Adaptive pointwise density estimation under local differential privacy

SANDRA SCHLUTTENHOFER∗  JAN JOHANNES†

June 16, 2022

Abstract

We consider the estimation of a density at a fixed point under a local differential privacy constraint, where the observations are anonymised before being available for statistical inference. We propose both a privatised version of a projection density estimator as well as a kernel density estimator and derive their minimax rates under a privacy constraint. There is a twofold deterioration of the minimax rates due to the anonymisation, which we show to be unavoidable by providing lower bounds. In both estimation procedures a tuning parameter has to be chosen. We suggest a variant of the classical Goldenshluger-Lepski method for choosing the bandwidth and the cut-off dimension, respectively, and analyse its performance. It provides adaptive minimax-optimal (up to log-factors) estimators. We discuss in detail how the lower and upper bound depend on the privacy constraints, which in turn is reflected by a modification of the adaptive method.

1 Introduction

In this paper we are interested in estimating the value \( f(t) \) of an unknown density \( f \) at a fixed point \( t \) under a local differential privacy constraint, where the raw sample of independent and identically distributed (iid) real random variables

\[
X_i \overset{iid}{\sim} f, \quad i \in \{1, \ldots, n\}, \tag{1.1}
\]

is anonymised before being available for statistical inference. The paper is organised as follows: in Section 2 we review pointwise density estimation and local differential privacy. We focus on a privatised kernel and projection density estimator in Section 3 and Section 4, respectively. In both cases the non-private estimator based on the raw sample equals a mean \( \frac{1}{n} \sum_{i=1}^{n} g(X_i) \) for a function \( g \) depending on the evaluation point \( t \) and a tuning parameter. We study a privatised version of these estimators, where the \( i \)-th data holder releases a sanitised version of \( g(X_i) \) rather than \( X_i \). The anonymisation is obtained by a general Laplace perturbation which guarantees local differential privacy in both cases. We derive an upper bound for their mean squared error and show a matching lower bound for the maximal mean squared error over Hölder classes when taking the infimum over all possible estimators and all local differential privatisation methods. The estimators attain the lower bound (up to a constant) only if the tuning parameter is chosen optimally. Since the necessary information for their optimal choice is typically inaccessible in practice, we propose a fully data-driven choice inspired by the work of Goldenshluger and Lepski [2011]. We establish an oracle inequality for the completely data-driven privatised kernel and projection density estimator. Comparing the upper bounds of their mean squared errors with

∗Aarhus Universitet, Institut for Matematik, Ny Munkegade 118, DK-8000 Aarhus C, Denmark, e-mail: schluttenhofer@math.au.dk
†Ruprecht-Karls-Universität Heidelberg, Institut für Angewandte Mathematik, MATHEMATIKON, Im Neuenheimer Feld 205, D-69120 Heidelberg, Germany, e-mail: johannes@math.uni-heidelberg.de
in a classical density estimation problem we have \( n \) real-valued observations \( (X_i)_{i \in \{1, \ldots, n\}} \), whose underlying distribution has a Lebesgue density \( f \). For a fixed point \( t \in \mathbb{R} \) one aims to estimate \( f(t) \in \mathbb{R} \). In this setting there exists a vast amount of estimators in the literature. We focus on two classical approaches, first we introduce kernel density estimators and secondly, projection density estimators. Let \( K \) be a kernel function, as usual we assume \( K \) to be square-integrable and bounded and to integrate to 1. The kernel density estimator for \( f(t) \) is given by

\[
\hat{f}_k(t) := \frac{1}{n} \sum_{i=1}^{n} K_h(X_i - t),
\]

where we use the notation \( K_h(x) := \frac{1}{h} K \left( \frac{x}{h} \right) \), \( x \in \mathbb{R} \), and \( h > 0 \) is a bandwidth. If \( f \) is a square-integrable density supported on \([0, 1]\) an alternative methods uses the projection density estimator. More precisely, for \( d \in \mathbb{N} \) let \( (\varphi_j)_{j \in \{1, \ldots, d\}} \) be an orthonormal system in \( L^2([0, 1]) \), the space of all square-integrable real-valued functions defined on \([0, 1]\). The projection density estimator for \( f(t) \) is given by

\[
\hat{f}_d(t) = \sum_{j=1}^{d} \hat{a}_j \varphi_j(t), \quad \hat{a}_j = \frac{1}{n} \sum_{i=1}^{n} \varphi_j(X_i).
\]

Both estimation strategies depend on a tuning parameter, the bandwidth \( h > 0 \) respectively the truncation parameter \( d \in \mathbb{N} \). Typically, the performance of an estimator \( T_n \) of \( f(t) \) is assessed by considering its estimation risk or mean squared error given by \( \mathbb{E}\left(|T_n - f(t)|^2\right) \).

Upper and lower bounds for the maximal estimation risk over different classes of densities have been studied extensively in the literature. It is well-known that the accuracy of both kernel and projection density estimators crucially depends on the choice of the tuning parameters. The data-driven choice of these parameters is subject of considerable literature. A well-studied data-driven approach for choosing the tuning parameters is the Goldenshluger-Lepski method (introduced in Goldenshluger and Lepski [2011]), which is based on finding an estimate of the mean squared error for each \( h \) resp. \( D \) in a collection and minimizing the estimate with respect to the smoothing parameters. We refer to the textbooks Tsybakov [2009] and Comte [2017] and the references therein.

**Local differential privacy.** In cases where the observations contain sensitive private information, the raw sample \( (X_i)_{i \in \{1, \ldots, n\}} \) is unavailable to the statistician. Instead, the data holder releases a privatized or sanitized sample \( (Z_i)_{i \in \{1, \ldots, n\}} \) that is obtained from \( (X_i)_{i \in \{1, \ldots, n\}} \) by a stochastic transformation \( Q \), called privacy mechanism, stochastic channel or data release mechanism. In the computer science literature, the samples \( (X_i)_{i \in \{1, \ldots, n\}} \) and \( (Z_i)_{i \in \{1, \ldots, n\}} \) are often called databases. Formally, let \( X \) and \( Z \) be defined on a common probability space with values in measurable spaces \( (\mathcal{X}, \mathcal{F}) \) and \( (\mathcal{Z}, \mathcal{G}) \), respectively. A local privacy mechanism \( Q \) is associated with a Markov kernel \( \kappa_Q : (\mathcal{X}, \mathcal{F}) \rightarrow [0, 1] \) with \( \kappa_Q(x, B) = \mathbb{P}(Z \in B \mid X = x) = Q(B \mid x) \).
for all $x \in \mathcal{X}$ and $B \in \mathcal{Z}$. In other words, the privacy mechanism $Q$ is the regular conditional distribution of $Z$ given $X$. We assume that the stochastic channel satisfies a privacy constraint, which we formalize next.

**Definition 2.1 ($\alpha$-differential privacy).** We call $Z$ a $\alpha$-differentially private view of $X$ with privacy parameter $\alpha \geq 0$ if the conditional distribution $Q$ satisfies

$$Q(B | x) \leq \exp(\alpha) \cdot Q(B | x')$$

for all $B \in \mathcal{Z}$ and $x, x' \in \mathcal{X}$. \hspace{1cm} (2.3)

The privacy mechanism $Q$ is then called $\alpha$-differentially locally private. We denote the set of all $\alpha$-differentially locally private mechanism by $\mathcal{Q}_\alpha$.

The sample $(Z_i)_{i \in \{1, \ldots, n\}}$ generated with $Q$ satisfying (2.3) is called a $\alpha$-differentially locally private (non-interactive) view of the raw sample $(X_i)_{i \in \{1, \ldots, n\}}$ in (1.1). The term locally refers to the fact that for the generation of the $i$th sanitized observation $Z_i$ the data holder only requires the $i$th raw observation $X_i$, thus, the raw data can be stored locally. In contrast to this, there also exists the concept of global differential privacy, where a data collector is entrusted with the data and generates a privatized database $(Z_i)_{i \in \{1, \ldots, n\}}$ based on the entire raw data set $(X_i)_{i \in \{1, \ldots, n\}}$. The privatization is called interactive if the $i$th data holder also has access to (already generated) sanitized observations $(Z_k)_{k \in \{1, \ldots, i-1\}}$. In the non-interactive case the data holders do not need to interact with each other in order to generate the private views.

**Related literature.** The concept of differential privacy was essentially introduced in the series of papers Dinur and Nissim [2003], Dwork and Nissim [2004] and Dwork [2006]. Dwork [2008] gives an overview of the early results in the field. First statistical results are derived in Wasserman and Zhou [2010] and Hall et al. [2013], where both papers work under global privacy constraints. Duchi et al. [2018] provide a toolbox of methods for deriving minimax rates of estimation under a local privacy constraint. Naturally, there exist many more concepts of privacy (smooth privacy, divergence-based privacy, approximate privacy to mention but a few), for a broad overview we refer the reader to Barber and Duchi [2014].

Let us now first heuristically explain the implications of the condition (2.3). A small value of $\alpha$ (close to 0) corresponds to a high privacy guarantee. In the extreme case $\alpha = 0$ the privacy mechanism $Q$ does not depend on the value of the input data $X$, in other words $Z$ and $X$ are independent. Hence, we achieve total privacy. Naturally, the privatized sample is useless for making inference on the distribution of $X$. Large values of $\alpha$ allow for low privacy, since a change in the original observation can then yield a completely different distribution for the output random variable and it is thus easier to draw conclusions about the raw data. Let us now formalize the effect (2.3) has on the information about concrete input data points. Assume we want to find out whether the original (raw) data comes from data holder 1 (with value $x$ with associated probability $P_0 = Q(\cdot | x)$) or from data holder 2 (with value $x' \neq x$ with associated probability $P_1 = Q(\cdot | x')$). This simple two-point testing task is solved using the Neyman-Pearson-Lemma. The privacy constraint gives a bound for the maximal power a test can achieve. The following proposition is a reformulation of Theorem 2.4. in Wasserman and Zhou [2010] and we state its proof in our setting for completeness.

**Proposition 2.2 (Plausible deniability).** Let $Z$ be an $\alpha$-differentially private view of $X$ obtained through the channel $Q$. Let $x \neq x'$. Any level-$\gamma$-test based on the observation $Z$
and the channel $Q$ for the task

$$H_0 : \{P_0 = Q(\cdot | x)\} \quad \text{against} \quad H_1 : \{P_1 = Q(\cdot | x')\}$$

has power bounded by $\gamma \exp(\alpha)$.

