Quantum Field Theory on the $q$–deformed Fuzzy Sphere

H. Steinacker

Abstract. We discuss the second quantization of scalar field theory on the $q$–deformed fuzzy sphere $S^2_{q,N}$ for $q \in \mathbb{R}$, using a path–integral approach. We find quantum field theories which are manifestly covariant under $U_q(su(2))$, have a smooth limit $q \to 1$, and satisfy positivity and twisted bosonic symmetry properties. Using a Drinfeld twist, they are equivalent to ordinary but slightly “nonlocal” QFT’s on the undeformed fuzzy sphere, which are covariant under $SU(2)$.

INTRODUCTION

In this paper, we first give a short introduction to the $q$–deformed fuzzy sphere, and then discuss some aspects of second quantization on this space. This is essentially a short introduction to the more extensive discussion in [1]. Much of the considerations concerning the second quantization generalize to other, higher–dimensional $q$–deformed spaces.

The $q$–deformed fuzzy sphere $S^2_{q,N}$ is a $q$–deformed version of the “ordinary” fuzzy sphere $S^2_N$ [2]. The algebra of functions on $S^2_{q,N}$ is isomorphic to the matrix algebra $Mat(N + 1, \mathbb{C})$, but viewed as a $U_q(su(2))$–module algebra. It admits additional structure compatible with covariance under the Drinfeld–Jimbo quantum group $U_q(su(2))$, such as an invariant integral and a differential calculus. It can be defined for both $q \in \mathbb{R}$ and $|q| = 1$, however we restrict ourselves to the case $q \in \mathbb{R}$ here. Then $S^2_{q,N}$ is precisely the “discrete series” of Podles spheres [3]. Moreover, we only consider scalar fields for simplicity. A much more detailed description of $S^2_{q,N}$ has been given in [4]. This space is of interest in the context of $D$–branes on the $SU_k(2)$ WZW model, as discussed by Alekseev, Recknagel and Schomerus [5]. These authors extract an “effective” algebra of functions on the $D$–branes from the OPE of the boundary vertex operators, which is twist–equivalent [4] to the space of functions on $S^2_{q,N}$ for $q$ a root of unity.
THE SPACE $S^2_{Q,N}$

Consider the spin $\frac{N}{2}$ representation of $U_q(su(2))$,

$$\rho : U_q(su(2)) \to Mat(N+1, \mathbb{C}),$$

which acts on $\mathbb{C}^{N+1}$. It can be used to define the quantum adjoint action of $U_q(su(2))$ on the set of matrices $Mat(N+1, \mathbb{C})$, by

$$u \triangleright_q M = \rho(u_1)M\rho(Su_2).$$

The usual matrix algebra $Mat(N+1, \mathbb{C})$ thereby becomes a $U_q(su(2))$–module algebra, which means that $u \triangleright_q (ab) = (u_1 \triangleright_q a)(u_2 \triangleright_q b)$ for $a, b \in Mat(N+1, \mathbb{C})$. Here $\Delta(u) = u(1) \otimes u(2)$ denotes the coproduct of $u \in U_q(su(2))$. $S^2_{q,N}$ is defined to be precisely this $U_q(su(2))$–module algebra $Mat(N+1, \mathbb{C})$, together with some additional structure. It is easy to see that under the (adjoint) action of $U_q(su(2))$, it decomposes into the irreducible representations

$$S^2_{q,N} = Mat(N+1, \mathbb{C}) = (1) \oplus (3) \oplus \ldots \oplus (2N+1), \quad (0.1)$$

where $(2K+1)$ is the spin $K$ representation of $U_q(su(2))$. This is the analog of the decomposition of functions on the sphere into spherical harmonics, which it is truncated on the fuzzy spheres. Let $\{x_i\}_{i=+-0}$ be the weight basis of the spin 1 components in $(0.1)$, so that $u \triangleright_q x_i = x_j \pi^i_j(u)$ for $u \in U_q(su(2))$. One can show that they satisfy the relations

$$\varepsilon^{ij}_k x_i x_j = \Lambda_N x_k, \quad g^{ij} x_i x_j = R^2.$$

Here

$$\Lambda_N = R \frac{[2]_{q^{N+1}}}{\sqrt{[N]_q[N+2]_q}},$$

$[n]_q = \frac{q^n-q^{-n}}{q-q^{-1}}$, and $\varepsilon^{ij}_k$ and $g^{ij}$ are the $q$–deformed invariant tensors. For example, $\varepsilon^{33}_3 = q^{-1} - q$, and $g^{11} = -q^{-1}$, $g^{00} = 1$, $g^{11} = -q$. In [4], these relations were derived using a Jordan–Wigner construction. For $q = 1$, the relations of $S^2_N$ are recovered.

