Algorithms for the Polar Decomposition in Certain Groups and the Quaternions

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Abstract

Constructive algorithms, requiring no more than $2 \times 2$ matrix manipulations, are provided for finding the entries of the positive definite factor in the polar decomposition of matrices in sixteen groups preserving a bilinear form in dimension four. This is used to find quaternionic representations for these groups analogous to that for the special orthogonal group. Furthermore, no eigencalculations are invoked. The groups to which the results are applicable include the symplectic group, and the groups whose signature matrices are $I_{2,2}$ and $I_{1,3}$ (i.e., the “Lorentz group”). This is achieved by characterizing positive definite matrices in these groups. Together with i) our earlier work on the real symplectic group; and ii) explicit isomorphisms, obtained by quaternionic algebra, between many of these groups, this characterization achieves the task at hand. A quaint, but key, observation is that for symmetric matrices in these groups, positive definiteness is equivalent to the positive definiteness of diagonal blocks. For the group whose signature matrix is $I_{2,2}$ a completion procedure based on this observation leads to said computation of the polar decomposition, while for the Lorentz group this is achieved by passage to its double cover. The latter is aided by the fact that the inversion of the covering map, when the target is a positive definite matrix, can be achieved essentially by inspection as we demonstrate. As byproducts we give a simple proof of the fact that positive definite matrices in some of these groups belong to the connected component of the identity, find an explicit expression for their logarithm, and find a simple characterization of the connected components of $G_{I_{2,2}}$ via certain determinants. A characterization of the symmetric matrices in the connected component of the identity of these groups in terms of their preimages in the corresponding covering group is also found. Finally, quaternionic representations of elements of these group is found.

Keywords: Positive definiteness, Schur Complements, Polar Decomposition, Double Cover, Lorentz Group, Quaternions, Algorithmic

Mathematics Subject Classification: 15Bxx
1 Introduction

This paper has as its motivation the following seemingly elementary question. What are the quaternionic representations of certain groups of real matrices in dimension four, analogous to the representation of the group $SO(4, \mathbb{R})$ by a pair of unit quaternions? Given the universal appeal and utility of the quaternions and its variants, with applications ranging from virtual reality to photonics and quantum computation (see, for instance, [4, 5, 12]), this is a natural line of investigation.

For other classes of matrices, such as symmetric, antisymmetric and those in the Lie algebras of the groups alluded to above, expressions for their quaternionic representations are known [14, 8, 15]. The groups of interest all arise as the isometry groups of non-degenerate real bilinear forms. In [2] this question was answered for the real symplectic group $Sp(4, \mathbb{R})$.

Note, by the “symplectic group”, reference is being made to the real symplectic group, and not the similarly denoted $Sp(n)$, which preserves the standard inner product on $\mathbb{H}^n$ (here $\mathbb{H}$ stands for the quaternions). The latter is, of course, trivially defined via quaternions. In the process of uncovering such a representation a fully constructive algorithm, not requiring any eigencalculations, for computing the polar decomposition of a matrix in $Sp(4, \mathbb{R})$ was provided in [2].

In physical applications where these groups arise, the positive definite factor in the polar decomposition can be interpreted as the “lossy” portion of such matrices, while the orthogonal factor is the conservative portion. Thus, for instance, if a matrix is in the Lorentz group, the positive definite factor is precisely the boost factor. Thus, methods which provide algorithms for the entries of the boost portion, starting from the matrix being decomposed, are obviously useful. The Lorentz group (and the related $SO(1, 2, \mathbb{R})$) have recently found many applications in optics (especially polarization optics), [3]. It is therefore expected that extracting the boost or spatial rotation factor of a Lorentz matrix will have applications to interpreting the lossy or lossless part of the corresponding optical element.

In this work, therefore, we will address algorithms for the polar decomposition for fifteen other matrix groups $G_M \subseteq M(4, \mathbb{R})$. Here $G_M = \{X : X^T M X = M\}$, with $M$ either one of the sixteen matrices belonging to the basis for $M(4, \mathbb{R})$, arising from the isomorphism of
the quaternion tensor product, $H \otimes H$, with $M(4, \mathbb{R})$, or $M = I_{1,3}$ (i.e., the group popularly called the Lorentz group). The matrices $M = I_4$ and $M = J_4$ belong to this basis and the corresponding cases have already been studied, whence the count of fifteen.

It is emphasized here that we are not merely providing the structural form of the factors in the polar decomposition. Rather, we provide algorithms to determine the entries of the positive definite factor in the polar decomposition, starting from the matrix being decomposed, in a fashion which requires nothing more than $2 \times 2$ manipulations. Once the positive definite factor has been found, finding the orthogonal factor is straightforward, since the inverse of a matrix $X$ in these groups is precisely $\pm MX^TM$. In addition to obtaining the polar decomposition and quaternionic representations, the following byproducts also emanate: i) A simple proof of the fact that positive definite matrices in $G_{I_{n,n}}$ belong to the connected component of the identity [Remark (3.3)], achieved via an explicit formula for their logarithm; ii) A characterization of the symmetric matrices in $SO^+(1,3,\mathbb{R})$ and $SO^+(2,2,\mathbb{R})$ in terms of their preimages in the corresponding covering groups [See Theorem (5.1) and Remark (5.1)] and iii) An elementary characterization of the connected components of $G_{I_{n,n}}$ analogous to the well-known such characterization for the Lorentz group, [see Theorem (5.3)].

In the process, we prove the following quaint fact which plays a useful role. A symmetric matrix in one of these groups $G_M$ is positive definite iff its diagonal blocks are themselves positive definite. As is well known, for a general real matrix this is only a necessary condition [see Theorem (3.1)].

It is worthwhile to mention a curious trichotomy here. Even though $Sp(4,\mathbb{R})$ is 10 dimensional, as opposed to the six dimensional $G_{I_{2,2}}$ or $G_{I_{1,3}}$, the quaternionic description of the former is more elegant than those of the latter two. In part due to this, one can use the quaternionic description of $Sp(4,\mathbb{R})$ itself to obtain the polar decomposition of matrices in it, whereas for the latter two groups it is easier to constructively obtain the polar decomposition by availing of the block structure of a matrix in one case and the corresponding covering spin group in the other. On the other hand obtaining a quaternionic representation for $G_{I_{2,2}}$, starting from the defining condition $X^TI_{2,2}X = I_{2,2}$, is arduous. For the group $G_{I_{1,3}}$ the obstacles corresponding to those for $G_{I_{2,2}}$ become even more formidable. Therefore, we resort to its double cover $SL(2,\mathbb{C})$. This is aided by the fact that finding the preimage in
of a positive definite matrix in the Lorentz group is significantly easier than finding the preimage of a general element in this group. In fact, it can be performed essentially by inspection. For $SO^+(2, 2, \mathbb{R})$ the corresponding statement does not hold, whence the need to go the completion procedure of Sections 3.1 and 3.2.

We conclude this introduction with some history of the linear algebraic applications of the isomorphism between $\mathbb{H} \otimes \mathbb{H}$ and $M(4, \mathbb{R})$. This isomorphism is central to the theory of Clifford algebras, [13], though it is not one of the standard isomorphisms of Clifford algebras with matrix algebras. However, its usage for linear algebraic (especially numerical linear algebraic) purposes is of relatively recent vintage. To the best of our knowledge the first instance seems to be the work of [11], where it was used in the study of linear maps preserving the Ky-Fan norm. Then in [6], this connection was used to obtain the Schur canonical form explicitly for real $4 \times 4$ skew-symmetric matrices. Next, is the work of [8, 14, 15], wherein this connection was put to innovative use for solving eigenproblems of several classes of structured matrices. In [17, 18], this isomorphism was used to explicitly calculate the exponentials of a wide variety of $4 \times 4$ matrices. In [19], explicit formulae for the minimal polynomials of classes of structured $4 \times 4$ matrices were obtained through this isomorphism. Finally, this isomorphism played an important role in studying reversion and Clifford conjugation on a wide variety of Clifford algebras without any intervention of the corresponding even algebras, [7, 9].

The balance of this manuscript is organized as follows. In the next section some notation and useful results on positive definite matrices and the algebra isomorphism between $\mathbb{H} \otimes \mathbb{H}$ and $M(4, \mathbb{R})$ are collected. We draw attention to Remark (2.1), Lemma (2.1), Remark (2.3) and Remark (2.4). These play an important role for what follows. The next section elucidates the block structure of positive definite matrices in the $G_M$’s considered here. Section 3.1 shows how to complete a matrix in $G_{I_{n,n}}$ to positive definiteness given one of its three blocks. Remark (3.3) is a byproduct of it and is used in Section 5 to characterize the connected components of $G_{I_{2,2}}$. This completion procedure is important in Subsection 3.2 which addresses the algorithmic determination of the positive definite factor in the polar decomposition of a matrix in $G_{I_{2,2}}$ and also for the quaternionic representation of matrices in $G_{I_{2,2}}$, described in Section 4. Section 5 studies positive definiteness in the Lorentz group, showing among other things that the only other symmetric matrices in the Lorentz group are all indefinite - see Theorem
A byproduct of the proof of this theorem is Algorithm (5.1) which shows that the preimage of a positive definite matrix in $SL(2, \mathbb{C})$, under the covering map, can be found by inspection. We also draw attention to Remark (5.1) which explains how much the analogous calculation can be carried out for $SO^+(2, 2, \mathbb{R})$. Theorem (5.3) is a byproduct of Remark (5.1). It presents a characterization of the connected components of $G_{I_{2,2}}$. The final section offers conclusions. An appendix provides the statement of our result from [2] on the quaternionic representation of symplectic, positive definite matrices for the purposes of comparison.

