Generalized path pairs and Fuss-Catalan triangles

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Abstract

Path pairs are a modification of parallelogram polyominoes that provide yet another combinatorial interpretation of the Catalan numbers. More specifically, the number of path pairs of length \( n \) and distance \( \delta \) corresponds to the \((n - 1, \delta - 1)\) entry of Shapiro’s so-called Catalan triangle.

In this paper, we widen the notion of path pairs \((\gamma_1, \gamma_2)\) to the situation where \(\gamma_1\) and \(\gamma_2\) may have different lengths, and then enforce divisibility conditions on runs of vertical steps in \(\gamma_2\). This creates a two-parameter family of integer triangles that generalize the Catalan triangle and qualify as proper Riordan arrays for many choices of parameters. In particular, we use generalized path pairs to provide a new combinatorial interpretation for all entries in every proper Riordan array \(R(d(t), h(t))\) of the form \(d(t) = C_k(t)^i\), \(h(t) = tC_k(t)^k\), where \(1 \leq i \leq k\) and \(C_k(t)\) is the generating function for some sequence of Fuss-Catalan numbers (some \(k \geq 2\)). Closed formulas are then provided for the number of generalized path pairs across an even broader range of parameters, as well as for the number of “weak” path pairs with a fixed number of non-initial intersections.

1 Introduction

The Catalan numbers are a seemingly ubiquitous sequence of positive integers whose \(n^{th}\) entry is \(C_n = \frac{1}{n+1} \binom{2n}{n}\). The Catalan numbers satisfy the recurrence \(C_{n+1} = \sum_{i+j=n} C_i C_j\) for all \(n \geq 0\), which translates to the ordinary generating function \(C(t) = \sum_{n=0}^{\infty} C_n t^n\) as the relation \(C(t) = tC(t)^2 + 1\). It follows that \(C(t) = \frac{1 - \sqrt{1 - 4t}}{2t}\).

Hundreds of combinatorial interpretations for the Catalan numbers have been compiled by Stanley [13]. One such interpretation identifies \(C_n\) with the number of parallelogram polyominoes with semiperimeter \(n + 1\). These are ordered pairs of lattice paths \((\gamma_1, \gamma_2)\) that satisfy all of the following:
1. Both $\gamma_1$ and $\gamma_2$ are composed of $n + 1$ steps from the step set 
\{ $E = (1, 0), N = (0, 1)$ \}, where $\gamma_1$ must begin with an $N$ step 
and $\gamma_2$ must begin with an $E$ step.

2. Both $\gamma_1$ and $\gamma_2$ begin at $(0, 0)$ and end at the same point.

3. $\gamma_1$ and $\gamma_2$ only intersect at their initial and final points.

See Figure 1 for an illustration of all parallelogram polyominoes with semiperimeter 4, noting that the number of such paths is $C_3 = 5$.

![Figure 1: The $C_3 = 5$ parallelogram polyominoes with semiperimeter 4, with the 
corresponding path pairs of length 3 (and $\delta = 1$) appearing as the bold edges.]

Generalizing the notion of parallelogram polyominoes are (fat) path pairs, as introduced by Shapiro [11] and developed by Deutsch and Shapiro [4]. A path pair 
of length $n$ is an ordered pair $(\gamma_1, \gamma_2)$ of lattice paths that satisfy all of the following:

1. Both $\gamma_1$ and $\gamma_2$ are composed of $n$ steps from the step set 
\{ $E = (1, 0), N = (0, 1)$ \}.

2. Both $\gamma_1$ and $\gamma_2$ begin at $(0, 0)$.

3. Apart from at $(0, 0)$, $\gamma_1$ stays strongly above $\gamma_2$.

Now consider the path pair $(\gamma_1, \gamma_2)$, and suppose that $\gamma_1$ terminates at $(x_1, y_1)$ 
while $\gamma_2$ terminates at $(x_2, y_2)$. Clearly $x_1 < x_2$ and $y_1 > y_2$. The path pair $(\gamma_1, \gamma_2)$ 
is said to have distance $\delta$ if $x_2 - x_1 = \delta$, and in this case we write $|\gamma_2 - \gamma_1| = \delta$. We henceforth use $P_{n, \delta}$ to denote the set of all path pairs of length $n$ and distance $\delta$.

There is a simple bijection between $P_{n, 1}$ and parallelogram polynomials of semiperimeter $n + 1$, via a map that adds an $E$ step to the end of $\gamma_1$ and an $N$ step 
to the end of $\gamma_2$. See Figure 1 for an illustration of the $n = 3$ case. It follows that 
$P_{n, 1} = C_n$ for all $n \geq 0$.

Enumeration of $P_{n, \delta}$ for all $\delta \geq 1$ and $n \geq 1$ was addressed by Shapiro [11], who identified $|P_{n, \delta}| = \frac{2^\delta}{2n}(\binom{2n}{n-\delta})$ with the $(n-1, \delta-1)$ entry of his so-called Catalan triangle. See Figure 2 for the first five rows of Shapiro’s Catalan triangle, an infinite lower-triangular matrix (with zero entries suppressed) whose entries $d_{i,j}$ are generated by the recurrence $d_{0,0} = 1$ and $d_{i,j} = d_{i-1,j-1} + 2d_{i-1,j} + d_{i-1,j+1}$ for all $i \geq 1, 0 \leq j \leq i$.

Shapiro’s Catalan triangle should not be confused with the “Catalan triangle” whose $(i, j)$ entry is the ballot number $d_{i,j} = \binom{2i}{i-j} \binom{2i-j}{i}$. We alternatively refer to this second infinite lower-triangular matrix as the ballot triangle. See Aigner [1] for connections between the ballot triangle and the Catalan triangle.
The Catalan triangle is a well-known example of a proper Riordan array. Given a pair of generating functions \( d(t) \) and \( h(t) \) such that \( d(0) \neq 0, h(0) = 0, \) and \( h'(0) \neq 0, \) the associated proper Riordan array \( \mathcal{R}(d(t), h(t)) \) is the infinite lower-triangular matrix whose \((i, j)\) entry is \( d_{i,j} = [t^i]d(t)h(t)^j \). Here we use the standard notation in which \([t^r]\) identifies the coefficient of \( t^r \) in a power series. It may be verified that Shapiro’s Catalan triangle is the proper Riordan array with \( d(t) = C(t)^2 \) and \( h(t) = tC(t)^2 \).

