AN $\epsilon$ CHARACTERIZATION OF A VERTEX FORMED BY TWO NON-OVERLAPPING GEODESIC ARCS ON SURFACES WITH CONSTANT GAUSSIAN CURVATURE

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Abstract. We determine a positive real number (weight) which corresponds to the intersection point (vertex) of two non-overlapping geodesic arcs, which depends on the two weights which correspond to two points of these geodesic arcs, respectively, and an infinitesimal number $\epsilon$. As a limiting case for $\epsilon \to 0$, the triad of the corresponding weights yields a degenerate weighted Fermat-Torricelli tree which coincides with these two geodesic arcs. By applying this process for a geodesic triangle on a circular cone, we derive an $\epsilon$ characterization of conical points in $\mathbb{R}^3$.

1. Introduction

Let $\triangle A_1A_2A_3$ be a geodesic triangle on a surface $S$ with constant Gaussian curvature in $\mathbb{R}^3$. We denote by $A_0$ a point on $S$, by $(a_{ij})_g$ the length of the geodesic arc $A_iA_j$ by $\angle A_iA_jA_k$ the angle formed by the geodesic arcs $A_iA_j$ and $A_jA_k$, at the vertex $A_j$, by $\vec{U}_{ij}$ is the unit tangent vector of the geodesic arc $A_iA_j$ at $A_i$, for $i, j, k = 0, 1, 2, 3$ and by $B_i$ a positive real number (weight), which corresponds to each vertex $A_i$, for $i = 1, 2, 3$.

We state the weighted Fermat-Torricelli problem for a $\triangle A_1A_2A_3$ on $S$.

Problem 1. Find a point $A_0$ (weighted Fermat-Torricelli point) on $S$, such that

$$f(A_0) = \sum_{i=1}^{3} B_i(a_{i0})_g \to \min. \quad (1.1)$$

The solution of the weighted Fermat-Torricelli problem is called a branching solution (weighted Fermat-Torricelli tree) and consists of the three geodesic branches $\{A_1A_0, A_2A_0, A_3A_0\}$ which meet at the weighted Fermat-Torricelli point $A_0$.

The following two propositions give a characterization of the weighted Fermat-Torricelli point $A_0$ on a smooth surface, which have been proved in [2,4].

Proposition 1 (Floating Case). [2,4] The following (I), (II), (III) conditions are equivalent:

(I) All the following inequalities are satisfied simultaneously:

$$\left\|B_2\vec{U}_{12} + B_3\vec{U}_{13}\right\| > B_1, \quad (1.2)$$
or \((1.7)\), we obtain a degenerate weighted Fermat-Torricelli tree

\[
\left\| B_1 \vec{U}_{21} + B_3 \vec{U}_{31} \right\| > B_2, \quad (1.3)
\]

\[
\left\| B_1 \vec{U}_{31} + B_2 \vec{U}_{32} \right\| > B_3, \quad (1.4)
\]

(II) The point \(A_0\) is an interior point of the triangle \(\triangle A_1A_2A_3\) and does not belong to the geodesic arcs \(A_iA_j\) for \(i, j = 1, 2, 3\).

(III) \(B_1 \vec{U}_{01} + B_2 \vec{U}_{02} + B_3 \vec{U}_{03} = 0\).

**Proposition 2** (Absorbed Case). \([2, 4]\) The following (I), (II) conditions are equivalent:

(I) One of the following inequalities is satisfied:

\[
\left\| B_2 \vec{U}_{12} + B_3 \vec{U}_{13} \right\| \leq B_1, \quad (1.5)
\]

or

\[
\left\| B_1 \vec{U}_{21} + B_3 \vec{U}_{23} \right\| \leq B_2, \quad (1.6)
\]

or

\[
\left\| B_1 \vec{U}_{31} + B_2 \vec{U}_{32} \right\| \leq B_3, \quad (1.7)
\]

(II) The point \(A_0\) is attained at \(A_1\) or \(A_2\) or \(A_3\), respectively.