2 Notation and Preliminary Observations

The following definitions, notations and results will be frequently met in this work:

- $M(4, \mathbb{R})$ (also denoted $gl(4, \mathbb{R})$) is the algebra of real $4 \times 4$ matrices.
- Let $M$ be an $n \times n$ real invertible matrix. Then $G_M = \{X \in M(n, \mathbb{R}) : X^T MX = M\}$. $G_M$ is a Lie group. There are obvious extensions to the complex case (both with transposition and Hermitian conjugation in the definition), but for the immediate purposes of this work, this definition of $G_M$ suffices.
- $I_{p,q} = \begin{pmatrix} I_p & 0_{p\times q} \\ 0_{q\times p} & I_q \end{pmatrix}$.
- $J_{2n} = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}$. $Sp(2n, \mathbb{R})$ is the standard notation for $G_{J_{2n}}$.
- We use the standard notation $SO(p,q,\mathbb{R})$ for the determinant 1 matrices in $G_{I_{p,q}}$ and $SO^+(p,q,\mathbb{R})$ for the connected component of the identity in $SO(p,q,\mathbb{R})$. The Lorentz group is then $SO^+(1,3,\mathbb{R})$.
- Essential use will be made of the following theorem (with statement adapted to the needs of this work, see [16]):

**Proposition 2.1** Let $X \in G_M$, where $M = cU$, with $c > 0$ and $U$ real orthogonal. If $X = QP$ is its polar decomposition, with $P$ positive definite and $Q$ real orthogonal, then $P$ and $Q$ are also in $G_M$. 
We next collect some definitions and results on real positive definite matrices. Most statements may be found in [10].

**Definition 2.1** Let \( Y \) be a real positive definite matrix. A real square matrix \( Z \) satisfying \( Y = Z^T Z \) is said to be a square root of \( Y \).

**Remark 2.1** Square roots of positive definite matrices are not unique. However, if \( Z_1 \) is a square root of \( Y \) then \( Z_2 \) is also a square root of \( Y \) iff there exists a real orthogonal matrix \( U \) such that \( Z_2 = UZ_1 \). For \( 2 \times 2 \) matrices, therefore,

\[
U = U_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}
\]  

if the determinants of \( Z_1 \) and \( Z_2 \) share the same sign, whereas,

\[
U = V_\theta = \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix}
\]  

if the determinants of \( Z_1 \) and \( Z_2 \) have opposite signs.

Among these square roots, there is a unique upper triangular one with positive diagonal entries (that provided by the Cholesky factorization), say \( Z_1 \) and a unique positive definite one, say \( Z_2 \). Thus, there is a real special orthogonal \( U \) such that \( X_2 = UX_1 \).

The following lemma shows, by way of variety, how to find \( X_2 \) in closed form for \( 2 \times 2 \) matrices without any recourse to any eigencalculations, though it can easily be computed in closed form by an orthogonal diagonalization.

**Lemma 2.1** If \( Y = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} \) is positive definite, then its unique positive definite square root is \( X_2 = \begin{pmatrix} \alpha \cos \theta & \alpha \sin \theta \\ \alpha \sin \theta & \beta \sin \theta + \gamma \cos \theta \end{pmatrix} \), where \( \alpha = \sqrt{a_{11}} \), \( \beta = \frac{a_{12}}{a_{11}} \), \( \gamma = \sqrt{\frac{\det(Y)}{a_{11}}} \), and \( \tan \theta = \frac{\beta}{\alpha+\gamma} \), where \( \theta \) is chosen to be in the first quadrant if \( a_{12} > 0 \) and in the fourth quadrant if \( a_{12} < 0 \). If \( a_{12} = 0 \), then \( \theta = 0 \).

**Proof:** The upper Cholesky factor of \( Y \) is \( X_1 = \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix} \). Therefore, since the determinants of \( X_1 \) and \( X_2 \) are both positive, \( X_2 = U_\theta X_1 \), with \( U_\theta \) as in Remark (2.1). The \((1,2)\) entry of \( UX_1 \) is \( \beta \cos \theta - \gamma \sin \theta \), while the \((2,1)\) entry is \( \alpha \sin \theta \). Since \( UX_1 \) has to be
symmetric, we find \( \tan \theta = \frac{\beta}{\alpha + \gamma} \). Next, since \( \det(X_2) = \det(X_1) \), we need the \((1,1)\) entry of \( UX_1 \) to be positive. But this entry is \( \alpha \cos \theta \). Since \( \alpha > 0 \), we need \( \cos \theta > 0 \). Since only \( \beta \) can be negative if at all (which happens precisely when \( a_{12} < 0 \)), we obtain the quadrant characterization of \( \theta \) in the statement. ♦

Use of the following remark will be made in Sec 3.2.

**Remark 2.2** Let \( A > 0 \) and \( A^2 - I \geq 0 \). Then \( A - I \geq 0 \) also. This is elementary, since \( \lambda \) is an eigenvalue of \( A \).

Next relevant definitions and results regarding quaternions and their connection to real matrices will be presented. Throughout \( \mathbf{H} \) will be denote the skew-field (the division algebra) of the quaternions, while \( \mathbf{P} \) stands for the purely imaginary quaternions, tacitly identified with \( \mathbb{R}^3 \).

**\( \mathbf{H} \otimes \mathbf{H} \) and \( M_4(\mathbb{R}) \):** The algebra isomorphism between \( \mathbf{H} \otimes \mathbf{H} \) and \( M_4(\mathbb{R}) \), which is used here is the following:

- Associate to each product tensor \( p \otimes q \in \mathbf{H} \otimes \mathbf{H} \) the matrix, \( M_{p \otimes q} \), of the map which sends \( x \in \mathbf{H} \) to \( px\bar{q} \), identifying \( \mathbb{R}^4 \) with \( \mathbf{H} \) via the basis \( \{1, i, j, k\} \). Here, \( \bar{q} \) is the conjugate of \( q \). This yields an algebra isomorphism between \( \mathbf{H} \otimes \mathbf{H} \) and \( M_4(\mathbb{R}) \).

- Define conjugation in \( \mathbf{H} \otimes \mathbf{H} \) by first defining the conjugate of a decomposable tensor \( a \otimes b \) as \( \bar{a} \otimes \bar{b} \), and then extending this to all of \( \mathbf{H} \otimes \mathbf{H} \) by linearity. Then \( M_{\bar{a} \otimes \bar{b}} = (M_{a \otimes b})^T \).

Thus, the most general element of \( M_4(\mathbb{R}) \) admits the quaternion representation \( a1 \otimes 1 + p \otimes i + q \otimes j + r \otimes k + s \otimes 1 + 1 \otimes t \), with \( a \in \mathbb{R} \) and \( p, q, r, s, t \in \mathbf{P} \). The summand \( a1 \otimes 1 + p \otimes i + q \otimes j + r \otimes k \) is the symmetric part of the matrix, while the summand \( s \otimes 1 + 1 \otimes t \) is the skew-symmetric part of the matrix. Expressions for \( a, p, q, r, s, t \) (which are linear in the entries of the matrix being represented) are easy to find, [14].

Finally \( 4a \) is the trace of the matrix.

**Remark 2.3** Several useful facts concerning the aforementioned isomorphism are collected below.

1. The sixteen matrices \( M_{e \otimes f} \), where \( e, f \in \{1, i, j, k\} \) form a basis, denoted \( \mathbf{B} \), of special orthogonal matrices for \( M(4, \mathbb{R}) \). Of these 10 are symmetric (\( M_{1 \otimes 1} \) and those for which neither \( e \) nor \( f \) equals 1) and the remaining six are antisymmetric.
2. Of special relevance to this work are \(M_i \otimes i\) which equals \(I_{2,2}\) and \(M_1 \otimes j\) which equals \(J_4\).
Other useful members of this basis are \(M_i \otimes j\), the \(4 \times 4\) flip matrix; \(M_i \otimes k\) and \(M_i \otimes 1\).
In [7, 9] essential use of \(M_1 \otimes i\), \(M_1 \otimes k\) and \(M_k \otimes 1\) was made to study algorithmically the spin groups.

3. One can find explicit congruences (simultaneously similarities) between the fifteen symmetric members of this basis, which do not equal \(I_4\). This task can be reduced to elementary algebraic calculations in \(H\). For instance, to find a similarity between \(M_i \otimes i\) and \(M_i \otimes j\), we find unit quaternions \(p, q\) satisfying \(\bar{p}ip = i\) and \(\bar{q}iq = j\). One choice is \(p = \frac{1}{\sqrt{2}}(j + k)\) and \(q = \frac{1}{\sqrt{2}}(i + j)\).

4. Similarly, we can find explicit special orthogonal similarities between any two of \(M_1 \otimes i\), \(M_1 \otimes j\), \(M_1 \otimes k\).
For instance, we find a unit quaternion \(q\) satisfying \(\bar{q}jq = i\). Then \(S^{-1}M_1 \otimes jS = M_1 \otimes i\),
where \(S = M_1 \otimes q\). One choice is \(q = \frac{1}{\sqrt{2}}(i + j)\) again. Similarly, we can find explicit special orthogonal similarities between any two of \(M_i \otimes 1\), \(M_j \otimes 1\), \(M_k \otimes 1\). Next notice that \(M_i \otimes 1 = -\text{diag}(J_2, J_2)\) and \(J_4 = M_1 \otimes j\). Since, \(P^T J_4 P = \text{diag}(J_2, J_2)\), where \(P\) is the permutation matrix which, in column form equals
\[
[e_1 | e_3 | e_2 | e_4]
\]
it is seen that one can find explicit similarities between any two of \(\{M_1 \otimes i, M_1 \otimes j, M_1 \otimes k, -M_i \otimes 1, -M_j \otimes 1, -M_k \otimes 1\}\).

