Correlated Bandits for Dynamic Pricing via the ARC algorithm

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Abstract

The Asymptotic Randomised Control (ARC) algorithm provides a rigorous approximation to the optimal strategy for a wide class of Bayesian bandits, while retaining reasonable computational complexity. In particular, it allows a decision maker to observe signals in addition to their rewards, to incorporate correlations between the outcomes of different choices, and to have nontrivial dynamics for their estimates. The algorithm is guaranteed to asymptotically optimise the expected discounted payoff, with error depending on the initial uncertainty of the bandit. In this paper, we consider a batched bandit problem where observations arrive from a generalised linear model; we extend the ARC algorithm to this setting. We apply this to a classic dynamic pricing problem based on a Bayesian hierarchical model and demonstrate that the ARC algorithm outperforms alternative approaches.

Keywords: multi-armed bandit, parametric bandit, generalised linear model, dynamic pricing
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1 Introduction

In the multi-armed bandit problem, a decision maker needs to sequentially decide between acting to reveal data about a system and acting to generate profit. The central idea of the multi-armed bandit is that the agent has $K$ ‘options’ or equivalently, a bandit with $K$ arms, and must choose which arm to play at each time. Playing an arm results in a reward generated from a fixed but unknown distribution which must be inferred ‘on-the-fly’.

In the classic multi-armed bandit problem, the reward of each arm is assumed to be independent of the others (Gittins and Jones [9], Agrawal [1], Lattimore and Szepesvári [13]) and it is the only observation obtained by the decision maker at each step. In practice, we often observe signals in addition to the rewards and there is often correlation between the distributions of outcomes for different choices.

For example, in a dynamic pricing problem (Dubé and Misra [6], Misra et al. [14]), an agent wants to fix the price of a single product, from a finite set of prices \{$c_1,\ldots,c_K$\} to maximise revenue. We know that when the price is high, the demand $p(c_k)$ (which we interpret as the chance that each customer will buy the product) is low, but each sale yields a higher return. The agent’s reward on a given day is $c_k S_k$ where $S_k|N \sim B(N, p(c_k))$ is the number of customers who buy the product and $N$ is the number of customers on a particular day. On each day, the agent wishes to choose $c_k$ to maximise $c_k E S_k = c_k p(c_k) E(N)$. Unfortunately, the agent does not know the true demand $p(c_k)$, and needs to infer it over time.

The situation above can be modeled as a multi-armed bandit problem, where each price corresponds to an arm of the bandit. However, this is not a classical bandit problem: First, we observe more than the reward, i.e. we observe both the number of customers entering the shop and the number of sales, while the reward only represents the number of sales. The second difference is more fundamental, in that we know that there is some correlation between the number of sales (and thus the rewards) of different price choices. For example, when the price $c_k$ increases, we expect the demand $p(c_k)$ to decrease.

In recent work, bandits with correlation have been considered (Filippi et al. [8] and Rusmevichientong and Tsitsiklis [15]). However, these approaches require the distribution of the reward to follow a Generalised

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Linear Model (GLM), and the reward is the only observation. In the case of the dynamic pricing problem described above, the numbers of customers on each day will vary and the reward is scaled with the price, making this an unnatural assumption.

In our earlier work [5], we considered a wide class of Bayesian multi-armed bandits as a stochastic control problem (equivalently, a Markov decision problem), allowing a wide range of flexibility in modelling. By applying an asymptotic expansion, together with a regularisation, this leads to the Asymptotic Randomised Control (ARC) algorithm. They also showed that the ARC algorithm gives a near optimal value for the total discounted reward problem [2.1]. However, their implementation is limited to the case where the estimation procedure admits a simple prior-posterior conjugate pair.

In this paper, we consider a general class of multi-armed bandit problems where the observations of each arm arrive from a parametric exponential family in batches, with a shared parameter, following a generalised linear model [1]. We also generalise the implementation of the ARC algorithm to tackle cases without exact prior-posterior conjugacy, by applying the Kalman filter to the generalised linear model as in Fahrmeir [7].

The paper proceeds as follows. In Section 2 we formulate dynamic online pricing as a generalised linear bandit model and give a description of how we can use large sample theory and Bayesian statistics to propagate our beliefs. We then give a brief description of bandit algorithms which may be considered as candidates for the generalised linear bandit problem and outline the implementation of the ARC algorithm in Section 3. Finally in Section 4, we use experimental data from Dubé and Misra [6] to illustrate the performance of each bandit algorithm in observations from a generalised linear model.

2 Dynamic Pricing and Generalised Linear Bandit Model

In September 2015 Dubé and Misra [6] ran an experiment, in collaboration with the business-to-business company ZipRecruiter.com, to choose an optimal price in an online sales problem. Their experiment ran in two stages, as an offline learning problem: first collecting data using randomly assigned prices, and then testing their optimal price.

Figure 1: (a): The relation between the price and the logit of Acquisition Rate. (b): The relation between the price and the expected reward per customer.

In contrast, Misra et al. [14] used the same data to illustrate dynamic online pricing as a classical multi-armed bandit problem and tackled this problem using a modification of the classical UCB algorithm.

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1Filippi et al. [8] consider the same observation sequence but they assume that the observation is the reward.
and UCB-tuned algorithm (Auer et al. [2]) where the demand correlation is not taken into consideration.

Figure 4(a) displays the relation between the logit of the acquisition rate (the proportion of the customers who subscribe), together with its best fit line. It is clear that the demands at different prices are related, which agrees with economic intuition. These data can be found in Table 5 of [6]. The expected reward for each customer \((c_k p(c_k))\) generated by the best fit line and the observed data is illustrated in Figure 4(b).

