A Note on $N = 2$ Superstrings

J. R. Bieńkowska  
*Enrico Fermi Institute*  
*The University of Chicago*  
*5640 South Ellis Ave.*  
*Chicago, IL 60637-1433, U.S.A.*

and

H. Lu  
*Center for Theoretical Physics*  
*Texas A & M University*  
*College Station, TX 77843, U.S.A.*

**ABSTRACT**

In this note we investigate the generalised critical $N = 2$ superstrings in $(1,2p)$ spacetime signature. We calculate the four-point functions for the tachyon operators of these theories. In contrast to the usual $N = 2$ superstring in $(2,2)$ spacetime, the four-point functions do not vanish. The exchanged particles of the four-point function are included in the physical spectrum of the corresponding theory and have vanishing fermion charge.

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1. Introduction

$N = 2$ superstrings have received a considerable amount of attention recently [1,2,3]. They are described by two-dimensional conformal field theories with local $N = 2$ super-Virasoro symmetry algebra. They have the maximal number of local supersymmetries which allow string theories to have a positive critical central charge. The critical value of central charge for the $N = 2$ super-Virasoro algebra is $c = 6$. Thus it can be realised by two $N = 2$ superfields, which then leads to the usual $N = 2$ critical string in $D = 2$ complex dimensions, i.e. four real dimensions. The existence of the complex structure which is required by the $N = 2$ worldsheet supersymmetry [4] makes the usual $N = 2$ superstring rather unattractive since it implies that the spacetime signature is $(2, 2)$. However, this argument can be relaxed by introducing a background charge lying in a certain direction without spoiling the local $N = 2$ supersymmetry [3]. Indeed, the $N = 2$ super-Virasoro algebra can be realised in terms of arbitrary number of $N = 2$ superfields and with background charges fixed so that the theory is critical. This gives rise to critical $N = 2$ superstrings in arbitrary numbers of complex dimensions. The existence of background charges can freeze one real coordinate, i.e. the momentum component in that direction is constrained by the physical-state conditions to take some specific values. If one starts from a theory with one complex time and chooses the background charge to lie in that direction, then the result of the coordinate freezing is a theory that effectively has only one real time direction.

Beside the fact that the generalised $N = 2$ superstrings make it possible to describe the critical theories effectively in Minkowskian spacetime signature, i.e. $(1, 2p)$ with $p \geq 2$, there are fundamental differences between the generalised $N = 2$ superstrings and the usual one in $(2, 2)$ spacetime. The usual $N = 2$ superstring is a highly-degenerate theory. The physical spectrum consists only of a massless scalar which turns out to be the Kähler potential for the four dimensional metric [1]. The corresponding interaction theory only involves upto three-point functions [1]. Vanishing of four-point functions and beyond is necessary for the consistency of the theory since higher-point functions tend to produce mass poles which do not exist in the usual $N = 2$ superstring. Since the usual $N = 2$ superstring turns out be a very interesting theory which provides a consistent quantum theory of self-dual gravity in four dimensions [1], it is most intriguing to study the more general cases in order to uncover the full richness of the $N = 2$ superstrings.

It has been shown in [3] that the physical spectrum of the critical $N = 2$ superstrings in $D \geq 3$ complex dimensions comprises an infinite tower of particles just like other string theories. The no-ghost theorem for the usual $N = 2$ string have been proven in [5]. Norms of low-lying physical states for generalised $N = 2$ superstring are calculated in [3] and the results indicate the unitarity of the theories. In this paper we shall first review the physical spectrum and then calculate the four-point functions for the tachyonic state and show that
they do not vanish in general. The mass poles from the gamma functions in the four-point functions are consistent with the physical-state conditions.

2. Physical Spectrum of Generalised $N = 2$ Superstrings

In this section, we shall review the discussion of ref. [3], where the critical $N = 2$ superstrings in $D \geq 3$ complex dimensions are constructed. Since we only study $N = 2$ superstrings in this paper, we shall no longer repeat “$N = 2$” unless there is ambiguity. The discussion of the physical spectrum applies to both open strings and close strings, for simplicity, we shall only consider the holomorphic sector in this section.

