EINSTEIN HYPERSURFACES OF DAMEK-RICCI SPACES

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Abstract. Einstein hypersurfaces are “very rare” in rank-one symmetric spaces. Damek-Ricci spaces may be viewed as the closest and the most natural generalisations of noncompact rank-one symmetric spaces. We prove that no Damek-Ricci space admits an Einstein hypersurface.

1. Introduction

The study of homogeneous manifolds is one of the main avenues in modern Riemannian geometry. In particular, the theory of homogeneous Einstein manifolds is a very active area, with many beautiful results and challenging conjectures. In comparison, the study of submanifolds of homogeneous spaces seems to be much less developed. In this paper, we investigate a question in the overlap of homogeneous geometry and submanifold geometry, the classification of Einstein hypersurfaces in Damek-Ricci spaces.

Einstein hypersurfaces in Riemannian manifolds are very rare (which is naively suggested by a parameter count), to the extent that the correct question may be to classify (say homogeneous) spaces admitting an Einstein hypersurface, rather than to classify Einstein hypersurfaces in a given space or class of spaces. Nevertheless, an explicit classification of Einstein hypersurfaces, to the best of our knowledge, is only known for two-point homogeneous spaces. By [Fia, Theorem 7.1], an Einstein hypersurface in a space of constant curvature is locally either totally umbilical, or totally geodesic, or is of conullity 1, or is the product of two spheres of the same Ricci curvature in the sphere. The classification for rank-one symmetric spaces of non-constant curvature is summarised in the following theorem (where we assume that the metric is always normalised in such a way that the sectional curvature lies in \([\frac{1}{4}, 1]\)).

Theorem 1.

1. There are no Einstein (real) hypersurfaces in the complex projective space and in the complex hyperbolic space [Kon, Theorem 4.3], [Mon, Corollary 8.2], [CR, Theorem 8.69].
2. There are no Einstein (real) hypersurfaces in the quaternionic hyperbolic space [OP, Corollary 1]. A connected (real) hypersurface in \(\mathbb{H}P^m, m \geq 2\), is Einstein if and only if it is a domain of a geodesic sphere of radius \(r \in (0, \pi)\), where \(\cos r = \frac{1-2m}{1+2m}\) [MP, Corollary 7.4].
3. There are no Einstein hypersurfaces in the Cayley hyperbolic plane. A connected hypersurface in the Cayley projective plane is Einstein if and only if it is a domain of a geodesic sphere of radius \(r \in (0, \pi)\) such that \(\cos r = -\frac{5}{11}\) [KNP2].

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Remark 1. An unfortunate typing error occurred in Example 1 of [KNP2]. The correct multiplicities of the principal curvatures are 8 for $\frac{1}{2} \cot \frac{1}{2} r$ and 7 for $\cot r$ (not vice versa, as published). And then the equation in the last line of Example 1 should read $1 + \cot^2 r - (7 \cot r + 4 \cot \frac{1}{2} r) \cot r = \frac{1}{2} + \frac{1}{2} \cot^2 \frac{1}{2} r - (7 \cot r + 4 \cot \frac{1}{2} r) \frac{1}{2} \cot \frac{1}{2} r$. This does not affect the rest of the paper, including the value of $r_0$ given in Example 1 of [KNP2].

In this paper, we study Einstein hypersurfaces of Damek-Ricci spaces. This class of spaces can be naturally considered as the closest generalisation of the class of noncompact rank-one symmetric spaces. A Damek-Ricci space is a solvable Lie group with a left-invariant metric whose Lie algebra is obtained by extending a generalised Heisenberg algebra by a derivation which acts as the identity on the centre $\mathfrak{z}$ of $\mathfrak{n}$ and as a $\frac{1}{2}$ times the identity on $\mathfrak{z}^\perp$ (see Section 2.2 for details). Any Damek-Ricci space has rank one (in the sense that any Cartan subalgebra of its Lie algebra is one-dimensional), and all noncompact rank-one symmetric spaces of non-constant curvature are Damek-Ricci. All Damek-Ricci spaces are Einstein solvmanifolds, and moreover, are harmonic manifolds. One of many equivalent definitions states that a Riemannian space is harmonic if a punctured neighbourhood of any point admits a harmonic function which depends only on the distance to that point. Any homogeneous harmonic space is either flat, or rank-one symmetric, or is a Damek-Ricci space by [Sz, Heb]; whether or not there exist nonhomogeneous harmonic spaces is an open question (for the current state of knowledge see [Kn] and references therein). For recent results and developments in geometry of Damek-Ricci spaces and their submanifolds we refer the reader to [CMO, DD, Kol].

By Theorem 1, noncompact rank-one symmetric spaces of non-constant curvature admit no Einstein hypersurfaces. We prove that this is still the case for Damek-Ricci spaces.

Theorem 2. A Damek-Ricci space admits no Einstein hypersurfaces.

2. Preliminaries

2.1. Einstein condition. Let $\overline{M}$ be an Einstein Riemannian manifold of dimension $n + 1 \geq 3$ and let $M$ be an Einstein hypersurface of $\overline{M}$. Denote $\langle \cdot, \cdot \rangle$ the metric tensor on $\overline{M}$ and the induced metric tensor on $M$. Denote $\nabla, \overline{\nabla}$ and $\nabla, R$ the Levi-Civita connection and the curvature tensor of $\overline{M}$ and of $M$ respectively; for $x \in M$ and $X \in T_x M$, the Jacobi operator $\overline{R}_X$ is defined by $\overline{R}_XY = \overline{R}(Y, X)X$ for $Y \in T_x \overline{M}$. Let $\xi$ be a unit normal vector field of $M$. For $x \in M$ we define the shape operator $S$ on $T_x M$ by $SX = -\nabla_X \xi$, so that $\langle \nabla_X Y, \xi \rangle = \langle SY, Y \rangle$, where $X, Y \in T_x M$. We will work on a small open, connected subset $\mathcal{M}$ of $M$ on which the multiplicities of the eigenvalues of $S$ are constant. On $\mathcal{M}$, we can choose a smooth orthonormal frame $X_i$ of eigenvectors of $S$, with the corresponding eigenvalues (principal curvatures) $\lambda_i$, $i = 1, \ldots, n$. Denote $H = \text{Tr} S = \sum_{i=1}^n \lambda_i$ the mean curvature of $M$.

By Gauss equation, $\langle \overline{R}(X_i, X_k, X_k, X_j) \rangle = R(X_i, X_k, X_k, X_j) + (\lambda_i^2 \delta_{ik} \delta_{jk} - \lambda_i \lambda_k \delta_{ij})$. Summing up by $k$ we obtain

\begin{equation}
\langle \overline{R}/\xi X_i, X_j \rangle = \alpha_i \delta_{ij}, \quad \text{where}
\end{equation}

\begin{equation}
\alpha_i = -\lambda_i^2 + H\lambda_i + C,
\end{equation}

and where $C$ is the difference of the Einstein constants of $\overline{M}$ and of $M$, and $\alpha_i$’s are the eigenvalues of the restriction of $\overline{R}/\xi$ to $T_x M$. Equivalently, (1) can be written as

\begin{equation}
\langle \overline{R}/\xi \rangle|_{T_x M} = -S^2 + HS + C \text{id}_{T_x M}.
\end{equation}
For an eigenvalue $\alpha$ of the restriction of $R_\xi$ to $T_xM$, denote $L_\alpha$ the corresponding eigenspace. Then for every $\alpha$, the eigenspace $L_\alpha$ is $S$-invariant (hypersurfaces with this property are called \textit{curvature-adapted}) and
\begin{equation}
S^2_{|L_\alpha} - HS_{|L_\alpha} + (\alpha - C)\text{id}_{L_\alpha} = 0.
\end{equation}

Codazzi equation takes the form
\begin{equation}
\overline{R}(X_k, X_i, X_j, \xi) = \delta_{ij}X_k(\lambda_j) - \delta_{kj}X_i(\lambda_j) + (\lambda_i - \lambda_j)\Gamma^j_{ki} - (\lambda_k - \lambda_j)\Gamma^i_{kj},
\end{equation}
where $\Gamma^j_{ki} = \langle \nabla_kX_i, X_j \rangle$ (note that $\Gamma^j_{ki} = -\Gamma^i_{kj}$).

Differentiating equation (1) in the direction of $X_k$ we obtain
\begin{equation}
\nabla_kR(X_i, \xi, \xi, X_i) = X_k(\alpha_i) + 2\lambda_k(R_X\xi, X_k),
\end{equation}
\begin{equation}
\Gamma^j_{ki}(\alpha_j - \alpha_i) = \lambda_k(R(\xi, X_j, X_i, X_k) + R(\xi, X_i, X_j, X_k)) - (\nabla_kR)(X_i, \xi, \xi, X_j),
\end{equation}
where $i \neq j$ (here and below we abbreviate $\nabla_{X_k}$ to $\nabla_k$).

Let the ambient space $M$ be a Lie group with a left-invariant metric. For a vector field $T$ (defined at $x$), we denote $\tilde{T}$ the left-invariant vector field on $\overline{M}$ such that $\tilde{T}(x) = T(x)$. Computing the covariant derivative of the curvature tensor for left-invariant vector fields we can write (6) and (7) in the following form, respectively:
\begin{equation}
\langle R_X\xi, \nabla_k\tilde{\xi} + \lambda_kX_k \rangle = -\frac{1}{2}X_k(\alpha_i),
\end{equation}
\begin{equation}
\Gamma^j_{ki}(\alpha_j - \alpha_i) = \langle R(\xi, X_j, X_i)X_k + R(\xi, X_i)X_j, \nabla_k\tilde{\xi} + \lambda_kX_k \rangle + (\alpha_j - \alpha_i)\langle \nabla_k\tilde{X}_i, X_j \rangle,
\end{equation}
where $i \neq j$. Note that the vector $\nabla_k\tilde{\xi} + \lambda_kX_k$ is the derivative of the \textit{Gauss map} of $M$ in the direction of $X_k$ [Rip, Proposition 3].

