SHORT INTERVALS ASYMPTOTIC FORMULAE FOR BINARY PROBLEMS WITH PRIME POWERS, II

ALESSANDRO LANGUASCO and ALESSANDRO ZACCAGNINI

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Abstract

We improve some results in our paper [A. Languasco and A. Zaccagnini, ‘Short intervals asymptotic formulae for binary problems with prime powers’, J. Théor. Nombres Bordeaux 30 (2018) 609–635] about the asymptotic formulae in short intervals for the average number of representations of integers of the forms $n = p_{1}^{\ell_{1}} + p_{2}^{\ell_{2}}$ and $n = p_{1}^{\ell_{1}} + m^{\ell_{2}}$, where $\ell_{1}, \ell_{2} \geq 2$ are fixed integers, $p, p_{1}, p_{2}$ are prime numbers and $m$ is an integer. We also remark that the techniques here used let us prove that a suitable asymptotic formula for the average number of representations of integers $n = \sum_{i=1}^{s} p_{i}^{\ell}$, where $s, \ell$ are two integers such that $2 \leq s \leq \ell - 1$, $\ell \geq 3$ and $p_{i}, i = 1, \ldots, s$, are prime numbers, holds in short intervals.

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1. Introduction

Let $N$ be a sufficiently large integer and $1 \leq H \leq N$. In our recent papers [5] and [7] we provided suitable asymptotic formulae in short intervals $[N, N + H]$ for the number of representations of an integer $n$ as a sum of a prime and a prime square, as a sum of a prime and a square, as a sum of two prime squares or as a sum of a prime square and a square. To describe these results, we need the following definitions. Let $\ell_{1}, \ell_{2} \geq 1$ be integers,

$$\lambda := 1/\ell_{1} + 1/\ell_{2} \quad \text{and} \quad c(\ell_{1}, \ell_{2}) := \frac{\Gamma(1/\ell_{1})\Gamma(1/\ell_{2})}{\ell_{1}\ell_{2}\Gamma(\lambda)} = c(\ell_{2}, \ell_{1}),$$

(1.1)

where $\Gamma$ is Euler’s function. Using this notation, we can say that our results in [5] and [7] are about $\lambda = 3/2$ and $\lambda = 1$ while here we are interested in the case $\lambda < 1$. We also recall that Suzuki [11, 12] has recently sharpened our results in [7] for the case $\lambda = 3/2$. In [8] we were able to get nontrivial results for the case $\lambda < 1$ but unfortunately in the unconditional case we were not able to address every possible
combination of \( \ell_1, \ell_2 \). The aim of this paper is to remove such limitations, thus getting nontrivial unconditional results for every \( \ell_1, \ell_2 \geq 2 \) such that \( \lambda < 1 \). Moreover, we will also improve a conditional result contained in [8] by extending its uniformity range. Such improvements follow from better estimates of the error term involved in the main-terms treatments and from using a lemma due to Tolev on a truncated mean-square average for the exponential sums over primes; see Lemmas 2.5 and 5.5. We recall here some definitions already given in [8]. Let

\[ A = A(N, d) := \exp \left( d \left( \frac{\log N}{\log \log N} \right)^{1/3} \right), \]

(1.2)

where \( d \) is a real parameter (positive or negative) chosen according to need, and

\[ R''_{\ell_1, \ell_2}(n) = \sum_{p_1^{\ell_1} + p_2^{\ell_2} = n} \log p_1 \log p_2. \]

(1.3)

The general shape of \( A \) depends on the saving over the trivial bound in the unconditional part of Lemma 2.4. In this case, due to the symmetry of the problem, we can assume that \( 2 \leq \ell_1 \leq \ell_2 \). We can now state the following result.

**Theorem 1.1.** Let \( N \geq 2, 1 \leq H \leq N, 2 \leq \ell_1 \leq \ell_2 \) be integers and \( \lambda < 1 \). Then, for every \( \varepsilon > 0 \), there exists \( C = C(\varepsilon) > 0 \) such that

\[ N + H \sum_{n=N+1}^{N+H} R''_{\ell_1, \ell_2}(n) = c(\ell_1, \ell_2)HN^{\lambda - 1} + O_{\ell_1, \ell_2}(HN^{\lambda - 1}A(N, -C(\varepsilon))) \]

uniformly for \( N^{1-5/(6\ell_2)+\varepsilon} \leq H \leq N^{1-\varepsilon} \), where \( \lambda \) and \( c(\ell_1, \ell_2) \) are defined in (1.1).

This should be compared with [8, Theorem 1.1]; here the uniformity on \( H \) is much larger so that Theorem 1.1 is nontrivial for every choice of \( 2 \leq \ell_1 \leq \ell_2 \) with \( \lambda < 1 \). We also remark that the uniformity level for \( H \) in Theorem 1.1 is the expected optimal one given the known density estimates for the nontrivial zeroes of the Riemann zeta function.

Assuming that the Riemann hypothesis (RH) holds, we get a nontrivial result for \( \sum_{n=N+1}^{N+H} R''_{\ell_1, \ell_2}(n) \) uniformly for every \( 2 \leq \ell_1 \leq \ell_2 \) and \( H \) in some range. We use throughout the paper the convenient notation \( f = \infty(g) \) for \( g = o(f) \).

**Theorem 1.2.** Let \( N \geq 2, 1 \leq H \leq N, 2 \leq \ell_1 \leq \ell_2 \) be integers, \( \lambda < 1 \) and assume that the Riemann hypothesis holds. Then

\[ \sum_{n=N+1}^{N+H} R''_{\ell_1, \ell_2}(n) = c(\ell_1, \ell_2)HN^{\lambda - 1} \]

\[ + O_{\ell_1, \ell_2}(H^2N^{\lambda - 2} + H^{1/2}N^{1/(\ell_1+1/(2\ell_2)-1/2)}(\log N)^3) \]

uniformly for \( \infty(N^{1-1/\ell_2}(\log N)^6) \leq H \leq o(N) \), where \( \lambda \) and \( c(\ell_1, \ell_2) \) are defined in (1.1).
This should be compared with [8, Theorem 1.2]. Here the second error term is improved and, as a consequence, the uniformity on $H$ is much larger and essentially optimal given the spacing of the sequences. If $\ell_1 = 2$, the log-power in the final result can be slightly improved by using Lemma 2.6 instead of Lemma 2.5 but, since the improvement is marginal, we did not insert this estimate in the proof of Theorem 1.2.

A slightly different problem is the one in which we replace a prime power with a power. Letting 

$$r'_{\ell_1, \ell_2}(n) = \sum_{\substack{p_1^\ell_1 + m^{\ell_2} = n \\ N/A \leq p_1^\ell_1, m^{\ell_2} \leq N}} \log p,$$

we lose the symmetry in $\ell_1, \ell_2$; hence, we just assume that $\ell_1, \ell_2 \geq 2$. Here we take $A$ as defined in (1.2) with a suitable $d > 0$. We need to change the setting and to use the finite sums, see Section 5, because, with the unique exception of the case $\ell = 2$, we cannot use the infinite series in this problem; this, for $\ell \neq 2$, is due to the lack of a suitable modular relation for the function $\omega_{\ell}(\alpha) = \sum_{m=1}^{\infty} e^{-m/|N|} e(m\alpha), \alpha \in [-1/2, 1/2]$. For technical reasons, in this case we need to localize the summands to get a sufficiently strong estimate in Lemma 5.4 below.

We have the following result.

**Theorem 1.3.** Let $N \geq 2$, $1 \leq H \leq N$, $\ell_1, \ell_2 \geq 2$ be integers and $\lambda < 1$. Then, for every $\varepsilon > 0$, there exists $C = C(\varepsilon) > 0$ such that

$$\sum_{n=H+1}^{N+H} r'_{\ell_1, \ell_2}(n) = c(\ell_1, \ell_2)HN^{\lambda-1} + O_{\ell_1, \ell_2}(HN^{\lambda-1}A(N, -C(\varepsilon)))$$

uniformly for $\max(N^{1-5/(6\ell_1)}, N^{1-1/\ell_2})N^\varepsilon \leq H \leq N^{1-\varepsilon}$, where $c(\ell_1, \ell_2)$ and $\lambda$ are defined in (1.1).

This should be compared with [8, Theorem 1.3]; here the uniformity on $H$ is much larger, so that Theorem 1.3 is nontrivial for every choice of $\ell_1, \ell_2 \geq 2$ with $\lambda < 1$. We also remark that the uniformity level for $H$ in Theorem 1.3 is the expected optimal one given the known density estimates for the nontrivial zeroes of the Riemann zeta function and the spacing of the sequences.