**Proof of Proposition 2.2.** The Neyman-Pearson Lemma states that the highest possible power (i.e. minimal type II error probability) is obtained by a test of the form

$$\Delta := 1_{\{dP_1 \geq \tau dP_0\}},$$

where $dP_1$ and $dP_0$ are densities with respect to an arbitrary dominating measure and the threshold $\tau$ is such that

$$P_0(\Delta = 1) = P_0\left(\frac{dP_1}{dP_0} \geq \tau \right) \leq \gamma.$$

Note that the distributions $P_0 = Q(\cdot | x)$ and $P_1 = Q(\cdot | x')$ satisfy $P_1(A) \leq \exp(\alpha) P_0(A)$ for any measurable set $A$. Hence, the power of the test is bounded by

$$1 - P_1(\Delta = 0) = P_1(\Delta = 1) \leq \exp(\alpha) P_0(\Delta = 1) \leq \exp(\alpha) \gamma,$$

which proves the result. □

We give two popular examples of privacy mechanisms that satisfy the privacy constraint (2.3).

**Example 2.3 (Perturbation approach, ”Adding noise”).** The perturbation approach consists of adding centred noise $\xi$ with Lebesgue density $h$ to the observations, i.e.

$$Z := X + \xi \quad \text{with} \quad \xi \sim h, \quad E\xi = 0.$$

Then the stochastic channel $Q$ has the density $q(z | x) = h(z - x)$ with respect to the Lebesgue measure. The most popular noise density is the Laplace density with appropriately chosen variance.

**Remark 2.4 (Naive privatization methods).** As described in Example 2.3 a standard technique for privatizing data is to add Laplace noise directly to the observations. Inferring on the density $f$ of $X$ based on observations of $X + \xi$ with privatization noise $\xi \sim h$ is essentially a de-convolution problem, since the density of $X + \xi$ equals the convolution of $f$ and $h$. Minimax rates for the pointwise density estimation in this setting are well-known and depend on the regularity of both the density of interest $f$ and the added noise $\xi$. By choosing a slightly more elaborate privacy mechanism we are able to significantly improve the private minimax rate. □

Let us now consider a more general form of a privacy mechanism.

**Example 2.5 (Exponential mechanism).** Let $\zeta : X \times Z \to [0, \infty)$ be a function and define the sensitivity of $\zeta$ by

$$\delta := \sup_{x,y \in X} \sup_{z \in Z} |\zeta(x, z) - \zeta(y, z)|$$

as the maximal change of $\zeta$ that can occur due to altering the input data. Define the density

$$h(z | x) = \frac{\exp(-\alpha \zeta(x, z))}{\int \exp(-\alpha \zeta(x, t)) dt}$$
and sample \( Z \sim h(\cdot \mid x) \). McSherry and Talwar [2007] show that the exponential mechanism yields a \( \alpha \)-differentially private channel.

Consider \( \mathcal{X} = [0, 1] \). Let us point out that for \( \zeta(x, z) = |x - z| \) (and \( \delta = 2 \)) the exponential mechanism in Example 2.5 corresponds to the perturbation approach in Example 2.3 with \( \xi \sim \text{Laplace}(0, \frac{1}{\alpha}) \). We briefly recall the Laplace distribution.

**Reminder 2.6 (Laplace distribution).** With \( \xi \sim \text{Laplace}(0, 1) \) we denote the distribution with probability density \( x \mapsto \frac{1}{2} \exp(-|x|). \) We have \( \mathbb{E} |\xi|^m = m! \) and \( b\xi + \mu \sim \text{Laplace}(\mu, b) \) for \( \mu \in \mathbb{R}, b > 0 \).

Thus, we have seen that perturbation with a Laplace distribution yields \( \alpha \)-differential privacy. By more generally choosing \( \zeta(x, z) = |g(x) - z| \) in Example 2.5 for a function \( g : [0, 1] \longrightarrow \mathbb{R} \) we obtain the following corollary.

**Corollary 2.7 (General Laplace perturbation).** Let \( g : \mathcal{X} \longrightarrow \mathbb{R} \) be a function. Let \( g(X) \) be the quantity that we want to anonymize. We define

\[
\Delta(g) := \sup_{x, x' \in \mathcal{X}} |g(x) - g(x')|.
\]

Let \( \alpha > 0 \) and \( \Delta(g) < \infty \). Then,

\[
Z = g(X) + b\xi, \quad \xi \sim \text{Laplace}(0, 1)
\]

with \( b \geq \frac{\Delta(g)}{\alpha} \) is an \( \alpha \)-differentially private view of \( g(X) \) and, hence, of \( X \).

Let us now come back to the kernel and projection estimators \( \hat{f}_h(t) \) and \( \hat{f}_d(t) \) given in (2.1) and (2.2), respectively. For \( t, x \in \mathbb{R}, h > 0 \) and \( d \in \mathbb{N} \) we define

\[
g_h(x) := K_h(x - t) \quad \text{and} \quad g_d(x) := \sum_{j=1}^d \varphi_j(x) \varphi_j(t).
\]  

(2.4)

Clearly, the estimators can be written as \( \hat{f}_h(t) = \frac{1}{n} \sum_{i=1}^n g_h(X_i) \) and \( \hat{f}_d(t) = \frac{1}{n} \sum_{i=1}^d g_d(X_i) \). Motivated by Remark 2.4 the \( i \)-th data holder is asked to release a sanitized version \( Z_{i,h} \) of \( g_h(X_i) \) resp. \( Z_{i,d} \) of \( g_d(X_i) \) rather than privatizing \( X_i \) directly. We will apply Corollary 2.7 to \( g = g_h \) for kernel density estimation in Section 3 and to \( g = g_d \) for projection density estimation in Section 4 to verify the privacy constraints. The private kernel and projection estimators are then given by

\[
\hat{f}_h(t) = \frac{1}{n} \sum_{i=1}^n Z_{i,h}, \quad \text{respectively} \quad \hat{f}_d(t) = \frac{1}{d} \sum_{i=1}^d Z_{i,d}.
\]  

(2.5)

**Private minimax theory.** Let \( E \) be a class of densities and let \( (X_i)_{i \in \{1, \ldots, n\}} \) be (non-private) random variables. Recall that \( \mathcal{Q}_n \) denotes the family of all privatization mechanisms satisfying \( \alpha \)-differential privacy. For \( Q \in \mathcal{Q}_n \) and the true density \( f \) we use the symbol \( \mathbb{E}_{f, Q} \) for the expectation associated with the distribution \( \mathbb{P}_{f, Q} \) of the private view \( (Z_i)_{i \in \{1, \ldots, n\}} \). If there is no confusion, we omit the index. We derive lower bounds for the private minimax risk given by

\[
\inf_{T_n} \inf_{Q \in \mathcal{Q}_n} \sup_{f \in \mathcal{P}} \mathbb{E}_{f, Q} \left( |T_n - f(t)|^2 \right).
\]
and a matching upper bound for the estimators in (2.5) combined with the privacy mechanism from Corollary 2.7. Roughly speaking, we are looking for an optimal combination of a privacy mechanism $Q$ and an estimation procedure $T_n$ rather than for a minimax-optimal estimator only. Comparing the private minimax risk and the classical minimax risk allows us to characterise the price to pay for data privacy.

**Related literature.** The first result for a projection approach for estimating a density under privacy constraints is due to Wasserman and Zhou [2010], Section 6. In a non-local setting they are able to achieve the minimax rate (with sample size replaced by the effective sample size $\alpha^2 n$) using a projection density estimator with Laplace perturbation of the coefficients. In a local setting Duchi et al. [2018] consider projection density estimation based on privatized views of the observations of the density with respect to $L^2([0,1])$ loss. The non-private minimax estimation risk is well-known to be of order $n^{-\frac{2}{2+\alpha}}$, where $\alpha$ is the smoothness parameter of a Sobolev ellipsoid. They show that the local $\alpha$-private minimax risk is of order $(\alpha^2 n)^{-\frac{2}{2+\alpha}}$, providing both a lower and an upper bound. Butucea et al. [2020] consider Besov ellipsoids with wavelet techniques combined with a Laplace perturbation approach. Also in this case, the privatization causes a deterioration of the order of the risk from $n^{-\frac{2}{2+\alpha}}$ to $(n(e^\alpha - 1)^2)^{-\frac{2}{2+\alpha}}$ ($\beta$ being the smoothness parameter of the Besov ellipsoid, we only state the dense zone here for illustration purposes). Note that this is comparable to the results of Duchi et al. [2018] since for small $\alpha$ we have $\alpha \approx e^\alpha - 1$. Kernel estimators are, for instance, treated in Hall et al. [2013] and Kroll [2019] under local differential approximate ($\alpha, \beta$)-privacy, which is a relaxation of the constraint we consider. These papers consider Laplace perturbation and Gaussian perturbation (which is only useful in the $(\alpha, \beta)$-differential privacy context). Kroll [2021] considers pointwise kernel density estimation under $\alpha$-differential privacy, but considers $\alpha$ to be a constant, thus not making the dependence of the minimax rates on the privacy parameter $\alpha$ explicit. Butucea and Issartel [2021] consider private estimation of nonlinear functionals. The results mentioned so far address estimation problems. Concerning testing tasks we mention two recent papers Lam-Weil et al. [2022] and Berrett and Butucea [2020].

**Adaptivity under privacy constraints.** Also in the private setting the optimal choice of the tuning parameters of the estimators in (2.5) depends on the regularity of the unknown function $f$. In the non-private setup a common method for choosing them is the Goldenshluger-Lepski method. Let us heuristically explain the idea behind it in the example of the kernel density estimator $\hat{f}_h(t)$ with bandwidth $h > 0$. Given a classical bias$^2$-variance-decomposition of the private risk of $\hat{f}_h(t)$ we construct estimators $\hat{R}(h) = V(h) + A(h)$, where $V(h)$ and $A(h)$ approximate the variance term respectively the bias$^2$-term. We choose the bandwidth that minimizes $\hat{R}(h)$ over a finite collection $\mathcal{H}$ of bandwidths. Compared to the non-private setup our proposed method differs in two ways. Firstly, the estimators $V(h)$ and $A(h)$ need to be adjusted to only make use of the available privatized observations. Secondly, we need access to estimators for several tuning parameters. Data holder $i$ is asked to release sanitized version of $g_b(X_i)$ for each $h \in \mathcal{H}$ with adjusted privacy parameter. The next lemma allows to guarantee privacy for several released observations. For a proof we refer to Lemma 2.16 in Kroll [2019] (where one sets $\beta_1 = \beta_2 = 0$).

