LIFESPAN OF SOLUTIONS FOR A WEAKLY COUPLED SYSTEM OF SEMILINEAR HEAT EQUATIONS

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Abstract. We introduce a straightforward method to analyze the blow-up of solutions to systems of ordinary differential inequalities, and apply it to study the blow-up of solutions to a weakly coupled system of semilinear heat equations. In particular, we give upper and lower estimates of the lifespan of the solution in the subcritical case.

1. Introduction

In this paper, we consider a weakly coupled system of semilinear heat equations

\[
\begin{align*}
\partial_t u - \Delta u &= F(u), \quad \text{for} \quad t \in [0, T), \quad x \in \mathbb{R}^n, \\
u(0, x) &= u_0(x), \quad \text{for} \quad x \in \mathbb{R}^n
\end{align*}
\]

with \(n \geq 1\). Here \(u = (u_1, u_2, \ldots, u_k) : [0, T) \times \mathbb{R}^n \to \mathbb{R}^k\), is an \(\mathbb{R}^k\)-valued unknown function with \(k \geq 1\). The nonlinearity \(F\) is defined as

\[F(u) = (F_1(u), F_2(u), \ldots, F_k(u)), \quad F_j(u) = |u_j+1|^{p_j}, \quad p_j \geq 1 \quad (j = 1, 2, \ldots, k),\]

where \(u_{k+1}\) is interpreted as \(u_1\). Also, \(u_0 = (u_{0,1}, \ldots, u_{0,k}) \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)\) is a given initial data.

The system (1.1) with \(k = 2\) was introduced by Escobedo and Herrero [4] as a simple model of a reaction-diffusion system, which can describe a heat propagation in a two-component combustible mixture. Later on, many authors studied the system (1.1) and determined the so-called critical exponent. Here the critical exponent is defined in the following way. Let \(\mathcal{P}\) be a \(k \times k\) matrix defined by

\[
\mathcal{P} = \begin{pmatrix}
0 & p_1 & 0 & \cdots & 0 \\
0 & 0 & p_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & p_{k-1} \\
p_k & 0 & \cdots & 0 & 0
\end{pmatrix}
\]

and let \(I\) be the \(k\)-th identity matrix. If \((p_1, \ldots, p_k) \neq (1, \ldots, 1)\), then it is easy to see that \(|\mathcal{P} - I| \neq 0\), which enables us to define

\[
\alpha = \sum_{1 \leq j \leq k} \alpha_j = (\mathcal{P} - I)^{-1} \cdot \mathbf{1}(1, \ldots, 1).
\]

Here for each \(j\), \(\alpha_j\) is explicitly given by

\[
\alpha_j = \frac{\sum_{k=0}^{j-2} \prod_{m=0}^{h} p_{j+m} + 1}{\prod_{m=1}^{j} p_m - 1},
\]

where for \(j > k\), \(p_j\) is interpreted as \(p_{j-k}\). Let \(\alpha_{\max} = \max_{1 \leq j \leq k} \alpha_j\). It is known that if \(\alpha_{\max} < n/2\), then the system (1.1) admits a unique global solution for small
initial data; if \( \alpha_{\text{max}} \geq n/2 \), then solutions to (1.1) blow up in a finite time for any nontrivial nonnegative initial data. In this sense, the relation \( \alpha_{\text{max}} = n/2 \) is called the critical exponent (see [14,15,18,21,22]). This is a natural extension of the pioneering works by [7,12,15,24] for the single semilinear heat equation because if \( k = 1 \), then \( \alpha = 1/(p-1) \) and the critical case is given when \( p = 1 + 2/n \), which is the well-known Fujita exponent.

The reason why the exponent \( \alpha \) is related to the critical exponent is explained by the scaling argument (the following argument is also found in [19]). If \( u \) is a solution to (1.1), then so is \( u^{(\lambda)} \) for any \( \lambda > 0 \), where for \( 1 \leq j \leq k \),

\[
(1.3) \quad u^{(\lambda)}_j := \lambda^{2\alpha_j} u_j(\lambda^2 t, \lambda x).
\]

Moreover, the \( L^1 \)-norm of the initial data \( \|u^{(\lambda)}_j(0, \cdot)\|_{L^1} = \lambda^{2\alpha_j-n}\|u_0,j\|_{L^1} \) is invariant under the critical condition \( \alpha_j = n/2 \).

Here, we remark that the invariant scaling transformation (1.3) implies that when \( k = 1 \), for any \( \lambda > 0 \),

\[
(1.4) \quad T_m(u_0)\|u_0\|_{L^1}^{1/(1/(p-1)-n/2)} = T_m(u_0)\lambda^{-2}\|u^{(\lambda)}(0, \cdot)\|_{L^1}^{1/(1/(p-1)-n/2)} = T_m(u^{(\lambda)}(0, \cdot))\|u^{(\lambda)}(0, \cdot)\|_{L^1}^{1/(1/(p-1)-n/2)}
\]

where

\[
T_m(u_0) := \sup\{T > 0; \text{With the initial data } u_0, \text{ there exists a unique solution } u \in C([0, T); L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)) \to (1.1)\}.
\]

Since \( \lambda > 0 \) is arbitrary, from (1.4), when \( k = 1 \), it is expected that for some \( 0 < c < C \),

\[
(1.5) \quad c\|u_0\|_{L^1}^{-1/(1/(p-1)-n/2)} \leq T_m(u_0) \leq C\|u_0\|_{L^1}^{-1/(1/(p-1)-n/2)}.
\]

Indeed, the first estimate of (1.5) holds for any \( n \geq 1 \) and non-negative \( u_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \) when \( p < p_F \), provided that \( \|u_0\|_{L^1} \) is replaced by \( \|u_0\|_{L^1 \cap L^\infty} \). For details, see Proposition 1.3 below.

