A discrete extension of $\Gamma_{1,p}^0(2)$ in $\text{Sp}(4, \mathbb{R})$ and the modular form of the Barth-Nieto quintic

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Abstract
In this note we construct a maximal discrete extension of $\Gamma_{1,p}^0(2)$, the paramodular group with a full level-2 structure. The corresponding Siegel variety parametrizes (birationally) the space of Kummer surfaces associated to $(1, p)$-polarized abelian surfaces with a level-2 structure. In the case $p = 3$ this is related to the Barth-Nieto quintic and in this case we also determine the space of cusp forms of weight 3.

1 Introduction
Let $p \geq 3$ prime. The paramodular group $\Gamma_{1,p}^0$ is defined as the subgroup

$$\Gamma_{1,p}^0 = \left\{ g \in \text{Sp}(4, \mathbb{Q}); \, g \in \left( \begin{array}{cccc} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \end{array} \right) \right\}$$

of $\text{Sp}(4, \mathbb{Q})$. This group acts on the Siegel upper halfspace

$$\mathbb{H}_2 := \{ \tau \in \text{Mat}(2, \mathbb{C}); \, \tau = ^t\tau, \, \text{Im} \tau > 0 \}$$

by

$$(A \, B \vert C \, D) : \begin{cases} \mathbb{H}_2 \rightarrow \mathbb{H}_2 \\ \tau \mapsto (A\tau + B)(C\tau + D)^{-1} \end{cases}$$

(where $A, B, C, D$ are $2 \times 2$-blocks). With this action, $\mathcal{A}_{1,p}^0 := \Gamma_{1,p}^0 \backslash \mathbb{H}_2$ is the moduli space of $(1, p)$-polarized abelian surfaces. Likewise, one obtains the moduli space of $(1, p)$-polarized abelian surfaces with level 2 structure $\mathcal{A}_{1,p}^0(2)$ by dividing $\mathbb{H}_2$ by the action of

$$\Gamma_{1,p}^0(2) = \left\{ g \in \Gamma_{1,p}^0; \, g - 1_4 \in \left( \begin{array}{cccc} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \end{array} \right) \right\}.$$
The group $\Gamma_{0,1,p}$ is conjugate via $R_p := \text{diag}(1,1,1,p)$ to the symplectic group $\tilde{\Gamma}_{0,1,p} = \text{Sp}(\Lambda_p, \mathbb{Z}) := \{ g \in \text{GL}(4, \mathbb{Z}) \ ; \ g \Lambda_p \cdot g = \Lambda_p \}$, where $\Lambda_p$ is the symplectic form $\Lambda_p = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & p \\ -1 & 0 & 0 & 0 \\ 0 & -p & 0 & 0 \end{pmatrix}$.

Under this isomorphism, $\Gamma_{0,1,p}(2)$ is identified with the group $\tilde{\Gamma}_{0,1,p}(2)$ consisting of all elements $g$ in $\text{Sp}(\Lambda_p, \mathbb{Z})$ with $g \equiv \mathbf{1}_4 \pmod{2}$. The case $p = 3$ is of special interest. Barth and Nieto showed in [BN] that the quintic $N = \left\{ \sum_{i=0}^{5} u_i = \sum_{i=0}^{5} \frac{1}{u_i} = 0 \right\} \subset \mathbb{P}^5$ parametrizes birationally the space of Kummer surfaces associated to abelian surfaces with $(1,3)$-polarization and a level 2 structure. Moreover, $N$ has a smooth model which is Calabi-Yau. From this, Barth and Nieto deduced that the space $A_{0,3}(2)$ also has a smooth model that is Calabi-Yau. So one may ask to determine the (up to a scalar) weight 3 cusp form with respect to the modular group $\Gamma_{0,1,p}(2)$. This was done in [GH1]. The cusp form in question was shown to be $\Delta_3^1$, where $\Delta_1$ is a cusp form of weight 1 with respect to the paramodular group $\Gamma_{0,3}$ with a character of order 6.