\section{Damek-Ricci spaces.}

Let $(\mathfrak{n}, \langle \cdot, \cdot \rangle)$ be a metric, two-step nilpotent Lie algebra with the centre $\mathfrak{z}$ and with $\mathfrak{v} = \mathfrak{z}^\perp$. For $Z \in \mathfrak{z}$, define $J_Z \in \mathfrak{so}(\mathfrak{v})$ by $\langle J_ZU, V \rangle = \langle [U, V], Z \rangle$ for $U, V \in \mathfrak{v}$. The metric algebra $(\mathfrak{n}, \langle \cdot, \cdot \rangle)$ is called a \textit{generalised Heisenberg algebra} if for all $Z \in \mathfrak{z}$, we have $J_Z^2 = -\|Z\|^2\text{id}_\mathfrak{v}$. Note that $\mathfrak{v}$ is a Clifford module over the Clifford algebra $\text{Cl}(\mathfrak{z}, -\langle \cdot, \cdot \rangle)$. Consider a one-dimensional extension $\mathfrak{s} = \mathfrak{n} \oplus \mathfrak{a}$ of a generalised Heisenberg algebra $\mathfrak{n}$, where $\mathfrak{a} = \mathbb{R}A$ and $[A, U] = \frac{1}{2}U$, $[A, Z] = Z$ for $U \in \mathfrak{v}$, $Z \in \mathfrak{z}$, and extend the inner product from $\mathfrak{n}$ to $\mathfrak{s}$ in such a way that $A \perp \mathfrak{v}$ and $\|A\| = 1$. Then $\mathfrak{s}$ is a metric, solvable Lie algebra.

The corresponding simply connected Lie group $S$ with the left-invariant metric defined by $\langle \cdot, \cdot \rangle$ is called a \textit{Damek-Ricci space}. Note that the hyperbolic space can be obtained by a similar construction, starting with an abelian algebra $\mathfrak{n}$; it is conventional to exclude this case hence assuming that both $\mathfrak{z}$ and $\mathfrak{v}$ have positive dimension.

\textbf{Remark 2.} We denote $d_3 = \dim \mathfrak{z}$ and $d_\mathfrak{v} = \dim \mathfrak{v}$. From the representation theory of Clifford modules we know that if $d_\mathfrak{v} = 2^{a+b}c$, where $0 \leq b \leq 3$ and $c$ is odd, then $d_3 \leq 8a + 2^b - 1$.

A Damek-Ricci space is symmetric in the following cases: $d_3 = 1$, $(d_3, d_\mathfrak{v}) = (7, 8)$, and $(d_3, d_\mathfrak{v}) = (3, 4m)$, when all the irreducible $4$-dimensional $\text{Cl}(\mathfrak{z})$-submodules of $\mathfrak{v}$ are isomorphic (that is, when $J_{Z_2}J_{Z_2}J_{Z_3} = \pm \text{id}_\mathfrak{v}$) for an orthonormal basis $\{Z_1, Z_2, Z_3\}$ for $\mathfrak{z}$. The corresponding Damek-Ricci space is rank-one symmetric and is isometric to the complex hyperbolic space, the Cayley hyperbolic plane and the quaternionic hyperbolic space respectively.
Let \( T_1, T_2 \in T_x\mathcal{S} \), with \( T_1 = V + Y + sA \), \( T_2 = U + X + rA \), where \( V, U \in \mathfrak{u} \), \( Y, X \in \mathfrak{z} \) and \( r, s \in \mathbb{R} \) (we identify \( T_x\mathcal{S} \) with \( \mathfrak{s} \) via left translation). Then by [BTV, §4.1.8, §4.1.6], for the Jacobi operator of \( \mathcal{S} \) at \( x \), the covariant derivative we have respectively

\[
\overline{R}_{T_1} T_2 = \frac{3}{4} J_x J_y V + \frac{3}{4} J_{[U,V]} V - \frac{3}{4} r J_y Y - \frac{3}{4} s J_x V - \frac{1}{4} \|T_1 \|^2 U + \left( \frac{1}{4} \langle X, Y \rangle + \frac{1}{4} \langle T_1, T_2 \rangle \right) V
\]

(10)

\[
\overline{\nabla}_{T_1} T_2 = -\frac{1}{2} J_x V \mp \frac{1}{2} J_y U - \frac{1}{2} s [U, V] - \frac{1}{2} r V - \frac{1}{2} [U, V] - r Y + \frac{1}{2} \langle U, V \rangle A + \langle X, Y \rangle A,
\]

(11)

where, as above, for a vector \( T \in T_x\mathcal{S} \) (or a vector field \( T \) on a neighbourhood of \( x \)) we denote \( \overline{T} \) the left-invariant vector field such that \( \overline{T}(x) = T(x) \).

In a generalised Heisenberg Lie algebra we have the following identities:

\[
[V, J_y V] = \|V\|^2 Y, \quad [V, J_y U] - [J_y V, U] = 2 \langle U, V \rangle Y, \quad \text{for } U, V \in \mathfrak{u}, \quad Y \in \mathfrak{z}.
\]

Following [BTV, §3.1.12], for nonzero vectors \( V \in \mathfrak{u} \) and \( Y \in \mathfrak{z} \) we define the operator \( K_{V,Y} \) on the subspace \( Y^\perp \cap \mathfrak{z} \) by

\[
K_{V,Y} X = \|V\|^{-2} \|Y\|^{-1} \langle V, J_x J_y V \rangle X.
\]

(12)

The operator \( K_{V,Y} \) is skew-symmetric, with all the eigenvalues of \( K_{V,Y}^2 \) lying in \([-1, 0] \). Furthermore,

\[
K_{V,Y}^2 X = -X \iff J_x J_y V = \|Y\| J_{K_{V,Y} X} V.
\]

(13)

Remark 3. On several occasions, we will use the following argument. Let \( \mathfrak{h} = \mathfrak{a} \oplus \mathfrak{v} \oplus \mathfrak{z} \) be a subalgebra of \( \mathfrak{s} \) such that \( J_y \mathfrak{v} \subset \mathfrak{v} \), where \( \mathfrak{v} \subset \mathfrak{u} \) and \( \mathfrak{z} \subset \mathfrak{z} \). Then by [Rou], the corresponding subgroup of \( \overline{\mathcal{M}} \) is totally geodesic and is a “smaller” Damek-Ricci space or the real hyperbolic space (when \( \mathfrak{v}' = 0 \) or \( \mathfrak{z}' = 0 \)). Note that by [KNP1], this construction gives “almost all” totally geodesic submanifolds of Damek-Ricci spaces. In particular, \( \mathfrak{h} \) is closed under \( \overline{R} \) and \( \overline{\nabla} \overline{R} \) (that is, \( \overline{R}(\mathfrak{h}, \mathfrak{h}) \mathfrak{h}, (\overline{\nabla}_h \overline{R})(\mathfrak{h}, \mathfrak{h}) \mathfrak{h} \subset \mathfrak{h} \)) and also \( \overline{\nabla}_h \overline{\nabla} \subset \mathfrak{h} \) for a left-invariant \( \overline{\nabla} \in \mathfrak{h} \).

Moreover, if that totally geodesic subgroup is a symmetric space, then it is rank-one symmetric (see Remark 2). Then we additionally have \( (\overline{\nabla}_h \overline{R})(\mathfrak{h}, \mathfrak{h}) \mathfrak{h} = 0 \), and also the following “duality” property: if unit vectors \( T_1, T_2 \in \mathfrak{h} \) are such that \( T_2 \) is an eigenvector of the Jacobi operator \( \overline{R}_{T_1} \), then \( T_1 \) is an eigenvector of the Jacobi operator \( \overline{R}_{T_2} \), with the same eigenvalue.

3. Proof of Theorem 2

Let \( \overline{M} \) be a Damek-Ricci space and \( M \) be an Einstein hypersurface in \( \overline{M} \). We adopt the notation of Section 2. We will work on a small open, connected neighbourhood \( \mathcal{M} \subset M \), and from now on, will replace \( M \) by \( \mathcal{M} \). Let \( \xi = V + Y + sA \) be a unit normal vector field of \( M \), where \( V \) and \( Y \) lie in the left-invariant subbundles \( \mathfrak{v} \) and \( \mathfrak{z} \) respectively, and \( s \) is a real function on \( M \). We can assume that on \( M \), the shape operator \( S \), the Jacobi operator \( \overline{R}_\xi \) and the operator \( K_{V,Y}^2 \) (defined by (12)) have constant number of pairwise distinct eigenvalues (and then the multiplicities of corresponding eigenvalues are also constant).

We split the proof of Theorem 2 into two cases. In Section 3.1 we consider the “special” cases, when one of the components \( V, Y \) or \( sA \) of \( \xi \) is locally zero. The proof in the “general” case, when all three are locally nonzero, is given in Section 3.2.
3.1. Special cases. In this section, we consider the cases when one of the components \( V, Y \) or \( sA \) of the unit normal vector \( \xi = V + Y + sA \) is zero at all points of \( M \).

