On the existence of A-loops with some commutative inner mappings and others of order 2

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Abstract
The existence of $A_\rho$-loops, $A_\lambda$-loops and $A_\mu$-loops that are neither extra loops nor CC-loops such that any two of their inner mappings $R(x, y), L(x, y)$ and $T(x)$ commute while the other one is of order 2 is shown.

1 INTRODUCTION
Let $L$ be a non-empty set. Define a binary operation $(\cdot)$ on $L$. If $x \cdot y \in L$ for all $x, y \in L$, $(L, \cdot)$ is called a groupoid. If the system of equations $a \cdot x = b$ and $y \cdot a = b$ have unique solutions for $x$ and $y$ respectively, then $(L, \cdot)$ is called a quasigroup. Furthermore, if there exists a unique element $e \in L$ called the identity element such that for all $x \in L$, $x \cdot e = e \cdot x = x$, $(L, \cdot)$ is called a loop. A detailed information on loop properties, types, concepts and applications are contained in [3, 1, 3, 8, 24] and [41].

The symmetric group of a loop $(L, \cdot)$ is denoted by $S(L, \cdot)$ and it is defined as the group of all permutations or self-bijections on $L$. The bijection $L_x : L \to L$ defined as $yL_x = x \cdot y$ for all $x, y \in L$ is called a left translation(multiplication) of $L$ while the bijection $R_x : L \to L$ defined as $yR_x = y \cdot x$ for all $x, y \in L$ is called a right translation(multiplication) of $L$. In a loop $(L, \cdot)$, the group generated by the set of left or right translations and their inverses is denoted by $M_\lambda(L, \cdot)$ or $M_\rho(L, \cdot)$ and called the left or right multiplication group of $(L, \cdot)$, while the group generated by the set of both left and right translations and their inverses is denoted by $M(L, \cdot)$ and called the multiplication group of $(L, \cdot)$. It is well known that the groups $M_\lambda(L, \cdot), M_\rho(L, \cdot)$ and $M(L, \cdot)$ are subgroups of $S(L, \cdot)$.

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The triple \((U, V, W)\) formed such that \(U, V, W \in S(L, \cdot)\) is called an autotopism of \(L\) if and only if \(xU \cdot yV = (x \cdot y)W\) for all \(x, y \in L\). The group of autotopisms of \(L\) is called the autotopism group and it is denoted by \(AUT(L, \cdot)\). If \(U = V = W\), then \(U\) is called an automorphism. The group of automorphisms on \(L\) is called the automorphism group and it is denoted by \(\text{Aut}(L)\).

If \(U \in S(L, \cdot)\) such that \((U, URc, URc) \in AUT(L, \cdot)\) or \((ULc, U, ULc) \in AUT(L, \cdot)\) for some \(c \in L\), then \(U\) is called a right or left pseudo-automorphism of the loop \(L\) with a right or left companion \(c\). The group formed by such permutations is called the right or left pseudo-automorphism group and it is denoted by \(PS_r(L, \cdot)\) or \(PS_\lambda(L, \cdot)\).

All elements \(\alpha \in M_\lambda(L, \cdot)\) or \(\alpha \in M_\rho(L, \cdot)\) or \(\alpha \in M(L, \cdot)\) such that \(e\alpha = e\) form a group called the left inner mapping group or right inner mapping group or inner mapping group of \((L, \cdot)\) and this is denoted by \(\text{Inn}_\lambda(L, \cdot)\) or \(\text{Inn}_\rho(L, \cdot)\) or \(\text{Inn}(L, \cdot)\).

The inner mapping \(R(x, y) = R_xR_yR_{xy}^{-1}\) is called a right inner mapping and it has been shown that they generate the group \(\text{Inn}_\rho(L, \cdot)\). A loop \(L\) is called a right \(A\)-loop \((A_\rho\text{-loop})\) if \(\text{Inn}_\rho(L, \cdot) \leq A(L, \cdot)\).

