A Santaló-type Inequality for the $\mathcal{J}$ Transform

D.I. Florentin, A. Segal

Abstract

This paper deals with an analog of the Mahler volume product related to the $\mathcal{J}$ transform acting in the class of geometric convex functions $\text{Cvx}_0(\mathbb{R}^n)$. We provide asymptotically sharp bounds for the quantity $s^\mathcal{J}(f) = \int e^{-\mathcal{J}f} \int e^{-f}$ and characterize all the extremal functions.

1 Introduction

The classical Blaschke-Santaló inequality states that the Mahler volume, an affine invariant functional given by

$$s(K) = \inf \left\{ \text{Vol}_n(K - x) \cdot \text{Vol}_n((K - x)^\circ) \mid x \in K \right\},$$

is maximized by ellipsoids. Here $K^\circ$ stands for the dual body of $K$, namely

$$K^\circ = \left\{ y \in \mathbb{R}^n \mid \forall x \in K, \langle x, y \rangle \leq 1 \right\}.$$

Analysis of the Mahler volume dates back to 1917, when Blaschke proved that in dimensions $n = 2, 3$, every convex body $K$ satisfies $s(K) \leq s(B_n^2)$ (see [7]). Santaló [21] extended this inequality to every dimension (see [17] for a simple proof due to Meyer and Pajor). However, finding the minimal value of the Mahler volume remains an open problem to this day. Mahler conjectured that for centrally symmetric bodies, the Mahler volume attains its minimum value on cubes, i.e. $s(K) \geq s(B_n^\infty) = \frac{4^n}{n!}$ (it is now known that any Hanner polytope has the same Mahler volume as the cube). The Mahler conjecture would imply that the minimum and maximum of $s(K)$ differ only by a factor of $c^n$, a fact which was verified in the celebrated Bourgain-Milman theorem [8]. They proved that there exists a universal constant $c$ such that:

$$c \leq \left( \frac{s(K)}{s(B_n^2)} \right)^{\frac{1}{n}},$$

thus settling the Mahler conjecture in the asymptotic sense. In recent years, several new proofs of the Bourgain-Milman inequality have been discovered, by Kuperberg.
Nazarov, and by Giannopoulos, Paouris, and Vritsiou, Kuperberg setting the value of the largest known constant $c$ at $\frac{\pi}{4}$. In the general case (where the body $K$ is not assumed to be centrally symmetric), minimizers are conjectured to be centered simplices. Mahler solved the planar case, and the (centrally symmetric) three dimensional case has been settled recently by Iriyeh and Shibata (see also for a simplified proof).

Similar questions were considered for the class of convex functions in $\mathbb{R}^n$, which we denote by $\text{Cvx}(\mathbb{R}^n)$. The analog of volume of a convex function $\varphi$ is usually defined to be the integral $\int_{\mathbb{R}^n} e^{-\varphi}$, and the dual of $\varphi$ is obtained by applying the Legendre transform

$$(L\varphi)(y) = \sup_{x \in \mathbb{R}^n} (\langle x, y \rangle - \varphi(x)).$$

The following infimum over translations is a functional analogue of the Mahler volume.

$$s_\mathcal{L}(\varphi) = \inf_{a \in \mathbb{R}^n} \left\{ \int e^{-T_a\varphi} \int e^{-\mathcal{L}T_a\varphi} \right\}.$$

Here (and only here) $T_a\varphi(x) = \varphi(x - a)$. Ball proved in for even functions, and Artstein, Klartag, and Milman proved in for any function, that

$$s_\mathcal{L}(\varphi) \leq s_\mathcal{L} \left( \frac{\cdot}{2} \right) = (2\pi)^n. \quad (1)$$

The lower bound for $s_\mathcal{L}$ was established by Klartag and Milman in for even functions, and by Fradelizi and Meyer in for all functions, namely

$$c^n \leq s_\mathcal{L}(\varphi), \quad (2)$$

where $c > 0$ is some universal constant. For a large family of convex functions there is another choice of duality besides the Legendre transform, namely the polarity transform $A$, which first appeared in. Artstein and Milman proved in that in $\text{Cvx}_0(\mathbb{R}^n)$, the class of non-negative convex functions vanishing at 0, the only dualities (i.e. order reversing involutions) are the Legendre transform $\mathcal{L}$ and the polarity transform $A$, which may be defined by

$$(A\varphi)(x) = \sup_y \frac{\langle x, y \rangle - 1}{\varphi(y)}.$$

One may similarly consider the Mahler volume with respect to $A$:

$$s_A(\varphi) = \int e^{-\varphi} \int e^{-A\varphi}.$$

It was shown in for even functions, that

$$c^n (\text{Vol}(B^n_2))^2 \leq s_A(\varphi) \leq (\text{Vol}(B^n_2) n!)^2 \left( 1 + \frac{C}{n} \right). \quad (3)$$
Finding maximizers and minimizers of $s^A$ (and even proving their existence) remains an open problem. The composition $J = A\mathcal{L} = \mathcal{L}A$ of the two order reversing involutions $\mathcal{L}$ and $A$ is an order preserving involution (see [4] and [1] for more details on the $J$ transform). Since $J$ is order preserving, the product $\int e^{-\varphi} \int e^{-J\varphi}$ cannot be bounded from above, or bounded away from zero. Thus, it makes sense to consider the following quantity, whenever $\int e^{-\varphi}$ and $\int e^{-J\varphi}$ are both positive and finite:

$$s^J(\varphi) = \frac{\int e^{-J\varphi}}{\int e^{-\varphi}}.$$

To avoid trivial exceptions we use the convention $0 \otimes \infty = 1$, thus defining $s^J$ on the whole of $Cvx_0(\mathbb{R}^n)$ (as it is not hard to verify that $\int e^{-\varphi} = 0$ if and only if $\int e^{-J\varphi} = 0$, and similarly $\int e^{-\varphi} = \infty$ if and only if $\int e^{-J\varphi} = \infty$). The purpose of this note is the study of the functional $s^J$. In our first theorem we describe all maximizers and minimizers of $s^J$. To this end we use the following notation. The class of all compact convex sets in $\mathbb{R}^n$, with non empty interior, which contain the origin is denoted by $\mathcal{K}^n$. For any $K \in \mathcal{K}^n$ and $a > 0$, let

$$\psi_{K,a} = \max \{1^\infty_K, a\cdot ||\cdot||_K\},$$

where $||\cdot||_K$ is the Minkowski functional of the body $K$, and the convex indicator function $1^\infty_K$ is defined to be $0$ on $K$ and $+\infty$ otherwise. It turns out (see e.g. the proof of Theorem 1.1) that the value of $s^J(\psi_{K,a})$ depends only on $a$, and not on the body $K$. Thus in the following theorem, which describes all the maximizers of $s^J$, the Euclidean ball $B^n_2$ may be replaced by any $K \in \mathcal{K}^n$.

