Geometric Quantization on the Superdisc

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Abstract

In this article we discuss the geometric quantization on a certain type of infinite dimensional super-disc. Such systems are quite natural when we analyze coupled bosons and fermions. The large-$N$ limit of a system like that corresponds to a certain super-homogeneous space. First, we define an example of a super-homogeneous manifold: a super-disc. We show that it has a natural symplectic form, it can be used to introduce classical dynamics once a Hamiltonian is chosen. Existence of moment maps provide a Poisson realization of the underlying symmetry super-group. These are the natural operators to quantize via methods of geometric quantization, and we show that this can be done.

1 Introduction

Geometric quantization is an interesting and useful program for quantizing systems whose phase spaces have a simple geometric description [16]. It is not always the case that the phase space has a nice geometric structure, and even if it does, the result of quantizing via this method does not actually solve the problem but in many cases just helps one to formulate it. The geometric approach to quantization goes back to works of Berezin [3, 4, 5, 6] and at about the same time appeared in the mathematics literature as well.

In this work we will extend our previous analysis [25] to the context of super-geometry. This is interesting in two ways, one is purely mathematical, it gives a natural way to construct unitary representations of the underlying symmetry group. The other one is the possibility of understanding physical systems which have coupled bosons and fermions. Super-geometry sets the natural arena for formulating and studying these problems. Our approach originates from ideas of Rajeev on the large-$N$ limit of field theories. Rajeev has shown that a proper large-$N$ limit of QCD in two dimensions has a natural phase space given by an infinite dimensional Grassmannian [24]. This general philosophy can be extended to other cases.
Whenever there is a mixture of fermions and bosons, the large-$N$ phase space is expected to be a certain kind of super-homogeneous manifold. In gauge theory, we have shown that this space is given by a certain kind of super-Grassmannian [19]. If instead we are looking at a fermionic system which has only a finite number of degrees of freedom coupled to a bosonic field theory, its large-$N$ limit can be formulated as a certain type of super-disc. This can be seen as follows: we get for such a system, in the language of creation and annihilation operators, bilinears of the form

$$N(p, q) = \frac{2}{N} : a^\dagger_\alpha(p) a(\alpha) : , \quad M_{ij} = \frac{2}{N} \chi^{\dagger}_i \chi_{\alpha j}, \quad Q_i(p) = \frac{2}{N} \chi^{\dagger}_i a(\alpha), \quad \bar{Q}_i(p) = \frac{2}{N} a^\dagger_\alpha(p) \chi_{\alpha i},$$  \hspace{1cm} (1)

where we have a normal ordering $: :$, for only the bosonic products and $\alpha$ denotes a “color” index. These operators are the natural ones for the large-$N$ phase space of the theory.

In general it may not be possible to express all the dynamical variables in terms of these bilinears, but if we restrict ourselves to the “color invariant” sector, these are the only ones we can compose. We note that this statement is strictly true when we look at a gauge theory in $1 + 1$ dimensions [19] but for that we need infinite degrees of freedom for the fermions, and that requires an analog of the Grassmannian. In some other cases this is only an approximation to the full model, the validity of which has to be tested depending on the specifics. As an example we write down a non-relativistic model, where a bosonic self-coupled field also couples with localized fermionic sources,

$$H = \int \left( \nabla \phi^\dagger \cdot \nabla \phi + m^2 \phi^\dagger \phi + \frac{\lambda^2}{2} (\phi^\dagger \phi)^2 : + g \sum_i \rho(x)(\phi \chi^\dagger_i(t) + \phi^\dagger \chi_{\alpha i}(t)) \right)$$  \hspace{1cm} (2)

These models may exhibit rather nontrivial dynamics, depending on the dimension we may need to renormalize the coupling constants. Our approach with Hilbert-Schmidt operators excludes cases which require renormalization, although a general super-disc is still present. The above operators actually provide a realization of the super-Lie algebra $U(H_-^c, H^c_+ | H^c_+)$ as we will see. In fact, one can see that many super-Lie algebras have natural realizations by fermionic and bosonic operators [1].

In this article we will only deal with the mathematical aspects of this problem and think of geometric quantization as a method for constructing the quantum Hilbert space where the dynamics takes place. Solving a specific model perhaps should be done first in the classical setting of the large-$N$ limit.

## 2 The Superdisc

In this section we present a brief definition of the superdisc which we denote by $D^1_I$ following the reference [4], we mostly adopt their conventions. As we will see there is a small difference between our approach and this reference. In the same reference there is a nice discussion of other cases, which one can generalize in the same way, but we choose to look at the above simpler case for the sake of clarity. The previous paper by the same authors [8] give a more detailed discussion of the $U(1, 1|1)$ case, since the general case in [4] is treated in a succinct manner, we prefer to give a detailed discussion and believe that some of the explicit
formulae could be useful for the reader. The physically interesting case requires an additional
 complication compared to the one in [4], one should look at an infinite Grassmann algebra.
 We will briefly discuss this generalization, yet the results are not so simple and as rigorous
 as in the finite dimensional one. Some other useful sources are the lectures of Kostant [20]
 and the books by Berezin [2] and Manin [21].

Let us consider two Hilbert spaces, $\mathcal{H}^e$ and $\mathcal{H}^o$ which correspond to the even and odd
 spaces respectively. In physically interesting cases they are either both separable infinite
dimensional, or the even one is separable infinite dimensional and the odd one is finite di-
 mensional. To keep the rigor we will only deal with $\mathcal{H}^o_\infty$ finite dimensional, but arbitrarily
 large. Let us assume that its dimension is $N$, later on we will extend this to infinite dimen-
sions. We will split the even space into positive and negative parts, each piece being infinite
dimensional, $\mathcal{H}^e = \mathcal{H}^e_+ \oplus \mathcal{H}^e_-$. We will really think of the odd part as the positive subspace
 and denote it as $\mathcal{H}^o_+$, this is just for convenience at the moment since we have not attached
 any physical significance to $D$.