The inner mapping \(L(x, y) = L_xL_yL_{yx}^{-1}\) is called a left inner mapping and it has been shown that they generate the group \(\text{Inn}_\lambda(L, \cdot)\). A loop \(L\) is called a left \(A\)-loop \((A_\lambda\text{-loop})\) if \(\text{Inn}_\lambda(L, \cdot) \leq A(L, \cdot)\).

The inner mapping \(T(x) = R_xL_x^{-1}\) is called a middle inner mapping and the group generated by these is denoted by \(\text{Inn}_\mu(L, \cdot)\) and called the middle inner mapping group. A loop \(L\) is called a middle \(A\)-loop \((A_\mu\text{-loop})\) if \(\text{Inn}_\mu(L, \cdot) \leq A(L, \cdot)\).

It has been shown in \cite{38} that the inner mapping group \(\text{Inn}(L, \cdot)\) of a loop \(L\) is generated by its left, right and middle inner mappings. So, if \(\text{Inn}(L, \cdot) \leq A(L, \cdot)\), \(L\) is called an \(A\)-loop, hence, \(L\) is an \(A\)-loop if and only if \(L\) is an \(A_\rho\text{-loop}\), \(A_\lambda\text{-loop}\) and an \(A_\mu\text{-loop}\). The study of \(A\)-loops started by Bruck and Paige in \cite{2}. Further studies on \(A\)-loops have been done by Osborn \cite{37}, Phillips \cite{39} and Drapal \cite{9}. The most interesting work on \(A\)-loops is Kinyon et. al. \cite{31} which gives the solution to the Osborn problem.

After the introduction of conjugacy closed loops (CC-loop) by Goodaire and Robinson \cite{22, 23}, a tremendous study of their properties and structural behaviours have been studied by Kunen \cite{32} and some recent works of Kinyon and Kunen \cite{28, 30}, Phillips et. al. \cite{29}, Drápal \cite{10, 11, 12, 13, 14}, Csörgő et. al. \cite{5, 19, 4} and Phillips \cite{40}. In \cite{28, 29, 30} and \cite{32}, it is proved and stated that in a CC-loop \(L\):

- \(R(x, y), (x, y) \in A(L)\), hence \(L\) is a both an \(A_\rho\text{-loop}\) and an \(A_\lambda\text{-loop}\),

- \(\text{Inn}_\lambda(L) = \text{Inn}_\rho(L)\) and \(\text{Inn}(L, \cdot) = \left\langle \{T(x) : x \in L\} \right\rangle\),

- \(R(x, y)R(u, v) = R(x, y)R(u, v)\) and \(R(x, y)L(u, v) = L(u, v)R(x, y)\), hence \(\text{Inn}_\rho(L)\) and \(\left\langle \{R(x, y), L(x, y) : x, y \in L\} \right\rangle\) are abelian groups.

If \(L\) is an extra loop, then the facts listed above and the ones below are true.

- \(R(x, y) = L(x, y) = R(y, x) = L(y, x), |R(x, y)| = 2\), hence \(\text{Inn}_\lambda(L) = \text{Inn}_\rho(L)\) are boolean groups,
• $T(x) \in A(L)$ if and only if $x \in N(L)$.

If $L$ is an A-loop then according to [2], $T(x)L(y, x) = L(y, x)T(x)$ and $T(x)R(x, y) = R(x, y)T(x)$.

The multiplication group and inner mapping group of loops have been studied by Drápal [16], [17], [18], [15], Drápal et. al. [20], [21], Kepka [25], [26], Kepka and Niemenmaa [27], Niemenmaa [33], [34], [35], Niemenmaa and Kepka [36], Csörgő and Kepka [6] in different fashions. The multiplication group structure determines the structure of a loop (e.g solvability of $M(L)$ implies the solvability of a finite loop $L$ [12]) while if $\text{Inn}(L)$ is of order $2p(p$ an odd prime), then $M(L)$ is solvable hence $L$ is solvable as well([2]).

The present study investigates the existence of $A_\rho$-loops, $A_\lambda$-loops, $A_\mu$-loops and A-loops that are neither extra loops nor CC-loops such that any two of their inner mappings $R(x, y), L(x, y)$ and $T(x)$ commute while the other one is of order 2.

