Complete hypersurfaces in Euclidean spaces with strong finite total curvature

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Abstract

We prove that strong finite total curvature (see definition in Section 2) complete hypersurfaces of \((n+1)\)-euclidean space are proper and diffeomorphic to a compact manifold minus finitely many points. With an additional condition, we also prove that the Gauss map of such hypersurfaces extends continuously to the punctures. This is related to results of White [21] and and Müller-Sverák [17]. Further properties of these hypersurfaces are presented, including a gap theorem for the total curvature.

1 Introduction

Let \( \phi: M^n \to \mathbb{R}^{n+1} \) be a hypersurface of the euclidean space \( \mathbb{R}^{n+1} \). We assume that \( M^n = M \) is orientable and we fix an orientation for \( M \). Let \( g: M \to S^1 \subset \mathbb{R}^{n+1} \) be the Gauss map in the given orientation, where \( S^1 \) is the unit \( n \)-sphere. Recall that the linear operator \( A: T_pM \to T_pM, \ p \in M \), associated to the second fundamental form, is given by

\[
\langle A(X), Y \rangle = -\langle \nabla_X N, Y \rangle, \quad X, Y \in T_pM,
\]

where \( \nabla \) is the covariant derivative of the ambient space and \( N \) is the unit normal vector in the given orientation. The map \( A = -dg \) is self-adjoint and its eigenvalues are the principal curvatures \( k_1, k_2, \ldots, k_n \).

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We say that the total curvature of the immersion is finite if \( \int_M |A|^n \, dM < \infty \), where \( |A| = \left( \sum_i k_i^2 \right)^{1/2} \), i.e., if \( |A| \) belongs to the space \( L^n(M) \). If \( \phi: M^n \rightarrow \mathbb{R}^{n+1} \) is a complete minimal hypersurface with finite total curvature then \( M \) is (equivalent to) a compact manifold \( \overline{M} \) minus finitely many points and the Gauss map extends to the punctures. This was proved by Osserman [18] for \( n = 2 \) (the equivalence here is conformal and the Gauss map extends to a (anti) holomorphic map \( \overline{g}: \overline{M}^2 \rightarrow S^1_1 \); the conformal equivalence had already been proved by Huber [13]). For an arbitrary \( n \), this was proved by Anderson [2] (here the equivalence is a diffeomorphism and the Gauss map extends smoothly).

When \( \phi \) is not necessarily minimal and \( n = 2 \), the above result, with the additional hypothesis that the Gauss curvature does not change sign at the ends, was shown to be surprisingly true by B. White [21]. The subject was taken up again by Müller-Šverák [17] who answered a question of [21] and obtained further information on the conformal behaviour of the ends.

The results of White [21] and Müller-Šverák [17] start from the fact that, since \( \int_{M^2} |A|^2 \, dM \geq 2 \int_{M^2} |K| \, dM \), finite total curvature for \( n = 2 \) implies, by Huber’s theorem, that \( M \) is homeomorphic to a compact surface minus finitely many points. For an arbitrary dimension, any generalization of Huber’s theorem should require stronger assumptions (see [6] and [7] for a discussion on the theme). Thus, for a generalization of [21] and [17] for \( n \geq 3 \) a further condition might be necessary to account for the lack of an appropriate generalized Huber theorem.

Here, we assume the hypothesis of strong finite total curvature, that is, we assume that \( |A| \) belongs to \( W^{1,q}_s \), a special Weighted Sobolev space (see Section 2 for precise definitions). We point out that the spaces \( W^{1,q}_s \) were used in a seminal work of R. Bartnik [3] for establishing a decay condition on the metric of an \( n \)-manifold, \( n \geq 3 \), in order to prove that the ADM-mass is well-defined. Following the ideas of [3], a lot of related papers also uses the norm of \( W^{1,q}_s \) to express decay assumptions (see for instance [14], [11], [19]).

We have proved the following results.

**THEOREM 1.1.** Let \( x: M^n \rightarrow \mathbb{R}^{n+1}, n \geq 3, \) be an orientable, complete hypersurface with strong finite total curvature. Then:

i) The immersion \( \phi \) is proper.

ii) \( M \) is diffeomorphic to a compact manifold \( \overline{M} \) minus a finite number of points \( q_1, \ldots q_k \).

Assume, in addition, that the Gauss-Kronecker curvature \( H_n = k_1 k_2 \ldots k_n \) of \( M \) does not change sign in punctured neighbourhoods of the \( q_i \)'s. Then:
iii) The Gauss map $g: M^n \to S_1^n$ extends continuously to the points $q_i$.

**Theorem 1.2.** Let $x: M^n \to \mathbb{R}^{n+1}$, $n \geq 3$, be an orientable complete hypersurface with strong finite total curvature. Assume that the set $N$ of critical values of the Gauss map $g$ is a finite union of submanifolds of $S_1^n$ with codimension $\geq 3$. Then:

i) The extended Gauss map $\bar{g}: \overline{M} \to S_1^n$ is a homeomorphism.

ii) If, in addition, $n$ is even, $M$ has exactly two ends.

**Remark.** The condition on $N$ can be replaced by a weaker condition on the Hausdorff dimension of $N$ and the rank of $g$ (See [15], Theorems B and C and Remark 6.7).

It follows from Theorem 1.1 that there is a computable lower bound for the total curvature of the non-planar hypersurfaces of the set $C^n$ defined in the statement below.

**Theorem 1.3.** (The Gap Theorem) Let $C^n$ be the set of strong finite total curvature complete orientable hypersurfaces $\phi: M^n \to \mathbb{R}^{n+1}$, $n \geq 3$, such that $H_n$ does not change sign in $M$. Then either $x(M^n)$ is a hyperplane, or

$$\int_M |A|^n dM > 2\sqrt{n!} (\sqrt{\pi})^{n+1} / \Gamma((n+1)/2),$$

where $\Gamma$ is the gamma function.

**Remark.** For the Gap Theorem it is not enough to requiring that $H_n$ does not change sign at the ends of the hypersurface. This should hold on the whole $M$. Consider the rotation hypersurfaces in $\mathbb{R}^{n+1}$ generated by the smooth curve $x^{n+1} = \varepsilon e^{-1/x^2}$, $\varepsilon > 0$, $(x_1, \ldots, x_n, x_{n+1}) \in \mathbb{R}^{n+1}$. It is easily checked that, for all $\varepsilon$, this hypersurface has strong finite total curvature and $H_n$ does not change sign at the (unique) end of the hypersurface. However, as $\varepsilon$ approaches zero, these hypersurfaces approach a hyperplane, and the lower bound for the total curvature of the family is zero.

