Artificial black holes. *

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To Misha Shubin on the occasion of his 65th birthday.

Abstract

We study black holes for the linear hyperbolic equations describing the wave propagation in the moving medium. Such black holes are called artificial since the Lorentz metric associated with the hyperbolic equation does not necessarily satisfy the Einstein equations. Artificial black holes also arise when we consider perturbations of the Einstein equations. In this paper we review results of [E2] and [E3] on the existence and the stability of black holes for the stationary wave equations in two space dimensions, and in the axisymmetric case.

1 Introduction.

Consider the wave equation of the form

\begin{equation}
\sum_{j,k=0}^{n} \frac{1}{\sqrt{(-1)^n g(x)}} \frac{\partial}{\partial x_j} \left( \sqrt{(-1)^n g(x)} g^{jk}(x) \frac{\partial u(x_0, x)}{\partial x_k} \right) = 0,
\end{equation}

where \( x = (x_1, ..., x_n) \in \mathbb{R}^n, x_0 \in \mathbb{R} \) is the time variable, the coefficients \( g^{jk}(x) \in C^\infty \) and are independent of \( x_0 \), \( g(x) = \det [g_{jk}(x)]_{j,k=0}^{n} \). \( [g_{jk}(x)]_{j,k=0}^{n} \) is the inverse to the matrix \( [g^{jk}(x)]_{j,k=0}^{n} \). We assume that \( [g_{jk}(x)]_{j,k=0}^{n} \) is a

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pseudo-Riemanian metric with the Lorentz signature $(1, -1, ..., -1)$. We also assume that

\begin{equation}
(1.2) \quad g^{00}(x) > 0, \ \forall x \in \mathbb{R}^n
\end{equation}

and

\begin{equation}
(1.3) \quad g^{jk}(x) - \delta_{jk} = O\left(\frac{1}{|x|}\right) \text{ when } |x| \to \infty.
\end{equation}

Equation (1.1) describes the wave propagation in a moving medium. As in [E1], [E2], we consider two examples:

**a) Propagation of light in a moving dielectric medium (cf. [G], [LP]).**

In this case equation (1.1) is called the Gordon equation and it has the form:

\begin{equation}
(1.4) \quad g^{jk}(x) = \eta^{jk} + (n^2(x) - 1)u^j u^k, \quad 0 \leq j, k \leq n, \ n = 3,
\end{equation}

\( \eta^{jk} = 0 \) when \( j \neq k \), \( \eta^{00} = 1 \), \( \eta^{jj} = -1 \) when \( 1 \leq j \leq 3 \), \( n(x) = \sqrt{\varepsilon(x)\mu(x)} \) is the refraction index, \( x_0 = ct \) where \( c \) is the speed of light in the vacuum, \( (u_0, u_1, u_2, u_3) \) is the four-velocity: \( u_0 = \left(1 - \frac{|w|^2}{c^2}\right)^{-\frac{1}{2}} \), \( u_j(x) = \frac{w_j(x)}{c} \left(1 - \frac{|w|^2}{c^2}\right)^{-\frac{1}{2}} \), \( 1 \leq j \leq 3 \). Here \( w = (w_1(x), w_2(x), w_3(x)) \) is the velocity of the flow, \( |w| = \sqrt{w_1^2 + w_2^2 + w_3^2} \).

**b) Acoustic waves in a moving fluid (cf. [V1]).**

In this case

\begin{equation}
(1.5) \quad g^{00} = \frac{1}{\rho c}, \quad g^{0j} = g^{j0} = \frac{1}{\rho c} v^j, \quad 1 \leq j \leq 3,
\end{equation}

\begin{equation}
(1.5) \quad g^{jk} = -\frac{1}{\rho c}(-c^2 \delta_{ij} + v^j v^k), \quad 1 \leq j, k \leq 3,
\end{equation}

where \( \rho \) is the density, \( c \) is the sound speed, \( v = (v^1, v^2, v^3) \) is the velocity.

Equations with metrics (1.4), (1.5) may have black holes (see §2 below). These black holes are called optical and acoustic black holes, respectively (cf.
[V1], [NVV], [U] and references there). They are called often artificial black holes since the metric in (1.1) not necessarily satisfies the Einstein equations. Physicists hope to create artificial black holes in the laboratory. The artificial black holes play role when one consider the perturbations of black holes of the general relativity such as the Schwarzschild and the Kerr black holes. We introduce the black holes in §2 and §3. In §4 we shall study the existence and the stability of black holes in the case of two space dimensions. In §5 we consider the axisymmetric case and in §6 we consider the inverse problems in the presence of black holes.

2 The black and white holes.

Let \( S_0(x) = 0 \) be a closed bounded smooth surface in \( \mathbb{R}^n \). Let \( \Omega_{\text{int}} \) and \( \Omega_{\text{ext}} \) be the interior and the exterior of \( S_0(x) = 0 \), respectively. The domain \( \Omega_{\text{int}} \times \mathbb{R} \) is called a black hole for (1.1) if no signals (disturbances) from \( \Omega_{\text{int}} \times \mathbb{R} \) can reach the exterior domain \( \Omega_{\text{ext}} \times \mathbb{R} \). Analogously, \( \Omega_{\text{int}} \times \mathbb{R} \) is a white hole for (1.1) if no signals (disturbances) from \( \Omega_{\text{ext}} \times \mathbb{R} \) can reach \( \Omega_{\text{int}} \times \mathbb{R} \). The surface \( \{ S_0(x) = 0 \} \times \mathbb{R} \) is called the event horizon if \( \Omega_{\text{int}} \times \mathbb{R} \) is either black or white hole. In order to find the conditions when \( \{ S_0(x) = 0 \} \times \mathbb{R} \) is an event horizon we need the notion of the forward domain of influence (cf. [CH]).