**Remark 2.4** Observing that \(G_{-M} = G_M\) and that if \(M_2 = S^T M_1 S\) for an explicit orthogonal \(S\) (as supplied by the previous two items of the previous remark), then \(X \in G_{M_2}\) iff \(SX^T S^T \in G_{M_1}\), we see that it suffices to explicitly compute the polar decomposition of matrices in \(G_{I_4} = O(4, R), G_{I_{2,2}}\) and \(G_{J_4} = Sp(4, R)\) to be able to compute the polar decomposition explicitly in any of the sixteen \(G_M\’s\), for \(M \in B\). Indeed, if \(X \in G_{M_2}\) and if the polar decomposition of \(SX^T S^T \in G_{M_1}\) is \(SX^T S^T = QP\), then as the matrices \(S^T QS\) and \(S^T PS\) are orthogonal and positive definite respectively, it follows from uniqueness in the polar decomposition for invertible matrices that \(X = (S^T QS)(S^T PS)\) is the polar decomposition of \(X\).

For the polar decomposition of \(O(4, R)\), there is nothing to do, while for \(Sp(4, R)\) this task was performed in [2]. Therefore, it remains to do the needful for the groups \(G_{I_{2,2}}\) and \(G_{I_{1,3}}\).
3 Positive Definiteness in Certain Groups

We begin with a characterization of symmetric matrices in $G_M$, when $M = I_{p,q}$.

**Lemma 3.1** Let $X = \begin{pmatrix} A & B \\ B^T & D \end{pmatrix}$ be a symmetric $n \times n$ real matrix. Let $A$ be $p \times p$ and $D q \times q$ with $p + q = n$. Then $X^T I_{p,q} X = I_{p,q}$ iff the following hold:

1. $A^2 - BB^T = I_p$.
2. $AB = BD$
3. $D^2 - B^T B = I_q$.

**Proof:** This is a straightforward block matrix calculation.

Next we will characterize when an $X$, as in Lemma (3.1) is positive definite. Clearly a necessary condition is that the diagonal blocks themselves are positive definite. For general $X$ this is not even remotely sufficient. But if $X$ is also in $G_M$, with $M = I_{p,q}$, then it is sufficient as the following result shows.

**Theorem 3.1** A symmetric $n \times n$ real symmetric matrix, satisfying $X^T I_{p,q} X = I_{p,q}$ is positive definite iff its diagonal square blocks are both positive definite.

**Proof:** The necessity is well known. Conversely, let $A > 0$ and $D > 0$. We need to show that the Schur complement $\Delta_A = D - B^T A^{-1} B > 0$ also. From 2) of Lemma (3.1) we get $AB = BD$ and hence $B = A^{-1} BD$. So $B$ intertwines $A^{-1}$ and $D^{-1}$ also.

Next, 3) of Lemma (3.1) gives $D^2 = I_q + BB^T$. Hence, $D = D^{-1} + B^T BD$. Hence, $D = D^{-1} + B^T A^{-1} B$. Therefore, $\Delta_A = D - B^T A^{-1} B = D^{-1} + B^T A^{-1} B - B^T A^{-1} B = D^{-1}$. As $D^{-1} > 0$ also, we obtain the desired conclusion. ♦

**Remark 3.1** One can also show that $\Delta_D$, the Schur complement of $D$, is $A^{-1}$.

**Remark 3.2** There are interesting analogues of Lemma (3.1) and Theorem (3.1) for other groups. For instance,

1. Though the group $G_{1,3}$ has been subsumed above, it is convenient to view matrices in it as $2 \times 2$ block matrices and the signature matrix $I_{1,3}$ as $diag(\sigma_2, -I_2)$. If one does
this a symmetric matrix is in $G_{t,3}$ iff

$$A\sigma_z A - BB^T = \sigma_z, A\sigma_z B = BD; D^2 - B^T\sigma_z B = I_2$$  \hspace{1cm} (3.3)$$

Using this one shows again that the Schur complement of $A$ is $D^{-1}$.

2. Similarly, let $X$ be $2n \times 2n$ real symmetric. Then $X$ is real symplectic iff $BA$ and $DB$ are symmetric and $AD = I_n + B^2$. Once again such an $X$ is positive definite iff its diagonal blocks are themselves positive definite. For instance, the Schur complement of $A$ is $D - B^T A^{-1} B = D + A^{-1} B A A^{-1} B = D + A^{-1} B^2 = D + A^{-1} (A D - I) = A^{-1}$.

3. Consider the $4 \times 4$ flip matrix $F_4$ (which is $M_{j@i}$). Its higher dimensional analogue is $F_{2n}$. Consider a $2n \times 2n$ symmetric matrix $X = \begin{pmatrix} A & B \\ B^T & D \end{pmatrix}$. Then $X^T F_{2n} X = F_{2n}$ iff we have the following conditions: i) $BF_n A$ and $DF_n B$ are antisymmetric and $BF_n B + AF_n D = F_n$.

Let $A > 0$. We will show that $\Delta_A > 0$ by showing that $F_n^T \Delta_A F_n > 0$. Now $F_n^T \Delta_A F_n = F_n \Delta_A F_n = F_n (D - B^T A^{-1} B) F_n$. But from $BF_n A = -AF_n B^T$, we get $BF_n = -AF_n B^T A^{-1}$ and hence $BF_n B F_n = -AF_n B^T A^{-1} F_n$.

On the other hand, $BF_n B = F - AF_n D$, whence $A^{-1} BF_n B F_n = A^{-1} (F_n^2 - A F_n D) F_n = A^{-1} - F_n D F_n$, since $F_n^2 = I_n$. Therefore, $F_n \Delta_A F_n = A^{-1} > 0$.

4. Let $M = M_i @ k = \begin{pmatrix} 0_2 & I_2 \\ I_2 & 0_2 \end{pmatrix}$. Its $2n$ dimensional generalization is $K_{2n} = \begin{pmatrix} 0_n & I_n \\ I_n & 0_n \end{pmatrix}$.

Now $X$ is symmetric and in $G_M$ iff $BA$ and $DB$ antisymmetric and $B^2 = -AD$, as a direct calculation shows. As before for $X$ to be positive definite we need $A > 0$ and $D > 0$. These conditions are also sufficient. For instance, the Schur complement of $A$ is $D - B^T A^{-1} B = D + A^{-1} B A A^{-1} B = D + A^{-1} B^2 = D + A^{-1} (I - AD) = A^{-1}$.

Here in the first equality we have used $B^T = -A^{-1} BA$ and in the penultimate equality $B^2 = I - AD$ has been employed.

5. In general the sixteen $M$’s arising from the $H \otimes H$ basis are all either block diagonal or block reverse diagonal. For symmetric $X \in G_M$, for the former class the pattern is that $\Delta_A$ is congruent to $D^{-1}$, while for the latter it is congruent to $A^{-1}$.
3.1 Completion to Positivity in $G_{I_{n,n}}$

Let us now discuss how one may “complete” $A$, $D$ or $B$ to obtain a positive definite matrix in $G_{I_{n,n}}$. The constructions in this section are crucial for finding the polar decomposition of a matrix in $G_{I_{2,2}}$ to be discussed in Section 3.2. In addition, they provide a constructive proof of the fact that positive definite matrices in $G_{I_{n,n}}$ belong to $SO^+(n, n, \mathbb{R})$. They also provide the quaternionic representation of a positive definite matrix in $G_{I_{2,2}}$.

- **Completion given $A$:** Suppose $A > 0$ and $A^2 - I_n \geq 0$. We first pick any square root $B^T$ of $A^2 - I_n$, i.e., a $n \times n$ $B$ satisfying $A^2 - I = BB^T$. Next let $D$ be the unique positive definite square root of $I_n + B^T B$. Clearly then the data satisfy requirements 1 and 3 of Lemma (3.1). The claim is that this data also satisfy requirement 2 of Lemma (3.1).

Towards that end note that $A^2 B = (I + BB^T)B = B + BB^T B$, while $BD^2 = B(I + B^T B) = B + BB^T B$. Thus, the $B$ constructed above automatically interwines $A^2$ and $B^2$.

Let $U, V$ be real orthogonal matrices which diagonalize $A$ and $D$ respectively. Then from $A^2 B = BD^2$, we get

$$
(U^T A^2 U)(U^T B V) = (U^T B V)(V^T D^2 V) 
$$

Let $\lambda_i, \mu_j$ be the eigenvalues of $A$ and $D$ respectively. Then Equation (3.4) is equivalent to

$$
(\lambda_i^2 - \mu_j^2)(U^T B V)_{ij} = 0 
$$

As $\lambda_i + \mu_j > 0$, we then obtain that $(\lambda_i - \mu_j)(U^T B V)_{ij} = 0$. This is, of course, equivalent to $(U^T A U)(U^T B V) = (U^T B V)(V^T D V)$, which in turn is equivalent to $AB = BD$.

- **Completion given $D$:** Let $D > 0$ and $D^2 - I \geq 0$. First pick $B$ to be any $n \times n$ square root of $D^2 - I_n$. Then pick $A$ to be the unique positive definite square root of $I + BB^T$.

