Research Article

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On pairs of equations in unlike powers of primes and powers of 2

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Abstract: In this paper, we obtained that when $k = 455$, every pair of large even integers satisfying some necessary conditions can be represented in the form of a pair of unlike powers of primes and $k$ powers of 2.

Keywords: Circle method, Linnik problem, Powers of 2

MSC: 11P32, 11P05, 11P55

1 Introduction

In 1951 and 1953, Linnik established the following “almost Goldbach” result that each large even integer $N$ is a sum of two primes $p_1, p_2$ and a bounded number of powers of 2, namely

$$ N = p_1 + p_2 + 2^{v_1} + \cdots + 2^{v_k}. \tag{1} $$

In 2002, Heath-Brown and Puchta [1] applied a rather different approach to this problem and showed that $k = 13$ and, on the GRH, $k = 7$. In 2003, Pintz and Ruzsa [10] established this latter result and announced that $k = 8$ is acceptable unconditionally. This paper is yet to appear in print. Elsholtz, in an unpublished manuscript, showed that $k = 12$; this was proved independently by Liu and Lü [9].

In 1999, Liu, Liu and Zhan [6] proved that every large even integer $N$ can be written as a sum of four squares of primes and a bounded number of powers of 2, namely

$$ N = p_1^2 + p_2^2 + p_3^2 + p_4^2 + 2^{v_1} + \cdots + 2^{v_k}. \tag{2} $$

Subsequently, Liu and Liu [4] got that $k = 8330$ suffices. Later Liu and Lü [7] improved the value of $k$ of (1.2) to 165, Li [3] improved it to 151 and Zhao [13] improved it to 46. Finally Platt and Trudgian [11] revised it to 45. In 2001, Liu and Liu [5] proved that every large even integer $N$ can be written as a sum of eight cubes of primes and a bounded number of powers of 2, namely

$$ N = p_1^3 + p_2^3 + \cdots + p_8^3 + 2^{v_1} + \cdots + 2^{v_k}. \tag{3} $$

The acceptable value was determined by Platt and Trudgian [11]. In 2011, Liu and Lü [8] considered a hybrid problem of (1.1), (1.2) and (1.3),

$$ N = p_1 + p_2^2 + p_3^3 + p_4^4 + 2^{v_1} + \cdots + 2^{v_k}. \tag{4} $$

They showed that $k = 161$ is acceptable and Platt and Trudgian [11] revised it to 156.

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Very recently, Kong [2] first considered the result on pairs of linear equations in four prime variables and powers of 2, in the form

\[
\begin{align*}
N_1 &= p_1 + p_2 + 2v_1 + \cdots + 2v_k, \\
N_2 &= p_3 + p_4 + 2v_1 + \cdots + 2v_k,
\end{align*}
\]

where \(k\) is a positive integer. She proved that the simultaneous equations (1.5) are solvable for \(k = 63\). Then Platt and Trudgian [11] revised it to 62.

In this paper, we shall consider the simultaneous representation of pairs of positive even integers \(N_2 \gg N_1 > N_2\), in the form

\[
\begin{align*}
N_1 &= p_1 + p_2^2 + p_3^3 + p_4^5 + 2v_1 + \cdots + 2v_k, \\
N_2 &= p_5 + p_6^2 + p_8^3 + 2v_1 + \cdots + 2v_k,
\end{align*}
\]

where \(k\) is a positive integer. Our result is stated as follows.

**Theorem 1.1.** For \(k = 455\), the equations (1.6) are solvable for every pair of sufficiently large positive even integers \(N_1\) and \(N_2\) satisfying \(N_2 \gg N_1 > N_2\).

We establish Theorem 1.1 by means of the circle method in combination with some new methods of using the the method of Lü [8].

**Notation.** Throughout this paper, the letter \(e\) denotes a positive constant which is arbitrarily small but may not be the same at different occurrences. And \(p\) and \(v\) denote a prime number and a positive integer, respectively.