The paper is organized as follows. In Section 2, we discuss (Proposition 2.2) the rate of decay at infinity of the second fundamental form of a hypersurface under the hypothesis of strong finite total curvature. In Section 3, we show that each end of such a hypersurface has a unique “tangent plane at infinity” (see the definition before Proposition 3.4) and in Section 4, we prove Theorems 1.1, 1.2 and the Gap Theorem.
2 The rate of decay of the second fundamental form

In the rest of this paper, we will be using the following notation for an immersion \( \phi: M^n \to \mathbb{R}^{n+1} \):

\[
\rho = \text{intrinsic distance in } M \\
d = \text{distance in } \mathbb{R}^{n+1}; \ 0 = \text{origin of } \mathbb{R}^{n+1}
\]

\[
D_p(R) = \{ x \in M; \rho(x, p) < R \}
\]

\[
D_p(R, S) = \{ x \in M; R < \rho(x, p) < S \}
\]

\[
B(R) = \{ x \in \mathbb{R}^{n+1}; d(x, 0) < R \}; \ S(R) = \partial B(R)
\]

\[
A(R, S) = \{ x \in \mathbb{R}^{n+1}; R < d(x, 0) < S \}.
\]

Without loss of generality, we assume that \( 0 \in \phi(M) \) and we choose a point \( 0 \in M \) such that \( 0 = \phi(0) \). For \( x \in M \), \( \rho_0(x) \) will denote the intrinsic distance in \( M \) from \( x \) to \( 0 \).

Now, we set the notation for the norms (see [3, (1.2)]) that will be used in the definition of strong finite total curvature.

Let \( \Omega \subset M \). Given any \( q > 0 \), we define the weighted space \( L^q_s(\Omega) \) of all measurable functions of finite norm

\[
||u||_{L^q_s(\Omega)} = \left( \int_{\Omega} |u|^q |\rho_0|^{-q s - n} \, dM \right)^{1/q}
\]

For positive integers \( k \), we introduce the weighted Sobolev space \( W^{k,q}_s(\Omega) \) of all measurable functions of finite norm

\[
||u||_{W^{k,q}_s(\Omega)} = \sum_{i=0}^{k} ||\nabla^i u||_{L^q_s(\Omega)}
\]

We say that the immersion has strong finite total curvature if

\[
|A| \in W^{-1,q}_s(M) \text{ for } q > n,
\]

that is, if

\[
\left( \int_{\Omega} |A|^q |\rho_0|^{q-n} \, dM \right)^{1/q} + \left( \int_{\Omega} ||\nabla A||_q |\rho_0|^{2q-n} \, dM \right)^{1/q} < \infty \text{ for } q > n.
\]

We remark that the function \( \rho_0 \) used above to define these norms could be replaced by the distance with respect to any point \( p \in M \). Here, we fixed the point \( 0 = \phi(0) \).
Then, in this paper, when we say that the immersion has strong finite total curvature we are implicitly assuming w.l.g. that \(0 \in \phi(M)\). We also remark that the weights used to define the norm \(||| \cdot |||_{W^{k,q}}\) makes it invariant by dilations (see the proof of Proposition 2.2).

The following Lemma will be repeatedly used in this and in the next sections.

**Lemma 2.1.** Let \(D \subset \mathbb{R}^{n+1}\) be a bounded domain with smooth boundary \(\partial D\). Let \((W_i)\) be a sequence of connected \(n\)-manifolds and let \(\phi: W_i \to \mathbb{R}^{n+1}\) be immersions such that \(x(\partial W_i) \cap D = \emptyset\) and \(x(W_i) \cap D = M_i\) is connected and nonvoid. Assume that there exists a constant \(C > 0\) such that \(\sup_{x \in M_i} |A_i(x)|^2 < C\) and that there exists a sequence of points \((x_i), x_i \in M_i\), with a limit point \(x_0 \in D\). Then:

i) A subsequence of \((M_i)\) converges \(C^1\) uniformly on the compact parts (see the definition below) to a union of hypersurfaces \(M_\infty \subset D\).

ii) If, in addition, \(\left(\int_{\Omega} |A|^q \alpha_i \, dM\right)^{1/q} + \left(\int_{\Omega} |\nabla |A|^q\beta_i \, dM\right)^{1/q} \to 0\), for sequences \((\alpha_i), (\beta_i)\) of continuous functions such that \(\inf_{x \in M_i} \{\alpha_i, \beta_i\} \geq \kappa > 0\). Then a subsequence of \(|A_i|\) converges to zero everywhere and \(M_\infty\) is a union of hyperplanes.

By \(C^1\) convergence to \(M_\infty\) we mean that for any \(m \in M_\infty\) and each tangent plane \(T_m M_\infty\) there exists an euclidean ball \(B_m\) around \(m\) so that, for \(i\) large, the image by \(\phi\) of some connected component of \(\phi^{-1}(B_m \cap M_i)\) can be graphed over \(T_m M_\infty\) by a function \(g_i^m\) and the sequence \(g_i^m\) converges \(C^1\) to the graph \(g_\infty\) of \(M_\infty\) over the chosen plane \(T_m M_\infty\).

**Proof.** From the uniform bound of the curvature \(|A_i|\^2\), we conclude the existence of a number \(\delta > 0\) such that for each \(p_i \in M_i\) and for each tangent space \(T_{p_i} M_i\), \(M_i\) can be graphed by a function \(f_i^p\) over a disk \(U_\delta(p_i) \subset T_{p_i} M_i\), of radius \(\delta\) and center \(p_i\) in \(T_{p_i} M_i\), and that such functions have a uniform \(C^1\) bound (independent of \(p_i\) and \(i\)). We want to show that we also have a uniform \(C^2\) bound.

Let \(q\) be a point in the part of \(M_i\) that is a graph over \(U_\delta(p_i)\) and let \(v \in T_q M_i\). Consider the plane \(P_q\) that contains the normal vector \(N_q(p)\) and \(v\) and take the curve \(C_i = P_q \cap M_i\). Parametrize \(C_i\) by \(c_i(t)\) with \(c_i(0) = q\), project it down to \(T_{p_i} M_i\) paralelly to the normal at \(p_i\). Let \(\tilde{c}_i(t)\) be this projection; then, \(c_i(t) = (\tilde{c}_i(t), f_i^p(\tilde{c}_i(t)))\) and the normal curvature of \(M_i\) in \(q\) along \(v\) is

\[
k_i^v(q) = \left( f_i^p(0) \right)' / \left( 1 + \left( (f_i^p)'(0) \right)^2 \right)^{3/2},
\]

where, e.g., \((f_i^p)'(t)\) means the derivative in \(t\) of \(f_i^p(\tilde{c}_i(t)) = f_i^p(t)\). It follows that we have a uniform estimate for second derivatives in any direction \(v\). By a standard procedure (see
interchangeable, a subsequence of $g$. By the theorem of Arzelá-Ascoli and the fact that, in this case, limits and derivatives are

By using the Cantor diagonal process, we obtain a sequence $M$ is uniform in the compact subsets of $\mathbb{C}$. Clearly $M_{\infty}$ has no boundary point in the interior of $D$. Thus $M_{\infty}$ extends to the boundary of $D$. Since the local convergence is uniform in compact subsets, it follows that the convergence to $M_{\infty}$ is uniform in the compact subsets of $M_{\infty}$. This completes the proof of (i) of Lemma 2.1.

