Decaying magnetohydrodynamics: effects of initial conditions

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We study the effects of random homogenous and isotropic initial conditions on decaying Magnetohydrodynamics (MHD). We show that for an initial distribution of velocity and magnetic field fluctuations, appropriately defined structure functions decay as power law in time. We also show that for a suitable choice of initial cross-correlations between velocity and magnetic fields even order structure functions acquire anomalous scaling in time whereas scaling exponents of the odd order structure functions remain unchanged. We discuss our results in the context of fully developed MHD turbulence.

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I. INTRODUCTION

In the absence of external forcing the fluid cannot maintain its steady state because of the continuous dissipation of energy. The decay of kinetic energy in incompressible fluid turbulence has been explored in several studies (see, e.g., [1–3]). These studies, principally numerical, indicate that kinetic energy decays as \( \sim (t - t_0)^{-\eta} \), where \( t_0 \) is the initial time, i.e. the time when the initial forcing is switched off. The value of \( \eta \) seems to depend on the initial state. Similarly, recent numerical studies of decaying MHD turbulence [3–6] also suggest that the total energy decays as power of time. The phenomena of decaying MHD turbulence is observed in many different situations. Star-forming clouds are astrophysical examples [3].

A. Decay of MHD turbulence in a shell model

In Ref. [3] we described a new shell model for MHD turbulence, which has all the conservation laws in the inviscid, unforced limit, has the right symmetries and reduces to the well-known GOY shell model for fluid turbulence in absence of any magnetic field. In this paper we use our shell model to study decaying MHD turbulence. We begin by solving this set of equations with the external forces and by using the numerical method described there. We evolve the shell velocities \( v_n \) and magnetic fields \( b_n \) till we achieve the nonequilibrium, statistical steady state whose properties are described in Ref. [3]. Once we attain this state, say at a time \( t = t_{ss} \), we use the last set of shell velocities and magnetic fields, i.e., \( v_n(t_{ss}) \) and \( b_n(t_{ss}) \), as the initial condition for our shell-model study of decaying turbulence in which we solve our shell-model equations but with all the forcing terms set equal to zero. We then use the resulting time series for \( v_n \) and \( b_n \) to determine the total energy \( E(t) \equiv \sum_k \langle |v(k, t)|^2 \rangle + \langle |b(k, t)|^2 \rangle \) and the total crosshelicity \( H^c(t) \equiv \langle |v(k, t) \cdot b(k, t)| \rangle \) where the origin of time is taken to be zero. We find that \( E(t) \sim H^c(t) \sim t^{-p} \), with \( p \approx 1.2 \); this is illustrated in the log-log plot of Fig.1. As we have noted above, it is relevant for initial conditions that have the correct statistical properties of homogenous, isotropic MHD turbulence which are well described by our shell model.

![FIG. 1. A log-log plot of decay of total energy and crosshelicity versus time.](image)
In a recent paper [8] the effects of random initial conditions on the decaying solutions of the unforced Navier-Stokes equation with hyperviscosity have been discussed: It has been shown that, for an initially Gaussian velocity distribution, equal-time two point correlation functions like $< u(x, t)u(x, t) >$ decay as $\sim t^{-1}$. It has also been shown that, for this particular choice of initial conditions, the theory remains finite (in a field-theoretic sense) just by renormalisation of the two-point vertex functions. However, it is more relevant to explore the statistical properties of decaying fluid turbulence with ordinary viscosity. It is also important to find out how the introduction of magnetic fields influences this decay. In this paper we study decaying MHD with ordinary viscosity; we work with both the 3dMHD equations and a $d$-dimensional generalisation of the 1dMHD model introduced in Ref. [9]. We show that both these models give similar results. In particular, we show that, in both these models, appropriately defined structure functions decay with a power law in time; this is modulated by a scaling function whose form depends on the initial state. In Section II, we describe the equations we work with and the initial conditions we use. For most of the calculations in this Chapter, we choose initial correlations which are analytic in the low-wavenumber limit. We choose ordinary viscosities for both velocity and magnetic fields. We calculate the dynamic exponent $z$ and show that it is unaffected by the presence of the nonlinearities, i.e., it remains 2. We next show that if there is no initial cross-correlation between velocity and magnetic fields, correlation functions defined at the same spatial point decay with power laws in time and, consequently, two point structure functions also decay likewise. Also, multipoint correlation functions (and structure functions) exhibit similar behaviour, with simple (and not anomalous) scaling in time. In other words, the exponents for high-order structure functions are simply related to those of two-point functions. Another interesting feature is that there is no change in the exponents with the introduction of the magnetic field (i.e., the exponents are the same for fluid and MHD turbulence); only the amplitudes of the correlation or structure functions change. In the Section III we show that in the presence of suitable cross correlations higher order correlations (and consequently structure functions) decay more slowly than when there is no cross correlation, i.e., they display anomalous scaling. In Section IV we extend and discuss our results in the context of the decay of fully developed MHD turbulence. In Section V we conclude and compare results obtained from both 3dMHD and our simplified model.