## 2 Outline of the method

Here we give an outline for the proof of Theorem 1.1.

In order to apply the circle method, we set

\[
P_i = N_i^{3/9-\epsilon}, \quad Q_i = N_i^{8/9+\epsilon}
\]

for \(i = 1, 2\). For any integers \(a_1, a_2, q_1, q_2\) satisfying

\[
1 \leq a_1 \leq q_1 < P_1, (a_1, q_1) = 1, \quad 1 \leq a_2 \leq q_2 < P_2, (a_2, q_2) = 1,
\]

we define the major arcs \(\mathcal{M}_1, \mathcal{M}_2\) and minor arcs \(C(\mathcal{M}_i), C(\mathcal{M}_j)\) as usual, namely

\[
\mathcal{M}_i = \bigcup_{q_i \in P_i} \bigcup_{1 \leq v_i < Q_i, (a_i, q_i) = 1} \mathcal{M}_i(a_i, q_i), \quad C(\mathcal{M}_i) = \left[ \frac{1}{Q_i}, 1 + \frac{1}{Q_i} \right] \setminus \mathcal{M}_i,
\]

where \(i = 1, 2\) and

\[
\mathcal{M}_i(a_i, q_i) = \left\{ a_i \in [0, 1] : \left| a_i - \frac{a_i}{q_i} \right| \leq \frac{1}{q_i Q_i} \right\}.
\]

It follows from \(2P_i \leq Q_i\) that the arcs \(\mathcal{M}_1(a_1, q_1)\) and \(\mathcal{M}_2(a_2, q_2)\) are mutually disjoint respectively. As in [12], let \(\delta = 10^{-6}\), and

\[
U_i = \left( \frac{N_i}{16(1+\delta)} \right)^{1/3}, \quad V_i = U_i^{5/6}
\]

for \(i = 1, 2\). We set

\[
f(a_i, N_i) = \sum_{p < N_i} (\log p) e(p a_i), \quad g(a_i, N_i) = \sum_{p^2 < N_i} (\log p) e(p^2 a_i),
\]
\[
S(a_i, U_i) = \sum_{p \sim U_i} (\log p)e(p^3a_i), \quad T(a_i, V_i) = \sum_{p \sim V_i} (\log p)e(p^3a_i),
\]

(10)

\[
G(a_i) = \sum_{v \sim L} e(2^v a_i), \quad \mathcal{E}_a := \{a_i \in [0, 1]: |G(a_i)| \geq \lambda L\}.
\]

(11)

where \(i = 1, 2, e(x) := \exp(2\pi ix)\) and \(L = \log_2 N_1\).

Let

\[
R(N_1, N_2) = \sum \log p_1 \log p_2 \cdots \log p_8
\]

be the weighted number of solutions of \((1.8)\) in \((p_1, \cdots, p_8, v_1, \cdots, v_8)\) with

\[
p_1 \leq N_1, \quad p_2^{1/2} \leq N_1, \quad p_3 \sim U_1, \quad p_4 \sim V_1, \quad p_5 \leq N_2, \quad p_6^2 \leq N_2, \quad p_7 \sim U_2, \quad p_8 \sim V_2, \quad v_j \leq L,
\]

for \(j = 1, 2, \cdots, k\). Then \(R(N_1, N_2)\) can be written as

\[
R(N_1, N_2) = \sum_{s=1}^{3} \sum_{t=1}^{3} R_{st}(N_1, N_2),
\]

where \(R_{st}(N_1, N_2)\) denotes the combination of \(s\)-th term in the first bracket and the \(t\)-th term in the second bracket.

We will establish Theorem 1.1 by estimating the term \(R_{st}(N_1, N_2)\) for all \(1 \leq s, t \leq 3\). We need to show that \(R(N_1, N_2) > 0\) for every pair of sufficiently large odd positive integers \(N_2 \gg N_1 > N_2\).

