Self-Force in the Radiation Reaction Formula

Adiabatic Approximation of a Metric Perturbation and an Orbit

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We investigate a calculation method for the gravitational evolution of an extreme mass ratio binary, i.e. a binary constituting of a galactic black hole and a stellar mass compact object. The inspiralling stage of this system is considered to be a possible source of detectable gravitational waves. Because of the extreme mass ratio, one may approximate such a system by a black hole geometry (a Kerr black hole) plus a linear metric perturbation induced by a point particle. With this approximation, a self-force calculation was proposed for a practical calculation of the orbital evolution, including the effect of the gravitational radiation reaction, which is now known as the MiSaTaQuWa self-force\(^1\). In addition, a radiation reaction formula was proposed\(^2\) as an extension of the well-known balance formula of Press and Teukolsky\(^3\). The radiation reaction formula provides us a convenient method to calculate an infinite time averaged loss of “the constants of motion” (i.e. the orbital energy, the \(z\) component of the angular momentum, and the Carter constant) through the gravitational radiation reaction, with which one may approximately calculate the orbital evolution. Because these methods are approximately equivalent, we investigate the consequence of the orbital evolution using the radiation reaction formula.

To this time, we have used the so-called adiabatic approximation of the orbital evolution and considered a method to evaluate the MiSaTaQuWa self-force by use of a linear metric perturbation. In this approach, we point out that there is a theoretical question concerning the choice of the gauge condition in the calculation of the MiSaTaQuWa self-force. Because of this gauge ambiguity, there is a case in which the MiSaTaQuWa self-force might not predict the orbital evolution in a physically expected manner, and this forces us to calculate a waveform only in the so-called dephasing time. We discuss the reason that such an unexpected thing happens and find that it is primarily because we consider the linear metric perturbation separately from the orbital evolution due to the self-force.

We propose a new metric perturbation scheme under a possible constraint of the gauge conditions in which we obtain a physically expected prediction of the orbital evolution caused by the MiSaTaQuWa self-force. In this new scheme of a metric perturbation, an adiabatic approximation is applied to both the metric perturbation and the orbit. As a result, we are able to predict the gravitational evolution of the system in the so-called radiation reaction time scale, which is longer than the dephasing time scale. However, for gravitational wave detection by LISA, this may still be insufficient. We further consider a gauge transformation in this new metric perturbation scheme, and find a special gauge condition with which we can calculate the gravitational waveform of a time scale long enough for gravitational wave detection by LISA.

§1. Introduction

Constructing a method for the accurate calculation of gravitational waveforms is an essential step toward success in gravitational wave detection. Since the Einstein equation is nonlinear, it is difficult to derive fully analytic solutions of every gravitational wave source, and we therefore consider an approximation method for

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a specific astrophysical source.

In this paper, we consider an extreme mass ratio binary system. X-ray and a variety of other observations suggest that there may exist supermassive black holes ($M \sim 10^6 - 10^8 M_\odot$) at the centers of galaxies. A number of stars have been observed at the galactic centers, and they interact with each other gravitationally. As a result of this multi-body interaction, we may expect that some galactic black holes may capture a stellar mass compact object ($m \sim 1 - 100 M_\odot$) deeply into its gravitational potential without tidal disruption. We refer to such a pair of objects as an extreme mass ratio binary system, and such systems are believed to be possible sources of gravitational waves detectable by LISA$^4$. Due to its orbital energy and angular momentum, the stellar mass compact object inspirals around the black hole for a long time, referred to as the ‘inspiral stage’ of the binary evolution. During this stage, the system has a huge quadrupole moment and emits detectable gravitational waves, while the system slowly loses its orbital energy and angular momentum through this emission of gravitational waves.

We consider a method to compute the gravitational waveform from the inspiral stage for such an extreme mass ratio binary. Because the black hole dominates the gravitational potential due to its huge mass, we assume that a metric perturbation provides a good approximation to describe the gravitational evolution of the system. We employ a Kerr black hole as the background because of the black hole uniqueness theorem in general relativity. Since the size of the stellar mass compact object is much smaller than the background curvature, it can be treated as a point particle to induce a metric perturbation$^5$. In this picture, the evolution of the binary can be interpreted as the orbital evolution of the particle around the black hole.

It has been asserted that, because we already have a convenient formula to calculate a linear metric perturbation in a Kerr background$^3$, the remaining issue in this problem is to develop a method to compute the orbit of a point particle. The orbital evolution can be derived by calculating the self-force, and a general formula for the self-force employing a linear metric perturbation scheme has been derived$^3$. Because the linear metric perturbation induced by a point particle diverges along the orbit, the implementation for the self-force involves a regularization calculation. We have made good progress in obtaining an exact derivation of the self-force$^1$ by implementing this regularization calculation. However, there remains a problem with this method when we consider the self-force of a general orbit in a Kerr background. A related method that employs the so-called radiation reaction formula has also been proposed$^2$. With this method, one can approximately derive the self-force, although it only contains the radiative part of the self-force. However, this method does not involve a complicated regularization calculation, and the numerical method has been established even for a general orbit in a Kerr background.

Although the calculation of the self-force seems to be a reasonable step for a gravitational waveform calculation, the self-force is entirely gauge dependent and there is a question of what physical content the self-force actually has. In fact, one can easily show that, with an extreme choice of the gauge condition, the self-force can be made to vanish over the entire time interval for which a usual metric perturbation scheme is valid in describing a gravitational evolution of the system. We discuss that,
although this is mathematically allowed in the usual metric perturbation scheme, the resulting description is technically disadvantageous from an energetics point of view. If we adopt such a gauge condition, we may eventually have to consider the gravitational evolution of the system in a nonperturbative manner, and we must use numerical relativity to calculate the nonlinear metric perturbation, to our knowledge.

Instead of numerical relativity, we consider the extension of the usual metric perturbation with a certain constraint on a gauge condition. A primary reason for such an unexpected consequence in the usual metric perturbation scheme is that the particle must move along a geodesic to the leading order of the perturbation. As a result, a linear metric perturbation cannot track gravitational evolution together with the orbital evolution affected by the self-force. We propose a new metric perturbation scheme in which the MiSaTaQuWa self-force does not vanish. In this approach, the orbital evolution caused by this self-force turns to be essential to calculate the gravitational waveform.

The new metric perturbation scheme may be able to predict the gravitational evolution of the system over a time scale longer than that of the usual metric perturbation. However, this time scale may not be sufficient to calculate a gravitational waveform for LISA. For this reason, we further consider a gauge transformation along an orbit so as to extend the validity of this new metric perturbation scheme. We find a special gauge condition with which the self-force contains only the radiation reaction component of the system, and, in this gauge condition, the adiabatic approximation of the linear metric perturbation can predict the evolution over a time scale long enough for LISA.

This paper is organized as follows. In Sec. we consider the gauge ambiguity of the MiSaTaQuWa self-force in the usual metric perturbation scheme and clarify the reason why such a problem could arise in the nonlinear theory of radiation reaction. In Sec. we propose a new metric perturbation scheme, with which one can treat the gravitational evolution of the system together with the orbital evolution. In Sec. we consider a further gauge transformation in the new metric perturbation scheme and propose the so-called radiation reaction gauge condition. We summarize our result in Sec. with emphasis on clarifying the validity of this new metric perturbation scheme.

§2. Gauge ambiguity in the self-force problem

Gravitational radiation reaction is a physically real phenomenon, since we can define a gauge invariant momentum flux of gravitational waves in an asymptotic flat region of a background metric. However, we find that a self-force is entirely gauge dependent consistently with a usual metric perturbation scheme and that it can be made to vanish through a special gauge choice along an orbit over the entire time scale for which the usual metric perturbation scheme is valid.

It is well-known that the self-force can be made to vanish at an orbital point by making the appropriate gauge choice, but we conjectured this might not be a problem, because the self-force would have non-vanishing components if we take the average of the self-force over a long time. In Ref., we show that this is true.
if we take the average of the self-force over an infinite time\footnote{It is important to note that this argument depends only on the properties of the orbital equation, and it does not depend on the metric perturbation scheme.}, and we prove that non-vanishing components are gauge invariant in agreement with the gravitational radiation reaction. However, in the usual metric perturbation scheme, one cannot consider an infinite time average of the self-force because it is valid only over a finite time interval.

In Subsec. 2.1, we consider a self-force in the usual metric perturbation scheme, and, in this scheme, we find that the average of the self-force can be made to vanish by making a special gauge choice along the orbit. This suggests that the relation between the self-force and the gravitational radiation reaction is entirely gauge dependent, and that it is not trivial as we previously believed. As a result, the so-called adiabatic approximation of the orbit\footnote{It is important to note that this argument depends only on the properties of the orbital equation, and it does not depend on the metric perturbation scheme.} would yield a gauge dependent prediction for the orbit. Because we expect that the orbital evolution caused by the self-force can be observed by the modulation of gravitational waves, a physically reasonable prediction of the orbit must be gauge invariant. This leads to the questions of how the self-force is related to the gravitational radiation reaction and what gauge conditions we should use to calculate the self-force. We discuss these questions in Subsec. 2.2

2.1. Metric perturbation

We assume that a regularization calculation for the self-force is formally possible by using a matched asymptotic expansion of the metric of the near-zone expansion and the metric of the far-zone expansion, as considered in Ref.\footnote{It is important to note that this argument depends only on the properties of the orbital equation, and it does not depend on the metric perturbation scheme.}. Because the resulting self-force is defined by regularizing the metric of the far-zone expansion, we consider only the metric of the far-zone expansion. The metric of the far-zone expansion consists of the sum of a regular vacuum background metric and its metric perturbation induced by a point particle moving in the background metric. We assume the point particle has an appropriate internal structure\footnote{It is important to note that this argument depends only on the properties of the orbital equation, and it does not depend on the metric perturbation scheme.} so that the metric of the far-zone expansion matches the metric of the near-zone expansion.

Suppose we use the usual metric perturbation scheme for the calculation of the metric of the far-zone expansion. Then the far-zone expansion is defined as a perturbation expansion with the small parameter $m/L$ where $m$ is the Schwarzschild radius of the particle and $L$ is the curvature scale of the background along the orbit. In the usual metric perturbation scheme, we expand the metric and the stress-energy tensor in term of this small parameter as

\begin{alignat}{2}
  g_{\mu\nu} &= g^{(bg)}_{\mu\nu} + (m/L)h^{(1)}_{\mu\nu} + (m/L)^2h^{(2)}_{\mu\nu} + \cdots, \\
  T^{\mu\nu} &= (m/L)T^{(1)\mu\nu} + (m/L)^2T^{(2)\mu\nu} + \cdots,
\end{alignat}

where $g^{(bg)}_{\mu\nu}$ is the vacuum background metric. For a valid perturbation, we must have

\begin{alignat}{2}
  O(1) > |(m/L)h^{(1)}_{\mu\nu}| > |(m/L)^2h^{(2)}_{\mu\nu}| > \cdots, \\
  O(1) > |(m/L)T^{(1)\mu\nu}| > |(m/L)^2T^{(2)\mu\nu}| > \cdots.
\end{alignat}
We also expand the Einstein equation in $m/L$, and schematically we have

\begin{align}
G^{(1)\mu\nu}[h^{(1)}] &= T^{(1)\mu\nu}, \\
G^{(1)\mu\nu}[h^{(2)}] + G^{(2)\mu\nu}[h^{(1)}, h^{(1)}] &= T^{(2)\mu\nu}, \\
\cdots,
\end{align}

where $G^{(1)\mu\nu}[h]$ and $G^{(2)\mu\nu}[h, h]$ are respectively the terms linear and quadratic in $h_{\mu\nu}$ of the Einstein tensor $G^{\mu\nu}[g + h]$.

