Penalized Likelihood Estimation in High-Dimensional Time Series Models and Its Application

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This paper presents a general theoretical framework of penalized quasi-maximum likelihood (PQML) estimation in stationary multiple time series models when the number of parameters possibly diverges. We show the oracle property of the PQML estimator under high-level, but tractable, assumptions, comprising the first half of the paper. Utilizing these results, we propose in the latter half of the paper a method of sparse estimation in high-dimensional vector autoregressive (VAR) models. Finally, the usability of the sparse high-dimensional VAR model is confirmed with a simulation study and an empirical analysis on a yield curve forecast.

Keywords: Penalized likelihood, diverging parameters, stationary process, vector autoregression, yield curve forecasting.

JEL classification: C13, C32. C55, C58

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1 Introduction

Many statistical models have been developed to capture the relationships between variables within a multiple time series, as well as their individual marginal behaviors. These types of models help researchers identify the probabilistic structure of the interdependent relationship between variables along the axis of time. Typical examples are vector autoregressive (VAR) models and multivariate generalized autoregressive conditional heteroskedasticity (MGARCH) models. Researchers may frequently be required to estimate such models under a high-dimensional setting. However, it is quite difficult to accurately estimate all the coefficients at the same time without some parameter restrictions, even when the dimension is not so high.

Attempts to address the challenges presented by high dimensionality have been addressed in many works. One solution is to utilize a factor structure of multiple time series. The most famous result may be the approximate factor models that stem from classical factor analysis; see the review by Bai and Ng (2008) and references therein. Another example that uses a factor structure is a variant of the dynamic Nelson–Siegel model, which attempts to parsimoniously model a term structure of interest rates; see the comprehensive book by Diebold and Rudebusch (2013). These factor-based approaches have gained great success in high-dimensional time series econometrics, specifically in forecasting. However, there are limits to what can be done by the factor models alone, and an alternative method, such as the direct use of a (large dimensional) VAR model, undoubtedly needs to be established. Accordingly, the present paper provides development in that direction. A penalized likelihood method is proposed for dimension reduction that is applicable to a variety of high-dimensional multivariate time series models.

Looking at the history of likelihood-based approaches, methods of model selection may have originally been informed by the use of information criteria, like the AIC and BIC. They have become very popular due to their tractability, but these methods are limited when
dealing with high-dimensional models since they demand an exhaustive search over all submodels. This difficulty led to a general penalized likelihood method that brought about simultaneous estimation and model selection. One of the most popular methods may be the $L_1$-penalization investigated by Tibshirani (1996). His approach was originally introduced in the context of penalized least squares and is well-known as the Lasso. For more information, see Fan et al. (2011).

After that, desirability of penalties increased, and their applicability to higher-dimensional models have been pursued. Fan and Li (2001) explored a statistically desirable concept called the oracle property, in which the estimator is asymptotically equivalent to the maximum likelihood estimator of a correct submodel. They also advocated the smoothly clipped absolute deviation (SCAD) penalty that gives an estimator with the oracle property, but they pointed out that the $L_1$-penalty might not produce the oracle property in general. Their results have been extended by many authors. In terms of the formulation of penalties, Lv and Fan (2009) proposed a wide class of folded-concave penalties, including the $L_1$-penalty and SCAD. Fan and Peng (2004) improved the SCAD-penalized likelihood to admit a dimensionality growing with the relatively slow rate $o(n^{1/3})$ or $o(n^{1/5})$, where $n$ is the sample size. Likewise, a faster rate has been examined by several authors as well. Kim et al. (2008) considered the case where the dimension is possibly larger than the sample size, growing at a polynomial rate, and Fan and Lv (2011) studied ultrahigh-dimensionality that grows non-polynomially fast. Such penalized estimation methods are definitely useful when managing high-dimensionality in statistics and econometrics studies. Nevertheless, all the results have been derived under i.i.d. assumptions and have not been extended to a general time series framework, to the best of our knowledge.

Responding to this context, the first half of this paper proposes a general framework of penalized quasi-maximum likelihood (PQML) estimation that can be applicable to many kinds of stationary multivariate time series models. As in the preceding research in the i.i.d. context, the number of parameters, $p$, is assumed to diverge with a rate proportional
to a polynomial of sample size $T$. Thus, $p$ is possibly larger than $T$, as long as the parameter vector is sparse. PQML finds an estimator that, by definition, maximizes a quasi-log-likelihood function with a penalty characterized by the broad class of folded-concave penalties introduced by Lv and Fan (2009). Hence, we can include many types of penalties other than the Lasso-type one. As a main theoretical contribution, we show the existence of this estimator and its oracle property under general high-level, but tractable, assumptions. The derivation of the results is based on Fan and Lv (2011), but the extension is not trivial since our setting is not limited to i.i.d. cases. Furthermore, contrary to the works introduced above, this paper employs a quasi-likelihood, which makes possible a wide range of applications—particularly in the analysis of VAR models.

Moving beyond general time series models to more specific ones, there are several articles that apply the penalized estimation method to obtain sparse estimates. Wang et al. (2007) adapted the Lasso to shrink the coefficients in a univariate regression model with autoregressive errors while assuming a fixed lag order. This work was extended by Nardi and Rinaldo (2011) to let the maximal lag grow. Recently, Medeiros and Mendes (2012) have further extended the result to permit more general processes, but still the method only applies to univariate models. Estimation of cointegrating regressions with a Lasso-type penalty has been studied by Mendes (2011)—considering a scalar-valued model with diverging parameters—and Liao and Phillips (2012)—investigating cointegrating rank and lag order selection of a vector error correction model. Again, these studies are also restricted to fixed dimensional models. Lasso-type estimation of VAR models has been studied by several authors, including Song and Bickel (2011), Audrino and Camponovo (2013), Basu and Michailidis (2013), and Kock and Callot (2014). Kock and Callot (2014), in particular, were theoretically successful in deriving oracle inequalities of estimated VAR coefficients with Lasso-type penalties. The latter half of the present paper relates to these works.

After establishing our framework, we demonstrate sparse estimation in a high-dimensional VAR model as a theoretical example of the results obtained in the first half of the paper. As
described above, there are several studies of sparse estimation in VAR models. Nevertheless, our result is not just a replication of those papers—it is an entirely new addition to the literature. First, contrary to a number of papers, our analysis allows the dimension and lag order to diverge, similar to Basu and Michailidis (2013) and Kock and Callot (2014). Thus, high-dimensional VAR models can be handled within our framework. Second, the errors are assumed to have only finite fourth moments, as opposed to the two papers just mentioned, in which the authors supposed Gaussian errors. This assumption is essential to their contributions and proofs, which emphasizes the difference of this paper. Third, our result includes many kinds of penalties, such as both SCAD and Lasso, while all the above papers employed only Lasso-type penalties. In addition, it is noteworthy that we can include analyses of both ordinary least squares (OLS) and generalized LS (GLS) estimations, thanks to the utilization of quasi-likelihood functions. In the well-known example of Zellner (1962), the OLS estimator is shown to be identical to the GLS estimator in an unrestricted regression in VAR models. However, when it comes to penalized estimation, a difference arises between these two methods in VAR estimation under parameter constraints. Note, though, that our developing theory is not limited to VAR estimation. Another possibility, such as an application to MGARCH, is briefly discussed in the end of this paper as well.

After verifying the oracle property of the PQML estimator in a high-dimensional VAR model, we proceed to a simulation study and give an empirical example. We then observe from a simulation study that the PQML estimation performs quite well compared to an unrestricted QML. At the same time, non-concave penalties, such as the SCAD, are observed to perform better than the $L_1$-penalty. Finally, the validity is further confirmed in terms of forecasting accuracy of the U.S. yield curve. The performance is measured through a comparison with the dynamic Nelson–Siegel model proposed by Diebold and Li (2006), which is representative of what exploits factor structures and is known to be superior among many other forecasting strategies, including simple VAR(1) and univariate AR predictions; see Diebold and Li (2006).
The remainder of the paper is organized as follows. In Section 2, we first define the model and PQML estimator, and then we introduce some assumptions required to derive asymptotic properties. Section 3 gives the main theoretical results; the weak oracle and oracle properties of the estimator are established. The results are applied to estimation of VAR models in Section 4. The finite sample performance of the VAR model estimation is investigated in Section 5. We also observe the usefulness of the method in real data analysis compared to another method in Section 6. Section 7 provides a summary of the paper and a discussion of further studies, including sparse MGARCH estimation. All the proofs and some lemmas are collected in the Appendices.

We conclude this section with the introduction of some notation. For some vector \( x \) and matrix \( A \), these \( i \)th and \( ij \)th elements are written as \( x_i \) and \( A_{i,j} \), respectively. The \( j \)th column (\( i \)th row) vector of \( A \) is similarly denoted as \( A_{\cdot j} \) (\( A_{i,\cdot} \)). \( \lambda_{\min}(A) \) and \( \lambda_{\max}(A) \) mean the minimum and maximum eigenvalues of \( A \), respectively. \( \|x\| \) is the Euclidean norm. \( \|x\|_\infty \) is the largest element of \( x \) in modulus. \( \|A\| \) represents the spectral norm, i.e., a square root of \( \lambda_{\max}(A^\top A) \). \( \|A\|_\infty \) refers to the operator norm induced by \( \|x\|_\infty \), or the largest absolute row sum.

2 Model and assumptions

In this section, we introduce the model and assumptions required for deriving the theoretical results. Our model extends penalized likelihood to general time series models and allows high dimensionality, as well as the opportunity to eliminate an i.i.d. assumption on the data. In Section 2.1, the model and PQML estimator are defined. Regarding the objective function, the penalty functions and log-likelihood function are considered in Sections 2.2 and 2.3, respectively.
2.1 Model setup

Consider a real vector, stationary, and ergodic time series \( \{y_t\} \) for \( t = 1, \ldots, T \). This process is assumed \( \mathcal{F}_t \)-measurable with a natural filtration \( \mathcal{F}_t := \sigma(Y_t) \), where \( Y_t = (y_t, y_{t-1}, \ldots, y_{t-r+1}) \) for some fixed \( r > 0 \) or \( Y_t = (y_t, y_{t-1}, \ldots) \). The former case corresponds to \( \text{VAR}(r) \) models and the latter to \( \text{MGARCH} \) models, for example. Let \( y_t \) have a continuous conditional density \( g(y_t | Y_{t-1}) \). However, \( g \) is usually unknown, so we must postulate a parametric family of measurable density functions, \( \{f(y_t | Y_{t-1} : \theta) : \theta \in \Theta \subseteq \mathbb{R}^p\} \), that may or may not include the true density \( g \). The parameter vector \( \theta \) is \( p \)-dimensional, and its “true value” \( \theta^0 \in \text{int}(\Theta) \) is naturally defined as the unique minimizer of the Kullback–Leibler divergence of \( g \) relative to \( f \). To get the estimator, we first define the quasi-log-likelihood function as

\[
L_T(\theta) = T^{-1} \sum_{t=1}^{T} \ell_t(\theta), \quad \text{where} \quad \ell_t(\theta) = \log f(y_t | Y_{t-1} : \theta).
\]

Regarding the parameter vector \( \theta^0 \), we consider the case where the dimension \( p \) diverges at the polynomial rate \( p = O(T^{\delta}) \) for some \( \delta \geq 0 \), whereas the number of included nonzero elements, \( q (\leq p) \), diverges at the slower polynomial rate \( q = O(T^{\delta_0}) \) for some \( (\delta \geq) \delta_0 \geq 0 \). Therefore, the vector \( \theta^0 \) is understood to be filled with many \( (p - q) \) zeros. It may be possible to allow exponentially diverging parameters, as in Fan and Lv (2011), by using a concentration inequality with exponentially decreasing bounds (such as the Azuma–Hoeffding inequality) even if the process is not i.i.d. Nevertheless, we do not take such an approach, because it is impossible to obtain such a sharp inequality without imposing additional assumptions in the process. Intuitively, a fast diverging rate is achieved at the cost of restricting the classes of processes.

