Hybrid Euler Method for Discretizing Continuous-Time Tomographic Dynamical System

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Abstract

To discretize a nonlinear differential equation, we have previously proposed a hybrid method constructed as a combination of the additive and multiplicative Euler methods. In this study, we formulate the vector field for which the hybrid Euler method is effective. Then, we evaluate the method through numerical and physical experiments for a tomographic dynamical system using, respectively, a sinogram synthesized by a digital phantom and a measured projection acquired from an X-ray computed tomography scanner. We found that the hybrid Euler method has an advantage over both the additive and multiplicative Euler methods.

1. Introduction

The Euler method is a well-known iterative algorithm for obtaining an approximate solution to a nonlinear differential equation. When considering the computational cost of the procedure, a larger step size of the first-order approximation satisfying a given tolerance of accuracy is desirable. Whereas the ordinary Euler method has an additive formula, it is reported that, for some cases, a discretization based on the multiplicative calculus gives a better result regarding the numerical stability for a relatively large step size [1, 2].

We previously presented a continuous-time nonlinear dynamical system for reconstructing tomographic images and a novel method for discretizing the differential equation [3]. The proposed discretization method is constructed as a combination of the additive and multiplicative Euler methods. In this study, we formulate the vector field for which the hybrid Euler method is effective. Moreover, we evaluate the method through numerical and physical experiments for a tomographic dynamical system using, respectively, a sinogram synthesized by a digital phantom and a measured projection acquired from an X-ray computed tomography scanner. Numerical stability was analyzed with the variation in step size used to discretize the dynamical system constructed with noise-free and noisy projection data. We found that the hybrid Euler method has an advantage over both the additive and multiplicative Euler methods. Consequently, we obtained an iterative algorithm for reconstructing high-quality images with less computation time.

2. Target System

Let us define a dynamical system described by a system of autonomous ordinary differential equations with variable \( x \in \mathbb{R}^J \):

\[
\frac{dx_j(t)}{dt} = x_j(t)(f_j(x(t)) + g_j(x(t))), \quad t \in \mathbb{R}
\]

for \( j = 1, 2, \ldots, J \), which is assumed to possess an equilibrium \( e \in \mathbb{R}^J_+ \), where \( f_j \) and \( g_j \) are sufficiently smooth functions satisfying \( f_j(e) = 0 \) and \( g_j(e) = 0 \) for any \( j \), and \( \mathbb{R}^J_+ \) denotes the set of positive real numbers. We consider an initial value problem for the system with an initial value given by \( x(0) =: x_0 \in \mathbb{R}^J_+ \). Then, we prove that the trajectory \( x(t) \) remains in the subspace \( \mathbb{R}^J_+ \) for all \( t \geq 0 \) according to the uniqueness of solutions to the initial value problem. The system in Eq. (1) appears in, for example, a method of solving inverse problems for image reconstruction in computed tomography [3].

3. Additive and Multiplicative Euler Methods

The forward Euler method is a first-order numerical procedure for obtaining an approximate solution to differential equations. Whereas the well-known Euler method has an additive formula, there is a discretization based on the multiplicative calculus. The additive and multiplicative Euler formula of the iterative variable \( z(k) \), \( k = 0, 1, 2, \ldots \), for the system in Eq. (1) are respectively given by

\[
z_j(k+1) = z_j(k)(1 + \delta (f_j(z(k))) + g_j(z(k)))
\]

and

\[
z_j(k+1) = z_j(k) \exp(\delta f_j(z(k))) \exp(\delta g_j(z(k)))
\]
for $j = 1, 2, \ldots, J$ with $z(0) = x^0$, where $\delta$ denotes the step size of each iteration.

4. Proposed Hybrid Euler Method

By combining the additive and multiplicative calculus for the functions $f$ and $g$ in Eq. (1), respectively, we obtain the iterative formula of a hybrid Euler method as

$$z_j(k + 1) = z_j(k)(1 + \delta f_j(z(k))) \exp(\delta g_j(z(k)))$$  \hspace{1cm} (4)

for $j = 1, 2, \ldots, J$ and $k = 0, 1, 2, \ldots$, by replacing $\exp(\delta f_j(z))$ as the multiplicative term in Eq. (3) with the additive term $(1 + \delta f_j(z))$, which is a first-order term in the Taylor series expansion of the function $\exp(\delta f_j(z))$ with $z$ in the neighborhood of $e$, while preserving the multiplication of $z_j$ for any $j$.

