1. Introduction

Throughout this work we deal with a natural number \( g \geq 2 \) and with an algebraically closed field \( k \) whose characteristic differs from 2. A hyperelliptic curve of genus \( g \) over \( k \) is a smooth curve of genus \( g \), that is a double cover of the projective line \( \mathbb{P}^1 \). The Riemann-Hurwitz formula implies that this covering should be ramified at \( 2g + 2 \) points.

Because of this explicit description, hyperelliptic curves have been studied for a long time from different points of view. Among recent advances, we want to mention the determination of all the possible automorphism groups of hyperelliptic curves (see [BS86], [BGG93], [Sha03]) as well as the extensive use of the Jacobian of hyperelliptic curves in cryptography (see [Sch85], [Can87], [Kob89], [Fre99], [Gau00], [Ked01], [Lan05], and the survey paper [JMS04]).

In this paper we are interested in the moduli space \( H_g \) of hyperelliptic curves and in the moduli stack \( \mathcal{H}_g \) of hyperelliptic curves, whose definitions we are going to briefly recall now.

The MODULI SCHEME \( H_g \) of hyperelliptic curves is defined as

\[
H_g = \left( \text{Sym}^{2g+2}(\mathbb{P}^1) - \Delta \right)/\text{PGL}_2,
\]

where \( \text{Sym}^{2g+2}(\mathbb{P}^1) \) is the \((2g+2)\)-th symmetric power of \( \mathbb{P}^1 \), \( \Delta \) is the closed subset where at the least two points coincide and the action of \( \text{PGL}_2 \) comes from its natural action on \( \mathbb{P}^1 \). Since a hyperelliptic curve over \( k \) is completely determined (up to isomorphism) by \( 2g + 2 \) points on \( \mathbb{P}^1 \) (up to isomorphism), over which the corresponding double cover of \( \mathbb{P}^1 \) ramifies, \( H_g \) has the property that its closed points parameterize isomorphism classes of hyperelliptic curves.

This modular variety has been studied from different points of view: Katsylo and Bogomolov proved its rationality (see [Kat84], [Bog86]), Avritzer and Lange considered various compactifications of \( H_g \) (that is an affine variety) comparing them with each other (see [AL02]).

Our new contribution to the study of \( H_g \) is the determination of the Picard group \( \text{Pic}(H_g) \) and of the divisor class group \( \text{Cl}(H_g) \). We prove that, away from some bad characteristic of the base field, \( \text{Pic}(H_g) \) is trivial (theorem 4.10), while \( \text{Cl}(H_g) \) is a cyclic group of order \( 4g + 2 \) if \( g \geq 3 \) and 5 if \( g = 2 \) (theorem 4.7). The fact that \( \text{Pic}(H_g) \neq \text{Cl}(H_g) \) indicates that \( H_g \) is a singular variety (although its explicit description as quotient imply that it’s a normal variety) and in fact we determine its smooth locus in proposition 4.5.

The MODULI FUNCTOR \( \mathcal{H}_g \) of hyperelliptic curves is the contravariant functor

\[
\mathcal{H}_g : \text{Sch}_k \to \text{Set}
\]

which associates to every \( k \)-scheme \( S \) the set

\[
\mathcal{H}_g(S) = \{ \mathcal{F} \to S \text{ family of hyperelliptic smooth curves of genus } g \}/\sim.
\]

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Arsie and Vistoli (see [AV04] and also [Vis98] only for \(g = 2\)) proved that \(H_g\) is a Deligne-Mumford stack isomorphic to a quotient stack, precisely

\[ H_g = \quotient{A_{sm}(2, 2g + 2)}{(GL_2/\mu_{g+1})} \]

where \(A_{sm}(2, 2g + 2)\) is the space of binary forms in two variables of degree \(2g + 2\) having only simple factors and \(GL_2/\mu_{g+1}\) acts as \((A) \cdot f(x) = f(A^{-1} \cdot x)\). Moreover they compute the equivariant Picard group \(\text{Pic}^{GL_2/\mu_{g+1}}(A_{sm}(2, 2g + 2))\) which in fact is isomorphic to the Picard group of the stack \(H_g\) (as defined functorially by Mumford in [Mum65]). In the case \(g = 2\), Vistoli proved (in [AV04]) that this group is generated by the first Chern class of the Hodge bundle but in the general case there wasn’t known such a functorial description.

In theorem 5.7, we provide an explicit functorial description of a generator of the Picard group of the stack. Moreover in theorem 5.8 we consider natural elements of the Picard group (obtaining by pushing-forward linear combinations of the relative canonical divisor and the relative Weierstrass divisor and then taking the determinant) and express them in terms of the generator found above. In particular we prove that the first Chern class of the Hodge bundle generates the Picard group if and only if 4 doesn’t divide \(g\) in which case it generates a subgroup of index 2 (corollary 5.9).

It is well known that \(H_g\) is a COARSE MODULI SCHEME for the functor \(H_g\), which means that there is a natural transformation of functors

\[ \Phi_H : H_g \to \text{Hom}(-, H_g) \]

satisfying two properties:

(i) \(\Phi_H(\text{Spec}(K)) : H_g(\text{Spec}(K)) \to \text{Hom}(\text{Spec}(K), H_g)\) is bijective for any algebraically closed field \(K = \overline{k} \supset k\).

(ii) (Universal property) If \(N\) is a scheme and \(\Psi : H_g \to \text{Hom}(-, N)\) is a natural transformation of functors, then there exists a unique morphism \(\pi : H_g \to N\) such that the corresponding natural transformation \(\Pi : \text{Hom}(-, H_g) \to \text{Hom}(-, N)\) satisfies \(\Psi = \Pi \circ \Phi_H\).

