Zero–one laws for eventually always hitting points in mixing systems

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Abstract

In this work we study the set of eventually always hitting points in shrinking target systems. These are points whose long orbit segments eventually hit the corresponding shrinking targets for all future times. We focus our attention on systems where translates of targets exhibit near perfect mutual independence, such as Bernoulli schemes and the Gauß map. For such systems, we present tight conditions on the shrinking rate of the targets so that the set of eventually always hitting points is a null set (or co-null set respectively).

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1 Introduction

Let \((X, \mu, T)\) be a measure preserving system, and let \(\mathcal{B} = \{B_n : n \in \mathbb{N}\}\) be a sequence of measurable subsets of \(X\). The hitting set \(H(\mathcal{B})\) is defined as the set of \(x \in X\) such that

\[ T^n x \in B_n \text{ for infinitely many } n \in \mathbb{N}. \]  

If \(\sum_n \mu(B_n)\) is finite, it follows from the Borel–Cantelli Lemma that \(H(\mathcal{B})\) has measure zero. Conversely, if \(\sum_n \mu(B_n)\) is infinite then in certain settings the hitting set \(H(\mathcal{B})\) has full measure. Results pertaining to this dichotomy, where \(\sum_n \mu(B_n)\) is either finite or infinite, are referred to as dynamical Borel-Cantelli lemmas.

The earliest result of this type is due to Kurzweil [Ku]. He proved that for \(X = [0,1]\) and \(T\) a rotation by \(\alpha\), \((1.2)\) holds for any sequence of nested intervals \((B_1 \supset B_2 \supset \ldots)\) if and only if \(\alpha\) is badly approximable. Later there was an important paper of Philipp [P], in which it is shown that \((1.2)\) holds in the cases where \(X = [0,1]\), \(B\) consists of (not necessarily nested) intervals, and \(T\) is either the map \(x \mapsto \beta x \mod 1\) or the Gauß map \(x \mapsto 1/x \mod 1\). See e.g. [S, KM, CK, HNPV, Ke, KY] for further results, and [A] for a survey.

Let us say that \((X,\mu,T,B)\) is a shrinking target system if the sets \(B_n\) are nested and

\[ \lim_{n \to \infty} \mu(B_n) = 0. \]  

For \(m \in \mathbb{N}\), write \(O_m(x) := \{T^0 x, T^1 x, \ldots, T^m x\}\) for the \(m\)-th orbit segment of a point \(x \in X\) under the transformation \(T\). Certainly, if \(x\) belongs to \(H(\mathcal{B})\) then \(O_m(x) \cap B_m \neq \emptyset\) for infinitely many \(m\). On the other hand, if \(O_m(x) \cap B_m \neq \emptyset\) for infinitely many \(m\) then either \(x \in H(\mathcal{B})\) or \(T^m x \in \bigcap_{n \in \mathbb{N}} B_n\). Thus, under the additional assumption \((1.3)\), \(H(\mathcal{B})\) essentially coincides with the set

\[ \{x \in X : O_m(x) \cap B_m \neq \emptyset \text{ infinitely often}\}. \]  

In this paper, we study a natural variation of the set defined in \((1.4)\). Following the terminology introduced by Kelmer [Ke], we define the eventually always hitting set \(\text{EAH}(\mathcal{B})\) to be the set of \(x \in X\) such that for all but finitely many \(m \in \mathbb{N}\) there exists \(n \in \{1, \ldots, m\}\) such that \(T^n x \in B_m\). Equivalently,

\[ \text{EAH}(\mathcal{B}) = \{x \in X : O_m(x) \cap B_m \neq \emptyset \text{ eventually always}\}. \]  

By comparing \((1.4)\) and \((1.5)\), we see that up to a set of measure zero the eventually always hitting property is a strengthening of \((1.1)\). It is also not hard to show that in any ergodic shrinking target system, the set of eventually always hitting points obeys a zero–one law (cf. Proposition \(\S\) and Corollary \(\S\) below). It is therefore natural to ask:

Under what conditions on the shrinking rate of the size of the targets in \(\mathcal{B}\) can one expect \(\text{EAH}(\mathcal{B})\) to have zero or full measure respectively?

This question has already been addressed for certain special classes of shrinking target systems. Bugeaud and Liao [BL] looked at maps \(x \mapsto \beta x \mod 1\) on \(X = \)
[0, 1] and computed the Hausdorff dimension of sets $\text{EAH}(\mathcal{B})$ for families of rapidly shrinking targets $\mathcal{B}$. In the set-up of [Ke], $X$ is the unit tangent bundle of a finite volume hyperbolic manifold of constant negative curvature, $T$ is the time-one map of the geodesic flow on $X$, and $\mathcal{B}$ consists of rotation-invariant subsets of $X$. Under these conditions, it was shown that $\text{EAH}(\mathcal{B})$ has full measure whenever the series $\sum_{j=1}^{\infty} \frac{1}{2^j \mu(B_j)}$ diverges. This was later generalized by Kelmer and Yu [KY] to higher rank homogeneous spaces, and by Kelmer and Oh [KO] to the set-up of actions on geometrically finite hyperbolic manifolds of infinite volume. More recently, several results in this direction were obtained by Kirsebom, Kunde and Persson [KKP] for some classes of interval maps, including the doubling map, some quadratic maps, the Gauß map, and the Manneville-Pomeau map.

1.1 Main technical result

Our main technical result concerns systems whose targets satisfy a long-term independence property that arises in connection with rapid mixing. In such cases, we give sufficient conditions for the set of eventually always hitting points to either have zero or full measure. The class of systems to which this applies contains several relevant examples, such as product systems, Bernoulli schemes and the Gauß map.

The long-term independence property that we impose in our theorem asserts, roughly speaking, that any target $B_m \in \mathcal{B}$ becomes “evenly spread out” under the transformation $T$ in the sense that
$$\mu(A \cap T^{-n} B_m) - \mu(A) \mu(B_m) \approx 0,$$
where $\eta : \mathbb{N} \to [0, 1]$ is some function that satisfies $\lim_{m \to \infty} \eta(m) = 0$, and $F : \mathbb{N} \to \mathbb{N}$ is some function that satisfies
$$F(m) \leq \frac{1}{(\log m)^{1+\delta} \mu(B_m)}$$
for some $\delta > 0$ and all large enough $m \in \mathbb{N}$. (1.7)

We also define the set
$$E_m := \{ x \in X : O_m(x) \cap B_m = \emptyset \},$$
which describes the collection of all points in $X$ for which none of the first $m$ iterates under the transformation $T$ visits the target $B_m$. Note that
$$X \setminus \text{EAH}(\mathcal{B}) = \limsup E_n := \bigcap_{n \in \mathbb{N}} \bigcup_{m \geq n} E_m.$$ (1.9)

Theorem 1. Let $(X, \mu, T, \mathcal{B})$ be a shrinking target system satisfying (1.6). If
$$\sum_{n=1}^{\infty} \frac{\mu(E_n)^{1-\varepsilon}}{n} < \infty$$
for some $\varepsilon > 0$, then $\text{EAH}(\mathcal{B})$ has full measure. On the other hand, if
$$\sum_{n=1}^{\infty} \frac{\mu(E_n)}{n} = \infty,$$
then $\text{EAH}(\mathcal{B})$ has zero measure.
1.2 Product systems

For our first application of Theorem 1, fix an arbitrary probability space \((Y, \nu)\), and let \(A_1 \supset A_2 \supset \ldots\) be a sequence of measurable subsets of \(Y\) with \(\mu(A_n) \to 0\) as \(n \to \infty\). Consider the shrinking target system \((X, \mu, T, \mathcal{B})\), where \(X := Y^{\mathbb{N}} \cup \{0\}\), \(\mu := \nu^\otimes \mathbb{N} \cup \{0\}\), \(T : X \to X\) denotes the left shift, and the shrinking targets \(\mathcal{B} := \{B_1 \supset B_2 \supset \ldots\}\) are defined as \(B_n := \{x \in X : x[0] \in A_n\}\). The elements in \(\mathcal{B}\) have the convenient property that

\[
\mu(B_n \cap T^{-k}B_m) = \mu(B_n)\mu(B_m), \quad \forall k, n, m \in \mathbb{N},
\]

which immediately implies that the shrinking target system \((X, \mu, T, \mathcal{B})\) satisfies condition (1.6) with \(\eta(m) = 0\) and \(F(m) = 0\) for all \(m \in \mathbb{N}\).

**Theorem 2.** Let \((X, \mu, T, \mathcal{B})\) be the shrinking target system described above. If

\[
\sum_{n=1}^{\infty} \frac{(1 - \mu(B_n))^{n(1-\varepsilon)}}{n} < \infty
\]

for some \(\varepsilon > 0\), then \(\text{EAH}(\mathcal{B})\) has full measure. On the other hand, if

\[
\sum_{n=1}^{\infty} \frac{(1 - \mu(B_n))^n}{n} = \infty,
\]

then \(\text{EAH}(\mathcal{B})\) has zero measure.

From Theorem 2, one can derive the following corollary.

**Corollary 3.** Let \((X, \mu, T, \mathcal{B})\) be as in Theorem 2. Suppose that there exists \(C > 1\) such that for all but finitely many \(m\) one has

\[
\mu(B_m) \geq \frac{C \log \log m}{m};
\]

then \(\text{EAH}(\mathcal{B})\) has full measure. If, on the other hand,

\[
\mu(B_m) \leq \frac{\log \log m}{m}
\]

for all but finitely many \(m\), then \(\text{EAH}(\mathcal{B})\) has zero measure.

1.3 Bernoulli schemes

Another class of systems that satisfy (1.6) for a natural choice of shrinking targets are Bernoulli schemes. Let \((X, T)\) denote the full symbolic shift in 2 letters that is, \(X := \{0, 1\}^\mathbb{N}\). Let \(T : X \to X\) be the left shift on \(X\), and denote by \(\mu\) the \((1/2, 1/2)\)-Bernoulli measure on \(X\). Given a non-decreasing unbounded sequence of indices \((r_m)_{m \in \mathbb{N}}\), consider the corresponding sequence of shrinking targets \(\mathcal{B} = \{B_1 \supset B_2 \supset \ldots\}\) defined as

\[
B_m := \{x \in X : x[0] = x[1] = \ldots = x[r_m - 1] = 0\} \quad \forall m \in \mathbb{N}.
\]

Note that \(\mu(B_m) = 2^{-r_m}\). It is then straightforward to verify that the resulting shrinking target system \((X, \mu, T, \mathcal{B})\) satisfies condition (1.6) with \(\eta(m) = 0\) and \(F(m) = r_m\) for all \(m \in \mathbb{N}\).

