RECTIFIABILITY AND MINKOWSKI BOUNDS FOR THE ZERO LOCI OF $\mathbb{Z}/2$ HARMONIC SPINORS IN DIMENSION 4

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Abstract. This article proves that the zero locus of a $\mathbb{Z}/2$ harmonic spinor on a 4-dimensional manifold is 2-rectifiable and has locally finite Minkowski content.

1. Introduction

1.1. Background. The notion of $\mathbb{Z}/2$ harmonic spinors was first introduced by Taubes [Tau12, Tau14] to describe the behaviour of certain non-convergent sequences of flat $PSL_2(\mathbb{C})$ connections on a three manifold. It also appears in the compactifications of the moduli spaces of solutions to Kapustin-Witten equations [Tan13], Vafa-Witten equations [Tau17], and Seiberg-Witten equations with multiple spinors [HW15, Tau16]. These equations may have important topological applications. For example, Witten [Wit14] has conjectured that the space of solutions to the Kapustin-Witten equations can be used to compute the Jones polynomials and the Khovanov homology for knots. Haydys [Hay17] conjectured a relation between the multiple spinor Seiberg-Witten monopoles, Fueter sections, and $G_2$ instantons. More recently, Doan and Walpuski [DW17] conjectured a relation between generalized Seiberg-Witten equations and counting of associative manifolds on $G_2$ manifolds.

All of these applications require better understanding of the compactifications for the relevant moduli spaces. The zero locus of $\mathbb{Z}/2$ harmonic spinors plays a crucial role in the description of the boundaries of the compactifications. It is the set of points where the sequence of solutions blow up after normalizations. Takahashi [Tak15, Tak17] studied the moduli spaces of $\mathbb{Z}/2$ harmonic spinors with additional regularity assumptions on the zero locus, where the zero locus was assumed to be a union of embedded circles in the case of dimension 3, and an embedded surface in the case of dimension 4. In general, the zero locus may not have this regularity. Taubes [Tau14] proved that the zero locus must have Hausdorff codimension at least 2. This article improves the regularity result by proving that the zero locus is rectifiable and has locally finite Minkowski content. The arguments are inspired by [DLMSV16], where a similar problem was studied for Dir-minimizing $Q$-valued functions. The proof relies on a general method developed recently by Naber and Valtorta [NV15].

1.2. Statement of results. Let $X$ be a 4-dimensional Riemannian manifold. Let $\mathcal{V}$ be a Clifford bundle over $X$. That is, $\mathcal{V}$ is a unitary vector bundle equipped with an extra structure $\rho \in \text{Hom}(TX, \text{Hom}(\mathcal{V}, \mathcal{V}))$, such that $\rho(e)^2 = -\|e\|^2 \cdot \text{id}$ and $\|\rho(e)(u)\| = \|e\| \cdot \|u\|$ for every $e \in T_pX$ and $u \in \mathcal{V}|_p$. Let $\nabla$ be a connection on $\mathcal{V}$ that is compatible with $(X, \mathcal{V}, \rho)$. Namely, for every pair of smooth vector fields $e$,
where $\rho''$, and every smooth section $u$ of $\mathcal{V}$, one has
\[ \nabla_e(\rho'' \cdot u) = \rho(\nabla_e') \cdot u + \rho'' \cdot \nabla_e(u). \]
The Dirac operator on $\mathcal{V}$ is defined by
\[ D(u) = \sum_{i=1}^{4} \rho(e_i) \nabla_{e_i} u, \]
where $\{e_i\}$ is a local orthonormal frame for $TX$.

Let $Q$ be a positive integer. For a vector space $E$, define $A_Q(E)$ to be the set of unordered $Q$-tuples of points in $E$. If $P_1, P_2, \ldots, P_Q$ are $Q$ points in $E$, use $\sum_{i=1}^{Q} [P_i] \in A_Q(E)$ to denote the $Q$-tuple given by the collection of $P_i$'s. If $E$ is endowed with a Euclidean metric, one can define a metric on $A_Q(E)$ by
\[ \text{dist}\left( \sum_{i} [P_i], \sum_{i} [S_i] \right) = \min_{\sigma \in P_Q} \sqrt{\sum_{i} |P_i - S_{\sigma(i)}|^2}, \]
where $P_Q$ is the permutation group of $\{1, 2, \ldots, Q\}$. If $T \in A_Q(E)$, define $|T| = \text{dist}(T, Q[0])$.

A map from $X$ is called a $Q$-valued section of $\mathcal{V}$ if it maps every $x \in X$ to an element of $A_Q(\mathcal{V}^{|_x})$. A $Q$-valued section is called continuous if it is continuous under local trivializations of $\mathcal{V}$.

**Definition 1.1.** Let $U$ be a continuous $2$-valued section of $\mathcal{V}$. Then $U$ is called a $\mathbb{Z}/2$ harmonic spinor if the following conditions hold.

1. $U$ is not identically $2[0]$.
2. Let $Z$ be the set of $U$ where $U = 2[0]$. For every $x \in X - Z$, there exists a neighborhood of $x$, such that on this neighborhood $U$ can be written as $U = \{u\} + \{-u\}$, where $u$ is a smooth section of $\mathcal{V}$ satisfying $D(u) = 0$.
3. Near a point $x \in X - Z$, write $U$ as $\{u\} + \{-u\}$, then the function $|\nabla u|$ is a well defined smooth function on $X - Z$. The section $U$ satisfies
\[ \int_{X - Z} |\nabla u|^2 < \infty. \]

This definition is equivalent to the definition of $\mathbb{Z}/2$ harmonic spinors given in [Tan14].

For a point $x \in X$ and $r > 0$, use $B_x(r)$ to denote the geodesic ball in $X$ with center $x$ and radius $r$. As in (1.5) of [Tan14], we make the following additional assumption on $U$.

**Assumption 1.2.** There exists a constant $\epsilon > 0$ such that the following holds. For every $x \in X$ with $U(x) = 2[0]$, there exist constants $C, r_0 > 0$, depending on $x$, such that
\[ \int_{B_{x}(r)} |U(y)|^2 \, dy < C \cdot r^{4+\epsilon}, \quad \text{for every } r \in (0, r_0). \]

Assume $U$ is a $\mathbb{Z}/2$ harmonic spinor, and let $Z$ be the set of $U$ where $U = 2[0]$. Taubes [Tan14] proved the following theorem.

**Theorem 1.3** (Taubes [Tan14]). If $U$ satisfies assumption 1.2, then the Hausdorff dimension of $Z$ is at most 2.

This article improves theorem 1.3 to the following result.
Theorem 1.4. If $U$ satisfies assumption 1.2, then $Z$ is a 2-rectifiable set. Moreover, for every compact subset $A \subset X$, there exist constants $C$ and $r_0$ depending on $A$ and $Z$, such that for every $r < r_0$,

$$\text{Vol} \left( \{ x : \text{dist}(x, A \cap Z) < r \} \right) < C \cdot r^2.$$ 

In other words, $Z$ is a 2-rectifiable set with locally finite 2 dimensional Minkowski content. Since the Minkowski content controls the Hausdorff measure, theorem 1.4 implies that $Z$ has locally finite 2 dimensional Hausdorff measure.

Theorem 1.4 immediately implies that the zero locus of a $\mathbb{Z}/2$ harmonic spinor on a 3-manifold is 1-rectifiable and has locally finite Minkowski content.

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2. $\mathbb{Z}/2$ harmonic spinors as Sobolev sections

Almgren [AJ00] developed a Sobolev theory for $Q$-valued functions on $\mathbb{R}^m$. For a quicker introduction, one can see for example [DLS11]. For an open set $\Omega \subset \mathbb{R}^m$, the space $W^{1,2}(\Omega, A_Q)$ is defined to be the space of $Q$ valued functions $T$ on $\Omega$, such that $|T| \in L^2(\Omega)$, and that $T$ has distributional derivatives which are also in $L^2(\Omega)$. The Sobolev theory extends to $Q$-valued sections of vector bundles without any difficulty. This section proves the following lemma.

Lemma 2.1. If $U$ is a $\mathbb{Z}/2$ harmonic spinor, then $U$ is in $W^{1,2}(X, A_2)$. Moreover, $D(U) = 0$ in the distributional sense.

This lemma allows us to study the compactness properties of $\mathbb{Z}/2$ harmonic spinors by the Sobolev theory for $Q$-valued functions.

We start with the following definition.

Definition 2.2. Let $T$ be a $Q$-valued section of $V$. It is called a smooth $Q$-valued section, if for every $x \in X$, there exists a neighborhood of $x$ on which $T$ can be written as

$$T = \sum_{i=1}^Q \lfloor f_i \rfloor,$$

where $f_i$’s are smooth sections of $V$.

If $T$ is a smooth $Q$-valued section and is locally written as $\sum_i \lfloor f_i \rfloor$, then the function $\sum_i |f_i|^2 + \sum_i |\nabla f_i|^2$ is well defined on $X$. In this case, the $W^{1,2}$ norm of $T$ is given by $(\int_X \sum_i |f_i|^2 + \sum_i |\nabla f_i|^2)^{1/2}$.

Proof of lemma 2.1. The proof is essentially the same as lemma 2.4 of [Tao14]. Let $\chi$ be a smooth non-increasing function on $\mathbb{R}$, such that $\chi(t) = 1$ when $t \leq 1$, and $\chi(t) = 0$ when $t \geq 2$. For $s > 0$, let $\tau_s = \chi(|\ln |U||/\ln s)$. Then $\tau_s(x) = 0$ when $|U(x)| \leq s^2$, and $\tau_s(x) = 1$ when $|U(x)| \geq s$.

The section $\tau_s U$ is a 2-valued smooth section of $V$. Recall that on $X - Z$, the $\mathbb{Z}/2$ harmonic spinor $U$ can be locally written as $U = [u] + [-u]$. Although $u$ is
Proof. Since Lemma 2.3. Let \( U \) be a \( Z/2 \) harmonic spinor. Then there exists a sequence of smooth sections \( U_i \), such that \( U_i = -U_i \), and
\[
\lim_{i \to \infty} U_i = U \text{ in } W^{1,2}.
\]

**Proof.** Since \( |U| \) and \( |\nabla U| \) are zero on the Lebesgue points of \( Z \), one has
\[
\|U\|_{W^{1,2}} = \int_{X-Z} (|U|^2 + |\nabla U|^2) = \sqrt{2} \int_{X-Z} (|u|^2 + |\nabla u|^2).
\]

Define \( \tau_s \) as in the proof of Lemma 2.1. It was proved previously that there is a sequence \( s_i \to 0 \), such that \( \tau_{s_i}U \) converges weakly to \( U \) in \( W^{1,2} \). As a consequence,
\[
\liminf_{i \to \infty} \|\tau_{s_i}U\|_{W^{1,2}} \geq \|U\|_{W^{1,2}}.
\]

On the other hand, by (1),
\[
\lim_{i \to \infty} \|\tau_{s_i}U\|_{W^{1,2}} = \sqrt{2} \int_{X-Z} (|u|^2 + |\nabla u|^2) = \|U\|_{W^{1,2}}.
\]

Therefore \( \tau_{s_i}U \) converges strongly to \( U \) in \( W^{1,2} \). \( \square \)

3. Frequency functions

The frequency functions were first introduced by Angenent [AJ79] to study the singular set of elliptic partial differential equations, and they were adapted by Taubes [Tau14] to study the zero loci of \( Z/2 \) harmonic spinors. This section recalls some results about the frequency functions from [Tau14].

Let \( U \) be a \( Z/2 \) harmonic spinor. On \( X-Z \) the section \( U \) can be locally written as \( U = [u] + [-u] \). As before, we will use notations like \( |u| \) and \( |\nabla u| \) to denote the corresponding functions on \( X-Z \) if they can be globally defined. The functions \( |u| \) and \( |\nabla u| \) extend to \( X \) by defining them to be zero on \( Z \).

The following \( C^0 \) estimate was established in [Tau14].
Lemma 3.1 ([Tau14], Lemma 2.3). Let $A \subset B$ be two open subsets of $X$, and assume the closure of $A$ is compact and contained in $B$. Then there exists a constant $K$, depending on $A$, $B$ and the norms of the curvatures of $X$ and $\mathcal{V}$, such that

$$\sup_{x \in A} |u(x)|^2 \leq K \int_B |u(x)|^2 \, dx.$$ 

Now introduce some notations. Fix a point $x_0 \in X$. Take $R > 0$ such that $B_{x_0}(500R) \subset X$ is complete, and that the injectivity radius of $X$ is greater than $1000R$ for every point in the ball $B_{x_0}(500R)$. 

Later on we will need to work on both the Euclidean space and the manifold $X$, so we need to differentiate the notations. We will use $B_x(r)$ to denote the geodesic ball on $X$ with center $x \in X$ and radius $r > 0$. Use $B_x(\tau_1, (3.6)$ and Lemma 3.2

Lemma 3.2

function on $X$ continuous with respect to $r$.

Section 3(a) of [Tau14] proved the following monotonicity properties for $x$ ball with center $1000$ $X$ ball on $K$ $A$ $B$ $B$ $A$ $x$, depending on $R$. Take $R$, use the normal coordinate centered at $x$, and define the frequency function

then $H$ is the largest and smallest eigenvalue of $g_x$. So that for every $x \in B_{x_0}(500R)$, $z \in \bar{B}(500R)$,

$$\left(\frac{11}{12}\right)^2 \leq \kappa_x(z) \leq K_x(z) \leq \left(\frac{12}{11}\right)^2$$

In order to prove theorem [14] one only needs to study the rectifiability and the Minkowski content of $Z \cap B_{x_0}(R/2)$.

For $x \in B_{x_0}(500R)$, $r \in (0, 500R]$, define the height function

$$H(x, r) = \int_{\partial B_x(r)} |u|^2,$$

then $H(x, r)$ is always positive [Tau14] Lemma 3.1]. Define

$$D(x, r) = \int_{B_x(r)} |\nabla u|^2,$$

and define the frequency function

$$N(x, r) = \frac{rD(x, r)}{H(x, r)}.$$ 

Section 3(a) of [14] proved the following monotonicity properties for $N$ and $H$:

Lemma 3.2 ([Tau14], (3.6) and Lemma 3.2). The functions $N$ and $H$ are absolutely continuous with respect to $r$, and there exist constants $\kappa > 0$ and $r_0 > 0$, depending only on the norms of curvatures of $X$ and $\mathcal{V}$ on $B_{x_0}(1000R)$, such that when $r \leq r_0$,

$$\frac{\partial}{\partial r} H \geq \frac{3}{r} H - \kappa r H,$$ (3)

$$\frac{\partial}{\partial r} N \geq -\kappa r (1 + N).$$ (4)

$$\left(\frac{N}{r} + \kappa r\right) \frac{H}{r^3} \geq \frac{\partial}{\partial r} \left(\frac{H}{r^3}\right) \geq \left(\frac{N}{r} - \kappa r\right) \frac{H}{r^3}$$ (5)

By shrinking the size of $R$, we assume without loss of generality that $r_0 = 500R$, hence inequalities (3), (4), and (5) hold for all $x \in B_{x_0}(500R)$ and $r \leq 500R$. 

