Noise Fit, Estimation Error and a Sharpe Information Criterion

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Abstract

When optimizing the Sharpe ratio over a k-dimensional parameter space the thus obtained in-sample Sharpe ratio tends to be higher than what will be captured out-of-sample. For two reasons: the estimated parameter will be skewed towards the noise in the in-sample data (noise fitting) and, second, the estimated parameter will deviate from the optimal parameter (estimation error). This article derives a simple correction for both. Selecting a model with the highest corrected Sharpe selects the model with the highest expected out-of-sample Sharpe in the same way as selection by Akaike Information Criterion does for the log-likelihood as measure of fit.

Keywords: Model Selection, Sharpe Ratio, Akaike Information Criterion, AIC, Backtesting, Noisefit, Overfit, Estimation Error, Sharpe Ratio Information Criterion, SRIC

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1. Introduction

A convenient measure for return predictability is the Sharpe ratio, sometimes called information ratio. The Sharpe ratio is the ratio of annualized mean (excess) returns over return volatility. It is a useful statistic as it summarizes the first two moments of the return distribution, is invariant under leverage, and is therefore, to a second order approximation, the relevant metric regardless of the investor’s risk-return preferences.

When fitting a model - that is when maximizing the Sharpe ratio over a set of parameters - the thus obtained in-sample Sharpe ratio tends to be higher than what can be expected on unseen data (out-of-sample). For two reasons: First, there is a certain amount of noise fitting: the free parameter will be skewed towards the noise in the in-sample returns. Second, the estimate of the optimal parameter contains some estimation error and the performance using the estimated parameter is therefore less than if the optimal parameter were known.

Consider a linear regression model for (excess) stock returns. Even if no predictability exists, the ex-post optimal regression coefficients will have positive in-sample performance (noise fit). Further, even if predictability exists, the estimated regression coefficients will almost surely deviate from the optimal coefficients so that the performance that can be captured out-of-sample is less than the theoretical possible optimum (estimation error).

In this paper we attempt to quantify these two effects. That is we address the following questions:

1. What is the best estimate of the true Sharpe ratio (i.e. if the optimal parameter were known) of a strategy given its in-sample fit?

2. What is best estimate for the out-of-sample Sharpe ratio if one additionally considers estimation error?

For ordinary least square regression, the analogue of the first question is answered by the adjusted $R^2$ as under the true model the expected value of the adjusted $R^2$ equals the true $R^2$ (assuming the true variance of the dependent variable is known and equal to the unbiased sample estimate).

In more general settings (e.g. regression or also the setting at hand, see below), the Akaike Information Criterion (AIC) asymptotically adjusts for both, noise-fit and estimation error, if the performance is measured in terms of the log-likelihood function. Choosing a model that minimizes AIC chooses the model with the highest estimated out-of-sample log-likelihood (see\(^1\) (Akaike, 1974, around equation (2)) or (Burnham and Anderson, 2002, page 61)).

Adjusting the Sharpe ratio for noise fit and estimation error turns out to be simple. Let

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\(^1\)Note that AIC is rooted in information theory, but the corresponding entropy measure can be interpreted as expected out-of-sample log-likelihood (see section 5).
be the in-sample Sharpe ratio maximized over a \( k \)-dimensional parameter space and \( T \) years of in-sample data. Let \( \hat{\tau} \) be the out-of-sample Sharpe ratio of the optimal in-sample parameter. The main result of this paper is that if we define

\[
SRIC = \hat{\rho} \left( 1 - \frac{k}{T\hat{\rho}^2} \right)
\]

where SRIC stands for Sharpe Information Criterion then - in the setting at hand - holds

\[
E[\hat{\tau}] = E[SRIC] + o(1/T)
\]

In particular, SRIC is an asymptotically unbiased estimator (of order \( 1/T \)) of the out-of-sample Sharpe ratio. Moreover, the formula is exact in the linear case with Gaussian noise.

The adjustment in SRIC can be decomposed into noise fit and estimation error

\[
\hat{\tau} \approx SRIC = \hat{\rho} - \frac{k}{2T\hat{\rho}} - \frac{k}{2T\hat{\rho}^2}
\]

Notice the simplicity of the expression, only involving the in-sample Sharpe, the number of parameter and the length of the in-sample period. For example, if there is a system with \( k = 5 \) parameter and an optimal in-sample Sharpe of \( \hat{\rho} = 1 \) over \( T = 10 \) years of data, then the estimated out-of-sample Sharpe would be \( 1 - \frac{5}{20} - \frac{5}{20} = 0.5 \).

The first application that we had in mind when starting to work on this paper, however, is model selection. As intuition would tell, both effects - noise fit and estimation error- are increasing in the number of parameter. Hence any model selection criterion needs to account for both. And, vice versa, a performance measure that corrects for both yields a meaningful model selection criterion when comparing models with different degrees of freedom.

Figure 1 illustrates the SRIC as a model selection criterion as well as an informant about out-of-sample performance in form of Sharpe indifference curves, i.e. combinations of measured in-sample Sharpe and number of parameters that lead to the same expected out-of-sample Sharpe (for \( T = 10 \) years of in-sample data).

For mean-variance utility rather than the Sharpe ratio as measure of fit, the estimated out-of-sample utility turns out (not surprisingly) to be a (linear transformation) of the Akaike Information criterion (AIC). Of course, this is not surprising as the log-likelihood of a Gaussian random variable is mean- variance.

A second application, not remote to model selection, are Sharpe Efficient Frontiers. Under Sharpe efficient frontiers we understand classical mean-variance efficient frontiers that

\footnote{Here, \( k \) is the number of parameters that influence the estimated Sharpe ratio, i.e. the leverage of a portfolio is not counted. For example the problem of choosing the (ex-post) optimal portfolio out of \( k \) assets possesses \( k - 1 \) parameter as one parameter is redundant and only determines the volatility. To be precise, the scale parameter not only determines the volatility but also the sign of the portfolio (long or short). This, however, is a discrete choice and does not matter asymptotically (for large \( T \)).}
Figure 1 (Sharpe Indifference Curves): In-sample Sharpe ratios that are needed for a fixed expected out-of-sample Sharpe depending on the number of parameter assuming 10 years of in-sample data.

account for noise fit and estimation error of the plug-in estimator for the mean return. A third application is more academic. By removing the noise fit (but not necessarily accounting for the noise) one can answer ex-post questions like "What has been the realized risk premium for trend following in the 90s? And is it significantly different from the 00s?". The fourth application is the most direct one as noted before: Estimation of the out-of-sample performance. The applications are briefly sketched in sections 5 and 6.

Additional things to highlight when it comes to summing up the contributions are: i) the simplicity of formula (2). ii) unlike AIC (at least directly) it allows to compare models estimated over different time periods. iii) There is leeway to incorporate transaction cost into the equation. iv) We also derive uncertainty bounds for noise fit and estimation error and last but not least v) due to the close relation to the AIC, this paper also provide an interpretation of the latter in terms noise-fit and estimation error. (Personally, the authors have learned a lot about AIC while thinking about this problem)

We organized the paper into two almost separate parts: The linear case and the non-linear case. The linear case (up to section 6) contains as little technical details as possible to maximize readability. This part is essentially self-contained, though sometimes we refer to results from the general non-linear case in order to avoid redundancy. The technically more involved general part follows but is not necessary for linear part.

2. LITERATURE

This paper is linked to three interrelated strands of literature.

First, there is a strand which deals with estimation of out-of-sample statistics. For instance West (1996) provides tools to calculate moments of smooth functions of out-of-
sample predictions. The suggested technique is to use a Taylor expansion around the true parameter which what we also do here. Hansen (2009) derives the joint limiting distribution of noise-fit and estimation error for a wide class of objective functions (but not the Sharpe ratio).

The second -related - strand deals with information criteria and model selection. The seminal work of Akaike (e.g. see Akaike (1974), Akaike (1998a), Akaike (1998b)) develops the the Akaike Information Criterion as a model selection criterion (AIC). Though the AIC is formulated in terms of information theory - the AIC minimizes the estimated Kullback-Leibler divergence between the selected and the true model, there is a different - in many contexts more intuitive - interpretation. The Kullback-Leibler divergence between the true and estimated model equals the difference between the out-of-sample and in-sample fit, when fit is measured in terms of log-likelihood. Hence choosing the minimum AIC model chooses the model with the highest estimated out-of-sample log-likelihood. See also Stone (1977) who shows that in a regression context minimizing the AIC is asymptotically equivalent to minimizing the cross validation error (sum of squared residuals). As the log-likelihood in a Gaussian setting is the same as mean-variance utility the AIC even possesses a financial interpretation in terms of mean-variance utility (see section 5 for details). The paper at hand can be seen as deriving the analogue of the AIC when performance is measured in terms of the Sharpe ratio rather than mean-variance-utility. Schwarz et al. (1978) derives the Bayesian-Information-Criterion (BIC) which maximizes (under a flat prior) the ex-post probability of selecting the true model. A good overview over different model selection criteria is provided by Burnham and Anderson (2002).

A third strand applies results from the other two to estimate or maximize out-of-sample performance in a financial context. For instance Kan and Zhou (2007) estimate the out-of-sample performance for plug-in estimators in a mean-variance portfolio optimization (Markovitz) setting. The authors proceed to suggest to shrink the the Sharpe-optimal portfolio towards non-return sensitive portfolios such as the minimum variance or equally weighted portfolio in order to achieve better out-of-sample performance. Bossaerts and Hillion (1999) empirically assess a range of model selection criteria for return prediction. Bailey and Lopez de Prado (2014) and Harvey and Liu (2015) derive a correction for the Sharpe ratio of a backtest that is obtained from \( N \) independent trials and Novy-Marx (2015) looks at critical values for selecting \( k \) best out of \( n \) independent signals.

We note that Lo (2002) accounts for serial correlation in monthly data when computing the annualized Sharpe ratios. This is different to what we do here.
3. Set-Up

Fix a probability space \((\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\). Let \(\Theta \subset \mathbb{R}^{k+1}\) be a \((k+1)\)-dimensional open set (parameter space). Suppose that for every \(\theta \in \Theta\) there is an investment strategy producing cumulative (excess) returns \(s^\theta_t\) over \(T\) years. Call \([0,T]\) the in-sample period.

For each \(\theta \in \Theta\) let \(\hat{\mu}(\theta, \omega)\) be the random variable describing the annualized realized mean (excess) return of strategy \(\theta\) and let \(\Sigma(\theta, \theta')\) be the annualized covariance of the strategies \(\theta\) and \(\theta'\). I.e.

**Notation 3.1.** Denote the annualized realized mean return by \(\hat{\mu} : \Theta \times \Omega \rightarrow \mathbb{R}\) and by \(\Sigma : \Theta \times \Theta \rightarrow \mathbb{R}\) the annualized covariance between strategy \(\theta\) and \(\theta'\). The realized mean return is a noisy observation of the true mean return \(\mu : \Theta \rightarrow \mathbb{R}\), that is

\[
\hat{\mu}(\theta, \omega) = \mu(\theta) + \nu(\theta, \omega)
\]

where \(\nu(\theta, \cdot)\) is a random variable for each \(\theta \in \Theta\) (random field) with covariance \(\text{cov}(\nu(\theta), \nu(\theta')) = \frac{1}{T} \Sigma(\theta, \theta')\) and \(\mu(\theta)\) to be interpreted as the true mean return.

Note that we made the assumption that \(\nu\) has the same covariance matrix as \(s^\theta\) scaled by \(\frac{1}{T}\). This will be the outcome/justified if the system returns are obtained by betting with different weights parametrized by \(\theta\) on the same stocks or markets. See the main section 7 for details. While the true returns are observed with noise, we assume that the covariance matrix can be observed without error.\(^4\)

For example, \(\theta \in [0,1]\) could describe the weighting between a stock-bond portfolio. E.g. \(\theta = 0.60\) would describe a 60 – 40 stock-bond portfolio mix. \(T\) could be the last 10 years. \(\hat{\mu}(\theta)\) the annualized realized returns of a \(\theta - (1 - \theta)\) mix and \(\Sigma(\theta, \theta')\) the annualized covariance of e.g. a 60 – 40 and a 50 – 50 portfolio. In a more complex example, \(\theta \in \mathbb{R}^2_+\) could describe the lookback or smoothing parameter in a trend system.

In the next sections (up to section 7, however, we deal with the simpler special case: The case in which \(s^\theta_t\) depends linearly on \(\theta\):

**Example 3.2 (Linear Case).** Set \(\Theta = \mathbb{R}^{k+1}\), \(\mu(\theta) = \mu^T \theta\), \(\Sigma(\theta, \theta') = \theta^T \Sigma \theta'\), where \(\mu \in \mathbb{R}^{k+1}\) and \(\Sigma \in \mathbb{R}^{k+1,k+1}\) is a symmetric positive definite matrix, and \(\hat{\mu} = \mu + \nu\) where \(\nu\) is a random variable with covariance matrix \(\frac{1}{T} \Sigma\) (with abuse of notation).

Applications of the linear case are ample, in particular within portfolio management. For instance when the vector \(\theta\) describes the portfolio weights on assets \(i = 1, \ldots k + 1\), or exposure to (tradable) risk factors.

\(^3\)We use the notation \(k + 1\) as one dimension will only influence the volatility and only \(k\) parameter will influence the Sharpe ratio.

