Some intrinsic properties of h-Randers conformal change

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Abstract

In the present paper we have considered h-Randers conformal change of a Finsler metric \( L \), which is defined as

\[
L(x, y) \rightarrow \tilde{L}(x, y) = e^{\sigma(x)} L(x, y) + \beta(x, y),
\]

where \( \sigma(x) \) is a function of \( x \), \( \beta(x, y) = b_i(x, y)y^i \) is a 1- form on \( M^n \) and \( b_i \) satisfies the condition of being an h-vector. We have obtained the expressions for geodesic spray coefficients under this change. Further we have studied some special Finsler spaces namely quasi-C-reducible, C-reducible, S3-like and S4-like Finsler spaces arising from this metric. We have also obtained the condition under which this change of metric leads a Berwald (or a Landsberg) space into a space of the same kind.

Mathematics subject Classification: 53B40, 53C60.

Keywords: h-vector; special Finsler spaces; geodesic; conformal change.
1 Introduction

Let $M^n$ be an n-dimensional differentiable manifold and $F^n$ be a Finsler space equipped with a fundamental function $L(x, y), (y^i = \dot{x}^i)$ of $M^n$. If a differential 1-form $\beta(x, y) = b_i(x)y^i$ is given on $M^n$, M. Matsumoto [7] introduced another Finsler space whose fundamental function is given by

$$\bar{L}(x, y) = L(x, y) + \beta(x, y)$$

This change of Finsler metric has been called $\beta$-change [10, 11].

The conformal theory of Finsler spaces was initiated by M.S. Knebelman [6] in 1929 and has been investigated in detail by many authors [1, 2, 3, 4]. The conformal change is defined as

$$\bar{L}(x, y) \rightarrow e^{\sigma(x)}L(x, y),$$

where $\sigma(x)$ is a function of position only and known as conformal factor.

In 1980, Izumi [3] introduced the h-vector $b_i$ which is v-covariantly constant with respect to Cartan’s connection $C_T$ (i.e. $b_i|_j = 0$) and satisfies the relation $LC^h_{ij}b_h = \rho h_{ij}$, where $C^h_{ij}$ are components of (h)hv-torsion tensor and $h_{ij}$ are components of angular metric tensor. Thus the h-vector is not only a function of coordinates $x^i$, but it is also a function of directional arguments satisfying $L\partial_j b_i = \rho h_{ij}$.

In the paper [14] S. H. Abed generalized the above two changes and have introduced another Finsler metric named as Conformal $\beta$- change and further Gupta and Pandey [15] renamed it Randers conformal change and obtained various important result in the field of Finsler spaces. Recently we [13] have generalized the metric given by S. H. Abed with the help of h-vector and have introduced another Finsler metric which is defined as

$$\bar{L}(x, y) = e^{\sigma(x)}L(x, y) + \beta(x, y),$$

(1.1)

where $\sigma(x)$ is a function of $x$ and $\beta(x, y) = b_i(x,y)y^i$ is a 1- form on $M^n$ and $b_i$ satisfies the condition of being an h-vector. We call the change $L(x, y) \rightarrow \bar{L}(x, y)$ as h-Randers conformal change. This change generalizes various types of changes. When $\beta = 0$, it reduces to a conformal change. When $\sigma = 0$, it reduces to a h-Randers change [9]. When $\beta = 0$ and $\sigma$ is a non-zero constant then it reduces to a homothetic change. When $b_i$ is function of position only and $\sigma = 0$, it reduces to Randers change[12]. When $b_i$ and $\sigma$ are functions of position only, it reduces to Randers conformal change
In the present paper we have obtained the expressions for geodesic spray coefficients under this change. Further we have studied some special Finsler spaces namely quasi C-reducible, C-reducible, S3-like and S4-like Finsler spaces arising from this metric. We have also obtained the conditions under which this change of metric leads a Berwald (or a Landsberg) space into a space of the same kind.