II. DECAYING MHD TURBULENCE: THE CASE OF ZERO INITIAL CROSS-CORRELATION OF THE VELOCITY AND MAGNETIC FIELDS

A. Construction of a simplified model

The dynamical equations which govern the time evolutions of the velocity and magnetic fields in a magnetised fluid are given by the equations of Magnetohydrodynamics, namely,

\[
\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}, \nabla)\mathbf{v} = -\frac{\nabla p}{\rho} + \frac{(\nabla \times \mathbf{b}) \times \mathbf{b}}{4\pi} + \nu \nabla^2 \mathbf{v} + \mathbf{f}_v, \tag{1}
\]

\[
\frac{\partial \mathbf{b}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{b}) + \mu \nabla^2 \mathbf{b} + \mathbf{f}_b, \tag{2}
\]

where $\mathbf{v}$ and $\mathbf{b}$ are the velocity and the magnetic fields, $\nu$ and $\mu$ are fluid and magnetic viscosities, $p$ and $\rho$ represent pressure and density, and $\mathbf{f}_v$ and $\mathbf{f}_b$ are external forces. We also impose the incompressibility condition $\nabla \cdot \mathbf{v} = 0$ and the divergence-free condition on the magnetic field $\nabla \cdot \mathbf{b} = 0$. In the absence of magnetic fields the MHD equations reduce to usual Navier-Stokes equation for fluid turbulence.

We now briefly present the ideas that go into the construction of the model, which has already been discussed in details in [8]. Since this model is an extension of Burgers equation for fluid dynamics to MHD, it should have the following properties:

- Both the equations should be Galilean invariant.
- The equations should be invariant under $t \rightarrow t; \mathbf{x} \rightarrow -\mathbf{x}, \mathbf{v} \rightarrow -\mathbf{v}$ and $\mathbf{b} \rightarrow \mathbf{b}$, i.e. the velocity field is odd parity and the magnetic field is even parity.
- The fields are irrotational vector fields in $d$-dimension (similar to Burgers velocity field).
- The equations in the ideal limit i.e. in absence of any viscosity and any external forcing should conserve all the scalar conserved quantities of 3dMHD namely the total energy and the cross-helicity. The third conserved quantity i.e. the magnetic helicity is zero for irrotational fields.
The above considerations lead to the following model equations

\[ \frac{\partial \mathbf{v}}{\partial t} + \frac{1}{2} \nabla v^2 + \frac{1}{2} \nabla b^2 = \nu \nabla^2 \mathbf{v} + \mathbf{f}_v, \]

\[ \frac{\partial \mathbf{b}}{\partial t} + \nabla (\mathbf{v} \cdot \mathbf{b}) = \mu \nabla^2 \mathbf{b} + \mathbf{f}_b, \]

whose relation to the 3dMHD equations is the same as the relation of the Burgers equation to the Navier-Stokes equation. Here \( \mathbf{v} \) and \( \mathbf{b} \) are the velocity and magnetic fields with magnitudes \( v \) and \( b \) respectively. These fields are \textit{irrotational}, i.e., \( \nabla \times \mathbf{v} = 0 \) and \( \nabla \times \mathbf{b} = 0 \). These equations together conserve the total energy \( 1/2 \int (v^2 + b^2), d^3x \) and crosshelicity \( \int \mathbf{v} \cdot \mathbf{b} d^3x \).

### B. Choice of initial conditions

We have \( < u_i(k, t = 0) > = < b_i(k, t = 0) > = 0 \) by homogeneity and isotropy. We choose the initial correlations to have the simple analytic forms

\[ < v_i(k, t = 0) v_j(k', t = 0) > = D_{ij} k_i k_j \delta(k + k'), \]

\[ < b_i(k, t = 0) b_j(k', t = 0) > = D_{ij} k_i k_j \delta(k + k'), \]

and

\[ < v_i(k, t = 0) b_j(k', t = 0) > = D_{ij}(k) \delta(k + k') \]

In \( k \)-space the 3dMHD equations are

\[ \partial_t v_i(p) + iM_{ijk}(p) \sum_q v_j(q) v_k(p - q) = iM_{ijl}(p) \sum_q b_j(q) b_l(p - q) / 4\pi \rho \]

\[ - \nu p^2 v_i(p) + f_i(p) \]

\[ \partial_t b_i(p) = - (i p \times \sum_q v(q) \times b(p - q))_i - \mu p^2 b_i(p) + g_i(p) \]

The quantities \( M_{ijl}(k) = P_{ijl} k_l + P_{ikl} k_j \) appear to include the incompressibility, \( P_{ij}(k) = \delta_{ij} - \frac{k_i k_j}{k^2} \) is the transverse projection operator and wavevector arguments \( p, q, k \) indicate spatial Fourier transforms. The initial conditions we use for 3dMHD are

\[ < u_i(k, t = 0) u_j(k', t = 0) > = D_{ui} k_i^2 P_{ij}(k) \delta(k + k'), \]

\[ < b_i(k, t = 0) b_j(k', t = 0) > = D_{bij} k_i^2 P_{ij}(k) \delta(k + k'), \]

\[ < u_i(k, t = 0) b_j(k', t = 0) > = D_{ij}(k) \delta(k + k') \]

which ensure the divergence-free conditions, namely,

\[ k_i < u_i(k) u_j(-k) > = 0 \]

\[ k_i < b_i(k) b_j(-k) > = 0. \]

We discuss the structure of \( D_{ij} \) later. For the time being, we choose \( D_{ij} = 0 \). These simple initial conditions, which are analytic in the \( k \to 0 \) limit, are enough to elucidate the issues raised in this Chapter.

