Chaotic dynamics around astrophysical objects with nonisotropic stresses

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The existence of chaotic behavior for the geodesics of the test particles orbiting compact objects is a subject of much current research. Some years ago, Guéron and Letelier [Phys. Rev. E 66, 046611 (2002)] reported the existence of chaotic behavior for the geodesics of the test particles orbiting compact objects like black holes induced by specific values of the quadrupolar deformation of the source using as models the Erez–Rosen solution and the Kerr black hole deformed by an internal multipole term. In this work, we are interested in the study of the dynamic behavior of geodesics around astrophysical objects with intrinsic quadrupolar deformation or nonisotropic stresses, which induces nonvanishing quadrupolar deformation for the nonrotating limit. For our purpose, we use the Tomimatsu-Sato spacetime [Phys. Rev. Lett. 29 1344 (1972)] and its arbitrary deformed generalization obtained as the particular vacuum case of the five parametric solution of Manko et al [Phys. Rev. D 62, 044048 (2000)], characterizing the geodesic dynamics throughout the Poincaré sections method. In contrast to the results by Guéron and Letelier we find chaotic motion for oblate deformations instead of prolate deformations. It opens the possibility that the particles forming the accretion disk around a large variety of different astrophysical bodies (nonprolate, e.g., neutron stars) could exhibit chaotic dynamics. We also conjecture that the existence of an arbitrary deformation parameter is necessary for the existence of chaotic dynamics.

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I. INTRODUCTION

Although it is very usual in the literature that the stress tensor of several astrophysics objects matter—neutron stars, exotic stars— is approximated by that of a perfect fluid, viz. nonviscous medium of total energy density $\varepsilon$ (mass density $\rho = \varepsilon/c^2$), in which all stresses are zero except for an isotropic pressure $P$. We have to notice that in astrophysics, the most common object are that which present nonvanishing quadrupolar deformation for the nonrotating limit, i.e., objects not deformed by current mass–rotation– either by arbitrary multipoles–arbitrary mass quadrupole or mass octupole– but intrinsically deformed. Besides, it is clear that the approximations is based on the fact that the shear stresses, e.g., those produced by elastic strain in the solid crust or by strong magnetic field, are generally negligible compared to the pressure, but it is also clear that when the pressure is assumed isotropic we refuse the possibility of intrinsic deformation because it could obey to nonisotropic stresses during its formation process.

With the aim to follow the usual premise that the exterior gravitational field of the relevant astrophysical objects can be modelled by stationary axially symmetric exact solutions to the Einstein–Maxwell field equations (see e.g. [1, 2] for the case of neutron stars) we adopt spacetimes endowed naturally in the Weyl–Lewis–Papapetrou line element [3]. This metric admits two Killing vectors, one timelike and other spacelike, which correspond to the desired symmetries mentioned above. In the literature, we can find several exact solutions to the stationary axis–symmetric Einstein–Maxwell system equations (see [4] for some of them), but just a few of them described objects intrinsically deformed. The most known are the corresponding members to the Tomimatsu–Sato family [3], [6] which quadrupole deformation in the nonrotating limit is $Q = -\frac{1}{3}m^3$, being $m = \frac{M}{\delta^2}$ where $M_0$ is the mass monopole of the source and $\delta$ a dimensionless parameter, taking the values $\delta = 1$ for the Kerr solution and $\delta = 2$ for the Tomimatsu–Sato $\delta = 2$ solution.

In order to clarify the physical interpretation of the Tomimatsu–Sato family (henceforth TS-family), we present briefly in the next paragraphs some interpretations given to it.

In the early 1970s, when the solution was obtained, Tomimatsu & Sato [7] and after Tanabe [8] showed that TS-family represents the gravitational field of rotating masses with angular momentum about the $z$-axis. But,
soon was realized that TS-family suffers of naked curvatures singularity. Papadopoulos and Xanthopoulos [9] tried to resurrect interest in these solutions putting them in other context, they modified slightly the TS-metrics by a suitable analytic continuation and concluded that these solutions represent cylindrical symmetric spacetime, interpreting this time the solution as a beamlike-shaped pulse of gravitational radiation scattered by a cosmic string. Kodama & Hikida [10] showed that the two points in the Weyl coordinates, which have been recognized as the directional singularities, are really two-dimensional surfaces and that these surfaces are horizons. They also showed that each of the two horizons has the topology of a sphere and concluded that this may indicates that TS-family describes the spacetime surrounding a new possibility of final states of gravitational collapse.