Then, as before $A^2 B = BD^2$. Therefore, just as in the item above, $AB = BD$.

- **Completion given $B$:** First pick $A$ to be the unique positive definite square root of $I + BB^T$ and then $D$ to be the unique positive definite square root of $I + B^T B$. Then
a calculation shows that $A^2B = BD^2$. Therefore, as in the item above, we also get $AB = BD$.

**Remark 3.3** It is perhaps folklore that the positive definite matrices in $G_{I_p,q}$ belong to $SO^+(p, q, \mathbb{R})$. Here we give a simple constructive proof, which also provides an explicit formula for “the” logarithm of such a matrix for the case of interest to us, namely $G_{I_n,n}$. Let $P$ be such a positive definite matrix. To show that it belongs to $SO^+(n, n, \mathbb{R})$, it suffices to express it as the exponential of an element in the Lie algebra of $SO^+(n, n, \mathbb{R})$. Since “the” logarithm of a positive definite matrix is also symmetric. We seek a matrix of the form

$$
\begin{pmatrix}
0 & X \\
X^T & 0
\end{pmatrix}.
$$

Let the $(1, 2)$ block of $P$ be $B$. By the discussion on completion given $B$, it is necessary and sufficient to find an $X$ so that the $(1, 2)$ block of $\exp\left[\begin{pmatrix} 0 & X \\ X^T & 0 \end{pmatrix}\right]$ equals $B$.

Let $B = UDV^T$ be the singular value decomposition of $B$, with $\sigma_i$ the singular values of $B$. We let $X = UEV^T$, where $E$ is a diagonal matrix of non-negative entries to be chosen. With this choice of $X$, a calculation which is omitted shows that if we pick $E = \text{diag}(y_1, \ldots, y_n)$, where $y_i = \ln\left[\frac{\sigma_i + \sqrt{\sigma_i^2 + 1}}{2}\right]$, then indeed the $(1, 2)$ block of $\exp\left[\begin{pmatrix} 0 & X \\ X^T & 0 \end{pmatrix}\right]$ is $B$ and hence that $P$ is the exponential of

$$
\begin{pmatrix}
0 & X \\
X^T & 0
\end{pmatrix}.
$$

Thus, $P \in SO^+(n, n, \mathbb{R})$.

Remark (3.3) will be used in Theorem (5.3) later to characterize the connected components of $G_{I_{2,2}}$.

### 3.2 Computing the polar decomposition in $G_{I_{2,2}}$:

We now specialize to $n = 2$. Then the constructions below lead to the positive definite factor in the polar decomposition of a matrix $X_{4 \times 4}$ satisfying $X^T I_{I_{2,2}} X = I_{I_{2,2}}$ without requiring even $2 \times 2$ eigencalculations. These manipulations carry over to $G_{I_{n,n}}$ or more generally $G_{I_{p,q}}$, except that we will need eigencalculations for $n > 2$, which are typically no longer possible in closed form.

So let $Z$ satisfy $Z^T I_{I_{2,2}} Z = I_{I_{2,2}}$. Then $X = Z^T Z$ is both positive definite and satisfies $X^T I_{I_{2,2}} X = I_{I_{2,2}}$. Therefore, $Y$, the unique positive definite square root also satisfies
\[ Y^T I_{2,2} Y = I_{2,2}. \]

Suppose

\[
X = \begin{pmatrix}
A & B \\
B^T & D
\end{pmatrix}
\]

Thus, \( A, B \) and \( D \) are known a priori and satisfy the conditions in Lemma (3.1) and Theorem (3.1). In particular \( A \) and \( D \) are positive definite with \( A^2 - I \) and \( D^2 - I \) both positive semidefinite. In light of Remark (2.2) it then follows that \( A - I \geq 0 \).

We then have to find

\[
P = \begin{pmatrix}
E & F \\
F^T & H
\end{pmatrix}
\]

with i) \( P^2 = X \); ii) \( E, F, H \) satisfying the conditions in Lemma (3.1) and Theorem (3.1).

The condition \( P^2 = X \) is equivalent to i) \( A = E^2 + FF^T \); ii) \( B = EF + FH \) and iii) \( D = F^T F + H^2 \).

Since necessarily \( E^2 - I = FF^T \), we find that necessarily \( A = I + (\sqrt{2}F)(\sqrt{2}F)^T \). Since, \( A - I \geq 0 \), we can find by inspection its lower Cholesky factor \( G \). Then the most general square root is \( G\theta_i \), where \( G = GR\theta \) with \( R\theta \) is either \( U\theta \) or \( V\theta \) as given by Equations (2.1) or (2.2) respectively.

Let \( F\theta_i = \frac{1}{\sqrt{2}} G\theta_i \), where \( \theta_1 \) corresponds to the choice of \( U\theta \) and \( \theta_2 \) to that of \( V\theta \). Correspondingly let \( E\theta_i, H\theta_i \) be the unique positive definite square roots of \( I + F\theta_i F\theta_i^T \) and \( I + F\theta_i^T F\theta_i \) respectively (these can be obtained in closed form, e.g., via Lemma (2.1). Note, thus that \( E\theta_i \) is independent of \( \theta \) while \( H\theta_i \) is a \( 2 \times 2 \) matrix whose entries are some explicitly computable trigonometric functions of \( \theta \).

By the Section 3.1, therefore the matrices

\[
\begin{pmatrix}
E\theta_i & F\theta_i \\
F\theta_i^T & H\theta_i
\end{pmatrix}
\]

are guaranteed to be positive definite and in \( G_{I_{2,2}} \). In particular, \( E\theta_i F\theta_i = F\theta_i H\theta_i, i = 1, 2 \).

Suppose at least one of the \( \theta_i \) can be chosen to satisfy \( E\theta_i F\theta_i + F\theta_i H\theta_i = B \) (we will see presently that precisely one of the \( \theta_i \) can be so picked). Then we claim that the remaining two conditions, viz., \( A = E\theta_i^2 + F\theta_i F\theta_i^T \); and \( D = F\theta_i^T F\theta_i + H\theta_i^2 \) will be satisfied.

For instance, to show the latter we proceed as follows. First observe that both \( D \) and \( F\theta_i^T F\theta_i + H\theta_i^2 \) are positive definite matrices (the latter because \( H\theta_i > 0 \)). So, as \( D \) is the
unique positive definite square root of $I + B^T B$, it suffices to demonstrate that
\[
(F_{\theta_i}^T F_{\theta_i} + H_{\theta_i}^2)^2 = I + B^T B \tag{3.6}
\]

Now a calculation, which repeatedly uses $H_{\theta_i}^2 = I + F_{\theta_i}^T F_{\theta_i}$ and $E_{\theta_i} F_{\theta_i} = F_{\theta_i} H_{\theta_i}$, confirms Equation (3.6). Similarly, $A = E_{\theta_i}^2 + F_{\theta_i} F_{\theta_i}^T$.

Now the equation $E_{\theta_i} F_{\theta_i} + F_{\theta_i} H_{\theta_i} = B$ is, of course,
\[
2E_{\theta_i} F_{\theta_i} = B
\]
which, in turn, is
\[
\sqrt{2}(I + 2GG^T)^{1/2}GR_{\theta_i} = B \tag{3.7}
\]

Now this last equation has at least one solution, corresponding to either $R_{\theta_1} = U_{\theta_1}$ or to $R_{\theta_2} = V_{\theta_2}$, because there is precisely one positive definite square root of $X^T X$ and this is guaranteed to be in $G_{I_{2,2}}$. In fact, if $\det(B) > 0$, then there is a solution $\theta_1$, while if $\det(B) < 0$, there is a solution $\theta_2$. If $\det(B) = 0$, then both equations may have to be addressed, with either both possessing a solution or one possessing a solution and the other none. In the case of multiple solutions, we are guaranteed that the corresponding $G_{\theta}$ is unique, because of the uniqueness of the positive definite square root.

Let this value of $\theta_1$ (or $\theta_2$, as the case may be) be denoted $\theta_0$. Then defining $F = F_{\theta_0}$, $E = E_{\theta_0}$ and $H_{\theta_0}$, it follows that
\[
P_{\theta_0} = \begin{pmatrix}
E_{\theta_0} & F_{\theta_0} \\
F_{\theta_0}^T & H_{\theta_0}
\end{pmatrix}
\]
is the positive definite factor in the polar decomposition of $X$.

Let us summarize the paragraphs above as an algorithm:

**Algorithm 3.1** Given $X \in G_{I_{2,2}}$, the following algorithm computes the positive definite factor, $P$, in the polar decomposition $X = PQ$.

1. Compute $X^T X = \begin{pmatrix} A & B \\ B^T & D \end{pmatrix}$. Thus, the blocks $A, B, D$ satisfy the conditions in Theorem (3.1). In particular, $A - I_2 \geq 0$.

2. Let $G$ be the lower triangular factor in the Cholesky factorization of $A - I$. Let $U_{\theta_1}$ be as in Equation (2.1) and $V_{\theta_2}$ be as in Equation (2.2).
3. If \( \det(B) > 0 \), then find \( \theta_1 \) from the equation \( \sqrt{2}(I_2 + \frac{1}{2}GG^T)^{1/2}GU_{\theta_1} = B \). If \( \det(B) < 0 \), then find \( \theta_2 \) from the equation \( \sqrt{2}(I_2 + \frac{1}{2}GG^T)^{1/2}GV_{\theta_2} = B \). If \( \det(B) = 0 \) then at least one of the two equations \( \sqrt{2}(I_2 + \frac{1}{2}GG^T)^{1/2}GU_{\theta_1} = B \) or \( \sqrt{2}(I_2 + \frac{1}{2}GG^T)^{1/2}GV_{\theta_2} = B \) will have a solution. Let the solution, \( \theta_1 \) or \( \theta_2 \), be denoted by \( \theta_0 \).

