Redundancy-Related Bounds for Generalized Huffman Codes

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Abstract—This paper presents new lower and upper bounds for the compression rate of optimal binary prefix codes on memoryless sources according to various nonlinear codeword length objectives. Like the most well-known redundancy bounds for minimum (arithmetic) average redundancy coding — Huffman coding — these are in terms of a form of entropy and/or the probability of the most probable input symbol. The bounds here improve on known bounds of the form $L \in [H, H+1]$, where $H$ is some form of entropy in bits (or, in the case of redundancy measurements, 0) and $L$ is the length objective, also in bits. The objectives explored here include exponential-average length, maximum pointwise redundancy, and exponential-average pointwise redundancy (also called $d$th exponential redundancy). These relate to queueing and single-shot communications, Shannon coding and universal modeling (worst-case minimax redundancy), and bridging the maximum pointwise redundancy problem with Huffman coding, respectively. A generalized form of Huffman coding known to find optimal codes for these objectives helps yield these bounds, some of which are tight. Related properties to such bounds, also explored here, are the necessary and sufficient conditions for the shortest codeword being a specific length.

Index Terms—Huffman codes, optimal prefix code, queueing, Rényi entropy, Shannon codes, worst case minimax redundancy.

I. INTRODUCTION

Since Shannon introduced information theory, we have had entropy bounds for the expected codeword length of optimal lossless fixed-to-variable-length binary codes. The lower bound is entropy, while upper bound — corresponding to a maximum average redundancy of one bit for an optimal code, thus yielding unit-sized bounds — follows from a suboptimal code, the Shannon code, for which the codeword length of an input of probability $p$ is $[-\log_2 p]$ [1]. Huffman found a method of producing an optimal code by building a tree in which the two nodes with lowest weight (probability) are merged to produce a node with their combined weight [2]. On the occasion of the twenty-fifth anniversary of the Huffman algorithm, Gallager introduced bounds in terms of the most probable symbol which improved on the unit-sized redundancy bound [3]. Both upper and lower bounds have since been improved given this most probable symbol [4]–[8] such that these bounds are tight when this symbol’s probability is at least 1/127 (and close-to-tight when it has lower probability). Such bounds are useful for quickly bounding the performance of an optimal code without running the algorithm that would produce the code. The bounds are for a fixed-sized input; asymptotic treatment of redundancy for block codes of growing size, based on binary memoryless sources, is given in [9].

Others have given consideration to objectives other than expected codeword length [10, §2.6]. Many of these nonlinear objectives, which have a variety of applications, also have unit-sized bounds but have heretofore lacked tighter closed-form bounds achieved using the most probable symbol and, if necessary, some form of entropy. We address such problems here, finding upper and lower bounds for the optimal codes of given probability mass functions for nonlinear objectives.

A lossless binary prefix coding problem takes a probability mass function $p = \{p_i\}$, defined for all $i$ in the input alphabet $\mathcal{X}$, and finds a binary code for $\mathcal{X}$. Without loss of generality, we consider an $n$-item source emitting symbols drawn from the alphabet $\mathcal{X} = \{1, 2, \ldots, n\}$ where $\{p_i\}$ is the sequence of probabilities for possible symbols ($p_i > 0$ for $i \in \mathcal{X}$ and $\sum_{i \in \mathcal{X}} p_i = 1$) in monotonically nonincreasing order ($p_i \geq p_j$ for $i < j$). Thus the most probable symbol is $p_1$. The source symbols are coded into binary codewords. The codeword $c_i \in \{0, 1\}^*$ in code $c$, corresponding to input symbol $i$, has length $l_i$, defining length vector $l$.

The goal of the traditional coding problem is to find a prefix code minimizing expected codeword length $\sum_{i \in \mathcal{X}} p_i l_i$, or, equivalently, minimizing average redun-
Given a vector of input probabilities \( \mathbf{p} = (p_i)_{i \in \mathcal{X}} \), the redundancy of a prefix code is defined as:

\[
\bar{R}(\mathbf{l}, \mathbf{p}) \triangleq \sum_{i \in \mathcal{X}} p_i l_i - H(\mathbf{p}) = \sum_{i \in \mathcal{X}} p_i (l_i + \log_2 p_i)
\]

where \( H(\mathbf{p}) = -\sum_{i \in \mathcal{X}} p_i \log p_i \) is the Shannon entropy. A prefix code is a code for which no codeword begins with another codeword.

This problem is equivalent to finding a minimum-weight external path among all rooted binary trees, due to the fact that every prefix code can be represented as a binary tree. In this tree representation, each edge from a parent node to a child node is labeled 0 (left) or 1 (right), with at most one of each type of edge per parent node. A leaf is a node without children; this corresponds to a codeword, and the codeword is determined by the path from the root to the leaf. Thus, for example, a leaf that is the right-edge (1) child of a left-edge (0) child of a left-edge (0) child of the root will correspond to codeword 001. Leaf depth (distance from the root) is thus codeword length. If we represent external path weight as \( \sum_{i \in \mathcal{X}} w(i) l_i \), the weights are the probabilities (i.e., \( w(i) = p_i \)), and, in fact, we refer to the problem inputs as \( \{w(i)\} \) for certain generalizations in which their sum, \( \sum_{i \in \mathcal{X}} w(i) \), need not be 1.

If formulated in terms of \( \mathbf{l} \), the constraints on the minimization are the integer constraint (i.e., that codes must be of integer length) and the Kraft inequality [11]; that is, the set of allowable codeword length vectors is:

\[
\mathcal{L}_n \triangleq \left\{ \mathbf{l} \in \mathbb{Z}_+^n \text{ such that } \sum_{i=1}^n 2^{-l_i} \leq 1 \right\}.
\]

Because Huffman’s algorithm [2] finds codes minimizing average redundancy \((1)\), the minimum-average redundancy problem itself is often referred to as the “Huffman problem,” even though the problem did not originate with Huffman himself. Huffman’s algorithm is a greedy algorithm built on the observation that the two least likely items will have the same length and can thus be considered siblings in the coding tree. A reduction can thus be made in which the two items of weights \( w(i) \) and \( w(j) \) can be considered as one with combined weight \( w(i) + w(j) \), and the codeword of the combined item determines all but the last bit of each of the items combined, which are differentiated by this last bit. This reduction continues until there is one item left, and, assigning this item the null string, a code is defined for all input items. In the corresponding optimal code tree, the \( i \)th leaf corresponds to the codeword of the \( i \)th input item, and thus has weight \( w(i) \), whereas the weight of parent nodes are determined by the combined weight of the corresponding merged item.

Shannon [1] had previously shown that an optimal \( l^{\text{opt}} \) must satisfy:

\[
H(\mathbf{p}) \leq \sum_{i \in \mathcal{X}} p_i l_i^{\text{opt}} < H(\mathbf{p}) + 1
\]
or, equivalently,

\[
0 \leq \bar{R}(l^{\text{opt}}, \mathbf{p}) < 1.
\]

Less well known is that simple changes to the Huffman algorithm solve several related coding problems which optimize for different objectives. We discuss three such problems, all three of which have been previously shown to satisfy redundancy bounds for optimal \( l \) of the form:

\[
\tilde{H}(\mathbf{p}) \leq \tilde{L}(\mathbf{l}, \tilde{l}) < \tilde{H}(\mathbf{p}) + 1
\]
or

\[
0 \leq \tilde{R}(l, \mathbf{p}) < 1
\]

for some entropy measure \( \tilde{H} \) and cost measure \( \tilde{L} \), or some redundancy measure \( \tilde{R} \).