In Ref. 5, it is found that $T^{(1)\mu\nu}$ is the usual stress-energy tensor of a monopole particle for a matched asymptotic expansion of the metric. It is important to note that the linearized Einstein equation is algebraically divergence free with respect to the background metric, i.e. $G^{(1)\mu\nu} = 0$. Thus, in this metric perturbation scheme, the particle must move along a geodesic in the background metric $g^{(bg)}_{\mu\nu}$ for a consistent solution of (2.5). The explicit form of $T^{(2)\mu\nu}$ can be derived by carrying out a further matched asymptotic expansion of the metric, but this has not been obtained yet. However, at least, it is clear that it has a term of the monopole particle coming from the deviation of the orbit from a geodesic because the MiSaTaQuWa self-force can be derived from LHS of (2.6) through a mass renormalization5). This shows that in the expansion of the stress-energy tensor (2.2), we assume an expansion of the orbit by $m/L$ as

\begin{equation}
(\tau) \rightarrow z^{(bg)}_{\mu}(\tau) + (m/L)z^{(1)}_{\mu} + \cdots,
\end{equation}

where $\tau$ is defined to be the proper time in the background metric, and $z^{(bg)}_{\mu}$ is a geodesic of the background metric given by $T^{(1)\mu\nu} = 0$. For a valid perturbation, we must have

\begin{equation}
O(1) > |(m/L)z^{(1)}_{\mu}| > \cdots.
\end{equation}

Intuitively, the orbit deviates from a geodesic as a result of the gravitational radiation reaction, and eventually the condition (2.8) will be violated. Then, beyond this time, this perturbation scheme would fail to approximate the system.

We consider a gauge transformation over a time interval during which the metric perturbation still approximates the system. The gauge transformation is defined as the small coordinate transformation $x^\mu \rightarrow \bar{x}^\mu = x^\mu + (m/L)\xi^\mu(x)$. The orbit transforms as $z^\mu(\tau) \rightarrow \tilde{z}^\mu(\tau) = z^\mu(\tau + (m/L)\delta\tau) + (m/L)\xi^\mu(z(\tau))$, where $\delta\tau$ is a function of $\tau$, so that $\tau$ remains the proper time of the new orbit $\tilde{z}^\mu(\tau)$ in the background metric. Applying this to the perturbation expansion of the orbit (2.7), we obtain the gauge transformation of $(m/L)z^{(1)}_{\mu}$ as

\begin{equation}
(m/L)z^{(1)}_{\mu}(\tau) \rightarrow (m/L)\tilde{z}^{(1)}_{\mu}(\tau) = (m/L)z^{(1)}_{\mu}(\tau) + (m/L)v^{(bg)}_{\mu}\delta\tau + (m/L)\xi^\mu(z^{(bg)}(\tau)),
\end{equation}

where $v^{(bg)}_{\mu} = dz^{(bg)}_{\mu}/d\tau$. We can eliminate $(m/L)z^{(1)}_{\mu}(\tau)$ if the gauge transformation along the orbit satisfies

\begin{equation}
(m/L)\xi^\mu(z^{(bg)}(\tau)) = -(m/L)z^{(1)}_{\mu}(\tau) + v^{(bg)}_{\mu}(m/L)\Delta\tau,
\end{equation}

where $\Delta\tau$ is the change in proper time.
with an arbitrary function $\Delta \tau$. There always exists a gauge transformation which satisfies this condition over the entire time interval during which the metric perturbation scheme is valid, because it is smaller than $O(1)$ in accordance with (2.8), and the orbit becomes a geodesic of the background metric under this gauge condition.

Because the MiSaTaQuWa self-force causes a self-acceleration of the leading order in $m/L$-expansion and causes the orbit to deviate from the geodesic, the relation $(m/L)z^{(1)}(\tau) = 0$ implies that the MiSaTaQuWa self-force entirely vanishes under this gauge condition over the entire time interval for which this metric perturbation scheme is valid. This extreme example suggests that the MiSaTaQuWa self-force is completely gauge dependent in this usual metric perturbation scheme, and, even the time average of the self-force over a long time scale could be gauge dependent in general, as long as we use this perturbation scheme.

At this stage, we see the mathematical reason why the self-force may not include the effect of gravitational radiation reaction. Because this perturbation scheme only allows a small deviation from the background geodesic, as expressed by (2.8), one could always eliminate this deviation by a gauge transformation, as in (2.9). We note that this problem cannot be solved by calculating a non-linear metric perturbation because the problem resides in the metric perturbation scheme itself. This shows that we must modify the metric perturbation scheme so that we can describe a non-perturbative orbital deviation from the background geodesic as $(m/L)z^{(1)}(\tau) \sim O(1)$. If the orbit can deviate from the geodesic nonperturbatively, it cannot be eliminated by the gauge transformation.

This extreme example suggests that the relation between the self-force and the gravitational radiation reaction is not trivial, as we believed before. By an adiabatic approximation of an orbit, we usually use the time averaged part of the self-force. If this is a component averaged over an infinite time, it is known to be gauge invariant, and we have a gauge invariant prediction for the orbit. However, because the metric perturbation is not valid over an infinite time scale, it is not valid to consider an infinite time average, and it seems reasonable to take a finite time average. Then the orbital prediction becomes gauge dependent. In this situation, if we still use a linear metric perturbation, the gravitational waveform at infinity would also be gauge dependent, although we would expect it to be gauge invariant because it is observable.

In the next subsection, we study this example from a different point of view, using a theory of a nonlinear metric perturbation. We expect that this extreme example may reveal to us the nature of the self-force, and, consequently we will be able to understand why and how we should take a certain class of gauge conditions for the self-force.

2.2. Energetics of the orbit and radiation

Some idea of why such an unexpected thing happens can be gained from Ref. 2. In Sec.III of Ref. 2, we argue that the orbital energy does not decrease monotonically as a result of the emission of gravitational waves. We conjecture that gravitational radiation has its own energy and that the self-force describes the interaction between the orbital energy and the radiation energy. In fact, one can define an effective stress-
energy tensor for the orbit and gravitational radiation, as argued in Ref.\(^5\). As in the previous subsection, we consider only the metric of the far-zone expansion, and we suppose that the metric consists of the sum of the vacuum background metric and its perturbation induced by a point particle with an appropriate internal structure\(^9\) as \(g_{\mu\nu} = g_{\mu\nu}^{(bg)} + h_{\mu\nu}\).

The Einstein equation for the far-zone expansion can be written formally as

\[
G_{\mu\nu}^{(1)}[h] + G_{\mu\nu}^{(2+)}[h] = T_{\mu\nu},
\]

where \(G_{\mu\nu}^{(1)}\) and \(G_{\mu\nu}^{(2+)}\) contain the linear terms and the rest of the Einstein tensor with respect to \(h_{\mu\nu}\), respectively. Because \(G_{\mu\nu}^{(1)}\) is algebraically divergence free, one can define a conserved stress-energy tensor in the background metric as

\[
T_{\mu\nu} = T_{\mu\nu} - G_{\mu\nu}^{(2+)}[h].
\]

We consider that the first and second terms of (2.11) represent the effective stress-energy tensors for the orbit and the gravitational radiation, respectively, and neither of these is conserved in the background metric by itself. Although the effective stress-energy tensor (2.11) is defined in a nonlinear manner in a general curved background, it reduces to the well-known gauge invariant stress-energy tensor for gravitational waves in the so-called short wavelength approximation and in the weak perturbation limit\(^6\).

We suppose that the background metric \(g_{\mu\nu}^{(bg)}\) is a Kerr black hole and that it has a Killing vector \(\xi_\mu\). Then we can define total, orbital and radiation energies in the background metric as

\[
E^{(tot)} = E^{(orb)} + E^{(rad)},
\]

\[
E^{(orb)} = \int d\Sigma_\mu \xi_\nu T^{\mu\nu}, \quad E^{(rad)} = -\int d\Sigma_\mu \xi_\nu G^{(2+)}{}^{\mu\nu}[h],
\]

where the surface integral is taken over the spacelike hypersurface bounded by the future horizon and the future null infinity of the background black hole metric, and we assume an appropriate regularization along the orbit of the particle, say, by a matched asymptotic expansion. We note that the radiation energy \(E^{(rad)}\) is not necessarily positive.

Integrating (2.11) over a small world tube surface around the orbit, one can derive the MiSaTaQuWa self-force\(^5\). This shows that the self-force describes the interaction between the orbital energy and the gravitational radiation energy. The gauge ambiguity of the self-force could be interpreted as an ambiguity in defining the orbital energy and the radiation energy of (2.12), since \(T^{\mu\nu}\) and \(G^{(2+)\mu\nu}[h]\) are gauge dependent. In order to see this, we apply the metric perturbation scheme presented in Subsec.2.1 to (2.12). This yields

\[
E^{(orb)} = (m/L)E^{(orb)(1)} + (m/L)E^{(orb)(2)} + \cdots,
\]

\[
(m/L)E^{(orb)(1)} = \int d\Sigma_\mu \xi_\nu (m/L)T^{(1)\mu\nu}, \quad \cdots
\]

\[
E^{(rad)} = (m/L)^2E^{(rad)(2)} + \cdots,
\]

\[
(m/L)^2E^{(rad)(2)} = -\int d\Sigma_\mu \xi_\nu (m/L)^2G^{(2+)\mu\nu}[h^{(1)}_1, h^{(1)}], \quad \cdots
\]
In the usual perturbation scheme, \((m/L)T^{(1)\mu \nu}\) becomes the conserved stress-energy tensor of a monopole particle moving along a background geodesic, and \(E^{(\text{orb})(1)}\) becomes a constant of motion. \((m/L)^2 T^{(2)\mu \nu}\) has not yet been derived explicitly, and calculations of \((m/L)^2 E^{(\text{orb})(2)}\) and \((m/L)^2 E^{(\text{rad})(2)}\) would involve regularization provided by the matched asymptotic expansion. However, it includes a contribution from the orbital deviation of the monopole particle relative to the background geodesic described by \((m/L)T^{(1)\mu \nu}\), as we discussed in the previous section. A gauge transformation of this part is defined by \(\mathcal{L}_{(m/L)\xi} (m/L)T^{(1)\mu \nu}\), and we find that the orbital energy transforms under gauge transformations as

\[
(m/L)^2 E^{(\text{orb})(2)} \rightarrow (m/L)^2 \tilde{E}^{(\text{orb})(2)} = (m/L)^2 E^{(\text{orb})(2)} + (m/L)^2 \delta E, \tag{2.15}
\]

where \(\delta E\) is arbitrary. Thus, the orbital energy is gauge dependent in the usual metric perturbation scheme.

By contrast, the balance formula considers the gauge invariant momentum flux of gravitational waves at the future null infinity and the future horizon of the background black hole geometry. Hence, it describes the radiation reaction to the total energy, \(E^{(\text{tot})}\), rather than that to the orbital energy, \(E^{(\text{orb})}\).