Guided by to Figure 4 it is reasonable to consider a logistic model for the probability of subscription; the reward, however, does not fit as naturally into a GLM framework. Given this model, the next question is how to use this model to sequentially choose prices. This requires us to combine the multi-armed bandit problem with generalised linear regression.

### 2.1 Multi-armed bandit problem

We now give a formal specification of our multi-armed bandit problem: Suppose that at each time \(t\), we need to choose one of the \(K\) arms. After choosing the \(k\)th arm at time \(t\), we observe a random variable \(Y_t^{(k)}\) whose distribution varies with \(k\) and collect a reward \(R_t^{(k)}(Y_t^{(k)})\). The collection of previous observations are used as historical data on which to base our next decision, and the problem repeats.

To provide a mathematical framework, let \((\Omega, \mathcal{P}, \mathcal{F})\) be a probability space equipped with a random variable \(\Theta\), a family of random variables \((Y_t^{(k)})_{t \in \mathbb{N}, k = 1, \ldots, K}\) and a sequence \((\zeta_t)_{t \in \mathbb{N}}\) with \(\zeta_t \sim_{i.i.d.} U[0, 1]\) independent of \((Y_t^{(k)}, \Theta)\). The random variable \(Y_t^{(k)}\) is interpreted as the observation when the \(k\)th arm is chosen at time \(t\), the process \((\zeta_t)\) is used to allow random decisions, and \(\Theta\) is a hidden parameter specifying the distribution of all \((Y_t^{(k)})\) in a Bayesian manner. We assume (for simplicity) that \((Y_t^{(1)}, \ldots, Y_t^{(K)})_{t \in \mathbb{N}}\) are conditionally independent for each \(t\) given \(\Theta\).

At time \(t\), we need to choose an action \(A_t\) taking values in \(\{1, \ldots, K\}\). We can incorporate additional randomisation by instead choosing a sequence \((U_t)\) taking values in \(\Delta^K := \{u \in [0, 1]^K : \sum_{i=1}^K u_i = 1\}\) where the \(k\)th component of \(U_t\) represents the (conditional) probability that we choose the \(k\)th arm. The corresponding actions and filtrations (representing historical observations) can be given by \(A_t = A(U_t, \zeta_t)\) where \(A(u, \zeta) := \text{sup}\{i : \sum_{k=1}^K u_k \geq \zeta\}\) and \(\mathcal{F}_t^U := \sigma(\zeta_1, Y_1^{(A_1)}, \ldots, \zeta_t, Y_t^{(A_t)})\).

The random variable \(U_{t+1}\) must be chosen based on information at time \(t\), that is, \(U\) is \((\mathcal{F}_t^U)_{t \geq 0}\)-predictable. We emphasise that, for any strategy \((U_t)\), the parameter \(\Theta\) is not generally \(\mathcal{F}_t^U\)-measurable for any \(t \geq 0\), that is, it is impossible to completely resolve our uncertainty over the distribution of outcomes.

The objective of the multi-armed bandit is to choose a strategy \(U\) to optimise some objective. This objective varies in the literature.

One common objective is to minimise the cumulative (frequentist) regret (e.g. Agrawal [1], Auer et al. [2], Kaufmann et al. [10], Lai and Robbins [12]):

\[
R(\Theta, T, (U_t)) := \sum_{t=1}^{T} \left( E \left[ R(A^*)(Y_t^{(A^*)}) \mid \Theta \right] - E \left[ R(A_t)(Y_t^{(A_t)}) \mid \Theta \right] \right),
\]

where \(A^* = \arg \max_E E \left[ R(k)(Y_t^{(k)}) \mid \Theta \right].\)

One may also minimise Bayesian regret (Russo and Roy [17, 18]):

\[
r(\pi, T, (U_t)) := E_\pi \left[ R(\Theta, T, (U_t)) \right],
\]

where \(\pi\) is a prior belief for the parameter \(\Theta\).

An alternative objective is to maximise the total discounted reward (Gittins and Jones [9], Ryzhov et al. [20], Cohen and Treetanthiploet [5]):

\[
V(\pi, \beta, (U_t)) := E_\pi \left[ \sum_{t=1}^{\infty} \beta^{t-1} R(A_t)(Y_t^{(A_t)}) \right] \quad (2.1)
\]

for a given discount rate \(\beta \in (0, 1)\).
2.2 Generalised Linear Bandit Model

In this section, we will focus on our dynamic pricing problem, where observations are sampled from an exponential family whose parameter depends on our decision.

At the beginning of each day, we need to choose a price from the set \{c_1, ..., c_K\}. On day \(t\), with chosen price \(c_k\), we observe \(N_t^{(k)}\) customers arriving at the store. In order to capture the relations between demands at different prices, we suppose that the probability that each customer buys the product can be modeled by a logistic model, i.e., the relation between the demand \(p(c_k)\) and the price \(c_k\) is given by

\[
\text{logit}(p(c_k)) = \Gamma_0 + \Gamma c_k = (\Gamma_0, \Gamma)^\top (1, c_k) =: \Theta^\top x^{(k)}
\]

where \(\text{logit}(p) = \log \left( \frac{p}{1-p} \right)\). The parameter \(\Theta = (\Gamma_0, \Gamma)\) is unknown. At the end of day \(t\), we observe \(Y_i^{(k)} := (N_t^{(k)}, \{Q_{i,t}^{(k)}\}_{i=1, ..., N_t^{(k)}})\), where \(Q_{i,t}^{(k)}\) takes values in \(\{0, 1\}\) and indicates whether the product is bought by the \(i\)th customer, and collect a reward \(R^{(k)}(Y_i^{(k)}) = c_k \sum_{i=1}^{N_t^{(k)}} Q_{i,t}^{(k)}\).

