Exact traversable wormhole solution in bumblebee gravity

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We find a new traversable wormhole solution in the framework of the bumblebee gravity theory in which the Lorentz symmetry violation arises from the dynamics of the bumblebee vector field non-minimally coupled to gravity. To this end, we check the flare-out and energy conditions (null, weak, and strong). Then we study the deflection angle of light in the weak limit approximation using the Gibbons-Werner method. In particular, we show that the bumblebee gravity effect leads to a non-trivial global topology of the wormhole spacetime. By using the Gauss-Bonnet theorem, it is shown that the obtained (non-asymptotically flat) wormhole solution yields a topological term in the deflection angle of light which is proportional to the coupling constant, but independent from the impact factor parameter. Significantly, it is shown that the bumblebee wormhole solutions, under specific conditions, support the normal matter wormhole geometries, which satisfy both the energy and the flare-out conditions.

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I. INTRODUCTION

The search for a theory of wormholes through Einstein’s general theory of relativity goes back to 1916 with the famous papers of Flamm [1]. The simplest possible solution of Einstein’s field equations is known as the Schwarzschild metric [2], which describes the gravitational field around a spherically symmetric static mass. If the mass (in fact the density) is sufficiently high, the solution describes a black hole – the Schwarzschild black hole. Flamm realized that Einstein’s equations also allow a second solution, which is presently known as a white hole. Those two solutions, describing two different regions of (flat) spacetime, are connected by a “spacetime tube”. This tube has nothing to state about where those regions of spacetime might be in the real world, the black hole "entrance" and white hole "exit" could be in different portions of the same universe or in entirely different universes. In 1935, Einstein and Rosen [3] further explored the theory inter-universe connections. In fact, their main aim was to try to understand the fundamental charged particles (protons, electrons etc.), in terms of spacetime tubes em-pierced by electric lines of force. Their study was entitled with “Einstein-Rosen bridge” by Wheeler, who would later call it as wormhole. It is worth noting that Wheeler [4] also coined the term black hole. Traversable wormholes have no horizon and allow two-way traveling [5] by connecting two different regions of spacetime in a Lorentzian geometry. Interest in traversable wormhole gained momentum following the paper of Morris, Thorne, and Yurtsever (MTY) [6]. With a traversable wormhole, an interstellar or inter-universe journey is possible [7, 8]. However, to construct such a traversable wormhole, one needs an exotic matter with a negative energy density and a large negative pressure, which should have higher value

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than the energy density. Meanwhile, Casimir effect [9] is a way of producing negative energy density. MTY also proved that traversable wormhole could be stabilized by using the Casimir effect. To this end, placing two sufficiently charged superconducting spheres to the traversable wormhole mouths is enough. On the other hand, in 2011, Kanti and Kleihaus [10] showed that it might be possible to construct a traversable wormhole with using normal matter by resorting to a form of string theory.

In literature many authors intensively have studied on various aspects of the traversable wormhole geometries within different modified gravity theories [11–60]. Among them a bumblebee gravity model dynamically violates the Lorentz symmetry and CPT [charge conjugation (C), parity transformation (P), and time reversal (T)] with its defined bumblebee vector field, which can also have the features of rotation and boost [61–66]. In fact, bumblebee gravity was first used by Kostelecky and Samuel in 1989 [67] as a simple model for investigating the consequences of spontaneous Lorentz violation. The mechanism of bumblebee is arose in the context of string theory and it leads to spontaneous breaking of Lorentz symmetry by tensor-valued fields acquiring vacuum expectation values [65]. The name bumblebee model, coined by Kostelecky, is based on an insect whose ability to fly has sometimes been questioned on theoretical grounds, but which nonetheless is able to fly successfully [63]. The forcefulness of the bumblebee vector field on the gravity has motivated us to construct traversable wormhole. Very recently, Schwarzschild-like bumblebee black hole solution has been obtained in [68]. From the point of string theory and loop quantum gravity theory, Lorentz symmetry breaking (LSB) is an interesting idea for exploring the tracks of the quantum gravity at low energy levels. LSB has been extensively studied in literature (see for example [69–74] and references therein).

