behaviour traders, \( \{(x^*, y^*)\} \) as a pointed space embedded inside the Riemann sphere of energy surface \( S^2 \):

\[
S^2 = X_t \prod X'_t \prod Y_t \prod Y'_t \prod x_t \prod x'_t \prod y_t \prod y'_t \tag{68}
\]

where \( x_t \in X_t, \quad y_t \in Y_t, \quad x'_t \in X'_t, \quad y'_t \in Y'_t \) are the Kolmogorov space for the time series data.

This is a new kind of super mathematics theory in probability theory with a new integral sign over the tangent of the supermanifold under Berezin coordinate transformation. The structure of supergeometry of the super-point induces a superstatistic of hidden ghost field in the financial market. We let \( y_t \) be an observed variable in the state space model and let \( x_t \) be a hidden variable of state. We denote \( \Phi_i(y_t) \in A \) as a ghost field and \( \Phi_i^+(x_t) \) as an anti-ghost field in finance with the relation of its ghost number

\[
\text{gh}(\Phi_i(y_t)) + \text{gh}(\Phi_i^+(x_t)) = -1. \tag{69}
\]

We have a parity \( p(\Phi_i(y_t)) = 1 - p(\Phi_i^+(x_t)) \) and \( p(\Phi_i(x_t)) = 1 - p(\Phi_i^+(x_t)) \) for both the state and space variable in the ghost field and the anti-ghost field.

### 2.4 Volatility Clustering Phenomena and General Equilibrium in Financial Market

We separate the system of financial market into two parts: the first is the state part \( \mathcal{Y}_t \) of hidden demand state, and the second is a space part \( \mathcal{Y}_t \) of observation of supply space of the state space model. In this paper, we use algebraic equations from algebraic topology and differential geometry as the main tool for defining a new mathematical object for arbitrage opportunity in the dynamic stochastic general equilibrium (DSGE) system of macroeconomics.

**Theorem 1.** *When the market is in equilibrium, we have*

\[
s^2 = 0 \leftrightarrow H^{-14}(A, s) = 0. \tag{70}
\]

**Proof.** See [13].

We divide the market into two separate sheets of D-brane and anti-D-brane of embedded indifference curve of supply curve and utility curve of demand. The interaction of two D-brane is induced from the trade-off between supply and demand as the general equilibrium point. We define the D-brane sheet of market is in real dimensions and the anti-self-duality (AdS) of D-brane to anti-D-brane is induced from the duality map from supply to demand.

Let OLS in superspace in time series data be written by

\[
y_t = \alpha_t + \beta_t x_t + \epsilon_t. \tag{71}
\]

Take a ghost functor \( \Phi_i, \Phi_i^+ : X \rightarrow (A, s) \simeq [X, S^\pm k] \)

\[
\Phi_i((y_t - \alpha_t) - \beta_t x_t \simeq \epsilon_t). \tag{72}
\]

Let \( \epsilon_t \) be the real present shock from economics and \( \epsilon_t^* \) be the expected shock in the future. In equilibrium, we assume a steady state of macroeconomics with no shock, i.e. \( \epsilon_t^2 = 0 \). Let \( 0 \) be the space of equilibrium and equivalent with moduli state space model of supply space \( \mathcal{Y}_t \) and
demand space \( X_t \) of market with \( \varepsilon_i^2 = \langle \varepsilon_i, \varepsilon_i \rangle \simeq Y_i/X_i \). Consider a short exact sequence of macroeconomics in general equilibrium with market risk \( \beta_t \) of sudden shock in the demand side and transfer into the supply side of economics, and let the systematics risk \( \sigma_t \) be a shock in both sides with a price-sticky market.

The moduli group \( \mathbb{Z}_2 \) defines a state-up and -down of the underlying financial time series data. When the market is in equilibrium, the short exact sequence will induce an infinite exact sequence of the market cocycle \( \beta_t \) and \( \alpha_t \).

In order to prove the existence of dark volatility in the financial market, we use the tensor correlation field. Let \( \tau \) be the time lag of the autocorrelation function of the volatility cluster

\[
C_{[\tau]} = \text{Corr}([r_{\tau}], [r_{\tau+t}]) = \frac{C}{\tau^\beta}
\]

with the power-law parameter \( \beta \leq 0.5 \). We define the time lag of the power-law function by \( x^*(\tau, \beta) := \tau^\beta \).

