Relating the wave-function collapse with Euler’s formula

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March 13, 2018

Abstract

One attractive interpretation of quantum mechanics is the ensemble interpretation, where Quantum Mechanics merely describes a statistical ensemble of systems rather than individual systems. However, such interpretation does not address why the wave-function plays a central role in the calculations of probabilities, unlike most other interpretations of quantum mechanics.

We first show that for a quantum system defined in a 2-dimensional real Hilbert space, the role of the wave-function is identical to the role of the Euler’s formula in engineering, while the collapse of the wave-function is identical to selecting the real part of a complex number.

We will then show that the wave-function is merely one possible parametrization of any probability distribution describing an ensemble: a surjective map from an hypersphere to the set of all possible probability distributions. The fact that the hypersphere is a surface of constant radius reflects the fact that the integral of the probability distribution is always 1. Any tranformation of a probability distribution is represented by a rotation of the hypersphere.

It is thus a very good parametrization which allows us to apply group theory to the hypersphere, despite the fact that a stochastic process is not always a Markov process. The collapse of the wave-function is required to compensate the fact that physical transformations on the probability distribution are not linear transformations.
The fundamental notions of calculus, namely differentiation and integration, are often viewed as being the quintessential concepts in mathematical analysis, as their standard definitions involve the concept of a limit. However, it is possible to capture most of the essence of these notions by purely algebraic means (almost completely avoiding the use of limits, Riemann sums, and similar devices), which turns out to be useful when trying to generalise these concepts.

T. Tao (2013) [1]

The consistent histories formalism has taught us that there are infinitely many incompatible descriptions of the world within quantum mechanics. Perhaps some simple criterion can be found to pick out one of these descriptions, by selecting one particular consistent set. Such a criterion should explain persistent quasiclassicality...

F. Dowker and A. Kent (1994) [2]

...decoherence occurs as a result of the entanglement of the measured system with its environment and results in the loss of phase relations between components of the wave function of the measured system.

Decoherence is essentially nothing else than quantum measurement, but considered from the point of view of its physical mechanism and resolved in time.

...a process leading to a quantum measurement is decoherence, but a continuous (repeated) quantum measurement may serve as a model for decoherence.

— M. Mensky (2000) [3]

...any mention of a collapse seems to have been avoided. It would be wrong to conclude hastily though that the decoherence point of view allows us to get rid of the projection postulate. Even if in this approach no explicit reference is made to a ‘collapse’ of the A or M systems, a projection of the wave function does occur in E [environment]. By tracing over the environment variables, we consider indeed that unread measurements have taken place in it and \( \rho_{A+M} \) [density matrix] is a weighted average of the projectors on the correlated ‘collapsed states’ of the A + M system.

The notion of collapse associated to measurement is thus deeply rooted in the quantum formalism, whether it is explicitly introduced as a postulate, or implicitly assumed in the rule defining the basic properties of partial density matrices.

— S. Haroche & J. Raimond (2006) [4]

1 Introduction

One attractive interpretation of quantum mechanics is the ensemble interpretation [5, 6], where Quantum Mechanics is merely a theory of statistical physics, describing a statistical ensemble of systems rather than an individual system.

Often quantum mechanics is interpreted instead as providing the probabilities of transition between different states of an individual system. This transition happens upon measurement, any measurement. The state of the system is defined by the wave-function which collapses to a different wave-function upon measurement. This raises a number of interpretation problems as to what do we mean by state of a system. If the state before measurement is A, but after measurement the state is B, what is then the state during the measurement: A and/or B or something else? Strangely, despite that we don’t know what happens during the measurement,
we know very well the transition probabilities—because once we assume that the state of the system is defined by the wave-function, there are not (many) alternatives to the Born’s rule defining the probabilities of transition as a function of the wave-function [7].

The ensemble interpretation avoids these interpretation problems: the state of the ensemble is unambiguously the probability distribution for the states of an individual system—as in statistical physics\(^1\). Then, there are several possible states of the ensemble, these different states are related by physical transformations. For instance, a translation in space-time or a rotation may change the state of the ensemble.

