On the asymptotic behavior of the hyperbolic Brownian motion

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Abstract

The main results in this paper concern large and moderate deviations for the radial component of a
$n$-dimensional hyperbolic Brownian motion (for $n \geq 2$) on the Poincaré half-space. We also investigate
the asymptotic behavior of the hitting probability $P_\eta(T^{(n)}_{\eta_1} < \infty)$ of a ball of radius $\eta_1$, as the distance
$\eta$ of the starting point of the hyperbolic Brownian motion goes to infinity.

Keywords: hitting probability, hyperbolic distance, large deviations, moderate deviations, Poincaré half-
space.

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1 Introduction

Random motions in hyperbolic spaces, i.e. Riemannian manifolds with constant negative curvature, have
been studied since the fifties and much attention has been placed on the so-called hyperbolic Brownian
motion on the Poincaré half-space $\mathbb{H}^n$; the interested reader can consult, for example, Gertsenshtein and
Vasiliev [15], Getoor [16], Gruet [18], Matsumoto and Yor [28], Lao and Orsingher [25], Byczkowski and
Malecki [3], Borodin [1]. Branching hyperbolic Brownian motion has been analyzed by Lalley and Sellke [24]
who investigated the connection between the birth rate and the underlying dynamics in supercritical and
subcritical cases. Also Kelbert and Suhov [23] and Karpelevich et al. [22] have studied the asymptotic be-
havior of the hyperbolic branching Brownian motion. Random walks on the geodesics of the hyperbolic plane
has been considered, for example, in Jørgensen [21] and Cammarota and Orsingher [4]. One-dimensional
and planar random motions in non-Euclidean spaces have also been analyzed in De Gregorio and Orsingher
[10].

There is a close link between one-dimensional disordered systems and Brownian diffusion on hyperbolic
spaces. For instance, Gertsenshtein and Vasiliev [15], in their pioneering work, have shown that the statistical
properties of reflection and transmission coefficients for waveguides with random inhomogeneities are directly
related to some random walk on the Poincaré half-plane.

Comtet and Monthus [7] showed how the one-dimensional classical diffusion of a particle in a quenched
random potential is directly related to Brownian motion on the hyperbolic plane. In particular the authors
discussed some functionals governing transport properties of a diffusion in a random Brownian potential, in
terms of hyperbolic Brownian motion.

The geodesic curves in Poincaré half-plane model are either half-circles with center lying on the $x$-axis
or vertical half-lines. For an optical non-homogenous media where light rays move with velocity $c(x, y) = y$
(independent from direction), on the base of Fermat’s principle, the possible paths for the light are those
curves $L$ which satisfy the equality

\[
\sin \alpha(y) = k, \quad (1.1)
\]

where $\alpha(y)$ is the angle between the vertical and the tangent to $L$ in the point with ordinate $y$. It is easy to
see that the circles with center on the $x$-axis and radius $\frac{1}{k}$ satisfy (1.1) (for $k = 0$ we get the vertical lines).
The hyperbolic geometrical optics where the ray trajectories are the geodesics in the Poincaré half-plane is

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analyzed for example in De Micheli et al. [12]. Scattered obstacles in the non-homogeneous medium cause random deviations in the propagation of light and this leads to the random model analyzed in Cammarota and Orsingher [4].

Hyperbolic Brownian motion has been revitalized by mathematical finance since some exotic derivatives have a strict connection with the stochastic representation of the hyperbolic Brownian motion. For example, Matsumoto and Yor [27] present some identities for the pricing formula of the Asian or average call option in the framework of the Black-Scholes model.

In this paper we present asymptotic results in the fashion of large deviations. The theory of large deviations gives an asymptotic computation of small probabilities on exponential scale and it is used in several fields, and, in particular, in physics; for instance a large number of authors (see e.g. Ellis [14]) saw large deviation theory as the proper mathematical framework in which problems of statistical mechanics can be formulated and solved. We remark that, in analogy with what happens in Monthus and Texier [29] for random walks on Bethe lattices, our asymptotic results can have interest in the study of some random walks in the context of polymer physics which admit the hyperbolic Brownian motion as a continuous approximation.

We conclude with the outline of the paper. In Section 2 we present some preliminaries on large deviations and on the hyperbolic Brownian motion. In Section 3 we prove large and moderate deviation results for the radial component of a $n$-dimensional hyperbolic Brownian motion (for $n \geq 2$). In Section 4 we investigate the asymptotic behavior of the hitting probabilities of an hyperbolic ball centered at the origin as the distance of the starting point of the process goes to infinity. Finally, in Section 5, we discuss some connections with the results in Hirao [20].

## 2 Preliminaries

We give some preliminaries on large deviations and on the hyperbolic Brownian motion.

### 2.1 Preliminaries on large deviations

We recall the basic definitions in Dembo and Zeitouni [11], pages 4–5. Let $X$ be a Hausdorff topological space with Borel $\sigma$-algebra $\mathcal{B}_X$. A lower semi-continuous function $I : X \to [0, \infty]$ is called rate function. A family of $X$-valued random variables $\{X(t) : t > 0\}$ satisfies the large deviation principle (LDP for short), as $t \to \infty$, with rate function $I$ and speed $v_t$ if: \(\lim_{t \to \infty} v_t = \infty\),

\[
\lim_{t \to \infty} \sup_{t} \frac{1}{v_t} \log P(X(t) \in F) \leq - \inf_{x \in F} I(x) \quad \text{for all closed sets } F
\]

and

\[
\lim_{t \to \infty} \inf_{t} \frac{1}{v_t} \log P(X(t) \in G) \geq - \inf_{x \in G} I(x) \quad \text{for all open sets } G.
\]

A rate function $I$ is said to be good if all the level sets $\{x \in X : I(x) \leq \gamma\}$ for $\gamma \geq 0$ are compact. We recall that the lower bound for open sets is equivalent to the following condition (see eq. (1.2.8) in Dembo and Zeitouni [11]):

\[
\lim_{t \to \infty} \inf_{t} \frac{1}{v_t} \log P(X(t) \in G) \geq -I(x) \quad \text{for all } x \in X \text{ such that } I(x) < \infty \quad \text{and}
\]

\[
\text{for all open sets } G \text{ such that } x \in G.
\]  