4. \( F_{\theta_0} = \frac{1}{\sqrt{2}}GR_{\theta_0} \), where \( R_{\theta_0} \) is \( U_{\theta_1} \) if \( \theta_0 = \theta_1 \) or \( V_{\theta_2} \) if \( \theta_0 = \theta_2 \). Let \( E_{\theta_0} \) be the unique positive definite square root of \( I_2 + \frac{1}{2}GG^T \) and let \( H_{\theta_0} \) be the unique positive definite square root of \( X_2 + F_{\theta_0}^TF_{\theta_0} \).

5. Then

\[
P = \begin{pmatrix}
E_{\theta_0} & F_{\theta_0} \\
F_{\theta_0}^T & H_{\theta_0}
\end{pmatrix}
\]

is the positive definite factor in the polar decomposition of \( X \).

**Remark 3.4**

- One could have also obtained the positive definite factor in the polar decomposition by using the constructive algorithm for diagonalizing a symmetric \( 4 \times 4 \) matrix in [14]. This would require solving certain analytic equations for three variables, whereas in the previous algorithm the number of unknowns is one, viz., one of the \( \theta_i \). Furthermore, only when \( \det(B) = 0 \) do these equations require any work beyond.

- The previous algorithm can, with some modifications, be extended to the Lorentz group (more precisely \( G_{I_{1,3}} \)). The main difference is that the \( F \) in Step2 would be rectangular, and hence instead of using \( U_{\theta} \) or \( V_{\theta} \) one would accordingly use \( 3 \times 3 \) orthogonal matrices.

This would therefore render the corresponding analytic equations to involve 3 variables - e.g., the Euler angles of these \( 3 \times 3 \) rotations. We will see however, that passage to its covering group provides a very tractable alternative to finding the polar decomposition in the Lorentz group. The pros and cons of a similar passage to the covering group of \( SO^+(2,2,\mathbb{R}) \) is discussed in Remark (5.1).

4 **Quaternion Representations of \( G_{I_{2,2}} \)**

To develop a quaternionic representation of an \( X \in G_{I_{2,2}} \), we invoke Proposition (2.1).

We therefore first obtain a quaternionic representation of the positive definite factor \( P \).
Proposition 4.1 Let $P$ be positive definite $P$ and belong to $G_{I_2,2}$. Suppose that its NE $2 \times 2$ block is $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then it has quaternionic representation $c(1 \otimes 1) + p \otimes i + q \otimes j + r \otimes k$, where

- $c = \frac{\alpha_1 \cos \theta_1 + \beta_1 \sin \theta_1 + \gamma_1 \cos \theta_1 + \alpha_2 \cos \theta_2 + \beta_2 \sin \theta_2 + \gamma_2 \cos \theta_2}{4}$
- $p = \begin{pmatrix} p_1 \\ \frac{b+c}{2} \\ \frac{d-a}{2} \end{pmatrix}$, where
  $p_1 = \frac{\alpha_1 \cos \theta_1 + \beta_1 \sin \theta_1 + \gamma_1 \cos \theta_1 - \alpha_2 \cos \theta_2 - \beta_2 \sin \theta_2 - \gamma_2 \cos \theta_2}{4}$
- $q = \begin{pmatrix} \frac{c-b}{2} \\ q_2 \\ \frac{a_2 \sin \theta_2 + a_1 \sin \theta_1}{2} \end{pmatrix}$ where
  $q_2 = \frac{\alpha_1 \cos \theta_1 - \beta_1 \sin \theta_1 - \gamma_1 \cos \theta_1 + \alpha_2 \cos \theta_2 - \beta_2 \sin \theta_2 - \gamma_2 \cos \theta_2}{4}$
- $r = \begin{pmatrix} \frac{a+d}{2} \\ \frac{a_2 \sin \theta_2 - a_1 \sin \theta_1}{2} \\ r_3 \end{pmatrix}$ with
  $r_3 = \frac{\alpha_1 \cos \theta_1 - \beta_1 \sin \theta_1 - \gamma_1 \cos \theta_1 - \alpha_2 \cos \theta_2 + \beta_2 \sin \theta_2 + \gamma_2 \cos \theta_2}{4}$

where i) $\alpha_1 = \sqrt{1 + a^2 + b^2}$; ii) $\beta_1 = \frac{ac + bd}{\alpha_1}$; iii) $\gamma_1 = \frac{[\frac{1}{\alpha_1}(1 + a^2 + b^2)(1 + c^2 + d^2) - (ac + bd)^2]^{1/2}}{\alpha_1}$; iv) $\alpha_2 = \sqrt{1 + a^2 + c^2}$; v) $\beta_2 = \frac{ab + cd}{\alpha_2}$; vi) $\gamma_2 = \frac{[\frac{1}{\alpha_2}(1 + a^2 + c^2)(1 + b^2 + d^2) - (ab + cd)^2]^{1/2}}{\alpha_2}$; vii) $\tan \theta_1 = \frac{\beta_1}{\alpha_1 + \gamma_1}$; viii) $\tan \theta_2 = \frac{\beta_2}{\alpha_2 + \gamma_2}$.

Proof: In Section 3.1, we saw that any real $B$ can serve as the NE block of a positive definite $P$ in $G_{I_2,2}$. In this case the NW block has to be the unique positive square root of $I + BB^T$ and the SE block has to be unique positive square root of $I + B^TB$. The result now follows from the formulae for the positive definite square root in Lemma (2.1). ♦

Next we address the representation of orthogonal matrices in $G_{I_2,2}$.

Proposition 4.2 Let $X$ be a $4 \times 4$ orthogonal matrix which is also in $G_{I_2,2}$. Then
1. If $X \in SO^+(2,2,\mathbb{R})$, then the quaternionic representation of it is $u \otimes v$, with $u = a + bi$ and $v = c + di$, with $u$ and $v$ both of unit length.

2. If $\det(X) = 1$, but $X$ does not belong to $SO^+(2,2,\mathbb{R})$, then the quaternionic representation of it is $u \otimes v$, with $u = \alpha j + \beta k$ and $v = \gamma j + \delta k$, with $u$ and $v$ both of unit length.

3. If $\det(X) = -1$, then $X$ has representation $\frac{1}{2}(1 \otimes 1 + i \otimes i + j \otimes j - k \otimes k)(u \otimes v)$, with $u$ and $v$ as in either 1) or 2) above.

**Proof:** We refer to Remark (5.1) to see that a special orthogonal matrix in $G_{I_{2,2}}$ must be block-diagonal, with both blocks orthogonal and both with determinant either 1 or both with determinant $-1$. The statement in 1) pertains to the determinant 1 case. The statement in 2) pertains to the determinant 1 case. Put differently, in the former case $u$ and $v$ both commute with $i$, while in the latter case they both anticommute with $i$.

The matrix $Y = I_{3,1}$ whose $2 \times 2$ block representation is $\text{diag}(I_2, \sigma_2)$, certainly has negative determinant and is in $G_{I_{2,2}} \cap O(4,\mathbb{R})$. So any orthogonal matrix with determinant $-1$ which is also in $G_{I_{2,2}}$ must be expressible as $ZY$, where $Z \in G_{I_{2,2}} \cap SO(4,\mathbb{R})$. Since the quaternionic representation of $Y$ is precisely $\frac{1}{2}(1 \otimes 1 + i \otimes i + j \otimes j - k \otimes k)$ the statement in 3) follows. ♦.

Combining the last two results one obtains a quaternionic representation of any matrix in $G_{I_{2,2}}$.

**Remark 4.1** Find the quaternionic representation of $Q$ from the entries of $Q$ is susceptible to the possibly folklore method of finding the pair of unit quaternions from the entries of a given special orthogonal matrix - see e.g., [5].

**Remark 4.2** The quaternionic representation of matrices in $Sp(4,\mathbb{R})$ was obtained by starting from the defining relations for $Sp(4,\mathbb{R})$. If we attempt to do this for $G_{I_{2,2}}$ then one is lead to a system of quadratic equations whose structure is not very transparent. Specifically, let us try to characterize real symmetric matrices in $G_{I_{2,2}}$. Such a matrix will have quaternionic representation $a1 \otimes 1 + p \otimes i + q \otimes j + r \otimes k$ with

$$a1 \otimes 1 + p \otimes i + q \otimes j + r \otimes k (i \otimes i) \otimes 1 + p \otimes i + q \otimes j + r \otimes k = i \otimes i$$
Then a direct, but elaborate, calculation shows that the above equation is equivalent to the following equations:

\[ ap_1 = (r \times q)_1 \quad (4.8) \]

\[ a(r \times i) + q_1 p + p_1 q = (p.q)i \quad (4.9) \]

\[ a(i \times q) + r_1 p + p_1 r = (p.r)i \quad (4.10) \]

\[ a^2 - \| p \| ^2 + \| q \| ^2 + \| r \| ^2 + 2p_1^2 - 2q_1^2 - 2r_1^2 = 1 \quad (4.11) \]

\[ p_1 \hat{p} = q_1 \hat{q} + r_1 \hat{r} \quad (4.12) \]

wherein the following notation has been employed: Identifying \( p, q, r \) with vectors in \( \mathbb{R}^3 \), we write \( \hat{p} \) etc., for their projection on to the \( \{ j, k \} \) plane.