Consider the Cauchy problem for the equation (1.1) in half-space \( x_0 > t_0 \) with initial conditions

\[
(2.1) \quad u(t_0, x) = \varphi_0(x), \quad u_{x_0}(t_0, x) = \varphi_1(x),
\]

where \( \text{supp} \varphi_k(x) \subset \overline{X}, \ k = 0, 1, \ x \in \mathbb{R}^n \). Denote by \( D_+(\varphi_0, \varphi_1) \) the support of \( u(x_0, x) \) for \( x_0 \geq t_0 \). Let \( D_+(X \times \{ x_0 = t_0 \}) \) be the closure of the union of \( D_+(\varphi_0, \varphi_1) \) over all \( \varphi_0, \varphi_1 \) with supports on \( \overline{X} \). Then \( D_+(X \times \{ x_0 = t_0 \}) \) is the forward domain of influence of \( X \times \{ x_0 = t_0 \} \). Therefore \( \Omega_{\text{int}} \times \mathbb{R} \) is a black hole if \( D_+(\Omega_{\text{int}} \times \{ x_0 = t_0 \}) \subset \overline{\Omega}_{\text{int}} \times \mathbb{R} \), and \( \Omega_{\text{int}} \times \mathbb{R} \) is a white hole if \( D_+(\Omega_{\text{ext}} \times \{ x_0 = t_0 \}) \subset \overline{\Omega}_{\text{ext}} \times \mathbb{R} \) for all \( t_0 \in \mathbb{R} \).

There is a geometric description of \( D_+(X \times \{ x_0 = t_0 \}) \).

Let \( [g_{jk}(x)]_{j,k=0}^n \) be the metric tensor corresponding to the operator (1, 1). Consider a curve in \( \mathbb{R}^{n+1} \):

\[
(2.2) \quad x_0 = x_0(s), \ x = x(s), \ s \geq 0, \ x_0(0) = y_0, \ x(0) = y.
\]
The curve \( (2.2) \) is called a time-like ray if
\[
(2.3) \quad \sum_{j,k=0}^{n} g_{jk}(x(s)) \frac{dx_j(s)}{ds} \frac{dx_k(s)}{ds} > 0, \quad \frac{dx_0(s)}{ds} > 0, \quad \text{for } s \geq 0.
\]

**Theorem 2.1.** (cf., for example, [CH]). The forward domain of influence \( D_+(X \times \{x_0 = t_0\}) \) is the closure of the union of all time-like rays starting at \( X \times \{x_0 = t_0\} \).

Let \( \{S_0(x) = 0\} \times \mathbb{R} \) be a characteristic surface for \( (1.1) \), i.e.
\[
(2.4) \quad \sum_{j,k=1}^{n} \gamma^{jk}(x) S_{0x_j}(x) S_{0x_k} = 0 \quad \text{when } S_0(x) = 0.
\]

We assume that \( S_{0x}(x) \) is the outward normal to \( S_0(x) = 0 \), \( S_{0x}(x) \neq 0 \) when \( S_0(x) = 0 \).

**Theorem 2.2.** (cf. [E2]). The domain \( \Omega \times \mathbb{R} \) is a black hole if \( (2.4) \) holds and
\[
(2.5) \quad \sum_{j=1}^{n} \gamma^{j0}(x) S_{0x_j}(x) < 0 \quad \text{when } S_0(x) = 0,
\]
and the domain \( \Omega \times \mathbb{R} \) is a white hole if \( (2.4) \) holds and if
\[
(2.6) \quad \sum_{j=1}^{n} \gamma^{j0}(x) S_{0x_j}(x) > 0 \quad \text{when } S_0(x) = 0.
\]

The proof of Theorem 2.2 based on the study of the time-like rays starting on \( S_0(x) = 0 \) was given in [E2].

One can also prove Theorem 2.2 using the energy-type estimates of the solutions of the equation \( (1.1) \) in the Sobolev spaces.

**Theorem 2.3.** (cf., for example, [E4]). Let \( (2.4) \) and \( (2.5) \) hold. Then for any solution \( u(x_0, x) \) of \( (1.1) \) we have
\[
(2.7) \quad \|u(x_0, \cdot)\|_{1, \Omega_{ext}}^2 + \|u_{x_0}(x_0, \cdot)\|_{0, \Omega_{ext}}^2 \leq C_T(\|u(t_0, \cdot)\|_{1, \Omega_{ext}}^2 + \|u_{x_0}(t_0, \cdot)\|_{0, \Omega_{ext}}^2),
\]
where \( T \) is arbitrary, \( t_0 \leq x_0 \leq T \), \( \|v\|_{p, \Omega_{ext}} \) is the norm in the Sobolev space \( H_p(\Omega_{ext}) \).
Having the estimate (2.7) it is easy to see that \( \Omega_{\text{int}} \times \mathbb{R} \) is a black hole:

Consider the Cauchy problem (1.1), (2.1) in the half-space \( x_0 > t_0 \), where \( \text{supp } \phi_k(x) \subseteq \overline{\Omega}_{\text{int}}, \ k = 0, 1. \) Then \( u(t_0, x) = u(x_0(t_0, x)) = 0 \) when \( x \in \Omega_{\text{ext}}. \) Applying the estimate (2.7) we get that \( u(x_0, x) = 0 \) in \( \Omega_{\text{ext}} \times (t_0, +\infty) \). Therefore \( \text{supp } u(x_0, x) \subseteq \overline{\Omega}_{\text{int}} \times [t_0, +\infty), \ i.e. \ D_+(\Omega_{\text{int}} \times \{x_0 = t_0\}) \subset \overline{\Omega}_{\text{int}} \times \mathbb{R} \) for any \( t_0, \ i.e. \ \Omega_{\text{int}} \times \mathbb{R} \) is a black hole.

