Determination the Parameters of Markowitz Portfolio Optimization Model

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Abstract
The main purpose of this study is the determination of the optimal length of the historical data for the estimation of statistical parameters in Markowitz Portfolio Optimization. We present a trading simulation using Markowitz method, for a portfolio consisting of foreign currency exchange rates and selected assets from the Istanbul Stock Exchange ISE 30, over the period 2001-2009. In the simulation, the expected returns and the covariance matrix are computed from historical data observed for past \( n \) days and the target returns are chosen as multiples of the return of the market index. The trading strategy is to buy a stock if the simulation resulted in a feasible solution and sell the stock after exactly \( m \) days, independently from the market conditions. The actual returns are computed for \( n \) and \( m \) being equal to 21, 42, 63, 84 and 105 days and we have seen that the best return is obtained when the observation period is 2 or 3 times the investment period.

Keywords: Markowitz Portfolio Optimization, Observation and Investment Period, Sliding Window Application to ISE 30, Parameter Determination.

1 Introduction

Portfolio optimization is the process of investing to financial instruments with the aim of optimizing certain criteria subject to a number of equality or inequality constraints. The output of the optimization process is a distribution of weights of these instruments in the portfolio. Markowitz portfolio optimization theory \([1]\) is considered to be the milestone of modern finance theory. In the basic application of the model, the aim is to achieve minimum risk subject to a specified level of expected return (or maximum return subject to a specified level of calculated risk) \([2]\).
The mathematics underlying the Markowitz portfolio optimization is the standard quadratic optimization problem subject to equality and inequality constraints. This is a well known problem that has implementations on many platforms. In the minimization of the risk for a predetermined level of target return, the objective function is the “risk” which is assumed to be represented by the quadratic form \( \frac{1}{2}X^TQX \) where \( Q \) is the covariance matrix of the assets in the portfolio and \( X \) is the solution vector that will give the weights of these assets in the portfolio. The crucial constraint expresses the requirement that the portfolio return \( R^TX \) be at least equal to a target return. \( R^TX \geq R_0 \). There may be other constraints that reflect the investors preferences or obligations towards certain groups of assets. Thus, once the covariance matrix \( Q \) and the expected return vector \( R \) have been chosen, the problem is straightforward. However, the optimal solution is very sensitive to these parameters and the determination of the covariance matrix and the expected return vector is crucial. Thus although the basic method is old and well established, the application of the method is still a current research interest.

In the literature, this problem is addressed either by a reformulation of the problem \[3\], by including the estimation risk into the problem \[4\], or by including the indeterminacy of the parameters into the model and using conic programming \[5\] and also by including the estimation of optimal portfolio return \[6\]. We also note that the portfolio optimization problem can be solved as a linear optimization instead of quadratic optimization \[7\].

In earlier investigations we noticed that the statistical parameters of market data were very sensitive to the length of the observation period and we decided to conduct a study for a quantitative measure of the effect of these fluctuations to overall performance of the method. We used historical data of various lengths to estimate the covariance matrix and the expected return vector; with the purpose of determining the optimal relative lengths of the past data to be used for parameter estimation and the mean trading period.

For this purpose, we applied the Markowitz Portfolio Optimization method to selected assets from the Istanbul Stock Exchange ISE 30, over the period 2001-2009 and we chose an investment strategy displaying the effects of the relative lengths of the observation and investment periods. This investment strategy is as follows: “Each day, if the ISE 30 index is down, no purchase is made. If it is up, a buy operation, targeting \( k = 2, 3 \) and \( 10 \) times market average, is made. This purchase is based on the output of the Markowitz Portfolio Optimization, whose parameters are obtained from historical data for past \( n \) days. The portfolio is sold after exactly \( m \) days, regardless of the market conditions and the net return is computed. This process is repeated each day.”

We applied this strategy to the data covering the period 2001-2009 for a range of observation and investment periods. The data and methodology is described in Section 2. The dependency of the parameters on the observation length is discussed in Section 3. The investment strategy and simulation results are presented in Section 4.

## 2 Data and Methodology

The Markowitz portfolio optimization method is a constrained optimization problem where the objective function is the risk of the portfolio \[8\]: in our investment strategy the inequality constraints force the portfolio return to be higher than a targeted return and the equality constraints consist of various technical restrictions. This is a typical quadratic constrained optimization problem that can be solved
with standard computational tools. We have used 14 stocks from ISE 30, and the exchange rates for Euro and USD, a total of 16 assets. Denoting the daily closing prices by \( P_t \), the daily returns \( R_t \) and the daily logarithmic returns \( r_t \) are defined respectively as below.

\[
R_t = \frac{P_t - P_{t-1}}{P_t}, \quad r_t = \ln(1 + R_t).
\]

In the following we work with the logarithmic return series for all assets.

