Polarization angle dependence of the breathing modes in confined one-dimensional dipolar bosons

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Probing the radial collective oscillation of a trapped quantum system is an accurate experimental tool to investigate interactions and dimensionality effects. We consider a fully polarized quasi-one dimensional dipolar quantum gas of bosonic dysprosium atoms in a parabolic trap at zero temperature. We model the dipolar gas with an effective quasi-one dimensional Hamiltonian in the single-mode approximation, and derive the equation of state using a variational approximation based on the Lieb-Liniger gas Bethe Ansatz wavefunction or perturbation theory. We calculate the breathing mode frequencies while varying polarization angles by a sum-rule approach, and find them in good agreement with recent experimental findings.

I. INTRODUCTION

Systems with long-range interactions present a host of exotic quantum states of matter including Wigner crystals [1, 2], Haldane Insulators [3] or Fulde-Ferrell-Larkin-Ovchinnikov phases [4], thanks to the interplay between quantum fluctuations and the frustrating effects of interactions. In particular, the advent of degenerate quantum gases consisting of atoms where strong dipolar forces provide the interactions, has even revealed the coexistence of both crystalline order and superfluidity, the so-called supersolidity [5–7].

Recently, the possibility of forming one-dimensional tubes of bosonic Dy atoms with tunable strength of the contact and dipolar interactions [8] has opened the fascinating perspective of investigating the interplay between quantum fluctuations, enhanced in reduced dimensionality, and interaction-driven fluctuations, leading to unconventional mechanism and the so-called scar states [9]. In fact, although in one dimension repulsive dipolar interaction decaying as $1/r^3$ at long distance are classified as finite-ranged interaction, they are expected to push bosonic systems to a regime of stronger interactions [10–12].

Since the majority of existing ultracold-gas experiments are carried out with spatially inhomogeneous systems, due to the presence of an external confining potential, exciting oscillations of the gas density distribution in such a confined geometry has been demonstrated to be a reliable, basic tool for investigating the spectrum of collective excitations and the phase diagram [13–15].

From this perspective, one-dimensional (1D) gases show their own peculiarities [16, 17]. A paradigmatic example is the exactly solvable Lieb-Liniger gas [18], where at infinite contact interaction strength $g_{1D} \to \infty$ the many-body excitation spectrum becomes identical to that of a free Fermi gas, known as the Tonks-Girardeau (TG) gas [19]. The presence of an external parabolic potential renders the low-lying part of the excitation spectrum discrete, where the simplest mode to be excited among the low-lying ones after small instantaneous changes of the trapping frequency $\omega_z$ is the so-called breathing (or compressional) mode, i.e. the uniform radial expansion and contraction of the system. The breathing-mode frequency $\omega_b$ depends on the interaction strength $g_{1D}$, the number of particles $N$ in the trap, and the gas temperature $T$. It has been previously shown that the frequency ratio $\omega_b/\omega_z$ presents two crossovers as a function of increasing interaction: from the value $2$ down to $\sqrt{3}$ while going from non-interacting to weakly interacting regime, and then back to $2$ after crossing towards the strongly interacting limit [20]. Theoretical descriptions based on local density approximation (LDA) [16], time-dependent Hartree method [21], and diffusion Quantum Monte Carlo simulations [22] have been produced following the system across the different regimes.

Here, we focus on the breathing mode of a one-dimensional dipolar quantum gas, and investigate the influence of both the dipole orientation and of the interplay between zero (contact) and finite-range (dipolar) interaction allowing for independent tuning of these two interactions. Our analysis is based on a sum-rule approach [16] that allows to extract the breathing mode frequency from ground-state density profiles obtained after solving the stationary generalized Gross-Pitaevskii equation. The latter is generalized by replacing the Hartree-term with the energy per unit length of the bulk quasi-one dimensional dipolar system, obtained using either a Bethe Ansatz wave-function in a variational calculation [23], or using a perturbative approach.

The results show that when dipolar interactions are attractive, the system manifests an incipient instability at
low-density and a sharp minimum is found in the breathing mode which is very peculiar of that finite-range interaction. In the repulsive regime an extension of the stability regime is instead observed. A good agreement with the experimental results reported in Ref. 9 and 24 is also found.

The paper is organized as follows. We introduce the model Hamiltonian and the generalized Gross-Pitaevskii equation in Sec. II. Then in Sec. III we discuss the equation of state by separating the short-range terms from the soft dipolar long range interaction in the single-mode approximation. In Sec. IV we present the results for the breathing mode by discussing the case of the repulsive and attractive interactions and follow the evolution of this quantity on varying the dipoles orientation $\theta$. Finally in Sec. V we give conclusions and discuss perspectives.

II. THE MODEL AND THE GENERALIZED GROSS-PITAEVSKII EQUATION

In highly elongated traps the atomic motion in the plane transverse to the longitudinal direction is described by the Hamiltonian:

$$H_\perp = \frac{p_i^2 + p_j^2}{2m} + \frac{\omega_\perp^2}{2}(x^2 + y^2),$$

(1)

where $m$ is the mass particle and $\omega_\perp$ is the confining harmonic oscillator frequency. When the frequency $\omega_\perp \gg \omega_{ho}$ is sufficiently larger than the longitudinal trapping frequency $\omega_{ho}$, the many-body wavefunction of the atoms can be projected on the ground-state manifold of the Hamiltonian (1) [25]. This the so-called single-mode approximation (SMA).

