Neutrino refractive effects during their decoupling era in the early universe

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There is an accepted approach to calculation of the neutrino flavor density-matrix in the halo of a supernova, in which neutrino amplitudes, not cross-sections, need to be followed carefully in the region above the region of frequent scatterings. We apply the same reasoning and techniques to the evolution of neutrino flavors and energy distributions in the early universe in the era of neutrino decoupling. It is likely that the resulting changes to the classic numbers that have been used in cosmology will be significantly greater than the ones that are found in recent updates of the theory in this region, in which only cross-sections were used. On a closely related issue: predictions for the production of sterile neutrinos, should they exist, will be changed significantly.

1. INTRODUCTION

There has been an explosion of recent works [1]-[7] with the aim of checking and improving the results of the standard calculations [8] of the neutrino distributions in the big bang during the era of neutrino decoupling. One of the attributes of this body of recent work is that it continues the tradition of using neutrino-neutrino cross-sections in the temperature region around T=1 MeV, instead of following amplitudes in detail. We refer to “refractive effects”, meaning as calculated from forward amplitudes, as are indices of refraction, but here much complicated by many-body effects that dominate at very high number densities.

All the experience from supernova halo calculations says we should be following the amplitudes. In the early universe the coherent amplitude approach is even more necessary, in our view of how the technology works. The basic equations for following amplitudes originate from the formalism of [8], and we follow it here. Finally, in this paper we show that “fast” collective processes [10]-[24] will dominate.

We begin our investigations by looking at a beam of $\bar{\nu}_e$'s moving upward colliding with a beam of $\nu_e$'s that moves downward, maximally violating the initial isotropy and near flavor balance that we shall demand later in this paper. Here, and in everything that follows, by “beams” we mean streams that are very nearly, but not quite, unidirectional. We would like to use all three active $\nu$'s, but because of computational complexities we are limited to two. For now $\nu_e$ is either $\nu_\mu$ or $\nu_\tau$. But we do maintain equal numbers of neutrinos and anti-neutrinos. Later we address initial conditions containing only small number differences or spectral differences between the flavors.

The present work will be applied roughly to the temperature range .1 MeV $< T < 10$ MeV and over time scales such that the $\nu$ masses can be neglected, both in their capacity as flavor mixers, and in the disruption of the coherence that is at the center of our methods. This time scale will be also many of orders of magnitude less than the mean time for ordinary scattering. For the initial neutrino states we take Fermi distributions in energy with chemical potentials equal to zero. Now consider reactions,

$$\nu_e(p) + \bar{\nu}_x(q) \leftrightarrow \bar{\nu}_x(q) + \nu_x(p), \quad (1)$$

and

$$\nu_e(p) + \bar{\nu}_x(q) \leftrightarrow \bar{\nu}_x(p) + \nu_x(q) \quad (2)$$

with momentum “preservation”, not just momentum conservation, and flavors are exchanged (or equivalently with a particle-anti-particle exchange).

Some of these processes were first discussed in ref. [13], demonstrating a fast mode exchange of particle and anti-particle identity in a two-beam simulation. These instabilities care not at all about individual particle energies, as long as all particles are sufficiently relativistic, but they care about the angle between beams through a factor of $1 - \cos \theta$.

And the instability is robust, in a sense that early “fast” process models , e.g. those of the present author [12], were not. In these latter models nature had to provide very specific forms of angular distributions for the instability to appear. The instability that we shall discuss thrives on angular distributions ranging from two beams head-on to isotropic.

But the instability is not seeded by the active neutrino mass matrix, as are those that are discussed in the supernova problem. We shall not plant seeds that violate lepton number conservation in bilinear terms in the fields. However by now we have worked through enough different “quantum break” scenarios [23,24,27], in which mean field equations are unstable in dense systems like the present one, to know that when an initial state is one of unstable equilibrium with number density $n$ and is unseeded, the evolution time is of the order $[nG_F]^{-1}$ times a factor that is at most proportional to the logarithm of the number of participating particles.