Many authors consider generalized versions of the Huffman algorithm [12]–[15]. These generalizations change the combining rule; instead of replacing items \( i \) and \( j \) with an item of weight \( w(i) + w(j) \), the generalized algorithm replaces them with an item of weight \( f(w(i), w(j)) \) for some function \( f \). Thus the weight of a combined item (a node) no longer need be equal to the sum of the probabilities of the items merged to create it (the sum of the leaves of the corresponding subtree). This has the result that the sum of weights in a reduced problem need not be 1, unlike in the original Huffman algorithm. In particular, the weight of the root, \( w_{\text{root}} \), need not be 1. However, we continue to assume that the sum of inputs to the coding problems will be 1 (with the exception of reductions among problems).

### A. Maximum pointwise redundancy

The most recent variation in problem objective we consider is the problem proposed by Drmota and Szpankowski [16]. Instead of minimizing average redundancy \( \bar{R}(l, \mathbf{p}) \triangleq \sum_{i \in \mathcal{X}} p_i (l_i + \log_2 p_i) \), here we minimize maximum pointwise redundancy

\[
R^*(l, \mathbf{p}) \triangleq \max_{i \in \mathcal{X}} (l_i + \log_2 p_i).
\]

Originally solved via an extension of Shannon coding, this was later noted to be solvable via a variation of Huffman coding [17] derived from that in [18], one using combining rule

\[
f^*(w(i), w(j)) \triangleq 2 \max(w(i), w(j)).
\]

The solution of this worst-case pointwise redundancy problem is relevant to optimizing maximal (worst-case)
minimax redundancy, a universal modeling problem for which the set \( \mathcal{P} \) of possible probability distributions results in a normalized “maximum likelihood distribution.” More recently Gawrychowski and Gabrie propose a different worst-case redundancy problem which also finds its solution in minimizing maximum pointwise redundancy, one for which normalization is not relevant and one which assumes all probability distributions are possible (rather than just a subset \( \mathcal{P} \) of the probability simplex) [19].

The first proposed algorithm for the maximum pointwise redundancy problem is closely related to Shannon coding. It is based on the Shannon code so that each codeword is the same length as or one bit shorter than the corresponding codeword in the Shannon code. This method is called “generalized Shannon coding.” (With proper tie-breaking techniques, the Huffman-like solution guarantees that each codeword, in turn, is no shorter than the corresponding codeword in the Shannon code.) As both are optimal, this makes no difference in the maximum pointwise redundancy.) In Section II we see that an optimal code for \( \mathcal{P} \) has redundancy \( R_{\text{opt}}^*(\mathcal{P}) \in [0, 1] \), an already-known bound improved upon therein. The upper bound is easily illustrated by noticing that a Shannon code (and thus a generalized Shannon code), unlike a Huffman code, guarantees that a given codeword has a length within one bit of the associated input symbol’s self-information, \(-\lg p_i\).

**B. \( d \)th exponential redundancy**

A spectrum of problems bridges the objective of Huffman coding with the objective optimized by generalized Shannon coding (and its Huffman-like alternative) using an objective proposed in [20] and solved for in [13]. In this particular context, the range of problems, parameterized by a variable \( d \), can be called \( d \)th exponential redundancy [17]. This is the minimization of the following:

\[
R_d^d(l, \mathcal{P}) \triangleq \frac{1}{d} \lg \sum_{i \in \mathcal{X}} p_i^{1 + d} \sum_{i \in \mathcal{X}} p_i^{2d(l + \lg p_i)}. 
\]

(3)

Although \( d > 0 \) is the case we consider most often here, \( d \in (-1, 0) \) is also a valid problem. If we let \( d \to 0 \), we approach the average redundancy (Huffman’s objective), while \( d \to \infty \) is maximum pointwise redundancy (with its Shannon-like solution) [17]. The combining rule, introduced in [13, p. 486], is

\[
f_d(w(i), w(j)) \triangleq \left( 2^d w(i)^{1 + d} + 2^d w(j)^{1 + d} \right)^{-\frac{1}{d}}. 
\]

(4)

As we show at end of Section II the upper bound for maximum pointwise redundancy also improves upon the already-known bound — \( R_{\text{opt}}^d(l, \mathcal{P}) \in [0, 1] \) — for this problem, as maximum pointwise redundancy is an upper bound for \( d \)th exponential redundancy. A lower bound is obtained by observing the reverse relationship with the average redundancy problem.

**C. Exponential average**

A related problem is one proposed by Campbell [21], [22]. This exponential problem, given probability mass function \( \mathcal{P} \) and \( a \in (0, \infty) \setminus \{1\} \), is to find a code minimizing

\[
L_a(l, \mathcal{P}) \triangleq \log_a \sum_{i \in \mathcal{X}} p_i a^i. 
\]

(5)

The solution to this [12], [13], [23] uses combining rule

\[
f_a(w(i), w(j)) \triangleq a w(i) + a w(j). 
\]

(6)

A change of variables transforms the \( d \)th exponential redundancy problem into (5) by assigning \( a = \lg d \) and using input weights \( w(i) \) proportional to \( p_i^{1 + d} \), which yields (4). We illustrate this precisely at the end of Section II in (15), which we use in Section III to find initial improved entropy bounds. These are supplemented by additional bounds for problems with \( a \in (0.5, 1) \) and \( p_1 \geq 2a/(2a + 3) \) at the end of Section III.

It is important to note here that \( a > 1 \) is an average of growing exponentials, while \( a < 1 \) is an average of decaying exponentials. Because of this, these two subproblems have different properties and have often been considered separately in the literature. The \( a > 1 \) variation of (5) was used in Humblet’s dissertation [24] for a queueing application originally proposed by Jelinek [25] and expounded upon in [26]. This problem is one in which overlow probability should be minimized, where the source produces symbols randomly and the codewords are temporarily stored in a finite buffer. The Huffman-like coding method was simultaneously published in [12], [13], [23]; in the last of these, Humblet noted that the Huffman combining method (6) finds the optimal code with \( a \in (0, 1) \) as well. An application for this decaying exponential variant involving single-shot communications has a communication channel with a window of opportunity of a total duration (in bits) distributed geometrically with parameter \( a \) [27]. The probability of successful transmission is

\[
P[\text{success}] = a^{L_a(l, \mathcal{P})}. 
\]

(7)

For \( a > 0.5 \), the unit-sized bound we improve upon is in terms of Rényi entropy, as in (13); the solution is trivial for \( a \leq 0.5 \), as we note at the start of Section III.
Note that $a \to 1$ corresponds to the usual linear expectation objective. Problems for $a$ near 1 are of special interest, since $a \downarrow 1$ corresponds to the minimum-variance solution if the problem has multiple solutions — as noted in [13], among others — while $a \uparrow 1$ corresponds to the maximum-variance solution.

