NOWHERE ZERO HARMONIC SPINORS
AND SELF-DUAL 2-FORMS

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Abstract. Let $M$ be a closed oriented 4-manifold, with Riemannian metric $g$, and a spin$^C$-structure induced by an almost-complex structure $\omega$. Each connection $A$ on the determinant line bundle induces a unique connection $\nabla^A$, and Dirac operator $D^A$ on spinor fields. Let $\sigma : W^+ \to \Lambda^+$ be the natural squaring map, taking self-dual (= positive) spinors to self-dual 2-forms.

In this paper, we characterize the self-dual 2-forms that are images of self-dual spinor fields through $\sigma$. They are those $\alpha$ for which (off zeros) $c_1(\alpha) = c_1(\omega)$, where $c_1(\alpha)$ is a suitably defined Chern class. We also obtain the formula: $\|\varphi\|^2 D^A \varphi = i(2d^* \sigma(\varphi) + \langle \nabla^A \varphi, i\varphi \rangle_R) \cdot \varphi$.

Using these, we establish a bijective correspondence between: 

\{ Kähler forms $\alpha$ compatible with a metric scalar-multiple of $g$, and with $c_1(\alpha) = c_1(\omega)$ \} and 
\{ gauge classes of pairs $(\varphi, A)$, with $\nabla^A \varphi = 0$ \}, as well as a bijective correspondence between: 
\{ symplectic forms $\alpha$ compatible with a metric conformal to $g$, and with $c_1(\alpha) = c_1(\omega)$ \} and 
\{ gauge classes of pairs $(\varphi, A)$, with $D^A \varphi = 0$, and $\langle \nabla^A \varphi, i\varphi \rangle_R = 0$, and $\varphi$ nowhere-zero \}.

Through [Wit94], the celebrated Seiberg–Witten monopole equations were introduced into 4-dimensional topology, thus bringing spin$^C$ geometry to the frontline of mathematical research.

For a closed oriented 4-manifold endowed with a fixed Riemannian metric $g$, the ingredients of the Seiberg–Witten equations are: a spin$^C$-structure with spinor bundles $W^\pm$, a connection $A$ on their determinant line bundle $L$, an associated Dirac operator $D^A : \Gamma(W^+) \to \Gamma(W^-)$, and a quadratic map $\sigma : W^+ \to \Lambda^+$ taking self-dual (= positive\(^1\)) spinors to self-dual 2-forms. Solutions to the Seiberg–Witten equations are gauge classes of pairs $(\varphi, A)$, made from a spinor field $\varphi \in \Gamma(W^+)$ and a connection $A$ on $L$. One of the Seiberg–Witten equations is $D^A \varphi = 0$, while the second equates $\sigma(\varphi)$ and part of the curvature of $A$.

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\item \textsuperscript{1}Instead of “self-dual spinor”, a more customary terminology would be “positive spinor”. We prefer the former, which is used in the classical paper [AHS78] and seems better suited to the peculiarities of dimension 4.
\end{itemize}
While this paper is not concerned with Seiberg–Witten theory, its focus is on the same creatures mentioned above. The paper grew out of an attempt to project the spinorial world down to the 4-manifold. The device mediating this descent was to be the squaring map $\sigma$.

In this paper we study the images $\sigma(\varphi)$ of gauge classes of pairs $(\varphi, A)$ satisfying the equation $D^A \varphi = 0$. First, we determine which self-dual 2-forms are images of self-dual spinor fields through $\sigma$. Then we determine when these 2-forms are Kähler or symplectic. The main results are summarized in Table 1, and proved as Theorems 4.1 and 4.3. (Note that we restrict ourselves to the special case of spin$^C$-structures that come from almost-complex structures.) An excellent general reference for related material (genuine spin-structures) is [LM89]. For the spin$^C$ point of view (and Seiberg–Witten theory), see [Mor96].

Table 1. There is a bijection, established through $\alpha = \sigma(\varphi)$, between the left and the right sides (when $H^2(M; \mathbb{Z})$ has no 2-torsion).

| Kähler forms $\alpha$ compatible with a metric multiple of $g$ and with $c_1(\alpha) = c_1(M)$. | Gauge classes of pairs $(\varphi, A)$ with $\nabla^A \varphi = 0$ and $\varphi$ not constantly-zero. |
| --- | --- |
| Symplectic forms $\alpha$ compatible with a metric conformal to $g$ and with $c_1(\alpha) = c_1(M)$. | Gauge classes of pairs $(\varphi, A)$ with $D^A \varphi = 0$ and $\langle \nabla^A \varphi, i\varphi \rangle_R = 0$ and $\varphi$ nowhere-zero. |
| Self-dual nowhere-zero 2-forms $\alpha$ with $c_1(\alpha) = c_1(M)$ | Gauge classes of pairs $(\varphi, A)$ with $D^A \varphi = 0$, and $\varphi$ nowhere-zero. |

Section 1 presents almost-complex structures as constant-length sections of $\Lambda^+$, then relates Chern classes of almost-complex structures with intersection theory in $\Lambda^+$. A main result is:

(Thm. 1.2) $c_1(\alpha) + c_1(\beta) \overset{\text{def}}{=} 2 \cdot \alpha \cap \beta|_M$

where on the right side we took intersections in the sphere bundle of $\Lambda^+$ and projected them as oriented surfaces in $M$.

Section 2 first reviews spin$^C$-structures associated to an almost-complex structure $\omega$, and defines the squaring map $\sigma : W^+ \rightarrow \Lambda^+$. Then it attacks the lifting problem for $\sigma$, that is, the problem of which self-dual 2-forms $\alpha$ are images $\sigma(\varphi)$ of spinor fields. The main result is that $\alpha$ can be lifted if and only if, off the zeros of $\alpha$, the class $c_1(\alpha)$ coincides with the Chern class $c_1(\omega)$ of the almost-complex structure $\omega$ that generated the spin$^C$-structure (Thms. 2.6 and 2.9).

Section 3 discusses connections on $W^\pm$ and defines the Dirac operator $D^A$. Then it works at separating the contributions of the Levi-Cività connection and of $A$ in building the Dirac operator. We obtain the formula

(Thm. 3.15) $\|\varphi\|^2 D^A \varphi = i(2 d^* \sigma(\varphi) + \langle \nabla^A \varphi, i\varphi \rangle_R) \cdot \varphi$
Section 4 gathers together all the results obtained along the way, and proves the statements from Table 1.

While this work was being completed, the paper [3.10] was electronically published. It contains two partial results in the direction of our work (see Remarks 3.10 and 3.19 for a discussion). Developed independently, our paper is strengthening their results.

1. Almost-Complex Structures

Let $M$ be a closed oriented 4-manifold, endowed with a fixed Riemannian metric $g$, and its Levi-Civita connection $\nabla$. The same notation $\nabla$ will denote the connections induced on the tensor bundles of $M$. Also, using the metric, we will systematically identify $T_M$ and $T_M^*$. Throughout the paper, an “$x$” will denote a generic point of $M$.

Let $\Lambda^+$ denote the 3-plane bundle of self-dual 2-forms on $M$, and $\Gamma(\Lambda^+)$ denote the space of self-dual 2-forms (sections in $\Lambda^+$). Also, let $\mathfrak{s}\Lambda^+$ denote the bundle of 2-spheres of radius $\sqrt[2]{2}$ in $\Lambda^+$. (The latter is also known as the twistor bundle of $M$.) At a point $x$, each $\alpha \in \mathfrak{s}\Lambda^+_x$ can be written as $\alpha = e_1 \wedge e_2 + e_3 \wedge e_4$ for some orthonormal orienting basis $\{e_k\}$ in $T_M|_x$. Therefore it defines an almost-complex structure $J_\alpha$ on $T_M|x$ by $J_\alpha(e_1) = e_2$ and $J_\alpha(e_3) = e_4$. The almost-complex structure $J_\alpha$ is orthogonal for $g|_x$, i.e. $g(J_\alpha v, J_\alpha w) = g(v, w)$. Conversely, given an orthogonal almost-complex structure $J$ on $T_M|x$, there is an associated 2-form $\alpha_J \in \mathfrak{s}\Lambda^+_x$ given by $\alpha_J|_x = e_1 \wedge e_2 + e_3 \wedge e_4$ for any orthonormal orienting basis $\{e_k\}$ with $J(e_1) = e_2$ and $J(e_3) = e_4$. (In general, a 2-form $\alpha \in \mathfrak{s}\Lambda^+_x$ and an orthogonal almost-complex structure $J$ on $T_M|x$ are associated through $\alpha(v, w) = g(Jv, w)$. From now on, all almost-complex structures considered will be orthogonal.)