We need the following lemmas to prove Theorem 1.1.

For Dirichlet character \(\chi \mod q\), let

\[
C_1(\chi, a) = \sum_{h=1}^{q} \overline{\chi}(h)e\left(\frac{ah}{q}\right), \quad C_1(q, a) = C_1(\chi^0, a),
\]

\[
C_2(\chi, a) = \sum_{h=1}^{q} \overline{\chi}(h)e\left(\frac{ah^2}{q}\right), \quad C_2(q, a) = C_2(\chi^0, a),
\]

\[
C_3(\chi, a) = \sum_{h=1}^{q} \overline{\chi}(h)e\left(\frac{ah^3}{q}\right), \quad C_3(q, a) = C_3(\chi^0, a),
\]

where the Ramanujan sum \(C_1(q, a) = \mu(q)\), \((a, q) = 1\). If \(\chi_1, \chi_2, \chi_3, \chi_4\) are characters mod \(q\), then we write

\[
B(n, q; \chi_1, \chi_2, \chi_3, \chi_4) = \sum_{(a, q)=1}^{q} C_1(\chi_1, a)C_2(\chi_2, a)C_3(\chi_3, a)C_3(\chi_4, a)e\left(-\frac{an}{q}\right),
\]

\[
B(n, q) = B(n, q; \chi^0, \chi^0, \chi^0, \chi^0),
\]

\[
A(n, q) = \frac{B(n, q)}{\varphi(q)}, \quad \mathcal{E}(n) = \sum_{q=1}^{\infty} A(n, q).
\]
Lemma 2.1. We have
\[ \text{meas}(E_1) \ll N_2^{-E(0)} , \]
with \( E(0.9457) > 109/126 + 10^{-10} \).

Proof. This is Lemma 4.4 in Liu and Lü [8].

Lemma 2.2. Let \( N_i \) be as in (2.1). Then for \( n \leq N_i \), we have
\[
\int_{N_i} f(a_1, N_i)g(a_1, N_i)S(a_1, U_i)T(a_1, V_i)e(-an)\,da = \frac{1}{2 \cdot 3^2} \mathcal{S}(n)J(n) + O(N_1^{10/9}L^{-1}).
\]
Here the singular series \( \mathcal{S}(n) \) satisfies \( \mathcal{S}(n) \gg 1 \) for \( n \equiv 0 \pmod{2} \). \( J(n) \) is defined as
\[
J(n) := \sum_{m_1, m_2, m_3, n = n} m_2^{-1/2} (m_3 m_4)^{-2/3},
\]
and satisfies \( N_1^{10/9} \ll J(n) \ll N_1^{10/9} \).

Proof. This is Lemma 4.2 in [8], we have
\[
\sum_{n_1 \in \mathfrak{B}(N_i), k} J(n_1)J(n_2) = \{2.3381\}^2 N_1^{10/9} N_2^{10/9} L^k.
\]

Lemma 2.3. For all integers \( n \equiv 0 \pmod{2} \), we have \( \mathcal{S}(n) \gg 0.2448 \).

Proof. This result can be found in Section 3 in Liu and Lü [8].

Lemma 2.4. Let \( \mathfrak{B}(N_i, k) = \{n_1 \geq 2 : n_1 = N_i - 2^v_1 - \cdots - 2^v_k\} \) with \( k \geq 2 \). Then for \( N_1 \equiv N_2 \equiv 0 \pmod{2} \), we have
\[
\sum_{n_1 \in \mathfrak{B}(N_i, k)} J(n_1)J(n_2) \geq (2.3381)^2 N_1^{10/9} N_2^{10/9} \sum_{(v)} 1,
\]
where \((v)\) means that \( v_1, \ldots, v_k \) satisfies
\[
1 \leq v_1, \ldots, v_k \leq \log_2(N_i/KL), \quad 2^{v_1} + \cdots + 2^{v_k} \equiv N_i \pmod{2}.
\]
Then following the argument of Lemma 4.1 in [8], we have
\[
\sum_{(v)} 1 \geq (1 - c)L^k.
\]
Then we get the proof of this lemma.

