COMPOSITENESS CONDITION IN THE
NAMBU-JONA-LASINIO MODEL

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The Nambu-Jona-Lasinio model is the special case of the renormalized Yukawa model with the compositeness condition. We use it to calculate the effective coupling constants in terms of the compositeness scale (momentum cut off) at the next-to-leading order in $1/N$. The next-to-leading correction is too large in the model of scalar composite, while that in the induced gauge theory is reasonably suppressed due to the gauge cancellation of the leading divergences.

1 Introduction

The Nambu-Jona-Lasinio model realizes tractable compositeness in a simple field-theoretical scheme. What is important there is the compositeness condition which connect the Nambu-Jona-Lasinio model to a special case of the renormalized Yukawa model. However, the relations used there are mostly based on arguments which hold only in the large $N$ limit, where $N$ is the number of the fermion species. In this talk, I would like to consider the compositeness condition $Z = 0$ at the next-to-leading order in $1/N$ expansion, where $Z$ is the wave-function renormalization constant of the ‘composite boson’.

About twenty years ago, Terazawa, Chikashige and myself considered a Nambu-Jona-Lasinio type model of the standard model with composite Higgs scalar and gauge bosons made of a quark-anti-quark pair. It becomes a spontaneously broken gauge theory at low energies. It provides various predictions on relations among the masses and coupling constants. According to them, at least one of the quarks should have a mass of the order of the weak interaction scale. It looked puzzling because the known quarks at that time were much smaller than the weak scale. Today, however, we know that the top quark has the mass of the order of the weak scale, and the sum rule becomes rather natural. This fact called the revived attentions to the NJL-type model of the spontaneously broken electroweak symmetry. Numerically, however,
it does not precisely hold. We need to consider how to make it more precise beyond the leading approximation in $1/N$.

2 Compositeness Condition in the NJL model

We consider the NJL model for the fermion $\psi = \{\psi_1, \psi_2, \ldots, \psi_N\}$ with $N$ colors given by the Lagrangian

$$L_{NJL} = \bar{\psi} i \gamma^\mu \partial_\mu \psi + f |\bar{\psi}_L \psi_R|^2$$

with $U(1) \times U(1)$ chiral symmetry, where $f$ is the coupling constant. In 3+1 dimensions, it is not renormalizable, and we assume a very large but finite momentum cutoff. This Lagrangian $L_{NJL}$ is known to be equivalent to

$$L'_{NJL} = \bar{\psi} i \gamma^\mu \partial_\mu \psi + (\bar{\psi}_L \phi \psi_R + h.c.) - \frac{1}{f} |\phi|^2$$

written in terms of the auxiliary boson field $\phi$. Now compare it with this Lagrangian of the renormalized Yukawa model

$$L_{Yukawa} = Z_\psi \bar{\psi}_i i \gamma^\mu \partial_\mu \psi_i + Z_g g_i (\bar{\psi}_i \phi \psi_i + h.c.) + Z_\phi |\partial_\mu \phi_i|^2 - Z_\mu \mu_i^2 |\phi_i|^2 - Z_\lambda \lambda_i |\phi_i|^4,$$

where $\psi_i$ and $\phi_i$ are the renormalized fermion and boson fields, respectively, $g_i$ and $\lambda_i$ are the renormalized coupling constants, $\mu_i$ is the renormalized mass of the field $\phi_i$, and $Z_\psi$, $Z_\phi$, $Z_g$, $Z_\mu$, and $Z_\lambda$ are the renormalization constants. We can see that, if

$$Z_\phi = Z_\lambda = 0,$$

the Lagrangian $L_{Yukawa}$ coincides with $L'_{NJL}$ where we identify $\psi, \phi, \phi_1$, and $f$ in $L'_{NJL}$, with $\sqrt{Z_\psi} \psi_i$, $(Z_g/Z_\psi) g_i \phi_i$, and $Z_2 g_i^2 / Z_\psi^2 Z_\mu^2 \mu_i^2$ in $L_{Yukawa}$, respectively. The condition (4) is called ‘compositeness condition’. Thus the Lagrangian for NJL model is the special case of the renormalized Yukawa model with the compositeness condition. The compositeness condition gives rise to relations among coupling constants $g_i$, $\lambda_i$, and the cut off $\Lambda$. If the chiral symmetry is spontaneously broken, they imply relations among the fermion mass $m_f$, the Higgs-scalar mass $M_H$, and the cut off $\Lambda$. Thus we can study everything in the NJL model by studying the well-understood Yukawa model, and by imposing the compositeness condition on the coupling constants and masses. Then what is urgent is to work out the compositeness condition, and solve it for the coupling constants.
2.1 Lowest order in \(1/N\)