Inequality (3) gives the following lemma
Lemma 3.3 (\cite{Tau14}, Lemma 3.1). There exists a constant $\kappa > 0$, such that when $s \leq r < 500R$,
$$H(x, r) \geq \left(\frac{r}{s}\right)^3 \cdot e^{-\kappa(r^2 - s^2)} \cdot H(x, s).$$

Inequality (4) gives

Lemma 3.4. There exists a constant $\kappa > 0$, such that when $s \leq r < 500R$,
$$N(x, r) \geq e^{-\kappa(r^2 - s^2)} N(x, s) - \kappa(r^2 - s^2).$$

Since $N(x, 500R)$ is continuous with respect to $x$, lemma 3.4 implies that $N(x, r)$ is bounded for all $x \in B_{x_0}(500R)$, $r \leq 500R$. Let $\Lambda$ be an upper bound for $N$. From now on $\Lambda$ will be treated as a constant. For the rest of this article, unless otherwise stated, $C$, $C_1$, $C_2$, $\cdots$ will denote positive constants that depend on $\Lambda$, $R$, and the norms of the curvatures of $X$ and $V$, but independent of $U$. The values of $C$, $C_1$, $C_2$, $\cdots$ may be different in different appearances.

If $|g| \leq C \cdot f$ for some constant $C$, we write $g = O(f)$.

Inequality (5) then implies that there exists a constant $C$ such that
$$\left| \frac{\partial}{\partial r} \left( \ln \left( \frac{H}{r^3} \right) \right) \right| = O\left( \frac{1}{r} \right).$$

Inequality (4) implies that there exists $C > 0$, such that whenever $r \geq s$,
$$N(x, r) \geq N(x, s) - C(r^2 - s^2).$$

4. Smoothed frequency functions

We need to use a modified version of frequency functions. Let $\phi$ be a non-increasing smooth function on $\mathbb{R}$ such that $\phi(t) = 1$ when $t \leq 3/4$, and $\phi(t) = 0$ when $t \geq 1$. From now on $\phi$ will be fixed, hence the values of $\phi$ and its derivatives are considered as universal constants. Following \cite{DLMSV16}, we define the smoothed frequency functions as follows.

Definition 4.1. For $x \in X$, let $\nu_x$ be the gradient vector field of the distance function $d(x, \cdot)$. For $x \in B_{x_0}(500R)$, $r \leq 500R$, introduce the following functions

$$D_\phi(x, r) = \int |\nabla u(y)|^2 \phi \left( \frac{d(x, y)}{r} \right) dy,$$
$$H_\phi(x, r) = -\int |u(y)|^2 d(x, y)^{-1} \phi' \left( \frac{d(x, y)}{r} \right) dy,$$
$$N_\phi(x, r) = \frac{r D_\phi(x, r)}{H_\phi(x, r)},$$
$$E_\phi(x, r) = -\int |\nabla \nu_x u(y)|^2 d(x, y) \phi' \left( \frac{d(x, y)}{r} \right) dy.$$

Inequality (6) has the following useful corollary.

Lemma 4.2. There exists a constant $C$ with the following property. Let $r \in (0, 32R]$. Assume $s_1 \leq 10r$, $s_2 \geq r/10$. Then for any two points $x$, $y$ with $d(x, y) \leq r$, one has
$$H_\phi(x, s_1) \leq C(H_\phi(y, s_2)).$$
Proof. Since the constant $K$ in lemma 3.1 only depends on the norms of the cur-
vatures and the sets $A, B$, a rescaling argument gives
\[
|u(z)|^2 \leq \frac{C_1}{r^4} \int_{B_x(r)} |u|^2, \quad \forall B_x(r) \subseteq B_{x_0}(500R).
\]
Therefore for every $z \in \partial B_{x}(s_1)$,
\[
|u(z)|^2 \leq \frac{C_2}{r^4} \int_{B_x(12r)} |u|^2.
\]
On the other hand, inequality (6) and lemma 3.3 gives
\[
\frac{1}{r^4} \int_{B_y(12r)} |u|^2 \leq \frac{C_3}{r^3} H(y, s_2).
\]
Therefore
\[
H(x, s_1) = O(H(y, s_2)).
\]
Apply (6) again, one obtains
\[
H(y, s_2) = O(H_\phi(y, s_2)),
\]
\[
H_\phi(x, s_1) = O(H(x, s_1)),
\]
hence the lemma is proved. \hfill \Box

Lemma 4.3. For $x \in B_{x_0}(32R)$, $r \leq 32R$, one has
\[
\int_{B_x(r)} |u(y)|^2 dy = O(r H_\phi(x, r)),
\]
\[
\int_{B_x(r)} |u(y)||\nabla u(y)| dy = O(H_\phi(x, r)),
\]
\[
\int_{B_x(r)} |\nabla u(y)|^2 dy = O\left(\frac{1}{r} H_\phi(x, r)\right).
\]
Proof. The first equation follows from inequality (6) and lemma 3.3. For the third,
\[
\int_{B_x(r)} |\nabla u(y)|^2 dy \leq D_\phi(x, 2r)
\]
\[
= \frac{1}{2r} N_\phi(x, 2r) H_\phi(x, 2r)
\]
\[
= O(\frac{1}{r} H_\phi(x, r)).
\]
The second equation then follows from Cauchy’s inequality. \hfill \Box

The main result of this section is the following proposition.

Proposition 4.4. The functions $D_\phi$, $H_\phi$, $N_\phi$, and $E_\phi$ are smooth in both variables. Assume $x \in B_{x_0}(32R)$, $r \leq 32R$, and $v \in T_x(X)$. Consider the normal coordinate centered at $x$ with radius $r$, extend the vector $v$ to a vector field on $B_x(r)$ by requiring that the coordinate functions of $v$ are constants. Then the following equations hold
\[
D_\phi(x, r) = -\frac{1}{r} \phi' \left( \frac{d(x, y)}{r} \right) \nabla_{\nu_x} u(y) \cdot u(y) dy + O(r H_\phi(x, r)),
\]
(7)
\[
\partial_r D_\phi(x, r) = \frac{2}{r} D_\phi(x, r) + \frac{2}{r^2} E_\phi(x, r) + O(H_\phi(x, r)),
\]
(8)
\[
\partial_r D_\phi(x, r) = -\frac{2}{r^2} \int \phi'(\frac{d(x, y)}{r}) \nabla_{\nu_x} u(y) \cdot \nabla u(y) \, dy + O(H_\phi(x, r)), \tag{9}
\]
\[
\partial_r H_\phi(x, r) = \frac{3}{r} H_\phi(x, r) + 2D_\phi(x, r) + O(rH_\phi(x, r)), \tag{10}
\]
\[
\partial_r H_\phi(x, r) = -2 \int u(y) \cdot \nabla u(y) \, dy + O(rH_\phi(x, r)). \tag{11}
\]

The smoothness of the functions follows from the fact that \( \phi \) is smooth and \(|u|\), \(|\nabla u|\) are both in \( L^2 \).

**Proof of (7).** It was proved in [Tan14, Section 2(c)] that
\[
\int_{\partial B_x(s)} \nabla_{\nu_x} u(y) \cdot u(y) \, dy = \int_{B_x(s)} |\nabla u(y)|^2 \, dy + \int_{B_x(s)} \langle u(y), R u(y) \rangle \, dy, \tag{12}
\]
where \( R \) is a bounded curvature term from the Weitzenböck formula.

Therefore, by lemma 4.3,
\[
D_\phi(x, r) = -\frac{1}{r} \int_0^r \phi'(\frac{s}{r}) \int_{B_x(s)} |\nabla u(y)|^2 \, dy \, ds
\]
\[
= -\frac{1}{r} \int_0^r \phi'(\frac{d(x, y)}{r}) \nabla_{\nu_x} u(y) \cdot u(y) \, dy + \frac{1}{r} \phi'(\frac{s}{r}) \int_{B_x(s)} \langle u, R u \rangle \, dy \, ds
\]
\[
= -\frac{1}{r} \int \phi'(\frac{d(x, y)}{r}) \nabla_{\nu_x} u(y) \cdot u(y) \, dy + O(rH_\phi(x, r)).
\]

\[\square\]

**Proof of (8).**
\[
\partial_r D_\phi(x, r) = -\frac{1}{r^2} \int |\nabla u(y)|^2 \phi'(\frac{d(x, y)}{r}) \cdot d(x, y) \, dy
\]
\[
= -\frac{1}{r^2} \int_0^r \phi'(\frac{s}{r}) \cdot s \int_{\partial B_x(s)} |\nabla u(y)|^2 \, dy \, ds. \tag{13}
\]

It was proved in [Tan14, Section 2(d)] that
\[
\int_{\partial B_x(s)} |\nabla u(y)|^2 \, dy = 2 \int_{\partial B_x(s)} |\nabla_{\nu_x} u(y)|^2 \, dy + \frac{2}{s} \int_{B_x(s)} |\nabla u(y)|^2 \, dy
\]
\[
+ \frac{2}{s} \int_{B_x(s)} \langle u(y), R u(y) \rangle \, dy - \int_{\partial B_x(s)} \langle R_1 u(y), \nabla u(y) \rangle \, dy + \int_{\partial B_x(s)} \langle u(y), R_2 u(y) \rangle \, dy,
\]

where \( R, R_1, R_2 \) are smooth tensors, \( R \) and \( R_2 \) are bounded, the norm of \( R_1 \) is bounded by \( C_1 \cdot r \).

Notice that
\[
- \int_0^r \phi'(\frac{s}{r}) \cdot s \int_{\partial B_x(s)} |\nabla_{\nu_x} u(y)|^2 \, dy \, ds = E_\phi(x, r),
\]
\[
- \frac{1}{r} \int_0^r \phi'(\frac{s}{r}) \int_{B_x(s)} |\nabla u(y)|^2 \, dy \, ds = D_\phi(x, r).
\]

Plug into equation (13), we have
\[
\partial_r D_\phi(x, r) = \frac{2}{r} D_\phi(x, r) + \frac{2}{r^2} E_\phi(x, r) - \frac{1}{r^2} \int_0^r \phi'(\frac{s}{r}) \cdot s \cdot \frac{2}{s} \int_{B_x(s)} \langle u(y), R u(y) \rangle \, dy.
\]
Hence the result is proved. □

Lemma 4.3 implies

\[-\frac{1}{r^2} \int_0^r \phi'(\frac{s}{r}) \cdot s \cdot \left[ \frac{2}{s} \int_{\partial B_x(s)} \langle u(y), \nabla u(y) \rangle \right] ds \]

On the other hand,

\[
\left| -\frac{1}{r^2} \int_0^r \phi'(\frac{s}{r}) \cdot s \cdot \left[ \frac{2}{s} \int_{\partial B_x(s)} \langle R_1 u(y), \nabla u(y) \rangle \right] ds \right| \\
\leq C_2 \cdot \int_0^r \left| \phi'(\frac{s}{r}) \right| \int_{\partial B_x(s)} |u(y)||\nabla u(y)| dy ds \\
\leq C_3 \int_{B_x(r)} |u(y)||\nabla u(y)| dy = O(H_\phi(x, r)).
\]

Hence the result is proved. □

Proof of 4.5. For a function $G(x, y)$ defined on $X \times X$ and a vector field $w$, use $\frac{\partial}{\partial x}$ to denote the directional derivative with respect to $x$, use $\frac{\partial}{\partial y}$ to denote the directional derivative with respect to $y$.

The first variation formula of geodesic lengths gives

\[
\frac{\partial}{\partial v} d(x, y) + \frac{\partial}{\partial v} d(x, y) = O(d(x, y)^2).
\]

We have

\[
\frac{\partial x}{\partial v} \phi(x, r) = \frac{1}{r} \int |\nabla u(y)|^2 \phi' \left(\frac{d(x, y)}{r}\right) \cdot \frac{\partial x}{\partial v} d(x, y) dy \\
= -\frac{1}{r} \int |\nabla u(y)|^2 \phi' \left(\frac{d(x, y)}{r}\right) \cdot \frac{\partial y}{\partial v} d(x, y) dy + O(r) \int_{B_x(r)} |\nabla u(y)|^2 \\
= -\int |\nabla u(y)|^2 \cdot \frac{\partial y}{\partial v} \phi \left(\frac{d(x, y)}{r}\right) dy + O(H_\phi(x, r)). \tag{14}
\]

One needs to establish the following lemma.

**Lemma 4.5.** Let $F$ be the curvature of $\mathcal{V}$, and $\{e_i\}$ be an orthonormal basis of $TX$. Let $\phi$ be a smooth function with supp $\phi \subset B_x(r)$. Then

\[
\int |\nabla u|^2 \partial_v \phi \\
= 2 \int \langle d\phi \otimes \nabla_v u, \nabla_u \rangle - 2 \sum_i \phi(F(v, e_i)u, \nabla_{e_i} u) - 2 \int \sum_i \phi(\nabla_{[v, e_i]} u, \nabla_{e_i} u) \\
- \int |\nabla u|^2 \phi \text{ div}(v) + 2 \sum_i \phi(\nabla_v u, \nabla_{v e_i} u) \\
+ 2 \sum_i \phi(\nabla_v u, \nabla_{e_i} u) \text{ div}(e_i) + 2 \int \phi(\nabla_v u, R_0 u),
\]

where $R_0$ is the curvature term in the Weitzenböck formula.
Proof of lemma 4.5. By lemma 2.3, there exists a sequence of smooth 2-valued section $U_i$, such that $U_i = -U_i$ and $U_i \to U$ in $W^{1,2}$. By partitions of unity, integration by parts works for $U_i$. For any $U_i$, locally write it as $[w] + [-w]$ where $w$ is a smooth section of $V$. Then

$$
\int |\nabla w|^2 \partial_v \varphi
= - \int \sum_i \varphi \nabla_v (\nabla_{e_i} w, \nabla_{e_i} w) - \int |\nabla w|^2 \varphi (v)
= -2 \int \sum_i \varphi \nabla_v (\nabla_{e_i} w, \nabla_{e_i} w) - 2 \int \sum_i \varphi (F(v, e_i) w, \nabla_{e_i} w)
- 2 \int \sum_i \varphi (\nabla_{[v,e_i]} w, \nabla_{e_i} w) - \int |\nabla w|^2 \varphi (v)
$$

Here $F$ denotes the curvature of $V$. For the first term in the formula above,

$$
\int \sum_i \varphi (\nabla_{e_i} \nabla_v w, \nabla_{e_i} w)
= - \int \sum_i (\nabla_{e_i} \varphi) \langle \nabla_v w, \nabla_{e_i} w \rangle - \int \sum_i \varphi (\nabla_v w, \nabla_{e_i} \nabla_{e_i} w)
- \int \sum_i \varphi (\nabla_v w, \nabla_{e_i} w) \text{div}(e_i)
= - \int \sum_i (\nabla_{e_i} \varphi) \langle \nabla_v w, \nabla_{e_i} w \rangle + \int \sum_i \varphi (\nabla_v w, \nabla_{e_i} \nabla w)
- \int \sum_i \varphi (\nabla_v w, \nabla_{e_i} w) \text{div}(e_i)
$$