2 h-Randers conformal change

Let the Cartan’s connection of Finsler space \( F^n \) be denoted by \( CT = (F^i_{jk}, G^i_j, C^i_{jk}) \). Since \( b_i(x, y) \) are components of h-vector, we have

\[
\begin{align*}
(a) \quad b^|_{ij} &= \dot{\gamma}_j b_i - b_h C^h_{ij} = 0 \\
(b) \quad LC^h_{ij} b_h &= \rho h_{ij}
\end{align*}
\]  

(2.1)

Hence we obtain

\[
\dot{\gamma}_j b_i = L^{-1} \rho h_{ij}
\]

(2.2)

Since \( h_{ij} \) are components of an indicatory tensor i.e. \( h_{ij} y^i = 0 \), we have \( \dot{\gamma}_i \beta = b_i \).

**Definition 2.1.** Let \( M^n \) be an \( n \)-dimensional differentiable manifold and \( F^n \) be a Finsler space equipped with a fundamental function \( L(x, y) \), \( (y^i = \dot{x}^i) \) of \( M^n \). A change in the fundamental function \( L \) by the equation (1.1) on the same manifold \( M^n \) is called h-Randers conformal change. A space equipped with fundamental metric \( \bar{L} \) is called h-Randers conformally changed Finsler space \( \bar{F}^n \).

Differentiating equation (1.1) with respect to \( y^i \), the normalized supporting element \( \bar{l}_i = \dot{\gamma}_i \bar{L} \) is given by

\[
\bar{l}_i = e^\sigma l_i + b_i,
\]

(2.3)

where \( l_i = \dot{\gamma}_i L \) is the normalized supporting element \( l_i \) of \( F^n \).

Differentiating (2.3) with respect to \( y^j \) and using (2.2) and the fact that \( \dot{\gamma}_j l_i = L^{-1} h_{ij} \), we get

\[
\bar{h}_{ij} = \phi h_{ij},
\]

(2.4)

where \( \phi = L^{-1} \bar{L}(e^\sigma + \rho) \) and \( h_{ij} = L \dot{\gamma}_i \dot{\gamma}_j L \) is the angular metric tensor in the Finsler space \( F^n \).
Since $h_{ij} = g_{ij} - l_i l_j$, from (2.3) and (2.4) the fundamental tensor $\bar{g}_{ij} = \partial_i \partial_j \frac{L^2}{2} = \bar{h}_{ij} + \bar{l}_i \bar{l}_j$ is given as

$$\bar{g}_{ij} = \phi g_{ij} + b_i b_j + e^\sigma (b_i l_j + b_j l_i) + (e^{2\sigma} - \phi) l_i l_j$$  \hspace{1cm} (2.5)

It is easy to see that the $\det(\bar{g}_{ij})$ does not vanish, and the reciprocal tensor with components $\bar{g}^{ij}$ of $F^n$, obtainable from $\bar{g}^{ij} \bar{g}_{jk} = \delta_i^k$, is given by

$$\bar{g}^{ij} = \phi^{-1} g^{ij} - \mu l^i l^j - \phi^{-2} (e^\sigma + \rho)(l^i b^j + l^j b^i),$$  \hspace{1cm} (2.6)

where $\mu = (e^\sigma + \rho)^2 \phi^{-3} (e^\sigma - b^2 - \phi)$, $b^2 = b_i b^i$, $b^i = g^{ij} b_j$ and $g^{ij}$ is the reciprocal tensor of $g_{ij}$ of $F^n$.

We have following lemma [13]:

**Lemma 2.1.** The scalar $\rho$ used in the condition of h-vector is a function of coordinates $x^i$ only.

From equations (1.1), (2.3) and lemma 2.1 we have

$$\partial_i \phi = L^{-1} (e^\sigma + \rho) m_i,$$  \hspace{1cm} (2.7)

where

$$m_i = b_i - (L^{-1} \beta) l_i$$  \hspace{1cm} (2.8)

Differentiating (2.4) with respect to $y^k$ and using (2.3), (2.4), (2.7) and the relation $\partial_i h_{ij} = 2 C_{ij} - L^{-1} (l_i h_{jk} + l_j h_{ik})$, the Cartan covariant tensor $\bar{C}_{ijk}$ is given by

$$\bar{C}_{ijk} = \phi C_{ijk} + \frac{(e^\sigma + \rho)}{2L} (h_{ij} m_k + h_{jk} m_i + h_{ki} m_j),$$  \hspace{1cm} (2.9)

where $C_{ijk}$ is (h)hv-torsion tensor of Cartan’s connection $\Gamma$ of Finsler space $F^n$.