2. THE MODEL

We wish to describe the development of systems of $\nu, \bar{\nu}$ that incorporate perfect particle-antiparticle neutrality and isotropy. We consider only time intervals that
are short on the scale of free paths for scattering, and in which the neutrino-neutrino interactions, if unstable, can dominate the dynamics. And in calculating coherent effects we must trace what each quantum state of the multi-particle system does, before adding up to see how the statistical ensemble (which is where the isotropy, e.g., is encoded) evolves.

Let \( a^\dagger \) be the annihilator for \( \nu_e \) with momentum \( \vec{p}_j \), in one incident \( \nu_e \) beam and \( b^k \) the annihilator for a state \( \vec{q}_k \) in the opposed \( \nu_e \) beam; with \( c^j, d^k \) being the annihilators for the respective same two momentum groups of the \( \nu_x \) system. We define the bilinears \( \sigma^j_+ = c^j a^j \) for \( j = 1 \) to \( N \), and \( \tau^k_\pm = \delta^j_k b_k \), for \( k = 1 \) to \( N \), together with their Hermitian conjugates, \( \sigma^j_\mp, \tau^k_\pm \). We shall also use \( \vec{\sigma}^j, \vec{\tau}^k \) as sets of 3-vectors that separately have the commutation relations of independent Pauli spin matrices, e.g. \( [\sigma^j_+, \sigma^k_-] = \delta^{j-k} \sigma^j_+ \) etc.

The standard model (Z exchange) \( \nu - \nu \), effective local Hamiltonian, for our system, leaving out inconsequential terms with no flavor dependence is,

\[
H_{\text{eff}} = \frac{4 \pi G_F}{V} \sum_{j,k} \left[ \sigma^+_j \tau^\pm_k + \sigma^\pm_j \tau^+_k + \frac{1}{2} \sigma^3_j \tau^3_k + \sigma^j_+ \tau^\pm_k + \sigma^\pm_j \tau^+_k - \frac{1}{2} \sigma^3_j \tau^3_k \right. \\
\left. - \sigma^+_j \tau^\pm_k - \sigma^\pm_j \tau^+_k + \frac{1}{2} \sigma^3_j \tau^3_k \right] (1 - \cos \theta_{j,k}).
\]

We refer to this as the effective Hamiltonian instead of \( -\mathcal{L}_T \), because the particle number and kinetic energies are taken as exactly conserved at every step of the process, so that the kinetic term in the Lagrangian is completely irrelevant. The dynamics is all in the flavors, and the wave functions are those of a periodic box of the corresponding \( N \) flavors. The indices \( j, k \) run over momentum states that are initially occupied (and eternally occupied in the framework of this paper). First we consider four beams in all: two in the up direction, initially containing respectively \( \nu_e \) and \( \bar{\nu}_e \), each beam with equal number \( N \); and two in the down direction with the same numbers of \( \nu_e \) and \( \bar{\nu}_e \). The initial states of each flavor The angular spreads are to be narrow enough that we can take \( 1 - \cos \theta = 2 \) when the beams meet roughly head-on.

It is straightforward to derive the equations of motion for the variables in (3), involving nothing more than using the commutation rules of Pauli matrices repeatedly. Under these conditions we can define collective \( \vec{\sigma}, \vec{\tau} \) operators as simple sums over the indices, and rescale them so that their algebra is the same as the Pauli matrices. Then we can rescale the time variable to eliminate the explicit coupling constant in the equations, with a time unit that is \( (\frac{4 \pi G_F}{V})^{-1} \). Then the equations of motion become

