Principal subspaces of higher-level standard $\hat{\mathfrak{sl}}(n)$-modules

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Abstract

Using completions of certain universal enveloping algebras, we provide a natural setting for families of defining relations for the principal subspaces of standard modules for untwisted affine Lie algebras. We also use the theory of vertex operator algebras and intertwining operators to construct exact sequences among principal subspaces of certain standard $\hat{\mathfrak{sl}}(n)$-modules, $n \geq 3$. As a consequence, we obtain the multigraded dimensions of the principal subspaces $W(k_1\Lambda_1 + k_2\Lambda_2)$ and $W(k_{n-2}\Lambda_{n-2} + k_{n-1}\Lambda_{n-1})$. This generalizes earlier work by Calinescu on principal subspaces of standard $\hat{\mathfrak{sl}}(3)$-modules.

1 Introduction

Principal subspaces of standard modules for affine Lie algebras were introduced and studied by Feigin and Stoyanovsky in [FS1]–[FS2]. In their work, motivated by the earlier work by Lepowsky and Primc [LP], Feigin and Stoyanovsky discovered that the multigraded dimensions (generating functions of dimensions of homogeneous subspaces) of principal subspaces of standard $\hat{\mathfrak{sl}}(2)$-modules are related to the Rogers-Ramanujan partition identities, and more generally, the Andrews-Gordon identities (cf. [A]). Multigraded dimensions for a more general class of principal subspaces were later studied by Georgiev in [G], where combinatorial bases were constructed for the principal subspaces of certain standard $\hat{\mathfrak{sl}}(n+1)$-modules, $n \geq 1$ (written for brevity as $\hat{\mathfrak{sl}}(n)$ in the title). More recently, combinatorial bases have been constructed for principal subspaces in more general lattice cases ([P], [MP]), for the principal subspaces of the vacuum standard modules for the affine Lie algebra $B_2^{(1)}$ [Bu], for principal subspaces in the quantum $\hat{\mathfrak{sl}}(n+1)$-case [Ko], and for certain substructures of principal subspaces ([Pr], [J1]–[J3], [T1]–[T4], [B2], [JP2]).

In [CLM1–CLM2], Capparelli, Lepowsky, and Milas interpreted the Rogers-Ramanujan and Rogers-Selberg recursions in terms of the multigraded dimensions of the principal subspaces of the standard $\hat{\mathfrak{sl}}(2)$-modules by using the vertex-algebraic structure of these modules, along with intertwining operators among them, to construct exact sequences. In [CLM1–CLM2] (as in [FS1–FS2]), the authors assumed certain presentations (generators and defining relations) for the principal subspaces of the standard $\hat{\mathfrak{sl}}(2)$-modules, presentations that can be derived from [LP]; the nontrivial part is the completeness of the relations. The question of proving in an a priori way that the relations assumed in these works were indeed a complete set of defining relations for the principal subspaces of the standard $\hat{\mathfrak{sl}}(2)$-modules was later addressed by
Calinescu, Lepowsky, and Milas in [CalLM1]–[CalLM2], where the authors gave such an a priori proof. These results were extended to the level 1 standard $\mathfrak{sl}(n+1)$-modules by Calinescu in [C2], and later to the level 1 standard modules for the untwisted affine Lie algebras of type ADE in [CalLM3]. In both [C2] and [CalLM3], the authors proved that the multigraded dimension of the principal subspace of the vacuum module satisfies a certain recursion, and using this recursion they found the multigraded dimensions of the principal subspaces of all the level 1 standard modules.

In the work [C1], Calinescu considered the principal subspaces of certain higher level standard $\mathfrak{sl}(3)$-modules. In this work, she conjecturally assumed presentations for certain principal subspaces, and using the theory of vertex operator algebras and intertwining operators, she constructed exact sequences among these principal subspaces. Using these exact sequences, along with the multigraded dimensions in [G], Calinescu was able to find the multigraded dimensions of principal subspaces which had not previously been studied. In [S] the presentations for the principal subspaces of all the standard $\mathfrak{sl}(3)$-modules were proved (including those assumed in [C1]).

Our main result in the present work is a natural generalization of [C1] to the case of $\mathfrak{sl}(n+1)$, $n \geq 2$. Although our methods recover the same information as in [CLM1]–[CLM2] when $n = 1$, we take $n \geq 2$ for notational convenience. In the case where $n = 2$, we recover the results in [C1] with a slight variant of the methods. Using the work of [G], we provide exact sequences among principal subspaces of certain standard $\mathfrak{sl}(n+1)$-modules. As a consequence, we obtain previously unknown multigraded dimensions of principal subspaces. In addition, as in [C1], we conjecturally assume presentations for certain principal subspaces, and use these to obtain exact sequences among a more general class of principal subspaces of certain standard $\mathfrak{sl}(n+1)$-modules. To state our main result, we let $\Lambda_0, \ldots, \Lambda_n$ denote the fundamental weights of $\mathfrak{sl}(n+1)$. The dominant integral weights $\Lambda$ of $\mathfrak{sl}(n+1)$ are $k_0 \Lambda_0 + \cdots + k_n \Lambda_n$ for $k_0, \ldots, k_n \in \mathbb{N}$, and we use $L(\Lambda)$ to denote the standard module with highest weight $\Lambda$, $W(\Lambda)$ to denote its principal subspace, and $\chi'_{W(\Lambda)}(x_1, \ldots, x_n, q)$ to denote its multigraded dimension. Our result states:

**Theorem 1.1** Let $k \geq 1$. For $k_1, k_2, k_{n-1}, k_n \in \mathbb{N}$ such that $k_1 + k_2 = k_{n-1} + k_n = k$ and $k_1 > 0$ and $k_n > 0$ the sequences

\[
0 \rightarrow W(k_1 \Lambda_1 + k_2 \Lambda_2) \xrightarrow{e^{\otimes k}} W(k_1 \Lambda_0 + k_2 \Lambda_1) \xrightarrow{1^{\otimes k_1-1} \otimes Y_{e^{h_1,x}} \otimes 1^{\otimes k_2}} W((k_1-1)\Lambda_0 + (k_2+1)\Lambda_1) \rightarrow 0
\]

and

\[
0 \rightarrow W(k_{n-1} \Lambda_{n-1} + k_n \Lambda_n) \xrightarrow{e^{\otimes k}} W(k_n \Lambda_0 + k_{n-1} \Lambda_n) \xrightarrow{1^{\otimes k_{n-1}} \otimes Y_{e^{h_n,x}} \otimes 1^{\otimes k_{n+1}}} W((k_n-1)\Lambda_0 + (k_{n-1}+1)\Lambda_n) \rightarrow 0
\]

are exact.
More generally, conjecturally assuming certain presentations, we obtain:

**Theorem 1.2** Let \( k \geq 1 \). For any \( i \) with \( 1 \leq i \leq n-1 \) and \( k_i, k_{i+1} \in \mathbb{N} \) such that \( k_i + k_{i+1} = k \), the sequences

\[
\begin{align*}
W(k_i \Lambda_i + k_{i+1} \Lambda_{i+1}) & \xrightarrow{\phi_i} W(k_i \Lambda_0 + k_{i+1} \Lambda_i) \\
& \xrightarrow{1^{\otimes k_i} - 1^{\otimes k_i+1}} W((k_i - 1) \Lambda_0 + (k_{i+1} + 1) \Lambda_i) \rightarrow 0
\end{align*}
\]

When \( k_i \geq 1 \), and

\[
\begin{align*}
W(k_i \Lambda_i + k_{i+1} \Lambda_{i+1}) & \xrightarrow{\psi_i} W(k_{i+1} \Lambda_0 + k_i \Lambda_{i+1}) \\
& \xrightarrow{1^{\otimes k_{i+1}} - 1^{\otimes k_i}} W((k_{i+1} - 1) \Lambda_0 + (k_i + 1) \Lambda_{i+1}) \rightarrow 0
\end{align*}
\]

when \( k_{i+1} \geq 1 \), are exact.

The maps \( \phi_i, \psi_i, e_{\omega_i}^{\otimes k_i}, e_{\omega_i}^{\otimes k_{i+1}} \), and \( \mathcal{Y}_c(e^{\lambda_i}, x) \) are maps naturally arising from the lattice construction of the level 1 standard modules and intertwining operators among these modules. As a consequence Theorem 1.1 we obtain results about multigraded dimensions, and we have the following theorem and its corollary:

**Theorem 1.3** Let \( k \geq 1 \). Let \( k_1, k_2, k_{n-1}, k_n \in \mathbb{N} \) with \( k_1 \geq 1 \) and \( k_n \geq 1 \), such that \( k_1 + k_2 = k \) and \( k_{n-1} + k_n = k \). Then

\[
\chi'_{W(k_1 \Lambda_1 + k_2 \Lambda_2)}(x_1, \ldots, x_n, q) =
\begin{align*}
&= x_1^{-k_1} \chi'_{W((k_1-1)\Lambda_0 + (k_2+1)\Lambda_1)}(x_1q^{-1}, x_2q, x_3, \ldots, x_n, q) \\
&\quad - x_1^{-k_1} \chi'_{W(k_1 \Lambda_0 + k_2 \Lambda_1)}(x_1q^{-1}, x_2q, x_3, \ldots, x_n, q)
\end{align*}
\]

and

\[
\chi'_{W(k_{n-1} \Lambda_{n-1} + k_n \Lambda_n)}(x_1, \ldots, x_n, q) =
\begin{align*}
&= x_n^{-k_n} \chi'_{W((k_{n-1} - 1)\Lambda_0 + (k_n - 1)\Lambda_n)}(x_1, \ldots, x_{n-1}q, x_nq^{-1}, q) \\
&\quad - x_n^{-k_n} \chi'_{W(k_{n-1} \Lambda_0 + k_n \Lambda_n)}(x_1, \ldots, x_{n-1}q, x_nq^{-1}, q)
\end{align*}
\]

Theorem 1.3 immediately gives us:

**Corollary 1.4** In the setting of Theorem 1.3, we have that

\[
\begin{align*}
\chi'_{W(k_1 \Lambda_1 + k_2 \Lambda_2)}(x_1, \ldots, x_n, q) =
&= \sum \left( \frac{q^{r_1(1)^2 + \cdots + r_1(k)^2} + \sum_{t=1}^{k} r_1(t) - r_1(k)}{(q)(q)^2 \cdots (q)(q)} \right) \left( \frac{q^{r_2(1)^2 + \cdots + r_2(k)^2} - r_2(k)}{(q)(q)^2 \cdots (q)(q)} \right) \times
\end{align*}
\]
\[
\times \cdots \times \left( \frac{q_{r_n}^{(1)^2} + \cdots + r_n^{(k)^2} - r_n^{(1)} \cdots - r_n^{(k)}}{(q^{(1)}_{r_1} - r_n^{(2)}) \cdots (q^{(k)}_{r_n} - r_n^{(1)}) (q_{r_n})} \right)^{-k_1 + \sum_{i=1}^k r_1^{(i)} \cdots \sum_{i=1}^n r_n^{(i)}},
\]

and

\[
\chi^{'W}(k_{n-1} A_{n-1} + k_n A_n)(x_1, \ldots, x_n, q) = \sum_{p_1^{(1)} \cdots p_n^{(k)} \geq 0} \frac{q^{\frac{1}{2} \sum_{i,m=1}^n \sum_{s,t=1}^i A_{i,m} B_{s,t} r_1^{(s)} p_m^{(t)}}}{\prod_{i=1}^n \prod_{s=1}^k (q_{p_i^{(s)}})} q^{n-1} q^{\sum_{t=1}^k p_2^{(t)} + \cdots + p_n^{(k)} - p_1^{(t)} - \cdots - p_1^{(k)}} \times (1 - q^{p_1^{(k_1+1)} + \cdots + p_1^{(k_2)}}) x_1^{k_1} \prod_{s=1}^n x_i^{\sum_{i=1}^s} p_i^{(s)}
\]

where the sums are taken over decreasing sequences \( r_1^{(1)} \geq r_2^{(2)} \geq \cdots \geq r_n^{(k)} \geq 0 \) for each \( j = 1, \ldots, n \).