In 2002, using the Sibgatullin’s integral method [11], Mielke, Manko and Sanabria–Gómez generalized a member of the vacuum TS-family, the Tomimatsu–Sato δ = 2 solution (henceforth TS2), to a most general spacetime in the electrovacuum case [6] possessing parameters for neglecting the high order multipoles, [6] reduces to model neutron stars close to the nonrotating limit, (i.e., $Q \leq \frac{1}{4} m^3$, which differs substantially of the spherical symmetry due to the high mass of the star.

In this point, following the concluding remarks by Kodama et al. [11] and Berti et al. [12], we assume that TS–family can be used to describe the topology of the spacetime around relevant astrophysical objects like neutron stars, strange quarks or any other exotic final states of the gravitational collapse. This last statement and the fact that the members of the TS–family have non-vanishing quadrupolar deformation constitute the initial point for our work because it enables us to consider the members of the TS–family as analytic closed form solutions for the gravitational field of relevant astrophysical objects with anisotropic stresses tensor.

After having an analytic closed form solution for the gravitational field of a source, one of the interesting topic to study is the motion of the particles orbiting this source. The construction of astrophysical models which are able to give us a complete description of realistic astronomical systems has to take into account the behavior of the surrounding matter in order to compare with astronomical observation and emit a proposition about the validity of the model. In this work, we are interested in the nature of the dynamics –chaotic or regular– of the test particles which orbit two specific members of the TS–family, the TS2 and the solution by Manko et al. in the vacuum case, viz. $Q = \mu = 0$.

In general relativity the study of stochastic motions in deterministic system –deterministic chaos– has followed two main branches. The first one is the study of the geodesic motion of test particles in a given gravitational field (Bombelli and Calzetta [14], Vieira and Letelier [15][16], Guéron and Letelier [17] and references therein). The other branch is the time evolution of the gravitational field itself (Motter and Letelier [18], Hobill, Burd and Coley [19], which is relevant in cosmology (see e.g. Motter [20]). This work is in line with the first scheme and is organized as follows. In Sec. II the TS2 solution and the Manko et al.’ solution are presented and a brief discussion of their features is given. In Sec. III the dynamics of geodesics of test particles is analyzed. Finally, in Sec. IV a brief discussion about the obtained results is presented.

II. THE PARTICULAR SPACE-TIMES

A. Case I: The Tomimatsu–Sato δ = 2 solution

The most simple form of the metric for a stationary axisymmetric space time was given by Papapetrou [18] and it can be written as

$$ds^2 = f(dt - \omega d\phi)^2 - f^{-1}[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\phi^2],$$

(1)

where $f$, $\omega$ and $\gamma$ are functions of the quasicylindrical Weyl-Lewis-Papapetrou coordinates $(\rho, z)$. The Weyl-Lewis-Papapetrou coordinates are related with the prolate spheroidal coordinates $(x, y)$ by means of the transformation

$$\rho^2 = k^2(x^2 - 1)(1 - y^2), \quad z = kxy,$$

(2)

with $x \geq 1$ and $-1 \leq y \leq 1$, then the metric is rewritten as

$$ds^2 = f(dt - \omega d\phi)^2 - k^2 f^{-1} [e^{2\gamma}(x^2 - y^2) \left( \frac{dx^2}{x^2 - 1} \right. + \left. \frac{dy^2}{1 - y^2} \right) + (x^2 - 1)(1 - y^2)d\phi^2].$$

(3)