More generally, we can fit the dynamic pricing problem into the GLM framework. Let \(\{x^{(1)}, ..., x^{(K)}\} \subseteq \mathbb{R}^d\) be a collection of features to be chosen by the agent, each corresponding to a choice \(\{1, ..., K\}\), and let \(\Theta\) be a random variable taking values in \(\mathbb{R}^d\) representing an unknown parameter. After choosing \(x^{(k)}\) at time \(t\), the agent observes \(N_t^{(k)}\) independent random variables \(\{Q_{i,t}^{(k)}\}_{i=1}^{N_t^{(k)}}\) where

\[
Q_{i,t}^{(k)} | \Theta \sim_{IID} h(q) \exp\left( \Phi^{(k)} q - G(\Phi^{(k)}) \right),
\]

(2.2)

\(\Phi^{(k)} = \phi(\Theta^\top x^{(k)})\) and \(Q_{i,t}^{(k)}\) is independent of \(N_t^{(k)}\). Here, \(h, \phi, G\) are known functions. For simplicity, we will assume that for each fixed \(k\), the processes \((N_t^{(k)})_{t=1}^\infty\) are IID with known distribution \(\mathcal{F}\). After observing \(Q_{i,t}^{(k)}\) and \(N_t^{(k)}\), we obtain a reward \(R^{(k)}(N_t^{(k)}, Q_{i,t}^{(k)}, ..., Q_{N_t^{(k)}}^{(k)})\).

Remark 1. A straightforward extension is to allow the observation \(Q\) to be such that \(T(Q)\) belongs to an exponential family for some function \(T\) in an arbitrary dimension. This follows the same analysis and will allow us to use the same model to consider e.g. pricing for multiple products.

2.2.1 Approximate posterior update

In order to implement a multi-armed bandit algorithm, we need an efficient way to update our estimate of the parameter \(\Theta\), together with its precision. As our observations are obtained in batches, we shall use a large sample approximation to update via a normal-normal conjugate model.

From (2.2), the mean and variance of \(Q\) given parameter \(\Theta = \theta\) are

\[
E_\theta(Q_{i,t}^{(k)}) = G'(\phi^{(k)}) \text{ and } \text{Var}_\theta(Q_{i,t}^{(k)}) = G''(\phi^{(k)}).
\]

where \(\phi^{(k)} = \phi(\theta^\top x^{(k)})\).

Suppose that the link function \(\phi\) is invertible and differentiable. If our model is non-degenerate, \(G'\) must also be invertible. We define \(\psi := (G' \circ \phi)^{-1}\). It then follows from the Central Limit Theorem and the Delta method that

\[
\sqrt{n}(\Psi_n - \theta^\top x^{(k)}) \xrightarrow{d} N\left(0, 1/[G''(\phi^{(k)})\phi'(\theta^\top x^{(k)})]^2\right),
\]

(2.3)

as \(n \to \infty\), where \(\Psi_n = \psi\left(\frac{1}{n} \sum_{i=1}^{n} Q_{i,t}^{(k)}\right)\). Moreover, by Slutsky’s lemma, \(\sqrt{n}V_n(\Psi_n - \theta^\top x^{(k)}) \xrightarrow{d} N(0, 1)\), where \(V_n := V(\Psi_n) := G''(\phi(\Psi_n))(\phi'(\Psi_n))^2\).

When \(n\) is not large, \(\psi(\frac{1}{n} \sum_{i=1}^{n} Q_{i,t}^{(k)})\) may not be well-defined for some values of \(Q\). This is the case for the logistic model when \(\frac{1}{n} \sum_{i=1}^{n} Q_{i,t}^{(k)} \in \{0, 1\}\). In order to avoid this degeneracy, if \(M_{t-1}\) is our running

\[\text{In fact, one may allow that the distribution of } (N_t^{(k)})_{t=1}^\infty \text{ is not known. In this case, all of our approximation arguments follow through with an extra posterior update step for the parameter of } (N_t^{(k)}) \text{. We will leave this to the reader in order to simplify our discussion.}\]
estimate of \( \theta \), we consider a linear expansion of \( \psi \) around \( \psi^{-1}(M_{t-1}^{-1}x^{(k)}) \), which approximates the expected value of \( Q_t^{(k)} \). This approach was used by Fahrmeir [7] to derive an extended Kalman filter with GLM observations, as in [2.4].

Suppose that the posterior of \( \Theta \) at time \( t-1 \) is

\[
\Theta | \mathcal{F}_{t-1}^U \sim N(M_{t-1}, \Sigma_{t-1}).
\]

Then, after observing \( \{Q_{t,i}^{(k)}\}_{i=1,...,N_t^{(k)}} \), the posterior can be approximately updated by the Kalman filter equations

\[
\begin{align*}
M_t &= \Sigma_t \left( \Sigma_{t-1}^{-1} M_{t-1} + S_t^{(k)} \Psi_t^{(k)} x^{(k)} \right), \\
\Sigma_t &= \left( \Sigma_{t-1}^{-1} + S_t^{(k)}(x^{(k)})^\top (x^{(k)})^\top \right)^{-1}, \\
\Psi_t^{(k)} &= M_{t-1}^\top x^{(k)} + (P_t^{(k)} - \tilde{P}_{t-1}^{(k)}) \psi' (\tilde{P}_{t-1}^{(k)}),
\end{align*}
\]

where \( S_t^{(k)} := N_t^{(k)} V(\Psi_t^{(k)}) \), \( \tilde{P}_{t-1}^{(k)} := \psi^{-1}(M_{t-1}^\top x^{(k)}) \), and \( P_t^{(k)} := \sum_{i=1}^{N_t^{(k)}} Q_{t,i}^{(k)}/N_t^{(k)} \).

