AROUND PRÜFER EXTENSIONS OF RINGS

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Abstract. The paper intends to apply the properties of Prüfer extensions, investigated in the Knebusch-Zhang book, to ring extensions $R \subseteq S$. The integral closure $\overline{R}$ of $R$ in $S$ is shown to be the intersection of all $T \in [R, S]$, such that $T \subseteq S$ is Prüfer. We are then able to establish an avoidance lemma for integrally closed subextensions. Rings of sections of the affine scheme defined by $R$ provide results on $S$-regular ideals. Some results on pullbacks characterizations of Prüfer extensions are given. We introduce locally strong divisors, examining the properties of strong divisors of a local ring and their links with Prüfer extensions. The locally strong divisors allow us to give characterizations of QR-extensions. We apply our results to Nagata extensions of rings. We also look at the Prüfer hull of a Nagata extension. We define quasi-Prüferian rings that may differ from quasi-Prüfer integral domains. We then derive some results on minimal and FCP extensions. Finally, we study the set of all primitive elements in an extension.

1. Introduction and Notation

We consider the category of commutative and unital rings, whose flat epimorphisms will be strongly involved, like localizations with respect to a multiplicatively closed subset.

If $R \subseteq S$ is a (ring) extension, we denote by $[R, S]$ the set of all $R$-subalgebras of $S$ and by $[R, S]_{fg}$ the set of all $T \in [R, S]$, such that $T$ is of finite type over $R$. Any undefined material is explained in the next subsection and in the following sections.

1.1. An overview of the paper. We present some properties of Prüfer extensions of rings and derive from them new results, using the properties and definitions of Knebusch and Zhang [26]. It is well known that Prüfer extensions are nothing but normal pairs. Prüfer

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2010 Mathematics Subject Classification. Primary: 13B02, 13B22, 13B40; Secondary: 13B30.

Key words and phrases. integral closure, Prüfer extension, pullback, QR-extension, quasi-Prüferian ring, Nagata extension, strong divisor, ring of sections.
extensions are defined by flat epimorphisms, while normal pairs are defined by the integrally closed property. We will deal with the Prüfer aspect, except in Section 6.

In Section 2 we give some recalls about Prüfer extensions. We also give rules on flat epimorphisms and direct limits, in order to make easier our proofs.

If $R \subset S$ is a ring extension, an ideal $I$ of $R$ is called $S$-regular by \cite{26} if $IS = S$. Such ideals are a useful concept in the next sections. Moreover, these ideals $I$ permit to factorize the extension through the ring of sections defined by the open subset associated to $I$. This is developed in Section 3, with some applications to Prüfer extensions. By the way, we give rules allowing to calculate rings of sections.

In Section 4, we show that the integral closure $\overline{R}$ of a ring extension $R \subset S$ is the set intersection of all $T \in [R, S]$ such that $T \subseteq S$ is Prüfer. This statement generalizes a classical result on integral closures.

As an application, we show that an avoidance lemma with respect to finitely many integrally closed subrings holds. The proof is not easy and uses Manis valuations. We also show an avoidance lemma with respect to finitely many flat epimorphisms. This is explained in Section 5.

Section 6 deals with pullbacks results. Olivier proved that integrally closed extensions are characterized by pullbacks in which some morphism is of the form $V \to K$, where $V$ is a semi-hereditary ring with total quotient ring $K$. We adapt this result to the Prüfer case and evidently reuse the normal pairs. Another result concerns a more classical situation.

In Section 7, we deal with extensions over local rings and introduce the strong divisors considered by \cite{26}. A strong divisor is a regular element $t$ of a ring $R$, such that $Rt$ is comparable to each ideal of $R$. The maximal Prüfer extension of a local ring $R$ is the localization of $R$ with respect to the multiplicatively closed subset of all strong divisors of $R$. We develop a theory of strong divisors. The most striking results are that a regular element $x$ of a local ring is a strong divisor if and only if $R \to Rx$ is Prüfer, and that an extension of finite type of $R$ is Prüfer if and only if it is of the form $R \to Rx$, where $x$ is a strong divisor.

QR-extensions $R \subset S$ are studied in Section 8: they are extensions such that each $T \in [R, S]$ is (isomorphic to) a localization. They are evidently Prüfer. We also look at the Bezout extensions of \cite{26} and examine the Bezout and Prüfer hull of an extension. Over a local ring or a Nagata ring $R(X)$, the Prüfer, Bezout and and QR properties are equivalent. To go further we have introduced locally strong divisors. As locally strong divisors appear each time we are dealing with Prüfer
extensions, we see that a ring $R$ admits non trivial Prüfer extensions if $R$ has locally strong divisors that are non-units. This concept is more stable than that of strong divisors. An interesting result is that QR-extensions are characterized by using locally strong divisors. Another one is that a QR-extension $R \subset S$ verifies that for each $s \in S$ there is a locally strong divisor $\rho$, such that $\rho s \in R$. The section ends on extensions whose supports are finite.

Section 9 is concerned with Nagata extensions $R(X) \subset S(X)$. We show that such an extension is Prüfer if and only if $R \subset S$ is Prüfer. We already said that over a Nagata ring the Prüfer and QR-concepts coincide. The Prüfer hull $\tilde{R}$ of an extension gives $\tilde{R}(X)$ for its Nagata extension in a lot of cases. It may be that the result holds for any extension but we do not know the answer. When $R$ is a local ring, we show that the strong divisors of $R(X)$ are in some sense the strong divisors of $R$.

We define in Section 10 quasi-Prüferian rings as rings $R$ such that $R \rightarrow R(X)$ is an $i$-extension. But our definition is not equivalent to $R \rightarrow \text{Tot}(R)$ is quasi-Prüfer in the sense of [12], contrary to the integral domains context, where we recover the classical notion of quasi-Prüfer rings. We get some results about these rings, largely inspired by the integral domain context. A sequence of statements in a theorem shows that a ring is quasi-Prüfer if it is quasi-Prüferian. The converse holds if the total quotient ring of the ring is zero-dimensional.

Section 11 is devoted to minimal or FCP extensions of a local ring that are either Prüfer or have the QR-property. A special attention is paid to $\mathcal{B}$-extensions (extensions that are locally determined in some sense).

The paper ends by considering the set of all primitive elements in an extension, a study initiated by Dobbs and Houston. There is a link with quasi-Prüfer extensions.

1.2. Basics concepts. As usual, $\text{Spec}(R)$ and $\text{Max}(R)$ are the set of prime and maximal ideals of a ring $R$ and $U(R)$ is the set of all its units.

We now give some notation for a ring morphism $f : R \rightarrow S$. We denote by $^af$ the spectral map $\text{Spec}(S) \rightarrow \text{Spec}(R)$. Then $\mathcal{X}_R(S)$ (or $\mathcal{X}(S)$) is the image of the map $^af$ and we say that $f$ is an $i$-morphism if $^af$ is injective. If $Q$ is a prime ideal of $S$ lying over $P$ in $R$, the ring morphism $R_P \rightarrow S_Q$ is called the local morphism at $Q$ of the morphism.

Then $(R : S)$ is the conductor of an extension $R \subseteq S$. The integral closure of $R$ in $S$ is denoted by $\overline{R}^S$ (or by $\overline{R}$ if no confusion can occur).
A local ring is here what is called elsewhere a quasi-local ring. For an extension $R \subseteq S$ and an ideal $I$ of $R$, we write $V_S(I) := \{P \in \text{Spec}(S) \mid I \subseteq P\}$ and $D_S(I)$ for its complement. If $R$ is a ring; then $Z(R)$ denotes the set of all its zero-divisors. The support of an $R$-module $E$ is $\text{Supp}_R(E) := \{P \in \text{Spec}(R) \mid E_P \neq 0\}$, and $\text{MSupp}_R(E) := \text{Supp}_R(E) \cap \text{Max}(R)$. When $R \subseteq S$ is an extension, we will set $\text{Supp}(T/R) := \text{Supp}_R(T/R)$ and $\text{Supp}(S/T) := \text{Supp}_R(S/T)$ for each $T \in [R, S]$, unless otherwise specified.

If $R \subseteq S$ is a ring extension and $\Sigma$ a mcs of $R$ (i.e. a multiplicatively closed subset of $R$), then $S_\Sigma$ is both the localization $S_\Sigma$ as a ring and the localization at $\Sigma$ of the $R$-module $S$; that is, $S \otimes_R \Sigma$.

Let $\Sigma_1$ and $\Sigma_2$ be two mcs of a ring $R$. We denote by $\Sigma_2/\Sigma_1$ the image of $\Sigma_2$ in $R_{\Sigma_1}$. We recall that $R_{\Sigma_1 \Sigma_2} = (R_{\Sigma_1})_{\Sigma_2/\Sigma_1}$. It follows that if $x \in R$ and $\Sigma$ is a mcs of $R$, then $(R_x)_\Sigma = (R_\Sigma)_x$.

Flat epimorphisms and their properties are the main tool in this paper. We use the theory that was developed by D. Lazard [29, Chapter IV]. The reader may also use the scholium of our paper [42].

When $R \to S$ and $R \to T$ are ring morphisms, we will write $S \cong_R T$, (or $S \cong T$) if there is an isomorphism of $R$-algebras $S \to T$. It may happens that $\cong$ is replaced with $=$.

Let $R \subseteq S$ be an extension. A chain of $R$-subalgebras of $S$ is a set of elements of $[R, S]$ that are pairwise comparable with respect to inclusion. We say that $R \subseteq S$ is chained if $[R, S]$ is a chain. We also say that the extension has FCP (or is an FCP extension) if each chain in $[R, S]$ is finite, or equivalently, the poset $[R, S]$ is Artinian and Noetherian. An extension is called FIP if $[R, S]$ has finitely many elements. An extension $R \subseteq S$ is called minimal if $[R, S] = \{R, S\}$. According to [19, Théorème 2.2], a minimal extension is either integral or a flat epimorphism. Finally, $|X|$ is the cardinality of a set $X$, $\subset$ denotes proper inclusion (contrary to [26] where $\subset$ denotes the large inclusion). A compact topological space does not need to be separated. For a positive integer $n$, we set $\mathbb{N}_n := \{1, \ldots, n\}$.

2. SOME DEFINITIONS, NOTATION AND USEFUL RESULTS

An extension $R \subseteq S$ is called Prüfer if $R \subseteq T$ is a flat epimorphism for each $T \in [R, S]$ (or equivalently, if $R \subseteq S$ is a normal pair) [26, Theorem 5.2, p. 47]. A Prüfer integral extension is trivial.

We denote by $Q(R)$ the complete ring of quotients (Utumi-Lambeck) of a ring $R$.

Definition 2.1. [26] A ring extension $R \subseteq S$ has:
A ring $R$ has:

1. A maximal flat epimorphic extension $R \subseteq \hat{R}^S$ (also termed the maximal flat epimorphic extension by some authors, like [29]).

2. A maximal Prüfer extension $R \subseteq \tilde{R}^S$.

We set $\hat{R} := \hat{R}^S$ and $\tilde{R} := \tilde{R}^S$, if no confusion can occur.

We also note the following known consequence:

**Proposition 2.2.** An extension $R \subseteq S$ is a flat epimorphism if and only if for each $s \in S$ there is some ideal $I$ of $R$ such that $IS = S$ and $Is \subseteq R$ (or equivalently $(R :_R s)S = S$).

**Corollary 2.3.** An extension $R \subseteq S$ is Prüfer if and only if $R[s] = (R :_R s)R[s]$ for each $s \in S$.

**Proof.** Use the definition of Prüfer extensions by flat epimorphisms. $\Box$

If an extension $R \subseteq S$ is Prüfer and $\Sigma$ is a mcs of $R$, then $R_\Sigma \subseteq S_\Sigma$ is Prüfer. We have a converse.

**Proposition 2.4.** [$42$, Proposition 1.1] An extension $R \subseteq S$ is Prüfer if and only if $R_M \subseteq S_M$ is Prüfer for each $M \in \text{Max}(R)$ (resp. for each $M \in \text{Spec}(R)$).

**Proposition 2.5.** [$42$, Corollary 3.15] Prüfer extensions are descended by faithfully flat morphisms.

**Proposition 2.6.** The Prüfer property of extensions $R \subseteq S$ is local on the spectrum; that is if $\text{Spec}(R) = \text{D}(r_1) \cup \cdots \cup \text{D}(r_n)$ for some elements $r_1, \ldots, r_n \in R$ and $R_{r_i} \subseteq S_{r_i}$ is Prüfer for each $i = 1, \ldots, n$, then $R \subseteq S$ is Prüfer.
Proof. The extension $R_{r_1} \times \cdots \times R_{r_n} \subseteq S_{r_1} \times \cdots \times S_{r_n}$ is Prüfer [26, Proposition 5.20, p.56]. To conclude use Proposition 2.5 since $R \to R_{r_1} \times \cdots \times R_{r_n}$ is faithfully flat. □

In [42], a minimal flat epimorphism is called a Prüfer minimal extension. An FCP Prüfer extension has FIP and is a tower of finitely many Prüfer minimal extensions [42, Proposition 1.3].

In [42], we defined an extension $R \subseteq S$ to be quasi-Prüfer if it can be factored $R \subseteq R' \subseteq S$, where $R \subseteq R'$ is integral and $R' \subseteq S$ is Prüfer. In this case $R'$ is necessarily $\overline{R}$. An FCP extension is quasi-Prüfer [42, Corollary 3.4].

An extension $R \subseteq S$ is called almost-Prüfer if it can be factored $R \subseteq S' \subseteq S$, where the first extension is Prüfer and the second is integral. In this case $S'$ is necessarily $\overline{R}$. An almost-Prüfer extension is quasi-Prüfer [42].

We now give some rules on flat epimorphisms. The following result of Lazard is a key result. Let $R$ be a ring. We denote by $\mathcal{FE}$ the collection of classes up to an isomorphism of flat epimorphisms whose domain is $R$ and by $\mathcal{X}$ the set of subsets of Spec($R$) that are affine schemes, when endowed with the induced sheaf. The elements of $\mathcal{X}$ are compact and stable under generization.

**Proposition 2.7.** [29, Proposition 2.5, p.112] The map $\mathcal{FE} \to \text{Spec}(R)$, defined by $T \mapsto \mathcal{X}(T)$ is a bijection onto $\mathcal{X}$. The inverse map is as follows: an affine scheme $X$ of Spec($R$) gives $R \to \Gamma(X)$, the ring of sections over $X$.

The next result, proved in [23, Proposition 3.4.10, p.242], will be useful in the sequel.

**Proposition 2.8.** ($\mathcal{L}$)-rule Let $R \to E$ be a ring morphism and $E = \lim E_i$ where each $E_i$ is an $R$-algebra, then $\mathcal{X}(E) = \cap \mathcal{X}(E_i)$.

We will use Proposition 2.7 under the following form.

**Proposition 2.9.** ($\mathcal{X}$)-rule Let $R \to E$ be a flat epimorphism and $R \to F$ a ring morphism.

1. There is a factorization $R \to E \to F$ if and only if $\mathcal{X}(F) \subseteq \mathcal{X}(E)$.

2. If $R \to F$ is a flat epimorphism, then $E \cong F$ if and only if $\mathcal{X}(F) = \mathcal{X}(E)$.

**Proof.** (1) The ring morphism $\alpha : F \to F \otimes_R E$ is a flat epimorphism. If $\mathcal{X}(F) \subseteq \mathcal{X}(E)$, then the spectral morphism of $\alpha$ is surjective, because there is a surjective map Spec($F \otimes_R E$) $\to$ Spec($E$) $\times_{\text{Spec}(R)}$ Spec($F$).
Corollaire 3.2.7.1, p.235]. It follows that $\alpha$ is a faithfully flat epi-
morphism, whence an isomorphism by [29, Lemme 1.2, p.109] and an
implication is proved. Its converse is obvious. Now (2) can be proved
by using (1). But it is also a consequence of Proposition 2.7. □

Corollary 2.10. (MCS)-rule Let $R \rightarrow E$ be a ring morphism.

1. If $E = R_\Sigma$ where $\Sigma$ is a mcs of $R$, then $X(E) = \cap[D(s)|s \in \Sigma]$.

2. If $E = \varprojlim R_{si}$, where $\{s_i\}$ is family of elements of $R$, then
$E = R_\Sigma$, where $\Sigma$ is the mcs of $R$ generated by the family.

Proof. The proof is a consequence of the above rules. □

3. S-regular ideals and rings of sections

If $I$ is an ideal of a ring $R$, then $\Gamma(D(I), R)$ (or $\Gamma(D(I)))$ denotes
the ring of sections of the scheme $\text{Spec}(R)$ over the open subset $D(I)$.
All that we need to know is that $\Gamma(D(R)) = R$, $\Gamma(\emptyset) = 0$ and if $f : R \rightarrow S$ is a ring morphism, there is a commutative diagram, because
$a f^{-1}(D(I)) = D(IS):

\[
\begin{array}{ccc}
R & \longrightarrow & S \\
\downarrow & & \downarrow \\
\Gamma(D(I)) & \longrightarrow & \Gamma(D(IS))
\end{array}
\]

We denote by $\text{Ass}(R)$ the set of all (Bourbaki) prime ideals $P
associated to the $R$-module $R$; that is, $P \in \text{Min}(V(0 : r))$ for some $r \in R$. Recall that a ring morphism $f : R \rightarrow S$ is called schematically dominant if
for each open subset $U$ of $\text{Spec}(R)$, the map $\Gamma(U, R) \rightarrow \Gamma(a f^{-1}(U), S)$
is injective [23, Proposition I.5.4.1]. The first author proved that a flat
ring morphism $f : R \rightarrow S$ is schematically dominant if and only if $\text{Ass}(R) \subseteq X(S)$ [10, Proposition 52]. Clearly if $\text{Min}(R) = \text{Ass}(R)$ (for
example, if $R$ is an integral domain) and $f$ is injective and flat, then $f$
is schematically dominant.

Lemma 3.1. A flat extension $R \subseteq S$ is schematically dominant.

Proof. If $P \in \text{Ass}(R)$, there is some $a \in R$, such that $P \in \text{Min}(V(0 : a))$. From $(0 : S a) \cap R = 0 : a$, we deduce that $R/(0 : a) \rightarrow S/(0 : S a)$
is injective and then $P/(0 : a)$ can be lifted up to a minimal prime ideal
$Q/(0 : S a)$. Hence $Q \in \text{Ass}(S)$ is above $P$. □

Let $R \subseteq S$ be an extension and an ideal $I$ of $R$. Then $I$ is called
S-regular if $IS = S$ [26]. Note that S-regular ideals play a prominent
role in [26]. They are involved in certain questions. For example, if $f : R \rightarrow S$ is a ring morphism, the fiber at a prime ideal $P$ of $R$
is $a f^{-1}(P)$. This fiber is homeomorphic to the spectrum of the ring
Therefore, the fiber is empty if and only if $S_P/PS_P$, which means that $PR_P$ is $S_P$-regular. If $f$ is a flat epimorphism, the fiber at $P$ is empty if and only if $S = PS$ [29, Proposition 2.4, p.111].

**Remark 3.2.** Let $f : R \hookrightarrow S$ be an extension.

(1) An ideal $I$ of $R$ is $S$-regular if and only if $\mathcal{X}(S) \subseteq D(I)$ [39, Lemma 2.3]. Such an ideal $I$ is dense; that is, $0 : I = 0$.

(1(a)) $I$ is $S$-regular if and only if $\sqrt{I}$ is $S$-regular, because $D(\sqrt{IS}) = af^{-1}(D(\sqrt{I})) = af^{-1}(D(I)) = D(IS)$.

(1(b)) $I$ is $S$-regular if and only if $IP$ is $S_P$-regular for each $P \in \text{Spec}(R)$. We need only to show that if the local condition holds, then $I$ is $S$-regular. Suppose that $IS \subset S$, then there is some prime ideal $Q$ of $S$, such that $IS \subseteq Q$. If $P = Q \cap R$, then $Q_P$ is a prime ideal of $S_P$, such that $IP_{S_P} \subseteq Q_P$, a contradiction.

