ON SHARP APERTURE-WEIGHTED ESTIMATES FOR SQUARE FUNCTIONS

ANDREI K. LERNER

ABSTRACT. Let $S_{\alpha,\psi}(f)$ be the square function defined by means of the cone in $\mathbb{R}_+^{n+1}$ of aperture $\alpha$, and a standard kernel $\psi$. Let $[w]_{A_p}$ denote the $A_p$ characteristic of the weight $w$. We show that for any $1 < p < \infty$ and $\alpha \geq 1$,

$$\|S_{\alpha,\psi}\|_{L^p(w)} \lesssim \alpha^n [w]_{A_p}^{\max(\frac{n}{2}, \frac{1}{p-1})}.$$ 

For each fixed $\alpha$ the dependence on $[w]_{A_p}$ is sharp. Also, on all class $A_p$ the result is sharp in $\alpha$. Previously this estimate was proved in the case $\alpha = 1$ using the intrinsic square function. However, that approach does not allow to get the above estimate with sharp dependence on $\alpha$. Hence we give a different proof suitable for all $\alpha \geq 1$ and avoiding the notion of the intrinsic square function.

1. Introduction

Let $\psi$ be an integrable function, $\int_{\mathbb{R}^n} \psi = 0$, and, for some $\varepsilon > 0$,

$$|\psi(x)| \leq \frac{c}{(1 + |x|)^{n+\varepsilon}} \quad \text{and} \quad \int_{\mathbb{R}^n} |\psi(x + h) - \psi(x)| dx \leq c|h|^\varepsilon. \quad (1.1)$$

Let $\mathbb{R}_+^{n+1} = \mathbb{R}^n \times \mathbb{R}_+$ and $\Gamma_\alpha(x) = \{(y, t) \in \mathbb{R}_+^{n+1} : |y - x| < \alpha t\}$. Set $\psi_t(x) = t^{-n}\psi(x/t)$. Define the square function $S_{\alpha,\psi}(f)$ by

$$S_{\alpha,\psi}(f)(x) = \left( \int_{\Gamma_\alpha(x)} |f \ast \psi_t(y)|^2 \frac{dydt}{t^{n+1}} \right)^{1/2} \quad (\alpha > 0).$$

We drop the subscript $\alpha$ if $\alpha = 1$.

Given a weight $w$, define its $A_p$ characteristic by

$$[w]_{A_p} = \sup_Q \left( \frac{1}{|Q|} \int_Q w dx \right) \left( \frac{1}{|Q|} \int_Q w^{-\frac{1}{p-1}} dx \right)^{p-1},$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$.

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It was proved in [13] that for any $1 < p < \infty$,
\[
\|S_{\psi}\|_{L^p(w)} \leq c_{p,n,\psi}[w]_{A_p}^{\max(\frac{1}{2}, \frac{1}{p-1})},
\]
and this estimate is sharp in terms of $[w]_{A_p}$ (we also refer to [13] for a
detailed history of closely related results).

Similarly one can show that
\[
\|S_{\alpha,\psi}\|_{L^p(w)} \leq c_{p,n,\psi}\gamma(\alpha)[w]_{A_p}^{\max(\frac{1}{2}, \frac{1}{p-1})} \quad (\alpha \geq 1, 1 < p < \infty);
\]
however, the sharp dependence on $\alpha$ in this estimate cannot be deter-
mined by means of the approach from [13]. The aim of this paper is to
find the sharp $\gamma(\alpha)$ in (1.3).

Let us explain first why the method from [13] gives a rough estimate
for $\gamma(\alpha)$. The proof in [13] was based on the intrinsic square function
$G_{\alpha,\beta}(f)$ by M. Wilson [19] defined as follows. For $0 < \beta \leq 1$, let $C_{\beta}$ be
the family of functions supported in the unit ball with mean zero and
such that for all $x$ and $x'$, $|\varphi(x) - \varphi(x')| \leq |x - x'|^\beta$. If $f \in L^1_{loc}(\mathbb{R}^n)$ and $(y, t) \in \mathbb{R}^{n+1}$, we define $A_{\beta}(f)(y, t) = \sup_{\varphi \in C_{\beta}} |f*\varphi_t(y)|$ and
\[
G_{\alpha,\beta}(f)(x) = \left( \int_{\Gamma_{\alpha}(x)} (A_{\beta}(f)(y, t))^2 dydt \right)^{1/2}.
\]
Set $G_{1,\beta}(f) = G_{\beta}(f)$.

The intrinsic square function has several interesting features (established in [19]). First, though $G_{\beta}(f)$ is defined by means of kernels with uniform compact support, it pointwise dominates $S_{\psi}(f)$. Also there is
a pointwise relation between $G_{\alpha,\beta}(f)$ with different apertures:
\[
G_{\alpha,\beta}(f)(x) \leq \alpha^{(3/2)n+\beta} G_{\beta}(f)(x) \quad (\alpha \geq 1).
\]
Notice that for the usual square functions $S_{\alpha,\psi}(f)$ such a pointwise
relation is not available.

In [13], (1.2) with $G_{\beta}(f)$ instead of $S_{\psi}(f)$ was obtained. Combining
this with (1.4), we would obtain that one can take $\gamma(\alpha) = \alpha^{(3/2)n+\beta}$
in (1.3) assuming that $\psi$ is compactly supported. For non-compactly
supported $\psi$ some additional ideas from [19] can be used that lead to
even worst estimate on $\gamma(\alpha)$. Observe also that it is not clear to us
whether (1.3) can be improved.

It is easy to see that the dependence $\gamma(\alpha) = \alpha^{(3/2)n+\beta}$ in (1.3) is far
from the sharp one. For instance, it is obvious that the information on
$\beta$ should not appear in (1.3). All this indicates that the intrinsic square
function approach is not suitable for our purposes in determining the
sharp $\gamma(\alpha)$. 

