THE ATTRACTORS FOR 2ND-ORDER STOCHASTIC DELAY LATTICE SYSTEMS

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Abstract. This paper deals with the long-time dynamical behavior of a class of 2nd-order stochastic delay lattice systems. It is shown under the dissipative and sublinear growth conditions that such a system possesses a compact global random attractor within the set of tempered random bounded sets. A numerical example is given to illustrate the obtained theoretical result.

1. Introduction. The purpose of this paper is to study the long-time dynamical behavior for the following 2nd-order stochastic delay lattice differential systems with additive white noise:

\begin{equation}
\begin{aligned}
\dot{u}_i + \alpha \dot{u}_i - (u_{i-1} - 2u_i + u_{i+1}) + \lambda u_i &= f_i(u_{it}) + g_i + a_i \dot{w}_i(t), \quad t > 0, \\
u_{i0} &= u_i(\theta), \quad \dot{u}_{i0} = \dot{u}_i(\theta), \quad \theta \in [-h, 0],
\end{aligned}
\end{equation}

where \(i \in \mathbb{Z}\) with \(\mathbb{Z}\) being the integer set, \(u_{it} = u_i(t + \theta) \ (\theta \in [-h, 0])\), \(\alpha, \lambda\) and \(h\) are positive constants, \(a = (a_i)_{i \in \mathbb{Z}} \in l^2\), \(g = (g_i)_{i \in \mathbb{Z}} \in l^2\), each \(f_i\) is a smooth function satisfying some dissipative condition, and \(w_i(t)\) denotes an independent two-side Brownian motion.

The various dynamical properties of deterministic lattice systems have caused wide concern in the past decades, and hence some significant related results have been presented in many references (see e.g., [4, 18, 19, 30, 31, 32]). Comparing with the deterministic models, it was found that stochastic models can describe the actual physical phenomena more completely. Hence, recently, the researchers started to work for the investigation of stochastic lattice systems. Since Bates et al. [3] initiated the study of stochastic lattice systems, there is by now a rather comprehensive mathematical literature on the existence of random attractors for stochastic lattice systems. For example, the first order lattice systems with an additive noise were considered in [21, 22, 23]. The existence results of random attractors of second-order lattice systems with additive noise were presented in [20]

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2. Preliminaries. In this section, for the subsequent analysis, we recall some concepts and the related results on stochastic dynamical system (SDS). These background knowledge also can be found in the references [1, 3]. Let \((X, \| \cdot \|_X)\) be a separable Banach space with Borel \(\sigma\)-algebra \(\mathcal{B}(X)\) and \((\Omega, \mathcal{F}, P)\) be a probability space. Then the following concepts and conclusions can be stated.

**Definition 2.2.** \((\Omega, \mathcal{F}, P, (\vartheta_t)_{t \in \mathbb{R}})\) is called a metric dynamical system if \(\vartheta : \mathbb{R} \times \Omega \to \Omega\) is \(\mathcal{B}(\mathbb{R} \times \mathcal{F}, \mathcal{F})\) measurable, \(\vartheta_0 = \text{Id}_\Omega\), \(\vartheta_{t+s} = \vartheta_t \circ \vartheta_s\) for all \(t, s \in \mathbb{R}\), and \(\vartheta_t P = P\) for all \(t \in \mathbb{R}\).

**Definition 2.2.** A continuous SDS on \(X\) over \((\Omega, \mathcal{F}, P, (\vartheta_t)_{t \in \mathbb{R}})\) is a mapping

\[S : [0, \infty) \times \Omega \times X \to X, \ (t, \omega, x) \to S(t, \omega)x\]

which is \((\mathcal{B}[0, \infty) \times \mathcal{F} \times \mathcal{B}(X), \mathcal{B}(X))\) measurable and satisfies, for \(P\)-a.e. \(\omega \in \Omega\),

1. \(S(t, \omega) : X \to X, \ x \mapsto S(t, \omega)x\) is continuous for every \(t \geq 0\);
2. \(S(0, \omega)\) is the identity operator on \(X\);
3. \(S(t+s, \omega) = S(t, \vartheta_s \omega) \circ S(s, \omega)\) for all \(t, s \geq 0\).

**Definition 2.3.** A random variable \(r(\omega) > 0\) is called tempered with respect to \((\vartheta_t)_{t \in \mathbb{R}}\), if for \(P\)-a.e. \(\omega \in \Omega\) \(\lim_{t \to \infty} e^{-\beta t} r(\vartheta_{-t} \omega) = 0\), for all \(\beta > 0\). A random set \(B(\omega)\) is called tempered if \(B(\omega)\) is contained in a ball with center zero and tempered radius \(r(\omega)\) for all \(\omega \in \Omega\).

**Proposition 2.1.** (cf. [11]) If \(r(\omega) > 0\) is tempered and \(r(\vartheta_t \omega)\) is continuous in \(t\) for \(P\)-a.e. \(\omega \in \Omega\), then

1. for any \(t \in \mathbb{R}\), \(r(\vartheta_t \omega)\) is tempered. Moreover, for any \(h > 0\), \(\max_{\theta \in [-h, 0]} r(\vartheta_{\theta \omega})\) is also tempered.
(2) for any $\beta > 0$ and P-a.e. $\omega \in \Omega$, $R(\omega) = \int_{-\infty}^{0} e^{\beta s r(\theta_s \omega)} ds < \infty$ is tempered, and $R(\theta_t \omega)$ is also continuous in $t$.

In what follows, $\mathcal{D}(X)$ denotes the set of all tempered random sets of $X$.

**Definition 2.4.** A random set $B_0(\omega)$ is called a random absorbing set in $\mathcal{D}(X)$ if for every $B(\omega) \in \mathcal{D}(X)$ and P-a.e. $\omega \in \Omega$, there exists $T_B(\omega) > 0$ such that $S(t, \theta_{-t} \omega)B_0(\omega) \subset B_0(\omega)$, $\forall t \geq T_B(\omega)$.

**Definition 2.5.** A random set $\mathcal{A}(\omega)$ is called a $\mathcal{D}$-random attractor set for $\{S(t, \omega)\}_{t \geq 0, \omega \in \Omega}$ if the following conditions are satisfied, for P-a.e. $\omega \in \Omega$,

1. $\mathcal{A}(\omega)$ is a compact set of $X$;
2. for all $t \geq 0$, $S(t, \omega)\mathcal{A}(\omega) = \mathcal{A}(\theta_t \omega)$;
3. for any $B(\omega) \in \mathcal{D}(X)$,
   \[ \lim_{t \to \infty} \text{dist}_X(S(t, \theta_{-t} \omega)B(\theta_{-t} \omega), \mathcal{A}(\omega)) = 0. \]

An existence result on the random attractor can be stated as follows.