However, the ensemble interpretation does not address the question why the wave-function plays a central role in the calculation of the probability distribution, unlike most other interpretations of quantum mechanics. By being compatible with most (if not all) interpretations of Quantum Mechanics, the ensemble interpretation is in practise a common denominator of most interpretations of Quantum Mechanics. It is useful, but it is not enough. For instance, the ensemble interpretation does not give any explanation as to why it looks like the electron’s wave-function interferes with itself in the double-slit experiment—that would imply that the wave-function describes (at least partially) an individual system. Moreover, the most prominent advocates of the ensemble interpretation were dissatisfied with the complementarity of position and momentum [9, 10], convincing themselves and others that the complementarity of position and momentum could not be satisfactorily explained by the the ensemble interpretation alone.

In this paper we will show that the wave-function is merely one possible parametrization of any probability distribution. The parametrization is a surjective map from an hypersphere to the set of all possible probability distributions. The fact that the hypersphere is a surface of constant radius reflects the fact that the integral of the probability distribution is always 1. Any transformation of a probability distribution is represented by a rotation of the hypersphere. It is thus a very good parametrization which allows us to apply group theory to the hypersphere, despite the fact that a stochastic process is not always a Markov process. The wave-function can be described as a multi-dimensional generalization of Euler’s formula, and its collapse as a generalization of taking the real part of Euler’s formula.

This is ironic, since Feynman described Euler’s formula as “our jewel” while the wave-function collapse certainly contributed for him to say “I think I can say that nobody understands Quantum Mechanics.” Besides the irony, the fact that the wave-function parametrizes any probability distribution means that the wave-function collapse is a feature of all random phenomena.

The above fact implies that alternatives to Quantum Mechanics motivated by a dissatisfaction with either the complementarity of position and momentum [10], or the wave-function collapse [11], may also feature complementarity of observables (possibly other than position and momentum) and wave-function collapse, once a parametrization with a wave-function is applied.

\(^1\)Note that the notion of probability distribution is not free from interpretation problems [8], but we believe these are intrinsic to any application of statistics.
But the above fact also implies that the wave-function collapse is not provoked by the interaction with the environment\(^2\); the wave-function collapse does not emerge from some particular cases of classical statistics [12]; Quantum Mechanics is *not* a generalization of the concept of probability algebra from commutative to non-commutative algebra [13]; and thus quantum computation/information is not fundamentally different from classical computation/information\(^3\). The wave-function is a possible parametrization for any theory of Statistics, including Statistical Physics.

This is comforting, since it is consistent with the empirical facts that Quantum Mechanics applies to a very wide range of physical systems, from the Hydrogen atom, a neutron star or the Universe; and that the collapse occurs upon measurement, any measurement.

The physical question is how the physical transformations affect the ensemble (and thus the wave-function), in particular whether there are viable alternatives to Quantum Mechanics which are deterministic [15] or at least verify the Markov property [16]; in such a case there would be other alternative parametrizations which also allows us to use methods of group theory and do not involve a wave-function and its collapse.

This also opens the door into applying the wave-function parametrization not just to quantum mechanics, but also to quantum statistical mechanics or even to other problems involving statistics other than quantum physics. For instance, sequential systems are a framework for machine learning that shares several features with Quantum Mechanics [17–19], (more) application of quantum methods to sequential systems thus seems straightforward. Other applications of quantum methods in statistics, either did not use the wave-function parametrization\(^4\); or they considered only deterministic scenarios [24, 25];

In Section 2, we will review the representation of an algebra of events in a real Hilbert space (which will be useful for the next sections), and we will argue that quantum mechanics is *not* a non-commutative version of probability theory; in Section 3 we will describe the relation between Euler’s formula and the parametrization of a probability distribution by a real wave-function; in Section 4 we will describe the parametrization of a discrete probability distribution by a real wave-function; in Section 5 we will address continuous probability distributions; in Section 6 we will address complex and quaternionic wave-functions.

Note that in this paper, the Hilbert space is always considered to be a separable Hilbert space [26].

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\(^2\)In a measurement there is always an interaction with the environment, therefore the environment necessarily affects the ensemble and it is possible that decoherence occurs. But such phenomena will be accounted for by the dependence of the ensemble from physical transformations. Note that a continuous (repeated) quantum measurement is a model of decoherence and thus decoherence does not avoid by itself the wave-function collapse [3, 4]. In case we opt for a model of decoherence which avoids collapse, then we are necessarily dealing with an alternative to Quantum Mechanics, such case was discussed above.

\(^3\)Different computers always have different properties, for instance different logic gates may enhance the performance of different algorithms [14]. But neglecting performance, the quantum bits can be constructed using classical bits and quantum logic gates can also be constructed using classical logic gates, with the Hadamard transform as an example.

