Asymptotics of Solution to the Nonstationary Schrödinger Equation

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\textbf{Abstract.} The Cauchy problem with a rapidly oscillating initial condition for the homogeneous Schrödinger equation was studied in [5]. Continuing the research ideas of this work and [3], in this paper we construct the asymptotic solution to the following mixed problem for the nonstationary Schrödinger equation:

\begin{align*}
L_0u & \equiv \partial_t u + \hbar^2 \partial_x^2 u - b(x,t)u = f(x,t), \quad (x,t) \in \Omega = (0,1) \times (0,T], \\
\left. u \right|_{t=0} &= g(x), \quad \left. u \right|_{t=0} = \left. u \right|_{t=1} = 0,
\end{align*}

where $h > 0$ is a Planck constant, $u = u(x,t,h)$, $b(x,t), f(x,t) \in C^\infty(\Omega)$, $g(x) \in C^\infty[0,1]$ are given functions.

The similar problem was studied in [7, 8] when the Plank constant is absent in the first term of the equation and asymptotics of solution of any order with respect to a parameter was constructed. In this paper, we use a generalization of the method used in [7].

1. Regularization of the Problem

For regularizations of the problem (1), we will introduce the following regulating variables

\begin{align*}
\tau_1 &= \frac{t}{\hbar^2}, \quad \tau_2 = \frac{is(x,t)}{\hbar}, \quad \xi_1 = \frac{x}{\sqrt{\hbar}}, \quad \xi_2 = \frac{1-x}{\sqrt{\hbar}}, \\
\eta_1 &= \frac{x}{\sqrt{\hbar}^3}, \quad \eta_2 = \frac{1-x}{\sqrt{\hbar}^3},
\end{align*}

where the existence of a smooth solution of the problem is assumed:

\begin{equation}
\partial_t s(x,t) - (\partial_x s(x,t))^2 - b(x,t) = 0, \quad s(x,t)|_{t=0} = 0.
\end{equation}

Instead of the desired function $u(x,t,h)$ we study the extended function $\tilde{u}(M,h)$, $M = (x,t,\xi, \eta, \tau)$, $\xi = (\xi_1, \xi_2)$, $\eta = (\eta_1, \eta_2)$, $\tau = (\tau_1, \tau_2)$ such that its constriction by regularizing variables coincides with the desired solution:

\begin{equation}
\tilde{u}(M,h)|_{x=\psi(x,t,h)} \equiv u(x,t,h),
\end{equation}

where $\chi = (\xi, \eta, \tau)$, $\psi(x,t,\eta) = (\frac{x}{\sqrt{\hbar}}, \frac{1-x}{\sqrt{\hbar}}, \frac{1-x}{\sqrt{\hbar}}, \frac{\sqrt{\hbar}i}{\hbar}s(x,t))$.

Using (2), from (3) we find

\begin{equation*}
\partial_t u \equiv (\partial_t \tilde{u} + \frac{1}{\hbar^2} \partial^2_{x_2} \tilde{u} + \frac{i\partial_x s(x,t)}{\hbar} \partial_{x_2} \tilde{u})|_{x=\psi(x,t,\eta)},
\end{equation*}

\begin{thebibliography}{10}
\bibitem{1} Regularization of the Problem
\bibitem{2} Asymptotics of Solution to the Nonstationary Schrödinger Equation
\bibitem{3} Asymptotics of Solution to the Nonstationary Schrödinger Equation
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\bibitem{5} Asymptotics of Solution to the Nonstationary Schrödinger Equation
\bibitem{6} Asymptotics of Solution to the Nonstationary Schrödinger Equation
\bibitem{7} Asymptotics of Solution to the Nonstationary Schrödinger Equation
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Email addresses: asan.omuraliev@manas.edu.kg (Asan Omuraliev), peyi1.esengul@manas.edu.kg (Peil Esengul Kyzy)
\[ \partial_t u \equiv (\partial_t \bar{u} + \frac{1}{\sqrt{t}} \sum_{l=1}^{2} \left[ \partial_{\xi} \bar{u} \right] + \frac{i \partial_{\eta} s}{h} \partial_{\xi} \bar{u}] )|_{\tau = \psi(x,t,\eta)}, \]