For the second term in the formula above, let $R_0$ be the curvature term in the Weitzenböck formula, then

$$
\int \sum_i \varphi (\nabla_v w, \nabla_{e_i} \nabla w) = \int \langle \varphi \nabla_v w, D^2 w - R_0 w \rangle
= - \int \varphi (\nabla_v w, R_0 w) + \int \langle \rho (\nabla \varphi) \nabla_v w, Dw \rangle - \int \langle \varphi (\nabla_v (D w), Dw) \rangle
- \int \langle \varphi (\nabla_v (D w), Dw) \rangle
- \frac{1}{2} \int \partial_v \varphi |Dw|^2 - \frac{1}{2} \int \varphi |Dw|^2 \text{div}(v)
$$

Therefore

$$
\int |\nabla w|^2 \partial_v \varphi
= -2 \int \sum_i \varphi (F(v, e_i) w, \nabla_{e_i} w) - 2 \int \sum_i \varphi (\nabla_{[v,e_i]} w, \nabla_{e_i} w) - \int |\nabla w|^2 \varphi \text{div}(v)
+ 2 \int \sum_i (\nabla_{e_i} \varphi) \langle \nabla_v w, \nabla_{e_i} w \rangle + 2 \int \sum_i \varphi (\nabla_v w, \nabla_{e_i} \nabla_{e_i} w) + 2 \int \sum_i \varphi (\nabla_v w, \nabla_{e_i} w) \text{div}(e_i)
$$
Take limit $U_i \to U$, one has
\[
\int |\nabla u|^2 \partial_v \varphi = -2 \sum_i \varphi(F(v, e_i)u, \nabla e_i u) - 2 \sum_i \varphi(\nabla_{[v, e_i]} u, \nabla e_i u) - \int |\nabla u|^2 \varphi \text{div}(v)
\]
\[
+ 2 \sum_i (\nabla_{e_i} \varphi)(\nabla v u, \nabla e_i u) + 2 \sum_i \varphi(\nabla_v u, \nabla v_{e_i} u) + 2 \sum_i \varphi(\nabla_v u, \nabla e_i u) \text{div}(e_i)
\]
\[
+ 2 \int \varphi(\nabla v u, \mathcal{R}_0 u) - 2 \int (\varphi(\nabla v u, Du) + 2 \int \varphi(\nabla v u, Du) + 2 \int \varphi(\nabla v u, \mathcal{R}_0 u)
\]
\[
+ \int \partial_v \varphi |Du|^2 - \int \varphi |Du|^2 \text{div}(v)
\]
\[
= -2 \sum_i \varphi(F(v, e_i)u, \nabla e_i u) - 2 \sum_i \varphi(\nabla_{[v, e_i]} u, \nabla e_i u) - \int |\nabla u|^2 \varphi \text{div}(v)
\]
\[
+ 2 \sum_i (\nabla_{e_i} \varphi)(\nabla v u, \nabla e_i u) + 2 \sum_i \varphi(\nabla_v u, \nabla v_{e_i} u) + 2 \sum_i \varphi(\nabla_v u, \nabla e_i u) \text{div}(e_i) + 2 \int \varphi(\nabla v u, \mathcal{R}_0 u)
\]

Notice that
\[
\sum_i (\nabla_{e_i} \varphi)(\nabla v u, \nabla e_i u) = (d \varphi \otimes \nabla v u, \nabla u),
\]
therefore the lemma is proved. \qed

Back to the proof of equation (10). Take $\varphi(y) = \phi(d(x, y)/r)$. By Lemma 4.3
\[
-2 \sum_i \varphi(F(v, e_i)u, \nabla e_i u) + 2 \int \varphi(\nabla v u, \mathcal{R}_0 u) = O(H_\phi(x, r)).
\]

On the other hand, $|\text{div}(v)| = O(r)$, and one can choose $\{e_i\}$ such that $|v, e_i| = O(r), |\text{div}(e_i)| = O(r), \text{and } |\nabla e_i e_i| = O(r)$. Thus by lemma 4.3
\[
-2 \sum_i \varphi(\nabla_{[v, e_i]} u, \nabla e_i u) - \int |\nabla u|^2 \varphi \text{div}(v) + 2 \sum_i \varphi(\nabla_v u, \nabla v_{e_i} e_i) + 2 \sum_i \varphi(\nabla_v u, \nabla e_i u) \text{div}(e_i) = O(H_\phi(x, r)).
\]

Equation (10) then follows immediately from equation (14) and lemma 4.5 \qed

Proof of (10). By [Tao14 Equation (2.11)],
\[
\partial_s H(x, s) = \frac{3}{8} H(x, s) + 2D(x, s) + \int_{B_x(s)} (u, \mathcal{R} u) + \int_{\partial B_x(s)} t |u|^2, \quad (15)
\]
where \( \mathcal{R} \) is a curvature term from the Weitzenböck formula, and \( t \) comes from the mean curvature of \( \partial B_x(s) \). The function \( t \) satisfies \( |t(y)| = O(d(x, y)) \). Notice that
\[
H_{\phi}(x, r) = \int_0^r -\phi'(s/r) \cdot \frac{1}{s} \cdot H(s) \, ds = \int_0^1 -\phi'(\lambda) \frac{1}{\lambda} \cdot H(\lambda r) \, d\lambda.
\]
Therefore
\[
\partial_r H_{\phi}(x, r) = \int_0^1 -\phi'(\lambda) \cdot (\partial_r H)(\lambda r) \, d\lambda
\]
\[
= \int_0^1 -\phi'(\lambda) \left[ \frac{3}{\lambda r} H(x, \lambda r) + 2D(x, \lambda r) + \int_{\partial B_x(\lambda r)} \langle u, \mathcal{R} u \rangle + \int_{\partial B_x(\lambda r)} t |u|^2 \right] \, d\lambda
\]
\[
= -\frac{1}{r} \int_0^r \phi'(s/r) \left[ \frac{3}{s} H(x, s) + 2D(x, s) + \int_{B_x(s)} \langle u, \mathcal{R} u \rangle + \int_{\partial B_x(s)} t |u|^2 \right] \, ds
\]
\[
= \frac{3}{r} H_{\phi}(x, r) + 2D_{\phi}(x, r) - \frac{1}{r} \int_0^r \phi'(s/r) \left[ \int_{B_x(s)} \langle u, \mathcal{R} u \rangle + \int_{\partial B_x(s)} t |u|^2 \right] \, ds
\]
\[
= \frac{3}{r} H_{\phi}(x, r) + 2D_{\phi}(x, r) + O(r H_{\phi}(x, r)).
\]

\( \square \)

Proof of (11). As in the proof of (9), for a function \( G(x, y) \), use \( \frac{\partial x}{\partial v} G \) to denote the directional derivative of \( G \) with respect to \( x \), and use \( \frac{\partial y}{\partial v} G \) to denote the directional derivative with respect to \( y \). Recall that we have
\[
\frac{\partial x}{\partial v} d(x, y) + \frac{\partial y}{\partial v} d(x, y) = O(d(x, y)^2),
\]
therefore
\[
\left( \frac{\partial x}{\partial v} + \frac{\partial y}{\partial v} \right) \left[ d(x, y)^{-1} \phi \left( \frac{d(x, y)}{r} \right) \right] = O(1).
\]
We have
\[
\partial_v H(x, r) = \int |u(y)|^2 \frac{\partial x}{\partial v} d(x, y)^{-1} \phi \left( \frac{d(x, y)}{r} \right) dy
\]
\[
= \int |u(y)|^2 \frac{\partial y}{\partial v} d(x, y)^{-1} \phi \left( \frac{d(x, y)}{r} \right) dy + O(\int_{B_x(r)} |u|^2)
\]
\[
= -\int \frac{\partial}{\partial v} |u(y)|^2 d(x, y)^{-1} \phi \left( \frac{d(x, y)}{r} \right) dy
\]
\[
- \int |u(y)|^2 d(x, y)^{-1} \phi \left( \frac{d(x, y)}{r} \right) \operatorname{div}(v) dy + O(r H_{\phi}(x, r))
\]
\[
= -2 \int u(y) \cdot \nabla_v u(y) d(x, y)^{-1} \phi \left( \frac{d(x, y)}{r} \right) dy + O(r H_{\phi}(x, r)).
\]
The last equality follows from \( |\operatorname{div}(v)| = O(r) \) and \( \int_{B_x(r)} |u|^2 = O(r H_{\phi}(x, r)) \). \( \square \)

Remark 4.6. When both \( X \) and \( V \) are flat, all the curvature terms in the computations above are zero. Therefore, proposition 4.4 becomes
\[
D_{\phi}(x, r) = -\frac{1}{r} \int \phi \left( \frac{d(x, y)}{r} \right) \nabla_{\nu_x} u(y) \cdot u(y) dy,
\]
\[
\partial_r D\phi(x, r) = \frac{2}{r} D\phi(x, r) + \frac{2}{r^2} E\phi(x, r)
\]

\[
\partial_r D\phi(x, r) = -\frac{2}{r} \int \phi' \left( \frac{d(x, y)}{r} \right) \nabla_{\nu^*} u(y) \cdot \nabla_v u(y) \, dy
\]

\[
\partial_r H\phi(x, r) = \frac{3}{r} H\phi(x, r) + 2D\phi(x, r)
\]

\[
\partial_r H\phi(x, r) = -2 \int u(y) \cdot \nabla_v u(y) \, d(x, y)^{-1} \phi' \left( \frac{d(x, y)}{r} \right) \, dy
\]

**Corollary 4.7.** Let \( \eta_x(y) = d(x, y) \cdot \nu_x(y) \). Under the assumptions of proposition 4.4 one has

\[
\partial_r N\phi(x, r) = \frac{2}{H\phi(x, r)} \int -\frac{1}{d(x, y)} \phi' \left( \frac{d(x, y)}{r} \right)
\]

\[
(\nabla_{\eta_x} u(y) - N\phi(x, r) u(y)) \cdot \nabla_v u(y) \, dy + O(r).
\]  

(16)

\[
\partial_r N\phi(x, r) = \frac{2}{rH\phi(x, r)} \int -\phi' \left( \frac{d(x, y)}{r} \right)
\]

\[
d(x, y)^{-1} |\nabla_{\eta_x} u(y) - N\phi(x, r) u(y)|^2 \, dy + O(r).
\]  

(17)

As a consequence, there exists a constant \( C \), such that \( (N\phi(x, r) + Cr^2) \) is increasing in \( r \).

**Proof.** The first equation follows immediately from proposition 4.4 by combining equations (9) and (11). For the first one, lemma 4.4 gives

\[
\partial_r N\phi(x, r) = \frac{2}{rH\phi(x, r)} \left( E\phi(x, r) - \frac{r^2 D\phi(x, r)^2}{H\phi(x, r)} \right) + O(r),
\]

and we have

\[
E\phi(x, r) = \frac{r^2 D\phi(x, r)^2}{H\phi(x, r)}
\]

\[
= E\phi(x, r) - 2r D\phi(x, r) N\phi(x, r) + N\phi(x, r)^2 H\phi(x, r)
\]

\[
= \int -\phi' \left( \frac{d(x, y)}{r} \right) d(x, y)^{-1} |\nabla_{\eta_x} u(y) - N\phi(x, r) u(y)|^2 \, dy + O(r^2 H\phi(x, r))
\]

Hence the second equation is verified.

\[\square\]

5. Compactness

This section proves a compactness result for \( \mathbb{Z}/2 \) harmonic spinors. Consider the ball \( \Omega = \mathcal{B}(5) \subset \mathbb{R}^4 \) centered at the origin. Let \( \mathcal{V} \) be a fixed trivial vector bundle on \( \Omega \). Assume \( g_n \) is a sequence of Riemannian metrics on \( \Omega \), \( A_n \) is a sequence of connection forms on \( \mathcal{V} \), and \( \rho_n \) is a sequence of Clifford bundle structures of \( \mathcal{V} \). Assume that \( (g_n, A_n, \rho_n) \) are compatible, and assume that \( (g_n, A_n, \rho_n) \) converge to \( (g, A, \rho) \) in \( C^\infty \). Assume \( g \) is the Euclidean metric on \( \mathcal{B}(5) \). Then for sufficiently large \( n \), the injectivity radius at each point in \( \mathcal{B}(2) \) is at least 2.5. Without loss of generality, assume that this property holds for every \( n \).

Fix \( \epsilon, \Lambda > 0 \). For every \( n \), assume \( U_n \) is a 2-valued section of \( \mathcal{V} \) defined on \( \mathcal{B}(5) \), with the following properties:
(1) The section $U_n$ is a $\mathbb{Z}/2$ harmonic spinor on $\bar{B}(5)$ with respect to $(g_n, A_n, \rho_n)$.

(2) $U_n$ satisfies assumption [12] with respect to $\epsilon$.

(3) Let $N^{(n)}_\phi$ be the smoothed frequency function for the extended $U_n$. Then whenever $N_\phi(x, r)$ is defined,

$$N^{(n)}_\phi(x, r) \leq \Lambda.$$ 

(4) Let $H^{(n)}_\phi$ be the smoothed height function of $U_n$, then $H^{(n)}_\phi(0, 1) = 1$.

The main result of this section is the following proposition.

**Proposition 5.1.** Let $U_n$ be given as above. Then there exits a subsequence of $\{U_n\}$, such that the sequence converges strongly in $W^{1, 2}((\bar{B}(2))$ to a section $U$. The section $U$ is a $\mathbb{Z}/2$ harmonic spinor on $\bar{B}(2)$ with respect to $(g, A, \rho)$, and $U$ satisfies assumption [12] for a possibly smaller value of $\epsilon$. Moreover, $U_n$ converges to $U$ uniformly on $\bar{B}(2)$.

