A generalized evidence distance

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Abstract

Dempster-Shafer theory of evidence (D-S theory) is widely used in uncertain information process. The basic probability assignment (BPA) is a key element in D-S theory. How to measure the distance between two BPAs is an open issue. In this paper, a new method to measure the distance of two BPAs is proposed. The proposed method is a generalized of existing evidence distance. Numerical examples are illustrated that the proposed method can overcome the shortcomings of existing methods.

Keywords: Dempster-Shafer theory of evidence, Basic probability assignment, Evidence distance, Similarity functions

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1. Introduction

Dempster-Shafer theory (D-S theory) is first proposed by Dempster, and later by Shafer improved \[1, 2\], has attracted more and more researchers, for its ability to handle uncertain information in many fields, such as combination of multi-classier, information fusion, target recognition, decision making, etc \[3, 4, 5, 6\].

Many complex, uncertain information can be well processed by the combination rule of D-S theory. However, when the basic probability assignments (BPAs) are high conflict, the combination rule of D-S theory will generate an invalid result of counter-intuition. Besides the coefficient $k$, evidence distance is another expression of measure the conflict of two BPAs. Recently, there are two distance functions of measure the distance of two BPAs \[7, 8, 9\]. One is proposed by Jousselme \[8\], another is proposed by Sunberg \[9\], based on Hausdorff distance \[10\]. The differences between the two methods are the distance functions. The main point of Jousselme’s method is the similarity matrix $D$, measured the conflict of focal elements of BPAs \[8\]. Jousselme’s method is effective in the case of stationary BPAs. While the BPAs are shifted, it can’t reflect the physical distance of two BPAs, intuitively. Sunberg’s method is designed specifically for orderable frames of discernment, applied a Hausdorff-based measure to account for the distance between focal elements \[9\]. It is suitable for varied BPAs, but can’t apply to varied masses. In other words, Jousselme’s method is suitable for fixed BPAs but mass changed, Sunberg’s method can be applied to fixed masses but BPAs
changed. To address these issues in existing methods, we propose a general-
ized method to measure the evidence distance, that integrates the merits of
the existing methods and overcomes the shortcoming of both.

The rest of the this paper is organized as follows. Section 2 presents some
preliminaries. The proposed method to measure the distance of two BPAs
and numerical examples and applications are presented in Section 3. A short
conclusion is drawn in the last section.

2. Preliminaries

In D-S theory, Let Θ = {θ₁, θ₂, · · · , θₙ} be the finite set of mutually
exclusive and exhaustive events. D-S theory is concerned with the set of all
subsets of Θ, which is a powerset of 2^|Θ|, known as the frame of discernment,
denoted as Ω = {∅, {θ₁}, {θ₂}, · · · , {θₙ}, {θ₁, θ₂}, · · · , {θ₁, θ₂ · · · , θₙ}}.

The mass function of evidence assigns probability to the subset of Ω,
also called basic probability assignment(BPA), which satisfies the follow-

The D-S rule of combination is the first one within the framework of
evidence theory which can combine two BPAs m₁ and m₂ to yield a new
BPA m. D-S rule of combination are presented as follow:

\[
m(A) = \frac{1}{1 - k} \sum_{B \cap C = A} m_1(B)m_2(C)
\]

with

\[
k = \sum_{B \cap C = \emptyset} m_1(B)m_2(C)
\]
Where $k$ is a normalization constant, called the conflict coefficient of BPAs.

There are two existing methods to measure the distance of two BPAs, one is proposed by Jousselme, another is proposed by Sunberg. The main points of the two methods are shown as follows.

The evidence distance proposed by Jousselme [8], are presented as follows:

$$d_{BPA}(m_1, m_2) = \sqrt{\frac{1}{2} (m_1 - m_2)^T D (m_1 - m_2)}$$

(3)

$D$ is an $2^N \times 2^N$ similarity matrix to measure the conflict of focal element in $m_1$ and $m_2$, where

$$D(A, B) = \frac{|A \cap B|}{|A \cup B|}$$

(4)

The another evidence distance proposed by Sunberg [9], are presented as follows:

$$d_{Haus}(m_1, m_2) = \sqrt{\frac{1}{2} (m_1 - m_2)^T D_H (m_1 - m_2)}$$

(5)

with

$$D_{H(i,j)} = S_H(A_i, A_j) = \frac{1}{1 + KH(A_i, A_j)}$$

(6)

Where $H(A_i, A_j)$ is the Hausdorff distance between focal elements $A_i$ and $A_j$. $K > 0$ is a user-defined tuning parameter ($K$ is set to be 1, simplified, the same as below). It is defined according to

$$H(A_i, A_j) = \max\{\sup_{b \subseteq A_i} \inf_{c \subseteq A_j} d(b, c), \sup_{c \subseteq A_j} \inf_{b \subseteq A_i} d(b, c)\}$$

(7)

Where $d(b, c)$ is the distance between two elements of the sets and can be defined as any valid metric distance on the measurement space [10]. In the
case where elements of the sets are real numbers, that is, the 1-dimensional Euclidean case, distance can be measured as the absolute value of the difference between the elements, the Hausdorff distance may be defined as

\[ H_{1-D}(A_i, A_j) = \max\{\{|\min(A_i) - \min(A_j)|, |\max(A_i) - \max(A_j)|\}\} \quad (8) \]