### 2.1 Overview of the Assets

We start by a general classification of the assets under consideration based on their expected return versus risk. For this, we compute the mean and the standard deviation of the logarithmic returns over the whole period as presented in Figure 1. From the figure, we can clearly see that the assets form clusters that we have denoted as groups 1-5.

![Figure 1: Return-Risk distribution of investment instruments. The first group includes currencies and it has low risk and low return values. Group 2 includes stocks from banking industry and it has the highest risk and highest return values excluding KRMD which seem like an outlier. Group 3 and Group 4 consist of industrial companies and they have higher return but lower/same risk compared with Group 5 which consists Telecommunication, Manufacturing, Energy companies and a Bank.](image)

As expected, the currencies (USD and Euro) form the “low risk-low return” group, denoted as Group 1. In this group, EURO and USD have the same risk value but EURO has higher return value than USD. The next group is the highest return cluster, Group 2, consisting of the banking stocks TEBNK and TSKB. This group has higher risk and higher return compared to the ISE 30 index. The risk and the return of the stocks which are in the groups of 3, 4 and 5 are closer to the values of the ISE 30 index. Except for ISCTR and TCELL, these groups consist of industrial manufacturing companies or energy corporations. The ISE 30 market index belongs to Group 4. The KRMD and DYHOL stocks that have either exceptionally high return-high risk or low return-moderate risk do not belong to any of these groups.
3 Statistical Analysis

In this section we shall study the statistical properties of the data. It is well known that the raw stock prices are non-stationary but logarithmic returns over relatively short periods of time are usually considered as stationary. We display in Figure 2, the logarithmic returns for stocks, currencies, the market index and the raw returns for all assets. The raw returns have clearly a rising trend over the second half of the decade and display a depression during the period of 2008 crisis. On the other hand, due to scaling, the effects of 2001 crisis are not that obvious in the price data while the are clearly observed in the return data. From this figure, one can see that ISE 30 average returns and standard deviations are quite volatile between 2001 and 2003, then move almost horizontally until the second half of 2008 where the global crisis shows its effects.

3.1 Observing Data Through Sliding Windows

As discussed in the previous section, it is clear that the time period 2001-2009 have regions of completely different character and data should definitely be split into smaller portions.

We should remind that our emphasis is in obtaining a measure of the lengths of relevant “past” and “future” intervals with respect to portfolio optimization. The key tool will be to observe the past over a “sliding window”. A sliding window is the observation of the data for past $n$ days applied each day starting from the $n + 1$ day. If the past data is taken “as is”, i.e, with a weight of 1 we say that we observe the data over a “rectangular window”. Based on theoretical or technical considerations, one can
multiply the observed data by certain functions to get different observation windows. As an example of technical requirements, in the case of Fourier analysis, one usually smooths out the discontinuities at the end points in order to get a better power spectrum. In the case of financial data, one usually argues that past market conditions have less effect on present and prefers to suppress past data usually by an exponential factor. In this work we modify the observation window by multiplying with the left half of the Gaussian Distribution Function (Normal Distribution) a

\[ f(X) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}. \]

We have computed the mean and the variance of the market index over 10-day sliding windows as a representation of almost instantaneous expected returns and standard deviations as displayed in Figure 3.

![Figure 3: 10-day Sliding window return and standard deviation being reference](image)

4 Simulation Results

In this section we first compare qualitatively the parameters as seen from sliding windows of various lengths with the 10-day sliding window is used as reference. We then describe the trading algorithm and present the simulation results.

4.1 Comparison of Sliding Windows

We now compare the means and standard deviations of the ISE 30 index, computed using sliding windows of various lengths with their instantaneous values represented by 10-day sliding windows. We give below
the comparisons for 50 and 100 day observation periods that will be used in trading.

Figure 4: 50/100-day sliding window application the solid continuous (green) line shows the 50/100-day period and the dots (blue points) show the 10-day sliding window thought to be a representative of the instantaneous values

By comparing these figures, it can be seen that for data observed through longer periods follow a smoother curve but they fail to react to abrupt changes in the market conditions. It is also clear that when the observation period gets longer, the statistical parameters follow smoother curve. Furthermore the extreme values of average returns and standard deviations turnout to be have lower values.

Sliding windows of 150-day, 200-day, 250-day and 300-day periods have been studied but they are not included in this paper as they failed to give reasonable results in trading simulations.

4.2 The Investment Strategy

The investment strategy that we have chosen, aims to compare the relative lengths of the observation and trading periods. It is based on the assumptions below.