\(^4\)This is for instance true in the continuous time framework below. But also in reality it is close to the truth even without continuous return observations as the estimated covariances is orders of magnitudes more precise than the estimated mean returns if the number of parameter is not too large.
In a different interpretation, rather than observing $T$ years of data and using for $\hat{\mu}$ the sample mean, $\hat{\mu}$ could be a view or prior belief of the investor. In this case $\nu$ models the uncertainty around the view and $T$ describes its precision (inverse of the variance).

We distinguish between the in-sample Sharpe ratio $\rho$ and the out-of-sample Sharpe ratio $\tau$.

**Definition 3.3 (In-sample Sharpe ratio).** The in-sample Sharpe ratio of strategy $\theta$ is

$$\rho(\theta) = \frac{\hat{\mu}(\theta)}{\sqrt{\Sigma(\theta, \theta)}}$$

The (unobserved) out-of-sample Sharpe ratio, denoted by $\tau$, follows by removing the noise term from the mean returns:

$$\tau(\theta) = \frac{\mu(\theta)}{\sqrt{\Sigma(\theta, \theta)}}$$

Consider an investor who maximizes the Sharpe ratio among all parameter $\theta$. As there is a difference between the in-sample and out-of-sample Sharpe, both maximizers are different. Use the following notations (assuming a maximizer exists) which are also summarized in Table 2:

**Notation 3.4.** Denote by $\hat{\theta}$ be a parameter that maximizes optimal in-sample Sharpe ratio\(^5\).

$$\hat{\theta} \in \arg \max_{\theta \in \Theta} \rho(\theta)$$

and let $\theta^*$ be a parameter maximizing the true Sharpe ratio.

$$\theta^* \in \arg \max_{\theta \in \Theta} \tau(\theta)$$

Abbreviate $\hat{\rho} = \rho(\hat{\theta}), \hat{\tau} = \tau(\hat{\theta}), \tau^* = \tau(\theta^*), \rho^* = \rho(\theta^*)$

Notation 3.4 leads to the decomposition that is central to this paper:

$$\hat{\tau} = \hat{\rho} - (\hat{\rho} - \rho^*) + \tau^* - \rho^* - (\tau^* - \hat{\tau})$$  \hspace{1cm} (3)

The decomposition (3) says that the out-of-sample Sharpe equals the in-sample Sharpe minus a correction for the noise fit and the estimation error plus a noise component. We record this decomposition in the next definition.

\(^5\)Note that due to the scale parameter, the maximizer cannot be unique, but the quantities of interest, $\rho(\hat{\theta}), \rho(\theta^*), \tau(\hat{\theta}), \tau(\theta^*)$ might be.
\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|p{6cm}|}
\hline
Value & Symbol & Parameter & Description \\
\hline
\rho(\theta) & \hat{\rho} & \hat{\theta} & Sharpe of optimal in-sample parameter applied to in-sample dataset \\
\rho(\theta^*) & \rho^* & \theta^* & Sharpe of optimal out-of-sample parameter applied to in-sample dataset \\
\tau(\theta) & \hat{\tau} & \hat{\theta} & Sharpe of optimal in-sample parameter applied to out-of-sample dataset \\
\tau(\theta^*) & \tau^* & \theta^* & Sharpe of optimal out-of-sample parameter applied to out-of-sample dataset \\
\hline
\end{tabular}
\caption{The four combinations of in- and out-of-sample sharpe and true and estimated parameter}
\end{table}

Definition 3.5. The in-sample Sharpe ratio can be decomposed into
\[ \hat{\rho} = \hat{\tau} + N + E + U \]
with \( E[U] = 0 \). where the following definitions are applied
\[
\begin{align*}
N &= \rho(\hat{\theta}) - \rho(\theta^*) \quad (\text{Noise Fit}) \\
E &= \tau(\theta^*) - \tau(\hat{\theta}) \quad (\text{Estimation Error}) \\
U &= \rho(\theta^*) - \tau(\theta^*) \quad (\text{Noise})
\end{align*}
\]

Naturally two questions emerge. First, (1) how informative is \( \hat{\rho} \), the maximal in-sample Sharpe ratio, for \( \tau^* \) the true optimum? Choosing the (ex post) optimal parameter \( \hat{\theta} \) will inevitably lead to some noise fit of the data and therefore its Sharpe ratio will (in expectation) overestimate the true risk adjusted return (noise fit). But by how much? And second, if the estimated system \( \hat{\theta} \), rather than the true system \( \theta^* \), is applied to unseen data, i.e. out-of-sample, what Sharpe ratio can be expected. That is (2) how severe is the estimation-error?

Both questions are approached below. As mentioned, we start with the simple linear case in which \( \mu \) and \( \Sigma \) are linear (resp. quadratic) in \( \theta \) before we extend the result to a more generalize case in section 4.

4. THE LINEAR CASE

This section deals with the simpler case in which the returns depend linearly on \( \theta \). The more general case will need some additional structural assumptions that the linear case allows to skip.

For this purpose let\(^6\) \( \Theta = \mathbb{R}^{k+1} \) and let there be a vector \( \mu \in \mathbb{R}^{k+1} \) and a positive definite matrix \( \Sigma \in \mathbb{R}^{k+1,k+1} \) which is assumed to be invertible (otherwise no optimal parameter exists

\(^6\)Remember that \( \theta \in \mathbb{R}^{k+1} \) counts as \( k \) parameters as one of the parameters only influences volatility (leverage) but not the Sharpe ratio.
or at least one is redundant) and a random variable $\nu(\omega) \in \mathbb{R}^{k+1}$, normally distributed with mean 0 and covariance matrix $\frac{1}{T}\Sigma$ such that for all $\theta, \theta' \in \Theta = \mathbb{R}^{k+1}$

\[
\begin{align*}
\mu(\theta) &= \mu^T \theta \quad \text{(annualized true mean)} \\
\nu(\theta) &= \nu^T \theta \quad \text{(annualized realized noise)} \\
\Sigma(\theta, \theta') &= \theta^T \Sigma \theta' \quad \text{(annualized covariance)} \\
\hat{\mu}(\theta) &= \hat{\mu}^T \theta = (\mu + \nu)^T \theta \quad \text{(annualized realized mean)}
\end{align*}
\]

with some abuse of notation. Alternatively, $\hat{\mu}$ could be a view on the mean return or a prior belief. In this case, $T$ would not be the length of the in-sample period but a scalar expressing the confidence of the view.

The assumptions of this section hold for example if $\theta$ is the weight vector of a long-short portfolio or in a linear regression context.

**Example 4.1 (Portfolio Choice).** Let there be $k + 1$ markets with returns $r_i$. Let

\[ s_\theta^\theta = \sum_{i=0}^{k} \theta_i r_i \]

be the return of the portfolio $\theta$. With $\hat{\mu}$ the realized mean return vector, $\Sigma$ the covariance matrix of $r_i$, $\mu$ the (unknown) true mean and $T$ the observation length (all annualized), the setting can be described as linear case.

**Example 4.2 (Regression).** Let there be $i = 1 \ldots N$ markets with return $r_i$. Suppose we want to estimate the following linear model

\[ r_t = \sum_{j=0}^{k} x_t^{i,j} \hat{\theta}_j + \varepsilon_t \]

where $x^{i,j}$ are some exogenous predetermined factors and where the random errors $\varepsilon_t^i$ have covariance $S_t$ (assumed to be known). We can write the prediction as $\hat{r}_t = \sum_{j=0}^{k} x_t^{i,j} \hat{\theta}_j = x_t \hat{\theta} \in \mathbb{R}^N$. A natural way to estimate $\theta$ would be minimize the sum of squares (regression).

\[
\begin{align*}
\min_{\theta} \sum_t (r_t - x_t \theta)^T S_t^{-1} (r_t - x_t \theta) \\
\Leftrightarrow \min_{\theta} -2 \sum_t r_t^T S_t^{-1} x_t \theta + \sum_t \theta^T x_t^T S_t^{-1} x_t \theta \\
\Leftrightarrow \max_{\theta} 2\hat{\mu}^T \theta - \theta^T \Sigma \theta \\
\text{with } \hat{\mu} = c \sum_t r_t^T S_t^{-1} x_t \text{ and } \Sigma = c \sum_t x_t^T S_t^{-1} x_t
\end{align*}
\]

\(^7\)Normality is assumed for simplicity as it fits in the framework of the general case.
where \( c \) is an annualization factor.

Hence the regression is the same as mean-variance optimization and the estimate \( \hat{\theta} \) obtained via regression also maximizes the Sharpe ratio of a strategy in which the weights are set to the covariance weighted prediction, i.e \( w_t = S_t^{-1} \hat{r}_t \). In other words regression and maximization of the Sharpe ratio are equivalent when it comes to parameter estimation. Both are different, however, when it comes to estimating the out-of-sample statistics and model selection. While for regression (mean-variance utility) the Akaike information criterion is the best estimate for out-of-sample performance, for the Sharpe ratio the best estimate is given by the SRIC (Theorem 4.3) developed in this paper.

Now observe that a parameter maximizing the in-sample Sharpe ratio as well as in-sample mean variance utility is \( \hat{\theta} = \Sigma^{-1} (\mu + \nu) \). Therefore

\[
\rho(\hat{\theta}) = \frac{\hat{\mu}^T \hat{\theta}}{\sqrt{\hat{\theta}^T \Sigma \hat{\theta}}}
\]

\[
= \frac{\sqrt{(\mu + \nu)^T \Sigma^{-1} (\mu + \nu)}}{\|\mu + \nu\|_{\Sigma^{-1}}} \tag{4}
\]

with \( \|x\|_{\Sigma^{-1}} = \sqrt{x^T \Sigma^{-1} x} \).

Similarly, for the optimal out-of-sample Sharpe ratio

\[
\tau(\theta^*) = \|\mu\|_{\Sigma^{-1}}
\]

with \( \theta^* = \Sigma^{-1} \mu \). Plugging \( \hat{\theta} \) into \( \tau \) leads to the out-of-sample Sharpe ratio \( \hat{\tau} \)

\[
\hat{\tau} = \frac{\mu^T \hat{\theta}}{\sqrt{\hat{\theta}^T \Sigma \hat{\theta}}} \quad \text{with} \quad \hat{\theta} = \Sigma^{-1} (\mu + \nu)
\]

\[
= \frac{\mu^T \Sigma^{-1} (\mu + \nu)}{\|\mu + \nu\|_{\Sigma^{-1}}} \tag{5}
\]

This shows that there is a geometric relationship between \( \tau^* \), \( \hat{\rho} \) and \( \hat{\tau} \). While \( \tau^* \) is the length of \( \mu \), \( \hat{\rho} \) is the length of \( \mu + \nu \). \( \hat{\tau} \) is obtained from projecting \( \mu \) on \( \mu + \nu \). For an illustration see Figure 3.

Due to this geometric relationship (Euclidean geometry is linear in squares), it is in fact easier to compute noise fit and estimation error for mean-variance utility rather than the Sharpe ratio (for this see the next subsection). But here, we are interested in the Sharpe ratio

**Theorem 4.3 (Estimation Error and Noise Fit Combined for the Sharpe Ratio).** Under the assumptions of this section holds

\[
\mathbb{E} [N + E + U] = \mathbb{E} \left[ \frac{k}{T \hat{\rho}} \right]
\]

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Figure 3: (Geometry of the Sharpe Ratio) The true out-of-sample Sharpe$\tau^* = \|\mu\|_{\Sigma^{-1}}$ equals the length of the vector $\mu$. The in-sample Sharpe $\hat{\rho} = \|\mu + \nu\|_{\Sigma^{-1}}$ equals the length of $\mu + \nu$. The out-of-sample Sharpe $\hat{\tau} = \mu^T \Sigma^{-1} (\mu + \nu) / \sqrt{(\mu + \nu)^T \Sigma^{-1} (\mu + \nu)}$ is obtained by projecting $\mu$ on the line generated by $\mu + \nu$.

In particular

$$E[\hat{\tau}] = E\left[\hat{\rho} \left(1 - \frac{k}{T \hat{\rho}^2}\right)\right]$$

(6)

so that $SRIC = \hat{\rho} - \frac{k}{T \hat{\rho}}$ is an unbiased estimator for the out-of-sample Sharpe $\hat{\tau}$.

Proof. See Appendix.

Theorem 4.3 puts noise-fit and estimation error together by showing that

$$SRIC = \hat{\rho} - \frac{k}{T \hat{\rho}}$$

is an unbiased estimator of the expected out-of-sample Sharpe ratio $E[\hat{\tau}]$. We give it the name SRIC for Sharpe Ratio Information Criterion for reasons that become clear in section 5. The Theorem is illustrated in Figures 4a to 4c. These figures show the (average) in-sample Sharpe, the in-sample Sharpe adjusted for noise-fit, the in-sample Sharpe adjusted for both, noise fit and estimation error, and the out-of-sample Sharpe when the true Sharpe is $\tau^* = 0$ (left hand side) or $\tau^* = 1$ (right hand side) for different numbers of parameters $k$ ($k = 1, 5, 10)$. All numbers are averages over 10,000 random draws. As the theorem claims, the in-sample Sharpe adjusted for both, noise-fit and estimation error, fits the out-of-sample Sharpe.
Figure 4a: (average) In-sample Sharpe ratio, adjustments for noise fit and estimation error and out-of-sample Sharpe when the true Sharpe is $\tau^* = 0$ (left hand side) or $\tau^* = 1$ (right hand side) when the number of parameters is $k = 1$. All numbers are averages over 10,000 random draws.