In [GH2] Gritsenko and Hulek showed that the Kummer surfaces which are associated to a $(1,p)$-polarized abelian and to its dual are isomorphic. This turns our attention to the Fricke-involution, which extends $\Gamma_{0,1,p}$ to $\Gamma_{1,1,p}$ and identifies a polarized abelian surface with its dual. So it is a natural question to ask if $\Gamma_{0,1,p}(2)$ can be extended uniquely to a group $\Gamma_{1,1,p}(2)$ in such a way that the diagram

$$\begin{array}{ccc}
\Gamma_{0,1,p}(2) & \hookrightarrow & \Gamma_{1,1,p}(2) \\
\downarrow & & \downarrow \\
\Gamma_{0,1,p} & \hookrightarrow & \Gamma_{1,1,p}
\end{array}$$

commutes. If so, is $\Delta_3^1$ still a cusp form with respect to $\Gamma_{1,3}(2)$? We will give answers to this question in this note.

2 The maximal discrete extension $\Gamma_{1,1,p}$ of $\Gamma_{0,1,p}$ in $\text{Sp}(4, \mathbb{R})$

A maximal discrete extension $\Gamma_{1,1,p}$ of $\Gamma_{0,1,p}$ in $\text{Sp}(4, \mathbb{R})$ is defined in [GH2]. This group still acts on $\mathbb{H}_2$ and the quotient $\Gamma_{1,1,p} \backslash \mathbb{H}_2$ has a moduli theoretic
meaning: $\Gamma^*_{1,p} \backslash \mathbb{H}_2$ is birationally the moduli space of Kummer surfaces associated to abelian surfaces with a $(1, p)$-polarisation (for details see [GH2]).

We will construct an extension of $\tilde{\Gamma}^{*}_{1,p}(2)$ in a natural way. For this, we summarize the construction of $\Gamma^*_{1,p}$. Let $x, y \in \mathbb{Z}$ with $xp - y = 1$ and consider the matrix

$$\hat{V}_p = \begin{pmatrix} px & -1 & 0 & 0 \\ -yp & p & 0 & 0 \\ 0 & 0 & p & yp \\ 0 & 0 & 1 & px \end{pmatrix}.$$ 

Let

$$V_p = \frac{1}{\sqrt{p}} \hat{V}_p \in \text{Sp}(4, \mathbb{R}).$$

Then it is easy to see that $V_p^2 \in \Gamma^{0}_{1,p}$ and $V_p \Gamma^{0}_{1,p} V_p = \Gamma^{0}_{1,p}$. So the matrix $V_p$ defines an involution modulo $\Gamma^{0}_{1,p}$ and $\Gamma^*_{1,p} := \langle \Gamma^{0}_{1,p}, V_p \rangle$ is a normal extension of $\Gamma^{0}_{1,p}$ with index 2. By [K] this is the only non-trivial discrete extension of $\Gamma^{0}_{1,p}$ in $\text{Sp}(4, \mathbb{R})$.

With

$$\bar{V}_p = \frac{1}{\sqrt{p}} \begin{pmatrix} 0 & 1 & 0 & 0 \\ p & 0 & 0 & 0 \\ 0 & 0 & 0 & p \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

the coset $V_p \Gamma^{0}_{1,p}$ can also be written as

$$V_p \Gamma^{0}_{1,p} = \bar{V}_p \Gamma^{0}_{1,p}.$$ 

To understand how $\bar{V}_p$ acts on $\mathcal{A}^{0}_{1,p}(2)$ let $E = \text{diag}(1, p)$ and $\tau = (\tau_1 \tau_2) \in \mathbb{H}_2$ be a point corresponding to the $(1, p)$-polarized abelian surface $X = \mathbb{C}^2 / L$, where the lattice $L$ is given by the normalized period matrix $\Omega = (E, \tau)$ and the hermitian form $H$, defining the polarization of $X$, is given by $(\text{Im} \tau)^{-1}$ with respect to the standard basis of $\mathbb{C}^2$. The polarization $H$ defines an isogeny

$$\lambda_H : \left\{ \begin{aligned} X & \to \widehat{A} = \text{Pic}^0 A \\ x & \mapsto T_x^* \mathcal{L} \otimes \mathcal{L}^{-1} \end{aligned} \right.$$ 

where $\mathcal{L}$ is a line bundle, which represents the polarization $H$ and $T_x$ is the translation by $x$. The map $\lambda_H$ depends only on the polarization, not on the choice of the line bundle $\mathcal{L}$. The kernel $\ker \lambda_H$ is (non-canonically) isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$, so this defines a quotient map

$$\lambda_p : X \to X / \ker \lambda_H = \widehat{X},$$

where $\widehat{X}$ is the dual abelian surface of $X$, which corresponds to the period matrix

$$\Omega' = \begin{pmatrix} p & 0 & \tau_1 & \tau_2 \\ 0 & 1 & \tau_2 & \tau_3/p \end{pmatrix}.$$ 