In conclusion, the bundle $\mathfrak{s}\Lambda^+$ can be seen as the bundle of orthogonal almost-complex structures on fibers of $T_M$. Global almost-complex structures on $M$ can be identified with global sections of $\mathfrak{s}\Lambda^+$.

An almost-complex structure $\alpha$ on $M$ makes $T_M$ a complex bundle. The complex-line bundle $K_\alpha^* = \text{det}_C(T_M, \alpha)$ is called the anti-canonical bundle of $\alpha$. Its Chern class $c_1(K_\alpha^*) = c_1(T_M, \alpha) \in H^2(M; \mathbb{Z})$ is called the Chern class of the almost-complex structure $\alpha$ and will be denoted by $c_1(\alpha)$. The bundle $K_\alpha^*$ can also be recovered as the oriented complement of $R\alpha$ in $\Lambda^+$, because of the oriented splitting

$$\Lambda^+ = R\alpha \oplus K_\alpha^*$$

Each nowhere-zero self-dual 2-form $\beta \in \Gamma(\Lambda^+)$ has a unique projection $\mathfrak{s}(\beta)$ in $\Gamma(\mathfrak{s}\Lambda^+)$ and thus defines an almost-complex structure with a well-defined Chern class $c_1(\beta)$.

In what follows we will study the relations between almost-complex structures, as sections in $\mathfrak{s}\Lambda^+$, and their Chern classes.

Let $\alpha, \beta \in \Gamma(\mathfrak{s}\Lambda^+)$ be two sections, seen as oriented 4-submanifolds in $\mathfrak{s}\Lambda^+$. Make one transverse to the other. Their intersection $\alpha \cap \beta$ will be an oriented surface in $\mathfrak{s}\Lambda^+$ that projects nicely to an oriented surface in $M$. The latter will be denoted by $\alpha \cap \beta|_M$. 

Looking at the signs of the intersections, one can see that we have: \( \alpha \cap \beta = \beta \cap \alpha \), and also: \( \alpha \cap \beta|_M = -((-\alpha) \cap (-\beta)|_M) \), where the minus represents reversal of orientation.

Let \( \alpha, \beta \in \Gamma(\mathfrak{s}\Lambda^+) \) be two almost-complex structures, with anti-canonical bundles \( K^*_\alpha \) and \( K^*_\beta \). Make \( \beta \) transverse to \( \alpha \) in \( \mathfrak{s}\Lambda^+ \), then project it on \( K^*_\alpha \) through the composition \( \mathfrak{s}\Lambda^+ \subset \Lambda^+ = \mathbb{R}\alpha \oplus K^*_\alpha \xrightarrow{\text{proj}} K^*_\alpha \) (see Figure 1). Call \( \beta_K \) the resulting section of \( K^*_\alpha \). Since \( \beta \) is transverse to \( \alpha \) in \( \mathfrak{s}\Lambda^+ \), it follows that the projection \( \beta_K \) is transverse to the zero section of \( K^*_\alpha \). The intersection points of \( \beta_K \) with the zero section in \( K^*_\alpha \) correspond to the intersection points of \( \beta \) with \( \alpha \) or with \( -\alpha \) in \( \mathfrak{s}\Lambda^+ \). More precisely, after matching signs:

\[
\beta_K \cap 0|_M = \beta \cap \alpha|_M - \beta \cap (-\alpha)|_M
\]

But the zero locus of a generic section in \( K^*_\alpha \) is the Poincaré-dual of the Chern class \( c_1(\alpha) \). We obtained:

**Lemma 1.1.** For all sections \( \alpha, \beta \) in \( \mathfrak{s}\Lambda^+ \), we have:

\[
c_1(\alpha) \equiv \beta \cap \alpha|_M - \beta \cap (-\alpha)|_M
\]

In particular: \( c_1(\alpha) \equiv \alpha \cap \alpha|_M \).

But \( \beta \cap (-\alpha)|_M = -((-\beta) \cap \alpha|_M) \), so we can write as well:

\[
c_1(\alpha) \equiv \alpha \cap \beta|_M + \alpha \cap (-\beta)|_M
\]

On the other hand, reversing the rôles of \( \alpha \) and \( \beta \) in the Lemma:

\[
c_1(\beta) \equiv \alpha \cap \beta|_M - \alpha \cap (-\beta)|_M
\]

Adding, we get:

**Theorem 1.2.** For all sections \( \alpha, \beta \in \Gamma(\mathfrak{s}\Lambda^+) \), we have

\[
c_1(\alpha) + c_1(\beta) \equiv 2 \cdot \alpha \cap \beta|_M
\]

A consequence is that a cocycle for the cohomology class \( c_1(\alpha) + c_1(\beta) \) can be written as: \( s^2 \mapsto 2(\alpha \cap \beta|_M) \cap s^2 \), for all singular 2-simplices \( s^2 \) in \( M \) transverse to the surface \( \alpha \cap \beta|_M \).

Without trouble, we can restate this as:
Corollary 1.3. A cocycle for \( c_1(\alpha) + c_1(\beta) \) can be written as:
\[
s^2 \mapsto 2\alpha(s^2) \cap \beta \quad \text{in} \ \mathfrak{s}\Lambda^+
\]
for all singular 2-simplices \( s^2 \) in \( M \) such that their image \( \alpha(s^2) \) is transverse to the 4-submanifold \( \beta \) in \( \mathfrak{s}\Lambda^+ \).

We conclude this section with a few definitions:

An almost-complex structure \( \alpha \in \Gamma(\mathfrak{s}\Lambda^+) \) is called a Kähler form compatible with the metric \( g \) if and only if \( \nabla \alpha = 0 \), where \( \nabla \) is the connection on \( \Lambda^+ \) induced from \( g \)'s Levi-Civită connection. Indeed, in that case \( J_\alpha \) is integrable, and \( (M, g, J_\alpha) \) is Kähler, with Kähler form \( \alpha \).

Let \( \alpha \) be a nowhere-zero self-dual 2-form with \( \nabla \alpha = 0 \). Then \( \alpha \) has constant length. Modifying the metric from \( g \) to \( g' \) by multiplying with a suitable constant, we can make \( \alpha \) of length \( \sqrt{2} \). Thus \( \alpha \) becomes a section in \( (\mathfrak{s}\Lambda^+)' \) and determines an almost-complex structure \( J'_\alpha \). Since the Levi-Civită connection stays the same, \( \nabla' = \nabla \), we conclude that \( (M, g', J'_\alpha) \) is Kähler. In conclusion:

Lemma 1.4. A nowhere-zero self-dual 2-form \( \alpha \in \Gamma(\Lambda^+) \) has \( \nabla \alpha = 0 \) if and only if \( \alpha \) is a Kähler form compatible with a metric scalar-multiple of \( g \).

A non-degenerate 2-form \( \alpha \) with \( d\alpha = 0 \) is called symplectic. If a symplectic form \( \alpha \) is self-dual and of constant length \( \sqrt{2} \), then \( \alpha \) is also called compatible with the metric \( g \). (And \( (M, g, \alpha) \) is called an almost-Kähler manifold.)

Lemma 1.5. A nowhere-zero self-dual 2-form \( \alpha \in \Gamma(\Lambda^+) \) has \( d\alpha = 0 \) if and only if \( \alpha \) is a symplectic form compatible with a metric conformal to \( g \).

Proof. A first thing to notice is that “nowhere-zero and self-dual” insures “non-degenerate”. When suitably rescaling the metric, one always preserves the bundle \( \Lambda^+ \), but can make \( ||\alpha|| \equiv \sqrt{2} \). \( \square \)

One goal of this paper is to investigate how certain spinor fields determine such Kähler or symplectic structures on \( M \).