\[
i \frac{d}{dt} \sigma^+_j = \sigma^+_j \tau^+ - \tau^+_j + \sigma^+_j \tau^3 - \tau^+_j \tau^3, \\
i \frac{d}{dt} \tau^+_j = \tau^+_j \sigma^+_j - \sigma^+_j \tau^+ + \tau^3 \tau^+_j - \tau^+ \tau^3, \\
i \frac{d}{dt} \sigma^+_j = \sigma^+_j \tau^3 + \tau^+_j \sigma^+_j - \tau^+_j \tau^3 + \tau^3 \tau^+_j, \\
i \frac{d}{dt} \tau^+_j = \tau^+_j \sigma^+_j - \sigma^+_j \tau^+ + \tau^3 \tau^+_j - \tau^+ \tau^3,
\]
and,

\[
i \frac{d}{dt} \sigma^+_j = 2(\sigma^+_j \tau^- - \sigma^-_j \tau^+ - 2(\sigma^+_j \tau^- - \sigma^-_j \tau^+), \\
i \frac{d}{dt} \tau^+_j = -2(\sigma^+_j \tau^- - \sigma^-_j \tau^+ - 2(\sigma^-_j \tau^- - \sigma^+_j \tau^+), \\
i \frac{d}{dt} \sigma^+_j = 2(\tau^- - \tau^+ - 2(\sigma^-_j \tau^- - \sigma^+_j \tau^+), \\
i \frac{d}{dt} \tau^+_j = -2(\sigma^-_j \tau^- - \sigma^+_j \tau^+ + 2(\sigma^-_j \tau^- - \sigma^+_j \tau^+.
\]

with the initial conditions given as discussed below.

Our first exercise of these equations will be to search for the instabilities that characterize fast evolution. We start at \( t = 0 \) in a flavor diagonal system, but then introduce tiny seeds in the variables \( \sigma^\pm \) and \( \tau^\pm \), and solve (4) as linear equations over the early times in which we can assume \( \sigma^\pm_j[0] = \tau^\pm_j[0] \approx 0 \). Then looking at (4) we see that the early change in the four variables \( \sigma^3, \tau^3, \tau^3, \tau^3 \) is quadratic in the seeds, so we can take them as fixed at their initial values for a short time, thus linearizing eqs. (4). Following this plan and taking initial values \( \sigma^3 = \tau^3 = 1, \tau^- = \tau^+ = -1 \), the implied matrix on the RHS of (4) has eigenvalues \( \pm \sqrt{2}i \) and 0, characterizing the “fast” mode of evolution.

In principle we could avoid explicit seeding to enable growth in the analytic way outlined in the simplest examples of refs. [25] and [26], from a “quantum break” calculation, in which we begin by dropping assumptions of the form \( \langle \sigma^+_j \tau^- \rangle = \langle \sigma^+_j \rangle \langle \tau^- \rangle \). This necessitates the introduction of new variables like \( \langle \sigma^+_j \tau^- \rangle \). In [27], in a condensed matter application to systems of 100’s of atoms, an analogous technique is attributed to Bogolyubov [28]. The method is very easy and beautiful for the simplest models in which we remove the anti-neutrino fields and the \( \sigma^3 \tau^3 \) couplings. It is straightforward but already fiercely complex for our present system. For now we use the results of our earlier analyses [25, 26] that find it is nearly equivalent to taking \( \sigma^\pm_j[t = 0] = -\tau^\pm_j[t = 0] = iN^{-1} \) where \( N \) is the number of particles (which could be 10^35), and will enter results only through its logarithm. Following the above plan we show in fig. 1 results for the values \( N = 10^3, 10^5, 10^7, 10^9 \).

The equal spacings indicate an \( N \) dependence of the turn-over time that is linear in \( \log N \). It is a total accident that the base of the logarithm is approximately 10, as one can see with one’s eyeballs from these curves.
Of course it is unpleasant to have this dependence on a non-intensive variable, and in earlier papers we have been slow in seeing what really happens. The behavior was originally built into the seeding that was described above, and confirmed in the solutions based on the exact equation of the wave-function for the N body system for two beams in collision (but for pitifully small N \leq 1000).