The main aim of the paper is to construct an exact solution of a traversable wormhole in the bumblebee gravity in which the Einstein field equations are in the influence of a spontaneous breaking of Lorentz symmetry and compute the weak deflection angle of the obtained bumblebee wormhole. To this end, we employ the Gibbons-Werner method [75]. In this method, the deflection angle, for the weak lensing limit, is calculated by the Gauss-Bonnet theorem defined for the background of optical geometry. It is important to highlight that non-singular domain considered is outside of the light ray, which means that Gauss-Bonnet theorem has a global impact [75–78]. Wormholes have been widely studied by many authors similarly to black holes, in particular the light deflection has attracted a lot of interest [79–93]. Another purpose of the present paper is to discover a traversable wormhole with normal matter, which satisfies energy and flare-out conditions in the bumblebee gravity. In the following sections, we shall explain how these goals are achieved.

The paper is organized as follows. In Sec. II, we briefly outline the bumblebee gravity and its corresponding Einstein’s field equations. In Sec. III, we present LSB wormhole solutions and study the flare-out conditions. We check the energy conditions of the bumblebee wormhole in Sec. IV. In the framework of the Gibbons-Werner method, Sec. V is devoted to the study of the deflection angle of light in the weak limit approximation. We give our conclusions and remarks in Sec. VI.

II. BUMBLEBEE GRAVITY

The action of the bumblebee gravity where the Lorentz violation arises from the dynamics of a single vector field, namely bumblebee field $B_{\mu
u}$, with a coupling constant controlling a non-minimal curvature-coupling $\xi$ is given by

$$S_B = \int d^4 x \sqrt{-g} \left[ \frac{1}{2\kappa} R + \frac{1}{2\kappa} \xi B^{\mu\nu} R_{\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} - V(B^\mu) \right] + \int d^4 x \mathcal{L}_M, $$

(2.1)

where the bumblebee field strength ($B_{\mu\nu}$) and the potential ($V$) are defined as follows

$$B_{\mu\nu} = \partial_{\mu} B_{\nu} - \partial_{\nu} B_{\mu},$$

(2.2)

$$V = V(B^\mu B_\mu \pm \tilde{b}^2).$$

(2.3)

Vacuum expectation value of the bumblebee field is governed by the following condition

$$V(B^\mu B_\mu \pm \tilde{b}^2) = 0.$$  

This automatically implies that

$$B^\mu B_\mu \pm \tilde{b}^2 = 0, $$

(2.4)

in which signs ($\pm$) signs in the potential determine the field type of $B^\mu$: timelike or spacelike. Solution of Eq. (2.4) is conditional on the field $B^\mu$ that acquires a non-null vacuum expectation value:

$$\langle B^\mu \rangle = b^\mu.$$  

(2.5)

Here, the non-null vector $b^\mu$ satisfies $b^\mu b_\mu = \mp \tilde{b}^2 = \text{constant}. Thus, } b^\mu$ spontaneously breaks the Lorentz symmetry. So that the curvature and torsion terminate.

Bumblebee modified Einstein field equations [68] are governed by

$$G_{\mu\nu} = \kappa T_{\mu\nu},$$

(2.6)
where the total energy momentum relation is given by [94]

\[ T_{\mu\nu} = T^M_{\mu\nu} + T^B_{\mu\nu}, \tag{2.7} \]

in which \( T^M_{\mu\nu} \) is the matter field, and the bumblebee energy-momentum tensor \( T^B_{\mu\nu} \) reads

\[
T^B_{\mu\nu} = -B_{\mu\alpha}B^\alpha - \frac{1}{4}B_{\alpha\beta}B^{\alpha\beta} - B_{\mu\alpha}B^\alpha - Vg_{\mu\nu} + 2V'B_{\mu}B_{\nu} \\
+ \frac{\xi}{\kappa} \left[ \frac{1}{2} B_aB^a g_{\mu\nu} - B^\mu B_{\nu} - B_{\alpha}B_{\alpha} \right] \\
+ \frac{1}{2} \nabla_\mu (B^a B_\nu) - \frac{1}{2} \nabla^2 (B_\mu B_\nu) \\
- \frac{\xi}{2} g_{\mu\nu} \nabla_\alpha \nabla_\beta (B^\alpha B^\beta). \tag{2.8} \]