For general information about Riordan arrays, see Rogers [10], Merlini et al. [9], or Shapiro et al. [12]. For a more focused discussion about how Riordan arrays similar to the Catalan triangle may be used to define so-called “Catalan-like numbers”, see Aigner [2].

Central to our work is the fact that every proper Riordan array \( \mathcal{R}(d(t), h(t)) \) possesses sequences of integers \( \{z_i\} \) and \( \{a_i\} \) such that

\[
d_{n,k} = \begin{cases} 
z_0d_{n-1,k} + z_1d_{n-1,k+1} + z_2d_{n-1,k+2} + \ldots & \text{for } k = 0 \text{ and all } n \geq 1; \\
\quad a_0d_{n-1,k-1} + a_1d_{n-1,k} + a_2d_{n-1,k+1} + \ldots & \text{for all } k \geq 1 \text{ and } n \geq 1. 
\end{cases} \tag{1}
\]

These sequences are referred to as the \( Z \)-sequence and the \( A \)-sequence of \( \mathcal{R}(d(t), h(t)) \), respectively. When represented as generating functions \( Z(t) = \sum_i z_i t^i \) and \( A(t) = \sum_i a_i t^i \), the \( Z \)- and \( A \)-sequences of a proper Riordan array are known to satisfy the relations

\[
d(t) = \frac{d(0)}{1-tZ(h(t))}, \quad h(t) = tA(h(t)). \tag{2}
\]

The defining recurrence of the Catalan triangle implies that it is a proper Riordan array with \( Z(t) = 2 + t \) and \( A(t) = 1 + 2t = t^2 = (1 + t)^2 \).

We pause to recap a few facts about the one-parameter Fuss-Catalan numbers, also known as the \( k \)-Catalan numbers, since they will play a major role in what follows. For any \( k \geq 2 \), the \( k \)-Catalan numbers are an integer sequence whose \( n^{th} \) entry is \( C_n^k = \binom{kn+1}{kn} \). Observe that the \( k = 2 \) case corresponds to the “original” Catalan numbers. For any \( k \geq 2 \), the \( k \)-Catalan numbers satisfy the recurrence

\[
C_{n+1}^k = \sum_{i_1 + \ldots + i_k = n} C_{i_1}^k \ldots C_{i_k}^k
\]

for all \( n \geq 0 \), implying that their generating functions \( C_k(t) = \sum_{n=0}^{\infty} C_n^k t^n \) satisfy \( C_k(t) = tC_k(t)^k + 1 \). For an introduction to the \( k \)-Catalan numbers, see Hilton and Pederson [8]. For a list of combinatorial interpretations for the \( k \)-Catalan numbers, see Heubach, Li and Mansour [7].

\[
\begin{array}{llllll}
1 & & & & & \\
2 & 1 & & & & \\
5 & 4 & 1 & & & \\
14 & 14 & 6 & 1 & & \\
42 & 48 & 27 & 8 & 1 & \\
\end{array}
\]

Figure 2: The first five rows of Shapiro’s Catalan triangle.
The goal of this paper is to simultaneously explore several generalizations of path pairs. Firstly, we eliminate the requirement that the two paths of \((\gamma_1, \gamma_2)\) have equal length, setting \(\epsilon = |\gamma_2| - |\gamma_1|\) and examining the full range of differences \(\epsilon \geq 0\) with \(|\gamma_1| \geq 0\). We also enforce conditions on the \(N\) steps of \(\gamma_2\) that are designed to mirror the generalization of the Catalan numbers to the \(k\)-Catalan numbers. We refer to the resulting combinatorial objects as \(k\)-path pairs of length \((n - \epsilon, n)\).

Section 2 focuses upon the enumeration of \(k\)-path pairs. In Subsection 2.1, we construct a two-parameter collection of infinite lower-triangular arrays \(A^{k,\epsilon}\), whose entries correspond to the number of \(k\)-path pairs of varying lengths and distances. For all \(0 \leq \epsilon \leq k - 1\), Theorem 2.2 identifies the triangle \(A^{k,\epsilon}\) with the proper Riordan array \(R(d(t), h(t))\) where \(d(t) = C_k(t)^{k-\epsilon}\) and \(h(t) = tC_k(t)^k\). In Subsection 2.2, we directly enumerate sets of \(k\)-path pairs for all \(k \geq 2\) and \(\epsilon \leq 0\). Theorem 2.5 uses the results of Subsection 2.2 to derive a closed formula for the size of all such sets, and Theorem 2.6 provides a significantly simplified formula within the range of \(0 \leq \epsilon \leq (k - 1)\delta\).

Section 3 introduces a related generalization where we now allow the two paths \((\gamma_1, \gamma_2)\) to intersect away from \((0,0)\), so long as \(\gamma_1\) stays weakly above \(\gamma_2\) for the entirety of its length. Theorem 3.2 applies the techniques of Section 2 to derive a closed formula for the number of “weak \(k\)-path pairs” whose paths intersect precisely \(m\) times away from \((0,0)\), assuming that we restrict ourselves to the range \(0 \leq \epsilon \leq (k - 1)\delta\).

### 2 Generalized \(k\)-Path Pairs

Take any pair of integers \(n, \epsilon\) such that \(0 \leq \epsilon < n\). Then define \(P_{n,\delta}\) to be the collection of ordered pairs \((\gamma_1, \gamma_2)\) of lattice paths that satisfy all of the following:

1. Both \(\gamma_1\) and \(\gamma_2\) begin at \((0,0)\) and use steps from \(\{E = (1,0), N = (0,1)\}\).
2. \(\gamma_2\) is composed of precisely \(n\) steps, the first of which is an \(E\) step.
3. \(\gamma_1\) is composed of precisely \(n - \epsilon\) steps, the first of which is a \(N\) step.
4. \(\gamma_1\) and \(\gamma_2\) do not intersect apart from at \((0,0)\).
5. The difference between the terminal \(x\) coordinates of \(\gamma_1\) and \(\gamma_2\) is \(\delta\).