Now we prove (ii) of Lemma 2.1. By (i), a subsequence of $M_i$ converges $C^1$ to a collection of hypersurfaces, $M_{\infty}$. As in the proof of (i), given $p \in M_{\infty}$, we can look upon the part of $M_i$ near $p$, for large $i$, as a graph of a function $g_i^p$ over $U_{\delta}^i(p) \subset T_p M_{\infty}$. The functions $g_i^p$ converge $C^1$ to the function $g^p$ that defines $M_{\infty}$ near $p$.

Let $G_i^p$ be the metric of $M_i$ restricted to $g_i^p(U_{\delta}^i(p))$, $G_{\infty}^p$ be the metric of $M_{\infty}$ restricted to $g^p(U_{\delta}^p(p))$ and let $E$ be the euclidean metric in $T_p M_{\infty}$. Notice that since the convergence $M_i \to M_{\infty}$ is $C^1$, $G_i^p$ converges to $G_{\infty}^p$. There exists a constant $\lambda_i > 0$ such that

$$\frac{1}{\lambda_i} E(X, X) \leq G_i^p(X, X) \leq \lambda_i E(X, X), \quad \text{for all } X \in T_p M_{\infty} \simeq \mathbb{R}^n.$$
Then $dM_i = \sqrt{\det(G)}dV \geq (\frac{1}{\lambda_i})^{n/2}dV$, where $dV$ is element of volume of $(T_pM_\infty, E) \simeq \mathbb{R}^n$. We obtain

$$\left( \int_{g_i^p(U_{\delta/2}(p))} |A|^q \alpha_i \, dM \right)^{1/q} + \left( \int_{g_i^p(U_{\delta/2}(p))} |\nabla|A|^{q} \beta_i \, dM \right)^{1/q} \geq \kappa(\frac{1}{\lambda_i})^{n/2} \left( \int_{U_{\delta/2}(p)} |A|^q \, dV \right)^{1/q} + \kappa(\frac{1}{\lambda_i})(n+q)/2 \left( \int_{U_{\delta/2}(p)} |\nabla|A|^{q} \, dV \right)^{1/q}$$

Since

$$\left( \int_{g_i^p(U_{\delta/2}(p))} |A|^q \alpha_i \, dM \right)^{1/q} + \left( \int_{g_i^p(U_{\delta/2}(p))} |\nabla|A|^{q} \beta_i \, dM \right)^{1/q} \to 0$$

we conclude that $|A_i| \to 0$ in the usual Sobolev space $W^{1,q}(U_{\delta/2}(p))$. Now, since $q > n$, it follows from the fact that the injection

$$W^{1,q}(U_{\delta/2}(p)) \hookrightarrow C^0(U_{\delta/2}(p), \mathbb{R})$$

is compact (see, for instance, [1], page 168) that a subsequence of $(|A_i|)_i$ (again denoted by $(|A_i|)_i$) converges to zero in $|||\cdot|||_{C^0}$.

Finally, we prove that $M_\infty$ is a collection of hyperplanes by using the fact that $|A_i| \to 0$ everywhere. Since we have not proved that the convergence is $C^2$, this is not immediate. An argument is as follows. Let $p \in M_\infty$ and again look at the part of $M_i$ near $p$ as a graph of a function $g_i^p$ over $U_{\delta/2}(p) \subset T_pM_\infty$ so that, as before, $g_i^p$ converges $C^1$ to $g^p$ that defines $M_\infty$ near $p$. Let $q \in U_{\delta/2}(p)$ and $w \in \mathbb{R}^n$, $|w| = 1$. Set $r(t) = q + tw \subset U_{\delta/2}(p)$, $c_i(t) = (r(t), g_i^p(r(t)))$ and $c(t) = (r(t), g^p(r(t)))$. The fact that $|A_i| \to 0$ is easily seen to imply that $(g_i^p)'(t) \to 0$ in $U_{\delta/2}(p)$ (See (2.1)).

We will prove that $M_\infty$ is a hyperplane over $U_{\delta/2}(p)$; since $p$ is arbitrary, this will yield the result. Since we have a bound for the second derivatives of $g_i^p$ in $U_\delta(p)$, we can use the Dominated Convergence Theorem and the fact that $(g_i^p)'(t) \to (g^p)'(t)$ to obtain

$$(g^p)'(t) - (g_i^p)'(0) = \lim_{j \to \infty} \{(g_i^p)'(t) - (g_i^p)'(0)\}$$

$$= \lim_{j \to \infty} \int_0^t (g_i^p)''(s) \, ds = \int_0^t \lim_{j \to \infty} (g_i^p)''(s) \, ds = 0,$$

Thus, $c(t)$ is a straight line and, since $w$ is arbitrary, $M_\infty$ is a hyperplane over $U_\delta(p)$, as we asserted. This concludes the proof of Lemma 2.1. \qed
Remark. For future use, we observe that in the proof that \( M_\infty \) is a hyperplane we only use that the convergence is \( C^1 \), that we have a bound for the second derivatives of \( g_i^p \) and that \( |A_i| \to 0 \) everywhere.

The proof of the following Proposition is inspired by that of [2], Proposition 2.2; for completeness, we present it here. Actually, the crucial point of the proof (Lemma 2.3 below), is also similar to the proof of Proposition 2 in Choi-Schoen [10].

**Proposition 2.2.** Let \( \phi: M^n \to \mathbb{R}^{n+1} \) be a complete immersion with strong finite total curvature. Then, given \( \varepsilon > 0 \) there exists \( R_0 > 0 \) such that, for \( r > R_0 \),

\[
\frac{r^2}{r^2 \sup_{x \in M - D_0(r)} |A|^2(x)} < \varepsilon.
\]

For the two lemmas below we use the following notation. We denote by \( h: X^n \to \mathbb{R}^{n+1} \) an immersion into \( \mathbb{R}^{n+1} \) of an \( n \)-manifold \( X^n = X \) with boundary \( \partial X \) such that there exists a point \( x \in X \) with \( D_{x}(1) \cap \partial X = \emptyset \).

**Lemma 2.3.** There exists \( \delta > 0 \) such that if

\[
\left( \int_{D_{x_1}(1)} |A|^q \mu \, dX \right)^{1/q} + \left( \int_{D_{x_1}(1)} |\nabla |A||^q \nu \, dX \right)^{1/q} < \delta,
\]

for any \( h: X^n \to \mathbb{R}^{n+1} \) as above and for any pair of continuous functions \( \mu, \nu: D_{x_1}(1) \to \mathbb{R} \) that satisfy \( \inf_{D_{x_1}(1)} \{\mu, \nu\} > c > 0 \), then

\[
\sup_{t \in [0,1]} \left[ t^2 \sup_{D_{x_1}(1-t)} |A_{h}|^2 \right] \leq 4.
\]

Here \( A_{h} \) is the linear map associated to the second fundamental of \( h \).

