CRITICAL GROUPS OF VAN LINT-SCHRIJVER CYCLOTOMIC STRONGLY REGULAR GRAPHS.

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Abstract. The critical group of a finite connected graph is an abelian group defined by the Smith normal form of its Laplacian. Let \( q \) be a power of a prime and \( H \) be a multiplicative subgroup of \( K = \mathbb{Z}_q \). By \( \text{Cay}(K, H) \) we denote the Cayley graph on the additive group of \( K \) with “connection” set \( H \). A strongly regular graph of the form \( \text{Cay}(K, H) \) is called a cyclotomic strongly regular graph. Let \( \ell > 2 \) and \( p \) be primes such that \( p \) is primitive \(( \text{mod} \; \ell)\). We compute the critical groups of a family of cyclotomic strongly regular graphs for which \( q = p^{\ell-1} \) (with \( t \in \mathbb{N} \)) and \( H \) is the unique multiplicative subgroup of order \( k = \frac{q-1}{\ell} \). These graphs were first discovered by van Lint and Schrijver in [24].

1. Introduction

Let \( \Gamma = (V, E) \) be a finite, simple, and connected graph. Let \( A \) be the adjacency matrix of \( \Gamma \) with respect to some arbitrary but fixed ordering of the vertex set \( V \). Define the matrix \( D \) to be the diagonal matrix of size \( |V| \) whose \( i \)-th diagonal entry is the valency of the \( i \)-th vertex of \( \Gamma \). The matrix \( L := D - A \) is called the Laplacian matrix of \( \Gamma \). By \( \mathbb{Z}^V \) we denote the free \( \mathbb{Z} \)-module with \( V \) as a basis set. By abuse of notation, we may consider \( L \) to be an element of \( \text{End}_\mathbb{Z}(\mathbb{Z}^V) \). The critical group \( C(\Gamma) \) is the finite part of the cokernel of \( L \).

This group is an invariant of \( \Gamma \). By Kirchhoff’s Matrix-tree theorem, it can be deduced that the order of \( C(\Gamma) \) is equal to the number of spanning trees of \( \Gamma \). (For instance, see [24].) The critical groups of various graphs arise in graph theory in the context of the chip firing game (cf. [4]), as the abelian sandpile group in statistical physics (cf. [9]), and also in arithmetic geometry (cf. [16]). Early works with computations of critical groups include [25] and [17]. In [25], the critical groups of Wheel graphs and complete bipartite graphs were computed. In the same paper, it was shown that the group depends only on the cycle matroid of the graph. The critical groups of complete bipartite graphs were computed independently in [17] as well.

Other papers that include computation of critical groups of families of graphs include [15], [8], [2], [14], [10], [6], and [18]. In [15], Lorenzini examined the proportion of graphs with cyclic critical groups among graphs with critical groups of particular order. There are relatively few classes of graphs with known critical groups. A particular class of graphs that has proved amenable to computations is the class of strongly regular graphs (for instance, see Section 3 of [15]). In this paper we describe the critical groups of the cyclotomic strongly regular graphs discovered in [24].

Consider a finite field \( K \) of characteristic \( p \) and a subgroup \( H \) of \( K^* \). By \( \text{Cay}(K, H) \) we denote the Cayley graph on the additive group of \( K \) with “connection” set \( H \). If \( \text{Cay}(K, H) \) is a strongly regular graph, then we speak of a cyclotomic strongly regular graph (cyclotomic SRG). The Paley graph is a well known example of a cyclotomic SRG. This family of SRGs has been studied extensively by many authors; see [24][5][19][12]. We refer the reader to section 4 of [26] for a survey on these graphs. If \( H \) is the multiplicative group of a non-trivial subfield of \( K \), then \( \text{Cay}(K, H) \) is a cyclotomic SRG. A graph of this form is called a subfield cyclotomic SRG. Other examples of cyclotomic SRGs are the semi-primitive cyclotomic SRGs. Consider a subgroup \( H \) of \( K^* \) with \( N := [K^*:H] > 1 \) and \( N \mid \frac{K^*}{p} \). Further assume that there exists an integer \( s \) such that \( p^s \equiv -1 \pmod{N} \). These arithmetic restrictions on \( H \) ensure that the adjacency matrix of the regular graph \( \text{Cay}(K, H) \) has exactly 3 eigenvalues and thus is a cyclotomic SRG (see for example, Section 4 of [26]). A graph of this form is called a semi-primitive cyclotomic SRG. According to a conjecture by Schmidt and White (Conjecture 4.4 of [19]), other than the above mentioned classes, there are only 11 sporadic examples of cyclotomic SRGs. In this paper we consider a class of semi-primitive cyclotomic SRGs discovered in [24].

Consider a pair of primes \( (p, \ell) \) with \( \ell \neq 2 \), and a positive integer \( t \in \mathbb{N} \). The graph \( G(p, \ell, t) \) denotes \( \text{Cay}(K, S) \), where \( K = \mathbb{F}_{p^{\ell-1}t} \) and \( S \) is the subgroup of index \( \ell \) in \( K^* \). Further assume (a) \( p^{\ell-1}t/2 \neq \ell - 1 \) whenever \( t \) is odd; and (b) \( p \) is primitive in \( \mathbb{Z}/2\mathbb{Z} \). The arithmetic constraint (a) is equivalent to the graph being connected, and (b) implies

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that $G(p, \ell, t)$ is a semi-primitive SRG. These semi-primitive cyclotomic SRGs were discovered in [24]. In this paper we describe the critical groups of this family of graphs. The construction of this family is similar to that of Paley and Peisert graphs. The critical group of the Paley graph was computed in [8], and that of Peisert graph was described in [20]. We extend the techniques used in [8] and [20] to compute the critical group of $G(p, \ell, t)$ (with $(p, \ell, t)$ satisfying arithmetic constraints (a) and (b)).  

We denote the critical group of $G(p, \ell, t)$ by $C$. Our results giving the elementary divisors of $C$ are stated in §4. As $C$ is an abelian group, we can find subgroups $C_p$ (the Sylow $p$-subgroup of $C$), $C_p'$ such that $C \cong C_p \oplus C_p'$ and $p \nmid |C_p'|$. We use different approaches to compute these two subgroups. Theorem 3 describes $C_p'$. We apply a standard method of diagonalizing the Laplacian using the character table of $C$. This is the undirected Cayley graph associated with $(p, \ell, t)$. Let $\mathbb{K}$ be the group of transformations $T_{(a,b,n)}$. The action of $G$ on $K$ is shown to be a permutation action of rank 3.

2. Definitions and Notation.

Let $(p, \ell)$ be a pair of primes such that $\ell > 2$ and $p$ is primitive $\pmod{\ell}$. Let $t \in \mathbb{Z}_{\geq 0}$ and $q = p^{\ell(1-t)}$. Moreover we assume that $\sqrt{q} \neq p^{(1-t)/2} \ell - 1$ whenever $t$ is odd. Consider the field $K = \mathbb{F}_q$ and the unique subgroup $S$ of $K^\times$ of order $k := (q-1)/\ell$. Then by $G(p, \ell, t)$ we denote the graph with vertex set $K$ and edge set $\{(x, y) \mid x, y \in K \text{ and } x-y \in S\}$. This is the undirected Cayley graph associated with $(K, S)$. By $A$ we denote the adjacency matrix of $G(p, \ell, t)$ with respect to some fixed but arbitrary ordering of the vertex set $K$. The Laplacian matrix $L$ of $G(p, \ell, t)$ is the matrix $KL - A$.

Given an Integral domain $\mathcal{R}$, by $\mathcal{R}^K$ we denote the $\mathcal{R}$-free module with $\{[x] \mid x \in K\}$ as a basis. Let $\mu_A, \mu_L$ be endomorphisms of $\mathcal{R}^K$ defined by $\mu_A([x]) := \sum_{s \in S} [x + s]$ and $\mu_L([x]) := k[x] - \sum_{s \in S} [x + s]$ respectively. The matrix representation of $\mu_L$ (respectively $\mu_A$) with respect to the basis set $\{[x] \mid x \in K\}$ is the Laplacian matrix $L$ (respectively $A$). The critical group $C$ of $G(p, \ell, t)$ is the finite part of the cokernel of $\mu_L : \mathbb{Z}^K \to \mathbb{Z}^K$. Let $C_p$ be the Sylow $p$-subgroup of $C$. Let $C_p'$ be the largest subgroup of $C$ whose order is not divisible by $p$. As $C$ is abelian, we have $C = C_p \oplus C_p'$.

3. Some properties of $G(p, \ell, t)$.

In section 2 of [24], the authors show that $G(p, \ell, t)$ is a strongly regular graph.

**Theorem** (van Lint-Schrijver). The graph $G(p, \ell, t)$ is a strongly regular graph with parameters

$$
\begin{pmatrix}
q, & \frac{q - 1}{\ell}, & \frac{q - 3\ell + 1 + (-1)^t(\ell - 1)(\ell - 2)\sqrt{q}}{\ell^2}, & \frac{q - \ell + 1 + (-1)^t(\ell - 2)\sqrt{q}}{\ell^2}
\end{pmatrix},
$$

where $q = p^{\ell(1-t)}$. The eigenvalues of the adjacency matrix $A$ of $G(p, \ell, t)$ are $k = \frac{q - 1}{\ell}$, $r_{x_1}$, $r_{x_2}$, with multiplicities 1, $k$, and $q - k - 1$ respectively. Here $r_{x_1} = \frac{-1 + (-1)^t\sqrt{q}}{\ell}$ and $r_{x_2} = r_{x_2} + (-1)^t\sqrt{q}$.