**Proposition 2.2.** (cf. [3]) Suppose that $B_0(\omega) \in \mathcal{D}(X)$ is a closed random absorbing set for the continuous SDS $\{S(t, \omega)\}_{t \geq 0, \omega \in \Omega}$, and satisfies that for a.e. $\omega \in \Omega$, each sequence $x_n \in S(t_n, \theta_{-t_n} \omega)B_0(\theta_{-t_n} \omega)$ with $t_n \to \infty$ has a convergent subsequence in $X$. Then the SDS $\{S(t, \omega)\}_{t \geq 0, \omega \in \Omega}$ has a unique $\mathcal{D}$-random attractor $\mathcal{A}(\omega)$ which is given by

\[ \mathcal{A}(\omega) = \bigcap_{s \geq 0} \bigcup_{t \geq s} S(t, \theta_{-t} \omega)B_0(\theta_{-t} \omega). \]

3. The 2nd-order stochastic delay lattice system. Let $l^2 = \{u = (u_i)_{i \in \mathbb{Z}} : \sum_{i \in \mathbb{Z}} u_i^2 < \infty\}$ be the separable Hilbert space with this inner product $(u, v) = \sum_{i \in \mathbb{Z}} u_i v_i$ and the corresponding norm $\|u\| = \sqrt{\sum_{i \in \mathbb{Z}} u_i^2}$, and $l^\infty = \{u = (u_i)_{i \in \mathbb{Z}} : \sup_{i \in \mathbb{Z}} |u_i| < +\infty\}$ the Banach space with norm $\|u\|_\infty = \sup_{i \in \mathbb{Z}} |u_i|$. On space $l^2$, we define linear operators $A$, $B$ and $B^*$:

\[ (Au)_i = -u_{i-1} + 2u_i - u_{i+1}, \quad (Bu)_i = u_{i+1} - u_{i}, \quad (B^* u)_i = u_{i-1} - u_i, \quad \forall u \in l^2, \quad i \in \mathbb{Z}, \]

which satisfy that

\[ (Au, v) = (BB^* u, v) = (B^* Bu, v) = (Bu, Bv) = (B^* u, B^* v), \quad \forall u, v \in l^2. \]

Moreover, in what follows, the following inner product and norm on $l^2$ will also be used:

\[ (u, v)_\lambda = (Bu, Bv) + \lambda(u, v), \quad \|u\|_\lambda^2 = (u, u)_\lambda = \|Bu\|^2 + \lambda \|u\|^2, \quad \forall u, v \in l^2. \]

Since

\[ \lambda \|u\|^2 \leq \|u\|_\lambda^2 \leq (4 + \lambda) \|u\|^2, \quad \forall u \in l^2, \]

norms $\|\cdot\|$ and $\|\cdot\|_\lambda$ are equivalent to each other. Denote $l^2_\lambda = (l^2, (\cdot, \cdot)_\lambda, \|\cdot\|_\lambda)$, then $H = l^2_\lambda \times l^2_\lambda$ is a separable Hilbert space endowed with inner product

\[ (\phi^{(1)}, \phi^{(2)})_H = (u^{(1)}, u^{(2)})_\lambda + (u^{(1)}, v^{(2)}), \quad \forall \phi^{(k)} = (u^{(k)}, v^{(k)})^T \in H, \quad k = 1, 2, \]

and norm

\[ \|\phi\|_H^2 = \sum_{i \in \mathbb{Z}} |\phi_i|^2_H = \sum_{i \in \mathbb{Z}} \left( (Bu_i^2 + \lambda u_i^2 + v_i^2) + (Bu_i^2 + \lambda u_i^2 + v_i^2) \right), \quad \forall \phi = (\phi_i)_{i \in \mathbb{Z}} = (u_i, v_i)_{i \in \mathbb{Z}} \in H. \]
In addition, space $H_0 = C([-h,0]; H)$ is endowed with norm $\|\phi\|_{H_0} = \max_{\theta \in [-h,0]} \|\phi(\theta)\|_H$.

In the following, we will also involve the probability space $(\Omega, \mathcal{F}, P)$ with $\Omega = \{\omega \in C(\mathbb{R}, l^2) : \omega(0) = 0\}$, the Borel $\sigma$-algebra $\mathcal{F}$ on $\Omega$ generated by the compact open topology and the corresponding Wiener measure $P$ on $\mathcal{F}$. Write

$$\partial_t \omega(t) = \omega(t) - \omega(t), \quad W(t) = W(t, \omega) = \sum_{i \in \mathbb{Z}} a_i w_i(t) e_i, \quad t \in \mathbb{R},$$

where $\{e_i\}_{i \in \mathbb{Z}}$ denotes a complete orthonormal basis in $l^2$. With these, we know that $(\Omega, \mathcal{F}, P, (\partial_t)_{t \in \mathbb{R}})$ is an ergodic metric dynamical system with the filtration $\mathcal{F}_t = \bigvee_{s \leq t} \mathcal{F}_s$, $t \in \mathbb{R}$, where $\mathcal{F}_s = \sigma\{W(t_2) - W(t_1) : s \leq t_1 \leq t_2 \leq t\}$ is the smallest $\sigma$-algebra generated by the random variable $W(t_2) - W(t_1)$.

For convenience, we express (1.1) as an abstract 1st-order stochastic functional differential equation in $H$. To this end, we set $u = (u_i)_{i \in \mathbb{Z}}, \quad u_t = (u_{it})_{i \in \mathbb{Z}}, \quad Au = ((Au)_i)_{i \in \mathbb{Z}}, \quad f(u_t) = (f_i(u_{it}))_{i \in \mathbb{Z}}$, and take

$$\bar{v} = \bar{u} + \epsilon u, \quad \epsilon = \min\{\frac{\alpha}{4}, \frac{\lambda}{2\alpha}\}.$$

In this way, system (1.1) can be written as

$$\begin{cases}
\dot{\varphi}(t) + D\varphi(t) = F(\varphi_t) + G(t), \quad t > 0, \\
\varphi_0 = (u_0, \bar{v}_0)^T = (u(\theta), \bar{u}(\theta) + \epsilon u(\theta))^T, \quad \theta \in [-h,0],
\end{cases}$$

where $\varphi = (u, \bar{v})^T, \varphi_t = (u_t, \bar{v}_t)^T, \quad F(\varphi_t) = (0, f(u_t) + g)^T, \quad G = (0, W)^T$ and

$$D\varphi = \begin{pmatrix}
\epsilon u - \bar{v} \\
Au + \lambda u + (\epsilon - \alpha)(\epsilon u - \bar{v})
\end{pmatrix}.$$

For the function $f$, we make the following assumptions:

(1) $f_i : C([-h,0]; \mathbb{R}) \to \mathbb{R}$ is continuous and $f_i(0) = 0$;

(2) $|f_i(\xi)| \leq M_{0,i} + M_{1,i} \max_{\theta \in [-h,0]} |\xi(\theta)|$ for all $\xi \in C([-h,0]; \mathbb{R})$, where $M_{r,i} \geq 0$,

$$\left(M_{r,i}\right)_{i \in \mathbb{Z}} \in l^2 \text{ and } M_r := \sqrt{\sum_{i \in \mathbb{Z}} M^2_{r,i}} (r = 0, 1);$$

(3) there exists an $L_f > 0$ such that

$$\|f(\xi) - f(\eta)\| \leq L_f \|\xi - \eta\|, \quad \forall \xi, \eta \in l^2.$$

An existence and uniqueness theorem of the solution of system (3.1) can be stated as follows.

**Theorem 3.1.** Let (I)-(III) hold. Then for any $T > 0$ and initial data $\varphi_0 \in H_0$, there exists a unique solution $\varphi(t) \in L^2(\Omega, C([0,T]; H))$ of equation (3.1) with $\varphi(t, \varphi_0) \in H_0$ for $t \in [0,T]$ and $\varphi_0(\cdot, \varphi_0) = \varphi_0$. Furthermore, $\varphi_t(\cdot, \omega, \varphi_0)$ depends continuously on the initial data $\varphi_0$ in $H_0$ for each $\omega \in \Omega$.