Equations (4.8) through (4.12) are significantly more complicated to analyse than those for \( Sp(4, \mathbb{R}) \).

5 The Lorentz Group and \( G_{I_{1,3}} \)

In this section we proceed differently for producing the polar decomposition and the quaternionic representation of \( G_{I_{1,3}} \). Namely, the fact that the positivity of the diagonal blocks being sufficient for positivity of a symmetric \( X \in G_{I_{1,3}} \) is combined with the covering of \( SO^+(1,3,\mathbb{R}) \) by \( SL(2,\mathbb{C}) \).

Let us recall this covering. Given a \( G \in SL(2,\mathbb{C}) \), \( \Phi(G) \) is the \( 4 \times 4 \) matrix representing the linear map which sends a \( 2 \times 2 \) Hermitian matrix \( X \) to \( GXG^* \) (written with respect to the basis \( \{ I_2, \sigma_x, \sigma_y, \sigma_z \} \)). Then \( \Phi(G) \) is in \( SO^+(1,3,\mathbb{R}) \). Furthermore, \( \Phi \) is onto and has kernel \( \{ \pm I_2 \} \).

We omit the explicit expression of \( \Phi(G) \). However, here are some consequences:

- i) The \( (1,1) \) entry of a matrix in \( SO^+(1,3,\mathbb{R}) \), as is well known, is always positive. Therefore, \( -I_4 \) cannot be in \( SO^+(1,3,\mathbb{R}) \). Hence \( X \) is in \( SO^+(1,3,\mathbb{R}) \), iff \( -X \) is in the other connected component of the determinant one matrices in \( G_{I_{1,3}} \).
• \( \Phi(G^*) = [\Phi(G)]^T \). This follows from the explicit expression for \( \Phi(G) \), but can also be shown by calculating the adjoint of the linear map \( X \rightarrow GXG^* \).

We can now characterize the symmetric matrices and also the positive definite matrices in \( SO^+(1,3,\mathbb{R}) \).

**Theorem 5.1** A symmetric matrix in \( SO^+(1,3,\mathbb{R}) \) is either positive definite or indefinite. The former are \( \Phi \) images of Hermitian matrices in \( SL(2,\mathbb{C}) \) and the latter are \( \Phi \) images of anti-Hermitian matrices in \( SL(2,\mathbb{C}) \). Furthermore, a positive definite matrix in \( SO^+(1,3,\mathbb{R}) \) is always expressible as \( \Phi(G) \) for a positive definite \( G \in SL(2,\mathbb{C}) \).

**Proof:** Every matrix in \( SO^+(1,3,\mathbb{R}) \) is a \( \Phi(G) \) for some \( G \in SL(2,\mathbb{C}) \). Let \( \Phi(G) \) be symmetric. Hence, \( \Phi(G^*) = \Phi(G) \) be virtue of the discussion above. Hence \( G^* = \pm G \). Let \( G^* = G \) first. So \( G = \begin{pmatrix} a & x + iy \\ x - iy & d \end{pmatrix} \). Now a calculation of \( \Phi(G) \), wherein repeated use of the fact that \( ad - (x^2 + y^2) = 1 \) is made, reveals it to be

\[
\Phi(G) = \begin{pmatrix}
\frac{(a+d)^2}{2} - 1 & (a + d)x & -(a + d)y & \frac{a^2 - d^2}{2} \\
(a + d)x & 1 + 2x^2 & -2xy & (a - d)x \\
-y(a + d) & -2xy & 1 + 2y^2 & y(d - a) \\
\frac{a^2 - d^2}{2} & (a - d)x & y(d - a) & \frac{(a-d)^2}{2} + 1
\end{pmatrix}
\]

To show that it is positive we just need to verify that the diagonal blocks are positive definite, thanks to 1) of Remark(3.2). Now the \((1,1)\) entry is positive (because it equals half of \( || G ||_F^2 \)). Next the determinant of the NW block is \( \frac{(a+d)^2}{2} - 1 - 2x^2 \). Using again the fact that \( ad - (x^2 + y^2) = 1 \), this equals \( \frac{a^2 + d^2 + 2y^2 - 2x^2}{2} \). Next \( a^2 + d^2 \geq 2ad = 2(1 + x^2 + y^2) \). So \( a^2 + d^2 + 2y^2 - 2x^2 \geq 2 + 4y^2 \). Hence the determinant of the NW block is positive. The \((3,3)\) entry is visibly positive, while the determinant of the SE block equals \( 1 + 2y^3 + \frac{(a-d)^2}{2} \) which is, of course, positive. Hence \( \Phi(G) \) is positive definite. Now a Hermitian \( G \) in \( SL(2,\mathbb{C}) \) is obviously either positive definite or negative definite. Hence, if we show that the \( \Phi(G) \) image of an anti-Hermitian matrix cannot be positive definite, then it will follow that \( X \in SO^+(1,3,\mathbb{R}) \) is positive definite iff it is expressible as \( \Phi(G) \) for a positive definite \( G \).

Now consider an anti-Hermitian \( G = \begin{pmatrix} ia & x + iy \\ -x + iy & id \end{pmatrix} \). We wish to show \( \Phi(G) \) is indefinite. Since the \((1,1)\) entry is positive for any \( \Phi(G) \), it cannot be negative definite. So
it suffices to show that is not positive definite. To this end we record the NW and SE blocks of $\Phi(G)$, making use of $x^2 + y^2 - ad = 1$:

$$NW = \begin{pmatrix} \frac{(a+d)^2}{2} + 1 & (a+d)y \\ (a+d)y & -1 + 2y^2 \end{pmatrix}$$

$$SE = \begin{pmatrix} -1 + 2x^2 & x(a-d) \\ x(a-d) & \frac{(a-d)^2}{2} - 1 \end{pmatrix}$$

We will now show that at least one of the determinants of these two blocks must be non-positive. Suppose, to the contrary, that both are positive. Then $-1 - \frac{(a+d)^2}{2} + 2y^2 > 0$ and $1 - \frac{(a-d)^2}{2} - 2x^2 > 0$. Subtracting the second inequality from the first one obtains $-2ad + 2y^2 + 2x^2 > 2$, which contradicts $-ad + x^2 + y^2 = 1$.

So $\Phi(G)$ is indefinite. ♦

The above proof yields the following algorithm to find a positive definite $G \in SL(2, \mathbb{C})$ satisfying $\Phi(G) = P$ for a positive definite $P \in SO^+(1,3,\mathbb{R})$ essentially by inspection, to wit:

**Algorithm 5.1** *Inversion of $\Phi : SL(2, \mathbb{C}) \rightarrow SO^+(1,3,\mathbb{R})$, when the target is a positive definite $P$:*

1. First suppose $P_{22} \neq 1$ and $P_{33} \neq 1$. Then let $x$ be one of the two square roots of $\frac{P_{33} - 1}{2}$ and $y$ one of the two square roots of $\frac{P_{22} - 1}{2}$. Corresponding to each such choice (there are four of them) solve the equations $a + d = -\frac{P_{34}}{y}$ and $a - d = \frac{P_{24}}{x}$. Precisely one of these pairs will have a solution $(a, d)$ with both $a$ and $d$ positive. Choose the corresponding $x, y, a, d$. Then $H = \begin{pmatrix} a & x + iy \\ x - iy & d \end{pmatrix}$ satisfies $\Phi(H) = G$.

2. If $P_{12} = 1$ and $P_{33} \neq 1$, pick $x = 0$ and $y$ to be one of the square roots of $\frac{P_{33} - 1}{2}$. Corresponding to each choice of $y$, solve the system $a + d = -\frac{P_{13}}{y}$, $d - a = \frac{P_{34}}{y}$. Precisely one of these will have a solution with both $a$ and $d$ positive. Choose the corresponding $y, a, d$ and let $H = \begin{pmatrix} a & iy \\ -iy & d \end{pmatrix}$. Then $\Phi(H) = G$.

3. If $P_{12} \neq 1$ but $P_{33} = 1$, then set $y = 0$ and let $x$ be one of the two square roots of $\frac{P_{22} - 1}{2}$. Corresponding to each choice of $x$ solve the system $a + d = \frac{P_{12}}{x}$, $a - d = \frac{P_{24}}{x}$. Precisely
one of these will have solution with both $a$ and $d$ positive. Choose this value of $x$ and let $H = \begin{pmatrix} a & x \\ x & d \end{pmatrix}$. Then $\Phi(H) = G$.

4. Finally, if both $P_{22} = 1$ and $P_{33} = 1$, then set $x = y = 0$, and let $\alpha$ be one of the two square roots of $2(P_{11} + 1)$ and $\beta$ one of the two square roots of $2(P_{44} - 1)$. For each choice solve $a + d = \alpha; a - d = \beta$. Precisely one of these will have a solution with both $a$ and $d$ positive. Let $H = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$. Then $\Phi(H) = G$.

With the algorithm above, we can now constructively find the polar decomposition of a matrix in the polar decomposition of any matrix in the Lorentz group and hence in all of $G_{1,3}$.

First consider a matrix $X \in SO^{+}(1,3;\mathbb{R})$. Then $X^T X$ is also in $X \in SO(1,3;\mathbb{R})$ and positive definite. Then one finds a $G \in SL(2,\mathbb{C})$, with $\Phi(G) = X^T X$. Let $H \in SL(2,\mathbb{C})$ be the unique positive definite square root of $G$. Thus $H^2 = G$, and hence $(\Phi(H))^2 = X^T X$.