The effective Hamiltonian in the projected subspace depends only on the coordinates along the $z$ axis. Its expression is [26, 27]

$$H_{1D} = -\frac{\hbar^2}{2m} \sum_i \frac{\partial^2}{\partial z_i^2} + g_{1D} \sum_{i<j} \delta (z_i - z_j)$$

$$+ \sum_i V_{ext}(z_i) + \sum_{i<j} V_{Q1D}(z_i - z_j),$$

(2)

where $V_{ext}(z) = \frac{1}{2}m\omega_{ho}^2z^2$ is the potential energy of the parabolic trap along the longitudinal $z$-direction, $g_{1D}$ is the contact interaction coming from Van der Waals or other short-ranged interactions, and the effective 1D dipole-dipole interaction $V_{Q1D}(z)$ in the single-mode approximation reads: [26]

$$V_{Q1D}(z/l_\perp) = V(\theta) \left[ V_{1D}^{DDI} \left( \frac{z}{l_\perp} \right) - \frac{8}{3} \delta \left( \frac{z}{l_\perp} \right) \right],$$

(3)

where

$$V(\theta) = \frac{\mu_0 \mu_D^2}{4\pi} \left[ 1 - \frac{3 \cos^2 \theta}{4l_\perp} \right]$$

(4)

encodes the sign and the effective strength of the interaction driven by the vacuum magnetic permeability $\mu_0$, the magnetic dipolar moment $\mu_D$ of the given atomic species, the angle $\theta$ between the dipoles orientation and the longitudinal $z$-axis, and the transverse oscillator length $l_\perp = \sqrt{\hbar/(m\omega_\perp)}$. The adimensional form of effective 1D dipolar potential $V_{1D}^{DDI}$ is:

$$V_{1D}^{DDI} \left( \frac{z}{l_\perp} \right) = -2 \left[ \frac{z_i - z_j}{l_\perp} \right] + 2 \left[ \frac{z_i - z_j}{l_\perp} \right]^2 + 2 \left[ \frac{z_i - z_j}{l_\perp} \right]^3 + \frac{1}{2} \left[ \frac{z_i - z_j}{l_\perp} \right]^4, \quad (5)$$

In the $^{162}$Dy case relevant to current experiments [9], $\mu_D = 9.93 \mu_B$ [8].

At zero temperature, the Gross-Pitaevskii theory [28–30] provides a good description of weakly-interacting three dimensional atomic Bose-Einstein condensates, yet it requires modifications either with strong interactions or reduced dimensionality. In the original form, without dipolar interaction, the energy functional in the Gross-Pitaevskii approximation is [28, 29]

$$F_{GP} = \int dz \left[ \frac{\hbar^2}{2m} \nabla \phi \nabla \phi^* + \frac{g_{1D}}{2} |\phi|^4 + \frac{1}{2} |\phi|^2 \left( \frac{1}{\mu} - \frac{1}{\mu_D} \right) \right],$$

(6)

where $\phi(z, t)$ is the BEC order parameter, $n(z, t) = |\phi(z, t)|^2$ is the boson density, and $\mu$ the chemical potential. In one dimension and in the case of hard-core bosons [19], Kolomeisky et al. have proposed a modification of the Gross-Pitaevskii equation to describe the Tonks-Girardeau regime [31], where the Hartree term $g_{1D}|\phi|^4/2$ is replaced by the energy density of the hard core boson (or free spinless fermion [19, 32]) gas i.e $\frac{1}{2} \hbar^2 \pi^2 \frac{1}{|\phi|^2}$. Such approach can be viewed as taking the classical limit in the bosonized Hamiltonian of spinless fermions with quadratic dispersion [33]. Afterwards, different proposals [25, 34, 35] have been offered to cover both the weakly and the strongly interacting regimes, one of them amounts to replace the Hartree term with an energy-density functional [25, 34] for the Lieb-Liniger gas that interpolates between the Hartree and the Tonks-Girardeau limits (see App. A). Indeed, in one dimension, the Lieb-Liniger gas is integrable by the Bethe Ansatz technique [36, 37] and an exact expression of the ground-state energy as a function of the boson density is available.

The ground-state energy density of the Lieb-Liniger gas reads

$$\epsilon_{LL}(n) = \frac{\hbar^2}{2m} n^3 \epsilon_{LL}(n);$$

(7)

where $\epsilon_{LL}(n)$ is an adimensional function that can be obtained from the Bethe Ansatz solution [38–40]. Using the ground-state energy ((7)) in the generalized GPE has been shown to reproduce [34] the results of the hydrodynamic approach [16, 17] for the lowest breathing mode (see Appendix A for details).
Along these lines, in this work we replace the Hartree-term in the Gross-Pitaevskii equation (6) with the energy per unit length of the bulk quasi-one dimensional dipolar system

\[ e(n) = \frac{\hbar^2}{2m} n^3 \varepsilon(n), \tag{8} \]

where \( \varepsilon(n) \) is obtained using either a Bethe Ansatz wavefunction in a variational calculation [23], or a perturbative approach that we detail in the next section.

The approximation to the energy functional now reads

\[ F_{GP} = \int dz \left[ \frac{\hbar^2}{2m} \nabla \phi \nabla \phi^* + (V_{ext}(z) - \mu)|\phi|^2 + e(|\phi|^2) \right], \tag{9} \]

yielding the equation of motion [34, 35] for \( \phi(z, \tau) \),

\[ i\hbar \partial_\tau \phi = \delta F_{GP} / \delta \phi^*, \quad \text{i.e.} \]

\[ i\hbar \partial_\tau \phi = \left[ -\frac{\hbar^2 \nabla^2}{2m} + (V_{ext}(z) - \mu) + \frac{\delta e(|\phi|^2)}{\delta \phi^*} \right] \phi, \tag{10} \]

with the wave function normalized to the number \( N \) of atoms in the trap, \( N = \int dz |\phi(z)|^2 \).