The case \(\epsilon = 0\) obviously corresponds to the original notion of path pairs. If \(\gamma_2\) terminates at \((x_2, y_2)\), then \(\gamma_1\) terminates at \((x_1, y_1) = (x_2 - \delta, y_2 + \delta - \epsilon)\). In particular, \(y_1 - y_2 \geq 0\) precisely when \(\delta \geq \epsilon\).

Now fix \(k \geq 2\), and consider some \((\gamma_1, \gamma_2) \in P_{n,\delta}\). The path pair \((\gamma_1, \gamma_2)\) is said to be a \(k\)-path pair of length \((n - \epsilon, n)\) and distance \(\delta\) if the bottom path \(\gamma_2 = E^1N^{b_1}E^1N^{b_2} \ldots E^1N^{b_m}\) satisfies \(b_i = (k - 2) \text{ mod } (k - 1)\) for all \(i\). Clearly, \(2\)-path pairs correspond to the notion of path pairs discussed above.

For any \(k\)-path pair \((\gamma_1, \gamma_2)\), the bottom path \(\gamma_2\) must decompose into a sequence of length-\((k - 1)\) subpaths, each of which is either \(N^{k-1}\) or \(E^1N^{k-2}\). In particular,
the length $n$ of $\gamma_2$ must be divisible by $k-1$. To avoid a large number of empty sets, we define $\mathcal{P}_{n,\delta}^{k,\epsilon}$ to be the collection of all $k$-path pairs of length $((k-1)n-\epsilon,(k-1)n)$ and distance $\delta$.

We continue to use the notation $\delta = |\gamma_2 - \gamma_1|$ for the distance of $k$-path pairs. For any $(\gamma_1, \gamma_2) \in \mathcal{P}_{n,\delta}^{k,\epsilon}$, it is always the case that $1 \leq \delta \leq n$, with the maximum distance of $n$ only being obtained by the pair with $\gamma_1 = N^{n-\epsilon}$ and $\gamma_2 = (EN^{k-2})^n$. It follows that the sets $\mathcal{P}_{n,\delta}^{k,\epsilon}$ encompass all nonempty collections of $k$-path pairs if we range over $1 \leq \delta \leq n$ and $0 \leq \epsilon \leq (k-1)n$.

2.1 Generalized $k$-Path Pairs with $0 \leq \epsilon \leq k-1$

In order to enumerate arbitrary $\mathcal{P}_{n,\delta}^{k,\epsilon}$, we fix $k,\epsilon$ and define a recurrence with respect to $n,\delta$. This recurrence will directly generalize Shapiro’s original recurrence for the Catalan triangle [11]. We begin with the range $0 \leq \epsilon \leq k-1$, where the recursion will eventually correspond to the $Z$- and $A$-sequences of a proper Riordan array.

**Theorem 2.1.** For any $k \geq 2$, $n \geq 1$, and $0 \leq \epsilon \leq k-1$,

$$|\mathcal{P}_{n,\delta}^{k,\epsilon}| = \begin{cases} \sum_{j=1}^{k} \binom{k}{j} |\mathcal{P}_{n-1,j}^{k,\epsilon}| - \sum_{j=1}^{\epsilon} \binom{\epsilon}{j} |\mathcal{P}_{n-1,j}^{k,\epsilon}| & \text{for } \delta = 1, \text{ and} \\ \sum_{j=0}^{k} \binom{k}{j} |\mathcal{P}_{n-1,\delta-j}^{k,\epsilon}| & \text{for } \delta > 1. \end{cases}$$

**Proof.** For any length-$(k-1)$ word $w$ in the alphabet $\{E, N\}$, define $U_w$ to be the set of all $(\gamma_1, \gamma_2) \in \mathcal{P}_{n,\delta}^{k,\epsilon}$ such that $\gamma_1$ terminates with $w$ and $\gamma_2$ terminates with $N^{k-1}$. If $w$ contains precisely $j$ instances of $E$, this implies $\gamma_1 = \eta_1 w$ and $\gamma_2 = \eta_2 N^{k-1}$ for some $(\eta_1, \eta_2) \in \mathcal{P}_{n-1,\delta-j}^{k,\epsilon}$. Similarly define $V_w$ to be all $(\gamma_1, \gamma_2) \in \mathcal{P}_{n,\delta}^{k,\epsilon}$ such that $\gamma_1$ terminates with $w$ and $\gamma_2$ terminates with $EN^{k-2}$. If $w$ contains precisely $j$ instances of $E$, then $\gamma_1 = \eta_1 w$ and $\gamma_2 = \eta_2 EN^{k-2}$ for some $k$-path pair $(\eta_1, \eta_2) \in \mathcal{P}_{n-1,\delta+j-1}^{k,\epsilon}$.

By construction, $\mathcal{P}_{n,\delta}^{k,\epsilon} = (\bigcup_w U_w) \cup (\bigcup_w V_w)$.

See Figure 3 for the general form of terminal subpaths in an element $(\gamma_1, \gamma_2)$ of $U_w$ or $V_w$. In both diagrams, $(a, b)$ is fixed as the terminal point of $\gamma_1$, whereas the final $k-1$ steps of $\gamma_1$ are determined by $w$ and lie within the dotted triangle in the upper-left of each image.