Suppose we seek for \( \gamma(\alpha) \) in the form \( \gamma(\alpha) = \alpha^r \). Then a simple observation shows that \( r \geq n \) for any \( 1 < p < \infty \). Indeed, consider the Littlewood-Paley function \( g^*_{\mu,\psi}(f) \) defined by
\[
g^*_{\mu,\psi}(f)(x) = \left( \int \int_{\mathbb{R}^{n+1}} \frac{t}{t + |x - y|} \left| f \ast \psi_t(y) \right|^2 \frac{dydt}{t^{n+1}} \right)^{1/2}.
\]
Using the standard estimate
\[
g^*_{\mu,\psi}(f)(x) \leq S_{\psi}(f)(x) + \sum_{k=0}^{\infty} 2^{-k\mu n/2} S_{2^{k+1},\psi}(f)(x),
\]
we obtain that \( 1.3 \) for some \( p = p_0 \) and \( \gamma(\alpha) = \alpha^{r_0} \) implies
\[
\|g^*_{\mu,\psi}\|_{L^{p_0}(w)} \lesssim \left( \sum_{k=0}^{\infty} 2^{-k\mu n/2} r_0^{2k} \right)^{\frac{1}{2}} w \max \left( \frac{1}{p_0}, \frac{1}{p_0 - 1} \right).
\]
This means that if \( \mu > 2r_0/n \), then \( g^*_{\mu,\psi} \) is bounded on \( L^{p_0}(w), w \in A_{p_0} \). From this, by the Rubio de Francia extrapolation theorem, \( g^*_{\mu,\psi} \) is bounded on the unweighted \( L^p \) for any \( p > 1 \), whenever \( \mu > 2r_0/n \). But it is well known that \( g^*_{\mu,\psi} \) is not bounded on \( L^p \) if \( 1 < \mu < 2 \) and \( 1 < p \leq 2/\mu \). Hence, if \( r_0 < n \), we would obtain a contradiction to the latter fact for \( p \) sufficiently close to 1.

Our main result shows that for any \( 1 < p < \infty \) one can take the optimal power growth \( \gamma(\alpha) = \alpha^n \).

**Theorem 1.1.** For any \( 1 < p < \infty \) and for all \( 1 \leq \alpha < \infty \),
\[
\|S_{\alpha,\psi}\|_{L^p(w)} \leq c_{p,n,\psi} \alpha^n \max \left( \frac{1}{2}, \frac{1}{p-1} \right).
\]

By \( 1.5 \), we immediately obtain the following.

**Corollary 1.2.** Let \( \mu > 2 \). Then for any \( 1 < p < \infty \),
\[
\|g^*_{\mu,\psi}(f)\|_{L^p(w)} \leq c_{p,n,\psi} \alpha^n \max \left( \frac{1}{2}, \frac{1}{p-1} \right).
\]

Observe that if \( \mu = 2 \), then \( g^*_{2,\psi} \) is also bounded on \( L^p(w) \) for \( w \in A_p \) (see [17]). However, the sharp dependence on \( \alpha \) in the corresponding \( L^p(w) \) inequality is unknown to us.

We emphasize that the growth \( \gamma(\alpha) = \alpha^n \) is best possible in the weighted \( L^p(w) \) estimate for \( w \in A_p \). In the unweighted case a better dependence on \( \alpha \) is known, namely, \( \|S_{\alpha,\psi}\|_{L^p} \leq c_{p,n,\psi} \alpha^\frac{n}{n-1} \), see [11, 18].

Some words about the proof of Theorem 1.1. As in [13], we use here the local mean oscillation decomposition. But in [13] we worked with the intrinsic square function, and due to the fact that this operator
is defined by uniform compactly supported kernels, we arrived to the operator
\[ A(f)(x) = \left( \sum_{j,k} (f_{\gamma Q_j^k})^2 \chi_{Q_j^k}(x) \right)^{1/2}, \]
where \( Q_j^k \) is a sparse family and \( \gamma > 1 \). This operator can be handled sufficiently easy.

Here we work with the square function \( S_{\alpha,\psi}(f) \) directly, more precisely we consider its smooth variant \( \tilde{S}_{\alpha,\psi}(f) \). Applying the local mean oscillation decomposition to \( \tilde{S}_{\alpha,\psi}(f) \), we obtain that \( S_{\alpha,\psi}(f) \) is essentially pointwise bounded by \( \alpha^n B(f) \), where
\[ B(f)(x) = \sum_{m=0}^{\infty} \frac{1}{2^n \delta} \left( \sum_{j,k} (f_{2^n Q_j^k})^2 \chi_{Q_j^k}(x) \right)^{1/2} \quad (\delta > 0). \]

Observe that this pointwise aperture estimate is interesting in its own right. In order to handle \( B \), we use a mixture of ideas from recent papers on a simple proof of the \( A_2 \) conjecture [14] and sharp weighted estimates for multilinear Calderón-Zygmund operators [5]. In particular, similarly to [14], we obtain the \( X^{(2)} \)-norm boundedness of \( B \) by \( A \) on an arbitrary Banach function space \( X \).

The paper is organized as follows. Next section contains some preliminary information. In Section 3, we obtain the main estimate, namely, the local mean oscillation estimate of \( \tilde{S}_{\alpha,\psi}(f) \). The proof of Theorem [14] is contained in Section 4. Section 5 contains some concluding remarks concerning the sharp aperture-weighted weak type estimates for \( S_{\alpha,\psi}(f) \).