Another problem we treat in this work is the question: is \(H_g\) a FINE moduli scheme for the functor \(H_g\)? And if not, how far is from being such? By definition, being a fine moduli scheme would mean that \(\Phi_H\) is an isomorphism of functors, or, in other words, that there exists a universal family of hyperelliptic curves \(F_g \to H_g\) such that every other family \(f : F \to S\) is obtained from this one by pulling back via the modular map \(\Phi_S(f) : S \to H_g\), i.e. \(F \cong F_g \times_{H_g} S\).

To attack this problem, we introduce a new moduli functor \(D_{2g+2}\) which is intermediate between \(H_g\) and \(H_g\) and is defined as the contravariant functor

\[ D_{2g+2} : \text{Sch}/k \to \text{Set} \]

which associates to every \(k\)-scheme \(S\) the set

\[ D_{2g+2}(S) = \left\{ C \to S \text{ family of } \mathbb{P}^1 \text{ and } D \subset C \text{ an effective Cartier divisor} \right. \]

\[ \text{finite and étale over } S \text{ of degree } 2g + 2 \]

Since, by general results of Lonstead and Kleiman ([LK79]), a hyperelliptic family is a double cover of a family of \(\mathbb{P}^1\) ramified along a Cartier divisor \(D\) as in the definition above, there is a natural transformation of functors \(\Phi : H_g \to D_{2g+2}\). Moreover, since over an algebraically closed field giving a hyperelliptic curve \(C\) is equivalent to give the \((2g + 2)\)-points (up to isomorphism) where the \(2 : 1\) map \(C \to \mathbb{P}^1\) ramifies, both these moduli functor have \(H_g\) as coarse moduli scheme.
We prove (theorem 6.2) that \( D_{2g+2} \) is actually an algebraic stack isomorphic to a quotient stack, precisely
\[
D_{2g+2} = [B_{sm}(2,2g+2)/(PGL_2)],
\]
where \( B_{sm}(2,2g+2) \) is the projective space of smooth binary forms in 2 variables of degree \( 2g+2 \) and the action of \( PGL_2 \) is defined by \( [A] \cdot [f(x)] = [f(A^{-1} \cdot x)] \).

Using this description as a stack, we compute the Picard group of \( D_{2g+2} \) (giving an explicit generator) and prove that the natural pull-back map \( \text{Pic}(D_{2g+2}) \to \text{Pic}(H_g) \) is an isomorphism for \( g \) even and an injection of index 2 for \( g \) odd (theorem 6.3).

Next, after this digression into the study of the auxiliary functor \( D_{2g+2} \), we return to the study of the finess of \( H_g \) for \( H_g \) and \( D_{2g+2} \). Since the existence of automorphisms is always one of the most serious obstructions to the finess of a moduli scheme, we restrict to the open subset \( H^0_g \) of hyperelliptic curves with automorphism group reduced to the hyperelliptic involution (which we call hyperelliptic curves without extra-automorphisms) as well as to the corresponding open substack \( \mathcal{H}^0_g := H_g \times_{H^0_g} H^0_g \) and \( D^0_{2g+2} = D_{2g+2} \times_{H^0_g} H^0_g \).

The first result is that \( H^0_g \) is actually a fine moduli scheme for the functor \( D^0_g \), that is over \( H^0_g \) there exists a universal family of \( \mathbb{P}^1 \) together with a universal Cartier divisor \( D \) as above (theorem 6.3). On the other hand, there doesn’t exist over \( H^0_g \) a universal family of hyperelliptic curves, thus \( H^0_g \) is not fine for \( \mathcal{H}^0_g \). In fact, in theorem 6.3 we prove that the set of families of hyperelliptic curves (without extra-automorphisms) over \( S \) with a fixed modular map (if non empty) is a principal homogeneous space for \( H^0_g(S, \mathbb{Z}/2\mathbb{Z}) \).

Then we deal with the existence of a tautological family of hyperelliptic curves over an open subset of \( H_g \) and we prove that a tautological family exists over an open subset if and only if \( g \) is odd (theorem 6.12). This was stated in the exercise 2.3 of the book of Harris-Morrison ([HM88]) but with a mistake: they say universal family but in fact it’s only a tautological family by what said before!

The paper is organized as follows. In section 2 we establish some basic properties of families of \( \mathbb{P}^1 \). We prove that such a family is always locally trivial in the étale topology (prop. 2.2) and we give several equivalent conditions for the local triviality in the Zariski topology (prop. 2.1). Also a cohomological interpretation is provided in terms of the Brauer group of the base. Surely these results are well known to the specialists but we include them here for the lack of an adequate bibliographical reference and also because they will play a great role in what follows.

In section 3, we first recall some classical basic facts about families of hyperelliptic curves (proved in [LK71]): the existence of a global hyperelliptic involution and of a family of \( \mathbb{P}^1 \) for which the initial family of hyperelliptic curves is a double cover, also we discuss some main properties of the Weierstrass divisor. Then we treat the question of the existence of a global \( g^2 \) (see the text for the precise definition). First, we give a criterion for this existence in terms of Zariski triviality of the underlying family of \( \mathbb{P}^1 \) (prop. 3.3), and then we prove that such a global \( g^2 \) always exists if \( g \) is even while for \( g \) odd we give a procedure of constructing families without such global \( g^2 \) (theorem 3.5). These results were proved by Mestroni-Ramanan ([MRS85]) as an application of their results on Poincaré bundles for families of curves. However, we believe that our approach is simpler and quite elementary.

Section 4 deals with the moduli space \( H_g \) as well as the open subset \( H^0_g \). First we study the locus \( H_g \setminus H^0_g \) of curves with extra-automorphisms determining the unique component of maximal dimension \( g \) (proposition 4.1). Then we prove that \( H^0_g \) is the smooth locus of \( H_g \) except in the case \( g = 2 \) where there is a unique singular point corresponding to the curve \( y^2 = x^5 - x \) (proposition 4.3). After these preliminary results, we prove the two main theorems of this section: the determination of \( \text{Cl}(H_g) \)
in theorem 4.7 (which turns out to be isomorphic to Pic(H^0_g)) and the determination of Pic(H^0) in theorem 4.10.