Integration. The unique invariant integral of a function $f \in S^2_{q,N}$ is given by its quantum trace over $Mat(N+1, \mathbb{C})$,

$$\int f := \frac{4\pi R^2}{[N+1]_q} \Tr_q(f) = \frac{4\pi R^2}{[N+1]_q} \Tr(f q^{-H}),$$

normalized such that $\int 1 = 4\pi R^2$. Here $H$ is the Cartan generator of $U_q(su(2))$. Invariance means that $\int u \triangleright_q f = \varepsilon(u) \int f$. 
Real structure. In order to define a real noncommutative space, one must specify a star structure on the algebra of functions. The star of an element $f$ is simply defined to be the hermitean adjoint of the matrix $f \in S_{q,N}^2 = Mat(N+1, C)$. In terms of the generators $x_i$, this becomes

$$x_i^* = g^{ij}x_j,$$

since $q$ is real.

$S_{q,N}^2$ admits additional structure, in particular a differential calculus. While the calculus is very interesting in the context of gauge theories, we shall not discuss it here. The interested reader is referred to [4]. However, we do need a Laplacian in order to write down Lagrangians and actions. While it can naturally be defined using the differential calculus as $\Delta = *_H d *_H d$, we give an ad–hoc definition here for simplicity. Assume that $\{\psi_{K,n}(x)\}_{K,n} \subset S_{q,N}^2$ is a weight basis of the spin $K$ representation of $U_q(gu(2))$, so that

$$u \triangleright_q \psi_{K,n}(x) = \psi_{K,m}(x) \pi_n^m(u).$$  \hspace{1cm} (0.2)

It can be normalized such that

$$\int \psi_{K,n}(x) \psi_{K',m}(x) = \delta_{K,K'} g^K_{n,m}.$$

The Laplacian is then given by

$$\Delta \psi_{K,n}(x) = \frac{1}{R^2} [K]_q [K+1]_q \psi_{K,n}(x).$$

**SCALAR FIELD THEORY ON $S_{Q,N}^2$**

We can now write down Lagrangians and actions defining scalar field theory on $S_{q,N}^2$. Consider for example

$$S[\Psi] = -\int \frac{1}{2} \Psi \Delta \Psi + \lambda \Psi^4 = S_{\text{free}}[\Psi] + S_{\text{int}}[\Psi]$$

where

$$\Psi(x) = \sum_{K,n} \psi_{K,n}(x) a^{K,n}. \hspace{1cm} (0.3)$$

The free action can be rewritten as

$$S_{\text{free}}[\Psi] = -\sum_{K,n} \frac{1}{2} D_K \pi_n^m a^{K,m} a^{K,n}.$$ 

In general, actions will be polynomials in the variables $a^{K,n}$ which are invariant under $\tilde{U}_q(su(2)).$

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1 the tilde labels objects associated with $\tilde{U}_q(su(2))$, which is another copy of $U_q(su(2))$ but with reversed coproduct, see [1].
We want to discuss the second quantization of such models, as in [1]. On the undeformed fuzzy sphere, this is fairly straightforward [7, 2]: the coefficients $a^{K,n}$ are considered as complex numbers or more precisely as coordinate functions\(^2\) on the representation space $R^{2K+1}$, so that the actions can be considered as polynomials in the algebra $\mathcal{A} = \otimes_{K=0}^{N} Fun(R^{2K+1})$. The “path integral” is then simply the product of the ordinary integrals over the coefficients $a^{K,n}$, i.e. over $\prod_{K} R^{2K+1}$. This defines a quantum field theory which has a $SO(3)$ rotation symmetry, because the path integral is invariant.

In the $q$–deformed case, this is not as easy, and needs some discussion. We certainly want the models to have a $U_q(su(2))$ symmetry at the quantum level. This means that the coefficients $a^{K,n}$ in (0.3) must be considered as representations of $U_q(su(2))$. In order to be able to do calculations, we also require that the $a^{K,n}$ generate some kind of algebra $\mathcal{A}$, this is almost a tautology. This strongly suggests that $\mathcal{A}$ should be a $U_q(su(2))$–module algebra. We do not have in mind here an algebra of field operators, which in fact would not be appropriate in the Euclidean case even for $q = 1$. Rather, $\mathcal{A}$ should be an analog of the algebra of coordinate functions on configuration space as above for $q = 1$, i.e. some deformed version of $\otimes_{K=0}^{N} Fun(R^{2K+1})$. Our goal is to define correlation functions of the fields (0.3), which after “Fourier transform” amounts to defining

\[
\langle a^{K_1,n_1} a^{K_2,n_2} \ldots a^{K_k,n_k} \rangle =: \langle P(a) \rangle \in \mathcal{C},
\]

perhaps by some kind of a path integral $\langle P(a) \rangle = \frac{1}{\mathcal{Z}} \int Da e^{-S[u]} P(a)$. $P(a)$ will denote some polynomial in the variables $a^{K,n}$ from now on.