2. Preliminaries

2.1. A weak type \((1, 1)\) estimate for square functions. It is well known that the operator \( S_{\alpha,\psi} \) is of weak type \((1, 1)\). However, we could not find in the literature the sharp dependence on \( \alpha \) in the corresponding inequality. Hence we give below an argument based on general square functions.

For a measurable function \( F \) on \( \mathbb{R}^{n+1}_+ \) define
\[ S_\alpha(F)(x) = \left( \int_{\Gamma_\alpha(x)} |F(y,t)|^2 \frac{dydt}{t^{n+1}} \right)^{1/2}. \]

**Lemma 2.1.** For any \( \alpha \geq 1 \),
\[ \|S_\alpha(F)\|_{L^{1,\infty}} \leq c_n \alpha^n \|S_1(F)\|_{L^{1,\infty}}. \]
Proof. We will use the following estimate which can be found in [18, p. 315]: if \( \Omega \subset \mathbb{R}^n \) is an open set and \( U = \{ x \in \mathbb{R}^n : M_\chi_\Omega(x) > 1/2\alpha^n \} \), where \( M \) is the Hardy-Littlewood maximal operator, then
\[
\int_{\mathbb{R}^n \setminus U} S_\alpha(F)(x)^2 \, dx \leq 2\alpha^n \int_{\mathbb{R}^n \setminus \Omega} S_1(F)(x)^2 \, dx
\]
(observe that the definitions of \( S_\alpha(F) \) here and in [18] are differ by the factor \( \alpha^{n/2} \).)

Let \( \Omega_\xi = \{ x : S_1(F)(x) > \xi \} \). Using the weak type \((1,1)\) of \( M \), Chebyshev’s inequality and the above estimate, we obtain
\[
|\{ x \in \mathbb{R}^n : S_\alpha(F)(x) > \xi \}| \leq |U_\xi| + |\{ x \in \mathbb{R}^n \setminus U_\xi : S_\alpha(F)(x) > \xi \}|
\leq c_n \alpha^n |\{ x : S_1(F)(x) > \xi \}| + \frac{1}{\xi^2} \int_{\mathbb{R}^n \setminus U_\xi} S_\alpha(F)(x)^2 \, dx
\leq c_n \alpha^n |\{ x : S_1(F)(x) > \xi \}| + \frac{2\alpha^n}{\xi^2} \int_{\mathbb{R}^n \setminus \Omega_\xi} S_1(F)(x)^2 \, dx.
\]
Further,
\[
\int_{\mathbb{R}^n \setminus \Omega_\xi} S_1(F)(x)^2 \, dx \leq 2 \int_{0}^{\xi} \lambda |\{ x : S_1(F)(x) > \lambda \}| \, d\lambda \leq 2\xi \| S_1(F) \|_{L^{1,\infty}}.
\]
Combining this with the previous estimate gives
\[
|\{ x : S_\alpha(F)(x) > \xi \}| \leq c_n \alpha^n |\{ x : S_1(F)(x) > \xi \}| + \frac{4\alpha^n}{\xi} \| S_1(F) \|_{L^{1,\infty}},
\]
which proves (2.1). \( \square \)

Note that the sharp strong \( L^p \) estimates related square functions of different apertures were obtained recently in [1].

By Lemma 2.1 and by the weak type \((1,1)\) of \( S_\psi(f) \) [9],
\[
\| S_{\alpha,\psi}(f) \|_{L^{1,\infty}} \leq c_{n,\psi} \alpha^n \| f \|_{L^1}.
\]

2.2. Dyadic grids and sparse families. Recall that the standard dyadic grid in \( \mathbb{R}^n \) consists of the cubes
\[
2^{-k}([0,1]^n + j), \quad k \in \mathbb{Z}, j \in \mathbb{Z}^n.
\]
Denote the standard grid by \( \mathcal{D} \).

By a general dyadic grid \( \mathcal{D} \) we mean a collection of cubes with the following properties: (i) for any \( Q \in \mathcal{D} \) its sidelength \( \ell_Q \) is of the form \( 2^k, k \in \mathbb{Z} \); (ii) \( Q \cap R \in \{ Q, R, \emptyset \} \) for any \( Q, R \in \mathcal{D} \); (iii) the cubes of a fixed sidelength \( 2^k \) form a partition of \( \mathbb{R}^n \).

Given a cube \( Q_0 \), denote by \( \mathcal{D}(Q_0) \) the set of all dyadic cubes with respect to \( Q_0 \), that is, the cubes from \( \mathcal{D}(Q_0) \) are formed by repeated
subdivision of $Q_0$ and each of its descendants into $2^n$ congruent sub-cubes. Observe that if $Q_0 \in \mathcal{D}$, then each cube from $\mathcal{D}(Q_0)$ will also belong to $\mathcal{D}$.

We will use the following proposition from [10].

**Proposition 2.2.** There are $2^n$ dyadic grids $\mathcal{D}_i$ such that for any cube $Q \subset \mathbb{R}^n$ there exists a cube $Q_i \in \mathcal{D}_i$ such that $Q \subset Q_i$ and $\ell Q_i \leq 6\ell Q$.

We say that $\{Q^k_j\}$ is a sparse family of cubes if: (i) the cubes $Q^k_j$ are disjoint in $j$, with $k$ fixed; (ii) if $\Omega_k = \bigcup_j Q^k_j$, then $\Omega_{k+1} \subset \Omega_k$; (iii) $|\bigcup_{k+1} \cap Q^k_j| \leq \frac{1}{2}|Q^k_j|$.