Section 5 deals with the stack H_g of hyperelliptic curves. First we recall (including a sketch of their instructive proofs) the results of Arsie and Vistoli: the description of H_g as a quotient stack (theorem 5.1) and the computation of its Picard group (theorem 5.4). After that, we provide a functorial description of a generator of the Picard group (theorem 5.7) and a description of other elements that one can naturally consider (theorem 5.8).

In section 6 we discuss how far is the moduli functor H_g to be finely represented by H_g. We introduce the intermediate algebraic stack D_{2g+2}: we describe it as a quotient (theorem 6.2), compute its Picard group and compare it with the Picard group of H_g (theorem 6.3). Next we prove that D_{2g+2} is indeed finely represented by H^0_g (theorem 6.4) and, using this, we study how many families of hyperelliptic curves there can be with the same modular map (theorem 6.5). Finally we treat the existence of a tautological family of hyperelliptic curves over an open subset of H_g (theorem 6.12).

The final section contains an application of the results of the preceding section to families of hyperelliptic curves with dominant and generically finite modular map. We prove that if such a family admits a global g_1^2 (that is the case, for example, if the family admits a rational section, see [GV]), then the degree of the modular map should be even. This is the analog for hyperelliptic curves of a result of Caporaso ([Cap03]) for families of generic smooth curves. In a forthcoming paper ([GV]), the authors will prove an analogous result for trigonal curves and formulate a conjecture for n-gonal curves.

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2. Families of P^1

We will call a smooth projective family of curves p : C → S of genus 0 a family of P^1 (sometimes it’s called a twisted P^1_S [LK79] or a conic bundle [Cil86]). Any such family may be embedded into the projectivization of p_*(\omega_C^{-1}/S), which is a vector bundle of rank 3 on S. So we obtain every family of P^1 as a family of conics into a Zariski locally trivial family of P^2 (see [Cil86] page 12-14)).

If the base S is irreducible, the pull-back C_\eta of this family to the generic point \eta := Spec(k(S)) \rightarrow S is a form of P^1_\eta (i.e. a variety which becomes isomorphic to P^1 over the algebraic closure k(S)). After we take the embedding given by the anticanonical line bundle, C_\eta becomes isomorphic to a conic inside P^2_\eta. Recall that a conic is isomorphic to P^1 if and only if it has a rational point and surely it acquires a rational point after a separable extension of the base field of degree 2 (consider the field extension given by cutting the conic with a line of P^2 which intersects the conic in two distinct points).

We want to study when p : C → S is Zariski locally-trivial.

Proposition 2.1. For the family p : C → S over an irreducible and smooth base S (with generic point \eta), the following conditions are equivalent:

(1) C is S-isomorphic to P(V) for some vector bundle sheaf V on S of rank 2.
(2i) C → S is Zariski locally trivial.
(2ii) There exists an open non-empty U ⊂ S such that C_U ≅ U × P^1.
Proof. We will prove several implications.

- The implications \((*) \Rightarrow (**)\) for \(* = 2, 4, 5\) are evident.
- The equivalences \((**) \iff (***)\) for \(* = 2, 3, 4, 5\) follow from the usual property of the generic point.
- The implications \((1) \Rightarrow (2), (2i) \Rightarrow (3i), (4*) \Rightarrow (5*)\) (for \(* = i, ii, iii\)) are evident. \((3i) \Rightarrow (4i)\) follows form the fact that \(Im(\sigma)\) is the support of a divisor on \(C_U\) of vertical degree 1.
- \((5*) \Rightarrow (4*)\) (for \(* = i, ii, iii\)) follows from the fact that the relative canonical \(\omega_{C/S}\) has vertical degree \(-2\) so that, taking an appropriate linear combination of it with \(M\), we obtain an invertible sheaf \(L\) with vertical degree 1.
- \((4ii) \Rightarrow (4i)\) and \((5ii) \Rightarrow (5i)\) are true because, thank to the smoothness of \(S\) and \(p\) (and hence of \(C\)), we can always extend \(L_U\) (or \(M_U\)) to an invertible sheaf on all \(C\) (simply taking the closure in \(C\) of the Weil=Cartier divisor in \(C_U\) corresponding to it) and the vertical degree will remain the same since it’s locally constant and the base is connected.
- \((4i) \Rightarrow (1)\) (see \([\text{LevK79}]\) prop. 3.3): Since the fibers of \(p\) are \(\mathbb{P}^1\), we have that \(R^1p_*(L) = 0\) and \(p_*(L)\) is a locally free sheaf of rank 2. The natural map \(p^*(p_*(L)) \to L\) is surjective since its restriction to every geometric fiber is surjective. Hence it determines an \(S\)-map \(\Phi : C \to \mathbb{P}(p_*(L))\) that, being an isomorphism on the fibers, is an isomorphism.

However the situation is different in the étale topology.

**Proposition 2.2.** The family \(p : C \to S\) is locally trivial in the étale topology.

**Proof.** Consider the family as a family of conics inside \(\mathbb{P}(p_*(\omega_{C/S}^{-1}))\). For any point \(x \in S\) we may choose a Zariski neighborhood \(U\) over which \(\mathbb{P}(p_*(\omega_{C/S}^{-1}))\) is trivial, i.e. there is an inclusion \(C_U \subset \mathbb{P}^2 \times U\). Choose a line \(l \subset \mathbb{P}^2\) that intersect the conic \(C_x\) in two different points. So there is an étale double cover over some smaller Zariski neighborhood \(x \in V \subset U\) corresponding to the intersection of \(C_U \cap (l \times V)\), over which the pull back of \(C_V\) is a Zariski locally trivial family of \(\mathbb{P}^1\) by \([2.1]\) since it has a section.