The aforementioned improved bounds are in all cases based on a given highest symbol probability, $p_1$. We also discuss the related issue of the length of the most likely codeword in these coding problems. These bounds are the first of their kind for nontraditional Huffman codes, bounds which are functions of both entropy (if applicable) and $p_1$, as in the traditional case. However, they are not the first improved bounds for such codes. More sophisticated bounds on the optimal solution for the exponential-average objective are given in [26] for $a > 1$; these appear as solutions to related problems rather than in closed form, however, and these problems require no less time or space to solve than the original problem. They are mainly useful for analysis. Bounds given elsewhere for a closely related objective having a one-to-one correspondence with [5] are demonstrated under the assumption that $p_1 \geq 0.4$ always implies $l_1$ can be 1 for the optimal code [28]. We show that this is not necessarily the case due to the difference between the exponential-average objective and the usual objective of an arithmetic average.

In the next section, we find tight exhaustive bounds for the values of optimal $R^*(l, p)$ and corresponding $l_1$ in terms of $p_1$, then find how we can extend these to exhaustive — but not tight — bounds for optimal $R^d(l, p)$. In Section III as previously noted, we investigate the behavior of optimal $L_a(p, l)$ and $l_1$ in terms of $p_1$. The main results are given as theorems, corollaries, and remarks immediately following them, and many are also illustrated as figures.

II. BOUNDS ON REDUNDANCY PROBLEMS

A. Shannon pointwise redundancy bounds

Shannon found redundancy bounds for $\bar{R} \text{opt}(p)$, the average redundancy $\bar{R}(l, p) = \sum_{i \in X} p_i l_i - H(p)$ of the average redundancy-optimal $l$. The simplest bounds for minimized maximum pointwise redundancy

$$R^*_\text{opt}(p) \triangleq \min_{l \in L_a} \max_{i \in X} (l_i + \lg p_i)$$

are quite similar to and can be combined with Shannon’s bounds as follows:

$$0 \leq \bar{R} \text{opt}(p) \leq R^*_\text{opt}(p) < 1 \quad (8)$$

The average redundancy case is a lower bound because the maximum ($R^*(l, p)$) of the values ($l_i + \lg p_i$) that average to a quantity ($\bar{R}(l, p)$) can be no less than the average (a fact that holds for all $l$ and $p$). The upper bound is due to Shannon code $l^0(p) \triangleq \lfloor -\lg p_i \rfloor$ resulting in $R^* \text{opt}(p) \leq R^*(l^0(p), p) = \max_{i \in X} (\lfloor -\lg p_i \rfloor + \lg p_i) < 1$.

A few observations can be used to find a series of improved lower and upper bounds on optimum maximum pointwise redundancy based on (8):

**Lemma 1**: Suppose we apply (2) to find a Huffman-like code tree in order to minimize maximum pointwise redundancy. Then the following holds:

1) Items are always merged by nondecreasing weight.
2) The weight of the root $w_{\text{root}}$ of the coding tree determines the maximum pointwise redundancy, $R^*(l, p) = \lg w_{\text{root}}$.
3) The total probability of any subtree is no greater than the total weight of the subtree.
4) If $p_1 \leq 2p_{n-1}$, then a minimum maximum pointwise redundancy code can be represented by a complete tree, that is, a tree with leaves at depth $\lfloor \lg n \rfloor$ and $\lceil \lg n \rceil$ only (with $\sum_{i \in X} 2^{-l_i} = 1$). (This is similar to the property noted in [29] for optimal-expected-length codes of sources termed quasi-uniform in [30].)

**Proof**: We use an inductive proof in which base cases of sizes 1 and 2 are trivial, and we use weight function $w$ instead of probability mass function $p$ to emphasize that the sums of weights need not necessarily add up to 1. Assume first that all properties here are true for trees of size $n-1$ and smaller. We wish to show that they are true for trees of size $n$.

The first property is true because $f^+(w(i), w(j)) = 2 \max(w(i), w(j)) > w(i)$ for any $i$ and $j$; that is, a compound item always has greater weight than either of the items combined to form it. Thus, after the first two weights are combined, all remaining weights, including the compound weight, are no less than either of the two original weights.

Consider the second property. After merging the two least weighted of $n$ (possibly merged) items, the property holds for the resulting $n-1$ items. For the $n-2$ untouched items, $l_i + \lg w(i)$ remains the same. For the two merged items, let $l_{n-1}$ and $w(n-1)$ denote the maximum depth/weight pair for item $n-1$ and $l_n$ and $w(n)$ the pair for $n$. If $l' \text{ and } w'$ denote the depth/weight pair of the combined item, then $l' + \lg w' = l_{n-1} + \lg(2 \max(w(n-1), w(n))) = \max(l_{n-1} + \lg w(n-1), l_n + \lg w(n))$, so the two trees have identical maximum redundancy, which is equal to $\lg w_{\text{root}}$ since the root node is of depth 0.

Consider, for example, $p = (0.5, 0.3, 0.2)$, which has optimal codewords with lengths $l = (1, 2, 2)$. The first combined pair has $l' + \lg w' = 1 + \lg 0.6 = \max(2 +$
lg 0.3, 2 + lg 0.2) = max(l_2 + lg p_2, l_3 + lg p_3). This value is identical to that of the maximum redundancy, lg 1.2 = lg w_{root}.

For the third property, the first combined pair yields a weight that is no less than the combined probabilities. Thus, via induction, the total probability of any (sub)tree is no greater than the weight of the (sub)tree.

In order to show the final property, first note that \( \sum_{i \in \mathcal{X}} 2^{-\lambda} = 1 \) for any tree created using the Huffman-like procedure, since all internal nodes have two children. Now think of the procedure as starting with a queue of input items, ordered by nondecreasing weight from head to tail. After merging two items, obtained from the head of the queue, into one compound item, that item is placed back into the queue as one item, but not necessarily at the tail; an item is placed such that its weight is no smaller than any item ahead of it and is smaller than any item behind it. In keeping items ordered, this results in a complete tree exists, but that, given an item, the total probability of any (sub)tree is no greater than the weight of the tree.

The first step of coding is to merge two nodes, like procedure, since all internal nodes have two children. After merging two items, obtained from the head of the queue, into one compound item, that item is placed back into the queue as one item, but not necessarily at the tail; an item is placed such that its weight is no smaller than any item ahead of it and is smaller than any item behind it. In keeping items ordered, this results in an optimal coding tree. A variant of this method can be used for linear-time coding [17].