From the definition of $m_i$, it is evident that

$$(a) \ m_i l^i = 0, \hspace{1cm} (b) \ m_i b^i = b^2 - \frac{\beta^2}{L^2} = b_i m^i,$$  \hspace{1cm} (2.10)

$$(c) \ g_{ij} m^i = h_{ij} m^i = m_j, \hspace{1cm} (d) \ C_{ih} m^h = L^{-1} \rho h_{ij}$$

From (2.1), (2.6), (2.9) and (2.10), we get

$$\bar{C}_{ij}^h = C_{ij}^h + \frac{1}{2L} (h_{ij} m^h + h_{jh} m_i + h_{ih} m_j) - \frac{1}{L} \left( \rho + \frac{L}{2L} (b^2 - \frac{\beta^2}{L^2}) \right) h_{ij} m^h + \frac{L}{L} m_i m_j l^h$$  \hspace{1cm} (2.11)
Proposition 2.1. Let \( \bar{F}^n = (M^n, \bar{L}) \) be an \( n \)-dimensional Finsler space obtained from the \( h \)-Randers conformal change of the Finsler space \( F^n = (M^n, L) \), then the normalized supporting element \( \bar{l}_i \), angular metric tensor \( \bar{h}_{ij} \), fundamental metric tensor \( \bar{g}_{ij} \) and (h)\( h \)-torsion tensor \( \bar{C}_{ijk} \) of \( \bar{F}^n \) are given by (2.3), (2.4), (2.5) and (2.9) respectively.

3 Geodesic Spray coefficients of \( \bar{F}^n \)

Let \( s \) be the arc-length of a curve \( x^i = x^i(t) \) on a differentiable manifold \( M^n \), then the equation of a geodesic [5] of \( F^n = (M^n, L) \) is written in the well-known form:

\[
\frac{d^2 x^i}{ds^2} + 2G^i(x, \frac{dx}{ds}) = 0, \quad (3.1)
\]

where functions \( G^i(x, y) \) are the geodesic spray coefficients given by

\[
2G^i = g^{ir}(y^j\partial_r \partial_j F - \partial_r F), \quad F = L^2.
\]

Now suppose \( \bar{s} \) is the arc-length of a curve \( \bar{x}^i = \bar{x}^i(t) \) on a differentiable manifold \( M^n \) in the Finsler space \( \bar{F}^n = (M^n, \bar{L}) \), then the equation of geodesic in \( \bar{F}^n \) can be written as

\[
\frac{d^2 x^i}{d\bar{s}^2} + 2\bar{G}^i(x, \frac{dx}{d\bar{s}}) = 0, \quad (3.2)
\]

where functions \( \bar{G}^i(x, y) \) are given by

\[
2\bar{G}^i = \bar{g}^{ir}(y^j\partial_r \partial_j \bar{F} - \partial_r \bar{F}), \quad \bar{F} = \bar{L}^2.
\]

Since \( d\bar{s} = \bar{L}(x, dx) \), this is also written as

\[
d\bar{s} = e^{\sigma(x)} L(x, dx) + b_i(x, y)dx^i = e^{\sigma(x)} ds + b_i(x, y)dx^i
\]

Since \( ds = L(x, dx) \), we have

\[
\frac{dx^i}{ds} = \frac{dx^i}{d\bar{s}} [e^{\sigma(x)} + b_i(x, y) \frac{dx^i}{ds}]
\]

(3.3)

Differentiating (3.3) with respect to \( s \), we have

\[
\frac{d^2 x^i}{ds^2} = \frac{d^2 x^i}{d\bar{s}^2} [e^{\sigma(x)} + b_i \frac{dx^i}{ds}]^2 + \frac{dx^i}{ds} \left( \frac{de^{\sigma(x)}}{ds} + \frac{db_i}{ds} \frac{dx^i}{ds} + b_i \frac{d^2 x^i}{d\bar{s}^2} \right)
\]
Substituting the value of $\frac{dx^i}{ds}$ from (3.3), the above equation becomes