In order to make the theory clear, the parameter vector \( \theta^0 = (\theta^0_1, \ldots, \theta^0_p)^\top \) should be decomposed into two subvectors. Let \( \mathcal{M}_0 \) denote the set of indices \( \{j \in \{1, \ldots, p\} : \theta^0_j \neq 0\} \) and \( \theta^0_{\mathcal{M}_0} \) be the \( q \)-dimensional vector composed of the nonzero elements \( \{\theta^0_j : j \in \mathcal{M}_0\} \). Similarly, we define \( \theta^0_{\mathcal{M}_0^c} \) as the \( (p - q) \)-dimensional zero vector. Without loss of generality, the vector is stacked like \( \theta^0 = (\theta^0_{\mathcal{M}_0}, \theta^0_{\mathcal{M}_0^c})^\top = (\theta^0_{\mathcal{M}_0}, 0^\top)^\top \).
In this paper, since the dimension $p$ diverges as $T$ tends to infinity, we consider the penalized quasi-maximum likelihood (PQML) estimation of $\theta^0$ in order to reduce irrelevant variables. Let $P_T(\theta) = \sum_{j=1}^{p} p_\lambda(|\theta_j|)$ be the penalty term that brings sparse estimation. The objective function of the PQML estimation is then defined as

$$Q_T(\theta) = L_T(\theta) - P_T(\theta).$$

(1)

The penalty function $p_\lambda$ is such as the $L_p$-penalty, with $0 < p < 1$, by Frank and Friedman (1993), the $L_1$-penalty (Lasso) by Tibshirani (1996), the SCAD penalty by Fan and Li (2001), the hierarchical penalty by Bickel et al. (2008), or the minimax concave penalty (MCP) by Zhang (2010). The tuning parameter $\lambda(= \lambda_T)$ determines the size of the model. We let $\lambda = O(T^{-\alpha})$, where a positive number $\alpha$ is specified later. The PQML estimator, $\hat{\theta}$, is defined by $Q_T(\hat{\theta}) = \max_{\theta \in \Theta} Q_T(\theta)$.

### 2.2 Penalty function

In this subsection, we start by introducing some notation and discussing the properties and assumptions of some types of the penalty functions in (1). Define half of the minimum signal as $d(= d_T) = \min_{j \in M_0} |\theta^0_j|/2$, and let $N_0 = \{\theta, \theta_0 \in \mathbb{R}^q : \|\theta - \theta_0\|_\infty \leq d\}$. Following Lv and Fan (2009), we let $\rho(x; \lambda) = p_\lambda(x)/\lambda$ and define the local concavity of $\rho$ at $x \in \mathbb{R}^r$ with $\|x\|_0 = r$ as

$$\kappa(\rho; x) = \lim_{\varepsilon \to 0^+} \max_{1 \leq j \leq r} \sup_{y_1, y_2 \in \Lambda_{\varepsilon, j}} \left\{ \frac{-\rho'(y_2; \lambda) - \rho'(y_1; \lambda)}{y_2 - y_1} : y_1 < y_2 \right\},$$

where $\Lambda_{\varepsilon, j} := (|x_j| - \varepsilon, |x_j| + \varepsilon)$. If the second derivative of $\rho(\cdot; \lambda)$ is continuous, we may easily see that $\kappa(\rho; x) = \max_{1 \leq j \leq r} -\rho''(|x_j|; \lambda)$. We sometimes drop $\lambda$ and write them simply as $\rho(x)$ and $\kappa(x)$. The next assumption, which was first introduced by Lv and Fan (2009) and was also used in Fan and Lv (2011), characterizes a broad class of penalties.

**Assumption 1** The penalty function $\rho(\cdot; \lambda)$ is increasing and concave on $[0, \infty)$, and it has a continuous derivative $\rho'(\cdot; \lambda)$ with $\rho'(0+; \lambda) > 0$. In addition, $\rho'(t; \cdot)$ is increasing on
(0,∞), and ρ′(0+;λ) is independent of λ.

Assumption 2 \( \lambda \sup_{\theta \in \mathbb{R}_0} \kappa(\theta) = o(1) \).

Assumption implies \( \kappa(x; \lambda) \geq 0 \) by concavity of \( \rho \) and is key to Lemma in Appendix A.1. Assumption 2 is required to verify a sufficient condition of Lemma which furthermore leads to the main theorem.

Fan and Li (2001) advocated penalty functions that endow estimators with three desirable properties: unbiasedness, sparsity, and continuity. It is known that the SCAD satisfies all of them simultaneously, but the \( L_1 \) and MCP fail to exhibit unbiasedness and continuity, respectively. All three penalties, however, satisfy Assumption 1.

We further give two sets of conditions on the penalty function \( p_\lambda \) to restrict the general class of penalties given by Assumption 1. The first one is used to obtain the so-called weak oracle property of the PQML estimator \( \hat{\theta} \) in the next section. This idea was first introduced by Lv and Fan (2009) and also obtained in Fan and Lv (2011). The second set of conditions is required to achieve the oracle property of \( \hat{\theta} \), a stronger result. (The role of positive constants \( m_1 \) and \( \beta \) below are understood in the context of Assumption 5 in the next subsection.)

Assumption 3 The penalty function \( p_\lambda \) satisfies the following properties:

(a) \( \lambda = O(T^{-\alpha}) \) for some \( \alpha \in (0, 1/2 - \delta_0/m_1 - \beta) \);

(b) \( d \geq T^{-\gamma} \log T \) for some \( \gamma \in (0, 1/2] \) and large \( T \);

(c) \( \lambda \rho'(d) = o(q^{-1/2}T^{-\gamma} \log T) \).

Assumption 4 The penalty function \( p_\lambda \) satisfies the following properties:

(a) \( d/\lambda \to \infty \) and \( \lambda = O(T^{-\alpha}) \) for some \( \alpha \in (0, 1/2 - \delta_0/2 - \beta) \);

(b) \( \lambda \rho'(d) = O(T^{-1/2}) \).
(b2) $\lambda \rho'(d) = o((qT)^{-1/2})$.

Assumption 4 is stronger than Assumption 3, enough to exclude the $L_1$-penalty, which is on a boundary of Assumption 1. For a SCAD or MCP, $p'(\lambda)(d)$ becomes exactly zero for a sufficiently large $T$ under Assumption 4(a), implying that 4(b2) and 4(b1) hold automatically. For the $L_1$-penalty, however, $\rho'(d) = 1$ holds identically, which indicates that this penalty fails to simultaneously satisfy 4(a) and 4(b1). Meanwhile, the $L_1$-penalty can be included in the class given by Assumption 3; the condition $\alpha \geq \delta_0 + \gamma$ is necessary for the $L_1$-penalty to satisfy Assumption 3(a) and (c) simultaneously. These features may be observed in the following example.

**Example 1**  
(a) The $L_1$-penalty is given by $p_\lambda(x) = \lambda |x|$, and we then obtain $p'_\lambda(|x|) = \lambda$ and $p''_\lambda(|x|) = 0$.

(b) The SCAD penalty is characterized by its derivative

$$p'_\lambda(x) = \lambda \left\{ 1(x \leq \lambda) + \frac{(a\lambda - x)}{(a - 1)\lambda} 1(x > \lambda) \right\}$$

for some $a > 2$. Then we have $p''_\lambda(|x|) = -(a - 1)^{-1}1\{|x| \in (\lambda,a\lambda)\}$.

(c) The MCP is defined through its derivative $p'_\lambda(x) = a^{-1}(a\lambda - x)_+$ for some $a \geq 1$. Thus, we have $p''_\lambda(|x|) = -a^{-1}1\{|x| < a\lambda\}$.

### 2.3 Likelihood function

We introduce some notation and assumptions on the likelihood function $L_T$. Hereafter, the likelihood $L_T$ is supposed to be twice continuously differentiable in a neighborhood $\Theta^0 \subset \Theta$ of $\theta^0$. Define the averages of the score vector and Hessian matrix as

$$S_T(\theta) = T^{-1} \sum_{t=1}^T s_t(\theta) \quad \text{and} \quad H_T(\theta) = T^{-1} \sum_{t=1}^T h_t(\theta),$$

where $s_t(\theta) = \partial \ell_t(\theta)/\partial \theta$ and $h_t(\theta) = \partial^2 \ell_t(\theta)/\partial \theta \partial \theta^\top$. For the development of some results later on, we need to define the averages of the “score subvector” $S_{M0T}(\theta)$ and the
“Hessian submatrix” $H_{\mathcal{M}T}(\theta)$, by way of setting $s_{\mathcal{M}0}(\theta) = \partial \ell_t(\theta) / \partial \theta_{\mathcal{M}0}$ (i.e., the sub-
vector of $s_t(\theta)$ that is composed of its first $q$ elements) and $h_{\mathcal{M}0}(\theta) = \partial^2 \ell_t(\theta) / \partial \theta_{\mathcal{M}0} \partial \theta_{\mathcal{M}0}^\top$ (i.e., the upper-left $q \times q$ submatrix of $h_t(\theta)$). We also define $S_{\mathcal{M}0}T(\theta)$ analogously. In
what follows, we occasionally suppress the argument $(\theta)$ when it is evaluated at $\theta^0 = (\theta_0^0, 0)$; for example, we will simply denote $S_T(\theta^0) = S_T(\theta_0^0, 0)$ as $S_T^0$. Define, moreover,

$I_{\mathcal{M}0T}(\theta) := E[T^{-1} \sum_{t=1}^T s_{\mathcal{M}0}(\theta) s_{\mathcal{M}0}^\top(\theta)], I_{\mathcal{M}0} := \lim_{T \to \infty} I_{\mathcal{M}0T}, J_{\mathcal{M}0T}(\theta) := -E[H_{\mathcal{M}0T}(\theta)],$
and $J_{\mathcal{M}0} := \lim_{T \to \infty} J_{\mathcal{M}0T}$. To derive the theoretical result, the score and Hessian require
some assumptions. Here, we introduce two sets of conditions. They are used to achieve the weak oracle and oracle properties, respectively, in the next section. The first one is necessary
for the weak oracle property.

**Assumption 5** The (quasi-)log-likelihood function $L_T$ satisfies the following conditions:

(a) For all $i \in \mathcal{M}_0$ and $T$, $E[T^{1/2} S_T^0]^{|m_1|} < \infty$ holds for some $m_1 > 0$.

(b) For all $i \in \mathcal{M}_0^c$ and $T$, $E[T^{1/2} S_T^0]^{|m_2|} < \infty$ holds for some $m_2 > 0$.

(c) For all $T$, $-H_{\mathcal{M}0}^0$ is a.s. positive definite, and $\lambda_{\min}(-H_{\mathcal{M}0}^0) =: C_1 = O_P(1)$.

(d) There is a neighborhood $\Theta_{\mathcal{M}0}^0 \subset \Theta$ of $\theta_{\mathcal{M}0}^0$ such that

$$
\|H_{\mathcal{M}0T}(\theta_1, 0) - H_{\mathcal{M}0T}(\theta_2, 0)\| \leq K_T \|\theta_1 - \theta_2\|_\infty
$$

holds for all $\theta_1, \theta_2 \in \Theta_{\mathcal{M}0}^0$ and for some $K_T = O_P(1)$.

(e) There is a neighborhood $\Theta_{\mathcal{M}0}^0 \subset \Theta$ of $\theta_{\mathcal{M}0}^0$ such that

$$
\sup_{\theta_1, \theta_2 \in \Theta_{\mathcal{M}0}^0} \|[(\partial / \partial \theta_{\mathcal{M}0}^\top) S_{\mathcal{M}0T}(\theta_1, 0)] [H_{\mathcal{M}0T}(\theta_2, 0)]^{-1}\| \leq \frac{c \rho'(0^+)}{\rho'(d)} \wedge O_P(T^{\beta})
$$

for some $c = O_P(1)$ that takes its value in $(0, 1)$ a.s. and some constant $\beta \in [0, 1/2)$.

Assumption 5(a) is needed for just a technical reason, but Assumption 5(b) is meaningful. A
higher moment condition in (b) allows estimation of a larger dimensional parameter vector.
If the dimension $p$ is large enough to exceed the sample size $T$, the number $m_2$ is required to be strictly greater than two. When a large $m_i$ ($i = 1, 2$) is concerned, it seems cumbersome to verify these conditions. However, Lemma 2 in Appendix A.1 is useful, as long as the score $S_{\mathcal{U}_0 T}$ obeys a martingale process. Assumption 5(c) means that $\lambda_{\min}(-H_{\mathcal{U}_0 T}^0)$ is a.s. positive and bounded away from zero, and not diverges. If the model is linear in parameter, the Hessian does not depend on the parameter, so that Assumption 5(d) automatically holds.

A similar condition is found in Wooldridge (1994, Theorem 8.1). Assumption 5(e) is similar to condition (16) of Fan and Lv (2011). The left-hand side is regarded as a regression coefficient of each irrelevant variable on important variables in the case of linear models. The right-hand side can diverge when a folded-concave penalty is concerned, but the upper bound becomes more restrictive when the $L_1$-penalty is used since the ratio $\rho'(0+)/\rho'(d)$ always becomes one.