There is a cohomological interpretation of this geometric picture. The family \(C \to S\) defines a class in \(H^2_{et}(S, PGL_2(O_S))\) by proposition \([2.2]\) and the family is Zariski locally trivial if and only if it comes from \(H^2_{Zar}(S, PGL_2(O_S))\). The short exact sequence of sheaves

\[
1 \to \mathcal{O}^*_S \to GL_2(O_S) \to PGL_2(O_S) \to 1
\]
gives rise to the following two exact sequences of sheaves (for the Zariski and the étale topology)

\[ H^1_{\text{Zar}}(S, \mathcal{O}_S^*) \rightarrow H^1_{\text{Zar}}(S, GL_2(\mathcal{O}_S)) \rightarrow H^1_{\text{Zar}}(S, PGL_2(\mathcal{O}_S)) \rightarrow H^2_{\text{Zar}}(S, \mathcal{O}_S^*) \]

\[ H^1_{\text{ét}}(S, \mathcal{O}_S^*) \rightarrow H^1_{\text{ét}}(S, GL_2(\mathcal{O}_S)) \rightarrow H^1_{\text{ét}}(S, PGL_2(\mathcal{O}_S)) \rightarrow H^2_{\text{ét}}(S, \mathcal{O}_S^*). \]

It’s well known that \( H^1_{\text{Zar}}(S, \mathcal{O}_S^*) = H^1_{\text{ét}}(S, \mathcal{O}_S^*) = \text{Pic}(S) \) (see [Mil, III, pag. 4.9]) and by descent theory (see [Mil, III, sect. 4]), there is an equality \( H^1_{\text{Zar}}(S, GL_2(\mathcal{O}_S)) = H^1_{\text{Zar}}(S, GL_2(\mathcal{O}_S)) \). Moreover for the regular scheme \( S \) the sheaf \( \mathcal{O}_S^* \) has a flasque resolution in the Zariski topology:

\[ 1 \rightarrow \mathcal{O}_S^* \rightarrow k(S)^* \rightarrow \bigoplus_{Y \in X^{(1)}} (i_Y)_* (\mathbb{Z}) \rightarrow 0, \]

where \( X^{(1)} \) denotes all the schematic points of codimension 1, and \( i_Y \) denotes the corresponding closed embedding. So \( H^2_{\text{Zar}}(S, \mathcal{O}_S^*) = 0. \) Thus we obtain two exact sequences:

(2.1) \( 0 \rightarrow \text{Pic}(S) \rightarrow H^1_{\text{Zar}}(S, GL_2(\mathcal{O}_S)) \rightarrow H^1_{\text{Zar}}(S, PGL_2(\mathcal{O}_S)) \rightarrow 0 \)

(2.2) \( 0 \rightarrow H^1_{\text{Zar}}(S, PGL_2(\mathcal{O}_S)) \rightarrow H^1_{\text{ét}}(S, PGL_2(\mathcal{O}_S)) \rightarrow H^2_{\text{ét}}(S, \mathcal{O}_S^*). \)

The first sequence says that every Zariski locally trivial family of \( \mathbb{P}^1 \) is the projectivization of a rank 2 vector bundle (proposition 2.1) while the second says that a family of \( \mathbb{P}^1 \) over \( S \) defines an element in the Brauer group of \( S \) which is trivial if and only if this family is Zariski locally trivial.

The same cohomological arguments work over the generic point Spec\( (k) \), where \( K = k(S) \). Hence the exact sequence (2.2) can be completed in the following way

\[ 0 \rightarrow H^1_{\text{Zar}}(S, PGL_2(\mathcal{O}_S)) \rightarrow H^1_{\text{ét}}(S, PGL_2(\mathcal{O}_S)) \rightarrow H^2_{\text{ét}}(S, \mathcal{O}_S^*) \]

\[ 0 \rightarrow H^1(Gal(K), PGL_2(\mathbb{K})) \rightarrow \text{Br}(K). \]

Since the map \( H^2_{\text{ét}}(S, \mathcal{O}_S^*) \rightarrow \text{Br}(k(S)) \) is injective (because \( S \) is smooth, see [Mil, III, Ex. 2.22]), this diagram says exactly that a family of \( \mathbb{P}^1 \) which is trivial on the generic point is Zariski locally trivial (see proposition 2.1).

Let us conclude this section with an example of a non-Zariski locally trivial family of \( \mathbb{P}^1 \).

**Example.** Consider the universal conic \( \mathcal{C} \rightarrow S \) where \( S \subset H^0(\mathbb{P}^2, \mathcal{O}(2)) \) is the open set of all smooth conics in \( \mathbb{P}^2 \). This family is canonically embedded into \( \mathbb{P}^2 \times S \) and is a non Zariski locally trivial family of forms of \( \mathbb{P}^1 \), i.e. it defines a non-trivial element in \( \text{Br}(k(S)) \) (see [Cil86, pag 16]).

### 3. Generalities about families of hyperelliptic curves

In this section, we recall first some known results about families of hyperelliptic curves \( \pi : \mathcal{F} \rightarrow S \), that are projective smooth morphisms whose geometric fibers are hyperelliptic curves of genus \( g \). Recall that we assume throughout this work that \( g \geq 2 \) even though many things remain true for \( g = 1 \) if one consider 1-pointed elliptic curves and family of elliptic curves endowed with a section (see [Mum65] for a detailed discussion of the elliptic case). Also recall that we work over an algebraically closed field \( k \) of characteristic different from 2 (to avoid problems with double covers).
Theorem 3.1. ([LK79] Theorem 5.5) For a family \( \pi : \mathcal{F} \to S \) of hyperelliptic curves, the following conditions hold (and characterize the hyperelliptic families):

(i) \( \mathcal{F} \) admits a global hyperelliptic involution \( i \), namely an involution over \( S \) which induces the hyperelliptic involution on every geometric fiber.