2.3. A “local mean oscillation decomposition”. The non-increasing rearrangement of a measurable function $f$ on $\mathbb{R}^n$ is defined by

$$f^*(t) = \inf \{\alpha > 0 : |\{x \in \mathbb{R}^n : |f(x)| < \alpha\}| < t\} \quad (0 < t < \infty).$$

Given a measurable function $f$ on $\mathbb{R}^n$ and a cube $Q$, the local mean oscillation of $f$ on $Q$ is defined by

$$\omega_\lambda(f; Q) = \inf_{c \in \mathbb{R}} (f - c)^*(\lambda|Q|) \quad (0 < \lambda < 1).$$

By a median value of $f$ over $Q$ we mean a possibly nonunique, real number $m_f(Q)$ such that

$$\max(|\{x \in Q : f(x) > m_f(Q)\}|, |\{x \in Q : f(x) < m_f(Q)\}|) \leq |Q|/2.$$

It is easy to see that the set of all median values of $f$ is either one point or the closed interval. In the latter case we will assume for the definiteness that $m_f(Q)$ is the maximal median value. Observe that it follows from the definitions that

$$(2.3) \quad |m_f(Q)| \leq (f^*|Q|)^*|\{Q/2\}|.$$

Given a cube $Q_0$, the dyadic local sharp maximal function $m_{\lambda,Q_0}^d f$ is defined by

$$m_{\lambda,Q_0}^d f(x) = \sup_{x \in Q_0} \omega_\lambda(f; Q').$$

The following theorem was proved in [13] (its very similar version can be found in [12]).

**Theorem 2.3.** Let $f$ be a measurable function on $\mathbb{R}^n$ and let $Q_0$ be a fixed cube. Then there exists a (possibly empty) sparse family of cubes $Q^k_j \in \mathcal{D}(Q_0)$ such that for a.e. $x \in Q_0$,

$$|f(x) - m_f(Q_0)| \leq 4m_{\frac{1}{2n+2}Q_0}^d f(x) + 2\sum_{k,j} \omega_{\frac{1}{2n+2}}(f; Q^k_j)\chi_{Q^k_j}(x).$$
3. A KEY ESTIMATE

In this section we will obtain the main local mean oscillation estimate of $S_{\alpha,\psi}$. We consider a smooth version of $S_{\alpha,\psi}$ defined as follows. Let $\Phi$ be a Schwartz function such that

$$\chi_{B(0,1)}(x) \leq \Phi(x) \leq \chi_{B(0,2)}(x).$$

Define

$$\tilde{S}_{\alpha,\psi}(f)(x) = \left( \int \int_{\mathbb{R}_{+}^{n+1}} \Phi \left( \frac{x - y}{t\alpha} \right) |f \ast \psi_t(y)|^2 \frac{dydt}{t^{n+1}} \right)^{1/2} (\alpha > 0).$$

It is easy to see that

$$S_{\alpha,\psi}(f)(x) \leq \tilde{S}_{\alpha,\psi}(f)(x) \leq S_{2\alpha,\psi}(f)(x).$$

Hence, by (2.2),

$$\| \tilde{S}_{\alpha,\psi}(f) \|_{L^{1,\infty}} \leq c_{n,\alpha} \| f \|_{L^1}.$$  

Lemma 3.1. For any cube $Q \subset \mathbb{R}^n$,

$$\omega_\lambda(\tilde{S}_{\alpha,\psi}(f)^2; Q) \leq c_{n,\lambda,\alpha} 2^n \sum_{k=0}^{\infty} \frac{1}{2^k \delta} \left( \frac{1}{|2^k Q|} \int_{2^k Q} |f| \right)^2,$$

where $\delta = \varepsilon$ from condition (1.1) if $\varepsilon < 1$, and $\delta < 1$ if $\varepsilon = 1$.

Proof. Given a cube $Q$, let $T(Q) = \{(y, t) : y \in Q, 0 < t < \ell_Q\}$, where $\ell_Q$ denotes the side length of $Q$. For $x \in Q$ we decompose $\tilde{S}_{\alpha,\psi}(f)(x)^2$ into the sum of

$$I_1(f)(x) = \int \int_{T(2^2 Q)} \Phi \left( \frac{x - y}{t\alpha} \right) |f \ast \psi_t(y)|^2 \frac{dydt}{t^{n+1}}$$

and

$$I_2(f)(x) = \int \int_{\mathbb{R}_{+}^{n+1} \setminus T(2^2 Q)} \Phi \left( \frac{x - y}{t\alpha} \right) |f \ast \psi_t(y)|^2 \frac{dydt}{t^{n+1}}.$$

Let us show first that

$$(I_1(f)\chi_Q)^*(\lambda|Q|) \leq c_{n,\lambda,\alpha} 2^n \sum_{k=0}^{\infty} \frac{1}{2^k \delta} \left( \frac{1}{|2^k Q|} \int_{2^k Q} |f| \right)^2.$$  

Using that $(a + b)^2 \leq 2(a^2 + b^2)$, we get

$$I_1(f)(x) \leq 2\left( I_1(f\chi_{4Q})(x) + I_1(f\chi_{\mathbb{R}^n \setminus 4Q})(x) \right).$$

Hence,

$$(I_1(f)\chi_Q)^*(\lambda|Q|) \leq 2((I_1(f\chi_{4Q}))^*(\lambda|Q|)/2) + (I_1(f\chi_{\mathbb{R}^n \setminus 4Q})\chi_Q)^*(\lambda|Q|/2)).$$
By (3.1),

\[
(I_1(f\chi_{4Q}))^*(\lambda|Q|/2) \leq (\tilde{S}_{\alpha,\psi}(f\chi_{4Q}))^*(\lambda|Q|/2)^2 \\
\leq c_{n,\lambda,\psi}2^n \left( \frac{1}{|4Q|} \int_{4Q} |f| \right)^2.
\]