Now take any length-$(k-1)$ word $w$ with precisely $j$ instances of $E$. Our strategy is to enumerate $U_w$ and $V_w$ via consideration of the injective maps $g_w : \mathcal{P}_{n-1,\delta+j}^{k,\epsilon} \rightarrow S$, $g_w(\eta_1, \eta_2) = (\eta_1 w, \eta_2 N^{k-1})$ and $h_w : \mathcal{P}_{n-1,\delta+j-1}^{k,\epsilon} \rightarrow S$, $h_w(\eta_1, \eta_2) = (\eta_1 w, \eta_2 EN^{k-2})$. Here $S$ denotes some collection of path-pairs whose elements may intersect apart from at $(0, 0)$. We clearly have $U_w \subseteq \text{Im}(g_w)$ and $V_w \subseteq \text{Im}(h_w)$ for any word $w$. We also have $U_w = \text{Im}(g_w)$ if and only if every path pair in $\text{Im}(g_w)$ is non-intersecting apart from $(0, 0)$, and $\text{Im}(h_w) = V_w$ if and only if every path pair in $\text{Im}(h_w)$ is non-intersecting apart from $(0, 0)$.
Begin with \( g_w \). The path pair \( g(\eta_1, \eta_2) = (\eta_1 w, \eta_2 N^{k-1}) \) can only feature an intersection away from \((0,0)\) if the final \( k - 1 \) steps of \( \eta_1 w \) pass through some northwest corner of \( \eta_2 N^{k-1} \). As seen in Figure 3, the largest possible \( y \)-coordinate for a northwest corner of \( \eta_2 N^{k-1} \) is \( b - \delta + \epsilon - 2k + 3 \), whereas the terminal point of \( \eta_1 \) has a \( y \)-coordinate of at least \( b - k + 1 \). Since we are assuming \( \epsilon \leq k - 1 \), we have \( \epsilon \leq \delta(k - 1) \) for all \( \delta \geq 1 \). It follows that \( b - \delta + \epsilon - 2k + 3 \leq b - k + 1 \) for all \( \delta \geq 1 \), with the case of \( b - d + \epsilon - 2k + 3 = b - k + 1 \) being impossible because the input path \((\eta_1, \eta_2)\) was assumed to be non-intersecting away from \((0,0)\). This implies that \( \eta_1 w \) cannot intersect \( \eta_2 N^{k-1} \) away from \((0,0)\) for any word \( w \).

Figure 3: Terminal subpaths for arbitrary \((\gamma_1, \gamma_2) \in U_w \) (left side) and arbitrary \((\gamma_1, \gamma_2) \in V_w \) (right side), as referenced in the proof of Theorem 2.1.

It follows that \( g_w \) represents a bijection from \( \mathcal{P}_{n-1,\delta+j}^{k,\epsilon} \) onto \( U_w \) for every word \( w \) when \( \epsilon \leq k - 1 \). Since there are \( \binom{k - 1}{j} \) words \( w \) with precisely \( j \) instances of \( E \), a total of \( \binom{k - 1}{j} \) sets \( U_w \) lie in bijection with \( \mathcal{P}_{n-1,\delta+j}^{k,\epsilon} \) for each \( 0 \leq j \leq j - 1 \). This gives

\[
\sum_w |U_w| = \sum_{j=0}^{k-1} \binom{k-1}{j} |\mathcal{P}_{n-1,\delta+j}^{k,\epsilon}| = \sum_{j=1}^{k} \binom{k-1}{j-1} |\mathcal{P}_{n-1,\delta+j-1}^{k,\epsilon}|. \tag{3}
\]

For \( h_w \), we separately consider the cases of \( \delta = 1 \) and \( \delta \geq 2 \). Begin by assuming \( \delta \geq 2 \). We once again note that \( h_w(\eta_1, \eta_2) = (\eta_1 w, \eta_2 EN^{k-2}) \) has intersections away from \((0,0)\) only when the final \( k - 1 \) steps of \( \eta_1 w \) intersect some northwest corner of \( \eta_2 EN^{k-2} \). From Figure 3, since \( \delta \geq 2 \) we see that the \( y \)-coordinate of such a corner can be at most \( b - \delta + \epsilon - 2k + 4 \). Our assumptions of \( \epsilon \leq k - 1 \) and \( \delta \leq 2 \) together ensure \( \epsilon \leq k - 3 + \delta \) and thus that \( b - \delta + \epsilon - 2k + 4 \leq b - k + 1 \), with the case of \( b - \delta + \epsilon - 2k + 4 = b - k + 1 \) being impossible because we’ve assumed that \((\eta_1, \eta_2)\) lacks intersections away from \((0,0)\). This implies that \( \eta_1 w \) cannot intersect \( \eta_2 EN^{k-2} \) away from \((0,0)\) for any word \( w \) when \( \delta \geq 2 \), and thus that \( h_w \) is a bijection from \( \mathcal{P}_{n-1,\delta+j-1}^{k,\epsilon} \) onto \( V_w \) for every word \( w \) when \( \delta \geq 2 \).

When \( \delta = 1 \), the map \( h_w \) may introduce new intersections. Fixing \( w \), either every image \( h_w(\eta_1, \eta_2) = (\eta_1 w, \eta_2 EN^{k-2}) \) will have an intersection away from \((0,0)\), or every image \( h_w(\eta_1, \eta_2) \) will lack such an intersection. That first subcase implies
that the corresponding set $V_w$ is empty, whereas that second subcase implies that $V_w$ is nonempty and in bijection with $\mathcal{P}_{n-1,\delta+j-1}^{k,\epsilon}$. We only need to enumerate how many words $w$ fall into each subcase (for each choice of $0 \leq j \leq k - 1$).

As seen on the right side of Figure 3, when $\delta = 1$ the final northwest corner of $\eta_2 E N^{k-2}$ occurs at $(a, b + \epsilon - k + 1)$. Fixing a word $w$ with precisely $j$ instances of $E$, we also see that $\eta_1$ terminates at $(a - j, b - k + j + 1)$. This means that $\eta_1$ can only pass through $(a, b + \epsilon - k + 1)$ if $j \leq \epsilon$. For any such $j \leq \epsilon$, there are precisely $\binom{k-1}{j}$ words $w$ in which this additional intersection occurs. As there are $\binom{k-1}{j}$ words $w$ with precisely $j$ instances of $E$, if $\epsilon \leq k - 1$ we know that $V_w$ is nonempty for precisely $\binom{k-1}{j} - \binom{j}{j}$ choices of $w$. Combining our results for $\delta \geq 2$ and $\delta = 1$ gives

$$\sum_w |V_w| = \begin{cases} \sum_{j=0}^{k-1} \binom{k-1}{j} |\mathcal{P}_{n-1,\delta+j-1}^{k,\epsilon}| & \text{for } \delta \geq 2, \\ \sum_{j=0}^{k-1} \left( \binom{k-1}{j} - \binom{\epsilon}{j} \right) |\mathcal{P}_{n-1,\delta+j-1}^{k,\epsilon}| & \text{for } \delta = 1. \end{cases} \tag{4}$$

