Is the quark- mixing matrix moduli symmetric?

S. Chaturvedi *
School of Physics, University of Hyderabad,
Hyderabad 500 046 India
Virendra Gupta †
Departamento de Física Aplicada, CINVESTAV-Unidad Mérida
A.P. 73 Cordemex 97310 Mérida, Yucatan, Mexico

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Abstract

If the unitary quark- mixing matrix, $V$, is moduli symmetric then it depends on three real parameters. This means that there is a relation between the four parameters needed to parametrize a general $V$. It is shown that there exists a very simple relation involving $|V_{11}|^2$, $|V_{33}|^2$, $\rho$ and $\eta$. This relation is compared with the present experimental data. It is concluded that a moduli symmetric $V$ is not ruled out.

*scsp@uohyd.ernet.in
†virendra@aruna.mda.cinvestav.mx
1 Introduction

It is well known that for three generations, the general parametrization \cite{1,2} of the Cabibbo-Kobayashi-Maskawa (CKM) quark-mixing matrix, $V$, depends on four parameters, namely, three angles and a phase. Experimental data gives the values of the moduli $|V_{ij}|$ and a particular parametrization of $V$ is needed to determine the complex matrix elements of the unitary matrix $V$. Four moduli (obtained from data) are needed to determine the four parameters in $V$.

The present data \cite{2}, gives us the ranges of $|V_{ij}|$. These are

$$V_{\text{EXP}} = \begin{pmatrix} 0.9741 - 0.9756 & 0.219 - 0.226 & 0.0025 - 0.0048 \\ 0.219 - 0.226 & 0.9732 - 0.9748 & 0.038 - 0.044 \\ 0.004 - 0.014 & 0.037 - 0.044 & 0.9990 - 0.9993 \end{pmatrix}. \quad (1)$$

It is clear from these values that there is a possibility that $V$ might turn out to be moduli symmetric. The ranges suggest that $|V_{ij}| = |V_{ji}|$ for $(i, j) = (1, 2)$ and $(2, 3)$, but it seems that $|V_{13}| \neq |V_{31}|$. However, the latter matrix elements are difficult to measure and may change in future. Since $V$ is unitary, it follows that

$$\Delta \equiv |V_{12}|^2 - |V_{21}|^2 = |V_{23}|^2 - |V_{32}|^2 = |V_{31}|^2 - |V_{13}|^2. \quad (2)$$

So either $V$ is completely moduli symmetric ($\Delta = 0$) or it is fully asymmetric ($\Delta \neq 0$). Recently an attempt to understand the smallness of this asymmetry (i.e. smallness of $\Delta$) has been made \cite{3}.

In this note, we explore the experimental consequences of a moduli symmetric $V$, denoted by $V_{\text{MS}}$. Since, $\Delta = 0$ for $V_{\text{MS}}$, this gives an extra condition\footnote{An explicit parametrization for $V_{\text{MS}}$ was considered in references 5 and 6. A relation involving $|V_{12}|, |V_{23}|$ and the parameters $\rho$ and $\eta$ of the unitarity triangle was pointed out.} and consequently a general parametrization of $V_{\text{MS}}$ contains only three real parameters \cite{3,4}.

The important point is that if $V = V_{\text{MS}}$ then there will be a relation between four measurables, which for a general $V$ would be independent. In this note we obtain a general relation and confront it with available data.

