ON THE FRTS APPROACH TO QUANTIZED CURRENT ALGEBRAS

JINTAI DING AND SERGEI KHOROSHKIN

Abstract. We study the possibility to establish $L$-operator’s formalism by Faddeev-Reshetikhin-Takhtajan-Semenov-Tian-Shansky (FRST) for quantized current algebras, that is, for quantum affine algebras in the "new realization" by V. Drinfeld with the corresponding Hopf algebra structure and for their Yangian counterpart. We establish this formalism using the twisting procedure by Tolstoy and the second author and explain the problems which FRST approach encounter for quantized current algebras. We show also that, for the case of $U_q(\hat{sl}_n)$, entries of the $L$-operators of FRTS type give the Drinfeld current operators for the non-simple roots, which we discovered recently. As an application we deduce the commutation relations between these current operators for $U_q(\hat{sl}_3)$.

1991 Math Subject Classification(s): 17B37

0. Introduction.

The current realization of the quantum affine algebra $U_q(\hat{g})$ and of the Yangian was obtained by Drinfeld [Dr1], which he called "new realization" of quantum affine algebras and of Yangians. The "new realization" came with its own Hopf algebra structure. In order to distinguish these Hopf algebras from the quantum affine algebras in Drinfeld-Jimbo formulation of quantized enveloppping Kac–Moody algebras, we use the name quantized current algebras and in the notations mark them with upper index $D$. These Hopf algebras and their generalizations are now extensively studied [DI1], [DI2], [EF1], [EF2].

In the paper [DK], we extended Drinfeld quantization of current algebras in several directions: we constructed current operators for nonsimple roots of $g$, defined the new braid group action in terms of currents and presented a description of the universal $R$-matrix in two equivalent forms: in a form of an infinite product and in a form of certain integrals over the current operators.

The aim of this paper is, in addition to [DK], to develop Faddeev-Reshetikhin-Takhtajan-Semenov-Tian-Shansky (FRST) approach, where the generators are gathered into $L$-operators which satisfy famous "RLT" relations, for quantized current algebras. Such a description of quantized envelopping algebras has at least two nice features: first, $L$-operators are group-like elements, which means that their entries obey simple comultiplication rules, and second, the Cartan-Weyl generators (for a specific ordering of roots) can be extracted from the $L$-operators.

In FRST approach, the $L$-operators are given by the projections of the tensor components of the universal $R$-matrix to certain finite-dimensional representations and all their properties follow immediately from the properties of the universal $R$-matrix. Unfortunately, this simple prescription does not work directly for quantized current algebras. The reason is as follows. Quantized current algebras are by definition topological algebras (otherwise the comultiplication structure, written in Laurent series, is not well defined), which act on the tensor category of highest weight representations. To the contrary, in the definition of tensor category of finite-dimensional modules, we come to problems of divergence.

However, the vector representation of $U_q^D(\hat{sl}_n)$ is well defined and we can project to it the components of the universal $R$-matrix. The corresponding $L$-operators look nice: they are triangular and Drinfeld current operators can be easily extracted from them. The image of the universal $R$-matrix
in tensor product of vector is also triangular with δ-function term out of diagonal. But it does not satisfy the Yang-Baxter equation.

In order to derive proper commutation relations between the entries of the $L$-operators, we use the results of [KT1], where the Hopf structure of $U_q^D(\hat{g})$ was obtained from the Hopf structure in $U_q(\hat{g})$ by two equivalent ways: either via twisting by certain factor of the universal $R$-matrix of $U_q(\hat{g})$ or as a limit of twists by automorphisms of affine shifts in the $(q)$-Weyl group of $U_q(\hat{g})$. We show that the commutation relations for Gauss coordinates of the $L$-operators for finitely twisted $U_q(\hat{g})$ have the limit in the topology of $U_q^D(\hat{g})$ and this limit is the defining relations for the currents of $U_q^D(\hat{g})$. This gives a way to restore the correct commutation relations for Gauss coordinates of the $L$-operators by looking back to the finite twist. We also prove that, if we multiply before the twist the initial $R$ matrix by the function $1 - u/v$, then the limit of the corresponding Yang-Baxter equations turns to the relations on $L$-operators with diagonal $R$-matrix, which are well defined and are the corollaries of the defining relations for the currents of $U_q^D(\hat{g})$.

The limiting procedure enables us to prove that the entries of the $L^\pm(z)$ produce the current operators for the non-simple roots discovered in [DK]; so we can use the commutation relations for the entries of $L$-operators in order to deduce the commutation relations the current operators for non-simple roots of $g$. We present the results of calculations for $U_q^D(\hat{sl}_3)$.

All the arguments above are valid as well for other types of quantized current algebras, e.g. for the Yangian type algebras and for elliptic algebras of dynamical type. We can also see from the ideology of twists by affine shifts, that in general the existence of Drinfeld current realization is in a sense equivalent to existence of nontrivial family of automorphisms of the basic solution of the Yang-Baxter equation. Only Baxter’s 8-vertex $R$-matrix does not have evident symmetries of such a type.

The exposition goes as follows. We first study the finite twisting coming from the Weyl group elements and derive the image of the twisted R-matrix of $U_q\hat{sl}_n$ on the fundamental representation $\mathbb{C}^\infty$. We check that they indeed satisfy Yang-Baxter equation. Taking certain limit, compatible with a topology of $U_q^D(\hat{sl}_n)$, we derive the R-matrix with entries of singular function $\delta(z)$ and triangular property. Then we will study the related quasi-triangular structure and explain the problem to use this R-matrix to formulate certain FRTS construction of quantized current algebras. Finally we proceed to derive the commutation relation between the Gauss coordinates and show their connection with the current operators for non-simple roots. We end with a discussion of other types of quintized current algebras.

1. Drinfeld realization as twisted quantum affine algebra.

This section is mainly to introduce the results from [KT1]. We will use basically the notation from [KT1], which we refer the reader to. In [KT1], an idea form Drinfeld [Dr3] and Reshetikhin [R], where Drinfeld studies the twisting of the Hopf algebra structure is utilized.

Let $A = (a_{ij})$ $i,j = 1,...,r$ be the Cartan matrix of simple Lie algebra $g$ of simple laced type. We define the quantized current algebra $U_q^D(\hat{g})$ as follows.

The Hopf algebra $U_q^D(\hat{g})$ is an associative algebra with unit 1 and the generators: $\varphi_i(m),\psi_i(-m)$, $x^\pm_i(l)$, for $i = i,...,r$, $l \in \mathbb{Z}$ and $m \in -\mathbb{Z}_+$ and a central element $c$. Let $z$ be a formal variable and $x^\pm_i(z) = \sum_{l\in\mathbb{Z}} x^\pm_i(l)z^{-l}$, $\varphi_i(z) = \sum_{m\in-\mathbb{Z}_+} \varphi_i(m)z^{-m}$ and $\psi_i(z) = \sum_{m\in\mathbb{Z}_+} \psi_i(m)z^{-m}$. In terms of
the formal variables, the defining relations are
\[
\begin{align*}
\varphi_i(z)\varphi_j(w) &= \varphi_j(w)\varphi_i(z), \\
\psi_i(z)\psi_j(w) &= \psi_j(w)\psi_i(z), \\
\varphi_i(z)\psi_j(w)\varphi_i(z)^{-1}\psi_j(w)^{-1} &= \frac{g_{ij}(\frac{z}{w}q^{-c})}{g_{ij}(\frac{z}{w}q^c)}, \\
\varphi_i(z)x_j^\pm(w)\varphi_i(z)^{-1} &= g_{ij}(\frac{z}{w}q^{\mp c})x_j^\pm(w), \\
\psi_i(z)x_j^\pm(w)\psi_i(z)^{-1} &= g_{ij}(\frac{w}{z}q^{\mp c})x_j^\pm(w), \\
[x_i^+(z), x_j^-(w)] &= \frac{\delta_{ij}}{q-q^{-1}} \left\{ \delta\left(\frac{z}{w}q^{-c}\right)\psi_i(wq^{c}) - \delta\left(\frac{z}{w}q^c\right)\varphi_i(zq^c) \right\}, \\
(zq-q^\pm a_{ij}w)x_i^\pm(z)x_j^\pm(w) &= (q^\pm a_{ij}z-wq^{-j})x_j^\pm(w)x_i^\pm(z), \\
[x_i^+(z), x_j^+(w)] &= 0 \quad \text{for } a_{ij} = 0, \\
x_i^+(z_1)x_i^+(z_2)x_j^+(w) - (q + q^{-1})x_i^+(z_1)x_j^+(w)x_i^+(z_2) + x_j^+(w)x_i^+(z_1)x_i^+(z_2) + \{z_1 \leftrightarrow z_2\} &= 0, \quad \text{for } a_{ij} = -1.
\end{align*}
\]
where
\[
\delta(z) = \sum_{k=\pm1} z^k, \quad \text{and} \quad g_{ij}(z) = \frac{q^{a_{ij}z} - 1}{z - q^{a_{ij}}} \quad |z| < |q^{a_{ij}}|.
\]

The Hopf algebra structure of \( U_q^D(\hat{g}) \) is given by the following formulae.

