THE NUMBER OF CLIQUES IN GRAPHS
OF GIVEN ORDER AND SIZE

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Abstract. Let $k_r(n,m)$ denote the minimum number of $r$-cliques in graphs with $n$ vertices and $m$ edges. For $r = 3, 4$ we give a lower bound on $k_r(n,m)$ that approximates $k_r(n,m)$ with an error smaller than $n^r / (n^2 - 2m)$.

The solution is based on a constraint minimization of certain multilinear forms. Our proof combines a combinatorial strategy with extensive analytical arguments.

Introduction

Our graph-theoretic notation follows [1]; in particular, an $r$-clique is a complete subgraph on $r$ vertices.

What is the minimum number $k_r(n,m)$ of $r$-cliques in graphs with $n$ vertices and $m$ edges? This problem originated with the famous graph-theoretical theorem of Turán [12] more than sixty years ago, but despite numerous attempts, never got a satisfactory solution; see [2], [5], [6], [7], [8], and [11] for some highlights of its long history. Most recently, the problem was discussed in detail in [1] and [9].

The best current result is due to Razborov [11]. Applying tools developed in [10], he achieved remarkable progress and essentially solved the problem for $r = 3$. However, so far his methods failed for larger $r$.

In this paper, using an alternative approach, we solve the problem for $r = 3, 4$. More precisely we study a class of multilinear forms and find their minima subject to certain constraints. As a consequence, for $r = 3, 4$, we obtain a lower bound on $k_r(n,m)$, approximating $k_r(n,m)$ with an error smaller than $n^r / (n^2 - 2m)$.

In our proof, a combinatorial main strategy cooperates with analytical arguments using Taylor’s expansion, Lagrange’s multipliers, compactness, continuity, and connectedness. We believe that such cooperation can be developed further and applied to other problems in extremal combinatorics.

We feel that our methods will help to find $k_r(n,m)$ for $r > 4$ as well. With this idea in mind we present all results as general as possible.

The main results of this paper, Theorems 1.2 and 1.3, are stated in Section 1 and proved in Sections 2 and 3, respectively. At the end of the paper we state a conjecture about the exact value of $k_r(n,m)$.
1. Main results

Suppose $1 \leq r \leq n$, let $[n] = \{1, \ldots, n\}$, and write $\binom{[n]}{r}$ for the set of $r$-subsets of $[n]$. Given a symmetric $n \times n$ matrix $A = (a_{ij})$ and a vector $x = (x_1, \ldots, x_n)$, set

$$L_r (A, x) = \sum_{\chi \in \binom{[n]}{r}} \prod_{i,j \in \chi, i < j} a_{ij} \prod_{i \in \chi} x_i.$$  

(1.1)

Define the set $\mathcal{A}(n)$ of symmetric $n \times n$ matrices $A = (a_{ij})$ by

$$\mathcal{A}(n) = \{A : a_{ii} = 0 \text{ and } 0 \leq a_{ij} = a_{ji} \leq 1 \text{ for all } i,j \in [n]\}.$$  

Our main goal is to find $\min L_r (A, x)$ subject to the constraints

$$A \in \mathcal{A}(n), \quad x \geq 0, \quad L_1 (A, x) = b, \quad L_2 (A, x) = c,$$  

where $b$ and $c$ are fixed positive numbers. Since every $L_s (A, x)$ is homogeneous of first degree in each $x_i$, for simplicity we assume that $b = 1$ and study

$$\min \{L_r (A, x) : (A, x) \in \mathcal{S}_n (c)\},$$  

where $\mathcal{S}_n (c)$ is the set of pairs $(A, x)$ defined as

$$\mathcal{S}_n (c) = \{(A, x) : A \in \mathcal{A}(n), x \geq 0, L_1 (A, x) = 1, L_2 (A, x) = c\}.$$  

Note that $\mathcal{S}_n (c)$ is compact since the functions $L_s (A, x)$ are continuous; hence $\min$ is defined whenever $\mathcal{S}_n (c)$ is nonempty. The following proposition, proved at the end of Section 2, describes when $\mathcal{S}_n (c) \neq \emptyset$.

**Proposition 1.1.** $\mathcal{S}_n (c)$ is nonempty if and only if $c < 1/2$ and $n \geq \lceil 1/(1 - 2c) \rceil$.

Hereafter we assume that $0 < c < 1/2$ and set $\xi (c) = \lceil 1/(1 - 2c) \rceil$.

To find $\min$, we solve a seemingly more general problem: for all $c \in (0, 1/2)$ and $3 \leq r \leq \xi (c) \leq n$, find

$$\varphi_r (n, c) = \min \{L_r (A, x) : r \leq k \leq n, (A, x) \in \mathcal{S}_k (c)\}.$$  

We obtain the solution of $\min$ by showing that, in fact, $\varphi_r (n, c)$ is independent of $n$.

To state $\varphi_r (n, c)$ precisely, we need some preparation. Set $s = \xi (c)$ and note that the system

$$\begin{align*}
\begin{cases}
(s - 1) x^2 + (s - 1) xy &= c, \\
(s - 1) x + y &= 1,
\end{cases}
\end{align*}$$  

(1.3)

(1.4)

has a unique solution

$$\begin{align*}
x &= \frac{1}{s} + \frac{1}{s} \sqrt{1 - \frac{2s}{s - 1} c}, \\
y &= \frac{1}{s} - \frac{1}{s} \sqrt{1 - \frac{2s}{s - 1} c}.
\end{align*}$$  

(1.5)

Write $x_s$ for the $s$-vector $(x, \ldots, x, y)$ and let $A_s \in \mathcal{A}(s)$ be the matrix with all off-diagonal entries equal to 1. Note that equations (1.3) and (1.4) give $(A_s, x_s) \in \mathcal{S}_s (c)$.

Setting $\varphi_r (c) = L_r (A_s, x_s)$, we arrive at the main result in this section.

**Theorem 1.2.** If $c \in (0, 1/2)$, $r \in \{3, 4\}$, and $r \leq \xi (c) \leq n$, then $\varphi_r (n, c) = \varphi_r (c)$.
Note first that the premise \( r \leq \xi (c) \) is not restrictive, as \( \varphi_r(n, c) = 0 \) whenever \( r > \xi (c) \). Indeed, assume that \( r > \xi (c) \) and write \( y \) for the \( r \)-vector \((x, \ldots, x, y, 0, \ldots, 0)\) whose last \( r - s \) entries are zero. Writing \( B \) for the \( r \times r \) matrix with \( A_s \) as a principal submatrix in the first \( s \) rows and with all other entries being zero, we see that \((B, y) \in S_r(c)\) and \( L_r(B, y) = 0\); hence \( \varphi_r(n, c) = 0\), as claimed.

Next, note an explicit form of \( \varphi_r(c) \):

\[
\varphi_r(c) = \left( \frac{s-1}{r} \right) x^r + \left( \frac{s-1}{r-1} \right) x^{r-1} y
\]

Thus Theorem 1.2 gives

\[
\sum_{i,j} a_{ij} = \begin{cases} 1, & \text{if } X \text{ induces an } r\text{-clique in } G, \\ 0, & \text{otherwise.} \end{cases}
\]

Hence, letting \( X = (1/n, \ldots, 1/n) \), we see that

\[
L_1(A, x) = 1, \quad L_2(A, x) = m/n^2, \quad \text{and } L_r(A, x) = k_r(G)/n^r;
\]

thus Theorem 1.2 gives

\[
k_r(n, m) \geq \frac{1}{\varphi_r(n, m/n^2)} n^r = \varphi_r(m/n^2) n^r.
\]

Now, setting \( s = \xi (m/n^2) = \left[1/ \left(1 - 2m/n^2\right)\right] \), from (1.6) we obtain an explicit form of this inequality:

\[
k_r(n, m) \geq \left( \frac{s}{r} \right) \frac{1}{s^r} \left( n - (r - 1) \sqrt{n^2 - \frac{2sm}{s - 1}} \right) \left( n + \sqrt{n^2 - \frac{2sm}{s - 1}} \right)^{r-1}.
\]

Inequality (1.7) turns out to be rather tight, as stated below and proved in Section 1.3.

