Drag Force on a High Porosity Aerogel in Liquid $^3$He

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Abstract. We discuss the drag force exerted on a high porosity aerogel in liquid $^3$He and its effect on sound propagation. The drag force is governed by distinct laws depending on the Knudsen number $Kn$, the ratio of the quasiparticle mean free path to a characteristic linear size of a body immersed in liquid. In the liquid $^3$He-aerogel system, the characteristic length scale is aerogel strand diameter, and both of the Knudsen limit ($Kn \gg 1$) and the hydrodynamic limit ($Kn \ll 1$) can be achieved in the normal phase by varying temperature. We show that the crossover between the two limits can be found as a drastic change in the temperature dependence of sound attenuation in the normal phase of liquid $^3$He in aerogel.

1. Introduction
The drag force plays a significant role for sound propagation in liquid $^3$He impregnated in high porosity aerogel [1–3]. If we model the aerogel as a collection of rigidly fixed impurities, the Landau theory predicts a few orders of magnitude larger value of ultrasound attenuation observed for 15 MHz longitudinal sound in 98 % silica aerogel [1,4]. This substantial discrepancy between the theory and the experiment is recovered when the collision drag effect is taken into account [1]: the drag force gives rise to simultaneous oscillation of aerogel and reduces frictional damping of sound. The collision drag effect affects also the sound velocity, since the dragged aerogel gives extra inertia to liquid motion.

In pure liquid $^3$He, the quasiparticle mean free path increases with decreasing temperature as $T^{-2}$. When a body is immersed in the liquid and the quasiparticle mean free path is sufficiently longer than the linear size of the body, the drag force is proportional to the cross-sectional area of the body. For example, the drag force exerted on a sphere of radius $a$ is given, in the long mean free path limit (the Knudsen limit), by $\pi a^2 n p_F v_{rel}$ [5] with $n$ the density of liquid $^3$He, $p_F$ the Fermi momentum, and $v_{rel}$ the flow velocity relative to the body. In the opposite limit of short mean free path (the hydrodynamic limit), the drag force obeys a different law, i.e., as is well known from hydrodynamics, the proportionality factor of the drag force to the relative velocity $v_{rel}$ is determined by the viscosity of liquid and the linear size of the body.

In the case of the liquid $^3$He-aerogel system, the corresponding crossover in the drag force is expected around a temperature at which the mean free path is comparable to the aerogel strand diameter. We shall show that the distinct laws for the drag force cause a drastic change in the temperature dependence of ultrasound attenuation in the normal phase of liquid $^3$He in aerogel.

2. Drag force in the Knudsen limit
Virtanen and Thuneberg [5] discussed the drag force in the Knudsen limit using the Landau transport equation together with a surface boundary condition. They derived a formula for the
drag force $F_0$ on a body of arbitrary shape. For a sphere of radius $a$ with specular surface, for example, their formula gives

$$F_0 = \sigma_{\text{tr}} n \rho_F v_{\text{rel}}$$

(1)

with $\sigma_{\text{tr}} = \pi a^2$ the transport cross-section.

Let us consider the drag force per unit volume, $F$, for dilute distribution of impurities of concentration $n_i$, as the simplest model of high porosity aerogel. From the above formula for $F_0$, we obtain

$$F = \frac{1}{\tau_{\text{tr}}} \frac{m^*}{m} \rho v_{\text{rel}};$$

(2)

where $\tau_{\text{tr}} = 1/n_i \sigma_{\text{tr}} v_F$ ($v_F = p_F/m^*$) is the transport mean free time and $\rho$ is the $^3\text{He}$ mass density. Equation (2) coincides with the drag force formula derived by Ichikawa et al. [1] from the impurity-quasiparticle collision integral for randomly distributed impurities. Using eq. (2), they discussed the effect of the drag force on sound propagation in normal liquid $^3\text{He}$ in aerogel and found good agreement with the ultrasound attenuation experiment [4] reported by the Northwestern group. It should be noted, however, that the highest temperature in the experiment [4] is well below the Knudsen-hydrodynamic crossover temperature. At temperatures above the crossover, eq. (2) is not a reasonable approximation for the drag force.

3. Drag force in the hydrodynamic limit

When the quasiparticle mean free path is sufficiently shorter than the linear size of a body, we can use the hydrodynamics of viscous fluid to calculate the drag force $F_0$, resulting in the famous relation

$$F_0 \propto \eta_0 v_{\text{rel}},$$

(3)

where $\eta_0$ is the viscosity of pure liquid $^3\text{He}$. The proportionality factor has dimension of length and carries information on the shape of the body. For a sphere of radius $a$, the factor is given by the Stokes formula $6\pi a$.

The drag force density for a distribution of the bodies obeys the Darcy law

$$F = \frac{\eta_0}{\kappa} v_{\text{rel}}$$

(4)

with $\kappa$ the permeability (with dimension of area) of the porous medium. In the case of the dilute distribution of spheres, the Stokes formula leads to the Darcy law with the permeability

$$\kappa_{\text{St}} = \frac{1}{6} a \tau_{\text{tr}},$$

(5)

where $l_{\text{tr}} = v_F \tau_{\text{tr}}$ is the transport mean free path.