Analogous result holds for the white hole.

**Theorem 2.4.** (cf., for example, [E4]). Let (2.4) and (2.6) hold. Then for any solution \( u(x_0, x) \) of (1.1) we have

\[
\|u(x_0, \cdot)\|^2_{1, \Omega_{\text{int}}} + \|u_{x_0}(x_0, \cdot)\|^2_{0, \Omega_{\text{int}}} \leq C_T(\|u(t_0, \cdot)\|^2_{1, \Omega_{\text{int}}} + \|u_{x_0}(t_0, \cdot)\|^2_{0, \Omega_{\text{int}}}),
\]

where \( T \) is arbitrary, \( t_0 \leq x_0 \leq T. \)

As in the case of Theorem 2.3 the estimate (2.8) implies that \( D_+(\Omega_{\text{ext}} \times \{x_0 = t_0\}) \subset \overline{\Omega}_{\text{ext}} \times \mathbb{R}, \ i.e. \ \Omega_{\text{int}} \times \mathbb{R} \) is a white hole.

### 3 The ergosphere.

The ergosphere is the surface \( S(x) = 0 \) where

\[
g_{00}(x) = 0. \tag{3.1}
\]

We assume that \( S(x) = 0 \) is a closed smooth surface, \( g_{00}(x) > 0 \) in the exterior of \( S(x) = 0 \) and \( g_{00}(x) < 0 \) in the interior of \( S(x) = 0 \) near \( S(x) = 0. \) We say that \( S(x) = 0 \) is a smooth surface if \( S_x(x) \neq 0 \) when \( S(x) = 0. \)

Let \( \Delta(x) = \det[g^{ik}(x)]_{i,k=1}^n. \) It is easy to show (cf. [E1]) that \( g_{00}(x) = 0 \) if and only if \( \Delta(x) = 0. \) In the case of the Gordon equation (cf. (1.4)) the equation of the ergosphere is

\[
|w(x)|^2 = \frac{c^2}{n^2(x)}. \tag{3.2}
\]

Note that \( (0, \xi) \) is not a characteristic direction for (1.1) for any \( \xi \neq 0 \) when \( g_{00}(x) > 0. \) It may happen that the ergosphere \( S(x) = 0 \) is also a characteristic surface, i.e. \( \{S(x) = 0\} \times \mathbb{R} \) is an event horizon. The celebrated example of such situation is the Schwarzschild black hole. The
Schwarzchchild metric has the following form in the Cartesian coordinates (cf. [V2]):

\[ ds^2 = (1 - \frac{2m}{R})dt^2 - dx^2 - dy^2 - dz^2 - \frac{4m}{R} dtdR - \frac{2m}{R} (dR)^2, \]

where \( R = \sqrt{x^2 + y^2 + z^2}. \) Therefore

\[ g_{00} = 1 - \frac{2m}{R} = 0 \]

is the ergosphere. Note that \( R = 2m \) is also a characteristic surface and \( \{ R = 2m \} \times \mathbb{R} \) is a black hole. We shall call the black hole such that the ergosphere is also an event horizon the Schwarzchchild type black holes.

Note that Schwarzchchild type black holes is unstable with respect to the perturbations of metrics (see §5 below). If we perturb the metric the ergosphere persists since we assume that the equation (3.1) is smooth, i.e. it has a non-vanishing normal at any point. However the perturbed surface may cease to be a characteristic surface and there is no characteristic surface near by (cf. §5).

It is easier to study the behavior of the solutions of (1.1) in the exterior of the Schwarzchield type black hole than in the case when the black hole is inside the ergosphere. For example, an important problem in the general relativity studied in [DR] is the uniform boundedness of solutions of (1.1) in the exterior of the black hole. Using the ideas from [DR] one can prove the following theorem:

**Theorem 3.1.** (cf. [E3]) Let \( \{ S_0(x) = 0 \} \times \mathbb{R} \) be the ergosphere and the boundary of a black hole, \( S_{0x}(x) \neq 0 \) when \( S_0(x) = 0 \). Consider the Cauchy problem for (1.1) in the exterior \( \Omega_{ext} \times (0, +\infty) \) of the black hole with the initial conditions

\[ u(0, x) = \varphi_0, \quad u_0(0, x) = \varphi_1(x), \quad x \in \Omega_{ext}, \]

where \( \varphi_0(x), \varphi_1(x) \) are smooth and rapidly decaying when \( |x| \to \infty \). Then \( u(x_0, x) \) is uniformly bounded in \( \Omega_{ext} \times (0, +\infty) \):

\[ |u(x_0, x)| \leq C. \]
4 The case of two space dimensions.