(2) If $I$ is $S$-regular, we have $\text{Spec}(S) = D(IS) = af^{-1}(D(I))$, so that there is a factorization $R \to \Gamma(D(I)) \to S$. If, in addition, $f$ is flat, then $f$ is schematically dominant (Lemma 3.1); so that, we can consider that there is a tower of extensions $R \subseteq \Gamma(D(I)) \subseteq S$. Moreover, $D(I)$ is an open subset which is (topologically) dense in $\text{Spec}(R)$, because a schematically dominant morphism is dominant [23, Proposition I.5.4.3], i.e. its spectral image is dense. The density follows from $\mathcal{X}(S) \subseteq D(I)$.

This result holds if the extension is Prüfer and then $R \to \Gamma(D(I))$ is Prüfer.

(3) We will use the following consequence of Proposition 2.7: If $I$ is an ideal of $R$, then $R \to \Gamma(D(I))$ is a flat epimorphism and $\mathcal{X}(\Gamma(D(I))) = D(I)$ if and only if $D(I)$ is an affine open subset of $\text{Spec}(R)$ (for example if $I$ is principal), in which case $D(I) = D(J)$ where $J$ is a finitely generated ideal.

We can say more after looking at the following recall adapted to ring morphisms (the reader is referred to [23, Definition I.4.2.1, p.260] for the definition of an open immersion of schemes). We will say that a ring morphism is an open immersion if the morphism of schemes associated is an open immersion.

**Proposition 3.3.** Let $f : R \to S$ be a ring morphism.

(1) [23, I.4.2.2] $f$ is an open immersion if and only if $\text{Spec}(S) \to \mathcal{X}(S)$ is a homeomorphism, $\mathcal{X}(S)$ is an open subset and the local morphisms of $f$ are isomorphisms.

(2) A flat epimorphism $R \to S$, such that $\mathcal{X}(S)$ is Zariski open is an open immersion.

(3) [22, Théorème 17.9.1, p.79] $f$ is an open immersion if and only if $f$ is a flat epimorphism of finite presentation.
An injective flat epimorphism of finite type is of finite presentation, whence is an open immersion.

**Proof.** We need only to prove (2) and using (1). Since $f$ is a flat epimorphism, its spectral map is an homeomorphism onto its image by [29, Corollaire 2.2, p.111] which is an open subset of the form $D(I)$. Moreover, the local morphisms of the map are isomorphisms. □

**Proposition 3.4.** Let $R \subseteq S$ be an injective flat epimorphism of finite type. Then $\mathfrak{X}(S)$ is an open affine subset $D(I)$, where $I$ is a $S$-regular ideal and there is an $R$-isomorphism $\Gamma(D(I)) \cong S$, $R \subseteq S$ is of finite presentation and $I$ is a dense ideal.

Conversely, if $D(I)$ is an open affine subset, where $I$ is a finitely generated dense ideal, then $R \to \Gamma(D(I))$ is an injective flat epimorphism, of finite type (presentation), such that $\mathfrak{X}(\Gamma(D(I))) = D(I)$.

**Proof.** To apply Proposition 2.7, we need only to look at injective flat epimorphisms of finite type $R \to S$. We know that such a ring morphism $f : R \subseteq S$ is of finite presentation according to Proposition 3.3(4). By the Chevalley Theorem, $\mathfrak{X}(S)$ is a Zariski quasi-compact open subset of $\text{Spec}(R)$, therefore of the form $D(I)$, where $I$ is an ideal of $R$, of finite type. We have $^a f^{-1}(D(I)) = D(IS) = \text{Spec}(S)$ because $\mathfrak{X}(S) = D(I)$, so that $IS = S$ and then $I$ is dense because it is $S$-regular. Moreover, $\Gamma(D(I)) \cong S$ by Proposition 2.9(2) because $\mathfrak{X}(S) = D(I) = \mathfrak{X}(\Gamma(D(I)))$.

Assume that the hypotheses of the converse hold. Since the morphism $R \to \Gamma(D(I))$ is an open immersion by Proposition 3.3, we get that $R \to \Gamma(D(I))$ is of finite presentation. Moreover, $0 : I = 0$ (which is equivalent to $\text{Ass}(R) \subseteq D(I)$ [29, Corollaire 1.14, p.93]), so that $R \to \Gamma(D(I))$ is injective, by [29, Proposition 3.3, p.96]. □

We note the following result:

**Proposition 3.5.** [26 Theorem 2.8, p.101, Theorem 2.6, p.100] Let $R \subseteq S$ be an extension which is a flat epimorphism. Then the extension is Prüfer if and only if for every finitely generated $S$-regular ideal $I$ of $R$, the ring $R/I$ is arithmetical (resp.; $I$ is locally principal).

**Proposition 3.6.** Let $R \subseteq S$ be a flat epimorphism. Then, $R \subseteq S$ is Prüfer if and only if for each $P \in \text{Spec}(R)$, the set of $S_P$-regular ideals of $R_P$ is a chain.

**Proof.** According to [42 Proposition 1.1(2)], the extension is Prüfer if and only if $R_P \subseteq S_P$ is Manis for each $P \in \text{Spec}(R)$ and equivalently $R_P \subseteq S_P$ is Prüfer-Manis. The result follows from [26 Theorem 3.5, p.190]. □
We recall that the dominion of a ring morphism \( f : R \to S \) is the subring \( \text{Dom}(f) = \{ x \in S \mid x \otimes 1 = 1 \otimes x \text{ in } S \otimes_R S \} \) of \( S \), which contains the subring \( f(R) \). Actually, \( \text{Dom}(f) \) is the kernel of the morphism of \( R \)-modules \( i_1 - i_2 : S \to S \otimes_R S \) where \( i_1, i_2 \) are the natural ring morphisms \( S \to S \otimes_R S \).

** Proposition 3.7.** If \( f : R \to S \) is a flat morphism and \( I \) an ideal of \( R \), such that \( \mathcal{X}_R(S) = D(I) \), then

1. \( \Gamma(D(I)) = \text{Dom}(f) \) and \( \Gamma(D(I)) \to S \) is an injective flat morphism.
2. If in addition \( f \) is a ring extension, then \( \widetilde{R} \subseteq \hat{R} \subseteq \Gamma(D(I)) \), each of the extensions in \( S \) being flat. In particular, if \( D(I) \) is affine, then \( \hat{R} = \Gamma(D(I)) \).
3. If \( g : R \to B \) is a flat morphism, setting \( C := S \otimes_R B \), then \( \mathcal{X}_B(C) = D(IB) \) and \( \Gamma(D(I)) \otimes_R B \cong \Gamma(D(IB)) \).
4. If \( P \) is a prime ideal of \( R \), then \( \Gamma(D(IP)) = (\Gamma(D(I)))_P \). In particular if \( P \in D(I) \), then \( (\Gamma(D(I)))_P = R_P \).
5. \( D(I) \subseteq \mathcal{X}(\Gamma(D(I))) \).
6. If \( \Gamma(D(I)) = \Gamma(D(I)) \), then \( D(I) = \mathcal{X}(\Gamma(D(I))) \), so that \( D(I) \) is an open affine subset if in addition \( R \to \Gamma(D(I)) \) is a flat epimorphism.

** Proof.** (1) is a translation of [39, Theorem 2.7]. The flatness of \( \Gamma(D(I)) \to S \) follows from [29, Proposition 3.1 (2), p.112].

(2) If \( f \) is a ring extension, observe that \( \hat{R} \subseteq \Gamma(D(I)) \), because \( R \to \hat{R} \) is an epimorphism and then \( y \otimes 1 = 1 \otimes y \) for each \( y \in \hat{R} \) [29, Lemme 1.0, p.108]. The flatness of the extensions \( \hat{R}, \hat{R} \subseteq S \) result from [29, Proposition 3.1(2), p.112]. At last, if \( D(I) \) is affine, then \( R \to \Gamma(D(I)) \) is a flat epimorphism by Remark [32,2], so that \( \hat{R} = \Gamma(D(I)) \).

(3) Because \( \text{Spec}(C) \to \text{Spec}(B) \times_{\text{Spec}(R)} \text{Spec}(S) \) is a surjective map [23, Corollaire 3.2.7.1, p.235], we have \( \mathcal{X}_B(C) = a^{-1}(D(I)) = D(IB) \). To conclude use (1) and the fact that a kernel tensorised by \( B \), which is flat over \( R \), is the kernel of the tensorised map.

(4) is gotten by taking \( B = R_P \) in (3).

(5) According to (4), an element \( P \) of \( D(I) \) is such that \( (\Gamma(D(I)))_P = R_P \). It follows that there is a prime ideal \( Q \) of \( \Gamma(D(I)) \) lying over \( P \).

(6) holds because \( I \) is \( \Gamma(D(I)) \)-regular. \( \square \)

We can apply the above result in the following three contexts, when \( I = (r_1, \ldots, r_n) \) is an ideal of finite type of \( R \) (the hypothesis of this result entails that \( \sqrt{I} = \sqrt{J} \), where \( J \) is an ideal of finite type). We can suppose that \( D(I) \neq \emptyset \) and that the set \( \{D(r_1), \ldots, D(r_n)\} \) is an
antichain; so that, the $r_i$'s cannot be nilpotent. The first sample is certainly the most interesting, because when $I = Rr$, we recover that $\Gamma(D(r)) = R_r$.

(1) We can consider the flat ring morphism $\varphi : R \to R_{r_1} \times \cdots \times R_{r_n} := S_I$, which is such that $X(S_I) = D(I)$. Actually, $\varphi$ is of finite presentation [23, Proposition 6.3.11, p.306] and its local morphisms are isomorphisms. But $\varphi$ may not be a flat epimorphism, when it is not an $i$-morphism.

In case $\{D(r_i)\}$ defines a partition on $D(I)$, $\varphi$ is an $i$-morphism, whence a flat epimorphism. In this case, $\Gamma(D(I)) = \prod R_{r_i}$.

(2) Let $F_I := R[X_1, \ldots, X_n]/(r_1X_1 + \cdots + r_nX_n - 1)$ be the forcing $R$-algebra with structural morphism $f_I$, associated to a finitely generated ideal $I = (r_1, \ldots, r_n)$ (it would be more correct to write: associated to the sequence $\{r_1, \ldots, r_n\}$). This ring is not zero, for otherwise $1 = (r_1X_1 + \cdots + r_nX_n - 1)P(X_1, \ldots, X_n)$, for some $P(X_1, \ldots, X_n) \in R[X_1, \ldots, X_n]$, implies that $1 - (r_1X_1 + \cdots + r_nX_n) \in U(R[X_1, \ldots, X_n])$, so that $r_1X_1 + \cdots + r_nX_n$ would be nilpotent and then also the $r_i$'s. Then $I$ is $F_I$-regular and for every ring morphism $R \to S$ for which $I$ is $S$-regular, there is a factorization $R \to F_I \to S$. But $F_I \to S$ does not need to be unique. According to [39, Theorem 2.7 and Remark 2.8(1)], the ring morphism $f_I$ is flat, $X(F_I) = D(I)$ and $\Gamma(D(I)) = \text{Dom}(f_I)$. Moreover, $\Gamma(D(I)) \to F_I$ is an injective flat ring morphism.

(3) The first author introduced in [38] the following construction that we adapt to the present context. Let $R$ be a ring and $I$ an ideal of $R$. Denoting by $C(p(X))$ the content of a polynomial $p(X) \in R[X]$, we consider the mcs $\Sigma := \{p(X) \in R[X] \mid D(I) \subseteq D(C(p(X)))\}$. Setting $R(D(I)) := R[X]_{\Sigma}$, we get a flat morphism $R \to R(D(I))$, such that $X(R(D(I))) = D(I)$.

4. Integral closures as intersections

We start by giving some results that do not seem to have been observed. They are consequences of a paper by P. Samuel [46]. Let $v$ be a valuation on a ring $R$. Following [26], we denote by $A_v$ the valuation ring of $v$.

**Lemma 4.1.** [46, Théorème 1(d)] An extension $R \subset S$, such that $S \setminus R$ is multiplicatively closed, is integrally closed. For example, $R \subseteq S$ is integrally closed if there is some valuation $v$ on $S$ such that $R = A_v$, the valuation ring of $v$.

We will use the next result.
Lemma 4.2. [46, Theorem 4] Let $R \subseteq S$ be an extension and $P$ a prime ideal of $R$. Due to Zorn Lemma, there is a maximal pair $(R', P')$ dominating $(R, P)$ and $S \setminus R'$ is a mcs.

A Manis valuation $v$ on a ring $S$ is a valuation such that $v : S \to \Gamma_v \cup \infty$ is surjective, where $\Gamma_v$ is the value group of $v$.

A ring extension $R \subseteq S$ is called Manis if $R = A_v$ for some Manis valuation $v$ on $S$. Prüfer-Manis extensions are defined as Prüfer extensions $R \subset S$ such that there is some Manis valuation $v$ on $S$ such that $A_v = R$ [26, Definition 1, p. 58].

By [26, Theorem 3.5, p.190], a flat epimorphism $R \subseteq S$ is Prüfer-Manis if and only if the set of all $S$-regular ideals of $R$ is a chain.

Lemma 4.3. [26, Theorem 3.3, p.187, Theorem 3.1, p.187, Proposition 5.1(iii), p. 46-47]. The following statements are equivalent for an extension $R \subseteq S$.

1. $R \subseteq S$ is Prüfer-Manis.
2. $S \setminus T$ is a mcs for each $T \in [R, S]$.
3. $R \subseteq S$ is integrally closed and chained.
4. $R \subseteq S$ is Prüfer and $S \setminus R$ is a mcs.

If the above condition (3) holds for an FCP extension, then $R \subseteq S$ has FIP.

By [26], we know that for a Prüfer extension $R \subset S$ and $P \in \text{Supp}(S/R)$, the subset $S_P \setminus R_P$ is multiplicatively closed. Also [26, Proposition 5.1(iii), p. 46-47] shows that if $U \in [R, S]$ and $S \setminus U$ is a mcs, then $U \subset S$ is Prüfer-Manis.

Corollary 4.4. A minimal extension $R \subset S$ is a flat epimorphism if and only if it is Prüfer and if and only if it is Prüfer-Manis.

Proof. The extension is a flat epimorphism if and only if it is integrally closed. To complete the proof it is enough to use [19, Proposition 3.1] which states that $S \setminus R$ is a mcs when $R \subset S$ is a flat epimorphism. □

Corollary 4.5. An FCP Prüfer extension has FIP and is a tower of finitely many Prüfer-Manis minimal extensions.

We will need the two following results. They generalize known results about the integral closure of an integral domain, which is the intersection of valuation rings.

Lemma 4.6. Let $R \subset S$ be an extension and $x \in S \setminus \overline{R}$. Then there is some $T \in [R, S]$, such that $T \subset S$ is Prüfer (respectively, Prüfer-Manis) and $x$ is not integral over $T$, and then $x \notin T$.  


Proof. It is enough to mimic the first part of the proof of [46, Théorème 8]. More precisely, let \( T \) be a maximal element of the \( \cup \)-inductive set \( \{ U \in [R,S] \mid x \notin U^S \} \). We intend to show that \( T \subseteq S \) is Prüfer-Manis. In view of the above results, we need only to show that any \( V \in [T,S] \) with \( V \neq S \) is such that \( S \setminus V \) is multiplicatively closed and then such that \( T \subseteq S \) is integrally closed. Now replace \( A \) with \( V \) in the second paragraph of the proof of [46, Théorème 8], and the result follows. \( \square \)

**Theorem 4.7.** Let \( R \subseteq S \) be an extension, then \( R^S \) is the intersection of all \( T \in [R,S] \) such that \( T \subseteq S \) is Prüfer (resp. Prüfer-Manis) and also the intersection of all \( U \in [R,S] \), such that \( S \setminus U \) is a mcs.

**Proof.** The second result is [46, Théorème 8]. Now Lemma 4.6 shows that \( R^S \) contains the intersection of all \( T \in [R,S] \) such that \( T \subseteq S \) is Prüfer (resp. Prüfer-Manis).

For the reverse inclusion, consider an element \( x \in S \), integral over \( R \). Then \( T \subseteq T[x] \) is integral and a flat epimorphism for any \( T \in [R,S] \) such that \( T \subseteq S \) is Prüfer. We deduce from [29, Lemme 1.2, p. 109], that \( T = T[x] \) and \( x \) belongs to \( T \). \( \square \)

**Remark 4.8.** As a consequence of the above Theorem, we get that an extension \( R \subseteq S \) is quasi-Prüfer if and only if the set of all \( T \in [R,S] \), such that \( T \subseteq S \) is Prüfer, has a smallest element.

### 5. Avoidance lemmata

Some of the following results are known in the context of integral domains and valuation domains. We will use the frame of their proofs but shorter different argumentations. Kostra proved the next Theorem [28, Lemma 2 and Theorem 2], in case \( S \) is a field. To prove it in our context, we follow the steps of his difficult proof by using Theorem 4.7.

If \( V \subseteq S \) is Prüfer-Manis, \( S \) is endowed with a valuation \( v : S \to \Gamma_v \cup \{\infty\} \), which is surjective. There is no need to consider invertible elements that may not exist but elements \( x \in V \), such that \( v(x) = 0 \), that is \( x \notin P_v \), the center of \( v \). Moreover, if \( v(x) > 0 \), then \( v(1 + x) = 0 \).

**Lemma 5.1.** Let \( R, T, V, V_1, \ldots, V_n \) be subrings of a ring \( S \), where \( n \) is a positive integer and such that \( V \subseteq S \) and \( V_i \subseteq S \) are Prüfer-Manis for each \( i \in \mathbb{N}_n \). Let \( v_i \) be the valuation associated to \( V_i \). Assume that there is some \( b \in [T \cap (\cap_{i \in \mathbb{N}_n} V_i)] \setminus V \). Then, there exists \( c \in [T \cap (\cap_{i \in \mathbb{N}_n} V_i)] \setminus V \) such that \( v_i(c) = 0 \) for any \( i \in \mathbb{N}_n \).

Moreover, for any \( W \in [R, S] \) such that \( W \subseteq S \) is Prüfer-Manis with \( b \notin W \), then \( c \notin W \).
Proof. We build by induction the sequence $S := \{b_i\}_{i=0}^n$ in the following way: set $b_0 := b$ and $b_k := 1 + \prod_{j=0}^{k-1} b_j$ for any $k \in \mathbb{N}_n$. Then, $b_k \in T \cap (\cap_{i \in \mathbb{N}_n} V_i)$ for any $k \in \{0, \ldots, n\}$, so that $v_i(b_k) \geq 0$ for any $i \in \mathbb{N}_n$ and any $k \in \{0, \ldots, n\}$.

If $b_l = b_k$ for some $k \neq l$, assume that $k > l$. Then, $b_k = 1 + b_k \prod_{j=0, j \neq l}^{k-1} b_j$, giving that $b_k(1 - \prod_{j=0, j \neq l}^{k-1} b_j) = 1$, so that $v_i(b_k) + v_i(1 - \prod_{j=0, j \neq l}^{k-1} b_j) = v_i(1) \ (*)$, with $b_k$ and $1 - \prod_{j=0, j \neq l}^{k-1} b_j$ both in $V_i$ for any $i \in \mathbb{N}_n$. It follows that $v_i(b_k) \geq 0$ and $v_i(1 - \prod_{j=0, j \neq l}^{k-1} b_j) \geq 0$. As $v_i(1) = 0$, $\ (*)$ leads to $v_i(b_k) = 0$ for any $i \in \mathbb{N}_n$ and the proof of the Lemma is gotten for $b_k$.

Assume now that $b_j \neq b_k$ for any $k, j \in \{0, \ldots, n\}$, $k \neq j$, so that $|S| = n + 1$.

We claim that for any $i \in \mathbb{N}_n$, there is at most one $b_i \in S$ such that $v_i(b_i) > 0 \ (**)$.