Thus, the modified Einstein’s field equations (2.6) with the bumblebee field can be expressed as follows [68]

\[ R_{\mu\nu} - 8\pi G \left[ T^M_{\mu\nu} + T^B_{\mu\nu} - \frac{1}{2} g_{\mu\nu} (T^M + T^B) \right] = 0, \tag{2.9} \]

which has the following explicit form:

\[
E^{\text{einstein}}_{\mu\nu} = R_{\mu\nu} - \kappa \left( T^M_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^M \right) - \kappa T^B_{\mu\nu} - 2\xi g_{\mu\nu} V' \\
+ \kappa B_aB^a g_{\mu\nu} V' - \frac{\xi}{4} g_{\mu\nu} \nabla^2 (B_aB^a) \\
- \frac{\xi}{2} g_{\mu\nu} \nabla_\alpha \nabla_\beta (B^\alpha B^\beta) = 0, \tag{2.10} \]

where \( T^M = g^{\mu\nu}T^M_{\mu\nu} \) and \( \kappa = 8\pi \). It can be easily checked that when the both bumblebee field \( B_\mu \) and potential \( V(B_\mu) \) are vanished, the original general relativity field equations are recovered.

Here, we focus on the vacuum solutions induced by the LSB, which is possible when the bumblebee field \( B_\mu \) remains frozen in its vacuum expectation value: \( b_\mu \).

Namely, we consider the case of Eq. (2.5), so that we have a vanishing potential: \( V = V' = 0 \) [95]. Thus, the bumblebee field strength (2.2) becomes

\[ b_{\mu\nu} = \partial_\mu b_\nu - \partial_\nu b_\mu. \tag{2.11} \]

### III. EXACT SOLUTION OF BUMBLEBEE WORMHOLE

In this section, we consider a static and a spherically symmetric traversable wormhole solution [5] without any tidal force

\[ ds^2 = -dt^2 + \frac{dr^2}{1 - \frac{\alpha}{W(r)}} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2, \tag{3.1} \]

where \( W(r) \) is the shape function of the wormhole. Furthermore, we set the bumblebee vector as follows

\[ b_\mu = \left[ 0, \frac{\alpha}{\sqrt{1 - \frac{\alpha}{W(r)}}, 0, 0 \right]. \tag{3.2} \]

The bumblebee modified Einstein’s field equations with the isotropic matter (\( T^M_{\mu\nu} = (-\rho, 0, 0, 0) \). We shall use the equation of state: \( P = \rho w \), in which \( \rho \) denotes the energy density of the matter field, \( P \) stands for the pressure, and \( w \) is the dimensionless \( \text{R} \) number. Thus, Eq. (2.10) yields the following three Einstein’s field equations in the bumblebee gravity theory for the wormhole metric (3.1):

\[ E^H_{tt} = -2\kappa \rho r^3 - 3\kappa w \rho r^3 + \frac{\chi a r}{d r} W(r) - \chi a W(r) = 0, \tag{3.3} \]

\[ E^H_{rr} = 2 \left( \frac{d}{d r} W(r) \right) r - 2W(r) + \kappa w \rho r^3 - 3\kappa a \rho r^3 + 3\chi a r \frac{d}{d r} W(r) - 3\chi a W(r) \tag{3.4} \]

\[ E^H_{\theta\theta} = \kappa w \rho r^3 - 3\kappa \rho r^3 + 2\chi a W(r) - 2\chi a r + \frac{d}{d r} W(r) r + W(r) = 0. \tag{3.5} \]

From Eqs. (3.3-3.5), we can easily obtain the density as follows

\[ \rho = \frac{\left( \frac{d}{d r} W(r) \right) r - W(r)}{\kappa r^3 (1 - 3w)}, \tag{3.6} \]

\[ \rho = \frac{-2r_0 l (-w - 1) l + 3w + 1)^{3/2}}{\kappa l^{3/2}}. \tag{3.7} \]

Solving the last equation, we find the shape function as follows

\[ W(r) = \frac{r^l + 1}{r^l + 1} + \frac{r_0 (r_0)}{l^{3/2}} \tag{3.8} \]

in which \( l = \xi \alpha \) and \( W(r_0) = r_0 \) are used, and with the
condition of \( w = -\frac{l+1}{3l+3} \):

\[
W(r) = \frac{1}{l+1} \left( lr + r_0 \left( \frac{r_0}{r} \right)^{\frac{5l+3}{3l+3}} \right). \tag{3.9}
\]

Note that every wormhole must have a minimum radius \( r = r_0 \neq 0 \), where a throat is located. It can be easily checked that \( g_{rr} \) diverges at \( r = r_0 \).