Definition 1.1 If $(L, \cdot)$ and $(G, \circ)$ are two distinct loops, then the triple $(U, V, W) : (L, \cdot) \rightarrow (G, \circ)$ such that $U, V, W : L \rightarrow G$ are bijections is called a loop isotopism if and only if $xU \circ yV = (x \cdot y)W \forall x, y \in L$.

Throughout, when $L_x : y \mapsto xy$ and $R_x : y \mapsto yx$ are respectively the left and right translations of a loop then the left and right translations of its loop isotope are denoted by $L'_x : y \mapsto xy$ and $R'_x : y \mapsto yx$ respectively.

Definition 1.2 Let $(B, \cdot)$ be a loop. If $x \in B$ and $\phi \in S(B, \cdot)$, then the mapping $\mu_x(\phi) : S(B, \cdot) \rightarrow S(B, \cdot)$ defined by $\mu_x(\phi) = \phi^{-1}L_x\phi L_x^{-1}$ is called the deviation of the mapping $\phi$ at $x$.

Furthermore, set $P(x, \phi) := L_x\phi - \phi L_x^{-1}\phi L_x\phi^{-1}L_x\phi$.

# 2 MAIN RESULTS

## 2.1 Deviation

Lemma 2.1 Let $(B, \cdot)$ be a loop with $\phi \in S(B, \cdot)$.

If $P(x, \phi) := 0$, then $\mu_x(\phi) = L_x^{-1}\phi L_x\phi^{-1}$.

Proof

If $P(x, \phi) := 0$, then $L_x\phi = \phi L_x^{-1}\phi L_x\phi^{-1}L_x\phi \implies \phi^{-1}L_x\phi L_x^{-1} = L_x^{-1}\phi L_x\phi^{-1} \implies \mu_x(\phi) = L_x^{-1}\phi L_x\phi^{-1}$ since $\mu_x(\phi) = \phi^{-1}L_x\phi L_x^{-1}$. 

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**Theorem 2.1** Let \((B, \cdot)\) be a loop with \(\phi \in S(B, \cdot)\) such that \(\phi : e \mapsto e\). If \(P(x, \phi) := 0\), then \(\phi \in A(B, \cdot)\).

**Proof**

\[ P(x, \phi) := 0 \implies L_x L_x = \phi L_x^{-1} \phi L_x \phi^{-1} L_x = y L_x = \phi L_x^{-1} \phi L_x \phi^{-1} L_x \implies (xy) = \phi L_x^{-1} \phi L_x \phi^{-1} L_x. \tag{1} \]

Let \(z = y \phi L_x^{-1} \implies x \cdot z = y \phi\). Put \(x = e\), then

\[ z = y \phi. \tag{2} \]

Now from equation (1), we have \((xy) = z \phi L_x \phi^{-1} L_x = (x \cdot z) \phi^{-1} L_x = x \phi \cdot (x \cdot z) \phi^{-1}\). So,

\[ (xy) = x \phi \cdot (x \cdot z) \phi^{-1}. \tag{3} \]

Let \(z' = (x \cdot z) \phi^{-1}\), then \(x \cdot z = z' \phi\). Using equation (2), \(x \cdot y \phi^2 = z' \phi\). Now, let \(x = e\), then \(y \phi^2 = z' \phi \implies z' = y \phi\).

From equation (3), \((xy) = x \phi \cdot z'\). So by equation (4),

\[ (xy) = x \phi \cdot y \phi \implies \phi \in A(B, \cdot). \]

**Theorem 2.2** Let \((B, \cdot)\) be a loop and \(\phi \in S(B, \cdot)\). The following are true.

1. \(\phi : e \mapsto e \iff \mu_x(\phi) : e \mapsto e \quad \forall x \in B\).
2. \(\phi \in PS_\lambda(B, \cdot) \iff \exists c \in B \quad \mu_x(\phi) = L(x, \phi, e)^{-1} \quad \forall x \in B\).
3. \(\phi \in A(B, \cdot) \iff \mu_x(\phi) = I \quad \forall x \in B\).

**Proof**

Recall that \(\mu_x(\phi) = \phi^{-1} L_x \phi L_x^{-1}\).