**Theorem 1.3.**

\[
k_r(n, m) < \varphi_r \left( \frac{m}{n^2} \right) n^r + \frac{n^r}{n^2 - 2m},
\]

Note, in particular, that if \( m < (1/2 - \varepsilon) n^2 \), then

\[
k_r(n, m) < \varphi_r \left( \frac{m}{n^2} \right) n^r + n^{r-2}/2\varepsilon,
\]

so the order of the error term is lower than expected.
Known previous results. For $n^2/4 \leq m \leq n^2/3$, inequality (1.7) was first proved by Fisher [7]. He showed that

$$k_3 (n, m) \geq \frac{9nm - 2n^3 - 2(n^2 - 3m)^{3/2}}{27} = \varphi_3 (m/n^2) n^3,$$

and outlined an example showing that, in fact,

$$k_3 (n, m) = \varphi_3 (m/n^2) n^3 + o (n^3).$$

Recently Razborov [11] extended Fisher’s result proving that

$$k_3 (n, \lceil cn^2 \rceil) \geq \varphi_3 (c) n^3,$$

$$k_3 (n, \lceil cn^2 \rceil) = \varphi_3 (c) n^3 + o (n^3)$$

for every fixed $c \in (0, 1/2)$.

It should be noted that neither Fisher nor Razborov discussed the order of the difference $k_3 (n, \lceil cn^2 \rceil) - \varphi_3 (c) n^3$, so Theorem 1.3 is new even for $r = 3$.

2. Proof of Theorem 1.2

The following technical lemma will be used in the proof of Theorem 1.2.

Lemma 2.1. Let $0 \leq c \leq a$ and $0 \leq d \leq b$. If $0 \leq x \leq \min (a, b)$ and $0 \leq y \leq \min (c, d)$, then

$$(a - c) (b - d) + x (c + d) + y (a + b) - (x + y)^2 \geq 0.$$

Proof. Set $P = x (c + d) + y (a + b) - (x + y)^2$. Since $(a - c) (b - d) \geq 0$, we may and shall suppose that $P < 0$. By symmetry, we also suppose that $a \geq b$. If $x + y \leq b$, by $c + d \leq a + b$ we have

$$P \geq (x + y) (c + d) + y (a + b - c - d) - (x + y)^2 \geq (x + y) (c + d) - (x + y)^2;$$

hence, $P < 0$ implies that $b > c + d$ and $P \geq b (c + d) - b^2$. Now the proof is completed by

$$(a - c) (b - d) + b (c + d) - b^2 = (a - b) (b - d) + cd > 0.$$

If $x + y > b$, by $c + d \leq a + b$, we have

$$P \geq b (c + d) + y (a + b) - (b + y)^2 = b (c + d) + y (a - b) - b^2 - y^2;$$

hence, $P < 0$ implies that $\min (c, d) > a - b$ and

$$P \geq b (c + d) + \min (c, d) (a - b) - b^2 - (\min (c, d))^2.$$

If $d \geq c$, we get

$$(a - c) (b - d) + P \geq (a - c) (b - d) - b (b - d) + c (a - c) \geq (a - c) (b - d) - b (b - d) + c (b - d) = (a - b) (b - d) \geq 0.$$

If $c \geq d$, we get

$$(a - c) (b - d) + P \geq (a - c) (b - d) + b (c + d) + d (a - b) - b^2 - d^2 = a (a - b) + c (c - d) \geq 0,$$

completing the proof of Lemma 2.1.

Next we show that $\varphi_r (n, c)$ increases in $c$ whenever $\varphi_r (n, c) > 0$. 

Proposition 2.2. Let \( c \in (0, 1/2) \) and \( 3 \leq r \leq \xi (c) \leq n \). If \( \varphi_r (n, c) > 0 \) and 
\( 0 < c_0 < c \), then \( \varphi_r (n, c) > \varphi_r (n, c_0) \).

Proof. Suppose that
\[
\xi (c) \leq k \leq n, \quad (A, \mathbf{x}) \in S_k (c), \quad \text{and} \quad \varphi_r (n, c) = L_r (A, \mathbf{x}).
\]
Setting \( \alpha = c_0 / c \), we see that \( \alpha A \in A (k) \) and
\[
L_2 (\alpha A, \mathbf{x}) = \alpha L_r (A, \mathbf{x}) = c_0;
\]
thus \( (\alpha A, \mathbf{x}) \in S_k (c_0) \). Hence we obtain
\[
\varphi_r (n, c) = L_r (A, \mathbf{x}) = \alpha^{-2} L_r (\alpha A, \mathbf{x}) > L_r (\alpha A, \mathbf{x}) \geq \varphi_r (n, c_0),
\]
completing the proof of Proposition 2.2. \( \square \)

Proof of Theorem 1.2. Define a set of \( n \)-vectors \( \mathcal{X} (n) \) by
\[
\mathcal{X} (n) = \{ (x_1, \ldots, x_n) : x_1 + \cdots + x_n = 1 \text{ and } x_i \geq 0, \ 1 \leq i \leq n \};
\]
that is to say, \( \mathbf{x} \in \mathcal{X} (n) \) is equivalent to \( \mathbf{x} \geq 0 \) and \( L_1 (A, \mathbf{x}) = 1 \).

We shall prove Theorem 1.2 by contradiction; thus, assume that the conclusion of the theorem fails, that is to say, assume there exist numbers \( c, r, n, \) a vector \( \mathbf{x} = (x_1, \ldots, x_n) \), and a matrix \( A = (a_{ij}) \) such that
\[
(2.1) \quad 0 < c < 1/2, \quad 3 \leq r \leq \xi (c) \leq n, \quad (A, \mathbf{x}) \in S_n (c)
\]
and
\[
(2.2) \quad \varphi_r (n, c) = L_r (A, \mathbf{x}) < \varphi_r (c).
\]

We may and shall assume that:

(i) \( n \) is the minimum integer with this property for all \( c \in (0, 1/2) \) and \( r \in [3, \xi (c)] \);

(ii) among all pairs \( (A, \mathbf{x}) \in S_n (c) \), \( A \) has the maximum number of zero entries.

Hereafter we shall refer to this assumption as the “main assumption”. The most important consequence of the main assumption is the following.

Claim 2.3. If \( (A, \mathbf{y}) \in S_n (c) \) and \( \varphi_r (n, c) = L_r (A, \mathbf{y}) \), then \( \mathbf{y} \) has no zero entries. \( \square \)

Next we introduce some notation and conventions to simplify the presentation. For short, for every \( i, j, \ldots, k \in [n] \), set
\[
C_i = \frac{\partial L_2 (A, \mathbf{x})}{\partial x_i}, \quad C_{ij} = \frac{\partial L_2 (A, \mathbf{x})}{\partial x_i \partial x_j}, \quad D_{ij \ldots k} = \frac{\partial L_r (A, \mathbf{x})}{\partial x_i \partial x_j \cdots \partial x_k},
\]
and note that
\[
(2.3) \quad C_{ij} = a_{ij} \quad \text{and} \quad \frac{\partial L_r (A, \mathbf{x})}{\partial a_{ij}} - a_{ij} = D_{ij} x_i x_j.
\]

Letting \( \mathbf{y} = (x_1 + \Delta_1, \ldots, x_n + \Delta_n) \), Taylor’s formula gives
\[
(2.4) \quad L_2 (A, \mathbf{y}) - L_2 (A, \mathbf{x}) = \sum_{i=1}^{n} C_i \Delta_i + \sum_{1 \leq i < j \leq n} C_{ij} \Delta_i \Delta_j
\]
and
\[
(2.5) \quad L_r (A, \mathbf{y}) - L_r (A, \mathbf{x}) = \sum_{s=1}^{r} \sum_{1 \leq i_1 < \cdots < i_s \leq n} D_{i_1 \ldots i_s} \Delta_{i_1} \cdots \Delta_{i_s}.
\]
We shall use extensively Lagrange multipliers. Since $x > 0$, by Lagrange’s method, there exist $\lambda$ and $\mu$ such that

\begin{equation}
D_i = \lambda C_i + \mu
\end{equation}

for all $i \in [n]$. Likewise, if $0 < a_{ij} < 1$, we have

\[
\frac{\partial L_r (A, x)}{\partial a_{ij}} = \lambda \frac{\partial L_2 (A, x)}{\partial a_{ij}} = \lambda x_i x_j,
\]

and so, in view of (2.3),

\begin{equation}
D_{ij} = \lambda a_{ij}
\end{equation}

whenever $0 < a_{ij} < 1$.

The rest of the proof is presented in a sequence of formal claims. First we show that $\varphi_r (n, c)$ is attained on a $(0, 1)$-matrix $A$.

**Claim 2.4.** Let $(A, x) \in S_n (c)$ satisfy (2.1) and (2.2), and suppose that $A$ has the smallest number of entries $a_{ij}$ such that $0 < a_{ij} < 1$. Then $A$ is a $(0, 1)$-matrix.

**Proof.** Assume for a contradiction that $i, j \in [n]$ and $0 < a_{ij} < 1$. By symmetry we suppose that $C_i \geq C_j$.