The super-Virasoro algebra is generated by a super energy-momentum tensor which can be realised in terms of free chiral superfields

$$
\Phi^{+}_\mu(z, \theta^+, \theta^-) = \phi_\mu(z) + \sqrt{2} \theta^- \psi_\mu(z) - \theta^+ \theta^- \partial \phi_\mu(z),
$$

$$
\Phi^{-}_\mu(z, \theta^+, \theta^-) = \bar{\phi}_\mu(z) + \sqrt{2} \theta^+ \bar{\psi}_\mu(z) + \theta^+ \theta^- \partial \bar{\phi}_\mu(z),
$$

where $\mu = (0, 1, \ldots, D - 1)$. The superfields satisfy the chirality conditions

$$
D^\pm \Phi^-_\mu = 0 = D^- \Phi^+_\mu,
$$

where

$$
D^\pm \equiv \frac{\partial}{\partial \theta^\pm} + \theta^\pm \partial.
$$

The operator-product expansions of these free superfields are given by

$$
\Phi^+_\mu(z_1, \theta^+_1, \theta^-_1) \Phi^-_\nu(z_2, \theta^+_2, \theta^-_2) \sim -\eta_{\mu \nu} \log(Z_1 - Z_2),
$$

where

$$
Z_1 - Z_2 \equiv z_1 - z_2 + \theta^+_1 \theta^-_1 + \theta^+_2 \theta^-_2 - 2 \theta^+_1 \theta^-_2.
$$

The super energy-momentum tensor $T(z, \theta^+, \theta^-)$ can then be written as

$$
T(z, \theta^+, \theta^-) = -\frac{1}{4} D^+ \Phi^+ \mu D^- \Phi^-_\mu + \frac{\alpha_0}{2} \partial \Phi^+_0 - \frac{\alpha_0}{2} \partial \Phi^-_0.
$$

Note that the background charge in expression (2.6) has been chosen to lie in the $\mu = 0$ direction for the reason explained in the introduction. The central charge of this realisation is

$$
c = 3(D - 2 \alpha_0^2).
$$

The anomaly-freedom condition $c = 6$ therefore requires that

$$
\alpha_0^2 = \frac{1}{2}(D - 2).
$$
We shall only consider in this paper the cases that $D \geq 3$ and thus it follows from (2.8) that the background charge is real.

The super energy-momentum tensor $T(z, \theta^+, \theta^-)$ may be expanded in components as

$$T(z, \theta^+, \theta^-) = \frac{1}{2}J(z) - \frac{1}{2}\theta^+ G^-(z) + \frac{1}{2}\theta^- G^+(z) + \theta^+ \theta^- T(z) .$$

(2.9)

The component currents $(J, G^+, G^-, T)$ have conformal spins $(1, \frac{3}{2}, \frac{3}{2}, 2)$ measured by the energy-momentum tensor $T(z)$. The explicit forms of these component currents are given by

$$J = -\psi^\mu \bar{\psi}_\mu - \alpha_0 \partial \phi_0 + \alpha_0 \partial \bar{\phi}_0 ,$$

$$G^+ = \sqrt{2} (\partial \bar{\phi}^\mu \psi_\mu - \alpha_0 \partial \psi_0) ,$$

$$G^- = \sqrt{2} (\partial \phi^\mu \bar{\psi}_\mu - \alpha_0 \partial \bar{\psi}_0) ,$$

$$T = \frac{1}{2} \psi^\mu \partial \bar{\psi}_\mu - \frac{1}{2} \bar{\psi}^\mu \psi_\mu - \partial \phi^\mu \partial \bar{\phi}_\mu + \frac{1}{2} \alpha_0 \partial^2 \phi_0 + \frac{1}{2} \alpha_0 \partial^2 \bar{\phi}_0 .$$

(2.10)

A physical state $|p\rangle$ satisfies the physical-state conditions

$$L_m |p\rangle = 0 = J_m |p\rangle \hspace{1cm} m \geq 0 ,$$

$$G^+_r |p\rangle = 0 = G^-_r |p\rangle \hspace{1cm} r > 0 .$$

(2.11)

(Since the intercepts of $J_0$ and $L_0$ are zero for the $N = 2$ super-conformal algebra, we include these in the physical-state conditions (2.11).) Physical states can be constructed by acting on an $SL(2, C)$-invariant vacuum $|0\rangle$ with physical operators (ground-state operators) $P(z)$, i.e. $|p\rangle \equiv P(0)|0\rangle$. The physical operators take the form

$$P(z) = R(z) e^{\beta \cdot \phi + \bar{\beta} \cdot \bar{\phi}} .$$

(2.12)