Although we have not yet proposed a new metric perturbation scheme in which a metric perturbation would be valid over a much longer time scale (see the next section), the orbital energy and the radiation energy are defined in a nonlinear manner as (2.12), and we can consider what happens when we keep the orbital energy constant under a gauge transformation while the total energy continues to decrease through the gravitational radiation reaction over a sufficiently long time. Because the radiation energy, \(E^{(\text{rad})}\), is not necessarily positive, it is easily seen that the amplitude of the metric perturbation would inevitably increase, because the radiation energy, \(E^{(\text{rad})}\), would decrease as a result of the radiation reaction. Although this is mathematically allowed, it creates two technical problems: 1) A linear approximation can no longer be used in calculating the evolution of the system gravitationally. 2) The background coordinate system cannot be used as an approximate reference to measure the orbit.

The above considerations reveal what is happening with regard to the behavior discussed in the previous subsection. It is reasonable to use a finite time averaged self-force with an arbitrary gauge condition for the adiabatic evolution of the orbit, and we would then have a gauge dependent prediction for the orbit. However, it is not reasonable to use a linear metric perturbation to calculate the gravitational waveform at infinity if we do not make an appropriate choice of the gauge condition for the self-force. We believe that the gauge invariant nature of the gravitational waveform at infinity can be recovered with an appropriate nonlinear calculation for the metric in this case.

\* The radiation reaction in the balance formula is gauge invariant, and one may expect that the total energy is also gauge invariant. However, an explicit computation to prove its gauge invariance is not trivial because the definition of the total energy involves a gauge dependent regularization calculation and a robust volume averaging operation.
In order to avoid the difficulty involved in dealing with a nonlinear metric perturbation, it is disadvantageous to use such a gauge condition, and we therefore assume consider that the orbital energy must decrease through the radiation reaction for a reasonable choice of gauge conditions, so that one can understand the orbit in the background coordinate system as an approximate reference. We define the averaged energetic gauge condition along the orbit by

\[
\langle \frac{d}{dt}E^{(\text{orb})} \rangle = \langle \frac{d}{dt}E^{(\text{tot})} \rangle,
\]

where \(<>\) represents a long-time average.

Although the above condition is a necessary condition for a reasonable gauge choice, this may not be a sufficient condition, because the orbit in a Kerr background is not characterized by the energy \(E^{(\text{orb})}\) alone. In order to confirm that a certain gauge condition is actually advantageous, it is necessary to demonstrate that we can actually construct a new metric perturbation scheme that is valid over a longer time scale. For this, the new metric perturbation scheme must allow a nonperturbative deviation of an orbit from a geodesic, and therefore it would not be constrained by (2.8). In the next section, we propose an adiabatic approximation of the metric perturbation in which the usual metric perturbation scheme is extended adiabatically.

§3. Adiabatic approximation

The adiabatic approximation of the orbital evolution is well known and is used in Ref.\(^7\)\(^8\). It consists of the following two steps:

1) From the fact that the self-force is weak, we can instantaneously approximate the orbit by a geodesic, and, instead of the orbital coordinates, we consider the evolution of geodesic “constants”.

2) We calculate the evolution of the geodesic “constants” caused by the gauge invariant momentum flux of gravitational waves induced by the approximated geodesic.

In a Kerr background, Step 2) is insufficient because we do not have a gauge invariant radiation reaction to the Carter “constant”. Instead of Step 2), the radiation reaction formula\(^2\) includes the following step:

2') We calculate the evolution of geodesic “constants” employing the gauge invariant “infinite time averaged” self-force induced by the approximated geodesic.

As shown in Ref.\(^10\), the “infinite time averaged” self-force acting on the orbital energy and the angular momentum is identical to the gauge invariant momentum flux of gravitational waves. Hence, 2') appears to be a generalization of 2). 2') seems more reasonable than 2) because 2') is based on the self-force. However, one cannot take an infinite time average in the usual metric perturbation scheme because the metric perturbation scheme is valid only in an finite time. Instead of 2'), we can consider the following step:

2") We calculate the evolution of geodesic “constants” employing the self-force induced by the approximated geodesic.

In this case, the self-force is entirely gauge dependent, and the orbital evolution
becomes gauge dependent as argued in the previous section.

Because general relativity allows an arbitrary coordinate system, an orbit is always “gauge dependent”. However, in a metric perturbation scheme, we use a fixed coordinate system of the background geometry, and we can define a gauge invariant object using the background metric as an approximate reference. To have a gauge dependent prediction for the orbit resulting from the self-force means that the coordinate system of the background geometry no longer acts as a proper reference for the full geometry. As shown in the previous section, this happens if the system cannot be described properly by a linear metric perturbation with an inappropriate gauge condition. This suggests that any argument for a self-force contains a gauge ambiguity until we consider the problem together with the evolution of the metric perturbation, so that we can carefully consider the validity of the metric perturbation during the orbital evolution. For this purpose, it is necessary to have a new metric perturbation scheme which predicts the orbital evolution consistently without the orbital constraint (2.8).

In Subsec.3.1, we propose an adiabatic approximation of the linear metric perturbation. In Subsec.3.2, we evaluate a metric perturbation in this new metric perturbation scheme based on the radiation reaction formalism. In Subsec.3.3, we investigate the orbital evolution and consider the validity of this metric perturbation. The reason for using the radiation reaction formula is that we have an explicit general form of the self-force in the radiation reaction formula, and we can identify the component of the self-force which corresponds to the gauge invariant “infinite time averaged” momentum flux of gravitational waves. (see (3.14)) Thus, one can easily formulate a new metric perturbation scheme under the averaged energetic gauge condition (2.16) using this explicit form.

3.1. Adiabatic linear metric perturbation

As we summarize in Appendix A, an inspiralling geodesic in a Kerr geometry is conveniently characterized by 7 constants with an orbital parameter \( \lambda \). We denote the orbital constants by \( \gamma = \{ \mathcal{E}^a; a = E, L, C; \lambda^b; b = r, \theta; C^c; c = t, \phi \} \). (See Appendix A for definitions.) In the usual metric perturbation scheme, a linear metric perturbation is induced by a particle moving along a geodesic \( \gamma \). Therefore, it is a function of \( \gamma \), and we write \( h^{(1)}_{\mu\nu}(x; \gamma) \).

We consider extending this linear metric perturbation by including the effect of the self-force. Due to the self-force, the “geodesic constants” evolve as functions of the orbital parameter \( \lambda \). We foliate the background spacetime into smooth spacelike hypersurfaces that intersect the orbit. We define the foliation function \( f(x) \) such that it is equal to the orbital parameter at the intersection with the orbit: \( f(z(\lambda)) = \)

\(^*\) The condition that the foliation surfaces are space-like is not important, as long as we consider the adiabatic approximation of the linear metric perturbation. For a well-defined nonlinear metric perturbation, there must be appropriate attenuation behavior of the adiabatic metric perturbation at the past null infinity. Otherwise, an asymptotic regularization is needed to calculate the nonlinear metric perturbation.
λ. Using the foliation function, we define an adiabatic linear metric perturbation as

\[ h^{\text{ad}(1)}_{\mu \nu}(x) = h^{(1)}_{\mu \nu}(x; \gamma(f(x))). \]  

(3.1)

Because there is an ambiguity in the foliation function, the adiabatic metric perturbation is not defined uniquely. However, our definition is sufficient for a leading correction to the gravitational evolution of the system.

Because the derivative acts on \( f(x) \), we have \([h^{(1)}_{\mu \nu; \lambda}(x; \gamma)]_{\gamma=\gamma(f(x))} \neq h^{\text{ad}(1)}_{\mu \nu; \lambda}(x)\). As a result, we cannot use the adiabatic linear metric perturbation as the metric of the far-zone expansion for the MiSaTaQuWa self-force in general. However, as long as \(|(d/d\lambda)\gamma(\lambda)| < O(1)\) is less severe than (2.8) for the usual metric perturbation scheme and the adiabatic linear metric perturbation is valid in a longer time scale, as we discuss in the following subsections.

An adiabatic linear metric perturbation does not satisfy the linearized Einstein equation (2.5). Hence, it is not defined by the usual metric perturbation scheme. Here we briefly discuss a systematic method to construct a post-adiabatic nonlinear metric perturbation. In addition to the small parameter used in the far-zone expansion \( m/L \), we introduce another small parameter for the adiabatic expansion \( \epsilon \sim |(d/d\lambda)\gamma| < O(1) \), and we expand the metric and the stress-energy tensor as

\[ g_{\mu \nu} = g_{\mu \nu}^{bg} + (m/L) h_{\mu \nu}^{(1),0} + (m/L) \epsilon h_{\mu \nu}^{(1),1} + (m/L) \epsilon^2 h_{\mu \nu}^{(1),2} + \cdots \]

\[ + (m/L)^2 h_{\mu \nu}^{(2),0} + (m/L)^2 \epsilon h_{\mu \nu}^{(2),1} + (m/L)^2 \epsilon^2 h_{\mu \nu}^{(2),2} + \cdots \]

\[ + \cdots, \quad (3.2) \]

\[ T^{\mu \nu} = (m/L) T^{(1),0}_{\mu \nu} + (m/L) \epsilon T^{(1),1}_{\mu \nu} + (m/L) \epsilon^2 T^{(1),2}_{\mu \nu} + \cdots \]

\[ + (m/L)^2 T^{(2),0}_{\mu \nu} + (m/L)^2 \epsilon T^{(2),1}_{\mu \nu} + (m/L)^2 \epsilon^2 T^{(2),2}_{\mu \nu} + \cdots \]

\[ + \cdots, \quad (3.3) \]

In the limiting case \( \epsilon \to 0 \), this metric perturbation scheme becomes the usual metric perturbation scheme, and therefore some results for the usual metric perturbation scheme may apply to \( h_{\mu \nu}^{(n,0)} \) and \( T^{(n,0)}_{\mu \nu} \). We define \( h_{\mu \nu}^{(1),0} \) to be \( h_{\mu \nu}^{\text{ad}(1)} \) and \( T^{(1,0)}_{\mu \nu} \) to be the stress-energy tensor of a monopole particle. In contrast to the usual metric perturbation scheme, \( T^{(1,0)}_{\mu \nu} \) does not necessarily satisfy the conservation law in the background metric by itself, because the violation of the conservation law for \( T^{(1,0)}_{\mu \nu} \) can be compensated for by a higher order expansion with respect to \( \epsilon \). Hence, the orbit of the particle described by \( T^{(1,0)}_{\mu \nu} \) is not necessarily a geodesic of the background, and the expansion of the orbit in the form (2.7) is not necessary in this scheme.

