The grammar of the Ising model: A new complexity hierarchy

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How complex is an Ising model? Usually, this is measured by the computational complexity of its ground state energy problem. Yet, this complexity measure only distinguishes between planar and non-planar interaction graphs, and thus fails to capture properties such as the average node degree, the number of long range interactions, or the dimensionality of the lattice. Herein, we introduce a new complexity measure for Ising models and thoroughly classify Ising models with respect to it. Specifically, given an Ising model we consider the decision problem corresponding to the function graph of its Hamiltonian, and classify this problem in the Chomsky hierarchy. We prove that the language of this decision problem is (i) regular if and only if the Ising model is finite, (ii) constructive context free if and only if the Ising model is linear and its edge language is regular, (iii) constructive context sensitive if and only if the edge language of the Ising model is context sensitive, and (iv) decidable if and only if the edge language of the Ising model is decidable. We apply this theorem to show that the 1d Ising model, the Ising model on generalised ladder graphs, and the Ising model on layerwise complete graphs are constructive context free, while the 2d Ising model, the all-to-all Ising model, and the Ising model on perfect binary trees are constructive context sensitive. We also provide a grammar for the 1d and 2d Ising model. This work is a first step in the characterisation of physical interactions in terms of grammars.

I. INTRODUCTION

Spin models are a powerful tool to model complex systems. While the paradigmatic spin model, the Ising model [1–3], was originally proposed as a stripped-off model of magnetism, it has since been used in a remarkable variety of settings, including as a toy model of matter in certain quantum gravity models [4], to model gases (via so-called lattice gas models) [5], in knot theory (via the connection of the Jones polynomial with the partition function of the Potts model in a certain parameter regime) [6], for artificial neural networks (stemming from Hopfield’s proposal) [7, 8], in ecology (e.g. to model the size of canopy trees) [9], to model flocks of birds [10], viruses as quasi-species [11, 12], genetic interactions [13], for protein folding [14–17] (together with its generalisation, the Potts model [18]), for economic opinions, urban segregation and language change [19], for random language models [20], social dynamics [21], earthquakes [22] and the US Supreme Court [23], to name some. The relevant questions differ for each of these applications—e.g. for artificial neural networks, one is interested in a “driven” Ising model, where the parameters are updated (corresponding to learning) and one may study convergence rates, whereas in complex systems [9], one may be interested in the behaviour of the Ising model at criticality. Whatever the focus may be, the fact is that this very simplified model provides insights into very different problems.

Depending on the application, the Ising model is considered on different families of interaction graphs, such as lattices of a certain dimensionality for magnetism, or (layerwise) complete graphs for artificial neural networks. How does one characterise these different Ising models? In particular, how can we measure the complexity of Ising models on different families of graphs? Traditionally, this is measured by the computational complexity of the ground state energy problem (GSE), which asks:

Given an interaction graph for \( n \) spins and an integer \( k \), does there exist a spin configuration with energy below \( k \)?

For the Ising model without fields, if the family of interaction graphs is planar, this problem is in P, and if it is non-planar, it is NP-complete [24, 25] [26]. These results have given rise to strong and fruitful ties between spin models (and, more generally, statistical mechanics) and computational complexity [27, 28]. For example, one can formulate many NP problems in terms of the GSE [29].

Yet, this measure is very coarse: It only classifies Ising models depending on whether they are defined on planar or non-planar graphs (resulting in a two-level hierarchy of P and NP-complete, respectively). It is insensitive to the dimensionality of the interaction graph (when considering lattices), the number of long range interactions, or the average node degree. This might be due to the facts that GSE only ‘cares’ about the low energy sector of the model, and that computational complexity tends to gloss over polynomial factors. Clearly, a 1d Ising model has a different local structure than a 2d Ising model, yet this distinction is invisible in the traditional measure. Can one devise a measure that captures the complexity of the local structure of an Ising model?

In this work, we introduce a new complexity measure for Ising models and thoroughly classify them in it. We do so in three steps. First, given an Ising model \( \mathcal{M} \) we define its language \( L_{\mathcal{M}} \) which encodes the function graph of its Hamiltonian \( H_{\mathcal{M}} \), that is, the set of all pairs of spin configurations and their energy,

\[
L_{\mathcal{M}} = \{ (x, H_{\mathcal{M}}(x)) \mid x \text{ is a spin configuration of } \mathcal{M} \}
\]  

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Second, we consider the problem of deciding $L_{\mathcal{M}}$, that is:

$$\text{Given } (x, E), \text{ is } x \text{ is a valid spin configuration of } \mathcal{M} \text{ and is } E = H_{\mathcal{M}}(x)?$$

Third, we measure the hardness of this problem by classifying $L_{\mathcal{M}}$ in a (refined) Chomsky hierarchy. This answers the question:

What is the simplest type of grammar (automaton) that generates (accepts) $L_{\mathcal{M}}$?

Note that in computational complexity one characterises the resources (in time, space or non-determinism) that a Turing machine needs to recognise the language. Since the language of most Ising models can be decided in polynomial time by a deterministic Turing machine, computational complexity is insufficient to distinguish among their local structures, whereas the Chomsky hierarchy can achieve that. The refinement of the Chomsky hierarchy stems from posing restrictions on the automata that accept context free and context sensitive languages, resulting in two levels called constructive context free and constructive context sensitive (Fig. 1), which we conjecture to coincide with context free and context sensitive, respectively.

We prove that (Theorem 1):

(i) $L_{\mathcal{M}}$ is regular if and only if $\mathcal{M}$ is finite;

(ii) $L_{\mathcal{M}}$ is constructive context free if and only if $\mathcal{M}$ is linear and $E_{\mathcal{M}}$ is regular;

(iii) $L_{\mathcal{M}}$ is constructive context sensitive if and only if $E_{\mathcal{M}}$ is context sensitive; and

(iv) $L_{\mathcal{M}}$ is decidable if and only if $E_{\mathcal{M}}$ is decidable.

This classification fully characterises the complexity of $L_{\mathcal{M}}$ in terms of properties of the interaction graphs of $\mathcal{M}$. Specifically, the edge language $E_{\mathcal{M}}$ encodes which spins interact, and its complexity captures how difficult it is to decide whether two spins interact or not. The remaining properties of $\mathcal{M}$ (being finite or linear) encode how the number of interactions grows with the system size.

We then apply this classification to common families of interaction graphs, and show that (Fig. 3):

(i) The language of the 1d Ising model with open or periodic boundary conditions, the Ising model on ladder graphs, and the Ising model on layerwise complete graphs is constructive context free. All of these Ising models are linear and their edge language is regular.

(ii) The language of the Ising model on perfect binary trees and the 2d Ising model is constructive context sensitive. All of these Ising models are linear and their edge language is context sensitive.

(iii) The language of the all-to-all Ising model is constructive context sensitive. This Ising model is not linear and its edge language is regular.

Note that a similar approach was recently proposed in [30], yet for a more general definition of spin model which only achieves a partial characterisation in the Chomsky hierarchy. The present focus on the Ising model allows us to promote the partial characterisation to a thorough classification. In particular, this work identifies which
properties of the interaction graphs play a role in the complexity of the model, and specifies how they interact (metaphorically) to increase the complexity.

This paper is structured as follows. In Section II we define the new complexity measure for Ising models. In Section III we define several properties of Ising models and state our main result, Theorem 1, which we prove in Appendix A. In Section IV we apply our main result to obtain the complexity of some well-known examples of Ising models, and in Section V we conclude and present an outlook. In Appendix B we list some basic definitions and results from formal language and automata theory, and in Appendix C we provide explicit grammars of the 1d and 2d Ising model.

II. A NEW COMPLEXITY MEASURE FOR ISING MODELS

Here we define the new complexity measure for Ising models. In Section II A we define the concept of an Ising model and explain how the function graph of its Hamiltonian is encoded as a formal language \( L_\mathcal{M} \). In Section II B we define how the complexity measure for \( \mathcal{M} \) is obtained by classifying \( L_\mathcal{M} \) in the Chomsky hierarchy.

A. The Ising model and its language

What is an Ising model? That depends on the context: the system size may be pre-determined, unspecified, or defined in the thermodynamic limit. Additionally, the couplings may be fixed or drawn from a probability distribution, in which case it is usually called a spin glass. In this work, an Ising model is defined as follows.

Definition 1 (Ising model). An Ising model \( \mathcal{M} \) is a pair \( \mathcal{M} = (N_\mathcal{M}, E_\mathcal{M}) \), where

\[
N_\mathcal{M} \subseteq \mathbb{N} \\
E_\mathcal{M} = \{(E_\mathcal{M})_n \mid n \in N_\mathcal{M}\}
\]

and

\[
(E_\mathcal{M})_n \subseteq \{(i, j) \mid i, j \in \{1, \ldots, n\}, i < j\}
\]

defines an undirected, ordered graph with vertex set \( V_n := \{1, \ldots, n\} \) that has no isolated vertices. An Ising model \( \mathcal{M} \) defines a Hamiltonian \( H_\mathcal{M} \):

\[
H_\mathcal{M} : \bigcup_{n \in N_\mathcal{M}} \{0, 1\}^n \to \mathbb{Z} \\
s_1 \ldots s_n \mapsto \sum_{(i, j) \in (E_\mathcal{M})_n} h(s_i, s_j)
\]

where \( h(s_i, s_j) = -1 \) if \( s_i = s_j \) and \( +1 \) else.

In words, \( N_\mathcal{M} \) specifies the system sizes for which \( \mathcal{M} \) is defined, and for each \( n \in N_\mathcal{M} \), the edge set \( (E_\mathcal{M})_n \) describes how the system of \( n \) spins interacts. Specifically, if there is an edge \( (i, j) \in (E_\mathcal{M})_n \), spins \( i \) and \( j \) interact. We require that no vertex is isolated (i.e. that every vertex is contained in at least one edge) because isolated vertices correspond to non-interacting spins, which do not contribute to the Hamiltonian.

This definition could be generalised to include non-constant couplings or higher order interactions, e.g. by using hyperedge-labeled hypergraphs (see e.g. \([31]\)), as done in \([30]\). Yet, in this work we focus on the constant coupling case in order to classify the complexity of Ising models solely based on their interaction structure.

Note that \( \mathcal{M} \) is generally defined for an infinite set of system sizes, and that it is not defined in the thermodynamic limit \( (n \to \infty) \). Both are crucial for encoding \( H_\mathcal{M} \) as a language, as finitely many system sizes would result in a finite language (which is trivially regular), and the thermodynamic limit would require infinite strings (precluding the use of formal languages).

Finally, note that in Definition 1 the vertices of the interaction graphs have an order, as imposing such an order is necessary when encoding graphs as strings, and thus a family of graphs as a language. Ultimately this is due to the fact that symbols in a string have a canonical order, while vertices in a graph do not. Disposing of the order of the vertices would require considering equivalence classes of encodings of graphs (where two strings are equivalent if they encode the same graph), and measuring the complexity of an Ising model would require a minimisation over all equivalent encodings of that Ising model. The latter would involve, in particular, solving the graph isomorphism problem. Alternatively, such an order could be disposed of by casting spin models as graph languages (see the Conclusions and Outlook).

In order to define the language of an Ising model, let \( u \) denote the unary encoding of integers

\[
u : \mathbb{Z} \to \{+, -\}^* \\
u(z) := \begin{cases} 
\epsilon & \text{if } z = 0 \\
+z & \text{if } z > 0 \\
-z & \text{else}
\end{cases}
\]

Note that here + and − are just symbols, not mathematical operations.

Definition 2 (Language of an Ising model). Let \( \mathcal{M} \) be an Ising model. The language of \( \mathcal{M} \), \( L_\mathcal{M} \), is defined as

\[
L_\mathcal{M} := \{s_1 \ldots s_n \bullet u(H_\mathcal{M}(s_1 \ldots s_n)) \mid n \in N_\mathcal{M}, s_i \in \{0, 1\}\}
\]

In words, \( L_\mathcal{M} \) encodes the function graph of \( H_\mathcal{M} \). Explicitly, we use the symbol \( \bullet \) as a separator between spin configurations and energies. Let \( \sigma \in \{+, -\} \). A string \( s_1 \ldots s_n \bullet \sigma^k \) is contained in \( L_\mathcal{M} \) if \( s_1 \ldots s_n \) is a spin configuration from the domain of \( H_\mathcal{M} \) and \( \sigma^k \) equals the unary encoding of \( H_\mathcal{M}(s_1 \ldots s_n) \).

Note that the energy is encoded in unary, as this leads to a more fine-grained classification in the Chomsky hierarchy. Specifically, encoding the energy in binary would
render the addition of individual energy contributions context sensitive and not context free, and we would lose one entire level of our complexity hierarchy (Fig. 1). The increase in complexity caused by a binary encoding has also been observed from a different angle in Ref. [30].

B. The complexity measure provided by the Chomsky hierarchy

We now classify \( L_R \) in the Chomsky hierarchy. To that end, we define two additional levels of the Chomsky hierarchy which are obtained by posing restrictions on the automata that accept context free and context sensitive languages, namely constructive context free and constructive context sensitive. We will conjecture that these two new levels are identical with context free and context sensitive, respectively.

PDAs and LBAs are defined in Definition 9 and Definition 11, respectively. Let us now define their constructive versions.

Definition 3 (Constructive automaton). Let \( M \) be an Ising model and \( L_R \) be its language.

(i) A PDA \( P \) that decides \( L_R \) is called constructive if for any \( n \in N_R \), there exists a unique partition of edges of \( (E_R)_{n} = \bigcup_{m=1}^{r_n} I_m \), such that on well-formed inputs \( s_1 \ldots s_n \bullet \sigma^k \), \( P \) operates as follows:

(a) First, \( P \) accumulates \( u(H_R(s_1 \ldots s_n)) \) on its stack. \( P \) iterates over \( m = 1, \ldots, r_n \), and in each step of the iteration it stores the states of the spins that interact according to \( I_m \),

\[
V_n|I_m := \{ i \in V_n \mid \exists j \in V_n \text{ s.t. } (i, j) \in I_m \text{ or } (j, i) \in I_m \} \tag{8}
\]

in its states. \( P \) then adds the unary encoding of the energy contribution of those spins

\[
H_R|I_m := \sum_{(i,j) \in I_m} h(s_i, s_j) \tag{9}
\]

to its stack.

(b) Second, \( P \) compares its stack content, \( u(H_R(s_1 \ldots s_n)) \) to the input energy \( \sigma^k \), and accepts if and only if they are equal.

(ii) A LBA \( M \) that decides \( L_R \) is called constructive if it uses a designated energy tape \( T_e \) to accumulate the energy \( H_R(s_1 \ldots s_n) \) in binary, compares the content of \( T_e \) to the input energy and accepts if and only if the two values coincide.

For the constructive PDA, note that there exists an upper bound for the size of \( I_m \), as by definition a constructive PDA must be able to store all spin states contained in \( V_n|I_m \) in its states.

In words, a constructive automaton works the way one naively expects: it adds up local energy contributions in a pre-determined way, and then compares the result to the input energy. In particular, constructive automata compute \( H_R \) as a function. Working with constructive automaton thus ensures that we consider the function problem of computing \( H_R \), although for technical reasons, we formulate it as a decision problem (deciding the function graph of \( H_R \)) so that we can work with the Chomsky hierarchy. We conjecture that the constructive condition on the automata is not necessary:

Conjecture 1. For every context free \( L_R \) there exists a constructive PDA that decides it. For every context sensitive \( L_R \) there exists a constructive LBA that decides it.

In this work, we do not assume that this conjecture is true, i.e. we explicitly state whenever we require that an automaton is constructive.

Constructive PDA and constructive LBA define two new complexity levels for \( L_R \). If \( L_R \) is decided by a constructive PDA, we say that \( L_R \) is constructive context free; if \( L_R \) is decided by a constructive LBA, we say that \( L_R \) is constructive context sensitive (cf. Fig. 1). Considering only languages of the type \( L_R \), constructive context free is a subset of context free, and constructive context sensitive is a subset of context sensitive. We now show that supplementing the Chomsky hierarchy with these two complexity levels still forms a hierarchy, i.e. that regular is a subset of constructive context free and context free is a subset of constructive context sensitive.

Proposition 1 (Refined Chomsky hierarchy). Let \( M \) be an Ising model and \( L_R \) be its language.

(i) If \( L_R \) is regular then it is constructive context free.

(ii) If \( L_R \) is context free then it is constructive context sensitive.

Proof. Starting with (i), if \( L_R \) is regular then by Theorem 1 (i), \( M \) is finite, i.e. there exists a maximum system size. Hence we can build a PDA \( P \) that on input \( s_1 \ldots s_n \bullet \sigma^k \) reads and stores all spin symbols. As there are finitely many, this can be done with a finite number of states. Next \( P \) adds the entire energy \( H_R(s_1 \ldots s_n) \) to its stack. This can be hardened in the transition rules. Finally \( P \) compares its stack to the input energy \( \sigma^k \) and accepts if and only if they are equal. Note that \( P \) is trivially constructive. For all system sizes, the partition of edges consists of one element only, namely the entire edge set.

Next we prove (ii). As \( L_R \) is context free, it has a context free grammar in Greibach normal form [32, Lecture 21]. From this grammar, one can build a PDA \( P \) without \( \epsilon \)-transitions that decides this language [32, Lecture 24]. Each transition rule of \( P \) pushes at most \( k \) symbols to the stack. As there are no \( \epsilon \)-transitions, \( P \) uses an amount of stack memory that is linear in the length of the input.
Definition 5 (Edge language). Let $\mathcal{M}$ be an Ising model. The edge language $E_{\mathcal{M}}$, as a formal language that encodes the family of interaction graphs of $\mathcal{M}$, is defined as

$$E_{\mathcal{M}} := \{0^{i-1}10^{j-i-1}10^n : (i,j) \in (E_{\mathcal{M}})_n \}$$  \hspace{1cm} (10)

In words, $E_{\mathcal{M}}$ directly encodes the interaction graphs of $\mathcal{M}$, as $w = w_1 \ldots w_n \in E_{\mathcal{M}}$ if and only if $n \in N_{\mathcal{M}}$ and $w$ is a row of the incidence matrix of the graph defined by $(E_{\mathcal{M}})_n$. So each string $w$ in the edge language specifies one edge of one interaction graph of $\mathcal{M}$, as well as its system size of the graph (encoded in the length of $w$).

The following proposition justifies the classification of the complexity of Ising models based on their edge language, as it states that not only $L_{\mathcal{M}}$ but also $E_{\mathcal{M}}$ characterise $\mathcal{M}$ uniquely.