The expressions in Corollary \[1.4\] can also be written as follows: As in \[C\], for \( s = 1, \ldots, k-1 \) and \( i = 1, \ldots, n \), set \( p_i^{(s)} = r_i^{(s)} - r_i^{(s+1)} \), and set \( p_i^{(k)} = r_i^{(k)} \). Also, let \( (A_{l,m})_{l,m=1}^n \) be the Cartan matrix of \( \mathfrak{sl}(n+1) \) and \( B_{st} := \min\{s, t\} \), \( 1 \leq s, t \leq k \). Then,

\[
\chi^{'W}(k_{1,2} + k_{2,1})(x_1, \ldots, x_n, q) = \sum_{p_1^{(1)} \cdots p_n^{(k)} \geq 0} \frac{q^{\frac{1}{2} \sum_{i,m=1}^n \sum_{s,t=1}^i A_{i,m} B_{s,t} r_1^{(s)} p_m^{(t)}}}{\prod_{i=1}^n \prod_{s=1}^k (q_{p_i^{(s)}})} q^{n-1} q^{\sum_{t=1}^k p_2^{(t)} + \cdots + p_n^{(k)} - p_1^{(t)} - \cdots - p_1^{(k)}} \times (1 - q^{p_1^{(k_1+1)} + \cdots + p_1^{(k_2)}}) x_1^{k_1} \prod_{s=1}^n x_i^{\sum_{i=1}^s} p_i^{(s)}
\]

where \( \overline{p}_1 = p_1^{(k_1+1)} + 2p_1^{(k_2)} + \cdots + kp_1^{(k)} \) and

\[
\chi^{'W}(k_{n-1} A_{n-1} + k_n A_n)(x_1, \ldots, x_n, q) = \sum_{p_1^{(1)} \cdots p_n^{(k)} \geq 0} \frac{q^{\frac{1}{2} \sum_{i,m=1}^n \sum_{s,t=1}^i A_{i,m} B_{s,t} r_1^{(s)} p_m^{(t)}}}{\prod_{i=1}^n \prod_{s=1}^k (q_{p_i^{(s)}})} q^{n-1} q^{\sum_{t=1}^k p_1^{(t)} + \cdots + p_n^{(k)} - p_1^{(t)} - \cdots - p_n^{(k)}} \times (1 - q^{p_n^{(k_n)}} + \cdots + p_n^{(k)}}) x_n^{k_1} \prod_{s=1}^n x_i^{\sum_{i=1}^s} p_i^{(s)}
\]
where \( \tilde{p}_n = p_n^{(k_n+1)} + 2p_n^{(k_n+2)} + \cdots + k_{n-1}p_n^{(k)} \).

Similar multigraded dimensions for different variants of principal subspaces have been studied in \[\text{AKS}\] and \[\text{FFJMM}\]. Modularity properties of certain multigraded dimensions, in the context of principal subspaces of standard modules, have been studied in \[\text{St}, \text{WZ}\], and more recently in \[\text{BCFK}\]. In \[\text{CalLM4}\], the authors have initiated the study of principal subspaces for standard modules for twisted affine Lie algebras, extending the past work of \[\text{CLM1}–\text{CLM2}, \text{CalLM1}–\text{CalLM3}\] to the case of the level 1 standard module for the twisted affine Lie algebra \( A_2^{(2)} \).

In \[\text{CLM1}–\text{CLM2}, \text{CalLM1}–\text{CalLM3}\], and \[\text{S}\], the annihilator of the highest weight vector of each principal subspace is written in terms of certain elements of \( \hat{U}(\hat{g}) \) which, when viewed as operators, annihilate the highest weight vector. An important set of these operators arises from certain null vector identities given by powers of vertex operators and are written as infinite formal sums of elements of \( \hat{U}(\hat{g}) \) also viewed as operators. The ideals which annihilate the highest weight vectors can be expressed using operators defined by certain truncations of these formal sums, in order to view these operators as elements of \( U(\hat{g}) \).

The remainder of the present work, including the Appendix, focuses on providing a more natural (without such truncations) setting for the annihilating ideals in order to give presentations of the principal subspaces of the standard modules. A construction of a completion of a universal enveloping algebra is used to give natural presentations for the defining annihilating ideals of principal subspaces. This completion was discussed in \[\text{CalLM1–CalLM3}\] but the details of this construction were not supplied. We also prove various properties of this completion and the defining ideals for principal subspaces, including their more natural definition inside this completion. These completions may be generalized to the twisted setting used in \[\text{CalLM4}\] (as in \[\text{LW}\], where similar completions were originally constructed in a general twisted or untwisted setting).

We now give a brief outline of the structure of the present work. In Section 2, we recall certain vertex operator constructions of the standard \( \hat{\mathfrak{sl}}(n+1) \)-modules and intertwining operators among these modules. In Section 3, we recall the definition of the principal subspace of a standard module, along with the definition of multigraded dimension. We also recall the presentations of the principal subspaces of standard \( \hat{\mathfrak{sl}}(n+1) \)-modules, including known and conjectured presentations. In Section 4, we reformulate these presentations in terms of a completion of a certain universal enveloping algebra. In Section 5, we construct exact sequences among principal subspaces of certain standard \( \hat{\mathfrak{sl}}(n+1) \)-modules and obtain the multigraded dimensions of \( W(k_1\Lambda_1 + k_2\Lambda_2) \) and \( W(k_{n-1}\Lambda_{n-1} + k_n\Lambda_n) \). In the Appendix, we recall a certain construction from \[\text{LW}\] and use it to construct the aforementioned completion of the universal enveloping algebra.

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2 Preliminaries

In this section we recall the vertex operator construction of the standard $\widehat{\mathfrak{sl}(n+1)}$-modules. We refer the reader to [CalLM3] and [S] for the full details of these constructions as they are used here. Let $\mathfrak{g} = \mathfrak{sl}(n+1)$, where $n \geq 2$ (the rank of $\mathfrak{g}$), and fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, a set of roots $\Delta$, a set of simple roots $\{\alpha_1, \ldots, \alpha_n\}$, and a set of positive roots $\Delta_+$. We use $\langle \cdot, \cdot \rangle$ to denote the Killing form and rescale it so that $\langle \alpha, \alpha \rangle = 2$ for each $\alpha \in \Delta$. Using this form, we identify $\mathfrak{h}$ with $\mathfrak{h}^\ast$. Let $\lambda_1, \ldots, \lambda_n \in \mathfrak{h} \cong \mathfrak{h}^\ast$ denote the fundamental weights of $\mathfrak{sl}(n+1)$. Recall that $\langle \lambda_i, \alpha_j \rangle = \delta_{ij}$ for each $i, j = 1, \ldots, n$. Denote by $Q = \sum_{i=1}^n \mathbb{Z} \alpha_i$ and $P = \sum_{i=1}^n \mathbb{Z} \lambda_i$ the root lattice and weight lattice of $\mathfrak{sl}(n+1)$, respectively.

For each root $\alpha \in \Delta$, fix a root vector $x_\alpha \in \mathfrak{sl}(n+1)$. Define

$$n = \sum_{\alpha \in \Delta_+} \mathbb{C} x_\alpha,$$

a nilpotent subalgebra of $\mathfrak{sl}(n+1)$.

We have the corresponding untwisted affine Lie algebra given by

$$\widehat{\mathfrak{sl}(n+1)} = \mathfrak{sl}(n+1) \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C} c,$$

where $c$ is a non-zero central element and

$$[x \otimes t^m, y \otimes t^p] = [x, y] \otimes t^{m+p} + m \langle x, y \rangle \delta_{m+p,0} c$$

for any $x, y \in \mathfrak{sl}(n+1)$ and $m, p \in \mathbb{Z}$. Adjoining the degree operator $d$

we obtain the affine Kac-Moody Lie algebra $\widehat{\mathfrak{sl}(n+1)} = \mathfrak{sl}(n+1) \oplus \mathbb{C} d$ (cf. [K]). We extend our form $\langle \cdot, \cdot \rangle$ to $\mathfrak{h} \oplus \mathbb{C} c \oplus \mathbb{C} d$.

Using this form, we may identify $\mathfrak{h} \oplus \mathbb{C} c \oplus \mathbb{C} d$ with $(\mathfrak{h} \oplus \mathbb{C} c \oplus \mathbb{C} d)^\ast$. The simple roots of $\mathfrak{sl}(n+1)$ are $\alpha_0, \alpha_1, \ldots, \alpha_n$ and the fundamental weights of $\mathfrak{sl}(n+1)$ are $\Lambda_0, \Lambda_1, \ldots, \Lambda_n$, given by

$$\alpha_0 = c - (\alpha_1 + \alpha_2 + \cdots + \alpha_n)$$

and

$$\Lambda_0 = d, \quad \Lambda_i = \Lambda_0 + \lambda_i$$

for each $i = 1, \ldots, n$. For each weight $\Lambda_i$, $i = 0, \ldots, n$, we denote by $L(\Lambda_i)$ the level 1 standard module with highest weight weight $\Lambda_i$, and we denote its highest weight vector by $v_{\Lambda_i}$.

We now recall the lattice vertex operator construction of the level 1 standard $\widehat{\mathfrak{sl}(n+1)}$-modules. $\mathfrak{sl}(n+1)$ has two important subalgebras:

$$\widehat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C} c$$

and the Heisenberg subalgebra

$$\widehat{\mathfrak{h}}_Z = \prod_{m \in \mathbb{Z} \setminus \{0\}} \mathfrak{h} \otimes t^m \oplus \mathbb{C} c$$
We use $U(\cdot)$ to denote the universal enveloping algebra. The induced module
\[ M(1) = U(\hat{h}) \otimes U(b \otimes C[t] \otimes C) \]
has a natural $\hat{h}$-module structure, where $b \otimes C[t]$ acts trivially and $c$ acts as identity on the one-dimensional module $C$.

We use the notation $C[S]$ to denote the group algebra of $S$ for any $S \subset P$, and define
\[ V_P = M(1) \otimes C[P], \]
\[ V_Q = M(1) \otimes C[Q] \]
and we set
\[ V_Qe^\lambda_i = M(1) \otimes C[Q]e^\lambda_i, \quad i = 1, \ldots, n \]
Given a Lie algebra element $a \otimes t^m \in \mathfrak{sl}(n+1)$, where $a \in \mathfrak{sl}(n+1), m \in \mathbb{Z}$, we will denote its action on an $\mathfrak{sl}(n+1)$-module using the notation $a(m)$. In particular, for $h \in \mathfrak{h}$ and $m \in \mathbb{Z}$, we have the operators $h(m)$ on $V_P$:
\[ h(0)(v \otimes \iota(e_\alpha)) = \langle h, \alpha \rangle (v \otimes \iota(e_\alpha)) \]
\[ h(m)(v \otimes \iota(e_\alpha)) = (h(m)v \otimes \iota(e_\alpha)), \]
making $V_P, V_Q, V_Qe^\lambda_1, \ldots, V_Qe^\lambda_n$ into $\mathfrak{h}$-modules. This action may be extended using vertex operators to make $V_P, V_Q$, and $V_Qe^\lambda_i$ into $\mathfrak{sl}(n+1)$-modules ([FLM], [FZ], cf. [LL]) with vertex operators satisfying
\[ Y(1 \otimes e^\alpha, x) = \sum_{n \in \mathbb{Z}} x_\alpha(n)x^{-n-1}. \]
In fact, we have that $L(\Lambda_i) \simeq V_Qe^\lambda_i$ for $i = 0, \ldots, n$.

We have natural operators
\[ e_\alpha \cdot e^\beta = \epsilon(\alpha, \beta)e^{\alpha+\beta} \]
for $\alpha, \beta \in P$, and that
\[ x_\alpha(m)e_\lambda = c(\alpha, \lambda)e_\lambda x_\alpha(m + \langle \alpha, \lambda \rangle) \]
for each $\alpha \in \Delta$ and $m \in \mathbb{Z}$, where
\[ \epsilon : P \times P \to \mathbb{Z}/2(n+1)^2\mathbb{Z} \]
is defined using a cocycle and
\[ c : P \times P \to \mathbb{Z}/2(n+1)^2\mathbb{Z} \]
is defined using a commutator map, both of which are obtained by taking a central extension of $P$ by a cyclic group of order $2(n+1)^2$ (cf. [FLM], [LL]).
As in [S], we need certain intertwining operators among level 1 standard modules. We have intertwining operators

\[ \mathcal{Y}(\cdot, x) : L(\Lambda_r) \to \text{Hom}(L(\Lambda_s), L(\Lambda_p)) \{x\}, \]

\[ w \mapsto \mathcal{Y}(w, x) = Y(w, x)e^{\pi \lambda_r c(\cdot, \lambda_r)} \]

of type

\[ \left( \begin{array}{c} L(\Lambda_p) \\ L(\Lambda_r) \\ L(\Lambda_s) \end{array} \right) \]

if and only if \( p \equiv r + s \mod (n + 1) \) (cf. [DL]). Given such an intertwining operator, we define a map

\[ \mathcal{Y}_c(\exp{\lambda_r}, x) : L(\Lambda_s) \to L(\Lambda_p) \]

by

\[ \mathcal{Y}_c(\exp{\lambda_r}, x) = \text{Res}_x x^{-1-\langle \lambda_r, \lambda_s \rangle} \mathcal{Y}(\exp{\lambda_r}, x). \]

For each \( \alpha \in \Delta_+ \), we have that

\[ [Y(e^\alpha, x_1), \mathcal{Y}_c(\exp{\lambda_r}, x_2)] = 0, \]

which implies

\[ [x_\alpha(m), \mathcal{Y}_c(\exp{\lambda_r}, x_2)] = 0 \]

for each \( m \in \mathbb{Z} \), a consequence of the Jacobi identity for intertwining operators among standard modules.