In section 7 (Theorem 7.8) we show that the bias correction in (6) can be split into noise fit and estimation error

$$\hat{\tau} \approx \text{SRIC} = \hat{\rho} - \frac{k}{2T\hat{\rho}} - \frac{k}{2T\hat{\rho}}$$

At this point, the reader will agree that the results can be generalized beyond the linear case, e.g. to the case in which $\theta$ is a smooth parametrization. In the long run, the estimated parameter will converge towards the true one and locally the behavior will be as in the linear case. This generalization will be dealt with in section 7.

4.1. MEAN VARIANCE UTILITY

We conclude the section by proving an analogous result for mean variance utility rather than the Sharpe ratio as measure of fit. These results are not new, for instance Kan and Zhou (2007) and Hansen (2009) show results very close to ours. However, we find it instructive to present them in a way consistent with the set-up here, in particular when we later draw comparisons to the Akaike Information Criterion (AIC).

For this let

$$\hat{u}(\theta) = 2\hat{\mu}(\theta) - \gamma \Sigma(\theta, \theta)$$

the in-sample and

$$u(\theta) = 2\mu(\theta) - \gamma \Sigma(\theta, \theta)$$
Figure 4b: (average) In-sample Sharpe ratio, adjustments for noise fit and estimation error and out-of-sample Sharpe when the true Sharpe is $\tau^* = 0$ (left hand side) or $\tau^* = 1$ (right hand side) when the number of parameters is $k = 5$. All numbers are averages over 10,000 random draws.

Figure 4c: (average) In-sample Sharpe ratio, adjustments for noise fit and estimation error and out-of-sample Sharpe when the true Sharpe is $\tau^* = 0$ (left hand side) or $\tau^* = 1$ (right hand side) when the number of parameters is $k = 10$. All numbers are averages over 10,000 random draws.
the out-of-sample mean variance-utility. The parametrization is such that for $\gamma = 1$ holds $\hat{u}(\hat{\theta}) = \rho(\hat{\theta})^2$.

The analogue of definition 4.4 is

**Definition 4.4.** The in-sample mean-variance utility can be decomposed into

$$\hat{u}(\hat{\theta}) = u(\hat{\theta}) + N_{\text{MV}} + E_{\text{MV}} + U_{\text{MV}}$$

with $N_{\text{MV}} = \hat{u}(\hat{\theta}) - \hat{u}(\theta^*)$ (Noise Fit), $E_{\text{MV}} = u(\theta^*) - u(\hat{\theta})$ (Estimation Error), $U_{\text{MV}} = \hat{u}(\theta^*) - u(\theta^*)$ (Noise).

Simple quadratic optimization shows that $\hat{\theta} = \frac{1}{\gamma} \Sigma^{-1} (\mu + \nu)$ maximizes in-sample utility $\hat{u}$, while $\theta^* = \frac{1}{2} \Sigma^{-1} \mu$ maximizes out-of-sample utility $u$. And that $\hat{u}(\hat{\theta}) = \frac{1}{\gamma} \rho^2$ as well $u(\theta^*) = \frac{1}{\gamma} \tau^2$.

Now it is straightforward to see that for mean-variance utility

$$E[N_{\text{MV}}] = E[\hat{u}(\hat{\theta})] - E[\hat{u}(\theta^*)]$$

$$= E\left[\frac{1}{\gamma} \rho(\hat{\theta})^2\right] - \frac{1}{\gamma} \tau(\theta^*)^2 + 0$$

$$= \frac{1}{\gamma} E\left[||\mu + \nu||_{\Sigma^{-1}}^2 - ||\mu||_{\Sigma^{-1}}^2\right]$$

$$= \frac{1}{\gamma} E\left[2 \mu^T \Sigma^{-1} \nu + \nu \Sigma^{-1} \nu\right]$$

$$= \frac{k + 1}{\gamma T}$$

Similarly

$$E[N_{\text{MV}} + E_{\text{MV}} + U_{\text{MV}}] = E[\hat{u}(\hat{\theta})] - E[u(\hat{\theta})]$$

$$= \frac{1}{\gamma} E\left[||\mu + \nu||_{\Sigma^{-1}}^2 - \left[2 \mu^T \Sigma^{-1} (\mu + \nu) - ||\mu + \nu||_{\Sigma^{-1}}^2\right]\right]$$

$$= \frac{2(k + 1)}{\gamma T}$$

This shows

**Theorem 4.5 (Noise-Fit and Estimation Error for Mean-Variance).** Under the assumptions of this section holds

$$E[N_{\text{MV}}] = \frac{k + 1}{\gamma T}$$
and

\[ \mathbb{E}[\mathcal{N}_{\text{MV}} + \mathcal{E}_{\text{MV}} + \mathcal{U}_{\text{MV}}] = \frac{2(k+1)}{\gamma T} \]

In particular \( \hat{u}(\hat{\theta}) - \frac{2(k+1)}{\gamma T} = \frac{1}{\gamma} \beta^2 - \frac{2(k+1)}{\gamma T} \) is an unbiased estimator for the out-of-sample utility \( u(\hat{\theta}) \).

The easiness of the proof was due to the beauty of the mean-variance utility resp. squared Sharpe as performance measure - its geometry is linear.

We will later show that the Akaike Information Criterion (AIC) can (after a linear transformation) be interpreted as \( \hat{u}(\hat{\theta}) - \frac{2(k+1)}{\gamma T} \) and therefore an unbiased estimator of the out-of-sample utility.

5. Model Selection (Sharpe Information Criterion)

5.1. Sharpe Information Criterion

In the previous sections, the objective was to quantify noise-fit and estimation error in order to gain insights about the out-of-sample Sharpe ratio. Now let \( \Theta = \Theta_1 \cup \ldots \cup \Theta_n \) with \( \Theta^i \subset \mathbb{R}^{k_i + 1} \) be a family of parameter spaces. Model selection is about selecting a pair \((i, \theta_i)\) with \( \theta_i \in \Theta^i \), that is a model \( i \in \{1, \ldots, n\} \) and a parametric fit \( \theta_i \).

A typical goal of model selection is to choose the model with the highest expected out-of-sample fit\(^8\). Theorem 7.8 therefore suggests to choose the model with the highest SRIC

\[ \text{SRIC}^i = \rho(\hat{\theta}_i) - \frac{k_i}{T \rho(\theta_i)} \]  

where \( \hat{\theta}_i \) maximizes \( \rho \) over \( \Theta^i \).

From the preceding sections it follows that choosing the model with the highest SRIC is choosing the model with the highest estimated out-of-sample performance. This in itself justifies the SRIC as model selection criterion. Moreover, in section 5.4 we show that this is exactly the same property that the Akaike Information Criterion (AIC) as model selection criterion has for log-likelihood rather than the Sharpe ratio as performance measure.

\(^8\)Some criteria have other goals. E.g. the Bayesian Information Criterion (BIC) maximizes the asymptotic posterior probability of choosing the true model.
5.2. Example: A Simulation

We illustrate the SRIC as a model selection criterion in the following simulation. We simulate 5 years of returns for 100 markets. The first 50 markets have a true Sharpe ratio uniformly distributed between $[-0.5, 0.5]$ and markets 51\ldots, 100 a true Sharpe of 0. We set $\Theta_i = \mathbb{R}^i$ where a model $i$ means choosing the best long-short weighting of the first $i$ markets. We then apply two model selection criteria: SRIC and AIC (see subsection 5.4 for details on the latter in this context). We thereafter compute the out-sample Sharpe ratio to judge the quality of the selected models. We finally average over $N = 100,000$ such simulations.

Figure 5 illustrates the results. On the left hand side one sees the histogram of the chosen dimension for SRIC and AIC. SRIC chooses higher dimensions (See Remark 5.1). The vertical lines mark the average. The right hand side shows the histogram of out-of-sample Sharpe ratios of the selected models. It is evident that in this example the average Sharpe ratio for SRIC (mean equals 0.893) is higher than for AIC (mean equals 0.364).

\begin{figure}
\centering
\includegraphics[width=\textwidth]{histograms}
\caption{Histograms of #Parameter chosen (Left) and OOS Sharpe (Right) for AIC (blue) and SRIC (red)}
\end{figure}

Figure 5 (Simulation): Model Selection by AIC (blue) and SRIC (red), based on 100,000 draws of 5 years of data. SRIC tends to choose a higher dimension and achieve a higher out-of-sample Sharpe.

5.3. Example: A Carry Strategy

We further illustrate the SRIC by a simple carry strategy. For the avoidance of any doubt, this is for illustrational purposes only, we would not recommend to trade this strategy.

For this, we use spot and forward prices for 20 different currencies from January 2000 to October 2015. Each month we (re-)build our portfolio, which consists of $k+1$ base strategies.

The first base strategy is a 12-month trend following strategy, where the weight is simply
the 12 month moving average return (divided by the current variance). The other strategies are carry strategies. The second base strategy is to set the weight equal to the current interest rate differential versus the US-Dollar (divided by the current variance), that is the difference between the (log) spot and the (log) forward price (lag 0). The third base-strategy is the same but using 1-month lagged interest rate differentials (lag 1). The fourth one uses 2-month lagged interest differentials and so on (lag 2). A model with dimension \( k \) is now a combination of the 12-month trend strategy and \( k \) lagged interest rate differential strategies.

By combining these base strategies to maximize the in-sample Sharpe ratio, we are essentially performing a regression of the currency returns on the predictions implied by the base strategies (factors). It is also intuitive that the additional information provided by including more and more lags diminishes and will at some point be outweighed by the cost in terms of overfitting (noise fit and estimation error).

The SRIC derived in the previous section can now serve as a model selection criterion. We illustrate this in Figure 6. The left hand side shows the cumulative returns of the optimal in-sample combination of the 12-month trend strategy and carry strategies with up to 2 lags. The right hand side shows the in-sample Sharpe ratio for strategies combining the trend base strategy with up to 5 lagged carry strategies. The in-sample Sharpe ratio increases with the dimension, but after correcting for noise fit and measurement error (SRIC) it reaches a peak at 2 lags. That is, in this example, the methodology would recommend to use a strategy which combines the trend base strategy with the lag 0 and lag 1 carry strategy.

**Figure 6:** Left: cumulative returns for different in-sample strategy. Right: in-sample Sharpe ratio, correction for noise fit and correction for noise fit and estimation error (SRIC).
5.4. Relation to AIC

In order to derive the AIC, we need to associate \( \theta \in \Theta \) with a prediction and derive the log-likelihood. We will do this in detail in section 4. Here we just anticipate Theorem 7.6 and state that (for an appropriate reference measure \( \mathbb{P}^0 \))

\[
\log \frac{d\mathbb{P}^\theta}{d\mathbb{P}^0} [ (p_t)_{t \in [0,1]} ] = T \gamma \hat{\mu}(\theta) - \frac{1}{2} T \Sigma(\theta, \theta) = \frac{T}{2} \hat{u}(\theta)
\]

In particular (not surprising in the Gaussian case) the log-likelihood is a multiple of mean variance utility \( \hat{u} \) and maximizing the former is equivalent to maximizing the latter. Hence

\[
AIC = -T \gamma \hat{u}(\hat{\theta}) + 2(k + 1)
\]

and therefore

\[
-\frac{2AIC}{T} = \gamma \hat{u}(\hat{\theta}) - \frac{2(k + 1)}{T}
\]

or

\[
-\frac{2AIC}{T} = \rho^2 - \frac{2(k + 1)}{T}
\]

That is, we can express the AIC in terms of mean variance utility or the squared Sharpe ratio.

Theorem 4.5 and later in the non-linear case Theorem 7.11 deliver the interpretation of the AIC as expected out-of-sample mean-variance-utility controlling for noise-fit and estimation error.

**Remark 5.1.** SRIC chooses a higher dimension than AIC, but both converge towards each other for \( T \to \infty \). To see the first statement, note that

\[
\frac{\partial \text{SRIC}}{\partial \rho} - \frac{\partial \text{SRIC}}{\partial k} = T \rho + \frac{k}{\rho} \geq T \rho = \frac{\partial \text{AIC}}{\partial \rho} - \frac{\partial \text{AIC}}{\partial k}
\]

To see the second statement, observe

\[
T (\text{SRIC})^2 = T \left( \hat{\rho} - \frac{k}{T \hat{\rho}} \right)^2
\]

\[
= T \hat{\rho}^2 - 2k + \left( \frac{k}{\hat{\rho}} \right)^2 \frac{1}{T}
\]

\[
= -AIC + \text{const} + \left( \frac{k}{\hat{\rho}} \right)^2 \frac{1}{T}
\]

Hence the difference is of lower order \( T^{-1} \) than \(-AIC = T \rho^2 - 2(k + 1)\) itself.
6. Other Applications

6.1. Application II: Out-Of-Sample-Efficient Frontier

In classic (unconstrained) portfolio optimization (compare to Markowitz (1952)), the portfolio weights are set to

$$w = \frac{1}{\gamma} \Sigma^{-1} \hat{\mu}$$

(8)

where $\gamma$ is parameter of risk aversion or a Langrange multiplier tuned to achieve a certain target volatility. The caveat of this approach is that estimation error in $\hat{\mu}$ (and for large dimensions also in $\Sigma$ but this is ignored here) deteriorates out-of-sample performance. As the loss due to noise fit and estimation error depends on the dimension of the asset-space so does the estimated out-of-sample ratio. At some point it will decline in the number of assets finally offsetting any diversification benefit (see e.g. Kan and Zhou (2007)).