\[
\partial_{t}^{2} u \equiv \left[ \partial_{t} \bar{u} + \frac{1}{h} \sum_{l=1}^{2} \left[ \partial_{\xi} \bar{u} + \frac{1}{h^2} \partial_{\xi} \bar{u} \right] + \frac{1}{\sqrt{h}} \bar{L}_{\xi} \bar{u} + \frac{1}{\sqrt{h}} \bar{L}_{\eta} \bar{u} + \right. \\
\left. + \left( \frac{i \partial_{\eta} s(x,t)}{h} \right) \partial_{\xi}^{2} \bar{u} + \frac{i}{h} \left( 2 \partial_{\xi} s \partial_{\tau_{2}} \bar{u} + \partial_{\xi}^{2} \partial_{\tau_{2}} \bar{u} \right) \right]|_{\tau = \psi(x,t,\eta)},
\]

(4)

On the basis of (1), (3), (4) for the extended function \( \bar{u}(M, h) \), we set the problem as:

\[
\bar{L}_{\eta} \bar{u} \equiv \frac{1}{h} T_{1} \bar{u} + D \bar{u} + \sqrt{h} L_{\xi} \bar{u} + h T_{2} \bar{u} + h \sqrt{h} L_{\xi} \bar{u} + h^{2} \partial_{\xi}^{2} \bar{u} = f(x,t), \quad M \in Q,
\]

(5)

where \( T_{1} \equiv \partial_{\eta} + \sum_{l=1}^{2} \partial_{\xi_{l}} \), \( T_{2} \equiv \partial_{\xi} + \sum_{l=1}^{2} \partial_{\xi_{l}} \), \( D \equiv -\partial_{\eta} s \partial_{\tau_{2}} + (\partial_{\xi} s)^{2} \partial_{\tau_{2}} + b(x,t) \). The following identity holds:

\[
(L_{\eta} \bar{u}(M,h))|_{\tau = \psi(x,t,\eta)} \equiv u(x,t,h).
\]

(6)

The solution of problem (5) is determined in the form of the following series

\[
u(M, h) = \sum_{k=0}^{\infty} h^{k/2} u_{k}(M).
\]

(7)

For the coefficients of this series, we obtain the following iterative problems:

\[
T_{1} u_{0}(M) = 0, v = 0,1, T_{1} u_{2}(M) = f(x,t) - D u_{0}(M),
\]

\[
T_{1} u_{k}(M) = -D u_{k-2} - L_{\tau_{2}} u_{k-3} - T_{2} u_{k-4} - L_{\xi} u_{k-5} - \partial_{\tau_{2}}^{2} u_{k-6}, \quad k \geq 3,
\]

\[
u_{0}(M)|_{\tau = \tau_{1} = \tau_{2} = 0} = g(x), u_{2}(M)|_{\tau = \tau_{1} = \tau_{2} = 0} = 0, u_{k}|_{\tau = 0, \xi = \eta = 0} = u_{k}|_{\tau = 1, \xi = \eta = 0} = 0.
\]

(8)

2. Solution of Iteration Problems

We introduce classes of functions in which the iterative problems are solved:

\[
U_{1} = \left\{ u^{1}_{1}(M) : u^{1} = v(x,t) + c(x,t) \exp(\tau_{2}) + \sum_{l=1}^{2} \omega^{l}(x,t) \text{erf}(\frac{\xi_{l}}{2\sqrt{h}}) \exp(\tau_{2}) \right\},
\]

\[
U_{2} = \left\{ u^{2}_{1}(M) : u^{2} = \sum_{l=1}^{2} Y^{l}(\eta), \quad N_{i} = (x,t,\tau_{1},\eta), \quad Y^{l}(\eta) \sim \exp\left(-\frac{\eta^{2}}{4\tau^{2}_{1}}\right), \quad \forall \eta, \tau_{1} \in (0,\infty) \right\}.
\]