We finally remark that even assuming the Riemann hypothesis we cannot improve the size of the error term in Theorem 1.3 because in the main-term evaluation we have a term of the size $HN^{\lambda-1}A^{-1/\ell_2}$, see (6.6) below; moreover, the magnitude of the error in the approximation in (5.2) is huge in the periphery of the arc, that is, for $\alpha$ ‘near’ $1/2$. So, under the assumption of the Riemann hypothesis, we can improve Theorem 1.3 essentially only for $\ell = 2$ by using the infinite series approach; but this result is already presented in [8].

The basic strategy for all of the proofs of our results is the same. We rewrite the quantity we are studying as a suitable integral of a product of exponential sums. We replace these by simpler approximations and then evaluate the ‘main term’ and estimate the error terms that arise in the approximations by means of the lemmas.
proved in the next section. The drawback of using finite sums instead of infinite series is that the main term has a more complicated shape and its treatment is less straightforward. The main new ingredient, and the reason why we can improve our earlier results, is a consequence of a result due to Tolev [13]: we need the two variants for infinite series and finite sums, which we state as Lemmas 2.5 and 5.5. In the proofs of Theorems 1.1–1.2, we also exploit the stronger error we have in (3.8), whereas in the remaining proofs we use the $L^2$ bound provided by Lemma 5.4 instead of the $L^\infty$ bound.

Using a similar argument, we can also prove the following two results about the number of representations of an integer as a sum of exactly $s$ summands, each one being an $\ell$th prime power. Let

$$r_{s,\ell}(n) = \sum_{\substack{i=1, \ldots, s \leq H(n) \leq N, \ell \geq 3}} \log p_1 \cdots \log p_s,$$

where $s \geq 2$, $\ell \geq 1$ are two integers, $N$ is a sufficiently large integer and $1 \leq H \leq N$ is an integer. The problem of obtaining an asymptotic formula for $r_{s,\ell}(n)$ is usually called the Waring–Goldbach problem. The history about the results on such a problem is a very long one; we refer to the surveys of Vaughan and Wooley [15] and Kumchev and Tolev [3] for an overview.

A simpler problem is to study the order of magnitude for an average of $r_{s,\ell}(n)$ because the averaging procedure lets us gain nontrivial information in cases in which the classical approaches fail, for example when $s \leq 4\ell \log \ell + O(1)$ for $\ell$ large or $s \leq H(\ell)$ for $4 \leq s \leq 10$, where $H(4) \leq 14$, $H(5) \leq 21$, $H(6) \leq 33$, $H(7) \leq 47$, $H(8) \leq 63$, $H(9) \leq 83$, $H(10) \leq 107$, according to [15, page 20]. Recently, Cantarini et al. [1] proved that a suitable asymptotic formula in short intervals holds for $\sum_{n=N+1}^{N+H} r_{s,\ell}(n)$ when $s = \ell + 1$ and $\ell \geq 2$, thus generalizing previous results by us (see [9] ($s = 4, \ell = 3$) and [5] ($s = 2, \ell = 2$)).

Here we restrict our attention to the more difficult case in which we have fewer summands, that is, when $2 \leq s \leq \ell - 1$, $\ell \geq 3$. Our unconditional result is as follows.

**Theorem 1.4.** Let $s$, $\ell$ be two integers such that $2 \leq s \leq \ell - 1$, $\ell \geq 3$, and let $N \geq 2$, $1 \leq H \leq N$ be integers. Then, for every $\varepsilon > 0$, there exists $C = C(\varepsilon) > 0$ such that

$$\sum_{n=N+1}^{N+H} r_{s,\ell}(n) = \frac{\Gamma(1 + 1/\ell)^s}{\Gamma(s/\ell)} H N^{s/\ell - 1} + O_{s,\ell} \left( H N^{s/\ell - 1} \exp \left( -C \left( \frac{\log N}{\log \log N} \right)^{1/3} \right) \right)$$

as $N \to \infty$ uniformly for $N^{1-5/(6\ell)+\varepsilon} \leq H \leq N^{1-\varepsilon}$, where $\Gamma$ is Euler’s function.

As an immediate consequence of Theorem 1.4, we can say that, for $N$ sufficiently large, every interval of size larger than $N^{1-5/(6\ell)+\varepsilon}$ contains the expected amount of integers which are a sum of exactly $s$ summands, $2 \leq s \leq \ell - 1$, each one being an $\ell$th prime power, $\ell \geq 3$. 
We remark that the uniformity level for $H$ in Theorem 1.4 is the expected optimal one given the known density estimates for the nontrivial zeroes of the Riemann zeta function.

Assuming that the Riemann hypothesis holds, we can improve the uniformity range of $H$ since in this case Lemma 2.4 below holds in the whole unit interval for $\xi$.

**Theorem 1.5.** Let $s$, $\ell$ be two integers such that $2 \leq s \leq \ell - 1$, $\ell \geq 3$, let $N \geq 2$, $1 \leq H \leq N$ be integers and assume that the Riemann hypothesis holds. Then

$$
\sum_{n=N+1}^{N+H} r_{s,\ell}(n) = \frac{\Gamma(1 + 1/\ell)^s}{\Gamma(s/\ell)} H N^{s/\ell-1} + O_{s,\ell}(H^2 N^{s/\ell-2} + H^{1/2} N^{s/\ell-1/(2\ell)-1/2} (\log N)^3)
$$

as $N \to \infty$ uniformly for $o((N^{1-1/\ell}(\log N)^6) \leq H \leq o(N)$, where $f = o(g)$ means $g = o(f)$ and $\Gamma$ is Euler’s function.

As an immediate consequence of Theorem 1.5, we can say that, for $N$ sufficiently large, every interval of size larger than $N^{1-1/\ell} + \varepsilon$ contains the expected amount of integers which are a sum of exactly $s$ summands, $2 \leq s \leq \ell - 1$, each one being an $\ell$th prime power, $\ell \geq 3$. We remark that in this case the $H$-level is essentially optimal given the spacing of the sequences.

Since the proofs of Theorems 1.4–1.5 can be obtained following the same argument used in the proofs of Theorems 1.1–1.2, we decided to just state them.

### 2. Setting and lemmas for Theorems 1.1–1.2

Let $\ell, \ell_1, \ell_2 \geq 2$ be integers, $e(\alpha) = e^{2\pi i \alpha}$ and $\alpha \in [-1/2, 1/2]$. For the proof of the first two theorems, it is convenient to use the original Hardy–Littlewood functions because the main-term contribution can be easier evaluated compared with the setting with finite exponential sums. Let

$$
\widetilde{S}_\ell(\alpha) = \sum_{n=1}^{\infty} \Lambda(n) e^{-n^\ell/N} e(n^\ell \alpha), \quad \widetilde{V}_\ell(\alpha) = \sum_{p=2}^{\infty} \log p \ e^{-p^\ell/N} e(p^\ell \alpha)
$$

and

$$
z = 1/N - 2\pi i \alpha.
$$

We now list some results we will use later. The lemmas in this section are mostly bounds for exponential sums of various types. We will use them in Section 3 after the dissection of the unit interval into subintervals where different tools are needed to evaluate the main term and estimate error terms.

**Lemma 2.1** [5, Lemma 3]. Let $\ell \geq 1$ be an integer. Then $|\widetilde{S}_\ell(\alpha) - \widetilde{V}_\ell(\alpha)| \ll \ell N^{1/(2\ell)}$.

**Lemma 2.2** [6, Lemma 2]. Let $\ell \geq 1$ be an integer, $N \geq 2$ and $\alpha \in [-1/2, 1/2]$. Then

$$
\widetilde{S}_\ell(\alpha) = \frac{\Gamma(1/\ell)}{\ell z^{1/\ell}} - \frac{1}{\ell} \sum_{\rho} z^{-p/\ell} \Gamma\left(\frac{p}{\ell}\right) + O_{\ell}(1),
$$

where $\rho = \beta + iy$ runs over the nontrivial zeroes of $\zeta(s)$. 

Lemma 2.3 [6, Lemma 4]. Let $N$ be a positive integer and $\mu > 0$. Then, uniformly for $n \geq 1$ and $X > 0$,

$$\int_{-X}^{X} z^{-\mu} e(-n\alpha) \, d\alpha = e^{-n/N} \frac{n^{\mu-1}}{\Gamma(\mu)} + O\left(\frac{1}{nX^\mu}\right),$$

where $\Gamma$ is Euler’s function.