Further, by (1.1), for \((y,t) \in T(2Q)\),

\[
|(f\chi_{\mathbb{R}^n\setminus 4Q})*\psi_t(y)| \leq c_{\psi}t^n \int_{\mathbb{R}^n\setminus 4Q} |f(\xi)| \frac{1}{(t + |y - \xi|)^{n+\varepsilon}} d\xi \\
\leq c_{n,\psi}(t/\ell_Q)^n \sum_{k=0}^\infty \frac{1}{2^{k\varepsilon}} \frac{1}{|2^k Q|} \int_{2^k Q} |f|.
\]

Hence, using Chebyshev’s inequality and that \(\int_{\mathbb{R}^n} \Phi(\frac{x-y}{t\alpha}) \, dx \leq c_n(t\alpha)^n\), we have

\[
(I_1(f\chi_{\mathbb{R}^n\setminus 4Q})\chi_Q)^*(\lambda|Q|/2) \\
\leq \frac{2}{\lambda|Q|} \int_{T(2Q)} \left( \int_{\mathbb{R}^n} \Phi(\frac{x-y}{t\alpha}) \, dx \right) |(f\chi_{\mathbb{R}^n\setminus 4Q})*\psi_t(y)|^2 \frac{dydt}{t^{n+1}} \\
\leq c_{n,\lambda,\psi}2^n (1/\ell_Q)^{2\varepsilon} \left( \sum_{k=0}^\infty \frac{1}{2^{k\varepsilon}} \frac{1}{|2^k Q|} \int_{2^k Q} |f| \right)^2 t^{2\varepsilon-1} dt \\
\leq c_{n,\lambda,\psi}2^n \left( \sum_{k=0}^\infty \frac{1}{2^{k\varepsilon}} \frac{1}{|2^k Q|} \int_{2^k Q} |f| \right)^2.
\]

By Hölder’s inequality,

\[
\left( \sum_{k=0}^\infty \frac{1}{2^{k\varepsilon}} \frac{1}{|2^k Q|} \int_{2^k Q} |f| \right)^2 \leq \left( \sum_{k=0}^\infty \frac{1}{2^{k\varepsilon}} \right)^2 \sum_{k=0}^\infty \frac{1}{2^{k\varepsilon}} \left( \frac{1}{|2^k Q|} \int_{2^k Q} |f| \right)^2.
\]

Combining this with the previous estimate and with (3.5) and (3.4) proves (3.3).

Let \(x, x_0 \in Q\), and let us estimate now \(|I_2(f)(x) - I_2(f)(x_0)|\). We have

\[
|I_2(f)(x) - I_2(f)(x_0)| \\
\leq \sum_{k=1}^\infty \int_{T(2^{k+1}Q) \setminus T(2^k Q)} |\Phi(\frac{x-y}{t\alpha}) - \Phi(\frac{x_0-y}{t\alpha})| |f*\psi_t(y)|^2 \frac{dydt}{t^{n+1}}.
\]

Suppose \((y, t) \in T(2^{k+1}Q) \setminus T(2^k Q)\). If \(y \in 2^k Q\), then \(t \geq 2^k \ell_Q\). On the other hand, if \(y \in 2^{k+1}Q \setminus 2^k Q\), then for any \(x \in Q\), \(|y - x| \geq
Since \( \int f \, dy = \frac{\ell_Q}{\alpha} \int_{2^k-2\ell_Q/\alpha}^{2^k+1\ell_Q} f(x) \, dx \), and \\
\begin{align*}
(2^k - 1/2)\ell_Q \quad \text{Hence, if} \quad t < \frac{1}{2a}(2^k - 1/2)\ell_Q, \text{then} \quad |y - x|/\alpha t > 2 \quad \text{and} \quad |y - x_0|/\alpha t > 2, \quad \text{and therefore,} \\
\Phi \left( \frac{x - y}{t\alpha} \right) - \Phi \left( \frac{x_0 - y}{t\alpha} \right) = 0.
\end{align*}

Using also that \\
\begin{align*}
\left| \Phi \left( \frac{x - y}{t\alpha} \right) - \Phi \left( \frac{x_0 - y}{t\alpha} \right) \right| &\leq \frac{\sqrt{n\ell_Q}}{\alpha t} \|\nabla \Phi\|_{L^\infty},
\end{align*}

we get \\
\begin{align*}
\left| \Phi \left( \frac{x - y}{t\alpha} \right) - \Phi \left( \frac{x_0 - y}{t\alpha} \right) \right| \chi_{\{T(2^{k+1}Q) \setminus T(2^kQ)\}}(y,t) \\
&\leq c_n \frac{\ell_Q}{\alpha t} \chi_{\{(y,t) : y \in 2^{k+1}Q, 2^k-2\ell_Q/\alpha \leq t \leq 2^{k+1}\ell_Q\}}(y,t).
\end{align*}

Hence, \\
\begin{align*}
\int_{T(2^{k+1}Q) \setminus T(2^kQ)} \left| \Phi \left( \frac{x - y}{t\alpha} \right) - \Phi \left( \frac{x_0 - y}{t\alpha} \right) \right| f \ast \psi_t(y) \, \frac{dydt}{t^{n+2}} \\
&\leq c_n \frac{\ell_Q}{\alpha t} \int_{2^k-2\ell_Q/\alpha}^{2^{k+1}\ell_Q} \int_{2^{k+1}Q} |f \ast \psi_t(y)|^2 \, \frac{dydt}{t^{n+2}} \leq c_n (J_1 + J_2),
\end{align*}

where \\
\begin{align*}
J_1 &= \frac{\ell_Q}{\alpha} \int_{2^k-2\ell_Q/\alpha}^{2^{k+1}\ell_Q} \int_{2^{k+1}Q} |(f \chi_{2^{k+2}Q}) \ast \psi_t(y)|^2 \, \frac{dydt}{t^{n+2}} \\
J_2 &= \frac{\ell_Q}{\alpha} \int_{2^{k-2}\ell_Q/\alpha}^{2^{k+1}\ell_Q} \int_{2^{k+1}Q} |(f \chi_{R^n \setminus 2^{k+2}Q}) \ast \psi_t(y)|^2 \, \frac{dydt}{t^{n+2}}.
\end{align*}