**Proof.** Suppose the lemma is false. Then there exist a sequence \( h_i: X_i \to \mathbb{R}^{n+1} \), a sequence of points \( x_i \in X_i \) with \( D_{x_i}(1) \cap \partial X_i = \emptyset \) and sequences \( (\mu_i)_i, (\nu_i)_i \), with \( \inf_{D_{x_i}(1)} \{\mu_i, \nu_i\} > c > 0 \) such that

\[
\left( \int_{D_{x_i}(1)} |A_i|^q \mu_i \, dX_i \right)^{1/q} + \left( \int_{D_{x_i}(1)} |\nabla |A_i||^q \nu_i \, dX_i \right)^{1/q} \to 0
\]

but

\[
\sup_{t \in [0,1]} \left[ t^2 \sup_{D_{x_i}(1-t)} |A_i|^2 \right] > 4,
\]

\[
8
\]
for all \(i\), where \(A_i = A_{h_i}\).

Choose \(t_i \in [0, 1]\) so that

\[
\frac{t_i^2}{t_i} \sup_{D_{x_i}(1-t_i)} |A_i|^2 = \sup_{t \in [0, 1]} \left[ \frac{t^2}{t} \sup_{D_{x_i}(1-t)} |A_i|^2 \right]
\]

and choose \(y_i \in D_{x_i}(1-t_i)\) so that

\[
|A_i|^2(y_i) = \sup_{D_{x_i}(1-t_i)} |A_i|^2.
\]

By using that \(D_{y_i}(t_i/2) \subset D_{x_i}(1-(t_i/2))\) we obtain

\[
\sup_{D_{y_i}(t_i/2)} |A_i|^2 \leq \sup_{D_{x_i}(1-(t_i/2))} |A_i|^2 \leq \frac{t_i^2}{t_i^2/4} \sup_{D_{x_i}(1-t_i)} |A_i|^2,
\]

hence, by the choice of \(y_i\), we have

\[
\sup_{D_{y_i}(t_i/2)} |A_i|^2 \leq 4|A_i|^2(y_i).
\]

(2.2)

We now rescale the metric defining \(ds_i^2 = |A_i|^2(y_i)\)\(ds_i^2\), that is, \(ds_i^2\) is the metric on \(X_i\) induced by \(\tilde{h}_i = d_i \circ h_i\), where \(d_i\) is the dilation of \(\mathbb{R}^{n+1}\) about \(h_i(y_i)\) (by translation, we may assume that \(h_i(y_i) = 0\)) by the factor \(|A_i|(y_i)\). The symbol \(\sim\) will indicate quantities measured with respect to the new metric \(ds_i^2\).

By assumption, \(|A_i|^2(y_i) > 4/t_i^2\). Thus

\[
\tilde{D}_{y_i}(1) = D_{y_i}([|A_i|(y_i)]^{-1}) \subset D_{y_i}(t_i/2) \subset D_{x_i}(1-t_i/2) \subset D_{x_i}(1).
\]

It follows that \(\tilde{D}_{y_i}(1) \cap \partial X_i = \emptyset\). Now, we use (2.2) and the fact that

\[
|\tilde{A}_i|(p) = [|A_i|(y_i)]^{-1}|A_i|(p)
\]

to obtain

\[
\sup_{\tilde{D}_{y_i}(1)} |\tilde{A}_i|^2 \leq 4.
\]

Therefore, the sequence \(\tilde{h}_i = \tilde{D}_{y_i}(1) \rightarrow \mathbb{R}^{n+1}\), \(\tilde{h}_i(y_i) = 0\), is a sequence of immersions with uniformly bounded second fundamental form.

By using that \(\tilde{D}_{y_i}(1) = D_{y_i}([|A_i|(y_i)]^{-1}) \subset D_{x_i}(1)\) we have

\[
\left(\int_{D_{x_i}(1)} |A_i|^q \mu_i \, dX_i\right)^{1/q} + \left(\int_{D_{x_i}(1)} |\nabla A_i|^q \nu_i \, dX_i\right)^{1/q} \geq \left(\int_{D_{y_i}([|A_i|(y_i)]^{-1})} |A_i|^q \mu_i \, dX_i\right)^{1/q} + \left(\int_{D_{y_i}([|A_i|(y_i)]^{-1})} |\nabla A_i|^q \nu_i \, dX_i\right)^{1/q}.
\]
Thus, we obtain
\[
\left( \left( \int_{\tilde{D}_n} |A_i|^q \mu_i^t A_i(y_i)^{q-n} d\tilde{X}_i \right)^{1/q} + \left( \int_{\tilde{D}_n} |A_i| |\nabla A_i|^q \nu_i A_i(y_i)^{2q-n} d\tilde{X}_i \right)^{1/q} \right) \leq \left( \left( \int_{D_n} |A_i|^q dX_i \right)^{1/q} \right) \leq 0.
\]

Since \(|A_i(y_i)| > \frac{2}{t_i} \geq 2\) we can use Lemma 2.1, with \(\alpha_i = \mu_i|A_i(y_i)|^{q-n}, \beta_i = \nu_i|A_i(y_i)|^{2q-n}\) and \(\kappa = 2c\), to conclude that a subsequence of \(|\tilde{A}_i|\) converges to zero. But \(|\tilde{A}_i|(y_i) = 1\), for all \(i\), hence \(|\tilde{A}_\infty|(y_\infty) = 1\). This is a contradiction, and completes the proof of Lemma 2.3.

**Lemma 2.4.** Given \(\varepsilon_1 > 0\), there exists \(\delta > 0\), such that if
\[
\left( \int_{D_n} |A|^q \mu^t dX \right)^{1/q} + \left( \int_{D_n} |\nabla A|^q \nu^t dX \right)^{1/q} < \delta,
\]
for any \(h: X^n \to \mathbb{R}^{n+1}\) as above and for any pair of continuous functions \(\mu, \nu: D_{x_i}(1) \to \mathbb{R}\) that satisfy \(\inf_{D_{x_i}(1)} \{\mu, \nu\} > c > 0\), then
\[
\sup_{D_{x_i}(1/2)} |A_h|^2 < \varepsilon_1.
\]

**Proof.** Suppose the lemma is false. Then there exist a sequence \(h_i: X_i \to \mathbb{R}^{n+1}\), a sequence of points \(x_i \in X_i\) with \(D_{x_i}(1) \cap \partial X_i = \emptyset\) and sequences \((\mu_i), (\nu_i)_i\), with \(\inf_{D_{x_i}(1)} \{\mu_i, \nu_i\} > c\) such that
\[
\left( \left( \int_{D_{x_i}(1)} |A|^q \mu_i^t dX_i \right)^{1/q} + \left( \int_{D_{x_i}(1)} |\nabla A|^q \nu_i dX_i \right)^{1/q} \right) \to 0 \quad (2.3)
\]
but
\[
\sup_{D_{x_i}(1/2)} |A_i|^2 \geq K^2, \quad (2.4)
\]
for some constant \(K\).