We now give a brief sketch of the proof of the above given in §2 of [24]. We recall from §2 that $K = \mathbb{F}_q$ and $S$ is the unique subgroup of $K^\times$ of size $k$. Given $a \in S$, $b \in K$, $n \in \mathbb{Z}$, define $T_{(a,b,n)} : K \to K$ by $T_{(a,b,n)}(x) := ax^n + b$. Let $G$ be the group of transformations $T_{(a,b,n)}$. The action of $G$ on $K$ is shown to be a permutation action of rank 3.
shown that the orbits of the natural action of $G$ on $K \times K$ are $\{(x, x) \mid x \in K\}$, $\Omega := \{(x, y) \mid x, y \in K \text{ and } x - y \in S \}$ and $\Delta := \{(x, y) \mid x, y \in K \text{ and } x - y \notin S \cup \{0\}\}$. The graph $G(p, \ell, t)$ is the graph with vertex set $K$ and edge set $\Omega$. Standard results on rank 3 permutation groups of even order show that $G(p, \ell, t)$ is a strongly regular graph.

Following notation in [22], we have $\mu_A([x]) = \sum_{s \in S} [x + s]$. Let $\hat{K}$ be the group of complex valued characters of $K$.

Given an additive character $\chi \in \hat{K}$, consider $[\chi] := \sum_{y \in K} \chi(y)[y]$ and $r_\chi = \sum_{s \in S} \chi(s)$. We have $\mu_A([\chi]) = r_\chi[\chi]$. By orthogonality of characters, we may observe that $[\chi][\chi'] \in \hat{K}$ is a basis of $\mathbb{C}^K$ and thus that all eigenvalues of $\mu_A$ are of the form $r_\chi$ for some additive character $\chi$. Consider the character $\chi_1$ defined by $\chi_1(x) := e^{2\pi i \frac{x}{p}}$. It is a well-known result that every character $\chi$ of the form $\chi = \chi_1 \alpha$, where $\alpha(x) = \chi(x(ax))$, for $a \in K$. Let $\alpha$ be a generator of $K^\times$. We observe that for all $s \in S$, we have $r_{\chi_1} = r_\chi$; and for all $b \notin S \cup \{0\}$, we have $r_{\chi_b} = r_{\chi_a}$. Thus the adjacency matrix has (at most) three eigenvalues $k = r_{\chi_0}, r_{\chi_1}, r_{\chi_a}$, with geometric multiplicities 1, $|S| = k$, and $q - k - 1 = |S \cup \{0\}|$, respectively. The parameters given in the Theorem above can now be deduced from the general theory of strongly regular graphs.

Now the eigenvalues of the Laplacian $L = kl - A$ are $0, u := k - r_{\chi_1}$, and $v := k - r_{\chi_a}$, with multiplicities 0, $k$, and $q - k - 1$ respectively. We can see that $v = \sqrt{q} \frac{\sqrt{q} + (-1)^{\ell+1}}{\ell}$ and $u = v + (-1)^{\ell} \sqrt{q}$. It is well known that the nullity of the Laplacian matrix of a graph is equal to the number of connected components. Clearly $v \neq 0$, and thus $G(p, \ell, t)$ is connected if and only if either $t$ is even, or $t$ is odd and $\sqrt{q} \neq \ell - 1$. We will assume throughout that $p^{(\ell-1)/2} \neq \ell - 1$ whenever $t$ is odd.

Given an element $a$ in an unramified extension of $\mathbb{Q}_p$, the $p$-adic valuation of $a$ is denoted by $v_p(a)$. Let $v_p(\ell - 1) = d$, then $v_p(a) = \frac{1}{2}(\ell - 1)t + d$ and $v_p(\ell) = \frac{1}{2}(\ell - 1)t$.

By Theorem 8.1.2 of [22], we have

$$L(L - (v + u)I) = vuI + L,$$

where $\mu = \frac{q - \ell + 1 - (-1)^{\ell}(\ell - 2) \sqrt{q}}{\ell^2}$. Observing that $LL = 0$, we see that the minimal polynomial of $L$ is $(x - u)(x - v) = 0$. Therefore $L$ is diagonalizable. As a consequence of Kirchhoff’s Matrix-Tree Theorem (cf. [22]), the order of critical group of $G(p, \ell, t)$ is $\frac{u^k\sqrt{q}^{k-1}}{q}$.

4. MAIN RESULTS

Let $(p, \ell)$ be a pair of primes with $\ell > 2$ and $p$ primitive modulo $\ell$. Given $t \in \mathbb{N}$, let $q = p^{(\ell - 1)/2}$ and $k = q - 1$. Let $C$ denote the critical group of $G(p, \ell, t)$. Let $C_p$ be the Sylow $p$-subgroup of $C$. Let $C_p'$ be the largest subgroup of $C$ whose order is not divisible by $p$. As $C$ is abelian, we have $C = C_p \oplus C_p'$.

The following theorem describes the Sylow $p$-subgroup $C_p$ of the critical group $C$ of $G(p, \ell, t)$.

**Theorem 1.** Consider the graph $G(p, \ell, t)$ with $\sqrt{q} = p^{(\ell-1)/2} \neq \ell - 1$ whenever $t$ is odd. Let $d$ denote $v_p(\ell - 1)$. Given integers $a, b$ not divisible by $q - 1$, let $c(a, b)$ denote the number of carries when adding the $p$-adic expansions of $a$ and $b$ (mod $q - 1$). Let $L$ be the Laplacian matrix and let $C$ be the critical group of $G(p, \ell, t)$. For $1 \leq i \leq k - 1$, let

$$\min(i) = \min \{(c(i + m, nk) \mid 0 \leq m \leq \ell - 1 \text{ and } 0 < n < \ell - 1)\}.$$  

Given a non-zero positive integer $j$, let $e_j$ be the multiplicity of $p^j$ as a $p$-elementary divisor of $C$. By $e_0$ we denote the $p$-rank of the Laplacian $L$ of $G(p, \ell, t)$. Then the following are true.

1. $e_0 = \left\{\left\lfloor \frac{i}{\ell} \right\rfloor \leq i \leq k - 1 \text{ and } \min(i) = 0 \right\} + 2$ and $e_{(\ell-1)/2} = \left\lfloor \frac{i}{\ell} \right\rfloor \min(i) = 0\right\}$.
2. $e_j = \left\lfloor \frac{i}{\ell} \right\rfloor \leq i \leq k - 1 \text{ and } \min(i) = j\right\}$ for $0 < j < \frac{(\ell-1)/2}{2}$.
3. $e_j = e_{(\ell-1)/2-j} < 0 < j < \frac{(\ell-1)/2}{2}$
4. **(4) If $p | \ell - 1$, then** $e_{\ell/2} = q + 1 - 2 \sum_{j=1}^{\ell/2} e_j$.
5. **(5) If $p | \ell - 1$, then**
   - **(a) $e_{(\ell-1)/2} = k + 2 - \sum_{j=1}^{\ell/2} e_j$ and**
   - **(b) $e_{\ell/2} = (\ell - 1)k - \sum_{j=1}^{\ell/2} e_j$**.
6. $e_j = 0$ for all other $j$. 

3
We prove the above Theorem in §7.
In the case of $G(p, 3, t)$, application of the transfer matrix method (cf. Section 4.7 of [21]) leads us to a recursive algorithm that outputs closed form expressions for multiplicities of $p$-elementary divisors of $C$. As a consequence, we also determine a closed form expression for the $p$-rank (i.e. $e_0$ in the context of the Theorem above) of the Laplacian. The following theorem gives a quick recursive algorithm to compute $p$-elementary divisors. The proof of the following result is in §8.

Let $P = \left(\left(\frac{p+1}{2}\right)^2 x^2y^2 + x^2 + xy + x + y + 1 + \left(\frac{p-2}{2}\right)^2 3xy\right), R = p^3 x^3 y^3$ and

\[ Q = \left(\left(\frac{p+1}{2}\right)^2 xy(3x^2y^2 + x^2 + xy + x + y + 1 + \left(\frac{p-1}{2}\right)^2 3x^2y^2)\right). \]

We define the polynomial $C(2t) \in \mathbb{C}[x, y]$ recursively as follows:

\[ C(2) = 2P, \]
\[ C(4) = 2(P^2 - 2Q), \]
\[ C(6) = 6R + 2(P^3 - 2QP) - 2PQ, \]
\[ \text{and } C(2t) = PC(2t - 2) - QC(2t - 4) + RC(2t - 6) \text{ for } t > 3. \]

**Theorem 2.** Let $C_p$ be the Sylow $p$-subgroup of the critical group of the graph $G(p, 3, t)$ (with $(p, t) \neq (2, 1)$). Given a non-zero positive integer $j$, let $e_j$ be the multiplicity of $p^j$ as a $p$-elementary divisor of $C$. By $e_0$ we denote the $p$-rank of the Laplacian $L$ of $G(p, 3, t)$. Let $e_{(a,b)}$ be the coefficient of $x^a y^b$ in $C(2t)$. Then the following are true. (Here $\delta_{ij}$ is the Kronecker delta function.)

1. $e_0 = e_{2t+1, p} + 2 = \left(\frac{p+1}{2}\right)^{2t+1} (2t+1 - 2).
2. For $a < t$, we have $e_a = e_{2t+1-a} = \sum_{a \leq j \leq t} e_{(a,b)}$.
3. $e_{t+1, p} = (k + 2 - \sum_{j < t} e_j) + (1 - \delta_{2p})(2k - \sum_{j < t} e_j)$.
4. $e_t = (1 - \delta_{2p})(k + 2 - \sum_{j < t} e_j) + (2k - \sum_{j < t} e_j)$.
5. $e_a = 0$ for all other $a$.