Let

\begin{equation}
f (\alpha) = \frac{a_{ij} \alpha^2 - (C_i - C_j) \alpha}{(x_i + \alpha)(x_j - \alpha)},
\end{equation}

and suppose that $\alpha$ satisfies

\begin{equation}
0 < \alpha < x_j \text{ and } 0 \leq a_{ij} + f (\alpha) \leq 1.
\end{equation}

Let $y_\alpha = (x_1 + \Delta_1, \ldots, x_n + \Delta_n)$, where

\begin{equation}
\Delta_i = \alpha, \quad \Delta_j = -\alpha, \quad \text{and } \Delta_l = 0 \text{ for } l \in [n] \setminus \{i, j\},
\end{equation}

and define the $n \times n$ matrix $B_\alpha = (b_{ij})$ by

\begin{equation}
b_{ij} = b_{ji} = a_{ij} + f (\alpha) \quad \text{and} \quad b_{pq} = a_{pq} \text{ for } \{p, q\} \neq \{i, j\}.
\end{equation}

Note that $B_\alpha \in A (n), y_\alpha \in X (n),$ and

\[
L_2 (B_\alpha, y_\alpha) - L_2 (A, y_\alpha) = f (\alpha) \frac{\partial L_2 (A, y_\alpha)}{\partial a_{ij}} = f (\alpha) (x_i + \alpha)(x_j - \alpha).
\]

Hence, Taylor’s expansion (2.8) and equation (2.8) give

\[
L_2 (B_\alpha, y_\alpha) - L_2 (A, y_\alpha) = L_2 (A, y_\alpha) - f (\alpha) (x_i + \alpha)(x_j - \alpha) = (C_i - C_j) \alpha - a_{ij} \alpha^2 + f (\alpha) (x_i + \alpha)(x_j - \alpha) = 0;
\]

thus $(B_\alpha, y_\alpha) \in S_n (c)$.

Note also that, in view of (2.8),

\[
L_r (B_\alpha, y_\alpha) - L_r (A, y_\alpha) = \frac{\partial L_r (A, y_\alpha)}{\partial a_{ij}} f (\alpha) = f (\alpha) y_i y_j \frac{D_{ij}}{a_{ij}} = f (\alpha) (x_i + \alpha)(x_j - \alpha) \frac{D_{ij}}{a_{ij}}.
\]
Hence Taylor’s expansion \(2.5\), Lagrange’s conditions \(2.6\) and \(2.7\), and equation \(2.8\) give

\[
L_r \left( B_\alpha, y_\alpha \right) - L_r \left( A, x_\alpha \right) = L_r \left( A, y_\alpha \right) - L_r \left( A, x_\alpha \right) + f \left( \alpha \right) \left( x_i + \alpha \right) \left( x_j - \alpha \right) \frac{D_{ij}}{a_{ij}}
\]

\[
= \left( D_i - D_j \right) \alpha - D_{ij} \alpha^2 + f \left( \alpha \right) \left( x_i + \alpha \right) \left( x_j - \alpha \right) \frac{D_{ij}}{a_{ij}}
\]

\[
= \lambda \left( C_i - C_j \right) \alpha - D_{ij} \alpha^2 + f \left( \alpha \right) \left( x_i + \alpha \right) \left( x_j - \alpha \right) \frac{D_{ij}}{a_{ij}}
\]

\[
= \frac{D_{ij}}{a_{ij}} \left( \left( C_i - C_j \right) \alpha - a_{ij} \alpha^2 + a_{ij} \alpha^2 - \left( C_i - C_j \right) \alpha \right) = 0.
\]

If there exists \( \alpha \in (0, x_j) \) such that \( a_{ij} + f \left( \alpha \right) = 0 \) or \( a_{ij} + f \left( \alpha \right) = 1 \), we see that the matrix \( B_\alpha \) has fewer entries belonging to \( (0, 1) \) than \( A \), contradicting the hypothesis and completing the proof. Assume therefore that \( 0 < a_{ij} + f \left( \alpha \right) < 1 \) for all \( \alpha \in (0, x_j) \). This condition implies that

\[ a_{ij} x_j = C_i - C_j, \]

for, otherwise, \( \lim_{\alpha \to x_j} |f \left( \alpha \right)| = \infty \), and so, either \( a_{ij} + f \left( \alpha \right) = 0 \) or \( a_{ij} + f \left( \alpha \right) = 1 \) for some \( \alpha \in (0, x_j) \).

Now, extending \( f \left( \alpha \right) \) continuously for \( \alpha = x_j \) by

\[
f \left( x_j \right) = \lim_{\alpha \to x_j} f \left( \alpha \right) = \lim_{\alpha \to x_j} \frac{a_{ij} \alpha \left( \alpha - x_j \right)}{\left( x_i + \alpha \right) \left( x_j - \alpha \right)} = - \frac{a_{ij} x_j}{x_i + x_j},
\]

and defining \( y_{x_j} \) by \(2.10\) and \( B_{x_j} \) by \(2.11\), we obtain

\[
L_r \left( B_{x_j}, y_{x_j} \right) - \varphi_r \left( n, c \right) = L_r \left( B_{x_j}, y_{x_j} \right) - L_r \left( A, x_\alpha \right) = 0,
\]

contradicting Claim \(2.3\) since the \( j \)’th entry of \( y_{x_j} \) is zero. This completes the proof of Claim \(2.4\). \( \square \)

Since \( A \) is a \((0, 1)\)-matrix with a zero main diagonal, it is the adjacency matrix of some graph \( G \) with vertex set \([n]\). Write \( E \left( G \right) \) for the edge set of \( G \) and let us restate the functions \( L_r \left( A, x_\alpha \right) \) in terms of \( G \). We have

\[
L_2 \left( A, x_\alpha \right) = \sum_{ij \in E \left( G \right)} x_i x_j
\]

and, more generally,

\[
L_r \left( A, x_\alpha \right) = \sum_{i_1, \ldots, i_r \in \varphi_r \left( n, c \right)} \{ x_i_1 \cdots x_i_r : \text{the set } \{ i_1, \ldots, i_r \} \text{ induces an } r\text{-clique in } G \}.
\]

To obtain a contradiction ending the proof of Theorem \(1.2\) we show that \( G \) is a complete graph, and that \( L_r \left( A, x_\alpha \right) = \varphi_r \left( c \right) \), thus contradicting \(2.2\).

**Proof that \( G \) is a complete graph.** For convenience we first outline this part of the proof. Write \( \overline{G} \) for the complement of \( G \) and \( E \left( \overline{G} \right) \) for the edge set of \( \overline{G} \). We assume that \( G \) is not complete and reach a contradiction by the following steps:

- if \( ij \in E \left( \overline{G} \right) \), then \( C_i \neq C_j \) (Claim \(2.5\));
- \( G \) has no vertex connected to every vertex other than itself (Claim \(2.9\));
- if \( ij \in E \left( G \right) \), then \( D_{ij} < \lambda \) (Claim \(2.7\));
- \( \overline{G} \) is triangle-free (Claim \(2.8\));
- $G$ is bipartite (Claims 2.9 and 2.10);
- $G$ contains induced 4-cycles (Claim 2.11);
- $G$ contains no induced 4-cycles (Claim 2.12 and the argument that follows it).

Now let us give the details.

**Claim 2.5.** If $ij \in E(G)$, then $C_i \neq C_j$.

*Proof.* Assume that $ij \in E(G)$ and $C_i = C_j$. Let $y = (x_1 + \Delta_1, \ldots, x_n + \Delta_n)$, where
\[
\Delta_i = -x_i, \quad \Delta_j = x_i, \quad \text{and} \quad \Delta_l = 0 \text{ for } l \in [n] \setminus \{i, j\}.
\]
Clearly, $y \in \mathcal{X}(n)$, and Taylor’s expansion (2.4) gives
\[
L_2(A, y) - L_2(A, x) = C_j x_i - C_i x_i = 0;
\]
thus, $(A, y) \in S_n(c)$. Taylor’s expansion (2.5) and Lagrange’s condition (2.6) give
\[
L_r(A, y) - L_r(A, x) = D_j x_i - D_i x_i = \mu (x_i - x_i) + \lambda (C_j - C_i) x_i = 0,
\]
contradicting Claim 2.3 as the $i$’th entry of $y$ is zero. The proof of Claim 2.5 is completed. □