**Proof.** Rewriting (3.1) as

$$\varphi(t) = \varphi(0) + \int_0^t (-D\varphi(s) + F(\varphi_s) + G(s))ds, \quad t > 0,$$
then, by (I) and (III), we have for any \( \varphi \in H_0 \) that
\[
\|F(\varphi_t)\|_H^2 = \|f(u_t) + g\|^2 \leq 2\|f(u_t)\|^2 + 2\|g\|^2 \leq 2\|f(u_t) - f(0)\|^2 + 2\|g\|^2 \\
\leq 2L_f^2 \|u\|_{C([0,T],[H^2])}^2 + 2\|g\|^2 \leq \frac{2L_f^2}{\lambda} \|\varphi\|_{H_0}^2 + 2\|g\|^2.
\]
Thus \( F \) maps the bounded sets of \( H_0 \) into the bounded sets of \( H \). Similarly, we have that
\[
\|D \varphi(t)\|_H^2 = \|eu - \bar{v}\|_H^2 + \|Au + \lambda u + (\epsilon - \alpha)(eu - \bar{v})\|_H^2 \\
\leq 2\epsilon^2 \|u\|_H^2 + 2\|\bar{v}\|_H^2 + 3\|Au\| + 2\lambda^2 \|u\|_H^2 + 3(\epsilon - \alpha)^2 \|eu - \bar{v}\|_H^2 \\
\leq (2\epsilon^2 + \frac{12}{\lambda} + 3\lambda + \frac{6\epsilon^2(\epsilon - \alpha)^2}{\lambda}) \|u\|_H^2 + (2(4 + \lambda) + 6(\epsilon - \alpha)^2) \|\bar{v}\|_H^2 \\
\leq L_D \|\varphi(t)\|_H^2 \leq L_D \|\varphi(t)\|_{H_0}^2,
\]
where \( L_D = \max \{2\epsilon^2 + \frac{12}{\lambda} + 3\lambda + \frac{6\epsilon^2(\epsilon - \alpha)^2}{\lambda}, 2(4 + \lambda) + 6(\epsilon - \alpha)^2\} \). In this way, the existence and uniqueness of the local solution of (3.1) can be proved by the standard technique. On the other hand, as shown in Lemma 4.1, we can find that \( \|\varphi_t\|_{H_0} \) is bounded in \( H_0 \). Therefore, the local solution \( \varphi(t) \) of (3.1) is global.

Next, we assume \( \varphi_0^{(1)}, \varphi_0^{(2)} \in H_0 \) and consider the solutions \( \varphi^{(1)}(t), \varphi^{(2)}(t) \) of (3.1) corresponding to the initial data \( \varphi_0^{(1)}, \varphi_0^{(2)} \), respectively. Then, it can be deduced from (3.1) that
\[
\frac{d}{dt} \|\varphi^{(1)}(t) - \varphi^{(2)}(t)\|_H^2 \\
= -2\langle D(\varphi^{(1)}(t) - \varphi^{(2)}(t)), \varphi^{(1)}(t) - \varphi^{(2)}(t) \rangle_H \\
+ 2\langle F(\varphi^{(1)}(t) - F(\varphi^{(2)}(t)), \varphi^{(1)}(t) - \varphi^{(2)}(t) \rangle_H \\
\leq \|D(\varphi^{(1)}(t) - \varphi^{(2)}(t))\|_H^2 + 2\|\varphi^{(1)}(t) - \varphi^{(2)}(t)\|_H^2 + \|F(\varphi^{(1)}(t) - F(\varphi^{(2)}(t))\|_H^2 \\
\leq (L_D + 2) \|\varphi^{(1)}(t) - \varphi^{(2)}(t)\|_H^2 + \|F(\varphi^{(1)}(t) - F(\varphi^{(2)}(t))\|_H^2.
\]
This implies that
\[
\|\varphi^{(1)}(t) - \varphi^{(2)}(t)\|_H^2 \\
\leq \|\varphi^{(1)}(0) - \varphi^{(2)}(0)\|_H^2 + (L_D + 2) \int_0^t \|\varphi^{(1)}(s) - \varphi^{(2)}(s)\|_H^2 ds \\
+ \int_0^t \|F(\varphi^{(1)}(s) - F(\varphi^{(2)}(s))\|_H^2 ds \\
\leq \|\varphi^{(1)}(0) - \varphi^{(2)}(0)\|_H^2 + (L_D + 2) \int_0^t \|\varphi^{(1)}(s) - \varphi^{(2)}(s)\|_H^2 ds \\
+ \frac{L_f^2}{\lambda} \int_0^{t+\theta} \|\varphi^{(1)}(s) - \varphi^{(2)}(s)\|_H^2 ds \\
\leq \|\varphi^{(1)}(0) - \varphi^{(2)}(0)\|_H^2 + (L_D + 2) \int_0^t \|\varphi^{(1)}(s) - \varphi^{(2)}(s)\|_H^2 ds \\
+ \frac{L_f^2}{\lambda} \int_{t-h}^t \|\varphi^{(1)}(s) - \varphi^{(2)}(s)\|_H^2 ds \\
\leq (1 + \frac{hL_f^2}{\lambda}) \|\varphi^{(1)}(0) - \varphi^{(2)}(0)\|_H^2 + (L_D + 2) \int_0^t \|\varphi^{(1)}(s) - \varphi^{(2)}(s)\|_H^2 ds.
\]
In terms of Gronwall inequality, it holds that
\[ \|\varphi^{(1)}(t) - \varphi^{(2)}(t)\|_H^2 \leq (1 + \frac{hL^2}{\lambda})e^{\frac{hL^2}{\lambda} + LD + 2}t\|\varphi^{(1)}_0 - \varphi^{(2)}_0\|_H^2, \quad \forall t \in [0, T], \]
namely,
\[ \sup_{t \in [0, T]} \|\varphi^{(1)}(t) - \varphi^{(2)}(t)\|_H^2 \leq (1 + \frac{hL^2}{\lambda})e^{\frac{hL^2}{\lambda} + LD + 2}T\|\varphi^{(1)}_0 - \varphi^{(2)}_0\|_H^2. \]
Therefore, the theorem is proven.

It is remarkable that \( W(t, \vartheta t \omega) \) is continuous in \( t \) and \( \omega \to W(t, \vartheta t \omega) \) is measurable for \( t \geq 0 \). Moreover, in terms of Theorem 3.1, the mapping \( \varphi_0 \to \varphi_t(\cdot, \omega, \varphi_0) \) is continuous for fixed \( t \) and \( \omega \). Hence, corresponding to the system (3.1), we can define the following continuous SDS for \( t \geq 0 \), \( \omega \in \Omega \), \( \varphi_0 \in H_0 \).

4. Existence of random attractor. This section will deal with the existence of a \( D \)-random attractor for the semigroup \( \{S(t, \omega)\}_{t \geq 0, \omega \in \Omega} \) in \( H_0 \). For this, we need to transform system (3.1) into a random evolution equation without noise. Firstly, we note that
\[ z(\vartheta t \omega) = -\alpha \int_{-\infty}^{0} e^{\alpha s} \vartheta t \omega(s)ds, \quad t \in \mathbb{R}, \]
is a solution of the following Itô equation
\[ dz + \alpha z dt = dW. \]
In fact, the mapping \( t \to z(\vartheta t \omega) \) is an Ornstein-Uhlenbeck process. Furthermore, there exists a \( \vartheta t \)-invariant set \( \Omega' \subseteq \Omega \) such that
(i) the mapping \( t \to z(\vartheta t \omega) \) is continuous for \( P \)-a.e. \( \omega \in \Omega' \);
(ii) the random variable \( \|z(\vartheta t \omega)\| \) is tempered (see e.g. [3]).
Denote
\[ \psi(t) = (u(t), v(t))^T = \varphi(t) + (0, -z(\vartheta t \omega))^T, \]
where \( \varphi(t) \) is the solution of (3.1). Then \( \psi(t) \) satisfies
\[ \begin{cases} \dot{\psi}(t) + D\psi(t) = F(\psi_t) + G_1(t), \quad t > 0, \\ \psi_0 = \varphi_0 - (0, z(\vartheta 0 \omega))^T, \end{cases} \]
where \( z(\vartheta \omega) = z(\vartheta \omega), \theta \in [-h, 0], \) for any \( t \geq 0 \), \( \psi_t = \varphi_t - (0, z(\vartheta \omega))^T, \)
\( F(\psi_t) = (0, f(u_t) + g)^T, G_1(t) = (z(\vartheta \omega), \epsilon z(\vartheta \omega))^T, \) and
\[ D\psi = \left( \begin{array}{c} \epsilon u - v \\ A u + \lambda u + (\epsilon - \alpha)(\epsilon u - v) \end{array} \right). \]

Now, we prove the existence of a random absorbing set in \( D(H_0) \) for \( \{S(t, \omega)\}_{t \geq 0, \omega \in \Omega} \).