**Coproduct \( \Delta \)**

\[
\begin{align*}
\Delta(q^c) &= q^c \otimes q^c, \\
\Delta(x_i^+(z)) &= x_i^+(z) \otimes 1 + \varphi_i(zq^{\frac{c}{2}}) \otimes x_i^+(zq^{\frac{c}{2}}), \\
\Delta(x_i^-(z)) &= 1 \otimes x_i^-(z) + x_i^-(zq^{-\frac{c}{2}}) \otimes \psi_i(zq^{\frac{c}{2}}), \\
\Delta(\varphi_i(z)) &= \varphi_i(zq^{-\frac{c}{2}}) \otimes \varphi_i(zq^{\frac{c}{2}}), \\
\Delta(\psi_i(z)) &= \psi_i(zq^{\frac{c}{2}}) \otimes \psi_i(zq^{-\frac{c}{2}}).
\end{align*}
\]

**Counit \( \varepsilon \)**

\[
\varepsilon(q^c) = 1 \quad \varepsilon(\varphi_i(z)) = \varepsilon(\psi_i(z)) = 1, \quad \varepsilon(x_i^\pm(z)) = 0.
\]

**Antipode \( a \)**

\[
\begin{align*}
a(q^c) &= q^{-c}, \\
a(x_i^+(z)) &= -\varphi_i(zq^{-\frac{c}{2}})^{-1}x_i^+(zq^{-c}), \\
a(x_i^-(z)) &= -x_i^-(zq^{-c})\psi_i(zq^{-\frac{c}{2}})^{-1}, \\
a(\varphi_i(z)) &= \varphi_i(z)^{-1}, \quad a(\psi_i(z)) = \psi_i(z)^{-1}.
\end{align*}
\]

The algebra \( U_q^D(\hat{g}) \) is a topological algebra. It means that \( U_q^D(\hat{g}) \) consists of certain series of the polynomials over the generators of \( U_q^D(\hat{g}) \) whose action on the highest weight modules of \( U_q(\hat{g}) \) is well defined and the topology is a formal topology defined by natural filtration on the series. One of the possibilities to describe precisely such a topology is given in [KT2]. It is based on the construction of a Cartan-Weyl basis for \( U_q(\hat{g}) \).
Let $e_{\pm \gamma}$ be the generators of Cartan-Weyl basis for a given normal ordering of positive roots (see details for affine case in [KT1]), $h$ be Cartan subalgebra of $U_q(\mathfrak{g})$ generated by the elements $k_{\alpha_i}^{\pm 1}$, $(i = 0, 1, \ldots, r)$. We consider formal series on the following monomials
\[ e_{-\beta}^{n_\beta} \cdots e_{-\gamma}^{n_\gamma} e_{\alpha}^{m_\alpha} e_{\gamma}^{m_\gamma} \cdots e_{\beta}^{m_\beta} \]  
with coefficients from $U_q(h)$, where $\alpha < \gamma < \cdots < \beta$ in a sense of the fixed normal ordering in the system $\Sigma$. Of positive roots with the condition that for any weight $\lambda \in h^*$ and for any positive integer $N$ there are only finitely many terms of total weight $\lambda$ satisfying the condition
\[ \sum_{i=0}^{r} \sum_{\alpha \in \Delta_+} (n_\alpha + m_\alpha) c_i^{(\alpha)} \leq N, \]
where $c_i^{(\alpha)}$ are coefficients in a decomposition of the root $\alpha$ with respect to the system of simple roots of $\mathfrak{g}$. The series over the monomials (2) with additional constraint
\[ \sum_{i=0}^{r} \sum_{\alpha \in \Delta_+} (n_\alpha + m_\alpha) c_i^{(\alpha)} > l. \]  
form a system of basic open sets in the algebra $U_q^D(\mathfrak{g})$.

In an analogous manner, the topology is introduced into the tensor square of $U_q^D(\mathfrak{g})$. Here we consider the series over monomials
\[ e_{-\beta}^{n_\beta} \cdots e_{-\gamma}^{n_\gamma} e_{-\alpha}^{m_\alpha} e_{\alpha}^{m_\alpha} e_{\gamma}^{m_\gamma} \cdots e_{-\beta}^{m_\beta} \]  
with coefficients from $U_q(h) \otimes U_q(h)$, such that for any two weights $\lambda, \mu \in h^*$ and for any positive integer $N$, there are only finitely many terms of total weight $\lambda \otimes \mu$ satisfying the condition
\[ \sum_{i=0}^{r} \sum_{\alpha \in \Delta_+} (n_\alpha' + n_\alpha + m_\alpha + m_\alpha') c_i^{(\alpha)} \leq N. \]
The series over the monomials (4) with additional constraint
\[ \sum_{i=0}^{r} \sum_{\alpha \in \Delta_+} (n_\alpha + m_\alpha + n_\alpha' + m_\alpha') c_i^{(\alpha)} > l. \]  
form a system of basic open sets in the algebra $U_q^D(\mathfrak{g}) \otimes U_q^D(\mathfrak{g})$.

Another way to describe the topological structure of $U_q^D(\mathfrak{g})$ is to use instead of Cartan-Weyl basis of $U_q(\mathfrak{g})$ the current type base introduced in [DK] or the crystal base by Kashiwara. For $U_q^D(sl_2)$ the Drinfeld generators are sufficient for the description of topology and of corresponding completion.

The Hopf algebra structure of $U_q^D(\mathfrak{g})$ can be obtained from the Hopf structure of quantum affine algebra $U_q(\mathfrak{g})$ by means of twisting of the Hopf structure [KT1]. Let us briefly remind the general idea of twistings [R].

Let $\mathfrak{A} := (\mathfrak{A}, \Delta, S, \varepsilon, \mathfrak{R})$ be a quasi-triangular Hopf algebra with a comultiplication $\Delta$, an antipode $S$, counit $\varepsilon$ and universal R-matrix $\mathfrak{R}$. Let also $F, F = \sum_i f_i \otimes f_i$ be an invertible element of some extension $T(\mathfrak{A} \otimes \mathfrak{A})$ of $\mathfrak{A} \otimes \mathfrak{A}$, such that the formula
\[ \Delta(F)(a) := F^{-1}(a)^{\Delta}F, \quad a \in \mathfrak{A}, \]
determine a new comultiplication, i.e. $\Delta(F)$ satisfies the coassociativity property
\[ (\Delta(F) \otimes \text{id})\Delta(F) = (\text{id} \otimes \Delta(F))\Delta(F). \]  
(6)
Then the comultiplication $\Delta^{(F)}$ is called the twisted coproduct. This comultiplication defines a new quasitriangular Hopf algebra $H^{(F)} = (\mathfrak{A}, \Delta^{(F)}, S^{(F)}, \varepsilon, \mathfrak{R}^{(F)})$. Here $S^{(F)}(a) = u^{-1}S(a)u$, where $u := ((id \otimes S)F) \circ 1 = \sum_i f_i S(f^i)$, and

$$\mathfrak{R}^{F} = (F^{21})^{-1}\mathfrak{R}.$$ 

The condition (6) is automatically satisfied if tensor $F$ satisfies the equalities:

$$(\varepsilon \otimes id)F = (id \otimes \varepsilon)F = 1,$$

$$\text{id} \otimes F)(\Delta \otimes \text{id})F = (F \otimes \text{id})(id \otimes \Delta)F,$$