We may denote the $\mathbb{Z}$ graded super-space as $\mathcal{H}$, which splits with respect to $\mathbb{Z}_2$ grading
 as $\mathcal{H}^e|\mathcal{H}^o$. It will be better to decompose this space as $\mathcal{H} = \mathcal{H}^e_+ \oplus \mathcal{H}^o_+|\mathcal{H}^o_+$. Let us introduce
 the set of complex super matrices $Z$ such that

$$Z = \begin{bmatrix} w & \theta \end{bmatrix}$$

where $w : \mathcal{H}^e_+ \to \mathcal{H}^e_+$ and $\theta : \mathcal{H}^o_+ \to \mathcal{H}^e_+$, furthermore we require the following convergence
 conditions $w \in \mathcal{I}_2$ and $\theta \in \mathcal{I}_2$, where $\mathcal{I}_2$ denotes the Hilbert-Schmidt ideal in this context. A
 super space is given by the algebra of smooth functions living on it, in any given super-chart
 $\mathcal{U}$ we have $C^\infty(\mathcal{U}) \approx C^\infty(\mathcal{U}) \otimes \Lambda(\mathbb{C}^n)$ for some $s$, and here $\mathcal{U}$ denotes the corresponding open
 set for the base manifold. (In [4] the underlying function algebra for the odd generators is
 chosen to be $\Lambda(\mathbb{C}^{mq})$). We will instead take the set of generators as $\Lambda(\mathbb{C}^n)$, and $\theta$ denotes the matrix of linear transformations from the super vector space $\mathcal{H}^o_+$ to $\mathcal{H}^e_+$.

Let us explain the meaning of these convergence conditions: if we expand the matrix $w$
 into a series

$$w = w_B + w_{a_1 a_2} \xi^a_1 \xi^a_2 + ...$$

(4)

where $\xi^a$ denotes half of the odd generators and this series terminates. There are also
 hermitian conjugates, that is we have a set of coordinates $\xi^a$ and $\xi^{a*}$. (Since the base manifold
 is contractable this expression is true, otherwise we need to assume it on any given chart).
 Then, we assume that each one of these matrices are in the Hilbert-Schmidt class, i.e.
 $w_B^1 w_B^2, w_{a_1 a_2}^1, ..., w_{1,2,...,r,a_1,a_2}^r \in \mathcal{I}_1$, here we use $\mathcal{I}_1$ to denote trace class operators. This
 decomposition is basis dependent, but the condition is basis invariant. It is possible to see
 this by looking at a change of basis which is given by an invertible super-matrix(non-type
 changing one):

$$(SwS^{-1})_{a_1 a_2 ... a_{2k}} B = S_B w_B S_B^{-1} ...$$

etc, and we see that each component is replaced by a sum, each term of which is conjugated
 by some bounded operators. The conjugated elements themselves are of Hilbert-Schmidt
 class. From this we conclude that our condition is basis independent. We point out that
some variants of this argument on Hilbert-Schmidt condition will be used over and over again. Same for \( \theta \) except that \( \theta \) only has odd terms. Notice that the second of these conditions is automatically true since the odd space is finite dimensional. In a more general case we will mention later on, there will be extra convergence conditions on the odd generators. In this setting \( w \) is even and \( \theta \) is odd. For computations it is sometimes better to decompose a given matrix into its ordinary part and its nilpotent part, just like a super number being decomposed into an ordinary complex number plus the rest. We use the terminology of deWitt and call it body and soul decomposition. For example \( w \) decomposed into an ordinary complex number plus the rest. We use the terminology of this setting \( w \) will mention later on, there will be extra convergence conditions on the odd generators. In is automatically true since the odd space is finite dimensional. In a more general case we except that \( \theta \) only has odd terms. Notice that we can interpret these to be the elements which generate the super-operators \( Z \) us define the restricted super-disc as the algebra of functions generated by the above set of super-operators \( Z \) with a further condition on \( w \),

\[
1 - w_B^\dagger w_B > 0
\]

Notice that we can interpret these to be the elements which generate the \( C^\infty \) functions on the superdisc. For later use we must give a meaning to \( Z^\dagger Z \), so we define it to be the tensor product, \( Z^\dagger Z = \begin{pmatrix} w^\dagger w & w^\dagger \theta \\ \theta^\dagger w & \theta^\dagger \theta \end{pmatrix} \). We do not demand any extra conditions on the \( \theta \) variable. The inverse of \( 1 - Z^\dagger Z \) can be computed; we write

\[
(1 - Z^\dagger Z)^{-1} = 1 + Z^\dagger (1 - ZZ^\dagger)^{-1} Z
\]

similarly for \( w_B \) we have \( (1 - w_B w_B^\dagger) = 1 + w_B (1 - w_B w_B^\dagger)^{-1} w_B^\dagger \) and the operator on the right is well-defined due to positivity condition, this means that the inverse on the left also exists. Since we use a finite dimensional odd-space we can define

\[
(1 - ZZ^\dagger)^{-1} = (1 - w_B w_B^\dagger - w_B^\dagger w_S - w_S^\dagger w_B - w_S^\dagger w_S^\dagger) \theta^\dagger \theta)^{-1} = [1 - (1 - w_B w_B^\dagger)^{-1}(w_B^\dagger w_S + w_S^\dagger w_B + w_S^\dagger w_S + \theta^\dagger \theta)]^{-1}(1 - w_B w_B^\dagger)^{-1},
\]

the first inverse in the last term can be expressed via a terminating expansion,

\[
[1 - (1 - w_B w_B^\dagger)^{-1}(w_B^\dagger w_S + w_S^\dagger w_B + w_S^\dagger w_S + \theta^\dagger \theta)]^{-1} = 1 + (1 - w_B w_B^\dagger)^{-1}(w_B^\dagger w_S + w_S^\dagger w_B + w_S^\dagger w_S + \theta^\dagger \theta) - ... + (1 - w_B w_B^\dagger)^{-1}(w_B^\dagger w_S + w_S^\dagger w_B + w_S^\dagger w_S + \theta^\dagger \theta)^s,
\]

where we assume that the degree of nilpotency of the supermatrix is \( s + 1 \). We note that this is a general fact, if the body of a matrix is invertible then the matrix is invertible. This series does not have to terminate in the infinite dimensional case, so one has to impose the invertibility condition separately, or assume that the infinite formal expansion can be given a meaning( see the book by deWitt \[10\]). The definition we propose later on may result in a deviation from the Kostant-Berezin-Leites definition \[2, 3\].

