κ-Poincaré phase space: speed of massless particles and relativity of spacetime locality

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Abstract. The study of phase-space constructions based on the properties of the κ-Poincaré Hopf algebra has been a very active area, mostly because of its possible applications in the phenomenology of Planck-scale-induced momentum dependence of the speed of ultrarelativistic particles. We here summarize, with emphasis on the emerging relativity of spacetime locality, some results relevant for this research program that were recently reported in arXiv:1006.2126 (PhysRevLett106,071301) arXiv:1102.4637 (PhysLettB700,150) and arXiv:1107.1724

1. Introduction

Over the last decade there has been a strong effort [1] aimed at seeking experimental evidence of Planck-scale ($M_p \sim 10^{19}$GeV) effects which could be motivated from the study of the quantum-gravity problem. One of the most studied opportunities concerns the possibility that the speed of massless particles (photons) might have a Planck-scale-induced dependence on wavelength/momentum, as suggested by several preliminary studies in some of the alternative directions of research on quantum gravity (see, e.g., Refs. [2, 3, 4, 5]). This is of particular interest in the context of observations of gamma-ray-bursts where some of the possible scenarios for the form of this momentum (and Planck-scale) dependence could have observably-large manifestations [1, 2, 3, 6, 7, 8, 9, 10].

From the conceptual perspective the results and models that shaped this research line have been primarily analyzed for what concerns the fate of Lorentz symmetry. A momentum-dependent speed of photons (and other particles, when their mass can be neglected) can evidently be accommodated in scenarios in which Lorentz symmetry is “broken”, violating the relativity of inertial frames [2, 3] (all see, for cases however without momentum dependence of the effect, Refs. [11, 12]). Starting with the proposal put forward in Refs. [13, 14] there has also been growing interest (also see, e.g., Refs. [15, 16, 17, 18, 19, 20] and references therein) in the possibility that Lorentz symmetry might be “deformed”, preserving the relativity of inertial frames at the “cost” of modifying the laws of transformation between observers.

We here focus on this deformed-Lorentz-symmetry scenario and particularly on the possibility that some viable phenomenological models of this type could be inspired [13, 19, 20] by the mathematical structure of the κ-Poincaré Hopf algebra[21, 22, 23]. Specifically, we here summarize the results of the studies originally reported in Refs. [24, 25, 26] where phase-space constructions inspired by κ-Poincaré were used to formalize the possibility of dependence on momentum of the speed of massless particles which in leading order are linear:

$$v_{m=0} \simeq 1 - \ell |p|$$

(1)
where $\ell$ is an inverse-momentum deformation scale (usually thought to be roughly of the order of $|\ell| \sim 1/M_p$) which can take both positive or negative values (in the sense that one can contemplate both models with $\ell M_p > 0$ and models with $\ell M_p < 0$), and it is assumed that (1) would hold for $|p| \ll \ell^{-1}$.

Besides the momentum dependence of the speed of photons we shall also put in focus the implications that the relevant deformations of Lorentz symmetry have for the relativity of spacetime locality [24, 25, 27]. We shall do this first, on the basis of Refs. [24, 25], in setups which are confined to the description of free particles, and then, in the part that summarizes the findings of Ref. [26], we shall describe interacting particles within the “relative-locality framework” proposed in Ref. [28], which is centered on the geometry of momentum space.

We work throughout at leading order in the deformation scale. This keeps formulas at reasonably manageable level, sufficiently characterizes the new concepts, and would be fully sufficient for phenomenology if indeed the deformation scale is roughly of the order of the huge Planck scale (in which case a leading-order analysis should be all we need for comparison to data we could realistically imagine to gather over the next few decades).

Some of the results are also discussed specifically for 1+1-dimensional cases, where all conceptual issues here relevant are already present and can be exposed more simply.

2. Deformations of Lorentz symmetry

The notion of deformed Lorentz symmetry which we shall here adopt is the one of the proposal “DSR” (“doubly-special”, or “deformed-special”, relativity), first introduced in Refs. [13, 14]). This proposal was put forward as a possible description of preliminary theory results suggesting that there might be violations of some special-relativistic laws in certain approaches to the quantum-gravity problem, most notably the ones based on spacetime noncommutativity and loop quantum gravity. The part of the quantum-gravity community interested in those results was interpreting them as a manifestation of a full breakdown of Lorentz symmetry, with the emergence of a preferred class of observers (an “ether”). But it was argued in Ref. [13] that departures from Special Relativity governed by a high-energy/short-distance scale may well be compatible with the Relativity Principle, the principle of relativity of inertial observers, at the cost of allowing some consistent modifications of relativistic kinematics and of the Poincaré/Lorentz transformations.

The main area of investigation of the DSR proposal has been for the last decade the possibility of introducing relativistically some Planck-scale-deformed on-shell relations. The DSR proposal was put forward [13] as a conceptual path for pursuing a broader class of scenarios of interest for fundamental physics, including the possibility of introducing the second observer-independent scale primitively in spacetime structure or primitively at the level of the (deformed) de Broglie relation between wavelength and momentum. However, the bulk of the preliminary results providing encouragement for this approach came from quantum-gravity research concerning Planck-scale departures from the special-relativistic on-shell relation, and this in turn became the main focus of DSR research.

This idea of deformed Lorentz symmetry is actually very simple, as we shall here render manifest on the basis of an analogy with how the Poincaré transformations came to be adopted as a deformation of Galilean transformations. Famously, as the Maxwell formulation of electromagnetism, with an observer-independent speed scale “$c$”, gained more and more experimental support (among which we count the Michelson-Morley results) it became clear that Galilean relativistic symmetries could no longer be upheld. From a modern perspective we should see the pre-Einsteinian attempts to address that crisis (such as the ones of Lorentz) as attempts to “break Galilean invariance”, i.e. preserve the validity of Galilean transformations as laws of transformation among inertial observers, but renouncing to the possibility that those transformations be a symmetry of the laws of physics. The “ether” would be a preferred frame
for the description of the laws of physics, and the laws of physics that hold in other frames would be obtained from the ones of the preferred frame via Galilean transformations. Those attempts failed. What succeeded is completely complementary. Experimental evidence, and the analyses of Einstein (and Poincaré) led us to a “deformation of Galilean invariance”: in Special Relativity the laws of transformation among observers still are a symmetry of the laws of physics (Special Relativity is no less relativistic than Galilean Relativity), but the special-relativistic transformation laws are a $c$-deformation of the Galilean laws of transformation with the special property of achieving the observer-independence of the speed scale $c$.

This famous $c$-deformation in particular replaces the Galilean on-shell relation $E = \text{constant} + \frac{p^2}{(2m)}$ with the special-relativistic version

$$E = \sqrt{c^2 p^2 + c^4 m^2},$$

and the Galilean composition of velocities $\mathbf{u} \oplus \mathbf{v} = \mathbf{u} + \mathbf{v}$ with the special relativistic law of composition of velocities

$$\mathbf{u} \oplus_c \mathbf{v} = \frac{1}{1 + \frac{\mathbf{u} \cdot \mathbf{v}}{c^2}} \left( \mathbf{u} + \frac{1}{\gamma_u} \mathbf{v} + \frac{1}{c^2} \frac{\gamma_u}{1 + \gamma_u (\mathbf{u} \cdot \mathbf{v})} \mathbf{u} \right) \quad (2)$$

where as usual $\gamma_u \equiv 1/\sqrt{1 - \mathbf{u} \cdot \mathbf{u}/c^2}$. The richness of the velocity-composition (2) is a necessary match for the demanding task of introducing an absolute scale in a relativistic theory. And it is unfortunate that undergraduate textbooks often choose to limit the discussion to the special case of (2) which applies when $\mathbf{u}$ and $\mathbf{v}$ are collinear:

$$\mathbf{u} \oplus_c \mathbf{v} \bigg|_{\text{collinear}} = \frac{\mathbf{u} + \mathbf{v}}{1 + \frac{\mathbf{u} \cdot \mathbf{v}}{c^2}}. \quad (3)$$

The invariance of the velocity scale $c$ of course requires that boosts act non-linearly on velocity space, and this is visible not only in (2) but also in (3). But also the non-commutativity and non-associativity of (2) (which are silenced in (3)) play a central role [29, 30, 31] in the logical consistency of Special Relativity as a theory enforcing relativistically the absoluteness of the speed scale $c$. For example, the composition law (2) encodes Thomas-Wigner rotations, and in turn the relativity of simultaneity.

