On Forgetting in Tractable Propositional Fragments

Yisong Wang
Department of Computer Science and Technology,
Guizhou University, China, 550025

Abstract

Distilling from a knowledge base only the part that is relevant to a subset of alphabet, which is recognized as forgetting, has attracted extensive interests in AI community. In standard propositional logic, a general algorithm of forgetting and its computation-oriented investigation in various fragments whose satisfiability are tractable are still lacking. The paper aims at filling the gap. After exploring some basic properties of forgetting in propositional logic, we present a resolution-based algorithm of forgetting for CNF fragment, and some complexity results about forgetting in Horn, renamable Horn, q-Horn, Krom, DNF and CNF fragments of propositional logic.

Keywords: Forgetting; CNF; Horn theories; Algorithms; Complexity

1 Introduction

Motivated from Lin and Reiter’s seminal work in first-order logic [1], the notion of forgetting – distilling from a knowledge base only the part that is relevant to a subset of the alphabet – has attracted extensive interests [2] [3]. A dual notion of forgetting in mathematical logic is called uniform interpolation [4]. In artificial intelligence, it has been studied under many different names including variable eliminating, irrelevance, independence, irredundancy, novelty, or separability [5].

In recent years, researchers have developed forgetting notions and theories in other non-classical logic systems, such as forgetting in logic programs under answer set/stable model semantics [6] [7] [8] [9] [10], forgetting in description logic [11] [12], and knowledge forgetting in modal logic [13] [14] [15]. It is commonly recognized that forgetting has both theoretical and practical interest as it can be
used for conflict solving [6, 3] and knowledge compilation [13, 16], and it is also closely related to other logical notions, including strongest necessary and weakest sufficient conditions [17], strongest and weakest definitions [18] and so on.

Though forgetting has been extensively investigated from various aspects of different logical systems, in standard propositional logic, a general algorithm of forgetting and its computation-oriented investigation in various fragments whose satisfiability are tractable are still lacking.

Firstly, the syntactic forgetting operator, which is defined as $\text{Forget}(\Sigma, p) = \Sigma[p/\top] \lor \Sigma[p/\bot]$ where $\varphi[p/\top]$ (resp. $\varphi[p/\bot]$) is obtained from $\varphi$ be replacing $p$ with $\top$ (resp. $\bot$), results in a disjunctive formula. Thus, it violates categoricity for non-disjunctive formulas, e.g., if $\Sigma$ is a conjunctive normal form (CNF) formula then $\text{Forget}(\Sigma, p)$ is not a CNF formula any longer. Though one can transform a CNF formula into an equivalent disjunctive normal form (DNF) formula, the translation will bring about exponential explosion if no fresh atoms are allowed.

Secondly, from the perspective of computation, Lang et al. have showed that deciding if a formula is independent of a set of atoms (called $\text{VAR-INDEPENDENCE}$) is co-NP-complete, and deciding if two formulas are equivalent on a common signature (called $\text{VAR-EQUIVALENCE}$) is $\Pi^P_2$-complete [2]. To our best knowledge, such two reasoning problems remain unknown for many interesting fragments of propositional logic, such as Horn [19], renamable Horn [20] (ren-Horn in short), q-Horn theories [21, 18] and so forth.

In the paper we mainly focus on CNF fragments of propositional logic, for which a resolution-based algorithm of forgetting is presented at first. Accordingly, we show that forgetting is categorical in the Horn, ren-Horn, q-Horn, double Horn [22] and Krom [23] (or 2-CNF) fragments. Namely, the result of forgetting from a Horn (resp. ren-Horn, q-Horn, double Horn and Krom [23] (or 2-CNF)) theory is Horn (resp. ren-Horn, q-Horn, double Horn and Krom) expressible.

More importantly, from the perspective of knowledge bases evolving, we are also interested in the following reasoning problems about forgetting, besides the $\text{VAR-INDEPENDENCE}$ and $\text{VAR-EQUIVALENCE}$ in [2], where $\text{Forget}(\varphi, V)$ stands for a result of forgetting $V$ from formula $\varphi$.

1. $\text{[VAR-INDEPENDENCE]}$ If a knowledge base $\Pi$ is independent of a set $V$ of atoms, i.e. $\text{Forget}(\Pi, V) \equiv \Pi$.

2. After a knowledge base $\Sigma$ has evolved from a knowledge base $\Pi$ by incorporating some knowledge additionally on a set $V$ of new propositions,

   [VAR-WEAK] if the restriction of $\Sigma$ on the signature of $\Pi$ is at most as strong as $\Pi$, i.e. $\Pi \models \text{Forget}(\Sigma, V)$.

   [VAR-STRONG] if the restriction of $\Sigma$ on the signature of $\Pi$ is at least as strong as $\Pi$, i.e. $\text{Forget}(\Sigma, V) \models \Pi$. 

2
Table 1: Complexity results

|                         | CNF           | Horn/ren-Horn/q-Horn | Krom | DNF           |
|-------------------------|--------------|----------------------|------|--------------|
| VAR-EQUIVALENCE         | $\Pi_2^c$-c  | co-NP-c              | P    | co-NP-c      |
| VAR-INDEPENDENCE        | co-NP-c      | P                    | P    | co-NP-c      |
| VAR-WEAK                | $\Pi_2^c$-c  | co-NP-c              | P    | co-NP-c      |
| VAR-STRONG              | co-NP-c      | P                    | P    | co-NP-c      |
| VAR-MATCH               | $\Pi_2^c$-c  | co-NP-c              | P    | co-NP-c      |
| VAR-ENTAILMENT          | $\Pi_2^c$-c  | co-NP-c              | P    | co-NP-c      |

[VAR-MATCH] if the restriction of $\Sigma$ on the signature of $\Pi$ perfectly matches $\Pi$, i.e. Forget($\Sigma$, $V$) $\equiv$ $\Pi$. This is also known as the forgetting result checking, i.e. if $\Pi$ is a result of forgetting $V$ from $\Sigma$.

(3) After two knowledge bases $\Pi$ and $\Sigma$ have evolved from a common knowledge base by incorporating some knowledge additionally on a set $V$ of new propositions,

[VAR-ENTAILMENT] if the restriction of one knowledge base on its original signature is at most as strong as that of the other, i.e. Forget($\Pi$, $V$) $|$= Forget($\Sigma$, $V$).

[VAR-EQUIVALENCE] if the restriction of the two knowledge bases on a common signature are equivalent, i.e. Forget($\Pi$, $V$) $\equiv$ Forget($\Sigma$, $V$).

We answer these problems for CNF, DNF, Horn, ren-Horn, q-Horn, and Krom fragments of propositional logic. The main complexity results are summarized in Table 1, from which one can see that for Krom (resp. DNF) fragments, all of the six problems are tractable (resp. co-NP-complete). While comparing Horn and its variants with CNF fragments, the corresponding complexity of the former is one level below the latter in the complexity hierarchy.

The rest of the paper are organized as follows. The basic notations of propositional logics and its fragments are briefly introduced in Section 2. Forgetting and its basic properties, algorithms and complexity are presented in Section 3. Related work and concluding remarks are discussed in Section 4 and 5 respectively.

2 Preliminaries

We assume a underlying propositional language $L_A$ with a finite set $A$ of atoms, called the signature of $L_A$. A literal is either an atom $p$ (called positive literal) or its negation $\neg p$ (called negative literal). The complement of a literal $l$ is $\neg l$. The formulas (of $L_A$) are defined as usual using connectives $\land, \lor, \rightarrow, \leftrightarrow$ and $\neg$.
We assume two propositional constants $\top$ and $\bot$ for tautology and contradiction respectively. A theory is a finite set of formulas. For a theory $\Sigma$, we use the following denotations:

- $\neg \Sigma = \{\neg \varphi | \varphi \in \Sigma\}$,
- $\wedge \Sigma = \wedge_{\varphi \in \Sigma} \varphi$,
- $\vee \Sigma = \vee_{\varphi \in \Sigma} \varphi$, and
- $\text{Var}(\Sigma)$ stands for the set of all atoms occurring in $\Sigma$.

An interpretation is a set of atoms, which assigns $true$ to the atoms in the set and $false$ to the others. The notion of satisfaction between an interpretation $I$ and a formula $\varphi$, written $I \models \varphi$, is inductively defined in the standard manner. In this case $I$ is called a model of $\varphi$. By $\text{Mod}(\varphi)$ we denote the set of models of $\varphi$.

A formula $\psi$ is a logical consequence of a formula $\varphi$, denoted by $\varphi \models \psi$, if $\text{Mod}(\varphi) \subseteq \text{Mod}(\psi)$. Two formulas $\varphi$ and $\psi$ are equivalent, written $\varphi \equiv \psi$, if $\varphi \models \psi$ and $\psi \models \varphi$. A formula $\psi$ is irrelevant to a set $V$ of atoms, denoted by $\text{IR}(\psi, V)$, if there is a formula $\varphi$ such that $\psi \equiv \varphi$ and $\text{Var}(\varphi) \cap V = \emptyset$. Otherwise, $\psi$ is relevant to $V$.

### 2.1 Clauses and terms

In the following we assume that $\neg \neg \varphi$ is shortten to $\varphi$ where $\varphi$ is a formula, unless explicitly stated otherwise. A clause is an expression of the form $l_1 \lor \cdots \lor l_n$ ($n \geq 0$) where $l_i$ ($1 \leq i \leq n$) are literals such that $l_i \neq \neg l_j$ for every $i, j$ ($1 \leq i < j \leq n$). It is an empty clause in the case $n = 0$, which means $false$. Dually, a term is an expression of the form $l_1 \land \cdots \land l_n$ ($n \geq 0$) where $l_i$ ($1 \leq i \leq n$) are literals such that $l_i \neq \neg l_j$ for every $i, j$ ($1 \leq i < j \leq n$). By abusing the notation, we identify a clause $l_1 \lor \cdots \lor l_n$ and a term $l_1 \land \cdots \land l_n$ with the set $\{l_1, \ldots, l_n\}$ when it is clear from its context.

A conjunctive normal form (CNF) formula is a conjunction of clauses, and a disjunctive normal form (DNF) formula is a disjunction of terms. A $k$-CNF (resp. $k$-DNF) formula is a CNF (resp. DNF) formula whose each clause (resp. term) contains no more than $k$ literals. In particular, 2-CNF formulas are called Krom formulas [23].

A prime implicate of a formula $\varphi$ is a clause $c$ such that $\varphi \models c$ and $\varphi \nmid c'$ for every proper subclause $c' \subset c$. Dually, a prime implicant of $\varphi$ is a term $t$ such that $t \models \varphi$ and $t' \nmid \varphi$ for each proper subterm $t' \subset t$. A CNF (resp. DNF) formula

The definition of relevant is equivalent with, but slightly different from, that of [24], in which $\psi$ is relevant to $V$ if there is a prime implicate of $\psi$ which mentions some atom from $V$. 