The hybrid Euler discretization is effective for the practical calculation of an initial value problem in Eq. (1) from the viewpoint of choosing a larger step size $\delta$ when either additive or multiplicative calculus is better for each term of the partial vector fields $x_j f_j(x)$ and $x_j g_j(x)$.

5. Simple Model

We introduce a simple toy model for which the hybrid method is effective. Consider the case $J = 1$ and the functions $f$ and $g$ in Eq. (1) defined as

$$f(x) := \frac{e}{x} - 1$$ \hspace{1cm} (5)

and

$$g(x) := \log \left( \frac{e}{x} \right)$$ \hspace{1cm} (6)

respectively. Note that $e$ is not only an equilibrium of the differential equation in Eq. (1) but also a fixed point of each of the difference equations in Eqs. (2), (3), and (4). The characteristic multiplier of the fixed point for each discrete-time dynamical system is analytically obtained as $1 - 2\delta$. Therefore, the fixed point is stable when $\delta < 1$ [4].

A numerical experiment was performed. Figure 1 shows a solution trajectory $x(t)$, $t \in [0, 6]$, to Eq. (1) (magenta solid line) and iterative points $z(k)$, $k = 0, 1, \ldots, 6$, obtained using Eqs. (2) (cyan filled circles), (3) (green filled circles), and (4) (blue filled circles) with $\delta = 0.5$. Here, $e = 0.5$ and $x^0$ was 0.1 and 1.6 as shown in Figs. 1(a) and (b), respectively. From the viewpoints of both the convergence to the stable fixed point and the accuracy of numerical integration for the differential equation, the additive (resp. multiplicative) and hybrid methods are better at an earlier time than the multiplicative (resp. additive) method as shown in Fig. 1(a) (resp. Fig. 1(b)). Therefore, the hybrid method is the best of the three methods regardless of the selection of initial state values. If the two forms of the right-hand sides in Eqs. (5) and (6) are exchanged with each other, another experiment shows that the corresponding hybrid method does not demonstrate good performance. The functions in Eqs. (5) and (6) indicate appropriate forms for which the hybrid method works well.

6. Practical Model

As a practical example, we treat a tomographic dynamical system.

6.1 Tomographic dynamical system

For given $y \in R_{++}^I$ and $A \in R^{I \times J}$, consider a dynamical system described in Eq. (1) with

$$f_j(x) = (1 - \alpha) \lambda_j \sum_{i=1}^I A_{ij} \left( \frac{y_i}{A_{ij} x} - 1 \right)$$ \hspace{1cm} (7)

and

$$g_j(x) = \alpha \lambda_j \sum_{i=1}^I A_{ij} \log \frac{y_i}{A_{ij} x}$$ \hspace{1cm} (8)

Figure 1: Trajectories $x(t)$ and iterative points $z(k)$ emanating from initial values (a) $x^0 = 0.1$ and (b) $x^0 = 1.6$ for simple model.
where
\[ \lambda_j = \left( \sum_{i=1}^{f} A_{ij} \right)^{-1} \] (9)

The system is derived to solve an inverse problem of image reconstruction in computed tomography, where an unknown variable \( x \) corresponds to the image density and \( y \) and \( A \) indicate a measured projection and a projection operator, respectively, modeled as
\[ y = Ax \] (10)

We previously presented [3] a tomographic dynamical system for optimizing the \( \alpha \)-skew \( J \)-divergence between the measured projection \( y \) and the forward projection \( Ax \), defined by
\[
J_{\alpha}(x) := \frac{1}{1-\alpha} KL(y, Ax) + \frac{\alpha}{1-\alpha} KL(Ax, y) \tag{11}
\]
\[ = \sum_{i=1}^{f} \left( 1 - \alpha \right) \left( y_i \log \frac{y_i}{Ax_i} + Ax_i - y_i \right) + \alpha \left( Ax_i \log \frac{Ax_i}{y_i} + y_i - Ax_i \right) \]