\(^4\)Analogies with the wave-function were made but unitarity was not preserved [20–23]
2 Review of projection-valued measures

The representation of an algebra of events in a real Hilbert space uses projection-valued measures [27–30]. A probability space consists of three parts: the set of possible states of a system; the set of events where each event is a subset of the set of possible states; and a probability distribution (also named a measure) which assigns a probability to each event.

The notion of probability is somewhat ambiguous [8], but it is useful to relate complex random phenomena with a simple standard random process. That the probability of an event is $5/567$ means that the likelihood of our event is the same as the likelihood of correctly guessing one unknown number in the interval $001 – 567$ in 5 attempts (standard random process). If the probability is a real number not rational, we can approximate any real probability by a rational number with infinitesimal error because the rational numbers are dense in the reals, therefore the relation to a simple standard random process is still possible.

A projection-valued measure assigns a self-adjoint projection operator of a real Hilbert space to each event, in such a way that the boolean algebra of events is represented by the commutative algebra of projection operators. Thus, intersection/union of events is represented by products/sums of projections, respectively.

Then, the state of the ensemble is a linear functional which assigns a probability to each projection.

Since the algebra is commutative, there is a basis where all the projections are diagonal. This leaves room for a non-commutative generalization of probability theory, since the state of the ensemble could also assign a probability to non-diagonal projections, these non-diagonal projections would generate a non-commutative algebra [13].

Consider for instance the projection $P_X$ to a region of space $X$ and a projection $UP_PU^\dagger$ to a region of momentum $P$, where $P_X$ and $P_P$ are diagonal. The projections $P_X$ and $UP_PU^\dagger$ are related by a Fourier transform $U$ and thus are diagonal in different basis and do not commute (they are complementary observables). Since we can choose to measure position or momentum, it seems that Quantum Mechanics is a non-commutative generalization of probability theory [13].

But due to the wave-function collapse, Quantum Mechanics is not a non-commutative generalization of probability theory despite the appearances: the measurement of the momentum is only possible if a physical transformation of the ensemble also occurs. Suppose that $E(P_X)$ is the probability that the system is in the region of space $X$, for the state of the ensemble $E$ diagonal (i.e. verifying $E(O) = 0$ for operators $O$ with null diagonal). Then we define $E_U(D) = E(UDU^\dagger)$ for diagonal operators $D$ and $E_U(O) = 0$ for operators $O$ with null diagonal (this is the collapse in action). Then $E_U(P_P) = E(UP_PU^\dagger)$ is the probability that the system is in the region of momentum $P$, for the state of the ensemble $E_U$. But the ensembles $E$ and $E_U$ are different, there is a physical transformation relating them.

Without collapse, we would have $E_U(O) = E(UOU^\dagger) \neq 0$ for operators $O$ with null-diagonal and we could talk about a common state of the ensemble $E$ assigning probabilities to a non-commutative algebra. But the collapse keeps Quantum Mechanics as a standard probability
theory, even when complementary observables are considered. This is a serious hint that the collapse plays a key role in the consistency of the theory, as we will see.

3 Euler’s formula for probabilities

Suppose that we have an oscillatory motion of a ball, with position \( x = \cos(t) \) and we want to make a translation in time, \( \cos(t) \to \cos(t + a) \). This is a non-linear transformation. However, if we consider not only the position but also the velocity of the ball, we have the “wave-function” given by the Euler’s formula \( q(t) = e^{it} \) and \( x \) is the real part of \( q \). Then, a translation is represented by a rotation \( q(t + a) = e^{ia}q(t) \). To know \( x \) after the translation, we need to take the real part of the wave-function \( e^{ia}q(t) \), after applying the translation operator.

Of course, \( \cos(t) \) is not positive and so it has nothing to do with probabilities. However, if we consider a 2-dimensional real vector space we can easily apply Euler’s formula to probabilities, in the sense that a transformation of the probability distribution corresponds to a rotation of the wavefunction and the collapse selects the diagonal part of the density matrix.

Given a 2-dimensional real wave-function \( \Psi(t) = \exp\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} t \right) \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) = \exp\left(\begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix} \right) \) (a real version of Euler’s formula), the density matrix is given by

\[
\Psi\Psi^\dagger = \begin{bmatrix} \cos^2(t) & \cos(t)\sin(t) \\ \cos(t)\sin(t) & \sin^2(t) \end{bmatrix}
\]

and the distribution function is given by the diagonal part of the density matrix \( F(\Psi\Psi^\dagger) = \begin{bmatrix} \cos^2(t) & 0 \\ 0 & \sin^2(t) \end{bmatrix} \).