Let \( ds^2 = x^2 - x^2^2 \) be a Minkowski metric for GARCH(1,1). Let \( x \) be the autocorrelation function \( x := C_{[\tau]}(\tau) \) of return with volatility cluster phenomena. Then there exists a dark volatility \( \sigma_i^2 < 0 \) with minus sign in the complex hidden time scale as a homogeneous coordinate in quaternionic projective space \( \mathbb{H}P^1 \) according to the equation of projective hyperplane in the Minkowski space \( x^2 - x^2^2 = 1 \).

Let \( x \) be the excess kurtosis, with \( \sigma^4 = \frac{u^2}{x} - 3 \). It is clear that \( \sigma^2 = \pm \sqrt{\frac{u^2}{x} - 3} \). Let us consider \( x := C_{[\tau]}(\tau) = \text{Corr}([r_{\tau}], [r_{\tau+t}]) = \frac{C}{\tau^\beta} \). So we have a rotated hyperbola graph, \( xx^* = C \), which we can transform by using the rotational eigenvector to the normal hyperbola graph \( x^2 - x^2^2 = C \). We normalise \( C^* \) to 1. If we use the Euclidean distance in superspace in time series data with the curvature constant over the Minkowski cone, we have no arbitrage opportunity with time series in the stationary state. Although in parallel transport of the connection \( A_\mu(t) \) is varying with time, we still have arbitrage opportunity in form of the Wilson loop over market cocycle \((a, \beta), W_{a,\beta}(A_\mu)\).

In the superspace of time series data \( X_t/Y_t \), where \( x := C_{[\tau]}(\tau) \in X_t \) and \( y := C_{[\tau+t]}(\tau) \in Y_t \), we have tensor correlation as network of behaviour traders \( x \otimes x^*, y \otimes y^*, x \otimes y, x^* \otimes y^* \) by

\[
x \otimes y = C_{[\tau]}(\tau) \otimes C_{[\tau+t]}(\tau) \simeq \text{Corr}([r_{\tau}], [r_{\tau+t}]) = \frac{C}{\tau^\beta} \cdot \frac{C}{x^*}.
\]

We have a triplet of tensor correlation with memory in constant value \( C \):

\[
x^*(x \otimes y) = C.
\]

This triplet has properties analogous to the gravitational field in the triplet of the Lie algebras in the Nahm equation for the monopole or the instanton in theoretical physics. We defined a new form of the Yang-Mills equation in the financial market as a Nahm equation in the financial market version as a main consequence of the proof of the volatility clustering phenomena. Let us denote \( x := D_t \), a demand field, \( y := S_t \) a supply field. Let the price be a moduli state space between the supply and demand:\n
\[
p_t = D_t/S_t.
\]

Let \( [D_t] := T^C_{C^x}, [S_t] := T^C_{C^y}, [p_t] := t^C_{C^y} \). The Nahm equations for the transition states in the financial market are

\[
\frac{dS_t}{d[S_t]} = [D_t, p_t],
\]

\[
\frac{dT_t}{d[S_t]} = [p_t, S_t],
\]

\[
\frac{dp_t}{d[S_t]} = [D_t, S_t].
\]

The three equations can be written together as

\[
\frac{dT^C_{C^x,x,y,y}}{d[S_t]} = \frac{1}{2} \sum_{j,k} \varepsilon_{ijk} [T^C_{C^x,y,y}, T^C_{C^x,x,y}] = \sum_{j,k} \varepsilon_{ijk} T^C_{C^x,y,y}, T^C_{C^x,x,y}.
\]