**Proof.** We remark that the proof is identical to the one of [6, Lemma 4] but in that case we just stated the lemma in the particular case $X = 1/2$. Now we need its full strength and hence, for completeness, we rewrite its proof. We start with the identity

$$\frac{1}{2\pi} \int_{\mathbb{R}} \frac{e^{iDx}}{(a + iu)^{\mu}} \, du = \frac{D^{\mu-1} e^{-aD}}{\Gamma(s)},$$

which is valid for $\sigma = \Re(s) > 0$ and $a \in \mathbb{C}$ with $\Re(a) > 0$ and $D > 0$. Letting $u = -2\pi\alpha$ and taking $s = \mu$, $D = n$ and $a = N^{-1}$,

$$\int_{\mathbb{R}} \frac{e(-n\alpha)}{(N^{-1} - 2\pi\alpha)^{\mu}} \, d\alpha = \int_{\mathbb{R}} z^{-\mu} e(-n\alpha) \, d\alpha = \frac{n^{\mu-1} e^{-n/N}}{\Gamma(\mu)}.$$

For $0 < X < Y$, let

$$I(X, Y) = \int_{X}^{Y} \frac{e^{iDu}}{(a + iu)^{\mu}} \, du.$$

An integration by parts yields

$$I(X, Y) = \left[ \frac{1}{iD} \frac{e^{iDu}}{(a + iu)^{\mu}} \right]_{X}^{Y} + \mu \frac{D}{D} \int_{X}^{Y} \frac{e^{iDu}}{(a + iu)^{\mu+1}} \, du.$$

Since $a > 0$, the first summand is $\ll_{\mu} D^{-1} X^{-\mu}$ uniformly. The second summand is

$$\ll_{\mu} \frac{\mu}{D} \int_{X}^{Y} \frac{du}{u^{\mu+1}} \ll_{\mu} D^{-1} X^{-\mu}.$$

The result follows. \qed

Lemma 2.4 [6, Lemma 3] and [5, Lemma 1]. Let $\varepsilon$ be an arbitrarily small positive constant, $\ell \geq 1$ be an integer, $N$ be a sufficiently large integer and $L = \log N$. Then there exists a positive constant $c_1 = c_1(\varepsilon)$, which does not depend on $\ell$, such that

$$\int_{-\xi}^{\xi} \left| \tilde{S}_{\ell}(\alpha) - \frac{\Gamma(1/\ell)}{\ell z^{1/\ell}} \right|^2 \, d\alpha \ll \xi N^{2/\ell - 1} A(N, -c_1)$$

uniformly for $0 \leq \xi < N^{-1+5/(6\varepsilon) - \varepsilon}$. Assuming the RH,

$$\int_{-\xi}^{\xi} \left| \tilde{S}_{\ell}(\alpha) - \frac{\Gamma(1/\ell)}{\ell z^{1/\ell}} \right|^2 \, d\alpha \ll \xi N^{1/\ell} \xi L^2$$

uniformly for $0 \leq \xi \leq 1/2$. 
Asymptotic formulae for binary problems

Some of the following lemmas hold for a real index \( k \) instead of an integral one \( \ell \); in general, we will always use \( k \) to denote a real index. The new ingredient we are using here is based on a lemma due to Tolev [13].

**Lemma 2.5 (Tolev).** Let \( k > 1, n \in \mathbb{N} \) and \( \tau > 0 \). We have
\[
\int_{-\tau}^{\tau} |\overline{S}_k(\alpha)|^2 \, d\alpha \ll_k (\tau N^{1/k} + N^{(2/k)-1}) L^3
\]
and
\[
\int_{-\tau}^{\tau} |\overline{V}_k(\alpha)|^2 \, d\alpha \ll_k (\tau N^{1/k} + N^{(2/k)-1}) L^3.
\]

**Proof.** We just prove the first part since the second one follows immediately by remarking that the primes are supported on a thinner set than the prime powers. Let \( P = (2NL/k)^{1/k} \). A direct estimate gives
\[
|\overline{S}_k(\alpha)| = \sum_{n \leq P} \Lambda(n) e^{-n^k/N} e(n^k \alpha) + O_k(L^{1/k}).
\]
Recalling that the prime number theorem implies \( \Lambda(n) \approx e(n^k \alpha) \) for \( n \leq t \), a partial integration argument gives
\[
\sum_{n \leq P} \Lambda(n) e^{-n^k/N} e(n^k \alpha) = -\frac{k}{N} \int_1^P t^{k-1} e^{-t^k/N} S_k(\alpha; t) \, dt + O_k(L^{1/k}).
\]
Using the inequality \((|a| + |b|)^2 \approx |a|^2 + |b|^2\), the Cauchy–Schwarz inequality and interchanging the integrals,
\[
\int_{-\tau}^{\tau} |\overline{S}_k(\alpha)|^2 \, d\alpha \ll_k \int_{-\tau}^{\tau} \left| \int_1^P \frac{1}{N} \int_1^P t^{k-1} e^{-t^k/N} S_k(\alpha; t) \, dt \right|^2 \, d\alpha + L^{2/k}
\]
\[
\ll_k \frac{1}{N^2} \left( \int_1^P t^{k-1} e^{-t^k/N} \, dt \right) \left( \int_1^P t^{k-1} e^{-t^k/N} \int_{-\tau}^{\tau} |S_k(\alpha; t)|^2 \, d\alpha \, dt \right) + L^{2/k}.
\]
Lemma 7 of Tolev [13] in the form given in [2, Lemma 5] on \( S_k(\alpha; t) \) implies that
\[
\int_1^P |S_k(\alpha; t)|^2 \, dt \ll_k (\tau t + t^{2-k})(\log t)^3.
\]
Using such an estimate and remarking that
\[
\int_1^P e^{-t^k/N} \, dt \ll_k N,
\]
\[
\int_{-\tau}^{\tau} |\overline{S}_k(\alpha)|^2 \, d\alpha \ll_k \frac{1}{N} \int_1^P (\tau t + t^{2-k})t^{k-1} e^{-t^k/N} (\log t)^3 \, dt + L^{2/k}
\]
\[
\ll_k (\tau N^{1/k} + N^{(2/k)-1}) L^3
\]
by a direct computation. This proves the first part of the lemma. \( \square \)

In the case \( \ell = 2 \) a slightly better final result can be obtained using

**Lemma 2.6 [7, Lemma 2].** Let \( \ell \geq 2 \) be an integer and \( 0 < \tau \leq 1/2 \). Then
\[
\int_{-\tau}^{\tau} |\overline{S}_\ell(\alpha)|^2 \, d\alpha \ll_\ell \tau N^{1/\ell} L + \begin{cases} L^2 & \text{if } \ell = 2, \\ 1 & \text{if } \ell > 2. \end{cases}
\]

The next lemma is a consequence of Lemmas 2.4–2.5.
Lemma 2.7. Let $N \in \mathbb{N}$, $k > 1$, $u \geq 1$ and $N^{-u} \leq \omega \leq N^{(1/k) - 1}/L$. Let further $I(\omega) := \{-1/2, -\omega\} \cup [\omega, 1/2]$. We have
\[
\int_{I(\omega)} |\overline{S}_k(\alpha)|^2 \frac{d\alpha}{|\alpha|} \ll_k \frac{N^{(2k) - 1}}{\omega} L^3 \quad \text{and} \quad \int_{I(\omega)} |\overline{V}_k(\alpha)|^2 \frac{d\alpha}{|\alpha|} \ll_k \frac{N^{(2k) - 1}}{\omega} L^3.
\]
Let further assume the Riemann Hypothesis, $\ell \geq 1$ be an integer and $N^{-u} \leq \eta \leq 1/2$. Then
\[
\int_{I(\eta)} |\overline{S}_\ell(\alpha) - \frac{\Gamma(1/\ell)}{\ell^{1/\ell}}|^2 \frac{d\alpha}{|\alpha|} \ll_{\ell} N^{1/\ell} L^3.
\]
Proof. By partial integration and Lemma 2.5,
\[
\int_\omega^{1/2} \frac{|\overline{S}_k(\alpha)|^2}{\alpha} d\alpha \ll \frac{1}{\omega} \int_0^{1/2} |\overline{S}_k(\alpha)|^2 d\alpha + \int_{-1/2}^{1/2} |\overline{S}_k(\alpha)|^2 d\alpha + \int_\omega^{1/2} \left( \int_{-\xi}^{\xi} |\overline{S}_k(\alpha)|^2 d\alpha \right) d\xi \ll_k \frac{L^3}{\omega} (\omega N^{1/k} + N^{(2k) - 1}) + N^{1/k} L^3 + L^3 \int_\omega^{1/2} \frac{N^{1/k} + N^{(2k) - 1}}{\xi^2} d\xi \ll_k N^{1/k} L^3 |\log(2\omega)| + \frac{N^{(2k) - 1}}{\omega} L^3 \ll_k \frac{N^{(2k) - 1}}{\omega} L^3
\]
since $N^{-u} \leq \omega \leq N^{(1/k) - 1}/L$. A similar computation proves the result in $[-1/2, -\omega]$ too. The estimate on $\overline{V}_k(\alpha)$ can be obtained analogously. The third estimate requires Lemma 2.4 instead of Lemma 2.5 but follows analogously. \( \square \)