(ii) There exists a well-defined finite, surjective \( S \)-morphism \( h : \mathcal{F} \to C \) of degree 2, for a certain family \( p : C \to S \) of \( \mathbb{P}^1 \) that restricts on each fiber to taking quotient w.r.t. the hyperelliptic involution. Moreover the family \( p : C \to S \) is uniquely determined up to \( S \)-isomorphisms.

(iii) The morphism \( h \) may be also described as the canonical morphism \( f : \mathcal{F} \to \mathbb{P}(p_*\omega_{\mathcal{F}/S}) \) whose image is isomorphic to a family \( p : C \to S \) of \( \mathbb{P}^1 \).

(iv) There exists a faithfully flat morphism \( T \to S \) and a finite faithfully flat \( T \)-morphism \( \mathcal{F}_T \to \mathbb{P}^1_T = \mathbb{P}^1 \times T \) of degree 2.

Lunstad and Kleiman studied also the Weierstrass subscheme \( W_{\mathcal{F}/S} \) of \( \mathcal{F} \to S \), namely the ramification divisor of the 2 : 1 of the \( S \)-map \( h : \mathcal{F} \to C \) of theorem 3.1(ii) endowed with the scheme structure defined by the 0-th Fitting ideal of \( \Omega^1_{\mathcal{F}/C} \).

Note that this is isomorphic to the branch divisor \( D := h(W_{\mathcal{F}/S}) \) on \( C \) of the map \( h \).

Theorem 3.2. ([LK79] Prop. 6.3, Prop. 6.5, Cor. 6.8, Theo. 7.3) The Weierstrass subscheme \( W_{\mathcal{F}/S} \subset \mathcal{F} \) of the family of hyperelliptic curves \( \mathcal{F} \to S \) satisfies the following:

(i) \( W_{\mathcal{F}/S} \) is the subscheme associated to an effective Cartier divisor on \( \mathcal{F} \) relative to \( S \).

(ii) \( W_{\mathcal{F}/S} \) is equal to the fixed point subscheme of \( \mathcal{F} \) with respect to the global hyperelliptic involution \( i \).

(iii) \( W_{\mathcal{F}/S} \) is finite and étale (since \( \text{char}(k) \neq 2 \) over \( S \) of degree \( 2g + 2 \).

(iv) If \( S \) is reduced, then a section \( \sigma \) of \( \pi : \mathcal{F} \to S \) is a Weierstrass section (i.e. \( \sigma(s) \) is a Weierstrass point of \( \mathcal{F}_s \) for every geometric point \( s \in S \)) if and only if it factors through \( W_{\mathcal{F}/S} \).

Remark 3.3. By the preceding results, a family \( \mathcal{F} \to S \) of hyperelliptic curves of genus \( g \) determines a family \( \mathcal{C} \to S \) of \( \mathbb{P}^1 \) together with a branch (Cartier) divisor \( D \) on \( \mathcal{C} \) which is finite étale over \( S \) of degree \( 2g + 2 \). Viceversa, by the classical theory of cyclic covers (see for example [Par91] or [AV04]), given the family \( \mathcal{C} \to S \) and the divisor \( D \) as above, we can construct a double \( S \)-cover of \( \mathcal{C} \) ramified exactly over \( D \) (which will be automatically a family of hyperelliptic curves of genus \( g \)) if and only if the Cartier divisor \( D \) is divisible by 2 in the Picard group of \( \mathcal{C} \).

An interesting problem for families of hyperelliptic curves is the existence of a global \( g^1_2 \), namely of an invertible sheaf \( G^1_2 \) on \( \mathcal{F} \) that restricts on every fiber of \( \mathcal{F} \to S \) to the hyperelliptic line bundle \( g^1_2 \). Clearly this \( G^1_2 \) is well-defined only up to tensoring with the pull-back of line bundles coming from \( S \) (see [Cil86, Lemma 2.1]). Although the uniqueness of the \( g^1_2 \) on every fiber of \( \mathcal{F} \to S \) could lead to think that a \( G^1_2 \) always exists, this is actually not the case! This strange phenomenon was already observed by N. Mestrano and S. Ramanan ([MR85, section 3]) as an application of their results on Poincaré bundles for families of hyperelliptic curves. Here we propose a different approach (simpler, as we believe) that is based on the following:

Proposition 3.4 (Criterion for the existence of a \( G^1_2 \)). Let \( \pi : \mathcal{F} \to S \) be a family of hyperelliptic curves and let \( p : \mathcal{C} \to S \) be a family of \( \mathbb{P}^1 \) corresponding to \( \mathcal{F} \). Assume \( S \) is smooth and irreducible with generic point \( \eta = \text{Spec}(k(S)) \). Then the following conditions are equivalent:

(i) There exists a \( G^1_2 \) on \( \mathcal{F} \).
(ii) There is a non-empty open subset $U \subset S$ such that the restriction $F_U \to U$ admits a $G^1_{2|U}$.

(iii) The hyperelliptic curve $F_\eta$ admits a $g^1_2$ defined over $k(S)$.

(iv) $p : C \to S$ is a Zariski locally trivial family of $\mathbb{P}^1$.

Proof. We will prove the following equivalences:

- $(i) \iff (ii)$: $(i) \Rightarrow (ii)$ is clear. Let’s prove the converse. Since $S$ and $\pi$ are smooth (and hence also $F$), we can extend the line bundle $G^1_{2|U}$ on $F_U$ to a line bundle $G^1_2$ on $F$ (simply take the closure of the Cartier=Weyl divisor associated to it) which will have vertical degree 2 everywhere (the vertical degree is locally constant and $S$ is irreducible). Now, by the semicontinuity of $h^0$ (see [Har, III.12.8]), $h^0(F_s, G^1_{2|F_s}) \geq 2$ for every geometric point of $S$. On the other hand for any non-zero effective divisor $E$ on an algebraic curve $C$ there is an inequality $h^0(C, O_C(E)) \leq \deg(E)$ and so in our case the equality holds. But then $G^1_{2|F_s}$ is the $g^1_2$ on the hyperelliptic curve $F_s$ (for every $s$) because this is the unique linear system of degree 2 and dimension 1.