**Proposition 2** (Uniqueness of the edge language). Let $\mathcal{M}$ and $\mathcal{M'}$ be two Ising models. The following are equivalent:

(i) $\mathcal{M} = \mathcal{M'}$

(ii) $L_{\mathcal{M}} = L_{\mathcal{M'}}$

(iii) $E_{\mathcal{M}} = E_{\mathcal{M'}}$

Proof. The two implications (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (i) are obvious. Thus we need to show (ii) $\Rightarrow$ (iii). If $L_{\mathcal{M}} = L_{\mathcal{M'}}$ then clearly $N_{\mathcal{M}} = N_{\mathcal{M'}}$. Now take any $n \in N_{\mathcal{M}}$. To show that $(E_{\mathcal{M}})_n = (E_{\mathcal{M'}})_n$ consider the function

$$C_{\mathcal{M}}(n,i,j) := -\frac{1}{4}
\left[H_{\mathcal{M}}(0^n) + H_{\mathcal{M}}(0^{i-1}10^{j-i-1}10^n) - H_{\mathcal{M}}(0^{i-1}10^n) - H_{\mathcal{M}}(0^{j-1}10^n)\right]$$  \hspace{1cm} (11)

Using Eq. (5) one readily concludes that

$$C_{\mathcal{M}}(n,i,j) = \begin{cases} 1 & \text{if } (i,j) \in (E_{\mathcal{M}})_n \\ 0 & \text{else} \end{cases}$$  \hspace{1cm} (12)

Thus

$$(E_{\mathcal{M}})_n = \{(i,j) : C_{\mathcal{M}}(n,i,j) = 1\}$$ \hspace{1cm} (13)

But as $L_{\mathcal{M}} = L_{\mathcal{M'}}$ is equivalent to $H_{\mathcal{M}} = H_{\mathcal{M'}}$, it is also the case that $C_{\mathcal{M}}(n,i,j) = C_{\mathcal{M'}}(n,i,j)$, and hence it follows that $(E_{\mathcal{M}})_n = (E_{\mathcal{M'}})_n$. \hfill $\square$

Next we consider how certain properties of the interaction graphs of an Ising model scale, i.e. how they change when changing the system size.

**Definition 6** (Finite, limited and linear Ising model). We call an Ising model $\mathcal{M}$

(i) finite if $N_{\mathcal{M}}$ is finite.

(ii) limited if there exists a natural number $b$ such that for all $n \in N_{\mathcal{M}}$ and any $(i,j) \in (E_{\mathcal{M}})_n$, if $j - i > b$ then either $i \leq b$ or $n - j \leq b$.

(iii) linear if there exists a natural number $k$ such that for all $n \in N_{\mathcal{M}}$, it is the case that $|E_{\mathcal{M}}| \leq kn$. 

Finally, we can define the complexity measure for Ising models: The complexity of an Ising model is obtained by classifying its language in the (refined) Chomsky hierarchy. The latter induces a complexity hierarchy of Ising models themselves.

**Definiton 3** (Complexity measure). Let $\mathcal{M}$ be an Ising model. We say that $M$ models itself.

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**III. FULL CLASSIFICATION OF ISING MODELS**

In this section we state and discuss our main result: a full classification of the complexity of Ising models based on properties of their interaction graphs (Theorem 1). First we define the relevant properties of interaction graphs, more precisely of families of interaction graphs (Section III A). Then we state Theorem 1 (Section III B), and provide a proof in Appendix A.

**A. Properties of Ising models**

Let us now introduce several properties of families of interaction graphs. These properties can be divided into two classes. The first class captures the complexity of the family of interaction graphs. The second class captures how certain properties of the individual graphs contained therein scale with the system size.

In order to quantify the complexity of the family of interaction graphs of an Ising model $\mathcal{M}$ we define the edge language of $\mathcal{M}$, $E_{\mathcal{M}}$, as a formal language that encodes the entire family of interaction graphs of $\mathcal{M}$. Consequently, classifying $E_{\mathcal{M}}$ in the Chomsky hierarchy measures the complexity of the family of interaction graphs of $\mathcal{M}$.

**Definition 5 (Edge language). Let $\mathcal{M}$ be an Ising model. The edge language of $\mathcal{M}$, $E_{\mathcal{M}}$, is defined as**

$$E_{\mathcal{M}} := \{0^{i-1}10^{j-i-1}10^n : (i,j) \in (E_{\mathcal{M}})_n \}$$  \hspace{1cm} (10)

$P$ can be simulated by a LBA $M$ simply by using an additional tape to simulate the linear bounded stack of $P$. We now show that this LBA can be assumed to be constructive. First note that when processing any well-formed input $s_1 \ldots s_n \bullet \sigma^k$, once the head of $P$ reaches $\bullet$, $P$ has stored the energy $H(\sigma^k)$ in its states or by using a combination of both, as otherwise $P$ could not decide if $w(H(s_1 \ldots s_n)) = \sigma^k$. Therefore, when $M$ stores $H(s_1 \ldots s_n)$ at some point during the computation and we can w.l.o.g. assume that $M$ stores this energy in binary, which makes it constructive. \hfill $\square$

The complexity of an Ising model is obtained by classifying its language in the (refined) Chomsky hierarchy. The latter induces a complexity hierarchy of Ising models. We say that $M$ models themselves. The following proposition justifies the classification of the complexity of Ising models based on their edge language, as it states that not only $L_{\mathcal{M}}$ but also $E_{\mathcal{M}}$ characterise $\mathcal{M}$ uniquely.

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Proof. The two implications (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (i) are obvious. Thus we need to show (ii) $\Rightarrow$ (iii). If $L_{\mathcal{M}} = L_{\mathcal{M'}}$, then clearly $N_{\mathcal{M}} = N_{\mathcal{M'}}$. Now take any $n \in N_{\mathcal{M}}$. To show that $(E_{\mathcal{M}})_n = (E_{\mathcal{M'}})_n$ consider the function

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Next we consider how certain properties of the interaction graphs of an Ising model scale, i.e. how they change when changing the system size.

**Definition 6** (Finite, limited and linear Ising model). We call an Ising model $\mathcal{M}$

(i) finite if $N_{\mathcal{M}}$ is finite.

(ii) limited if there exists a natural number $b$ such that for all $n \in N_{\mathcal{M}}$ and any $(i,j) \in (E_{\mathcal{M}})_n$, if $j - i > b$ then either $i \leq b$ or $n - j \leq b$.

(iii) linear if there exists a natural number $k$ such that for all $n \in N_{\mathcal{M}}$, it is the case that $|E_{\mathcal{M}}| \leq kn$. 

In words, $E_{\mathcal{M}}$ directly encodes the interaction graphs of $\mathcal{M}$, as $w = w_1 \ldots w_n \in E_{\mathcal{M}}$ if and only if $n \in N_{\mathcal{M}}$ and $w$ is a row of the incidence matrix of the graph defined by $(E_{\mathcal{M}})_n$. So each string $w$ in the edge language specifies one edge of one interaction graph of $\mathcal{M}$, as well as the system size of that interaction graph (encoded in the length of $w$).
For every Ising model the number of edges scales at most quadratically with the system size (because the complete graph has $\binom{n}{2}$ edges). The properties (i) finite and (iii) linear fine-grain the scaling of the number of edges—in a truncated scaling (finite), and a linear scaling.

Property (ii) (limited) captures the scaling of the maximal interaction range. Intuitively, an Ising model is limited if there is an upper bound on its interaction range, i.e., $a$ such that there is no edge $(i, j)$ where $i$ and $j$ are separated by at least $a$ other vertices (i.e., $|j - i| > a$). With an exception: if $i$ is within the first $b$ vertices or $j$ is within the last $b$ vertices, then either of them can have long-range edges, that is, they can be linked to other vertices which are further away than $b$ (see Fig. 2).

Finally, the properties finite, limited, linear form a hierarchy. Clearly, every finite Ising model is limited. Proving that every limited Ising model is linear follows from a simple counting argument: If $M$ is limited, considering $n > 2b$ vertices, the first and last $b$ vertices are included in at most $2bn$ edges, and the remaining vertices are included in at most $(n - 2b)2b$ additional edges. In total we have

$$|\langle E_M \rangle_n| \leq 4bn - 4b^2$$

(14)

As per definition $b$ is independent of the system size, this shows that $M$ is linear.

Overall we have two hierarchies that capture properties of the family of interaction graphs of an Ising model (Fig. 3). The first hierarchy classifies Ising models based on the complexity of their family of interaction graphs; specifically, it encodes the interaction graphs as a language $E_M$ and classifies it in the Chomsky hierarchy (purple shapes of Fig. 3). The second hierarchy classifies Ising models based on the scaling of the number of edges, as well as the scaling of the maximal interaction range with the system size (red shapes of Fig. 3).

**B. Main result**

We are now ready to state the full classification of the complexity of Ising models based on the properties introduced above.

**Theorem 1** (Main result). Let $M$ be an Ising model.

(i) $L_M$ is regular if and only if $M$ is finite;

(ii) $L_M$ is constructive context free if and only if $M$ is linear and $E_M$ is regular;

(iii) $L_M$ is constructive context sensitive if and only if $E_M$ is context sensitive;

(iv) $L_M$ is decidable if and only if $E_M$ is decidable.

While this theorem is proven in Appendix A, the statements can be intuitively understood as follows.

(i). A finite Ising model only contains a finite number of system sizes, thus both $L_M$ and $E_M$ are finite languages, which are trivially regular. Conversely, by the pumping lemma for regular languages, it follows that infinite Ising models cannot have a regular language.

(ii). The essence of the argument is the following. If $M$ is linear and $E_M$ is regular, a constructive PDA for $L_M$ can be built from a FSA for $E_M$. Deciding $L_M$ amounts to deciding $E_M$ and adding the individual energy contributions. The PDA decides $E_M$ by running the FSA in its states. As $M$ is linear, also adding the individual energy contributions is possible for a PDA. Note that if $M$ is not finite, adding the individual energy contributions requires at least a PDA, i.e. cannot be achieved by a FSA. Consequently, deciding the edges can at most require a FSA, since PDAs can use FSAs but not PDAs as subroutines. Conversely, if $M$ were not linear, then already adding the individual energy contributions would require a LBA (independently of the complexity of $E_M$), as it is the case for the all-to-all Ising model (Fig. 3 and Section IV F). Regularity of $E_M$ can be proven by using the constructive PDA for $L_M$ to compute Eq. (11). So this PDA can be modified to decide $E_M$. From the fact that the PDA is constructive it then follows that the modified
PDA only ever uses a single cell of stack memory, so it
effectively is a FSA.

(iii). In contrast to PDAs, LBAs can use LBAs as
subroutines. This is the key difference between (ii) and
(iii). It follows that there is no separation in the com-
plexity of \( E_M \) and \( L_M \), i.e. in contrast to (ii), they can
be of the same complexity. The constructive LBA that
decides \( L_M \) can be built by using a LBA for deciding \( E_M \),
to select the individual edges that contribute to the en-
ergy, and another LBA to sum up these individual energy
contributions. Conversely, if \( L_M \) is constructive context
sensitive then a LBA for \( E_M \) can be built by modifying
the LBA for \( L_M \) so that it computes Eq. (11).

(iv). Turing machines can also use Turing machines
as subroutines. A Turing machine that decides \( L_M \) can be
built in the same way as the LBA in the previous case,
and also the converse direction of (iv) works the same
way as that of (iii).

Let us highlight a corollary of the proof of Theorem 1,
which will prove useful in the examples of Section IV:

**Corollary 1.** If \( M \) is not limited then \( L_M \) is not con-
structive context free.

**Proof.** From Theorem 1 (ii) we know that if \( L_M \) is con-
structive context free then \( M \) is linear and \( E_M \) is regular.
From Appendix A 2 (c), it follows that \( M \) is limited. The
corollary states the contrapositive. \( \square \)

### IV. EXAMPLES: THE COMPLEXITY OF ISING
**MODELS**

We now consider various Ising models and compute
their complexity by applying Theorem 1. Specifically, we
consider the 1d Ising model (Section IV A) the 1d Ising
model with periodic boundary conditions (Section IV B),
the Ising model on ladder graphs (Section IV C), the Ising
model on layerwise complete graphs (Section IV D), the
2d Ising model (Section IV E), the all-to-all Ising model
(Section IV F), and the Ising model on perfect binary
trees (Section IV G). The results are summarised in Ta-
ble I.

We remark that we provide a grammar for the 1d and
2d Ising model in Appendix C. These constitute inde-
pendent proofs of their context freeness and context sen-
sitiveness, respectively. In fact, Appendix C provides a
total of two alternative proofs of the context freeness of
the 1d Ising model.

#### A. 1d Ising model

The most straightforward example of a constructive
context free Ising model uses 1-dimensional chains as
interaction graphs (Fig. 4a). We denote this model as
\( M_{1d} := (N_{1d}, E_{1d}) \), defined by

\[
N_{1d} := \{ n \in \mathbb{N} \mid n \geq 2 \}
\]

\[
(E_{1d})_n := \{ (i, i + 1) \mid 1 \leq i \leq n - 1 \}
\]
| Ising model $\mathcal{M}$ | $L_{\mathcal{M}}$ | $E_{\mathcal{M}}$ | Finite | Limited | Linear |
|-------------------------|----------------|--------------|--------|---------|--------|
| $\mathcal{M}_{1d}$     | Constructive context free | Regular | No   | Yes   | Yes   |
| $\mathcal{M}_{\text{circ}}$ | Constructive context free | Regular | No   | Yes   | Yes   |
| $\mathcal{M}_{\text{ladder}}$ | Constructive context free | Regular | No   | Yes   | Yes   |
| $\mathcal{M}_{\text{layer}}$ | Constructive context free | Regular | No   | Yes   | Yes   |
| $\mathcal{M}_{2d}$     | Constructive context sensitive | Context sensitive | No   | No   | Yes   |
| $\mathcal{M}_{\text{all}}$ | Constructive context sensitive | Context sensitive | No   | No   | Yes   |
| $\mathcal{M}_{\text{tree}}$ | Constructive context sensitive | Context sensitive | No   | No   | Yes   |

Table I: The Ising models (first column) considered in Section IV, their complexity (second column) and the properties that determine their complexity (remaining columns).

We now use Theorem 1 (ii) to prove that the language of $\mathcal{M}_{1d}$ is constructive context free. To that end, let us show that it is linear and its edge language is regular. Linearity is immediate, as for any $n \in N_{1d}$, $|E_{1d}| = n - 1$. Also regularity of $E_{1d}$ can be concluded straightforwardly, as $E_{1d} = 0^*110^*$ (where $0^*$ denotes the concatenation of any number of 0s, including the empty one). Since $\mathcal{M}_{1d}$ is clearly not finite, by Theorem 1 (i) $L_{\mathcal{M}_{1d}}$ is not regular.

### B. 1d Ising model with periodic boundary conditions

A second Ising model with constructive context free language is obtained by taking circles as interaction graphs (Fig. 4b). We denote this model as $\mathcal{M}_{\text{circ}} := (N_{\text{circ}}, E_{\text{circ}})$, where

$$N_{\text{circ}} := \{n \in \mathbb{N} \mid n \geq 3\}$$

$$E_{\text{circ}} := \langle E_{1d} \rangle_n \cup \{1, n\}$$

(16)

Again, as $|E_{\text{circ}}| = n$, $\mathcal{M}_{\text{circ}}$ clearly is linear. Besides,

$$E_{\text{circ}} = E_{1d} \cup 10^*1$$

(17)

is a union of regular languages, so it is regular. So according to Theorem 1 (ii) $L_{\text{circ}}$ is constructive context free. As $\mathcal{M}_{\text{circ}}$ is not finite, according to Theorem 1 (i) $L_{\text{circ}}$ is not regular.

### C. Ising model on a ladder graph

Another class of Ising models with constructive context free languages is obtained by considering generalised ladder graphs as interaction graphs. For each such model the interaction graphs are given by a family of d-dimensional lattices, such that all lattices of the family have equal size along all but one dimension, i.e. increasing the system size amounts to adding spins along one distinguished dimension. It follows from Theorem 1 (ii) that all these Ising models are constructive context free.

To illustrate this, we consider 2-dimensional ladders with constant width $k$ (Fig. 4c). The corresponding Ising model $\mathcal{M}_{\text{ladder}}$ is defined by

$$N_{\text{ladder}} := \{ik \mid i \geq 2\}$$

$$(E_{\text{ladder}})_{ik} := (E_{\text{ladder}})^{\text{vert}}_{ik} \cup (E_{\text{ladder}})^{\text{hor}}_{ik}$$

(18)

where

$$(E_{\text{ladder}})^{\text{vert}}_{ik} := \{(jk + l, jk + l + 1) \mid 0 \leq j \leq i - 1, 1 \leq l \leq k - 1\}$$

(19)

contains the vertical edges and

$$(E_{\text{ladder}})^{\text{hor}}_{ik} := \{(jk + l, (j + 1)k + l) \mid 0 \leq j \leq i - 2, 1 \leq l \leq k\}$$

(20)

contains the horizontal edges. $\mathcal{M}_{\text{ladder}}$ is linear, as $|E_{\text{ladder}}| = 2ik - i - k < 2n$. Moreover, its edge language $E_{\text{ladder}}$ is regular, since it can be written as a finite union of regular expressions

$$E_{\text{ladder}} = E_{\text{ladder}}^{\text{vert}} \cup E_{\text{ladder}}^{\text{hor}}$$

$$E_{\text{ladder}}^{\text{vert}} := \bigcup_{1 \leq l \leq k - 1} \left( (0^k)^*0^l10^l110^k - 1 - 10^k(0^k)^* \right)$$

$$E_{\text{ladder}}^{\text{hor}} := \bigcup_{1 \leq l \leq k} (0^k)^*0^l110^k - 110^k - 10^k(0^k)^*$$

(21)

From Theorem 1 (ii) it follows that $L_{\text{ladder}}$ is constructive context free. In addition, $\mathcal{M}_{\text{ladder}}$ is not finite, so by Theorem 1 (i) $L_{\text{ladder}}$ is not regular.