In this case, we show not only that an optimal complete tree exists, but that, given an n-item tree, all items that finish at level \( \lceil \log n \rceil \) appear closer to the head of the queue than any item at level \( \lceil \log n \rceil - 1 \) (if any), using a similar approach to the proof of Lemma 2 in [27]. Suppose this is true for every case with \( n - 1 \) items for \( n > 2 \), that is, that all nodes are at levels \( \lceil \log(n - 1) \rceil \) or \( \lceil \log n \rceil - 1 \), with the latter items closer to the head of the queue than the former. Consider now a case with \( n \) nodes. The first step of coding is to merge two nodes, resulting in a combined item that is placed at the end of the combined-item queue, as we have asserted that \( p_1 \leq 2p_{n-1} = 2 \max(p_{n-1}, p_n) \). Because it is at the end of the queue in the \( n - 1 \) case, this combined node is at level \( \lceil \log(n - 1) \rceil \) in the final tree, and its children are at level \( 1 + \lceil \log(n - 1) \rceil = \lceil \log n \rceil \). If \( n \) is a power of two, the remaining items end up on level \( \lceil \log n \rceil = \lceil \log(n - 1) \rceil \), satisfying this lemma. If \( n - 1 \) is a power of two, they end up on level \( \lceil \log(n - 1) \rceil = \lceil \log n \rceil \), also satisfying the lemma. Otherwise, there is at least one item ending up at level \( \lceil \log n \rceil = \lceil \log(n - 1) \rceil \) near the head of the queue, followed by the remaining items, which end up at level \( \lceil \log n \rceil = \lceil \log(n - 1) \rceil \). In any case, all properties of the lemma are satisfied for \( n \) items, and thus for any number of items.

We can now present the improved redundancy bounds.

**Theorem 1:** For any distribution in which \( p_1 \geq 2/3 \), \( R_{\text{opt}}^*(p) = 1 + \max(p_1, p_n) \). If \( p_1 \in [0.5, 2/3] \), then \( R_{\text{opt}}^*(p) \in [1 + \max(p_1, 2 + \max(1 - p_1))] \) and these bounds are tight. Define \( \lambda \triangleq -\max(1 - p_1) \). Thus \( \lambda \) satisfies \( p_1 \in [2^{-\lambda}, 2^{-\lambda+1}] \), and \( \lambda > 1 \) for \( p_1 \in (0, 0.5) \); in this range, the following bounds for \( R_{\text{opt}}^*(p) \) are tight:

\[
\begin{array}{c|c}
 p_1 & R_{\text{opt}}^*(p) \\
 \hline
 \left[ \frac{1}{2}, \frac{1}{2}^\lambda \right) & \left[ \lambda + \max(p_1, 1 + \max(1 - p_1)) \right] \\
 \left[ \frac{1}{2}^\lambda, \frac{1}{2} \right] & \left[ \log \frac{1 - p_1}{\lambda + \max(p_1, 1 + \max(1 - p_1))} \right] \\
 \left[ \frac{1}{2}, 1 \right] & \left[ \lambda + \max(p_1, 1 + \max(1 - p_1)) \right] \\
\end{array}
\]

**Proof:** The key here is generalizing the unit-sized Shannon code:

1) **Upper bound:** Before we prove the upper bound, note that, once proven, the tightness of the upper bound in \( [0.5, 1) \) is shown via

\[
p = (p_1, 1 - p_1 - \epsilon, \epsilon)
\]

for which the bound is achieved in \( [2/3, 1) \) for any \( \epsilon \in (0, (1 - p_1)/2] \) and approached in \( [0.5, 2/3] \) as \( \epsilon \downarrow 0 \).

Let us define what we call a first-order Shannon code:

\[
l_1^0(p) = \begin{cases} 
\lambda \equiv \lceil -\log p_1 \rceil \smallskip 
\lceil -\log \left( p_1 \left( \frac{1 - 2^{-\lambda}}{1 - 2^{-\lambda+1}} \right) \right) \rceil, & \text{if } i = 1 \\
\lceil -\log \left( p_1 \left( \frac{1 - 2^{-\lambda}}{1 - 2^{-\lambda+1}} \right) \right) \rceil, & \text{if } i \in \{2, 3, \ldots, n\} 
\end{cases}
\]

This code, previously presented in the context of finding average redundancy bounds given any probability [31], improves upon the original “zero-order” Shannon code \( l^0(p) \) by taking the length of the first codeword into account when designing the rest of the code. The code satisfies the Kraft inequality, and thus, as a valid code, its redundancy is an upper bound on the redundancy of an optimal code. Note that

\[
\max_{i > 1}(l_1^0(p) + \log p_i) = \max_{i > 1} \left( \left\lceil \log \left( \frac{1 - p_1}{1 - 2^{-\lambda+1}} \right) \right\rceil + \log p_i \right)
\]

\[
< 1 + \log \frac{1 - p_1}{1 - 2^{-\lambda+1}}.
\]

There are two cases:

a) \( p_1 \in [2/(2^\lambda + 1), 1/(2^\lambda - 1)] \): In this case, the maximum pointwise redundancy of the first item is no less than \( 1 + \log((1 - p_1)/(1 - 2^{-\lambda})) \), and thus \( R_{\text{opt}}^*(p) \leq R_{\text{opt}}^*(l_1^0(p), p) = \lambda + \log p_1 \). If \( \lambda > 1 \) and \( p_1 \in [2/(2^\lambda + 1), 1/(2^\lambda - 1)] \), consider probability mass function

\[
p = \left( p_1, \frac{1 - p_1 - \epsilon}{2^\lambda - 2}, \ldots, \frac{1 - p_1 - \epsilon}{2^\lambda - 2}, \epsilon \right)
\]

where \( \epsilon \in (0, 1 - p_1 - 2^\lambda) \). Because \( p_1 \geq 2/(2^\lambda + 1) \), \( 1 - p_1 - 2^\lambda \leq (1 - p_1 - \epsilon)/(2^\lambda - 2) \), and \( p_{n-1} \geq p_n \). Similarly, \( p_1 < 1/(2^\lambda - 1) \) assures that \( p_1 \geq p_2 \), so the probability mass function is monotonic. Since \( 2p_{n-1} > \)
by Lemma 1 an optimal code for this probability mass function is \( l_i = \lambda \) for all \( i \), achieving \( R^*(l, p) = \lambda + \lg p_1 \), with item 1 having the maximum pointwise redundancy.

b) \( p_1 \in [1/2^\lambda, 2/(2^\lambda + 1)] \): In this case, (9) immediately results in \( R^*_\text{opt}(p) \leq R^*(l^1(p), p) < 1 + \lg((1-p_1)/(1-2^{-\lambda})) \). The probability mass function

\[
p = \left( p_1, \frac{1 - p_1 - \epsilon}{2^\lambda - 1}, \ldots, \frac{1 - p_1 - \epsilon}{2^\lambda - 1}, \epsilon \right)
\]

illustrates the tightness of this bound for \( \epsilon \downarrow 0 \). This is a monotonic probability mass function for sufficiently small \( \epsilon \), for which we also have \( p_1 < 2p_{n-1} \), so (again from Lemma 1) this results in an optimal code with \( l_i = \lambda \) for \( i \in \{1, 2, \ldots, n-1\} \) and \( l_{n-1} = l_n = \lambda + 1 \), and thus the bound is approached with item \( n-1 \) having the maximum pointwise redundancy.