Let $S$ be an ergosphere, i.e. $\Delta(x) = g^{11}(x)g^{22}(x) - (g^{12}(x))^2 = 0$, $x = (x_1, x_2) \in \mathbb{R}^2$. We assume that $S$ is a smooth Jordan curve. Let $S_1$ be another closed Jordan curve inside $S$. Denote by $\Omega$ the region between $S$ and $S_1$. We assume that $\Delta(x) < 0$ in $\Omega \setminus S$. Let $K_+(y)$ be the half-cone of all forward time-like directions at $y \in S_1$, i.e. $K_+(y) = \{(\dot{x}_0, \dot{x}_1, \dot{x}_2) \in \mathbb{R}^3 : \sum_{j,k=0}^2 g_{jk}(y)\dot{x}_j\dot{x}_k > 0, \dot{x}_0 > 0\}$. Let $N(y)$ be the outward normal to $S_1$. We assume that either $(\dot{x}_0, \dot{x}_1, \dot{x}_2) \cdot (0, N(y)) > 0$ or $(\dot{x}_0, \dot{x}_1, \dot{x}_2) \cdot (0, N(y)) < 0$ for all $y \in S_1$ and all $(\dot{x}_0, \dot{x}_1, \dot{x}_2) \in K_+(y)$.

Remark 4.1 The interior of $S_1$ is called a trapped region if $(\dot{x}_0, \dot{x}_1, \dot{x}_2) \cdot (0, N(y)) < 0$ for all $y \in S_1$ and $(\dot{x}_0, \dot{x}_1, \dot{x}_2) \in K_+(y)$.

The main result of [E2] is the following theorem:

Theorem 4.1. Suppose the ergosphere $S$ is not characteristic for any $x \in S$, i.e.

\begin{equation}
(\dot{x}_0, \dot{x}_1, \dot{x}_2) \cdot (0, N(y)) > 0
\end{equation}

or

\begin{equation}
(\dot{x}_0, \dot{x}_1, \dot{x}_2) \cdot (0, N(y)) < 0
\end{equation}

for all $y \in S_1$ and all $(\dot{x}_0, \dot{x}_1, \dot{x}_2) \in K_+(y)$.

Sketch of the proof of Theorem 4.1

Since $\Delta(x) < 0$ in $\Omega$ there exist two families $S^\pm(x) = \text{const}$ of characteristic curves

\begin{equation}
\sum_{j,k=1}^2 g^{jk}(x)\nu_j(x)\nu_k(x) \neq 0, \quad \forall x \in S,
\end{equation}

where $\nu(x) = (\nu_1(x), \nu_2(x))$ is the normal to $S$ at $x \in S$. Suppose the condition (4.1) is satisfied on $S_1$. Then there exists a smooth Jordan curve $S_0(x) = 0$ between $S$ and $S_1$ such that $\{S_0(x) = 0\} \times \mathbb{R}$ is a characteristic surface, i.e. $\{S_0(x) = 0\} \times \mathbb{R}$ is an event horizon.
and
\[(4.3) \quad f_1^\pm(x)S_{x_1}^\pm(x) + f_2^\pm(x)S_{x_2}^\pm(x) = 0 \quad \text{in } \overline{\Omega}.\]

Consider two systems of differential equations:
\[(4.4) \quad \frac{dx^\pm(\sigma)}{d\sigma} = f^\pm(x(\sigma)), \quad \sigma \geq 0, \quad x^\pm(0) = y \in S.\]

Let
\[(4.5) \quad \frac{dx_j(s)}{ds} = 2 \sum_{k=0}^{2} g^{jk}(x(s))\xi_k(s), \quad x_j(0) = y_j, \quad 0 \leq j \leq 2,
\]
\[\frac{d\xi_p(s)}{ds} = -\sum_{j,k=0}^{2} g_{xp}(x(s))\xi_j(s)\xi_k(s), \quad \xi_p(0) = \eta_p, \quad 0 \leq p \leq 2,\]
be the equations of null-bicharacteristics for (1.1). Note that \(\frac{d\xi_0(s)}{ds} = 0\), i.e. \(\xi_0(s) = \eta_0\) for all \(s \geq 0\) and we take \(\eta_0 = 0\). Therefore (4.5) is a null-bicharacteristic if \(\sum_{j,k=1}^{2} g^{jk}(y)\eta_j\eta_k = 0\).

It can be shown that the curves \(x = x^\pm(\sigma)\) of (4.4) are the projections on \((x_1, x_2)\)-plane of some forward null-bicharacteristics. Since \(\frac{dx_0(s)}{ds} \neq 0\) on these bicharacteristics we can use \(x_0\) as a parameter instead of \(\sigma\). It appears that for one family (say \(x = x^+(\sigma)\)) \(\sigma\) is decreasing when \(x_0\) is increasing and for another family (\(x = x^-(\sigma)\)) \(\sigma\) is increasing when \(x_0\) is increasing.

The condition (4.1) is equivalent to the condition that the projections on \((x_1, x_2)\)-plane of all forward null-bicharacteristics are either leaving \(\Omega\) when \(x_0\) is increasing or are entering \(\Omega\) when \(x_0\) is increasing. Suppose for the definiteness that the projections of all null-bicharacteristics are leaving \(\Omega\) when \(x_0\) is increasing. Then the trajectory \(x = x^-(\sigma)\) starting on \(S\) can not reach \(S_1\) (cf. [E2]). Therefore by the Poincare-Bendixson theorem there exists a limit cycle \(S_0(x) = 0\), i.e. a closed Jordan curve in \(\Omega\) that is a characteristic curve. Therefore \(\{S_0(x) = 0\} \times \mathbb{R}\) is an event horizon, i.e. a boundary of either black or white hole.