One can also twist the coalgebraic structure of the quasi-triangular Hopf algebra $H = (\mathfrak{A}, \Delta, S, \varepsilon, \mathfrak{R})$ by using any automorphism in the algebraic sector $\mathfrak{A}$. Namely, let $\omega : \mathfrak{A} \to \mathfrak{A}$ be an algebra automorphism of $\mathfrak{A}$. The maps $\Delta^{(\omega)} : \mathfrak{A} \to \mathfrak{A} \otimes \mathfrak{A}$ and $S^{(\omega)} : \mathfrak{A} \to \mathfrak{A}$, 

$$\Delta^{(\omega)}(a) := (\omega \otimes \omega)\Delta(\omega^{-1}a) \quad S^{(\omega)}(a) := \omega S(\omega^{-1}a) \quad a \in \mathfrak{A},$$

define the quasitriangular Hopf algebra $H^{(\omega)} = (\mathfrak{A}, \Delta^{(\omega)}, S^{(\omega)}, \varepsilon, \mathfrak{R}^{(\omega)})$ where

$$\mathfrak{R}^{(\omega)} = (\omega \otimes \omega)\mathfrak{R}.$$ 

Let $\mathfrak{g}$ be any contragredient Lie algebra of finite growth with a symmetrizable Cartan matrix $A$. Denote by $\Delta_+$ the reduced system of positive roots with $\mathfrak{g}$, $\alpha_1, \ldots, \alpha_r$ being positive simple roots and $\delta$ being the minimal positive imaginary root. For the quantum algebra $U_q(\mathfrak{g})$ with the coproduct rule as

$$\Delta(e_{\alpha_i}) = e_{\alpha_i} \otimes 1 + k_{\alpha_i} \otimes e_{\alpha_i},$$

$$\Delta(e_{-\alpha_i}) = 1 \otimes e_{-\alpha_i} + e_{-\alpha_i} \otimes k_{\alpha_i}^{-1},$$

we have a family of twisting, which can be described in both languages given above.

For a fixed normal ordering in $\Delta_+$ and for any $\gamma \in \Delta_+$, $\gamma < \delta$ we put

$$F_{\gamma} := \prod_{\beta \gamma} R_{\beta}^{21}$$

(7)

where $R_{\beta} = \exp_{q^{(\beta, \beta)}}(c(\beta) e_{\beta} \otimes e_{-\beta})$; $e_{\pm, \beta}$ are Cartan-Weyl generators which correspond to this ordering of the root system; $c(\beta)$ is a constant which is equal to $q^{-1} - q$ in simple laced case in which we are interested in. It is well known that the tensor $F_{\gamma}$ defines a twisting of the Hopf structure for $U_q(\mathfrak{g})$ (that is, the coassociativity condition (6) is satisfied).

Let now $\gamma \in \Delta_+$ is such that the initial segment $\gamma_1, \ldots, \gamma_n = \gamma$ of the normal ordering is finite. Then $\gamma$ uniquely defines the element $\omega_\gamma$ of the Weyl group of $\mathfrak{g}$. The element $\omega_\gamma$ together with its reduced decomposition $\omega_\gamma = s_{\alpha_1} \cdots s_{\alpha_n}$ can be defined by induction:

$$\omega_{\gamma_1} = id,$$  

$$\omega_{\gamma_{n+1}} = \omega_{\gamma_k} s_{\alpha_k} \iff \omega_{\gamma_k}(\alpha_{i_k}) = \gamma_k.$$ 

and uniquely defines the automorphism $\bar{\omega}_\gamma$ of $U_q(\mathfrak{g})$ as

$$\bar{\omega}_\gamma = T_{\alpha_1}^{-1} \cdots T_{\alpha_n}^{-1}$$

where $T_{\alpha_k}$ are Lusztig automorphisms. It is also known that the twisting by $F_{\gamma}$ coincides with twisting by Lusztig automorphism $\bar{\omega}_\gamma$.

We are specially interested in the twist by a $q$-Weyl group element $\mathfrak{T}_N^q$ for affine quantum algebras. Here $t_\delta$ is a translation by imaginary root $\delta$:

$$t_\delta(\pm\alpha_k) = \pm\alpha_k \pm \delta.$$ 

**Proposition 1.** [KT]. (i) Quantum affine algebra $U_q(\mathfrak{g})$ twisted by a tensor $F_\delta$, where $\delta$ is minimal imaginary root, is isomorphic to Drinfeld’s realization $U_q^{D}(\mathfrak{g})$ of quantum affine algebra

(ii) The twisting by $F_\delta$ is equivalent to a limit of twisting by Lusztig automorphism $\bar{\omega}_N^q$, $N \to \infty$ in the topology of $U_q^D(\mathfrak{g}) \otimes U_q^D(\mathfrak{g})$ described above.
In particular, this proposition allows to describe the universal \( R \)-matrix for \( U_q(\hat{\mathfrak{g}}) \). The universal \( R \)-matrix for \( U_q(\hat{\mathfrak{g}}) \) admits natural triangular decomposition \([KT1], [KST]\):

\[
\mathcal{R} = \mathcal{R}_+ \mathcal{R}_0 \mathcal{R}_- \tag{8}
\]

with

\[
\mathcal{R}_+ = \prod_{\beta < \delta} R_{\beta}, \quad \mathcal{R}_0 = R_{\delta} K, \quad \mathcal{R}_- = K^{-1} \prod_{\beta > \delta} R_{\beta} K,
\]

where \( K = q^{-t_i \otimes t_i} \), \( t_i \) and \( t^i \) are dual bases of extended Cartan subalgebra and \( R_{\delta} \) is a tensor, corresponding to imaginary root vectors, see precise expression in \([KT1], [KST]\), where one should change in notations \( q \) to \( q^{-1} \) in order to adapt them to the definition of \( U_q^D(\hat{\mathfrak{g}}) \). The Proposition \([1]\) implies that the universal \( R \)-matrix \( \mathcal{R}^D \) for Drinfeld’s realization has a form

\[
\mathcal{R}^D = \mathcal{R}_0 \mathcal{R}_- \mathcal{R}_+^2. \tag{9}
\]

Another presentation of the universal \( R \)-matrix for \( U_q^D(\hat{\mathfrak{g}}) \) in a form of integral over current operators is given in \([DK]\).

2. The singular \( R \)-matrix

Let us compute the projections of the universal \( R \)-matrices \((\hat{\mathfrak{g}}_2 \otimes \hat{\mathfrak{g}}_2)^N \mathfrak{R}\) and of \( \mathcal{R}^D \) onto tensor products of fundamental representations of \( U_q(\hat{\mathfrak{sl}}_2) \).