III. EQUATION OF STATE

We start our analysis by recalling the method used in Ref. 8 to reduce the system with dipolar interaction (3) to an integrable Lieb-Liniger model. First, in the Hamiltonian (2) all the short-range contact interactions are isolated. Then, besides the van der Waals \( g_{1D} \) and the contact interaction in Eq. (2), a contact term \( AV(\theta) \) that effectively accounts for the short-range part of the interaction \( V_{DDI}(r) \) is added. The effective Lieb-Liniger Hamiltonian reads

\[ H_{Q1D}^{LL} = \frac{\hbar^2}{2m} \sum_i \frac{\partial^2}{\partial x_i^2} \]

\[ + \left[ g_{1D} + V(\theta)(A - \frac{8}{3})l_\perp \right] \sum_{i<j} \delta(x_i - x_j), \tag{11} \]

where the normalized strength of the short-range part of the interaction can be approximately taken as \( A = \int_{\sqrt{8\pi}}^{\infty} dz V_{DDI}^{1D}(n) \approx 3.6 \), in the single-mode approximation and independently of the density of atoms [8]. The nonzero \( A \) takes care of the shortest-ranged part \( (|z| < \sqrt{2\pi l_\perp}) \) of the dipolar potential (5) leaving the longer-ranged \( \sim 1/z^3 \) integrability-breaking tail as a possible perturbation.

Taking \( A = 0 \) would amount to neglect the short-range part of the dipolar interaction (5) and thus approximate repulsive or attractive dipolar interactions with an attractive or repulsive contact interaction, respectively [26]. Obviously, such an approximation is unphysical. The effect of making \( A > 0 \) is to counterbalance the attractive contact term coming from the single-mode approximation. When \( A > 8/3 \), stability is enlarged in the repulsive case, while in the attractive case instability can be obtained for \( g_{1D} \) not sufficiently repulsive.

A reliable estimate of \( A \) can be determined via a variational Bethe-Ansatz (VBA) wavefunction approach [23], where, for each density, this effective contact interaction is determined by the minimization of the energy per particle using the Bethe-Ansatz wavefunction of the Lieb-Liniger model as trial wavefunction.

The dimensionless coupling \( \gamma \) for the Lieb-Liniger Hamiltonian defined in (11) is

\[ \gamma = \frac{1}{n} \frac{m}{\hbar^2} g_{Q1D}(\theta) = \frac{2}{n a_{Q1D}} \]

\[ = \frac{2}{n} \left[ -\frac{1}{a_{1D}} + \frac{a_d 1 - 3 \cos \theta^2}{4} \right] \left( A - \frac{8}{3} \right), \tag{12} \]

where \( g_{1D} = -2\hbar^2/(ma_{Q1D}) \) and with \( a_d = \mu_0 \mu_1^2/(8\pi l_\perp^2) \), the dipolar length. In this work we will focus on the region where \( a_{1D} < 0 \), so that the contact interaction strength \( g_{1D} \) is positive.

In previous modelizations [8], the basic assumptions were that (i) \( A \) was independent of the density and the scattering length \( a_{1D} \), and (ii) the tail of the dipolar interaction was negligible. To start with, let us include the tail of \( V_{DDI}^{1D}(z/l_\perp) \) by means of a perturbative approach.

We write the original Hamiltonian (2) as the sum of the integrable Lieb-Liniger Hamiltonian (11) and a correction term \( \delta V \)

\[ H = H_{Q1D}^{LL}(\gamma) + \sum_{i<j} \delta V(z_i - z_j), \tag{13} \]

\[ \delta V(z) = V(\theta) \left[ V_{DDI}^{1D}(z_i - z_j) - A l_\perp \delta(z_i - z_j) \right]. \tag{14} \]

In order to estimate the effect of the interaction \( \delta V(z) \), we resort to perturbation theory (PT). In particular, we will consider two benchmark values for \( A, A = 3.6 \) as in Ref. 8, and \( A = 0 \), which amounts to treat the whole \( V(\theta) \left[ V_{DDI}^{1D}(z_i - z_j) \right] \) at perturbative level. At first order, the energy per \( N \) particles is:

\[ \frac{E_{pt}}{N} = \epsilon_{pt}(n) \approx \frac{E_{LL}(\gamma)}{N} + \frac{n}{2} \int dz \delta V(z) g_{LL}(z), \tag{15} \]

with \( E_{LL} \) the Lieb-Liniger ground-state energy for the Hamiltonian (11) evaluated at \( \gamma \), while \( g_{LL}(z) \) is the pair correlation function [41, 42] of the Lieb-Liniger gas. Using (15), we obtain an equation of state \( \epsilon_{pt}(n) \) that depends on the chosen \( A \), besides \( a_d l_\perp, |a_{1D}| l_\perp \) and \( \theta \).

In the rest of the paper, we compare the results obtained with the following three approximations for the equation of state: from \( \epsilon_{LL}(n) \), perturbation theory based on (15) with \( A = 0 \) and \( A = 3.6 \), and variational Bethe Ansatz as in Ref. 23, that gives a variational estimate for the ground-state energy independent of the approximation [8] chosen for \( A \).

