Exploring two non-perturbative definitions of $c_A^{\oplus}$

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We present two determinations of the coefficient $c_A$ in quenched QCD, needed to build the $O(a)$ improved axial current. The first condition used is the requirement that the PCAC quark mass, as a function of $x_0$, stays flat for a non-trivial spatial phase for the fermions in the Schrödinger functional. The second condition is that the PCAC relation for the ground-state and the first excited state at finite $L$ give the same quark mass. Our results confirm previous findings that in the quenched theory the intrinsic $O(a)$ ambiguity of $c_A$ gets relevant around $\beta \approx 6.0$.

1. INTRODUCTION

The improved axial current

$$A_{ij}^I(x) \equiv A_0(x) + a c_A \frac{\partial_0 + \partial_5}{2} P(x)$$  \hspace{1cm} (1)

with $A_0(x) = \bar{\psi}_i \gamma_\mu \gamma_5 \gamma_\nu \psi_j$ and $P(x) = \bar{\psi}_i \gamma_\mu \psi_j$ is designed to modify the scaling behavior of on-shell quantities from $O(a)$ to $O(a^2)$, if $c_A$ and $c_{SW}$ are chosen appropriately. Therefore, an accurate determination of the coefficient $c_A$ is an important ingredient in the improvement programme à la Symanzik. Here, we present results in the quenched setting, but the real motivation is to find a criterion which is practical in a dynamical setting, where large cut-off effects have been found$^1$.

Our data have been generated in the Schrödinger functional (SF) setup: we use a $T \times L^3$ box, applying some Dirichlet type boundary conditions in the time direction (i.e. at $x_0=0, T$) while keeping the gauge-field periodic in space and the fermions periodic up to a phase: $U_\mu(x_0, x+L_\mu) = U_\mu(x)$, $\psi_\mu(x_0, x+L_\mu) = \exp(i\theta)\psi_\mu(x)$. The initial pion with flavor content $ij$ is created at $x_0=0$ through the boundary operator

$$O_{ij} \equiv \frac{a^6}{L^3} \sum_{\mathbf{u}, \mathbf{v}} \bar{\zeta}_i(\mathbf{u}) \gamma_5 \zeta_j(\mathbf{v}) \omega(\mathbf{u}-\mathbf{v})$$  \hspace{1cm} (2)

by the analogous operator $O_{ij}^I$ at $x_0 = T$. This means that we consider the SF correlators

$$f_X(x_0, T, L) = -\frac{L^3}{2} \langle X(x) O \rangle$$  \hspace{1cm} (3)

$$f_1(T, L) = -\frac{1}{2} \langle O' O \rangle$$  \hspace{1cm} (4)

with $X$ either $A_0$ or $P$ to get the PCAC mass $m=r+a c_A s$ with

$$r(x_0) = \frac{1}{2} \frac{\langle \partial_0 + \partial_5 \rangle f_A(x_0)}{2 f_P(x_0)}$$  \hspace{1cm} (5)

$$s(x_0) = \frac{\left( \frac{1}{2} \partial_0 + \partial_5 \right) f_P(x_0)}{2 f_P(x_0)}$$  \hspace{1cm} (6)

Requiring $m$ constant at fixed $\beta$ leads to a definition of the improvement coefficient through $c_A \equiv \Delta r/\Delta s$, where the difference may be w.r.t.

- (0) $\theta$ at fixed $x_0=T/2$ (ALPHA $^2$)
- (1) $x_0$ at fixed $\theta$ (LANL $^3$, “slope criterion”)
- (2) state in $O[\omega]$ (UKQCD $^4$, “gap criterion”)

and depending on the choice, physical quantities differ by $O(a^2)$ effects. This means that there is an intrinsic $O(a)$ ambiguity in $c_A$ itself which, already for $\beta \approx 6$, is not such a small effect $^3$. Here we investigate (in a quenched setting) which one, out of (0)-(2), might be a promising criterion for $N_f=2$, with a view on the following wishlist: (i) no high-energy state involved, i.e. no $x_0<r_0$ (say) used, (ii) large “sensitivity”, i.e. not too small value of $\Delta s$, (iii) affordable numerical effort, i.e. not requiring large volume, (iv) “scalability”, i.e. allowing to move to another $\beta$ while...
keeping physics in units of $r_0$ constant. Both the ambiguity of improvement coefficients and how to deal with it have been discussed in [5].

We emphasize that the SF states generated by the boundary operator [2] are multiplicatively renormalizable. The associate $Z$-factor cancels in the ratios (5, 6), and everything is scalable.