**Theorem 1.1.** For every $n \geq 1$ there exists $a_n > 0$ such that for every $\varphi \in Cvx_0(\mathbb{R}^n)$, we have

$$s^J(J\psi_{B^n_2,a_n}) \leq s^J(\varphi) \leq s^J(\psi_{B^n_2,a_n}).$$

Equality holds only if $\varphi = \psi_{K,a}$ for some $K \in \mathcal{K}^n$, $a > 0$. Moreover, $\lim_{n \to \infty} na_n = 1$.

Note that $J\psi_{K,a} = \frac{1}{a}K_{\frac{1}{a}}$ (see e.g. [4]). We will see in Lemma 2.4 below, that $s^J$ attains the same value on $\psi_{1K,a}$ and on $\psi_{K,a}$, thus Theorem 1.1 implies that

$$\psi_{K,a} \text{ is a maximizer } \iff \psi_{K,a} \text{ is a minimizer.}$$

Our second theorem provides asymptotically sharp bounds on the extremal values of $s^J$.

**Theorem 1.2.** There exist universal constants $c, C > 0$ such that for $n$ large enough:

$$\left(1 + \frac{C}{n}\right)n! \leq \max_{\varphi \in Cvx_0(\mathbb{R}^n)} \{s^J(\varphi)\} \leq \left(1 + \frac{C}{n}\right)n!.$$
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2 Preliminaries - basic properties of $s^J$

In this section we analyze the action of $J$ using properties of the point map which induces it, and prove that $J$ commutes with the action of symmetrizing a function. This allows us to apply the known bounds for $s^L$ and $s^A$ and show that $s^J$ is bounded (with a non optimal constant). More importantly, it sets the groundwork to reducing the problem of finding optimal bounds for $s^J$ (and extremal functions), to a certain two dimensional optimization problem. We begin by considering the function $F: \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^n \times \mathbb{R}^+$ given by

$$F(x, z) = \left(\frac{x}{z}, \frac{1}{z}\right).$$

The map $F$ is an involution that induces the $J$ transform (see [4]), in the following sense. Let $\text{epi}(\phi) = \{(x, z) \in \mathbb{R}^n \times \mathbb{R}^+ : \phi(x) < z\}$ denote the epi-graph of a function $\phi$. Then for any $\varphi \in \text{Cvx}_0(\mathbb{R}^n)$ we have:

$$\text{epi}(J\varphi) = F(\text{epi} \varphi),$$

For any $z \geq 0$, let $H_z = \{(x, z) \in \mathbb{R}^n \times \mathbb{R}^+ : \phi(x) < z\}$ denote the hyperplane at height $z$, and let $L_z(\varphi) = \{x \in \mathbb{R}^n : (x, z) \in \text{epi}(\varphi)\} \subseteq \mathbb{R}^n$ denote the corresponding slice of $\text{epi}(\varphi)$. Note that the map $F$, restricted to $H_z$, is simply a dilation by $\frac{1}{z}$. Consequently, we have the following simple relation between level sets of $\varphi$ and $J\varphi$:

$$L_{\frac{1}{z}}(J\varphi) = L_z(\varphi).$$

(4)

Consider the measures $\mu, \nu$ on $\mathbb{R}^n \times \mathbb{R}^+$ with densities $e^{-z}, e^{-\frac{1}{z}z-(n+2)}$ respectively. It was shown in [2] that for any $\varphi \in \text{Cvx}_0(\mathbb{R}^n)$,

$$\mu(\text{epi} \varphi) = \int_{\mathbb{R}^n} e^{-\varphi}, \quad \nu(\text{epi} \varphi) = \int_{\mathbb{R}^n} e^{-J\varphi}.$$

(6)
Since the densities of $\mu$ and $\nu$ only depend on $z$ (and not on $x$), and the volumes of $L_z(S_u\varphi)$ and $L_z(\varphi)$ are equal, we have
\[
\int_{\mathbb{R}^n} e^{-\varphi} = \int_{\mathbb{R}^n} e^{-S_u\varphi}, \quad \int_{\mathbb{R}^n} e^{-J\varphi} = \int_{\mathbb{R}^n} e^{-J S_u\varphi}.
\tag{7}
\]
Applying $n$ successive Steiner symmetrizations in directions $e_1, \ldots, e_n$ results in an unconditional function, i.e. a function satisfying $\eta(\pm x_1, \ldots, \pm x_n) = \eta(x_1, \ldots, x_n)$. In particular $\eta$ is even, so we can easily conclude that the ratio $s^J$ is bounded above and below. Recall that $J = \mathcal{L} \circ \mathcal{A}$, so we may write:
\[
s^J(\eta) = \frac{\int e^{-J \eta} \int e^{-A \eta}}{\int e^{-\eta} \int e^{-A \eta}} = \frac{s^\mathcal{L}(A \eta)}{s^\mathcal{A}(\eta)} = \frac{s^\mathcal{A}(\mathcal{L} \eta)}{s^\mathcal{A}(\eta)}.
\]
This implies
\[
\max \left\{ \frac{\min\{s^\mathcal{L}\}}{\max\{s^\mathcal{A}\}}, \frac{\min\{s^\mathcal{A}\}}{\max\{s^\mathcal{L}\}} \right\} \leq s^J(\eta) \leq \min \left\{ \frac{\max\{s^\mathcal{L}\}}{\min\{s^\mathcal{A}\}}, \frac{\max\{s^\mathcal{A}\}}{\min\{s^\mathcal{L}\}} \right\}.
\]
Using the bounds (1), (2), and (3) yields
\[
s^J(\eta) \leq C^n n!
\]
for some $C > 1$. However, these bounds are far from optimal. In fact, we shall see that the maximal value of $s^J$ is not much larger than $n! = s^J(1_\mathbb{R}^\infty)$. To summarize, we have shown the following.