In this paper we prove LDPs with $X = \mathbb{R}$. We start with Proposition 3.2 where we have $v_t = t$. We also study the moderate deviations, i.e. we prove Proposition 3.3 which provides a class of LDPs where $v_t = t^{1 - 2\beta}$, varying $\beta \in (0, 1/2)$, and the rate function does not depend on $\beta$. In some sense the moderate deviations fill the gap between an asymptotic normality result (i.e. Theorem 2.1 in Matsumoto [27]) for $\beta = \frac{1}{2}$, and a convergence in probability to a constant (i.e. Corollary 5.7.3 in Davies [8]) - together with a centering of the random variables - for $\beta = 0$. We remark that we have a quadratic rate function which vanishes at zero. To better explain this concept, we recall the basic result on moderate deviations for the empirical means $\{\frac{X_1 + \cdots + X_n}{n} : n \geq 1\}$ of i.i.d. centered and $\mathbb{R}^d$-valued random variables $\{X_n : n \geq 1\}$ (see e.g. Theorem 3.7.1 in Dembo and Zeitouni [11]) which fill the gap between the central limit theorem and the law of the large numbers. In such a case we have the LDP for $\left\{\sqrt{n a_n} \frac{X_1 + \cdots + X_n}{n} : n \geq 1\right\}$ for $\{a_n : n \geq 1\}$ such that $a_n \to 0$ and $na_n \to \infty$ (as $n \to \infty$) with speed $v_n = \frac{1}{a_n}$, then one can check that $a_n$ plays the role of $t^{2\beta - 1}$ in Proposition 3.3 (actually $t^{2\beta - 1} \to 0$ and $t \cdot t^{2\beta - 1} \to \infty$ as $t \to \infty$).
2.2 Preliminaries on the hyperbolic Brownian motion

For \( n \geq 2 \), let \( \mathbb{H}^n = \{ z = (x, y) : x \in \mathbb{R}^{n-1}, y > 0 \} \) be the upper half-space with origin \( O_n = (0, \ldots, 0, 1) \) endowed with the hyperbolic Riemannian metric

\[
ds^2 = \frac{dx^2 + \cdots + dx_{n-1}^2 + dy^2}{y^2}.
\]

The hyperbolic distance \( \eta(z, z') \) between \( z = (x, y) \) and \( z' = (x', y') \) in \( \mathbb{H}^n \) is given by the formula

\[
cosh \eta(z, z') = \frac{|x - x'|^2 + y^2 + y'^2}{2yy'}
\]

where \( |x - x'| \) is the Euclidean distance between \( x, x' \in \mathbb{R}^{n-1} \), and the volume element is given by

\[
dv = y^{-n} dx dy = \sinh^{n-1} \eta \, d\eta \, d\Omega_n
\]

where \( d\Omega_n \) is the surface element of the \( n \)-dimensional unit sphere. The Laplace-Beltrami operator in \( \mathbb{H}^n \) is

\[
\Delta_n = y^2 \left( \sum_{i=1}^{n-1} \frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y^2} \right) - \frac{(n-2)}{2} y \frac{\partial}{\partial y}.
\]

(see, for example, Chavel [6] page 265).

The hyperbolic Brownian motion is a diffusion governed by the generator \( \Delta_n/2 \). We denote by \( k_n(z, z', t) \) the heat kernel, with respect to the volume element \( dv \). Since the Laplace-Beltrami operator is invariant under diffeomorphism, \( k_n(z, z', t) \) is a function of \( \eta(z, z') \) and we write \( k_n(\eta, t) \) for \( k_n(z, z', t) \). Therefore the transition density \( p_n(\eta, t) \) of the radial component of hyperbolic Brownian motion is given by

\[
p_n(\eta, t) = \frac{2\pi^{n/2}}{\Gamma(n/2)} k_n(\eta, t) \sinh^{n-1} \eta \, d\eta \quad (\eta > 0, \ t > 0).
\]

The classical formulae for the heat kernel, which are of different forms for odd and even dimensions \( n \), are well known together with the recurrence Millson’s formula with respect to the dimension \( n \) (see e.g Davies and Mandouvalos [9]). The analogue formulae for the hyperbolic heat kernel can be found in several references: see, e.g., Buser [2] Theorem 7.4.1, Chavel [6] Section X.2, Terras [30] Section 3.2 and Helgason [19] page 29.

In detail we have

\[
k_2(\eta, t) = \frac{e^{-t}}{2^{5/2}(\pi t)^{3/2}} \int_0^\infty \frac{\varphi e^{-\frac{\varphi^2}{2}}}{\sqrt{\cosh \varphi - \cosh \eta}} d\varphi,
\]

\[
k_3(\eta, t) = \frac{e^{-t}}{2^3(\pi t)^{3/2}} \frac{\eta e^{-\frac{\eta^2}{2}}}{\sinh \eta}
\]

and, in general, for all \( n \geq 2 \), closed form expressions are not available for \( k_n = k_n(\eta, t) \). Therefore we will use some known sharp bounds, see e.g. Theorem 5.7.2 in Davies [8] or Theorem 3.1 in Davies and Mandouvalos [9] (note that \( n \) in that reference is replaced by \( n - 1 \) in this paper). Let \( h_n \) be defined by

\[
h_n(\eta, t) := t^{-n/2} \exp \left( -\frac{(n-1)^2 t}{4} - \frac{(n-1)\eta}{2} - \frac{\eta^2}{4t} \right) (1 + \eta + t)^{(n-3)/2} (1 + \eta);
\]

we have that \( k_n(\eta, t) \sim h_n(\eta), \) i.e. there exists \( c_n \in (1, \infty) \) such that we have

\[
c_n^{-1} h_n(\eta, t) \leq k_n(\eta, t) \leq c_n h_n(\eta, t)
\]

for all \( t > 0 \) and \( \eta > 0 \). Then, since \( (\sinh \eta)^{n-1} \sim \left( \frac{\eta}{1 + \eta} \right)^{n-1} e^{(n-1)\eta} \), if we set

\[
g_n(\eta, t) := t^{-n/2} \left( \frac{\eta}{1 + \eta} \right)^{n-1} \exp \left( -\frac{(n-1)\eta}{4t} \right) (1 + \eta + t)^{(n-3)/2} (1 + \eta),
\]

we can say that

\[
(\sinh \eta)^{n-1} k_n(\eta, t) \sim g_n(\eta, t)
\]

uniformly in \( \eta \) and \( t, \) i.e. there exists a constant \( d_n \in (1, \infty) \) such that

\[
d_n^{-1} g_n(\eta, t) \leq (\sinh \eta)^{n-1} k_n(\eta, t) \leq d_n g_n(\eta, t)
\]

for all \( t > 0 \) and \( \eta > 0 \).
3 Large and moderate deviations

In this section we prove asymptotic results for the radial component of a $n$-dimensional hyperbolic Brownian motion (for $n \geq 2$), which will be denoted by $\{D_n(t) : t \geq 0\}$. More precisely we prove two LDPs: the first one (Proposition 3.2) concerns the convergence in probability to a constant, the second one (Proposition 3.3) concerns the moderate deviation regime. Both proofs are divided in two parts.

1. The proof of the lower bound for open sets, and we refer to condition (2.1) with appropriate choices of $(X(t) : t > 0)$, $v_t$ and $I$.

2. The proof of the upper bound for closed sets, and we often refer to an upper bound for the moment generating function $E[e^{\text{h}D_n(t)}]$ for all $\lambda \in \mathbb{R}$ (actually we have to consider $\lambda \neq 0$) which is given in the next Lemma 3.1.