Let further
\[
U(\alpha, H) := \sum_{1 \leq m \leq H} e(m\alpha).
\]
We also have the usual numerically explicit inequality
\[
|U(\alpha, H)| \leq \min(H; |\alpha|^{-1}); \tag{2.2}
\]
see, for example, Montgomery [10, page 39]. Using (2.2),

Lemma 2.8. Let $H \geq 2$, $\mu \in \mathbb{R}$, $\mu \geq 1$. Then
\[
\mathcal{U}(\mu, H) := \int_{-1/2}^{1/2} |U(\alpha, H)|^\mu d\alpha \ll \mu \begin{cases} \log H & \text{if } \mu = 1, \\ H^\mu & \text{if } \mu > 1. \end{cases} \tag{2.3}
\]

Combining (2.2), Lemmas 2.4 and 2.7, we get the following result.

Lemma 2.9. Let $\ell \geq 1$ be an integer, $N$ be a sufficiently large integer and $L = \log N$. Assume the Riemann hypothesis. We have
\[
\mathcal{E}_\ell(H, N) := \int_{-1/2}^{1/2} \left| \overline{S}_\ell(\alpha) - \frac{\Gamma(1/\ell)}{\ell^{1/\ell}} \right|^2 |U(-\alpha, H)| d\alpha \ll_{\ell} N^{1/\ell} L^3. \tag{2.4}
\]

Combining (2.2), Lemmas 2.5, 2.7 and 2.1, we get the following result.
Lemma 2.10. Let \( k > 1 \), \( N \) be a sufficiently large integer, \( L = \log N \) and \( N^{1-(1/k)}L \ll H \leq N \). We have

\[
S_k(H, N) := \int_{-1/2}^{1/2} |\tilde{S}_k(\alpha)|^2 |U(-\alpha, H)| \, d\alpha \ll_k H N^{(2/k)-1} L^3
\]  
(2.5)

and

\[
V_k(H, N) := \int_{-1/2}^{1/2} |\tilde{V}_k(\alpha)|^2 |U(-\alpha, H)| \, d\alpha \ll_k H N^{(2/k)-1} L^3.
\]  
(2.6)

3. Proof of Theorem 1.1

Due to the symmetry of the summands we may let \( 2 \leq \ell_1 \leq \ell_2 \) and \( \lambda < 1 \), where \( \lambda \) is defined in (1.1); we will see at the end of the proof how the conditions in the statement of this theorem follow. Assume that

\[
B = B(N, \varepsilon) = N^\varepsilon
\]  
(3.1)

and let \( H > 2B \). Basically, we now replace \( \tilde{V}_\ell \) by \( \tilde{S}_\ell \) at the centre of the integration interval, that is, on \([-B/H, B/H]\). Then we bound the error term and the contribution of the remainder of the integration range by means of several lemmas proved in Section 2. We have

\[
\sum_{n=N+1}^{N+H} e^{-n/N} R''_{\ell_1, \ell_2}(n) = \int_{-1/2}^{1/2} \tilde{V}_{\ell_1}(\alpha) \tilde{V}_{\ell_2}(\alpha) U(-\alpha, H) e(-N\alpha) \, d\alpha
\]

\[
= \int_{-B/H}^{B/H} \tilde{S}_{\ell_1}(\alpha) \tilde{S}_{\ell_2}(\alpha) U(-\alpha, H) e(-N\alpha) \, d\alpha
\]

\[+ \int_{I(B/H)} \tilde{S}_{\ell_1}(\alpha) \tilde{S}_{\ell_2}(\alpha) U(-\alpha, H) e(-N\alpha) \, d\alpha
\]

\[+ \int_{-1/2}^{1/2} (\tilde{V}_{\ell_1}(\alpha) - \tilde{S}_{\ell_1}(\alpha)) \tilde{V}_{\ell_2}(\alpha) U(-\alpha, H) e(-N\alpha) \, d\alpha
\]

\[+ \int_{-1/2}^{1/2} (\tilde{V}_{\ell_2}(\alpha) - \tilde{S}_{\ell_2}(\alpha)) \tilde{S}_{\ell_1}(\alpha) U(-\alpha, H) e(-N\alpha) \, d\alpha,
\]  
(3.2)
say, where \( I(B/H) := [-1/2, -B/H] \cup [B/H, 1/2] \).

3.1. Estimate of \( I_2 \). By the Cauchy–Schwarz inequality, (2.2) and Lemma 2.7,

\[
I_2 \ll \left( \int_{I(B/H)} |\tilde{S}_{\ell_1}(\alpha)|^2 \left| \frac{d\alpha}{|\alpha|} \right| \right)^{1/2} \left( \int_{I(B/H)} \left| \tilde{S}_{\ell_2}(\alpha) \right|^2 \left| \frac{d\alpha}{|\alpha|} \right| \right)^{1/2} \ll_{\ell_1, \ell_2} \frac{HN^{1-1/L^3}}{B},
\]  
(3.3)

provided that \( H \gg \max(N^{1-1/\ell_1}; N^{1-1/\ell_2})BL = N^{1-1/\ell_2} BL \) since \( \ell_2 \geq \ell_1 \).
3.2. Estimate of $I_3$ and $I_4$. Using Lemma 2.1, the Cauchy–Schwarz inequality, (2.6) and (2.3),

$$I_3 \ll \ell_1 \ell_2 N^{1/(2\ell_1)} \mathcal{V}_{\ell_2}(H, N)^{1/2} U(1, H)^{1/2} \ll \ell_1 \ell_2 H^{1/2} N^{1/(2\ell_1)+1/\ell_2-1/2} L^2,$$

(3.4)

provided that $H \gg N^{1-1/\ell_2} L$.

Using Lemma 2.1, the Cauchy–Schwarz inequality, (2.5) and (2.3),

$$I_4 \ll \ell_1 \ell_2 N^{1/(2\ell_2)} S\ell_1(H, N)^{1/2} U(1, H)^{1/2} \ll \ell_1 \ell_2 H^{1/2} N^{1/(\ell_1+1/2\ell_2)-1/2} L^2,$$

(3.5)

provided that $H \gg N^{1-1/\ell_1} L$. Hence, using (3.4)–(3.5) and recalling that $\ell_1 \leq \ell_2$,

$$I_3 + I_4 \ll \ell_1 \ell_2 H^{1/2} N^{1/(\ell_1+1/2\ell_2)-1/2} L^2,$$

(3.6)

provided that $H \gg N^{1-1/\ell_2} L$.

3.3. Evaluation of $I_1$. We now obtain the main term. From now on, we denote

$$\widetilde{E}_\ell(\alpha) := S_\ell(\alpha) - \frac{\Gamma(1/\ell)}{\ell^{1/\ell}}.$$

In the formula below, we see that the main term arises from the product of the two terms $\Gamma(1/\ell)/(\ell^{1/\ell})$. The other terms give a smaller contribution since they contain at least one factor $\widetilde{E}_\ell$, which is small on average by Lemma 2.4. Recalling (3.2), by (2.1),

$$I_1 = \frac{\Gamma(1/\ell_1)\Gamma(1/\ell_2)}{\ell_1 \ell_2} \int_{-B/H}^{B/H} z^{-1/\ell_1-1/\ell_2} U(-\alpha, H)e(-N\alpha) \, d\alpha$$

$$+ \frac{\Gamma(1/\ell_1)}{\ell_1} \int_{-B/H}^{B/H} z^{-1/\ell_1} \widetilde{E}_{\ell_2}(\alpha) U(-\alpha, H)e(-N\alpha) \, d\alpha$$

$$+ \frac{\Gamma(1/\ell_2)}{\ell_2} \int_{-B/H}^{B/H} z^{-1/\ell_2} \widetilde{E}_{\ell_1}(\alpha) U(-\alpha, H)e(-N\alpha) \, d\alpha$$

$$+ \int_{-B/H}^{B/H} \widetilde{E}_{\ell_1}(\alpha) \widetilde{E}_{\ell_2}(\alpha) U(-\alpha, H)e(-N\alpha) \, d\alpha$$

$$= I_1 + I_2 + I_3 + I_4,$$

(3.7)

say. We now evaluate these terms.