**Remark 2.2.** Denoting
\[
\lambda_n = \sup_{\varphi \in \text{Cvx}_0(\mathbb{R}^n)} s^J(\varphi),
\]
we have
\[
n! \leq \lambda_n \leq C^n n!
\tag{8}
\]

Another important property of the $J$ transform is that its volume is preserved under rearrangement. For a convex function $\varphi \in \text{Cvx}_0(\mathbb{R}^n)$ we define the symmetric rearrangement of $\varphi$ to be the function $\varphi^* : \mathbb{R}^n \rightarrow \mathbb{R}^+$ satisfying that $L_z(\varphi^*)$ is the Schwartz symmetrization of $L_z(\varphi)$, i.e. a centered Euclidean ball with the same volume.
\[
\int_{\mathbb{R}^n} e^{-\varphi} = \int_0^\infty \text{Vol}_n (x : e^{-\varphi} \geq t) \, dt = \int_0^\infty \text{Vol}_n \left( x : e^{-\varphi} \geq e^{-z} \right) e^{-z} \, dz
\]
\[
= \int_0^\infty \text{Vol}_n (x : \varphi \leq z) e^{-z} \, dz = \int_0^\infty \text{Vol}_n (L_z(\varphi)) e^{-z} \, dz.
\]
A similar formula holds for the volume of $J \varphi$, as a weighted integral of the $n$-dimensional volumes of its level sets.
Lemma 2.3. Let $\varphi \in \text{Cvx}_0(\mathbb{R}^n)$. Then
\[ \int_{\mathbb{R}^n} e^{-J\varphi} = \int_0^\infty \text{Vol}_n(L_z(\varphi)) e^{-\frac{1}{z}z^{-(n+2)}} dz. \]

Proof. We use (6) to get
\[ \int_{\mathbb{R}^n} e^{-J\varphi} = \int_{\text{epi}(\varphi)} e^{-\frac{1}{z}z^{-(n+2)}} dz dx = \int_0^\infty \text{Vol}_n(L_z(\varphi)) e^{-\frac{1}{z}z^{-(n+2)}} dz. \]

From Lemma 2.3 we conclude that the functional $s^J(\varphi)$ depends only on the volumes of level sets of $\varphi$. Thus, by replacing the level sets of $\varphi$ with balls, we conclude that it is enough to maximize $s^J$ over spherically symmetric functions. Given a spherically symmetric function $\varphi : \mathbb{R}^n \to \mathbb{R}^+$, let $\psi = \varphi|_\mathbb{R}^+$ denote its restriction to a ray. Since $\varphi(x)$ depends only on $|x|$ we have:
\[ \text{Vol}(J\varphi) = \int e^{-J\varphi} = \int_{\mathbb{R}^n} dx \int_0^\infty e^{-\frac{1}{z}z^{-(n+2)}} dz = n\kappa_n \int_0^\infty r^{n-1} dr \int_0^\infty e^{-\frac{1}{z}z^{-(n+2)}} dz. \]

Thus we define the 2-dimensional measure $\nu_2$ on the first quadrant by
\[ d\nu_2 = n\kappa_n r^{n-1} e^{-\frac{1}{z}z^{-(n+2)}} dz dr. \]
We have seen that $\text{Vol}(J\varphi) = \nu(\text{epi}(\varphi)) = \nu_2(\text{epi}(\psi))$. Similarly, defining
\[ d\mu_2 = n\kappa_n r^{n-1} e^{-z} dz dr, \]
yields $\text{Vol}(\varphi) = \mu(\text{epi}(\varphi)) = \mu_2(\text{epi}(\psi))$. Therefore we define
\[ s_n^J(\psi) = \frac{\nu_2(\text{epi}(\psi))}{\mu_2(\text{epi}(\psi))} = s^J(\varphi), \quad (9) \]
and conclude that
\[ \lambda_n = \sup_{\varphi \in \text{Cvx}_0(\mathbb{R}^n)} s^J(\varphi) = \sup_{\psi \in \text{Cvx}_0(\mathbb{R}^+)} s_n^J(\psi). \quad (10) \]

Another useful property of the ratio $s^J$ is its invariance under rescaling. Namely, defining $\varphi_a \in \text{Cvx}_0(\mathbb{R}^n)$ by $\varphi_a(x) := \varphi(ax)$, we have:

Lemma 2.4. Let $\varphi \in \text{Cvx}_0(\mathbb{R}^n)$ and $a > 0$. Then, $s^J(\varphi_a) = s^J(\varphi)$.

Proof. Since $L_z(\varphi_a) = \frac{1}{a}L_z(\varphi)$, we have
\[ \text{Vol}_n(L_z(\varphi_a)) = \frac{1}{a^n} \text{Vol}_n(L_z(\varphi)). \]
By Lemma 2.3 we get $\text{Vol}(J\varphi_a) = \frac{1}{a^n} \text{Vol}(J\varphi)$. Similarly, $\text{Vol}(\varphi_a) = \frac{1}{a^n} \text{Vol}(\varphi)$, and the proof follows. \qed
3 Characterization of maximizers of $s^J$

We know by (8) that $s^J$ is bounded from above and below. In this section we prove Theorem 1.1 by showing that $s^J$ attains its maximum (which implies that its minimum is also attained). The maximizer is found by reducing the problem to a two dimensional optimization problem, then using the scaling invariance established in Lemma 2.4, thus restricting to the class $\text{Cvx}_{0,z}(\mathbb{R}^+) \subset \text{Cvx}_0(\mathbb{R}^+)$ defined as follows, for $z > 0$.

\[
\text{Cvx}_{0,z}(\mathbb{R}^+) = \{ \varphi \in \text{Cvx}_0(\mathbb{R}^+) : (1, z) \in \partial \text{epi} (\varphi) \},
\]

or equivalently, $\varphi \in \text{Cvx}_0(\mathbb{R}^+) \iff \text{L}_z(\varphi) = [0, 1]$. By Lemma 2.4 we have

\[
\lambda_n = \sup_{\varphi \in \text{Cvx}_0(\mathbb{R}^+)} s_n^J(\varphi) = \sup_{\varphi \in \text{Cvx}_{0,z}(\mathbb{R}^+)} s_n^J(\varphi). \tag{11}
\]

Moreover, any $\varphi \in \text{Cvx}_{0,z}(\mathbb{R}^+)$ satisfies

\[
\hat{\min} \{ 1^{\infty}_{[0,1]}, l_z \} \leq \varphi \leq \max \{ 1^{\infty}_{[0,1]}, l_z \},
\]

where $l_z$ is a linear function with slope $z$, and $\hat{\min}(\eta, \xi)$ is defined to be the largest convex function smaller than $\min(\eta, \xi)$. This implies the existence of positive constants $c = c(n, z)$, and $C = C(n, z)$ such that

\[
c(n, z) \leq \mu_2(\text{epi} (\varphi)) \leq C(n, z). \tag{12}
\]

We define a signed measure $\Delta$ on $\mathbb{R}^+ \times \mathbb{R}^+$ by

\[
\Delta = \nu_2 - \lambda_n \mu_2. \tag{13}
\]

We get, for any $\varphi \in \text{Cvx}_0(\mathbb{R}^+)$ that

\[
\Delta(\text{epi} (\varphi)) = \nu_2(\text{epi} (\varphi)) - \lambda_n \mu_2(\text{epi} (\varphi)) \leq 0. \tag{14}
\]

**Remark 3.1.** Clearly, $\Delta(\text{epi} (\varphi)) = 0$ if and only if $s_n^J(\varphi) = \lambda_n$, i.e. if and only if $\varphi \in \text{Cvx}_0(\mathbb{R}^+)$ is a maximizer of $s_n^J$.