$$
\frac{d^2x^i}{ds^2} = \frac{d^2x^i}{ds^2}[e^{\sigma(x)}] + b_i \frac{dx^i}{ds} \bigg[ \frac{de^{\sigma(x)}}{ds} + \frac{db_i}{ds} \frac{dx^i}{ds} + \frac{d^2x^i}{ds^2} \bigg]
$$

which we shall denote by

$$
(3.4)
$$

Now differentiating equation (1.1) with respect to $x^i$ we have

$$
\partial_i \bar{L} = e^\sigma A_i + B_i,
$$

where $A_i = L \partial_i \sigma + \partial_i L$ and $B_i = \partial_i \{b_r(x,y)\}y^r$.

Differentiating above equation with respect to $y^j$ we have

$$
\partial_j \partial_i \bar{L} = e^\sigma \partial_j A_i + \partial_j B_i,
$$

where $\partial_j A_i = l_j \partial_i \sigma + \partial_j \partial_i L$ and $\partial_j B_i = \partial_j \{\partial_i b_r(x,y)\}y^r + \partial_i b_r(x,y)\delta^i_j$.

Since

$$
2\bar{G}_r = y^j (l_r \partial_j \bar{L} + \bar{L} \partial_j \partial_i \bar{L}) - \bar{L} \partial_r \bar{L}
$$

therefore using equations (2.3), (3.5) and (3.6) we have

$$
2\bar{G}_r = 2e^{2\sigma} G_r + y^j \{2e^{2\sigma} l_r \partial_j \sigma + e^\sigma (l_r B_j + b_r A_j) + b_j + e^\sigma \sigma \partial_j A_i + e^\sigma \partial_j B_i \} - (e^{2\sigma} L^2 \partial_i \sigma + e^\sigma LB_r + \beta e^\sigma A_r + \beta B_r),
$$

where

$$
G_r = y^j \{l_r \partial_j L + \bar{L} \partial_j \partial_i \bar{L}\} - \bar{L} \partial_r \bar{L}
$$

is the spray coefficients for the Finsler space $F^n$.

Using equations (2.6) and (3.7) we have

$$
\bar{G}^i = JG^i + M^i,
$$

where $G^i = g^{ir} G_r$, $J = \frac{1}{\phi}$, and $M^i = \frac{1}{2} e^{2\sigma} G_r \{-\mu l^r \nu - \phi^{-2}(e^\sigma + \rho)(l^r b^l + l^l b^r) + \frac{1}{2} [\phi^{-1} g^{ir} - \mu l^r \nu - \phi^{-2}(e^\sigma + \rho)(l^l b^r + l^r b^l)]\}j^j \{2e^{2\sigma} l_r \sigma \partial_j \sigma + e^\sigma (l_r B_j + b_r A_j) + b_r B_j + e^\sigma \partial_r B_j + \partial_r A_j \} - (e^{2\sigma} L^2 \partial_i \sigma + e^\sigma LB_r + \beta e^\sigma A_r + \beta B_r)$.

**Theorem 3.1.** Let $\hat{F}^n = (M^n, \bar{L})$ be an $n$-dimensional Finsler space obtained from the $h$-Randers conformal change of the Finsler space $F^n = (M^n, L)$, then the the geodesic spray coefficients $\bar{G}^i$ for the Finsler space $\hat{F}^n$ are given by (3.8) in the terms of the geodesic spray coefficients $G^i$ of the Finsler space $F^n$. 

6
Corollary 3.1. Let $\tilde{F}^n = (M^n, \tilde{L})$ be an $n$-dimensional Finsler space obtained from the $h$-Randers conformal change of the Finsler space $F^n = (M^n, L)$, then the equation of geodesic of $\tilde{F}^n$ is given by (3.2), where $\frac{d^2 \mathbf{x}^i}{ds^2}$ and $\bar{G}$ are given by (3.4) and (3.8) respectively.

4 C-reducibility of $\tilde{F}^n$

Following Matsumoto [8], in this section we shall investigate special cases of the Finsler space with $h$-Randers conformally changed Finsler space $\tilde{F}^n$.