There is a natural super-operator on the space \( \mathcal{H} \) given with respect to the above direct sum as:

\[
J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.
\]

Similar to the finite dimensional case, we have an action of a certain super-pseudounitary group on the super-disc \( D_1^I \). Let us define the set of superoperators \( g: \mathcal{H} \rightarrow \mathcal{H} \) with a
bounded inverse, such that they leave the operator $J$ invariant:

$$gJg^\dagger = J.$$  \hspace{1cm} (8)

Let us explicitly write this condition in a block decomposition:

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$ \hspace{1cm} (9)

and here $A: \mathcal{H}_-^e \to \mathcal{H}_-^e$, $B: \mathcal{H}_+^e|\mathcal{H}_0^o \to \mathcal{H}_-^e$, $C: \mathcal{H}_-^e \to \mathcal{H}_+^e|\mathcal{H}_0^o$, finally $D: \mathcal{H}_+^e|\mathcal{H}_0^o \to \mathcal{H}_+^e|\mathcal{H}_0^o$. This representation is better suited for our needs. We have then

$$AA^\dagger - BB^\dagger = 1 \quad CA^\dagger = DB^\dagger \quad DD^\dagger - CC^\dagger = 1.$$ \hspace{1cm} (10)

Using the invertibility we see that $g^\dagger Jg = J$ is also true, hence we get

$$A^\dagger A - C^\dagger C = 1 \quad A^\dagger B = C^\dagger D \quad D^\dagger D - B^\dagger B = 1.$$ \hspace{1cm} (11)

The first one means in terms of body and soul decomposition,

$$A_BA_B^\dagger - B_BB_B^\dagger = 1 \quad A_S^\dagger A_B + A_B^\dagger A_S + A_S^\dagger A_S + C_B^\dagger C_S + C_S^\dagger C_B + C_S^\dagger C_S = 0$$ \hspace{1cm} (12)

and similarly for the others. This means that the body parts of these matrices obey exactly the usual pseudounitary conditions hence we can do everything in the same way like the non-supercase. Among these set of operators we pick the ones which satisfy a convergence condition, written with respect to the direct sum decomposition $\mathcal{H}_-^e \oplus \mathcal{H}_+^e|\mathcal{H}_0^o$:

$$g = \begin{pmatrix} \mathcal{B} & \mathcal{I}_2 & \mathcal{I}_2 \\ \mathcal{I}_2 & \mathcal{B} & \mathcal{B} \\ \mathcal{I}_2 & \mathcal{B} & \mathcal{B} \end{pmatrix},$$ \hspace{1cm} (13)

and these conditions are imposed on the components of each term, i.e. if we expand the upper corner, $\beta = \beta_{\alpha}\xi^{a} + \beta_{\alpha_{1}\alpha_{2}a_{3}}\xi^{a_{1}}\xi^{a_{2}}\xi^{a_{3}} + \ldots$, each term belongs to $\mathcal{I}_2$ and similarly for the other parts. We may also economically express these in the form $[J, g] \in \mathcal{I}_2$, with the above interpretation for the ideal. Therefore we can summarize the above set of operators in the form of a group:

$$U_1(\mathcal{H}_-^e, \mathcal{H}_+^e|\mathcal{H}_0^o) = \{ g|g^{-1} \text{ exists, } [J, g] \in \mathcal{I}_2 \text{ and } gJg^\dagger = J \},$$ \hspace{1cm} (14)

where the ideal condition refers to our convention. The main point is to show that the convergence conditions hold after the multiplication. This follows the same line of arguments as before, if one writes explicitly the components, we see that each one is a finite sum of Hilbert-Schmidt operators. We leave it to the reader to check the details. This group is one possible super version of the pseudounitary group. We refer to this set the restricted super-pseudounitary group.

Just like the classical case, the restricted super-pseudounitary group has an action on the super-disc $D^1_I$. This action is written in the super-operator language exactly as in the classical case:

$$Z \mapsto (AZ + B)(CZ + D)^{-1},$$ \hspace{1cm} (15)
where we use
\[
g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}
\]

We need to clarify the action of \( C \), if we denote \( C \) as \( \begin{pmatrix} c \\ \gamma \end{pmatrix} \),
\[
CZ = C \otimes Z = \begin{pmatrix} cw & c\theta \\ \gamma w & \gamma \theta \end{pmatrix},
\]
which shows that the action is well-defined and the resulting operator goes from \( \mathcal{H}_+^c|\mathcal{H}_+^g \) to \( \mathcal{H}_+^c|\mathcal{H}_+^g \), thus we can add \( D \) to this. Let us note that the inverse on the right exists, this is because the even part has an inverse and we can define the inverse by a terminating expansion. Just for an illustration we give the explicit version, the reader who is familiar with this kind of manipulations is advised to skip this part: We would like to show that \( CZ + D \) has an inverse. We know that \( D^{-1} \) is well-defined, hence it is better to look at \( D^{-1}CZ + 1 \). We use the following formula for the inverse of a supermatrix:
\[
\left( \begin{array}{cc} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{array} \right)^{-1} = \left( \begin{array}{cc} (\tilde{A} - \tilde{B}\tilde{D}^{-1}\tilde{C})^{-1} & -\tilde{A}^{-1}\tilde{B}(\tilde{D} - \tilde{C}\tilde{A}^{-1}\tilde{B})^{-1} \\ -\tilde{D}^{-1}\tilde{C}(\tilde{A} - \tilde{B}\tilde{D}^{-1}\tilde{C})^{-1} & (\tilde{D} - \tilde{C}\tilde{A}^{-1}\tilde{B})^{-1} \end{array} \right)
\]
(18)
This can be written in the following form:
\[
\begin{pmatrix}
(1 + (d_{11} - \delta_{12}d_{22}^{-1}\delta_{21})^{-1}cw - d_{11}^{-1}\delta_{12}(d_{22} - \delta_{21}d_{11}^{-1}\delta_{12})^{-1}\gamma w & \ast \ast \ast \\
\ast \ast \ast & 1 - d_{22}^{-1}\delta_{21}(d_{11} - \delta_{12}d_{22}^{-1}\delta_{21})^{-1}c\theta + (d_{22} - \delta_{21}d_{11}^{-1}\delta_{12})^{-1}\gamma \theta
\end{pmatrix}
\]
(19)
To prove the invertibility we do not need the explicit forms of the off-diagonal components, this is why they are not shown in the above matrix. The lower diagonal block is invertible, due to the nilpotency of the part added to 1. Hence we need to check only the upper diagonal block (actually, this is a general result). To do this we recall that the super-pseudounitarity means, \( DD^\dagger = 1 + CC^\dagger \), written in terms of components, the upper block gives us \( d_{11}d_{11}^\dagger = cc^\dagger + 1 - \delta_{12}\delta_{12}^\dagger \). This means that we can define an inverse square root of the above matrix; for this we use the following integral representation,
\[
T^{-1/2} = \frac{1}{\pi} \int_0^\infty \frac{d\lambda}{\lambda^{1/2}} (T + \lambda I)^{-1}
\]
(20)
This formula is used for a positive operator and can be extended to the super-case when the body of the super-matrix is positive (this is the case for us as we will see shortly). As a result we get,
\[
(d_{11}d_{11}^\dagger)^{-1/2} = \frac{1}{\pi} \int_0^\infty \frac{d\lambda}{\lambda^{1/2}} [(c_Bc_B^\dagger + \lambda 1 + 1)^{-1} + (c_Bc_B^\dagger + \lambda 1 + 1)^{-1} f_S(c_Bc_B^\dagger + \lambda 1 + 1)^{-1} + ... -(-1)^r((c_Bc_B^\dagger + \lambda 1 + 1)^{-1}f_S(c_Bc_B^\dagger + \lambda 1 + 1)^{-1}] .
\]
here we use \( f_S = c_Bc_B^\dagger + c_Sc_S^\dagger + c_Sc_S^\dagger + \delta_{12}\delta_{12}^\dagger \) which is a nilpotent matrix and we assumed that it has degree \( r + 1 \). The first term is the usual term \( (1 + c_Bc_B^\dagger)^{-1/2} \), the others are nilpotent contributions. Hence when we write this first diagonal block in this form,
\[
1 + u(1 + c_Bc_B^\dagger)^{-\frac{1}{2}}c_Bw_B + \text{nilpotent parts}
\]
(21)
where $u$ is a unitary piece that we cannot determine—stripped off from its possible nilpotent part. It is enough to show that this leading part is invertible, but that is the same as in non-super case: $\|w_B^\dagger c_B^\dagger (1 + c_Bc_B^\dagger)^{-1/2}w^\dagger u(1 + c_Bc_B^\dagger)^{-1/2}c_Bw_B\| \leq \|w_B^\dagger w_B\| < 1$, this implies that the series expansion will converge and we have an invertible element. This concludes our demonstration. Of course we have done more than just showing that the inverse is well-defined, we also got an expansion of the inverse, which is useful to show the convergence condition in the infinite dimensional case. There is a simpler way to show the invertibility, which we repeat here for clarity,