**Remark 4.2** Since conditions (4.1), (4.2) hold when we slightly perturb the metric the black and white holes obtained by Theorem (4.1) are stable.

**Example 4.1** (cf. [V1]) Consider the acoustic equation with the metric (1.5) when \(n = 2, \rho = c = 1,\)
\[(4.6) \quad v(x) = (v^1(x), v^2(x)) = \frac{A}{r} \hat{r} + \frac{B}{r} \hat{\theta},\]
where \( r = |x| = \sqrt{x_1^2 + x_2^2} \), \( \hat{r} = \frac{x}{|x|} \), \( \hat{\theta} = (-\frac{x_2}{|x|}, \frac{x_1}{|x|}) \), \( A > 0 \), \( B > 0 \) are constants. We assume that \( \sqrt{A^2 + B^2} \geq |x| > r_1 \), where \( r_1 < A \). The ergosphere in this case is \( r = \sqrt{A^2 + B^2} \). The differential equations (4.4) have the following form in the polar coordinates \((r, \theta)\):

\[
\frac{dr}{ds} = A^2 - r^2, \quad \frac{d\theta}{ds} = \frac{AB}{r} + \sqrt{A^2 + B^2 - r^2},
\]

and

\[
\frac{dr}{ds} = -1, \quad \frac{d\theta}{ds} = \frac{1 - \frac{B^2}{r^2}}{\frac{AB}{r} + \sqrt{A^2 + B^2 - r^2}}.
\]

It follows from (4.7) that \( r = A \) is a limit cycle and \( \{r < A\} \times \mathbb{R} \) is a white hole.

**Example 4.2** Consider the same situation as in Example 4.1 with \( B = 0 \) and the domain \( r < A \). Then \( r = A \) is the ergosphere and \( \{r = A\} \times \mathbb{R} \) is the event horizon. Since \( A > 0 \) we have that \( \{r < A\} \times \mathbb{R} \) is a white hole. Note that the equations (4.4) have the following form in the polar coordinates:

\[
\frac{dr}{d\theta} = \pm \sqrt{A^2 - r^2}, \quad r(\theta_0) = A.
\]

It has a solution \( r = A \), which is the event horizon, and it also has other solutions \( r = A \cos(\theta - \theta_0) \) that touch the event horizon at \( \theta = \theta_0 \). In general situation when the ergosphere coincides with the event horizon the solution of (4.4) are also tangent to the event horizon.

**Remark 4.3** When \( B \neq 0 \) is small Example 4.1 can be viewed as a perturbation of Example 4.2. Note that when \( B \) is small the stable event horizon \( r = A \) will be close to the ergosphere \( r = \sqrt{A^2 + B^2} \). A similar situation will happen in the general case:

If the left hand side of the condition (4.2) is small then the stable event horizon obtained in Theorem 4.1 will be close to the ergosphere \( S \). If the left hand side of (4.2) changes sign on \( S \) (for example, if \( B \) in (4.6) depends on \( \theta \) and changes sign when \( 0 \leq \theta \leq 2\pi \)) there will be no event horizon near \( S \).
5 Axisymmetric metrics and rotating black holes.

Let \((\rho, \varphi, z)\) be the cylindrical coordinates in \(\mathbb{R}^3\):

\[
(5.1) \quad x = \rho \cos \varphi, \quad y = \rho \sin \varphi, \quad z = z.
\]

A stationary axisymmetric metric in \(\mathbb{R}^3 \times \mathbb{R}\) is the metric that does not depend on \(t\) and \(\varphi\). For the convenience, we shall use the following notations:

\[
(5.2) \quad y_0 = t, \quad y_1 = \rho, \quad y_2 = z, \quad y_3 = \varphi.
\]

Then the stationary axisymmetric metric has the form:

\[
(5.3) \quad ds^2 = \sum_{j,k=0}^{3} g_{jk}(\rho, z)dy_jdy_k,
\]

where \(g_{jk}(\rho, z)\) are smooth and even in \(\rho\). Denote

\[
[g^{jk}(\rho, z)]_{j,k=0}^{3} = \left( [g_{jk}(\rho, z)]_{j,k=0}^{3} \right)^{-1}.
\]

The ergosphere is given by the equation

\[
(5.4) \quad g_{00}(\rho, z) = 0,
\]

or, equivalently:

\[
(5.5) \quad \Delta(\rho, z) = \det [g^{jk}(\rho, z)]_{j,k=1}^{3} = 0.
\]

We will be looking for the rotating black and white holes, i.e. when the event horizon has the form:

\[
(5.6) \quad \{ S(\rho, z) = 0 \} \times S^1 \times \mathbb{R},
\]

where \(S(\rho, z) = 0\) is a closed smooth curve in the \((\rho, z)\)-plane, even in \(\rho, \varphi \in S^1, \ t \in \mathbb{R}, \ S^1\) is the unit circle. More precisely, we have to take in \((5.5)\) the restriction of the curve \(S(\rho, z) = 0\) to the half-plane \(\rho \geq 0\) but we did not indicate this in \((5.6)\) for the simplicity of notation. Since \((5.6)\) is the event horizon it must be a characteristic surface, i.e.

\[
(5.7) \quad \sum_{j,k=1}^{2} g^{jk}(\rho, z)S_{y_j}(\rho, z)S_{y_k}(\rho, z) = 0 \quad \text{on} \quad S(\rho, z) = 0.
\]
Here \( y_1 = \rho, \ y_2 = z \). Therefore \( \{ S(\rho, z) = 0 \} \times \mathbb{R} \) is the event horizon for the tensor \( [g^{jk}(\rho, z)]_{j,k=0}^2 \), i.e. for the case of two dimensions considered in the previous section.