We extend the operators \( e_\lambda, \lambda \in P \), to operators on \( V^\otimes_k \), \( k \) a positive integer, by defining:

\[ e_\lambda^\otimes_k = e_\lambda \otimes \cdots \otimes e_\lambda : V^\otimes_k \to V^\otimes_k. \]

For any standard \( \widehat{\mathfrak{sl}(n+1)} \)-module \( L(\Lambda) \) of positive integral level \( k \), its highest weight \( \Lambda \) is of the form

\[ \Lambda = k_0 \Lambda_0 + \cdots + k_n \Lambda_n \]

for some nonnegative integers \( k_0, \ldots, k_n \) satisfying \( k_0 + \cdots + k_n = k \). Consider the space

\[ V^\otimes_k = \bigotimes_{k \text{ times}} V_P, \]

and let

\[ v_{i_1, \ldots, i_k} = v_{\Lambda_{i_1}} \otimes \cdots \otimes v_{\Lambda_{i_k}} \in V^\otimes_k, \]

where exactly \( k_i \) indices are equal to \( i \) for each \( i = 0, \ldots, n \). Then, we have that \( v_{i_1, \ldots, i_k} \) is a highest weight vector for \( \widehat{\mathfrak{sl}(n+1)} \), and

\[ L(\Lambda) \simeq U(\widehat{\mathfrak{sl}(n+1)}) \cdot v_{i_1, \ldots, i_k} \subset V^\otimes_k \]

(cf. [K]). We also have a natural vertex operator algebra structure on \( L(k \Lambda_0) \) and \( L(k \Lambda_0) \)-module structure on \( L(\Lambda) \) given by:
Theorem 2.1 ([FZ], [DL], [Li1]; cf. [LL]) The standard module \( \Lambda \) has a natural vertex operator algebra structure. The level \( k \) standard \( \hat{\mathfrak{sl}}(n+1) \)-modules provide a complete list of irreducible \( \Lambda \)-modules.

Let \( \omega \) denote the Virasoro vector in \( \Lambda \). We have a natural representation of the Virasoro algebra on each \( \Lambda \) given by

\[
Y_{\Lambda}(\omega, x) = \sum_{m \in \mathbb{Z}} \Lambda(m)x^{-m-2}
\]  
(2.8)

The operators \( L(0) \) defined in (2.8) provide each \( \Lambda \) with a grading, which we refer to as the weight grading:

\[
\Lambda = \bigoplus_{s \in \mathbb{Z}} \Lambda(s + h_{\Lambda})
\]  
(2.9)

where \( h_{\Lambda} \in \mathbb{Q} \) and depends on \( \Lambda \). In particular, we have the grading

\[
\Lambda(0) = \bigoplus_{s \in \mathbb{Z}} \Lambda(s).
\]  
(2.10)

We denote the weight of an element \( a \cdot v_{\Lambda} \in W(\Lambda) \) by \( \text{wt}(a \cdot v_{\Lambda}) \). We will also write

\[
\text{wt}(x_{\alpha}(m)) = -m,
\]

where we view \( x_{\alpha}(m) \) both as an operator and as an element of \( U(\hat{\mathfrak{sl}}(n+1)) \).

We also have \( n \) distinct charge gradings on each \( \Lambda \) of level \( k \), given by the eigenvalues of the operators \( \lambda_i(0) \) for \( i = 1, \ldots, n \):

\[
\Lambda = \bigoplus_{r_1, \ldots, r_n, s \in \mathbb{Z}} L(\Lambda)_{r_1 + \langle \lambda_1, \Lambda \rangle, \ldots, r_n + \langle \lambda_n, \Lambda \rangle, s + h_{\Lambda}}.
\]  
(2.11)

We call these the \( \lambda_i \)-charge gradings. An element of \( \Lambda \) with \( \lambda_i \)-charges \( n_i \) for \( i = 1, \ldots, n \) has total charge \( \sum_{i=1}^{n} n_i \). The gradings (2.9) and (2.11) are compatible, and we have that

\[
\Lambda = \bigoplus_{r_1, \ldots, r_n, s \in \mathbb{Z}} L(\Lambda)_{r_1 + \langle \lambda_1, \Lambda \rangle, \ldots, r_n + \langle \lambda_n, \Lambda \rangle, s + h_{\Lambda}}.
\]  
(2.12)

3 Principal subspaces and ideals

We are now ready to define our main object of study. Consider the \( \hat{\mathfrak{sl}}(n+1) \)-subalgebra

\[
\bar{n} = n \otimes \mathbb{C}[t, t^{-1}].
\]  
(3.1)

The Lie algebra \( \bar{n} \) has the following important subalgebras:

\[
\bar{n}_- = n \otimes t^{-1} \mathbb{C}[t^{-1}]
\]

We denote the weight of an element \( a \cdot v_{\Lambda} \in W(\Lambda) \) by \( \text{wt}(a \cdot v_{\Lambda}) \). We will also write

\[
\text{wt}(x_{\alpha}(m)) = -m,
\]

where we view \( x_{\alpha}(m) \) both as an operator and as an element of \( U(\hat{\mathfrak{sl}}(n+1)) \).

We also have \( n \) distinct charge gradings on each \( \Lambda \) of level \( k \), given by the eigenvalues of the operators \( \lambda_i(0) \) for \( i = 1, \ldots, n \):

\[
\Lambda = \bigoplus_{r_1, \ldots, r_n, s \in \mathbb{Z}} L(\Lambda)_{r_1 + \langle \lambda_1, \Lambda \rangle, \ldots, r_n + \langle \lambda_n, \Lambda \rangle, s + h_{\Lambda}}.
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\[
\Lambda = \bigoplus_{r_1, \ldots, r_n, s \in \mathbb{Z}} L(\Lambda)_{r_1 + \langle \lambda_1, \Lambda \rangle, \ldots, r_n + \langle \lambda_n, \Lambda \rangle, s + h_{\Lambda}}.
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3 Principal subspaces and ideals

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\]  
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The Lie algebra \( \bar{n} \) has the following important subalgebras:

\[
\bar{n}_- = n \otimes t^{-1} \mathbb{C}[t^{-1}]
\]
and

\[ \bar{\mathfrak{n}}_+ = \mathfrak{n} \otimes \mathbb{C}[t] \]

Let \( U(\bar{n}) \) be the universal enveloping algebra of \( \bar{n} \). We recall that \( U(\bar{n}) \) has the decomposition:

\[
U(\bar{n}) = U(\bar{n}_-) \oplus U(\bar{n})\bar{n}_+.
\] (3.2)

Given a \( \widehat{\mathfrak{sl}}(n+1) \)-module \( L(\Lambda) \) of positive integral level \( k \) with highest weight vector \( v_\Lambda \), the principal subspace of \( L(\Lambda) \) is defined by:

\[
W(\Lambda) = U(\bar{n}) \cdot v_\Lambda.
\]

\( W(\Lambda) \) inherits the grading (2.12), and we have that

\[
W(\Lambda) = \bigoplus_{r_1, \ldots, r_n, s \in \mathbb{Z}} W(\Lambda)_{r_1 + \langle \lambda_1, \Lambda \rangle, \ldots, r_n + \langle \lambda_n, \Lambda \rangle; s + h_\Lambda}.
\] (3.3)

For convenience, we will use the notation

\[
W(\Lambda)_{r_1, \ldots, r_n; s} = W(\Lambda)_{r_1 + \langle \lambda_1, \Lambda \rangle, \ldots, r_n + \langle \lambda_n, \Lambda \rangle; s + h_\Lambda}
\]

As in [CLM1]–[CLM2], [CaLM3], [C1]–[C2], define the multigraded dimension of \( W(\Lambda) \) by:

\[
\chi_{W(\Lambda)}(x_1, \ldots, x_n, q) = \text{tr}_{W(\Lambda)} x_1^{\lambda_1} \cdots x_n^{\lambda_n} q^{L(0)}
\]

and its modification

\[
\chi'_{W(\Lambda)}(x_1, \ldots, x_n, q) = x_1^{-\langle \lambda_1, \Lambda \rangle} \cdots x_n^{-\langle \lambda_n, \Lambda \rangle} q^{-h_\Lambda} \chi_{W(\Lambda)}(x_1, \ldots, x_n, q) \in \mathbb{C}[[x_1, \ldots, x_n, q]]
\]

In particular, we have that

\[
\chi'_{W(\Lambda)}(x_1, \ldots, x_n, q) = \sum_{r_1, \ldots, r_n, s \in \mathbb{N}} \dim(W(\Lambda)_{r_1, \ldots, r_n; s}) x_1^{r_1} \cdots x_n^{r_n} q^s.
\]

For each such \( \Lambda \), we have a surjective map

\[
F_\Lambda : \hat{\mathfrak{g}} \twoheadrightarrow L(\Lambda)
\]

\( a \mapsto a \cdot v_\Lambda \) (3.4)

and its surjective restriction \( f_\Lambda \):

\[
f_\Lambda : U(\bar{n}) \twoheadrightarrow W(\Lambda)
\]

\( a \mapsto a \cdot v_\Lambda \). (3.5)

A precise description of the kernels \( \text{Ker} f_\Lambda \) for every each \( \Lambda = \sum_{i=0}^n k_i \Lambda_i \) gives a presentation of the principal subspaces \( W(\Lambda) \) for \( \hat{\mathfrak{sl}}(n+1) \), as we will now discuss.
Define the formal sums
\[ R^i_t = \sum_{m_1 + \cdots + m_n = -t} x_{\alpha_i}(m_1)x_{\alpha_i}(m_2) \cdots x_{\alpha_i}(m_{k+1}) \]  (3.6)
and their truncations
\[ R^i_{M,t} = \sum_{m_1 + \cdots + m_{k+1} = -t, \atop m_1, \ldots, m_{k+1} \leq M} x_{\alpha_i}(m_1) \cdots x_{\alpha_i}(m_{k+1}) \]  (3.7)
for \( t \in \mathbb{Z}, M \in \mathbb{Z} \) and \( i = 1, \ldots, n \). Note that each \( R^i_{M,t} \in U(\bar{n}) \) and the infinite sum \( R^i_t \notin U(\bar{n}) \), but \( R^i_t \) is still well-defined as an operator on \( W(\Lambda) \), since, when acting on any element of \( W(\Lambda) \), only finitely many of its terms are nonzero. Let \( J \) be the left ideal of \( U(\bar{n}) \) generated by the elements \( R^i_{-1,t} \) for \( t \geq k + 1 \) and \( i = 1, 2 \):
\[ J = \sum_{i=1}^{n} \sum_{t \geq k+1} U(\bar{n})R^i_{-1,t}. \]  (3.8)

Define a left ideal of \( U(\bar{n}) \) by:
\[ I_{k_0} = J + U(\bar{n})\bar{n} \]
and for each \( \Lambda = \sum_{i=0}^{k} k_i \Lambda_i \), define
\[ I_{\Lambda} = I_{k_0} + \sum_{\alpha \in \Delta_+} U(\bar{n})x_{\alpha}(-1)^{k+1 - \langle \alpha, \Lambda \rangle}. \]

**Conjecture 3.1** For each \( \Lambda = k_0 \Lambda_0 + \cdots + k_n \Lambda_n \) with \( k_0, \ldots, k_n \in \mathbb{N}, k \geq 1, \) and \( k_0 + \cdots + k_n = k \), we have that \( \text{Ker} f_{\Lambda} = I_{\Lambda} \).

For \( \Lambda = k_0 \Lambda_0 + k_i \Lambda_i \), we have
\[ \text{Ker} f_{k_0 \Lambda_0 + k_i \Lambda_i} = I_{k_0 \Lambda_0} + U(\bar{n})x_{\alpha_i}(-1)^{k_0 + 1}. \]  (3.9)

In the cases that \( g = \mathfrak{sl}(2) \) and \( k \geq 1 \) or \( g \) is of type ADE and \( k = 1 \), Conjecture 3.1 has been proved in \cite{CalLM1}-\cite{CalLM3}. In the case that \( g = \mathfrak{sl}(3) \) and \( k \geq 1 \) this conjecture has been proved in \cite{S}. The presentations (3.9) are suggested by the bases found in \cite{G}, but an a priori proof is lacking. The proof of (3.9) will appear in future work.