The identity

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p & 0 & \tau_1 & \tau_2 \\ 0 & 1 & \tau_2 & \tau_3/p \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \bar{V}_p(\tau)$$
Lemma 1

Let \( \varphi(p) : \begin{cases} A_{1,p}^0 & \to A_{1,p}^0 \\ (X,H) & \mapsto (\hat{X},\hat{H}) \end{cases} \)

which maps an abelian surface to its dual.

Since it is easier to work with matrices with entries in \( \mathbb{Z} \), let us consider

\[ \tilde{W}_p = R_p \hat{V}_p R_p^{-1} = \frac{1}{\sqrt{p}} \begin{pmatrix} 0 & 1 & 0 & 0 \\ p & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \]

We define \( \Gamma^*_{1,p} := \langle \hat{\Gamma}_{1,p}^0, \tilde{W}_p \rangle \). Recall that \( \hat{\Gamma}_{1,p}^0(2) \) is the kernel of the surjection \( \pi : \hat{\Gamma}_{1,p}^0 \to \text{Sp}(4,\mathbb{Z}_2) \approx S_6 \), \( \pi(M) = \overline{M} \). (Here and henceforth we write \( \overline{M} \) for reduction modulo 2 of an integer-valued matrix \( M \).

**Lemma 1** Let \( \iota = \begin{pmatrix} (1 & 0) \\ (0 & 1) \end{pmatrix} \).

The map

\[ \pi^* \begin{cases} \hat{\Gamma}_{1,p}^0 & \to \text{Sp}(4,\mathbb{Z}_2) \\ g & \mapsto \begin{cases} \pi(g) & \text{if } g \in \hat{\Gamma}_{1,p}^0 \\ \pi(g \cdot \tilde{W}_p) \cdot \iota & \text{if } g \in \hat{\Gamma}_{1,p}^0 \setminus \hat{\Gamma}_{1,p}^0 \end{cases} \end{cases} \]

is a homomorphism which extends the map \( \pi \).

**Proof.** It is easy to see that the equation

\[ \iota \cdot \pi^*(\tilde{W}_p \cdot g \cdot \tilde{W}_p) \cdot \iota = \pi^*(g) \quad \text{for all } g \in \hat{\Gamma}_{1,p}^0 \] (*)

holds. Namely, let \( g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \hat{\Gamma}_{1,p}^0 \), \( A = (a_{ij})_{1 \leq i,j \leq 2}, \ldots, D = (d_{ij})_{1 \leq i,j \leq 2} \).

By [HKW, Proposition I.1.16] we have \( a_{21} \equiv b_{21} \equiv c_{21} \equiv d_{21} \equiv 0 \mod p \) and