4
is prime, if it contains only prime implicates (resp. implicants). By $\Pi(\psi)$ (resp. $\text{IP}(\psi)$) we denote the set of prime implicates (resp. implicants) of formula $\psi$.

In the following we shall identify a theory $\Sigma$ with the formula $\bigwedge \Sigma$ when there is no confusion. The following lemma is well-known [25].

**Lemma 1** Let $\Sigma$ be a theory and $\varphi$ be a term. Then

1. $\bigwedge \Pi(\Sigma) \equiv \bigvee \text{IP}(\Sigma) \equiv \Sigma$.
2. $\varphi$ is a prime implicant of $\Sigma$ iff $\neg \varphi$ is a prime implicate of $\neg \Sigma$.
3. If $\Pi \equiv \Sigma$ then $\Pi(\Sigma) = \Pi(\Pi)$ and $\text{IP}(\Sigma) = \text{IP}(\Pi)$.

Two clauses $c, c'$ are resolvable, if there is an atom $p$ such that $p, \neg p \in c \cup c'$ and $c^* = (c \cup c') \setminus \{p, \neg p\}$ is a legal clause, viz, $c^*$ contains no pair of complement literals. In this case we denote the clause $c \cup c' \setminus \{p, \neg p\}$ by $\text{res}(c, c')$, which is called their resolvent; otherwise, $\text{res}(c, c')$ is undefined. It is well-known that all prime implicates of a CNF formula $\varphi$ can be generated by resolution.

### 2.2 Horn formulas and its variants

In the following, by $\text{Pos}(c)$ (resp. $\text{Neg}(c)$) we denote the set of atoms occurring positively (resp. negatively) in the clause or term $c$. In this sense a clause $c$ can be written as $\text{Pos}(c) \cup \neg \text{Neg}(c)$.

A clause $c$ is Horn [19] if $|\text{Pos}(c)| \leq 1$. Here $|D|$ denotes the number of elements in the set $D$. A Horn formula is a conjunction of Horn clauses. A formula $\varphi$ is Horn expressible if there is a Horn formula $\psi$ such that $\psi \equiv \varphi$. A Horn formula $\varphi$ is double Horn [22] if there is a Horn formula $\psi$ such that $\psi \equiv \neg \varphi$, i.e., the negation of $\varphi$ is also Horn expressible.

Given a formula $\varphi$ and $V \subseteq A$, we denote $\text{ren}(\varphi, V)$ the result of replacing every occurrence of atom $p \in V$ in $\varphi$ by $\neg p$ and $\neg \neg p$ is shortened to $p$. For instance $\text{ren}(p_1 \lor \neg p_2 \lor \neg p_3, \{p_1, p_2\})$ is the formula $\neg p_1 \lor p_2 \lor \neg p_3$. A CNF formula $\varphi$ is Horn renamable [20] iff there exists a Horn renaming for it, i.e., $\text{ren}(\varphi, V)$ is a Horn formula for some $V \subseteq A$.

**Definition 1** ([21, 18]) A CNF theory $\Sigma$ has a QH-partition iff there exists a partition $\{Q, H\}$ of $\text{Var}(\Sigma)$ s.t for every clause $\delta$ of $\Sigma$, the following conditions hold:

1. $|\text{Var}(\delta) \cap Q| \leq 2$.
2. $|\text{Pos}(\delta) \cap H| \leq 1$.
3. If $|\text{Pos}(\delta) \cap H| = 1$ then $\text{Var}(\delta) \cap Q = \emptyset$. 

5
A CNF theory $\Sigma$ is $q$-Horn iff there exists a $q$-Horn renaming for it \cite{21}, i.e., there is a set $V \subseteq A$ such that replacing in $\Sigma$ every occurrence of $p \in V$ by $\neg p$ leads to a CNF theory having a QH-partition $\{Q, H\}$. Here $\neg \neg p$ is shorten to $p$. It is not difficult to see that, every Horn theory is Horn renamable, every Horn renamable theory is $q$-Horn ($Q = \emptyset$), and every 2-CNF theory is also $q$-Horn ($H = \emptyset$). A CNF formula $\varphi$ is Krom (resp. ren-Horn and $q$-Horn) expressible if there is Krom (resp. ren-Horn and $q$-Horn) formula $\psi$ such that $\varphi \equiv \psi$.

In terms of Lemma\cite{1} the following lemma are well-known.

**Lemma 2** Let $\Sigma$ be a CNF theory. The following conditions are equivalent.

(i) $\Sigma$ is Horn expressible.

(ii) $Pl(\Sigma)$ is a Horn theory.

(iii) $M_1 \models \Sigma$ and $M_2 \models \Sigma$ imply $M_1 \cap M_2 \models \Sigma$, i.e. $Mod(\Sigma)$ is closed under intersection.

It is known that it is tractable to recognize if a CNF theory is ren-Horn \cite{20, 26}, $q$-Horn \cite{27}, or double-Horn \cite{22}, and the satisfiability of ren-Horn, $q$-Horn and double Horn formulas are all tractable.

**Proposition 1** Let $\Sigma$ be a CNF theory, $V \subseteq A$ and $c_1, c_2$ two resolvable clauses of $\Sigma$. We have the following.

(i) $res(\text{ren}(c_1, V), \text{ren}(c_2, V)) = \text{ren}(\text{res}(c_1, c_2), V)$.

(ii) If two subsets $Q, H$ of $A$ with $Q \cap H = \emptyset$ and $\text{Var}(c_1 \cup c_2) \subseteq Q \cup H$ satisfy the conditions (i), (ii) and (iii) of Definition\cite{1} for both $c_1$ and $c_2$, then $Q$ and $H$ satisfy the same conditions for $\text{res}(c_1, c_2)$ as well.

**Proof:** Without loss of generality, suppose $c_1 = \{p\} \cup c'_1$ and $c_2 = \{-p\} \cup c'_2$.

(i) Note that $\text{res}(c_1, c_2) = c'_1 \cup c'_2$, $\text{ren}(c_1, V) = \text{ren}(p, V) \cup \text{ren}(c'_1, V)$ and $\text{ren}(c_2, V) = \text{ren}(-p, V) \cup \text{ren}(c'_2, V)$. Due to the fact that $\text{ren}(c_1, V)$ and $\text{ren}(c_2, V)$ are resolvable and $\text{res}(\text{ren}(c_1, V), \text{ren}(c_2, V)) = \text{ren}(c'_1, V) \cup \text{ren}(c'_2, V)$, it follows that $\text{res}(\text{res}(c_1, V), \text{res}(c_2, V)) = \text{res}(\text{res}(c_1, c_2), V)$.

(ii) We consider the following two cases:

(a) $p \in Q$. We have the following:

- Note that $p \in \text{Var}(c_1) \cap \text{Var}(c_2)$ and $|\text{Var}(c_i) \cap Q| \leq 2$ for $i = 1, 2$ by the condition (i) in Definition\cite{1}. It shows that $|\text{Var}(c'_1 \cup c'_2) \cap Q| \leq 2$.

- By $\text{Var}(c_i) \cap Q \neq \emptyset$ for $i = 1, 2$ we have that $|\text{Pos}(c_i) \cap H| = 0$ due to the fact $|\text{Pos}(c_i) \cap H| \leq 1$ and $|\text{Pos}(c_i) \cap H| \neq 1$ according to the conditions (ii) and (iii) of Definition\cite{1}. It follows $|\text{Pos}(c'_1 \cup c'_2) \cap H| = 0$. 

6
(b) $p \notin Q$ i.e. $p \in H$. Now we have the following:

- Since $p \in Pos(c_1) \cap H$ we have $Var(c_1) \cap Q = \emptyset$ by conditions (ii) and (iii) of Definition 1. It implies that $|Var(c_1 \cup c_2) \cap Q| = |Var(c_2) \cap Q| \leq 2$ by condition (i) of Definition 1. Thus $|Var(c_1 \cup c_2) \cap Q| \leq 2$.

- Note that $|Pos(c_1) \cap H| \leq 1$ by condition (ii) of Definition 1 and $p \in Pos(c_1) \cap H$. It shows that $|Pos(c_1) \cap H| = 1$ and $Pos(c_1) \cap H = \emptyset$, thus $|Pos(c_1' \cup c_2') \cap H| \leq 1$ due to $|Pos(c_2) \cap H| \leq 1$ by condition (ii) of Definition 1.

- In the case $|Pos(c_1' \cup c_2') \cap H| = 1$ we have that $|Pos(c_2') \cap H| = 1$ due to $Pos(c_1') \cap H = \emptyset$, which shows that $|Pos(c_2') \cap H| = 1$ by condition (ii) of Definition 1 and then $Var(c_2') \cap Q = \emptyset$. Recall that $Var(c_1) \cap Q = \emptyset$ (see the proof in the first item). Thus $Var(c_1 \cup c_2) \cap Q = \emptyset$, then $Var(c_1' \cup c_2') \cap Q = \emptyset$.

It completes the proof.

Let $\Sigma$ be a CNF theory. We define

\begin{align*}
res_0^\Sigma &= \Sigma, \\
res_{\Sigma}^{n+1} &= res_\Sigma^n \cup \{res(c, c')|c, c' \in res_\Sigma^n \text{ and } c, c' \text{ are resolvable}\}.
\end{align*}

**Theorem 1** Let $V \subseteq A$ and $\Sigma$ a CNF theory.

(i) If $\text{ren}(\Sigma, V)$ is a Horn theory then $\text{ren}(\text{res}_\Sigma^0, V)$ is a Horn theory for $n \geq 0$.

(ii) If the partition $\{Q, H\}$ of $\text{Var}(\text{ren}(\Sigma, V))$ satisfies the conditions (i), (ii) and (iii) of Definition 1 for every clause of $\Sigma$, then $\{Q, H\}$ satisfies the same conditions for every clauses in $\text{res}_n^{\text{ren}(\Sigma, V)}$ for $n \geq 0$.

**Proof:** We prove the theorem by induction on $n$.

(i) Base: it trivially holds for $n = 0$ due to $res_\Sigma^0 = \Sigma$.

Step: Suppose that $\text{ren}(\text{res}_\Sigma^n, V)$ is a Horn formula. For any $c \in \text{res}_\Sigma^{n+1} \setminus \text{res}_\Sigma^n$, $c = \text{res}(c_1, c_2)$ for some clauses $c_1, c_2$ of $\text{res}_\Sigma^n$. According to (i) of Proposition 1 we have $\text{ren}(c, V) = \text{ren}(\text{res}(c_1, c_2), V) = \text{res}(\text{ren}(c_1, V), \text{ren}(c_2, V))$. It follows that $\text{ren}(c, V)$ is a Horn clause since the resolvent of two Horn clauses is a Horn clause.

(ii) Base: it trivially holds for $n = 0$ due to $\text{res}_n^{\text{ren}(\Sigma, V)} = \text{ren}(\Sigma, V)$.