The inverse weighted Fermat-Torricelli problem on \(S\) states that:

**Problem 2.** \([2, 4]\) Given a point \(A_0\) which belongs to the interior of \(\triangle A_1A_2A_3\) on \(M\), does there exist a unique set of positive weights \(B_i\), such that

\[B_1 + B_2 + B_3 = c = \text{const},\]

for which \(A_0\) minimizes

\[f(A_0) = B_1(a_{10})_g + B_2(a_{20})_g + B_3(a_{30})_g\]

A positive answer w.r. to the inverse weighted Fermat-Torricelli problem on a \(C^2\) complete surface is given by the following proposition \([2, 4]\):

**Proposition 3.** \([2, 4]\) The weight \(B_i\) are uniquely determined by the formula:

\[
B_i = \frac{C}{1 + \frac{\sin \angle A_i A_0 A_j}{\sin \angle A_i A_0 A_k} + \frac{\sin \angle A_i A_0 A_k}{\sin \angle A_j A_0 A_k}}, \quad (1.8)
\]

for \(i, j, k = 1, 2, 3\) and \(i \neq j \neq k\).

If \(B_1, B_2, B_3\) satisfy the inequalities of the floating case, we derive a weighted Fermat-Torricelli tree \(\{A_1A_0, A_2A_0, A_3A_0\}\).

The location of the weighted Fermat-Torricelli (floating) tree for geodesic triangles on the \(K\) plane (sphere \(S^2\), hyperboloid \(H^2\)) is given in \([5, 6]\) and an analytical solution of the weighted Fermat-Torricelli problem for an equilateral geodesic triangle with equal lengths \(\frac{\pi}{4}\) is given in \([8]\), for the weighted floating case. Concerning, the solution of the weighted Fermat-Torricelli problem for geodesic triangles on flat surfaces of revolution (Circular cylinder, circular cone) we refer to \([7]\).

If \(B_1, B_2, B_3\) satisfy one of the inequalities of the absorbed case \((1.5)\) or \((1.6)\) or \((1.7)\), we obtain a degenerate weighted Fermat-Torricelli tree \(\{A_2A_1, A_1A_3\}, \{A_1A_2, A_2A_3\}\) and \(\{A_1A_3, A_3A_2\}\), respectively.

For instance, if \((1.5)\) is valid, the minimum value of \(B_1\) is determined by:
\[ B_1^2 = B_2^2 + B_3^2 + 2B_2B_3 \cos \angle A_2A_1A_3 \]

or

\[ B_1 = f(B_2, B_3). \]

Thus, we consider the following problem:

**Problem 3.** How can we determine the values of \( B_2, B_3 \), such that \( f(B_2, B_3) \) gives the minimum value of \( B_1 \) that corresponds to the vertex \( A_1 \) on a surface with constant Gaussian curvature \( S \)?

In this paper, we determine the value of \( B_1 \) by introducing an infinitesimal real number \( \epsilon \), \( (\epsilon \text{ characterization of } A_1) \) such that: \( \angle A_1A_2A_0 = \|\epsilon\|, \angle A_2A_0A_3 = \angle A_2A_1A_3 + 2\epsilon, \angle A_0A_3A_1 = k\|\epsilon\| \) for a rational number \( k \), by applying the solution of the inverse weighted Fermat-Torricelli problem on \( S \). By setting \( A_1 \) to be the vertex of a (right) circular cone, we give an \( \epsilon \) characterization of the vertices of a triangle in \( \mathbb{R}^3 \).

2. **An \( \epsilon \) characterization of the vertices of a triangle in \( \mathbb{R}^2 \)**

Let \( A_0 \) be an interior point of \( \triangle A_1A_2A_3 \) in \( \mathbb{R}^2 \).

We denote by \((a_{ij})_0\) the length of the linear segment \( A_iA_j \).

We set \( \angle A_1A_2A_0 = \epsilon, \angle A_2A_0A_3 = \angle A_2A_1A_3 + 2\epsilon, \angle A_0A_3A_1 = k\|\epsilon\| \).

**Theorem 1.** The weight \( B_1 = B_1(\epsilon) \) are uniquely determined by the formula:

\[ B_1 = \frac{C}{1 + \sin \angle A_1A_2A_3 + \sin \angle A_1A_0A_3}, \quad (2.1) \]

\[ B_2 = \frac{C}{1 + \sin \angle A_2A_1A_3 + \sin \angle A_2A_0A_3}, \quad (2.2) \]

\[ B_3 = \frac{C}{1 + \sin \angle A_3A_1A_2 + \sin \angle A_3A_0A_2}, \quad (2.3) \]

where

\[ \angle A_2A_0A_3 = \angle A_2A_1A_3 + 2\epsilon \]

\[ \angle A_1A_0A_2 = \angle A_1A_0A_2(\epsilon) = \arccot\left(\frac{(a_{13}0) + (a_{12})_0 \cos(\angle A_2A_1A_3 + 2\epsilon)}{(a_{12})_0 \sin(\angle A_2A_1A_3 + 2\epsilon)}\right), \quad (2.5) \]

\[ \angle A_1A_0A_3 = 2\pi - \angle A_2A_1A_3 - 2\epsilon - \angle A_1A_0A_2(\epsilon). \quad (2.6) \]