By Theorem (5.1), $\Phi(H)$ is positive definite and hence must be the positive part $P$ of the polar decomposition $X = VP$ of $X$. Now finding $V$ is routine. Furthermore, since $V$ is also in $SO^{+}(1,3;\mathbb{R})$, it must be special orthogonal and also satisfy $VI_{1,3} = I_{1,3}V$. Therefore, its first column and row must be the first standard unit vector. Hence $V$ is an (standard) embedding of a matrix in $SO(3,\mathbb{R})$ into a $4 \times 4$ matrix. Thus $V$ is represented by $v \otimes v$ for some unit quaternion $v$.

Next, let $X \in SO(1,3;\mathbb{R})$ have determinant 1 and a negative $(1,1)$ entry. Hence $-X \in SO^{+}(1,3,\mathbb{R})$. Computing $-X = VP$, we find $X = (-V)P$ is the polar decomposition of $-X$. Thus, in particular the orthogonal factor has quaternionic representation $-u \otimes u$ for some unit quaternion $u$.

Suppose $X \in G_{1,3}$ has determinant equal to $-1$ and its $(1,1)$ entry is positive. Then $X = I_{1,3}Y$, where $Y \in SO^{+}(1,3,\mathbb{R})$. Clearly, $Y^T Y = X^T X$ and thus the positive definite factor in $X$’s polar decomposition coincides with that of $Y$, while its orthogonal factor is clearly $I_{1,3}V$, where $V$ is that of $Y$. Hence its quaternionic representation is $\frac{1}{2}(1 \otimes 1 - i \otimes i - j \otimes j - k \otimes k)v \otimes v$, for a unit quaternion $v$.

Finally, let $X \in G_{1,3}$ has determinant equal to $-1$ and suppose its $(1,1)$ entry is negative. Then $-X = I_{1,3}Y$, with $Y \in SO^{+}(1,3;\mathbb{R})$. Once again the positive definite factor of $X$ coincides with that of $Y$, and its orthogonal factor is $-I_{1,3}V$, where $V$ is that of $Y$. So its
The quaternionic representation is 
\( -\frac{1}{2}(1 \otimes 1 - i \otimes i - j \otimes j - k \otimes k)v \otimes v \), for a uniq quaternion \( v \).

The foregoing arguments are now summarized as an algorithm:

**Algorithm 5.2**

1. Let \( X_{11} > 0 \) and \( \det(X) = 1 \). Compute \( X^TX \). Use Algorithm (5.1) to find the positive definite \( G \in SL(2, \mathbb{C}) \) satisfying \( \Phi(G) = X^TX \).

2. Compute the unique positive definite square root \( H \in SL(2, \mathbb{C}) \) of \( G \). Find \( P = \Phi(H) \in SO^+(2, 2, \mathbb{R}) \). Then \( P \) is the positive definite factor in the polar decomposition of \( X = VP \). The orthogonal factor \( V \) equals \( I_{1,3}PI_{1,3} \). Since \( I_{1,3} \) is diagonal, this computation requires only \( 2 \times 2 \) matrix multiplications.

3. Next let \( X_{11} < 0 \) and \( \det(X) = 1 \). Then \( -X \in SO^+(2, 2, \mathbb{R}) \). Apply Steps 1 and 2 to \( -X \) to find the polar decomposition \( -X = VP \). Then \( X = (-V)P \) is the polar decomposition of \( X \).

4. Finally let \( \det(X) = -1 \). Then \( I_{1,3}X \in SO(2, 2, \mathbb{R}) \). Apply the previous steps to find the polar decomposition \( I_{1,3}X = VP \). Then \( X = I_{1,3}VP \) is the polar decomposition of \( X \).

Next we find the quaternionic representation of any matrix in \( G_{I_{1,3}} \):

**Theorem 5.2** Let \( X \in G_{I_{1,3}} \). Then

1. If \( X_{11} > 0 \) and \( \det(X) = 1 \), then \( X \)’s quaternionic representation is \( [u \otimes u][u \otimes u] + p \otimes i + q \otimes j + r \otimes k \), where \( u \) is a unit quaternion and \( p, q, r \) and \( c \) are as follows:

   \[
   p = \begin{bmatrix}
   x^2 \\
   (a^2 - d^2 - 4xy)/4 \\
   ((a - d)x + (a + d)y)/2
   \end{bmatrix}
   \]

   \[
   q = \begin{bmatrix}
   (d^2 - a^2 - 4xy)/4 \\
   y^2 \\
   ((a + d)x + (d - a)y)/2
   \end{bmatrix}
   \]

   \[
   r = \begin{bmatrix}
   ((a - d)x - (a + d)y)/2 \\
   (y(d - a) - (a + d)x)/2 \\
   (a - d)^2/4
   \end{bmatrix}
   \]
\[ c = (a + d)^2 / 4 \]

with \( ad - (x^2 + y^2) = 1 \) and \( a > 0 \).

2. If \( X_{11} < 0 \) and \( \det(X) = 1 \), then the representation of \( X \) is \(-[u \otimes u][c1 \otimes 1 + p \otimes i + q \otimes j + r \otimes k]\) with \( p, q, r, c \) and \( u \) as in Case 1.

3. If \( X_{11} > 0 \) and \( \det(X) = -1 \), then the representation is \( \frac{1}{2}(1 \otimes 1 - i \otimes i - j \otimes j - k \otimes k)[u \otimes u][c1 \otimes 1 + p \otimes i + q \otimes j + r \otimes k] \), with \( p, q, r, c \) and \( u \) as in Case 1.

4. If \( X_{11} < 0 \) and \( \det(X) = -1 \), then the representation is \( -\frac{1}{2}(1 \otimes 1 - i \otimes i - j \otimes j - k \otimes k)[u \otimes u][c1 \otimes 1 + p \otimes i + q \otimes j + r \otimes k] \), with \( p, q, r, c \) and \( u \) as in Case 1.

**Remark 5.1** A natural question that arises is if one could not have profitably used the covering \( SL(2, \mathbb{R}) \times SL(2, \mathbb{R}) \to SO^+(2, 2, \mathbb{R}) \) to compute the polar decomposition in \( SO^+(2, 2, \mathbb{R}) \), in a manner analogous to that for the Lorentz group. The principal obstruction is that the inversion of the covering map when the target in \( SO^+(2, 2, \mathbb{R}) \) is a positive definite matrix is no longer any simpler than it is for a general matrix in \( SO^+(2, 2, \mathbb{R}) \), in sharp contrast to the Lorentz case, [1]. Nevertheless this exercise reveals the following facts of independent interest:

1. \( \Phi(A^T, B^T) = [\Phi(A, B)]^T \) as before.

2. If \( A, B \in SL(2, \mathbb{R}) \) are both positive definite or both negative definite, then \( \Phi(A, B) \) is positive definite and only then is a symmetric matrix in \( SO^+(2, 2, \mathbb{R}) \) positive definite.

3. Unlike the case for the Lorentz group, there are negative definite matrices in \( SO^+(2, 2, \mathbb{R}) \) and these are precisely the image of a pair \((A, B)\) where one is positive definite and the other negative definite.

4. If both \( A \) and \( B \) are antisymmetric, then \( \Phi(A, B) \) is either \( I_{2,2} \) or \( -I_{2,2} \). Therefore, unlike the Lorentz group, \( -I_4 \in SO^+(2, 2, \mathbb{R}) \).

5. A determinant one matrix in \( G_{I_{2,2}} \) is special orthogonal iff it is block-diagonal. This is because these two conditions imply that \( XI_{2,2} = I_{2,2}X \). Obviously both blocks have to
be orthogonal. Such a matrix is in $SO^+(2, 2, \mathbb{R})$ iff both these diagonal blocks are in $SO(2, \mathbb{R})$. If both blocks have determinant equal to $-1$, then the matrix cannot be in $SO^+(2, 2, \mathbb{R})$. This follows from the explicit form of the covering homomorphism $\Phi : \text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R}) \to \text{SO}^+(1, 2, \mathbb{R})$. Indeed, the only solutions $(A, B)$ to $\Phi(A, B) = X$, with $X$ block-diagonal, and both blocks orthogonal and with determinant equal to $-1$ turn out to be complex. The same reasoning extends to $SO(n, n, \mathbb{R})$. Specifically, the map which assigns to a matrix $X$ the determinants of its diagonal blocks is continuous. For $X \in SO(n, n, \mathbb{R}) \cap SO(2n, \mathbb{R})$ this function assumes only the values $(1, 1)$ and $(-1, -1)$. Therefore, the these determine the two connected components of $SO(n, n, \mathbb{R}) \cap SO(2n, \mathbb{R})$.

This circumstance therefore distinguishes the two connected components of $\{X : X \in G_{t_{n, n}, \text{det}(X) = 1}\}$ - the orthogonal factor in the polar decomposition of $SO^+(n, n, \mathbb{R})$ is block-diagonal with both blocks in $SO(n, \mathbb{R})$, whereas the orthogonal factor in the polar decomposition of matrices in the other connected component have both diagonal blocks orthogonal with determinant $-1$.

We will content ourselves with the proof of the fact that if $A$ and $B$ are positive definite then so is $\Phi(A, B)$. To that end we let $A = \begin{pmatrix} x_1 & x_2 \\ x_2 & x_8 \end{pmatrix}$ and $B = \begin{pmatrix} x_3 & x_4 \\ x_4 & x_6 \end{pmatrix}$ be a pair of positive definite matrices in $\text{SL}(2, \mathbb{R})$. Following [9] we embed this pair concentrically in $M(4, \mathbb{R})$ as follows:

$$C = \begin{pmatrix} x_1 & 0 & 0 & x_2 \\ 0 & x_3 & x_4 & 0 \\ 0 & x_5 & x_6 & 0 \\ x_7 & 0 & 0 & x_8 \end{pmatrix}$$

(This unusual embedding is what naturally results from the iterative constructions in Clifford algebras, [9].)