Step: Suppose that $Q$ and $H$ satisfy the same conditions for every clauses in $\text{res}_n^{\text{ren}(\Sigma, V)}$. For every clause $c \in \text{res}_n^{\text{ren}(\Sigma, V)} \setminus \text{res}_n^{\text{ren}(\Sigma, V)}$, there are two resolvable clauses $c_1, c_2 \in \text{res}_n^{\text{ren}(\Sigma, V)}$ such that $c = \text{res}(c_1, c_2)$. In terms of (ii) of Proposition 1 $Q$ and $H$ satisfy the conditions (i), (ii) and (iii) of Definition 1 for the clause.
c. Thus \( \{Q, H\} \) satisfies the same conditions for every clauses in \( \text{res}^{n+1}_{\text{ren}(\Sigma, V)} \).

Together with Lemma 2 and the fact that \( |\text{res}(c_1, c_2)| \leq 2 \) if \( |c_i| \leq 2 \) (1 \( \leq i \leq 2 \)), the theorem above implies:

**Corollary 2** Let \( V \subseteq A \) and \( \Sigma \) a CNF theory. If \( \Sigma \) is a Horn (resp. ren-Horn and q-Horn) theory then \( \Pi(\Sigma) \) is a Horn (resp. ren-Horn and q-Horn) theory.

As illustrated by the following example, the reverse of the above corollary do not generally hold even if \( \Sigma \) is Horn expressible.

**Example 1** Let \( \Sigma = (p \lor q) \land (\neg p \lor \neg q) \land (p \lor \neg q) \). Since \( \text{Mod}(\Sigma) = \{\{p\}\} \) (over the signature \{p, q\}), \( \Sigma \) is Horn expressible but it is not a Horn formula. In fact, \( \Pi(\Sigma) = \{p, \neg q\} \), which is a Horn theory. However \( \Sigma \) is not Horn renamable as we have that \( \text{ren}(\Sigma, V) \) is not a Horn formula for any \( V \subseteq \{p, q\} \).

Let \( \Pi = (p \lor q \lor r) \land (p \lor q \lor \neg r) \land (\neg p \lor \neg q \lor r) \land (\neg p \lor \neg q \lor \neg r) \land (p \lor \neg q) \). We have that \( \Pi(\Pi) = \{p, \neg q\} \). It is evident that \( \Pi(\Pi) \) is a 2-CNF formula, thus a q-Horn formula. However, one can verify that \( \Pi \) is not a q-Horn formula.

Let \( M, X \) be two sets of atoms. We denote \( M \div X \) the symmetric difference \( (M \setminus X) \cup (X \setminus M) \). For a collection \( \mathcal{M} \) of interpretations, we denote \( \mathcal{M} \div X = \{M \div X | M \in \mathcal{M}\} \).

**Proposition 2** Let \( \Sigma \) be a formula and \( V \subseteq A \). Then \( \text{Mod}(\Sigma) \div V = \text{Mod}(\text{ren}(\Sigma, V)) \).

**Proof:** (\( \Rightarrow \)) Let \( M \in \text{Mod}(\Sigma) \div V \). There exists \( M' \models \Sigma \) such that \( M = (M' \setminus V) \cup (V \setminus M') \). Suppose \( M \not\models \text{ren}(\Sigma, V) \). It follows that \( M \not\models \text{ren}(c, V) \) for some clause \( c \in \Sigma \). By \( M' \models c \) we have that \( M' \models l \) for some literal \( l \) in \( c \). Evidently, if \( \text{Var}(l) \not\subseteq V \) then \( l \) is also a literal of \( \text{ren}(c, V) \) and \( M \models l \), thus \( M \models \text{ren}(c, V) \). In the case \( \text{Var}(l) \subseteq V \), we consider the two cases, where \( p \) is an atom:

- \( l = p \). It shows that \( p \in M' \) and then \( p \not\in M \). Thus \( M \models \text{ren}(c, V) \) due to \( M \models \neg p \).
- \( l = \neg p \). It shows \( p \not\in M' \) and then \( p \in M \). Thus \( M \models \text{ren}(c, V) \) due to \( M \models p \).

Either of the above two cases result in a confliction.

(\( \Leftarrow \)) Let \( M \in \text{Mod}(\text{ren}(\Sigma, V)) \). We have that \( (M \setminus V) \cup (V \setminus M) \models \text{ren}(\text{ren}(\Sigma, V), V) \), which implies \( (M \setminus V) \cup (V \setminus M) \models \Sigma \), i.e. \( M \in \text{Mod}(\Sigma) \). 

The following corollary easily follows from the proposition above.

**Corollary 3** Let \( \Sigma \) be a CNF theory. Then \( \Sigma \) is Horn renamable iff there exists \( V \subseteq A \) such that \( \text{Mod}(\Sigma) \div V \) is closed under intersection.
3 Forgetting

Starting with the basic notations and properties of forgetting, we will consider a general algorithm for computing forgetting results of CNF theories, and computational complexity on various reasoning problems relating to forgetting.

Let $\Sigma$ be a propositional formula, we denote $\Sigma[p/\top]$ (resp. $\Sigma[p/\bot]$) the formula obtained from $\Sigma$ by substituting all occurrences of $p$ with $\top$ (true) (resp. $\bot$ (false)). For instance, if $\Sigma = \{p \supset q, (q \land r) \supset s\}$, then $\Sigma[q/\top] \equiv \{r \supset s\}$ and $\Sigma[q/\bot] \equiv \{\neg p\}$.

3.1 Basic properties

Let $M, N$ be two interpretations and $V \subseteq \mathcal{A}$. $M$ and $N$ are $V$-bisimilar, written $M \sim_V N$, if and only if $M \setminus V = N \setminus V$.

Definition 2 ([1]) Let $\varphi$ be a formula and $V \subseteq \mathcal{A}$. A formula $\psi$ is a result of forgetting $V$ from $\varphi$ iff, for every model $M$ of $\psi$, $\varphi$ has a model $M'$ such that $M \sim_V M'$.

The syntactic counterpart of forgetting is a binary operator, written $\text{Forget}(\cdot, \cdot)$, which is defined recursively as:

$$\text{Forget}(\varphi, \emptyset) = \varphi,$$

$$\text{Forget}(\varphi, \{p\}) = \varphi[p/\top] \lor \varphi[p/\bot],$$

$$\text{Forget}(\varphi, V \cup \{p\}) = \text{Forget}(\text{Forget}(\varphi, \{p\}), V)$$

where $\varphi$ is a formula and $V \subseteq \mathcal{A}$.

Due to the fact that if $\varphi'$ and $\psi'$ is a result of forgetting $V$ from $\varphi$ and $\psi$ respectively, then $\varphi' \equiv \psi'$, by abusing the notation, we will denote $\text{Forget}(\varphi, V)$ the result of forgetting $V$ from $\varphi$ when there is no ambiguity.

The following proposition easily follows from the definition of forgetting, cf, Propositions 17 and 21 of [2].

Proposition 3 Let $\psi, \phi$ be two formulas and $V \subseteq \mathcal{A}$. Then we have

(i) $\text{Forget}(\psi \lor \phi, V) \equiv \text{Forget}(\psi, V) \lor \text{Forget}(\phi, V)$.

(ii) $\text{Forget}(\psi \land \phi, V) \equiv \text{Forget}(\psi, V) \land \phi$ if $\text{IR}(\phi, V)$.

To establish a semantic characterization of forgetting, we introduce the notion of extension. Let $M$ be an interpretation and $V \subseteq \mathcal{A}$. The extension of $M$ over $V$, written $M|_V$, is the collection $\{X \subseteq \mathcal{A} | X \sim_V M\}$. The extension of a collection $\mathcal{M}$ of interpretations is $\bigcup_{M \in \mathcal{M}} M|_V$. The following lemma establishes the
semantic characterization of the syntactic forgetting, which says that \( \varphi \) is a result of forgetting \( V \) from \( \psi \) if and only if the models of \( \varphi \) consist of the \( V \)-extensions of models of \( \psi \).

The following proposition is a variant of Corollary 1 of [2] and an extension of Corollary 5 of [2].

**Proposition 4** Let \( \varphi, \psi \) be two formulas and \( X \subseteq A \). Then \( \varphi \equiv \text{Forget}(\psi, V) \) if and only if \( \text{Mod}(\varphi) = \text{Mod}(\psi)_{tv} \).

**Proof:** (\( \Rightarrow \)) On the one hand, for every \( M \in \text{Mod}(\varphi) \), there exists \( M' \in \text{Mod}(\psi) \) such that \( M \sim_{V} M' \) by Definition \( 2 \) i.e. \( M \in \text{Mod}(\psi)_{tv} \). On the other hand, if \( M \in \text{Mod}(\psi)_{tv} \) then there exists \( M' \in \text{Mod}(\psi) \) such that \( M \sim_{V} M' \), which shows that \( M \models \varphi \) by Definition \( 2 \) again. Thus \( \text{Mod}(\varphi) = \text{Mod}(\psi)_{tv} \).

(\( \Leftarrow \)) Note that \( \text{Mod}(\varphi) = \text{Mod}(\psi)_{tv} \) implies, for every \( M \models \varphi \), there exists a mode \( M' \models \psi \) such that \( M \sim_{V} M' \). Thus \( \varphi \) is a result of forgetting \( V \) from \( \psi \) by Definition \( 2 \) i.e. \( \varphi \equiv \text{Forget}(\psi, V) \).

The following theorem shows that the forgetting is closely connected with prime implicants and implicates.

**Theorem 4** Let \( \Pi, \Sigma \) be two theories and \( V \) a set of atoms. The following conditions are equivalent to each other:

(i) \( \Sigma \equiv \text{Forget}(\Pi, V) \).

(ii) \( \Sigma \equiv \{ \psi | \Pi \models \psi \text{ and } IR(\psi, V) \} \).

(iii) \( \Sigma \equiv \bigvee \{ t | t \in IP(\Pi) \text{ and } \text{Var}(t) \cap V = \emptyset \} \).

(iv) \( \Sigma \equiv \{ c | c \in PI(\Pi) \text{ and } \text{Var}(c) \cap V = \emptyset \} \).

**Proof:** (i) \( \iff \) (ii). It is trivial if \( \Pi \equiv \bot \). Suppose \( \Pi \) is not falsity. Let \( \Pi' = \{ \psi | \Pi \models \psi \text{ and } IR(\psi, V) \} \). It is sufficient to prove \( \text{Forget}(\Pi, V) \equiv \Pi' \). On the one side, \( M \models \text{Forget}(\Pi, V) \) implies \( \exists M' \models \Pi \text{ such that } M \sim_{V} M' \). It follows that \( M' \models \Pi' \). On the other side, \( M' \models \Pi' \) implies \( M' \) can be modified to a model \( M \) of \( \Pi \) where \( M \sim_{V} M' \). It shows that \( M' \in \text{Mod}(\Pi)_{tv} \).