2) Lower bound: Consider all optimal codes with \( l_1 = \mu \) for some fixed \( \mu \in \{1, 2, \ldots\} \). If \( p_1 \geq 2^{-\mu} \), 
\[ R^*(l, p) \geq l_1 + \lg p_1 = \mu + \lg p_1 . \]
If \( p_1 < 2^{-\mu} \), consider the weights at level \( \mu \) (i.e., \( \mu \) edges below the root). One of these weights is \( p_1 \), while the rest are known to sum to a number no less than 1 - \( p_1 \). Thus at least one weight must be at least \((1-p_1)/(2^\mu-1) \) and 
\[ R^*(l, p) \geq \mu + \lg((1-p_1)/(2^\mu-1)) . \]
Thus,
\[ R^*_\text{opt}(p) \geq \mu + \lg \max \left( p_1, \frac{1-p_1}{2^\mu-1} \right) \]
for \( l_1 = \mu \), and, since \( \mu \) can be any positive integer,
\[ R^*_\text{opt}(p) \geq \min_{\mu \in \{1, 2, 3, \ldots\}} \left( \mu + \lg \max \left( p_1, \frac{1-p_1}{2^\mu-1} \right) \right) \]
which is equivalent to the bounds provided.

For \( p_1 \in [1/(2^{\mu+1} - 1), 1/2^\mu) \) for some \( \mu \), consider
\[
\left( p_1, \frac{1 - p_1}{2^\mu+1 - 2}, \ldots, \frac{1 - p_1}{2^\mu+1 - 2}, \frac{2^\mu - 1}{2^\mu+1 - 2} \right).
\]

By Lemma 1, this has a complete coding tree — in this case with \( l_1 \) one bit shorter than the other lengths — and thus achieves the lower bound for this range (\( \lambda = \mu+1 \)). Similarly
\[
\left( p_1, \frac{2^\mu - 1}{2^\mu+1 - 2}, \frac{2^\mu - 1}{2^\mu+1 - 2}, \frac{2^\mu - 1}{2^\mu+1 - 2} \right)
\]
has a fixed-length optimal coding tree for \( p_1 \in [1/2^\mu, 1/(2^\mu - 1)] \), achieving the lower bound for this range (\( \lambda = \mu \)).

Note that the unit-sized bounds of (8) are identical to the tight bounds at (negative integer) powers of two. In addition, the tight bounds clearly approach 0 and 1 as \( p_1 \downarrow 0 \). This behavior is in stark contrast with average redundancy, for which bounds get closer, not further apart, illustrated by Gallager’s redundancy bound \( (R^*_\text{opt}(p) \leq p_1 + 0.086) \) which cannot be significantly improved for small \( p_1 \) [8]. Moreover, approaching 1, the upper and lower bounds on minimum average redundancy coding converge but never merge, whereas the minimum maximum redundancy bounds are identical for \( p_1 \geq 2/3 \).

**B. Minimized maximum pointwise redundancy codeword lengths**

In addition to finding redundancy bounds in terms of \( p_1 \), it is also often useful to find bounds on the behavior of \( l_1 \) in terms of \( p_1 \), as was done for optimal average redundancy in [32].

**Theorem 2:** Any optimal code for probability mass function \( p \), where \( p_1 \geq 2^{-\nu} \), must have \( l_1 \leq \nu \). This bound is tight, in the sense that, for \( p_1 < 2^{-\nu} \), one can always find a probability mass function with \( l_1 > \nu \). Conversely, if \( p_1 \leq 1/(2^\nu - 1) \), there is an optimal code with \( l_1 \geq \nu \), and this bound is also tight.

**Proof:** Suppose \( p_1 \geq 2^{-\nu} \) and \( l_1 \geq 1 + \nu \). Then \( R^*_\text{opt}(p) = R^*(l, p) \geq l_1 + \lg p_1 \geq 1 \), contradicting the unit-sized bounds of (8). Thus \( l_1 \leq \nu \).
For tightness of the bound, suppose \( p_1 \in (2^{-\nu - 1}, 2^{-\nu}) \) and consider \( n = 2^\nu + 1 \) and
\[
p = \left( \frac{p_1, 2^{-\nu - 1}, \ldots, 2^{-\nu - 1}, 2^{-\nu} - p_1}{n - 2} \right).
\]
If \( l_1 \leq \nu \), then, by the Kraft inequality, one of \( l_2 \) through \( l_{n-1} \) must exceed \( \nu \). However, this contradicts the unit-sized bounds of (8). For \( p_1 = 2^{-\nu - 1} \), a uniform distribution results in \( l_1 = \nu + 1 \). Thus, since these two results hold for any \( \nu \), this extends to all \( p_1 < 2^{-\nu - 1} \), and this bound is tight.

Suppose \( p_1 \leq 1/(2^\nu - 1) \) and consider an optimal length distribution with \( l_1 < \nu \). Consider the weights of the nodes of the corresponding code tree at level \( l_1 \). One of these weights is \( p_1 \), while the rest are known to sum to a number no less than \( 1 - p_1 \). Thus there is one node of at least weight
\[
\frac{1 - p_1}{2^l - 1} \geq \frac{1 - p_1}{2^l - 1 - 2^l - 1} \nu
\]
and thus, taking the logarithm and adding \( l_1 \) to the right-hand side,
\[
R^*(l, p) \geq \nu - 1 + \log \frac{1 - p_1}{2^\nu - 1 - 1}.
\]
Note that \( l_1 + 1 + \log p_1 \leq \nu + \log p_1 \leq \nu - 1 + \log(1 - p_1)/2^\nu - 1 \)), a direct consequence of \( p_1 \leq 1/(2^\nu - 1) \). Thus, if we replace this code with one for which \( l_1 = \nu \), the code is still optimal. The tightness of the bound is easily seen by applying Lemma 1 to distributions of the form
\[
p = \left( \frac{1 - p_1, 1 - p_2, \ldots, 1 - p_1}{2^\nu - 2} \right)
\]
for \( p_1 \in (1/(2^\nu - 1), 1/2^\nu - 1) \). This results in \( l_1 = \nu - 1 \) and thus \( R^*_\nu(p) = \nu + \log(1 - p_1) - \log(2^\nu - 2) \), which no code with \( l_1 > \nu - 1 \) could achieve.

In particular, if \( p_1 \geq 0.5 \), \( l_1 = 1 \), while if \( p_1 \leq 1/3 \), there is an optimal code with \( l_1 > 1 \).