$$
(CZ + D) = \begin{pmatrix} cw + d_{11} & c\theta + d_{12} \\ \gamma w + d_{21} & \gamma\theta + d_{22} \end{pmatrix},
$$

and as we have observed for invertibility it is enough to know the invertibility of the body parts, we have $d_{11}d_{11} = c\gamma + 1 - \delta_{12}\delta_{12}$, the body parts satisfy $(d_{11})B(d_{11})_B = 1 + c_Bc_B^\dagger$, and using the same argument as before this implies that the body is invertible (we already know $d_{22}$ is invertible).

One can check that the resulting operator $Z'$ is an element of the super-disc. We briefly indicate how this is done: The convergence conditions are easy since we have $Z \in \mathcal{I}_2$ and $B \in \mathcal{I}_2$. If we want to show that the resulting operator satisfies $1 - w_B'^\dagger w_B' > 0$, we look at $(AZ + B)(CZ + D)^{-1})_{B'}$, this comes from $w'$,

$$w' = (Aw + b)(cw + d_{11} - (c\theta + d_{12})(\gamma\theta + d_{22})^{-1}(\gamma w + d_{21}))^{-1},$$

which has body part:

$$w_B' = (ABw_B + b_B)(c_Bw_B + (d_{11})B)^{-1}.
$$

We have from the pseudo-unitarity conditions, $AB_A^\dagger B - b_Bb_B^\dagger = 1$, $c_BA_B^\dagger = (d_{11})Bb_B^\dagger$, $(d_{11})B(d_{11})_B - c_Bc_B^\dagger = 1$. But these are exactly the conditions for the ordinary pseudo-unitary group $U_1(\mathcal{H}^c, \mathcal{H}^c_+)$, hence the positivity condition follows as in the ordinary case. Of course the point is to show that the action is transitive, and hence to prove that the super-disc is a homogeneous manifold. Let us go over this point as well using similar techniques to the above proof. To prove this it is enough to show that the action is transitive over the generating set of elements for the $C^\infty(D^\dagger_1)$ we introduced, $Z = [w \theta]$. (Notice that a super manifold is really defined through the algebra of functions living on it.) Let us show that we can obtain all the generators starting from $Z = 0$ using the group action. Recall that the pseudo-unitarity imposes the following conditions,

$$AA^\dagger - BB^\dagger = 1 \quad CA^\dagger = DB^\dagger \quad D^\dagger D - B^\dagger B = 1,
$$

the last one uses the opposite multiplication. For any $Z = BD^{-1}$, if we insert this into the last one we see that $D = (1 - Z^\dagger Z)^{-1/2}U$, where $U$ is an arbitrary super-unitary element acting on the same space, is a solution. Later on we will prove that this square root makes sense and the body belongs to the desired class, but first we will present the formal solution in the super-matrix form:

$$
g = \begin{pmatrix} (1 - ZZ^\dagger)^{-1/2}V & Z(1 - Z^\dagger Z)^{-1/2}U \\ (1 - Z^\dagger Z)^{-1/2}V & (1 - Z^\dagger Z)^{-1/2}U \end{pmatrix}.
$$
where $V \in U(H^c_-)$ and $U \in U(H^c_+|H^o_+)$. In fact this shows the ambiguity in the solution to be exactly the subset we mode out with. Let us prove the claim using the integral form of the square root of the matrix, we begin with $A,
abla$
\[
(1 - ZZ^\dagger)^{-1/2} = \frac{1}{\pi} \int_0^\infty \frac{d\lambda}{\lambda^{1/2}} (\lambda 1 + 1 - \theta^\dagger w - \theta^\dagger)^{-1} 
\]
\[
= (1 - w_Bw_B^\dagger)^{-1/2} + \frac{1}{\pi} \int_0^\infty \frac{d\lambda}{\lambda^{1/2}} (\lambda 1 + 1 - \theta^\dagger w w_B^\dagger)^{-1} f_S(\lambda 1 + 1 - \theta^\dagger w_B^\dagger)^{-1} + ... 
\]
\[
- (1)^{\dagger} \frac{1}{\pi} \int_0^\infty \frac{d\lambda}{\lambda^{1/2}} (\lambda 1 + 1 - \theta^\dagger w_B^\dagger)^{-1} [f_S(\lambda 1 + 1 - \theta^\dagger w_B^\dagger)^{-1}]^r,
\]
where $f_S = w_S w_B^\dagger + w_B^\dagger w_S^\dagger + \theta \theta^\dagger$. Notice that everything is well-defined here. Let us now indicate that $D$ is well-defined, we do this for the upper corner only.