Let $X$ be the complex character table of $K$ and $A$ the adjacency matrix of $G(p, \ell, t)$. Then all the entries of $X$ lie in $\mathbb{Z}[\zeta]$ for some primitive $\rho$th root of unity $\zeta$. We have by character orthogonality $\frac{1}{q} XAX^T = I$ and

\[ \frac{1}{q} XAX^T = \text{diag}(r_\psi \phi), \] (4.2)

where $\psi$ runs over additive characters of $K$ and $r_\psi$ is as defined in §3. Note that $r_\phi$ is an eigenvalue of $A$. We note that every prime $m \neq p$ is unramified in $\mathbb{Q}[\zeta]$. Let $m$ be a prime lying over $p$, then the relation (4.2) shows similarity of matrices over the local PID $(\mathbb{Z}[\zeta])_m$. We can now conclude that $L = kI - A$ is similar to $\text{diag}(0, u \ldots u, v \ldots v)$, where $u$ is $\ell$ times $q-k-1$ times over $(\mathbb{Z}[\zeta])_m$, for all primes $m \neq p$. This similarity implies $(\mathbb{Z}[\zeta])_m$-equivalence of matrices. We have now proved the following result.

**Theorem 3.** Consider the graph $G(p, \ell, t)$ with $p^{\ell-1/2} \neq \ell - 1$. Let $C_{p'}$ be the largest subgroup of $C$ whose order is not divisible by $p$. Then $C_{p'} \cong \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^k \times \left(\frac{\mathbb{Z}}{p\mathbb{Z}}\right)^{q-k-1}$ that is coprime to $p$, and $u'$ is the biggest divisor of $u = v + (-1)^{\ell-1} \sqrt{q}$ that is coprime to $p$.

**Example 1.** Implementing the Recursion in (4.1) in a computer algebra system such as Sage, we can compute $C(8)$. Now application of Theorems 2 and 3 yield the critical groups of the family of graphs $(G(p, 3, 4))_p$, with $p$ running over primes primitive (mod 3).

The 2-part of the critical group of $G(2, 3, 4)$ is $\prod_{i=1}^{9} \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^{e_i}$, where $[e_i]_p^0 = [32, 8, 16, 84, 1, 16, 8, 32, 28]$. The 2-complement of the critical group of $G(2, 3, 4)$ is $\mathbb{Z}/15\mathbb{Z}$.

The Sylow $p$-subgroup of the critical group of $G(p, 3, 4)$ (with $p \neq 2$) $\prod_{i=1}^{8} \left(\frac{\mathbb{Z}}{p^i\mathbb{Z}}\right)^{e_i(p)}$, where
\begin{enumerate}
\item \(e_8(p) = 510 \left(\frac{p + 1}{3}\right)^8 - 2\),
\item \(e_1(p) = e_2(p) = 256/6561p^6 + 1040/6561p^7 + 1120/6561p^5 - 784/6561p^3 - 2240/6561p^3 - 784/6561p^3 + 1120/6561p^2 + 1040/6561p + 256/6561\),
\item \(e_2(p) = e_3(p) = 776/6561p^6 + 592/6561p^7 - 2248/6561p^5 - 1904/6561p^4 - 1904/6561p^3 - 2248/6561p^2 + 592/6561p + 776/6561\),
\item \(e_3(p) = e_5(p) = 304/2187p^6 - 448/2187p^7 - 128/2187p^5 + 608/2187p^5 - 32/2187p^4 + 608/2187p^4 - 128/2187p^3 - 448/2187p + 304/2187\),
\item \(e_4(p) = 871/2187p^6 - 352/2187p^5 + 448/2187p^6 - 544/2187p^5 - 56/2187p^4 + 448/2187p^4 - 352/2187p + 871/2187\).
\end{enumerate}

The \(p\)-complement of the critical group of \(G(p, 3, 4)\) (with \(p \neq 2\)) is \(\mathbb{Z}/u'v'\mathbb{Z}\), where \(u' = \frac{p^3 - 1}{3}\) and \(v' = \frac{p^3 + 2}{3}\).

**Remark.** For a fixed \(t\), Theorem 1 implies that the multiplicities of the \(p\)-elementary divisors of the Laplacian of \(G(p, 3, t)\) are polynomial expressions in \(p\) of degree \(2t\). We were however unable to extend the techniques in \(\S 8\) to prove similar results in the general case.

## 5. Smith normal form

Let \(\mathcal{R}\) be a Principal Ideal Domain, \(p \in \mathcal{R}\) a prime, and \(Z : \mathcal{R}^n \to \mathcal{R}^m\) be a linear transformation. By the structure theorem for finitely generated modules over PIDs, we have \([\alpha_i]_{i \in \mathbb{N}} \subset \mathcal{R} \setminus \{0\}\) such that \(\alpha_i | \alpha_{i+1}\) and

\[
coker(Z) \cong \mathcal{R}^{n-r} \oplus \bigoplus_{i=1}^{s} \mathcal{R}/\alpha_i \mathcal{R}.
\]

Let \([Z]\) denote the matrix representation of \(Z\) with respect to the standard basis. Then the above equation tells us that we can find \(P \in \text{GL}_n(\mathcal{R})\), and \(Q \in \text{GL}_m(\mathcal{R})\) such that

\[
P[Z]Q = \begin{bmatrix}
Y \\
0_{(m-r) \times (m-1)} \\
0_{(n-r) \times (m-1)}
\end{bmatrix},
\]

where \(Y = \text{diag}(\alpha_1, \ldots, \alpha_s)\). The diagonal form \(P[Z]Q\) is called the Smith normal form of \(Z\). Its uniqueness (up to multiplication of \(\alpha_i\) by units) is also guaranteed by the aforementioned structure theorem. By invariant factors (elementary divisors) of \(Z\), we mean the invariant factors (respectively elementary divisors) of the module \(\text{coker}(Z)\).

The following is a well known result (for eg. see Theorem 2.4 of [22]) that gives a description of the Smith normal form in terms of minor determinants.

**Lemma 4.** Let \(Z\), \([Z]\), and \([\alpha_i]_{1 \leq i \leq s}\) be as described above. Given \(1 \leq i \leq s\), let \(d_i(Z)\) be the GCD of all \(i \times i\) minor determinants of \([Z]\), and let \(d_0(Z) = 1\). We then have \(\alpha_i = d_i([Z])/d_{i-1}([Z])\).

Define \(e_i(Z) = ||\alpha_i| v_i(\alpha_i) = j||\). Now \(e_j(Z)\) is the multiplicity of \(p^j\) as \(p\)-elementary divisors of the \(\mathcal{R}\)-module \(\text{coker}(Z)\). If \(R = \mathbb{Z}\), then \(e_j(Z)\) is the multiplicity of \(p^j\) as an elementary divisor of the abelian group \(\text{coker}(Z)\).

Let \(\mathcal{R}_p\) be the \(p\)-adic completion of \(\mathcal{R}\). We have

\[
\mathcal{R}_p^n/T(\mathcal{R}_p^m) \cong \mathcal{R}_p^{n-r} \oplus \bigoplus_{j>0} (\mathcal{R}_p/p\mathcal{R}_p)^{e_j(0)}.
\]

Define \(M_j(Z) := \{x \in \mathcal{R}_p^n : Z(x) \in \mathcal{R}^{p^j}\}\). We have \(\mathcal{R}_p^m = M_0(Z) \supset M_1(Z) \supset \cdots \supset M_n(Z) \supset \cdots\).

Let \(\mathbb{F} = \mathcal{R}_p/p\mathcal{R}_p\). If \(M \subset \mathcal{R}_p^m\) is a submodule, define \(\overline{M} = (M + p\mathcal{R}_p^m)/p\mathcal{R}_p^m\). Then \(\overline{M}\) is an \(\mathbb{F}\)-vector space. The following Lemma follows from the structure theorem.

**Lemma 5.** \(e_j(Z) := \dim(M_j(Z)/M_{j+1}(Z))\).

So we have,

\[
\dim(M_j(Z)) - \dim(\ker(Z)) = \sum_{i < j} e_i(Z). \tag{5.1}
\]

The following is Lemma 3.1 of [11]. We include a short proof for the convenience of the reader.
Lemma 6. Let $Z$, $v$, $M_i(Z)$, and $e_i(Z)$ be as defined above. Let $\kappa(Z)$ be the $v$-adic valuation of the product of a complete set of non-zero invariant factors of $Z$, counted with multiplicities. Suppose that we have two sequences of integers $0 < t_1 < t_2 \ldots < t_j$ and $s_1 > s_2 \ldots > s_j > s_{j+1} = \dim(\ker(Z))$ satisfying the following conditions.

1. $\dim(M_i(Z)) \geq s_i$ for $1 \leq i \leq j$
2. $\kappa(Z) = \sum_{i=1}^{j} (s_i - s_{i+1}) t_i$

Then the following hold.

(a) $e_0(Z) = m - s_1$.
(b) $e_j(Z) = s_1 - s_{i+1}$.
(c) $e_a(Z) = 0$ for $a \notin \{t_1 \ldots t_i, \ldots, t_j\}$.