C. \( d^\text{th} \) exponential redundancy bounds

We now briefly address the \( d^\text{th} \) exponential redundancy problem. Recall that this is the minimization of (3),
\[
R^d(p, l) = \frac{1}{d} \log \sum_{i \in X} p_i 2^{d(l_i + \log p_i)}.
\]
A straightforward application of Lyapunov’s inequality for moments yields \( R^d(p, l) \leq R^d(p, l) \) for \( d^\prime \leq d \), which, taking limits to 0 and \( \infty \), results in
\[
0 \leq \tilde{R}(p, l) \leq R^d(p, l) \leq R^*(p, l) < 1
\]
for any valid \( l, p \), and \( d > 0 \), resulting in an extension of (8),
\[
0 \leq \tilde{R}_\nu(p) \leq R^d_\nu(p) \leq R^*_\nu(p) < 1
\]
where \( R^*_\nu(p) \) is the optimal \( d^\text{th} \) exponential redundancy, an improvement on the bounds found in [17]. This leads directly to:

Corollary 1: The upper bounds of Theorem 1 are upper bounds for \( R^*_\nu(p) \) with any \( d \), while the lower bounds of average redundancy (Huffman) coding [6] are lower bounds for \( R^*_\nu(p) \) with \( d > 0 \). These lower bounds are
\[
\tilde{R}_\nu(p) \geq \xi - (1 - p_1) \log(2^\xi - 1) - H(p_1) \quad (10)
\]
where
\[
\xi = \left[ \log \frac{1 - 2^\frac{1}{2} - \frac{d}{d-1}}{1 - 2^\frac{1}{2} - \frac{d}{d-1}} \right]
\]
for \( p_1 \in (0, 1) \) and
\[
H(x) \triangleq -x \log x - (1 - x) \log(1 - x).
\]
This result is illustrated in Fig. 2 showing an improvement on the original unit bounds for values of \( p_1 \) other than (negative integer) powers of two.

III. BOUNDS ON EXPONENTIAL-AVERAGE PROBLEMS

A. Previously known exponential-average bounds

While the average, maximum, and \( d^\text{th} \) average redundancy problems yield performance bounds in terms of \( p_1 \) alone, here we simply seek to find any bounds
on $L_\alpha(p, l)$ in terms of $p_1$ and an appropriate entropy measure (introduced below).

Note that $a \leq 0.5$ is a trivial case, always solved by a finite unary code,

$$c^\alpha(n) \triangleq (0, 10, 110, \ldots, 1^{n-1}0, 1^{n-1}).$$

This can be seen by applying the exponential version of the Huffman algorithm; at each step, the combined weight will be the lowest weight of the reduced problem, being strictly less than the higher of the two combined weights, thus leading to a unary code.

For $a > 0.5$, there is a relationship between this problem and Rényi entropy. Rényi entropy [33] is defined as

$$H_\alpha(p) \triangleq \frac{1}{1-\alpha} \log \sum_{i=1}^n p_i^\alpha$$

for $\alpha > 0$, $\alpha \neq 1$. It is often defined for $\alpha \in \{0, 1, \infty\}$ via limits, that is,

$$H_0(p) \triangleq \lim_{\alpha \to 0^+} H_\alpha(p) = \log \|p\|$$

(the logarithm of the cardinality of $p$),

$$H_1(p) \triangleq \lim_{\alpha \to 1^-} H_\alpha(p) = -\sum_{i=1}^n p_i \log p_i$$

(the Shannon entropy of $p$), and

$$H_\infty(p) \triangleq \lim_{\alpha \to \infty} H_\alpha(p) = -\log p_1$$

(the min-entropy).

Campbell first proposed exponential utility functions for coding in [21], [22]. He observed the simple lower bound for $a > 0.5$ in [22]; the simple upper bound has subsequently shown, e.g., in [34, p. 156] and [26]. These bounds are similar to the minimum average redundancy bounds. In this case, however, the bounds involve Rényi’s entropy, not Shannon’s.

Defining

$$\alpha(a) \triangleq \frac{1}{\log 2a} = \frac{1}{1 + \log a} \quad \text{and}$$

$$L^\alpha_{\text{opt}}(p) \triangleq \min_{l \in L_n} L_\alpha(p, l)$$

the unit-sized bounds for $a > 0.5$, $a \neq 1$ are

$$0 \leq L^\alpha_{\text{opt}}(p) - H_\alpha(a) < 1. \quad (13)$$

In the next subsection we show how this bound follows from a result introduced there.

As an example of these bounds, consider the probability distribution implied by Benford’s law [35], [36]:

$$p_i = \log_{10}(i + 1) - \log_{10}(i), \ i = 1, 2, \ldots 9 \quad (14)$$

that is,

$$p \approx (0.30, 0.17, 0.12, 0.10, 0.08, 0.07, 0.06, 0.05, 0.05).$$

At $a = 0.6$, for example, $H_\alpha(a)(p) = 2.259\ldots$, so the optimal code cost is between 2.259 and 3.260. In the application given in [27] with (7), this corresponds to an optimal solution with probability of success (codeword transmission) between 0.189 and 0.316. Running the algorithm, the optimal lengths are $l = (1, 2, 3, 4, 5, 6, 7, 8, 8)$, resulting in cost $2.382\ldots$ (probability of success 0.296\ldots). At $a = 2$, $H_\alpha(a)(p) = 3.026\ldots$, so the optimal code cost is between 3.026 and 4.027, while the algorithm yields an optimal code with $l = (2, 3, 3, 3, 4, 4, 4, 4, 4)$, resulting in cost $3.099\ldots$.

Note that the optimal cost in both cases is quite close to entropy, indicating that better upper bounds might be possible. In looking for better bounds, recall first that — as with the exponential Huffman algorithm — (13) applies for both $a \in (0.5, 1)$ and $a > 1$. Improved bounds on the optimal solution for the $a > 1$ case are given in [26], but not in closed form, while closed-form bounds for a related objective are given in [28]. However, the proof for those bounds is incorrect in that it uses the assumption that we will always have an exponential-average-optimal $l_1$ equal to 1 if $p_1 \geq 0.4$. We shortly disprove this assumption for $a > 1$, showing the need for modified entropy bounds. Before this, we derive bounds based on the results of the prior section.

### B. Better exponential-average bounds

Because any exponential-average minimization can be transformed into a $d^\text{th}$ minimization problem, we can apply Corollary [10] for the exponential-average problem with $a > 1$ and a similar result where $a \in (0.5, 1)$: Given an exponential-average minimization problem with $p$ and $a$, if we define $\tilde{\alpha} \triangleq \alpha(a) = 1/(1 + \log a)$ and

$$\hat{p}_i \triangleq \frac{p_i^\tilde{\alpha}}{\sum_{j=1}^n p_j^\tilde{\alpha}} = \frac{p_i^\tilde{\alpha}}{2(1-\tilde{\alpha})H_{\tilde{\alpha}}(p)}$$

we have

$$R_{\text{exp}}^\alpha(\hat{p}, l) = \frac{1}{\log a} \log \sum_{i=1}^n p_i^{1+d \alpha(i)}$$

$$= \log_{\alpha(a)} \sum_{i=1}^n p_i \alpha(i) - \log_{\alpha(a)} \left( \sum_{i=1}^n p_i^\tilde{\alpha} \right)^\tilde{\alpha}$$

$$= L_\alpha(l, p) - H_{\tilde{\alpha}}(p)$$

(15)

where $H_{\tilde{\alpha}}(p)$ is Rényi entropy, as in (12). This transformation — shown previously in [26] — provides a reduction of exponential-average minimization to $d^\text{th}$
into the above inequality. Similarly, for \( a \leq 1 \), we have

\[
\hat{\alpha} \left( p_1^2 \alpha^2 H_\alpha(p) \right) \leq L_\alpha^{opt}(p) - H_\alpha(p) \leq \omega^* \left( p_1^2 \alpha^2 H_\alpha(p) \right)
\]

Similarly, for \( a \in (0.5, 1) \), we have

\[
L_\alpha^{opt}(p) \leq H_\alpha(p) + \hat{\alpha} \left( p_1^2 \alpha^2 H_\alpha(p) \right)
\]

**Proof:** The \( a > 1 \) case is a direct result of Corollary 1 and equation (15). The \( a < 1 \) case is similar in nature: Lyapunov’s inequality for moments yields \( R^d(p, l) \leq \hat{R}(p, l) \) for \( d < 0 \) for all \( l \) and thus \( R_\alpha^{opt}(p) \leq \hat{R}_\alpha^{opt}(p) \leq \hat{\omega}(p_1) \). Equation (15) turns this into the above inequality.