The density of the signed measure $\Delta$ is given by

\[
d\Delta = n\kappa_n r^{n-1} \left( e^{-\frac{1}{2} z^{-(n+2)}} - \frac{\lambda_n}{z} e^{-z} \right) dz dr = n\kappa_n r^{n-1} m(z) dz dr,
\]

where $m(z) = e^{-\frac{1}{2} z^{-(n+2)}} - \frac{\lambda_n}{z} e^{-z}$.

**Lemma 3.2.** The function $m : \mathbb{R}^+ \rightarrow \mathbb{R}$ changes sign at three points $z_1 < z_2 < z_3$. 

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Proof. Consider the function \( f(z) = e^{\frac{1}{n+2}(z^{-\frac{1}{2}})} - \lambda_n^{\frac{1}{n+2}} z \). Since
\[
m(z) = 0 \iff e^{-\frac{1}{2}z^{-(n+2)}} = \lambda_n e^{-z} \iff e^{\frac{1}{n+2}(z^{-\frac{1}{2}})} = \lambda_n^{\frac{1}{n+2}} z,
\]
we conclude that \( m \) and \( f \) have the same roots. Since
\[
f''(z) = \frac{e^{\left(\frac{z^{-1/2}}{n+2}\right)}}{z^4(n+2)^2} \cdot \left((z^2 + 1)^2 - 2(n+2)z\right),
\]
we get
\[
f''(z) = 0 \iff z^4 + 2z^2 + 1 = 2(n+2)z.
\]
Since \( z \mapsto z^4 + 2z^2 + 1 \) is strictly convex and \( z \mapsto 2(n+2)z \) is linear, \( f''(z) \) has at most two roots, which implies that \( f \) has at most four roots. Moreover, \( m(0^+) = -\lambda_n \) is negative and \( m(\infty) = 0^+ \), which implies that one of the two following holds.

1. There exists \( z_0 > 0 \) such that \( m \leq 0 \) on \( [0, z_0] \) and \( 0 \leq m \) on \( [z_0, \infty) \).

2. There exist \( 0 < z_1 < z_2 < z_3 \) such that \( m \leq 0 \) on \( [0, z_1] \cup [z_2, z_3] \) and \( 0 \leq m \) on \( [z_1, z_2] \cup [z_3, \infty) \).

Next, we shall exclude the first case. Consider the function \( l(x) = z_0 x \) defined on \( \mathbb{R}^+ \), and for any \( \varphi \in \text{Cvx}_{0,z_0}(\mathbb{R}^+) \), define the sets:
\[
X = \text{epi} (\varphi) \cap \text{epi} (l), \quad Y = \text{epi} (\varphi) \setminus \text{epi} (l), \quad Z = \text{epi} (l) \setminus \text{epi} (\varphi).
\]
By convexity, \( Y \subset \{(x, z) : z \in [0, z_0]\} \), we have \( \Delta(Y) < 0 \). Similarly, \( \Delta(Z) > 0 \). Therefore
\[
\Delta(\text{epi} (\varphi)) = \Delta(X) + \Delta(Y) < \Delta(X) < \Delta(X) + \Delta(Z) = \Delta(\text{epi} (l)) = \\
\nu \left( \text{epi} \left( \| \cdot \|_{z_0 B} \right) \right) - \lambda_n \mu \left( \text{epi} \left( \| \cdot \|_{z_0 B} \right) \right) = \frac{\kappa}{z_0^2} (1 - n! \lambda_n) < 0.
\]
However, by (11), for every $\varepsilon > 0$ there exists $\varphi \in \text{Cvx}_{0,z_0}(\mathbb{R}^+)$ such that

$$\Delta(\text{epi}(\varphi)) > -\varepsilon \mu_2(\text{epi}(\varphi)).$$

Combining the above with (12) we obtain

$$0 > \Delta(\text{epi}(l)) > \Delta(\text{epi}(\varphi)) > -\varepsilon \mu_2(\text{epi}(\varphi)) > -\varepsilon C(n,z_0),$$

thus $\frac{z_0}{z_0}(1 - n! \lambda_n) \in (-\varepsilon C(n,z_0), 0)$ for any $\varepsilon > 0$, which is a contradiction. We are left with the second case, and the proof is complete.

The exclusion of the “one root case” in Lemma 3.2 is based on improving (i.e. increasing) the measure $\Delta$ of an epi-graph, by means of intersecting it with a ray (while relying on the convexity of the epi-graph). We next extend this idea to improve the measure $\Delta$ of an epi-graph, using the three roots of $m$. We use it to show that a maximizer of $s_n^T$ must be of the form

$$T_{a,b,x_0}(x) = \left\{ \begin{array}{ll} ax & : 0 \leq x \leq x_0 \\ (a + b)(x - x_0) + ax_0 & : x_0 \leq x \end{array} \right\},$$

where $a \in [0, \infty)$, $b \in [0, \infty]$, $x_0 \in [0, \infty)$. To this end we will define a mapping which assigns to each function in $\text{Cvx}_{0}(\mathbb{R}^+)$ a function of the form $T_{a,b,x_0}$.

**Definition 3.3.** The map $T : \text{Cvx}_{0}(\mathbb{R}^+) \to \text{Cvx}_{0}(\mathbb{R}^+)$ is defined as follows. Let $x_1, x_2, x_3$ be such that $L_{z_i}(\varphi) = [0, x_i]$, where $z_1, z_2, z_3$ are the three points where $m$ changes sign. If $\varphi \equiv 0$, set $T(\varphi) := \varphi$. Otherwise $x_1 \leq x_2 \leq x_3 < \infty$. Set

$$a = \frac{z_1}{x_1} > 0, \quad b = \frac{z_3 - z_2}{x_3 - x_2} - a \geq 0,$$

$$L_1(x) = ax, \quad L_2(x) = (a + b)(x - x_2) + z_2.$$

Set $T(\varphi) := \max\{L_1, L_2\}$ (see Figure 2).