Define

\[
\Delta_1(\rho, z) = \det[g^{jk}(\rho, z)]_{j,k=1}^2 = g^{11}(\rho, z)g^{22}(\rho, z) - (g^{12}(\rho, z))^2.
\]

We shall call the curve \( \Delta_1(\rho, z) = 0 \) the restricted ergosphere since it is the ergosphere of the two-dimensional problem for \( [g^{jk}(\rho, z)]_{j,k=0}^2 \). We can extend all results of §4 to the case of rotating black and white holes. For example, let the curve \( \Delta_1(\rho, z) = 0 \) be a Jordan curve such that

\[
\sum_{j,k=1}^2 g^{jk}(\rho, z)\nu_j(\rho, z)\nu_k(\rho, z) \neq 0 \quad \text{on} \quad \Delta_1(\rho, z) = 0,
\]

and let \( S_1 \) be a Jordan curve inside \( \Delta_1(\rho, z) = 0 \).

Suppose conditions (4.1) are satisfied where the matrix \( [\tilde{g}_{jk}(\rho, z)]_{j,k=0}^2 \) is the inverse to \( [g^{jk}(\rho, z)]_{j,k=0}^2 \). Then there exists a Jordan curve \( S_0(\rho, z) = 0 \) between \( \Delta_1 = 0 \) and \( S_1 \) such that \( \{ S_0(\rho, z) = 0 \} \times S^1 \times \mathbb{R} \) is the event horizon in \( \mathbb{R}^3 \times \mathbb{R} \).

Consider now the problem of the stability of the black and white holes with respect to the perturbations of metrics.

The famous example of an axisymmetric metric is the Kerr metric. The Kerr metric in the Kerr-Schild coordinates has the form (see [V2]):

\[
ds^2 = dt^2 - dx^2 - dy^2 - dz^2 - \frac{2mr^3}{r^4 + a^2 z^2} \left[ dt + \frac{r(xdx + ydy)}{r^2 + a^2} + \frac{a(ydx - xdy)}{r^2 + a^2} + \frac{zdz}{r} \right]^2,
\]

where

\[
r(x, y, z) = \sqrt{\frac{(R^2 - a^2) + \sqrt{(R^2 - a^2)^2 + 4a^2 z^2}}{2}}, \quad R^2 = x^2 + y^2 + z^2.
\]

It follows from (5.4) and (5.9) that the ergosphere is

\[
r^4 + a^2 z^2 - 2mr^3 = 0.
\]
One can show that (5.11) consists of two curves in \((\rho, z)\)-plane

\[
\begin{align*}
(5.12) & \quad r - \left( m + \sqrt{m^2 - \frac{a^2 z^2}{r^2}} \right) = 0, \\
(5.13) & \quad r - \left( m - \sqrt{m^2 - \frac{a^2 z^2}{r^2}} \right) = 0,
\end{align*}
\]

Equation (5.12) defines the outer ergosphere and (5.13) defines the inner ergosphere for the Kerr metric.

Compute the restricted ergosphere \(\Delta_1(\rho, z)\) (cf. (5.8)) for the Kerr metric. The inverse to the Kerr metric tensor has the form:

\[
\eta^{jk} + \frac{2mr^3}{r^4 + a^2 z^2} l^j l^k,
\]

where

\[
(l^0, l^1, l^2, l^3) = \left( -1, \frac{rx + ay}{r^2 + a^2}, \frac{ry - ax}{r^2 + a^2}, \frac{z}{r} \right).
\]

In the \((\rho, z, \varphi)\) coordinates we have

\[
\begin{align*}
(5.15) & \quad g^{ik}(\rho, z) = \xi^{ik} + \frac{2mr^3}{r^4 + a^2 z^2} m^i m^k,
\end{align*}
\]

where \((m^0, m^1, m^2, m^3) = (-1, \frac{r^0}{r^2 + a^2}, \frac{z}{r^2 + a^2}, \frac{a}{r^2 + a^2})\), \(\xi^{ik}\) is \(\eta^{ik}\) in the cylindrical coordinates, \(\xi^{00} = 1, \, \xi^{11} = \xi^{22} = -1, \, \xi^{33} = -\frac{1}{r^2},\) \(\xi^{jk} = 0\) for \(j \neq k\). Therefore

\[
(5.16) \quad \Delta_1(\rho, z) = 1 - \frac{2mr^5 \rho^2}{(r^4 + a^2 z^2)(r^2 + a^2)^2} - \frac{2mrz^2}{r^4 + a^2 z^2}.
\]

The equation \(\Delta_1 = 0\) for the Kerr metric can be substantially simplified.

**Proposition 5.1.** (cf. [E3]) The equation \(\Delta_1(\rho, z) = 0\) is equivalent to two equations \(r - r_+ = 0\) and \(r - r_- = 0\) where \(r_\pm = m \pm \sqrt{m^2 - a^2}\).

