Duality and quasiparticles in the Calogero-Sutherland model:
Some exact results

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Abstract

The quantum-mechanical many-body system with the potential proportional to the pairwise inverse-square distance possesses a strong-weak coupling duality. Based on this duality, particle and/or quasiparticle states are described as SU(1,1) coherent states. The constructed quasiparticle states are of hierarchical nature.

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I. INTRODUCTION

The one-dimensional pairwise inverse-square distance interaction of $N$ particles, and their generalizations under the name of Calogero-Sutherland-Moser models (CSMM) \cite{1} \cite{2} have so far appeared in a variety of different physical contexts. They are related to the random matrix model \cite{4} and the two-dimensional Yang-Mills theory \cite{5}. They also represent an example of generalized exclusion statistics \cite{6}, and quantum spin chains with long-range

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interactions \cite{1}. CSMM’s describe edge states in the quantum Hall system \cite{8} and the Chern-Simons theory \cite{9}.

We still lack a local canonical field-theoretical formulation of CSMM’s \cite{10}, but a collective-field theory \cite{11} can be established \cite{12,13} in the large-$N$ limit, and it connects CSMM’s with 2d gravity \cite{14}. A deeper understanding of the models can be gained by exploring various solutions. CSMM’s are exactly solvable and integrable, both classically and quantum-mechanically. From the quantum Lax formulation \cite{15} we can find infinitely many commuting conserved operators, and the underlying algebraic structure should reveal the large degeneracy structure of CSMM’s. Its eigenfunctions are known to be symmetric polynomials \cite{16,17} but, to these days, the only explicit form have been the original wave functions found by Calogero \cite{1}. In the collective-field formalism it has been found that there exist static solitons in the Bogomol’nyi limit and moving solitons as solutions of the equations of motion \cite{18,19,13}.

We would like to show that a specific CSMM on the line (called the Calogero-Moser model) with the Hamiltonian (\(\hbar = m = 1\))

\[
H_{CM} = \frac{1}{2} \sum_{i=1}^{N} p_i^2 + \frac{\lambda(\lambda - 1)}{2} \sum_{i \neq j}^{N} \frac{1}{(x_i - x_j)^2}
\]

(1)

admits a family of new solutions describing the quasiparticles. We shall construct the dynamics of quasiparticle states using the subgroup SU(1,1) \cite{20} of the \(W_\infty\) algebra \cite{15}. It has been conjectured \cite{21} that Calogero scattering eigenfunctions are generalized coherent states related to the representations of SU(1,1). A proper treatment of the center-of-mass degrees of freedom is important in order to preserve the translational invariance of the model. Using the weak-strong coupling duality \cite{22} between particles and quasiparticles, we can extend the coherent-state representation to the case with both particles and quasiparticles.

II. SU(1,1) ALGEBRA

Let us first construct the representation of a spectrum generating algebra for the Hamiltonian \cite{1} as a generator. In fact, there is a larger symmetry group \cite{15}, but here we
deal only with quasiparticles related to the subgroup \( SU(1,1) \). It is convenient to extract
the Jastrow factor from the wave function and perform a similarity transformation of the
Hamiltonian into
\[
\prod_{i<j} (x_i - x_j)^{-\lambda} (-H_{CM}) \prod_{i<j} (x_i - x_j)^{\lambda} = \frac{1}{2} \sum_{i=1}^{N} \partial_i^2 + \frac{\lambda}{2} \sum_{i \neq j} \frac{1}{x_i - x_j} (\partial_i - \partial_j). \tag{2}
\]
Owing to the translational invariance of the model we should introduce completely translationally invariant variables
\[
\xi_i = x_i - X, \quad \partial_{\xi i} \xi_j = \delta_{ij} - \frac{1}{N}. \tag{3}
\]
Here we have introduced the center-of-mass coordinate \( X = \frac{1}{N} \sum_{i=1}^{N} x_i \) and its canonical
conjugate \( \partial_X = \sum_{i=1}^{N} \partial_i = iP_{\text{tot}} \). As an \( SU(1,1) \) generator we take Hamiltonian with
eliminated center-of-mass degrees of freedom:
\[
T_+(x, \lambda) = \frac{1}{2} \sum_{i=1}^{N} \left( \partial_i - \frac{1}{N} \partial_X \right)^2 + \frac{\lambda}{2} \sum_{i \neq j} \frac{1}{x_i - x_j} (\partial_i - \partial_j) = \frac{1}{2} \sum_{i=1}^{N} \partial_i^2 + \frac{\lambda}{2} \sum_{i \neq j} \frac{1}{\xi_i - \xi_j} (\partial_{\xi i} - \partial_{\xi j}). \tag{4}
\]
Owing to the scale and special invariance we introduce two additional generators:
\[
T_0(x, \lambda) = -\frac{1}{2} \left( \sum_{i=1}^{N} x_i \partial_i - X \partial_X + E_0 - \frac{1}{2} \right) = -\frac{1}{2} \left( \sum_{i=1}^{N} \xi_i \partial_{\xi i} + E_0 - \frac{1}{2} \right),
\]
\[
T_-(x, \lambda) = \frac{1}{2} \sum_{i=1}^{N} x_i^2 - \frac{N}{2} X^2 = \frac{1}{4N} \sum_{i \neq j} (x_i - x_j)^2 = \frac{1}{2} \sum_{i=1}^{N} \xi_i^2, \tag{5}
\]
and, after performing some calculation, we can verify
\[
[T_+, T_-] = -2T_0, \quad [T_0, T_{\pm}] = \pm T_{\pm}. \tag{6}
\]
This is the usual \( SU(1,1) \) conformal algebra [15,21], with the Casimir operator \( \hat{C} = T_+ T_- - T_0(T_0 - 1) \). In the definition (3) of the operator \( T_0 \) the constant \( E_0 \) is \( E_0 = \frac{1}{2} N(N-1) + \frac{N}{2} \) for
consistency reasons, and \(-1/2\) appears after removing the center-of-mass degrees of freedom.