3. The generalized evidence distance and Applications

3.1. New evidence distance

Both Jousselme’s and Sunberg’s methods can measure the distance of two BPAs, in D-S theory, but the two existing methods take effect only under the special situations, while the cases changed, counter-intuitive results will be presented. In view of these situations, we propose a new method that synthesize the merits of both Jousselme and Sunberg proposed, suitable for the both special cases, overcome the shortcomings of the both existing methods. This new metric proposed mirrors the quadratic form from structure of Jousselme but replaces the distance function. The new metric \( d_{New} \) is defined as follows:

\[ d_{New}(m_1, m_2) = \sqrt{\frac{1}{2}(m_1 - m_2)^T D_{New}(m_1 - m_2)} \quad (9) \]

with

\[ D_{New(i,j)} = \alpha D_{(i,j)} + (1 - \alpha) D_{H(i,j)} \quad (10) \]

Where \( D_{(i,j)} \) is the similarity matrix of Jousselme in Eq.(4), \( D_{H(i,j)} \) is the distance matrix of Sunberg in Eq.(7). The parameter \( \alpha \) is a user-defined
weighted normalized coefficient, $\alpha \in [0, 1]$. In this paper, $\alpha$ is set to be 0.5. And the new metric $d_{\text{New}}$ satisfies the follow conditions:

$$\text{while } \alpha = 1, \quad d_{\text{New}}(m_1, m_2) = d_{\text{BPA}}(m_1, m_2) \quad (11)$$

$$\text{while } \alpha = 0, \quad d_{\text{New}}(m_1, m_2) = d_{\text{Haus}}(m_1, m_2) \quad (12)$$

Thus, both the two methods\cite{8, 9} are special cases of the proposed distance function. The new proposed method of evidence distance is generalized.

3.2. Numerical examples and Applications

3.2.1. Fixed masses of varied BPAs

We use an example of orderable sets in \cite{9}. BPA $A$ is held stationary while the position of BPA $B$ is shifted right along the real line, such that the absolute value of the midpoints of each BPA take on different values (For more detail, please refer to \cite{9}). In this example, the two BPAs are constructed as follows:

The first BPA $A$ is shown as

$$m_A(2) = 0.1, \quad m_A(2, 2.3) = 0.2, \quad m_A(2, 2.3, 2.5) = 0.4,$$

$$m_A(2, 2.3, 2.5, 2.7) = 0.2, \quad m_A(2, 2.3, 2.5, 2.7, 3) = 0.1$$

The second BPA $B$ is given as

$$m_B(i) = \frac{1}{3}, \quad m_B(i, 0.5 + i) = \frac{1}{3}, \quad m_B(i, 0.5 + i, 1 + i) = \frac{1}{3}$$

Where $i$ is an integer, varied from 2 to 12, means that $B_1$ shifted right along the real line to $B_2$. The distance of the two BPAs are graphically illustrated in the Fig.1 through different methods, respectively. It should be
pointed that, while the BPAs are varied, Jousselme’s method can’t reflect the metric of two BPAs effectively, both Sunberg and the new proposed methods can measure the distance of the two BPAs, clearly.

![Graph showing distance metrics](image)

Figure 1: Values for distance metrics as one BPA is held stationary, and another BPA is shifted right to different locations, in the section 3.2.1. Note that minimum distances are nonzero due to the difference in structure between the BPAs.

### 3.2.2. Varied masses of fixed BPAs

We use an example of varied elements in [11]. Let $\Omega$ be a frame of discernment with 20 elements (or any number of elements that is pre-defined). The first BPA $m_1$ is shown as

$$m_1(\{2, 3, 4\}) = 0.05, m_1(\{7\}) = 0.05, m_1(\Omega) = 0.1, m_1(A) = 0.8$$

Where $A$ is a subset of $\Omega$. The second BPA $m_2$ is given as

$$m_2(\{1, 2, 3, 4, 5\}) = 1$$
There are 20 cases where subset $A$ increments one more element at a time, starting from Case 1 with $A = \{1\}$ and ending with Case 20 when $A = \Omega$. The metric of two BPAs are graphically illustrated in Fig.2 by the means of different methods, respectively. In this example, we notice that the method proposed by Sunberg is inability for the cases of focal elements varied, both Jousselme and our methods can deal with this situation, easily.

![Figure 2: The X-axis shows the sizes of subset $A$, when subset $A$ changes, and the Y-axis show the different evidence distances, in the section 3.2.2 used different methods.](image)

4. Conclusion

The evidence distance of BPAs plays a key role in D-S theory. In this paper, two evidence distances are illustrated the shortcomings in some situations. We propose a generalized method to measure the distance of two BPAs. It is generalized and flexible, owing to the adjustable parameter $\alpha$. 
Jousselme and Sunberg’s methods are the special cases of this proposed generalized method. It integrates the merits and overcomes the shortcomings of existing method. Numerical examples are demonstrated that the new proposed method can measure the distance of two BPAs, effectively.

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