1. \(e \mu_x(\phi) = e \iff e \phi^{-1} L_x \phi L_x^{-1} = e \iff e \phi^{-1} L_x \phi = e L_x \phi \iff (x \cdot e \phi^{-1}) = x \phi \iff x \cdot e \phi^{-1} = e \iff e \phi = e\).

2. \(\phi \in PS_\lambda(B, \cdot)\) with a left companion \(c \in B\) if and only if

\[ c \cdot (x \cdot y) \phi = (c \cdot x) \phi \cdot y \phi \iff L_{(c-x) \phi} = \phi^{-1} L_x \phi L_c. \tag{5} \]

\[ L(x, y) = L_x L_y L_{yx}^{-1}, \text{ so } L(x, \phi, c) = L_{x \phi} L_c L_{(c-x) \phi}^{-1}. \text{ Thus, computing and using equation (5),} \]

\[ \mu_x(\phi)L(x, \phi, c) = \left(\phi^{-1} L_x \phi L_x^{-1}\right) \left(L_x \phi L_c L_{(c-x) \phi}^{-1}\right) = \phi^{-1} L_x \phi L_c L_{(c-x) \phi}^{-1} = L_{(c-x) \phi} L_{(c-x) \phi}^{-1} = I. \]
which implies $\mu_x(\phi)L(x,\phi,c) = I \implies \mu_x(\phi) = L(x,\phi,c)^{-1}$.

Conversely, if $\mu_x(\phi) = L(x,\phi,c)^{-1}$, then $\mu_x(\phi)L(x,\phi,c) = I$. So,

$$\left(\phi^{-1}L_x\phi L_x^{-1}\right)\left(L_x\phi L_c L_{(c,x\phi)}^{-1}\right) = I \implies \phi^{-1}L_x\phi L_c L_{(c,x\phi)}^{-1} = I \implies \phi^{-1}L_x\phi L_c = L_{(c,x\phi)}$$

implies $\phi \in PS_\lambda(B,\cdot)$ with a left companion $c \in B$ by following equation (4).

3. Following 2., $\phi \in A(B,\cdot) \iff \phi \in PS_\lambda(B,\cdot)$ such that $c = e$. $L(x,\phi,e) = I$,

$$\therefore \phi \in A(B,\cdot) \iff \mu_x(\phi) = I \forall x \in B.$$

**Theorem 2.3** Let $(B,\cdot)$ be a loop with $\phi \in S(B,\cdot)$ such that $\phi : e \mapsto e$. If $P(x,\phi) := 0$, then $|\phi| = 2$.

**Proof**

By Theorem 2.1, $\phi \in A(B,\cdot)$. So following Theorem 2.2 and Lemma 2.1, $\mu_x(\phi) = L_{x\phi}^{-1}\phi L_x \phi^{-1} = I$. Thus, $L_{x\phi}^{-1}\phi L_x \phi^{-1} = I \implies L_x = L_{x\phi} \phi \implies yL_x = yL_{x\phi} \phi \implies x \cdot y\phi = (x\phi \cdot y)\phi = x\phi^2 \cdot y\phi \implies x = x\phi^2 \implies \phi^2 = I$.

### 2.2 Isotopic Characterization Of A-loops

**Theorem 2.4** Let $(G,\cdot)$ and $(H,\circ)$ be any two distinct quasigroups. If $A,B,C : G \rightarrow H$ are permutations, then the following conditions are equivalent:

1. the triple $\alpha = (A,B,C)$ is an isotopism of $G$ upon $H$.
2. $R'_{xB} = A^{-1}R_x C \forall x \in G$.
3. $L'_{yA} = B^{-1}L_y C \forall y \in G$.

**Proof**

$(1 \iff 2)$ If $\alpha = (A,B,C) : (G,\cdot) \rightarrow (H,\circ)$ is an isotopism, then $xA \circ yB = (x \cdot y)C \iff xAR'_{yB} = xR_y C \iff AR'_{yB} = R_y C \iff R'_{yB} = A^{-1}R_y C$.