---

1The same results hold for shifts on \(\{0, \ldots, b - 1\}^{\mathbb{N}}\) for any integer \(b > 2\); we chose to restrict ourselves to the case \(b = 2\) to slightly simplify the presentation.
Theorem 4. Let \((X, \mu, T, B)\) be as above, and assume that either one of the following two conditions is satisfied:

\[ \exists D > 2 \text{ such that } \mu(B_m) \geq \frac{D \log \log m}{m} \text{ for all but finitely many } m \in \mathbb{N}; \quad (1.11) \]

\[ \exists \tau > 0 \text{ such that } \mu(B_m) \leq \frac{1}{(\log m)^\tau} \text{ for all but finitely many } m \in \mathbb{N}. \quad (1.12) \]

If

\[ \sum_{n=1}^{\infty} \frac{(1 - \mu(B_n)) n (1-\epsilon)}{n} < \infty \]

for some \(\epsilon > 0\), then \(\text{EAH}(B)\) has full measure. On the other hand, if

\[ \sum_{n=1}^{\infty} \frac{(1 - \mu(B_n)) n}{n} = \infty, \]

then \(\text{EAH}(B)\) has zero measure.

In analogy to Corollary 3, we can derive from Theorem 4 the following corollary.

Corollary 5. Let \((X, \mu, T, B)\) be as in Theorem 4. If \(1.11\) holds, then \(\text{EAH}(B)\) has full measure. If, on the other hand,

\[ \mu(B_m) \leq \frac{2 \log \log m}{m} \]

for all but finitely many \(m\), then \(\text{EAH}(B)\) has zero measure.

1.4 The Gauß map

The Gauß map is a map \(T\) on the interval \(X := [0,1]\) defined as

\[ T(x) := \begin{cases} \frac{1}{x} - \lfloor \frac{1}{x} \rfloor & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases} \]

There is an explicit \(T\)-invariant Borel probability measure on \([0,1]\) called the Gauß measure (cf. [EW, Lemma 3.5]):

\[ \mu(B) := \frac{1}{\log 2} \int_B \frac{dx}{1 + x}, \quad \text{for all measurable } B \subset [0,1]. \]

The Gauß map and the Gauß measure are tightly connected to the theory of continued fractions. Any irrational number \(x \in [0,1]\) has a unique simple continued fraction expansion

\[ x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}, \quad a_1, a_2, \ldots \in \mathbb{N}, \]

which we write as \([a_1, a_2, \ldots]\). Note that if \(x = [a_1, a_2, \ldots]\), then \(T(x) = [a_2, a_3, \ldots]\). Thus \(T\) acts as the left shift on the continued fraction representation of a number. This identification leads us to a natural shrinking target problem where the
targets are determined by digit restrictions in the continued fraction expansion. Let \((k_m)_{m \in \mathbb{N}}\) be a non-decreasing sequence of natural numbers, and consider the sequence of shrinking targets \(B = \{B_1 \supset B_2 \supset \ldots\}\) given by

\[ B_m := \{[a_1, a_2, \ldots] : a_1 \geq k_m\} \]

for all \(m \in \mathbb{N}\). Note that \(B_m\) coincides with the interval \([0, 1/k_m]\), and

\[
\mu(B_m) = \frac{\log(1 + 1/k_m)}{\log 2}.
\] (1.13)

We show in Section 6 that the shrinking target system \((X, \mu, T, \mathcal{B})\) satisfies condition \((1.6)\) for any \(F(m)\) that satisfies \((1.7)\) and \(\eta(m) = O\left(\exp(-C\sqrt{F(m)})\right)\) for some universal constant \(C > 0\). Combining this with Theorem 1 allows us to derive the following result.

**Theorem 6.** Let \((X, \mu, T, \mathcal{B})\) be as described above, and assume that either there exists \(\sigma < 1\) such that \(k_m \leq \frac{2m}{\log \log m}\) for all but finitely many \(m \in \mathbb{N}\), or there exists \(\tau > 0\) such that \(k_m \geq (\log m)^\tau\) for all but finitely many \(m \in \mathbb{N}\). If

\[
\sum_{n=1}^{\infty} \frac{(1 - \mu(B_n))^{n\log 2(1 - \varepsilon)}}{n} < \infty
\]

for some \(\varepsilon > 0\), then \(\text{EH}(\mathcal{B})\) has full measure. On the other hand, if

\[
\sum_{n=1}^{\infty} \frac{(1 - \mu(B_n))^{n\log 2(1 + \varepsilon)}}{n} = \infty,
\]

for some \(\varepsilon > 0\), then \(\text{EH}(\mathcal{B})\) has zero measure.

**Corollary 7.** Let \((X, \mu, T, \mathcal{B})\) be as described above. If there exists \(C_1 > \frac{1}{\log 2}\) such that for all but finitely many \(m\) one has

\[
\mu(B_m) \geq \frac{C_1 \log \log m}{m},
\]

then \(\text{EH}(\mathcal{B})\) has full measure. If, on the other hand, there exists \(C_2 < \frac{1}{\log 2}\) such that

\[
\mu(B_m) \leq \frac{C_2 \log \log m}{m}
\]

for all but finitely many \(m\), then \(\text{EH}(\mathcal{B})\) has zero measure.

## 2 General properties of \(\text{EH}\) sets

Before embarking on the proofs of Theorems 1, 2, 4, and 6, we gather in this section some general results regarding \(\text{EH}\) sets that apply to all ergodic shrinking target systems. In Subsection 2.1 below, we show that all \(\text{EH}\) sets obey a zero–one law. Thereafter, in Subsections 2.2 and 2.3 we present general necessary and sufficient conditions for \(\text{EH}\) sets to have full measure.
2.1 The zero–one law for eventually always hitting points

We begin with showing that \( \text{EAH} \) sets are essentially invariant, a result that was obtained independently in [KKP].

**Proposition 8.** Let \( (X, \mu, T, \mathcal{B}) \) be a (not necessarily ergodic) shrinking target system. Then \( \text{EAH}(\mathcal{B}) \) is essentially invariant under \( T \), that is,

\[
\mu\left(\text{EAH}(\mathcal{B}) \triangle T^{-1}\text{EAH}(\mathcal{B})\right) = 0.
\]

**Proof.** Let \( Y := \{ x \in X : T^k x \in \mathcal{B}_n \text{ for infinitely many } n \} \). Since \( \mathcal{B}_n \) are nested, we have that \( Y = \bigcap_{n \in \mathbb{N}} T^{-1}\mathcal{B}_n \) and, using \( \mu(\mathcal{B}_n) \to 0 \) as \( n \to \infty \), it follows from the monotone convergence theorem that \( Y \) has zero measure.

We now claim that if \( x \in \text{EAH}(\mathcal{B}) \setminus Y \), then \( T^k x \in \text{EAH}(\mathcal{B}) \). To verify this claim recall that

\[
x \in \text{EAH}(\mathcal{B}) \iff O_n(x) \cap \mathcal{B}_n \neq \emptyset \text{ for all but finitely many } n.
\]

Note also that \( O_n(x) = \{ T^k x \} \cup O_{n-1}(T^k x) \). Therefore, if \( x \notin Y \), then \( O_n(x) \cap \mathcal{B}_n \) is non-empty for cofinitely many \( n \) if and only if \( O_{n-1}(T^k x) \cap \mathcal{B}_n \neq \emptyset \) for cofinitely many \( n \). Hence

\[
x \in \text{EAH}(\mathcal{B}) \setminus Y \implies O_n(x) \cap \mathcal{B}_n \neq \emptyset \text{ for all but finitely many } n
\]

\[
\implies O_{n-1}(T^k x) \cap \mathcal{B}_n \neq \emptyset \text{ for all but finitely many } n
\]

\[
\implies O_{n-1}(T^k x) \cap \mathcal{B}_{n-1} \neq \emptyset \text{ for all but finitely many } n
\]

\[
\implies O_{n}(T^k x) \cap \mathcal{B}_n \neq \emptyset \text{ for all but finitely many } n,
\]

where in the second to last implication we have used that \( \mathcal{B}_n \subset \mathcal{B}_{n-1} \). This proves that if \( x \in \text{EAH}(\mathcal{B}) \setminus Y \) then \( T^k x \in \text{EAH}(\mathcal{B}) \). Therefore

\[
\text{EAH}(\mathcal{B}) \setminus Y \subset T^{-1}\text{EAH}(\mathcal{B}).
\]

Since \( \mu(Y) = 0 \) and \( T \) is measure preserving, we conclude that

\[
\mu\left(\text{EAH}(\mathcal{B}) \triangle T^{-1}\text{EAH}(\mathcal{B})\right) = \mu\left(\left(\text{EAH}(\mathcal{B}) \setminus Y\right) \triangle T^{-1}\text{EAH}(\mathcal{B})\right)
\]

\[
= \mu\left(\left(T^{-1}\text{EAH}(\mathcal{B})\right) \setminus \left(\text{EAH}(\mathcal{B}) \setminus Y\right)\right)
\]

\[
= \mu(T^{-1}\text{EAH}(\mathcal{B})) - \mu(\text{EAH}(\mathcal{B}) \setminus Y)
\]

\[
= 0.
\]

This finishes the proof. \( \square \)

In the presence of ergodicity, all essentially invariant sets are trivial. Therefore Proposition 8 implies the following corollary.

**Corollary 9.** If \( (X, \mu, T, \mathcal{B}) \) is an ergodic shrinking target system, then \( \text{EAH}(\mathcal{B}) \) is either a null set or a co-null set.
2.2 General sufficient condition for $\mu(\text{EAH}(B)) = 1$

For $m, n \in \mathbb{N}$ define

$$E_{n,m} := \{ x : O_n(x) \cap B_m = \emptyset \}, \quad (2.1)$$

which can also be written as

$$E_{n,m} = \bigcap_{i=1}^{n} T^{-i} B_m^c. \quad (2.2)$$

The following result is taken from [Ke] and plays an important role in our proof of Theorem 1:

**Lemma 10** ([Ke, Lemma 13]). Suppose there exists a non-decreasing sequence $m_j$ such that $\sum_{j=0}^{\infty} \mu(E_{m_j-1,m_j}) < \infty$. Then $\mu(\text{EAH}(B)) = 1$.

