Delocalization and spin-wave dynamics in ferromagnetic chains with long-range correlated random exchange

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We study the one-dimensional quantum Heisenberg ferromagnet with exchange couplings exhibiting long-range correlated disorder with power spectrum proportional to $1/k^\alpha$, where $k$ is the wave-vector of the modulations on the random coupling landscape. By using renormalization group, integration of the equations of motion and exact diagonalization, we compute the spin-wave localization length and the mean-square displacement of the wave-packet. We find that, associated with the emergence of extended spin-waves in the low-energy region for $\alpha > 1$, the wave-packet mean-square displacement changes from a long-time super-diffusive behavior for $\alpha < 1$ to a long-time ballistic behavior for $\alpha > 1$. At the vicinity of $\alpha = 1$, the mobility edge separating the extended and localized phases is shown to scale with the degree of correlation as $E_c \propto (\alpha - 1)^{1/3}$.

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I. INTRODUCTION

It is well established that one-electron eigenstates in chains with uncorrelated disorder are exponentially localized. However, several theoretical investigations have shown that a series of one-dimensional versions of the Anderson Model exhibit a breakdown of the Anderson's localization as a function of the degree of correlation and the localization length.

It is also known that the magnon equation of motion for ferromagnetic spin chains with uncorrelated random nearest-neighbor exchange couplings can be exactly mapped onto an electronic chain with a particular form of off-diagonal disorder where the hopping integrals appear correlated in pairs. By using a perturbation approach combined with a scaling hypothesis it was demonstrated that the singularities of the density of states and the localization length of a random ferromagnetic Heisenberg chain depend on the distribution of exchange couplings. For uncorrelated random exchange couplings $J \in [0, 1]$ with a probability distribution function $P(J) = (1 - \delta)J^{-\delta}$, it was demonstrated that for $\delta < -1$, which implies $< 1/J >$ and $< 1/J^2 >$ finite, the density of states $\rho(E)$ diverges as $1/E^{1/2}$ and the localization length as $1/E$. Distinct regimes emerge when either of the above two moments diverges, thus generalizing other studies.

In this paper, we study the nature of the spin-wave modes of a quantum Heisenberg ferromagnetic chain with long-range correlated random exchange couplings $J_n$ assumed to have spectral power density $S \propto 1/k^\alpha$. A previous study reported some finite-size scaling evidences of the emergence of a phase of extended low-energy excitations. Here, we will use a renormalization group technique to provide accurate estimates for the mobility-edge energy as a function of the degree of correlation and to obtain the scaling behavior governing the vanishing of the mobility edge in the vicinity of $\alpha = 1$. Further, we also study the quantum diffusion of the wave-packets in these chains using direct integration of the motion equations and exact diagonalization to investigate the possible emergence of a new dynamical regime associated with the...
occurrence of extended low-energy spin waves for \( \alpha > 1 \).

II. THE MODEL AND RENORMALIZATION-GROUP CALCULATION

We consider a Hamiltonian model describing a spin\(-1/2\) quantum ferromagnetic Heisenberg chain of \( N \) sites with random exchange couplings \( J_n \):

\[
H = \sum_{n=-N/2}^{N/2} J_n S_n \cdot S_{n+1},
\]

where \( S_n \) represents the quantum spin operator at site \( n \) and open boundary conditions are used. We take the exchange couplings \( J_n \) connecting sites \( n \) and \( n+1 \) to be correlated in such a sequency to describe the trace of a fractional Brownian motion:

\[
J_n = \sum_{k=1}^{N/2} \left[ k^{-\alpha} \left( \frac{2\pi}{N} \right)^{(1-\alpha)} \right]^{1/2} \cos \left( \frac{2\pi nk}{N} + \phi_k \right),
\]