First suppose that \( \xi \) has no \( v \)-component. Then \( TM \) contains the left-invariant subbundle \( v \), and so, by Frobenius Theorem, it also contains \([v, v] = 3\). It follows that \( TM = v \oplus 3 \), and so \( M \) is a domain on a nilpotent group, the generalised Heisenberg group from which the Damek-Ricci space has been constructed (see Section 2.2). But the latter is never Einstein [BTV, § 3.1.7] (and in general, a nilpotent group with a left-invariant metric can only be Einstein if it is abelian [Mil, Theorem 2.4]).

Now suppose that on \( M \), the unit normal vector field \( \xi \) has no \( A \)-component. Then \( \xi = V + Y \) (and we may assume that \( V \neq 0 \)) and \( TM \) contains the left-invariant vector field \( A \). Let \( T = U + X + rA \) be a tangent vector field on \( M \), where \( U \in v \), \( X \in 3 \), \( r \in \mathbb{R} \). Then by (11), \( \nabla_{T}A, \xi = (-\frac{1}{2}V - Y, T) \), and so \( SA = \frac{1}{2}\|Y\|^2V - \frac{1}{2}\|V\|^2Y \). Note that by (11), \( \nabla_{A}v \subset v \) and \( \nabla_{A}3 \subset 3 \). It follows that \( \langle \nabla_{A}A, SA \rangle = \frac{1}{2}\|Y\|^2\langle A, V \rangle - \frac{1}{2}\|V\|^2\langle A, Y \rangle = \frac{1}{4}(\|Y\|^2A(\|V\|^2) - \|V\|^2A(\|Y\|^2)) = \frac{1}{4}A(\|V\|^2A) \). On the other hand, \( \langle \nabla_{A}A, SA \rangle = -\|SA\|^2 = -\frac{1}{4}\|Y\|^2\|V\|^2 \), and so \( A(\|V\|^2) = -\|Y\|^2\|V\|^2 \). Now from (3) we obtain \( \langle R_{\xi}A, A \rangle = -\|SA\|^2 + H(SA, A) + C = C - \frac{1}{4}\|Y\|^2\|V\|^2 \), and so from (10), \( C = \frac{1}{4}\|Y\|^2\|V\|^2 - \frac{1}{2}\|V\|^2 - \|Y\|^2 = -\frac{1}{4}(2 - \|V\|^2)^2 \). It follows that \( \|V\|^2 \) is a constant and so from the above, \( Y = 0 \). Then \( C = \frac{1}{4} \) and \( \xi = V \), so \( TM \) contains the left-invariant subbundle \( 3 \). Let \( Z \in 3 \) be a unit, left-invariant vector field and \( T = U + X + rA \) be a tangent vector field on \( M \), with \( U \in v \), \( X \in 3 \) \( r \in \mathbb{R} \). Then from (11) we have \( \langle \nabla_{T}Z, \xi \rangle = \frac{1}{2}\langle J_{Z}V, U \rangle \), and so \( SZ = \frac{1}{2}J_{Z}V \). But now from (3) we get \( \langle R_{\xi}Z, Z \rangle = -\|SZ\|^2 + H(SZ, Z) + C = -\frac{1}{4} \), while from (10), \( \langle R_{\xi}Z, Z \rangle = -\frac{1}{4} \), a contradiction.

The last case to consider is a little more involved. We have the following.

Proposition 1. In a Damek-Ricci space, there is no Einstein hypersurface whose normal vector field \( \xi \) locally has no \( 3 \)-component.

Proof. By assumption, we have \( \xi = V + sA \), and we may also assume that both \( s \) and \( V \) are locally nonzero.

From [BTV, Theorem 4.2(v)], the restriction of the Jacobi operator \( R_{\xi} \) to \( TM \) has two eigenvalues, \(-1\) and \( -\frac{1}{4} \), with corresponding eigenspaces

\[
L_{-1} = \{ sZ + J_{Z}V \mid Z \in 3 \}, \quad L_{-\frac{1}{4}} = \{ \|V\|^2Z - sJ_{Z}V \mid Z \in 3 \} \oplus \mathbb{R}Q \oplus p,
\]

where \( Q = \|V\|^{-1}(sV - \|V\|^2A) \) and \( p = \{ U \in v \mid \langle U, V \rangle = 0, [U, V] = 0 \} \).

We have the following.

Lemma 1. The functions \( s, \|V\|^2, H \) and all the eigenvalues of \( S \) are constant and

\[
s^2 = 2C + 1, \quad \|V\|^2 = -2C, \quad H = -Cs^{-1}.
\]

The eigenspaces of \( S \) and the corresponding eigenvalues are given by

\[
\mathcal{V}_{-} = \{ sZ + J_{Z}V \mid Z \in 3 \} = L_{-1} \quad \rho_{-} = \frac{4s^2}{2s},
\]

\[
\mathcal{V}_{1} = \{ \|V\|^2Z - sJ_{Z}V \mid Z \in 3 \} \oplus \mathbb{R}Q \oplus p_{1} \quad \rho_{1} = \frac{s}{2s},
\]

\[
\mathcal{V}_{2} = p_{2} \quad \rho_{2} = \frac{1-2s^2}{2s},
\]

where \( p_{1} \oplus p_{2} = p \).
Proof. The tangent bundle $TM$ contains the left-invariant subbundle $\mathfrak{z}$. By (11), for any nonzero $Z \in \mathfrak{z}$ and a tangent vector $T = U + X + rA$, where $U \in \mathfrak{v}$, $X \in \mathfrak{z}$, $r \in \mathbb{R}$, we have $\langle \nabla_{T}Z, \xi \rangle = \frac{1}{2}(J_{2}Z, U) + s(X, Z) = \frac{1}{2}J_{2}Z + sZ, T)$, and so $SZ = \frac{1}{2}J_{2}Z + sZ$. We have $Z = sSZ + J_{2}ZV + (||Z\|^2Z - sJ_{2}ZV)$, and so by (14), $R_{\xi}Z = -sSZ + J_{2}ZV - \frac{1}{2}(||Z\|^2Z - sJ_{2}ZV) = (-s^2 - \frac{1}{4}||V\|^2Z - \frac{3}{4}sJ_{2}ZV)$. Acting by both sides of (3) on $Z$ we get $S_{J_{2}}ZV = (\frac{1}{2}||V\|^2 + 2sH + 2C)Z + (\frac{1}{2}s + H)J_{2}Z$. As $S$ is symmetric, from $\langle SZ, J_{2}ZV \rangle$ we obtain $H = -Cs^{-1}$, and so $S_{J_{2}}ZV = \frac{1}{2}(1-s^2)Z + (\frac{1}{2}s-Cs^{-1})J_{2}ZV$. It follows that the subspace $\text{Span}(Z, J_{2}ZV)$ is $S$-invariant.

By (14), it is also $R_{\xi}$-invariant, and the restriction of $R_{\xi}$ to it has two different eigenvalues. As any eigenvector of $S$ is an eigenvector of $R_{\xi}$ by (3), we obtain that both $SZ + J_{2}ZV$ and $||V\|^2Z - sJ_{2}ZV$ are eigenvectors of $S$. We have $S(sSZ + J_{2}ZV) = \frac{1}{2}(1+s^2)Z + (s-Cs^{-1})J_{2}ZV$, and so $s^2 = 2C+1$ and the eigenvalue of $S$ corresponding to $sSZ + J_{2}ZV$ is $\rho_\ast = \frac{1+s^2}{2s}$; moreover, $L_{-1}$ is the eigenspace of $S$ with eigenvalue $\rho_\ast$. Furthermore, $||V\|^2Z - sJ_{2}ZV$ is an eigenvector of $S$, with eigenvalue $\rho_1 = \frac{1}{2}s$. Then the subspace $L_{-1}$ is the orthogonal sum of two eigenspaces $\mathcal{V}_1$ and $\mathcal{V}_2$ with corresponding eigenvalues satisfying the equation $-\frac{1}{4} = -\rho_1^2 - Cs^{-1}\rho + C$ which gives $\rho_1$ as above and $\rho_2 = \frac{1-s^2}{2s}$.

As $s^2 = 2C + 1$, we obtain that $s, ||V||, H$ and the eigenvalues of $S$ are constant.

It remains to show that $Q \in \mathcal{V}_1$. We have $\langle \nabla_{Q}Q, Q \rangle = ||V||^{-1}(\langle \nabla_{Q}V + s\nabla_{Q}A, AV - ||V\|^2A \rangle = ||V||^{-1}(||V||^2(\langle \nabla_{Q}V, A \rangle + s^2(\nabla_{Q}A, V)) = -\frac{1}{8}s$ from (11). Then $\langle SQ, Q \rangle = \frac{1}{8}s = \rho_1$, which implies $S = \rho_1Q$, as the restriction of the second fundamental form $\langle ST, T \rangle$ to the unit sphere of $L_{-1}$ has two extremal values, $\rho_1$ and $\rho_2$. □

Next we need the following fact.

**Lemma 2.** We have $d_2 > d_1 + \frac{1}{2}d_0$, where $d_2 = \dim \mathcal{V}_2$. In particular, the subspace $p_2$ is nonzero and $\rho_1 \neq \rho_2$.

Proof. We have $H = d_3\rho_- +d_0\rho_2 + (d_0 - d_2)\rho_1$, and so by (16) we obtain $(1 + d_3 + d_0 - 3d_2)s^2 + (d_3 + d_2 - 1) = 0$. As $s \in (0, 1)$, we get $d_2 > \frac{1}{3}(1 + d_3 + d_0)$ and $d_2 > d_1 + \frac{1}{2}d_0$. □

We now take $X_k = P, X_j = P' \in \mathcal{V}_2 \subset p$ and $X_i = sZ + J_{2}ZV \in \mathcal{V}_2$ in (9). Then $\alpha_j = -\frac{1}{4}, \alpha_i = -1$ and $\lambda_k = \rho_2$. From (11) and (16)) we find $\nabla_{k}\xi + \lambda_kX_k = \frac{1-3s^2}{2s}P', \langle \nabla_{k}\xi, X_j \rangle = -\frac{1}{2}s\langle J_{2}ZP, P' \rangle$.