2.3 General necessary condition for $\mu(\text{EAH}(B)) = 1$

The next result establishes a necessary condition for EAH sets to have full measure, conditional under the assumption that the sets $E_n$ are asymptotically independent.

**Theorem 11.** Let $(m_j)_{j \in \mathbb{N}}$ be a non-decreasing sequence and $(X, \mu, T, \mathcal{B})$ a shrinking target system with the property that

$$\mu(E_{m_s} \cap E_{m_t}) = (1 + o_{t \to \infty}(1)) \mu(E_{m_s}) \mu(E_{m_t}) 1 - 2^{n-\ell+1} + O(\mu(E_{m_s}) v_t), \quad (2.3)$$

where $(v_t)_{t \in \mathbb{N}}$ is a sequence of non-negative numbers satisfying $\sum_{t \in \mathbb{N}} v_t < \infty$. If $\mu(\text{EAH}(B)) = 1$ then necessarily $\sum_{j=1}^{\infty} \mu(E_{m_j}) < \infty$.

For the proof of Theorem 11 we need the following lemma.

**Lemma 12.** Let $0 < q_i < 1$ for $i = 1, \cdots, N$. Then

$$\sum_{m>n}^{N} \left(q_n q_m^{1-2^{n-m+1}} - q_n q_m\right) = O \left(\sum_{n=1}^{N} q_n\right).$$

**Proof.** Recall Bernoulli’s inequality, which asserts that $(1 + y)^r - 1 \leq ry$ for all $r \in (0, 1)$ and $y > -1$. If we apply this inequality with $y = \frac{1}{q_m} - 1$ and $r = 2^{n-m+1}$ we obtain

$$\left(\frac{1}{q_m}\right)^{2^{n-m+1}} - 1 \leq 2^{n-m+1} \left(\frac{1}{q_m} - 1\right) \leq \frac{1}{2^{m-n-1}q_m}.$$ 

This gives

$$\sum_{m>n}^{N} \left(q_n q_m^{1-2^{n-m+1}} - q_n q_m\right) = \sum_{m>n}^{N} q_n q_m \left(\left(\frac{1}{q_m}\right)^{2^{n-m+1}} - 1\right) \leq \sum_{m>n}^{N} \frac{q_n}{2^{m-n+1}} = O \left(\sum_{n=1}^{N} q_n\right).$$

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Proof of Theorem 1.1. By way of contradiction, assume that \( \sum_{j=1}^{\infty} \mu(E_{m_j}) = \infty \). Let \( 1_j = 1_{E_{m_j}} \) denote the indicator function of \( E_{m_j} \), and define \( q_j := \mu(E_{m_j}) \). Consider the normalized deviation

\[
D_N = \frac{\sum_{j=1}^{N} 1_j}{\sum_{j=1}^{N} q_j} - 1.
\]

Its \( L^2 \) norm is

\[
\|D_N\|_2^2 = \frac{2 \sum_{t>s} (1_s - q_s, 1_t - q_t)}{\left(\sum_{j=1}^{N} q_j\right)^2} + \frac{\sum_{j=1}^{N} (1_j - q_j, 1_j - q_j)}{\left(\sum_{j=1}^{N} q_j\right)^2}
\]

\[
= 2 \sum_{t>s} \frac{(1_s - q_s, 1_t - q_t)}{\left(\sum_{j=1}^{N} q_j\right)^2} + o_{N \to \infty}(1)
\]

\[
= 2 \sum_{t>s} \frac{\mu(E_{m_s} \cap E_{m_t}) - q_t q_t}{\left(\sum_{j=1}^{N} q_j\right)^2} + o_{N \to \infty}(1).
\]

Fix \( \varepsilon > 0 \). As guaranteed by (2.3), there exists \( m \in \mathbb{N} \) such that for all \( s, t \in \mathbb{N} \) with \( t \geq m \) one has

\[
\mu(E_{m_s} \cap E_{m_t}) \leq (1 + \varepsilon) q_s q_t^{1-2^{s-t+1}} + O(q_s v_t).
\]

Hence

\[
\|D_N\|_2^2 \leq 2 \sum_{m<s<N} \frac{(q_s q_t^{1-2^{s-t+1}} (1+\varepsilon) - q_s q_t + O(q_s v_t))}{\left(\sum_{j=1}^{N} q_j\right)^2} + o_{N \to \infty}(1)
\]

\[
= \frac{2(1+\varepsilon) \sum_{m<s<N} q_s q_t^{1-2^{s-t+1}}}{\left(\sum_{j=1}^{N} q_j\right)^2} + \frac{2\varepsilon \sum_{m<s<N} q_s q_t}{\left(\sum_{j=1}^{N} q_j\right)^2} + O\left(\frac{\sum_{m<s<N} q_s v_t}{\left(\sum_{j=1}^{N} q_j\right)^2}\right) + o_{N \to \infty}(1)
\]

\[
\leq \frac{2(1+\varepsilon) \sum_{m<s<N} q_s q_t^{1-2^{s-t+1}}}{\left(\sum_{j=1}^{N} q_j\right)^2} + \varepsilon + O\left(\frac{\sum_{m<s<N} v_t}{\left(\sum_{j=1}^{N} q_j\right)^2}\right) + o_{N \to \infty}(1).
\]

Since by assumption \( \sum_{j=1}^{\infty} q_j = \infty \) and \( \sum_{t=1}^{\infty} v_t < \infty \), the term \( O\left(\frac{\sum_{m<s<N} v_t}{\left(\sum_{j=1}^{N} q_j\right)^2}\right) \) goes to 0 as \( N \to \infty \). Also, using Lemma 12 we can control the term

\[
\frac{2(1+\varepsilon) \sum_{m<s<N} (q_s q_t^{1-2^{s-t+1}} - q_s q_t)}{\left(\sum_{j=1}^{N} q_j\right)^2}.
\]

Indeed,

\[
\frac{2(1+\varepsilon) \sum_{m<s<N} (q_s q_t^{1-2^{s-t+1}} - q_s q_t)}{\left(\sum_{j=1}^{N} q_j\right)^2} \leq \frac{2(1+\varepsilon) \sum_{1<s<N} (q_s q_t^{1-2^{s-t+1}} - q_s q_t)}{\left(\sum_{j=1}^{N} q_j\right)^2}
\]

\[
= O\left(\frac{2(1+\varepsilon) \sum_{1<s<N} q_s q_t}{\left(\sum_{j=1}^{N} q_j\right)^2}\right)
\]

\[
= O\left(\frac{1}{\left(\sum_{j=1}^{N} q_j\right)^2}\right)
\]

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This proves that \( \|D_N\|^2 \leq \varepsilon + o_{N \to \infty}(1) \). Since \( \varepsilon \) was chosen arbitrarily, we obtain \( \|D_N\|^2 = o_{N \to \infty}(1) \). The decay of the \( L^2 \)-norm of \( D_N \) implies that \( \limsup E_{m_j} \) has full measure. Therefore \( \mu(\limsup E_n) = 1 \), which, in view of (1.9), contradicts \( \mu(\text{AH}(B)) = 1 \). □

3 Proof of the main technical result

This section is dedicated to the proof of Theorem 1, and is divided into three subsections. In Subsection 3.1 we study the asymptotic behavior of \( \mu(E_{m_{j-1},m_j}) \) for certain lacunary sequences \((m_j)\), which is needed for the proof of Theorem 1 in combination with Lemma 10. In Subsection 3.2 we proceed to study the asymptotic independence of \( E_{m_j} \) along dyadic sequences \((m_j)\), which we need for the application of Theorem 11. Finally, in Subsection 3.3 we combine all these results to finish the proof of Theorem 1.

3.1 Estimates for the measure of \( E_{m_{j-1},m_j} \)

**Proposition 13.** Let \((X,\mu,T,\mathcal{B})\) be a shrinking target system satisfying (1.6). Let \( m,n,k \in \mathbb{N} \) with \( kn \leq m \). Then

\[
\mu(E_{kn,m}) = (1 + o_{m \to \infty}(1)) \cdot \mu(E_{n,m})^k + O(F(m)\mu(B_m)).
\] (3.1)

Here, the implicit constant in \( O(F(m)\mu(B_m)) \) may depend on \( k \), but is otherwise universal.

For the proof of Proposition 13 it will be convenient to write \( E_{n,m}^* \) for the set

\[
E_{n,m}^* := \begin{cases} T^{-F(m)}E_{n-F(m),m}, & \text{if } n > F(m), \\ X, & \text{otherwise.} \end{cases}
\] (3.2)

Note that \( E_{n,m}^* \) always contains \( E_{n,m} \) as a subset. This inclusion follows quickly from the definition of \( E_{n,m} \) (cf. (2.1) and (2.2)), because

\[
E_{n,m} = \bigcap_{j=1}^{n} T^{-j}B_m^c \subset \bigcap_{j=F(m)+1}^{n} T^{-j}B_m^c = T^{-F(m)}E_{n-F(m),m} = E_{n,m}^*.
\]

In general, this inclusion is proper and the sets \( E_{n,m} \) and \( E_{n,m}^* \) are not identical. However, they are approximately the same. Indeed, since we are only interested in the case where the quantity \( F(m) \) is much smaller than \( m \), the difference in measure between \( E_{n,m} \) and \( E_{n,m}^* \) becomes negligible (as we will see in the proofs of Proposition 13 and Lemma 16 below). For that reason, we suggest to think of \( E_{n,m}^* \) as an approximation of \( E_{n,m} \).