\[
D = (1 - Z^\dagger Z)^{-1/2} = \frac{1}{\pi} \int_0^\infty \frac{d\lambda}{\lambda^{1/2}} \left( \begin{array}{ccc}
\lambda & 1 - w^\dagger w & -w^\dagger \theta \\
-\theta^\dagger w & \lambda & 1 - \theta^\dagger \theta \\
\lambda & 1 - \theta^\dagger \theta & \lambda
\end{array} \right)^{-1} \tag{28}
\]
As a result of this expression we see that all the elements are well-defined and belong to the correct classes, in fact we can write the expansion for $d_{11},$

\[
d_{11} = (1 - w_B^\dagger w_B)^{-1/2} + \frac{1}{\pi} \int_0^\infty \frac{d\lambda}{\lambda^{1/2}} (\lambda 1 + 1 - \theta^\dagger w w_B)^{-1} 
\]
\[
\times (w_B^\dagger w_S + w_S^\dagger w_B + w_S^\dagger w_B^\dagger + w^\dagger \theta (\lambda 1 - \theta^\dagger \theta)^{-1} \theta^\dagger w) (\lambda 1 + 1 - \theta^\dagger w_B)^{-1} + ....
\]
where the series terminates. we see that everything is well-defined. One can see that the rest of it can be done in a simple way since the expressions for $B, C$ have explicit multiplicative factors of $Z$ which is in the Hilbert-Schmidt class, so we skip the details for brevity. Let us also check again the stability subgroup of $Z = 0$ is given by $U(H^c_-) \times U(H^c_+|H^o_+)$. For $Z = 0$, $Z' = BD^{-1}$, if we set this to zero, since $D$ is invertible, we get $B = 0$. From the invariance of $J$ we get $AA^\dagger = 1$, and this together with $AC^\dagger - BD^\dagger = 0$ implies $C = 0$. The result of this is the diffeomorphism we are after:

\[
D^1 = U_1(H^c_-, H^c_+|H^o_+) / U(H^c_-) \times U(H^c_+|H^o_+). \tag{29}
\]
We emphasize that the explicit coordinate $Z$ shows that this is a super-complex manifold, the group action point of view instead shows that this space is a super homogeneous space.

### 3 Supersymplectic Structure

In this section we will discuss the classical mechanics on this super disc. There is a natural supersymplectic structure, it is homogeneous and further more it is Kähler. This is a natural choice from the point of view of geometry and as we will see it also provides us a natural method of quantization, which is an extension of the Bargmann representation

\[^1\text{Just for fun, we suggest the reader to show the following identity, which gives an alternative proof of the existence,}
\]
\[
(1 - Z^\dagger Z)^{-1/2} = 1 + Z^\dagger ((1 - ZZ^\dagger)^{-1/2} - \frac{1}{2} \int_0^1 dt (1 - tZZ^\dagger)^{-1/2}) Z
\tag{27}
\]
to this case\cite{23}. The analysis of symmetric domains and the use of Toeplitz operators in the quantization problem is thoroughly discussed in the book by Upmeier \cite{29}. We also recommend the articles by Borthwick et al\cite{9}.

It will be simpler to use the following super operator to show that the disc is a supersymplectic space:

\[
\Phi = -1 + 2 \begin{pmatrix}
(1 - ZZ^\dagger)^{-1} & -(1 - ZZ^\dagger)^{-1}Z \\
Z^\dagger(1 - ZZ^\dagger)^{-1} & -Z^\dagger(1 - ZZ^\dagger)^{-1}Z
\end{pmatrix},
\]

(30)

Notice that this operator is well-defined on \(H^c_+ \oplus H^c_+|H^0\). The reader can check that

\[
\Phi^2 = 1, \quad J\Phi^\dagger J = \Phi.
\]

(31)

An important point is that the action of the group on \(Z\) becomes very simple on \(\Phi, Z \rightarrow g \circ Z\) induces \(\Phi \rightarrow g\Phi g^{-1}\)(see Appendix). \(Z = 0\) corresponds to \(\Phi = J\), and we can check that \(\Phi(Z) = g(Z)Jg(Z)^{-1}\)(see Appendix). We may define a symplectic form on \(D^1\) using \(\Phi\); formally,

\[
\Omega = \frac{i}{4} \text{Str}\Phi d\Phi \wedge d\Phi.
\]

(32)

What we mean by this two form is that if we take two vector fields \(V_u, V_v\), which are generated by the action of the super-Lie group, we get a number:

\[
\Omega(V_u, V_v) = \frac{i}{8} \text{Str}[J, g^{-1}u g]_s[J, g^{-1}v g]_s.
\]

(33)

Using the above formal expression, we see that \(\Omega\) is closed and furthermore it is homogeneous. This easily follows from the transformation of \(\Phi\) under the group action. One can actually see this by looking at the explicit form of it. The nondegeneracy and super-Kähler structures are best understood around \(J\), then we use the homogeneity to distribute this form over all the manifold. When we restrict ourselves to the point \(J\):

\[
\Omega|_{Z=0} = i \text{Str} \begin{pmatrix}
-dZ \wedge dZ^\dagger & 0 \\
0 & dZ^\dagger \wedge dZ
\end{pmatrix},
\]

(34)

here the two wedge products have different meanings: \(dZ \wedge dZ^\dagger = dw \wedge dw^\dagger + d\theta \wedge d\theta\), and \(dZ^\dagger \wedge dZ = \begin{pmatrix} dw^\dagger \wedge dw & dw^\dagger \wedge d\theta \\
d\theta^\dagger \wedge dw & d\theta^\dagger \wedge d\theta
\end{pmatrix}\). Hence we can rewrite this expression as,

\[
\Omega|_{Z=0} = i \text{Str} \begin{pmatrix}
-dw \wedge dw^\dagger - d\theta \wedge d\theta^\dagger & 0 \\
0 & (dw^\dagger \wedge dw \wedge d\theta \wedge d\theta)
\end{pmatrix}.
\]

(35)

By expanding the trace,

\[
\Omega|_{Z=0} = i [-\text{Tr} dw \wedge dw^\dagger - \text{Tr} d\theta \wedge d\theta^\dagger + \text{Tr} \begin{pmatrix} dw^\dagger \wedge dw & dw^\dagger \wedge d\theta \\
d\theta^\dagger \wedge dw & d\theta^\dagger \wedge d\theta
\end{pmatrix}]
\]

\[
= i [-\text{Tr} dw \wedge dw^\dagger - \text{Tr} d\theta \wedge d\theta^\dagger + \text{Tr} dw^\dagger \wedge dw - \text{Tr} d\theta^\dagger \wedge d\theta]
\]

\[
= -2i \text{Tr} dw \wedge dw^\dagger - 2i \text{Tr} d\theta \wedge d\theta^\dagger = -2i \text{Tr} dZ \wedge dZ^\dagger.
\]

(This incidentally shows that the form is super-Kähler\cite{13, 14, 21}). By contracting this with two vector fields at the origin, we get

\[
\Omega(V_u, V_v)|_{Z=0} = -2i[\text{Tr}(b_1b_2^\dagger - b_2b_1^\dagger) + \text{Tr}(\beta_1\beta_2^\dagger + \beta_1^\dagger\beta_2)].
\]

(36)
Notice that we use the vector $[b \beta]$ for the component $u_{12}$ of the Lie algebra element (this could be somewhat confusing but we try to avoid the proliferation of indices). Using the above form it is possible to prove the nondegeneracy, this is given in the Appendix. The symplectic form above provides us with a Poisson structure.