Proof. We have

$$\kappa(Z) = \sum_{i \geq 1} ie_i(Z) \geq \sum_{k=1}^{j-1} \left( \sum_{i \leq t_i} \sum_{i \leq j} ie_i(Z) \right) + \sum_{i \geq t_j} ie_i(Z) \geq \sum_{k=1}^{j-1} \left( t_k \sum_{i \leq j} ie_i(Z) \right) + t_j \sum_{i \geq t_j} e_i(Z).$$

(5.2)

Application of equation (5.1) given above yields

$$\sum_{k=1}^{j-1} t_k \sum_{i \leq j} e_i(Z) + t_j \sum_{i \geq t_j} e_i(Z) = \sum_{k=1}^{j-1} \left( t_k (\dim(M_k(Z)) - \dim(M_{i+1}(Z))) \right) + t_j \left( \dim(M_j(Z)) - \dim(\ker(Z)) \right).$$

Now application of conditions (1) and (2) in the statement gives us

$$\sum_{k=1}^{j-1} t_k (\dim(M_k(Z)) - \dim(M_{i+1}(Z))) + t_j \left( \dim(M_j(Z)) - \dim(\ker(Z)) \right) \geq \sum_{i=1}^{j} (s_i - s_{i+1}) t_i = \kappa(Z).$$

(5.3)

So the inequalities (5.2) and (5.3) are in fact equations and thus the lemma follows. □

The following result is 12.8.4 of [7].

Lemma 7. Let $Z : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and $\phi \in \mathbb{R}$ be an eigenvalue for $Z$, with geometric multiplicity $c$. Then $\dim(M_{\phi}(Z)) \geq c$.

Proof. Let $Fr(R)$ be the field of fractions of $\mathbb{R}_p$. We can extend $Z$ to a unique element of $\text{End}_{Fr(R)}(Fr(R)^n)$. For convenience, let us denote this element by $Z$ as well. Consider the eigenspace $V_\phi = \{ x \in Fr(R)^n | Z(x) = \phi x \}$. Then $V_\phi \cap \mathbb{R}_p^n$ is a pure $\mathbb{R}_p$-submodule ($\mathbb{R}_p$-direct summand of $\mathbb{R}_p^n$ of rank $c = \dim(V_\phi)$. It is clear that $V_\phi \cap \mathbb{R}_p^n \subset M_d(Z)$. As $V_\phi \cap \mathbb{R}_p^n$ is pure, we have $V_\phi \cap \mathbb{R}_p^n \subset M_{\phi}(Z)$. □

6. Character sums and block diagonal form of $L$.

We recall the definition of the graph $G(p, \ell, t)$, its adjacency matrix $A$ and Laplaceian matrix $L$. This is the graph with vertex set $K = \mathbb{F}_q$ (with $q = p^{(\ell-1)}$) and edge set $\{(x, y) | x, y \in K$ and $x - y \in S\}$, where $S$ is the subgroup of $K^\times$ with index $\ell$. By $C$, we denote the critical group of $G(p, \ell, t)$. We saw in [3] that $L$ has eigenvalues $0, \nu = \sqrt{q + (-1)^{\ell+1}}$ and $u = v + (-1)^{\ell} \sqrt{q}$, with multiplicities $1, q - k - 1$ and $k$ respectively.

Let $\xi$ be a primitive $(q - 1)$-st root of unity in the algebraic closure of $\mathbb{Q}_p$. Then $\mathbb{Q}_p(\xi)$ is the unique unramified extension of degree $(\ell - 1) t$ over $\mathbb{Q}_p$. Let $R$ be the ring of integers in $\mathbb{Q}_p(\xi)$, then $pR$ is maximal in $R$. Now the regular module $R^\times$ decomposes further into a direct sum of $R^\times$-invariant submodules of rank 1, according to the characters $T^i$, $i = 0, \ldots, q - 2$. The component $T^i$ is spanned by $f_i := \sum_{x \in K^x} T^i(x^{-1})[x]$. Therefore $\{1, f_1, \ldots, f_{q-2}, [0]\}$ is a basis for $R^\times$, where $1 := f_0 + [0] = \sum_{x \in K} [x]$. 

6
Given an $R$-free $RS$-module $M$ and a character $\chi: S \to R^*$, the isotypic component of $M$ corresponding to $\chi$ is the $RS$-submodule $M_\chi := \{m \in M \mid sm = \chi(s)m \text{ for all } s \in S\}$. For $0 < j < k - 1$, let $N_j$ denote the $R$-submodule of $R^K$ with $\{f_{j+m_0} \mid 0 \leq m \leq \ell - 1\}$ as a basis set. Define $N_0$ to be the $R$-submodule of $R^K$ with $\{1, [0], f_s, \ldots, f_{(\ell-1)k}\}$ as a basis set. Then $N_i$ is the isotypic component for the character $T|_{IS}$ of the group $S$. We now have

$$R^K = N_0 \oplus N_1 \ldots \oplus N_{k-1}. \quad (6.1)$$

Since $S$ is a group of automorphisms for $G(p, \ell, t)$, the $R$-linear maps $\mu_A$ and $\mu_L$ are in fact $RS$-module endomorphisms. It follows that $\mu_A$ and $\mu_L$ preserve the decomposition (6.1). For $0 < i < k - 1$, let $L_i$ denote the matrix of $\mu_{L|_{N_i}}$ with respect to the ordered basis $(f_{j+m_0}) (0 \leq m \leq \ell - 1)$. Let $L_0$ be the matrix of $\mu_{L|_{N_0}}$ with respect to the ordered basis $(1, [0], f_s, \ldots, f_{(\ell-1)k})$. So with respect to the ordered basis $(1, [0], f_s, \ldots, f_{(\ell-1)k})$, the matrix representation of the $R$-linear map $\mu_L$ is $\text{diag}(L_0, L_1, \ldots, L_{k-1})$. We proved the following Lemma.

**Lemma 8.** As $R$-matrices, $L$ is similar to the block diagonal matrix $\text{diag}(L_0, L_1, \ldots, L_{k-1})$.

Following conventions in [1], we extend the $T^n$s to $K$. As per this convention, the character $T^0$ maps every element of $K$ to 1, while $T^{n-1}$ maps 0 to 0. All other characters map 0 to 0. For two integers $a, b$ the Jacobi sum $J(T^a, T^b)$ is $\sum_{x \in K} T^a(x)T^b(1-x)$. We refer the reader to Chapter 2 of [3] for formal properties of Jacobi sums. Following the conventions established, for $a \not= 0 \pmod{q-1}$, we have $J(T^a, T^0) = 0$ and $J(T^a, T^{n-1}) = -1$.

The following Lemma describes action of $L_i$ on $N_i$.

**Lemma 9.** (1) If $k \nmid i$, we have $\mu_{L_i}(f_i) = \frac{1}{\ell} \left( qf_i - \sum_{m=1}^{\ell-1} J(T^{-i}, T^{-mk})f_{i+m} \right)$.

(2) For $1 \leq j \leq \ell - 1$, we have $\mu_{L_i}(f_jk) = \frac{1}{\ell} \left( 1 + qf_jk - \sum_{m \neq j, 0} J(T^{-jk}, T^{-mk})f_{jk+m} - q[0] \right)$.

(3) $\mu_{L_i}([0]) = \frac{1}{\ell} \left( q[0] - \sum_{m=1}^{\ell-1} f_{mk} - 1 \right)$.

(4) $\mu_{L_i}(1) = 0$.

**Proof.** For $x \in K$, we have $\mu_A[x] = \sum_{y \in S} [x + y]$.

Let $\delta_S$ denote the characteristic function of $S$, treated as a subset of $K$. We now have $\mu_A[x] = \sum_{z \in K} \delta_S(z-x)[z]$. Writing $\delta_S$ as a linear combination of characters of $S$, we have $\delta_S = \frac{1}{\ell} \left( \sum_{m=0}^{\ell-1} T^{mk} - \delta_0 \right)$. Here $\delta_0$ is 1 at 0 and 0 elsewhere.

We have

$$\ell \mu_A(f_i) = \mu_A \left( \sum_{x \in K^x} T^{-i}(x) [x] \right)$$

$$= \ell \sum_{x \in K^x} T^{-i}(x) \sum_{z \in K} \delta_S(z-x)[z]$$

$$= \sum_{x \in K^x} T^{-i}(x) \sum_{z \in K} T^0(z-x)[z] + \sum_{m=1}^{\ell-1} \sum_{x \in K^x} T^{-i}(x) T^{mk}(z-x)[z] - \sum_{x \in K^x} T^{-i}(x) \sum_{z \in K} \delta_0(z-x)[z]. \quad (6.2)$$

From definition of $f_i$ and $\delta_0$, we have $\sum_{x \in K^x} T^{-i}(x) \sum_{z \in K} \delta_0(z-x)[z] = f_i$.

We recall from character theory that for a character $\chi$ of $K^x$,

$$\sum_{x \in K^x} \chi(x) = \begin{cases} q-1 & \text{if } \chi \text{ is trivial, and} \\ \ell & \text{otherwise.} \end{cases} \quad (6.3)$$

Using (6.3), we see that

$$\sum_{x \in K^x} T^{-i}(x) \sum_{z \in K} T^0(z-x)[z] = \left( \sum_{x \in K^x} T^{-i}(x) \right) \left( \sum_{z \in K} [z] \right) = 0 \quad (6.4)$$
We now turn our attention to \[ \sum_{x \in K^r} T^{-i}(x) \sum_{z \in K} T^{mk}(z-x)[z], \] with \( 1 \leq m \leq \ell - 1 \). We have

\[
\sum_{x \in K^r} T^{-i}(x) \sum_{z \in K} T^{mk}(z-x)[z] = \sum_{(x,z) \in K^r \times K^r} T^{-i}(x)T^{mk}(z-x)[z] + \sum_{x \in K^r} T^{-i}(x)T^{mk}(-x)[0]
\] (6.5)

For \( z \in K^r \), we have \( T^{-i}(x)T^{mk}(z-x) = T^{i}(-z)T^{mk}(z)(1 - (x/z)) \). Now, we have

\[
\sum_{(x,z) \in K^r \times K^r} T^{-i}(x)T^{mk}(z-x)[z] = \sum_{(z,y) \in K^r \times K^r} T^{i-mk}(z-1)T^{-i}(y)T^{mk}(1 - y)[z]
\] = \( \sum_{z \in K^r} T^{-i}(y)T^{mk}(1 - y) \) \( \sum_{y \in K^r} T^{i-mk}(z-1)[z] \) = \( J(T^{-i}, T^{mk})f_{i-mk} \).