The advantage of using \( E_{n,m}^* \) over \( E_{n,m} \) is that for any \( \ell \in \mathbb{N} \) with \( \ell \geq n - F(m) \) and any set \( C \in \Xi_{\ell,m} \) one has

\[
|\mu(C \cap T^{-\ell}E_{n,m}) - \mu(C)\mu(E_{n,m}^*)| \leq \eta(m)\mu(C)\mu(E_{n,m}^*),
\] (3.3)

which follows directly from (1.6) by choosing \( A = C \) and \( B = E_{n-F(m),m} \).
Proof of Proposition 13. Recall that \( E_{n,m} = \bigcap_{i=1}^{n} T^{-i}B^c_m \). We can split off the first \( F(m) \) terms in this intersection and thus write \( E_{n,m} \) as the intersection of two sets,

\[
E_{n,m} = \bigcap_{i=1}^{\min\{n,F(m)\}} T^{-i}B^c_m \cap E^*_{n,m},
\]

where \( E^*_{n,m} \) is as defined in (3.2). We can think of \( E^*_{n,m} \) as the “main part” of \( E_{n,m} \) and of \( R \) as the “remainder”. Since

\[
E_{kn,m} = \bigcap_{j=1}^{kn} T^{-j}B^c_m = \bigcap_{j=0}^{k-1} T^{-jn} \left( \bigcap_{i=1}^{n} T^{-i}B^c_m \right) = \bigcap_{j=0}^{k-1} T^{-jn} E_{n,m},
\]

we can now write

\[
\mu(E_{kn,m}) = \mu \left( \bigcap_{j=0}^{k-1} T^{-jn} E_{n,m} \right) = \mu \left( R' \cap \bigcap_{j=0}^{k-1} T^{-jn} E^*_{n,m} \right),
\]

where \( R' = \bigcap_{j=0}^{k-1} T^{-jn} R \). From this it follows that

\[
\mu(E_{kn,m}) \leq \mu \left( \bigcap_{j=0}^{k-1} T^{-jn} E^*_{n,m} \right), \tag{3.4}
\]

which provides us with a suitable upper bound on \( \mu(E_{kn,m}) \). We also want to find a good lower bound for \( \mu(E_{kn,m}) \). Observe that the measure of \( R \) can trivially be bounded from below by \( \mu(R) \geq 1 - F(m) \mu(B_m) \). Therefore, we can bound the measure of \( R' \) from below by \( \mu(R) \geq 1 - k \mu(R') \geq 1 - k F(m) \mu(B_m) \). This gives the estimate

\[
\mu(E_{kn,m}) = \mu \left( R' \cap \bigcap_{j=0}^{k-1} T^{-jn} E^*_{n,m} \right) \geq \mu \left( \bigcap_{j=0}^{k-1} T^{-jn} E^*_{n,m} \right) - k F(m) \mu(B_m). \tag{3.5}
\]

To finish the proof, we only have to apply (3.5) \((k - 1)\) times to find that

\[
\mu \left( \bigcap_{j=0}^{k-1} T^{-jn} E^*_{n,m} \right) \geq (1 - \eta(m))^{k-1} \mu(E^*_{n,m})^k \tag{3.6}
\]

and

\[
\mu \left( \bigcap_{j=0}^{k-1} T^{-jn} E^*_{n,m} \right) \leq (1 + \eta(m))^{k-1} \mu(E^*_{n,m})^k. \tag{3.7}
\]

Finally, since \( \mu(E^*_{n,m}) = \mu(E_{n,m}) + O\left( F(m) \mu(B_m) \right) \), we obtain

\[
\mu(E^*_{n,m})^k = \mu(E_{n,m})^k + O(k F(m) \mu(B_m)) \tag{3.8}
\]

Putting together (3.4), (3.5), (3.6), (3.7), and (3.8) proves (3.1). \( \square \)
From Proposition 13 we can now derive the following corollary.

**Corollary 14.** For any shrinking target system \((X, \mu, T, B)\) that satisfies \([1.6]\) and any \(m, n, k \in \mathbb{N}\) with \(kn \leq m\),

\[
\mu(E_{kn,m}) = (1 + o_{m \to \infty}(1)) \mu(E_{(k+1)n,m})^{\frac{1}{k+1}} + O\left((F(m) \mu(B_m))^{\frac{1}{k+1}}\right).
\]

**Proof.** By Proposition 13,

\[
\mu(E_{kn,m}) = (1 + o(1)) \mu(E_{n,m})^k + O(F(m) \mu(B_m))
\]

\[
= \left((1 + o(1)) \mu(E_{n,m})^{k+1}\right)^{\frac{k}{k+1}} + O\left(F(m) \mu(B_m)\right)
\]

\[
= \left((1 + o(1)) \mu(E_{(k+1)n,m}) + O(F(m) \mu(B_m))\right)^{\frac{k}{k+1}} + O\left(F(m) \mu(B_m)\right)
\]

\[
= (1 + o(1)) \mu(E_{(k+1)n,m})^{\frac{k}{k+1}} + O\left(F(m) \mu(B_m)\right)^{\frac{k}{k+1}}.
\]

This finishes the proof.

**Proposition 15.** Let \((X, \mu, T, B)\) be a shrinking target system satisfying \([1.6]\). Let \(k \geq 2\) and define

\[
m_j := k \left\lfloor \frac{(k+1)^{\frac{j+1}{k}}}{k^{\frac{j+1}{k}}} \right\rfloor.
\]

Then \(\mu(E_{mj-1,m_j}) = (1 + o_{j \to \infty}(1)) \mu(E_{mj+1,m_j})^{\frac{1}{k+1}} + O\left((F(m_j) \mu(B_{m_j}))^{\frac{1}{k+1}}\right).
\]

**Proof.** Set \(n_j := \left\lfloor \frac{(k+1)^{\frac{j+1}{k}}}{k^{\frac{j+1}{k}}} \right\rfloor\). Then \(n_{j-1} = kn_{j-1}\).

Observe that

\[
n_{j+1} - \frac{k+1}{k} n_{j-1} = \left\lfloor \frac{(k+1)^{\frac{j+1}{k}}}{k^{\frac{j+1}{k}}} \right\rfloor - \frac{k+1}{k} \left\lfloor \frac{(k+1)^{\frac{j-1}{k}}}{k^{\frac{j-1}{k}}} \right\rfloor
\]

\[
= - \left\lfloor \frac{(k+1)^{\frac{j+1}{k}}}{k^{\frac{j+1}{k}}} \right\rfloor + \frac{k+1}{k} \left\lfloor \frac{(k+1)^{\frac{j-1}{k}}}{k^{\frac{j-1}{k}}} \right\rfloor
\]

\[
= O(1),
\]

and therefore \(kn_{j+1} - (k+1)n_{j-1} = O(k)\).

Observe also that \((k+2)n_{j-1} = \frac{k+2}{k+1} \frac{k+1}{k} n_{j-1}\) and hence \(|m_{j+1} - (k+2)n_{j-1}|\) is bounded from above by \(2k\). Since \(k\) is fixed, we will view \(O(k)\) as \(O(1)\). It follows that \(|m_{j+1} - (k+1)n_{j-1}| = O(1)\) and hence

\[
\mu(E_{mj+1,m_j}) = \mu(E_{(k+1)n_{j-1},m_j}) + O(\mu(B_{m_j})).
\]

In view of Corollary 14 we obtain

\[
\mu(E_{mj-1,m_j}) = \mu(E_{knj-1,m_j})
\]

\[
= (1 + o(1)) \mu(E_{(k+1)n_{j-1},m_j})^{\frac{1}{k+1}} + O\left((F(m_j) \mu(B_{m_j}))^{\frac{1}{k+1}}\right)
\]

\[
= (1 + o(1)) \left(\mu(E_{mj+1,m_j}) + O(\mu(B_{m_j}))\right)^{\frac{k}{k+1}}
\]

\[
+ O\left((F(m_j) \mu(B_{m_j}))^{\frac{k}{k+1}}\right)
\]

\[
= (1 + o(1)) \mu(E_{mj+1,m_j})^{\frac{k}{k+1}} + O\left((F(m_j) \mu(B_{m_j}))^{\frac{k}{k+1}}\right).
\]

This completes the proof.

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3.2 Independence of dyadic samples

**Lemma 16.** Let \((X, \mu, T, \mathcal{B})\) be a shrinking target system satisfying (1.6). For every \(s \in \mathbb{N}\) let \(m_s\) be a number in \(\{2^s, 2^{s+1}\}\). Then, for all \(t > s\), the intersection of \(E_{m_s}\) with \(E_{m_t}\) has measure bounded from above by

\[
\mu(E_{m_s} \cap E_{m_t}) \leq (1 + o_t \to \infty(1)) \mu(E_{m_s}) \mu(E_{m_t})^{1 - \frac{2^{s+1}}{2^s}} + O(\mu(E_{m_s}) (F(m_t) \mu(B_{m_t}))^{1 - \frac{2^{s+1}}{2^s}}).
\]

Proof. It follows from the definition of \(E_{m_s}\) and \(E_{m_t}\) (cf. (2.1) and (2.2)) and the fact that \(B_{m_t} \subset B_{m_s}\) that

\[
E_{m_s} \cap E_{m_t} = E_{m_s} \cap T^{-m_s} E_{m_t - m_s, m_t}.
\]

Since \(E_{m_t - m_s, m_t}\) is a subset of \(E_{m_t}^{* - m_{s, m_t}}\), we trivially have

\[
\mu(E_{m_s} \cap E_{m_t}) = \mu(E_{m_s} \cap T^{-m_s} E_{m_t - m_s, m_t}) \leq \mu(E_{m_s} \cap T^{-m_s} E_{m_t}^{* - m_{s, m_t}}).
\]

(3.9)

It follows from (1.6) that

\[
\mu(E_{m_s} \cap T^{-m_s} E_{m_t}^{* - m_{s, m_t}}) \leq (1 + \eta(m_t)) \mu(E_{m_s}) \mu(E_{m_t}^{* - m_{s, m_t}}).
\]

(3.10)

Putting together (3.9) and (3.10) we obtain

\[
\mu(E_{m_s} \cap E_{m_t}) \leq (1 + \eta(m_t)) \mu(E_{m_s}) \mu(E_{m_t}^{* - m_{s, m_t}}).
\]

(3.11)

Let \(k := \lfloor m_t / m_s \rfloor - 1\). In light of (3.11) we see that for the proof of Lemma 16 it is beneficial to find a good upper bound on the measure of the set \(E_{m_t - m_s, m_t}^{*}\), preferably in terms of the measure of \(E_{m_t}\). In order to find such an upper bound, we will first prove the following inequality:

\[
\frac{\mu(E_{m_t - m_s, m_t}^{*})}{1 + \eta(m_t)} \leq \mu(E_{m_s, m_t}^{*}).
\]

(3.12)

Since \(k m_j \leq m_t - m_s\), the set \(E_{m_t - m_s, m_t}^{*}\) is a subset of \(E_{k m_j, m_t}^{*}\) and hence \(\mu(E_{m_t - m_s, m_t}^{*}) \leq \mu(E_{k m_j, m_t}^{*})\). Therefore, instead of (3.12) it suffices to show

\[
\frac{\mu(E_{k m_j, m_t}^{*})}{1 + \eta(m_t)} \leq \mu(E_{m_s, m_t}^{*}).
\]

(3.13)

Note that

\[
E_{k m_s, m_t}^{*} = \bigcap_{i=F(m_t)+1}^{k m_s} T^{-i} B_{m_t} = \bigcap_{i=F(m_t)+1}^{m_s} T^{-i} B_{m_t} \cap \bigcap_{i=m_s+1}^{2m_s} T^{-i} B_{m_t} \cap \ldots \cap \bigcap_{i=(k-1)m_s+1}^{k m_s} T^{-i} B_{m_t}.
\]

Also observe that for any \(\ell \in \{1, \ldots, k - 1\},

\[
\bigcap_{i=\ell m_s+1}^{(\ell+1)m_s} T^{-i} B_{m_t} \subset T^{-\ell m_s} \left( \bigcap_{i=F(m_t)+1}^{m_s} T^{-i} B_{m_t} \right) = T^{-\ell m_s} E_{m_s, m_t}^{*}.
\]

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This proves that
\[ E_{km,s,m_t}^* \subset \bigcap_{\ell=0}^{k-1} T^{-\ell m_s} E_{m,s,m_t}^*. \]

If we now apply property (1.6) to \( \mu \left( \bigcap_{\ell=0}^{k-1} T^{-\ell m_s} E_{m,s,m_t}^* \right) \) \((k-1)\) times, then we see that
\[ \mu(E_{km,s,m_t}^*) \leq \mu \left( \bigcap_{\ell=0}^{k-1} T^{-\ell m_s} E_{m,s,m_t}^* \right) \leq (1 + \eta(m_t))^{k-1} \mu(E_{m,s,m_t}^*). \]

This completes the proof of (3.13), and hence also of (3.12).