2. The Lifting Problem

\( \textbf{Spin}\mathbb{C}-\text{structures.} \) Since our manifold is oriented Riemannian, the bundle \( T_M \) has structure group \( SO(4) = SU(2)^+ \times SU(2)^- / \pm 1 \). The complex spin group can be described as \( Spin^\mathbb{C}(4) = \mathbb{S}^1 \times SU(2)^+ \times SU(2)^- / \pm 1 \), and projects canonically to \( SO(4) \). The manifold \( M \) is said to admit a spin\(^{\mathbb{C}}\)-structure if and only if the \( SO(4) \)-cocycle of \( T_M \) can be lifted to a \( Spin^\mathbb{C}(4) \)-cocycle. On 4-manifolds, spin\(^{\mathbb{C}}\)-structures always exist.

Suppose given an almost-complex structure \( \omega \in \Gamma(\mathfrak{s}\Lambda^+) \). Then the \( SO(4) \)-cocycle of \( T_M \) can be reduced to an \( U(2) \)-cocycle. More, since \( U(2) = \mathbb{S}^1 \times SU(2)^+ / \pm 1 \subset SU(2)^+ \times SU(2)^- / \pm 1 = SO(4) \), and \( Spin^\mathbb{C}(4) = \mathbb{S}^1 \times SU(2)^+ \times SU(2)^- / \pm 1 \), there is an obvious canonical map \( U(2) \rightarrow Spin^\mathbb{C}(4) \) covering the inclusion \( U(2) \subset SO(4) \). Then one can use this map to lift the \( U(2) \)-cocycle of \( T_M \) to a \( Spin^\mathbb{C}(4) \)-cocycle. Therefore, each almost-complex structure induces a canonical spin\(^{\mathbb{C}}\)-structure on \( M \).

Choose some spin\(^{\mathbb{C}}\)-structure on \( M \), i.e. a \( Spin^\mathbb{C}(4) \)-cocycle for \( T_M \). Then the natural projections \( Spin^\mathbb{C}(4) \rightarrow U(2)^\pm = \mathbb{S}^1 \times SU(2)^\pm / \pm 1 \) induce \( U(2) \)-cocycles.
for hermitian complex-plane bundles \( W^\pm \), called the **bundles of self-dual/anti-self-dual spinors**. The complex-line bundle \( L = \det_\mathbb{C} W^\pm \) is called the **determinant bundle** of the spin\(^\mathbb{C} \)-structure. If the spin\(^\mathbb{C} \)-structure comes from an almost-complex structure \( \omega \), then its determinant bundle \( L \) is exactly the anti-canonical bundle \( K^*_\mathcal{O} \) of the almost-complex structure.

A spin\(^\mathbb{C} \)-structure also determines a **Clifford multiplication** map \( T_M \times \mathcal{W}^+ \to \mathcal{W}^- \), denoted by \( (v, \varphi) \mapsto v \cdot \varphi \). It is modelled on quaternionic multiplication. In particular, if \( v \cdot \varphi = 0 \), then \( v = 0 \) or \( \varphi = 0 \). Clifford multiplication induces an action \( \Lambda^+ \times \mathcal{W}^+ \to \mathcal{W}^+ \). When the spin\(^\mathbb{C} \)-structure is associated to an almost-complex structure \( \omega \), the Clifford action of the 2-form \( \omega \) splits \( \mathcal{W}^+ \) into its \( \pm 2i \)-eigenbundles as

\[
\mathcal{W}^+ = \mathbb{C} \oplus K^*_\mathcal{O}^+.
\]

On the anti-self-dual side, we have that \( \mathcal{W}^- \approx (T_M, \omega) \) (as complex bundles).

Let \( \text{End}_\mathbb{C}(\mathcal{W}^+) \) denote the space of **traceless** \( \mathbb{C} \)-endomorphisms of \( \mathcal{W}^+ \). The Clifford action \( \Lambda^+ \times \mathcal{W}^+ \to \mathcal{W}^+ \) identifies \( \text{End}_\mathbb{C}(\mathcal{W}^+) \approx \Lambda^+ \otimes \mathbb{C} \). For every \( \varphi \in \mathcal{W}^+|_x \), we have that \( \varphi \otimes \varphi^* \) is an endomorphism of \( \mathcal{W}^+ \) with traceless part

\[
\varphi \otimes \varphi^* - \frac{1}{4} \|\varphi\|^2 \text{id}.
\]

The latter corresponds to a purely imaginary 2-form \( i\sigma(\varphi) \in i\Lambda^+|_x \). This defines the **squaring map**

\[
\sigma : \mathcal{W}^+ \to \Lambda^+
\]

It is uniquely characterized by:

\[
\sigma(\varphi) \cdot \varphi = -i\frac{\|\varphi\|^2}{2} \varphi.
\]

If the spin\(^\mathbb{C} \)-structure comes from an almost-complex structure \( \omega \), then:

\[
\varphi \in \mathbb{C} \subset \mathcal{W}^+ \text{ of length } 2 \implies \omega \cdot \varphi = -2i \cdot \varphi \implies \sigma(\varphi) = +\omega
\]

\[
\varphi \in K^* \subset \mathcal{W}^+ \text{ of length } 2 \implies \omega \cdot \varphi = +2i \cdot \varphi \implies \sigma(\varphi) = -\omega
\]

The squaring map has the property that \( \sigma(e^i\varphi) = \sigma(\varphi) \). Thus, if we restrict \( \sigma \) to a sphere in a fiber \( \mathcal{W}^+|_x \), then it behaves like a Hopf map \( S^3 \to S^2 \). And since \( \|\sigma(\varphi)\| = \frac{1}{\sqrt{\lambda}} \|\varphi\|^2 \), one can further look at \( \sigma \) fiber-wise as a sort of squared-cone on the Hopf map. More precisely, if \( h : S^3 \to S^2 \) is the Hopf map between the sphere (of radius 2) in \( \mathcal{W}^+|_x \) and the sphere (of radius \( \sqrt{2} \)) in \( \Lambda^+|_x \), then

\[
\sigma(a\varphi) = a^2 \cdot h(\varphi)
\]

for all \( a \in \mathbb{R} \) and \( \varphi \in S^3 \).

**Remark 2.1.** A generic spin\(^\mathbb{C} \)-structure has the following property: \( \mathcal{W}^+ \) admits a nowhere-zero section if and only if the spin\(^\mathbb{C} \)-structure comes from some almost-complex structure on \( M \). More, in this case there is a **distinguished spinor field** \( \mathfrak{z} \in \Gamma(\mathcal{W}^+) \) of constant-length, which is uniquely characterized by the property that, for any \( v \in T_M \), we have \( v \cdot \mathfrak{z} = v \) (equality seen through \( \mathcal{W}^- \approx T_M \)). In particular, in the splitting \( \mathcal{W}^+ = \mathbb{C} \oplus K^* \) we have \( \mathbb{C} = \mathbb{C} \cdot \mathfrak{z} \), and also \( \sigma(\mathfrak{z}) = \frac{1}{4} \omega \).

From now on, we **fix** an almost-complex structure \( \omega \in \Gamma(s\Lambda^+) \) and its associated spin\(^\mathbb{C} \)-structure, with spinor bundles \( \mathcal{W}^\pm \), and determinant bundle \( K^* = K^*_\mathcal{O} \).

The squaring map is surjective fiber-wise, but not surjective on sections: There are section in \( \Lambda^+ \) that are not images through \( \sigma \) of sections of \( \mathcal{W}^+ \). This **lifting problem** for \( \sigma \) will be investigated in the rest of this section. The topological obstruction will turn out to be \( c_1(\alpha) - c_1(\omega) \).
The setting. For every vector bundle $E \to B$, we will denote by $sE \to B$ a sphere bundle (of some chosen radius) of $E$. Since the lifting problem is topological in nature, we will blissfully ignore for a while all positive scalars involved in the expressions for the squaring map $\sigma$ (such as $\frac{\|x\|^2}{2\sqrt{2}}$, $\sqrt{2}$, etc.), and consider $\sigma$ simply as a fiber-wise Hopf map from the 3-sphere bundle $sW^+$ to the 2-sphere bundle $s\Lambda^+$.