The aim of this paper is to prove the blow-up of solutions to (1.1) by a straightforward approach of ordinary differential equation (ODE). Specifically, a blow-up of solutions to (1.1) follows from the study of the following ODE system:

\[
(1.7) \quad \frac{df}{dt}(t) = \tilde{F}(t, f(t)), \quad \text{for } 0 < t < T.
\]

A general approach to study ODE systems is to find some function \( G : \mathbb{R}^k \to \mathbb{R} \) such that a single ODE of \( G \) follows from (1.7). In particular, we may find a function \( G \) satisfying that

\[
(1.8) \quad \frac{d}{dt}G(f(t)) \geq CG(f(t))^{\gamma},
\]

with some positive constants \( C \) and \( \gamma \), which may imply that \( G \) blows up at a finite time. For example, Mochizuki [17] showed that solutions to (1.1) blow up when \( k = 2 \), by studying (1.7) with \( \tilde{F} = F \) and \( G(f) = f_1 \). Indeed, Mochizuki obtained an ODE for \( f_1 \) by connecting two ODEs of (1.7), with the following identity:

\[
(1.9) \quad \frac{1}{p_2 + 1} \frac{d}{dt}(f_1^{p_2+1}) = f_1^{p_2} f_2^{p_2} = \frac{1}{p_1 + 1} \frac{d}{dt}(f_2^{p_1+1}).
\]
By combining (1.7) and (1.9), one formally has

\[(1.10) \quad \frac{d}{dt}f_1(t) = f_2(t)^{p_1} \sim f_1(t)^{\frac{p_1(p_1+1)}{p_1+1}} = f_1(t)^{\frac{1}{\alpha}+1},\]

from which a sharp lifespan estimate for (1.1) is obtained. Here we say that a lifespan estimate is sharp if \(p_j = p < p_F\) and \(u_{0,j} = v_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)\) for any \(j\), then \(T_m(u_0)\) satisfies (1.10). However, for \(k \geq 3\), there is no identity like (1.9) which unites equations of (1.7). In the case where \(k \geq 3\), Filo and Quittner [6] studied (1.1) by an ordinary differential inequality (1.8) with \(G(f) = \sum_{j=1}^{k} f_j\) and discussed the blow-up rate of solutions near blow-up time (See Lemma 2.1 in [6]). However, with this choice of \(G\), it seems difficult to obtain the sharp lifespan estimate for (1.7) and consequently also for (1.1) as well. On the other hand, Wang [23] obtained the blow-up rate of solutions to (1.1) on a bounded domain in \(\mathbb{R}^n\) with \(k \geq 3\) by (1.8) with \(G(f) = \prod_{j=1}^{k} f_j\) (See the proof of Theorem 1 in [23]). However, it also seems difficult to get the sharp lifespan estimate with \(G(f) = \prod_{j=1}^{k} f_j\).

In this paper, so as to obtain a sharp lifespan estimate, we avoid using a function \(G\) such as \(G(f) = \prod_{j=1}^{k} f_j\) or \(G(f) = \sum_{j=1}^{k} f_j\). On the other hand, we introduce a concatenation of equations of (1.7) derived by weakly coupled interaction and show that for each \(j\),

\[\frac{d}{dt}f_j(t) \geq C f_j(t)^{\frac{1}{\alpha_j}+1}.\]

This is a natural extension of the idea of (1.10).

Before stating our main results, we recall the local existence of the solution.

**Proposition 1.1.** For any \(p_j \geq 1\) (\(j = 1, \ldots, k\)) and \(u_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)\), there exist \(T > 0\) and a unique solution \(u \in C([0, T); L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n))]\) to (1.1), which satisfies (1.1) in the classical sense for \(0 < t < T\).

Proposition 1.1 may be obtained by a simple modification of the proof of Theorem 2.1 in [4]. Next we define lifespan of solutions to (1.1), in a similar manner to (1.5), by

\[T_m := \sup\{T > 0; \text{There exists a unique solution,} \ u \in C([0, T]; L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)))\} \to \text{(1.1)}\].

As a corollary of Proposition 1.1 we have the following blow-up alternative.

**Corollary 1.2.** If \(T_m < \infty\), then we have \(\lim_{t \to T_m} - 0 \|u(t)\|_{L^1 \cap L^\infty} = \infty\).

Moreover, from Proposition 1.1 and a standard a priori estimate, we have the lower bound of the lifespan.

**Proposition 1.3.** In addition to the assumptions of Proposition 1.1, we further assume \((p_1, \ldots, p_k) \neq (1, \ldots, 1)\). Then, there exist constants \(\varepsilon_0 > 0\) and \(C > 0\) such that for any \(u_0\) satisfying \(\|u_0\|_{L^1 \cap L^\infty} \leq \varepsilon_0\), the lifespan satisfies

\[T_m \geq \begin{cases} \infty & (\alpha_{\text{max}} < n/2), \\ C \|u_0\|_{L^1 \cap L^\infty}^{-1/(\alpha_{\text{max}} - n/2)} & (\alpha_{\text{max}} > n/2). \end{cases}\]

We give a proof of Proposition 1.3 in Section 4.

**Remark 1.1.** In the critical case where \(\alpha_{\text{max}} = n/2\), we can also have the estimate \(T_m \geq \exp(C \|u_0\|_{L^1 \cap L^\infty}^{-(\min_j p_j - 1)})\) in the same way. However, it seems not optimal and we do not pursue here the critical case.
In order to state the main blow-up result of this paper, we introduce some notation. Let \( \lambda > 0 \) and nonnegative \( \phi \) satisfy \( \| \psi \|_{L^1} = 1 \) and

\[
\frac{\lambda}{2} \psi(x) = \begin{cases} \Delta \psi(x) > 0, & \text{if } |x| \leq 1, \\ 0, & \text{if } |x| \geq 1. \end{cases}
\]

Namely, \( \lambda/2 \) is a positive eigenvalue of Dirichlet Laplacian on the unit disc, and \( \psi \) is a null-extension of normalized positive eigenfunction. We put \( \phi_R(x) = \psi(x/R)^2 \) with \( R > 0 \). Then

\[
\begin{cases} R^{-2} \lambda \phi_R(x) \geq -\Delta \phi_R(x), & \text{if } |x| \leq R, \\ \phi_R(x) = \nabla \phi_R(x) = 0, & \text{if } |x| \geq R. \end{cases}
\]

At last, for solutions to (1.1), we define

\[
U_{j,R}(t) = \int_{\mathbb{R}^n} u_j(t,x) \phi_R(x) dx \quad (j = 1, \ldots, k).
\]

Then we have the following estimate.

**Theorem 1.4.** Let \( p_j \geq 1 \) for any \( 1 \leq j \leq k \) but let \( p_j > 1 \) for some \( j \). Let \( u_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \) satisfy that \( u_0,j \) is nonnegative for any \( j \). Let \( u \in C([0,T_m); L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)) \) be the corresponding solution of (1.1). If there exists \( j_0 \in \{1, \ldots, k\} \) such that \( u_{0,j_0} \neq 0 \) and

\[
\alpha_{j_0} > \frac{n}{2},
\]

then \( u \) cannot exist globally and there exists \( R_0 > 0 \) such that

\[
T_m \leq C_0 U_{j_0,R_0}(0)^{-1/(\alpha_{j_0}-n/2)}.
\]

with some constant \( C_0 \).

**Remark 1.2.** (i) \( R_0 \) is determined by

\[
U_{j_0,R_0}(0) = C_1 R_0^{-2\alpha_{j_0}+n}
\]

with a certain constant \( C_1 \). (1.11) implies existence of \( R_0 \) satisfying (1.13) for any nonnegative \( u_{0,j_0} \in L^1(\mathbb{R}^n) \backslash \{0\} \), since \( f : R \mapsto U_{j_0,R}(0) \) is an increasing function for \( R \geq 0 \) with \( \lim_{R \to 0} f(R) = 0 \).

