The 1D Bose Gas with Weakly Repulsive Delta Interaction

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We consider the asymptotic solutions to the Bethe ansatz equations of the integrable model of interacting bosons in the weakly interacting limit. In this limit we establish that the ground state maps to the highest energy state of a strongly-coupled repulsive bosonic pairing model.

KEYWORDS: Integrable models, 1D Bose gas, Bethe ansatz equations, BCS pairing models

1. Introduction

There has been a revival of interest in the exactly solved 1D model of interacting bosons.\textsuperscript{1} In part this is due to the experimental realisation of a quasi-1D quantum gas of bosons at ultracold temperatures.\textsuperscript{2–6} The essential point is that the interactions between the trapped atoms can be tuned to bring about a continuous passage from the weakly interacting regime to the strongly interacting regime. In this way the full subtleties of quantum many-body physics are observed, from Bose-Einstein condensation in the weak coupling regime to the pronounced fermionic behaviour of the Tonks-Girardeau gas in the strong coupling regime.

Recently it was observed\textsuperscript{7} with the help of numerical analysis that the Bethe ansatz roots for the ground state of the exactly solved 1D Bose gas in the weak coupling limit satisfy a similar set of equations as an exactly solved BCS boson pairing model in the strong coupling limit. Here we further clarify this correspondence in two ways: first we derive, for arbitrary particle number, the system of equations satisfied by the ground state roots of the 1D Bose gas in the weak coupling limit. Then we show that the mapping to a BCS type system is precise for certain bosonic systems where the Cooper pairs are formed from integer spin particles.

2. Weak coupling limit

The Hamiltonian

\[ \mathcal{H} = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + 2c \sum_{j<k}^{N} \delta(x_j - x_k) \] (1)
describing \( N \) interacting bosons on a periodic interval of length \( L \) has been extensively studied.\(^1\,8–13\) The eigenstates have energy and momenta given by

\[
E = \sum_{j=1}^{N} k_j^2, \quad P = \sum_{j=1}^{N} k_j, \quad (2)
\]

where \( k_j \) satisfy the Bethe ansatz equations (BAE)

\[
\exp(ik_jL) = -\prod_{l=1}^{N} \frac{k_j - k_l + i c}{k_j - k_l - i c}, \quad j = 1, \ldots, N. \quad (3)
\]

For repulsive interactions \((c > 0)\) it is known that the Bethe roots \( k_j \) are real and distinct.\(^13\) Moreover, the eigenspectrum is positive, i.e., \( \mathcal{E}_0 \leq \mathcal{E}_1 \leq \mathcal{E}_2 \ldots \), where \( \mathcal{E}_0 \) is the ground state.

Asymptotic solutions to the BAE (3) are possible in two limiting cases: \( Lc \ll 1 \) and \( Lc \gg 1 \), corresponding to weak and strong delta interaction. In the limit \( Lc \gg 1 \),\(^7\) which we do not discuss here, the asymptotic solutions to the BAE (3) describe the Tonks-Giradeau gas.\(^14,15\) In the limit \( Lc \ll 1 \), numerical checks suggest that the momenta \( k_j \) are proportional to the square root of \( c \).\(^7\) It follows that, to order \( c^2 \), the BAE (3) reduce to

\[
\exp(ik_jL) \approx 1 - 2\sum_{\ell=1}^{N} \frac{c^2}{(k_j - k_\ell)^2} - 4\sum_{\ell=1}^{N-1} \sum_{\ell' = 2}^{N} \frac{c}{(k_j - k_\ell)} \frac{c}{(k_j - k_\ell')}, \quad (4)
\]

in which the summations exclude \( \ell = j \) and \( \ell' = j \). Indeed, to order \( c^2 \),

\[
\cos(k_jL) \approx 1 - 2\sum_{\ell=1}^{N} \frac{c^2}{(k_j - k_\ell)^2} - 4\sum_{\ell=1}^{N} \sum_{\ell' = 2}^{N} \frac{c}{(k_j - k_\ell)} \frac{c}{(k_j - k_\ell')}, \quad (5)
\]

\[
\sin(k_jL) \approx 2\sum_{\ell=1}^{N} \frac{c}{(k_j - k_\ell)}. \quad (6)
\]