From [BTV, §4.1.7] we obtain $\Pi(\xi, P', P, X_i) = \frac{1}{2}\langle J_{2}ZP, P' \rangle$, and so by the first Bianchi identity, $\Pi(\xi, X_i, P', P) = -\frac{1}{2}\langle J_{2}ZP, P' \rangle$. Then (9) gives $\Gamma_{ki}^j = \frac{2s^2-1}{2s^2}\langle J_{2}ZP, P' \rangle$, and so from (5) we get $(1 - 3s^2)\langle J_{2}ZP, P' \rangle = 0$, for all $P, P' \in \mathcal{V}_2$ and $Z \in \mathfrak{z}$.

Now if $s^2 = \frac{1}{3}$, then from (16) we get $\rho_1 = \rho_2$ which contradicts Lemma 2. If $s^2 \neq \frac{1}{3}$, then the subspace $\mathcal{V}_2 \subset \mathfrak{v}$ is an isotropic subspace of $J_{2}Z$, and so $d_2 \leq \frac{1}{2}d_0$ which again contradicts Lemma 2. □

### 3.2. General case.

In this section, we prove Theorem 2 assuming that on $M$, all three components of the unit normal vector field $\xi = V + Y + sA$ are nonzero (where $Y \in \mathfrak{z}, V \in \mathfrak{v}$). By Theorem 1 we can assume that $\overline{\mathcal{M}}$ is not symmetric.
Let $K = K_{V,Y}$ be the operator defined by (12). By [BTV, Theorem 4.2(vi)], the restriction of the Jacobi operator $\overline{\mathcal{T}}_\xi$ to $T_z M$ has eigenvalues $-1, -\frac{1}{2}$, and (if $K^2 \neq -\text{id}$), some eigenvalues lying in $(-1,0) \setminus \{-\frac{1}{2}\}$.

The eigenspaces $L_{-1}$ and $L_{-\frac{1}{2}}$ of $\overline{\mathcal{T}}_\xi$ are constructed as follows. We define the subspaces $s_4 = \text{Span}(A,V,Y,J_Y V)$ and $p = \{U \in \mathfrak{v} \mid [U,V] = [U,J_Y V] = 0\} \subset \mathfrak{v}$, and the unit vector $T^0 = \|Y\|^{-1}(J_Y V + s_Y - \|Y\|^2 A) \in s_4$. Furthermore, let $\tilde{3}_1 \subset \tilde{3}$ be the $(-1)$-eigenspace of $K^2$ and $v_1 = J_{z_1} V$. Then $L_{-1} \oplus L_{-1/4} \oplus \mathbb{R} \xi = s_4 \oplus p \oplus \tilde{3}_1 \oplus v_1$ and we have

$$L_{-1} = \mathbb{R} T^0 \oplus \{\|Y\|^2 - 1\} Z + J_{|Y|^2 K_Z - s_Z 2 V} | Z \in \tilde{3}_1\},$$

$$(17) \quad L_{-1/4} = (s_4 \cap \text{Span}(\xi, T^0)) \oplus p \oplus \{\|Y\|^2 Z + J_{|Y|^2 K_Z - s_Z 2 V} | Z \in \tilde{3}_1\}.$$  

We note that the subspaces $\tilde{3}_1, v_1$ and $p$ can be trivial. Also note that $\tilde{3}_1$ is $K$-invariant, and by (13), $J_Z J_Y V = \|Y\| J_{K_Z} V$, for $Z \in \tilde{3}_1$ (and so also $J_{K_Z} J_Y V = -\|Y\| J_{Z} V$). The subspace $p$ is $J_Y$-invariant. We also have $v_1 = J_{z_1} V \cap J_z J_Y V$ and $p = (J_Y V + J_{z_1} V)^\perp \cap \mathfrak{v}$, so for $d_p = \dim p$ and $d_{-1} = \dim \tilde{3}_1$ we get

$$d_v = d_p + 2d_3 - d_{-1}.$$  

We start with the following fact.

**Proposition 2.** Let $M$ be an Einstein hypersurface of the Damek-Ricci space $\overline{M}$. Suppose that all three components $V, Y$ and $s A$ of the unit normal vector field $\xi = V + Y + s A$ are locally nonzero. Then

(a) The vector $T^0$ is an eigenvector of $S$.

(b) The operator $K^2$ has an eigenvalue different from $-1$.

**Proof.** We first note that if no eigenvalue of $K^2$ equals $-1$, then there is nothing to prove. Indeed, assertion (b) follows immediately, as otherwise $d_4 = 1$, and so $\overline{M}$ is a symmetric space isometric to the complex hyperbolic space (see Remark 2). Assertion (a) also follows, as then $L_{-1} = \mathbb{R} T^0$ by (17). As $L_{-1}$ is $S$-invariant by (3), $T^0$ is an eigenvector of $S$.

For the rest of the proof of the proposition we assume that $-1$ is an eigenvalue of $K^2$. Since $d_{-1}$ is even (as $\tilde{3}_1$ is $K$-invariant), we have $d_{-1} \geq 2$ and so $d_3 \geq 3$.

By (8) we have $\langle \overline{\mathcal{T}}_{T^0} \xi, \overline{\nabla}_{k} \xi + \lambda_k X_k \rangle = 0$, for any $T \in L_{-\frac{1}{2}}$. Furthermore, by Remark 3, for any (nonzero) $P \in p$, the subspace $\mathfrak{h} = s_4 \oplus \text{Span}(P, J_Y P)$ is a subalgebra tangent to the totally geodesic subgroup $M' \subset \overline{M}$ which is isometric to the complex hyperbolic space $\mathbb{C} H^3$; then by the duality property, we get $\overline{\mathcal{T}}_{T^0} \xi = -\frac{1}{2} \|T\|^2 \xi$, for all $T \in L_{-\frac{1}{2}} \cap \mathfrak{h}$. A similar argument applies to the subspace $\mathfrak{h} = s_4 \oplus \text{Span}(Z, K_Z, J_Z V, J_{K_Z} V)$; for a nonzero $Z \in \tilde{3}_1$, it is tangent to the totally geodesic subgroup isometric to the quaternionic hyperbolic plane $\mathbb{H}^2$, and so we obtain $\overline{\mathcal{T}}_{T^0} \xi = -\frac{1}{2} \|T\|^2 \xi$, for all $T \in L_{-\frac{1}{2}} \cap \mathfrak{h}$. It follows from (18) that we can take $T \in s_4 \cap L_{-\frac{1}{2}}$ in the equation $\langle \overline{\mathcal{T}}_{T^0} \xi, \overline{\nabla}_{k} \xi + \lambda_k X_k \rangle = 0$. Let $T = \|Y\|^2 Z + J_{|Y|^2 K_Z - s_Z 2 V} + P$, where $P \in p$ and $Z \in \tilde{3}_1$. From (10) we obtain $\overline{\mathcal{T}}_{T^0} \xi = \frac{3}{2} \|V\|^2 (J_Y J_Z + \|Y\| J_{K_Z}) P - \frac{1}{2} \|T\|^2 \xi$, and so

$$0 = \langle (J_Y J_Z + \|Y\| J_{K_Z}) P, \overline{\nabla}_{k} \xi + \lambda_k X_k \rangle = \langle (J_Y J_Z + \|Y\| J_{K_Z}) P, (N_3 + S) X_k \rangle,$$

for all $k$, where $N_3$ is the Nomizu operator on $T_z \overline{M}$ defined by $N_3 T' = \overline{\nabla}_{T'} \xi$, for $T' \in T_z \overline{M}$. Then the latter equation gives $(N_3^2 + S)(J_Y J_Z + \|Y\| J_{K_Z}) P = 0$, for all $P \in p$ and all $Z \in \tilde{3}_1$. Denote $W = (J_Y J_Z + \|Y\| J_{K_Z}) P \in \mathfrak{v}$. Computing the Nomizu operator $N_3$ from (11) we find
\[ N^I_W = \frac{1}{2} J_Y W - \frac{1}{2} s W - \frac{1}{2} [V, W], \] and so we obtain \[ SW = -\frac{1}{2} J_Y W + \frac{1}{2} s W + \frac{1}{2} [V, W]. \] But \( p \) is \( J_Y \)-invariant. Replacing \( P \) by \( J_Y P \) we obtain \[ S J_Y W = \frac{1}{2} \|Y\|^2 W + \frac{1}{2} s J_Y W + \frac{1}{2} [V, J_Y W]. \] As the operator \( S \) is symmetric, we get \[ \langle S J_Y W, W \rangle = \frac{1}{2} \|Y\|^2 \|W\|^2 = -\frac{1}{2} \|Y\|^2 \|W\|^2, \] which implies \( W = 0 \), that is, \[ (J_Y J_Z + \|Y\| J_{KZ}) P = 0, \quad \text{for all} \ P \in p, \ Z \in j_{-1}. \]