A. Isotropy

In the above picture we now include a factor 1 − cos θ_{j,k} with beams in many directions ±\hat{n}_j chosen to mimic isotropy, and the initial conditions, for example with \nu^{(j)}_e in “up” and \bar{\nu}^{(j)}_e in “down”, but now with “up” meaning the direction in which the original +\hat{z} axis ended up after the particular rotation that brought it to a new patch of solid angle, indexed by (j). The plots for transition probability are similar to those of fig. 1, but with a time scale greater by a factor of two. Thus near perfect isotropy, which we demand be maintained in the early universe applications, is achieved with the mere averaging of 1 − cos θ_{j,k} over the sphere.

B. Different initial flavor distributions

For both simplicity of description and necessity in our simulations, we here revert to the (almost) up and down case to discuss different flavor arrangements. The conclusions should be applicable to the isotropic cases as well. The initial conditions for \{\sigma_3, \tau_3, \bar{\sigma}_3, \bar{\tau}_3\}, the operators that are diagonal in the flavor basis, with \sigma_3 the “up” component and \tau_3 the “down”. We find complete flavor turnover for values \{1, 0, 0, −1\}, \{0, 1, −1, 0\}, and for their negatives, we are looking for the fast process that could apply to a system with all four possible flavors having nearly the same occupancy. So we try \{1, −1, −1, −1\} in which all four states appear, but some are going up and some are going down; there is still total turnover.

Now in our multi-beam equations with N_a = 4, we go to four times the number of beams in each direction, with respective initial conditions \{1, 0, 0, −1\}, \{−1, 0, 0, 1\}, \{0, 1, −1, 0\}, \{0, −1, 1, 0\} . This would be the ordinary total equilibrium case, if we stipulated that each sub-beam has the same Fermi distribution in momenta, with zero chemical potential. Now each sub-beam turns over separately to its negative. And we see that nothing has happened when we add up the results for any variable. Since each beam begins as a Fermi distribution in absolute momenta with zero chemical potential, and since the sets of absolute momenta in any two beams are nearly disjoint, we appear to have started in an equilibrium state, and we expect it to stay where it is.

But something actually did happen: The individual momenta, q_i in each sub-beam are drawn from that Fermi distribution at random, and the beams are almost completely disjoint in their exact momentum list. Therefore in the common turnover time for the sub-beams we
come back to the same “state”, as defined by their flavors, but one in which the sub-beam of a particular flavor is microscopically constituted from a nearly completely different set of momenta.

Now suppose that we begin with a small deviation from equilibrium, for example in the $\nu_e$, $\bar{\nu}_e$ spectra, caused by interactions with $e_\pm$ as the $\nu$’s gradually decouple from the electrons. From the above we can infer that this distortion would be shared with $\nu_x$, $\bar{\nu}_x$‘s almost instantly over one single cycle of the fast mechanism. We say “almost instantly” because, at in the range $T=1$-$3$ MeV the period for the near square wave response is in the range of $10^{-9}$ times the neutrino-neutrino collision time. And we might go on to say, e.g., “then probably the small corrections to the entropy conserving (11/4)$^{1/3}$ ratio of neutrino to photon temperature will be completely eliminated”. We need to do a 3-flavor extension of the present work, before making absolute claims about rapidly bringing the $\nu_e$, $\nu_x$, $\bar{\nu}_x$, numbers into near exact equality, but we think that this is highly likely.

C. Initiation and extinction

Much more analysis should be devoted to initiation and extinction, before reaching firm conclusions, however. It may be that after the first sub-system reversal, as we get to the time when the system approaches its completely flipped over initial state ($\zeta = 1$ in fig. 1, half way through this plateau) the story ends due to tiny random fluctuations in the environment. We have found that even a small change in $t=0$ conditions, like $\{1, 0, 0, -1\} \rightarrow \{1, 0, 0, -0.98\}$, which leaves the first 1/2 cycle intact ends in immediate frequency doubling in the subsequent cycles. And, plausibly, more drastic effects will disrupt the near return to the original state, with its off-diagonal terms of order $N^{-1}$.