(i) \( \iff \) (iii). \( \text{Forget}(\Sigma, V) \)
\[ \equiv \bigvee \{ t \in IP(\Sigma) \text{ and } \text{Var}(t) \cap V = \emptyset \} \]
\[ \equiv \bigvee \{ t \in IP(\Sigma) \text{ and } \text{Var}(t) \cap V = \emptyset \} \text{ by (i) of Proposition 8} \]
\[ \equiv \bigvee \{ t \in IP(\Sigma) \text{ and } \text{Var}(t) \cap V = \emptyset \} \text{ by (ii) of Proposition 8} \]

(i) \( \iff \) (iv). It is proved by Theorem 37 of [24], and can follows from Propositions 19 and 20 of [2].
Actually, (i)⇔(iv) is mentioned as a fact in [17], which states that \( \text{Forget}(\Sigma, V) \) is equivalent to the conjunction of prime implicates of \( \Sigma \) that do not mention any propositions from \( V \). In terms of Corollary 2 and the theorem above, we have the following corollary.

**Corollary 5** Let \( \Sigma \) be a CNF theory and \( V \subseteq A \). If \( \Sigma \) is a Horn (resp. Krom, ren-Horn and q-Horn) expressible then \( \text{Forget}(\Sigma, V) \) is a Horn (resp. Krom, ren-Horn and q-Horn) expressible.

### 3.2 A resolution-based algorithm

Given a set \( \Pi \) of clauses and an atom \( p \), the *unfolding* of \( \Pi \) w.r.t. \( p \), written \( \text{unfold}(\Pi, p) \), is the set of clauses obtained from \( \Pi \) by replacing every clause \( c \in \Pi \) such that \( p \in \text{Pos}(c) \) with the clauses

\[
\text{res}(c, c_i) \quad (1 \leq i \leq k)
\]

where \( c_1, \ldots, c_k \) are all the clauses of \( \Pi \) such that \( p \in \text{Neg}(c_i) \) and, the two clauses \( c \) and \( c_i \) are resolvable for every \( i \) (1 \( \leq i \leq k \)). In particular, if \( k = 0 \) then \( \text{unfold}(\Pi, p) \) is obtained from \( \Pi \) by simply removing all the clauses that contain the positive literal \( p \).

The *strong unfolding* of \( \Pi \) w.r.t. an atom \( p \), denoted \( \text{sunfold}(\Pi, p) \), is obtained from \( \text{unfold}(\Pi, p) \) by removing all clauses containing \( \neg p \).

**Example 2** Let us consider the below two CNF theories.

\[
\Pi = \{ p \lor q \lor \neg a, \quad p \lor \neg q, \quad b \lor \neg p, \quad c \lor \neg p \}\.
\]

\[
\Sigma = \{ p \lor \neg a, \quad p \lor \neg q \lor \neg b, \quad q \lor \neg p, \quad c \lor \neg p \}\.
\]

We have that

\[
\text{sunfold}(\Pi, p) = \{ b \lor q \lor \neg a, \quad c \lor q \lor \neg a, \quad b \lor \neg q, \quad c \lor \neg q \}\,
\]

\[
\text{sunfold}(\Pi, q) = \{ p \lor \neg a, \quad b \lor \neg p, \quad c \lor \neg p \},
\]

\[
\text{sunfold}(\text{sunfold}(\Pi, p), q) = \{ b \lor \neg a, \quad c \lor \neg a, \quad b \lor c \lor \neg a \},
\]

\[
\text{sunfold}(\text{sunfold}(\Pi, q), p) = \{ p \lor \neg a, \quad c \lor \neg a \},
\]

\[
\text{sunfold}(\Sigma, p) = \{ q \lor \neg a, \quad c \lor \neg a, \quad c \lor \neg a \land b \},
\]

\[
\text{sunfold}(\Sigma, q) = \{ p \lor \neg a, \quad c \lor \neg p \},
\]

\[
\text{sunfold}(\text{sunfold}(\Sigma, p), q) = \{ c \lor \neg a, \quad c \lor \neg a \lor \neg b \},
\]

\[
\text{sunfold}(\text{sunfold}(\Sigma, q), p) = \{ c \lor \neg a \}.
\]

Though \( \text{sunfold}(\text{sunfold}(\Pi, p), q) \neq \text{sunfold}(\text{sunfold}(\Pi, q), p) \), we will see that the two theories are equivalent, i.e., having same models. □
As demonstrated by Theorem 4, forgetting results always exist, as every formula can be translated into an equivalent CNF theory. The below proposition shows that forgetting in CNF theories can be achieved by unfolding.

**Theorem 6** Let \( \Pi \) be a CNF theory and \( p \in A \). Then \( \text{Forget}(\Pi, p) \equiv \text{sunfold}(\Pi, p) \).

**Proof:** Without loss of generality, we assume that \( \Pi \) contains no tautology. Note that if the clause \( c : A \cup \neg B \) in \( \Pi \) satisfies \( p \notin A \cup B \) then \( c \in \text{sunfold}(\Pi, p) \) and \( \text{Forget}(\Pi, p) \models c \) by (ii) of Proposition 3. Thus we can assume \( p \in A \cup B \) for every clause \( A \cup \neg B \) of \( \Pi \).

Let \( c_i (1 \leq i \leq n) \) be all the clauses of \( \Pi \) such that \( p \in c_i \), and \( c'_j (1 \leq j \leq m) \) be all the clauses of \( \Pi \) such that \( \neg p \in c'_j \).

The direction from left to right is clear by (ii) of Theorem 4, i.e., \( \text{Forget}(\Pi, p) \models \text{sunfold}(\Pi, p) \), since \( \Pi \models \text{res}(c_i, c'_j) \) for every \( i, j (1 \leq i \leq n, 1 \leq j \leq m) \) whenever \( c_i, c'_j \) are resolvable.

To prove the other direction, it is sufficient to show that for every model \( M \) of \( \text{sunfold}(\Pi, P) \), there exists a model \( M' \) of \( \Pi \) such that \( M' \models \neg p \). We prove this by contradiction. Without loss of generality, let \( M \models \text{sunfold}(\Pi, p), p \notin M, M' = M \cup \{p\}, M \models \Pi \) and \( M' \models \Pi \). It follows that \( M \models c_i \) for some \( i (1 \leq i \leq n) \) and \( M' \models \neg c'_j \) for some \( j (1 \leq j \leq m) \). Let us consider the following two cases:

1. \( c_i \) and \( c'_j \) are not resolvable. It shows that there is an atom \( q \) different from \( p \) such that \( q, \neg q \in c_i \cup c'_j \). Recall that \( c_i, c'_j \) are not tautology. In the case \( q \in M \) we have that \( q \in c'_j \) and \( \neg q \in c_i \) as \( M \models \neg c_i \). It shows that \( M \models c'_j \), thus \( M' \models c'_j \), a contradiction. In the case \( q \notin M \) we have that \( q \in c_i \) and \( \neg q \in c'_j \) as \( M \models \neg c_i \). It follows that \( M \models c'_j \), thus \( M' \models c'_j \), a contradiction.

2. \( c_i \) and \( c'_j \) are resolvable. It shows that the resolvent \( \text{res}(c_i, c'_j) = (c_i \setminus \{p\}) \cup (c'_j \setminus \{\neg p\}) \) belongs to \( \text{sunfold}(\Pi, p) \). Note that \( M \models \neg c_i \) implies \( M \models c_i \setminus \{p\} \). It follows that \( M \models c'_j \setminus \{\neg p\} \) since \( M \models \text{res}(c_i, c'_j) \), thus \( M' \models c'_j \setminus \{\neg p\} \) and \( M' \models c'_j \) by \( c'_j \setminus \{\neg p\} \models c'_j \), a contradiction. \( \blacksquare \)

**Proposition 5** Let \( \Pi \) be a CNF theory, \( p, q \) two atoms. Then we have that

\[
\text{sunfold}(\text{sunfold}(\Pi, p), q) \equiv \text{sunfold}(\text{sunfold}(\Pi, q), p).
\]

**Proof:** By Theorem 6 we have that

\[
\text{sunfold}(\text{sunfold}(\Pi, p), q)
\]

\[
\equiv \text{sunfold}(\text{Forget}(\Pi, p), q)
\]

\[
\equiv \text{Forget}(\text{Forget}(\Pi, p), q)
\]

\[
\equiv \text{Forget}(\Pi, \{p, q\})
\]

\[
\equiv \text{Forget}(\text{Forget}(\Pi, q), p)
\]
≡ Forget\left( sunfold(\Pi, q), p \right)
≡ Forget\left( Forget(\Pi, q), p \right)
≡ sunfold\left( sunfold(\Pi, p), q \right).

In terms of the above proposition, the unfolding is independent of the ordering of atoms to be strongly unfolded. We define unfolding a set of atoms as following,

\begin{align*}
sunfold(\Pi, \emptyset) &= \Pi, \\
sunfold(\Pi, V \cup \{p\}) &= sunfold(sunfold(\Pi, p), V)
\end{align*}

where \( \Pi \) is a CNF theory and \( V \subseteq A \).

It follows that, by Theorem 6 and Proposition 5,

**Corollary 7** Let \( \Pi \) be a CNF theory and \( V \subseteq A \). \( \text{Forget}(\Pi, V) \equiv sunfold(\Pi, V) \).

In terms of Corollaries 2 and 7, we have

**Corollary 8** Let \( \Sigma \) be a CNF theory and \( V \subseteq A \). If \( \Sigma \) is a Horn (resp. Krom, ren-Horn and q-Horn) theory then \( \text{sunfold}(\Sigma, V) \) is a Horn (resp. Krom, ren-Horn and q-Horn) theory.

The strong unfolding provides alternative approach of evaluating forgetting. In particular, strong unfolding results of CNF theories are in CNF as well. If \( \Pi \) is a Horn theory then \( \text{Forget}(\Pi, V) \) is also Horn which can be achieved by strong unfolding. It distinguishes from the syntactic approach \( \text{Forget}(\Pi, p) = \Pi[p/\bot] \lor \Pi[p/\top] \), which is not in CNF, though it can be transformed into CNF (with possibly much more expense).

Based on the notion of strong unfolding, we present the algorithm for computing forgetting results of CNF theories in Algorithm 1. The following proposition asserts the correctness.

**Proposition 6** Let \( \Pi, V, \Sigma \) be as in Algorithm 1 Then \( \Sigma \equiv \text{Forget}(\Pi, V) \).

**Proof:** It follows from that the lines 3-9 of Algorithm compute \( \text{Forget}(\Pi, p) \). □

The algorithm remains the potentiality of heuristics. For example, one can forget the atoms one by one in a specific order, and similarly choose two specific clauses to do resolution sequentially. In addition, to save space, one can add the condition \( \Sigma \not\models \text{res}(c, c') \) at line 7 of the algorithm. While checking the condition is intractable generally, however, it is tractable for some special CNF theories, including Horn, ren-Horn, q-Horn and Krom ones.