The TS metrics were obtained from a solution to the Ernst equation in the vacuum case [21], which is given
where $p = (1 - q^2)^{1/2}$, $q = J/m^2$ and $k = mp/\delta$. The $\delta$ parameter is dimensionless, taking the values $\delta = 1$ and $\delta = 2$ for the Kerr and the TS2 solutions, respectively. The metric functions derived from (4) are 

$$f = \frac{A}{B}, \quad \omega = \frac{2mqC(1 - y^2)}{A}, \quad e^{2\gamma} = \frac{A}{p^2(x^2 - y^2)^2},$$

where

$$A = p^4(x^2 - 1)^4 + q^4(1 - y^2)^4 - 2p^2q^2(x^2 - 1)(1 - y^2) \times \{2(x^3 - 1)^2 + 2(1 - y^2)^2 + 3(x^2 - 1)(1 - y^2)\},$$
$$B = \{p^2(x^2 + 1)(x^2 - 1) - q^2(y^2 + 1)(1 - y^2) + 2px(x^2 - 1)^2 + 4q^2y^2 \{p(x(x - 1) + (p + 1)(1 - y^2)^2) + q^2(x(x - 1) + (y^2 - 1)^2).$$

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$$B = \{p^2(x^2 + 1)(x^2 - 1) - q^2(y^2 + 1)(1 - y^2) + 2px(x^2 - 1)^2 + 4q^2y^2 \}$$
$$\times \{p(x(x - 1) + (p + 1)(1 - y^2)^2) + q^2(x(x - 1) + (y^2 - 1)^2).$$

The physical sense of the parameters $m$ and $J$ is derived from the Simon’s multipole moments [22]. The first three of the relativistic multipole moments calculated from (11) with the aid of the Hoenselaers-Perjé procedure [23] are

$$P_0 = \frac{2m}{\delta}, \quad P_1 = \frac{4J}{\delta^2}, \quad P_2 = -\frac{2}{\delta^3} \left(1 + \frac{3J^2}{m^3}\right)m^3.$$

The terms $P_0$ and $P_2$ denote the monopole and the quadrupole of mass, respectively. They are related to the mass and the deformation of the source. On the other hand, the term $P_1$ describes the angular momentum. From (10) is obvious that $P_0$, $P_2$, and $P_1$ are determined only by two parameters, $m$ and $J$, and means that the source only has mass and arbitrary angular momentum.

Defining the parameter $j$ as $J/m^2$ the quadrupole of mass for the case $\delta = 2$ can be written as $P_2 = Q = -0.25m^3(1 + 3j^2)$. From the previous expression it is clear that the quadrupole deformation of TS2 solution always is negative. For such reason we can affirm that the TS2 solution describes the spacetime around an oblate source.

### B. Case II: Solution by Manko, Mielke and Sanabria–Gómez

This solution has five relevant and independent parameters: $m$ the gravitational mass, $a$ the specific angular momentum ($a = J/m$), $Q$ the electric charge, $b$ a parameter related with the mass quadrupole moment, and $\mu$ a parameter related with the dipolar magnetic moment. In addition, it has a remarkable feature, which is that the quadrupolar deformation,

$$Q = m(\delta - d - a(a - b)),$$

with

$$\delta = \frac{\mu^2 - m^2b^2}{m^2 - (a - b)^2 - Q^2}$$
$$d = \frac{1}{4}[m^2 - (a - b)^2 - Q^2],$$

depends directly on electromagnetic parameter $Q$ and $\mu$. This means that a test particle “sees” a source with a different quadrupolar deformation than the real one. This feature could implies that the electromagnetic field can induce chaos in the dynamic geodesic for uncharged particles orbiting this source [24]. In this work, we are interested in the influence of the real deformation of the mass and not in the effective deformation. For that reason, we will choose $Q = 0$ and $\mu = 0$, then the metric functions take the form

$$f = \frac{E}{D}, \quad e^{2\gamma} = \frac{E}{16\kappa^2(x^2 - y^2)^4}, \quad \omega = -\frac{(1 - y^2)F}{E},$$