2. THETA CRITERION

The old ALPHA criterion [2] resulted in rather small $\Delta s$ values. Furthermore, for $N_f=2$ several $\theta$-angles mean several simulations. Therefore, we didn’t investigate (0) further.

3. SLOPE CRITERION

The slope criterion (1) requires only one $\theta$-value, but for completeness we decided to test it for $\theta=0, \pi/3, 2\pi/3, \pi$.

Fig. 1 shows $r(x_0)$ and $s(x_0)$ in the SF. For $\theta=\pi$ there is a good sensitivity. Using $\Delta r(x_0) = r(x_0) - r(x_{ref})$ with $x_{ref}$ around the extremum (and ditto for $\Delta s$) the recipe $c_A = \Delta r/\Delta s$ yields a local $c_A(x_0)$. The remnant dependence on $x_{ref}$ was checked to be small. This and the dependence on $\theta, L, T, \kappa, \omega$ represent genuine $O(a)$ effects on $c_A$.

Fig. 2 displays $-c_A(x_0)$ determined via this “slope criterion”. For $\theta=\pi$ and small enough $L$ there is an early plateau which is a sign that all states but the lowest two have disappeared. The corresponding value for $c_A$ seems consistent with the old ALPHA determination [2].

4. GAP CRITERION

From a transfer matrix analysis one gets

$$f_X \approx \frac{L^3}{2} \rho \xi e^{-M_{\pi x_0}} \{1 + \eta_X^0 e^{-\Delta x_0} + \eta_X^0 e^{-M_G(T-x_0)}\}$$

with the matrix element $\xi = \langle 0, 0 | \mathbf{X} | \pi, 0 \rangle$ and known representations of $\rho, \eta_X^0$ in terms of states $|Q, n\rangle$ with a given set of quantum numbers and excitation level. In (7) $\Delta$ denotes the gap in the corresponding (here: pseudoscalar) channel and $M_G$ the mass of the lowest $(0^{+\cdots})$ glueball state. The important point is that the coefficients $\rho, \eta_X^0$ depend on the initial state $|i, \pi\rangle$ and hence on the wave function $\omega$, while $\xi, \eta_X^0$ do not. This creates the possibility to linearly combine the correlators (7) over several $\omega$ to build one which is clearly dominated by either the ground-state or the first excited state. After checking that the
corresponding effective masses are indeed distinct (i.e. $M_\pi < M_\star$), one may define $\Delta r$ as the difference of the expressions $r(x_0)$ w.r.t. these two linear combinations (at a given $x_0, \theta$) and ditto for $\Delta s$. This amounts to an operational definition – time slice by time-slice – of $c_A$ according to the “gap criterion” (2).

Fig. 3 shows $r(x_0)$ and $s(x_0)$ from the ground-state and the first excited state at $\beta = 6.0, \kappa = 0.13415, \theta = 0$. The plateau is reached at $x_0 \simeq 0.75$ fm, and typical values are somewhat smaller than the original ALPHA value at $\beta = 6.0$ [2]. The absolute value $|c_A| = -c_A$ that we get decreases with $L$, but for $L > 3r_0$ this effect seems not particularly pronounced any more. We take this as a sign that there is good hope to determine $c_A$ in a $(1.5 \text{ fm})^4$ box in the $N_f = 2$ theory.

5. SUMMARY

We have tested several improvement conditions that might be used to determine $c_A$ non-perturbatively in an unquenched setting. The $O(a)$ ambiguity of $c_A$ that has been pointed out previously [3,4] has been confirmed via, (a) extension of (1) to several $\theta$ and, (b) refinement of (2) with a wave function projection technique which achieves dominance by ground- or 1st excited state in the region 0.5 fm ... 1.0 fm. We hope this proves sufficient to determine $c_A$ with $N_f = 2$.

REFERENCES

1. R. Sommer, these proceedings.
2. M. Lüscher, S. Sint, R. Sommer, P. Weisz and U. Wolff, Nucl. Phys. B 491, 323 (1997) [hep-lat/9609035].
3. T. Bhattacharya, S. Chandrasekharan, R. Gupta, W. J. Lee and S. R. Sharpe, Phys. Lett. B 461, 79 (1999) [hep-lat/9904011].
4. S. Collins, C. T. Davies, G. P. Lepage and J. Shigemitsu [UKQCD collaboration], Phys. Rev. D 67, 014504 (2003) [hep-lat/0110159].
5. M. Guagnelli, R. Petronzio, J. Rolf, S. Sint, R. Sommer and U. Wolff [ALPHA Collaboration], Nucl. Phys. B 595, 44 (2001) [hep-lat/0009021].