THEORY OF PARABOLIC ARCS IN INTERSTELLAR SCINTILLATION SPECTRA

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Received 2004 June 22; accepted 2005 September 27

ABSTRACT

Interstellar scintillation (ISS), observed as time variation in the intensity of a compact radio source, is caused by small-scale structure in the electron density of the interstellar plasma. Dynamic spectra of ISS show modulation in radio frequency and time. Here we relate the (two-dimensional) power spectrum of the dynamic spectrum—the secondary spectrum—to the scattered image of the source. Recent work has identified remarkable parabolic arcs in secondary spectra. Each point in a secondary spectrum corresponds to interference between points in the scattered image with a certain Doppler shift and a certain delay. The parabolic arc corresponds to the quadratic relation between differential Doppler shift and delay through their common dependence on scattering angle. We show that arcs will occur in all media that scatter significant power at angles larger than the rms angle. Thus, effects such as source diameter, steep spectra, and dissipation scales, which truncate high angle scattering, also truncate arcs. Arcs are equally visible in simulations of nondispersive scattering. They are enhanced by anisotropic scattering when the spatial structure is elongated perpendicular to the velocity. In weak scattering the secondary spectrum is directly mapped from the scattered image, and this mapping can be inverted. We discuss additional observed phenomena including multiple arcs and reverse arclets oriented oppositely to the main arc. These phenomena persist for many refractive scattering times, suggesting that they are due to large-scale density structures, rather than low-frequency components of Kolmogorov turbulence.

Subject headings: ISM: general — ISM: structure — pulsars: general — scattering

1. INTRODUCTION

Phenomena caused by scattering of pulsar radiation in the interstellar medium (ISM) have long been used to study small-scale structure in the electron density. This microstructure covers a range of scales extending from $\lesssim 10^3$ km to at least 10 AU and follows a Kolmogorov-like wavenumber spectrum. That is, the slope of the three-dimensional wavenumber spectrum is often estimated to be roughly $-11/3$, although some lines of sight at some epochs show evidence of excess power in the wavenumber spectrum on scales near 1 AU. The chief observable phenomena are intensity fluctuations in time and frequency (interstellar scintillation [ISS]), angular broadening (“seeing”), pulse broadening, and arrival time variations. These effects have been well studied in the three decades since pulsars were discovered. Scintillations comprise both diffractive ISS (DISS) and refractive ISS (RISS), which are fast and slow in time, respectively. Although RISS has also been recognized in compact extragalactic radio sources, their larger angular sizes suppress the DISS that probes the smallest scales in the medium. Thus, pulsars are the primary source of information of the smallest scale structure in the ISM.

DISS intensity variations are usually quantified using the dynamic spectrum, the intensity versus time and frequency, $S(\nu, t)$. Its two-dimensional correlation function is typically used to estimate the characteristic bandwidth and timescale for the scintillations. Frequency-dependent refraction by large-scale irregularities can also modulate these diffractive quantities (see, e.g., Bhat et al. 1999). Observers have noted occasional periodic fringes in the dynamic spectrum, which they quantified using a two-dimensional Fourier analysis of the dynamic spectrum (Ewing et al. 1970; Roberts & Ables 1982; Hewish et al. 1985; Cordes & Wolszczan 1986; Wolszczan & Cordes 1987; Rickett et al. 1997). We now refer to the two-dimensional power spectrum of $S(\nu, t)$ as the “secondary” spectrum of the scintillations. Fringes in the dynamic spectrum appear as discrete features in the secondary spectrum and are typically explained as interference between two or more scattered images. Recent studies of secondary spectra with much higher dynamic range have identified the dramatic “parabolic arc” phenomenon (Stinebring et al. 2001, hereafter Paper I). Arcs appear as enhanced power (at very low levels) along parabolic curves extending out from the origin well beyond the normal diffractive feature.

The primary goals of this paper are to understand how arcs are produced and to explore those aspects of the scattering medium that influence arcs. Our results indicate how observations may be inverted into constraints on both the medium and radio sources. Applications to data will be presented in subsequent papers. Here we develop a theory that relates the secondary spectrum to the underlying scattered image of the pulsar. We find that the arc phenomenon is a generic feature of forward scattering, and we identify conditions that enhance or diminish arcs. We first drafted our paper in 2001 following the publication of Paper I; that draft was widely distributed and stimulated work on arcs by Walker et al. (2004). They used methods closely related to ours and reached conclusions largely consistent with ours. Our work now also includes formal analysis of arcs under the asymptotic strong and weak scintillation regimes and results from a full electromagnetic simulation in the intermediate regime. In §2 we show examples of the variety of phenomena observed in arcs and list their salient features. Then in §3 we introduce the general theory of secondary spectra through its relation to angular scattering and provide examples that lead to generalizations of the arc phenomenon. In §4 we discuss cases relevant to the interpretation of observed

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phenomena. In § 5 we discuss the scattering physics and show examples of parabolic arcs from a full screen simulation of the scattering, and we end with discussion and conclusions in § 6.

2. OBSERVED PHENOMENA

The continuum spectrum emitted by a pulsar is deeply modulated by interference that changes with time due to motions of the observer, medium, and pulsar. The dynamic spectrum, $S(u, t)$, is the main observable quantity for our study. It is obtained by summing over the on-pulse portions of several to many pulse periods in a multichannel spectrometer covering a total bandwidth up to ~100 MHz, for durations ~1 hr. We compute its two-dimensional power spectrum $\tilde{S}(f_x, f_t) = \left| \tilde{S}(f_x, f_t) \right|^2$, the secondary spectrum, where the tilde indicates a two-dimensional Fourier transform and $f_x$ and $f_t$ are variables conjugate to $\nu$ and $t$, respectively. The total receiver bandwidth and integration time define finite ranges for the transform.

Examples of dynamic and secondary spectra are shown in Figures 1–3. The data for these figures were obtained at the Arecibo Observatory at a range of epochs and center frequencies that are listed in Table 1 along with other observational details, including the integration time per spectrum ($\Delta t$), the total bandwidth ($\Delta \nu$), the number of frequency channels across the bandwidth ($N_{\text{ch}}$), and the corresponding Nyquist frequencies in the secondary spectra, $f_{\text{Nyquist}}(\Delta \nu) = 1/2\Delta \nu$ and $f_{\text{Nyquist}}(\Delta \nu) = N_{\text{ch}}/2\Delta \nu$. We also give the strength of arc features in the secondary spectrum (see below) and the corresponding figure. The arc strength is the ratio of the arc amplitude to the peak in the secondary spectrum. We give two values: one for the weakest part of the arc seen and the other for the portion of the arc near the origin of the secondary spectrum.

Dynamic spectra often show randomly distributed diffractive maxima (scintles) in the frequency-time plane. In addition, organized patterns such as tilted scintles, periodic fringe patterns, and a loosely organized crisscross pattern have been observed and studied (e.g., Hewish et al. 1985; Cordes & Wolszczan 1986). In an analysis of four bright pulsars observed with high dynamic range at Arecibo, we discovered that the crisscross pattern has a distinctive signature in the secondary spectrum (Paper I). Faint but clearly visible power extends away from the origin in a parabolic pattern or arc, the curvature of which depends on observing frequency and pulsar. The observational properties of these scintillation arcs are explored in more detail in Hill et al. (2003, hereafter Paper II) and Hill et al. (2005, hereafter Paper III). Here we summarize the major properties of scintillation arcs in observations in order to provide a context for their theoretical explanation. Some of this summarizes earlier work, while other properties are new features manifested in Figures 1–3:

1. Scintillation arcs are faint, but they are ubiquitous and persistent when the secondary spectrum has adequate dynamic range and high frequency and time resolution. Table 1 indicates that arcs range in strength from about 0.1 to $10^{-5}$ of the feature at the origin of the secondary spectrum.

2. The arcs often have a sharp outer edge, although in some cases (e.g., Fig. 1c) the parabola is a guide to a diffuse power distribution. There is usually power inside the parabolic arc (Fig. 1), but the power falls off rapidly outside the arc except in cases where the overall distribution is diffuse.

3. The arc outlines are parabolic with minimal tilt or offset from the origin: $f_x = af_t^2$. Although symmetrical outlines are the norm, there are several examples of detectable shape asymmetries in our data.

4. In contrast to the symmetrical shape typical of the arc outline, the amplitude of the arc can be highly asymmetric in $f_t$ for a given $f_x$ and can show significant substructure. An example is shown in Figure 2. The timescale for change of this substructure is not well established, but some patterns and asymmetries have persisted for months.

5. A particularly striking form of substructure consists of inverted arclets with the same value of $|a|$ and with apices that lie along or inside the main arc outline (Figs. 2b and 2c). Paper III explores the motion of arclets along the main parabola and demonstrates, in one case at least, that the material causing the arclets is nearly stationary in the scattering screen.

6. Although a single scintillation arc is usually present for each pulsar, there is one case (PSR B1133+16) in which multiple scintillation arcs (with different $a$-values) are seen (Fig. 3). At least two distinct $a$-values (and perhaps as many as four) are traceable over decades of time.

7. Arc curvature accurately follows a simple scaling law with radio frequency: $a \propto \nu^{-2}$ (Paper II). In contemporaneous month-long observations over the range 0.4–2.2 GHz, scintillation arcs were present at all frequencies if they were visible at any for a given pulsar, and the scintillation arc structure became sharper and better defined at the higher frequencies.

8. The arc curvature parameter $a$ is constant at the 5%–10% level for ~20 yr for the half-dozen pulsars for which we have long-term data spans.

In the following sections we identify the conditions under which the various arc forms can arise and thus explain or otherwise address all of the points listed above. We do so semiquantitatively. We defer a quantitative fitting of the theory to the observations to a later paper. We start with an approach based on the addition of intensity fringe patterns from each pair of scattered waves (§§ 3 and 4). We postpone to § 5 and the appendices a more formal analysis of the asymptotic strong and weak scintillation regimes.

3. THEORY OF SECONDARY SPECTRA AND SCINTILLATION ARCS

3.1. Introduction

Although the parabolic arc phenomenon is striking and unexpected, it can be understood in terms of a simple model based on a “thin screen” of ionized gas containing fluctuations in electron density on a wide range of scales. Incident radiation is scattered by the screen and then travels through free space to the observer, where interference between components of the scattered wave causes intensity variations (scintillation). The arcs arise in small-angle scattering from the square law dependence of the delay on angle of scattering and do not depend on the dispersive nature of the interstellar plasma. The screen model is widely used, since it includes most of the physics of scattering in an extended inhomogeneous medium and is mathematically tractable. As seen below, there is, in fact, observational evidence that it applies to the ISS on some lines of sight, a conclusion that follows from the sharpness of arcs observed from some pulsars.

To get interference, the screen must be illuminated by a radiation field of high spatial coherence, i.e., radiation from a point-like source. However, the source can be temporally incoherent because the different components of the temporal Fourier transform each contribute nearly identical interference patterns, as in interferometry. Consider two scattered waves arriving from directions $\theta_1$ and $\theta_2$, which interfere to produce a two-dimensional fringe pattern. The pattern is sampled in time and frequency as the observer

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4 Of 12 bright ($S_{400} > 60$ mJy) pulsars that we have observed at Arecibo, 10 display scintillation arcs, and the other 2 have features related to arcs.
moves through it, creating a sinusoidal fringe pattern in the dynamic spectrum whose phase varies slowly with observing frequency. The fringe appears as a single Fourier component in the secondary spectrum. Under the small-angle scattering of the ISM its coordinate is the differential geometric delay \( \frac{\theta_2 - \theta_1}{C} \) and its \( f_t \) coordinate is the fringe rate \( \propto V_\perp \cdot (\theta_2 - \theta_1) \), where \( V_\perp \) is an appropriate transverse velocity (see below). A quadratic relationship \( f_t \propto f_t^2 \) results naturally from their quadratic and linear dependencies on the angles. When one of the interfering waves is undeviated (e.g., \( \theta_1 = 0 \)), we immediately get the simple parabola \( f_t \propto f_t^2 \).