Furthermore the Eq. (3.9) is not asymptotically flat when \( (r \to \infty) \):

\[
\lim_{r \to \infty} \frac{W(r)}{r} \to \frac{l}{l+1} + \lim_{r \to \infty} \left( \frac{r_0}{r} \right)^{1 + \frac{5l+3}{3l+3}}. \tag{3.10}
\]

From the last equation we see that the first term is independent of \( r \), while the second term is asymptotically flat only if \( 1 + \frac{5l+3}{3l+3} > 0 \). In general, however it is known that the isotropic matter wormholes are not asymptotically flat, due to the radial coordinate in the shape function. In our wormhole model the spacetime is not asymptotically flat. This reflects the nontrivial topology of the spacetime due to the LSB effects, such solutions are similar to the topological defects and dilaton fields [97–99]. However, in principle, the solution obtained should be matched to an exterior vacuum solution (for details, a reader may refer to [51]).

The Ricci scalar results in

\[
R = \frac{3 \left( \frac{d}{dr} W(r) \right)^2 r^2 - 2 W(r) \left( \frac{d}{dr} W(r) \right) r + 3 \left( W(r) \right)^2}{2 r^6}, \tag{3.11}
\]

which at \( r = r_0 \) results as follows

\[
R|_{r_0} = \frac{18 l^2 + 24 l + 12}{r_0^4 (3 l + 1)^2}. \tag{3.12}
\]

In a similar way the Kretschmann scalar is found to be

\[
K = 2 \left( \frac{d}{dr} W(r) \right)^2 r^2 - 2 W(r) \left( \frac{d}{dr} W(r) \right) r + 3 \left( W(r) \right)^2 \frac{r^6}{r^6}, \tag{3.13}
\]

thus yielding

\[
K|_{r_0} = \frac{36 l^2 + 24 l + 12}{r_0^4 (3 l + 1)^2}. \tag{3.14}
\]

It is clear that singularity is the point \( l = -\frac{1}{3} \) from both the Ricci scalar \( R \) and the Kretschmann scalar \( K \).

**A. Flare-out conditions:**

Traversability of a wormhole is determined by the flare-out conditions [5]. We can easily visualize the spatial geometry of the wormhole using an embedding diagram. The metric with \( l = l_0 \) (constant) reduces to the following form at the equatorial plane \( \theta = \frac{\pi}{2} \) [96]:

\[
ds^2 = \frac{dr^2}{1 - \frac{W(r)}{r}} + r^2 d\theta^2. \tag{3.15}
\]

Then, we embed the wormhole geometry into a Euclidean 3-space:

\[
d\sigma^2 = dz^2 + dr^2 + r^2 d\theta^2, \tag{3.16}
\]

which can be rewritten as follows

\[
d\sigma^2 = \left[ 1 + \left( \frac{dz}{dr} \right)^2 \right] dr^2 + r^2 d\theta^2, \tag{3.17}
\]

where

\[
\frac{dz}{dr} = \pm \frac{1}{\sqrt{r^2 W(r) - 1}}. \tag{3.18}
\]

We can now calculate the proper radial distance, which ought to be real and finite:

\[
d(r) = \int_0^r \frac{dr}{\sqrt{1 - \frac{W(r)}{r}}} \tag{3.19}
\]

We deduce from the above equation that

\[
\sqrt{1 - \frac{W(r)}{r}} > 0. \tag{3.20}
\]
It is worth to recall that there is a coordinate singularly at the throat of the wormhole. The flare-out conditions for Eq. (3.9) thus can be written as \[5, 17\]:

\[ W(r) - r \leq 0, \]

and

\[ \frac{dW(r)}{dr} - W(r) < 0. \]

Thus, we have

\[ W' = \frac{3l + 3}{3l + 1} < 1. \]

The formation of wormhole is depicted where the flare-out conditions are satisfied for above condition. On the other hand, white region shows that there is no wormhole. Furthermore, we have plotted the \( W' \) in the Fig. (2) which exhibit that flare-out conditions are thoroughly satisfied.

