A polynomial-time algorithm for deciding the Hilbert Nullstellensatz over \( \mathbb{Z}_2 \). A proof of \( P = \text{NP} \) hypothesis

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Abstract

Let \( P \) be the class of polynomial-time decision problems and \( \text{NP} \) be the class of nondeterministic polynomial time decision problems. We prove the following.

**Theorem 3.** The classes \( P \) and \( \text{NP} \) are equivalent. That is, \( P = \text{NP} \).

Theorem 3 gives a positive answer to the question

Does \( P = \text{NP} \),

see S. Cook, *The P versus NP problem*, Official problem description, www.claymath.org/millennium-problems. Crucial for its proof is Theorem 2, from which it follows that the \( \text{NP} \)-complete problem of deciding the Hilbert Nullstellensatz over \( \mathbb{Z}_2 \) belongs to the class \( P \).

**Theorem 2.** There is a constructive algorithm for deciding the Hilbert Nullstellensatz over \( \mathbb{Z}_2 \), where \( \mathbb{Z}_2 \) is the space of all complex numbers with integer real and imaginary parts. The number \( s(n, m_\sigma) \) of basic steps of the algorithm, where \( n \) is the number of variables and \( m_\sigma \) is the total length of input polynomials, satisfies the inequality

\[
s(n, m_\sigma) \leq c_2 m_\sigma^2 \log m_\sigma + \min\{m_\sigma, (d_1)^3\} + \sum_{\ell=1}^{n-2} N^{(\ell)} \min\{[(m_\sigma^{(\ell)})^2, (d_{\ell+1})^2]\}
\]

\[
+ N^{(n-1)} \min\{m_\sigma, d_n\}
\]

where \( c_2 \) is an absolute constant, \( \{d_\ell\}_{\ell=1}^n \) are the maximal partial degrees in \( \{z_\ell\}_{\ell=1}^n \), respectively, \( m_\sigma^{(\ell)} \) is the total length of the \( (z_1, z_2, \ldots, z_\ell) \)-polynomial coefficients in front of \( z_{\ell+1}^j \), \( j = 0, 1, \ldots, d_{\ell+1} \), from all \( (z_1, z_2, \ldots, z_{\ell+1}) \)-sub-monomials, \( \ell = 1, 2, \ldots, n-1 \), and \( N^{(\ell)} \) is the number in the natural order of the major \( (z_1, z_2, \ldots, z_\ell) \)-sub-monomial, \( \ell = 1, 2, \ldots, n-1 \).

**Keywords:** Complexity classes, P versus NP problem, Completeness, Hilbert Nullstellensatz.

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1 Introduction

The P versus NP problem (Cook [6], Fortnow [7]) is formulated using the notion of formal computer. The standard way to define the class P is through the notion of Turing machine (Turing [18]), which historically is the first definition of formal computer. Informally, P is the class of decision problems, solvable in polynomial time (PT), that is, in number of steps, bounded by a fixed polynomial of the length of input. More precisely, the class P is defined in terms of finite alphabets and languages. Languages are finite non-empty sets of finite strings with at least two elements, based on the corresponding alphabet. Having a Turing machine M and a language, we define by $L(M)$ the set of all strings of the language acceptable by M. These are all (input) strings of the language, for which M halts in finite number of steps in a fixed state, called $q_{\text{accept}}$. If we restrict ourselves on strings of length at most $n$ and denote by $T_M(n)$ the longest run of M for such input strings, then we say that M runs in polynomial time (PT) if there exists a $k$ such that for all $n$, $T_M(n) \leq n^k + k$, and define the class P as follows:

$$P := \{ L \mid L = L(M) \text{ for some Turing machine } M \text{ that runs in PT} \}$$ (Cook [6]).

Let us describe more in detail the notion of Turing machine M. It deals with (finite) alphabets $\Gamma$ (including the blank symbol b) and subsets $\Sigma \subset \Gamma$ ($\Sigma$ not containing b), languages $\Gamma^*$ and $\Sigma^*$ (finite sets of strings over $\Gamma$ and $\Sigma$, respectively), and states and configurations (Cook [6]). Let us imagine a tape head, moving on an infinite tape consisting of cells containing symbols from $\Gamma$. The way this head is moving on the tape and what is reading/printing is completely determined by a finite set of states $Q$ and a transition function $\delta$,

$$\delta : (Q \setminus \{ q_{\text{accept}}, q_{\text{reject}} \} \times \Gamma \rightarrow Q \times \Gamma \times \{-1,1\},$$