When the scattering is weak, there will be a significant undeviated contribution resulting in such a simple parabolic arc.

**Fig. 1.**—Dynamic and secondary spectrum pairs shown for four pulsars that exhibit the scintillation arc phenomenon. The gray scale for the dynamic spectrum is linear in flux density. For the secondary spectrum, the logarithm of the power is plotted, and the gray scale extends from the noise floor (white) to 5.5 decades above it (black). The dispersion measures of these pulsars range from 3.2 to 48.4 pc cm\(^{-3}\), and the scattering measures range from \( 10^{-4.5} \) to \( 10^{-3.6} \) kpc m\(^{-20/3}\), with PSR B1929+10 and PSR B1737+13 representing the extremes in scattering measure. The data shown here and in Figs. 2 and 3 were obtained from the Arecibo Observatory.
Fig. 2.—Three observations of PSR B0834+06 taken within 2 weeks of each other. The asymmetry with respect to the conjugate time axis is present, in the same sense, in all three observations. The broad power distribution at 430 MHz in (a) is much sharper 1 day later at 1175 MHz (panel [b]); however, a more diffuse component has returned 14 days later (panel [c]). Note that the scales for the delay axis in (b) and (c) differ from that in (a). The inverted parabolic arclets noticeable in (b) and (c) are a common form of substructure for this and several other pulsars. The gray scales are the same as in Fig. 1.

Fig. 3.—One of the pulsars in our sample, PSR B1133+16, shows multiple scintillation arcs on occasion. The broad, asymmetric power distribution in (a) has numerous arclets at 321 MHz. Panels (b) and (c) are at frequencies above 1 GHz. Panel (b) shows two clear arcs (along with a horizontal line at t due to narrowband radio frequency interference and the sidelobe response of power near the origin). Four months later (panel [c]), only the outer of these two arcs, widened by the a \propto \nu^{-2} scaling, is visible. The cause of the diagonal line from the origin is unknown. The gray scales are the same as in Fig. 1.
However, we also find that an arc appears in strong scattering due to the interference of waves contained within the normal scattering disk with a faint halo of waves scattered at much larger angles. The faint halo exists only for scattering media having wavenumber spectra less steep than (wavenumber)^{-4}, including the Kolmogorov spectrum, as we show in §4.2.

While the relation of f_s to time delay is well known through the Fourier relationship of the pulse broadening to the radio spectrum of intensity scintillation, the physical interpretation of f_s can be viewed in several ways besides the fringe rate. Scintillation versus time is often Fourier analyzed into a spectrum versus frequency (f_s) in Hz; in turn, this is simply related to spatial wavenumber (k) as f_s = k · V_s/(2π) and hence to angle of arrival (k = kθ) where k = 2π/λ. It can also be thought of as the beat frequency due to the different Doppler shifts of the two scattered waves. Thus, the secondary spectrum can be considered as a differential delay Doppler spectrum, similar to that measured in radar applications.

### 3.2. A Simple Screen Theory

In this section we develop a simple approximate theory that gives the relationship between the scattered image and the secondary spectrum S_2. We do so by taking a probabilistic approach that assumes that the components of the scattered image have randomly distributed phases. Later, in §5, we discuss the theory more formally and show that explicit results are obtained in the limits of strong and weak scintillation (Appendices C and D). We find that the weak scintillation result is simpler, being a second moment of the electric field since it only involves the interference of scattered waves with the unscattered field. However, pulsars are typically observed in strong scintillation, and the strong scintillation limit gives exactly the same result as the approximate theory, which we now develop.

In Appendix A we use the Fresnel-Kirchoff diffraction integral to analyze the following geometry: a point source at z = 0, an observer at z = D, and a thin screen in between at z = sD. The screen changes only the phase of incident waves, but it can both diffract and refract radiation from the source. For a single frequency emitted from the source, consider spherical waves coming from two points on the screen subtending angles θ_1 and θ_2. They sum and interfere at the observer’s location with a phase difference Φ. The resulting intensity is I = I_1 + I_2 + 2(1/2I_1I_2)^{1/2} cos Φ, where I_1 and I_2 are the intensities from each wave (e.g., Cordes & Wolszczan 1986).

The total phase difference Φ = Φ_s + φ includes a contribution from geometrical path length differences Φ_s and a dispersive contribution from the screen phase, φ, which can include both small- and large-scale structures that refract and diffract radiation. We expand Φ to first order in time and frequency increments:

\[ \Phi \sim \Phi_0 + 2\pi(f_s δt + f_c δν), \]

where δν = ν - ν_0, δt = t - t_0, and t_0, ν_0 define the center of the observing window; f_s = (1/2π)k̇Φ is the fringe rate or differential Doppler shift, and f_c = (1/2π)k̇Φ̇ is the differential group delay. In Appendix A we discuss calculations of the partial derivatives. In general, they contain a contribution from derivatives of the screen phase, which are related to dispersive time delays. In practice, we find that the dispersive contribution to f_c is dominated by the geometric contribution, so we henceforth drop the dispersive terms. In §5.2 we give a brief consideration of the effects of dispersion.

As already reported in Paper I, based on derivations that have lead to the present paper, the fringe frequencies are

\[ f_ν = \left[ \frac{D(1 - s)}{2cs} \right] \left( \theta_2^2 - \theta_1^2 \right), \]

\[ f_t = \left( \frac{1}{λs} \right) (\theta_2 - \theta_1) · V_⊥. \]

Here λ is the wavelength at the center of the band, and V_⊥ is the velocity of the point in the screen intersected by a straight line from the pulsar to the observer, given by a weighted sum of the velocities of the source, screen, and observer (e.g., Cordes & Rickett 1998):

\[ V_⊥ = (1 - s)V_{p⊥} + sV_{obs⊥} - V_{screen⊥}. \]

It is also convenient to define an effective distance d_e,

\[ d_e = Ds(1 - s). \]

The results of Walker et al. (2004) are the same as equations (2) and (3). In their analysis they approximate the Fresnel-Kirchoff integral as a sum over stationary phase points whose mutual interference causes the arc phenomenon. As they note, their approach is only appropriate in the regime of strong scintillation. As we do, they ignore the dispersive delays relative to the geometric delays.

The two-dimensional Fourier transform of the interference term cos Φ(δν, δt) is a pair of delta functions placed symmetrically.

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**TABLE 1**

| Name         | MJD   | ν_0 (MHz) | Δν (MHz) | N_{ih} | f_s (Nyquist) (μs) | Arc Strength | Figure |
|--------------|-------|-----------|----------|--------|-------------------|--------------|--------|
| B0823+26     | 52773 | 1175      | 50       | 1024   | 10.2              | 10^{-5} to 10^{-3.2} | 1a     |
| B0834+06     | 46156 | 430       | 10       | 252    | 12.6              | 10^{-4.8} to 10^{-1.9} | 1b     |
| B1737+13     | 52472 | 1410      | 100      | 1024   | 5.1               | 10^{-3.3} to 10^{-0.8} | 1c     |
| B1929+10     | 52521 | 1175      | 100      | 1024   | 5.1               | 10^{-3.3} to 10^{-3.5} | 1d     |
| B0834+06     | 52435 | 430       | 10       | 1024   | 51.2              | 10^{-4.4} to 10^{-1.3} | 2a     |
| B0834+06     | 52434 | 1175      | 100      | 1024   | 5.1               | 10^{-3.4} to 10^{-3.5} | 2b     |
| B1133+16     | 52838 | 321       | 12.5     | 1024   | 41.0              | 10^{-4.4} to 10^{-1.2} | 2c     |
| B1133+16     | 52448 | 1175      | 100      | 1024   | 5.1               | 10^{-5.3} to 10^{-3.0} | 3b     |
| B1133+16     | 52566 | 1420      | 100      | 1024   | 5.1               | 10^{-4.6} to 10^{-2.6} | 3c     |

**Note.**—All observations were sampled at Δt = 10 s, which yields a Nyquist Doppler frequency (fringe rate) of 50 mHz.
about the origin of the delay–fringe rate plane. The secondary spectrum at \((f_s, f_t)\) is thus the summation of the delta functions from all pairs of angles subject to equations (2) and (3), as we describe in the next section.

Scintillation phenomena either can be described in terms of the Fresnel-Kirchoff diffraction integral, which sums spherical waves from all points on the screen, or it can be described by the mutual interference of components of an angular spectrum of plane waves. These are formally quite different descriptions of scintillation, since a particular plane wave is due to a single spatial Fourier component in the screen, while a particular spherical wave represents the scattering from a single point on the screen. Nevertheless, the time delay due to a scattered plane wave arriving at an angle \(\theta\) and the time delay due to a spherical wave arriving from a point \(r\) in the screen are both given by equation (2), where \(\theta\) is either the angle of arrival of the plane wave or the apparent angular position of the point on the screen \(\theta = r/(D - sD)\). Similarly, the fringe rate is given by equation (3) in both descriptions. Thus, although these two equations were derived using the diffraction integral, from this point on we discuss the secondary spectrum in terms of the angular spectrum.

3.3. Secondary Spectrum in Terms of the Scattered Brightness

In this section we examine how the secondary spectrum depends on the shape of the scattered image, without considering the associated physical conditions in the medium that lead to the image. We postpone until §5 a discussion of the scattering physics and the influence of the integration time. The integration time is important in determining whether the scattered brightness is a smooth function or is broken into speckles, as discussed by Narayan & Goodman (1989). This in turn influences whether the secondary spectrum takes a simple parabolic form or becomes fragmented.

A simple way to obtain the secondary spectrum (presented here) is to analyze an arbitrary scattered image by treating its scattered brightness distribution, \(B(\theta)\), as a probability density function (PDF) for the angles of scattering. In the continuous limit the secondary spectrum is the joint PDF of \(f_s, f_t\) subject to the constraints of equations (2) and (3). We use dimensionless variables for the delay and fringe rate:

\[
p = \theta_s^2 - \theta_t^2 = \left[ \frac{2cs}{D(1 - s)} \right] f_s,
\]

\[
q = (\theta_s - \theta_t) \cdot \hat{V}_\perp = \left( \frac{ks}{V_\perp} \right) f_t,
\]

where \(\hat{V}_\perp\) is a two-dimensional unit vector for the transverse effective velocity. It is also useful to normalize angles by the characteristic diffraction angle, \(\theta_0\), so in some contexts discussed below we let \(\theta \rightarrow \theta/\theta_0\), so that \(p \rightarrow f_s/\theta_0\) and \(q \rightarrow 2\pi \Delta t_2 f_t\), where \(\Delta t_2\) is the pulse broadening time and \(\Delta t_2\) is the diffraction scintillation time. The conditional probability for \(p, q\) is \(\delta(p - \tilde{p})\delta(q - \tilde{q})\), where \(\tilde{p}, \tilde{q}\) are the values for a particular pair of angles, \(\theta_1, \theta_2\), as given in equations (6) and (7). The total PDF is the integral of the conditional PDF over the pair of vector angles multiplied by the PDFs of those angles. We use an angular distribution \(B(\theta)\) as the PDF of the vector angle and call the PDF of \(p, q\) the secondary spectrum:

\[
S_2(p, q) = \int d\theta_1 d\theta_2 \delta(\theta_1 - \theta) \delta(\theta_2 - \theta) \delta(p - \tilde{p})\delta(q - \tilde{q}).
\]

Implicit in the integration used here is that we can add the intensity patterns from each pair of interfering waves. For this to be true, the phase coherence of the waves incident on the scattering screen (which is necessary in order that there be interference) needs to be randomized by the screen’s phase perturbations. Such a situation applies by definition in the strong scattering regime. We obtain the same result as equation (8) for the ensemble average secondary spectrum using physical optics in the limit of strong diffractive scattering in Appendix C. However, this is a more restrictive condition than the randomly phased angular spectrum assumption used here.