We start from \( \mathfrak{sl}_2 \) case. We fix \( q \) in a region \(|q| < 1\). With the definition of the quantum algebras from \([KT1]\), the projection of \( \rho(z_1) \otimes \rho(z_2) \) of the universal \( R \)-matrix of \( U_q(\hat{\mathfrak{sl}}_2) \), where \( \rho \) is two-dimensional representation of \( U_q(\hat{\mathfrak{sl}}_2) \) and \( z_1 \) and \( z_2 \) are formal variables, gives the following for \( z = z_1/z_2\):

\[
R(z) = r(z) \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & q(1-z) & (1-q^2) & 0 \\
0 & (1-q^2)z & q(1-z) & 0 \\
0 & 0 & 1 & 0
\end{pmatrix}
\]

where

\[
r(z) = q^{-\frac{1}{2}} \frac{(zq^2; q^4)^2}{(z; q^4)(zq^2; q^4)}.
\]

Note that \( r(z) \) has a pole at \( z = 1 \). The triangular decomposition of the universal \( R \)-matrix yields the triangular decomposition of the matrix \( R(z) \) \([KST]\):

\[
R(z) = r(z) \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & (q^{-1-q})_{1-z} & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & q(1-z) & 0 & 0 \\
0 & 0 & q(1-z) & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

The tensor \( F_N \) of the type \((7)\) defining the twist by \( \hat{\mathfrak{g}}_2^N \), is equal to

\[
\prod_{0 \leq k \leq 2N-1} \exp q^2(q^{-1} - q) e_{\alpha+k \delta} \otimes e_{-\alpha-k \delta} = \prod_{0 \leq k \leq 2N-1} \exp q^2(q^{-1} - q) x^+_k \otimes x^-_k.
\]

Its projection is a triangular matrix \( G(z) \)

\[
G(z) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & (q^{-1-q})(1-z^{2N}) & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

Thus the image \( R_{2N}(z) \) of twisted universal \( R \)-matrix \( \mathcal{R}_\delta^{2N} \) can be written as
\[ R_{2N}(z) = r(z) \begin{pmatrix} 1 & 0 & (q^{-1}qz^{2N})z & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & q(1-q^{2z}) & 0 & 0 \\ 0 & 0 & q(1-z) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \times \]

\[
= q^{-1}(zq^2; q^4)^2 \begin{pmatrix} 1/(1-z) & 0 & 0 & 0 \\ 0 & 1/(1-q^{2z}) & 0 & 0 \\ 0 & 0 & 1/(1-q^{2z}) & 0 \\ 0 & 0 & 0 & 1/(1-z) \end{pmatrix}.
\]

**Remark.** We would like to take certain the limit of the expression above, for which we need to define the topology on the space \( \text{End}(\rho \otimes \rho) \otimes [z, z^{-1}] \). We define the topology by defining the limit of a series of formal power series expression as the expression with each coefficient of \( z^n \) as the limit of the each coefficient of \( z^n \) of the series. Clearly, this is the only reasonable topology we can have here. However, then, there appears a subtle point about taking the the limit of the operator \( R_{2N}(z) \). There are two ways to take the limit namely take the limit of the expression above directly or take the limit of each component of the Gauss decomposition then multiply all the components together again. These two limits are different. The reason for such a disagreement comes from the fact that, under the topology we use, the following formula is not valid:

\[
\lim_{n \to \infty} f(n, z) \lim_{n \to \infty} g(n, z) \neq \lim_{n \to \infty} f(n, z)g(n, z),
\]

even both \( \lim_{n \to \infty} f(n, z) \) and \( \lim_{n \to \infty} g(n, z) \) exist. For example:

\[
\lim_{n \to +\infty} n \sum_{m \in \mathbb{Z}_+} z^m = 0, \quad \lim_{n \to +\infty} n \sum_{m \in \mathbb{Z}_+} z^m = \delta(z),
\]

but

\[
\lim_{n \to +\infty} (z^n \sum_{m \in \mathbb{Z}_+} z^m)(z^{-n} \sum_{m \in \mathbb{Z}_+} z^m) = (\Sigma_{m \in \mathbb{Z}_+} z^m)^2.
\]

Here, we would take the second kind of limit namely the limit of the product of the all the components of the Gauss decomposition, which coincides with the topology we define in Section 1 for affine quantum algebras. Then we have

\[
R^D(z) = q^{-1}(zq^2; q^4)^2 \begin{pmatrix} 1/(1-z) & 0 & 0 & 0 \\ 0 & 1/(1-q^{z^2}) & 0 & 0 \\ 0 & 0 & 1/(1-q^{z^2}) & 0 \\ 0 & 0 & 0 & 1/(1-z) \end{pmatrix}.
\]

We see, in accordance with Proposition 1 that

\[
R^D(z) = (R^+_+(z))^{-1} R(z) R^+_+(z)^2
\]

where \( R^+_+(z) \) is the image of the first factor of Gauss decomposition for \( R(z) \); it coincides with the image of tensor \( F_\delta \). So we prove

**Proposition 2.** The image \( \rho(z_1) \otimes \rho(z_2) R^D \) of the universal \( R \)-matrix for \( U_q^D(\mathfrak{sl}_2) \) is given by the formula (11):

\[
R^D(z) = \rho(z_1) \otimes \rho(z_2) R^D.
\]
We also have
\[
((R^D)^{21})^{-1}(z) = r^{-1}(z) \begin{pmatrix}
1 - z & 0 & 0 & 0 \\
0 & \frac{q^{-1}(1-z)^2}{1-q^2z} & (1-z)\delta(z) & 0 \\
0 & 0 & q(1-q^2z) & 0 \\
0 & 0 & 0 & 1 - z
\end{pmatrix}
\]
which is diagonal.

The twisted operator $R^D$ is well defined if we let it act on $V_a \otimes V_b$, where $V_a$ and $V_b$ are the highest weight modules. This operator may not well defined if we choose any $V_a \otimes V_b$. The example above is just a case that the projection of $R^D$ is well defined.

A similar case is for $U_q^D(\mathfrak{sl}_n)$. Let $V$ be the $n$-dimensional fundamental representation, the projection of $R$, the universal R-matrix gives us:
\[
R(z) = r(z) \left( \sum_{i=1}^{n} E_{ii} \otimes E_{jj} + \frac{z - 1}{qz - q^{-1}} \sum_{i,j=1, i \neq j}^{n} E_{ij} \otimes E_{ji} \right) \times 
\left( \sum_{i=1}^{n} E_{ii} \otimes E_{jj} + \frac{q^{-1}z - q}{z - 1} \sum_{i,j=1, i < j}^{n} E_{ii} \otimes E_{jj} + \frac{z - 1}{qz - q^{-1}} \sum_{i,j=1, i > j}^{n} E_{ii} \otimes E_{jj} \right) 
\left( \sum_{i=1}^{n} E_{ii} \otimes E_{jj} + \frac{(q - q^{-1})z}{z - 1} \sum_{i,j=1}^{n} E_{ij} \otimes E_{ji} \right) 
\]
where $r(z) = q^{-\frac{1}{2}} \left( \frac{(zq^n q^{-n})^2}{(zq^n)(zq^{-n})} \right)$ and $E_{ij} \in \text{End}(V)$.

The triangular decomposition of the universal R-matrix yields the triangular decomposition of the matrix $R(z)$:
\[
R(z) = r(z) \left( \sum_{i=1}^{n} E_{ii} \otimes E_{jj} + \frac{(q - q^{-1})}{z - 1} \sum_{i,j=1}^{n} E_{ij} \otimes E_{ji} \right) \times 
\left( \sum_{i=1}^{n} E_{ii} \otimes E_{jj} + \frac{q^{-1}z - q}{z - 1} \sum_{i,j=1, i < j}^{n} E_{ii} \otimes E_{jj} + \frac{z - 1}{qz - q^{-1}} \sum_{i,j=1, i > j}^{n} E_{ii} \otimes E_{jj} \right) \times 
\left( \sum_{i=1}^{n} E_{ii} \otimes E_{jj} + \frac{(q - q^{-1})z}{z - 1} \sum_{i,j=1}^{n} E_{ij} \otimes E_{ji} \right) 
\]

The tensor $F_N$ defining the twist by $\widetilde{t}_3^N$, is equal to ordered product of $R_\beta$ over real roots $\beta$ such that $t_{2N\beta}(\beta) \in -\Delta_\beta$. These roots are $\varepsilon_j - \varepsilon_i + k\delta$, where $i > j$, $0 \leq k < 2N(i - j)$. Its projection is a triangular matrix $G_N(z)$
\[
G_N(z) = Id \otimes Id - \frac{(q - q^{-1})(1 - z^{2N(i-j)})}{1 - z} \sum_{i,j=1}^{n} E_{ij} \otimes E_{ji} 
\]
and the image $R_{2N}(z)$ of twisted universal R-matrix $R_{2N}^N$ looks as
\[
R_{2N}(z) = r(z) \left( \sum_{i=1}^{n} E_{ii} \otimes E_{jj} + z^{2N(i-j)} \frac{(q - q^{-1})}{z - 1} \sum_{i,j=1}^{n} E_{ij} \otimes E_{ji} \right) \times 
\left( \sum_{i=1}^{n} E_{ii} \otimes E_{jj} + \frac{q^{-1}z - q}{z - 1} \sum_{i,j=1, i < j}^{n} E_{ii} \otimes E_{jj} + \frac{z - 1}{qz - q^{-1}} \sum_{i,j=1, i > j}^{n} E_{ii} \otimes E_{jj} \right) \times 
\left( \sum_{i=1}^{n} E_{ii} \otimes E_{jj} + z^{2N(i-j)+1} \frac{(q - q^{-1})}{z - 1} \sum_{i,j=1}^{n} E_{ij} \otimes E_{ji} \right) 
\]
and, finally,
Again, we have that