Using the above equality along with (6.2), (6.4), (6.5), we have

\[
\ell \mu_l(f_i) = \sum_{m=0}^{\ell-1} J(T^{-i}, T^{mk})f_{i-mk} \] (6.6)

As \( -1 \in S \) and \([K^r : S] = k\), we have \( T^{mk}(-x) = T^{mk}(x) \). Thus the middle sum above is \( \sum_{m=0}^{\ell-1} J(T^{-i}, T^{mk}(x))[0] \). Using this along with (6.3) in (6.6), we conclude that

a) if \( k \not\mid i \), we have \( \ell \mu_l(f_i) = \sum_{m=0}^{\ell-1} J(T^{-i}, T^{mk})f_{i-mk} - f_i \), and;

b) for \( 1 \leq j \leq \ell - 1 \), we have \( \ell \mu_l(f_{jk}) = \sum_{m=1}^{\ell-1} J(T^{-jk}, T^{mk})f_{jk-mk} + (q-1)[0] - f_{jk} \).

Using \( L = Kl - A \) now readily yields (1). From the general theory of Jacobi sums, we have for any character \( \lambda \), \( J(\lambda, \lambda^{-1}) = -\lambda(-1) \). Since \( -1 \in S \), we have \( T^{jk}(-1) = 1 \), and therefore we have \( J(T^{-mk}, T^{mk}) = -1 \). Thus \( \ell \mu_l(f_{jk}) = \sum_{m=1}^{\ell-1} J(T^{-jk}, T^{mk})f_{jk-mk} + (q-1)[0] - f_{jk} = \sum_{m\not\equiv j-0} J(T^{-jk}, T^{mk})f_{jk-mk} + q[0] \). Now the (2) follows by using \( L = Kl - A \).

The proof of the remaining statements is straightforward.

We observed in \S 3 that \( L \) is diagonalizable and thus so are \( L_i \)'s. We recall from \S 3 that the eigenvalues of \( L \) are \( 0, u \) and \( v \), with multiplicities \( 1, k \) and \( (\ell - 1)k \) (same as \( q - k - 1 \)), respectively. Again from \S 3 we know that the nullity of \( L \) is 1. Now since the nullity of \( L_0 \) is 1 (c.f. Lemma 8), all other \( L_i \)'s are invertible. It follows that for \( i \not\equiv 0 \), the characteristic polynomial of \( L_i \) is a polynomial of the form \((x - u)^k(x - v))^k \) with \( a + b = \ell \). By Lemma 9 and diagonalizability of \( L_i \), we have \( q = tr(L_i) = au + bv \). It now follows that \( a = 1 \) and \( b = \ell - 1 \). By similar arguments, we may show that the eigenvalues of \( L_0 \) are \( 0, u \) and \( v \) with geometric multiplicities \( 1, 1 \) and \( \ell - 1 \), respectively. We have proved the following Lemma.

**Lemma 10.**

1. For \( i \not\equiv 0 \), the eigenvalues of \( L_i \) are \( u \) and \( v \) with geometric multiplicities \( 1, 1 \) and \( \ell - 1 \), respectively.
2. The eigenvalues of \( L_0 \) are \( 0, u \) and \( v \) with geometric multiplicities \( 1, 1 \) and \( \ell - 1 \), respectively.

7. The Sylow p-subgroup of the critical group of \( G(p, \ell, t) \)

By Lemma 8 it is clear that finding the elementary divisors of \( R \)-matrices \( L_i \)'s will determine the \( p \)-elementary divisors of the critical group. As \( L \) is a unit in \( R \), the Smith normal form of \( L \) is the same as that of \( \ell L_i \). Lemma 9 shows that the any entry of \( \ell L_i \) is either \( q \) or is a Jacobi sum of the form \( J(T^{-(i+mk)}, T^{mk}) \), where \( 0 \leq m \leq \ell - 1 \) and \( 0 < n \leq \ell - 1 \). In the context of Lemma 4 it is worth investigating the \( p \)-adic valuations of Jacobi sums.

An integer \( n \) not divisible by \( q - 1 \) has, when reduced modulo \( q - 1 \), a unique \( p \)-digit expansion \( a \equiv a_0 + a_1p + \ldots + a_{(l-1)-1}p^{(l-1)-1} \) (mod \( q - 1 \)), where \( 0 \leq a_i \leq p - 1 \). We represent this expansion by the tuple of digits \((a_0, \ldots, a_i, \ldots, a_{(l-1)-1})\). By \( s(a) \) we denote the sum \( \sum a_i \). For example, 1 has the expansion \((1, \ldots, 0, \ldots 0)\) and \( s(1) = 1 \).

Applying Stickelberger's theorem on Gauss Sums [23] and the well know relation between Gauss and Jacobi sums we can deduce the following theorem.
Theorem 11. Let \( q \) be a power of a prime \( p \) and let \( a \) and \( b \) be integers not divisible by \( q - 1 \). If \( a + b \equiv 0 \pmod{q - 1} \), then we have
\[
\nu_p(J(T^{-a}, T^{-b})) = \frac{s(a) + s(b) - s(a + b)}{p - 1}.
\]
In other words, the \( p \)-adic valuation of \( J(T^{-a}, T^{-b}) \) is equal to the number of carries, when adding \( p \)-expansions of \( a \) and \( b \) modulo \( q - 1 \).

Given \( b \in \mathbb{Z} \), by \([b]\) denote the unique positive integer less than \( \ell \) satisfying \( b \equiv [b] \pmod{\ell} \). We can now see that
\[
k = \frac{q - 1}{\ell} = \frac{p^{\ell-1} - 1}{\ell} \times \frac{p^{(\ell-1)i} - 1}{p^{\ell-1} - 1}
= \frac{\ell - 2}{i = 0} \frac{[p i]}{p} \frac{p^{(\ell-1)i} - 1}{p^{\ell-1} - 1} \times \frac{\ell - 1}{p^{(\ell-1)i}}.
\]
Thus in the notation we adopted, the tuple for \( k \) is the tuple in which the string
\[
\left( \frac{[p^{(\ell-1)i}] - 1}{\ell} \ldots, \frac{[p]}{\ell} \ldots, \frac{p - [p]}{\ell} \right)
\]
repeats \( t \) times. As \( p \) is primitive modulo \( \ell \), we have \([p i]0 \leq i \leq \ell - 2 = \{1, 2, \ldots, \ell - 1\} \). We can now conclude that
\[
s(k) = \ell \sum_{i = 0}^{\ell - 2} \frac{[p i]}{\ell} \frac{p^{(\ell-1)i} - 1}{p^{\ell-1} - 1} = \frac{\ell - 2}{\ell} (p - 1).
\]

Now for \( 0 \leq i, j \leq \ell - 1 \), we can find \( r_{i,j} \in \mathbb{Z} \) such that \([p i [p i] = [p^{i+j}] + r_{i,j} \ell \). Given any \( m \in \{1, 2, \ldots, \ell - 1\} \), as \( p \) is primitive \( \pmod{\ell} \), there is a positive integer \( j \) such that \([p i] = m \). We have \( mk = \left( \frac{[p i]}{\ell} \right) \frac{p^{(\ell-1)i}}{p^{\ell-1} - 1} \). Now we have
\[
\left( \frac{[p i]}{\ell} \frac{p^{(\ell-1)i} - 1}{p^{\ell-1} - 1} \right) \times \frac{p^{(\ell-1)i} - 1}{p^{\ell-1} - 1} = \sum_{i = 0}^{\ell - 2} \frac{[p i]}{p^{\ell-1} - 1} \frac{p^{(\ell-1)i} - 1}{p^{\ell-1} - 1} \times \frac{\ell - 1}{p^{(\ell-1)i}}
\]

Thus in the notation we adopted, the tuple for \([p i]k \) is the tuple in which the string
\[
\left( \frac{[p^{(\ell-1)i} - 2]}{\ell} \ldots, \frac{[p^{i+j}]}{\ell} \ldots, \frac{[p i]}{\ell} \ldots, \frac{p - [p]}{\ell} \right)
\]
repeats \( t \) times. So the digits of \( mk = \left( \frac{[p i]}{\ell} \right) \) can be obtained by permuting the digits of \( k \), and thus \( s(mk) = s(k) = \frac{\ell - 1}{\ell} (p - 1) \).

Given \( a, b \) as described in the theorem above, by \( c(a, b) \) we denote \( v_p(J(T^{-a}, T^{-b})) \). Then by Lemma 9 the off-diagonal entries of \( L_i \) (with \( i \neq 0 \)) are \( u_{mk} \ell (i, m) \) for some units \( u_{mk} \) of \( \mathbb{R} \), and the diagonal entries are all \( q/\ell \). Lemma 10 shows that \( L_i \) satisfies \( (x - u)(x - v) = 0 \). We make use of this to arrive at the following lemma.