The equivalence of equation (8) obtained using the probabilistic approach with the ensemble average of the physical optics approach elucidates how we should picture \(B(\theta)\). In the ensemble average case, \(B(\theta)\) is a smooth function that has a single smooth maximum in real physical cases involving random media with stationary statistics. For the probabilistic case, no assumption is made about integration over an ensemble or over an infinite time interval, so \(B(\theta)\) in that case can include stochastic fluctuations that appear only over the finite time interval used to form the secondary spectrum; thus, \(B(\theta)\) can include speckles and longer lived multiple images that are inferred to exist using scintillation data. Equation (8) is identical to equation (30) of Walker et al. (2004), obtained for each stationary phase point that contributes to the Fresnel-Kirchoff integral. They assumed that such points are described by a probability density function that takes the place of our \(B(\theta)\). We note that our approach is somewhat more general since it only depends on the randomly phased assumption for the angular spectrum.

By definition the secondary spectrum is symmetric through the origin \((p \rightarrow -p, q \rightarrow -q)\). Equation (8) shows that it is essentially a distorted autocorrelation of the scattered image. With no loss of generality we simplify the analysis by taking the direction of the velocity to be the \(x\)-direction. The fourfold integration in equation (8) may be reduced to a double integral by integrating the delta functions over, say, \(\theta_2\) to obtain

\[
S_2(p, q) = \frac{1}{2} \int d\theta_1 d\theta_1 H(U) U^{-1/2} B(\theta_1, \theta_1) \times \left[ B\left(q + \theta_1, \sqrt{U}\right) + B\left(q + \theta_1, -\sqrt{U}\right) \right],
\]

where

\[
U \equiv p - q^2 - 2q\theta_1 + \theta_1^2
\]

and \(H(U)\) is the unit step function. With this form, the integrand is seen to maximize at the singularity \(U = 0\), which yields a quadratic relationship between \(p\) and \(q\). For an image offset from the origin by angle \(\theta_0\), e.g., \(B(\theta) \rightarrow B(\theta - \theta_0)\), the form for \(S_2\) is similar except that \(U = p - q^2 - 2q\theta_1 + \theta_1^2 + \theta_0^2\) and the \(\theta_1\) arguments of \(B\) in square brackets in equation (9) become \(\pm \sqrt{U} - \theta_0\). The (integrable) singularity at \(U = 0\) makes the form of equation (9) inconvenient for numerical evaluation. In Appendix B we use a change of variables to avoid the singularity, giving a form (eq. [84]) that can be used in numerical integration. However, this does not remove the divergence of \(S_2\) at the origin \(p = q = 0\). In Appendix B we show that inclusion of

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4 In our treatment, we interchangeably use “scattered brightness distribution,” “scattered image,” and “PDF for the angle of scattering” while recognizing that it is as yet undetermined as to whether \(B(\theta)\) is a smooth function or one that is broken into speckles. Refining our picture of the properties of \(B(\theta)\) is a goal of analyses of secondary spectra that goes beyond the purview of the present paper.
finite resolutions in p or q, which follow from the finite extent of the original dynamic spectrum in time and frequency, avoids the divergence at the origin. Hence, the dynamic range of the secondary spectrum is strongly influenced by resolution effects.

The relation \( p = q^2 \) follows from the singularity \( U = 0 \) after integration over angles in equation (9) and is confirmed in a number of specific geometries discussed below. This relation becomes \( a = df_p^2 \) in dimensional units, with

\[
a = \frac{Dp(1 - s)}{2c} \left( \frac{\lambda}{V_\perp} \right)^2 = 0.116 \left[ \frac{s(1 - s)}{1/4} \right] D_{\text{ps}} V_\perp^2 V_{100}^2 \mathrm{s}^{-3},
\]

(11)

where \( V_\perp = (100 \text{ km s}^{-1}) V_{100} \) and \( \nu \) is in GHz. For screens halfway between source and observer \( s = \frac{1}{2} \), this relation yields values for \( a \) that are consistent with the arcs evident in Figures 1–3 and also shown in Papers I and II. As noted in Paper I, \( a \) does not depend on any aspect of the scattering screen save for its fractional distance from the source, \( s \), and it maximizes at \( s = \frac{1}{2} \) for fixed \( V_\perp \).

When a single arc occurs and other parameters in equation (11) are known, \( s \) can be determined to within a pair of solutions that is symmetric about \( s = \frac{1}{2} \). However, when \( V_\perp \) is dominated by the pulsar speed, \( s \) can be determined uniquely (see Paper I).

### 3.4. Properties of the Secondary Spectrum

To determine salient properties of the secondary spectrum that can account for many of the observed phenomena, we consider special cases of images for which equation (9) can be evaluated. As noted above, the effects of finite resolution are important both observationally and computationally and are discussed in Appendix B.

#### 3.4.1. Point Images

A point image produces no interference effects, so the secondary spectrum consists of a delta function at the origin, \( (p, q) = (0, 0) \). Two point images with amplitudes \( a_1 \) and \( a_2 \) produce fringes in the dynamic spectrum, which give delta functions of amplitude \( a_1 a_2 \) at a position given by equations (6) and (7) and its counterpart reflected through the origin and also a delta function at the origin with amplitude \( 1 - 2a_1 a_2 \). Evidently, an assembly of point images gives a symmetrical pair of delta functions in \( p, q \) for each pair of images.

#### 3.4.2. One-Dimensional Images

Arc features can be prominent for images elongated in the direction of the effective velocity. Consider an extreme but simple case of a one-dimensional image extended along the x-axis only.

\[
B(\theta) = g(\theta_p) \delta(\theta_p),
\]

(12)

and where \( g \) is an arbitrary function. The secondary spectrum is

\[
S_2(p, q) = (2|q|)^{-1} g \left( \frac{p - q^2}{2q} \right) g \left( \frac{p + q^2}{2q} \right)
\]

(13)

By inspection, parabolic arcs extend along \( p = \pm q^2 \) with amplitude \( S_2 \propto g(q)/|q| \) along the arc, becoming wider in \( p \) at large \( |q| \). However, for the particular case where \( g \) is a Gaussian function, the product of the \( q \) functions in equation (13) \( \propto \exp \left[ -\frac{1}{2}(p^2 + q^2) \right] \), which cuts off the arcs very steeply (see § 4.1). For images that decline more slowly at large angles, the arc features can extend far from the origin. If the image is elongated at an arbitrary angle \( \phi \) to the velocity, the mathematics is unchanged except that \( q \) becomes \( q \sin \phi \) in equation (13). This simply narrows the arc in fringe rate, and in the extreme case of an image elongated transverse to \( x \), there is only a ridge along the \( q \)-axis. These examples suggest that prominent arcs are expected when images are highly anisotropic in a direction approximately aligned with the effective velocity, as we confirm below for more general image shapes (see also § 3.2 of Walker et al. 2004).

#### 3.4.3. Images with a Point and an Extended Component

While equation (8) is a complete description of \( S_2 \) as an integral over all pairs of interfering waves, it is illuminating to examine the contribution from a single angle of arrival interfering with all points in a broadened image, since this naturally gives rise to a parabolic arc. Moreover, weak scattering, which yields an image comprising both an unscattered and a scattered component, is described with this model. So consider a scattered image consisting of a point source at \( \theta_p \) and an arbitrary, two-dimensional image component, \( g(\theta) \),

\[
B(\theta) = a_1 \delta(\theta - \theta_p) + a_2 g(\theta).
\]

(14)

The secondary spectrum consists of the self-interference terms from each image component and their cross interference term. The latter, which is our chief interest, is of the form

\[
\Delta S_2(p, q) = \frac{1}{2} a_1 a_2 U^{-1/2} H(U) \sum_{\pm} g(q + \theta_p \pm, \pm \sqrt{U}) + \text{S.O.,}
\]

(15)

where \( U = p - q^2 - 2q\theta_p + \theta_p^2 \), \( H(U) \) is the unit step function, and \( \text{S.O.} \) implies an additional term that is symmetric through the origin, obtained by letting \( p \rightarrow -p \) and \( q \rightarrow -q \) in the first component. A parabolic arc is defined by the \( H(U)/U^{-1/2} \) factor. As \( U \rightarrow 0 \) the arc amplitude is \( a_1 a_2 U^{-1/2} g(q + \theta_p, 0) \). Considering \( g \) to be centered on the origin with width \( W_p \) in the \( x \)-direction, the arc extends to \( |q + \theta_p| \leq W_p/2 \). Of course, if the point component has a small but nonzero diameter, the amplitude of the arc would be large but finite. If the point is at the origin, the arc is simply \( p = q^2 \) as already discussed. In such cases equation (15) can be inverted to estimate \( g(\theta) \) from measurements of \( \Delta S_2 \). The possibility of estimating the two-dimensional scattered brightness from observations with a single dish is one of the intriguing aspects of the arc phenomenon.

With the point component displaced from the origin, the apex of the parabola is shifted from the origin to

\[
(p, q)_{\text{apex}} = (-\theta_p^2, -\theta_p).
\]

(16)

By inspection of equation (15), the condition \( U = 0 \) implies that the arc is dominated by contributions from image components scattered parallel to the velocity vector at angles \( \theta = (q + \theta_p, 0) \). Thus, images elongated along the velocity vector produce arcs enhanced over those produced by symmetric images. This conclusion is general and is independent of the location, \( \theta_p \), of the point component.

Figure 4 shows parabolic arc lines \( (U = 0) \) for various assumed positions for the point component. Equation (16) gives a negative delay at the apex, but since \( S_2 \) is symmetric about the origin, there is also a parabola with a positive apex, but with reversed curvature. When \( \theta_p = 0 \), the apex must lie on the basic arcs \( p = \pm q^2 \) (left panel), and otherwise the apex must lie inside...
them (right panel). Such features suggest an explanation for the reversed arclets that are occasionally observed, as in Figure 2. We emphasize that the discussion here has been restricted to the interference between a point and an extended component, and one must add their self-interference terms for a full description. The foregoing analysis evidently can be applied to a point image and an ensemble of subimages. The general nature of $S_2$ is similar in that it is enhanced along the curve where $U = 0$, which enhances subimages lying along the velocity vector.

4. SECONDARY SPECTRA FOR CASES RELEVANT TO THE INTERSTELLAR MEDIUM

We now compute the theoretical secondary spectrum for various simple scattered brightness functions commonly invoked for interstellar scattering. The computations include self-interference and also the finite resolution effects, discussed in Appendix B. Similar results were obtained in § 3 of Walker et al. (2004).

4.1. Elliptical Gaussian Images

Measurements of angular broadening of highly scattered OH masers (Frail et al. 1994), Cyg X-3 (Wilkinson et al. 1994), pulsars (Gwinn et al. 1993), and active galactic nuclei (e.g., Spangler & Cordes 1988; Desai & Fey 2001) indicate that the scattering is anisotropic and, in the case of Cyg X-3, consistent with a Gaussian scattering image, which probably indicates diffractive scales smaller than the “inner scale” of the medium. Dynamic spectra cannot be measured for such sources because the time and frequency scales of the diffractive scintillations are too small and are quenched by the finite source size. However, some less scattered pulsars may have measurable scintillations that are related to an underlying Gaussian image.

The secondary spectrum for an elliptical Gaussian image cannot be solved analytically, although equation (B4) can be reduced to a one-dimensional integral that simplifies the numerical evaluation. Figure 5 shows some examples. Figure 5a is for a single circular Gaussian image. It does not exhibit arclike features, although it “bulges” out along the dashed arc line ($p = q^2$). Figure 5b is for an elliptical Gaussian image with a 2:1 axial ratio parallel to the velocity vector. The 3:1 case (Fig. 5c) shows the deep “valley” along the delay ($p$) axis, which is characteristic of images elongated parallel to the velocity. Such deep valleys are frequently seen in the observations and provide evidence for anisotropic scattering. Notice, however, that the secondary spectrum can be strong outside of the arc line, where the contours become parallel to the $p$-axis. Inside the arc line the contours follow curves like the arc line. The 3:1 case with the major axis transverse to the velocity (Fig. 5d) shows enhancement along the delay axis with no bulging along the arc line.