We now give a partial proof of this conjecture, using the quasiparticle bases for principal subspaces obtained in \cite{G} to obtain the second term in (3.9) for the cases \( i = 1 \) and \( i = n \). As in \cite{G}, for each \( i = 1, \ldots, n \), we define \( n_{\alpha_i} = \mathbb{C}x_{\alpha_i} \) and \( \bar{n}_{\alpha_i} = n_{\alpha_i} \otimes \mathbb{C}[t, t^{-1}] \). For each \( i = 1, \ldots, n \), define the operators
\[ x_{M\alpha_i}(m) = \sum_{m_1 + \cdots + m_M = m} x_{\alpha_i}(m_1) \cdots x_{\alpha_i}(m_M), \]
called quasiparticles of color \( i \) and charge \( M \) in \([G]\). These operators act in a well defined way on any element of \( W(\Lambda) \), \( \Lambda \) a dominant integral weight, in the sense that, when applied to an element of \( W(\Lambda) \), only finitely many terms are nonzero. We will consider \( \Lambda \) of the form

\[
\Lambda = k_0 \Lambda_0 + k_j \Lambda_j
\]

for \( j = 1, \ldots, n \) and \( k_0, k_j \in \mathbb{N} \) with \( k_0 + k_j = k \). Define

\[
J_t := \begin{cases} 
0 & \text{for } 0 < t \leq k_0 \\
J & \text{for } k_0 < t \leq k = k_0 + k_j.
\end{cases}
\]

In \([G]\), Georgiev proved that \( W(\Lambda) = U(\bar{n}_{\bar{\alpha}}) \cdots U(\bar{n}_{\bar{\alpha}_1}) \cdot v_\Lambda \)

and also that the set of operators

\[
\mathcal{B}_{W(\Lambda)} := \bigcup_{0 \leq n,(n_{k_0}) \leq \cdots \leq n_{k_1} \leq k} \left\{ x_{n_{k_0},n_{k_1}} \alpha_n(m_{r_0}^{(1)},m_{r_1}^{(1)}) \cdots x_{n_1,n_{k_1}} \alpha_1(m_{r_0}^{(1)},m_{r_1}^{(1)}) \cdots x_{n_1,n_1} \alpha_1(m_{r_0}^{(1)},m_{r_1}^{(1)}) \right\}
\]

where \( r_0^{(1)} := 0 \), forms a basis for \( W(\Lambda) \) when applied to \( v_\Lambda \). It is important to notice that, for each \( j = 1, \ldots, n \),

\[
\mathcal{B}_{W(k\Lambda_j)} \subset \mathcal{B}_{W(k_0\Lambda_0 + (k-1)\Lambda_j)} \subset \cdots \subset \mathcal{B}_{W(k\Lambda_0)}
\]

Consider the maps

\[
f_{k\Lambda_0} : U(\bar{n}) \rightarrow W(k\Lambda_0)
\]

and

\[
1^{\otimes k_0} \otimes \mathcal{Y}_e(e^{\lambda_j}, x)^{k_j} : W(k\Lambda_0) \rightarrow W(k_0\Lambda_0 + k_j\Lambda_j).
\]

Composing these maps, we obtain (after multiplication by a scalar):

\[
f_{k_0\Lambda_0 + k_j\Lambda_j} = (1^{\otimes k_0} \otimes \mathcal{Y}_e(e^{\lambda_j}, x)^{k_j}) \circ f_{k\Lambda_0},
\]

which implies that

\[
\text{Ker} f_{k\Lambda_0} \subset \text{Ker} f_{k_0\Lambda_0 + k_j\Lambda_j}
\]

Using these maps, along with the bases above, we obtain the following proposition:
Proposition 3.2 Fix $k_0, k_1, k_n, k \in \mathbb{N}$ with $k_0 + k_1 = k_0 + k_n = k > 0$. Then

$$\text{Ker} f_{k_0A_0 + k_1A_1} = \text{Ker} f_{kA_0} + U(\tilde{n}) x_{\alpha_1} (-1)^{k_0+1}$$

(3.12)

and

$$\text{Ker} f_{k_0A_0 + k_nA_n} = \text{Ker} f_{kA_0} + U(\tilde{n}) x_{\alpha_n} (-1)^{k_0+1}.$$  

(3.13)

Proof: We prove (3.12). Note that (3.13) will follow by an identical proof when the bases (3.10) are rewritten as subsets of $U(\tilde{n}_{\alpha_1}) \cdots U(\tilde{n}_{\alpha_n})$.

First, the fact that

$$U(\tilde{n}) x_{\alpha_1} (-1)^{k_0+1} \subset \text{Ker} f_{k_0A_0 + k_1A_1}$$

follows immediately from the fact that

$$x_{\alpha_1} (-1)^2 \cdot v_{\Lambda_0} = 0$$

and

$$x_{\alpha_1} (-1) \cdot v_{\Lambda_1} = 0,$$

and, along with (3.11) gives us that

$$\text{Ker} f_{kA_0} + U(\tilde{n}) x_{\alpha_1} (-1)^{k_0+1} \subset \text{Ker} f_{k_0A_0 + k_1A_1}$$

We now show the reverse inclusion. Using the spanning argument for (3.10) in [G], we may write $a \cdot v_{kA_0} \in \text{span}(B_W(kA_0)) \cdot v_{kA_0}$ as

$$a \cdot v_{kA_0} = (b + cx_{\alpha_1} (-1)^{k_0+1}) \cdot v_{kA_0}$$

(3.14)

for some $b \in \text{span}(B_W(k_0A_0 + k_1A_1)) \subset \text{span}(B_W(kA_0))$ and $c \in U(\tilde{n}_{\alpha_1}) \cdots U(\tilde{n}_{\alpha_n})$. Suppose that $a \in \text{Ker} f_{k_0A_0 + k_1A_1}$. If $a \in \text{Ker} f_{kA_0}$, we are done. So, suppose that $a \cdot v_{kA_0} \neq 0$. Then, applying the map $(1 \otimes k_0 \otimes \gamma c(x^{A_1}, x^{k_1}))$ and using (3.14) we have

$$0 = a \cdot v_{k_0A_0 + k_1A_1}$$

$$= (b + cx_{\alpha_1} (-1)^{k_0+1}) \cdot v_{k_0A_0 + k_1A_1}$$

$$= b \cdot v_{k_0A_0 + k_1A_1},$$

which implies that $b = 0$, so that $a \cdot v_{kA_0} = cx_{\alpha_1} (-1)^{k_0+1} \cdot v_{kA_0}$, and $a \in \text{Ker} f_{kA_0} + U(\tilde{n}) x_{\alpha_1} (-1)^{k_0+1}$.

Remark 3.3 As an alternative to the spanning argument in [G], we may obtain (3.14) by solving a linear system of equations. To obtain this system of equations, it suffices to rewrite only elements of $B_W(kA_0) \setminus B_W(k_0A_0 + k_1A_1)$. Given such an element $a \in B_W(kA_0) \setminus B_W(k_0A_0 + k_1A_1)$, expand all the terms and look at all summands in $a \cdot v_{kA_0}$ not in $U(\tilde{n}_{\alpha_1}) \cdots U(\tilde{n}_{\alpha_n}) x_{\alpha_1} (-1)^{k_0+1}$, $v_{kA_0}$ and match them up with linear combinations of expanded elements from $B_W(k_0A_0 + k_1A_1)$, $v_{kA_0}$ with the same charges and weight. The term $cx_{\alpha_1} (-1)^{k_0+1}$ arises from adding in terms
from $U(\tilde{n}_\alpha)\cdots U(\tilde{n}_{\alpha_1})x_{\alpha_1}(-1)^{k_0+1}\cdot v_{k\Lambda_0}$ to make both sides of (3.14) equal. Take, for example, the operator
\[ x_{2\alpha_1}(-6)x_{3\alpha_1}(-5) \in \mathfrak{B}_W(k\Lambda_0), \]
where we assume $k > 5$. We have, by definition, that
\[ x_{2\alpha_1}(-6)x_{3\alpha_1}(-5) \notin \mathfrak{B}_W(\Lambda_0+(k-1)\Lambda_1). \]
Applying this operator to $v_{\Lambda_0+(k-1)\Lambda_1}$, we have that
\[ x_{2\alpha_1}(-6)x_{3\alpha_1}(-5) \cdot v_{\Lambda_0+(k-1)\Lambda_1} \neq 0. \]
We may write $x_{2\alpha_1}(-6)x_{3\alpha_1}(-5) \cdot v_{k\Lambda_0}$ as a linear combination of
\[ x_{5\alpha_1}(-11) \cdot v_{k\Lambda_0}, \ x_{\alpha_1}(-3)x_{4\alpha_1}(-8) \cdot v_{k\Lambda_0}, \ x_{\alpha_1}(-4)x_{4\alpha_1}(-7) \cdot v_{k\Lambda_0} \in \mathfrak{B}_W(\Lambda_0+(k-1)\Lambda_1) \cdot v_{k\Lambda_0} \]
and an element of the form $U(\tilde{n}_\alpha)\cdots U(\tilde{n}_{\alpha_1})x_{\alpha_1}(-1)^2 \cdot v_{k\Lambda_0}$, giving precisely an expression of the form (3.14), as follows:
\[
\begin{align*}
  x_{2\alpha_1}(-6)x_{3\alpha_1}(-5) \cdot v_{k\Lambda_0} & = \left(\frac{1}{2}\right)x_{\alpha_1}(-3)x_{4\alpha_1}(-8) - \frac{1}{10}x_{5\alpha_1}(-11) + 2x_{\alpha_1}(-4)x_{4\alpha_1}(-7) \cdot v_{k\Lambda_0} + cx_{\alpha_1}(-1)^2 \cdot v_{k\Lambda_0} \\
 & \quad \text{for some } c \in U(\tilde{n}_\alpha)\cdots U(\tilde{n}_{\alpha_1}).
\end{align*}
\]

4 A reformulation of the presentation problem

In this section we reformulate Conjecture 3.1 along with all known presentations of principal subspaces, in terms of a natural completion of $U(\tilde{n})$, which we denote by $\widetilde{U}(\tilde{n})$. A version of this completion was constructed in [LW], and we recall this construction, suitably adapted to our present setting, in the appendix. In this section only, for $\alpha \in \Delta$ and $n \in \mathbb{Z}$, will use the notation $x_\alpha(n)$ for completion elements $X_\alpha(n)$ from the appendix, and no confusion should arise.

We may define a natural “lifting” of the maps $f_\Lambda$:
\[
\tilde{f}_\Lambda : \widetilde{U}(\tilde{n}) \rightarrow W(\Lambda) \quad (4.1)
\]
\[ a \mapsto a \cdot v_\Lambda. \]

Indeed, given $a \in \widetilde{U}(\tilde{n})$, we may uniquely express $a$ as $a = b + c$ for some $b \in U(\tilde{n}_-)\$ and $c \in \widetilde{U}(\tilde{n})\tilde{n}_+$ (by (A.7)), and define $a \cdot v_\Lambda = b \cdot v_\Lambda$. That is, we let $c$ act as 0.

We now reformulate Conjecture 3.1 in terms of finding $\text{Ker} \tilde{f}_\Lambda$. Recall the formal sums
\[
R_t^i = \sum_{m_1+\cdots+m_{k+1}=-t} x_{\alpha_1}(m_1)\cdots x_{\alpha_1}(m_{k+1}),
\]
which are well defined as operators on each $W(\Lambda)$. It is important to note that each $R_t^i$ is not an element of $\widetilde{U}(\tilde{n})$, so we seek natural representatives for $R_t^i$ in $\widetilde{U}(\tilde{n})$, in the sense that, when viewed as operators on $W(\Lambda)$, these representatives are equal to $R_t^i$. 

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Let $\mathcal{A}$ denote the set of finite sequences of integers. Given a sequence of integers $A = (m_1, \ldots, m_{k+1}) \in \mathcal{A}$, define a function

$$
\# : \mathbb{Z} \times A \rightarrow \mathbb{N} \quad (n, A) \mapsto \text{number of occurrences of } n \text{ in } A.
$$

For any sequence in $(m_1, \ldots, m_{k+1}) \in \mathcal{A}$, define

$$A_{m_1, \ldots, m_{k+1}} = \{ \#(n, A) | n \in \mathbb{Z} \setminus \{0\} = \{n_1, \ldots, n_j\} \}
$$

where $n_1, \ldots, n_j$ are positive integers and $n_1 + \cdots + n_j = k + 1$. Define integers

$$c_{m_1, \ldots, m_{k+1}} = \binom{k+1}{n_1, \ldots, n_j} = \frac{(k+1)!}{(n_1)! \cdots (n_j)!}.
$$

We define

$$R^i_t = R^i_{-1,t} + \sum_{m_1 \leq \cdots \leq m_{k+1}, \atop m_1 + \cdots + m_{k+1} = -t, \atop m_{k+1} \geq 0} c_{m_1, \ldots, m_{k+1}} x_{\alpha_i}(m_1) \cdots x_{\alpha_i}(m_{k+1}), \quad (4.3)
$$

which is clearly in $\tilde{U}(\tilde{n})$. We may also write, for each $R^i_t$,

$$R^i_t = R^i_{M,t} + \sum_{m_1 \leq \cdots \leq m_{k+1}, \atop m_1 + \cdots + m_{k+1} = -t, \atop m_{k+1} \geq M + 1} c_{m_1, \ldots, m_{k+1}} x_{\alpha_i}(m_1) \cdots x_{\alpha_i}(m_{k+1}), \quad (4.4)
$$

and, as elements of $\tilde{U}(\tilde{n})$, (4.3) and (4.4) are equal.