Since \( \cos^2(t) + \sin^2(t) = 1 \) and \( 0 < \cos(t) < 1 \), we see that the wave-function parametrizes all distribution functions for a system with 2 states. Moreover, any transformation of the distribution function is represented by a rotation of the wave-function \( \Psi(t + a) = \exp\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} a \right) \Psi(t) \), for some \( a \).

The question remaining is whether the wave-function parametrization works for systems with more than 2 states.

4 Discrete probability distribution

Consider now a system with a discrete set of possible states, indexed by an integer \( n > 0 \). Defining \( S_n = \{k : k \geq n\} \) as the event which contains all states with index starting at \( n \), we can write the probability distribution as \( P(n) = P(S_n)P(n|S_n) = \left( \prod_{k=1}^{n-1} P(S_{k+1}|S_k) \right) P(n|s_n) \).

That is, as a product of the probabilities \( P(S_{k+1}|S_k) \) and \( P(k|S_k) \), which verify \( P(k|s_k) + P(S_{k+1}|S_k) = 1 \).

Consider now a countable orthonormal basis for a separable Hilbert space \( \{\ell_n\} \), indexed by an integer \( n > 0 \). We can parametrize a normalized vector in the Hilbert space \([31] \), as \( v_n = c_n \ell_n + s_n v_{n+1} \), with \( c_n^2 + s_n^2 = 1 \). We call to \( v_1 \) the wave-function. Then, the projection to the linear space generated by \( v_n \) is:

\[
v_n v_n^\dagger = c_n^2 \ell_n^\dagger + s_n^2 v_{n+1}^\dagger + c_n s_n (\ell_n v_{n+1}^\dagger + v_n^\dagger) \]

6
We consider a basis where all \( l_n t_n \) are diagonal. The operator \( v_n v_n^\dagger \) is a projection thanks to the off-diagonal terms \( c_n s_n (l_n v_n^\dagger + v_n l_n^\dagger) \). If the off-diagonal terms are suppressed (collapsed) \( \text{diag}(v_n v_n^\dagger) = c_n^2 l_n t_n + s_n^2 v_{n+1} v_{n+1}^\dagger \), then \( \text{diag}(v_1 v_1^\dagger) \) is a diagonal operator which represents the probability distribution \( P(n) \) in the Hilbert space.

That is, \( P(n) = \text{tr}(\text{diag}(v_1 v_1^\dagger) l_n t_n) \) and \( P(O) = 0 \) for operators \( O \) with null-diagonal.

Thus, the wave-function \( v_1 \) allows us to apply Lie group theory to Quantum Mechanics. To represent physical transformations as rotations of the wave-function. The collapse of the wave-function is required to compensate the fact that physical transformations on the probability distribution are not a linear representation of the Lie group. Thus the collapse establishes the correct relation between the physical transformations and the probability distribution.

Quantum mechanics is thus, not a generalization of probability theory, but a generalization of the theory of linear representations of groups and the action of the group in a probability distribution.

## 5 Continuous probability distribution

Remember that a continuous probability distribution is a probability distribution that has a cumulative distribution function that is continuous.

The wave-function may struggle to describe a system with a continuous set of possible states, due to normalization issues. A generalization of Hilbert space—the rigged Hilbert space—was proposed \([32, 33]\); a more economical solution is to consider the framework of quantum statistical mechanics and use the density matrix built from projection-valued measures to select regions in the continuous set of possible states of the system, see Section 2. However, since we claim that we can parametrize quantum statistical mechanics with a wave-function defined in a Hilbert space, then we need a third alternative.

First, it is important to understand when is the rigged Hilbert space necessary. Suppose we have a sequence of normalized wave functions \( \psi_k \) that are null everywhere except in a small region of \( X_k \) the continuous space of states—where they are constant—, where \( X_{k+1} \) is contained in \( X_k \) and the sequence of regions converges to a point \( x \). For \( k \) as large as we want, \( X_{k+l} \) with \( l \) sufficiently large will cover a very small subspace of \( X_k \). Therefore, \( \lim_{l \to +\infty} \psi_k^\dagger \psi_{k+l} = 0 \) and so the sequence of normalized wave functions \( \psi_k \) is not a Cauchy sequence. Therefore, it does not need to converge to an element of the Hilbert space, despite that the Hilbert space is complete. And indeed the sequence does not converge to an element of the Hilbert space, since the point \( x \) has null measure and thus an element which is null everywhere except in \( x \) is equivalent to the null element. But this applies also to the density matrix and thus to the continuous probability distribution: we cannot have a density matrix or a continuous probability distribution which is null everywhere except at a point \( x \).