Once again noting that $\mathcal{P}_{n,\delta}^{k,\epsilon} = (\bigcup_w U_w) \cup (\bigcup_w V_w)$, for $\delta \geq 2$ we have

$$|\mathcal{P}_{n,\delta}^{k,\epsilon}| = \sum_w |U_w| + \sum_w |V_w| = \sum_{j=1}^{k} \binom{k-1}{j-1} |\mathcal{P}_{n-1,\delta+j-1}^{k,\epsilon}| + \sum_{j=0}^{k-1} \binom{k-1}{j} |\mathcal{P}_{n-1,\delta+j-1}^{k,\epsilon}|$$

$$= \sum_{j=0}^{k} \left( \binom{k-1}{j-1} + \binom{k-1}{j} \right) |\mathcal{P}_{n-1,\delta+j-1}^{k,\epsilon}| = \sum_{j=0}^{k} \binom{k}{j} |\mathcal{P}_{n-1,\delta+j-1}^{k,\epsilon}|.$$

For $\delta = 1$, the facts that $0 \leq \epsilon \leq k - 1$ and $|\mathcal{P}_{n-1,0}^{k,\epsilon}| = 0$ prompt the similar result

$$|\mathcal{P}_{n,1}^{k,\epsilon}| = \sum_w |U_w| + \sum_w |V_w|$$

$$= \sum_{j=1}^{k} \binom{k-1}{j-1} |\mathcal{P}_{n-1,j}^{k,\epsilon}| + \sum_{j=0}^{k-1} \left( \binom{k-1}{j} - \binom{\epsilon}{j} \right) |\mathcal{P}_{n-1,j}^{k,\epsilon}|$$

$$= \sum_{j=0}^{k} \left( \binom{k-1}{j-1} - \binom{k-1}{j} \right) |\mathcal{P}_{n-1,j}^{k,\epsilon}| - \sum_{j=0}^{k-1} \binom{\epsilon}{j} |\mathcal{P}_{n-1,j}^{k,\epsilon}|$$

$$= \sum_{j=1}^{k} \binom{k}{j} |\mathcal{P}_{n-1,j}^{k,\epsilon}| - \sum_{j=1}^{\epsilon} \binom{\epsilon}{j} |\mathcal{P}_{n-1,j}^{k,\epsilon}|.$$

It should be noted that the methods from Theorem 2.1 may be extended to a somewhat broader range of parameters than $\epsilon \leq k - 1$. In particular, the summation of (3) may be shown to hold for all $\epsilon \leq (k - 1)\delta$, whereas the $\delta \geq 2$ summation of
may be shown to hold for all $\epsilon \leq (k - 1)(\delta - 1)$. Sadly, developing a general recursive relation for the full $\epsilon = \delta(k - 1)$ range of Theorem 2.6 is extremely involved. The enumerative usage of those recursions is also limited when $\epsilon > k - 1$, as they no longer qualify as the $A$- and $Z$-sequences of a proper Riordan array. As such, we delay the $\epsilon > k - 1$ case until Subsection 2.2, where generating function techniques may be applied to directly derive closed formulas from pre-existing results for the general case.

For each choice of $k \geq 2$ and $0 \leq \epsilon \leq k - 1$, the recursive relations of Theorem 2.1 may be used to generate an infinite lower-triangular matrix $A^{k,\epsilon}$ whose $(i, j)$ entry is $a_{i,j}^{k,\epsilon} = |P_{i+1,j+1}^{k,\epsilon}|$. These $A^{k,\epsilon}$ qualify as proper Riordan arrays:

**Theorem 2.2.** For any $k \geq 2$ and $0 \leq \epsilon \leq k - 1$, the integer triangle $A^{k,\epsilon}$ with $(i, j)$ entry $|P_{i+1,j+1}^{k,\epsilon}|$ is the proper Riordan array $R(C_k(t)^k - \epsilon, tC_k(t)^k)$, where $C_k(t)$ is the generating function for the $k$-Catalan numbers.

**Proof.** By Theorem 2.1, the array $A^{k,\epsilon}$ has $A$-sequence $A(t) = (1 + t)^k$ and $Z$-sequence $Z(t) = \frac{(1 + t)^k - (1 + t)^\epsilon}{t}$. The $k$-Catalan relation $C_k(t) = tC_k(t)^k + 1$ may then be used to verify the identities of (2):

$$d(0) = \frac{1}{1 + tZ(h(t))} = \frac{1}{1 - t^{1 + \epsilon} C_k(t)^k} = \frac{1}{1 - \frac{C_k(t)^k - C_k(t)^\epsilon}{C_k(t)^k}} = C_k(t)^\epsilon = d(t).$$

Every integer triangle $A^{k,\epsilon}$ is a Fuss-Catalan triangle of the type introduced by He and Shapiro [5], seeing as they all take the form $R(C_k^\delta, C_k^\gamma)$ for some $k \geq 2$ and some $\delta, \gamma > 0$. Many specific triangles $A^{k,\epsilon}$ also correspond to Riordan arrays that are well-represented in the literature. The triangle $A^{2,0}$ is Shapiro’s Catalan triangle, while $A^{2,0}$ and $A^{2,1}$ are two of the admissible matrices discussed by Aigner [1]. More generally, whenever $\epsilon = 0$ the triangle $A^{k,\epsilon}$ is a renewal array with “identical” $A$- and $Z$-sequences, as investigated by Cheon, Kim and Shapiro [3]. For additional results of this type, see He and Sprugnoli [6].

In a slight deviation from He and Shapiro [5], we refer to $A^{k,\epsilon}$ as the $(k, \epsilon)$-Catalan triangle. See Figure 4 for all $(k, \epsilon)$-Catalan triangles with $k = 2, 3, 4$.

One immediate consequence of Theorem 2.2 is a closed formula for the size of every set $\mathcal{P}_{k,\epsilon}^{n,\delta}$ when $0 \leq \epsilon \leq k - 1$. Observe that every cardinality $|\mathcal{P}_{n,\delta}^{k,\epsilon}| = \binom{k\delta - \epsilon}{kn - \epsilon}$ from Corollary 2.3 is the Raney number $R_{k,k\delta-\epsilon}(n - \delta)$. As defined by Hilton and Pedersen [8], the Raney numbers (two-parameter Fuss-Catalan numbers) are defined to be $R_{k,\epsilon}(n) = \left\lfloor n \right\rfloor C_k(t)^\epsilon$, with the original $k$-Catalan numbers corresponding to $C_k^n = R_{k,1}(n) = R_{k,k}(n - 1)$.