With the modified Nahm equation for the interaction of the behaviour of the traders from the supply and the demand sides in the financial market, we let a pair of the costates in the market from the pair of the supply and demand sides to be the modified Lax pair equations for the pairs of the market state in the financial market as the costates between the fundamentalist and the chartist with the optimistic and the pessimistic forward looking of the price under the risk field \( \sigma^2 \). It is trivial that the modified Nahm equations for biology can be written in Lax pairs of the demand pairs, \( D \)-pairs wave function and the supply pairs, \( S \)-pairs wave function, as follows: Let

\[
\psi_{D \text{-pairs}} = T^C_{1_1} + iT^C_{2_2}, \quad \psi_{(D,S) \text{-pairs}} = -2iT^C_{3_3},
\]

\[
\psi_{S \text{-pairs}} = T^C_{1_1} - iT^C_{2_2}.
\]
and
\[ \psi_{k, CO}^{D,p} = \psi_{D,p}^{D}\text{–pairs} + k\psi_{D,S}^{(D,S)\text{–pairs}} + k^2\psi_{S}^{S\text{–pairs}}, \] (81)
\[ \psi_{k, CO}^{S,p} = \frac{1}{2} \frac{d\psi_{k, CO}^{D,p}}{d|s|} = \frac{1}{2} \psi_{D,S}^{(D,S)\text{–pairs}} + k\psi_{S}^{S\text{–pairs}}, \] (82)

where \( k \) is a D-brane of the market coupling constant in the Chern-Simons current as a transition state in the quantum price orbital, \( J^\mu = k \). Notice that dark volatility induces from the supply a short term \( \alpha_t^2 \) market the transition state in the supply side of the financial market. The system of Nahm equations is equivalent to the Lax pair equation of the financial market. The co-state \( \psi_{D, S}^{(D,S)\text{–pairs}} := (\psi_{k, CO}^{D,p}, \psi_{k, CO}^{S,p}) \) is a transition between the costate of the interaction of the behaviour field of the trader from the supply and the demand side in the financial market by the supply-demand pairs in the general equilibrium of the market costate with \( (\alpha_t^2, \sigma_t^2) \) in the financial market.

These systems of equations have a property of the Chern-Simons current in the time series data as the eigenvalues of the Dirac operator \( D \) for the financial market
\[ D\psi_{D,S}^{D,S}\text{–pairs} = J^\mu \psi_{D,S}^{D,S}\text{–pairs}, \] (83)
where the critical current with some threshold of the risk aversion for an arbitrage opportunity of the behaviour of the trader is denoted by \( J^\mu_c \). We have
\[ \sin \psi_{D,S}^{D,S}\text{–pairs} = \frac{J^\mu}{J^\mu_c} = \psi_{xx}^{(D,S)\text{–pairs}} - \psi_{yy}^{(D,S)\text{–pairs}} = \frac{\sigma_t^2}{\sin^2 \Psi - \sigma_t^2} \psi_{D,S}^{(D,S)\text{–pairs}} = D\psi_{D,S}^{D,S}\text{–pairs}. \] (84)

We define a price as a transition state of an orbit state in the fibre space with the co-states of the transformation of the behaviour field of the traders from the demand side of the market to the supply side by new coordinates \((p_{CO}^{O_1}, p_{CO}^{O_2}) \in (O_X, O_Y)\) with
\[ p_{CO}^{O_1} = \frac{D_t + S_t}{2}, \quad p_{CO}^{O_2} = \frac{D_t - S_t}{2}, \] (85)
\[ p_{CO}^{O_1'} = \frac{D_t' + S_t'}{2}, \quad p_{CO}^{O_2'} = \frac{D_t' - S_t'}{2}. \]

The equation can be transformed to
\[ \psi_{CO}^{D-pairs} = \sin \psi_{CO}^{D-pairs}, \quad \psi_{CO}^{S-pairs} = \sin \psi_{CO}^{S-pairs}. \] (86)

The super stationary state of the demand pair solutions for the pairing of \((O_D, O_D')\) and \((O_S, O_S')\) states in \((O_D', O_S')\) pairs can be solved by using the Paterson-Godazzi equation for an arbitrage opportunity. One part of the Lax pair equation for an arbitrage is the Paterson-Godazzi equation for the demand side, which has the pairing of \((O_D, O_D')\) solution by
\[ \psi_{D-pair}(D, S) := 4 \arctan e^{\kappa(D - p_S) + \alpha_t}, \]
\[ \psi_{S-pair}(D', S') := 4 \arctan e^{\kappa(D' - p_S') + \alpha_t'}, \] (87)