If $v_i(b_k) = 0$ for any $k \in \{0, \ldots, n\}$, then $\ (**)$ holds. Otherwise, let $j_0$ be the least integer of $\{0, \ldots, n\}$ such that $v_i(b_{j_0}) \neq 0$, that is $v_i(b_{j_0}) > 0$. It follows that for any $k \geq j_0$, we have $v_i(\prod_{j=0}^{k} b_j) > 0$, so that $v_i(b_{k+1}) = v_i(1 + \prod_{j=0}^{k} b_j) = v_i(1) = 0$. Since $v_i(b_k) = 0$ for any $k < j_0$, we get that $v_i(b_k) \neq 0$ if and only if $k = j_0$. Then $\ (**)$ holds. Hence, $\{|b_j \in S \ | \ \exists i \in \mathbb{N}_n \text{ such that } v_i(b_j) \neq 0\} \leq n < |S|$. It follows that there exists some $c := b_k \in [T \cap (\cap_{i \in \mathbb{N}_n} V_i)]$ such that $v_i(c) = 0$ for any $i \in \mathbb{N}_n$.

It remains to show that $c \notin V$. We prove by induction on $j \in \{0, \ldots, k\}$ that $b_j \notin V$ for any $j \in \{0, \ldots, k\}$. This is satisfied for $j = 0$ since $b_0 = b$. Assume that $b_j \notin V$ for any $j \in \{0, \ldots, l\}$ where $l < k$. But $S \setminus V$ is a mcs, so that $\prod_{j=0}^{l} b_j \notin V$, which implies that $b_{l+1} \notin V$ and then $c = b_k = 1 + \prod_{j=0}^{k-1} b_j \notin V$.

Now, let $W \in [R, S]$ be such that $W \subseteq S$ is Prüfer-Manis with $b \notin W$. We follow the proof of [28] Remark, page 173]. We consider the previous sequence \{b_j\} with $b_0 := b$ and $b_j = 1 + \prod_{i=0}^{j-1} b_i$. We still have $c \in T \cap (\cap_{i \in \mathbb{N}_n} V_i)$. Obviously, since $b \notin W$, so is any $b_i$, and then $b_k = c \notin W$ because $S \setminus W$ is an mcs. \hfill \Box

**Theorem 5.2.** Let $R, B_1, \ldots, B_n$ be subrings of a ring $S$, where $n$ is a positive integer, $n > 1$. If the $B_i$'s are integrally closed in $S$, except at most two of them, and $R \subseteq B_1 \cup \cdots \cup B_n$, then $R$ is contained in some of the subrings $B_i$.

**Proof.** First, we may remark that $R \subseteq B_1 \cup B_2$ implies that $R$ is contained in one of the subrings $B_1, B_2$ by an obvious property of additive subgroups. So, we may assume that $n \geq 3$ with $B_i$ integrally closed
in $S$ for any $i \geq 3$. There is no harm to assume that $n$ is the least integer such that $R \subseteq B_1 \cup \cdots \cup B_n$, that is $R \not\subseteq \cup_{i \in \mathbb{N}_n, i \neq j} B_i$ for each $j \in \mathbb{N}_n$ (*).

To prove the Theorem, it is enough to show that if $R$ is not contained in any of the subrings $B_i$, we get a contradiction, that is $R \not\subseteq B_1 \cup \cdots \cup B_n$, or equivalently, there exists some $x \in R \setminus (B_1 \cup \cdots \cup B_n)$. This $x$ is gotten after five steps.

**Step 1.** Assume that $R \subseteq B_1 \cup \cdots \cup B_n$ with $R$ not contained in any of the subrings $B_i$. According to (*), for any $j \in \mathbb{N}_n$, there exists $a_j \in (R \cap B_j) \setminus (\cup_{i \in \mathbb{N}_n, i \neq j} B_i)$.

Fix some $i \in \mathbb{N}_n$, $i \neq j$, $i > 2$. Since $B_i \subseteq S$ is integrally closed, by Theorem 4.7, there exists a family $\{V_{k,i}\} \subseteq [B_i, S]$ such that $V_{k,i} \subseteq S$ is Prüfer-Manis, with $B_i = \cap V_{k,i}$. Let $v_{k,i}$ be the Manis valuation associated to $V_{k,i}$. As $a_j \notin B_i$, there exists some $V_{j,i}$ such that $a_j \notin V_{j,i}$.

Moreover, $a_j \in V_{k,j}$ for any $k$ if $j \geq 2$.

Set $M := \{V_{j,i} \mid i > 2, i \neq j\}$. Then, $B_3 \cup \cdots \cup B_n \subseteq \cup_{i > 2, i \neq j} V_{j,i} = \cup_{V_{j,i} \in M} V_{j,i}$. For each $ak$, set $M^{(k)} := \{V_{j,i} \in M \mid a_k \in V_{j,i}\}$ so that $V_{k,j} \in M^{(j)}$ for any $k$ if $j \geq 2$.

**Step 2.** If $M^{(k)} \neq \emptyset$, then $a_k \in [R \cap (\cap V_{j,i}^{(k)}) \setminus V_{k,i}]$. It follows from Lemma 5.1 that there exists $c_k \in R$ such that $v_{j,i}(c_k) = 0$ for any $V_{j,i} \in M^{(k)}$ and $c_k \notin V_{k,i}$. In particular, $c_k \in V_{j,i}$ for any $V_{j,i} \in M^{(k)}$. If $M^{(k)} = \emptyset$, set $c_k := a_k \in R$. Since $a_k \notin V_{j,i}$ for any $V_{j,i} \in M$, it follows that $c_k \notin V_{j,i}$ for any $V_{j,i} \in M$.

**Step 3.** Set $d_0 := \prod_{k=1}^{n} c_k$. Then, $d_0 \in R$. We claim that $d_0 \notin V$, for any $V \in M$. Let $V \in M$. Then, there exist $j_0, j_0 > 2, i_0 \neq j_0$ such that $V = V_{j_0, i_0}$, so that $a_{j_0} \notin V$. Whatever is $M^{(j_0)}$, we have that $c_{j_0} \notin V$. It is obvious if $M^{(j_0)} \neq \emptyset$. If $M^{(j_0)} = \emptyset$, then, $c_{j_0} \notin V_{j,i}$ for any $V_{j,i} \in M$. In particular, $c_{j_0} \notin V$. It follows that $v_{j_0, i_0}(c_{j_0}) < 0$.

Consider $c_k$ for some $k \neq j_0$. If $c_k \notin V$, then $v_{j_0, i_0}(c_k) < 0$. If $c_k \in V$, we cannot have $M^{(k)} = \emptyset$, so that $M^{(k)} \neq \emptyset$. If $V \in M^{(k)}$, then, $v_{j_0, i_0}(c_k) = 0$ and $a_k \in V_{j_0, i_0}$. If $V \notin M^{(k)}$, then $a_k \notin V_{j_0, i_0}$ and $c_k \notin V_{j_0, i_0}$ by Lemma 5.1 which leads to $v_{j_0, i_0}(c_k) < 0$. In any case $v_{j_0, i_0}(c_k) \leq 0$.

To conclude $v_{j_0, i_0}(d_0) = \sum_{k=1}^{n} v_{j_0, i_0}(c_k) \leq v_{j_0, i_0}(c_{j_0}) < 0$. This implies that $d_0 \notin V$ for any $V \in M$, and then $d_0 \notin B_3 \cup \cdots \cup B_n$.

Set $M_0 := \{V_{1,i} \mid i > 2\} \cup \{V_{2,i} \mid i > 2\} \cup \{V_{3,i} \mid i > 3\}$, with $\{V_{3,i} \mid i > 3\} = \emptyset$ if $n = 3$. Obviously, $M_0 \subseteq M$, so that $d_0 \notin V_{j,i}$ for any $V_{j,i} \in M_0$.

Let $t_1, t_2 \in \mathbb{N}$, $t_1 \neq t_2$. We claim that $v_{j,i}(d_0^{1}) \neq v_{j,i}(d_0^{2})$ for any $V_{j,i} \in M_0$. Assume that $t_1 > t_2$ and set $t := t_1 - t_2$, that is $t_1 = t + t_2$. It follows that $d_0^{1} = d_0^{2} d_0^{3}$, so that $v_{j,i}(d_0^{1}) = v_{j,i}(d_0^{2}) + v_{j,i}(d_0^{3})$. 
Now, $v_{j,i}(d_0^{i}) = v_{j,i}(d_0^{0})$ implies $v_{j,i}(d_0^0) = 0$, that is $d_0^0 \in V_{j,i}$. But $V_{j,i} \subseteq S$ is Prüfer-Manis, and then integrally closed, so that $d_0 \in V_{j,i}$, a contradiction. Then, $v_{j,i}(d_0^0) \neq v_{j,i}(d_0^0)$.

Let $l \in \{1,2,3\}$ and consider the corresponding $a_l$ defined at the beginning of the proof. Then, there exists at most one $t_{j,i,l} \in \mathbb{N}$ such that $v_{j,i}(a_l) = v_{j,i}(d_0^{j,i,l})$. If there does not exist such $t_{j,i,l}$, we have $v_{j,i}(a_l) \neq v_{j,i}(d_0)$. In this case, set $t_{j,i,l} = 1$. It follows that in any case and for any $t > t_{j,i,l}$, we have $v_{j,i}(a_l) \neq v_{j,i}(d_0^l)$.

Let $t_0 := \sup\{1 + t_{j,i,l} \mid j,l \in \{1,2,3\}, i \in \{3,\ldots,n\}, i > j\}$.

Then, $v(d_0^0) \neq v(a_l)$ for any $V \in \mathcal{M}_0$ and any $l \in \{1,2,3\}$.

**Step 4.** Set $d := d_0^0$. Then, $v(d) \neq v(a_l)$ for any $V \in \mathcal{M}_0$ (**). Moreover, for any $V \in \mathcal{M}$, we have $d_0 \not\in V$, which implies $d \not\in V$ since $V \subset S$ is integrally closed. In particular, $d \not\in B_3 \cup \cdots \cup B_0$, but $d_0 \in R$ implies $d \in R \subseteq B_3 \cup B_2 \cup B_3 \cup \cdots \cup B_n$, so that $d \in B_1 \cup B_2$. Now, $B_1 \cup B_2 = (B_1 \cap B_2) \cup (B_1 \setminus B_2) \cup (B_2 \setminus B_1)$.

**Step 5.** We are going to consider the three possible cases for $d$.

1. $d \in B_1 \cap B_2$.

   Set $x := a_3 + d \in R$. Since $a_3 \in (R \cap B_3) \setminus (\cup_{i \in \mathbb{N}_n, i \neq 3} B_i)$, we have $a_3 \not\in B_1 \cup B_2$, so that $x \not\in B_1 \cup B_2$. Moreover, $d \not\in B_3$, which implies that $x \not\in B_3$. Let $i > 3$.

   If $v_{3,i}(d) < v_{3,i}(a_3)$, then $v_{3,i}(x) = v_{3,i}(a_3 + d) = v_{3,i}(d) < 0$ because $d \not\in V_{3,i}$. Then, $x \not\in B_i$.

   If $v_{3,i}(d) \geq v_{3,i}(a_3)$, then $v_{3,i}(d) > v_{3,i}(a_3)$ by (**), so that $v_{3,i}(x) = v_{3,i}(a_3) < 0$ because $a_3 \not\in V_{3,i}$. Then, $x \not\in B_i$.

   It follows that $x \not\in B_1 \cup B_2 \cup B_3 \cup \cdots \cup B_n$, a contradiction.

2. $d \in B_1 \setminus B_2$.

   Set $x := a_2 + d \in R$. Since $a_2 \not\in B_1$, we have $x \not\in B_1$ and since $a_2 \in B_2$, this implies that $x \not\in B_2$, so that $x \not\in B_1 \cup B_2$. Let $i > 2$.

   If $v_{2,i}(d) < v_{2,i}(a_2)$, then $v_{2,i}(x) = v_{2,i}(d) < 0$ because $d \not\in V_{2,i}$. Then, $x \not\in B_i$.

   If $v_{2,i}(d) \geq v_{2,i}(a_2)$, then $v_{2,i}(d) > v_{2,i}(a_2)$ by (**), so that $v_{2,i}(x) = v_{2,i}(a_2) < 0$ because $a_2 \not\in V_{2,i}$. Then, $x \not\in B_i$.

   It follows that $x \not\in B_1 \cup B_2 \cup B_3 \cup \cdots \cup B_n$, a contradiction.

3. $d \in B_2 \setminus B_1$.

   The proof is similar as in (2) by changing $B_1$ and $B_2$.

   To conclude, we get a contradiction in any case, so that there exists some $i$ such that $R \subseteq B_i$.

\[\Box\]

**Proposition 5.3.** Let $R \subseteq S$ be a Prüfer extension and $U, B_1, \ldots, B_n \in [R,S]$ such that $B_1 \cap \cdots \cap B_n \subseteq U$ and $U \subseteq S$ is Prüfer-Manis. Then there is some $i$ such that $B_i \subseteq U$. 

**Proof.** Actually, this result is given under an equivalent form in [27, Theorem 1.4, p.4]. Let $B_1,\ldots,B_n \in [R,S]$ be such that $B_1 \cap \cdots \cap B_n \subset U$. Then, we have $U = U(B_1 \cap \cdots \cap B_n) = UB_1 \cap \cdots \cap UB_n$ by [26, Theorem 1.4(4), p.86-87]. Now $[U,S]$ is a chain [26, Theorem 3.1, p. 187]; so that, $U = UB_i$ for some $i$ and then $B_i \subseteq U$. □

Note that if the extension $R \subseteq S$ is Prüfer-Manis, so is $U \subseteq S$ for any $U \in [R,S]$ [26, Corollary 3.2, P. 187].

Gotlieb proved the following result for a ring extension $R \subset K$ where $R$ is an integral domain with quotient field $K$ [21, Theorem 6].

**Theorem 5.4.** Let $R \subset S$ be an extension and $T, T_1, \ldots, T_n \in [R,S]$, such that $T = R\Sigma$, where $\Sigma$ is a mcs of $R$ and $R \to T_i$ is a flat epimorphism for $i = 1, \ldots, n$ such that $T \subseteq T_1 \cup \ldots \cup T_n$. Then $T$ is contained in some $T_i$.

**Proof.** Assume that $T$ is not contained in any $T_i$. By the (X)-rule, there are prime ideals $P_i \in \mathcal{X}(T_i) \setminus \mathcal{X}(T)$ and $\mathcal{X}(T) = \{P \in \Spec(R) \mid P \cap \Sigma = \emptyset\}$. We set $I := P_1 \cap \cdots \cap P_n$. We deduce from $T \subseteq T_1 \cup \ldots \cup T_n$ that $I \cap \Sigma = \emptyset$. There exists some prime ideal $P$ of $R$ such that $I \subseteq P$ and $P \cap \Sigma = \emptyset$. Then some $P_i$ is contained in $P$: so that, $P_i \in \mathcal{X}(T)$. Hence we get a contradiction. □

### 6. Pullback results

Consider the following pullback diagram (D) in the category of commutative unital rings:

$$
\begin{array}{ccc}
R & \xrightarrow{i} & S \\
\downarrow f & & \downarrow g \\
V & \xrightarrow{j} & K
\end{array}
$$

where $i$ and $j$ are ring extensions. It can be considered as a composite of the two diagrams:

- **Ker(D):**
  $$
  \begin{array}{ccc}
  \downarrow & & \\
  R/\Ker(f) & \longrightarrow & S/\Ker(g)
  \end{array}
  \quad \text{and} \quad
  \begin{array}{ccc}
  \downarrow & & \\
  f(R) & \longrightarrow & g(S)
  \end{array}
  $$

- **Im(D):**
  $$
  \begin{array}{ccc}
  \downarrow & & \\
  V & \longrightarrow & K
  \end{array}
  $$

The first diagram is a pullback because $\Ker(f) = \Ker(g)$ thanks to the pullback diagram (D). It follows that $R = f(R) \times_{g(S)} S$. 

It is of the form\[ A \longrightarrow B \]
where \( I \) is an ideal shared by the rings \( A \) and \( B \). We recall that in this case, \( A \subseteq B \) is Prüfer if and only if \( A/I \subseteq B/I \) is Prüfer ([26, Proposition 5.8, p.52]).

It is easy to prove that the second diagram is a pullback and is such that \( f(R) \) is isomorphic to \( V \cap g(S) \).

Recall that a ring \( R \) is called semi-hereditary if each of its finitely generated ideals is a projective \( R \)-module.

Olivier proved that an extension of rings \( R \subset S \) is integrally closed if and only if there is a pullback diagram (\( D \)), where \( V \) is a semi-hereditary ring with an (absolutely flat) total quotient ring \( K \) [35, Corollary p.56] or [33, Théorème de Ker Chalon (2.1)]. In this case, we call (\( DO \)) the diagram (\( D \)). Therefore, the Prüfer property is not descended in pullbacks, since \( V \subset K \) is Prüfer [13, Theorem 2] and there are integrally closed extensions that are not Prüfer.

On the other hand we have a pullback example provided by the following result.

**Proposition 6.1.** [13, Theorem 6.8 and Theorem 6.10] If \( R \) is a local ring, an extension \( R \subseteq S \) is Prüfer if and only if there exists \( P \in \text{Spec}(R) \) such that \( S = R_P, P = SP \) and \( R/P \) is a valuation domain. Under these conditions, \( S/P \) is the quotient field of \( R/P \) and \( P \) is a divided prime ideal of \( R \) (i.e. comparable to each ideal of \( R \)). In particular, \( [R,S] \) is a chain.

**Proof.** To complete the proof, observe that there is an order isomorphism \([R,S] \to [R/P,S/P]\) given by \( T \mapsto T/P \) for \( T \in [R,S] \). \( \Box \)

We now use Olivier’s result to find a characterization of Prüfer extensions.

**Theorem 6.2.** Let \( R \subset S \) be an integrally closed extension and (\( DO \)) the pullback diagram where \( V \) is semi-hereditary with total quotient ring \( K \). Then, \( R \subset S \) is Prüfer if and only if \( g(T)V \cap g(S) = g(T) \) for each \( T \in [R,S] \) or equivalently, the following diagram (\( D_T \)) is a pullback,

\[
\begin{array}{c}
T \longrightarrow S \\
\downarrow \quad \downarrow \\
g(T)V \longrightarrow K
\end{array}
\]

In that case, we have \( R = V \times_{g(T)V} T \) and \( g(T)V \cong V \otimes_R T \).
Proof. We use the characterization of Prüfer extensions by normal pairs and flat epimorphisms. Suppose that \((D_T)\) is a pullback. Since an over-ring of a semi-hereditary ring is semi-hereditary ([6, Corollary p.143]), Olivier’s result implies that \(T \subset S\) is integrally closed. Hence \(R \subset S\) is Prüfer. We now prove the converse. Suppose that \(R \subset S\) is Prüfer. Then \(R \subset T\) is a flat epimorphism. Tensoring the diagram \((D)\) by \(\otimes_R T\), we get another pullback diagram because the pullback \(R\) is a kernel of a morphism of \(R\)-modules and \(T\) is flat over \(R\). We next identify the rings of the new pullback. We have clearly \(T \cong R \otimes_R T\). Moreover we also have \(S \otimes_R T \cong S\). This is a consequence of [47, Satz 2.2 (d)] which states that if \(M\) is a \(T\)-module and \(R \to T\) an epimorphism, then \(M \otimes_R T \cong M\) (an isomorphism of \(T\)-modules). We next show that \(V \otimes_R T \cong g(T)V\). Consider the natural map \(V \otimes_R T \to K\); its image is \(g(T)V\). Then \(V \to V \otimes_R T\) is a flat epimorphism deduced from \(R \to T\) by the base change \(R \to V\) and \(V \to V \otimes_R T \to g(T)V\) is injective. It follows that \(V \otimes_R T \to g(T)V\) is an isomorphism, because a flat epimorphism is essential by [29, Lemme 1.2, p.109]. Then we show that \(K \otimes_R T \cong K\). We first observe that \(K \to K \otimes_T T\) is a flat epimorphism whose domain is an absolutely flat ring. This map is surjective. To see this, if \(J\) is the kernel of the morphism, then \(K/J \to K \otimes_T T\) is a faithfully flat epimorphism because \(K/J\) is absolutely flat whence is an isomorphism by [29, Lemme 1.2, p.109]. Moreover, \(V \to V \otimes_R T\) identifies to \(V \to g(T)V\); whence is injective. As \(V \to K\) is flat, the map \(K \to (V \otimes_R T) \otimes_V K \cong K \otimes_R T\) is injective, so that \(K \cong K \otimes_R T\).

