ABSTRACT

A simple, semi-analytic representation is developed for nuclear burning in Type Ia supernovae in the special case where turbulent eddies completely disrupt the flame. The speed and width of the "distributed" flame front are derived. For the conditions considered, the burning front can be considered as a turbulent flame brush composed of corrugated sheets of well-mixed flames. These flames are assumed to have a quasi-steady state structure similar to the laminar flame structure, but controlled by turbulent diffusion. Detonations cannot appear in the system as long as distributed flames are still quasi-steady state, but this condition is violated when the distributed flame width becomes comparable to the size of the largest turbulent eddies. When this happens, a transition to detonation may occur. For current best estimates of the turbulent energy, the most likely density for the transition to detonation is in the range $0.5 - 1.5 \times 10^7$ g cm$^{-3}$. 

Subject headings: hydrodynamics — shock waves — supernovae: general — turbulence

1. INTRODUCTION

One of the greatest uncertainties in how a Chandrasekhar mass white dwarf explodes as a Type Ia supernova is whether and how an initially subsonic burning front, a deflagration, makes a transition to a supersonic detonation. A related question involves the characteristics of nuclear burning in a medium where turbulence has become so strong that hot ash and cold fuel can be cominled before burning. The conditions for the latter, known as "burning in the distributed regime," have long been known to both the combustion and astronomical communities (Pope 1987; Niemeyer & Kerstein 1997; Khokhlov et al. 1997; Niemeyer & Woosley 1997; Peters 2000). When the laminar flame grows thick enough and the turbulent intensity great enough, the "Gibson" length—i.e., the length scale that turns over due to turbulent eddies as fast as a laminar flame crosses it—becomes smaller than the flame thickness itself. That is,

$$ \frac{L_{\text{Gib}}}{\epsilon_{\text{lam}}} \gtrsim r_{\text{turb}}(L_{\text{Gib}}). \quad (1) $$

Approximate conditions for entering the distributed regime have been given for an exploding white dwarf in Figure 2 of Niemeyer & Woosley (1997), and those conditions define the applicability of this paper. For those conditions, $L_{\text{Gib}}$ is also much greater than the Kolmogorov length, $L_{\text{kol}} = L(\text{Re})^{-3/4} \sim 10^{-4}$ to $10^{-5}$ cm, where the turbulence is dissipated. Here $L$ is the integral length scale of turbulence ($\sim 10^6$ cm) and Re is the Reynolds number ($\sim 10^{14}$). The inequality $L_{\text{kol}} \ll L_{\text{Gib}} \ll \epsilon_{\text{lam}}$ is thus satisfied, where $\epsilon_{\text{lam}}$ is the thickness of the laminar flame (typically $\sim$ centimeters).

Deep in the distributed regime, turbulence is more effective at transporting both heat and composition, even on scales as small as the laminar flame width, than conduction and diffusion (the conductive and radiative opacities here are comparable; Timmes 2000). As a consequence, the concept of a laminar flame, one whose width and speed are determined by the near-equality of burning and diffusion timescales (Landau & Lifshitz 1959; Timmes & Woosley 1992), breaks down. Fuel and hot ash are cominled, and the definition of the width of them must be modified (Lisewski et al. 2000b). If that width grows large enough and the burning rapid enough, a transition to detonation can occur (Khokhlov et al. 1997; Niemeyer & Woosley 1997; Lisewski et al. 2000a). Such a transition is impossible within the extent of a laminar flame; it can only occur in a turbulently stirred one. Here the physics of that transition is explored. Kolmogorov scaling is assumed throughout. Expressions are derived for the turbulent flame speed and its width ($\S 2$). A hypothetical "steady state" is posited, in which fuel burns at a rate-balancing turbulent mixing. The width of the flame grows as the density declines because the temperature of the ash, to which the burning rate in the mixture is very sensitive, is lower there. The turbulent flame moves at a rate given by the length scale at which turbulent diffusion matches burning and, as the width of the burning mixture becomes greater, so too does its speed. Thus, in supernovae, the distributed burning flame moves faster as the density decreases—the opposite of what happens for a laminar flame. On larger scales, turbulence also folds these (turbulently broaded) flames and, just as in the laminar case, there is an overall "flame brush" (Damköhler 1940; Hillebrandt & Niemeyer 2000) whose motion determines the total rate of burning.

As the density continues to decline, however, the turbulently mixed flame grows ever broader, eventually approaching the size of the integral length scale of the turbulence. Qualitatively, this is the largest scale at which the anisotropic shear and instabilities introduced by flotation produce a nearly isotropic cascade of turbulence with constant energy density. Technically, it is the distance scale beyond which the self-correlation of the velocity components vanishes. For a typical Type Ia explosion, plumes of size $\sim 100$ km float at speeds in excess of 1000 km s$^{-1}$. Empirically, from numerical simulations, the size of the isotropically stirred region is $\sim 10$ km, although certainly variations of a factor of several around this are allowed. Also important is the velocity at the integral length scale, $u_x$, which specifies the energy density in the turbulence. This is typically some fraction, $\sim 10\%$, of the rotation speed, but again, large variations are expected, up to the rotation speed itself.

When the width of the turbulently mixed flame becomes comparable to the integral length scale of the turbulence itself, the
largest turbulent speeds have thus become an appreciable fraction of the sound speed. Recent studies by Röpke (2007) show that the highest turbulent speeds can approach 1000 km s\(^{-1}\) at densities \(\sim 10^7\) g cm\(^{-3}\); the sound speed there is about 4000–5000 km s\(^{-1}\). As long as the assumed steady state solution persists, supersonic burning remains, by definition, impossible for subsonic turbulence. However, once there is only a single flame or two in the integral length scale, the steady state assumption certainly breaks down. A large eddy and its accompanying cascade stirs the mixture, but then burning goes on at a nonsteady rate before another eddy happens in the same region. In §§ 4.2 and 4.3, it is shown that this sets the stage for a detonation. The conditions are restrictive and require turbulent energies corresponding to close to 1000 km s\(^{-1}\) on a length scale of 10 km and a density between 0.5 \(\times\) \(10^7\) and 1.5 \(\times\) \(10^7\) g cm\(^{-3}\). This is a much more restrictive condition than just “entering the distributed burning regime.”