where $\delta(q, s) = (q', s', h)$ has the interpretation that, if at a certain step the machine M is in state q with a head, pointing to a cell containing symbol $s \in \Gamma$, then the transition function $\delta$ decides in a unique way $\delta(q, s)$ to (over)write $s' \in \Gamma$ onto the current cell and to move the tape head to the left (if $h = -1$) or to the right (if $h = 1$), choosing thus a unique new state $q'$. In the beginning, the tape stores only a compact (with no blank symbols, or empty) string $w$ (input data) from $\Sigma^*$ and the head is positioned on the left most symbol of $w$. In this way, M is passing from a state to state with the help of the transition function $\delta$, in the end finding itself in one of the following situations - it either halts in one of the two halting configurations $q_{\text{accept}}$ and $q_{\text{reject}}$, or infinitely passes from configuration to configuration. In more precise words, the set $L(M)$ defined above is the set of all decidable input strings of $\Sigma^*$. That is, there exists a Turing machine M that halts in $q_{\text{accept}}$ for all input strings from $L(M)$. In addition, P is the class of all those $L(M)$, for which the halting accomplishes in a polynomial time.

Next we extend the scope of polynomial-time decision problems P to the nondeterministic polynomial time class of languages NP, using the definition of checking relation (Cook [6]). Given finite alphabets $\Sigma$ and $\Sigma_1$, a checking relation $R$ is defined as $R \subseteq \Sigma^* \times \Sigma_1^*$. A language
Let \( L_R \) over the alphabet \( \Sigma \cup \Sigma_1 \cup \{\#\} \) is associated with every binary relation \( R \) of this type, where \( \# \notin \Sigma \) and
\[
L_R := \{w\#y \mid R(w, y)\}.
\]

**Definition 1.** A language \( L \) over \( \Sigma \) is in the class \( \text{NP} \), if and only if there is a \( k \in \mathbb{N} \) and a polynomial-time checking relation \( R \) such that for all \( w \in \Sigma^* \),
\[
w \in L \iff \exists y \text{ such that } (|y| \leq |w|^k \text{ and } R(w, y)),
\]
where \( |w| \) and \( |y| \) are the lengths of \( w \) and \( y \), respectively.

**Main Problem** (Cook \[6\]).

Does \( P = \text{NP} \)? (1)

The importance of the Main Problem (1) stems from the remarkable practical consequences of a constructive proof of \( P = \text{NP} \) in cryptography (see i.e. Horie and Watanabe \[11\], De Abhishek Kumarasubramanian and Venkatesan \[1\]), operations research (see i.e. Robinson \[15\]), life sciences (see i.e. Berger and Leighton \[3\]), etc. In order to prove the equivalence of the classes \( P \) and \( \text{NP} \), we need the definition of \( \text{NP} \)-completeness. A notion of \( \text{NP} \)-completeness was first introduced by Cook in \[5\], where it was shown that several natural problems, including Satisfiability problem (Aho, Hopcroft and Ullman \[2\]), 3-SAT problem (\[2\]) and Subgraph isomorphism problem (Ullmann \[19\]), are \( \text{NP} \)-complete. A year later Karp \[12\] introduced the notation \( P \) and \( \text{NP} \), which is commonly in use, and redefined \( \text{NP} \)-completeness in terms of polynomial-time reducibility, definition that has become standard. Independently of Cook and Karp, Levin \[13\] defined the notion of \text{Universal search problem}, similar to the \( \text{NP} \)-complete problem, and gave six examples, including Satisfiability problem. Problems related to the importance of (1), as well as to the consequences of its solution, were also considered before 1971 (Nash \[14\], Hartmanis \[9\]). The standard definition of \( \text{NP} \)-completeness is based on the notion of \( p \)-reducibility.

**Definition 2.** Let \( L_i \) be languages over \( \Sigma_i \), \( i = 1, 2 \). Then \( L_1 \leq_p L_2 \) or \( L_1 \) is \( p \)-reducible to \( L_2 \) if and only if there is a polynomial time function \( f : \Sigma_1^* \rightarrow \Sigma_2^* \) such that \( x \in L_1 \iff f(x) \in L_2 \), for all \( x \in \Sigma_1^* \).

**Definition 3.** A language \( L \) is \( \text{NP} \)-complete if and only if \( L \) is in \( \text{NP} \), and \( L' \leq_p L \) for every language \( L' \) in \( \text{NP} \).

There are hundreds of known \( \text{NP} \)-complete problems. Their importance for resolving the Main Problem (1) is based on the following statement (Cook \[6\]), and especially on its part (c).

**Proposition 1.** (a) If \( L_1 \leq_p L_2 \) and \( L_2 \in P \), then \( L_2 \in P \).
(b) If \( L_1 \) is \( \text{NP} \)-complete, \( L_2 \in \text{NP} \) and \( L_1 \leq_p L_2 \), then \( L_2 \) is \( \text{NP} \)-complete.
(c) If \( L \) is \( \text{NP} \)-complete and \( L \in P \), then \( P = \text{NP} \).