**Remark 3.4.** Note that if the equation $L_1(x_0) = L_2(x_0)$ determines $x_0$ uniquely, then $T(\varphi) = T_{a,b,x_0}$, and $ax_0 \in [z_1, z_2]$. If however, $L_1 = L_2$ has more than one solution, then it is not hard to see that $\varphi$ is linear on $[0, x_3]$, $b = 0$, and $T(\varphi) = L_1 = L_2$. Thus we may write in this case too, that $T(\varphi) = T_{a,0,x_0}$ and $ax_0 \in [z_1, z_2]$ (say, $x_0 := x_1$).

The mapping $T$ improves upon the measure $\Delta$ of an epi-graph. More precisely,

**Lemma 3.5.** Let $\varphi \in \text{Cvx}_{0}(\mathbb{R}^+)$. Then, $\Delta(\varphi) \leq \Delta(T(\varphi))$. Moreover, if $\varphi$ is not of the form $T_{a,b,x_0}$ then the inequality is strict.
Proof. In Lemma 3.2 we have seen that there exist three distinct points \( z_1 < z_2 < z_3 \) in which \( m \) changes sign. This implies that the sign of the density of the signed measure \( \Delta \) is fixed on each of the following slabs:

\[
S_1 = \{ (x, z) : 0 \leq z \leq z_1 \}
\]
\[
S_2 = \{ (x, z) : z_1 \leq z \leq z_2 \}
\]
\[
S_3 = \{ (x, z) : z_2 \leq z \leq z_3 \}
\]
\[
S_4 = \{ (x, z) : z_3 \leq z \}. 
\]

By construction, we have

\[
\text{epi} (T(\varphi)) \cap S_1 \subseteq \text{epi} (\varphi) \cap S_1
\]
\[
\text{epi} (T(\varphi)) \cap S_2 \supseteq \text{epi} (\varphi) \cap S_2
\]
\[
\text{epi} (T(\varphi)) \cap S_3 \subseteq \text{epi} (\varphi) \cap S_3
\]
\[
\text{epi} (T(\varphi)) \cap S_4 \supseteq \text{epi} (\varphi) \cap S_4. 
\]

Since the signed measure \( \Delta \) is negative on \( S_1, S_3 \), and positive on \( S_2, S_4 \), the statement follows. If one of the above inclusions is strict, then (since the sets are convex) \( \Delta(\varphi) < \Delta(T(\varphi)) \).

Lemma 3.6. If \( \varphi \) is not of the form \( T_{a,b,x_0} \), then \( \varphi \) is not a maximizer of \( s_n^\varphi \).

Proof. If \( \varphi \) is a maximizer of \( s_n^\varphi \) then by Remark 3.1, \( \Delta(\varphi) = 0 \). If \( \varphi \) is not of the form \( T_{a,b,x_0} \), then by Lemma 3.3 we have

\[
0 = \Delta(\varphi) < \Delta(T(\varphi)),
\]

which is a contradiction to \( \Delta(\varphi) = 0 \). 

Figure 2: Definition of \( T \): \( \varphi \) in red, \( T(\varphi) \) in blue.
We next show that the supremum of \( s_n^f \) over functions of the form \( T_{a,b,x_0} \) is not (strictly) smaller than the supremum over all of \( \text{Cvx}_0(\mathbb{R}^+) \).

**Lemma 3.7.** For every \( n \geq 1 \):

\[
\sup_{a,b,x_0} s_n^f(T_{a,b,x_0}) = \sup_{\varphi \in \text{Cvx}_0(\mathbb{R}^+)} s_n^f(\varphi).
\]

**Proof.** Denote

\[
\delta_n = \sup s_n^f(T_{a,b,x_0}).
\]

Assume that \( \delta_n < \lambda_n \). Recall that by [12], for any \( \psi \in \text{Cvx}_0(\mathbb{R}^+) \),

\[
\hat{\min}\{1_{[0,1]}, l_{z_1}\} \leq \psi \leq \max\{1_{[0,1]}, l_{z_1}\} \quad \Rightarrow \quad 0 < c(n, z_3) < \mu_2(\psi) < C(n, z_1) \quad (15)
\]

Choose \( \varepsilon = (\lambda_n - \delta_n) \frac{c(n, z_3)}{2c(n, z_1)} > 0 \), and let \( \bar{\varphi} \in \text{Cvx}_0(\mathbb{R}^+) \) be such that \( s_n^f(\bar{\varphi}) > \lambda_n - \varepsilon \).

If \( T_{a,b,x_0} = T(\bar{\varphi}) \), then for \( \varphi = \bar{\varphi}_{x_0} \) one has

\[
T(\varphi) = T(\bar{\varphi}_{x_0}) = (T(\bar{\varphi}))_{x_0} = T_{x_0a,x_0b,1} \equiv T_{a,b,1},
\]

and \( s_n^f(\varphi) = s_n^f(\bar{\varphi}) > \lambda_n - \varepsilon \). Moreover,

\[
\hat{\min}\{1_{[0,1]}, l_{z_1}\} \leq \varphi \leq \max\{1_{[0,1]}, l_{z_1}\},
\]

which by (15) implies that \( \frac{\mu_2(\varphi)}{\mu_2(T(\varphi))} \leq \frac{C(n, z_1)}{c(n, z_3)} \). Since \( \nu_2(T(\varphi)) \leq \delta_n \mu_2(T(\varphi)) \), we get

\[
(\delta_n - \lambda_n) \mu_2(T(\varphi)) \geq \nu_2(T(\varphi)) - \lambda_n \mu_2(T(\varphi)) = \Delta(T(\varphi)) \geq \Delta(\varphi) = \nu_2(\varphi) - \lambda_n(\varphi) > -\varepsilon \mu_2(\varphi) \geq -\varepsilon \frac{C(n, z_1)}{c(n, z_3)} \mu_2(T(\varphi)).
\]

The latter implies that \( (\lambda_n - \delta_n) < \varepsilon \frac{C(n, z_1)}{c(n, z_3)} = \frac{1}{2}(\lambda_n - \delta_n) \), which is a contradiction. \( \square \)