Equipped with this quick reminder of some features of the transition from Galilean Relativity to Special Relativity we can now quickly summarize the logical ingredients of a DSR framework. The analogy is particularly close in cases where the DSR-deformation of Lorentz symmetry is introduced primitively at the level of the on-shell relation. To see this let us consider an on-shell relation

$$m^2 = p_0^2 - p^2 + \Delta(E, \mathbf{p}; \ell) \quad (4)$$

where $\Delta$ is the deformation and $\ell$ is the deformation scale. Evidently when $\Delta \neq 0$ such an on-shell relation (4) is not Lorentz invariant. If we insist on this law and on the validity of classical (undeformed) Lorentz transformations between inertial observers we clearly end up with a preferred-frame picture, and the Principle of Relativity of inertial frames must be abandoned: the scale $\ell$ cannot be observer independent, and actually the whole form of (4) is subject to vary from one class of inertial observers to another. The other option [13] in such cases is the DSR option of enforcing the relativistic invariance of (4), preserving the relativity of inertial frames, at the cost of modifying the action of boosts on momenta. Then in such theories both the velocity scale $c$ (here mute only because of the choice of dimensions) and the energy scale $\ell$ play the same role [13, 16] of invariant scales of the relativistic theory which govern the form of boost transformations.

Several examples of boost deformations adapted in the DSR sense to modified on-shell relations
have been analyzed in some detail (see e.g. Refs. [15, 16, 17, 18, 19, 20, 32]). Clearly these DSR-deformed boosts \( N_j \) must be such that they admit the deformed shell, \( p_0^2 - p^2 + \Delta(E, p, M_*, \ell) \), as an invariant, and in turn the law of composition of momenta must also be deformed [13], \( p_\mu \oplus N k_\mu \), since it must be covariant [13, 32] under the action of the (DSR-deformed) boost transformations.

All this is evidently analogous to corresponding aspects of Galilean Relativity and Special Relativity: of course in all these cases the on-shell relation is boost invariant (but respectively under Galilean boosts, Lorentz boosts, and DSR-deformed Lorentz boosts); for Special Relativity the action of boosts evidently must depend on the speed scale \( c \) and must act non-linearly on velocities (since it must enforce observer independence of \( c \)-dependent laws), and for DSR relativity the action of boosts evidently must depend on both the speed scale \( c \) and the scale \( \ell \), acting non-linearly both on velocities and momenta, since it must enforce observer independence of \( c \)-dependent and \( \ell \)-dependent laws (the scale \( \ell \) is endowed in DSR with properties that are completely analogous to the familiar properties of \( c \); DSR-relativistic theories have both \( c \) and \( \ell \) as relativistic-invariant scales).

3. \( \kappa \)-Poincaré phase-space construction with Minkowski coordinates

Let us start, in this section, by summarizing the main results of Ref. [24], where a relativistic description of momentum dependence of the speed of massless particles was given within a classical-phase-space construction inspired by properties of the \( \kappa \)-Poincaré Hopf algebra [21, 22, 23].

At the quantum level the \( \kappa \)-Poincaré Hopf algebra is intimately related to the \( \kappa \)-Minkowski noncommutative spacetime [22, 23], on which we shall return later. But for classical-phase-space constructions one can consider standard Minkowski spacetime coordinates in combination with a description of relativistic transformations inspired by properties of the \( \kappa \)-Poincaré Hopf algebra. So in this section we assume trivial Poisson brackets for the spacetime coordinates, \( \{x, t\} = 0 \).

Actually Ref. [24] analyzed a 3-parameter family of phase spaces suitable for a DSR-relativistic implementation of the speed law (1). But in order to highlight the possible connection with the \( \kappa \)-Poincaré Hopf algebra we here specialize to the case involving the following description of the boost and translation generators in a 1+1-dimensional context:

\[
\{E, p\} = 0, \quad \{N, E\} = p, \quad \{N, p\} = E + \frac{\ell}{2} p^2, \quad (5)
\]

which is inspired by the so-called \( \text{“} \kappa \)-Poincaré bicrosproduct basis” \( \text{”} \) introduced in Ref. [22].

From (5) one sees that the on-shell relation must be of the form [24]

\[
\mathcal{C} = E^2 - p^2 + \ell E p^2. \quad (6)
\]

Following again Ref. [24] we derive the worldlines within a Hamiltonian setup. One starts introducing canonical momenta conjugate to the coordinates \( x \) and \( t \):

\[
\{p, x\} = -1, \quad \{E, t\} = 1, \quad (7)
\]

\[
\{p, t\} = 0, \quad \{E, x\} = 0. \quad (8)
\]

The on-shell relation \( \mathcal{C} \) then will play the role of Hamiltonian. Hamilton’s equations give the conservation of \( p \) and \( E \) along the worldlines

\[
\dot{p} = \frac{\partial \mathcal{C}}{\partial x} = 0, \quad \dot{E} = -\frac{\partial \mathcal{C}}{\partial t} = 0, \quad (8)
\]
where $\dot{f} \equiv \partial f / \partial \tau$ and $\tau$ is an auxiliary worldline parameter.

The worldlines can then be obtained observing that
\[
\dot{x} = -\frac{\partial C}{\partial p} \Rightarrow x(\tau) = x^{(0)} + (2p - 2\ell \gamma_2 E)p \tau
\]
\[
\dot{t} = \frac{\partial C}{\partial E} \Rightarrow t(\tau) = t^{(0)} + (2E + \ell p^2) \tau
\]
Eliminating the parameter $\tau$ and imposing the Hamiltonian constraint $C = 0$ (massless case) one finds that
\[
x = x^{(0)} + \frac{p}{\sqrt{p^2 + m^2}}(t - t^{(0)}) - \ell p(t - t^{(0)})
\]
which reproduces (1) in the limit $m \to 0$.

The compatibility between boost transformations and form of the on-shell relation is encoded in the requirement that the boost charge is conserved
\[
\dot{N} = \{C, N\} = \frac{\partial C}{\partial E} \frac{\partial N}{\partial \tau} - \frac{\partial C}{\partial p} \frac{\partial N}{\partial x} = 0
\]
This leads us to adopt the following representation of boosts
\[
N = -tp + xE + \ell x \left( E^2 + \frac{1}{2}p^2 \right)
\]
The DSR-relativistic covariance of this setup, ensured by construction, can also be verified by computing explicitly [24] the action of an infinitesimal deformed boost with rapidity vector $\xi$
\[
p' = p - \xi E - \ell \xi \left( E^2 + \frac{1}{2}p^2 \right)
\]
\[
t' = t - \xi x - 2\ell \xi Ex
\]
\[
x' = x - t\xi + \ell \xi xp
\]
Using these one easily verifies that when Alice has the particle on the worldline (10) Bob sees the particle on the worldline
\[
x' = x'^{(0)} + \frac{p}{\sqrt{m^2 + p^2}}(t' - t'^{(0)}) - \ell p'(t' - t'^{(0)})
\]
consistently with the (DSR-)relativistic nature of the framework.

Our next task is to summarize the evidence, first noticed indeed in Ref. [24], that DSR-relativistic properties of the type illustrated by this phase-space construction may lead to the novel notion of a “relativity of spacetime locality”.

Relativistically, as stressed in Ref. [24], the absoluteness of locality is codified in the fact that when one observer established that two events coincide then all other observers agree that the two events coincide.

It had been known for several years [33, 35, 34, 36] that a revision of the notion of locality is necessary in DSR-relativistic frameworks with momentum-dependent speed of massless particles. But a logically consistent picture did not arise until the study of Ref. [24] showed that the needed revision of locality does not take the form of a total breakdown of locality, but rather just the weakening of locality that intervenes in going from an absolute-locality picture to a picture in which locality is a relativistic notion, assessed relatively to a given observer.

We shall now summarize some of the results of Ref. [24] which characterize this novel notion, but before we do that let us remind our readers of the analogy they should notice with the emergence of relative simultaneity in special relativity. In Galilean relativity all velocities are relative and
time is absolute. And it is well establish that the fact that Special Relativity introduce an absolute velocity scale $c$ came at the cost of rendering simultaneity relative. It is therefore not surprising if a DSR-relativistic framework, now endowing also to the momentums scale $|\ell|^{-1}$ with the role of absolute scale then must pay the price of rendering something relative, and this is where the relativity of locality is relevant.

A quick and efficient characterization of the relativity of spacetime locality is obtained using the derivation of worldlines discussed above to obtain the results shown in Fig. 1.

