Covariant feedback Control of arbitrary qudit state against Depolarizing Noise is impossible

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In this paper, we prove that it is impossible to construct a covariant quantum feedback control instruments which aim to correct the channel noise imposed on an unknown qubit. The proof is based on the searching for the optimal quantum control protocol and it turns out there exist no better complete positive and covariant quantum operations which provide a higher fidelity than the trivial Identity operators. The generalization of the investigation to bipartite entangled pure state is also included.

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The laws of quantum mechanics impose a number of no-go theorems on many Quantum Information Processing. Examples of such impossible tasks can be provided by the famous no-cloning theorem\textsuperscript{[1,2]}, no-deletion principle\textsuperscript{[3]} or by the theorem on nonexistence of the universal-NOT gates\textsuperscript{[4]}. Despite its discouraging impossibility, lots of attention have been paid to construction of the optimal and approximate quantum instruments for duplicating\textsuperscript{[5]}, deleting\textsuperscript{[6]} and complementing\textsuperscript{[7]} an unknown quantum state. These considerations are of great interest since they provide great insight into the fundamental limits on the distribution and manipulation of quantum information and can become of practical relevance for analyzing the security of quantum cryptography and quantum communications.

In this work, we give an investigation of quantum feedback control of a completely unknown qubit and provide another no-go theorem. Our main result is that, by driving the optimal and covariant correction maps for correcting errors of the completely unknown quantum pure state which was previously subjected to the depolarizing noise, there is no better other quantum operations maps that will provide a higher fidelity than the trivial Identity operators.

In the literature, counteracting the effects on quantum systems of noise imposed by environment is also a central problem for quantum information technology. In 1983, V. P. Belavkin investigated the feedback control of quantum system and proposed a theoretic framework for control model\textsuperscript{[8]}. Since this seminal work, the model of quantum feedback control has received a wide and extensive attention\textsuperscript{[9,10,11,12,13,14,15,16,17,18,19]}. The methods utilized in these models can be generally divided into two categories. One is to investigate the continuous time feedback, which includes: the stabilization of a single state of a driven and damped two-level atom\textsuperscript{[11,11]}, the maintenance of the coherence of a noisy qubit using tracking control\textsuperscript{[12]}, the state preparation and stabilization onto eigenstates of a continuous measured observable in higher-dimensional system\textsuperscript{[13]}, and more recently, the correctability of quantum subsystems under continuous dynamics evolution has also been studied\textsuperscript{[14]}.

The other is to study the problem in a discrete time setting which considerably simplifies the problem and most clearly illustrates the essential concepts. For example, in Ref.\textsuperscript{[15]}, Barnum and Knill gave a near-optimal correction procedure to recover the ensembles of orthogonal states from a general noisy process. Similar problems have also been considered in Ref.\textsuperscript{[16]} where Ticozzi et al considered the suppression of the unwanted dynamics of a single qubit by applying both dynamic decoupling and feedback methods and in Ref.\textsuperscript{[17]} where the criteria for quantum information to be perfectly corrigible and the related feedback operation is obtained. Moreover, experiments set-up on quantum control of a single photonic qubit, have already been reported\textsuperscript{[18]}.

The quantum feedback control problem is typically the following. Suppose a pure quantum state is initially prepared and later, damped and disturbed by the environmental noise. One performs some kinds of quantum operations such as measurement and feedback control on the damped state to achieve optimal or near-optimal preservation of the initial pure state. Particularly, in Ref.\textsuperscript{[19]}, the optimal control of a qubit against the dephasing channel noise is considered. It was shown there that one can apply a measure and feedback-control quantum operation and further correct, at least to some extent, the unknown qubit if it was originally prepared in one of two non-orthogonal states and subsequently subjected to the dephasing noise. However, in this paper, we find that the construction of a similar dephasing-noise-counteracting feedback control is impossible, for a completely unknown quantum pure state which resides in the arbitrary $D$-dimensional Hilbert Space $\mathcal{H}_D$.