with

$$E = \{4|\kappa|^2(x^2 - 1) + \delta(1 - y^2)^2 + (a - b)[(a - b)(d - \delta) - m^2b(1 - y^2)^2] + 16\kappa^2(x^2 - 1)(1 - y^2)^2 + 2\delta y^2 + m^2b^2\}^2,$$
$$D = \{4(\kappa^2x^2 - \delta y^2)^2 + 2\kappa mx|2\kappa^2(x^2 - 1) + (2\delta + ab - b^2)(1 - y^2)^2 + (a - b)[(a - b)(d - \delta) - m^2b(1 - y^2)^2] - m^2b(1 - y^2)^2 + 4\kappa^2(x^2 - 1) + \kappa x(a - b) - mb - 2mb(1 - y^2)^2 + [(a - b)\{(d - \delta) - m^2b(1 - y^2)^2\}]$$
$$+ m^2b^2\}$$
$$\times \{\kappa mx + 2\kappa x + 2\delta y^2 + 2m^2y^2\}$$
$$+ m^2b^2\}$$
$$\times \{\kappa mx + 2\kappa x + 2\delta y^2 + 2m^2y^2\}$$
$$+ \{4|\kappa|^2(x^2 - 1) + \delta(1 - y^2)^2\}$$
$$+ (a - b)[(a - b)(d - \delta) - m^2b(1 - y^2)^2]$$
$$+ (4(2\kappa mx + 2\kappa x + 2\delta y^2 + (1 - y^2)^2)$$
$$+ (1 - y^2)^2\{(a - b)(m^2b^2 - 4\delta d) - (4\kappa mx + 2m^2)\}[(a - b)(d - \delta) - m^2b]^2\}.\)
III. GEODESICS DYNAMICS FOR TEST PARTICLES

Following an standard procedure, we define the Lagrangian function $\mathcal{L}$ as $2\mathcal{L} = g_{\mu\nu}dx^\mu/d\tau dx^\nu/d\tau$, for the stationary axisymmetric metric (3), we obtain

$$2\mathcal{L} = f(\dot{t} - \omega \dot{\phi})^2 - \frac{k^2}{f} \left[ e^{2\gamma}(x^2 - y^2) \left( \frac{\dot{x}^2}{x^2 - 1} + \frac{\dot{y}^2}{1 - y^2} \right) + (x^2 - 1)(1 - y^2) \dot{\phi}^2 \right],$$

(11)

where the overdot indicates derivation respect the proper time $\tau$. Using the Hamiltonian formalism we found that the motion equations for the test particle are given by

$$\dot{x} = -\frac{f(x^2 - 1)}{k^2 e^{2\gamma}(x^2 - y^2)} P_x, \quad \dot{y} = -\frac{f(1 - y^2)}{k^2 e^{2\gamma}(x^2 - y^2)} P_y,$$

(12)

$$\dot{P}_x = -\frac{1}{2} \left\{ \frac{E^2}{f} - \frac{f(L + E\omega)^2}{k^2(x^2 - 1)(1 - y^2)} \right. - \frac{k^2}{f} \left[ P_x^2(x^2 - 1) + P_y^2(1 - y^2) \right] \right\},$$

(13)

$$\dot{P}_y = -\frac{1}{2} \left\{ \frac{E^2}{f} - \frac{f(L + E\omega)^2}{k^2(x^2 - 1)(1 - y^2)} \right. - \frac{k^2}{f} \left[ P_x^2(x^2 - 1) + P_y^2(1 - y^2) \right] \right\},$$

(14)

with

$$E = f(\dot{t} - \omega \dot{\phi}),$$

(15)

$$L = -\omega f(\dot{t} - \omega \dot{\phi}) - \frac{k^2}{f} (x^2 - 1)(1 - y^2) \dot{\phi}.$$   

(16)

The constants of integration $E$ and $L$ are related to the energy and to the angular momentum of the test particle, respectively. In the case of timelike geodesic, the lagrangian $\mathcal{L}$ satisfies $\mathcal{L} = 0.5$, this relation allows us to define an effective potential, which explicitly is

$$\Phi(x, y) = \frac{f}{k^2 e^{2\gamma}(x^2 - y^2)} \left[ \frac{E^2}{f} - \frac{f(L + E\omega)^2}{k^2(x^2 - 1)(1 - y^2)} - 1 \right].$$

By the $\Phi$ definition the motion must be restricted to the region $\Phi \geq 0$. With the aim of study the dynamic of geodesics is necessary to be sure that the test particles motion is in a confinement region. The existence of such regions is determined throughout the condition $\Phi \geq 0$.