**Claim 2.6.** $G$ has no vertex connected to every vertex other than itself.

*Proof.* By symmetry, suppose that the vertex $n$ is connected to every vertex of $G$ other than itself. Set
\[
y = \frac{1}{1 - x_n} (x_1, \ldots, x_{n-1})
\]
and let $B$ be the principal submatrix of $A$ in the first $(n - 1)$ columns. Clearly,
\[
L_1(B, y) = \frac{x_1 + \cdots + x_{n-1}}{1 - x_n} = 1,
\]
\[
L_2(B, y) = \frac{c - x_n (1 - x_n) (1 - x_n)^2}{(1 - x_n)^2}.
\]
and
\[
L_r(A, x) = x_n L_{r-1}(B, y) (1 - x_n)^{r-1} + L_r(B, y) (1 - x_n)^r.
\]
Let
\[
c' = \frac{c - x_n (1 - x_n)}{(1 - x_n)^2} = L_2(B, y).
\]
Since $B \in A(n - 1)$, the main assumption applied to the matrix $B$ and the vector $y$ gives
\[
L_{r-1}(B, y) \geq \varphi_{r-1}(c') \quad \text{and} \quad L_r(B, y) \geq \varphi_r(c').
\]
Note that $\varphi_{r-1}(c')$ and $\varphi_r(c')$ are defined with the same $s = \xi(c')$ and the same $x, y$ given by (1.5). Thus, write $z$ for the $(n - 1)$-vector
\[
(1 - x_n) (x, \ldots, x, y, 0, \ldots, 0)
\]
whose last $n - 1 - s$ entries are zero, and define the $(n - 1) \times (n - 1)$ matrix $B' = (b_{ij})$ by
\[
b_{ij} = \begin{cases} 
1 & \text{if } 1 \leq i \leq s \text{ and } 1 \leq j \leq s \text{ and } i \neq j, \\
0 & \text{otherwise}.
\end{cases}
\]
We see that
\begin{align*}
L_1 (B', z) &= 1 - x_n, \\
L_2 (B', z) &= c' (1 - x_n)^2 = c - x_n (1 - x_n), \\
L_{r-1} (B', z) &= \varphi_{r-1} (c') (1 - x_n)^{r-1} \leq L_{r-1} (B, y) (1 - x_n)^{r-1}, \\
L_r (B', z) &= \varphi_r (c') (1 - x_n)^r \leq L_r (B, y) (1 - x_n)^r.
\end{align*}

Replace \((x_1, \ldots, x_{n-1})\) by \(z\) and write \(x'\) for the resulting vector. Also, replace the principal submatrix of \(A\) in the first \((n - 1)\) columns by \(B'\) and write \(A'\) for the resulting matrix. We easily see that
\begin{align*}
L_1 (A', x') &= L_1 (B', z) + x_n = 1, \\
L_2 (A', x') &= x_n L_1 (B', z) + L_2 (B', z) = x_n (1 - x_n) + c' (1 - x_n)^2 = c, \\
L_r (A', x') &= x_n L_{r-1} (B', z) + L_r (B', z) \\
&\leq x_n L_{r-1} (B, y) (1 - x_n)^{r-1} + L_r (B, y) (1 - x_n)^r \\
&= L_r (A, x) \geq L_r (A', x').
\end{align*}

Thus we have \((A', x') \in S_n (c)\) and so, \(L_r (A', x') = L_r (A, x)\). By the main assumption, \(z\) has no zero entries; hence, \(B'\) is the adjacency matrix of a complete graph, and so \(G\) is complete too, a contradiction ending the proof of Claim 2.6 \(\Box\)

**Claim 2.7.** If \(ij \in E (G)\), then \(D_{ij} < \lambda\).

**Proof.** Assume that \(ij \in E (G)\) and \(D_{ij} \geq \lambda\). Claim 2.4 implies that there exists a vertex \(p\) such that \(ip \in E (G)\); by Claim 2.5 suppose that \(C_i > C_p\). For every \(\alpha \in (0, x_p]\), let \(y_\alpha = (y_1, \ldots, y_n)\), where
\[
y_i = x_i + \alpha, \quad y_p = x_p - \alpha, \quad \text{and} \quad y_l = x_l \text{ for all } l \in [n] \setminus \{i, p\}.
\]

Let
\begin{equation}
(2.14) \quad f (\alpha) = \frac{(C_p - C_i) \alpha}{y_i y_j}
\end{equation}

and define the \(n \times n\) matrix \(B_\alpha = (b_{rs})\) by
\[
b_{ij} = b_{ji} = 1 + f (\alpha) \quad \text{and} \quad b_{rs} = a_{rs} \text{ for } \{r, s\} \neq \{i, j\}.
\]

For \(\alpha\) sufficiently small, \(-1 < f (\alpha) < 0\), and so \(B_\alpha \in A (n)\) and \(y_\alpha \in \mathcal{X} (n)\). Taylor’s expansion \((2.3)\) and equation \((2.14)\) give
\[
L_2 (B_\alpha, y_\alpha) - L_2 (A, x) = L_2 (B_\alpha, y_\alpha) - L_2 (A, y_\alpha) + L_2 (A, y_\alpha) - L_2 (A, x) \\
= f (\alpha) y_i y_j + \alpha (C_i - C_p) = 0;
\]

thus, \((B_\alpha, y_\alpha) \in S_n (c)\).

Since \(\{i_1, \ldots, i_{r-2}\} \cap \{i, p\} = \emptyset\), we see that
\[
D_{ij} = \sum \{x_{i_1} \cdots x_{i_{r-2}} : \{i, j, i_1, \ldots, i_{r-2}\} \text{ induces an } r\text{-clique}\}
\]
\[
= \sum \{y_{i_1} \cdots y_{i_{r-2}} : \{i, j, i_1, \ldots, i_{r-2}\} \text{ induces an } r\text{-clique}\},
\]

and so,
\[
L_r (B_\alpha, y_\alpha) - L_r (A, y_\alpha) = f (\alpha) D_{ij} y_i y_j = D_{ij} (C_p - C_i) \alpha.
\]
Hence, Taylor’s expansion (2.5), Lagrange’s condition (2.6), and equation (2.14) give
\[
L_r (B_\alpha, y_\alpha) - L_r (A, x) = L_r (B_\alpha, y_\alpha) - L_r (A, y_\alpha) + L_r (A, y_\alpha) - L_r (A, x)
= D_i \alpha - D_p \alpha + D_{ij} (C_p - C_i) \alpha
= \lambda (C_i - C_p) \alpha - D_{ij} (C_i - C_p) \alpha
= \alpha (C_i - C_p) (\lambda - D_{ij}).
\]

Since \( L_r (B_\alpha, y_\alpha) \geq L_r (A, x), \) \( \alpha (C_i - C_p) > 0, \) and \( D_{ij} \geq \lambda, \) we see that \( L_r (B_\alpha, y_\alpha) = L_r (A, x). \) If there exists \( \alpha \in (0, x_p] \) such that \( f (\alpha) = -1, \) then the \((0, 1)\)-matrix \( B_\alpha \) has more zero entries than \( A, \) contradicting the main assumption. Otherwise for \( \alpha = x_p, \) we have \( y_p = 0, \) contradicting Claim 2.9 and completing the proof of Claim 2.7. □

Claim 2.8. The graph \( \overline{G} \) is triangle-free.

Proof. Assume the assertion is false and let \( i, j, k \in [n] \) be such that \( ij, ik, jk \in E (\overline{G}) \). Let the line given by
\[
(C_i - C_k) x + (C_j - C_k) y = 0
\]
intersect the triangle formed by the lines \( x = -x_i, y = -x_j, x + y = x_k \) at some point \((\alpha, \beta).\) Let \( y = (x_1 + \Delta_1, \ldots, x_n + \Delta_n), \) where
\[
\Delta_i = \alpha, \quad \Delta_j = \beta, \quad \Delta_k = -\alpha - \beta, \quad \text{and} \quad \Delta_l = 0 \text{ for } l \in [n] \setminus \{i, j, k\}.
\]
Clearly, \( y \in \mathcal{X} (n); \) Taylor’s expansion (2.4) and equation (2.15) give
\[
L_2 (A, y) - L_2 (A, x) = C_i \alpha + C_j \beta - C_k (\alpha + \beta) = 0;
\]
thus \( (A, y) \in \mathcal{S}_n (c). \) Taylor’s expansion (2.3), Lagrange’s condition (2.6), and equation (2.15) give
\[
L_r (A, y) - L_r (A, x) = D_i \alpha + D_j \beta - D_k (\alpha + \beta)
= \mu (\alpha + \beta - \alpha - \beta) + \lambda ((C_i - C_k) \alpha + (C_j - C_k) \beta) = 0,
\]
contradicting Claim 2.9 as \( y \) has a zero entry. The proof of Claim 2.8 is completed. □

Using the following claim, we shall prove that \( \overline{G} \) is a specific bipartite graph.