From these spaces we construct a new space:

\[
U = U_{1} \oplus U_{2};
\]

then the function \( u_{k}(M) \in U \) has the form

\[
u_{4}(M) = v_{4}(x,t) + \sum_{l=1}^{2} Y^{l}(\eta) + [c(x,t) + \sum_{l=1}^{2} \omega^{l}(x,t) \text{erf}(\frac{\xi}{2\sqrt{h}})] \exp(\tau_{2}), \quad k \geq 0.
\]

(9)
Theorem 2.1. If the given functions are smooth, the problem (2) has a smooth solution and the right-hand side of the equation

$$T_1 u_k(M) = H_k(M)$$

belongs to $U_2$, then the equation (10) is solvable in $U$.

Proof. We substitute the function $u_k(M) \in U$ from (9) into (10); then, with respect to $Y_k^l(N_i)$, we obtain the equation

$$T_1 Y_k^l(N_i) = H_k(M), \quad T_1 \equiv i\partial_{\tau} - \partial^2_{\eta^l}. $$

Since the right-hand side of $H_k(M) \in U_2$, this equation, with the appropriate boundary conditions, has a solution of the form

$$Y_k^l(N_i) = d_k^l(x, t) \text{erfc}(\frac{\eta}{2\sqrt{\tau}}) + \frac{2}{\sqrt{\tau}} \int_0^\infty \int_0^\infty \frac{H_k^l(t)}{\sqrt{\tau_1 - \tau}} \left[ \exp\left(-\frac{(\eta - s)^2}{4i(\tau_1 - \tau)}\right) - \exp\left(-\frac{(\eta + s)^2}{4i(\tau_1 - \tau)}\right) \right] ds d\tau $$

The theorem is proved. □

Theorem 2.2. Let the conditions of Theorem 2.1 hold. Then equation (10) under additional conditions

1) $u_k(M)|_{x=x_{l-1}=0} = g(x), u_k(M)|_{x=x_l\in\mathbb{R}, \eta=0} = 0$, $l = 1, 2,$

2) $H(M) \equiv -D u_{k-2} - L_\eta u_{k-3} - T_2 u_{k-4} - L_\xi u_{k-5} - \partial^2_{\xi} u_{k-6} \in U_2,$

3) $L_\eta u_k = 0, \quad L_\xi u_k = 0$

has a unique solution.

Proof. By Theorem 2.1, equation (10) has solutions $u_k(M) \in U$. Since the function $u_k(M)$ satisfies conditions 1), we obtain

$$Y_k^l(N_i)|_{x=x_{l-1}=0} = 0, \quad Y_k^l(N_i)|_{\eta=0} = d_k^l(x, t),$$

$$d_k^l(x, t)|_{x=x_{l-1}} = -v_k(l-1, t), \quad d_k^l(x, t)|_{x=x_{l}} = d_k^l(x),$$

$$\omega_k^l(x, t)|_{x=x_{l-1}} = \omega_k^{l,0}(x), \quad \omega_k^l(x, t)|_{x=x_{l}} = -c_k(l-1, t), \quad l = 1, 2. $$

There $d_k^l(x), \omega_k^{l,0}(x)$ are arbitrary functions.

We calculate the actions of the operators $D, L_\eta, T_2, L_\xi, \partial^2_{\xi}$ on the function $u_k(M) \in U$ with allowance for (2), and we obtain

$$Du_{k-2}(M) = b(x, t) Y_{k-2}^l + b(x, t) v_{k-2}(x, t),$$

$$L_\eta u_{k-3}(M) = 2 \sum_{l=1}^{2} (-1)^{l-1} \partial^2_{\eta^l} Y_{k-3}^l,$$

$$T_2 u_{k-4}(M) = i\partial_{\tau} v_{k-4} + i\partial_{\tau} Y_{k-4}^l + \sum_{l=1}^{2} i\partial_{\tau} \omega_k^{l,4}(x, t) \text{erfc}(\frac{\xi_l}{2\sqrt{\tau}}) + i\partial_{\tau} c_k(x, t) \exp(\tau_2),$$