It therefore makes sense to consider an estimate of the out-of sample Sharpe ratio rather than the naive in-sample estimate as the slope of efficient frontier, e.g. SRIC.

We illustrate this application in Figure 7 in form of a simulation. We simulate $T = 10$ years of returns for $K = 20$ assets with a Sharpe ratio of 1 and mutual correlation of 0.5. We then compute the naive in-sample Sharpe and the estimated out-of-sample Sharpe (SRIC) for the best in-sample portfolio built out of the first $k = 1, \ldots, K$ assets. We average over $N = 10,000$ simulations. The left hand picture shows the (average) naive mean-variance-efficient frontier and the out-of-sample-efficient frontier (SRIC) for $k = 1, 10, 20$. The right hand side shows the SRIC for $k = 1, \ldots, 20$.

It can be seen that even though we use 10 years of data with and individual Sharpe of 1, the SRIC peaks at only 4 assets. The reason is that estimation error is so severe that the diversification effect of additional correlated assets is quickly offset. A better approach would therefore be to first build independent base portfolios (e.g. based on principal components) and then successively build the portfolio by including the next base portfolios as long as the SRIC is improved.

6.2. Application III: Ex-Post Analysis

Another application of the adjustments derived in this paper is more academic. By removing the noise fit questions like the following can be answered: How high has a specific risk premium (e.g. for trend following) been in a specific period. For example: How high has the return to a trend following strategy been in the 90s compared to 2000s? Was it different or did the time-scale of trends change?

To illustrate this, we build a simple trend following system on 43 Futures markets across
all sectors (Bonds, Foreign Exchange, Equities, Metals, Energy and Soft Commodities). We set the weight equal to the $x$-day exponential moving average of past returns and aggregate across sectors (scaling sectors to the same level of volatility by a rolling 1 year volatility).

Looking back 5 years of data each month, we choose the time-scale $x$ which maximized the Sharpe ratio (hence there are $k = 1$ parameter). This is not a linear parameter dependency so we are not in the set-up of the linear case here but refer to the general case section 7. We then calculate the in-sample Sharpe and remove the expected noise fit via Theorem 7.8. This is illustrated in Figure 8. The blue line is the in-sample Sharpe ratio (rolling 5 years). The black line the estimate of the true Sharpe ratio (i.e. adjusted for noise fit). The grey area is the 80%-confidence set$^9$ around the noise fit adjustment (note the confidence set for the true-Sharpe ratio are larger as they would include the in-sample noise $\mathcal{U}$ as well). For completeness, the red line denotes the SRIC and the rose-colored area the 80%-confidence set around the combination of noise fit and estimation-error. The green line is the Sharpe of the last 5 years returns where each month the time-scale $x$ is set to the prior 5 years optimum (rolling out-of-sample Sharpe).

It can be seen that even after accounting for the uncertainty of the noise-fit, the Sharpe ratio of a simple trend strategy declines over the years. Though (based on the graph) it cannot be ruled out that this is due to noise $\mathcal{U}$, the graph indicates that a change in the time-scale $x$ is not the culprit.

Note that this application is akin to testing for joint predictability of a set of factors (not checking significance of individual coefficients)

$^9$For computations of confidence sets see Theorem 7.10
6.3. Application IV: Estimation of Out-Of-Sample-Sharpe

The most obvious application of the SRIC is of course as estimator of the out-of-sample Sharpe ratio.

We illustrate this again by a simple trend following system. Unlike in the last section, however, we fix the time-scale to $x = 200$ days. Instead, we optimize on a rolling basis (5 year lookback) over the sector weights. The cumulative returns for this are shown in Figure 9.

Figure 10 shows the rolling 5-year in-sample Sharpe ratio (blue), the adjustment for noise-fit (black), the SRIC (red), the 5 year rolling out-of-sample Sharpe (rolling optimal weights for the last 5 years, green) and the next 5-years out-of-sample Sharpe (magenta). Hence the magenta line equals the green line shifted 5 years back. Note how the out-sample-Sharpe gets unstable at the end of the period as there are less then 5 are remaining for computation. For completeness, the yellow line denotes the rolling Sharpe of the equally weighted systems.

Several things are remarkable. First the uncertainties around noise-fit and - estimation error (even excluding the uncertainty of the noise) are large, i.e. the 80% confidence sets are wide. Second, the forward looking 5 year out-of-sample Sharpe (magenta line) wobbles around its estimate (red) as should be for an estimator.
7. THE NON-LINEAR CASE

7.1. NON-LINEAR CASE (STRUCTURAL ASSUMPTIONS)

In order to quantify the noise fit some structure on the relation between the parameter $\theta$ and the noise it can fit is needed. This relation is implicit in the linear case (e.g. $\text{cov}(\nu(\theta), \nu(\theta')) = \Sigma(\theta, \theta')/T$), but demands more attention in the general case.

Roughly speaking, the assumption that we need is that the system returns $s^\theta$ are obtained by betting the weight $w_t$ (not necessarily observed) on markets with return $r_t$.

For that let $p_t$ be a process of cumulated excess returns on $m$ abstract markets given by a solution to

$$\frac{dp_t}{p_t} = Y_t dt + S_t dW_t$$

with a $m$-dimensional Brownian Motion $W$ and a deterministic process $Y = (Y_t)_{t \in \mathbb{R}}$ and $S_t \in \mathbb{R}^{m,m}$ invertible. Here $Y$ is the (unknown) predictable component of the excess returns. For illustrational purposes assume that $S_t = I_t$, the identity matrix. This is in fact without loss of generality after a potential rotation and change of leverage of the markets $p_t$ for $X$.

Let

$$X : [0, T] \times \Theta \to \mathbb{R}^m \hspace{1cm} \text{with } X(\theta) \in L^2([0, T]) \hspace{1cm} \forall \theta \in \Theta$$

be a parametrized deterministic process $X = (X_t)_{t \geq 0}$, the (parametrized) predictions.

The assumption is now that the system returns are given by betting $w_t = X_t/p_t$ on the
Figure 10 (Estimation of Out-of-Sample Sharpe): rolling 5 year in-sample Sharpe (blue), estimated out-of-sample Sharpe (red) and out-of-sample Sharpe (green (backlooking) and magenta (forward looking))

markets \( p_t \) (Note that these weights maximize the Sharpe ratio if the expected return at \( t \) is \( X_t \)). Doing so leads to the cash-flow

\[
ds_t^\theta = w_t^T dp_t = X_t(\theta)^T Y_t dt + X_t(\theta)^T dW_t
\]

Let \( \Sigma(\theta, \theta') \) be the (annualized) quadratic covariation of \( s^\theta \), that is

\[
\Sigma(\theta, \theta') = \frac{1}{T} \int_0^T X_t^T(\theta) X_t(\theta') dt \quad (9)
\]

The (annualized) realized returns are then given by

\[
\hat{\mu}(\theta) = \frac{1}{T} \int_0^T X_t(\theta)^T Y_t dt + \frac{1}{T} \int_0^T X_t(\theta)^T dW_t = \mu(\theta) + \nu(\theta)
\]

where \( \nu(\theta) \) is a random variable with covariance \( \frac{1}{T} \Sigma \). Equations (9) and (10) have a specific structure via \( X \), which we need in the following. More specifically, we assume the following structural assumptions on \( \mu, \nu \) and \( \Sigma \) are met:

**Assumption 7.1 (Structural Assumption). Assume there exists \( m \in \mathbb{N} \), a process \( Y \in L^2([0,T], \mathbb{R}^m) \) and for very \( \theta \in \Theta \) a process \( X(\theta) \in L^2([0,T], \mathbb{R}^m) \) (all possibly unobserved)
such that

\[
\begin{align*}
\mu(\theta) &= \frac{1}{T} \langle X(\theta), Y \rangle \\
\nu(\theta) &= \frac{1}{T} \langle X(\theta) \cdot W \rangle \\
\Sigma(\theta, \theta') &= \frac{1}{T} \langle X(\theta), X(\theta') \rangle
\end{align*}
\]

where \( W \) is a \( m \)-dimensional Brownian Motion and \( \langle \cdot, \cdot \rangle \) denotes the inner product of \( L^2([0, T], \mathbb{R}^m) \)

\[
\langle X(\theta), X(\theta') \rangle = \int_0^T X_t(\theta)^T X_t(\theta') dt
\]

And \( X \cdot W \) denotes stochastic integration, i.e.

\[
X(\theta) \cdot W = \int_0^T X_t(\theta)^T dW_t
\]

Note that the linear model from section 4 is a special case with \( X(\theta) = X\theta \) and \( X \) is the Cholesky decomposition of \( \Sigma \) (\( Y \) and \( W \) can be backed out).

**Example 7.2.** For example let \( p_t = (p_t^{S&P500}, p_t^{US10}, p_t^{GSCI}) \) be the cumulative return\(^{10}\) of the S&P 500 Index, US 10-Year Treasuries and a Commodity Index. Here, \( W \) is the noise in these markets and \( Y \) is its predictable part. E.g. \( X(\theta) \) could be smoothed past returns and \( \theta \) is the smoothing parameter. The perfect system would be to bet \( Y \) units on these markets, but the investor might not know \( Y \). Instead he looks at betting the predictions (or weights) \( X(\theta) \) and looks for the system \( \theta \in \Theta \) that captures most of \( Y \).

**Remark 7.3.** The choice of a continuous-time model over a discrete-time model has mainly technical reasons: With continuous-time, the covariance matrix is observable. All results, however, also hold for a discrete time model assuming the covariance matrix is known (as can be easily seen by embedding a discrete time model into a continuous time). In this case, we also wouldn’t need Brownian motion as noise term, as long as the noise possesses sufficiently high moments.

**Remark 7.4.** Note that here the strategy predictions \( X \) are assumed to be deterministic, i.e. exogenous to the price process \( p_t \). This is quite often not strictly the case as many predictions are obtained by using past prices as input data. To relax this assumption would come at the price of much higher technical burden. But it also would not necessarily be the right thing to do. The analysis here fixes the prediction (and therefore the weights and risk levels), regards it as exogenous, and examines what happens if one only removes or re-samples the noise (disregarding that the noise might be part of the prediction). This is essentially conditioning on the realization of the prediction.

\(^{10}\)To be precise: rescaled to unit volatility and rotated so that the components are independent to fulfill the 'wlog assumption'
For technical reasons, we make the assumption that one of the parameters is a scale (leverage parameter). This is without loss of generality when looking at the Sharpe ratio as performance measure:

**Assumption 7.5 (Scale Parameter).** For any \( \theta \in \Theta \) and \( \lambda > 0 \) there is \( \theta' \in \Theta \) such that \( \mu(\theta') = \lambda \mu(\theta) \), \( \hat{\mu}(\theta') = \hat{\lambda} \hat{\mu}(\theta) \), \( \Sigma(\theta', \cdot) = \lambda \Sigma(\theta, \cdot) \).

Note that due to Assumption 7.5 a maximizer of mean variance utility also maximizes the Sharpe ratio and vice versa: for each maximizer of the Sharpe ratio a proper rescale maximizes mean-variance utility. Hence we can assume that \( \hat{\theta} \) maximizes both, \( \rho \) and \( \hat{u} \), and similarly \( \theta^* \) maximizes \( \tau \) and \( u \).

For the computation of the AIC we need the log-utility of a model \( \theta \). As this section laid down a predictive model underlying the assumptions, it is the right place to keep of hold of Theorem 7.6.

**Theorem 7.6.** Under model \( \theta \) in (12) the log likelihood (density with respect to an appropriate reference measure \( \mathbb{P}^0 \)) of the realized prices is given by

\[
\log \frac{d \mathbb{P}^\theta}{d \mathbb{P}^\theta_0} \left[ \left( p_t \right)_{t \in [0,1]} \right] = T \hat{\mu}(\theta) - \frac{1}{2} T \Sigma(\theta, \theta) = T \gamma \hat{u}(\theta)
\]

**Proof.** See appendix.

We are now almost ready to formulate the main theorem in a more general version. But beforehand we must make some technical assumptions.

### 7.2. Technical Assumption

Denote by \( \Theta^*_T \) be the parameter set such that the variance is bounded by \( (\tau^*_T)^2 + 1 \).

\[
\Theta^*_T = \Theta \cap \{ \theta \in \Theta \mid \Sigma(\theta, \theta) \leq (\tau^*_T)^2 + 1 \}
\]

We consider the problem for \( T \to \infty \). Therefore all variables will depend on \( T \) and therefore would need a subscript \( T \) - e.g. \( \rho_T \) - though we will drop it sometimes for better readability.

We make the following technical assumptions:

**Assumption 7.7 (Technical Assumptions).**

1. **(Uniqueness of Optimal Parameter)** \( \theta^*_T = \arg\max \tau_T(\theta) \) s.t. \( \Sigma(\theta, \theta) = \tau(\theta)^2 \) exists and is unique. Further it holds: For every sequence \( \theta^n_T \in \Theta \) scaled so that \( \Sigma(\theta^n_T, \theta^n_T) = \tau_T(\theta^n_T)^2 \) with \( \lim_{n \to \infty} \tau_T(\theta^n_T) = \tau_T(\theta^*_T) \) holds \( \theta^n_T \xrightarrow{\text{d}} \theta^*_T \) uniformly in \( T \). More precisely: For any \( \epsilon > 0 \), there is a \( \delta > 0 \) so that for all \( \theta \) with \( \Sigma(\theta, \theta) = \tau_T(\theta^*_T)^2 \), \( \| \tau_T(\theta) - \tau_T(\theta^*_T) \| < \delta \) implies \( \| \theta - \theta^*_T \| < \epsilon \) and \( \delta \) can be chosen independently of \( T \).
2. (Differentiability) There is $\epsilon > 0$ such that for all $\theta \in B(\theta^*_T, \epsilon) \cap \Theta^T_1$, the process $X(\theta)$ is three times continuously differentiable with uniformly (in $T$) bounded derivatives and so that the derivative of the stochastic integral commutes with the stochastic integral of the derivative. I.e $D^i(X \cdot W) = (D^iX) \cdot W$ for $i = 1, 2, 3$.