**Lemma 2.8 (Composition lemma).** Let $Z_1, Z_2$ be $\alpha_1$– respectively $\alpha_2$– differentially private views of $X$, which are conditionally on $X$ independent. Then, $Z = (Z_1, Z_2)$ is a $(\alpha_1 + \alpha_2)$-differentially private view of $X$.

Let us briefly point out how to apply the composition lemma in our setting. The $i$-th data holder publishes $Z = (Z_{i,h}, h \in \mathcal{H})$ with adapted privacy parameter $\alpha/|\mathcal{H}|$, where $|\mathcal{H}|$ denotes
the number of elements in $\mathcal{H}$. Then $Z$ fulfils $\alpha$-differential privacy.

**Related literature.** Butucea et al. [2020] consider density estimation using wavelets. As usual in the context of wavelet estimation using non-linear thresholding they are able to obtain an adaptive estimator. Kroll [2019] and Kroll [2021] also address adaptivity issues. In the kernel density estimation case Kroll [2021] suggests a privatised version of Lepski’s method, which is different from ours, both in the formulations and the proof techniques and it requires the a priori knowledge of an upper bound for the sup-norm of the true density. Moreover, both the privacy parameter $\alpha$ and the size of the collection of bandwidths $|\mathcal{H}|$ are considered to be constants and do, thus, not appear in the minimax rate. In our study, we make this dependence on $\alpha$ and $|\mathcal{H}|$ explicit.

3 Privatised kernel density estimation

3.1 Upper bound for KDE

In this section we consider a privatised version of the kernel density estimator (2.1). Note that the estimator is a mean over evaluated kernel functions. We, therefore, consider the following privacy mechanism. Let $t \in \mathbb{R}$ and $h \in (0,1)$ be fixed. For $i \in \{1, \ldots, n\}$ the $i$-th data holder releases

$$Z_{i,h} = g_h(X_i) + \frac{2\|K\|_\infty}{\alpha h} \xi_{i,h}, \quad \xi_{i,h} \overset{\text{iid}}{\sim} \text{Laplace}(0,1),$$

(3.1)

where $g_h(x) := K_h(x - t)$, $x \in \mathbb{R}$. Corollary 2.7 shows that $Z_{i,h}$ is an $\alpha$-differentially private view of $X_i$, since

$$\Delta(g_h) = \sup_{x,x' \in [0,1]} |K_h(x - t) - K_h(x' - t)| \leq \frac{2}{h} \|K\|_\infty.$$

Based on the private views $Z_{1,h}, \ldots, Z_{n,h}$ a natural estimator of $f_h(t)$ is given as in (2.5) by $\hat{f}_h(t) = \frac{1}{n} \sum_{i=1}^n Z_{i,h}$. Note that due to the centredness of the added Laplace noise, we have

$$\mathbb{E}(\hat{f}_h(t)) = \frac{1}{n} \sum_{i=1}^n \mathbb{E}Z_{i,h} = \frac{1}{n} \sum_{i=1}^n \mathbb{E}K_h(X_i - t) = \mathbb{E}(\hat{f}_h(t)),$$

where $\hat{f}_h$ denotes the non-private estimator defined in (2.1) and we denote $f_h(t) = \mathbb{E}(\hat{f}_h(t))$. For simplicity of notation we use the symbol $\mathbb{E}$ without an index for the expectation over the randomness in the data and in the privatisation. The following Proposition 3.1 presents an upper bound on the mean squared error. The upper bound is given by a classical bias$^2$-variance trade-off, which contains the standard variance term of order $\frac{1}{nh}$ and an additional variance term of order $\frac{8}{n^3h^2}$.

**Proposition 3.1 (Risk bound).** Consider the privacy mechanism (3.1) and the estimator (2.5). For $\alpha \in (0,1)$, $n \in \mathbb{N}$ and $h \in (0,1)$ we have

$$\mathbb{E} \left( |\hat{f}_h(t) - f(t)|^2 \right) \leq |f_h(t) - f(t)|^2 + \frac{\|f\|_\infty \|K\|_2^2}{nh} + \frac{8 \|K\|_\infty^2}{nh^2}.$$

**Proof.** Recall the notation $f_h(t) = \mathbb{E}\hat{f}_h(t)$. We have classical bias$^2$-variance decomposition of the mean squared error

$$\mathbb{E} \left( |\hat{f}_h(t) - f(t)|^2 \right) = |f_h(t) - f(t)|^2 + \mathbb{E} \left( \left| \hat{f}_h(t) - f_h(t) \right|^2 \right).$$
Let us control the variance term. Recall that we denote \( \hat{f}_h(t) = \frac{1}{n} \sum_{i=1}^n K_h(X_i - t) \). Then due to the independence of the Laplace noise and the original observations, we have

\[
\text{var}(\hat{f}_h(t)) = \text{var}(\hat{f}_h(t)) + \frac{1}{n} \frac{4}{\alpha^2 h^2} \text{var}(\xi_{1,h}) \\
\leq \frac{\|f\|_\infty \|K\|_2^2}{nh} + \frac{8 \|K\|_\infty^2}{n\alpha^2 h^2},
\]

which ends the proof. \( \Box \)

Let us illustrate the result Proposition 3.1 for a class of densities with smoothness \( s > 0 \). The optimal bandwidth is obtained by balancing the bias\(^2\)-term and the variance terms. Here and subsequently for two real sequence \((a_n)_{n \in \mathbb{N}}\) and \((b_n)_{n \in \mathbb{N}}\) we write \( a_n \lesssim b_n \) if there exists an universal numerical constant \( C > 0 \) such that \( a_n \leq C b_n \) for all \( n \in \mathbb{N} \). If \( a_n \lesssim b_n \) and \( b_n \lesssim a_n \) then we write \( a_n \sim b_n \).

**Corollary 3.2.** Consider the following class of densities

\[
\mathcal{E} = \left\{ f : \mathbb{R} \rightarrow \mathbb{R}, f \text{ is a density, } \|f\|_\infty \leq c, \sup_{h \in (0,1)} h^{-2\beta} (f_h(t) - f(t))^2 \leq c \right\}
\]

for a universal constant \( c > 0 \). With \( h = h_1^* \lor h_2^* \), \( h_1^* \sim n^{-\frac{1}{2\beta+1}}, h_2^* \sim (\alpha^2 n)^{-\frac{1}{2\beta+2}} \) we obtain

\[
\sup_{f \in \mathcal{E}} \mathbb{E} \left( \left| \hat{f}_h(t) - f(t) \right|^2 \right) \lesssim (c + c \|K\|_2^2 + \|K\|_\infty^2) \left\{ n^{-\frac{2\beta}{2\beta+1}} \lor (\alpha^2 n)^{-\frac{2\beta}{2\beta+2}} \right\}.
\]

As expected (see related literature in Section 2) there is a twofold deterioration in the rate in the private regime. Firstly, the exponent changes from \(-\frac{2\beta}{2\beta+1}\) to \(-\frac{2\beta}{2\beta+2}\). Secondly, in the private regime the effective sample size is given by \( \alpha^2 n \), which makes the influence of the privacy parameter explicit. For example the Hölder class of smoothness \( \beta > 0 \) (we refer to Definition 2.3 in Comte [2017]) is covered by Corollary 3.2.

### 3.2 Lower bound

Let \( \beta, L > 0 \). The Hölder class \( \Sigma(\beta, L) \) on \( \mathbb{R} \) is the set of \( l = \lfloor \beta \rfloor \)-times differentiable functions \( f : \mathbb{R} \rightarrow \mathbb{R} \) such that the \( l \)-th derivative satisfies the Hölder equation

\[
\left| f^{(l)}(x) - f^{(l)}(x') \right| \leq L |x - x'|^{\beta - l}, \quad \forall x, x' \in \mathbb{R}.
\]

We additional introduce the class of Hölder smooth densities

\[
\mathcal{P}(\beta, L) = \{ f : \mathbb{R} \rightarrow \mathbb{R}, f \text{ is a density, } f \in \Sigma(\beta, L) \}
\]

Proposition 1.2 in Tsybakov [2009] shows that for \( f \in \mathcal{P}(\beta, L) \) we have \( (f(t) - f_h(t))^2 \leq c h^{2\beta} \), where \( c \) is a constant only dependent on \( K, L \) and \( \beta \). That is, the Hölder class of densities fits into the framework of Corollary 3.2. We provide the following lower bound for privatised pointwise density estimation over a Hölder class. We point out that for small \( \alpha \) we have \( \alpha \approx \exp(\alpha) - 1 \), hence, the lower bound matches the upper bound given in Corollary 3.2. Recall that \( \mathcal{Q}_\alpha \) denotes the family of all privatisation mechanisms satisfying \( \alpha \)-differential privacy. For \( Q \in \mathcal{Q}_\alpha \) and the true density \( f \) we use the symbol \( \mathbb{E}_{f,Q} \) for the expectation of the privatised data.
Theorem 3.3 (Lower bound).
Let $\beta, L > 0$. For a constant $C > 0$ only depending on $\beta$ and $L$ we have
\[
\liminf_{n \to \infty} \inf_{Q \in \mathcal{Q}_n} \inf_{f \in \mathcal{P}(\beta, L)} \sup_{t \in P} \mathbb{E}_f, Q \left( \left( n(\exp(\alpha) - 1)^2 \right)^{\frac{2\beta}{\beta+2}} |T_n - f(t)|^2 \right) \geq C,
\]
where $\inf_{T_n}$ denotes the infimum over all possible estimators based on the privatised data.