To derive the oracle property, we need a slightly different set of conditions. For any matrix $A$ and vector $x$ such that $Ax$ is well-defined, let $\|A\|_2,\infty := \max_{\|x\|=1} \|Ax\|_\infty$.

**Assumption 6** The (quasi-)log-likelihood function $L_T$ satisfies Assumption 5(b), (c), and the following conditions:

(d) There is a neighborhood $\Theta^0_{\mathcal{U}_0} \subset \Theta$ of $\theta^0_{\mathcal{U}_0}$ such that

$$\|H_{\mathcal{U}_0 T}(\theta_1, 0) - H_{\mathcal{U}_0 T}(\theta_2, 0)\| \leq K_T \|\theta_1 - \theta_2\|$$

holds for all $\theta_1, \theta_2 \in \Theta^0_{\mathcal{U}_0}$ and for some $K_T = O_p(1)$.

(e) There is a neighborhood $\Theta^0_{\mathcal{U}_0} \subset \Theta$ of $\theta^0_{\mathcal{U}_0}$ such that, for some $\beta \in [0, \infty)$,

$$\sup_{\theta_1 \in \Theta^0_{\mathcal{U}_0}} \|(\partial / \partial \theta_{\mathcal{U}_0}^\top) S_{\mathcal{U}_0 T}(\theta_1, 0)\|_2,\infty = O_p(T^\beta).$$

The role of Assumption 6(d) is the same as that of Assumption 5(d). Condition (e) restricts the asymptotic behavior of the lower-left $(p - q) \times q$ submatrix of $H_T(\theta)$ and is similar to condition (27) of Fan and Lv (2011).
The next assumption is required to obtain asymptotic normality of the estimator with the oracle property, in addition to Assumption 6.

**Assumption 7** The (quasi-)log-likelihood function $L_T$ satisfies the following conditions:

(a) For all $T$, $I_0^T$ is positive definite and $\lambda_{\text{max}}(I_0^T) \leq C_2^2$ for some constant $C_2 < \infty$.

(b) For any vector $a \in \mathbb{R}^q$ such that $\|a\| = 1$, the score vector admits a central limit theorem $T^{1/2} a^T I_0^{-1/2} S_0^T S_0 a \to_d N(0, 1)$.

In many cases, a score vector obeys the martingale process, so that a central limit theorem for a strictly stationary and ergodic martingale difference sequence is available for Assumption 7(b) under general conditions.

### 3 Theoretical results

In this section, we establish the oracle property. This property states that the PQML estimator is asymptotically equivalent to the QML estimator that is obtained with the correct zero restrictions. The first theorem is the weaker version of the oracle property. A similar result was first obtained by Lv and Fan (2009) and also Fan and Lv (2011) under i.i.d. conditions.

**Theorem 1 (Weak oracle property)** Suppose that Assumptions 1, 2, 3, and 5 hold. If $p = O(T^\delta)$ and $q = O(T^{\delta_0})$ satisfy the conditions

$$\delta \in [0, m_2(1/2 - \alpha)), \quad \delta_0 \in [0, (1/2 - \gamma)/(1/2 + 1/m_1)], \quad \delta_0 \in [0, \gamma),$$

then there exists a local maximizer $\hat{\theta} = (\hat{\theta}_0^T, \hat{\theta}_c^T)_c$ of $Q_T(\theta)$ such that the following properties are satisfied:

(a) (Sparsity) $\hat{\theta}_c = 0$ with probability approaching one;

(b) (Rate of convergence) $\|\hat{\theta}_0 - \theta_0^0\|_\infty = O_p(T^{-\gamma \log T})$. 

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Theorem 1 is somewhat different from the weak oracle property developed by Fan and Lv (2011) in that ours is based on the asymptotic result. Also, Theorem 1 gives an opportunity for the PQML estimator to achieve model selection consistency even when the $L_1$-penalty is employed. To see this, let $Q_{L_1T}(\theta)$ denote the objective function with $p_\lambda$ given by the $L_1$-penalty.

**Corollary 1 (L$_1$-penalized QML estimator)** Suppose all the assumptions in Theorem 7 hold. Then there exists a local maximizer $\hat{\theta} = (\hat{\theta}_{M_0}^\top, \hat{\theta}_{M_0^c}^\top)^\top$ of $Q_{L_1T}(\theta)$ such that the properties of Theorem 7(a) and (b) hold.

If the (quasi-)likelihood function is globally concave, then the objective function $Q_{L_1n}$ is as well because the $L_1$-penalty is globally concave. Hence, the local maximizer is extended to the unique global one in this case.

If we employ a SCAD-type penalty, a stronger and more desirable result can be obtained. This result is called the oracle property, as studied by Fan and Li (2001).

**Theorem 2 (Oracle property)** Suppose that Assumptions 1, 2, 4(a)(b1), and 6 hold. If $\delta < m_2(1/2 - \alpha)$ is true, then there exists a local maximizer $\hat{\theta} = (\hat{\theta}_{M_0}^\top, \hat{\theta}_{M_0^c}^\top)^\top$ of $Q_T(\theta)$ such that the following properties are satisfied:

(a) (Sparsity) $\hat{\theta}_{M_0^c} = 0$ with probability approaching one;

(b) (Rate of convergence) $\|\hat{\theta}_{M_0} - \theta_{M_0}^0\| = O_p((q/T)^{1/2})$.

In addition, suppose Assumption 7 holds. If Assumption 4(b1) is strengthened with 4(b2) and $\delta_0 < 1/2$, then, for any vector $a \in \mathbb{R}^q$ that satisfies $\|a\| = 1$, we have:

(c) (Asymptotic normality) $T^{1/2}a^\top \hat{\theta}_{M_0}^0 - H_{M_0}^{-1/2}H_{M_0} \frac{1}{T} T_{M_0^c} (\hat{\theta}_{M_0} - \theta_{M_0}^0) \rightarrow_d N(0, 1)$.

This property means that the model selection is consistent in the sense that $\hat{\theta}_{M_0^c} = 0$ with probability approaching one, and the estimator has the same asymptotic efficiency as the (infeasible) MLE obtained with advance knowledge of the true submodel. Thanks to the
property, we can estimate high-dimensional models without irksome tests for zero restrictions on the parameters.

As is described in Section 2.2, a SCAD-type penalty automatically satisfies not only Assumption 4(b1) but also Assumption 4(b2), as long as we choose an adequate $\lambda$ that satisfies Assumption 4(a) with sufficiently large $T$. Again, the $L_1$-penalty fails to satisfy Assumption 4.

When a fixed $q$ is supposed, result (c) reduces to

$$T^{1/2}(\hat{\theta}_{M_0} - \theta_{M_0}^0) \overset{d}{\to} N\left(0, (J_{M_0}^0)^{-1}I_{M_0}^0(J_{M_0}^0)^{-1}\right). \tag{2}$$

This asymptotic covariance matrix of the estimator can be consistently estimated by a standard procedure, as in Wooldridge (1994, Sec. 4.5). Specifically, it is given by $\hat{J}^{-1}_T\hat{I}_T\hat{J}^{-1}_T$, say, where $\hat{J}_T$ is an averaged Hessian matrix $H_T(\theta)$ evaluated by $\hat{\theta}$, and $\hat{I}_T$ is a conventional long-run variance estimator of $s_0^T$ made of a score vector $s_t(\theta)$ evaluated by $\hat{\theta}$.

If the conditional density $g(y_t|Y_t)$ depends on the infinitely past observations, or $Y_t = (y_t, y_{t-1}, \ldots)$, it is necessary to replace them with some fixed initial values, say $\tilde{Y}_t$, to perform the optimization. We denote $\tilde{\ell}_t$ as the $\ell_t$ with density conditional on $\tilde{Y}_t$. The maximizer of the objective function based on $\tilde{\ell}_t$, instead of $\ell_t$, is then asymptotically equivalent to $\hat{\theta}$ provided that

$$\lim_{T \to \infty} \sup_{\theta \in \Theta} \left| \ell_t(\theta) - \tilde{\ell}_t(\theta) \right| = 0,$$

in probability. Such a refinement for infinite initial values is typically made in estimation of (multivariate) GARCH models; see Section 4.2 of Comte and Lieberman (2003), for instance. (See also Hafner and Preminger, 2009 and Ling and McAleer, 2010.)

## 4 Application to VAR

So far, we have studied a general method of PQML estimation for high-dimensional time series. In this section, we see how the theory can be applied to estimation of a large di-
dimensional VAR(r) model. There are several works on estimation in high-dimensional VAR models, including recent works by Basu and Michailidis (2013) and Kock and Callot (2014). However, we will see that our result here is not just a replication of those studies but is entirely novel to the literature. First, our result requires only a finite fourth moment for the error term, while they supposed Gaussian errors. This assumption is essential to their results as it admits the dimension $p$ to grow non-polynomially fast; but Gaussian VAR models are sometimes too restrictive to capture various economic behavior. On the contrary, we will keep the generality of VAR models, including non-Gaussian ones; nevertheless, dimension $p$ grows at a polynomial rate. Second, our result can employ many non-concave penalties other than the Lasso, which broadens the opportunities for empirical analysis. To the best of our knowledge, all the research on high-dimensional VAR estimation is based on Lasso-type procedures. Third, our strategy includes both OLS and GLS estimations, since quasi-likelihood is permitted. This will briefly be considered at the end of this section.

4.1 VAR model and assumptions

Consider a centered vector process $y_t$ characterized by the following $k$-dimensional VAR(r) model:

$$y_t = \Phi^0 \top x_t + \epsilon_t, \quad t = 1, \ldots, T,$$

where $\Phi^0 \top = (\Phi_1^0, \ldots, \Phi_r^0)$ is the parameter matrix, $x_t = (y_{t-1}^\top, \ldots, y_{t-r}^\top)^\top$, and $\epsilon_t$ is an error term with mean zero and finite positive definite covariance matrix $\Sigma_\epsilon$. A constant vector may also be included in model (3), but we omit it to keep the discussion clear. The autoregressive order $r$ and the dimension $k$ are allowed to increase as $T$ tends to infinity. More precisely, if we let $\theta^0 = \text{vec}(\Phi^0 \top) \in \mathbb{R}^p$ with $p := k^2 r$, the dimension $p$ may diverge with the rate $p = O(T^{\delta})$ for some $\delta \geq 0$. The non-sparsity dimension in $\theta^0$, $q (\leq p)$, is also assumed to possibly diverge at the rate $q = O(T^{\delta_0})$ for some $\delta_0 \geq 0$. Define, furthermore, $Y_t = (y_t, x_t^\top)^\top$ and $\mathcal{F}_t = \sigma(Y_t)$.
Given \( Y_0 \), the quasi-Gaussian log-likelihood function (up to a constant term) is given by

\[
L_T(\theta) = \frac{1}{T} \sum_{t=1}^{T} \ell_t(\theta) = \sum_{t=1}^{T} \left\{ y_t - (x_t^\top \otimes I_k) \theta \right\}^\top \Sigma^{-1} \left\{ y_t - (x_t^\top \otimes I_k) \theta \right\},
\]

where \( \Sigma \) is a finite and positive definite weighting matrix and \( I_k \) is the \( k \times k \) identity matrix.

The PQML estimator is then obtained through (1) and (4) with a SCAD-type penalty that satisfies Assumptions 1 and 4. Note that the matrix \( \Sigma \) is determined by a researcher; if one defines \( \Sigma = I_k \), it leads to the OLS, and if one sets \( \Sigma = \Sigma_\varepsilon \), though it is usually unknown, it would reduce to the infeasible GLS.