### D. Ising model on layerwise complete graph

Also layerwise complete graphs, which are used in many neural network models, define a class of Ising models with constructive context free language. To see this, consider interaction graphs composed of $i$ layers of $k$ vertices, such that there is no edge between vertices within the same layer, and any two vertices from neighbouring layers are connected (Fig. 4d). The corresponding Ising model $\mathcal{M}_{\text{layer}}$ is defined as

$$N_{\text{layer}} := \{ik \mid i \geq 2\}$$

$$(E_{\text{layer}})_{ik} := \{(jk + l, (j + 1)k + r) \mid 0 \leq j \leq i - 2, 1 \leq l \leq k, 1 \leq r \leq l\}$$

(22)

Since

$$|E_{\text{layer}}| = (i - 1)k^2 < kn$$

(23)
Figure 4: Interaction graphs of $M_{1d}$ (4a), $M_{circ}$ (4b), $M_{ladder}$ (4c), and $M_{layer}$ (4d). All these Ising models are constructive context free. Intuitively, this is because their interaction graphs all have one distinguished dimension along which an elementary building block (that contains a constant number $k$ spins) is repeated ($i$ times) in a periodic fashion. In (4a) and (4b) there is only one dimension, in (4c) and (4d) the distinguished dimension is indicated as “direction of growth”. This property is made precise in Theorem 1 (ii): constructive context free Ising models are uniquely characterised by $E_M$ being regular and $M$ being linear (or limited according to Appendix A 2 (c)).

Using Theorem 1 (ii) it follows that $L_{layer}$ is constructive context free. As $M_{layer}$ is not finite, by Theorem 1 (i) we conclude that $L_{layer}$ is not regular.

### E. 2d Ising model

Let us now show that the Ising model on 2d square lattices has a constructive context sensitive language. We denote the 2d Ising model as $M_{2d}$, and define it as

$$N_{2d} := \{n^2 \mid n \geq 2\}$$

$$(E_{2d})_{n^2} := (E_{2d})_{n^2}^{\text{hor}} \cup (E_{2d})_{n^2}^{\text{ver}}$$

The edge set of size $n^2$ is split in horizontal and vertical edges:

$$(E_{2d})_{n^2}^{\text{hor}} := \{(i, i+1) \mid 1 \leq i \leq n^2 - 1, i \not\in n\mathbb{N}\}$$

$$(E_{2d})_{n^2}^{\text{ver}} := \{(i, i+n) \mid 1 \leq i \leq n^2 - n\}$$

Its family of interaction graphs can be seen in Fig. 4c, with the difference that for $M_{2d}$, when increasing the system size both dimensions are scaled up simultaneously.

Using Theorem 1(iii) we now prove that $L_{2d}$ is constructive context sensitive by showing that its edge language $E_{2d}$ is context sensitive. To this end, we build a LBA that decides $E_{2d}$. Asserting that the input $w_1 \ldots w_m$ is well-formed, i.e. of the form $0^{*}10^{*}10^{*}$, can be achieved by a FSA, which can be simulated by the LBA. We can thus w.l.o.g. assume that the input is well-formed. Next the LBA checks if $m = n^2$ for some natural number $n$. This is done by iterating over natural numbers $n$, starting with $n = 1$. In each step of the iteration the LBA computes $n^2$ and checks if this matches the length of the
input \( m \). If \( n^2 = m \) this subroutine terminates, if \( n^2 < m \) the LBA moves on with the next natural number \( n + 1 \), and if \( n^2 > m \) the LBA rejects the input, as then \( m \) is not a square number, i.e. not in \( N_{2d} \). Explicitly, we use \((n + 1)^2 = n^2 + 2n + 1\) to compute the square numbers. The LBA uses an additional tape \( T_n \) to store \( n \) in unary. Initially \( n = 1 \) and the head of the LBA is placed over the first cell of the input tape. The LBA enters a loop: The head moves \( 2n + 1 \) cells to the right on the input tape. Note that this places the head over cell \((n + 1)^2\). The LBA now checks if the current cell is empty. If yes, then \( m \not\in N_{2d} \) and the LBA rejects the input. If no, the LBA checks if the next cell is empty. If no, it increases \( n \) by one in the additional tape, and starts again. If yes, it accepts.

Now the LBA traverses the input until its head reaches the first 1. While doing so it uses another additional tape \( T_f \) to count the position of that first 1 in the input string, \( f \). Then the LBA counts the number of 0s between the first and the second 1, \( z \), and stores it in another additional tape \( T_z \). Finally, it accepts the input if either \( z = n - 1 \), corresponding to a vertical edge, or if \( z = 0 \) and there exists no \( k \) satisfying \( f = kn \) (this can be done since \( n \) is written on \( T_n \)), corresponding to a horizontal edge.

Note that, as
\[
| (E_{2d})^z \| = 2n(n - 1) < 2n^2
\]
\( \mathcal{M}_{2d} \) is linear. However, for any \( n \in \mathbb{N} \),
\[
(n, 2n) \in (E_{2d})^z
\]
and hence \( \mathcal{M}_{2d} \) is not limited. So by Corollary 1 \( L_{2d} \) is not constructive context free.

We now show that \( L_{2d} \) is not context free by using the pumping lemma for context free languages [32]. Assume that \( L_{2d} \) was context free and let \( p \) be the pumping length of \( L_{2d} \). Now consider
\[
l := 0^p \cdot \text{	extcircled{E}} - 2p(p - 1) \] (29)
Note that \( l \in L_{2d} \), as a configuration of \( p^2 \) spins has \( 2p(p - 1) \) edges. When writing \( l = uvwxy \), \( v \) must be a non-empty string of \( 0 \) symbols (let \(|v| =: k\)), while \( x \) must be a non-empty string of \( - \) symbols. Otherwise pumping up \( l \) would yield a mismatch between spin configuration and energy. Since \(|vwxy| \leq p\) we also have that \( k \leq p \), so \( uv^2wx^2y \) yields a configuration with \( p^2 + k \) spins. But \( p^2 < p^2 + k < (p + 1)^2 \), so \( p^2 + k \not\in N_{2d} \) and thus \( uv^2wx^2y \notin L_{2d} \). Hence \( L_{2d} \) is not context free.

\section*{F. All-to-all Ising model}

Also the all-to-all Ising model (Fig. 5a), i.e. the Ising model with complete interaction graphs, has a constructive context sensitive language. We denote this model as \( \mathcal{M}_{\text{all}} \), and define it by
\[
N_{\text{all}} := \{ n \mid n \geq 2 \}
\]
\[
(E_{\text{all}})_n := \{ (i, j) \mid 1 \leq i < j \leq n \}
\]
(30)
Its edge language \( E_{\text{all}} \) is regular, since
\[
E_{\text{all}} = 0^*10^*10^*
\]
(31)
Thus \( E_{\text{all}} \) is in particular context sensitive and by Theorem 1 (iii), \( L_{\text{all}} \) is constructive context sensitive. Since \( \mathcal{M}_{\text{all}} \) is not linear and, by Theorem 1 (ii), \( L_{\text{all}} \) is not constructive context free. In fact, by the pumping lemma [32], \( L_{\text{all}} \) can be proven to be not context free. Assume \( L_{\text{all}} \) was context free and denote its pumping length by \( p \). Now take
\[
l := 0^p \cdot \text{	extcircled{E}} - \frac{p(p - 1)}{2} \in L_{\text{all}}
\]
(33)
Writing \( l = uvwxy \) as required by the pumping lemma, \( v \) must be a non-empty string of \( 0 \)s and \( x \) must be a non-empty string of \( - \) symbols. Pumping up once yields \( k := |v| \) new spin symbols and thus increases the overall energy by
\[
e := kp + \frac{1}{2}k(k - 1)
\]
(34)
Hence it must be the case that \( x = -e \). Pumping up a second time additionally adds
\[
k(p + k) + \frac{1}{2}k(k - 1) = e + k^2
\]
(35)
more pair interactions but only \( e \) more \( - \) symbols. Thus there is a mismatch between spin configuration and energy, and hence \( uv^3wx^3y \notin L_{\text{all}} \). Therefore \( L_{\text{all}} \) is not context free.

\section*{G. Ising model on perfect binary trees}

Next we consider the Ising model \( \mathcal{M}_{\text{tree}} \) that uses perfect binary trees as interaction graphs (Fig. 5b). This model is defined by
\[
N_{\text{tree}} := \{ 2^n - 1 \mid n \geq 2 \}
\]
\[
(E_{\text{tree}})^{2^n - 1} := (E_{\text{tree}})^{2^n - 1}_{\text{left}} \cup (E_{\text{tree}})^{2^n - 1}_{\text{right}}
\]
(36)
where the edge set of size \( 2^n - 1 \) is split into those that connect the parent vertex to its left child vertex and those that connect the parent vertex to its right child vertex:
\[
(E_{\text{tree}})^{2^n - 1}_{\text{left}} := \{ (i, 2i) \mid 1 \leq i \leq 2^n - 1 \}
\]
\[
(E_{\text{tree}})^{2^n - 1}_{\text{right}} := \{ (i, 2i + 1) \mid 1 \leq i \leq 2^n - 1 \}
\]
(37)
In order to apply Theorem 1 (iii) we need to prove that $E_{\text{tree}}$ is context sensitive. To this end, consider the language that only encodes the system sizes $N_{\text{tree}}$.

$$N_{\text{tree}} := \{w_1 \ldots w_{2^n-1} \mid w_i \in \{0, 1\}, \ n \geq 2\} \tag{38}$$

We now show that this language is context sensitive by constructing a LBA that decides it. Given an input string

$$w_1 \ldots w_m \in \{0, 1\}^* \tag{39}$$

the LBA checks if $m = 2^n - 1$ for some natural number $n$. It does so by using an additional tape $T_{2^n}$ to store $2^n$ in unary. It starts with $n = 1$ (so that $2^n = 2$), and the head placed on the first cell of the input tape. Then it enters the following loop. It moves the head on the input tape $2^n$ cells to the right (this is possible because $2^n$ is stored on the additional tape). If the current cell is the last non-empty cell, it accepts. If the cell is empty, it rejects. Else, it doubles the number of symbols on the additional tape (so that it now contains $2^n+1$), moves its head back to the beginning of the input tape, and continues with the first step of the loop. This shows that $N_{\text{tree}}$ is context sensitive.

Next, note that the edge language is given by

$$E_{\text{tree}} = (E_{\text{tree}}^\text{left} \cup E_{\text{tree}}^\text{right}) \cap N_{\text{tree}} \tag{40}$$

where

$$E_{\text{tree}}^\text{left} := \{0^{i-1}10^{i-1}10^n \mid i \geq 1\}$$

$$E_{\text{tree}}^\text{right} := \{0^{i-1}10^{i-1}10^n \mid i \geq 1\} \tag{41}$$

Both $E_{\text{tree}}^\text{left}$ and $E_{\text{tree}}^\text{right}$ are context free, as can be seen by constructing two PDAs $P_{\text{left}}$ and $P_{\text{right}}$ that accept these two languages, respectively. (This can also directly be seen from the fact that both languages are essentially of the form $\{a^n b^n \mid n \geq 1\}$. $P_{\text{left}}$ uses its stack to count the number of zeros in front of the first 1, and then it compares this number against the number of zeros in front of the second 1. If the two numbers coincide and the string contains no further 1, it accepts, else it rejects. $P_{\text{right}}$ does the same, except for ignoring the first symbol after the first 1 if it is 0 and rejecting if it is 1.

Finally, from (40) and the closure properties of context sensitive languages [32], it follows that $E_{\text{tree}}$ is context sensitive. Hence by Theorem 1 (iii), $L_{\text{tree}}$ is context sensitive. $M_{\text{tree}}$ is not limited, as for any $n \geq 2$, the edge

$$(2^n-1, 2^n-2) \in (E_{\text{tree}}^\text{left})_{2^n-1} \tag{42}$$

is long-range. Hence, by Corollary 1, $L_{\text{tree}}$ is not context free.

Moreover, $L_{\text{tree}}$ is not context free. This can be proven with the pumping lemma of context free languages [32]. Assume $L_{\text{tree}}$ was context free and let $p$ be its pumping length. Take $n$ to be the smallest natural number that satisfies $2^n-1 \geq p$. Consider

$$l = 0^{2^n-1} \cdot 2^{n-2} \in L_{\text{tree}} \tag{43}$$

Writing $l = uvwxy$, $v$ must be a non-empty string of 0s and $x$ a non-empty string of − symbols. Then pumping up once yields a string that corresponds to configuration of $2^n-1 + k$ spins, where $k = |v|$. As $k \leq p \leq 2^n-1$ it follows that $2^n-1 + k < 2^{n+1}-1$. Additionally using that $k > 0$ shows that $2^n-1 + k \notin N_{\text{tree}}$ and hence $uv^2wx^2y \notin L_{\text{tree}}$. Thus $L_{\text{tree}}$ is not context free.

V. CONCLUSIONS AND OUTLOOK

In this work we have introduced a new complexity measure for Ising models and fully classified Ising models according to it (Theorem 1). The complexity measure consists of classifying the decision problem corresponding to the function graph of the Hamiltonian of an Ising...
model in the Chomsky hierarchy. In order to establish this classification, we have identified certain properties of interaction graphs of Ising models. These properties can be divided into two classes: those that capture the complexity of interaction graphs (viz. the complexity of the edge language, Definition 5), versus those that capture the scaling of interaction graphs (viz. finite, limited and linear, Definition 6). In our main result we have unveiled which properties of interaction graphs correspond to which complexity level of an Ising model in a one-to-one manner. We have then used the classification of Theorem 1 to compute the complexity of the 1d Ising model, the Ising model on ladder graphs, on layerwise complete graphs, the 2d Ising model, the all-to-all Ising model, and the Ising model on perfect binary trees (Table 1). We find a different easy-to-hard threshold than for the computational complexity of the ground state energy problem.

Among other things, this work raises the question of whether constructive context free and constructive context sensitive differ from context free and context sensitive. We conjecture this is not the case (Conjecture 1). Constructive automata first compute the energy of the input spin configuration and then compare it to the input energy. For this reason, working with constructive automata ensures that we characterise the function problem of computing the Hamiltonian (although for technical reasons we phrase it as the decision problem of deciding its function graph). Therefore, Conjecture 1 ultimately concerns the question of how function problems differ from their corresponding decision problems when considering PDAs or LBAs.

Furthermore, it would be interesting to compare our complexity measure to the complexity of GSE rigorously, namely to investigate how the non-planarity required for the NP-completeness of GSE relates to the properties of Definition 6 and the complexity of the edge language (Definition 5). Specifically, in our measure we observe that both the Ising model on ladder graphs (irrespective of the dimension of the ladder) and the Ising model on layerwise complete graphs are “easy”; yet, both of these families contain an infinite number of non-planar graphs (The ladder graphs only if the dimension is at least 3). On the other hand, both the 2d Ising models and the Ising model on perfect binary trees are “hard” in our measure, while they both contain only planar graphs. It is clear that the two measures capture different properties; it is however not clear to us how the number of crossings needs to scale with the system size for the GSE to be NP-complete and thus how the two measures exactly differ.

A different way of comparing the two measures consists of investigating the computational complexity of deciding $L_{\mathcal{G}}$, that is, the time resources a Turing machine needs to decide $L_{\mathcal{G}}$—this is done in [30] for general spin models. Conversely, one could classify the language of GSE in the Chomsky hierarchy, and thereby unveil the grammar (i.e. local structure) of the set of yes instances of the ground state energy problem.

Encoding the Hamiltonian of an Ising model as a formal language enforces a total order of the spins, as mentioned in Section II. This could be avoided by using graph languages and graph grammars instead of string languages and string grammars. A graph language is a set of graphs, and graph grammars generalise the production rule of string grammars to operate directly on graphs [33]. While encoding Ising models as graph languages is more natural, graph grammars lack the well-studied complexity hierarchy of string grammars.

From a broader perspective, this work —together with [30]— establishes a new connection between spin models and theoretical computer science. Among other reasons this connection is motivated by the recent discovery that certain spin models such as the 2d Ising model with fields or the 3d Ising model are universal [34]. This notion of universality has several similarities to that of a universal Turing machine [35]. In [34] it is proven that a spin model is universal if and only if it is closed and its GSE is NP-complete. The complexity measure for Ising models introduced here could help obtain another characterisation of universal Ising models—for instance by translating closure into properties of the interaction graphs. This would lead to a better understanding of spin model universality and thereby also its relation to Turing machine universality. We are currently working on a categorical framework for universality which allows to compare notions of universality [36].

This work is a first step in the characterisation of physical interactions in terms of grammars, and an invitation to characterising interactions beyond the Ising model, as well as other systems with a local structure, in the light of grammars.

ACKNOWLEDGEMENTS

We thank Sebastian Stengele for many discussions, as well as everyone in the group—Tomáš Gonda, Andreas Klingler and Mirte van der Eyden. We also thank David Bänisch for joint work on graph grammars and graph languages. We thank Thomas Müller, Hadil Karawani and Sahra Styger from the University of Konstanz for many discussions about counterfactuals. We acknowledge funding from Austrian Science Fund (FWF) via the FWF START Prize Y1261-N.

Appendix A: Proof of Theorem 1

Here we prove Theorem 1 with one subsection for each statement.

1. Proof of Theorem 1(i)

If $N_{\mathcal{G}}$ is finite then so is $L_{\mathcal{G}}$. Thus $L_{\mathcal{G}}$ is trivially regular [32].
In order to prove the “only if” direction using the pumping lemma for regular languages [32], we prove that an Ising model with infinite $\mathcal{N}_R$ cannot be regular. To this end, assume such a $L_R$ was regular and let $p$ be its pumping length. As $M$ is infinite, there exists a configuration of length $q > p$. Hence, the string

$$0^q \bullet -e \in L_R$$

with $e := |(E_R)_{q}|$ is contained in $L_R$. Note that $e \leq \frac{1}{2}q(q-1)$, and hence, for any $n > \frac{1}{2}q(q-1) + 1$ we have $|E_n| > e$. Pumping up $k$ times yields a string of the form $0^{q+kp} \bullet -e$. Now choosing $k$ large enough such that

$$q + kp > \frac{1}{2}q(q-1) + 1$$

this string is not contained in $L_R$, since $|E_{q+kp}| > e$, so $H_R(0^{q+kp}) \neq -e$. Thus an infinite Ising model cannot have a regular language.

2. Proof of Theorem 1(ii)

In order to prove that

$$L_R \text{ is constructive context free } \iff M \text{ is linear and } E_R \text{ is regular}$$

we prove the following four statements:

(a) $L_R$ is context free $\implies M$ is linear

(b) $L_R$ is constructive context free $\implies E_R$ is regular

(c) $M$ is linear and $E_R$ is regular $\implies M$ is limited

(d) $M$ is limited and $E_R$ is regular $\implies L_R$ is constructive context free

Combining (a) and (b) (and the fact that constructive context free is included in context free) yields the forward direction of the statement, and combining (c) and (d) yields the other direction.