3. (Exclude Pathological Examples/ Finiteness of the Search Space) There exists $C \in \mathbb{R}$ such that

$$E \left[ \sup_{\theta \in \Theta^T_1} \left\| \frac{1}{\sqrt{T}} D^iX(\theta) \cdot W \right\| \right] \leq C \quad i = 0, 1, 2, 3$$

and

$$\sup_{\theta \in \Theta^T_1} E \left[ \left\| \frac{1}{\sqrt{T}} D^iX(\theta) \cdot W \right\|^2 \right] \leq C \quad i = 0, 1, 2, 3$$

4. (Invertibility or non-Degeneracy) $\left( \frac{1}{T} \langle DX, DX \rangle \right)^{-1}$ exists in a neighborhood around $\theta^*_T$ and its (matrix) norm is uniformly (in $T$) bounded on sets for which $\Sigma(\theta, \theta) = c$ constant.

5. (Boundedness) The true Sharpe ratio $\tau_T(\theta)$ is (uniformly) bounded, i.e. there exists a constant $C$ such that $|\tau_T(\theta)| \leq C$ for all $T \in \mathbb{R}^+$ and all $\theta \in \Theta$. Also, the true Sharpe ratio is bounded away from 0. That is there exists $\epsilon > 0$ such that $\tau_T(\theta^*_T) \geq \epsilon$ for all $T$.

6. (non-Degeneracy of Unpredictable Component) Further assume that

$$\langle D^2X(\theta^*), X(\theta^*) - Y \rangle = 0$$

(11)

The interpretation of Assumption 7.7 is as follows. 1. ensures that there is only one optimal parameter (and other ones do not come arbitrarily close). This will justify a Taylor expansion around the unique optimal parameter. 2. demands the predictions to depend smoothly on the parameter so that the Taylor expansion can be performed to a sufficient order. See Hutton and Nelson (1984) for conditions sufficient for differentiation and stochastic integration to commute (In this case boundedness of $X(\theta)$ (uniformly in $T$) would be sufficient). 3. limits the size of the search space after application of a concentration inequality. 4. and 5. ensure that the parameters are non-degenerated and well behaved. 6. ensures that varying the parameter around the optimal one does not (infinitesimally) move one closer to the unpredictable component $X(\theta^*) - Y$ (note that optimality of $\theta^*$ already implies $\langle D^2X(\theta^*), X(\theta^*) - Y \rangle \leq 0$). In other words what ever is unpredictable $(X(\theta^*) - Y)$ stays the same locally around $\theta^*_T$. Note that condition (11) is fulfilled when $Y = X(\theta^*)$.

### 7.3. Noise-fit and Estimation Error for the Sharpe ratio

In analogy to the linear case (Theorem 4.3) holds (main theorem)
Theorem 7.8 (Noise Fit and Estimation Error for the Sharpe). Under Assumptions 7.5, 7.1 and 7.7 holds. If $1/\hat{\rho} \in L^q$ for some $q > 1$, then

1. (Noise Fit)
\[
E[N] = E \left[ \frac{k}{2T\rho(\hat{\theta})} \right] + o \left( \frac{1}{T} \right)
\]

2. (Noise Fit and Estimation Error for the Sharpe)
\[
E[N + E + U] = E \left[ \frac{k}{T\rho(\hat{\theta})} \right] + o \left( \frac{1}{T} \right)
\]

Remark: If $1/\hat{\rho} \notin L^q$, then the formulas are true if $1/T\hat{\rho}$ is replaced by $1/T\hat{\rho} \wedge C$ for some constant $C$

Corollary 7.9 (Estimator for Out of Sample Sharpe). In particular, under the conditions of Theorem 7.8: If $1/\hat{\rho} \in L^q$

1. (Sharpe corrected for Noise Fit)
\[
\tau(\theta^*) = E \left[ \rho(\hat{\theta}) \left( 1 - \frac{1}{2T\rho(\hat{\theta})^2} \right) \right] + o(T^{-1})
\]

2. (Sharpe corrected for Noise Fit and Estimation Error)
\[
E[\tau(\hat{\theta})] = E \left[ \rho(\hat{\theta}) \left( 1 - \frac{k}{T\rho(\hat{\theta})^2} \right) \right] + o(T^{-1})
\]

Remark: If $1/\hat{\rho} \notin L^q$, the corollary is true if $1/T\hat{\rho}$ is capped, e.g. $\left( 1 - \frac{k}{T\rho(\hat{\theta})^2} \right)$ is replaced by $\left( 1 - \frac{k}{T\rho(\hat{\theta})^2} \right)^+$.

Proof. Immediate from Theorem 7.8

The theorem is a generalization of the linear case. All interpretations, e.g. the Sharpe indifference curves (Figure 1 and 4) apply.
7.3.1. Confidence Sets and Hypothesis Tests

We now want to estimate the uncertainty around the point estimates $\hat{\tau}_T$. For this we record the following theorem

**Theorem 7.10. (Asymptotic Distributions)** For the noise holds

$$
\mathcal{U} \overset{\mathcal{D}}{=} T^{-\frac{1}{2}}\mathcal{N}(0, 1)
$$

where $\mathcal{N}(0, 1)$ denotes a standard normal distribution. For the noise-fit and estimation error holds asymptotically

$$
\mathcal{N} \overset{\mathcal{D}}{=} \frac{1}{2T\hat{\rho}}\chi^2(k) + R_N^N
$$

$$
\mathcal{E} \overset{\mathcal{D}}{=} \frac{1}{2T\hat{\rho}}\chi^2(k) + R_E^E
$$

$$
\mathcal{N} + \mathcal{E} \overset{\mathcal{D}}{=} \frac{1}{T\hat{\rho}}\chi^2(k) + R_{N+E}^{N+\xi}
$$

with $T\mathbb{E} \left[ |R_N^N| \right] \to 0$, $T\mathbb{E} \left[ |R_E^E| \right] \to 0$ and $T\mathbb{E} \left[ |R_{N+E}^{N+\xi}| \right] \to 0$ for $T \to \infty$. Here $\chi^2(k)$ denotes a $\chi^2$-distributed random variable with $k$ degrees of freedom.

*Proof.* See Appendix.

7.4. Noise-fit and Estimation Error for Mean Variance Utility

Similar, for mean-variance utility as performance measure

**Theorem 7.11 (Noise Fit and Estimation Error for Mean Variance).** Under Assumptions 7.5, 7.1 and 7.7 holds

1. *(Noise Fit)*

$$
\mathbb{E}[\mathcal{N}_{MV}] = \frac{k+1}{\gamma T} + o\left(\frac{1}{T}\right)
$$

2. *(Mean Variance Corrected for Noise Fit)* In particular

$$
u(\theta^*) = \mathbb{E}\left[\hat{u}(\hat{\theta}) - \frac{k+1}{\gamma T}\right] + o\left(\frac{1}{T}\right)
$$

3. *(Noise Fit and Estimation Error for the Sharpe)*

$$
\mathbb{E}[\mathcal{N}_{MV} + \mathcal{E}_{MV}] = \frac{2k+2}{\gamma T} + o\left(\frac{1}{T}\right)
$$
4. (Mean Variance Corrected for Noise Fit and Estimation Error) In particular

\[ \mathbb{E} \left[ u(\hat{\theta}) \right] = \mathbb{E} \left[ \hat{u}(\hat{\theta}) - \frac{2k + 2}{\gamma T} \right] + o \left( \frac{1}{T} \right) \]

Proof. See Appendix.

8. Extensions

In this section, we discuss potential extensions of the result.

First of all it would be nice to incorporate transaction cost and estimate the out-of-sample Sharpe net of transaction cost. In the easiest case the transaction cost are just a constant (independent of the parameter \( \theta \)). In this case \( \rho \) and \( \tau \) as defined in the paper would be the gross Sharpe ratios while \( \rho - c \) and \( \tau - c \) would be the net Sharpe ratios. Hence with constant transaction cost the analysis in this paper applies. Note that holding the net Sharpe \( \tau^* - c \) fix, higher transaction cost \( c \) benefit as they tend to reduce the noise fit and estimation error.

Second, here we chose to consider a continuous time setting with Gaussian noise. This is not necessary, all results apply in discrete time with non-Gaussian noise provided two conditions are fulfilled. i) The noise needs to have sufficiently bounded higher moments ii) The covariance matrix \( \Sigma \) needs to be known (and not estimated).

Third, it would be interesting to consider the case in which the covariance \( \Sigma \) is only estimated and not known. A natural modeling choice would be that \( \Sigma_{true}^{-1} \Sigma_{estimated} \) is Wishart-distributed. One can then look at the loss of going from \( \hat{\tau}(\Sigma_{estimated}) \) to \( \hat{\tau}(\Sigma_{true}) \).

A fourth extension could be to soften the differentiability assumptions as it seems that all what is needed is to control the local maxima of the random field \( X(\theta) \cdot W \).

A fifth possibility to consider would be to consider stochastic predictors \( X \). This would demand more techniques as e.g. \( \langle X, X \rangle \) would then be random as well. However, even with stochastic predictions, taking deterministic predictions \( X \) is the right thing to do if the interest lies in removing the effect of the noise in the market returns \( \nu \) rather than the noise in the strength and direction of the prediction \( X \). The interpretation would be that assuming an exogenous \( X \) is essentially conditioning on the realization of the prediction and then resampling the noise in the market returns.
9. Conclusion

In this paper, we derived \textit{SRIC}, an analogon to the Akaike information criterion if the performance measure is the Sharpe ratio rather than log-likelihood. SRIC adjusts the in-sample Sharpe ratio by the numbers of parameter maximized upon and the length of the in-sample period by a simple formula. It can be interpreted as (asymptotically) correcting the in-sample Sharpe for noise-fit and estimation error and therefore is an (asymptotically unbiased) estimator of the out-of-sample Sharpe ratio.

In the linear Gaussian case the corrections are exact, in the non-linear case asymptotically of order \(1/T\). The results are useful when it comes to model selection (i.e. comparing models of different dimensions) and whenever estimating the Sharpe ratio net of noise fit or estimation error is of interest, e.g. in portfolio management.

10. Appendix

10.1. Proofs

10.1.1. Proof of Theorem 4.3

\textit{Proof of Theorem 4.3}. We have show

\[
\mathbb{E}[\hat{\rho} - \hat{\tau}] = \mathbb{E}\left[\frac{k}{T\hat{\rho}}\right]
\]

Using (4) and (5) this amounts to showing

\[
\mathbb{E}\left[\frac{\nu^T\Sigma^{-1}(\mu + \nu)}{\|\mu + \nu\|_{\Sigma^{-1}}}\right] = \mathbb{E}\left[\frac{k}{T\|\mu + \nu\|_{\Sigma^{-1}}}\right]
\]

Without loss of generality (after a reparametrization) we can assume that \(\Sigma = I\), the identity matrix. We then have to show

\[
\mathbb{E}\left[\frac{\nu^T(\mu + \nu)}{\|\mu + \nu\|}\right] = \mathbb{E}\left[\frac{k}{T\|\mu + \nu\|}\right]
\]

We write

\[
\mathbb{E}\left[\frac{\nu^T(\mu + \nu)}{\|\mu + \nu\|}\right] = \sum_{i=1}^{k+1} \mathbb{E}\left[\frac{\nu_i(\mu_i + \nu_i)}{\|\mu + \nu\|}\right] = \sum_{i=1}^{k+1} I_i
\]
Now integrating out $\nu_i$,

\[
I_i = \mathbb{E} \left[ \int_{-\infty}^{\infty} \frac{\nu_i (\mu_i + \nu_i)}{\|\mu + \nu\|} \sqrt{\frac{T}{2\pi}} e^{-T\nu_i^2/2} d\nu_i \right]
\]

where expectation is over $\nu_j, j \neq i$

\[
= \mathbb{E} \left[ \int_{-\infty}^{\infty} \frac{1}{T} \sqrt{\frac{T}{2\pi}} g(\nu_i) f'(\nu_i) d\nu_i \right]
\]

with

\[
f(\nu_i) = -e^{-T\nu_i^2/2}
\]

\[
g(\nu_i) = \frac{\nu_i + \mu_i}{\|\mu + \nu\|}
\]

so that

\[
f'(\nu_i) = \nu_i T e^{-T\nu_i^2/2}
\]

\[
g'(\nu_i) = \frac{1}{\|\mu + \nu\|} - (\nu_i + \mu_i) \frac{1}{\|\mu + \nu\|^3} (\nu_i + \mu_i)
\]

\[
= \frac{\sum_{j \neq i} (\mu_j + \nu_j)^2}{\|\mu + \nu\|^3}
\]