The structure of our paper is organized as follows. We will first give a brief introduction to the quantum depolarizing channel and also the figure of metric which characterizes the optimality of our noise-counteracting maps. Then, we will exploit the group symmetry property and obtain the optimal covariant control of the unknown qudit. Finally, following such a framework, we also give a generalization from the single party quantum state to the bipartite pure state, and consider the optimal quantum recovery of maximal entanglement.
FIG. 1: Feedback control of single qudit. Suppose a pure quantum state is initially prepared and later, damped and disturbed by the environmental noise. One performs some kinds of feedback control operations to counteract the channel noise.

The noise model we are interested in is the depolarizing channel which is an important type of quantum noise[20]. Imagine we randomly take an arbitrary state $\rho$, from the entire Hilbert Space $H_D$, and with a probability $p$ that the qubit is depolarized. Described by the Complete Positive Trace Preserving (CPTP) map $D_p$, the noise can be characterized by[20]:

$$D_p(\rho) = \frac{I}{D} + (1-p)\rho. \quad (1)$$

Our main task at hand is to search for a quantum correction procedure $C$ which measures the disturbed state and later feedback-controls the state to preserve the initial pure state optimally. We give a description of the measurement and controlling scheme in Fig.1.

To continue our discussion, we need a figure of metric that quantifies the performance of our control operation. For ease, we will use the Uhlmann’s fidelity[21]. This is a good evaluation of the resemblance of two quantum state. For the original pure state $\psi$ and the corrected state $\rho' = C(D_p(\rho))$, the fidelity can be given by:

$$F_\rho = |\langle \psi | C(D_p(\rho)) | \psi \rangle|. \quad (4)$$

The problem is to look for the optimal operation $C$ which exhibits the best possible performance. According the Kraus’s representation[22], an arbitrary quantum operation $C$ can be represented with a set of CPTP maps $\{C_h\}$ with $h$ indicating the operation outcome. Furthermore, written with Kraus operator, each map can then be written with $C_h(\rho) = \sum \mu A_{h\mu} \rho A^\dagger_{h\mu}$ and can provide the result $h$ with a probability $p_h = \text{Tr} \left[ \rho \sum \mu A^\dagger_{h\mu} A_{h\mu} \right]$. Note that the overall map $C$ must be trace preserving, which imposes the constraints that

$$\int d\mu \sum \mu A^\dagger_{h\mu} A_{h\mu} = I. \quad (5)$$

In general, optimization of the quantum operation $C$ is equivalent to search for the optimal operation $\{A_{h\mu}\}$ over all CPTP maps acting on a single qubit. This is quite intractable in practice. However, by introducing an isomorphism and utilizing the group covariance property of our problem, we will find such a problem can be evaluated as a maximal eigenvalue of certain operator. In Ref.[23], Jamiołkowski established an isomorphism between the completely positive map $C_h$ on Hilbert Space $H$ and positive semi-definite operators $R_h$ on $H_{out} \otimes H_{in}$. Such a correspondence is one-to-one and its inverse can be given by

$$C_h(\rho) = \text{Tr}_{\text{in}}[(I_{out} \otimes \rho) R_h], \quad R_h = (C_h \otimes I)[I], \quad (6)$$

in which the relevant Hilbert Space $H$ is $D$-dimensional and $|I\rangle = \sum |i\rangle_{out} |i\rangle_{in}$ is the unnormalized maximal entangled state in $H_{out} \otimes H_{in}$. With such an isomorphism, the trace preserving constraints for $R_h$ should be

$$\text{Tr}_{\text{out}}[\int_h dh R_h] = I_{\text{in}}, \quad (7)$$

and the average fidelity in Eq.(4) follows:

$$F = \int d\mu \int_h dh \text{Tr}[\rho_g \otimes D_p(\rho_g)^\dagger R_h], \quad (8)$$

where $\rho_g = |\psi_g \rangle \langle \psi_g |$.