(ii) Theorem 1.4 seems to be sharp from the viewpoint of the scaling (see 1.6).

(iii) The proof of Theorem 1.4 implies that

\[
U_{j,R_0}(t) \geq C_j (T_0 - t)^{-\alpha_j}
\]

for any \( 1 \leq j \leq k \) when \( t \in (0,T_0) \) close to \( T_0 \), where \( C_j \) is a positive constant and \( T_0 \) is the RHS of (1.12). However, Theorem 1.4 does not give the exact blow-up rate because \( T_0 \) is nothing but an upper bound of blow-up time (see 2).

In Section 2, we prepare blow-up results for a system of ODEs. Then, in Section 3, we show Theorem 1.4 by combining a test function method developed by [3, 8, 11, 16, 25] with the ODE argument discussed in Section 2. Section 4 is devoted to the proof of Proposition 1.3.
2. ODE ARGUMENT

In this section, for \( k \geq 2, \ p_j \geq 1 \ (j = 1, \ldots, k) \) with \((p_1, \ldots, p_k) \neq (1, \ldots, 1)\), and \( \bar{\lambda} \geq 0 \), we consider the following ODE system:

\[
\begin{cases}
\frac{d}{dt} f_j(t) \geq \tilde{C}_j f_{j+1}(t)^{p_j}, & \text{for } 1 \leq j \leq k - 1, \ t \in [0, T), \\
\frac{d}{dt} f_k(t) \geq \tilde{C}_k e^{-\bar{\lambda} t} f_1(t)^{p_k}, & \text{for } t \in [0, T).
\end{cases}
\]  

(2.1)

where \( \tilde{C}_j > 0 \) for any \( j \).

In order to state the blow-up statement for (2.1), we introduce the following notation. For \( 1 \leq j < k \), let

\[
P_{k-j} = \sum_{h=0}^{j} \prod_{m=0}^{h} p_{k-j+m} + 1, \quad Q_{k-j} = \sum_{h=0}^{j-1} \prod_{m=0}^{h} p_{k-j+m} + 1,
\]

and let

\[
P_k = p_k + 1, \quad Q_k = 1.
\]

Then \((P_{k-j})_{j=1}^{k-1}\) and \((Q_{k-j})_{j=1}^{k-1}\) satisfy that for any \( 1 \leq j < k \),

\[
P_{k-j} = p_{k-j} P_{k-j+1} + 1, \quad Q_{k-j} = p_{k-j} Q_{k-j+1} + 1,
\]

and

\[
P_{k-j} - Q_{k-j} = \prod_{m=k-j}^{k} p_t.
\]

(2.2)

By (1.2), (2.4), and the definition of \( Q_1 \) and \( P_2 \),

\[
\alpha_1 = \frac{Q_1}{P_1 - Q_1 - 1}, \quad \alpha_2 = \frac{P_2}{P_1 - Q_1 - 1}.
\]

(2.5)

For \( 1 \leq j < k \), we put

\[
A_{k-j} = P_{k-j}^{-1} \bar{C}_{k-j} \prod_{h=0}^{j-1} \left( P_{k-h}^{-1} \bar{C}_{k-h} \right)^{p_{k-h}^{-1}} \prod_{h=k}^{j-1} p_t,
\]

\[
L_{k-j} = \bar{\lambda} \prod_{m=k-j}^{k} p_m.
\]

(2.6)

and let

\[
A_k = P_k^{-1} \bar{C}_k, \quad L_k = \bar{\lambda}.
\]

(2.7)

Then for \( 1 \leq j \leq k - 1 \),

\[
A_{k-j} = P_{k-j}^{-1} \bar{C}_{k-j} A_{k-j+1}^{p_{k-j}}, \quad L_{k-j} = p_{k-j} L_{k-j+1}.
\]

(2.8)

Moreover, let

\[
\bar{C} = L_1^{-1} (P_1 - Q_1 - 1)^{2 - p_1} (P_2 - 1)^{1/Q_1} (P_1^{1/Q_1} A_1^{1/Q_1}).
\]

(2.9)

Now we are in the position to state our blow-up statement for (2.1).
Proposition 2.1. If \( f_j(0) \geq 0 \) for \( j \neq 2 \) and
\[
(2.9) \quad f_2(0) > C_1^{-\alpha_2} A_2^{\alpha_2/(\alpha_1 p_2)} C_1^{-\alpha_2 Q_2/(\alpha_1 p_2)},
\]
then solutions \( f_j \) to \((\text{2.1})\) satisfy that
\[
(2.10) \quad f_1(t) \geq C_1^{-1/\alpha_1}(e^{-L_1 t/Q_1} - e^{-L_1 \tilde{T}_0/Q_1})^{-\alpha_1} - A_2^{-1/p_2} C_1^{-Q_2/p_2} f_2(0)^{-1/\alpha_2},
\]
where
\[
\tilde{T}_0 = -L_1^{-1} Q_1 \log(1 - C_1^{-1} A_2^{1/(\alpha_1 p_2)} C_1^{-Q_2/(\alpha_1 p_2)} f_2(0)^{-1/\alpha_2}).
\]

Remark 2.1. Under the condition \((\text{2.7})\), we have \( \tilde{T}_0 > 0 \).

Remark 2.2. Proposition \((\text{2.7})\) implies that if \( t \) is close to \( \tilde{T}_0 \), the minorant of \( f_1(t) \) takes the form of \((\tilde{T}_0 - t)^{-\alpha_1} \). Then if \( f_k \) has a sense when \( t \) is close to \( \tilde{T}_0 \), we have
\[
\frac{d}{dt} f_k(t) \geq C(\tilde{T}_0 - t)^{-p_k \alpha_1},
\]
and therefore \( f_k \) may satisfy
\[
f_k(t) \geq C(\tilde{T}_0 - t)^{-\alpha_k}
\]
for \( t \) close to \( \tilde{T}_0 \). Indeed,
\[
p_k \alpha_1 - 1 = p_k \sum_{h=0}^{k-2} \frac{\prod_{\ell=0}^{h} p_{k+1+\ell} + 1}{\prod_{\ell=1}^{k} p_{\ell} - 1} - 1
= \frac{\sum_{h=0}^{k-3} \prod_{\ell=0}^{h+1} p_{k+\ell} + p_k + 1}{\prod_{\ell=1}^{k} p_{\ell} - 1}
= \frac{\sum_{h=0}^{k-2} \prod_{\ell=0}^{h} p_{k+\ell} + 1}{\prod_{\ell=1}^{k} p_{\ell} - 1} = \alpha_k,
\]
where we put \( p_{k+j} = p_j \) for \( 1 \leq j < k \). Similarly, we have
\[
f_j(t) \geq C(\tilde{T}_0 - t)^{-\alpha_j}
\]
for \( t \) close to \( \tilde{T}_0 \) as long as \( f_j \) has a sense.