The violation of the conservation law presents a difficulty with regard to deriving equations for post-adiabatic linear metric perturbations, i.e., \( h_{\mu \nu}^{(0,m)} \) with \( m \neq 1 \). Schematically, an adiabatic linear metric perturbation satisfies the equation

\[ G^{(1),0}_{\mu \nu}[(m/L)h^{\text{ad}(1)}] = (m/L) T^{(1,0)}_{\mu \nu} + \epsilon A^{(1,0)}_{\mu \nu}[(m/L)h^{\text{ad}(1)}], \quad (3.4) \]
where $A^{(1,1)\mu\nu}$ appears because $T^{(1,0)\mu\nu}$ does not satisfy the conservation law, while the LHS of (3-4) is algebraically divergence free. One may expect that the relation $G^{(1)\mu\nu}[(m/L)\epsilon h^{(1,1)}] = (m/L)\epsilon T^{(1,1)\mu\nu} - (m/L)\epsilon A^{(1,0)\mu\nu}[(m/L)h^{a(d)}]$ holds for the first post-adiabatic linear metric perturbation $h^{(1,1)}$, where $(m/L)\epsilon T^{(1,1)\mu\nu}$ can be derived appropriately with a matched asymptotic expansion. However, this equation might be inconsistent, because the RHS is not necessarily divergence free, while the LHS is algebraically divergence free. We suppose that, as in the case of (3-4), there is a rank-two symmetric tensor that is $O((m/L)^2)$ and cancels the divergence of the RHS. As a result, we obtain the following equations for the post-adiabatic linear metric perturbations:

$$G^{(1)\mu\nu}[(m/L)\epsilon h^{(1,n)}] = (m/L)\epsilon T^{(1,n)\mu\nu} - \epsilon A^{(1,n-1)\mu\nu}[(m/L)\epsilon_{n-1} h^{(1,n-1)}] + \epsilon A^{(1,n)\mu\nu}[(m/L)\epsilon_{n} h^{(1,n)}].$$  (3.5)

We believe that equations for nonlinear metric perturbations would be derived in a similar way.

### 3.2. Adiabatic extension of the radiation reaction formula

In order to see that the new metric perturbation scheme is valid over a longer time scale than the usual metric perturbation scheme, we extend the argument of the radiation reaction formula. In the radiation reaction formula, a component of a self-force averaged over an infinite time is derived in the usual metric perturbation scheme from the general form of the self-force. It is shown that the loss of orbital energy and angular momentum due to the self-force averaged over an infinite amount of time is equal to that by the momentum flux caused by gravitational waves. Since the self-force is entirely gauge dependent in the usual metric perturbation scheme, we can choose a gauge condition such that the rest of the self-force is small. Then we can say that the self-force of the radiation reaction formula is defined by the averaged energetic gauge condition (2-16) over a time scale for which the usual metric perturbation scheme is valid. We consider application of the adiabatic extension of the linear metric perturbation (3-1) in this gauge condition. Because of the averaged energetic gauge condition (2-16), the radiation energy (2-12) does not decrease by radiation reaction, and therefore we expect that the adiabatic metric perturbation scheme is valid over a time scale longer than that for the usual metric perturbation scheme.

Using the result given in Appendix A, the stress-energy tensor of a geodesic $\gamma = \{E^a, \lambda^b, C^c\}$ can be defined as

$$T_{\gamma}^{\alpha\beta}(x) = \int d\lambda s^{\alpha\beta}(\lambda)\delta(t - z^t)\delta(r - z^r)\delta(\theta - z^\theta)\delta(\phi - z^\phi),$$  (3.6)

$$s^{\alpha\beta}(\lambda) = \sum_{n_r,n_\theta} s^{\alpha\beta(n_r,n_\theta)} \exp[in_r \chi_r + in_\theta \chi_\theta],$$  (3.7)

where the expansion coefficients $s^{\alpha\beta(n_r,n_\theta)}$ are functions of $E^a$. Because a Kerr black hole is a stationary and axisymmetric solution, we can assume that a Green function...
for a linear metric perturbation in the Boyer-Lindquist coordinates can be defined as
\[ G_{\mu\nu,\alpha\beta}(x, x') = \sum_{\omega, m} g_{\mu\nu,\alpha\beta}(r, \theta, \theta', \theta') \exp[-i\omega(t - t') + im(\phi - \phi')] . \tag{3.8} \]

The radiation reaction formula uses a linear metric perturbation derived with the Green method. It is schematically written as
\[ h_{\mu\nu}(x; \gamma) = \sum_{\omega, m, n_r, n_\theta} k_{\omega, m, n_r, n_\theta}(\gamma) \]
\[ \times h_{\mu\nu,\omega, m, n_r, n_\theta}(\mathcal{E}^a; r, \theta) \exp[-i\omega t + im\phi] . \tag{3.9} \]

\[ k_{\omega, m, n_r, n_\theta}(\gamma) = \int d\lambda \hat{k}_{\omega, m, n_r, n_\theta}(\mathcal{E}^a) \exp[i\omega\kappa_t - im\kappa_\phi - in_r\chi_r - in_\theta\chi_\theta] \]
\[ = \frac{2\pi}{\langle \hat{Z}^1 \rangle} \hat{k}_{\omega, m, n_r, n_\theta}(\mathcal{E}^a)e^{i(\omega C^t - mC^a)}e^{-2\pi i(n_r \hat{\Omega}_r \lambda^t + n_\theta \hat{\Omega}_\theta \lambda^\theta)} \]
\[ \times \delta(\omega - 2\pi \Omega_{m, n_r, n_\theta}) , \tag{3.10} \]
\[ \Omega_{m, n_r, n_\theta} = m\Omega_\phi + n_r\Omega_r + n_\theta\Omega_\theta , \tag{3.11} \]
\[ \hat{\Omega}_\phi = \frac{\hat{Z}^\phi}{\langle \hat{Z}^1 \rangle} , \quad \hat{\Omega}_r = \frac{\hat{Z}^r}{\langle \hat{Z}^1 \rangle} , \quad \hat{\Omega}_\theta = \frac{\hat{Z}^\theta}{\langle \hat{Z}^1 \rangle} . \tag{3.12} \]

Here, \( \Omega_\phi, \Omega_r \) and \( \Omega_\theta \) are principal frequencies of the geodesic \( \gamma \), and they are observable in the form of a spectral information concerning gravitational waves at infinity.

Although (3.9) looks general, the derivation of the linear metric perturbation constrains the gauge condition, because one may still add a gauge mode which is not a function of \( t - t' \) or \( \phi - \phi' \) to the Green function. We call this a physically reasonable class of gauge conditions, because one can easily read the principal frequencies from the resulting metric perturbation (3.9), and because it satisfies the averaged energetic gauge condition (2.16).

For the adiabatic approximation of the linear metric perturbation (3.1), we regard \( \gamma \) as dynamical variables. With the foliation function \( f(x) \), we have
\[ h_{\mu\nu}^{ad(1)}(x) = \sum_{\omega, m, n_r, n_\theta} k_{\omega, m, n_r, n_\theta}(\gamma(f(x))) \]
\[ \times h_{\mu\nu,\omega, m, n_r, n_\theta}(\mathcal{E}^a(f(x)); r, \theta) \exp[-i\omega t + im\phi] . \tag{3.13} \]

The adiabatic linear metric perturbation satisfies the equation (3.4). Using the explicit form (3.13) of the adiabatic linear metric perturbation, it is seen that \( A^{(1,0)\mu\nu}[h^{ad(1)}] \) is linear or quadratic in \((d/d\lambda)\gamma(f)\) or linear in \((d^2/df^2)\gamma(f)\). We find that the adiabatic linear metric satisfies the Einstein equation to an accuracy of \( O(m/L) \) when \( |(d/d\lambda)\gamma| \) and \( |(d^2/d\lambda^2)\gamma| \) are small. In the next subsection, we investigate the evolution of \( \gamma \) due to the adiabatic linear metric perturbation and show that the assumption of the adiabaticity is consistent.

\[ ^{1} \] Because the original radiation reaction formula uses a symmetry property of the linear metric perturbation itself rather than the metric perturbation derived with the Green function (3.8), we present a derivation of the self-force in Appendix E.
3.3. Adiabatic evolution of an orbit

Here, we study the orbital evolution caused by the adiabatic metric perturbation (3.13). Under the assumption that \(|(d/d\lambda)\gamma|\) and \(|(d^2/d\lambda^2)\gamma|\) are small, (3.13) approximates the linear metric perturbation, and we can still use the general expression of the self-force given in Ref.2 by simply changing \(\gamma\) to \(\gamma(\lambda)\). This yields

\[
\frac{d}{d\lambda} \mathcal{E}^a = \sum_{n_r,n_\theta} \mathcal{E}^{a(n_r,n_\theta)}(\mathcal{E}^a(\lambda)) \exp[im_r \bar{\chi}_r + im_\theta \bar{\chi}_\theta],
\]

(3.14)

where \(\bar{\chi}_b = 2\pi \bar{\Omega}_b(\mathcal{E}^a(\lambda))(\lambda + \lambda_b(\lambda))\). For the orbital evolution, Ref.2 uses the usual metric perturbation scheme and treats the orbital evolution by expanding the orbit as (2.7), which is constrained by (2.8). In the framework of the adiabatic metric perturbation, we cannot use (2.7). Therefore we must treat the orbit in a nonperturbative manner. With the idea of the adiabatic evolution of the orbit, we consider the evolution of the orbital “constants” instead of the evolution of the coordinates.

Although Ref.2 uses the perturbative expansion of the orbit (2.7), the extrapolation of its result suggests

\[
\mathcal{E}^a - \mathcal{E}^a_0 \sim O((m/L)(t/L)),
\]

\[
\lambda^b - \lambda^b_0 \sim O((m/L)(t/L)^2),
\]

\[
C^c - C^c_0 \sim O((m/L)(t/L)^2),
\]

(3.15)

where \(\{\mathcal{E}^a_0, \lambda^b_0, C^c_0\}\) are the initial values. Thus, the evolution of the primary “constants” \(\mathcal{E}^a\) becomes nonperturbative after the radiation reaction time, \(T_{rad} \sim O(L(m/L)^{-1})\), whereas the evolution of the secondary “constants” \((\lambda^b, C^c)\) becomes nonperturbative after the dephasing time, \(T_{dep} \sim O(L(m/L)^{-1/2})(< T_{rad})\). In Appendix C, we derive evolution equations for the secondary “constants”, assuming that the evolution of the primary “constants” is perturbative, and we find that the approximation (3.15) obtained with the usual metric perturbation scheme is qualitatively correct.

Equation (3.15) shows that the assumption of the adiabaticity is constrained by the conditions \(|(d/d\lambda)(\lambda^b, C^c)| < O(1)\) and \(|(d^2/d\lambda^2)(\lambda^b, C^c)| < O(1)\), and the adiabaticity holds on the radiation reaction time scale, where the primary “constants” still evolve perturbatively. Hence the adiabatic linear metric perturbation satisfies the Einstein equation to an accuracy of \(O(m/L)\) over the radiation reaction time scale \(t < T_{rad}\).

In Appendix D, we integrate the evolution equations in the case that the orbit is either eccentric in the equatorial plane \((\theta = \pi/2)\) or circular in an inclined precessing plane \((r = \text{const})\). In either case, it is shown that the circular (equatorial) orbit remains circular (equatorial) under the influence of the self-force. In Appendix E, we find that the self-force in the adiabatic linear metric perturbation scheme can be used to predict the orbit over the radiation reaction time scale. The dominant part of the orbit is described by only the zero mode of the self-force, \(\hat{\mathcal{E}}^{a(0,0)}\), which is the gauge invariant component of the self-force in the usual metric perturbation scheme. This is a reasonable result of the adiabatic metric perturbation scheme. Because the adiabatic linear metric perturbation (3.13) is a valid metric perturbation
on this time scale, the background metric accurately approximates the full geometry, and the orbit using the background coordinates as an approximate reference must be gauge invariant to leading order. We also find that this is observable. Because the adiabatic metric perturbation scheme describes the metric perturbation together with the orbital evolution caused by the self-force, the modulation of gravitational waves induced by the radiation reaction can be read off of (3.13). In (3.13), we have

$$k(\omega; m, n_r, n_\theta)(\gamma(f)) \sim e^{-i2\pi(n_r\tilde{\Omega}_r + n_\theta\tilde{\Omega}_\theta)(f)}e^{i(\omega C^c(f) - mC^b(f))},$$

(3.16)

and the nonperturbative evolution of ($\chi$, $C^c$) can be observed as a modulation of the wave phase.