Next, consider the trivial identity
\[ \mu(E_{m_t-m,s,m_t}^*) = \frac{\mu(E_{m,s,m_t}^*) \mu(T^{-m_s} E_{m_t-m,s,m_t}^*)}{\mu(E_{m,s,m_t}^*)}. \]  
(3.14)

Using (3.12) we get
\[ \frac{\mu(E_{m,s,m_t}^*) \mu(T^{-m_s} E_{m_t-m,s,m_t}^*)}{\mu(E_{m,s,m_t}^*)} \leq (1 + \eta(m_t)) \mu(E_{m,s,m_t}^*) \mu(T^{-m_s} E_{m_t-m,s,m_t}^*)^{1-\frac{1}{k}}. \]

Using property (1.6) once more, we conclude
\[ \left( \mu(E_{m,s,m_t}^*) \mu(T^{-m_s} E_{m_t-m,s,m_t}^*) \right)^{1-\frac{1}{k}} \leq (1 + \eta(m_t)) \mu(E_{m,s,m_t}^*) \mu(T^{-m_s} E_{m_t-m,s,m_t}^*)^{1-\frac{1}{k}}. \]  
(3.15)

Similar to the proof of Proposition 13 we can approximate \( E_{m_t-m,s,m_t}^* \) with \( E_{m_t-m,s,m_t}^* \) and \( E_{m,s,m_t}^* \) with \( E_{m,s,m_t}^* \) by enduring an error of \( \mathcal{O}(F(m_t)\mu(B_{m_t})) \), which gives
\[ \mu(E_{m,s,m_t}^* \cap T^{-m_s} E_{m_t-m,s,m_t}^*) = \mu(E_{m,s,m_t}^* \cap T^{-m_s} E_{m_t-m,s,m_t}^*) + \mathcal{O}(F(m_t)\mu(B_{m_t})) \]
\[ = \mu(E_{m,s,m_t}^*) + \mathcal{O}(F(m_t)\mu(B_{m_t})). \]

It follows that
\[ \mu(E_{m,s,m_t}^* \cap T^{-m_s} E_{m_t-m,s,m_t}^*)^{1-\frac{1}{k}} = \mu(E_{m,s,m_t}^*)^{1-\frac{1}{k}} + \mathcal{O}\left( (F(m_t)\mu(B_{m_t}))^{1-\frac{1}{k}} \right). \]  
(3.16)

Combining (3.11), (3.12), (3.13), (3.15) and (3.16) yields
\[ \mu(E_{m_t} \cap E_{m_t}) \leq (1 + o_{t \to \infty}(1)) \mu(E_{m,s}^*) \mu(E_{m_t}^{1-\frac{1}{k}}) + \mathcal{O}\left( (\mu(E_{m,s} F(m_t)\mu(B_{m_t}))^{1-\frac{1}{k}} \right). \]

Finally, since \( k = \ceil{m_t/m_s} - 1 \geq \left[ 2^t/(2^{s+1} - 1) \right] - 1 = 2^t/2^{s+1} - 1 \), one has
\[ 1 - \frac{1}{k} \geq 1 - \frac{2^{s+1}}{2^{t}}. \]

This finishes the proof. \( \square \)
3.3 Proof of Theorem 1

We need one more lemma before proving Theorem 1.

Lemma 17. Suppose that (1.7) holds. Let \( \sigma > 1 \) and let \( (m_j)_{j \in \mathbb{N}} \) be a sequence of natural numbers such that

\[
m_{j+1}/m_j \geq \sigma \quad \text{for all large enough } j \in \mathbb{N}.
\]  

(3.18)

Define \( v_j := F(m_j)\mu(B_{m_j}) \). Then

\[
\sum_{j \in \mathbb{N}} v_j^{k+\frac{1}{k+\pi}} < \infty
\]

for all \( k \geq 2\delta^{-1} \).

**Proof.** In view of (3.18), there exists some \( c > 0 \) such that \( m_j \geq c\sigma^j \) for all large enough \( j \in \mathbb{N} \). Hence from (1.7) we can conclude that

\[
v_j \leq \frac{1}{(\log m_j)^{1+\delta}} \leq \frac{1}{j^{1+\delta}} \quad \text{for all but finitely many } j.
\]

Since \( \sum_{j \in \mathbb{N}} \left(\frac{1}{j^{1+\delta}}\right)^{k+\pi} < \infty \) for all \( k \) with \( k \geq 2\delta^{-1} \), the claim follows. \( \square \)

**Proof of Theorem 1**. First assume there exists \( \varepsilon > 0 \) such that

\[
\sum_{n=1}^{\infty} \frac{\mu(E_n)^{1-\varepsilon}}{n} < \infty.
\]

By assumption, there exists \( \delta > 0 \) such that \( F(m) \leq (\log^{1+\delta}(m)\mu(B_m))^{-1} \) for all but finitely many \( m \in \mathbb{N} \). Fix such a \( \delta \). Pick now \( k \in \mathbb{N} \) with \( 1/k < \min\{\varepsilon, \delta/2\} \).

Next let \( (m_j)_{j \in \mathbb{N}} \) be defined as in Proposition 15, that is, \( m_j := k \left( \frac{k+1}{k+\pi} \right)^j \).

It is easy to check that (3.18) holds, and, by combining Proposition 15 with Lemma 17 we see that the series \( \sum_{j \in \mathbb{N}} \mu(E_{m_{j-1},m_j}) \) converges if and only if so does the series \( \sum_{j \in \mathbb{N}} \mu(E_{m_{j+1},m_j})^{k+\pi} \). We now have

\[
\sum_{n=1}^{\infty} \frac{\mu(E_n)^{1-\varepsilon}}{n} \geq \sum_{j=1}^{\infty} \sum_{n \in [m_j,m_{j+1})} \frac{\mu(E_n)^{1-\varepsilon}}{n}
\]

\[
\geq \sum_{j=1}^{\infty} \frac{1}{m_{j+1}} \sum_{n \in [m_j,m_{j+1})} \mu(E_n)^{1-\varepsilon}.
\]

For any \( n \) with \( m_j \leq n < m_{j+1} \) we have \( \mu(E_n) \geq \mu(E_{m_{j+1},m_j}) \). Therefore

\[
\sum_{n=1}^{\infty} \frac{\mu(E_n)^{1-\varepsilon}}{n} \geq \sum_{j=1}^{\infty} \frac{1}{m_{j+1}} \sum_{n \in [m_j,m_{j+1})} \mu(E_{m_{j+1},m_j})^{1-\varepsilon}
\]

\[
\geq \sum_{j=1}^{\infty} \frac{m_{j+1}-m_j}{m_{j+1}} \mu(E_{m_{j+1},m_j})^{1-\varepsilon}
\]

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\[ \frac{1}{2^k} \sum_{j=1}^{\infty} \mu(E_{m_{j+1}, m_j})^{\frac{1}{\gamma+1}}. \]

Since \( \sum_{n=1}^{\infty} \frac{\mu(E_n)^{1-\epsilon}}{n} < \infty \), we conclude that \( \sum_{j=1}^{\infty} \mu(E_{m_{j+1}, m_j})^{\frac{1}{\gamma+1}} < \infty \), and therefore also \( \sum_{j \in \mathbb{N}} \mu(E_{m_{j-1}, m_j}) < \infty \). In view of Lemma 10, this implies that \( \text{EAH}(\mathcal{B}) \) has full measure, which completes the proof of the first part of Theorem 1.

For the second part, assume \( \sum_{n=1}^{\infty} \frac{\mu(E_n)}{n} = \infty \). For every \( j \in \mathbb{N} \) let \( m_j \) be a number in \([2^j, 2^{j+1})\) that satisfies \( \mu(E_{m_j}) = \max_{n \in [2^j, 2^{j+1})} \mu(E_n) \).

Then
\[
\sum_{n=1}^{\infty} \frac{\mu(E_n)}{n} = \sum_{j=0}^{\infty} \sum_{n \in [2^j, 2^{j+1})} \frac{\mu(E_n)}{n} \leq \sum_{j=0}^{\infty} \sum_{n \in [2^j, 2^{j+1})} \frac{1}{2^j} \mu(E_n) \leq \sum_{j=0}^{\infty} \mu(E_{m_j}).
\]

It follows that \( \sum_{j=1}^{\infty} \mu(E_{m_j}) = \infty \). In view of Lemma 16 and Lemma 17 we see that (2.3) is satisfied. Thus, by Theorem 11, we conclude that \( \mu(\text{EAH}(\mathcal{B})) \) is not equal to 1. Therefore, since \( \mu(\text{EAH}(\mathcal{B})) \) is essentially invariant (see Proposition 8), we must have \( \mu(\text{EAH}(\mathcal{B})) = 0. \)

4 Shrinking target systems with independent targets

Let us now show how Theorem 2 and the corresponding Corollary 3 follow from the results we have obtained so far.

Proof of Theorem 2. It follows immediately from property (1.10) and the definition of \( E_m \) (see (1.8)) that
\[ \mu(E_m) = \left(1 - \mu(B_m)\right)^m. \]
Hence Theorem 2 follows from Theorem 1.