**Corollary 2.3.** For any $k \geq 2$ and $0 \leq \epsilon \leq k - 1$,

$$|\mathcal{P}_{n,\delta}^{k,\epsilon}| = \binom{n-\delta}{kn-\epsilon} C_k(t)^{k\delta-\epsilon} = \frac{k\delta - \epsilon}{kn - \epsilon} \binom{kn - \epsilon}{n - \delta}.$$
\[
\begin{array}{cccc}
\epsilon = 0 & \epsilon = 1 & \epsilon = 2 & \epsilon = 3 \\
1 & 1 & 1 & 1 \\
2 & 1 & 1 & 1 \\
k = 2 & 5 & 4 & 1 & 2 & 3 & 1 & 14 & 14 & 6 & 1 & 5 & 9 & 5 & 1 & 14 & 48 & 27 & 8 & 1 & 14 & 28 & 20 & 7 & 1 \\
1 & 1 & 1 & 1 \\
k = 3 & 12 & 6 & 1 & 7 & 5 & 1 & 3 & 4 & 1 & 55 & 33 & 9 & 1 & 30 & 25 & 8 & 1 & 12 & 18 & 7 & 1 & 273 & 182 & 63 & 12 & 1 & 143 & 130 & 52 & 11 & 1 & 55 & 88 & 42 & 10 & 1 \\
1 & 1 & 1 & 1 \\
k = 4 & 22 & 8 & 1 & 15 & 7 & 1 & 9 & 6 & 1 & 4 & 5 & 1 & 140 & 60 & 12 & 1 & 91 & 49 & 11 & 1 & 52 & 39 & 10 & 1 & 22 & 30 & 9 & 1 & 969 & 456 & 114 & 16 & 1 & 612 & 357 & 99 & 15 & 1 & 340 & 272 & 85 & 14 & 1 & 140 & 200 & 72 & 13 & 1 \\
\end{array}
\]

Figure 4: Top five rows for all \((k, \epsilon)\)-Catalan triangles \(A^{k,\epsilon}\) with \(k = 2, 3, 4\).

Proof. By the definition of \(A^{k,\epsilon}\) we have
\[
a^{k,\epsilon}_{i,j} = \left\lfloor t^i \right\rfloor C_k(t)^{k-\epsilon} (tC_k(t)^k)^j = \left\lfloor t^i \right\rfloor C_k(t)^{k-\epsilon+j}.
\]
The corollary then follows from the fact that \(|P_{n,\delta}^{k,\epsilon}| = a^{k,\epsilon}_{n-1,\delta-1} - 1|.

\[\square\]

2.2 Generalized \(k\)-Path Pairs, all \(\epsilon \geq 0\)

If \(\epsilon > k - 1\), there need not be a bijection between \(P_{n,\delta}^{k,\epsilon}\) and some Raney number \(R_{k,r}(n) = [t^n]C_k(t)^r\). This implies that the cardinalities \(|P_{n,\delta}^{k,\epsilon}|\) cannot be organized into any Fuss-Catalan triangle. One may still define an infinite lower-triangular array \(A^{k,\epsilon}\) whose \((i,j)\) entry is \(a^{k,\epsilon}_{i,j} = |P_{n-1,\delta+1}^{k,\epsilon}|\), but for \(\epsilon > k - 1\) we always have \(a^{k,\epsilon}_{0,0} = 0\) and the resulting arrays never qualify as a proper Riordan array.

For general \(\epsilon\), we still have the following decomposition for \(|P_{n,\delta}^{k,\epsilon}|\):

**Proposition 2.4.** Fix \(n \geq 1, 1 \leq \delta \leq n,\) and \(0 \leq \epsilon \leq (k-1)n\). For any pair of non-negative integers \(\epsilon_1, \epsilon_2\) such that \(\epsilon = (k-1)\epsilon_1 + \epsilon_2\),

\[
|P_{n,\delta}^{k,\epsilon}| = \sum_{i=1}^{\delta} \binom{\epsilon_1}{\delta - i} |P_{n-\epsilon_1,i}^{k,\epsilon_2}|.
\]

Proof. As seen in Figure 5, for any \((\gamma_1, \gamma_2) \in P_{n,\delta}^{k,\epsilon}\) we may divide \(\gamma_2\) into an initial subpath \(\eta_1\) of length \(n - (k-1)\epsilon_1\) and a terminal subpath \(\eta_2\) of length \((k-1)\epsilon_1\). As the length of \(\eta_1\) is divisible by \(k-1\), it is always the case that \((\gamma_1, \eta_1) \in P_{n-\epsilon_1,i}^{k,\epsilon_2}\) for some \(1 \leq i \leq \delta\).

Then consider the map \(f : P_{n,\delta}^{k,\epsilon} \to \bigcup_{i=1}^{\delta} P_{n-\epsilon_1,i}^{k,\epsilon_2}\) where \(f(\gamma_1, \gamma_2) = (\gamma_1, \eta_1)\). This map is clearly surjective. For any \(1 \leq i \leq \delta\) and any \((\gamma_1, \eta_1) \in P_{n-\epsilon_1,i}^{k,\epsilon_2}\), every way of
appending precisely $\delta - i$ copies of $E^1 N^{k-2}$ and $\epsilon_1 - \delta + i$ copies of $N^{k-1}$ to the end of $\eta_1$ (in any order) produces an element of $P_{n,\delta}^{k,\epsilon}$. It follows that the inverse image $f^{-1}(\gamma_1', \gamma_2')$ of every $(\gamma_1', \gamma_2') \in P_{n-\epsilon_1,i}^{k,\epsilon_2}$ has size $(\epsilon_1 - i)$. Ranging over $1 \leq i \leq \delta$ gives the required summation.