This work bears some similarity to that of Khokhlov et al. (1997), Niemeyer & Woosley (1997), and Lisewski et al. (2000a), but considers more carefully both the conditions in the mixed region and the need for a turbulent flame that is already moving at a fraction of the sound speed before a transition to detonation can occur. The conditions required are thus more precisely determined and much more constrained. The necessity of a separation between the carbon-burning flame and the oxygen-burning flame and the breaking of the steady state assumption at the integral length scale are emphasized. Some of the conclusions are similar to those of Kerstein (2001) for low Prandtl number flames, but are more extensively discussed in the astrophysical context.

2. A SIMPLE MODEL FOR BURNING IN THE DISTRIBUTED REGIME

The basic parameters of any flame—its width and speed—can be estimated from the rates for fuel consumption and fuel advection into the burning region. Although turbulence is stochastic and the distribution of heat in a stirred region is not nearly as smooth as when huge numbers of electrons and photons participate in conduction and diffusion, there is still, on average, something like a steady state. In that steady state, the rate at which fuel (carbon) is brought into the burning region by turbulent eddies balances the rate of consumption within that region. The whole mixed-up, burning ensemble moves through the fuel with a typical speed that is essentially the size of the region divided by its turbulent turnover time. That is,

\[
\int \frac{dn_{12}}{dt} dV = \int n_{12} v_{\text{turb}} dA,
\]

where \(n_{12} = \rho N_A Y_{12}\) is the number density of carbon nuclei as a function of location, \(\rho\) is the density, \(N_A\) is Avogadro’s number, \(Y_{12}\) is the mass fraction of carbon, \(X_{12}\) is divided by 12, \(v_{\text{turb}}\) is the velocity of the burning turbulent front normal to the area \(A\), and \(V\) is the volume bounded by that area. Here we consider a one-dimensional flame in plane geometry. For the low densities of interest, reactions beyond carbon burning, i.e., oxygen burning, occur so far behind the carbon-burning flame as to be negligible on the scale of the problem. Let the thickness of the mixed region be \(\lambda\); then

\[
\int_0^{\lambda} \rho(l') \frac{dY_{12}(l')}{dt} dl' = \rho_{\text{fuel}} Y_{12} v_{\text{turb}}.
\]

The left-hand side gives the rate of carbon destruction (in g cm\(^{-2}\) s\(^{-1}\)) by nuclear reactions, while the right-hand side is the rate of advection into the burning region by turbulent eddies with characteristic length scale \(\lambda\) and speed \(v_{\text{turb}}\). Here \(\rho_{\text{fuel}}\) is the density in the unmixed fuel, and \(Y_{12}^0\) is the carbon abundance there. The burning rate is

\[
\frac{dY_{12}}{dt} = -2\rho Y_{12} R_{12,12}(\rho, T),
\]

where \(R_{12,12}\) is the rate factor for the carbon-fusion reaction. Because of the temperature sensitivity of this rate, about \(T^{20}\), most of the carbon consumption goes on in a narrow region where the mass fraction is low and the temperature high (Bell et al. 2004), closer to the hot ash than to the cold fuel. The rapid rate of burning there has to balance, on average, what is advected into the larger region. In order to obtain \(v_{\text{turb}}\) it is thus necessary to specify \(\rho(l'), T(l'), Y_{12}(l'),\) and \(v_{\text{turb}}\).

For \(v_{\text{turb}}\), it is appropriate to take the velocity of the turbulent eddy with length scale \(\lambda\). Assuming Kolmogorov scaling and a turbulent energy input on the integral length scale \(L\), corresponding to velocity \(u_L\),

\[
v_{\text{turb}} = \frac{\lambda}{L} u_L,
\]

where, from typical numerical simulations, \(u_L\) is in the range \(10^{-7}–10^{-6}\) cm s\(^{-1}\) for \(L = 10^6\) cm (Röpke 2007).

All speeds are very subsonic, so to good approximation, the pressure in the fuel, ash, and mixture is constant. Provided that mixing is faster than burning and conduction, the temperature and density in the mixture are thus uniquely defined by the fractions of ash and fuel that are mixed and the compositions of each. In fact, some burning does occur during the mixing, and this affects the temperature distribution, but to first order, the temperature obtained by burning to a certain carbon mass fraction is the same as that obtained by mixing cold fuel with ash of higher energy and lower carbon abundance to obtain that same final mass fraction. This is not precise because carbon burns to different products at different temperatures, so the energy released is not a linear function of carbon consumed, but the difference is not large.

Next we seek a description of how temperature and density vary in the mixed region. This requires an Ansatz for how the carbon mass fraction varies. To illustrate the procedure, assume an initial composition of 50% C and 50% O at a density of \(1.0 \times 10^7\) g cm\(^{-3}\) and temperature \(6.0 \times 10^8\) K. This is a typical temperature in the outer parts of the white dwarf when it runs away, but the answer will not depend on the exact value because the pressure is not very sensitive to the temperature, and the internal energy of the ash is much higher than that of the fuel. The pressure in this fuel is \(9.046 \times 10^{23}\) dyn cm\(^{-2}\), and its internal energy is \(e = 1.706 \times 10^{17}\) ergs g\(^{-1}\) (here and throughout the paper we employ the Helmholtz equation-of-state routine of Timmes & Swesty 2000). Now this fuel is mixed with a small amount of ash so that the temperature rises a small increment, \(\delta T\), and the carbon fraction goes down. For this new state, \((T, P)\), we iterate on the density and internal energy until a solution is found with the same pressure as before. The new density is lower and its formation required expansion. Its new energy is the heat brought in by the mixing minus the energy lost to \(P dV\) work. That is,