IV. ENERGY CONDITIONS

The energy conditions for the wormhole in Eq.(3.1) and Eq. (3.9) are defined in \[100\] as follows:

A. Null energy condition:

\[ \rho + p \geq 0. \] \hspace{1cm} (4.1)

\[ \rho = \frac{(4l + 2) r_0}{(3l + 1)kr^3} \left( \frac{r_0}{r} \right) \frac{5l - 3}{3l + 1} \geq 0 \] \hspace{1cm} (4.2)

At \( l = -\frac{1}{2} \), the null energy condition \( \rho + p \geq 0 \) becomes zero. It is clear from Fig. (3) that null energy condition for the bumblebee wormhole is satisfied. Moreover, there is interactive plot in \[101\] for null energy condition for bumblebee wormhole to see effect of the parameter \( l \).

B. Weak energy condition:

\[ \rho \geq 0, \hspace{0.5cm} \rho + p \geq 0. \] \hspace{1cm} (4.3)

\[ \rho = \frac{r_0 (5l + 3)}{(3l + 1)kr^3} \left( \frac{r_0}{r} \right) \frac{5l - 3}{3l + 1} \geq 0 \] \hspace{1cm} (4.4)

In Fig. (3) and Fig. (4), show that null and weak energy conditions for the bumblebee wormhole are satisfied when the physical parameters are fixed to \( r_0 = 1, \ l = -2 \) and \( \kappa = 1 \), respectively. There are interactive plots in \[101\] and \[102\] for weak energy condition for bumblebee wormhole to see effect of the parameter \( l \).
C. Strong energy condition:

\[
\rho + 3p \geq 0, \quad \rho + p \geq 0. \tag{4.5}
\]

\[
\rho + 3p = \frac{2r_0 l}{(3 l + 1) \times r^3} \left( \frac{r_0}{r} \right)^{-\frac{3l-3}{2l+1}} \geq 0. \tag{4.6}
\]

From Fig. (3) and Fig. (5), we observe that strong energy condition for the bumblebee wormhole is satisfied for \( r_0 = 1, l = -2 \) and \( \kappa = 1 \), respectively. There is an interactive plot in [103] for strong energy condition for bumblebee.

\[3 \rho + \rho \]

**Figure 5:** The plot of \( \rho + 3p \) is satisfied for \( r_0 = 1, l = -2 \) and \( \kappa = 1 \). There is interactive plot in [103] for strong energy condition for bumblebee wormhole.

V. DEFLECTION OF LIGHT

In this section, we shall explore the effect of bumblebee gravity in the gravitational lensing of the spacetime of the wormhole metric in Eq. (3.1) and Eq. (3.8). For simplicity, the Lagrangian is chosen in the equatorial plane. Thus, we get

\[2 \mathcal{L} = -\dot{t}^2 + \frac{(1 + l)^2}{1 - \left( \frac{r_0}{r} \right)^{1 + \frac{3l-3}{2l+1}} + r^2 \dot{\phi}^2}. \tag{5.1}\]

There are two constants of motion (energy and angular momentum) for a massless particle, which are defined as follows

\[E = -g_{\mu \nu} \dot{x}^\mu U^\nu = \frac{dt}{d\lambda}, \tag{5.2}\]

\[L = g_{\mu \nu} \Phi^{\mu \nu} U^\nu = r^2 \frac{d\phi}{d\lambda}. \tag{5.3}\]

in which \( \lambda \) denotes the affine parameter along the light ray. Note that \( K^{\mu} \) and \( \Phi^{\mu \nu} \) are the timelike Killing vector and rotational Killing vector, respectively. One can define the impact parameter of the light ray as

\[b = \frac{L}{E} = r^2 \frac{d\phi}{dt}. \tag{5.4}\]

From the above relations, one can find the following differential equation for the light ray

\[\left( \frac{dr}{d\phi} \right)^2 + \frac{r^2}{B(r)} = \frac{r^4}{b^2 B(r)}, \tag{5.5}\]

in which

\[B(r) = \frac{\left( 1 + l \right)}{1 - \left( \frac{r_0}{r} \right)^{1 + \frac{3l-3}{2l+1}} + \frac{3}{2l+1}}. \tag{5.6}\]