(Normal ordering is understood.) The operators $R(z)$ can be classified by their eigenvalues $q$ and $n$ under $J_0$ and $L_0$ respectively. The eigenvalue $q$ measures the fermion charge ($U(1)$ charge) of the operator $R(z)$; each $\psi_\mu$ in a monomial in $R(z)$ contributes +1, each $\bar{\psi}_\mu$ contributes −1, and $\partial \phi_\mu$ and $\partial \bar{\phi}_\mu$ contribute 0. The eigenvalue $n$ measures the conformal dimension of the operator $R(z)$, i.e. the level number. The corresponding vertex operators, which are defined as total derivatives under the super energy-momentum tensor, are given by

$$V(z) = \frac{1}{2} (G^+_1 G^-_1 - G^-_1 G^+_1) P(z) ,$$

(2.13)

At level $n = 0$, $R$ is just the identity operator, with $q = 0$, and $P(z)$ is the “tachyon” ground-state operator. At level $n = \frac{1}{2}$, $R$ can be $\xi_\mu \psi^\mu$, with $q = +1$; or $\xi_\mu \bar{\psi}^\mu$, with $q = -1$. At level $n = 1, q$ can be $-2$, 0, +2. In general, at level $n$, $q$ takes the values

$$q = -2n, \ -2n + 2, \ldots, \ 2n - 2, \ 2n .$$

(2.14)
For physical states with level number $n$ and fermion charge $q$, the $J_0$ and $L_0$ constraints in (2.11) give

$$J_0 : \quad 0 = q + \alpha_0 (\beta_0 - \bar{\beta}_0)$$

$$L_0 : \quad 0 = n - \beta^\mu \bar{\beta}_\mu + \frac{1}{2} \alpha_0 (\beta_0 + \bar{\beta}_0).$$

In the usual discussion of the $N = 2$ superstring, for which $D = 2$ and hence from (2.8) the background charge $\alpha_0$ is zero, equation (2.15a) implies that for all physical states the fermionic charge of $R(z)$ must be zero. This implies that in this case there are no physical states occurring at levels with $n$ a half-integer. (In fact, as discussed in [1,5,6], all the higher-level states satisfying conditions (2.15a,b) are longitudinal, and hence have no physical degrees of freedom.) It follows from equation (2.15a) that the real momentum component $\bar{\beta}_0 - \beta_0$ is “frozen” to the value

$$\bar{\beta}_0 - \beta_0 = \frac{q}{\alpha_0}.$$ (2.16)

Thus the real time direction ($\phi_0 - \bar{\phi}_0$) is not a physical-observable coordinate; effectively one is left with one real time coordinate ($\phi_0 + \bar{\phi}_0$). (Momentum-freezing is a genuine phenomenon for $W$-string theories with non-linear local symmetry algebras, namely $W$ algebras [7]. It is remarkable that such phenomenon occurs also in the linear superstrings.)

3. Four-point Function of Tachyonic physical operators

Having obtained a complete set of physical operators in section 2, we shall turn our attention to interaction theories. We shall concentrate on four-point function of the tachyonic physical state, i.e. level-zero state. The corresponding vertex operator can be evaluated from (2.13) as

$$V_0(z) = (\beta \cdot \partial \bar{\phi} - \bar{\beta} \cdot \partial \phi - 2(\beta \cdot \bar{\psi})(\bar{\beta} \cdot \psi)) e^{\beta \cdot \phi + \bar{\beta} \cdot \bar{\phi}},$$

where “•” stands for the contraction of $\mu$ indices. In section 2 we choose the component form to construct the physical spectrum. It is more convenient to work with the superfield language to calculate the $n$-point functions. In superfield form, the vertex operators in (2.13) can be expressed equivalently as

$$V(z) = \int d^2 \theta V(z, \theta^\pm)$$

$$= \int d^2 \theta R(D^+ \Phi^+, D^- \Phi^-) e^{\bar{\beta} \cdot \Phi^+ + \beta \cdot \Phi^-},$$