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In order to use Proposition 1(c) to give an answer "Yes" to the Main Problem 1, we are going to consider a particular NP-complete problem, namely the problem for deciding the Hilbert Nullstellensatz over $\mathbb{Z}_2$ (Smale [16]). As it is mentioned by Smale [16], there is no sense to make use of the formal definition of Turing machine for this problem. From practical point of view, it is much more useful to show existence of a polynomial-time constructive algorithm for solving it (Cook [6]). Therefore, in the next section we will introduce our own definitions and notations, describing a kind of a formal computer, more practically oriented, but resembling that of a Turing machine.

## 2 Definitions and notations

- Let $n \in \mathbb{N}$. In what follows, $c$ will denote different positive constants.
- $\mathcal{P}_n$ will be the set of all complex polynomials of $n$ variables $z := z^{(n)} := (z_1, z_2, \ldots, z_n)$.
- Let there be given a sequence of polynomials $\{f_i\}_{i=1}^k \subset \mathcal{P}_n$, $f_i := f_i(z) := \sum_{\alpha \in A_i} c_{\alpha} z^\alpha = \sum_{\alpha \in A_i} c_{\alpha} \prod_{j=1}^n z_j^{\alpha_j}$, where $\{c_{\alpha}\}_{\alpha \in A_i}$ is the set of non-zero coefficients of $f_i$, $i = 1, 2, \ldots, k$.
- The number of monomials in a polynomial from the sequence $\{f_i\}_{i=1}^k$ will be denoted by $m_i$, $i = 1, 2, \ldots, k$, and the total number of monomials by $m_\sigma := \sum_{i=1}^k m_i$. Let also, for $j = 1, 2, \ldots, n$, $d_j$ be the maximal degree of $z_j$ in all monomials from $\{f_i\}_{i=1}^k$.
- A basic step will be called one of the following: assignment of a complex value to a storage cell, carrying out one of the arithmetic operations $\{+,-,\ast,\div\}$, comparison of the (real) values of two storage cells.
- We will be interested in solving identities of the form $\sum_{i=1}^k f_i(z)g_i(z) \equiv 1$, where $\{g_i\}_{i=1}^k \subset \mathcal{P}_n$, $g_i := g_i(z) := \sum_{\beta \in B_i} c_{\beta} z^\beta$, are unknown polynomials. The minimal number of basic steps for finding all polynomials in a sequence $\{g_i(z)\}_{i=1}^k$ or proving its non-existence will be denoted by $s(n, m_\sigma)$.
- Coefficients $\{(c_{\alpha})_{\alpha \in A_i}\}_{i=1}^k$ will be stored in $m_\sigma$ records, following for every $f_i$, $i = 1, 2, \ldots, k$, the natural order of monomials (to be explained more precisely in the next section), accompanied by records of the corresponding exponents $\alpha$.

## 3 Main result

The following theorem provides a criterion for that when two polynomials from $\mathcal{P}_n$ do not have a common zero.

**Hilbert Nullstellensatz [over $\mathbb{C}$]** (Hilbert [10], Smale [16, 17]). Let there be given a sequence of polynomials $\{f_i\}_{i=1}^k \subset \mathcal{P}_n$. The polynomials $\{f_i\}_{i=1}^k$ do not possess a common zero.
zero $\zeta \in \mathbb{C}^n$ if and only if there exists a corresponding sequence $\{g_i\}_{i=1}^k \subseteq \mathcal{P}_n$ such that
\[
\sum_{i=1}^k f_i(z)g_i(z) \equiv 1, \quad z \in \mathbb{C}^n.
\] (2)

This statement raises naturally the question of how difficult it is to use in practice the criterion from Hilbert Nullstellensatz over $\mathbb{C}$. Is it possible a decision to be made in a polynomial time, or in other words, does the decision problem Problem 1 below belong to the class $\mathbf{P}$? The following conjecture was stated by Smale [16]:

**Conjecture 1.** There is no polynomial time algorithm for deciding the Hilbert Nullstellensatz over $\mathbb{C}$.

We are going to question this conjecture by exploring the following

**Problem 1.** Given polynomials $\{f_i\}_{i=1}^k \subseteq \mathcal{P}_n$, check if they possess a common zero $\zeta \in \mathbb{C}^n$ and find a upper bound for the minimal number of basic steps to do this.

Let $\mathbb{Z}_2$ be the space of all complex numbers with integer real and imaginary parts. One can also consider the analogue of Problem 1 over $\mathbb{Z}_2$. Since the latter is $\mathbf{NP}$-complete, it can serve as a tool for resolving the Main Problem (1):

**Problem 2.** Given polynomials $\{f_i\}_{i=1}^k$, whose coefficients are in $\mathbb{Z}_2$, check if they possess a common zero $\zeta \in (\mathbb{Z}_2)^n$ and find a upper bound for the minimal number of basic steps to do this.

The following two theorems present the results from our considerations of Problems 1 and 2 respectively.