### C. Calculation of effective correlation and structure functions within a one-loop perturbation theory: temporal scaling

The formal solutions of the Eqs. (3) and (4) are given by


\[
\begin{align*}
\frac{1}{\nu k^2} &= \frac{1}{\nu k^2} + \frac{D_{uu}}{\nu k^2} \int d^3 p \frac{p_m k.(k-p)}{2\nu k.p} \left( \frac{k^2}{2\nu p^2 - 2\nu k.p} \left( 1 - \frac{k^2}{\nu (p^2 + (k-p)^2)} \right) \right) \\
&\quad - \frac{1}{2\nu p^2} \left( 1 - \frac{k^2}{2\nu p^2} \right) + \frac{D_{bb}}{\nu k^2} \int d^3 p \frac{p_m k.(k-p)}{2\mu k.p} \left( \frac{1}{2\mu p^2} \left( 1 - \frac{\nu k^2}{2\mu p^2} \right) \right). 
\end{align*}
\tag{19}
\]

We see that, in the long-wavelength limit, 1-loop corrections have no infrared divergence at \( d = 3 \); in other words \( z \) remains 2. In the case of the 3dMHD equations, the forms of the integrals remain unchanged, only changes appear in the coefficients of the integrals where, in place of \( p_m \), appropriate projection operators are present.
is no diagramatic correction to viscosities in the long-wavelength limit in 3dMHD. Hence the exponent $z$ is unaffected by the nonlinear terms. Therefore, without any loss of generality, we can put $\nu = \mu = 1$.

We now calculate the effective $D_{uu}$ and $D_{bb}$ for $d = 3$ at the one-loop level. The diagrams shown in Fig.3 contribute at the 1-loop level to $D_{uu}$. These contributions are proportional to

\[
k_i k_j (D_{uu}^2 + D_{bb}^2) \int d^3p \frac{p^2 p^2 (p^2 - k^2)^2}{k^4 p^8} \sim k_i k_j (D_{uu}^2 + D_{bb}^2) \int \Lambda^{-5} d^3p, \tag{20}\]

which does not have any infrared divergence in any dimension. The one-loop diagram contributing to $D_{bb}$ has the same form. For 3dMHD the forms of the one-loop integrals do not change; only the coefficients have appropriate factors of projection operators instead of $k_i k_j$. On purely dimensional grounds for $d = 3$,

\[
\langle u_i(x, t) u_j(x, t) \rangle \sim \int d^3 k \langle u_i(k, t) u_j(-k, t) \rangle \sim D_{uu} \int d^3 k e^{-\nu k^2 t} e^{-\nu k^2 t} k_i k_j \sim D_{uu} t^{-\frac{2}{3}} \delta_{ij} \tag{21}\]

and

\[
\langle b_i(x, t) b_j(x, t) \rangle \sim \int d^3 k \langle b_i(k, t) b_j(-k, t) \rangle D_{bb} \sim \int d^3 k e^{-\nu k^2 t} e^{-\nu k^2 t} k_i k_j \sim D_{bb} t^{-\frac{5}{9}} \delta_{ij}. \tag{22}\]

Similar quantities obtained from 3dMHD also show the same exponents but with different amplitudes. Note that the temporal dependences can be scaled out of the momentum integrals easily as the lower and upper limits of the momentum integrals can be extended to 0 and $\infty$, respectively.

It is easy to see that all multi-point correlation functions also do not exhibit any infrared divergence in three dimensions: Consider, e.g., 3-point functions. Even though $\langle u_i(k_1, t) u_j(k_2, t) u_s(k_3, t) \rangle = 0$ at $t = 0$, it becomes non-zero at $t > 0$ as can be easily seen at the one-loop level. $\langle u_i(k_1, t) u_j(k_2, t) u_s(k_3, t) \rangle$ has a tree-level contribution and next a one-loop contribution. The tree-level part being proportional to $\langle u_i(k_1, 0) u_j(k_2, 0) u_s(k_3, 0) \rangle$ is zero because we have chosen a Gaussian distribution for the initial conditions. The one-loop diagrams are shown in Fig.3.