\[
\delta q = e(T_0 + \delta T) - e(T_0) + P \left( \frac{1}{\rho + \delta \rho} - \frac{1}{\rho} \right) \\
\approx e(T_0 + \delta T) - e(T_0) - P \left( \frac{\delta \rho}{\rho^2} \right).
\]
Here \( \delta p \) is inherently negative, so the pressure term is positive. The composition of the mixture will also have changed by an amount that depends on the composition of the ash. Here we adopt the ash composition given by following isobaric burning to completion off line using a small seven-isotope network. For the conditions given above, a typical ash composition will be 57% O, 16% Mg, 26% Si, and 1% S. At a density of \( 3 \times 10^7 \) g cm\(^{-3} \), the composition would have been slightly different: 57% O, 8% Mg, 33% Si, and 2% S, but the \( Q \)-value is not very sensitive to the difference. The change in nuclear-binding energy between the ash and fuel (50% each \(^{12}\)C and \(^{16}\)O) is thus \( Q = 3.10 \times 10^{17} \text{ ergs g}^{-1} \). Substantially different numbers characterize an initially carbon-rich composition. For a fuel that is 75% carbon and 25% oxygen, the ash composition at \( 10^7 \) g cm\(^{-3} \) is 38% O, 15% Mg, 45% Si, and 2% S, implying \( Q = 4.70 \times 10^{17} \text{ ergs g}^{-1} \).

The fraction of carbon in the mixture is

\[
Y_{12}(T + \delta T) = Y_{12}(\text{fuel}) \left( 1 - \frac{\delta q}{Q} \right),
\]

since, by definition, \( Y_{12}(\text{ash}) = 0 \). The other composition variables are similarly interpolated between their initial (fuel) and final (ash) values, based on the change in energy. Using this new composition, the density is again iterated to find the isobaric state appropriate to the new temperature and self-consistent mixed composition. The process is continued for about 1000 steps until the carbon abundance is zero. The outcome of this calculation is a set of temperatures, densities, and compositions for the mixture consistent with the pressure in the fuel.

To reach closure, it remains to specify how the carbon abundance varies within the mixed region. In reality, the carbon mass fraction will be heterogeneous, reflecting the operation of numerous eddies on all scales and the large Lewis number. Stirring is more effective on small scales, so the most natural distribution would be a “noisy” staircase function, or even a homogeneous mixture (Kerstein 2001). We return to this picture in \( \S \) 4.2. For now, however, a simple approximation is made that the carbon abundance is distributed linearly within the stirred flame. That is, for \( 0 \leq \lambda' / \lambda \leq 1 \),

\[
Y_{12} = Y_{12}^0 \left( 1 - \frac{\lambda'}{\lambda} \right). \tag{8}
\]

Such a linear approximation is consistent with multidimensional simulations so far (Fig. 31 of Bell et al. 2004) and gives equations that are easy to manipulate and understand.

Given \( Y_{12}(\lambda'), \rho(Y_{12}), T(Y_{12}), L, \) and \( u_L \), one is now equipped to solve equation (3) for a unique value of \( \lambda \). Some results are given in Figures 1 and 2 and Table 1, the latter showing a dramatic dependence of the burning-front width and speed on the density and turbulent energy. For the lowest densities and highest turbulent energies considered, the mixed region becomes comparable to the integral length scale, 10 km, and the burning can approach a fraction of the sound speed. The numbers that give \( \lambda \approx 10 \) km in Table 1 are not physical unless equation (5) and the assumption of homogeneous, isotropic turbulence can be extrapolated to these larger scales, and the velocity continues to increase above \( u_L \). In those cases where \( u_L \) is already \( 10^8 \) cm s\(^{-1} \), that is doubtful. The flame widths and speeds are also smaller for carbon-rich mixtures. Burning more carbon raises the temperature of the ash and mixture and makes the burning region smaller.

It is important that the flame has separated into two components. If one added the energy generation from oxygen burning, the temperatures would be higher and the width of the burning region much smaller. Since, as we shall see, only the largest values for \( \lambda \) in Table 1 imply a possible transition to detonation, a necessary condition for a delayed detonation transition is that the oxygen and carbon-burning flames have split and are widely separated.

3. APPROXIMATIONS TO THE TURBULENT FLAME SPEED

The speed of a laminar flame is proportional to the square root of the heat diffusion coefficient. In the distributed regime one expects a similar relation with the turbulent diffusion coefficient,
\( D_{\text{turb}} \sim v_{i} \lambda \), substituting for the radiative one (e.g., Röpke & Hillebrandt 2005). Here \( v_{i} \) is the turbulent eddy speed on the scale of the flame width, \( \lambda \), and hence \( D_{\text{turb}} \sim \lambda^{1/3} u_{L}^{-2/3} \). The width of the flame is given by equating the nuclear and diffusion times,

\[
\tau_{\text{nuc}} \approx \frac{\lambda^{2}}{D_{\text{turb}}} = \frac{L^{1/3} \lambda^{2/3}}{u_{L}},
\]

and hence

\[
\lambda \approx \left( \frac{\tau_{\text{nuc}} u_{L}}{L} \right)^{3/2}.
\]

This relation is well known in the chemical combustion community (Peters 1999, 2000; Kerstein 2001). Here \( \tau_{\text{nuc}} \) is the average nuclear burning time in the region defined by equation (2). For a given density and turbulent energy, this suggests a scaling \( \lambda \propto u_{L}^{2/3} \), which agrees with the values in Table 1. Such a scaling is also expected because the left-hand side of equation (3) is proportional to \( \lambda \) while the right-hand side depends on \( \lambda^{2/3} u_{L} \).

Equation (10) also states that the turbulent flame speed, \( v_{\text{turb}} \), is the square root of the energy dissipated by the turbulent cascade, \( u_{L}^{2} / L \), in a nuclear timescale. For a given turbulent energy, to get the flame to move faster one must increase the nuclear timescale, i.e., slow the burning.