Let $\alpha : M \to s\Lambda^+$ be a section. We try to lift $\alpha$ to a section $\varphi : M \to sW^+$ such that $\sigma(\varphi) = \alpha$.

First note that $\sigma : sW^+ \to s\Lambda^+$ is a circle-bundle over $s\Lambda^+$, made from patching Hopf fibrations $S^3 \to S^2$. Therefore it has a well-defined Chern class $c_1(\sigma) = c_1(sW^+ \to s\Lambda^+) \in H^2(s\Lambda^+; \mathbb{Z})$.

Pull-back the bundle $\sigma : sW^+ \to s\Lambda^+$ through $\alpha : M \to s\Lambda^+$ to get the circle-bundle $\alpha^*(sW^+) \to M$, fitting in the diagram:

$$
\begin{array}{ccc}
\alpha^*(sW^+) & \longrightarrow & sW^+ \\
\downarrow & & \downarrow \sigma \\
M & \longrightarrow & s\Lambda^+
\end{array}
$$

A lift of $\alpha : M \to s\Lambda^+$ to a section $\varphi : M \to sW^+$ is equivalent to a section in the circle-bundle $\alpha^*(sW^+) \to M$. But the latter bundle has a section if and only if it is trivial. That means: if and only if its Chern class is zero. But

$$
c_1(\alpha^*(sW^+) \to M) = \alpha^*[c_1(\sigma)]
$$

where the $\alpha^*$ on the right is the induced morphism $\alpha^* : H^2(s\Lambda^+; \mathbb{Z}) \to H^2(M; \mathbb{Z})$. Therefore:

**Lemma 2.2.** A nowhere-zero self-dual 2-form $\alpha$ can be lifted to a spinor field if and only if $\alpha^*[c_1(\sigma)] = 0$ in $H^2(M; \mathbb{Z})$.

**Remark 2.3.** For example, take the two distinguished 2-forms $\omega$ and $-\omega$. Since $W^+ = \mathbb{C} \oplus K^*$ and $\sigma(s\mathbb{C}) = \omega$ and $\sigma(sK^*) = -\omega$, we must have the pull-back diagrams:

$$
\begin{array}{ccc}
s\mathbb{C} & \longrightarrow & sW^+ \\
\downarrow & & \downarrow \sigma \\
M & \longrightarrow & s\Lambda^+
\end{array}
\quad
\begin{array}{ccc}
sK^* & \longrightarrow & sW^+ \\
\downarrow & & \downarrow \sigma \\
M & \longrightarrow & s\Lambda^+
\end{array}
$$

and therefore: $\omega^*(c_1(\sigma)) = 0$ and $(-\omega)^*(c_1(\sigma)) = c_1(K^*)$. And indeed, $\omega$ lifts immediately to the distinguished spinor $\mathfrak{1}$ (see 2.4), while in general $-\omega$ cannot be lifted at all.

In what follows, we will first identify $c_1(\sigma)$ as the Poincaré-dual of $-\omega$ in $s\Lambda^+$, and then show that $2\alpha^*[c_1(\sigma)] = c_1(\alpha) - c_1(\omega)$.

**A section in $\sigma$.** Think of $\sigma : sW^+ \to s\Lambda^+$ as a bundle made up from patching Hopf fibrations over each 2-sphere fiber in $s\Lambda^+$ (see Figure 2.2). Since $\Lambda^+ = \mathbb{R} \omega \oplus K^*$, we can imagine that, as we move among the 2-spheres of $s\Lambda^+$, the poles $\omega$ and $-\omega$ stay still, while the equator moves around and twists. Since $W^+ = \mathbb{C} \oplus K^*$ (with $\mathbb{C} = \mathbb{C} \cdot \mathfrak{1}$), we can also imagine that, as we move among the 3-spheres of $sW^+$, the point determined by the spinor field $\mathfrak{1}$ (see 2.4) stays still as well. The
circle $S^1 \cdot \beta_x = s\mathbb{C}|_x$ of $sW^+|_x$ is sent through $\sigma$ to the $\omega_x$-pole in $s\Lambda^+|_x$. The circle $sK^*|_x$ is sent through $\sigma$ to $-\omega_x$.

![Diagram](image)

**Figure 2. A piece of $\sigma$ as a Hopf fibration**

To determine the Chern class of the circle-bundle $\sigma : sW^+ \to s\Lambda^+$, we will build a section $s$ of its associated complex-line bundle $L \to s\Lambda^+$, by patching sections over each 2-sphere of $s\Lambda^+$.

Lift the pole $\omega_x$ of the 2-sphere $s\Lambda^+|_x$ to $\beta_x$, and thus define $s(\omega) = \beta$. Further, since $\beta_x$ and the circle $\sigma^{-1}(-\omega_x)$ over $-\omega_x$ are well-determined in each 3-sphere, we can choose uniformly a disk $D_x$ in $sW^+|_x$, centered at $\beta_x$ and bounded by the circle $\sigma^{-1}(-\omega_x)$.

Over each point $\lambda$ in the 2-sphere $s\Lambda^+|_x$ lies a circle $\sigma^{-1}\lambda$ in the 3-sphere $sW^+|_x$. If we chose the disk $D_x$ nicely enough, the circle $\sigma^{-1}\lambda$ cuts $D_x$ in exactly one point. Define $s(\lambda)$ to be that intersection point: $s(\lambda) = \sigma^{-1}\lambda \cap D_x$.

The only exception is when $\lambda = -\omega_x$. The circle over $-\omega_x$ lies entirely in $D_x$ as its boundary. Define $s(-\omega_x) = 0$. To make the resulting $s$ continuous, modify the lengths of all other $s(\lambda)$’s, so that, as $\lambda$ approaches $-\omega_x$ in $s\Lambda^+|_x$, $s(\lambda)$ will approach 0.

This defines $s$ in one Hopf patch of $\sigma$. We repeat the same procedure over the whole $s\Lambda^+$. With some care, in the end we have a well-defined continuous section $s : s\Lambda^+ \to L$ that has zeros exactly along the image of $-\omega : M \to s\Lambda^+$. That means:

**Lemma 2.4.** The Chern class $c_1(\sigma) \in H^2(s\Lambda^+; \mathbb{Z})$ of the bundle $\sigma : sW^+ \to s\Lambda^+$ is the Poincaré-dual of $-\omega_*[M] \in H_4(s\Lambda^+; \mathbb{Z})$. Or, in short:

$$c_1(\sigma) \overset{\text{def}}{=} -\omega$$

**and lifting.** Since $c_1(\sigma) \overset{\text{def}}{=} -\omega$, that means that $c_1(\sigma) \in H^2(s\Lambda^+; \mathbb{Z})$ can be represented by the cocycle: $s^2 \mapsto s^2 \cap (-\omega)$, for all singular 2-simplices $s^2$ in $s\Lambda^+$ that are transverse to $-\omega$. 
For a section $\alpha : M \rightarrow s\Lambda^+$, the class $\alpha^*[c_1(\sigma)] \in H^2(M; \mathbb{Z})$ is then represented by the cocycle:

$$s^2 \mapsto \alpha(s^2) \cap (-\omega) \quad \text{in } s\Lambda^+$$

for all singular 2-simplices $s^2$ in $M$ whose images through $\alpha$ are transverse to the 4-submanifold $-\omega$ of $s\Lambda^+$.

But remember from Corollary 2.7 that this is exactly half a cocycle that represents the class

$$c_1(\alpha) + c_1(-\omega)$$

Noticing that $c_1(-\omega) = -c_1(\omega)$, we have:

**Lemma 2.5.** For every section $\alpha : M \rightarrow s\Lambda^+$, we have

$$2\alpha^*[c_1(\sigma)] = c_1(\alpha) - c_1(\omega)$$

Remember now that a section $\alpha$ in the sphere bundle $s\Lambda^+$ can be lifted to $s\mathcal{W}^+$ if and only if $\alpha^*[c_1(\sigma)] = 0$ (Lemma 2.2). That implies $2\alpha^*[c_1(\sigma)] = 0$, and so $c_1(\alpha) = c_1(\omega)$. Therefore:

**Theorem 2.6.** Let $\alpha \in \Gamma(s\Lambda^+)$. If $\alpha$ can be lifted to a spinor field, then $c_1(\alpha) = c_1(\omega)$. If $H^2(M; \mathbb{Z})$ has no 2-torsion, then the converse is also true.