4.2. Media with Power-Law Structure Functions

Here we consider brightness distributions associated with wavenumber spectra having a power-law form. For circularly symmetric scattering, the phase structure function depends only on the magnitude of the baseline so the visibility function is $\Gamma(b) = \exp \left[ -\frac{1}{2} D_\theta(b/s_0) \right]$. A suitable power-law form for the structure function is $D_\theta(b) = b^{\nu} s_0^{\nu}$, where $s_0$ is the spatial scale of the intensity diffraction pattern, defined such that $\left[ \Gamma(s_0) \right]^2 = e^{-1}$ (e.g., Cordes & Rickett 1998). The corresponding image is

$$B(\theta) = 2\pi \int_0^\infty b \, db \, \Gamma(b) J_0(2\pi b\theta/\lambda),$$

where $J_0$ is the Bessel function. Computations are done in terms of an image normalized so that $B(0) = 1$, a scaled baseline $\eta = b/s_0$, and a scaled angular coordinate $\psi = \theta/\theta_d$, where $\theta_d \approx \lambda/(2\pi s_0)$ is the $e^{-1}$ scattered angular width.
4.2.1. Single-Slope Power Laws

First, we consider phase structure functions of the form $D_\phi \propto b^\alpha$ with $\alpha = 1, 1.5, 1.67, 1.95, \text{ and } 2$ (thickest to thinnest lines and as labeled). The inner scale is assumed to be negligible, and the images all have the same $e^{-1}$ width (before integration). The $\alpha = 2$ case corresponds to an image that is Gaussian in form. For cases with $\alpha < 2$, the scattered images consist of a core component superposed with a much larger halo component. Such cases produce strong arcs in secondary spectra. For $\alpha = 1$, the halo component (defined as the flux for $\theta > \theta_\alpha$, say) contains 10% of the total flux density, while for $\alpha = 1$ it is 40%. These halos contain enough flux to account for the arc strengths given in Table 1. It is clear that the shallower spectra have more power in their halos and so cause more prominent arcs.

The secondary spectra corresponding to the cases in Figure 6 (left panel) are shown in Figure 7. Arcs are most prevalent for the smallest value of $\alpha$, become less so for larger $\alpha$, and are nonexistent for $\alpha = 2$. Thus, the observation of arcs in pulsar secondary spectra rules out the underlying image having the form of a symmetric Gaussian function and so puts an upper limit on an inner scale in the medium.

Arcs for all of the more extended images, including the $\alpha = 1.95$ case, appear to be due to the interference of a central “core” ($\psi < 1$) component with a weaker “halo” ($\psi \gg 1$) evident in Figure 6 (both panels). This interpretation is confirmed by the work of Codona et al. (1986). Their equation (43) gives the cross spectrum of scintillations between two frequencies in the limit of large wavenumbers in strong scintillation. When transformed, the resulting secondary spectrum is exactly the form of an unscattered core interfering with the scattered angular spectrum. The halo brightness at asymptotically large angles scales as $\psi^{-\alpha+2}$. Along the $p$-axis, $S_2(p, 0) \propto p^{-(\alpha+2)/2}$. Only through a detailed quantitative analysis can the value of $\alpha$ be inferred from observations of the secondary spectrum. As this analysis is non-trivial, we defer it to another publication.

Fig. 6.—Integrated images $I(\theta)$ for scattering media that produce circularly symmetric, scattered brightness distributions. Left: Cases for phase structure functions $D_\phi \propto b^\alpha$ with $\alpha = 1, 1.5, 1.67, 1.95, \text{ and } 2$ (thickest to thinnest lines and as labeled). The inner scale is assumed to be negligible, and the images all have the same $e^{-1}$ width (before integration). The $\alpha = 2$ case corresponds to an image that is Gaussian in form. For cases with $\alpha < 2$, the scattered images consist of a core component superposed with a much larger halo component. Such cases produce strong arcs in secondary spectra. Right: Cases for a Kolmogorov wavenumber spectrum having different fractional inner scales, $\zeta$, as defined in the text and as labeled for each curve. A sufficiently large inner scale diminishes the image halo component, thus truncating arcs in secondary spectra.

Fig. 7.—Secondary spectra corresponding to the image cases shown in the left panel of Fig. 6. (a) $\alpha = 1.0$; (b) $\alpha = 1.67$; (c) $\alpha = 1.95$; (d) $\alpha = 2.0$. The contour separation is one decade and the outermost contour is six decades from the peak of the secondary spectrum. The dashed line shows the parabola, $p = q^2$. 
4.2.2. Kolmogorov Spectra with an Inner Scale

A realistic medium is expected to have a smallest ("inner") scale in its density fluctuations. Depending on its size, the inner scale may be evident in the properties of pulsar scintillations and angular broadening. Angular broadening measurements, in particular, have been used to place constraints on the inner scale for heavily scattered lines of sight (Moran et al. 1990; Molnar et al. 1995; Wilkinson et al. 1994; Spangler & Gwinn 1990). For scintillations, the inner scale can alter the strength of parabolic arcs, thus providing an important method for constraining the inner scale for lines of sight with scattering measures much smaller than those on which angular broadening measurements have been made. We consider an inner scale, \( l_1 \), that cuts off a Kolmogorov spectrum and give computed results in terms of the normalized inner scale \( \zeta = l_1 / l_0 \). The structure function scales asymptotically as \( D_0 \propto b^2 \) below \( l_1 \) and \( \propto b^{5/3} \) above.

In the right panel of Figure 6 we show integrated images for different values of \( \zeta \). For \( \zeta \ll 1 \), the inner scale is negligible and the image falls off relatively slowly at large \( \psi \), showing the extended halo \( \propto \psi^{-11/3} \), as for the Kolmogorov spectrum with no inner scale. For \( \zeta = 2 \), the image falls off similarly to the shown Gaussian form (dashed line) and thus does not have an extended halo. Secondary spectra are shown in Figure 8 for four values of inner scale. As expected, the arcs are strongest for the case of negligible inner scale and become progressively dimmer and truncated as \( \zeta \) increases and the image tends toward a Gaussian form. The appearance of strong arcs in measured data indicates that the inner scale must be much less than the diffraction scale, or \( l_1 \ll 10^4 \) km, for lines of sight to nearby pulsars, such as those illustrated in Figures 1–3. This implies \( n_e \gg 0.005 \) cm\(^{-3} \) if the inner scale is due to cyclotron damping, as is the case in the solar wind.

4.2.3. Anisotropic Kolmogorov Spectrum

Figure 9 shows the results for an anisotropic Kolmogorov spectrum with a 3:1 axial ratio for different orientation angles of the image with respect to the velocity. In the top left panel the scattered image is elongated parallel to the velocity, and the arc is substantially enhanced with a deep valley along the \( p \)-axis. The deep valley in the parallel case occurs because along the \( p \)-axis \( q = \theta_2 - \theta_1 = 0 \), so \( p = \theta_2^p - \theta_1^p \), in which case the secondary spectrum falls steeply like the brightness distribution along its narrow (\( y \)) dimension. By comparison, along the arc itself, the secondary spectrum probes the wide dimension and thus receives greater weight. As the orientation changes with respect to the velocity in the other panels, the arc diminishes in contrast and disappears at 90°. However, it is still detectable for orientations \( \leq 60° \), which would have a probability of \( 1/2 \) for a random distribution.

5. SCATTERING THEORY

5.1. Scattering Regimes and the Asymptotic Limits

In prior sections we related \( f_s \) and \( f_t \) to the angles of scattering through equations (2) and (3). The intensity of the scattered waves at each angle was represented by a scattered brightness.
function $B(\theta)$ with little discussion of the scattering physics that relates $B(\theta)$ to the properties of the ISM.

In discussing the scattering physics, we must consider the relevant averaging interval. Ensemble average results can be found in the limits of strong and weak scintillation, as we describe in Appendices C and D. In both cases the analysis depends on the assumption of normal statistics for the scattered field and the use of the Fresnel-Kirchoff diffraction integral to obtain a relation with the appropriate ensemble average $B(\theta)$. However, for strong scattering we adopt the common procedure of approximating the observed spectrum by the ensemble average in the diffusive limit (after removal of the slow changes in pulse arrival time due to changing dispersion measure). In both cases $B(\theta)$ is a smooth function of angle (except for the unscattered component in weak scintillation).

An instantaneous angular spectrum is a single realization of a random process. Thus, it will exhibit “speckle;” i.e., the components of the angular spectrum are statistically independent, having an exponential distribution. This is the “snapshot” scattered image, as described by Goodman & Narayan (1989) for a very short integration from a filled aperture telescope. The speckle is reduced when the integration time is longer than the diffusive scintillation time.

We want to consider the effect of image speckle on the secondary spectrum, but first we have to recognize that the secondary spectrum will itself exhibit speckle versus delay and Doppler shift, since it is obtained by spectral analysis from a random process. Although we compute the secondary spectrum from a dynamic spectrum assembled over $\sim 1$ hr, which is longer than the typical diffraction time, the samples in the secondary spectrum still only have 2 degrees of freedom and so will have exponential statistics.

Now consider the influence of speckle versus angle. Since we do not use a simple telescope or imaging array, we cannot define an equivalent imaging instrument: our “scattered image” consists of the amplitude of the waves received as a function of the angle of arrival at a point observer along a 1 hr track. Since this image comes from a single realization of a random process, it will also show full speckle versus angle, albeit over resolution cells that vary over the field of view. An approximate understanding of the effect of such speckle in the image on the second-

angle of arrival at a point observer along a 1 hr track. Since this consists of the amplitude of the waves received as a function of the secondary spectrum of a random process models a primary dynamic spectrum, which is then subject to the same secondary spectral analysis as used in the observations.

The results, while calculated for a plane wave source, can be mapped to a point source using the well-known scaling transformations for the screen geometry (i.e., $z \rightarrow d_\star$, as defined in eq. [5]). While the screen geometry is idealized, the simulation uses an electromagnetic calculation and properly accounts for dispersion, refraction, diffraction, and interference. Furthermore, the finite size of the region simulated corresponds to the finite integration time in the observations. A scalar field is used since the angles of scattering are extremely small and magnetoionic effects are negligible in this context (see Simonetti et al. 1984).

5.2. Interstellar Conditions Responsible for Arcs

In § 3 we found two conditions that emphasize the arcs. The first is the interference between an undeviated wave and scattered waves, and the second is the enhancement of arcs when the waves are scattered preferentially parallel to the scintillation velocity. We now discuss how these conditions might occur.

5.2.1. Core-Halo Conditions

The basic arc, $p = q^2$, is formed by interference of a core of undeviated or weakly deviated waves with widely scattered waves. There are two circumstances where a significant fraction of the total flux density can come from near $\theta = 0$. One is in strong scintillation, when the “central core” of the scattered brightness interferes with a wider low-level halo, as discussed in § 4.2 for a Kolmogorov density spectrum and other power-law forms with $\beta < 4$ and negligible inner scale. In this case the core of the scattered brightness is not a point source but is merely much more compact than the extended halo radiation. Thus, the arc is a smeared version of the form in equation (15), which has no power outside the basic arc line. Examples were shown in Figures 7 and 8.

The other case is when the scattering is weak enough that a fraction of the flux density is not broadened significantly in angle. We characterize the strength of scintillation by $m_{\text{sc}}$, the normalized rms intensity (scintillation index) under the Born approximation. This can be written in terms of the wave function $D_\phi(r_F)$ at the Fresnel scale, $r_F \equiv (d_\star/k)^{1/2}$. For a simple Kolmogorov spectrum $D_\phi(b) = (b/s_0)^{1/2}$ we find $m_{\text{sc}}^2 = 0.77(r_F/s_0)^{3/2}$. Note that a distinction should be made between strength of scintillation and the strength of scattering. While strong scintillation ($m_{\text{sc}}^2 \gtrsim 1$) corresponds to a change in phase of more than 1 rad across a Fresnel scale, strong scattering is when the overall rms phase perturbations are large compared to 1 rad, which applies in the ISM for all radio frequencies relevant to radio astronomical observations.