Lemma 3.7 implies that the maximal value of \( s_n^f \) on \( \text{Cvx}_0(\mathbb{R}^+) \) can be found by studying a function of two variables \( F : [0, \infty) \times [0, \infty] \rightarrow \mathbb{R}^+ \) given by

\[
F(a, b) = s_n^f(T_{a,b,1}) = \int_{\frac{a}{2}}^{a} e^{-\frac{1}{2} z - (n+2)} \left( \frac{z}{a} \right)^n dz + \int_{a}^{\infty} e^{-\frac{1}{2} z - (n+2)} \left( \frac{z+b}{a+b} \right)^n dz
\]

which is understood for \( a = 0 \) and \( b = \infty \) by taking a limit. Note that \( F \) is a rational function of \( b \) with coefficients that are smooth in \( a \), as such is continuous on \([z_1, z_2] \times [0, \infty] \). It is easy to verify that when \( a \neq 0 \) and \( b < \infty \) we may write

\[
F(a, b) = \frac{(a + b)^n e^{-\frac{1}{2} z} + a n \int_{a}^{\infty} e^{-\frac{1}{2} z - (n+2)}(z+b)^n dz}{(a + b)^n \int_{a}^{\infty} e^{-z} z^n dz + a n \int_{a}^{\infty} e^{-z}(z+b)^n dz}
\]

Note that by Remark 3.4, it suffices to look for a maximum of \( F \) when \( a \in [z_1, z_2] \).
Lemma 3.8. There exist $a \in [z_1, z_2]$, $b \in [0, \infty]$ such that
\[ s_n^J(T_{a,b,1}) = \lambda_n. \]

Proof. Let \( \{a_k\} \subset [z_1, z_2] \), \( \{b_k\} \subset [0, \infty] \) be two sequences with \( s_n^J(T_{a_k,b_k,1}) \nearrow \lambda_n \).

Since \( a_k \in [z_0, z_1] \), there exists a subsequence \( \{a_{k_l}\} \) such that \( a_{k_l} \rightarrow a \) for some \( a \in [z_1, z_2] \). In addition, there exists a subsequence \( \{b_{k_{lm}}\} \) of \( \{b_{k_l}\} \) that converges to some \( b \in [0, \infty] \). Continuity of \( F \) implies that \( s_n^J(T_{a,b,1}) = \lambda_n \). \( \square \)

We have seen that \( s_n^J \) has a maximizer of the form \( T_{a,b,1} \). The next two lemmas provide bounds on the parameters \( a, b \) of such a maximizer.

Lemma 3.9. Let \( \alpha \in (\frac{1}{2}, 1) \). There exists \( n_0 \) such that if \( n > n_0 \) and \( T_{a,b,1} \) is a maximizer of \( s_n^J \), then:
\[ \frac{1}{n + \alpha} \leq a \leq \frac{1}{n - \alpha} \]

Proof. By remark 3.4, \( a \in [z_1, z_2] \), thus we need to estimate \( z_1, z_2 \), the first two roots of \( m(z) \). We do this by considering the following family of functions.
\[ m_\lambda(z) = e^{-\frac{1}{2}z - (n+2)} - \lambda e^{-z}. \]

For any \( \lambda \in \left[ \frac{1}{n}, \lambda_n \right] \) we may repeat the proof of Lemma 3.2 and deduce that \( m_\lambda \) has three sign changes, denoted by \( z_1(\lambda) < z_2(\lambda) < z_3(\lambda) \). Since \( m_\lambda(z) \) is decreasing in \( \lambda \), we get by (8) that
\[ m(z) = m_{\lambda_n}(z) \leq m_n!(z), \]
thus \( a \in [z_1, z_2] = [z_1(\lambda_n), z_2(\lambda_n)] \subseteq [z_1(n!), z_2(n!)] \). Note that \( m_n!(\frac{1}{n}) > 0 \). To check the sign of \( m_n! \) at the points \( \left( \frac{1}{n+\alpha} \right) \) we note that:
\[ \frac{(1 \pm \frac{1}{n+\alpha})^n}{e^{n^\alpha}} = e^{n \log(1 \pm \frac{1}{n+\alpha}) + \frac{1}{n+\alpha}} = e^{-\frac{1}{2}[n^{2\alpha-1} + \pi(n^{2\alpha-1})]}, \]
which tends to 0 for \( \alpha \in (\frac{1}{2}, 1) \). Thus,
\[ m_n! \left( \frac{1}{n+\alpha} \right) = \left( \frac{n}{e} \right)^n \left( n^\alpha \right)^2 e^{-\frac{1}{2}[n^{2\alpha-1} + \pi(n^{2\alpha-1})]} - \sqrt{2\pi n} (1 + o(1)) < 0 \]
for \( n \) large enough. Since \( m_n! \left( \frac{1}{n+\alpha} \right) \) are negative and \( m_n! \left( \frac{1}{n} \right) \) is positive, we get
\[ a \in [z_1(n!), z_2(n!)] \subseteq \left[ \frac{1}{n+n^{1-\alpha}}, \frac{1}{n-n^{1-\alpha}} \right]. \]
\( \square \)

Lemma 3.10. The function \( F(a, b) = s_n^J(T_{a,b,1}) \) is maximized only on points of the form \((a, \infty)\).
Proof. We know by Lemma 3.8 that \( F \) attains a maximum, and we will show that it does not attain a maximum at any point \((a, b)\), where \( b \in [0, \infty) \). We shall do this by showing that \( F_b(a, b) > 0 \) for \( b \neq \infty \), provided that the following conditions hold:

\[
F(a, b) = \lambda_n, \quad (17)
\]

\[
F_a(a, b) = \frac{\partial F}{\partial a}(a, b) = 0. \quad (18)
\]

This would imply that \( F \) has no (global) maximum points in \([z_1, z_2] \times [0, \infty)\). Recall that by (16) we may write, for \( a \neq 0, b \neq \infty \):

\[
F(a, b) = \frac{(a + b)^n N_1 + a^n N_2}{(a + b)^n M_1 + a^n M_2},
\]

where

\[
N_1 = e^{-\frac{1}{a}}, \quad N_2 = \int_a^\infty (z + b)^n e^{-\frac{1}{z}} z^{-(n+2)} dz,
\]

\[
M_1 = \int_0^a z^n e^{-z} dz = \gamma(n + 1, a), \quad M_2 = \int_a^\infty (z + b)^n e^{-z} dz.
\]

Computing partial derivatives with respect to \( a \) and \( b \) yields:

\[
\frac{\partial N_1}{\partial a} = \frac{1}{a^2} e^{-\frac{1}{a}}, \quad \frac{\partial N_2}{\partial a} = -(a + b)^n e^{-\frac{1}{z}} a^{-(n+2)},
\]

\[
\frac{\partial M_1}{\partial a} = a^n e^{-a}, \quad \frac{\partial M_2}{\partial a} = -(a + b)^n e^{-a}.
\]

and

\[
\frac{\partial N_1}{\partial b} = 0, \quad \frac{\partial N_2}{\partial b} = n \int_a^\infty (z + b)^{n-1} e^{-\frac{1}{z}} z^{-(n+2)} dz
\]

\[
\frac{\partial M_1}{\partial b} = 0, \quad \frac{\partial M_2}{\partial b} = n \int_a^\infty (z + b)^{n-1} e^{-z} dz
\]