Proof of Theorem 3.3. We use a general reduction scheme for lower bounds based on two hypotheses (cf. Tsybakov [2009], Section 2.2). Note that for $\psi_n = (n(\exp(\alpha) - 1)^2)^{\frac{2\beta}{\beta+2}}$ due to Markov’s inequality it is sufficient to prove
\[
\inf_{T_n} \inf_{Q \in \mathcal{Q}_n} \inf_{f \in \mathcal{P}(\beta, L)} \sup_{t \in P} \mathbb{E}_f, Q \left( \psi_n |T_n - f(t)|^2 \right) \geq A^2 \inf_{T_n} \inf_{Q \in \mathcal{Q}_n} \max_{f \in P} \mathbb{P}(|T_n - f| \geq A\psi_n) \geq C > 0,
\]
where $f_0, f_1$ are two hypotheses in $\mathcal{P}(\beta, L), A > 0$. Below we construct $f_0, f_1$ such that

(i) $f_0, f_1 \in \mathcal{P}(\beta, L),$

(ii) $(f_0(t) - f_1(t))^2 \geq 4s_n^2, s_n = A\psi_n,$

(iii) $\sup_{Q \in \mathcal{Q}_n} \text{KL}(\mathbb{P}_{f_1, Q}, \mathbb{P}_{f_0, Q}) \leq c < \infty,$

for $n$ large enough. Part(iii) of Theorem 2.2. in Tsybakov [2009] implies that
\[
\sup_{f \in \mathcal{P}(\beta, L)} \mathbb{P}_f(|T_n - f| \geq s_n) \geq \max_{j \in \{0, 1\}} \mathbb{P}_{f_j}(|T_n - f_j| \geq s_n) \geq \max \left( \left( \frac{1}{2} \exp(-c) \right)^{\frac{1}{2}}, \left( \frac{1}{2} \right) \right).
\]

Let us now construct $f_0, f_1$ such that (i),(ii) and (iii) are satisfied. Let $\varphi_\sigma$ be the density of a normal distribution with mean 0 and variance $\sigma^2$. We choose the hypotheses
\[
f_0(x) := \varphi_\sigma(x), \quad \text{and} \quad f_1(x) := \varphi_\sigma(x) + \frac{L}{2} h_\beta^2 H \left( \frac{x - l}{h_n} \right), \quad x \in \mathbb{R},
\]
where $h_n = (n(\exp(\alpha) - 1)^2)^{-\frac{1}{\beta+2}}$ and $H : \mathbb{R} \to [0, \infty)$ satisfies $H \in \Sigma(\beta, \frac{1}{2}) H(0) > 0, \|H\|_\infty < \infty, \|H\|_1 < \infty$ and $\int H(u)du = 0$. One can e.g. choose $H(x) = K(x) - K(x - 1)$ with $K$ as in equation (2.34) of Tsybakov [2009]. Let us check that $f_0, f_1$ satisfy the desired constraints.

(i) $f_0, f_1 \in \mathcal{P}(\beta, L)$

Clearly, $f_0 \in \Sigma(\beta, \frac{1}{2})$ for $\sigma$ large enough and, hence, $f_0 \in \mathcal{P}(\beta, L)$. Moreover, for $l = |\beta|, x, x' \in \mathbb{R}$ we have
\[
f_1(l)(x) = \varphi_\sigma(l)(x) + \frac{L}{2} h_\beta^2 H \left( \frac{x - l}{h_n} \right). \quad \text{Thus we obtain for} \quad x, x' \in \mathbb{R}
\]
\[
|f_1(l)(x) - f_1(l)(x')| = \left| \varphi_\sigma(l)(x) - \varphi_\sigma(l)(x') \right| + \frac{L}{2} h_\beta^{-1} \left| H(l) \left( \frac{x - l}{h_n} \right) - H(l) \left( \frac{x' - l}{h_n} \right) \right|
\]
\[
\leq \frac{L}{2} |x - x'|^{1-\beta-l} + \frac{L}{2} h_\beta^{-1} \left| \frac{x - x'}{h_n} \right|^{\beta-l} = L |x - x'|^{\beta-l}.
\]

Furthermore, since $H$ integrates to 0 and is bounded, $f_1$ is a density for $n$ large enough.
As illustrated in H ⊆ bandwidth, which is inspired by the method of Goldenshluger and Lepski [2011], and show it’s
This finishes the proof.

We mimic the notation of Comte [2017] and denote
more elements in the collection of bandwidths. We propose the following bandwidth selection

Due to the adaptation, we choose the bandwidth ˆh := 600 and ˆc := 432. Here ˆA(h) and ˆV(h) corresponds to a bias^2- respectively variance-estimate for the estimator ˆf_h(t), where the variance term is scaled up by a log-term due to the adaptation. We choose the bandwidth ˆh as a minimiser

\[ \hat{h} \in \arg\min_{h \in \mathcal{H}} \{ \hat{A}(h) + \hat{V}(h) \}. \] (3.2)

In contrast to the (non-private) bandwidth selection procedure presented in Comte [2017], we have a random variance estimate ˆV(h) that does not require an a priori knowledge of an upper bound for \( \|f\|_\infty \). The classical Goldenshluger-Lepski method also uses a double convolution term as an estimate of the bias^2, which requires knowledge of the entire function t \mapsto K_h(X_i - t) for each h \in \mathcal{H}. Since this is not available in the private setting, the bias^2 estimate had to be modified. The next theorem shows the optimality of our proposed procedure. Let us define the deterministic counterparts of ˆA(h) and ˆV(h), i.e. a bias^2- and a variance-term

\[ \text{bias}^2(h) := \sup_{\eta \leq h} |f_\eta(t) - f(t)| \]

\[ V(h) := \left( c_1 \frac{\sigma_h^2}{n} + c_2 \frac{1}{nh} \right) \log n, \quad \text{with } \sigma_h^2 := E Z_{i,h}^2 \]
with $c_1$ and $c_2$ as above. Note that (up to constants) $V(h)$ is bounded by the variance term appearing in Proposition 3.1 with an additional log-term that is due to the adaptation. The values for the universal constants $c_1$ and $c_2$, though convenient for deriving the theory, are far too large in practice. Typically, their values are determined by means of preliminary simulations as proposed in Comte et al. [2006], for example.

**Theorem 3.4 (Oracle inequality).** Let $\alpha \in (0, 1)$ and $\mathcal{H} \subseteq (0, 1)$ be a collection of bandwidths satisfying $|H| \leq n$ such that $n h \geq \max (\log n, 1)$ for all $h \in \mathcal{H}$. Let $\hat{h}$ be defined by (3.2), then

$$E \left( \left| \hat{f}_h(t) - f(t) \right|^2 \right) \leq 16 \min_{h \in \mathcal{H}} \left\{ \text{bias}^2(h) + V(h) \right\} + \frac{C}{n \alpha^2},$$

where $C$ is a constant only depending on $\|K\|_\infty$.

Before proving the oracle inequality, let us derive an immediate implication in the context of Corollary 3.2. An easy computation shows the following corollary and we omit its proof.

**Corollary 3.5.** Consider the class $\mathcal{E}$ as in Corollary 3.2. Let $\mathcal{H} := \left\{ \frac{1}{n^k}, k \in \{1, \ldots, n\} \right\}$ and define $\hat{h}$ by (3.2). There exists a constant $c$ only depending on $\|K\|_\infty$ such that for all $n \in \mathbb{N}$ we have

$$\sup_{f \in \mathcal{E}} E \left( \left| \hat{f}_h(t) - f(t) \right|^2 \right) \leq c \left\{ \left( \frac{n}{\log n} \right)^{-\frac{2\beta}{n^2+2}} \right\}.$$

Proof of Theorem 3.4. Due to the key argument given in Lemma 3.6 we have for each $h \in \mathcal{H}$

$$\left| \hat{f}_h(t) - f(t) \right|^2 \leq 16 \text{bias}^2(h) + \frac{4}{3} V(h) + 4 \hat{V}(h)$$

$$+ 28 \max_{\eta \in \mathcal{H}} \left\{ \left| f_\eta(t) - \hat{f}_\eta(t) \right|^2 - \frac{V(\eta)}{3} \right\} + 8 \max_{\eta \in \mathcal{H}} \left\{ V(\eta) - \hat{V}(\eta) \right\}.$$ \hspace{1cm} (3.3)

The last term is bounded as in Lemma 3.7. Furthermore, we have $E \hat{V}(h) \leq 2V(h)$, since $E \sigma_h^2 = \sigma_h^2$. It remains to bound the expectation of the second last term. Note that we have

$$\max_{\eta \in \mathcal{H}} \left\{ \left| f_\eta(t) - \hat{f}_\eta(t) \right|^2 - \frac{V(\eta)}{3} \right\} = \left\{ \left( f_\eta(t) - \hat{f}_\eta(t) \right)^2 - \frac{V(\eta)}{3} \right\}.$$ \hspace{1cm} (3.3)

For $\eta \in \mathcal{H}$ and $i \in \{1, \ldots, n\}$ we write $S_{i, \eta} := g_\eta(X_i) = K_\eta(X_i - t)$ such that

$$\hat{f}_\eta(t) - f_\eta(t) = \frac{1}{n} \sum_{i=1}^n (S_{i, \eta} - E S_{i, \eta}) + b_\eta \frac{1}{n} \sum_{i=1}^n \xi_{i, \eta}.$$
The claim follows from Lemmata 3.8 and 3.9 combined with (3.3), $|\mathcal{H}| \leq n$ and $b_n^2 \leq \frac{4|\mathcal{H}|^2}{n^2} n^4$.