The nuclear timescale is very sensitive to the temperature in the mixed region, which is highly variable; but near \( 3 \times 10^{9} \) K, it is approximately (Woosley et al. 2004)

\[
\tau_{\text{nuc}} \approx \frac{C_{p} \rho T}{n \dot{\rho}_{\text{nuc}}},
\]

where \( \dot{\rho}_{\text{nuc}} \propto \rho X_{12}^{2} T^{n} \). The heat capacity, due to a combination of semidegenerate electrons and radiation, increases very roughly as \( T^{2} \) while, for barrier penetration, \( n \) in the temperature range near 2–3 billion K is

\[
n = 28.05 T_{9}^{-1/3} - \frac{2}{3} \approx 20.
\]

Thus, \( \tau_{\text{nuc}} \) is roughly proportional to \( \rho^{-1} X_{12}^{-2} T^{-17} \) and, since the temperature in the burning region is proportional to \( T_{\text{ash}} \), the flame width

\[
\lambda \propto u_{L}^{3/2} L^{-1/2} \tau_{\text{nuc}}^{3/2} = \left( \frac{u_{L}}{10^{8} \text{ cm s}^{-1}} \right)^{3/2} \left( \frac{T_{\text{ash}}}{2.79 \times 10^{9} \text{ K}} \right)^{-25.5} \left( \frac{\rho_{\text{fuel}}}{10^{7} \text{ g cm}^{-3}} \right)^{-3/2} \left( \frac{0.8}{X_{12}} \right)^{3} \text{ km},
\]

where the proportionality has been normalized to the numerical results in Table 1 for the fiducial values. This is an overall good fit for other densities and compositions in the range of interest.

The turbulent speed, \( v_{\text{turb}} \), is given either by equation (5) evaluated for the flame width derived above, or by

\[
v_{\text{turb}} \approx \frac{\lambda}{\tau_{\text{nuc}}} \approx \frac{u_{L}^{3/2}}{L^{1/2}} \tau_{\text{nuc}}^{1/2} \approx 790 \left( \frac{T_{\text{ash}}}{2.79 \times 10^{9} \text{ K}} \right)^{-8.5} \left( \frac{\rho_{\text{fuel}}}{10^{7} \text{ g cm}^{-3}} \right)^{-1/2} \left( \frac{0.5}{X_{12}} \right)^{1/2} \left( \frac{10 \text{ km}}{L} \right)^{1/2} \text{ km s}^{-1}.
\]

In addition, \( v_{\text{turb}} \) should not be greater than \( u_{L} \). Note the strong reciprocal dependence on the temperature of the ash and hence on the density and energy yield of the burning. It is this dependence that is chiefly responsible for the spreading and acceleration of the turbulent flame at low density. If oxygen burned as

---

**Table 1: Turbulent Flame Properties**

| \( X_{12} \) | \( \rho_{\text{fuel}} \) | \( u_{L} \) at 10 km | \( T_{\text{ash}} \) | \( \rho_{\text{ash}} \) | \( \lambda \) | \( v_{\text{turb}} \) |
|---|---|---|---|---|---|---|
| 0.5 | 0.6 | 2.45 | 0.284 | 1.0E7 | | |
| 0.6 | 3 | 2.45 | 0.284 | 6.1E6 | 5.5E7 | |
| 0.8 | 3 | 2.64 | 0.400 | 5.19E5 | 2.4E7 | |
| 0.8 | 10 | 2.64 | 0.400 | 3.16E6 | 1.5E8 | |
| 1 | 1 | 2.79 | 0.523 | 1.55E4 | 2.5E6 | |
| 1 | 3 | 2.79 | 0.523 | 8.30E4 | 1.3E7 | |
| 1 | 10 | 2.79 | 0.523 | 4.92E5 | 9.0E7 | |
| 2 | 1 | 3.30 | 1.18 | 6.82E1 | 4.1E5 | |
| 2 | 3 | 3.30 | 1.18 | 3.54E2 | 2.1E6 | |
| 2 | 10 | 3.30 | 1.18 | 2.16E3 | 1.3E7 | |
| 2.3 | 1 | 3.40 | 1.39 | 2.43E4 | 2.9E5 | |
| 3 | 1 | 3.63 | 1.89 | 3.62E2 | 1.5E5 | |
| 3 | 10 | 3.63 | 1.89 | 1.15E4 | 4.9E6 | |
| 0.75 | 0.6 | 2.68 | 0.23 | 3.85E4 | 3.4E6 | |
| 0.6 | 10 | 2.68 | 0.23 | 1.22E6 | 1.1E8 | |
| 0.8 | 1 | 2.89 | 0.33 | 2.99E3 | 1.4E6 | |
| 0.8 | 10 | 2.89 | 0.33 | 9.44E4 | 4.6E7 | |
| 1 | 1 | 3.07 | 0.43 | 4.38E2 | 7.6E5 | |
| 1 | 10 | 3.07 | 0.43 | 1.38E4 | 2.4E7 | |
| 2 | 1 | 3.69 | 1.00 | 1.60E0 | 1.2E5 | |
| 2 | 10 | 3.69 | 1.00 | 5.07E1 | 3.7E6 | |
| 3 | 10 | 4.09 | 1.62 | 2.47E0 | 1.3E6 | |
well as carbon, $T_{\text{ash}}$ would be much greater, and the flame speeds and widths would be drastically reduced. The fast speeds and broad widths derived here rely on the oxygen flame lagging far behind the carbon flame, i.e., outside the mixed region. For the low densities we consider, this is the case. Note also that $v_{\text{turb}}$ is enormously greater than the laminar flame speed, which at $10^7$ g cm$^{-3}$ is only 3000 cm s$^{-1}$ (Bell et al. 2004).

The quantity $T_{\text{ash}}$ can be computed off line as a function of initial density and composition (Timmes & Woosley 1992). Approximate values obtained using a small seven-isotope network are given in Table 2. The values in the table can be approximately fit by an expression of the form

$$T_{\text{ash}} \approx 2.79 \times 10^9 \left( \frac{\rho_{\text{fuel}} X_{12}}{5 \times 10^9 \text{ g cm}^{-3}} \right)^{0.25}.$$  \hspace{1cm} (15)

Substituting this in equations (14) and (15), one has

$$\lambda \approx 4.9 \left( \frac{u_L}{10^8 \text{ cm s}^{-1}} \right)^{3/2} \left( \frac{10 \text{ km}}{L} \right)^{1/2} \times \left( \frac{\rho_{\text{fuel}}}{10^7 \text{ g cm}^{-3}} \right)^{-7.9} \left( \frac{0.5}{X_{12}} \right)^{9.4} \text{ km},$$  \hspace{1cm} (16)

$$v_{\text{turb}} \approx 790 \left( \frac{u_L}{10^8 \text{ cm s}^{-1}} \right)^{3/2} \left( \frac{\rho_{\text{fuel}}}{10^7 \text{ g cm}^{-3}} \right)^{-2.6} \times \left( \frac{0.5}{X_{12}} \right)^{3.1} \left( \frac{10 \text{ km}}{L} \right)^{1/2} \text{ km s}^{-1}.$$  \hspace{1cm} (17)

For a given turbulent energy and carbon fraction, this implies a flame speed that scales roughly as $\rho^{-2.6}$.