Lemma 4.1. Let (I)-(III) hold, and assume that
\[ 2M_tch < \sqrt{\lambda}, \quad 0 < \frac{\epsilon}{2} - \eta, \]
where \( \eta \in (\eta_0, \eta_1) \) and \( \eta_1, \eta_2 \) are two solutions of the equation \( \sqrt{\lambda} \eta e^{-\eta h} = 2M_t \).
Then, there exists a random absorbing set \( B_0(\omega) \in D(H_0) \) for \( \{S(t, \omega)\}_{t \geq 0, \omega \in \Omega} \), that is, for any \( B(\omega) \in D(H_0) \) and \( P \)-a.e. \( \omega \in \Omega \), there exists \( T_B(\omega) > 0 \) such that
\[ S(t, \vartheta \omega)B(\vartheta \omega) \subset B_0(\omega), \quad \forall t \geq T_B(\omega). \]
Proof. Taking the inner product $(\cdot, \cdot)_H$ on both sides of (4.1) with $\psi(t)$ follows that
\[
\frac{1}{2} \frac{d}{dt} \|\psi(t)\|_H^2 + (D\psi(t), \psi(t))_H = (F(\psi), \psi(t))_H + (G_1(t), \psi(t))_H.
\]
(4.3)
In the following, we estimate the various terms in (4.3). Firstly, when setting $\epsilon = \min \left\{ \frac{\alpha}{2}, \frac{\lambda}{2} \right\}$, it holds that
\[
(D\psi(t), \psi(t))_H = \epsilon \|u(t)\|_H^2 + (\alpha - \epsilon) \|v(t)\|^2 - \epsilon(\alpha - \epsilon) \|u(t), v(t)\|
\geq \epsilon \|u(t)\|_H^2 + (\alpha - \epsilon) \|v(t)\|^2 - \frac{\epsilon \alpha}{\sqrt{\lambda}} \|u(t)\| \|v(t)\|
\geq \epsilon \|u(t)\|_H^2 + 3 \frac{\alpha}{4} \|v(t)\|^2 - \frac{\epsilon}{2} \|u(t)\|_H^2 - \frac{\epsilon \alpha^2}{2 \lambda} \|v(t)\|^2
\geq \frac{\epsilon}{2} \|u(t)\|_H^2 + \frac{\alpha}{2} \|v(t)\|^2 \geq \frac{\epsilon}{2} \|\psi(t)\|_H^2.
\]
Secondly, it follows from condition (II) that
\[
2(F(\psi), \psi(t))_H = 2(f(u) + \psi(t)) + 2(g, \psi(t))
\leq 2 \sum_{i \in I} \left( M_{0, i} + M_{1, i} \max_{\theta \in [-\epsilon, \epsilon]} |u_{i, \theta}| \right) \|v_i(t)\| + 2 \sum_{i \in I} |g_i| \|v_i(t)\|
\leq 2 M_0 \|v(t)\| + 2 \|u_i\| \|c_{l, \theta} = 2 \epsilon \|v(t)\| + 2 \|g_i\| \|v_i(t)\|
\leq \frac{\epsilon}{4} \|v(t)\|^2 + \frac{8}{\epsilon} (M_0^2 + |v(t)|^2) + 2 \|M_i\| \|v_i(t)\|
\leq \frac{\epsilon}{4} \|v(t)\|^2 + \frac{8}{\epsilon} (M_0^2 + |g(t)|^2) + \frac{8}{\epsilon} \|v(t)\|^2
\leq c_1 \|z(\theta_\omega)\|^2 + \frac{\epsilon}{2} \|u(t)\|_H^2 + \frac{\epsilon}{4} \|v(t)\|^2, \text{ where } c_1 = \frac{2}{\epsilon} (4 + \lambda) + 4 \epsilon.
\]
Substituting the above estimates into (4.3) gives that
\[
\frac{d}{dt} \|\psi(t)\|_H^2 \leq -\frac{\epsilon}{2} \|\psi(t)\|_H^2 + \frac{8}{\epsilon} (M_0^2 + |g(t)|^2) + \frac{2M_1}{\sqrt{\lambda}} \|\psi_l\|_H^2 + c_1 \|z(\theta_\omega)\|^2. \quad (4.4)
\]
Multiplying (4.4) by $e^{\eta t}$ and using (4.2) derive that
\[
\frac{d}{dt} (e^{\eta t} \|\psi(t)\|_H^2) \leq (\eta - \frac{\epsilon}{2}) e^{\eta t} \|\psi(t)\|_H^2 + \frac{8 e^{\eta t}}{\epsilon} (M_0^2 + |g(t)|^2)
+ \frac{2M_1 e^{\eta t}}{\sqrt{\lambda}} \|\psi_l\|_H^2 + c_1 e^{\eta t} \|z(\theta_\omega)\|^2
\leq \frac{8 e^{\eta t}}{\epsilon} (M_0^2 + |g(t)|^2) + \frac{2M_1 e^{\eta t}}{\sqrt{\lambda}} \|\psi_l\|_H^2 + c_1 e^{\eta t} \|z(\theta_\omega)\|^2.
\]
It deduces by integrating the above inequality on $[0, t]$ that
\[
e^{\eta t} \|\psi(t)\|_H \leq \|\psi(0)\|_H + \frac{8 e^{\eta t}}{\epsilon \eta} (M_0^2 + |g(t)|^2)
+ \frac{2M_1}{\sqrt{\lambda}} \int_0^t e^{\eta s} \|\psi_l\|_H^2 ds + c_1 \int_0^t e^{\eta s} \|z(\theta_\omega)\|^2 ds.
\]
Noticing that \( \| \psi(\theta) \|_H \leq \| \psi_0 \|_{H_0} \) for fixed \( \theta \in [-h, 0] \), hence we have for all \( t \geq 0 \),
\[
e^{\eta t} \| \psi(t + \theta) \|_H^2 \leq e^{-\eta \theta} \| \psi(0) \|_H^2 + \frac{8e^{-\eta \theta}e^{\eta(t+\theta)}}{c\eta} (M_0^2 + \| g \|^2) e^{\eta t} + 2M_1 e^{-\eta \theta} \sqrt{\lambda} \int_0^{t+\theta} e^{\eta \tau} \| \psi_s \|_{H_0}^2 d\tau + c_1 e^{\eta \theta} \int_0^t e^{\eta \tau} \| \psi_s \|_{H_0}^2 d\tau
\]
\[
\leq e^{\eta h} \| \psi_0 \|_{H_0}^2 + \frac{8e^{\eta t}}{c\eta} (M_0^2 + \| g \|^2) + \frac{2M_1 e^{\eta h}}{\sqrt{\lambda}} \int_0^t e^{\eta \tau} \| \psi_s \|_{H_0}^2 d\tau + c_1 e^{\eta h} \int_0^t e^{\eta \tau} \| \psi_s \|_{H_0}^2 d\tau.
\]
This implies that
\[
e^{\eta t} \| \psi_t \|^2_{H_0} \leq e^{\eta h} \| \psi_0 \|_{H_0}^2 + \hat{C}_1 e^{\eta t} + L \int_0^t e^{\eta \tau} \| \psi_s \|_{H_0}^2 d\tau + c_1 e^{\eta h} \int_0^t e^{\eta \tau} \| z(\vartheta_s \omega) \|^2 d\tau.
\]
Applying Gronwall inequality to the above inequality yields that
\[
e^{\eta t} \| \psi_t \|^2_{H_0} \leq e^{\eta h} \| \psi_0 \|_{H_0}^2 + \hat{C}_1 e^{\eta t} + c_1 e^{\eta h} \int_0^t e^{\eta \tau} \| z(\vartheta_s \omega) \|^2 d\tau
\]
\[
+ L \int_0^t \left( e^{\eta \theta} \| \psi_0 \|_{H_0}^2 + \hat{C}_1 e^{\eta t} + c_1 e^{\eta h} \int_0^\tau e^{\eta \xi} \| z(\vartheta_s \omega) \|^2 d\xi \right) e^{L(t-s)} d\tau
\]
\[
\leq 2e^{\eta h} \| \psi_0 \|_{H_0}^2 e^{Lt} + \frac{\hat{C}_1 \eta}{\eta - L} e^{\eta t} + 2c_1 e^{\eta h} e^{Lt} \int_0^t e^{(\eta - L)s} \| z(\vartheta_s \omega) \|^2 ds,
\]
and hence
\[
\| \psi_t(\cdot, \vartheta_s \omega, \psi_0(\omega)) \|^2_{H_0} \leq 2e^{\eta h} \| \psi_0(\omega) \|^2_{H_0} e^{-(\eta - L)t} + \frac{\hat{C}_1 \eta}{\eta - L}
\]
\[
+ 2c_1 e^{\eta h} \int_0^t e^{(\eta - L)(s-t)} \| z(\vartheta_s \omega) \|^2 ds.
\]
Replacing \( \omega \) by \( \vartheta_s \omega \) in (4.7), we get that
\[
\| \psi_t(\cdot, \vartheta_s \omega, \psi_0(\vartheta_{-s} \omega)) \|^2_{H_0}
\]
\[
\leq 2e^{\eta h} \| \psi_0(\vartheta_{-s} \omega) \|^2_{H_0} e^{-(\eta - L)t} + \frac{\hat{C}_1 \eta}{\eta - L} + 2c_1 e^{\eta h} \int_0^t e^{(\eta - L)(s-t)} \| z(\vartheta_s \omega) \|^2 ds
\]
\[
\leq 2e^{\eta h} \| \psi_0(\vartheta_{-s} \omega) \|^2_{H_0} e^{-(\eta - L)t} + \frac{\hat{C}_1 \eta}{\eta - L} + 2c_1 e^{\eta h} \int_{-\infty}^0 e^{(\eta - L)s} \| z(\vartheta_s \omega) \|^2 ds.
\]
Lemma 4.2. Assume that the conditions of Lemma 4.1 hold, and