**Lemma 3.6 (Key argument).** For all collections $\mathcal{H} \subseteq (0, \infty)$ and all $h \in \mathcal{H}$ we have

$$
\left| \hat{f}_h(t) - f(t) \right|^2 \leq 16\text{bias}^2(h) + \frac{4}{3} V(h) + 4\hat{V}(h) + 28 \max_{\eta \in \mathcal{H}} \left\{ |f_\eta(t) - \hat{f}_\eta(t)|^2 - \frac{V(\eta)}{3} \right\}_+ + 8 \max_{\eta \in \mathcal{H}} \{ V(\eta) - \hat{V}(\eta) \}._+
$$

**Proof of Lemma 3.6.** We have $\left| \hat{f}_h(t) - f(t) \right|^2 \leq 2 \left| \hat{f}_h(t) - \hat{f}_h(t) \right|^2 + 2 \left| \hat{f}_h(t) - f(t) \right|^2$ for all $h \in \mathcal{H}$. Furthermore, for $h \in \mathcal{H}$ we obtain

$$
\left| \hat{f}_h(t) - \hat{f}_h(t) \right|^2 = \left| \hat{f}_h(t) - \hat{f}_h(t) \right|^2 - \hat{V}(\hat{h} \vee h) - \hat{V}(\hat{h} \wedge h) + \hat{V}(\hat{h} \wedge h) + \hat{V}(\hat{h} \wedge h)
\leq \hat{A}(\hat{h} \vee h) + \hat{V}(\hat{h} \vee h) + \hat{V}(\hat{h} \wedge h)
\leq \hat{A}(\hat{h} \vee h) + \hat{V}(\hat{h} \vee h) + \hat{A}(\hat{h} \wedge h) + \hat{V}(\hat{h} \wedge h)
\leq 2 \left( \hat{A}(h) + \hat{V}(h) \right),
$$

where we exploited the definition of $\hat{A}$, the positivity of both $\hat{A}$ and $\hat{V}$ and the minimising property of $\hat{h}$. The term $\hat{A}(h)$ can further be bounded by

$$
\hat{A}(h) \leq \max_{\eta \in \mathcal{H}} \max_{\eta \leq h} \left\{ 3 |\hat{f}_h(t) - f(h)|^2 + 3 |f_\eta(t) - f_\eta(t)|^2 + 3 |f_\eta(t) - \hat{f}_\eta(t)|^2 - \hat{V}(\eta) - \hat{V}(h) \right\}_+
\leq \max_{\eta \in \mathcal{H}} \max_{\eta \leq h} \left\{ 3 |\hat{f}_h(t) - f(h)|^2 + 3 |f_\eta(t) - f_\eta(t)|^2 - V(\eta) - V(h) \right\}_+
+ 3 \max_{\eta \in \mathcal{H}} \max_{\eta \leq h} \left\{ V(\eta) - \hat{V}(\eta) \right\}_+ + \left\{ V(h) - \hat{V}(h) \right\}_+
\leq 3 \max_{\eta \in \mathcal{H}} \max_{\eta \leq h} \left\{ |f_\eta(t) - \hat{f}_\eta(t)|^2 - \frac{V(\eta)}{3} \right\}_+ + 3 \left\{ |f_\eta(t) - \hat{f}_\eta(t)|^2 - \frac{V(h)}{3} \right\}_+ + 3\text{bias}^2(h)
+ 2 \max_{\eta \in \mathcal{H}} \max_{\eta \leq h} \left\{ V(\eta) - \hat{V}(\eta) \right\}_+
\leq 6 \max_{\eta \in \mathcal{H}} \max_{\eta \leq h} \left\{ |f_\eta(t) - \hat{f}_\eta(t)|^2 - \frac{V(\eta)}{3} \right\}_+ + 3\text{bias}^2(h) + 2 \max_{\eta \in \mathcal{H}} \max_{\eta \leq h} \left\{ V(\eta) - \hat{V}(\eta) \right\}_+.\quad (3.5)
$$

Moreover, we bound

$$
\left| \hat{f}_h(t) - f(t) \right|^2 \leq 2 \left| \hat{f}_h(t) - \hat{f}_h(t) \right|^2 + 2 |f_\eta(t) - f(t)|^2
\leq 2 \left\{ |f_\eta(t) - \hat{f}_\eta(t)|^2 - \frac{V(\eta)}{3} \right\}_+ + \frac{2V(h)}{3} + 2\text{bias}^2(h).
$$

Combining the bounds (3.4), (3.5) and (3.6) yields the desired result. 

**Lemma 3.7 (Concentration of $\hat{V}$).** We have

$$
\mathbb{E} \max_{\eta \in \mathcal{H}} \left\{ V(\eta) - \hat{V}(\eta) \right\}_+ \leq \frac{C}{n\alpha^2}.
$$
where \( C \) is a constant only depending on \( \| K \|_\infty \).

**Proof.** We bound the maximum by the sum,

\[
\mathbb{E} \max_{\eta \in \mathcal{H}} \left\{ V(\eta) - \tilde{V}(\eta) \right\} \leq \sum_{\eta \in \mathcal{H}} \mathbb{E} \left\{ V(\eta) - \tilde{V}(\eta) \right\} = c_1 \frac{\log n}{n} \sum_{\eta \in \mathcal{H}} \mathbb{E} \left\{ \sigma_\eta^2 - 2\sigma_\eta^3 \right\}.
\]

Consider the event \( \Omega_\eta := \left\{ \left| \sigma_\eta^2 - \hat{\sigma}_\eta^2 \right| \leq \frac{\sigma_\eta^2}{2} \right\} \) and its complement \( \Omega^c_\eta \). On \( \Omega_\eta \) we have \( \sigma_\eta^2 \leq 2\hat{\sigma}_\eta^2 \) and thus,

\[
\left\{ \sigma_\eta^2 - 2\hat{\sigma}_\eta^2 \right\} = \left\{ \sigma_\eta^2 - 2\sigma_\eta^2 \right\} + \mathbf{1}_{\Omega^c_\eta} < 2\sigma_\eta^2 \left| \frac{\hat{\sigma}_\eta^2}{\sigma_\eta^2} - 1 \right| \mathbf{1}_{\Omega^c_\eta}.
\]

Let \( m \in \mathbb{N} \), to be chosen below. On \( \Omega^c_\eta \) we have \( \left| \frac{\hat{\sigma}_\eta^2}{\sigma_\eta^2} - 1 \right| > \frac{1}{2} \), hence we obtain

\[
\mathbb{E} \max_{\eta \in \mathcal{H}} \left\{ V(\eta) - \tilde{V}(\eta) \right\} \leq 2c_1 \frac{\log n}{n} \sum_{\eta \in \mathcal{H}} \mathbb{E} \left\{ \frac{\hat{\sigma}_\eta^2}{\sigma_\eta^2} \right\}^{2m-1}.
\] (3.7)

Let us find an upper bound for \( \mathbb{E} \left( \left| \frac{\hat{\sigma}_\eta^2}{\sigma_\eta^2} - 1 \right| \right)^m \). Note that with \( U_{i,\eta} = \frac{Z_{i,\eta}}{\sigma_\eta} - 1 \) we have \( \frac{\hat{\sigma}_\eta^2}{\sigma_\eta^2} - 1 = \frac{1}{n} \sum_{i=1}^n U_{i,\eta} \), where \( U_{i,\eta}, i \in \{1, \ldots, n\} \) are independent random variables with \( \mathbb{E} U_{i,\eta} = 0 \). We apply Petrov’s inequality (Proposition 5.2), which shows that for \( m \geq 2 \)

\[
\mathbb{E} \left( \left| \frac{\hat{\sigma}_\eta^2}{\sigma_\eta^2} - 1 \right| \right)^m = \mathbb{E} \left| \frac{1}{n} \sum_{i=1}^n U_{i,\eta} \right|^m \leq c_m n^{-m/2-1} \sum_{i=1}^n \mathbb{E} |U_{i,\eta}|^m
\] (3.8)

for a constant \( c_m \) only depending on \( m \). It remains to find an upper bound for \( \mathbb{E} |U_{i,\eta}|^m \). First note that due to Jensen’s inequality \( \mathbb{E} |U_{i,\eta}|^m = \mathbb{E} \left| \frac{Z_{i,\eta}}{\sigma_\eta} - 1 \right|^m \leq 2m \left( 1 + \mathbb{E} \left| \frac{Z_{i,\eta}}{\sigma_\eta} \right| \right)^m \). Moreover,

\[
\mathbb{E} \left| \frac{Z_{i,\eta}}{\sigma_\eta} \right|^m \leq 2m \frac{\mathbb{E} S_{1,\eta}^2 + \mathbb{E} T_{1,\eta}^2}{\sigma_\eta^2}.
\] (3.9)

Since for \( X \sim \text{Laplace}(0,1) \) we have \( \mathbb{E} |X|^m = m! \) (cp. Reminder 2.6), we obtain

\[
\mathbb{E} T_{1,\eta}^2 = b_0^2 (2m)! \quad \text{and} \quad \mathbb{E} S_{1,\eta}^2 = \mathbb{E} K_\eta^2 (X_1 - t) \leq \frac{1}{\eta^2m} \| K \|_\infty^2.
\]

Inserting these bounds into (3.9) and exploiting \( \sigma_\eta^2 / b_0^2 \geq 2 \) yields

\[
\mathbb{E} \left| \frac{Z_{i,\eta}}{\sigma_\eta} \right|^m \leq 2m \| K \|_\infty^{2m} \eta^{-2m} + b_0^2 (2m)! \| K \|_\infty^{2m} \eta^{-2m} b_0^{-2m} + (2m)!
\]

Recall that \( b_0^2 = \frac{4|\mathcal{H}|^2 \| K \|_\alpha^2}{\alpha^2 \sigma_\eta^2} \) and that \( \alpha / |\mathcal{H}| \leq 1 \). Therefore, we obtain

\[
\mathbb{E} \left| \frac{Z_{i,\eta}}{\sigma_\eta} \right|^m \leq 4^{-2m} + (2m)!
\]

and with (3.8) finally

\[
\mathbb{E} \left( \left| \frac{\hat{\sigma}_\eta^2}{\sigma_\eta^2} - 1 \right| \right)^m \leq c_m n^{-m/2-1} \sum_{i=1}^n 2^m \left( 1 + 4^{-2m} + (2m)! \right) \leq C_m n^{-m/2},
\]
where $C_m$ is a constant only dependent on $m$. Inserting this bound into (3.7) we obtain
\[
\mathbb{E} \max_{\eta \in \mathcal{H}} \left\{ V(\eta) - \hat{V}(\eta) \right\} + \leq 2c_1 \frac{\log n}{n} \frac{\sigma^2}{\eta} \sum_{\eta \in \mathcal{H}} \frac{n}{m^{\eta/2}}
\]
Since $n\eta \geq 1$ and $|\mathcal{H}| \leq n$ we have
\[
\sigma^2 \leq \frac{1}{\eta^2} \| K \|_2^2 + 8\frac{|\mathcal{H}|^2}{\alpha^2 \eta^2} \| K \|_2^2 \leq n^2 \| K \|_2^2 + 8n^4 \| K \|_2^2 \frac{1}{\alpha^2},
\]
i.e. $n^{-4} \sigma^2 \leq \| K \|_\infty (1 + \frac{8}{\alpha^2})$. Hence with $m = 12$ we obtain the claim.