**Remark 4.1** As mentioned above, the formal sums

$$R^i_t = \sum_{m_1 + \cdots + m_{k+1} = -t} x_{\alpha_i}(m_1) \cdots x_{\alpha_i}(m_{k+1}).
$$

are not elements of $\tilde{U}(\tilde{n})$. Informally, $R^i_t$ is in a sense a “limit” of (4.4), i.e.

$$R^i_t = \lim_{M \to \infty} \left( R^i_{M,t} + \sum_{m_1 \leq \cdots \leq m_{k+1}, \atop m_1 + \cdots + m_{k+1} = -t, \atop m_{k+1} \geq M + 1} c_{m_1, \ldots, m_{k+1}} x_{\alpha_i}(m_1) \cdots x_{\alpha_i}(m_{k+1}) \right),
$$

where infinitely many relations in $\tilde{I}$ need to be applied to obtain $R^i_t$ from $R^i_t$. However, as operators on $W(\Lambda)$, $R^i_t$ and $R^i_t$ are equal.
Lemma 4.2 Let $\alpha \in \Delta_+$ and $m \in \mathbb{N}$. Then, for any $i = 1, \ldots, n$ and $t \in \mathbb{Z}$ we have that
\[
R_i x_\alpha(-m) = x_\alpha(-m) R_i^i + x_\alpha(0) R_{i+m} + c
\]
for some $c \in \widetilde{U(n)} \bar{n}_+$. In particular,
\[
R_i x_\alpha(-m) \in I_{k\Lambda_0} + \widetilde{U(n)} \bar{n}_+.
\]

Proof: First, suppose that $\alpha + \alpha_i \in \Delta_+$. We may write
\[
R_i = R_{m,t}^i + \sum_{m_1, \ldots, m_{k+1} \leq m+1} c_{m_1, \ldots, m_{k+1}} x_{\alpha_i(m_1)} \cdots x_{\alpha_i(m_{k+1})}.
\]

By definition of $\widetilde{U(n)} \bar{n}_+$,
\[
\sum_{m_1, \ldots, m_{k+1} \leq m+1} c_{m_1, \ldots, m_{k+1}} x_{\alpha_i(m_1)} \cdots x_{\alpha_i(m_{k+1})} x_\alpha(-m) \in \widetilde{U(n)} \bar{n}_+.
\]

For $R_{m,t}^i x_\alpha(-m)$, we may write
\[
R_{m,t}^i x_\alpha(-m) = \sum_{m_1, \ldots, m_{k+1} \leq m} x_{\alpha_i(m_1)} \cdots x_{\alpha_i(m_{k+1})} x_\alpha(-m)
\]
\[
= \sum_{j=1}^{k+1} \sum_{m_1, \ldots, m_{k+1} \leq m} C_{\alpha, \alpha_i} x_{\alpha_i(m_1)} \cdots x_{\alpha_i(m_{j-1})} x_{\alpha_i(m_j - m)} \cdots x_{\alpha_i(m_{k+1})}
\]
\[
+ x_\alpha(-m) R_{m,t}^i
\]
\[
= \sum_{j=1}^{k+1} \sum_{m_1, \ldots, m_{k+1} \leq m} C_{\alpha, \alpha_i} x_{\alpha_i(m_1)} \cdots x_{\alpha_i(m_{j-1})} x_{\alpha_i(m_j)} \cdots x_{\alpha_i(m_{k+1})}
\]
\[
+ b + x_\alpha(-m) R_{m,t}^i
\]
for some $b \in U(n) \bar{n}_+$. We have that
\[
\sum_{j=1}^{k+1} \sum_{m_1, \ldots, m_{k+1} \leq m} C_{\alpha, \alpha_i} x_{\alpha_i(m_1)} \cdots x_{\alpha_i(m_j)} \cdots x_{\alpha_i(m_{k+1})} + b + x_\alpha(-m) R_{m,t}^i
\]
\[
= [x_\alpha(0), R_{m+t}^i] + b + x_\alpha(-m) R_{m,t}^i,
\]
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establishing our claim when $\alpha + \alpha_i \in \Delta_+$. If $\alpha + \alpha_i \notin \Delta_+$ the claim is clear since

$$R_t^i x_{\alpha}(-m) = x_{\alpha}(-m)R_t^i \in I_{k\Lambda_0} + \widetilde{U(n)n_+},$$

concluding our proof.

Using an almost identical argument, we have that

**Corollary 4.3** If $a \in U(n_-)$ and $t \in \mathbb{Z}$, we have that

$$R_t^ia \in I_{k\Lambda_0} + \widetilde{U(n)n_+}.$$

**Proof:** It suffices to show that the claim holds for monomials

$$x_{\beta_1}(-m_1) \ldots x_{\beta_j}(-m_j) \in U(n).$$

This follows immediately using the same argument as above, and writing

$$R_t^ia = R_t^{i_1 + \cdots + i_j, t} + \sum_{m_1 \leq \cdots \leq m_{k+1}, m_1 + \cdots + m_{k+1} = -t, m_{k+1} \geq m_1 + \cdots + m_j + 1} c_{m_1, \ldots, m_{k+1}, x_{\alpha_1}(m_1) \cdots x_{\alpha_i}(m_{k+1}).$$

As in [C1]-[C2] and [CallM3], let $J$ be the two sided ideal of $\widetilde{U(n)n_+}$ generated by the $R_t^i$, $i = 1, \ldots, n$ and $t \geq k + 1$. As in [CallM3], we have the following theorem:

**Theorem 4.4** We may describe $I_{k\Lambda_0}$ by:

$$I_{k\Lambda_0} \equiv J \mod \widetilde{U(n)n_+}. \quad (4.5)$$

and moreover, for $I_\Lambda$, we have:

$$I_\Lambda \equiv J + \sum_{\alpha \in \Delta_+} U(n)x_{\alpha}(-1)^{k+1-\langle \alpha, \Lambda \rangle} \mod \widetilde{U(n)n_+}. \quad (4.6)$$

**Proof:** We first show that

$$I_{k\Lambda_0} \subseteq J \mod \widetilde{U(n)n_+}.$$

Indeed, any element $a \in I_{k\Lambda_0}$ may be written as

$$a = \sum_{i=1}^{n} a_i R_{-1, t}^i + b$$

for some $a_i \in U(n)$ and $b \in U(n)n_+$. It suffices to show that each $a_i R_{-1, t}^i \in J + \widetilde{U(n)n_+}$. Indeed, we may write $R_{-1, t}^i = R_t^i + c$ for some $c \in U(n)n_+$, and we clearly have that $a_i R_{-1, t}^i = a_i R_t^i + a_ic \in J + \widetilde{U(n)n_+}$. 

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It remains to show that \( J \subset I_{k\Lambda_0} \) modulo \( \widetilde{U(\bar{n})}\bar{n}_+ \).

It suffices to prove

\[
aR^i_t b \in I_{k\Lambda_0} + \widetilde{U(\bar{n})}\bar{n}_+
\]

for all \( a, b \in \widetilde{U(\bar{n})} \). By (A.7), we may write \( b = b_1 + b_2 \) for some \( b_1 \in U(\bar{n}_-) \) and \( b_2 \in \widetilde{U(\bar{n})}\bar{n}_+ \).

Clearly \( aR^i_t b_2 \in \widetilde{U(\bar{n})}\bar{n}_+ \), and so

\[
aR^i_t b \equiv aR^i_t b_1 \text{ modulo } \widetilde{U(\bar{n})}\bar{n}_+.
\]

By Corollary 4.3 we have that \( R^i_t b_1 \in I_{k\Lambda_0} + \widetilde{U(\bar{n})}\bar{n}_+ \), so it suffices to show that

\[
aR^i_{-1,t} \in I_{k\Lambda_0} + \widetilde{U(\bar{n})}\bar{n}_+.
\]

Using the notation from the appendix, we have that \( a = [\mu] \) for some \( \mu \in F(\Delta_+) \), and we may write

\[
\mu = \sum_{c \in \text{Supp}(\mu)} \mu(c)X(c) = \sum_{c \in \text{Supp}_1(\mu)} \mu(c)X(c) + \sum_{c \in \text{Supp}(\mu) \setminus \text{Supp}_1(\mu)} \mu(c)X(c).
\]

The sum \( \sum_{c \in \text{Supp}_1(\mu)} \mu(c)X(c) \) is finite, so we have that

\[
\sum_{c \in \text{Supp}_1(\mu)} [\mu(c)X(c)]R^i_{-1,t} \in I_{k\Lambda_0},
\]

and by definition of \( \widetilde{U(\bar{n})}\bar{n}_+ \) we have that

\[
\sum_{c \in \text{Supp}(\mu) \setminus \text{Supp}_1(\mu)} [\mu(c)X(c)]R^i_{-1,t} \in \widetilde{U(\bar{n})}\bar{n}_+,
\]

establishing

\[
I_{k\Lambda_0} \equiv J \text{ modulo } \widetilde{U(\bar{n})}\bar{n}_+.
\]

The fact that

\[
I_{\Lambda} \equiv J + \sum_{\alpha \in \Delta_+} U(\bar{n})x_\alpha(-1)^{k+1-\langle \alpha, \Lambda \rangle} \text{ modulo } \widetilde{U(\bar{n})}\bar{n}_+
\]

follows immediately, establishing our theorem.

As a consequence of Theorem 4.4, along with the results of [CalLM1] - [CalLM3] and [S], we have that:

**Theorem 4.5** In the case where \( g = \mathfrak{sl}(n+1) \) with:
• $n = 1$ and $\Lambda = k_0 \Lambda_0 + k_1 \Lambda_1$ with $k_0 + k_1 = k \geq 1$
• $n = 2$ and $\Lambda = k_0 \Lambda_0 + k_1 \Lambda_1 + k_2 \Lambda_2$ with $k_0 + k_1 + k_2 = k \geq 1$
• $n \geq 3$ and $\Lambda = \Lambda_i$ with $i = 0, \ldots, n$

or $\mathfrak{g}$ is of type $D$ or $E$ with $k = 1$ we have that

$$\text{Ker} f_\Lambda \equiv \hat{I}_\Lambda \text{ modulo } U(\frak{n})\frak{n}_+.$$

We reformulate Conjecture 3.1 as follows:

**Conjecture 4.6** Suppose $\mathfrak{g} = \mathfrak{sl}(n+1)$, $k_0, \ldots, k_n, k \in \mathbb{N}$ with $k \geq 1$ and $k_0 + \cdots + k_n = k$. For each $\Lambda = k_0 \Lambda_0 + \cdots + k_n \Lambda_n$, we have that

$$\text{Ker} f_\Lambda \equiv \hat{I}_\Lambda \text{ modulo } U(\frak{n})\frak{n}_+$$

or that

**Conjecture 4.7** In the context of Conjecture 4.6, for each $\Lambda = k_0 \Lambda_0 + \cdots + k_n \Lambda_n$, we have that

$$\text{Ker} f_\Lambda = \hat{I}_\Lambda.$$

### 5 Exact sequences and multigraded dimensions

In this section, we construct exact sequences among the principal subspaces of certain standard modules, and use these to find multigraded dimensions.

Given $\lambda \in P$ and character $\nu : Q \to \mathbb{C}^*$, we define a map $\tau_{\lambda,\nu}$ on $\frak{n}$ by

$$\tau_{\lambda,\nu}(x_\alpha(m)) = \nu(\alpha) x_\alpha(m - \langle \lambda, \alpha \rangle)$$

for $\alpha \in \Delta_+$ and $m \in \mathbb{Z}$. It is easy to see that $\tau_{\lambda,\nu}$ is an automorphism of $\frak{n}$. The map $\tau_{\lambda,\nu}$ extends canonically to an automorphism of $U(\frak{n})$, also denoted by $\tau_{\lambda,\nu}$, given by

$$\tau_{\lambda,\nu}(x_{\beta_1}(m_1) \cdots x_{\beta_r}(m_r)) = \nu(\beta_1 + \cdots + \beta_r) x_{\beta_1}(m_1 - \langle \lambda, \beta_1 \rangle) \cdots x_{\beta_r}(m_r - \langle \lambda, \beta_r \rangle) \quad (5.1)$$

for $\beta_1, \ldots, \beta_r \in \Delta_+$ and $m_1, \ldots, m_r \in \mathbb{Z}$. In particular, we have that

$$e_{\lambda}^\otimes k(a \cdot v_\Lambda) = \tau_{\lambda, e_{-\lambda}}(a) \cdot e_{\lambda}^\otimes k v_\Lambda \quad (5.2)$$

where $\lambda \in P$, $\Lambda$ is a dominant integral weight of $\frak{sl}(n+1)$, and $e_{-\lambda}(\alpha) = e(-\lambda, \alpha)$ for all $\alpha \in \Delta_+$.

For each $j = 1, \ldots, n$, set $\omega_j = \alpha_j - \lambda_j$. For each $1 \leq i \leq n - 1$ and $k_i, k_{i+1} \in \mathbb{N}$ with $k_i + k_{i+1} = k \geq 1$, define maps

$$\phi_i = e_{\omega_i}^\otimes k \circ (1 \otimes k_i \otimes \mathcal{Y}_c(e_{\lambda_{i-1}}, x) \otimes e_{\lambda_{i+1}}, x)$$

$$\psi_i = e_{\omega_{i+1}}^\otimes k \circ (1 \otimes k_i \otimes \mathcal{Y}_c(e_{\lambda_{i+2}}, x) \otimes e_{\lambda_{i+1}}, x)$$

In the case that $i = 1$, we take $\phi_1 = e_{\omega_1}^\otimes k$ and in the case that $i = n - 1$ we take $\psi_{n-1} = e_{\omega_n}^\otimes k$.  