![Figure 1](image-url)  
**Figure 1.** A case with two ultrasoft (red) worldlines, still transforming as if boosts were undeformed, one with ultrahard momentum $p_v = 0.1/\ell$ (violet), and one with momentum $p_b = p_v/2$ (blue). According to Alice (whose lines are solid, while boosted Bob has dashed lines) three of the worldlines give a distant coincidence of events, while two of the worldlines cross in the origin.

In figure we simply characterize “events” as crossings of worldlines (or crossing or a particle worldline with the worldline of an observer). As shown by two of the worldlines in Fig. 1, when an observer Alice is local to a coincidence of events (i.e. there are two events both occurring in the origin of her coordinate system) all observers that are purely boosted with respect to Alice, and therefore share her origin of the coordinate system, will also describe those two events as coincident. So in our DSR framework one finds that locality, a coincidence of events, preserves its objectivity if assessed by a nearby observer.

The element of nonlocality that is actually produced by DSR-deformed boosts is seen by focusing on the “burst” of three photon worldlines also shown in Fig. 1, whose crossings establish a coincidence of events for Alice far from her origin, an aspect of locality encoded in a “distant coincidence of events”. The objectivity of such distant coincidences of events is partly spoiled by the DSR deformation: as shown in Fig. 1 the coincidence is only approximately present in the coordinates of an observer boosted with respect to Alice. We should however stress that even the distant coincidence is objective up to a very good approximation if indeed $|\ell|^{-1}$ is of the order of the Planck scale (corresponding to the Planck length $\sim 10^{-35}m$). On terrestrial scales one might imagine hypothetically to observe a certain particle decay with two laboratories, with a large relative boost of, say, $\xi \sim 10^{-5}$, with idealized absolute accuracy in tracking back to the decay region the worldlines of two particles that are the decay products. As one easily checks from (14)-(15) the peculiar sort of departures from absolute locality that is codified in the “relativity of spacetime locality” which we just described has magnitude governed by $\xi Lp$. Therefore even if the distance $L$ between decay region and observers if of, say, $10^3m$ and the
decay products have momenta of, say, 100 GeV, one ends up with an apparent nonlocality of the decay region which is of only $\sim 10^{-18} m$.

Going beyond terrestrial scales another context where estimates might be interesting is the one of a typical observation of a gamma-ray burst, with particles of GeV momenta that travel for, say, $10^{17}$ s before reaching our telescopes. In such a context, for two satellite-telescopes with a relative boost of, say, $\xi \sim 10^{-4}$, the loss of coincidence of events at the source is of order $\sim 1 Km$, well below the sharpness we are able to attribute to the location of a gamma-ray burst.

We should also stress that actually in such a DSR framework two relatively boosted observers should not dwell about distant coincidences, but rather express all observables in terms of local measurements (which is anyway what should be done in a relativistic theory). For example, for the burst of three photons shown in Fig. 1 the momentum dependence of the speed of photons is objectively manifest (manifest both for Alice and Bob) in the linear correlation between arrival times and momentum of the photons (which we highlight in a panel contained in Fig. 1).

Amusingly it appears that coincident events were viewed as cumbersome already by Einstein, as shown by a footnote in the famous 1905 paper [37]:

“We shall not discuss here the imprecision inherent in the concept of simultaneity of two events taking place at (approximately) the same location, which can be removed only by abstraction.”

4. $\kappa$-Poincaré phase-space construction with “$\kappa$-Minkowski coordinates”

4.1. A $\kappa$-Minkowski coordinate velocity

Our next task is to summarize the results of Ref. [25] with a $\kappa$-Poincaré phase-space construction using “$\kappa$-Minkowski coordinates”, meaning that we shall assume “$\kappa$-Minkowski Poisson brackets” for the spacetime coordinates

$$\{x, t\} = -\ell x.$$  

We label these as “$\kappa$-Minkowski Poisson brackets” because they mimic the noncommutativity $[\hat{x}, \hat{t}] = -\ell \hat{x}$ of the $\kappa$-Minkowski spacetime [22, 23].

Notice that one goes from the standard coordinates of the previous section to the “$\kappa$-Minkowski coordinates” of (16) by using the redefinition $x \to x$, $t \to t + \ell x P$, which was already considered for other reasons in Ref. [27]. And we stress that the physical aspects of this sort of relativistic theories are all codified in the readout of clocks local to the emission of particles and (appropriately synchronized) clocks local to the detection of particles. So the redefinition $x \to x$, $t \to t + \ell x P$, must be armless: the redefinition is moot at $x = 0$ so it has no effect for the times an observer assigns to emission or detection events that she witnesses as “nearby observer” (in her origin with $x = 0$). We must expect that the results in this section will confirm the results of the previous section, although in a peculiar disguise.

The description of space and time translations is obtained from (7) taking only into account the redefinition $x \to x$, $t \to t + \ell x P$:

$$\{E, t\} = 1, \quad \{E, x\} = 0,$$

$$\{p, t\} = \ell p, \quad \{p, x\} = -1.$$  

Notice that the difference between these translations and the ones of (7) all resides in $\{p, t\} = \ell p$. But this must be taken into account consistently throughout [25] the analysis. We shall see that it brings about changes for the derivation of the “coordinate velocity”, and, as observed in Ref. [25], it also changes the form of the map between observers in such a way that the “coordinate velocity” does not coincide with the physical velocity.

In comparing the content of this section and of the previous one it is necessary to take into account of this difference of translation generators, whereas also in this section we use the same
description of boost transformations of Eq. (5) (and of course we still maintain the commutativity of translations):

\[ \{ E, p \} = 0, \quad \{ N, E \} = p, \quad \{ N, p \} = E + \ell E^2 + \frac{\ell}{2} p^2, \]  

which of course also preserves the form of the on-shell relation

\[ m^2 = E^2 - p^2 + \ell E p^2. \] (20)

It is easy to check that (16),(17),(18),(5) satisfy all Jacobi identities.

Let us use again the on-shell relation as Hamiltonian of evolution of the observables on the worldline of a particle in terms of the worldline parameter \( \tau \). Again Hamilton’s equations give the conservation of \( p \) and \( E \) along the worldlines

\[ \dot{p} = \frac{\partial C}{\partial x} = 0, \quad \dot{E} = -\frac{\partial C}{\partial t} = 0. \]

One then takes into account (18) in the derivation of the equations of motion:

\[ i = \{ C, t \} = \frac{\partial C}{\partial E} \{ E, t \} + \frac{\partial C}{\partial p} \{ p, t \} = 2E - \ell p^2, \]
\[ \dot{x} = \{ C, x \} = \frac{\partial C}{\partial E} \{ E, x \} + \frac{\partial C}{\partial p} \{ p, x \} = 2p - 2\ell Ep. \]

This evidently implies \( t(\tau) = t_0 + (2E - \ell p^2) \tau \) and \( x(\tau) = x_0 + (2p - 2\ell Ep) \tau \), from which, eliminating the parameter \( \tau \) and imposing the Hamiltonian constraint \( C = m^2 \), one finds

\[ x(p, x_0, t_0; t) = x_0 + \left( \frac{p}{\sqrt{p^2 + m^2}} - \ell p \left( 1 - \frac{p^2}{p^2 + m^2} \right) \right) (t-t_0). \]

In particular, for massless particles these worldlines give a momentum-independent coordinate velocity:

\[ x(p, x_0, t_0; t) = x_0 + (t - t_0)p/|p|. \] (21)

This momentum-independent coordinate velocity had been derived in several previous studies (see, e.g., Refs. [38, 39, 40, 41, 43]), and these studies, which were unaware of the possibility of relative locality, emphasized a possible contrast between this result of a possible analysis of classical phase spaces inspired by \( \kappa \)-Minkowski and arguments based on the quantum properties of \( \kappa \)-Minkowski spacetime, which had found [5, 42] the momentum dependence of the speed of massless particles.

4.2. What about Bob?

Awareness of the possibility of a relativity of spacetime locality, in the sense of Ref. [24] (here summarized in the previous section) immediately changes one’s perspective on the momentum independence of the coordinate velocity of massless particles found in Eq. (21).

Let us expose this issue by following the reasoning of Ref. [25].