3.4. Computation of the main term $I_1$. By Lemma 2.3, (1.1) and using $e^{-n/N} = e^{-1} + O(H/N)$ for $n \in [N+1, N+H]$, $1 \leq H \leq N$, a direct calculation gives

$$I_1 = c(\ell_1, \ell_2) \sum_{n=N+1}^{N+H} e^{-n/N} n^{\ell_1-1} + O_{\ell_1, \ell_2} \left( \frac{H}{N} \left( \frac{H}{B} \right)^{\ell_1} \right)$$

$$= c(\ell_1, \ell_2) \sum_{n=N+1}^{N+H} n^{\ell_1-1} + O_{\ell_1, \ell_2} \left( \frac{H}{N} \left( \frac{H}{B} \right)^{\ell_1} + H^{2}N^{\ell_1-2} \right)$$

$$= c(\ell_1, \ell_2) \frac{HN^{\ell_1-1}}{e} + O_{\ell_1, \ell_2} \left( \frac{H}{N} \left( \frac{H}{B} \right)^{\ell_1} + H^{2}N^{\ell_1-2} + N^{\ell_1-1} \right).$$

(3.8)
3.5. Estimate of $I_4$. Denote
\[ \mathcal{E}_\ell(B/H, N) := \int_{-B/H}^{B/H} |\widetilde{E}_\ell(\alpha)|^2 \, d\alpha \ll_\ell N^{2/\ell - 1} A(N, -c_1), \tag{3.9} \]
in which the estimate follows from Lemma 2.4 for $H \gg N^{1-1/\ell + \epsilon}$. Using (1.1), the Cauchy–Schwarz inequality and (3.9),
\[ I_4 \ll_{\ell_1, \ell_2} H \int_{-B/H}^{B/H} |\widetilde{E}_\ell(\alpha)||\widetilde{E}_{\ell_2}(\alpha)| \, d\alpha \ll_{\ell_1, \ell_2} H \mathcal{E}_{\ell_1}(B/H, N)^{1/2} \mathcal{E}_{\ell_2}(B/H, N)^{1/2} \]
\[ \ll_{\ell_1, \ell_2} H N^{\lambda-1} A(N, -c_1/2), \tag{3.10} \]
provided that $H \gg N^{1-5/(6\ell_2) + \epsilon} B$.

3.6. Estimate of $I_2$. Denote
\[ \mathcal{S}_\ell(B/H, N) := \int_{-B/H}^{B/H} |\tilde{S}_\ell(\alpha)|^2 \, d\alpha \ll_\ell N^{2/\ell - 1} L^3, \tag{3.11} \]
in which the estimates follow from Lemma 5.5, provided that $H \gg N^{1-1/\ell} B$. Remark that $|z|^{-1/\ell} \ll_\ell |S_\ell(\alpha)| + |E_\ell(\alpha)|$, using the Cauchy–Schwarz inequality, (3.9) and (3.11),
\[ I_2 \ll_{\ell_1, \ell_2} H \int_{-B/H}^{B/H} |\tilde{S}_\ell(\alpha)||\widetilde{E}_{\ell_2}(\alpha)| \, d\alpha + H \int_{-B/H}^{B/H} |\tilde{E}_\ell(\alpha)||\widetilde{E}_{\ell_2}(\alpha)| \, d\alpha \]
\[ \ll_{\ell_1, \ell_2} H \mathcal{S}_{\ell_1}(B/H, N)^{1/2} \mathcal{E}_{\ell_2}(B/H, N)^{1/2} + H \mathcal{E}_{\ell_1}(B/H, N)^{1/2} \mathcal{E}_{\ell_2}(B/H, N)^{1/2} \]
\[ \ll_{\ell_1, \ell_2} H N^{\lambda-1} A(N, -c_1/4), \tag{3.12} \]
provided that $N^{1-5/(6\ell_2) + \epsilon} B \leq H \leq N^{1-\epsilon}$.

3.7. Estimate of $I_3$. It is very similar to the $I_2$: we just need to interchange $\ell_1$ with $\ell_2$, thus getting that there exists $C = C(\epsilon) > 0$ such that
\[ I_3 \ll_{\ell_1, \ell_2} H N^{\lambda-1} A(N, -C), \tag{3.13} \]
provided that $N^{1-5/(6\ell_2) + \epsilon} B \leq H \leq N^{1-\epsilon}$.

3.8. Final words. Summarizing, recalling that $2 \leq \ell_1 \leq \ell_2$ and $\lambda < 1$, by (3.2)–(3.5), (3.7)–(3.8), (3.10), (3.12)–(3.13) and by optimizing the choice of $B$ as in (3.1), we have that there exists $C = C(\epsilon) > 0$ such that
\[ \sum_{n=N+1}^{N+H} e^{-n/N} R'_{\ell_1, \ell_2}(n) = \frac{c(\ell_1, \ell_2)}{e} H N^{\lambda-1} + O_{\ell_1, \ell_2}(H N^{\lambda-1} A(N, -C)) \tag{3.14} \]
uniformly for $N^{1-5/(6\ell_2) + \epsilon} \leq H \leq N^{1-\epsilon}$. From $e^{-n/N} = e^{-1} + O(H/N)$ for $n \in [N+1, N + H]$, $1 \leq H \leq N$,
\[ \sum_{n=N+1}^{N+H} R''_{\ell_1, \ell_2}(n) = c(\ell_1, \ell_2) H N^{\lambda-1} + O_{\ell_1, \ell_2}(H N^{\lambda-1} A(N, -C)) + O\left(\frac{H}{N} \sum_{n=N+1}^{N+H} R'_{\ell_1, \ell_2}(n)\right). \]
Using $e^{n/N} \leq e^2$ and (3.14), the last error term is $\ll \ell_1 \ell_2 H^2 N^{1-2}$. Hence,
\[
\sum_{n=N+1}^{N+H} R''_{\ell_1 \ell_2} (n) = c(\ell_1, \ell_2) H N^{\ell_1-1} + O_{\ell_1 \ell_2} (H N^{\ell_1} A(N, -C))
\]
uniformly for $N^{1-5/6\ell_2} + \epsilon \leq H \leq N^{1-\epsilon}$ and $2 \leq \ell_1 \leq \ell_2$. Theorem 1.1 follows.

4. Proof of Theorem 1.2

In this section we assume that the Riemann hypothesis holds. In this case, we do not need a different argument for ‘centre’ and ‘periphery’ of the integration interval since Lemma 2.4 is valid throughout $[-1/2, 1/2]$. Recalling (1.3),
\[
\sum_{n=N+1}^{N+H} e^{-n/N} R''_{\ell_1 \ell_2} (n) = \int_{-1/2}^{1/2} \overline{V}_{\ell_1} (\alpha) \overline{V}_{\ell_2} (\alpha) U(-\alpha, H) e(-N\alpha) \, d\alpha
\]
\[= \int_{-1/2}^{1/2} \overline{S}_{\ell_1} (\alpha) \overline{S}_{\ell_2} (\alpha) U(-\alpha, H) e(-N\alpha) \, d\alpha
\]
\[+ \int_{-1/2}^{1/2} (\overline{V}_{\ell_1} (\alpha) - \overline{S}_{\ell_1} (\alpha)) \overline{V}_{\ell_2} (\alpha) U(-\alpha, H) e(-N\alpha) \, d\alpha
\]
\[+ \int_{-1/2}^{1/2} (\overline{V}_{\ell_2} (\alpha) - \overline{S}_{\ell_2} (\alpha)) \overline{S}_{\ell_1} (\alpha) U(-\alpha, H) e(-N\alpha) \, d\alpha,
\]
\[= J_1 + J_2 + J_3,
\]
say.

4.1. Estimate of $J_2$ and $J_3$. The quantities $J_2$ and $J_3$ are equal to $I_3$ and $I_4$ of Section 3.2. Hence, by (3.6),
\[J_2 + J_3 \ll \ell_1 \ell_2 H^{1/2} N^{1/\ell_1 + 1/(2\ell_2) - 1/2} L^2,
\]
provided that $H \gg N^{1-1/\ell_1} L$.