Definition 4.1. A Finsler space $(M^n, L)$ with dimension $n \geq 3$ is said to be quasi-$C$-reducible if the Cartan tensor $C_{ijk}$ satisfies

$$C_{ijk} = Q_{ij}C_k + Q_{jk}C_i + Q_{ki}C_j, \quad (4.1)$$

where $Q_{ij}$ is a symmetric indicatory tensor.

Substituting $h = j$ in equation (2.11) we get

$$\bar{C}_i = C_i + \frac{(n + 1)}{2L} m_i \quad (4.2)$$

Using equations (2.9) and (4.2), we have

$$\bar{C}_{ijk} = \phi C_{ijk} + \frac{\phi}{(n+1)} \pi (ijk) \{ h_{ij} (\bar{C}_k - C_k) \};$$

where $\pi (ijk)$ represents cyclic permutation and sum over the indices $i, j$ and $k$.

The above equation can be written as

$$\bar{C}_{ijk} = \phi C_{ijk} + \frac{\phi}{(n+1)} \pi (ijk) (h_{ij} \bar{C}_k) - \frac{\phi}{(n+1)} \pi (ijk) (h_{ij} C_k)$$

Thus

Lemma 4.1. In an $h$-Randers conformally changed Finsler space $\bar{F}^n$, the Cartan’s tensor can be written in the form

$$\bar{C}_{ijk} = \pi (ijk) (H_{ij} C_k) + V_{ijk}, \quad (4.3)$$

where $\bar{H}_{ij} = \frac{h_{ij}}{(n+1)}$ and $V_{ijk} = \phi C_{ijk} - \frac{\phi}{(n+1)} \pi (ijk) (h_{ij} C_k)$.

Since $\bar{H}_{ij}$ is a symmetric and indicatory tensor, so from the above lemma and (4.1) we get
Theorem 4.1. An h-Randers conformally changed Finsler space $\tilde{F}^n$ is quasi-C-reducible if the tensor $V_{ijk}$ of equation (4.3) vanishes identically.

We obtain a generalized form of Matsumoto’s result known [8] as a corollary of the above theorem

Corollary 4.1. If $F^n$ is Riemannian then an h-Randers conformally changed Finsler space $\tilde{F}^n$ is always a quasi-C-reducible Finsler space.

Definition 4.2. A Finsler space $(M^n, L)$ of dimension $n \geq 3$ is called C-reducible if the Cartan tensor $C_{ijk}$ is written in the form

$$C_{ijk} = \frac{1}{(n+1)}(h_{ij}C_k + h_{ki}C_j + h_{jk}C_i) \quad (4.4)$$

Now from equation (2.9) and definition of C-reducibility we have

$$\phi C_{ijk} = \pi_{(ijk)}(\tilde{h}_{ij}N_k), \quad (4.5)$$

where $N_k = \frac{1}{(n+1)}C_k - \frac{1}{2n}m_k$. Conversely, if (4.5) is satisfied for certain covariant vector $N_k$ then from (2.9) we have

$$\tilde{C}_{ijk} = \frac{1}{(n+1)}\pi_{(ijk)}(\tilde{h}_{ij}\tilde{C}_k) \quad (4.6)$$

Thus we have

Theorem 4.2. An h-Randers conformally changed Finsler space $\tilde{F}^n$ is C-reducible iff equation (4.5) holds good.

Corollary 4.2. If the Finsler space $F^n$ is C-reducible Finsler space, then an h-Randers conformally changed Finsler space $\tilde{F}^n$ is always a C-reducible Finsler space.

5 Some Important tensors of $\tilde{F}^n$

The $v$-curvature tensor [8] of Finsler space with fundamental function $L$ is given by

$$S_{hijk} = C_{ijr}C_{hk}^r - C_{ikr}C_{hj}^r$$

Therefore the $v$-curvature tensor of an h-Randers conformally changed Finsler space $F^n$ will be given by

$$\tilde{S}_{hijk} = \tilde{C}_{ijr}\tilde{C}_{hk}^r - \tilde{C}_{ikr}\tilde{C}_{hj}^r \quad (5.1)$$
From equations (2.9) and (2.11) we have

\[ \tilde{C}_{ijr} \tilde{C}_{rhk} = \phi \left[ C_{ijr} C_{rhk} + \left( \frac{\rho}{L \ell} - \frac{m^2}{4L^2} \right) h_{ik} h_{ij} + \frac{1}{2L} (C_{ijk} m_h + C_{ijh} m_k + C_{ihk} m_j) \right. \]

\[ \left. + \frac{1}{4L^2} \left( h_{ik} m_j m_k + h_{jk} m_i m_h + h_{ki} m_j m_h \right) \right] \]

where \( h_{jr} C_{rhk} = C_{jhr} = h^r C_{rhk} \), \( m_i m^i = m^2 \).