The solution of which determines the asymptotic roots of the BAE (3). From Eqs. (5) and (6) the Bethe roots are seen to satisfy

\[
k_j = \frac{2\pi d_j}{L} + \frac{2c}{L} \sum_{\ell \neq j}^{N} \frac{1}{k_j - k_\ell}, \quad j = 1, \ldots N. \quad (7)
\]

Here \( d_j = 0, \pm 1, \pm 2, \ldots \) denotes excited states and the summation excludes \( j = \ell \).

The asymptotic equations are closely related to those appearing in Stieltjes problems.\(^19\) The ground state has zero total momentum with \( d_j = 0 \) for \( j = 1, \ldots, N \), with the ground state energy per particle

\[
\frac{\mathcal{E}_0}{N} = \frac{c(N-1)}{L}, \quad (8)
\]

following directly from Eq. (7). The algebraic equations (7) for the ground state are given in Gaudin.\(^11\) In this way Gaudin showed that the \( k_j \) are roots of Hermite polynomials of
degree $N$, namely $H_N(k) = 0$. They are also related to roots of the Laguerre polynomial. These connections provide a systematic way for studying quantities such as the momentum distribution function and correlations. The normalised momentum density distribution is given by the semi-circle law

$$n(k) = \frac{L}{2N\pi c} (4\rho - k^2)^{\frac{1}{2}},$$

where $\rho = N/L$. The stronger the interaction strength, the larger the momentum distribution region. This reveals a significant signature of the 1D Bose gas in the weakly repulsive regime $Lc \ll 1$. Remarkably, this behaviour was recently observed in the experiments for weakly interacting bosons. If $c = 0$ all the particles condense in the ground state at zero temperature.

Now consider the excitations above the ground state, which have total momentum $P = 2n\pi/L$ with $n = 0, \pm 1, \pm 2, \ldots$. As an example of the solutions obtained from the asymptotic equations (7), consider the numerical data for four bosons given in Table I. Each state is characterised by the quantum numbers $d_j$, with total momentum $P = \sum_{j=1}^{N} 2d_j\pi/L$. The excitation energies are at least doubly degenerate in view of the counterpart assignments $-d_j$ with momenta $-k_j$. Shown for direct comparison are the numerical solutions from the full BAE (3). Clearly the agreement is excellent.

Let us consider the lowest excited state $E_1$, with total momentum $P = 2\pi/L$, in more detail. Without loss of generality, we choose an assignment $d_1 = 1$ and $d_j = 0$ for $j = 2, \ldots, N$. Using Eq. (7) we can approximately calculate the lowest excitation energy. Specifically,

$$\frac{E_1}{N} = \sum_{j=1}^{N} \frac{k_j^2}{L} \approx \frac{2\pi k_1}{LN} + \frac{c(N - 1)}{L},$$

from which we can infer that

$$k_1 \approx \frac{2\pi}{L} + \frac{c(N - 1)}{\pi},$$

$$\frac{E_1}{N} \approx \frac{c(N - 1)}{L} + \frac{4\pi^2}{L^2N} + \frac{2c(N - 1)}{LN},$$

which are valid for $N/L$ finite. These approximations are superior to those given earlier. It is clearly seen from Eq. (12) that the energy gap will vanish in the limit $N \to \infty$. For the values of Table I, Eqs. (11) and (12) give $k_1 \approx 6.30705$ and $E_1 \approx 39.9284$, which are in good agreement with the numerical results obtained from Eqs. (7) and (3). As a further example, consider $N = 8$ with $c = 0.025$ and $L = 2$. In this case Eqs. (11) and (12) give $k_1 \approx 3.1973$ and $E_1 \approx 10.7446$. These results are to be compared with $k_1 \approx 3.1693$ and $E_1 \approx 10.3065$, which follow from Eq. (7), and $k_1 \approx 3.1966$ and $E_1 \approx 10.7403$, obtained from the BAE (3).