Therefore, we have proved that there is a pullback diagram \((D_T)\). To complete the proof, it is enough to consider \(\text{Im}(D_T)\), in which case the pullback condition on \(T\) can be written \(g(T)V \cap g(S) = g(T)\).

Nevertheless, we give some example of pullbacks where the ascent property holds.

**Proposition 6.3.** Let \(I\) be an ideal of a ring \(S\) and set \(S' = S/I\). Denote by \(\varphi\) the canonical map \(S \to S/I\). Let \(R'\) be a subring of \(S'\)

\[
\begin{array}{ccc}
R & \longrightarrow & S \\
\downarrow & & \downarrow \\
R' & \longrightarrow & S'
\end{array}
\]

and \(R\) the pullback ring in the following diagram: \(R \subset S\) is Prüfer if and only of \(R' \subset S'\) is Prüfer.

Proof. Clearly \(I\) is an ideal shared by \(R\) and \(S\). Now, observe that \(R'\) identifies to \((R + I)/I \cong R/(I \cap R) \cong R/I\). It is then enough to apply [26, Proposition 5.8, p.52].
7. The case of a local base ring

When the base ring $R$ is local, we already gave a characterization of Prüfer extensions in Proposition 6.1.

**Definition 7.1.** An extension $R \subseteq S$ is called **module-distributive** if $R \cap (X + Y) = (R \cap X) + (R \cap Y)$ for each pair of $R$-submodules $(X,Y)$ of $S$ (cf. [26, p.119]). We say that $R \subseteq S$ is distributive if the lattice $[R,S]$ endowed with compositum and intersection as laws is distributive.

[26, Theorem 5.4, p.121] shows that an extension is $R \subseteq S$ module-distributive if and only if it is Prüfer. As a consequence we get that the set of $R_P$-submodules of $S_P/R_P$ is a chain [16, Corollary 2], when the extension is Prüfer. This gives a stronger result than that of Proposition 6.1. Moreover, we see that a Prüfer extension is both module-distributive and distributive. For the distributivity, use [41, Proposition 5.18] since a Prüfer extension $R \subset S$ is arithmetical (that is $R_M \subset S_M$ is chained for any $M \in \text{Max}(R)$).

In order to get more results, we introduce the following considerations.

In view of [26, Proposition 5.2, p.119], an ideal $I$ of a ring $R$ is called **distributive** if $I + (J \cap K) = (I + J) \cap (I + K)$ for all ideals $J,K$ of $R$. When $R$ is local, an ideal $I$ is distributive if and only if $I$ is comparable to each ideal (principal) ideal of $R$. In this case ($R$ is local), we will call $I$ a **strong divisor** if in addition $0 : I = 0$. The following result will be useful.

**Proposition 7.2.** [26, Example 5.1, p.119] Let $R \subseteq S$ be a Prüfer extension. An $S$-regular ideal $I$ of $R$ is distributive. In particular, such an ideal is a strong divisor if $R$ is local.

We can translate some results of [16, Lemma 1.1, Corollary] as follows. Let $R$ be a ring and set $\Sigma := \{\sigma \in R \backslash Z(R) \mid R\sigma$ is distributive\}. Then $\Sigma$ is a saturated mcs of $R$. Moreover, if $\mathcal{T}$ is a mcs of $R$, such that $\mathcal{T} \subseteq R \backslash Z(R)$ and $R \subseteq R_{\mathcal{T}}$ is Prüfer, then $\mathcal{T} \subseteq \Sigma$.

We next examine the local case. We may find in [26, p.123] the following definition and result.

**Definition 7.3.** A **strong divisor** $t$ of a local ring $R$ is an element $t$ of $R$, such that the ideal $Rt$ is a strong divisor. The set $\Delta(R)$ of all strong divisors of $R$ is a saturated mcs of $R$ and $U(R) \subseteq \Delta(R)$.

We observe that for $t \in \Delta(R)$, the open subset $D(t) = \{Q \in \text{Spec}(R) \mid Q \subset Rt\}$. 
Recall that a ring $R$ has a maximal Pr"ufer extension $R \subseteq \tilde{\mathbb{P}}(R) := \tilde{R}^{Q(R)}$ where $Q(R)$ is the complete ring of quotients of $R$ (Utumi-Lambeck) [26]. Then $\tilde{\mathbb{P}}(R)$ is called the Pr"ufer hull of $R$.

It is known that a Pr"ufer extension $R \subset S$, where $R$ is local, is a QR-extension; that is, is such that each $T \in [R, S]$ verifies $T \cong R_\Sigma$ (an isomorphism of $R$-algebras) for some mcs $\Sigma$ of $R$. For more information, see Proposition 8.3. Next result refines this observation.

**Proposition 7.4.** Let $R$ be a local ring and $R \subset S$ an extension.

1. $\tilde{\mathbb{P}}(R) = R_{\Delta(R)}$.
2. An extension $R \subset S$ is Pr"ufer if and only if $S = R_\Sigma$, for some mcs $\Sigma \subseteq \Delta(R)$ and, if and only if $R \subseteq Rs$ (i.e. $s^{-1}$ exists and belongs to $R$), for each $s \in S \setminus R$. In this case $S \subseteq \text{Tot}(R)$.

**Proof.** The proof is a consequence the following facts: $R_{\Delta(R)}$ is the Pr"ufer hull of $R$. If $R \subset S$ is Pr"ufer, there is some mcs $\Sigma \subseteq \Delta(R)$ such that $S = R_\Sigma$ in which case $R \subseteq R_\Sigma \subseteq R_{\Delta(R)}$ [26, Remark 5.9, Proposition 5.10 p.123]. The last assertion is [9, Proposition 3.1]. □

**Lemma 7.5.** Let $R \subset R_\Sigma := S$ be an extension of finite type, where $\Sigma$ is a mcs of $R$. Then there is some $x \in \Sigma$ such that $S = R_x$.

**Proof.** It follows from [7, Theorem 1.1], that $R \subset R_\Sigma := S$ is of finite presentation, because it is an injective flat epimorphism of finite type. Therefore, according to the (MCS)-rule, $\mathfrak{X}(S) = \cap [D(r) | r \in \Sigma]$ is an open subset of the patch topology (constructible topology) by the Chevalley Theorem and is even open because a flat morphism of finite presentation is open for the Zariski topology. As the patch topology is compact and the sets $D(r)$ for $r \in \Sigma$ are closed in this topology, we get that $\mathfrak{X}(S)$ is the intersection of finitely many $D(r_i)$ for $i = 1, \ldots, n$ with $r_i \in \Sigma$. Setting $x = r_1 \cdots r_n$, we get that $\mathfrak{X}(S) = D(x)$ and then $S = R_x$ by the ($\mathfrak{X}$)-rule. □

The next result is now clear.

**Proposition 7.6.** An extension $R \subset S$ of finite type over a local ring $R$ is Pr"ufer if and only if there is some $s \in \Delta(R)$ such that $S = R_s$.

**Proof.** Suppose that $R \subset S$ is Pr"ufer then $S = R_\Sigma$ for some mcs $\Sigma \subseteq \Delta(R)$ (Proposition 7.4). We deduce from Lemma 7.5 that $S = R_s$ for some $s \in \Delta(R)$. The converse is obvious. □

The following results will be useful.

**Proposition 7.7.** Let $R$ be a local ring and $x \in R$ a regular element. Then $x$ is a strong divisor if and only if $R \subseteq R_x$ is Pr"ufer.
Proof. Proposition 7.6 gives one implication. Suppose that $R \subset R_s$ is Prüfer. From Proposition 7.6 we deduce that $R_s = R_s$ for some strong divisor $s \in R$. It follows that $\sqrt{Rx} = \sqrt{Rs}$ and then $s^n = yx$ for some $n \in \mathbb{N}$ and $y \in R$. Therefore, $x$ is a strong divisor.

Example 7.8. Let $R$ be a local arithmetical ring. The set of all its ideals is a chain. It follows that each regular element $x$ of $R$ is a strong divisor and then $R \subset R_s$ is Prüfer.

Proposition 7.9. Let $f : R \to R'$ be a faithfully flat ring morphism between local rings and $x \in R$. If $f(x)$ is a strong divisor, so is $x$.

Proof. Observe that $x$ is regular in $R$. To conclude, use Proposition 2.5 because $R'_f(x) = R_x \otimes_R R'$.

Let $R \subset S$ be an extension and $\Delta$ a mcs of $R$. The large quotient ring $R_{[\Delta]}$ of $R$ (in $S$) with respect to $\Delta$ is the set of all $x \in S$ such that there is some $s \in \Delta$ with $sx \in R$. In case $\Delta = R \setminus P$, where $P$ is a prime ideal of $R$, we set $R_{[P]} := R_{\Delta}$.

Proposition 7.10. Let $R$ be a local ring and $R \subset S$ a flat extension, then $\tilde{R} = R_{[\Delta(R)]} = R_{\Sigma}$, where $\Sigma := \Delta(R) \cap U(S)$.

Proof. By Proposition 7.4 there is some multiplicatively closed subset $\Theta$ of $\Delta(R)$ such that $\tilde{R} = R_{\Theta}$. We have clearly $\Theta \subseteq U(S)$; so that $\Theta \subseteq \Sigma$. It follows that $R_{\Theta} \subseteq R_{\Sigma}$, while $R \subseteq R_{\Sigma}$ is Prüfer and therefore $R_{\Theta} = R_{\Sigma}$.

Now let $z \in R_{[\Delta(R)]}$, there is some $t \in \Delta(R)$ such that ($\ast$): $x = tz \in R$. Since $Rt$ is a strong divisor, $Rt$ and $Rx$ are comparable. Moreover, since $R \subset S$ is flat, $t$ is also regular in $S$.

If $Rx \subset Rt$, then $x = at$, so that $z = a \in R_t$, because $t$ is regular.

If $Rt \subset Rx$, then $t = bx$ and since $\Delta(R)$ is saturated, we get that $x \in \Delta(R)$ and $x$ is regular. We deduce from ($\ast$) that $bz = 1$ in $S$. It follows that $z \in U(S)$ and $z = b^{-1}$, with $b \in \Delta(R) \cap U(S)$, so that $z \in R_{\Sigma}$.

To conclude, we have $R_{[\Delta(R)]} \subseteq R_{\Sigma}$. As the reverse inclusion is obvious, we get finally that $R_{[\Delta(R)]} = R_{\Sigma}$.

If $Q$ is a prime ideal of a ring $R$, we denote by $Q^\uparrow$ its generization i.e $\{ P \in \text{Spec}(R) \mid P \subset Q \}$. The first author defined a prime $g$-ideal as a prime ideal $Q$ such that $Q^\uparrow$ is an open subset of $\text{Spec}(R)$ [37]. If $Q$ is a $g$-ideal of $R$, then $Q$ is a Goldman ideal of $R$; that is $R/P \subset \kappa(P)$ is of finite type as an algebra. [37].

Proposition 7.11. Let $s$ be a non-unit strong divisor of a local ring $R$ and $R \subset S := R_s$ the Prüfer extension associated. Then $P = \cap[R_s^n \mid
$n \in \mathbb{N}$} is a prime g-ideal, $S = R_P$, $PS = P$ is a divided prime ideal of $R$ and $R/P$ is a valuation domain with quotient field $S/P$. We will denote by $P_s$ the ideal $P$.

**Proof.** There exists $P \in \text{Spec}(R)$ such that $S = R_P$, $PS = P$ is a divided prime ideal of $R$ and $R/P$ is a valuation domain with quotient field $S/P$. According to Proposition 6.1, we aim to show that $I = P$. We have $I \subseteq P$, because if not, there is some $x \in I$ and $xy = s^p = bs^{p+1}$ for some $y, b \in R$ and $p \in \mathbb{N}$. Since $s$ is regular, it follows that $s$ is a unit, a contradiction. Now let $x \in P$ and suppose that $x \not\in I$. Then $x \not\in Rs^n$ for some positive integer $n$. Because $s^n$ is a strong divisor, we get $Rs^n \subseteq Rx$ and then $x$ belongs to $<s>$, a contradiction. Now $P$ is a prime g-ideal, because $P^i = D(s)$ is an open subset. □

If $R \subset S$ is a Prüfer extension of finite type over a local ring, there is some $s \in \Delta(R)$ such that $S = Rs$ by Proposition 7.1.

**Remark 7.12.** We use the notation of Proposition 7.11. It is easy to prove that $P = RsP$, because $s$ is regular and $P = \cap[Rs^n \mid n \in \mathbb{N}]$. Therefore, if $(R, M)$ is Noetherian and local and $s$ is not a unit, from $P = MP$ we deduce that $P = 0$ and $R$ needs to be an integral domain, so that $R$ is a Noetherian valuation domain, that is a discrete valuation domain, and $S$ is the quotient field of $R$. Another consequence is that if $R$ is not an integral domain, the only strong divisors of $R$ are the units.

The next result is now clear.

**Proposition 7.13.** Let $R \subset S$ be an extension over a local ring. The set of rings $\mathcal{F} := \{Rs \in [R, S] \mid s \in \Delta(R) \cap U(S)\}$ is a chain and $\bar{R}$ is the set union of all elements of $\mathcal{F}$. It follows that $\bar{R} \hookrightarrow \text{Tot}(R)$.

**Definition 7.14.** We say that two ideals $I$ and $J$ of a ring $R$ are equivalent if $\sqrt{I} = \sqrt{J}$ (equivalently $D(I) = D(J)$). We also say that two elements $x, y$ of $R$ are equivalent if $D(x) = D(y)$ and we write $x \simeq y$. This condition is equivalent to $Rx \cong Ry$ and also to $\sqrt{Rx} = \sqrt{Ry}$. Note that if $x$ is a strong divisor and $x \simeq y$, then $y$ is a strong divisor because $\sqrt{Rx} = \sqrt{Ry}$ and the set of all strong divisors is a saturated mcs.

**Remark 7.15.** We reconsider the context of Proposition 7.11 and we set $\delta(R) := \Delta(R) \setminus U(R)$. There is a surjective map $\delta(R) \to \{P_s \mid s \in\}$.
\( \delta(R) \), defined by \( s \mapsto P_s \). Setting \( \Delta \text{Spec}(R) := \{ P_s \mid s \in \delta(R) \} \), there is therefore a bijective map \( (\delta(R)/\sim) \to \Delta \text{Spec}(R) \).

Then \( \Delta \text{Spec}(R) \) is a chain. It follows that the set intersection of all its elements is a prime ideal \( \mathcal{R} \) that could be called the strong radical of the local ring \( R \). Now, according to Proposition 7.13 and the \((\text{MCS})\)-rule, \( \mathcal{X}(\mathbb{P}(R)) = \cap [D(s) \mid s \in \Delta(R)] = \cap [P_s^\perp \mid s \in \delta(R)] = \mathcal{R}^\perp \). It follows that \( \mathbb{P}(R) = R_{\mathcal{R}} \). If \( R \subset S \) is a ring extension, then \( \tilde{R} = \mathbb{P}(R) \cap S = R_{[R]} \).

We think that the set \( \Delta \text{Spec}(R) \) deserves a deeper study, especially with respect to some classes of rings.

**Proposition 7.16.** Let \( R \subset S \) be a Prüfer extension, where \( R \) is local and \( I \) an ideal of finite type of \( R \). Then \( I \) is \( S \)-regular if and only if \( I = R \rho \) where \( \rho \) is a strong divisor of \( R \), invertible in \( S \).

**Proof.** Assume that \( I \) is \( S \)-regular. From \( IS = S \) we deduce that \( I \) is a principal ideal \( R \rho \) by [26, Theorem 1.13, p. 91 and Proposition 2.3, p.97], because \( IS = S \) means that \( I \) is \( S \)-regular and, \( R \) being local, is \( S \)-invertible, whence principal of the form \( I = R \rho \). An appeal to Proposition 7.2 yields that \( R \rho \) is a strong divisor and \( S \rho = S \) shows that \( \rho \) is invertible in \( S \). The converse is obvious. \( \Box \)

**8. QR-extensions**

We first give some notation and definitions for an extension \( R \subset S \). For \( T \in [R,S] \), we set \( \Sigma_T := U(T) \cap R \), which is a mcs of \( R \) whose elements are regular and such that \( R \subset R_{\Sigma_T} \subseteq T \).

A Prüfer extension \( R \subset S \) is called Bezout, if each finitely generated \( S \)-regular ideal of \( R \) is principal [26 Definition 1; Theorem 10.2, p.145].

Let \((R,M)\) be a local ring, then an extension \( R \subset S \) is Bezout if and only if it is Prüfer, and if and only if \((R,M)\) is Manis in \( S \) [26, Scholium 10.4 p.147].

We call QR-extension any extension \( R \subset S \) such that each \( T \in [R,S] \) is of the form \( T \cong R_{\Sigma} \) (an isomorphism of \( R \)-algebras) for some mcs \( \Sigma \) of \( R \), in which case the elements of \( \Sigma \) are regular, invertible in \( S \) and \( T = R_{\Sigma_T} \). It is easy to show that \( R \subset S \) is a QR-extension if and only if the defining property holds for each \( T \in [R,S]_{fg} \). Moreover, an extension \( R \subset S \) is a QR-extension if and only if it is Prüfer and each finitely generated \( S \)-regular ideal \( I \) of \( R \) satisfies \( \sqrt{I} = \sqrt{R_x} \) for some \( x \in R \) (which implies that \( D(I) = D(x) \) is (special) affine) [26, Proposition 4.15, p.116].

A Prüfer extension does not need to be a QR-extension: look at the example [26, Section 4, Examples].
Proposition 8.1. A Bezout extension $R \subseteq S$ is a QR-extension.

**Proof.** We first observe that a subextension $R \subseteq T$ is Bezout. Then [26, Proposition 10.16, p.152] shows that $T = R_\Sigma$, for some mcs $\Sigma$ of $R$ and therefore the extension is QR. \qed

Corollary 8.2. Each extension $R \subseteq S$ has a unique Bezout subextension $R \subseteq T$, that contains any $T' \in [R, S]$, such that $R \subseteq T'$ is Bezout. Then $T$ is called the Bezout hull of $R$ and denoted here by $\beta(R)$.

**Proof.** It is enough to use [26, Theorem 10.14, p.151] \qed

Moreover, we have the next result.

Proposition 8.3. Let $R \subseteq S$ be an extension where projective $R$-modules of rank one are free. Then $R \subseteq S$ is Prüfer if and only if it is a QR-extension, and if and only if $R \subseteq S$ is Bezout. If the above statements hold, then a finitely generated $S$-regular ideal $I$ of $R$ is of the form $I = R\rho$, where $\rho$ is a locally strong divisor.

**Proof.** The first equivalence is [26, Proposition 4.16 p.116]. The second is a consequence of [26, Proposition 2.3, p.97] because under the hypotheses on $R$, a Prüfer extension is Bezout and the converse holds for an arbitrary ring $R$. The last statement is a consequence of Proposition 7.16. \qed

The condition on projective modules that are involved in this paper are either $R$ is semilocal or a Nagata ring $A(X)$ [18]. In particular we recover Proposition 6.1 in case $R$ is a local ring.

We will need an extension of the notion of strong divisors. A regular element of a ring $R$ is called a *locally strong divisor* (shorten in *lsd*) if $R \subseteq R_\delta$ is Prüfer. In order to justify this definition, we recall that an extension $R \subseteq S$ is Prüfer if and only if all its localizations by a prime ideal of $R$ are Prüfer. Hence if $x \in R$ is a lsd and $P$ is a prime ideal of $R$, then $x/1 \in R_P$ is a strong divisor. For the converse, use that if $x \in R$ is regular in every ring $R_P$, where $P$ is a prime ideal, so is $x$ because $R \rightarrow \prod [R_M \mid M \in \text{Max}(R)]$ is injective. The set of all locally strong divisors is a saturated mcs $\Lambda \Delta$. Clearly, a strong divisor of a local ring is a lsd. Now if $R \subseteq S$ is a ring extension, we denote by $\lambda\delta(R)$ the ring $R_{\Lambda \Delta \cap U(S)}$.