Looking for a possible critical point, we assume (18) holds, and use the relation \((a + b)^n \frac{\partial N_1}{\partial a} + a^n \frac{\partial N_2}{\partial a} = 0\), and similarly \((a + b)^n \frac{\partial M_1}{\partial a} + a^n \frac{\partial M_2}{\partial a} = 0\), to get:

\[
[(a + b)^{n-1} N_1 + a^{n-1} N_2] [(a + b)^n M_1 + a^n M_2] =
\]

\[
= [(a + b)^{n-1} M_1 + a^{n-1} M_2] [(a + b)^n N_1 + a^n N_2],
\]

or equivalently \( N_1 M_2 = M_1 N_2 \), which together with (17) implies that

\[
\frac{N_1}{M_1} = \frac{N_2}{M_2} = \lambda_n. \quad (19)
\]
Next we wish to show that $\frac{\partial F}{\partial b} > 0$, so we may use (19) to write:

$$((a + b)^n M_1 + a^n M_2)^2 \frac{\partial F}{\partial b} =$$

$$= \left[ n(a + b)^{n-1}N_1 + na^n \int_a^\infty (z + b)^{n-1}e^{\frac{-1}{2}z^{-(n+2)}}dz \right][(a + b)^n M_1 + a^n M_2]$$

$$- \left[ n(a + b)^{n-1}M_1 + na^n \int_a^\infty (z + b)^{n-1}dz \right][(a + b)^n N_1 + a^n N_2]$$

$$= na^n [(a + b)^n M_1 + a^n M_2] \left[ \int_a^\infty (z + b)^{n-1} \left(e^{\frac{-1}{2}z^{-(n+2)}} - \lambda_n e^{-z}\right)dz \right].$$

Recall that $m(z) = e^{-\frac{1}{2}z^{-(n+2)}} - \lambda_n e^{-z}$, thus

$$\frac{\partial F}{\partial b} > 0 \iff \int_a^\infty (z + b)^{n-1}m(z)dz > 0 \quad (20)$$

By (19) we have $\int_a^\infty (z + b)^n m(z)dz = N_2 - \lambda_n M_2 = 0$, which we may write as

$$P_{n-1}(b) := \int_a^\infty (z + b)^{n-1}m(z)dz = \lambda_n \int_a^\infty (z + b)^{n-1}m(z)dz.$$

By (20), it suffices to show that $P_{n-1}(b) < 0$. Note that $P_{n-1}(b) = \sum_{i=0}^{n-1} c_k b^k$ is a polynomial of degree $n - 1$ with coefficients

$$c_k = \binom{n-1}{k} \int_a^\infty z^{n-k}m(z)dz, \quad k = 0, 1, \ldots n - 1.$$

It is not hard to check that

$$\binom{n-1}{k}^{-1} c_k = \int_a^\infty z^{n-k}e^{\frac{-1}{2}z^{-(n+2)}}dz - \lambda_n \int_a^\infty z^{n-k}e^{-z}dz =$$

$$= \int_0^{1/a} t^k e^{-t} dt - \lambda_n \int_a^\infty z^{n-k}e^{-z}dz =$$

$$= \gamma \left( k + 1, \frac{1}{a} \right) - \lambda_n \Gamma(n - k + 1, a).$$

Since $k \leq n - 1$ we have $a \leq 1 \leq n - k$ and

$$\Gamma(n - k + 1, a) \geq \Gamma(n - k + 1, n - k) \geq \frac{1}{2} (n - k)! \geq \frac{1}{2}.$$

For the second inequality $\Gamma(m + 1, m) \geq \frac{m!}{2}$ see e.g. [19, Equation 8.10.13]. Thus, for $n \geq 2$ we have, for every $k \in \{0, \ldots, n - 1\}$

$$\gamma \left( k + 1, \frac{1}{a} \right) - \lambda_n \Gamma(n - k + 1, a) < k! - \frac{1}{2} n! \leq (n - 1)! - \frac{1}{2} n! \leq 0,$$
which means $c_k < 0$. We conclude that $P_{n-1}(b) < 0$ for all $b > 0$, which implies that
\[ \int_a^\infty (z + h)^{n-1}m(z)dz > 0, \]
i.e. the derivative $\frac{\partial F}{\partial b}$ is positive at any point $(a, b)$ satisfying (17) and (18).

We shall now prove the main theorem.

**Proof of Theorem 1.1.** By (10), existence of a maximizer for $s^J$ is equivalent to existence of a maximizer for $s^J_n$, which is verified in Lemma 3.8. Thus the statement of Theorem 1.1 will follow, if we show that a maximizer must be of the form $\psi_{K,a}$. Let $\varphi \in \text{Cvx}_0(\mathbb{R}^n)$ be a maximizer of $s^J$. As before, let $\psi = \varphi^*|_{\mathbb{R}^+}$, where $\varphi^*$ is the symmetric of rearrangement $\varphi$. Then $\psi$ is a maximizer of $s^J_n$, and by Lemma 3.6, $\psi = T_{a,x_0}^\infty$, for some $a \in (0, \infty)$, $b \in [0, \infty]$, $x_0 \in (0, \infty)$. Note that $\psi_{x_0} = T_{a,0,x_0,1}$, which by Lemma 3.10 implies $b = \infty$. Thus $\psi = T_{a,x_0,\infty}^\infty$. Let $\varphi^* = \max \left\{ 1_{x_0}^aB_2^n, \frac{a}{x_0} \right\}$. The level sets of $\varphi^*$ are given by
\[ L_z(\varphi^*) = \begin{cases} \frac{n+1}{a}zB_2^n & : \ 0 \leq z \leq a \\ \frac{n}{x_0B_2^n} & : \ a \leq z \end{cases} \]
This implies that $z \mapsto (\text{Vol}_n(L_z(\varphi)))^{\frac{1}{n}}$ is linear on $[0, a]$ and constant on $[a, \infty)$. By the equality condition of the Brunn-Minkowski inequality applied to the sets $K := L_a(\varphi)$ and $L_z(\varphi)$, all level sets of $\varphi$ are homothetic to $K$. We get
\[ L_z(\varphi) = \begin{cases} \frac{n}{a}K & : \ 0 \leq z \leq a \\ \frac{n}{a}K & : \ a \leq z \end{cases} \]
In other words $\varphi = \psi_{K,a}$ for some positive number $a$, as required. Note that since $\psi_{x_0} = T_{a,0,\infty}^\infty$ maximizes $s^J_n$, we get from Lemma 3.9 that
\[ \frac{n}{n + \frac{n^2}{3}} \leq na \leq \frac{n}{n - \frac{n^2}{3}}, \]
and the proof is complete.