After having established the representation of \( SU(1,1) \) algebra, we show that the
Calogero solutions are completely determined assuming the zero-energy solutions are known:
\[ T_+(x, \lambda) P_m(x_1, x_2, \ldots, x_N) = 0, \]

\[ T_0(x, \lambda) P_m(x_1, x_2, \ldots, x_N) = \mu_m P_m(x_1, x_2, \ldots, x_N), \quad \mu_m = -\frac{1}{2}(m + E_0 - \frac{1}{2}). \]  

(7)

Calogero has proved that the functions \( P_m(x_1, x_2, \ldots, x_N) \) are scale- and translationally invariant homogeneous multivariable polynomials of degree \( m \), written in the center-of-mass frame. There are no general explicit representations of these polynomials, except in the case including quasiparticles, which we derive below. Let us assume that a nonzero eigenstate of the operator \( T_+(x, \lambda) \) is of the general form

\[ \Psi(T_-, T_0, T_+) P_m(x_1, x_2, \ldots, x_N) = \sum_{p,q,n} c_{pqn} T_+^p T_0^q T_+^n P_m(x_1, x_2, \ldots, x_N) = \Psi(T_-, T_0) P_m(x_1, x_2, \ldots, x_N). \]  

(8)

Using (7) we can derive the formula

\[ [T_+, f(T_-)] = T_- f''(T_-) - 2 f'(T_-) T_0. \]  

(9)

Using (7) and the eigenvalue equation

\[ -T_+ \Psi_m(T_-) P_m(x_1, x_2, \ldots, x_N) = E \Psi_m(T_-) P_m(x_1, x_2, \ldots, x_N) \]  

(10)

we obtain the Calogero solution (in his notation \( p = \sqrt{2E}, \ r^2 = 2T_- \)):

\[ \Psi_m(T_-) P_m \sim T_-^{(1-m-E_0+1/2)/2} Z_{m+E_0-3/2} (2\sqrt{ET_-}) P_m(x_1, x_2, \ldots, x_N) \]

\[ \sim r^{-(m+E_0-3/2)} Z_{m+E_0-3/2} (pr) P_m(x_1, x_2, \ldots, x_N). \]  

(11)

We can also show that already at the first level the Calogero Hamiltonian \( \mathcal{H} \) for symmetric wave function possesses a quasiparticle solution. For \( P_\kappa(\xi_i) = \prod_i^N \xi_i^\kappa \), with \( \kappa = 1 - \lambda - 1/N \), the equation

\[ T_+ \prod_{i=1}^N \xi_i^\kappa \Psi(T_-) = -E \prod_{i=1}^N \xi_i^\kappa \Psi(T_-) \]  

(12)

has a Bessel function quasihole solution sitting at the origin of the Calogero system. Up to the \( 1/N \) correction for \( \lambda \), this is the type of solution already found in the collective-field approach [18].
III. DUALITY

The Calogero-Sutherland model is a rare quantum-mechanical system where a strong-weak coupling duality exists [22,16] relating various physical quantities for the constants of interaction $\lambda$ and $1/\lambda$. Depending on the parameters, the duality exchanges particles with quasiparticles. We show how to find a solution of the model with quasiparticles using duality relations. From the collective-field-theory approach to the problem we know that a wave function describing the holes (or lumps) has a prefactor of the form

$$V^\kappa(x - z) = \prod_{i,\alpha=1}^{N,M} (x_i - Z_\alpha)^\kappa, \quad \alpha = 1, \ldots, M,$$

(13)

where $Z_\alpha$ denotes $M$ zeros of the wave function, describing positions of $M$ quasiparticles.