1) Observation period is past “p” days; investment period is future “q” days. At day “i,” the investor buys shares according to his/her observations for past “p” days and holds these shares for “q” days. The investor sells these at the “(i + q)th” day, regardless of the market conditions. Investor calculates the daily return by dividing the difference between selling price and buying price to buying price. And this result is divided by “q” to obtain a daily average to be compared with
other investment periods’ results.

2) The average return vector and covariance matrix are computed by observing the past \( p \) days.

3) Target return is assumed to be a multiple of the ISE-30 index at day \( i \). If return of the ISE 30 index is negative that day no purchase is made.

4) The quadratic optimization program is run with these parameters. No short-sell is allowed. In there is no feasible solution, no purchase is made.

5) There is no transaction cost. The sum of the weights of the instruments included in our portfolio is equal to 1.

4.3 Simulation Results

We have run the trading algorithm described above by choosing target returns that are \( k \) times the return of the ISE 30 index, for \( k = 2, k = 3 \) and \( k = 10 \). We had also studied higher multiples to see how the algorithm performs under extreme conditions but as \( k = 10 \) is sufficiently representative of the extreme conditions, these are not included here.

We present in Figure 5 the graphs of the daily returns computed as described above, for \( k = 2 \). Depending on the parameters we set the green which shows the ISE 30 index performance and the blue line shows the performance of our portfolio under the same conditions. Here we see that the algorithm results in trading the most of the days and also extreme gains/losses occur to our portfolio performance at 2003 and 2005, except further situations our portfolio outperformed in the same way as the ISE 30 index as expected.

![Figure 5: Investment Strategy of the Program when Target Return \( k=2 \), Observations Period \( n=50 \) days, Investment Period \( t=21 \) days](image)

Figure 6 displays the trading results for \( k = 10 \), which is an extreme target return we show how investment strategy acts if we only change the rate of target return. For \( k = 10 \) same computations are
made and one can see that the number of days which buy/sell operations are made decreased. In addition the volatility and risk of operations increased. For the situation \( k = 2 \), return values approximately range from \(-0.028\) to \(0.039\) and for the situation \( k = 10 \), return values approximately range from \(-0.11\) to \(0.13\).

Figure 6: Investment Strategy of the Program when Target Return \( k=10 \), Observations Period \( n=50\) days, Investment Period \( t=21\) days

We have used observation and investment periods over a wide range but we just present the results that are meaningful in Table 1 below.

Table 1: Average Reference and Real Return Rates Depending on Sliding Window Lengths

| Observation Period(Days) | 21 | 42 | 63 | 84 | 105 |
|-------------------------|----|----|----|----|-----|
| 50                      | 6.11 | 6.64 | 5.67 | 8.27 | Reference  |
| 100                     | 5.63 | 4.35 | 6.11 | 7.58 | Real |
| 150                     | 5.71 | 3.12 | 6.42 | 7.21 | Reference  |
| Reference               | Real | Reference | Real | Reference | Real |
| 50                      | 6.11 | 4.68 | 5.67 | 6.33 | Reference  |
| 100                     | 5.63 | 2.27 | 6.11 | 4.9 | Real |
| 150                     | 5.71 | 1.38 | 6.42 | 3.25 | Reference  |
| Reference               | Real | Reference | Real | Reference | Real |
| 50                      | 6.11 | 1.38 | 5.67 | 7.39 | Reference  |
| 100                     | 5.63 | 0.65 | 6.11 | 3.25 | Real |
| 150                     | 5.71 | 0.38 | 6.42 | 2.52 | Reference  |
| Reference               | Real | Reference | Real | Reference | Real |

The return rates given in this table are obtained by observing past \( n = 50, 100 \) or 150 days from a sliding window; the investment is then made to assets for investment periods of \( t = 21, 42, 63, 84 \) or 105 days and this operation is repeated for all applicable days between 2001-2009.

One can see that the best result corresponds to the case where the investor aims \( k = 2 \) times of
reference return, the observation period is 50-days and trading period is 42-days. Even if exactly k-
times of the targeted returns cannot be achieved, the real returns are very close to reference returns or
sometimes higher at lower targets. For higher returns the real returns usually decrease in all cases.

5 Conclusion

In this paper we made a simulation for the performance of a portfolio under various choices of target
returns and investment periods, with the emphasis on the lengths of observation periods for the
determination of statistical parameters. Within the framework of our trading strategy, we have tested
extreme cases by aiming target returns up to 100 times the market return and we have seen that target
returns should be a moderate multiple (2-3 times) of the market return in order to achieve the aimed
returns.

For each of these cases, we increased the observation periods from 21 days to 105 days, by 21-day
steps, and we increased the investment periods 10 days to 300 days, by 10-day steps, but we presented
here only the ones that lead to most reasonable results.

We finally note that our investment strategy doesn’t allow short selling and this has to be ensured
in the simulations.
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