Hence, via integration by parts

\[
I_i = \mathbb{E} \left[ \int_{-\infty}^{\infty} \frac{1}{T} \sqrt{\frac{T}{2\pi}} g'(\nu_i) (-f(\nu_i)) d\nu_i \right]
\]

\[
= \frac{1}{T} \mathbb{E} \left[ \int_{-\infty}^{\infty} \sqrt{\frac{T}{2\pi}} \frac{\sum_{j \neq i} (\mu_j + \nu_j)^2}{\|\mu + \nu\|^3} e^{-T\nu_i^2/2} d\nu_i \right]
\]

\[
= \frac{1}{T} \mathbb{E} \left[ \frac{\sum_{j \neq i} (\mu_j + \nu_j)^2}{\|\mu + \nu\|^3} \right]
\]

By symmetry, we have
\[
\sum_{i=1}^{k+1} I_i = \frac{1}{T} \sum_{i=1}^{k+1} \mathbb{E} \left[ \frac{\sum_{j \neq i} (\mu_j + \nu_j)^2}{\|\mu + \nu\|^3} \right] \\
= \frac{1}{T} \mathbb{E} \left[ \frac{k}{\|\mu + \nu\|^3} \right] \\
= \frac{1}{T} \mathbb{E} \left[ \frac{k}{\|\mu + \nu\|} \right]
\]

what we had to show.

\[\square\]

10.1.2. Proof of Theorem 7.6

Proof of Theorem 7.6. To proof this, we need to compute the log-likelihood of prediction \( \theta \). By (9), the dynamics of the market prices are given by

\[
\frac{dp_t}{p_t} = Y_t dt + dW_t
\]

with unknown \( Y_t \). Now \( \theta \) parametrizes different models \( \mathbb{P}^\theta \) for \( p_t \)

\[
\frac{dp_t}{p_t} = X_t(\theta) dt + dW^\theta_t 
\]

with \( W^\theta \) a Brownian Motion under \( \mathbb{P}^\theta \). The question is: What is the (log)likelihood of the realized market price process \( p \) under the model \( \theta \)? The answer is given by the help of the Girsanov Theorem. First, due to continuous time, each likelihood (density) is zero. So the only meaningful definition of the log likelihood function is as a relative density with respect to a reference probability, e.g. towards the probability distribution of a Brownian Motion. For this purpose let

\[
\frac{dp_t}{p_t} = dW^0_t 
\]

be a geometric Brownian Motion under the probability distribution \( \mathbb{P}^0 \).

Let \( Z_T = T \hat{\mu}(\theta) - \frac{1}{2} T \Sigma(\theta, \theta) \)
Then

\[ Z_T = \int_0^T X_t(\theta)^T Y_t dt + \int_0^T X_t(\theta)^T dW_t^\theta \\
- \frac{1}{2} \int_0^T X_t(\theta)^T X_t(\theta) dt \\
= \int_0^T X_t(\theta)^T dW_t^0 - \frac{1}{2} \int_0^T X_t(\theta)^T X_t(\theta) dt \]

Hence by the (multivariate) Girsanov Theorem (see any book on stochastic analysis, e.g. Kallenberg (2002) or Øksendal (2003)), \( \tilde{W}_t = W_t^0 - \text{diag}([Z, W_0]_t) = W_t^0 - \int_0^T X_t(\theta) dt \) is a Brownian Motion under \( Q^\theta = \exp(Z_T)_P^0 \). In particular under \( Q^\theta \):

\[ \frac{dp_t}{p_t} = dW_t^0 = d\tilde{W}_t + X_t(\theta) dt \]

where \( \tilde{W}_t \) a Brownian Motion. Hence \( P^\theta \overset{D}{=} Q^\theta = e^{Z_T} P^0 \).

\[ \Box \]

10.1.3. Proof of Theorem 7.8 (Sketch)

We first sketch the proof on a high level and then fill the gaps on a detail level.

We will drop the index \( T \) from \( \theta^*_T, \hat{\theta}_T, \hat{\rho}_T, \hat{\tau}_T \) and \( \tau^*_T \) to simplify notation. Also, we use constants \( C, \alpha > 0 \) (that do not depend on \( T \)) and sets \( A_T \in F_T \) that might change during the steps in the proof, but keep their properties (such as independence of \( T \)).

The idea of the proof is quite simple. With increasing observation time \( T \) the estimated parameter gets closer to the optimal parameter \( \theta^*_T \). Around \( \theta^*_T \) the behavior is close to linear. This - together with a control of large deviations - will proof the theorem.

Though this intuition is simple, tedious derivatives are are necessary, which is why we only sketch the proof here. The arising gaps are filled in the next section.

Proof of Theorem 7.8 (Sketch). Define the auxiliary function

\[ F(\theta, u) = \frac{\mu(\theta) + u\nu(\theta)}{\sqrt{\Sigma(\theta, \theta)}} \]  

(14)

Note that \( F \) interpolates between the true Sharpe ratio \( \tau \) and the in-sample Sharpe ratio \( \rho \), i.e. \( F(\theta, 1) = \rho(\theta) \) and \( F(\theta, 0) = \tau(\theta) \).

The first step is to show the following Lemma
Lemma 10.1 (Taylor Approximation). Under Assumptions 7.5, 7.1 and 7.7 holds the following: Let \( \hat{\rho} \in L^q \) for some \( q > 1 \), then there are \( C, \alpha > 0 \) and sets \( A_T \in F_T \) with \( \mathbb{P} [ A_T^C ] \leq C e^{-\alpha T^{1/4}} \) (the non-extreme events) such that on \( A_T \) holds

\[
F(\hat{\theta}, u) = F(\theta^*, 0) + F_u(\theta^*, 0) u + u F_{u, \theta}(\theta^*, 0) \Delta_1 \theta + F_\theta(\theta^*, 0) \Delta_1 \theta \\
+ (\Delta_1 \theta)^T F_{\theta, \theta}(\theta^*, 0) \Delta_1 \theta + \bar{R}
\]

with

\[
\Delta_1 \theta = \left( \frac{1}{T} DX(\theta^*), DX(\theta^*) \right)^{-1} \frac{1}{T} (DX(\theta^*) \cdot W)
\]

and for the remainder \( R \) holds

\[
\lim_{T \to \infty} \mathbb{E} [ T \bar{R} ] = 0 \tag{15}
\]

The Lemma allows to write \( F \) in a Taylor series:

\[
F(\hat{\theta}, u) = F(\theta^*, 0) + F_u(\theta^*, 0) u + u F_{u, \theta}(\theta^*, 0) \Delta_1 \theta + F_\theta(\theta^*, 0) \Delta_1 \theta \\
+ (\Delta_1 \theta)^T F_{\theta, \theta}(\theta^*, 0) \Delta_1 \theta + \bar{R} = \tilde{F}(\hat{\theta}, u) + R
\]

with \( R \), a remainder Term.

Now let there be some sets \( A_T \in F_T \) with \( \mathbb{P} [ A_T^C ] \leq C e^{-\alpha T^{1/4}} \) for some \( C, \alpha > 0 \) (the non-extreme events) to be chosen in the appendix. Then we will show that

\[
\mathbb{E} [ \mathcal{N} + \mathcal{E} + \mathcal{U} ] = \mathbb{E} [ \hat{\rho} - \hat{\tau} ]
\]

\[
= \mathbb{E} \left[ F(\hat{\theta}, 1) - F(\hat{\theta}, 0) \right]
\]

\[
= \mathbb{E} \left[ F(\hat{\theta}, 1) - F(\hat{\theta}, 0); A_T \right] + o(T^{-1}) \quad \text{by Lemma 10.4} \tag{16}
\]

\[
= \mathbb{E} \left[ \tilde{F}(\hat{\theta}, 1) - \tilde{F}(\hat{\theta}, 0); A_T \right] + o(T^{-1}) \quad \text{by Lemma 10.11} \tag{17}
\]

\[
= \mathbb{E} \left[ \frac{k}{T \theta^*}; A_T \right] + o(T^{-1}) \quad \text{by Lemma 10.10} \tag{18}
\]

\[
= \mathbb{E} \left[ \frac{k}{T \hat{\rho}}; A_T \right] + o(T^{-1}) \quad \text{by Lemma 10.12} \tag{19}
\]

\[
= \mathbb{E} \left[ \frac{k}{T \hat{\rho}} \right] + o(T^{-1}) \quad \text{by Lemma 10.4} \tag{20}
\]

And analogously for \( \mathbb{E} [ \mathcal{N} ] = \mathbb{E} \left[ F(\hat{\theta}, 1) - F(\theta^*, 0) \right] \).
10.1.4. Proof of Theorem 7.8 (Details)

Here, we fill the gaps in the sketchy proof of Theorem 7.8, that is Lemma 10.1, Lemma 10.4, Lemma 10.10, Lemma 10.11 and Lemma 10.12.

Assume that $\hat{\theta}$ (wherever it exists) is scaled so that $\rho(\hat{\theta})^2 = \Sigma(\hat{\theta}, \hat{\theta})$ and $\tau(\theta^*)^2 = \Sigma(\theta^*, \theta^*)$.

This is without loss of generality due to Assumption 7.5.

Throughout the proof, we make use of the well-known (e.g. Massart (2003)) concentration inequality for Gaussian processes:

**Theorem 10.2 (Concentration Inequality for Gaussian processes).** Let $\{Y_t\}_{t \in T}$ be a almost surely continuous centered Gaussian process on some set $T \subset \mathbb{R}^p$. Then for all $\lambda$

$$\mathbb{P} \left[ \left| \sup_{t \in T} Y_t - \mathbb{E} \left[ \sup_{t \in T} Y_t \right] \right| > \lambda \right] \leq 2 \exp \left( -\frac{\lambda^2}{2 \sup_{t \in T} \mathbb{E} [Y_t^2]} \right)$$

(21)

**Proof.** See Massart (2003)[Proposition 3.19] or Boucheron et al. (2013)[Theorem 5.8].

Combining the differentiability of $X \cdot W$ with Theorem 10.2 gives over large deviations, that is let

$$Z_T(\theta) = \frac{1}{T} \left( D^i X(\theta) \cdot W \right)_T$$

for some $i=0,1,2,3$ and

$$Z_T^* = \sup_{\theta \in B_{\varepsilon}(\theta^*) \cap \Theta_1} Z_T(\theta)$$

(22)

Then

**Lemma 10.3.** There is $\varepsilon > 0$, $\alpha > 0$, $C > 0$ (independent of $T$) such that for every $\theta \in B_{\varepsilon}(\theta^*)$ and $\lambda > 0$ and $T \geq 0$

holds

$$\mathbb{P} \left[ |Z_T^*| > \lambda \right] \leq C e^{-\alpha \lambda^2 T}$$

Further, for every $2 \geq q > 1$ there exists $C > 0$ and $T_0$ such that for $T > T_0$

$$\mathbb{E} \left[ |Z_T^*|^q ; |Z_T^*| > \lambda \right] \leq C e^{-\alpha \lambda^q T}$$

**Proof.** For the first part apply Assumption 7.7 part 3) and the Concentration Inequality (Theorem 10.2). To wit, note that $\mathbb{E} \left[ |Z_T^*| \right] \leq C_1 / \sqrt{T}$ for some constant $C$ by Assumption 7.7. Now fix $\lambda$ and let $T \geq 4C_1^2 / \lambda^2$. Then $\mathbb{E} \left[ |Z_T^*| \right] \leq C_1 / \sqrt{T} \leq \lambda / 2$ and therefore

$$\mathbb{P} \left[ |Z_T^*| > \lambda \right] \leq \mathbb{P} \left[ |Z_T^* - \mathbb{E} [Z_T^*]| > \lambda / 2 \right] \leq 2 e^{-\alpha \lambda^2 T}$$

for some $\alpha > 0$ by the Concentration Inequality.
For $T \leq 4C_1^2/\lambda^2$, on the other hand, we have for $C_2 = e^{\alpha_2 C_1^2}$

$$C_2 e^{-\alpha_2 T} \geq C_2 e^{-\alpha_4 C_1^2} \geq 1$$

and therefore trivially $\mathbb{P}[|Z_T^*| > \lambda] \leq C_2 e^{-\alpha_2 T}$. 

As $C_2$ was chosen independently of $\lambda$, we can combine both estimates and get (uniformly across $\lambda$) with $C = \max[2, C_2]$

$$\mathbb{P}[|Z_T^*| > \lambda] \leq C e^{-\alpha_2 T}$$

which proofs the claim.

For the second note that

$$\mathbb{E}[|Z_T^*|^q; |Z_T^*|^q > \lambda] = \int_\lambda^\infty \mathbb{P}[|Z_T^*|^q > x] dx + \lambda \mathbb{P}[|Z_T^*|^q > \lambda]$$

$$= \int_\lambda^\infty \mathbb{P}[|Z_T^*|^q > x] dx + \lambda \mathbb{P}[|Z_T^*|^q > \lambda]$$

$$\leq C \int_\lambda^\infty e^{-\lambda T^2} dx + \lambda C e^{-\lambda T^2}$$

$$\leq C e^{-\lambda T^2} \left( \int_0^\infty e^{-\lambda T^2} dx + \lambda \right)$$

for appropriate $C_2, \alpha_2 > 0$

The benefit of Lemma 10.3 is that it allows to focus only on small neighborhoods around the optimal parameter, as the expectation on large deviations converges quickly to zero.

Next observe the following auxiliary Lemma.