By Lemma 2.3 (with \(t = 1/2\)), we have, for \(i\) sufficiently large,
\[
\sup_{D_{x_i}(1/2)} |A_i|^2 \leq 16.
\]

By (2.3) and Lemma 2.1, a subsequence of \(|A_i|\) converges to zero. This is a contradiction to (2.4) and proves Lemma 2.4.
Proof of Proposition 2.2. We first rescale the immersion $\phi$ to $\tilde{\phi} = d_{2/r} \circ \phi$, where $d_{2/r}$ is the dilation by the factor $2/r$. Thus the metric induced by $\tilde{x}$ in $M$ is $d\tilde{s}^2 = (4/r^2)ds^2$, where $ds^2$ is the metric induced by $\phi$. We will denote the quantities measured relative to the new metric by the superscript $\sim$. Notice that the second fundamental form $\tilde{A}$ satisfies $|\tilde{A}|^2 = \frac{r^2}{4} |A|^2$.

Therefore, Proposition 2.2 will be established once we prove that given $\epsilon > 0$ there exists $R_0$ such that, for $r > R_0$,

$$\sup_{M - \tilde{D}_0(2)} |\tilde{A}|^2 < \epsilon/4.$$ 

Given the above $\epsilon$, set $\epsilon_1 < \epsilon/4$ and let $\delta > 0$ be given by Lemma 2.4. Since $M$ has strong finite total curvature, there exists $R_0$ such that, for $r > R_0$,

$$\delta > \left( \int_{D_0(r/2, \infty)} |A|^q |\rho_0|^{q-n} \, dM \right)^{1/q} + \left( \int_{D_0(r/2, \infty)} |\nabla|A||^q |\rho_0|^{2q-n} \, dM \right)^{1/q}$$

$$= \left( \int_{\tilde{D}_0(1, \infty)} |\tilde{A}|^q |\tilde{\rho}_0|^{q-n} \, d\tilde{M} \right)^{1/q} + \left( \int_{\tilde{D}_0(1, \infty)} |\nabla|\tilde{A}||^q |\tilde{\rho}_0|^{2q-n} \, d\tilde{M} \right)^{1/q}$$

For $x \in M - \tilde{D}_0(2)$, we have $\tilde{D}_x(1) \subset \tilde{D}_0(1, \infty)$ and then $\inf_{\tilde{D}_x(1)} \tilde{\rho}_0 > 1$. Now, Lemma 2.4 with $\mu = |\tilde{\rho}_0|^{q-n}$ and $\nu = |\tilde{\rho}_0|^{2q-n}$, and the above inequality imply that

$$\sup_{\tilde{D}_x(1/2)} |\tilde{A}|^2 < \epsilon_1,$$

hence

$$\sup_{M - \tilde{D}_0(2)} |\tilde{A}|^2 \leq \epsilon_1 < \epsilon/4.$$ 

This completes the proof of Proposition 2.2.

3 Uniqueness of the tangent plane at infinity

The proof of our Theorem 1.1 depend on a series of lemmas and a crucial proposition to be presented in a while. In this section, $\phi: M^n \to \mathbb{R}^{n+1}$ will always denote a complete hypersurface such that $\phi(M^n)$ passes through the origin 0 of $\mathbb{R}^{n+1}$, with strong finite total curvature.

The following lemma is similar to Lemma 2.3 in Anderson [2].
Lemma 3.1. Let $\phi : M^n \to \mathbb{R}^{n+1}$ be as above and let $r(p) = d(\phi(p), 0)$, where $p \in M$ and $d$ is the distance in $\mathbb{R}^{n+1}$. Then $\phi$ is proper and the gradient $\nabla r$ of $r$ in $M$ satisfies

$$\lim_{r \to \infty} |\nabla r| = 1.$$ 

In particular, there exists $r_0$ such that if $r > r_0$, $\nabla r \neq 0$, i.e., the function $r$ has no critical points outside the ball $B(r_0)$.

Proof. If the immersion is not proper, we can find a ray $\gamma(s)$ issuing from 0 and parametrized by the arc length $s$ such that as $s$ goes to infinity the distance $r(\gamma(s))$ is bounded. Let such a ray be given and set $T = \gamma'(s)$. Let

$$X = (1/2)\nabla r^2 = r \nabla r,$$

be the position vector field, where $\nabla r$ is the gradient of $r$ in $\mathbb{R}^{n+1}$. Then

$$T\langle X, T \rangle = \langle \nabla_T X, T \rangle + \langle X, \nabla_T T \rangle = 1 + \langle X, \nabla_T T \rangle.$$ 

Since $\gamma$ is a geodesic in $M$, the tangent component of $\nabla_T T$ vanishes and

$$\nabla_T T = \langle \nabla_T T, N \rangle N = -\langle \nabla_T N, T \rangle N = \langle A(T), T \rangle N.$$ 

It follows, by Cauchy-Schwarz inequality, that

$$|\langle X, \nabla_T T \rangle| \leq |X| |A(T)| |T| \leq |X| |A|,$$

hence

$$T\langle X, T \rangle \geq 1 - |X| |A|.$$ 

By using Proposition 2.2 with $\varepsilon = 1/m^2$, and the facts that $r = |X(s)| \leq s$ and that $\gamma$ is a minimizing geodesic, we obtain

$$T\langle X, T \rangle(s) \geq 1 - \frac{1}{m}, \quad (3.1)$$ 

for all $s > R_0$, where $R_0$ is given by Proposition 2.2. Integration of (3.1) from $R_0$ to $s$ gives

$$\langle X, T \rangle(s) \geq \left(1 - \frac{1}{m}\right)(s - R_0) + \langle X, T \rangle(R_0). \quad (3.2)$$

Because $r(s) = |X(s)| \geq \langle X, T \rangle(s)$, we see from (3.2) that $r$ goes to infinity with $s$. This is a contradiction and proves that $M$ is properly immersed.
Now let \( \{p_i\} \) be a sequence of points in \( M \) such that \( \{r(p_i)\} \to \infty \). Let \( \gamma_i \) be a minimizing geodesic from 0 to \( p_i \), and denote again by \( \gamma(s) \) the ray which is the limit of \( \{\gamma_i\} \). For each \( \gamma_i \), we apply the above computation, and since

\[
\langle X_i, T_i \rangle(s) = \langle r_i \nabla r_i, T_i \rangle(s) \leq r_i |\nabla r_i|(s),
\]

we have

\[
|\nabla r_i|(s) \geq \frac{\langle X_i, T_i \rangle(s)}{s} \geq \left(1 - \frac{1}{m}\right) \left(\frac{s - R_0}{s}\right) + \frac{\langle X_i, T_i \rangle(R_0)}{s},
\]

hence, for the ray \( \gamma(s) \),

\[
|\nabla r|(s) \geq \left(1 - \frac{1}{m}\right) \left(\frac{s - R_0}{s}\right) + \frac{\langle X, T \rangle(R_0)}{s}. \tag{3.3}
\]

By taking the limit in (3.3) as \( s \to \infty \), we obtain that \( \lim_{s \to \infty} |\nabla r| \geq 1 - \frac{1}{m} \). Since \( m \) and the sequence \( \{p_i\} \) are arbitrary, and \( |\nabla r| \leq 1 \), we conclude that \( \lim_{r \to \infty} |\nabla r| = 1 \), and this completes the proof of Lemma 3.1.