\( R^D(z) = q^{-\frac{1}{2}} \frac{(zq^n; q^{2n})^2}{(z q^{2n}; q^{2n})^2} \left( \sum_{i=1}^{n} \frac{1}{(1-z)} E_{ii} \otimes E_{ii} + \frac{q^{-1}z - q}{(z-1)^2} \sum_{i<j}^{n} E_{ii} \otimes E_{jj} + \frac{1}{(qz - q^{-1})} \sum_{i>j=1}^{n} E_{ii} \otimes E_{jj} + \delta(z) \sum_{i>j}^{n} E_{ij} \otimes E_{ji} \right) \) \hspace{1cm} (14)

Again, we have that

\( R^D(z) = \rho(z_1) \otimes \rho(z_2) R^D. \)

We summarize the calculations in the following proposition:

**Proposition 3.** (i) The image \( R^D(z) \) of the universal \( R \)-matrix in tensor product of fundamental representations of \( U_q(\mathfrak{sl}_n) \) is given by the formula (14).

(ii) The matrix \( R^D(z) \) is a product of the limit \( (N \to \infty) \) of all the components of the Gauss decomposition of the \( R \)-matrices \( R_{2N}(z) \).

(iii) The \( R \)-matrices \( R_{2N}(z) \) satisfy the Yang-Baxter equation. They are connected to the standard trigonometric solutions \( R(z) \) of the Yang-Baxter equation by means of the relations

\[ R_{2N}(z) = G_N^{-1}(z) R(z) G_N^{21}(z) \] \hspace{1cm} (15)

or

\[ R_{2N}(z) = U^{2N}(z) R(z) U^{-2N}(z). \] \hspace{1cm} (16)

where \( G_N(z) \) is given by the relation (12) and \( U(z) \) be the following diagonal matrix:

\[ U(z) = \sum_{i,j=1}^{n} z^{i-j} E_{ii} \otimes E_{jj} \] \hspace{1cm} (17)

One may treat the statement (iii) of the theorem as an existence of one-parameter family of symmetries for the trigonometric Yang-Baxter equation.

However this singular \( R \)-matrix, in some sense, should not be called \( R \)-matrix

**Proposition 4.** The matrix \( R^D(z) \) does not satisfies Yang-Baxter equation.

It can be checked directly that both \( R^D_{12}(z) R^D_{13}(zw) R^D_{23}(w) \) and \( R^D_{23}(w) R^D_{13}(zw) R^D_{12}(z) \) diverge. Beyond this, we also know that Drinfeld comultiplication is not really well defined on finite dimensional representations [D13] as well as the topology defined in section 1 does not work in the case of finite dimensional representations. Thus, we need to deal with convergence problem of the twisted quasitriangular structure if we want to use \( L \)-operator’s approach which is based on the properties of finite-dimensional representations.

### 3. Quasitriangular Hopf Algebra Structure, FRTS Realization and Current Operators for Non-Simple Roots

Let us first remind the technique of introducing the \( L \)-operators from the universal \( R \)-matrix [FRT], [FR]. We can introduce the parameter dependence of the universal \( R \)-matrix for the algebra \( U_q(\mathfrak{g}) \) and to reformulate the quasitriangular Hopf structure in the same manner as for quantum affine algebra \( U_q(\hat{\mathfrak{g}}) \) [FR]. This is done by means of automorphisms \( D_z \), which acts on Chevalley generators as

\[ D_z(e_{\alpha}) = z^{\delta_{\alpha,0}} e_{\alpha}, D_z(f_{\alpha}) = z^{-1} \delta_{\alpha,0} f_{\alpha}. \]

The action of \( D_z \) is equivalent to conjugation by \( z^d \), where \( d \) is the grading element:

\[ [d, e_{\pm\alpha}] = \pm \delta_{\alpha,0} e_{\pm\alpha}. \]
We define the map $\Delta(a) = (D_z \otimes id)\Delta(a)$ and $\Delta'(a) = (D_z \otimes id)\Delta'(a)$, where $a \in U_q(\hat{g})$ and $\Delta'$ denotes the opposite comultiplication. Then we put

$$ R^D(z) = (D_z \otimes id)q^{d\otimes c+c\otimes d}R^D, $$

where $R^D$ is as defined in (9) and $c$ is the central element of $U^D_q(\hat{g})$. We can use the tensor $R^D(z)$ for the description of quasitriangular structure of $U^D_q(\hat{g})$ on the category of highest weight modules. It means that on the tensor category of the highest weight representations of $U_q(\hat{g})$, the operator $R^D(z) \in U_q(\hat{g}) \otimes U_q(\hat{g}) \otimes \mathbb{C}[z]$ satisfies

$$ R^D(z)\Delta_z(a) = (D_{q^{1/2}}^{-1} \otimes D_{q^{-1}}^{-1})\Delta'_z(a)R^D(z), $$

$$ (\Delta \otimes I)R^D(z) = R^D_{13}(zq^{c^2})R^D_{23}(z), $$

$$ (I \otimes \Delta)R^D(z) = R^D_{13}(zq^{-c^2})R^D_{12}(z), $$

$$ R^D_{12}(z)R^D_{13}(zq^{c^2}w)R^D_{23}(w) = R^D_{23}(w)R^D_{13}(zq^{-c^2}w)R^D_{12}(z). $$

Here $c_1 = c \otimes 1$, $c_2 = 1 \otimes c$.

It follows from the fact that the projection of any of the operators $R^+$, $R^0$ and $R^-$ are finite expressions when they act on any vector $v_1 \otimes v_2$ for $v_1$ and $v_2$ in any two highest weight representations $V_1$ and $V_2$.

Let now $V$ be a finite dimensional representation of $U_q(\hat{g})$. Let us assume that the image of $R^D(\hat{z})$ on $V \otimes V$, which we is denoted by $R^D(\hat{z})$ is well defined. Let

$$ L^-(z) = (id \otimes \pi_V)(R^D(\hat{z})^{-1}), \quad L^+(z) = (id \otimes \pi_V)(R^D_{21}(z)). $$

Then both operators are well defined ans $U^D_q(\hat{g})$ as an algebra is generated by operator entries of $L^+(z)$ and $L^-(z)$ for the case $g$ is $\mathfrak{sl}_n$. However, in general the equalities

$$ R^D(z/w)L^+_1(z)L^+_2(w) = L^+_2(w)L^+_1(z)R^D(z/w), $$

$$ R^D(wq^{-c}/z)L^+_1(z)L^+_2(w) = L^+_2(w)L^+_1(z)R^D(wq^c/z), $$

are not valid, in particular, both sides of (19) diverge. Therefore, they can not be used to describe the commutation relations between the generators of the algebra $U^D_q(\hat{g})$. One can see this at the simplest example of $U^D_q(\mathfrak{sl}_2)$. Nevertheless, the analysis of the twisting procedure from the previous section allows to restore the full set of commutation relations between the entries of $L$-operators (18) and to prove that they are the defining relations of the algebra $U^D_q(\hat{g})$. In the following we show this for the algebra $U^D_q(\mathfrak{sl}_n)$. From now on, in this section, we will deal with $U^D_q(\mathfrak{sl}_n)$.