$$L_\xi u_{k-5}(M) = 2 \sum_{l=1}^{2} (-1)^{l-1} \partial_{\xi} \omega_k^{l-5}(x, t) \partial_{\xi} \text{erfc}(\frac{\xi_l}{2\sqrt{\tau}}),$$

$$L_x u_{k-6}(M) = \partial_{x}^2 v_{k-6}(x, t) + \sum_{l=1}^{2} \partial_{x}^2 \omega_k^{l-6}(x, t) \text{erfc}(\frac{\xi_l}{2\sqrt{\tau}}).$$
Using these relations and ensuring condition 2), we set

\[ L_1 u_{k-5}(M) = 0, \quad L_2 u_{k-3}(M) = 0, \]

\[ b(x, t) \nu_0(x, t) + i \partial_t \nu_{-4}(x, t) + \partial_x^2 \nu_{-6} = 0, \quad \partial_t u_{k-5}(x, t) = 0, \]

\[ i \partial_t c_{k-4}(x, t) + \partial_x^2 c_{k-6}(x, t) = 0, \quad i \partial_t \alpha_{k-4}(x, t) + \partial_x^2 \alpha_{k-6}(x, t) = 0. \]

With such a choice of the functions entering into the function \( u_k(M) \), equation (10) takes the form

\[ T_{1k} Y_k^l(N_i) = b(x, t) Y_{k-2}^l(N_i), \]

of the solution, which, under the boundary conditions from (11), can be written in the form

\[ Y_k^l(N_i) = d_k(x, t) \text{erfc} \left( \frac{\eta t}{2 \sqrt{\tau_1}} \right) + \frac{1}{2 \sqrt{	au_1}} \int_0^{\tau_1} \int_0^\infty b(x, t) Y_{k-2}^l(\cdot) \left[ \exp(-\frac{(\eta t - s)^2}{4(\tau_1 - \tau)}) - \exp(-\frac{(\eta t + s)^2}{4(\tau_1 - \tau)}) \right] ds dt. \]  

(13)

The function \( d_k^l(x, t) \) stands with the factor of the function \( \text{erfc} \left( \frac{\eta t}{2 \sqrt{\tau_1}} \right) \). Since \( \text{erfc} \left( \frac{\eta t}{2 \sqrt{\tau_1}} \right) \big|_{\tau_1=0} = 0 \) is the value of the function \( d_k^l(x, t) \) for \( t = 0 \) arbitrarily chosen and this arbitrary function ensures the condition \( L_7 Y_{k-3}^l(N_i) = 0 \). The initial condition for this equation is determined from the relation

\[ Y_{k-3}^l(N_i)|_{t=\tau_1, \eta=0} = d_k^l(x, t)|_{t=0} = -v_{k-3}(l - 1, t). \]

Thus the function \( Y_k^l(N_i) \) is uniquely defined. Solving equations (12) with the corresponding initial conditions from (11). The function \( \alpha_k^l(x, t) \) is expressed in terms of an arbitrary function \( \alpha_k^0(x) \), which ensures the condition \( L_7 u_k(M) = 0 \). This uniquely determines all functions occurring in \( u_k(M) \) from (9). The theorem is proved.

We solve the iterative problems (8) in the class of functions \( II \). By Theorem 2.1, problem (8) for \( k = 0, 1 \) has a solution of the form (9) if the function \( Y_k^l(N_i) \) is a solution of equation

\[ i \partial_t Y_k^l = \partial_x^2 Y_k^l, \quad \nu = 0, 1 \]

for initial and boundary conditions in (8):

\[ Y_k^l(N_i)|_{t=0} = Y_k^l(N_i)|_{\eta=0} = d_k^l(x, t) = -v_k(l - 1, t), \quad d_k^l(x, t)|_{t=0} = d_k^0(x), \]

\[ c_0(x, 0) = g(x) - v_0(x, 0), \quad \alpha_k^l(x, t)|_{t=0} = \alpha_k^0(x), \quad c_1(x, 0) = v_1(x, 0), \]

\[ \alpha_k^l(x, t)|_{t=\tau_1, \eta=0} = -c_0(l - 1, t). \]