In weak scintillation (but strong scattering) there is indeed an “unscattered core,” defined such that the differential propagation delay over the core does not exceed, say, a quarter of the wave period at the center frequency ($\nu_0$), i.e., $D\theta^2(1 - s)/2cs < 1/(4\nu_0)$. Such a core represents the fact that there is a flux density that has a mean amplitude larger than its rms variation. Put another way, an interferometer (baseline $b$) located near the screen would detect a large amplitude of correlated flux density whose phase would wander slowly. If averaged coherently for a long time, the ensemble average visibility would tend to $\exp \left[ -D_\phi(b)/2 \right]$, but over a time short compared to the Fresnel time there would be an effective unresolved core. In weak scintillation there is a large fraction of the flux density in the core. But even when $m_{\text{sc}}^2 > 1$ and the scattering disk size exceeds the Fresnel scale, the core...
component can be significant, since it is that portion of the scattering disk smaller than the first Fresnel zone. In such a case the core fraction $\sim (\theta_{0}/\theta_{F})^{2}$, where $\theta_{F}$ is the angle subtended by the Fresnel scale and $\theta_{0}$ is the scattering angle. It can also be written as $\sim (s_{0}/r_{F})^{2}/m_{B}^{-2.4}$. It is the interference of this unscattered core with the scattered waves that causes a prominent arc to be visible in weak scintillation.

Figure 10 (top panels) shows a simulation in weak scintillation with $m_{B}^{2} = 0.1$, in which $S_{2}$ drops steeply outside the arc $p = q^{2}$. Even sharper edges are obtained when $S_{2}$ is analyzed as a function of wavelength rather than frequency, as is clear from the theory in Appendix D. We have also simulated weak scintillation in a nondispersive phase screen and find similar arcs, showing that the basic phenomenon is not dependent on the dispersive nature of the scattering medium.

Weak scintillation arcs have a sharp outer edge because they are dominated by the interference between scattered waves and the unscattered core, as described by equation (15) with a point component at the origin. In sufficiently weak scintillation we can ignore the mutual interference between scattered waves since they are much weaker than the core. The result is that the secondary spectrum is analogous to a hologram in which the unscattered core serves as the reference beam. The secondary spectrum can then be inverted to recover the scattered brightness function using equation (15), which is the analog of viewing a holo- graphic image. The inversion, however, is not complete in that there is an ambiguity in the equation between positive and negative values of $\theta_{p}$. Nevertheless, the technique opens the prospect of mapping a two-dimensional scattered image from observations with a single dish at a resolution of milliarcseconds.

As the scintillation strength increases, the core becomes less prominent, and the parabolic arc loses contrast. However, we can detect arcs at very low levels, and they are readily seen at large values of $m_{B}^{2}$. Medium strong scintillation is shown in Figure 10 (bottom panels), which is a simulation of a screen with an isotropic Kolmogorov spectrum with $m_{B}^{2} = 10$ for which $s_{0} = 0.22r_{F}$. These results are comparable to several of the observations shown in § 2. In particular, Figures 1a and 1b show arcs with partially filled interiors as in this simulation. We also note that several of the observations (e.g., Figs. 1d, 3b, and 3c) show sharp-edged and symmetric parabolic arcs, which appear to be more common for low or intermediate strength of scintillation (as characterized by the apparent fractional diffractive bandwidth). This fits with our postulate that the arcs become sharper as the strength of scintillation decreases.

Since in weak scintillation the Born approximation applies, more arcs can be caused by two or more screens separated along the line of sight. We have confirmed this by simulating waves passing through several screens, each of which yields a separate arc with curvature as expected for an unscattered wave incident on each screen. This is presumably the explanation for the multiple arcs seen in Figure 3 for pulsar B1133+16. A related question concerns what constraints we can put on the thickness of the regions that cause the scattering. The question is clearest in weak scintillation where the components at different distance add linearly. The curvature is linearly proportional to distance, so it is smoothed by the fractional thickness of the layer. Thus, the arc thickness in delay is a constant fraction of the mean delay. We expect a similar smoothing in stronger scattering. In asymptotic strong scintillation the arcs already have less contrast and we have not attempted to compute the effects of an extended scattering layer. In a future publication we expect to obtain constraints on the layer thicknesses from observations using simulation and detailed model fitting in both weak and strong scintillation.

In terms of normalized variables $p$, $q$, the half-power widths of $S_{2}$ are approximately unity in strong scattering, and yet we can see the arcs out to $q \gg 1$. As we noted earlier, this corresponds to scattering angles well above the diffractive angle $\theta_{F}$ and so probes scales much smaller than those probed by normal analysis of diffractive ISS. Our simulations confirm that with an inner scale in the density spectrum having a wavenumber cutoff $k_{\text{inner}}$, the arc will be reduced beyond where $q \sim k_{\text{inner}}s_{0}$. However, to detect such a cutoff observationally requires very high sensitivity and a large dynamic range in the secondary spectrum.

5.2.2. Anisotropy, Arclets, and Image Substructure

The enhancement in arc contrast when the scintillation is extended along the direction of the effective velocity was discussed in § 3. The enhancement is confirmed by comparing the $m_{B}^{2} = 10$ simulations in the bottom panels of Figure 10 and the top panels of Figure 11. In the anisotropic case there is very little power inside the arc, which thus stands out with greater contrast. This simulation bears a strong resemblance to the observation in Figure 1c, in which there is little power inside the arc. We made a quantitative comparison of this observation with the simulation as follows. We compared slices through $S_{2}$ at fixed delay and
found that the observations had a deeper depression on the delay axis than both the isotropic simulations and the asymptotic theory. However, they were reasonably consistent with the anisotropic simulation at $m_0^2 = 10$ (Fig. 11, top panels). We studied the scaling with delay by summing $S_2$ over Doppler frequency ($f_v$) and overplotted observations, simulation, and asymptotic theory against delay ($f_v$). All three curves followed an asymptotic power-law decay with indices in the range $2.0 – 2.4$, which are close to the theoretical value of $7/3$. This shows that our simulations and asymptotic theory are both useful in modeling the observations and suggests that they will be valuable in a comprehensive comparison with the observations, which will be the subject of a further paper.

On close examination of the anisotropic simulation, we also see that the arc consists of finely spaced parallel arcs that are in turn crossed by reverse arclets, reminiscent of some of the observations. As noted above, these could be caused by the mutual interference of substructure in the angular spectrum $B(\theta)$. The question is, what causes the substructure? One possibility is that it is stochastic speckle-like substructure in the image of a scatter-broadened source. Alternatively, there could be discrete features in $B(\theta)$ (“multiple images”) caused by particular structures in the ISM phase.

In our simulations, the screen is stochastic with a Kolmogorov spectrum and so it should have speckle but no deterministic multiple images. The diffractive scale, $s_0$, is approximately the size of a stationary phase point on the screen and, also, approximately the size of a constructive maximum at the observer plane. There are, thus, $(r_y/s_0)^d$ such points across the two-dimensional scattering disk that contribute to an individual point at the observer plane. This large number of speckles ($\sim 400$ in these simulations) has the effect of averaging out details in the secondary spectrum, particularly in the isotropic scattering case.

The effect of anisotropy in the coherence scale is to broaden the image along one axis with respect to the other. An image elongated by a ratio of $R = \theta_L/\theta_s \approx s_{0,y}/s_{0,x} > 1$ would have $N_x \approx (r_y/s_{0,y})^2$ speckles in the x-direction but only $N_y \approx (r_y/s_{0,y})^2$ in the y-direction. In short, the image would break into a line of many elliptical speckles (elongated in $y$) distributed predominantly along $x$. In comparing the anisotropic and isotropic simulations (Fig. 11, top panels; Fig. 10, bottom panels), it appears that the arclets are more visible under anisotropy. We suggest that the changed substructure in the image is responsible. Substructure may consist of short-lived speckles or longer lived multiple sub-images whose relative contributions depend on the properties of the scattering medium. In the simulations the arclets appear to be independent from one realization to the next, as expected for a speckle phenomenon. However, in some of the observations arclets persist for as long as a month and exhibit a higher contrast than in the simulation. See §2, Figure 2, and recent results from Paper III. These require long-lived multiple images in the angular spectrum and so imply deterministic structures in the ISM plasma; in other words, they cannot be due to spatially homogeneous turbulence.

In summary, we conclude that the occasional isolated arclets (as in Fig. 2) must be caused by fine substructure in $B(\theta)$. Whereas some of these may be stochastic as in the speckles of a scattered image, the long-lived arclets require the existence of discrete features in the angular spectrum, which cannot be part of a Kolmogorov spectrum. These might be more evidence for discrete structures in the medium at scales $\gtrsim 1$ AU, which have been invoked to explain fringing episodes (e.g., Hewish et al. 1985; Cordes & Wolszczan 1986; Wolszczan & Cordes 1987; Rickett et al. 1997) and, possibly, extreme scattering events (Fiedler et al. 1987; Romani et al. 1987) in quasar light curves.

5.2.3. Asymmetry in the Secondary Spectrum

As summarized in §2, the observed arcs are sometimes asymmetric, as well as exhibiting reverse arclets (e.g., Fig. 2). A scattered brightness that is asymmetric in $\theta_s$ can cause $S_2$ to become asymmetric in $q$. We do not expect a true ensemble average brightness to be asymmetric, but the existence of large-scale gradients in phase will refractively shift the seeing image, with the frequency dependence of a plasma.

Simulations can be used to study refractive effects, as in Figure 11 (bottom panels), which shows $S_2$ for a screen with an isotropic Kolmogorov spectrum and a linear phase gradient. The phase gradient causes sloping features in the primary dynamic spectrum and asymmetry in the secondary spectrum. The apex of the parabolic arc is shifted to negative $q$ value and becomes brighter for positive $q$. Thus, we suggest refraction as the explanation of the occasional asymmetry observed in the arcs. Relatively large phase gradients are needed to give as much asymmetry as is sometimes observed. For example, a gradient of 2 rad per $s_0$, which shifts the scattering disk by about its diameter, was included in the simulation of Figure 11 to exemplify image wandering. In a stationary random medium with a Kolmogorov
spectrum, such large shifts can occur depending on the outer scale, but they should vary only over times long compared with the refractive scintillation timescale. This is a prediction that can be tested.

In considering the fringe frequencies in Appendix A, we only included the geometric contribution to the net phase and excluded the plasma term. We have redone the analysis to include a plasma term with a large-scale gradient and curvature in the screen phase added to the small-scale variations that cause diffusive scattering. We do not give the details here and present only the result and a summary of the issues involved.

We analyzed the case where there are large-scale phase terms following the plasma dispersion law, which shift and stretch the unscattered ray, as in the analysis of Cordes et al. (1986). In the absence of diffusive scattering these refractive terms create a shifted stationary phase point (i.e., ray center) at $\hat{\theta}$ and a weak modulation in amplitude due to focusing or defocusing over an elliptical region. With the diffusive scattering also included we find that the fringe frequency $q$ is unaffected by refraction but the delay $p$ becomes

$$p = (\theta_2 + \hat{\theta})^2 - (\theta_1 + \hat{\theta})^2 - \frac{i\lambda}{2\pi} d_s \cdot C \cdot \left[ (\theta_2 - \hat{\theta}) \cdot (\theta_2 - \hat{\theta}) - (\theta_1 - \hat{\theta}) \cdot (\theta_1 - \hat{\theta}) \right],$$  

where $C$ is a $2 \times 2$ matrix of second derivatives that describes the quadratic dependence of the refracting phase from the image center.

A related question is whether the shift in image position due to a phase gradient also shifts the position of minimum delay, in the fashion one might expect from Fermat’s principle that a ray path is one of minimum delay. However, since Fermat’s minimum delay is a phase delay and our variable $f_s$ or $p$ is a group delay, its minimum position is shifted in the opposite direction by a plasma phase gradient. This is shown by the first two terms, which respectively go to zero at $\theta_{1,2} = -\hat{\theta}$. However, we do not pursue this result here, since the formulation of §3 does not include the frequency dependence of the scattered brightness function, which would be required for a full analysis of the frequency dependence of the plasma scattering. It is thus a topic for future theoretical study, but meanwhile the simulations provide the insight that indeed plasma refraction can cause pronounced asymmetry in the arcs.