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\textbf{Theorem 5.1} For every \( k_i, k_{i+1} \in \mathbb{N} \) with \( k_i + k_{i+1} = k \) and \( k \geq 1 \), we have

\[
\phi_i : W(k_i \Lambda_i + k_{i+1} \Lambda_{i+1}) \to W(k_i \Lambda_0 + k_{i+1} \Lambda_i)
\]  

(5.3)

and

\[
\psi_i : W(k_i \Lambda_i + k_{i+1} \Lambda_{i+1}) \to W(k_{i+1} \Lambda_0 + k_i \Lambda_{i+1})
\]  

(5.4)

Moreover, for \( r_1, \ldots, r_n, s \in \mathbb{Z} \),

\[
\phi_i : W(k_i \Lambda_i + k_{i+1} \Lambda_{i+1})_{r_1, \ldots, r_n; s} \to W(k_i \Lambda_0 + k_{i+1} \Lambda_i)_{r_1, \ldots, r_i + k_i, \ldots, r_n; s - r_{i-1} - r_{i+1} + k_i}
\]  

(5.5)

and

\[
\psi_i : W(k_i \Lambda_i + k_{i+1} \Lambda_{i+1})_{r_1, \ldots, r_n; s} \to W(k_{i+1} \Lambda_0 + k_i \Lambda_{i+1})_{r_1, \ldots, r_i + k_{i+1}, \ldots, r_n; s - r_i + r_{i+1} - r_{i+2} + k_i}
\]  

(5.6)

where we take \( r_0 = r_{n+1} = 0 \).

\textbf{Proof:} We prove only (5.3) since (5.4) follows analogously. Let \( a \cdot v_{k_i \Lambda_i + k_{i+1} \Lambda_{i+1}} \in W(k_i \Lambda_i + k_{i+1} \Lambda_{i+1}) \) for some \( a \in U(\mathfrak{n}) \). We have that

\[
\phi_i(a \cdot v_{k_i \Lambda_i + k_{i+1} \Lambda_{i+1}}) = \phi_i(a \cdot (e^{\lambda_i} \otimes \cdots \otimes e^{\lambda_i})),
\]

\[
= \left( e^{\otimes k_i} \circ (1^{\otimes k_i} \otimes \mathcal{V}^c(e^{\lambda_i-1}) \otimes \delta^{k_{i+1}}) \right) (a \cdot (e^{\lambda_i} \otimes \cdots \otimes e^{\lambda_i})),
\]

\[
= e^{\otimes k_i} (a \cdot (e^{\lambda_i} \otimes \cdots \otimes e^{\lambda_i})),
\]

\[
= c_1 \tau_{\omega_1, c - \omega_1} (a) \cdot (e^{\lambda_i} \otimes \cdots \otimes e^{\lambda_i}),
\]

\[
= c_2 \tau_{\omega_1, c - \omega_1} (a) \cdot (1 \otimes \cdots \otimes 1 \otimes e^{\lambda_i} \otimes \cdots \otimes e^{\lambda_i}),
\]

\[
\in W(k_i \Lambda_0 + k_{i+1} \Lambda_i)
\]

for some constants \( c_1, c_2 \in \mathbb{C} \). The fourth equality follows from the fact that \( \lambda_i - 1 + \lambda_{i+1} + \omega_i = \lambda_i \). This concludes our proof.

Using the presentations (3.9), we construct exact sequences which give the multigraded dimensions of certain principal subspaces (compare to [C1]).
Theorem 5.2 Let $k \geq 1$. For $k_1, k_2, k_{n-1}, k_n \in \mathbb{N}$ such that $k_1 + k_2 = k_{n-1} + k_n = k$ and $k_1 > 0$ and $k_n > 0$. Then following sequences are exact:

$$0 \longrightarrow W(k_1 \Lambda_1 + k_2 \Lambda_2) \xrightarrow{e_{\omega_1}^k} W(k_1 \Lambda_0 + k_2 \Lambda_1) \xrightarrow{1^\otimes k_1 - 1 \otimes Y_c(e^{\lambda_1}, x) \otimes 1^\otimes k_2} W((k_1 - 1) \Lambda_0 + (k_2 + 1) \Lambda_1) \longrightarrow 0$$

and

$$0 \longrightarrow W(k_{n-1} \Lambda_{n-1} + k_n \Lambda_n) \xrightarrow{e_{\omega_i}^k} W(k_n \Lambda_0 + k_{n-1} \Lambda_n) \xrightarrow{1^\otimes k_n - 1 \otimes Y_c(e^{\lambda_n}, x) \otimes 1^\otimes k_{n-1}} W((k_n - 1) \Lambda_0 + (k_{n-1} + 1) \Lambda_n) \longrightarrow 0$$

**Proof:** We prove that (5.9) is exact. The exactness of (5.10) can be proved analogously. The fact that $e_{\omega_i}^k$ is injective is clear, since its left inverse is $e_{-\omega_i}^-1$. We first show that $\text{Im}(e_{\omega_1}^k) \subset \text{Ker}(1^\otimes k_1 - 1 \otimes Y_c(e^{\lambda_1}, x) \otimes 1^\otimes k_2)$. Suppose that $w \in \text{Im}(e_{\omega_1}^k)$. We have that

$$(1^\otimes k_1 - 1 \otimes Y_c(e^{\lambda_1}, x) \otimes 1^\otimes k_2)(w) = vx_{\alpha_1}(-1)^{k_1} \cdot v(k_1 - 1)\Lambda_0 + (k_2 + 1)\Lambda_1 = 0$$

for some $v \in U(\mathfrak{n})$, and so $w \in \text{Ker}(1^\otimes k_1 - 1 \otimes Y_c(e^{\lambda_1}, x) \otimes 1^\otimes k_2)$. Hence $\text{Im}(e_{\omega_1}^k) \subset \text{Ker}(1^\otimes k_1 - 1 \otimes Y_c(e^{\lambda_1}, x) \otimes 1^\otimes k_2)$.

We now show that $\text{Ker}(1^\otimes k_1 - 1 \otimes Y_c(e^{\lambda_1}, x) \otimes 1^\otimes k_2) \subset \text{Im}(e_{\omega_1}^k)$ by characterizing the elements of each set. If $w \in \text{Ker}(1^\otimes k_1 - 1 \otimes Y_c(e^{\lambda_1}, x) \otimes 1^\otimes k_2)$, we may write $w = f_{k_1\Lambda_0 + k_2\Lambda_1}(u)$ for some $u \in U(\mathfrak{n})$. We have that

$$(1^\otimes k_1 - 1 \otimes Y_c(e^{\lambda_1}, x) \otimes 1^\otimes k_2)(f_{k_1\Lambda_0 + k_2\Lambda_1}(u)) = 0$$

iff $f_{(k_1 - 1)\Lambda_0 + (k_2 + 1)\Lambda_1}(u) = 0$ and by (3.12) we have

$$f_{(k_1 - 1)\Lambda_0 + (k_2 + 1)\Lambda_1}(u) = 0 \iff u \in \text{Ker}f_{(k_1 - 1)\Lambda_0 + (k_2 + 1)\Lambda_1} = \text{Ker}f_{k\Lambda_0} + U(\mathfrak{n})x_{\alpha_1}(-1)^{k_1}$$

so that

$$w = f_{k_1\Lambda_0 + k_2\Lambda_1}(u) \in \text{Ker}(1^\otimes k_1 - 1 \otimes Y_c(e^{\lambda_1}, x) \otimes 1^\otimes k_2) \iff u \in \text{Ker}f_{(k_1 - 1)\Lambda_0 + (k_2 + 1)\Lambda_1}.$$
which implies
\[ u - v_{\alpha_{1}}(-1)^{k_{1}} \in \text{Ker} f_{k_{1}\Lambda_{0} + k_{2}\Lambda_{1}}. \]

We therefore have that
\[ w = f_{k_{1}\Lambda_{0} + k_{2}\Lambda_{1}}(u) \in \text{Im}(e_{\omega_{1}}^{\otimes k}) \text{ iff } u \in \text{Ker} f_{k_{1}\Lambda_{0} + k_{2}\Lambda_{1}} + U(\bar{n})x_{\alpha_{1}}(-1)^{k_{1}}. \]

By (3.12) we have that
\[ \text{Ker} f_{(k_{1} - 1)\Lambda_{0} + (k_{2} + 1)\Lambda_{1}} = \text{Ker} f_{k_{1}\Lambda_{0} + U(\bar{n})x_{\alpha_{1}}(-1)^{k_{1}}} \subset \text{Ker} f_{k_{1}\Lambda_{0} + k_{2}\Lambda_{1}} + U(\bar{n})x_{\alpha_{1}}(-1)^{k_{1}}, \]

we have that
\[ w = f_{k_{1}\Lambda_{0} + k_{2}\Lambda_{1}}(u) \in \text{Ker} \left( 1^{\otimes k_{1} - 1} \otimes \mathcal{Y}(e^{\Lambda_{1}}, x) \otimes 1^{\otimes k_{2}} \right) \]
\[ \iff u \in \text{Ker} f_{(k_{1} - 1)\Lambda_{0} + (k_{2} + 1)\Lambda_{1}} \]
\[ \iff u \in \text{Ker} f_{k_{1}\Lambda_{0} + k_{2}\Lambda_{1}} + U(\bar{n})x_{\alpha_{1}}(-1)^{k_{1}} \]
\[ \iff w = f_{k_{1}\Lambda_{0} + k_{2}\Lambda_{1}}(u) \in \text{Im}(e_{\omega_{1}}^{\otimes k}), \]

completing our proof.

Using the conjecture presentations (3.9), an almost identical proof gives:

**Theorem 5.3** Let \( k \geq 1 \). For any \( i \) with \( 1 \leq i \leq n - 1 \) and \( k_{i}, k_{i+1} \in \mathbb{N} \) such that \( k_{i} + k_{i+1} = k \), the sequences:

\[ W(k_{i}\Lambda_{i} + k_{i+1}\Lambda_{i+1}) \xrightarrow{\phi_{i}} \]
\[ W(k_{i}\Lambda_{0} + k_{i+1}\Lambda_{i}) \]
\[ W((k_{i} - 1)\Lambda_{0} + (k_{i+1} + 1)\Lambda_{i}) \rightarrow 0, \]

when \( k_{i} \geq 1 \), and

\[ W(k_{i}\Lambda_{i} + k_{i+1}\Lambda_{i+1}) \xrightarrow{\psi_{i}} \]
\[ W(k_{i+1}\Lambda_{0} + k_{i}\Lambda_{i+1}) \]
\[ W((k_{i+1} - 1)\Lambda_{0} + (k_{i} + 1)\Lambda_{i+1}) \rightarrow 0, \]

when \( k_{i+1} \geq 1 \), are exact.

**Remark 5.4** It is important to note that (5.9), (5.10), (5.11), and (5.12) are fundamentally different from the exact sequences used in [C1] and [CalLM2]. In [C1] and [CalLM2], exact sequences are constructed using intertwining operators among level \( k \) standard modules. The sequences (5.9), (5.10), (5.11), and (5.12) only require intertwining operators among level 1 standard modules and recover the same information about multigraded dimensions, as we will see below.
Remark 5.5 Notice that, in general, the first maps in [5.11] and [5.12] are not injective, in contrast with the corresponding maps in [CLM1]–[CLM2], [C1]–[C2], and [CalLM1]–[CalLM3]. In fact, there are only a few cases where injectivity holds, namely, in the case of Theorem 5.2 and in the corollary below.

Corollary 5.6 For each \( i = 1, \ldots, n \) the following sequences are exact:

\[
0 \longrightarrow W(k\Lambda_i) \xrightarrow{e_i^k} W(k\Lambda_0) \xrightarrow{1 \otimes^{k-1} \otimes Y_{\xi}(e_{\lambda_i}, x)} W((k-1)\Lambda_0 + \Lambda_i) \longrightarrow 0
\]

Remark 5.7 These exact sequences in Corollary 5.6 are the level \( k \) analogues of the exact sequences found in [CalLM3].

We now use the exact sequences (5.9) and (5.10) to obtain the multigraded dimensions \( \chi'_{W(k_1\Lambda_0+k_2\Lambda_2)}(x_1, \ldots, x_n, q) \) and \( \chi'_{W(k_{n-1}\Lambda_{n-1}+k_n\Lambda_n)}(x_1, \ldots, x_n, q) \).