In order to observe the theoretical result clearly, we introduce a permutation matrix \( P \); that is, \( P \) is a matrix with a single one in each row and column and all other elements zero, satisfying \( P^\top \theta^0 = (\theta_0^\top, 0^\top)^\top \). Since any permutation matrix \( P \) satisfies the orthogonality condition \( PP^\top = I_p \), we can put \( PP^\top \) between \( (x_t^\top \otimes I_k) \) and \( \theta \) in (4). With simple algebra, the corresponding score and Hessian evaluated at \( \theta = \theta^0 \) are then given by

\[
s_t^0 = P^\top (x_t \otimes \Sigma^{-1}) \varepsilon_t \quad \text{and} \quad h_t^0 = -P^\top (x_t x_t^\top \otimes \Sigma^{-1}) P,
\]

respectively. Therefore, we can always find \( s_t^0 \), whose elements are ordered like \( (s_{0,0}^0, s_{0,1}^0) \), by employing some \( P \). We further let \( K_{\theta_0}^\top = (I_q, O_{q,p-q}) \) and \( K_{\theta_0}^\top = (O_{p-q,q}, I_{p-q}) \), where \( O_{a,b} \) is the \( a \times b \) zero matrix, and define \( P_{\theta_0}^\top := K_{\theta_0}^\top P^\top (q \times p) \) and \( P_{\theta_0}^\top := K_{\theta_0}^\top P^\top ((p-q) \times p) \). Then there exists a \( P \) (and hence \( P_{\theta_0} \)) such that the following holds:

\[
s_{0,\theta_0}^0 = K_{\theta_0}^\top s_{0,\theta_0}^0 = P_{\theta_0}^\top (x_t \otimes \Sigma^{-1}) \varepsilon_t; \quad \text{(5)}
\]

\[
h_{0,\theta_0}^0 = K_{\theta_0}^\top h_{0,\theta_0}^0 K_{\theta_0}^\top = -P_{\theta_0}^\top (x_t x_t^\top \otimes \Sigma^{-1}) P_{\theta_0}; \quad \text{(6)}
\]

Similarly, we can define \( s_{0,\theta_0}^0 \) and \( h_{0,\theta_0}^0 \) by using \( P_{\theta_0}^\top \) instead of \( P_{\theta_0} \) in (5) and (6). Notice that this manipulation is no more than an auxiliary routine to promote understanding of the following proposition. Thus, it is not necessary to specify such a \( P \) in an empirical analysis.

To utilize the results obtained in Section 3, we constrain model (3) to satisfy the next assumption.
Assumption 8 For each $k$ and $r$, the following conditions hold:

(a) The error term $\varepsilon_t = (\varepsilon_{1t}, \ldots, \varepsilon_{kt})^\top$ is a sequence of i.i.d. random vectors having a continuous distribution with $E|\varepsilon_{i_1t}\varepsilon_{i_2t}\varepsilon_{i_3t}\varepsilon_{i_4t}| \leq c_{\varepsilon}$ for some constant $c_{\varepsilon} < \infty$ and for all $t$ and $i_1, i_2, i_3, i_4 \in \{1, \ldots, k\}$;

(b) For all $k$ and $r$, $\det(I_k - \Phi_1^{0}z - \cdots - \Phi_r^{0}z^r) \neq 0$ for $|z| \leq 1$.

(c) For each $\ell \in \{1, \ldots, k\}$, all the elements of the row vector $(\Sigma^{-1})_{\ell}$ are zero except the finite numbers of them. The number does not depend on $k$.

Assumption 8(a) is standard in stationary multivariate time series analysis to achieve asymptotic results; see Lütkepohl (2005, p. 73). This condition is also used to verify Assumption 5(a)(b) with $m_1 = m_2 = 4$. A higher moment condition is, of course, required if one wants to make Assumption 5(a)(b) hold with $m_1$ and $m_2$ larger than four. Assumption 8(b) is essential to ensure that model (3) has no unit root and is stable, in the sense of Lütkepohl (2005, p. 15). This condition together with Assumption 5(a) guarantees stationarity of the model. Assumption 8(c) restricts the asymptotic behavior towards $k$’s direction for some technical reasons.

4.2 Result

Now we have made the preparations to derive the oracle property for the PQML estimator of VAR model (3).

Proposition 1 Suppose that Assumptions [7][4] and [8] are satisfied. Then all the conditions in Assumptions [6] and [7] are true with $m_1 = m_2 = 4$ and $\beta = \delta_0/2$, and the result of Theorem 2 holds for PQML estimator $\hat{\Theta}$ of model (3) if $\delta < 4(1/2 - \alpha)$.

We treat $q$ as a finite number in Proposition 1 to construe the subsequent asymptotic result. The asymptotic distribution is then given by (2) with $I_{\mathcal{M}_0}^{0} = P_{\mathcal{M}_0}^\top (\Gamma \otimes \Sigma^{-1}) P_{\mathcal{M}_0}$ and $J_{\mathcal{M}_0}^{0} = P_{\mathcal{M}_0}^\top (\Gamma \otimes \Sigma^{-1}) P_{\mathcal{M}_0}$, where $\Gamma = E[x_t x_t^\top]$. 

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In a multiple equation model, as was made well known by Zellner (1962), the OLS and GLS estimators are identical as long as the regressors are the same in all equations. The typical example is a VAR model. However, the assumption collapses once parameter restrictions are constrained, and these estimators do not become identical; see Lütkepohl (2005, Sec. 3.2.1 and Sec. 5.2). Proposition 1 is a comparable result. If \( p > q \), the asymptotic variance of \( \hat{\theta}_0 \) is found to be

\[
(P_{\mathcal{M}_0}^\top (\Gamma \otimes \Sigma^{-1})P_{\mathcal{M}_0})^{-1}P_{\mathcal{M}_0}^\top (\Gamma \otimes \Sigma^{-1} \Sigma_{\varepsilon} \Sigma^{-1})P_{\mathcal{M}_0} (P_{\mathcal{M}_0}^\top (\Gamma \otimes \Sigma^{-1})P_{\mathcal{M}_0})^{-1}.
\]  

(7)

Hence, if \( \Sigma \) is set to \( I \) or \( \Sigma_{\varepsilon} \) in (7), we have

\[
(P_{\mathcal{M}_0}^\top (\Gamma \otimes I)P_{\mathcal{M}_0})^{-1}P_{\mathcal{M}_0}^\top (\Gamma \otimes \Sigma_{\varepsilon})P_{\mathcal{M}_0} (P_{\mathcal{M}_0}^\top (\Gamma \otimes I)P_{\mathcal{M}_0})^{-1}
\]

if \( \Sigma = I \);

\[
(P_{\mathcal{M}_0}^\top (\Gamma \otimes \Sigma_{\varepsilon}^{-1})P_{\mathcal{M}_0})^{-1}
\]

if \( \Sigma = \Sigma_{\varepsilon} \).

On the other hand, when we consider the case \( p = q \), we have \( K_{\mathcal{M}_0} = I_q \) and hence \( P_{\mathcal{M}_0} = P \), which corresponds to the case where no penalty is constrained. With simple algebra, it is easy to see that (7) reduces to \( P^\top (\Gamma^{-1} \otimes \Sigma_{\varepsilon})P \), meaning that the resulting estimators are the same, notwithstanding the choice of \( \Sigma \). This signifies that the PQML method is essentially the same as the parameter-restricted QML; a point of difference is that the parameters’ constraint to zero is automatically imposed.

5 Simulation study

In this section, we check the estimation accuracy of the PQML estimates of the VAR model analyzed in Section 4. VAR model (3) is now specified with \( k = 8 \), \( r = 2 \), and parameter
matrices given by

\[
\Phi_0^1 = \begin{pmatrix}
.7 & .1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & .4 & .1 & 0 & 0 & 0 & 0 & 0 \\
.6 & -.2 & .6 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -.2 & .4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & .3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
\Phi_0^2 = \begin{pmatrix}
-.2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & .2 & .1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -.3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

We may easily confirm that this formulation entails Assumption 8. The error term \( \epsilon_t \) is assumed to be distributed as a serially uncorrelated multivariate normal \( N(0, \Sigma_\epsilon) \), where the covariance matrix \( \Sigma_\epsilon \) is specified through decomposition \( UU^T \), with

\[
U = \begin{pmatrix}
.5 & .1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & .3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & .9 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & .2 & .4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -.2 & .3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & .3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & .3 & 0
\end{pmatrix}.
\]

Validity of the PQML estimation is measured by comparison with other methods, including the usual unrestricted and correctly zero-restricted QML estimations; the former appears to have inefficient estimates, and the latter, though infeasible, is expected to result in highly efficient estimation. All the strategies are based on the QML in the sense that \( \Sigma \) in (4) is given by \( I_8 \). That is, the unrestricted and restricted QML estimates agree with the unrestricted and restricted OLS estimates, respectively. The penalty terms in the PQML
estimations include the SCAD, MCP, and Lasso, and the tuning parameter $a$ is set to 2.5 and 20 for SCAD and 1.5 and 20 for MCP. These PQML estimates are computed using the coordinate descent algorithm provided by the \texttt{R} package \texttt{ncvreg}; see Breheny and Huang (2011) for detail. Note, however, that the package automatically includes the intercept in the model, so that the code is adjusted to meet the present no-intercept model. We must also choose a tuning parameter $\lambda$. Since we have not revealed an optimal way to do so, we use the conventional method of selection by 10-fold cross-validation, which leaves out $1/10$ of the data. If the data were i.i.d., the issues on tuning parameter selection could be addressed by several methods, elaborated in other works; see Fan and Tang (2013) and the references therein.

Estimation accuracy is checked by observing three measures: a root mean squared error (RMSE), standard deviation (STDEV) and success rate on model selection, and sign consistency (MSSC) for PQML estimates. These statistics are constructed on the basis of 100, 300, 500, and 1,000 observations, with 1,000 replications each. As the parameter vector $\text{vec}(\Phi_1^0, \Phi_2^0)$ is large (128-dimensional), it is difficult to display all the results. Hence, they are evaluated by the distance between vectors measured by the $L_2$-norm.

The simulation results are summarized in Table II. At first glance, the three penalized estimations greatly succeed in gaining efficiency in regard to RMSE and STDEV, compared to the conventional MLE. In addition, these penalized estimates possess high success rates for model selection (MSSC), except when nonzero entries are estimated with $n = 100$ and small $a$’s. In this case, some nonzeros are misconstrued as zeros. Such a defective behavior is, however, eliminated immediately as the sample size gets larger. Taking a closer look at the columns of SCAD and MCP, we notice that, for small $a$, MSSC rates tend to be better and biases are reduced in exchange for sacrificing, to some extent, the values of RMSE and STDEV; and, for large $a$, the reverse behavior is observed. Although the Lasso exhibits a good performance in terms of RMSE and STDEV, especially in the cases of moderate sample size, it seems to cause a slightly larger bias relative to the SCAD and MCP, which is a
well-known phenomenon in the literature; see Fan and Li (2001), for instance. Moreover, the MSSC rate of the Lasso is not improved even when the number of observations is increased.

6 Real data analysis

The usefulness of the proposed sparse VAR (sVAR) estimation method is clearly observed when looking at its performance in forecasting the term structures of government bond yields. We use zero-coupon government bond yields of the U.S. at a monthly frequency from January 1986 to December 2007 with 8 maturities: 3, 6, 12, 24, 36, 60, 84, and 120 months. This dataset is constructed by Wright (2011) and can be downloaded from his website. The performance is confirmed by comparing the out-of-sample forecasting accuracy of our method to that obtained from the dynamic Nelson–Siegel (DNS) model accommodated by Diebold and Li (2006), as it is a representative model utilizing a factor structure to reduce the high-dimensionality (it is briefly explained later). Both the models are estimated recursively, using data from January 1986 to the time that the $h$-month-ahead forecast is made, beginning in January 2001 and extending through December 2007. Therefore, we obtain $72 - h + 1$ forecast values for each model. The forecasting horizon, $h$, is given by 1, 3, 6, and 12 months, respectively. The accuracy of the forecast is measured by the RMSE relative to the actually observed yields.

Now we explain the two models, DNS and sVAR. The first model, DNS, has achieved dimensionality reduction via factor structure of yields and is well-known in the econometric literature for its superiority in forecasting; see, for example, the empirical examples in Diebold and Li (2006) and Kargin and Onatski (2008). It is a suitable competitor for sVAR, since it has the best performance among several models, including simple time series models like VAR(1). DNS models yield $y_{\tau t}$ of maturity $\tau$ at time $t$ as

$$y_{\tau t} = \beta_{1t} + \beta_{2t} \left( \frac{1 - e^{-\eta_{\tau} t}}{\eta_{\tau}} \right) + \beta_{3t} \left( \frac{1 - e^{-\eta_{\tau} t}}{\eta_{\tau}} - e^{-\eta_{\tau} t} \right).$$
The coefficients $\beta_1t$, $\beta_2t$, and $\beta_3t$ may be interpreted as latent dynamic factors corresponding to the level, slope, and curvature, respectively, because of the construction of their loadings. $\eta_t$ is a sequence of tuning parameters. By the construction of the model, prediction of the yields is equivalent to that of the $\beta$’s in the sense that estimated yields are obtained immediately once $\beta$’s are determined. For more information about the model and its variants, see Diebold and Rudebusch (2013). According to a recommendation of Diebold and Li (2006), $\eta_t$ is determined by maximizing the loading on the medium-term factor for each period (although Diebold and Li, 2006 fixed the value at 0.0609 for simplicity); see Diebold and Li (2006, pp. 346–347) and Guirreri (2010, p. 9). The strategy of parameter estimation is basically the same as in Diebold and Li (2006) and is implemented using the R package YieldCurve. These estimated parameters are modeled as univariate AR models, with $\beta_{it}$ regressing on 1 and $\beta_{i,t-h}$ for $i = 1, 2, 3$. We then obtain $h$-step-ahead forecasts $\hat{\beta}_{i,t+h}$ by $\beta_{it}$.