**Lemma 3.1** Let $d_n$ be as in (2.2) and let $\lambda \in \mathbb{R}$ be arbitrarily fixed. Moreover we set $\kappa(\lambda) := \lambda(\lambda + n - 1)$ and $m_n := \left\lfloor \frac{n-1}{2} \right\rfloor$. Then, for all $t > 0$, we have

$$
E[e^{\lambda D_n(t)}] \leq d_n \int_{0}^{\infty} e^{\lambda \eta} e^{(n-1)\eta} \exp\left(-\frac{(\eta-(n-1)t)^2}{4t}\right)(1+\eta+t)^{(n-1)/2}(1+\eta)d\eta
$$

where $W_i$ is a Normal distributed random variable with mean $\lambda$ and variance $\frac{1}{2t}$.

**Proof.** First, by (2.2), we have

$$
E[e^{\lambda D_n(t)}] \leq d_n \int_{0}^{\infty} e^{\lambda \eta} \exp\left(-\frac{(\eta-(n-1)t)^2}{4t}\right)(1+\eta+t)^{(n-1)/2}(1+\eta)d\eta
$$

thus, by the change of variable $\alpha = \frac{\eta-(n-1)t}{2t}$ and some computations, we obtain

$$
E[e^{\lambda D_n(t)}] \leq d_n t^{-n/2} \int_{0}^{\infty} e^{\lambda \eta} \exp\left(\lambda \eta - \lambda(\lambda + n - 1)t - \frac{(\eta-(n-1)t)^2}{4t}\right)(1+\eta+t)^{(n-1)/2}d\eta
$$

$$
= 2d_n t^{-n/2+1} e^{\lambda(\lambda + n - 1)} A(\lambda, t, n)
$$

where

$$
A(\lambda, t, n) := \int_{(1-n)/2}^{\infty} \exp\left(-\lambda^2 t^2\right)(1+2\alpha t + nt)^{(n-1)/2}d\alpha.
$$

Finally, since

$$
A(\lambda, t, n) \leq \int_{(1-n)/2}^{\infty} \exp\left(-\lambda^2 t^2\right)(1+2\alpha t + nt)^{m_n}d\alpha
$$

$$
= \sum_{j=0}^{m_n} \binom{m_n}{j}(1+nt)^{m_n-j}(2t)^j \int_{(1-n)/2}^{\infty} \exp\left(-\lambda^2 t^2\right)\alpha^j d\alpha
$$

$$
\leq \sum_{j=0}^{m_n} \binom{m_n}{j}(1+nt)^{m_n-j}(2t)^j \int_{-\infty}^{\infty} \exp\left(-\lambda^2 t^2\right)\alpha^j d\alpha
$$

and

$$
\int_{-\infty}^{\infty} \exp\left(-\lambda^2 t^2\right)\alpha^j d\alpha = \int_{-\infty}^{\infty} \exp\left(-\frac{(\lambda t)^2}{2(1/2t)}\right)\alpha^j d\alpha = \sqrt{2\pi(1/(2t))}E[|W_i|^j],
$$

we conclude by putting together the inequalities for $E[e^{\lambda D_n(t)}]$ and $A(\lambda, t, n)$.

We start with the LDP associated to the convergence in probability of $\frac{D_n(t)}{t}$ to $n-1$ (as $t \to \infty$).
Proposition 3.2 The family $\left\{ \frac{D_n(t)}{t} : t > 0 \right\}$ satisfies the LDP with good rate function $I_1$ defined by

$$I_1(x) = \begin{cases} 
\frac{(x-(n-1))^2}{4} & \text{if } x \geq 0 \\
\infty & \text{if } x < 0, 
\end{cases}$$

and speed function $v_t = t$.

\textbf{Proof.} The proof is divided in two parts.

1) Lower bound for open sets. We have to check that

$$\liminf_{t \to \infty} \frac{1}{t} \log P \left( \frac{D_n(t)}{t} \in G \right) \geq - \frac{(x-(n-1))^2}{4}$$

for all $x \geq 0$ and for all open sets $G$ such that $x \in G$. We have two cases. 

\textit{Case} $x > 0$. Let $\varepsilon > 0$ be such that $(x-\varepsilon, x+\varepsilon) \subset G$; moreover we can choose $\varepsilon \in (0, x)$. Then, by (2.2) and by considering the change of variable $\alpha = \frac{x}{t}$, we have

$$P \left( \frac{D_n(t)}{t} \in G \right) \geq P \left( \frac{D_n(t)}{t} \in (x-\varepsilon, x+\varepsilon) \right)$$

$$\geq d_n^{-1} \int_{x-\varepsilon}^{x+\varepsilon} \left( \frac{\alpha x}{1+\alpha t} \right)^{n-1} \exp \left( -\frac{(\alpha - (n-1))^2}{4t} \right) (1 + \alpha t)^{(n-3)/2} d\alpha$$

$$\geq d_n^{-1} t^{-n/2+1} \int_{x-\varepsilon}^{x+\varepsilon} \left( \frac{\alpha x}{1+\alpha t} \right)^{n-1} \exp \left( -\frac{(\alpha - (n-1))^2}{4t} \right) (1 + \alpha t)^{(n-3)/2} d\alpha$$

$$\geq d_n^{-1} t^{-n/2+1} 2\varepsilon t \int_{x-\varepsilon}^{x+\varepsilon} \left( \frac{\alpha x}{1+\alpha t} \right)^{n-1} \exp \left( -\frac{(\alpha - (n-1))^2}{4t} \right) (1 + \alpha t)^{(n-3)/2} d\alpha$$

thus

$$\liminf_{t \to \infty} \frac{1}{t} \log P \left( \frac{D_n(t)}{t} \in G \right) \geq - \max \left\{ \frac{(x-\varepsilon-(n-1))^2}{4}, \frac{(x+\varepsilon-(n-1))^2}{4} \right\},$$

and we conclude by letting $\varepsilon$ go to zero.

Case $x = 0$. Let $\{x_k : k \geq 1\}$ and $\{\delta_k : k \geq 1\}$ be two sequences of positive numbers such that $\lim k \to \infty x_k = 0$ and, for all $k \geq 1$, $(x_k - \delta_k, x_k + \delta_k) \subset G$. Then, for each fixed $k \geq 1$, we have

$$\liminf_{t \to \infty} \frac{1}{t} \log P \left( \frac{D_n(t)}{t} \in G \right) \geq \liminf_{t \to \infty} \frac{1}{t} \log P \left( \frac{D_n(t)}{t} \in (x_k - \delta_k, x_k + \delta_k) \right) \geq - \frac{(x_k - (n-1))^2}{4}$$

by the first part of the proof concerning $x > 0$ (we have $x_k$ in place of $x$ and $(x_k - \delta_k, x_k + \delta_k)$ in place of $G$) and we complete the proof (for the case $x = 0$) letting $k$ go to infinity.