In Fig. 1 we show the energies \( \epsilon(n) \) within three different approximations: \( \epsilon_{LL}(n) \) using Eq. (11) and \( A = 3.6, \)
$\epsilon_{PT}(n)$ with $A = 3.6$, and finally the variational Bethe Ansatz. Results are shown for three selected scattering lengths $a_{1D} = -100a_0$, $-1000a_0$ and $-5000a_0$ and for $\theta = \pi/2$, i.e. for repulsive interaction. We choose $a_d = 195a_0$, $l_\perp = 57.3nm$ and $a_{ho} = 24000a_0$, to make contact with recent experimental works [8, 9]. For small scattering lengths, the equations of state from the perturbative approach using $A = 3.6$ and from the variational Bethe Ansatz are in good agreement with each other. Equations of state within these two approximations, visibly depart from $\epsilon_{LL}$ based on Eq. (11) at small and intermediate densities and on increasing the scattering length as expected since the dipolar interactions becomes more dominant.

The typical situation for the attractive interaction, ie. for $\theta = 0$, is displayed in Fig. 2, where we show the data for $a_{1D} = -1000a_0$ and $-5000a_0$, and compare the energy results coming from variational Bethe Ansatz, perturbation theory using $A = 3.6$ for all densities, and using $A = 0.0$, that is treating the whole dipolar interaction as a perturbation. At low and intermediate densities, energies obtained within PT with $A = 0$ largely deviates from VBA results and within themselves. Only at very large densities these differences reduce, and results from PT with $A = 0$ are closer to the variational results. It should be kept in mind that those differences are strongly reduced by the $n^3$ factor in the energy per volume $E/V = n^3\epsilon(n)$.

These results emphasize that using a single effective contact interaction $A$, independent of both density and scattering length, can yield an inaccurate equation of state, especially with attractive dipolar interactions. The

FIG. 1. Energy per unit length $\epsilon(n)$ in units of $\hbar^2n^3/(2m)$ within three different approximations: $\epsilon_{LL}(n)$ with $A = 3.6$ (dashed lines); perturbation theory using $A = 3.6$ (solid lines) and variational Bethe Ansatz (solid dots). The results are shown for three selected scattering lengths $a_{1D} = -100a_0$, $-1000a_0$ and $-5000a_0$, respectively the black, red and blue data and for $\theta = \pi/2$.

FIG. 2. Energy per unit length $\epsilon(n)$ in units of $\hbar^2n^3/(2m)$ within three different approximations: using perturbation theory with $A = 3.6$ (solid lines) and using $A = 0.0$ (dashed lines) compared with those within the variational Bethe Ansatz (solid dots). Data are shown for two selected scattering lengths $a_{1D} = -1000a_0$ and $-5000a_0$, represented by the red and blue data points, respectively. Results are for $\theta = 0$.

VBA approach, by optimizing the parameter $A$ to minimize the ground state energy, takes care of these uncertainties. The relevance of such differences for the calculation of breathing mode frequency in a trapped system will be considered in the next section.

IV. THE BREATHING MODE

We evaluate the frequency of the lowest radial compressional oscillation by a sum-rule approach [16], that allows to compute the breathing mode frequency from ground-state density profiles obtained by solving the stationary generalized Gross-Pitaevskii equation using standard imaginary time evolution algorithms [43]. The breathing mode $\omega_b$ is obtained as the response of the gas to a change of the trap frequency $\omega_{ho}$:

$$\omega_b^2 = -2\langle \sum_{i=1}^N z_i^2 \rangle \left[ \frac{\partial^2}{\partial \omega_{ho}^2} \right]^{-1}.$$  \hspace{1cm} (16)

It is convenient (see App. B) to study the evolution of the breathing mode as a function of $A = Na_{1D}^2/a_{ho}^2$, with $N$ the number of particles in the trap. By solving the time-dependent generalized Gross-Pitaevskii equation, we have verified that, after initially exciting the mode by external radial compression of the trap, in the limit $|a_{1D}| \to 0$ $(\omega_{b}/\omega_{ho})^2 \equiv 4$.

We estimate the breathing mode using the different approximations described above, starting from the case $\theta = \pi/2$ (see Fig. 3).
In the region of small scattering lengths \(|a_{1D}|\), the breathing modes are dominated by the van der Waals repulsive contact interaction and dipolar interactions are marginally relevant: all the approximations, even completely neglecting the dipolar interaction, predict similar results. On increasing \(|a_{1D}|\), apart from using PT with \(A = 0\) that fails when \(g_{Q1D}(\pi/2)\) becomes negative, all the other approximations shown in Fig. 3 are very close to each other. The important effect of dipolar interaction becomes visible for very large \(|a_{1D}|\) values, where it enlarges the region of stability and, for \(|a_{1D}| \to \infty\), the breathing-mode frequency reaches a plateau. This behavior can be already observed within the approximations of [8] since with \(A = 3.6\), according to Eq. (11), \(g_{Q1D}(\pi/2)\) saturates in that limit.

We note that when \(\Lambda < 10^3\), the estimates of the breathing mode frequency from both the PT using \(A = 3.6\) and the VBA are compatible with the one obtained by dropping entirely the dipolar interaction. This last modelization however would predict that the breathing mode reaches the non-interacting limit at large \(\Lambda > 10^4\), i.e. \((\omega_b/\omega_{ho})^2 \to 4\), at variance with the other two approximations that predict a plateau at a lower \((\omega_b/\omega_{ho})^2 \approx 3.2\), hinting at the persistence of interactions.

At large \(|a_{1D}|\), even when the predictions for the breathing mode are all compatible, we can trace a difference in the density profile, as illustrated in Fig. 4 at \(\Lambda = 434\), where we show it using VBA and LL.