By using the Woodbury identity, (2.4) simplifies to

\[
\begin{align*}
M_t &= M_{t-1} + R_t^{(k)} \left( \Psi_t^{(k)} - M_{t-1}^\top x^{(k)} \right) \Sigma_{t-1} (x^{(k)}), \\
\Sigma_t &= \Sigma_{t-1} - R_t^{(k)} \Sigma_{t-1} (x^{(k)}) (x^{(k)})^\top \Sigma_{t-1} - 1.
\end{align*}
\]

where \( R_t^{(k)} = S_t^{(k)} / \left( S_t^{(k)} (x^{(k)})^\top \Sigma_{t-1} (x^{(k)}) + 1 \right) \).

### 3 Multi-armed bandit algorithms

There are a number of approaches commonly used to solve the multi-armed bandit problem. In this section, we present a few approaches which can be adapted for application to the multi-armed bandit with generalised linear observation. We will implement these using the approximate posterior \( \Theta | \mathcal{F}_{t}^U \sim N(M_t, \Sigma_t) \), where \( (M_t, \Sigma_t) \) is given by (2.5).

#### 3.1 A selection of multi-armed bandit algorithms

- **\( \epsilon \)-Greedy** (Vermorel and Mohri [22]): With probability \( 1 - \epsilon \), we choose an arm based on its maximal expected reward and with probability \( \epsilon \), we choose an arm uniformly at random.

- **Explore-then-commit (ETC)** (Rusmevichientong and Tsitsiklis [15]): Let \( \epsilon \in (0, 1) \). We choose an arm uniformly at random in the first \( \lfloor \epsilon T \rfloor \) trials and then choose an arm which maximises the estimated expected reward for each remaining trial.

- **Thompson Sampling (TS)** (Thompson [21], Russo et al. [19]): We select according to the posterior probability that each arm is ‘best’. In particular, we choose a randomised strategy \( U_t \) such that

\[
U_t^{TS} = \mathbb{P}(A_t^{TS} = i | \mathcal{F}_{t-1}^U) = \mathbb{P}(A^* = i | \mathcal{F}_{t-1}^U)
\]

where \( A^* := \arg \max_{i \in \mathcal{A}} \mathbb{E}(R^{(k)}(Y^{(k)})) | \Theta \).

This algorithm can be implemented by sampling \( \hat{\Theta}_t \) at time \( t \) from its posterior distribution (i.e. conditional on \( \mathcal{F}_{t-1}^U \)). We can then choose \( A_t^{TS} = \arg \max_{i \in \mathcal{A}} \mathbb{E}(R^{(k)}(Y^{(k)})) | \Theta = \hat{\Theta}_t \).

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3One may also see our parameter update (2.5) as the Laplace approximation method for Thompson Sampling (see [19], Chapter 5)
• **Bayesian Upper Confidence Bound (Bayes-UCB)** (Kaufmann et al. [10], Filippi [8])\(^4\): This is an optimistic strategy to choose the arm based on an upper bound of the reward.

\[
A_t^{\text{Bayes-UCB}} = \arg \max_i Q\left(p, R^{(k)}(Y_t^{(k)}) \bigg| F_{t-1}^U\right),
\]

where \(Q(p, X)\) is the \(p\)-quantile of \(X\).

**Remark 2.** Kaufmann et al. [10] prove a theoretical guarantee of optimal order for the classical (independence) Bernoulli bandit when \(p = 1 - 1/(t(\log(T))^c)\) and \(c \geq 5\); their simulations suggest that \(c = 0\) performs well in practice. We will use these values to implement the Bayes-UCB in our simulations.

• **Knowledge Gradient (KG)** (Ryzhov et al. [20]): At each time, we pretend that we will only update our posterior once, immediately after the current trial. This simplifies the objective function (2.1) and suggests the choice

\[
A_t^{\text{KG}} = \arg \max_k \mathbb{E}\left[R^{(k)}(Y_t^{(k)}) + \left(\frac{\beta}{1 - \beta}\right)\tilde{V}_{t+1}^{(k)} \bigg| F_t^U, A_t = k\right],
\]

where \(\tilde{V}_{t+1}^{(k)} = \mathbb{E}\left[\max_j R^{(j)}(Y_{t+1}^{(j)}) \big| F_t^U, A_t = k\right]\).

The expectation can be computed using an appropriate quadrature or Monte Carlo approach if it is not available explicitly.

• **Information-Directed Sampling (IDS)** (Russo and Roy [18], Kirschner and Krause [11]): This algorithm chooses based on the information ratio between the single-step regret and the information gain.

We define

\[
\Delta_t(u) := \sum_{k \in A} u_k \mathbb{E}\left(R^{(A^*)}(Y_t^{(A^*)}) - R^{(k)}(Y_t^{(k)}) \bigg| F_{t-1}\right)
\]

where \(A^* := \arg \max_{A \in \mathcal{A}} \mathbb{E}\left(R^{(A)}(Y^{(A)})\big| \Theta\right)\).