2) Upper bound for closed sets. The upper bound

$$\limsup_{t \to \infty} \frac{1}{t} \log P \left( \frac{D_n(t)}{t} \in F \right) \leq - \inf_{x \in F} I_1(x)$$

for all closed sets $F$

trivially holds if $n-1 \in F$ or $F \cap [0, \infty)$ is empty. Thus, from now on, we assume that $n-1 \notin F$ and $F \cap [0, \infty)$ is nonempty. We also assume that both $F \cap [0, n-1)$ and $F \cap (n-1, \infty)$ are nonempty; actually at least one of the two sets is nonempty and, if one of them would be empty, the proof presented below could be readily adapted. We define

$$\hat{x} := \sup(F \cap [0, n-1]), \quad \check{x} := \inf(F \cap (n-1, \infty))$$

and we have $0 \leq \check{x} < x - 1 < \hat{x}$, and $F \subset (\check{x}, \hat{x}) \cup [\hat{x}, \infty)$. Then

$$P \left( \frac{D_n(t)}{t} \in F \right) \leq P \left( \frac{D_n(t)}{t} \leq \check{x} \right) + P \left( \frac{D_n(t)}{t} \geq \hat{x} \right)$$
and, by Lemma 1.2.15 in Dembo and Zeitouni [11], we get
\[
\limsup_{t \to \infty} \frac{1}{t} \log P \left( \frac{D_n(t)}{t} \in F \right) \\
\leq \max \left\{ \limsup_{t \to \infty} \frac{1}{t} \log P \left( \frac{D_n(t)}{t} \leq \hat{x} \right), \limsup_{t \to \infty} \frac{1}{t} \log P \left( \frac{D_n(t)}{t} \geq \hat{x} \right) \right\}.
\]

Thus, if we prove
\[
\limsup_{t \to \infty} \frac{1}{t} \log P \left( \frac{D_n(t)}{t} \leq \hat{x} \right) \leq - \inf_{x \leq \hat{x}} I_1(x) = - \frac{(\hat{x} - (n - 1))^2}{4}
\]
(3.1)
and
\[
\limsup_{t \to \infty} \frac{1}{t} \log P \left( \frac{D_n(t)}{t} \geq \hat{x} \right) \leq - \inf_{x \geq \hat{x}} I_1(x) = - \frac{(\hat{x} - (n - 1))^2}{4},
\]
(3.2)
we conclude the proof because we get
\[
\limsup_{t \to \infty} \frac{1}{t} \log P \left( \frac{D_n(t)}{t} \in F \right) \leq \max \left\{ - \inf_{x \leq \hat{x}} I_1(x), - \inf_{x \geq \hat{x}} I_1(x) \right\} \\
\leq - \min \left\{ \inf_{x \leq \hat{x}} I_1(x), \inf_{x \geq \hat{x}} I_1(x) \right\} = - \inf_{x \in \mathbb{R}} I_1(x).
\]

Proof of (3.1). The case \( \hat{x} = 0 \) is trivial because we have \( -\infty \leq - \frac{(n-1)^2}{4} \) noting that \( P \left( \frac{D_n(t)}{t} \leq 0 \right) = 0 \). For \( \hat{x} > 0 \), by (2.2), we have
\[
P \left( \frac{D_n(t)}{t} \leq \hat{x} \right) \leq d_n \int_{0}^{\hat{x} \sqrt{t}} t^{-n/2} \left( \frac{\eta}{1+\eta} \right)^{n-1} \exp \left( - \frac{(\eta - (n-1)t)^2}{4t} \right) (1 + \eta)^{-t(n-3)/2} (1 + \eta) d\eta \\
\leq d_n t^{-n/2} \hat{x} \exp \left( - \frac{(\hat{x} - (n-1)t)^2}{4} \right) (1 + t \hat{x} + t)^{n-3}/2 (1 + t \hat{x}),
\]
and we obtain
\[
\limsup_{t \to \infty} \frac{1}{t} \log P \left( \frac{D_n(t)}{t} \leq \hat{x} \right) \leq - \frac{(\hat{x} - (n - 1))^2}{4}.
\]

Proof of (3.2). Firstly, by Markov’s inequality, for all \( \lambda > 0 \) we have
\[
P \left( \frac{D_n(t)}{t} \geq \hat{x} \right) \leq e^{-\lambda\hat{x}} 2\sqrt{2\pi(1/(2t))} d_n t^{-n/2+1} e^{\kappa(\lambda)t} \sum_{j=0}^{m_n} \binom{m_n}{j} (1 + nt)^{m_n-j} (2t)^j \mathbb{E}[|W_1|^j].
\]
Moreover, by Lemma 3.1, we get
\[
P \left( \frac{D_n(t)}{t} \geq \hat{x} \right) \leq e^{-\lambda\hat{x}} 2\sqrt{2\pi(1/(2t))} d_n t^{-n/2+1} e^{\kappa(\lambda)t} \sum_{j=0}^{m_n} \binom{m_n}{j} (1 + nt)^{m_n-j} (2t)^j \mathbb{E}[|W_1|^j].
\]
Now let \( t_0 > 0 \) be arbitrarily fixed; then, for all \( t > t_0 \), we have
\[
\mathbb{E}[|W_1|^j] = \frac{1}{\sqrt{2\pi(1/(2t))}} \int_{-\infty}^{\infty} \exp \left( - \frac{(\lambda - \alpha)^2}{2(1/(2t))} \right) |\alpha|^j d\alpha \\
\leq \sqrt{\frac{t}{t_0}} \frac{1}{\sqrt{2\pi(1/(2t_0))}} \int_{-\infty}^{\infty} \exp \left( - \frac{(\lambda - \alpha)^2}{2(1/(2t_0))} \right) |\alpha|^j d\alpha = \sqrt{\frac{t}{t_0}} \mathbb{E}[|W_{t_0}|^j].
\]
In conclusion we have
\[
\limsup_{t \to \infty} \frac{1}{t} \log P \left( \frac{D_n(t)}{t} \geq \hat{x} \right) \leq - \lambda\hat{x} + \kappa(\lambda),
\]
and therefore
\[
\limsup_{t \to \infty} \frac{1}{t} \log P \left( \frac{D_n(t)}{t} \geq \hat{x} \right) \leq \inf_{\lambda > 0} \{ - \lambda\hat{x} + \kappa(\lambda) \} = - \inf_{x \geq \hat{x}} I_1(x)
\]
because \( \inf_{\lambda > 0} \{ - \lambda\hat{x} + \kappa(\lambda) \} = - \frac{(\hat{x} - (n-1))^2}{4} = -I_1(\hat{x}) \) (actually the infimum is attained at \( \lambda = \frac{\hat{x} - (n-1)}{2} \)). □
Remark 3.1 We can state the analogue of Proposition 3.2 for a centered Euclidean $n$-dimensional Brownian motion $\{B_n(t) : t \geq 0\}$. One can easily check that $\left\{ \frac{\|B_n(t)\|}{t} : t > 0 \right\}$ satisfies the LDP with good rate function $J_1$ defined by

$$J_1(x) := \begin{cases} \frac{x^2}{2} & \text{if } x \geq 0 \\ \infty & \text{if } x < 0 \end{cases}$$

with a standard application of the Gärtner Ellis Theorem (see e.g. Theorem 2.3.6 in Dembo and Zeitouni [11]) and of the contraction principle (see e.g. Theorem 4.2.1 in Dembo and Zeitouni [11]). Thus, in some sense (see also the note just after the statement of Corollary 5.7.3 in Davies [8]), the hyperbolic Brownian motion has an implicit drift directed away from the origin because $I_1(x)$ is a quadratic rate function (on $[0, \infty)$) which vanishes at $x = n - 1$, while $I_1(x)$ vanishes at $x = 0$.