Let $\mathcal{R}$ be the universal R-matrix for $U_q(\mathfrak{sl}_n)$, and $R^\mathcal{R}_{2N}$ be the twisted universal R-matrix. Let $V$ be a $n$ dimensional representation of fundamental representation of $U_q(\hat{g})$. The image $R^\mathcal{R}_{2N}(\hat{z})$ of $R^\mathcal{R}_{2N}$ on $V \otimes V$, is given by the relation (13). Let

$$ L^-(2N)(z) = (id \otimes \pi_V)(R^\mathcal{R}_{2N}(\hat{z})^{-1}), \quad L^+(2N)(z) = (id \otimes \pi_V)(R^\mathcal{R}_{2N_{21}}(z)). $$

Both operators are well defined. As it follows from the definitions and from Proposition 3, the $L$-operators $L^\pm(2N)(z)$ satisfy the relations:

$$ R^\mathcal{R}_{2N}(z/w)(L^\pm(2N)_1)_1(z)(L^\pm(2N)_2)_2(w) = (L^\pm(2N)_1)_2(w)(L^\pm(2N)_2)_1(z)R^\mathcal{R}_{2N}(z/w), $$

$$ R^\mathcal{R}_{2N}(wq^{-c}/z)(L^+(2N)_1)_1(z)(L^-(2N)_2)_2(w) = (L^-(2N)_1)_2(w)(L^+(2N)_2)_1(z)R^\mathcal{R}_{2N}(wq^c/z). $$

Moreover, on the category of highest weight representations,

$$ \lim_{N \to \infty} L^\pm(2N)(z) = L^\pm(z). $$
We know also from the previous section, that
\[
\lim_{N \to \infty} R_{2N}(z/w) = R^D(z).
\]
but still both sides of (21), (22) diverge. Nevertheless, their matrix coefficients in highest weight module converge, so we have:
\[
\langle v, R^D(z/w)L_1^+(z)L_2^+(w)u \rangle = \langle v, L_1^+(w)L_2^+(z)R^D(z/w)u \rangle,
\]
\[
\langle v, R^D(wq^{-c}/z)L_1^+(z)L_2^+(w)u \rangle = \langle v, L_2^+(w)L_1^+(z)R^D(wq^c/z)u \rangle.
\]
Here \( u \in U \) is a vector of highest weight module \( U \), \( v \in U^* \) and both sides of the equalities are treated as analytic functions and \( R^D(z/w) \) is a diagonal matrix, whose diagonal entries are the same as that of \( R^D(z/w) \). As a corollary, we have the following statement.

**Proposition 5.** In the category of the highest weight representations,
\[
(z-w)R^D(z/w)L_1^+(z)L_2^+(w) = (z-w)R^D(z/w)L_2^+(w)L_1^+(z),
\]
\[
(z-q^cw)(z-q^{-c}w)R^D(wq^{-c}/z)L_1^+(z)L_2^+(w) = (z-q^cw)(z-q^{-c}w)R^D(wq^c/z)L_2^+(w)L_1^+(z),
\]
Here \((z-w)R(z/w)) \), \((z^2-q^{2c}w^2)R^D(wzq^c/z)\) are diagonal and \( z,w \) are both formal variables.

**Proof.** First we know that the equality above is true on the level of matrix coefficients as analytic functions. The limit of (21) and (22) diverge, it tells us that there is no poles on both side of the equations above, thus they are equal.

**Remark.** For the case of \( \hat{\mathfrak{sl}}_2 \) and \( \hat{\mathfrak{sl}}_3 \), one can show by calculation that (20) is actually valid. It seems this should be true for any \( n \), which we do not know how to prove in a simple way.

Now we come to the Gauss coordinates of the \( L \)-operators. Let
\[
L^+(z) = \begin{pmatrix}
1 & 0 & 0 \\
\vdots & \ddots & \vdots \\
0 & k_n^-(z) & e_{1,n}(z) + e_{n-1,n}(z) & 1 \\
\end{pmatrix}
\]
\[
L^-(z) = \begin{pmatrix}
1 & f_{2,1}(z) & f_{n,1}(z) & k_1^+(z) & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & \vdots & f_{n,n-1}(z) & 1 & k_n^+(z) \\
\end{pmatrix}
\]
that is,
\[
L^+(z) = \left( \sum_{i=1}^{n} k_i^- E_{ii} \right) \left( 1 + \sum_{i<j} e_{i,j} E_{ji} \right), \quad L^-(z) = \left( 1 + \sum_{i>j} f_{i,j} E_{ji} \right) \left( \sum_{i=1}^{n} k_i^+ E_{ii} \right),
\]
and let also
\[
L^+_{(2N)}(z) = \left( 1 + \sum_{i>j} \tilde{f}_{i,j} E_{ji} \right) \left( \sum_{i=1}^{n} \tilde{k}_i^- E_{ii} \right) \left( 1 + \sum_{i<j} \tilde{e}_{i,j} E_{ji} \right),
\]
\[
L^-_{(2N)}(z) = \left( 1 + \sum_{i>j} \tilde{f}_{i,j} E_{ji} \right) \left( \sum_{i=1}^{n} \tilde{k}_i^+ E_{ii} \right) \left( 1 + \sum_{i<j} \tilde{e}_{i,j} E_{ji} \right),
\]
be Gauss decomposition of finitely twisted \( L \)-operators. Let us introduce the currents
\[
x^+_i(z) = (q^{-1} - q)^{-1} e_{i,i+1} (zq^i), \quad x^-_i(z) = (q^{-1} - q)^{-1} f_{i+1,i} (zq^i),
\]
\[
\psi_i(z) = k_i^{-1} (zq^i q^{-c/2})^{-1} (k_i^- (zq^i q^{-c/2})), \quad \phi(z) = (k_{i+1}^+ (zq^i q^{-c/2}))^{-1} (k_i^+ (zq^i q^{-c/2})),
\]
and analogously for the finitely twisted $L$-operators:

$$
\tilde{x}_i^\pm(z) = (q^{-1} - q)^{-1}\tilde{e}_{i+1}(zq^i), \quad \tilde{x}_i^\pm(z) = (q^{-1} - q)^{-1}\tilde{f}_{i+1,i}(zq^i),
$$

$$
\tilde{\psi}_i(z) = \tilde{k}_{i+1}(zq^i q^{-c/2})^{-1}(\tilde{k}_i^{-1}(zq^i q^{-c/2})), \quad \tilde{\phi}(z) = (\tilde{k}_{i+1}(zq^i q^{-c/2})^{-1}(\tilde{k}_i^+(zq^i q^{-c/2})).
$$

**Theorem 6.** (i) The commutation relations between the currents $\tilde{x}_i^\pm(z)$, $\tilde{\psi}_i(z)$ and $\tilde{\phi}(z)$ encoded in the relations (21), (22) have well defined limit where they coincide with the relations on $x_i^\pm(z)$, $\psi_i(z)$ and $\phi(z)$ in the definition of the algebra $U_q^D(\hat{\mathfrak{sl}}_n)$.

(ii) The comultiplication structure of $U_q^D(\hat{\mathfrak{sl}}_n)$ (see section 1) coincides with the following comultiplication rule of the $L$-operators:

$$
\Delta'(L^\pm(z)) = L^\pm(zq^{-c_2}) \otimes L^\pm(z), \quad \Delta'(L^-(z)) = L^-(z) \otimes L^-(zq^{c_1})
$$

The proof follows from the results in [KT1]. There are no convergency problems since the topology of $U_q^D(\hat{\mathfrak{sl}}_n)$ is compatible with Gauss decomposition of the universal $R$-matrix. However the direct calculations can be treated as another direct proof of the results in [KT1].

**Remark.** One may notice that in [DF1], we shift the positive half and the negative half currents coming from FRTS realizations, which is not necessary here. The reason is that the operator $K$, which is one of the factor inside the universal $R$-matrix includes operator similar to $D_1$, which automatically shifts the half current just as in [DF1]. Let us also point out that the comultiplication formula for $L^\pm$ makes the the Drinfeld comultiplication transparent.

Clearly, we know now everything about the diagonal entries and the ones right above and below the diagonal entries of $L^\pm(z)$. The question to write down the complete commutation relation of all the current operators of $L^\pm(z)$ arises.