To move on, we will consider the covariant quantum measurement [25, 21, 27]. The term covariant implies that the performance of the measurement does not depend on the specific choice of input state and has been proven to be optimal in state estimation[28] and quantum cloning[24] process. We observe that such an optimality is also true with respect to our quantum control problem. This observation allows us to greatly reduce the complexity of the optimization. In fact, for an arbitrary non-covariant quantum measurement $\{R_h\}$ with
the average fidelity $F$, one can also construct a covariant instrument $\{R_h\}$:

\begin{align}
\bar{R}_0 &= \int dG |U^*_h \otimes U^*_n| R_h(U_h \otimes U_n^*), \\
\bar{R}_h &= (U_h \otimes U_n^*) \bar{R}_0(U_h^* \otimes U_n),
\end{align}

which preserve the same average fidelity as $\{R_h\}$, i.e., $\bar{F} = F$. Thus, the optimal average fidelity for covariant quantum control $\bar{R}$ is also the optimal bound for non-covariant maps. Therefore, in looking for the optimal quantum control, there will be no loss of generality if we restrict our search within covariant maps.

In the rest of the paper, we will omit the tilde “$\sim$” for clarity. By considering the covariant map in Eq. (10), the fidelity in Eq. (8) can be given by

\begin{align}
F = \int dG \text{Tr} [\rho_g \otimes D_p(\rho_g)^* R_0] \\
= \text{Tr} [\Upsilon R_0],
\end{align}

where $\Upsilon = \int dG \rho_g \otimes D_p(\rho_g)^*$ and can be easily calculated. Actually,

\begin{align}
\Upsilon &= \int dG U_g \otimes (g) \left[ |0\rangle \langle 0| \otimes \left( \frac{p}{D} I + (1-p)|0\rangle \langle 0| \right) \right] U_g^* \otimes U^\tau.
\end{align}

Using Schur’s lemma for reducible group representations, it can be obtained that

\begin{align}
\Upsilon = \frac{1-p}{D(D+1)} |I\rangle \langle I| + \frac{D+1}{D^2(D+1)} I \otimes I,
\end{align}

where $|I\rangle = \sum_{i=0}^{D-1} |i\rangle |i\rangle$.

Similarly, following the Schur’s lemma for irreducible group representations, we have the identity $\int dG U_g^{*} X U_g = \frac{1}{D} \text{Tr} [X |I\rangle \langle I|]$ and the condition Eq. (12) can be equivalently reduced to

\begin{align}
\text{Tr}[R_0] = D.
\end{align}

The optimal fidelity can now be easily cast to looking for a positive operator $R_0$ which satisfies Eq. (13) and maximizes $F$. Then, $R_0$ will provide a covariant instrument that achieves the optimal feedback control. It follows that the fidelity is bounded above by the maximum eigenvalue $\lambda_{\text{max}}$ of the operator $\Upsilon$. One can check that, for any value of $p_1$, the maximum eigenvalue and the corresponding eigenvector is given by

\begin{align}
\lambda_{\text{max}} = \frac{D(1-p) + p}{D^2}, \lambda_{\text{max}} = \frac{1}{\sqrt{D}} |I\rangle,
\end{align}

Thus, the optimal operator $R_0$ can be chosen as $R_0^{(\text{opt})} = |I\rangle \langle I|$ and the corresponding fidelity is $F^{(\text{opt})} = 1 - p + \frac{p}{D}$. From Eq. (15), it follows that the optimal feedback control can be achieved by an instrument with Kraus operators $A_g = I$, which denotes the trivial identity maps and confirms us that the optimal reversal and feed-back control is to do nothing. That is to say, there exist no better complete positive and covariant quantum operations which provide a higher average fidelity.

Before concluding our paper, it should also be noted that such a framework of deriving optimal covariant feedback control of single partite pure state can be directly generalized to the bipartite entangled cases. Bipartite entanglement state, especially, the maximal entanglement state, is a fundamental resource for quantum teleportation, quantum cryptography and for quantum communication [31]. Consider the following problem: Alice, who owns a pair of maximally entangled state $|\Psi\rangle_{AB}$, wants to establish some quantum entanglement by sending one of the entanglement particles $B$ to remote Bob via the noisy quantum channel. A question that naturally arises in this field is whether there exist nontrivial CP maps that could increase the amount of shared entanglement.