Claim 2.9. Let the vertices \( i, j, k \) satisfy \( ij \in E (G), ik \in E (\overline{G}), jk \in E (\overline{G}). \) Then
\[
(C_i - C_k) (C_j - C_k) > 0.
\]

Proof. Note first that by Claim 2.8, we have \( C_i \neq C_k \) and \( C_j \neq C_k. \) Consider the hyperbola defined by
\[
(C_i - C_k) x + (C_j - C_k) y + xy = 0,
\]
and write \( H \) for its branch containing the origin. Obviously \( (C_i - C_k) (C_j - C_k) < 0 \) implies that \( \alpha \beta > 0 \) for all \( (\alpha, \beta) \in H. \)

Suppose that \( (\alpha, \beta) \in H \) is sufficiently close to the origin and let \( y = (x_1 + \Delta_1, \ldots, x_n + \Delta_n), \)
where
\[
\Delta_i = \alpha, \quad \Delta_j = \beta, \quad \Delta_k = -\alpha - \beta, \quad \text{and} \quad \Delta_l = 0 \text{ for } l \in [n] \setminus \{i, j, k\}.
\]
Clearly, \( y \in \mathcal{X}(n) \); Taylor’s expansion (2.24) and equation (2.16) give
\[
L_2(A, y) - L_2(A, x) = C_i \alpha + C_j \beta - C_k (\alpha + \beta) + \alpha \beta = 0;
\]
thus \((A, y) \in S_n(c)\). Taylor’s expansion (2.25), Lagrange’s condition (2.6), and equation (2.16) give
\[
L_r(A, y) - L_r(A, x) = D_i \alpha + D_j \beta - D_k (\alpha + \beta) + D_{ij} \alpha \beta
= \lambda(C_i \alpha + C_j \beta - C_k (\alpha + \beta)) + D_{ij} \alpha \beta = (D_{ij} - \lambda) \alpha \beta.
\]
Since \(D_{ij} < \lambda\) and \(L_r(A, y) \leq L_r(A, x)\), we see that \(\alpha \beta < 0\). Thus,
\[
(C_i - C_k)(C_j - C_k) > 0,
\]
completing the proof of Claim 2.11. \(\square\)

Claim 2.10. \(\overline{G}\) is a bipartite graph and its vertex classes \(U^+\) and \(U^-\) can be selected so that \(C_u > C_w\) for all \(u \in U^+\) and \(w \in U^-\) such that \(uw \in E(\overline{G})\).

Proof. Since \(C_i \neq C_j\) for every \(ij \in E(\overline{G})\), if \(\overline{G}\) has an odd cycle, there exist three consecutive vertices \(i, k, j\) along the cycle such that \((C_i - C_k)(C_j - C_k) < 0\). Since \(\overline{G}\) is triangle-free, \(ij \in E(G)\); hence the existence of the vertices \(i, j, k\) contradicts Claim 2.9. Thus, \(\overline{G}\) is bipartite.

Claim 2.9 implies that for every \(u \in [n]\), the value \(C_u - C_v\) has the same sign for every \(v\) such that \(uv \in E(\overline{G})\). Let \(U^+\) be the set of vertices for which this sign is positive, and let \(U^- = [n] \setminus U^+\). Clearly, for every \(uv \in E(\overline{G})\), if \(u \in U^+\), then \(v \in U^-\), and if \(u \in U^-\), then \(v \in U^+\). Hence, \(U^+\) and \(U^-\) partition properly the vertices of \(\overline{G}\), completing the proof of Claim 2.10. \(\square\)

Hereafter we suppose that the vertex classes \(U^+\) and \(U^-\) of \(\overline{G}\) are selected to satisfy the condition of Claim 2.10. Note that \(U^+\) and \(U^-\) induce complete graphs in \(G\).

For convenience, an induced 4-cycle in \(G\) will be denoted by a quadruple \((i, j, k, l)\), where \(i, j, k, l\) are the vertices of the cycle, arranged so that \(i, j \in U^+\), \(k, l \in U^-\), \(ik \notin E(G)\), and \(jl \notin E(G)\).

Claim 2.11. \(G\) contains an induced 4-cycle \((i, j, k, l)\).

Proof. Assume the assertion is false. For every vertex \(v\), write \(N(v)\) for the set of its neighbors in the vertex class opposite to its own class.

If there exist \(i, j \in U^+\) such that \(N(i) \cap N(j) \neq \emptyset\) and \(N(j) \cap N(i) \neq \emptyset\), taking \(l \in N(i) \setminus N(j)\) and \(k \in N(j) \setminus N(i)\), we see that \(i, j, k, l\) induce a 4-cycle in \(G\); thus we will assume that \(N(i) \subset N(j)\) or \(N(j) \subset N(i)\) for every \(i, j \in U^+\). This condition implies that there is a vertex \(i \in U^+\) such that \(N(v) \subset N(i)\) for every \(v \in U^+\). By symmetry, there is a vertex \(k \in U^-\) such that \(N(v) \subset N(k)\) for every \(v \in U^-\).

If \(N(i) = U^-\), then \(i\) is connected to every vertex other than itself, contradicting Claim 2.6. Hence there exists \(l \in U^- \setminus N(i)\) and, by symmetry, there exists \(j \in U^+ \setminus N(k)\). Note first that \(N(j) = \emptyset\) and \(N(l) = \emptyset\). Hence, adding the edge \(jl\) to \(E(G)\), we see that \(L_j(A, x)\) remains the same, while \(L_2(A, x)\) increases, contradicting that \(\varphi_r(n, c)\) is increasing in \(c\) (Proposition 2.22) and completing the proof of Claim 2.11. \(\square\)

Claim 2.12. If \((i, j, k, l)\) is an induced 4-cycle in \(G\), then \(D_{ij} + D_{kl} < D_{jk} + D_{ik}\).
Proof: Indeed, let \( L \) be the line defined by
\[
(2.17) \quad (C_i - C_k) x + (C_j - C_l) y = 0.
\]
Since \( i, j \in U^+ \) and \( k, l \in U^- \), we have \( C_i > C_k \) and \( C_j > C_l \); thus \( xy < 0 \) for all \((x, y) \in L\). Suppose that \( \alpha \in (0, x_k), \; \beta \in (-x_j, 0), \) and \((\alpha, \beta) \in L\). Let
\[
y_{\alpha} = (x_1 + \Delta_1, \ldots, x_n + \Delta_n),
\]
where
\[
\Delta_i = \alpha, \quad \Delta_j = \beta, \quad \Delta_k = -\alpha, \quad \Delta_l = -\beta, \quad \text{and} \quad \Delta_h = 0 \text{ for } h \in [n] \setminus \{i, j, k, l\}.
\]
Clearly, \( y_{\alpha} \in \mathcal{X}(n) \); Taylor’s expansion (2.14) and equation (2.17) give
\[
L_2 (A, y_{\alpha}) - L_2 (A, x) = (C_i - C_k) \alpha + (C_j - C_l) \beta + (C_i x_k + C_l x_j - C_k x_j - C_l x_k) \alpha \beta
\]
thus \((A, y_{\alpha}) \in \mathcal{S}_n(c)\). Taylor’s expansion (2.5), Lagrange’s condition (2.6), and equation (2.17) give
\[
L_r (A, y_{\alpha}) - L_r (A, x) = D_r \alpha + D_j \beta - D_k \alpha - D_l \beta + (D_{ij} - D_{jk} + D_{kl} - D_{li}) \alpha \beta
\]
\[
= \lambda(C_i x_k + C_j x_l - C_j x_l + C_l x_k) + (D_{ij} - D_{jk} + D_{kl} - D_{li}) \alpha \beta
\]
\[
= (D_{ij} - D_{jk} + D_{kl} - D_{li}) \alpha \beta.
\]
Since \( L_r (A, y_{\alpha}) \geq L_r (A, x) \) and \( \alpha \beta < 0 \), we find that \( D_{ij} + D_{kl} \leq D_{jk} + D_{li} \). If \( D_{ij} + D_{kl} = D_{jk} + D_{li} \), setting
\[
\alpha = \min \left\{ x_k, \frac{C_j - C_l}{C_i - C_k} x_j \right\},
\]
we see that \( L_r (A, y_{\alpha}) = L_r (A, x) \) and either the \( k \)'th or the \( j \)'th entry of \( y_{\alpha} \) is zero, contradicting Claim 2.3. Hence, \( D_{ij} + D_{kl} < D_{jk} + D_{li} \), completing the proof of Claim 2.12. \( \square \)