5.2.4. Arcs from Extended Sources

Like other scintillation phenomena, arcs will be suppressed if the source is extended. In particular, the lengths of arcs depend on the transverse spatial extent over which the scattering screen is illuminated by coherent radiation from the source. For a source of finite angular size $\theta_{as}$ as viewed at the screen, the incident wave field is spatially incoherent on scales larger than $b_{as} \sim \lambda/(\theta_{as})$. An arc measured at a particular fringe rate $f_j$ represents the interference fringes of waves separated by a baseline $b = (D - D_s)(\theta_2 - \theta_1)$ at the screen, corresponding to a fringe rate $f_j = b \cdot V_s / i d_s$. The fringes are visible only if the wave field is coherent over $b$ and are otherwise suppressed. Thus, arcs in the secondary spectrum $S_2$ will be cut off for fringe rates $f_j \approx f_{s, sou}$, where

$$f_{s, sou} = \frac{s V_s}{D_s \theta_{sou}}. \quad (19)$$

Here we distinguish between the source size $\theta_{sou}$ viewed at the observer and its size viewed at the screen, $\theta_{as} = \theta_{sou}/s$. For ISS of a distant extragalactic source, the factor $s$ approaches unity but can be much smaller for a Galactic pulsar. Equation (19) indicates that longer arcs are expected for more compact sources, larger effective velocities, and scattering screens nearer to the observer. Equivalently, using $f_s = s f_j$, the arc length can be measured along the $f_s$ axis with a corresponding cutoff,

$$f_s = \frac{s}{V_0} \left( \frac{\theta_i}{\theta_{sou}} \right)^2, \quad (20)$$

where $\theta_i = [kD(1 - s)]^{-1/2}$ is the effective Fresnel angle.

It is useful to measure the arc’s extent in $f_s$ in terms of the characteristic DISS timescale, $\Delta f_s \approx s_0/V_1$, and to relate the characteristic diffraction scale $s_0$ to the isoplanatic angular scale, $\theta_{iso} \approx s_0/(D - D_s)$. The product of maximum fringe rate and DISS timescale is then

$$f_{s, sou} \Delta f_s \approx \frac{s_0}{d_s \theta_{as}} \approx \frac{\theta_{iso}}{\theta_{sou}}. \quad (21)$$

The isoplanatic angular scale defines the source size that will quench DISS by $\sim 50\%$. We note that equation (21) is consistent with the extended source result for scintillation modulations (Salpeter 1967). Thus, the length of the arc along the $f_s$ axis in units of the reciprocal DISS timescale is a direct measure of the ratio of isoplanatic angular scale to source size. The long arcs seen therefore demonstrate that emission regions are much smaller than the isoplanatic scale, which is typically $\lesssim 10^{-6}$ arcsec for measurements of dynamic spectra.

The theoretical analysis under weak scintillation conditions is given in §D2, where it is seen that the squared visibility function provides a cutoff to the point-source secondary spectrum $S_2$. A remarkable result from this analysis is that measurements of $S_2$ of an extended source can be used, in principle, to estimate the squared visibility function of the source in two dimensions, if the underlying secondary spectrum, $S_2$, for a point source is already known.

The effects of an extended source can also be analyzed in asymptotic strong scintillation. This requires the same type of analysis as for the frequency decorrelation in scintillation of an extended source. Chashei & Shishov (1976) gave the result for a medium modeled by a square law structure function of phase. Codona et al. (1986) gave results for screens with a power-law spectrum of phase in both weak and strong scintillation. We have used their analysis to obtain an expression for the secondary spectrum in the strong scattering limit. The result is that the source visibility function appears as an additional factor $|V|^2$ inside the brightness integral of equation (8) (with arguments depending on $p$, $q$ and other quantities).

It is clear that the detection and measurement of arcs from pulsars can put constraints on the size of their emitting regions. This is intimately related to estimating source structure from their occasional episodes of interstellar fringing (e.g., Cordes & Wolszczan 1986; Wolszczan & Cordes 1987; Smirnova et al. 1996; Gupta et al. 1999). These observers detected changes in the phase of the fringes versus pulsar longitude and so constrained any spatial offset in the emitting region as the star rotates. They essentially measured the phase of the “cross secondary spectrum” between the ISS at different longitudes, at a particular $f_\gamma, f_s$. Clearly
one could extend this to study the phase along an arc in \( f_o, f_t \). Such studies require high signal-to-noise ratio data with time and frequency samplings that resolve scintillations in dynamic spectra, which can be obtained on a few pulsars with the Arecibo and Green Bank Telescopes. The future Square Kilometer Array, with \( \sim 20 \) times the sensitivity of Arecibo, would allow routine measurements on large samples of pulsars.

ISS has been seen in quasars and active galactic nuclei (sometimes referred to as intraday variations), but few observations have had sufficient frequency coverage to consider the dynamic spectrum and test for arcs. However, spectral observations have been reported for the quasars J1819+385 (Macquart & de Bruyn 2005) and B0405−385 (L. Kedziora-Chudczer et al. 2006, in preparation). A preliminary analysis of the data from J1819+385 by B. J. Rickett et al. (2006, in preparation) showed no detectable arc, from which they set a lower limit on the source size. New observations over a wide well-sampled range of frequencies will allow better use of this technique.

6. DISCUSSION AND CONCLUSIONS

It is evident that we have only begun to explain the detailed structures in the parabolic arcs observed in pulsar scintillation. However, it is also clear that the basic phenomenon can be understood from a remarkably simple model of small-angle scattering from a thin phase-changing screen and does not depend on the dispersive nature of the refractive index in the screen. Interference fringes between pairs of scattered waves lie at the heart of the phenomenon. The \( f_o \) coordinate of the secondary spectrum is readily interpreted as the differential group delay between the two interfering waves, and the coordinate \( f_t \) is interpreted as their fringe rate or, equivalently, the differential Doppler frequency, which is proportional to the difference in angles of scattering projected along the direction of the scintillation velocity.

We have developed the theory by modeling the interference for an arbitrary angular distribution of scattering. We have given the ensemble average secondary spectrum in asymptotic strong and weak scintillation, and we have used a full phase screen simulation to test the results under weak and intermediate strength of scintillation. The results are mutually consistent. A simple parabolic arc with apex at the origin of the \((f_o, f_t)\)-plane arises most simply in weak scintillation as the interference between a scattered and an “uncattered” wave. The secondary spectrum is then of second order in the scattered field and maps to the two-dimensional wavenumber spectrum of the screen phase, although with an ambiguity in the sign of the wavenumber perpendicular to the velocity. Remarkably, this gives a way to estimate two-dimensional structure in the scattering medium from observations at a single antenna, in a fashion that is analogous to holographic reconstruction.

In strong scattering the parabolic arcs become less distinct since the interference between two scattered waves has to be summed over all possible angles of scattering, making it a fourth-order quantity in the scattered field. Nevertheless, the arc remains visible when the scattered brightness has a compact core and a “halo” of low-level scattering at relatively large angles. Media with shallow power-law wavenumber spectra (including the Kolmogorov spectrum) have such extended halos, and the detection of arcs provides a powerful probe of structures 10 or more times smaller than those probed by normal ISS and can thus be used to test for an inner scale cutoff in the interstellar density spectrum.

The prominence of arcs depends on the anisotropy of the scattering medium, as well as on the slope and inner scale of its wavenumber spectrum. Arcs become more prominent when the scattering is anisotropic and enhanced preferentially along the scintillation velocity. However, in simulations, prominent arcs are seen over quite a wide range of angles about this orientation. Scattering that is enhanced parallel to the scintillation velocity corresponds to spatial structures that are elongated transverse to the velocity vector. Thus, the common detection of arcs may provide evidence for anisotropy in the interstellar plasma, and with careful modeling observations should yield estimates for the associated axial ratios.

There are several details of the observed arcs for which we have only tentative explanations. We can understand the existence of discrete reverse arcs as due to discrete peaks in the scattered brightness interfering with an extended halo. Such isolated peaks are to be expected in short-term integrations due to speckle in the scattered image. However, observations with only a few isolated reverse arcs (and, particularly, arclets that persist for days to weeks) imply only a few discrete peaks in the scattered image, while normal speckle is expected to give multiple bright points with a much higher filling factor. This is a topic for further investigation.

Another observational detail is that on some occasions the arc power distribution is highly asymmetrical in fringe frequency. This can only be caused by asymmetry in the scattered brightness relative to the velocity direction. Our proposed explanation is that it is due to large-scale gradients in the medium that cause the image to be refractively shifted. The simulations demonstrate that this explanation is feasible, but considerably more work needs to be done to interpret what conditions in the ISM are implied by the unusual asymmetric arcs.

Our theoretical analysis is based on a thin-screen model, and future theoretical work is needed on the arc phenomena with multiple screens (such as might cause the multiple arcs in Fig. 3) and with an extended scattering medium. While the extension to an extended medium or multiple screens is relatively straightforward in weak scintillation, it is more difficult in strong scintillation. Adding the effect of a source with a finite diameter is also important since pulsar emission regions may have finite size and the detection of arcs from quasars provides the prospect of a more powerful probe of their angular structure than from simple analysis of their scintillation light curves. In addition to these extensions of our analysis, future work will include a detailed study of the inverted arclet phenomenon, exploiting the arc phenomenon to determine the anisotropy and inner scale of scattering irregularities and using the multiple arc phenomenon to aid modeling of the local ISM, for which the weak scattering regime is especially relevant.

We acknowledge helpful discussions with D. Melrose and M. Walker. D. R. S. wishes to thank Oberlin students H. Barnor, D. Berwick, A. Hill, N. Hinkel, D. Reeves, and A. Webber for assistance in the preparation of this work. This work was supported by the National Science Foundation through grants to Cornell (AST 98-19931 and AST 02-06036), Oberlin (AST 00-98561), and UCSD (AST 99-88398). This work was also supported by the National Astronomy and Ionosphere Center, which operates the Arecibo Observatory under a cooperative agreement with the Arecibo Observatory. The Australia Telescope National Facility provided hospitality for D. R. S. during preparation of this paper.
APPENDIX A
FRINGE FREQUENCIES FROM A PLASMA SCREEN

Consider the following thin-screen geometry: a point source at \((r_0,0)\), a thin screen in the plane \((r',D_s)\), and an observer at \((r,D)\), where \(r, r'\), and \(r\) are two-dimensional vectors. The screen changes the phase of incident waves and thus diffracts and refracts radiation from the source.

The Kirchoff diffraction integral (KDI) gives the wave field at \(r\) as

\[
\varepsilon(\nu, r) = (i\lambda d_e)^{-1} \int dr' e^{i\Phi(\nu, r)},
\]

using the effective distance \(d_e\) as defined in equation (5) and where \(\Phi = \Phi_g + \phi_d\) is the sum of the geometric phase,

\[
\Phi_g = \frac{k}{2} \left[ D_s^{-1} |r' - r_0|^2 + (D - D_s)^{-1} |r - r_0|^2 \right],
\]

and a diffractive phase \(\phi_d(r')\) that scatters radiation. Frequency scalings are \(\Phi_g \propto \nu\) and \(\phi_d \propto \nu^{-1}\).