Let us now prove each of the statements.

(a) If $L_R$ is context free, then $M$ is linear

As by assumption $L_R$ is context free, we claim that so is the language containing the configuration of minimal energy for each system size,

$$(L_R)_{\text{min}} := \{0^n - e_n \mid n \in N_R\}$$

where $e_n := |(E_R)_{n}|$. This holds since $(L_R)_{\text{min}}$ can be obtained from $L_R$ by first intersecting with the regular language $0^* \bullet -*$ and then applying the homomorphism that maps $\bullet$ to the empty string and acts as identity on $\{0,1,+,-\}$. Since the class of context free languages is closed both with respect to intersections with regular languages and homomorphisms [32], this proves the claim that $(L_R)_{\text{min}}$ is context free.

As $(L_R)_{\text{min}}$ is context free, its image under the Parikh map

$$P(0^n - e_n) = (n, e_n)$$

is a semilinear subset of $\mathbb{N}^2$, i.e. a union of finitely many linear subsets $U_1, \ldots, U_r \subset \mathbb{N}^2$ [32]. We now construct a natural number $k$ such that for all $n \in N_R$, $e_n \leq kn$. Take any $(n, e_n)$ from the image of $P$. Then there is an $i \leq r$ such that $(n, e_n) \in U_i$. As $U_i$ is linear, there exist $u_0 \in \mathbb{N}^2, u_1, \ldots, u_d \in \mathbb{N}^2 \setminus \{(0,0)\}$, such that any element in $U_i$ can be written as

$$u_0 + \lambda_1 u_1 + \ldots + \lambda_d u_d$$

with $\lambda_j$ natural numbers. Thus, denoting $u_j = (v_j, w_j)$ we in particular have

$$e_n = \frac{w_0 + \lambda_1 w_1 + \ldots + \lambda_d w_d}{v_0 + \lambda_1 v_1 + \ldots + \lambda_d v_d}$$

Now note that for any $u_j$ it holds that $v_j$ is strictly positive. For assume that $v_j = 0$. Then, by the linearity of $U_i$,

$$0^{n+\lambda_0} - e_n + \lambda w_j \in L_R$$

so a single spin configuration, $0^n$, would have energies $-e_n$ and $-e_n + \lambda w_j$. In other words, the relation between spin configuration and energy would no longer be functional. Moreover, $v_0$ cannot be zero either, by Definition 2.

To finish the proof, take

$$k_i := \max\left\{ \frac{w_j}{v_j} \mid l, j \leq d \right\}$$

Then it is easy to see that

$$\frac{e_n}{n} \leq k_i$$

Defining $k$ to be the maximum taken over $\{k_i \mid i \leq r\}$ shows that for any $n \in N_R$, $|(E_R)_{n}| \leq kn$ and hence proves the claim.

(b) If $L_R$ is constructive context free, then $E_R$ is regular

As $L_R$ is constructive context free there exists an constructive PDA $P$ that accepts $L_R$. We prove the claim by first using $P$ to construct a second PDA $P_C$ that decides $E_R$, and showing that there exists a finite bound on the stack memory of $P_C$. Since a finite stack can be simulated by a FSA (by increasing the number of states), $P_C$ can be transformed into a FSA, which proves the claim.

So let us consider a potential edge

$$\langle i, j - 1, n - j \rangle := 0^{i-1}10^{j-1}10^n - j$$

(A10)
In order to decide if it is in $E_R$, $P_C$ computes $C_R(n,i,j)$ defined in Eq. (11), by simulating $P$'s computation on the four input spin configurations
\[
0^{i-1}10^{j-1}10^{n-j}, \quad 0^{i-1}10^{n-i}, \quad 0^{j-1}10^{n-j}, \quad 0^n
\] (A11)
and summing the four energies appropriately.

We now prove that, since $P$ is constructive, $P_C$ can be taken to be a FSA. Let $(I_m)_{m=1,...,r}$ be the unique partition of $(E_R)$, that witnesses that $P$ is constructive. At step $m$ of the main iteration of $P$ (cf. Definition 3 (a)), $P$ computes the energy contribution from interactions contained in $I_m$ and adds it to its stack. Consequently, $P_C$ computes
\[
C_R(n,i,j)_{I_m} := -\frac{1}{4} \left[ H_R |_{I_m} (0^n) + H_R |_{I_m} (0^{i-1}10^j - 110^{n-j}) - H_R |_{I_m} (0^{i-1}10^n) - H_R |_{I_m} (0^{j-1}10^{n-j}) \right] \tag{A12}
\]
and adds the result to its stack. By Definition 3 the energy that each $I_m$ contributes is upper bounded by the number of states of $P$, and so in particular it is finite. Thus, summing up the four terms of Eq. (A12) can be done in the states of $P_C$, and we can assume that $P_C$ only uses its stack to accumulate
\[
\sum_{m=1}^{r} C_R(n,i,j)_{I_m} \tag{A13}
\]
By construction $C_R(n,i,j)_{I_m}$ is +1 if $(i,j) \in I_m$ and 0 else. Thus, a finite stack suffices to compute (A13), and hence this can be done in the states of $P_C$. This makes $P_C$ a FSA.

Finally, if $n \notin N_R$, $P_C$ rejects by construction, as so does $P$. If $n \in N_R$, $P_C$ accepts the input if and only if $C_R(n,i,j) = 1$. So $P_C$ correctly decides $E_R$, which proves that $E_R$ is regular.

(c) If $M$ is linear and $E_R$ is regular, then $M$ is limited

We prove that if $E_R$ is regular and $M$ is not limited, then $M$ cannot be linear. To this end, for any natural number $k$, assuming $E_R$ is regular, we construct a natural number $l$ such that $M$ contains more than $kl$ edges of length $l$, i.e. it is not linear.

By assumption $E_R$ is regular. Let $F$ be a FSA that accepts it and denote the number of states of $F$ by $b$. Consider an edge $(p,q,r) \in E_R$ with $p > b$. When accepting $(p,q,r)$ there has to be at least one state that $F$ enters twice before reaching the first 1 with its head. Thus, $F$ contains a loop in its transition rules. Denote the number of transitions that are contained in this loop by $w_p$. Then, for any natural number $n_p$,
\[
(p + n_p w_p, q, r) \in E_R \tag{A14}
\]
By a similar reasoning, for an edge $(p,q,r)$ with $q > b$, $F$ must enter a loop after reading the first 1 and before reading the second 1. Denote the length of the corresponding loop by $w_q$. Then for any natural number $n_q$,
\[
(p, q + n_q w_q, r) \in E_R \tag{A15}
\]
Similarly, for an edge $(p,q,r)$ with $r > b$, $F$ enters a loop after reading the second 1. Denote the number of transitions in this loop as $w_r$. Then for any natural number $n_r$,
\[
(p, q, r + n_r w_r) \in E_R \tag{A16}
\]
Now, since $M$ is not limited, there exists an edge $(p,q,r) \in E_R$ with $p,q,r > b$. By the above reasoning, there exist natural numbers $w_p,w_q,w_r$ such that for any $n_p,n_q,n_r$,
\[
(p + n_p w_p, q + n_q w_q, r + n_r w_r) \in E_R \tag{A17}
\]
Take any natural number $m$ and define
\[
l_m := p + q + r + 2 + m w_p w_q w_r \tag{A18}
\]
We will now show that for any $k$, choosing $m$ appropriately, there are more than $kl_m$ words of length $l_m$ and hence $M$ is not linear. To this end, take any $m_p,m_q,m_r$ that satisfy $m_p + m_q + m_r = m$. Then, the edge
\[
(p + m_p w_p w_q w_r, q + m_q w_p w_q w_r, r + m_r w_p w_q w_r) \tag{A19}
\]
is contained in $E_R$ and has length $l_m$. Thus the number of edges of length $l_m$ is at least as big as the number of triples $(m_p,m_q,m_r)$ that sum to $m$,
\[
\left| \{ (m_p,m_q,m_r) \mid m_p + m_q + m_r = m \} \right| = \sum_{m_p=0}^{m} \sum_{m_q=0}^{m-m_p} 1 = 1 + \frac{3}{2} m + \frac{1}{2} m^2 \tag{A20}
\]
So, while $l_m$ grows linearly with $m$, the number of words of length $l_m$ grows at least quadratically with $m$. Thus, for any $k \in \mathbb{N}$, choosing $m$ appropriately yields more than $kl_m$ words of length $l_m$. Hence $M$ is not linear.

(d) If $M$ is limited and $E_R$ is regular, then $L_R$ is constructive context free

We prove the claim by building a constructive PDA that accepts $L_R$. Let $F$ be a FSA that accepts $E_R$ and let $b$ denote the number of states of $F$. As a first step, we use $F$ to decompose $E_R$ into 8 disjoint subsets, represented by eight finite sets (A25). In the second step, for each of these sets, we build a PDA that computes the energy contribution corresponding to the edges in that set. Putting together these contributions shows that $L_R$ can be recognised by a constructive PDA.
Decomposing $E_R$. Take any edge $(p, q, r) \in E_R$. If $F$ enters a loop when processing $\rho$, we can w.l.o.g. assume that this loop is irreducible in the sense that it contains each state at most once; otherwise we decompose it until it is irreducible. Denote the number of states of this loop by $w_p$. Then we can write $p = v_p + n_p w_p$ for some natural number $n_p$. Note that

\[(v_p + nw_p, q, r) \in E_R\]  \hspace{1cm} (A21)

for any natural number $n$. We call $(w_p, v_p)$ the 1-loop-parameters of $(p, q, r)$ (1- to indicate that the loop occurs in $p$ and not in $q$ or $r$) and say that $(p, q, r)$ is 1-periodic if there exist 1-loop-parameters $(w_p, v_p)$ and a natural number $n_p$ such that $p = v_p + n_p w_p$. Next, we define the set of 1-loop-parameters that correspond to valid edges in $E_R$, requiring that all such 1-loop-parameters describe irreducible loops,

\[P_1 := \{(w_p, v_p) \mid (w_p, v_p) \text{ 1-loop-parameters of } E_R\}\]  \hspace{1cm} (A22)

Note that $v_p, w_p \leq b$ as otherwise the loop would not be irreducible. Thus, $P_1$ is a finite set.

If $(p, q, r)$ is not 1-periodic, we say it is 1-finite. We define

\[P_1 := \{p \mid \exists q, r \text{ s.t. } (p, q, r) \in E_R \text{ and } \exists v_p : (p, v_p) \in P_1\}\]  \hspace{1cm} (A23)

Note that any $p \in P_1$ must satisfy $p \leq b$, so $P_1$ is also a finite set. Note also that, by construction, any edge $(p, q, r) \in E_R$ is either 1-finite or 1-periodic, i.e. either $p \in P_1$ or $p = w_p n + v_p$ for a unique $(w_p, v_p) \in P_1$ and $n \in \mathbb{N}$.

In exactly the same way we define 2-periodicity, 2-finiteness, 3-periodicity and 3-finiteness of an edge $(p, q, r)$, where periodicity or finiteness refers to $q$ and $r$, respectively, as well as 2-loop-parameters and 3-loop-parameters and the corresponding sets

\[Q_1 := \{(w_q, v_q) \mid (w_q, v_q) \text{ 2-loop-parameters of } E_R\}\]
\[Q_1 := \{q \mid \exists p, r \text{ s.t. } (p, q, r) \in E_R \text{ and } \exists v_q : (q, v_q) \in Q_1\}\]
\[R_1 := \{(w_r, v_r) \mid (w_r, v_r) \text{ 3-loop-parameters of } E_R\}\]
\[R_1 := \{r \mid \exists p, q \text{ s.t. } (p, q, r) \in E_R \text{ and } \exists v_r : (r, v_r) \in R_1\}\]  \hspace{1cm} (A24)

In addition, we define sets of combinations of $p, q, r$ that lead to valid edges in $E_R$,

\[E_{\text{ff}} := \{(p, q, r) \in P_1 \times Q_1 \times R_1 \mid (p, q, r) \in E_R\}\]
\[E_{\text{ff}} := \{((w_p, v_p), q, r) \in P_1 \times Q_1 \times R_1 \mid (p, v_p, q, r) \in E_R\}\]
\[E_{\text{ff}} := \{(p, (w_q, v_q), r) \in P_1 \times Q_1 \times R_1 \mid (p, q, v_r) \in E_R\}\]
\[E_{\text{ff}} := \{((w_p, v_p), (w_q, v_q), r) \in P_1 \times Q_1 \times R_1 \mid (v_p, v_q, r) \in E_R\}\]
\[E_{\text{ff}} := \{((w_p, v_p), (w_q, v_q), (w_r, v_r)) \in P_1 \times Q_1 \times R_1 \mid (p, v_p, q, v_q, v_r) \in E_R\}\]
\[E_{\text{ff}} := \{((w_p, v_p), (w_q, v_q), (w_r, v_r)) \in P_1 \times Q_1 \times R_1 \mid (v_p, v_q, v_r) \in E_R\}\]  \hspace{1cm} (A25)

Note that each of these sets is finite, and that they are all disjoint. Note also that if $M$ is limited then $E_{\text{ff}}$ is empty. So this decomposes $E_R$ into 7 disjoint nonempty subsets. Explicitly, define their union

\[\mathcal{E} := E_{\text{ff}} \cup E_{\text{ff}} \cup E_{\text{ff}} \cup E_{\text{ff}} \cup E_{\text{ff}} \cup E_{\text{ff}} \cup E_{\text{ff}}\]  \hspace{1cm} (A26)

For each such set, any of its elements describes a subset $I_e \subseteq E_R$ of edges of $M$. For $(p, q, r) \in E_{\text{ff}}$ this is a singleton $I_{(p,q,r)} = \{(p, q, r)\}$, but for elements of any set other than $E_{\text{ff}}$, $I_e$ is infinite. For example, for $((w_p, v_p), (w_q, v_q), (w_r, v_r)) \in E_{\text{ff}}$ we have

\[I_{((w_p, v_p), (w_q, v_q), (w_r, v_r))} = \{(v_p + nw_p, v_q + mw_q, r) \mid n, m \in \mathbb{N}\}\]  \hspace{1cm} (A27)

In other words, an edge $(p', q', r')$ is contained in $I_{((w_p, v_p), (w_q, v_q), (w_r, v_r))}$ if and only of

\[p' = v_p \mod w_p\]
\[q' = v_q \mod w_q\]
\[r' = r\]  \hspace{1cm} (A28)

By construction, any edge $(p', q', r') \in E_R$ is described by a unique $e \in \mathcal{E}$. Thus we obtain a partition of $E_R$ into a finite number of disjoint subsets:

\[E_R = \bigcup_{e \in \mathcal{E}} I_e\]  \hspace{1cm} (A29)

Building the PDAs. In order to build a constructive PDA $P$ that accepts $L_M$, we build a constructive PDA $P_e$ for every $e \in \mathcal{E}$. The idea is the following. First, since $E_R$ is regular, well-formedness of the input can easily be checked by a FSA, and thus also be simulated in the states of $P_e$. So henceforth we shall assume that all inputs are well-formed, i.e. of the form

\[s_1 \ldots s_n \bullet a^k\]  \hspace{1cm} (A30)

Given a well-formed input, $P_e$ accumulates the energy contributions that correspond to edges in $I_e$ on its stack.
that is, \( P_c \) computes \( H_{\mathcal{A}}|_{I_c}(s_1 \ldots s_n) \). The required constructive PDA \( P \) for \( L_{\mathcal{A}} \) is obtained by running all \( P_c \)'s in parallel while providing access to the same stack, so that \( P \) accumulates
\[
\sum_{c \in \mathcal{E}} H_{\mathcal{A}}|_{I_c}(s_1 \ldots s_n) \quad (A31)
\]
on its stack. Since \( \mathcal{E} \) is finite, there is a finite number of PDAs \( P_c \), and hence their parallel simulation can be performed by a PDA, \( P \). Moreover, since \( \bigcup_{c \in \mathcal{E}} I_c \) is a partition of \( E_{\mathcal{A}} \) into disjoint subsets, equation (A31) equals \( H_{\mathcal{A}}(s_1 \ldots s_n) \). Finally \( P \) compares its stack content to the input energy \( \sigma^d \) and accepts if and only if the two values are equal. All PDAs \( P_c \) are built such that \( P \) is constructive.

Let us construct the PDAs \( P_c \) for any \( c \in \mathcal{E} \). We will start by considering \( c \in E_{\mathsf{iff}} \), and then continue with the following cases of (A25).

1. The PDA for \( E_{\mathsf{iff}} \). We consider \( (p, q, r) \in E_{\mathsf{iff}} \) and construct the PDA \( P_{(p,q,r)} \). We have that \( I_{(p,q,r)} = \{(p, q, r)\} \) contains an interaction between \( s_i \) and \( s_j \) if and only if
   \[
   \begin{align*}
   i &= p + 1 \\
   j &= p + q + 2 \\
   n &= p + q + r + 2
   \end{align*}
   \quad (A32)
   \]
The PDA starts by reading the first \( p + q + r + 2 \) symbols of its input and storing them in its state. Then it checks if the next input symbol is \( \bullet \). If yes, then \( n = p + q + r + 2 \) and it adds \( h(s_i, s_j) \) to its stack. The relevant spin values are stored in its states and the value \( h(s_i, s_j) \) can be hard-wired into the transition rules. If it reads \( \bullet \) during any other step of the computation, it rejects.

2. The PDA for \( E_{\mathsf{iff}} \). We now consider \(((w_p, v_p), q, r) \in E_{\mathsf{iff}} \) and construct the PDA \( P_{(w_p, v_p), q, r} \). \( s_i \) and \( s_j \) interact if and only if
   \[
   \begin{align*}
   i &= v_p + 1 \mod w_p \\
   j &= i + q + 1 \\
   n &= j + r
   \end{align*}
   \quad (A33)
   \]
If, for a given \( n \), Eq. (A33) has a solution, this solution is unique. Hence the PDA needs to compute at most one interaction, and works as follows. The PDA reads the first \( r + q + 2 \) spin symbols \( s_1 \ldots s_{r+q+2} \) and stores them in its states. Then it iteratively reads the next input symbol, stores it in its states and removes the left-most of the currently stored input symbols—we call this the main iteration. Note that at any given time, the stored spins are \( s_1 \ldots s_{r+q+r+1} \). To test if \( i = v_p + 1 \mod w_p \), it uses a counter \( c_{w_p} \), initialised at 1, which is updated as
\[
c_{w_p} \mapsto c_{w_p} + 1 \mod w_p \quad (A34)
\]

at each step of the main iteration. Thus, if \( i \) solves the first equation of Eq. (A33), \( c_{w_p} = v_p + 1 \). If this is the case, the PDA has stored \( s_1 \ldots s_n \), as by Eq. (A33) \( n = i + r + q + 1 \). It then checks if the next input symbol is \( \bullet \). If yes, it adds \( h(s_i, s_j) \) to its stack where, according to Eq. (A33), \( j = n - r \). This is possible since at this step of the computation both \( s_i \) and \( s_j \) are stored in the states of the PDA. If no, it continues with the main iteration. If it reaches \( \bullet \) at any other step of the computation, it rejects.