It happens that \(r = r_\pm\) are two event horizons, \(r - r_+ = 0\) is called the outer event horizon and \(r - r_- = 0\) is the inner event horizon. More exactly, \(\{r = r_\pm\} \times S^1 \times \mathbb{R}\) are the event horizons. Therefore \(\Delta_1(\rho, z) = 0\) relates explicitly the event horizons of the Kerr metric to the metric tensor.
Definition 5.1 Let \( \{ \psi = 0 \} \times S^1 \times \mathbb{R} \) be the event horizon for the metric \([g_{jk}]_{j,k=0}^3\). We say that this event horizon is stable in the class of axisymmetric metrics if any smooth family \([g_{\varepsilon jk}]\) of axisymmetric metrics, \(0 \leq \varepsilon \leq \varepsilon_0\), \(g_{0jk} = g_{jk}\), has a smooth family of event horizons \(\{ \psi_\varepsilon = 0 \} \times S^1 \times \mathbb{R}\) such that \(\psi_0 = \psi\). Otherwise we say that \(\{ \psi = 0 \} \times S^1 \times \mathbb{R}\) is an unstable event horizon.

We restrict perturbations to a more narrow class of axisymmetric metrics of the form:

\[
g_{\varepsilon jk} = \xi_{jk} + v^j_\varepsilon(\rho, z)v^k_\varepsilon(\rho, z)\tag{5.17}
\]

Note that the Kerr metric and the metrics (1.4), (1.5) have the form (5.17).

Proposition 5.2. (cf. [E3]) Let \(\Delta_1 = 0\) be the restricted ergosphere, and let \(\Delta_1 = 0\) be a characteristic curve, i.e. \(\{ \Delta_1 = 0 \} \times S^1 \times \mathbb{R}\) is an event horizon. Then this event horizon is unstable when we consider perturbations in the class of the metrics of the form (5.17). In particular, the outer and the inner event horizons for the Kerr metric are unstable.

Sketch of the proof of Proposition 5.2

Denote \(\Delta_1^\varepsilon = g_{\varepsilon 11}g_{\varepsilon 22} - (g_{\varepsilon 12})^2\). Then \(\Delta_1^\varepsilon = 0\) is a smooth perturbation of restricted ergosphere, \(\Delta_0^\varepsilon = \Delta_1\). We can choose perturbations of the form (5.17) such that \(\Delta_1^\varepsilon = 0\) will not be a characteristic curve for \(0 < \varepsilon \leq \varepsilon_0\). Moreover, one can choose \([g_{\varepsilon jk}]\) such that there is no characteristic curve near \(\Delta_1^\varepsilon = 0\) (cf. [E3]).

In the next proposition we shall prove that there is a rich class of perturbations \([g_{\varepsilon jk}]\), \(0 \leq \varepsilon \leq \varepsilon_0\), of the Kerr metric that have a smooth family of event horizons \(\{ \Delta_1^\varepsilon = 0 \} \times S^1 \times \mathbb{R}\) such that \(\{ \Delta_0^\varepsilon = 0 \} \times S^1 \times \mathbb{R}\) is the Kerr event horizon.

Proposition 5.3. (cf. [E3]) Let \(\Delta_1 = 0\) be a restricted ergosphere and \(\{ \Delta_1 = 0 \} \times S^1 \times \mathbb{R}\) is an event horizon. Let \(\Delta_1^\varepsilon = 0\) be arbitrary family of closed, even in \(\rho\), smooth curves, \(0 \leq \varepsilon \leq \varepsilon_0\), such that \(\Delta_0^\varepsilon = \Delta_1\). Then there exists a family of metrics \([g_{\varepsilon jk}]\), \(0 \leq \varepsilon \leq \varepsilon_0\), \([g_{0jk}] = [g_{jk}]\), of the form (5.17) such that \(\Delta_1^\varepsilon = 0\) are restricted ergospheres for \([g_{\varepsilon jk}]\) and \(\{ \Delta_1^\varepsilon = 0 \} \times S^1 \times \mathbb{R}\) are event horizons for \([g_{\varepsilon jk}]\).

Remark 5.1 We shall call the event horizons obtained in Theorem 4.1 the stable event horizons and the event horizons that coincide with \(\{ \Delta_1 = 0\}\) the unstable event horizons.
0} \times S^1 \times \mathbb{R}$ the Schwarzschild type event horizons. According to this definition the Kerr event horizon is a Schwarzschild type event horizon.

Fix some axisymmetric metric $[g_{jk}]$ that has a Schwarzschild type event horizon $\{\Delta_1 = 0\} \times S^1 \times \mathbb{R}$. The proposition 5.3 shows that in any neighborhood of $[g_{jk}]$ there are metrics having Schwarzschild type event horizons close to $\{\Delta_1 = 0\} \times S^1 \times \mathbb{R}$. Also it follows from the Remark 4.3 that in any neighborhood of $[g_{jk}]$ (in particular, in any neighborhood of the Kerr metric) there are stable event horizons close to $\{\Delta_1 = 0\} \times S^1 \times \mathbb{R}$.

6 Determination of the ergosphere by the boundary measurements.

Let $u(x_0, x)$ be the solution of (1.1) in a cylinder $\Omega \times \mathbb{R}$ satisfying the zero initial conditions

\begin{equation}
(6.1) \quad u = 0 \quad \text{for} \quad x_0 \ll 0, \quad x \in \Omega,
\end{equation}

and the boundary condition

\begin{equation}
(6.2) \quad u|_{\partial \Omega \times \mathbb{R}} = f.
\end{equation}

Here $\Omega$ is a smooth bounded domain in $\mathbb{R}^n$, $f$ is a smooth function with a compact support in $\partial \Omega \times \mathbb{R}$. The solution of the initial-boundary problem (1.1), (6.1), (6.2) exists and is unique assuming that $\partial \Omega \times \mathbb{R}$ is not characteristic at any point and $g^{00}(x) > 0$ on $\overline{\Omega}$.