For an illustration, we begin with the lowest-order contributions in \(1/N\) expansion. In the Yukawa model, the boson self-energy part and the four-boson vertex part are given by the diagrams

\[
\begin{align*}
\cdots \hbar \cdots &= g_t^2 NI p^2, & \cdots X \cdots &= (Z_\phi - 1)p^2, \\
\hbar &= g_t^4 NI, & \hbar &= (Z_\lambda - 1)\lambda_r,
\end{align*}
\]

where the solid (dotted) line indicates the fermion (boson) propagator, and

\[
I = \begin{cases} 
\frac{1}{16\pi^2} \frac{1}{\epsilon} & \text{(dimensional regularization, } \epsilon = \frac{4 - d}{2}, \ d : \text{ dimension)} \\
\frac{1}{16\pi^2} \log \Lambda^2 & \text{(Pauli Villars regularization, } \Lambda : \text{ regulator mass)}
\end{cases}
\]

The renormalization constants \(Z_\phi\) and \(Z_\lambda\) should be chosen as

\[
Z_\phi = 1 - g_t^2 NI, \quad Z_\lambda \lambda_r = \lambda_r - g_t^4 NI,
\]

so as to cancel out all the divergences. Then the compositeness condition is obtained by putting \(Z_\phi = Z_\lambda = 0\), and it is easily solved to give

\[
g_t^2 = \frac{1}{NI}, \quad \lambda_r = \frac{1}{NI}
\]

If the chiral symmetry is spontaneously broken (i.e. if \(\mu_r^2 < 0\)), the physical fermion mass \(m_f\) and the physical Higgs mass \(M_H\) are given by

\[
m_f = g_t \langle \phi \rangle / \sqrt{NI}, \quad M_H = 2\sqrt{\lambda_r} \langle \phi \rangle = 2\langle \phi \rangle / \sqrt{NI},
\]

and hence we have \(2m_f = M_H\). These reproduce the well known results of the lowest order Nambu-Jona-Lasinio model.

2.2 Next-to-leading order in \(1/N\)

Now we turn to the next-to-leading order in \(1/N\). In the Yukawa model, the boson self-energy part is given by the diagram...
... ... + the counter terms for all the subdiagram divergences,

where ••••• stands for •••••• + ...

The renormalization constant $Z_\phi$ is calculated to be

$$Z_\phi = 1 - g_r^2 NI - \frac{1}{N} (1 - g_r^2 NI) \log(1 - g_r^2 NI)$$

(9)

so as to cancel out all the divergences there. The logarithm arises from the

infinite sum over the fermion loop insertions into the internal boson line. Similarly, the four boson vertex part is given by the diagrams

and the counter terms for all the subdiagram divergences, where ••••• stands for

••••• with arbitrary permutations of four external boson lines. The renormal-

ization constant $Z_\lambda$ is calculated to be

$$Z_\lambda = \lambda_r - g_r^4 NI + 8 g_r^4 I + \frac{20 (\lambda_r - g_r^2)^2 I}{1 - g_r^2 NI}$$

$$- \frac{1}{N} \left[ 2 g_r^2 (1 - g_r^2 NI) + 20 (\lambda_r - g_r^2) \right] \log(1 - g_r^2 NI)$$

(10)

so as to cancel out all the divergences here, where the logarithm again arises

from the infinite sum over the fermion loop insertions into the internal boson

lines. The compositeness condition is given by putting these expressions van-

ishing. Though it looks somewhat complex at first sight, it can be solved by

iteration to give the very simple solution

$$g_r^2 = \frac{1}{NI} \left[ 1 - \frac{1}{N} + O\left(\frac{1}{N^2}\right) \right], \quad \lambda_r = \frac{1}{NI} \left[ 1 - \frac{10}{N} + O\left(\frac{1}{N^2}\right) \right].$$

(11)

If the chiral symmetry is spontaneously broken the masses of the physical fermion and physical Higgs scalar are given by

$$m_f = g_r \langle \phi \rangle = \frac{\langle \phi \rangle}{\sqrt{NI}} \left[ 1 - \frac{1}{2N} + O\left(\frac{1}{N^2}\right) \right],$$

$$M_H = 2 \sqrt{\lambda_r} \langle \phi \rangle = \frac{2 \langle \phi \rangle}{\sqrt{NI}} \left[ 1 - \frac{5}{N} + O\left(\frac{1}{N^2}\right) \right],$$

hence

$$\frac{M_H}{m_f} = 2 \left[ 1 - \frac{9}{2N} + O\left(\frac{1}{N^2}\right) \right].$$

(12)

For the case of $N = 3$ of the practical interest, the corrections turn out to be too large, and the coupling constant $\lambda$ is negative, which implies that the
Higgs potential is unstable. The origin of this large negative contributions is traced back to the boson loop diagrams. A possible way getting rid of these difficulties is to assume that the cutoff $\Lambda_\phi$ for the composite $\phi$ is much smaller than the cutoff $\Lambda$ for the elementary fermions. In this case, the correction terms are suppressed by the small factor $r = \log \Lambda_\phi / \log \Lambda$.