The secondary spectrum is the distribution of conjugate frequencies \(f_r, f_c\) produced by all pairs of exit points from the screen. Consider the relative phase, \(\Delta \Phi = \Phi_2 - \Phi_1\), between two components of the radiation field that exit the phase screen at two different points, \(r_1, r_2\), that correspond to deviation angles as viewed by the observer, \(\theta_{1,2} = r_{1,2}/(D - D_s)\). The combined radiation from the two points will oscillate as a function of time, frequency, and spatial location. For fixed location on axis \((r = 0)\) and using the effective velocity (eq. [4]) to map spatial offsets at the screen to time, we can expand \(\Delta \Phi\) in time and frequency offsets:

\[
\Delta \Phi(\nu, t) = \Delta \Phi(\nu_0, 0) + 2\pi i [f_r(\nu - \nu_0) + f_c t],
\]

where (using \(\partial_t \equiv \partial/\partial t\), etc.) the fringe frequencies are

\[
f_r = (2\pi)^{-1} \partial_t \Delta \Phi(\nu, t),
\]

\[
f_c = (2\pi)^{-1} \partial_\nu \Delta \Phi(\nu, t).
\]

Here we use only the geometric phase to calculate the fringe frequencies. The result is given by equations (2) and (3) in terms of the two apparent angles \(\theta_{1,2}\) and the effective velocity \(V_e\) (eq. [4]). Since the delay is defined in terms of the frequency derivative of the phase, \(f_c\) is the difference in the group delay. While the distinction is unimportant for the geometric phase, it makes a difference for the dispersive plasma contributions. When the analysis is done including the derivatives in these plasma terms, the equation for \(f_r\) is modified, but there is no change in the equation for \(f_c\), as mentioned at the end of § 5.2. The importance of the dispersive plasma contribution to \(f_c\) can be characterized by the ratio of the difference in dispersive delay to the difference in geometric delay across the scattering disk. For a Kolmogorov screen this ratio is \(\sim 0.5\left(r_f/s_0\right)^{0.33}\), where \(r_f\) and \(s_0\) are the Fresnel scale and diffractive scale, respectively. As noted in § 5.2, the scintillations are strong when \(r_f/s_0 > 1\), and so strong scintillation gives the condition for ignoring the dispersive plasma contribution to \(f_c\). However, if the scattering medium has a steep spectrum (or has an inner scale cutoff), the phase structure function becomes quadratic and then plasma and geometric delays are of the same magnitude for all strengths of scattering (see also § 2.2 of Walker et al. 2004).

In Appendices C and D a derivation is given of the secondary spectrum in the limits of strong and weak scintillation, respectively. In the strong limit we find that it is given explicitly by the double integral over the observed angles of the integrand in equation (8). We note that the integrand is the scattered brightness function, defined as a spectrum of plane waves. In contrast, the discussion given above in terms of the KDI is based on spherical waves emanating from points in the screen. While one cannot equate the apparent angular position of points on the screen, \(\theta_{1,2} = r_{1,2}/(D - D_s)\), to the angles of arrival of plane wave components in the angular spectrum, one can obtain identical equations for \(f_r\) and \(f_c\) by considering an integral over plane waves emanating from a screen, which is illuminated by a point source. The method is similar to the KDI analysis above, except that one expands the propagation phase of each plane wave component as a function of frequency and time. Its derivatives give \(f_r\) and \(f_c\), precisely, as in equations (2) and (3).

APPENDIX B

FINITE RESOLUTION AND NUMERICAL ISSUES FOR THE SECONDARY SPECTRUM

Empirically, the secondary spectrum is estimated over a finite total bandwidth \(B\) and integration time \(T\) with finite resolution in frequency and time. These in turn set finite resolutions in \(f_r\) and \(f_c\) and so in \(p\) and \(q\). The integral expressions for the secondary spectrum such as equation (8) diverge at the origin of the \(p-q\) plane since they ignore resolution effects.

Finite resolution in \(p\) can be included by replacing the Dirac delta functions in equation (8) with rectangular functions of unit area whose limiting forms are delta functions. Performing the integrations over \(d\theta_2\) then yields a form for \(S_2\).

\[
S_2(p, q) \approx \sum_{\pm} \int d\theta_1 B(\theta_1) B(\theta_1 + q, \pm \sqrt{U}) H' \left( U - \frac{\Delta p}{2} \right) \frac{\sqrt{U + \Delta p/2} - \sqrt{U - \Delta p/2}}{\Delta p},
\]
where $U = (\theta^2 + p - q^2 - 2q\theta_1)\sqrt{\theta}$ and the summation is over the two ideal solutions $\theta_2, = \pm \sqrt{U}$, and we have ignored the variation in $B$ over the range $\Delta p$ near each solution; $H'$ is a modified unit step function with a transition width $\Delta p$. As $\Delta p \to 0$, the factors involving $\Delta p$ tend toward a delta function. For finite $\Delta p$, however, $S_2(0, 0)$ remains finite.

In terms of the bandwidth $B$ and time span $T$, the resolutions in $p$ and $q$ are (when angles are normalized by the diffraction angle, $\theta_d$)

$$
\Delta p = \frac{1}{B \tau_d} = \frac{2\pi \epsilon}{N_\nu},
$$

$$
\Delta q = \frac{2\pi \Delta t_d}{T} = \frac{2\pi \epsilon}{N_\nu},
$$

where $N_\nu$ and $N_\nu$ are the number of distinct “scintles” along the time and frequency axes, respectively (here $\epsilon \sim 0.2$ is a constant that quantifies the filling factor of scintles in the dynamic spectrum; e.g., Cordes & Lazio 2002). The resolutions of $p$ and $q$ therefore are determined by how many scintles are contained in the dynamic spectrum, which in turn determine the statistical robustness (through $N^{-1/2}$ effects) of any analyses of a particular dynamic spectrum. For typical dynamic spectra, $N_\nu$ and $N_\nu$ are each $\approx 10$, so $\Delta p$ and $\Delta q$ are each $\lesssim 0.1$. Observationally, the individual channel bandwidth and sampling time are also important since they determine the Nyquist points in $p$ and $q$.

In computing the secondary spectrum from the integral in equation (9), one also needs to resolve the (integrable) singularity $U^{-1/2}$. This can be achieved by changing variables to $s_{x,y} = (\theta_{x,y} + \theta_{x,y})/\sqrt{|q|}$ and $d_{x,y} = (\theta_{2x,y} - \theta_{1x,y})/2\sqrt{|q|}$ and integrating over the delta functions. Then letting $x = s_y$ and $y = -d_y$, we obtain

$$
S_2(p, q) = \int \int dx \, dy \, B[X_-, \sqrt{|q|(x + y)}] B[X_+, \sqrt{|q|(x - y)}],
$$

$$
X_\pm = \frac{p \pm q^2}{2q} + 2 \text{sgn}(q)xy,
$$

where $\text{sgn}(q)$ is the sign of $q$. Note that symmetry of $S_2$ upon letting $p \to -p$ and $q \to -q$ is demonstrated by also letting $y \to -y$.

**Appendix C**

**The Secondary Spectrum in the Strong Scattering Limit**

Here we present the secondary spectrum expected in the asymptotic limit case of strong scintillations from a single phase screen. The intensity from a point source recorded at position $r$ and frequency $\nu$ is the squared magnitude of the phasor $\varepsilon$ for the electric field: $I(r, \nu) = |\varepsilon(r, \nu)|^2$, where the dependence on source position $r$, is suppressed. From the dynamic spectrum of a pulsar we can define the correlation of intensity versus offsets in both space and frequency. After subtracting the mean, $\Delta I = I - \langle I \rangle$, the correlation function is

$$
R_{\Delta I}(\Delta r, \Delta \nu) = \langle \Delta I(r, \nu) \Delta I(r + \Delta r, \nu + \Delta \nu) \rangle.
$$

Under asymptotic conditions of strong scattering the phasor $\varepsilon$ becomes a Gaussian random variable with zero mean and random phase, the real and imaginary parts of $\varepsilon$ are uncorrelated, and the fourth moment can be expanded in products of second moments. It follows that

$$
R_{\Delta I}(\Delta r, \Delta \nu) = |\Gamma(r, \Delta r, \nu, \nu + \Delta \nu)|^2,
$$

where

$$
\Gamma(r, \Delta r, \nu, \nu + \Delta \nu) = \langle \varepsilon(r, \nu)\varepsilon^*(r + \Delta r, \nu + \Delta \nu) \rangle.
$$

When the field $\varepsilon$ is due to a point source scattered by a phase screen at distance $D_s$ from the source and $D - D_s$ from the observer (with $D$ the pulsar distance), the second moment is the product (see Lambert & Rickett 1999):

$$
\Gamma(r, \Delta r, \nu, \nu + \Delta \nu) = \Gamma_{\text{point}} \Gamma_r \Gamma_D.
$$

Here $\Gamma_{\text{point}}$ is simply due to the spherical wave nature of a point source, which is essentially unity for typical pulsar observations, and $\Gamma_r$ is due to the wandering of dispersive travel times about its ensemble average as the electron column density changes, $\Gamma_r = \exp(-\pi^2 \Delta \nu^2 \tau_r^2)$. $\Gamma_D$ is the diffractive second moment, which is given in terms of the scattered angular spectrum $B(\theta)$ at the radio frequency $\nu$:

$$
\Gamma_D(\Delta r, \Delta \nu) = \int d\theta \, B(\theta) \exp\left[-2\pi i \Delta \nu \theta^2 D(1 - s)/(2cs)\right] \exp(i k \theta \cdot \Delta r)
$$

(recall that $s = D_s/D$). The phase term in the first exponential is proportional to the extra delay $\theta^2 D(1 - s)/(2cs)$ for waves arriving at the observer at an angle $\theta$; this quadratic relation between time delay and angle of arrival gives rise to the quadratic features in the secondary spectrum. In single-dish pulsar observations the spatial offset $\Delta r$ is sampled by a time offset $t$ times the relative velocity of
the diffraction pattern past the observer. Such observations are in a short-term regime in which the dispersive delay is essentially constant over the integration time and so observations are characterized by $\Gamma_D$, the diffractive second moment at $\Delta r = Vt/D_s$; the distance ratio is needed since $V_s$ is the effective screen velocity. The secondary spectrum is the double Fourier transform of the correlation function $R_{\Delta r}$.

$$S_2(f_\nu, f_\nu) = \int dt \int d\Delta \nu e^{2\pi i t \nu_1 + 2\pi i \Delta \nu} R_{\Delta r}(Vt/D_s, \Delta \nu).$$

(C6)

In the short-term regime, this equation is evaluated using equation (C5) for $\Gamma_D$ in place of $\Gamma$ in equation (C2). Integration over $t$ and $\Delta \nu$ and conversion to the scaled variables of § 3 yield equation (8) in the main text.

**APPENDIX D**

**THE WEAK SCINTILLATION LIMIT**

We now examine the secondary spectrum in the limit of weak scintillation due to a plasma phase screen with a power-law wavenumber spectrum.

**D.1. POINT SOURCE**

Scintillation is said to be weak when the point-source, monochromatic scintillation index (rms/mean intensity) is much less than 1. This applies near a phase screen since intensity fluctuations only build up as a wave travels beyond the screen. In this regime the KDI (eq. [A1]) can be approximated by a first-order expansion of $\exp(i\varphi_d(r'))$, where the screen phase at $r'$ is written as the screen phase at the observer coordinate $r$ plus the difference in phase between $r'$ and $r$. This allows a linearization of the problem, even though the rms variation in overall screen phase may be very large, as expected for a power-law spectrum.

Various authors have described the frequency dependence of scintillation under these conditions. Codona et al. (1986), in particular, give a thorough analysis applicable to evaluating the secondary spectrum. They obtain expressions for the correlation of intensity fluctuations between different observing wavelengths ($\lambda_1, \lambda_2$), the quantity in our equation (C1). Under weak scintillation conditions the result is most simply expressed in terms of its two-dimensional Fourier transform over $\kappa$, the cross spectrum of intensity fluctuations:

$$P_{\Delta \kappa}(\kappa, \lambda_1, \lambda_2) = \int d\kappa R_{\Delta \kappa}(\Delta \kappa, \lambda_1, \lambda_2) \exp(ik\cdot\Delta r)/(4\pi^2).$$

(D1)

Here we find it convenient to work in terms of observing wavelength rather than frequency because it simplifies the weak scintillation results.