Recall the example of Benford’s distribution in (14) for \( a = 2 \). In this case, the bounds improve from [3.026…, 4.026…] to [3.039…, 3.910…] using the \( \omega^* \) from Theorem 1 and \( \hat{\omega} \) from [6] as given in (10) here. For \( a = 0.6 \), the bounds on cost are reduced from [2.259…, 3.259…] to [2.259…, 2.783…] using \( \hat{\omega} \) as given as (10) in [3];

\[
\hat{R}_\alpha^{opt}(p) \leq 2 - H(p_1) - p_1
\]

where recall from (11) that \( H(x) = -x \log x - (1 - x) \log(1 - x) \).

Although the bounds derived from Huffman coding are close for \( a \approx 1 \) (the most common case), these are likely not tight bounds; we introduce another bound for \( a < 1 \) after deriving a certain condition in the next section.

**C. Exponential-average shortest codeword length**

Techniques for finding Huffman coding bounds do not always translate readily to exponential generalizations because Rényi entropy’s very definition [33] involves a relaxation of a property used in finding bounds such as Gallager’s entropy bounds [3], namely

\[
H_1[p_1, (1 - t)p_1, p_2, \ldots, p_n] = H_1[p_1, p_2, \ldots, p_n] + p_1 H_1(t, 1 - t)
\]

for Shannon entropy \( H_1 \) and \( t \in [0, 1] \). This fails to hold for Rényi entropy. The penalty function \( L_\alpha \) differs from the usual measure of expectation in an analogous fashion, and we cannot know the weight of a given subtree in the optimal code (merged item in the coding procedure) simply by knowing the sum probability of the items included. However, we can find improved bounds for the exponential problem when we know that \( l_1 = 1 \); the question then becomes when we can know this given only \( a \) and \( p_1 \):

**Theorem 3:** There exists an optimal code with \( l_1 = 1 \) for \( a \) and \( p \) if either \( a \leq 0.5 \) or both \( a \in (0.5, 1] \) and \( p_1 \geq a/(2a + 3) \). Conversely, given \( a \in (0.5, 1] \) and \( p_1 < 2a/(2a + 3) \), there exists a \( p \) such that any code with \( l_1 = 1 \) is suboptimal. Likewise, given \( a > 1 \) and \( p_1 < 1 \), there exists a \( p \) such that any code with \( l_1 = 1 \) is suboptimal.

**Proof:** Recall that the exponential Huffman algorithm combines the items with the smallest weights, \( w' \) and \( w'' \), yielding a new item of weight \( w = aw' + aw'' \), and this process is repeated on the new set of weights, the tree thus being constructed up from the leaves to the root. This process makes it clear that, as mentioned, the finite unary code (with \( l_1 = 1 \)) is optimal for all \( a \leq 0.5 \). This leaves the two nontrivial cases.

1) \( a \in (0.5, 1] \): This is a generalization of [4] and is only slightly more complex to prove. Consider the coding step at which item 1 gets combined with other items; we wish to prove that this is the last step. At the beginning of this step the (possibly merged) items left

Fig. 3. Minimum \( p_1 \) sufficient for the existence of an optimal \( l_1 \) not exceeding 1.
to combine are \{1\}, \(S'_2\), \(S'_3\), \(s'_k\), where we use \(S'_k\) to denote both a (possibly merged) item of weight \(w(S'_k)\) and the set of (individual) items combined in item \(S'_k\). Since \{1\} is combined in this step, all but one \(S'_k\) has at least weight \(p_1\). Note too that all weights \(w(S'_k)\) must be less than or equal to the sums of probabilities \(\sum_{i \in S'_j} p_i\).

Then

\[
\frac{2a(k-1)}{2a+3} \leq (k-1)p_1
\]

\[
< p_1 + \sum_{k=2}^n w(S'_k)
\]

\[
\leq p_1 + \sum_{j=2}^n \sum_{i \in S'_j} p_i
= \sum_{i=1}^n p_i
= 1
\]

which, since \(a > 0.5\), means that \(k < 5\). Thus, because \(n < 4\) always has an optimal code with \(l_1 = 1\), we can consider the steps in exponential Huffman coding at and after which four items remain, one of which is item \{1\} and the others of which are \(S'_4\), \(S'_3\), and \(S'_2\). We show that, if \(p_1 \geq 2/(2a+3)\), these items are combined as shown in Fig. 4.

We assume without loss of generality that weights \(w(S'_4)\), \(w(S'_3)\), and \(w(S'_2)\) are in descending order. From

\[
w(S'_4) + w(S'_3) + w(S'_2) \leq \sum_{i=2}^n p_i
\]

\[
\leq \frac{2a+3}{2a+3}
\]

\[
w(S'_4) \geq w(S'_3),
\]

and

\[
w(S'_2) \geq w(S'_4)
\]

it follows that \(w(S'_3) + w(S'_2) \leq 2/(2a+3)\). Consider set \(S'_2\). If its cardinality is 1, then \(w(S'_2) \leq p_1\), so the next step merges the least two weighted items \(S'_4\) and \(S'_3\). Since the merged item has weight at most \(2a/(2a+3)\), this item can then be combined with \(S'_2\), then \{1\}, so that \(l_1 = 1\). If \(S'_2\) is a merged item, let us call the two items (sets) that merged to form it \(S'_3\) and \(S'_4\), indicated by the dashed nodes in Fig. 4. Because these were combined prior to this step,

\[
w(S'_2) + w(S'_3) \leq w(S'_3) + w(S'_4)
\]

so

\[
w(S'_4) \leq aw(S'_3) + aw(S'_4) \leq \frac{2a}{2a+3}.
\]

Thus \(w(S'_4)\), and by extension \(w(S'_3)\) and \(w(S'_1)\), are at most \(p_1\). So \(S'_4\) and \(S'_3\) can be combined and this merged item can be combined with \(S'_2\), then \{1\}, again resulting in \(l_1 = 1\).

This can be shown to be tight by noting that, for any \(\epsilon \in (0, (2a-1)/(8a+12))\),

\[
p^{(\epsilon)} = \left(\frac{2a}{2a+3} - 3\epsilon, \frac{2a}{2a+3} + \epsilon, \frac{2a}{2a+3} + \epsilon, \frac{2a}{2a+3} + \epsilon\right)
\]

achieves optimality only with length vector \(l = (2, 2, 2, 2)\). The result extends to smaller \(p_1\).