It suffices to formalize a simultaneous emission occurring in the origin of an observer Alice. This will be described by Alice in terms of two worldlines, a massless particle with momentum \( p_1 \) (\( > 0 \)) and a massless particle with momentum \( p_2 \) (\( > 0 \)), which actually coincide because of the momentum independence of the coordinate velocity:

\[ x_{p_1}^A(t^A) = t^A, \quad x_{p_2}^A(t^A) = t^A. \] (22)
It is useful to focus on the case of \( p_1 \) and \( p_2 \) such that \( |p_1| \ll |p_2| \), and \( |\ell p_1| \simeq 0 \) (the particle with momentum \( p_1 \) is soft enough that it behaves as if \( \ell = 0 \)) while \( |\ell p_2| \neq 0 \), in the sense that for the hard particle the effects of \( \ell \)-deformation are not negligible.

A central role in our analysis is played by the translation transformations codified in (17),(18). These allow us to establish how the assignment of coordinates on points of a worldline differs between two observers connected by a generic translation \( T_{a_t,a_x} \), with component \( a_t \) along the \( t \) axis and component \( a_x \) along the \( x \) axis.

\[
x' = x - a_t \{E, x\} + a_x \{p, x\} , \\
t' = t - a_t \{E, t\} + a_x \{p, t\} .
\]

Using these we can look at the two Alice worldlines, given in (22), from the perspective of a second observer, Bob, at rest with respect to Alice at distance \( a \) from Alice (Bob = \( T_{a,a} \) Alice), local to a detector that the two particles eventually reach. Of course, since we have seen that the coordinate velocity is momentum independent, according to Alice’s coordinates the two particles reach Bob simultaneously.

But can this distant coincidence of events be trusted? The two events which according to the coordinates of distant observer Alice are coincident are the crossing of Bob’s worldline with the worldline of the particle with momentum \( p_1 \) and the crossing of Bob’s worldline with the worldline of the particle with momentum \( p_2 \). This is a coincidence of events distant from observer Alice, which in presence of the relativity of spacetime locality cannot be trusted.

To clarify the situation we should look at the two worldlines from the perspective of Bob, the observer who is local to the detection of the particles.

Evidently these Bob worldlines are obtained from Alice worldlines using the translation transformation codified in (17),(18). Acting on a generic Alice worldline \( x^A(p^A, x_0^A, t_0^A; t^A) \) this gives a Bob worldline \( x^B(p^B, x_0^B, t_0^B; t^B) \) as follows:

\[
p^B = p^A - a \{E, p^A\} + a \{p, p^A\} = p^A , \\
x_0^B = x_0^A - a \{E, x_0^A\} + a \{p, x_0^A\} = x_0^A - a , \\
t_0^B = t_0^A - a \{E, t_0^A\} + a \{p, t_0^A\} = t_0^A - a + \ell ap .
\]

And specifically for the two worldlines of our interest, given for Alice in (22), one then finds

\[
x_{p_1}^B(t^B) = t^B - \ell ap_1 \simeq t^B , \\
x_{p_2}^B(t^B) = t^B - \ell ap_2 .
\]

We have found that, because of the peculiarities of translational symmetries of the \( \kappa \)-Minkowski quantum spacetime, the two worldlines, which were coincident according to Alice, are distinct worldlines for Bob. According to Bob, who is at the detector, the two particles reach the detector at different times: \( t^B \simeq 0 \) for the soft particle and \( t^B = \ell ap_2 \) for the hard particle. And these are the two massless particles which, according to the observer Alice who is at the emitter, were emitted simultaneously.
Figure 2. Two simultaneously-emitted massless particles of different momentum in $\kappa$-Minkowski are detected at different times. The figure shows how the simultaneous emission of two such particles and their non-simultaneous detection who is at the emitter, and in the “$\kappa$-Minkowski coordinates” of observer Bob (right panel), who is at the detector.

Reassuringly this result of Ref. [25] for momentum dependence of time of detection of simultaneously-emitted massless particles,

$$\Delta t = \ell \Delta p,$$

is in perfect agreement with the result already obtained in Ref. [24] (here summarized in the previous section) and with the findings of Refs. [5, 42], which, using completely independent arguments, had found a result for the physical velocity in $\kappa$-Minkowski that is in perfect agreement (including signs and numerical factors) with these quantitative prediction for the momentum dependence of time of detection of simultaneously-emitted massless particles.

It is perhaps worth highlighting the soundness of the operative procedure by which we have determined this correlation between momentum of simultaneously-emitted particles and times of detection. Our procedure rests safely on the robust shoulders of the procedure for determining the physical velocity measured by inertial observers in classical Minkowski spacetime, and this connection is allowed by the fact that the properties of infrared massless particles (properties of massless particles in the infrared limit) are unaffected by the $\kappa$-Poincaré deformation. The distant synchronization of the clocks on emitter/Alice and on detector/Bob is evidently a special-relativistic synchronization, relying on exchanges of infrared massless particles. And we implicitly assumed that the relative rest of Alice and Bob is established by exchanges of infrared massless particles, so that indeed it can borrow from the special-relativistic operative definition of inertial observers in relative rest. By construction the distance $a$ between Alice and Bob is also defined operatively just like in special relativity, with the only peculiarity that Alice and Bob should determine it by exchanging infrared massless particles. This setup guarantees that infrared massless particles are timed and observed in $\kappa$-Minkowski exactly as in classical Minkowski spacetime. The new element of the $\kappa$-Minkowski relativistic theory, concerning the “hard” (“high-momentum”) massless particles, then also acquires a sound operative definition by our procedure centered on the simultaneous emission (with simultaneity prudently established by the local observer/emitter Alice) of an infrared and a hard massless particle, then comparing the arrival times at observer/detector Bob (times of arrival prudently established according to the local observer, indeed Bob).

In the idealized setting of a sharply flat spacetime our procedure is applicable for any arbitrarily high value of the distance $a$. But of course for most realistic applications one will

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be interested in contexts where sharp flatness of spacetime cannot a priori be assumed, and evidently in such more general cases the limit \( a \to 0 \) of our procedure should be relied upon. Note however that the characterization of observations (local) and inferences (distant) given by the coordinates of Alice and Bob, which we summarized in Fig. 2, evidently remains valid even for small values of \( a \): no matter how close Alice and Bob are, one still has that in Alice’s coordinates the detections at Bob appear to be simultaneous (while Bob, local to the detections, establishes that they are not simultaneous) and that in Bob’s coordinates the emission at Alice appears to be not simultaneous (while Alice, local to the emissions, establishes that they are simultaneous).

5. \( \kappa \)-Poincaré inspired relative-locality momentum space

5.1. The “relative-locality framework”

In the previous two sections we summarized the results that establish the momentum dependence of the speed of massless particles in certain \( \kappa \)-Poincaré inspired phase space, and we saw already a few aspects of the relativity of spacetime locality. All this however was confined to formalizations that can only accommodate free particles, which evidently is not fully satisfactory. On the other hand for nearly 15 years it was unknown how to formulate interactions on a \( \kappa \)-Poincaré phase space. One of the major obstructions for progress in this direction came from the fact that several arguments (recently reviewed in Ref. [26]) suggest that the law of composition of momenta on a \( \kappa \)-Poincaré phase space of the type we are studying should be of the type

\[
(p \oplus q)_\mu = p_\mu + q_\mu - \ell \delta_\mu^j p_\alpha q_\beta + \cdots
\]

and it was unclear how to implement this in a consistent manner.

The answer to this puzzle came only recently and in a round-about manner. First came the proposal, in Refs. [28, 45], of the “relative locality framework” which can accommodate a vast class of proposals for modifications of the on-shell relation and modifications of the law of composition of momenta, while providing indeed a coherent description of interactions among particles. Then came the realization [26] (also see Ref. [46]) that a particular application of the relative-locality framework would allow to address the longstanding issues for the description of interactions on \( \kappa \)-Poincaré phase space of the type we are studying should be of the type

where the right-hand side assumes momenta are small with respect to \(|\ell|^{-1}\) and \( \Gamma^\alpha_\mu^\beta \) are the (\( \ell \)-rescaled) connection coefficients on momentum space evaluated at \( p_\mu = 0 \).
Already in Ref. [28] an action principle was given that could govern a single-interaction process on the relative-locality momentum space

\[ S = \int ds \left( x^\mu p_\mu^J - \mathcal{N}_J \mathcal{C}^J (p) \right) - \xi^\mu \mathcal{K}_\mu \]

The bulk part of this action ends up characterizing [28] the propagation of the particles, with the Lagrange multipliers \( \mathcal{N}_J \) enforcing the on-shell relations \( \mathcal{C}^J \) in turn derived from the metric on momentum space as the distance of \( k_\mu \) from the origin of momentum space. The form of the boundary term \( \xi^\mu \mathcal{K}_\mu(s_0) \) is such that [28] the Lagrange multipliers \( \xi^\mu \) enforce the condition \( \mathcal{K}_\mu(s_0) = 0 \), so that by taking for \( \mathcal{K}_\mu \) a suitable composition of the momenta \( k_\mu \), \( p_\mu \), \( q_\mu \) the boundary terms enforces a law of conservation of momentum at the interaction.