**Proof.** From \( \triangle A_2A_0A_3 \), we obtain:

\[ \angle A_1A_2A_3 - \epsilon + \angle A_2A_1A_3 + 2\epsilon + \angle A_1A_3A_2 - k\epsilon = \pi, \]

which yields

\[ k = 1. \]

By applying the law of sines in \( \triangle A_0A_2A_1, \triangle A_0A_3A_1 \), we derive:
\[
\frac{(a_{01})_0}{\sin \epsilon} = \frac{(a_{12})_0}{\sin \angle A_1 A_0 A_2} = \frac{(a_{13})_0}{-\sin(\angle A_2 A_1 A_3 + 2\epsilon + \angle A_1 A_0 A_2)}. \tag{2.7}
\]

From (2.7), we obtain (2.5).

By replacing (2.5), (2.4), (2.6) in (1.8), we obtain \( B_i = B_i(\epsilon) \), for \( i = 1, 2, 3 \).

\[ \square \]

**Corollary 1.** For \( \epsilon \to 0 \), we derive a degenerate weighted Fermat-Torricelli tree \( \{A_2 A_1, A_1 A_3\} \).

3. **An \( \epsilon \) characterization of the vertices of a geodesic triangle on the \( K \)-plane**

Let \( \triangle A_1 A_2 A_3 \) be a geodesic triangle on the \( K \) plane. The \( K \) plane is a sphere \( S_K^2 \) of radius \( R = \frac{1}{\sqrt{|K|}} \) and a hyperbolic plane \( H_K^2 \) with constant Gaussian curvature \( -K \) for \( K < 0 \) in \( \mathbb{R}^3 \).

We set

\[ \kappa = \begin{cases} \sqrt{|K|} & \text{if } K > 0, \\ i\sqrt{-K} & \text{if } K < 0. \end{cases} \]

The unified law of cosines and law of sines on the \( K \) plane is given in [1].

We set \( \angle A_1 A_2 A_0 = ||\epsilon||, \angle A_2 A_0 A_3 = \angle A_2 A_1 A_3 + 2\epsilon, \angle A_0 A_3 A_1 = \frac{||\epsilon||}{2} \).

We set

\[ \epsilon = \begin{cases} ||\epsilon|| & \text{if } K > 0, \\ -||\epsilon|| & \text{if } K < 0. \end{cases} \]

**Theorem 2.** The weight \( B_i = B_i(\epsilon) \) are uniquely determined by the formula:

\[
B_1 = \frac{C}{1 + \frac{\sin \angle A_1 A_0 A_3}{\sin \angle A_2 A_0 A_3} + \frac{\sin \angle A_1 A_0 A_3}{\sin \angle A_2 A_0 A_3}}, \tag{3.1}
\]

\[
B_2 = \frac{C}{1 + \frac{\sin \angle A_2 A_0 A_3}{\sin \angle A_1 A_0 A_3} + \frac{\sin \angle A_2 A_0 A_3}{\sin \angle A_1 A_0 A_3}}, \tag{3.2}
\]

\[
B_3 = \frac{C}{1 + \frac{\sin \angle A_3 A_0 A_3}{\sin \angle A_1 A_0 A_2} + \frac{\sin \angle A_3 A_0 A_3}{\sin \angle A_1 A_0 A_2}}, \tag{3.3}
\]

where

\[
\angle A_2 A_0 A_3 = \angle A_2 A_1 A_3 + 2\epsilon \tag{3.4}
\]

\[
\angle A_1 A_0 A_2 = \arccot \left( \frac{\sin(\kappa(a_{13})_g) + 2 \sin(\kappa(a_{12})_g) \cos(\frac{||\epsilon||}{2}) \cos(\angle A_2 A_1 A_3 + 2\epsilon)}{2 \sin(\kappa(a_{12})_g) \cos(\frac{||\epsilon||}{2}) \sin(\angle A_2 A_1 A_3 + 2\epsilon)} \right), \tag{3.5}
\]

\[
\angle A_1 A_0 A_3 = 2\pi - \angle A_2 A_1 A_3 - 2\epsilon - \angle A_1 A_0 A_2(\epsilon). \tag{3.6}
\]