Lemma 12. Given \( j < \frac{(\ell - 1)i}{2} \) and \( 0 < i \leq k - 1 \), the multiplicity of \( p^i \) as an elementary divisor of \( L_i \) is the same as that of \( p^{s(a, b)} \).
Proof. As $L_i$ satisfies $(x - u)(x - v) = 0$, we have $(L_i)(L_i - (v + u)I) = vuI$. Let $P$ and $Q$ be $R$-matrices such that $PL_iQ$ is the Smith normal form of $L_i$. Now consider $PL_iQQ^{-1}(L_i - (v + u)I)P^{-1} = vuI$. This shows that the multiplicity of $p^{v(u)+1}$ as an elementary divisor of $L_i$ is the same as the multiplicity of $p^j$ as an elementary divisor of $L_i - (v + u)I$. Since $L_i$ and $L_i - (v + u)I$ are congruent modulo $p^{v(u)+j}$ for $0 < j < (\ell - 1)/2$, the multiplicity of $p^j$ as an elementary divisor of $L_i - (v + u)I$ is the same as the multiplicity of $p^j$ as an elementary divisor of $L_i$. \hfill $\Box$

We now compute the Smith normal forms of $L_i$'s.

Lemma 13. \begin{enumerate}[(i)]  
  \item For $0 < i \leq \ell - 1$, the Smith normal form of $L_i$ over $R$ is the diagonal matrix $$\text{diag}(p^{\min(i)}, p^{(\ell-1)/2}, \ldots, p^{(\ell-1)/2}), p^{v(u)-\min(i)})$$ where $v(u)$ is the Smith normal form of $(v(u) - \min(i))$. We have $\text{dim}(L_i) = \text{dim}(\text{Im}(L_i)) = \text{dim}(L_i - (v + u)I)$. As a consequence of Kirchhoff’s Matrix-Tree theorem, we have $\kappa(L) = v_p(\det(L)) = v_p(\ell) + (\ell - 1)\kappa(v_p(v) - v_p(q))$. Lemma 10 implies that for $i \neq 0$, we have $\kappa(L_i) = v_p(\det(L_i)) = v_p(u) + (\ell - 1)v_p(v)$. Application of Lemma 8 gives us $\kappa(L) = \kappa(L) - \sum_{i \neq 0} \kappa(L_i) = v_p(u) + (\ell - 3)v_p(v)$.

  \item The Smith normal form of $L_0$ over $R$ is $$\text{diag}(1, 1, p^{(\ell-1)/2}, p^{v(u) - \min(i)}, 0).$$
\end{enumerate}

Proof. Given an $R$-matrix $X$, by $\kappa(X)$, we denote the $p$-adic valuation of the product of a complete set of non-zero invariant factors of $X$, counted with multiplicities. By the notation in §5, $e_i(X)$ denotes the multiplicity of $p^i$ as an elementary divisor of $X$. Following the notation in §5 we consider the vector spaces $M_i(X)$ with $y \in \mathbb{Z}_p$. We use Lemma 6 to probe our results.

As a consequence of Kirchhoff’s Matrix-Tree theorem, we have $\kappa(L) = v_p(\det(L)) = \frac{\ell^d}{q} \nu(\ell - 1)\kappa(v_p(v) - v_p(q))$. Lemma 10 implies that for $i \neq 0$, we have $\kappa(L_i) = v_p(\det(L_i)) = v_p(u) + (\ell - 1)v_p(v)$. Application of Lemma 8 gives us $\kappa(L) = \kappa(L) - \sum_{i \neq 0} \kappa(L_i) = v_p(u) + (\ell - 3)v_p(v)$.

(i) By Theorem 11 we have $e_i(\ell - 1)/2$. Let $\sigma(\beta_1, \beta_2, \ldots, \beta_\ell)$ be the Smith normal form of $L_i$. Then by Lemma 9 and Lemma 14 it follows that $\min(i) = v_p(\beta_1)$. By definition of $\min(i)$ it follows that $\min(i) = \ell - 1$. We can now conclude that $\min(i) = (\ell - 1)/2$. Now apply Lemma 6 to conclude that $\dim(M_{\min(i)}(L_i)) = \ell - 1$. By Lemma 7 we have $\dim(M_{\min(i)}(L_i)) = 1$ and thus $\dim(M_{\min(i)}(L_i)) = 1$. Lemma 10 tells us that geometric multiplicity of $v$ as an eigenvalue of $L_i$ is $\ell - 1$. Now Lemma 7 implies that $\dim(M_{\ell - 1/2}(L_i)) = \ell - 1$. Applying (5.4), we have $\dim(M_{\ell - 1/2}(L_i)) = \ell - 1$. Therefore by Lemma 6 setting $j = 3$, $s_1 = \ell$, $s_2 = \ell - 1$, $s_3 = 1$, $s_4 = \kappa(L_i) = 0$, $t_1 = \min(i)$, $t_2 = (\ell - 1)/2$, and $t_3 = v_p(u)$, we have $\min(\ell - 1/2) = 1$. Since $\min(i) = (\ell - 1)/2$, we have $e_i(\ell - 1/2)(L_i) = \ell - 1$, $e_i(\ell - 1/2)(L_i) = 1$, and $e_i(\ell - 1/2)(L_i) = 0$ for all other $i$. Thus we have (i).

(ii) By Lemma 9 and Theorem 11 there are units $v_{(i, i)}$ in $R$ such that the matrix $L_0$ is

\[
\begin{pmatrix}
q & \sqrt{v(1)} & \cdots & \sqrt{v(1)} & \sqrt{v(1)} \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\sqrt{v(1)} & \cdots & \cdots & \cdots & \sqrt{v(1)} \\
q & \cdots & \cdots & \cdots & q \\
1 & \cdots & \cdots & 1 & -1
\end{pmatrix}
\]

The determinant of the $2 \times 2$ minor

\[
\begin{pmatrix}
q & -1 \\
1 & -1
\end{pmatrix}
\]

of $\lambda L_0$ is a unit in $R$. Observe that any $3 \times 3$ minor of $\lambda L_0$ has $p$-valuation of at least $v_p(q)$. Now applying Lemma 11 yields that the multiplicity of $p^1 = 0$ as an elementary divisor of $L_0$ is 2, that is $e_0(L_0) = 2$. Now Lemma 8 implies that $\dim(M_0(L_0)) = 2$, and thus we have $\dim(M_0(L_0)) = \ell + 1 - 2 = \ell - 1$. By Lemma 11 and Lemma 7 we have $\dim(M_{v_p(v)}(L_0)) = \ell - 1$. Since $M_1(L_0) \supset M_{v_p(v)}(L_0)$, we have $\dim(M_{v_p(v)}(L_0)) = \ell - 1$. Lemma 2 implies that $\text{Im}(L_0)$ is generated by $1$ and $f_{\ell}(1)$. Therefore $\dim(\text{Im}(L_0)) = 2$.

As $L_0$ is $0$ by (3.1) the restriction of $L$ to $\text{Im}(L)$ satisfies $LL - v + uI = vuI$. As $\text{Im}(L_0) \subset \text{Im}(L)$, we can conclude that $\text{Im}(L_0) \subset M_{v_p(v)}(L_0) \subset M_{v_p(v)}(L_0)$. Indeed we have $\kappa(L_0) = v_p(v)(\ell - 1/2) + v_p(u) (2 - 1) = \frac{(\ell - 1)/2}{2}(\ell - 1/2) + v_p(u) (2 - 1)$. Now application of Lemma 6 yields (ii).
7.1. **Proof of Theorem I** Lemma 8 shows the Laplacian matrix \( L \) is similar over \( R \) to the block diagonal matrix \( \text{diag}(L_0, L_1, L_2 \ldots L_{k-1}) \). Results (1), (2), (3), and (6) now follow by applying Lemma |13|

If \( p | \ell - 1 \), we have \( d = v_p(\ell - 1) = 0 \) and \( v_p(u) = (\ell - 1)u \). From (5.1), we have \( q - 1 = \sum_{j \mid \ell - 1} e_j + e_{\ell - 1} \). Now application of (1) and (3) yields (4).

If \( p | \ell - 1 \), then \( v_p(u) > (\ell - 1)u \). Now by Lemma 8 and Lemma 13, we deduce that \( e_{v_p(u)} = [i | 1 \leq i \leq k - 1 \text{ and } \min(i) = \sum_{j \mid \ell - 1} e_j \). Theorem 1 implies that, for \( 0 < \alpha < t \), we have \( e_\alpha = [i | \min(i) = a] \), and \( e_0 = [i | \min(i) = 0] + 2 \). Thus the solution to this problem will immediately provide us with the p-elementary divisors of the critical group of graphs of the form \( G(p, 3, t) \).

Every integer \( a \) that is not divisible by \( q - 1 \), when reduced modulo \( q - 1 \), has a unique \( p \)-adic expansion \( a \equiv \sum_{m=0}^{2s(a)} a_mp^m \) (mod \( q - 1 \)), where \( 0 \leq a_m \leq p - 1 \). By \( s(a) \), we denote \( \sum a_m \). The \( p \)-adic expression for \( k \) is \( k = \sum_{m=0}^{r(m - 1)} p^{2m} + \left( p^2 - 1 \right) p^{2m+1} \). Thus we have \( s(k) = s(2k) = r(p - 1) \).

We may observe from Theorem 11 that

\[
c(a, b) = \frac{s(a) + s(b) - s(a + b)}{p - 1}
\]

Given \( j \in \mathbb{Z} \), by \( \overline{j} \) denote the unique element of \( \{0, 1, \ldots, q - 2\} \) satisfying \( j \equiv \overline{j} \) (mod \( q - 1 \)). The following holds from (8.1).

**Lemma 14.** Given an integer \( j \neq 0 \) (mod \( q - 1 \)) and \( m = -1, 1 \), the following hold.

1. \( c(j, mk) + c(j + mk, -mk) = 2t \)
2. \( c(j, mk) + c(j + mk, mk) = t + c(j, -mk) \)
3. \( c(j, mk) = c(-j - mk, mk) \)

Let \( j \in \{0, 1, \ldots, q - 2\} \setminus \{k, 2k\} \), define \( g(j) := \{c(j, k), c(j, 2k)\} \). For every \( j \), there is a unique \( \phi(j) \in \{1, 2, \ldots, k - 1\} \) such that \( j - \phi(j) \in \{0, k, 2k\} \). Note that \( \phi^{-1}(i) = [i, i + k, i + 2k] \).