4.2. Evaluation of $J_1$. Here we obtain the main term essentially as above, but we can deal with the whole integration interval at once. Hence,
\[J_1 = \frac{\Gamma(1/\ell_1) \Gamma(1/\ell_2)}{\ell_1 \ell_2} \int_{-1/2}^{1/2} z^{-1/\ell_1 - 1/\ell_2} U(-\alpha, H) e(-N\alpha) \, d\alpha
\]
\[+ \frac{\Gamma(1/\ell_1)}{\ell_1} \int_{-1/2}^{1/2} z^{-1/\ell_1} \overline{E}_{\ell_2} (\alpha) U(-\alpha, H) e(-N\alpha) \, d\alpha
\]
\[+ \frac{\Gamma(1/\ell_2)}{\ell_2} \int_{-1/2}^{1/2} z^{-1/\ell_2} \overline{E}_{\ell_1} (\alpha) U(-\alpha, H) e(-N\alpha) \, d\alpha
\]
\[+ \int_{-1/2}^{1/2} \overline{E}_{\ell_1} (\alpha) \overline{E}_{\ell_2} (\alpha) U(-\alpha, H) e(-N\alpha) \, d\alpha
\]
\[= J_1 + J_2 + J_3 + J_4,
\]
say. Now we evaluate these terms.
4.3. Computation of \( J_1 \). By Lemma 2.3, (1.1) and using \( e^{-n/N} = e^{-1} + O(H/N) \) for \( n \in [N + 1, N + H] \), \( 1 \leq H \leq N \), a direct calculation gives

\[
J_1 = c(\ell_1, \ell_2) \sum_{n=N+1}^{N+H} e^{-n/N} n^{\ell_1 - 1} + O(\ell_1, \ell_2) \left( \frac{H}{N} \right)
\]

\[
= \frac{c(\ell_1, \ell_2)}{e} \sum_{n=N+1}^{N+H} n^{\ell_1 - 1} + O(\ell_1, \ell_2) \left( \frac{H}{N} + H^2 N^{\ell_1 - 2} \right)
\]

\[
= c(\ell_1, \ell_2) \frac{HN^{\ell_1 - 1}}{e} + O(\ell_1, \ell_2) \left( \frac{H}{N} + H^2 N^{\ell_1 - 2} + N^{\ell_1 - 1} \right).
\]

4.4. Estimate of \( J_4 \). Hence, by the Cauchy–Schwarz inequality and (2.4),

\[
J_4 \ll_{\ell_1, \ell_2} E_{\ell_1}(H, N)^{1/2} E_{\ell_2}(H, N)^{1/2} \ll_{\ell_1, \ell_2} N^{3/2} L^3.
\]

4.5. Estimate of \( J_2 \). Remark that \( |z|^{-1/\ell} \ll_{\ell} |S_{\ell}(\alpha)| + |\tilde{E}_{\ell}(\alpha)| \), using the Cauchy–Schwarz inequality, (2.4) and (2.5),

\[
J_2 \ll_{\ell_1, \ell_2} \int_{-1/2}^{1/2} |S_{\ell_1}(\alpha)||\tilde{E}_{\ell_2}(\alpha)||U(-\alpha, H)| d\alpha + \int_{-1/2}^{1/2} |\tilde{E}_{\ell_1}(\alpha)||\tilde{E}_{\ell_2}(\alpha)||U(-\alpha, H)| d\alpha
\]

\[
\ll_{\ell_1, \ell_2} S_{\ell_1}(H, N)^{1/2} E_{\ell_2}(H, N)^{1/2} + E_{\ell_1}(H, N)^{1/2} E_{\ell_2}(H, N)^{1/2}
\]

\[
\ll_{\ell_1, \ell_2} H^{1/2} N^{1/\ell_1 + 1/(2\ell_2) - 1/2} L^3,
\]

provided that \( H \gg N^{1-1/\ell_1} L \).

4.6. Estimate of \( J_3 \). The estimate of \( J_3 \) is very similar to the \( J_2 \); we just need to interchange \( \ell_1 \) with \( \ell_2 \). We obtain

\[
J_3 \ll_{\ell_1, \ell_2} H^{1/2} N^{1/\ell_1 + 1/(2\ell_2) - 1/2} L^3,
\]

provided that \( H \gg N^{1-1/\ell_2} L \).

4.7. Final words. Summarizing, recalling \( 2 \leq \ell_1 \leq \ell_2 \), by (1.1) and (4.1)–(4.2),

\[
\sum_{n=N+1}^{N+H} e^{-n/N} R_{\ell_1, \ell_2}''(n) = c(\ell_1, \ell_2) \frac{HN^{\ell_1 - 1}}{e}
\]

\[
+ O(\ell_1, \ell_2) \left( \frac{H}{N} + H^2 N^{\ell_1 - 2} + H^{1/2} N^{1/\ell_1 + 1/(2\ell_2) - 1/2} L^3 \right),
\]

which is an asymptotic formula for \( o(N^{1-1/\ell_2} L^6) \leq H \leq o(N) \). From \( e^{-n/N} = e^{-1} + O(H/N) \) for \( n \in [N + 1, N + H] \), \( 1 \leq H \leq N \),

\[
\sum_{n=N+1}^{N+H} R_{\ell_1, \ell_2}''(n) = c(\ell_1, \ell_2) H N^{\ell_1 - 1} + O(\ell_1, \ell_2) (H^2 N^{\ell_1 - 2} + H^{1/2} N^{1/\ell_1 + 1/(2\ell_2) - 1/2} L^3)
\]

\[
+ O \left( \frac{H}{N} \sum_{n=N+1}^{N+H} R_{\ell_1, \ell_2}''(n) \right).
\]

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Using $e^{\alpha/N} \leq e^2$ and (4.3), the last error term is $\ll_{\ell_1, \ell_2} H^2 N^{\lambda_2-2}$. Hence,

$$
\sum_{n=N+1}^{N+H} R''_{\ell_1, \ell_2}(n) = c(\ell_1, \ell_2) H N^{\lambda_1-1} + O_{\ell_1, \ell_2}(H^2 N^{\lambda_2-2} + H^{1/2} N^{1/\ell_1 + 1/(2\ell_2)-1/2} L^3)
$$

uniformly for every $2 \leq \ell_1 \leq \ell_2$ and $\infty(N^{1-1/\ell_2} L^6) \leq H \leq o(N)$. Theorem 1.2 follows.

5. Setting and lemmas for Theorem 1.3

We also need similar lemmas for the finite sums since we will use them for proving the third result. Let $k > 0$ be a real number and

$$
S_k(\alpha) := \sum_{N/|A| \leq m^k N} \Lambda(m) e(m^k \alpha), \quad V_k(\alpha) := \sum_{N/|A| \leq p^k N} \log p e(p^k \alpha),
$$

$$
T_k(\alpha) := \sum_{N/|A| \leq m^k N} e(m^k \alpha), \quad f_k(\alpha) := (1/k) \sum_{N/|A| \leq m^k N} m^{(1/k)-1} e(m\alpha),
$$

(5.1)

where $A$ is defined in (1.2) with $d > 0$. We need this parameter because, if we chose $A = N$ in the definition of $f_k$ above, the $L^2$ bound in Lemma 5.4 would become too weak. We remark that we can choose $d$ in such a way that the constant $C(\varepsilon)$ in the statement of Theorem 1.3 is independent of $\ell_1$ and $\ell_2$. By Vaughan [14, Lemmas 2.8 and 4.1],

$$
|T_k(\alpha) - f_k(\alpha)| \ll_k (1 + |\alpha| N)^{1/2}.
$$

(5.2)

We recall that $\varepsilon > 0$ and we let $L = \log N$. Now we recall some lemmas from [8].

**Lemma 5.1** [8, Lemma 2]. Let $k > 0$ be a real number. Then $|S_k(\alpha) - V_k(\alpha)| \ll_k N^{1/(2k)}$.

We need the following lemma, which collects the results of [4, Theorems 3.1 and 3.2]; see also [6, Lemma 1].

**Lemma 5.2.** Let $k > 0$ be a real number and $\varepsilon$ be an arbitrarily small positive constant. Then there exists a positive constant $c_1 = c_1(\varepsilon)$, which does not depend on $k$, such that

$$
\int_{-1/K}^{1/K} |S_k(\alpha) - T_k(\alpha)|^2 \, d\alpha \ll_k N^{(2/k)-1} \left( A(N, -c_1) + \frac{KL^2}{N} \right)
$$

uniformly for $N^{1-5/(6k)+\varepsilon} \leq K \leq N$. Assuming further the RH,

$$
\int_{-1/K}^{1/K} |S_k(\alpha) - T_k(\alpha)|^2 \, d\alpha \ll_k \frac{N^{1/k} L^2}{K} + KN^{(2/k)-2} L^2
$$

uniformly for $N^{1/(1/k)} \leq K \leq N$.