\[ \text{Then for any } \varepsilon > 0, \text{ it follows from (4.8) that} \]

\[
\|\varphi(t, \omega, \varphi_0(\omega) - (0, z(\theta_{-t}\omega))^T) + (0, z(\theta_{-t}\omega))^T, \|H_0^2 = 2\|\varphi_0(\theta_{-t}\omega)\|^2 + 2\|z(\theta_{-t+\theta}\omega)\|^2 e^{-(\eta-L)t} + \frac{\tilde{C}_1\eta}{\eta-L} + 2c_1e^{\eta h} \int_{-\infty}^{0} e^{(\eta-L)s} \|z(\theta_s\omega)\|^2 ds + 2 \max_{\theta \in [-h,0]} \|z(\theta_0\omega)\|^2. \]

By assumption, \( B(\omega) \in D(H_0) \) is tempered. On the other hand, by Proposition 2.1, we know that \( \max_{\theta \in [-h,0]} \|z(\theta_0\omega)\|^2, \max_{\theta \in [-h,0]} \|z(\theta_{-t+\theta}\omega)\|^2 \) and \( \int_{-\infty}^{0} e^{(\eta-L)s} \|z(\theta_s\omega)\|^2 ds \) are also tempered. Thus, if \( \varphi_0(\theta_{-t}\omega) \in B(\theta_{-t}\omega) \), then there exists \( T_B(\omega) > 0 \) such that, for all \( t \geq T_B(\omega) \),

\[
\|\varphi(t, \omega, \varphi_0(\omega) - (0, z(\theta_{-t}\omega))^T) + (0, z(\theta_{-t}\omega))^T, \|H_0^2 \leq \frac{2\tilde{C}_1\eta}{\eta-L} + 2c_1e^{\eta h} \int_{-\infty}^{0} e^{(\eta-L)s} \|z(\theta_s\omega)\|^2 ds + 2 \max_{\theta \in [-h,0]} \|z(\theta_0\omega)\|^2 \equiv \tilde{R}_0(\omega),
\]

which means that \( B_0(\omega) = \{ \xi \in H_0 : \|\xi\|_{H_0} \leq R_0(\omega) \} \) is a random absorbing set for \( \{ S(t,\omega) \}_{t \geq 0, \omega \in \Omega} \). Therefore the lemma is proven.

In order to obtain the existence of a random attractor, we need to use an estimate of the tails of solutions.

**Lemma 4.2.** Assume that the conditions of Lemma 4.1 hold, and \( \varphi_0(\omega) \in B_0(\omega) \). Then for any \( \varepsilon > 0 \), there exist \( T(\varepsilon, \omega) \) and \( K(\varepsilon, \omega) \) such that the solution \( \varphi(t, \omega, \varphi_0(\omega)) \) of (3.1) satisfies

\[
\max_{\theta \in [-h,0]} \sum_{|i| > K(\varepsilon, \omega)} \|\varphi_i(t + \theta, \omega, \varphi_0(\theta_{-t}\omega))\|^2_H < \varepsilon, \quad \forall t \geq T(\varepsilon, \omega).
\]

**Proof.** Let \( \rho(s) \in C^1([0,\infty), \mathbb{R}) \) be a smooth increasing function with

\[
\rho(s) = 0, \quad 0 \leq s < 1; \quad 0 \leq \rho(s) \leq 1, \quad 1 \leq s < 2; \quad \rho(s) = 1, \quad s \geq 2,
\]

and assume that there exists a constant \( c_2 \) such that \( |\rho'(s)| \leq c_2 \) for all \( s > 0 \). Write

\[
\rho_{K,i} = \rho\left(\frac{|i|}{K}\right), \quad \rho_{K,i} \psi = (\rho_{K,i} \psi_i)_{i \in \mathbb{Z}} = (\rho_{K,i} u_i, \rho_{K,i} v_i)^T_{i \in \mathbb{Z}},
\]

and

\[
\rho_{K,i} \hat{\psi} = (\rho_{K,i} \hat{\psi}_i)_{i \in \mathbb{Z}} = (\rho_{K,i} u_i, \rho_{K,i} v_i)^T_{i \in \mathbb{Z}},
\]

where \( K \) is a fixed positive integer. Taking the inner product \((\cdot, \cdot)_H\) of (4.1) with \( \rho_K \hat{\psi}(t) \), we get that

\[
(\hat{\psi}(t), \rho_K \hat{\psi}(t))_H + (D \hat{\psi}(t), \rho_K \hat{\psi}(t))_H = (F(\psi_i), \rho_K \hat{\psi}(t))_H + (G_1(t), \rho_K \hat{\psi}(t))_H. \quad (4.10)
\]