### 3. Link to the BCS model

The weak coupling limit provides a direct link between the integrable 1D Bose gas and the integrable BCS pairing models. In this regime the Bethe roots for the ground state are of
Table I. The ground state and the leading five excitations of four weakly interacting bosons with $L = 1$ and $c = 0.025$. For each state, solutions of the asymptotic equations (7) are shown on the first line. The second line shows solutions obtained from the full BAE (3).

| $n$ | $E_n$ | $P$ | $k_4$ | $k_3$ | $k_2$ | $k_1$ | $(d_1, d_2, d_3, d_4)$ |
|-----|-------|-----|-------|-------|-------|-------|---------------------|
| 0   | 0.30000 | 0   | -0.36910 | -0.11731 | 0.11731 | 0.36910 | (0,0,0,0) |
| 0   | 0.29938 | 0   | -0.36872 | -0.11719 | 0.11719 | 0.36872 |
| 1   | 39.88783 | $2\pi$ | -0.28162 | -0.00793 | 0.26576 | 6.30379 | (1,0,0,0) |
| 1   | 39.92754 | $2\pi$ | -0.28133 | -0.00792 | 0.26548 | 6.30697 |
| 2   | 79.37604 | $4\pi$ | -0.17378 | 0.14205 | 6.13794 | 6.45378 | (1,1,0,0) |
| 2   | 79.45587 | $4\pi$ | -0.17361 | 0.14190 | 6.14129 | 6.45679 |
| 3   | 79.42614 | 0   | -6.29985 | -0.15791 | 0.15791 | 6.29985 | (1,-1,0,0) |
| 3   | 79.50607 | 0   | -6.30303 | -0.15775 | 0.15775 | 6.30303 |
| 4   | 118.7646 | $6\pi$ | -0.02380 | 6.01424 | 6.28793 | 6.56162 | (1,1,1,0) |
| 4   | 118.8844 | $6\pi$ | -0.02378 | 6.01771 | 6.29111 | 6.56452 |
| 5   | 118.8647 | $2\pi$ | -6.29590 | -0.00793 | 6.13393 | 6.44990 | (1,1,-1,0) |
| 5   | 118.9846 | $2\pi$ | -6.29907 | -0.00793 | 6.13727 | 6.45292 |

the form
\[
k_1 = -k_2 = \sqrt{E_1}, \quad k_3 = -k_4 = \sqrt{E_2}, \ldots, \quad k_{2M-1} = -k_{2M} = \sqrt{E_M}, \tag{13}
\]
where the $E_i$ satisfy the equations (cf (7))
\[
-\frac{L}{2c} + \sum_{j \neq i}^{M} \frac{2}{E_i - E_j} = -\frac{1}{2E_i}, \tag{14}
\]
for $i = 1, \ldots, M$ and the total number of bosons are even, i.e. $N = 2M$.

For an odd number of bosons, $N = 2M + 1$, the ground state Bethe roots are given by
\[
k_1 = 0, \quad k_2 = -k_3 = \sqrt{E_1}, \quad k_4 = -k_5 = \sqrt{E_2}, \ldots, \quad k_{2M} = -k_{2M+1} = \sqrt{E_M}, \tag{15}
\]
where the $E_i$ satisfy the equations
\[
\sum_{j \neq i}^{M} \frac{2}{E_i - E_j} - \frac{L}{2c} = -\frac{3}{2E_i}, \tag{16}
\]
for $i = 1, \ldots, M$.