2 The relation

There is a lot of interest in measuring the quantities connected with the unitarity relation or triangle, viz.,

$$V_{11}V_{13}^* + V_{21}V_{23}^* + V_{31}V_{33}^* = 0, \quad (3)$$
Define $z_i = V_{1i}V_{3i}^*; \ i = 1, \ 2, \ 3$ then Eq.(3) can be written as
\[ -z_1/z_2 - z_3/z_2 = 1. \] (4)

This defines a triangle. Define the complex numbers\[2\]
\[ -z_1/z_2 = \bar{\rho} + i\eta, \] (5)
so using Eq.(4),
\[ -z_3/z_2 = (1 - \bar{\rho}) - i\eta \] (6)

This notation like that for the angles of the triangle has become standard. The angles $\alpha = \arg(-z_3/z_1)$, $\beta = \arg(-z_2/z_3)$, and $\gamma = \arg(-z_1/z_2)$ of the triangle satisfy
\[ \sin \alpha = \frac{\sin \beta}{\sqrt{\rho^2 + \eta^2}} = \frac{\sin \gamma}{\sqrt{(1-\rho)^2 + \eta^2}}, \] (7)

and
\[ \tan \gamma = \frac{\eta}{\rho}. \] (8)

To obtain the desired relation we note that from Eqs.(5-6)
\[ \frac{(1 - \bar{\rho})^2 + \eta^2}{\rho^2 + \eta^2} = \frac{|V_{33}V_{31}|^2}{|V_{11}V_{13}|^2} = \frac{|V_{33}|^2}{|V_{11}|^2} = r \] (9)

The last equality follows if $V$ is moduli symmetric since then $|V_{13}|^2 = |V_{31}|^2$. Thus, for $V_{MS}$, the four independent quantities $\bar{\rho}, \eta, |V_{11}|$ and $|V_{33}|$ are related.

To compute the ratio $r$, we convert the ranges for $|V_{ij}|$ given in Eq.(11) into a central value with errors. This procedure gives, $|V_{11}| = 0.97485 \pm 0.00075$, $|V_{33}| = 0.99915 \pm 0.00015$ $|V_{13}| = 0.00365 \pm 0.00115$ and $|V_{31}| = 0.009 \pm 0.005$. Using these we find for $V_{MS}$, $r_{MS} = |V_{33}|^2/|V_{11}|^2 = 1.05048 \pm 0.00165$, otherwise $r = |V_{31}V_{33}|^2/|V_{13}V_{11}|^2 = 6.38683 \pm 8.15826$. The extremely large error in $r$ reflects the large errors in $|V_{13}|$ and $|V_{31}|$.

According to the relation in Eq.(2), $\bar{\rho}$ and $\eta$ lie on the circle
\[ (\bar{\rho} + 1/(r - 1))^2 + \eta^2 = (\sqrt{r}/(r - 1))^2 \] (10)

The circles for $r_{MS}$ are plotted in Fig 1. The relevant portion in the first quadrant is shown since $\bar{\rho}$ and $\eta$ are both positive. It should be noted for $r = 1$ the circle degenerates into a straight line $\bar{\rho} = 1/2$. As $r$ increases, the radius increases and the centre approaches the origin along negative $\bar{\rho}$-axis. For $r = 6.38683$ the centre of the circle is at $(-0.185638, 0)$ and the radius is 0.469148. Since there is a large error in $r$ (for the asymmetric case), it is clear that the range of values for $r$ contain those for $r_{MS}$. Given the present
data it seems that the possibility that $V$ is moduli symmetric is not ruled out. Our point here is that Eq.(10) provides a very simple and direct way to check if $V$ is moduli symmetric or not. One has to await more accurate data for $|V_{13}|$ and $|V_{31}|$ to come to a definitive conclusion in this regard.

From Eqs.(7,8), we can determine

$$\sin 2\beta = 2\eta (1 - \rho) \left(1 - \rho^2 + \eta^2\right).$$

(11)

The curve in Eq.(11) represents the product of straight lines given by

$$\eta = \tan \beta (1 - \rho),$$

(12)

$$\eta = \cot \beta (1 - \rho).$$

(13)