**Theorem 2.5.** Fix $n \geq 1$, $1 \leq \delta \leq n$, and $0 \leq \epsilon \leq (k-1)n$. For any pair of non-negative integers $\epsilon_1, \epsilon_2$ such that $\epsilon = (k-1)\epsilon_1 + \epsilon_2$ and $0 \leq \epsilon_2 \leq k-1$,

$$|P_{n,\delta}^{k,\epsilon}| = (t^{n-\epsilon_1})^{\delta} \sum_{i=1}^{\delta} \binom{\epsilon_1}{\delta - i} t^i C_k(t)^{ki-\epsilon_2}$$

$$= \sum_{i=1}^{\delta} \frac{ki - \epsilon_2}{k(n - \epsilon_1) - \epsilon_2} \binom{\epsilon_1}{\delta - i} \binom{k(n - \epsilon_1) - \epsilon_2}{n - \epsilon_1 - i}.$$

Beyond the $\epsilon \leq k-1$ case of Subsection 2.1, there are several situations where the general identity of Theorem 2.5 simplifies to give an enumeration equivalent to Corollary 2.3.

**Theorem 2.6.** Fix $n \geq 1$ and $0 \leq \epsilon \leq (k-1)n$, and take any pair of non-negative integers $\epsilon_1, \epsilon_2$ such that $\epsilon = (k-1)\epsilon_1 + \epsilon_2$ and $0 \leq \epsilon_2 \leq k-1$. For all $\delta > \epsilon_1$, as well as for all $0 \leq \epsilon \leq (k-1)\delta$, we have

$$|P_{n,\delta}^{k,\epsilon}| = [t^{n-\delta}] C_k(t)^{k\delta-\epsilon} = \frac{k\delta - \epsilon}{kn - \epsilon} \binom{kn - \epsilon}{n - \delta}.$$
\textbf{Proof.} Beginning with Theorem 2.5, when \( \delta - \epsilon_1 > 0 \) we may rewrite the bounds of the summation and then perform the change of variables \( j = \epsilon_1 - \delta + i \) to give

\[
|P_{n,\delta}^{\epsilon_1}| = [t^{n-\epsilon_1}] \sum_{i=1}^{\delta} \binom{\epsilon_1}{\delta - i} t^i C_k(t)^{ki-\epsilon_2} = [t^{n-\epsilon_1}] \sum_{i=\delta-\epsilon_1}^{\delta} \binom{\epsilon_1}{\delta - i} t^i C_k(t)^{ki-\epsilon_2} = [t^{n-\epsilon_1}] \sum_{j=0}^{\epsilon_1} \binom{\epsilon_1}{j} t^{\delta + j - \epsilon_1} C_k(t)^{kj + \delta - \epsilon_1 - \epsilon_2} = [t^{n-\epsilon_1}] t^{\delta - \epsilon_1} C_k(t)^{\delta - k\epsilon_1 - \epsilon_2} \sum_{j=0}^{\epsilon_1} \binom{\epsilon_1}{j} (tC_k(t))^j.
\]

Recognizing the binomial expansion and applying the identity \( C_k(t) = tC_k(t) + 1 \) yields

\[
|P_{n,\delta}^{\epsilon_1}| = [t^{n-\delta}] C_k(t)^{\delta - k\epsilon_1 - \epsilon_2} (1 + tC_k(t)^{\delta - \epsilon_1}).
\]

For the second range of parameters given, we separately consider \( \epsilon < (k-1)\delta \) and \( \epsilon = (k-1)\delta \). For the first subcase we always have \( \epsilon < (k-1)\delta \leq (k-1)\delta + \epsilon_2 \) and \( \epsilon - \epsilon_2 = (k-1)\epsilon_1 < (k-1)\delta \), which implies \( \epsilon_1 < \delta \) and allows us to apply our first result. When \( \epsilon = (k-1)\delta \) we may choose \( \epsilon_1 = \delta - 1 \) and \( \epsilon_2 = k - 1 \), which again implies \( \epsilon_1 < \delta \).

\[\Box\]

\section{Weak \( k \)-Path Pairs}

In this section, we loosen our restriction that generalized \( k \)-path pairs \((\gamma_1, \gamma_2)\) cannot intersect apart from \((0,0)\) and merely require that \( \gamma_1 \) stays weakly above \( \gamma_2 \). Formally, for any \( k \geq 2 \) and any set of non-negative integers \( n, \epsilon, \delta \) such that \( 0 \leq \epsilon \leq (k-1)n \) and \( 0 \leq \delta \leq n \), we define \( \widetilde{P}_{n,\delta}^{k,\epsilon} \) to be the collection of ordered pairs \((\gamma_1, \gamma_2)\) of lattice paths that satisfy all of the following:

1. Both \( \gamma_1 \) and \( \gamma_2 \) begin at \((0,0)\) and use steps from \( \{E = (1,0), N = (0,1)\} \).
2. \( \gamma_2 \) is composed of precisely \((k-1)n\) steps, the first of which is an \( E \) step.
3. \( \gamma_1 \) is composed of precisely \((k-1)n - \epsilon\) steps, the first of which is an \( N \) step.
4. \( \gamma_1 \) stays weakly above \( \gamma_2 \).
5. The difference between the terminal \( x \) coordinates of \( \gamma_1 \) and \( \gamma_2 \) is \( \delta \).
6. \( \gamma_2 = E^1 N^{b_1} E^1 N^{b_2} \ldots E^1 N^{b_m} \) satisfies \( b_i = (k-2) \mod (k-1) \) for all \( i \).
We refer to any element $(\gamma_1, \gamma_2) \in \widetilde{P}^{k,\epsilon}_{n,\delta}$ as a weak \emph{k-path pair} of distance $\delta$. Notice that $\delta = 0$ is now possible when we also have $\epsilon = 0$, corresponding to the case where $\gamma_1$ and $\gamma_2$ terminate at the same point. We refer to this special case of $\delta = \epsilon = 0$ as a closed (weak) \emph{k-path pair}. All nonempty sets $\widetilde{P}^{k,\epsilon}_{n,\delta}$ fall within the ranges $0 \leq \delta \leq n$ and $0 \leq \epsilon \leq (k - 1)n$.

Elements of $(\gamma_1, \gamma_2) \in \widetilde{P}^{k,\epsilon}_{n,\delta}$ may then be subdivided according to the number of intersections between $\gamma_1$ and $\gamma_2$. We let $\widetilde{P}^{k,\epsilon}_{n,\delta,\epsilon}$ denote the collection of $(\gamma_1, \gamma_2) \in \widetilde{P}^{k,\epsilon}_{n,\delta}$ where $\gamma_1$ and $\gamma_2$ intersect precisely $m$ times away from $(0, 0)$, and we define such path pairs to be weak $k$-path pairs with $m$ returns. It is easy to show that $\widetilde{P}^{k,\epsilon}_{n,\delta,m}$ is empty unless $0 \leq m \leq n$, and that $\epsilon$ places further restrictions on which $m$ are possible. For example, $m = n$ is only possible when $\epsilon = 0$.