**Proof.** Fix a trivialization of $\mathcal{V}$, and fix $s \in (0, 0.5)$. The bound on $N^{(n)}_\phi$ and the assumption that $H^{(n)}_\phi(0, 1) = 1$ implies that $\|U\|_{L^2(\bar{B}(2 + s/2))} \leq C_1$ for some constant $C_1$. The upper bound on $N_\phi$ then implies $\|\nabla A_n U\|_{L^2(\bar{B}(2 + s/2))} \leq C_2$. Since $A_n \to A$ in $C^\infty$, this implies $U_n$ is bounded in $W^{1, 2}(\bar{B}(2 + s/2))$. Therefore, there is a subsequence of $\{U_n\}$ which converges weakly in $W^{1, 2}(\bar{B}(2 + s/2))$ and converges strongly in $L^2(\bar{B}(2 + s/2))$. To avoid complicated notations, the subsequence is still denoted by $\{U_n\}$. Denote the limit of $\{U_n\}$ on $\bar{B}(2 + s/2)$ by $U$. Let $H^{(n)}_\phi$, $D^{(n)}_\phi$, $N^{(n)}_\phi$ be the smoothed frequency functions for $U_n$, let $H_\phi$, $D_\phi$, $N_\phi$ be the corresponding functions for $U$. Since $U_n \to U$ strongly in $L^2$, one has $H_\phi(0, 1) = 1$, thus $U$ is not identically 2\([0]$. By [Tan14] Section 3(e)], there exists constants $K > 0$ and $\alpha \in (0, 1)$, depending on $\epsilon$, $\Lambda$, $R$ and the $C^1$ norms of the curvatures of $\{g_n\}$ and $A_n$, such that

$$\|U_n\|_{C^{\alpha}(\bar{B}(2 + s/2))} \leq K. $$

By the Arzela-Ascoli theorem, there exists a further subsequence of $\{U_n\}$ which converges uniformly to $U$ on $\bar{B}(2 + s/2)$. Still denote this subsequence by $\{U_n\}$. Since solutions to the Dirac equation are closed under $C^0$ limits, $U$ is a $\mathbb{Z}/2$ harmonic spinor. $U$ is also Hölder continuous, so it satisfies assumption [12].

Locally write $U_n$ as $[u_n] + [-u_n]$, and write $U$ as $[u] + [-u]$. The weak convergence of $U_n$ to $U$ implies

$$\liminf_{n \to \infty} \int_{\bar{B}(2)} |\nabla A_n u_n|^2 \geq \int_{\bar{B}(2)} |\nabla A u|^2.$$ 

We want to prove that

$$\lim_{n \to \infty} \int_{\bar{B}(2)} |\nabla A_n u_n|^2 = \int_{\bar{B}(2)} |\nabla A u|^2.$$ 

Assume the contrary, then there exists a subsequence of $n$ such that

$$\int_{\bar{B}(2)} |\nabla A_n u_n|^2 \geq \int_{\bar{B}(2)} |\nabla A u|^2 + \delta$$

for some $\delta > 0$. Since $\int_{\bar{B}(r)} |\nabla A u|^2$ is continuous in $r$, and $\int_{\bar{B}(r)} |\nabla A_n u_n|^2$ is non-decreasing in $r$ for every $n$, there exists $r \in (2, 2 + s/2)$ and $\sigma \in (1, (2 + s/2)/r)$,
such that for every \( t \in [2, r] \),
\[
\int_{B(t)} |\nabla A_n u_n|^2 \geq \int_{B(\sigma t)} |\nabla A u|^2 + \delta/2 \tag{18}
\]
Use \( B_n(t) \) to denote the geodesic ball of center 0 and radius \( t \) with metric \( g_n \). Since \( g_n \to g \), we have \( \bar{B}(t) \subset B_n(\sigma t) \) for sufficiently large \( n \). Equation (18) then gives
\[
\int_{B_n(\sigma t)} |\nabla A_n u_n|^2 \geq \int_{B(\sigma t)} |\nabla A u|^2 + \delta/2, \quad \text{for } t \in [2, r] \tag{19}
\]
when \( n \) is sufficiently large.

By equation (15), for every \( t \),
\[
\partial_t H^{(n)}(0, t) = \frac{3}{t} H^{(n)}(0, t) + 2D^{(n)}(0, t) + \int_{B_n(t)} \langle u, R^{(n)}u \rangle + \int_{\partial B_n(t)} t^{(n)}|u|^2,
\]
\[
\partial_t H(0, t) = \frac{3}{t} H(0, t) + 2D(0, t) + \int_{\bar{B}(t)} \langle u, R u \rangle + \int_{\partial \bar{B}(t)} t|u|^2,
\]
where \( R^{(n)} \) and \( t^{(n)} \) are bounded terms that are uniformly convergent to \( R \) and \( t \) as \( n \) goes to infinity. The uniform convergence of \( |u_n| \) and \( g_n \) then imply
\[
\lim_{\sigma \to \infty} \int_{2\sigma}^{\infty} D^{(n)}(0, t) \, dt = \int_{2\sigma}^{\infty} D(0, t) \, dt,
\]
which contradicts (19). In conclusion,
\[
\lim_{n \to \infty} \int_{\bar{B}(2)} |\nabla A_n u_n|^2 = \int_{\bar{B}(2)} |\nabla A u|^2.
\]
Since \((A_n, g_n) \to (A, g) \) in \( C^\infty \), this implies
\[
\lim_{n \to \infty} \|U_n\|_{W^{1, 2}(\bar{B}(2))} = \|U\|_{W^{1, 2}(\bar{B}(2))},
\]
therefore \( U \) convergence strongly to \( U \) in \( W^{1, 2}(\bar{B}(2)) \). \( \square \)

**Corollary 5.2.** Let \( \sigma > 1 \). Let \( g_\ast \) be a metric on \( \mathbb{R}^4 \) given by a constant metric matrix, such that all eigenvalues of the matrix are in the interval \( [\sigma^{-2}, \sigma^2] \).

Assume \( \{(g_n, A_n, \rho_n)\}_{n \geq 1} \) is a sequence of geometric data on \( \bar{B}(5\sigma^2) \), and assume \( (g_n, A_n, \rho_n) \) converge to \((g_\ast, A, \rho) \) in \( C^\infty \). Let \( U_n \) be a \( \mathbb{Z}/2 \) harmonic spinor on \( \bar{B}(5\sigma^2) \) with respect to \((g_n, A_n, \rho_n) \), such that the sequence \( U_n \) satisfies conditions (2) to (4) listed before proposition 5.1. Then a subsequence of \( U_n \) converges to a \( \mathbb{Z}/2 \) harmonic spinor in \( W^{1, 2}(\bar{B}(2)) \) with respect to \((g, A, \rho) \). The limit \( U \) satisfies assumption 5.4 and the sequence \( U_n \) converges to \( U \) uniformly.

**Proof.** Take a linear map \( T : \mathbb{R}^4 \to \mathbb{R}^4 \) such that \( T^\ast(g_\ast) \) is the Euclidean metric. Then \((T^\ast g_n, T^\ast A_n, T^\ast \rho_n, T^\ast U_n) \) gives a sequence of \( \mathbb{Z}/2 \) harmonic spinor on \( \bar{B}(5\sigma) \). Since \( T^\ast g_n \) converges to the Euclidean metric, one can apply lemma 5.1 and find a convergent subsequence on \( \bar{B}(2\sigma) \). Now pull back by \( T^{-1} \), one obtains a convergent subsequence of \( U_n \) on \( \bar{B}(2) \). \( \square \)
6. Frequency pinching estimates

For \( x \in B_{x_0}(32R) \) and \( 0 < s < r \leq 32R \), define
\[
W^r_{s}(x) = N_\phi(x, r) - N_\phi(x, s).
\]
This section proves the following estimate

**Proposition 6.1.** There exists a constant \( C \) with the following property. Let \( r \in (0, 8R] \). Assume \( x_1, x_2 \in B_{x_0}(32R) \), such that \( d(x_1, x_2) \leq r/4 \). Let \( x \) be a point on the short geodesic \( \gamma \) bounded by \( x_1 \) and \( x_2 \). Let \( v \) be a unit tangent vector of \( \gamma \) at \( x \). Then
\[
d(x_1, x_2) \cdot |\partial_v N_\phi(x, r)| \leq C \left[ \sqrt{|W^4_{r/4}(x_1)|} + \sqrt{|W^4_{r/4}(x_2)|} + r \right].
\]

The proof is adapted from the arguments in [DLMSV16, Section 4]. First, one needs to prove the following lemma.

**Lemma 6.2.** There exists a constant \( C \), such that for every \( x \in B_{x_0}(32R) \) and \( r \leq 8R \), one has
\[
\int_{B_{x}(3r) \cup B_{x}(r/3)} \| \nabla_{\eta'} u(y) - N_\phi(x, d(x, y))u(y) \|^2 dy \leq CrH_\phi(x, r)(W^4_{r/4}(x) + Cr^2).
\]

**Proof.** By equation (17),
\[
\int_{r/4}^{4r} \frac{\partial_s N_\phi(x, s)}{sH_\phi(x, s)} ds + O(s^2)
\]
\[
= \int_{r/4}^{4r} \frac{2}{sH_\phi(x, s)} \int \phi'(\frac{d(x, y)}{s}) d(x, y)^{-1} |\nabla_{\eta'} u(y) - N_\phi(x, s)u(y)|^2 dy ds
\]
\[
\geq C_1 rH_\phi(x, r) \int_{r/4}^{4r} \int \phi'(\frac{d(x, y)}{s}) d(x, y)^{-1} |\nabla_{\eta'} u(y) - N_\phi(x, s)u(y)|^2 dy ds
\]
\[
\geq C_1 rH_\phi(x, r) \int_{r/3}^{4r} \int \phi'(\frac{d(x, y)}{s}) d(x, y)^{-1} |\nabla_{\eta'} u(y) - N_\phi(x, s)u(y)|^2 dy ds
\]
\[
= (A)
\]
For every pair \((y, s)\) in the support of the integration in \((A)\), one has \( d(x, y) \in [r/4, 4r] \), hence
\[
|N_\phi(x, s) - N_\phi(x, d(x, y))| \leq W^4_{r/4}(x) + C_2 r^2.
\]
Therefore,
\[
(A) \geq \frac{1}{C_1 rH_\phi(x, r)} \int_{r/3}^{4r} \int \phi'(\frac{d(x, y)}{s}) d(x, y)^{-1} |\nabla_{\eta'} u(y) - N_\phi(x, d(x, y))u(y)|^2 dy ds
\]
\[
- \frac{C_3(W^4_{r/4}(x) + C_2 r^2)}{rH_\phi(x, r)} \int_{r/3}^{4r} \int \phi'(\frac{d(x, y)}{s}) d(x, y)^{-1} \left[ |\nabla u(y)||u(y)|d(x, y) + |u(y)|^2 \right] dy ds.
\]
By lemma 4.3 \( II = O(rH_\phi(x, 4r)) = O((rH_\phi(x, r)) \). By Fubini’s theorem,
\[
I = \int_{B_x(4r)} |\nabla_{\eta'} u(y) - N_\phi(x, d(x, y))u(y)|^2 \int_{r/3}^{4r} \phi'(\frac{d(x, y)}{s}) d(x, y)^{-1} ds dy
\]
\[
= e^{\phi'((rH_\phi(x, r))} \int_{B_x(4r)} \left[ |\nabla u(y)||u(y)|d(x, y) + |u(y)|^2 \right] dy.
\]
Then there exists a constant $\Theta$ such that for every point $x$ in $B_x$ and every vector field $\eta$ on $B_x$, we have:

$$
\inf_{\{y,d(x,y)\in[r/3,3r]\}} \int_{r/3}^{4r} -\phi\left(\frac{d(x,y)}{s}\right) d(x,y)^{-1} ds > 0.
$$

Therefore,

$$
I \geq \frac{1}{C_1} \int_{B_x(3r) - B_x(r/3)} |\nabla_n u(y) - N_\phi(x,d(x,y))u(y)|^2 dy.
$$

In conclusion,

$$(A) \geq \frac{1}{C_5 r H_\phi(x,r)} \int_{B_x(3r) - B_x(r/3)} |\nabla_n u(y) - N_\phi(x,d(x,y))u(y)|^2 dy
- C_6 (W_{r/4}^4(x) + C_2 r^2),$$

hence

$$
C_7 r H_\phi(x,r)(W_{r/4}^4(x) + C_2 r^2) \geq \int_{B_x(3r) - B_x(r/3)} |\nabla_n u(y) - N_\phi(x,d(x,y))u(y)|^2 dy.
$$

One also needs the following technical lemma.

**Lemma 6.3.** Assume $M$ is a compact manifold, possibly with boundary. Let $\varphi^\zeta : \Omega \subset \overline{B_{x_0}(64R)} \to \mathbb{R}^4$ be a smooth family of smooth embeddings, parametrized by $\zeta \in M$. For every $\zeta \in M$ and $x \in B_{x_0}(64R)$, one can define a vector field $\eta^\zeta_x$ on $B_{x_0}(64R)$ as follows. For every $y \in B_{x_0}(64R)$, let

$$
\eta^\zeta_x(y) = [(\varphi^\zeta)_*(y)]^{-1}(\varphi^\zeta(y) - \varphi^\zeta(x)).
$$

Then there exists a constant $\Theta > 0$, depending on $\varphi$, such that

$$
|\eta^\zeta_x(y) - \eta_x(y)| \leq \Theta \cdot d(x,y)^2.
$$

**Proof.** Fix $x$, compute the covariant derivatives of $\eta^\zeta_x$ and $\eta_x$ at $x$. Since both vector fields are zero at $x$, their covariant derivatives at $x$ are independent of the connections. Let $\xi \in T_x X$. Taking derivative in the Euclidean coordinates $\varphi^\zeta$, one obtains $\nabla_\xi (\eta^\zeta_x)(x) = \xi$. Taking derivative in the normal coordinates centered at $x$, one obtains $\nabla_\xi (\eta_x)(x) = \xi$. Therefore, $\eta^\zeta_x$ and $\eta_x$ have the same derivatives at $x$. Since we are working on compact manifolds, $|\eta^\zeta_x(y) - \eta_x(y)| \leq \Theta \cdot d(x,y)^2$ for some constant $\Theta$ independent of $x$. 