3. The PDA for \( E_{\mathsf{iff}} \). We now consider \((p, (w_q, v_q), r) \in E_{\mathsf{iff}} \) and construct the PDA \( P_{(p, (w_q, v_q), r)} \). \( s_i \) and \( s_j \) interact if and only if
   \[
   \begin{align*}
   i &= p + 1 \\
   j &= i + v_q + 1 \mod w_q \\
   n &= j + r
   \end{align*}
   \quad (A35)
   \]
The PDA starts by moving its head \( p \) symbols to the right. Next it stores \( s_{p+1} \) in its states, since according to Eq. (A35), \( i = p + 1 \). It now enters the main iteration: It stores the next \( r \) spin symbols in its states. Its head is now placed over \( s_{p+r+2} \) and it currently stores \( s_{p+1} \ldots s_{i+r} \). At each step of the main iteration, it reads the next spin symbol, stores it in its states and deletes the left-most spin symbol from its states. In addition to that, it uses a counter \( c_{w_q} \) that is initialised at \( c_{w_q} = i \mod w_q \). At each step of the main iteration the counter is updated as
\[
c_{w_q} \mapsto c_{w_q} + 1 \mod w_q \quad (A36)
\]
If during this main iteration the leftmost stored spin symbol is \( s_j \) then the counter is \( c_{w_q} = j \mod w_q \). Once it reaches \( \bullet \) the leftmost stored spin symbol \( s_j \) satisfies \( j = n - r \), i.e. solves the last equation in Eq. (A35). If additionally \( c_{w_q} = i + v_q + 1 \), it also solves the second equation in Eq. (A35). The PDA then adds \( h(s_i, s_{n-r}) \) to the stack. This is possible since both \( s_i \) and \( s_{n-r} \) are then stored in the states. If \( c_{w_q} \neq j \mod w_q \), the input is rejected. If during any other step of the computation the PDA reaches \( \bullet \), the input is rejected.

4. The PDA for \( E_{\mathsf{iff}} \). For \((p, (w_r, v_r)) \in E_{\mathsf{iff}} \) we construct the PDA \( P_{(p, (w_r, v_r))} \). \( s_i \) and \( s_j \) interact if and only if
   \[
   \begin{align*}
   i &= p + 1 \\
   j &= i + q + 1 \\
   n &= j + v_r \mod w_r
   \end{align*}
   \quad (A37)
   \]
Now both spin states \( s_i, s_j \) can be read from the beginning of the spin configuration. The PDA starts by storing the first \( p + q + 2 \) spins from its input in its states. Next it counts if \( n = p + q + 2 + v_r \mod w_r \) again using \( w_r \) of its states as a counter modulo \( w_r \). If yes, it adds \( h(s_i, s_j) \) to its stack; if no, it rejects.
are stored then

i

\begin{equation}
\begin{aligned}
& i = v_p + 1 \mod w_p \\
& j = i + v_q + 1 \mod w_q \\
& n = j + r 
\end{aligned}
\tag{A38}
\end{equation}

Let \( l := \text{lcm}(w_p, w_q) \), \( g := \text{gcd}(w_p, w_q) \). By the Chinese remainder theorem, if

\begin{equation}
\begin{aligned}
& v_p + 1 = j - v_q - 1 \mod g 
\end{aligned}
\tag{A39}
\end{equation}

Eq. (A38) has a unique solution modulo \( l \), else it has no solution. In particular, if there exists a solution, then there is a unique one with \( i \leq l \). All further solutions \( i' \) are given as \( i' = i + ml \) where \( m \) satisfies

\begin{equation}
\begin{aligned}
& i + ml \leq n - r - v_q - 1 
\end{aligned}
\tag{A40}
\end{equation}

The PDA first non-deterministically guesses both \( s_{n-r} \) and the unique \( i \leq l \) solving Eq. (A38). Then it iterates over the input and adds all \( h(s_{i'}, s_j) \) for \( i' = i + ml \) satisfying (A40) to its stack.

More precisely, it starts by non-deterministically guessing the state of spin \( s_{n-r} \). Next it reads the first \( r + 1 \) symbols of the input and stores them in its states. Now the main iteration starts. At each step of the main iteration, the PDA reads the next symbol from the input, stores this symbol in its states and deletes the left-most of its stored symbols. Additionally, it uses two counters, \( c_{w_p} \) and \( c_l \). Both counters are initialised as one. After each step of the iteration, the counters are updated as

\begin{equation}
\begin{aligned}
& c_{w_p} \mapsto c_{w_p} + 1 \mod w_p \\
& c_l \mapsto c_l + 1 \mod l 
\end{aligned}
\tag{A41}
\end{equation}

Note that both these counters correspond to the position of the left-most stored spin symbol, i.e. when \( s_{i'} \) is stored then \( c_{w_p} = i \mod w_p \) and \( c_l = i \mod l \). The main iteration stops once \( c_l = 0 \).

If, during the main iteration, \( c_{w_p} = v_p + 1 \), the PDA non-deterministically branches into the two options of the current position \( i \) either solving or not solving Eq. (A38). If it guesses that \( i \) does not solve Eq. (A38), it continues the main iteration with \( i + 1 \). If it guesses that \( i \) solves Eq. (A38) it adds \( h(s_i, s_{n-r}) \) to the stack. Next it sets \( c_l \) to zero, and sets an additional counter \( c_{w_q} \) modulo \( w_q \) to zero, too. It continues iterating over the remaining input, still updating \( c_l \) as before. Additionally, it now updates \( c_{w_q} \mapsto c_{w_q} + 1 \mod w_q \) instead of \( c_{w_p} \). Note that now both counters, \( c_l \) and \( c_{w_q} \) correspond to the position of the left-most stored spin symbol relative to \( s_i \), i.e. when \( s_{i'} \) is stored, the counters correspond to \( c_l = i' - i \mod l \), and similarly for \( c_{w_q} \). If, at any time \( c_l = 0 \), it adds the corresponding energy \( h(s_{i'}, s_{n-r}) \) to the stack, as in that case \( i' = i + ml \) and as \( i \) solves Eq. (A38) so does \( i' \). Given that the non-deterministic guess of \( i \) solving Eq. (A38) was right, the PDA hence accumulates the energy contributions of all solutions of Eq. (A38) on its stack.

Finally, once the head reaches \( \bullet \), it has \( s_{n-r} \ldots s_n \) stored in its states and the second counter yields \( c_{w_q} = j - i \mod w_q \). This allows the PDA to verify its two non-deterministic guesses. If \( c_{w_q} = v_q + 1 \) and the initial guess of \( s_{n-r} \) was correct, it accepts; else it rejects.

6. The PDA for \( E_{17} \). We now consider \( ((w_p, v_p), ((w_q, v_q), ((w_r, v_r)))) \in E_{17} \) and construct the PDA \( P_{((w_p, v_p), ((w_q, v_q), ((w_r, v_r))))} \). \( s_i \) and \( s_j \) interact if and only if

\begin{equation}
\begin{aligned}
& i = p + 1 \\
& j = i + v_q + 1 \mod w_q \\
& n = j + v_r \mod w_r 
\end{aligned}
\tag{A42}
\end{equation}

Using again the Chinese remainder Theorem with \( l := \text{lcm}(w_q, w_r) \), Eq. (A42) either has a unique solution modulo \( l \), or there exists no solution. The PDA can now be built similarly to the previous case, the only difference is that, as \( j = i + q + 1 \), there is no need to apply non-determinism to obtain \( s_i \).

More precisely, the PDA first traverses the input string, while keeping track of the current head position, using two modulo counters, \( c_{w_q} \) and \( c_l \), that are both initialised as \( c_{w_p} = c_l = 1 \). This process stops when \( c_l = 0 \). Whenever \( c_{w_p} = v_q + 1 \) it non-deterministically guesses if the current position \( i \) solves Eq. (A42). If no, it continues traversing the input; if yes, it sets \( c_l \) to zero, reads and stores the next \( q + 1 \) spin symbol \( s_i \ldots s_{i+q+1} \). Then it adds \( h(s_i, s_{i+q+1}) \) to its stack and further initialised an additional modulo \( w_r \) counter \( c_{w_r} \) at zero. Now it iterates over the remaining input. At each step it deletes the leftmost stored spin and stores the next symbol from the input. Additionally, the two counters are updated.

Note that when the stored spins are \( s_{i'} \ldots s_{i'\ldots q+r} \), the values of the two counters are \( c_l = i' - i \mod l \), \( c_{w_r} = i' - i \mod w_r \). While \( c_l \) is used to obtain all further solutions \( i' = i + ml \) to Eq. (A42), \( c_{w_r} \) allows the PDA to validate the non-deterministic guess of \( i \) being a valid solution. If at any time \( c_l = 0 \) the PDA adds \( h(s_{i'\ldots q+r+1}) \) to the stack. Finally, when it reaches \( \bullet \), \( c_w = n - i' - q - 1 \mod w_r \). Hence, \( i \) solves Eq. (A42) if and only if \( c_w = v_r \). If this holds, the PDA accepts; else the initial guess was wrong and it rejects.

7. The PDA for \( E_{17} \). We now consider \( ((p, w_q, v_q), ((w_r, v_r))) \in E_{17} \) and build the PDA \( P_{((p, w_q, v_q), ((w_r, v_r))))} \). \( s_i \) and \( s_j \) interact if and only if

\begin{equation}
\begin{aligned}
& i = p + 1 \\
& j = i + v_q + 1 \mod w_q \\
& n = j + v_r \mod w_r 
\end{aligned}
\tag{A43}
\end{equation}

As before, let \( l := \text{lcm}(w_q, w_r) \). The PDA starts by traversing the input until it reaches \( s_{p+1} \). It then stores \( s_{p+1} \) in its states. Next it initialises two modulo counters
and adds \( c_{w_j} = v_q + 1 \), it non-deterministically guesses if the current position \( j \) solves Eq. (A43). If no, it continues traversing the input as before: if yes it sets \( c_l = 0 \), initializes a second modulator counter \( c_{w_j} \), and adds \( h(s_{p+1}, s_j) \) to its stack. The PDA then traverses the remaining input.

Whenever for the current head position \( j' \), \( c_l = 0 \) it adds \( h(s_{p+1}, s_j') \) to its stack. Finally, when its head reaches \( \ast \), it accepts its unmarked spin at position \( q \) and energy tapes \( E_{mix}, l, i, j \).

The proof follows a similar line of reasoning as that of (iii). If \( L_{eR} \) is decidable there exists a Turing machine \( M_H \) that decides it. From this we can build a second Turing machine \( M_H \) that computes \( H_{eR} \) as a function, i.e. that on input \( s_1 \ldots s_n \) with \( n \in N_{eR} \), accepts a single edge, in which case \( M_H \) rejects the input.

Conversely, if \( E_R \) is context sensitive, there exists an LBA \( M_{E_R} \) that accepts \( L_{eR} \). According to the Immerman–Szelepcsenyi theorem there exists another LBA \( M_{E_R} \) that accepts the complement of \( E_R \). We now build a constructive LBA \( M_{E_R} \) that accepts \( L_{eR} \) as follows.

\( M \) uses a specific tape \( T_{S_E} \) to simulate \( M_E \) and a second tape \( T_{S_E} \) to simulate \( M_E \). In addition, \( M \) has the usual input tape \( T_{in} \) and energy tape \( T_{E} \). On input \( s_1 \ldots s_n \), \( M \) iterates over all possible pairs \((i, j)\) that satisfy \( i < j \leq n \). This can be achieved by marking the input on \( T_{in} \) appropriately; explicitly, by marking it as

\[
s_1' s_2'' s_3 \ldots s_n \otimes e
\]

in the beginning, then moving the \( s_j'' \) mark one step to the right after each iteration. Once \( \ast \) is reached the \( s_j'' \) mark is removed, \( M \) moves the \( s_i' \) mark one position to the right and marks the spin symbol to the right of it as \( s_i'' \).

At every step of the iteration, with \( s_i' \) and \( s_j'' \) marked, \( M \) copies the entire input spin configuration

\[
s_1 \ldots s_i' \ldots s_j'' \ldots s_n
\]

both to \( T_{S_E} \) and \( T_{S_E} \), and replaces each unmarked spin with a 0 and each marked spin with a 1. Thereby the edge between \( i \) and \( j \) is written to these two tapes.

Now \( M \) simulates \( M_E \) with \( T_{S_E} \) as input and \( M_E \) with \( T_{S_E} \) as input. If \( M_E \) accepts, \( \mathcal{M} \) contains an interaction of \( s_i' \) and \( s_j'' \), so \( M \) adds \( h(s_i, s_j) \) in binary to its energy tape. If \( M_E \) accepts, \( \mathcal{M} \) contains no interaction of \( s_i' \) and \( s_j'' \), so \( M \) moves to the next pair of spins without adding \( h(s_i, s_j) \) to \( T_E \). After that, \( M \) cleans both \( T_{S_E} \) and \( T_{S_E} \), before it continues with the next step of the iteration.

When the iteration terminates, \( M \) has stored \( H_{eR}(s_1 \ldots s_n) \) in binary on \( T_E \) and compares this to the input energy \( e \). If the computed and the input energy coincide, it accepts; otherwise, it rejects.

If \( n \notin N_{eR} \), the iteration terminates without \( M_E \) accepting a single edge, in which case \( M \) rejects the input. This case can be checked by, prior to the iteration over possible edges, writing a distinguished symbol to the energy tape that is removed once the first energy is added.
uses the decider for $E_R$, $M_E$, to decide if $(i,j) \in (E_R)n$. If yes, $M$ adds $h(s_i, s_j)$ to its energy tape $T_E$ and continues the iteration. If no, $M$ continues the iteration without adding the energy. Once the iteration over edges terminates, $M$ has $H_R(s_1 \ldots s_n)$ stored on its energy tape. Hence $L_R$ can be decided by letting $M$ accept the input if and only if there was at least one edge accepted by $M_E$, to ensure that $n \in N_R$, and additionally input energy and computed energy are equal.

Appendix B: Formal language theory toolbox

Let $\Sigma$ be a finite set, called the alphabet. Let $\Sigma^*$ denote the free monoid over $\Sigma$, with unit being the empty string $\epsilon$. In the context of formal language theory $\Sigma^*$ is often called the Kleene star of $\Sigma$. $\Sigma^*$ contains all finite strings that can be formed with symbols from $\Sigma$, including the empty string. A formal language $L$ over $\Sigma$ is a subset of $\Sigma^*$.

While most languages can only be characterised in an extensive way, namely by specifying the (infinite) set $L \subseteq \Sigma^*$ itself, some admit a finite description. There are two ways of providing this finite description: by providing a grammar $G$ that generates $L$, or by constructing an automaton that accepts $L$.

**Definition 7 (Grammar).** A grammar is a 4-tuple $G = (S, T, NT, P)$, where

- $S \in NT$ is a distinguished symbol, called the start symbol of $G$;
- $T$, $NT$ are disjoint, finite sets, whose elements are called terminal and non-terminal symbols respectively;
- $P \subseteq (T \cup NT)^* \times (T \cup NT)^*$ is a finite set of production rules. For $(\alpha, \beta) \in P$ we write $\alpha \rightarrow \beta$ and for $\{(\alpha, \beta), (\alpha, \beta') \} \subset P$ we write $\alpha \rightarrow \beta | \beta'$.

Given a string $w \in (T \cup NT)^*$, a production rule $\alpha \rightarrow \beta \in P$ is applied to it by replacing an occurrence of $\alpha$ as a substring of $w$ with $\beta$. If a string $w$ can be obtained from another string $w'$ by repeated application of production rules of $G$ we say that $w$ can be derived from $w'$ by means of $G$, and write $w' \Rightarrow^* G w$. The language $L(G)$ that a grammar $G$ generates is the set of all terminal strings that can be derived from the start symbol $S$,

$$L(G) := \{ w \in T^* | S \Rightarrow^* G w \} \quad (B1)$$

Grammars can be classified according to the form of the production rules they contain. The most famous such classification is the Chomsky hierarchy.

regular if $\alpha \in NT$ and $\beta = \epsilon$ or $\beta \in T$ or $\beta = bB$ with $b \in T$ and $B \in NT$;

context free if $\alpha \in NT$;

context sensitive if $aBc \rightarrow adc$, where $a,c \in (T \cup NT)^*, B \in NT$, and $c \neq d \in (T \cup NT)^*$;

unrestricted in any case.

This hierarchy of grammars can be lifted to a hierarchy of languages, by calling a formal language $L$ regular if there exists a regular grammar with $L = L(G)$, and similar for context free and context sensitive. The class of languages corresponding to unrestricted grammars is called recursively enumerable. If both a language and its complement are recursively enumerable, then it is called decidable.

For every level in the Chomsky hierarchy, there exists a type of automaton (i.e. a model of computation) that accepts the languages from that level: regular languages are accepted by finite state automata (FSA), context free languages are accepted by pushdown automata (PDA), context sensitive languages are accepted by linear bounded automata (LBA), and recursively enumerable languages are accepted by Turing machines (TM). Proving that a language is accepted by a certain type automaton is equivalent to proving that it is in the corresponding level. We now review these automata (see e.g. [32, 39]).

A FSA can be imagined as a machine with one tape, and a head that scans one cell of the input tape at a time. The FSA has a finite number of states in its head as memory. The computation starts with the input written on the tape and the head placed over the first input symbol. At each computation step, it reads the symbol that its head is currently placed over, and, depending on the symbol on the tape and the current state, transitions to a new state. The head then moves to the next input symbol. A FSA can neither change the direction of its head movement nor overwrite the tape.