Lemma 2.5. Let \( f(a_i, N_i), g(a_i, N_i), S(a_i, N_i), T(a_i, V_i) \) be defined by (2.3) and (2.4), \( C(M_i) \) by (2.1). Then
\[
\sup_{a \in C(M_i)} |f(a_i, N_i)| \ll N_1^{17/18 + \varepsilon}, \quad \sup_{a \in C(M_i)} |g(a_i, N_i)| \ll N_1^{6/9 + \varepsilon},
\]
\[
\sup_{a \in C(M_i)} |S(a_i, U_i)| \ll N_1^{5/18 + \varepsilon}, \quad \sup_{a \in C(M_i)} |T(a_i, V_i)| \ll N_1^{13/42 + \varepsilon}.
\]

Proof. The proof of this lemma can be found in [8], which is based on the estimate of exponential sums over primes.
Lemma 2.6. Let \( f(a_i, N_i), g(a_i, N_i), S(a_i, N_i) \) and \( T(a_i, V_i) \) be defined by (2.3) and (2.4), \( G(a_i) \) by (2.5). Then we have
\[
\int_0^1 |f(a_i, N_i)g(a_i, N_i)S(a_i, U_i)T(a_i, V_i)G^2(2a_i)|\,da_i \leq 170.1881N_i^{10/9}L^2.
\]

Proof. From the definition of \( G(a_i) \), Lemma 10 in [1], Lemma 2.3 and 2.5 in [8], we have
\[
\int_0^1 |f(a_i, N_i)G(2a_i)|^2\,da_i \leq 12.3238c_0N_iL^2,
\]
\[
\int_0^1 |g(a_i, N_i)G(2a_i)|^4\,da_i \leq c_1 \frac{\pi^2}{16}N_iL^4,
\]
\[
\int_0^1 |S(a_i, U_i)T(a_i, V_i)|^4\,da_i \leq 0.3591N_i^{13/9},
\]
where
\[
c_0 = 0.6601, \quad c_1 \leq \left( \frac{32^3 \cdot 101 \cdot 1.6207}{3} + \frac{8 \cdot \log^2 2}{\pi^2} \right) \cdot (1 + \varepsilon)^9.
\]

Then we have
\[
\int_0^1 |f(a_i, N_i)g(a_i, N_i)S(a_i, U_i)T(a_i, V_i)G^2(2a_i)|\,da_i \leq 170.1881N_i^{10/9}L^2.
\]

Thus we can get the proof of this lemma. \( \square \)

3 Proof of Theorem 1.1

In this section, we will give the proof of Theorem 1.1.

We begin with the estimate for \( R_{11}(N_1, N_2) \). Applying Lemmas 2.2, 2.3 and 2.4 and introducing the notation \( B(N_1, k) \), we can get
\[
R_{11}(N_1, N_2) = \int_{N_1} f(a_1, N_1)g(a_1, N_1)S(a_1, U_1)T(a_1, V_1)G^k(a_1)\,da_1
\]
\[
\times \int_{N_2} f(a_2, N_2)g(a_2, N_2)S(a_2, U_2)T(a_2, V_2)G^k(a_2)\,da_2
\]
\[
= \sum_{n_1 \in \mathbb{Z}[N_1 \setminus N_1]} \int_{N_1} f(a_1, N_1)g(a_1, N_1)S(a_1, U_1)T(a_1, V_1)e(-a_1n_1)\,da_1
\]
\[
\times \int_{N_2} f(a_2, N_2)g(a_2, N_2)S(a_2, U_2)T(a_2, V_2)e(-a_2n_2)\,da_2
\]
\[
\times \int_{\mathcal{N}_2} f(\alpha_2, N_2)g(\alpha_2, N_2)(\alpha_2, U_2)T(\alpha_2, V_2)e(-\alpha_2n_2)da_2
\]
\[
\geq \left(\frac{1}{2 - 3^2}\right)^2 \sum_{n_1 \in \mathbb{N}(n_1, k_1), \alpha \in \mathbb{N}(n_2, k_2)} \mathcal{G}(n_1)\mathcal{G}(n_2)f(n_1)f(n_2) + O(N_1^{10/9}N_2^{10/9}L^{k-1})
\]
\[
\geq \frac{\pi^2}{16} \cdot (0.2448)^2 \cdot 5.4671N_1^{10/9}N_2^{10/9}L^k,
\]