where $R(D^+ \Phi^+, D^- \Phi^-)$ is the corresponding super differential polynomial on superfields $\Phi^+_\mu$ and $\Phi^-_{\mu}$. When $R$ is just the identity operator, it gives rise to the vertex operator for the tachyonic state in superfield form

$$V_0(z, \theta^\pm) = e^{\beta \cdot \Phi^+ + \bar{\beta} \cdot \Phi^-}.$$ (3.3)
Integrating out the $\theta^\pm$ coordinates leads to the vertex operator (3.1). The free superfields $\Phi^+_\mu$ and $\Phi^-_\mu$ satisfy the OPEs given by (2.4) and (2.5). The n-point correlation functions are then given by

$$
\langle V_0(z_1, \theta^+_1) V_0(z_2, \theta^+_2) \ldots V_0(z_n, \theta^+_n) \rangle = \prod_{i<j} (Z_i - Z_j)^{-(\beta_i \cdot \bar{\beta}_j + \bar{\beta}_i \cdot \beta_j)},
$$

(3.4)

Where $(Z_i - Z_j)$ is defined in (2.5). The $\beta^\mu_i$ and $\bar{\beta}^\mu_i$ in (3.4) satisfy the momentum-conservation law

$$
\sum^n_i \beta^\mu_i = -\alpha^\mu \quad \text{and} \quad \sum^n_i \bar{\beta}^\mu_i = -\alpha^\mu,
$$

(3.5)

where $\alpha^\mu \equiv (\alpha_0, 0, 0, \ldots, 0)$. The momentum-conservation law is modified in $\mu = 0$ direction by the background charge.

So far our discussion applies to both open strings and closed strings. We shall calculate the four-point amplitude only for closed superstrings. Including the anti-holomorphic sector, the tachyonic vertex operator for the closed string is given by

$$
V_0(z, \bar{z}, \theta^\pm, \bar{\theta}^\pm) = e^{\bar{\beta} \cdot \left( \Phi^+(z, \theta^\pm) + \Phi^-(z, \theta^\pm) \right) + \beta \cdot \left( \Phi^-(z, \bar{\theta}^\pm) + \Phi^+(z, \bar{\theta}^\pm) \right)},
$$

(3.6)

where $\Phi^\pm_\mu(z, \bar{z})$ are the anti-holomorphic superfields and $(z, \bar{z})$ are anti-holomorphic supercoordinates. To calculate the n-point amplitudes, one need to integrate out all the supercoordinates on the super Riemann sphere. Since there are three conformal Killing vectors and two super-conformal Killing spinors, we can fix three bosonic coordinates to $\infty, 1,$ and 0, and set two pairs of fermionic coordinates to zero. Thus the three-point function is given by

$$
A_3 = \int d^2 \theta d^2 \bar{\theta} \langle V_0(z_1 = \infty, 0, 0) V_0(z_2 = 1, \theta^\pm, \bar{\theta}^\pm) V_0(z_3 = 0, 0, 0) \rangle = (\beta_2 \cdot \bar{\beta}_3 - \beta_3 \cdot \bar{\beta}_2)^2.
$$

(3.7)

Similarly, the four-point function is given by

$$
A_4 = \int d^2 z d^2 \theta_2 d^2 \theta_3 d^2 \bar{\theta}_3 \left\langle V_0(\infty, 0, 0) V_0(1, \theta^\pm_2, \bar{\theta}^\pm_3) V_0(z, \theta^\pm_3, \bar{\theta}^\pm_3) V_0(0, 0, 0) \right\rangle
= \int d^2 z \left| \frac{t(t+1)}{(1-z)^2} + \frac{c_{12}c_{34}}{z} + \frac{c_{23}c_{41}}{(1-z)^2} \right|^2 \frac{2}{|z| 1-z}^{-2},
$$

(3.8)

where $c_{ij} = \bar{\beta}_i \beta_j - \beta_i \bar{\beta}_j$ is a square-root of the three-point function given in (3.7) and $s = \bar{\beta}_1 \beta_2 + \beta_1 \bar{\beta}_2$, $t = \bar{\beta}_2 \beta_3 + \beta_2 \bar{\beta}_3$ and $u = \bar{\beta}_1 \beta_3 + \beta_1 \bar{\beta}_3$. Using our definition of Mandelstam variables it is easy to check that even with the presence of the background charge, the evaluation of the four-point function (3.8) gives the same form as the one of the usual D=2 critical superstring in [1]. The result is

$$
A_4 = \frac{\pi F^2 \Gamma(1-s) \Gamma(1-t) \Gamma(1-u)}{\Gamma(s) \Gamma(t) \Gamma(u)},
$$