We conclude with the LDPs concerning the moderate deviation regime.

**Proposition 3.3** For all $\beta \in (0, 1/2)$, the family $\left\{ t^{(2\beta-1)/2} \left( \frac{D_n(t)-(n-1)t}{\sqrt{t}} \right) : t > 0 \right\}$ satisfies the LDP with rate function $I_2$ defined by $I_2(x) = \frac{x^2}{4}$ (for $x \in \mathbb{R}$) and speed function $v_t = t^{1-2\beta}$.

**Proof.** Firstly, in order to have simpler formulae, we remark that

$$t^{(2\beta-1)/2} \frac{D_n(t)-(n-1)t}{\sqrt{t}} = t^{\beta-1}(D_n(t)-(n-1)t).$$

The proof is divided in two parts.

1) **Lower bound for open sets.** We have to check that

$$\liminf_{t \to \infty} \frac{1}{t^{1-2\beta}} \log P \left( t^{\beta-1}(D_n(t)-(n-1)t) \in G \right) \geq -\frac{x^2}{4}$$

for all $x \in \mathbb{R}$ and for all open sets $G$ such that $x \in G$. We can find $\varepsilon > 0$ such that $(x-\varepsilon, x+\varepsilon) \subset G$. Then we have

$$P \left( t^{\beta-1}(D_n(t)-(n-1)t) \in G \right) \geq P \left( t^{\beta-1}(D_n(t)-(n-1)t) \in (x-\varepsilon, x+\varepsilon) \right) \geq P \left( D_n(t) \in \left( \frac{x-\varepsilon}{t^{\beta-1}} + (n-1)t, \frac{x+\varepsilon}{t^{\beta-1}} + (n-1)t \right) \right)$$

and, from now on, we take $t$ large enough to have $\frac{x-\varepsilon}{t^{\beta-1}} + (n-1)t > 0$. Now, by (2.2) and by considering the change of variable $\alpha = t^{\beta-1}(\eta-t(n-1))$, we have

$$P \left( t^{\beta-1}(D_n(t)-(n-1)t) \in G \right) \geq d_n^{-1} \int_{\frac{x-\varepsilon}{t^{\beta-1}} + (n-1)t}^{\frac{x+\varepsilon}{t^{\beta-1}} + (n-1)t} t^{-n/2} \left( \frac{\eta}{1+\eta} \right)^{n-1} \cdot \exp \left( -\frac{(\eta-(n-1)t)^2}{4t} \right) \left( 1 + \frac{\eta}{1+\eta} \right)^{(n-3)/2} d\eta$$

$$\geq d_n^{-1} t^{-n/2+1-\beta} \int_{x-\varepsilon}^{x+\varepsilon} \left( \frac{\alpha}{1+\alpha} \right)^{(n-3)/2} d\alpha.$$

Moreover there exists $\alpha_{\varepsilon,t} \in (x-\varepsilon, x+\varepsilon)$ such that

$$P \left( t^{\beta-1}(D_n(t)-(n-1)t) \in G \right) \geq d_n^{-1} t^{-n/2+1-\beta} \left( \frac{\alpha_{\varepsilon,t}}{1+\alpha_{\varepsilon,t}} \right)^{(n-3)/2} \cdot \exp \left( -\frac{\alpha_{\varepsilon,t}^2}{4t^{2\beta-1}} \right) \left( 1 + \frac{\alpha_{\varepsilon,t}}{1+\alpha_{\varepsilon,t}} + tn \right)^{(n-3)/2} d\alpha.$$

this yields

$$\liminf_{t \to \infty} \frac{1}{t^{1-2\beta}} \log P \left( t^{\beta-1}(D_n(t)-(n-1)t) \in G \right) \geq - \max \left\{ \frac{(x-\varepsilon)^2}{4}, \frac{(x+\varepsilon)^2}{4} \right\},$$
and we conclude by letting $\varepsilon$ go to zero.

2) Upper bound for closed sets. The upper bound
\[
\limsup_{t \to \infty} \frac{1}{t^{1-2\beta}} \log P \left( t^{\beta-1}(D_n(t) - (n-1)t) \in F \right) \leq -\inf_{x \leq \hat{x}} I_2(x) \quad \text{for all closed sets } F
\]
trivially holds if $0 \in F$ or $F$ is empty. Thus, from now on, we assume that $0 \notin F$ and $F$ is nonempty. We also assume that both $F \cap (-\infty, 0)$ and $F \cap (0, \infty)$ are nonempty; actually at least one of the two sets is nonempty and, if one of them would be empty, the proof presented below could be readily adapted. We define
\[
\hat{x} := \sup(F \cap (-\infty, 0)) \quad \text{and} \quad \hat{x} := \inf(F \cap (0, \infty))
\]
and we have $\hat{x}, \tilde{x} \in F$, $\hat{x} < 0 < \tilde{x}$, and $F \subset (-\infty, \tilde{x}] \cup [\hat{x}, \infty)$. Then
\[
P \left( t^{\beta-1}(D_n(t) - (n-1)t) \in F \right) \leq P \left( t^{\beta-1}(D_n(t) - (n-1)t) \leq \hat{x} \right) + P \left( t^{\beta-1}(D_n(t) - (n-1)t) \geq \tilde{x} \right)
\]
and, by Lemma 1.2.15 in Dembo and Zeitouni [11], we get
\[
\limsup_{t \to \infty} \frac{1}{t^{1-2\beta}} \log P \left( t^{\beta-1}(D_n(t) - (n-1)t) \leq \hat{x} \right) \leq \max \left\{ \limsup_{t \to \infty} \frac{1}{t^{1-2\beta}} \log P \left( t^{\beta-1}(D_n(t) - (n-1)t) \leq \hat{x} \right), \right. \\
\left. \limsup_{t \to \infty} \frac{1}{t^{1-2\beta}} \log P \left( t^{\beta-1}(D_n(t) - (n-1)t) \geq \tilde{x} \right) \right\}.
\]
Thus, if we prove
\[
\limsup_{t \to \infty} \frac{1}{t^{1-2\beta}} \log P \left( t^{\beta-1}(D_n(t) - (n-1)t) \leq \hat{x} \right) \leq -\inf_{x \leq \hat{x}} I_2(x) = -\frac{\tilde{x}^2}{4} \tag{3.3}
\]
and
\[
\limsup_{t \to \infty} \frac{1}{t^{1-2\beta}} \log P \left( t^{\beta-1}(D_n(t) - (n-1)t) \geq \tilde{x} \right) \leq -\inf_{x \geq \tilde{x}} I_2(x) = -\frac{\hat{x}^2}{4}, \tag{3.4}
\]
we conclude the proof because we get
\[
\limsup_{t \to \infty} \frac{1}{t^{1-2\beta}} \log P \left( t^{\beta-1}(D_n(t) - (n-1)t) \in F \right) \leq \max \left\{ -\inf_{x \leq \hat{x}} I_2(x), -\inf_{x \geq \tilde{x}} I_2(x) \right\} \leq -\min \left\{ \inf_{x \leq \hat{x}} I_2(x), \inf_{x \geq \tilde{x}} I_2(x) \right\} = -\inf_{x \in F} I_2(x).
\]