**Lemma 3.8 (Concentration for $\sum_{i=1}^n S_{i,\eta}$).**
Recall the definition $V_S(\eta) = \left( c_1 \frac{\mathbb{E} (S_{\eta,1})}{\eta} + c_2 \frac{\mathbb{E} (S_{\eta,n})}{n^\eta} \right) \log n$. We have
\[
\mathbb{E} \left\{ \frac{1}{n} \sum_{i=1}^n (S_{i,\eta} - \mathbb{E} S_{i,\eta}) \right\}^2 \leq \frac{V_S(\eta)}{6} \leq 128 \| K \|_\infty \frac{1}{n^2}.
\]

**Proof of Lemma 3.8.** We have
\[
\mathbb{E} \left\{ \frac{1}{n} \sum_{i=1}^n (S_{i,\eta} - \mathbb{E} S_{i,\eta}) \right\}^2 \leq \frac{V_S(\eta)}{6} \leq 128 \| K \|_\infty \frac{1}{n^2}.
\]

Theorem 5.1 yields for $x = \sqrt{\frac{V_S(\eta)}{6}} + x$ the following
\[
\mathbb{P} \left( \frac{1}{n} \sum_{i=1}^n (S_{i,\eta} - \mathbb{E} S_{i,\eta}) \right) \geq \sqrt{\frac{V_S(\eta)}{6} + x}
\leq 2 \exp \left( -\frac{n}{4\sqrt{x}} \frac{V_S(\eta)}{6} + x \right), \exp \left( -\frac{n}{4b} \sqrt{\frac{V_S(\eta)}{6} + x} \right)
\leq 2 \exp(-3 \log n) \max \left\{ \exp \left( -\frac{n}{4\sqrt{x}} \right), \exp \left( -\frac{n}{4\sqrt{2b}} \sqrt{x} \right) \right\},
\]
since $\sqrt{2\sqrt{x} + y} \geq \sqrt{x} + \sqrt{y}$ for $x, y \geq 0$ and $n\eta \geq \log n$. Finally, we obtain
\[
\mathbb{P} \left( \frac{1}{n} \sum_{i=1}^n (S_{i,\eta} - \mathbb{E} S_{i,\eta}) \right) \geq \sqrt{\frac{V_S(\eta)}{6} + x} \leq \frac{2}{n^3} \max \left\{ \exp(-\tau_1 x), \exp(-\tau_2 \sqrt{x}) \right\}. \quad (3.11)
\]
with $\tau_1 = \frac{n}{4\sqrt{2b}}$ and $\tau_2 = \frac{n}{4\sqrt{2b}}$. Note that we have $\int_0^\infty \exp(-\tau_1 x) dx = \frac{1}{\tau_1} \leq 4 \| K \|_\infty n$ and $\int_0^\infty \exp(-\tau_2 \sqrt{x}) dx = \frac{2}{\tau_2} \leq 4^3 \| K \|_\infty n$, where we again exploited that $n\eta \geq 1$. Hence, inserting the bound (3.11) into (3.10) and integrating yields the result. 

14
Lemma 3.9 (Concentration for $\sum_{i=1}^n \xi_{i, \eta}$). Recall the definition $V_\xi = \frac{2c_1 n}{n} \log n$. We have

$$E \left\{ \left| \frac{1}{n} \sum_{i=1}^n \xi_{i, \eta} \right|^2 - \frac{V_\xi}{6} \right\} + \leq \frac{32}{n^6}.$$

Proof of Lemma 3.9. We have

$$E \left\{ \left| \frac{1}{n} \sum_{i=1}^n \xi_{i, \eta} \right|^2 - \frac{V_\xi}{6} \right\} = \int_0^\infty P \left( \left| \frac{1}{n} \sum_{i=1}^n \xi_{i, \eta} \right| \geq \sqrt{\frac{V_\xi}{6} + x} \right) \, dx. \tag{3.12}$$

Consequently, the tail bound Proposition 5.3 for $\varepsilon = \sqrt{\frac{V_\xi}{6} + x}$ with $V_\xi = \frac{2c_1 n}{n} \log n$ yields

$$P \left( \left| \frac{1}{n} \sum_{i=1}^n \xi_{i, \eta} \right| \geq \sqrt{\frac{V_\xi}{6} + x} \right) \leq 2 \max \left\{ \exp \left( -\frac{n\varepsilon^2}{16} \right), \exp \left( -\frac{n\varepsilon}{2} \right) \right\} \leq 2 \max \left\{ \exp \left( -\frac{n}{16} \left( \frac{V_\xi}{6} + x \right) \right), \exp \left( -\frac{n}{2} \sqrt{\frac{V_\xi}{6} + x} \right) \right\} \leq 2 \exp(-5 \log n) \max \left\{ \exp \left( -\frac{n}{16} x \right), \exp \left( -\frac{n}{2} \sqrt{x} \right) \right\},$$

where we used $\sqrt{2\sqrt{x} + y} \geq \sqrt{x} + \sqrt{y}$ for $x, y \geq 0$. Hence, for $\tau_1 = \frac{1}{16}$ and $\tau_2 = \frac{n}{2\sqrt{2}}$ we obtain

$$P \left( \left| \frac{1}{n} \sum_{i=1}^n \xi_{i, \eta} \right| \geq \sqrt{\frac{V_\xi}{6} + x} \right) \leq \frac{2}{n^3} \max \left\{ \exp(-\tau_1 x), \exp(-\tau_2 \sqrt{x}) \right\}. \tag{3.13}$$

Note that we have $\int_0^\infty \exp(-\tau_1 x) \, dx = \frac{1}{\tau_1} = \frac{10}{n}$ and $\int_0^\infty \exp(-\tau_2 \sqrt{x}) \, dx = \frac{2}{\tau_2} = \frac{8}{n}$. Thus, inserting the bound (3.13) into (3.12) and integrating yields the result. 

4 Privatized projection density estimation

4.1 Upper bound for PDE

In this section we consider a privatized version of the projection density estimator (2.2) where we assume the true density to belong to $L^2([0, 1])$. Here and subsequently, let $\{\varphi_j\}_{j \in \mathbb{N}}$ denote an arbitrary orthonormal system of $L^2([0, 1])$ satisfying the following assumption.

**Assumption 4.1.** There is a finite constant $\phi_o$ such that $\left\| \sum_{j=1}^d \varphi_j^2 \right\|_\infty \leq d\phi_o$ for all $d \in \mathbb{N}$.

According to Lemma 6 of Birgé and Massart [1997] our Assumption 4.1 is exactly equivalent to following property: there exists a positive constant $\phi_o$ such that for any $h \in \text{lin}(\{\varphi_i\}_{i \in \{1, \ldots, d\}})$ we have $\|h\|_2^2 \leq \phi_o d \|h\|_2^2$. Typical examples are bounded basis, such as the trigonometric basis, or bases satisfying the condition: there exists a positive constant $C$ such that for any $c \in \mathbb{R}^d$, $\|\sum_{i=1}^d c_i \varphi_i\|_\infty^2 \leq Cd^2 |c|_\infty^2$ where $|c|_\infty = \max\{|c_i|_{i \in \{1, \ldots, d\}}$. Birgé and Massart [1997] show that the last property is satisfied for piecewise polynomials, splines and wavelets.
Consider the following class of densities

$$\mathcal{E} = \left\{ f \in L^2([0, 1]) : f \text{ is a density}, \|f\|_\infty \leq c, \sup\{d^2|f_d(t) - f(t)|^2 : d \in \mathbb{N}\} \leq c \right\}$$
for a universal constant $c > 0$. With $d = d_1^* \land d_2^*, d_1^* \sim n^{\frac{1}{2m+1}}, d_2^* \sim (\alpha^2 n)^{\frac{1}{2m+2}}$ we obtain

$$\sup_{f \in \mathcal{E}}(f \hat{d}(t) - f(t))^2 \lesssim (c + c_0^2 + \phi_0^2) \left\{ n^{-\frac{2m}{2m+1}} \lor (\alpha^2 n)^{-\frac{2m}{2m+2}} \right\}.$$ 

We observe the same two-fold deterioration in the rate in the private regime as in kernel density estimation. As an example for a class of densities that is covered by Corollary 4.3 we refer to Definition 1.11. in Tsybakov [2009] (Sobolev spaces).

### 4.2 Lower bound

Let $\beta \geq 1$ be an integer and $L > 0$. The Sobolev class $\mathcal{W}(\beta, L)$ is the set of all $(\beta - 1)$-times differentiable functions $f : [0, 1] \to \mathbb{R}$ such that the $(\beta - 1)$-th derivative is absolutely continuous and satisfies $\int (f(\beta)(x))^2 \, dx \leq L^2$ (cp. Section 1.7.1 in Tsybakov [2009]). We denote by $\mathcal{P}_\mathcal{W}(\beta, L)$ the set of densities in $\mathcal{W}(\beta, L)$.

**Theorem 4.4 (Lower bound).**

Let $\beta, L > 0$. For a constant $C > 0$ only depending on $\beta$ and $L$ we have

$$\liminf_{n \to \infty} \inf_{T_n} \inf_{Q \in \mathcal{Q}_\alpha} \sup_{f \in \mathcal{P}_\mathcal{W}(\beta, L)} \mathbb{E}_{f, Q} \left( n(\exp(\alpha) - 1)^2 \right)^{\frac{2m}{2m+2}} \| T_n - f(t) \|^2 \geq C,$$

where $\inf_{T_n}$ denotes the infimum over all possible estimators based on the privatised data.