\footnote{This is easy to understand: Calculation of the exponents through the one-loop method depends upon the power-counting of the one-loop integrals which are same for both the 3dMHD equations and our simplified model as the projection operator $P_I$ that appears in the 3dMHD equations is dimensionless and the nonlinearities in both the equations do not renormalise due to the Galilean invariance of our simplified model and 3dMHD equations (which in turn keeps the power-counting structure same). Amplitudes of course depend upon the specific models. Similarly, energy spectra obtained in the steady-state from different models which give same power-counting in the one-loop integrals, when driven by stochastic forces with correlations having same scaling, have same scaling properties, see, for e.g. Refs. [1][2].}
The 1-loop contribution is however nonzero. It has the form

\[
(u_1(k_1, t)u_j(k_2, t)u_s(k_3, t)) \sim \exp[\nu(-k_1^2 - k_2^2 - (k_1 + k_2)^2)t]k_1k_2(-k_1s - k_2s)D_{uuu},
\]

(23)

where \(D_{uuu}\) is the value of the one-loop integral in the limit \(k_1, k_2 \rightarrow 0\), i.e.,

\[
D_{uuu} \sim \int d^d q(D_{uuu}^3 + D_{bb}^3)
\]

(24)

which is infrared (IR) finite. We can interpret this one-loop result by saying that the nonlinearities in Eqs. (3) and (4) are absent but there are nonzero initial three-point correlations at \(t = 0\), i.e., the effective linear system behaves as if

\[
< u_i(k_1, 0)u_j(k_2, 0)u_s(k_3, 0) > \equiv k_1k_2k_3D_{uuu} \delta(k_1 + k_2 + k_3),
\]

(25)

\[
< b_i(k_1, 0)b_j(k_2, 0)b_s(k_3, 0) > \equiv k_1k_2k_3D_{bb} \delta(k_1 + k_2 + k_3).
\]

(26)

Hence

\[
< u_i(x_1, t)u_j(x_2, t)u_s(x_3, t) > \\
\equiv \int d^3 k_1 d^3 k_2 \exp[i k_1.(x_1 - x_2) + k_2.(x_1 - x_3)](u_i(k_1, t)u_j(k_2, t)u_s(-k_1 - k_2, t) > \\
= \int d^3 k_1 d^3 k_2 \exp(-k_1^2 - k_2^2 - (k_1 + k_2)^2)t \exp[i k_1.(x_1 - x_2) + k_2.(x_1 - x_3)]
\]

\[
k_1k_2k_3D_{uuu} \sim t^{-9/2} A_{ijs} \frac{(x_1 - x_2)^{x_1 - x_3} + \sqrt{t}}{\sqrt{t}}.
\]

(27)

The last step can be easily obtained by using the dimensionless variables \(y_1 = k_1^2t, y_2 = k_2^2t\).

Next we consider the four-point functions: We concentrate on \( < u_i(x_1, t)u_j(x_2, t)u_s(x_3, t)u_l(x_4, t) > \); the diagramatic representations up to the one-loop contributions are given in Fig. [5].
The one-loop integrals in the long-wavelength limit, are
\[ D_{uuuu} \sim \int d^3q \left( \frac{D_{uu}^4}{\nu^4} + \frac{D_{bb}^4}{\mu^4} \right). \]  
(28)

These one-loop contributions can be interpreted as in the previous case of three-point functions i.e. as if there were initial four-point correlations in absence of the non-linearities:
\[ \langle u_i(k_1,0)u_j(k_2,0)u_s(k_3,0)u_l(k_4,0) \rangle \equiv D_{uuuu}(k_1+k_2+k_3+k_4). \]  
(29)

Similar results can be easily obtained for other four-point functions involving velocity and magnetic fields. Hence,
\[ <u_i(x_1,t)u_j(x_2,t)u_s(x_3,t)u_l(x_4,t)> = D_{uuuu}^2 \int d^3k_1 d^3k_2 d^3k_3 d^3k_4 \exp[-2\nu(k_1^2 + k_2^2)t] \exp[i\mathbf{k}_1 \cdot \mathbf{x}_1 - \mathbf{k}_2 \cdot \mathbf{x}_2 + i\mathbf{k}_3 \cdot \mathbf{x}_3 - i\mathbf{k}_4 \cdot \mathbf{x}_4] \]
\[ + D_{uuuu}^2 \int d^3k_1 d^3k_2 d^3k_3 d^3k_4 \exp[-\nu(k_1^2 + k_2^2 + k_3^2 + k_4^2)t] \exp[i\mathbf{k}_1 \cdot \mathbf{x}_1 - i\mathbf{k}_4 \cdot \mathbf{x}_4 + i\mathbf{k}_2 \cdot \mathbf{x}_2 - i\mathbf{k}_3 \cdot \mathbf{x}_3] \]
\[ \sim D_{uu}^2 t^{-5} + D_{uuuu}^2 t^{-13/2}. \]  
(30)

In the infinite-Reynolds-number limit, i.e., in the limit \( \nu, \mu \to 0 \), \( D_{uuuu} \to \infty \) and hence the second term in the last line of Eq.(30) dominates. As in the case for the three-point functions four-point functions, are of the form
\[ <u_i(x_1,t)u_j(x_2,t)u_s(x_3,t)u_l(x_4,t)> \sim t^{-13/2} A_{ijkl}(\frac{x_1-x_2}{\sqrt{t}}, \frac{x_2-x_3}{\sqrt{t}}, \frac{x_3-x_1}{\sqrt{t}}). \]  
(31)