**Proof.** By applying the law of sines in \( \triangle A_0 A_2 A_1 \), we get:

\[
\frac{\sin(\kappa(a_{10})_g)}{\sin ||\epsilon||} = \frac{\sin(\kappa(a_{12})_g)}{\sin \angle A_2 A_0 A_1} \tag{3.7}
\]

or
\[
\frac{\sin(\kappa(a_{10}g))}{\sin \frac{\beta}{2}} \cdot \frac{1}{2 \cos \frac{\beta}{2}} = \frac{\sin(\kappa(a_{12}g))}{\sin \angle A_2 A_0 A_1}.
\]

(3.8)

By applying the law of sines in \(\triangle A_0 A_1 A_3\), we get:
\[
\frac{\sin(\kappa(a_{10}g))}{\sin \frac{\beta}{2}} = -\frac{\sin(\kappa(a_{13}g))}{\sin(\angle A_2 A_1 A_3 + 2\epsilon + \angle A_2 A_0 A_1)}.
\]

(3.9)

By replacing (3.9) in (3.8), we derive:
\[
-\frac{\sin(\kappa(a_{13}g))}{\sin(\angle A_2 A_1 A_3 + 2\epsilon + \angle A_2 A_0 A_1)} \frac{1}{2 \cos \frac{\beta}{2}} = \frac{\sin(\kappa(a_{12}g))}{\sin \angle A_2 A_0 A_1}.
\]

(3.10)

By solving (3.10) w.r. to \(\angle A_2 A_0 A_1\), we obtain (3.5).

\[\square\]

4. An \(\epsilon\) characterization of the vertices of a geodesic triangle on flat surfaces of revolution

In this section, we shall give an \(\epsilon\) characterization of the vertices of geodesic triangles on flat surfaces of revolution (Circular Cylinder \(S\) and Circular Cone \(C\)) which are flat Euclidean Surfaces with zero Gaussian curvature.

4.1. An \(\epsilon\) characterization of the vertices of a geodesic triangle on a circular cylinder.

The parametric form of a (right) circular cylinder \(S\) of unit radius and axis (axis of revolution) the \(z\)-axis:
\[
\vec{r}(u, v) = (\cos v, \sin v, u).
\]

The geodesics of the circular cylinder are the straight lines on the circular cylinder parallel to the \(z\)-axis, the circles obtained by intersecting the circular cylinder with planes parallel to the \(xy\)-plane and circular helixes of the parametric form \(\vec{r}(t) = (\cos t, \sin t, b t + c)\).

Let \(\triangle A_1 A_2 A_3\) be a geodesic triangle on \(S\) which is composed of three circular helixes.

We set \(A_1 = (1, 0, 0)\), \(A_2 = (\cos \varphi_2, \sin \varphi_2, z_2)\) \(A_3 = (\cos \varphi_3, \sin \varphi_3, z_3)\) and \(\vec{r}_{ij} = (\cos t, \sin t, b_{ij} t)\) the circular helix on \(S\) from \(A_i\) to \(A_j\) for \(i, j = 1, 2, 3\), \(i \neq j\) and \(\varphi_2, \varphi_3 \in (0, \pi)\).

The coefficient \(b_{ij}\) is called the step of the helix from \(A_i\) to \(A_j\). The step of the helices \(b_{12}\) from \(A_1\) to \(A_2\) and \(b_{13}\) from \(A_1\) to \(A_3\) are given by:
\[
b_{12} = \frac{z_2}{\varphi_2}
\]

and
\[
b_{13} = \frac{z_3}{\varphi_3}
\]

A cylindrical law of cosines for geodesic triangles on \(S\) composed of three circular helixes is given in [7].

Proposition 4. [7] The following formula holds for \(\triangle A_1 A_2 A_3\) on \(S\):
\[
(1 + b_{23}^2)(\varphi_2 - \varphi_3)^2 = (1 + b_{12}^2)\varphi_2^2 + (1 + b_{13}^2)\varphi_3^2 - 2\sqrt{(1 + b_{12}^2)(1 + b_{13}^2)\varphi_2\varphi_3 \cos \alpha_{213}}.
\]

(4.1)