Let \(H\) be an \((F_t^U)\)-adapted process taking values in \(\mathbb{R}^K\) whose \(k\)th component represents the information gain when the \(k\)th arm is chosen. Define \(G_t(u) := \sum_{k=1}^K u_k H_t^{(k)}\).

The IDS algorithm chooses an arm randomly according to the probabilities \(U_t^{\text{IDS}} = \arg \min_{u \in \Delta^K} \left(\frac{\delta_t(u)^2}{\log(|u|)}\right)\).

In Russo and Roy [18], \(H_t^{(k)}\) is defined to be the improvement in the Shannon entropy of \(A^*\)'s posterior when the \(k\)th arm is chosen at time \(t\). However, computing this \(H_t^{(k)}\) is expensive. Thus, in our simulation, we will consider

\[
H_t^{(k)} = \mathbb{E}\left(H(\Theta_{t-1}) - H(\Theta_t) \bigg| F_{t-1}, A_t = k\right)
\]

where \(H(\Theta_t)\) is the Shannon entropy of the posterior distribution of \(\Theta\) at time \(t\). This information gain is considered in Kirschner and Krause [11].

• **Asymptotic Randomised Control (ARC)** [5]: We choose an arm to maximise the asymptotic value of our decision. We will summarise the required calculation in Section 3.2.

In our simulation, we will also consider the classical UCB and the UCB-tuned algorithm (Auer et al. [2]) as candidates, as considered in Misra et al. [14]. These algorithms are similar to the Bayes-UCB algorithm but consider the problem from a frequentist perspective and ignore the correlation between outcomes. In particular, we do not use [2,5] to propagate our belief but we record the reward of each arm separately. The reader may refer to Misra et al. [13] or a survey by Burtini et al. [3].

\[^4\]Filippi [8] consider the UCB algorithm with correlation under frequentist perspective where an additional rescaling on the bandwidth appears due to the presence of the link function. Nonetheless, their formulation only restricts to the case where the reward is the only observation.
3.2 Implementation of the ARC algorithm

The key idea of the ARC algorithm is to give an estimate of the optimal solution to (2.1) via a Markov decision process with \((m, \Sigma)\) as an underlying state. A smooth approximation is obtained by introducing a preference for random decisions in the objective function (2.1), in particular adding a reward \(\lambda H(A_t)\) to \(R^{(k)}\) in (2.1), where \(H\) is a smooth entropy function (e.g. Shannon entropy, which we use here). The scale of this preference is controlled through the parameter \(\lambda\), which is determined dynamically in order to have a negligible effect when uncertainty is low. This approximation results in a semi-index strategy which amounts to computing the solution \(a \in \mathbb{R}^K\) to the fixed point equation

\[
a = f + \left( \frac{\beta}{1 - \beta} \right) L^\lambda(a)
\]

where the term \(f = \left[ \mathbb{E}(R^{(k)}(Y^{(k)})|\mathcal{F}^t) \right]_{k=1,...,K}\) is the vector of expected rewards over one time step (quantifying the gain from immediate exploitation) and \(L^\lambda(a)\) is an exploration term, given in (3.1) below. Intuitively, the term \(L^\lambda(a)\) combines the derivatives of \(f\), with respect to the parameter estimate and its precision, to give an approximation of the expected increase in future payoffs which would be generated from the observations \(Y^{(k)}\), separately from the immediate reward.

With this interpretation of \(f\) and \(L^\lambda\), the components of \(a\) can be interpreted as measuring the immediate reward and the increase in total reward arising from each choice, taking into account the effect of learning on future rewards. The entropy term results in the ARC algorithm applying a softmax function to \(a\), yielding conditional probabilities of choosing each arm, rather than a deterministic choice. We refer to [5, Section 6] for a more detailed overview of this algorithm.

The implementation of the ARC algorithm requires the computation of the dynamics of the posterior parameter and the derivative of the expected one-period cost with respect to the underlying state (i.e., the posterior parameters). In our model, we use (2.5) to simplify our posterior dynamic for the generalised linear bandit, allowing us to estimate the relevant terms in the ARC algorithm for this case. The implementation is given by the following steps:

**Step I: Estimate the dynamics of posterior parameter.** The ARC algorithm requires the computation of how we expect the parameter estimate and its precision to change. In the case of interest, we will treat the posterior mean \(m\) and variance \(\Sigma\) of the parameter \(\Theta\) as the ‘estimator’ and ‘precision’ in the ARC algorithm.

Let \(m\) and \(\Sigma\) be the posterior mean and variance conditional on \(\mathcal{F}^t\) and let \(M^{(k)}\) and \(\Sigma^{(k)}\) be their update after we choose the \(k\)th arm. As discussed in Section 2.2.1, we can update \(M^{(k)}\) and \(\Sigma^{(k)}\) by (2.5). From (2.3), we estimate \(\Psi^{(k)}(\Theta) \approx N(\Theta^\top x^{(k)}, 1/S^{(k)})\). Assuming the posterior \(\Theta \sim N(m, \Sigma)\) gives \(\Theta^\top x^{(k)} \sim N(m^\top x^{(k)}, (x^{(k)})^\top \Sigma (x^{(k)})^{-1})\). Hence, the conditional distribution of \(\Psi^{(k)}(m, \Sigma)\) can be approximated by \(\Psi^{(k)}(m, \Sigma) \approx N\left(m^\top x^{(k)}, \frac{S^{(k)}(x^{(k)})^{-1} \Sigma (x^{(k)})^{-1} + 1}{S^{(k)}(x^{(k)})^{-1} \Sigma (x^{(k)})^{-1}}\right)\).