- $(ii) \iff (iii)$: Clear from the usual property of the generic point.

- $(iii) \iff (iv)$: Consider the diagram over the generic point $k(S)$

\[
\begin{array}{ccc}
F_\eta & \xrightarrow{h} & C_\eta \\
\pi \downarrow & & \downarrow p \\
& k(S) & \\
\end{array}
\]

In view of proposition 2.1, we have to prove that $F_\eta$ has a $g^1_2$ defined over $k(S)$ if and only if $C_\eta \cong \mathbb{P}^1$. Now, if $C_\eta \cong \mathbb{P}^1$ then $h^*(O_{\mathbb{P}^1}(1))$ provides the required $g^1_2$ on $F_\eta$. Conversely, if the $g^1_2$ of $F_\eta$ is defined over $k(S)$ then $V := \pi_*(g^1_2) = H^0(g^1_2)$ is a vector space over $k(S)$ of dimension 2 and, by construction, $C_\eta \cong \mathbb{P}(V) = \mathbb{P}^1$.

Now using this criterion, we can analyze the existence of a global $g^1_2$ (which we call $G^1_2$) for families of hyperelliptic curves.

**Theorem 3.5.** Let $F \to S$ be a family of hyperelliptic curves of genus $g$ and let $C \to S$ be the associated family of $\mathbb{P}^1$. Then the following holds:

- (i) $2G^1_2$ (namely an invertible line bundle that restricts to twice the $g^1_2$ on every fiber of the family) is always defined globally on $F \to S$.

- (ii) If $g$ is even, a $G^1_2$ is defined globally, or equivalently by the criterion 3.4, the associated family $C \to S$ is Zariski locally trivial.

- (iii) Vice versa, for $g$ odd, given any family of $\mathbb{P}^1$ (maybe not Zariski locally trivial) and a divisor étale and finite of degree $2g + 2$ over the base, it possible, after restricting the base to an open subset, construct above it a family of hyperelliptic curves of genus $g$. Hence if we start with a family of $\mathbb{P}^1$ non Zariski locally trivial, the resulting family of hyperelliptic curves will not admit a $G^1_2$ by criterion 3.4.
Proof. Keep in mind the following diagram

$$\begin{array}{ccc}
F & \xrightarrow{h} & C \\
\downarrow & & \downarrow \\
\pi & & \rightarrow \\
\downarrow & & \downarrow \\
S & \xleftarrow{p} & D
\end{array}$$

(i) Since $\omega_{C/S}^{-1}$ restrict to $O(2)$ on every fiber of $p$, the pull-back $h^*(\omega_{C/S}^{-1})$ restricts to $2g_2^1$ on every fiber of $\pi$ and hence it’s the desired $2G_2^1$.

(ii) By the remark $3.3$, the Cartier divisor $D$ is divisible by 2 in the Picard group of $C$. This means that there exists a line bundle on $C$ of the vertical degree $g+1$, which is odd since $g$ is even. But then by proposition $2.1(5i)$, $C \to S$ is Zariski locally trivial and by the criterion $3.4$ there exists a $G_2^1$ on $F \to S$.

(iii) Let $p : C \to S$ a family of $\mathbb{P}^1$ and let $D$ a divisor above it that is étale and finite over $S$ of degree $2g+2$. Clearly $O_C(D)$ and $(\omega_{C/S}^{-1})^2$ have the same degree on the fibers so that:

$$O_C(D) \otimes (\omega_{C/S}^{-1})^2 = p^*(L)$$

for some line bundle coming from the base. Taking an open subset $U$ of $S$ such that $L|_U$ is trivial, we get that $D$ is a square in the Picard group of $C_U$ and therefore, by remark $3.3$ we can construct the required family of hyperelliptic curves.

$\square$

4. Moduli space of hyperelliptic curves and its Picard group

Recall that the moduli scheme $H_g$ parameterising isomorphism classes of hyperelliptic curves is an integral subscheme of $M_g$ of dimension $2g-1$ that can be realized as

$$(4.1) \quad H_g = (\text{Sym}^{2g+2}(\mathbb{P}^1) - \Delta)/\text{PGL}_2$$

where $\text{Sym}^{2g+2}(\mathbb{P}^1)$ is the $(2g+2)$-th symmetric product of $\mathbb{P}^1$, $\Delta$ is the closed subset where at least two points coincide and the action of $\text{PGL}_2$ comes from the natural action on $\mathbb{P}^1$.

Equivalently, since we can identify the $(2g+2)$-th symmetric product of $\mathbb{P}^1$ as the projective space $\mathbb{B}(2, 2g+2)$ of binary forms of degree $2g+2$ in two variables, we have the alternative description

$$(4.2) \quad H_g = \mathbb{B}_{sm}(2, 2g+2)/\text{PGL}_2$$

where $\mathbb{B}_{sm}(2, 2g+2)$ denotes the open subset of smooth binary forms (i.e. with all the roots distinct) and the action of $\text{PGL}_2$ is defined as $[A] \cdot [f(x)] = [f(A^{-1}x)]$.

We indicate with $H_0^g$ the open subset of $H_g$ consisting of hyperelliptic curves with no extra-automorphisms apart from the hyperelliptic involution. Let $\mathbb{B}_{sm}(2, 2g+2)^0$ denote the preimage of $H_0^g$ inside $\mathbb{B}_{sm}(2, 2g+2)$.