Similar equations have arisen in a number of contexts. Of particular interest here is the connection between Eq. (14) and Richardson’s equations for the BCS pairing model in the strong coupling limit. We now make the connection with these ‘pairing interaction’ Hamiltonians more precise. In terms of the $su(2)$ algebra with commutation relations
\[
[S^z, S^\pm] = \pm S^\pm, \quad [S^+, S^-] = 2S^z, \tag{17}
\]
we consider the class of Hamiltonians acting on the $L$-fold tensor product of $su(2)$-modules, (not necessarily finite-dimensional,) with lowest weight $-S$. Specifically,

$$\mathcal{H} = \sum_{j=1}^{\mathcal{L}} 2\epsilon_j (S_j^z + SI) - g \sum_{j,k=1}^{\mathcal{L}} S_j^+ S_k^-,$$  \hspace{2cm} (18)

where $I$ is the identity operator. The single particle energy levels $\epsilon_j$ and the coupling constant $g$ are arbitrary real parameters. The energy levels of the Hamiltonians (18) are $\mathcal{E} = 2\sum_{j=1}^{m} E_j$ where the $\{E_j\}$ satisfy the Bethe equations

$$\frac{2}{g} + \sum_{k=1}^{\mathcal{L}} \frac{2S}{E_j - \epsilon_k} = \sum_{\ell \neq j}^{m} \frac{2}{E_j - E_\ell}.$$  \hspace{2cm} (19)

Suppose each energy level $\epsilon_j$ can be occupied by a spin $s$ particle so the degeneracy is $2s+1$. We introduce the creation and annihilation operators $a_\sigma, a_\sigma^\dagger$, where $\sigma = -s, -s+1, \ldots, s$ is the spin label, satisfying the usual boson commutation (resp. fermion anticommutation) relations for integer (resp. half-odd integer) values of $s$. We can define a representation of $su(2)$ through

$$S^- = -\frac{1}{2} \sum_{\sigma = -s}^{s} (-1)^{s-\sigma} a_\sigma a_{\sigma}, \quad S^+ = \frac{1}{2} \sum_{\sigma = -s}^{s} (-1)^{s-\sigma} a_\sigma^\dagger a_{\sigma},$$

$$S^z = \frac{1}{4} \sum_{\sigma = -s}^{s} (2a_\sigma^\dagger a_\sigma + (-1)^{2s} I),$$  \hspace{2cm} (20)

satisfying (17) where $\overline{\sigma} = -\sigma$. In each case the vacuum state $|0\rangle$ is a lowest weight state of weight $-S = (-1)^{2s} \frac{1}{4}(2s + 1)$. In the fermionic case (where $s$ is a half-odd integer) these representations are finite-dimensional and $S = \frac{1}{4}, 1, \frac{3}{4}, \ldots$. In the bosonic case (where $s$ is an integer) the representations are infinite-dimensional and $S = -\frac{1}{4}, -\frac{3}{4}, -\frac{5}{4}, \ldots$. For the bosonic case, we can also consider lowest weight states of the form

$$a_{\sigma_1} \, |0\rangle, \quad a_{\sigma_1}^\dagger a_{\sigma_2} \, |0\rangle, \quad a_{\sigma_1}^\dagger a_{\sigma_2}^\dagger a_{\sigma_3} \, |0\rangle, \quad \text{etc.}$$  \hspace{2cm} (21)

provided $\sigma_i \neq \overline{\sigma_j}$ for all $i, j$. When there are $z$ bosons in a lowest weight state of the form (21) the lowest weight is given by $-S = \frac{1}{4}(2z + (-1)^{2s}(2s + 1))$ giving rise to the sequence $S = -\frac{1}{2} - \frac{1}{4}, -\frac{1}{2}, -\frac{3}{4}, -\frac{1}{2} + \frac{5}{4}, \ldots$. Under these representations (18) describes pairing Hamiltonians for spin $s$ particles, and the integer $m$ in (19) denotes the number of Cooper pairs in the corresponding eigenstate. The usual reduced BCS model for spin $\frac{1}{2}$ fermions corresponds to the case $s = \frac{1}{2}$ giving $S = \frac{1}{4}$.\textsuperscript{17,22}