**Definition 8.** (Finite state automaton) A finite state automaton is a 5-tuple $F = (Q, \Sigma, \delta, q_0, A)$, where

- $Q$ and $\Sigma$ are finite sets called the states and the input alphabet;
- $\delta : Q \times \Sigma \rightarrow Q$ is called the transition function;
- $q_0 \in Q$ is the start state;
- $A \subseteq Q$ are the accept states.

The transition function encodes one computation step of the FSA $F$: When in state $q$ upon reading $s$, $F$ transitions to state $q' = \delta(q, s)$. On input $w_1 \ldots w_n \in \Sigma^*$, $F$ starts in state $q_0$. It then processes the entire input: for each input symbol it uses $\delta$ and the current state to compute the new state; then it moves on with the next input symbol. After processing the entire input, if $F$ is in a state $f \in A$, the input is accepted by $F$; else, $F$ rejects.
A PDA can be imagined as a FSA which additionally has access to a stack. Specifically, at each step of the computation, the head of the PDA reads the current symbol on the tape, pops a symbol from the top of the stack, and pushes a finite number of symbols onto the stack. Then the head moves to the next symbol.

**Definition 9** (Pushdown automaton). A pushdown automaton is a 7-tuple $P = (Q, \Sigma_{in}, \Sigma_{stack}, \delta, q_0, Z, A)$, where

- $Q, \Sigma_{in}$ and $\Sigma_{stack}$ are finite sets called the states, input alphabet and stack alphabet;
- $\delta \subseteq (Q \times (\Sigma_{in} \cup \{\epsilon\}) \times \Sigma_{stack}) \times (Q \times \Sigma^*_{stack})$ is a finite set called the transition relation;
- $q_0 \in Q$ is the initial state;
- $Z \in \Sigma_{stack}$ is the initial stack symbol;
- $A \subseteq Q$ are the accept states.

The transition relation models one step of the computation: When in state $q$, upon reading $x$ and popping $s$, $P$ transitions to state $q'$ and pushes $s'$ to the stack, where $q'$ and $s'$ are such that

$$(q, x, s, q', s') \in \delta \quad \text{(B3)}$$

Then $P$ moves its head to the next input symbol. If there are multiple such $q'$, $s'$, $P$ branches its computation to pursue all such options simultaneously.

$P$ is called deterministic if for any $q \in Q$, $x \in \Sigma_{in}$ and $s \in \Sigma_{stack}$, there exist a unique $q' \in Q$, $s' \in \Sigma^*_{stack}$, such that either

$$(q, x, s, q', s') \in \delta \quad \text{or} \quad (q, \epsilon, s, q', s') \in \delta \quad \text{(B4)}$$

Otherwise $P$ is called non-deterministic.

An input string $w_1 \ldots w_n \in \Sigma_{in}$ is accepted by $P$ if with its head placed over the first symbol $w_1$, in state $q_0$ and with its stack containing only one symbol $Z$, after processing the entire input as dictated by $\delta$, there is at least one computation path that leads to a state $f \in A$. Otherwise the input is rejected by $P$.

The most powerful notion of machine that we will encounter in this work is that of a Turing machine (TM). A TM can be imagined as a machine with a finite number of states and an input tape. In contrast to a PDA, a TM can overwrite the input tape, and move left or right.

**Definition 10** (Turing machine). A Turing machine (TM) is a 7-tuple $M = (Q, \Sigma_{in}, \Sigma_{tape}, \delta, q_0, A, B)$, where

- $Q, \Sigma_{tape}$ are finite sets called the states and the tape alphabet;
- $\Sigma_{in} \subseteq \Sigma_{tape}$ is the input alphabet;
- $\delta \subseteq (Q \times \Sigma_{tape}) \times (Q \times \Sigma_{tape} \times \{L, R\})$ is the transition relation;
- $q_0 \in Q$ is the start state;
- $A \subseteq Q$ are the final states;
- $B \in \Sigma_{tape}$ is the blank symbol that represents an empty input cell.

When in state $q$ upon reading $s$, $M$ transitions to state $q'$, overwrites $s$ with $s'$ and moves its head one step in the direction specified by $m \in \{L, R\}$, where $(q', s', m)$ are specified by $\delta$,

$$(q, s, q', s', m) \in \delta \quad \text{(B5)}$$

If there are multiple such options $M$ branches its computation path to carry them out simultaneously.

$M$ is called deterministic if for each $(q, s) \in Q \times \Sigma_{tape}$ there exists at most one $(q', s', m) \in Q \times \Sigma_{tape} \times \{L, R\}$ such that (B5) holds. Otherwise $M$ is called non-deterministic.

If for a given state and input symbol $(q, s)$ there is no $(q', s', m)$ satisfying (B5), $M$ is said to halt in state $q$. An input string $w_1 \ldots w_n \in \Sigma_{in}$ is accepted by $M$ if $M$ when started in state $q_0$ with $w_1 \ldots w_n$ written on its input tape and its head placed over the first cell, after repeatedly performing the transitions as specified by $\delta$, after a finite number of steps there is at least one computation path that leads to $M$ halting in a final state.

Whereas a TM may use an unbounded amount of tape to carry out the computation, for the weaker notion of a linear bounded automaton (LBA) the accessible tape is limited to the cells which are initially used by the input string.

**Definition 11** (Linear bounded automaton). A linear bounded automaton is a 9-tuple $L = (Q, \Sigma_{in}, \Sigma_{tape}, \delta, q_0, A, B, \bot_L, \bot_R)$, where

- $(Q, \Sigma_{in}, \Sigma_{tape}, \delta, q_0, A, B)$ is a Turing machine
- $\bot_L, \bot_R \in \Sigma_{tape}$ are two special symbols that satisfy
  $$(q, \bot_L, q', s', m) \in \delta \Rightarrow s' = \bot_L \text{ and } m = R \quad \text{(B6)}$$
  $$(q, \bot_R, q', s', m) \in \delta \Rightarrow s' = \bot_R \text{ and } m = L$$

The special symbols $\bot_L, \bot_R$ serve as left and right endmarkers of the tape. Throughout the computation, $L$ neither overwrites these endmarkers nor moves its head past them. Other than that, the computation works exactly like that of a TM.

Relaxing the definition of the LBA such that the accessible tape space is a linear function of the input length, or allowing the LBA to perform its computation on multiple tapes does not change the class of problems it can solve [40, Theorem 12]. Hence, the class of context sensitive languages is identical with the complexity class NLINSSPACE of problems that can be solved in linear space on a non-deterministic Turing machine [41, Theorem 3.33].

Finally, we stress that most languages do not have a grammar (or, equivalently, are not recognised by a Turing
machine), as there are uncountably many languages but countably many grammars (or Turing machines). Explicitly, the number of languages over a finite alphabet $\Sigma$ is $|\varphi(\Sigma^*)| = 2^{(|\Sigma|)}$, whereas the number of grammars (or Turing machines) is $|\mathbb{N}|$.

**Appendix C: The grammar of the 1d and 2d Ising model**

Here we provide grammars of the 1d and 2d Ising model (Appendix C2 and Appendix C3, respectively). Before presenting the grammar of the 1d Ising model, we present a connection to lattice paths which provides an intuitive and transparent picture of the grammar (Appendix C1).

While we have already proven that $L_{1d}$ is context free and $L_{2d}$ is context sensitive by means of our main theorem, the following provides independent proofs of the same statements, and an alternative perspective on the same facts—if the automaton recognising a language is a ‘passive’ description of this language, the grammar generating it can be seen an ‘active’ description thereof. As a matter of fact, the connection to lattice paths of Appendix C1 provides yet another proof of the context freeness of $L_{1d}$. So, in total, we provide three proofs of the context freeness of the language of the 1d Ising model: by applying our main theorem to the 1d Ising model (Section IV A), via the connection to lattice paths (Appendix C1), and by providing a context free grammar (Appendix C2).

We remark that proving the complexity of an Ising model by providing a grammar only works for certain examples. This underlines the generality and usefulness of Theorem 1.

1. **The 1d Ising model and lattice paths**

Here we provide another proof (besides Section IV A and Appendix C2) that the language of the 1d Ising model, $L_{1d}$, is context free. We do so by establishing a connection between $L_{1d}$ and lattice paths. We construct a corresponding context free language of lattice paths, $L_p$, that is related to $L_{1d}$ by a mapping (called a generalised string homomorphism) which guarantees that the two languages, $L_{1d}$ and $L_p$, have the same complexity in the Chomsky hierarchy. Thus context freeness of $L_{1d}$ is implied by context freeness of $L_p$.

We conjecture that the connection between $L_{1d}$ and lattice paths is a consequence of the Chomsky–Schützenberger representation theorem, which states that every context free language can be constructed from a language of matched brackets of different types in a natural way [32, Supp. Lec. G]. The lattice paths used in this section can be seen as a variant of such languages of matched brackets.

**Figure 6: Representation of the lattice path $\downarrow\uparrow\downarrow\uparrow\downarrow$.**

**Definition 12** (Lattice path). A lattice path with steps $\uparrow$ and $\downarrow$ is a string $l \in \{\uparrow, \downarrow\}^*$.

We draw a lattice path by taking the lattice $\mathbb{Z}^2$, starting at $(0, 0)$ and then traversing $l$, consecutively moving one step in $(+1, +1)$ direction for each $\uparrow$ and one step in $(+1, -1)$ direction for each $\downarrow$ that we encounter (see Fig. 6).

We assign an energy to such a lattice path $l$ by taking the difference between the number of up-steps $\#\uparrow(l)$ and the number of down-steps $\#\downarrow(l)$,

$$H_p : \{\uparrow, \downarrow\}^* \rightarrow \mathbb{Z}$$

$$l \mapsto \#\uparrow(l) - \#\downarrow(l) \quad (C1)$$

With this, we define a formal language that encodes $H_p$ as

$$L_p := \{l \cdot u(H_p(l)) \mid l \in \{\uparrow, \downarrow\}^\ast\} \quad (C2)$$

This language is related to $L_{1d}$ by two generalised string homomorphisms (GSH) (see [42] and [40, Chapter 11.2]). A generalised string homomorphism can be realised as a machine that computes an output string from a given input string. For our purpose it suffices to consider a deterministic GSH.

**Definition 13** (Generalised string homomorphism). A generalised string homomorphism (GSH) is a 5-tuple $M = (Q, \Sigma_{in}, \Sigma_{out}, \delta, q_0)$, where

- $Q, \Sigma_{in}$ and $\Sigma_{out}$ are finite sets called the states, input alphabet and output alphabet;
- $q_0 \in Q$ is the start state;
- $\delta : Q \times \Sigma_{in} \rightarrow Q \times \Sigma_{out}$ is the transition function.

When in state $q \in Q$, upon reading $x \in \Sigma_{in}$, $M$ outputs $s \in \Sigma_{out}$ and transitions to state $q'$ before moving on with the next symbol, where $(q', s) := \delta(q, x)$. Given an input string $w \in \Sigma_{in}^\ast$, by starting in state $q_0$ and processing $w$ in this fashion, $M$ produces an output string. Thereby it defines a map which by abuse of notion will also be denoted as $M$

$$M : \Sigma_{in}^\ast \rightarrow \Sigma_{out}^\ast \quad (C3)$$
Figure 7: The transition function of GSH $M_0$. All transition rules that are not drawn lead to the only missing state, $R_0$ and output $\bullet$, irrespective of the input.

Note that a GSH with only one state equals a conventional string homomorphism. A multistate GSH can additionally use its states to alter how symbols from $\Sigma_{in}$ are mapped to strings in $\Sigma_{out}$.

A GSH is called $\epsilon$-free if for every $w \in \Sigma_{in}$ it outputs at least one symbol.

**Lemma 1 (Theorem 11.1 in [40])**. Let $L \subseteq \Sigma_{\ast}$ be a context-free language and $M = (Q, \Sigma_1, \Sigma_2, \delta, q_0)$ an $\epsilon$-free GSH, then also the image of $L$ under $M$ is context free.

GSH can be specified by a graph where every node corresponds to a state, and a directed edge from $q_1$ to $q_2$ with label $w \mid x$ denotes that $\delta(q_1, w) = (q_2, x)$ (see Fig. 7 and [40, Figure 11.1]).

The correspondence between $L_p$ and $L_{1d}$ is provided by two GSH $M_0$ and $M_1$, where $M_0$ ($M_1$) maps lattice paths to spin configurations that start with $s_1 = 0$ ($s_1 = 1$). They are defined by

$$M_s := (Q_s, \{\uparrow, \downarrow, \bullet, +, -\}, \{0, 1, \bullet, +, -\}, \delta_s, S_s) \quad (C4)$$

where $s \in \{0, 1\}, M_s$ uses states

$$Q_s := \{S_s, I^0_s, I^1_s, E_s, R_s\} \quad (C5)$$

The transition function $\delta_0$ is shown in Fig. 7, and $\delta_1$ only differs in the treatment of the first input symbols

$$\delta_1(I_1, \uparrow) := (I^0_1, 10)$$
$$\delta_1(I_1, \downarrow) := (I^1_1, 11) \quad (C6)$$

Note that these mappings are not possible by conventional string homomorphisms, as deciding if a symbol $l_i = \uparrow$ should be mapped to a spin state $s_{i+1} = 0$ or $s_{i+1} = 1$ requires knowledge of the previous spin state $s_i$.

**Example 1.** Consider the lattice path $\downarrow \uparrow \uparrow \uparrow \downarrow$, shown in Fig. 6. Clearly $H_p(l) = 0$. Applying $M_0$ and $M_1$ to the corresponding string in $L_p$ yields

$$M_0(\downarrow \uparrow \uparrow \uparrow \downarrow) = 0011011\bullet$$
$$M_1(\downarrow \uparrow \uparrow \uparrow \downarrow) = 1100100\bullet \quad (C7)$$

Both $M_0$ and $M_1$ are injective when restricted to $L_p$. Moreover,

$$L_{1d} = M_0(L_p) \cup M_1(L_p) \quad (C8)$$

We thus have a 2-to-1 correspondence between $L_{1d}$ and $L_p$. The lattice path that corresponds to a spin configuration can be interpreted as uncovering the information about how the global energy of the spin configuration is composed locally. The fact that this correspondence is 2-to-1 follows from the bit-flip symmetry of the Ising Hamiltonian $H_R$: for any spin configuration $s_1 \ldots s_n \in S_R$,

$$H_R(\bar{s}_1 \ldots \bar{s}_n) = H_R(s_1 \ldots s_n) \quad (C9)$$

where $\bar{s}$ denotes application of the bit-flip operation, i.e. $0 \rightarrow 1$ and $1 \rightarrow 0$. Also the two maps $M_s$ are precisely related by this flip operation. Moving from spin configurations to lattice paths can thus be seen as modding out this bit flip symmetry.

**Proposition 3.** $L_{1d}$ is context free.

**Proof.** We first prove that $L_p$ is context free. The claim then follows from Lemma 1 and the fact that the union of context free languages is itself context free [40, Theorem 6.1].

We start by considering the subset of lattice paths with energy 0

$$L^0_p := H_p^{-1}(\{0\}) \quad (C10)$$

Note that for any $l = l_1 \ldots l_n \in L^0_p$ one of the following three cases holds true

1. $n = 2$ and either $l = \uparrow \downarrow$ or $l = \downarrow \uparrow$
2. $l_1 \neq l_n$ and thus $l_2 \ldots l_{n-1} \in L^0_p$
3. there exists $w, w' \in L^0_p$ with $l = ww'$

$L^0_p$ is generated by the grammar

$$G^0_p := (S^0_p, \{\uparrow, \downarrow\}, \{S^0_p\}, P^0_p) \quad (C11)$$

where the production rules $P^0_p$ are given as

$$S^0_p \rightarrow \uparrow \downarrow \mid \downarrow \uparrow \mid S^0_p \downarrow \mid \downarrow S^0_p \mid S^0_p S^0_p \quad (C12)$$

Clearly all production rules produce an equal number of $\uparrow$ and $\downarrow$ symbols, so by induction over the derivation length of any word $w$ in the language (i.e. the number of production rules applied to derive $w$ from $S^0_p$), it follows that $L(G^0_p) \subseteq L^0_p$.
As all strings in $L_p^0$ admit one of the three forms above, and all these forms are captured by the production rules, by induction over the length of strings in $L_p^0$, it follows that all such strings can be derived by $P_p^0$, so $L_p^0 \subseteq L(G_p^0)$. Thus

$$L_p^0 = L(G_p^0)$$  \hspace{1cm} (C13)

We now construct a grammar for $L_p$,

$$G_p := (S_p, T_p, NT_p, P_p)$$  \hspace{1cm} (C14)

with terminal symbols

$$T_p := \{\uparrow, \downarrow, \bullet, +, -\}$$  \hspace{1cm} (C15)

non-terminal symbols

$$NT_p := \{S_p, S_p^+, S_p^-, S_p^0, I_p^+, I_p^-\}$$  \hspace{1cm} (C16)

and production rules $P_p$

\begin{align*}
S_p & \rightarrow \quad S_p^0 \bullet \mid S_p^+ \mid S_p^- & \hspace{1cm} (C17a) \\
S_p^+ & \rightarrow \quad \uparrow I_p^+ \mid S_p^0 \uparrow I_p^+ \mid \uparrow S_p^0 I_p^+ \mid \uparrow S_p^0 \uparrow I_p^+ & \hspace{1cm} (C17b) \\
S_p^- & \rightarrow \quad \downarrow I_p^- \mid S_p^0 \downarrow I_p^- \mid \downarrow S_p^0 I_p^- \mid \downarrow S_p^0 \downarrow I_p^- & \hspace{1cm} (C17c) \\
I_p^+ & \rightarrow \quad \bullet \mid \uparrow I_p^+ \mid \uparrow S_p^0 I_p^+ & \hspace{1cm} (C17d) \\
I_p^- & \rightarrow \quad \bullet \mid \downarrow I_p^- \mid \downarrow S_p^0 I_p^- & \hspace{1cm} (C17e)
\end{align*}

Additionally, $P_p$ contains all productions in $P_p^0$.

First, observe that all terminal expressions that can be derived with the production rules $P_p$ are well-formed, i.e. they contain exactly one • symbol, only ↑, ↓ symbols left of it and a potentially empty string that consists of either only + or only − symbols to the right of it. Besides, all rules in $P_p$ produce an equal number of ↑ and +, or ↓ and −. Thus, again by induction over the derivation length it follows that $L(G_p) \subseteq L_p$.