One can solve this equation by introducing a new variable, let us say \( u(\phi) \), which is related with the radial coordinate as \( r = \frac{1}{u(\phi)} \). If we use the following identity:

\[\dot{r} \frac{\dot{r}}{\dot{\phi}} = \frac{dr}{d\phi} = -\frac{1}{u^2} \frac{du}{d\phi}, \tag{5.7}\]

then in the large \( r \) limit, it is possible to recover the following relation

\[d^2u \frac{du}{d\phi^2} + \beta u = 0. \tag{5.8}\]

Furthermore, we can approximate \( \beta = (1 + l)^{-1} \), since in the weak limit approximation \( B(r) \rightarrow 1 + l \), as \( r \rightarrow \infty \). The solution of the last differential equation is given by

\[u(\phi) = A_1 \sin(\sqrt{\beta} \phi) + A_2 \cos(\sqrt{\beta} \phi). \tag{5.9}\]

When using the following initial conditions \( u(\phi) = 0 \) and \( u(\phi = \pi/2) = \frac{1}{b} \), we find

\[u(\phi) = \frac{\sin(\sqrt{\beta} \phi)}{b} \left( \sin(\frac{\sqrt{\beta} \pi}{2}) \right)^{-1}. \tag{5.10}\]

Moreover, one can use \( \sin(\sqrt{\beta} \pi) \simeq 1 \) and in sequel derives the light ray expression:

\[r = \frac{b}{\sin(\sqrt{\beta} \phi)}. \tag{5.11}\]

This equation is important in computing the deflection angle in the GBT. Next, let us find the wormhole optical metric by letting \( ds^2 = 0 \), which corresponds to

\[dr^2 = \frac{(1 + l) dr^2}{1 - \left( \frac{r_0}{r} \right)^{1 + \frac{3l-3}{2l+1}}} + r^2 d\phi^2. \tag{5.12}\]
It is also possible to write down the wormhole optical metric in terms of new coordinates \( x^i \):
\[
d s^2 = h_{ab} \, d x^a d x^b = d \zeta^2 + \mathcal{H}^2(\zeta) \, d \varphi^2,
\] where
\[
d \zeta = \frac{\sqrt{1 + l \, d r}}{\sqrt{1 - \left(\frac{\varphi}{\pi}\right)^2 \, d r}}. \quad \mathcal{H} = r.
\]

The Gaussian optical curvature (GOC) \( \mathcal{H} \) can be found to be (see for details, [75])
\[
\mathcal{H} = -\frac{1}{\mathcal{H}} \left[ \frac{d r \, d \varphi}{d \zeta} \, \frac{d \mathcal{H}}{d r} + \left( \frac{d r}{d \zeta} \right)^2 \frac{d^2 \mathcal{H}}{d r^2} \right] = -\frac{(1 + \Xi)}{2r^2} \left( \frac{r_0}{r} \right)^{1+\Xi},
\]
where
\[
\Xi = \frac{5wl + 3l + 3w + 1}{wl - l + 3w + 1}.
\]

Alternatively, one can approximate the above equation by expanding in series around \( l \), so we find
\[
\mathcal{H} \approx -\frac{r_0^2}{r^4} - \frac{r_0^2}{4} \left[ 4 \ln \left(\frac{r_0}{r}\right) \left(1 + \frac{1}{w} - \frac{1}{w+1}\right) \right] \frac{1}{r^8(3w+1)}.
\]

The key point in this method is that we choose a non-singular domain outside the light ray, say \( \mathcal{A}_R \), which is bounded by \( d\mathcal{A}_R = \gamma_h \cup C_R \). The famous GBT in the context of the optical geometry is stated as follows
\[
\int\int_{\mathcal{A}_R} \mathcal{H} \, d\sigma + \int_{\mathcal{A}_R} \kappa \, d t + \sum_k \psi_k = 2\pi \chi(\mathcal{A}_R),
\]
where \( \kappa \) gives the geodesic curvature, \( d\sigma \) is the optical surface element, and \( \psi_k \) stands for the exterior angle at the \( k \)-th vertex. We set the Euler characteristic number to one, i.e., \( \chi(\mathcal{A}_R) = 1 \). Thus, the geodesic curvature is defined by [75]
\[
\kappa = h \left( \nabla \gamma, \dot{\gamma} \right),
\]
in which the unit speed condition is selected as \( h(\gamma, \dot{\gamma}) = 1 \). For a very large radial coordinate \( R \to \infty \), our two jump angles (at the source \( \gamma \), and observer \( \theta \)), yield \( \psi_0 + \psi_5 \to \pi \) [75]. Thus, the GBT simplifies to
\[
\int\int_{\mathcal{A}_R} \mathcal{H} \, d\sigma + \int_{\mathcal{A}_R} \kappa \, d t \to \int_{\mathcal{A}_\infty} \mathcal{H} \, d\sigma + \int_{0}^{\pi + \hat{\kappa}} \, d\varphi = \pi.
\]