Using equations (5.1) and (5.2) we have

\[ \bar{S}_{hijk} = \phi \left[ S_{hijk} + \left( \frac{\rho}{L \ell} - \frac{m^2}{4L^2} \right) \left\{ h_{ik} h_{ij} - h_{kj} h_{ik} \right\} + \frac{1}{4L^2} \left\{ h_{ik} m_j m_k - h_{kj} m_i m_k \right\} \right] \]

**Proposition 5.1.** In an h-Randers conformally changed Finsler space \( \tilde{F}^n \) the v-curvature tensor \( \bar{S}_{hijk} \) is given by (5.3).

It is well known\(^8\) that the v-curvature tensor of any three-dimensional Finsler space is of the form

\[ L^2 S_{hijk} = S (h_{ij} h_{ik} - h_{kk} h_{ij}), \quad (5.4) \]

where scalar \( S \) in (5.4) is a function of \( x \) alone.

Owing to this fact M. Matsumoto defined the S3-like Finsler space as

**Definition 5.1.** A Finsler space \( F^n (n \geq 3) \) is said to be S3-like Finsler space if the v-curvature tensor is of the form (5.4).

The v-curvature tensor of any four-dimensional Finsler space may be written as \(^8\):

\[ L^2 S_{hijk} = \Theta_{(jk)} \{ h_{ij} K_{ki} + h_{ik} K_{hj} \}, \quad (5.5) \]

where \( K_{ij} \) is a (0, 2) type symmetric Finsler tensor field which is such that \( K_{ij} y^j = 0 \) and the symbol \( \Theta_{(jk)} \{ ... \} \) denotes the interchange of \( j, k \) and subtraction. The definition of S4-like Finsler space is given as

**Definition 5.2.** A Finsler space \( F^n (n \geq 4) \) is said to be S4-like Finsler space if the v-curvature tensor is of the form (5.5).

From equation (5.3) we have
Lemma 5.1. The v-curvature tensor \( \bar{S}_{hijk} \) of a h-Randers conformally changed Finsler space can be written as
\[
\bar{S}_{hijk} = \bar{S}(\bar{h}_{ij} \bar{h}_{ik} - \bar{h}_{ik} \bar{h}_{ij}) + U_{hijk},
\]
where \( \bar{S} = -\frac{1}{\phi}\left(\frac{\rho}{LL} - \frac{m^2}{4L^2}\right) \) and \( U_{hijk} = \phi\left[ S_{hijk} + \frac{1}{4L^2}\{ h_{ij}m_km_j - h_{ik}m_im_j + h_{ik}m_m_k - h_{ij}m_hm_k \} \right] \).

From lemma (5.1) and definition of S3-like Finsler space we have

Theorem 5.1. An h-Randers conformally changed Finsler space \( \bar{F}^n \) is S3-like if the tensor \( U_{hijk} \) of equation (5.6) vanishes identically.

From equation (5.3) we have

Lemma 5.2. The v-curvature tensor \( \bar{S}_{hijk} \) of an h-Randers conformally changed Finsler space can also be written as
\[
\bar{S}_{hijk} = \Theta_{ijk}(\bar{h}_{ij}K_{ij} + \bar{h}_{ik}K_{hj}) + \phi S_{hijk},
\]
where \( K_{ij} = \frac{1}{4L^2}m_i m_j - \frac{1}{2}\left( \frac{\rho}{LL} - \frac{m^2}{4L^2} \right) h_{ij} \).

Thus from lemma (5.2) and definition of S4-like Finsler space we have

Theorem 5.2. If the v-curvature tensor of Finsler space \( F^n \) vanishes identically then an h-Randers conformally changed Finsler space \( \bar{F}^n \) is S4-like Finsler space.