The remaining component of the orbit involves nonzero modes of the self-force (3.14), $\mathcal{E}^a(n_r, n_\theta)$, with $(n_r, n_\theta) \neq (0, 0)$. They impart an $O((m/L))$ contribution to the orbit on the radiation reaction time scale. This is the same order as the metric perturbation, and therefore these modes are interpreted as a gauge. In the next section, we show that the nonzero mode of the self-force $\mathcal{E}^a(n_r, n_\theta)$, with $(n_r, n_\theta) \neq (0, 0)$ is gauge dependent in the adiabatic metric perturbation scheme. We also find that the adiabatic nonlinear metric perturbation imparts an $O((m/L)^2((t/L)^2)$ contribution to the orbit. Well within the radiation reaction time scale, this is smaller than $O(1)$, and it is consistent with the fact that the adiabatic linear metric perturbation satisfies the Einstein equation to an accuracy of $O(m/L)$ on the radiation reaction time scale.

Although the above may seem quite reasonable, there is an additional concern when we consider a general orbit. By looking at (3.14), one can easily understand the reason why the gauge invariant component of the self-force $\mathcal{E}^a(0,0)$ imparts a dominant contribution to the orbit. In the adiabatic metric perturbation scheme, $\mathcal{E}^a(0,0)$ slowly changes over the radiation reaction time scale and it coherently contributes to the orbital evolution. Contrastingly, the nonzero modes of the self-force, $(n_r, n_\theta) \neq (0, 0)$ give a small contribution to the orbital evolution, because the phase $n_r\tilde{\chi}_r + n_\theta\tilde{\chi}_\theta$ changes on the dynamical time scale in general, and the contribution cancels over a longer time scale. This is true for a circular or equatorial orbit. However, there could be a case in which the phase $n_r\tilde{\chi}_r + n_\theta\tilde{\chi}_\theta$ of a nonzero mode becomes almost stationary.

This problem appears as a discrepancy between the gauge invariant quantity in the radiation reaction formula and in the balance formula in the framework of the usual metric perturbation scheme. In Ref(2), the gauge invariant component of the self-force is derived by taking an infinite time average over the self-force. The infinite time averaged self-force is derived in the usual metric perturbation as

$$\left\langle \begin{array}{c} d \mathcal{E}^a \\ \Delta \end{array} \right\rangle = \sum \mathcal{E}^a(n_r, n_\theta),$$

where the sum is taken over the values of $n_r$ and $n_\theta$ such that $n_r\tilde{\Omega}_r + n_\theta\tilde{\Omega}_\theta = 0$. On the other hand, Ref(10) shows that the zero mode, $\mathcal{E}^a(0,0)$, is equal to the gauge invariant momentum flux of gravitational waves. In order to resolve this disagreement, it is assumed in Ref(2) that $\tilde{\Omega}_r / \tilde{\Omega}_\theta$ is irrational. In this case, the infinite time averaged component of the self-force becomes equal to the gauge invariant momentum flux of gravitational waves. However, this assumption is not strictly valid. If there is a nonzero mode for which $n_r\tilde{\Omega}_r + n_\theta\tilde{\Omega}_\theta = 0$, the
phase $n_r \tilde{\chi}_r + n_\theta \tilde{\chi}_\theta$ becomes almost stationary, and it may contribute to the orbital evolution. In this case, we may need to develop another formula, since we can only calculate $\hat{\mathcal{E}}^{a(0,0)}$ using the radiation reaction formula $\mathcal{H}$.  

By extending the analysis of Appendic D, we conjecture that such a problem might not arise in the adiabatic metric perturbation scheme and that all we need for an orbital prediction is $\hat{\mathcal{E}}^{a(0,0)}$. The primary “constants” for a general orbit during the radiation reaction time, $t < O(L(m/L)^{-1})$, evolve according to

$$\mathcal{E}^a = \mathcal{E}_0^a + \int_{\lambda_0}^\lambda d\lambda \sum_{n_r, n_\theta} \hat{\mathcal{E}}^{a(n_r, n_\theta)}(E_0) \exp[i n_r \tilde{\chi}_r + i n_\theta \tilde{\chi}_\theta] + O(\mu^2 t). \quad (3.17)$$

During the dephasing time interval $0 < t < O(L(m/L)^{-1/2})$, we have $\tilde{\chi}_b \sim \tilde{\Omega}_b \lambda + \text{const}$. As a result, the phase of any nonzero mode with $|n_r \tilde{\Omega}_r + n_\theta \tilde{\Omega}_\theta| < O(1/L(m/L)^{1/2})$ becomes stationary and gives a coherent $O((m/L)^{1/2})$ contribution to the primary “constants” given in (3.17). Beyond the dephasing time, $\tilde{\chi}_b$ begins to evolve in a nonperturbative manner, and, even if $|n_r \tilde{\Omega}_r + n_\theta \tilde{\Omega}_\theta|$ is small, $n_r \tilde{\chi}_r + n_\theta \tilde{\chi}_\theta$ would not be stationary unless $(n_r, n_\theta) = (0, 0)$. During the time interval satisfying $O(L(m/L)^{-1/2}) < t < O(L(m/L)^{-2/3})$, a mode for which $|(n_r \tilde{\Omega}_r + n_\theta \tilde{\Omega}_\theta) \hat{\mathcal{E}}^{a(0,0)}| < O(1/L(m/L)^2)$ would have a stationary phase. Extending this approximation, during the time interval $O(L(m/L)^{-n/(n+1)}) < t < O(L(m/L)^{-(n+1)/(n+2)})$, a mode for which $|(n_r \tilde{\Omega}_r + n_\theta \tilde{\Omega}_\theta) \hat{\mathcal{E}}^{a(0,0)}|^n < O(1/L(m/L)^{n+1})$ would have a stationary phase, and it would contribute an $O((m/L)^{(n+2)}) < O(1)$ term to (3.17) during this time interval. Thus, only the zero mode $\hat{\mathcal{E}}^{a(0,0)}$ could impart a coherent contribution of $O(1)$ over the radiation reaction time scale.

Before further investigating the gauge condition, we discuss the validity of the adiabatic metric perturbation for an application to the LISA project. The LISA project is planning to detect gravitational waves for several years, and we need theoretical templates of gravitational waveforms valid over such time intervals. It is estimated that the mass ratio of a possible target binary would be $\sim 10^{-5}$. Suppose $m/L = 10^{-5}$ with a dynamical time scale $L = 100s$. Then the radiation reaction time $T_{\text{rad}}$ would be on the order of several months. Hence, a theoretical prediction obtained using the adiabatic linear metric perturbation may not be reliable enough for the LISA project.

In the next section, we argue that fixing the gauge condition improves the validity of an adiabatic linear metric perturbation, and with this improvement one can calculate gravitational waveforms with the adiabatic metric perturbation in a more reliable manner.

§4. Radiation reaction gauge

In this section, we consider a further gauge transformation of an adiabatic metric perturbation. It is found in Ref. [3] that in the usual metric perturbation scheme, the gauge transformation of the self-force acting on the primary “constants” can be written in terms of a total derivative along the orbit. For this reason, the gauge transformation of the primary “constants” can be expressed locally. Because the
argument used here depends on only the properties of the orbital equation, the same result can be used for an adiabatic metric perturbation. However, for technical simplicity, here we first consider the gauge transformation in the usual metric perturbation scheme. Then we extend the result to the case of an adiabatic metric perturbation.

Suppose we transform the coordinates as \( x^\mu \rightarrow x^\mu + \xi^\mu \). Then, to leading order in the gauge vector \( \xi^\mu \), we have the primary “constants” transform as

\[
\delta \mathcal{E}^{E/L} = -\eta^{E/L}_{\alpha \beta} v^\beta \xi^\alpha, \quad \delta \mathcal{C}^C = -\eta^C_{\alpha \beta} v^\beta v^\gamma \xi^\alpha, \quad (4.1)
\]

where \( v^\alpha \) is the 4-vector of the orbit, \( \eta^{E/L}_{\alpha \beta} \) are the Killing vectors used to define the orbital energy and the \( z \) component of the angular momentum, and \( \eta^C_{\alpha \beta} \) is the Killing tensor used to define Carter constant. In the usual metric perturbation scheme, the primary “constants” evolve according to the relation

\[
\mathcal{E}^a = \mathcal{E}^a_0 + \mathcal{E}^a(0,0) \lambda + \sum_{(n_r, n_\theta) \neq (0,0)} \frac{1}{i(n_r \tilde{\Omega}_r + n_\theta \tilde{\Omega}_\theta)} \mathcal{E}^a(n_r, n_\theta) \exp[i(n_r \tilde{\Omega}_r + in_\theta \tilde{\Omega}_\theta) \lambda], \quad (4.2)
\]

where we have excluded the exceptional case that \( \tilde{\Omega}_r / \tilde{\Omega}_\theta \) is rational, as in Ref.3).

We now consider the elimination of the last oscillating part of (4.2) through a gauge transformation. Although \( \xi^\alpha \) is a vector field defined in the entire spacetime, the following analysis considers the condition for \( \xi^\alpha \) along the orbit. Therefore, we consider \( \xi^\alpha \) to be a function of the orbital parameter as \( \xi^\alpha = \xi^\alpha(z(\lambda)) \).