For \( 0 \leq a \leq t \), we define \( Y_a := \{j | g(j) = \{a, b\} \text{ for some } b \text{ such that } a \leq b \leq t \} \) and \( R_a = \{i | 1 \leq i \leq k - 1 \text{ and } \min(i) = a \} \). From Theorem 11 we have i) \( e_a = |R_a| \), for \( 0 < a < t \) and ii) \( e_0 = |R_0| + 2 \).

**Lemma 15.** Given \( Y_a \) and \( \phi \) defined above and \( a < t \), the following are true.

1. If \( \phi(y) \) is the restriction of \( \phi \) to \( Y_a \), then \( \phi(y)(Y_a) = R_a \).
2. Let \( i \in R_a \). If \( j \in \phi^{-1}(i) \) and \( m \in \{1, 2\} \) such that \( c(j, mk) = a \), then
(a) $j + m, k \not\in \phi^{-1}_c(i)$
(b) $\phi_c(i) = \{j\}$ if and only if $t \not\in g(j)$;
(c) and $\phi_c^1(i) = \{j, j - m, k\}$ if and only if $t \in g(j)$.

(3) For $0 \leq a < t$, we have $|R_a| = |Y_a| - \frac{1}{2}||g(j)| = |a, t|| = |Y_a| - ||g(j)| = |a||$

Proof. 1) Let $m \in \{1, 2\}$ and $j \in Y_a$ such that $c(j, mk) = a$ and $c(j, -mk) = b$. Then by Lemma 14, we have 
$\{c(j + mk, mk)\} | m \leq 2$, and $n = 1, 2 \} = \{a, b, t - a + b, t - b - a, 2t - a, 2t - b\}$. Since $a \leq b \leq t$, we have 
$\phi_1(Y_a) \subset \phi_1(R_a)$. If $j \in \phi_1(R_a)$, then there exists $j \in \{i, i + k, i + 2k\}$ and $m \in \{1, 2\}$ such that $c(j, mk) = a$.
Using $a = \min (c(i + mk, mk) | m \leq 2, n = 1, 2)$, and Lemma 14, we have $c(j, mk) \leq c(j, -mk, -mk) = t + c(j, mk) - c(j, -mk)$. Thus we have $c(j, -mk) \leq t$ and therefore $j \in Y_a$ and $\phi_1(j) = i$.

2) From Lemma 14, we have $c(j, mk) + c(j + mk, -mk) = 2t$ and $c(j, mk) + c(j + mk, mk) = t + c(j, -mk)$. As $c(j, mk) = a < t$, Lemma 14 implies $c(j + mk, -mk) = 2t - a > t$. As $j \in Y_a$, we have $c(j, -mk) \geq c(j, mk)$ and thus $c(j + mk, -mk) = t + c(j, -mk) = t + c(j, mk) - c(j, -mk)$. Thus $j + mk \not\in \phi_1(j)$. We have $\phi_1(j) \subset \{j, j - mk\}$.

As $j \in Y_a$, we have that $a = c(j, mk) \leq c(j, -mk) \leq t$. Now, application of Lemma 14 yields $c(j - mk, mk) = 2t - c(j, mk) \geq t$. By Lemma 14, we have $c(j - mk, -mk) = t + c(j, mk)$, and thus $c(j - mk, -mk) = a$ if and only if $c(j, -mk) = t$. Therefore $j - mk \in Y_a$ is and only if $c(j, -mk) = t$. Thus (2) is true.

3) From (1) and (2), we have $|R_a| = \sum_{j \in \phi_1^{-1}(i)} |Y_a| - \frac{1}{2}||g(j)| = |a, t||$. Given $j \in \{g(j) = |a, t||$, let $m \in \{1, 2\}$ such that $c(j, mk) = a$. Then we have $c(j, -mk) = t$. By Lemma 14, we have $c(j, mk) + t = c(j, -mk) + c(-j, mk, -mk)$ and thus $c(-j, mk, -mk) = c(j, mk).$ As $c(j, mk) = a$. Using $c(j, mk) = c(-j, mk, mk)$ and $c(-j, -mk, -mk) = c(j, -mk, -mk)$ from Lemma 14, we have $c(-j, -mk, -mk) = c(j, mk) = c(j, -mk, -mk) = c(-j, -mk, -mk)$. Thus if $j \in \{g(j) = |a, t||$, then $-j - mk \in \{g(j) = |a, t||$. Now the map $\lambda : \{g(j) = |a, t|| \to \{g(j) = |a, t||$ defined by $\lambda(j) = -j - mk$ is well-defined. For $j \in \{g(j) = |a, t||$, we have $\lambda^{-1}(j) \subset \{-j, -j + k, -j + 2k\}$. Given $m \in \{1, 2\}$, Lemma 14 gives us $c(-j, mk) = c(j, -mk, mk) = 2t - c(j, -mk) = 2t - a, c(-j + mk, -mk) = c(j + mk, mk) = t + c(j, -mk) - c(j, mk) = t + a - a = t$, and $c(-j + mk, -mk) = c(j, -mk) = a$. Thus the map $\lambda$ is a 2 to 1 map and therefore $\frac{1}{2}||g(j)| = |a, t|| = ||g(j)| = |a||$. □

**Corollary 16.** $e_0 = \left(\frac{p + 1}{3}\right)^{2t} - 2$.

Proof. Lemma 15 and Theorem 1 imply $e_0 = |R_0| + 2 = |Y_0| - \{g(j) = |0|| + 2$. We recall that $k = \sum_{m=0}^{r-1} \left(\frac{2^{-1}}{2}\right) p^{2m} + \left(\frac{2^{-1}}{2}\right) p^{2m+1}$ and $2k = \sum_{m=0}^{r-1} \left(\frac{2^{-1}}{2}\right) p^{2m} + \left(\frac{2^{-1}}{2}\right) p^{2m+1}$ Therefore set $\{g(j, k), c(j, 2k)\} = (0, b)$ is made up of numbers of the form $j = \sum_{m=0}^{r-1} \{2m \alpha 2m + \alpha_{m+1} p^{2m+1}$ satisfying: i) $0 \leq a_{2m} < \frac{2^{-1}}{2}$, ii) $0 \leq a_{2m+1} < 2^{(r+1)}$, and iii) $j \not\in |k, 2k|$. Thus this set has size $2\left(\frac{p + 1}{3}\right)^{2t} - 2$. Similar arguments yield $||j| \neq 0$ and $c(j, k), c(j, 2k)) = (b, 0)) = 2\left(\frac{p + 1}{3}\right)^{2t} - 2$ and $||j| \neq 0$ and $g(j) = |0||) = \left(\frac{2^{-1}}{2}\right)^{2t} - 1$. The result now follows by the principle of inclusion-exclusion. □

For $0 < a < t$, Lemma 15 shows that $e_a = |R_a|$. We will use the transfer method to compute $|R_a|$. We construct a weighted digraph $D$, and change the problem of computing $e_a$ to that of counting closed walks in $D$ of certain length and weight.

Let $D$ be a digraph with vertex set $V$, edge set $E$, and with a weight function $wt : E \to \mathbb{R}$ with values in some commutative ring $\mathbb{R}$. By $M$, we denote the adjacency matrix of $D$ with respect to the weight $wt$. Given $n \in \mathbb{Z}_{>0}$, let $C(n) = \sum_{i} wt(\delta)$, where the sum is over closed walks in $D$ of length $n$. The following Lemma which is Corollary 4.7.3 of [21] gives us the generating function $\sum_{n \geq 1} C(n)x^n$.

**Lemma 17.** Let $T(z) = det(I - zM)$, then $\sum_{n \geq 1} C(n)x^n = -\frac{T'(z)}{T(z)}$.

Consider $A_1 = \{(\alpha, \gamma, \delta) | (\alpha, \gamma, \delta) \in [0, 1, \ldots, p - 1] \times [1, 2] \times [1, 2]\}$ and $A_2 = \{|(\alpha, \gamma, \delta) | (\alpha, \gamma, \delta) \in [0, 1, \ldots, p - 1] \times [1, 2] \times [1, 2]\}$. We construct a bipartite digraph $D = (A_1 \cup A_2, E)$. There is an arc $e \in E$ from $(\alpha, \gamma, \delta) \in A_1$ to

\[ [\alpha', \gamma', \delta] \in A_2 \] if an only if

\[
\alpha + \frac{2p - 1}{3} + \gamma = \beta + p\gamma' \\
\text{and} \\
\alpha + \frac{p - 2}{3} + \delta = \epsilon + p\delta'
\]

for some \( \beta, \epsilon \in \{0, 1, \ldots, p - 1\} \). There is an arc \( e_1 \in E \) from \([\alpha, \gamma, \delta] \in A_2\) to \((\alpha', \gamma', \delta') \in A_1\) if and only if

\[
\alpha + \frac{p - 2}{3} + \gamma = \beta + p\gamma' \\
\text{and} \\
\alpha + \frac{2p - 1}{3} + \delta = \epsilon + p\delta'
\]

for some \( \beta, \epsilon \in \{0, 1, \ldots, p - 1\} \). The arcs in \( D \) of type \( e_1 \) and \( e_2 \) are assigned label \( \alpha \) and weights \( wt(e) = wt(e_1) = x^\alpha y^\beta \). So we have a weight function \( wt : E \to \mathbb{C}[x, y] \) on \( D \). The weight of a walk on \( D \) will be the products of the weights of its arcs.