Select an induced 4-cycle \((i, j, k, l)\) and let us investigate \( D_{ij}, D_{kl}, D_{jk}, D_{li} \) in the light of Claim 2.12. We have
\[
D_{ij} = \sum \left\{ x_{i_1} \cdots x_{i_{r-2}} : \{i, j, i_1, \ldots, i_{r-2}\} \text{ induces an } r\text{-clique} \right\},
\]
\[
D_{kl} = \sum \left\{ x_{i_1} \cdots x_{i_{r-2}} : \{k, l, i_1, \ldots, i_{r-2}\} \text{ induces an } r\text{-clique} \right\},
\]
\[
D_{jk} = \sum \left\{ x_{i_1} \cdots x_{i_{r-2}} : \{j, k, i_1, \ldots, i_{r-2}\} \text{ induces an } r\text{-clique} \right\},
\]
\[
D_{li} = \sum \left\{ x_{i_1} \cdots x_{i_{r-2}} : \{j, k, i_1, \ldots, i_{r-2}\} \text{ induces an } r\text{-clique} \right\}.
\]
First note that if a product \( x_{i_1} \cdots x_{i_{r-2}} \) is present in any of the above sums, then
\[
\{i_1, \ldots, i_{r-2}\} \cap \{i, j, k, l\} = \emptyset.
\]
Also, a product \( x_{i_1} \cdots x_{i_{r-2}} \) is present in both \( D_{ij} \) and \( D_{kl} \) exactly when it is present in both \( D_{jk} \) and \( D_{li} \). Hence, Claim 2.12 implies that there exists a set \( \{i_1, \ldots, i_{r-2}\} \) such that either \( \{j, k, i_1, \ldots, i_{r-2}\} \) or \( \{i, l, i_1, \ldots, i_{r-2}\} \) induces an \( r \)-clique, but neither \( \{i, j, i_1, \ldots, i_{r-2}\} \) nor \( \{k, l, i_1, \ldots, i_{r-2}\} \) induces an \( r \)-clique. This is a contradiction for \( r = 3 \), as either \( \{p, i, j\} \) or \( \{p, k, l\} \) induces a triangle for every vertex \( p \notin \{i, j, k, l\} \).

Now let \( r = 4 \). We shall reach a contradiction by proving that
\[
D_{ij} + D_{kl} \geq D_{jk} + D_{li}.
\]
Let \( D_{ij}^* \) be the sum of all products \( x_{p_1} x_{p_2} \) present in \( D_{ij} \) but not present in any of \( D_{jk}, D_{kl}, D_{li} \). Defining the sums \( D_{jk}^*, D_{kl}^*, \) and \( D_{li}^* \) likewise, we see that
\[
D_{ij} + D_{kl} - D_{jk} - D_{li} = D_{ij}^* + D_{kl}^* - D_{jk}^* - D_{li}^*.
\]
so it suffices to prove that

\[ D_{ij}^* + D_{ik}^* - D_{jk}^* - D_{li}^* \geq 0. \]

To this end, write \( \Gamma(u) \) for the set of neighbors of a vertex \( u \) and set

\[
\begin{align*}
A &= \Gamma(i) \setminus \Gamma(k), \quad B = \Gamma(j) \setminus \Gamma(l), \quad X = A \cap B, \\
C &= \Gamma(k) \setminus \Gamma(i), \quad D = \Gamma(l) \setminus \Gamma(j), \quad Y = C \cap D,
\end{align*}
\]

\[
a = \sum_{p \in A} x_p, \quad b = \sum_{p \in B} x_p, \quad c = \sum_{p \in C} x_p, \quad d = \sum_{p \in D} x_p, \quad x = \sum_{p \in X} x_p, \quad y = \sum_{p \in Y} x_p.
\]

Observe that \( A, B \) and \( X \) are subsets of \( U^+ \setminus \{i, j\} \), while \( C, D \) and \( Y \) are subsets of \( U^- \setminus \{k, l\} \). For the reader’s sake, here is an alternative view on \( A, B, C, D, X, \) and \( Y \):

\[
\begin{align*}
A \setminus X &= \Gamma(i) \cap \Gamma(j) \cap \Gamma(l) \setminus \Gamma(k), \quad B \setminus X = \Gamma(i) \cap \Gamma(j) \cap \Gamma(k) \setminus \Gamma(l), \\
C \setminus Y &= \Gamma(k) \cap \Gamma(l) \cap \Gamma(j) \setminus \Gamma(i), \quad D \setminus Y = \Gamma(k) \cap \Gamma(l) \cap \Gamma(i) \setminus \Gamma(j), \\
X &= \Gamma(i) \cap \Gamma(j) \setminus (\Gamma(k) \cup \Gamma(l)), \quad Y = \Gamma(k) \cap \Gamma(l) \setminus (\Gamma(i) \cup \Gamma(j)).
\end{align*}
\]

Let the product \( x_p x_q \) be present in \( D_{jk}^* \); by definition, \( \{j, k, p, q\} \) induces a 4-clique, but neither \( \{i, j, p, q\} \) nor \( \{k, l, p, q\} \) induce a 4-clique. Clearly, \( p \) and \( q \) belong to different vertex classes of \( G \), say \( p \in U^+ \) and \( q \in U^- \). Since \( i, j, \) and \( k \) are joined to \( p \), we must have \( pl \notin E(G) \), and so \( p \in B \setminus X \); likewise we find that \( q \in C \setminus Y \). Thus

\begin{equation}
(2.18)
D_{jk}^* \leq \sum_{u \in B \setminus X} x_u \sum_{u \in C \setminus Y} x_u = (b - x)(c - y),
\end{equation}

and by symmetry,

\begin{equation}
(2.19)
D_{il}^* \leq \sum_{u \in A \setminus X} x_u \sum_{u \in D \setminus Y} x_u = (a - x)(d - y).
\end{equation}

For every pair \((p, q)\) satisfying

\[ p \in X, \ q \in B \setminus X, \quad \text{or} \quad p \in A \setminus X, \ q \in X, \quad \text{or} \quad p \in A \setminus X, \ q \in B \setminus X, \]

we see that \( \{i, j, p, q\} \) induces a 4-clique, but \( p \) is not joined to \( k \) and \( q \) is not joined to \( l \); thus \( x_p x_q \) is present in \( D_{ij}^* \). Therefore,

\begin{equation}
(2.20)
D_{ij}^* \geq \sum_{u \in X} x_u \sum_{u \in B \setminus X} x_u + \sum_{u \in A \setminus X} x_u \sum_{u \in X} x_u + \sum_{u \in X} x_u \sum_{u \in B \setminus X} x_u = ab - x^2,
\end{equation}

and by symmetry,

\begin{equation}
(2.21)
D_{kl}^* \geq \sum_{u \in Y} x_u \sum_{u \in D \setminus Y} x_u + \sum_{u \in C \setminus Y} x_u \sum_{u \in Y} x_u + \sum_{u \in Y} x_u \sum_{u \in D \setminus Y} x_u = cd - y^2.
\end{equation}

Now adding (2.20) and (2.21), and subtracting (2.18) and (2.19), we obtain

\[
D_{ij}^* + D_{kl}^* - D_{jk}^* - D_{li}^* \geq ab - x^2 + cd - y^2 - (b - x)(c - y) - (a - x)(d - y)
\]

\[
= (a - c)(b - d) + x(c + d) + y(a + b) - (x + y)^2.
\]
Hence, using \( x \leq \min(a, b), y \leq \min(c, d), \) and the inequalities
\[
a - c = \sum_{u \in \Gamma(i) \setminus \Gamma(k)} x_u + \sum_{u \in \Gamma(i) \setminus \Gamma(l)} x_u - \sum_{u \in \Gamma(k) \setminus \Gamma(i)} x_u - \sum_{u \in \Gamma(l) \setminus \Gamma(i)} x_u = C_i - C_k > 0,
\]
\[
b - d = \sum_{u \in \Gamma(j) \setminus \Gamma(l)} x_u + \sum_{u \in \Gamma(j) \setminus \Gamma(l)} x_u - \sum_{u \in \Gamma(l) \setminus \Gamma(j)} x_u - \sum_{u \in \Gamma(l) \setminus \Gamma(j)} x_u = C_j - C_l > 0,
\]
Lemma 2.1 implies that \( D_{ij}^* + D_{kl}^* - D_{jk}^* - D_{il}^* \geq 0, \) as required.

This finishes the proof that \( G \) is a complete graph for \( r = 3, 4. \)

**Proof of** \( L_r(A, x) = \varphi_r(c). \) We know now that \( G \) is a complete graph. We have to show that \( n = \xi(e) \) and \((x_1, \ldots, x_n) = (x, \ldots, x, y), \) where \( x \) and \( y \) are given by (1.3). Our proof is based on the following assertion.