**Remark 8.4.** Let $f : R \rightarrow S$ be a ring morphism.

1. If $f$ is a flat morphism and $x \in R$ is such that $f(x)$ is a lsd, then so is $x$. Indeed for $Q \in \text{Spec}(S)$ lying over $P$, then $R_P \rightarrow S_Q$ is faithfully flat.
(2) If $f$ is a flat epimorphism and $x \in R$ is a lsd so is $f(x)$, because for each $Q \in \text{Spec}(S)$ and $P := f^{-1}(Q)$, the natural map $R_P \to S_Q$ is an isomorphism.

**Theorem 8.5.** Let $R \subset S$ be an extension. Then $R \subset S$ is a QR-extension if and only if each $T \in [R, S]$ of finite type over $R$ is of the form $T = R_s$ for some lsd $s \in R$. In particular, if these conditions hold, each $T \in [R, S]$ is of the form $T = R_T$, where $T \subseteq \Lambda \Delta$ is a mcs.

**Proof.** One implication is clear. Suppose that the extension is QR. To see that the condition holds, it is enough to suppose that it is of finite type. According to Lemma 7.5, there is some $s \in R$ such that $S = R_s$. Then $R \subseteq S$ is Prüfer, whence so is $R_P \subseteq S_P$ for each prime ideal $P$ of $R$ and $S_P = (R_P)s/1$. We have also $S_P = (R_P)y$, where $y \in \Delta(R_P)$ by Proposition 7.6. It follows that $D(s/1) = D(y)$ and by Definition 7.14 $s/1$ is a strong divisor. The last statement follows from the $\langle \text{MCS} \rangle$-rule applied to the flat epimorphism $R \subseteq T$, since $T$ is a union of finitely generated QR-extensions. □

**Theorem 8.6.** Any extension $R \subset S$ has a QR-hull; that is, there exists a largest QR-extension $\chi(R) \in [R, S]$, contained in $\bar{R}$. As a consequence, $\chi(R)$ is the compositum of all QR-extensions in $[R, S]$.

**Proof.** Let $X$ be the set of all QR-extensions in $[R, S]_{fg}$, which is directed upwards: take $T, U \in X$. They are of the form $R_x$ and $R_y$, where $x$ and $y$ are regular in $R$, because they are units in $S$. Then we have $R_x, R_y \subseteq R_{xy}$. We can now use the proof of [8, Theorem 5] which holds for an arbitrary extension $R \subset S$ and show that $R_{xy} \in X$.

Denote by $\chi(R)$ the set union of the elements of $X$. Since a QR-extension in $[R, S]$ is a union of finitely generated QR-extensions, it is contained in $\chi(R)$. To complete the proof, observe that an element of $[R, \chi(R)]_{fg}$ is contained in an element of $X$, whence is in $X$, from which we infer that $R \subseteq \chi(R)$ is a QR-extension. □

Actually, the proof of Davis shows that the set of all elements $x \in R$ such that $R \to R_x$ is a QR-extension is a mcs $\Omega(R)$ (also denoted $\Omega$) contained in the mcs $\Lambda \Delta$. Moreover, in case $R$ is either local or a Nagata ring, projective $R$-modules of rank one are free, so that an extension $R \subset S$ is Prüfer if and only if it is a QR-extension by Proposition 8.3, giving $\Omega \cap U(S) = \Lambda \Delta \cap U(S)$. An application of the $\langle \text{X} \rangle$-rule gives the following result.

**Corollary 8.7.** If $R \subset S$ is an extension, then $\chi(R) = R_{\Omega \cap U(S)}$. It follows that an extension $R \subset S$ is a QR-extension if and only if for each $s \in S$ there is some $\rho \in \Omega \cap U(S)$ such that $\rho s \in R$. 
We remark that $\beta(R) \subseteq \chi(R) \subseteq \lambda\delta(R) \subseteq \tilde{R}$.

**Proposition 8.8.** An extension $R \subset S$ is a QR-extension if and only if each $S$-regular finitely generated ideal is equivalent to a principal ideal of $R$ and there exists a mcs $\Sigma \subseteq \Lambda\Delta \cap U(S)$ such that $S = R\Sigma$. If these conditions hold, then $S = R_{\Lambda\Delta \cap U(S)}$.

**Proof.** Assume first that $R \subseteq S$ is a QR-extension. By the recall before Proposition 8.1, each $S$-regular finitely generated ideal is equivalent to a principal ideal of $R$. According to Theorem 8.5, there exists a mcs $\Sigma$, whose elements are some lsd of $R$, and such that $S = R\Sigma$. It follows that $S_M = (R\Sigma)_M = (R_M)_{\Sigma'}$, where $\Sigma'$ is a mcs whose elements are some lsd of $R_M$. Then, Proposition 8.1 implies that $R_M \subseteq S_M$ is Prüfer. Since this holds for any $M \in \text{Max}(R)$, we get that $R \subset S$ is Prüfer, and then a QR-extension by the recall before Proposition 8.1.

If these conditions hold, set $\Sigma' := \Lambda\Delta \cap U(S) \subset S$. Since $\Sigma'$ is a mcs whose elements are units of $S$, it follows that $R_{\Sigma'} \subseteq S$. But $\Sigma \subseteq \Sigma'$ implies $S = R\Sigma \subseteq R_{\Sigma'} \subseteq S$, so that $S = R_{\Sigma'}$. □

We end this section by considering ring extensions $R \subset S$ that are flat epimorphisms, such that the support $\text{Supp}(S/R)$ of the $R$-module $(S/R)$ is finite. We recall that $R \subset S$ is a flat epimorphism $\iff$ for all $P \in \text{Spec}(R)$, either $R_P = S_P$ is an isomorphism or $S = PS$, these two conditions being mutually exclusive [29, Proposition 2.4, p.112].

It is known that the support $\text{Supp}(S/R)$ of the $R$-module $S/R$ is the set of all $P \in \text{Spec}(R)$, such that $PS = S$. Therefore, each element of the support is $S$-regular. Moreover, the support is closed because as any support, it is stable under specialization. Hence the support equals to $V(J)$, where $J$ is the intersection of all elements $P_1, \ldots, P_n$ of the support. Now each $P_i$ is the radical of an $S$-regular finitely generated ideal, as an examination of the proof of [2 Corollary 13] by Abbas and Ayache shows. Moreover, assume that $R \subset S$ is a QR-extension. Using [26 Proposition 4.15, p.116], we get that $P_i$ is of the form $\sqrt{R x_i}$ for some $x_i \in R$. Then $J = \sqrt{R x}$ where $x = x_1 \cdots x_n$. Now if $I$ is an $S$-regular finitely generated ideal and $Q$ is a prime ideal of $R$ containing $I$, then $Q$ is $S$-regular. Reasoning as above we see that $\sqrt{I} = \sqrt{R y}$, for some $y \in R$. Taking into account the characterization of QR-extensions at the beginning of the section, we see that we have proved the following result:
Proposition 8.9. Let \( R \subset S \) be a Prüfer extension where \( \text{Supp}(S/R) \) is finite (in particular, if \( R \subset S \) has FCP). We set \( J := \bigcap \{ P \mid P \in \text{Supp}(S/R) \} \). The following statements are equivalent:

1. \( R \subset S \) is a QR-extension;
2. Each element of \( \text{Supp}(S/R) \) is equivalent to a principal ideal;
3. Each \( S \)-regular finitely generated ideal of \( R \) is equivalent to a principal ideal.

In case one of the above statement holds, \( J \) is a \( S \)-regular ideal equivalent to a principal ideal \( Rx \) and \( \Gamma(D(J)) = Rx \).

Proof. We only need to prove the following. By the flatness of the extension, \( JS = \bigcap \{ PS \mid P \in \text{Supp}(S/R) \} = S \). \( \square \)

Remark 8.10. Proposition 20 of [2] states that if, in addition to the above hypotheses, \( S \) is an integral domain, then each \( T \in [R,S] \) is of the form \( Rx \) for some \( x \in R \). This proves that the extension is strongly affine. Actually, in the proof of [2, Proposition 20], we can replace the Kaplansky transform of an ideal by a ring of sections.

9. Nagata extensions

We start this section by recalling some facts about Nagata rings, that are explained in [14, Section 3]. Let \( R \) be a ring and \( R[X] \) the polynomial ring in the indeterminate \( X \) over \( R \). We denote by \( C(p) \) the content of any polynomial \( p \in R[X] \). Then \( \Sigma_R := \{ p \in R[X] \mid C(p) = R \} \) is a saturated mc of \( R[X] \), each of whose elements is a non-zero-divisor of \( R[X] \). The Nagata ring of \( R \) is defined to be \( R(X) := R[X]_{\Sigma_R} \). Its main properties that are used in this section are the following. If \( I \) is an ideal of \( R \), we set \( I(X) = IR(X) \), which is an ideal of \( R(X) \). Now if \( P \) is a prime ideal of \( R \), then \( P(X) \) is a prime ideal of \( R(X) \) lying over \( P \). The inclusion map \( R \hookrightarrow R(X) \) is a faithfully flat ring homomorphism, and \( \text{Max}(R(X)) = \{ M(X) \mid M \in \text{Max}(R) \} \). Hence \( R(X) \) is local if \( R \) is local. In fact, \( R[X] \setminus \Sigma_R = \bigcup \{ M[X] \mid M \in \text{Max}(R) \} \). Also, note that any ring homomorphism \( f : R \to S \) extends to a ring homomorphism \( f_e : R[X] \to S[X] \) that fixes \( X \), and \( f_e \) in turn induces a ring homomorphism \( f_{\text{nag}} : R(X) \to S(X) \) that also fixes \( X \). By the remark before [12, Proposition II.9], if \( f \) is an extension, then so is \( f_{\text{nag}} \). Finally, if \( f : R \to S \) is a ring homomorphism, we will say that \( R(X) \otimes_R S = S(X) \) canonically (with respect to \( f \)) if the ring homomorphism \( g : R(X) \otimes_R S \to S(X) \) that is induced by \( f \) is an isomorphism. We will also say that \( f \) verifies the property (\( T \)). An FCP extension \( f \) verifies (\( T \)) [14, Proposition 3.2].

One sees easily that \( R(X)_{\text{Max}(R)} = R_M(X) \) for each \( M \in \text{Spec}(R) \).
Proposition 9.1. [38, Section IV, Proposition 4 and Proposition 7] A ring morphism \( f : R \to S \) verifies (\( T \)) when it is either integral (no necessarily injective) or a flat epimorphism. Consequently, for each \( P \in \Spec(R) \), the ring morphism \( R_P \to S_P \) verifies (\( T \)) (we say that the morphism verifies (\( T_P \))).

We deduce from the above proposition:
(A) If \( \Sigma \) is a mcs of \( R \) then \( R_\Sigma(X) = (R(X))_\Sigma \).
(B) If \( I \) is an ideal of \( R \), then \( (R/I)(X) = R(X)/I(X) \).

Proposition 9.2. If \( f : R \to S \) verifies (\( T \)) and (\( T_M \)) for some maximal ideal \( M \) of \( R \), then \( R(X)_M(X) \to S(X)_M(X) \) identifies to \( R_M(X) \to S_M(X) \).

Proof. We have \( S_M(X)^{T_M} S_M \otimes_{R_M} R_M(X) = (S \otimes_R R_M) \otimes_{R_M} R_M(X) = S \otimes_R R_M(X) = S \otimes_R R(X)_M(X) = S \otimes_R R(X) \otimes_R (X)_M(X) \)

Lemma 9.3. Let \( R \subseteq S \) be an extension and \( Q \) a prime ideal of \( S \), lying over \( P \) in \( R \). Then \( Q(X) \) is lying over \( P(X) \) and the natural map \( R(X)_{P(X)} \to S(X)_{Q(X)} \) identifies to \( R_P(X) \to S_Q(X) \).

Proof. That \( Q(X) \) is lying over \( P(X) \) is an easy consequence of the following fact: if \( p \in \bar{R}[X] \) and \( q \in \bar{R}[X] \) is such that \( C(q) = R \), then \( C(pq) = C(p) \). This follows from the Lemma of Dedekind-Mertens.

Proposition 9.4. [42 Corollary 3.15] Let \( R \subseteq S \) be an extension of rings, \( R \to R' \) a faithfully flat base change and \( S' := R' \otimes_R S \). If \( R' \to S' \) is Prüfer, then \( R \to S \) is Prüfer.

Proposition 9.5. Let \( R \to S \) be a ring extension. The following statements are equivalent:

1. \( R \to S \) is a flat epimorphism;
2. \( R(X) \to S(X) \) is a flat epimorphism;
3. \( R[X]_\Sigma = S(X) \), where \( \Sigma \subseteq R[X] \) is the mcs of all \( p(X) \in R[X] \) whose contents in \( S \) are \( S \).

Proof. (1) is equivalent to (3) by [36, Lemma 3.11]. Now \( R(X) \to S(X) \) is a flat epimorphism if \( R \to S \) is a flat epimorphism [14, Lemma 3.1(g)] and Proposition 9.1. To prove the converse we use [42, Scholium A (1)] which states that a ring morphism \( A \to B \) is a flat epimorphism if and only if its local morphisms are isomorphisms and \( A \to B \) is an \( i \)-morphism. We then consider a prime ideal \( Q \) of \( S \) lying over \( P \). Suppose that \( R(X) \to S(X) \) is a flat epimorphism. From Lemma
we get that \( R_P(X) \cong R(X)_{P(X)} \rightarrow S_Q(X) \cong S(X)_{Q(X)} \), and
deduce that \( R_P(X) \rightarrow S_Q(X) \) is an isomorphism, whence \( R_P \rightarrow S_Q \)
is injective. Identifying this map with an extension \( R_P \subset S_Q \), we get that \( R_P(X) = S_Q(X) \). It follows from [14]
Lemma 3.1(f)] that \( R_P = S_Q \cap R_P(X) = S_Q \cap S_Q(X) = S_Q \). The injectivity of \( S \rightarrow \text{Spec}(R) \)
may be proved as follows. If \( Q \) and \( Q' \) are prime ideals of \( S \) lying over \( P \) in \( R \), then \( Q(X) \) and \( Q'(X) \) are lying over \( P(X) \).

**Theorem 9.6.** An extension \( R \subset S \) is (quasi-)Prüfer if and only if \( R(X) \subset S(X) \) is (quasi-)Prüfer. In that case, \( \overline{R(X)} = \overline{R(X)} = \overline{R} \otimes_R R(X) \) and \( S(X) \cong S \otimes_R R(X) \).

*Proof.* Assume that \( R \subset S \) is Prüfer, we intend to show that so is \( R(X) \subset S(X) \). We can suppose that \( (R, M) \) is local according to Proposition 9.2 in order to use Proposition 2.4 and Proposition 6.1 Then it is enough to know the following facts: \( V(X) \) is a valuation domain if so is \( V \) and \( R[X]_{P[X]} \cong R(X)_{P(X)} \cong R_P(X) \) [14 Theorem 4 and Lemma 2]; \( R(X)/P(X) \cong (R/P)(X) \) for \( P \in \text{Spec}(R) \). Now suppose that \( R(X) \subset S(X) \) is Prüfer, then \( R \subset S \) is a flat epimorphism by Proposition 9.5. Hence, \( S(X) \cong R(X) \otimes_R S \) by Proposition 9.1. It follows that Proposition 9.4 entails that \( R \subset S \) is Prüfer since \( R \rightarrow R(X) \) is faithfully flat. For the quasi-Prüfer case, it is enough to use the Prüfer case and Proposition 9.1 because \( \overline{R(X)} = \overline{R(X)} \).

**Corollary 9.7.** An element \( x \) of ring \( R \) is a lsd of \( R \) if and only if \( x \) is a lsd of the ring \( R(X) \).

*Proof.* Obvious by Theorem 9.6 and Proposition 7.7.

**Proposition 9.8.** If \( R \subset S \) is an extension, then \( R(X) \subset S(X) \) is a Prüfer extension if and only if it is a QR-extension, and in this case \( S(X) = R(X)_{\Delta(R(X)) \cup U(S(X))} \).

*Proof.* If \( R(X) \subset S(X) \) is Prüfer, it is a QR-extension because over \( R(X) \) each projective module of rank one is free (Propositions 8.3, 8.8 and [18 Theorem 3.1]).

**Proposition 9.9.** Let \( R \subset S \) be a Prüfer extension of finite type. Then \( S(X) = R(X)_{g(x)} \) for some lsd \( g(X) \in R(X) \).

*Proof.* We know that \( R(X) \subset S(X) \) is Prüfer. To complete the proof, use the above result and Theorem 8.3.

If \( R \) is a local ring, within a unit, a non-unit strong divisor of the local ring \( R(X) \) can be written \( p(X)/1 \) where \( p(X) \in R[X] \) is regular with content \( c(p(X)) \neq R \). Then \( R(X)p(X)/1 \) is comparable to each
ideal. We note that if \( x \) is an element of \( R \), which is a strong divisor in \( R(X) \), it is a strong divisor of \( R \) by Proposition 7.9

**Theorem 9.10.** Let \( R \) be a local ring, \( p(X)/1 \) a non unit strong divisor of \( R(X) \) and \( I \) the content ideal of \( p(X) = \sum_{i=0}^{n} a_iX^i \in R[X] \). Then \( I = R \rho \) where \( \rho \) is a strong divisor of \( R \) such that \( p(X)/1 \sim \rho \) in \( R(X) \).

In fact, \( I(X) = R(X)p(X)/1 \). If \( p(X)/1 \in R(X)a_i \) for some \( a_i \), then \( I = Ra_i \). Moreover \( X_R(R(X)p(X)/1) = D(I) \).

**Proof.** We start by showing that \( I(X) = R(X)p(X)/1 \). We have that \( I = \sum_{i=0}^{n} Ra_i \) and \( p(X)/1 \in I(X) \). Moreover, \( p(X)/1 \) being a strong divisor of \( R(X) \), the ideal \( R(X)p(X)/1 \) is comparable to each \( R(X)a_i \). Assume first that \( a_i \in p(X)R(X) \) for each \( a_i \). Then, \( I(X) \subseteq R(X)p(X)/1 \subseteq I(X) \) gives \( I(X) = R(X)p(X)/1 \). Assume now that \( p(X)/1 \in R(X)a_i \) for some \( a_i \). Then \( p(X)/1 = a_iq(X)/s(X) \) for some \( q(X), s(X) \in R[X] \) with \( c(s) = R \) and \( a_i \) is regular. This implies that \( p(X)s(X) = a_iq(X) \) in \( R[X] \). By the Lemma of Dedekind-Mertens, \( c(ps) = c(p) = I = a_i c(q) \subseteq Ra_i \). Moreover, \( a_i c(q) = c(p) = Ra_i \) leads to \( c(q) = R \). To conclude, \( a_i/1 = p(X)s(X)/q(X) \) and we recover \( I(X) = R(X)p(X)/1 \). In both cases, \( I(X) = R(X)p(X)/1 \) shows that \( I(X) \) is a strong divisor of \( R(X) \).

We claim that \( I \) is a strong divisor of \( R \). Let \( x \in R \). Then \( xR(X) \) is comparable to \( I(X) \). If \( x \in I(X) \), then \( x/1 = r(t)/t(X) \), with \( r(X) \in I[X] \) and \( t(X) \in R[X] \) with \( c(t) = R \). It follows that \( xt(X) = r(X) \) in \( R[X] \), so that \( c(xt) = Rx = c(r) \subseteq I \). If \( I(X) \subseteq xR(X) \), let \( y \in I \). We have \( y/1 = xu(X)/v(X) \), with \( u(X), v(X) \in R[X] \) and \( c(v) = R \), so that \( yc(v) = xu(X) \in R[X] \), giving \( yc(v) = Rx = xc(u) \subseteq Rx \). Then, \( I \subseteq Rx \). In any case, \( Rx \) is comparable to \( I \), and \( I \) is a strong divisor of \( R \).

Setting \( S := R(X)p(X)/1 \), we have that \( R(X) \subseteq S \) is Prüfer. Because \( p(X) \in IR[X] \), \( I(X) \) is \( S \)-regular and then we have a tower of extensions \( R(X) \subseteq \Gamma(D(I(X))) \subseteq S \) by Remark 3.2.2. It follows that \( R(X) \subseteq \Gamma(D(I(X))) \) is Prüfer. Because \( R \rightarrow R(X) \) is faithfully flat, \( \Gamma(D(I(X))) \cong \Gamma(D(I)) \otimes_R R(X) \) by Proposition 3.7.3. According to Proposition 2.5, faithfully flat morphisms descend Prüfer extensions and then \( R \rightarrow \Gamma(D(I)) \) is Prüfer. Since \( I(X) = R(X)p(X)/1 \), we get \( IS = S \), then \( \Gamma(D(IS)) = \Gamma(D(S)) = S \).