Given \( a, b \in \{0, 1, 2, \ldots, 2t + 1\} \), let \( E_{(a, b)} \) be the set of closed walks of length \( 2t \) and weight \( x^\alpha y^\beta \). A closed walk of length \( 2t \) with its initial vertex in \( A_1 \) is said to be of type \( A_1 \), and is of type \( A_2 \) otherwise. Let \( Y_{a,b} = \{ j \in \{1, 2, \ldots, q - 2\} \setminus \{k, 2k\} | g(j) = [a, b]\} \). Let \( a_0, a_1, \ldots, a_{2t} \) be the labels of arcs of a walk \( w \in \cup E_{(a, b)} \), then define \( \psi(w) = \sum a_i p^i \). When \([a, b] \cap \{0, 2t\} = \emptyset \), we have \( \psi(E_{(a, b)}) \subset Y_{a,b} \). By the \( p \)-ary add-with-carry-algorithm described in Theorem 4.1 of \([13]\), given \( j \in Y_{a,b} \), there exist carry sequences \((\gamma_0, \gamma_1, \ldots, 2\gamma_{t-1})\) and \((\delta_0, \delta_1, \ldots, \delta_{2t-1})\) with\( \gamma_i, \delta_i \in \{1, 2\} \) such that

\[
a_i + \frac{2p - 1}{3} + \gamma_i = b_i + \gamma_{i+1}p \\
\text{and} \\
a_i + \frac{p - 2}{3} + \gamma_i = b_i + \gamma_{i+1}p
\]

for even \( i \) and;

\[
a_i + \frac{2p - 1}{3} + \delta_i = d_i + \delta_{i+1}p \\
\text{and} \\
a_i + \frac{p - 2}{3} + \delta_i = d_i + \delta_{i+1}p
\]

for odd \( i \).

Here \( j = \sum a_i p^i \), \( j + k = \sum b_i p^i \) and \( j + 2k = d_i p^i \). We can now see that there are exactly two closed walks, one of each type which map to \( j \) under \( \psi \). If \( w(j, A_1) \) (respectively \( w(j, A_2) \)) is the walk of type \( A_1 \) (respectively type \( A_2 \)) such that \( \psi(w(j, A_1)) = j \) (respectively \( \psi(w(j, A_2)) = j \)), then \( wt(w(j, A_1)) = x^{(j+2k)} \) (respectively \( wt(w(j, A_2)) = x^{(j+2k)} \)). We can now conclude that for \( a \neq b \), the restriction of \( \psi \) is a bijection from \( E_{(a, b)} \) to \( Y_{a,b} \). Applying Lemma \([15, 3]\) gives us

\[
e_a = \sum_{b=a+1}^t |E_{(a, b)}|,
\]

for all \( 0 < a < t \).

We observe that for all \( \alpha, \alpha' \in \{0, 1, \ldots, p - 1\} \) and \( \gamma, \delta \in \{1, 2\} \times \{1, 2\} \), there is no arc from \( [\alpha, \gamma, \delta] \) (resp. \( [\alpha', \gamma, \delta] \)) to \([\alpha', 0, 1]\) (resp. \( [\alpha', 1, 0] \)). We may also conclude that

1. there is an edge from \( (\alpha, \gamma, \delta) \) to \([\alpha', 0, 0]\) if and only if \( 0 \leq \alpha < \frac{p+1}{3} - \gamma \);
2. there is an edge from \( (\alpha, \gamma, \delta) \) to \([\alpha', 1, 0]\) if and only if \( \frac{p+1}{3} - \gamma \leq \alpha < \frac{2(p+1)}{3} - \delta \);
3. there is an edge from \( (\alpha, \gamma, \delta) \) to \([\alpha', 1, 1]\) if and only if \( \frac{2(p+1)}{3} - \delta \leq \alpha < p \);
4. there is an edge from \( (\alpha, \gamma, \delta) \) to \([\alpha', 0, 0]\) if and only if \( 0 \leq \alpha < \frac{p+1}{3} - \delta \);
5. there is an edge from \( (\alpha, \gamma, \delta) \) to \([\alpha', 1, 0]\) if and only if \( \frac{p+1}{3} - \delta \leq \alpha < \frac{2(p+1)}{3} - \gamma \);
6. and there is an edge from \( [\alpha, \gamma, \delta] \) to \([\alpha', 1, 1]\) if and only if \( \frac{2(p+1)}{3} - \gamma \leq \alpha < p \).

Let \( M \) be the adjacency matrix of the weighted digraph \( D \) and let \( U \) be the \( \mathbb{C}(x, y) \) vector space generated by the vertex set \( A_1 \cup A_2 \) (of \( D \)) as a basis. By abuse of notation, we may assume \( M \in \text{End}(U) \).

Let \( h_1 := \sum_{(\gamma, \delta) \alpha' < \frac{p+1}{3} - \gamma} [\alpha', \gamma, \delta], h_2 := \sum_{(\gamma, \delta) \gamma < \frac{p+1}{3} - \gamma} [\alpha', \gamma, \delta], h_3 := \sum_{(\gamma, \delta) \alpha' < \frac{2(p+1)}{3} - \delta} [\alpha', \gamma, \delta], h_4 := \sum_{(\gamma, \delta) \gamma < \frac{2(p+1)}{3} - \delta} [\alpha', \gamma, \delta] \),

\( h'_1 := \sum_{(\gamma, \delta) \gamma < \frac{p+1}{3} - \gamma} (\alpha', \gamma, \delta), h'_2 := \sum_{(\gamma, \delta) \gamma < \frac{2(p+1)}{3} - \delta} (\alpha', \gamma, \delta), \) and \( h'_3 := \sum_{(\gamma, \delta) \gamma < \frac{2(p+1)}{3} - \delta} (\alpha', \gamma, \delta) \).

We can see that \( M(A_1 \cup A_2) = \{h_1, h_2, h_3, h'_1, h'_2, h'_3\} \). We also have,
Let \( W \) be the subspace of \( U \) generated by \( \{ h_i, \ h'_i \mid i = 1, 2, 3 \} \), then \( M(U) = W \). The set \( \beta = \{ h_i, \ h'_i \mid i = 1, 2, 3 \} \) is linearly independent, and thus is a basis for \( W \). Let \( M_{[\beta]} \) be the matrix representation of \( M_{|W} \) with respect to the basis \( \beta \) of \( W \). From above we see that \( M_{[\beta]} \) is

\[
M(h_1) = \begin{bmatrix}
0 & 0 & 0 & p+1 & p+1 & p-2 \\
0 & 0 & 0 & p+1 & p-2 & p+1 \\
0 & 0 & 0 & p-2 & p+1 & p-1 \\
0 & 0 & 0 & p+1 & p-1 & p-1 \\
p+1 & p+1 & p+1 & 0 & 0 & 0 \\
p+1 & p+1 & p+1 & 0 & 0 & 0 \\
\end{bmatrix},
\]

As \( \det(M_{[\beta]}) = -p^2x^3y^3 \neq 0 \), we have \( W \cap \ker(M) = \{ 0 \} \) and thus \( U = \ker(M) \oplus W \).

Thus the characteristic polynomial of \( M \) is \( f(z) = z^{8p-6}\det(zI - M_{[\beta]}) \). Careful computation shows that \( \det(zI - M_{[\beta]}) = z^6 - Pz^4 + Qz^2 - R \),

where \( P = \left( \frac{p+1}{x} \right)^2 (x^2y^2 + x^2y + x + y + 1) + \left( \frac{p-2}{x} \right)^2 3xy \),

\( Q = \left( \frac{p+1}{y} \right)^2 (xy)(x^2y^3 + x^2y + x + y + 1) + \left( \frac{p-2}{y} \right)^2 3x^2y^2 \),

and \( R = p^2x^3y^3 \). Thus \( f(z) = z^{8p} - Pz^{8p-2} + Qz^{8p-4} - Rz^{8p-6} \).

Let \( C(n) = \sum_{\omega} \mathsf{wt}(\omega) \), where the sum is over closed walks in \( D \) of length \( n \). As \( D \) is a bipartite graph, we have \( C(n) = 0 \) for all odd \( n \). By Lemma 17, we have

\[
\sum_{t \geq 1} C(2t)z^{2t} = -\frac{zT'(z)}{T(z)},
\]

where \( T(z) = \det(I - zM) \). The characteristic polynomial of \( M \) was computed above to be \( z^{8p} - Pz^{8p-2} + Qz^{8p-4} - Rz^{8p-6} \), and thus we have

\[
\sum_{t \geq 1} C(2t)z^{2t} = \frac{2Pz^2 - 4Qz^2 + 6Rz^6}{1 - (Pz^2 - Qz^4 + Rz^6)}.
\]

Let \( C(2t) = 0 \) for \( t \leq 0 \). We have \( \sum_{t \geq 1} (C(2t) - PC(2t - 2) + QC(2t - 4) - RC(2t - 6))z^t = 2Pz - 4Qz^2 + 6Rz^3 \). Thus we have

\[
C(2) = 2P \\
C(4) = 2(P^2 - 2Q), \\
C(6) = 6R + 2(P^3 - 2QP) - 2PQ,
\]

and \( C(2t) = PC(2t - 2) - QC(2t - 4) + RC(2t - 6) \) for \( t > 3 \).
The coefficient of $x^ay^b$ in $C(2t)$ is $|E_{(a,b)}|$. Given $a < t$, we have from (8.2) that $e_a = \sum_{a < b \leq t} |E_{(a,b)}|$. Application of Theorem[1] and Corollary[16] yields Theorem[2].

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