\[ \iota \cdot \pi^*(\tilde{W}_p \cdot g \cdot \tilde{W}_p) \cdot \iota = \iota \cdot \pi(\tilde{W}_p \cdot g \cdot \tilde{W}_p) \cdot \iota = \iota \cdot \pi(\tilde{W}_p \cdot g \cdot \tilde{W}_p) \cdot \iota = \iota \cdot \pi \left( \begin{pmatrix} \frac{1}{p} & 0 & 0 & 0 \\ 0 & \frac{1}{p} & 0 & 0 \\ 0 & 0 & \frac{1}{p} & 0 \\ 0 & 0 & 0 & \frac{1}{p} \end{pmatrix} \cdot \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & C & 0 \\ 0 & 0 & 0 & D \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{p} & 0 & 0 & 0 \\ 0 & \frac{1}{p} & 0 & 0 \\ 0 & 0 & \frac{1}{p} & 0 \\ 0 & 0 & 0 & \frac{1}{p} \end{pmatrix} \right) \cdot \iota = \iota \cdot \begin{pmatrix} a_{22} & a_{21} & b_{22} & b_{21} \\ p a_{12} & a_{11} & b_{12} & b_{11} \\ c_{22} & c_{21} & d_{22} & d_{21} \\ p c_{12} & c_{11} & d_{12} & d_{11} \end{pmatrix} \cdot \iota = \pi^*(g). \]
Since \( \tilde{\Gamma}_{1,p}^\circ \) is normal in \( \tilde{\Gamma}_{1,p} \) and \( \tilde{W}_p^2 = 1_4 \), \((*)\) is equivalent to
\[
\iota \cdot \pi^*(\tilde{W}_p \cdot h) = \pi^*(h \cdot \tilde{W}_p) \cdot \iota \quad \text{for all } h \in \tilde{\Gamma}_{1,p} \setminus \tilde{\Gamma}_{1,p}^\circ.
\]

Let \( g \in \tilde{\Gamma}_{1,p}^\circ, h_1, h_2 \in \tilde{\Gamma}_{1,p}^\circ \). Then
\[
(i) \quad \pi^*(g \cdot h_1) = \pi(g \cdot h_1 \cdot \tilde{W}_p) \cdot \iota \\
= \pi(g) \cdot \pi(h_1 \cdot \tilde{W}_p) \cdot \iota \\
= \pi^*(g) \cdot \pi^*(h_1) \\
(ii) \quad \pi^*(h_1 \cdot g) = \pi(h_1 \cdot g \cdot \tilde{W}_p) \cdot \iota \\
= \pi(h_1 \cdot \tilde{W}_p \cdot W_p \cdot g \cdot \tilde{W}_p) \cdot \iota \\
= \pi(h_1 \cdot \tilde{W}_p) \cdot \pi(W_p \cdot g \cdot \tilde{W}_p) \cdot \iota \\
= \pi^*(h_1) \cdot \iota \cdot \pi(W_p \cdot g \cdot \tilde{W}_p) \cdot \iota \\
(*) \quad \pi^*(h_1) \cdot \pi^*(g) \\
(iii) \quad \pi^*(h_1 \cdot h_2) = \pi^*(h_1 \cdot \tilde{W}_p \cdot \tilde{W}_p \cdot h_2) \\
= \pi^*(h_1 \cdot \tilde{W}_p) \cdot \pi^*(\tilde{W}_p \cdot h_2) \\
= \pi^*(h_1 \cdot \tilde{W}_p) \cdot \pi^*(\tilde{W}_p \cdot h_2) \\
= \pi^*(h_1) \cdot \iota \cdot \pi^*(\tilde{W}_p \cdot h_2) \\
(**) \quad \pi^*(h_1) \cdot \pi^*(h_2 \cdot \tilde{W}_p) \cdot \iota \\
= \pi^*(h_1) \cdot \pi^*(h_2).
\]

\( \Box \)

**Proposition 1** There is exactly one group \( \tilde{\Gamma}_{1,p}(2) \) such that the diagram

\[
\begin{array}{cccccccc}
1 & & 1 & & & & 1 & & 1 \\
\downarrow & & & & \downarrow & & & & \downarrow \\
1 & \longrightarrow & \tilde{\Gamma}_{1,p}(2) & \longrightarrow & \tilde{\Gamma}_{1,p}(2) & \longrightarrow & \mathbb{Z}_2 & \longrightarrow & 1 \\
\downarrow & & & & \downarrow & & & & \downarrow \\
1 & \longrightarrow & \tilde{\Gamma}_{1,p}^\circ & \longrightarrow & \tilde{\Gamma}_{1,p}^\circ & \longrightarrow & \mathbb{Z}_2 & \longrightarrow & 1 \\
\downarrow & & & & \downarrow & & & & \downarrow \\
\Sp(4, \mathbb{Z}_2) & \longrightarrow & \Sp(4, \mathbb{Z}_2) & & & & & & & \\
\downarrow & & & & & & \downarrow & & & & \downarrow \\
1 & & 1 & & & & & & &
\end{array}
\]