The second model, sVAR, demonstrates the difference of yields directly as a VAR model of lag order 12, with the assumption that the coefficient matrices are sparse. The model is estimated with a SCAD-penalized QML, reducing the high-dimensionality and performing model selection. The computation is done with the aid of the R package ncvreg, as in the preceding section. In this case, however, the tuning parameter $\lambda$ is simply set to $(8T)^{-0.4}/4$ because cross-validation for the present large model, which has $8^2 \times 12 = 768$ coefficient parameters of interest, is extremely time-consuming. The forecasting strategy is quite simple: differenced yields are predicted recursively over $h = 1$ to 12, with the estimated parameters fixed at each time $t$ that the forecast is made. It should be noted that we do not compare to a simpler and more parsimonious model such as VAR(1), since it has already been reported to be inferior to the DNS model in Diebold and Li (2006).

The results are summarized in Tables 2 and 3. The former reports RMSEs of out-of-sample predicted yields (relative to the actual ones) obtained by each model, and the latter shows the ratios of RMSEs-by-sVAR to RMSEs-by-DNS. The impact of the results is highly impressive and straightforward: that is, the superiority of the sVAR model can easily...
be observed. In fact, RMSEs-by-sVAR are uniformly lower than those by DNS in all the maturities $\tau$ for each prediction period $h$. Although there are many versions that improve the original DNS, sVAR is undoubtedly a strong competitor in terms of forecasting.

7 Conclusions

We present our conclusions in two parts. First, we consider another potential application of our method in the estimation of high-dimensional MGARCH in Section 7.1, and Section 7.2 presents concluding remarks.

7.1 Another potential application

Regarding the application of the general theory, this paper has focused only on a VAR model, in spite of the potential applicability to the other stationary models of high-dimensionality. For instance, we may apply the same theory to estimation of high- (or even moderate-) dimensional MGARCH models.

Although there are several variants of MGARCH, the so-called BEKK(1,1) model is known to be comparatively general among them. This particular model supposes that the conditional variance $\Sigma_t$ of a $k$-dimensional stationary process $y_t$ is modeled by

$$\Sigma_t = CC^\top + A^\top y_{t-1} y_{t-1}^\top A + B^\top \Sigma_{t-1} B.$$  

The parameter $C$ is a $k \times k$ lower triangle matrix and $A$ and $B$ are $k \times k$ matrices which vary freely, so that the number of parameters to be estimated amounts to $(5k + 1)k/2$. One characteristic that helps avoid estimating many parameters is that the model is made to specify parameter restrictions in advance. For example, constant conditional correlation GARCH restricts the parameters to make the model possess constant correlations, while modeling the conditional variances by univariate GARCH models; see, for example, Bauwens et al. (2006), Silvennoinen and Teräsvirta (2009), and Francq and Zakoïan (2010). However, such
modeling may be too restrictive in that the parameter constraint is imposed somewhat artificially.

Of course, while there are many improved models in that direction, it seems more appropriate that the effective parameters be selected by observed data in a general model such as BEKK(1,1). This direction could be pursued with the method proposed in the paper. In such a case, we must take care when imposing penalties on the parameters. If all the parameters are equally constrained, it might be possible that some diagonal elements of $\Sigma_t$, which should not be zero, might be forced into becoming so.

### 7.2 Concluding remarks

In this paper, we have first developed a general framework of the penalized quasi-maximum likelihood (PQML) estimation that results in sparse estimates of high-dimensional parameter vectors in stationary time series models. As far as theoretical results, the so-called oracle properties of the estimators are explored under the assumption that the number of parameters of interest grows at a polynomial rate. It should be underscored that SCAD-type penalties are recommended instead of the $L_1$ penalty, which is frequently used in the time series context, because of the bias reduction. The theoretical results in the first half of the paper have been derived under the assumption of high-level conditions, and hence a researcher should verify the conditions for each model he/she wants to consider when using this approach.

Second, large but sparse VAR models have been examined as an application of the results achieved in the first half. Specifically, assumptions under which the oracle property holds have been suggested. We may hereby avoid irksome tests for zero restrictions of coefficient matrices and accept instead a high-dimensional VAR model, as long as the model is sparse. The validity and superiority has been confirmed through a Monte Carlo simulation and an empirical example. In particular, we have observed from the empirical analysis that, even when the dimension is large, the PQML estimation sufficiently reduces the innate trouble in
VAR models, which can often suffer terribly from high-dimensionality.

It goes without saying that there are many issues that must be addressed in the future. In particular, there is a limitation in regards to the use of the tuning parameter $\lambda$. In the case of the i.i.d. assumption, much attention has been paid to how to determine $\lambda$; see Fan and Tang (2013) and references therein. However, this paper, which has treated dependent data, has presented no guideline on the choice of “good” tuning parameters $\lambda$ (and $a$). Future research is necessary to indicate the best way to set tuning parameters in empirical studies.
A Appendices

A.1 Useful lemmas

In Lemma 1 below, let $\hat{\mathcal{M}} := \text{supp}(\hat{\theta})$, a set of indices corresponding to all nonzero components of $\hat{\theta}$, and $\hat{\theta}_{\hat{\mathcal{M}}}$ denote a subvector of $\hat{\theta}$ formed by its restriction to $\hat{\mathcal{M}}$. The other symbols are defined analogously. Let $\odot$ denote the Hadamard product. The sign function $\text{sgn}(\cdot)$ is applied coordinate-wise.

**Lemma 1** Suppose that Assumption [7] holds. Then $\hat{\theta}$ is a strict local maximizer of the penalized quasi-likelihood (PQL) $Q_T(\theta)$ defined in (1) if

$$S_{\hat{\mathcal{M}}^c}(\hat{\theta}) - \lambda_T \rho'(|\hat{\theta}_{\hat{\mathcal{M}}}|) \odot \text{sgn}(\hat{\theta}_{\hat{\mathcal{M}}}) = 0; \quad (8)$$

$$\|S_{\hat{\mathcal{M}}^c}(\hat{\theta})\|_\infty < \lambda_T \rho'(0+); \quad (9)$$

$$\lambda_{\min}[-H_{\hat{\mathcal{M}}^c}(\hat{\theta})] > \lambda_T \kappa(\rho; \hat{\theta}_{\hat{\mathcal{M}}}). \quad (10)$$

**Proof** The proof follows from Fan and Lv (2011). We first consider the PQL $Q_T(\theta)$ defined in (1) in the constrained $\|\hat{\theta}\|_0$-dimensional subspace $\mathcal{S} := \{\theta \in \mathbb{R}^p : \theta^c = 0\}$ of $\mathbb{R}^p$, where $\theta^c$ denotes the subvector of $\theta$ formed by components in $\hat{\mathcal{M}}^c$. It follows from condition (10) that $Q_T(\theta)$ is strictly concave in a ball $\mathbb{N}_0 \subset \mathcal{S}$ centered at $\hat{\theta}$. This, along with (8), entails that $\hat{\theta}$, as a critical point of $Q_T(\theta)$ in $\mathcal{S}$, is the unique maximizer of $Q_T(\theta)$ in $\mathbb{N}_0$.

It remains to show that the sparse vector $\hat{\theta}$ is indeed a strict local maximizer of $Q_T(\theta)$ on the whole space $\mathbb{R}^p$. Take a small ball $\mathbb{N}_1 \subset \mathbb{R}^p$ centered at $\hat{\theta}$ such that $\mathbb{N}_1 \cap \mathcal{S} \subset \mathbb{N}_0$. We then need to show that $Q_T(\hat{\theta}) > Q_T(\gamma_1)$ for any $\gamma_1 \in \mathbb{N}_1 \setminus \mathbb{N}_0$. Let $\gamma_2$ be the projection of $\gamma_1$ onto $\mathcal{S}$, so that $\gamma_2 \in \mathbb{N}_0$. This ensures that $Q_T(\hat{\theta}) > Q_T(\gamma_2)$ unless $\gamma_2 = \hat{\theta}$, since $\hat{\theta}$ is a strict maximizer of $Q_T(\theta)$ in $\mathbb{N}_0$. Thus it suffices to prove $Q_T(\gamma_2) > Q_T(\gamma_1)$.

By the mean value theorem, we have

$$Q_T(\gamma_1) - Q_T(\gamma_2) = \frac{\partial Q_T(\gamma_0)}{\partial \gamma}(\gamma_1 - \gamma_2),$$

where $\gamma_0$ lies on the line segment from $\gamma_1$ to $\gamma_2$. Since $\gamma_0 \in \mathcal{S}$, we can apply the concavity of $Q_T(\theta)$ to obtain $Q_T(\gamma_2) > Q_T(\gamma_1)$.
where the vector $\gamma_0$ lies between $\gamma_1$ and $\gamma_2$. Note that the components of $\gamma_1 - \gamma_2$ are zero for their indices in $\hat{\mathcal{M}}$ and $\text{sgn}(\gamma_{0j}) = \text{sgn}(\gamma_{1j})$ for $j \in \hat{\mathcal{M}}$. Therefore, we have

$$\frac{\partial Q_T(\gamma_0)}{\partial \gamma_j}(\gamma_1 - \gamma_2) = S_T(\gamma_0)\top (\gamma_1 - \gamma_2) - \lambda_T [\rho'(|\gamma_0|) \odot \text{sgn}(\gamma_0)]\top (\gamma_1 - \gamma_2)$$

$$= S_{\hat{\mathcal{M}}_c\top}(\gamma_0)\top \gamma_{1c} - \lambda_T \sum_{j \in \hat{\mathcal{M}}_c} \rho'(|\gamma_{0j}|)|\gamma_{1j}|,$$

(11)

where $\gamma_{1c}$ is a subvector of $\gamma_1$ formed by the components in $\hat{\mathcal{M}}_c$. It follows from the concavity of $\rho$ in Assumption 4 that $\rho'$ is decreasing on $[0, \infty)$. By condition (9) and the continuity of $S_n(\cdot)$, there exists some $\delta > 0$ such that, for any $\theta$ in a ball in $\mathbb{R}^p$ centered at $\hat{\theta}$ with radius $\delta$,

$$\| S_{\hat{\mathcal{M}}_c\top}(\theta)\|_\infty < \lambda_T \rho'(\delta).$$

(12)

We further shrink the radius of ball $\mathbb{N}_1$ to less than $\delta$ so that $|\gamma_{0j}| \leq |\gamma_{1j}| < \delta$ for $j \in \mathcal{M}_c$, and (12) holds for any $\theta \in \mathbb{N}_1$. Since $\gamma_0 \in \mathbb{N}_1$, it follows from (12) that (11) is strictly less than

$$\lambda_T \rho'(\delta)\|\gamma_{1c}\|_1 - \lambda_T \rho'(\delta)\|\gamma_{1c}\|_1 = 0$$

because of the monotonicity of $\rho'$. Thus $Q_T(\gamma_2) > Q_T(\gamma_1)$ holds and the proof completes.

□

**Lemma 2** Let $w_t$ be a martingale difference sequence with $\mathbb{E}|w_t|^m \leq C_w$ for all $t$, where $m > 2$ and $C_w$ is a constant. Then we have

$$T^{-m/2} \mathbb{E} \left( \sum_{t=1}^T w_t \right)^m < \infty.$$
Proof of Lemma 2 A Marcinkiewicz–Zygmund inequality for martingales (e.g., Rio, 2013, p. 61) states that

\[ E \left( \sum_{t=1}^{T} w_t \right)^m \leq \left\{ 4m(m-1) \right\}^{m/2} T^{(m-2)/2} \sum_{t=1}^{T} E|w_t|^m \]

holds for \( m > 2 \). Thus, by the condition \( E|w_t|^m \leq C_w \) for all \( t \), we can easily observe that

\[ T^{-m/2} E \left( \sum_{t=1}^{T} w_t \right)^m \leq \left\{ 4m(m-1) \right\}^{m/2} T^{-1} \sum_{t=1}^{T} E|w_t|^m \leq \left\{ 4m(m-1) \right\}^{m/2} C_w. \]

This completes the proof. \( \square \)

A.2 Proofs of theorems

Remember that \( \theta_{\mathcal{M}} = (\theta_1, \ldots, \theta_q)^\top \). For notational simplicity, we sometimes write, for example, \( Q_T((\theta_{\mathcal{M}}^\top, \theta_{\mathcal{M}_0}^\top)) \) as \( Q_T(\theta_{\mathcal{M}}, \theta_{\mathcal{M}_0}) \).