Using either \(\epsilon_{VBA}(n)\) or \(\epsilon_{PT}(A = 3.6, n)\) yields a density at the center of the trap that ranges from \(z \approx 2.6\) to \(3.1\mu m^{-1}\) (see the black curves in the main panel of Fig. 4), while the density value at the center of the trap is almost doubled for the Lieb-Liniger gas without dipolar interaction described by \(\epsilon_{LL}(n)\). The first estimates are in agreement with the averaged density at the center of trap as measured in Ref. 8. In the inset we show the variation of the density at the center of the trap as a function of \(\Lambda\) for the Lieb-Liniger equation of state and the VBA. The values become notably different for \(\Lambda \gtrsim 10\), whereas (see Fig. 3) the behavior of the breathing mode becomes qualitatively different for the two approximations only for \(\Lambda \gtrsim 500\). The behavior of the density profiles shows that the physics of the system at large scattering lengths is different in the presence of repulsive dipolar interactions, and could be used as a sensitive indicator together with the frequency of the breathing mode.

Turning to the attractive case, i.e. \(\theta = 0\), on increasing \(|a_{1D}|\) the \(g_{Q1D}(\theta)\) in Eq. (13) becomes rapidly small and negative, and the key issue is to what extent the system of dipolar gas is still stable against possible collapse [44], the formation of a solitonic/droplets phase [35] or a gas/droplets coexistence [45]. The predictions for the breathing modes are qualitatively different from the repulsive case, since both the VBA and the estimates with \(A = 3.6\), with or without correction to the first order, predict that for \(\Lambda > 2\) the breathing mode rapidly decreases to reach a minimum with \((\omega_b^2/\omega_{ho}^2 < 3)\), after which it rapidly increases until the overall effective interaction becomes negative. Due to negative \(g_{Q1D}\), also the LL model using \(A = 3.6\) (Eq. (11)) predicts an instability,
yet at a higher value of $\Lambda$ than the two other approximations with $3 < \omega_b^2/\omega_{ho}^2 < 4$.

The important discrepancy for $\Lambda > 1$ between the $A = 3.6$ approximation and its first-order correction, is suggesting a breakdown of this approximation. If we contrast with the VBA, we observe that the latter predicts a deeper minimum of the breathing mode frequency than the $A = 3.6$ approximation, even with first-order corrections. Comparing the densities at the center of the trap (see inset of Fig. 5), we note that differences in density are becoming noticeable already for $\Lambda \sim 0.1$, suggesting again that the density profile is more sensitive to the presence of dipolar interaction than the breathing mode. In any case, all the approximations confirm that we are approaching an instability at intermediate values of $\Lambda \sim 1$. Of course, only a direct comparison with experimental data could permit to identify which approximation is the most suitable for other predictions, as we will see later on.

Comparing the repulsive and attractive cases, we see that attractive dipolar interactions produce stronger qualitative effects on the behavior of the breathing mode or on the density profile at a given $a_{1D}$. In addition, differences between the VBA and the perturbation theory with $A = 3.6$ are also more significant in the presence of repulsive interactions. Such observation is in agreement with the behavior of the energy density represented in Figs. 1 and 2, where the differences between the approximations manifest themselves for lower $|a_{1D}|$ in the attractive case.

Finally, we show in Fig. 6 the effect of changing the polarization angle $\theta$, while keeping fixed the scattering length $|a_{1D}|$. For a large range of scattering lengths the effect of varying the angle is very small and visible only just before the system becomes unstable. For the largest scattering length, and attractive interaction, the breathing mode rapidly grows, signaling the instability, as previously found.

We conclude our discussion by contrasting the proposed approximation with the experimental data from Ref. 9 and 24, as shown in Fig. 7. We note that for the repulsive case ($\theta = \pi/2$) all the experimental points are in very good agreement with the VBA prediction both with $N = 25$ and $N = 40$, that are the minimum and maximum number of particles in the trap characterizing the experiment. Our findings suggest that for $\Lambda \geq 1$ the dipolar interaction is efficient in enhancing the region of stability of the interacting regime and in inhibiting the increase of the breathing mode frequency towards the non-interacting limit. The agreement with the VBA predictions is also confirmed for the attractive case ($\theta = 0$) where the curve with $N = 80$ and $N = 50$ agrees well with the experimental points. The comparison with the LL theory, using $A = 3.6$ and neglecting the tail interaction (Eq. (11)) clearly shows that despite its correct qualitative behavior the VBA description is needed to make contact with experimental findings. The comparison with the experiments clearly indicate that the system crosses over an instability point for $\Lambda$ of the order 1. Whether this instability is due to the formation of simple bound states [44] or droplets formation [35] needs further investigations, in particular from the experimental point of view.

V. CONCLUSIONS

In conclusion, we have considered the energy density of a gas of dipolar bosons in a tight transverse trapping using either the approximation of Ref.8, supplemented either by first order perturbation theory, or a variational approximation [23]. We have found that in the case of repulsive dipolar interactions, the two approaches were in good agreement with each other. We have used the energy densities under the different approximations to predict the breathing mode frequencies of the trapped dipolar gas. When dipolar interactions become attractive, the results of the two approximations become quite different, especially at low density. This gives rise to notable differences in the frequency of the breathing mode, the variational method giving a stronger dip before the instability. In all cases, observing the effect of the dipolar interaction requires to weaken enough the contact interaction that is competing with it. Contrasting with experimental results, we have shown that the variational predictions are especially compatible with the present mea-
measurements [9, 24]. However, except than in the attractive case, the experimental results are also compatible with a pure contact interaction.

As already noticed in our previous work [10], this can be considered as a further proof that to all relevant purposes the nature of $1/r^3$ power-law interactions in 1D, in the ground state repulsive branch, can be viewed as short-range interactions [46]. It would be worthwhile that future experiments attempt to explore the region with $\Lambda > 400$ in the repulsive regime, where deviations for the pure contact interaction are expected, and the range $0.1 < \Lambda < 1$ in the attractive case, where deviations from the pure contact interaction are maximal, and the difference between the two approximations considered here are the most visible.