(15)

The solution of equation (13) with boundary conditions (14) has the form

\[ Y_k^l(N_i) = d_k^l(x, t) \text{erfc} \left( \frac{\eta t}{2 \sqrt{\tau_1}} \right). \]

(16)

For \( \tau_1 = 0 \), we have \( \text{erfc} \left( \frac{\eta t}{2 \sqrt{\tau_1}} \right) = 0 \); therefore, by its factor we chose an arbitrary function \( d_k^l(x, t) \) and the function \( \alpha_k^0(x) \) is taken as the value for \( t = 0 \). Following Theorem 2.2, this function will be used to make zero \( L_7 u_k(M) = 0 \). We substitute (14) into the equation for \( Y_k^l(N_i) \) from (12); then, with respect to \( d_k^l(x, t) \), we obtain equation

\[ \partial_t d_k^l(x, t) + \partial_x^2 d_k^l(x, t) = 0. \]

Solving it under the initial condition \( d_k^l(x, t)|_{t=0} = d_k^0(x) \), we define

\[ d_k^l(x, t) = d_k^0(x) + \nu_k^l(x, t). \]  

(17)
Now substitute in \( L_\eta u_b(M) \), then taking into account (17) with respect to \( d_{k-4}^{(0)}(\alpha) \), we obtain a differential equation. The initial condition for it is determined from the relation with respect to \( Y_2'(N_l) \) is the one entering into (14)

\[
d_{k-4}(x,t)|_{l=1} = (d_{k-4}^{(0)}(\alpha) + P_{k-4}(x,t))|_{l=1} = -v_{k-4}(l-1, t).
\]  

(18)

Thus the function \( Y_2'(N_l) \) is uniquely defined. Consider equation (8) for \( k = 2 \). Assuring solvability in \( U \), according to Theorem 2.1, we require condition

\[
F_2(M) = f(x,t) - Du_0 \in U_2;
\]

then equation (8), \( k = 2 \) is solvable if \( Y_2'(N_l) \) and is a solution of the equation

\[
i\partial_t Y_2' = d_{0}^2 Y_2' + F_2(N_l).
\]

Providing condition (19), following Theorem 2.2, we obtain

\[
b(x,t)v_0(x,t) = -f(x,t);
\]

the right-hand side is rewritten as

\[
F_2(N_l) = -b(x,t)Y_0'(N_l).
\]

Equation (20) has the solution of the form (13) under the appropriate conditions from (14). In the next step, the right-hand side of equation (8), with \( k = 3 \), has the form

\[
F_3(M) = -Du_1 - L_\eta u_0.
\]

According to Theorems 2.1 and 2.2, we get

\[
L_\eta u_0 = 2 \sum_{l=1}^{2} (-1)^{l-1} \partial_x d_{0}^{(0)}(x) \partial_y \left( \text{erf}(\frac{\eta}{2\sqrt{t_1}}) \right) = 0, \text{ or } (d_{0}^{(0)}(x))' = 0
\]

\[
v_1(x,t) = 0.
\]

Whence we determine

\[
d_{0}^{(0)}(x) = -v_0(l-1, t),
\]

the value of \( d \) is determined in the next step from the problem

\[
\partial_{x}d_{0}^{(0)}(x, t) = 0, \quad d_{0}^{(0)}(x, t)|_{l=0} = d_{0}^{(0)}(x).
\]

Notice that the function \( u_b(M) \) with odd indices vanishes. Indeed, the free term of the next iteration equation for \( k = 4 \) has the form

\[
F_4(M) = -Du_2 - L_\eta u_1 - T_1 u_0.
\]