By definition, the geodesic curvature for \( \gamma_h \) is set to zero. Then, we are left with a contribution from the curve \( C_R \) located at a coordinate distance \( R \) from the coordinate system chosen at the wormhole center in the equatorial plane. In short, we need to compute
\[
\kappa(C_R) = |\nabla \gamma \cdot \dot{C_R}|.
\]

In component notation, the radial part can be written as
\[
\left( \nabla_{\gamma} \dot{C_R} \right)^T = \dot{C_R}^\varphi \left( \partial_{\varphi} C_R^\varphi \right) + \Gamma_{\rho\varphi}^{(op)} \left( \dot{C_R}^\rho \right)^2.
\]

With the help of the unit speed condition, after computing the Christoffel symbols that are related to our optical metric in the large coordinate radius \( R \), we get
\[
\lim_{R \to \infty} \kappa(C_R) = \lim_{R \to \infty} \left| \nabla \gamma \cdot \dot{C_R} \right|, \to \frac{1}{\sqrt{1 + l \, r}}.
\]

To understand the meaning of above equation, we write the optical metric, in this limit, for a constant \( R \). We find
\[
\lim_{R \to \infty} \frac{d t}{d \varphi} \to R \, d \varphi.
\]

Combining the last two equations together, we obtain
\[
\kappa(C_R) \, d t = \frac{1}{\sqrt{1 + l \, r}} \, d \varphi.
\]

This equation indicates that our wormhole geometry is non-asymptotically flat, and consequently the optical metric is not asymptotically Euclidean. Using this result from GBT, we can express the deflection angle as follows
\[
\hat{\kappa} = \left( \sqrt{1 + l - 1} \right) \pi - \sqrt{1 + l} \int_{0}^{\pi} \int_{\frac{b}{\sin \left(\frac{\varphi}{\sqrt{1 + l}}\right)}}^{\infty} \mathcal{H} \, d\sigma,
\]
where the light ray \( r(\varphi) = \frac{b}{\sin \left(\frac{\varphi}{\sqrt{1 + l}}\right)} \) (\( b \) is now interpreted as the impact parameter [84]) can be approximated with the closest approach distance from the wormhole in the first order approximation. The first term of Eq. (5.26) can be approximated as
\[
\left( \sqrt{1 + l - 1} \right) \pi \approx \frac{l \pi}{2} - \frac{l^2 \pi}{8} + ...
\]

The surface can also be approximated to
\[
d\sigma = \sqrt{h} \, d\zeta \, d\varphi \approx \sqrt{1 + l \, r} \, d r \, d\varphi.
\]

Finally, the total deflection angle is found to be
\[
\hat{\kappa} \approx \frac{l \pi}{2} - \frac{l \pi}{2} \int_{0}^{\frac{1}{\sqrt{1 + l}} \left( \frac{r_0}{r} \right)^{1+\Xi}} \left[ \frac{1 + \Xi}{2r} \right] d r \, d\varphi.
\]

Evaluating the last integral, we find
\[
\hat{\kappa} \approx \frac{l \pi}{2} + \frac{(1 + l) \pi}{2} \left( \frac{r_0}{b} \right)^{1+\Xi} \Gamma \left( \frac{2+\Xi}{2} \right) - \frac{l^2 \pi}{8} \Gamma \left( \frac{2+\Xi}{2} \right).
\]
with the condition of $1 + \Xi > 0$. Performing a series expansion, we can write the last equation as follows

$$\alpha \simeq \frac{l \pi}{2} + \frac{\pi r_0^2}{4 b^2} + \frac{5 \pi r_0^4 l}{8 b^2} - \frac{\pi r_0^2 l w}{b^2} + \frac{\pi r_0^2 l \ln\left(\frac{r_0}{b}\right)}{4 b^2} - \frac{\pi r_0^2 l \ln 2}{b^2} + \frac{2 \pi r_0^2 l w \ln 2}{b^2} - \frac{2 \pi r_0^2 l w \ln\left(\frac{r_0}{b}\right)}{b^2}.$$  (5.31)

Note that for vanishing bumblebee gravity, $l = 0$, it reduces to the Einstein’s gravity and the deflection angle becomes $\alpha \simeq \frac{\pi r_0^2}{4 b^2}$, which is in agreement with the Ellis wormhole [80]. On the other hand, if $1 + \Xi \leq 0$, we can only incorporate finite distance corrections in the deflection angle of light.