For our purposes here, and for the work reported in Ref. [26] which we here summarize, it is important to notice that the on-shell relation we used in the previous two sections for our discussions of a \( \kappa \)-Poincaré phase space can be formulated in this framework by adopting a de Sitter metric on momentum space [46, 26]. And the law of composition of momenta (23) which is of interest for the development of this phase-construction can be faithfully described as a legitimate (but non-metric and torsionful) choice of affine connection on a de-Sitter momentum space [46, 26]. So one can rely on the powerful machinery of the relative-locality framework for introducing interactions on a \( \kappa \)-Poincaré phase space.

In the rest of this section we shall summarize the findings Ref. [26], which were based on this strategy of description of interactions on a \( \kappa \)-Poincaré phase space.

In this section we adopt the conventions of Ref. [26], which also allow an easier comparison to other results on the relative-locality framework. With respect to the conventions used in the previous two sections this consists in mapping

\[ E \rightarrow p_0 \ , \quad p \rightarrow -p_1 \ , \quad t \rightarrow x^0 \ , \quad x \rightarrow x^1 \ , \quad (24) \]

together with a sign change for Poisson brackets

\[ \{ , \} \rightarrow - \{ , \} \ . \quad (25) \]

We also often use in this section the compact notation for which \( p_\mu \equiv (p_0, p_1) \) and \( p^\mu \equiv (p^0, p^1) \), and then the map (24) turns to be convenient\(^1\) for expressing the action of momenta, when they act as generators of translations, in compact form, as in

\[ x^\mu \rightarrow x^\mu + b^\nu \{ p_\nu, x^\mu \} \ . \quad (26) \]

5.2. Causally connected interactions and translations generated by total momentum

The primary objective of the study reported in Ref. [26] was to provide a prescription for achieving translational invariance of the description, within the relative-locality framework, of cases with causally-connected interactions. It was shown in Ref. [26] that there are alternative ways to codify momentum-conservation laws in boundary conditions, but the requirement of translational invariance of the description of causally-connected interactions severely restricts the number of options that are available. Successfully finding ways to satisfy these restrictions was the main achievement reported in Ref. [26], but we shall here not focus on that issue. Rather we take the translationally-invariant description devised in Ref. [26] as an established starting point, and summarize here instead the parts of Ref. [26] which are relevant for the issues here of

\(^1\) One can also think that if \( (E, p, t, x) \equiv (p^0, p^1, x^0, t^1) \), with up indexes, the map (24) is equivalent to lower the indexes of the momenta with a Lorentzian metric \( \{ -1, 1 \} \).
interest concerning the speed of massless particles on \(\kappa\)-Poincaré phase spaces and the relativity of spacetime locality.

For this we only need the part of Ref. [26] which concerned a single pair of causally-connected interactions, as in the case we here show in Fig. 3.

Following Ref. [26] we describe the case shown in Fig. 3 through the action

\[
\mathcal{S}^{(2)} = \int_{-\infty}^{s_0} ds \left( \mathcal{L}_{\mu}(k_\mu - \ell z^1 \dot{k}_0 + \mathcal{N}_k \mathcal{C}_\kappa[k]) \right) + \int_{s_0}^{s_1} ds \left( \mathcal{L}_{\mu}(\dot{p}_\mu - \ell x^1 p_0 + \mathcal{N}_p \mathcal{C}_\kappa[p]) \right) + \int_{s_1}^{+\infty} ds \left( \mathcal{L}_{\mu}(\dot{p}_\mu - \ell x^1 p_0 + \mathcal{N}_p \mathcal{C}_\kappa[p]) \right)
\]

\[
+ \int_{s_0}^{+\infty} ds \left( \mathcal{L}_{\mu}(\dot{q}_\mu - \ell y^1 q_0 + \mathcal{N}_q \mathcal{C}_\kappa[q]) \right) - \xi_{\mu} \mathcal{K}_{\mu}^{[0]}(s_0) - \xi_{\mu} \mathcal{K}_{\mu}^{[1]}(s_1),
\]

(27)

where the boundary terms at endpoints of worldlines a given, following the prescription of Ref. [26], in terms of

\[
\mathcal{K}^{[0]} = k - (p \oplus q)
\]

\[
\mathcal{K}^{[1]} = (p \oplus q) - (p' \oplus p'' \oplus q),
\]

i.e.

\[
\mathcal{K}^{[0]}_{\mu} = k_\mu - (p \oplus q)_\mu = k_\mu - p_\mu - q_\mu - \ell \delta^{1}_{\mu} p_0 q_0,
\]

\[
\mathcal{K}^{[1]}_{\mu} = (p \oplus q)_\mu - (p' \oplus p'' \oplus q)_\mu = p_\mu - p'_\mu - p''_\mu - \ell \delta^{1}_{\mu} (-q_0 p_1 + p'_0 q_0 + p''_0 q_1 + p''_0 q_1).
\]

(28)

This formulation ensures the availability of a relativistic description of distant observers, i.e. the invariance under translation transformations.

In order to see that this is the case it suffices to note down the equations of motion (and constraints) that follow from the action \(\mathcal{S}^{(2)}\):

\[
\dot{p}_\mu = 0, \quad \dot{q}_\mu = 0, \quad \dot{k}_\mu = 0, \quad \dot{p}'_\mu = 0, \quad \dot{p}''_\mu = 0,
\]

\[
\mathcal{C}_\kappa[p] = 0, \quad \mathcal{C}_\kappa[q] = 0, \quad \mathcal{C}_\kappa[k] = 0, \quad \mathcal{C}_\kappa[p'] = 0, \quad \mathcal{C}_\kappa[p''] = 0,
\]

\[
\mathcal{K}^{[0]}_{\mu}(s_0) = 0, \quad \mathcal{K}^{[1]}_{\mu}(s_1) = 0.
\]

(29)
\[ \dot{x}^\mu = N_p \left( \frac{\delta C_\kappa}{\delta p_\mu} + \ell \delta_0^\mu \frac{\delta C_\kappa}{\delta p_1} \right) = \delta_0^\mu N_p \left( 2p_0 - \ell p_1 \right) - 2 \delta_1^\mu N_p \left( p_1 - \ell p_0 p_1 \right), \]
\[ \dot{y}^\mu = N_q \left( \frac{\delta C_\kappa}{\delta q_\mu} + \ell \delta_0^\mu \frac{\delta C_\kappa}{\delta q_1} \right) = \delta_0^\mu N_q \left( 2q_0 - \ell q_1 \right) - 2 \delta_1^\mu N_q \left( q_1 - \ell q_0 q_1 \right), \]
\[ \dot{z}^\mu = N_k \left( \frac{\delta C_\kappa}{\delta k_\mu} + \ell \delta_0^\mu \frac{\delta C_\kappa}{\delta k_1} \right) = \delta_0^\mu N_k \left( 2k_0 - \ell k_1 \right) - 2 \delta_1^\mu N_k \left( k_1 - \ell k_0 k_1 \right), \]
\[ \dot{x}^\nu = N_p \left( \frac{\delta C_n}{\delta p_\mu} + \ell \delta_0^\nu \frac{\delta C_n}{\delta p_1} \right) = \delta_0^\nu N_p \left( 2p_0 - \ell p_1^2 \right) - 2 \delta_1^\nu N_p \left( p_1^1 - \ell p_0 p_1^1 \right), \]
\[ \dot{x}^\nu = N_p \left( \frac{\delta C_n}{\delta p_\mu} + \ell \delta_0^\nu \frac{\delta C_n}{\delta p_1} \right) = \delta_0^\nu N_p \left( 2p_0 - \ell p_1^2 \right) - 2 \delta_1^\nu N_p \left( p_1^1 - \ell p_0 p_1^1 \right), \] and the conditions at the \( s = s_0 \) and \( s = s_1 \) boundaries:

\[ z^\mu (s_0) = \xi^\mu \left( \frac{\delta K^{00}}{\delta k^\mu} + \ell \delta_0^\mu \frac{\delta K^{00}}{\delta k_1} \right) = \xi^\mu_0 + \ell \delta_0^\mu \xi^1_0 k_1, \]
\[ x^\mu (s_0) = -\xi^\nu \left( \frac{\delta K^{00}}{\delta p^\mu} + \ell \delta_0^\mu \frac{\delta K^{00}}{\delta p_1} \right) = \xi^\mu_0 + \ell \delta_0^\mu \xi^1_0 (p_1 + q_1), \]
\[ x^\mu (s_1) = \xi^\nu \left( \frac{\delta K^{11}}{\delta p^\mu} + \ell \delta_0^\mu \frac{\delta K^{11}}{\delta p_1} \right) = \xi^\mu_1 + \ell \delta_0^\mu \xi^1_1 (p_1 + q_1), \]
\[ y^\mu (s_0) = -\xi^\nu \left( \frac{\delta K^{00}}{\delta q^\mu} + \ell \delta_0^\mu \frac{\delta K^{00}}{\delta q_1} \right) = \xi^\mu_0 + \ell \delta_0^\mu \xi^1_0 q_1 + \ell \delta_0^\mu \xi_1^0 p_0, \]
\[ x^\mu (s_1) = -\xi^\nu \left( \frac{\delta K^{11}}{\delta q^\mu} + \ell \delta_0^\mu \frac{\delta K^{11}}{\delta q_1} \right) = \xi^\mu_1 + \ell \delta_0^\mu \xi^1_1 (p_1^1 + q_1), \]
\[ x^\mu (s_1) = -\xi^\nu \left( \frac{\delta K^{11}}{\delta p^\mu} + \ell \delta_0^\mu \frac{\delta K^{11}}{\delta p_1} \right) = \xi^\mu_1 + \ell \delta_0^\mu \xi^1_1 (p_1^1 + q_1) + \ell \delta_1^\nu \xi^1_1 p_0. \]

One then can easily check \cite{26} that the following translation transformations, generated by the total momentum,

\[ z^0_B (s) = z^0_A (s) + b^\mu \{ k^\mu, z^0 \} = z^0_A (s) - b^1 k_1, \]
\[ z^1_B (s) = z^1_A (s) + b^\mu \{ k^\mu, z^1 \} = z^1_A (s) - b^1, \]
\[ x^0_B (s) = x^0_A (s) + b^\mu \{ p^\mu + q^\mu, x^0 \} = x^0_A (s) - b^1 (p_1 + q_1), \]
\[ x^1_B (s) = x^1_A (s) + b^\mu \{ p^\mu + q^\mu, x^1 \} = x^1_A (s) - b^1, \]
\[ y^0_B (s) = y^0_A (s) + b^\mu \{ p^\mu + q^\mu, y^0 \} = y^0_A (s) - b^1 (-q_1), \]
\[ y^1_B (s) = y^1_A (s) + b^\mu \{ p^\mu + q^\mu, y^1 \} = y^1_A (s) - b^1 (-q_1), \]

leave the equations of motion (30) unchanged and leave the boundary conditions (32) unchanged.
5.3. Free-particle limit
In relation to the discussion we offered in the previous two sections, an important observation reported in Ref. [26] concerns the fact that in the relative-locality framework a particle is still “essentially free” when its interactions only involve exchanges of very small fractions of its momentum. So in this limit applications of the relative-locality framework to the case of a $\kappa$-Poincaré-inspired momentum space should reproduce the results we summarized in the previous two sections. This is indeed what was found in Ref. [26], following a strategy of analysis that shall now revisit. One can consider the situation shown in Figure 4.

![Figure 4](image)

**Figure 4.** Schematics of a pion decaying into a soft and a hard photon, with the hard photon ultimately detected through an interaction in which it exchanges a small part of its momentum with a particle in a detector (hard worldlines in solid blue, soft worldlines in dotted red).

To give some tangibility to the situation shown in Figure 4 one could view, for example, the incoming blue worldline $p,x$ as a highly boosted pion, which decays producing a hard photon $(p',x')$ and a very soft photon $(k,z)$; then the hard photon propagates freely until it exchanges a small amount of momentum with a soft particle $(q,y)$. One can then ask if and how, for fixed time of emission (first interaction), the time of detection (second interaction) of the hard photon depends on its momentum $p'$.

An action which is suitable for the relative-locality-framework description of the process shown in Figure 4 is [26]

$$S^{\kappa(2)} = \int_{s_0}^{+\infty} ds \left( z^\mu k_\mu - \ell z^1 k_z + N_k C_\kappa [k] \right) + \int_{-\infty}^{s_0} ds \left( x^\mu p_\mu - \ell x^1 p_z + N_p C_\kappa [p] \right) + \int_{s_1}^{+\infty} ds \left( x'^\mu p''_\mu - \ell x'^1 p''_z + N_p C_\kappa [p''] \right) + \int_{-\infty}^{s_1} ds \left( y^\mu q_\mu - \ell y^1 q_z + N_q C_\kappa [q] \right) + \int_{s_1}^{+\infty} ds \left( y'^\mu q'_\mu - \ell y'^1 q'_z + N_q C_\kappa [q'] \right) - \epsilon_\mu^{\kappa(0)}(s_0) - \epsilon_\mu^{\kappa[1]}(s_1),$$

(33)

where

$$K_\mu^{[0]}(s_0) = (q \oplus p)_\mu - (q \oplus p' \oplus k)_\mu = p_\mu - p'_\mu - k_\mu - \ell \delta_\mu^1 (-q_0 p_1 + q_0 p'_1 + q_0 k_1 + p'_0 k_1),$$

$$K_\mu^{[1]}(s_1) = (q \oplus p' \oplus k)_\mu - (p'' \oplus q' \oplus k)_\mu = q_\mu + p'_\mu - p''_\mu - q'_\mu - \ell \delta_\mu^1 (-q_0 p'_1 - q_0 k_1 - p''_0 k_1 + p''_0 k_1 + q'_0 k_1 + q'_0 k_1).$$

(34)
From this action one obtains easily the equations of motion and the constraints,

\[ \dot{p}_\mu = 0, \quad \dot{q}_\mu = 0, \quad \dot{k}_\mu = 0, \quad \dot{p}_\mu' = 0, \quad \dot{p}_\mu'' = 0, \]

\[ C_\nu [p] = 0, \quad C_\nu [q] = 0, \quad C_\nu [k] = 0, \quad C_\nu [p'] = 0, \quad C_\nu [p''] = 0, \]

\[ \dot{z}^\mu - N_k \left( \frac{\delta C_\nu [k]}{\delta k_\mu} + \ell \delta_0^\mu \frac{\delta C_\nu [k]}{\delta q_1} k_1 \right) = 0, \quad \dot{y}^\mu - N_q \left( \frac{\delta C_\nu [q]}{\delta q_\mu} + \ell \delta_0^\mu \frac{\delta C_\nu [q]}{\delta q_1} q_1 \right) = 0, \]

\[ \dot{x}^\mu - N_p \left( \frac{\delta C_\nu [p]}{\delta p_\mu} + \ell \delta_0^\mu \frac{\delta C_\nu [p]}{\delta p_1} p_1 \right) = 0, \quad \dot{x}^{\nu'} - N_{p'} \left( \frac{\delta C_\nu [p']}{\delta p'_\mu} + \ell \delta_0^\nu \frac{\delta C_\nu [p']}{\delta p'_1} p'_1 \right) = 0, \]

and the boundary conditions:

\[ z^\mu (s_0) = -\xi_0^\nu \left( \frac{\delta K_\nu [0]}{\delta k_\mu} + \ell \delta_0^\nu \frac{\delta K_\nu [0]}{\delta q_1} k_1 \right), \]

\[ x^\mu (s_0) = \xi_0^\nu \left( \frac{\delta K_\nu [0]}{\delta p_\mu} + \ell \delta_0^\nu \frac{\delta K_\nu [0]}{\delta p_1} p_1 \right), \]

\[ x^{\nu'} (s_0) = -\xi_0^\nu \left( \frac{\delta K_\nu [0]}{\delta p'_\mu} + \ell \delta_0^\nu \frac{\delta K_\nu [0]}{\delta p'_1} p'_1 \right), \]

\[ x^{\nu'} (s_1) = \xi_1^\nu \left( \frac{\delta K_\nu [1]}{\delta p'_\mu} + \ell \delta_0^\nu \frac{\delta K_\nu [1]}{\delta p'_1} p'_1 \right), \]

\[ y^{\mu'} (s_1) = \xi_{1'}^\mu \left( \frac{\delta K_{\mu'} [1]}{\delta q_{\mu'}} + \ell \delta_0^\mu \frac{\delta K_{\mu'} [1]}{\delta q_{1'}} q_{1'} \right), \]

\[ y^{\nu'} (s_1) = -\xi_{1'}^\nu \left( \frac{\delta K_{\nu'} [1]}{\delta q_{\nu'}} + \ell \delta_0^\nu \frac{\delta K_{\nu'} [1]}{\delta q_{1'}} q_{1'} \right). \]