One can define classical dynamics on this superspace, given an even Hamiltonian, a physical observable, $H$. The time evolution of any observable $O$ is given by

$$\frac{\partial O}{\partial t} = \{H, O\}_s \quad (37)$$

One can naturally ask if there are moment maps which generate the group action. It is not possible to use $F_u = \frac{i}{2} \text{Str} \Phi u$ due to divergence of the trace, but it is possible to do a vacuum subtraction and get a convergent one. To prove this we use a rearrangement of the formula for $\Phi$:

$$\Phi(Z) = J + \left( \begin{array}{cc} 2Z(1 - Z^t Z)^{-1} Z^t & -2(1 - ZZ^t)^{-1} Z \\ 2Z^t (1 - ZZ^t)^{-1} & -2Z^t (1 - ZZ^t)^{-1} Z \end{array} \right), \quad (38)$$

If we look at now the difference, $\Phi(Z) - J$ the last part remains. The diagonal parts of this operator are better behaved than the off-diagonal parts, $Z(1 - Z^t Z)^{-1} Z^t \in \mathcal{I}_1$ in our sense as one can see. and similarly for the other one. The off-diagonal parts are actually in $\mathcal{I}_2$. So when we look at $(\Phi(Z) - J)u$, we see that,

$$\left( \begin{array}{ccc} \mathcal{I}_1 & \mathcal{I}_2 & \mathcal{I}_2 \\ \mathcal{I}_2 & \mathcal{I}_1 & \mathcal{I}_1 \end{array} \right) \left( \begin{array}{ccc} \mathcal{B} & \mathcal{I}_2 & \mathcal{I}_2 \\ \mathcal{I}_2 & \mathcal{B} & \mathcal{B} \end{array} \right) = \left( \begin{array}{ccc} \mathcal{I}_1 & \mathcal{I}_2 & \mathcal{I}_2 \\ \mathcal{I}_2 & \mathcal{I}_1 & \mathcal{I}_1 \end{array} \right). \quad (39)$$

Hence a conditional trace exists: if we throw away the nontrace parts, $\text{Str}_J(\Phi(Z) - J)u = \frac{1}{2} \text{Str}[(\Phi(Z) - J)u + J(\Phi(Z) - J)uJ]$ is actually convergent. We see that this is very similar to the ordinary disc case in [25].

A general discussion shows that we get a Poisson realization of the super-Lie algebra through the moment maps:

$$\{F_u, F_v\}_s = F_{[u,v]_s} + \Sigma_s(u,v) \quad (40)$$

It is possible to find this central term by evaluating everything at the origin, $\Phi = J$:

$$\Sigma_s(u,v) = \frac{i}{8} \text{Str} \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{array} \right) \left[ \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{array} \right), u \right] \left[ \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{array} \right), v \right]_s$$

$$= \frac{i}{2} \text{Str}_J[J,u]v,$$

It is interesting to write down the central term explicitly:

$$\Sigma_s(u,v) = -2i \text{Str} \left( \begin{array}{ccc} b_1 b_2^t + \beta_1 \beta_2^* \\ b_1^t b_2 \\ \beta_1^t \beta_2 \end{array} \right)$$

$$= -2i \left( \text{Tr}(b_2 b_1^t) + \text{Tr}(\beta_1 \beta_2^*) - \text{Tr}(\beta_1^t \beta_2) \right)$$
and we see that at each step the diagonals are in $I_1$ and hence the traces are all well-defined. This is equal to the symplectic form at the origin we computed before using the explicit coordinate $Z$ as it should be. This type of central term is expected when there are bosons and fermions mixed. An interesting discussion of such central extensions from the Fock space point of view is given in [13]. In [11] $Z_2$ graded Schwinger terms for neutral particles are worked out, in [12] current super-algebras are studied, providing a generalization of Mickelsson-Rajeev cocycle [22]. The use of pseudodifferential operators in this reference we believe is better motivated in these higher dimensional cases. There should be a similar extension of our results using this restricted class of operators.

We are therefore equipped with a powerful geometric setting to develop our geometric quantization program.

4 Geometric Quantization

Our presentation here will be somewhat more concise, most of the computations can be done similar to our previous work, except one has to watch for the signs. The technical details and explanation of the main ideas are already given in [20], we recommend the examples in [13, 14], and one can read a more general program in [30] (we believe it is interesting to follow the philosophy of the last reference).

We can follow exactly the same steps in [25] introduce a prequantization line bundle (for ordinary geometric quantization we refer to [16, 17, 18, 31]), and we introduce a super-one-form on this bundle:

$$\Theta_s = \frac{1}{\hbar} (\text{Str}(1-Z^\dagger Z)^{-1}dZ^\dagger Z - \text{Str}(1-Z^\dagger Z)^{-1}Z^\dagger dZ).$$

(41)

This is use to define the covariant derivate as in the nonsuper case [4],

$$\nabla_V = L_V^s + \Theta_s(V),$$

(42)

where we used a superscript to denote the super-Lie derivative. For any given super-function, we have the vector field generated from the symplectic form,

$$\Omega(V_f,*) = -df$$

(43)

using this vector field a prequantization operator is obtained,

$$\tilde{f} = -i\hbar \nabla_{V_f} + f$$

(44)

This gives us a representation of the Poisson brackets:

$$\{\tilde{f}, \tilde{g}\}_s = -i\hbar [\tilde{f}, \tilde{g}]_s.$$ 

(45)

\footnote{strictly speaking in the model of super-sections this acts on the a prolongation, $\Gamma(M, \wedge (\mathbb{C}^N) \otimes K)$ where $K$ is a prequantum complex line bundle on the base $M$.}
As in the ordinary case, we need to restrict the prequantum Hilbert space, since the prequantization map does not lead to an irreducible representation. We will choose superholomorphic functions,
\[ \nabla_{Z^\dagger} \psi(Z, Z^\dagger) = 0. \] (46)

The super analysis is designed to provide a complete analogy to the usual analysis, hence most of what we said follows from a routine yet long (and care required due to signs) computations.