In order to prove Proposition \((\text{2.1})\) at first, we recall the comparison principle for ODE systems.

Lemma 2.2 \((\text{13})\). Let \( F_j : [0, T) \times \mathbb{R} \rightarrow \mathbb{R} \) satisfy that if \( x > y \),
\[
F_j(t, x) > F_j(t, y)
\]
holds for any \( 1 \leq j \leq k \) and \( t \in [0, T) \). Let \((f_j)_{j=1}^{k}, (g_j)_{j=1}^{k} \subset C^1([0, T) ; \mathbb{R})\) satisfy that for any \( 1 \leq j \leq k \),
\[
\frac{d}{dt} f_j(t) \geq F_j(t, f_j(1)(t)), \quad \frac{d}{dt} g_j(t) \leq F_j(t, g_j(1)(t)), \quad \text{for } t \in [0, T)
\]
and
\[
f_j(0) \geq g_j(0),
\]
where \( f_{k+1} = f_1 \) and \( g_{k+1} = g_1 \). If \( f_l(0) > g_l(0) \) for some \( 1 \leq l \leq k \), then \( f_j(t) > g_j(t) \) holds for any \( t \in (0, T) \) and \( 1 \leq j \leq k \).

For completeness, we prove Lemma \((\text{2.2})\).
Proof. Without loss of generality, we assume \( f_k(0) > g_k(0) \). This implies \( f_k(t) > g_k(t) \) for \( t \in [0, \tau_0) \) with some \( \tau_0 > 0 \). Therefore, the inequality \( \mathcal{F}_{k-1}(t, f_k(t)) > \mathcal{F}_{k-1}(t, g_k(t)) \) holds for \( t \in (0, \tau_0) \) and this leads to

\[
\frac{d}{dt}(f_{k-1}(t) - g_{k-1}(t)) \geq \mathcal{F}_{k-1}(t, f_k(t)) - \mathcal{F}_{k-1}(t, g_k(t)) > 0
\]

for \( t \in (0, \tau_0) \). Hence, the inequality above and \( f_{k-1}(0) > g_{k-1}(0) \) imply that \( f_{k-1}(t) > g_{k-1}(t) \) holds for \( t \in (0, \tau_0) \). Repeating this argument, we see that \( f_j(t) > g_j(t) \) holds for any \( j \) and \( t \in (0, \tau_0) \). We suppose that for some \( j_0 \in \{1, \ldots, k\} \) and some \( t \in (0, T) \), \( f_{j_0}(t) = g_{j_0}(t) \) holds. Then, we define

\[
\tau_{j_0} = \inf\{0 < t < T; f_{j_0}(t) = g_{j_0}(t)\}.
\]

We note that \( \tau_{j_0} > \tau_0 \). By the continuity of \( f_{j_0} \) and \( g_{j_0} \),

\[
\frac{d}{dt}f_{j_0}(\tau_{j_0}) - \frac{d}{dt}g_{j_0}(\tau_{j_0}) \leq 0.
\]

Since

\[
\frac{d}{dt}(f_{j_0}(t) - g_{j_0}(t)) \geq \mathcal{F}_{j_0}(t; f_{j_0+1}(t)) - \mathcal{F}_{j_0}(t; g_{j_0+1}(t)),
\]

we have \( f_{j_0+1}(\tau_{j_0}) \leq g_{j_0+1}(\tau_{j_0}) \). Now, we can also define

\[
\tau_{j_0+1} = \inf\{0 < t < T; f_{j_0+1}(t) = g_{j_0+1}(t)\}
\]

and we obtain \( \tau_{j_0+1} < \tau_{j_0} \). Repeating this procedure, we define \( \tau_j = \inf\{0 < t < T; f_{j}(t) = g_{j}(t)\} \) satisfying that \( \tau_{j_0} > \tau_{j_0+1} > \cdots > \tau_k > \tau_{j_0} > \cdots > \tau_{j_0-1} \) (when \( j_0 = 1, \tau_1 > \cdots > \tau_k \)). However, the same argument implies \( \tau_{j_0-1} < \tau_{j_0} \) (when \( j_0 = 1, \tau_k > \tau_1 \)), which is a contradiction and we have the assertion.

For the proof of Proposition 2.4, the next lemma plays a critical role. For simplicity, hereinafter, we denote \( \frac{d}{dt}f \) as \( f' \).

Lemma 2.3. Let \( T > 0 \) and \( C, \tilde{\lambda}, p, q \geq 0 \). Let \( f \in C^2([0, T]; [0, \infty)) \) and \( g \in C^1([0, T]; [0, \infty)) \) satisfy that

\[
\begin{align*}
C f(t)^p &\leq e^{\tilde{\lambda} t} g(t) f'(t)^q, \quad \text{for } t \in [0, T), \\
&f'(t), f''(t) \geq 0, \quad \text{for } t \in [0, T).
\end{align*}
\]

Then

\[
\frac{C}{p+1} (f(t)^{p+1} - f(0)^{p+1}) \leq e^{\tilde{\lambda} t} g(t) f'(t)^{q+1} - g(0) f'(0)^{q+1}.
\]

Proof. By using integration by parts,

\[
\frac{C}{p+1} (f(t)^{p+1} - f(0)^{p+1})
\]

\[
= C \int_0^t f(\tau)^p f'(\tau)d\tau
\]

\[
\leq \int_0^t e^{\tilde{\lambda} \tau} g'(\tau) f'(\tau)^{q+1}d\tau
\]

\[
= e^{\tilde{\lambda} t} g(t) f'(t)^{q+1} - g(0) f'(0)^{q+1} - \int_0^t g(\tau) (e^{\tilde{\lambda} \tau} f'(\tau)^{q+1})'d\tau
\]

\[
\leq e^{\tilde{\lambda} t} g(t) f'(t)^{q+1} - g(0) f'(0)^{q+1}.
\]