where we used \( \frac{n_2}{N_1} = 1 + O(L^{-1}) \) for \( n_1 \in \mathcal{B}(n, k) \).

Now we turn to give an upper bound for \( R_{12}(N_1, N_2) \). The estimate for \( R_{21}(N_1, N_2) \) is similar. By Cauchy’s inequality, we can get
\[
|G(\alpha_1 + \alpha_2)| \leq \sqrt{|G(2\alpha_1)G(2\alpha_2)|}.
\]

For \( \alpha \in C(M_2) \setminus \mathcal{E}_A \) and sufficiently large \( N_1 \), we have
\[
|G(2\alpha_1)| \leq |G(\alpha_1)| + 2 \leq 2L + 2 \leq (1 + o(1))L.
\]

Then using the definition of \( \mathcal{E}_A \), the trivial bound of \( G(\alpha_1) \), Lemmas 2.1, 2.5 and 2.6, we have
\[
R_{12}(N_1, N_2)
= \int_{\mathcal{N}_1} \int_{\mathcal{N}_2} f(\alpha_1, N_1)g(\alpha_1, N_1)(\alpha_1, U_1)T(\alpha_1, V_1)
\times \int_{\mathcal{N}_2} f(\alpha_2, N_2)g(\alpha_2, N_2)(\alpha_2, U_2)T(\alpha_2, V_2)e(-\alpha_1n_1 - \alpha_2n_2)da_1da_2
\leq \int_{\mathcal{N}_1} f(\alpha_1, N_1)g(\alpha_1, N_1)(\alpha_1, U_1)T(\alpha_1, V_1)G^{k/2}(\alpha_1)|da_1
\times \int_{\mathcal{N}_2} f(\alpha_2, N_2)g(\alpha_2, N_2)(\alpha_2, U_2)T(\alpha_2, V_2)G^{k/2}(\alpha_2)|da_2
\leq L^{k/2-2} \int_{\mathcal{N}_1} f(\alpha_1, N_1)g(\alpha_1, N_1)(\alpha_1, U_1)T(\alpha_1, V_1)G^{2}(\alpha_1)|da_1
\times \int_{\mathcal{N}_2} f(\alpha_2, N_2)g(\alpha_2, N_2)(\alpha_2, U_2)T(\alpha_2, V_2)G^{k/2}(\alpha_2)|da_2
\leq N_1^{10/9}L^{k/2}L^{k/2} \max_{\alpha_1 \in C(M_2)} |f(\alpha_1, N_2)g(\alpha_1, N_2)(\alpha_2, U_2)T(\alpha_2, V_2)| \left( \int_{\mathcal{E}_A} 1da_2 \right)
\leq N_1^{10/9}L^{k}N_2^{10/9}\left( \frac{1}{\text{meas}(\mathcal{E}_A)} \right)
\leq N_1^{10/9}L^{k}N_2^{10/9}N_2^{10/9}N_2^{10/9} \leq N_1^{10/9}N_2^{10/9}L^{k-1}.
\]

Similarly, we can get
\[
R_{21}(N_1, N_2) \ll N_1^{10/9}N_2^{10/9}L^{k-1}.
\]