Before end of the section, we formally analyze the computational costs.
Algorithm 1: An Algorithm for Forget(Π, V)

**input**: A set Π of clauses and a set V of atoms

**output**: The result of forgetting V in Π

1. \[ S \leftarrow \{ c | c \in \Pi \text{ and } V \cap \text{Var}(c) = \emptyset \}; \]
2. \[ \Pi \leftarrow \Pi \setminus S; \]
3. \[ \Pi' \leftarrow \{ c | c \in \Pi \text{ and } p \in \text{Var}(c) \}; \]
4. \[ \Sigma \leftarrow \Pi \setminus \Pi'; \]
5. \[ \text{foreach } (c \in \Pi' \text{ s.t } p \in \text{Pos}(c)) \text{ do} \]
6. \[ \text{foreach } (c' \in \Pi' \text{ s.t } p \in \text{Neg}(c') \text{ and } c, c' \text{ are resolvable}) \text{ do} \]
7. \[ \Sigma \leftarrow \Sigma \cup \text{res}(c, c'); \]
8. \[ \Pi \leftarrow \Sigma; \]
9. \[ \text{end} \]
10. \[ \text{end} \]
11. \[ \text{return } \Sigma \cup S \]

Proposition 7 Let Π be a CNF theory and V ⊆ A where |Π| = n and |V| = k. The time and space complexity of Algorithm 1 are \( O(n^{2k}) \).

**Proof**: It follows from that the lines 5-9 of the algorithm, which is to compute sunfold(Π, p), is bounded by \( O(|\Pi|^2) \), and the size of sunfold(Π, p) is bounded by \( O(|\Pi|^2) \) as well.

One can evidently note that, if k is given as a fixed parameter then sunfold(Π, V) can be computed in polynomial time in the size of Π. The following example shows that an exponential explosion of Forget(Π, V) is inescapable even if Π is a Horn theory.

Example 3 Let Π be the Horn theory consisting of

\[ p \lor \neg q_1 \lor \ldots \lor \neg q_n, \quad q_1 \lor \neg r_1, \quad q_1 \lor \neg r'_1, \quad \ldots, \quad q_n \lor \neg r_n, \quad q_n \lor \neg r'_n. \]

It is not difficult to see that, for each subset I of \( N = \{1, \ldots, n\} \),

\[ \Pi \models \left( \bigvee_{i \in I} \neg r_i \right) \lor \left( \bigvee_{j \in (N \setminus I)} \neg r'_j \right) \lor p. \]
Thus \(\text{Forget}(\Pi, \{q_1, \ldots, q_n\})\) is in exponential size of \(\Pi\) since there are \(2^n\) number of subsets of \(N\). And as a matter of fact, there is no Horn theory that is in polynomial size of \(\Pi\) and is equivalent to \(\text{Forget}(\Pi, \{q_1, \ldots, q_n\})\) since 
\[
\left( \bigvee_{i \in I} \neg r_i \right) \lor \left( \bigvee_{j \in (N \setminus I)} \neg r'_j \right) \lor p \text{ is a prime implicate of } \Pi.
\]

Note that, in the case \(\Pi\) is a Krom theory, there are at most \(O(m^2)\) number clauses where \(m = |\text{Var}(\Pi)|\). Thus \(|\Sigma|\) in the line 7 of Algorithm 1 is bounded by \(O(n^2)\) where \(n = |\Pi|\). Then the overall time and space complexity is \(O(kn^2)\) whenever \(\Pi\) is a Krom theory where \(k = |V|\).

### 3.3 Complexities

In the following we consider the complexities of reasoning problems on forgetting for various fragments of propositional logic.

#### 3.3.1 DNF, CNF and arbitrary theories

**Proposition 8** Let \(\Pi, \Sigma\) be two (CNF) theories, and \(V \subseteq A\). We have that

(i) deciding if \(\Pi \models \text{Forget}(\Sigma, V)\) is \(\Pi_2^P\)-complete,

(ii) deciding if \(\text{Forget}(\Pi, V) \models \Sigma\) is co-NP-complete,

(iii) deciding if \(\text{Forget}(\Pi, V) \models \text{Forget}(\Sigma, V)\) is \(\Pi_2^P\)-complete.

**Proof:** (i) Membership. In the case \(\Pi \not models \text{Forget}(\Sigma, V)\), there exists a model \(M\) of \(\Pi\) such that \(M \not models \text{Forget}(\Sigma, V)\), i.e. for every model \(M'\) of \(\Sigma\) such that \(M \sim_V M', M' \not models \Sigma\), which can be done in polynomial time in the size of \(\Sigma\) and \(V\) by calling a nondeterministic Turing machine.

Hardness. It follows from the fact that \(T \models \text{Forget}(\Sigma, V)\) iff \(\text{Forget}(\Sigma, V)\) is valid, i.e. \(\forall V' \exists V' \Sigma\) is valid, where \(V' = \text{Var}(\Sigma) \setminus V\). The latter is \(\Pi_2^P\)-complete even if \(\Sigma\) is a CNF theory, as every formula can be translated into a CNF theory with auxiliary variables that preserves the satisfiability, informally \(\forall V' \exists V' \Sigma\) can be translated polynomially into \(\forall V' \exists V' \Sigma'\) such that (a) \(\Sigma'\) is a CNF theory, and (b) \(\forall V' \exists V' \Sigma'\) is valid iff \(\forall V' \exists V' \Sigma'\) is valid, where \(V'\) is the introduced auxiliary variables [28].

(ii) Membership. If \(\text{Forget}(\Pi, V) \not models \Sigma\) then there exists two sets \(M\) and \(M'\) such that \(M \models \Pi, M' \not models \Sigma\) and \(M \sim_V M'\). It is in polynomial time to guess such \(M, M'\) and check the conditions \(M \models \Pi, M' \not models \Sigma\) and \(M \sim_V M'\). Hence the problem is in co-NP.

Hardness. \(\text{Forget}(\Pi, V) \models \bot\) if and only if \(\Pi \models \bot\), i.e. \(\Pi\) has no model, which is co-NP-hard. Thus the problem is co-NP-complete.
(iii) Membership. If \( \text{Forget}(\Pi, V) \not\models \text{Forget}(\Sigma, V) \) then there exist an interpretation \( M \) such that \( M \models \text{Forget}(\Pi, V) \) but \( M \not\models \text{Forget}(\Sigma, V) \), i.e., there is \( M' \sim_V M \) with \( M' \models \Pi \) but \( M'' \not\models \Sigma \) for every \( M'' \sim_V M \). It is evident that guessing such \( M, M' \) with \( M \models \text{Forget}(\Pi, V) \) and checking \( M' \models \Pi \) are feasible, while checking \( M'' \not\models \Sigma \) for every \( M'' \sim_V M \) can be done in polynomial time in the size of \( V \) and \( \Sigma \) by call a nondeterministic Turing machine. Thus the problem is in \( \Pi^P_2 \).

Hardness. It follows from (i) due to the fact that \( \text{Forget}(\Pi, V) \models \text{Forget}(\Sigma, V) \) iff \( \Pi \models \text{Forget}(\Sigma, V) \).

The proposition implies:

**Corollary 9** Let \( \Pi, \Sigma \) be two (CNF) theories, and \( V \subseteq A \). Then

(i) deciding if \( \Pi \equiv \text{Forget}(\Sigma, V) \) is \( \Pi^P_2 \)-complete,

(ii) deciding if \( \text{Forget}(\Pi, V) \equiv \text{Forget}(\Sigma, V) \) is \( \Pi^P_2 \)-complete, and

(iii) deciding if \( \text{Forget}(\Pi, V) \equiv \Pi \) is co-NP-complete.

In the case \( \Pi \) is an arbitrary propositional formula, (ii) and (iii) of the corollary corresponds to \textsc{Var-Equivalence} and \textsc{Var-Independence} in [2], in which it is proved to be the same complexity as that of CNF theory case, respectively. Note that the inverse of item (iii) is the relevance problem, i.e., if a formula \( \Pi \) is relevant to \( V \), which is NP-hard (cf. Theorem 50 of [24]).

Recall that \( \text{Forget}(\varphi, p) = \varphi[p/\top] \lor \varphi[p/\bot] \) for a given formula \( \varphi \) and an atom \( p \). According to (i) of Proposition 4 when \( \varphi \) is a term \( l_1 \land \cdots \land l_n \), \( \text{Forget}(\varphi, V) \) is the term obtained from \( \varphi \) by replacing \( l_i \) \( (1 \leq i \leq n) \) with \( \top \) if \( \text{Var}(l_i) \subseteq V \). E.g., \( \text{Forget}(p \land \neg q, \{p\}) \equiv \neg q \) and \( \text{Forget}(p \land \neg q, \{q\}) \equiv q \). It implies that if \( \Pi \) is a DNF theory then \( \text{Forget}(\Pi, V) \) can be computed in linear time in the size of \( \Pi \) by (i) of Proposition 3.

**Proposition 9** Let \( \Pi, \Sigma \) be two DNF theories, and \( V \subseteq A \). The following problems are co-NP-complete:

(i) deciding if \( \Pi \models \text{Forget}(\Sigma, V) \),

(ii) deciding if \( \text{Forget}(\Pi, V) \models \Sigma \),

(iii) deciding if \( \text{Forget}(\Pi, V) \models \text{Forget}(\Sigma, V) \).

**Proof:** (i) Membership. It is obvious that if \( \Pi \not\not\models \text{Forget}(\Sigma, V) \) then there exists a set \( M \) of atoms such that \( M \models \Pi \) and \( M \not\not\models \text{Forget}(\Sigma, V) \). As \( \text{Forget}(\Sigma, V) \) is
computable in polynomial time, the checking $M \models \Pi$ and $M \not\models \text{Forget}(\Sigma, V)$ is feasible in polynomial time as well. Hence the problem is in co-NP.

Hardness. Let $\Pi \equiv \top$. Note that $\top \models \text{Forget}(\Sigma, V)$ iff $\text{Forget}(\Sigma, V)$ is valid. As $\text{Forget}(\Sigma, V)$ is still a DNF theory whose validity is co-NP-hard, it shows that the problem is co-NP-hard as well.

(ii) and (iii) can be similarly proved as that of (i).

The proposition above implies

**Corollary 10** Let $\Pi, \Sigma$ be two DNF theories, and $V \subseteq A$. The following problems are co-NP-complete.

1. deciding if $\Pi \equiv \text{Forget}(\Sigma, V)$,
2. deciding if $\text{Forget}(\Pi, V) \equiv \text{Forget}(\Sigma, V)$,
3. deciding if $\text{Forget}(\Pi, V) \equiv \Pi$.