**Warning 2.7.** For the simplicity of later statements, from now on we will assume that $H^2(M; \mathbb{Z})$ has no 2-torsion. Under this assumption, the statement above would be restated as: “$\alpha$ can be lifted if and only if $c_1(\alpha) = c_1(\omega)$”.

Passing from section in $s\Lambda^+$ to nowhere-zero sections in $\Lambda^+$ is immediate. Each nowhere-zero section $\alpha : M \rightarrow \Lambda^+$ has a well-defined projection $s(\alpha) : M \rightarrow s\Lambda^+$. Then $\alpha$ can be lifted if and only if $s(\alpha)$ can be lifted, using a formula such as $\sigma(a\varphi) = a^2 \sigma(\varphi)$ to adjust the lengths. And Chern classes are, by definition, unchanged by $\alpha \mapsto s(\alpha)$.

**Corollary 2.8.** A nowhere-zero self-dual 2-form $\alpha$ lifts to a spinor field if and only if $c_1(\alpha) = c_1(\omega)$.

If $\alpha : M \rightarrow \Lambda^+$ has zeros, repeat the entire discussion on $M \setminus \{\text{zeros}\}$. We get that $\alpha|_{\text{off zeros}}$ can be lifted to some $\varphi|_{\text{off zeros}}$ if and only if $c_1(\alpha|_{\text{off zeros}}) = c_1(\omega)|_{\text{off zeros}}$. But $\alpha|_{\text{off zeros}}$ can be lifted if and only if $\alpha$ can be lifted: Indeed, since $\sigma(0) = 0$ and $||\sigma(\varphi)|| = \frac{1}{2\sqrt{2}} ||\varphi||^2$, the partial lift $\varphi|_{\text{off zeros}}$ can be completed with zeros to get a lift of the whole $\alpha$. We proved:

**Corollary 2.9.** A self-dual 2-form $\alpha$ can be lifted to a continuous spinor field if and only if $c_1(\alpha|_{\text{off zeros}}) = c_1(\omega)|_{\text{off zeros}}$ in $H^2(M \setminus \{\text{zeros}\}; \mathbb{Z})$ (assuming $H^2(M \setminus \{\text{zeros}\}; \mathbb{Z})$ has no 2-torsion).

Note that this procedure creates only a *continuous* spinor field, which might be not smooth at zeros.

### 3. The Splitting Formula

**Connections and Dirac operators.** Pick some spin$\mathbb{C}$-structure with spinor bundles $\mathcal{W}^\pm$ and with determinant bundle $L$. Then the Levi-Civita connection $\nabla$ on $T_M$ and a choice of unitary connection $A$ on the determinant line bundle $L$ will determine a unique connection $\nabla^A$ on the spinor bundles $\mathcal{W}^\pm$. 
Let $\text{Conn}(L)$ denote the space of unitary connections on $L$. It is an affine space with model $\text{i} \Gamma(T^*_M)$: any two connections differ by an imaginary 1-form. If $A = A_0 + 2i\theta$, then the corresponding spinorial connections have $\nabla^A = \nabla^{A_0} + i\theta$.

All spinorial connections $\nabla^A$ on $\mathcal{W}^\pm$ are compatible with the Hermitian metrics:

$$\partial_X \langle \varphi, \psi \rangle = \langle \nabla^A_X \varphi, \psi \rangle + \langle \varphi, \nabla^A_X \psi \rangle.$$  

The connections $\nabla^A$ are also compatible with Clifford multiplication:

$$\nabla^A (v \cdot \varphi) = (\nabla v) \cdot \varphi + v \cdot (\nabla^A \varphi)$$  

for every $v \in \Gamma(T_M)$ and $\varphi \in \Gamma(\mathcal{W}^\pm)$. A similar relation holds for Clifford multiplication with 2-forms.

The **Dirac operator** $D^A$ is defined by the composition:

$$D^A : \Gamma(\mathcal{W}^+) \xrightarrow{\nabla^A} \Gamma(\mathcal{W}^+ \otimes T^*_M) \xrightarrow{\cdot} \Gamma(\mathcal{W}^-)$$

or, equivalently, by the formula:

$$D^A \varphi = \sum e_k \cdot \nabla_{e_k}^A \varphi$$

for any local orienting orthonormal frame $\{e_k\}$ in $T_M$. A spinor field $\varphi \in \Gamma(\mathcal{W}^+)$ such that $D^A \varphi = 0$ is called **harmonic** for $A$.

Since, for functions $f : M \to \mathbb{C}$, we have $D^A (f \cdot \varphi) = df \cdot \varphi + f : D^A \varphi$, it follows that the symbol of $D^A$ is the Clifford multiplication. Thus $D^A$ is elliptic. It has the following **strong unique continuation property** (see [Ato97], or [BBW93] ch. 8):

**Theorem 3.1.** If $\varphi \in \Gamma(\mathcal{W}^+)$ has a zero of infinite order, and $D^A \varphi = 0$ for some $A \in \text{Conn}(L)$, then $\varphi = 0$. Consequently, if $\varphi \in \Gamma(\mathcal{W}^+)$ is zero on an open set, and $D^A \varphi = 0$ for some $A$, then $\varphi = 0$. In particular, if $D^A \varphi = 0$, then either $\varphi = 0$, or $\text{supp} \varphi = M$.

**Remark 3.2.** This does not mean that the zeros are generic (in this case, isolated). In general, the zero set is **countably 2-rectifiable** (countable union of Lipschitz images of pieces of a plane), so in particular its Hausdorff dimension could be as big as 2. See [Bar97] for a proof.

The Dirac operator depends on the connection $A \in \text{Conn}(L)$ as follows:

$$D^{A+2i\theta} \varphi = D^A \varphi + i\theta \cdot \varphi$$

**Remark 3.3.** The **Seiberg–Witten equations** are: $D^A \varphi = 0$ and $\sigma(\varphi) = F_A^+$, where $iF_A$ is the curvature 2-form of $A$. See [Don96] or [Mor96] for an introduction to this theory.

The gauge group $\mathcal{G}(L)$ of the unitary bundle $L$ is the group of its automorphisms. It can be described as $\mathcal{G}(L) = \{g : M \to \mathbb{S}^1\}$, acting on $L$ by obvious multiplication. Its action on $L$ induces an action on $\text{Conn}(L)$ by $g \cdot A = A + 2gdg^{-1}$. The gauge group $\mathcal{G}(L)$ also acts on $\mathcal{W}^\pm$ by the obvious $(g \cdot \varphi)|_x = g(x) \cdot \varphi|_x$.

Since $\sigma(\varphi) = \sigma(\psi)$ if and only if $\varphi = g \cdot \psi$ for some $g \in \mathcal{G}(L)$, we conclude that the map $\sigma : \Gamma(\mathcal{W}^+) \to \Gamma(\Lambda^+)$ factors through the orbit space $\Gamma(\mathcal{W}^+) / \mathcal{G}(L)$.

The Dirac operator, seen as a map $\Gamma(\mathcal{W}^+) \times \text{Conn}(L) \to \Gamma(\mathcal{W}^-)$, with $(\varphi, A) \mapsto D^A \varphi$, is $\mathcal{G}(L)$-invariant as well: $D^{g \cdot A} (g \cdot \varphi) = g \cdot D^A \varphi$. Therefore, the space of pairs $(\varphi, A)$ with $D^A \varphi = 0$ is a gauge-invariant subspace of $\Gamma(\mathcal{W}^+) \times \text{Conn}(L)$.

The starting point for proving the results in Table II are the following simple Lemmata. First, a uniqueness result:
Lemma 3.4. If \( \varphi \in \Gamma(W^+) \) admits a connection \( A \in \text{Conn}(L) \) such that \( D^A \varphi = 0 \), then \( A \) is unique, unless \( \varphi = 0 \).