Indeed our favored picture is that at random times there will be these pop-up short range events in every direction that do the equilibration little by little, in each direction. Surely thinking about incubation and extinction of our modes is easier in this venue than it is in the supernova application. The early universe has the advantage, because of its translational invariance, isotropy and strict balancing of number of $\nu$‘s and $\bar{\nu}$‘s.

On the general subject of initiation and extinction, there is a lot that still needs to be understood. One observation, backed up by simulations, lets us relax a little: In our computational approach it may appear that we add another beam of just one $\nu$ with flavor $\nu_e$ and momentum $q$, just going up. We evolve it in flavor in the familiar way. The back-reaction on the rest of the system, in which even in our simulations we take $10^7$ particles in a beam (it is the number of beams that is limited), is spread over all of these particles and is negligible. Then we easily find that in the same turn-over time for everyone else the $\nu_e$ has become an equal mixture of $\nu_x$ and $\bar{\nu}_x$.

The usual picture for sterile coupling, with a prescribed sterile mass that is much greater than a neutrino mass and a mixing angle $\theta$ (say just for the $\nu_e$) that is small, is that the oscillations are interrupted by a collision of the $\nu_e$ in the medium, shaking loose the sterile component, which we name $\nu_s$, to live forever. But in the domain $T=1$-$3$ MeV the average time between collisions is of order $10^8$ times our turnover times.

Can our repeated sudden flavor changes in, say, a mode $q$ lead to the conversion to a sterile in this same mode? We assume e.g. that the coupling of the sterile is to the mode $\nu_e$ and not to $\nu_x$. We have introduced the additional variables needed to add the sterile mass and coupling terms into $H_{\text{eff}}$, and derived a bigger equations set. Four new variables are required to achieve a closed set of first order equations. Details are given in a supporting document for this paper. For results we have found a thicket of possible behaviors, of which we plot just one

D. Adding active-sterile coupling

Still in the up-and down simplification we go to the equilibrium world of the 4-beam calculation described above, in which the net result was that nothing happened, except some effective shuffling of momenta. Now we add another beam of just one $\nu$ with flavor $\nu_e$ and momentum $q$, just going up. We evolve it in flavor in the familiar way. The back-reaction on the rest of the system, in which even in our simulations we take $10^7$ particles in a beam (it is the number of beams that is limited), is spread over all of these particles and is negligible. Then we easily find that in the same turn-over time for everyone else the $\nu_e$ has become an equal mixture of $\nu_x$ and $\bar{\nu}_x$.
in fig. 2, where we choose (very roughly) a sterile mass of 1KeV, a mixing parameter \( \sin \theta = 10^{-5} \), and temperature region \( T = 1-3 \) MeV (to determine the appropriate \( \nu \) number densities).

![FIG. 2:](image)

Dashed curve shows the development of \( \sigma_3 \) coming from solutions of (4) and (5); essentially one of the curves of fig.1 extended over six periods, at which point the stimulating fast background oscillation turns itself off. The solid curve is for \(-10^9 \times \rho \) that a sterile \( \nu \) has been created by that time.

The plots show the dramatic effect of suddenly ending the driving oscillations, and apparently leaving in their place some long term sterile probability that qualifies as real production, in view of the constant component. They also show the chaos wrought on the active-sterile oscillation by the short burst of non-linear oscillations for the small effects of existing corrections to the complete equilibrium calculation, with its \((11/4)^{1/3}\) temperature ratio, since faster processes should contribute to faster equilibration. Probably equalization of neutrino flavors and their spectra will follow as well. The subject demands being picked up by a strong group with vastly better computing facilities, but it is still straightforward and could be solved expeditiously, as needed for possible data confrontation.

Pursuing the possibility of making steriles, or of putting limits on sterile parameters, appears to be a harder slog, because it depends more critically on initiation and extinction issues. But it certainly demands further attention.

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