**Proof of Theorem 4.4.** It is easily verified that the Hölder class $\Sigma(\beta, L)$ restricted to $[0, 1]$ is contained in the Sobolev class $\mathcal{W}(\beta, L)$ (cf. also Section 1.7.1 in Tsybakov [2009]). Hence, the densities constructed in the reduction scheme in the proof of the lower bound of Theorem 4.4 are contained in $\mathcal{P}_\mathcal{W}(\beta, L)$, if we replace the normal density $\phi_\sigma$ by the uniform density to guarantee that their support is contained in $[0, 1]$. The proof then follows line by line the proof of Theorem 4.4. \hfill \square

### 4.3 Adaptation for privatised PDE

Similar to Section 3.3 we propose the following dimension selection method. Let $\mathcal{D} \subseteq \mathbb{N}$ be a finite collection of dimension. As before we adjust the privacy parameter $\alpha$ in (4.1) to $\frac{N}{|\mathcal{D}|}$ and for $d \in \mathcal{D}$ define

$$\hat{A}(d) := \max_{D \in \mathcal{D}} \left\{ \left| \hat{f}_d(t) - f_D(t) \right|^2 - \left( \hat{V}(d) + \hat{V}(D) \right) \right\} +,$$

$$\hat{V}(d) := \left( 2c_1 \frac{\hat{d}_2^2}{n} + c_2 \frac{d}{n} \right) \log n, \quad \text{with} \quad \hat{d}_2 := \frac{1}{n} \sum_{i=1}^{n} Z_{i,d},$$

where $c_1 := 600$ and $c_2 := 432$ as in Section 3.3. We choose the dimension $\hat{d}$ as a minimiser

$$\hat{d} \in \arg \min_{d \in \mathcal{D}} \left\{ \hat{A}(d) + \hat{V}(d) \right\}. \quad (4.2)$$

Let us define the deterministic counterpart of $V(d)$, i.e. a bias$^2$- and a variance-term

$$\text{bias}^2(d) := \sup_{D \geq d} \left| f_D(t) - f(t) \right|^2$$

$$V(d) := \left( c_1 \frac{\sigma_2^2}{n} + c_2 \frac{d}{n} \right) \log n, \quad \text{with} \quad \sigma_2^2 := \mathbb{E} Z_{1,d}^2$$

17
with \( c_1 \) and \( c_2 \) as above. We see at once that identifying \( \frac{1}{d} \) with \( h \) in the results of Section 3.3, the following oracle inequality is proven by exactly the same methods as Theorem 3.4. For the convenience of the reader we repeat the relevant steps of the proof, thus making our exposition complete.

**Theorem 4.5 (Oracle inequality).** Let \( \alpha \in (0, 1) \) and \( D \subseteq N \) be a collection of dimension parameters satisfying \(|D| \leq n\) such that \( n/d \geq \max(\log n, 1)\) for all \( d \in D \). Let \( \hat{d} \) be defined by (4.2), then there is a constant \( C \) depending only on \( \phi_0 \) such that

\[
\mathbb{E} \left( \left| \hat{f}(t) - f(t) \right|^2 \right) \leq 16 \min_{d \in D} \left\{ \text{bias}(d) + V(d) \right\} + \frac{C \phi_0^2}{n \alpha^2}.
\]

Similarly to Section 3.3 before proving the oracle inequality, we first provide an immediate implication in the context of Corollary 4.3 and omit its proof.

**Corollary 4.6.** Consider the class \( \mathcal{E} \) as in Corollary 4.3. Let \( D := \{1, \ldots, n\} \) and define \( \hat{d} \) by (4.2). There exists a constant \( c \) only depending on \( \phi_0 \) such that for all \( n \in \mathbb{N} \) we have

\[
\sup_{f \in \mathcal{E}} \mathbb{E} \left( \left| \hat{f}(t) - f(t) \right|^2 \right) \lesssim c \left\{ \left( \frac{n}{\log n} \right)^{- \frac{2 \alpha^2}{n+2}} \vee \left( \frac{n \alpha^2}{\log n} \right)^{- \frac{2 \alpha^2}{n+2}} \right\}.
\]

**Proof of Theorem 4.5.** Identifying \( h = d^{-1} \) the key argument given in Lemma 3.6 for each \( d \in D \) implies

\[
\left| \hat{f}(t) - f(t) \right|^2 \leq 16 \text{bias}(d)^2 + \frac{4}{3} V(d) + 4 \hat{V}(d)
\]

\[
+ 28 \max_{D \in D} \left\{ \left| f_D(t) - \hat{f}_D(t) \right|^2 - \frac{V(D)}{3} \right\}_+ + 8 \max_{D \in D} \left\{ V(D) - \hat{V}(D) \right\}_+.
\]

The last term is bounded as in Lemma 4.7. Furthermore, we have \( \mathbb{E} \hat{V}(d) \leq 2V(d) \), since \( \mathbb{E} \hat{\sigma}^2_d = \sigma^2_d \). It remains to bound the expectation of the second last term. Note that we have

\[
\mathbb{E} \max_{D \in D} \left\{ \left| f_D(t) - \hat{f}_D(t) \right|^2 - \frac{V(D)}{3} \right\}_+ \leq \sum_{D \in D} \mathbb{E} \left\{ \left| f_D(t) - \hat{f}_D(t) \right|^2 - \frac{V(D)}{3} \right\}_+. \tag{4.3}
\]

For \( d \in D \) and \( i \in \{1, \ldots, n\} \) we write \( S_{i,d} := g_d(X_i) \) and \( b_{i,d} := 2\phi_0|D|d/\alpha \) such that

\[
\hat{f}_d(t) - f_d(t) = \frac{1}{n} \sum_{i=1}^n (S_{i,d} - \mathbb{E}S_{i,d}) + b_{i,d} - \frac{1}{n} \sum_{i=1}^n \xi_{i,d}.
\]

We also write \( V(d) = V_S(d) + b_d^2 V_\xi \) with \( V_S(d) = \left( c_1 \frac{\mathbb{E} S_{i,d}^2}{n} + c_2 \frac{d}{n} \right) \log n \) and \( V_\xi = \frac{2c_1}{n} \log n \). Hence,

\[
\mathbb{E} \left\{ \left| f_d(t) - \hat{f}_d(t) \right|^2 - \frac{V(d)}{3} \right\}_+
\]

\[
\leq 2\mathbb{E} \left\{ \left( \frac{1}{n} \sum_{i=1}^n (S_{i,h} - \mathbb{E}S_{i,h}) \right)^2 - \frac{V_S(d)}{6} \right\}_+ + 2b_d^2 \mathbb{E} \left\{ \left( \frac{1}{n} \sum_{i=1}^n \xi_{i,d} \right)^2 - \frac{V_\xi}{6} \right\}_+.
\]

The claim follows from Lemmata 4.8 and 3.9 combined with (3.3), \(|D| \leq n\) and \( b_d^2 \leq 4\phi_0^2 \alpha^{-2} n^4 \). \(\blacksquare\)
Lemma 4.7 (Concentration of $\hat{V}$). There is a universal constant $C$ such that
\[
E \max_{d \in \mathcal{D}} \left\{ V(d) - \hat{V}(d) \right\}_+ \leq \frac{C \phi_0^2}{n\alpha^2}.
\]

Proof. The proof follows line by line the proof of Lemma 3.7. Similar to (3.7) for $m \in \mathbb{N}$, to be chosen below, we conclude
\[
E \max_{d \in \mathcal{D}} \left\{ V(d) - \hat{V}(d) \right\}_+ \leq 2c_1 (\log n)n^{-1} \sum_{d \in \mathcal{D}} \sigma_d^2 2^{m-1} E \left( \left| \frac{\sigma_d^2}{\sigma_d^2} - 1 \right|^{m} \right). \tag{4.4}
\]
Moreover, we have $\sigma_d^2 \sigma_d^{-2} - 1 = \frac{1}{n} \sum_{i=1}^{n} U_{i,d}$, where $U_{i,d} := Z_{i,d}^2 \sigma_d^{-2} - 1$, $i \in \{1, \ldots, n\}$ are independent random variables with $E U_{i,d} = 0$. Consequently, making again use of (3.8) we obtain for $m \geq 2$
\[
E \left( \left| \frac{\sigma_d^2}{\sigma_d^2} - 1 \right|^{m} \right) = E \left( \frac{1}{n} \sum_{i=1}^{n} U_{i,d} \right)^{m} \leq c_m n^{-m/2-1} \sum_{i=1}^{n} \left( 1 + E \left| \frac{Z_{i,d}}{\sigma_d^2} \right|^{m} \right) \tag{4.5}
\]
for a constant $c_m$ only depending on $m$. Since
\[
ES_{1,d}^{2m} = E g_d^{2m}(X_1) \leq E \left( \sum_{j=1}^{d} \phi_j^2(X_1) \sum_{l=1}^{d} \phi_l^2(t) \right)^m \leq E \left( \sum_{j=1}^{d} \phi_j^2 \right)^{2m} \leq d^{2m} \phi_0^{2m}
\]
as in the proof of Lemma 3.7, from the bound (3.9) we obtain (exploiting $\sigma_d^2 = E(S_{1,d}^2) + 2\alpha_d^2 \geq 2\alpha_d^2$)
\[
E \left| \frac{Z_{i,d}}{\sigma_d^2} \right|^{m} \leq 2^{m} d^{2m} \phi_0^{2m} \sigma_d^{-2m} (2m)! \leq d^{2m} \phi_0^{2m} b_d^{2m} (2m)!,
\]
which together with $\alpha_d^2 = 4|\mathcal{D}|^2 \phi_0^2 d^2 \alpha^{-2}$ and $\alpha/|\mathcal{D}| \leq 1$ implies for all $d \in \mathcal{D}$
\[
E \left| \frac{Z_{i,d}}{\sigma_d^2} \right|^{m} \leq 4^{-2m} + (2m)!,
\]
Combining the last bound and (4.5) there is a constant $C_m$ only depending on $m$ such that
\[
E \left( \left| \frac{\sigma_d^2}{\sigma_d^2} - 1 \right|^{m} \right) \leq C_m n^{-m/2},
\]
Inserting this bound into (4.4) we obtain
\[
E \max_{d \in \mathcal{D}} \left\{ V(d) - \hat{V}(d) \right\}_+ \leq C_m (\log n)n^{-1-m/2} \sum_{d \in \mathcal{D}} \sigma_d^2
\]
Since $nd^{-1} \geq 1$ and $|\mathcal{D}| \leq n$ we have
\[
\sigma_d^2 \leq d^2 \phi_0^2 + 8\phi_0^2 |\mathcal{H}|^2 d^2 \alpha^{-2} \leq (n^2 + 8n^4 \alpha^{-2})\phi_0^2,
\]
i.e. $n^{-4} \sigma_d^2 \leq \phi_0^2 \left( 1 + 8\alpha^{-2} \right)$. Hence with $m = 12$ we obtain the claim. \hfill \Box
Lemma 4.8 (Concentration for $\sum_{i=1}^{n} S_{i,d}$).
Recall the definition $V_S(d) = \left( c_1 n^{-1} \mathbb{E} S_{i,d}^2 + c_2 d n^{-1} \right) \log n$. We have
\[
\mathbb{E} \left\{ \left| \frac{1}{n} \sum_{i=1}^{n} (S_{i,d} - \mathbb{E} S_{i,d}) \right|^2 - \frac{V_S(d)}{6} \right\} + \leq 128\phi_n^2 n^{-2}.
\]