It follows immediately from these considerations that $\mathcal{A}$ cannot be commutative, because the coproduct of $U_q(su(2))$ is not cocommutative. In particular, the $a^{K,n}$ cannot be ordinary complex numbers. Therefore an ordinary integral over commutative modes $a^{K,n}$ would violate $U_q(su(2))$ invariance at the quantum level. In some sense, this means that on $q$–deformed spaces, a second quantization is required by consistency. There is one more essential requirement: $\mathcal{A}$ should have the same Poincaré series as classically, i.e. the dimension of the space of polynomials at a given degree should be the same as in the undeformed case. This is in fact precisely the content of a symmetrization postulate, and it is of course an essential physical requirement at least for low energies, in order to have the correct number of degrees of freedom. It means that the “amount of information” contained in the $n$–point functions should be the same as for $q = 1$, so that a smooth limit $q \to 1$ is conceivable. In other words, we want to consider ordinary bosons\(^3\). While some proposals have been given in the literature [8] how to define QFT on spaces with quantum group symmetry, none of them seems to satisfy these requirements.

On a more formal level, we impose the following requirements [1]:

1. **Covariance:**

\[
\langle u \circ_q P(a) \rangle = \epsilon_q(u) \langle P(a) \rangle,
\]

which means that the $\langle P(a) \rangle$ are invariant tensors of $\tilde{U}_q(su(2))$.

\(^2\) with star structure $a^*_i = g^{ij} a_j$
\(^3\) we do not consider fermions here
(2) Hermiticity: 
\[ \langle P(a) \rangle^* = \langle P^*(a) \rangle \]
for a suitable involution $*$ on $A$.

(3) Positivity: 
\[ \langle P(a)^* P(a) \rangle \geq 0, \]

(4) Symmetry
under permutations of the fields, by which we mean that the polynomials in the $a^{K,n}$ can be ordered as usual, i.e. the Poincaré series of $A$ should be underformed.

A slight refinement will be needed later.

Unfortunately, there is no obvious candidate for an associative $U_q(su(2))$–module algebra $A$ with the same Poincaré series as $\otimes_{K=0}^N \text{Fun}(\mathbb{R}^{2K+1})$ (except for small $N$).

We will therefore construct a suitable quasiassociative algebra $A$ which is a star–deformation of the commutative $\otimes_{K=0}^N \text{Fun}(\mathbb{R}^{2K+1})$. We want to emphasise that quasiassociativity is in no way inconsistent with the usual axioms of quantum mechanics, because the algebra $A$ will not be interpreted as algebra of observables; it is only a tool which is useful to calculate correlation functions, just like Grassman variables are used to calculate fermionic correlation functions. In fact, it is possible to avoid the use of quasiassociative algebras altogether, see [1]. Any lingering doubts can be eliminated by showing the equivalence of our models to ordinary QFT on the undeformed fuzzy sphere, with slightly deformed interactions.

The chosen approach is rather general and is applicable in a more general context, such as for higher–dimensional theories.

The quasiassociative star product

As discussed, we assume that the coefficients $a^{K,n}$ transform in the spin $K$ representation of $\tilde{U}_q(su(2))$,
\[ u \triangleright a^{K,n} = \pi^n_m (\hat{S} u) a^{K,m}. \] (0.5)