For a nonzero \( Z \in j_{-1} \), the symmetric operator \( F_Z = \|Z\|^2 J_Y J_Z J_{KZ} \) on \( v \) satisfies \( F_Z^2 = \text{id}_v \). Denote \( \mathcal{V}_\pm \) the \((\pm 1)\)-eigenspaces of \( F_Z \) respectively. Then \( \mathcal{V}_+ \) contains the vectors \( V, J_Y V, J_Z V, J_{KZ} V \), and also the subspace \( p \), by (20). If \( d_3 = 3 \) we have \( j = \text{Span}(Y, Z, KZ) \), and so \( v = \text{Span}(V, J_Y V, J_Z V, J_{KZ} V) \oplus p = \mathcal{V}_+ \). But then \( F_Z = \text{id}_v \), and so by Remark 2, the Damek-Ricci space \( M \) is symmetric (and is isometric to the quaternionic hyperbolic space). It follows that \( d_3 > 3 \). Then for any \( X \in j, X \perp Y, Z, K \), the operator \( F_Z \) anti-commutes with \( J_X \), and so \( \text{dim} \mathcal{V}_+ = \frac{1}{2} d_3 \), and moreover, \( J_X V, J_Y J_X V \in \mathcal{V}_- \). Thus \( p = \mathcal{V}_+ \cap (\text{Span}(V, J_Y V, J_Z V, J_{KZ} V))^\perp \). It follows that \( p \) is \( J_Z \)-invariant, for any \( Z \in j_{-1} \) (and is \( J_Y \)-invariant, by definition), and that \( d_p = \frac{1}{2} d_3 - 4 \), which by (19) gives \[ 2d_3 - \frac{1}{2} d_3 - 4 = d_{-1} \geq 2. \]

From Remark 2 we find that this inequality is only possible in the following cases: \((d_3, d_6) = (5, 8), (6, 8), (7, 8), (7, 16), (8, 16)\). We can exclude the case \((d_3, d_6) = (7, 8)\), as then by Remark 2 the Damek-Ricci space \( M \) is isometric to the Cayley hyperbolic plane.

Furthermore, the case \((d_3, d_6) = (8, 16)\) is also not possible, as the following argument shows. If \((d_3, d_6) = (8, 16)\), the Clifford module \( v \) can be identified with the octonion plane \( \mathbb{O}^2 \), and \( j \), with the algebra of octonions \( \mathbb{O} \). For \( Z \in j = \mathbb{O} \) and \( W = (W_1, W_2) \in v = \mathbb{O}^2 \), we have \( J_Z W = (ZW_2, -Z^* W_1) \). Then the fact that \( d_{-1} > 0 \) implies that there exist two orthonormal, unit octonions \( Z, Z' \) such that \( -ZV_1 = Z'V_2 \) and \( ZV_2 = Z'V_1 \). It follows that \( \|V_1\| = \|V_2\| \neq 0 \) and that \( V_1 \perp V_2 \). From the first equation, \( Z' = -\|V_2\|^{-2} (ZV_1) V_2 = \|V_2\|^{-2} (ZV_2) V_1^* \), as \( V_1 \perp V_2 \). Then the equation is automatically satisfied. Therefore we have \( J_Z J_Y V = \|Y\| J_{||V_2||^2 - z (ZV_2) V_1^*} V \), for all unit imaginary octonions \( Z \) such that the octonion \( ||V_2||^2 - z (ZV_2) V_1^* \) is unit, orthogonal to \( Z \) and imaginary. Then the first two conditions follow from the fact that \( \|V_2\| \perp V_1 \) and \( V_1 \perp V_2 \); the third one gives \( Z \perp V_1 V_2^* \) (note that \( V_1 V_2^* \perp 1 \)). It follows that \( J_Z J_Y V = \|Y\| J_{||V_2||^2 - z (ZV_2) V_1^*} V \) for all \( Z \) in the six-dimensional subspace \( \text{Span}(V_1, V_2) \cup \mathbb{O} \), and so \( d_{-1} = 6 \) which violates the inequality (21).

We therefore have three cases to consider: \((d_3, d_6, d_{-1}, d_p) = (5, 8, 2, 0), (6, 8, 4, 0), (7, 16, 2, 4)\), where the values for \( d_{-1} \) and \( d_p \) are obtained from (19), (21) (note that an argument similar to the above shows that the case \((d_3, d_6, d_{-1}, d_p) = (7, 16, 2, 4)\) is also impossible if the two irreducible Clifford submodules of \( v \) are non-isomorphic, but we will not need this fact in the rest of the proof).

As in all three cases, \( d_{-1} < d_3 - 1 \), the operator \( K^2 \) has at least one eigenvalue other than \(-1\) which completes the proof of assertion (b). It remains to prove assertion (a).

We first consider the first and the third case simultaneously; in both cases, \( d_{-1} = 2 \). Let \( Z \) be a nonzero vector from \( j_{-1} \), so that \( j_{-1} = \text{Span}(Z, KZ) \). Consider the subspace \( h = s_4 \oplus (j_{-1} \oplus v_{-1}) \oplus p \). If \((d_3, d_6, d_{-1}, d_p) = (5, 8, 2, 0)\), we have \( p = 0 \), and so \( h \) is tangent to a
totally geodesic subgroup of $\mathcal{M}$ isometric to $\mathbb{H}H^2$. If $(d_j, d_p, d_-1, d_p)$ is $J_{-1}$- and $J_{1}$-invariant from the above, and so $\mathfrak{p} = \text{Span}(P, J_{-1}P, J_1P, J_{K_2}P)$ for some (any) nonzero $P \in \mathfrak{p}$. It follows from (20) and Remark 3 that $\mathfrak{h}$ is tangent to a totally geodesic subgroup of $\mathcal{M}$ isometric to $\mathbb{H}H^3$.

By (17,18) we have $L_{-1}, L_{-1/4} \subset \mathfrak{h}$ (in fact, $\mathfrak{h} = L_{-1} \oplus L_{-1/4} \oplus \mathbb{R} \xi$). In equation (9), take $X_i \in L_{-1}, X_j, X_k \in L_{-1/4}$. As $\mathfrak{h}$ is tangent to a totally geodesic symmetric space, we have $(\nabla_k \mathcal{R})(X_i, \xi, \xi, X_j) = 0$, and so (7) gives $\lambda_k \mathcal{R}(\xi, X_i)X_j + \mathcal{R}(\xi, X_j)X_i, X_k) = \frac{3}{4} \Gamma^{\xi}_{k j}$. Moreover, by the duality property get $\frac{\mathcal{R}(\xi, X_j, X_k, X_i) + \mathcal{R}(\xi, X_k, X_j, X_i)}{2} = 0$. Then by the first Bianchi identity, $\langle \mathcal{R}(\xi, X_j)X_i + \mathcal{R}(\xi, X_j)X_i, X_k) = -3\mathcal{R}(\xi, X_j, X_k, X_i)$. We obtain

$$\Gamma_{k j} = 4\lambda_k \mathcal{R}(\xi, X_j, X_k, X_i).$$

Interchanging $j$ and $k$ and substituting into (5) we find

$$(22) \quad \left(\frac{1}{2} + (\lambda_j - \lambda_i)\lambda_k + (\lambda_k - \lambda_i)\lambda_j\right) \mathcal{R}(\xi, X_j, X_k, X_i) = 0.$$

The term $\mathcal{R}(\xi, X_j, X_k, X_i)$ can be computed from the curvature tensor of $\mathbb{H}H^3$ (respectively of $\mathbb{H}H^2$). Let $X_{i_1}, X_{i_2}, X_{i_3} \in L_{-1}$ be an orthonormal basis of eigenvectors of $S$. For every $i \in I = \{i_1, i_2, i_3\}$, there is a unique complex structure $J_i$ on $\mathfrak{h}$ such that $J_i^2 = X_i$, and $\text{Span}(J_{i_1}', J_{i_2}', J_{i_3}')$ is the quaternionic structure on $\mathfrak{h}$. The subspace $L_{-1/4}$ is $J_{-1}$-invariant for all $i \in I$; denote $J_i$ the restriction of $J_i$ to $L_{-1/4}$. Then we have $J_{i_1}', J_{i_2}', J_{i_3}' = \pm \text{id}_{L_{-1/4}}$ and $\mathcal{R}(\xi, X_j, X_k, X_i) = -\frac{1}{4} \langle J_{i_1}'X_j, X_k \rangle$.

Let $S'$ be the restriction of $S$ to $L_{-1/4}$ (recall that $L_{-1/4}$ is $S$-invariant by (3)). Then (22) gives $\langle J_iX_j, X_k \rangle + 4\langle J_iS'X_j, S'X_k \rangle - 2\lambda_i(\langle J_iS'X_j, X_k \rangle + \langle J_1X_j, S'X_k \rangle) = 0$, for all $i \in I$. As this is satisfied for any $X_j, X_k \in L_{-1/4}$, we obtain

$$(23) \quad J_i + 4S'J_iS' - 2\lambda_i(\langle J_iS' + S'J_i \rangle = 0,$$

for all $i \in I$.