Let us first estimate \( J_1 \). Using Minkowski’s integral inequality, we obtain \\
\begin{align*}
J_1 &\leq \frac{\ell_Q}{\alpha} \left( \int_{2^{k+2}Q} |f(y)| \left( \int_{2^{k+2}Q} \int_{2^{k+1}Q} \psi_t(y - \xi) \, \frac{dydt}{t^{n+2}} \right)^{1/2} \, d\xi \right)^2 \\
&\leq c_n,psi 2n^{-k} \left( \int_{2^{k+2}Q} |f(y)| \, dy \right)^2 \int_{2^{k-2}\ell_Q/\alpha}^{2^{k+1}\ell_Q} \frac{dt}{t^{2n+2}}.
\end{align*}

Since \\
\begin{align*}
\int_{2^{k+1}Q} \psi_t(y - \xi)^2 \, dy &\leq \frac{\|\psi\|_{L^\infty}}{t^n} \|\psi_t\|_{L^1} = \frac{\|\psi\|_{L^\infty}^2 \|\psi\|_{L^1}}{t^n},
\end{align*}

we get \\
\begin{align*}
J_1 &\leq c_n,psi 2n^{-k} \left( \int_{2^{k+2}Q} |f(y)| \, dy \right)^2 \int_{2^{k-2}\ell_Q/\alpha}^{2^{k+1}\ell_Q} \frac{dt}{t^{2n+2}} \\
&\leq c_n,psi 2n^{-k} \left( \frac{1}{|2^{k+2}Q|} \int_{2^{k+2}Q} |f(y)| \, dy \right)^2.
\end{align*}
We turn to the estimate of $J_2$. By (1.1), for $(y,t) \in T(2^{k+1}Q)$,
\[
| (f \chi_{R^n \setminus 2^{k+2}Q} \ast \psi_t)(y) | \leq c_\psi t^\varepsilon \int_{R^n \setminus 2^{k+2}Q} |f(\xi)| \frac{1}{(t + |y - \xi|)^{n+\varepsilon}} d\xi \\
\leq c_{n,\psi}(t/\ell_Q)^\varepsilon \sum_{i=k}^{\infty} \frac{1}{2^{i\varepsilon}} \frac{1}{|2^iQ|} \int_{2^iQ} |f|.
\]
Therefore,
\[
J_2 \leq c_{n,\psi} \frac{\ell_Q}{\alpha} \left( \sum_{i=k}^{\infty} \frac{1}{2^{i\varepsilon}} \frac{1}{|2^iQ|} \int_{2^iQ} |f| \right)^2 \frac{1}{\ell_{2^{k-2}\ell_Q/\alpha}} \int_{2^{k+1}Q} dy dt \\
\leq c_{n,\psi} \alpha^{n-2 \varepsilon \cdot 2^{(2\varepsilon-1)k}} \left( \sum_{i=k}^{\infty} \frac{1}{2^{i\varepsilon}} \frac{1}{|2^iQ|} \int_{2^iQ} |f| \right)^2.
\]
Combining the estimates for $J_1$ and $J_2$, we obtain
\[
|I_2(f)(x) - I_2(f)(x_0)| \leq c_{n,\psi} \alpha^{2n} \sum_{k=1}^{\infty} \frac{1}{2^k} \left( \frac{1}{|2^kQ|} \int_{2^kQ} |f(\xi)| d\xi \right)^2 \\
+ c_{n,\psi} \alpha^{n-2 \varepsilon \cdot 2^{(2\varepsilon-1)k}} \left( \sum_{i=k}^{\infty} \frac{1}{2^{i\varepsilon}} \frac{1}{|2^iQ|} \int_{2^iQ} |f| \right)^2.
\]
By Hölder’s inequality,
\[
\sum_{k=1}^{\infty} \frac{2^{2\varepsilon k}}{2^k} \left( \sum_{i=k}^{\infty} \frac{1}{2^{i\varepsilon}} \frac{1}{|2^iQ|} \int_{2^iQ} |f| \right)^2 \\
\leq c_\varepsilon \sum_{k=1}^{\infty} \frac{2^{2\varepsilon k}}{2^k} \sum_{i=k}^{\infty} \frac{1}{2^{i\varepsilon}} \left( \frac{1}{|2^iQ|} \int_{2^iQ} |f| \right)^2 \\
\leq c_\varepsilon \sum_{k=1}^{\infty} \gamma(k, \varepsilon) \left( \frac{1}{|2^kQ|} \int_{2^kQ} |f| \right)^2,
\]
where
\[
\gamma(k, \varepsilon) = \begin{cases} 
\frac{2^{2\varepsilon k}}{2^k}, & \varepsilon < 1 \\
\frac{k}{2^k}, & \varepsilon = 1.
\end{cases}
\]
Therefore,
\[
|I_2(f)(x) - I_2(f)(x_0)| \leq c_{n,\psi} \alpha^{2n} \sum_{k=1}^{\infty} \gamma(k, \varepsilon) \left( \frac{1}{|2^kQ|} \int_{2^kQ} |f| \right)^2.
\]
From this and from \( (3.3) \),
\[
\omega_\lambda(\tilde{S}_\alpha, \psi(f^2) \| Q \|) \leq c_{n, \lambda, \psi} 2^n \sum_{k=0}^\infty \gamma(k, \varepsilon) \left( \frac{1}{\| 2^k Q \|} \int_{2^k Q} |f| \right)^2 ,
\]
which completes the proof. \( \square \)

4. **Proof of Theorem 1.1**

4.1. **Several auxiliary operators.** Given a sparse family \( S = \{ Q_j \} \in D \), define
\[
T_{2,m}^S f(x) = \left( \sum_{j,k} (f_{2^m Q_j})^2 \chi_{Q_j}(x) \right)^{1/2}.
\]

In the case when \( m = 0 \), the following result was proved in [4].