**Lemma 10.4.** Let $Z_T$ be a sequence of random variables uniformly (in $T$) bounded in $L^q$ with $q > 1$. Let $A_T \in \mathcal{F}_T$ be events such that $\mathbb{P}[A_C^T] \leq C e^{-\alpha T}$ for some $C, \alpha > 0$. Then there exists $C', \alpha' > 0$ such that $\mathbb{E}[Z_T, A_C^T] \leq C' e^{-\alpha' T}$

**Proof.** For any $\lambda > 0$

$$\mathbb{E}[Z_T; A_C^T] = \mathbb{E}[Z_T; A_C^T \cap \{Z_T \leq \lambda\}] + \mathbb{E}[Z_T; A_C^T \cap \{Z_T > \lambda\}]$$

Now choose for example $\lambda = e^{+\alpha T/2}$

With these auxiliary observations, it suffices to focus on the behavior within neighborhoods of the optimal parameter. The next step is to derive a differential equation for the optimal parameter.

**Lemma 10.5.** There are events\textsuperscript{11} $A_T \in \mathcal{F}_T$ with $\mathbb{P}[A_C^T] \leq C e^{-\alpha T}$ such that on $A_T$ a Sharpe

\textsuperscript{11}The notation is such that $A_C^C = \Omega \setminus A$
optimal in-sample parameter $\hat{\theta}$ is given by

$$
\hat{\theta} = \theta^* + \Delta_1 \theta + R_0
$$

(23)

with

$$
\Delta_1 \theta = \langle DX(\theta^*), DX(\theta^*) \rangle^{-1} (DX(\theta^*) \cdot W)
$$

The remainder has the form

$$
R_0 = \int_0^1 \tilde{\theta}'(s) - \tilde{\theta}'(0) ds
$$

where $\tilde{\theta}'(u)$ is determined by the following differential equation

$$
\tilde{\theta}'(u) =
\begin{array}{l}
\left[ \frac{1}{T} \left( DX(\tilde{\theta}(u)), DX(\tilde{\theta}(u)) \right) - \frac{1}{T} \left( Y - X(\tilde{\theta}(u)), D^2 X(\tilde{\theta}(u)) \right) - \frac{u}{T} D^2 X(\tilde{\theta}(u)) \cdot W \right]^{-1} \\
\frac{1}{T} DX(\tilde{\theta}(u)) \cdot W
\end{array}
$$

(24)

$\hat{\theta}$ fulfills $\Sigma(\hat{\theta}, \hat{\theta}) = \rho(\hat{\theta})^2$.

**Proof.** Let $\varepsilon > 0$ be such that $Z \cdot W$ is differentiable for $Z = X, \dot{Z} = DX$ and $Z = D^2 X$ in $B_0(\theta_T^*) \ (\text{Assumption 7.7 (Differentiability))}$. Let $A_T = \{ \hat{\theta} \in B_0(\theta_T^*) \}$. By assumption 7.7 1) (Convergence) and the Concentration inequality (Lemma 10.3), $\mathbb{P} [ A_T ] \leq C e^{-\alpha T}$.

Now consider the auxiliary problem

$$
\text{(AUX 1)} \quad \max_{\hat{\theta} \in \Theta} G(\hat{\theta}; u) \quad u \in [0, 1]
$$

with $G(\hat{\theta}, u) = 2\mu(\hat{\theta}) + 2u\nu(\hat{\theta}) - \Sigma(\hat{\theta}, \hat{\theta}')$

Intuitively, the noise $\nu$ is replaced by $u\nu$ for $u \in [0, 1]$. This allows to consider the solutions $\tilde{\theta}(u)$ for all levels of noise between 0 and $\nu$. Now fix $u$ and consider the first order condition for $\tilde{\theta}(u)$:

$$
\text{(FONC)} \quad 0 = \left\langle DX(\tilde{\theta}), Y - X(\tilde{\theta}) \right\rangle + uDX(\tilde{\theta}) \cdot W
$$

(25)

Differentiate the (FONC) with respect to $u$ to find the ODE for $\tilde{\theta}'(u)$.

$$
\tilde{\theta}'(u)^T = \frac{d}{du} \tilde{\theta}'(u)^T =
\begin{array}{l}
\left[ \frac{1}{T} \left( DX(\tilde{\theta}(u)), DX(\tilde{\theta}(u)) \right) - \frac{1}{T} \left( Y - X(\tilde{\theta}(u)), D^2 X(\tilde{\theta}(u)) \right) - \frac{u}{T} D^2 X(\tilde{\theta}(u)) \cdot W \right]^{-1} \\
\frac{1}{T} DX(\tilde{\theta}(u)) \cdot W
\end{array}
$$

It is easy to see that an optimum of (AUX1) with $u = 1$ also maximizes the in-sample Sharpe ratio $\rho$ and is scaled so that $\Sigma(\tilde{\theta}, \tilde{\theta}) = \rho(\tilde{\theta})^2$. For $u = 0$ there is no noise and the
optimal solution is \( \tilde{\theta}(0) = \theta^* \). Hence

\[
\dot{\theta} = \tilde{\theta}(1) = \theta^* + \tilde{\theta}'(0) + \int_0^1 \tilde{\theta}'(s) - \tilde{\theta}'(0) ds
\]

\[
= \theta^* + \Delta_1 \theta + R_0
\]

The proof is finished by showing existence of a solution to the ODE on a set \( A_T \) of sufficient mass. This is the assertion of the next lemma.

**Lemma 10.6.** Outside of events \( A_T^C \in \mathcal{F}_T \) with \( \mathbb{P}[A_T^C] \leq Ce^{-\alpha T} \) a solution \( \tilde{\theta}(u) \) to the differential equation (24) exists for \( u \in [0, 1] \) Moreover there exist \( C, \alpha, \bar{\lambda} > 0 \) such that for any \( 0 < \lambda \leq \bar{\lambda} \)

\[
\mathbb{P} \left[ \sup_{u \in [0,1]} |\tilde{\theta}_u - \tilde{\theta}_0| \geq \lambda \right] \leq Ce^{-\alpha \lambda^2 T} \tag{26}
\]

**Proof.** Note that the ODE is solved conditional on \( \mathcal{F}_T \), i.e. it is an ordinary differential equation with realized random coefficients. By Lemma 10.3 and Assumption 7.7 (Invertibility) there exists \( \varepsilon > 0 \) such that the vectorfield defining the ODE is differentiable and its derivative is bounded in \( \frac{1}{T} D^i X \cdot W \) with \( (i = 0, 1, 2, 3) \) within \( B_\varepsilon(\theta^*_T) \cap \Theta_T^1 \). That is let \( V(\theta, u; T) \) denote the right hand side of equation (24), then for all \( \theta \in B_\varepsilon(\theta^*_T) \cap \Theta_T^1 \), we have for some \( L \in \mathbb{R} \)

\[
\left| \frac{d}{d\theta} V(\theta, u; T) \right| \leq \frac{L}{4} \sum_{i=0}^{3} \frac{1}{T} D^i X \cdot W \tag{27}
\]

and

\[
\left| \frac{d}{du} V(\theta, u; T) \right| \leq \frac{L}{4} \sum_{i=0}^{3} \frac{1}{T} D^i X \cdot W \tag{28}
\]

and

\[
|V(\theta, 0; T)| \leq \frac{L}{2T} |DX \cdot W| \tag{29}
\]

for some \( L > 0 \) (independent of \( T \)). Let \( A_T^{\delta, \varepsilon} = \{|\frac{1}{T} D^i X(\theta) \cdot W| < \delta \text{ for all } i = 0, 1, 2, 3 \text{ and } \theta \in B_\varepsilon(\theta^*_T) \cap \Theta_T^1\} \) On all \( A_T^{\delta, \varepsilon} \) we have

\[
\left| \tilde{\theta}'(0) \right| \leq \frac{1}{2} L \delta \text{ and } \tag{30}
\]

\[
\left| \frac{d}{d\theta} V(\theta, u; T) \right| \leq L \delta \text{ and } \tag{31}
\]

so by Gronwall’s Lemma

\[
\sup_{u \in [0,1]} |\tilde{\theta}_u - \tilde{\theta}_0| \leq L \delta e^{L \delta} \tag{32}
\]
To see this let \( g(s) = \bar{\theta}(s) - \bar{\theta}(0) \). Then

\[
\begin{align*}
g(t) &= \int_0^t g'(s)ds \\
&= \int_0^t \bar{\theta}'(s) - \bar{\theta}(0)ds + t\theta'(0) \\
&= \int_0^t \left( \mathbb{V}(\bar{\theta}(s), s; T) - \mathbb{V}(\bar{\theta}(0), 0; T) \right) ds + t\theta'(0) \\
&= \int_0^t \left( \mathbb{V}(\bar{\theta}(s), s; T) - \mathbb{V}(\bar{\theta}(0), s; T) + \mathbb{V}(\bar{\theta}(0), s; T) - \mathbb{V}(\bar{\theta}(0), 0; T) \right) ds + t\theta'(0)
\end{align*}
\]

Hence on \( A_T^{\delta,\varepsilon} \):

\[
|g(t)| \leq \int_0^t (\beta g(s) + \beta s) ds + \frac{1}{2}t\beta, \quad \text{with } \beta = \delta L
\]

\[
= \int_0^t \beta g(s)ds + \frac{1}{2}t\beta + \frac{1}{2}t^2\beta
\]

\[
\leq \beta e^{\beta t} \quad \text{for } t \leq 1 \text{ by Gronwall}
\]

This is 32. In particular, there exists \( \delta > 0 \) so that on \( A_T^{\delta,\varepsilon} \) there exists a solution \( \bar{\theta} \) of the ODE within \( \mathcal{B}_\varepsilon(\theta^*_T) \) for \( u \in [0,1] \). By Lemma 10.3 \( A_T = A_T^{\delta,\varepsilon} \) possesses the desired properties. Moreover, by equation (32)

\[
\sup_{u \in [0,1]} \left| \bar{\theta}_u - \bar{\theta}_0 \right| \geq \lambda \text{ on some } A_T^{\delta,\varepsilon} \text{ for some } \lambda > 0 \text{ implies } \delta \geq \min(1, \lambda e^{L/\varepsilon}). \text{In particular, if } \lambda \leq \bar{\lambda} = L/e^L \text{ then } \delta \geq \lambda e^{L/\varepsilon}. \text{Hence}
\]

\[
P \left[ \sup_{u \in [0,1]} \left| \bar{\theta}_u - \bar{\theta}_0 \right| \geq \lambda \right] \leq P \left[ \left( A_T^{\lambda e^{L/\varepsilon}} \right)^C \right] \leq C e^{-\alpha \lambda^2 T} \tag{33}
\]

for some \( C, \alpha > 0 \) by Lemma 10.3.

Let \( \Delta \theta = \hat{\theta} - \theta^* \) be the full difference of the optimal in- and out-of-sample parameter so that \( \Delta \theta = \Delta_1 \theta + R_0 \)

Write (14) as

\[
F(\theta, u) = \frac{1}{\sqrt{T}} \lambda(\theta) \langle \mathbf{X}(\theta), \mathbf{Y} + uW \rangle \quad \text{with}
\]

\[
\lambda(\theta) = \sqrt{\langle \mathbf{X}(\theta), \mathbf{X}(\theta) \rangle^{-1}}
\]
and expand it into a Taylor-like expression

\[
F(\hat{\theta}, u) = F(\theta^*, 0) + F_u(\theta^*, 0)u + uF_{u,\theta}(\theta^*, 0)\Delta_1\theta + F_{\theta}(\theta^*, 0)\Delta_1\theta
+ \frac{1}{2}\Delta_1\theta^T F_{\theta,\theta}(\theta^*, 0)\Delta_1\theta + R
\]

where \( \tilde{F} \) is defined via the last equality and the remainder term

\[
R = R_1 + R_2 + R_3 + R_4 \quad \text{where}
\]

\[
R_1 = uF_{u,\theta}(\theta^*, 0)R_0
R_2 = \frac{1}{2}R_0^T F_{\theta,\theta}(\theta^*, 0)R_0
R_3 = \Delta_1\theta^T F_{\theta,\theta}(\theta^*, 0)R_0
R_4 = \Delta\theta^T \left( \int_0^1 \int_0^s F_{\theta,\theta}(\theta^* + t\Delta\theta, 0) - F_{\theta,\theta}(\theta^*, 0)dt ds \right) \Delta\theta
\]

The following observation is helpful

**Remark 10.7.** After a reparametrization of the parameter space around \( \theta_T^* \), we can assume that \( D_0X(\theta^*) = X(\theta^*) \) and \( D_iX(\theta^*) \perp D_jX(\theta^*) \) \( \forall i \neq j \) and \( ||D_iX(\theta^*)|| = 1 \) for \( i \neq 0 \). Note that in this case

\[
\langle DX(\theta^*), DX(\theta^*) \rangle = \begin{pmatrix} \X(\theta^*), X(\theta^*) \\ 0 \end{pmatrix} \in \mathbb{R}^{k+1,k+1}
\]

and

\[
\Delta_1\theta = \begin{pmatrix} \frac{X(\theta^*) \cdot W}{\langle X(\theta^*), X(\theta^*) \rangle} \\ D_1X(\theta^*) \cdot W \\ \ldots \\ D_kX(\theta^*) \cdot W \end{pmatrix}
\]

In particular

\[
\langle DX(\theta^*), \Delta_1\theta, X(\theta^*) \rangle = X(\theta^*) \cdot W
\]

and

\[
\langle DX(\theta^*), W \rangle^T \Delta_1\theta = \frac{(X \cdot W)^2}{\langle X(\theta^*), X(\theta^*) \rangle} + \sum_{i=1}^{k} (D_iX(\theta^*) \cdot W)^2 \tag{34}
\]

**Lemma 10.8.** The following relations hold

\[
\langle DX(\theta^*), X(\theta^*) \rangle = \langle DX(\theta^*), Y \rangle \tag{35}
\]

\[
\langle X(\theta^*), X(\theta^*) \rangle = \langle X(\theta^*), Y \rangle \tag{36}
\]

\[
\langle X(\theta^*), DX(\theta^*) \Delta_1\theta \rangle = X(\theta^*) \cdot W \tag{37}
\]

\[
\frac{1}{T} \langle X(\theta^*), X(\theta^*) \rangle = \tau(\theta^*)^2 \tag{38}
\]
Proof. As $\theta^*$ solves by definition
\[
\max_{\theta \in \Theta} 2 \langle X(\theta), Y \rangle - \langle X(\theta), X(\theta) \rangle
\]
the first order condition implies (35). Now by Assumption 7.5 there is one scale parameter, i.e there is $s \in \mathbb{R}^{k+1}$ such that $D_s X = X$. This applied to (35) delivers (36). (37) follows from Remark 10.7. (38) holds by definition.