For the terms of (4.10), it is easy to check that

\[
(\hat{\psi}(t), \rho_K \hat{\psi}(t))_H = \frac{1}{2} \frac{d}{dt} \sum_{i \in \mathbb{Z}} \rho_{K,i} |\psi_i(t)|^2_H = \frac{1}{2} \frac{d}{dt} \|\rho_K \hat{\psi}(t)\|_H^2,
\]

\[ 
\]
\[ (D\psi(t), \rho_K\psi(t))_H \geq \frac{\epsilon}{2} \| \rho_K^{\frac{1}{2}} \psi(t) \|_H^2 - \sum_{i \in Z} (\rho_{K,i+1} - \rho_{K,i})(Bu_i(t), v_{i+1}(t)) \]
\[ \geq \frac{\epsilon}{2} \| \rho_K^{\frac{1}{2}} \psi(t) \|_H^2 - \frac{c_2}{K} \| \psi(t) \|_H^2, \]
and
\[ 2(F(\psi_1), \rho_K\psi(t))_H + 2(G_1(t), \rho_K\psi(t))_H \leq \frac{\epsilon}{2} \| \rho_K^{\frac{1}{2}} \psi(t) \|_H^2 + \frac{8}{\epsilon} \sum_{i \in Z} \rho_{K,i}(M_{0,i}^2 + |g_i|^2) \]
\[ + \frac{2M_1}{\sqrt{\lambda}} \| \rho_K^{\frac{1}{2}} \psi(t) \|_{H_0} + c_1 \sum_{i \in Z} \rho_{K,i} |z_i(\vartheta_1 \omega)|^2. \]
Substituting the above inequalities into (4.10) gives that
\[ \frac{d}{dt} \| \rho_K^{\frac{1}{2}} \psi(t) \|_{H_0}^2 \leq -\frac{\epsilon}{2} \| \rho_K^{\frac{1}{2}} \psi(t) \|_{H_0}^2 + \frac{8}{\epsilon} \sum_{i \in Z} \rho_{K,i}(M_{0,i}^2 + |g_i|^2) \]
\[ + \frac{2M_1}{\sqrt{\lambda}} \| \rho_K^{\frac{1}{2}} \psi(t) \|_{H_0} + c_1 \sum_{i \in Z} \rho_{K,i} |z_i(\vartheta_1 \omega)|^2. \]
Then, making use of the arguments in the Lemma 4.1, we obtain
\[ e^{nt} \| \rho_K^{\frac{1}{2}} \psi(t) \|_{H_0}^2 \leq e^{n\theta} \| \rho_K^{\frac{1}{2}} \psi_0 \|_{H_0}^2 + \tilde{C}_2 e^{nt} + L \int_0^t e^{ns} \| \rho_K^{\frac{1}{2}} \psi(s) \|_{H_0}^2 ds \]
\[ + \frac{2c_2 e^{nh}}{K} \int_0^t e^{ns} \| \psi(s) \|_{H_0}^2 ds + c_1 e^{nt} \int_0^t e^{ns} \sum_{i \in Z} \rho_{K,i} |z_i(\vartheta_s \omega)|^2 ds, \]
where \( L \) is indicated in (4.6) and \( \tilde{C}_2 = \frac{8}{\epsilon} \sum_{i \in Z} \rho_{K,i}(M_{0,i}^2 + |g_i|^2). \) Using Gronwall inequality yields that
\[ e^{nt} \| \rho_K^{\frac{1}{2}} \psi(t) \|_{H_0}^2 \leq e^{n\theta} \| \rho_K^{\frac{1}{2}} \psi_0 \|_{H_0}^2 + \tilde{C}_2 e^{nt} + \frac{2c_2 e^{nh}}{K} \int_0^t e^{ns} \| \psi(s) \|_{H_0}^2 ds \]
\[ + c_1 e^{nt} \int_0^t e^{ns} \sum_{i \in Z} \rho_{K,i} |z_i(\vartheta_s \omega)|^2 ds + L \int_0^t e^{nt} \| \rho_K^{\frac{1}{2}} \psi_0 \|_{H_0}^2 e^{L(t-s)} ds \]
\[ + L \int_0^t \tilde{C}_2 e^{ns} e^{L(t-s)} ds + L \int_0^t \frac{2c_2 e^{nh}}{K} \int_0^s e^{nt} \| \psi(s) \|_{H_0}^2 d \tau e^{L(t-s)} ds \]
\[ + c_1 e^{nt} \int_0^t e^{ns} \sum_{i \in Z} \rho_{K,i} |z_i(\vartheta_s \omega)|^2 d \tau e^{L(t-s)} ds \]
\[ \leq 2e^{nt} \| \rho_K^{\frac{1}{2}} \psi_0 \|_{H_0}^2 e^{Lt} + \frac{\tilde{C}_2 \eta}{\eta - L} e^{nt} + \frac{4c_2 e^{nh} e^{Lt}}{K} \int_0^t e^{(\eta - L)s} \| \psi(s) \|_{H_0}^2 ds \]
\[ + c_1 e^{nt} e^{(\eta - L)t} \int_0^t e^{-(\eta - L)s} \sum_{i \in Z} \rho_{K,i} |z_i(\vartheta_s \omega)|^2 ds, \]
and by which, for all \( t \geq 0, \) it holds that
\[ \| \rho_K^{\frac{1}{2}} \psi(t) \|_{H_0}^2 \]
\[ \leq 2e^{nt} \| \rho_K^{\frac{1}{2}} \psi_0 \|_{H_0}^2 e^{-(\eta - L)t} + \frac{\tilde{C}_2 \eta}{\eta - L} + \frac{4c_2 e^{nh} e^{-(\eta - L)t}}{K} \int_0^t e^{(\eta - L)s} \| \psi(s) \|_{H_0}^2 ds \]
\[ + c_1 e^{nt} e^{-(\eta - L)t} \int_0^t e^{-(\eta - L)s} \sum_{i \in Z} \rho_{K,i} |z_i(\vartheta_s \omega)|^2 ds. \]
Replacing $\omega$ with $\vartheta_{-t}\omega$, we obtain
\[
\|\rho_{c_1}^{\tilde{\gamma}t} \psi_t(\cdot, \vartheta_{-t}\omega, \psi_0(\vartheta_{-t}\omega))\|_{H_0}^2 \\
\leq 2e^{nh} \|\rho_{c_1}^{\tilde{\gamma}t} \psi_0(\vartheta_{-t}\omega)\|_{H_0}^2 e^{-(\eta-L)t} + \frac{\tilde{C}_2 \eta}{\eta - L} \\
+ 4c_2 e^{nh} e^{-(\eta-L)t} \int_0^t e^{(\eta-L)s} \|\psi(s, \vartheta_{-s}\omega, \psi_0(\vartheta_{-s}\omega))\|_{H}^2 ds \\
+ 2c_1 e^{nh} \int_{-\infty}^0 e^{(\eta-L)s} \sum_{i \in \mathbb{Z}} \rho_{c_1}^{\tilde{\gamma}t} |z_i(\vartheta_{s}\omega)|^2 ds.
\]

In what follows, we estimate the terms on the right-hand side of (4.11). Since $\|\varphi_0(\vartheta_{-t}\omega)\|^2$, $\max_{\theta \in [-h,0]} \|z_{-t+\theta}(\omega)\|^2$ are tempered, we have that, for every $\varepsilon > 0$, there exists $T_1(\varepsilon, \omega) > 0$ such that for all $t \geq T_1(\varepsilon, \omega)$,
\[
2c_1 e^{nh} \int_{-\infty}^0 e^{(\eta-L)s} \sum_{i \in \mathbb{Z}} \rho_{c_1}^{\tilde{\gamma}t} |z_i(\vartheta_{s}\omega)|^2 ds < \varepsilon/4, \quad \forall K \geq K_1(\varepsilon, \omega).
\]

Also, by $M_0$, $g \in l^2$, there exists $K_1(\varepsilon, \omega) > 0$ such that
\[
\frac{\tilde{C}_2 \eta}{\eta - L} < \frac{\varepsilon}{4}, \quad \forall K \geq K_1(\varepsilon, \omega).
\]

In light of $\int_{-\infty}^0 e^{(\eta-L)s} \|z(\vartheta_s\omega)\|^2 ds < \infty$ and the Lebesgue theorem of dominated convergence, there exists $K_2(\varepsilon, \omega) > 0$ such that
\[
2c_1 e^{nh} \int_{-\infty}^0 e^{(\eta-L)s} \sum_{i \in \mathbb{Z}} \rho_{c_1}^{\tilde{\gamma}t} |z_i(\vartheta_{s}\omega)|^2 ds < \varepsilon/4, \quad \forall K \geq K_2(\varepsilon, \omega).
\]