$(1 \iff 3)$ If $\alpha = (A,B,C) : (G,\cdot) \rightarrow (H,\circ)$ is an isotopism, then $xA \circ yB = (x \cdot y)C \iff yBL'_{xA} = yL_x C \iff BL'_{xA} = L_x C \iff L'_{xA} = B^{-1}L_x C$.

Finally, $1 \iff 2$ and $1 \iff 3 \Rightarrow 2 \iff 3$. Hence the statements 1, 2 and 3 are equivalent to each other.

**Theorem 2.5** Let $G = (\Omega,\cdot)$ and $G' = (\Omega,\circ)$ be any two distinct loops. If $A,B,C \in S(\Omega)$ such that $P(x,\phi) := 0 \forall \phi \in \{A,B,C\}$, then the following are equivalent for all $x \in \Omega$. 
1. \((A, B, C)\) is an isotopism of \(G\) upon \(G'\).

2. \(\mu_x(A) = CL^{-1}_{xA}B^{-1}ALxA^{-1}\).

3. \(\mu_x(A) = L^{-1}_{xA}ABL'_xA(AC)^{-1}\).

4. \(\mu_x(B) = CL^{-1}_{xABA}LxB^{-1}\).

5. \(\mu_x(B) = L^{-1}_{xB}B^2L'_xA(BC)^{-1}\).

6. \(\mu_x(C) = CL^{-1}_{xCA}B^{-1}CLxC^{-1}\).

7. \(\mu_x(C) = L^{-1}_{xC}CBL'_xA^{-2}\).

8. \((I, AB, \mu_{xA^{-1}}(A)AC)\) is an isotopism of \(G\) upon \(G'\).

9. \((B^{-1}A, B^2, \mu_{xB^{-1}}(B)BC)\) is an isotopism of \(G\) upon \(G'\).

10. \((C^{-1}A, CB, \mu_{xC^{-1}}(C)C^2)\) is an isotopism of \(G\) upon \(G'\).

**Proof**

1 \(\Leftrightarrow\) 2, 1 \(\Leftrightarrow\) 3, 1 \(\Leftrightarrow\) 4, 1 \(\Leftrightarrow\) 5, 1 \(\Leftrightarrow\) 6, 1 \(\Leftrightarrow\) 7, 1 \(\Leftrightarrow\) 8, 1 \(\Leftrightarrow\) 9 and 1 \(\Leftrightarrow\) 10 are achieved by using Theorem 2.4, Lemma 2.1 and Definition 1.2.

**Theorem 2.6** Let \(G = (\Omega, \cdot)\) and \(G' = (\Omega, \circ)\) be two distinct loops.

1. If \(A \in S(\Omega)\) such that \(A : e \mapsto e\) and \(P(x, A) := 0 \ \forall \ x \in \Omega\), then the following are equivalent:
   
   (i) \((A, B, C)\) is an isotopism of \(G\) upon \(G'\).
   
   (ii) \((I, AB, AC)\) is an isotopism of \(G\) upon \(G'\).
   
   (iii) \(ALxAAC = BL'_xA\).
   
   (iv) \(A = CL^{-1}_{xBA}LxA\).

2. If \(B \in S(\Omega)\) such that \(B : e \mapsto e\) and \(P(x, B) := 0 \ \forall \ x \in \Omega\), then the following are equivalent:
   
   (i) \((A, B, C)\) is an isotopism of \(G\) upon \(G'\).
   
   (ii) \((BA, I, BC)\) is an isotopism of \(G\) upon \(G'\).
   
   (iii) \(B = CL^{-1}_{xBA}LxA\).

3. If \(C \in S(\Omega)\) such that \(C : e \mapsto e\) and \(P(x, C) := 0 \ \forall \ x \in \Omega\), then the following are equivalent:
   
   (i) \((A, B, C)\) is an isotopism of \(G\) upon \(G'\).
(ii) \((CA, CB, I)\) is an isotopism of \(G\) upon \(G'\).

(iii) \(L_x = CBL'_xCA\).

Proof
The proof lies wholly on Theorem 2.5. And the outcomes are achieved by using Theorem 2.2, Theorem 2.3 and Theorem 2.1.