The rigged Hilbert space is no longer necessary once we consider a wave-function which is non-null in a set with non-null measure. And this wave-function can be the limit of a sequence
of linear combinations of simple wave-functions, because this is a Cauchy sequence. The simple wave-functions are null everywhere except in a region of the continuous set of possible states of a system where they are constant; they will produce a probability distribution proportional to a projection-valued measure, up to a normalization factor.

The resulting sequence of wave-functions will generate a corresponding sequence of probability distributions. Both will converge. Thus, in the continuous case we just need to replace in the previous section the indexes corresponding to discrete states by indexes corresponding to discrete regions forming a partition of the continuous space of states; the orthonormal basis is replaced by the simple normalized wave functions (described above) corresponding to the regions of the partition.

6 Complex and Quaternionic Hilbert spaces

While the parametrization with a real wave-function is always possible, it may not be the best one. Suppose that we have two phenomena affecting the ensemble, one of them can be described by classical statistical methods. Then we may want to use the framework of quantum statistical mechanics, with the phenomena that can be described by methods of classical statistics acting directly on the probability distribution (after collapse). We will describe in the following two such cases.

As we have seen, the wave-function parametrization allows us to apply group theory to the states of the ensemble, since unitary transformations (i.e. a multi-dimensional rotation) preserve the properties of the parametrization (in particular the conservation of total probability).

We have then real unitary representations since the wave-function considered so far is real. The (real version of the) Schur’s lemma [27–29] implies that the set of operators commuting with irreducible representations forms a real associative division algebra—such algebra is isomorphic to either: the real numbers, the complex numbers or the quaternions.

Therefore, the density matrix can commute with a set of operators isomorphic to the real numbers, complex numbers or the quaternionic numbers. This allows us to optionally define wave-functions in complex and quaternionic Hilbert spaces. These cases are equivalent to a parametrization by a real wave-function, followed by averaging the density matrix in such a way that the averaged density matrix commutes with a set of operators isomorphic to the complex numbers or quaternions [27, 28].

Let us consider the quaternionic case (it will be then easy to see how are things in the complex case). We have a discrete state space defined by two real numbers \( n, m \), with \( 1 \leq m \leq 4 \) and with identical probabilities for \( m \) \( P(n, m|n) = \frac{1}{4} \).

Then, a more meaningful parametrization reflecting by construction the particular structure of the probability distribution uses a quaternionic wave function. Namely, \( \{l_n\} \) is an orthonormal basis of quaternionic wave-functions and

\[
v_n^\dagger v_n = c_n^2 l_n^\dagger l_n + s_n^2 v_{n+1}^\dagger v_{n+1} + c_n s_n (l_n v_{n+1}^\dagger + v_{n+1} l_n^\dagger)
\]

8
Note that there is a basis where $l_n l_n^\dagger$ is real diagonal and thus upon collapse $v_n v_n^\dagger$ becomes real diagonal as well.

The complex case is just the above case with complex numbers replacing quaternions and a state space which is the union of 2 identical spaces. The continuous case is analogous, if we use the solution proposed in Section 5.

7 Another proof using the GNS construction

What we showed in the previous sections, namely that the wave-function is one possible parametrization of any probability distribution, is a major claim. And so it should be supported by major evidence. We present in this section an alternative, cleaner proof (but also less down-to-earth) of this claim, based on the GNS construction\(^5\).

In fact, the conclusion that the wave-function is one possible parametrization of any probability distribution could in principle be reached at least as early as in 1947 by Irving Segal [35] or by someone familiar with the GNS construction in the meantime. But apparently this did not happen, because (as we will see below) the GNS construction alone comes close but it is not sufficient to show that the wave-function is one possible parametrization of any probability distribution. Thus some motivation to get to such conclusion is required. Unfortunately such motivation never existed before, because the GNS construction was introduced as a point in favor of the prejudice that quantum mechanics is a non-commutative generalization of probability theory \(^6\). In reference [19] in the context of sequential systems for machine learning, it was showed that an infinite-dimensional wave-function is one possible parametrization of any finite-dimensional probability distribution; but the authors of reference [19]—who apparently were not aware of the GNS construction—they could not show that a finite-dimensional wave-function is one possible parametrization of any finite-dimensional probability distribution (as we do here) and left the relation of their framework with quantum mechanics for future work.