\[
g_c^2 = \frac{1}{NI} \left[ 1 - \frac{r}{N} + O\left(\frac{r^2}{N^2}\right) \right], \quad \lambda_r = \frac{1}{NI} \left[ 1 - \frac{10r}{N} + O\left(\frac{r^2}{N^2}\right) \right]. \tag{13}
\]

It is straightforward to extend these results to the models of larger chiral symmetries. For example, for $SU(2) \times SU(2)$ for the pions and $\sigma$ meson, the correction to $g_c$ is absent at this order, and that for $\lambda_r$ is $6/N$. For $SU(2) \times U(1)$ of the electroweak symmetry, they are $3/2N$ and $12/N$, respectively. The corrections for the cases of the general number $F$ of flavors are also calculated. In any case, the corrections are too large for $N = 3$.

### 3 Compositeness Condition in the induced gauge theory

We can apply this method to the induced gauge theory, namely, the gauge theory with a composite gauge field. It is given by the strong coupling limit $f \to \infty$ of the vector-type four Fermi interaction model.

\[
\mathcal{L}_{4F} = \bar{\psi}_j (i\partial - m) \psi - f (\bar{\psi} \gamma_\mu \psi)^2,
\]  

where $\psi = \{\psi_1, \psi_2, \cdots, \psi_N\}$, $f$ is the coupling constant, and $m$ is the mass of $\psi$. The Lagrangian is equivalent to the linearized one written in terms of the auxiliary vector-boson field $A_\mu$. We can see that it is the special case of the renormalized gauge theory with the compositeness condition $Z_3 = 0$, where $Z_3$ is the wave-function renormalization constant of the gauge field identified with $A_\mu$. In the gauge theory, $Z_3$ should be chosen so as to cancel out the divergences in the gauge boson self-energy part. At the leading and next-to-leading order, it is given by the following diagrams.

\[\text{...} \quad \text{...} \quad \text{...} \quad \text{...} \quad \text{...} \]

After a lengthy calculation, we obtain

\[
Z_3 = 1 - \frac{e_c^2 N}{12\pi^2} - \frac{3e_c^2}{16\pi^2} \left[ 1 + \frac{12\pi^2 \epsilon}{e_c^2 N} \ln \left( 1 - \frac{e_c^2 N}{12\pi^2 \epsilon} \right) \right], \tag{15}
\]

cThis section reviews Ref. 6.
where $\epsilon_r$ is the renormalized effective coupling constant. Then, the compositeness condition $Z_3 = 0$ is solved to give the simple solution

$$e_r^2 = \frac{12\pi^2\epsilon}{N} \left[ 1 - \frac{9\epsilon}{4N} + O\left(\frac{1}{N^2}\right) \right].$$

(16)

The correction term $9\epsilon/4N$ is naturally suppressed by the small factor $\epsilon$. It justifies the lowest order approximation of this model unlike in the case of the aforementioned NJL model of the scalar composite. The origin of the suppression factor is traced back to the gauge cancellation of the leading divergence in the next-to-leading order diagrams. So far we assumed that all the fermions have the same charges for simplicity. If the charges $Q_j$ are different, the expression is modified as follows.

$$e_r^2 = \frac{12\pi^2\epsilon}{\sum_j Q_j^2} \left[ 1 - \frac{9\epsilon \sum_j Q_j^4}{4(\sum_j Q_j^2)^2} \right].$$

(17)

If we apply this to the quantum electrodynamics with 3 generations of quarks and leptons, $\epsilon$ is estimated to be $6\times10^{-3}$, which implies the next-to-leading order correction amounts only to 0.1% of the lowest order term.

4 Summary

The Nambu-Jona-Lasinio model is the special case of the renormalized Yukawa model with the compositeness condition $Z_\phi = Z_\lambda = 0$. We used it to calculate the effective coupling constants in terms of the compositeness scale (momentum cut off) at the next-to-leading order in $1/N$. The next-to-leading correction to the coupling constant $g_r^2$ is $1/N$, and that to the coupling constant $\lambda_r$ is $10/N$. For $N = 3$ of our practical interests, the corrections are too large, and $\lambda < 0$, as implies unstable Higgs potential. For induced gauge theory, the compositeness condition $Z_3 = 0$ implies that the next-to-leading correction term is $9\epsilon/4N$, which is naturally suppressed by the small factor $\epsilon$. Interesting extensions to the nonabelian gauge theories are now under investigation. In this case, if the corresponding elementary gauge theory is asymptotically free, the next-to-leading corrections according to the compositeness condition become too large to justify the $1/N$ expansion. Finally we comment that, the compositeness condition holds independently of choice of the renormalization conditions, because the renormalization is multiplicative. It implies the NJL model is at the fixed point in the renormalization flow of the Yukawa model (with a fixed cutoff). This is consistent with the fact that the compositeness condition is a relation among the observable renormalized quantities.
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