Let $i = 0, 1, ..., n - 1$ be the nodes of the Dynkin diagram of $\hat{\mathfrak{sl}}_n$. Let $\alpha_i, i = 1, ..., n - 1$ be the simple roots of $\mathfrak{sl}_n$ corresponding to the nodes $i = 1, ..., n - 1$. Let $s_{\beta}$ denotes the Weyl group element corresponding to the root $\beta$ of $\mathfrak{sl}_n$. Let us fix the following reduced decomposition of the longest element $\omega_0$ of the Weyl group $W = S_n$ of Lie algebra $\mathfrak{sl}_n$:

$$
\omega_0 = (s_{\alpha_1}s_{\alpha_2}...s_{\alpha_{n-1}})(s_{\alpha_1}s_{\alpha_2}...s_{\alpha_{n-2}})...(s_{\alpha_1}s_{\alpha_2})s_{\alpha_1}. \quad (25)
$$

**Theorem 7.** The entries of the second factor of $L^+(z)$ and the first factor of $L^-(z)$ are the same as the current operators for all roots of $\mathfrak{sl}_n$ corresponding to the decomposition (25) up to the shifts of spectral parameters defined in [DK].

**Proof.** First, for any highest weight module $V_\lambda$ and $v \in V_\lambda$, we have $L^{\pm 2N}(z)v = L^\pm(z)v$, when $N$ is big enough. With this, we can prove the theorem for the case of $\mathfrak{sl}_3$ by using the commutation relations for $L^{\pm 2N}(z)$ as in [DF1], then we factor out $k^\pm(z)$ operators and take the limit.

Below, we give the list of the commutation relations for the case of $U_q(\mathfrak{sl}_3)$. Let $K^+_{i,i+1}(z) = k^+_{i+1}(zq^{-c/2})^{-1}k^+_{i}(zq^{-c/2})$, $K^-_{i,i+1}(zq^{-c/2}) = k^-_{i+1}(zq^{-c/2})^{-1}k^-_{i}(zq^{-c/2})$. The currents $e_{i,j}(z)$ and $f_{i,j}(z)$ are as defined in (24). First, the currents, corresponding to simple roots of $\mathfrak{sl}_3$, satisfy the relations given in definition of $U_q^D(\hat{\mathfrak{g}})$, if we shift them accordingly as in Theorem 6. We give the rest in the following:

$$
e_{1,2}(z_1)e_{2,3}(z_2) - \frac{zq-wq^{-1}}{z-w}e_{2,3}(z_2)e_{1,2}(z_1) = \delta(z_1/z_2)e_{1,3}(z_1),$$

$$
f_{3,2}(z_1)f_{2,1}(z_2) - \frac{zq-wq^{-1}}{z-w}f_{2,1}(z_2)f_{3,2}(z_1) = \delta(z_1/z_2)f_{3,1}(z_1),$$
\[ [e_{1,3}(z), f_{3,1}(w)] = -\frac{1}{q - q^{-1}} \left\{ \delta(\frac{z}{w}q^{-c})K_{1,2}(wq\frac{1}{w})K_{2,3}(wq\frac{1}{w}) - \delta(\frac{z}{w}q^{-c})K_{1,2}^+(wq\frac{1}{w})K_{2,3}^+(wq\frac{1}{w}) \right\}, \]
\[(zq - wq^{-1})e_{1,3}(z)e_{1,3}(w) = (zq^{-1} - wq)e_{1,3}(w)e_{1,3}(z),\]
\[(zq^{-1} - wq)f_{3,1}(z)f_{3,1}(w) = (zq^{-1} - wq)f_{3,1}(w)f_{3,1}(z),\]
\[[e_{1,2}(z), f_{3,1}(w)] = \delta(\frac{z}{w}q^{-c})K_{1,2}^+(zq\frac{1}{w})f_{3,1}(z),\]
\[[e_{2,3}(z), f_{3,1}(w)] = \delta(\frac{z}{w}q^{-c})f_{2,1}(w)K_{2,3}^+(wq\frac{1}{w}),\]
\[[f_{2,1}(z), e_{1,3}(w)] = \delta(\frac{w}{z}q^{-c})e_{2,3}(zq^{-c})K_{1,2}^+(zq\frac{1}{w}),\]
\[[f_{3,2}(z), e_{1,3}(w)] = \delta(\frac{w}{z}q^{-c})K_{1,2}^+(wq\frac{1}{w})e_{1,2}(w),\]
\[\frac{(zq^{-1} - zq)}{z - w} f_{2,1}(z)f_{3,1}(w) = f_{3,1}(w)f_{2,1}(z),\]
\[\frac{z - w}{zq - wq^{-1}} f_{3,1}(w)f_{3,2}(z) = f_{3,2}(z)f_{3,1}(w),\]
\[e_{1,2}(z)e_{1,3}(w) = \frac{(zq^{-1} - zq)}{z - w} e_{1,3}(w)e_{1,2}(z),\]
\[e_{1,3}(w)e_{2,3}(z) = \frac{z - w}{zq - wq^{-1}} e_{2,3}(z)e_{1,3}(w).\]

The last four relations imply the cubic Serre relation. Using induction, for general \(n\), one can derive all the commutation relations for the entries of \(L^\pm(z)\) of formulas for \(U_q(\mathfrak{sl}_n)\).

4. Discussion

This paper is more to use the known results to explain the situation related to the quasitriangular structure for the Drinfeld realization and ask the proper questions. The quasitriangular structure for the can not be recover completely Drinfeld realization on the category of finite dimensional representation. The difficulty comes from the fact that we have not been able to properly control the convergence problem in such a situation. However we want to emphasize that such a problem is not just for this case but rather general when we study the current type of realizations of quantized algebras. The similar situation happens to other cases, such as Yangian, the new elliptic algebras \([KLP] [EF]\). We will discuss the situation looking at the twisting of the R-matrix on the finite dimensional representations.

For instance, for deriving current realization of the double of the Yangian for \(\mathfrak{sl}_2\), we start from the application of tensor product of two-dimensional representation to the universal \(R\)-matrix and this gives the answer

\[ R(u) = r(u) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & u & u - i\hbar & 0 \\ 0 & u - i\hbar & u & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \]

with

\[ r(u) = \frac{\Gamma\left(1 - \frac{u}{2\hbar}\right) \Gamma\left(-\frac{u}{2\hbar}\right)}{\Gamma^2\left(\frac{1}{2} - \frac{u}{2\hbar}\right)} \]
The same procedure gives
\[
R^D(u) = \frac{\Gamma^2(1 - \frac{u}{2 \pi i})}{\Gamma^2(\frac{1}{2} - \frac{u}{2 \pi i})} \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & u^{-i \hbar} & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]
and repeat all the arguments presented in previous sections. Analogously for the scaled elliptic algebra \([\text{KLF}]\). Note that for a final result we do need the universal \(R\)-matrix, which is not known in this case. What is important is that the scalar factor before \(R\)-matrix, determined from crossing and unitary conditions, has a pole when \(u = 0\). The recent work by \([\text{EF}1]\) can also be put in the same framework, where a Gauss decomposition of dynamic elliptic \(R\)-matrix is given.

\[
R(\zeta', \zeta) = \sum_{1}^{n} \frac{A(\zeta', \zeta)}{(\zeta - \zeta')} \left( \frac{1}{(\zeta - \zeta')} E_{1,1} \otimes E_{1,1} + \frac{1}{(\zeta - \zeta')} E_{2,2} \otimes E_{2,2} + \frac{\theta(\zeta')}{\theta(\zeta') \delta(\zeta', \zeta) E_{1,1} \otimes E_{1,1}} \delta(\zeta', \zeta) E_{1,1} \otimes E_{1,1} \right)
\]