Proof of Corollary 3. First assume for all but finitely many \( m \in \mathbb{N} \) that
\[ \mu(B_m) \geq \frac{C \log \log m}{m}. \]
Choose any \( b \in (1, C) \). Using the inequality \( (1 + x) \leq e^x \), which holds for all real numbers \( x \), we obtain (with \( x = -(C \log(k \log b/2))/[b^k] \)) that for all sufficiently large \( k \),
\[
\left(1 - \frac{C \log \log [b^k]}{[b^k]}\right)^{[b^k]} \leq \left(1 - \frac{C \log \log (b^k/2)}{[b^k]}\right)^{[b^k]} = \left(1 - \frac{C \log \left(\frac{k \log b}{2}\right)}{[b^k]}\right)^{[b^k]} \leq e^{-C \log \left(\frac{k \log b}{2}\right)} = \frac{e^{-C \log \left(\frac{k \log b}{2}\right)}}{k^C}. \]
Then
\[
\sum_{m=1}^{\infty} \frac{(1 - \mu(B_m))m^{(1-\varepsilon)}}{m} \leq \sum_{m=1}^{\infty} \left( 1 - \frac{C\log m}{m} \right)^{m(1-\varepsilon)} \leq \sum_{k=1}^{\infty} \left( 1 - \frac{C\log [b^k]}{[b^k]} \right)^{[b^k](1-\varepsilon)} = O \left( \sum_{k=1}^{\infty} \frac{1}{k^{C(1-\varepsilon)}} \right).
\]

Since \( \sum_{k=1}^{\infty} \frac{1}{k^{C(1-\varepsilon)}} < \infty \) for sufficiently small \( \varepsilon \), it follows from Theorem 2 that \( \text{EAH}(\mathcal{B}) \) has full measure.

The second part follows from an analogous calculation where instead of the inequality \((1+x) \leq e^x\) one uses the inequality \((1+x) \geq e^x - x^2\), which holds for all \( x \in (-1/2, 0) \). Indeed,
\[
\begin{align*}
\sum_{m=1}^{\infty} \frac{\mu(E_m)}{m} &= \sum_{m=1}^{\infty} \frac{(1 - \mu(B_m))m}{m} \\
&\geq \sum_{m=1}^{\infty} \frac{(1 - \log \log m)^m}{m} \\
&\geq \frac{1}{2} \sum_{k=1}^{\infty} \left( 1 - \frac{\log k}{2k} \right)^{2k} \\
&\geq \frac{1}{2} \sum_{k=1}^{\infty} \left( e^{-\frac{\log k}{2k} - \frac{\log^2 k}{2k^2}} \right)^{2k} \\
&= \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{k} e^{-\frac{\log^2 k}{2k}} = \infty.
\end{align*}
\]

Therefore, by Theorem 2 \( \text{EAH}(\mathcal{B}) \) has zero measure. \( \Box \)

5 Bernoulli schemes and a proof of Theorem 4

In this section we give a proof of Theorem 4. Let \((r_n)_{n \in \mathbb{N}} \) and \((X, \mu, T, \mathcal{B})\) be as in Subsection 1.3. Given a point \( x \in X = \{0, 1\}^{\mathbb{N}} \cup \{0\} \) we denote by \( x[1, \ldots, n] \) the word \( x[1]x[2] \ldots x[n] \).

In order to derive Theorem 4 from Theorem 1 we first need to understand the measure of the set \( E_n = \bigcap_{j=1}^{n} T^{-j}B_{r}^{c} \). Note that \( E_n \) consists exactly of all the points \( x \in \{0, 1\}^{\mathbb{N}} \cup \{0\} \) with the property that the word \( x[1, n + r] \) does not contain \( r_n \) consecutive zeros. To estimate \( \mu(E_n) \), it will therefore be convenient to beforehand estimate the average number of zeros in \( x[1, n + r] \). For each \( n \geq 1 \) and \( x \in X \), let \( V_n(x) := \max \{ \text{number of consecutive zeros in } x[1, \ldots, n] \} \). Let \( \log_2 x := \frac{\log x}{\log 2} \). Our main tool is the following estimate from [FS].

**Proposition 18 ([FS Proposition V.1.]).** Let \( a(n) := 2^{[\log_2 n]} \). One has
\[
\mu(V_n < [\log_2 n] + h) = \exp \left( - a(n)2^{-h-1} \right) + O \left( \frac{\log n}{\sqrt{n}} \right).
\]

Note that
\[
E_{n,m} = \{ x \in X : V_{n+r_m}(x) < r_m \}. \tag{5.1}
\]
Using this, we can get the first order asymptotics for $\mu(E_{n,m})$.

**Theorem 19.** One has

$$
\mu(E_{n,m}) = \exp \left( - \frac{2}{7} \mu(B_m) \right) (1 + o_{m \to \infty}(1)) + O \left( \frac{\log n}{\sqrt{n}} \right).
$$

**Proof.** We will write $h_{n,m} := r_m - \lfloor \log_2 (n + r_m) \rfloor$. In view of (5.1),

$$
\mu(E_{n,m}) = \exp \left( - a(n + r_m)^2 - h_{n,m} - 1 \right) + O \left( \frac{\log(n + r_m)}{\sqrt{n + r_m}} \right).
$$

We can replace $O \left( \frac{\log(n + r_m)}{\sqrt{n + r_m}} \right)$ with $O \left( \frac{\log n}{\sqrt{n}} \right)$. Thus,

$$
\mu(E_{n,m}) = \exp \left( - a(n + r_m)^2 - h_{n,m} - 1 \right) + O \left( \frac{\log n}{\sqrt{n}} \right).
$$

$$
= \exp \left( - a(n + r_m)^2 - \log_2 (n + r_m) - r_m + 1 \right) + O \left( \frac{\log n}{\sqrt{n}} \right)
$$

$$
= \exp \left( - (n + r_m) 2^{-r_m - 1} \right) + O \left( \frac{\log n}{\sqrt{n}} \right)
$$

$$
= \exp \left( - n 2^{-r_m - 1} \right) (1 + o_{m \to \infty}(1)) + O \left( \frac{\log n}{\sqrt{n}} \right)
$$

$$
= \exp \left( - \frac{n}{2} \mu(B_m) \right) (1 + o_{m \to \infty}(1)) + O \left( \frac{\log n}{\sqrt{n}} \right).
$$

From this the claim follows. \[\square\]

Choosing $n = m$ in Theorem 19 yields the following corollary.

**Corollary 20.** One has

$$
\mu(E_m) = \exp \left( - \frac{m}{2} \mu(B_m) \right) (1 + o_{m \to \infty}(1)) + O \left( \frac{\log m}{\sqrt{m}} \right).
$$

**Remark 21.** Theorem 19 can also be useful to estimate the measure of sets of the from $E_{m_{j-1},m_j}$, which are of interest because of Lemma 10. For the proof of Theorem 4, which we will present at the end of this section, we are particularly interested in the case where

$$
m_j = \lfloor b^j \rfloor
$$

for some $b > 1$. In this case, it follows from Theorem 19 that

$$
\mu(E_{m_{j-1},m_j}) = \exp \left( - \frac{m_{j-1}}{2} \mu(B_{m_j}) \right) (1 + o_{m \to \infty}(1)) + O \left( \frac{j}{b^j} \right). \tag{5.2}
$$

Since $m_{j-1} \geq \frac{m}{c^j}$ for all but finitely many $j$ as long as $c > b$, we deduce from (5.2) that

$$
\mu(E_{m_{j-1},m_j}) \leq \exp \left( - \frac{m_j}{2c} \mu(B_{m_j}) \right) (1 + o_{m \to \infty}(1)) + O \left( \frac{j}{b^j} \right) \tag{5.3}
$$

for all $c > b$.  

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Theorem 22. Let \((X, \mu, T, \mathcal{B})\) be a shrinking target system. If there exists \(\varepsilon > 0\) such that
\[
\sum_{m=1}^{\infty} \frac{(1 - \mu(B_m))^{\frac{m(1-\varepsilon)}{2}}}{m} < \infty,
\]
then there exists \(\varepsilon' > 0\) such that
\[
\sum_{m=1}^{\infty} \exp \left( -\frac{m}{T} \mu(B_m) \right)^{1-\varepsilon'} < \infty.
\]
Also, if
\[
\sum_{m=1}^{\infty} \frac{(1 - \mu(B_m))^{\frac{m}{T}}}{m} = \infty
\]
then
\[
\sum_{m=1}^{\infty} \exp \left( -\frac{m}{T} \mu(B_m) \right) = \infty.
\]

Proof. Using the basic inequality \((1 + x)^r \leq \exp(rx)\) it is straightforward to show that
\[
(1 - \mu(B_m))^{\frac{m}{T}} \leq \exp \left( -\frac{m}{T} \mu(B_m) \right).
\]
From this the implication
\[
\sum_{m=1}^{\infty} \frac{(1 - \mu(B_m))^{\frac{m}{T}}}{m} = \infty \implies \sum_{m=1}^{\infty} \exp \left( -\frac{m}{T} \mu(B_m) \right) = \infty
\]
follows. Vice versa, applying the inequality \((1 + x)^{\delta r} \geq \exp(rx)\), which holds for every \(\delta < 1\) and all negative \(x\) that are sufficiently close to 0, for \(\delta = \frac{1 - \varepsilon}{1 - \varepsilon'}\) we get
\[
(1 - \mu(B_m))^{\frac{m(1-\varepsilon)}{2}} \geq \exp \left( -\frac{m}{T} \mu(B_m) \right)^{1-\varepsilon'},
\]
where \(\varepsilon' > 0\) can be any number that is strictly smaller than \(\varepsilon\). This implies
\[
\sum_{m=1}^{\infty} \frac{(1 - \mu(B_m))^{\frac{m(1-\varepsilon)}{2}}}{m} < \infty \implies \sum_{m=1}^{\infty} \exp \left( -\frac{m}{T} \mu(B_m) \right)^{1-\varepsilon'} < \infty.
\]

Proof of Theorem 4. Recall that by assumption either (1.11) or (1.12) are satisfied. We will show below that (1.12) forces conditions (1.6) and (1.7) to be satisfied for an appropriate choice of \(F\) and \(\eta\), which will allow us to derive the conclusion of Theorem 4 from Theorem 1. On the other hand, under the assumption (1.11) we cannot guarantee that (1.7) is satisfied, because the measure of the targets \(B_m\) might not shrink sufficiently fast. In this case, instead of using Theorem 1, our argument will build on Remark 21 together with Lemma 10.