Corollary 2.1 Let \(G = (\Omega, \cdot)\) and \(G' = (\Omega, \circ)\) be two distinct isotopic loops with different identity elements such that the triple \((A, B, C)\) is the isotopism between \(G\) and \(G'\).

1. If \(A \in S(\Omega)\) such that \(A : e \mapsto e\) and \(P(x, A) := 0 \ \forall \ x \in \Omega\), \((A, B, C)\) is an isotopism of \(G\) upon \(G'\) if and only if \(L'_x = B^{-1}L_xAC\). Hence,

(i) \(C = BL'_e, \ B = CL'_e\) and \(L'^2_e = I\).

(ii) \(C^{-1}B = B^{-1}C\) and \(CB = BC\).

2. If \(B \in S(\Omega)\) such that \(B : e \mapsto e\) and \(P(x, B) := 0 \ \forall \ x \in \Omega\), then \(C = BL'_eA\).

3. If \(C \in S(\Omega)\) such that \(C : e \mapsto e\) and \(P(x, C) := 0 \ \forall \ x \in \Omega\), then \(C = BL'_eA\).

Proof
The proof of this is a consequence of Theorem 2.6 by replacing \(x \in \Omega\) with the identity element \(e\) of \(G\).

The tables below summarize the important results of this subsection as shown in Theorem 2.6 and Corollary 2.1.

| Hypothesis | Hypothesis | Inference |
|------------|------------|-----------|
| \(P(x, \cdot) := 0\) | \(\cdot : e \mapsto e\) | \(C = BL'_e, B = CL'_e\) \(C^{-1}B = B^{-1}C, CB = BC\) |
| \(A\) | \(A\) | |
| \(B\) | \(B\) | \(C = BL'_e\) |
| \(C\) | \(C\) | \(C = BL'_e\) |

| Hypothesis | Hypothesis | Hypothesis | Inference |
|------------|------------|------------|-----------|
| \(\cdot : e \mapsto e\) | \(P(x, \cdot) := 0\) | Isotopism | Equivalent Isotopism |
| \(A\) | \(A\) | \(A,B,C\) | \(I, AB, AC\) |
| \(B\) | \(B\) | \(A,B,C\) | \(BA, I, BC\) |
| \(C\) | \(C\) | \(A,B,C\) | \(CA, CB, I\) |

Theorem 2.7 Let \(G = (\Omega, \cdot)\) and \(G' = (\Omega, \circ)\) be two distinct isotopic loops.

1. Under the triple \((R(x, y), L(u, v), T(z))\),

   (a) if \(P(z, R(x, y)) := 0\) then,
Under the triple

(i) \( G \) is an \( A_{\mu} \)-loop and \(|R(x,y)| = 2\).

(ii) \( \text{Inn}_\mu(G) = \langle L(x,y)L'_e : x, y \in \Omega \rangle \) and \( \text{Inn}_\lambda(G) = \langle T(x)L'_e : x \in \Omega \rangle \).

(iii) \( T(z)L(x,y) = L(x,y)T(z) \) and \( T(z)^{-1}L(x,y) = L(x,y)^{-1}T(z) \), hence \( L(x,y)^2 = T(z)^2 \).

(iv) the triple \( (I, R(x,y)L(u,v), R(x,y)T(z)) \) is an isotopism from \( G \) to \( G' \).

(b) if \( P(z, L(x,y)) := 0 \) then,

(i) \( G \) is an \( A_{\lambda} \)-loop and \(|L(x,y)| = 2\).

(ii) \( \text{Inn}_\mu(G) = \langle L(x,y)L'_e : x, y \in \Omega \rangle \).

(iii) the triple \( (L(u,v)R(x,y), I, L(u,v)T(z)) \) is an isotopism from \( G \) to \( G' \).

(c) if \( P(z, T(x)) := 0 \) then,

(i) \( G \) is an \( A_{\mu} \)-loop and \(|T(x)| = 2\).

(ii) \( \text{Inn}_\mu(G) = \langle L(x,y)L'_e : x, y \in \Omega \rangle \).