Here the function \(A(\zeta', \zeta)\) has a pole at \(\zeta = \zeta'\) \([\text{EF}1]\). Let \(A(\zeta, \zeta') = \hat{A}(\zeta, \zeta')\), where \(\hat{A}(\zeta, \zeta')\) is neither 0 nor a pole at \(\zeta - \zeta' = 0\). Similar procedure would give us the singular \(R\)-matrix:

\[
\hat{A}(\zeta, \zeta') = \sum_{1}^{n} \frac{\hat{A}(\zeta, \zeta')}{(\zeta - \zeta')} \left( \frac{1}{(\zeta - \zeta')} E_{1,1} \otimes E_{1,1} + \frac{1}{(\zeta - \zeta')} E_{2,2} \otimes E_{2,2} + \frac{\theta(\zeta')}{\theta(\zeta') \delta(\zeta', \zeta) E_{1,1} \otimes E_{1,1}} \delta(\zeta', \zeta) E_{1,1} \otimes E_{1,1} \right)
\]

We can see that this singular \(R\)-matrix does not depend on the dynamic parameter anymore. This idea can be further applied to other cases, especially the structure related to Elliptic \(R\)-matrices to derive Drinfeld type of realizations using Gauss decomposition. The importance of the new interpretation of Drinfeld realization from the point of view of the structure is that it gives us a better way to understand the meaning related to the Gauss decomposition of the FRTS realizations \([\text{DF}]\) \([\text{KLP}]\) \([\text{EF}1]\) \([\text{KT}3]\) and the related quasi-triangular Hopf algebra structure for Drinfeld type of realizations. Further more, we might use our idea to to study the algebra \([\text{DI}]\), namely to understand those algebra’s quasitriangular structure, which is one of the motivations of this paper.

The entries of the \(L^\pm(z)\) are also of great interest. From the calculation, we can see that the operator \(f_1(z)\) and \(e_1(z)\) commute with themselves for the case of \(U_q(\hat{\mathfrak{sl}}_n)\). This operator actually is used in the quantum semi-infinite construction \([\text{DFe}3]\). For the case of \(U_q(\hat{\mathfrak{sl}}_n)\), the matrix operator \(L^\pm(z)\) are triangular, whose entries should give us naturally the current operators that correspond to other non-simple roots. Those current operator will enable us to derive a better description of the integrable conditions \([\text{DM}]\) as in the classical case and give the semi-infinite construction \(U_q(\hat{\mathfrak{sl}}_n)\). One more possible application is that the current operators corresponding to the longest root may naturally provide us a way to derive representations of quantum toroidal algebras.

Another immediate application is to use the idea to derive the intertwiners corresponding to the Drinfeld realizations \([\text{DI}2]\) and the other way around. Namely, we can compose the intertwiners with the twister \(D\) to derive the intertwiners corresponding to the Drinfeld comultiplication and vice versa. Combining this with the results \([\text{FR}]\), we can obtain a q-KZ equation corresponding to the Drinfeld comultiplication \([\text{DFe}2]\). This new interpretation will definitely help us to understand problem related to the intertwiners corresponding to the Drinfeld comultiplication, for example, why certain types of the intertwiners corresponding to the Drinfeld comultiplication do not exist \([\text{DI}]\).
For the case of $U_q(\mathfrak{sl}_2)$ [Kd], a FRTS realization using a diagonal R-matrix is given by a diagonal R-matrix. There is no details about the main results Proposition 3.1 about the FRTS realization, which mathematically looks impossible.

Acknowledgements. The authors are thankful to Prof. B. Feigin for useful discussions. S.Kh would like to thank Prof. T. Miwa and RIMS for hospitality in Kyoto, where the work was finished. S.Kh. was supported in part by RFBR grant 98-01-00344, grant 96-15-96455 for support of scientific schools and INTAS grant 930166-ext and by Award No. RM2-150 of the U.S. Civilian Research & Development Foundation (CRDF) for the Independent States of the Former Soviet Union. J. D. would like to thank the hospitality of the Feigins during his visit in Moscow, where this work was started.

References

[DFe1] J. Ding, B. Feigin Quantum current operators (III): Commutative quantum current operators, semi-infinite construction and functional models, [qalg/9612009].

[DFe2] J. Ding, B. Feigin Drinfeld realization and its KZ-equation, in preparation.

[DF] J. Ding, I. B. Frenkel Isomorphism of two realizations of quantum affine algebra $U_q(\mathfrak{gl}(n))$, CMP, 156, 1993, 277-300 Physics

[D1] J. Ding, K. Iohara Generalization and Deformation of Drinfeld quantum affine algebras, [q-alg/9608002].

[Di1] J. Ding, K. Iohara Drinfeld comultiplication and vertex operators, [q-alg/9608003].

[DK] J. Ding and S. Khoroshkin Weyl group extension of quantized current algebras q-alg/9804xxx.

[DM] J. Ding and Miwa Zeros and poles of quantum current operators and the condition of quantum integrability, [q-alg/9608000].

[Dr1] V. G. Drinfeld New realization of Yangian and quantum affine algebra, Soviet Math. Doklady, 36, 1988, 212-216

[Dr2] V. G. Drinfeld Hopf algebra and the quantum Yang-Baxter Equation Dokl. Akad. Nauk. SSS 283, 1985

[Dr3] Drinfeld, V.G. Quasi-Hopf algebra. Algebra and Analysis (Petersburg Math. Journ.) (1990), 1419-1457.

[Dr4] Drinfeld, V.G. Quantum groups. Proc. ICM-86 (Berkeley USA) vol.1, 798-820. Amer. Math. Soc. (1987).

[EF1] B. Enriquez, G. Felder Elliptic quantum groups $E_{\tau,\eta}(\mathfrak{sl}_2)$ and quasi-Hopf algebras, [q-alg/9703018].

[EF2] B. Enriquez, G. Felder A construction of Hopf algebra cocycles for the double Yangian $DY(SL_2)$, [q-alg/9703012].

[FR] I.B. Frenkel, N.Yu. Reshetikhin Quantum affine algebras and holomorphic difference equation, Comm. Math. Phys. 146, 1992, 1-60

[FRT] L. D. Faddeev, N. Yu, Reshetikhin, L. A. Takhtajan Quantization of Lie groups and Lie algebras, Yang-Baxter equation in Integrable Systems, (Advanced Series in Mathematical Physics 10) World Scientific, 1989, 299-309.

[Kd] R. Kedem Singular R-matrices and Drinfeld’s comultiplication, [q-alg/96011001].

[KLP] S. Khoroshkin, D. Lebedev, S. Pakuliak Elliptic algebra $A_{q,\rho}(\mathfrak{sl}_2)$ in the scaling limit, q-alg/9702000

[KST] S.Khoroshkin, A. Stolin, V. Tolstoy Generalized Gauss decomposition of trigonometric R-matrices, Modern Phys. Lett. A, 19, 1995, 1375-1392.

[KT1] Khoroshkin, S.M., and Tolstoy, V.N. Twisting of quantum (super)algebras. Connection of Drinfeld’s and Cartan-Weyl realizations for quantum affine algebras, [hep-th/9404036].

[KT2] Khoroshkin, S.M., and Tolstoy, V.N. Universal R-matrix for quantized (super)algebras, Commun. Math. Phys. 141, 1991, 599-617.

[KT3] Khoroshkin, S.M., and Tolstoy, V.N. Yangian Double, Lett. Math. Phys. 36, 1996, 373-402.

[LS] Levendorskii, S.Z., and Soibelman, Ya.S. Some application of quantum Weyl groups. The multiplicative formula for universal R-matrix for simple Lie algebras. Geom. and Phys 7:4 (1990), 1-14.

[L] Lustzig, G. Canonical bases arising from quantized enveloping algebras, J. Amer. Math. Soc., 3, 1990, 447-498.

[R] Reshetikhin, N.Yu. Multiparameter quantum groups and twisted quasitriangular Hopf algebras, Lett. Math. Phys. 20, 1990, 331-335.

[RS] N.Yu. Reshetikhin, M.A. Semenov-Tian-Shansky Central Extensions of Quantum Current Groups, LMP, 19, 1990

JINTAI DING, RIMS, KYOTO UNIVERSITY, KYOTO 606, JAPAN

SERGEI KHOROSHKIN, ITEP, 117259 MOSCOW, RUSSIA

15