Denote by $\Lambda f$ the following operator (the DN operator):

\begin{equation}
(6.3) \quad \Lambda f = \sum_{j,k=1}^{n} g^{jk}(x) \frac{\partial u}{\partial x_j} \nu_k(x) \left( \sum_{\nu_1, \ldots, \nu_n=1}^{n} g^{pr}(x) \nu_\nu \nu_r \right)^{-\frac{1}{2}} \Bigg|_{\partial \Omega \times \mathbb{R}},
\end{equation}

where $(\nu_1(x), \ldots, \nu_n(x))$ is the outward unit normal to $\partial \Omega$. Let $\Gamma$ be an open subset of $\partial \Omega$. We say that boundary measurements on $\Gamma \times (0, T)$ are taken if we know $\Lambda f$ on $\Gamma \times (0, T)$ for all $f$ with support in $\overline{\Gamma} \times [0, T]$.

The inverse boundary value problem is the determination of $[g_{jk}(x)]_{j,k=0}^n$ knowing the boundary measurements on $\Gamma \times (0, T)$. Let

\begin{equation}
(6.4) \quad \dot{x} = \varphi(x), \quad \dot{x}_0 = x_0 + a(x),
\end{equation}
where \( \hat{x} = \varphi(x) \) is a diffeomorphism of \( \Omega \) onto a new domain \( \tilde{\Omega} \), \( a(x) \in C^\infty(\Omega) \). We assume that

\begin{equation}
(6.5) \quad \varphi(x) = x \quad \text{on} \quad \Gamma, \quad a(x) = 0 \quad \text{on} \quad \Gamma.
\end{equation}

Note that (6.4) transform (1.1) to an equation of the same form with a new tensor \( [\hat{g}^{jk}(\hat{x})]_{j,k=0}^n \) isometric to the old one. It follows from (6.5) that DN operator \( \Lambda \) does not change under the change of variables (6.4), (6.5). If there exists an event horizon inside \( \Omega \times \mathbb{R} \) then we cannot determine the metric inside the event horizon since any change of metric inside the event horizon will not change boundary measurements. But we can try to recover the event horizon itself (up to diffeomorphism (6.4), (6.5)).

This is an open problem. We can prove only that the boundary measurements allow to determine the ergosphere.

**Theorem 6.1.** (cf. [E3]) Consider the wave equation (1.1). Assume that \( g^{00}(x) > 0 \) on \( \Omega \) and the normal to \( \partial \Omega \) is not characteristic at any point of \( \partial \Omega \). Let \( \Delta(x) = 0 \) be the ergosphere, \( \Delta(x) = 0 \) is a smooth closed surface, \( \Delta(x) > 0 \) in \( \tilde{\Omega} \) outside of \( \Delta(x) = 0 \). Let \( \Gamma \) be an open subset of \( \partial \Omega \). Then the boundary measurements on \( \Gamma \times (0, +\infty) \) determine \( \Delta(x) = 0 \) up to the change of variables (6.4), (6.5).

Note that for the proof of Theorem 6.1 it does not matter whether the ergosphere is an event horizon or not. The proof is an extension of the proof of Theorem 2.3 in [E1].

We will determine the ergosphere by determining the metric in \( \Omega \cap \Omega_{\text{ext}} \), where \( \Omega_{\text{ext}} \) is the exterior of \( \Delta(x) = 0 \).

We start with the determination of the metric in a small neighborhood of \( \Gamma \) and gradually continue to recover the metric deeper in \( \Omega \). As we progress the time interval \( (0, T) \) needed to reach the point \( x \in \Omega \) increases when the point approaches the ergosphere. One can show that \( T \to +\infty \) when \( x \to \{ \Delta(x) = 0 \} \). This is the reason why one needs the unlimited time interval \( (0, +\infty) \) to recover the ergosphere.

**Remark 6.1** Let \( L(x, \frac{\partial}{\partial x}, \frac{\partial}{\partial x_0})u(x_0, x) = 0 \) be the equation (1.1) in \( \mathbb{R}^{n+1} \). Making the Fourier transform in \( x_0 \) we get

\begin{equation}
(6.6) \quad L(x, \frac{\partial}{\partial x}, ik) \hat{u}(k, x_0) = 0, \quad x \in \mathbb{R}^n.
\end{equation}

Suppose that

\begin{equation}
(6.7) \quad L(x, \frac{\partial}{\partial x}, ik) = -\Delta - k^2 \quad \text{for} \quad |x| > R.
\end{equation}
Let $a(\theta, \omega, k)$ be the scattering amplitude for the operator (6.6). It is well known that the scattering amplitude given for all $k > 0$, $\theta \in S^{n-1}$, $\omega \in S^{n-1}$, determines the DN operator on the $\{|x| = R\} \times [0, +\infty)$. Therefore by the Theorem 6.1 the scattering amplitude determines the ergosphere. Note that when (6.7) holds $a(\theta, \omega, k)$ is real analytic in $(\theta, \omega, k)$. Therefore it is enough to know $a(\theta, \omega, k)$ in an arbitrary neighborhood of some point $(\theta_0, \omega_0, k_0)$ to determine the ergosphere.

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