In this following, let’s simulate such a problem with a maximal entanglement state which is randomly chosen from $\mathcal{H}_D \otimes \mathcal{H}_D$ and with the depolarizing channel. Using the notation in Ref. [32], the set of all the possible maximally entangled state can also be generated from the fixed state $\frac{1}{\sqrt{D}} |I\rangle |I\rangle$:

\begin{align}
\Omega = \{ |\Psi\rangle_{AB} = \frac{1}{\sqrt{D}} U_g \otimes |I\rangle |I\rangle, U_g \in SU(D) \}.
\end{align}

The depolarizing channel, which takes effect only on the second particle $B$, can be formulated by

\begin{align}
D_p^{(B)} (|\Psi_g\rangle \langle \Psi_g|) = U_g \otimes I \left( \frac{1-p}{D} |I\rangle \langle I| + \frac{p}{D^2} I \right) U_g^\dagger \otimes I.
\end{align}

Without loss of generality, we still consider the covariant operator $\{R_h^{(\text{ent})}\} \in \mathcal{H}_{\text{out}} \otimes \mathcal{H}_{\text{in}} \triangleq \mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)} \otimes \mathcal{H}^{(3)} \otimes \mathcal{H}^{(4)}$,

\begin{align}
R_h^{(\text{ent})} = (U_h^{(1)} \otimes U_h^{(2)}) R_0^{(\text{ent})} (U_h^{(1)*} \otimes U_h^{(2)*}),
\end{align}

with the trace preserving condition $\text{Tr}[R_0] = D^2$. In this way, we have the average fidelity:

\begin{align}
F^{(\text{ent})} = \int dG \text{Tr} \left[ |\Psi_h\rangle \langle \Psi_h| \otimes D_p^{(B)} (|\Psi_h\rangle \langle \Psi_h|)^* R_0^{(\text{ent})} \right] \\
= \text{Tr} [\Xi R_0^{(\text{ent})}],
\end{align}

with $\Xi$ is a integral which can be still straightforwardly evaluated following the reducible group representation [30].

\begin{align}
\Xi = \frac{D^2 - p}{D^2(D^2 - 1)} I + \frac{1 - p}{D^2(D^2 - 1)} |I\rangle_{13} \langle I| \otimes |I\rangle_{24} \langle I| \\
- \frac{1 - p}{D^2(D^2 - 1)} \left( I^{(13)} \otimes |I\rangle_{24} \langle I| + |I\rangle_{13} \langle I| \otimes I^{(24)} \right).
\end{align}
The operator $\Xi$, whose maximal eigenvalue determines the maximal achievable fidelity can be easily calculated and we have $\lambda_{\text{max}}(\text{ent}) = 1 - p + \frac{p}{\sqrt{p^2 + 1}}$ and $|\lambda_{\text{max}}(\text{ent})\rangle = \frac{1}{\sqrt{p^2 + 1}}|I\rangle_{13}|I\rangle_{24}$. However, the maximal fidelity $F_{\text{opt}}(\text{ent}) = 1 - p - \frac{p}{\sqrt{p^2 + 1}}$ is no larger than the case when no quantum recovery operation is done. This coincides with the fact that the optimal quantum operation is $R_{0,\text{opt}}(\text{ent}) = D^2|\lambda_{\text{max}}(\text{ent})\rangle\langle\lambda_{\text{max}}(\text{ent})|$. and in Kraus’s form we have $A_0 = I^{(AB)}$.

In summary, by means of group covariant technology, we give a thorough and analytically proof of the existence of the nontrivial control schemes for depolarizing channel. The quantum states we investigated include both single quantum state and bipartite quantum entangled state. However, it should be noted that whether these results can be generalized to the multi-copy quantum state is still open. This is quite an interesting but relative intricate problem, which may require a further and systematic investigation in the future.

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