By Theorems 2.1 and 2.2, this equation has a solution in \( U \) if

\[
-b(x,t)v_2(x,t) = \partial_{x}v_0(x,t),
\]

\[
\partial_{x}d_{0}^{(0)}(x, t) = 0, \quad d_{0}^{(0)}(x, t)|_{l=0} = d_{0}^{(0)}(x),
\]

\[
(d_1^{(1)}(x,t)|_{l=1} = -v_1(l-1, t), \quad d_{1}^{(1)}(x,t)|_{l=0} = d_{1}^{(0)}(x).
\]

\[
\partial_{x}a_{0}^{(1)}(x,t) = 0, \quad a_{0}^{(1)}(x,t)|_{l=0} = a_{0}^{(0)}(x), \quad \partial_{x}c_0(x,t) = 0, \quad c_0(x,t)|_{l=0} = g(x) - v_0(x,0),
\]

\[
a_{0}^{(2)}(x,t)|_{l=1} = -c_0(l-1, t).
\]

Taking into account that \( v_2(x,t) = 0 \), we find \( d_{1}^{(1)}(x,t) = 0 \), and from the remaining problems we define \( v_2(x,t), a_{0}^{(1)}(x,t), c_0(x,t) \). Further, repeating this process, we successively determine all the coefficients of the partial sum.
Lemma 2.3. For the function
\[ \text{erfc}(\frac{\xi}{\sqrt{4it}}) = \frac{2}{\sqrt{\pi}} \int_{\frac{\xi}{\sqrt{4it}}}^{\infty} e^{-s^2} ds \]
it holds
\[ \text{erfc}(\frac{\xi}{\sqrt{4it}}) < c \exp(-\frac{\xi^2}{4it}). \]

Proof. We make the change of variables \( s = y + \frac{\xi}{\sqrt{4it}}, \) \( dy = ds, \) and considering that \( \frac{1}{\sqrt{i}} = \frac{2}{\sqrt{2}} (1 - i) \) we get
\[
erfc(\frac{\xi}{\sqrt{4it}}) = \frac{2}{\sqrt{\pi}} \int_{0}^{\infty} e^{-y^2 - \frac{\xi^2}{\sqrt{4it}}(1 - iy)} dy = \]
\[
= \frac{2}{\sqrt{\pi}} \exp(-\frac{\xi^2}{4it}) \int_{0}^{\infty} e^{-y^2 - \sqrt{\frac{2}{i}} \xi y + \sqrt{\frac{2}{i}} \xi y} dy = \]
\[
= \frac{2}{\sqrt{\pi}} \exp(-\frac{\xi^2}{4it}) \int_{0}^{\infty} e^{-y^2} \left[ \cos(\sqrt{\frac{2}{i}} \xi y) + \sin(\sqrt{\frac{2}{i}} \xi y) \right] dy.
\]
Using Hölder’s inequality we have
\[
erfc(\frac{\xi}{\sqrt{4it}}) \leq \frac{2}{\sqrt{\pi}} \exp(-\frac{\xi^2}{4it}) \left( \int_{0}^{\infty} \left| \exp(-y^2 - \sqrt{\frac{2}{i}} \xi y) \right| dy \right)^\frac{1}{2} \]
\[
\times \left( \int_{0}^{\infty} \left| \exp(-y^2 - \sqrt{\frac{2}{i}} \xi y) \right| \left| \cos(\sqrt{\frac{2}{i}} \xi y) + \sin(\sqrt{\frac{2}{i}} \xi y) \right|^2 dy \right)^\frac{1}{2} = \]
\[
= \frac{2}{\sqrt{\pi}} \exp(-\frac{\xi^2}{4it}) \int_{0}^{\infty} e^{-y^2} \sqrt{\frac{2}{i}} \xi y dy.
\]
Replacing the integral by the formula 7.4.2 of [1], we find
\[
erfc(\frac{\xi}{\sqrt{4it}}) \leq \frac{2}{\sqrt{\pi}} \exp(-\frac{\xi^2}{4it}) \sqrt{\pi} \exp(\frac{\xi^2}{2t}) \int_{0}^{\infty} e^{-\frac{1}{2} s^2} ds.
\]
Using inequality 4 from §4.8.5 in [4], we obtain
\[
erfc(\frac{\xi}{\sqrt{4it}}) \leq \exp(-\frac{\xi^2}{4it}) \frac{1}{\sqrt[(\pi - 2)^2 + 1]} = c \exp(-\frac{\xi^2}{4it}).
\]
\[ \square \]