commutes with exact rows and columns.
Proof. Let \( \varphi \) be a homomorphism such that the diagram
\[
\begin{array}{ccc}
\Gamma_1^0 & \xrightarrow{\varphi} & \Gamma_1^*\\
\pi & & \varphi \\
\text{Sp}(4, \mathbb{Z}_2) & \xrightarrow{} & \text{Sp}(4, \mathbb{Z}_2)
\end{array}
\]
commutes. It is enough to show that one necessarily has \( \varphi(\widetilde{W}_p) = \pi^*(\widetilde{W}_p) = \iota \).
First, \( \varphi(\widetilde{W}_p) \) is an involution in \( \text{Sp}(4, \mathbb{Z}_2) \) since \( \widetilde{W}_p^2 = 1_4 \). Moreover, we have
\[
\varphi(g \cdot \widetilde{W}_p \cdot g^{-1}) = \pi(g) \cdot \varphi(\widetilde{W}_p) \cdot \pi(g)^{-1} \quad \forall \ g \in \Gamma_1^0,
\]
so
\[
g \in \text{centr}(\widetilde{W}_p, \Gamma_1^*) \implies \pi(g) \in \text{centr}(\varphi(\widetilde{W}_p), \text{Sp}(4, \mathbb{Z}_2))
\]
(\( \text{centr}(x, G) \) means the centraliser of \( x \) in \( G \)). The matrices
\[
h_1 = \begin{pmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & p & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\quad \text{and} \quad
h_2 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 \\
p & 0 & 0 & 1
\end{pmatrix}
\]
are in \( \text{centr}(\widetilde{W}_p, \Gamma_1^*) \). Since
\[
\pi(h_1) = \begin{pmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]
must lie in \( \text{centr}(\varphi(\widetilde{W}_p), \text{Sp}(4, \mathbb{Z}_2)) \), it follows (with \( \varphi(\widetilde{W}_p) = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \)), that \( C = 0_2 \). Likewise, one argues with \( \pi(h_2) \) that necessarily \( B = 0_2 \) holds.
Since \( \varphi(\widetilde{W}_p) \) is an involution in \( \text{Sp}(4, \mathbb{Z}_2) \), we deduce that \( A \) (and hence \( D \)) has to be an involution in \( \text{SL}(2, \mathbb{Z}_2) \) (i.e. equal to \( 1_2 \), \( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \), \( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \) or \( \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \)).
Now it is easy to see that necessarily \( A = D = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) holds. To see this first assume \( A = 1_2 \) (and consequently \( D = 1_2 \)). Then necessarily \( \pi(\widetilde{W}_p \cdot h \cdot \widetilde{W}_p) = \pi(h) \) must hold for all \( h \in \Gamma_1^0 \). But this is not the case for
\[
\begin{pmatrix}
1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -p \end{pmatrix}
\in \Gamma_1^0.
\]
Similarly one argues with
\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
p & 1 & 0 & 0 \\
0 & 0 & 1 & -1 \\
0 & 0 & 0 & 1
\end{pmatrix}
\in \Gamma_1^0
\]
to exclude the cases \( A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) and \( A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \). This shows that the only possibility is \( \Gamma_1^0(2) = \ker(\pi^*) \).
\[\square\]
For future use we note that the kernel \( \ker(\pi^*) \) is generated by \( \Gamma_1^0(2) \) and \( \kappa_p \) with \( \kappa_p = \widetilde{W}_p \cdot g \) and \( g \in \Gamma_1^0 \). \( \pi^*(g) = \iota \). Of course we have
\[ G = \langle \Gamma_{1,3}^0(2), \tilde{W}_p \cdot g_1 \rangle = \langle \Gamma_{1,3}^0(2), \tilde{W}_p \cdot g_2 \rangle \text{ for } g_1, g_2 \in \tilde{\Gamma}_{1,p}^0 \text{ with } \pi^*(g_1) = \pi^*(g_2) = \iota, \text{ so we can choose} \]

\[ \tilde{\kappa}_p = \tilde{W}_p \cdot \left( \begin{array}{cccc}
 p-1 & 2-p & 0 & 0 \\
 0 & 1-p & 0 & 0 \\
 0 & 0 & p(2-p) & 1-p \\
 0 & 0 & 0 & p(2-p) \end{array} \right) \]

\[ = \frac{1}{\sqrt{p}} \tilde{V}_p, \quad \tilde{V}_p = \left( \begin{array}{cccc}
 p & 1-p & 0 & 0 \\
 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & p(2-p) \end{array} \right) \]

as a generator.