We can solve for this holomorphicity condition as in the ordinary case:
\[ \nabla_{Z^\dagger} \psi = 0, \quad \psi(Z, Z^\dagger) = s\det(1 - Z^\dagger Z)\Psi(Z) \] (47)

where \( \Psi(Z) \) denotes a superholomorphic function on the disc. We define the superdeterminant (or Berezinian) as
\[ s\det(\begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{pmatrix}) = \det(\begin{pmatrix} \tilde{A} - \tilde{B}\tilde{D}^{-1}\tilde{C} & (\det \tilde{D})^{-1} \end{pmatrix}, \] (48)

where the operator is written according to the even and odd decomposition of the super Hilbert space. The infinite dimensionality of the underlying space requires the full operator to be of the form \( 1 + \mathcal{I} \), otherwise one has to use a conditional determinant. The resulting operators for the moment maps acting on holomorphic sections will be exactly the same as in the ordinary case,
\[ \hat{F}_u \Psi(Z) = -i\hbar[\mathcal{L}_{\nu_u}^a - \frac{1}{\hbar}\text{Str}(u_{21}Z)]\Psi(Z) \] (49)

where we have used the same letters to denote the components of the Lie algebra elements, \( u = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \), not to bring new notation. Holomorphicity is clearly preserved and these are the correct operators to start a quantization program.

These moment maps can be integrated to a representation of a central extension of the super-pseudounitary group:
\[ \rho(g^{-1})\Psi(Z) = s\det^{-\frac{1}{\hbar}}[(D^{-1}CZ + 1)\Psi((AZ + B)(CZ + D)^{-1})] \] (50)

This is a well-defined representation, let us see that the determinant exits:
\[ \begin{pmatrix} (d^{-1})_{11} & (\delta^{-1})_{12} \\ (\delta^{-1})_{21} & (d^{-1})_{22} \end{pmatrix} \begin{pmatrix} cw & c\theta \\ \gamma w & \gamma \theta \end{pmatrix} = \begin{pmatrix} B & B \\ B & B \end{pmatrix} \begin{pmatrix} \mathcal{I}_{1} & \mathcal{I}_{1} \\ \mathcal{I}_{1} & \mathcal{I}_{1} \end{pmatrix} = \begin{pmatrix} \mathcal{I}_{1} & \mathcal{I}_{1} \\ \mathcal{I}_{1} & \mathcal{I}_{1} \end{pmatrix} \]. (51)

This shows that the determinant is absolutely convergent–independent of the basis chosen–. The central term of the representation is given by,
\[ c_S(g_1, g_2) = s\det^{\frac{1}{\hbar}}[(D_1D_2)^{-1}C_1B_2 + 1], \] (52)

derivation of this supercentral term does not present any more difficulties than the ordinary case (see the appendix of [PS]), convergence issue follows the same lines as the above one, we leave the details to the reader.
5 Infinite dimensional case

We will propose a way of extending our results when \( \mathcal{H}^0 \) is infinite dimensional. In this section we will not repeat the previous arguments, since some of them are direct generalizations and some of them require a much deeper study. We plan to come back to those issues in another publication, so in this section we only give a sketch of ideas. While we were working on this problem, we became aware of a rather similar set of ideas by Schmidt in [26]. Which set of ideas are more appropriate for our problem is not so clear to us at this moment, so we follow our point of view, we plan to take a more detail study of all these issues in the future.

First we change our notion of a super-number:

\[
  z = z_B + \sum_{N=0}^{\infty} \sum_{a_1 < a_2 < \ldots < a_N} z_{a_1 a_2 \ldots a_N} \xi^{a_1} \xi^{a_2} \ldots \xi^{a_N},
\]

where we assume that the sums have square integrable coefficients \( \sum_N \sum_{a_1 < a_2 < \ldots < a_N} |z_{a_1 a_2 \ldots a_N}|^2 < \infty \). This makes the product of two super numbers well-defined, hence behaves much better than the formal sums, it is physically more transparent as well. The product becomes

\[
  (zt)_{a_1 a_2 \ldots a_N} = \sum_n z_{a_1 a_2 \ldots a_n} t_{a_{n+1} \ldots a_N},
\]

here \( (\ldots) \) denotes an appropriate symmetrization of the indices, due to the ordering of the generators(keeping the previous ordering in mind). From a more abstract point of view when we look at the algebra of smooth functions on this flat space we get \( C^\infty(F) \approx \bigoplus I_2 \wedge^k \mathcal{H} \) and this is what defines the cartesian product of supernumbers. We will naturally represent the right hand side as the naive Fock space of the Hilbert space: \( \mathcal{F}(|\mathcal{H}|) = \bigoplus I_2 \wedge^k \mathcal{H} \). (This is not the Fock space corresponding to the Dirac sea, it is the naive one). We look at again the matrix algebra modeled on these super-numbers, they will be transformations from \( Z : \mathcal{H}^e_+|\mathcal{H}^o_+ \rightarrow \mathcal{H}^e_- \) written explicitly, \( Z = Z_B + \sum_N \sum_{a_1 < a_2 < \ldots < a_N} Z_{a_1 a_2 \ldots a_N}, \) matrix coefficients satisfying,

\[
  \sum_N \sum_{a_1 < a_2 < \ldots < a_N} ||Z_{a_1 a_2 \ldots a_N}||_2^2 < \infty
\]

where \( ||*||_2 \) denotes the norm in the Hilbert-Schmidt ideal. This implies that we have a space of matrices which is modeled on \( I_2 \otimes \mathcal{F}(|\mathcal{H}|) \). We may use the above convergence condition to get an inner product:

\[
  <Z, W > = \sum_N \sum_{a_1 < a_2 < \ldots < a_N} \text{Tr} Z_{a_1 a_2 \ldots a_N}^\dagger W_{a_1 a_2 \ldots a_N},
\]

We note that this abstract space is still a Hilbert space with the above inner product, and indeed that will equip us with all the luxuries of Hilbert spaces. We can prove by using standard techniques that the product of two such matrices, \( ZW \) is still in the above class, i.e.

\[
  \sum_N \sum_{a_1 < a_2 < \ldots < a_N} ||(ZW)_{a_1 a_2 \ldots a_N}||_2^2 = \sum_N \sum_{a_1 < a_2 < \ldots < a_N} \sum_n ||\sum Z_{a_1 a_2 \ldots a_n} W_{a_{n+1} \ldots a_N}||_2^2 < \infty.
\]
There is the same type possible reorderings of the indices in this expression. The rest will follow exactly the same lines as before, the convergence conditions should be checked much more carefully this time.