We will show that all three $\lambda_i$, $i \in I$, are equal (then the claim of assertion (a) follows trivially). Assume they are not all equal. From (4) we have $S'^2 - HS' - (C + \frac{1}{2}) \text{id}_{L_{-1/4}} = 0$, and $\lambda_i^2 - H\lambda_i - (C + 1) = 0$ for $i \in I$. If $S' = \tau \text{id}$ for some $\tau \in \mathbb{R}$, then from (23) we get $4\tau^2 - 4\lambda_i\tau + 1 = 0$, contradicting our assumption. If $S'$ is not a multiple of the identity, but commutes with one of $J_i$, $i \in I$, then from (23) we obtain $\text{id} + 4S'^2 - 4\lambda_iS' = 0$, and so $(2C + 1) \text{id} + 2(H - \lambda_i)S' = 0$. Then $C = -\frac{1}{2}$ and $\lambda_i = H$, which contradicts the fact that $\lambda_i^2 - H\lambda_i - (C + 1) = 0$. So we may assume that no $J_i, i \in I$, commutes with $S$. Multiplying (23) by $S'$ from the left we get $-(H\lambda_i + C)S'J_i + (H\lambda_i + 3C)J_iS' - (2C\lambda_i + H)J_i = 0$. Adding the transposed gives $(H\lambda_i + 2C)(J_iS' - S'J_i) = 0$, and so $H\lambda_i + 2C = 0$, for all $i \in I$. If $H \neq 0$, then all three $\lambda_i$’s are equal. If $H = 0$, then $C = 0$, and so $\lambda_i$ is $\pm 1$ and $S'^2 = \frac{1}{2} \text{id}$, so that the eigenvalues of $S'$ are $\pm \frac{1}{2}$. For each $i \in I$, equation (23) gives $(S' - \frac{1}{2}\lambda_i \text{id})J_i(S' - \frac{1}{2}\lambda_i \text{id}) = 0$, and so the $(-\frac{1}{2}\lambda_i)$-eigenspace of $S'$ must be $J_i$-isotropic. In particular, its dimension is at most $\frac{1}{2} \dim L_{-1/4}$. As not all the $\lambda_i$’s are the same, both $(\pm \frac{1}{2})$-eigenspaces of $S'$ have dimension $\frac{1}{2} \dim L_{-1/4}$. For each $i \in I$, one of these subspaces is $J_i$-isotropic; but as $J_i$ is orthogonal, the other subspace must also be $J_i$-isotropic. It follows that $L_{-1/4}$ splits into the direct sum of two subspaces of dimension $\frac{1}{2} \dim L_{-1/4}$ which are isotropic relative to each of the operators $J_i, i \in I$. Then each of $J_i$ interchanges these subspaces which contradicts the fact that $J_iF, J_{i_2}F, J_{i_3}F = \pm \text{id}$. 


To complete the proof it remains to prove assertion (a) in the case \((d_3, d_6, d_1, d_p) = (6, 8, 4, 0)\). In that case, \(p\) is trivial. Moreover, for any nonzero \(Z \in \mathfrak{z}_{-1}\), the subspace \(\mathfrak{h}(Z) = \mathfrak{a}_4 \oplus (\text{Span}\,(Z, KZ) \oplus \text{Span}(J_{2}Z, J_{KZ}V))\) is tangent to a totally geodesic subgroup of \(\mathcal{M}\) isometric to \(\mathbb{H}H^2\) (note that \(\mathfrak{h}(Z) = \mathfrak{h}(KZ)\)).

In equation (9), take \(X_i \in L_{-1}, X_j, X_k \in L_{-1/4}\). By (17) we have \(X_i = a_i T^0 + (||V||^2 - 1)Z' + J_{1||Y||KZ\sim-Z}V\) for some \(a_i \in \mathbb{R}\) and \(Z' \in \mathfrak{z}_{-1}\) such that \(1 = ||X_i||^2 = a_i^2 + ||Z'||^2(1-||V||^2)\). We may assume that \(Z' \neq 0\) (otherwise \(T^0\) is an eigenvector of \(S\) and we are done). Let a nonzero \(Z'' \in \mathfrak{z}_{-1}\) be such that \(Z'' \perp Z', KZ', KZ''\). Note that \(Z = \text{Span}(Z', KZ', Z'', KZ'')\), as \(d_{-1} = 4\). Let \(X_j = X_j + X''_j, X_k = X_k' + X''_k\), where \(X_j', X_k' \in \mathfrak{h}(Z')\) and \(X''_j, X''_k \perp \mathfrak{h}(Z')\). Note that \(X''_j, X''_k \in \mathfrak{h}(Z'')\) and that \(X_j', X_k', X''_j, X''_k \in L_{-1/4}\). We have \((\nabla_{X'} \mathcal{R})(X_i, \xi, X_j) = (\nabla_{X''} \mathcal{R})(X_i, \xi, X_j) + (\nabla_{X'} \mathcal{R})(X_i, \xi, X'_j) + (\nabla_{X''} \mathcal{R})(X_i, \xi, X''_j).\) But \((\nabla_{X'} \mathcal{R})(X_i, \xi)\xi = 0, as X'_j, X_k \in \mathfrak{h}(Z')\) which is tangent to a totally geodesic symmetric space. A similar argument applied to \(X_i, \xi, X'_j, X_k \in \mathfrak{h}(Z'')\) (the second Bianchi identity) shows that \((\nabla_{X''} \mathcal{R})(X_i, \xi, X_j) = 0\). And then \((\nabla_{X''} \mathcal{R})(X_i, \xi, X_j) = 0, as X''_j, X''_k \in \mathfrak{h}(Z'').\) Now given any \(T \in L_{-1/4}\) we can find \(Z_T \in \mathfrak{z}_{-1}\) such that \(T \in \mathfrak{h}(Z_T)\). By the duality property (see Remark 3), we have \(\mathcal{R}_T \xi = -\frac{1}{4} ||T||^2 \xi). Taking T = X_j + X_k and polarising we obtain \(\mathcal{R}_T(X_i, X_j, X_k) + \mathcal{R}(X_j, X_k, X_i) = 0, and so by the first Bianchi identity, \(\mathcal{R}_T(X_j, X_k, X_i) = \frac{1}{2} \mathcal{R}(X_i, X_j, X_k).\) Therefore \(\Gamma_{ij} = -2\lambda_i \mathcal{R}(X_i, X_j, X_k).\) Interchanging \(j\) and \(k\) and substituting into (5) we obtain

\[
\left(\frac{1}{2} + \lambda_i - \lambda_j\right)\lambda_k (\lambda_k - \lambda_j) \mathcal{R}(X_i, X_j, X_k) = 0.
\]

As above, we denote \(S'\) the restriction of \(S\) to \(L_{-1/4}\) (recall that \(\dim L_{-1/4} = 6\)). The operator \(S'\) is symmetric and satisfies the equation \(S'^2 - HS' - (C + \frac{1}{4}) \text{id}_{L_{-1/4}} = 0\) by (4). For a (unit) eigenvector \(X_i \in L_{-1} - S\) with corresponding eigenvalue \(\lambda_i\) we denote \(J_i\) the skew-symmetric operator on \(L_{-1/4}\) defined by \((J_i T_1, T_2) = 2 \mathcal{R}(X_i, T_1, T_2)\), for \(T_1, T_2 \in L_{-1/4}\). Then (24) takes the form

\[
J_i + 4 S' J_i S' - 2 \lambda_i (J_i S' + S' J_i) = 0
\]

similar to (23). But the structure of the \(J_i\)'s in this case is more complicated as in the previous case. We have \(\mathcal{R}(\xi, X_i, X_j, X_k) = \mathcal{R}(\xi, X_i, X'_j, X''_k) + \mathcal{R}(\xi, X_i, X''_j, X'_k)\) (the other two terms are zeros, as \(\xi, X_i, X'_j, X''_k \in \mathfrak{h}(Z') \perp X''_j, X''_k\). Then, similar to the previous case, \(\mathcal{R}(\xi, X_i, X'_j, X''_k) = \frac{1}{2} (J'_i X'_j, X'_k)\), where \(J'_i\) is the restriction of one of the complex structures (belonging to the quaternionic structure on \(\mathfrak{h}(Z')\) and uniquely defined by the fact that it maps \(\xi\) to \(X_i\) to the 4-dimensional subspace \(L_{-1/4} \cap \mathfrak{h}(Z')\). Note that \(J^i_0 = -\text{id}\) on that subspace. To compute \(\mathcal{R}(\xi, X_i, X'_j, X''_k)\) we decompose \(X_i\) as \(X_i = a_i T^0 + X'_i\), where \(X'_i = (||V||^2 - 1)Z' + J_{1||Y||KZ\sim-Z}V\), as above. Note that \(\mathcal{R}(\xi, X'_i, X''_j, X''_k) = 0, as \(\xi, X''_j, X''_k \in \mathfrak{h}(Z'') \perp X'_i\). So \(\mathcal{R}(\xi, X'_i, X''_j, X''_k) = a_i \mathcal{R}(\xi, T^0, X''_j, X''_k) = \frac{1}{4} a_i (J^0 X''_j, X''_k)\), where \(J^0\) is a skew-symmetric operator on the 2-dimensional subspace \(L_{-1/4} \cap \mathfrak{h}(Z'')\) such that \((J^0)^2 = -\text{id}\) on that space. Thus, relative to the orthogonal decomposition \(L_{-1/4} = (L_{-1/4} \cap \mathfrak{h}(Z')) \oplus (L_{-1/4} \cap \mathfrak{h}(Z''))^{1}\), the matrix of \(J_i\) has the form \(J'_i \oplus (-a_i, 0)\) (but note that this decomposition itself depends on \(X_i\)). We note that \(J_i\) is nonsingular if and only if \(a_i \neq 0\) (and has rank 4 otherwise).
We now analyse equation (25) in several possible cases. Note that if all the eigenvalues \( \lambda_i \)'s of the restriction of \( S \) on \( L_{-1} \) are equal, there is nothing to prove, so we will assume that they are not. Then this restriction has two different eigenvalues which both satisfy the equation \( \lambda_i^2 - H\lambda_i - (C + 1) = 0 \), by (2). The 5-dimensional space \( L_{-1} \) splits into orthogonal sum of corresponding eigenspaces. Note that if for all \( X_i \) in one of these eigenspaces we have \( a_i = 0 \), then the other eigenspace contains \( T^0 \) and we are done. Otherwise, we can choose an orthonormal basis \( \{X_i\} \), \( i \in I = \{i_1, i_2, i_3, i_4, i_5\} \), for \( L_{-1} \) such that \( a_i \neq 0 \) for all \( i \in I \), and so the operators \( J_i \) in (25) are nonsingular.