**Proof of proposition 6.1.** Assume that $v$ points from $x_1$ towards $x_2$. Extend $v$ to a vector field on $B_x(r)$, such that the coordinates of $v$ are constant under the normal coordinate centered at $x$. Now apply lemma 6.3. Let $M = \overline{B_{x_0}(32R)}$. For every $\zeta \in \overline{B_{x_0}(32R)}$, let $\varphi^\zeta$ be the exponential map centered at $\zeta$. Then for every $z \in B_x(r)$,

$$
v(z) = \frac{\eta^\zeta_{x_1}(z) - \eta^\zeta_{x_2}(z)}{\varphi^\zeta_{x_1}(x_1) - \varphi^\zeta_{x_2}(x_2)}. 
$$

By lemma 6.3,

$$
|\eta^\zeta_{x_1}(z) - \eta_{x_1}(z)| = O(r^2), \quad |\eta^\zeta_{x_2}(z) - \eta_{x_2}(z)| = O(r^2)
$$

Notice that since $\varphi^\zeta$ is the exponential map centered at $x$,

$$
|\varphi^\zeta(x_1) - \varphi^\zeta(x_2)| = d(x_1,x_2).
$$
Combine (20), (21) and (22) together, one obtains

\[
\left| u(z) - \frac{\eta_{x_1}(z) - \eta_{x_2}(z)}{d(x_1, x_2)} \right| = O(r^2/d(x_1, x_2)).
\]

Define

\[
E_l(z) = \nabla_{\eta_{x_1}} u(z) - N_{\phi}(x_1, d(z, x_l)) u(z) \quad \text{for } l = 1, 2.
\]

Then

\[
d(x_1, x_2) \nabla_v u(z) = \nabla_{\eta_{x_1}} u(z) - \nabla_{\eta_{x_2}} u(z) + O(r^2|\nabla u|) = (N_{\phi}(x_1, d(z, x_1)) - N_{\phi}(x_2, d(z, x_2))) u(z) =: E(z)
\]

\[
+ E_1(z) - E_2(z) + O(r^2|\nabla u|).
\]

To simplify notations, define the measure

\[
d\mu_x = -d(x, y)^{-1} \phi'(\frac{d(x, y)}{r}) dy.
\]

Using (16), one can write

\[
d(x_1, x_2) \cdot \partial_v N_{\phi}(x, r)
\]

\[
= O(r^2) + \frac{2}{H_{\phi}(x, r)} \int \nabla_{\eta_{x}} u(y) \cdot (E_1 - E_2 + E_3 + O(r^2|\nabla u|)) d\mu_x
\]

\[- \frac{2}{H_{\phi}(x, r)} \int u N_{\phi}(x, r) \cdot (E_1 - E_2 + E_3 + O(r^2|\nabla u|)) d\mu_x
\]

\[
= \frac{2}{H_{\phi}(x, r)} \int \nabla_{\eta_{x}} u(y) \cdot (E_1 - E_2) d\mu_x - \frac{2N_{\phi}(x, r)}{H_{\phi}(x, r)} \int u \cdot (E_1 - E_2) d\mu_x
\]

\[
= (A)
\]

\[
+ \frac{2}{H_{\phi}(x, r)} \int E_3 u(\nabla_{\eta_{x}} u - N_{\phi}(x, r) u) d\mu_x + O(r^2)
\]

\[
= (C)
\]

To bound (C), notice that

\[
E_3(z) = N_{\phi}(x_1, r) - N_{\phi}(x_2, r) + [N_{\phi}(x_1, d(z, x_1)) - N_{\phi}(x_1, r)]
\]

\[
- [N_{\phi}(x_2, d(z, x_2)) - N_{\phi}(x_2, r)].
\]

By (7),

\[
\int u \cdot \nabla_{\eta_{x}} u d\mu_x = r D_{\phi}(x, r) + O(r^2 H_{\phi}(x, r))
\]

\[
= N_{\phi}(x, r) H_{\phi}(x, r) + O(r^2 H_{\phi}(x, r))
\]

\[
= N_{\phi}(x, r) \int |u|^2 d\mu_x + O(r^2 H_{\phi}(x, r)).
\]

Hence

\[
\int u \cdot (\nabla_{\eta_{x}} u - N_{\phi}(x, r) u) d\mu_x = O(r^2 H_{\phi}(x, r)),
\]
then

\[ \int \mathcal{E} u \cdot (\nabla_{\eta_x} u - N_\phi(x, r)u) d\mu_x = O(r^2 H_\phi(x, r)). \]

By lemma 4.3

\[ 2 \int |u| (|\nabla_{\eta_x} u| + |N_\phi(x, r)||u|) d\mu_x = O(H_\phi(x, r)). \]

In addition, notice that

\[ \sup_{z \in \text{supp } \mu_x} |\mathcal{E}_4(z)| + |\mathcal{E}_5(z)| \leq W_{r/4}^{4r}(x_1) + W_{r/4}^{4r}(x_2) + C_1 r^2. \]

Therefore,

\[ \int (|\mathcal{E}_4| + |\mathcal{E}_5|) \cdot |u(\nabla_{\eta_x} u - N_\phi(x, r)u)| d\mu_x \]

\[ \leq C_2 H_\phi(x, r)(W_{r/4}^{4r}(x_1) + W_{r/4}^{4r}(x_2) + C_1 r^2). \]

As a result,

\[ (C) \leq C_3 (W_{r/4}^{4r}(x_1) + W_{r/4}^{4r}(x_2) + C_4 r^2). \]

To bound (A), use Cauchy’s inequality to obtain

\[ (A) \leq \frac{C_5}{H_\phi(x, r)} \left( \int_{B_x(r)} |\nabla u|^2 dy \right)^{1/2} \left( \int_{B_x(r) - B_x(3r/4)} (\mathcal{E}_1^2 + \mathcal{E}_2^2) dy \right)^{1/2} \]

\[ \leq \frac{C_6}{r^{1/2}} \left( \int_{B_x(r) - B_x(3r/4)} (\mathcal{E}_1^2 + \mathcal{E}_2^2) dy \right)^{1/2}. \]

Now apply lemma 6.2

\[ \int_{B_x(r) - B_x(3r/4)} \mathcal{E}_1^2 dy \leq \int_{B_x(5r/4) - B_x(3r/4)} \mathcal{E}_1^2 dy \]

\[ \leq C_7 r H_\phi(x_1, r)(W_{r/4}^{4r}(x_1) + C_7 r^2). \]

A similar estimate works for the integral of \( \mathcal{E}_2 \). Therefore

\[ (A) \leq C_8 \left[ \sqrt{|W_{r/4}^{4r}(x_1)|} + \sqrt{|W_{r/4}^{4r}(x_2)|} + r \right]. \]

Similarly, applying Cauchy’s inequality on (B) leads to

\[ (B) \leq \frac{C_9}{r H_\phi(x, r)} \left( \int_{B_x(r)} |u|^2 dy \right)^{1/2} \left( \int_{B_x(r) - B_x(3r/4)} (\mathcal{E}_1^2 + \mathcal{E}_2^2) dy \right)^{1/2} \]

\[ \leq \frac{C_{10}}{r^{1/2}} \left( \int_{B_x(r) - B_x(3r/4)} (\mathcal{E}_1^2 + \mathcal{E}_2^2) dy \right)^{1/2}. \]

Lemma 6.2 then gives

\[ (B) \leq C_{11} \left[ \sqrt{|W_{r/4}^{4r}(x_1)|} + \sqrt{|W_{r/4}^{4r}(x_2)|} + r \right]. \]

and the proposition is proved. \( \square \)

**Corollary 6.4.** Assume \( x_1, x_2 \in B_{x_0}(32R) \), assume \( r \in (0, 8R] \). If \( d(x_1, x_2) \leq r/4 \), then

\[ |N_\phi(x_1, r) - N_\phi(x_2, r)| \leq C \left[ \sqrt{|W_{r/4}^{4r}(x_1)|} + \sqrt{|W_{r/4}^{4r}(x_2)|} + r \right]. \]
7. $L^2$ APPROXIMATION BY PLANES

This section establishes a distortion bound in the spirit of [NV15]. Assume $U$ satisfies assumption 1.2 with respect to $\epsilon > 0$. In this section, the constants $C, C_1, C_2, \cdots$ will denote constants that depend on $\Lambda, R$, the $C^1$ norms of the curvatures, as well as $\epsilon$. The techniques in this section were developed by [NV15], and the presentation here is adapted from section 5 of [DLMSV16].

**Definition 7.1.** Suppose $\mu$ is a Radon measure on $\mathbb{R}^4$. For $x \in \mathbb{R}^4$, $r > 0$, define

$$D^2_{\mu}(x, r) = \inf_L r^{-4} \int_{B_x(r)} \text{dist}(y, L)^2 d\mu(y),$$

where $L$ is taken among the set of 2-dimensional affine subspaces.

For a measure $\mu$ supported in $Z$, we wish to bound the value of $D^2_{\mu}(x, r)$ in terms of the frequency functions. However, we have to be careful, since $X$ is a Riemannian manifold, but $D^2_{\mu}(x, r)$ is only defined for Euclidean spaces. We identify $B_{x_0}(32R)$ with $\bar{B}(32R)$ using the exponential map centered at $x_0$. From now on, we will work on the Euclidean space using this identification.

The main result of this section is the following

**Proposition 7.2.** There exists a positive constant $R_0 \leq R$ and a constant $C$ with the following property. Let $\mu$ be a Radon measure supported in $Z$. For $x \in \bar{B}(R)$ and $r \leq R_0$, one has

$$D^2_{\mu}(x, r/8) \leq \frac{C}{r^2} \int_{B_x(r/8)} (W_{r/4}^4(z) + Cr^2) d\mu(z).$$

First, observe that the function $D^2_{\mu}(x, r)$ has the following geometric interpretation. Assume $\mu(\bar{B}_x(r)) > 0$, let

$$\bar{z} = \frac{1}{\mu(B_x(r))} \int_{B_x(z)} z d\mu(z),$$

Define a non-negative bilinear form $b$ on $\mathbb{R}^4$ as

$$b(v, w) = \int_{B_x(r)} ((z - \bar{z}) \cdot v) ((z - \bar{z}) \cdot w) d\mu(z).$$

Let $0 \leq \lambda_1 \leq \cdots \leq \lambda_4$ be the eigenvalues of $b$, then

$$D^2_{\mu}(x, r) = r^{-4} (\lambda_1 + \lambda_2).$$

Let $v_i$ be an eigenvector with eigenvalue $\lambda_i$, a straightforward argument of linear algebra shows that

$$\int_{B_x(r)} ((z - \bar{z}) \cdot v_i) z d\mu(z) = \lambda_i v_i.$$

(23)

The following lemma can be understood as a version of Poincaré inequality for $Z/2$ harmonic spinors.

**Lemma 7.3.** There exist constants $C, R_0 > 0$ with the following property. Let $v_1, v_2, v_3$ be orthonormal vectors in $\mathbb{R}^4$. Let $x \in B(R)$, $r \leq R_0$. Assume $Z \cap B_x(r/8) \neq \emptyset$, then

$$\int_{B_x(5r/4) - B_x(3r/4)} \sum_{j=1}^3 |\nabla v_i u(z)|^2 dz \geq \frac{H_\phi(x, r)}{Cr}.$$
Proof. Assume such constants do not exist. Then there exists a sequence
$$\{(x_n, r_n, U_n, v_1^{(n)}, v_2^{(n)}, v_3^{(n)})\}_{n \geq 1},$$
such that $r_n \leq \frac{1}{n}$, the vectors $v_1^{(n)}, v_2^{(n)}, v_3^{(n)}$ are orthonormal in $\mathbb{R}^4$,
$$\int_{B_{x_n}(5r_n/4) - B_{x_n}(3r_n/4)} \sum_{j=1}^{3} |\nabla v_j^{(n)} u(z)|^2 \, dz \leq \frac{H_\varphi(x_n, r_n)}{nr_n}, \quad (24)$$
and $Z \cap \bar{B}_{x_n}(r_n/8) \neq \emptyset$.

Let $\sigma = (12/11)^2$. Rescale the ball $B_{x_n}(5\sigma^2 r_n)$ to $B(5\sigma^2)$, and normalize the restriction of $U$. By assumption (2), the pull back metrics $g_n$ are given by matrix-valued functions on $\bar{B}(5\sigma^2)$ with eigenvalues bounded by $1/\sigma^2$ and $\sigma^2$. There is a subsequence of the pull backs of $(g_n, A_n, \rho_n, v_1^{(n)}, v_2^{(n)}, v_3^{(n)})$ that converges to some data set $(g, A, \rho, v_1, v_2, v_3)$ in $C^\infty$, and since $r_n \to 0$, the limit data set $(g, A, \rho)$ is invariant under translations. By corollary 5.2 after taking a subsequence, the rescaled $U_n$ converges to a $\mathbb{Z}/2$ harmonic spinor $U^*$ on $B(2)$ with respect to $(g, A, \rho)$, which satisfies assumption 1.2.

The assumption that $Z \cap \bar{B}_{x_n}(r_n/8) \neq \emptyset$ implies that $U^*$ has at least one zero point in $\bar{B}(1/8)$. Inequality (24) gives
$$\int_{B(5/4) - B(3/4)} \sum_{j=1}^{3} |\nabla v_j u^*(z)|^2 \, dz = 0$$
Theorem 1.3 implies that $U^*$ is not identically zero on $B(5/4) - B(3/4)$. Since $U^*$ solves the Dirac equation on non-zero points, the unique continuation property implies that $|U|$ is constant in 3 linearly independent directions in $B(5/4) - B(3/4)$, hence theorem 1.3 implies that $U$ is everywhere non-zero in $B(5/4)$, and that is a contradiction.

Now one can give the proof of proposition 7.2. The proof is adapted from the proof of proposition 5.3 in [DLMSV16].

Proof of proposition 7.2. Let $R_0$ be given by lemma 7.2, and assume $r \leq R_0$. Without loss of generality, assume that $D_\mu^2(x, r/8) > 0$. In particular, $\mu(B(x,r/8)) > 0$, thus $Z \cap B_x(r/8) \neq \emptyset$. Let
$$\bar{z} = \frac{1}{\mu(B_x(r/8))} \int_{B_x(r/8)} zd\mu(z).$$
Let $0 \leq \lambda_1 \leq \cdots \leq \lambda_4$ be the corresponding eigenvalues, then $D_\mu^2(x, r/8) > 0$ implies $\lambda_2 > 0$. Let $v_1$ be the unit eigenvector with eigenvalue $\lambda_1$. Let $\text{grad} u(z)$ be the vector in $T_z\mathbb{R}^4 \otimes \mathcal{V}$, such that for every $v \in T_z\mathbb{R}^4$,
$$\langle v, \text{grad} u(z) \rangle_{\mathbb{R}^4} = \nabla_v u(z).$$
By (2), $\|\text{grad} u(z)\|_{\mathbb{R}^4} \leq (12/11)\|\nabla u\|_X$. Equation (28) gives
$$-\lambda_1 v_1 \cdot \text{grad} u(y) = \int_{B_x(r/8)} ((z - \bar{z}) \cdot v_1)((y - z) \cdot \text{grad} u(y) - \alpha u(y)) \, d\mu(z)$$
for any constant $\alpha$. By Cauchy’s inequality
$$\lambda_1^2 |v_1 \cdot \text{grad} u(y)|^2$$
By lemma 7.3, this implies

\[ \lambda_i \int_{B_x(r/8)} |(y - z) \cdot \nabla u(y) - \alpha u(y)|^2 \, d\mu(z) \]

Therefore, when \( \lambda_i \neq 0 \),

\[ \lambda_i |v_i \cdot \nabla u(y)|^2 \leq \int_{B_x(r/8)} |(y - z) \cdot \nabla u(y) - \alpha u(y)|^2 \, d\mu(z). \]

Integrate with respect to \( y \) on \( B_x(5r/4) - B_x(3r/4) \), and sum up \( i = 2, 3, 4 \),

\[ \int_{B_x(5r/4) - B_x(3r/4)} \sum_{i=2}^{4} \lambda_i |v_i \cdot \nabla u(y)|^2 \, dy \]

\[ \leq 3 \int_{y \in B_x(5r/4) - B_x(3r/4)} \int_{z \in B_x(r/8)} |(y - z) \cdot \nabla u(y) - \alpha u(y)|^2 \, d\mu(z) \, dy \]

\[ \leq 3 \int_{z \in B_x(r/8)} \int_{y \in B_x(11r/8) - B_x(5r/8)} |(y - z) \cdot \nabla u(y) - \alpha u(y)|^2 \, dy \, d\mu(z). \] (25)

On the other hand,

\[ r^2 D^2_{\mu}(x, r) \sum_{i=2}^{4} |v_i \cdot \nabla u(y)|^2 = r^{-2}(\lambda_1 + \lambda_2) \sum_{i=2}^{4} |v_i \cdot \nabla u(y)|^2 \]

\[ \leq \frac{2}{r^2} \sum_{i=2}^{4} \lambda_i |v_i \cdot \nabla u(y)|^2 \]

Therefore

\[ r^2 D^2_{\mu}(x, r) \int_{B_x(5r/4) - B_x(3r/4)} \sum_{i=2}^{4} |v_i \cdot \nabla u(y)|^2 \, dy \]

\[ \leq \frac{2}{r^2} \int_{B_x(5r/4) - B_x(3r/4)} \sum_{i=2}^{4} \lambda_i |v_i \cdot \nabla u(y)|^2 \, dy \]

By lemma 7.3 this implies

\[ r^2 H_{\phi}(x, r) D^2_{\mu}(x, r) \leq \frac{C_1}{r} \int_{B_x(5r/4) - B_x(3r/4)} \sum_{i=2}^{4} \lambda_i |v_i \cdot \nabla u(y)|^2 \, dy \]

Therefore inequality (25) gives

\[ r^2 H_{\phi}(x, r) D^2_{\mu}(x, r) \leq \frac{3C_1}{r} \int_{B_x(r/8)} \int_{B_x(11r/8) - B_x(5r/8)} |(y - z) \cdot \nabla u(y) - \alpha u(y)|^2 \, dy \, d\mu(z). \] (26)

where the constant \( C_1 \) is independent of \( \alpha \).