**Theorem 1.** There is a constructive algorithm for deciding Problem 1. The number $s(n, m_\sigma)$ of basic steps of the algorithm satisfies the inequality
\[
s(n, m_\sigma) \leq c_1 \left\{ m_\sigma^2 \log m_\sigma + \min\{ [m_\sigma(1)]^3, (d_1)^3 \} + \sum_{\ell=1}^{n-2} N^{(\ell)} \min\{ [m_\sigma(\ell+1)]^2, (d_{\ell+1})^2 \} \right. \\
\left. + N^{(n-1)} \min\{ m_\sigma, d_n \} \right\}
\]
where $c_1$ is an absolute constant, $\{d_\ell\}_{\ell=1}^n$ are the maximal partial degrees in $\{z_\ell\}_{\ell=1}^n$, respectively, $m_\sigma(\ell)$ is the total length of the $(z_1, z_2, \ldots, z_\ell)$-polynomial coefficients in front of $z_j^{d_{\ell+1}}$, $j = 0, 1, \ldots, d_{\ell+1}$, from all $(z_1, z_2, \ldots, z_{\ell+1})$-sub-monomials, $\ell = 1, 2, \ldots, n-1$, and $N^{(\ell)}$ is the number in the natural order of the major $(z_1, z_2, \ldots, z_\ell)$-sub-monomial, $\ell = 1, 2, \ldots, n-1$. 

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Theorem 2. There is a constructive algorithm for deciding Problem 2. The number \( s(n, m_\sigma) \) of basic steps of the algorithm satisfies the inequality

\[
s(n, m_\sigma) \leq c_2 \{ m_\sigma^2 \log m_\sigma + \min \{ [m_\sigma^{(1)}]^3, (d_1)^3 \} + \sum_{\ell=1}^{n-2} N^{(\ell)} \min \{ [m_\sigma^{(\ell+1)}]^2, (d_{\ell+1})^2 \} \}
+ N^{(n-1)} \min \{ m_\sigma, d_n \}\]

where \( c_2 \) is an absolute constant and \( m_\sigma^{(\ell)} \) and \( N^{(\ell)} \) are defined as in Theorem 1.

Remark 1. In Smale [16, Conjecture], it was conjectured also that there is no polynomial-time algorithm for deciding Problem 2, which is equivalent to a negative answer \( P \neq NP \) to the Main Problem 1.

By using Theorem 2, we disprove the conjecture mentioned in Remark 1 and thus, prove the main result of the paper:

Theorem 3. The classes \( P \) and \( NP \) are equivalent. That is, \( P = NP \).

Theorem 3 answers the question 1 of the Main Problem with \textit{Yes}. Observe that, the opinion of most of the experts was exactly the opposite. A series of polls was conducted since 2002, in which the percentage of those voting for \( P \neq NP \) reached 99% in 2019 (see Gasarch [8]).

Proof of Theorem 4. Given a sequence \( \{ f_i \}_{i=1}^k \subset \mathcal{P}_n \), we will show how to check if there exists a sequence \( \{ g_i \}_{i=1}^k \subset \mathcal{P}_n \) satisfying (2) (and how to find it, if it exists), estimating simultaneously the minimal number of steps \( s(n, m_\sigma) \). This will be done by reduction with respect to the number of variables \( n \), followed by induction. Let us call the problem for doing this a \((n, m_\sigma)\)-problem. Recall that \( m_\sigma \) is the total number of monomials \( m_\sigma = \sum_{i=1}^k m_i \) in the polynomials \( \{ f_i \}_{i=1}^k \) and let us explain also what we mean by \textit{natural order} in the set of \( n \)-variate monomials: This means that amongst two monomials, the one with lower total degree is located to the left from the other, and if the total degrees of two monomials coincide and \( j \in \{ 1, 2, \ldots, n \} \) is the maximal index such that their exponents on the variables \( (z_1, z_2, \ldots, z_{j-1}) \) coincide, then the one possessing a higher exponent of \( z_j \), is located to the left from the other. In this way, the monomials are ordered as follows:

\[
1 \prec z_1 \prec z_2 \prec \cdots \prec z_n \prec z_1^2 \prec z_1z_2 \prec z_2^2 \prec \cdots \prec z_{n-1}z_n \prec z_n^2 \prec \cdots .
\]