Combining the two previous lemmas, we get the following result.
**Lemma 5.3** [8, Lemma 4]. Let \( k > 0 \) be a real number and \( \varepsilon \) be an arbitrarily small positive constant. Then there exists a positive constant \( c_1 = c_1(\varepsilon) \), which does not depend on \( k \), such that

\[
E_k(N, K) := \int_{-1/K}^{1/K} |V_k(\alpha) - T_k(\alpha)|^2 \, d\alpha \ll (A(N, -c_1) + \frac{KL^2}{N})
\]

uniformly for \( N^{1-5/(6k)+\varepsilon} \leq K \leq N \). Assuming further the RH,

\[
E_k(N, K) \ll \frac{N^{1/k}L^2}{K} + KN^{(2/k)-2}L^2,
\]

uniformly for \( N^{1-(1/k)} \leq K \leq N \).

**Lemma 5.4** [8, Lemma 6]. Let \( k > 0 \) be a real number and recall that \( A \) is defined in (1.2). Then

\[
F_k(N, A) := \int_{-1/2}^{1/2} |f_k(\alpha)|^2 \, d\alpha \ll_k \begin{cases} A^{1-(2/k)} & \text{if } k > 2, \\ \log A & \text{if } k = 2, \\ 1 & \text{if } 0 < k < 2. \end{cases}
\]

The new ingredient we are using here is based on Tolev’s lemma [13] in the form given in [2, Lemma 5].

**Lemma 5.5** (Tolev). Let \( k > 1, N \in \mathbb{N} \) and \( \tau > 0 \). Then

\[
\int_{-\tau}^{\tau} |V_k(\alpha)|^2 \, d\alpha \ll_k (\tau N^{1/k} + N^{(2/k)-1})L^3
\]

and

\[
\int_{-\tau}^{\tau} |T_k(\alpha)|^2 \, d\alpha \ll_k (\tau N^{1/k} + N^{(2/k)-1})L.
\]

The last lemma is a consequence of Lemma 5.5 and its proof follows the line of Lemma 2.7.

**Lemma 5.6.** Let \( N \in \mathbb{N}, k > 1, c \geq 1 \) and \( N^{-c} \leq \omega \leq N^{1/k-1}/L \). Let further \( I(\omega) := [-1/2, -\omega] \cup [\omega, 1/2] \). Then

\[
\int_{I(\omega)} |V_k(\alpha)|^2 \frac{d\alpha}{|\alpha|} \ll_k \frac{N^{(2/k)-1}}{\omega}L^3 \quad \text{and} \quad \int_{I(\omega)} |T_k(\alpha)|^2 \frac{d\alpha}{|\alpha|} \ll_k \frac{N^{(2/k)-1}}{\omega}L.
\]

**6. Proof of Theorem 1.3**

Assume that \( \ell_1, \ell_2 \geq 2 \) and \( \lambda < 1 \), where \( \lambda \) is defined in (1.1). We will see at the end of the proof how the conditions in the statement of this theorem follow; we remark that in this case we cannot interchange the roles of \( \ell_1, \ell_2 \). Assume that

\[
B = B(N, \varepsilon) = N^\varepsilon
\]

(6.1)
and let $H > 2B$. We have

$$
\sum_{n=N+1}^{N+H} r'_{\ell_1, \ell_2}(n) = \int_{-1/2}^{1/2} V_{\ell_1}(\alpha) T_{\ell_2}(\alpha) U(-\alpha, H) e(-Na) \, d\alpha
$$

$$
= \int_{-B/H}^{B/H} V_{\ell_1}(\alpha) T_{\ell_2}(\alpha) U(-\alpha, H) e(-Na) \, d\alpha
+ \int_{I(B/H)} V_{\ell_1}(\alpha) T_{\ell_2}(\alpha) U(-\alpha, H) e(-Na) \, d\alpha,
$$

(6.2)

where $I(B/H) := [-1/2, -B/H] \cup [B/H, 1/2]$. By (2.2), the Cauchy–Schwarz inequality and Lemma 5.6,

$$
\int_{I(B/H)} V_{\ell_1}(\alpha) T_{\ell_2}(\alpha) U(-\alpha, H) e(-Na) \, d\alpha
\lesssim \left( \int_{I(B/H)} |V_{\ell_1}(\alpha)|^2 \frac{d\alpha}{|\alpha|} \right)^{1/2} \left( \int_{I(B/H)} |T_{\ell_2}(\alpha)|^2 \frac{d\alpha}{|\alpha|} \right)^{1/2}
\lesssim \frac{HN^{\lambda-1}L^2}{B},
$$

(6.3)

provided that $H \gg \max(N^{1-1/\ell_1}; N^{1-1/\ell_2})BL$. By (6.2)–(6.3),

$$
\sum_{n=N+1}^{N+H} r'_{\ell_1, \ell_2}(n) = \int_{-B/H}^{B/H} V_{\ell_1}(\alpha) T_{\ell_2}(\alpha) U(-\alpha, H) e(-Na) \, d\alpha + O_{\ell_1, \ell_2} \left( \frac{HN^{\lambda-1}L^2}{B} \right).
$$

Hence, recalling (5.1),

$$
\sum_{n=N+1}^{N+H} r'_{\ell_1, \ell_2}(n) = \int_{-B/H}^{B/H} f_{\ell_1}(\alpha) f_{\ell_2}(\alpha) U(-\alpha, H) e(-Na) \, d\alpha
+ \int_{-B/H}^{B/H} f_{\ell_2}(\alpha)(V_{\ell_1}(\alpha) - f_{\ell_1}(\alpha)) U(-\alpha, H) e(-Na) \, d\alpha
+ \int_{-B/H}^{B/H} f_{\ell_1}(\alpha)(T_{\ell_2}(\alpha) - f_{\ell_2}(\alpha)) U(-\alpha, H) e(-Na) \, d\alpha
+ \int_{-B/H}^{B/H} (V_{\ell_1}(\alpha) - f_{\ell_1}(\alpha))(T_{\ell_2}(\alpha) - f_{\ell_2}(\alpha)) U(-\alpha, H) e(-Na) \, d\alpha
+ O_{\ell_1, \ell_2} \left( \frac{HN^{\lambda-1}L^2}{B} \right)
= I_1 + I_2 + I_3 + I_4 + E,
$$

(6.4)

say. We now evaluate these terms.

6.1. Computation of the main term $I_1$. Recalling Definition (1.1) and $I(B/H) = [-1/2, -B/H] \cup [B/H, 1/2]$, a direct calculation, (2.2), the Cauchy–Schwarz
inequality and Lemma 5.4 give

\[ I_1 = \sum_{n=1}^{H} \int_{-1/2}^{1/2} f_{\ell_1}(\alpha) f_{\ell_2}(\alpha) e(-(n + N)\alpha) \, d\alpha + O_{\ell_1, \ell_2} \left( \int_{|B/H|} |f_{\ell_1}(\alpha) f_{\ell_2}(\alpha)| \, d\alpha \right) \]

\[
= \frac{1}{\ell_1 \ell_2} \sum_{n=1}^{H} \sum_{m_1, m_2 = n+N, N/A \leq m_1 \leq N} m_1^{1/\ell_1 - 1} m_2^{1/\ell_2 - 1} + O_{\ell_1, \ell_2} \left( \frac{H}{B} F_{\ell_1}(N, A)^{1/2} F_{\ell_2}(N, A)^{1/2} \right) \]

\[
= M_{\ell_1, \ell_2}(H, N) + O_{\ell_1, \ell_2} \left( \frac{H}{B} N^{d - 1} A^{-1/\ell_1} \right). \tag{6.5} \]

say. Recalling Vaughan [14, Lemma 2.8], we can see that the order of magnitude of the main term \( M_{\ell_1, \ell_2}(H, N) \) is \( c(\ell_1, \ell_2)HN^{1/\ell_1} \). We first complete the range of summation for \( m_1 \) and \( m_2 \) to the interval \([1, N]\). The corresponding error term is

\[
\ll_{\ell_1, \ell_2} \sum_{n=1}^{N/A} \sum_{m_1, m_2 = n+N, N/A \leq m_1 \leq N} m_1^{1/\ell_1 - 1} m_2^{1/\ell_2 - 1} \ll_{\ell_1, \ell_2} \sum_{n=1}^{N/A} m_1^{1/\ell_1 - 1} (n + N - m)^{1/\ell_1 - 1}
\]

\[
\ll_{\ell_1, \ell_2} HN^{1/\ell_1 - 1} \sum_{n=1}^{N/A} m_1^{1/\ell_1 - 1} \ll_{\ell_1, \ell_2} HN^{d - 1} A^{-1/\ell_2}. \]