Theorem 5.8 Let \( k \geq 1 \). Let \( k_1, k_2, k_{n-1}, k_n \in \mathbb{N} \) with \( k_1 \geq 1 \) and \( k_n \geq 1 \), such that \( k_1 + k_2 = k \) and \( k_{n-1} + k_n = k \). Then

\[
\chi'_{W(k_1\Lambda_0+k_2\Lambda_2)}(x_1, \ldots, x_n, q) = \chi'_{W(k_{n-1}\Lambda_{n-1}+k_n\Lambda_n)}(x_1, \ldots, x_n, q)
\]

and

\[
\chi'_{W(k_{n-1}\Lambda_{n-1}+k_n\Lambda_n)}(x_1, \ldots, x_n, q) = \chi'_{W(k_{n-1}\Lambda_{n-1}+k_n\Lambda_n)}(x_1, \ldots, x_n, q)
\]

Proof: It is easy to see that the maps used in (5.9) and (5.10) have the property that:

\[
W(k_1\Lambda_0 + k_2\Lambda_2)_{r_1, \ldots, r_n, \delta} \xrightarrow{1 \otimes^{k-1} \otimes Y_{\xi}(e_{\lambda_i}, x) \otimes 1 \otimes^{k_2}} W((k_1 - 1)\Lambda_0 + (k_2 + 1)\Lambda_1)_{r_1, \ldots, r_n, \delta}
\]

and

\[
W(k_{n-1}\Lambda_{n-1} + k_n\Lambda_n)_{r_1, \ldots, r_n, \delta} \xrightarrow{1 \otimes^{k-1} \otimes Y_{\xi}(e_{\lambda_i}, x) \otimes 1 \otimes^{k_n}} W((k_n - 1)\Lambda_0 + (k_{n-1} + 1)\Lambda_n)_{r_1, \ldots, r_n, \delta}.
\]

Combining this fact with the exactness of (5.9) and (5.10), along with (5.5) and (5.7) give

\[
\chi'_{W(k_1\Lambda_0+k_2\Lambda_2)}(x_1, \ldots, x_n, q) = \chi'_{W(k_{n-1}\Lambda_{n-1}+k_n\Lambda_n)}(x_1, \ldots, x_n, q)
\]

with the following analogues of the exact sequences found in [CalLM3].
and

\[ \chi'_{W(k_n \Lambda_0 + k_{n-1} \Lambda_n)}(x_1, \ldots, x_n, q) = \]
\[ = x_n^{k_n} q^{k_n} \chi'_{W(k_{n-1} \Lambda_{n-1} + k_n \Lambda_n)}(x_1, \ldots, x_{n-1} q^{-1}, x_n q, q) \]
\[ + \chi'_{W((k_n-1) \Lambda_0 + (k_{n-1}+1) \Lambda_n)}(x_1, \ldots, x_n, q). \]

which may be rewritten as

\[ \chi'_{W(k_1 \Lambda_1 + k_2 \Lambda_2)}(x_1 q, x_2 q^{-1}, x_3, \ldots, x_n, q) = \]
\[ = x_1^{-k_1} q^{-k_1} \chi'_{W(k_1 \Lambda_0 + k_2 \Lambda_1)}(x_1, \ldots, x_n, q) \]
\[ - x_1^{-k_1} q^{-k_1} \chi'_{W((k_1-1) \Lambda_0 + (k_2+1) \Lambda_1)}(x_1, \ldots, x_n, q) \]

and

\[ \chi'_{W(k_{n-1} \Lambda_{n-1} + k_n \Lambda_n)}(x_1, \ldots, x_{n-1} q^{-1}, x_n q, q) = \]
\[ = x_n^{-k_n} q^{-k_n} \chi'_{W(k_n \Lambda_0 + k_{n-1} \Lambda_{n-1})}(x_1, \ldots, x_n, q) \]
\[ - x_n^{-k_n} q^{-k_n} \chi'_{W((k_n-1) \Lambda_0 + (k_{n-1}+1) \Lambda_{n-1})}(x_1, \ldots, x_n, q). \]

Making the substitutions

\[ x_1 \mapsto x_1 q^{-1}, \ x_2 \mapsto x_2 q \]

in (5.17) and

\[ x_n \mapsto x_n q^{-1}, \ x_{n-1} \mapsto x_{n-1} q \]

in (5.18) immediately proves our theorem.

We now use Theorem 5.8 to write down explicit expressions for

\[ \chi'_{W(k_1 \Lambda_1 + k_2 \Lambda_2)}(x_1, \ldots, x_n, q) \]

and

\[ \chi'_{W(k_{n-1} \Lambda_{n-1} + k_n \Lambda_n)}(x_1, \ldots, x_n, q). \]

In [G], Georgiev obtained:

\[ \chi'_{W(k_2 \Lambda_2 + k_1 \Lambda_1)}(x_1, \ldots, x_n, q) = \]
\[ = \sum \left( \frac{(q_1^{(1)})^2 + \ldots + r_1^{(k)^2} + \sum_{t=1}^k r_1^{(t) \delta_{i_t} j_t}}{(q_1^{(1)})^{(2)}(1) - r_1^{(2)}(1) \cdots (q_1^{(k)})^{(2)}(k)} \right) \times \]
\[ \times \cdots \times \left( \frac{(q_n^{(1)})^2 + \ldots + r_n^{(k)^2} - r_{n-1}^{(1)} - \ldots - r_{n-1}^{(k)} + \sum_{t=1}^k r_n^{(t) \delta_{i_t} j_t}}{(q_n^{(1)})^{(2)}(1) - r_n^{(2)}(1) \cdots (q_n^{(k)})^{(2)}(k)} \right) \]

where the sums are taken over decreasing sequences \( r_j^{(1)} \geq r_j^{(2)} \geq \ldots \geq r_j^{(k)} \geq 0 \) for each \( j = 1, \ldots, n \) and \( j_t = 0 \) for \( 0 \leq t \leq k_0 \) and \( j_t = j \) for \( k_0 < t \leq k, j = 1, \ldots, n \), where \( (q)_r = \prod_{i=1}^r (1 - q^i) \) and \( (q)_0 = 1 \). In particular, we have that

\[ \chi'_{W(k_1 \Lambda_0 + k_2 \Lambda_1)}(x_1, \ldots, x_n, q) = \]
Corollary 5.9

Applying Theorem 5.8 to these expressions immediately gives:

and

Applying Theorem 5.8 to these expressions immediately gives:

Corollary 5.9 In the setting of Theorem 5.8, we have

and

where the sums are taken over decreasing sequences \( r_j^{(1)} \geq r_j^{(2)} \geq \cdots \geq r_j^{(k)} \geq 0 \) for each \( j = 1, \ldots, n \).
Remark 5.10 The expressions in Corollary 5.9 can also be written as follows: As in [G], for $s = 1, \ldots, k - 1$ and $i = 1, \ldots, n$, set $p_i^{(s)} = r_i^{(s)} - r_i^{(s+1)}$, and set $p_i^{(k)} = r_i^{(k)}$. Also, let $(A_{lm})_{l,m=1}^n$ be the Cartan matrix of $\mathfrak{sl}(n+1)$ and $B^{st} := \min\{s, t\}$, $1 \leq s, t \leq k$. Then,

$$
\chi_W^{(k_1\Lambda_1+k_2\Lambda_2)}(x_1, \ldots, x_n, q) = 
\sum_{p_1^{(1)}, \ldots, p_n^{(k)} \geq 0} \frac{q^{\frac{1}{2} \sum_{l,m=1}^n \sum_{s,t=1}^l A_{lm} B^{st} p_i^{(s)} p_m^{(t)}}}{\prod_{i=1}^n \prod_{s=1}^k (q)_{p_i^{(s)}}^k} q^p_1 q^{\sum_{t=1}^k p_2^{(t)} + \cdots + p_k^{(k)} - p_1^{(1)} - \cdots - p_k^{(k)}} \times
\sum_{p_1^{(1)}, \ldots, p_n^{(k)} \geq 0} \prod_{i=1}^n \prod_{s=1}^k (q)_{p_i^{(s)}}^k
$$

where $\tilde{p}_1 = p_1^{(k_1+1)} + 2p_1^{(k_1+2)} + \cdots + k_2p_1^{(k)}$ and

$$
\chi_W^{(k_1\Lambda_1+k_2\Lambda_2)}(x_1, \ldots, x_n, q) = 
\sum_{p_1^{(1)}, \ldots, p_n^{(k)} \geq 0} \frac{q^{\frac{1}{2} \sum_{l,m=1}^n \sum_{s,t=1}^l A_{lm} B^{st} p_i^{(s)} p_m^{(t)}}}{\prod_{i=1}^n \prod_{s=1}^k (q)_{p_i^{(s)}}^k} q^p_1 q^{\sum_{t=1}^k p_2^{(t)} + \cdots + p_k^{(k)} - p_1^{(1)} - \cdots - p_k^{(k)}} \times
\sum_{p_1^{(1)}, \ldots, p_n^{(k)} \geq 0} \prod_{i=1}^n \prod_{s=1}^k (q)_{p_i^{(s)}}^k
$$

where $\tilde{p}_n = p_n^{(k_1+1)} + 2p_n^{(k_1+2)} + \cdots + k_{n-1}p_n^{(k)}$.

Remark 5.11 Corollary 5.9 above is a natural $\mathfrak{sl}(n+1)$-analogue of Corollary 4.1 in [CI]. The multigraded dimension for $\chi_W^{(k_1\Lambda_1+k_2\Lambda_2)}$ in [CI] can be recovered from the expression above for $\chi_W^{(k_1\Lambda_1+k_2\Lambda_2)}$ by taking $n = 2$.

Remark 5.12 Throughout this work we assume that $n \geq 2$ for notational convenience. In the case that $n = 1$ (that is, when $\mathfrak{g} = \mathfrak{sl}(2)$), the above results recover the recursions and multigraded dimensions found in [CLM1]-[CLM2].

A Appendix

Above, we needed the construction of a completion of a certain universal enveloping algebra. In this section, working in a natural generality, we recall a construction in [LW] and use it to construct a natural completion of the universal enveloping algebra of a certain type of subalgebra.
of an affine Lie algebras associated to a finite dimensional semisimple Lie algebra. We also prove a natural decomposition of this completion, which is needed above.

Let \( g \) be a finite dimensional semisimple Lie algebra. Fix a Cartan subalgebra \( h \subset g \), a set of roots \( \Delta \), a set of simple roots \( \Pi = \{ \alpha_1, ..., \alpha_n \} \), a set of positive roots \( \Delta_+ \), and a symmetric invariant nondegenerate bilinear form \( \langle \cdot, \cdot \rangle \), normalized so that \( \langle \alpha, \alpha \rangle = 2 \) for long roots \( \alpha \in \Delta \).

For each \( \alpha \in \Delta_+ \), let \( x_\alpha \in g \) be a root vector associated to the root \( \alpha \). We have that

\[
[x_\alpha, x_\beta] = C_{\alpha, \beta} x_{\alpha + \beta} \tag{A.1}
\]

for some constants \( C_{\alpha, \beta} \in \mathbb{C} \). Let \( S \subset \Delta_+ \) be a nonempty set of positive roots such that if \( \alpha, \beta \in S \) and \( \alpha + \beta \in \Delta_+ \), then \( \alpha + \beta \in S \). Define the nilpotent subalgebra \( n_S \subset g \) by

\[
n_S = \sum_{\alpha \in S} \mathbb{C} x_\alpha.
\]

In the case that \( S = \Delta_+ \), we write \( n_S = n \).