Reduction) In this first part of the algorithm, the original \((n, m_\sigma^{(n)})\)-problem will be reduced to a \((1, m_\sigma^{(1)})\)-problem, passing through problems of types \((\ell - 1, m_\sigma^{(\ell - 1)})\), \( \ell = n, n - 1, \ldots, 3 \), where \( m_\sigma^{(\ell - 1)} \leq m_\sigma \) is the total length of the coefficients \( \{ f_{ij}^{(\ell)} (z^{(\ell - 1)}) \}_{i=1}^{k^{(\ell)}} \) in front of \( z_i^j \), \( j = 0, 1, \ldots, d_\ell \), \( \ell = 2, 3, \ldots, n \), in all \((z_1, z_2, \ldots, z_{\ell-1})\)-sub-monomials from \( \{ f_i \}_{i=1}^k \). Let also \( N^{(\ell - 1)} \), be the number in the \textit{natural order} of the major \((z_1, z_2, \ldots, z_{\ell-1})\)-sub-monomial, \( \ell = 2, 3, \ldots, n \), in \( \{ f_i \}_{i=1}^k \). The unknown polynomials \( \{ g_{ij}^{(\ell)} (z^{(\ell - 1)}) \}_{i=1}^{k^{(\ell)}} \) will be considered as
linear combination of all \((z_1, z_2, \ldots, z_{\ell-1})\)-monomials in the natural order up to that with number \(N^{(\ell-1)}, \ell = 2, 3, \ldots, n\). Thus, the numbers \(N^{(\ell-1)}, \ell = 2, 3, \ldots, n\), serve as bounds for the total degrees of \(\{g_{ij}(z^{(\ell-1)})\}_{k=1}^k\). Let us mention here, that estimates for the total degrees of the unknown polynomials, involving the \(n\)-th power of the maximal total degree of the polynomials \(f_i\) are obtained in Brownawell [4]. To match the representations of \(\{g_{ij}(z^{(\ell-1)})\}\), the polynomials \(\{f_{ij}(z^{(\ell-1)})\}\) will also be considered as linear combination of all \((z_1, z_2, \ldots, z_{\ell-1})\)-monomials in the natural order up to that with number \(N^{(\ell-1)}\), taking into account that the missing monomials have coefficients equal to 0. The superscript \((\ell)\) of \(k\) will be used only to make the exposition more transparent. In fact, the estimation of the number of basic steps requires only the quantities \(m_{\sigma}(\ell)\) and the number of variables \(n\). Hence, for \(\ell = n\),

\[
f_i^{(n)} = \sum_{j=0}^{d_n} f_{ij}^{(n)}(z^{(n-1)}) z_n^j, \quad g_i^{(n)} = \sum_{j=0}^{d_n} g_{ij}^{(n)}(z^{(n-1)}) z_n^j, \quad i = 1, 2, \ldots, k = k^{(n)}.
\]

**Remark 2.** During the reduction part of the algorithm, at every level \(\ell = n, n-1, \ldots, 2\), non-zero coefficients of the polynomials \(\{f_{ij}(z^{(\ell-1)})\}_{i=1}^k\) will be sorted at the cost of less than or equal to

\[
\sum_{n=1}^{cm^2} \log m_{\sigma} (3)
\]

basic steps. The order of the variables \(\{z_j\}_{j=1}^n\) will be discussed in Remark [4].

The left hand side of (2) is a linear combination of 1, \(z_n, \ldots, z_n^{2d_n}\). Consequently, the following identities must be fulfilled in \(\mathbb{C}^{n-1}\):

\[
\begin{cases}
\sum_{i=1}^k f^{(n)}_{i0} g_{00}^{(n)} + \sum_{i=1}^k f^{(n)}_{i1} g_{11}^{(n)} + \sum_{i=1}^k f^{(n)}_{i2} g_{22}^{(n)} + \cdots \\
\sum_{i=1}^k f^{(n)}_{id} g_{dd}^{(n)} \\
\sum_{i=1}^k f^{(n)}_{i0} g_{00}^{(n)} + \sum_{i=1}^k f^{(n)}_{i1} g_{11}^{(n)} + \sum_{i=1}^k f^{(n)}_{i2} g_{22}^{(n)} + \cdots \\
\sum_{i=1}^k f^{(n)}_{id} g_{dd}^{(n)} \equiv 0
\end{cases}
\]

This is the system that we need to solve or disprove. In the reduction section we will be interested only in the first \((\equiv 1)\) identities of systems like [4]. The treatment of the rest \((\equiv 0)\) identities will be left for the next section Induction. As we already mentioned above, passing through problems of types \((\ell, m^{(\ell)}_{\sigma})\), \(\ell = n - 1, n - 2, \ldots, 2\), we will inevitably arrive at an \((1, m^{(1)}_{\sigma})\) problem:

\[
\sum_{i=1}^k f^{(1)}_i(z_1) g^{(1)}_i(z_1) \equiv 1, \quad f^{(1)}_i(z_1) = \sum_{u=1}^{d_i} a_{iu} z_1^u, \quad g^{(1)}_i(z_1) = \sum_{j=0}^{d_i} b_{ij} z_1^j, \quad i = 1, 2, \ldots, k^{(1)}.
\]