where \( k \) is the wave-vector of the modulations on the random coupling landscape and \( \phi_k \) are \( N/2 \) random phases uniformly distributed in the interval \([0, 2\pi]\). The exponent \( \alpha \) is directly related to the Hurst exponent \( H \) \((\alpha = 2H + 1)\) of the rescaled range analysis, which describes the self-similar character of the series and the persistent character of its increments: for \( \alpha > 2 \) (\( H > 1/2 \)) the increments are persistent, while for \( \alpha < 2 \) (\( H < 1/2 \)) they are anti-persistent. In the case of \( \alpha = 2 \) (\( H = 1/2 \)) the sequency of exchange couplings resembles the trace of the usual Brownian motion, while for \( \alpha = 0 \) (\( H = -1/2 \)) one recovers the uncorrelated random exchange Heisenberg model. The coupling distribution is Gaussian for \( \alpha = 0 \) but assumes a non-Gaussian form for finite \( \alpha \) once the presence of long-range correlations implies in the lack of self-averaging and the breakdown of the central limit theorem. In order to avoid a vanishing exchange coupling we shift all couplings generated by Eq. (2) such to have average value \( \langle J_n \rangle = -4.5 \) and variance \( \langle \Delta J_n \rangle = 1 \). Note that, in such case, all moments of the resulting distribution are finite. In order to keep the variance size independent, the normalization factor scales with the chain size. A detailed finite-size scaling analysis has shown how such normalization procedure is reflected in the main character of the one-magnon excitations [14]. Indeed, without such rescaling of the potential the disorder width diverges for any \( \alpha \neq 0 \) and all states are expected to remain localized [13].

The ground state of the system contains all spins pointing in the same direction. If a spin deviation occurs at a site \( n \), this excited state is described by

\[
\phi_n = S_n^+ |0>,
\]

where the operator \( S_n^+ \) creates a spin deviation at site \( n \) and \( |0> \) denotes the ground state. The eigenstates of the Hamiltonian are therefore a linear combination of \( \phi_n \), i.e., \( \Phi = \sum c_n \phi_n \), the coefficients \( c_n \) satisfying the equation below [14 12 13]:

\[
(J_{n-1} + J_n)c_n - J_{n-1}c_{n-1} - J_n c_{n+1} = 2Ec_n,
\]

where \( E \) is the excitation energy.

In order to study the properties of one-magnon states, we apply a decimation renormalization-group technique. The method is based on the particular form assumed by the equation of motion satisfied by the Green’s operator matrix elements \( [G(E)]_{i,j} = \langle i|1/(E-H)|j \rangle \) [19]:

\[
(E - \epsilon_{n+\mu}^0)[G(E)]_{n+\mu,n} = \delta_{\mu,0} + J_{n+\mu,n+\mu-1}^0 [G(E)]_{n+\mu-1,j} + J_{n+\mu,n}^0,
\]

where \( \epsilon_n^0 = (J_{n-1} + J_n)/2 \) and \( J_{n,n+1}^0 = J_{n+1,n} = J_n/2 \). After eliminating the matrix elements associated with a given site, the remaining set of equations of motion can be expressed in the same form as the original one, but with renormalized parameters:

\[
\epsilon_{N}^{(N-1)}(E) = \epsilon_N + J_{N-1,N} \frac{1}{E - \epsilon_{N-1}^{(N-2)}} J_{N-1,N},
\]

\[
j_{0,N}^{(eff)}(E) = \frac{1}{E - \epsilon_{N-1}^{(N-2)}} J_{N-1,N},
\]

where, after \( N - 1 \) decimations, \( \epsilon_N^{(N-1)} \) denotes the renormalized diagonal element at site \( N \) and \( j_{0,N}^{(eff)} \) indicates the effective renormalized exchange coupling connecting the sites \( 0 \) and \( N \).

To investigate the localized/delocalized nature of the spin-wave modes, we compute the inverse of the excitation width or the Lyapunov coefficient \( \gamma(E) \) (inverse localization length). The Lyapunov coefficient is asymptotically related to the effective exchange coupling by [4 9]:

\[
\gamma(E) = -\lim_{N \to \infty} \frac{1}{N} \ln |j_{0,N}^{(eff)}(E)|.
\]

After a linear regression of \( \ln |j_{0,N}^{(eff)}(E)| \) versus \( N \) we have a direct extrapolation of the Lyapunov coefficient in the thermodynamical limit. We computed \( \gamma(E) \) for distinct values of the exponent \( \alpha \) and \( N = 10^5 \) sites. In addition, the density of states (DOS) was calculated by using the numerical Dean’s method [20]. In Fig. 1 we show the normalized DOS for chains with \( N = 10^5 \) sites. Notice that it becomes less rough as \( \alpha \) is increased and its singularity at the bottom of the band is not affected by the imposed long-range correlation in the coupling constants (see insets). For \( \alpha = 1.5 \) it consists of a non-fluctuating part near the bottom of the band with the same form as that of the pure chain \( (J_n = \text{constant}) \). Previous studies have pointed out that the smoothing of the DOS is usually connected with the emergence of delocalized states [21].