□
Proof of Proposition 2.1: By Lemma 2.2, it is enough to show (2.10) for
\[
\begin{align*}
\begin{cases}
g'_j(t) = \tilde{C}_j g_{j+1}(t)^{p_j}, & \text{for } 1 \leq j \leq k - 1, \quad t \in [0,T), \\
g'_k(t) = \tilde{C}_k e^{-\lambda t} g_1(t)^{p_k}, & \text{for } t \in [0,T),
\end{cases}
\end{align*}
\]
g_j(0) = 0 for \(j \neq 2 \) and
\[
(2.11) \quad \tilde{C}_2^{-\alpha_2} A_2^{\alpha_2/(\alpha_1 P_2)} \tilde{C}_1^{-\alpha_2 Q_2/(\alpha_1 P_2)} < g_2(0) < f_2(0).
\]
Again by Lemma 2.2, we remark that \(g_j(t) \geq 0\) for any \(1 \leq j \leq k\). By Lemma 2.3 and \(\tilde{C}_k g_1^{p_k} = e^{\lambda t} g_k\), we deduce
\[
\frac{\tilde{C}_k}{p_k + 1} g_1(t)^{p_k + 1} \leq e^{\lambda t} g_k(t) g'_1(t),
\]
which is rewritten as
\[
(2.12) \quad A_k g_1(t)^{p_k} \leq e^{L_k t} g_k(t) g'_1(t)^Q_k.
\]
Here, we have used the fact that
\[
g''_1(t) = p_1 \tilde{C}_1 g_2(t)^{p_1 - 1} g'_1(t) = p_1 \tilde{C}_1 \tilde{C}_2 g_2(t)^{p_1 - 1} g_3(t)^{p_2} \geq 0.
\]
Taking the \(p_{k-1}\)-th power in both sides of (2.12), multiplying the both sides by \(\tilde{C}_{k-1}\), and by (2.2), (2.3), (2.6), (2.7), and using \(g'_{k-1}(t) = \tilde{C}_{k-1} g_k(t)^{p_{k-1}}\), we have
\[
P_{k-1} A_{k-1} g_1(t)^{p_{k-1} - 1} \leq e^{L_{k-1} t} g_{k-1}'(t) g_1(t)^{Q_{k-1} - 1}.
\]
Again, by Lemma 2.3, we see that
\[
A_{k-1} g_1(t)^{p_{k-1}} = e^{L_{k-1} t} g_{k-1}(t) g_1'(t)^{Q_{k-1}}.
\]
Repeating this argument, we have
\[
A_2 g_1(t)^{p_2} \leq e^{L_2 t} g_2(t) g'_1(t)^{Q_2} - g_2(0) g'_1(0)^{Q_2} = e^{L_2 t} g_2(t) g'_1(t)^{Q_2} - \tilde{C}_1^{Q_2} g_2(0)^{Q_1}.
\]
Here, we put
\[
\tilde{g}_1(t) = g_1(t) + A_2^{-1/p_2} \tilde{C}_1^{Q_2/p_2} g_2(0)^{Q_1/p_2}.
\]
Then, we have
\[
(2.13) \quad A_2 \tilde{g}_1(t)^{p_2} = A_2 \left( g_1(t) + A_2^{-1/p_2} \tilde{C}_1^{Q_2/p_2} g_2(0)^{Q_1/p_2} \right)^{p_2} \\
\leq 2^{p_2 - 1} \left( A_2 g_1(t)^{p_2} + \tilde{C}_1^{Q_2} g_2(0)^{Q_1} \right) \\
\leq 2^{p_2 - 1} e^{L_2 t} g_2(t) \tilde{g}_1(t)^{Q_2}.
\]
Then, taking the \(p_1\)-th power in both sides of (2.13), and multiplying the both sides by \(\tilde{C}_1\), we have
\[
2^{-p_1(p_2 - 1)} P_1 A_1 \tilde{g}_1(t)^{p_1} \leq e^{L_1 t} \tilde{g}_1(t)^{Q_1}.
\]
Therefore, we obtain
\[
\tilde{g}_1(t) \geq 2^{-p_1(p_2 - 1)/Q_1} P_1^{1/Q_1} A_1^{1/Q_1} e^{-L_1 t/Q_1} \tilde{g}_1(t)^{(p_1 - 1)/Q_1},
\]
which and (2.14), (2.17), and $(P_3 - 1)/Q_1 > 1$ imply that

$$
\overline{g}_1(t) \geq \left( \overline{g}_1(0) - (P_3 - Q_1 - 1)/Q_1, \overline{C}(1 - e^{-L_1 t/Q_1}) \right)^{-Q_1/(P_3 - Q_1 - 1)}
$$

$$
= \left( \overline{g}_1(0)^{-1/\alpha_1} - \overline{C}(1 - e^{-L_1 t/Q_1}) \right)^{-\alpha_1}
$$

$$
= \overline{C}^{-1/\alpha_1} \left( e^{-L_1 t/Q_1} - e^{-L_1 \tilde{T}_1/Q_1} \right)^{-\alpha_1},
$$

where

$$
\tilde{T}_1 = -L_1^{-1} Q_1 \log \left( 1 - \overline{C}^{-1} A_2^{1/(\alpha_1 P_2)} \overline{C}_1^{-Q_2/(\alpha_1 P_2)} g_2(0)^{-1/\alpha_2} \right).
$$

Indeed, we compute

$$
\overline{g}_1(0)^{-1/\alpha_1} = A_2^{1/(\alpha_1 P_2)} \overline{C}_1^{-Q_2/(\alpha_1 P_2)} g_2(0)^{-Q_1/(\alpha_1 P_2)}
$$

$$
= A_2^{1/(\alpha_1 P_2)} \overline{C}_1^{-Q_2/(\alpha_1 P_2)} g_2(0)^{-\alpha_1 (P_3 - Q_1 - 1)/P_2}
$$

$$
= A_2^{1/(\alpha_1 P_2)} \overline{C}_1^{-Q_2/(\alpha_1 P_2)} g_2(0)^{-1/\alpha_2}.
$$

We also remark that (2.11) implies positiveness of $\tilde{T}_1$. From (2.14), the definition of $\overline{g}_1(t)$, and Lemma 2.2, we have

$$
f_1(t) \geq g_1(t) \geq \overline{C}^{-1/\alpha_1} \left( e^{-L_1 t/Q_1} - e^{-L_1 \tilde{T}_1/Q_1} \right)^{-\alpha_1} - A_2^{-1/P_2} \overline{C}_1^{Q_2/P_2} g_2(0)^{Q_1/P_2}.
$$

Finally, by taking the limit $g_2(0) \to f_2(0)$, the RHS of the inequality above converges to that of (2.11). We note that $g_2(0) < f_2(0)$ leads to $\tilde{T}_1 > \tilde{T}_0$, and $\tilde{T}_1 \to \tilde{T}_0$ by letting $g_2(0) \to f_2(0)$.

\[ \square \]

3. Proof of Theorem 1.4

In this section, we prove Theorem 1.4. Here we restate Theorem 1.4 with more details.