We denote by \((a_{ij})_S\) the length of the geodesic arc from \(A_i\) to \(A_j\).
Theorem 3. The weight $B_i = B_i(\epsilon)$ are uniquely determined by the formula:

$$B_1 = \frac{C}{1 + \sin \angle A_1 A_0 A_3 + \sin \angle A_1 A_0 A_2},$$  \hspace{1cm} (4.2)

$$B_2 = \frac{C}{1 + \sin \angle A_1 A_0 A_3 + \sin \angle A_1 A_0 A_2},$$  \hspace{1cm} (4.3)

$$B_3 = \frac{C}{1 + \sin \angle A_1 A_0 A_2 + \sin \angle A_1 A_0 A_3},$$  \hspace{1cm} (4.4)

where

$$\angle A_2 A_0 A_3 = \angle A_2 A_1 A_3 + 2\epsilon$$  \hspace{1cm} (4.5)

$$\angle A_1 A_0 A_2 = \angle A_1 A_0 A_2(\epsilon) = \arccot\left(-\frac{\sqrt{1 + b_{12}^2 \varphi_3} + \sqrt{1 + b_{12}^2 \varphi_2} \cos(\angle A_2 A_1 A_3 + 2\epsilon)}{\sqrt{1 + b_{12}^2 \varphi_2} \sin(\angle A_2 A_1 A_3 + 2\epsilon)}\right),$$  \hspace{1cm} (4.6)

$$\angle A_1 A_0 A_3 = 2\pi - \angle A_2 A_1 A_3 - 2\epsilon - \angle A_1 A_0 A_2(\epsilon).$$  \hspace{1cm} (4.7)

Proof. Unrolling the cylinder $S$ in terms of the vertex $A_1$, we derive an isometric mapping from $S$ to $\mathbb{R}^2$, which yields:

$$(a_{ij})_S = (a_{ij})_0, \text{ for } i, j = 1, 2, 3.$$  \hspace{1cm} (4.8)

From (4.8), we have:

$$(a_{12})_S = (a_{12})_0$$  \hspace{1cm} (4.9)

and

$$(a_{13})_S = (a_{13})_0$$  \hspace{1cm} (4.10)

The Euclidean distances $(a_{12})_0$ and $(a_{13})_0$ are given by:

$$(a_{12})_0 = \sqrt{z_2^2 + \varphi_2^2}$$  \hspace{1cm} (4.11)

or

$$(a_{12})_0 = \sqrt{b_{12}^2 + 1\varphi_2}$$  \hspace{1cm} (4.12)

and

$$(a_{13})_0 = \sqrt{z_3^2 + \varphi_3^2}$$  \hspace{1cm} (4.13)

or

$$(a_{13})_0 = \sqrt{b_{13}^2 + 1\varphi_3}$$  \hspace{1cm} (4.14)

Thus, the length of the circular helix with parametric form $\vec{r}_{12}$ from $A_1$ to $A_2$ and with parametric form $\vec{r}_{13}$ from $A_1$ to $A_3$ is given by:

$$(a_{12})_S = \sqrt{1 + b_{12}^2 \varphi_2}.$$  \hspace{1cm} (4.15)

and

$$(a_{13})_S = \sqrt{1 + b_{13}^2 \varphi_3}.$$  \hspace{1cm} (4.16)
By replacing (4.15) and (4.16) in 2.5, we obtain (4.6) and by applying Theorem 1, we derive (4.2), (4.3) and (4.4). The weights $B_1, B_2, B_3$ depend on the angle $\angle A_2 A_1 A_3, \epsilon$ and the step of the helices $b_{12}$ and $b_{13}$.

4.2. An $\epsilon$ characterization of the vertices of a geodesic triangle on a circular cone. We consider the parametric form of a (right) circular cone $C$ with a unit base radius.

$$\vec{r}(u, v) = \left( (1 - \frac{u}{H}) \cos v, (1 - \frac{u}{H}) \sin v, u \right), \quad 0 < u \leq H, 0 < v < 2\pi.$$ 

The geodesic equations on $C$ are given in [3, Exercise 5.2.14, Subsection 5.6.2 pp. 222, pp. 247-248].

Let $\triangle A_1 A_2 A_3$ be a geodesic triangle on $C$.

We denote by $P$ the center of the unit bases circle of $S'$, $H$ the distance $AP$, by $A_i p$ the intersection of the line defined by the linear segment $AA_i$ with the unit bases circle $c(P, 1)$ for $i = 0, 1, 2, 3$, by $\varphi_2$ the angle $\angle A_1 PA_2 p$ by $\varphi_3$ the angle $\angle A_1 PA_3 p$ and by $\varphi_0$ the angle $\angle A_1 PA_0 p$.

By unrolling the circular cone $C$ w.r. to $A_1 A$, (cut along $A_1 A$) we derive an isometric mapping from $C$ to $\mathbb{R}^2$.