Next we give an upper bound for \( R_{13}(N_1, N_2) \). By Lemma 2.6, using the trivial bound \( |G(2\alpha)| \leq L \) when \( \alpha \in M_1 \) and the bound \( |G(2\alpha)| \leq (1 + o(1))L \) when \( \alpha \in C(N_2) \setminus \mathcal{E}_A \), we have
\[
|R_{13}(N_1, N_2)|
= \int_{\mathcal{N}_1} \int_{\mathcal{N}_2} f(\alpha_1, N_1)g(\alpha_1, N_1)(\alpha_1, U_1)T(\alpha_1, V_1)
\times \int_{\mathcal{N}_2} f(\alpha_2, N_2)g(\alpha_2, N_2)(\alpha_2, U_2)T(\alpha_2, V_2)e(-\alpha_1n_1 - \alpha_2n_2)da_1da_2|
\leq N_1^{10/9}L^{k}N_2^{10/9}N_2^{10/9} \ll N_1^{10/9}N_2^{10/9}L^{k-1}.
\]
We give the estimate for
\[
\int_{\mathcal{M}_1} |f(a_1, N_1)g(a_1, N_1)S(a_1, U_1)T(a_1, V_1)|d\alpha_1
\times \int_{\mathcal{C}(N_2)\setminus E_3} |f(a_2, N_2)g(a_2, N_2)S(a_2, U_2)T(a_2, V_2)|d\alpha_2
\leq L^{k/2-2} \int_{\mathcal{M}_1} |f(a_1, N_1)g(a_1, N_1)S(a_1, U_1)T(a_1, V_1)|d\alpha_1
\times (\lambda L)^{k/2-2} \int_{\mathcal{C}(N_2)\setminus E_3} |f(a_2, N_2)g(a_2, N_2)S(a_2, U_2)T(a_2, V_2)|d\alpha_2
\leq (170.1881)^{2} \lambda^{k/2-2} N_1^{10/9} N_2^{10/9} L^k.
\]

We can obtain the estimate for \( R_{31}(N_1, N_2) \) analogously,
\[
|R_{31}(N_1, N_2)| \leq (170.1881)^{2} \lambda^{k/2-2} N_1^{10/9} N_2^{10/9} L^k. \tag{18}
\]

We give the estimate for \( R_{22}(N_1, N_2) \) by the trivial bound for \( G(\alpha) \), Lemma 2.5 and the definition of \( E_4 \),
\[
R_{22}(N_1, N_2) \leq \int_{\mathcal{M}_1 \cap E_4} \int_{\mathcal{M}_1 \cap E_4} f(a_1, N_1)g(a_1, N_1)S(a_1, U_1)T(a_1, V_1)
\times f(a_2, N_2)g(a_2, N_2)S(a_2, U_2)T(a_2, V_2)G^k(\alpha_1 + \alpha_2)e(-\alpha_1 N_1 - \alpha_2 N_2)d\alpha_1 d\alpha_2
\ll \int_{\mathcal{M}_1 \cap E_4} |f(a_1, N_1)g(a_1, N_1)S(a_1, U_1)T(a_1, V_1)|d\alpha_1
\times \int_{\mathcal{M}_2 \cap E_4} |f(a_2, N_2)g(a_2, N_2)S(a_2, U_2)T(a_2, V_2)|d\alpha_2
\ll N_1^{10/9} L^{k/2-1} N_2^{10/9} L^{k/2-1} \ll N_1^{10/9} N_2^{10/9} L^{-k}. \tag{19}
\]