To prove that $L_p \subseteq L(G_p)$, note that the production rules Eq. (C17a) separate the strictly positive, strictly negative and zero energy case, so these three cases can be treated independently. The lattice paths with zero energy can of course be obtained by the rules in $P_p^0$. For a lattice path of strictly positive energy,

$$l_1 \ldots l_n \bullet +^k \in L_p$$  \hspace{1cm} (C18)

one can always separate a maximal prefix with energy 1, i.e. there always exists an $i \in \{2, \ldots, n\}$ such that $l_1^{(i)} := l_1 \ldots l_i$ has energy +1. If $k = 1$ then $i = n$; else, $l_{i+1} = \uparrow$ as otherwise $i$ would not be maximal. Iteratively separating maximal prefixes with energy +1 from $l_1 \ldots l_n$ yields

$$l_1 \ldots l_n = l_1^{(1)} \ldots l^{(k)}$$  \hspace{1cm} (C19)

such that also for all $i = 2, \ldots, k - 1$, $l^{(i)}$ is a maximal energy +1 prefix of $l^{(i+1)} \ldots l^{(k)}$. An example is shown in Fig. 8. The first such maximal prefix $l_1^{(1)} := l_1 \ldots l_i$ of energy +1 admits one of the following forms.

\[ \bullet \]

\[ \uparrow \]

\[ \downarrow \]

\[ I_p^+ \]

\[ I_p^- \]

\[ \uparrow I_p^+ \]

\[ \downarrow I_p^- \]

\[ \uparrow S_p^0 I_p^+ \]

\[ \downarrow S_p^0 I_p^- \]

\[ l^{(i)} = \uparrow \]

\[ l^{(i)} = w \uparrow \text{ with } w \in L_p^0 \]

\[ l^{(i)} = \downarrow w \text{ with } w \in L_p^0 \]

\[ l^{(i)} = w \uparrow w' \text{ with } w, w' \in L_p^0 \]

These 4 possibilities are precisely covered by the four production rules in Eq. (C17b). As after that, all further maximal prefixes $l^{(i)}$, with $i = 2, \ldots, k$ necessarily start with ↑, there are only two possibilities for these

\[ l^{(i)} = \uparrow \]

\[ l^{(i)} = \uparrow w \text{ with } w \in L_p^0 \]

These two options can be generated by the two production rules in Eq. (C17d).

This works in the same way for lattice paths with negative energy. Proceeding by induction over the energy, treating positive and negative energies separately, we see that the production rules in $P_p$ derive any string in $L_p$ and hence $L_p \subseteq L(G_p)$. Thus,

$$L_p = L(G_p)$$  \hspace{1cm} (C20)

As $G_p$ is context free, so is $L_p$.

The central idea of these grammars is the following. At the first step of each derivation, they distinguish between zero, positive or negative energies. In the zero energy case, only strings that keep the overall energy zero can be appended. In the strictly positive (negative) case, only strings that increase (decrease) the overall energy can be appended. This property is somewhat counterintuitive but necessary, as within context free grammars it is not possible to define production rules that cancel positive against negative energies:

$$+ - \rightarrow \epsilon$$  \hspace{1cm} (C21)
2. A context free grammar for the 1d Ising model

Here we provide a context free grammar for the language of the 1d Ising model, $L_{1d}$. We start by providing a context free grammar for the zero energy spin configurations (Proposition 4), which will act as the backbone of the grammar for $L_{1d}$ (Proposition 5). The ideas behind the grammar for $L_{1d}$ are very similar to those behind the grammar for lattice paths $L_p$.

**Proposition 4** (Grammar for $L^0_{1d}$). Let $L^0_{1d}$ be the set of zero energy spin configurations of the Ising model $M_{1d}$

\[
L^0_{1d} := H_{1d}^{-1}(\{0\})
\]  

(C22)

Let $i, j \in \{0, 1\}$ with $i \neq j$, whenever they appear in the same production rule. The context free grammar

\[
G^0_{1d} := (S^0_{1d}, T^0_{1d}, NT^0_{1d}, P^0_{1d})
\]  

(C23)

with start symbol $S^0_{1d}$, terminal symbols

\[
T^0_{1d} := \{0, 1\}
\]  

(C24)

non-terminal symbols

\[
NT^0_{1d} := \{S^0_{1d}, S_{ii}, S_{ij}, \bar{S}_{ii}, \bar{S}_{ij}\}
\]  

(C25)

and production rules $P^0_{1d}$

\[
S^0_{1d} \rightarrow S_{ii} | S_{ij}
\]  

(C26a)

\[
S_{ii} \rightarrow iS_{ij}i | iS_{ij}i | S_{ii}\bar{S}_{ii} | S_{ij}\bar{S}_{ij}
\]  

(C26b)

\[
S_{ij} \rightarrow ijj | ijj | iS_{jj}j | iS_{jj}j | S_{ij}\bar{S}_{jj} | S_{ii}\bar{S}_{ij}
\]  

(C26c)

\[
\bar{S}_{ii} \rightarrow S_{ji}i | S_{ji}i | \bar{S}_{ii}\bar{S}_{ii} | \bar{S}_{ij}\bar{S}_{ij}
\]  

(C26d)

\[
\bar{S}_{ij} \rightarrow jj | jj | S_{jj}j | S_{ij}j | S_{ij}\bar{S}_{jj} | S_{ii}\bar{S}_{ij}
\]  

(C26e)

generates $L^0_{1d}$.

In words, (C26b) generates the zero energy spin configurations that start and end in the same state (called $i$). (C26c) generates the zero energy spin configurations that start and end in a different state ($i$ and $j$). So $S_{ii}$ stands for any zero energy spin configuration that starts and ends in $i$, and $S_{ij}$ stands for any zero energy spin configuration that starts in $i$ and ends in $j$. $\bar{S}_{ii}$ stands for any zero energy spin configuration that starts and ends in $i$ but is ‘missing an $i$ on the left’, and similarly for $S_{ij}$.

**Proof.** We start by proving that $L(G^0_{1d}) \subseteq L^0_{1d}$ by induction over the derivation length. First, note that any spin configuration that can be derived from $S_{ij}$ starts with $i$ and ends with $j$. The only derivations of length 2 are

\[
S^0_{1d} \rightarrow S_{01} \rightarrow 001 | 011
\]  

(C27)

\[
S^0_{1d} \rightarrow S_{10} \rightarrow 100 | 110
\]

All these configurations have energy 0. Now consider an arbitrary configuration of length $n$. We only show explicitly the case where the derivation starts with $S^0_{1d} \rightarrow S_{ii}$, as the other case works in exactly the same way. There are two cases of how the derivation can continue. In the first case the derivation is of the form

\[
S^0_{1d} \rightarrow S_{ii} \rightarrow iS_{ij}i | iS_{ij}i
\]  

(C28)

Then the induction hypothesis applies to the remaining productions that transform $S_{ij}$ or $\bar{S}_{ij}$ into terminal symbols. Thus, $S_{ij}$ will be transformed into a spin configuration of energy 0 that starts with $j$ and ends with $i$, while $\bar{S}_{ij}$ will be transformed into a spin configuration of energy 0 that starts with $i$ and ends with $j$. Since $i \neq j$, in either case, the first and last contributions to the energy of $iS_{ij}i$ or $iS_{ij}i$ cancel. Hence, also the derivation of length $n + 1$ yields a final configuration with energy 0.

In the second case the derivation is of the form

\[
S^0_{1d} \rightarrow S_{ii} \rightarrow S_{ii}\bar{S}_{ii} | S_{ij}\bar{S}_{jj}
\]  

(C29)

Note that a configuration $s_1 \ldots s_k$ can be derived from $\bar{S}_{ij}$ if and only if $s_1 \ldots s_k$ can be derived from $S_{ij}$, as can readily be seen from comparing the production rules (C26e) and (C26c). Thus the induction hypothesis applies not only to the productions that follow for $S_{ii} | S_{ij}$ but also to those that follow for $S_{ii} | S_{ij}$ once the missing $i \mid j$ is appended. This case corresponds to two zero energy configurations that are glued together, overlapping in a single spin symbol. Therefore also this case yields a configuration with zero energy.

In order to prove $L^0_{1d} \subseteq L(G^0_{1d})$ we proceed by induction over the word length, i.e. the length of the spin configuration. $L^0_{1d}$ contains exactly four configurations of length three, namely 001, 011, 100, 110, which can all be derived from $S^0_{1d}$. Consider now an arbitrary configuration of $n$ spins, $s = s_1 \ldots s_n \in L^0_{1d}$. There are now only two cases. The first one is

\[
h(s_1, s_2) + h(s_{n-1}, s_n) = 0
\]  

(C30)

In this case, $s_2 \ldots s_{n-1}$ has zero energy and can thus by assumption be derived from $S_{ii} | S_{ij}$, so that $s_1 \ldots s_n$ can be derived by pre-composing the derivation of $s_2 \ldots s_{n-1}$ with an additional production. Note that $h(s_1, s_2)$ can be either $+1$ or $-1$ and $s_2 \ldots s_{n-1}$ can be derived from either $S_{ii}$ or $S_{ij}$. If $h(s_1, s_2) = -1$ and $s_2 \ldots s_{n-1}$ can be derived from $S_{ii}$ then $s_1 = s_2$ and $s_2 = s_{n-1}$. Since $h(s_{n-1}, s_n) = +1$, we have $s_n \neq s_1$. Hence, we can derive $s_1 \ldots s_n$ as

\[
S^0_{1d} \rightarrow S_{ij} \rightarrow iS_{ij}j
\]  

(C31)

where $i = s_1$ and $j = s_n$. The other three possibilities work similarly.

In the second case,

\[
h(s_1, s_2) = h(s_{n-1}, s_n)
\]  

(C32)

As $H_{1d}(s_1s_2) = h(s_1, s_2)$ but $H_{1d}(s_1 \ldots s_{n-1}) = -h(s_1, s_2)$, there must be a zero energy sub-configuration
\[ S_{1d}^0 \to S_{ii} \to S_{ii} \hat{S}_{ii} \]  
(C33)

where \( i = s_1 \). All other possibilities of \( s_1 = s_i \neq s_n \), \( s_1 = s_n \neq s_i \) and \( s_1 \neq s_i = s_n \) work similarly. Thus the configuration can be derived by means of the grammar \( G_{1d}^0 \). This proves that \( L_{1d}^0 \subseteq L(G_{1d}^0) \) and therefore, in total, that \( L_{1d} = L(G_{1d}^0) \). \( \square \)

We now provide a context free grammar for the 1d Ising model, in which the production rules that generate zero energy spin configurations act as building blocks for other energies.

**Proposition 5** (Grammar for \( L_{1d} \)). The context free grammar

\[ G_{1d} := (S_{1d}, T_{1d}, NT_{1d}, P_{1d}) \]  
(C34)

with start symbol \( S_{1d} \), terminal symbols

\[ T_{1d} := \{0,1,\bullet,+,−\} \]  
(C35)

non-terminal symbols

\[ NT_{1d} := \{S_{1d}^0, S_{ii}, \hat{S}_{ii}, S_{ij}, \hat{S}_{ij}, S_+, S_−, I^+_i, I^-_i\} \]  
(C36)

and production rules \( P_{1d} \) including \( P_{1d}^0 \) and

\[ S_{1d} \to S_{1d}^0 \bullet \mid S_+ \mid S_− \]  
(C37a)

\[ S_+ \to ijI^+_j + \mid \]  
(C37b)

\[ iS_{ij}I^+_i + \mid iS_{ij}I^+_i + \mid \]  
\[ S_{ii}I^+_j + \mid S_{ij}I^+_i + \mid \]  
\[ S_{ij}I^+_i + \mid S_{ij}I^+_j + \mid \]  
\[ S_{ij}I^+_i + \mid S_{ij}I^+_j + \mid \]  
\[ S_− \to iiI^−_i − \mid \]  
(C37c)

\[ iS_{ii}I^−_i − \mid iS_{ii}I^−_i − \mid \]  
\[ S_{ii}I^−_i − \mid S_{ii}I^−_i − \mid \]  
\[ S_{ij}I^−_i − \mid S_{ij}I^−_i − \mid \]  
\[ S_{ij}I^−_i − \mid S_{ij}I^−_i − \mid \]  
\[ I^+_i \to \bullet \mid jI^+_j + \mid S_{ij}I^+_i + \mid S_{ij}I^+_j + \]  
(C37d)

\[ I^-_i \to \bullet \mid iI^-_i − \mid S_{ii}I^-_i − \mid S_{ij}I^-_j − \]  
(C37e)

where as before \( i, j \in \{0,1\} \) with \( i \neq j \), generates \( L_{1d} \).

In words, \( S_+ \) and \( S_− \) stand for any string in \( L_{1d} \) with positive and negative energy, respectively. The symbols \( I^\pm_i \) keep track of the last spin symbol in the configuration \( i \) and keep track of whether the whole string has positive or negative energy (indicated with + or − as a superindex). As before, \( S_{id} \) stands for any spin configuration starting and ending with i, and similarly for \( S_{ii} \). Since \( S_{ij} \) and \( S_{ij} \) are also subject to the production rules of the zero energy case, zero energy spin configurations can be inserted at any position. The main idea is that any spin configuration can be split into adjacent segments of zero energy; the only contribution to the final energy comes from the adjacency of these segments. More specifically, if the total energy of this spin configuration is \( +k \), this spin configuration will admit a splitting into \( k+1 \) zero energy non-overlapping contiguous segments, so that every border of segments contributes +1.

**Proof.** We first prove that \( L(G_{1d}) \subseteq L_{1d} \). It is straightforward to see that all terminal expressions that can be derived by means of \( G_{1d} \) are well-formed, i.e. of the form \( s_\bullet \sigma^k \), where \( s \in \{0,1\}^* \) and \( \sigma \in \{+,−\} \). We have to show that for any such terminal expression \( \sigma^k = u(H_{1d}(s)) \).

Consider first the rules in (C37b). Every such rule has a right hand side of the form \( C^+_j j^+_j + \), where the first part, \( C^+_j \), is either a spin configuration with energy +1 that has its last spin in state \( j \), i.e. \( C^+_j = ij \) or is a non-terminal that eventually must be replaced by such a spin configuration, such as \( C^+_j = iS_{ij} \). (Recall that \( S_{ij} \) stands for a zero energy spin configuration that starts and ends with spins in state \( j \).) The middle part, \( I^+_i \), stores the last symbol of the previous configuration \( C^+_j \), and the last part correctly accounts for the energy of \( C^+_j \). Whenever a rule in (C37d) is used to append further spin symbols, the overall energy is increased by +1, which is also accounted for by simultaneously adding a + symbol at the correct position. Thus all terminal expressions that can be derived from \( S_\bullet \) yield a valid spin configuration energy pair. The same argument holds for derivations by rules (C37c) and (C37e) and by Proposition 4 also for all terminal expression that can be derived from \( S_{1d}^0 \bullet \). Thus \( L(G_{1d}) \subseteq L_{1d} \).

We now prove that \( L_{1d} \subseteq L(G_{1d}) \). Take any \( s_\bullet \sigma^k \in L_{1d} \) with \( s = s_1 \ldots s_n \); we now show how this string can be derived by means of \( G_{1d} \), by treating the three cases \( \sigma = +, \sigma = − \) and \( k = 0 \) separately. If \( k = 0 \) then the claim follows from Proposition 4. The remaining two cases both follow from the same arguments, so we only show \( \sigma = + \) explicitly.

We proceed by induction over \( k \). We first consider the base case, \( k = 1 \), where we select a maximal zero energy sub-configuration \( s_1 \ldots s_i \), where \( s_{i+1} \neq s_i \) as otherwise the sub-configuration would not be maximal. We first consider the case in which \( i = 1 \). Since \( s_1 \neq s_2 \), the remaining configuration \( s_2 \ldots s_n \) is of zero energy. If, moreover, \( i + 1 = n \), then the string is of the form \( ij \bullet + \) and can be derived as

\[ S_{1d} \to S_+ \to s_1 s_2 I^+_1 \to s_1 s_2 \bullet + \]  
(C38)

If \( i + 1 \neq n \), then \( s_2 \ldots s_n \) can be derived from \( S_{2s_n} \), by Proposition 4, and the string \( s_1 \ldots s_n \bullet + \) can be derived
as
\[ S_{1d} \rightarrow S_+ \rightarrow s_1S_{2d}nI_{s_1}^+ \rightarrow \]
\[ s_1S_{2d}n \bullet + \rightarrow \ldots \rightarrow s_1 \ldots s_n \bullet + \quad (C39) \]

We now consider the case \( i > 1 \). By Proposition 4, \( s_1 \ldots s_i \) can be derived from \( S_{1,i} \), and we have two cases. In the first case, \( i + 1 = n \) and we can derive \( s_1 \ldots s_n \bullet + \) from \( S_{1,i}S_{i+1} \), and hence the whole configuration as
\[ S_{1d} \rightarrow S_+ \rightarrow S_{1,i}S_{i+1}nI_{s_1}^+ \rightarrow \]
\[ S_{1,i}S_{i+1}n \bullet + \rightarrow \ldots \rightarrow s_1 \ldots s_n \bullet + \quad (C40) \]
The second case consists of \( i + 1 \neq n \), but then also \( s_1 \ldots s_n \) has zero energy and hence can be derived from \( S_{1,i} \). Thus we can derive \( s_1 \ldots s_n \bullet + \) as
\[ S_{1d} \rightarrow S_+ \rightarrow S_{1,i}S_{i+1}nI_{s_1}^+ \rightarrow S_{1,i}S_{i+1}n \bullet + \rightarrow \ldots \rightarrow s_1 \ldots s_i S_{i+1}n \bullet + \rightarrow \ldots \rightarrow s_1 \ldots s_n \bullet + \quad (C41) \]

We now prove the induction step. For a configuration of energy \( k \) we split off a maximal sub-configuration of energy \( k-1 \), \( s_1 \ldots s_i \), which can be derived from \( S_+ \) by the induction assumption. W.l.o.g. we assume that the last step of this derivation consists of \( I_{s_1}^+ \rightarrow \bullet \), and thus there exists a sub-derivation
\[ S_{1d} \rightarrow S_+ \rightarrow \ldots \rightarrow s_1 \ldots s_i I_{s_i}^+ \rightarrow s_{i+1} \ldots s_n \bullet \rightarrow s_{i+1} \ldots s_n I_{s_n}^+ \rightarrow s_n \bullet + \quad (C42) \]
The second case consists of \( i + 1 \neq n \), therefore \( s_n \neq s_i \) and thus there exists a derivation
\[ I_{s_i}^+ \rightarrow s_n I_{s_n}^+ \rightarrow s_n \bullet + \quad (C43) \]
which when composed with the previous sub-derivation yields a derivation of \( s_1 \ldots s_n \bullet + \cdot \). If \( i + 1 \neq n \) then \( s_i + 1 \ldots s_n \) is of zero energy and can thus be derived from \( S_{1,i} \). From the previous sub-derivation together with
\[ I_{s_i}^+ \rightarrow S_{s_1}I_{s_1}^+ \rightarrow S_{s_1}I_{s_1}^+ \rightarrow s_1 \ldots s_n \bullet + \quad (C44) \]
we obtain a derivation of \( s_1 \ldots s_n \bullet + \cdot \). Thus \( L_{1d} \subseteq L(G_{1d}) \) and therefore, in total, it is proven that \( L_{1d} \subseteq L(G_{1d}) \).