The duality is displayed by the following relations:

\begin{align*}
P_{\text{tot}}(x)V^\kappa(x - z) &= -P_{\text{tot}}(z)V^\kappa(x - z), \\
T_0(x, \lambda)V^\kappa(x - z) &= \left\{ -T_0(z, \frac{\kappa^2}{\lambda}) - \frac{1}{2} [\kappa NM + \epsilon_0(N, \lambda) + \epsilon_0(M, \frac{\kappa^2}{\lambda})] \right\} V^\kappa(x - z), \\
T_+(x, \lambda)V^\kappa(x - z) &= \left\{ -\frac{\lambda}{\kappa} T_+(z, \frac{\kappa^2}{\lambda}) + \frac{1 + \lambda/\kappa}{2} \sum_{i,\alpha}^{N,M} \frac{\kappa(\kappa - 1)}{(x_i - Z_\alpha)^2} \right\} V^\kappa(x - z),
\end{align*}

(14)

where the operator $T_{0,\pm}(z, \frac{\kappa^2}{\lambda})$ denotes an operator with the same functional dependence on $Z_\alpha$ as that of the operator $T_{0,\pm}(x, \lambda)$ on $x_i$, with the coupling constant $\lambda$ changed into $\kappa^2/\lambda$. In addition to these duality relations we take that the center-of-mass coordinates are identical, namely, $X = Z$, $(Z = \frac{1}{M} \sum_{\alpha=1}^{M} Z_\alpha)$ at the end of calculations. The duality also places an interesting restriction on the number of quasiparticles $M$. Namely, the number of quasiparticles is determined by the coupling constant $\lambda$ and is proportional to the number of particles $N$:

$$M = -\frac{\lambda N}{\kappa}. \tag{15}$$

Here we have manifest duality: if we interchange particles and quasiparticles ($N \leftrightarrow M$), then $\lambda$ goes to $\kappa^2/\lambda$. For $\kappa = 1$, the relation (13) was conjectured in Ref. [22]. Let us introduce new collective generators for the system of particles and quasiparticles:
\( T_+ = T_+(x, \lambda) + \frac{\lambda}{\kappa} T_+(z, \frac{\kappa^2}{\lambda}) - \frac{(\lambda + \kappa)(\kappa - 1)}{2} \sum_{i,\alpha}^{N,M} \frac{1}{(x_i - Z_\alpha)^2}, \)

\( T_0 = T_0(x, \lambda) + T_0(z, \frac{\kappa^2}{\lambda}), \)

\( T_- = T_-(x, \lambda) + \frac{\kappa}{\lambda} T_-(z, \frac{\kappa^2}{\lambda}). \) (16)

It can be easily checked that the above generators satisfy the \( SU(1,1) \) conformal algebra. In terms of the generators (16), the duality relation (14) turns out to be a sufficient condition for solving the Calogero model with quasiparticles:

\[
T_+ V^\kappa (x - z) = 0,
\]

\[
T_0 V^\kappa (x - z) = -\left(\frac{N + M}{4}\right)(\kappa + 1) - 2 V^\kappa (x - z).
\] (17)

We interpret the operator \( T_+ \) as a Hamiltonian for a more general problem. After performing a similarity transformation of \( T_+ \), we obtain the master Hamiltonian for particles and quasiparticles,

\[
H(x, z) = -\frac{1}{2} \sum_{i=1}^{N} \frac{\partial_i^2}{\bar{\lambda}} + \frac{\lambda(\lambda - 1)}{2} \sum_{i \neq j}^{N} \frac{1}{(x_i - x_j)^2}
\]

\[
+ \frac{\lambda}{2\kappa} \sum_{a=1}^{M} \frac{\partial_a^2}{\bar{\lambda}} + \frac{\kappa^2}{2\lambda} \left(\frac{\kappa^2}{\lambda} - 1\right) \sum_{a \neq \beta}^{M} \frac{1}{(Z_a - Z_\beta)^2} - \frac{1}{2} \left(1 + \frac{\lambda}{\kappa}\right) \sum_{i,\alpha}^{N,M} \frac{\kappa(\kappa - 1)}{(x_i - Z_\alpha)^2}.
\] (18)