For \( R_{23}(N_1, N_2) \), we can easily get
\[
R_{23}(N_1, N_2) \leq \int_{\mathcal{M}_1 \cap E_4} \int_{\mathcal{M}_1 \cap E_4} f(a_1, N_1)g(a_1, N_1)S(a_1, U_1)T(a_1, V_1)
\times f(a_2, N_2)g(a_2, N_2)S(a_2, U_2)T(a_2, V_2)G^k(\alpha_1 + \alpha_2)e(-\alpha_1 N_1 - \alpha_2 N_2)d\alpha_1 d\alpha_2
\ll \int_{\mathcal{M}_1 \cap E_4} |f(a_1, N_1)g(a_1, N_1)S(a_1, U_1)T(a_1, V_1)|d\alpha_1
\times \int_{\mathcal{M}_2 \cap E_4} |f(a_2, N_2)g(a_2, N_2)S(a_2, U_2)T(a_2, V_2)|d\alpha_2
\ll N_1^{10/9} L^{k/2-1} N_2^{10/9} L^{k/2} \ll N_1^{10/9} N_2^{10/9} L^{-k}. \tag{20}
\]

Similarly, we have
\[
R_{32}(N_1, N_2) \ll N_1^{10/9} N_2^{10/9} L^{-k}. \tag{21}
\]

In the end, we provide the upper bound for \( R_{33}(N_1, N_2) \).
\[
|R_{33}(N_1, N_2)| \leq \int_{\mathcal{C}(N_2) \setminus E_4} \int_{\mathcal{C}(N_2) \setminus E_4} f(a_1, N_1)g(a_1, N_1)S(a_1, U_1)T(a_1, V_1)
\times f(a_2, N_2)g(a_2, N_2)S(a_2, U_2)T(a_2, V_2)G^k(\alpha_1 + \alpha_2)e(-\alpha_1 N_1 - \alpha_2 N_2)d\alpha_1 d\alpha_2
\ll N_1^{10/9} L^{k/2-1} N_2^{10/9} L^{k/2} \ll N_1^{10/9} N_2^{10/9} L^{-k}. \tag{22}
\]
Therefore, we solve the inequality
\[ x_f(a_2, N_2)g(a_2, N_2)S(a_2, U_2)T(a_2, V_2)G^k(a_1 + a_2)e(-a_1N_1 - a_2N_2)da_1da_2 |\]
\[ \leq \int_{C(N_1)} |f(a_1, N_1)g(a_1, N_1)S(a_1, U_1)T(a_1, V_1)G^{k/2}(2a_1)| da_1 \times \int_{C(N_2)} |f(a_2, N_2)g(a_2, N_2)S(a_2, U_2)T(a_2, V_2)G^{k/2}(2a_2)| da_2 \]
\[ \leq (AL)^{k/2 - 2} \times 170.1881N_1^{10/9}L^2 \left( (AL)^{k/2 - 2} \times 170.1881N_2^{10/9}L^2 \right) \]
\[ \leq \lambda^{k-4}(170.1881)^2N_1^{10/9}N_2^{10/9}L^k. \]

Combining (3.1)-(3.9), we can obtain
\[ R(N_1, N_2) > R_{11}(N_1, N_2) - R_{13}(N_1, N_2) - R_{31}(N_1, N_2) - R_{33}(N_1, N_2) + O(N_1^{10/9}N_2^{10/9}L^{k-1}) \]
\[ + O(N_1^{10/9}N_2^{10/9}L^{k-1}) \]
\[ > \frac{\pi^2}{16} \cdot 0.2448 \cdot 5.4671N_1^{10/9}N_2^{10/9}L^k - 2 \times (170.1881)^2 \lambda^{k/2 - 2}N_1^{10/9}N_2^{10/9}L^k \]
\[ - \lambda^{k-4}(170.1881)^2N_1^{10/9}N_2^{10/9}L^k + O(N_1^{10/9}N_2^{10/9}L^{k-1}). \]

Therefore, we solve the inequality
\[ R(N_1, N_2) > 0 \]
and get \( k \geq 455 \). Consequently, we deduce that every pair of large odd integers \( N_1, N_2 \) satisfying \( N_2 \gg N_1 > N_2 \) and \( N_1 \equiv N_2 \equiv 0(\text{mod } 2) \) can be written in the form of (1.3) for \( k \geq 455 \). Thus Theorem 1.1 follows.

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