Here \( A_{ijkl} \) and \( A_{ijkl} \) are third and fourth rank tensors respectively, and \( <u_i(x,t)^4> \sim t^{-13/2} \). To study decaying MHD, we extend the definition of equal-time structure functions as follows:
\[ S_u^n(r,t) \equiv <||u_i(x_1,t) - u_i(x_2,t)||^n> \]  
(32)
and
\[ S_b^n(r,t) \equiv <||b_i(x_1,t) - b_i(x_2,t)||^n>, \]  
(33)
where \( r \equiv |x_1 - x_2| \) and \( t \) denotes the time that has elapsed after the forcing terms are switched off. We now look at the general forms of these structure functions. Consider the three-point function first:

\[
S^3_u(r, t) \equiv \langle |u_i(x_1, t) - u_i(x_2, t)|^3 \rangle. \tag{34}
\]

On expanding the expressions for, \( S^3_u(r, t) \) we get the following kind of terms: (i) \( \langle u_i(x_1, t)^3 \rangle \) and (ii) \( \langle u_i(x_1, t)^2 u_i(x_2, t) \rangle \). The first one is given by

\[
< u_i(x_1, t)^3 > = \int d^3k_1 d^3k_2 k_{1i} k_{2i} \exp[-\{k_1^2 - k_2^2 - (k_1 + k_2)^2\} t] D_{uuu} \sim t^{-9/2}, \tag{35}
\]

whereas the second term has the form

\[
< u_i(x_1, t)^2 u_i(x_2, t) > = \int d^3k_1 d^3k_2 k_{1i} k_{2i} \exp[-\{k_1^2 - k_2^2 - (k_1 + k_2)^2\} t] \exp[i k_1 \cdot (x_1 - x_2)] D_{uuu} \sim t^{-9/2} \phi(r/\sqrt{T}), \tag{36}
\]

with \( r \equiv |x_1 - x_2| \). Thus, it is easy to see that \( S^3_u(r, t) \) has the structure

\[
S^3_u(r, t) \equiv t^{-9/2} F^u_n(r/\sqrt{T}), \tag{37}
\]

with \( F^u_n(r/\sqrt{T}) \) is a function with dimensionless argument \( r/\sqrt{T} \). We can easily extend our calculation above to see that

\[
S^n_u(r, t) \sim t^{-z^u_n} F^n_u(r/\sqrt{T}), \tag{38}
\]

\[
S^b_n(r, t) \sim t^{-z^b_n} F^n_b(r/\sqrt{T}), \tag{39}
\]

where \( F^u_n \) and \( F^b_n \) are functions with dimensionless argument \( r/\sqrt{T} \). It is obvious that all these higher-order correlation functions do not exhibit any anomalous scaling in time; their temporal behaviours can be obtained by simple dimensional arguments from the behaviours of the two-point functions. It is easy to see that

\[
S^n_u \sim t^{-(n-1)/2 - n/2} F^n_u(r/\sqrt{T}), \tag{40}
\]

\[
S^b_n \sim t^{-(n-1)/2 - n/2} F^n_b(r/\sqrt{T}), \tag{41}
\]

i.e., \( z^u_n = z^b_n = (n - 1)/2 + n/2 \), at the level of our one-loop approximation.

\[\text{III. NON-ZERO INITIAL CROSS-CORRELATIONS}\]

\[\text{A. Form of the initial cross correlations}\]

What happens if we make \( D_{ij} \) non-zero? Let us first establish the form of \( D_{ij} \): Recall that \( \langle u_i(x_1, t) b_j(x_2, t) \rangle \) has odd parity (since \( u \) is a polar vector whereas \( b \) is an axial vector). Hence it is purely imaginary and is an odd function of \( k \). (From now on we will work only in three dimension.) Thus we choose

\[
D_{ij} = i D_{ab} \epsilon_{ijp} k^p k^a, \tag{42}
\]

where \( \epsilon_{ijp} \) is the totally antisymmetric tensor. This is the simplest analytic form with the desired structure. We consider the cases \( \alpha = 0 \) and 2.
B. The temporal decay of correlation and structure functions

It is easy to see that the value of $z$ does not change if $D_{ub} \neq 0$ as no new diagram appears. However, an additional diagram (Fig. 4.6) contributes to the effective $D_{uu}$ and $D_{bb}$ if $D_{ub} \neq 0$.

\begin{equation}
\frac{k_1k_2}{16k^4}D_{ub}^2 \int \frac{d^3q}{(2\pi)^3} \frac{2\delta_{sn}q_snq_nq^\alpha}{q^2},
\end{equation}

FIG. 6. 1-loop diagram contributing to the renormalisation of $D_{uu}$ and $D_{bb}$, which arises when $D_{ub} \neq 0$ (see text).