2) \(a > 1\): Given \(a > 1\) and \(p_1 < 1\), we wish to show that a probability distribution always exists such that there is no optimal code with \(l_1 = 1\). We first show that, for the exponential penalties as for the traditional Huffman penalty, every optimal \(l\) can be obtained via the (modified) Huffman procedure. That is, if multiple length vectors are optimal, each optimal length vector can be obtained by the Huffman procedure as long as ties are broken in a certain manner.

Clearly the optimal code is obtained for \(n = 2\). Let \(n'\) be the smallest \(n\) for which there is an \(l\) that is optimal but cannot be obtained via the algorithm. Since \(l\) is optimal, consider the two smallest probabilities, \(p_{n'}\) and \(p_{n'-1}\). In this optimal code, two items having these probabilities (although not necessarily items \(n'-1\) and \(n'\)) must have the longest codewords and must have the same codeword lengths. Were the latter not the case, the codeword of the more probable item could be exchanged with one of a less probable item, resulting in a better code. Were the former not the case, the longest codeword length could be decreased by one without violating the Kraft inequality, resulting in a better code. Either way, the code would no longer be optimal. So clearly we can find two such smallest items with largest codewords (by breaking any ties properly), which, without loss of generality, can be considered siblings. This means that the problem can be reduced to one of size \(n' - 1\) via the exponential Huffman algorithm. But since all problems of size \(n' - 1\) can be solved via the algorithm, this is a contradiction, and the Huffman algorithm can thus find any optimal code.

Note that this is not true for minimizing maximum pointwise redundancy, as the exchange argument no
This page contains a technical discussion about entropy bounds in information theory. The text refers to previous theorems and derivations, and introduces new results. The mathematical expressions involve logarithms, summations, and probabilities, typical in such contexts. The discussion includes theorems, corollaries, and derivations, all related to the study of entropy in coding theorems. The notation and symbols are common in information theory, specifically for the study of entropy bounds and their applications.

D. Exponential-average bounds for $a \in (0.5, 1)$, $p_1 \geq 2a/(2a + 3)$

Entropy bounds derived from Theorem 3, although rather complicated, are, in a certain sense, tight.

**Corollary 3:** For $l_1 = 1$ (and thus for all $p_1 \geq 2a/(2a + 3)$) and $a \in (0.5, 1)$, the following holds, where $\tilde{\alpha} = \alpha(a) \triangleq 1/(1 + \log a)$:

$$\sum_{i=1}^{n} p_i a_i^\alpha > a^{2} \left( a^{\tilde{\alpha} H_{\tilde{\alpha}}(p)} - p_1^{\tilde{\alpha}} \right)^{1/2} + a p_1$$

or, equivalently,

$$L_a(p) < 1 + \log_a \left( a \left[ a^{\tilde{\alpha} H_{\tilde{\alpha}}(p)} - p_1^{\tilde{\alpha}} \right]^{1/2} + p_1 \right)$$

and

$$\sum_{i=1}^{n} p_i a_i^\alpha \leq a^{2} \left( a^{\tilde{\alpha} H_{\tilde{\alpha}}(p)} - p_1^{\tilde{\alpha}} \right)^{1/2} + a p_1$$

or, equivalently,

$$L_a(p) \geq 1 + \log_a \left( a \left[ a^{\tilde{\alpha} H_{\tilde{\alpha}}(p)} - p_1^{\tilde{\alpha}} \right]^{1/2} + p_1 \right)$$

Note that this upper bound is tight for $p_1 \geq 0.5$, in the sense that, given values for $a$ and $p_1$, we find $p$ to make the inequality arbitrarily close. Probability distribution $p = (p_1, 1 - p_1 + \epsilon, \epsilon)$ does this for small $\epsilon$, while the lower bound is tight (in the same sense) over its full range, since $p = (p_1, (1 - p_1)/4, (1 - p_1)/4, (1 - p_1)/4, (1 - p_1)/4)$ achieves it (with a zero-redundancy subtree of the weights excluding $p_1$).

**Proof:** We apply the simple unit-sized coding bounds (13) for the subtree that includes all items but item $\{1\}$. Let $B = \{2, 3, \ldots, n\}$ with $p_i^B = P[i \mid i \in B] = p_i/(1 - p_1)$ and with Rényi $\alpha$-entropy

$$H_{\tilde{\alpha}}(p^B) = \frac{1}{\tilde{\alpha} - 1} \log \sum_{i=2}^{n} \left( \frac{p_i}{1 - p_1} \right)^{\tilde{\alpha}}$$

$H_{\tilde{\alpha}}(p^B)$ is related to the entropy of the original source $p$ by

$$2^{(1-\tilde{\alpha}) H_{\tilde{\alpha}}(p)} = (1 - p_1)^{\tilde{\alpha} 2^{1-\tilde{\alpha}} H_{\tilde{\alpha}}(p^B)} + p_1$$

or, equivalently, since $2^{1-\tilde{\alpha}} = a^{\tilde{\alpha}}$,

$$a^{H_{\tilde{\alpha}}(p^B)} = \frac{1}{1 - p_1} \left[ a^{\tilde{\alpha} H_{\tilde{\alpha}}(p)} - p_1^{\tilde{\alpha}} \right]^{1/2}$$

(16)

Applying (13) to subtree $B$, we have

$$a^{H_{\tilde{\alpha}}(p^B)} \geq \frac{1}{(1 - p_1) a} \sum_{i=2}^{n} p_i a_i^\alpha > a^{H_{\tilde{\alpha}}(p^B) + 1}$$

The bounds for $\sum_i p_i a_i^\alpha$ are obtained by substituting (16), multiplying both sides by $(1 - p_1) a$, and adding the contribution of item $\{1\}$, $a p_1$. 

A Benford distribution (14) for $a = 0.6$ yields $H_{\alpha(a)}(p) \approx 2.260$. Since $p_1 > 2a/(2a + 3)$, $l_1$ is 1 and the probability of success is between 0.250 and...
0.298; that is, $L_a^{\text{opt}} \in [2.372 \ldots, 2.707 \ldots])$. Recall that the bounds found using [15] were $P[\text{success}] \in (0.241, 0.316)$ and $L_a^{\text{opt}} \in [2.259 \ldots, 2.783 \ldots]$, an improvement on the unit-sized bounds, but not as good as those of Corollary [5]. The optimal code $l = (1, 2, 3, 4, 5, 6, 7, 8, 8)$ yields a probability of success of $0.296$ ($L_a^{\text{opt}} = 2.382 \ldots$).

Note that these arguments fail for $a > 1$ due to the lack of sufficient conditions for $l_1 = 1$. For $a < 1$, other cases likely have improved bounds that could be found by bounding $l_1$ — as with the use of lengths in [37] to come up with general bounds [7], [8] — but new bounds would each cover a more limited range of $p_1$ and be more complicated to state and to prove.

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