We see that the quasiparticle mass is \( \kappa/\lambda \), and the factor in front of the interaction term is \( \kappa(\kappa - 1)/2 \) times the inverse reduced mass of the particles and quasiparticles (remember that we set the particle mass to one). This is the second hierarchical level Hamiltonian with the solution for the given energy \( E = k^2/2 \)

\[
\Psi(x, z; k) = \prod_{i<j}^{N} (x_i - x_j)^\lambda \prod_{a<\beta}^{M} (Z_a - Z_\beta)^{\kappa^2/\lambda} (kR)^{-b} Z_b(kR) \prod_{i,\alpha}^{N,M} (x_i - Z_\alpha)^{\kappa},
\] (19)

where the index of the Bessel function is \( b = \kappa MN + E_0(N, \lambda) + E_0(M, \kappa^2/\lambda) - 2 \). The solution has been found following the procedure outlined in Eqs.(8-11). To get more insight into the hierarchical Hamiltonian we can eliminate quasiparticle degrees of freedom by performing the appropriate derivatives in \( T_+ \) and putting \( Z_\alpha = 0 \). The remaining equation is the Calogero first hierarchical level for quasiparticles situated at the origin (12).
IV. COHERENT STATES

Let us show that the Calogero solution (11) let in the operator form is a generalized Barut-Girardello coherent state of the \( SU(1,1) \) group [20]. In this case the coherent state is defined as an eigenvector of the "annihilation" operator \( T_+(x, \lambda) \). The Calogero solution (11) can be recast in the form of a coherent state by specifying the coefficients \( c_{pq0} \):

\[
\Psi(T_-, T_0) = \sum_{p=0}^{\infty} \frac{E_p}{p!} T_p \prod_{i=1}^{p} F(T_0 - i + 1) = \exp(ET_- F(T_0)). \tag{20}
\]

The operator \( A^+ \equiv T_- F(T_0) \) plays the role of a "creation" operator and the function \( F(T_0) \) can be determined such that \( A^+ \) is the canonical conjugate of \( T_+(x, \lambda) \), i.e., \([T_+, A^+] = 1\).

Using the \( SU(1,1) \) algebra and demanding that the canonical commutation relation is valid for the zero energy solutions, we obtain

\[
A^+ = T_- \frac{T_0 + 1 - \frac{m + E_0 - 1/2}{2}}{C + T_0(T_0 - 1)}. \tag{21}
\]

Now, the translationally invariant Calogero eigenfunctions can be written as

\[
\Phi(x_1, x_2, ..., x_N) = \exp\left(\frac{k^2}{2} A^+\right) P_m(x_1, x_2, ..., x_N)
\sim (kr)^{-b} Z_b(kR) P_m(x_1, x_2, ..., x_N), \tag{22}
\]

where \( Z_b \) is a Bessel function, \( b = m + E_0 - \frac{3}{2} \), and \( r^2 = 2T_- \).

Finnaly, following the steps outlined above, we construct the canonically conjugate operator \( A^+ \)

\[
A^+ = T_- \frac{\kappa + 1}{C + T_0(T_0 - 1)}, \tag{23}
\]

and then a coherent state of particles and quasiparticles follows as

\[
\Psi(x, z; k) = \prod_{i<j} (x_i - x_j)^\lambda \prod_{\alpha<\beta} (Z_\alpha - Z_\beta)\kappa^{2/\lambda} \exp\left(\frac{k^2}{2} A^+(x, z)\right) V^\kappa(x - z). \tag{24}
\]

Applying the operator \( A^+ \) to \( V^\kappa \) we obtain the solution (13).
V. CONCLUSION

In summary, we have constructed the master Hamiltonian for particles and their dual quasiparticles. The solutions of this Hamiltonian in the operator form has been found to be generalized $SU(1,1)$ coherent states. Formally, this is also a solution to the Calogero two-family problem [1] with different masses and coupling constants. Owing to duality, the construction of the master Hamiltonian and its solutions are of hierarchical nature [23]. In fact, each new family represents a new hierarchical level and is obtained by introducing a new prefactor (in the wave function) and extending the $SU(1,1)$ generators by corresponding terms for quasiparticles. For example, the third family introduced by $V^\gamma(z - y)$, where $y$ represents positions of new quasiparticles, will have the mass $\kappa/\gamma$ and the statistical factor $\frac{\gamma^2}{\kappa^2\lambda}$. There are possible more complicated constructions owing to the fact that higher dynamical groups then $SU(1,1)$ exists for the Calogero model. We may observe that in many respects (duality, hierarchy, statistics) the CSMM very much resembles 2-dimensional quasiparticles appearing in the fractional quantum Hall effect. An interesting open question is how to formulate this hierarchy in field theory context.

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