Let us remark that all the points of $\mathbb{B}_{sm}(2, 2g+2)$ are stable for the action of $\text{PGL}_2$ and with finite stabilizers (see [311] prop. 4.1]), so that the quotient $\pi : \mathbb{B}_{sm}(2, 2g+2) \to \mathbb{B}_{sm}(2, 2g+2)/\text{PGL}_2 = H_g$ is a geometric quotient. Moreover, the action is free exactly on $\pi^{-1}(H_0^g) = \mathbb{B}_{sm}(2, 2g+2)^0$, i.e. on the forms whose corresponding $(2g+2)$-uples of points don’t have non-trivial automorphisms.
In the next proposition, we determine the dimension of the closed subset $H^\text{aut}_g = H_g \backslash H^0_g$ of hyperelliptic curves with extra-automorphisms and study the component of maximal dimension.

**Proposition 4.1.** The locus $H^\text{aut}_g = H_g \backslash H^0_g$ has dimension $g$ (and hence codimension $g-1$). Moreover it has a unique irreducible component of maximal dimension made by curves which have an extra-involution (besides the hyperelliptic one), acting on the $2g + 2$ ramification points as a product of $g + 1$ commuting transpositions.

**Proof.** The automorphism group $\text{Aut}(C)$ of a hyperelliptic curve $C$ always contains the hyperelliptic involution $i$ as a central element. Consider the group $G = \text{Aut}(C)/(i)$. There is a canonical inclusion inside the symmetric group $G \subset S_{2g+2}$, since every automorphism of a hyperelliptic curve must preserve the ramification divisor. Hence the variety $H^\text{aut}_g$ decomposes into the strata

$$H^\text{aut}_g = \bigcup_{\text{primes } p \leq 2g+2} H^\text{aut, }p_g,$$

where $H^\text{aut, }p_g$ denotes the set of hyperelliptic curves such that there exists an element of order $p$ in the corresponding group $G$. There is a canonical finite map $H^\text{aut, fixed}_g \rightarrow H^\text{aut, }p_g$, where $H^\text{aut, fixed}_g$ is the moduli space of isomorphism classes of pairs: a curve $C$ from $H^\text{aut}_g$ and a fixed element $\sigma$ of order $p$ in $G$.

Since $\sigma \in G$ is induced by an automorphism of $\mathbb{P}^1$ preserving the ramification divisor, we see that in fact $H^\text{aut, fixed}_g$ is the moduli space of isomorphism classes of pairs consisting of an automorphism $\tau$ of $\mathbb{P}^1$ of order $p$ and a reduced effective divisor $D$ of degree $2g+2$ on $\mathbb{P}^1$, stable under $\tau$. Now consider the natural quotient map

$$\pi: \mathbb{P}^1 = \mathbb{P}^1_1 \rightarrow \mathbb{P}^1_2 = \mathbb{P}^1/\langle \tau \rangle.$$

The fact that $p$ is prime and the Riemann–Hurwitz formula imply that there is only one opportunity for the ramification structure of $\pi$: a cyclic ramification of order $p$ at two points $x_1, x_2 \in \mathbb{P}^1_1$. Moreover, there are three opportunities for the divisor $D \subset \mathbb{P}^1_1$:

1) $D$ contains no points among $x_1$ and $x_2$, 
2) $D$ contains only one point among $x_1$ and $x_2$, 
3) $D$ contains both points $x_1$ and $x_2$.

Thus we get one more stratification:

$$H^\text{aut, fixed}_g = \bigcup_{i=0,1,2} H^\text{aut, fixed, }i_g$$

according to the three cases above.

It is easy to see that in fact $H^\text{aut, fixed, }i_g$ is parametrizing isomorphism classes of pairs, consisting of two non-intersecting reduced effective divisors of degrees $2$ and $(2g+2-i)/p$ on the projective line $\mathbb{P}^1_1$ (in this case $2g+2-i$ must be divisible by $p$). Thus, since each such configuration of points on $\mathbb{P}^1_1$ has a finite stabilizer in the automorphism group $PGL_2$, we get the equality

$$\dim H^\text{aut, fixed, }i_g = 2 + \frac{2g+2-i}{p} - 3 = \frac{2g+2-i}{p} - 1.$$

Now notice that the case $p = 2$ and $i = 1$ is impossible because of the divisibility condition. Further, if $p \geq 3$, or $p = 2$ and $i = 2$, then

$$\frac{2g+2-i}{p} - 1 \leq \frac{2g+2}{3} - 1 \leq g - 1,$$

or

$$\frac{2g+2-2}{2} - 1 = g - 1,$$
automorphism group $\text{Aut}(C)$.

Consider few particular cases. The inertia group of all the ramification points should be the same, namely $p$. If $C$ is geometrically the condition above means that the curve $\tilde{C}$ has an element $\tilde{\sigma}$ in the automorphism group $\text{Aut}(C)$ itself (not only in $G$). Indeed, consider the composition

$$\varphi : C \xrightarrow{\tilde{\sigma}} \mathbb{P}^1_{\mathbb{F}_p} \xrightarrow{\varphi_1} \mathbb{P}^1_{\mathbb{F}_p}.$$  

This map is a Galois map of degree 4 with Galois group $H$ generated in $\text{Aut}(C)$ by any preimage $\tilde{\sigma} \in \text{Aut}(C)$ of $\sigma \in G$ and $i$. Moreover, it is easy to see that the ramification of $\varphi$ is formed only by pairs of double points. If $H \cong \mathbb{Z}/4\mathbb{Z}$, then the inertia group of all the ramification points should be the same, namely $(i)$. This would mean that the map $\pi : \mathbb{P}^1_{\mathbb{F}_p} = C/(i) \to \mathbb{P}^1_{\mathbb{F}_p}$ should be unramified, that is actually not true. Hence $H \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, and so $\sigma \in G$ is of order two.

Viceversa, if $\text{Aut}(C)$ has an element $\sigma \neq i$ of order 2 then $i = 0$, otherwise $\varphi$ would have a point from $D \setminus \{x_1, x_2\}$, having ramification of order 4, contradicting with the isomorphism $H \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Note that $H^2_{2, \text{fixed}, 0}$ is irreducible and moreover, from the explicit geometric description of the ramification of the covering $C \to \mathbb{P}^1_{\mathbb{F}_p}$, it follows that $\sigma \in G \subset S_{2g+2}$ must be the product of $g+1$ commuting transpositions. Thus we get the required statement.