The solution to the equation system (12)-(14), could be found using a symplectic Runge-Kutta method. That method let us find the numerical solution of the system, given the constants $E, L$ and the initial conditions $x(0), y(0), P_x(0)$ and $P_y(0)$. By the existence of the integral of motion $\mathcal{L} = \mathcal{H} = 0.5$ with $\mathcal{H}$ the hamiltonian of the system, if we have $E, L$ and $x(0), y(0), P_x(0)$, the momentum $P_y(0)$ will be determined for this equation. The values of $x(0)$ and $y(0)$ are selected in such form that satisfies the condition $\Phi \geq 0$ for confined motions, then the only arbitrary parameter is $P_x(0)$. These constants of motion indicate to us that the geodesic motion is performed in a three dimensional effective phase space in which the Poincaré section method is an adequate tool to study the motion (see Letelier & Gueron [17, 22]).

A. Case I: TS2 solution

First we shall to consider the TS2 solution. In this case, the $k$ parameter introduced in (3) is completely determined by the mass and the angular momentum of the source, implying that we only can vary the energy $E$, and the angular momentum $L$, of the test particle. For this solution, we only found bounded region of motion like that shown in Fig.1(a). Any configuration with two or more bounded regions of motion was not found. In Fig.1(a), the curve $\Phi = 0$ in the plane $xy$ for $E = 0.94$, $L = -3.12$ is plotted and it is observed that there just exists one bounded region and one escape region. In Fig.1(b)

![Fig. 1](image.png)

FIG. 1: (a) Boundary contour $\Phi = 0$ using $E = 0.94$, $L = -3.12$. There is one escape zone in the left-hand side of the picture, which correspond to small values of $x$, and a closed zone of bounded motion to the right. (b) Poincaré sections in the plane $xP_x$ for the values defined in (a). We have regular motion, we can see that the motion in the bounded region of the Fig.1(a) is completely regular such as that which occurs in the Schwarzschild, Kerr and Kerr–Newman black holes [26]. The geodesics for this solution were studied using surface sections for many different values of $E$ and $L$.

The numerical results suggest the existence of only integrable geodesics.

B. Case II: Manko et al. Solution

In this case, we used the available information present in the literature for typical values for the multipolar struc-
ture of neutron stars. In particular, we took numerical data from the Berti and Stergioulas work \cite{12}. In \cite{12}, Berti and Stergioulas solved in a numerical way the full Einstein equations to determine the spacetime for rapidly rotating neutron star along of sequences of constant rest mass for selected equation of state (henceforth EOS) denoted as EOS A \cite{27}, EOS AU \cite{28}, EOS FPS \cite{29}, EOS L \cite{30} and EOS APRb \cite{31}.

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{fig2(a)}\hspace{1cm}\includegraphics[width=0.4\textwidth]{fig2(b)}
\caption{(a) Boundary contour $\Phi = 0$ for $M = 1.840M_\odot$, $J = 3.683$, $b = -0.3792$, $E = 0.1$ and $L = -5.6$. There is two escape zones, and a closed zone of bounded motion. (b) Poincaré sections in the plane $xP_x$ for the values defined in (a). We see only regular motions.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{fig3(a)}\hspace{1cm}\includegraphics[width=0.4\textwidth]{fig3(b)}
\caption{(a) Boundary contour $\Phi = 0$ for $M = 1.936M_\odot$, $J = 4.498$, $b = -0.3080$, $E = 0.95$ and $L = -9.0$. There is one escape zone, and a closed zone of bounded motion. (b) Poincaré sections in the plane $xP_x$ for the values defined in (a). The geodesics are only regular.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{fig4(a)}\hspace{1cm}\includegraphics[width=0.4\textwidth]{fig4(b)}
\caption{(a) Boundary contour $\Phi = 0$ for $M = 1.936M_\odot$, $J = 4.498$, $b = 0.8$, $E = 0.96$ and $L = -9.9$. There are two small escape zones, and two closed zones of bounded motion. (b) Poincaré sections in the plane $xP_x$ for the values defined in (a). We see chaotic motion to the left-hand side and regular motion to the right-hand side.}
\end{figure}

Additionally, they matched the Manko et. al. solution \cite{8} to the numerical solutions imposing the condition that the mass–quadrupole moment of the numerical and analytic spacetimes be the same. Under this condition, the $b$ parameter is determined from the numerical data. From \cite{12} we can see that the Manko et al. solution fixes better for the EOS FPS, for that reason we took all the values presented by Berti and Stergioulas for each sequence of mass of the EOS FPS founding bounding potentials like the presented in Fig2(a) and Fig3(a).