Let $\varphi$ be the algebra (not coalgebra!)–isomorphism [9]
\[ \varphi : \tilde{U}_q(su(2)) \to U(su(2))[[h]], \]

where $q = e^h$. Moreover, let $\mathcal{F} = \mathcal{F}_1 \otimes \mathcal{F}_2 \in U(su(2))[[h]] \otimes U(su(2))[[h]]$ be the “Drinfeld–twist” [9] which relates the Hopf algebras $\tilde{U}_q(su(2))$ and $U(su(2))$, and satisfies among others
\[ \mathcal{F} = 1 \otimes 1 + o(h), \]
\[ (\varepsilon \otimes 1) \mathcal{F} = 1 = (1 \otimes \varepsilon) \mathcal{F}, \]
\[ (\varphi \otimes \varphi) \hat{\Delta}_q(u) = \mathcal{F} \Delta(\varphi(u)) \mathcal{F}^{-1}, \]
\[ (\varphi \otimes \varphi) \mathcal{R}_\mathcal{F} = \mathcal{F}_1 q^\frac{1}{2} \mathcal{F}^{-1}. \]

for any $u \in \tilde{U}_q(su(2))$. Using this twist, there is an action of $U(su(2))$ on the coefficients $a^{K,n}$, by $u \triangleright a^{K,n} = \varphi^{-1}(u) \triangleright a^{K,n}$. Hence we can consider the usual commutative algebra
\( \mathcal{A}^K := \text{Fun}(\mathbb{R}^{2K+1}) \) generated by the \( a^{K,n} \), and view it as a \( U(su(2)) \)–module algebra \( (\mathcal{A}^K, \cdot, \triangleright) \). We now then define a new multiplication on the same space \( \mathcal{A} \) by

\[
a \ast b := (\mathcal{F}_1^{-1} \triangleright a) \cdot (\mathcal{F}_2^{-1} \triangleright b) = \cdot (\mathcal{F}^{-1} \triangleright (a \otimes b)) \tag{0.7}
\]

for any \( a, b \in \mathcal{A}^K \). This is analogous to the Moyal product in deformation quantization. It is easy to verify that it satisfies

\[
u \tilde{\triangleright} q (a \ast b) = \ast (\tilde{\Delta} q (\nu) \tilde{\triangleright} q (a \otimes b)),
\]

which means that \( (\mathcal{A}^K, \ast, \tilde{\triangleright} q) \) is a \( \tilde{\Delta} q (\text{su}(2)) \)–module algebra. It follows from (0.6) that if \( a \) is invariant under \( \tilde{\Delta} q (\text{su}(2)) \), then it is also central in \( (\mathcal{A}^K, \ast) \). Moreover, the following commutation relations are derived in [4]:

\[
a_i \ast a_j - a_k \ast a_l \tilde{R}^{lk}_{ij} = 0, \tag{0.8}
\]

were \( \tilde{R}^{lk}_{ij} \) is obtained from the universal \( \mathcal{R} \) matrix of \( \tilde{U}_q(su(2)) \). This new product is not associative, but quasiassociative:

\[
(a \ast b) \ast c = (\tilde{\phi}_1 \triangleright a) \ast ((\tilde{\phi}_2 \triangleright b) \ast (\tilde{\phi}_3 \triangleright c)).
\]

Here

\[
\tilde{\phi} := (1 \otimes \mathcal{F})[(\text{id} \otimes \Delta) \mathcal{F}][(\Delta \otimes \text{id}) \mathcal{F}^{-1}](\mathcal{F}^{-1} \otimes 1)
\]

is the coassociator, which is invariant under \( U_q(su(2)) \) and closely related to the KZ equation [9]. It is much easier to work with than the Drinfeld-twist \( \mathcal{F} \), which in fact is never needed explicitly.

Finally, \( (\mathcal{A}, \ast, \tilde{\triangleright} q) \) is defined as in (0.7), applied to any element of \( \mathcal{A} = \otimes K \text{Fun}(\mathbb{R}^{2K+1}) \). Polynomials \( P_\ast (a) \) must now be given including some “bracketing”. Nevertheless, the Poincaré series of \( \mathcal{A} \) is undeformed, because the vector space \( \mathcal{A} \) is undeformed, and the new product preserves the grading. Different bracketings can always be related using the coassociator.

Invariant actions are now considered of the form

\[
S_{\text{int}}[\Psi] = \int \Psi(x) \ast (\Psi(x) \ast \Psi(x)) = I^{(3)}_{K,K',K'';n,m,l} a^{K,n} \ast (a^{K',m} \ast a^{K'',l}), \tag{0.9}
\]

which are invariant polynomials in \( \mathcal{A} \).