Applying Lemma 10.8 yields the following relations

**Lemma 10.9.** The following holds
\[
\begin{align*}
\mathbb{E}[\langle DX\Delta_1 \theta, X \rangle (X \cdot W)] &= \langle X, X \rangle \quad (39) \\
\mathbb{E}[\langle DX \cdot W \rangle \Delta_1 \theta] &= k + 1 \quad (40) \\
\mathbb{E}[\langle DX\Delta_1 \theta, X \rangle^2] &= \langle X, X \rangle \quad (41) \\
\mathbb{E}[\langle DX\Delta_1 \theta, DX\Delta_1 \theta \rangle] &= k + 1 \quad (42)
\end{align*}
\]

**Proof.**

Now,

**Lemma 10.10.** There are $C, \alpha > 0$ and sets $A_T$ with $\mathbb{P}[A_T^c] \leq Ce^{-\alpha T}$ such that
\[
\mathbb{E}\left[\tilde{F}(\hat{\theta}, 1); A_T\right] = F(\theta^*, 0) + \frac{k}{2T^{\tau_T}} + o(e^{-\alpha T}) \quad (43)
\]
and
\[
\mathbb{E}\left[\tilde{F}(\hat{\theta}, 0); A_T\right] = F(\theta^*, 0) - \frac{k}{2T^{\tau_T}} + o(e^{-\alpha T}) \quad (44)
\]

**Proof.** Observe that (see computations in the Appendix 10.2)
\[
\begin{align*}
F_u(\theta^*, 0) &= +T^{-\frac{1}{2}} \lambda \langle X \cdot W \rangle \\
F_{u,u}(\theta^*, 0) &= 0 \\
F_{u,\theta}(\theta^*, 0) &= -T^{-\frac{1}{2}} \lambda^3 \langle DX, X \rangle \langle X \cdot W \rangle + T^{-\frac{1}{2}} \lambda \langle DX \cdot W \rangle \\
F_{\theta}(\theta^*, 0) &= 0 \\
F_{\theta,\theta}(\theta^*, 0) &= T^{-\frac{1}{2}} \lambda^3 \langle DX, X \rangle^2 - T^{-\frac{1}{2}} \lambda \langle DX, DX \rangle
\end{align*}
\]

Let $A_T$ be as in the previous lemmata Now take expectations:

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$$\mathbb{E} \left[ \tilde{F}(\hat{\theta}, 1); A_T \right]$$

$$= \mathbb{E} \left[ \tilde{F}(\hat{\theta}, 1) \right] + o(e^{-\alpha T}) \text{ by Lemma 10.4}$$

$$= F(\theta^*, 0) + T^{-\frac{3}{4}} \mathbb{E} \left[ \lambda (X \cdot W) \right]$$

$$+ T^{-\frac{3}{4}} \mathbb{E} \left[ -\lambda^3 (DX \Delta_1 \theta, X) (X \cdot W) + \lambda (DX \cdot W) \Delta_1 \theta \right]$$

$$+ \frac{1}{2} T^{-\frac{3}{4}} \mathbb{E} \left[ \lambda^3 (DX \Delta_1 \theta, X)^2 - \lambda (DX \Delta_1 \theta, DX \Delta_1 \theta) \right] + o(e^{-\alpha T})$$

$$= F(\theta^*, 0) + 0 - \lambda T^{-\frac{3}{4}} + \lambda T^{-\frac{3}{4}} (k + 1) + \frac{\lambda}{2} T^{-\frac{3}{4}} - \frac{\lambda}{2} T^{-\frac{3}{4}} (k + 1) + o(e^{-\alpha T})$$

$$= F(\theta^*, 0) + \frac{k}{2T^{\tau^*}} + o(e^{-\alpha T}) \text{ as } \lambda(\theta^*) = \frac{1}{\tau^* \sqrt{T}}$$

And similar for $$\mathbb{E} \left[ \tilde{F}(\hat{\theta}, 0); A_T \right].$$
Proof. Note that $\hat{\rho} = F(\hat{\theta}, 1)$ and $\tau^* = F(\theta^*, 0)$ and the difference is controllable. For $\|\hat{\rho} - \tau^*\|$ small enough, we have

$$\left\| \frac{1}{\hat{\rho}} - \frac{1}{\tau^*} \right\| \leq C \|\hat{\rho} - \tau^*\|$$

So set, for some $\varepsilon > 0$ small enough $A' = \{\|\hat{\rho} - \tau^*\| \leq \varepsilon\} \cap A_T$. Then

$$\mathbb{E} \left[ \left\| \frac{1}{2T\hat{\rho}} - \frac{1}{2T\tau^*} \right\| \right] = \mathbb{E} \left[ \left\| \frac{1}{2T\hat{\rho}} - \frac{1}{2T\tau^*} \right\|; A_T \right] + o(T^{-1}) \quad \text{by Lemma 10.4}$$

$$\leq \frac{1}{2T} \mathbb{E} \left[ \|\hat{\rho} - \tau^*\|; A_T \right] + o(T^{-1}) = o(T^{-1})$$

Proof of Theorem 7.10. For $\mathcal{U}$ the claim holds by definition. Now consider $\mathcal{N} + \mathcal{E}$. One can write

$$\mathcal{N} + \mathcal{E} = F_{u,\theta}(\theta^*, 0) \Delta \theta$$

$$= F_{u,\theta}(\theta^*, 0) \Delta_1 \theta + F_{u,\theta}(\theta^*, 0) (\Delta \theta - \Delta_1 \theta)$$

Now observe that $F_{u,\theta}(\theta^*, 0) \Delta_1 \theta = \frac{1}{T\tau^*} \chi^2(k)$ and let $A_T$ be as in the previous proof.

$$\mathbb{E} [F_{u,\theta}(\theta^*, 0) (\Delta \theta - \Delta_1 \theta)]$$

$$= \mathbb{E} [F_{u,\theta}(\theta^*, 0) R_0, A_T] + o(T^{-1}) \quad \text{by Lemma 10.4}$$

$$= o(T^{-1}) \text{ as in Lemma 10.11}$$

Via the same argument as in Lemma 10.12 one gets

$$\mathbb{E} \left[ \left\| \left( \frac{1}{T\tau^*} - \frac{1}{T\hat{\rho}} \right) \chi^2(k) \right\| \right] = o(T^{-1})$$

finishing the proof for $\mathcal{N} + \mathcal{E}$. Analogous arguments hold for $\mathcal{N}$ and $\mathcal{E}$. \qed

Proof of Theorem 7.11. It suffices to proof the theorem for $\gamma = 1$ (as $\gamma$ only scales the exposure in form of a multiplicative factor). Let

$$G(\theta, u) = \frac{2}{T} \langle (X(\theta), Y) + uX(\theta) \cdot W \rangle - \frac{1}{T} \langle X(\theta), X(\theta) \rangle$$
Hence
\[
G_u(\theta, u) = \frac{2}{T} X \cdot W \\
G_\theta(\theta, u) = \frac{2}{T} \langle DX, Y \rangle - \frac{2}{T} \langle DX, X \rangle + \frac{2}{T} u DX \cdot W \\
G_{\theta, u}(\theta, u) = \frac{2}{T} DX \cdot W \\
G_{\theta, \theta}(\theta, u) = \frac{2}{T} \langle D^2X, Y - X \rangle - \frac{2}{T} \langle DX, DX \rangle + \frac{2}{T} u D^2X \cdot W
\]

In particular
\[
G_u(\theta^*, 0) = \frac{2}{T} X \cdot W \\
G_\theta(\theta^*, 0) = 0 \\
G_{\theta, u}(\theta^*, 0) = \frac{2}{T} DX \cdot W \\
G_{\theta, \theta}(\theta^*, 0) = -\frac{2}{T} \langle DX, DX \rangle
\]

So that
\[
\mathbb{E} [G_u(\theta^*, 0)] = 0 \\
\mathbb{E} [G_\theta(\theta^*, 0) \Delta_1 \theta] = 0 \\
\mathbb{E} [G_{\theta, u}(\theta^*, 0) \Delta_1 \theta] = \frac{2(k + 1)}{T} \\
\mathbb{E} [\Delta_1 \theta^T G_{\theta, \theta}(\theta^*, 0) \Delta_1 \theta] = -\frac{2(k + 1)}{T}
\]

Hence
\[
\mathbb{E} [N_{MV}] = \mathbb{E} \left[ G(\hat{\theta}, 1) - G(\theta^*, 1) \right] \\
= \mathbb{E} \left[ G_\theta(\theta^*, 0) \Delta_1 \theta + G_{\theta, u}(\theta^*, 0) \Delta_1 \theta + \frac{1}{2} \Delta_1 \theta^T G_{\theta, \theta}(\theta^*, 0) \Delta_1 \theta + R \right] \\
= \frac{k + 1}{T} + o(T^{-1})
\]

and as in the proof of Theorem 7.8 the remainder fulfills \( \mathbb{E} [TR] \to 0 \).

Similarly
\[
\mathbb{E} [E_{MV}] = \mathbb{E} \left[ G(\theta^*, 0) - G(\hat{\theta}, 0) \right] = \frac{k + 1}{T} + o(T^{-1})
\]

\[\square\]
10.2. SOME COMPUTATIONS

Let

\[ F(\theta, u) = \lambda(\theta) (\langle X(\theta), Y \rangle + u \langle X(\theta) \cdot W \rangle) \quad \text{with} \quad \lambda(\theta) = \frac{1}{\sqrt{\langle X(\theta), X(\theta) \rangle}} \]

Then we get the following derivatives

\[ D\lambda = -\lambda^3 \langle DX, X \rangle \]
\[ D^2\lambda = 3\lambda^5 \langle DX, X \rangle^2 - \lambda^3 \langle D^2X, X \rangle - \lambda^3 \langle DX, DX \rangle \]

and

\[ F_u(\theta, u) = +\lambda (X \cdot W) \]
\[ F_{u,u}(\theta, u) = 0 \]
\[ F_{u,\theta}(\theta, u) = D\lambda X \cdot W + \lambda DX \cdot W \]
\[ = -\lambda^3 \langle DX, X \rangle (X \cdot W) + \lambda (DX \cdot W) \]
\[ F_\theta(\theta, u) = D\lambda (\langle X, Y \rangle + u (X \cdot W)) + \lambda (\langle DX, Y \rangle + u DX \cdot W)) \]
\[ = -\lambda^3 \langle DX, X \rangle (X, Y) + u (X \cdot W) + \lambda (DX, Y) + u (DX \cdot W)) \]
\[ F_{\theta,\theta}(\theta, u) = D^2\lambda (\langle X, Y \rangle + u (X \cdot W)) + \]
\[ D\lambda (\langle DX, X \rangle + u (DX \cdot W)) + \]
\[ D\lambda (\langle DX, Y \rangle + u (DX \cdot W)) + \]
\[ \lambda (\langle D^2X, Y \rangle + u (D^2X \cdot W)) \]

In particular

\[ F_u(\theta^*, 0) = +\lambda (X \cdot W) \]
\[ F_{u,u}(\theta^*, 0) = 0 \]
\[ F_{u,\theta}(\theta^*, 0) = -\lambda^3 \langle DX, X \rangle (X \cdot W) + \lambda (DX \cdot W) \]
\[ F_\theta(\theta^*, 0) = 0 \]
\[ F_{\theta,\theta}(\theta^*, 0) = D^2\lambda \langle X, Y \rangle + D\lambda \langle DX, Y \rangle + D\lambda \langle DX, Y \rangle + \lambda \langle D^2X, Y \rangle \]
\[ = 3\lambda^3 \langle DX, X \rangle^2 - \lambda \langle D^2X, X \rangle - \lambda \langle DX, DX \rangle + \]
\[ -\lambda^3 \langle DX, X \rangle \langle DX, Y \rangle - \lambda^3 \langle DX, Y \rangle^2 \]
\[ + \lambda \langle D^2X, Y \rangle \] using Lemma 10.8 and \[ \lambda(\theta) = \frac{1}{\sqrt{\langle X(\theta), X(\theta) \rangle}} \]
\[ = \lambda^3 \langle DX, X \rangle^2 - \lambda \langle DX, DX \rangle + \lambda \langle D^2X, Y - X \rangle \]
\[ = \lambda^3 \langle DX, X \rangle^2 - \lambda \langle DX, DX \rangle \] by assumption 7.7 6)
10. References

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