The algebra of projection-valued measures $A$ associated to a measurable space $X$ is a commutative real C* algebra. The expectation value $E$ is a positive linear functional. The expectation value allows us to define the bilinear form $\langle a, b \rangle = E(ab)$ where $a, b \in A$.

This bilinear form is not yet an inner-product, since it is only positive semi-definite. However, the set of projections with null expectation value $I_E$ is a linear subspace of $A$. Thus, the completion of the quotient $A/I_E$ is an Hilbert space (with inner product given by the above bilinear form, this is the GNS construction). The vector $v_0$ corresponding to the identity 1 of the algebra is a cyclic vector. The projection-valued measures correspond to the projection of

\(^5\)For more information on the Gelfand-Naimark-Segal (GNS) construction see Ref. [34] for the case of a complex algebra and Ref. [29] for the case of a real algebra

\(^6\)The focus of most people using the GNS construction (among them Irving Segal, as the title of Ref. [35] makes clear) is on the irreducible representations of the (usually non-commutative) algebra of observables. Since the irreducible representations of a commutative algebra (our case) are one-dimensional, the irreducible representations are of little help to conclude that the wave-function is one possible parametrization of any probability distribution.
Thus the Hilbert space corresponding to $A/I_E$ is the space of square-integrable functions in the region of $X$ where the expectation value is not null. We need to go now beyond the GNS construction and consider instead the Hilbert space of square-integrable functions in all $X$, since we still have that $<v_0, P_Y v_0> = E(P_Y)$; but in this case $v_0$ is not necessarily a cyclic vector, since the projections of $v_0$ in subspaces of $X$ do not necessarily constitute a basis of the Hilbert space.

### 8 Deterministic transformations

The great advantage of the wave-function parametrization is that the space of wave-functions is a multi-dimensional sphere, which is an homogeneous space for the group of rotations. This means that for any wave-functions $\psi, \phi$, there is a unitary operator $U$ such that $\psi = U\phi$. This leaves us in a good position to use group theory to describe the physical transformations of the probability distribution.

However the precise relation between the wave-functions and the elements of a group of physical transformations is not obvious (see next section). We will use the deterministic transformations to help us.

A deterministic transformation acts as $E(P_A) \rightarrow E(P_B)$ where $A, B$ are events, for any probability distribution $E$ and event $A$. When $E$ is concentrated in the neighborhood of a single outcome (say $A$), we have effectively a deterministic case and this transformation ($A \rightarrow B$) conserves the determinism, thus it is a deterministic transformation.

Note that above, $P_A$ and $P_B$ necessarily commute. On the other hand, if the transformation is such that $E(P_A) \rightarrow E(U P_A U^\dagger)$ where $P_A$ and $U P_A U^\dagger$ do not commute, then the transformation cannot be deterministic. Consider the discrete case with $E(P_n)$ given by $Tr(P_m P_n) = \delta_{mn}$ up to a normalization factor, for instance. Then $Tr(P_m P_n) \rightarrow Tr(P_m U P_n U^\dagger) = U_{nm}^2$. If the transformation would be deterministic, then necessarily $U_{nm}^2 = \delta_{kn}$ for some $k = f(n)$ dependent on $n$, and so $U P_n U^\dagger = P_l$ with $l = f^{-1}(n)$ would commute with $P_n$. In the continuous case, we can consider a discrete partition of the set of outcomes and we get to a similar conclusion.

We get to the conclusion that a transformation $U$ is deterministic if and only if $P_A$ and $U P_A U^\dagger$ commute for all events $A$. Thus, the complementarity of two observables (e.g. position and momentum) is due to the stochastic (i.e. non-deterministic) nature of the physical transformation relating the two observables. This clarifies that probability theory has no trouble in dealing with non-commuting observables. Note that quantum mechanics is not a generalization of probability theory, but it is definitely a generalization of classical mechanics since it involves non-deterministic physical transformations. For instance, the time evolution may be non-deterministic unlike in classical mechanics.
9 Physical transformations

We now associate one wave-function to each element of a group of physical transformations, proceeding in two steps. First we associate one orthogonal basis of wave-functions to each element of a group and then we select one wave-function from each basis.