**4 Asymptotically sharp bounds for $s^J$**

In this section we prove Theorem 1.2. In our estimates we use the Gamma function and the incomplete Gamma functions, defined as follows. The Gamma function is given by:
\[ \Gamma(n + 1) = \int_0^\infty t^n e^{-t}dt. \]
The incomplete Gamma functions are defined by:

$$\gamma(n+1, a) = \int_0^a t^n e^{-t} dt,$$

$$\Gamma(n+1, a) = \int_a^\infty t^n e^{-t} dt.$$ 

Replacing the exponent $e^{-t}$ by its maximal and minimal values on $[0, a]$ yields, for any $a \leq 1$:

$$e^{-a} \frac{a^{n+1}}{n+1} \leq \gamma(n+1, a) \leq \frac{a^{n+1}}{n+1}.$$ 

(21)

The following lemma is an upper bound on $\Gamma(n+1, a)$.

**Lemma 4.1.** Let $n \geq 1$ and $0 < t < 2(n+1)$. Then

$$\int_0^{n+1+t} s^n e^{-s} ds \geq \left( 1 - e^{-\frac{t^2}{8(n+1)}} \right) n!$$ 

(22)

**Proof.** Let $X$ be a random variable with density $\frac{s^n e^{-s}}{n!}$, so that $E[X] = n + 1$. For $\theta \in (0, \frac{1}{2})$, let $Y = e^{\theta(X-E[X])}$, so that $E[Y] = \left( \frac{1}{e^{\theta(1-\theta)}} \right)^{n+1} \leq e^{2\theta^2(n+1)}$. Choose $a = e^{\theta t}$ and use Markov’s inequality for $Y$ to get

$$P(X \geq n + 1 + t) = P(X - E[X] \geq t) = P(Y \geq a) \leq \frac{E[Y]}{a} \leq e^{2\theta^2(n+1) - \theta t}$$

Optimizing for $\theta$, we choose $\theta = \frac{t}{4(n+1)} < \frac{1}{2}$ to get

$$\int_{n+1+t}^\infty \frac{s^n e^{-s} ds}{n!} = P(X \geq n + 1 + t) \leq e^{-\frac{t^2}{8(n+1)}},$$

which completes the proof. \[ \square \]

**Remark 4.2.** Lemma 4.1 is a special case of a more general concentration property due to Klartag, where $e^{-s}$ is replaced by an arbitrary log concave function (see Lemmas 4.3 and 4.5 in [13]).

By Lemma 3.10 we get that, defining $G : (0, \infty) \rightarrow (0, \infty)$ by

$$G(a) = s_n^\gamma(T_{a,\infty,1}) = \frac{e^{-\frac{1}{a}} + a^n \int_a^\infty e^{-\frac{1}{z}} z^{-(n+2)} dz}{\gamma(n+1,a)} + a^n e^{-a},$$

we have $\lambda_n = \max_{a \in [z_1, z_2]} G(a)$. The curious reader may verify, similarly to the way (19) is obtained in the proof of Lemma 3.10 that $G'(a) = 0$ and $G(a) = \lambda_n$ imply

$$\lambda_n = \frac{e^{-\frac{1}{a}}}{\gamma(n+1,a)},$$

(23)
\[ \lambda_n = \int_a^\infty \frac{e^{-\frac{1}{z}z^{-(n+2)}}}{e^{-a}} \, dz = e^a \gamma(n+1, \frac{1}{a}). \quad (24) \]

We shall use (24), together with the bounds on the partial Gamma functions, to prove the asymptotically sharp bounds on \( \lambda_n \).

**Proof of Theorem 1.2.** We want to show that there exist positive constants \( c, C \) such that for \( n \) large enough:

\[
\left( 1 + \frac{c}{n} \right) n! \leq \lambda_n \leq \left( 1 + \frac{C}{n} \right) n!.
\]

By Lemma 3.9, for any \( \alpha \in \left( \frac{1}{2}, 1 \right) \) there exists \( n_0 = n_0(\alpha) \) such that \( n > n_0 \) implies \( a < \frac{1}{n-n^\alpha} \). Thus, there exists a universal constant \( C > 0 \) such that

\[ e^a < 1 + \frac{C}{n}. \]

Combining the above with (24) yields the required upper bound:

\[ \lambda_n < \left( 1 + \frac{C}{n} \right) n!. \]

For the lower bound, we choose a point \( a = \frac{1}{2(n+1)} \) and show that \( G(a) \geq \left( 1 + \frac{c}{n} \right) n! \).

Note that \( \frac{1}{3n} < a < \frac{1}{2n} \). By (21) we have

\[ \gamma(n+1, a) \leq \frac{ae^a}{(n+1)} < \frac{1}{n^2}. \]

We use (22), and the inequalities \( e^x > 1 + x \) and \( \frac{1}{x+1} > 1 - x \) for positive \( x \), to get

\[
G(a) = \frac{e^{-\frac{1}{a}} + a^n \int_a^\infty e^{-\frac{1}{z}z^{-(n+2)}}dz}{\gamma(n+1, a) + a^n e^{-a}} > \frac{a^n \int_a^\infty e^{-\frac{1}{z}z^{-(n+2)}}dz}{\gamma(n+1, a) + a^n e^{-a}} = \\
= \frac{e^a \int_0^\infty s^n e^{-s}ds}{\gamma(n+1, a) + 1} \left( 1 + a \right) \left( 1 - \frac{\gamma(n+1, a)}{a^n e^{-a}} \right) \int_0^{2(n+1)} s^n e^{-s}ds > \\
> \left( 1 + \frac{1}{3n} \right) \left( 1 - \frac{1}{n^2} \right) \left( 1 - e^{-\frac{n+1}{8}} \right) n!
\]

which for \( n \) large enough implies \( G(a) > \left( 1 + \frac{c}{n} \right) n! \).

\[ \square \]
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Dan Florentin  
Department of Mathematical Sciences  
Kent State University, Kent, OH, 44242, USA  
*email*: danflorentin@gmail.com

Alexander Segal  
Afeka Academic College of Engineering, Tel Aviv, 69107, Israel  
*email*: segalale@gmail.com