Thus, we get:

$$(a_{ij})_0 = (a_{ij})_0 \quad (4.17)$$

By setting $A_1 = (0, 0)$, we obtain:

$$\varphi = \frac{2\pi}{\sqrt{1 + H^2}} \quad (4.18)$$

$$\angle A_1 AA_2 = \frac{\varphi_2}{\sqrt{1 + H^2}}, \quad (4.19)$$

$$\angle A_1 AA_3 = \frac{\varphi_3}{\sqrt{1 + H^2}} \quad (4.20)$$

and

$$\angle A_1 AA_0 = \frac{\varphi_0}{\sqrt{1 + H^2}}. \quad (4.21)$$

where

$$A_i A = \sqrt{1 + H^2}. \quad (4.22)$$

**Theorem 4.** The weight $B_i = B_i(\epsilon)$ are uniquely determined by the formula:

$$B_1 = \frac{C}{1 + \frac{\sin \angle A_1 A_2 A_3}{\sin \angle A_2 A_0 A_3} + \frac{\sin \angle A_1 A_3 A_2}{\sin \angle A_3 A_0 A_2}}, \quad (4.23)$$

$$B_2 = \frac{C}{1 + \frac{\sin \angle A_2 A_0 A_3}{\sin \angle A_1 A_0 A_3} + \frac{\sin \angle A_2 A_3 A_0}{\sin \angle A_3 A_1 A_0}}, \quad (4.24)$$

$$B_3 = \frac{C}{1 + \frac{\sin \angle A_3 A_0 A_2}{\sin \angle A_1 A_0 A_2} + \frac{\sin \angle A_3 A_2 A_0}{\sin \angle A_2 A_1 A_0}}, \quad (4.25)$$

where

$$\angle A_2 A_0 A_3 = \angle A_2 A_1 A_3 + 2\epsilon. \quad (4.26)$$
\[ \angle A_1 A_0 A_2 = \angle A_1 A_0 A_2(\epsilon) = \arccot\left(-\frac{(a_{13})_c + (a_{12})_c \cos(\angle A_2 A_1 A_3 + 2\epsilon)}{(a_{12})_c \sin(\angle A_2 A_1 A_3 + 2\epsilon)}\right), \] 
(4.27)

and

\[ \angle A_1 A_0 A_3 = 2\pi - \angle A_2 A_1 A_3 - 2\epsilon - \angle A_1 A_0 A_2(\epsilon). \]
(4.28)

\[ (a_{12})_c = \sqrt{(1 + H^2) + (A_2 A)^2 - 2\sqrt{1 + H^2}(A_2 A) \cos(\frac{\varphi_2}{\sqrt{1 + H^2}})}, \]
(4.29)

\[ (a_{13})_c = \sqrt{(1 + H^2) + (A_3 A)^2 - 2\sqrt{1 + H^2}(A_3 A) \cos(\frac{\varphi_3}{\sqrt{1 + H^2}})}. \]
(4.30)

**Proof.** From the law of cosines in \( \triangle A_1 A A_2 \), we get:

\[ (a_{12})_c = \sqrt{(A_1 A) + (A_2 A)^2 - 2(A_1 A)(A_2 A) \cos(\angle A_1 A A_2)}. \]
(4.31)

By replacing (4.22) and (4.19) in (4.31), we obtain (4.29). From the law of cosines in \( \triangle A_1 A A_3 \), we get:

\[ (a_{13})_c = \sqrt{(A_1 A)^2 + (A_3 A)^2 - 2A_1 A(A_3 A) \cos(\angle A_1 A A_3)}. \]
(4.32)

By replacing (4.22) and (4.20) in (4.32), we obtain (4.30). Unrolling \( C \) along \( A_1 A \) yields an isometric mapping from \( C \) to \( \mathbb{R}^2 \) and by applying theorem 1, we obtain (4.33), (4.34) and (4.35). The weights \( B_1, B_2, B_3 \) depend on \( \epsilon, \varphi_2, \varphi_3 \) and \( H \).

\[ \angle A_2 A_0 A_3 = \angle A_2 A_1 A_3 + 2\epsilon \]
(4.36)

Remark 1. The deviation of \( \|B_1 - B_1(\epsilon)\| \) where

\[ B_1 = \sqrt{B_2(\epsilon)^2 + B_3(\epsilon)^2 + 2B_2(\epsilon)B_3(\epsilon) \cos \angle A_3 A_1 A_2} \]
gives an error estimate which depends on \( \epsilon \).
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