**Lemma 4.1.** For any \( 1 < p < \infty \),
\[
\| T_{2,0}^S \|_{L^p(w)} \leq c_{n,p} [w]_{A_p}^{\max\left( \frac{1}{2}, \frac{1}{p-1} \right)}.
\]

Given a sparse family \( S = \{ Q_j^k \} \in D \), define
\[
M_{i,m}^S(f, g)(x) = \sum_{j,k} (f_{P_{i,m}^m Q_j}) \left( \frac{1}{\| P_{i,m}^m Q_j \|} \int_{P_{i,m}^m Q_j} g \right) \chi_{P_{i,m}^m Q_j}(x).
\]

Applying Proposition 2.2, we decompose the cubes \( Q_j^k \) into \( 2^n \) disjoint families \( F_i \) such that for any \( Q_j^k \in F_i \) there exists a cube \( P_{i,m}^m \in D_i \) such that \( 2^m Q_j^k \subset P_{i,m}^m \) and \( \ell_{i,m}^m \leq 6 \ell_{2m} Q_j^k \). Hence,
\[
M_{i,m}^S(f, g)(x) \leq 6^{2n} \sum_{i=1}^{2^n} M_{i,m}^S(f, g)(x),
\]

where
\[
M_{i,m}^S(f, g)(x) = \sum_{j,k} (f_{P_{i,m}^m Q_j}) \left( \frac{1}{\| P_{i,m}^m Q_j \|} \int_{P_{i,m}^m Q_j} g \right) \chi_{P_{i,m}^m Q_j}(x).
\]

The following statement was obtained in [5].

**Lemma 4.2.** Suppose that the sum defining \( M_{i,m}^S(f, g) \) is finite. Then there are at most \( 2^n \) cubes \( Q_\nu \in D_i \) covering the support of \( M_{i,m}^S(f, g) \)
so that for any \( Q_\nu \) there are two sparse families \( S_{i,1} \) and \( S_{i,2} \) from \( \mathcal{D}_i \) such that for a.e. \( x \in Q_\nu \),

\[
\mathcal{M}_{i,m}^S(f, g)(x) \leq c_n m \sum_{\kappa=1}^{2} \sum_{Q_j^\kappa \in S_{i,\kappa}} f_{Q_j^\kappa} g_{Q_j^\kappa} \chi_{Q_j^\kappa}(x).
\]

Observe that the proof of Lemma 4.2 is based on Theorem 2.3 along with [14, Lemma 3.2]. Formally Lemma 4.2 follows from [5, Lemma 4.2] taking there \( m = 2 \) (which corresponds to a bilinear case) and \( l = m \), and from the subsequent argument in [5, Section 4.2].

Let \( X \) be a Banach function space, and let \( X' \) denote the associate space (see [2, Ch. 1]). Given a Banach function space \( X \), denote by \( X(2) \) the space endowed with the norm

\[
\|f\|_{X(2)} = \||f|^2\|_X^{1/2}.
\]

It is well known [16, Ch. 1] that \( X(2) \) is also a Banach space.

**Lemma 4.3.** For any Banach function space \( X \),

\[
\sup_{S \in \mathcal{D}} \|T_{2,m}Sf\|_{X(2)} \leq c_n m^{1/2} \max_{1 \leq i \leq 2} \sup_{S \in \mathcal{D}_i} \|T_{2,0}Sf\|_{X(2)}.
\]

**Proof.** By the standard argument, one can assume that the sum defining \( T_{2,m}Sf \) is finite. Fix \( S \in \mathcal{D} \). By duality, there exists \( g \geq 0 \) with \( \|g\|_{X'} = 1 \) such that

\[
(4.2) \quad \|T_{2,m}Sf\|_{X(2)}^2 = \int_{\mathbb{R}^n} (T_{2,m}Sf)^2 g \, dx = \sum_{j,k} (f_{2^mQ_j^k})^2 \int_{Q_j^k} g \quad = \quad \int_{\mathbb{R}^n} \mathcal{M}_{m}^S(f, g) f \, dx.
\]

Observe that the sum defining \( \mathcal{M}_{m}^S(f, g) \) is finite. By Lemma 4.2 and by Hölder’s inequality,

\[
\int_{Q_\nu} \mathcal{M}_{i,m}^S(f, g) f \, dx \leq c_n m \sum_{\kappa=1}^{2} \sum_{Q_j^\kappa \in S_{i,\kappa}} (f_{Q_j^\kappa})^2 \int_{Q_j^\kappa} g \\
\quad \leq c_n m \sum_{\kappa=1}^{2} \int_{\mathbb{R}^n} (T_{2,0}S_{i,\kappa}f)^2 g \, dx \\
\quad \leq 2c_n m \sup_{S \in \mathcal{D}_i} \|T_{2,0}Sf\|_{X(2)}^2.
\]

Summing up over \( Q_\nu \) and using (4.1), we obtain

\[
\int_{\mathbb{R}^n} \mathcal{M}_{m}^S(f, g) f \, dx \leq c_n m \max_{1 \leq i \leq 2} \sup_{S \in \mathcal{D}_i} \|T_{2,0}Sf\|_{X(2)}^2.
\]
This along with (4.2) completes the proof. □