As is shown in Appendix F, the gauge equation (4.1) becomes

\[
\delta \mathcal{E}^a = \sum_{b=E, L, C} A^a_b \frac{d}{d\lambda} \xi^b + B^a_b \xi^b, \quad (4.3)
\]

where \( A \) is a regular 3 \( \times \) 3-symmetric matrix and \( B \) is a 3 \( \times \) 3 anti-symmetric matrix. With these properties, one can formally integrate the gauge equation as

\[
\xi = C^{-1} \int d\lambda CA^{-1} \delta \mathcal{E}, \quad (4.4)
\]

where \( C \) is a matrix schematically defined by

\[
C = \exp(\int d\lambda A^{-1} B). \quad (4.5)
\]

We recall that, because the components of the matrix \( A^{-1} B \) are functions of \( r(\lambda) \) and \( \theta(\lambda) \), their components can be Fourier-decomposed as \( \exp(in_r \chi_r + in_\theta \chi_\theta) \) (see Appendix A). Applying an appropriate mathematical transformation, the matrix \( C \) becomes

\[
C = \exp(ic_1 u_1 + ic_2 u_2), \quad (4.6)
\]

where \( u_1 \) and \( u_2 \) are two independent elements of the \( so(3) \)-algebra, and two coefficients \( c_1 \) and \( c_2 \) are functions of \( \lambda \) that satisfy

\[
c_{1/2} = \dot{\chi}_{1/2} \lambda + \sum c^{(n_r, n_\theta)}_{1/2} \exp(in_r \chi_r + in_\theta \chi_\theta). \quad (4.7)
\]
For this reason, the matrix $C$ can be decomposed into discrete Fourier modes, and we schematically write

$$C = \sum_\omega C_\omega \exp(i\omega \lambda).$$

(4.8)

In order to eliminate the oscillating term of (4.2), we define $\delta E^a$ for (4.4) as

$$\delta E^a = \sum_{n_r,n_\theta} E^{a(n_r,n_\theta)} \exp[in_r \chi_r + in_\theta \chi_\theta] - \delta E^a_0,$$

(4.9)

where $\delta E^a_0$ is a constant. If we set $\delta E^a_0 = 0$, we would have $\xi^a \sim O((m/L)\lambda)$, due to the integration over $\lambda$ in (4.4), and as a result, the gauge transformation of the metric perturbation would grow as $O((m/L)(t/L))$ around the orbit. Such a gauge transformation would cause the metric perturbation to become invalid in a short time in the adiabatic metric perturbation scheme. In order to avoid this, we choose $\delta E^a_0 \neq 0$, so that the gauge transformation (4.4) becomes oscillatory. Then we have

$$E^a = (E^a_0 + \delta E^a_0) + \dot{E}^{a(0,0)} \lambda.$$

(4.10)

This shows that nonzero modes, $\dot{E}^{a(n_r,n_\theta)}$ with $(n_r,n_\theta) \neq (0,0)$, are completely gauge dependent. We recall that the initial value $E^a_0$ has a $O(m/L)$ ambiguity when we include the metric perturbation. Therefore, $\delta E^a_0$ can be renormalized into the initial value, $E^a_0$.

We now consider the self-force in the adiabatic metric perturbation scheme. The self-force under this gauge condition becomes

$$\frac{d}{d\lambda} E^a = \dot{E}^{a(0,0)}.$$  

(4.11)

We refer to this gauge condition as the “radiation reaction gauge”, as the self-force is written solely in terms of the gauge invariant dissipative term. It is interesting to note that a previous calculation employing the balance formula uses the same equation. However, it was considered to be an approximation and a validity of this approximation has been discussed.

The application of this result to a motion of a spinning particle is interesting. A spinning particle does not move along a geodesic due to the spin-curvature coupling, and one may regard this effect as a force:

$$\frac{D}{d\lambda} v^a = F_{\text{spin}}^{a\alpha}.$$  

(4.12)

However, this is a conservative effect, and the conservative force can be eliminated by choosing the radiation reaction gauge, as long as the force is on the order of the metric perturbation, $F_{\text{spin}}^{a\alpha} \sim O(m/L)$. This is the case for a spinning compact object, and the result suggests that we may not observe the spin effect by gravitational wave detection. Although the force resulting from the spin-curvature coupling can be
eliminated, this does not mean that there is no such effect. As (4.10) shows, it is simply absorbed into the initial value. As a result, it alters the condition for the last stable orbit of the binary.

It is interesting that the radiation reaction gauge changes the validity argument for an adiabatic linear metric perturbation. Because \( \frac{d}{d\lambda} \mathcal{E}^a \) does not have nonzero modes, we have \( \frac{d^2}{d\lambda^2} \mathcal{E}^a \sim O((m/L)^2) \) under this gauge condition. We also have \( ((d^2/d\lambda^2)\lambda^b, (d^2/d\lambda^2)C^c) \sim O(m/L) \) and the validity of the adiabatic linear metric perturbation depends on only \( ((d/d\lambda)\lambda^b, (d/d\lambda)C^c) \).

Here we use the post-Newtonian approximation for \( (d/d\lambda)\mathcal{E}^a = \dot{\mathcal{E}}^a(0,0) \), because \( \dot{\mathcal{E}}^a(0,0) \) is a thoroughly studied gauge invariant quantity. Although the post-Newtonian expansion is not a good approximation for a strongly gravitating system, such as an extreme mass ratio binary system, an extrapolation is believed to give a good approximation of the result. In the perturbation limit, we have \( \dot{\mathcal{E}}^a(0,0) \sim O((m/L)v^5) \), where \( v \) is a typical velocity of the system, and \( v^2 \) is estimated to be less than 0.1 during the inspiral stage of the binary. We have \( ((d/d\lambda)\lambda^b, (d/d\lambda)C^c) \sim O((m/L)v^5(t/L)) \), and therefore the assumption of adiabaticity holds for the post-Newtonian radiation reaction time scale, \( t < T_{P,N,rad} \sim O(Lv^{-5}(m/L)^{-1}) \), over which the adiabatic linear metric perturbation could be a solution of the Einstein equation to an accuracy of \( O(m/L) \).

As in the previous section, we suppose \( m/L = 10^{-5} \) with a dynamical time scale \( L = 100s \) for a possible target of the LISA project, and we have \( T_{P,N,rad} \) on the order of several years. Hence, we believe that the adiabatic linear metric perturbation in the radiation reaction gauge should give us a reliable prediction of gravitational waveforms for the LISA project.

§5. Summary

In this paper, we have considered the so-called self-force problem in the calculation of gravitational waves from an extreme mass ratio binary. A strategy previously employed for this problem was that, because we already have a method to calculate gravitational waves for a given stress-energy tensor, we calculate a self-force in order to derive the orbit of a particle. A general form of a self-force called the MiSa-TaQuWa self-force, was derived in the usual metric perturbation scheme. We find that, in the usual metric perturbation scheme, the self-force is completely gauge dependent, and it can even be the case that the self-force vanishes.

In Sec.2 we argue that this is because the usual metric perturbation scheme is defined to allow an arbitrary gauge condition, and as a result, an inappropriate choice of the gauge condition would eventually lead to a nonperturbative evolution of the metric from an energetic point of view. Such a consequence cannot be avoided in the usual metric perturbation scheme, because the scheme requires that the linear metric perturbation must be induced by a background geodesic, and therefore the metric perturbation cannot be tracked with the orbital evolution caused by the self-force. Although the nonperturbative evolution of the metric could be calculated using numerical relativity, it would require a very large computational cost, and for this reason, we consider it advantageous to modify the metric perturbation scheme with
a constraint of a gauge condition for the self-force. In Sec. 3, we propose an adiabatic metric perturbation scheme. In this new metric perturbation scheme, the so-called adiabatic approximation is applied to both the metric perturbation and the orbit. As a result, the metric perturbation is derived together with the orbital evolution caused by the gravitational self-force. We find that the new metric perturbation scheme is valid over the radiation reaction time scale, whereas the usual metric perturbation scheme is valid over only the dephasing time scale, which is shorter than the radiation reaction time scale by the square root of the mass ratio ($\sim 10^{-2.5}$).

The adiabatic approximation has been successfully applied to linear metric perturbations, but it is not a simple problem to apply it to post-adiabatic metric perturbations because the linearized Einstein operator is algebraically divergence free, as we discussed briefly in Sec. 3. Although it is valid for sufficiently small ($d/d\lambda$)$\gamma$ and $(d^2/d\lambda^2)\gamma$, this may pose a question regarding the convergence of the adiabatic extension.

For the self-force problem, there are two approaches: One employs the self-force calculation, which involves an explicit regularization calculation, and the other employs the radiation reaction formula, which is a nontrivial extension of the balance formula. In this paper, we have defined the adiabatic metric perturbation scheme using the radiation reaction formula. One may ask whether an adiabatic extension is possible for the self-force calculation. We believe that this would be nontrivial, because, for a technical reason, the self-force calculation cannot be easily carried out with a linear metric perturbation taking the form (3-9). When the background is a Schwarzschild black hole, one may use the Zerilli-Regge-Wheeler formalism to calculate a metric perturbation. In this case, one can transform the metric perturbation into the form (3-9). However, when the background is a Kerr black hole, a method to calculate an inhomogeneous metric perturbation has only been proposed and it has not yet been implemented numerically. It is not clear that the metric perturbation in this method can be transformed into the form (3-9).

Because all we need for the orbital evolution in the end is a gauge invariant quantity $\dot{E}^a(0,0)$, one may consider the possibility of calculating it by taking a long time average of the result of the self-force calculation assuming that the self-force calculation is implemented under an appropriate gauge condition. Although this may sound a plausible approach, the self-force averaged over a long time $T$ becomes

$$\left\langle \frac{d}{d\lambda}E^a \right\rangle_T \sim \sum \dot{E}^a(n_r,n_{\theta}),$$

where the sum is taken over $n_r$ and $n_{\theta}$ for which $|n_r\tilde{\Omega}_r + n_{\theta}\tilde{\Omega}_{\theta}| < 1/T$ (see (3-14)). Thus, there is a case in which this approach fails to provide $\dot{E}^a(0,0)$.

By contrast, the radiation reaction formula requires only a homogeneous metric perturbation and a conventional method for this calculation is known. Because the necessary calculation is only a slight extension of the balance formula, we consider the numerical technique for this calculation to be established. A homogeneous metric perturbation can be derived by operating with a tensor differential operator on the homogeneous Teukolsky function, and an analytic method for the homogeneous Teukolsky function is known. Because of its fast convergence, this analytic
method gives a more accurate result in a more efficient way.\textsuperscript{[5]}

In Sec.\textsuperscript{[3]} we introduced the radiation reaction gauge, in which the self-force is expressed exactly by the radiation reaction formula (4.11). We recall that the energetic argument given in Sec.\textsuperscript{[2]} shows that the metric could be nonperturbative with an inappropriate choice of the gauge condition for the self-force. Because the usual metric perturbation scheme allows an arbitrary gauge condition, it is valid only over a short time scale. Contrastingly, the validity of the adiabatic metric perturbation depends on the gauge condition, because it solves the metric perturbation and the orbit at the same time. We find that the adiabatic metric perturbation under the radiation reaction gauge condition is valid over a longer time scale than that under a general gauge condition for the radiation reaction formula (3.9). We also note that the evaluation of the validity might not be correct for a highly eccentric orbit.

It is also interesting that a previous study using the balance formula\textsuperscript{[7],[8]} uses the equation (4.11) by the assumption of the adiabatic approximation. In this approach, the adiabatic approximation holds if the orbital period is sufficiently shorter than the dephasing time. However, the analysis in Sec.\textsuperscript{[3]} shows that one can consistently eliminate the oscillating part of the self-force through the gauge, as long as it is small. Hence, the condition for (4.11) must be the same for the adiabatic metric perturbation scheme.