Let us first deal with (1.11). Let \(c\) be such that \(1 < c < D/2\), and define \(\eta := \frac{D}{4} - \frac{c}{2}\). Since \(\mu(B_m) \geq \frac{2(c+2n)\log\log m}{m}\) for all but finitely many \(m\), it follows
that $\frac{m}{2} \mu(B_m) \geq (c + 2\eta) \log \log m \geq (c + \eta) \log \left( \frac{\log m}{\log c} \right)$. Applying $\exp(-x/c)$ to both sides of this inequality yields

$$\exp \left( -\frac{m}{2c} \mu(B_m) \right) \leq \left( \frac{\log c}{\log m} \right)^{1 + \frac{2}{c}}$$

(5.4)

for all but finitely many $m$. Let $b \in (1, c)$ be arbitrary. Combining (5.4) with the first part of Theorem 22 shows that there exists $\varepsilon > 0$ such that

$$\sum_{m=1}^{\infty} \frac{(1 - \mu(B_m))^{m(1-\varepsilon)}}{m} < \infty.$$

This means we are in the first case of Theorem 3. After substituting $m_j = \lfloor b^j \rfloor$ for $m$ in (5.4), we are left with

$$\exp \left( -\frac{m_j}{2} \mu(B_m) \right) \leq \frac{1}{j^{1+\varepsilon}}.$$

Combining this with (5.3) shows that $\sum_{j \in \mathbb{N}} \mu(E_{m_{j-1}, m_j}) < \infty$. In light of Lemma 10, this proves that $\text{EAH}(B)$ has full measure.

Next, we deal with (1.12). Pick $F(m) = r_m$ and $\eta = 0$. Choose $\delta > 0$ sufficiently small such that such that $\tau \geq \frac{1}{1-\delta}$. Since

$$\mu(B_m) \log_2 \left( \frac{1}{\mu(B_m)} \right) \leq \mu(B_m) \log_2 \left( \frac{1}{\mu(B_m)} \right) \leq \mu(B_m)^{1-\delta}$$

for all but finitely many $m \in \mathbb{N}$ (because $\lim_{m \to \infty} \mu(B_m) = 0$), we deduce that

$$F(m) \mu(B_m) = \mu(B_m) \log_2 \left( \frac{1}{\mu(B_m)} \right) \leq \mu(B_m)^{1-\delta} \leq \left( \frac{1}{\log m} \right)^{1-\delta} \leq \frac{1}{(\log m)^{1+\delta}}.$$

Hence $F$ satisfies (1.7). By construction, the shrinking target system also satisfies (1.6). In light of Corollary 20 we have

$$\sum_{m=1}^{\infty} \frac{\mu(E_m)^{1-\varepsilon}}{m} < \infty \iff \sum_{m=1}^{\infty} \frac{\exp \left( -\frac{m}{2} \mu(B_m) \right)^{1-\varepsilon}}{m} < \infty$$

as well as

$$\sum_{m=1}^{\infty} \frac{\mu(E_m)}{m} = \infty \iff \sum_{m=1}^{\infty} \frac{\exp \left( -\frac{m}{2} \mu(B_m) \right)}{m} = \infty.$$

Hence Theorem 4 follows directly from Theorem 1 together with Theorem 22.

Corollary 5 can be derived from Theorem 4 the same way that Corollary 3 was derived from Theorem 2. Therefore we omit its proof.
6 Gauß map and Gauß measure

In this section let $(X, \mu, T, \mathcal{B})$ denote the shrinking target system considered in Subsection 1.4, where $X$ is the interval $[0, 1]$, $T: [0, 1] \to [0, 1]$ is the Gauß map, $\mu$ is the Gauß measure, and $\mathcal{B} = \{B_1 \supset B_2 \supset \ldots\}$ was defined as

$$B_m := \{[a_1, a_2, \ldots] : a_1 \geq k_m\} = [0, 1/k_m]$$

for all $m \in \mathbb{N}$.

We begin by showing that for this shrinking target system condition (1.6) holds for any $F(m)$ that satisfies (1.7) and $\eta(m) = O(\exp(-C \sqrt{F(m)})$ for some universal constant $C > 0$. The following result of Phillipp will be crucial for making this deduction.

**Lemma 23 ([P]).** There exists a constant $\lambda \in (0, 1)$ such that for all sets of the form $A = \{[a_1, a_2, \ldots] : a_1 = r_1, \ldots, a_n = r_n\}$, where $k, n, r_1, \ldots, r_n \in \mathbb{N}$ are arbitrary, and all measurable set $B \subset [0, 1]$ one has

$$\mu(A \cap T^{-n-k} B) = \mu(A) \mu(B) \left(1 + O\left(\lambda^{\sqrt{k}}\right)\right).$$

Since sets of the form $\{[a_1, a_2, \ldots] : a_1 = r_1, \ldots, a_n = r_n\}$ form an algebra that contains $\Xi_{n,m}$, it does indeed follow from Lemma 23 that (1.6) holds for any $F(m)$ that satisfies (1.7) and $\eta(m) = O(\exp(-C \sqrt{F(m)})$ for some universal constant $C > 0$.

Recall the definition of $E_{n,m}$ from (2.1) and (2.2).

**Proposition 24.** There exist constants $C, D \geq 1$ such that

$$\left(1 - (\log 2)\mu(B_m) - C\mu(B_m)^2\right)^n \leq \mu(E_{n,m}) \leq D \left(1 - (\log 2)\mu(B_m) + C\mu(B_m)^2\right)^n$$

for all $n \leq m \in \mathbb{N}$.

For the proof of Proposition 24 we will need the following lemma.

**Lemma 25.** Let $\mu$ denote the Gauß measure on $[0, 1]$, and $\lambda$ the Lebesgue measure on $[0, 1]$. We have the following estimates:

(i) $\mu(B_m) = \frac{1}{\log 2} \frac{1}{k_m} + O\left(\frac{1}{k_m^2}\right)$.

(ii) For all $0 \leq a < b \leq 1$ and all $i \geq k_m$ we have

$$\frac{1}{i+a} - \frac{1}{i+b} = \frac{b-a}{i^2} + O\left(\frac{b-a}{i^3}\right).$$

(iii) $\sum_{i=k_m}^{\infty} \frac{1}{i} = \frac{1}{k_m} + O\left(\frac{1}{k_m^2}\right)$.

**Proof.** Part (i) is immediate from (1.13). Part (ii) follows from a straightforward calculation:

$$\frac{1}{i+a} - \frac{1}{i+b} = \frac{b-a}{(i+a)(i+b)} = \frac{b-a}{i^2} + O\left(\frac{b-a}{i^3}\right).$$

Finally, for (iii) we have

$$\sum_{i=k_m}^{\infty} \frac{1}{i^2} = \sum_{i=k_m}^{\infty} \left(\int_{i}^{i+1} \frac{dx}{x^2} + O\left(\frac{1}{i^3}\right)\right)$$
\[
\int_{k_m}^{\infty} \frac{dx}{x^2} + O\left(\frac{1}{k_m^2}\right)
= \frac{1}{k_m} + O\left(\frac{1}{k_m^2}\right).
\]

**Proof of Proposition 24.** Consider the set \( \tilde{E}_{n,m} := \bigcap_{i=0}^{n-1} T^{-i} B_m^c \). Since \( E_{n,m} = T^{-1} \tilde{E}_{n,m} \) and since the Gauß measure is invariant under \( T \), the measure of \( E_{n,m} \) with respect to \( \mu \) equals the measure of \( \tilde{E}_{n,m} \) with respect to \( \mu \). Also, \( \mu \) is absolutely continuous with respect to the Lebesgue measure \( \lambda \), and so there exists some constant \( D \geq 1 \) for which

\[
\frac{1}{D} \lambda(\tilde{E}_{n,m}) \leq \mu(\tilde{E}_{n,m}) \leq D \lambda(\tilde{E}_{n,m}), \quad \forall m, n \in \mathbb{N}.
\]

Therefore, instead of (6.1), it suffices to prove the existence of some constant \( C \geq 1 \) such that

\[
(1 - \log 2 \mu(B_m) - C \mu(B_m)^2)^n \leq \lambda(\tilde{E}_{n,m}) \leq (1 - \log 2 \mu(B_m) + C \mu(B_m)^2)^n.
\]

Moreover, using part (i) of Lemma 25, we see that (6.2) is equivalent to

\[
(1 - \lambda(B_m) - C \lambda(B_m)^2)^n \leq \lambda(\tilde{E}_{n,m}) \leq (1 - \lambda(B_m) + C \lambda(B_m)^2)^n.
\]

Next, we observe that

\[
\lambda(\tilde{E}_{n,m}) = \lambda \left( \bigcap_{i=0}^{n-1} T^{-i} B_m^c \right)
= \lambda \left( B_m^c \cap T^{-n} \tilde{E}_{n-1,m} \right)
= \lambda(\tilde{E}_{n-1,m}) - \lambda \left( B_m \cap \tilde{E}_{n-1,m} \right).
\]

Since \( B_m^c = (\frac{1}{k_m}, 1] \subset [0, 1] \) is an interval and the Gauß map \( T \) is piecewise continuous, the set \( \tilde{E}_{n-1,m} \) is a union of disjoint intervals \((a_t, b_t)\), i.e.,

\[
\tilde{E}_{n-1,m} = \bigcup_t (a_t, b_t).
\]

Using the definition of the Gauß map, the pre-image of \( \tilde{E}_{n-1,m} \) under the transformation \( T \) can then be described as

\[
T^{-1} \tilde{E}_{n-1,m} = \bigcup_t \bigcup_{i \in \mathbb{N}} \left( \frac{1}{i + b_t}, \frac{1}{i + a_t} \right).
\]

Since \( B_m \) consists only of numbers whose first coefficient in its continued fraction expansion is bigger than \( k_m \), we deduce that

\[
B_m \cap T^{-1} \tilde{E}_{n-1,m} = \bigcup_t \bigcup_{i \geq k_m} \left( \frac{1}{i + b_t}, \frac{1}{i + a_t} \right).
\]
Since the intervals \( \left( \frac{1}{1+2i}, \frac{1}{1+a_1} \right) \) are all disjoint, we have

\[
\lambda(B_m \cap T^{-1}\tilde{E}_{n-1,m}) = \sum_{\ell} \sum_{i \geq k_m} \mu \left( \frac{1}{i+b_\ell}, \frac{1}{i+a_\ell} \right).
\]

Using part \( \text{(iii)} \) of Lemma 25 we see that

\[
\lambda(B_m \cap T^{-1}\tilde{E}_{n-1,m}) = \sum_{\ell} \sum_{i \geq k_m} \frac{\lambda((a_\ell,b_\ell))}{i^2} + O \left( \frac{\lambda((a_\ell,b_\ell))}{i^3} \right)
= \lambda(\tilde{E}_{n-1,m}) \left( \sum_{i \geq k_m} \frac{1}{i^2} + O \left( \frac{1}{i^3} \right) \right).
\]

By part \( \text{(iii)} \) of Lemma 25 we get

\[
\sum_{i \geq k_m} \frac{1}{i^2} + O \left( \frac{1}{i^3} \right) = \frac{1}{k_m} + O \left( \frac{1}{k_m} \right) = \lambda(B_m) + O(\lambda(B_m)^2),
\]

and then,

\[
\lambda(B_m \cap T^{-1}\tilde{E}_{n-1,m}) = \mu(\tilde{E}_{n-1,m}) \left( \lambda(B_m) + O(\lambda(B_m)^2) \right).
\]

By combining (6.5) with (6.3), we finally get

\[
\lambda(\tilde{E}_{n,m}) = \lambda(\tilde{E}_{n-1,m}) \left( 1 - \lambda(B_m) + O(\lambda(B_m)^2) \right).
\]

Iterating (6.6) \( (n-1) \) times finishes the proof of (6.3).