Our goal now is to check if the following system

\[
\begin{cases}
\sum_{i=1}^k a_{00} b_{00}^{(1)} = 1 \\
\sum_{i=1}^k (a_{01} b_{11}^{(1)} + a_{i1} b_{i0}^{(1)}) = 0 \\
\sum_{i=1}^k (a_{02} b_{12}^{(1)} + a_{i2} b_{i1}^{(1)} + a_{i2} b_{i0}^{(1)}) = 0 \\
\cdots \\
\sum_{i=1}^k a_{id} b_{id}^{(1)} = 0
\end{cases}
\]

(5)
does or does not possess a solution and find it, if it exists. It is a linear system of \(2d_1 + 1\) equations with respect to \(k^{(1)}(d_1 + 1)\) unknowns \(\{b_{ij}\}_{j=0}^{d_1}, i = 1, 2, \ldots, k^{(1)}\), having \(m^{(1)}\) non-zero coefficients amongst all \(\{a_{ij}\}_{j=0}^{d_1}, i = 1, 2, \ldots, k^{(1)}\). Consequently, at most \(k^{(1)}(d_1 + 1) - (2d_1 + 1) = (k^{(1)} - 2)(d_1 + 1) + 1\) of the unknowns could be free, or in other words, the solution, if it exists, is at most \(\{(k^{(1)} - 2)(d_1 + 1) + 1\}\)-dimensional linear manifold. Observe that therefore, \(d_1\) is the optimal choice for the degree of the unknown polynomials, since in the lowest non-trivial case \(k^{(1)} = 2\) the manifold is of dimension 1. By using (a modified) Gauss elimination, we will solve the system or will prove that it does not have a solution.

Note that the single terms in every row are combined in \(k^{(1)}\)-tuple sums. We take first the tuple \(\{a_{i0}\}_{i=1}^{k^{(1)}}\). Going through the equations of the system, pivot coefficients are chosen from the current tuple \(\{a_{i0}\}_{i=1}^{k^{(1)}}\) in ascending order of their first subscript, until the non-zeros in it are used up. Then we consider the next \(k^{(1)}\)-tuple sum from the current equation. If, in our case, there are \(l_0\) non-zeros in \(\{a_{i0}\}_{i=1}^{k^{(1)}}\), at the equation number \(l_0 + 1\) we will consider the \(k^{(1)}\)-tuple sum \(\sum_{i=1}^{k^{(1)}} a_{i1}b_{i0-1}\) and will look for pivots in \(\{a_{i1}\}_{i=1}^{k^{(1)}}\), respectively.

If all coefficients in a sum \(\sum_{i=1}^{k^{(1)}} a_{i1}b_{ij2}\) are zero, this means that none of the polynomials \(\{f_1^{(1)}(z_1)\}_{i=1}^{k^{(1)}}\) contains a term with \(z_1^{j2}\). If this is true for \(j_1 = 0\), the process is terminated with a result "No solution at Level 1, Equation 1". If not, we solve the equation number 1 with respect to one of the unknowns, say \(b_{10}\), representing it in terms of those \(\{b_{i0}\}_{i=2}^{k^{(1)}}\) such that the corresponding \(a_{i0}\) are non-zero. Then take the next non-zero coefficient \(a_{i0}\) if any, solve equation number 2 with respect to \(b_{i01}\) and thus, represent \(b_{i01}\) as a linear function of \(b_{10} = b_{10}(b_{20}, \ldots, b_{k^{(1)}0}), \{b_{i0}\}_{i=2}^{k^{(1)}}\) and \(\{b_{i1}\}_{i=1}^{k^{(1)}}\), \(i \neq i_0\). Let us suppose that there is not such an \(a_{i0}\). Then we look for non-zeros in \(\{a_{i1}\}_{i=1}^{k^{(1)}}\). If all terms in this set are also zero, this means that the second equation is of the form \(a_{10}b_{11} = 0\), from where \(b_{11} = 0\). Further, the equation number 3 is under consideration and we do the same with the sums \(\sum_{i=1}^{k^{(1)}} a_{i0}b_{12}\), \(\sum_{i=1}^{k^{(1)}} a_{i1}b_{11}\) and \(\sum_{i=1}^{k^{(1)}} a_{i2}b_{i0}\), taken in this order. The manipulation of the third equation is finished again with one of the same results – either a new unknown is represented as a linear function of some other unknowns (including \(b_{10}\) and for ex. \(b_{i01}\), or due to luck of sufficiently many non-zero \(a_{i1}\)’s, some of the next unknowns is equal to zero. After finishing with all equations, we will have a set of at most \(2d_1 + 1\) unknowns \(\{b_{ij}\}_{j=1}^{\sigma}\), such that \(b_{i1j1} = b_{i1j1}(\text{list of variables}(1)), \ldots, b_{i1j\sigma} = b_{i1j\sigma}(\text{list of variables}(\sigma))\). It is important to realize at this stage, that if the coefficients of \(2\) depend linearly on free parameters, any of these parameters should be equal to 0. Consequently, we can take \(b_{i1j1}\) and substitute by 0 those of the free parameters, which are in \(\text{list of variables}(1)\), set the same for \(b_{i2j2}\) (including also the value \(b_{i1j1}\), if it is in \(\text{list of variables}(2)\), and so on, until we get to \(b_{i\sigma j\sigma}\). Thus, if the process is not terminated, the unique solution of the system \(5\) will be found. The next step is to find the polynomials from the upper level systems by using the solution of the \((1, m^{(1)})\)-problem.