**Proposition 3.1.** Let $u_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ satisfy $u_{0,j} \geq 0$ for any $1 \leq j \leq k$. We assume $p_j \geq 1$ (j = 1, ..., k) and $(p_1, ..., p_k) \neq (1, ..., 1)$. Let $u \in C([0, T]; L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n))$ be a classical solution of (1.1). We further assume that there exists $j_0 \in \{1, ..., k\}$ such that $\alpha_{j_0} > \frac{n}{2}$ and $u_{0,j_0} \neq 0$. Then, for some constants $\overline{C}_1, \overline{C}_2, \overline{C}_3$, we have

$$
U_{j_0 - 1, R_0}(t) + \overline{C}_2
$$

$$
\geq \overline{C}_1 e^{-2R_0^2 t} (e^{-2\lambda \max_{m \neq j_0 - 2} p_m R_0^{-2} t/\alpha_{j_0 - 1}} - e^{-2\lambda \max_{m \neq j_0 - 2} p_m R_0^{-2} t/\alpha_{j_0 - 1}})^{-\alpha_{j_0 - 1}}
$$

with $T_0$ satisfying

$$
0 < T_0 \leq \overline{C}_3 U_{j_0, R}(0)^{-1/(\alpha_{j_0 - n/2})}
$$

and $R_0$ defined by

$$
U_{j_0, R_0} = 2^{2\alpha_{j_0} n} \overline{C}_1^{2\alpha_{j_0} n - n} R_0^{-2\alpha_{j_0} n + n},
$$

where we interpret $U_{0, R} = U_{k, R}$, $\alpha_0 = \alpha_k$, $p_0 = p_k$, and $p_{-1} = p_{k-1}$.

Here, we remark that Proposition 3.1 implies $T_m \leq T_0$, that is, the assertion of Theorem 1.4.
Proof. Without loss of generality, we assume \( j_0 = 2 \). Recall \( \| \phi \|_{L^1} = 1 \). Then by the Hölder inequality, we have

\[
\int_{\mathbb{R}^n} u_{j+1}(t,x) \phi_{R_0}(x) dx \leq \left( \int_{\mathbb{R}^n} \phi_{R_0}(x) dx \right)^{1/p_j'} \left( \int_{\mathbb{R}^n} |u_{j+1}(t,x)|^{p_j} \phi_{R_0}(x) dx \right)^{1/p_j}.
\]

Here, \( p_j' \) is the Hölder conjugate of \( p_j \), that is, \( p_j' = p_j/(p_j - 1) \) if \( p_j > 1 \) and \( p_j' = \infty \) if \( p_j = 1 \). Hereafter, we interpret that \( 1/p_j' = 0 \) if \( p_j' = \infty \). We deduce that

\[
U_{j,R_0}'(t) = \int_{\mathbb{R}^n} \partial_t u_j(t,x) \phi_{R_0}(x) dx
= \int_{\mathbb{R}^n} \Delta u_j(t,x) \phi_{R_0}(x) dx + \int_{\mathbb{R}^n} |u_{j+1}(t,x)|^{p_j} \phi_{R_0}(x) dx
\]

\[
= \int_{\mathbb{R}^n} u_j(t,x) \Delta \phi_{R_0}(x) dx + \int_{\mathbb{R}^n} |u_{j+1}(t,x)|^{p_j} \phi_{R_0}(x) dx
\geq -2\lambda R_0^{-2} \int_{\mathbb{R}^n} u_j(t,x) \phi_{R_0}(x) dx + \int_{\mathbb{R}^n} |u_{j+1}(t,x)|^{p_j} \phi_{R_0}(x) dx
\geq -2\lambda R_0^{-2} U_{j,R_0}(t) + \int_{\mathbb{R}^n} |u_{j+1}(t,x)|^{p_j} \phi_{R_0}(x) dx.
\]

Combining this and (3.3), we see that for any \( j \in \{1, \ldots, k\} \), \( U_{j,R_0} \) satisfies

\[
U_{j,R_0}' + 2\lambda R_0^{-2} U_{j,R_0} \geq R_0^{-n(p_j-1)} U_{j+1,R_0}^{p_j}.
\]

Furthermore, since \( U_{j,R} \geq 0 \), we immediately obtain from the above inequality that

\[
U_{j,R_0}' + \Lambda_j R_0^{-2} U_{j,R_0} \geq R_0^{-n(p_j-1)} U_{j+1,R_0}^{p_j},
\]

where

\[
\Lambda_k = 2\lambda, \quad \Lambda_j = p_j \Lambda_{j+1} \quad (j = 1, \ldots, k-1).
\]

Put \( \widetilde{U}_j = e^{\Lambda_j R_0^{-2} t} U_{j,R_0} \) for \( 1 \leq j \leq k \). Then, by (3.3), for \( 1 \leq j \leq k-1 \), we have

\[
\widetilde{U}_j' = e^{\Lambda_j R_0^{-2} t} \left( U_{j,R_0}' + \Lambda_j R_0^{-2} U_{j,R_0} \right)
\geq R_0^{-n(p_j-1)} e^{\Lambda_j R_0^{-2} t} U_{j+1,R_0}^{p_j}
= R_0^{-n(p_j-1)} e^{-(p_j \Lambda_{j+1} - \Lambda_j) R_0^{-2} t} \widetilde{U}_{j+1}^{p_j}
= R_0^{-n(p_j-1)} \widetilde{U}_{j+1}^{p_j},
\]

and

\[
\widetilde{U}_k' = e^{\Lambda_k R_0^{-2} t} \left( U_{k,R_0}' + \Lambda_k R_0^{-2} U_{k,R_0} \right)
\geq R_0^{-n(p_k-1)} e^{\Lambda_k R_0^{-2} t} U_{1,R_0}^{p_k}
= R_0^{-n(p_k-1)} e^{-(p_k \Lambda_1 - \Lambda_k) R_0^{-2} t} \widetilde{U}_{1}^{p_k}.
\]

Then we apply Proposition 2.1 with \( f_j = \widetilde{U}_j, \ C_j = R_0^{-n(p_j-1)}, \ \bar{\lambda} = (p_k \Lambda_1 - \Lambda_k) R_0^{-2} \).