### 3.3.2 Horn theories and its variants

For a Horn formula $\Sigma$, its dependency graph is the directed graph $G(\Sigma) = (V, E)$, where $V = A$ and $(a_i, a_j) \in E$ if there is a Horn clause $c \in \Sigma$ such that $\neg a_i \in c$ and $a_j \in c$. A Horn formula $\Sigma$ is acyclic if $G(\Sigma)$ has no directed cycle.

**Theorem 11** Let $\Pi, \Sigma$ be Horn (resp. ren-Horn and q-Horn) theories and $V \subseteq A$.

1. The problem of deciding if $\Pi \models \text{Forget}(\Sigma, V)$ is co-NP-complete, even if $\Pi$ and $\Sigma$ are acyclic.
2. The problem of deciding if $\text{Forget}(\Pi, V) \models \Sigma$ is tractable.
3. The problem of deciding if $\text{Forget}(\Pi, V) \models \text{Forget}(\Sigma, V)$ is co-NP-complete, even if $\Pi$ and $\Sigma$ are acyclic.

**Proof:** (i) Membership. Note that $\Pi \not\models \text{Forget}(\Sigma, V)$ iff there is a prime implicate $c$ of $\Sigma$ such that $\text{Var}(c) \cap V = \emptyset$ and $\Pi \not\models c$, the latter holds iff $\Pi \cup \neg c$ has a model, where $\neg c = \{ \neg l \mid l$ is a disjunct of $c \}$. In the case $\Pi$ is q-Horn, $\Pi \cup \neg c$ is q-Horn and its satiability checking is tractable [21]. One can guess such a prime implicate $c$ and check if $\Pi \not\models c$ in polynomial time in the size of $\Pi$ and $\Sigma$. Thus the problem is in co-NP even if $\Pi, \Sigma$ are q-Horn theories.

Hardness. Let $\gamma = c_1 \wedge \cdots \wedge c_m$ be a 3CNF formula over atoms $x_1, \ldots, x_n$, where $c_i = l_{i,1} \lor l_{i,2} \lor l_{i,3}$. The below construction is quite similar to the one used in the proof of Theorem 4.1 [29]. We introduce for each clause $c_i$ a new atom $y_i$, for each atom $x_j$ a new atom $x'_j$ (which intuitively corresponds to $\neg x_j$), and a
special atom \( z \). The Horn theory \( \Pi = \{ \neg x_i \vee \neg x'_i | 1 \leq i \leq n \} \) and \( \Sigma \) contains \( \Pi \) and additional the below clauses:

\[
\neg z \vee y_1, \\
\neg y_i \vee \neg l^*_{i,j} \vee y_{i+1} \text{ for all } i = 1, \ldots, m - 1, \text{ and } j = 1, 2, 3, \\
\neg y_m \vee \neg l^*_{m,j} \text{ for } j = 1, 2, 3
\]

where \( l^* = x \) if \( l \) is a positive literal \( x \), and \( l^* = x' \) if \( l \) is a negative literal \( \neg x \).

It is clear that both \( \Pi \) and \( \Sigma \) are acyclic Horn formulas, thus Horn renamable and q-Horn formulas. We claim that \( \gamma \) is satisfiable iff \( \Pi \not\models \text{Forget}(\Sigma, V) \) where \( V = \{ y_1, \ldots, y_m \} \). It is easy to see that \( \Sigma \) has a prime implicate \( c \) such that \( \text{Var}(c) \cap V = \emptyset \) and \( c \notin \Pi \) iff \( \Pi \not\models \text{Forget}(\Sigma, V) \).

On the one hand, let \( \sigma \) be a satisfying assignment of \( \gamma \). Then we arbitrarily choose from each \( c_i \) a literal \( l_{i,j_i} \) satisfied by \( \sigma \). It follows that \( c = \neg z \vee (\bigvee_{1 \leq i \leq m} \neg l^*_{i,j_i}) \) is an implicate of \( \Sigma \) where \( j_i \in \{1, 2, 3\} \) and \( c \) contains at most one literal in \( \{ \neg x_i, \neg x'_i \} \) for every \( i \) \( (1 \leq i \leq n) \). As \( \text{Var}(c) \cap V = \emptyset \), and \( \bigvee_i \neg l^*_{i,j_i} \) is not an implicate of \( \Pi \) since there is no subclauses of it is generated by the resolution procedure for \( \Pi \), we have that \( c \) is a prime implicate of \( \Sigma \) and \( \Pi \not\models c \). Thus \( \Pi \not\models \text{Forget}(\Sigma, V) \).

On the other hand, there exists a prime implicate \( c \) of \( \Sigma \) such that both \( \Pi \not\models c \) and \( \text{Var}(c) \cap V = \emptyset \) due to \( \Pi \not\models \text{Forget}(\Sigma, V) \). This prime implicate \( c \) can only be generated from the Horn clauses in \( \Sigma \setminus \Pi \) and has the form \( \neg z \vee (\bigvee_{1 \leq i \leq m} \neg l^*_{i,j_i}) \) where \( j_i \in \{1, 2, 3\} \). As \( \neg x_i \vee \neg x'_i \in \Pi \), we have \( \neg x_i \vee \neg x'_i \not\models c \) for every \( i \) \( (1 \leq i \leq n) \) due to \( \Pi \not\models c \). It shows that \( c \) mentions at most one atom in \( \{ x_i, x'_i \} \) for every \( i \). Therefore \( c \) corresponds to a satisfying assignment for \( \gamma \).

(ii) In the case that \( \Sigma \) is unsatisfiable, i.e. \( \Sigma \equiv \bot \), \( \text{Forget}(\Pi, V) \equiv \bot \) iff \( \Pi \equiv \bot \). In this case the problem is tractable. Suppose \( \Sigma \) is satisfiable. We have \( \text{Forget}(\Pi, V) \models \Sigma \) iff \( \text{Forget}(\Pi, V) \models c \) for every clause \( c \) of \( \Sigma \). In the case \( \text{Var}(c) \cap V \neq \emptyset \), we have \( \text{Forget}(\Pi, V) \not\models c \). in the case \( \text{Var}(c) \cap V = \emptyset \), \( \text{Forget}(\Pi, V) \models c \) iff \( \Pi \cup \neg c \) is unsatisfiable, which is tractable even if \( \Pi \) is a q-Horn theory \( [21] \).

(iii) Membership. If \( \text{Forget}(\Pi, V) \not\models \text{Forget}(\Sigma, V) \) then there exists a prime implicate \( c \) of \( \Sigma \) such that \( \Pi \not\models c \) and \( \text{Var}(c) \cap V = \emptyset \). Thus it is in co-NP.

Hardness. It follows from (i) since \( \text{Forget}(\Pi, V) \models \text{Forget}(\Sigma, V) \) iff \( \Pi \models \text{Forget}(\Sigma, V) \).

Accordingly, we have the following corollary.

**Corollary 12** Let \( \Pi, \Sigma \) be two Horn (resp. ren-Horn and q-Horn) theories and \( V \subseteq A \).

(i) The problem of deciding if \( \Pi \equiv \text{Forget}(\Sigma, V) \) is co-NP-complete.
(ii) The problem of deciding if $\text{Forget}(\Pi, V) \equiv \text{Forget}(\Sigma, V)$ is co-NP-complete.

(iii) The problem of deciding if $\text{Forget}(\Pi, V) \equiv \Pi$ is tractable.

**Proof:**

(i) As $\Pi \neq \text{Forget}(\Sigma, V)$ iff $\Pi \not\models \text{Forget}(\Sigma, V)$ or $\text{Forget}(\Sigma, V) \not\models \Pi$, the latter is tractable by (ii) of Theorem 11 while the former is in co-NP. Hardness follows from (i) of Theorem 11. Thus the problem is co-NP-complete.

(ii) Membership is easy. Hardness follows from (iii) of Theorem 11.

(iii) It follows from the facts that $\text{Forget}(\Pi, V) \equiv \Pi$ iff $\text{Forget}(\Pi, V) \models \Pi$, and (ii) of Theorem 11. □

The item (iii) in the above corollary shows that the problem of deciding whether $\Pi$ is relevant to $V$ is tractable if $\Pi$ is a q-Horn theory. Thus it generalizes Theorem 51 of [24] for Horn theories.

### 3.3.3 Krom theories

Note that, for every Krom theory $\Sigma$ and $V \subseteq \mathcal{A}$. It is evident that

$$\text{Forget}(\Sigma, V) \equiv \{l_1 \lor l_2 | \text{Var}(\{l_1, l_2\}) \subseteq \text{Var}(\Sigma) \setminus V \text{ and } \Sigma \models l_1 \lor l_2\}.$$  

It implies that $\text{Forget}(\Sigma, V)$ can be computed in polynomial time in the size of $\Sigma$ and $V$ since $\Sigma \models l_1 \lor l_2$ is tractable [23] and there are at most $O(|\text{Var}(\Sigma) \setminus V|^2)$ number of such clauses. The following corollary follows.

**Corollary 13** Let $\Pi, \Sigma$ be two Krom theories and $V \subseteq \mathcal{A}$. All of the following problems are tractable:

(i) deciding if $\Pi \models \text{Forget}(\Sigma, V)$,

(ii) deciding if $\text{Forget}(\Pi, V) \models \Sigma$,

(iii) deciding if $\text{Forget}(\Pi, V) \models \text{Forget}(\Sigma, V)$,

(iv) deciding if $\Pi \equiv \text{Forget}(\Sigma, V)$,

(v) deciding if $\text{Forget}(\Pi, V) \equiv \text{Forget}(\Sigma, V)$,

(vi) deciding if $\text{Forget}(\Pi, V) \equiv \Pi$.

### 4 Related Work

In the section we consider the applications of forgetting, including uniform interpolation [30], strongest necessary and weakest sufficient conditions [17], and strongest and weakest definitions [18].
4.1 Uniform interpolation

Let $\alpha, \beta$ be two formulas. If $\alpha \models \beta$, an interpolant for $(\alpha, \beta)$ is a formula $\gamma$ s.t
\[
\alpha \models \gamma \quad \text{and} \quad \gamma \models \beta
\] (1)

where $\text{Var}(\gamma) \subseteq \text{Var}(\alpha) \cap \text{Var}(\beta)$.

A logic $\mathcal{L}$ with inference $\models_{\mathcal{L}}$ is said to have the interpolation property if an interpolant exists for every pair of formulas $(\alpha, \beta)$ such that $\alpha \models_{\mathcal{L}} \beta$. A logic $\mathcal{L}$ has uniform interpolation property iff for any formula $\alpha$ and $V$ a set of atoms, there exists a formula $\gamma$ such that $\text{Var}(\gamma) \subseteq \text{Var}(\alpha) \setminus V$, and for any formula $\beta$ with $\text{Var}(\beta) \cap V = \emptyset$,
\[
\alpha \models_{\mathcal{L}} \beta \quad \text{iff} \quad \gamma \models_{\mathcal{L}} \beta.
\] (2)

It is easy to see that uniform interpolation is a strengthening of interpolation. A well-known result is that propositional logic has uniform interpolation property, while first-order logic does not [30].