VI. CONCLUSION

Searching the way to construct traversable wormhole solution, which satisfies the energy conditions become the most interesting application of the general relativity theory. To find some realistic matter source which keeps the wormhole throat open and thus the interstellar or the inter-universe travel might become possible, the modified theories of gravity are thought to be a new remedy. In this regard, we have considered the bumblebee gravity to have traversable wormhole solution that satisfy the null, weak, and strong energy conditions.

In this paper, we have first derived the modified Einstein’s field equations for the Lorentzian wormhole in the bumblebee gravity. Next, using the associated field equations with bumblebee gravity, we have obtained the new traversable wormhole solution with the exact shape function (3.9) and $w = -\frac{l+1}{3+3}$. Then, physical features of the obtained wormhole are studied in detail. Singularity of the solution is analyzed by computing the Ricci and Kretschmann scalars. It is seen that the singularity appears when $l = -\frac{1}{3}$. Afterwards, we have checked the flare-out conditions $\mathcal{W} < 1$ for the obtained bumblebee wormhole solution. We have shown that the flare-out conditions are satisfied if $\frac{3l+3}{3l+1} < 1$ where it is plotted in Fig. (2).

In section IV, we have checked the energy conditions (null, weak, and strong) for the bumblebee wormhole. To this end, we have done a study on graphics. In Figs. (3, 4, 5), we have analyzed the null, weak and strong energy conditions of the bumblebee wormhole for the values of $r_0 = 1$, $l = -2$ and $\kappa = 1$. We have seen that all energy conditions for the bumblebee wormhole are satisfied when $r_0 = 1$, $l = -2$ and $\kappa = 1$. We have also plotted energy conditions manipulated as interactive in [101–103].

Another important point to be noted is that under the LSB effect the global topology of the wormhole spacetime changes. The limit of $\frac{\mathcal{W}(r)}{r}$ at spatial infinity was found to be $\lim_{r \to r_0} \frac{\mathcal{W}(r)}{r} \to \frac{1}{r_0 l}$, which shows that our wormhole solution is non-asymptotically flat. The deflection of light has been computed by applying the GBT to the bumblebee wormhole expressed in the optical geometry. It is seen that bumblebee parameter effects the geodesic optical curvature which modifies the final result for the deflection angle. Due to the nontrivial global topology, we have shown that the deflection of light is changed by $\delta \kappa = \pi/2$, which is purely topological term and independent from the impact factor $b$. In addition, we have incorporated the bumblebee effects in the total deflection angle by deriving the light ray equation, which modifies the straight line approximation in the domain of integration. In other words, the total deflection angle does not only depend on the geometry of the bumblebee traversable wormhole (i.e., throat radius $r_0$), but it varies with the coupling constant $l$, and state parameter $w$. Finally, in the case of $l = 0$, we recover the Ellis wormhole deflection angle as being reported in the literature [80].

In short, the bumblebee wormhole that we have constructed satisfies the energy conditions with normal matter and flare-out conditions near the throat. In the near future, we plan to extend to add new sources (scalar, electromagnetic etc.) to the bumblebee gravity. By this way, we want to obtain new spacetime solutions and analyze their physical features [104].

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Interactive plot of $\rho + p$ of bumblebee wormhole
https://www.wolframcloud.com/objects/b9c3ae67-7f31-466f-ae9d-7e2c1f08139f

Interactive plot of energy density $\rho$ of bumblebee wormhole
https://www.wolframcloud.com/objects/72a4aa0e-8fac-4a66-a07d-e4cc53333ded

Interactive plot of $\rho + 3p$ of bumblebee wormhole
https://www.wolframcloud.com/objects/6bd52e2c-f49c-41b2-8d28-e2153c4c10b1

Github page of bumblebee model of gravity project
https://bumblebeeogr.github.io/BIAS/