4. THE TRANSITION FROM A DEFLAGRATION TO A DETONATION

4.1. Spontaneous Detonation

One of the simplest ways a transition to detonation could happen in an exploding white dwarf, which is included here only because it seems to have been overlooked, is if frictional heating—the dissipation of the turbulent energy on the Kolmogorov scale—heats a region of fuel to the flash point.

Consider a region of fuel close to a large rising element of ash. The rise of a burning plume injects turbulent energy at some characteristic length scale. The velocities are initially anisotropic, but after the energy cascades down approximately one decade in length scale, that energy resides in isotropic Kolmogorov turbulence (Zingale et al. 2005). Typical turbulent speeds at length scales of $\sim 10$ km where the Kolmogorov cascade might begin are, at reasonably late times in the explosion, in the range $(1-10) \times 10^7$ cm s$^{-1}$ (Schmidt et al. 2006; Röpke 2007).

This energy cascades downward to the Kolmogorov length, $10^{-4}$ to $10^{-3}$ cm, where it dissipates as heat. The amount of heat dissipated is $u_{\text{turb}}^2 / L \rho g$ s$^{-1}$, i.e., the conserved quantity in Kolmogorov turbulence. For speeds $10^7-10^8$ cm s$^{-1}$ on a length scale of 10 km, that corresponds to $10^{15}-10^{16}$ ergs g$^{-1}$. Most of this dissipation occurs inside the ash (Schmidt et al. 2006), in part because the flame spreads at a speed comparable to that of the largest eddies. However, there may be small regions near the floating ash, unresolved in current studies, perhaps within the Kelvin-Helmholtz rolls that bound the rising plumes or in the wakes of detached bubbles, where a locally large concentration of turbulent energy is dissipated in the fuel. The dissipation might be particularly large in vortex tubes shed by the rising plumes.

Because of its low heat capacity, $\leq 2 \times 10^{17}$ ergs g$^{-1}$ is necessary to raise the fuel temperature to the point where it will burn supersonically. If the explosive burning region is larger than a critical mass, a detonation will occur (Niemeyer & Woosley 1997; Dursi & Timmes 2006). For the highest turbulent energies considered, this would only take about 0.2 s of uninterrupted dissipation, significantly less than the expansion time of the star. For regions in close proximity to an active flame, the frictionally heated fuel will probably be burned before it can run away. However, in the trailing wake of rising bubbles, there might be time for viscous dissipation in unburned fuel to ignite new burning. If the temperature gradient in this new region is sufficiently small, the ignition could have a supersonic phase velocity.

4.2. Detonation in Fuel-Ash Mixtures

In the absence of this viscous ignition (§ 4.1), or ignition by compression (Arnett & Livne 1994; Plewa et al. 2004; Röpke et al. 2007), carbon detonation can only occur in a mixture of hot ash and cold fuel in the distributed regime. But there can be no detonation within the mixture as long as a steady state subsonic flame exists, with burning balancing the average rate at which fuel is heated by either conduction or mixing. This is always true in the case of laminar flames, where the thickness is much less than the critical mass for detonation, but it remains true for steady state flames in the distributed regime. Although the widths of the flames in Table 1 can become quite large, the turbulent eddy that sets the timescale for burning in $\lambda$ is itself subsonic. Burning occurs at a rate just sufficient to balance the advancement of the mixing and cannot become supersonic.

However, this steady state is a fiction, useful only for obtaining rough estimates for the size and speed of the mixed burning region. As long as the mixing region defined by the turbulent integral length scale contains many flames, fluctuations in the burning rate will average out. The situation remains closely analogous to the flamelet regime. Many flame surfaces combine to make a “flame brush” with fractal dimension $D = 2.36$. For the allowed range of length scales, the burning region moves at a speed given by the largest turbulent eddies, and the individual flame speeds—be they turbulent or laminar—are not important.

The situation changes, however, when the entire integral length scale of the turbulence contains only one or a few flames (Table 1). The large eddies driving the mixing are random. Occasionally, a long time may elapse before a new eddy arrives.

| $\rho_L$ | $X_{12} = 0.50$ | $X_{12} = 0.75$ |
|---------|----------------|----------------|
| 0.6     | 2.45           | 2.68           |
| 0.8     | 2.64           | 2.89           |
| 1.0     | 2.79           | 3.07           |
| 1.2     | 2.92           | 3.22           |
| 1.4     | 3.03           | 3.36           |
| 1.6     | 3.13           | 3.48           |
| 1.8     | 3.22           | 3.59           |
| 2.0     | 3.30           | 3.69           |
| 2.2     | 3.37           | 3.78           |
| 2.4     | 3.44           | 3.86           |
| 2.6     | 3.51           | 3.94           |
| 2.8     | 3.57           | 4.02           |
| 3.0     | 3.63           | 4.09           |
| 3.2     | 3.68           | 4.16           |
Within the region stirred by this large eddy, layers exist of nearly isothermal mixtures of fuel and ash. Such layers were seen by Lisewski et al. (2000b) and were responsible for the “micro-explosions” observed in their simulations, but because of the small dimensions of the mixed flames studied in that paper by Lisewski et al., that burning never approached sonic speeds.

4.3. Conditions for Detonation

A necessary condition for detonation is that sustained burning inside a distributed flame width occurs faster than its sound-crossing time. This is a condition on the sonic length scale,

\[ r_{\text{sonic}} = c_s \tau_{\text{nuc}}(T') \leq \lambda < L, \]

where \( T' \) is some temperature \( 0 < T < T_{\text{ash}} \) in the isobaric mixture. If \( T' \) in this equation were the same as the temperature in \( \tau_{\text{nuc}} \) in equation (10), one would require supersonic turbulent motion in order to initiate a detonation, since that would imply \( L \geq \lambda \geq \left( \frac{c_s^3}{u_L} \right) L \).