Notice that

\[ A(z, r) \leq 3 \left( \int_{B_x(11r/8) - B_x(5r/8)} |\eta_z(y) \cdot \nabla u(y) - N_{\phi}(z, d(z, y)) u(y)|^2 \, dy \right) \]

where

\[ A(z, r) \leq 3 \left( \int_{B_x(11r/8) - B_x(5r/8)} |N_{\phi}(z, d(z, y)) u(y)|^2 \, dy \right) \]

\[ = A_1(z, r) \]
by lemma 6.2, Notice that by (2), we have $\bar{\phi}$

$$\sum_{k=1}^{m} \int_{B_k(11r/8)-B_k(5r/8)} |(y - z) - \eta_k(y)|^2|\text{grad } u(y)|^2 dy$$

$$= : A_2(z, r)$$

$$+ \int_{B_k(11r/8)-B_k(5r/8)} (N_\phi(z, \delta(z, y)) - \alpha)^2|u(y)|^2 dy$$

$$=: A_3(z, r)$$

To bound $B_3$, first break it into two parts

$$A_3(z, r) \leq C_3 \int_{B_k(3r/2)-B_k(r/2)} (N_\phi(z, d(z, y)) - N_\phi(z, r))^2|u(y)|^2 dy$$

$$=: A_4(z, r)$$

$$+ C_4 \int_{B_k(3r/2)-B_k(r/2)} (N_\phi(z, r) - \alpha)^2|u(y)|^2 dy$$

$$=: A_5(z, r)$$

Notice that by (2), we have $B_k(11r/8)-B_k(5r/8) \subset B_k(3r/2)-B_k(r/2)$. Therefore, by lemma 6.3

$$A_1(z, r) \leq C_2rH_\phi(z, r)(W_{r/4}^r(z) + C_2r^2).$$

By lemma 6.3 and lemma 4.3,

$$A_2(z, r) = O(r^4 \int_{B_k(3r/2)} |\nabla u|^2) = O(r^3 H_\phi(x, r)).$$

To bound $A_3(z, r)$, first break it into two parts

$$A_3(z, r) \leq C_3 \int_{B_k(3r/2)-B_k(r/2)} (N_\phi(z, d(z, y)) - N_\phi(z, r))^2|u(y)|^2 dy$$

$$=: A_4(z, r)$$

$$+ C_4 \int_{B_k(3r/2)-B_k(r/2)} (N_\phi(z, r) - \alpha)^2|u(y)|^2 dy$$

$$=: A_5(z, r)$$

Here the balls $B_k(3r/2)$ and $B_k(r/2)$ are the geodesic balls on $X$, and the measure $dy$ is the volume form of $X$. The monotonicity of $N_\phi$ implies that

$$A_4(z, r) \leq (W_{r/4}^r(z) + C_5r^2) \int_{B_k(3r/2)} |u(y)|^2 dy$$

$$\leq C_6rH_\phi(x, r)(W_{r/4}^r(z) + C_5r^2).$$

Now take $p \in B_k(r/8)$, such that

$$|W_{r/4}^r(p)| = \inf_{q \in B_k(r/8)} |W_{r/4}^r(q)|,$$

and take $\alpha = N_\phi(p, r)$. Then by lemma 6.3 for $z \in B_k(r/8)$,

$$A_5(z, r) \leq \int_{B_k(3r/2)-B_k(r/2)} (C_7(\sqrt{|W_{r/4}^r(z)|} + \sqrt{|W_{r/4}^r(p)|} + r))^2|u(y)|^2 dy$$

$$\leq C_8(W_{r/4}^r(z) + C_9r^2) \int_{B_k(3r/2)-B_k(r/2)} |u(y)|^2 dy$$

$$\leq C_9rH_\phi(x, r)(W_{r/4}^r(z) + C_9r^2)$$

In conclusion,

$$A(z, r) \leq C_{10}rH_\phi(x, r)(W_{r/4}^r(z) + C_{11}r^2).$$

Therefore proposition 7.2 follows from inequality 26. \qed
8. Approximate spines

Definition 8.1. Given a set of points \( \{p_i\}_{i=0}^k \subset \mathbb{R}^4 \) and a number \( \beta > 0 \), one says that \( \{p_i\}_{i=0}^k \) is \( \beta \)-linearly independent, if for every \( j \in \{0, 1, \cdots, k\} \), the distance between \( p_j \) and the affine subspace spanned by \( \{p_i\}_{i=0}^k \setminus \{p_j\} \) is at least \( \beta \).

Given a set \( F \subset \mathbb{R}^4 \), one says that \( F \) \( \beta \)-spans a \( k \)-dimensional affine subspace, if there exit \( (k+1) \) points in \( F \) that are \( \beta \)-linearly independent.

Lemma 8.2. If \( F \) is a bounded set that does not \( \beta \)-span a \( k \)-dimensional affine space, then there exists a \( (k-1) \)-dimensional affine space \( V \), such that \( F \) is contained in the \( 2\beta \)-neighborhood of \( V \).

Proof. For \( k \) points \( \{q_1, \cdots, q_k\} \) in \( \mathbb{R}^4 \), let \( V(q_1, \cdots, q_k) \) be the volume of the \( (k-1) \) dimensional simplex spanned by these points. Let \( \{p_1, \cdots, p_k\} \subset F \) be \( k \) points in \( F \) such that
\[
V(p_1, \cdots, p_k) \geq \frac{1}{2} \sup_{q_1, \cdots, q_k \in F} V(q_1, \cdots, q_k).
\]
(27)

If the volume \( V(p_1, \cdots, p_k) \) is zero, then \( F \) is contained in a \( (k-1) \)-dimensional affine subspace, and the statement is trivial. If the volume is positive, then the set \( \{p_1, \cdots, p_k\} \) spans a \( k-1 \) dimensional affine space \( V \). If \( F \) is contained in the \( 2\beta \) neighborhood of \( V \), then the statement is verified. Otherwise, there exists a point \( p_{k+1} \in F \), such that the distance of \( p_{k+1} \) and \( V \) is greater than \( 2\beta \). Let \( d_j \) be the distance between \( p_j \) and the affine subspace spanned by \( \{p_i\}_{i=0}^{k+1} \setminus \{p_j\} \), then \( d_{k+1} \geq 2\beta \). By \((27)\), \( 2d_j \geq d_{k+1} \) for every \( j \). Therefore \( \{p_1, \cdots, p_{k+1}\} \) is \( \beta \)-linearly independent, and that contradicts the assumption on \( F \). \( \square \)

As in section 7 use the normal coordinate centered at \( x_0 \) to identify \( B_{x_0}(32R) \) with the ball \( \bar{B}(32R) \) in \( \mathbb{R}^4 \). Recall that by assumption \((2)\),
\[
\left(\frac{11}{12}\right)^2 \leq \kappa_{x_0}(z) \leq K_{x_0}(z) \leq \left(\frac{12}{11}\right)^2,
\]
where \( \kappa_{x_0}(z) \) and \( K_{x_0}(z) \) are the upper and lower bound of the eigenvalues of the metric matrix at \( z \in \bar{B}_x(32R) \).

The compactness property of \( \mathbb{Z}/2 \) harmonic spinors leads to the following lemma.

Lemma 8.3. Let \( \beta, \tilde{\beta}, \tilde{\tilde{\beta}} \in (0, 1) \) be given. Then there exists \( \delta > 0 \), depending on \( \beta, \tilde{\beta}, \tilde{\tilde{\beta}} \), the upper bound \( \Lambda \) of the frequency function, the value of \( R \), the curvatures of \( X \) and \( Y \), and the constant \( \epsilon \) in assumption \((4)\), such that the following holds. If \( x \in \bar{B}(R) \), \( r \leq \delta \), and \( \{p_1, p_2, p_3\} \) is a set of \( \beta r \)-linearly independent points in \( \bar{B}_x(r) \), such that
\[
N_\phi(p_i, 2r) - N_\phi(p_i, \tilde{\beta}r) < \delta \quad i = 1, 2, 3.
\]
Let \( V \) be the affine space spanned by \( p_1, p_2, p_3 \). Then the set \( Z \cap \bar{B}_x(r) \) is contained in the \( \beta r \) neighborhood of \( V \cap \bar{B}_x(r) \).

Proof. Assume such \( \delta \) does not exist. Then there exist sequences \( \{p_i^{(n)}\}_{i=1}^3 \), \( x_n \), and \( r_n \), such that \( r_n \to 0 \), the points \( \{p_i^{(n)}\}_{i=1}^3 \) are contained in \( \bar{B}_{x_n}(r_n) \) and are \( \tilde{\beta} r_n \)-linearly independent, and
\[
N_\phi(p_i^{(n)}, 2r_n) - N_\phi(p_i^{(n)}, \tilde{\beta}r_n) < \frac{1}{n} \quad i = 1, 2, 3,
\]
and there exists \( y_n \in Z \) such that the distance from \( y_n \) to the affine space spanned by \( \{p_i^{(n)}\}_{i=1}^3 \) is greater than \( \beta r_n \)
Let $\sigma = 12/11$. Rescale the balls $\bar{B}_{x}(10\sigma^{2}r_{n})$ to radius $10\sigma^{2}$, and normalize the section $U$. Corollary 5.2 then gives a limit section $U^{*}$ which satisfies the following properties:

1. $U^{*}$ is a $\mathbb{Z}/2$ harmonic spinor on $\bar{B}(4)$, with respect to a translation-invariant metric, the trivial connection on $\mathcal{V}$, and a translation-invariant Clifford multiplication. $U^{*}$ satisfies assumption 1.2.

2. There exist points $p_{1}^{*}, p_{2}^{*}, p_{3}^{*} \in B(1)$, such that they are $\beta$-linearly independent, and
   \[ N_{\phi}(p_{i}^{*}, 2) - N_{\phi}(p_{i}^{*}, \tilde{\beta}) = 0 \quad i = 1, 2, 3, \]
   \[ (28) \]

3. Let $V^{*}$ be the affine space spanned by $\{p_{i}^{*}\}_{i=1}^{3}$. There exists a point $q \in \bar{B}(1)$ in the zero set of $U^{*}$, such that the distance from $q$ to $V^{*} \cap \bar{B}(1)$ is at least $\beta$.

Since $U^{*}$ is defined on a flat manifold with flat bundle, remark 4.6 indicates that for $U^{*}$,
\[ \partial_{\nu}N_{\phi}(x, r) = \frac{2}{rH_{\phi}(x, r)} \int_{\mathcal{V}} -\phi' \left( \frac{d(x, y)}{r} \right) d(x, y)^{-1} |\nabla_{\nu} u(y) - N_{\phi}(x, r)u(y)|^{2} dy. \]

Therefore equation (28) implies that for $i \in \{1, 2, 3\}$, the section $U^{*}$ is homogeneous on $\bar{B}_{p_{i}^{*}}(2) - \bar{B}_{p_{i}^{*}}(\tilde{\beta})$ with respect to the center $p_{i}^{*}$. The unique continuation property for solutions to the Dirac equation implies that $U^{*}$ is homogeneous on $\bar{B}(2)$ with respect to $p_{i}^{*}$. An elementary argument (see for example [DLMSV16, Lemma 6.8]) then shows that the section $U^{*}$ is zero on the affine space $V^{*}$, and that $U^{*}$ is invariant in the directions parallel to $V^{*}$. Therefore, property (3) of $U^{*}$ implies that $U^{*}$ is zero on a 3-dimensional affine subspace, which contradicts theorem 1.3.

Similarly, one has

**Lemma 8.4.** Let $\beta, \tilde{\beta}, \tilde{\beta} \in (0, 1)$ and $\tau > 0$ be given. Then there exists $\delta > 0$, depending on $\beta, \tilde{\beta}, \tilde{\beta}, \tau$, the upper bound $\Lambda$ of the frequency function, the value of $R$, the curvatures of $X$ and $\mathcal{V}$, and the constant $\epsilon$ in assumption 1.2 such that the following holds. Assume $x \in \bar{B}(R)$, and $r \leq \delta$, and $\{p_{1}, p_{2}, p_{3}\}$ is a set of points in $\bar{B}_{x}(r)$ that is $\beta r$-linearly independent, such that
\[ N_{\phi}(p_{i}, 2r) - N_{\phi}(p_{i}, \tilde{\beta}r) < \delta \quad i = 1, 2, 3. \]

Let $V$ be the affine space spanned by $\{p_{i}\}$. Then for all $y, y' \in \bar{B}_{x}(r) \cap Z$, one has
\[ |N_{\phi}(y, \beta r) - N_{\phi}(y', \beta r)| < \tau. \]

**Proof.** Assume such $\delta$ does not exist, then arguing as before, one obtains a 2-valued section $U^{*}$ on $\bar{B}(4)$ with the following properties:

1. $U^{*}$ is a $\mathbb{Z}/2$ harmonic spinor on $\bar{B}(4)$, with respect to a translation-invariant metric, the trivial connection on $\mathcal{V}$, and a translation-invariant Clifford multiplication. $U^{*}$ satisfies assumption 1.2.

2. There exist points $p_{1}^{*}, p_{2}^{*}, p_{3}^{*} \in B(1)$, such that they are $\tilde{\beta}$-linearly independent, and
   \[ N_{\phi}(p_{i}^{*}, 2) - N_{\phi}(p_{i}^{*}, \tilde{\beta}) = 0 \quad i = 1, 2, 3, \]
   \[ (29) \]

3. Let $Z^{*}$ be the zero set of $U^{*}$. There exist $y, y' \in \bar{B}(1) \cap Z^{*}$, such that
   \[ |N_{\phi}(y, \beta) - N_{\phi}(y', \beta)| \geq \tau. \]
9. Rectifiability and the Minkowski bound

This section only concerns estimates on the Euclidean space. To simplify notations, for the rest of this section, use $B_x(r)$ and $B(r)$ to denote the Euclidean balls.