(iii) the triple \( (T(z)R(x,y), T(z)L(u,v), I) \) is an isotopism from \( G \) to \( G' \).

2. Under the triple \( (L(x,y), R(u,v), T(z)) \),

(a) if \( P(z, L(x,y)) := 0 \) then,

(i) \( G \) is an \( A_{\lambda} \)-loop and \(|L(x,y)| = 2\).

(ii) \( \text{Inn}_\mu(G) = \langle L(x,y)L'_e : x, y \in \Omega \rangle \) and \( \text{Inn}_\lambda(G) = \langle T(x)L'_e : x \in \Omega \rangle \).

(iii) \( T(z)R(x,y) = R(x,y)T(z) \) and \( T(z)^{-1}R(x,y) = R(x,y)^{-1}T(z) \), hence \( R(x,y)^2 = T(z)^2 \).

(iv) the triple \( (I, L(x,y)R(u,v), L(x,y)T(z)) \) is an isotopism from \( G \) to \( G' \).

(b) if \( P(z, R(x,y)) := 0 \) then,

(i) \( G \) is an \( A_{\mu} \)-loop and \(|R(x,y)| = 2\).

(ii) \( \text{Inn}_\mu(G) = \langle R(x,y)L'_e : x, y \in \Omega \rangle \).

(iii) the triple \( (R(x,y)L(u,v), I, R(x,y)T(z)) \) is an isotopism from \( G \) to \( G' \).

(c) if \( P(z, T(x)) := 0 \) then,

(i) \( G \) is an \( A_{\mu} \)-loop and \(|T(x)| = 2\).

(ii) \( \text{Inn}_\mu(G) = \langle R(x,y)L'_e : x, y \in \Omega \rangle \).

(iii) the triple \( (T(z)L(x,y), T(z)L(u,v), I) \) is an isotopism from \( G \) to \( G' \).

3. Under the triple \( (T(z), R(x,y), L(u,v)) \),

(a) if \( P(y, T(x)) := 0 \) then,

(i) \( G \) is an \( A_{\mu} \)-loop and \(|L(x,y)| = 2\).

(ii) \( \text{Inn}_\lambda(G) = \langle R(x,y)L'_e : x, y \in \Omega \rangle \) and \( \text{Inn}_\mu(G) = \langle L(x,y)L'_e : x \in \Omega \rangle \).

(iii) \( R(x,y)L(u,v) = L(u,v)R(x,y) \) and \( L(u,v)^{-1}R(x,y) = R(x,y)^{-1}L(u,v) \), hence \( R(x,y)^2 = L(u,v)^2 \).

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Corollary 2.2

Let $G = (Ω, ·)$ and $G' = (Ω, ◦)$ be two distinct isotopic loops.

1. Under the triple $(R(x, y), L(u, v), T(z))$, if $P(z, R(x, y)) := 0$, $P(z, L(x, y)) := 0$ and $P(z, T(x)) := 0$ then,
   
   (a) $G$ is an $A$-loop and $|R(x, y)| = 2$.
   (b) $T(z)L(x, y) = L(x, y)T(z)$.
   (c) $\text{Inn}_u(G) = \langle L(x, y)L'_e : x, y \in Ω \rangle$ and $\text{Inn}_G(G) = \langle T(x)L'_e : x \in Ω \rangle$.
   (d) the triples $(I, L(x, y)R(u, v), L(x, y)T(z))$ and $(T(z)L(x, y), T(z)L(u, v), T(z)L(x, y), I, L(x, y)T(z))$ are isotopisms from $G$ to $G'$.

2. Under the triple $(L(x, y), R(u, v), T(z))$, if $P(z, R(x, y)) := 0$, $P(z, L(x, y)) := 0$ and $P(z, T(x)) := 0$ then,
   
   (a) $G$ is an $A$-loop and $|R(x, y)| = 2$.
   (b) $T(z)R(x, y) = R(x, y)T(z)$.
   (c) $\text{Inn}_u(G) = \langle R(x, y)L'_e : x, y \in Ω \rangle$ and $\text{Inn}_G(G) = \langle T(x)L'_e : x \in Ω \rangle$.
   (d) the triples $(I, L(x, y)R(u, v), L(x, y)T(z))$, $(R(x, y)L(u, v), I, R(x, y)T(z))$ and $(T(z)L(x, y), T(z)L(u, v), T(z)L(x, y), I, L(x, y)T(z))$ are isotopisms from $G$ to $G'$.