The fallacy in this argument is that \( \lambda \) is some approximate length scale in a fictitious steady state, which never exists at any one place and time, while \( r_{\text{sonic}} \) can vary greatly depending on the instantaneous local values in a given flame. Occasionally the distribution of temperature inside the mixed flame is such that, including the effects of induction, it burns much faster than steady state. Table 3, calculated using a small seven-isotope network, gives the characteristics of burning on different timescales. It can be assumed that, as we shall find is necessary, a high degree of turbulence exists such that \( L/u_L \sim 0.01 \) s (i.e., \( u_L \sim 10^6 \) cm s\(^{-1}\), where \( L \sim 10^6 \) cm). Mixing can go on without appreciable burning as long as the temperature \( T \) remains cool enough that \( \tau_{\text{nuc}}(T) \geq L/u_L \). Here a small margin of error is included, and the temperature is calculated such that half the fuel would burn in 0.02 s. This is \( T_{0.02} \) in Table 3, and the corresponding carbon mass fraction is \( X_{12,0.02} \). The mixture and timescale are calculated for isobaric fuel-ash mixtures with an initial carbon mass fraction of either 0.5 or 0.75. Because carbon burns to different compositions at different temperatures, this does not give exactly the same values as burning a given composition in place without mixing, but for the mixing that goes on before burning, this is the correct procedure.

These mixed plasmas are then allowed to burn without further intervention. Because of the high temperature sensitivity of the reaction rate, most of the energy from burning is released after mixing during this phase of “inductive burning.” Toward the end, the burning can become supersonic for a region, \( r_{\text{sonic}} \), given by the sound speed and burning time. The minimal burning time, hence the smallest \( r_{\text{sonic}} \), called here \( r_{\text{sonic}}^{\text{min}} \), is evaluated by two criteria. One is that 60% of the initial carbon has burned. This corresponds to the onset of the rapid rise in, e.g., Figure 1. A second value, the actual minimum value of \( r_{\text{sonic}} \), comes from numerically evaluating the point where the energy generation is a maximum for an isobaric mixture of carbon and ash starting at the initial temperature, density, and carbon mass fraction. The initial temperature matters little. The first choice gives a lower temperature and hence larger \( r_{\text{sonic}} \) and is thus more difficult to achieve. However, it gives a larger carbon mass fraction, which decreases the necessary mass for detonation. This first set of numbers was plotted for \( X_{12,0.02} = 0.5 \) in Figure 4.

For fuel densities below \( 7 \times 10^6 \) g cm\(^{-3}\) and \( X_{12} = 0.50 \), the ash temperature is just too low to burn in less than 0.02 s, no matter what the mixture. The sonic radius is much larger than

| \( X_{12} \) | \( \rho_7 \) (10\(^7\) g cm\(^{-3}\)) | \( X_{12,0.02} \) | \( T_{0.02} \) (10\(^5\) K) | \( \rho_{0.02} \) (10\(^7\) g cm\(^{-3}\)) | \( X_{12,\text{max}} \) | \( T_{\text{max}} \) (10\(^5\) K) | \( \rho_{\text{max}} \) (10\(^7\) g cm\(^{-3}\)) | \( \tau_{\text{min},\text{min}} \) (s) | \( r_{\text{sonic}}^{\text{min}} \) (cm) |
|---|---|---|---|---|---|---|---|---|---|
| 0.5 | 0.8 | 0.26 | 2.09 | 0.53 | 0.20 | 2.22 | 0.50 | 1.7E-3 | 7.5E5 |
| 1.0 | 0.36 | 1.77 | 0.78 | 0.20 | 2.26 | 0.66 | 7.7E-4 | 3.5E5 |
| 1.5 | 0.40 | 1.65 | 1.27 | 0.20 | 2.43 | 0.45 | 1.1E-4 | 5.4E4 |
| 2.0 | 0.41 | 1.60 | 1.76 | 0.20 | 2.56 | 1.46 | 2.9E-5 | 1.4E4 |
| 2.5 | 0.43 | 1.56 | 2.26 | 0.20 | 2.67 | 1.87 | 1.0E-4 | 5.4E3 |
| 3.0 | 0.43 | 1.54 | 2.75 | 0.20 | 2.77 | 2.12 | 4.8E-5 | 2.5E3 |
| 0.8 | 0.26 | 2.09 | 0.53 | 0.13 | 2.36 | 0.47 | 1.1E-4 | 5.4E3 |
| 1.0 | 0.36 | 1.77 | 0.78 | 0.12 | 2.46 | 0.61 | 4.7E-4 | 2.2E5 |
| 1.5 | 0.40 | 1.65 | 1.27 | 0.11 | 2.70 | 0.97 | 6.0E-5 | 3.0E4 |
| 2.0 | 0.41 | 1.60 | 1.76 | 0.10 | 2.88 | 1.35 | 1.5E-5 | 5.4E3 |
| 2.5 | 0.43 | 1.56 | 2.26 | 0.10 | 3.02 | 1.73 | 5.1E-6 | 2.7E3 |
| 3.0 | 0.43 | 1.54 | 2.75 | 0.098 | 3.14 | 2.04 | 2.4E-6 | 1.3E3 |

Table 3: Conditions for Supersonic Burning
The carbon abundance affects the transition in several ways, sometimes subtly. One might expect that, due to its larger energy release, burning a more “incendiary” mixture of carbon and oxygen—one with a larger carbon mass fraction—would make it somehow more likely to detonate. In fact, equation (22) shows the opposite behavior: a weak reciprocal dependence of the detonation density on the local carbon abundance. This is because the burning of a carbon-rich composition produces a hotter

$$L_1, \text{ with characteristics given in Table 3 (} T_{\text{max}}, \rho_{\text{max}} \text{) was embedded in a comparable size region, } L_2, \text{ where the temperature declined gradually to the cold fuel value. A sample setup is shown in Figure 3. It is important that the isothermal region is surrounded by a region where the temperature changes gradually so that a supersonic phase velocity can develop. This gradual change in temperature seems reasonable given the turbulent mixing that occurs on all length scales.}

The results for some trial detonations are given in Table 4 and Figure 4. It seems that homogeneously mixed regions of size greater than a few times the sonic radius are capable of igniting detonations. The condition \( r_{\text{sonic}} \ll \lambda \) is not only necessary for detonation, but is sufficient. The fact that the size is somewhat larger than \( r_{\text{sonic}} \) is not alarming because ignition may involve several adjacent mixed layers.