Moreover, it follows from (4.8) that
\[
\frac{4c_2 e^{nh} e^{-(\eta-L)t}}{K} \int_0^t e^{(\eta-L)s} \|\psi(s, \vartheta_{-s}\omega, \psi_0(\vartheta_{-s}\omega))\|_{H}^2 ds \\
\leq 4c_2 e^{nh} e^{-(\eta-L)t} \int_0^t e^{(\eta-L)s} \left(2c_1 e^{nh} \|\psi_0(\vartheta_{-s}\omega)\|_{H_0}^2 e^{-(\eta-L)s} + \frac{\tilde{C}_1 \eta}{\eta - L}\right) ds \\
+ \frac{4c_2 e^{nh} e^{-(\eta-L)t}}{K} \int_0^t e^{(\eta-L)s} 2c_1 e^{nh} \int_{-\infty}^0 e^{(\eta-L)\tau} \|z(\vartheta_s\omega)\|^2 d\tau ds \\
\leq \frac{8c_2 e^{2nh}}{K} \|\psi_0(\vartheta_{-t}\omega)\|^2_{H_0} e^{-(\eta-L)t} + \frac{4c_2 \eta e^{nh} \tilde{C}_1}{(\eta - L)^2 K} \\
+ \frac{8c_1 c_2 e^{2nh}}{(\eta - L) K} \int_{-\infty}^0 e^{(\eta-L)s} \|z(\vartheta_s\omega)\|^2 ds,
\]

which shows that there exist $T_2(\varepsilon, \omega) > 0$ and $K_3(\varepsilon, \omega) > 0$ such that, for $t \geq T_2(\varepsilon, \omega)$, $K \geq K_3(\varepsilon, \omega)$,
\[
\frac{4c_2 e^{nh} e^{-(\eta-L)t}}{K} \int_0^t e^{(\eta-L)s} \|\psi(s, \vartheta_{-s}\omega, \psi_0(\vartheta_{-s}\omega))\|_{H}^2 ds < \varepsilon/4.
\]

Therefore, when setting
\[
T(\varepsilon, \omega) = \max\{T_1(\varepsilon, \omega), T_2(\varepsilon, \omega)\}, \quad K_4(\varepsilon, \omega) = \max\{K_1(\varepsilon, \omega), K_2(\varepsilon, \omega), K_3(\varepsilon, \omega)\},
\]
we have 
\[ \| \rho^2 \psi(t, \vartheta, \psi_0) \|_{H^0}^2 < \varepsilon, \forall t \geq T(\varepsilon, \omega), \ K \geq K_4(\varepsilon, \omega). \]

Further, by Dini theorem there is a constant \( K_5(\varepsilon, \omega) > 0 \) such that 
\[ \max_{\theta \in [-h,0]} \sum_{i \in \mathbb{Z}} \rho_{K,i} |z_i(\vartheta_\omega)|^2 < \varepsilon, \forall K \geq K_5(\varepsilon, \omega), \]
which implies that, for all \( t \geq T(\varepsilon, \omega) \) and \( K \geq K_5(\varepsilon, \omega) = \max\{K_4(\varepsilon, \omega), K_5(\varepsilon, \omega)\} \), the following inequality holds 
\[ \max_{\theta \in [-h,0]} \sum_{|i| \geq 2K} |\phi_i(t + \theta, \vartheta, \psi_0(\vartheta_\omega))|^2_{H^0} \leq 2 \max_{\theta \in [-h,0]} \sum_{|i| \geq 2K} \rho_{K,i} |z_i(\vartheta_\omega)|^2 + 2 \sum_{|i| \geq 2K} \rho_{K,i} |z_i(\vartheta_\omega)|^2 
= 2 \rho^2 \psi(t, \vartheta, \psi_0(\vartheta_\omega)) \|_{H^0}^2 + 2 \sum_{|i| \geq 2K} \rho_{K,i} |z_i(\vartheta_\omega)|^2 < 4\varepsilon. \]

This completes the proof of this lemma. \( \square \)

Next, we state the main result of this section.

**Theorem 4.3.** Under the same conditions of Lemma 4.1, the SDS \( \{S(t, \omega)\}_{t \geq 0, \omega \in \Omega} \) defined by (3.1) has a unique \( D \)-random attractor \( \mathcal{A}(\omega) \).

**Proof.** In view of Proposition 2.2 and Lemma 4.1, it suffices to prove that, for a.e. \( \omega \in \Omega \), each sequence \( \varphi_{t_n}, \vartheta_{t_n}, \psi_0(\vartheta_{t_n}) = S(t_n, \vartheta_{t_n}, \psi_0(\vartheta_{t_n})) \) has a convergent subsequence in \( H_0 \) provided \( t_n \to \infty \) and \( \varphi_0(\vartheta_{t_n}) \in B_0(\vartheta_{t_n}) \).

By (4.9) we have for all \( \theta \in [-h,0] \) and \( n > 0 \) that
\[ \| \varphi(t_n + \theta, \vartheta_{t_n}, \psi_0(\vartheta_{t_n})) \|_H \leq \| \varphi_{t_n}, \vartheta_{t_n}, \psi_0(\vartheta_{t_n}) \|_{H_0} \leq C, \]
where \( C > 0 \) is given constant. For a fixed \( \theta \in [-h,0] \), we can find a subsequence \( \{\varphi(t_n + \theta, \vartheta_{t_n}, \psi_0(\vartheta_{t_n}))\} \) and \( \mu(\theta) \in H \) such that
\[ \varphi(t_n + \theta, \vartheta_{t_n}, \psi_0(\vartheta_{t_n})) \to \mu(\theta) \] weakly in \( H \) as \( n \to \infty \).

In fact, the above convergence is also strong by Lemma 4.2. Now that, for any \( \varepsilon > 0 \), there exist \( N(\varepsilon, \omega) > 0 \) and \( K(\varepsilon, \omega) > 0 \) such that
\[ \sum_{|i| > K(\varepsilon, \omega)} \| \varphi_i(t_n + \theta, \vartheta_{t_n}, \psi_0(\vartheta_{t_n})) \|_{H^0}^2 < \varepsilon, \sum_{|i| > K(\varepsilon, \omega)} \| \mu_i(\theta) \|_{H^0}^2 < \varepsilon, \]
and for \( n \geq N(\varepsilon, \omega) \) it holds that
\[ \sum_{|i| \leq K(\varepsilon, \omega)} \| \varphi_i(t_n + \theta, \vartheta_{t_n}, \psi_0(\vartheta_{t_n})) - \mu_i(\theta) \|_{H^0}^2 < \varepsilon. \]

Hence, a combination of (4.12) and (4.13) yields for \( n \geq N(\varepsilon, \omega) \) that
\[ \| \varphi(t_n + \theta, \vartheta_{t_n}, \psi_0(\vartheta_{t_n})) - \mu(\theta) \|_{H^0}^2 \]
\[ \leq \sum_{|i| > K(\varepsilon, \omega)} \| \varphi_i(t_n + \theta, \vartheta_{t_n}, \psi_0(\vartheta_{t_n})) - \mu_i(\theta) \|_{H^0}^2 
+ \sum_{|i| \leq K(\varepsilon, \omega)} \| \varphi_i(t_n + \theta, \vartheta_{t_n}, \psi_0(\vartheta_{t_n})) - \mu_i(\theta) \|_{H^0}^2 \leq 5\varepsilon. \]
This derives for any \( \theta \in [-h, 0] \) that \( \varphi(t_n + \theta, \vartheta_{-t_n}\omega, \varphi_0(\vartheta_{-t_n}\omega)) \to \mu(\theta) \) strongly in \( H \) as \( n \to \infty \).