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**Appendix A: Generalized Gross-Pitaevskii for the Lieb-Liniger trapped system**

We replace the Hartree term in Eq. (6) with the ground-state density energy of the Lieb-Liniger gas Eq. (7), so that the energy functional and the equation of motion reads:

$$F_{GP} = \int dz \left[ \frac{\hbar^2}{2m} \left( \nabla \phi \right)^2 + V_{ext}(z) \right| \phi \right|^2 + e(\phi^2) \right]$$

$$i\hbar \delta \phi = \left[ \frac{-\hbar^2 \nabla^2}{2m} + V_{ext}(z) + \frac{1}{\phi} \delta \phi^* \right] \phi,$$

where

$$\frac{1}{\phi} \delta \phi^* = \frac{\hbar^2}{2m} \left( 3n^2 \epsilon_{LL}(\gamma[n]) - \frac{2n}{a_{1D}} \frac{d \epsilon_{LL}(\gamma[n])}{d\gamma} \right),$$

with $\epsilon_{LL}(\gamma[n])$ the adimensional ground-state density energy functional for the Lieb-Liniger as for example from the works 38–40. We use a normalized wave-function as well as harmonic-oscillator units, that means that lengths and energies are respectively expressed in units of $\hbar/\omega_{ho}$ and $\hbar \omega_{ho}$, so that:

$$n = \frac{N}{\omega_{ho}} |\psi|^2$$

$$\gamma = \frac{2\omega_{ho}}{N\omega_{ho}} = \frac{2}{N^2 \lambda |\psi|^2}$$

$$\lambda = \frac{a_{1D}}{N \omega_{ho}},$$

where $\lambda$ is the Hartree parameter. In these units, Eq. A3 becomes

$$\frac{1}{\phi} \delta \phi^* = \frac{3}{2} N^2 |\psi|^4 \epsilon(\gamma[n]) - \frac{1}{\lambda} |\psi|^2 \epsilon'(\gamma[n]),$$

**FIG. 6.** Squared breathing mode frequency over trapping frequency squared $\omega_b^2/\omega_{ho}^2$ as a function of the polarization angle $\theta$ for selected values of the scattering length $a_{1D}/\omega_{ho} = -100, -2000, -5000, -7000$, represented by red, dark-green, blue and black solid dots, respectively. The solid line joining the data are only guides to the eye. The vertical blue dashed line splits the regions with negative $V(\theta) < 0$ (left) and positive $V(\theta) > 0$ (right). Data refer to estimates based on the Variational Bethe-Ansatz equation of state.

**FIG. 7.** Squared breathing mode frequency over trapping frequency squared $\omega_b^2/\omega_{ho}^2$ as a function of $\Lambda = N(a_{1D}/\omega_{ho})^2$, using different approximations, for the attractive case (upper panel) and the repulsive one (lower panel). Dark-red triangles represent the experimental data taken from Ref. 9 and 24. All results are shown for two values of $N$, $N = 50, 80$ (upper panel) and $N = 25, 40$ (lower panel) and lines are only a guide to the eyes. The dashed black and red lines represent estimates using the Lieb-Liniger model. Black and red solid squares are estimates based on the Variational Bethe-Ansatz (VBA) equation of state.
and it covers the strong and the weak interaction cases. Indeed, in the weak-interacting limit
\[
\lim_{\gamma \to 0} \frac{\delta \epsilon(n)}{\delta \phi^*} \to \frac{3}{\lambda} |\psi|^2 - \frac{2}{\lambda} |\psi|^2 = \frac{2}{\lambda} |\psi|^2
\]
we recover the usual Gross-Pitaevskii equation
\[
\rho \partial_t \psi = \left[ \frac{1}{2} \left( -\nabla^2 + x^2 \right) + \frac{\pi^2}{2} N^2 |\psi|^2 + \right] \psi, \quad (A8)
\]
while in the strong interacting limit
\[
\lim_{\gamma \to \infty} \frac{\delta \epsilon(n)}{\delta \phi^*} \to \frac{3}{2} N^2 |\psi|^4 \epsilon(\gamma[n]) = \frac{\pi^2}{2} N^2 |\psi|^4
\]
we get back to the proposal from Kolomeisky et al. [31] to describe the Tonks-Girardeau gas
\[
\rho \partial_t \psi = \left[ \frac{1}{2} \left( -\nabla^2 + x^2 \right) + \frac{\pi^2}{2} N^2 |\psi|^2 + \right] \psi. \quad (A9)
\]
Results for the breathing modes using this approach, for different number of particles in the trap, are shown in Fig. 8 and compared with the usual Gross-Pitaevskii equation (GPE)

\[
\begin{align*}
\lambda &= a_{1D}/(N a_{ho}) \\
\text{GGPE N=4} & \quad \text{GGPE N=10} \\
\text{GGPE N=15} & \quad \text{GGPE N=20} \\
\text{GGPE N=25} & \quad \text{GGPE N=30} \\
\end{align*}
\]

\[\text{FIG. 8. Ratio of breathing mode frequency to trap frequency squared } \omega_b^2/\omega_{ho} \text{ as a function of the Hartree parameter } \lambda \text{ for different numbers of particles in the trap, namely } N = 4, 10, 15, 20, 25 \text{ and } 30 \text{ using the generalized Gross-Pitaevskii equation Eq. (A2). The red solid line represents the Hartree approximation, independent of the number of particles.}\]