In Fig2(a), we present the boundary contour for $\Phi = 0$ in the $xy$ plane for $M = 1.840M_\odot$, $J = 3.683$, $b = -0.3792$, $E = 0.1$ and $L = -5.6$, observing that exists two escape regions and a bounded region. In Fig3(a) boundary contour for $\Phi = 0$ in the plane $xy$ for $M = 1.936M_\odot$, $J = 4.498$, $b = -0.3080$, $E = 0.95$ and $L = -9.0$ is presented. This configuration is very similar to that presented in the first case and is the commonest shape found. The geodesic behavior inside of the bounded region for these potentials is presented in the Fig2(b) and Fig3(b), respectively. From there can be seen that the motion is completely regular.

In order to found chaotic behavior, we let ourself modify slighty the values of the $b$ parameter changing the quadrupole deformation and also the properties of the test particle throughout the change in its energy and angular momentum values. Introducing this changes we found potential like the presented in Fig4(a) and Fig5(a).

In Fig4(a), we present the boundary contour for $M = 1.936M_\odot$, $J = 4.498$, $b = 0.8$, $E = 0.96$ and $L = -9.9$ observing that there are two very small escape regions and two disconnected bounded regions. In Fig5(a) boundary contour for $\Phi = 0$ in the plane $xy$ for $M = 1.936M_\odot$, $J = 4.498$, $b = 0.8$, $E = 0.971$ and $L = -9.3$ is presented. This configuration conserves the tiny scape regions but
the confined regions are connected. The geodesic behavior inside of the bounded region in the phase space for these potentials is presented in the Fig.4(b) and Fig.5(b), respectively. From there one can notice the presence of a mixed phase space –chaotic and regular– to the left but only regular to the right in both cases.

IV. CONCLUDING REMARKS

We have constructed the Poincaré sections for a lot of possible combinations of energy and angular momentum (and deformation parameter in the Manko et al. case) which confine the motion of the test particles orbiting around two types of sources.

We concluded that is apparently impossible to find chaotic geodesics around a source described for the TS solution and that the stability of the geodesics does not depend on any way of the relative spin direction of the center of attraction nor of the angular momentum of the test particle. This result shows numerical evidences of the existence of only integrable geodesics for this system. In other words, the case of the test particle turning around a Tomimatsu-Sato source type is completely integrable. This work attempts to complete the study of the geodesic dynamics in the well known trilogy of axially symmetric solutions (Schwarzschild, Kerr, Tomimatsu Sato $\delta = 2$).

On the other hand, we use the Manko et al. solution with the idea of analyze the geodesic dynamics of test particles in rapidly rotating neutron stars with equation of state FPS. For this we took the values presented by Berti and Stergioulas for the parameters describing the multipolar structure of neutron stars. In this case we only found regular geodesics like in the precedent metric. In order to find chaotic behavior, we modify the values of the deformation parameter. Introducing these changes we found potentials with two bounded regions, in which the Poincaré sections exhibit mixed phase spaces, i.e., phase spaces when some of the KAM tori survives inside a chaotic sea.

The main result in the Letelier’s work was the existence of chaotic geodesics in the geometry that characterizes the prolate case. In the oblate case these orbits appear to be regular. In our case we found chaotic geodesics (in the Manko’s solution) only in the geometry that characterizes the oblate case. In [17] Letelier et al. analyzed the influence of the introduction of multipolar terms corresponding to the quadrupole deformation in the Schwarzschild and Kerr solution finding chaotic behavior for some values of the deformation. Considering the former statement and based on the numerical evidence of our work we could conjecture that the parameter of arbitrary deformation is which what induces the ergodic motion for uncharged test particles orbiting general relativistic vacuum sources.

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