Codona et al. (1986) give an expansion for the cross spectrum applicable to low wavenumbers in their equation (27). This is the product of the wavenumber spectrum of the screen phase with the two-wavelength “Fresnel filter” and with an exponential cutoff applicable to strong refractive scintillation, which can be ignored in weak scintillation. Their results are given for a nondispersive phase screen and a plane incident wave; when converted to a plasma screen and the point-source geometry described in previous sections, the result is

$$P_{\Delta \kappa}(\kappa, \lambda_1, \lambda_2) = (\lambda_1 \lambda_2/\lambda_0^2) P_\kappa(\kappa) 2[\cos(\kappa^2 d_s \lambda_d/4\pi) - \cos(\kappa^2 d_s \lambda_0/2\pi)],$$

(D2)

where $d_s = Ds(1-s)$, $\lambda_d = \lambda_1 - \lambda_2$, $\lambda_0 = (\lambda_1 + \lambda_2)/2$, and $P_\kappa(\kappa)$ is the wavenumber spectrum of screen phase at $\lambda_0$. Note that with $\lambda_d = 0$ the difference in the two cosine functions becomes the well-known $\sin^2$ Fresnel filter.

Defining the secondary spectrum as the Fourier transform with respect to time difference and wavelength difference instead of frequency difference, we obtain

$$S_2(f_s, f_\nu) = \int d\lambda_d e^{2\pi i f_\nu \lambda_d} \int d\kappa P_{\Delta \kappa}(\kappa, \lambda_1, \lambda_2) \delta(f_\nu - \kappa \cdot V_s/2\pi),$$

(D3)

where we have not included the finite resolution effects or the integration with respect to the mean wavelength $\lambda_0$ over the total bandwidth $B$. Substituting from equation (D2), we obtain

$$S_2(f_s, f_\nu) = (\lambda_0^2)^2 \int d\lambda_d \left[1 - (\lambda_d/2\lambda_0)^2\right] \exp[2\pi i f_\nu \lambda_d]$$

$$\times \int d\kappa P_\kappa(\kappa, \kappa_\nu) \delta(f_\nu - \kappa \cdot V_s/2\pi) [\cos(\kappa^2 d_s \lambda_d/4\pi) - \cos(\kappa^2 d_s \lambda_0/2\pi)].$$

(D4)

For small fractional bandwidths we approximate $1 - (\lambda_d/2\lambda_0)^2 \approx 1$, and, taking the $x$-axis along $V_s$, the integrals can be evaluated to give

$$S_2(f_s, f_\nu) = \frac{8\pi^3 H^{\kappa^2}_{V_s d_e \kappa_{yp}}}{V_s d_e \kappa_{yp}} \left[P_\kappa(\kappa_x = \frac{2\pi f_\nu}{V_s}, \kappa_{yp}) + P_\kappa(\kappa_x = \frac{2\pi f_\nu}{V_s}, -\kappa_{yp})\right] - \delta(f_s) P_\nu(f_\nu).$$

(D5)
Here $H(u)$ is the unit step function,

$$\kappa_{yp} = \sqrt{8\pi^2|f_\perp|/d_e - (2\pi f_\perp/V_\perp)^2},$$  \hspace{1cm} (D6)

and $P_w(f_\parallel)$ is

$$P_w(f_\parallel) = \frac{4\pi}{V_\perp} \int d\kappa_y \cos \left[ \frac{d_e \lambda_\theta}{2\pi} \left( \frac{2\pi f_\perp^2}{V_\perp^2} + \kappa_y^2 \right) \right] P_\phi \left( \frac{2\pi f_\parallel}{V_\perp}, \kappa_y \right).$$  \hspace{1cm} (D7)

It is closely related to the normal weak scintillation spectrum at wavelength $\lambda_\theta$, with the difference that the Fresnel filter $[\sin^2(\cdot) - \frac{1}{2}]$ function is replaced here by $\sin^2(\cdot)$. Excluding the $P_w$ term in equation (D5), we see that $S_2$ diverges along the parabolic curve where $\kappa_{yp} = 0$, creating a parabolic arc, and is cut off by the step function outside that curve. With circularly symmetric scintillation, $P_\phi$ is a function only of $|\kappa|^2 = 8\pi^2|f_\parallel|/d_e$ and so the dependence on $f_\parallel$ is purely through the known arc enhancement factor $1/\kappa_{yp}$ using equation (D6). Thus, a measurement of $S_2(f_\parallel, f_\parallel)$ can be inverted to estimate $P_\phi(\kappa_x, \kappa_y)$, and so we have a direct method of estimating the phase spectrum of the medium (averaged over positive and negative $\kappa$ values). This is analogous to the reconstruction of an image from a hologram.

The final result (eq. [D5]) can be viewed as the interference of scattered waves with an unscattered wave. To see this, compare $S_2(f_\parallel, f_\parallel)$ (excluding the $P_w$ term) with the interference result in equation (15) discussed in § 3.4.3 with $\theta_p = 0$. First, we transform into scaled variables ($\rho, \phi$) and assume small fractional differences in wavelength, for which $f_\parallel \lambda_\theta \sim f_\parallel \nu_\theta$. Hence, we can express $\kappa_{yp}$ in equation (D6) as

$$\kappa_{yp} = \sqrt{|\rho| - q^2/\lambda_\theta^2},$$  \hspace{1cm} (D8)

so $H(\kappa_{yp}^2/\kappa_{yp})$ becomes $H(U)/\sqrt{U(\lambda_\theta)}$, where $U = |\rho| - q^2$, and $\lambda_\theta$ is the diffractive scale as defined in § 4.2. In equation (15), the brightness function $g$ represents the scattered waves that interfere with an undeviated plane wave, corresponding to the mean intensity in weak scintillation.

**D.2. EXTENDED SOURCE**

A temporally and spatially incoherent extended source at a distance $D$ from the observer is described by its brightness distribution $B_{\text{so}}(\theta_p)$. Hence, we can simply add the intensity patterns due to each point component at $\theta_p$ to obtain the well-known convolution result:

$$I_{\text{ext}}(r) = \int d\theta_p I(r + \theta_p D(1 - s), 0, \lambda) B_{\text{so}}(\theta_p),$$  \hspace{1cm} (D9)

where $I(r, \rho, \phi)$ is the intensity pattern for a point source at $r$. This convolution can also be expressed in the wavenumber ($\kappa$) domain as a product using the source visibility function $V(u = \kappa D(1 - s)/2\pi)$, where $u$ is the baseline scaled by the wavelength. Combining this relation with the point-source expressions, we find that the integrand in equation (D4) is multiplied by the product $V_1(u = \kappa D(1 - s)/2\pi) V_2(u = \kappa D(1 - s)/2\pi)$, where $V_1, V_2$ are visibilities at $\lambda_1, \lambda_2$. Now consider the wavelength dependence of the visibility function. If the source brightness distribution is independent of wavelength (i.e., fixed angular size), then $V_1(u) = V_2(u)$. Consequently, in equation (D5) $P_\phi$ is simply multiplied by $|V|^2$ to give

$$S_2(f_\parallel, f_\parallel) = \frac{8\pi^2}{V_\perp d_e \kappa_{yp}} H\left(\kappa_{yp}\right) \sum_{\pm} \left[V^2 f_\parallel [(1 - s)/V_\perp, \pm \kappa_{yp} (1 - s)/2\pi]\right]^2 P_\phi \left( \frac{2\pi f_\parallel}{V_\perp}, \pm \kappa_{yp} \right) \delta(f_\parallel) P_{w,\text{ext}}(f_\parallel),$$  \hspace{1cm} (D10)

where the summation is over two equal and opposite values for $\kappa_{yp}$. In this equation the $P_{w,\text{ext}}$ function is similarly modified by the visibility function but is of no immediate interest here. Our discussion shows that the secondary spectrum $S_2(f_\parallel, f_\parallel)$ in the weak scintillation regime can be inverted to estimate the product of the medium phase spectrum by the squared visibility function of the source, in two dimensions. Since there are several lines of evidence supporting a Kolmogorov model for the phase spectrum, we have a new way of estimating the squared visibility function of a source. This allows a form of imaging from spectral observations with a single dish.

**REFERENCES**

Bhat, N. D. R., Gupta, Y., & Rao, A. P. 1999, ApJ, 514, 249

Chashei, I. V., & Shishov, V. I. 1976, Soviet Astron., 20, 13

Codona, J. L., Creamer, D. B., Flatte, S. M., Frehlich, R. G., & Henyey, F. S. 1986, Radio Sci., 21, 805

Coles, Wm. A., Filice, J. P., Frehlich, R. G., & Yadlowsky, M. 1995, Appl. Opt., 34, 2089

Cordes, J. M., & Lazio, T. J. W. 2002, preprint (astro-ph/0207156)

Cordes, J. M., Pidwerbetsky, A., & Lovelace, R. V. E. 1986, ApJ, 310, 737

Cordes, J. M., & Rickett, B. J. 1998, ApJ, 507, 846

Cordes, J. M., & Wolszczan, A. 1986, ApJ, 307, L27

Desai, K. M., & Fey, A. L. 2001, ApJS, 133, 395

Ewing, M. S., Batchelor, R. A., Friefeld, R. D., Price, R. M., & Staelin, D. H. 1970, ApJ, 162, L169

Fiedler, R. L., Dennison, B., Johnston, K. J., & Hewish, A. 1987, Nature, 326, 675

Frail, D. A., Diamond, P. J., Cordes, J. M., & van Langevelde, H. J. 1994, ApJ, 427, L43

Goodman, J., & Narayan, R. 1989, MNRAS, 238, 995

Gupta, Y., Bhat, N. D. R., & Rao, A. P. 1999, ApJ, 520, 173

Gwinn, C. R., Bartel, N., & Cordes, J. M. 1993, ApJ, 410, 673

Hewish, A., Wolszczan, A., & Graham, D. A. 1985, MNRAS, 213, 167

Hill, A. S., Stonebring, D. R., Asplund, C. T., Berwick, D. E., Everett, W. B., & Hinkel, N. R. 2005, ApJ, 619, L171 (Paper III)
Hill, A. S., Stinebring, D. R., Barnor, H. A., Berwick, D. E., & Webber, A. B.  
2003, ApJ, 599, 457 (Paper II)  
Lambert, H. C., & Rickett, B. J. 1999, ApJ, 517, 299  
Macquart, J.-P., & de Bruyn, G. 2005, preprint (astro-ph/0510495)  
Molnar, L. A., Mutel, R. L., Reid, M. J., & Johnston, K. J. 1995, ApJ, 438, 708  
Moran, J. M., Greene, B., Rodriguez, L. F., & Backer, D. C. 1990, ApJ, 348, 147  
Narayan, R., & Goodman, J. 1989, MNRAS, 238, 963  
Rickett, B. J. 1990, ARA&A, 28, 561  
Rickett, B. J., Lyne, A. G., & Gupta, Y. 1997, MNRAS, 287, 739  
Roberts, J. A., & Ables, J. G. 1982, MNRAS, 201, 1119  
Romani, R. W., Blandford, R. D., & Cordes, J. M. 1987, Nature, 328, 324  
Salpeter, E. E. 1967, ApJ, 147, 433  
Simonetti, J. H., Cordes, J. M., & Spangler, S. R. 1984, ApJ, 284, 126  
Smirnova, T. V., Shishov, V. I., & Malofeev, V. M. 1996, ApJ, 462, 289  
Spangler, S. R., & Cordes, J. M. 1988, ApJ, 332, 346  
Spangler, S. R., & Gwinn, C. R. 1990, ApJ, 353, L29  
Stinebring, D. R., McLaughlin, M. A., Cordes, J. M., Becker, K. M., Goodman, J. E. E., Kramer, M. A., Sheckard, J. L., & Smith, C. T. 2001, ApJ, 549, L97 (Paper I)  
Walker, M. A., Melrose, D. B., Stinebring, D. R., & Zhang, C. M. 2004, MNRAS, 354, 43  
Wilkinson, P. N., Narayan, R., & Spencer, R. E. 1994, MNRAS, 269, 67  
Wolszczan, A., & Cordes, J. M. 1987, ApJ, 320, L35