We henceforth call a closed $k$-path pair with only $m = 1$ return as an irreducible (closed) $k$-path pair. Any weak $k$-path pair $(\gamma_1, \gamma_2) \in \widetilde{P}^{k,\epsilon}_{n,\delta,\epsilon}$ with precisely $m$ returns may be uniquely decomposed into a sequence of subpath pairs $(\gamma_{1,1}, \gamma_{2,1}), \ldots, (\gamma_{1,m+1}, \gamma_{2,m+1})$ such that $(\gamma_{1,i}, \gamma_{2,i})$ corresponds to an irreducible $k$-path pair for each $1 \leq i \leq m$ (after translating each subpath pair so that it begins at the origin). If $(\gamma_1, \gamma_2)$ is a closed $k$-path pair, then the final subpath pair $(\gamma_{1,m+1}, \gamma_{2,m+1})$ is empty. Otherwise, that final subpath pair corresponds to some $k$-path pair $(\gamma'_1, \gamma'_2) \in \widetilde{P}^{k,\epsilon}_{n',\delta}$ for some $n' > 0$.

To enumerate $\widetilde{P}^{k,\epsilon}_{n,\delta}$ and the $\widetilde{P}^{k,\epsilon}_{n,\delta,m}$, we begin by enumerating irreducible $k$-path pairs:

**Proposition 3.1.** Fix $k \geq 2$. For any $n \geq 1$,

$$|\widetilde{P}^{k,0}_{n,0,0}| = [t^{n-1}]C_k(t)^{k-1} = \frac{k - 1}{kn - 1} \binom{kn - 1}{n - 1}.$$  

\textit{Proof.} For any $(\gamma_1, \gamma_2) \in \widetilde{P}^{k,0}_{n,0,0}$, observe that the final step of $\gamma_1$ must be an $E$ step. This means that $\widetilde{P}^{k,0}_{n,0,0}$ lies in bijection with $P^{k,1}_{n,1}$, via the map that deletes the final step of $\gamma_1$. The result then follows from Corollary 2.3. \hfill \square

Observe that $\widetilde{P}^{2,0}_{n,0,0}$ is equivalent to the original notion of parallelogram polyominoes with semiperimeter $n$. Proposition 3.1 recovers this preexisting combinatorial interpretation of the Catalan numbers as $|\widetilde{P}^{2,0}_{n,0,1}| = [t^{n-1}]C(t) = C_{n-1}$. For any $k \geq 2$, one could define the elements of $\widetilde{P}^{2,0}_{n,0,1}$ as $k$-parallelogram polyominoes with semiperimeter $(k - 1)n$, although for $k > 2$ these objects do not provide a combinatorial interpretation for the $k$-Catalan numbers.

The primary application of Proposition 3.1 is that it may be used to quickly enumerate any collection $\widetilde{P}^{k,\epsilon}_{n,\delta,m}$, assuming $\epsilon$ and $\delta$ fall within the range proscribed by Theorem 2.6:

**Theorem 3.2.** Fix $n \geq 1$ and $k \geq 2$. For any non-negative integers $\delta, \epsilon, m$ such that $\epsilon = \delta = 0$ or $0 \leq \epsilon \leq (k - 1)\delta$,

$$|\widetilde{P}^{k,\epsilon}_{n,\delta,m}| = [t^{n-\delta-m}]C_k(t)^{k\delta - \epsilon + (k - 1)m} = \frac{k\delta - \epsilon + (k - 1)m}{kn - \epsilon - m} \binom{kn - \epsilon - m}{n - m - \delta}.$$
Proof. By Proposition 3.1, for any \( k \geq 2 \) the generating function of irreducible \( k \)-path pairs is \( \sum_{i=0}^{\infty} |\tilde{P}_{k,0,0}^{i}| t^i = tC_k(t)^{k-1} \). From Theorem 2.6, when \( 0 \leq \epsilon < (k-1)\delta \) we also have the generating function \( \sum_{i=0}^{\infty} |\tilde{P}_{k,\epsilon,\delta}^{i}| t^i = t^\delta C_k(t)^{k\delta - \epsilon} \). We treat the two cases of the theorem statement separately.

For the \( \epsilon = \delta = 0 \) case, every element of \( \tilde{P}_{k,0,0}^{i,m} \) may be uniquely decomposed into a sequence of \( m \) non-empty irreducible \( k \)-path pairs. It follows that
\[
\sum_{i=0}^{\infty} |\tilde{P}_{k,0,0}^{i,m}| t^i = (tC_k(t)^{k-1})^m = t^m C_k(t)^{(k-1)m}.
\]
In this case we then have
\[
|\tilde{P}_{n,0,0}^{k,0}| = [t^n] t^m C_k(t)^{(k-1)m} = [t^n] C_k(t)^{(k-1)m}.
\]

For the \( 0 < \epsilon < (k-1)\delta \) case, every element of \( \tilde{P}_{k,\epsilon,\delta}^{i,m} \) may be uniquely decomposed into a sequence of \( m \) non-empty irreducible \( k \)-path pairs and an element of \( P_{n',\epsilon}^{k,\delta} \) for some \( 0 < n' < n - m \). Here we have
\[
\sum_{i=0}^{\infty} |\tilde{P}_{k,\epsilon,\delta}^{i,m}| t^i = (tC_k(t)^{k-1})^m t^\delta C_k(t)^{k\delta - \epsilon} = t^\delta + m C_k(t)^{k\delta - \epsilon + (k-1)m}.
\]
For this second case we then have
\[
|\tilde{P}_{n,\epsilon,\delta}^{k,\delta}| = [t^n] t^{\delta + m C_k(t)^{k\delta - \epsilon + (k-1)m}} = [t^n - m] C_k(t)^{k\delta - \epsilon + (k-1)m}.
\]

\[\square\]

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(Received 7 July 2020; revised 1 Feb 2021)