Following\textsuperscript{17} we consider the large $g$ limit. Let $r \leq m$ denote the number of roots $E_j$ which diverge as $g \to \infty$. For these roots we have to lowest order in $g^{-1}$

$$\frac{2}{g} + \frac{n}{E_j} = \sum_{\ell \neq j}^{r} \frac{2}{E_j - E_\ell}.$$  \hspace{2cm} (22)

where $j = 1, \ldots, r$ and

$$n = 2S\mathcal{L} + 2r - 2m.$$  \hspace{2cm} (23)
The remaining roots $E_j$, $j = r+1, \ldots, m$, are small in comparison with $g$. Comparison between (14), (16) and (22) shows we may identify $E_j = k_{2j-1}^2 = k_{2j}^2$ for $N$ even and $E_j = k_{2j}^2 = k_{2j+1}^2$ for $N$ odd, with

$$
\frac{2}{g} = \frac{L}{2c}, \quad r = M,
$$

where $n = -\frac{1}{2}$ for $N$ even, and $n = -\frac{3}{2}$ for $N$ odd. The fact that $\mathcal{L}$, $m$ are positive integers, $r$ is a non-negative integer and $r \leq m$ imposes severe restrictions on the allowed solutions of (23). For $n = -\frac{1}{2}$ the only solution is $\mathcal{L} = 1$, $m = M$ and $S = -\frac{1}{4}$ ($s = 0$, $z = 0$). For $n = -\frac{3}{2}$ we can have $\mathcal{L} = 3$, $m = M$ and $S = -\frac{1}{4}$ ($s = 0$, $z = 0$). In all cases $s$ is an integer so the mapping is to a bosonic pairing model. Moreover, since $g$ is positive, and the bosonic representations (20) are non-unitary (specifically $(S^+)^\dagger = -S^\dagger$) these systems describe repulsive pairing interactions. In the strong coupling limit, all energy levels may collapse into one multiply-degenerate level for $N$ even and into three energy levels for $N$ odd. In comparison with large pairing scattering energies, the level spacing is negligible. Therefore the multiply-degenerate levels are a reasonable expectation in the strong coupling limit.

Multiplying each of (22) by $E_j$ and taking the sum gives

$$
E(r) = \sum_{j=1}^{r} E_j = gr(r - n - 1)
= -gr(2\mathcal{L}r + r - 2m + 1).
$$

The energy function $E(r)$ has zeroes at 0 and $2m - 2\mathcal{L} - 1$ and attains the maximum value at $m - \mathcal{L} - \frac{1}{2}$. Note that for bosonic pairing models $-\mathcal{S} \geq \frac{1}{4} + \frac{z}{4}$ ($z = 0$ in this case), which imposes the lower bound $m - \frac{1}{4}$ for the value at which the maximum occurs. The energy gaps between successive levels are found to be

$$
\Delta(r) = E(r) - E(r + 1)
= 2g(\mathcal{L}r + r + 1 - m),
$$

and in particular

$$
\Delta(m-1) = 2g\mathcal{L},
\Delta(m-2) = 2g(\mathcal{L} - 1),
\Delta(m-3) = 2g(\mathcal{L} - 2).
$$

We see that $\Delta(r) < 0$ if $r \in \{0, 1, \ldots, m - 1\}$ and $\mathcal{S} \leq -\frac{3}{4} - \frac{1}{4}$. Thus $E(r)$ takes its maximal value when $r = m = M$. From Eqs. (14) and (16), the ground state energy per particle of the weakly interacting Bose gas follows as

$$
\frac{E_0}{N} = \sum_{i=1}^{N} \frac{k_i^2}{N} = \frac{2E(M)}{N} = \frac{c(N-1)}{L}.
$$
This agrees with Eq. (8) and the results of\textsuperscript{1,23,24} among others. We see that it also coincides with the highest energy per particle state of the strongly coupled BCS boson pairing model (18).

In conclusion we have considered the asymptotic solutions to the Bethe ansatz equations for the the weakly interacting 1D Bose gas. We have established that the ground state maps to the highest energy state of a strongly-coupled repulsive bosonic pairing model. However, the precise link between the integrable boson model with weakly repulsive delta interaction and the standard BCS fermionic model is quite subtle and deserves further investigation.

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