Experimentally, different groups and different decay modes give a wide range of values for $\sin 2\beta$. Using the average of all modes and groups [7], $\sin 2\beta = 0.699 \pm 0.054$, the straight lines in Eq.(12, 13) are also plotted in Fig.1. It is interesting to note that the circles for $r = r_{MS} = 1.05048 \pm 0.00165$ and the line in Eq(13) with $\cot \beta = 2.45368 \pm 0.265066$ have a small region of intersection around $\eta = 0.447577$, $\eta = 1.36391$. However, this is excluded by constraints from other data [2]. For the general case, taking $1/2$ the error into account, that is $r = 6.38683 \pm 4.07913$, one finds that there is a large region of intersection region with the lines in Eq.(12) with $\tan \beta = 0.407551 \pm 0.044027$, though there is no intersection with Eq.(13). As one can see the lines corresponding to Eq.(12) with $\tan \beta = 0.407551 \pm 0.044027$ have a small region of intersection with the circles for $r = r_{MS} = 1.05048 \pm 0.00165$ around the point $\eta = 0.492779$ and $\eta = 0.260728$ keeping open the possibility that $V$ be moduli symmetric.

Further, we note that $\sin^2 \gamma / \sin^2 \beta = r$ so that knowledge of $\beta$ from $\sin 2\beta$ enables one to obtain angles $\alpha$ and $\gamma$. The values of the angles for $r = r_{MS}$ and the general $r$ are given in Table I. The values in the two columns, as expected, are fairly different. The point to note is that for the moduli symmetric case, since $r_{MS} \approx 1$, one expects $\beta \approx \gamma$, unlike the general or asymmetric $V$ where the two angles can be quite different (viz. Table I). It is very interesting to note the value of $\sin 2\alpha$ (which is in the process of being measured) in the two cases. From Table I, for the central values, one expects $\sin 2\alpha = -0.9974$ for the moduli symmetric case in contrast to a value of 0.163 for an asymmetric $V$. An experimental value of $\sin 2\alpha$ near $-1$ would favour a moduli symmetric $V$.

In conclusion, we have pointed out that a simple, model independent relation between $\eta, \eta, |V_{11}|$ and $|V_{33}|$ provides a direct test of the moduli
symmetry of $V$. The present data does not rule out such a symmetry. For a conclusive answer we must await future data.

In our view a moduli symmetric quark-mixing matrix would be far more elegant and physically interesting than one with a tiny, difficult to explain, asymmetry.

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Table I Numerical values of the unitarity triangle angles with errors corresponding to $r = r_{MS}$ and for general $r$. Note that in the latter case we have taken the error in $r$ to be half of its actual value.

| ANGLES | $r = r_{MS} = 1.05048 \pm 0.00165$ | $r = 6.38683 \pm (8.15826)/2$ |
|--------|-----------------------------------|---------------------------------|
| $\beta$ | $22.1734 \pm 2.16325^\circ$ | $22.1734 \pm 2.16325^\circ$ |
| $\gamma$ | $22.7568 \pm 2.17246^\circ$ | $72.5159 \pm 58.4675^\circ$ |
| $\alpha$ | $137.07 \pm 3.06581^\circ$ | $85.3107 \pm 58.5075^\circ$ |
Figure 1: Plots of $\eta$ versus $\bar{\rho}$: (a) General Case: The lower three curves represent Eq(10) for $r = 6.38683 + (8.15826)/2, 6.38683$ and $6.38683 - (8.15826)/2$. They are parts of circles of radii 0.341763, 0.469148, 1.1617 with centres at $(-0.105643, 0), (-0.185638, 0), (-0.764701, 0)$ respectively. (b) Moduli Symmetric Case: The upper three curves again represent Eq(10) for $r = 1.05048 + 0.00165, 1.05048$ and $1.05048 - 0.00165$. The radii of these circles are 19.6765, 20.3037, 20.9733 with centres at $(-19.1828, 0), (-19.8098, 0), (-20.4792, 0)$ respectively. (c) The lower pair of straight lines corresponds to Eq.(12) with $\tan \beta = 0.407551 \pm 0.044027$ while the upper pair of straight lines corresponds to Eq.(13) with $\cot \beta = 2.45368 \pm 0.265066$. 