It follows that \( R \rightarrow \Gamma(D(I)) \) is of finite presentation because this property is descended by faithfully flat ring morphism 3.4. Proposition 5.3]. Therefore, there is a strong divisor \( t \in R \), such that \( \Gamma(D(I)) = R_t \) by Proposition 7.6. Since \( I(X) \) is a principal ideal generated by a regular element, it is free, and then a flat ideal. But \( I(X) \cong I \otimes_R R(X) \).
by flatness of $R \to R(X)$. Because faithfully flat morphisms descend flatness \cite[Proposition 4.1]{34}, $I$ is a flat ideal of finite type, whence is free over $R$. Therefore, $I$ is a principal ideal $R\rho$. Then, $I(X) = \rho R(X) = R(X)p(X)/1$ shows that $\rho$ is a strong divisor of $R$.

The image of $\text{Spec}(S) \to \text{Spec}(R)$ is $D(I)$, because $\mathfrak{a}g(D(p(X))) = D(I)$. We get $X(R(S) \subseteq D(I)$ by Remark 3.2(1). Let $P \in D(I)$, so that there exists some coefficient of $p(X)$ which is not in $P$, giving $p(X)/1 \not\in P(X)$, which is a prime ideal of $R(X)$. Then, there exists $Q \in \text{Spec}(S)$ lying over $P(X)$, and then over $P \in \text{Spec}(R)$. It follows that $P \in X(R(S)$, giving $X(R(S) = D(I)$.

**Remark 9.11.** If $R \subseteq S$ is a Prüfer extension of finite type (for example an FCP extension or a minimal extension) and $R$ is a local ring, there is a strong divisor $\rho$ of $R$, such that $S = R\rho$. It follows that $S(X) = R(X)\rho$. As $R(X) \subseteq S(X)$ is of finite type and Prüfer, $S(X)$ is of the form $R(X)p(X)/1$, where $p(X)/1$ is a strong divisor of $R(X)$. We therefore have $p(X)/1 \simeq \rho$. We recover (Proposition 9.9) under a more precise form.

Here is an example illustrating Proposition 9.10.

Let $R$ be a local arithmetical ring. According to Example 7.8, each of its regular elements is a strong divisor. Now let $p(X)/1$ be a regular element of $R(X)$, where $p(X) \in R[X]$ has a content $\neq R$. If $a_0, \ldots, a_n$ are the coefficients of $p(X)$, one of them, say $a_i$, is a multiple of all the others. It follows that $p(X) = a_i q(X)$ where the content of $q(X)$ is $R$ and $a_i$ is regular. Therefore, $a_i$ is a strong divisor and $p(x)/1 \simeq a_i$. Note that $R(X)$ is local and arithmetical \cite[Theorem 3.1]{1}.

We now give some conditions implying that the Prüfer hull commutes with the formation of Nagata rings. We don’t know if these conditions are superfluous.

We say that a ring $R$ is quasi-Prüferian if $R \to R(X)$ is an $i$-morphism. These rings will be studied in Section 10.

**Proposition 9.12.** Let $R \subseteq S$ be a ring extension, where $R$ is quasi-Prüferian and $\tilde{R} \subseteq S$ is lying over (for example if the extension is almost-Prüfer), then $\tilde{R}(X) = \tilde{R}(X)$.

**Proof.** If $R$ is quasi-Prüferian so is $\tilde{R}$, because $R \subseteq \tilde{R}$ is a flat epimorphism, according to Proposition 10.3(1). Therefore $\text{Spec}(\tilde{R}(X)) \to \text{Spec}(\tilde{R})$ is bijective. The extension $\tilde{R}(X) \to \tilde{R}(X)$ is Prüfer, whence a flat epimorphism, which is surjective if it has lying-over. A prime ideal of $\tilde{R}(X)$ is of the form $P(X)$, where $P$ is a prime ideal of $\tilde{R}$. Let $Q$
be a prime ideal of \( S \) above \( P \). Then \( Q(X) \) contracts to \( P(X) \), which gives a prime ideal of \( \overline{R(X)} \) lying over \( P(X) \).

**Proposition 9.13.** Let \( R \subseteq S \) be a ring extension such that the map \( \psi : [R, S] \to [\overline{R(X)}, S(X)] \) defined by \( T \mapsto T(X) \) is bijective. Then, \( R(X) = \overline{R(X)} \).

**Proof.** According to Theorem 9.6, \( \overline{R(X)} \subseteq \overline{R(X)} \) is Prüfer. Then, \( \overline{R(X)} \subseteq \overline{R(X)} \). Assume that \( R(X) \neq \overline{R(X)} \); so that, there exists \( T \in [R, S] \), \( T \neq \overline{R} \) such that \( T(X) = \overline{R(X)} \) because \( \psi \) is bijective. Using again Theorem 9.6, it asserts that \( R \subseteq T \) is Prüfer, leading to \( T \subseteq \overline{R} \), and then to \( T(X) = R(X) \subseteq \overline{R(X)} \subseteq \overline{R(X)} \). To conclude, \( R(X) = \overline{R(X)} \). □

**Proposition 9.14.** Let \( R \subseteq S \) be a ring extension, where \( R \) is local quasi-Prüferian. Then, \( R(X) = \overline{R(X)} \).

**Proof.** Since \( R(X) \subseteq \overline{R(X)} \) is Prüfer and \( R(X) \) is local, Proposition 6.1 says that there exists \( Q' \in \text{Spec}(R(X)) \) such that \( R(X) = (R(X))_{Q'} \). In the same way, there exists \( P' \in \text{Spec}(R(X)) \) such that \( \overline{R(X)} = (R(X))_{P'} \). We still have \( \overline{R(X)} \subseteq R(X) \), so that \( Q' \subseteq P' \). Set \( P := P' \cap R \) and \( Q := Q' \cap R \). As \( R \) is quasi-Prüferian, it follows that \( P' = P(X) \) and \( Q' = Q(X) \), with \( Q \subseteq P \). Then, \( \overline{R(X)} = (R(X))_{P'} = R_P(X) \).

A same reasoning gives \( R(X) = (R(X))_{Q'} = R_Q(X) \subseteq S(X) \). In particular, \( \overline{R} = \overline{R(X)} \cap S = R_P(X) \cap S = R_P \). Now, \( R_Q \subseteq R_Q(X) \subseteq S(X) \) shows that \( R_Q \in [R, S] \). But, \( R \subseteq R_Q \) is Prüfer, since so is \( R(X) \subseteq R_Q(X) = \overline{R(X)} \). Then, we have \( R_Q \subseteq R_P \subseteq R_Q \) which implies \( R_Q = R_P \), and then \( R_Q(X) = R_P(X) \), that is \( \overline{R(X)} = \overline{R(X)} \). □

**Proposition 9.15.** If \( R \subseteq S \) is almost-Prüfer, then \( R(X) \subseteq S(X) \) is almost-Prüfer, so that \( \overline{R(X)} = \overline{R(X)} \).

**Proof.** If \( R \subseteq S \) is almost-Prüfer, then \( R \subseteq \overline{R} \) is Prüfer and \( \overline{R} \subseteq S \) is integral; so that, \( R(X) \subseteq \overline{R(X)} \) is Prüfer and \( \overline{R(X)} \subseteq S(X) \) is integral, whence \( R(X) \subseteq S(X) \) is almost-Prüfer with \( R(X) = \overline{R(X)} \). □

In case \( R \subseteq S \) is an FCP extension, we get more results. A minimal extension \( R \subseteq S \) is such that there exists a maximal ideal \( M \) of \( R \) satisfying \( \text{Supp}(S/R) = \{ M \} \). Such a prime ideal \( M \) is called the crucial (maximal) ideal \( C(R, S) \) of \( R \subset S \) [13, Theorem 2.1].
Lemma 9.16. Let $R \subset S$ be an FCP ring extension such that $\widetilde{R} = R$. Then, $\widetilde{R}(X) = R(X)$.

Proof. If $R(X) \not= \widetilde{R}(X)$, there is some $T' \in [R(X), \widetilde{R}(X)]$ such that $R(X) \subset T'$ is Prüfer minimal. Set $M' := C(R(X), T')$ which is in $\text{MSupp}(S/R(X))$. Let $M \in \text{MSupp}(S/R)$ be such that $M' = MR(X)$ [14, Lemma 3.3]. Since [42, Proposition 4.18(2)] asserts that $M' \not\in \text{MSupp}(R(X)/R(X)) = \text{MSupp}(\widetilde{R}(X)/R(X))$, this gives that $M \not\in \text{MSupp}(S/R)$, which entails that $M \in \text{MSupp}(S/R)$. By [43, Lemma 1.8], there exists $T \in [R, \widehat{R}]$ such that $M = C(R, T)$ with $R \subset T$ Prüfer minimal, a contradiction. □

Proposition 9.17. If $R \subset S$ has FCP, then, $\widetilde{R}(X) = \widetilde{R}(X)$.

Proof. Because $R \subset \widetilde{R}$ is Prüfer, $R(X) \subset \widetilde{R}(X)$ is Prüfer by Proposition 9.6. Then, $\widetilde{R}(X) \subset \widetilde{R}(X)$. Assume that $\widetilde{R}(X) \not= \widetilde{R}(X)$ and set $T := \widetilde{T}$, so that $T = \widehat{T}$, giving $T(X) = T(X) = \widetilde{R}(X)$ by Lemma 9.16. Hence $\widetilde{T}(X) \subset R(X)$ is a Prüfer extension, contradicting the definition of $T(X)$. So, $\widetilde{R}(X) = \widetilde{R}(X)$. □

10. Quasi-Prüferian rings

We call a ring $R$ quasi-Prüfer if $R \subset \text{Tot}(R)$ is quasi-Prüfer in order to have a coherent definition (R is classically called Prüfer if $R \subset \text{Tot}(R)$ is Prüfer). Trivially, a total quotient ring is quasi-Prüfer. Proposition 10.1 gives another example.

We recall that a ring $R$ is called McCoy (or satisfies the condition (A)) if each finitely generated ideal $I$ of $R$ contained in $Z(R)$ is such that $0 : I \not= 0$ [30].

It is easy to prove that if $R \rightarrow S$ is an injective flat epimorphism and $R$ is McCoy, then $S$ is McCoy. Indeed, a finitely generated ideal of $S$ is of the form $IS$ for some finitely generated ideal $I$ of $R$. It is well known that $R[X]$ is McCoy for any ring $R$, whence so is $R(X)$ [24, Corollary 1].

We will need the following definitions and results:

Let $M$ be an $R$-module and $P \in \text{Spec}(R)$. Then $P$ is called an attached prime ideal of $M$ (a strong Krull prime ideal by [25]) if for each finitely generated ideal $I \subseteq P$ there is some $x \in M$, such that $I \subseteq 0 : x \subseteq P$. The set of all attached prime ideals of $M$ is denoted by $\text{Att}(M)$. We recall that $Z(R) = \cup[P \mid P \in \text{Att}(R)]$. 
(1) An element of $\text{Att}(R[X])$ is of the form $P[X]$ for some $P \in \text{Att}(R)$ \cite[Theorem 2.5]{25}. Therefore, the set of zero-divisors of $R[X]$ is $Z(R[X]) = \bigcup \{P[X] \mid P \in \text{Att}(R)\}$.

(2) The mcs $S := \{p(x) \in R[X] \mid c(p(X)) = R\}$ is contained in $R[X] \setminus Z(R[X])$ and the elements of $\text{Att}(R[X])$ are of the form $P(X)$, where $P \in \text{Att}(R)$. It follows that $R[X] \subseteq R(X) \subseteq \text{Tot}(R[X])$ and then $\text{Tot}(R(X)) = \text{Tot}(R[X])$.

Lucas proved that a ring $R$ is McCoy if and only if $\text{Tot}(R)(X) = \text{Tot}(R[X]) (= \text{Tot}(R[X]))$ by \cite[Proposition 4.1]{30} and the above result (2).

**Proposition 10.1.** Let $R$ be a quasi-Pr"ufer McCoy ring, then $\text{Tot}(R(X)) \cong \text{Tot}(R)(X)$ and $R(X)$ is quasi-Pr"ufer.

**Proof.** We have $\text{Tot}(R(X)) = (\text{Tot}(R))(X)$, so that $R(X) \subseteq \text{Tot}(R(X))$ is quasi-Pr"ufer, because deduced from the flat epimorphism $R \rightarrow \text{Tot}(R)$ by the base change $R \rightarrow R(X)$. \hfill $\square$

Quasi-Pr"ufer rings defined in the book \cite{17} do not coincide with our quasi-Pr"ufer rings. They are called elsewhere quasi-Pr"uferian, a terminology we adopt.

**Definition 10.2.** \cite{5} A ring $R$ is called quasi-Pr"uferian if for each prime (resp.; maximal) ideal $M$ of $R$, any prime ideal $Q$ of $R[X]$, such that $Q \subseteq M[X]$ is of the form $(Q \cap R)[X]$. It is easy to show that a ring $R$ is quasi-Pr"uferian if and only if the faithfully flat ring morphism $R \rightarrow R(X)$ is an $i$-extension (because any maximal ideal of $R(X)$ is of the form $MR(X)$ for some $M \in \text{Max}(R)$). Another characterization is $\text{Spec}(R(X)) = \{P(X) \mid P \in \text{Spec}(R)\}$. We will use the following characterization: a ring $R$ is quasi-Pr"uferian if and only if $R \rightarrow R(X)$ has the incomparability property (the INC-property).

The class of quasi-Pr"uferian rings is stable under the formation of factor rings and the formation of Nagata rings. Moreover, a ring $R$ is quasi-Pr"uferian if and only if $R_P$ is quasi-Pr"uferian for each $P \in \text{Spec}(R)$. Over an integral domain the two classes, quasi-Pr"ufer and quasi-Pr"uferian, coincide \cite[Section 6.5]{17}. It follows that a ring which is locally a quasi-Pr"ufer domain is quasi-Pr"uferian.

The stability of the class of quasi-Pr"uferian rings under various operations does not seem to be valid for the class of quasi-Pr"ufer rings. For example, the formation of total quotient rings does not commute with localizations. We will show at the end of the Section (Example 10.8) that these classes are different.

We will use and sometimes generalize some results of \cite{3} holding in the integral domain context.
We intend now to generalize [3, Lemma 2.3 and Theorem 2.7].

**Proposition 10.3.** Let \( f : R \to S \) be a ring morphism.

1. If \( R \) is quasi-Prüferian and \( f \) is either an integral morphism or a flat epimorphism, then \( S \) is quasi-Prüferian.
2. If \( S \) is quasi-Prüferian and \( f \) is injective and integral, then \( R \) is quasi-Prüferian.
3. If \( R \) is quasi-Prüferian and \( f \) is a quasi-Prüfer extension, then \( S \) is quasi-Prüferian.

**Proof.**

(1) If \( R \) is quasi-Prüferian and \( f \) is either integral or a flat epimorphism, then \( S(X) \cong R(X) \otimes_R S \) by Proposition 9.1, so that \( R(X) \to S(X) \) has the INC-property and so has \( R \to R(X) \). It follows easily that \( S \to S(X) \) has the INC-property.

(2) Suppose that \( f \) is integral and injective, then so is \( R(X) \to S(X) \). Therefore, each couple \( P \subseteq P' \) of prime ideals of \( R(X) \) can be lifted up to a couple \( Q \subseteq Q' \) of prime ideals of \( S(X) \), by the lying-over and going-up properties of an integral extension. Because \( S \to S(X) \) has the INC-property, as well as \( R \to S \), we can assert that \( R \to R(X) \) has the INC-property.

(3) We have a tower of extensions \( R \subseteq \overline{R} \subseteq S \), where the last extension is Prüfer.

As usual if \( P \) is a prime ideal of a ring \( R \), a prime ideal \( Q \) of \( R[X] \) is called an upper of \( P \) if \( Q \neq P[X] \) and \( Q = Q \cap R \).

**Theorem 10.4.** Each next statement on a ring \( R \) implies the following:

1. \( R \) is quasi-Prüferian,
2. If \( P \in \text{Spec}(R) \) and \( M \in \text{Max}(R) \), then no upper of \( P \) is contained in \( M[X] \),
3. If \( M \) is a maximal ideal of \( R \), then no upper of a minimal prime ideal of \( R \) is contained in \( M[X] \),
4. Any flat injective epimorphism \( R \subseteq S \) is quasi-Prüfer,
5. \( R \subseteq \text{Tot}(R) \) is quasi-Prüfer, i.e. \( R \) is quasi-Prüfer.

**Proof.** We will follow the scheme of the proof of [3, Theorem 2.7]. We first prove the contrapositive of (4) \( \Rightarrow \) (3). Suppose that an injective flat epimorphism \( R \subseteq S \) is not quasi-Prüfer. Then there exists some \( u \in S \) such that \( R \subseteq R[u] \) does not satisfies INC. Hence by [10, Theorem 2.3], there exist distinct prime ideals \( Q'_1 \subset Q'_2 \) of \( R[u] \) and a maximal ideal \( M \) of \( R \), such that \( Q'_1 \cap R = Q'_2 \cap R = M \). Consider the ring morphism \( e : R[X] \to R[u] \), defined by \( X \mapsto u \) and set \( Q_i := e^{-1}(Q'_i) \). Let also be a minimal prime ideal \( N' \subset Q'_1 \) of \( R[u] \) and set \( N = e^{-1}(N') \). Now because \( R[u] \subseteq S \) is injective there is a minimal
prime ideal \( W \) of \( S \) lying over \( N \). Because \( R \subset S \) is flat, whence has the going-down property, we get that \( W \cap R = N \cap R \) is a minimal prime ideal \( \Pi \) of \( R \). As \( Q_1 \subset Q_2 \) are distinct prime ideals (\( e \) is a surjective map) of \( R[X] \) lying over \( M \), it follows that \( Q_1 = M[X] \) (it is enough to consider the field \( R/M \)).

Suppose that \( N \) is not an upper of \( \Pi \), then \( N = \Pi[X] \). We then have a factorization \( R/\Pi \to (R/\Pi)[X] = R[X]/\Pi[X] = R[X]/N \cong R[u]/N' \to S/W \) into injective morphisms. An appeal to [29, Corollaire 3.2(ii), p. 114] yields that \( R/\Pi \to (R/\Pi)[X] \) is a faithfully flat epimorphism, whence an isomorphism, an absurdity. Since \( Q_1 = M[X] \supset N \), then (3) fails.

**Lemma 10.5.** A ring \( R \), with \( R(X) \) treed, is treed and quasi-Prüferian.

**Proof.** [3, Proposition 2.2] is valid for arbitrary rings.

**Proposition 10.6.** Let \( R \) be a ring such that \( \text{Tot}(R) \) is zero-dimensional. Then \( R \) is quasi-Prüferian if and only if \( R \) is quasi-Prüfer. In this case, \( R(X) \) is treed and \( \text{Tot}(R(X)) \cong \text{Tot}(R) \otimes_R R(X) \cong (\text{Tot}(R))(X) \).

**Proof.** If \( R \) is quasi-Prüferian, then \( R \) is quasi-Prüfer by Theorem [10.4]. To prove the converse, and according to Lemma [10.3] we can suppose that \( R \) is Prüfer. Then we can reduce to the case where \( R \) is local, because for any prime ideal \( P \) of \( R \), we have \( (R(X))_P \cong R_P(X) \) and \( \text{Tot}(R)_P \cong \text{Tot}(R_P) \), so that \( \text{Tot}(R_P) \) is zero-dimensional. The proof of the last statement is as follows. There is an injective ring morphism \( \text{Tot}(R) \to \text{Tot}(R_P) \) because \( R_P \) is flat over \( R \). This morphism induces another one \( \text{Tot}(R) \otimes_R R_P \to \text{Tot}(R_P) \). There is a factorization \( \text{Tot}(R) \to \text{Tot}(R) \otimes_R R_P \to \text{Tot}(R_P) \). The first morphism is a flat epimorphism and the composite is injective. We get that \( \text{Tot}(R) \otimes_R R_P \to \text{Tot}(R_P) \) is injective by [29, Lemme 3.4, p.114]. To conclude, we use the following fact: an injective flat epimorphism, whose domain is zero-dimensional, is an isomorphism [29, Lemme 1.2, p. 109], whence the injective flat epimorphism \( \text{Tot}(R) \otimes_R R_P \to \text{Tot}(R_P) \) is an isomorphism.