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Appendix A

Geodesic

We consider inspiralling geodesics in a Kerr geometry.\textsuperscript{[2]} A geodesic equation is a set of 2nd order differential equations. Therefore, geodesics are characterized by 6 integral constants. 3 of them are constants of motion, the orbital energy ($E$), the $z$-component of the angular momentum ($L$) and the Carter constant ($C$). In order to distinguish the rest of the constants, we call $E$, $L$ and $C$ the primary constants and denote them by $E^a$, $a = \{E, L, C\}$ when it is appropriate. Using the primary constants, the geodesic equation becomes

$$
\left( \frac{dr}{d\lambda} \right)^2 = (E(r^2 + a^2) - aL)^2 - (r^2 + (L - aE)^2 + C)\Delta,
$$

\textsuperscript{*} A rather sophisticated analysis using Hamilton-Jacobi formalism is given Ref.\textsuperscript{[19]}. Here we summarize another derivation given in Ref.\textsuperscript{[2]} because it is convenient for the analysis given in this paper.
\[ \left( \frac{d\theta}{d\lambda} \right)^2 = C - a^2(1 - E^2) \cos^2 \theta - L^2 \cot^2 \theta, \quad (A.2) \]
\[ \frac{dt}{d\lambda} = -a(aE \sin^2 \theta - L) + (r^2 + a^2)(E(r^2 + a^2) - aL)/\Delta, \quad (A.3) \]
\[ \frac{d\phi}{d\lambda} = -(aE - L\cosec^2 \theta) + (a/\Delta)(E(r^2 + a^2) - aL), \quad (A.4) \]
\[ \Delta = r^2 - 2Mr + a^2. \quad (A.5) \]

Here we use \( \lambda \) as an orbital parameter which has a one-to-one relation with the proper time, \( \tau \), given by \( \tau = \int d\lambda (r^2 + a^2 \cos^2 \theta)^2 \).

For inspiraling geodesics, \( r/\theta \)-motion is bounded and the solutions of (A.1) and (A.2) can be expanded in discrete Fourier series as
\[ z^b = \sum_n Z^{b(n)} \exp\{in\chi_b\}, \quad \chi_b = 2\pi \tilde{\Omega}_b(\lambda + \lambda^b), \quad (A.6) \]
where \( b = \{r, \theta\} \), with \( r = z^r \) and \( \theta = z^\theta \), and \( Z^{b(n)} \) and \( \tilde{\Omega}_b \) are functions of \( E^a \). Here we have two integral constants \( \lambda^b \), but, since we can freely choose the zero point of \( \lambda, \lambda^r - \lambda^\theta \) alone specifies the geodesic. (A.3) and (A.4) are easily integrated, and we obtain
\[ z^c = \kappa_c + \sum_{b,n} Z^{c(n)}_b \exp\{in\chi_b\}, \quad \kappa_c = < \dot{Z}^c > \lambda + C^c, \quad (A.7) \]
where \( c = \{t, \phi\} \) with \( t = z^t \) and \( \phi = z^\phi \), and \( < \dot{Z}^c > \) and \( Z^{c(n)}_b \) are functions of \( E^a \). Here we have two integral constants \( C^c \) which specify the geodesic. We call \( \lambda^b \) and \( C^c \) the secondary constants of the geodesic.

Because of the periodicity, one can freely add \( 1/\tilde{\Omega}_b \) to \( \lambda^b \). In general, the ratio of \( \tilde{\Omega}_r \) to \( \tilde{\Omega}_\theta \) is irrational, and one can take \( \lambda^r - \lambda^\theta \) as small as possible using this freedom. The constants \( C^c \) can be set zero by using the \( t/\phi \)-translation symmetry of the Kerr geometry.

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**Appendix B**

**Self-Force**

We argue that a metric perturbation defined by (3.9) yields a general form of the self-force given in Ref.[2]. In Ref.[2], a general form of the self-force (3.14) is derived, using a symmetry property of the Kerr geometry and a symmetry property of a geodesic. Because the self-force is induced by a geodesic, it is a function of the geodesic parameter \( \lambda \) and the geodesic constants \( \gamma = \{E^a, \lambda^b, C^c\} \) (see Appendix [A] for definitions). Employing the symmetry of the Kerr geometry, Ref.[2] finds that the self-force can be independent of \( C^c \). Then by a symmetry property of a geodesic, the self-force can be expanded in a discrete Fourier series of terms of the form \( \exp\{n_r(\lambda - \lambda^r) + n_\theta(\lambda - \lambda^\theta)\} \). In this appendix, we demonstrate that the gauge condition for the metric perturbation (3.9) actually respects the symmetry of the Kerr geometry by showing that the \( C^c \) derivative of the self-force vanishes.
We assume that the self-force acting on the orbital energy, angular momentum and Carter constant can be derived using a regularization calculation as

\[
\frac{d}{d\lambda} \Delta^a(\lambda) = \lim_{x \to z(\lambda; \gamma)} F^a[h(x; \gamma) - h^{\text{sing}}(x; \gamma)] ,
\]

where \( F^a[h] \) is a tensor derivative operator acting on the metric perturbation \( h_{\mu\nu} \). \( x^{\mu} \) and \( z^{\mu}(\lambda; \gamma) \) are a field point and an orbital point of the geodesic \( \gamma \) respectively. \( h_{\mu\nu}(x; \gamma) \) is defined by (3.9). \( h^{\text{sing}}(x; \gamma) \) is called the “singular part” of the metric perturbation, and it is defined in Ref. under the harmonic gauge condition.

Using the temporal and rotational Killing vectors \( \xi^{\mu}(c) \), where \( c = \{ t, \phi \} \), the symmetry property of the Kerr geometry is defined in terms of the Lie derivative along the Killing vectors as

\[
\mathcal{L}_{\xi^{(c)}} g^{\text{kerr}}_{\mu\nu} = 0 .
\]

The property (3.8) of the full Green function for a metric perturbation is expressed as

\[
(\mathcal{L}_{\xi^{(c)}} + \mathcal{L}'_{\xi^{(c)}}) G_{\mu\nu \alpha\beta}(x, z) = 0 ,
\]

where \( \mathcal{L}_{\xi^{(c)}} \) acts on \( x \) and the indices \( \mu \) and \( \nu \), and \( \mathcal{L}'_{\xi^{(c)}} \) acts on \( z \) and the indices \( \alpha \) and \( \beta \). The property (3.8) is consistent with the linearized Einstein operator \( G^{(1)}_{\mu\nu} \) since it commutes with the Killing vectors, i.e.

\[
[\mathcal{L}_{\xi^{(c)}}, G^{(1)}_{\mu\nu}] = 0 .
\]

It is possible to derive \( h^{\text{sing}}_{\mu\nu}(x; \gamma) \) using the Green method with a singular Green function, and an explicit form of the singular Green function in the harmonic gauge condition is constructed using a bi-tensor expansion in Ref. Because bi-tensors are defined in terms of background geometric quantities, the singular Green function in the harmonic gauge condition can also satisfy the relation

\[
(\mathcal{L}_{\xi^{(c)}} + \mathcal{L}'_{\xi^{(c)}}) G^{\text{sing}}_{\mu\nu \alpha\beta}(x, z) = 0 .
\]

As in Ref., we assume that a gauge transformation acts only on the full Green function. Then, the singular Green function has the property (B-5) in general.

Because the singular behavior of the full Green function cancels with that of the singular Green function, one can define a regular Green function by

\[
G^{\text{reg}}_{\mu\nu \alpha\beta}(x, z) = G_{\mu\nu \alpha\beta}(x, z) - G^{\text{sing}}_{\mu\nu \alpha\beta}(x, z) ,
\]

\footnote{Strictly speaking, there are two types of gauge transformations for the MiSaTaQuWa self-force. One is for the metric of the far-zone expansion and is discussed in Ref. The other is for the metric of the near-zone expansion. The metric of the near-zone expansion is defined as the sum of the black hole metric as a point particle and its perturbation. In Ref., we choose the gauge condition for which the particle is located at \( r = 0 \) of the Schwarzschild coordinates when the \( L = 1 \) mode of the black hole perturbation vanishes. With this condition, the singular part is uniquely determined.}
and the self-force is simply derived as
\[ \frac{d}{d\lambda} \mathcal{E}^a(\lambda) = F^a[h^{reg}](z(\lambda); \gamma), \]
\[ h^{reg}_{\mu\nu}(x; \gamma) = \int dz^4 C^{reg}_{\mu\nu \alpha\beta}(x, z) T^{\alpha\beta}(z; \gamma), \] 
where \( T^{\alpha\beta} \) is the usual stress-energy tensor of a point particle moving along the geodesic \( \gamma \).

Using (B.3) and (B.5), and integrating by parts, we have
\[ \mathcal{L}_{\xi(c)} h^{reg}_{\mu\nu}(x; \gamma) = \int dz^4 C^{reg}_{\mu\nu \alpha\beta}(x, z) \mathcal{L}'_{\xi(c)} T^{\alpha\beta}(z; \gamma). \] 
From the result of Appendix A, we find
\[ \mathcal{L}'_{\xi(c)} T^{\alpha\beta}(z; \gamma) = -\frac{\partial}{\partial C} T^{\alpha\beta}(z; \gamma). \] 
we find that the self-force is independent of \( C \).

**Appendix C**

--- Evolution Equations ---

The evolution equations for the primary “constants” are given by the self-force (3.14). In this appendix, we derive the evolution equations for the secondary “constants” \( (\lambda^b, C^c) \). For convenience, we define the orbit as
\[ z^b = \sum_n Z_b^{(n)} \exp[in\tilde{\chi}_b], \quad z^c = \tilde{\kappa}_c + \sum_{b,n} Z_b^{(n)} \exp[in\tilde{\chi}_b], \] 
where \( Z_b^{(n)} \) and \( Z_b^{(n)} \) are the functions of \( \mathcal{E}^a \) given in (A.6) and (A.7), and consider the evolution equations for \( (\tilde{\chi}_b, \tilde{\kappa}_c) \). We can derive the evolution for \( (\lambda^b, C^c) \) by using \( \tilde{\chi}_b = 2\pi \tilde{\Omega}_b(\lambda + \lambda^b) \) and \( \tilde{\kappa}_c = <Z^c > \lambda + C^c \) (see (A.6) and (A.7)).

Using \( (\tilde{\chi}_b, \tilde{\kappa}_c) \), the self-force (3.14) is written as
\[ \frac{d}{d\lambda} \mathcal{E}^a = \tilde{F}^a = \sum_{n_r, n_\theta} \tilde{\mathcal{E}}^{a(n_r, n_\theta)}(\mathcal{E}^a) \exp[in_r\tilde{\chi}_r + in_\theta\tilde{\chi}_\theta]. \] 
It is important to note that (A.1)-(A.4) hold for non-geodesic orbits, because they are derived from the definitions of \( (E, L, C) \), whether \( (E, L, C) \) are constant or not. Together with (C.2), (A.1) and (A.2) give the evolution equations for \( \tilde{\chi}_b \), and we obtain the evolution equations for \( \tilde{\kappa}_c \) from (A.3) and (A.4).
We rewrite (A-1) and (A-2) as
\[
\left( \frac{dz^b}{d\lambda} \right)^2 = Z^b(\mathcal{E}, z^b). \tag{C.3}
\]
For convenience, we define
\[
\delta_\lambda z^b = \sum_n in2\pi \tilde{\Omega}_b Z^{(n)}_b \exp[in\tilde{\chi}_b], \quad \delta_a z^b = \sum_n Z^{(n)}_{b,a} \exp[in\tilde{\chi}_b], \tag{C.4}
\]
where \(,a\) represents the derivative with respect to \(\mathcal{E}^a\). Then, we have
\[
\frac{dz^b}{d\lambda} = \frac{\delta_\lambda z^b}{2\pi \tilde{\Omega}_b} \left( \frac{d\tilde{\chi}_b}{d\lambda} \right) + \sum_a (\delta_a z^b) \tilde{F}^a, \tag{C.5}
\]
and, from (C.3), we obtain the evolution equations for \(\tilde{\chi}_b\) as
\[
\frac{d\tilde{\chi}_b}{d\lambda} = 2\pi \left( 1 - \frac{\sum_a \delta_a z^b \tilde{F}^a}{\delta_\lambda z^b} \right) \tilde{\Omega}_b, \tag{C.6}
\]
where we have used \((\delta_\lambda z^b)^2 = Z^b(\mathcal{E}, z^b)\).