3. Under the triple $(T(z), R(x, y), L(u, v))$, if $P(z, R(x, y)) := 0$, $P(z, L(x, y)) := 0$ and $P(z, T(x)) := 0$ then,
   
   (a) $G$ is an $A$-loop and $|R(x, y)| = 2$.
   (b) $R(x, y)L(u, v) = L(u, v)R(x, y)$.
   (c) $\text{Inn}_u(G) = \langle R(x, y)L'_e : x, y \in Ω \rangle$ and $\text{Inn}_G(G) = \langle L(x, y)L'_e : x \in Ω \rangle$.
   (d) the triples $(I, T(z)R(x, y), T(z)L(u, v))$, $(R(x, y)T(z), I, R(x, y)L(u, v))$ and $(L(x, y)T(z), L(x, y)R(u, v), I)$ are isotopisms from $G$ to $G'$.

Proof

This is proved using Theorem 2.6 and Corollary 2.1.

Corollary 2.2

Let $G = (Ω, ·)$ and $G' = (Ω, ◦)$ be two distinct isotopic loops.

1. Under the triple $(R(x, y), L(u, v), T(z))$, if $P(z, R(x, y)) := 0$, $P(z, L(x, y)) := 0$ and $P(z, T(x)) := 0$ then,
   
   (a) $G$ is an $A$-loop and $|R(x, y)| = 2$.
   (b) $T(z)L(x, y) = L(x, y)T(z)$.
   (c) $\text{Inn}_u(G) = \langle L(x, y)L'_e : x, y \in Ω \rangle$ and $\text{Inn}_G(G) = \langle T(x)L'_e : x \in Ω \rangle$.
   (d) the triples $(I, L(x, y)R(u, v), L(x, y)T(z))$, $(R(x, y)L(u, v), I, R(x, y)T(z))$ and $(T(z)L(x, y), T(z)L(u, v), T(z)L(x, y), I)$ are isotopisms from $G$ to $G'$.

2. Under the triple $(L(x, y), R(u, v), T(z))$, if $P(z, R(x, y)) := 0$, $P(z, L(x, y)) := 0$ and $P(z, T(x)) := 0$ then,
   
   (a) $G$ is an $A$-loop and $|R(x, y)| = 2$.
   (b) $T(z)R(x, y) = R(x, y)T(z)$.
   (c) $\text{Inn}_u(G) = \langle R(x, y)L'_e : x, y \in Ω \rangle$ and $\text{Inn}_G(G) = \langle T(x)L'_e : x \in Ω \rangle$.
   (d) the triples $(I, L(x, y)R(u, v), L(x, y)T(z))$, $(R(x, y)L(u, v), I, R(x, y)T(z))$ and $(T(z)L(x, y), T(z)L(u, v), T(z)L(x, y), I)$ are isotopisms from $G$ to $G'$.

3. Under the triple $(T(z), R(x, y), L(u, v))$, if $P(z, R(x, y)) := 0$, $P(z, L(x, y)) := 0$ and $P(z, T(x)) := 0$ then,
   
   (a) $G$ is an $A$-loop and $|R(x, y)| = 2$.
   (b) $R(x, y)L(u, v) = L(u, v)R(x, y)$.
   (c) $\text{Inn}_u(G) = \langle R(x, y)L'_e : x, y \in Ω \rangle$ and $\text{Inn}_G(G) = \langle L(x, y)L'_e : x \in Ω \rangle$.
   (d) the triples $(I, T(z)R(x, y), T(z)L(u, v))$, $(R(x, y)T(z), I, R(x, y)L(u, v))$ and $(L(x, y)T(z), L(x, y)R(u, v), I)$ are isotopisms from $G$ to $G'$.

Proof

This follows directly from Theorem 2.4.
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