Now we are concerned with \((v)hv\)-torsion tensor \( P_{ijk} \). With respect to the Cartan’s connection \( CT \), \( L_{ji} = 0, L_{ij} = 0, h_{ij|k} = 0 \) hold good \[8\].

Taking h-covariant derivative of the equation (2.9) we have
\[
\bar{C}_{ijk|h} = L^{-1}\bar{L}(e^\sigma_{|h} + \rho_{|h})\{ C_{ijk} + \frac{1}{2L}(h_{ij}m_k + h_{jk}m_i + h_{ki}m_j) \} + \phi\left[ C_{ijk|h} + \frac{1}{2L}(h_{ij}m_k|_h + h_{jk}m_i|_h + h_{ki}m_j|_h) \right]
\]
where \( m_{i|h} = b_{i|h} - L^{-1}l_ib_{|h}y^r \).

Lemma 5.3. The h-covariant derivative of the Cartan tensor \( \bar{C}_{ijk} \) of an h-Randers conformally changed Finsler space \( F^n \) can be written as
\[
\bar{C}_{ijk|h} = \phi C_{ijk|h} + V_{ijkh},
\]
where \( V_{ijkh} = L^{-1}\bar{L}(e^\sigma_{|h} + \rho_{|h})\{ C_{ijk} + \frac{1}{2L}(h_{ij}m_k + h_{jk}m_i + h_{ki}m_j) \} + \phi\left( h_{ij}m_k|_h + h_{jk}m_i|_h + h_{ki}m_j|_h \right) \).
The \((v)hv\)-torsion tensor \(P_{ijk}\) of the Cartan connection \(C\) is written in the form
\[
P_{ijk} = C_{ijk|0},
\]
where the subscript ‘0’ means the contraction with respect to the supporting element \(y^i\).

From the equation (5.8), the \((v)hv\)-torsion tensor \(\bar{P}_{ijk}\) is given by
\[
\bar{P}_{ijk} = \phi P_{ijk} + L^{-1}\bar{L}(e^\sigma\sigma_{0} + \rho_{0})\{C_{ijk} + \frac{1}{2\bar{L}}(h_{ij}m_k + h_{jk}m_i + h_{ki}m_j)\} + \frac{\phi}{2\bar{L}}\{h_{ij}m_k|0 + h_{jk}m_{i|0} + h_{ki}m_{j|0}\}.
\]

Thus we have

**Proposition 5.2.** The \((v)hv\)-torsion tensor \(\bar{P}_{ijk}\) of an \(h\)-Randers conformally changed Finsler space can be written in the form of (5.10).

From the equation (5.10) we have

**Lemma 5.4.** The \((v)hv\)-torsion tensor \(\bar{P}_{ijk}\) of an \(h\)-Randers conformally changed Finsler space can also be written as
\[
\bar{P}_{ijk} = \phi P_{ijk} + W_{ijk},
\]
where
\[
W_{ijk} = L^{-1}\bar{L}(e^\sigma\sigma_{0} + \rho_{0})\{C_{ijk} + \frac{1}{2\bar{L}}(h_{ij}m_k + h_{jk}m_i + h_{ki}m_j)\} + \frac{\phi}{2\bar{L}}\{h_{ij}m_k|0 + h_{jk}m_{i|0} + h_{ki}m_{j|0}\}.
\]

We have

**Definition 5.3.** A Finsler space is called a Berwald space if \(C_{ijk|h} = 0\) holds good.

**Definition 5.4.** A Finsler space is called a Landsberg space if \(P_{ijk} = 0\) holds good.

In view of above definition (5.3) and the lemma (5.3) we have

**Theorem 5.3.** If a Finsler space \(F^n\) is a Berwald space and the tensor \(V_{ijkh}\) of equation (5.9) vanishes identically then an \(h\)-Randers conformally changed Finsler space \(\bar{F}^n\) is a Berwald space.

In view of above definition (5.4) and the lemma (5.4) we have

**Theorem 5.4.** If a Finsler space \(F^n\) is a Landsberg space and the tensor \(W_{ijk}\) of equation (5.11) vanishes identically then an \(h\)-Randers conformally changed Finsler space \(\bar{F}^n\) is a Landsberg space.
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