**Corollary 26.** There exists \( D \geq 1 \) such that for every \( \varepsilon > 0 \) the inequalities

\[
\frac{1}{D} \left( 1 - \mu(B_m) \right)^{m(\log 2)(1+\varepsilon)} \leq \mu(E_m) \leq D \left( 1 - \mu(B_m) \right)^{m(\log 2)(1-\varepsilon)}
\]

hold for all but finitely many \( m \in \mathbb{N} \).

**Proof.** Take \( C \) and \( D \) as in Proposition 24. In view of that proposition, it remains to show that

\[
\left( 1 - \mu(B_m) \right)^{m(\log 2)(1+\varepsilon)} \leq \left( 1 - (\log 2)\mu(B_m) - C\mu(B_m)^2 \right)^m
\]

as well as

\[
\left( 1 - \mu(B_m) \right)^{m(\log 2)(1-\varepsilon)} \geq \left( 1 - (\log 2)\mu(B_m) + C\mu(B_m)^2 \right)^m.
\]

This follows from the inequalities

\[
(1 - x)^r(1+\varepsilon) \leq 1 - rx - Cx^2 \quad \text{and} \quad (1 - x)^r(1-\varepsilon) \geq 1 - rx + Cx^2,
\]

which hold for all \( r > 0 \) and all sufficiently small positive \( x \).

**Remark 27.** We are also interested in using Proposition 24 to estimate the measure of sets of the form \( E_{m_j-1,m_j} \). In particular, if

\[
m_j = \lfloor b^j \rfloor
\]

for some \( b > 1 \), then

\[
\mu(E_{m_j-1,m_j}) = \left( 1 - (\log 2)\mu(B_{m_j}) + O(\mu(B_{m_j})^2) \right)^{m_j-1}
\]

\[
\leq \left( 1 - \mu(B_{m_j}) \right)^{m_j(\log 2)/c}
\]

as long as \( c > b \).
Proof of Theorem 6. We begin with the case where there exists $\sigma < 1$ such that $k_m \leq \frac{\sigma m}{\log \log m}$ for all but finitely many $m \in \mathbb{N}$. Let $\sigma'$ be any number satisfying $\sigma < \sigma' < 1$, and define $C := \frac{\sigma'}{\sigma \log 2}$. Since $\mu(B_m) \geq \frac{\sigma' \log m}{\sigma m \log 2}$ for all but finitely many $m$, it follows that

$$\mu(B_m) \geq \frac{\sigma' \log m}{\sigma m \log 2} = \frac{C \log \log m}{m}.$$  

Then, repeating an analogous argument to the one used in the proof of Corollary 3, we can show that

$$\sum_{j \in \mathbb{N}} \left(1 - \mu(B_{m_j})\right)^{m_j/(\log 2/c)} < \infty$$

for any $b, c \in [1, C]$ with $b < c$, where $m_j = \lfloor b^j \rfloor$. In view of (6.7), this means that $\sum_{j \in \mathbb{N}} \mu(E_{m_j-1,m_j}) < \infty$. Using Lemma 10 we conclude that $\text{EAH}(B)$ has full measure.

Next, we deal with the case where there exists $\tau > 0$ such that $k_m \geq (\log m)^\tau$ for all but finitely many $m \in \mathbb{N}$. Set $F(m) = \frac{1}{(\log m)^{\tau/2}} \mu(B_m)$ and note that $F(m)$ converges to $\infty$ as $m \to \infty$, because of the assumption that $k_m \geq (\log m)^\tau$. Moreover, $F$ satisfies (1.7) by construction and, as explained at the beginning of this section, (1.6) is satisfied for $\eta(m) = O\left(\exp(-C \sqrt{F(m)})\right)$. Here, it is important that $\lim_{m \to \infty} F(m) = \infty$, since this implies $\lim_{m \to \infty} \eta(m) = 0$. In light of Corollary 26 we have

$$\sum_{m=1}^\infty \frac{(1 - \mu(B_m))^{m/(\log 2)(1-\epsilon)}}{m} < \infty \implies \sum_{m=1}^\infty \frac{\mu(E_m)^{1-\frac{\epsilon}{2}}}{m} < \infty$$

as well as

$$\sum_{m=1}^\infty \frac{(1 - \mu(B_m))^{m/(\log 2)(1+\epsilon)}}{m} < \infty \implies \sum_{m=1}^\infty \frac{\mu(E_m)}{m} = \infty.$$ 

Hence Theorem 6 follows directly from Theorem 1.

We omit the proof of Corollary 7, since it can be derived from Theorem 6 in the same way that Corollary 3 was derived from Theorem 2.

7 Further explorations and open questions

There are still a multitude of intriguing questions surrounding the behavior of eventually always hitting sets. We begin with the following.

Question 28. Is it possible to remove $\epsilon$ in Theorems 4, 6, 8 and 10?

It is not at all clear if one should expect Question 28 to have an affirmative answer. Indeed, even in the simplest case of Theorem 2 where translates of targets exhibit perfect mutual independence, it remains questionable whether

$$\sum_{n=1}^\infty \frac{(1 - \mu(B_n))^n}{n} < \infty \iff \text{EAH}(B) \text{ has full measure.}$$
It seems that an equally plausible possibility (which is also consistent with our results) is

\[ \sum_n \mu(B_n)(1 - \mu(B_n))n \begin{cases} < \infty & \iff \text{EAH}(\mathcal{B}) \text{ has full measure.} \\ = \infty & \end{cases} \]

Another intriguing question concerns rotations on the torus. Fix \( \alpha \in [0,1) \) and consider the shrinking target system where \( X \) equals the torus \( T \), the transformation is given by \( T(x) = x + \alpha \mod 1 \), \( \mu \) is Lebesgue measure, and

\[ \mathcal{B} = (B_n) \text{ with } B_n := \{ x \in T : \|x\|_T < \psi(n) \} \tag{7.1} \]

where \( \psi : \mathbb{N} \to [0,1] \) is some non-increasing function. In this case, the set of eventually always hitting points can be written as

\[ \text{EAH}(\mathcal{B}) = \{ y \in T : \min_{1 \leq k \leq n} \|k\alpha - y\|_T < \psi(n) \text{ eventually always} \}. \tag{7.2} \]

In [KL] the Hausdorff dimension of \( \text{EAH}(\mathcal{B}) \) was computed for the cases where \( \psi(n) = n^{-\tau} \) for some \( \tau > 0 \). Closely related to the study of (7.2) are also questions regarding inhomogeneous versions of Dirichlet’s classical approximation theorem addressed in [KW].

As was mentioned in Section 1, Kurzweil [Ku] proved that when \( \alpha \) is badly approximable the hitting set \( H(\mathcal{B}) \) for \( \mathcal{B} \) as in (7.1) obeys the zero–one law

\[ \sum_n \psi(n) \begin{cases} < \infty & \iff H(\mathcal{B}) \text{ has full measure.} \\ = \infty & \end{cases} \]

More recently, an extension of Kurzweil’s result to arbitrary \( \alpha \in [0,1) \) was given by Fuchs and Kim [FK]:

\[ \sum_{k=1}^{\infty} \left( \sum_{n=q_k}^{q_{k+1} - 1} \min\{\psi(n), \|q_k \alpha\|_T\} \right) \begin{cases} < \infty & \iff H(\mathcal{B}) \text{ has full measure,} \\ = \infty & \end{cases} \]

where \( p_k/q_k \) denote the principal convergents of \( \alpha \).

By Corollary 9 we know that \( \text{EAH}(\mathcal{B}) \) also obeys a zero–one law. This leads to the following question.

**Question 29.** For a fixed \( \alpha \) (at least in the case when \( \alpha \) is badly approximable) what are necessary and sufficient conditions on \( \psi \) so that the set \( \text{EAH}(\mathcal{B}) \) as in (7.2) is a null set (or co-null set respectively)?

Another classical type of shrinking target systems are \( \beta \)-transformations. Let \( X = T \) and, for \( \beta > 1 \), consider the map \( T_\beta(x) = \beta x \mod 1 \) alongside the shrinking targets on the torus \( T \) given by (7.1). In this set-up,

\[ \text{EAH}(\mathcal{B}) = \{ y \in T : \min_{1 \leq k \leq n} \|T_\beta^k(y)\|_T < \psi(n) \text{ eventually always} \}. \tag{7.3} \]

The Hausdorff dimension of the set \( \text{EAH}(\mathcal{B}) \) in (7.3) was studied in [BL]. Unlike rotation by \( \alpha \), the map \( T_\beta \) is highly mixing, which suggests the following question.

**Question 30.** Does \( T_\beta \) and \( \mathcal{B} \) as above satisfy condition (1.6), perhaps with some additional assumptions on \( \psi \)?

An affirmative answer to Question 30 could lead to a better understanding of necessary and sufficient conditions for \( \text{EAH}(\mathcal{B}) \) in (7.3) to have full or zero measure respectively, similar in spirit to Theorems 4 and 6.
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