where \( p_t = \frac{d\theta}{ds} = \frac{d\theta}{dx} \) is a parameter for the price as a transition market state by changing the coordinate of the \( D \) coordinate in the demand side of the superspace of the financial market \( S \) and \( D \) is the coordinate in the supply side. The new parameter \( \sigma_t^2 \) is a hidden risk fear field or the dark volatility in this paper. It comes from an evolutionary factor of an adaptive behaviour of the traders of the risk aversion and who want to pursue an arbitrage opportunity. We let the transition of the states between the active price and the passive price while the prices are matching in the double auction market defined by the phase parameters of the market cocycle \((a, \beta)\), \( \alpha^2 = \frac{1}{1-p^2}, \beta^2 = \frac{1}{1-p^2} \), where the extra parameter \( k \) comes from the coupling constant of the behaviour of the traders with the Chern-Simons current \( J^\mu = J^\mu \sin \Psi \).

The \((O_D, O_D')\) pairs and \((O_S, O_S')\) pairs of the solution have a positive root and a negative root for \((a, \beta)\) cocycle as a twist in the variable to produce an arbitrage opportunity from the general equilibrium in the mean reversion as the super stationary state with the Chern-Simons current for the financial market. The phase shift of the price quantum in the lax pairs \((\psi^{D-pair}, \psi^{S-pair})\) takes the solution \( \psi^{D-pair} = 0 = \psi^{S-pair} \) to an adjacent with \( \psi^{D-pair} = 2\pi = \psi^{S-pair} \) as a boundary condition for the price quantum with \( (\psi^{D-pair}, \psi^{S-pair}) = (0, 0)(mod 2\pi) \).

3 Results and Conclusion

In this paper, we introduced a new approach to obtain the description of the dynamics of the market panic in the financial world by using the cohomology theory and the Yang-Mills theory over the traditional time series model, the so-called GARCH(1,1). Because the financial events studied by the social sciences evidence cannot be directly transported into the theory of Yang-Mills field for the financial market, we needed to add some more assumptions also to soften some of the the assumptions of the Yang-Mills equations for their suitable use in the financial market with the equilibrium point in the Minkowski spacetime. We explained stylised facts in the financial time series data using Yang-Mills equation for the financial market. We used Jacobian flow to construct
the Minkowski metric with the hidden state in the time series data as the spinor field. We used ghost and anti-ghost fields in the Grothendieck cohomology to explain the interaction of the behaviour of the traders from the supply and demand side of the market in an infinite cohomology sequence. We proved the power law in the volatility clustering phenomena by using the inversion invariant in the Yang-Mills theory. The result of this work is a new type of GARCH(1,1) model in the superspace of the time series data, the so-called Minkowski GARCH(1,1). This theory is used to explain the spectrum of the nonstationary and nonlinear financial time series data as an analogy with the spectrum sequence of the behaviour of the trader as pairs of the quantum field states in the underlying time series data in the spacetime geometry of the Kolmogorov space in time series data. We proved all the possibilities of the existence of the prior problem and all the possibilities of the theorem to solve with the precise proving procedure. A collection of the theories for solving this problem we called the evolution feedback path. The result of this solution can be used to predict the financial time series with a very high precision without any end effect by using a mixture of filtering technique and tools of theoretical physics. In this paper, we discussed a stylised fact on the long memory process of the volatility cluster phenomena by using the Minkowski metric for GARCH(1,1). Also we presented the result of the minus sign of the volatility in the reversed direction of the timescale. It was named the dark volatility or the hidden risk fear field. These situations have been extensively studied, and the correlations have been found to be a very powerful tool. Yet most natural processes in the financial market are the nonstationary. In particular, in times of a financial crisis, with some accident events or economic shock news, stationarity is lost. The precise definition of a nonstationary state of times series can help analyse the non-equilibrium state in the financial market. As an example, we may think about the financial market state from a mathematical point of view. There is no classical stochastic process that will match the real financial data, because there is no single Kolmogorov space describing the whole financial market. Dynamics of the market panic (in the financial world) from the point of view of new financial model was presented. We hope that the precise definition of the Kolmogorov topological space can lead to a better understanding of the macroeconomic models. Suitable algebraic reconstruction of the space of time series can help with the analysis of the prior effect and end effect of time series models. The presented approach, together with the close connection with of empirical mode decomposition and intrinsic time scale decomposition of correlation matrix, can serve as the main tool for the detection of market crash over time series data, as was introduced in [21].

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