In Fig. 2 we show the plot of \( \gamma \) versus \( E \) for \( \alpha = 0 \) (uncorrelated random exchanges). In this case, the Lyapunov coefficient is finite for all energies, except for \( E = 0 \).
as usual. This behavior remains qualitatively unaltered for $0 < \alpha \leq 1$, therefore implying in the absence of extended spin waves in this regime. For $\alpha = 0$, we have also observed that, near the bottom of the band, the Lyapunov coefficient vanishes as $\gamma \propto E^\nu$ with $\nu = 1.0$, in agreement with Ref. [10] for probability distribution functions with finite $< 1/J >$ and $< 1/J^2 >$ (see inset of Fig. 2). The picture is qualitatively different for $\alpha > 1$. In Fig. 3(a) we plot $\gamma$ versus $E$ for $\alpha = 1.5$. The Lyapunov coefficient vanishes within a finite range of energy values, thus confirming the presence of low-energy extended spin waves. In all chains studied with sizes ranging from $10^5$ up to $10^8$ sites, the $\gamma(E)$ curves appear to be the same, indicating that the extended phase of magnons is stable in the thermodynamical limit. The phase diagram in the $(E_c, \alpha)$ plane is shown in Fig. 3(b), with $E_c$ (given in units of $D\Delta J$) denoting the mobility edge and statistical errors are smaller than the symbol sizes. The data analysis (see inset) suggests that, at the vicinity of $\alpha = 1$, the mobility edge depends on the correlation exponent as $E_c \propto (\alpha - 1)^\gamma$, with $\gamma = 1/3$.

III. SPIN-WAVE DYNAMICS

In order to investigate the spin-wave dynamics, we compute the time dependence of the mean-square displacement of the wave-packet. Let us consider an excitation initially localized at site $n_0$, represented at $t = 0$ by its eigenfunction $\phi_n(t = 0) = \delta_{n,n_0}$. Its time evolution is described by the Schrödinger equation ($\hbar = 1$):

$$i \frac{d\phi_n(t)}{dt} = H\phi_n(t),$$

whose time-dependent wave-function can be written in terms of the computed eigenvectors $V^{(j)}$ and eigenvalues $E_j$ of $H$ as

$$\phi_n(t) = \sum_j^N V_n^{(j)} V_{n_0}^{(j)} \exp(-iE_j t).$$

(10)

The same time-dependent wave-function can be obtained by integrating the equations of motion. The second moment of the corresponding spatial probability distribution is then given by

$$\sigma^2(t) = \sum_n (n - n_0)^2 \phi_n(t)\phi_n^*(t).$$

(11)

From the mean-square displacement $\sigma^2(t)$ we can estimate the wave-packet spread in space at a time $t$. For any $\alpha \geq 0$, we find ballistic behavior $[\sigma(t)^2 \propto t^2]$ for initial times, indicating that the disorder have not yet been realized by the spin waves [11]. In the case of uncorrelated random exchange ($\alpha = 0$) the self-expanded chain was used to minimize end effects. When the probability of finding the particle at the ends of the chain exceeded $10^{-100}$ we added new sites thus expanding the chain. In Fig. 4 we show the mean squared displacement versus time for $\alpha = 0$ as obtained by integrating the equations of motion. In this case, for longer times the wave-packet presents a super-diffusive spread $[\sigma(t)^2 \propto t^{3/2}]$ in agreement with previous studies [11] for uncorrelated random exchange distribution with $< 1/J >$ finite. In Fig. 5 we plot the data obtained by integrating the equations of motion for $\alpha = 0.5$ and 20000 sites. As indicated, the initial ballistic motion extends over longer times, although a super-diffusive motion still takes place after this initial transient. Finally, Fig. 6 shows that for $\alpha = 1.5$ the wave-packet displays only a ballistic spread. In this case our calculation was performed by using both numerical integration of the equations of motion for $N = 20000$ sites and exact diagonalization for $N = 2000, 4000$ and 8000 sites, in which case the end effect is present (see inset).