Proof of Theorem 1 Let \( \mathcal{M}_0 = \text{supp}(\theta^0) \). Consider the events

\[ \mathcal{E}_T^1 = \{ \| S_{\mathcal{M}_0}^0 \|_\infty \leq (q^{2/m_1} / T)^{1/2} \log^{1/m_1} T \} \quad \text{and} \quad \mathcal{E}_T^2 = \{ \| S_{\mathcal{M}_0}^0 \|_\infty \leq \lambda \rho'(0+) \log^{-1} T \}, \]

where \( q = O(T^{\delta}) \) and \( \lambda = O(T^{-\alpha}) \). It follows from Bonferroni’s inequality and Markov’s inequality together with Assumption 5 (a)(b) that

\[
P(\mathcal{E}_T^1 \cap \mathcal{E}_T^2) \geq 1 - \sum_{i \in \mathcal{M}_0} P(\| T^{1/2} S_{\mathcal{M}_0}^0 \|_1 > q^{1/m_1} \log^{1/m_1} T) - \sum_{i \in \mathcal{M}_0} P(\| T^{1/2} S_{\mathcal{M}_0}^0 \|_1 > T^{1/2 - \alpha} \rho'(0+))
\]

\[
\geq 1 - q \max_{i \in \mathcal{M}_0} E[\| T^{1/2} S_{\mathcal{M}_0}^0 \|_{m_1}] - (p - q) \max_{i \in \mathcal{M}_0} E[\| T^{1/2} S_{\mathcal{M}_0}^0 \|_{m_2}]
\]

\[
T^{m_2(1/2 - \alpha)} \rho'(0+) \log^{m_2} T
\]

\[
= 1 - O(\log^{-1} T) - O(T^{\delta - m_2(1/2 - \alpha)} \log^{m_2} T),
\]

where the last two terms are \( o(1) \) because of the condition \( \delta < m_2(1/2 - \alpha) \). Under the event \( \mathcal{E}_T^1 \cap \mathcal{E}_T^2 \), we will show that there exists a solution \( \hat{\theta} \in \mathbb{R}^p \) to conditions (8)–(10) with \( \text{sgn}(\hat{\theta}) = \text{sgn}(\theta^0) \) and \( \| \hat{\theta} - \theta^0 \|_\infty = O(T^{-\gamma} \log T) \) for some \( \gamma \in (0, 1/2] \).
Step 1. We first prove that, for a sufficiently large $T$, equation (8) has a solution $\hat{\theta}_M$ inside the hypercube

$$\mathbb{N} = \{ \theta_M \in \mathbb{R}^q : \| \theta_M - \theta_M^0 \|_\infty = T^{-\gamma} \log T \}$$

when we suppose $M = M_0$. Define the function $\Psi : \mathbb{R}^q \to \mathbb{R}^q$ by

$$\Psi(\theta_M) = S_{\theta_M} T(\theta_M, 0) - \lambda \rho'(\| \theta_M \|) \odot \text{sgn}(\theta_M).$$

(14)

Then condition (8) is equivalent to $\Psi(\hat{\theta}_M) = 0$. To show that the solution is in the hypercube $\mathbb{N}$, we expand $\Psi(\theta_M)$ around $\theta_M^0$. Function (14) is written as

$$\Psi(\theta_M) = S_{\theta_M} T(\theta_M, 0) + H_{\theta_M} T(\theta_M^*, 0)(\theta_M - \theta_M^0) - \lambda \rho'(\| \theta_M \|) \odot \text{sgn}(\theta_M)$$

$$= H_{\theta_M} T(\theta_M - \theta_M^0) + [S_{\theta_M} T - \lambda \rho'(\| \theta_M \|) \odot \text{sgn}(\theta_M)]$$

$$+ [H_{\theta_M} T(\theta_M^*, 0) - H_{\theta_M} T(\theta_M - \theta_M^0)]$$

$$=: H_{\theta_M} T(\theta_M - \theta_M^0) + v_T + w_T,$$

(15)

where $\theta_M^*$ lies on the line segment joining $\theta_M$ and $\theta_M^0$. Since the matrix $H_{\theta_M}^0$ is invertible by Assumption 5(c), (15) is further written as

$$\tilde{\Psi}(\theta_M) := H_{\theta_M}^{-1} \Psi(\theta_M)$$

$$= \theta_M - \theta_M^0 + H_{\theta_M}^{-1} v_T + H_{\theta_M}^{-1} w_T =: \theta_M - \theta_M^0 + \tilde{v}_T + \tilde{w}_T$$

(16)

when $T$ is sufficiently large. We then bound the last two terms, $\tilde{v}_T$ and $\tilde{w}_T$, in (16).

First, we deal with $\tilde{v}_T$. For any $\theta_M \in \mathbb{N}$, it holds that

$$\min_{j \in M} | \theta_j | \geq \min_{j \in M} | \theta_j^0 | - d_T = d_T \geq T^{-\gamma} \log T$$

(17)

by Assumption 3(b), and $\text{sgn}(\theta_M) = \text{sgn}(\theta_M^0)$. Using the monotonicity of $\rho'(\cdot)$ in Assumption 1 with (17) and Assumption 3(c), we obtain

$$\| \lambda \rho'(\| \theta_M \|) \odot \text{sgn}(\theta_M) \|_\infty \leq \lambda \rho'(d) = o(q^{-1/2} T^{-\gamma} \log T).$$
This, along with the property of matrix norms and Assumption 5(c), entails that on the event $E_1^c$,
\[
\|\tilde{v}_T\|_\infty = \|H_{\mathcal{M}_0}^{-1}[S_{\mathcal{M}_0T}^0 - \lambda \rho^r(\|\theta_{\mathcal{M}_0}\|) \odot \text{sgn}(\theta_{\mathcal{M}_0})]\|_\infty \\
\leq \|H_{\mathcal{M}_0}^{-1}\| \|S_{\mathcal{M}_0T}^0 - \lambda \rho^r(\|\theta_{\mathcal{M}_0}\|) \odot \text{sgn}(\theta_{\mathcal{M}_0})\|_\infty \\
\leq q^{1/2}\|H_{\mathcal{M}_0}^{-1}\| (\|S_{\mathcal{M}_0T}^0\|_\infty + \|\lambda \rho^r(\|\theta_{\mathcal{M}_0}\|) \odot \text{sgn}(\theta_{\mathcal{M}_0})\|_\infty) \\
\leq q^{1/2}O_p(1) \left( (q^{2/m_1}/T)^{1/2} \log T + o(T^{-\gamma} \log T) \right) \\
= o_p(T^{-\gamma} \log T),
\]
where the last equality follows from $q = O(T^{\delta_0})$ and $\delta_0 \leq (1/2 - \gamma)/(2 + 1/m_1)$.

Next, we consider $\tilde{w}_T$. By the property of norms and Assumption 5(c)(d), we have
\[
\|\tilde{w}_T\|_\infty = \|H_{\mathcal{M}_0}^{-1}[H_{\mathcal{M}_0T}(\theta_{\mathcal{M}_0}^*, 0) - H_{\mathcal{M}_0T}(\theta_{\mathcal{M}_0}^0, 0)](\theta_{\mathcal{M}_0} - \theta_{\mathcal{M}_0}^0)\|_\infty \\
\leq q^{1/2}\|H_{\mathcal{M}_0}^{-1}\| \|H_{\mathcal{M}_0T}(\theta_{\mathcal{M}_0}^*, 0) - H_{\mathcal{M}_0T}(\theta_{\mathcal{M}_0}^0, 0)\||\theta_{\mathcal{M}_0} - \theta_{\mathcal{M}_0}^0\|_\infty \\
\leq qO_p(1)\|H_{\mathcal{M}_0T}(\theta_{\mathcal{M}_0}^*, 0) - H_{\mathcal{M}_0T}(\theta_{\mathcal{M}_0}^0, 0)\||\theta_{\mathcal{M}_0} - \theta_{\mathcal{M}_0}^0\|_\infty \\
\leq qO_p(1)K_T \|\theta_{\mathcal{M}_0}^* - \theta_{\mathcal{M}_0}^0\|_\infty \|\theta_{\mathcal{M}_0} - \theta_{\mathcal{M}_0}^0\|_\infty.
\]
Therefore, since $K_T = O_p(1)$ and $q = O(T^{\delta_0})$ with $\delta_0 < \gamma$,
\[
\|\tilde{w}_T\|_\infty = qO_p(T^{-2\gamma} \log^2 T) = o_p(T^{-\gamma} \log T)
\]
holds, provided that $\theta_i - \theta_i^0 = T^{-\gamma} \log T$ for all $i \in \mathcal{M}_0$. By (16), (18), and (19), for a sufficiently large $T$ and for all $i \in \mathcal{M}_0$, we thus have
\[
\tilde{\Psi}_i(\theta_{\mathcal{M}_0}) \geq T^{-\gamma} \log T - \|\tilde{v}_T\|_\infty - \|\tilde{w}_T\|_\infty \geq 0 \quad \text{if} \quad \theta_i - \theta_i^0 = T^{-\gamma} \log T; \quad \text{(20)}
\]
\[
\tilde{\Psi}_i(\theta_{\mathcal{M}_0}) \leq -T^{-\gamma} \log T + \|\tilde{v}_T\|_\infty + \|\tilde{w}_T\|_\infty \leq 0 \quad \text{if} \quad \theta_i - \theta_i^0 = -T^{-\gamma} \log T. \quad \text{(21)}
\]
By the continuity of $\tilde{\Psi}(\cdot)$ and inequalities (20) and (21), an application of Miranda’s existence theorem shows that the equation $\tilde{\Psi}(\theta_{\mathcal{M}_0}) = 0$ has a solution $\tilde{\theta}_{\mathcal{M}_0}$ in $\mathbb{N}$. Clearly, $\tilde{\theta}_{\mathcal{M}_0}$ also solves the equation $\Psi(\theta_{\mathcal{M}_0}) = 0$, in regard to the first equality in (16). Thus, we have
shown that (8) indeed has a solution \( \hat{\theta}_{\mathcal{H}_0} \) in \( \mathbb{N} \).