### 4.4. Dependence on the Carbon Abundance

The carbon abundance affects the transition in several ways, sometimes subtly. One might expect that, due to its larger energy release, burning a more “incendiary” mixture of carbon and oxygen—one with a larger carbon mass fraction—would make it somehow more likely to detonate. In fact, equation (22) shows the opposite behavior: a weak reciprocal dependence of the detonation density on the local carbon abundance. This is because the burning of a carbon-rich composition produces a hotter

$$L_1, \text{ with characteristics given in Table 3 (} T_{\text{max}}, \rho_{\text{max}} \text{) was embedded in a comparable size region, } L_2, \text{ where the temperature declined gradually to the cold fuel value. A sample setup is shown in Figure 3. It is important that the isothermal region is surrounded by a region where the temperature changes gradually so that a supersonic phase velocity can develop. This gradual change in temperature seems reasonable given the turbulent mixing that occurs on all length scales.}

The results for some trial detonations are given in Table 4 and Figure 4. It seems that homogeneously mixed regions of size greater than a few times the sonic radius are capable of igniting detonations. The condition \( r_{\text{sonic}} \ll \lambda \) is not only necessary for detonation, but is sufficient. The fact that the size is somewhat larger than \( r_{\text{sonic}} \) is not alarming because ignition may involve several adjacent mixed layers.

$$L_1, \text{ with characteristics given in Table 3 (} T_{\text{max}}, \rho_{\text{max}} \text{) was embedded in a comparable size region, } L_2, \text{ where the temperature declined gradually to the cold fuel value. A sample setup is shown in Figure 3. It is important that the isothermal region is surrounded by a region where the temperature changes gradually so that a supersonic phase velocity can develop. This gradual change in temperature seems reasonable given the turbulent mixing that occurs on all length scales.}

The results for some trial detonations are given in Table 4 and Figure 4. It seems that homogeneously mixed regions of size greater than a few times the sonic radius are capable of igniting detonations. The condition \( r_{\text{sonic}} \ll \lambda \) is not only necessary for detonation, but is sufficient. The fact that the size is somewhat larger than \( r_{\text{sonic}} \) is not alarming because ignition may involve several adjacent mixed layers.

$$L_1, \text{ with characteristics given in Table 3 (} T_{\text{max}}, \rho_{\text{max}} \text{) was embedded in a comparable size region, } L_2, \text{ where the temperature declined gradually to the cold fuel value. A sample setup is shown in Figure 3. It is important that the isothermal region is surrounded by a region where the temperature changes gradually so that a supersonic phase velocity can develop. This gradual change in temperature seems reasonable given the turbulent mixing that occurs on all length scales.}

The results for some trial detonations are given in Table 4 and Figure 4. It seems that homogeneously mixed regions of size greater than a few times the sonic radius are capable of igniting detonations. The condition \( r_{\text{sonic}} \ll \lambda \) is not only necessary for detonation, but is sufficient. The fact that the size is somewhat larger than \( r_{\text{sonic}} \) is not alarming because ignition may involve several adjacent mixed layers.
ash, and the fuel-ash mixture maintains a thin burning width until a lower density. This is unavoidable.

Unfortunately, this is the opposite behavior of that postulated by Umeda et al. (1999a) in an attempt to explain the preponderance of bright Type Ia supernovae in late-type galaxies (Filippenko 1989; Branch et al. 1996a, 1996b). The transition densities computed here are also lower, and the dependence on density comparatively weak.

However, there is another possibility. Equation (22) is a necessary but not sufficient condition for detonation. To detonate, one also needs a critical mass. Table 4 shows that when the homogeneously mixed region (Table 3) is 2–3 times $r_{\text{min}}$, detonation usually happens. Only one model in this paper strictly satisfies the criteria $L_{1} \sim 3 \rho_{\text{fuel}}^{1/3} \leq \lambda$ for $u_{L} \leq 10^{9}$ cm s$^{-1}$ at 10 km and that is the model with $X_{12} = 0.75$ and $\rho_{\text{fuel}} = 6 \times 10^{6}$ g cm$^{-3}$. Turning the density down to find a solution where this works for $X_{12} = 0.5$ requires mixed regions that are either considerably larger than the integral length scale or so cold that they do not detonate at all. Certainly the accuracy of this study is not enough to rule out models with $X_{12} = 0.5$, which do satisfy this condition within a factor of about 2 ($X_{12} = 0.5$, $\rho_{\text{fuel}} = 10^{7}$ g cm$^{-3}$, for example, but a possibility is that detonation requires a certain minimum value of carbon mass fraction). This would have major ramifications. Models with less than critical $X_{12}$ would have to be pure deflagrations and would have distinctly different properties. A lot more study is needed before this possibility is taken too seriously.

Alternatively, there may be other explanations for the preponderance of bright Type Ia supernovae in late-type galaxies. Perhaps the carbon-oxygen ratio in the outer layers of the white dwarf at the time it explodes correlates differently with stellar evolution than Umeda et al. assume. In this regard it is noteworthy that more massive white dwarfs, which are derived from larger stars, do have lower carbon abundances in their interiors when they explode (see Figs. 6 and 12 of Umeda et al. 1999b). The carbon abundance, and other factors, may also influence the conditions that are known to have a major effect on the supernova brightness.

5. CONCLUSIONS

Using simple scaling arguments, the steady state width and speed of a turbulent flame in the distributed regime have been derived as a function of turbulent energy, density, and carbon mass fraction. This speed is much faster than the laminar speed at the same density and displays a sensitive reciprocal dependence on the density. All such steady state flames are subsonic as long as the turbulence driving them remains subsonic. The bulk propagation of the burning in the distributed regime, prior to any detonation, will still be governed by the motion of the large turbulent eddies, not the speed of the individual flames.