**Proposition 10** If $\Sigma$ is a double Horn theory and $V \subseteq A$ then $\text{Forget}(\Sigma, V)$ is a double Horn theory.

**Proof:** Firstly $\text{Forget}(\Sigma, V)$ is Horn expressible by Corollary 5. We show that $\neg \text{Forget}(\Sigma, V)$ is Horn expressible by contradiction in the following. Suppose that there exist two interpretations $X, Y$ such that
\[
X \not\models \text{Forget}(\Sigma, V), \ Y \not\models \text{Forget}(\Sigma, V), \ X \cap Y \models \text{Forget}(\Sigma, V).
\]

Note that $\text{Forget}(\Sigma, V)$ is irrelevant to $V$. Thus $I \models \text{Forget}(\Sigma, V)$ if and only if $I \setminus V \models \text{Forget}(\Sigma, V)$. For this reason, we assume $X \cap V = \emptyset$ and $Y \cap V = \emptyset$. The following three conditions hold:

(a) $X' \not\models \Sigma$ for any $X \subseteq X' \subseteq X \cup V$.

(b) $Y' \not\models \Sigma$ for any $Y \subseteq Y' \subseteq Y \cup V$.

(c) There exists $Z \models \Sigma$ for some $X \cap Y \subseteq Z \subseteq X \cap Y \cup V$.

The conditions (a) and (b) imply $X' \cap Y' \not\models \Sigma$ since $\Sigma$ is a double Horn formula. It is evident that $X \cap Y \subseteq X' \cap Y' \subseteq (X \cup V) \cap (Y \cup V) = X \cap Y \cup V$. This contradicts with condition (c). 

Together with Corollary 8, the proposition above implies:

**Corollary 14** The Horn, Krom, double Horn, ren-Horn and $q$-Horn fragments of propositional logic have uniform interpolation property.
4.2 Strongest necessary and weakest sufficient conditions

Let $T$ be a theory, $V \subseteq \text{Var}(T)$ and $q \in \text{Var}(T) \setminus V$. A formula $\varphi$ of $V$ is a necessary condition of $q$ on $T$ if $T \models q \supseteq \varphi$. It is a strongest necessary condition (SNC) if it is a necessary condition and for any other necessary condition $\varphi', T \models \varphi' \supseteq \varphi$. A formula $\psi$ of $V$ is a sufficient condition of $q$ on $T$ if $T \models \psi \supseteq q$. It is a weakest sufficient condition (WSC) if it is a sufficient condition and, for any other sufficient condition $\psi'$, $T \models \psi' \supseteq \psi$ \cite{17}.

**Theorem 15 (Theorem 2 of \cite{17})** Let $T$ be a theory, $V \subseteq \text{Var}(T)$, $q \in \text{Var}(T) \setminus V$, and $V' = \text{Var}(T) \setminus (V \cup \{q\})$.

- The strongest necessary condition of $q$ on $V$ under $T$ is $\text{Forget}(T[q/T], V')$.
- The weakest sufficient condition of $q$ on $V$ under $T$ is $\neg \text{Forget}(T[q/\bot], V')$.

Note that $T[q/T]$ is a Horn (resp. Krom, ren-Horn and q-Horn) theory if $T$ is a Horn (resp. Krom, ren-Horn and q-Horn) theory. In terms of Corollary \cite{14} the SNC of $q$ under $T$ is Horn (resp. Krom, ren-Horn and q-Horn) expressible if $T$ is a Horn (resp. Krom, ren-Horn and q-Horn) theory.

The following example shows that the weakest sufficient condition on Horn (resp. Krom) formulas may be not Horn (resp. Krom) expressible.

**Example 4** Let’s consider the following two theories.

1. Let $\Sigma = (\neg p \lor \neg r) \land (\neg q \lor r) \land (\neg s \lor r) \land \neg t$, which is a Horn formula. We have that $\text{Forget}(\Sigma[t/\bot], r) \equiv (\neg p \lor \neg q) \land (\neg p \lor \neg s)$. Thus $\neg \text{Forget}(\Sigma, r) \equiv p \land (q \lor s)$, which is evidently not Horn expressible. That is the weakest sufficient condition of $t$ on $\{p, q, s\}$ under $\Sigma$ is not Horn expressible.

2. Let $\Pi = (p_1 \lor p_2) \land (\neg p_1 \lor p_3) \land (\neg p_2 \lor \neg p_3) \land \neg q$, which is a Krom formula. Note that $\text{Forget}(\Pi[q/\bot], \emptyset) \equiv (p_1 \lor p_2) \land (\neg p_1 \lor p_3) \land (\neg p_2 \lor \neg p_3)$. Thus $\neg \text{Forget}(\Pi[q/\bot], \emptyset) \equiv (p_1 \lor p_2 \lor \neg p_3) \land (p_1 \lor \neg p_2 \lor p_3) \land (\neg p_2 \lor \neg p_3)$. It is not a Krom formula. Actually, the clause $\neg p_1 \lor p_2 \lor \neg p_3$ is a prime implicate of $\neg \text{Forget}(\Pi[q/\bot], \emptyset)$.

**Theorem 16** Let $T, \varphi$ be two formulas, $V \subseteq \text{Var}(T)$, $q \in \text{Var}(T) \setminus V$.

(i) Deciding if $\varphi$ is a necessary (sufficient) condition of $q$ under $T$ is co-NP-complete.

(ii) Deciding if $\varphi$ is a necessary (sufficient) condition of $q$ under $T$ is tractable if $T$ and $\varphi$ are Horn (resp. ren-Horn and q-Horn) formulas.

(iii) Deciding if $\varphi$ is a strongest necessary (weakest sufficient) condition of $q$ under $T$ is $\Pi^P_2$-complete.
(iv) Deciding if \( \varphi \) is a strongest necessary (weakest sufficient) condition of \( q \) under \( T \) is co-NP-complete if \( T \) and \( \varphi \) are Horn (resp. ren-Horn and q-Horn) formulas.

**Proof:** (i) \( T \models q \supset \varphi \) iff \( T \land q \land \neg \varphi \) is unsatisfiable. This is in co-NP and co-NP-hard, i.e. deciding if \( \varphi \) is a necessary condition of \( q \) under \( T \) is co-NP-complete. The case of sufficient condition is similar.

(ii) \( T \models q \supset \varphi \) iff \( T \land q \land \neg \varphi \) is unsatisfiable for every clause \( c \) of \( \varphi \), which is tractable even \( T \) and \( \varphi \) are q-Horn formulas. Thus deciding if \( \varphi \) is a necessary condition of \( q \) under \( T \) is tractable. Similarly \( T \models \varphi \supset q \) iff \( T \land \varphi \land \neg q \) is unsatisfiable even if \( T \) and \( \varphi \) are q-Horn formulas.

(iii) In terms of Theorem 15, \( \varphi \) is a strongest necessary condition of \( q \) under \( T \) iff \( \varphi \equiv \text{Forget}(T[q/T], V') \) where \( V' = \text{Var}(T) \setminus (V \cup \{q\}) \). It is in \( \Pi_2^p \) and \( \Pi_2^p \)-hard by (i) of Corollary 9.

(iv) Recall that \( \varphi \) is a strong necessary condition of \( q \) under \( T \) if and only if \( \varphi \equiv \text{Forget}(T[q/T], V') \) by (i) of Theorem 15 where \( V' = \text{Var}(T) \setminus (V \cup \{q\}) \). Thus it is in co-NP when \( \varphi \) and \( T \) are q-Horn formulas and is co-NP-hard when \( \varphi \) and \( T \) are Horn formulas by (i) of Corollary 12.

**Proposition 11** Let \( T \) and \( \varphi \) be two Krom formulas, \( V \subseteq \text{Var}(T) \), \( q \in \text{Var}(T) \setminus V \).

(i) Deciding if \( \varphi \) is a strongest necessary condition of \( q \) under \( T \) is tractable.

(ii) Deciding if \( \varphi \) is a weakest sufficient condition of \( q \) under \( T \) is tractable.

**Proof:** Firstly, according to Theorem 4 one can compute \( \text{Forget}(T[q/T], V') \) in polynomial time in the size of \( T \) and \( V \) where \( V' = \text{Var}(T) \setminus (V \cup \{q\}) \). It is evident that \( \Sigma = \text{sunfold}(\text{Forget}[q/T], V') \) and \( \Sigma' = \text{Forget}(T[q/\bot], V') \) are Krom theories.

(i) It follows from the facts that checking equivalence for Krom theories is tractable and \( \varphi \) is a strongest condition of \( q \) under \( T \) iff \( \varphi \equiv \Sigma \) by (i) of Theorem 15.

(ii) \( \varphi \) is a weakest sufficient condition of \( q \) under \( T \) iff \( \varphi \equiv \neg \Sigma' \)

iff \( \neg \Sigma' \models \varphi \land \Sigma' \models \varphi \)

iff \( \varphi \land \Sigma' \) is unsatisfiable and \( \neg \Sigma' \models l_1 \lor l_2 \) for every conjunct \( l_1 \lor l_2 \) of \( \varphi \).

It is evident that checking satisfiability of \( \varphi \land \Sigma' \) is tractable since \( \varphi \land \Sigma' \) is a Krom formula. Note further that \( \neg \Sigma' \models l_1 \lor l_2 \)

iff \( \neg \Sigma' \land \neg l_1 \land \neg l_2 \) is unsatisfiable

iff \( \Sigma'' = \neg (\Sigma'[-l_1/T][-l_2/T]) \) is unsatisfiable

iff \( s_1 \land s_2 \) is unsatisfiable for every disjunct \( s_1 \land s_2 \) of \( \Sigma'' \), which is a 2-DNF formula.

\[ \square \]
4.3 Strongest and weakest definitions

Definability is acknowledged as an important logical concept when reasoning about knowledge represented in propositional logic. Informally speaking, an atom \( p \) can be “defined” in a given formula \( \Sigma \) in terms of a set \( X \) of atoms whenever the knowledge of the truth values of \( X \) enables concluding about the truth value of \( p \), under the condition of \( \Sigma \) [18].

Definition 3 ([18]) Let \( \Sigma \) be a formula, \( p \in A \), \( X \subseteq A \) and \( Y \subseteq A \).

- \( \Sigma \) defines \( p \) in terms of \( X \), denoted by \( X \sqsubseteq \Sigma p \), iff there exists a formula \( \Psi \) over \( X \) such that \( \Sigma \models \Psi \iff p \).

- \( \Sigma \) defines \( Y \) in terms of \( X \), denoted by \( X \sqsubseteq \Sigma Y \), iff there exists a formula \( \Psi \) over \( X \) such that \( \Sigma \models \Psi \iff p \) for every \( p \in Y \).