Similar to the previous case, if \( S' = \tau \) id for some \( \tau \in \mathbb{R} \), then from (25) we get \( 4\tau^2 - 4\lambda_i \tau + 1 = 0 \) which contradicts the assumption that not all \( \lambda_i \), \( i \in I \), are equal. If \( S' \) is not a multiple of the identity, but commutes with one of \( J_i \), \( i \in I \), then (25) gives \( i \)d + 4\( S'^2 - 4\lambda_i S' = 0 \) (as \( J_i \) is nonsingular), and so \( (2C + 1) \)d + 2(\( H - \lambda_i \)S' = 0. Then \( C = -\frac{1}{2} \) and \( \lambda_i = H \), which contradicts the fact that \( \lambda_i^2 - H\lambda_i - (C + 1) = 0 \). We may therefore assume that no \( J_i \), \( i \in I \), commutes with \( S \). Multiplying (25) by \( S' \) from the left and adding the transposed we obtain \( (H\lambda_i + 2C)(J_iS' - S'J_i) = 0 \), and so \( H\lambda_i + 2C = 0 \), for all \( i \in I \). As not all the \( \lambda_i \)'s are equal, we get \( H = C = 0 \), and so \( \lambda_i = \pm 1 \) and \( S'^2 = \frac{1}{2} \)id, so that the eigenvalues of \( S' \) are \( \pm \frac{1}{2} \). Then for each \( i \in I \), equation (25) gives \( (S' - \frac{1}{2}\lambda_i \)id)\( J_i(S' - \frac{1}{2}\lambda_i \)id) = 0, and so the \( \pm \lambda_i \)-eigenspace of \( S' \) must be \( L_i \)-isotropic. Since \( J_i \) is nonsingular, the dimension of that eigenspace is at most \( 3 = \frac{1}{2} \dim L_{-1/4} \). As not all the \( \lambda_i \)'s are the same, both \( (\pm \frac{1}{2}) \)-eigenspaces of \( S' \) have dimension 3. But as the operators \( J_i \) are not orthogonal, we need an argument different from the one above to get a contradiction.

Recall that \( H = \text{Tr} S = \sum_{i=1}^{14} \lambda_i \). Out of 14 eigenvalues \( \lambda_i \) of \( S \), we have five eigenvalues of the restriction of \( S \) to \( L_{-1} \), each of which being \( \pm 1 \), and two eigenvalues \( \pm \frac{1}{2} \), each of multiplicity 3, which are the eigenvalues of \( S' \), the restriction of \( S \) to \( L_{-1/4} \). As \( H = 0 \), the sum of the remaining three eigenvalues (which we label \( \lambda_1, \lambda_2, \lambda_3 \)) must be an odd integer. They are constructed as follows. The skew-symmetric operator \( K \) on the 5-dimensional space \( \mathfrak{g} \cap Y^\perp \) has a 1-dimensional kernel. Let \( Z_0 \) be a unit vector in that kernel; note that its orthogonal complement in \( \mathfrak{g} \cap Y^\perp \) is precisely \( \mathfrak{g} \cap Y \). According to [BTV, Theorem 4.2(vi)], the 3-dimensional subspace \( \text{Span}(Z_0, J_2Z_0, J_3Z_0) \) of \( \mathcal{R}_C \)-invariant, and the restriction of \( \mathcal{R}_C \) to it has three pairwise different eigenvalues \( \alpha_i \), \( i = 1, 2, 3 \), which are the roots of the equation \( f(t) = 0 \), where \( f(t) = t^3 + \frac{3\alpha_2}{5}t^2 + \frac{9\alpha_2}{10}t + q^2 \) and \( q^2 = \frac{1}{16} - \frac{27}{64}||V||^4||Y||^2 \) (it is easy to see that the right-hand side of the latter equation is always positive, and so we can take \( q \in (0, \frac{1}{4}) \)). Up to relabelling, one has \( -1 < \alpha_1 < -\frac{3}{4} < \alpha_2 < -\frac{3}{4} < \alpha_3 < 0 \). As \( C = H = 0 \) in our case, (2) gives \( \lambda_1 = \pm \sqrt{-\alpha_1} \), for \( i = 1, 2, 3 \), and so \( \lambda_1 \leq 0 \), \( \lambda_2 \leq \lambda_3 \), \( \lambda_1 \leq \lambda_2 \leq \lambda_3 \) (up to changing the sign of \( \xi \)). Then from the same inequalities it follows that we must have \( \lambda_1 = \sqrt{-\alpha_1} \) and either \( \lambda_2 = \sqrt{-\alpha_2} \), \( \lambda_3 = -\sqrt{-\alpha_3} \), or \( \lambda_2 = -\sqrt{-\alpha_2} \), \( \lambda_3 = \sqrt{-\alpha_3} \). In the first case, we get \( \sqrt{-\alpha_1} + \sqrt{-\alpha_2} = 1 + \sqrt{-\alpha_3} \). Squaring both sides and using the fact that \( \alpha_1 + \alpha_2 + \alpha_3 = -\frac{3}{2} \) and \( \alpha_2 \alpha_3 = -q^2 \) we obtain \( \alpha_3 = \frac{3}{4} = (\sqrt{-\alpha_3})^{-1/2}(q + \alpha_2) \). The left-hand side is positive, but the right-hand side is negative. To see that, we note that \( f(0) = q^2 \geq 0 \), but \( f(q) = q(-q - \frac{27}{64}(q - \frac{1}{4}) < 0 \) (as \( q \in (0, \frac{1}{4}) \)). So \( \alpha_3 \), the biggest root of \( f \) lies in \( (-q, 0) \).

In the second case, we have \( \sqrt{-\alpha_1} + \sqrt{-\alpha_3} = 1 + \sqrt{-\alpha_2} \), and so by a similar calculation, \( \alpha_2 + \frac{3}{4} = (\sqrt{-\alpha_2})^{-1/2}(q + \alpha_2) \). Squaring both sides we get \( \alpha_2^2 + \frac{3}{2} \alpha_2^2 + (2q + \frac{45}{10})\alpha_2 + q^2 = 0 \). As \( f(\alpha_2) = 0 \), we get \( 2(q - \frac{1}{4})\alpha_2 = 0 \), a contradiction. \( \square \)
With Proposition 2, we can now complete the proof of Theorem 2.

By [BTV, Theorem 4.2(vi)], the eigenspaces of $\overline{R}_\xi$ with corresponding eigenvalues different from $-1$ and $-\frac{1}{4}$ are constructed as follows. For each eigenvalue $\mu \neq -1$ of $K^2$, we consider the corresponding eigenspace $\mathfrak{z}_\mu \subset \mathfrak{g} \cap \mathfrak{y}^\perp$ and the subspace $\mathfrak{v}_\mu = \text{Span}(J_{ZV}, J_{KZV}, J_{ZJYV} | Z \in \mathfrak{z}_\mu)$. Note that $\mathfrak{z}_\mu$ is $K$-invariant (and is of even dimension if $\mu \neq 0$). Each of the (pairwise orthogonal) subspaces $\mathfrak{z}_\mu \oplus \mathfrak{v}_\mu$ splits into the orthogonal sum of three eigenspaces $L_{\alpha_{\mu,l}}$, $l = 1, 2, 3$, of $\overline{R}_\xi$. The eigenvalues $\alpha_{\mu,l}$ satisfy the equation $(\alpha_{\mu,l} + 1)(\alpha_{\mu,l} + \frac{1}{4}) = \frac{27}{64}|Y|^2||Y|^2(1+\mu)$; they are pairwise nonequal and lie in $(-1, 0) \setminus \{-\frac{1}{4}\}$.

For the rest of the proof, we choose and fix a particular eigenvalue $\mu \in (-1, 0]$ of $K^2$ (note that at least one such eigenvalue exists by Proposition 2(b)). To simplify notation, we will drop $\mu$ from the subscripts, so that we will write $\alpha_l$, $l = 1, 2, 3$, for the corresponding $\alpha_{\mu,l}$.

The eigenspaces $L_{\alpha_l}$, $l = 1, 2, 3$, are given by

$$L_{\alpha_l} = \{ \eta_l \nu_l Z + 3\nu_l J_{ZJY} V - 9|V|^2|Y||J_{KZV} - 3s\eta_l J_{ZV} | Z \in \mathfrak{z}_\mu \},$$

where $\eta_l = 4\alpha_l + 1$, $\nu_l = \eta_l + 3|V|^2$.

According to (3), each of the eigenspaces $L_{\alpha_l}$, $l = 1, 2, 3$, is $S$-invariant. For each $l = 1, 2, 3$, there is a linear bijection $\psi_l$ between $\mathfrak{z}_\mu$ and $L_{\alpha_l}$ which send $Z \in \mathfrak{z}_\mu$ to the vector $\psi_l(Z) = \eta_l \nu_l Z + 3\nu_l J_{ZJY} V - 9|V|^2|Y||J_{KZV} - 3s\eta_l J_{ZV}$. Moreover, $\psi_l$ is a homothety as $\|\psi_l(Z)\|^2 = |Z|^2(\eta_l \nu_l)^2 + (3\nu_l)^2|Y|^2|V|^2 - 81|V|^6|Y|^2 + (3s\eta_l)^2|V|^2)^2 - 54\eta_l|V|^2|Y|\langle J_{ZJY}V, J_{KZV} \rangle$.