We next consider the \(t/\phi\)-motion. We rewrite (A-3) and (A-4) as
\[
\frac{dz^c}{d\lambda} = \sum_b Z^c_b(\mathcal{E}, z^b). \tag{C.7}
\]
For convenience, we define
\[
\delta_b z^c = \sum_n in2\pi \tilde{\Omega}_b Z^{(n)}_b \exp[in\tilde{\chi}_b], \quad \delta_a z^c = \sum_{b,n} Z^{(n)}_{b,a} \exp[in\tilde{\chi}_b]. \tag{C.8}
\]
Then, we have
\[
\frac{dz^c}{d\lambda} = \frac{d\tilde{\kappa}_c}{d\lambda} + \sum_b \left( \frac{d\tilde{\chi}_b}{d\lambda} \right) \frac{\delta_b z^c}{2\pi \tilde{\Omega}_b} + \sum_a (\delta_a z^c) \tilde{F}^a, \tag{C.9}
\]
and, from (C.7), we obtain the evolution equations for \(\tilde{\kappa}_c\) as
\[
\frac{d\tilde{\kappa}_c}{d\lambda} = \langle \dot{Z}^c \rangle + \sum_a \left( -\delta_a z^c + \sum_b \frac{\delta_a z^b \delta_b z^c}{\delta_\lambda z^b} \right) \tilde{F}^a, \tag{C.10}
\]
where we have used \(\langle \dot{Z}^c \rangle + \sum_b \delta_b z^c = \sum_b Z^c_b(\mathcal{E}, z^b)\).

In summary, we obtain the evolution equations for \((\lambda^b, C^c)\) as
\[
\frac{d\lambda^b}{d\lambda} = -\sum_a \left( \delta_a z^b + \frac{\tilde{\Omega}_{b,a}}{\Omega_b} (\lambda + \lambda_b) \right) \tilde{F}^a, \tag{C.11}
\]
\[
\frac{dC^c}{d\lambda} = -\sum_a \left( \delta_a z^c - \sum_b \frac{\delta_a z^b \delta_b z^c}{\delta_\lambda z^b} + \langle \dot{\tilde{Z}}^c \rangle \right) \tilde{F}^a. \tag{C.12}
\]
It is notable that, although we have the freedom to choose the zero point for \(\lambda\) and to add \(1/\tilde{\Omega}_b\) to \(\lambda^b\), the last terms of (C.11) and (C.12) eventually grow linearly in \(\lambda\). Therefore, \(((d/d\lambda)\lambda^b, (d/d\lambda)c^c)\) are estimated to be \(O((m/L)(t/L))\), and similarly \(((d^2/d\lambda^2)\lambda^b, (d^2/d\lambda^2)c^c)\) behaves as \(O((m/L)(t/L))\).
We consider integration of the orbital equations (C.2), (C.6) and (C.10) in order to understand the properties of orbits. Here we consider only either an eccentric orbit in the equatorial plane ($\theta = \pi/2$) or a circular orbit in an inclined precessing plane ($r = \text{const}$). In this case, it is sufficient to consider just one of $\tilde{\chi}_b$ (or $\lambda_b$). We also use the fact that the primary “constants” can be treated in a perturbative manner over the radiation reaction time scale.

With an appropriate scaling, the evolution equations (C.2), (C.6) and (C.10) can be reduced to

$$\frac{dx}{dt} = A + O(\mu^2), \quad A = \sum_n a_ne^{iny}, \quad \text{(D.1)}$$

$$\frac{dy}{dt} = 1 + x + B + O(\mu^2), \quad B = \sum_n b_ne^{iny}, \quad \text{(D.2)}$$

$$\frac{dz}{dt} = 1 + x + C + O(\mu^2), \quad C = \sum_n c_ne^{iny}, \quad \text{(D.3)}$$

where $(x, y, z; t)$ corresponds to $(\mathcal{E}^a, \tilde{\chi}^b, \tilde{\kappa}^c; \lambda)$. Here, $\mu = m/L < 1$ is used as an index of the approximation. Thus, $(a_n, b_n, c_n) \sim O(\mu)$. The $O(\mu^2)$ terms of (D.1)–(D.3) come from the nonlinear metric perturbation.

For simplicity, we assume the initial values $x = y = z = 0$ at $t = 0$. From (D.1) and (D.2), we have

$$\frac{dx}{dy} = A + O(\mu^2),$$

and therefore we have

$$x = A^{[1]} + O(\mu^2y), \quad \text{(D.4)}$$

where

$$A^{[1]} = a_0y + \sum_n a_n^{[1]}e^{iny},$$

$$a_0^{[1]} = -\sum_{n \neq 0} a_n^{[1]}, \quad a_n^{[1]} = \frac{1}{in}a_n, \quad (n \neq 0) \quad \text{(D.6)}$$

From (D.3) and (D.2), we have

$$\frac{dz}{dy} = 1 - B + C + O(\mu^2y),$$

and therefore we have

$$z = y - B^{[1]} + C^{[1]} + O(\mu^2y^2), \quad \text{(D.8)}$$
where

\[ B^{[1]} = b_0 y + \sum_n b_n^{[1]} e^{in\phi}, \]
\[ b_0^{[1]} = -\sum_{n \neq 0} b_n^{[1]}, \quad b_n^{[1]} = \frac{1}{in}b_n, \quad (n \neq 0) \quad \text{(D.9)} \]
\[ C^{[1]} = c_0 y + \sum_n c_n^{[1]} e^{in\phi}, \]
\[ c_0^{[1]} = -\sum_{n \neq 0} c_n^{[1]}, \quad c_n^{[1]} = \frac{1}{in}c_n. \quad (n \neq 0) \quad \text{(D.10)} \]

(D.5) and (D.8) show that the evolution of \( x \) and \( z \) with respect to \( y \) is dominantly determined by \( a_0, b_0 \) and \( c_0 \) when \( y > 1 \) and \( y > \mu \), respectively, and the effect of \( a_n, b_n \) and \( c_n \) with \( n \neq 0 \) is only \( O(\mu) \). We also note that the effect of the self-force by a nonlinear metric perturbation becomes \( O(1) \) when \( y \sim \mu^{-1} \), and therefore the approximation loses validity.

(D.2) becomes

\[ (1 - A^{[1]} - B) \frac{dy}{dt} = 1 + O(\mu^2 t), \quad \text{(D.11)} \]

and we can integrate to obtain

\[ y - A^{[2]} - B^{[1]} = t + O(\mu^2 t^2), \quad \text{(D.12)} \]

where we define \( A^{[2]} \) and \( B^{[1]} \) as

\[ A^{[2]} = a_0 y^2/2 + a_0^{[1]} y + \sum_n a_n^{[2]} e^{in\phi}, \]
\[ a_n^{[2]} = \frac{1}{in}a_n^{[1]} (n \neq 0), \quad a_0^{[2]} = -\sum_{n \neq 0} a_n^{[2]} . \quad \text{(D.13)} \]

We can rewrite (D.12) as

\[ y = \frac{1 - a_0^{[1]} - b_0}{a_0} \left[ 1 - \frac{2a_0}{(1 - a_0^{[1]} - b_0)^2}(t + D) \right] + O(\mu^2 t^2), \quad \text{(D.14)} \]

where we define \( D = \sum_n (a_n^{[2]} + b_n^{[1]}) e^{in\phi} \).

(D.14) shows that the evolution of \( y \) with respect to \( t \) is dominantly determined by \( a_0, a_0^{[1]} \) and \( b_0 \) when \( t > \mu \), and the contribution from \( a_n, b_n \) and \( c_n \) with \( n \neq 0 \) is smaller by a factor of \( O(\mu/t) \) relative to the leading order. The approximation becomes invalid when \( t \sim \mu^{-1} \) due to the non-linear self-force. Since \( y \sim \mu^{-1} \) for \( t \sim \mu^{-1} \), the linear self-force can be used to obtain a prediction of the orbital evolution for the time satisfying \( t < \mu^{-1} \). One can further extend (D.14) as

\[ y = \frac{1}{1 - a_0^{[1]} - b_0} (t + D) + \frac{a_0}{2(1 - a_0^{[1]} - b_0)^3} (t + D)^2 + \cdots + O(\mu^2 t^2), \quad \text{(D.15)} \]
\[ = t + (a_0^{[1]} + b_0)t + a_0 t^2/2 + a_0^{[3]} t^2/2 + \cdots + O(\mu, \mu^2 t^2). \quad \text{(D.16)} \]
Since $a_0[i]$ depends on the choice of initial values, this shows that only $a_0$ is essentially important in (D-1). This also suggests that (3-15) is a qualitatively good approximation over the radiation reaction time scale, because $a_0 t^2/2 > a_0^2 t^3/2$.

**Appendix E**

### Gauge Transformation

To evaluate (4-1), we choose the gauge transformation vector as

$$\xi^\alpha = \xi^E \eta^E \alpha + \xi^L \eta^L \alpha + \xi^C \eta^C \alpha \beta v^\beta .$$  \hspace{1cm} (E.1)

Then, the transformation of the primary constants defined in the perturbed space-time becomes

$$\delta \xi^a = \sum_{b=E,L,C} (A^a_b \frac{d}{d\lambda} \xi^b + B^a_b \xi^b),$$  \hspace{1cm} (E.2)

where the $3 \times 3$ matrices $A^a_b$ and $B^a_b$ are defined as

$$A^a_b = \begin{pmatrix}
-\eta^E \eta^E \alpha \\
-\eta^E \eta^L \alpha \\
-\eta^E \eta^C \alpha \beta v^\beta \\
-\eta^L \eta^E \alpha \\
-\eta^L \eta^L \alpha \\
-\eta^L \eta^C \alpha \beta v^\beta \\
-\eta^C \eta^E \alpha \beta v^\beta \\
-\eta^C \eta^L \alpha \beta v^\beta \\
-\eta^C \eta^C \alpha \beta v^\beta \gamma \\
\end{pmatrix} \frac{d\lambda}{d\tau},$$  \hspace{1cm} (E.3)

$$B^a_b = \begin{pmatrix}
0 \\
v_\alpha [\eta^E, \eta^L] \alpha \\
-\eta^C \eta^E \alpha \frac{D}{D\tau} \eta^E \beta \\
0 \\
v_\alpha [\eta^E, \eta^L] \alpha \\
-\eta^C \eta^L \alpha \frac{D}{D\tau} \eta^L \beta \\
0 \\
-\eta^C \eta^C \alpha \frac{D}{D\tau} \eta^C \beta \gamma \\
\end{pmatrix}.$$  \hspace{1cm} (E.4)

For a general orbit, one can assume that the symmetric matrix $A$ has an inverse matrix, since the appropriate linear combinations of $\eta^E \alpha$, $\eta^L \alpha$ and $\eta^C \alpha \beta v^\beta$ give three independent vectors. When the orbit is equatorial (or quasi-circular), the symmetric matrix could be singular. However, since it is sufficient to consider $\{E, L\}$ (or $\{E, C\}$) in this case, the relevant $2 \times 2$ sub-matrix of $A$ still have an inverse.

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