Therefore, (2.5) yields an approximate innovation representation

\[
\Delta M^{(k)} \approx \left( \frac{S^{(k)}(x^{(k)})^{-1} \Sigma (x^{(k)})^{-1} + 1}{S^{(k)}(x^{(k)})^{-1} \Sigma (x^{(k)})^{-1}} \right)^{1/2} \Sigma (x^{(k)}) Z,
\]

where \(Z \sim N(0, 1)\).

As \(\Theta \sim N(m, \Sigma)\), when \(\Sigma\) is small, we may estimate \(S^{(k)} \approx n_k V(m^\top x^{(k)})\) where \(n_k = \mathbb{E}(N^{(k)})\). Hence, in the notation of [3], the dynamics of our state \((m, \Sigma)\) can be approximated by

\[
\begin{align*}
\mathbb{E}_{m, \Sigma}(\Delta M^{(k)}) &\approx \bar{\mu}^{(k)} (m, \Sigma) := 0, \\
\text{Var}_{m, \Sigma}(\Delta M^{(k)}) &\approx (\bar{\sigma} \bar{\sigma}^\top)_{(k)} (m, d) := \omega(m, \Sigma, k) \Sigma (x^{(k)})^-1 \Sigma, \\
\mathbb{E}_{m, \Sigma}(\Delta \Sigma^{(k)}) &\approx \bar{\delta}^{(k)} (m, d) := -\omega(m, \Sigma, k) \Sigma (x^{(k)})^-1 \Sigma \Sigma (x^{(k)})^-1 \Sigma,
\end{align*}
\]

where \(\omega(m, \Sigma, k) := (n_k V(m^\top x^{(k)})(x^{(k)})^{-1} \Sigma (x^{(k)})+1)^{-1}\).

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This fixed-point calculation is the only computationally expensive part of the algorithm, and standard methods can be applied.
**Step II:** Compute the expected reward $f$ and learning function $L^\lambda$ using the estimated dynamics. We next compute the expected reward given the (estimate) posterior parameter, that is $f : \mathbb{R}^l \times S_+^l \rightarrow \mathbb{R}^K$ with components

$$f_k(m, \Sigma) := \mathbb{E}
\left( R^{(k)}(Y^{(k)}) \bigg| \Theta \sim N(m, \Sigma) \right),$$

where $S_+^l$ is the family of positive definite $\mathbb{R}^{l \times l}$ matrices, and the learning function $L^\lambda : \mathbb{R}^K \times \mathbb{R}^l \times S_+^l \rightarrow \mathbb{R}^K$ with components

$$L_k^\lambda(a, m, \Sigma) := \langle B^\lambda(a, m, \Sigma); \mathbb{E}_{m,\Sigma}\left(\Delta\Sigma^{(k)}\right) \rangle + \langle M^\lambda(a, m, \Sigma); \mathbb{E}_{m,\Sigma}\left(\Delta M^{(k)}\right) \rangle + \frac{1}{2}\langle \Xi^\lambda(a, m, \Sigma); \mathbb{V}_{m,\Sigma}\left(\Delta M^{(k)}\right) \rangle,$$

where we define

$$B^\lambda := \sum_k \nu_k^\lambda(a)(\partial_m f_k), \quad M^\lambda := \sum_k \nu_k^\lambda(a)(\partial_m f_k),$$

$$\Xi^\lambda := \sum_k \nu_k^\lambda(a)(\partial_m f_k) + \frac{1}{\lambda} \sum_{j,k} \eta_{j,k}(a)(\partial_m f_k)(\partial_m f_k)^\top,$$

$$\nu_k^\lambda(a) := \exp(a_k/\lambda)/\sum_j \exp(a_j/\lambda), \quad \eta_{j,k}(a) := \nu_k^\lambda(a)(\delta(j = k) - \nu_k^\lambda(a)).$$

The operator $\langle \cdot; \cdot \rangle$ is interpreted as an inner product of tensors, i.e. if $u, v \in \mathbb{R}^l$, then $\langle u; v \rangle = u \cdot v$ and if $u, v \in \mathbb{R}^{l \times l}$, then $\langle u; v \rangle = \text{Tr}(uv)$.

The ARC algorithm focuses on the regime where $\Sigma$ is small. Hence, we estimate our expected reward by $\tilde{f} : \mathbb{R}^l \rightarrow \mathbb{R}^K$ with components

$$f_k(m, \Sigma) \approx \tilde{f}_k(m) := \mathbb{E}
\left( R^{(k)}\left(N^{(k)}(Q_i^{(k)})_{i=1}^{N^{(k)}}\right) \bigg| \Theta = m \right) = h_k(m^\top x^{(k)}),$$

for some function $h_k$. The last equality follows from the fact that $\{Q_i^{(k)}|\Theta = m\}_{i=1,...,\infty} \sim \text{IID}\ g(q; m^\top x^{(k)})$ and $N^{(k)}$ is independent of $\{Q_i^{(k)}\}_{i=1,...,\infty}$.

Using the estimates $\tilde{\mu}^{(k)}$, $(\tilde{\sigma} \tilde{\sigma}^\top)^{(k)}$, $\tilde{\sigma}^{(k)}$, and $\tilde{f}_k$, we can approximate $L^\lambda$ in the ARC algorithm by

$$L_k^\lambda := \frac{1}{2} \nu(m, \Sigma, k) \left[ \sum_{j=1}^K \nu_j^\lambda(a)(h_j''(m^\top x^{(j)}))(r_{kj}(\Sigma))^2 + \frac{1}{\lambda} \left( \sum_{j=1}^K \nu_j^\lambda(a)(h_j'(m^\top x^{(j)}))r_{kj}(\Sigma) \right)^2 - \left( \sum_{j=1}^K \nu_j^\lambda(a)(h_j'(m^\top x^{(j)}))r_{kj}(\Sigma) \right)^2 \right],$$

where $r_{kj}(\Sigma) := (x^{(j)} \Sigma x^{(k)})$.