Appendix B: Breathing mode of an inhomogeneous Tomonaga-Luttinger liquid

Here, we briefly recall the relevant parameters to study the evolution of the breathing mode in a trapped Tomonaga-Luttinger liquid [11, 17, 47]. The Hamiltonian of the inhomogeneous Tomonaga-Luttinger liquid reads
\[
H = \int_{-R}^{R} \frac{dx}{2\pi} \left[ u(x) K(x) (\pi \Pi)^2 + \frac{u(x)}{K(x)} (\partial_x \phi)^2 \right], \quad (B1)
\]
where [11]
\[
\begin{align*}
u(x) K(x) &= \frac{\hbar \pi \rho(x)}{m}, \quad (B2) \\
u(x) K(x) &= \frac{1}{\pi \hbar} \left( \frac{\partial \mu}{\partial \rho} \right)_{\rho=\rho(x)}, \quad (B3)
\end{align*}
\]
with \(\rho(x)\) the density of atoms at position \(x\), \(\mu\) the chemical potential, \(m\) the mass of a single atom, and \(2R\) the dimension of the trapped atomic cloud. Using the equations of motion method, one obtains [11, 17, 47]
\[
\partial_t \phi = \frac{u(x) K(x) \partial_x \left( \frac{u(x)}{K(x)} (\partial_x \phi) \right)}{m}, \quad (B4)
\]
\[
\rho(x) \partial_x \left( \frac{\partial \mu}{\partial \rho} \partial_x \phi \right), \quad (B5)
\]
The breathing modes are obtained by looking for solutions of (B4) of the form \(\phi(x, t) = \phi_n(x) e^{i\omega_n t}\) subject to the boundary conditions \(\phi_n(\pm R) = 0\). The local chemical potential in a harmonic trap is fixed by the equation
\[
\mu(\rho(x)) = \frac{1}{2} m \omega_{ho}^2 (R^2 - x^2). \quad (B6)
\]
In the case of the Lieb-Liniger gas, the energy per unit length is given by (8)
\[
e(\rho) = \frac{\hbar^2 \rho^3}{m} \epsilon(\rho a_{1D}), \quad (B7)
\]
therefore it is convenient to use a reduced density \(\nu(x) = a_{1D} \rho(x)\), and write \(\mu(\rho) = \partial_\nu \rho(\rho)\) in the form
\[
\mu = \frac{\hbar^2}{m a_{1D}^2} \nu(\nu), \quad (B8)
\]
so that after inverting (B6) we find
\[
\nu = \psi^{-1} \left( \frac{a_{1D}^2 (R^2 - x^2)}{a_{ho}^2} \right), \quad (B9)
\]
where we have introduced the trapping length \(a_{ho} = \sqrt{\hbar/(m \omega_{ho})}\). If we consider the total number of particles \(N\), we have
\[
N = \int_{-R}^{R} \rho(x) dx, \quad (B10)
\]
and injecting (B9), we find that
\[
\frac{Na_{1D}}{R} = \int_{-1}^{1} du \psi^{-1} \left( \frac{a_{1D}^2 R^2 (1 - u^2)}{a_{ho}^2} \right). \quad (B11)
\]
Solving that equation yields
\[
R = \frac{a_{ho}^2}{a_{1D}^2} G(\Lambda), \quad (B12)
\]
with \( \Lambda = Na_{1D}^2/a_0^2 \). Introducing the dimensionless variable \( \xi = a_{1D}x/a_0^2 \), we can rewrite the density and the chemical potential in the form:

\[
\rho(x) = a_{1D}^{-1}F_1[G(\Lambda)^2 - \xi^2] \tag{B13}
\]

\[
\partial_x \mu(\rho(x)) = \frac{\hbar^2}{ma_{1D}}F_2[G(\Lambda)^2 - \xi^2], \tag{B14}
\]

and obtain the dimensionless eigenvalue equation

\[
F_1[G(\Lambda)^2 - \xi^2]\partial_x \{ F_2[G(\Lambda)^2 - \xi^2]\partial_x \phi_n \} = -(\omega_n/\omega_0)^2\phi_n, \tag{B15}
\]

with boundary conditions \( \phi_n(\xi = \pi G(\Lambda)) = 0 \). So, in the case of the Lieb-Liniger gas the eigenvalues \((\omega_n/\omega_0)^2\) depend only on the parameter \( \Lambda \). Obviously, this is not going to be the case in the dipolar gas where the ground state energy per unit length depends also on the dimensionless ratios \( a_{1D}/a_d, a_{1D}/a_L \) and the angle \( \theta \). However, in a limit where the dipolar interaction can be replaced by an effective contact interaction, the same kind of scaling will hold.