3. A context sensitive grammar for the 2d Ising model

We now provide a non-contracting grammar \( G_{2d} \) that generates the language of the 2d Ising model \( L_{2d} \). A grammar is called non-contracting if for any of its production rules \( \alpha \rightarrow \beta \), it holds that \( |\alpha| \leq |\beta| \). By a standard construction, every non-contracting grammar can be transformed into a context sensitive grammar that generates the same language (see [43, Lemma 1] and also [37, Theorem 11]).

**Proposition 6** (Grammar for \( L_{2d} \)). The non-contracting grammar
\[
G_{2d} := (S_{2d}, T_{2d}, NT_{2d}, P_{2d})
\]
with start symbol \( S_{2d} \), terminal symbols
\[
T_{2d} := \{0, 1, \bullet, +, -\}
\]
non-terminal symbols
\[
NT_{2d} := \{S_{2d}, S_s, (s^c)^\prime, (s^c)^\prime\prime, I_1, I_2, \perp, (s^c)^\perp, s^c, s^e\}
\]
and production rules \( P_{2d} \)

\[
S_{2d} \rightarrow S_0 \mid S_+ \mid S_-
\]
\[
S_1 \rightarrow (s^c)^\perp I_1
\]
\[
I_0 \rightarrow s^c I_1
\]
\[
s^c \rightarrow (s^c)^\perp I_1
\]
\[
(s^c)^\perp \rightarrow (s^c)^\prime
\]
\[
(s^c)^\prime \rightarrow (s^c)^\prime\prime
\]
\[
(s^c)^\prime\prime \rightarrow (s^c)^\perp
\]
\[
(s^c)^\perp \rightarrow I_1
\]
\[
I_1 \rightarrow I_1 \perp s_1 s_2
\]
\[
I_2 \rightarrow I_2 \perp s_1 s_2
\]
\[
\perp \rightarrow \perp
\]
\[
\perp \rightarrow I_1
\]
\[
\perp \rightarrow I_2
\]

where \( e^c \in \{0, 1, 2\} \) and \( e^c \in \{-2, -1, 0\} \), generates \( L_{2d} \).

The following proof provides explanations of the use of each of these rules.

**Proof.** We first show that \( L(G_{2d}) \subseteq L_{2d} \). To this end, we take any terminal expression \( x \) that can be obtained from \( G_{2d} \) and show that \( x \in L_{2d} \), i.e. that
\[
x = s_1 \ldots s_n \bullet u(H_{2d}(s_1 \ldots s_n^2))
\]
for some \( n^2 \in N_{2d} \).

First, note that the beginning of every derivation consists of
\[
S_{2d} \rightarrow S_1 \rightarrow (s^c)^\perp I_1
\]
Next there are only three production rules that can follow: the two in (C48c), and (C48q). If (C48q) was next
then there would be no option left to convert the non-terminal spin symbols to terminal spin symbols. Thus for every derivation the first part must take the form

\[ S_{2d} \rightarrow S_i \rightarrow \ldots \rightarrow (s_0^0)^\dagger s_0^0 \ldots s_{n-1}^0 \rightarrow s_n^0 \dagger \bullet_i \quad \text{(C51)} \]

where in order to indicate their position in the square lattice, the spin symbols are labeled \( s_n \ldots s_1 \) as they will reverse order during the remaining steps of the derivation. The symbol \( \perp \) is used to indicate a line break. Note that line breaks are not present in the final spin configuration, but can only be inferred from the number of spins contained in the given configuration. Consequently \( \perp \) is a non-terminal symbol, i.e. will be removed by some production before we arrive at the terminal expression.

Next observe that for every \( s_0^0 \) and every \( \perp \) we can generate a new \( s_2^h(s_1, s_2) \) right of the \( \perp \) symbol, i.e. in the next line, by using production rule (C48d). In fact this is the only option available in the next step of the derivation. We can either continue applying production rules to the spin symbols of the first line, those that are left of the first \( \perp \), or we move on with the second line, i.e. those symbols that are placed to the right of the first \( \perp \). As the final result of any derivation that eventually yields a terminal expression is independent of this choice, w.l.o.g. we continue by applying rules to the first line.

The only possibility for the next rule is (C48e). This rule allows the symbols \((s')^r\) to be moved left of the symbols \( \bar{s}_i \), such that it is possible to continue with rule (C48d). Proceeding by using only these two rules one obtains the following expression

\[ (s_0^0)^\dagger (s_1^e)^r \ldots (s_{n-1}^0)\perp s_{2n-1}^0 \ldots s_{n+1}^0 \perp n-2 \bullet_i \quad \text{(C52)} \]

Next one must apply production rule (C48f) \((n-1)\)-times to move all spin symbols symbols \((s')^r\) to the left of \( \bar{s}_i \). Once this is done they are marked as \((s')^n\). Now it is possible to apply rule (C48g) to obtain in total

\[ (s_1^e)^r \ldots (s_{n-1}^0)^n s_{2n}^0 \dagger s_{2n-1}^0 \ldots s_{n+1}^0 \perp n-2 \bullet_i \quad \text{(C53)} \]

Note that all these productions really must be applied to the spin symbols of the given line as otherwise it is not possible to get rid of the non-terminal symbol \( \perp \) and hence not possible to obtain a terminal expression.

Proceeding in exactly the same fashion for all the remaining lines of the spin configuration, except for the last one, yields the intermediate expression

\[ (s_1^e)^r \ldots (s_{n-1}^0)^n s_{2n}^0 \dagger \ldots \]

\[ (s_{n+1}^0)^r \ldots (s_{2n-1}^0)^n s_{2n}^0 \dagger \ldots \]

\[ \ldots \]

\[ (s_{n}^0)^r \dagger \ldots \]

\[ s_{2n}^0 \dagger \ldots \dagger \perp s_{n+1}^0 \dagger s_{n+2}^0 \dagger \ldots \]

To get rid of the non-terminal \((s_0^0)^\perp\) in the last line one additionally has to use (C48h) and (C48i). Together with the previous rules this transforms the last line into

\[ (s_0^0)^\dagger s_{n-1}^0 \ldots (s_{n-1}^0)^n s_{n-1}^0 \quad \text{(C55)} \]

Observe that all spin symbols except for those in the last line now store an energy. It is straightforward to conclude that for spin \( i \leq n^2 - n \) the energy that is stored by the corresponding symbol \( e_i \) equals \( h(s_i, s_{i+n}) \). Put differently, all energy contributions that correspond to vertical edges (see Fig. 4c), i.e. those in \((E_{2d})_{n^2}^{\text{hor}}\) defined in Eq. (26) are correctly taken into account. To that end assume spin \( i \) is the \( r \)-th spin in line \( j \), with \( 1 \leq r \leq n \) and \( 1 \leq j \leq n - 1 \), so \( i = (j-1)n + r \). Thus, before rules (C48d), (C48e), (C48f) and (C48g) are applied to the spins in line \( j \) and consequently they reverse order, spin \( i \) is located at position \( \hat{r} := n - r + 1 \)

\[ \ldots (s_{(j-1)n+n}^0)^\dagger s_{(j-1)n+n-1}^0 \ldots \]

\[ \ldots (s_i^0)^\dagger s_{(j-1)n+n-1}^0 \perp j \bullet_i \quad \text{(C56)} \]

Now in line \( j \) there are \( n - \hat{r} = r - 1 \) spin symbols on the right hand side of \( s_i^0 \). Applying rules (C48d) and (C48e) to these yields \( r - 1 \) spin symbols in line \( j + 1 \). Next, rule (C48d) must be applied to spin \( i \)

\[ s_i^0 \perp \rightarrow (s_i^0)^{h(s_i,s_i)} \perp s_i^0 \quad \text{(C57)} \]

and after that also the remaining \( n - r + 1 \) spins of line \( j \) are treated in the same fashion. Moreover, in order to obtain a terminal expression, either rules (C48d), (C48e), (C48f) and (C48g), if \( j + 1 < n \) or rules (C48e), (C48f) and (C48h) or (C48i), if \( j + 1 = n \) must be applied to line \( j + 1 \). In any case also the spins in line \( j \) reverse order, such that once the configuration admits its final order there are \( r - 1 \) spin symbols left of \( s_i^0 \), i.e. we have \( i = jn + r = i + n \). Note that the symbols of the last line at that point do not store an energy.

The intermediate expression obtained so far is a string that consists of \( n^2 \) spin symbols followed by a \( \bullet \), all spin symbols except for the last ones in each line, \( s_{kn}^0 \) are marked by a \((n)^r\) and each spin correctly stores one of its at most two contributions to the overall energy. In order to remove the \((n)^r\) mark, one has to apply rule (C48j) in each row independently, starting from the last two spins of each row, successively moving to the left one spin at a time. Note that then all energy contributions \( h(s_i, s_{i+1}) \), i.e. those that are correspond to horizontal edges contained in \((E_{2d})_{n^2}^{\text{ver}}\) defined in Eq. (26) (see also Fig. 4c) are correctly taken into account and stored in the spin symbols. Hence after completing this step, we have an intermediate configuration of spin symbols \( s_i^e \), where \( e_i \) equals the energy that spin \( s_i \) contributes. Summing the stored energies over all spins thus yields the correct energy of the spin configuration, i.e.

\[ \sum_{i=1}^{n^2} e_i = H_{2d}(s_1 \ldots s_{n^2}) \quad \text{(C58)} \]

Finally, the production rules (C48k) and (C48l) allow one to move the stored energies freely between spin symbols, and further, if the two energies of two neighbouring spin symbols differ in their sign, they allow to add these and
store the sum at the first symbol, whereas the second now stores zero as energy. Using these rules, it is possible to achieve one of the following cases, and which case applies solely depends on the sign of the overall energy of the configuration:

1. \( \forall i = 1, \ldots, n^2, \ e_i \geq 0 \iff H_{2d}(s_1 \ldots s_{n^2}) > 0 \)
2. \( \forall i = 1, \ldots, n^2, \ e_i \leq 0 \iff H_{2d}(s_1 \ldots s_{n^2}) < 0 \)
3. \( \forall i = 1, \ldots, n^2, \ e_i = 0 \iff H_{2d}(s_1 \ldots s_{n^2}) = 0 \)

The three cases correspond to the three symbols \( \bullet, \cdot, \cdot_0 \). Note that the distinction between the three options \( i = +, -, 0 \), i.e. the sign of the energy of the final spin configuration, is made in the very first step of each production. Rules (C48n), (C48o) and (C48p) ensure that only those energies that are consistent with the chosen option can be separated from the spin symbols, yielding the terminal spin symbols \( s_i \). Thereby this entire mechanism prevents the total energy of a terminal expression that can be derived from \( S_{2d} \) from containing both symbols + and −. Only in the last step the final rule (C48q) can be used, as otherwise it is not possible to obtain terminal spin symbols.

In total, any terminal expression that can be derived by \( G_{2d} \) is of the form

\[
s_1 \ldots s_{n^2}: \bullet u(H(s_1 \ldots s_{n^2}))
\]

for some \( n^2 \in N_{2d} \), and hence an element of \( L_{2d} \).

On the other hand, proving that \( L_{2d} \subseteq L(G_{2d}) \) is straightforward. First note that rule (C48c) can be applied an arbitrary number of times, yielding a configuration of arbitrary size \( n^2 \in N_{2d} \). Second, observe that for both productions that invoke a new spin symbol, rule (C48d) and rule (C48g), the state of the new spin can be chosen arbitrarily. Last, by the previous arguments, the productions ensure that all terminal expressions have the correct energy right of the \( \bullet \) sign. Thus, all elements of \( L_{2d} \) can be derived by \( G_{2d} \).

\[\square\]

[1] E. Ising, Beitrag zur Theorie des Ferromagnetismus, Zeitschrift für Physik 31, 253 (1925).
[2] R. K. Pathria, Statistical Mechanics (Elsevier, Second edition, 1996).
[3] B. McCoy and T. T. Wu, The Two-Dimensional Ising model, 2nd ed. (Dover, 2013).
[4] J. Ambjørn, K. N. Anagnostopoulos, R. Loll, and I. Pushinka, Shaken, but not stirred—Potts model coupled to quantum gravity, Nucl. Phys. B 807, 251 (2009).
[5] D. Chandler, Introduction to Modern Statistical Mechanics (Oxford University Press, 1987).
[6] L. H. Kauffman, Knots and Physics (World Scientific, Singapore, 2001).
[7] J. J. Hopfield, Neural networks and physical systems with emergent collective computational abilities, Proc. Natl. Acad. Sci. 79, 2554 (1982).
[8] D. J. Amit, H. Gutfreund, and H. Sompolinsky, Spin-glass models of neural networks, Phys. Rev. A 32, 1007 (1985).
[9] R. V. Solé and B. Goodwin, Signs of Life: How complexity pervades biology (Basic Books, 2000).
[10] W. Bialeka, A. Cavagna, I. Giardina, T. Mora, E. Silvestri, M. Viale, and A. M. Walczak, Statistical mechanics for natural flocks of birds, Proc. Natl. Acad. Sci. 109, 4786 (2012).
[11] P. W. Anderson, Suggested model for prebiotic evolution: The use of chaos, Proc. Natl. Acad. Sci. USA 80, 3386 (1983).
[12] P. Tarazona, Error thresholds for molecular quasispecies as phase transitions: From simple landscapes to spin-glass models, Phys. Rev. A 45, 6038 (1992).
[13] T. R. Lezon, J. R. Banavar, M. Cieplak, A. Maritan, and N. V. Fedoroff, Using the principle of entropy maximization to infer genetic interaction networks from gene expression patterns, Proc. Natl. Acad. Sci. 103, 50 (2006).
[14] A. Bak and J. S. Høy, One-dimensional Ising model applied to protein folding, Physica A 323, 504 (2003).
[15] M. Ekeberg, C. Löökvist, Y. Lan, M. Weigt, and E. Aurell, Improved contact prediction in proteins: Using pseudolikelihoods to infer Potts models, Phys. Rev. E 87, 012707 (2013).
[16] I. Leuthäusser, An exact correspondence between Eigen’s evolution model and a two-dimensional Ising system, J. Chem. Phys. 84, 1884 (1986).
[17] I. Leuthäusser, Statistical mechanics of Eigen’s evolution model, J. Stat. Mech. 48, 343 (1987).
[18] F. Rizzato, A. Coucke, E. de Leonardis, J. P. Barton, J. Tubiana, R. Monasson, and S. Cocco, Inference of compressed Potts graphical models, Phys. Rev. E 101, 012309 (2020).
[19] D. Stauffer, Social applications of two-dimensional Ising models, Am. J. Phys. 76, 10.1119/1.2779882 (2008).
[20] E. DeGiuli, Random Language Model, Phys. Rev. Lett. 122, 128301 (2019).
[21] C. Castellano, S. Fortunato, and V. Loretto, Statistical physics of social dynamics, Rev. Mod. Phys. 81, 591 (2009).
[22] A. Jiménez, K. F. Tiamo, and A. M. Posadas, An Ising model for earthquake dynamics, Nonlin. Processes Geophys. 14, 5 (2007).
[23] E. D. Lee, C. P. Brodersz, and W. Bialek, Statistical Mechanics of the US Supreme Court, J. Stat. Phys. 160, 275 (2015).
[24] F. Barahona, On the computational complexity of Ising spin glass models, J. Phys. A 15, 3241 (1982).
[25] S. Istrail, Statistical Mechanics, three-dimensionality and NP-completeness: Universality of Intractability of the Partition Functions of the Ising Model Across Non-Planar Lattices, in STOC’00 (Portland, Oregon: ACM Press, 2000) pp. 87–96.
[26] P (NP) is the class of decision problems solvable in polynomial time by a (non)-deterministic Turing machine.
[27] C. Moore and S. Mertens, The nature of computation (Oxford University Press, 2011).
[28] M. Mezard and A. Montanari, *Information, Physics, And Computation* (Oxford Graduate Texts, 2009).
[29] A. Lucas, Ising formulations of many NP problems, *Frontiers in Physics* 2, 1 (2014).
[30] S. Stengele, D. Drexel, and G. De las Cuevas, Classical spin Hamiltonians are context-sensitive languages, (2022), arXiv:2006.03529.
[31] G. De las Cuevas, A quantum information approach to statistical mechanics – a tutorial, *J. Phys. B* 46, 243001 (2013).
[32] D. C. Kozen, *Automata and Computability* (Springer, 1997).
[33] G. Rozenberg, ed., *Handbook of graph grammars and computing by graph transformations* (World Scientific, 1997).
[34] G. De las Cuevas and T. S. Cubitt, Simple universal models capture all classical spin physics, *Science* 351, 1180 (2016).
[35] G. De las Cuevas, Universality everywhere implies undecidability everywhere, *FQXi Essay* (2020).
[36] S. Stengele, T. Reinhart, T. Gonda, and G. De las Cuevas, A framework for universality across disciplines, In preparation (2022).
[37] N. Chomsky, Formal Properties of Language, in *Handbook of Mathematical Psychology*, edited by D. Luce (John Wiley & Sons., 1963) p. 2.
[38] N. Chomsky, *Aspects of the theory of syntax* (MIT Press, 1965).
[39] G. Rozenberg, G. R. A. Salomaa, and A. Salomaa, *Handbook of Formal Languages: Volume 1. Word, Language, Grammar*, Handbook of Formal Languages (Springer, 1997).
[40] J. E. Hopcroft and J. D. Ullman, *Introduction to Automata Theory, Languages, and Computation* (Addison-Wesley Publishing Company, 1979).
[41] J. Rothe, *Complexity Theory and Cryptology: An Introduction to Cryptocomplexity* (2005).
[42] J. W. Thatcher, Generalized sequential machine maps, *JCSS* 4, 339 (1970).
[43] N. Chomsky, On certain formal properties of grammars, *Inf. Control* 2, 137 (1959).