**Step III:** Use the ARC algorithm for the generalised linear bandit. Finally, to apply the ARC algorithm, one also needs to choose a rescaling function to encourage randomisation when estimates are uncertain. Here, we choose $\lambda_\rho(m, \Sigma) := \rho \|\Sigma\|$, using the Euclidean norm $\|\Sigma\| = \sqrt{\text{Tr}(\Sigma \Sigma)}$. We refer to [5] for further discussion of the effect of this choice and alternatives.

The implementation of this algorithm is then given by the pseudocode of Algorithm 1. To run the ARC algorithm, we need to choose the parameters $\rho > 0$ and $\beta \in (0, 1)$. Here $\rho$ determines the randomness of our early choices, encouraging early exploration, while $\beta$ is a discount factor, which is used to value the future reward. These parameters can be chosen by the user.\footnote{It is suggested by Russo [3] for a related approach that $\beta = 1 - 1/T$, where $T$ is the total number of trials, might be an appropriate choice.}
Algorithm 1 ARC Algorithm

**Input:** $\rho, \beta, m, \Sigma, T$

Let $f$ and $L^\lambda$ be function given in (3.2) and (3.3)

for $t = 1, \ldots, T$ do

Define $\lambda = \rho \| \Sigma \|$ 

Solve $a = \tilde{f}(m) + (\frac{\beta_1 - \beta}{\beta}) \tilde{L}^\lambda(a, m, \Sigma)$ for $a$

Sample $A_t$ with $P(A_t = k) = \nu_k^\lambda(a)$

Obtain observations $(N, Q)$ and collect the reward

Update the posterior parameter $(m, \Sigma)$ as in (2.5)

end for

3.3 Theoretical Guarantee of the ARC algorithm

It was shown in [5] that for a smooth function $f$ and $\tilde{V}((\pi, \beta, (U_t))) := E_{\pi}[\sum_{t=1}^{\infty} \beta^{t-1} f(A_t, M_{t-1}, \Sigma_{t-1})]$, (3.4)

we have

$$|\tilde{V}((\pi, \beta, (U_t^{\text{ARC}}))) - \sup_{U_t} \tilde{V}((\pi, \beta, (U_t)))| = O(\|\Sigma_0\|).$$

In addition, they also showed that $\|\Sigma_t\| \to 0$ as $t \to \infty$ $P_\pi$-a.s. (given that $G$ is sufficiently nice).

For the multi-armed bandit problem, one may wish to consider the total discounted reward (2.1). However, when we do not have a prior-posterior conjugate pair, which is the case for this GLM framework, the equality between (2.1) and (3.4) cannot be guaranteed. Nonetheless, given that the batch size is relatively large, the difference between (2.1) and (3.4) is small.

4 Simulation of a bandit environment

As the strategy followed determines the observations available, we cannot directly use historical data to test the algorithm. However, we can use the data to construct an environment in which to run tests. We take a Bayesian view and build a simple hierarchical model (with an improper uniform prior). Effectively, this assumes that our observations come from an exchangeable copy of the world we would deploy our bandits in, with the same (unknown) realised value of $\Theta$. Then we will use Laplace approximation, as in Russo et al. [19, Chapter 5] or Chapelle and Li [4], to obtain a posterior sample to simulate the markets.

Remark 3. It is worth emphasising that when implementing the algorithm in the simulation, we do not assume that our algorithms know the distribution that we use to simulate the parameter, $\Theta$. Instead the simulations are initialised with an almost uninformative prior.

To construct a posterior for $\Theta$ given historical data, we assume that we have a collection of observations $(q_i^{(1)})_{i=1}^{r_1}, (q_i^{(2)})_{i=1}^{r_2}, \ldots, (q_i^{(K)})_{i=1}^{r_K}$ from an exponential family modeled by (2.2). We denote the corresponding log-likelihood function of $\Theta$ by $\ell(\theta; (q_i^{(k)}))$.

Let $\hat{\theta}$ be the maximum likelihood estimator. i.e. $\hat{\theta} = \arg \max_{\theta} \ell(\theta; (q_i^{(k)}))$. Then we may approximate the log-likelihood function by

$$\ell(\theta; (q_i^{(k)})) \approx \frac{1}{2} (\theta - \hat{\theta})^\top \partial^2_\theta \ell(\hat{\theta}; (q_i^{(k)}))(\theta - \hat{\theta}) + c,$$

for $c$ independent of $\theta$. Therefore, starting from the uninformative (improper, uniform) prior, we can estimate the posterior of the parameter $\Theta$ by

$$\Theta \sim N\left(\hat{\theta}, \left(-\partial^2_\theta \ell(\hat{\theta}; (q_i^{(k)}))\right)^{-1}\right),$$

(4.1)
In particular, if the parameter for the exponential family is given in its canonical form, i.e., when \( \phi \) in (2.2) is the identity, then the observed Fisher information is

\[-\partial^2 \ell (\hat{\theta}; (y_i^{(k)})) = \sum_{k=1}^{K} r_k G'' (\hat{\theta}^T x^{(k)}) (x^{(k)})^T.\]

We will use the simulated values of \( \Theta \) as the (hidden) realisations with which to test our algorithms.