Proof of Lemma 4.8. We have
\[
\mathbb{P} \left\{ \left| \frac{1}{n} \sum_{i=1}^{n} (S_{i,d} - \mathbb{E} S_{i,d}) \right|^2 - \frac{V_S(d)}{6} \right\} + = \int_{0}^{\infty} \mathbb{P} \left\{ \left| \frac{1}{n} \sum_{i=1}^{n} (S_{i,d} - \mathbb{E} S_{i,d}) \right| \geq \sqrt{\frac{V_S(d)}{6} + x} \right\} dx. \quad (4.6)
\]

We note that $S_{i,d}, i \in \{1, \ldots, n\}$, are i.i.d. with $|S_{i,d}|^2 \leq d \phi_n^2 =: b$ and $\text{var}(S_{i,d}) \leq \mathbb{E} S_{i,d}^2 =: \nu^2$. We can write $V_S(d) = \left( c_1 \frac{\nu^2}{n} + c_2 \frac{\phi_n}{n} \right) \log n$ Consequently, the Bernstein type inequality Theorem 5.1 yields for $\varepsilon = \sqrt{\frac{V_S(d)}{6} + x}$ the following
\[
\mathbb{P} \left\{ \left| \frac{1}{n} \sum_{i=1}^{n} (S_{i,d} - \mathbb{E} S_{i,d}) \right| \geq \sqrt{\frac{V_S(d)}{6} + x} \right\} \leq 2 \max \left\{ \exp \left( -\frac{n}{4\nu^2} \left( \frac{V_S(d)}{6} + x \right) \right), \exp \left( -\frac{n}{4b} \sqrt{\frac{V_S(d)}{6} + x} \right) \right\} \leq 2 \exp(-3\log n) \max \left\{ \exp \left( -\frac{n}{4\nu^2} x \right), \exp \left( -\frac{n}{4\sqrt{2b}} \sqrt{x} \right) \right\},
\]
since $\sqrt{2} \sqrt{x+y} \geq \sqrt{x} + \sqrt{y}$ for $x, y \geq 0$ and $n d^{-1} \geq \log n$. Finally, we obtain
\[
\mathbb{P} \left\{ \left| \frac{1}{n} \sum_{i=1}^{n} (S_{i,d} - \mathbb{E} S_{i,d}) \right| \geq \sqrt{\frac{V_S(d)}{6} + x} \right\} \leq \frac{2}{n^3} \max \left\{ \exp(-\tau_1 x), \exp(-\tau_2 \sqrt{x}) \right\}. \quad (4.7)
\]
with $\tau_1 = \frac{n}{4\nu^2}$ and $\tau_2 = \frac{n}{4\sqrt{2b}}$. Note that we have $\int_{0}^{\infty} \exp(-\tau_1 x) dx = \frac{1}{\tau_1} \leq 4\phi_n^2 n$ and $\int_{0}^{\infty} \exp(-\tau_2 \sqrt{x}) dx = \frac{2}{\tau_2} \leq 4^2 \phi_n^2 n$, where we again exploited that $n d^{-1} \geq 1$. Hence, inserting the bound (4.7) into (4.6) and integrating yields the result. \hfill \Box

5 Appendix

Theorem 5.1 (Bernstein inequality, Comte [2017], Appendix B, Lemma B.2).
Let $(U_i)_{i \in \{1, \ldots, n\}}$ be independent and identically distributed random variables and let $\nu^2$ and $b$ be such that $\text{var}(U_1) \leq \nu^2$ and $|U_1| \leq b$. Then, for $\varepsilon > 0$ we have
\[
\mathbb{P} \left\{ \left| \frac{1}{n} \sum_{i=1}^{n} (U_i - \mathbb{E} U_i) \right| \geq \varepsilon \right\} \leq 2 \max \left\{ \exp \left( -\frac{n\varepsilon^2}{4\nu^2} \right), \exp \left( -\frac{n\varepsilon}{4b} \right) \right\}.
\]
Proposition 5.2 (Petrov’s inequality, Petrov [1995], Theorem 2.10). Let \((U_i)_{i∈\{1,...,n\}}\) be independent with expectation zero. Then for \(m \geq 2\) there exists a constant \(c_m\) only depending on \(m\) such that

\[
E \left( \left| \sum_{i=1}^n U_i \right|^m \right) \leq c_m n^{m/2-1} \sum_{i=1}^n E (|U_i|^m).
\]

Proposition 5.3 (Tail bound for Laplace random variables). Let \((ξ_i)_{i∈\{1,...,n\}}\) be independent and identically \(\text{Laplace}(0,b)\)-distributed random variables. Then, for all \(ε \geq 0\) we have

\[
P \left( \left| \frac{1}{n} \sum_{i=1}^n ξ_i \right| \geq ε \right) \leq 2 \max \left\{ \exp \left( -\frac{nε^2}{16b^2} \right), \exp \left( -\frac{nε}{2b} \right) \right\}.
\]

Proof of Proposition 5.3. We aim to apply Lemma 8.2. in Birgé [2001]. Straightforward calculations show that for \(ξ_i \sim \text{Laplace}(0,b)\) and \(0 < t < \frac{1}{b}\) we have \(E\exp(tξ_i) = \frac{1}{1-t^2b^2}\). Hence, it follows

\[
\log \left( E \left( \exp(tS) \right) \right) = \sum_{i=1}^n \log \left( E \left( \exp(tξ_i) \right) \right) \leq \frac{nt^2b^2}{1-tb}.
\]

which implies for \(S = \sum_{i=1}^n ξ_i\) that

\[
\log \left( E \left( \exp(tS) \right) \right) = \sum_{i=1}^n \log \left( E \left( \exp(tξ_i) \right) \right) \leq \frac{nt^2b^2}{1-tb}.
\]

Thus, we can apply Lemma 8.2. from Birgé [2001] with \(a = \sqrt{n}b\) and \(b = b\). We obtain

\[
P \left( \frac{1}{n} \sum_{i=1}^n ξ_i \geq \frac{2b}{\sqrt{n}} + \frac{b}{n} x \right) \leq \exp(-x).
\]

The claim now follows from the symmetry of the centered Laplace distribution and simple rearranging of this inequality.
References

R. F. Barber and J. C. Duchi. Privacy and statistical risk: Formalisms and minimax bounds. arXiv:1412.4451, 2014.

T. Berrett and C. Butucea. Locally private non-asymptotic testing of discrete distributions is faster using interactive mechanisms. Advances in Neural Information Processing Systems, 33: 3164–3173, 2020.

L. Birgé. An alternative point of view on Lepski’s method. Institute of Mathematical Statistics Lecture Notes - Monograph Series, 36:113–133, 2001.

L. Birgé and P. Massart. From model selection to adaptive estimation. In Festschrift for Lucien Le Cam, pages 55–87. Springer, 1997.

L. D. Brown and M. G. Low. A constrained risk inequality with applications to nonparametric functional estimation. The Annals of Statistics, 24(6):2524–2535, 1996.

C. Butucea and Y. Issartel. Locally differentially private estimation of nonlinear functionals of discrete distributions. arXiv:2107.03940, 2021.

C. Butucea, A. Dubois, M. Kroll, and A. Saumard. Local differential privacy: Elbow effect in optimal density estimation and adaptation over Besov ellipsoids. Bernoulli, 26(3):1727–1764, 2020.

F. Comte. Nonparametric estimation. Master and Research. Spartacus-Idh, Paris, 2017.

F. Comte, Y. Rozenholc, and M.-L. Taupin. Penalized contrast estimator for adaptive density deconvolution. Canadian Journal of Statistics, 34(3):431–452, 2006.

I. Dinur and K. Nissim. Revealing information while preserving privacy. In Proceedings of the twenty-second ACM SIGMOD-SIGACT-SIGART symposium on Principles of database systems, pages 202–210, 2003.

J. C. Duchi, M. I. Jordan, and M. J. Wainwright. Minimax optimal procedures for locally private estimation. Journal of the American Statistical Association, 113(521):182–201, 2018.

C. Dwork. Differential privacy, in automata, languages and programming. Lecture Notes in Computer Science, 4052:112, 2006.

C. Dwork. Differential privacy: A survey of results. In International conference on theory and applications of models of computation, pages 1–19. Springer, 2008.

C. Dwork and K. Nissim. Privacy-preserving datamining on vertically partitioned databases. In Annual International Cryptology Conference, pages 528–544. Springer, 2004.

A. Goldenshluger and O. Lepski. Bandwidth selection in kernel density estimation: Oracle inequalities and adaptive minimax optimality. The Annals of Statistics, 39(3):1608–1632, 2011.

R. Hall, A. Rinaldo, and L. Wasserman. Differential privacy for functions and functional data. Journal of Machine Learning Research, 14(Feb):703–727, 2013.

M. Kroll. Pointwise adaptive kernel density estimation under local approximate differential privacy. arXiv:1907.06233, 2019.

M. Kroll. On density estimation at a fixed point under local differential privacy. Electronic Journal of Statistics, 15(1):1783–1813, 2021.
J. Lam-Weil, B. Laurent, and J.-M. Loubes. Minimax optimal goodness-of-fit testing for densities and multinomials under a local differential privacy constraint. Bernoulli, 28(1):579–600, 2022.

B. Laurent, C. Ludena, and C. Prieur. Adaptive estimation of linear functionals by model selection. Electronic Journal of Statistics, 2:993–1020, 2008.

F. McSherry and K. Talwar. Mechanism design via differential privacy. In 48th Annual IEEE Symposium on Foundations of Computer Science (FOCS’07), pages 94–103. IEEE, 2007.

V. V. Petrov. Limit theorems of probability theory. Sequences of independent random variables. Oxford Studies in Probability. Clarendon Press., Oxford, 4. edition, 1995.

A. B. Tsybakov. Introduction to Nonparametric Estimation. Springer New York, 2009.

L. Wasserman and S. Zhou. A statistical framework for differential privacy. Journal of the American Statistical Association, 105(489):375–389, 2010.