Evidently here the issue of interest is primarily contained in the dependence of the time of detection at a given detector of simultaneously-emitted particles on the momenta of the particles and on the specific properties of the interactions involved in the analysis. We start by noticing that for the particle of worldline \( x^\nu \), we have

\[ x^1 (s) = x^1 (\bar{s}) + v^1 (x^0 (s) - x^0 (\bar{s})), \]

which in the massless case (and whenever \( m/p_1^2 \ll |\ell p_1| \)) takes the simple form

\[ x^1 (s) = x^1 (\bar{s}) - \frac{p_1}{|p_1|} (x^0 (s) - x^0 (\bar{s})). \]

In obtaining (36) we used the on-shell relation

\[ p_0 = \sqrt{p_1^2 + m^2} = \frac{\ell}{2} p_1, \]

and the fact that for \( m/p_1^2 \ll |\ell p_1| \) (consistently again with our choice of conventions, which is such that \( v^1 > 0 \iff p^1 < 0 \))

\[ v^1 = \frac{x^1}{x^0} = -\frac{p_1}{p_0} (1 - \ell p_0 + \frac{\ell p_1^2}{2p_0}) = -\frac{p_1}{|p_1|}. \]
As in the previous section we have here momentum-independent coordinate speeds for massless particles, so in particular according to Alice’s coordinates two massless particles of momenta $p_A^0$ and $p_B^0$ simultaneously emitted at Alice (in Alice’s spacetime origin) appear to reach detector Bob simultaneously, apparently establishing a coincidence of detection events. But once again the presence of relative locality evidently requires that in order to establish the dependence of the time of detection on the momentum of the massless particles we must again transform the relevant worldlines to the corresponding description by an observer Bob local to the detection. Let us then return to the two-interaction process of Fig. 4 and take as our hard massless particle of momentum $p_B^0$ the particle in that process which we had originally labeled as having momentum $p'_1$. For the process of Fig. 4 the description of the transformation from Alice’s to Bob’s worldlines is [26]

\[
\begin{align*}
  z_B^0(s) &= z_A^0(s) + b'^{(q + p' + k)}(s), \\
  z_B^1(s) &= z_A^1(s) + b'^{(q + p' + k)}(s), \\
  x_B^0(s) &= x_A^0(s) + b'^{(q + p)\mu}(s), \\
  x_B^1(s) &= x_A^1(s) + b'^{(q + p)\mu}(s), \\
  y_B^0(s) &= y_A^0(s) + b'^{(p + q + k)\mu}(s), \\
  y_B^1(s) &= y_A^1(s) + b'^{(p + q + k)\mu}(s).
\end{align*}
\]

Using these transformation laws it is easy to recognize that, having dropped the negligible “soft terms” from small momenta, indeed we are obtaining results that are fully consistent with the ones obtained in the Hamiltonian description of free particles. To see this explicitly let us consider the situation where, simultaneously to the interaction emitting the hard particle $x', p'$ in Alice origin, we also have the emission of a soft photon $x_s, p_s$.

And as observer Bob let us take one who is reached in its spacetime origin in Alice origin, we also have the emission of a soft photon $x_s, p_s$.

From a relative-locality perspective the setup we are arranging is such that “Alice is an emitter” (the spatial origin of Alice’s coordinate system is an ideally compact, infinitely small, emitter) and “Bob is a detector” (the spatial origin of Bob’s coordinate system is an ideally compact, infinitely small, detector). The two worldlines we focus on, a soft and a hard worldline, both originate from Alice’s spacetime origin (they are both emitted by Alice, in the spatial origin of Alice’s frame of reference, and both at time $t_{Alice} = 0$) and both end up being detected by Bob, but, while by construction the soft particle reaches Bob’s spacetime origin, the time at which the hard particle reaches Bob spatial origin is to be determined.

Reasoning as usual at first order in $\ell$, it is easy to verify that Bob describes the “interaction coordinate” $\xi_B^{[1]}(s)$ of the interaction at $s = s_1$ as coincident with the $s = s_1$ endpoints of the worldlines $x_B^0, x_B^1; x_B^0, x'_B, p_B, q, y, q', y'$:

\[
\xi_B^{[1]} = x_B^0(s_1) = x_B^1(s_1) = y_B^0(s_1) = y_B^1(s_1).
\]
We take into account that there are no relative-locality effects in the description given by Bob whenever an interaction occurs "in the vicinity of Bob": our leading-order analysis assumes the available sensitivity is sufficient to expose manifestations of relativity of locality of order \( \ell p_h L \) (where \( L \) is the distance from the interaction-event to the origin of the observer and \( p_h \) is a "suitably high" momentum), with \( L \) set in this case by the distance Alice-Bob, so even a hard-particle interaction which is at a distance \( d \) from the origin of Bob will be treated as absolutely local by Bob if \( d \ll L \).

According to this both "detection events" are absolutely local for observer Bob: of course this is true for the event of detection of the soft photon \( x_s, p_s \) (which we did not even specify since its softness ensures us of its absolute locality) and it is also true for the interaction-event of "detection near Bob" of the hard particle \( x', p' \). Ultimately this allows us to handle the time component of the coordinate fourvector (38) as the actual delay that Bob measures between the two detection times:

\[
\Delta t = \xi_B^{[1]}(s_1) = x_B^0(s_1) = x'^0_B(s_1) = y_B^0(s_1) = y'_B^0(s_1).
\]

From the equations (37) relative to the worldline \( x', p' \), it follows that

\[
x_A^1(s_1) = x_B^1(s_1) + b^1 = b^1,
\]

from which, considering the worldlines (36), it follows that (assuming indeed \( m/|p_1|^2 \ll |\ell p'_1| \)) Alice "sees" the \( s = s_1 \) endpoint of the worldline \( x', p' \) at the coordinates

\[
x_A^{\mu}(s_1) = x_B^{\mu}(s_1) + b^\mu = b^\mu = (b, b).
\]

And then, from the equations (37) and (39), it follows that Bob measures the delay [26]

\[
\Delta t = x_B^0(s_1) = x_A^0(s_1) - b^0 - \ell b^1 p'_1 = \ell b|p'_1|,
\]

in agreement with the results we summarized in the previous two sections. The strategy of this analysis is schematically reported in Figure 5.

### 5.4. Role of “hard interactions”

The results of Ref. [26] which we summarized in the previous subsection concerned the limit in which a particle exchanged among interactions propagates “essentially free”, originating from an interaction in which it is the only outgoing hard particle and ending into an interaction in which it is the only incoming hard particle. We shall now summarize the corresponding results of Ref. [26] which concern more general cases. Our main objective in this subsection is to highlight the results of Ref. [26] which establish a possible dependence of certain aspects of the propagation of a particle on the actual emission and detection interaction which that particle connects.

For these purposes it suffices to modify the analysis summarized in the previous subsection in rather minor way: the new features become evident when [26] at least one of the interactions (emission and/or detection) involves at least 3 hard particles in total, among in and out particles. As an example of this situation, Ref. [26] considers the case of a ultraenergetic particle at rest decaying into two particles, both hard, one of which is the particle detected at our observatory.

As shown in Fig. 6 the analysis can be arranged in exactly the same way as in the previous subsection, with a trivalent vertex for the emission interaction and a four-valent vertex for the detection. And the kinematics at the four-valent vertex is left unchanged, involving a soft particle in the in state and a soft particle in the out state. What changes in this case is the kinematics of the emission vertex, now assuming that all particles involved are hard.
Figure 5. Schematic description of the time delay discussed in this subsection. The various agents in the analysis are shown both as described by Alice (left panel) and as described by Bob (right panel). These are spacetime graphs (in a 2D spacetime) showing the actual worldlines of particles. In addition to the two hard interactions we are considering (qualitatively described already in Figure 4), we also show (as the orange-dotted worldline) a soft photon going from Alice’s origin to Bob’s origin. As shown in the figure the derivation is arranged so that all emissions and detections occur in the spatial origin of either Alice or Bob (but because of the relative locality according to Alice the hard detections at/near Bob would be nonlocal interactions and according to Bob the hard emissions at Alice would be nonlocal processes). We also show, as the bulky green dots, the results obtained in Ref. [26] for the formal positions of the interaction points, as coded in the formal “interaction coordinates” $\xi^\mu$.