**Remark 3.** The manipulation of the lowest level linear system \(5\) takes at most

\[
C \min\{(m^{(1)})^2, (d_1)^2\}
\]

basic steps.
Remark 4. In order to be used at the next levels, the solution process of (5)

\[ \mathbf{Ab} = (1, 0, \ldots, 0)^t \Rightarrow \mathbf{b}^* = \mathbf{b}^*(A) \]  \hspace{1cm} (7)

should be implemented in a separate module for an arbitrary matrix \( A \) with \( 2d_1 + 1 \) rows and \( m_{\sigma}^{(1)} \) non-zero elements, where \( m_{\sigma}^{(1)}, d_1 \), the indeces and the values of the non-zero elements of \( A \) are the parameters of the module.

**Induction** Let us consider the system

\[
\begin{pmatrix}
\sum_{i=1}^{k(2)} f_{i0}^{(2)} (z_1) g_{i0}^{(2)} (z_1) & \equiv & 1 \\
\sum_{i=1}^{k(2)} f_{i0}^{(2)} (z_1) g_{i1}^{(2)} (z_1) + \sum_{i=1}^{k(2)} f_{i1}^{(2)} (z_1) g_{i0}^{(2)} (z_1) & \equiv & 0 \\
\sum_{i=1}^{k(2)} f_{i0}^{(2)} (z_1) g_{i2}^{(2)} (z_1) + \sum_{i=1}^{k(2)} f_{i1}^{(2)} (z_1) g_{i1}^{(2)} (z_1) + \sum_{i=1}^{k(2)} f_{i2}^{(2)} (z_1) g_{i0}^{(2)} (z_1) & \equiv & 0 \\
\sum_{i=1}^{k(2)} f_{id_2}^{(2)} (z_1) g_{id_2}^{(2)} (z_1) & \equiv & 0
\end{pmatrix}, \hspace{1cm} (8)
\]

that is a consequence of the \((2, m_{\sigma}^{(2)})\)-problem

\[
\sum_{i=1}^{k(2)} f_i^{(2)} (z_1, z_2) g_i^{(2)} (z_1, z_2) \equiv 1, \hspace{1cm} (9)
\]

as well as (3) is a consequence of the first identity in (8). Polynomials \( \{g_{i0}\}_{i=1}^{k(2)} \) are already calculated from (5). Let us add the first row of (8) to the second one, and then the second one to the third one, in order to get

\[
\sum_{i=1}^{k(2)} f_{i0}^{(2)} g_{i0}^{(2)} \equiv 1 \hspace{1cm} (10)
\]

\[
\sum_{i=1}^{k(2)} f_{i0}^{(2)} \left[ g_{i0}^{(2)} + g_{i1}^{(2)} \right] + \sum_{i=1}^{k(2)} f_{i1}^{(2)} g_{i0}^{(2)} \equiv 1 \hspace{1cm} (11)
\]

\[
\sum_{i=1}^{k(2)} f_{i0}^{(2)} \left[ g_{i0}^{(2)} + g_{i1}^{(2)} + g_{i2}^{(2)} \right] + \sum_{i=1}^{k(2)} f_{i1}^{(2)} \left[ g_{i0}^{(2)} + g_{i1}^{(2)} \right] + \sum_{i=1}^{k(2)} f_{i2}^{(2)} g_{i0}^{(2)} \equiv 1 \hspace{1cm} (12)
\]

for the first three identities of the system. Identities (11) and (12) are \((2, m_{\sigma}^{(2)})\)-problems, similar to (10). Here is how many steps we need to solve (11) and find \( g_{i1}^{(2)} \). As we know from (6), its solution is found in at most

\[ c \min \{ (m_{0,\sigma}^{(1)} + m_{1,\sigma}^{(1)})^2, (d_1)^2 \} \]

basic steps, where \( m_{0,\sigma}^{(1)} \) and \( m_{1,\sigma}^{(1)} \) are the total lengths of univariate non-zero polynomials of \( z_1 \) in front of \( z_0^2 \) and \( z_1^2 \), respectively. We need also at most \( N^{(1)} \) steps to find the new
functions $g_{il}^{(2)}$, if they exist. As usual, if the solution does not exist, the process is terminated with a message "No solution at Level 2, Identity 2". A similar estimation is valid for the third row. Representing the rest of the rows in the same way, and summing up the corresponding estimates for the number of steps yields the following total estimate for level 2:

$$
c \left( \sum_{l=0}^{\min\{m_{\ell}^{(2)},d_2\}} \min \left\{ \left( \sum_{l_1=0}^{l} m_{l_1;\sigma}^{(1)} \right)^2, (d_1)^2 \right\} + N^{(1)} \min\{m_{\sigma}^{(2)}, d_2\} \right) \leq c \left\{ \min\{m_{\sigma}^{(3)} \}, (d_2)^3 \} + N^{(1)} \min\{m_{\sigma}^{(2)}, d_2\} \right\} + N^{(1)} \min\{m_{\sigma}^{(2)}, d_2\} \right\} (13)