3 The modular form \( \Delta_1^3 \)

It was shown in [GN] that

\[ \Delta_1(\tau) = q^{1/6} r^{1/2} s^{1/2} \prod_{n \geq 1} (1 - q^n r^1 s^m) f(nm,l) \]

with \( q = e^{2\pi i r_1}, \ r = e^{2\pi i r_2}, \ s = e^{2\pi i r_3} \) and

\[ \sum_{n \geq 0} f(n,l) q^n r^l = r^{-1} \left( \prod_{n \geq 1} (1 + q^n r)(1 + q^n r^{-1})(1 - q^{2n-1} r^2)(1 - q^{2n-1} r^{-2}) \right)^2 \]

is a cusp form of weight one with respect to \( \Gamma_{1,3}^0 \) with a character \( \chi_6 \) of order 6. Then \( \Delta_1^3 \) is a cusp form with respect to \( \Gamma_{1,3}^* \) with a character \( \chi_{(2,-)} \) of order two and \( \chi_{(2,-)}(V_3) = -1 \). The character \( \chi_{(2,-)}|_{\Gamma_{1,3}^*} \) arises in the following way:

\[ 1 \longrightarrow \Gamma_{1,3}^0(2) \longrightarrow \Gamma_{1,3}^0 \longrightarrow \text{Sp}(4, \mathbb{Z}) \simeq S_6 \longrightarrow 1 \]

\[ \chi_{(2,-)} \downarrow \text{sgn} \]

\[ \{ \pm 1 \} \]

(for all this see (see [GH2])).

**Proposition 2** Let \( S_3(\Gamma_{1,3}^*(2)) \) be the vectorspace of cusp forms of weight 3 with respect to \( \Gamma_{1,3}^*(2) \). Then

\[ S_3(\Gamma_{1,3}^*(2)) = \mathbb{C} \cdot \Delta_1^3. \]

**Proof.** Consider \( \Gamma_{1,3}^*(2) = \langle \Gamma_{1,3}^0(2), R_3^{-1} \cdot \tilde{\kappa}_3 \cdot R_3 \rangle \). From the diagram above we obtain \( \chi_{(2,-)}|_{\Gamma_{1,3}^0(2)} \equiv 1 \). We have \( R_3^{-1} \cdot \tilde{\kappa}_3 \cdot R_3 = V_3 \cdot g \) with

\[ g = \left( \begin{array}{cccc}
 2 & -1 & 0 & 0 \\
 3 & -2 & 0 & 0 \\
 0 & 0 & 2 & 3 \\
 0 & 0 & -1 & -2 \end{array} \right) \in \Gamma_{1,3}^0. \]
Moreover we have $\pi(R_3 \cdot g \cdot R_3^{-1}) = \iota$ and it is easy to see that $\text{sgn}(\iota) = -1$, so

$$\chi_{(2,-)}(R_3^{-1} \cdot \kappa_3 \cdot R_3) = \chi_{(2,-)}(\tilde{V}_3) \cdot \chi_{(2,-)}(g) = -1 \cdot \text{sgn}(\iota) = 1.$$  

This shows that $\Delta_1^3$ is a cusp form of weight 3 with respect to $\Gamma_{1,3}(2)$. By [GH2] and [BN] the moduli space $\Gamma_{1,3}(2) \backslash \mathbb{H}_2$ is birationally equivalent to a Calabi-Yau variety. Hence the result follows from Freitag’s extension theorem (cf. [F, Hilfssatz 3.2.1]) which says that for any discrete group $\Gamma$ the space $S_3(\Gamma)$ is isomorphic to $H^{3,0}(\mathcal{A}(\Gamma))$ for any smooth projective model $\mathcal{A}(\Gamma)$ of $\mathcal{A}(\Gamma) = \Gamma \backslash \mathbb{H}_2$.

□

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