On the other hand, making use of the integral representation of solutions, we obtain for any \( s_1, s_2 \in [-h, 0] \) that
\[
\| \varphi(t_n + s_1, \vartheta_{-t_n}\omega, \varphi_0(\vartheta_{-t_n}\omega)) - \varphi(t_n + s_2, \vartheta_{-t_n}\omega, \varphi_0(\vartheta_{-t_n}\omega)) \|_H \\
\leq \| \psi(t_n + s_1, \vartheta_{-t_n}\omega, \psi_0(\vartheta_{-t_n}\omega)) - \psi(t_n + s_2, \vartheta_{-t_n}\omega, \psi_0(\vartheta_{-t_n}\omega)) \|_H \\
+ \| z(\vartheta_{s_1}\omega) - z(\vartheta_{s_2}\omega) \| \\
\leq \int_{s_2}^{s_1} \left( \| D(\psi(t_n + s, \vartheta_{-t_n}\omega, \psi_0(\vartheta_{-t_n}\omega))) \|_H + \| F(\psi(t_n + s, \vartheta_{-t_n}\omega, \varphi_0(\vartheta_{-t_n}\omega))) \|_H \\
+ \| G_1(s) \|_H \right) ds + \| z(\vartheta_{s_1}\omega) - z(\vartheta_{s_2}\omega) \|.
\]
It is obvious that \( D \) is a linear operator from \( H \) into itself and the operator \( F: H_0 \to H \) is bounded. From (4.8), \( \psi(t, \vartheta_{-t}\omega, \psi_0(\vartheta_{-t}\omega)) \) is uniformly bounded in \( H_0 \) for all \( t \geq 0 \). Hence we have
\[
\lim_{|s_1 - s_2| \to 0} \int_{s_2}^{s_1} \left( \| D(\psi(t_n + s, \vartheta_{-t_n}\omega, \psi_0(\vartheta_{-t_n}\omega))) \|_H \\
+ \| F(\psi(t_n + s, \vartheta_{-t_n}\omega, \varphi_0(\vartheta_{-t_n}\omega))) \|_H \right) ds = 0.
\]
Also, we have that
\[
\lim_{|s_1 - s_2| \to 0} \int_{s_2}^{s_1} \| G_1(s) \|_H ds \leq \sqrt{4 + \lambda + \epsilon^2} \lim_{|s_1 - s_2| \to 0} \int_{s_2}^{s_1} \| z(\vartheta_{s_1}\omega) \| ds \\
\leq \sqrt{4 + \lambda + \epsilon^2} \max_{\theta \in [-h, 0]} \| z(\vartheta_{\theta}\omega) \| \lim_{|s_1 - s_2| \to 0} |s_1 - s_2| = 0.
\]
Moreover, condition (i) gives that
\[
\lim_{|s_1 - s_2| \to 0} \| z(\vartheta_{s_1}\omega) - z(\vartheta_{s_2}\omega) \| = 0.
\]
In conclusion, it holds for any \( s_1, s_2 \in [-h, 0] \) that
\[
\lim_{|s_1 - s_2| \to 0} \| \varphi(t_n + s_1, \vartheta_{-t_n}\omega, \varphi_0(\vartheta_{-t_n}\omega)) - \varphi(t_n + s_2, \vartheta_{-t_n}\omega, \varphi_0(\vartheta_{-t_n}\omega)) \|_H = 0,
\]
which is the required equicontinuity. Therefore, by the well-known Ascoli-Arzelà theorem, we conclude that there exists a subsequence \( \{ \varphi_{t_n^k}(\cdot, \vartheta_{-t_n^k}\omega, \psi_0(\vartheta_{-t_n^k}\omega)) \} \) of \( \{ \varphi_{t_n}(\cdot, \vartheta_{-t_n}\omega, \psi_0(\vartheta_{-t_n}\omega)) \} \) such that
\[
\varphi_{t_n^k}(\cdot, \vartheta_{-t_n^k}\omega, \varphi_0(\vartheta_{-t_n^k}\omega)) \to \mu(\cdot) \text{ strongly in } H_0.
\]
This completes the proof. \( \square \)

5. Numerical illustration. In this section, we present a numerical example to support our conclusions. Especially, taking using of several numerical experiments, we will test the influence of initial conditions to the system.

Consider the 2nd-order stochastic delay lattice differential equation:
\[
\dot{u}_i + 4\dot{u}_i - (u_{i-1} - 2u_i + u_{i+1}) + 8u_i + f_i(u_{it}) + \dot{w}_i(t) = 0, \quad t > 0, \quad i \in \mathbb{Z}, \quad (5.1)
\]
where the maximum delay $h = 0.1$ and

$$f_i(u_{it}) = \frac{u_i(t - 0.05)}{10(1 + \|u_i(t - 0.05)\|^2)} + \frac{1}{t} \int_{-0.1}^{0} u_i(t + \theta)d\theta.$$  

It is easy to verify that the above equation satisfies the conditions of Theorem 4.3 since $\epsilon = 1$, $M_1 = \frac{\sqrt{2}}{h}$, $\eta = 0.45 \in (0.0514, 72.5711)$, in which the endpoints 0.0514 and 72.5711 are two approximate solutions of the equation $\sqrt{\eta e^{-\eta h}} = 2M_1$, and

$$0.6973 = 2M_1\epsilon h < \sqrt{\lambda} = 2.8284, \quad 0 < \frac{\epsilon}{2} - \eta = 0.05.$$ 

Hence, by Theorem 4.3 system (5.1) has a $\mathcal{D}$-random attractor.

Combining the first and the second-order difference-quotient approximate formula and the compound trapezoidal rule, we can obtain the following finite difference method for (5.1):

$$u_i^{n+1} - 2u_i^n + u_i^{n-1} + 2\left(\frac{u_i^{n+1} - u_i^{n-1}}{k} + (u_i^{n-1} - 2u_i^n + u_i^{n+1}) + 8u_i^n + \frac{u_i^{n-m}}{10(1+\|u_i^{n-m}\|^2)}\right)$$ 

$$+ \frac{k}{2\pi} (u_i^{n-2m} + u_i^n + 2\sum_{j=1}^{m-1} u_i^{n-j}) + \Delta_i^n = 0, \quad i \in \mathbb{Z},$$  

where $k = h/2m$ denotes the computational stepsize in time, $m$ is a given positive integer, $u_i^n$ is an approximation to $u_i(t_n)$, and $\Delta_i^n = u_i(t_{n+1}) - u_i(t_n)$, which is independent $N(0,k)$-distributed Gaussian random variable.

Setting $i = 1, 2, \ldots, 100$ and $m = 100$ and then using the numerical scheme (5.2) to solve the equations (5.1) with different initial functions: $u_i(t) = \frac{\partial}{\partial t}u_i(t) = \exp(t) \cos\left(\frac{\pi}{50}\right)$ and $u_i(t) = \frac{\partial}{\partial t}u_i(t) = \exp(t) \sin\left(\frac{\pi}{50}\right)$ ($t \in [-h, 0]$), respectively, we can obtain the corresponding numerical solutions. The 3D figures of the numerical solutions are plotted in Figure 1 and 2, respectively. And a detailed information of the numerical solutions at time $t = 10$ is shown in Figure 3. From the numerical results, we can find that the numerical solutions with different initial conditions arrive at the same stable state as the time increases, which implies the stochastic delay lattice systems have a $\mathcal{D}$-random attractor. Hence, the previous theoretical results are further verified by the above numerical experiments.

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Figure 1. Numerical simulation for the equation (5.1) with $u_i(t) = \frac{\partial}{\partial t} u_i(t) = \exp(t) \cos\left(\frac{i\pi}{50}\right)$

Figure 2. Numerical simulation for the equation (5.1) with $u_i(t) = \frac{\partial}{\partial t} u_i(t) = \exp(t) \sin\left(\frac{i\pi}{50}\right)$

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Figure 3. Numerical solutions with different initial conditions at time $t = 10$

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