This diagram yields the following contribution

\begin{equation}
\epsilon_{lms}\epsilon_{lmn} = (d - 1)\delta_{sn} = 2\delta_{sn}. 
\end{equation}

Naive power counting shows that this contribution is finite for $d = 3$, hence $D_{uu}$ and $D_{bb}$ are not renormalised even if $D_{ub} \neq 0$, for both $\alpha = 0$ and 2. The same conclusion holds for 3dMHD as the one-loop integral remains finite; only the coefficients change because of factors of $P_{ij}(k)$. It is quite interesting to note that for, $D_{ub} \neq 0$, there is a diagramatic correction to $D_{ub}$.

\begin{equation}
\frac{k_1k_2}{16k^4}D_{uu}D_{ub} \int \frac{d^3q}{(2\pi)^3} \frac{q^\alpha\epsilon_{lms}q_snq_m}{q^4},
\end{equation}

FIG. 7. 1-loop diagramatic correction to $D_{ub}$.

the form of the correction is

\begin{equation}
\frac{k_1k_2}{16k^4}D_{uu}D_{ub} \int \frac{d^3q}{(2\pi)^3} \frac{q^\alpha\epsilon_{lms}q_snq_m}{q^4},
\end{equation}

however this correction vanishes for both $\alpha = 0$ and 2 since the integrand is odd in $q$. Hence there is no renormalisation of $D_{ub}$ and two-point structure functions Eqs.(4.22) and (4.23) scale as $t^{-5/2}$ just as they do when $D_{ub} = 0$. 
C. Higher-order correlation and structure functions: anomalous temporal scaling

We now consider the \( n \)-point correlation functions with \( n > 2 \). None of the three-point functions have any infrared divergence. Hence they exhibit simple scaling, i.e., \(< u_i(x,t)^3 > \sim t^{-9/2} \). Let us consider the four-point functions namely,

\[
\begin{align*}
&< u_i(x_1,t) u_j(x_2,t) u_s(x_3,t) u_l(x_4,t) >, \\
&< b_i(x_1,t) b_j(x_2,t) b_s(x_3,t) b_l(x_4,t) >, \\
&< u_i(x_1,t) u_j(x_2,t) b_s(x_3,t) b_l(x_4,t) >, \\
&< u_i(x_1,t) u_j(x_2,t) u_s(x_3,t) b_l(x_4,t) >.
\end{align*}
\]

(46)

At the 1-loop level, all of them they will have finite parts (coming from initial auto correlations of velocity and magnetic fields). However, the first three of them will have new, infrared-divergent, one-loop contributions if \( D_{ub} \neq 0 \).

![Diagram](image)

FIG. 8. Divergent one-loop contributions to \(< u_i(x_1,t) u_j(x_2,t) u_s(x_3,t) u_l(x_4,t) >, \ < u_i(x_1,t) u_j(x_2,t) b_s(x_3,t) b_l(x_4,t) >, \ < b_i(x_1,t) b_j(x_2,t) b_s(x_3,t) b_l(x_4,t) >, \ < b_i(x_1,t) b_j(x_2,t) u_s(x_3,t) b_l(x_4,t) > \) respectively (see text).

The analytic structure of the divergent 1-loop integral is illustrated for one of these correlation functions in Fig.8 whose contribution is

\[
\langle u_i(k_1) u_j(k_2) u_s(k_3) u_l(k_4) \rangle \sim \frac{k_1 k_2 k_3 k_4}{k^8} \int \frac{d^3p}{(2\pi)^3} \epsilon_{a c m} \epsilon_{b d n} e_{c d f} e_{e a t} \frac{p_m p_n p_f p_t p^\alpha}{p^8}.
\]

(47)

Isotropy yields

\[
p_m p_n p_f p_t = A p^4 [\delta_{mn} \delta_{ft} + \delta_{mf} \delta_{nt} + \delta_{mt} \delta_{fn}],
\]

(48)

so summing over \((m,n)\) and \((f,t)\) separately we obtain

\[
A = \frac{1}{d^2 + 2d} = 1/15
\]

(49)

for \( d = 3 \). Hence naive power counting for the integral in Eq.(47) yields, for \( \alpha = 0 \),

\[
\sim \frac{k_1 k_2 k_3 k_4}{15 k^8} D_{ub}^4 \int \frac{d^3p}{p^8} \sim \frac{k_1 k_2 k_3 k_4 \Lambda^{-1}}{15 k^8}
\]

(50)

where \( \Lambda \) is a momentum scale arising from the lower limit of the integral. Hence it is linearly infrared (IR) divergent for \( \alpha = 0 \). For \( \alpha = 2 \) there is no IR divergence as is the case with \( D_{ub} = 0 \). We analyse the case \( \alpha = 0 \) in detail: There is a disconnected diagram also that contributes to \( < u_i(x_1,t) u_j(x_2,t) u_s(x_3,t) u_l(x_4,t) > \), which yields
Hence we see that these structure functions exhibit anomalous dynamical scaling if $D_{ub} \neq 0$. Similarly, it follows that the even-order exponents are
\begin{equation}
\Delta_{2n}^u = \Delta_{2n}^b = \frac{3}{2}(2n - 1) + n - \Delta_{2n}^{an}, \quad n \geq 2,
\end{equation}
where $\Delta_{2n}^{an} = (2n - 3)/2$ is the *anomalous* part of the exponent $\Delta_{2n}$. By contrast the odd order exponents
\begin{equation}
\Delta_{n}^u = \Delta_{n}^b = (n - 1)3/2 + n/2, \quad \text{for any odd } n,
\end{equation}
do not have an anomalous part. In Fig. 9 we present plots of $\Delta_{n}$ versus $n$ for both even and odd $n$. 