Indeed, we first remark that \( \widetilde{U}_2(0) > 0 \) holds since we assume that \( u_{0,2} \geq 0 \) and
\( u_{0,2} \neq 0 \) (see also Remark 1.2 (i) and \((3.3)\)). Next, we check the condition \((2.9)\). We note that \(p_k \alpha_1 - \lambda_k = 2\lambda(\prod_{j=1}^{k} p_j - 1) = 2\lambda(P_1 - Q_1 - 1)\), and hence,

\[
(3.6) \quad \frac{L_1}{Q_1} = \frac{\tilde{\alpha}}{Q_1} \prod_{m=1}^{k-1} p_m = 2\lambda R_0^2 \frac{P_1 - Q_1 - 1}{Q_1} \prod_{m=1}^{k-1} p_m = 2\lambda R_0^2 \alpha_1 \prod_{m=1}^{k-1} p_m.
\]

Here, we have used \((2.5)\). From here, \(C\) denotes general constants independent of \(R_0\). If \(k = 2\), \(A_2\) is computed as

\[
A_2 = P_2^{-1} \tilde{C} = CR_0^{-n(p_2 - 1)} = CR_0^{-n(P_2 - Q_2 - 1)},
\]

otherwise,

\[
A_2 = P_2^{-1} \tilde{C} = \prod_{k=0}^{k-3} \prod_{h=k}^{h-1} \prod_{m=0}^{m-1} p_m
\]

\[
= CR_0^{-n(p_2 - 1 + \sum_{h=k}^{h-1} (p_k - h - 1) \prod_{m=0}^{m-2} p_{m+2})}
\]

\[
= CR_0^{-n(P_2 - Q_2 - 1)}
\]

Here, we have used \((2.4)\). We recall that by \((2.6)\) and \((2.8)\),

\[
\tilde{C} = C\tilde{\alpha}^{-1} CR_0^{n/p_1}/Q_1.
\]

From this and \(\tilde{\lambda}^{-1} = CR_0^{2}\), RHS of \((2.9)\) is calculated as

\[
(3.9) \quad \tilde{C}^{-\alpha_2} A_2^{\alpha_2/(\alpha_1 P_2)} \tilde{C}_1^{-\alpha_2 Q_2/(\alpha_1 P_2)} = C \left( R_0^{-2} \tilde{C}_1^{-1/Q_1 - Q_2/(\alpha_1 P_2)} A_2^{1/(\alpha_1 P_2) - p_1/Q_1} \right)^{\alpha_2}
\]

Let us further compute the RHS. We also directly obtain

\[
-\frac{1}{Q_1} \frac{Q_2}{\alpha_1 P_2} = \frac{1}{Q_1} \frac{Q_2}{P_2} \left( \frac{P_1 - 1}{Q_1} - 1 \right)
\]

\[
= -\frac{1}{Q_1} \frac{Q_2}{P_2} \left( \frac{P_1 - 1}{Q_1} - 1 \right)
\]

\[
= -\frac{1}{Q_1} \frac{Q_1 - 1}{Q_1} + \frac{Q_2}{P_2} = \frac{Q_2 - P_2}{P_2}.
\]

\[
1 - \frac{p_1}{Q_1} = \frac{P_1 - Q_1 - 1}{Q_1} - \frac{p_1}{Q_1} = -\frac{1}{P_2}.
\]

Since \(\tilde{C}_1 = CR^{-n(P_1 - 1)}\),

\[
(3.10) \quad \tilde{C}_1^{-1/Q_1 - Q_2/(\alpha_1 P_2)} = CR_0^{-n(P_1 - 1)(Q_2 - P_2)/P_2} = CR_0^{-n(P_1 - Q_1 - (P_2 - Q_2))/P_2}.
\]

Moreover, \((3.7)\) and \((3.8)\) imply

\[
(3.11) \quad A_2^{1/(\alpha_1 P_2) - p_1/Q_1} = CR_0^{n(P_2 - Q_2 - 1)/P_2}.
\]

From \((3.9)\) - \((3.11)\), that the LHS of \((3.9)\) is simply expressed as

\[
\left( \tilde{C}^{-1} A_2^{1/(\alpha_1 P_2)} \tilde{C}_1^{-Q_2/(\alpha_1 P_2)} \right)^{\alpha_2} = C \left( R_0^{-2n(P_1 - Q_1 - 1)/P_2} \right)^{\alpha_2} = CR_0^{-2n + 2n}.
\]
Here, in the constant $C$ on the RHS depends only on $(p_1, \ldots, p_k)$. Let us rewrite the identity above as
\[(3.12) \quad (\tilde{C}^{-1} A_2^{1/(\alpha_1 P_2)} C_1^{-Q_2/(\alpha_1 P_2)})^{\alpha_2} = (C_1 R_0^{-1})^{2\alpha_2 - n}.
\]
This and (3.3) imply
\[\tilde{C}^{-1} A_2^{1/(\alpha_1 P_2)} C_1^{-Q_2/(\alpha_1 P_2)} U_{2, R_0}(0)^{-1/\alpha_2} = (C_1 R_0^{-1})^{2 - n/\alpha_2} U_{2, R_0}(0)^{-1/\alpha_2} = \frac{1}{2}.
\]
Thus, the assumption (2.9) holds, and we can apply Proposition 2.1. The assertion (2.10) of Proposition 2.1 with (3.6) lead to the estimate (3.1) with $T_0$ satisfying
\[T_0 = -L_1^{-1} Q_1 \log \left(1 - \tilde{C}^{-1} A_2^{1/(\alpha_1 P_2)} C_1^{-Q_2/(\alpha_1 P_2)} U_{2, R_0}(0)^{-1/\alpha_2}\right)
\leq CR_0^2
\leq CU_{2, R_0}(0)^{-1/(\alpha_2 - n/2)}.
\]
Here, the first identity follows from the assertion of Proposition 2.1, the second inequality is due to (3.6), and the third inequality is due to (3.3). This completes the proof.

4. LOWER BOUND OF THE LIFESPAN

In this section, we give the proof of Proposition 1.3. The global existence for the case when $\alpha_{\max} < n/2$ may be shown by the argument of [1,20,22]. However, for reader’s convenience, we give a proof here. By Proposition 1.1 we construct the local solution $u \in C([0, T_m); L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n))$ of the integral equation
\[u_j(t) = e^{t \Delta} u_{0,j} + \int_0^t e^{(t-\tau)\Delta} |u_{j+1}|^{p_j} \, d\tau.
\]
Let $l_j \leq n/2$ determined later and we define
\[M(t) := \sup_{\tau \in [0,t]} \sum_{j=1}^k \left\{ (1 + \tau)^{l_j} \|u_j(\tau)\|_{L^\infty} + (1 + \tau)^{l_j - n/2} \|u_j(\tau)\|_{L^1} \right\}.
\]
It is well known that for $1 \leq p \leq q \leq \infty$,
\[(4.1) \quad \|e^{t \Delta} f\|_{L^q} \leq C t^{-\frac{q}{p} + \frac{1}{2} - \frac{n}{2}} \|f\|_{L^p}.
\]
By using (4.1),
\[(1 + t)^{l_j - n/2} \|u_j(t)\|_{L^1} \leq C (1 + t)^{l_j - n/2} \|u_{0,j}\|_{L^1} + C (1 + t)^{l_j - n/2} \int_0^t \|u_{j+1}(\tau)\|_{L^{p_j}}^{p_j} \, d\tau
\leq C (1 + t)^{l_j - n/2} \|u_{0,j}\|_{L^1}
+ C (1 + t)^{l_j - n/2} \int_0^t (1 + \tau)^{-l_j + n/2} M(\tau)^{p_j} \, d\tau
\leq C (1 + t)^{l_j - n/2} \|u_{0,j}\|_{L^1} + CL_j(t) M(t)^{p_j},
\]
where