with the parameter \( \alpha \) in the interval \([0,1]\). The premetric measure \( J_{\alpha} \) means the average of two mutually alternative Kullback–Leibler divergences [5], \( KL(y, Ax) \) and \( KL(Ax, y) \), when \( \alpha = 0.5 \). We proved the stability of the equilibrium \( e \) of Eq. (1) by applying the Lyapunov theorem to \( J_{\alpha} \) as a Lyapunov function. For practical application to image reconstruction with a large pixel size, the numerical cost of obtaining a solution to the initial value problem of the dynamical system should be reduced. To reconstruct a high-quality image with a short computation time, we need an effective Euler method with a numerical discretization. The approach of creating an iterative reconstruction algorithm by discretizing a continuous-time dynamical system based on an objective function as a Lyapunov function is useful for radiation dose reduction as well as high-quality image reconstruction [6, 7, 8, 9].

6.2 Experiment

Numerical and physical experiments were carried out on the continuous-time dynamical system and its three types of discretization. The step size \( \delta \) for the Euler methods and the parameter \( \alpha \) in Eqs. (7), (8), and (11) were selected to be 1 and 0.5, respectively.

6.2.1 Digital phantom

We performed an experiment using a Shepp–Logan phantom with \( J = 16,384 \) (128 \( \times \) 128 pixels), as shown in Fig. 2(a), and projection data with \( J = 33,120 \) (184 acquisition bins and 180 projection directions over 180 degrees) stored in a sinogram, as illustrated in Fig. 2(b). The projection \( y \) was created using \( y = Ae + \sigma \), where \( e \) and \( \sigma \) respectively denote the phantom and a white Gaussian noise resulting in 30 dB of signal-to-noise ratio (SNR).

Figure 2: (a) Shepp–Logan phantom image and (b) sinogram

Figure 3 shows the result of the numerical simulation of solving the differential equation in Eq. (1) (magenta line) and for the two Euler methods in Eq. (2) (cyan filled circles) and Eq. (4) (blue filled circles) with the nonlinear functions \( f \) and \( g \) in Eqs. (7) and (8), respectively. The iterative solution to Eq. (3) diverges at the second iteration and gives a huge value of the objective function. We see that the hybrid method approximates the solution to the differential equation, whereas the iterative points obtained by the multiplicative and additive Euler methods diverge and oscillate, respectively.

\[ J_{0.5}(x(t)), J_{0.5}(z(k)) \]

Figure 3: Transition of objective functions \( J_{0.5}(x(t)) \) and \( J_{0.5}(z(k)) \) for numerical phantom model

6.2.2 Physical phantom

We examined a physical experiment using a measured projection acquired from an X-ray computed tomography scanner (Toshiba Medical Systems, Japan) with a body phantom [10] (Kyoto Kagaku, Japan). The scan conditions include a tube voltage of 100 kVp and a tube current of 50 mA. The sinogram with \( J = 574,200 \) (957 acquisition bins and 600 projection directions over 180 degrees) is shown in Fig. 4. The number of image pixels for reconstruction was set to \( J = 454,276 \) (674 \( \times \) 674 pixels). Figure 5 illustrates...
the time course of the objective functions along the solution trajectories to Eqs. (1), (2) and (4). The legend is the same as that used in Fig. 3. The result shows a similar behaviour to that in Fig. 3 and the solution to Eq. (3) diverges at the second iteration, indicating that the hybrid method exhibits the best performance. Here, as shown for the convergence process of the hybrid method, there is a bias to the value of the objective function for the differential equation. The reason why it remains at a high number of iterations is because the first-order approximation with a large step size results in the accumulation of error. The insufficient accuracy of the computation when $\delta = 1$ can be confirmed by the fact that a smaller step size, e.g., $\delta = 0.1$, gives an exact approximation for the numerical integration. It is noteworthy that the hybrid Euler method has a stable stationary solution even for a relatively large step size, which is advantageous for an image reconstruction method with fast computation. An example of the reconstructed image at the 10th iteration of the hybrid algorithm is shown in Fig. 6.

7. Conclusion

We evaluated the hybrid Euler method constructed as a combination based on additive and multiplicative calculus. We gave a simple vector field for which the hybrid Euler method is effective. Through numerical and physical experiments for a tomographic dynamical system, we found that the hybrid Euler method has an advantage over both the additive and multiplicative Euler methods.