It is known that if both \( \varphi \) and \( \psi \) (over a same signature \( X \)) are definitions of \( p \) in \( \Sigma \) then \( \Sigma \models \varphi \iff \psi \), and additionally both \( \varphi \land \psi \) and \( \varphi \lor \psi \) are definitions of \( p \) in \( \Sigma \). In this situation, the strongest (resp. weakest) definition of \( p \) in \( \Sigma \) exist, they are denoted by \( \text{Def}_{X,l}^\Sigma(p) \) and \( \text{Def}_{X,u}^\Sigma(p) \) respectively. In terms of Corollary 9 of [3] and Theorem 10 of [18], if \( \Sigma \) defines \( p \) in terms of \( X \) then \( \text{Def}_{X,l}^\Sigma(p) \) (resp. \( \text{Def}_{X,u}^\Sigma(p) \)) is equivalent to the strongest necessary (resp. weakest sufficient) condition of \( p \) under \( \Sigma \). Thus according to Theorem 16 and Proposition 11 we have the following:

Corollary 17 Let \( \Sigma, \varphi \) be two formulas, \( X \subseteq A \), \( p \in A \) and \( \text{Var}(\varphi) \subseteq X \).

(i) The problem of deciding if \( \varphi \) is a strongest (resp. weakest) definition of \( p \) (in terms of \( X \)) in \( \Sigma \) is \( \Pi_2^P \)-complete.

(ii) The problem of deciding if \( \varphi \) is a strongest (resp. weakest) definition of \( p \) (in terms of \( X \)) in \( \Sigma \) is co-NP-complete if both \( \Sigma \) and \( \varphi \) are Horn (resp. ren-Horn and q-Horn) formulas.

(iii) deciding if \( \varphi \) is a strongest (resp. weakest) definition of \( p \) (in terms of \( X \)) in \( \Sigma \) is tractable if both \( \Sigma \) and \( \varphi \) are Krom formulas.

5 Concluding Remarks

As mentioned in the introduction, forgetting is closely connected with many other logical concepts. Quite late, the notion of relevance was quantitatively investigated [31], and the notion of independence was applied to belief change [32], which is a long-standing and vive topic in AI [33]. The main concerned Horn,
Krom and other fragments of propositional logic are also ubiquitous in AI [34, 35, 36, 37, 38].

In the paper we have firstly presented a resolution-based algorithm for computing forgetting results of CNF fragments of propositional logic. Though the algorithm is generally expensive even for Horn fragment as it is theoretically intractable, it opens a heuristic potentiality, e.g. choosing different orders of atoms to forget, and choosing different orders of resolvable clauses to do resolution. To investigate the effectiveness of the algorithm, heuristics and extensive experiments are worthy of studying.

What’s more, when concerning the dynamics of knowledge base, we considered various reasoning problems about forgetting in the fragments of propositional logic whose satisfiability are tractable. In particular, we concentrated on Horn, re-namable Horn, q-Horn and Krom theories. The considered reasoning problems include VAR-EQUIVALENCE, VAR-INDEPENDENCE, VAR-WEAK, VAR-STRONG, VAR-MATCH and VAR-ENTAILMENT. Although some of the problems have been partially solved, e.g., VAR-EQUIVALENCE and VAR-INDEPENDENCE for propositional logic are proved in [2], this is the first comprehensive study on these problems for CNF, Horn, ren-Horn, q-Horn, Krom and DNF fragments, to our knowledge. It motivates us to consider these reasoning problems for forgetting in non-classical logical systems, such as model logic S5 in particular.

It deserves our further effort to investigate the knowledge simplification or compilation [16] in other logical formalisms, logic programming under stable model semantics, particularly.

Acknowledgement This work was supported by the National Natural Science Foundation of China under grants 60963009,61370161 and Stadholder Foundation of Guizhou Province under grant (2012)62.

References

[1] Fangzhen Lin and Ray Reiter. Forget it! In In Proceedings of the AAAI Fall Symposium on Relevance, pages 154–159, 1994.

[2] Jérôme Lang, Paolo Liberatore, and Pierre Marquis. Propositional independence: Formula-variable independence and forgetting. Journal of Artificial Intelligence Research, 18:391–443, 2003.

[3] Jérôme Lang and Pierre Marquis. Reasoning under inconsistency: A forgetting-based approach. Artificial Intelligence, 174(12-13):799–823, 2010.
[4] Albert Visser. Uniform interpolation and layered bisimulation. In Gödel’96, pages 139–164, 1996.

[5] Daniel G. Bobrow, Devika Subramanian, Russell Greiner, and Judea Pearl, editors. Special issue on relevance 97 (1-2). Artificial Intelligence Journal, 1997.

[6] Yan Zhang and Norman Y. Foo. Solving logic program conflict through strong and weak forgettings. Artificial Intelligence, 170(8-9):739–778, 2006.

[7] Thomas Eiter and Kewen Wang. Semantic forgetting in answer set programming. Artificial Intelligence, 172(14):1644–1672, 2008.

[8] Ka-Shu Wong. Forgetting in Logic Programs. PhD thesis, The University of New South Wales, 2009.

[9] Yisong Wang, Yan Zhang, Yi Zhou, and Mingyi Zhang. Forgetting in logic programs under strong equivalence. In Principles of Knowledge Representation and Reasoning: Proceedings of the Thirteenth International Conference, pages 643–647, Rome, Italy, 2012. AAAI Press.

[10] Yisong Wang, Kewen Wang, and Mingyi Zhang. Forgetting for answer set programs revisited. In IJCAI 2013, Proceedings of the 23rd International Joint Conference on Artificial Intelligence, pages 1162–1168, Beijing, China, 2013. IJCAI/AAAI.

[11] Zhe Wang, Kewen Wang, Rodney W. Topor, and Jeff Z. Pan. Forgetting for knowledge bases in dl-lite. Annuals of Mathematics and Artificial Intelligence, 58(1-2):117–151, 2010.

[12] Carsten Lutz and Frank Wolter. Foundations for uniform interpolation and forgetting in expressive description logics. In IJCAI 2011, Proceedings of the 22nd International Joint Conference on Artificial Intelligence, pages 989–995, Barcelona, Catalonia, Spain, 2011. IJCAI/AAAI.

[13] Yan Zhang and Yi Zhou. Knowledge forgetting: Properties and applications. Artificial Intelligence, 173(16-17):1525–1537, 2009.

[14] Kaile Su, Abdul Sattar, Guanfeng Lv, and Yan Zhang. Variable forgetting in reasoning about knowledge. Journal of Artificial Intelligence Research, 35:677–716, 2009.

[15] Yongmei Liu and Ximing Wen. On the progression of knowledge in the situation calculus. In IJCAI 2011, Proceedings of the 22nd International Joint
Conference on Artificial Intelligence, pages 976–982, Barcelona, Catalonia, Spain, 2011. IJCAI/AAAI.

[16] Meghyn Bienvenu, Hélène Fargier, and Pierre Marquis. Knowledge compilation in the modal logic S5. In Maria Fox and David Poole, editors, AAAI. AAAI Press, 2010.

[17] Fangzhen Lin. On strongest necessary and weakest sufficient conditions. Artificial Intelligence, 128(1-2):143–159, 2001.

[18] Jérôme Lang and Pierre Marquis. On propositional definability. Artif. Intell., 172(8-9):991–1017, 2008.

[19] Alfred Horn. On sentences which are true of direct unions of algebras. The Journal of Symbolic Logic, 16(1):14–21, 1951.

[20] Harry R. Lewis. Renaming a set of clauses as a horn set. J. ACM, 25(1):134–135, 1978.

[21] Endre Boros, Yves Crama, and Peter L. Hammer. Polynomial-time inference of all valid implications for horn and related formulae. Annals of Mathematics and Artificial Intelligence, 1:21–32, 1990.

[22] Thomas Eiter, Toshihide Ibaraki, and Kazuhisa Makino. Double horn functions. Inf. Comput., 144(2):155–190, 1998.

[23] M. R. Krom. The decision problem for formulas in prenex conjunctive normal form with binary disjunctions. The Journal of Symbolic Logic, 35(2):210–216, 1970.

[24] Gerhard Lakemeyer. Relevance from an epistemic perspective. Artif. Intell., 97(1-2):137–167, 1997.

[25] Pierre Marquis. Handbook of Defeasible Reasoning and Uncertainty Management Systems: Algorithms for Defeasible and Uncertain Reasoning, volume 5, chapter Consequence finding algorithms, pages 41–145. Kluwer Academic Publishers, 1999.

[26] V. Chandru, Collette R. Coullard, Peter L. Hammer, M. Montanuz, and Xiaorong Sun. On renamable horn and generalized horn functions. Annals of Mathematics and Artificial Intelligence, 1(1-4):33–47, 1990.

[27] Endre Boros. Recognition of q-horn formulae in linear time. Discrete Applied Mathematics, 55(1):1 – 13, 1994.
[28] Hans Kleine Büning and Uwe Bubeck. *Handbook of Satisfiability*, chapter 23 Theory of Quantified Boolean Formulas, pages 735–760. IOS Press, 2009.

[29] Thomas Eiter and Kazuhisa Makino. On computing all abductive explanations from a propositional horn theory. *J. ACM*, 54(5), 2007.

[30] Giovanna D’Agostino. Interpolation in non-classical logics. *Synthese*, 164(3):421–435, 2008.

[31] Xin Liang, Zuoquan Lin, and Jan Van den Bussche. Quantitatively evaluating formula-variable relevance by forgetting. In *Canadian Conference on AI*, volume 7884 of *Lecture Notes in Computer Science*, pages 271–277, Regina, SK, Canada, 2013. Springer.

[32] Pierre Marquis and Nicolas Schwind. Lost in translation: Language independence in propositional logic – application to belief change. *Artificial Intelligence*, 206(0):1 – 24, 2014.

[33] Carlos E. Alchourrön, Peter Gärdnfor and David Makinson. On the logic of theory change: Partial meet contraction and revision functions. *Journal of Symbolic Logic*, 50(2):510–530, 1985.

[34] Uwe Bubeck, Hans Kleine Büing, and Xishun Zhao. Quantifier rewriting and equivalence models for quantified horn formulas. In Fahiem Bacchus and Toby Walsh, editors, *Theory and Applications of Satisfiability Testing*, volume 3569 of *Lecture Notes in Computer Science*, pages 386–392. Springer Berlin Heidelberg, 2005.

[35] Paolo Liberatore. Redundancy in logic II: 2CNF and horn propositional formulae. *Artificial Intelligence*, 172(2-3):265–299, 2008.

[36] Maonian Wu, Dongmo Zhang, and Mingyi Zhang. Language splitting and relevance-based belief change in horn logic. In *AAAI*, San Francisco, California, USA, 2011. AAAI Press.

[37] James P. Delgrande and Renata Wassermann. Horn clause contraction functions. *Journal of Artificial Intelligence Research*, 48:475–511, 2013.

[38] James P. Delgrande and Pavlos Peppas. Belief revision in horn theories. *Artificial Intelligence*, 218:1–22, 2015.