4. Ultrasound attenuation

Now we discuss the effect of the drag force on sound propagation in normal liquid $^3\text{He}$ in aerogel. We are interested in sound frequency $\omega/2\pi$ of order 10 MHz, which has been used in the ultrasound attenuation experiments so far reported. We shall compare our result with the experiment by Choi et al. [6] who performed longitudinal ultrasound (9.5 MHz) attenuation measurements on liquid $^3\text{He}$ in 98% aerogel over a wide temperature range up to 200 mK.
Since $\tau_{tr}$ is of order 1 ns and therefore $\omega \tau_{tr} \ll 1$, the motion of liquid $^3$He in aerogel is governed by the Navier-Stokes equation

$$\rho \partial_t \mathbf{v} = -\nabla P + \eta (\Delta \mathbf{v} + \frac{1}{3} \nabla \text{div} \mathbf{v}) - \mathbf{F},$$

(6)

where

$$\eta = \frac{1}{5} \frac{m^*}{m} \rho v_F^2 \tau_\eta$$

(7)

is the viscosity and $P$ is pressure. The viscous relaxation time $\tau_\eta$ is limited by impurity scattering at low temperatures,

$$\frac{1}{\tau_{\eta}} = \frac{1}{\tau_{in}} + \frac{1}{\tau_{el}},$$

(8)

where $1/\tau_{in} \propto T^2$ comes from inelastic quasiparticle collisions and $1/\tau_{el} = \text{const.}$ from elastic impurity scattering.

Since the wave length of the sound considered is much longer than the average distance between aerogel strands, the equation of motion of dragged aerogel can be written, within continuum approximation, in the form [1,2]

$$\rho_a \partial_t^2 \mathbf{u}_a = \rho_a c_a^2 \Delta \mathbf{u}_a + \mathbf{F},$$

(9)

where $\rho_a$, $\mathbf{u}_a$, and $c_a$ are the mass density, the displacement vector, and the sound velocity of aerogel, respectively.

One can parameterize the drag force density $\mathbf{F}$ as

$$\mathbf{F} = \frac{1}{\tau_f} \rho (\mathbf{v} - \partial_t \mathbf{u}_a).$$

(10)

Here $\tau_f$ is a relaxation time characterizing the frictional damping of sound and is given by $\tau_{tr} m/m^*$ in the Knudsen limit and by $\rho c / \eta_0$ in the hydrodynamic limit.

It is straightforward to obtain, from the above set of equations, the velocity $c$ and attenuation $\alpha$ of longitudinal sounds. The results are

$$c^2 = \frac{c_1^2}{1 + \rho_a/\rho},$$

(11)

$$\alpha = \frac{\omega^2}{2c} \left[ \frac{(\rho_a/\rho)^2}{1 + \rho_a/\rho} \tau_f + \frac{4}{3} \frac{\eta}{\rho c_1^2} \right],$$

(12)

where $c_1$ is the first sound velocity in pure liquid $^3$He. In Eq. (12), the first term $\propto \tau_f$ and the second term $\propto \eta$ describe the frictional and viscous effects on the sound attenuation, respectively.

In Fig. 1, we plot the relevant length scales in 98 % aerogel and the temperature dependence of the sound attenuation at $\omega/2\pi = 9.5$ MHz. In these plots, we used the parameters for $^3$He at a pressure 30 bars [7,8]. The parameters associated with the aerogel are chosen as $l_{el} = v_F \tau_{el} = 160 \text{ nm}$, $l_{tr} = 100 \text{ nm}$, $a = 1.5 \text{ nm}$, and $\rho_a = 0.044 \text{ g/cm}^3$ [6]. The attenuation is obtained from eq. (12) using the following interpolation formula for $1/\tau_f$:

$$\frac{1}{\tau_f} = \frac{\eta_0 / \kappa \rho}{1 + C \text{Kn}} = \frac{1}{\tau_{tr}} \frac{m^*}{m} \frac{1}{1 + 5 \kappa / l_{tr} l_{in}},$$

(13)
Figure 1. (a) Relevant length scales in 98% aerogel as a function of temperature $T$.
(b) Temperature dependence of the sound attenuation $\alpha$ calculated with the interpolation formula Eq. (13) (black curve). The gray and dashed curves represent the attenuation with the Knudsen formula $\tau_f = \tau \nu / m^a$ and with the hydrodynamic one $\tau_f = \rho_k / \eta_0$, respectively.

where $\text{Kn} = l_{\text{in}} / a = v_F \tau_{\text{in}} / a$ is the Knudsen number and $C = a l_{\text{in}} / 5 \kappa$ is a temperature independent constant. For evaluating the permeability, we have used the Stokes formula $\kappa_S$. The attenuation curve is almost flat below 3 mK, reflecting the temperature independence of $\tau_\eta$ and $\tau_f$ at the low temperatures. With increasing temperature, $\tau_\eta$ decreases but $\tau_f$ is still almost independent of temperature, so that the attenuation decreases monotonically up to $T \sim 40$ mK.

As we increase temperature further, the attenuation takes a minimum and then increases rapidly as $T^2$. This upturn is due to the crossover of the drag force from the Knudsen limit $l_{\text{in}} \gg a$ to the hydrodynamic limit $l_{\text{in}} \ll a$.

The structure around 40 mK is very similar to that observed by Choi et al. [6]. The overall temperature dependence is also in good agreement with the experiment, except for the high temperature limit behavior where the theoretical curve does not show the observed $T^{0.7}$ dependence but shows $T^2$ dependence. At the high temperatures, since the mean free path is very short, the effect of the detailed structure of aerogel should be examined. This is a challenging problem beyond the scope of the present work.

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