We claim that \( R \) is treed if \( R \subset \text{Tot}(R) \) is Prüfer when \( R \) is local. In this case, there exists some divided prime ideal \( P \) of \( R \) such that \( \text{Tot}(R) = R_P \) and \( R/P \) is a valuation domain. As \( T := R_P \) is zero-dimensional, \( P \) is a minimal prime ideal of \( T \). In fact, \( \text{Spec}(T) = \{ P \} \). Then \( P \) is also a minimal prime ideal of \( R \) because \( T = R_P \). Moreover, \( R \to T \) being injective, \( P \) is the unique minimal prime ideal of \( R \). At last, \( \text{Spec}(R) = V(P) \) is a chain because \( R/P \) is a valuation domain, so that \( R \) is treed.
The same proof works for $R(X)$, which is also local and such that $R(X) \subset T(X)$ is Prüfer. To begin with, $T$ being zero-dimensional, so is $T(X)$, because the (minimal prime ideals) maximal ideals of $T(X)$ are of the form $MT(X)$ where $(M \in \Min(T)) M \in \Max(T)$. Then there is a factorization $R(X) \rightarrow \Tot(R) \otimes_R R(X) \rightarrow \Tot(R(X))$, where the first morphism is a flat epimorphism and the composite is an injective flat epimorphism. It follows that the last morphism, being a flat injective epimorphism, is an isomorphism because its domain is zero-dimensional. Therefore, $T(X) \cong T \otimes_R R(X) \cong \Tot(R(X))$, leading to $R(X) \rightarrow \Tot(R(X))$ is Prüfer, with $\Tot(R(X))$ zero-dimensional. Mimicking the proof we got for $R$, it follows that $R(X)$ is treed and we can apply Lemma 10.5 to get that $R$ is quasi-Prüferian.

**Remark 10.7.** If we suppose that $\dim(M(R)) = 0$, then $M(R_P) \cong (M(R))_P$. To see this it is enough to use [29, Proposition 3.5, p.115]. Suppose now in addition that $R \subset M(R)$ is quasi-Prüfer. Mimicking the proof of Proposition 10.6 we can also show that $R$ is quasi-Prüferian. The reader may find in [29] many contexts in which $\Tot(R) = M(R)$.

**Example 10.8.** A total quotient ring (which is a (quasi-)Prüfer ring) need not to be quasi-Prüferian. To see this we consider the following example given by Lucas [31, Example 2.11]. There is a total quotient ring $R$ which is not locally Prüfer because there is a prime ideal $P$ of $R$ such that $R_P$ is not Prüfer but is integrally closed (actually, an UFD). Suppose that quasi-Prüfer rings are quasi-Prüferian, then $R_P$ would be quasi-Prüferian, since quasi-Prüfer. As it is integrally closed, it would be Prüfer, a contradiction.

11. Prüfer FCP extensions over a local ring

Clearly, a minimal extension is a flat epimorphism if and only if it is Prüfer. So we will call Prüfer minimal such extensions. We note as a first result the following Proposition, which results from Proposition 8.9.

**Proposition 11.1.** A Prüfer minimal extension with crucial ideal maximal $M$ is a QR-extension if and only if $M$ is equivalent to a principal ideal.

Proposition 6.1 take the following form, observing that a Prüfer extension is integrally closed.

**Proposition 11.2.** [13, Theorem 6.8 and Theorem 6.10] If $R$ is a local ring, an extension $R \subseteq S$ is Prüfer FCP (resp.; minimal) if and only...
if there exists $P \in \text{Spec}(R)$ such that $S = R_P$, $P = SP$ and $R/P$ is a finite dimensional (resp.; one-dimensional) valuation domain. Under these conditions, $S/P$ is the quotient field of $R/P$ and $P$ is a divided prime ideal of $R$.

The conductor of a ring extension $R \subset S$ is denoted by $(R : S)$. The following Corollary recalls, for a Prüfer minimal extension $R \subset S$, the link between the crucial ideal maximal $C(R,S)$ and $(R : S)$.

**Corollary 11.3.** If $R \subset S$ is a Prüfer minimal extension with crucial ideal maximal $M$, then, $P := (R : S)$ is a prime ideal of $R$, $P \subset M$, and there is no prime ideal of $R$ contained strictly between $P$ and $M$.

**Proof.** First, $P := (R : S)$ is a prime ideal of $R$ by [19, Lemme 3.2]. Moreover, $P_M = P_R M = (R_M : S_M)$ with $R_M \neq S_M$ shows that $P \subset M$. At last, $R_M \subset S_M$ is also a Prüfer minimal extension. Then, according to Proposition 11.2, $R_M/P_M$ is a one-dimensional valuation domain, so that there is no prime ideal of $R_M$ contained strictly between $P_M$ and $MR_M$ giving that there is no prime ideal of $R$ contained strictly between $P$ and $M$. □

**Corollary 11.4.** Let $R \subset S$ be a Prüfer minimal extension over a local ring $(R,M)$. Then, with the notation of Proposition 11.2, each element $t \in M \setminus P$ is a strong divisor of $R$, $S \cong R_t$, $P = \cap [Rt^n | n \in \mathbb{N}]$ and $M = \sqrt{Rt}$.

**Proof.** Because $t \notin P$, we have $t \in U(S)$ and then a factorization $R \subset R_t \subset S$, so that $S = R_t$ by minimality. By (Proposition 7.6), $t$ is a strong divisor. The third statement follows from Corollary 7.11. Because $R/P$ is one-dimensional, $M$ is the only prime ideal containing $P$, so that $M = \sqrt{Rt}$. □

**Proposition 11.5.** Let $R \subset S$ be a ring extension where $(R,M)$ is a local ring. The following statements are equivalent:

1. $R \subset S$ is a Prüfer minimal extension;
2. there is a strong divisor $a \in R \setminus U(R)$ such that $S = R_a$ and $\sqrt{Ra} \subseteq \sqrt{Rb} \Rightarrow \sqrt{Ra} = \sqrt{Rb}$ (or equivalently, $D(a) \subseteq D(b) \Rightarrow D(a) = D(b)$) for each $b \in R \setminus U(R)$;
3. there is a strong divisor $a \in R \setminus U(R)$ such that $S = R_a$ and $M = \sqrt{Ra}$;
4. there is a strong divisor $a \in R \setminus U(R)$, such that $S = R_a$, and such that $D(a)$ is an open affine subset, maximal in the set of proper open affine subsets.

**Proof.** We have clearly (2) $\iff$ (3) by (Corollary 11.3), once (1) $\iff$ (2) is proved. We then prove that (1) is equivalent to (2). Suppose that
$R \subset S$ is a minimal extension, that is a flat epimorphism. Then $R \subset S$
 is clearly a Pr"ufer extension. By Proposition 7.10 there is a mcs $\Sigma$ of $R$, whose elements
are strong divisors and such that the extension identifies with $R \subset R_\Sigma$. Picking an arbitrary
element $a \in \Sigma$, we get a factorization $R \to R_a \to R_\Sigma$. Its factors are injective because the flat
epimorphism $R \to R_a$ verifies [29, Lemme 3.4, p.114].

As $a$ is not invertible, $R \neq R_a$ implies $S = R_a$, by minimality of $R \subset S$. In the same way a
coloration $R \subset R_b \subset R_a$ implies $R_b = R_a$, or equivalently $D(a) \subset D(b) \Rightarrow D(a) = D(b)$, which means that
$\sqrt{Ra} \subset \sqrt{Rb} \Rightarrow \sqrt{Ra} = \sqrt{Rb}$.

We prove the converse. Observe that for any mcs $\Sigma$ of $R$ and $a \in R$
such that there is a factorization $R \to R_\Sigma \to R_a$, we have $D(a) \subset \cap\{D(\sigma) \mid \sigma \in \Sigma\}$. Suppose that $R \subset S$
verifies the conditions of the proposition. Then $S = R_a$, where $a \in R \setminus U(R)$ is a strong divisor and
then $R \subset S$ is Pr"ufer, so that any subextension $R \subset T \subset S$ is Pr"ufer. By Proposition 7.10 we get that $T = R_\Sigma$, for some mcs $\Sigma$ of $R$. The
above observation shows that $D(a) \subset D(\sigma)$ for any $\sigma \in \Sigma$. The last
condition entails that $\Sigma = \{ a \}$, and then $T = R_a$. Therefore, $R \subset S$
is minimal and a flat epimorphism.

Clearly (4) implies (2). The converse is a consequence of the following
facts. If $D(a) \subset D(I)$, where $D(I)$ is an open affine subset different from
$\text{Spec}(R)$, we have a factorization $R \subset \Gamma(D(I)) \subset R_a$ and $R \to \Gamma(D(I))$
is an injective flat epimorphism whose spectral image is $D(I)$.

Lemma 11.6. Let $R$ be a ring and $a \in R$. Then, there exists some
$M \in \text{Max}(R)$ such that $M = \sqrt{Ra}$ if and only if $\text{Supp}(R_a/R) = \{ M \}$.

Proof. Let $M \in \text{Max}(R)$. Then $M = \sqrt{Ra} \iff M$ is the only $P \in \text{Spec}(R)$
containing $a$ $\iff$ for any $P \in \text{Spec}(R) \setminus \{ M \}$, $a \not\in P$ and
$a \in M$ $\iff$ for any $P \in \text{Spec}(R) \setminus \{ M \}$, $a/1 \in U(R_P)$ and $a/1 \notin U(R_M)$$\iff$ for any $P \in \text{Spec}(R) \setminus \{ M \}$, $R_P = (R_a)_P$ and $R_M \neq (R_a)_M \iff \text{Supp}(R_a/R) = \{ M \}$.\hfill $\square$

Proposition 11.7. Let $R \subset S$ be a ring extension. The following
statements are equivalent:

1. $R \subset S$ is a minimal QR-extension;
2. there exists some $M \in \text{Max}(R)$ such that $\text{Supp}(S/R) = \{ M \}$
   and there is a lsd $a \in R \setminus U(R)$ such that $S = R_a$;
3. there is a lsd $a \in R \setminus U(R)$ such that $S = R_a$ and there exists
   some $M \in \text{Max}(R)$ such that $M = \sqrt{Ra}$;

If these conditions are satisfied, then $M \not\subset \cup\{ P \in \text{Max}(R) \mid P \neq M \}$.

Proof. (1) $\Rightarrow$ (2) Assume that $R \subset S$ is a minimal QR-extension.
Then, there exists some $M \in \text{Max}(R)$ such that $\text{Supp}(S/R) = \{ M \}$.
Since \( R \subset S \) is a minimal QR-extension, Theorem \[8.5\] asserts that there is a lsd \( a \in R \setminus U(R) \) such that \( S = R_a \), since \( S \) is of finite type over \( R \). It follows that \( \text{Supp}(R_a/R) = \{M\} \), which gives \( M = \sqrt{Ra} \) by Lemma \[11.6\] so that \( a \in M \setminus \bigcup \{P \in \text{Max}(R) \mid P \neq M\} \), giving \( M \not\subseteq \bigcup \{P \in \text{Max}(R) \mid P \neq M\} \), proving the last assertion.

(2) \( \Rightarrow \) (1) Assume that there exists some \( M \in \text{Max}(R) \) such that \( \text{Supp}(S/R) = \{M\} \) and there is a lsd \( a \in R \setminus U(R) \) such that \( S = R_a \). These two conditions lead to \( M = \sqrt{Ra} \) by Lemma \[11.6\]. As \( a/1 \) is a strong divisor of \( R_M \) and \( S_M = (Ra)_M = (R_M)_{(a/1)} \), Proposition \[11.5\] shows that \( R_M \subset S_M \) is minimal Prüfer. Moreover, \( R_P = S_P \) for any \( P \in \text{Max}(R) \), \( P \neq M \) implies that \( R \subset S \) is minimal Prüfer. At last, Theorem \[8.5\] shows that \( R \subset S \) is a QR-extension, since minimal.

(2) \( \Leftrightarrow \) (3) by Lemma \[11.6\].

**Corollary 11.8.** A minimal Prüfer extension \( R \subset S \) such that \( S = R_s \) for some lsd \( s \) of \( R \) is a QR-extension.

**Proof.** Since \( R \subset S \) is minimal, there exists some \( M \in \text{Max}(R) \) such that \( \text{Supp}(S/R) = \{M\} \). Then Proposition \[11.7\] shows that \( R \subset S \) is a QR-extension. \( \square \)

**Example 11.9.** Set \( R := \mathbb{Z} \), \( P := p\mathbb{Z} \) where \( p \) is a prime integer and \( S := \mathbb{Z}_p \). Obviously, \( \text{Supp}(S/R) = \{P\} \), so that \( S = R_p \) and \( p = \sqrt{Rp} \). Then, \( p/1 \) is a strong divisor of \( R_P \) and \( p/1 \in U(R_M) \) for any \( M \in \text{Spec}(R) \setminus \{P\} \), showing that \( p \) is a lsd of \( R \). We recover the fact that \( \mathbb{Z} \subset \mathbb{Z}_p \) is a minimal Prüfer QR-extension.

Next result shows that Prüfer FCP extensions can be described in a special manner.

**Proposition 11.10.** [12, Proposition 1.3] Let \( R \subset S \) be an FCP extension. Then \( R \subset S \) is integrally closed \( \Leftrightarrow \) \( R \subset S \) is Prüfer \( \Leftrightarrow \) \( R \subset S \) is a tower of Prüfer minimal extensions.

**Theorem 11.11.** An FCP QR-extension \( R \subset S \) admits a tower of Prüfer minimal extensions \( R \subset R_1 \subset \cdots \subset R_i \subset R_{i+1} \subset \cdots \subset R_n = S \), where \( R_{i+1} = (R_i)_{a_i} = R_{a_i} \) for some lsd \( a_i \in R \) and \( S = R_{a_1 \ldots a_n} = R_{a_n} \). The integer \( n \) is independent of the sequence and is equal to \( |\text{Supp}(S/R)| \).

**Proof.** There is a tower of Prüfer minimal extensions \( R_i \subset R_{i+1} \) by Proposition \[11.10\] because a QR-extension is Prüfer. Therefore, each \( T \in [R, S] \) is a localization \( R_a \), for some lsd \( a \in R \) by Theorem \[8.5\] and \( R_i \subset R_{i+1} \) identifies to \( R_i \subset R_{a_i} \) for some lsd \( a_i \in R \setminus U(R_i) \).

Then by minimality, we get that \( R_{i+1} = R_{a_i} = (R_i)_{a_i} \) and the result follows. The last result is [13, Proposition 6.12]. \( \square \)
The above result applies when \( R \subseteq S \) is an FCP extension \( A(X) \subseteq B(X) \), or equivalently, \( A \subseteq B \) has FCP [14, Theorem 3.9].

**Proposition 11.12.** Let \( R \subset S \) be an FCP Prüfer extension over a local ring \((R,M)\).

1. There is a sequence of Prüfer minimal extensions between local rings \( R \subset R_1 \subset \cdots \subset R_i \subset R_{i+1} \subset \cdots \subset R_n = S \), where \( R_{i+1} = (R_i)_{a_i} = R_{a_i} \) for some \( a_i \in \Delta(R) \) and \( S = R_{a_1 \cdots a_n} = R_{a_n} \). In fact \([R,S] = \{R_i\}_{i=0}^n\). Moreover, \( R \subset S \) is a QR-extension.

2. There is some subset \( \{P_0, P_1, \cdots, P_n\} \) of Spec\((R)\), with \( P_0 = M \) and \( P_i \subset P_{i-1} \) for \( i = 1, \cdots, n \), such that \( R_i = R_{P_i} \), \( P_iR_i = P_i \) and \( R/P_i \) is a valuation domain whose dimension is \( i \).

3. For all \( i = 1, \cdots, n \) and \( t \in P_{i-1} \setminus P_i \), we have \( R_i = R_t \). Moreover, each element of \( M \setminus P_n \) is a strong divisor.

4. Any finitely generated \( S \)-regular ideal is equivalent to a principal ideal.

**Proof.** (1) We know that \( R \subset S \) is chained because \( R \) is local [13, Theorem 6.10], and each \( R_i \) is local by Proposition [11.2]. Moreover, this proposition shows that for each \( i \in \mathbb{N}_n \), there exists \( P_i \in \text{Spec}(R) \) such that \( R_i = R_{P_i} \), \( P_iR_i = P_i \) and \( R/P_i \) is a \( i \)-dimensional valuation domain. Therefore, \( R \subset S \) is a QR-extension. Then apply Theorem [11.11].

(2) Since \( R \subset R_1 \) is Prüfer minimal, \( P_0 := M = \mathcal{C}(R,R_1) \). As \( S = R_n \), it follows that \( R/P_n \) is a \( n \)-dimensional valuation domain and \( \{P_0, P_1, \cdots, P_{n-1}\} = \text{Supp}(S/R) \) according to [13, Proposition 6.12] with \( P_i \subset P_{i-1} \) for each \( i \in \mathbb{N}_n \).

(3) Let \( t \in P_{i-1} \setminus P_i \). Then, \( t \) is a unit of \( R_{P_i} = R_i \) and \( R_t \subseteq R_i \). As \( t \in P_{i-1} \), this implies that \( t \) is not a unit of \( R_{P_{i-1}} = R_{t-1} \), so that \( R_t \nsubseteq R_{t-1} \). But \( [R, S] \) is a chain, which leads to \( R_t = R_i \).

Let \( x \in M \setminus P_n \). Since \( \{P_0, P_1, \cdots, P_n\} \) is a chain, there exists some \( i \in \mathbb{N}_n \) such that \( x \in P_{i-1} \setminus P_i \). Then, \( R_x = R_i \) by the first part of (3). To end, Proposition [7.6] and Definition [7.14] show that \( x \) is a strong divisor.

(4) Since \( R \subset S \) is a QR-extension, any finitely generated \( S \)-regular ideal is equivalent to a principal ideal according to the recall at the beginning of Section 8.

We end this section by a generalization of Proposition [11.12] to \( \mathcal{B} \)-extensions. We recall that an extension \( R \subset S \) is a \( \mathcal{B} \)-extension if the map \( \beta : [R,S] \to \prod[[R_M, S_M] \mid M \in \text{MSupp}(S/R)] \) defined by \( T \mapsto (T_M)_{M \in \text{MSupp}(S/R)} \) is bijective. Actually, an FCP extension \( R \subseteq S \) is a
**Proof.**  (3). \[ \text{B-extension if and only if } R/P \text{ is a local ring for each } P \in \text{Supp}(S/R) \] \cite[Proposition 2.21]{44}. The following lemma gives a first special case of $\text{B}$-extension. For any extension $R \subseteq S$, the length $\ell[R, S]$ of $[R, S]$ is the supremum of the lengths of chains of $R$-subalgebras of $S$. Notice that if $R \subseteq S$ has FCP, then there does exist some maximal chain of $R$-subalgebras of $S$ with length $\ell[R, S]$ \cite[Theorem 4.11]{14}.

**Lemma 11.13.** Let $R \subset S$ be an FCP Prüfer extension such that $|\text{MSupp}(S/R)| = 1$. Then, $R \subset S$ is a $\text{B}$-extension where $\text{Supp}(S/R)$ and $[R, S]$ are chains with $n := |\text{Supp}(S/R)| = |[R, S]| - 1$. There is a tower of Prüfer minimal extensions $R \subset R_{1} \subset \cdots \subset R_{i} \subset R_{i+1} \subset \cdots \subset R_{n} = S$, and for each $\beta$, $R_{i} = S/R_{i}$, such that $[R, S] = \{R_{i}\}_{i=0}^{n}$. We define as follows a subset $\{P_{0}, P_{1}, \ldots, P_{n}\}$ of $\text{Spec}(R)$ by $\text{Supp}(S/R) = \{P_{0}, P_{1}, \ldots, P_{n-1}\}$ and $P_{n} := R \cap (R_{n-1} : S)$, where $P_{i} \subset P_{i-1}$ for each $i \in \mathbb{N}_{n}$. In particular, $\text{Supp}(R_{i}/R) = \{P_{0}, P_{1}, \ldots, P_{i-1}\}$.