4.2. Proof of Theorem 1.1. Let $Q \in \mathcal{D}$. Applying Theorem 2.3 along with Lemma 3.1, we get that there exists a sparse family $S = \{Q_k^j\} \in \mathcal{D}(Q)$ such that for a.e. $x \in Q$,

$$|\tilde{S}_{\alpha,\psi}(f)(x)^2 - m_Q(\tilde{S}_{\alpha,\psi}(f))^2| \leq c_{n,\psi} \alpha^{2n} \left( Mf(x)^2 + \sum_{m=0}^{\infty} \frac{1}{2m+\delta} (T_{2,m}^S f(x))^2 \right).$$

Hence,

$$|\tilde{S}_{\alpha,\psi}(f)^2 - m_Q(\tilde{S}_{\alpha,\psi}(f))^2|^{1/2} \leq c_{n,\psi} \alpha^n (Mf(x) + T(f)(x)),$$

where

$$T(f)(x) = \sum_{m=0}^{\infty} \frac{1}{2m+\delta} T_{2,m}^S f(x).$$

Assuming, for instance, that $f \in L^1$, and using (2.3) and (3.1), we get

$$\lim_{|Q| \to \infty} m_Q(\tilde{S}_{\alpha,\psi}(f))^2 = 0.$$

Therefore, letting $Q$ to anyone of $2^n$ quadrants and using Fatou’s lemma, by (4.3) we obtain

$$(4.4) \quad \|\tilde{S}_{\alpha,\psi}(f)\|_{L^p(w)} \leq c_{n,\psi} \alpha^n (\|Mf\|_{L^p(w)} + \|T(f)\|_{L^p(w)}).$$

Combining Lemma 4.1 and Lemma 4.3 with $X = L^{3/2}(w)$ yields

$$\|T(f)\|_{L^3(w)} \leq \sum_{m=0}^{\infty} \frac{1}{2m+\delta/2} \|T_{2,m}^S f\|_{L^3(w)} \leq c_n \sum_{m=0}^{\infty} \frac{m^{1/2}}{2m+\delta/2} \max_{1 \leq i \leq 2^n} \sup_{S \in \mathcal{G}_i} \|T_{2,0}^S f\|_{L^3(w)}$$

$$\leq c_{n,\delta}[w]_{A_3}^{1/2} \|f\|_{L^3(w)}.$$

Hence, by the sharp version of the Rubio de Francia extrapolation theorem (see [6] or [7]),

$$(4.5) \quad \|T(f)\|_{L^p(w)} \leq c_{n,p,\delta}[w]_{A_p}^{\max(\frac{1}{2}, \frac{1}{p-1})} \|f\|_{L^p(w)} \quad (1 < p < \infty).$$

Thus, applying this result along with Buckley’s estimate $\|M\|_{L^p(w)} \leq c_{n,p}[w]_{A_p}^{-1}$ (see [3]) and (4.4), we get

$$\|S_{\alpha,\psi}\|_{L^p(w)} \leq \|\tilde{S}_{\alpha,\psi}\|_{L^p(w)} \leq c_{n,p,\psi} \alpha^n [w]_{A_p}^{\max(\frac{1}{2}, \frac{1}{p-1})},$$

and therefore, the proof is complete.
5. Concluding remarks

In a recent work [11], the following weak type estimate was obtained for \( G_\beta(f) \) (and hence for \( S_\psi(f) \)): if \( 1 < p < 3 \), then

\[
\|G_\beta(f)\|_{L^p,\infty(w)} \lesssim [w]_{A_p}^{\max\left(\frac{1}{2} \frac{1}{p} \right)} \Phi_p([w]_{A_p}) \|f\|_{L^p(w)},
\]

where \( \Phi_p(t) = 1 \) if \( 1 < p < 2 \) and \( \Phi_p(t) = 1 + \log t \) if \( p \geq 2 \). The proof was based on the local mean oscillation decomposition technique along with the estimate

\[
(5.1) \quad \|T \mathcal{S}_{0,f}\|_{L^p,\infty(w)} \lesssim [w]_{A_p}^{\max\left(\frac{1}{2} \frac{1}{p} \right)} \Phi_p([w]_{A_p}) \|f\|_{L^p(w)}.
\]

Since the space \( L^{p,\infty}(w) \) is normable if \( p > 1 \) (see, e.g., [2, p. 220]), combining Lemma 4.3 with \( X = L^{1+\varepsilon,\infty}(w), \varepsilon > 0 \), and (5.1) yields for \( 2 < p < 3 \) that

\[
(5.2) \quad \|T f\|_{L^{p,\infty}(w)} \lesssim [w]_{A_p}^{\max\left(\frac{1}{2} \frac{1}{p} \right)} \Phi_p([w]_{A_p}) \|f\|_{L^p(w)}.
\]

Hence, exactly as above, by (4.3) (and by the weak type estimate for \( M \) proved in [3]), we obtain

\[
\|S_{\alpha,\psi}(f)\|_{L^{p,\infty}(w)} \lesssim \alpha^n [w]_{A_p}^{\max\left(\frac{1}{2} \frac{1}{p} \right)} \Phi_p([w]_{A_p}) \|f\|_{L^p(w)} \quad (2 < p < 3).
\]

We emphasize that our approach does not allow to extend this estimate to \( 1 < p \leq 2 \). This is clearly related to the same problem with (5.2). The limitation \( 2 < p < 3 \) in (5.2) is due to Lemma 4.3 where the condition that \( X \) is a Banach function space was essential in the proof. This raises a natural question whether Lemma 4.3 holds under the condition that \( X \) is a quasi-Banach space. Observe that the same question can be asked regarding a recent estimate related \( X \)-norms of Calderón-Zygmund and dyadic positive operators [15].

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Department of Mathematics, Bar-Ilan University, 52900 Ramat Gan, Israel
E-mail address: aklerner@netvision.net.il