**Quantization**

The path integral should be a “functional” on \( \mathcal{A} \) which is invariant under \( \tilde{U}_q(su(2)) \). As in deformation quantization, we view \( \mathcal{A}^K \) as the vector space of complex–valued functions on \( \mathbb{R}^{2K+1} \), and consider the usual classical integral over \( \mathbb{R}^{2K+1} \). Observe that it is also invariant under the action \( \tilde{\triangleright} q (0.5) \) of \( \tilde{U}_q(su(2)) \), because the algebra structure
does not enter here at all. Explicitly, let \( \int d^{2K+1}a^K f \) be the ordinary integral of \( f \in \mathcal{A}^K \) over \( \mathbb{R}^{2K+1} \). The path integral is then defined as
\[
\int \mathcal{D}\Psi f[\Psi] := \int \prod_K d^{2K+1}a^K f[\Psi],
\]
where \( f[\Psi] \in \mathcal{A} \) denotes any integrable function (in the usual sense) of the variables \( a^{K,n} \). It is by construction invariant under \( \tilde{U}_q(su(2)) \).

Correlation functions can now be defined as functionals of “bracketed polynomials” \( P_*(a) = a^{K_1,n_1} \ast (a^{K_2,n_2} \ast \ldots \ast a^{K_l,n_l}) \) in the field coefficients by
\[
\langle P_*(a) \rangle := \frac{\int \mathcal{D}\Psi e^{-S[\Psi]} P_*(a)}{\int \mathcal{D}\Psi e^{-S[\Psi]}}.
\]  
(0.10)

This is natural, because all invariant actions \( S[\Psi] \) commute with the generators \( a^{K,n} \). Strictly speaking there should be a factor \( \frac{1}{\hbar} \) in front of the action, which we shall omit. Invariance of the action \( S[\Psi] \in \mathcal{A} \) implies that
\[
\langle u \tilde{\otimes}_q P_*(a) \rangle = \varepsilon_q(u) \langle P_*(a) \rangle.
\]

By construction, the number of independent modes of a polynomial \( P_*(a) \) with given degree is the same as for \( q = 1 \). One can in fact order them, using quasiassociativity together with the commutation relations (0.8). Therefore the symmetry requirement (4) above is satisfied. Using a suitable formalism, one can show that the requirements (2) and (3) are satisfied as well, see [1].

The field theories defined in this way are equivalent to ordinary QFT’s on the undeformed fuzzy sphere, with slightly nonlocal interactions. Consider an interaction term of the form (0.9). If we write down explicitly the definition of the \( \ast \) product of the \( a^{K,n} \) variables, then it can be viewed as an interaction term of \( a^{K,n} \) variables with a tensor which is invariant under the undeformed \( U(su(2)) \), obtained from the \( \tilde{U}_q(su(2)) \)--invariant tensor by multiplication with the twist \( F = 1 + o(\hbar) \). In other words, the above actions can also be viewed as actions on the undeformed fuzzy sphere \( S^2_{q=1,N} \), with interactions which are slightly “nonlocal” in the sense of \( S^2_{q=1,N} \), i.e. they are given by traces of products of matrices only to the lowest order in \( \hbar \). Upon spelling out the \( \ast \) product in the correlation functions (0.10) as well, they can be considered as ordinary correlation functions of a slightly nonlocal field theory on \( S^2_{q=1,N} \), disguised by the transformation \( F \). Therefore \( q \)--deformation simply amounts to some kind of nonlocality of the interactions. A similar interpretation is well–known in the context of field theories on spaces with a Moyal product.

Finally, it is possible to calculate correlators in perturbation theory, and to derive an analog of Wicks theorem. For lack of space, the reader is referred to [1].

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REFERENCES

1. H. Grosse, J. Madore, H. Steinacker, hep-th/0103164.
2. J. Madore, “The Fuzzy Sphere”, Class. Quant. Grav. 9, 69 (1992).
3. P. Podleś, Lett. Math. Phys. 14, 193 (1987).
4. H. Grosse, J. Madore, H. Steinacker, hep-th/0005273, to appear in J. Geom. Phys.
5. A. Yu. Alekseev, A. Recknagel, V. Schomerus, JHEP 9909, 023 (1999), hep-th/9908040.
6. A. Yu. Alekseev, A. Recknagel, V. Schomerus, J. High-Energy Phys. 0005 (2000) 010, hep-th/0003187.
7. H. Grosse, C. Klimcik, P. Presnajder, Int. J. Theor. Phys. 35, 231 (1996), hep-th/9505175.
8. R. Oeckl, Commun. Math. Phys. 217 (2001) 451-473; M. Chaichian, A. Demichev, P. Presnajder, J.Math.Phys. 41 (2000) 1647-1671.
9. V. G. Drinfel’d, Leningrad Math. J. 2, No.4, 829 (1991); V. G. Drinfel’d, Leningrad Math. J. 1, No.6, 1419 (1991).