Let us also show the corresponding estimate for the system of level 3, that is, the system \( \mathfrak{I} \) for \( n = 3 \). Observe that, the actual length of that system, as well as the additional arithmetic operations for finding the new functions $g_{il}^{(3)}$ from identities number \( l + 1 \), depend only on the parameters $m_{\sigma}^{(3)}, d_3 $ and $N^{(2)}$. Therefore, using (13), considerations similar to those for (8) will lead to an expression of the form

$$
\leq c \left\{ \min\{(m_{\sigma}^{(3)})^4, (m_{\sigma}^{(2)})^4, (m_{\sigma}^{(1)})^3, (d_3)^4, (d_2)^4, (d_1)^3 \} + N^{(1)} \min\{(m_{\sigma}^{(3)})^2, (m_{\sigma}^{(2)})^2, (d_3)^2, (d_2)^2 \} + N^{(2)} \min\{m_{\sigma}^{(3)}, d_3 \} \right\}
$$

Performing in a similar manner induction up to \( n \) gives the following estimate for the total number of basic steps of the process (without counting sorting operations)

$$
\leq c \left\{ \min\{(m_{\sigma}^{(n)})^{n+1}, (m_{\sigma}^{(n-1)})^{n+1}, \ldots, (m_{\sigma}^{(1)})^3, (d_n)^{n+1}, (d_{n-1})^{n+1}, \ldots, (d_1)^3 \}\right. \\
+ N^{(1)} \min\{(m_{\sigma}^{(n)})^n, (m_{\sigma}^{(n-1)})^n, \ldots, (m_{\sigma}^{(2)})^2, (d_n)^n, (d_{n-1})^n, \ldots, (d_2)^2 \} \\
+ \ldots \\
+ N^{(n-1)} \min\{m_{\sigma}^{(n)}, d_n \} \right\}. \quad (14)
$$

Consequently, combining (3) and (14), it follows for every \( n \geq 1 \) that

$$
s(n, m_{\sigma}) \leq c \{ m_{\sigma}^2 \log m_{\sigma} + \min\{m_{\sigma}^{(1)} \}, (d_1)^3 \} + \sum_{\ell=1}^{n-2} N^{(l)} \min\{m_{\sigma}^{(l+1)} \}, (d_{\ell+1})^2 \} + N^{(n-1)} \min\{m_{\sigma}^{(n)}, d_n \} \}
$$

Theorem \( \mathfrak{I} \) is proved.

**Remark 5.** It is clear now, that in the beginning the variables \( \{z_j\}_{j=1}^n \) can be sorted so, that the expression in the outer curly brackets in (14) to be minimal.

**Remark 6.** In order to keep the number of basic steps in the frames of the upper bound (14), system solution processes at any level \( \ell, \ell = 2, 3, \ldots, n - 1 \), should also be implemented in separate modules, similarly to the case \( \ell = 1 \) (see Remark \( \mathfrak{I} \)). Note though, that these modules are to be prepared "on the fly" and should be normally quite ordinary in size, since their size depends rather on the total input length \( m_{\sigma} \) than on the number of variables \( n \).
Proof of Theorem 2. The proof follows that of Theorem 1 with the only difference that we use $Q_2$-arithmetic and at every division-step normalize the result to a simple fraction. This can only lead to an increase of $c_1$ with a factor that comes from the application of the Euclidean algorithm, which is bound by a constant multiple of $\max_{1 \leq i \leq k} \max_{\alpha \in A_i} \log |c_\alpha|$. □

Proof of Theorem 3. It follows from Proposition (1)(c), Theorem 2 and the fact that the Hilbert Nullstellensatz over $Z_2$ is an NP-complete problem. □

Remark 7. Obviously, the values of the constants $c_1$ and $c_2$ in Theorems 1 and 2, respectively, lie within some ordinary limits. More precise estimates for them are not subject of this paper and can be obtained empirically by applying the corresponding algorithms to polynomials of different degrees and with different numbers of non-zero coefficients.

Remark 8. Note that the aim of the present paper is to validate theoretically the existence of the algorithms from Theorems 1 and 2, to explain their complexity and milestones, but not to present precise descriptions of them.

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