**Claim 2.13.** Let \( x_3 \geq x_2 \geq x_1 > 0 \) be real numbers satisfying
\[
(2.22) \quad x_1 + x_2 + x_3 = a, \\
(2.23) \quad x_1 x_2 + x_2 x_3 + x_3 x_1 = b,
\]
and let \( x_1 x_2 x_3 \) be the minimum subject to (2.22) and (2.23). Then \( x_2 = x_3. \)

**Proof.** First note that the hypothesis implies that
\[
(2.24) \quad a^2/4 < b \leq a^2/3.
\]
Indeed, the second of these inequalities follows from Maclaurin’s inequality; assume for a contradiction that the first one fails. Then, selecting a sufficiently small \( \varepsilon > 0 \) and setting
\[
y_1 = \varepsilon, \quad y_2 = \frac{a - \varepsilon - \sqrt{(a + \varepsilon)^2 - 4(b + \varepsilon^2)}}{2}, \quad y_3 = \frac{a - \varepsilon + \sqrt{(a + \varepsilon)^2 - 4(b + \varepsilon^2)}}{2},
\]
we see that \( y_1, y_2, y_3 \) satisfy (2.22), (2.23), and
\[
y_1 y_2 y_3 = \varepsilon (b - a \varepsilon + \varepsilon^2) < \varepsilon b.
\]
Thus, \( \min x_1 x_2 x_3, \) subject to (2.22) and (2.23), cannot be attained for positive \( x_1, \) \( x_2, x_3, \) a contradiction, completing the proof of (2.24).

By Lagrange’s method there exist \( \eta \) and \( \theta \) such that
\[
x_1 x_2 = \eta + \theta (x_1 + x_2) = \eta + \theta (a - x_3), \\
x_1 x_3 = \eta + \theta (x_1 + x_3) = \eta + \theta (a - x_2), \\
x_2 x_3 = \eta + \theta (x_2 + x_3) = \eta + \theta (a - x_1).
\]
If \( \theta = 0 \) we see that \( x_1 = x_2 = x_3, \) completing the proof. Suppose \( \theta \neq 0 \) and assume for a contradiction that \( x_2 < x_3. \) We find that
\[
x_1 (x_3 - x_2) = \theta (x_3 - x_2), \\
x_2 (x_3 - x_1) = \theta (x_3 - x_1),
\]
and so, \( x_1 = x_2. \) Solving the system (2.22), (2.23) with \( x_1 = x_2, \) we obtain
\[
x_3 = \frac{a}{3} + \frac{2}{3} \sqrt{a^2 - 3b}, \quad x_1 = x_2 = \frac{a}{3} - \frac{1}{3} \sqrt{a^2 - 3b},
\]
implying that
\[
(2.25) \quad x_1 x_2 x_3 = \left(\frac{a}{3} + \frac{2}{3} \sqrt{a^2 - 3b}\right) \left(\frac{a}{3} - \frac{1}{3} \sqrt{a^2 - 3b}\right)^2.
\]
If $b = a^2/3$, we see that $x_1 = x_2 = x_3$, completing the proof, so suppose that $b < a^2/3$. We shall show that $\min x_1x_2x_3$, subject to (2.22) and (2.23), is smaller than the right-hand side of (2.25). Indeed, setting

$$y_1 = \frac{a}{3} - \frac{2}{3} \sqrt{a^2 - 3b}, \quad y_2 = y_3 = \frac{a}{3} + \frac{1}{3} \sqrt{a^2 - 3b},$$

in view of (2.24), we see that $y_1$, $y_2$, $y_3$ satisfy (2.22) and (2.23). After some algebra we obtain

$$y_1y_2y_3 - x_1x_2x_3 = -\frac{4}{27} (a^2 - 3b)^{3/2} < 0.$$

This contradiction completes the proof of Claim 2.13.

Claim 2.13 implies that, out of every three entries of $x$, the two largest ones are equal; hence all but the smallest entry of $x$ are equal. Writing $y$ and $x$ for the smallest and largest entries of $x$, we see that $y$ and $x$ satisfy

$$\left(\begin{array}{c}
\frac{n-1}{2} \end{array}\right) x^2 + nxy = c, \\
(n-1)x + y = 1, \\
y \leq x,$$

and so,

$$y = \frac{1}{n} - \sqrt{1 - \frac{2n}{n-1} c}, \quad x = \frac{1}{n} + \frac{1}{n} \sqrt{1 - \frac{2n}{n-1} c}.$$ 

Since the condition $1 - 2nc / (n-1) \geq 0$ gives

$$n \geq \frac{1}{1 - 2c},$$

and $y > 0$ gives

$$1 - 2c < \frac{1}{n} + \frac{1}{n^2} < \frac{1}{n-1},$$

we find that $n = \xi(c)$, completing the proof of Theorem 1.2.

2.1. Proof of Proposition 1.1. Suppose that $S_n(c)$ is nonempty and that

$$A \in \mathcal{A}(n), \quad x \geq 0, \quad L_1(A,x) = 1, \quad \text{and} \quad L_2(A,x) = c.$$ 

Then

$$c = \sum_{1 \leq i < j \leq n} a_{ij}x_i x_j \leq \sum_{1 \leq i < j \leq n} x_i x_j = \frac{1}{2} \left( \sum_i x_i \right)^2 - \frac{1}{2} \sum_i x_i^2 \leq \frac{n-1}{2n} < \frac{1}{2},$$

and so, $c < 1/2$ and $n \geq 1 / (1 - 2c)$; thus $n \geq \lceil 1 / (1 - 2c) \rceil$.

On the other hand, if $c < 1/2$ and $n \geq \lceil 1 / (1 - 2c) \rceil$, let $A \in \mathcal{A}(n)$ be the matrix with all off-diagonal entries equal to 1, and let $x, y$ satisfy

$$\left(\begin{array}{c}
\frac{n-1}{2} \end{array}\right) x^2 + (n-1)xy = c, \\
(n-1)x + y = 1.$$ 

Writing $x$ for the $n$-vector $(x, \ldots, x, y)$, we see that $L_1(A,x) = 1$ and $L_2(A,x) = c$; thus $S_n(c)$ is nonempty, completing the proof.
3. Upper bounds on \( k_r(n,m) \)

In this section we prove Theorem 1.3. We start with some facts about Turán graphs.

The \( s \)-partite Turán graph \( T_s(n) \) is a complete \( s \)-partite graph on \( n \) vertices with each vertex class of size \( \lfloor n/s \rfloor \) or \( \lceil n/s \rceil \). Setting \( t_s(n) = e(T_s(n)) \), after some algebra we obtain

\[
t_s(n) = s - \frac{1}{2s}n^2 - \frac{t(s-t)}{2s},
\]

where \( t \) is the remainder of \( n \) mod \( s \), and hence,

\[
s - \frac{1}{2s}n^2 - \frac{s}{8} \leq t_s(n) \leq s - \frac{1}{2s}n^2.
\]

It is known that the second one of these inequalities can be extended for all \( 2 \leq r \leq s \):

\[
k_r(T_s(n)) \leq \binom{s}{r} \left( \frac{n}{s} \right)^r.
\]

The Turán graphs play an exceptional role for the function \( k_r(n,m) \); indeed, a result of Bollobás [2] (see also, [3], Ch. 6) implies that if \( G \) is a graph with \( n \) vertices and \( t_s(n) \) edges, then \( k_r(G) \geq k_r(T_s(n)) \); hence,

**Fact 3.1.** \( k_r(n,t_s(n)) = k_r(T_s(n)) \). \( \square \)

Thus to simplify our presentation, we assume that \( n \geq s \geq r \geq 3 \) are fixed integers and \( m \) is an integer satisfying \( t_{s-1}(n) < m \leq t_s(n) \).

First we define a class of graphs \( H(n,m) \) giving upper bounds on \( k_r(n,m) \).

**The graphs** \( H(n,m) \). We shall construct a graph \( H(n,m) \) with \( n \) vertices and \( m \) edges, where \( n, s, \) and \( m \) satisfy \( n \geq s \geq 3 \) and \( t_{s-1}(n) < m \leq t_s(n) \). Note that the construction of \( H(n,m) \) is independent of \( r \).

First we define a sequence of graphs \( H_0, \ldots, H_{\lfloor n/s \rfloor} \) satisfying

\[
t_{s-1}(n) = e(H_0) < e(H_1) < \cdots < e(H_{\lfloor n/s \rfloor}) = t_s(n),
\]

and then we construct \( H(n,m) \) using \( H_0, \ldots, H_{\lfloor n/s \rfloor} \).

**The graphs** \( H_0, \ldots, H_{\lfloor n/s \rfloor} \). For every \( 0 \leq i \leq \lfloor n/s \rfloor \), let \( H_i \) be the complete \( s \)-partite graph of order \( n \) with vertex classes \( I, V_1, \ldots, V_{s-1} \) such that \( |I| = i \) and

\[
\lfloor (n-i)/(s-1) \rfloor = |V_1| \leq \cdots \leq |V_{s-1}| = \lfloor (n-i)/(s-1) \rfloor.
\]

Note that \( H_0 \) is the \((s-1)\)-partite Turán graph \( T_{s-1}(n) \), but it is convenient to consider it to be \( s \)-partite with an empty vertex class \( I \). Note also that \( H_{\lfloor n/s \rfloor} = T_s(n) \). Put in a different terminology: \( H_i \) is the join of an edgeless graph \( K_i \) with the Turán graph \( T_s(n-i) \).