Proof of (3.3). Firstly, by Markov’s inequality, for all $\lambda < 0$ we have
\[
P \left( t^{\beta-1}(D_n(t) - (n-1)t) \leq \hat{x} \right) = P(D_n(t) \leq \hat{x}t^{1-\beta} + t(n-1)) \leq \mathbb{E}[e^{\lambda D_n(t)}]e^{-\lambda (\hat{x}t^{1-\beta} + t(n-1))}.
\]
Moreover, by Lemma 3.1, we get
\[
P \left( t^{\beta-1}(D_n(t) - (n-1)t) \leq \hat{x} \right) \leq e^{-\lambda (\hat{x}t^{1-\beta} + t(n-1))} \cdot \sum_{j=0}^{m} \binom{m}{j} \left( \frac{\lambda}{2} \right)^j (1 + nt)^{m-j} (2t)^j \mathbb{E}[|W_t|^j].
\]
We remark that, if we argue as in the proof of (3.2) (in the proof of Proposition 3.2), we obtain the estimate
\[
\mathbb{E}[|W_t|^j] \leq \sqrt{\frac{\pi}{nt}} \mathbb{E}[|W_{t_0}|^j]
\]
for some arbitrarily fixed $t_0 > 0$ and for $t > t_0$. Finally we take $\lambda = \frac{\tilde{x}}{2} t^{-\beta}$, and we have
\[
\limsup_{t \to \infty} \frac{1}{t^{1-2\beta}} \log P \left( t^{\beta-1}(D_n(t) - (n-1)t) \leq \hat{x} \right) \leq \limsup_{t \to \infty} \frac{1}{t^{1-2\beta}} \left\{ \frac{\tilde{x}}{2} t^{-\beta} (\hat{x}t^{1-\beta} + t(n-1)) + \lambda \left( \frac{\tilde{x}}{2} t^{-\beta} \right) t \right\} = \limsup_{t \to \infty} \left\{ -\frac{\tilde{x}^2}{2} - \frac{\tilde{x}}{2} (n-1)t^\beta + \frac{\tilde{x}}{2} t^\beta (\frac{\tilde{x}}{2} t^{-\beta} + n-1) \right\} = -\frac{\tilde{x}^2}{2} + \frac{\tilde{x}^2}{4} = -\frac{\tilde{x}^2}{4}.
\]
Proof of (3.4). Firstly, by Markov’s inequality, for all \( \lambda > 0 \) we have
\[
P(\tau_{r}^{(n)} - (n-1)t \geq \tilde{x}) = P(D_{n}(t) \geq \tilde{x}t^{1-\beta} + t(n-1)) \leq \mathbb{E}[\lambda D_{n}(t)] e^{-\lambda (\tilde{x}t^{1-\beta} + t(n-1))}.
\]
Moreover, by Lemma 3.1, we get
\[
P(\tau_{r}^{(n)} - (n-1)t \geq \tilde{x}) \leq e^{-\lambda (\tilde{x}t^{1-\beta} + t(n-1))} 2^{2\pi(\frac{1}{(2t)})d_{n}t^{-n/2+1}} e^{\kappa(\lambda)t} \cdot \sum_{j=0}^{m_{n}} \left( m_{n} \right) (1 + nt)^{m_{n}-j}(2t)^{j} \mathbb{E}[|W|^{j}].
\]
Finally (arguing as for the proof of (3.3) above) we take \( \lambda = \frac{\tilde{x}}{2} t^{-\beta} \), and we have
\[
\limsup_{t \to \infty} \frac{1}{t^{1-2\beta}} \log P(\tau_{r}^{(n)} - (n-1)t \geq \tilde{x}) \\
\leq \limsup_{t \to \infty} \frac{1}{t^{1-2\beta}} \left\{ \frac{-\tilde{x}^{2}}{2} (\tilde{x}t^{1-\beta} + t(n-1)) + \kappa \left( \frac{\tilde{x}}{2} t^{-\beta} \right) t \right\} \\
= \limsup_{t \to \infty} \left\{ \frac{-\tilde{x}^{2}}{2} - \frac{\tilde{x}^{2}}{2} (n-1)t^{\beta} + \frac{\tilde{x}^{2}}{2} \left( \frac{\tilde{x}}{2} t^{-\beta} + n-1 \right) \right\} = -\frac{\tilde{x}^{2}}{2} + \frac{\tilde{x}^{2}}{4} = -\frac{\tilde{x}^{2}}{4}. \quad \square
\]

4 On the asymptotic behavior of some hitting probabilities

Let \( \tau_{r_{1}}^{(n)} \) be the first hitting time of the \( n \)-dimensional Euclidean Brownian motion on the sphere of radius \( r_{1} \) centered at the origin. Moreover, for \( r > r_{1} \), let \( P_{r}(\tau_{r_{1}}^{(n)} < \infty) \) be the hitting probability when the Euclidean Brownian motion starts at some point having Euclidean distance \( r \) from the origin. It is well known that
\[
P_{r}(\tau_{r_{1}}^{(n)} < \infty) = \begin{cases} 1 & \text{if } n = 2 \\
\frac{\tilde{x}^{2-n}}{r_{1}^{n-2}} & \text{if } n \geq 3 \quad \text{(for } r > r_{1}).
\end{cases}
\]
Thus, as in immediate consequence, we have a polynomial decay as \( r \to \infty \) in the fashion of large deviations, i.e.
\[
\lim_{r \to \infty} \frac{1}{\log r} \log P_{r}(\tau_{r_{1}}^{(n)} < \infty) = -(n-2);
\]
moreover the decay rate \( w_{r}(n) := n - 2 \) is increasing with \( n \) and actually one expects that, the larger is the dimension of the space, the faster is the decay of \( P_{r}(\tau_{r_{1}}^{(n)} < \infty) \) as \( r \to \infty \).