**Remark.** Related to Lemma 3.1, Bessa, Jorge and Montenegro \[5\] proved independently that for an immersion \( \phi: M^n \to \mathbb{R}^N \) (of arbitrary codimension) for which the norm \( |\alpha| \) of the second fundamental form \( \alpha \) satisfies

\[
\lim_{r \to \infty} \sup_{p \in M - D_0(r)} r^2 |\alpha|^2 < 1
\]

it holds that \( \phi \) is proper and that the distance function \( r = d(\phi(p), 0), p \in M \), has no critical point outside a certain ball.

Now, let \( r_0 \) be chosen so that \( r \) has no critical points in \( W = \phi(M) - (B(r_0) \cap \phi(M)) \). By Morse Theory, \( x^{-1}(W) \) is homeomorphic to \( \phi^{-1}[\phi(M) \cap S(r_0)] \times [0, \infty) \). Let \( V \) be a connected component of \( \phi^{-1}(W) \), to be called an *end* of \( M \). It follows that \( M \) has only a finite number of ends. In what follows, we identify \( V \) and \( \phi(V) \).

Let \( r > r_0 \) and set

\[
\Sigma_r = \frac{1}{r} [V \cap S(r)] \subset S(1),
\]

\[
V_r = \frac{1}{r} [V \cap B(r)] \subset B(1).
\]

Denote by \( A_r \) the second fundamental form of \( V_r \). Then

\[
|A_r|^2(x) = r^2 |A|^2(rx).
\]
Lemma 3.2. For \( r > r_0 \), \( V \cap B(r) \) is connected.

Proof. Notice that \( V = S \times [0, \infty) \) where \( S \) is a connected component of \( M \cap S(r_0) \).
Assume that \( V \cap B(r) \) has two connected components, \( V_1 \) and \( V_2 \). Since \((V_1 \cup V_2) \cap S(r_0)\) is connected, either \( V_1 \cap S(r_0) \) or \( V_2 \cap S(r_0) \) is empty. Assume it is \( V_2 \cap S(r_0) \).

Let \( p \in V_2 \). Since all the trajectories of \( \nabla r \) start from \( V_1 \cap S(r_0) \), there exists a trajectory \( \varphi(t) \) with \( \varphi(0) \in V_1 \cap S(r_0) \) and \( \varphi(t_2) = p \). Thus, there exist \( t_0, t_1 \in [0, t_2] \), such that a trajectory of \( \nabla r \) satisfies \( |\varphi(t_0)| = |\varphi(t_1)| = r \). We claim that this implies the existence of a critical point of \( r \) at some point of \( \varphi(t) \).

Indeed, let \( f(t) = r(\varphi(t)) \). Then \( f : \mathbb{R} \to \mathbb{R} \) is a smooth function with \( f(t_0) = f(t_1) \). Thus, there exists \( \bar{t} \in [t_0, t_1] \) with \( f'\left(\bar{t}\right) = 0 \).

Therefore, \( 0 = f'(\bar{t}) = |\nabla r(\bar{t})|^2 \)

and this proves our claim.

Thus we have reached a contradiction and this proves the lemma.

Lemma 3.3. Let \( 0 < \delta < 1 \) be given and fix a ring \( A(\delta, 1) \subset B(1) \). Then, given \( \varepsilon > 0 \), there exists \( r_1 \) such that, for all \( r > r_1 \) and all \( x \in V_r \cap A(\delta, 1) \), we have

\[
|A_r|^2(x) < \varepsilon.
\]

Proof. By Proposition 2.2 there exists \( r_0 \) such that for \( r > r_0 \)

\[
r^2 \sup_{x \in M - D_0(r)} |A|^2(x) < \delta^2 \varepsilon.
\]

(3.4)

Take \( r_1 = r_0 / \delta \). Then, for \( r > r_1 \) and \( x \in V_r \cap A(\delta, 1) \),

\[
r|x| > r\delta > r_0.
\]

Thus, by (3.4), for all \( x \in V_r \cap A(\delta, 1) \) and \( r > r_1 \),

\[
r^2|x|^2 \left[ \sup_{y \in M - D_0(r|x|)} |A|^2(y) \right] < \delta^2 \varepsilon.
\]

(3.5)

Now, by using again Proposition 2.2 and (3.5), we obtain that

\[
|A_r|^2(x) = r^2|A|^2(rx) \leq r^2 \sup_{y \in M - D_0(r|x|)} |A|^2(y) < \frac{\delta^2 \varepsilon}{|x|^2} < \varepsilon,
\]

for all \( x \in V_r \cap A(\delta, 1) \) and \( r > r_1 \), and this proves Lemma 3.3. 

\[\square\]
By Lemma 3.3, we see that \(|A_r|^2 \to 0\) uniformly in the ring \(A(\delta, 1)\). It follows from this and the fact that \(V_r\) is connected that we can apply Lemma 2.1(i) and conclude that a subsequence \(V_{r_i}\) of \(V_r\), \(r_i \to \infty\), converges \(C^1\) to a union of hypersurfaces \(\pi\) in \(A(\delta, 1)\). Again, since \(|A_r| \to 0\) uniformly, \(\pi\) is a union of \(n\)-planes in \(A(\delta, 1)\) (see Remark after the proof of Lemma 2.1). Since \(\delta\) is arbitrary, a subsequence again denoted by \(V_{r_i}\) converges to \(\pi\) in \(B(1) - \{0\}\) and the \(n\)-planes in \(\pi\) all pass through the origin \(0\). Thus, each two of them intersect along a linear \((n-1)\)-subspace \(L\) and the hypersurfaces \(\Sigma_{r_i} \subset S^n(1)\), given by the inverse images of the regular values \(r_i\) of the distance function \(r\), converge to a family \(\Sigma_\infty\) of equators of \(S^n(1)\) each two of which intersect along \(L \cup S^n(1)\). We claim that \(\Sigma_\infty\) contains only one equator. In fact, for \(r_i\) large enough, by the basic transversality theorem ([12] Chapter 3, Theorem 2.1), \(\Sigma_{r_i}\) has a self intersection close to \(L \cup S^n(1)\) and this contradicts the fact that \(\Sigma_{r_i}\) is an embedded hypersurface. It follows that \(\pi\) is a single \(n\)-plane passing through \(0\), possibly with multiplicity \(m \geq 1\). Since \(\Sigma_\infty\) covers \(S^n(1)\), which is simply-connected, \(m = 1\). Thus \(V\) is embedded and \(\pi\) is a single plane that passes through the origin.