![Figure 9](image_url)

**FIG. 9.** A plot showing the dependence of $\Delta_{n}$ over $n$; the line gives the exponents for the odd order structure functions for all values of $D_{ub}$ and for the even order structure functions when $D_{ub} = 0$; we also show $\Delta_{4}$ and $\Delta_{6}$ in this plot (circles) when $D_{ub} \neq 0$ (for $\alpha = 0$).
IV. EFFECTS OF INITIAL CONDITIONS WITH SINGULAR AMPLITUDES FOR THE VARIANCE

In this Section we choose initial conditions of the following form:

\begin{align}
\langle u_i(\mathbf{k},0)u_j(\mathbf{k}',0) \rangle &= D_{uu} \frac{k_i k_j}{k^{s+2}} \delta(\mathbf{k} + \mathbf{k}'), \\
\langle b_i(\mathbf{k},0)b_j(\mathbf{k}',0) \rangle &= D_{bb} \frac{k_i k_j}{k^{s+2}} \delta(\mathbf{k} + \mathbf{k}'), \\
\text{and} \\
\langle u_i(\mathbf{k},0)b_j(\mathbf{k}',0) \rangle &= i D_{ub} \epsilon_{ijp} \frac{k_p}{k^{s+1}} \delta(\mathbf{k} + \mathbf{k}').
\end{align}

We choose $y \geq y$. The effective fluid viscosity $\tilde{\nu}$, and two-point correlations $\tilde{D}_{uu}$ and $\tilde{D}_{bb}$ are given by

\begin{align}
\tilde{D}_{uu} &= D_{uu} + \Delta \frac{D_{uu}^2}{\nu^{s+1}}, \\
\tilde{D}_{bb} &= D_{bb} + \Delta \frac{D_{bb}^2}{\nu^{s+1}}, \\
\tilde{D}_{ub} &= D_{ub},
\end{align}

where $\Delta$ is just a numerical constant. The self-consistent solution is given by $\nu, \mu \sim k^{\frac{1-s}{2}}$, with the long-wavelength behaviours of $D_{uu}$, $D_{bb}$ and $D_{ub}$ unchanged. Now we can easily calculate the decay of the total velocity and magnetic energy, and total cross helicity; these are:

\begin{align}
E^v(t) &\sim t^{-\frac{3-2s}{s+2}}, \\
E^b(t) &\sim t^{-\frac{1}{2}},
\end{align}

and

\begin{equation}
H^c(t) \sim t^{-\frac{3-2s}{s+2}}.
\end{equation}

Here $E^v = \int d^d k \frac{1}{2} \langle |v(k)|^2 \rangle$, $E^b = \int d^d k \frac{1}{2} \langle |b(k)|^2 \rangle$, $H^c = \int d^d k |v(k), b(k)|$ are total velocity-field and magnetic-field energies, and total cross helicity respectively. While calculating the temporal decay of these quantities it has been implicitly assumed that the lower and the upper limits of the momentum integrals can be extended upto 0 and $\infty$, respectively; if this not possible temporal behaviour cannot be obtained by scaling (see next Section).

A. Decay of fully developed MHD turbulence

In fully developed 3dMHD turbulence one finds, in the inertial range,

\begin{align}
\langle v_i(\mathbf{k},t)v_j(\mathbf{k}',t) \rangle &\sim P_{ij}(\mathbf{k})k^{-11/3}\delta(\mathbf{k} + \mathbf{k}'), \\
\langle b_i(\mathbf{k},t)b_j(\mathbf{k}',t) \rangle &\sim P_{ij}(\mathbf{k})k^{-11/3}\delta(\mathbf{k} + \mathbf{k}').
\end{align}

Of course, a study of higher-order correlation functions reveals that the velocity and magnetic-field distributions are not Gaussian. If, however, for the purpose of analytical tractability, we assume that the initial distributions are Gaussian with the variances as (for our Burges-type model)