Proof. Assume \( D^A \varphi = 0 \). If \( \varphi \) is not constantly-zero, then, by the unique continuation property, we must have \( \text{supp} \varphi = M \). Suppose \( D^{A+2i\theta} \varphi = 0 \). Since \( D^{A+2i\theta} \varphi = D^A \varphi + i\theta \cdot \varphi \), we have \( \theta \cdot \varphi = 0 \). But a Clifford product is zero if and only if one of the factors is zero. Thus, since \( \text{supp} \varphi = M \), we deduce that \( \theta = 0 \) almost-everywhere. By continuity, \( \theta \equiv 0 \). \( \square \)

And an existence result:

Lemma 3.5. If \( \varphi \in \Gamma(W^+) \) is nowhere-zero, then there is a connection \( A \in \text{Conn}(L) \) such that \( D^A \varphi = 0 \).

Proof. Choose a random connection \( A \). For any other connection \( A + 2i\theta \), we have \( D^{A+2i\theta} \varphi = D^A \varphi + i\theta \cdot \varphi \). Since \( \varphi \) is nowhere-zero, and since Clifford multiplication is modelled on quaternionic multiplication, the equation \( i\theta \cdot \varphi = -D^A \varphi \) can always be solved for a \( \theta \in \Gamma(T^*_M) \), and then \( D^{A+2i\theta} \varphi = 0 \). \( \square \)

Therefore, each spinor field \( \varphi \) admits at most one connection \( A \) so that \( D^A \varphi = 0 \).

We can now easily prove the third line of Table 1.

Lemma 3.6. \( \alpha = \sigma(\varphi) \) is a bijective correspondence between:

\[
\begin{align*}
\text{Self-dual 2-forms } \alpha & \quad \text{Gauge classes of pairs } (\varphi, A) \smallskip \\
\text{nowhere-zero} & \quad \text{with } D^A \varphi = 0 \\
\text{and with } c_1(\alpha) = c_1(\omega) & \quad \text{and } \varphi \text{ nowhere-zero.}
\end{align*}
\]

Proof. As we assumed that \( H^2(M; \mathbb{Z}) \) has no 2-torsion, the existence of a \( \varphi \) with \( \sigma(\varphi) = \alpha \) is equivalent to \( c_1(\alpha) = c_1(\omega) \). Lemmata 3.4 and 3.5 ensure that there is a unique \( A \) such that \( D^A \varphi = 0 \). And everything is gauge-invariant. \( \square \)

Splitting the connection. In what follows, we obtain formulae that completely separate the contributions of the Levi-Civita connection \( \nabla \) and of \( A \) to the spinorial connection \( \nabla^A \) and to the Dirac operator \( D^A \). These results hold for general spinor structures, not necessarily associated with an almost-complex structure.

The bundle \( W^+ \) has a natural Hermitian metric, denoted by \( \langle \cdot, \cdot \rangle_\mathbb{C} \). The real part of this metric is a real metric \( \langle \cdot, \cdot \rangle_{\mathbb{R}} = \text{Re} \langle \cdot, \cdot \rangle_\mathbb{C} \). In what follows, \( \langle \cdot, \cdot \rangle \) will always denote this real metric, and the symbol \( \perp \) will denote orthogonality with respect to this real metric (and so: \( v \perp iv \)). Also, all projections will be taken with respect to this real metric.

If we change the connection on \( K^* = \det \mathbb{C} W^+ \) from \( A \) to \( A + 2i\theta \), then the induced connection on \( W^+ \) changes from \( \nabla^A \) to \( \nabla^A + i\theta \), and so \( \nabla^A \varphi \) changes to \( \nabla^A X \varphi + i\theta(X) \cdot \varphi \). Therefore \( \nabla^A \varphi \) changes only in its component along \( i\varphi \), while the rest remains fixed, no matter what connection \( A \) we choose. And indeed:

Proposition 3.7. For every spinor structure and connection \( A \) we have:

\[
\|\varphi\|^2 \cdot \text{proj}_{i\varphi^\perp} (\nabla^A \varphi) = i(\nabla \sigma(\varphi)) \cdot \varphi
\]

where \( \nabla \) is the connection on \( \Lambda^+ \) induced by the Levi-Civita connection.

Proof. Remember that

\[
\sigma(\varphi) \cdot \varphi = -i\frac{1}{2} \|\varphi\|^2 \cdot \varphi
\]
Apply $\nabla^A$: we get $\nabla^A(\sigma(\varphi) \cdot \varphi) = -i\frac{1}{2} \nabla^A(\varphi \cdot \varphi)$. Then use the compatibility of the connections with the Clifford multiplication:

$$\nabla^A(\sigma(\varphi)) \cdot \varphi + \sigma(\varphi) \cdot (\nabla^A \varphi) = -i\frac{1}{2} \Delta \varphi^2 \cdot \varphi - i\frac{1}{2} \varphi \cdot \varphi \cdot \nabla \varphi$$

and rearrange terms:

$$(\sigma(\varphi) + i\frac{1}{2} \varphi^2) \cdot \nabla^A \varphi = -i\frac{1}{2} \Delta \varphi^2 \cdot \varphi - (\nabla \sigma(\varphi)) \cdot \varphi$$

Since $\varphi \mapsto \sigma(\varphi) \cdot \varphi$ is traceless on $\mathcal{W}^+$, it must act on each $\varphi \in \mathbb{C} \varphi$ by: $\varphi \mapsto -i\frac{1}{2} \varphi^2 \cdot \varphi$, while on each $\varphi \in (\mathbb{C} \varphi)^\perp$ it must act by: $\varphi \mapsto +i\frac{1}{2} \varphi^2 \cdot \varphi$. Therefore $\sigma(\varphi) + i\frac{1}{2} \varphi^2$ acts on $\mathcal{W}^+$ by killing $\mathbb{C} \varphi$, and by multiplying $(\mathbb{C} \varphi)^\perp$ with $i \varphi^2$. Therefore on $\mathcal{W}^+$ we have $\sigma(\varphi) + i\frac{1}{2} \varphi^2 = i \varphi^2 \cdot \text{proj} \ (\mathbb{C} \varphi)^\perp$, and thus:

$$i \varphi^2 \cdot \text{proj} \ (\mathbb{C} \varphi)^\perp (\nabla^A \varphi) = -i\frac{1}{2} \varphi^2 \cdot \varphi - (\nabla \sigma(\varphi)) \cdot \varphi$$

On the other hand, $d \varphi^2 = d \langle \varphi, \varphi \rangle_R = 2 \langle \nabla^A \varphi, \varphi \rangle$, and therefore $d \varphi^2 \cdot \varphi = 2 \langle \nabla^A \varphi, \varphi \rangle \varphi = 2 \varphi \cdot \varphi^2 \langle \nabla^A \varphi, \frac{1}{\sqrt{2}} \varphi \rangle = 2 \varphi \cdot \varphi \cdot \text{proj} \ (\mathbb{C} \varphi)^\perp \nabla \varphi$. Consequently:

$$i \varphi^2 \cdot \text{proj} \ (\mathbb{C} \varphi)^\perp (\nabla^A \varphi) = -i \varphi^2 \cdot \text{proj} \ (\mathbb{C} \varphi)^\perp (\nabla^A \varphi) - (\nabla \sigma(\varphi)) \cdot \varphi$$

and rearranging terms:

$$i \varphi^2 \cdot \text{proj} \ (\mathbb{C} \varphi)^\perp (\nabla^A \varphi) = -i \varphi^2 \cdot \text{proj} \ (\mathbb{C} \varphi)^\perp (\nabla^A \varphi) - (\nabla \sigma(\varphi)) \cdot \varphi$$

Since $\text{proj} \ (\mathbb{C} \varphi)^\perp + \text{proj} \varphi = \text{proj} \varphi$, the above is equivalent to the statement of the proposition.

It should be clear that the formula above (and those following) can be used carelessly only off the zeros of $\varphi$.

**Corollary 3.8.** For every spin$^C$-structure and every unitary connection $A$, we have: If $\nabla^A \varphi = 0$, then $\nabla X \sigma(\varphi) = 0$.