The \(n\)-plane \(\pi\) spanned by \(\Sigma_\infty\) is called the tangent plane at infinity of the end \(V\) associated to the sequence \(\{r_i\}\). A crucial point in the proof of Theorem 1.1 is to show that this plane does not depend on the sequence \(\{r_i\}\). Here we use for the first time the hypothesis on \(H_n\).

**Proposition 3.4.** Each end \(V\) of \(M\) has a unique tangent plane at infinity.

**Proof.** Suppose that \(\{s_i\}\) and \(\{r_i\}\), \(s_i, r_i \to \infty\), are sequences of real numbers and that \(\pi_1\) and \(\pi_2\) are distinct tangent planes at infinity associated to \(\{s_i\}\) and \(\{r_i\}\), respectively. We can assume that the sequences satisfy

\[
s_1 < r_1 < s_2 < r_2 < \cdots < s_i < r_i < \ldots.
\]

Let \(K\) be the closure of \(B(3/4) - B(1/4)\) and let \(N_1\) be the normal to \(\pi_1\), obtained as the limit of the normals to

\[
K \cap \left\{ \frac{1}{s_i} V \right\} \to \frac{1}{s_i}(V \cap s_i K).
\]

Similarly, let \(N_2\) be the normal to \(\pi_2\) obtained as the limit of the normals to \(K \cap \{(1/r_i)V\}\).

Now let \(U_1\) and \(U_2\) be neighborhoods in \(S^n(1)\) of \(N_1\) and \(N_2\), respectively, such that \(U_1 \cap U_2 = \emptyset\). Thus, there exists an index \(i_0\) such that, for \(i > i_0\), the normals to \(K^1_i = (s_i K) \cap V\) are in \(U_1\) and the normals to \(K^2_i = (r_i K) \cap V\) are in \(U_2\). If \(K^1_i \cap K^2_i \neq \emptyset\), for some \(i > i_0\), this contradicts the fact that \(U_1 \cap U_2 = \emptyset\), and the proposition is proved.

Thus we may assume that, for all \(i > i_0\), \(K^1_i \cap K^2_i = \emptyset\). In this case, we have
(1/4)r_i > (3/4)s_i; here, and in what follows, we always assume i > i_0. Set

\[ W_i = V \cap \left( B \left( \frac{1}{4} r_i \right) - B \left( \frac{3}{4} s_i \right) \right). \]

Since \( H_n \) does not change sign in \( V \), we have that (3.6, Theorem II) \( g(\partial W_i) \supset \partial(g(W_i)). \)

Since \( g \left( S \left( \frac{1}{4} r_i \right) \cap V \right) \subset U_2, \)
\[ g \left( S \left( \frac{3}{4} s_i \right) \cap V \right) \subset U_1, \]
we have \( g(\partial W_i) \subset U_1 \cup U_2. \) Thus

\[ \partial(g(W_i)) \subset g(\partial W_i) \subset U_1 \cup U_2 . \quad (3.6) \]

We claim that there exists a point \( x \in \text{Int}(W_i) \) with \( H_n(x) \neq 0. \) Suppose that

\[ \{ x \in \text{Int} W_i ; H_n(x) \neq 0 \} = \emptyset. \quad (3.7) \]

Since \( g(W_i) \) is connected and has nonvoid intersection with \( U_1 \) and \( U_2 \) which are disjoint, there is a point \( x_0 \in \text{Int} W_i \) such that \( g(x_0) \notin U_1 \cup U_2. \) Let \( \text{rank} A(x_0) = m. \) By (3.7), \( m < n. \) Since the \( k_i \)'s are continuous, there is a neighborhood \( V \) of \( x_0 \) such that if \( x \in V, \)

\[ n > \text{rank} A(x) \geq m, \]
where the left hand inequality follows from (3.7). This implies that either \( \text{rank} A \) is constant and equal to \( m \) in a neighborhood of \( x_0 \) or in each neighborhood of \( x_0 \) there is a point such that the rank of \( A \) at this point is greater than \( m. \) In view of (3.7), the latter implies that we can find such a point, to be called \( y_0, \) so that about \( y_0 \) there is a neighborhood with rank \( A = m_0 > m. \)

In both cases, we obtain a point and a neighborhood of this point for which rank \( A \) is constant. Without loss of generality, we can assume this point to be \( y_0. \) Notice that we can assume \( g(y_0) \notin U_1 \cup U_2. \) By the Lemma of Chern-Lashof (9, Lemma 2), there passes through \( y_0 \) a piece \( L^p \) of a \( p \)-dimensional plane, \( p = n - m_0, \) along which \( g \) is constant. If \( L^p \) intersects \( \partial W_i, \) \( g(y_0) \in g(\partial W_i) \subset U_1 \cup U_2, \) and this contradicts the choice of \( y_0. \) If not, a point \( \bar{y}_0 \) in \( \partial L^p \) has again rank \( A = m_0 \) (9, Lemma 2), and arbitrarily close to \( \bar{y}_0, \) we have a point \( y_1 \) and a neighborhood of \( y_1 \) whose rank is \( m_1 > m_0. \) Thus, we can repeat the process.

After a finite number of steps, the process will lead either to finding a point with rank \( A = n, \) what contradicts (3.7), or to finding a piece \( L \) of a plane of appropriate dimension
with the property that \( L \cap \partial W_i \neq \emptyset \). As we have seen above, this is again a contradiction and proves our claim.

Thus, we can assume that there is a point \( x \in \text{Int}(W_i) \) with \( H_n(x) \neq 0 \). Then \( g(W_i) \) contains an open set around \( g(x) \). We can assume that \( U_1 \) and \( U_2 \) are small enough so that \( g(x) \notin U_1 \cup U_2 \). Since \( g(W_i) \) is connected and has nonvoid intersection with both \( U_1 \) and \( U_2 \), the fact that there are interior points in \( g(W_i) \) and (3.6) imply that

\[
g(W_i) \supset S^n(1) - \{U_1 \cup U_2\}. \tag{3.8}
\]

On the other hand, because

\[
(\Sigma k_i^2)^q > C k_1^2 \ldots k_n^2,
\]

for a constant \( C = C(n) \), we have that

\[
|H_n| < \frac{1}{\sqrt{C}} |A|^q.
\]

Furthermore, since \( \phi \) has strong finite total curvature,

\[
\int_{W_i} |A|^q |\rho_0|^{n-q} \, dM \to 0, \quad i \to \infty.
\]

Therefore, since

\[
\text{Area } g(W_i) \leq \int_{W_i} |H_n| \, dM < \left( \frac{1}{\sqrt{C}} \right) \int_{W_i} |A|^q |\rho_0|^{n-q} \, dM,
\]

we have that \( \text{Area } g(W_i) \to 0 \). This a contradiction to (3.8), and completes the proof of Proposition 3.4.