An orthogonal basis is essentially an unitary matrix, which is the principal homogeneous space (i.e. torsor [36]) of the group of rotations. A torsor is isomorphic to the set of elements of a group, however it is not a group because we do not know which element of the torsor corresponds to the identity. Thus, in order to associate one basis to each element of the group of rotations, we just need to choose which basis corresponds to the identity.

From the point of view of physics, we should choose a deterministic basis to correspond to the identity transformation, since the purpose of a probability theory is to describe a random process which affects an otherwise deterministic system. For instance, the dice is a deterministic system as long as we do not throw it. Since we are working with separable Hilbert spaces, we can always choose a discrete set of outcomes. We can then use the unitary transformations to access a continuous set of outcomes, without determinism. Note that a continuous probability distribution cannot be deterministic by definition.

We now select only one deterministic wave-function to correspond to the identity transformation. We need to do this because a sphere is not a principal homogeneous space of the group of rotations, it is only an homogeneous space, meaning that the unitary operator $U$ such that $\psi = U\phi$ is not necessarily unique. The key point is that the deterministic wave-functions are all related by deterministic transformations (see the previous section). Thus we can choose any deterministic wave-function without loss of generality. For instance, suppose we select as the reference a dice in state 1, but the initial state (i.e. before we apply the random process) of the dice is in fact 2. Then the wave-function corresponding to this process is $U_g\psi_1 = U_g (U_2\psi_1)$ (where $\psi_1(n) = \delta_{1n}$), transforming state 1 into state 2 and then applying the random process $g$. This is equivalent to selecting the wave-function $(U_2\psi_1)$ as a reference and then applying the random process $g$.

Note that selecting only one column of the unitary matrix, $U_{n1}$ has the same degrees of freedom as a wave-function. By selecting as a group representation a subset of the unitary matrices, we associate a wave-function (and thus a probability distribution) to each element of a group of transformations (possibly non-deterministic transformations), $\psi_g = U_g\psi_1$. Thus, the wave-function parametrization allows us to use group theory in probability theory in all stochastic processes, even in processes without the Markov property. It may be tempting to identify $U_{nm}^2$ with the conditional probability of $n$ given $m$, but that is not correct since $U_{nm}^2$ stands for the probability of the event $n$ as given by the probability distribution $m$; we can use the wave-function parametrization for any probability distribution, including conditional probability, thus $U_{nm}^2$, a priori has little to do with whether the probability is conditional or not.

Note that we have not used Wigner’s theorem [37], since we associate a wave-function
(instead of a unitary matrix) with a physical transformation. Wigner’s theorem is most useful in the context of a non-commutative generalization of probability theory, which is not the case of Quantum Mechanics due to the wave-function collapse.

10 A comment on conditioned probability and the random walk

It is well-known that quantum mechanics can be described as the Wick-rotation of a Wiener stochastic process [38]. In other words, the time evolution in Quantum Mechanics is a Wiener process for imaginary time. This is the origin of the Feynman’s path integral approach to Quantum Mechanics and Quantum Field Theory.

Since the Wiener process is one of the best known Lévi processes—a Lévi process is the continuous-time analog of a random walk—this fact often leads to an identification of Quantum Mechanics with a random walk. In particular, it often leads to an identification of the probabilities calculated in Quantum Mechanics with conditioned probabilities—the next state in a random walk is conditioned by the previous state.

Certainly, the usefulness of group theory is common to both a random walk and to Quantum Mechanics and this unavoidably leads to similarities between a random walk and Quantum Mechanics. However, imaginary time is very different from real time and thus the probabilities calculated in Quantum Mechanics are not necessarily conditioned probabilities in a random walk.

In order to relate a random walk with Quantum Mechanics correctly, we need the probability distribution for the complete paths of the random walk. Then, we can use a wave-function parametrization of the probability distribution for the complete paths of the random walk. Finally, we can apply quantum methods to this wave-function. The result is a Quantum Stochastic Process [39], which is not a generalization of a stochastic process due to the wave-function collapse, but merely the parametrization of a stochastic process with a wave-function.

Acknowledgments

The author acknowledges Zita Marinho for convincing him to study applications of quantum methods in problems other than quantum physics, and for discussions about the use of Hilbert spaces in machine learning.

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