In this section we investigate the same kind of problem for the hyperbolic Brownian motion. Let \( T_{\eta_{1}}^{(n)} \) be the first hitting time of the \( n \)-dimensional hyperbolic Brownian motion on the hyperbolic sphere of radius \( \eta_{1} \) centered at the origin \( O_{n} \). Moreover, for \( \eta > \eta_{1} \), let \( P_{\eta}(T_{\eta_{1}}^{(n)} < \infty) \) be the hitting probability when the hyperbolic Brownian motion starts at some point having hyperbolic distance \( \eta \) from \( O_{n} \). It is known (see Corollary 3.1 and Corollary 3.2 in Cammarota and Orsingher [5]) that, for \( \eta > \eta_{1} \), we have
\[
P_{\eta}(T_{\eta_{1}}^{(2)} < \infty) = \frac{\log \tanh \frac{\eta}{2}}{\log \tanh \frac{\eta_{1}}{2}}, \quad P_{\eta}(T_{\eta_{1}}^{(3)} < \infty) = \frac{1 - \coth \frac{\eta}{2}}{1 - \coth \eta_{1}},
\]
and, if we consider the values \( \{c(n, k) : k \in \{0, \ldots, \frac{n-4}{2}\} \} \) defined by
\[
c(n, 0) = 1 \quad \text{and} \quad c(n, k) = \frac{(n-3)(n-5)\cdots(n-2k-1)}{(n-4)(n-6)\cdots(n-2k-2)},
\]
we have
\[
P_{\eta}(T_{\eta_{1}}^{(4)} < \infty) = \sum_{k=0}^{\frac{n-4}{2}} (-1)^{k} c(n, k) \frac{\cosh \eta}{\sinh^{n-2k-3} \eta} + (-1)^{\frac{n-4}{2}} \frac{\cosh \eta}{(n-4)!} \log \tanh \frac{\eta}{2}, \quad \text{for } n \in \{4, 6, 8, \ldots\}
\]
and
\[
P_{\eta}(T_{\eta_{1}}^{(n)} < \infty) = \sum_{k=0}^{\frac{n-5}{2}} (-1)^{k} c(n, k) \frac{\cosh \eta_{1}}{\sinh^{n-2k-4} \eta_{1}} + (-1)^{\frac{n-5}{2}} \frac{\cosh \eta_{1}}{(n-4)!} \log \tanh \frac{\eta_{1}}{2}, \quad \text{for } n \in \{5, 7, 9, \ldots\}.
\]
An inspection for small values of \( n \) (some details will be presented below, and one could have an idea on how the methods can be adapted when \( n \) is larger) lead us to think that, for all \( n \geq 2 \), we have

\[
\lim_{\eta \to \infty} \frac{1}{\eta} \log P_\eta(T_{n1}^{(6)}) < \infty = -(n - 1). \tag{4.1}
\]

Then we would have an exponential decay as \( \eta \to \infty \) in the fashion of large deviations and, as happens for \( w_\varepsilon(n) := n - 2 \) above, the rate \( w_\varepsilon(n) := n - 1 \) is linearly increasing with \( n \). This kind of exponential decay has some analogy with some asymptotic results in the literature for the logarithm of the level crossing probabilities of real valued stochastic processes with a stability condition, which is satisfied when we have a drift directed away from the level to cross (actually, as pointed out in Remark 3.1, the hyperbolic Brownian motion has an implicit drift directed away from the origin). Here we refer to the quite general result in Duffy et al. [13] (Theorem 2.2), but we have in mind the simple case where \( \nu \) and \( a \) are the identity function, \( V = A = 1 \) and \( J \) is convex (this situation comes up in several cases; for the discrete time random walks with light tail increments see Theorem 1 in Lehtonen and Nyrhinen [26]).

Now, we provide some details on the computations of the limit (4.1) for \( n \in \{2, \ldots, 7\} \). We start with the cases \( n \in \{2, 4, 6\} \). When \( n \) is even, we have to handle the logarithmic term in the expression of \( P_\eta(T_{n1}^{(4)} < \infty) \); in view of this, since \( \log \tanh \frac{\eta}{2} = -\log \frac{1 + e^{-2\eta}}{1 - e^{-2\eta}} \), by eq. (1.513.1) in Gradshteyn and Ryzhik [17] with \( x = e^{-\eta} \), we get

\[
\log \tanh \frac{\eta}{2} = -2 \sum_{k=1}^{\infty} \frac{(e^{-\eta})^{2k-1}}{2k - 1}.
\]

For \( n = 2 \) and \( n = 4 \) we have to take into account

\[
e^{-\eta} \leq \sum_{k=1}^{\infty} \frac{(e^{-\eta})^{2k-1}}{2k - 1} \leq \frac{e^{-\eta}}{1 - e^{-2\eta}}. \tag{4.2}
\]

Actually (4.1) holds with \( n = 2 \) by noting that

\[
\lim_{\eta \to \infty} \frac{1}{\eta} \log P_\eta(T_{n1}^{(2)} < \infty) \geq \lim_{\eta \to \infty} \frac{1}{\eta} \log(2e^{-\eta}) = -1
\]

and

\[
\limsup_{\eta \to \infty} \frac{1}{\eta} \log P_\eta(T_{n1}^{(2)} < \infty) \leq \limsup_{\eta \to \infty} \frac{1}{\eta} \log \left( \frac{2 - e^{-\eta}}{1 - e^{-2\eta}} \right) = -1;
\]

moreover

\[
\lim_{\eta \to \infty} \frac{1}{\eta} \log P_\eta(T_{n1}^{(4)} < \infty) = \lim_{\eta \to \infty} \frac{1}{\eta} \log \left( \frac{2 - e^{-\eta} + e^{-3\eta}}{(e^{-\eta} - e^{-3\eta})^2} - 2 \sum_{k=1}^{\infty} \frac{(e^{-\eta})^{2k-1}}{2k - 1} \right)
\]

and, again, we can prove (4.1) with \( n = 4 \) by deriving upper and lower bounds in terms of (4.2). For \( n = 6 \) we have

\[
\lim_{\eta \to \infty} \frac{1}{\eta} \log P_\eta(T_{n1}^{(6)} < \infty) = \lim_{\eta \to \infty} \frac{1}{\eta} \log \left( \frac{8 - e^{-\eta} + e^{-3\eta}}{(e^{-\eta} - e^{-3\eta})^2} - 3 \frac{e^{-\eta} + e^{-3\eta}}{(e^{-\eta} - e^{-3\eta})^2} + 3 \sum_{k=1}^{\infty} \frac{(e^{-\eta})^{2k-1}}{2k - 1} \right)
\]

and we are able to prove (4.1) by deriving upper and lower bounds as in the previous cases, by means of the following relationships