We deal with the main term \( M_{\ell_1, \ell_2}(H, N) \) using Vaughan [14, Lemma 2.8], which yields the \( \Gamma \) factors hidden in \( c(\ell_1, \ell_2) \):

\[
\frac{1}{\ell_1 \ell_2} \sum_{n=1}^{H} \sum_{m_1, m_2 = n+N, 1 \leq m_1 \leq N, 1 \leq m_2 \leq N} m_1^{1/\ell_1 - 1} m_2^{1/\ell_2 - 1}
\]

\[
= \frac{1}{\ell_1 \ell_2} \sum_{n=1}^{H} \sum_{m_1 = n+N} m_1^{1/\ell_2 - 1} (n + N - m)^{1/\ell_1 - 1}
\]

\[
= c(\ell_1, \ell_2) \sum_{n=1}^{H} [(n + N)^{\ell_1 - 1} + O((n + N)^{1/\ell_1 - 1} + N^{1/\ell_2 - 1} n^{1/\ell_1})]
\]

\[
= c(\ell_1, \ell_2) \sum_{n=1}^{H} (n + N)^{1/\ell_1 - 1} + O_{\ell_1, \ell_2}(HN^{1/\ell_1 - 1} + H^{1/\ell_1 + 1} N^{1/\ell_2 - 1})
\]

\[
= c(\ell_1, \ell_2)HN^{1/\ell_1 - 1} + O_{\ell_1, \ell_2}(H^2 N^{d - 2} + HN^{1/\ell_1 - 1} + H^{1/\ell_1 + 1} N^{1/\ell_2 - 1}).
\]

Summarizing,

\[
M_{\ell_1, \ell_2}(H, N) = c(\ell_1, \ell_2)HN^{1/\ell_1 - 1} + O_{\ell_1, \ell_2}(H^2 N^{d - 2} + HN^{1/\ell_1 - 1} + H^{1/\ell_1 + 1} N^{1/\ell_2 - 1} + HN^{d - 1} A^{-1/\ell_2}). \tag{6.6} \]
Combining (6.5)–(6.6) and using (1.2) and (3.1),

\[ I_1 = c(\ell_1, \ell_2)HN^{\lambda-1} + O_{\ell_1, \ell_2}(HN^{\lambda-1}A(N, -C)) \]

for a suitable choice of \( C = C(\varepsilon) > 0 \), provided that \( H \ll N^{1-\varepsilon} \).

**6.2. Estimate of \( I_2 \).** Using (5.2),

\[ |V_\ell(\alpha) - f_\ell(\alpha)| \leq |V_\ell(\alpha) - T_\ell(\alpha)| + O_{\ell}(1 + |\alpha|N)^{1/2}. \]  

(6.7)

Hence,

\[ I_2 \ll \int_{-B/H}^{B/H} |f_\ell(\alpha)||V_\ell(\alpha) - T_\ell(\alpha)||U(-\alpha, H)| \, d\alpha \]

\[ + \int_{-B/H}^{B/H} |f_\ell(\alpha)|(1 + |\alpha|N)^{1/2}|U(-\alpha, H)| \, d\alpha = E_1 + E_2, \]  

(6.8)

say. Letting

\[ W(N, H, B) := \int_{-B/H}^{B/H} (1 + |\alpha|N)|U(-\alpha, H)|^2 \, d\alpha \]

\[ \ll \frac{H^2}{N} + H^2N \int_{1/N}^{1/H} \alpha \, d\alpha + N \int_{1/H}^{B/H} \frac{d\alpha}{\alpha} \]

\[ \ll NL, \]  

(6.9)

in which we used (2.2), using the Cauchy–Schwarz inequality, (6.9) and Lemma 5.4,

\[ E_2 \ll F_{\ell_2}(N, A)^{1/2}W(N, H, B)^{1/2} \ll_{\ell_2} \left(\frac{N}{A}\right)^{1/f_2-1/2}(\log A)^{1/2}(NL)^{1/2} \]

\[ \ll_{\ell_2} N^{1/f_2}A^{1/2-1/f_2}L^{1/2}(\log A)^{1/2}, \]  

(6.10)

where \( A \) is defined in (1.2). Using (2.2), the Cauchy–Schwarz inequality, (6.9) and Lemmas 5.3–5.4,

\[ E_1 \ll HF_{\ell_2}(N, A)^{1/2}E_{\ell_1}(K, B/H)^{1/2} \]

\[ \ll_{\ell_1, \ell_2} H\left(\frac{N}{A}\right)^{1/f_2-1/2}(\log A)^{1/2}N^{1/f_1-1/2}A(N, -c_1/2) \]

\[ \ll_{\ell_1, \ell_2} HN^{\lambda-1}A(N, -C) \]  

(6.11)

for a suitable choice of \( C = C(\varepsilon) > 0 \), provided that \( N^{1-5/(6\ell_1+\varepsilon)}B \leq H \leq N^{1-\varepsilon} \). Summarizing, by (1.1) and (6.8)–(6.11), we obtain that there exists \( C = C(\varepsilon) > 0 \) such that

\[ I_2 \ll_{\ell_1, \ell_2} HN^{\lambda-1}A(N, -C), \]  

provided that \( N^{1-5/(6\ell_1+\varepsilon)}B \leq H \leq N^{1-\varepsilon} \).
6.3. Estimate of $I_3$. Using (5.2),

$$I_3 \ll \ell_2 \int_{-B/H}^{B/H} |f_{\ell_1}(\alpha)||1 + |\alpha|N\rangle^{1/2}|U(-\alpha, H)| d\alpha$$

and the right-hand side is similar to $E_2$ of Section 6.2; hence, arguing as for (6.10),

$$I_3 \ll \ell_1, \ell_2 N^{1/(\ell_1)} A^{1/2-1/\ell_1} L^{1/2} (\log A)^{1/2},$$

(6.12)

where $A$ is defined in (1.2).

6.4. Estimate of $I_4$. By (3.7) and (6.7), we can write

$$I_4 \ll \ell_1, \ell_2 \int_{-B/H}^{B/H} |V_{\ell_1}(\alpha) - T_{\ell_1}(\alpha)||1 + |\alpha|N\rangle^{1/2}|U(-\alpha, H)| d\alpha$$

$$+ \int_{-B/H}^{B/H} (1 + |\alpha|N)|U(-\alpha, H)| d\alpha$$

$$= R_1 + R_2,$$

(6.13)
say. By the Cauchy–Schwarz inequality, Lemma 5.3 and arguing as in (6.10),

$$R_1 \ll E_{\ell_1} K, B/H \rangle^{1/2} W(N, H, B)^{1/2} \ll \ell_1 N^{1/\ell_1} A(N, -C)$$

for a suitable choice of $C = C(\varepsilon) > 0$, provided that $N^{1-5/(6\ell_1)+\varepsilon} B \leq H \leq N^{1-\varepsilon}$.

Moreover, by (2.2),

$$R_2 \ll H \int_{-1/N}^{1/N} d\alpha + HN \int_{1/N}^{1/H} \alpha d\alpha + N \int_{1/H}^{B/H} d\alpha \ll \frac{NB}{H}.$$  (6.14)

Summarizing, by (1.1) and (6.13)–(6.14),

$$I_4 \ll \ell_1, \ell_2 H N^{\lambda-1} A(N, -C)$$

(6.15)

for a suitable choice of $C = C(\varepsilon) > 0$, provided that $N^{1-5/(6\ell_1)+\varepsilon} B \leq H \leq N^{1-\varepsilon}$ and $H \gg N^{1-1/\ell_2+\varepsilon}$.

6.5. Final words. Summarizing, recalling that $\ell_1, \ell_2 \geq 2$ and $\lambda < 1$, by (6.4)–(6.12), (6.15) and by optimizing the choice of $B$ as in (6.1), we have that there exists $C = C(\varepsilon) > 0$ such that

$$\sum_{n=N+1}^{N+H} r'_{\ell_1, \ell_2}(n) = c(\ell_1, \ell_2) H N^{\lambda-1} + O(\ell_1, \ell_2) (H N^{\lambda-1} A(N, -C))$$

uniformly for $\max(N^{1-5/(6\ell_1)}, N^{1-1/\ell_2}) N^\varepsilon \leq H \leq N^{1-\varepsilon}$. Theorem 1.3 follows.
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ALESSANDRO LANGUASCO, Università di Padova,
Dipartimento di Matematica ‘Tullio Levi-Civita’, Via Trieste 63,
35121 Padova, Italy
e-mail: alessandro.languasco@unipd.it

ALESSANDRO ZACCAGNINI, Università di Parma,
Dipartimento di Scienze Matematiche,
Fisiche e Informatiche, Parco Area delle Scienze 53/a,
43124 Parma, Italy
e-mail: alessandro.zaccagnini@unipr.it