**Proof.** From $|\text{MSupp}(S/R)| = 1$, we deduce that $R \subset S$ is a $\text{B}$-extension by \cite[Proposition 2.21]{44}. If $\{M\} := \text{MSupp}(S/R)$, the map $\beta : [R, S] \to [R_{M}, S_{M}]$ defined by $\beta(T) = T_{M}$ for any $T \in [R, S]$ is bijective. But, $R_{M} \subset S_{M}$ is chained by Proposition 11.12, whence $R \subset S$ is chained. According to \cite[Theorem 3.10]{45}, $\text{Supp}(S/R)$ has a least element $P_{0}$ and $\text{Supp}(S/R) = V(P)$ is chained. Moreover, $|\text{Supp}(S/R)| = |[R, S]| - 1 = \ell[R, S]$ by \cite[Proposition 6.12]{13}. If $n := |[R, S]| - 1$, there is a sequence of Prüfer minimal extensions $R \subset R_{1} \subset \cdots \subset R_{i} \subset R_{i+1} \subset \cdots \subset R_{n} = S$ such that $[R, S] = \{R_{i}\}_{i=0}^{n}$ since $R \subset S$ is chained. Moreover, there is some subset $\{P_{0}, P_{1}, \ldots, P_{n}\}$ of $\text{Spec}(R)$ such that $\text{Supp}(S/R) = \{P_{0}, P_{1}, \ldots, P_{n-1}\}$ where $P_{i} \subset P_{i-1}$ for each $i \in \mathbb{N}_{n-1}$. In particular, $\text{Supp}(R_{i}/R) = \{P_{0}, P_{1}, \ldots, P_{i-1}\}$ for each $i \in \mathbb{N}_{n}$. In fact, we have $P_{i} = R \cap C(R_{i}, R_{i+1})$ for each $i = 0, \ldots, n-1$ by \cite[Corollary 3.2]{13} and also $P_{i} = R \cap (R_{i-1} : R_{i})$ for each $i \in \mathbb{N}_{n-1}$ by Corollary 11.3. \[ \square \]

**Proposition 11.14.** Let $R \subset S$ be an FCP Prüfer extension such that $|\text{MSupp}(S/R)| = 1$. Set $[R, S] = \{R_{i}\}_{i=0}^{n}$, $P_{n} := R \cap (R_{n-1} : S)$ and $\text{Supp}(S/R) = \{P_{0}, P_{1}, \ldots, P_{n-1}\}$ as defined in Lemma 11.13. The following conditions are equivalent:

1. $R \subset S$ is a QR-extension;
2. for each $i \in \mathbb{N}_{n}$, there is some lsd $a_{i} \in R$ such that $R_{i} = R_{a_{i}}$;
3. for each $i \in \mathbb{N}_{n}$, there is some $a_{i} \in R$ such that $P_{i-1} = \sqrt{R_{a_{i}}}$.

If these conditions hold, then, for each $i \in \mathbb{N}_{n}$, we have $R_{i} = (R_{i-1})_{a_{i}}$ and $a_{i} \in P_{i-1} \setminus P_{i}$. Moreover, $a_{i}$ satisfies (2) if and only if it satisfies (3).

**Proof.** (1) $\iff$ (2) by Theorem 8.5 and (3) $\iff$ (1) by Proposition 8.9.
Assume that (2) holds with \(a_i\) an lsd such that \(R_i = R_{a_i}\). Obviously, \(R_i = (R_{i-1})_{a_i} \tag{\ast}\).

Let \(P \in \text{Spec}(R)\). Then, \(a_i \in P\) implies that \(PR_i = R_i\), so that \(P \in \text{Supp}(R_i/R)\) = \(\{P_0, P_1, \ldots, P_{i-1}\}\). Then, there is some \(j < i\) such that \(P = P_j\) and \(a_i \notin P_j\). To prove that \(a_i \in P_{i-1}\), we localize the extension \(R \subset S\) at \(M\). Set \(R'_j := (R_j)_M\), \(P'_j := P_j \cap S\) for each \(j = 0, \ldots, n\) and \(a'_i := a_i/1\). Then, \(R'_i \subset S\) is an FCP Pr"ufer extension over the local ring \((R', M')\) with \(n = \ell[R', S']\). Using Proposition \[11.12\] we get that \(\{P'_0, P'_1, \ldots, P'_n\}\) is the subset of \(\text{Spec}(R')\) such that \(R'_i = R'_i P'_i\), \(P'_i R'_i = P'_i\). It follows that \(\ast\) gives \(R'_i = R'_{a'_i} = (R'_{i-1})_{a'_i}\), with \(R'_{i-1}\) a local ring with maximal ideal \(P'_{i-1}\). Then \(a'_i \in P'_{i-1}\) since \(R'_i = (R'_{i-1})_{a'_i} \neq R'_{i-1}\) which gives \(a_i \in P_{i-1} \setminus P_i\). To conclude, \(P_{i-1} = \sqrt{Ra_i}\) as the least prime ideal of \(R\) containing \(a_i\), and also the least element of \(\text{Supp}(R_i/R)\). It follows that \(a_i\) satisfies (3).

Conversely, if there is some \(b_i\) such that \(P_{i-1} = \sqrt{Rb_i}\), then \(\sqrt{Ra_i} = \sqrt{Rb_i}\) implies \(R_i = R_{a_i} = R_{b_i}\) by Definition \[7.14\] \(\square\)

In order to generalize Lemma \[11.13\] to an arbitrary FCP Pr"ufer \(\mathcal{B}\)-extension, we need the following definition introduced in \[45\]. Let \(R \subset S\) be an FCP \(\mathcal{B}\)-extension and \(M \in \text{MSupp}(S/R)\). The elementary splitter \(\sigma(M) := T\), associated to \(M\), is defined by \(\text{MSupp}(T/R) = \{M\}\) and \(\text{MSupp}(S/T) = \text{MSupp}(S/R) \setminus \{M\}\). Such a \(T\) always exists (see \[45\] Theorem 4.6 and the paragraph after Corollary 5.5)).

**Proposition 11.15.** Let \(R \subset S\) be an FCP Pr"ufer \(\mathcal{B}\)-extension and QR-extension. Let \(T \in [R, S]\) and set \(\text{MSupp}(T/R) := \{M_1, \ldots, M_n\}\). For each \(i \in \mathbb{N}_n\), let \(P_i\) be the least element of \(\text{Supp}(T/R)\) contained in \(M_i\). Then, there exists some lsd \(t \in R\) such that \(T = R_t\) and \(\sqrt{Rt} = \cap[P_i \mid i \in \mathbb{N}_n]\).

**Proof.** Since \(R \subset S\) is an FCP Pr"ufer \(\mathcal{B}\)- and QR-extension, so is \(R \subset T\) by \[45\] Proposition 3.5] for the \(\mathcal{B}\)-extension property. Set \(\text{MSupp}(T/R) = \{M_1, \ldots, M_n\}\). For each \(i \in \mathbb{N}_n\), according to \[45\] Theorem 3.10], there is a least element \(P_i\) of \(\text{Supp}(T/R)\) contained in \(M_i\). The same reference gives that \(V(P_i)\) is a chain, whose greatest element is \(M_i\) and least element is \(P_i\).

For each \(i \in \mathbb{N}_n\), set \(T_i := \sigma(M_i)\), so that \(|\text{MSupp}(T_i/R)| = 1\). Moreover, for any \(P \in V(P_i)\), we have \(P \in \text{Supp}(T_i/R)\). Then, \(\text{Supp}(T_i/R) = V(P_i)\) because \(\text{MSupp}(T_i/R) = \{M_i\}\), and we can apply Proposition \[11.14\]. For each \(i \in \mathbb{N}_n\), we have \(T_i = R_{a_i}\) for some lsd \(a_i \in R\) and \(P_i = \sqrt{R_{a_i}}\) for each \(i \in \mathbb{N}_n\). Set \(t := a_1 \cdots a_n\), which is still an lsd. Now, \[45\] Proposition 5.11] asserts that \(T = \prod[T_i \mid i \in \mathbb{N}_n]\).
\[\mathbb{N}_n = \prod[R_{a_i} \mid i \in \mathbb{N}_n] = R_t.\] Moreover, \[\sqrt{Rt} = \sqrt{\prod[R_{a_i} \mid i \in \mathbb{N}_n]} = \cap[\sqrt{R_{a_i}} \mid i \in \mathbb{N}_n] = \cap[P_i \mid i \in \mathbb{N}_n].\] □

12. The set of all primitive elements

Let \( R \subseteq S \) be an extension. An element \( s \in S \) is called primitive (over \( R \)) if there exists a polynomial \( p(X) \in R[X] \) whose content is \( R \) and such that \( p(s) = 0 \). An extension \( R \subseteq S \) is called a \( \mathcal{P} \)-extension if all the elements of \( S \) are primitive over \( R \). Important examples of \( \mathcal{P} \)-extensions are given by the Prüfer extensions of \cite{26} (equivalently normal pairs \cite[Theorem 5.2, p.47]{26}). We recall that an element \( s \) of \( S \) is primitive if and only if \( R \subseteq R[s] \) has the INC property \cite[Theorem 2.3]{10} and that an extension \( R \subseteq S \) is a \( \mathcal{P} \)-extension if and only if \( R \subseteq S \) is an INC pair \cite[Corollary 2.4]{10}. We proved that an INC pair is nothing but a quasi-Prüfer extension \cite[Theorem 2.3]{42}. It follows easily that an extension is quasi-Prüfer if and only if it is a \( \mathcal{P} \)-extension. Therefore an FCP extension is a \( \mathcal{P} \)-extension \cite[Corollary 3.4]{42}.

For an extension \( R \subseteq S \), we denote by \( \mathcal{P}_S(R) \) the set of all elements of \( S \) that are primitive over \( R \), a set studied by Dobbs-Houston in \cite{11}. We defined in \cite[Theorem 3.18]{42} the quasi-Prüfer closure (or hull) \( \overline{R} \) of an extension \( R \subseteq S \). This closure is the greatest quasi-Prüfer subextension \( R \subseteq T \) of \( R \subseteq S \) and is equal to \( \overline{R} \). It follows that \( \overline{R} \subseteq \mathcal{P}_S(R) \). Obviously, \( \overline{R} \subseteq \mathcal{P}_S(R) \).

**Proposition 12.1.** Let \( R \subseteq S \) be an extension. Then \( \mathcal{P}_S(R) \) is a ring if and only if \( \mathcal{P}_S(R) = \overline{R} \).

**Proof.** It is enough to show that if \( \mathcal{P}_S(R) \) is a ring, it is contained in \( \overline{R} \), that is \( \overline{R} \subseteq \mathcal{P}_S(R) \) is Prüfer. But we may assume that \( R \subseteq S \) is integrally closed because an element of \( S \) primitive over \( R \) is primitive over \( \overline{R} \).

By \cite[Proposition 2.6]{11}, we can also assume that \( R \) is local. Let \( s \in \mathcal{P}_S(R) \). Then, either \( s \in R \) or \( s \) is a unit of \( S \) such that \( s^{-1} \in R \) according to \cite[Corollary 2.5]{11}. If \( s \in R \), then \( R[s] = R[s^2] \). If \( s \) is a unit of \( S \) such that \( s^{-1} \in R \), then \( s^{-1} \in R[s^2] \) implies \( s \in R[s^2] \), and we still have \( R[s] = R[s^2] \). Therefore, for each \( s \in \mathcal{P}_S(R) \), we have \( R[s^2] = R[s] \). We deduce from \cite[Chapter 1, Theorem 5.2]{26}, that the ring \( \mathcal{P}_S(R) \) is Prüfer over \( R \) and then \( \mathcal{P}_S(R) = \overline{R} \). □

It may happen that an extension \( R \subseteq S \) is such that \( \mathcal{P}_S(R) = S \). For example, if we denote by \( \text{Tot}(R) \) the total ring of quotients of a
ring $R$, then $\mathcal{P}_{\text{Tot}(R)} R$ is a ring if and only if $\mathcal{P}_{\text{Tot}(R)} R = \text{Tot}(R)$. [11] Corollary 2.9. If $R \subseteq S$ defines a lying-over pair, then $\mathcal{P}_S(R) = R$. [11] Proposition 3.11. The next result generalizes some results of [11].

**Corollary 12.2.** An extension $R \subseteq S$ is such that $\mathcal{P}_S(R) = S$ if and only if $R \subseteq S$ is quasi-Prüfer.

**References**

[1] D.D. Anderson, D.F. Anderson and R. Markanda, The rings $R(X)$ and $R < X >$, *J. Algebra*, 95, (1985), 96–115.
[2] R. Abbas and A. Ayache, On questions related to normal pairs, *Comm. Algebra*, 49 (3), (2021), 956–966.
[3] D. F. Anderson, D. E. Dobbs and M. Fontana, On treed Nagata rings, *J. Pure Appl. Algebra*, 61, (1989), 107–122.
[4] J. T. Arnold, On the ideal theory of the Kronecker function ring and the domain $D(X)$, *Canadian J. Math.*, 21, (1969), 558–563.
[5] A. Ayache, P.J. Cahen and O. Echi, Anneaux quasi-Prüfériens et P-anneaux, *Bollettino U.M.I.; (7)* 10-B, (1996), 1–24.
[6] V. P. Camillo A note on semi-hereditary rings, *Arch. Math.*, Vol. XXIV, (1973), 142–143.
[7] S.H. Cox, Jr and D.E. Rush, Finiteness in flat modules and algebras, *J. Algebra*, 32, (1974), 44–50.
[8] E. D. Davis, Overrings of commutative rings. III: Normal pairs, *Trans. Amer. Math. Soc.*, 182, (1973), 175–185.
[9] T. M. K. Davison, Distributive homomorphisms of rings and modules, *J. Reine Angew. Math.*, 271, (1974), 28–34.
[10] D. E. Dobbs, *Lying-over pairs of commutative rings*, Can. J. Math., (XXXIII) (2) (1981), 454–475.
[11] D. E. Dobbs and E. Houston, *On sums and products of primitive elements*, Comm. Algebra (45)(2017), 357–370.
[12] D. E. Dobbs, B. Mullins, G. Picavet and M. Picavet-L’Hermitte, On the FIP property for extensions of commutative rings, *Comm. Algebra*, 33 (2005), 3091–3119.
[13] D. E. Dobbs, G. Picavet and M. Picavet-L’Hermitte, Characterizing the ring extensions that satisfy FIP or FCP, *J. Algebra*, 371 (2012), 391–429.
[14] D. E. Dobbs, G. Picavet and M. Picavet-L’Hermitte, Transfer results for the FIP and FCP properties of ring extensions, *Comm. Algebra*, 43 (2015), 1279–1316.
[15] S. Endo, On semi-hereditary rings, *J. Math. Soc. Japan* 13, No 2 (1961), 109–119.
[16] V. Erdődgu, The distributive hull of a ring, *J. Algebra*, 132, (1990), 263–269.
[17] M. Fontana, J. A. Huckaba and I. J. Papick, Prüfer domains, Monographs and Textbooks in Pure and Applied Mathematics, 203. Marcel Dekker, Inc., New York, (1997).
[18] D. Ferrand, Trivialisation des modules projectifs. La méthode de Kronecker, *J. Pure Appl. Algebra*, 24 (1982), 261–264.
[19] D. Ferrand and J.-P. Olivier, Homomorphismes minimaux d’anneaux, *J. Algebra*, 16 (1970), 461–471.
[20] R. Gilmer and J. Ohm, Integral domains with quotient overrings, *Math. Ann.*, 153, (1964), 813–818.
[21] C. Gotlieb, Finite unions of overrings of an integral domain, *J. Commut. Algebra*, DOI 10.1216/jca.2020.12.87.
[22] A. Grothendieck, Éléments de Géométrie Algébrique. IV. Étude locale des schémas et des morphismes de schémas IV. Inst. Hautes Études Sci. Publ. Math. 32, (1967), 361 pp.
[23] A. Grothendieck and J. Dieudonné, Éléments de Géométrie Algébrique I, Springer Verlag, Berlin, (1971).
[24] J.A. Huckaba and J.M. Keller, Annihilation of ideals in commutative rings, *Pacific. J. Math.*, 83, No. 2, (1979), 375–379.
[25] J. Iroz and D.E. Rush, Associated prime ideals in non-Noetherian rings, *Can. J. Math.*, Vol. XXXVI, No 2, (1984), 344–360.
[26] M. Knebusch and D. Zhang, Manis Valuations and Prüfer Extensions I, Springer, Berlin (2002).
[27] M. Knebusch and T. Kaiser, Manis Valuations and Prüfer Extensions II, Springer, Cham Heidelberg, 2014.
[28] J. Kostra, The covering of rings by integrally closed rings, *Mathematica Slovaca*, 34, no 2, (1984), 171–176.
[29] D. Lazard, Autour de la platitude, *Bull. Soc. Math. France*, 97, (1969), 81–128.
[30] T. G. Lucas, Two annihilator conditions: Property (A) and (A.C.), *Comm. Algebra*, 14(3), (1986), 557–580.
[31] T. G. Lucas, Some results on Prüfer rings, *Pacific J. Math.*, 124, (1986), 333–343.
[32] K. Morita, Flat modules, Injective modules and quotient rings, *Math. Z.*, 120 (1971), 25–40.
[33] J.P. Olivier, Fermeture intégrale et changements de base absolument plats, *Colloque d’Algèbre Commutative* (Rennes 1972), Exp. No 9, 13 pp. Publ. Sém. Math. Univ. Rennes, Année 1972, Univ. Rennes, Rennes, (1972).
[34] J.P. Olivier, Descente de quelques propriétés élémentaires par morphismes purs, *An. Acad. Brasil. Ci.* 45 (1973), 17–33.
[35] J.P. Olivier, Going up along absolutely flat morphisms, *J. Pure Appl. Algebra*, 30 (1983), 47–59.
[36] L. Paudel and S. Tchamna, Kronecker function rings of ring extensions, *J. Algebra Appl.*, Vol. 16, No 11, (2018), 1850021.
[37] G. Picavet, Autour des idéaux premiers de Goldman d’un anneau commutatif, *Ann. Sci. Univ. Clermont*, 57, Math. No. 11 (1975), 73–90.
[38] G. Picavet, Propriétés et applications de la notion de contenu, *Comm. Algebra*, 13, (1985), 2231–2265.
[39] G. Picavet, Geometric subsets of a spectrum, Commutative ring theory and applications (Fez, 2001), 387–417, Lecture Notes in Pure and Appl. Math., 231, Dekker, New York, 2003.
[40] G. Picavet, Recent progress on submersions: a survey and new properties, *Algebra*, doi.org/10.1155/2013/128064, (2913), 2013, Article ID 128064, 14 p. (2013).
[41] G. Picavet and M. Picavet-L’Hermitte, Some more combinatorics results on Nagata extensions, *Palest. J. Math.*, 5 (2016), 49–62.

[42] G. Picavet and M. Picavet-L’Hermitte, Quasi-Prüfer extensions of rings, pp. 307–336, in: *Rings, Polynomials and Modules*, Springer, 2017.

[43] G. Picavet and M. Picavet-L’Hermitte, Rings extensions of length two, *J. Algebra Appl.*, 18 (2019 no 9), 1950174, 34pp.

[44] G. Picavet and M. Picavet-L’Hermitte, Boolean FIP ring extensions, *Comm. Algebra*, 48 (2020), 1821–1852.

[45] G. Picavet and M. Picavet-L’Hermitte, *Splitting ring extensions*, arXiv:2107.04102v1 [math.AC].

[46] P. Samuel, La notion de place dans un anneau. *Bull. Soc. Math. France*, 85, (1957), 123–133.

[47] H. H. Storrer, Epimorphismen von kommutativen Ringen, *Comment. Math. Helv.*, 43 (1968), 378–401.