### 4.1 Dynamic Pricing Simulation

Finally, we can implement the multi-armed bandit algorithm to run a simulation for the dynamic online pricing problem.

Recall that in this model, at each time \( t \), we need to choose \( x^{(k)} = (1, c_k) \in \mathbb{R}^2 \) where \( c_k \) is the price of the product. We then observe the number of customers \( N_{t,1}^{(k)} \) and whether each customer buys the product or not in terms of a binary random variable \( (Q_{t,1}^{(k)})_{i=1}^{N_{t,1}^{(k)}} \). The reward that we receive in each step is given by

\[ R^{(k)} (N_1^{(k)}, Q_{1,1}^{(k)}, \ldots, Q_{N_{1}^{(k)}}) := c_k \sum_{i=1}^{N_{1}^{(k)}} Q_{i,1}^{(k)}. \]

We will model \( (Q_{t,1}^{(k)}) \) by a logistic model, i.e.,

\[ G(z) = \log(1 + e^z) \quad \text{and} \quad \phi(z) = z \quad \text{for the functions} \quad \phi \quad \text{and} \quad G \quad \text{in (2.2)}. \]

We can now write down the expected reward,

\[ E_{\theta} \left( R^{(k)} (N_1^{(k)}, Q_{1,1}^{(k)}, \ldots, Q_{N_{1}^{(k)}}) \right) = n_k c_k G' (\theta^T x^{(k)}), \]

where \( n_k = E(N_{1}^{(k)}) \). In particular, the function \( h_k \) in (3.2) and (3.3) is given by

\[ h_k(y) = n_k c_k G'(y). \]

#### 4.1.1 Market data and simulation environment

In Dubé and Misra [6], in stage one of their experiment, they randomly assigned one of ten experimental pricing cells to 7,867 different customers who reached Ziprecruiter’s paywall. The exact numbers of customers that were assigned to each price are not reported. Hence, we will assume that there are exactly 787 customers for each price. We then use their reported subscription rate to estimate the exact numbers of customers who decided to subscribe given each price.

Using this data, we apply (4.1) to infer an approximate posterior distribution:

\[ \Theta | (y_i^{(k)}) \sim N \left( \begin{pmatrix} -6.42 \times 10^{-1} \\ -4.03 \times 10^{-3} \end{pmatrix} , \begin{pmatrix} 1.90 \times 10^{-3} & -8.86 \times 10^{-6} \\ -8.86 \times 10^{-6} & 6.82 \times 10^{-8} \end{pmatrix} \right). \]

(4.2)

In order to compare performance, we will consider each algorithm over a period of one year (365 days) and only allow the agent to change the price at the end of each day. We assume that a common price must be shown to all customers on each day. We will also assume that the chosen price does not affect the number of customers reaching the paywall, i.e., we assume that \( N_{t,1}^{(k)} \equiv N_t \).

We run \( 5 \times 10^3 \) independent simulations of the different market situations, where for each simulation the parameter \( \Theta \) is sampled from (4.2). We also independently sample \((N_i)_{i=1}^{365} \sim \text{Poisson}(270)\) to represent the number of visitors on each day. The simulated subscription probability and the simulated expected revenue per customer, for each price level, are illustrated in Figure [2].

#### 4.1.2 Simulation Results

We apply each algorithm described in Section 3 with \( m_0 = (0, 0) \) and \( \Sigma_0 = I_2 \) as an initial prior for \( \Theta \) and use (2.6) to propagate the prior.

To assess the performance of each algorithm, one often compares the cumulative pseudo-regret of each algorithm given the true parameter \( \Theta \):

\[ R(\theta, T, (A_t)) := \sum_{t=1}^{T} \left( \max_k h_k(\theta) - h_{A_t}(\theta) \right). \]

\[ \text{In [6], they observed that ZipRecruiter.com had roughly 8,000 visitors per month. Hence, it is reasonable to assume that there are roughly 270 visitors per day.} \]
Figure 2: (a): The average subscription rate in our simulation, with two standard deviation bands. (b): The average reward per customer in our simulation, with two standard deviation bands.

where \( h_k(\theta) := \mathbb{E}[R^{(k)}(Y_1^{(k)}) | \Theta = \theta] = nc_kG'(\theta^T x^{(k)}) \), \( G(y) = \log(1 + e^y) \), \( (A_t) \) is the sequence of actions that the algorithm chooses and \( n = \mathbb{E}(N^{(k)}) \).

Figure 3 shows the mean, median, 0.75 quantile and 0.90 quantile of cumulative pseudo-regret of each algorithm described in Section 3. We see that most algorithms outperform the classical UCB and the UCB-tuned used in Misra et al. [14], which is unsurprising as these approaches ignore correlation between demand at different prices. We also see that the ARC algorithm outperforms all other algorithms both on average and in extreme cases (as shown by the quantile plots).

In addition to the regret criteria, when displaying the price, the agent may not want to change the price too frequently. This is why the Explore-Then-Commit (ETC) algorithm is often preferred when considering a pricing problem. Figure 4 shows that the ARC algorithm and KG algorithm typically require a small number of price changes but still achieve a reasonably low regret (as shown in Figure 3) whereas, Thompson sampling and the UCB type algorithms require a larger number of price changes.
Figure 3: (a): Mean, (b): Median, (c): 0.75 quantile and (d): 0.90 quantile of cumulative expected-expected pseudo regret
Figure 4: (a): Mean, (b): Median, (c): 0.75 quantile and (d): 0.90 quantile of the number of price changes

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