\text{2If $y > s$ then one-loop effective $D_{uu}$ and $D_{bb}$ will scale as $k^{-y}$.}
\[ \langle u_i(\mathbf{k}, t = 0) u_j(\mathbf{k'}, t = 0) \rangle = D_{uu} k_i k_j |k|^{-11/3 - 2}, \]
\[ \langle b_i(\mathbf{k}, t = 0) b_j(\mathbf{k'}, t = 0) \rangle = D_{bb} k_i k_j |k|^{-11/3 - 2}, \]

such that \( |u(k, t = 0) u(-k, t = 0)| \geq | \sim | b(k, t = 0) b(-k, t = 0) | \sim k^{-11/3} \) as observed in fully developed MHD turbulence, we can calculate \( z \) and effective two-point correlations as we did above in Sec (4.2.3). For simplicity we give details only for our simple Burgers-type model [Eqs. (3)]. The effective \( \nu, D_{uu} \) and \( D_{bb} \) are given by

\[
\begin{align*}
\frac{1}{\nu k^2} & = \frac{1}{\nu k^2} + \frac{D_{uu}}{\nu k^2} \int d^4 p \frac{p_i p_j p^{-11/3 - 2} k_i (k - p_j) k_j}{2 \nu k p} (1 - \frac{k^2}{2 \nu p^2 - (k - p)^2}) \\
& - \frac{1}{2 \nu \mu^2} \frac{1}{(1 - \frac{k^2}{2 \nu p^2}) + \frac{D_{bb}}{\nu k^2} \int d^4 p \frac{p_i p_j p^{-11/3 - 2} k_i (k - p_j) k_j}{2 \nu k p} (1 - \frac{k^2}{2 \nu p^2 - (k - p)^2}) \\
& + \frac{1}{2 \nu \mu^2} (1 - \frac{k^2}{2 \nu p^2}) \\
\dot{D}_{uu} & = D_{uu} + \delta \frac{D_{uu}^2}{\nu^2 k^8/3} + \delta \frac{D_{bb}^2}{\mu^2 k^8/3} \\
\dot{D}_{bb} & = D_{bb} + 2 \delta \frac{D_{bb} D_{uu}}{\mu^2 k^8/3}.
\end{align*}
\]

Here \( \delta \) is a numerical constant. The self-consistent solutions of these equations are given by \( \nu, \mu \sim k^{-4/3} \) and \( D_{uu} = D_{bb} \), implying that \( z = 2/3 \); this is the same as in the steady state. With this self-consistent solution, we see that the 1-loop corrections to \( D_{uu} \) and \( D_{bb} \) are as singular as the bare quantities. Hence we ignore their renormalisations. When \( D_{ab} \) is non-zero, it will contribute to the 1-loop corrections of \( D_{aa} \) and \( D_{bb} \) with the same low-wavenumber singularity. \( D_{ab} \) itself does not have any diagramatic correction, as in the case discussed in Section III.

The decay of two-point correlation functions will be of the form

\[ \langle u_i(\mathbf{x}, t)^2 \rangle \sim \int d^3 k \exp(-2 \nu k^2/3 t) k^{-22/3} \]

It is easy to see that, because of the infrared divergence of the integral, the lower limit cannot be extended to 0, so it will depend upon a lower cut-off (the inverse of which is the integral scale). Consequently the \( t \)-dependence cannot be scaled out as in the previous case. This problem is reminiscent of the sweeping divergence that appears in a perturbative DRG calculation of fully developed turbulence or MHD turbulence \[12\]. In fact this problem begins when \( < u_i(\mathbf{k}, 0) u_j(\mathbf{k'}, 0) > \sim k^{4-y-d} \) and \( y = 3 \); then the above integral becomes log-divergent. This type of initial state can be prepared by randomly stirring the system with a stochastic force with a variance \( k^{4-y-d} \) with \( y = 3 \).

In a DRG analysis, sweeping divergences appear when \( y \geq 3 \) (but K41 scaling obtains for \( y = 4 \)) \[12\]. Higher-order correlation functions have more severe divergences, hence their decay also depend strongly upon the integral scale.

V. CONCLUDING REMARKS

We have examined the decay of the total kinetic and magnetic energies, and the crosshelicity in our shell model, These quantities decay with power laws in time. In order to understand this behaviour analytically, we have employed a one-loop perturbation theory. In the first part of our calculations, we have chosen random initial conditions with the statistical properties given in Eqs.(5),(6) and (10),(11) (with no crosscorrelations). We have shown that such initial conditions lead to power-law decay of temporal correlations and appropriately defined structure functions. For initial correlations analytic in the long-wavelength limit no higher order structure functions show any anomalous scaling. However, on introducing crosscorrelations \( < u_i(\mathbf{k}, 0) b_j(\mathbf{k'}, 0) > \) (Eq.4.43) higher-order structure functions exhibit anomalous scaling. This highlights the importance of crosscorrelations in the statistical properties of MHD turbulence. We have also extended our calculations to the case of initial correlations with singular variances and discussed the difficulties in the decay of fully developed MHD turbulence. We have argued, that these difficulties are related to the sweeping effect which arises in the study of the stochastically driven MHD turbulence (in the statistically steady state). It is also worth noting that we have obtained similar temporal behaviour from both 3dMHD and our simplified Burgers-type model. It would be interesting to see if our results can be checked experimentally and numerically.
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