**Proof.** We need only remark that, since $\partial_X \|\varphi\|^2 = \partial_X \langle \varphi, \varphi \rangle_R = 2 \langle \nabla_X \varphi, \varphi \rangle$, if $\nabla^A_X \varphi = 0$, then $\partial_X \|\varphi\|^2 = 0$, and so the length of $\varphi$ is constant in the $X$-direction. If the length is constantly 0, then $\sigma(\varphi) = 0$ and the statement is trivial. Otherwise, use Proposition 3.7 above.

Remembering Lemma 1.4, we get:

**Corollary 3.9.** If a non-trivial spinor field $\varphi \in \Gamma(\mathcal{W}^+)$ admits a connection $A$ such that $\nabla^A \varphi = 0$, then $\sigma(\varphi)$ is a Kähler form compatible with a metric scalar-Kähler.

**Remark 3.10.** This last result was obtained independently in [Blair]. Note also that this is only the spin$^C$ version of a similar statement holding for genuine spin-structures: If $M$ admits a spin-structure with a parallel spinor field, then $M$ is Kähler. The parallel spinor field reduces the holonomy of $M$ to SU(2), and so $M$ is also Ricci-flat, see [Hit74] (or 9.18 in [LM89, p. 344]). Because of the extra $S^1$-freedom of spin$^C$-structures, in our case the holonomy reduces only to $U(2)$.

The part of $\nabla^A \varphi$ that was left undetermined by Proposition 3.7 is its component along $i\varphi$. But since $\|\varphi\|^2 \text{proj}_{i\varphi} \nabla^A \varphi = \langle \nabla^A \varphi, i\varphi \rangle_R$, we get:

**Corollary 3.11.** $\|\varphi\|^2 \nabla^A_X \varphi = i \langle \nabla_X \sigma(\varphi) \rangle \cdot \varphi + \langle \nabla^X_A \varphi, i\varphi \rangle_R \cdot i\varphi$
The first term on the right side depends only on the Levi-Civitá connection on $M$. The second depends only on the choice of $A$. By changing $A$ to $A + 2i\theta$, the term $\langle \nabla^A_X \varphi, i\varphi \rangle$ will change to $\langle \nabla^A_X \varphi, i\varphi \rangle + \|\varphi\|^2 \theta(X)$. For a fixed nowhere-zero $\varphi$, suitable choices of $A$ can make $\langle \nabla^A_X \varphi, i\varphi \rangle$ anything one might want. In particular — zero:

**Lemma 3.12.** For every spin$^C$-structure and every nowhere-zero spinor field $\varphi \in \Gamma(\mathcal{W}^+)$, there is a unique connection $A$ such that the induced connection $\nabla^A$ on $\mathcal{W}^+$ has

$$\langle \nabla^A_X \varphi, i\varphi \rangle_{\mathbb{R}} = 0$$

and so: $\|\varphi\|^2 \nabla^A \varphi = i(\nabla (\sigma)) \cdot \varphi$.

**Proof.** The existence part was argued. Uniqueness: If both $A$ and $A + 2i\theta$ satisfy the above, then $i\theta(X) \cdot \varphi = 0$, but since $\text{supp} \varphi = M$, we must have $\theta = 0$. $\square$

**Remark 3.13.** In §3.11, we keep $\varphi$ fixed and vary $A$. Then we can see that the condition $\langle \nabla^A_X \varphi, \varphi \rangle_{\mathbb{R}} = 0$ is equivalent to the covariant derivative $\nabla^A \varphi |_x$ being **minimal**. Thus, we could call a connection $A$ with $\langle \nabla^A_X \varphi, \varphi \rangle_{\mathbb{R}} = 0$ a **minimal connection** for $\varphi$.

**Splitting the Dirac operator.** Let $\{e_k\}$ be any orthonormal orienting local frame in $T_M$. Then:

$$\|\varphi\|^2 D^A \varphi = \sum e_k \cdot \|\varphi\|^2 \nabla^A_{e_k} \varphi = \sum e_k \cdot (i \nabla^A_{e_k} \sigma(\varphi) \cdot \varphi + \langle \nabla^A_{e_k} \varphi, i\varphi \rangle \cdot i\varphi)$$

$$= i \sum e_k \cdot (\nabla^A_{e_k} \sigma(\varphi) \cdot \varphi) + i \sum \langle \nabla^A_{e_k} \varphi, i\varphi \rangle e_k \cdot \varphi$$

But Clifford multiplication has the property that

$$v \cdot (\alpha \cdot \psi) = (v \wedge \alpha - v \lhd \alpha) \cdot \varphi$$

for every $v \in T_M = T^*_M$ and $\alpha \in \Lambda^* (M)$, where $\lhd$ is the interior product $(v \lhd \alpha)(X) = \alpha(v, X)$ (see [LM89] p. 25). Also, since the Levi-Civitá connection $\nabla$ is torsion-free, we have that

$$\sum e_k \wedge \nabla^A_{e_k} \alpha = d\alpha \quad \sum e_k \lhd \nabla^A_{e_k} \alpha = -d^*\alpha$$

where $d$ is exterior differentiation of forms, and $d^* = -\ast d\ast$ is the formal adjoint of $d$ with respect to the metric $g$ (see [LM89] p. 123). We obtain:

$$\|\varphi\|^2 D^A \varphi = i(\langle \varphi | (d + d\ast)\sigma(\varphi) + \langle \nabla^A_X \varphi, i\varphi \rangle \rangle \cdot \varphi$$

where we think of $\langle \nabla^A_X \varphi, i\varphi \rangle$ as the 1-form $X \mapsto \langle \nabla^A_X \varphi, i\varphi \rangle$.

Starting from the Clifford relation $-e_1 e_2 e_3 e_4 \cdot \varphi = \varphi$ for $\varphi \in \mathcal{W}^+$, it is easy to prove that:

**Lemma 3.14.** For every $\beta \in \Lambda^3 (M)$ and every $\varphi \in \mathcal{W}^+$, we have $\beta \cdot \varphi = -(\ast \beta) \cdot \varphi$. Therefore, for every $\alpha \in \Gamma(\Lambda^+) \text{ and } \varphi \in \Gamma(\mathcal{W}^+)$, we have $d\alpha \cdot \varphi = d^*\alpha \cdot \varphi$

Therefore:

**Theorem 3.15 (The splitting formula).**

For every spin$^C$-structure and every connection $A$, we have:

$$\|\varphi\|^2 D^A \varphi = i(2d^*\sigma(\varphi) + \langle \nabla^A_X \varphi, i\varphi \rangle) \cdot \varphi$$
Remark 3.16. This formula is most useful off the zeros of $\varphi$. Since the structure of the zeros is rather wild in general (see Remark 3.2), one encounters big difficulties when trying to obtain something useable at the zeros.

Some consequences of 3.15 are:

**Corollary 3.17.** If $\varphi \in \Gamma(W^+)$ is a spinor field, and $A$ is a connection such that $
abla A \varphi, i\varphi = 0$, then: \[ ||\varphi||^2 D A \varphi = 2 d^* \sigma(\varphi) \cdot \varphi. \] In particular, if $DA \varphi = 0$, then $\sigma(\varphi)$ is closed.

**Proof.** For the second statement, since $DA$ is elliptic, it has the unique continuation property. Thus, if $DA \varphi = 0$, then either $\varphi = 0$ and the conclusion is trivial, or $\text{supp} \varphi = M$. Then, off the zeros of $\varphi$, $d^* \sigma(\varphi) = 0$, and thus by continuity over all $M$. And $d^* = -*d$ on $\Gamma(L^+)$. \[ \square \]

Further, remembering Lemma 1.5:

**Corollary 3.18.** If a nowhere-zero spinor field $\varphi$ has a connection $A$ such that $DA \varphi = 0$ and $\langle \nabla A \varphi, \varphi \rangle_R = 0$, then $\sigma(\varphi)$ is symplectic.

**Remark 3.19.** A similar statement was proved independently in [BLPR01]. There it was proved that: If $DA \varphi = 0$ and $\langle \nabla A \varphi, \varphi \rangle_C = 0$, then $\sigma(\varphi)$ is symplectic.

Since the latter statement uses the complex inner product, it is more restrictive; notice that the $\varphi$'s satisfying the latter statement must have constant length, while those satisfying the Corollary above need not.