But $\langle J_{ZJY}V, J_{KZV} \rangle = \langle KZV, J_{ZJY}V \rangle = |Y|^2|V|^2|KZ|^2 = -\mu|Y|^2|V|^2|Z|^2$ by (12). It follows that for each $l = 1, 2, 3$, the operator $S_l$ on $\mathfrak{z}_\mu$ defined by $\psi_l(S_l Z) = S(\psi_l(Z))$ is symmetric. Moreover, we have $-S_2^2 + HS_1 + (C - \alpha_l)I_{\mathfrak{z}_\mu} = 0$ by (4).

By Proposition 2(a), $T^0 \in L_{-1}$ is an eigenvector of $S$. Denote $\lambda_l$ the corresponding eigenvalue. We take $X_k = T^0$ and $X_l = \psi_l(Z_l) \in L_{\alpha_l}$, $X_j = \psi_j(Z_j) \in L_{\alpha_j}$, where $\alpha_l \neq \alpha_j$, in (9). Using [BTV, § 4.1.7] and (11) we compute $\Gamma_{ki}^j$.

Now taking the same $X_k, X_l, X_j$ and the equation obtained from (9) by first interchanging $i$ and $j$ we can similarly compute $\Gamma_{ik}^j$. Substitute the expression for $\Gamma_{ki}^j$ and $\Gamma_{ik}^j$ in (5) and multiply both sides by $\eta_j = (4\alpha_j + 1)$. Adding to the resulting equation the same equation with $i$ and $j$ interchanged we arrive (after some computer assisted, but straightforward calculation) to the equation

$$m_1(KZ_i, Z_j) + m_2(\lambda_i - \lambda_j)(KZ_i, Z_j) + m_3(Z_i, Z_j) + m_4(\lambda_i - \lambda_j)(Z_i, Z_j) = 0,$$

with the coefficients $m_a$, $a = 1, 2, 3, 4$, given by

$$m_1 = 9|V|^2|Y|(\eta_l - \eta_j)(q - \sigma_2),$$

$$m_2 = 36|V|^2|Y|(s - \lambda_1)(\sigma_2(\sigma_1 + 6) - 2q),$$

$$m_3 = 8|V|^2(2\sigma_2^2 + 3\sigma_2\sigma_1) + q(2\sigma_1^2 - 8\sigma_2 - 3|V|^2\sigma_1),$$

$$m_4 = 2(\eta_l - \eta_j)(9|V|^2(s^2 - |Y|^2 - 2s\lambda_1)\sigma_2 - q(12s^2 + 3|V|^2 - 12s\lambda_1 + \sigma_1),$$

where $\sigma_1 = \eta_l + \eta_j$, $\sigma_2 = \eta_l \eta_j$ and $q = 27|V|^4||Y|^2(1 + \mu)$. Note that computationally, it is easier to work with $\eta_l$ than with $\alpha_l$. The numbers $\eta_1, \eta_2, \eta_3$ are three pairwise different roots of the polynomial $p(t) = t^2(t + 3) - q$; the expressions for $m_a$ in (27) are reduced modulo $p$. Note that $\sigma_1 = -3 - \eta_k$, $\sigma_2 = \eta_k(\eta_k + 3)$, where $\{i, j, k\} = \{1, 2, 3\}$.

Equation (26) gives $(m_1 K_{\mu} + m_2(K_{\mu} S_i - S_j K_{\mu}) + m_3 I_{\mathfrak{z}_\mu} + m_4(S_i - S_j))Z_i, Z_j = 0$, where $K_{\mu}$ is the restriction of $K$ to $\mathfrak{z}_\mu$. As both $\psi_i$ and $\psi_j$ are linear bijections, this implies $m_1 K_{\mu} + \ldots = 0$.
$m_2(K_\mu S_i - S_j K_\mu) + m_3 d_{y_\mu} + m_4 (S_i - S_j) = 0$ which is equivalent to $S_j (m_2 K_\mu + m_4 d_{y_\mu}) = m_1 K_\mu + m_2 S_j + m_3 d_{y_\mu} + m_4 S_i$. Multiplying both sides from the left by $m_2 K_\mu - m_4 d_{y_\mu}$, we obtain a symmetric operator on the left-hand side, while on the right-hand side we get $(m_1 m_2 K_\mu - m_3 m_4) d_{y_\mu} + (m_2^2 \mu - m_1^2) S_i + (m_2 m_3 - m_1 m_4) K_\mu$. Suppose that $K_\mu \neq 0$. From (27), $m_2 m_3 - m_1 m_4 = 0$ is a symmetric polynomial in $\eta_i$, $\eta_j$. Using the fact that $\sigma_i = -3 - \eta_k$ and $\sigma_2 = \eta_k(\eta_k + 3)$, where $\{i, j, k\} = \{1, 2, 3\}$, and reducing modulo $p(\eta_k) = 0$ we obtain $0 = m_2 m_3 - m_1 m_4 = A_2 \eta_k^2 + A_1 \eta_k + A_0$, where the coefficients $A_2, A_1, A_0$ depend only on $\parallel V \parallel, \parallel Y \parallel, s, \mu$ and $A_2 = q(1 - 3 \parallel V \parallel^2) + 9(1 + 5 \parallel V \parallel^2)(1 - \parallel V \parallel^2)$. As this is satisfied for all $k = 1, 2, 3$ and as $\eta_1, \eta_2, \eta_3$ are pairwise different roots of the equation $\Phi(\eta_k) = 0$, it remains to consider the case $K_\mu = 0$. Then also $\mu = 0$. Using (27) and the fact that $\sigma_1 = -3 - \eta_k$, $\sigma_2 = \eta_k(\eta_k + 3)$ we can write equation (26) in the following form (after multiplying by $\eta_k$ and reducing modulo $p(\eta_k) = 0$):

$$
\Phi(\eta_k) d_{y_\mu} + (\eta_i - \eta_j) \Psi(\eta_k) (S_i - S_j) = 0,
$$

where

$$
\Phi(t) = 3s((3 \parallel V \parallel^2 + 2)t^2 + 6(\parallel V \parallel^2 + 1)t - (9 \parallel V \parallel^2 + 2q))
$$

$$
\Psi(t) = 2t^2 + 6((\parallel V \parallel^2 - 3s^2 + 4s\lambda_1)t + 18\parallel V \parallel^2(s^2 - 2s\lambda_1 - \parallel V \parallel^2).
$$

From (28) we obtain that the cyclic sum of the expressions $(\eta_i - \eta_k)(\eta_k - \eta_j)\Psi(\eta_k)\Psi(\eta_j)\Phi(\eta_k)$ by $i, j, k$ is zero. As $\eta_i, \eta_j, \eta_k$ are three pairwise different roots of the equation $\Phi(t) = 0$, we obtain

$$(q - 4)(s(3 \parallel V \parallel^2 - 2)\lambda_1 + 2s(1 - \parallel V \parallel^2)) = 0.
$$

Note that $q - 4 < 0$ (as $s \neq 0$) and also $\parallel V \parallel^2 \neq \frac{2}{3}$ (as otherwise $s = 0$). We find $\lambda_1 = \frac{2s(1 - \parallel V \parallel^2)}{2 - 3\parallel V \parallel^2}$. Furthermore, if $\Phi(\eta_k) = 0$, then $\Phi(\eta_k) = 0$ from (28), but this cannot be satisfied simultaneously with $p(\eta_k) = 0$. It follows that $\Psi(\eta_k) \neq 0$, and so from (28), $S_i$ and $S_j$ commute. Multiplying both sides of (28) by $S_i + S_j - H d_{y_\mu}$ we obtain from (3) $\Phi(\eta_k) (S_i + S_j - H d_{y_\mu} + (\eta_i - \eta_j)(\alpha_j - \alpha_i) \Psi(\eta_k) d_{y_\mu} = 0$. It follows that $\Phi(\eta_k)\Phi(\eta_j) (S_i + S_j - H d_{y_\mu} - \frac{1}{4}(\eta_i - \eta_j)^2 \Phi(\eta_j)\Psi(\eta_j) d_{y_\mu} = 0$. Subtracting the same equation with $j$ and $k$ interchanged we get $\Phi(\eta_k)\Phi(\eta_j) (S_j - S_k - \frac{1}{4}(\eta_i - \eta_j)^2 \Phi(\eta_j)\Psi(\eta_j) d_{y_\mu} = 0$, and so by (28),

$$
4\Phi(\eta_k)\Phi(\eta_j) (\eta_j - \eta_k) \Psi(\eta_i) ((\eta_k - \eta_j)^2 \Phi(\eta_k)\Psi(\eta_j) - (\eta_i - \eta_j)^2 \Phi(\eta_j)\Psi(\eta_k)) = 0.
$$

The left-hand side is symmetric in $\eta_j, \eta_k$. Using the fact that $\eta_j + \eta_k = -3 - \eta_i$ and $\eta_j + \eta_k = \eta_i(\eta_k + 3)$, that $p(\eta_k) = 0$ and substituting the expression for $\lambda_1$ from the above we get $2\parallel V \parallel^4 + (2 - \parallel V \parallel^2)^2 + 8\parallel Y \parallel^2(1 - 3\parallel V \parallel^2) = 0$. But the left-hand side is easily seen to be positive when $\parallel V \parallel^2 + \parallel Y \parallel^2 < 1$, a contradiction.

This completes the proof of Theorem 2.

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