\[
e^{-\eta} + \frac{e^{-3\eta}}{3} \leq \sum_{k=1}^{\infty} \frac{(e^{-\eta})^{2k-1}}{2k - 1} \leq e^{-\eta} + \frac{e^{-3\eta}}{3} + \frac{e^{-5\eta}}{1 - e^{-2\eta}}.
\]

instead of (4.2). Finally the cases \( n \in \{3, 5, 7\} \):

\[
\lim_{\eta \to \infty} \frac{1}{\eta} \log P_\eta(T_{n1}^{(3)} < \infty) = \lim_{\eta \to \infty} \frac{1}{\eta} \log(c\thetah \eta - 1)
\]

\[
\lim_{\eta \to \infty} \frac{1}{\eta} \log \left( \frac{e^{-\eta} + e^{-3\eta}}{(e^{-\eta} - e^{-3\eta})^2} - 1 \right) = \lim_{\eta \to \infty} \frac{1}{\eta} \log \left( \frac{2}{e^{2\eta} - 1} \right) = -2;
\]
\[
\lim_{\eta \to \infty} \frac{1}{\eta} \log P_\eta(T_{\eta_1}^{(5)} < \infty) = \lim_{\eta \to \infty} \frac{1}{\eta} \log \left( \frac{\cosh \eta}{\sinh \eta} + 2 \left( 1 - \frac{\cosh \eta}{\sinh \eta} \right) \right)
= \lim_{\eta \to \infty} \frac{1}{\eta} \log \left( 4 \frac{e^\eta + e^{-\eta}}{(e^\eta - e^{-\eta})^3} - 4 \frac{e^{-\eta}}{e^\eta - e^{-\eta}} \right)
= \lim_{\eta \to \infty} \frac{1}{\eta} \log \left( \frac{e^{-\eta} + o(e^{-\eta})}{(e^\eta - e^{-\eta})^3} \right) = -4;
\]

\[
\lim_{\eta \to \infty} \frac{1}{\eta} \log P_\eta(T_{\eta_1}^{(7)} < \infty) = \lim_{\eta \to \infty} \frac{1}{\eta} \log \left( \frac{\cosh \eta}{\sinh \eta} - \frac{4}{3} \frac{\cosh \eta}{\sinh \eta} - 8 \left( 1 - \frac{\cosh \eta}{\sinh \eta} \right) \right)
= \lim_{\eta \to \infty} \frac{1}{\eta} \log \left( 16 \frac{e^\eta + e^{-\eta}}{(e^\eta - e^{-\eta})^3} - 16 \frac{e^\eta + e^{-\eta}}{(e^\eta - e^{-\eta})^3} + 16 \frac{e^{-\eta}}{e^\eta - e^{-\eta}} \right)
= \lim_{\eta \to \infty} \frac{1}{\eta} \log \left( \frac{e^{-\eta} + o(e^{-\eta})}{(e^\eta - e^{-\eta})^3} \right) = -6.
\]

5 On a recent LDP in the literature

Theorem 1.1 in Hirao [20] provides the LDP for \{d_\eta(Z_\eta^z(t),z) \colon t > 0\}, where \{Z_\eta^z(t) \colon t \geq 0\} is a Brownian motion on a hyperbolic space \(H^n(F)\) with distance function \(d_\eta\) (several choices of \(F\) are allowed), and \(Z_\eta^z(0) = z\). Here we want to illustrate the relationship between Theorem 1.1 in Hirao [20] when \(F\) concerns the parameters \((k,m) = (0, n-1)\) in eq. (1) in Hirao [20], and Proposition 3.2 in this paper. Actually, if we denote the rate function for the first LDP by \(\Lambda^*\), we have \(\Lambda^*(x) = 2I_1(x)\) for all \(x \geq 0\).

Firstly we should have \(\Lambda^*(x) = I_1(x)\) for all \(x \geq 0\) if we consider \(\frac{1}{2}\) in place of \(t\) because the exponential part of function \(h_{k,m}^{(k,m)}\) in eq. (4) in Hirao [20] with \((k,m) = (0, n-1)\) would coincide with the one of \(h_\eta(\eta, \frac{1}{2})\) in this paper. Furthermore the proof of Theorem 1.1 in Hirao [20] shows the existence of the limit

\[
\lim_{t \to \infty} \frac{1}{t} \log \mathbb{E} \left[ e^{\lambda d_\eta(Z_\eta^z(t),z)} \right] = \frac{1}{2} \lambda(\lambda + 2k + m) =: \Lambda(\lambda) \quad (5.1)
\]

for all \(\lambda \in \mathbb{R}\) and, by an application of the Gärtner Ellis Theorem, the LDP holds with the good rate function \(\Lambda^*\) defined by

\[
\Lambda^*(x) := \sup_{\lambda \in \mathbb{R}} \{ \lambda x - \Lambda(\lambda) \}. \quad (5.2)
\]

Thus, by (5.1) and (5.2), the rate function in Theorem 1.1 in Hirao [20] is

\[
\Lambda^*(x) = \frac{1}{2} \left( x - \frac{2k + m}{2} \right)^2
\]

for all \(x \in \mathbb{R}\).

We remark that, since the random variables \(\{d_\eta(Z_\eta^z(t),z) \colon t > 0\}\) should be nonnegative, there is a slight inexactness in this proof. Actually, if a family of nonnegative random variables \(\{Z(t) \colon t > 0\}\) satisfies the LDP with a rate function \(I\), the lower bound for the open set \(G = (-\infty, 0)\) in the definition of LDP would yield \(I(x) = \infty\) for \(x \in (-\infty, 0)\); moreover, for the same reason, the limit (5.1) fails because the function \(\Lambda\) should be nondecreasing. We think that the proof could be corrected showing that we have

\[
\lim_{t \to \infty} \frac{1}{t} \log \mathbb{E} \left[ e^{\lambda d_\eta(Z_\eta^z(t),z)} \right] = \begin{cases} \frac{1}{2} \lambda(\lambda + 2k + m) & \text{if } \lambda \geq - \frac{2k + m}{2} \\ \frac{1}{2} \left( \frac{2k + m}{2} \right)^2 & \text{if } \lambda < - \frac{2k + m}{2} \end{cases} =: \Lambda(\lambda) \quad (5.3)
\]

in place of (5.1); actually, in such a case, the hypotheses of Gärtner Ellis Theorem would be satisfied and the LDP would hold with rate function \(\Lambda^*\) defined by (5.2). Moreover the function \(\Lambda\) in (5.3) is nondecreasing, and (5.2) and (5.3) yield

\[
\Lambda^*(x) = \begin{cases} \frac{1}{2} \left( x - \frac{2k + m}{2} \right)^2 & \text{if } x \geq 0 \\ -\infty & \text{if } x < 0 \end{cases}
\]

(thus \(\Lambda^*(x) = \infty\) for \(x \in (-\infty, 0)\)).
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