Definition 9.1. Let $Z$ be a Borel subset of $\overline{B}(R) \subset \mathbb{R}^4$. A function $\mathcal{I}(x, r)$ defined for $x \in Z$ and $r \leq 128R$ is called a taming function for $Z$, if the following conditions hold.

1. $\mathcal{I}(x, r)$ is non-negative, bounded, continuous, and is non-decreasing in $r$.
2. Let $\beta, \bar{\beta} \in (0, 1)$ and $\tau > 0$ be given. Then there exists $\epsilon(\beta, \bar{\beta}, \tau) > 0$, depending on $\beta, \bar{\beta}, \tau$, such that the following holds. Assume $x \in \overline{B}(R), r \leq R$, and \{p_1, p_2, p_3\} is a set of points in $B_x(r)$ that is $\bar{\beta}r$-linearly independent, such that
   \[ \mathcal{I}(p_i, 2r) - \mathcal{I}(p_i, \bar{\beta}r/2) < \epsilon(\beta, \bar{\beta}, \tau) \quad i = 1, 2, 3. \]

Then for all $y, y' \in \overline{B}_x(r) \cap Z$, one has
   \[ |\mathcal{I}(y, \beta r/2) - \mathcal{I}(y', \beta r/2)| < \tau. \]

3. There exists a constant $C$, such that for every Radon measure $\mu$ supported in $Z$, the following inequality holds for every $x \in B(2R)$ and $r \leq 2R$:
   \[ D^2_\mu(x, r) \leq \frac{C}{r^2} \int_{\overline{B}_x(r)} |\mathcal{I}(z, 32r) - \mathcal{I}(z, 2r)| d\mu(z). \]

The following result follows almost verbatim from sections 7 and 8 of [DLMSV16], and a large part of the arguments originated from [NV15]. Nevertheless, a proof is provided here for the reader’s convenience.

Theorem 9.2 ([NV15], [DLMSV16]). Assume $Z$ is a Borel subset of $B(R)$ and there exists a taming function $\mathcal{I}(x, r)$ for $Z$. Then the set $Z \cap B(R/2)$ is 2-rectifiable and has finite 2-dimensional Minkowski content.

The proof of theorem 9.2 makes use of the following Reifenberg-type theorem. We state the theorem for the cases of dimension 4 and codimension 2.

Theorem 9.3 ([NV15], Theorem 3.4). There exist universal constants $K_0 > 0$ and $\delta_0 > 0$ such that the following holds. Assume \{B_z, (r_i)\} is a collection of balls in $B(2R)$, such that \{B_z, (r_i/4)\} are disjoint. Define a measure $\mu = \sum_i r_i^2 \delta_z$. Suppose
   \[ \int_{B_z(r)} \int_0^r \frac{D^2_\mu(z, s)}{s} ds d\mu(z) < \delta_0 r^2 \]
   for every $B_x(r) \subset B(2R)$, then $\mu(B(R)) \leq K_0 R^2$.

Proof of theorem 9.3. Assume $B_z(r) \subset B(R)$. If one rescales $B_z(r)$ to $B(R)$, then the function $\mathcal{I}(y, s) = \mathcal{I}(x + (ry)/R, sr/R)$ is a taming function for $[(A - x) \cdot (R/r)] \cap B(R)$ with the same function $\epsilon(\beta, \bar{\beta}, \tau)$ and constant $C$. Therefore definition 9.1 is invariant under rescaling, thus one only needs to consider the case for $R = 2$. 
Let $\beta = 1/10$. Let $\bar{\beta} \leq 1/100$ be a positive universal constant, let $\tau > 0$ be a constant that is defined by $\bar{\beta}$ and $C$, and let $\delta > 0$ be a constant that is defined by $\bar{\beta}, \tau$, the function $\epsilon$ and the constant $C$. The exact values for $\bar{\beta}, \tau$ and $\delta$ will be determined later in the proof.

Let $\Lambda$ be an upper bound of $I$, namely $\Lambda \geq \sup_{x \in A, x \leq 128R} I(x, r) = \sup_{x \in A} I(x, 256)$.

Define
\[
D_\delta(r) = B(R/2) \cap \{x \in Z | I(x, \beta r/2) \geq \Lambda - \delta\}.
\]
Define
\[
W_{r_i}^x(x) = I(x, r_1) - I(x, r_2).
\]
If $\{B_{x_i}(r_i)\}$ is a family of balls, we call the sum $\sum_i r_i^2$ its 2-dimensional volume.

**Step 1.** First, require that $\delta < \epsilon(\beta, \bar{\beta}, \tau)$. For $B_x(r) \subset B(2)$, and a set $A \subset Z \cap B_x(r)$, define an operator $F_A$, which turns $B_x(r)$ into a finite set of balls. It has the property that either $F_A(B_x(r)) = \{B_x(r)\}$, or $F_A(B_x(r))$ is a family of balls with radius $\beta r$. In either case, the balls in $F(B_x(r))$ will cover the set $A$. The operator $F_A$ is defined as follows. If $A \cap D_\delta(r)$ does not $\beta r$-span a 2-dimensional affine space, then it is called “bad”. Otherwise, it is called “good”. In the bad case, define $F_A(B_x(r)) = \{B_x(r)\}$. In the good case, cover $A$ by a family of balls $\{B_{x_i}(\beta r)\}$ with the following properties

1. The distance between $x_i$ and $x_j$ is at least $\beta r/2$ for all $i \neq j$,
2. Each $x_i$ is an element of $A$.

Define $F_A(B_x(r))$ to be the family $\{B_{x_i}(\beta r)\}$.

Obviously the descriptions above do not uniquely specify the operator $F_A$. When there are more than one possibilities, choose one arbitrarily.

If $B_x(r)$ is a good ball, let $p_1, p_2, p_3 \in D_\delta(r) \cap B_x(r)$ be three points that $\bar{\beta} r$ span a plane, let $F(B_x(r)) = \{B_{x_i}(\beta r)\}$. By condition (2) of definition 9.1
\[
|I(x_i, \beta r/2) - I(p_i, \beta r/2)| \leq \tau.
\]
Therefore
\[
I(x_i, \beta r/2) \geq \Lambda - \delta - \tau \tag{30}
\]

The operator $F_A$ can be extended to act on a collection of balls. Assume $\{B_{x_i}(r)\}_{i=1}^n$ is a collection of balls with the same radius. Let $A \subset \bigcup B_{x_i}(r) \cap Z$. Assume $\{B_{x_i}(r)\}_{i=1}^k$ are the good balls, and $\{B_{x_i}(r)\}_{i=k+1}^n$ are the bad balls. Then there exists a collection of balls $\{B_{y_j}(\beta r)\}$, such that

1. $\{B_{y_j}(\beta r)\}$ covers $\bigcup_{i=1}^k (A \cap B_{x_i}(r))$,
2. $|y_j - y_i| \geq \beta r/2$, for all $j \neq i$.
3. $y_j \in \bigcup_{i=1}^k A \cap B_{x_i}(r)$, for $\forall j$.

Inequality 30 still holds when $x_i$ is replaced by $y_j$. Define $F_A\{B_{x_i}(r)\}$ to be the union of $\{B_{y_j}(\beta r)\}$ and $\{B_{x_i}(r)\}_{i=k+1}^n$.

**Step 2.** Let $N > 0$ be a positive integer. Let $A_0(x, r) = Z \cap B_x(r)$. Apply the operator $F_{A_0}$ to $B_x(r)$ to obtain a set of balls, which we denote by $S_1(x, r)$. Assume $S_1(x, r)$ splits two sets $S_1(x, r) = S_{1,g}(x, r) \cup S_{1,b}(x, r)$, where $S_{1,g}(x, r)$ is the collection of good balls and $S_{1,b}(x, r)$ is the collection of bad balls. Let
\[
A_1(x, r) = A_0(x, r) - \bigcup_{B_{x_i}(r_i) \in S_{1,b}(x, r)} B_{x_i}(r_i).
\]
Apply $\mathcal{F}_{A_1(x,r)}$ to $S_{1,g}(x,r)$ and obtain a new set of balls

$$S_2(x,r) = \mathcal{F}_{A_1(x,r)}(S_{1,g}(x,r)) \cup S_{1,b}(x,r).$$

Similarly, write $S_2(x,r) = S_{2,g}(x,r) \cup S_{2,b}(x,r)$, and define

$$A_2(x,r) = A_1(x,r) - \bigcup_{B_{x_i}(r_i) \in S_{2,b}(x,r)} B_{x_i}(r_i),$$

and define $S_3 = \mathcal{F}_{A_2(S_{2,g})} \cup S_{2,b}$. Repeat the procedure $N$ times to obtain a set of balls $S_N(x,r)$.

The family $S_N(x,r)$ has the following property. If $B_{x_1}(r_1)$ and $B_{x_2}(r_2)$ are two distinct elements of $S_N(x,r)$, then

$$|x_1 - x_2| \geq (r_1 + r_2)/4. \quad (31)$$

Inequality $(31)$ can be proved by induction. For $N = 1$, it follows from the definition of $\mathcal{F}_A$. Assume $(31)$ holds for $N - 1$, and write $S_N = \mathcal{F}_{A_{N-1}}(S_{N-1,g}) \cup S_{N-1,b}$. Let $B_{x_1}(r_1), B_{x_2}(r_2) \in S_N$. If both $B_{x_1}(r_1), B_{x_2}(r_2) \in \mathcal{F}_{A_{N-1}}(S_{N-1,b})$, then $(31)$ follows from the definition of $\mathcal{F}$. If both $B_{x_1}(r_1), B_{x_2}(r_2) \in S_{N-1,b}$, then $(31)$ follows from the induction hypothesis. If $B_{x_1}(r_1) \in \mathcal{F}_{A_{N-1}}(S_{N-1,g})$, $B_{x_2}(r_2) \in S_{N-1,b}$, then $x_1 \not\in B_{x_2}(r_2)$. By the construction of $\mathcal{F}$, one has $r_1 > \beta r_2$. Since $\beta = 1/10$, one has $|x_1 - x_2| \geq r_2 \geq (r_1 + r_2)/2$.

By $(31)$, either $S_N = \{B_{x}(r)\}$, or

$$I(x_1, r_1/2) \geq \Lambda - \delta - \tau, \quad \forall B_{x}(r) \in S_N. \quad (32)$$

**Step 3.** We claim that there exists a universal constant $K_1 > 1$, such that for $\tau$ and $\delta$ sufficiently small, we have

$$\sum_{B_{x_i}(r_i) \in S_N(x,r)} r_i^2 < K_1 r^2. \quad (33)$$

Without loss of generality, assume $S_N(x,r) \neq \{B_{x}(r)\}$. Let $r_j = \beta^{N-j} r$. Define Radon measures

$$\mu = \sum_{B_y(s) \in S_N(x,r)} s^2 \delta_y, \quad \mu_j = \sum_{B_y(s) \in S_N(x,r), s \leq r_j} s^2 \delta_y.$$ 

Notice that by $(31)$, there exists a universal constant $K_2$ such that

$$\mu_0\left((B_{y}(r_j))\right) \leq K_2 r_j^2, \quad \forall x. \quad (34)$$

Let $K_0$ be the constant given by theorem 9.3, let $K_3 = \max\{K_0, K_2\}$. We prove that if $\tau, \delta$ is chosen sufficiently small, then for every $j = 0, 1, \cdots, N - 3$, and every $B_y(r_j) \subset B_x(2r)$, one has

$$\mu_j(B_y(r_j)) \leq K_3 r_j^2. \quad (35)$$

The claim is proved by induction on $j$. The case for $j = 0$ follows from $(34)$. Assume that the claim is proved for $0, 1, \cdots, j$, and $j < N - 3$. Then there exists a universal constant $M > 1$, such that for every $y \in B_x(2r)$, $k \leq j + 1$, and $s \in [r_k/2, 2r_k],$

$$\mu_{k+3}(B_y(s)) \leq M (K_3 + 1) s^2 \quad (36)$$

We want to use theorem 9.3 and $(34)$ to prove

$$\mu_{j+1}(B_y(r_{j+1})) \leq K_3 r_{j+1}^2, \quad \forall B_y(r_{j+1}) \subset B_x(2r).$$
If \( \mu_{j+1}(B_y(r_{j+1})) = 0 \), the inequality is trivial. From now on assume \( \mu(B_y(r_{j+1})) > 0 \). Since \( r_{j+1} \leq r_{N-3} = r/8 \), and supp \( \mu \subset B_x(r) \), we have \( B_y(4r_{j+1}) \subset B_x(2r) \).

Notice that for \( B_x(s_i) \in S_N \), if \( t < \min_{k} |x_i - x_k| \), then

\[
D^2_i(x_i, t) = 0.
\]

Define

\[
W_{2t}^{22t}(x_i) = \begin{cases} 0 & \text{if } t < s_i/4, \\ W_{2t}^{22t}(x_i) & \text{if } t \geq s_i/4. \end{cases}
\]

Inequality (31) and condition (3) of definition 9.1 gives

\[
D^2(p, t) \leq C \int_{B_4(t)} \frac{W_{2t}^{22t}(p)}{t^3} d\mu(p)
\]

(37)

for every \((q, t)\).

For \( B_z(s) \subset B_y(2r_{j+1}) \), assume \( s \in \{r_k/2, 2r_k\} \) for \( k \leq j + 1 \). Inequality (37) gives

\[
\int_{B_z(s)} \int_{B_4(t)} \frac{D^2_{j+1}(q, t)}{t} dt d\mu_{j+1}(q)
\]

\[
\leq C \int_{B_z(s)} \int_{B_4(t)} \frac{W_{2t}^{22t}(p)}{t^3} d\mu_{j+1}(p) dt d\mu_{j+1}(q)
\]

\[
\leq C \int_{B_z(s)} \int_{B_4(t)} \frac{W_{2t}^{22t}(p)}{t^3} d\mu_{k+3}(p) dt d\mu_{k+3}(q)
\]

(38)

\[
\leq C \int_{B_z(s)} \int_{B_4(t)} \frac{W_{2t}^{22t}(p)}{t^3} d\mu_{k+3}(q) ds d\mu_{k+3}(p)
\]

\[
\leq CM(K_3 + 1) \int_{B_z(s)} \int_{0}^{s} \frac{W_{2t}^{22t}(p)}{t} dt d\mu_{k+3}(p),
\]

(39)

where inequality (38) follows from (31). For \( p \in \text{supp} \mu_{j+1} \), let \( s_p \) be the radius of ball in \( S_N \) with center \( p \). If \( s \geq s_p/4 \), then

\[
\int_{0}^{s} \frac{W_{2t}^{22t}(p)}{t} dt = \int_{s_p/4}^{s} \frac{W_{2t}^{22t}(p)}{t} dt = \int_{s_p/4}^{32s} \mathcal{I}(p, t) dt - \int_{s_p/4}^{16s_p} \mathcal{I}(p, t) dt
\]

\[
\leq W_{s_p/2}^{22t}(p) \int_{2}^{32} \frac{1}{t} dt \leq \ln(16) (\delta + \tau).
\]

(40)

The last inequality above follows from (32). Therefore, the right hand side of (39) is bounded by

\[
CM(K_3 + 1) \int_{B_z(s)} \int_{0}^{s} \frac{W_{2t}^{22t}(p)}{t} dt d\mu_{k+3}(p)
\]

\[
\leq CM(K_3 + 1) \mu_{k+3}(B_z(s)) \ln(16) (\tau + \delta) \leq 4CM^2(K_3 + 1)^2 \ln(16)(\tau + \delta) s^2
\]

Let \( \delta_0 \) be the constant given by theorem 9.3. Take

\[
\tau < \frac{\delta_0}{8CM^2(K_3 + 1)^2 \ln(16)},
\]
and
\[ \delta < \frac{\delta_0}{8CM^2 (K_3 + 1)^2 \ln(16)}. \]
then the conditions of theorem 9.3 are satisfied, therefore \( \mu_{j+1}((B_y(r_{j+1})) \leq K_0 r_{j+1}^2 \).
By induction, (3) is proved. Inequality (3) then follows from (3) by the the case of \( j = N - 3 \).