Remember from 2.4 the spinor field $\mathfrak{z}$. We have:

**Corollary 3.20.** Choose a spin$^C$-structure associated with some almost-complex structure $\omega$. Then for the distinguished spinor field $\mathfrak{z} \in \Gamma(W^+)$ and the unique connection $A$ with $\langle \nabla A \mathfrak{z}, i\mathfrak{z} \rangle = 0$, we have $DA \mathfrak{z} = 0$ if and only if $\omega$ is symplectic.

**Remark 3.21.** Since $\mathfrak{z}$ has constant length, the condition $\langle \nabla A \mathfrak{z}, i\mathfrak{z} \rangle = 0$ is equivalent to $\nabla A \mathfrak{z} \perp C \mathfrak{z}$. Thus the above is essentially the same as C. Taubes' Lemma 1 from [Taf93]. It is the easiest of the starting steps of the long investigation from [Taf00]. There, by using the (perturbed) Seiberg–Witten equations $DA \varphi = 0$ and $\sigma(\varphi) = F_A^+ - F_A^0 + \frac{1}{4} r \omega$, where $A_0$ has $\langle \nabla A_0 \mathfrak{z}, i\mathfrak{z} \rangle = 0$, and $r$ is a positive parameter, Taubes shows roughly that, in the symplectic case, Seiberg–Witten solutions correspond to holomorphic curves in $(M, \omega)$.

4. The Correspondence

In this section, we prove the results announced in Table 1. Remember from 2.7 that $H^2(M; \mathbb{Z})$ is assumed to have no 2-torsion. Proving is now just a matter of gathering what was spread around in the paper:

**Theorem 4.1.** $\alpha = \sigma(\varphi)$ is a bijective correspondence between:

| Kähler forms $\alpha$ compatible with a metric scalar-multiple of $g$ and with $c_1(\alpha) = c_1(\omega)$. | Gauge classes of pairs $(\varphi, A)$ with $\nabla A \varphi = 0$, and $\varphi$ not constantly zero. |

**Proof.** A Kähler form for a metric scalar multiple of $g$ means a nowhere-zero self-dual 2-form $\alpha$ with $\nabla \alpha = 0$ (Lemma 1.4). By Theorem 2.6 it can be lifted to...
some \( \varphi \) if and only if \( c_1(\alpha) = c_1(\omega) \). From left to right, suppose given such a Kähler form \( \alpha \). Lift it to a spinor field \( \varphi \) with \( \sigma(\varphi) = \alpha \). The connection \( A \) such that \( \langle \nabla^A \varphi, i \varphi \rangle \equiv 0 \) (Lemma 3.12) has \( \| \varphi \|^2 \nabla^A \varphi = i(\nabla \sigma(\varphi)) \cdot \varphi \), so \( \nabla^A \varphi = 0 \). And everything is gauge-invariant. From right to left, use Corollary 3.18. \( \square \)

**Remark 4.2.** The above statement can be widened to account for all Kähler forms compatible with a metric *conformal to* \( g \) (not just scalar-multiple of \( g \)). In one direction, if \( \varphi \) is a spinor field and \( \nabla^A \) a connection on \( \mathcal{W}^+ \) such that

\[
v \cdot \nabla^A w \varphi = w \cdot \nabla^A v \varphi
\]

for all \( v, w \in T_M \), then \( \sigma(\varphi) \) is a Kähler form compatible with a metric conformal to \( g \). But to go in the other direction and achieve a bijection similar to the one above, one must admit more connections than just the \( \nabla^A \)'s. More precisely, one must first embed \( \mathcal{W}^+ \) into a suitable larger bundle, and then accept certain unitary connections \( \nabla \) on it that do not necessarily preserve \( \mathcal{W}^+ \). In that setting, there is a bijection between: on one hand, Kähler forms \( \alpha \) compatible with a metric conformal to \( g \) and with \( c_1(\alpha) = c_1(\omega) \), and, on the other hand, gauge classes of pairs \( (\varphi, \nabla) \) (with \( \varphi \in \Gamma(\mathcal{W}^+) \) and \( \nabla \) an extended connection) such that

\[
v \cdot \nabla^A_w \varphi = w \cdot \nabla^A_v \varphi \quad \text{for all} \quad v, w \in T_M.
\]

The discussion of such a statement belongs to a different set of ideas, and thus will not be presented here.

**Theorem 4.3.** \( \alpha = \sigma(\varphi) \) is a bijective correspondence between:

- Symplectic forms \( \alpha \) compatible with a metric conformal to \( g \) and with \( c_1(\alpha) = c_1(\omega) \).
- Gauge classes of pairs \( (\varphi, A) \), with \( \mathcal{D}^A \varphi = 0 \) and \( \langle \nabla^A \varphi, i \varphi \rangle_R = 0 \) and \( \varphi \) nowhere-zero.

**Proof.** Again, \( \alpha \) is nowhere-zero and self-dual, so \( c_1(\alpha) = c_1(\omega) \) is equivalent to the existence of a \( \varphi \) with \( \sigma(\varphi) = \alpha \). From left to right, given \( \alpha \) with \( d \alpha = 0 \), lift it to some \( \varphi \). Choose \( A \) such that \( \langle \nabla^A \varphi, i \varphi \rangle \equiv 0 \). From 3.17, we have \( \| \varphi \|^2 \mathcal{D}^A \varphi = 2 d^* \sigma(\varphi) \cdot \varphi \), and thus \( \mathcal{D}^A \varphi = 0 \). And everything is gauge-invariant. From right to left, use Corollary 3.18. \( \square \)

As an amusing application of some methods from this paper, we briefly re-prove below a classic property of Betti numbers for Kähler manifolds. (We entertain the hope that our better peers might use techniques from this paper to also uncover some new properties of Kähler or symplectic 4-manifolds.)

**Corollary 4.4.** If \( M \) is Kähler, then \( b_2^+ \) must be odd and \( b_1 \) even.

**Proof.** Being nowhere-zero, the distinguished spinor field \( \mathfrak{z} \) allows us to identify \( \mathbb{R} \oplus \Lambda^+ \) with \( \mathcal{W}^+ \) (as real bundles) through the Clifford action \( \xi \mapsto \xi \cdot \mathfrak{z} \). Then one can compute

\[
\mathcal{D}^A(\xi \cdot \mathfrak{z}) = ((d + d^*)\xi) \cdot \mathfrak{z} + \sum e_k \cdot \xi \cdot \nabla^A e_{k \mathfrak{z}}
\]

But \( \xi = f + \alpha \) has \( (d + d^*)\xi = df + d\alpha + d^* \alpha \), and \( d\alpha \cdot \mathfrak{z} = d^* \alpha \cdot \mathfrak{z} \), and so

\[
((d + d^*)\xi) \cdot \mathfrak{z} = (df + 2 d^* \alpha) \cdot \mathfrak{z}
\]

Assume now that \((M, g, \omega)\) is Kähler. Then, since \( \sigma(\mathfrak{z}) = \frac{1}{4} \omega \), there is a connection \( A_0 \) such that \( \nabla^{A_0} \mathfrak{z} \equiv 0 \). In conclusion:

\[
\mathcal{D}^{A_0}((f + \alpha) \cdot \mathfrak{z}) = (df + 2 d^* \alpha) \cdot \mathfrak{z}
\]
This identifies $D^{A_0} : \Gamma(W^+) \rightarrow \Gamma(W^-)$ with $d \oplus 2d^* : \Gamma(\mathbb{R} \oplus \Lambda^+) \rightarrow \Gamma(\Lambda^1)$.

In particular, their kernels must be isomorphic. Since $d$ and $d^*$ have orthogonal images in $\Gamma(\Lambda^1)$, we conclude that $df + 2d^*\alpha = 0$ only when $df = 0$ and $d^*\alpha = 0$.

Thus, the kernel of $d \oplus 2d^*$ has dimension $b_0 + b_2^+ = 1 + b_2^+$. On the other hand, $D^{A_0}$ is $\mathbb{C}$-linear, and thus its kernel must be even-dimensional. Therefore $b_2^+$ must be odd.

And finally, since $M$ admits almost-complex structures, it has $b_1 + b_2^+$ odd, and so $b_1$ must be even.

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