Lemma 2.4. Let
\[ F(\xi, t) \leq c \exp(-\frac{\xi^2}{4it}). \] (L − 1)

Then for the integral
Proof. Consider

\[ I(\xi, t) = \frac{2}{\sqrt{\pi}} \int_{0}^{\infty} \int_{0}^{\infty} \frac{F(s, \tau)}{\sqrt{t-\tau}} \left[ \exp\left(-\frac{(\xi-s)^2}{4it-\tau}\right) - \exp\left(-\frac{(\xi+s)^2}{4i(t-\tau)}\right) \right] ds\,d\tau = \]

we have

\[ I(\xi, t) \leq \exp\left(-\frac{\xi^2}{4it}\right). \quad (L \ - \ 3) \]

\[ \frac{4c}{\sqrt{\pi}} \int_{0}^{\infty} \int_{-\infty}^{\infty} F(\xi, \tau) \left[ \exp\left(-\frac{(\xi-s)^2}{4it-\tau}\right) - \exp\left(-\frac{(\xi+s)^2}{4i(t-\tau)}\right) \right] ds\,d\tau = \]

With regard to (L-3) we rewrite this as

\[ I(\xi, t) \leq \frac{4c}{\sqrt{\pi}} \int_{0}^{\infty} \int_{-\infty}^{\infty} F(\xi, \tau) e^{-z^2} \left[ \exp\left(-\frac{(\xi-2\sqrt{i(t-\tau)}z)^2}{4it}\right) \right] ds\,d\tau \]

\[ \leq \frac{4c}{\sqrt{\pi}} \int_{0}^{\infty} \int_{-\infty}^{\infty} \exp\left(-z^2 - \frac{(\xi-2\sqrt{i(t-\tau)}z)^2}{4it}\right) dz\,d\tau \]

Using the formula 3.323.3 from [2] we obtain

\[ I(\xi, t) \sim \frac{4c}{\sqrt{\pi}} \int_{0}^{\infty} \exp\left(-\frac{\xi^2}{4\tau}\right) \exp\left(-\frac{\xi^2}{4i\tau}\right) \frac{\sqrt{\pi}}{\sqrt{\tau}} \,d\tau = \]

\[ = \frac{4c}{\sqrt{\pi}} \int_{0}^{\infty} \exp\left(-\frac{\xi^2}{4\tau}\right) \exp\left(\frac{(t-\tau)\xi^2}{4rt\tau}\right) \frac{\sqrt{\tau}}{\sqrt{\tau}} \,d\tau = \]

\[ = \frac{4c}{\sqrt{\pi}} \int_{0}^{\infty} \exp\left(-\frac{\xi^2}{4\tau}\right) \exp\left(\frac{(t-\tau)\xi^2}{4rt\tau}\right) \frac{\sqrt{\tau}}{t} \,d\tau = \]
\[ c \left( \frac{\xi^2}{4t} \right) \int_0^t \sqrt{\tau} d\tau = \exp \left( \frac{-\xi^2}{4t} \right). \]

\[ u_{n,h}(M) = \sum_{k=0}^n h^k u_{2k}(M) \quad (L - 4) \]

Producing a restriction by means of the regularizing functions, on the basis of (6), for the remainder term

\[ R_n(x, t, h) = u(x, t, \epsilon) - u_{n,h}(M) \mid_{x = \psi(x, t, h)} \]

we obtain the problem

\[ L_n R_n = h^{n+1} g_{2n}(x, t, h), \quad R_n(x, t, h) \mid_{t = 0} = R_n(x, t, h) \mid_{t = 1} = 0, \]

where \( |g_{2n}(x, t, h)| < c \). Using the maximum principle and following [6]. We get the estimate

\[ |R_n(x, t, h)| < ch^{n+1}. \]

**Theorem 2.5.** Let the given functions be sufficiently smooth. Then the problem (1) has an asymptotic solution that is representable in the form (L-4) for \( \chi = \psi(x, t, \eta) \) and for all \( n = 0, 1, 2, ..., 0 < h < h_0 \) holds.

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