In contrast with the case of uncorrelated random exchange couplings [11], in which an asymptotic super-diffusive behavior sets up whenever $< 1/J >$ is finite, we find a crossover with increasing $\alpha$ from super-diffusive to ballistic asymptotic regimes induced by long-range correlations in the exchange couplings. The ballistic regime for $\alpha > 1$ can be understood following arguments similar to those used in Ref. [2] [11]. Exploring the exact mapping of the magnon problem onto a paired electronic one, the diffusion coefficient $D$ can be estimated by integrating $v(k)\lambda(k)$ over the extended states that effectively participate in the transport, where $v(k)$ and $\lambda(k)$ are respectively the velocity and mean free-path of the excitation mode with wavenumber $k$. In a finite chain all extended modes have $\lambda(k) \approx N$ and travel with finite velocity since in the electronic problem the DOS is non-singular near the band center. Once there is a finite fraction of states that are delocalized, the integration runs over a finite wavenumber and, interchanging $N$ and $t$, the diffusion coefficient results $D \propto t$. Consequently, the mean square displacement $[\sigma^2(t)] = Dt \propto t^2$ confirming the ballistic nature of the wave-packet spread found in our numerical analysis.

IV. CONCLUSIONS

In summary, we have studied the one-dimensional quantum Heisenberg ferromagnet with exchange couplings exhibiting long-range correlated disorder with spectral power density proportional to $1/k^\alpha$. By using a decimation renormalization-group technique we have found further evidences suggesting that this magnetic system displays a phase of extended spin waves in the low-energy region for $\alpha > 1$ ($H > 0$). The mobility edge separating low-energy extended and high-energy localized states was shown to depend on the degree of correlation in a very special manner. Finally, through integration of the equations of motion and exact diagonalization, we have also computed the mean-square displacement of the spin-wave packet. For $0 < \alpha \leq 1$, we
have found long-time super-diffusion, in agreement with previous works for uncorrelated random exchange distribution with $<1/J>$ finite. However, for strong correlations ($\alpha > 1$) a long-time ballistic regime was numerically observed which is associated with the emergence of extended excitations. We believe that the reported results might be useful to stimulate further theoretical and experimental investigations of spin-wave dynamics on correlated ferromagnetic chains and non-periodic ferromagnetic super-lattices.

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Fig. 2 Lyapunov coefficient $\gamma$ versus energy $E$ for $\alpha = 0$ (uncorrelated random exchange model) and $N = 10^5$ sites from the renormalization procedure. The Lyapunov coefficient is finite for non-zero energies (localized states), and vanishes as $\gamma \propto E^\nu$, $E \to 0$, with $\nu = 1.0$ (inset).

Fig. 3 (a) Lyapunov coefficient $\gamma$ versus energy $E$ for $\alpha = 1.5$ and $N = 10^5$ sites from the renormalization procedure. The Lyapunov coefficient vanishes within a finite range of energy values revealing the presence of extended low-energy spin waves. (b) $(E_c, \alpha)$ phase diagram, where $E_c$ is the mobility edge (in units of $\Delta J$) for $N = 10^5$ sites. The phase of extended spin-waves emerges for $\alpha > 1$ and $E < E_c$.

Fig. 4 Mean squared displacement $\sigma^2$ versus time $t$ for $\alpha = 0$ from the integration of the equations of motion. The self-expanded chain was used to minimize end effects. The spread of the wavepacket depicts a crossover from a initial ballistic ($\sigma^2 \propto t^2$) to a super-diffusive ($\sigma^2 \propto t^{3/2}$) behavior.

Fig. 5 Mean squared displacement $\sigma^2$ versus time $t$ for $\alpha = 0.5$ and $N = 20000$ sites from the integration of the equations of motion. A longer living ballistic motion ($\sigma^2 \propto t^2$) is found but still followed by a crossover to the super-diffusive regime ($\sigma^2 \propto t^{3/2}$).

Fig. 6 Mean squared displacement $\sigma^2$ versus time $t$ for $\alpha = 1.5$ for $N = 20000$ sites from the integration of the equations of motion. Inset: results using exact diagonalization for $N = 2000$ (dotted line), 4000 (dashed line) and 8000 (dot-dashed line) sites. Ballistic behavior ($\sigma^2 \propto t^2$) is found for all times.