\[ L_j(t) = \begin{cases} 
(1 + t)^{l_j - n/2} & (-l_j + 1)p_j + n/2 < -1), \\
(1 + t)^{l_j - n/2} \log(1 + t) & (-l_j + 1)p_j + n/2 = -1), \\
(1 + t)^{l_j - l_j + 1}p_j + 1 & (-l_j + 1)p_j + n/2 > -1) 
\]

and we have used the fact that, by the interpolation and Young inequality

\[
\|u_j+1(\tau)\|_{L^p_j}^{P_j} \leq \|u_j+1(\tau)\|_{L^{P_j}}^{P_j-1} \|u_j+1(\tau)\|_{L^1} = (1 + \tau)^{-p_j} ((1 + \tau)^{l_j+1} \|u_j+1(\tau)\|_{L^\infty})^{P_j-1} ((1 + \tau)^{l_j+1-n/2} \|u_j(\tau)\|_{L^1}) \leq (1 + \tau)^{-p_j} \|u_j+1(\tau)\|_{L^{P_j}}^{P_j}.
\]

Next, we consider the estimate for \( \|u_j(t)\|_{L^\infty} \). First, for \( 0 < t \leq 1 \), we apply \( L^\infty_L \) estimate to obtain

\[
\|u_j(t)\|_{L^\infty} \leq C \|u_{0,j}\|_{L^\infty} + C \int_0^t \|u_j(\tau)\|_{L^\infty}^{P_j} d\tau \leq C \|u_{0,j}\|_{L^\infty} + C \int_0^t (1 + \tau)^{-l_j+1} M(\tau)^{P_j} d\tau \leq C \|u_{0,j}\|_{L^\infty} + C M(t)^{P_j}.
\]

For \( t \geq 1 \), we apply \( L^1_L \) estimate to obtain

\[
(1 + t)^{l_j} \|u_j(t)\|_{L^1} \leq C (1 + t)^{l_j} t^{-n/2} \|u_{0,j}\|_{L^1} + C (1 + t)^{l_j} \int_0^{t/2} (t - \tau)^{-n/2} \|u_j+1(\tau)\|_{L^1}^{P_j} d\tau \leq C (1 + t)^{l_j-n/2} \|u_{0,j}\|_{L^1} + C (1 + t)^{l_j-n/2} \int_0^{t/2} (1 + \tau)^{-l_j+1} M(\tau)^{P_j} d\tau \leq C (1 + t)^{l_j-n/2} \|u_{0,j}\|_{L^1} + C L_j(t) M(t)^{P_j}.
\]

We remark that, the definition of \( L_j(t) \) implies that for any \( t > 0 \),

\[
(1 + t)^{l_j} \int_{t/2}^t (1 + \tau)^{-l_j+1} M(\tau)^{P_j} d\tau \leq C (1 + t)^{l_j-l_j+1} M(t)^{P_j} \leq C L_j(t).
\]

Therefore, we conclude

\[
M(t) \leq C_0 \|u_0\|_{L^1 \cap L^\infty} + C_1 \max_{1 \leq j \leq k} (L_j(t) M(t)^{P_j})
\]

with some constants \( C_0, C_1 > 0 \).

Now, we determine \( l_j \) (\( 1 \leq j \leq k \)) in the following way. Let \( \mathbf{1} = (1, \ldots, 1) \), \( l = (l_1, \ldots, l_k) \), \( \alpha = (\alpha_1, \ldots, \alpha_k) \).
Therefore \( p \leq (4.4) \), for any \( 1 \leq j \leq k \).

Moreover, we remark that for any \( \alpha \) (4.3)

Indeed, if there exists some \( \alpha \) (4.4)

Then, it is obvious that \( l_j < n/2 \) for \( 1 \leq j \leq k \). Therefore, by the definition of \( L_j(t) \), in any case we have \( L_j(t) \leq C \) with some constant \( C > 0 \) independent of \( t \geq 0 \). Thus, from (4.2), we have the a priori estimate

which enables us to prove the small data global existence of the solution. Namely, in this case \( T = \infty \) holds for small initial data.

**Case 2:** When \( \alpha_{\max} > n/2 \), we determine \( l \) by the relation

\[
l_j - l_{j+1}p_j + 1 = (\alpha_{\max} - n/2)(p_j - 1) \quad (1 \leq j \leq k),
\]

that is,

\[
l = \alpha - (\alpha_{\max} - n/2)I.
\]

In this case, again,

(4.3) \[ l_j = \alpha_j - \alpha_{\max} + n/2 \leq n/2. \]

Moreover, we remark that for any \( j \),

(4.4) \[ -l_{j+1}p_j + n/2 > -1. \]

Indeed, if there exists some \( j_0 \) such that

\[-l_{j_0}p_{j_0} + n/2 \leq -1,\]

then

\[ 0 \geq l_{j_0} - n/2 \geq l_{j_0} - p_{j_0}l_{j_0+1} + 1 = (\alpha_{\max} - n/2)(p_{j_0} - 1) \geq 0. \]

Therefore \( p_{j_0} = 1 \), \( l_{j_0} = n/2 \), and \( l_{j_0+1} \geq n/2 + 1 \), which contradicts (4.3). By (4.4), for any \( 1 \leq j \leq k \),

\[
L_j(t) = (1 + t)^{(\alpha_{\max} - n/2)(p_j - 1)}.
\]

Therefore, by (4.2) we conclude

\[
M(t) \leq C_0\|u_0\|_{L^1 \cap L^\infty} + C_1 \max_{1 \leq j \leq k} (1 + t)^{(\alpha_{\max} - n/2)(p_j - 1)} M(t)^{p_j}.
\]

We take again \( T_1 \) as the smallest time such that \( M(t) = 2C_0\|u_0\|_{L^1 \cap L^\infty} \) (we note that if such a time does not exist, we have \( T = \infty \)). Then, substituting \( t = T_1 \) in the above inequality, we have

\[
C_0\|u_0\|_{L^1 \cap L^\infty} \leq C_1 \max_{1 \leq j \leq k} (1 + T_1)^{(\alpha_{\max} - n/2)(p_j - 1)} (2C_0\|u_0\|_{L^1 \cap L^\infty})^{p_j},
\]

which and smallness of \( u_0 \) imply that

\[
T_1 > T_1 \geq C\|u_0\|_{L^1 \cap L^\infty}^{-1/(\alpha_{\max} - n/2)}.
\]

This completes the proof.
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