As the density nears $1 \times 10^{7}$ g cm$^{-3}$, however, the width of the flame approaches the integral length scale, and the unsteady nature of the burning becomes important. For sufficiently intense turbulence, detonation becomes possible (see also Lisewski et al. 2000a). In order for detonation to occur, several criteria must be satisfied simultaneously.

First, the carbon- and oxygen-burning flames must separate spatially. If both fuels burn simultaneously, the resulting ash is too hot, and the fuel-ash mixture too combustible. This makes the average width of the mixed burning region narrow and prevents the detonation due to flame broadening described here.

Second, the speed of the largest turbulent eddies must approach the sound speed. That is, the maximum speed below which the assumption of isotropic Kolmogorov turbulence is valid must not be too subsonic. It is this speed that sets the characteristic scale of the problem. Some additional increase in flame speed may be achieved by unsteady burning, but the amplification is not likely to be large enough to bridge orders of magnitude. It is the large turbulent energies reported by Röpke (2007; ~20% sonic at the integral length scale) that makes things work.

Third, the size of the mixture of fuel and ash must become as large as the largest eddies. That is, the flame width must approach the integral length scale. For white dwarfs this is some multiple of 10 km. The most likely detonation site will be at the merger of several such flames.

These conditions may occasionally all be satisfied at the low densities encountered by the flame as it moves to the surface layers and the white dwarf begins to come apart. However, just reaching a particular density is not adequate. Nor is it sufficient simply to move into a region where the burning is distributed (e.g., the Gibson length is smaller than the flame thickness; Röpke & Niemeyer 2007). Most of the models in Table 1 are in the distributed regime, but only those where the flame width $\lambda$ approaches the integral length scale $L$ can detonate. Detonation thus requires the right combination of turbulent energy and density, and such conditions are rare. Whether they are sufficiently rare as sometimes not to happen is beyond the scope of this paper.

It is interesting, however, that the range of detonation densities derived here, $(0.5–1.5) \times 10^{7}$ g cm$^{-3}$, is what has been invoked for some time in order to achieve good agreement with nucleosynthesis, spectra, and light curves in artificially parameterized descriptions of the explosion (Hoflich et al. 1995).

Many of the approximations in this paper and its conclusions warrant careful checking by numerical experiments that this paper—it is hoped—will motivate.

The author gratefully acknowledges helpful conversations on the subject of the paper with John Bell, Martin Lisewski, Jens Niemeyer, Fritz Roepke, Mike Zingale, and especially Alan Kerstein. This research has been supported by the NASA Theory Program (NGG 05-GG08G) and the DOE SciDAC Program (DE-FC02-06ER41438).

REFERENCES

Arnett, D., & Livne, E. 1994, ApJ, 427, 330
Bell, J. B., Day, M. S., Rendleman, C. A., Woosley, S. E., & Zingale, M. 2004, ApJ, 608, 883
Branch, D., Romanishin, W., & Baron, E. 1996a, ApJ, 465, 73
Branch, D., Romanishin, W., & Baron, E. 1996b, ApJ, 467, 473
Damköhler, G. 1940, Z. Elektrochern, 46, 601
Dursi, L. J., & Timmes, F. X. 2006, ApJ, 641, 1071
Filippenko, A. 1989, PASP, 101, 588
Hillebrandt, W., & Niemeyer, J. 2000, ARA&A, 38, 191
Hoffich, P., Khokhlov, A. M., & Wheeler, J. C. 1995, ApJ, 444, 831
Kerstein, A. 2001, Phys. Rev. E., 64, 066306
Khokhlov, A., Oran, E. S., & Wheeler, J. C. 1997, ApJ, 478, 678
Landau, L., & Lifshitz, F. M. 1959, Fluid Mechanics (London: Pergamon)
Lisewski, A. M., Hillebrandt, W., & Woosley, S. E. 2000a, ApJ, 538, 831
Lisewski, A. M., Hillebrandt, W., Woosley, S. E., Niemeyer, J. C., & Kerstein, A. R. 2000b, ApJ, 537, 405
Niemeyer, J. C., & Kerstein, A. R. 1997, NewA, 2, 239
Niemeyer, J. C., & Woosley, S. E. 1997, ApJ, 475, 740
Peters, N. 1999, J. Fluid Mech., 384, 107
———. 2000, in Turbulent Combustion, ed. N. Peters (Cambridge: Cambridge
Univ. Press), 320
Plewa, T., Calder, A. C., & Lamb, D. Q. 2004, ApJ, 612, L37
Pope, S. B. 1987, Ann. Rev. Fluid Mech., 19, 237
Röpke, F. 2007, ApJ, 668, 1103
Röpke, F. K., & Hillebrandt, W. 2005, A&A, 429, L29
Röpke, F. K., & Niemeyer, J. C. 2007, A&A, 464, 683
Röpke, F. K., Woosley, S. E., & Hillebrandt, W. 2007, ApJ, 660, 1344
Schmidt, W., Niemeyer, J. C., Hillebrandt, W., & Röpke, F. K. 2006, A&A,
450, 283

Timmes, F. X. 2000, ApJ, 528, 913
Timmes, F. X., & Swesty, F. D. 2000, ApJS, 126, 501
Timmes, F. X., & Woosley, S. E. 1992, ApJ, 396, 649
Umeda, H., Nomoto, K., Kobayashi, C., Hachisu, I., & Kato, M. 1999a, ApJ,
522, L43
Umeda, H., Nomoto, K., Yamaoka, H., & Wanajo, S. 1999b, ApJ, 513, 861
Weaver, T. A., Woosley, S. E., & Zimmerman, G. B. 1978, ApJ, 225, 1021
Woosley, S. E., Heger, A., & Weaver, T. A. 2002, Rev. Mod. Phys., 74, 1015
Woosley, S. E., Wunsch, S., & Kuhlen, M. 2004, ApJ, 607, 921
Zingale, M., Woosley, S. E., Rendleman, C. A., Day, M. S., & Bell, J. B. 2005,
ApJ, 632, 1021