[1] H. J. Schulz, Phys. Rev. Lett. 71, 1864 (1993).
[2] S. Capponi, D. Poilblanc, and T. Giamarchi, Phys. Rev. B 61, 134110 (2002).
[3] E. G. Dalla Torre, E. Berg, and E. Altman, Phys. Rev. Lett. 97, 260401 (2006).
[4] X. Wei, C. Gao, R. Asgari, P. Wang, and G. Xianlong, Physical Review A 98, 023631 (2018), arXiv: 1806.01582.
[5] L. Tanzi, E. Lucioni, F. Famà, J. Catani, A. Fioretti, C. Gabbanini, R. N. Bisset, L. Santos, and G. Modugno, Phys. Rev. Lett. 122, 130405 (2019).
[6] F. Böttcher, J.-N. Schmidt, J. Hertkorn, M. Guo, T. Langen, and T. Pfau, Phys. Rev. X 9, 011051 (2019).
[7] L. Chomaz, D. Petter, P. Ilzhöfer, G. Natale, A. Trautmann, C. Politi, G. Durastante, C. Gabbanini, R. N. Bisset, L. Santos, and G. Modugno, Phys. Rev. Lett. 122, 130405 (2019).
[8] Y. Tang, W. Kao, K.-Y. Li, S. Seo, K. Mallayya, M. Rigol, S. Gopalakrishnan, and B. L. Lev, Phys. Rev. X 8, 021030 (2018), arXiv: 1707.07031.
[9] W. Kao, K.-Y. Li, K.-Y. Lin, S. Gopalakrishnan, and B. L. Lev, (2020), arXiv:2002.10475.
[10] R. Citro, E. Orignac, S. De Palo, and M.-L. Chiofalo, Phys. Rev. A 75, 051602(R) (2007), cond-mat/0611667.
[11] R. Citro, S. De Palo, E. Orignac, P. Pedri, and M. Chiofalo, New J. Phys. 10, 045011 (2008).
[12] T. Roscilde and M. Boninsegni, New J. Phys. 12, 033032 (2010).
[13] M.-O. Mewes, M. R. Andrews, N. J. van Druten, D. M. Kurn, D. S. Durfee, C. M. Townsend, and W. Ketterle, Phys. Rev. Lett. 77, 988 (1996).
[14] D. J. Jin, J. R. Ensher, M. R. Matthews, C. E. Wieman, and E. A. Cornell, Phys. Rev. Lett. 77, 420 (1996).
[15] A. Altmeyer, S. Riedl, C. Kohstall, M. J. Wright, R. Geursen, M. Bartenstein, C. Chin, J. H. Denschlag, and R. Grimm, Phys. Rev. Lett. 98, 040401 (2007).
[16] C. Menotti and S. Stringari, Phys. Rev. A 66, 043610 (2002).
[17] D. Petrov, D. Gangardt, and G. Shlyapnikov, J. Phys. IV (France) 116, 3 (2004).
[18] E. H. Lieb and W. Liniger, Phys. Rev. 130, 1605 (1963).
[19] M. Girardeau, J. Math. Phys. 1, 516 (1960).
[20] E. Halir, M. Gustavsson, M. Mark, J. G. Danzl, R. Hart, H. C. Naegele, and G. Pupillo, Science 325, 1224 (2009).
[21] R. Schmitz, S. Krönke, L. Cao, and P. Schmelcher, Phys. Rev. A 88, 043601 (2013).
[22] A. I. Gudyma, G. E. Astrakharchik, and M. B. Zvonarev, Phys. Rev. A 92, 021601 (2015).
[23] S. De Palo, R. Citro, and E. Orignac, Physical Review B 101, 045102 (2020), arXiv:1910.05965.
[24] B. Lev and al., Private communication (2020).
[25] V. Dunjko, V. Lorent, and M. Olshanii, Phys. Rev. Lett. 69, 851 (2001).
[26] F. Deuretzbacher, J. C. Cremon, and S. M. Reimann, Phys. Rev. A 81, 063616 (2010), [Erratum: Phys. Rev. A 87, 039903(E) (2013)].
[27] S. Sinha and L. Santos, Phys. Rev. Lett. 99, 140406 (2007).
[28] L. P. Pitaevskii, Sov. Phys. JETP 13, 451 (1961), russian original: ZhETF, v. 40, p. 646 (1961).
[29] C. Menotti and S. Stringari, Bose-Einstein Condensation (Clarendon Press, Oxford, 2003).
[30] B. Lev and al., Private communication (2020).
[31] E. P. Gross, J. Math. Phys. 4, 195 (1963).
[32] L. Pitaevskii and S. Stringari, Bose-Einstein Condensation (Clarendon Press, Oxford, 2003).
[33] E. B. Kolomeisky, T. J. Newman, J. P. Straley, and X. Qi, Phys. Rev. Lett. 85, 1146 (2000).
[34] M. Minguzzi, P. Vignolo, M. L. Chiofalo, and M. P. Tosi, Phys. Rev. A 64, 033605 (2001).
[35] E. Bettelheim, A. G. Abanov, and P. B. Wiegmann, J. Phys. A 41, 392003 (2008).
[36] P. Öhberg and L. Santos, Phys. Rev. Lett. 89, 240402 (2002).
[37] R. Oldziejewski, W. Górecki, K. Pawlowski, and K. Rzązewski, Phys. Rev. Lett. 124, 090401 (2020), arXiv:1908.00108.
[38] E. H. Lieb and W. Liniger, Phys. Rev. 130, 1605 (1963).
[39] E. H. Lieb, Phys. Rev. 130, 1616 (1963).
[40] G. Lang, F. Hekking, and A. Minguzzi, SciPost Phys. 3, 003 (2017).
[41] Z. Ristivojevic, Phys. Rev. B 100, 081110 (2019).
[42] M. Marino and T. Reis, J. Stat. Phys. 177, 1148 (2019), arXiv:1905.09575.
[43] J.-S. Caux and P. Calabrese, Phys. Rev. A 74, 031605(R) (2006), arXiv:cond-mat/0603654.
[44] A. Cherny and J. Brand, Phys. Rev. A 79, 043607 (2009).
[45] R. K. Kumar, L. E. Young-S., D. Vudragović, A. Balaz, P. Muruganandam, and S. Adhikari, Computer Physics Communications 195, 117128 (2015).
[46] J. B. McGuire, J. Math. Phys. 5, 622 (1964).
[45] Y. Kora and M. Boninsegni, Phys. Rev. A 101, 023602 (2020).
[46] M. Dalmonte, G. Pupillo, and P. Zoller, Phys. Rev. Lett. 105, 140401 (2010).
[47] C. Menotti and S. Stringari, Phys. Rev. A 66, 043610 (2002).