Figure 6. Schematic description of a case where a hard ultrarelativistic particle originating from a hard emission interaction (one hard particle in, two hard particles out) is detected in a soft interaction (only one hard particle in and only one hard particle out). Solid-blue lines are for hard particles, dashed-red lines are for soft particles.

In this case it is actually important to consider at least two possibilities [26], since this allows to put in focus some of the relevant issues. The first possibility is encoded in the following choice...
of boundary conditions at endpoints of worldlines:

\[
K^{[0]}_{\mu}(s_0) = (q \otimes p)_\mu - (q \otimes p' \otimes k)_\mu = p_\mu - p'_\mu - k_\mu - \ell \delta^1_\mu(-q_0 p_1 + q_0 p'_1 + q_0 k_1 + p_0 k_1),
\]

\[
K^{[1]}_{\mu}(s_1) = (q \otimes p' \otimes k)_\mu - (p'' \otimes q' \otimes k)_\mu
= q_\mu + p'_\mu - p''_\mu - q'_\mu - \ell \delta^1_\mu(-q_0 p'_1 - q_0 k_1 - p'_0 k_1 + p''_0 q'_1 + p''_0 k_1 + q_0 k_1).
\]

The worldlines seen by observer/detector Bob, distant from the emission, that follow from this choice of boundary terms have already been given in Eq. (36). The main difference between the situation in the previous subsection and this situation is that the “primary”, the particle incoming to the emission interaction, is at rest, with \( p_1 = 0 \), which also implies that the two outgoing particles of the emission interaction must both be hard. For the worldlines involved in the emission interaction this leads to

\[
x^0_B(s) = x^0_A(s) + b^\mu((q \otimes p)_\mu, x^0) = x^0_A(s) - b^0 - \ell b^1 p_1 = x^0_A(s) - b^0,
\]

\[
x^1_B(s) = x^1_A(s) + b^\mu((q \otimes p')_\mu, x^1) = x^1_A(s) - b^1 - \ell q_0 \simeq x^1_A(s) - b^1,
\]

\[
x^0_B(s) = x^0_A(s) + b^\mu((q \otimes p' \otimes k)_\mu, x^0) = x^0_A(s) - b^0 - \ell b^1(k_1 + p'_1) = x^0_A(s) - b^0,
\]

\[
x^1_B(s) = x^1_A(s) + b^\mu((q \otimes p' \otimes k)_\mu, x^1) = x^1_A(s) - b^1 - \ell q_0 \simeq x^1_A(s) - b^1,
\]

\[
z^0_B(s) = z^0_A(s) + b^\mu((q \otimes p' \otimes k)_\mu, z^0) = z^0_A(s) - b^0 - \ell b^1 k_1,
\]

\[
z^1_B(s) = z^1_A(s) + b^\mu((q \otimes p' \otimes k)_\mu, z^1) = z^1_A(s) - b^1 - \ell(p'_0 + q_0) \simeq z^1_A(s) - b^1 - \ell b^1 p'_0.
\]

And from this one easily sees that the particle \( p', x' \), the particle then detected at Bob, translates classically, without any deformation term. So this time we have that no momentum dependence of the times of arrival is predicted

\[ t_{\text{detection}} = x^0_B(s_1) = x^0_A(s_1) - b^0 = 0. \]

As announced, in this case one must however consider at least one alternative possibility for the choice of boundary conditions. As observed in Ref. [26] in cases, like the emission interaction we are presently contemplating, in which an interaction involves only hard particles the noncommutativity of the composition law can play a highly non-trivial role. As a result there is particular interest in analyzing [26] the following alternative choice of \( K \)'s for the boundary terms

\[
K^{[0]}_{\mu} = (p + q)_\mu - (k + p' + q)_\mu = p_\mu - p'_\mu - k_\mu - \ell \delta^1_\mu(-p_0 q_1 + k_0 p'_1 + k_0 q_1 + p'_0 q_1),
\]

\[
K^{[1]}_{\mu} = (k + p' \otimes q)_\mu - (k + p'' \otimes q')_\mu
= p'_\mu + q_\mu - p''_\mu - q'_\mu - \ell \delta^1_\mu(-k_0 p'_1 - k_0 q_1 - p'_0 q_1 + k_0 p''_1 + k_0 q'_1 + p''_0 q'_1).
\]

Focusing again on the worldline \( x', p' \) detected at Bob one then finds [26]

\[
x^0_B(s) = x^0_A(s) + b^\mu((k \otimes p' \otimes q)_\mu, x^0) = x^0_A(s) - b^0 - \ell b^1(q_1 + p'_1) \simeq x^0_A(s) - b^0 - \ell b^1 p'_1,
\]

\[
x^1_B(s) = x^1_A(s) + b^\mu((k \otimes p' \otimes q)_\mu, x^1) = x^1_A(s) - b^1 - \ell b^1 k_0.
\]

And from the equation of motion (36) one now deduces that

\[ x^1_B(s) = x^0_B(s) - \ell b^1(k_0 - p'_1), \]
which in turn implies that the time of detection at Bob of the particle with worldline $x', p'$ is [26]

$$t_{\text{detection}} = x_B(0) = -\ell b^1 (p'_1 - k_0) = 2\ell b^1 |p'_1| .$$

(47)

The dependence of the time of detection on the momentum of the particle being detected is back! And this dependence is twice as strong as the dependence on momentum found in the previous subsection!

So these results of Ref. [26] expose, within the relative-locality framework, a rather striking aspect of the dependence of the propagation of a particle on the actual emission and detection interaction which that particle connects. For what concerns times of detection of simultaneously emitted massless particles of momentum $p'_1$, emitted from a source at a distance $L$ from the detector one encounters 3 situations:

- **Case A** the emission interaction involves only one hard incoming particle and one hard outgoing particle, all other particles in the emission interaction being soft:
  the times of arrival have a dependence on momentum governed by

  $$t_{\text{detection}} = \ell L |p'_1|$$

  and this result is independent of the position occupied by the momentum $p'_\mu$ in our noncommutative composition law

- **Case B** the emission interaction is the decay of a ultra-high-energy particle at rest, involves a total of 3 hard particles, and the momentum $p'_\mu$ appears in the composition of momenta to the left of a hard particle:
  the times of arrival have no dependence on momentum

  $$t_{\text{detection}} = 0$$

- **Case C** the emission interaction is the decay of a ultra-high-energy particle at rest, involves a total of 3 hard particles, and the momentum $p'_\mu$ appears in the composition of momenta to the right of a hard particle:
  the times of arrival have the following dependence on momentum

  $$t_{\text{detection}} = 2\ell L |p'_1|$$

(twice as large as in the case A).

6. Closing remarks

We have here summarized some of the recent results [24, 25, 26] which further contributed to establishing the possibility of relying on $\kappa$-Poincaré phase spaces for the description of a momentum dependence of the speed of massless particles. Awareness of the possibility of a relativity of spacetime locality evidently proves to be a key resource from this perspective. And the recently-proposed relative-locality framework has remarkable potentialities for further empowering this approach (and many other approaches). It will be interesting to explore the implications of a possible relativity of spacetime locality for other scenarios being considered from a DSR/deformed-Lorentz-symmetry perspective (see, e.g., Refs. [19, 20, 50, 51]).

The fact that effects of the type here discussed are subjectable to experimental testing, even if $|\ell|^{-1}$ is as big as the Planck scale [1, 2, 3, 6, 7, 8, 9, 10], must remain the main goal of this research programme, even though it is amusing that in ongoing analyses of the anomaly tentatively reported by the OPERA collaboration [52] some of the concepts that were here relevant are contributing to valuable clarifications [53, 54, 55].

Specifically for what concerns $\kappa$-Poincaré-inspired research future studies should give high priority to the exploration of formulations that are alternative to the one adopted in
Refs. [24, 25, 26], which are here summarized. Refs. [24, 25, 26] all worked throughout consistently inspired by the so-called time-to-the-right ordering convention [22], which is by far the choice most frequently preferred in the literature. But a lot remains to be understood concerning the relationship between this ordering prescription and other ordering prescription, such as the ones discussed in Refs. [56, 57].

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