The disc is defined as $1 - \hat{w}_B w_B > 0$, and $Z = [w \theta]$. where each one of these supermatrices satisfy the above condition for being in $\mathcal{I}_2$. We can define the same symplectic form,

$$\Omega(V_u, V_v) = \frac{i}{8} \text{Str} J[[J, g^{-1} u g], [J, g^{-1} v g]]_s$$

(58)

here each term is in $\mathcal{I}_2$.

The rest of the arguments apart from the convergence issues are exactly the same, so we leave the details to a future work.

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7 Appendix

For completeness we define here $A^\alpha$ where $0 < \alpha < 1$ and the body of the super operator is positive,

$$A^\alpha = \frac{\sin \pi \alpha}{\pi} \int_0^\infty \frac{d\lambda}{\lambda^{1-\alpha}} (\lambda 1 + A)^{-1}. \quad (59)$$

The advantage of this expression is that we may actually expand the inverse and obtain a series for the super operator. Note that there is no simple recursive process when $\alpha$ is not a rational number.

In this part we will give a proof of the following transformation rule: $Z \mapsto g \circ Z$ implies $\Phi \mapsto g^\dagger \Phi g^{-1}$. First we note that when $Z \mapsto (AZ + B)(CZ + D)^{-1}$, we have $(1 - Z^\dagger Z)^{-1} \mapsto (CZ + D)(1 - Z^\dagger Z)^{-1}(CZ + D)^{-1}$. Next we rewrite $\Phi(Z)$:

$$\Phi = -1 + 2 \left( K^{-1} \begin{pmatrix} K^{-1} Z & -K^{-1} Z \\ Z^\dagger K^{-1} & -Z^\dagger K^{-1} Z \end{pmatrix} = 1 + 2 \begin{pmatrix} Z^\dagger S^{-1} Z & -Z S^{-1} \\ S^{-1} Z^\dagger & -S^{-1} \end{pmatrix} \right), \quad (60)$$

where $K = (1 - ZZ^\dagger)$ and $S = (1 - Z^\dagger Z)$. Using the above observation we see that

$$\Phi(g \circ Z) = 1 + 2 \left( \begin{pmatrix} (AZ + B)S^{-1}(AZ + B)^\dagger & -(AZ + B)S^{-1}(CZ + D)^\dagger \\ (CZ + D)S^{-1}(AZ + B)^\dagger & -(CZ + D)S^{-1}(CZ + D)^\dagger \end{pmatrix} \right). \quad (61)$$

One can see that the above expression can be written as:

$$\Phi(g \circ Z) = 1 + 2 \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} Z^\dagger S^{-1} Z & -Z S^{-1} \\ S^{-1} Z^\dagger & -S^{-1} \end{pmatrix} \begin{pmatrix} A^\dagger & -C^\dagger \\ -B^\dagger & D^\dagger \end{pmatrix}, \quad (62)$$
this is precisely what we claimed. Next point to check is $\Phi(Z) = g(Z)Jg(Z)^{-1} = g(Z)g(Z)^\dagger J$:

$$g(Z)g(Z)^\dagger = \begin{pmatrix} K^{-1/2} & ZS^{-1/2} \\ S^{-1/2}Z^\dagger K^{1/2} & S^{-1/2} \end{pmatrix} \begin{pmatrix} K^{-1/2} & K^{1/2}ZS^{-1} \\ S^{-1/2}Z^\dagger & S^{-1/2} \end{pmatrix} = \begin{pmatrix} K^{-1} & ZS^{-1}Z^\dagger + 1 - 1 \\ 2ZS^{-1} & S^{-1}Z^\dagger KZS^{-1} + 1 \end{pmatrix}.$$

Multiply this with $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, then in the last line use $-(S^{-1}Z^\dagger KZS^{-1} + S^{-1}) = -(S^{-1}Z^\dagger KK^{-1}Z + S^{-1})$, which gives $-(S^{-1}Z^\dagger Z + S^{-1}) = S^{-1}(1 - Z^\dagger Z) - S^{-1} - S^{-1}$. This gives $1 - 2S^{-1}$, then the result follows.

We will prove the non-degeneracy of the super-two form;

$$\Omega(V_u, V_v)|_{Z=0} = -2i\text{Tr}(b_1b_2^\dagger - b_2b_1^\dagger) - 2i\text{Tr}(\beta_1\beta_2^\dagger + \beta_2\beta_1^\dagger). \quad (63)$$

Here we write $V_u(Z) = u_{11}Z - Zu_{22} - Z u_{21} + u_{21}$, and similarly for the $V_u(Z)^\dagger$. Furthermore we write for $u_{12} = [b \beta]$, hoping that the use of the same letters for the Lie algebra elements will not cause any confusion. Let us expand each term as a super-matrix(ignoring the multiplicative factor $-2i$),

$$\text{Tr}((b_1)_B(b_2)^\dagger_B -(b_2)_B(b_1)^\dagger_B) = 0$$

$$\text{Tr}((b_1)_{a_1a_2}(b_2)^\dagger_B - (b_2)_{a_1a_2}(b_1)^\dagger_B)\xi^{a_1}\xi^{a_2} = 0$$

$$\text{Tr}((b_1)_B(b_2)^\dagger_{a_1a_2} -(b_2)_B(b_1)^\dagger_{a_1a_2})\xi^{a_1}\xi^{a_2} = 0$$

$$\text{Tr}((b_1)_{a_1a_2a_3a_4}(b_2)^\dagger_B\xi^{a_1}\xi^{a_2}\xi^{a_3}\xi^{a_4} + (b_1)_{a_1a_2}(b_2)^\dagger_{a_3a_4}\xi^{a_1}\xi^{a_2}\xi^{a_3}\xi^{a_4} + (b_1)_B(b_2)^\dagger_{a_1a_2a_3a_4}\xi^{a_1}\xi^{a_2}\xi^{a_3}\xi^{a_4}$$

$$- \text{Tr}((b_2)_{a_1a_2}(b_1)^\dagger_{a_3a_4}\xi^{a_1}\xi^{a_2}\xi^{a_3}\xi^{a_4} + (b_2)_B(b_1)^\dagger_{a_1a_2a_3a_4}\xi^{a_1}\xi^{a_2}\xi^{a_3}\xi^{a_4} + (b_2)_{a_1a_2a_3a_4}(b_1)^\dagger_B\xi^{a_1}\xi^{a_2}\xi^{a_3}\xi^{a_4})$$

$$+ (b_2)_{a_1a_2a_3a_4}(b_1)_{B}\xi^{a_1}\xi^{a_2}\xi^{a_3}\xi^{a_4}) = 0$$

... where the dots refer to the continuation of this expansion. From these relations we conclude that an iterative process gives us the required nondegeneracy.

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