It seems helpful to think that \( H_{i+1} \) is obtained from \( H_i \) as follows:

(a) select a class \( V_j \) with \( |V_j| = \lfloor (n-i)/(s-1) \rfloor \) and choose \( u \in V_j \);
(b) remove all edges joining \( u \) to vertices in \( I \);
(c) move \( u \) from \( V_j \) to \( I \), keeping all edges incident to \( u \).

In particular, we see that

\[
e(H_{i+1}) - e(H_i) = \lfloor (n-i)/(s-1) \rfloor - i > 0,
\]

implying in turn (3.3).
Constructing $H(n,m)$. Let $I, V_1, \ldots, V_{s-1}$ be the vertex classes of $H_i$. Select $V_j$ with $|V_j| = [(n-i)/(s-1)]$, select a vertex $u \in V_j$, let $l = [(n-i)/(s-1)] - 1$, and suppose that $V_j \setminus \{u\} = \{v_1, \ldots, v_l\}$. Do the following steps:

(a) remove all edges joining $u$ to vertices in $I$;
(b) move $u$ from $V_j$ to $I$, keeping all edges incident to $u$;
(c) for $m = e(H_i) + 1, \ldots, e(H_{i+1})$ join $u$ to $v_{m-e(H_i)}$ and write $H(n,m)$ for the resulting graph.

Two observations are in place: first, $e(H(n,m)) = m$, and second, $H(n,e(H_i)) = H_i$ for every $i = 1, \ldots, \lfloor n/s \rfloor$.

Note also that every additional edge in step (c) increases the number of $r$-cliques by $k_{r-2}(H')$, where $H'$ is the fixed graph induced by the set $[n] \setminus (I \cup V_j)$. We thus make the following

Claim 3.2. The function $k_r(H(n,m))$ increases linearly in $m$ for $e(H_i) \leq m \leq e(H_i)$.

We also need the following upper bound on $k_r(H_i)$.

Claim 3.3.

$$k_r(H_i) \leq \binom{s-1}{r-1} \left( \frac{n-i}{s-1} \right)^{r-1} + \binom{s-1}{r} \left( \frac{n-i}{s-1} \right)^r.$$

Proof. Let $I, V_1, \ldots, V_{s-1}$ be the vertex classes of $H_i$. Since the sizes of the sets $V_1, \ldots, V_{s-1}$ differ by at most 1, we see that the set $V_1 \cup \cdots \cup V_{s-1}$ induces the Turán graph $T_{s-1}(n-i)$. Hence a straightforward counting gives

$$k_r(H_i) \leq k_{r-1}(T_{s-1}(n-i)) + k_r(T_{s-1}(n-i)),$$

and the claim follows from inequality (3.2).

Proof of Theorem 1.3. Assume that $x$ is a real number satisfying

$$\frac{s-2}{2(s-1)} n^2 < x \leq \frac{s-1}{2s} n^2$$

and define the functions $p = p(x)$ and $q = q(x)$ by

$$p \geq q,$$

$$p \geq q,$$

$$p \geq q,$$

$$pq = x.$$

We note that

$$p(x) = \frac{1}{s} \left( n + \sqrt{n^2 - \frac{2s}{s-1} x} \right),$$

$$q(x) = \frac{1}{s} \left( n - (s-1) \sqrt{n^2 - \frac{2s}{s-1} x} \right).$$

Set

$$f(x) = \left( \frac{s-1}{r} \right)^r q + \left( \frac{s-1}{r} \right) p^r.$$
and note that \( f(x) = \varphi_t(x/n^2) n^r \); hence, to prove Theorem 1.3 it is enough to show that if
\[
\frac{s-2}{2(s-1)} n^2 < m \leq \frac{s-1}{2s} n^2,
\]
then
\[
(3.8) \quad k_r(n,m) \leq f(m) + \frac{n^r}{n^2 - 2m}.
\]

We first introduce the auxiliary function \( \tilde{f}(x) \), defined for \( x \in [t_{s-1}(n), t_s(n)] \) by
\[
\tilde{f}(x) = \begin{cases} 
 f(x + \frac{s-1}{8}), & \text{if} \ t_{s-1}(n) < x \leq \frac{s-1}{2s} n^2 - \frac{s-1}{8}; \\
 f\left(\frac{s-1}{2s} n^2\right), & \text{if} \ \frac{s-1}{2s} n^2 - \frac{s-1}{8} < x \leq t_s(n).
\end{cases}
\]

To finish the proof of Theorem 1.3 we first show that
\[
(3.9) \quad k_r(H(n,m)) \leq \tilde{f}(m),
\]
and then derive (3.8) using Taylor’s expansion and the fact that \( k_r(n,m) \leq k_r(H(n,m)) \).

**Claim 3.4.** If \( m = e(H_i) \), then
\[
k_r(H_i) \leq f\left(m - t_{s-1}(n-i) + \frac{s-2}{2(s-1)} (n-i)^2\right).
\]

**Proof.** Indeed, as mentioned above, in the graph \( H_i \), the set \( V_1 \cup \cdots \cup V_{s-1} \) induces a \( T_{s-1}(n-i) \); hence,
\[
m = i(n-i) + t_{s-1}(n-i),
\]
and so,
\[
i(n-i) + \frac{s-2}{2(s-1)} (n-i)^2 = m - t_{s-1}(n-i) + \frac{s-2}{2(s-1)} (n-i)^2.
\]
Set
\[
m_1 = m - t_{s-1}(n-i) + \frac{s-2}{2(s-1)} (n-i)^2 = i(n-i) + \frac{s-2}{2(s-1)} (n-i)^2
\]
and note that \( i = q(m_1) \). In view of Claim 3.3 we obtain
\[
k_r(H_i) \leq \left(\frac{s-1}{r-1}\right)^r \left(\frac{n-i}{s-1}\right)^{r-1} i + \left(\frac{s-1}{r}\right)^r \left(\frac{n-i}{s-1}\right)^r = f(m_1),
\]
completing the proof.

**Claim 3.5.** \( f'(x) = \binom{s-2}{r-2} p^{r-2} \).

**Proof.** From (3.7) we have
\[
f(x) = \binom{s-1}{r-1} \left(\frac{s-r}{r} p^r + p^{r-1} q\right),
\]
and so,
\[
f'(x) = \binom{s-1}{r-1} \left((s-r)p^{r-1}p' + (r-1)p^{r-2}qp' + p^{r-1}q'\right).
\]
From (3.5) and (3.6) we have

\[(s - 1)p' + q' = 0\]

and

\[(s - 1)((s - 2)pp' + p'q + pq') = (s - 1)p'(q - p) = x' = 1.\]

Now the claim follows after some algebra. □

We immediately see that \(f(x)\) is increasing. Also, since \(p(x)\) is decreasing, \(f'(x)\) is decreasing too, implying that \(f(x)\) is concave. This, in turn, implies that \(\hat{f}(x)\) is concave.

For every \(i = 1, \ldots, \lfloor n/s \rfloor\), by Claim 3.4, we have

\[k_r(H_i) \leq f(m_1) \leq \hat{f}(m),\]

and since, by Claim 3.2, \(k_r(H(n, m))\) is linear in \(m\) for \(m \in \left[e(H_i), e(H_{i+1})\right]\), inequality (3.9) follows.

To finish the proof of (3.8), note that since \(f(x)\) is concave, Taylor’s formula implies that

\[
\hat{f}(m) \leq f\left(m + \frac{s - 1}{8}\right) \leq f(m) + \frac{s - 1}{8}f'(m) = f(m) + \frac{s - 1}{8}\left(s - 2\right)p^{r-2}
\leq f(m) + \frac{s - 1}{8}\left(s - 2\right)\left(\frac{n}{s - 1}\right)^{r-2} < f(m) + sn^{r-2} \leq f(m) + \frac{n^r}{2m^2},
\]

completing the proof of Theorem 1.3. □

A conjecture about the exact value of \(k_r(n, m)\). It should be noted that Theorems 1.2 and 1.3 give only an approximation of \(k_r(n, m)\), and the exact determination of \(k_r(n, m)\) seems to require new ideas. Having defined the graphs \(H(n, m)\), we would suggest the following natural conjecture.

**Conjecture 3.6.** \(k_r(n, m) = k_r(H(n, m))\).

**Acknowledgements**

Thanks are due to Cecil Rousseau for helpful discussions, to Alex Razborov for pointing out some mistakes in the initial version of the manuscript, and to the referee for a number of suggestions.

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