We have the corresponding untwisted affine Lie algebra given by

\[
\hat{g} = g \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C} c,
\]

where \( c \) is a non-zero central element and

\[
[x \otimes t^m, y \otimes t^p] = [x, y] \otimes t^{m+p} + m(x, y) \delta_{m+p,0} c
\]

for any \( x, y \in g \) and \( m, p \in \mathbb{Z} \) and

\[
\bar{n}_S = n_S \otimes \mathbb{C}[t, t^{-1}],
\]

a Lie subalgebra of \( \hat{g} \). The Lie algebra \( \bar{n}_S \) has the following important subalgebras:

\[
\bar{n}_S^- = n_S \otimes t^{-1} \mathbb{C}[t^{-1}]
\]

and

\[
\bar{n}_S^+ = n_S \otimes \mathbb{C}[t].
\]

Let \( U(\bar{n}_S) \) be the universal enveloping algebra of \( \bar{n}_S \). Using the Poincare-Birkhoff-Witt theorem, it is easy to see that \( U(\bar{n}_S) \) has the decomposition

\[
U(\bar{n}_S) = U(\bar{n}_S^-) \oplus U(\bar{n}_S)\bar{n}_S^+ \tag{A.2}
\]

Let \( M(S) \) denote the free monoid on \( \mathbb{Z} \times S \). We may write

\[
M(S) = \bigcup_{n \geq 0} M(S)_n
\]

where

\[
M(S)_n = \mathbb{Z}^n \times S^n
\]

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and composition of elements $\circ$ is given by juxtaposition:
\[
(n_1, \ldots, n_k; \gamma_1, \ldots, \gamma_k) \circ (m_1, \ldots, m_l; \beta_1, \ldots, \beta_l) = (n_1, \ldots, n_k, m_1, \ldots, m_l; \gamma_1, \ldots, \gamma_k, \beta_1, \ldots, \beta_l)
\]
where
\[
(n_1, \ldots, n_k; \gamma_1, \ldots, \gamma_k) \in M(S)_k,
\]
\[
(m_1, \ldots, m_l; \beta_1, \ldots, \beta_l) \in M(S)_l,
\]
and
\[
(n_1, \ldots, n_k, m_1, \ldots, m_l; \gamma_1, \ldots, \gamma_k, \beta_1, \ldots, \beta_l) \in M(S)_{k+l}.
\]
As in [LW], define, for $n \geq 0$, a map
\[
\tau : \mathbb{Z}^n \rightarrow \mathbb{Z}^n
\]
(A.3)
\[
(i_1, \ldots, i_n) \mapsto (i_1 + \cdots + i_n, i_1 + \cdots + i_n, \ldots, i_n).
\]
For any $b = (n_1, \ldots, n_k; \beta_1, \ldots, \beta_k) \in M(S)_k$ and $i \in \mathbb{Z}$, we write
\[
b \leq i \text{ if } \tau(n_1, \ldots, n_k) \leq (i, \ldots, i).
\]
In other words, we have
\[
n_1 + \cdots + n_k \leq i,
\]
\[
n_2 + \cdots + n_k \leq i,
\]
\[
\vdots
\]
\[
n_k \leq i.
\]

The set $\text{Map}(M(S), \mathbb{C})$ of all functions
\[
f : M(S) \rightarrow \mathbb{C}
\]
has the structure of an algebra given by taking the identity element to be the function which is 1 on $M(S)_0$ and 0 elsewhere, and by setting
\[
(r \mu)(a) = r(\mu(a)),
\]
\[
(\mu_1 + \mu_2)(a) = \mu_1(a) + \mu_2(a),
\]
and
\[
(\mu_1 \mu_2)(a) = \sum_{a=loc} \mu_1(b) \mu_2(c)
\]
for $r \in \mathbb{C}$, $\mu, \mu_1, \mu_2 \in \text{Map}(M(S), \mathbb{C})$, and $a \in M(S)$. As in [LW], for each $\mu \in \text{Map}(M(S), \mathbb{C})$ and $i \in \mathbb{Z}$, we define sets
\[
\text{Supp}(\mu) = \{a \in M(S) | \mu(a) \neq 0\}
\]
and
\[
\text{Supp}_i(\mu) = \{a \in M(\Delta_+) | a \leq i\} \cap \text{Supp}(\mu).
\]
Note that if \( i \leq j \) then
\[
\text{Supp}_i(\mu) \subset \text{Supp}_j(\mu)
\]
and that
\[
\text{Supp}(\mu) = \cup_{i \in \mathbb{Z}} \text{Supp}_i(\mu).
\]
Define \( F(S) \subset \text{Map}(M(S), \mathbb{C}) \) by
\[
F(S) := \{ \mu : M(S) \to \mathbb{C} \mid \text{Supp}_i(\mu) \text{ is finite for all } i \in \mathbb{Z} \}
\]
and \( F_0(S) \subset F(S) \) by
\[
F_0(S) := \{ \mu \in F(S) \mid \text{Supp}(\mu) \text{ is finite} \}.
\]
We have:

**Proposition A.1** ([LW], Proposition 4.2) The set \( F(S) \) is a subalgebra of \( \text{Map}(M(S), \mathbb{C}) \), and \( F_0(S) \subset F(S) \) is a subalgebra of \( F(S) \). Moreover, \( F_0(S) \) is the free algebra on \( \mathbb{Z} \times S \).

For each \( a \in M(S) \), define maps \( X(a) \in F_0(S) \) by
\[
X(a)(b) = \delta_{a,b}.
\]
In particular, for \((n; \beta) \in M(S)_1\), write
\[
X_\beta(n) = X((n; \beta))
\]
and extend this so that for any \( a = (n_1, \ldots, n_k; \beta_1, \ldots, \beta_k) \in M(S) \)
\[
X(a) = X_{\beta_1}(n_1) \ldots X_{\beta_k}(n_k).
\]
For any \( \mu \in \text{Map}(M(S), \mathbb{C}) \), we may write
\[
\mu = \sum_{a \in \text{Supp}(\mu)} \mu(a)X(a).
\]
Consider the ideal \( I_S \) of \( F_0(S) \) generated by
\[
[X_\alpha(n), X_\beta(m)] - C_{\alpha,\beta}X_\alpha+\beta(m+n)
\]
for \( \alpha, \beta \in S \) and \( m, n \in \mathbb{Z} \), where \( C_{\alpha,\beta} \) are the structure constants given by (A.1).

**Proposition A.2** We have \( U(\tilde{n}_S) \simeq F_0(S)/I_S \).

**Proof:** Let \( T(\tilde{n}_S) \) denote the tensor algebra over \( \tilde{n}_S \). Let \( \phi \) be the bijection
\[
\phi : \mathbb{Z} \times S \to \tilde{n}_S\quad (n, \beta) \mapsto x_\beta(n).
\]
Since $F_0(S)$ is the free algebra on $\mathbb{Z} \times \Delta_+$ and $T(\bar{n}_S)$ is the free algebra on $\bar{n}_S$, we extend $\phi$ to a map of free algebras

$$
\phi : F_0(S) \rightarrow T(\bar{n}_S) \tag{A.5}
$$

$$
X_{\beta_1}(n_1) \cdots X_{\beta_k}(n_k) \mapsto x_{\beta_1}(n_1) \cdots x_{\beta_k}(n_k),
$$

extended linearly to all of $F_0(S)$. The fact that $\phi$ is an algebra isomorphism is clear. The proposition follows immediately.

We now impose similar natural relations on $F(S)$. Consider the ideal $\tilde{I}_S$ of $F(S)$ generated by

$$
[X_\alpha(n), X_\beta(m)] - C_{\alpha,\beta}X_{\alpha+\beta}(m+n)
$$

for $\alpha, \beta \in S$ and $m, n \in \mathbb{Z}$, where $C_{\alpha,\beta}$ are the structure constants (A.1).

**Definition A.3** Define the completion of $U(\bar{n}_S)$ by:

$$
\widehat{U(\bar{n}_S)} := F(S)/\tilde{I}_S. \tag{A.6}
$$

Denote by $[\mu]$ the coset of $\mu \in F(S)$ in $\widehat{U(\bar{n}_S)}$.

We now introduce some important substructures of $\widehat{U(\bar{n}_S)}$ and prove some useful facts about these substructures. Let

$$
M(S)_- = \{(m_1, \ldots, m_k; \beta_1, \ldots, \beta_k) \in M(S) | k \in \mathbb{N}, m_i \leq -1 \text{ for each } i = 1, \ldots, k\}.
$$

Define

$$
\widehat{U(\bar{n}_S_-)} = \{a \in \widehat{U(\bar{n}_S)} \mid a = [\mu] \text{ for some } \mu \in F(S) \text{ with } \text{Supp}(\mu) \subset M(S)_-\}.
$$

**Lemma A.4** We have that

$$
\widehat{U(\bar{n}_S_-)} \simeq U(\bar{n}_S_-).
$$

**Proof:** Suppose $[\mu] \in \widehat{U(\bar{n}_S_-)}$ for some $\mu$ with $\text{Supp}(\mu) \subset M(S)_-$. We may write

$$
\mu = \sum_{a \in \text{Supp}(\mu)} \mu(a)X(a)
$$

and so

$$
[\mu] = \sum_{a \in \text{Supp}(\mu)} [\mu(a)X(a)].
$$

By definition, $\text{Supp}_-(\mu)$ is finite, so that there are finitely many

$$
a = (m_1, \ldots, m_n; \beta_1, \ldots, \beta_n) \in \text{Supp}(\mu), \ k \in \mathbb{N}
$$
such that
\[ m_1 + \cdots + m_k \leq -1 \]
\[ m_2 + \cdots + m_k \leq -1 \]
\[ \vdots \]
\[ m_k \leq -1. \]
Since each such \( m_i \leq -1, \ i = 1, \ldots k, \) have have that \( \text{Supp}_n(\mu) = \text{Supp}_{-1}(\mu) \) for all \( n \geq 0. \) In particular, we have that
\[ \text{Supp}(\mu) = \bigcup_{n \in \mathbb{Z}} \text{Supp}_n(\mu) = \text{Supp}_{-1}(\mu) \]
and so \( \text{Supp}(\mu) \) is finite and \( \mu \in F_0(S). \) By the proof of Proposition A.2, we have that
\[ \widetilde{U}(\bar{n}_S) \simeq U(\bar{n}_S), \]
concluding our proof.

Let
\[ M(S)_+ = \{(m_1, \ldots, m_k; \beta_1, \ldots, \beta_k) \in M(S) \mid k \geq 1 \text{ and there exists } i \leq k \text{ with } m_i + \cdots + m_k \geq 0\} \]
We define
\[ U(\bar{n}_S)\bar{n}_S^+ = \{a \in U(\bar{n}_S) \mid a = [\mu] \text{ for some } \mu \in F(S) \text{ with } \text{Supp}(\mu) \subset M(S)_+\}. \]

**Remark A.5** The space \( U(\bar{n}_S)\bar{n}_S^+ \) is the collection of all elements of \( \widetilde{U}(\bar{n}_S) \) which have at least one representation as an “infinite sum” of elements of \( U(\bar{n}_S)\bar{n}_S^+. \) Indeed, any element \( X(a) \in U(\bar{n}_S) \) with \( a \in M(S)_+ \) can be written as
\[ X(a) = X(b)X(c), \]
where \( i \leq k \) and
\[ b = (m_1, \ldots, m_{i-1}; \beta_1, \ldots, \beta_{i-1}), \]
\[ c = (m_i, \ldots, m_k; \beta_i, \ldots, \beta_k), \]
and \( m_i + \cdots + m_k \geq 0. \) By (A.2), \( X(c) \in U(\bar{n}_S)\bar{n}_S^+ \) and \( X(b) \in U(\bar{n}_S), \) and so \( X(a) \in U(\bar{n}_S)\bar{n}_S^+. \)

**Proposition A.6** The space \( U(\bar{n}_S) \) has the decomposition
\[ U(\bar{n}_S) = U(\bar{n}_S^-) \oplus U(\bar{n}_S)\bar{n}_S^+ \quad \text{(A.7)} \]
Proof: Given any $u \in U(\tilde{n}_S)$, using (A.2) we may write

$$u = u_1 + u_2$$

where $u_1 \in U(\tilde{n}_{S-})$ and $u_2 \in U(\tilde{n}_S)\tilde{n}_{S+}$. Suppose $[\mu] \in \widehat{U}(\tilde{n}_S)$ for some $\mu \in F(S)$. Writing

$$\mu = \sum_{a \in \text{Supp}(\mu)} \mu(a)X(a),$$

we have

$$[\mu] = \sum_{a \in \text{Supp}(\mu)} [\mu(a)X(a)]$$

and each $[\mu(a)X(a)] \in U(\tilde{n}_S)$. Since $\mu \in F(S)$, there are only finitely many $a \in \text{Supp}(\mu)$ such that $a \in \text{Supp}_{-1}(\mu)$, so that, ranging over all $k \in \mathbb{Z}$, there are only finitely many $a = (m_1, \ldots, m_k; \beta_1, \ldots, \beta_k)$ with

$$m_1 + \cdots + m_k \leq -1$$
$$m_2 + \cdots + m_k \leq -1$$
$$\vdots$$
$$m_k \leq -1.$$

For these finitely many $a \in \text{Supp}_{-1}(\mu)$, we write

$$[\mu(a)X(a)] = [\mu_{1,a}] + [\mu_{2,a}]$$

for some $[\mu_{1,a}] \in U(\tilde{n}_{S-})$ and $[\mu_{2,a}] \in U(\tilde{n})\tilde{n}_{S+}$. By definition of $U(\tilde{n}_{S-})$, we have that

$$\sum_{a \in \text{Supp}_{-1}(\mu)} [\mu_{1,a}] \in U(\tilde{n}_{S-})$$

since the sum is finite, and

$$\sum_{a \in \text{Supp}_{-1}(\mu)} [\mu_{2,a}] + \sum_{a \in \text{Supp}(\mu) \setminus \text{Supp}_{-1}(\mu)} [\mu(a)X(a)] \in U(\tilde{n})\tilde{n}_{S+},$$

since

$$\text{Supp}(\mu) \setminus \text{Supp}_{-1}(\mu) \subset M(S)_+.$$ 

This shows $[\mu] \in U(\tilde{n}_{S-}) + U(\tilde{n})\tilde{n}_{S+}$. The fact that $U(\tilde{n}_{S-}) \cap U(\tilde{n})\tilde{n}_{S+} = 0$ follows from the fact that $U(\tilde{n}_{S-}) \cap U(\tilde{n})\tilde{n}_{S+} = 0$, proving our proposition.


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