**Step 4.** By lemma 8.2 the result obtained from the previous steps can be summarized as follows. For any integer \( N > 0 \), and any ball \( B_x(r) \), there is a covering of \( Z \cap B_x(r) \) by a family of balls \( S_N(x,r) = \{ B_x(r_i) \}_{i} \), such that the following properties hold:

1. The radius of each ball is at least \( \beta^N r \).
2. For any ball \( B_x(r_i) \in S_N \), either \( r_i = \beta^N r \), or \( r_i = \beta^j r \) for some integer \( j < N \), and \( B_x(r_i) \cap D_s(r_i) \) is contained in the \( 2\beta r_i \) neighborhood of a line.
3. \( \sum_i r_i^2 \leq K_1 r^2 \).

As a consequence,

**Lemma 9.4.** There exists a universal constant \( K_1 > 1 \), and a constant \( \delta \), such that the following property holds. For any \( B_x(r) \subset B(2) \), and \( s \in (0, r) \), there exists a covering of \( Z \cap B_x(r) \) by balls \( S = \{ B_x(r_i) \}_{i} \), such that

1. The radius of each ball is at least \( \beta s \).
2. For any ball \( B_x(r_i) \in S \), either \( r_i \leq s \), or \( B_x(r_i) \cap D_s(r_i) \) is contained in the \( 2\beta r_i \) neighborhood of a line.
3. \( \sum_i r_i^2 \leq K_1 r^2 \).

**Step 5.** We prove the following lemma

**Lemma 9.5.** There exists a universal constant \( K_4 \), and a constant \( \delta \), such that the following property holds. For any \( B_x(r) \subset B(2) \), and \( s \in (0, r) \), there exists a splitting of \( Z \) into \( Z = \bigcup_i \mathcal{E}_i \), and a family of balls \( S = \{ B_x(r_i) \}_{i} \), such that

1. \( \mathcal{E}_i \subset B_{x_i}(r_i) \).
2. The radius of each ball is at least \( 4\beta s \).
3. For any ball \( B_x(r_i) \in S \), either \( r_i \in [4\beta s, s] \), or \( B_x(r_i) \cap D_s(r_i) = \emptyset \).
4. \( \sum_i r_i^2 \leq K_4 r^2 \).

**Proof of lemma 9.5.** Notice that by the assumptions on \( \beta \) and \( \bar{\beta} \), we have \( 4\bar{\beta} < \beta \).

If \( \{ B_x(r_i) \}_{i} \) is a covering of \( Z \cap B_x(r) \) that satisfies the three properties given by lemma 9.4 with respect to \( s \), we say that \( \{ B_x(r_i) \}_{i} \) is an \( s \)-admissible covering of \( B_x(r) \cap Z \). Fix \( s > 0 \), by lemma 9.4 \( s \)-admissible coverings of \( B_x(r) \cap Z \) exist.

Let \( \{ B_x(r_i) \} \) be an \( s \)-admissible covering of \( B_x(r) \cap Z \). Let \( \mathcal{E}_i = Z \cap B_{x_i}(r_i) \). Then the family \( \{ \mathcal{E}_i, B_{x_i}(r_i) \} \) satisfies conditions (1), (2) of lemma 9.5 and \( \sum_i r_i^2 \leq K_1 r^2 \). However, it may not satisfy condition (3). In the following, we will give a procedure to adjust the family, such that at each step the covering still satisfies property (2) of \( s \)-admissibility, and after finitely many steps of adjustments, the family will satisfy property (3) of lemma 9.5. At the same time, \( \sum_i r_i^2 \) is being controlled throughout the adjustments.

Assume \( \{ B_x(r_i) \} \) is an \( s \)-admissible covering of \( B_x(r) \cap Z \), and \( \mathcal{E}_i \subset B_{x_i}(r_i), B_x(r) \cap Z = \bigcup \mathcal{E}_i \). Assume \( \{ \mathcal{E}_0, B_{x_0}(r_0) \} \) does not satisfy property (3) of lemma 9.5. Then \( r_0 > s \).
By property (2) of s-admissibility, $B_{x_0}(r_0) \cap D_\delta(r_0)$ is contained in the $2\beta r_0$ neighborhood of a line. Thus one can cover $B_{x_0}(r_0) \cap D_\delta(r_0)$ by a family of no more than $[10/\beta]$ balls with radius $4\beta r_0$. Let $\{B_{y_j}(t_j)\}$ be this family. If $4\beta r_0 > s$, apply lemma 9.4 again to each ball $B_{y_j}(t_j)$ and replace it with an s-admissible covering of $B_{y_j}(t_j) \cap D_\delta(r_0)$. Otherwise keep the family $\{B_{y_j}(t_j)\}$ as it is. Let $\{B_{z_j}(l_j)\}$ be the result of this procedure. Then $\{B_{z_j}(l_j)\}$ covers $B_{x_0}(r_0) \cap D_\delta(r_0)$, and it has the following properties

(1) $4\beta s \leq l_j \leq 4\beta r_0$ for each $j$,
(2) $\sum_j l_j^2 \leq (10/\beta) \cdot K_1 \cdot (4\beta r_0)^2$.

Take $\beta \leq 1/(320K_1)$, then $\sum_j l_j^2 \leq \frac{1}{4} r_0^2$.

The adjustment of the family $\{\mathcal{E}_i, B_{x_i}(r_i)\}$ is defined as follows. First, remove $(\mathcal{E}_0, B_{x_0}(r_0))$ from the family, and add $(\mathcal{E}_0 \setminus D_\delta(r_0), B_{x_0}(r_0))$ into the family. Next, add the family $\{\mathcal{E}_0 \cap B_{z_j}(l_j), B_{z_j}(l_j)\}$ constructed from the previous paragraph into this family.

This adjustment replaces an element $(\mathcal{E}_0, B_{x_0}(r_0))$ which does not satisfy property (3) of lemma 9.4 by a family of balls, such that the biggest ball in this family has the same radius $r_0$ and satisfies property (3). The rest of the balls have radius in the interval $[4\beta s, 4\beta r_0]$ and their 2-dimensional volume is bounded by $\frac{1}{4} r_0^2$. Moreover, the new family still satisfies property (2) of lemma 9.4. Therefore, after finitely many times of adjustments, we will obtain a family that satisfies conditions (1), (2), (3), with 2-dimensional volume

$$\sum_i r_i^2 \leq 2K_1 r^2,$$

hence the lemma is proved. \hfill \Box

**Step 6.** Given $s \in (0, 1)$, we use lemma 9.5 to construct a covering of $Z \cap B(1)$ by a family of balls $\{B_{x_i}(r_i)\}$ with radius $r_i \in [4\beta s, s]$, such that the 2-dimensional volume of the covering is bounded.

We call a family $\{\mathcal{E}_i, B_{x_i}(r_i)\}$ a split-covering of a set $A$, if $\mathcal{E}_i \subset B_{x_i}(r_i)$, and $A = \bigcup \mathcal{E}_i$.

If a split-covering of $Z \cap B(r)$ satisfies the properties given by lemma 9.5, we say that it is strongly s-admissible.

Let $\mathcal{S}$ be a strongly s-admissible split-covering of $Z \cap B(1)$. For every $B_{x_i}(r_i) \in \mathcal{S}$, if $r_i \leq s$, we say it is of type I. Otherwise, we say it is of type II. Assume $B_{x_i}(r_i)$ is a ball of type II, then the function $I(x, r)$ is at most $\Lambda - \delta$ for $x \in \mathcal{E}_i$, $r_i \leq \beta r_i/2$. There exists a universal constant $L$ such that $\mathcal{E}_i$ can be covered by $L$ balls $B_{y_j}(\beta r_i/512)$ with radius $(\beta r_i/512)$. Therefore, for each ball, the set $\mathcal{E}_i \cap B_{y_j}(\beta r_i/512)$ has a strongly s-admissible split-covering, with $\Lambda$ replaced by $\Lambda - \delta$.

Change $(B_{x_i}(r_i), \mathcal{E}_i)$ to the union of the $L$ strongly s-admissible split-coverings of $\mathcal{E}_i \cap B_{y_j}(\beta r_i/512)$, we obtain a split-covering of $\mathcal{E}_i$ with 2-dimensional volume at most $LR_4(\beta r_i/512)^2$. Define an operation $\mathcal{G}$ on $\mathcal{S}$, such that $\mathcal{G}(\mathcal{S})$ is constructed from $\mathcal{S}$ by replacing every type II element in $\mathcal{S}$ with the union of the $L$ split-coverings described above.

Notice that for the balls $B_{y_j}(\beta r_i/512)$, the upper bound $\Lambda$ is replaced by $\Lambda - \delta$. Therefore, this procedure can only be carried for at most $N = [\frac{s}{4}]$ times. After that, every ball in $\mathcal{G}(N)(\mathcal{S})$ is of type I. Namely, every ball in $\mathcal{G}(N)(\mathcal{S})$ has radius in the interval $[4\beta s, s]$. 
Let $V_n$ be the 2 dimensional volume of $G(n)(S)$, then we have

$$V_{n+1} \leq (1 + LK_4(\beta/512)^2)V_n.$$ 

Therefore the total 2-dimensional volume of $G(n)(S)$ is bounded by

$$V_n \leq (1 + LK_4(\beta/512)^2)^N K_4.$$ 

Since $s$ can be taken to be arbitrarily small, the Minkowski content of $Z \cap B(1)$ is bounded by a constant $K$ depending on $\Lambda$, $\epsilon$ and $C$.

By rescaling, we conclude that the Minkowski content of $Z \cap B_x(r)$ is bounded by $K r^2$. Since the Minkowski content bounds the Hausdorff measure, there exists a constant $K'$ depending on $\Lambda$, $\epsilon$ and $C$, such that

$$\mathcal{H}_2(Z \cap B_x(r)) \leq K' r^2. \quad (41)$$

**Step 7.** So far we have been treating theorem 9.3 as a “black box”, and we used it to prove an upper bound for the Minkowski content of $Z$. It turns out that a more careful look at the proof of theorem 9.3 also renders a rectifiable map for $Z$, hence it concludes the proof of theorem 9.2.

Another way to show the rectifiability of $Z$ without opening the “black box” is to cite the following theorem of Azzam and Tolsa.

**Theorem 9.6** ([AT15], Corollary 1.3). Assume $S \subset B(2)$ is a $\mathcal{H}_2$-measurable set and has finite Hausdorff measure, let $\lambda$ be the restriction of $\mathcal{H}_2$ to $S$. Assume that for $\lambda$-a.e. $z$,

$$\int_0^1 \frac{D^2(z, s)}{s} ds < +\infty,$$

then $S$ is 2-rectifiable.

Now invoke theorem 9.6 and let $S$ be the set $Z$. By (41),

$$\int_{B(1)} \int_0^1 \frac{D^2(z, s)}{s} ds d\lambda(z) \leq C \int_{B(1)} \int_0^1 \frac{W^{32s}(p)}{s^3} d\lambda(p) ds d\lambda(z) \leq C \int_{B(2)} \int_0^1 \frac{W^{32s}(p)}{s^3} d\lambda(z) ds d\lambda(p) \leq CK' \int_{B(2)} \int_0^1 \frac{W^{32s}(p)}{s} ds d\lambda(p)$$

The same estimate as (40) gives

$$\int_0^1 \frac{W^{32s}(p)}{s} ds \leq \ln(16)\Lambda.$$ 

Thus

$$CK' \int_{B(2)} \int_0^1 \frac{W^{32s}(p)}{s} ds d\lambda(p) \leq 4C(K')^2 \ln(16)\Lambda < \infty.$$ 

Therefore, the conditions of theorem 9.6 are satisfied for $Z \cap B(1)$, hence $Z \cap B(1)$ is a rectifiable set, and the result is proved.

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1 As Aaron Naber kindly pointed out to the author, this argument could be misleading, because it actually takes an unnecessary detour when all the proofs are unfolded. Nevertheless, it may serve the readers who want to verify the result and are willing to take the established theorems for granted.
Proof of theorem 1.4. Let $R_0$ be the constant given by proposition 7.2. Cover $B_{x_0}(R)$ by finitely many Euclidean balls of radius $R_0/32$. Let $B_{x_1}(R_0/32)$ be such a ball, we claim that there exists a constant $C$ such that

$$I(x, r) = N_\phi(x, r) + Cr^2$$

is a taming function for $Z \cap B_{x_1}(R_0/16)$ on the ball $B_{x_1}(R_0/16)$.

In fact, it follows from the definition that $N_\phi(x, r)$ is non-negative and continuous. By equation (17), there exists $C_1 > 0$ such that $I_1(x, r) = N_\phi(x, r) + C_1r^2$ is increasing in $r$. By proposition 7.2 there exists $C_2$, such that for $I_2(x, r) = I_1(x, r) + C_2r^2$, one has

$$D^2_\mu(x, r) \leq C_1 \int_{B_{x_1}(r)} [I_2(32r) - I_2(2r)]d\mu(x)$$

for every Radon measure supported in $Z \cap B_{x_1}(R_0)$ and $r \leq 8R_0$, thus $I_2$ satisfies condition (3) of definition 9.1.

Notice that since $I_1(x, r)$ is increasing in $r$, for $\beta > 0$, the inequality

$$I_2(x, 2r) - I_2(x, \beta r) < \delta$$

implies that $r < \sqrt{\delta/(4C_2)}$. Therefore, lemma 8.4 implies $I_2$ satisfies condition (2) of definition 9.1.

In conclusion, $I_2(x, r)$ is a taming function for $Z$ on $B_{x_1}(R_0/16)$, therefore theorem 1.4 follows from theorem 9.2. □

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