4 Proofs of Theorems 1.1, 1.2 and 1.3

Proof of Theorem 1.1 (i) has already been proved in Lemma 3.1. To prove (ii), we apply to each end \( V_i \) the inversion \( I : \mathbb{R}^{n+1} - \{0\} \to \mathbb{R}^{n+1} - \{0\}, I(x) = x/|x|^2 \). Then \( I(V_i) \subset B(1) - B(0) \) and as \( |x| \to \infty \) in \( V_i \), \( I(x) \) converges to the origin 0. It follows that each \( V_i \) can be compactified with a point \( q_i \). Doing this for each \( V_i \), we obtain a compact manifold \( \overline{M} \) such that \( \overline{M} - \{q_1, \ldots, q_k\} \) is diffeomorphic to \( M \). This prove (ii).

To prove (iii), we use again the above inversion and observe that, by Proposition 3.4, as \( |x| \to \infty \) in \( V_i \), the normals at \( I(x) \) converge to a unique normal \( p_i \in S^n_1 \) (namely, to the normal of the unique plane at infinity of \( V_i \)). Thus we obtain a continuous extension \( \overline{g} : \overline{M} \to S^n_1 \) of \( g \) by setting \( \overline{g}(q_i) = p_i \). This proves (iii).
Proof of Theorem 1.2(i). We first observe that $S^n_1 - (N)$ is still simply-connected. This comes from the fact that a closed curve $C$ in $S^n_1 - (N)$ is homotopic to a simple one and a disk generated by such a curve can, by transversality, be made disjoint of $N$ by a small perturbation. Thus $C$ is homotopic to a point in $S^n_1 - (N)$.

Next, the restriction map

$$\tilde{g}: M - \overline{g}^{-1}(N \cup \{p_i\}) \to S^n_1 - (N \cup \{p_i\})$$

where $p_i$ is defined in the proof of Theorem 1.1, is clearly proper and its Jacobian never vanishes. In this situation, it is known that the map is surjective and a covering map ([22], Corollary 1). Since $S^n_1 - (N \cup \{p_i\})$ is simply-connected, $\tilde{g}$ is a global diffeomorphism.

To complete the proof we must show that if $g(n_1) = g(n_2) = p, n_1, n_2 \in \overline{g}^{-1}(N \cup \{p_i\})$ then $n_1 = n_2$. Suppose that $n_1 \neq n_2$. Let $W \subset S^n(1)$ be a neighborhood of $p$. By continuity, there exist disjoint neighborhoods $U_1$ of $n_1$ and $U_2$ of $n_2$ in $\overline{M}$ such that $\overline{g}(U_1) \subset W$ and $\overline{g}(U_2) \subset W$. Choose $t \in \overline{g}(U_1) \cap \overline{g}(U_2)$, $t \notin N \cup \{p_i\}$. Then, there exist $r_1 \in U_1$ and $r_2 \in U_2$ such that $\tilde{g}(r_1) = \tilde{g}(r_2) = t$. But this contradicts the fact that $\tilde{g}$ is a diffeomorphism and concludes the proof of (i).

(ii) We will use a result of Barbosa, Fukuoka and Mercuri [4]. By using Hopf’s theorem that the Euler characteristic $\chi(M)$ of $\overline{M}$ is equal to the sum of the indices of a vector field, the following expression is obtained in [4] Theorem 2.3: if $n$ is even,

$$\chi(M) = \sum_{i=1}^{k} (1 + I(q_i)) + 2d\sigma.$$

Here $I(q_i)$ is the multiplicity of the end $V_i$ (since $n \geq 3$, $I(q_i) = 1$ in our case), $\sigma$ is $\pm 1$ depending on the sign of $H_n$, $k$ is the number of ends and $d$ is the degree of the Gauss map $\overline{g}$. From Theorem 1.2 (i), $\overline{g}$ is a homeomorphism. Thus, $d = 1$ and, since $n$ is even, $\chi(M) = 2$. It follows that

$$2 = 2k + 2\sigma.$$

Thus $k = 2$ and $\sigma = -1$, and the result follows.

Proof of the Gap Theorem. First, we easily compute that

$$|A|^{2n} > (n!)H_n^2.$$

Thus, since $H_n$ is the determinant of the Gauss map $g: M^n \to S^n_1$, we obtain

$$\int_M |A|^n dM > \sqrt{n!} \int_M |H_n|dM = \sqrt{n!} \text{ area of } g(M) \text{ with multiplicity.}$$
The extended map \( \overline{g}: \overline{M} \to S^n_1 \), which is given by Theorem 1.1, has a well defined degree \( d \), hence
\[
\text{area } g(M) = \text{area } \overline{g}(\overline{M}) = d \text{ area } S^n_1.
\]
Now, assume that \( \phi(M) \) is not a hyperplane. We claim that \( d \neq 0 \). To see that, we first show that there exists a point in \( M \) where \( H_n \neq 0 \).

Suppose the contrary holds. Then, since \( \phi(M) \) is not a hyperplane, there is a point \( x_\ell \in M \) such that rank \( A \) at \( x_\ell \) is \( \ell \), \( 0 < \ell < n \). Thus, by using the Lemma of Chern-Lashof (9, Lemma 2) in the same way as we did in Proposition 3.4, we arrive, after a finite number of steps, at one of the two following situations. Either we find a point where \( H_n \neq 0 \), which is a contradiction, or we find an open set \( U_j \subseteq M \), whose points satisfy rank \( A = j \geq \ell \), \( j < n \), foliated by \((n - j)\)-planes the leaves of which extend to infinity. In the second situation, observe that the Gauss map on each leaf is constant and, since there is only one normal at infinity for each end, the normal map is constant on \( U_j \). Thus \( U_j \) is a piece of a hyperplane, and we find again a contradiction, this time to the fact that \( n > j \geq \ell > 0 \).

Therefore, there exists a point \( x_0 \in M \) with \( H_n(x_0) \neq 0 \). Then, for a neighborhood \( V \) of \( x_0 \), we have that \( H_n(x) \neq 0 \), \( x \in V \), and that \( g(V) \subset S^n_1 \) is a neighborhood of \( g(x_0) \). By Sard’s theorem, the set of critical values of \( g \) has measure zero, hence some point of \( g(V) \) is a regular value. It follows that the Gauss map \( g \) has regular values whose inverse images are not empty. Since \( H_n \) does not change sign, this prove our claim.

Furthermore the area \( \sigma_n \) of a unit sphere of \( \mathbb{R}^{n+1} \) is given by
\[
\sigma_n = \frac{2(\sqrt{\pi})^{n+1}}{\Gamma((n+1)/2)};
\]
here \( \Gamma \) is the gamma function, which, in the present case is given by
\[
\Gamma((n+1)/2) = ((n-1)/2)! \text{, if } n \text{ is odd}
\]
\[
\Gamma((n+1)/2) = \frac{(n-1)(n-3)\ldots1}{2^{n/2}}\sqrt{\pi}, \text{ if } n \text{ is even}.
\]

It follows that, for all non-planar \( x \in C^n \),
\[
\int_M |A|^n dM > 2\sqrt{n!}(\sqrt{\pi})^{n+1}/\Gamma((n+1)/2). \]

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