**Step 2.** Let \( \hat{\theta} = (\hat{\theta}_{\mathcal{H}_0}^\top, \hat{\theta}_{\mathcal{H}_0^c}^\top)^\top \in \mathbb{R}^p \) with \( \hat{\theta}_{\mathcal{H}_0} \in \mathbb{N} \) as a solution to (8) and \( \hat{\theta}_{\mathcal{H}_0^c} = 0 \). Next, we show that \( \hat{\theta} \) satisfies (9) on the event \( \mathcal{E}_T^2 \). By the triangle inequality and mean value theorem, we have

\[
\lambda^{-1} \| S_{\mathcal{H}_0^c} T(\hat{\theta}) \|_\infty \leq \lambda^{-1} \| S_{\mathcal{H}_0^c} T \|_\infty + \lambda^{-1} \| S_{\mathcal{H}_0^c} T(\hat{\theta}) - S_{\mathcal{H}_0^c} T \|_\infty \\
\leq \rho(0+) \log^{-1} T + \lambda^{-1} \| (\partial / \partial \theta_{\mathcal{H}_0}) S_{\mathcal{H}_0^c} T(\hat{\theta}_{\mathcal{H}_0}^*, 0)(\hat{\theta}_{\mathcal{H}_0} - \theta_{\mathcal{H}_0}^0) \|_\infty, \tag{22}
\]

where \( \hat{\theta}_{\mathcal{H}_0}^* \) lies on the line segment joining \( \hat{\theta}_{\mathcal{H}_0} \) and \( \theta_{\mathcal{H}_0}^0 \). The first term of the upper bound in (22) is negligible, so that it suffices to show the second term is less than \( \rho'(0+) \). Since \( \hat{\theta}_{\mathcal{H}_0} \) solves the equation \( \Psi(\theta_{\mathcal{H}_0}) = 0 \) in (14), we get

\[
S_{\mathcal{H}_0^c} T + H_{\mathcal{H}_0 T}(\hat{\theta}_{\mathcal{H}_0}^*, 0)(\hat{\theta}_{\mathcal{H}_0} - \theta_{\mathcal{H}_0}^0) - \lambda \rho'(|\hat{\theta}_{\mathcal{H}_0}|) \odot \text{sgn}(\hat{\theta}_{\mathcal{H}_0}) = 0,
\]

with \( \hat{\theta}_{\mathcal{H}_0}^* \) lying between \( \hat{\theta}_{\mathcal{H}_0} \) and \( \theta_{\mathcal{H}_0}^0 \). Note that \( H_{\mathcal{H}_0 T}(\hat{\theta}_{\mathcal{H}_0}^*, 0) \) is invertible with probability approaching one from Assumption 5(c)(d). Hence, from Assumption 5(e), the last term of (22) is evaluated as

\[
\lambda^{-1} \|(\partial / \partial \theta_{\mathcal{H}_0}) S_{\mathcal{H}_0^c} T(\hat{\theta}_{\mathcal{H}_0}^*, 0)[H_{\mathcal{H}_0 T}(\hat{\theta}_{\mathcal{H}_0}^*, 0)]^{-1}[S_{\mathcal{H}_0^c} T - \lambda \rho'(|\hat{\theta}_{\mathcal{H}_0}|) \odot \text{sgn}(\hat{\theta}_{\mathcal{H}_0})]\|_\infty \\
\leq \lambda^{-1} \sup_{\theta_1, \theta_2 \in \mathbb{N}} \| (\partial / \partial \theta_{\mathcal{H}_0}) S_{\mathcal{H}_0^c} T(\theta_1, 0)[H_{\mathcal{H}_0 T}(\theta_2, 0)]^{-1}\|_\infty \| S_{\mathcal{H}_0^c} T \|_\infty + \| \lambda \rho'(|\theta_{\mathcal{H}_0}|) \|_\infty \\
\leq \lambda^{-1} \frac{\| \rho'(0+) \|}{\rho'(d)} O_p(T^\beta) \left( (q^{2/m_1} / T)^{1/2} \log^{1/2} T + \lambda \rho'(d) \right) \\
= \lambda^{-1} O_p(T^\beta) (q^{2/m_1} / T)^{1/2} \log^{1/2} T + c \rho'(0+). \tag{23}
\]

Since the first term in the final equation of (23) is equal to \( O_p(T^{\alpha + \beta + \delta_0 / m_1 - 1/2} \log^{1/2} T) \), which is \( o_p(1) \) by Assumption 5(a), (23) is eventually less than \( \rho'(0+) \). This verifies condition (9).

Finally, condition (10) is guaranteed by Assumption 5(c)(d) for a sufficiently large \( T \). Therefore, by Lemma 1, we have shown that \( \hat{\theta} = (\hat{\theta}_{\mathcal{H}_0}^\top, \hat{\theta}_{\mathcal{H}_0^c}^\top)^\top \) is a strict local maximizer of
the penalized likelihood \( Q_T(\theta) \) in (11) with \( \|\hat{\theta} - \theta^0\|_\infty = O(T^{-\gamma} \log T) \) and \( \hat{\theta}_{\mathcal{M}^c} = 0 \) under the event \( E_T^1 \cap E_T^2 \). Thus, the proofs of (a) and (b) are completed by (13). □

**Proof of Theorem 2**

First, we show results (a) and (b) through the following two steps.

**Step 1.** We consider \( Q_T(\theta) \) in the correctly constrained space \( \{ \theta \in \mathbb{R}^p : \theta_{\mathcal{M}} = 0 \in \mathbb{R}^{p-q} \} \) and focus on the \( q \)-dimensional intrinsic subspace \( \{ \theta_{\mathcal{M}} \in \mathbb{R}^q \} \). The corresponding PQL is given by

\[
Q_T(\theta_{\mathcal{M}}, 0) = L_T(\theta_{\mathcal{M}}, 0) - \sum_{j=1}^{q} p_\lambda(|\theta_j|).
\]

We now show the existence of a strict local maximizer \( \hat{\theta}_{\mathcal{M}} \) of \( Q_T(\theta_{\mathcal{M}}, 0) \) such that \( \|\hat{\theta}_{\mathcal{M}} - \theta_{\mathcal{M}}^0\| = O_p((q/T)^{1/2}) \). To this end, it is sufficient to prove that, for any given \( \varepsilon > 0 \), there is a large constant \( C \) such that

\[
P \left( \sup_{|u|=C} Q_T(\theta_{\mathcal{M}}^0 + u(q/T)^{1/2}, 0) < Q_T(\theta_{\mathcal{M}}^0, 0) \right) \geq 1 - \varepsilon.
\]

This implies that, with probability tending to one, there is a local maximizer \( \hat{\theta}_{\mathcal{M}} \) of \( Q_T(\theta_{\mathcal{M}}, 0) \) in the ball \( \{ \theta_{\mathcal{M}} \in \mathbb{R}^q : \|\theta_{\mathcal{M}} - \theta_{\mathcal{M}}^0\| \leq C(q/T)^{1/2} \} \). Recall that \( \rho(t) = p_\lambda(t)/\lambda \). We have

\[
R_T(u) := Q_T(\theta_{\mathcal{M}}^0 + u(q/T)^{1/2}, 0) - Q_T(\theta_{\mathcal{M}}^0, 0)
\]

\[
= \left( L_T(\theta_{\mathcal{M}}^0 + u(q/T)^{1/2}, 0) - L_T(\theta_{\mathcal{M}}^0, 0) \right)
\]

\[
- \lambda_T \sum_{j=1}^{q} \left( \rho(|\theta_j^0 + u_j(q/T)^{1/2}|) - \rho(|\theta_j^0|) \right) =: (I) + (II).
\]

First, we evaluate \((II)\). The Taylor expansion gives

\[
\rho(|\theta_j^0 + u_j(q/T)^{1/2}|) - \rho(|\theta_j^0|) = \rho'(\theta_j^{0*}) (|\theta_j^0 + u_j(q/T)^{1/2}| - |\theta_j^0|)
\]

\[
\leq \rho'(d_T)(q/T)^{1/2} |u_j|,
\]

where \( |\theta_j^{0*}| \) lies between \( |\theta_j^0| \) and \( |\theta_j^0 + u_j(q/T)^{1/2}| \), and the last inequality follows from the monotonicity of \( \rho'(\cdot) \) and the triangle inequality. By (26), the Cauchy–Schwarz inequality, and Assumption 4(b1), we have

\[
|(II)| \leq \lambda_T \rho'(d_T)(q/T)^{1/2} \sum_{j=1}^{q} |u_j|
\]

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\[ \leq \lambda_T \rho'(d_T)(q/T)^{1/2}q^{1/2}\|u\| = O(q/T)\|u\|. \]

Next, we consider \((I)\). By the Taylor expansion, we have

\[ (I) = (q/T)^{1/2}S^0_{\theta^0_\mathcal{M}_0}u + \frac{q/T}{2}u^\top H_{\mathcal{M}_0T}(\theta^{0**}_{\mathcal{M}_0}, 0)u =: (I_1) + (I_2), \]

where the vector \(\theta^{0**}_{\mathcal{M}_0}\) lies between \(\theta^0_{\mathcal{M}_0}\) and \(\theta^0_{\mathcal{M}_0} + uT^{-1/2}\). By the Cauchy–Schwarz inequality and Assumption 6(a), together with the Markov inequality, we obtain

\[
|I_1| \leq (q/T)^{1/2}\|S^0_{\theta^0_\mathcal{M}_0}\||u||u| \\
\leq (q/T)^{1/2}q^{1/2}\|S^0_{\theta^0_\mathcal{M}_0}\|\|u\| = (q/T)^{1/2}q^{1/2}O_T(T^{-1/2})\|u\| = O_P(q/T)\|u\|,
\]

whereas, by Assumptions 5(c) and 6(d), we get

\[
(I_2) = \frac{q/T}{2}u^\top \left[ H^0_{\mathcal{M}_0T} + \left\{ H_{\mathcal{M}_0T}(\theta^{0**}_{\mathcal{M}_0}, 0) - H^0_{\mathcal{M}_0T} \right\} \right] u \leq -\frac{q/T}{2}C_1(1 + o_P(1))\|u\|^2,
\]

where \(C_1 > 0\) a.s.) is the minimum eigenvalue of \(-H^0_{\mathcal{M}_0T}\). Because \((I_2)\) dominates \((II)\) and \((I_1)\) when a large value of \(\|u\|\) is taken, \(\sup_{\|u\| = C_T} \|u\|\) tends to negativity as \(T\) grows large. Thus, (25) holds.

**Step 2.** To complete the proof of (a) and (b), it remains to show that \(\hat{\theta}^0 := (\hat{\theta}^{0}_{\mathcal{M}_0}, 0)\) is indeed a strict local maximizer of \(Q_T(\theta)\) in \(\mathbb{R}^{p}\). From Lemma 1, it suffices to check conditions (8), (9), and (10) while setting \(\hat{\theta} = \hat{\theta}^0\), but condition (8) is clearly satisfied by the argument so far and the Karush–Kuhn–Tucker condition. Condition (10) is also satisfied by Assumptions 5(c) and 6(d). To verify (9), we consider the event

\[ E^3_T = \{ \|S^0_{\theta^0_\mathcal{M}_0}\|_\infty \leq \lambda_T \rho'(0+) / 2 \}. \]

By the Markov inequality and Assumptions 6(b), we have

\[
P(E^3_T) \geq 1 - \sum_{j \in \mathcal{M}_0} P(|T^{1/2}S^0_{\theta^0_\mathcal{M}_0}| > T^{1/2} \lambda_T \rho'(0+) / 2) \\
\geq 1 - (p - q)O(T^{m_2(1/2 - \alpha)}) = 1 - O(T^{\delta - m_2(1/2 - \alpha)}), \tag{27}
\]

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Thus, it suffices to check whether condition (9) holds on the event $\mathbb{E}_T^3$ by a similar argument as in (22) and (23). From Assumption 6(e), we obtain

$$
\lambda^{-1} \| S_{\hat{\theta}}(\hat{\theta}) \|_{\infty} \\
\leq \lambda^{-1} \| S_{\hat{\theta}}^0 \|_{\infty} + \lambda^{-1} \| S_{\hat{\theta}}(\hat{\theta}) - S_{\hat{\theta}}^0 \|_{\infty} \\
\leq \rho(0+) / 2 + \lambda^{-1} \| (\partial / \partial \theta_{\mathcal{M}_0}) S_{\hat{\theta}}^0(\hat{\theta}_{\mathcal{M}_0}, 0)(\hat{\theta}_{\mathcal{M}_0} - \theta_{\mathcal{M}_0}^0) \|_{\infty} \\
\leq \rho(0+) / 2 + \lambda^{-1} \sup_{\theta_1} \| (\partial / \partial \theta_{\mathcal{M}_0}) S_{\hat{\theta}}^0(\theta_1, 0) \|_{2,\infty} \| (\hat{\theta}_{\mathcal{M}_0} - \theta_{\mathcal{M}_0}^0) \| \\
= \rho(0+) / 2 + O_p(T^{\alpha+\beta})(q/T)^{1/2},
$$

(28)

where $\hat{\theta}_{\mathcal{M}_0}^\dagger$ lies on the line segment joining $\hat{\theta}_{\mathcal{M}_0}$ and $\theta_{\mathcal{M}_0}^0$. The last term in (28) is $o_p(1)$ by Assumption 4(a), which completes the proof of (a) and (b).