(3.9)
and the identity
\[ s + t + u = 0 \] (3.10)
still holds in the presence of the background charge. The prefactor \( F \) in (3.9) takes the same form as the one in the usual superstring
\[ F = 1 - \frac{c_{12}c_{34}}{su} - \frac{c_{23}c_{41}}{tu}. \] (3.11)

Using the modified momentum-conservation law (3.5) and the intercept condition (2.15a,b), one can rewrite \( c_{34} = c_{13} + c_{23} \) and \( c_{41} = c_{12} + c_{13} \). Substituting these into (3.11) and using (3.10), one has
\[ F = \frac{4}{stu} \left( (\beta_1 \cdot \bar{\beta}_2)(\beta_2 \cdot \bar{\beta}_3)(\beta_3 \cdot \bar{\beta}_1) + (\bar{\beta}_1 \cdot \beta_2)(\bar{\beta}_2 \cdot \beta_3)(\bar{\beta}_3 \cdot \beta_1) \right). \] (3.12)

The form of the four-point function is identical for the usual super-string and the generalised superstrings. However the mass-shell conditions and the spacetime dimensions are different. These differences play important rôles of the final result of the four-point functions. In the case of usual superstring where spacetime signature is \((2,2)\) and hence there is no background charge, the prefactor \( F \) vanishes due to the on-shell condition of the external momenta \( \beta_i \cdot \bar{\beta}_i = 0 \). For the generalised superstring, the prefactor is not zero in general. For example, when \( \beta_1 = \beta_2 = \beta_3 = \beta \) and \( \bar{\beta}_1 = \bar{\beta}_2 = \bar{\beta}_3 = \bar{\beta} \), the prefactor \( F \) is proportional to \((\beta \cdot \bar{\beta})^3\) which does not vanish since the mass-shell condition is \( \beta \cdot \bar{\beta} = \alpha_0 \beta_0 \).

The gamma functions in (3.9) will produce mass-poles in the scattering amplitudes. For \( s \)-channel the singularity occurs when \( s = n \geq 1 \), which gives exactly the intercept conditions (2.15a,b) with \( q = 0 \) and \( n \) integers. Similar analysis applies to \( t \)- and \( u \)-channels as well. Since the fermion charge of the tachyonic vertex operator is zero, owing to the charge conservation, the exchange-particle channels only involve the physical states with vanishing fermion charge \( q = 0 \). In fact, from (2.16), the the fermion charge is proportional to the momentum. The charge conservation originates from the momentum conservation.

4. Conclusions

In this paper, we have looked at details of the generalised critical \( N = 2 \) superstrings in \((1,2p)\) spacetime signature. We calculate the four-point function of the theories. The form of the four-point function is similar to the one of the usual \( N = 2 \) superstring. However, since the spacetime signatures are different and the external momenta subject to different on-shell conditions, the final implications of the four-point function calculation are different. The four-point function vanishes in the usual \( N = 2 \) superstring, while it is non-zero in the generalised \( N = 2 \) superstrings. The mass poles in the gamma functions only involve the physical
states with vanishing fermion charge, which is consistent with the charge conservation the theories possess.

The charge conservation however does not exclude the possibility of propagation of $q$-charged states in loops. It can be easily checked that the momentum conservation constraint (3.5) applied to the scattering process of three physical states with charges $(q, -q, 0)$ does not exclude the possibility of propagating the $q \neq 0$ states in loops. However we can compactify the frozen time coordinate coupled to $(\bar{\beta}_0 - \beta_0)$ momentum on a circle with radius $R \neq \frac{n}{m} \alpha_0$, where $n, m$ are natural numbers. The quantised momentum satisfies $\alpha_0(\bar{\beta}_0 - \beta_0) \neq m$ (m integers except for zero). The constraint that $q$ charge has to be integer for a physical state then automatically reduces the physical states space to the states with $q = 0$. This condition also excludes all possible tachyons from the physical states spectrum of the generalised N=2 superstrings [3].

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