Then the covering map is the map which sends $(A, B)$ to the matrix of the linear map $X \to CXC^{-1}$, with respect to the basis $\{\sigma_z \otimes \sigma_x, \sigma_x \otimes I_2, \sigma_z \otimes i\sigma_y, i\sigma_y \otimes I_2\}$. We don’t need the full matrix to prove our claim, since a matrix in $SO^+(2, 2, \mathbb{R})$ is positive definite iff its
diagonal blocks are. Accordingly we display only these blocks. They are

\[
NW = \frac{1}{2} \begin{pmatrix}
  x_1x_6 + x_3x_8 + 2x_2x_4 & 2x_2x_4 \\
  2x_2x_4 & x_1x_3 + x_6x_8 - 2x_2x_4
\end{pmatrix}
\]

and

\[
SE = \frac{1}{2} \begin{pmatrix}
  x_1x_6 + x_3x_8 - 2x_2x_4 & 2x_2x_4 \\
  2x_2x_4 & x_1x_3 + x_6x_8 + 2x_2x_4
\end{pmatrix}
\]

Now \( x_1x_6 + x_3x_8 \geq 2\sqrt{x_1x_6 x_3x_8} - x_2^2x_4^2 \), after a calculation which uses
\[1 - x_1x_8 = -x_6^2 \] and \( 1 - x_3x_6 = -x_2^2 \). Obviously this is positive since \( x_1x_8 > x_2^2 \) and \( x_3x_6 > x_2^2 \).

One similarly shows that the determinant of the SE block is also positive. Thus, \( \Phi(A, B) \) is positive definite.

The following result which stems from Remark (5.1) is worth recording separately:

**Theorem 5.3** Let \( X \in G_{1,n,n} \) with its diagonal \( n \times n \) blocks denoted \( A \) and \( D \). The four connected components of \( G_{1,n,n} \) admit the following description:

- \( \det(X) = 1, \det(A) > 0 \) and \( \det(D) > 0 \). This is \( SO^+(n,n,\mathbb{R}) \).

- \( \det(X) = 1, \det(A) < 0 \) and \( \det(D) < 0 \).

- \( \det(X) = -1, \det(A) > 0 \) and \( \det(D) < 0 \).

- \( \det(X) = -1, \det(A) < 0 \) and \( \det(D) > 0 \).

**Proof:** Let \( X = VP \) be the polar decomposition of \( X \). By Remark (3.3), \( P \) always belongs to \( SO^+(n,n,\mathbb{R}) \) and thus since \( SO^+(n,n,\mathbb{R}) \) is a group, we see that if \( X \in SO^+(n,n,\mathbb{R}) \), then so is \( V \). Hence by 5) of Remark (5.1), \( V \) is block-diagonal with both blocks having positive determinant. Since both diagonal blocks of \( P \) have positive determinants, it follows that the same holds for \( X \). Next let \( \det(X) = 1 \), but let \( X \) not belong to \( SO^+(n,n,\mathbb{R}) \). If \( X = VP \), then it follows that \( V \) cannot be in \( SO^+(n,n,\mathbb{R}) \), and hence by 5) of Remark (5.1), \( V \) is block-diagonal with both blocks having negative determinants. Hence, the same holds for \( X \).

Next, let \( \det(X) = -1 \). Then obviously, \( X = \text{diag}(I_n, I_{n,1})Y \), with \( Y \in SO(n,n,\mathbb{R}) \). Hence the remaining statements follow. \( \diamond \).
6 Conclusions

In this work algorithms for computing the entries of the matrices in a polar decomposition of matrices preserving the non-degenerate bilinear form with signature matrices $I_{2,2}$ and $I_{1,3}$, as well as their quaternionic representations were provided. These methods eschewed any eigenvalue calculations. In the process an interesting characterization of positive definite matrices and methods for completion to positivity in these groups were obtained. In addition, an explicit formula for “the” logarithm of a positive definite matrix in $G_{In,n}$ was obtained.

One question suggested by this work that seems worth pondering over is the following.

Is it true, in general, that the preimage of a positive definite matrix in $SO^+(p, q, R)$ in its covering group can also be chosen to be positive definite?

7 Appendix

In this appendix the quaternionic characterization of symmetric and positive definite matrices in $Sp(4, R)$ is provided for purposes of comparison.

Theorem 7.1 Let $X$ be a $4 \times 4$ symplectic matrix which is also symmetric. Then it admits the quaternion representation $X = a1 \otimes 1 + p \otimes i + q \otimes j + r \otimes k$, with $aq = r \times p$, $p.q = 0 = r.q$, and $a$ satisfying the constraint $a^2 - p.p + q.q - r.r = 1$. If $a = \frac{1}{4}Tr(X) \neq 0$, then $X$ is symplectic iff $aq = r \times p$ and $a^2 - p.p + q.q - r.r = 1$. Such an $X$ is positive definite in addition, iff i) $a > 0$ and ii) $2a^2 - 2(q.q) + 1 > 0$. In particular, a symmetric, symplectic matrix with $a > 0, q = 0$ is always positive definite.

It is primarily the fact that the vector $q$ is essentially the cross-product of $p$ and $r$ that made the above quaternionic representation as the most convenient starting point for the algorithmic determination of the polar decomposition in the symplectic group in [2].

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9 Competing Interests

On behalf of each author, the corresponding author states that there is no conflict of interest.

References

[1] F. Adjei, N. Dabkowski & V. Ramakrishna, “Inversion of the Covering Map for Indefinite Spin Groups - I”, in preparation.

[2] Y. Ansari & V. Ramakrishna, “On the Non-compact Portion of $Sp(4,\mathbb{R})$ via Quaternions”, J. Phys A: Math. Theor, 41, 335203, 1-12, (2008).

[3] S. Baskal and Y. Kim, “Lorentz Group in Ray and Polarization Optics”, n Mathematical Optics: Classical, Quantum and Imaging Methods Vasudevan Lakshminarayanan, Taylor and Francis, New York, (2012).

[4] G. Chen, D. Church, B. Englert, C. Henkel, “An Elementary, First Principles Approach to the the Indefinite Spin Groups”, Advances in Applied Clifford Algebras, 27, No 2, 1283-1311, (2017).

[5] T. Constantinescu, V. Ramakrishna, N. Spears, L. R. Hunt, J. Tong, I. Panahi, G. Kannan, D. L. MacFarlane, G. Evans & M. P. Christensen, “Composition methods for four-port couplers in photonic integrated circuitry”, Journal of Optical Society of America A, 23 (2006), 2919-2931.

[6] D. Hacon, “Jacobi’s Method for Skew-Symmetric Matrices”, 14, 619 - 628, (1993).

[7] E. Herzig, V. Ramakrishna & M. Dabkowski, “Note on Rotation, Reversion and Exponentiation in Dimensions Five and Six”, Journal of Geometry and Symmetry in Physics, 35, doi: 10.7546, pp 61-101, (2014).

[8] H. Fassbender, D. Mackey & N. Mackey, “Hamilton and Jacobi come full circle: Jacobi algorithms for structured Hamiltonian eigenproblems”, Linear Algebra and Its Applications, 332, 37-80 (2001).
[9] E. Herzig & V. Ramakrishna, “An Elementary, First Principles Approach to the the Indefinite Spin Groups”, *Advances in Applied Clifford Algebras*, **27**, No 2, 1283-1311, (2017).

[10] R. A. Horn & C. R. Johnson, *Matrix Analysis*, II edition, Cambridge University Press (2012).

[11] C. R. Johnson, T. Laffe & C. K. Li, “Linear Transformations on $M_n(\mathbb{R})$ that Preserve the Ky Fan Norm and a Remarkable Special Case when $(n, k) = (4, 20)$”, *Linear and Multilinear Algebra*, **23**, 285-298, (1988).

[12] J. Kuiper, *Quaternions and Rotation Sequences*, Princeton University Press (1999).

[13] P. Lounesto, *Clifford Algebras and Spinors*, II edition, Cambridge University Press (2002).

[14] N. Mackey, “Hamilton and Jacobi Meet Again: Quaternions and the Eigenvalue Problem”, *Siam J. Matrix Analysis*, **16**, 421-435, (1995).

[15] D. Mackey, N. Mackey & S. Dunleavy, “Structure Preserving Algorithms for Perplectic Eigenproblems, *Electronic J of Linear Algebra*, **13**, 10-39, (2005).

[16] D. Mackey, N. Mackey & F. Tisseur, “Structured Factorizations in Scalar Product Spaces”, *SIAM J. Matrix Analysis*, **27**, 821-850, (2006).

[17] V. Ramakrishna & F. Costa, ‘On the Exponential of Some Structured Matrices’, *J. Phys A: Math & General*, **37**, 11613-11627, (2004).

[18] V. Ramakrishna & H. Zhou, “On the Exponential of Matrices in $su(4)$”, *J. Phys. A - Math & General*, **39** 3021-3034, (2006).

[19] V. Ramakrishna, Y. Ansari & F. Costa, “Minimal Polynomials of Some Matrices Via Quaternions”, *Advances in Applied Clifford Algebras*, **22**, 159-183, (2012).