Finally, we prove (c). To this purpose, it suffices to show the asymptotic normality of $\hat{\theta}_{\mathcal{M}_0}$. By the first order condition $(\partial / \partial \theta_{\mathcal{M}_0}) Q_T(\hat{\theta}_{\mathcal{M}_0}, 0) = 0$ and Taylor expansion of the likelihood function, we have, for any vector $a \in \mathbb{R}^q$ such that $\|a\| = 1$,

$$
-T^{1/2} a^T I_{\mathcal{M}_0}^{-1/2} H_{\mathcal{M}_0}^{-1} (\hat{\theta}_{\mathcal{M}_0} - \theta_{\mathcal{M}_0}^0) \\
= T^{1/2} a^T I_{\mathcal{M}_0}^{-1/2} S_{\mathcal{M}_0}^0 - T^{1/2} a^T I_{\mathcal{M}_0}^{-1/2} H_{\mathcal{M}_0}^{-1/2} \lambda_T \rho'(|\hat{\theta}_{\mathcal{M}_0}|) \odot \text{sgn}(\hat{\theta}_{\mathcal{M}_0}) \\
+ T^{1/2} a^T I_{\mathcal{M}_0}^{-1/2} [H_{\mathcal{M}_0}^{-1/2}(\hat{\theta}_{\mathcal{M}_0}^\dagger, 0) - H_{\mathcal{M}_0}^{-1/2}(\hat{\theta}_{\mathcal{M}_0} - \theta_{\mathcal{M}_0}^0)] \\
=: (III_1) + (III_2) + (III_3),
$$

where $\hat{\theta}_{\mathcal{M}_0}^\dagger$ lies between $\hat{\theta}_{\mathcal{M}_0}$ and $\theta_{\mathcal{M}_0}^0$. The proof is completed if we prove $(III_2)$ and $(III_3)$ are $o_p(1)$ because $(III_1)$ is asymptotically standard normal by the direct use of Assumption 7(b) and the Slutzky lemma. By the Cauchy–Schwarz inequality, Assumptions 7(a), and 4(b2), we obtain

$$
|\langle III_2 \rangle| = T^{1/2} \| a^T I_{\mathcal{M}_0}^{-1/2} \lambda_T \rho'(|\hat{\theta}_{\mathcal{M}_0}|) \odot \text{sgn}(\hat{\theta}_{\mathcal{M}_0}) \| \\
\leq T^{1/2} q^{1/2} \| I_{\mathcal{M}_0}^{-1/2} \lambda_T \rho'(d) \| \leq T^{1/2} q^{1/2} C_2 o((qT)^{-1/2}) = o(1).
$$
Similarly, by the Cauchy–Schwarz inequality, Assumptions 6(c), and 7(a), we obtain
\[
\big(III_3\big) = T^{1/2}\left| a_{\theta_0}^{\top}H_{\theta_0}(\hat{\theta}_0^\top,0) - H_{\theta_0}(\hat{\theta}_0^\top,0) \right| \\
\leq T^{1/2}\left| I_{\theta_0}^{\top}H_{\theta_0}(\hat{\theta}_0^\top,0) - H_{\theta_0}(\hat{\theta}_0^\top,0) \right| \\
\leq T^{1/2}C_T\left\| \hat{\theta}_0^\top - \theta_0^\top \right\| \left\| \hat{\theta}_0 - \theta_0 \right\| = O_p(T^{1/2})O_p(q/T) = o_p(1),
\]
and the proof is completed. □

Proof of Proposition 1 The proof is completed if Assumptions 5(a)–(c), 6(d)(e), and 7(a)(b) are verified. First, we show that Assumption 5(a)(b) are true for \( m_1 = m_2 = 4 \). Using the property of the vec-operator and Kronecker product (e.g., Lütkepohl, 2005, A.12.1), and letting vec\((\epsilon_t x_t^\top) = (w_{1t}, \ldots, w_{pt})^\top = w_t \) and \( \sigma^{ij} \) be the \((i,j)\)th element of \( \Sigma^{-1} \), we see that
\[
s_t^0 = P^\top(x_t \otimes \Sigma^{-1})\epsilon_t = P^\top(I_{kr} \otimes \Sigma^{-1})w_t \\
= P^\top \sum_{j=1}^k (\sigma^{1j}w_{jt}, \ldots, \sigma^{kj}w_{jt}, \sigma^{1j}w_{k+j,t}, \ldots, \sigma^{kj}w_{k+j,t}, \ldots, \sigma^{1j}w_{(k-1)j,t}, \ldots, \sigma^{kj}w_{(k-1)j,t})^\top.
\]
Hence, it suffices to consider a typical element \( s_{hk+\ell,t}^0 = \sum_{j=1}^k \sigma^{kj}w_{hk+j,t} \), where \( \ell \in \{1, \ldots, k\} \) and \( h \in \{0, 1, \ldots, kr - 1\} \), and prove that \( E|T^{-1/2}\sum_{t=1}^T s_{hk+\ell,t}^0|^4 < \infty \). Note that \( \sum_{t=1}^T s_{hk+\ell,t}^0 \) is a martingale because \( s_{hk+\ell,t}^0 \) is a martingale difference sequence with respect to \( \mathcal{F}_{t-1} \).

This proof is thus completed if we show \( E|s_{hk+\ell,t}^0|^4 < \infty \) in view of Lemma 2. To see this, we notice that \( \sigma^{1j}, \ldots, \sigma^{kj} \) are zero except a finite number of them, by Assumption 8(c). Let \( \mathcal{I}_\ell := \{ j \in \{1, \ldots, k\} : \sigma^{kj} \neq 0 \} \) and \( \tilde{k} := \|\mathcal{I}_\ell\| \). Applying the \( C_\gamma \)-inequality repeatedly, we have
\[
E|s_{hk+\ell,t}^0|^4 = E\left| \sum_{j=1}^k \sigma^{kj}w_{hk+j,t} \right|^4 = E\left| \sum_{j \in \mathcal{I}_\ell} \sigma^{kj}w_{hk+j,t} \right|^4 \\
\leq 2^{3k} \sum_{j \in \mathcal{I}_\ell} E|\sigma^{kj}w_{hk+j,t}|^4 \leq 2^{3k} \tilde{k} \max_{j \in \mathcal{I}_\ell} |\sigma^{kj}|^4 \max_{j \in \mathcal{I}_\ell} E|w_{hk+j,t}|^4 < \infty,
\]
since \( E|w_{hk+j,t}|^4 < \infty \) is easily observed from the law of iterated expectations under Assumption 8(a)(b). Thus, the result follows.

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Next, we check Assumptions 5(c) and 6(d), but 6(d) is clear because the Hessian does not depend on $\theta$. We then have $H_{\mathcal{M}_0T}(\theta) = H_{\mathcal{M}_0T}^0$ for all $\theta$. 5(c) is also clear from the construction of this Hessian together with Assumption 8(a) and Rule (6) in Lütkepohl (2005, p. 661).

Next, we verify Assumption 6(e) with $\beta = \frac{\delta_0}{2}$. To this end, it is sufficient to show, for every $i \in \{1, \ldots, p\}$, $\max_{1 \leq j \leq q} \|v\| = \frac{1}{2} \sup_{1 \leq j \leq q} |H_{ijT}|$. Since $H_{ijT}^0 = O_p(1)$ under Assumption 8(a)(b) and $q = O(T^{\delta_0})$, the result follows.

Finally, we observe Assumption 7(a)(b) are satisfied. Since $\Gamma = E[x_t x_t^\top]$, $\Sigma^{-1}$ and $\Sigma_\varepsilon$ are finite and positive definite, $\Gamma \otimes \Sigma^{-1}\Sigma_\varepsilon\Sigma^{-1}$ is finite and positive definite. Thus, $I^0_{\mathcal{M}_0} = P_{\mathcal{M}_0}^\top (\Gamma \otimes \Sigma^{-1}\Sigma_\varepsilon\Sigma^{-1})P_{\mathcal{M}_0}$ is finite and positive definite. Since $a^\top I^0_{\mathcal{M}_0}^{-1/2} S_{\mathcal{M}_0}^{-1/2}$ is a stationary ergodic martingale difference sequence with

$$\text{Var}(a^\top I^0_{\mathcal{M}_0}^{-1/2} S_{\mathcal{M}_0}^{-1/2}) = a^\top I^0_{\mathcal{M}_0}^{-1/2} I^0_{\mathcal{M}_0} I^0_{\mathcal{M}_0}^{-1/2} a = a^\top a = 1$$

for any $a \in \mathbb{R}^q$ such that $\|a\| = 1$, we have $T^{1/2}a^\top I^0_{\mathcal{M}_0}^{-1/2} S_{\mathcal{M}_0}^{-1/2} \rightarrow_d N(0, 1)$. □

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Table 1: Estimation accuracy

\begin{tabular}{lcccccccc}
\hline
 & Oracle & MLE & SCAD & MCP & Lasso & & & \\
\hline
$T \setminus a$ & & & & & & & & \\
100 & 0.374 & 1.634 & 0.984 & 0.863 & 1.043 & 0.865 & 0.871 & \\
300 & 0.214 & 0.885 & 0.559 & 0.494 & 0.596 & 0.494 & 0.524 & \\
500 & 0.164 & 0.678 & 0.416 & 0.378 & 0.450 & 0.378 & 0.411 & \\
1000 & 0.116 & 0.478 & 0.251 & 0.262 & 0.281 & 0.262 & 0.295 & \\
\hline
100 & 0.373 & 1.632 & 0.837 & 0.701 & 0.904 & 0.705 & 0.698 & \\
300 & 0.214 & 0.884 & 0.510 & 0.426 & 0.540 & 0.426 & 0.435 & \\
500 & 0.164 & 0.678 & 0.394 & 0.331 & 0.419 & 0.331 & 0.346 & \\
1000 & 0.116 & 0.478 & 0.264 & 0.229 & 0.276 & 0.230 & 0.250 & \\
\hline
100 & 89.2 & 84.9 & 91.4 & 85.7 & 83.1 & & & \\
( nonzero) & (66.5) & (82.7) & (52.8) & (81.6) & (84.4) & & & \\
(zero) & (92.3) & (85.2) & (96.5) & (86.3) & (82.9) & & & \\
300 & 91.0 & 85.2 & 94.5 & 86.3 & 81.2 & & & \\
( nonzero) & (86.8) & (95.0) & (78.9) & (94.7) & (95.2) & & & \\
(zero) & (91.5) & (83.9) & (96.5) & (85.2) & (79.3) & & & \\
500 & 91.9 & 85.8 & 95.5 & 86.9 & 80.2 & & & \\
( nonzero) & (93.3) & (97.5) & (86.8) & (97.3) & (97.9) & & & \\
(zero) & (91.8) & (84.2) & (96.7) & (85.5) & (77.9) & & & \\
1000 & 94.6 & 87.9 & 97.3 & 88.8 & 79.7 & & & \\
( nonzero) & (98.1) & (99.2) & (95.8) & (99.1) & (99.5) & & & \\
(zero) & (94.1) & (86.4) & (97.5) & (87.4) & (77.1) & & & \\
\hline
\end{tabular}
### Table 2: RMSEs of out-of-sample forecasts

|   | DNS       |       |       |       |       |       |       |       |
|---|-----------|-------|-------|-------|-------|-------|-------|-------|
|   | h \ τ     | 3     | 6     | 12    | 24    | 36    | 60    | 84    | 120   |
|---|-----------|-------|-------|-------|-------|-------|-------|-------|-------|
| 1 | 0.519     | 0.559 | 0.760 | 1.018 | 1.156 | 1.220 | 1.145 | 0.984 |
| 3 | 0.784     | 0.873 | 1.128 | 1.438 | 1.564 | 1.548 | 1.400 | 1.164 |
| 6 | 1.102     | 1.224 | 1.511 | 1.838 | 1.923 | 1.813 | 1.603 | 1.311 |
| 12| 1.806     | 1.952 | 2.134 | 2.287 | 2.252 | 2.023 | 1.769 | 1.454 |

|   | sVAR      |       |       |       |       |       |       |       |
|---|-----------|-------|-------|-------|-------|-------|-------|-------|
| 1 | 0.185     | 0.168 | 0.219 | 0.284 | 0.308 | 0.310 | 0.295 | 0.271 |
| 3 | 0.328     | 0.343 | 0.404 | 0.496 | 0.521 | 0.501 | 0.461 | 0.414 |
| 6 | 0.614     | 0.628 | 0.669 | 0.745 | 0.752 | 0.688 | 0.610 | 0.524 |
| 12| 1.128     | 1.153 | 1.153 | 1.126 | 1.055 | 0.895 | 0.770 | 0.647 |

### Table 3: Ratios of RMSEs of out-of-sample forecasts

|   | sVAR/DNS  | 3     | 6     | 12    | 24    | 36    | 60    | 84    | 120   |
|---|-----------|-------|-------|-------|-------|-------|-------|-------|-------|
| 1 | 0.356     | 0.301 | 0.288 | 0.279 | 0.266 | 0.254 | 0.258 | 0.275 |
| 3 | 0.418     | 0.393 | 0.358 | 0.345 | 0.333 | 0.324 | 0.329 | 0.356 |
| 6 | 0.557     | 0.513 | 0.443 | 0.405 | 0.391 | 0.379 | 0.381 | 0.400 |
| 12| 0.625     | 0.591 | 0.540 | 0.492 | 0.468 | 0.442 | 0.435 | 0.445 |