There exists a combinatorial proof of a weaker variant of proposition 4.1 that we will describe now.

**Proposition 4.3**’. The closed subset $H^g_{\text{aut}} = H^g_{\text{aut}} \setminus H^0_{\text{aut}}$ of hyperelliptic curves with extra-automorphisms has codimension at least 2 for $g \geq 2$, and is of codimension 1 for $g = 2$. Moreover, in latter case the divisorial component is formed by hyperelliptic curves of genus 2 with an extra involution, whose action on the six ramification points is conjugated to $(12)(34)(56)$.

**Proof.** The main ingredient of the proof is the following purely combinatorial lemma.

**Lemma 4.2.** Let $\rho : M \to M$ be a permutation of the finite set $M$, whose cardinality is at least 6. Suppose that $\rho$ has at most two fixed points, and for $|M| = 6$ the permutation $\rho$ is not conjugated to $(12)(34)(56)$. Then there are two 4-tuples $N_1, N_2 \subset M$ such that

$$|N_1 \cap \rho(N_1)| \neq |N_2 \cap \rho(N_2)|,$$

and $|N_i \cap \rho(N_i)| < 4$ for $i = 1, 2$ (here $|\cdot|$ denotes the cardinality of a set).

**Proof.** We treat different cases according to the cycle decomposition of $\rho$. First, we bound the length of cycles of $\rho$, then we bound their number, and finally we consider few particular cases.

**Case 1** Suppose that there exists at least one cycle of length at least 4, i.e. there exists $x \in M$ such that $x_1 = x, x_2 = \rho(x), x_3 = \rho^2(x)$ and $x_4 = \rho^3(x)$ are all different. Take two arbitrary elements $y, z \in M \setminus \{x_1, x_2, x_3, x_4\}$. One can check that $N_1 = \{x_1, x_2, y, z\}$ and $N_2 = \{x_1, x_3, y, z\}$ fit both conditions of lemma 4.2 having $k_1 = |\{x_1, y, z\} \cap \{\rho(y), \rho(z)\}|, k_2 = |\{x_1, y, z\} \cap \{\rho(y), \rho(z)\}| + 1 \leq 3.$
Case 2 Assume that there are at least four cycles. Let us take elements $x_i \in M$, $i = 1, 2, 3, 4$ to be in different cycles, and such that $\rho(x_1) \neq x_1, \rho(x_2) \neq x_2$. Let $l \leq 2$ be equal to the number of fixed points among $x_3$ and $x_4$. Then for $N_1 = \{x_1, x_2, x_3, x_4\}$ and $N_2 = \{x_1, \rho(x_1), x_3, x_4\}$ there are equalities $k_1 = l$ and $k_2 = l + 1$ or $l + 2$, if $\rho^2(x_1) \neq x_1$ or $\rho(x_1) = x_1$, respectively. For the case, when $\rho(x_1) = x_1$ and $l + 2 = 4$, i.e. when both points $x_3$ and $x_4$ are fixed, take $N_2 = \{x_1, \rho(x_1), x_2, x_3\}$. Then $k_2 = 3$.

Case 3 Now let us suppose that there are not more than three cycles of length at most 3. If there are two cycles $(x_1, x_2, x_3)$ and $(y_1, y_2, y_3)$ of length 3, then put $N_1 = \{x_1, x_2, x_3, y_1\}$, $N_2 = \{x_1, x_2, y_1, y_2\}$, having $k_1 = 3$, $k_2 = 2$. Otherwise from the conditions on $\rho$ and $M$ we conclude that the lengths of cycles could be equal to $\{3, 2, 2\}$ or $\{3, 2, 1\}$. In that case consider the first cycle $(x_1, x_2, x_3)$ and two points $y, z$ from another two cycles such that $\rho(y) \neq y$. For $N_1 = \{x_1, x_2, x_3, y\}$ and $N_2 = \{x_1, x_2, y, z\}$ we get $k_1 = 3$ and $k_2 = 1$ or $k_2 = 2$.

Now let $\sigma$ be an auxiliary automorphism of the hyperelliptic curve $C$ such that the induced permutation of $2g + 2$ ramification points is not conjugated to $(12)(34)(56)$ for $g = 2$. Then by lemma 4.2 there are two 4-tuples $N_1$ and $N_2$, consisting of ramification points, such that

$$k_1 = |N_1 \cap \rho(N_1)| \neq |N_2 \cap \rho(N_2)| = k_2,$$

and $k_i < 4$. Thus the point $x_C$ in $\text{Sym}^{2g+2}(\mathbb{P}^1) - \Delta = \mathbb{P}^6 - \Delta$, corresponding to $C$, must lie in both closed subsets $D_{k_1}$ and $D_{k_2}$, defined in the following way: $D_k$ consists of points $x \in \text{Sym}^{2g+2}(\mathbb{P}^1) - \Delta$ such that in the corresponding $(2g+2)$-tuple of points on $\mathbb{P}^1$ there are two 4-tuples with the same double ratio and intersecting by $k$ points.

**Lemma 4.3.** The closed subset $D_k$ is a divisor, for $k \neq 4$. Moreover, if $k_1 \neq k_2$, and $k_i < 4$ for $i = 1, 2$, then $D_{k_1}$ and $D_{k_2}$ intersect transversely.

**Proof.** Suppose that $k_1 > k_2$, and $x \in D_{k_1} \cap D_{k_2}$. For a $(2g+2)$-tuple $M_x$ of points on $\mathbb{P}^1$, corresponding to $x$, there exists at least one pair of 4-tuples $(N_1, N_2)$ with the same double ratio, such that $|N_1 \cap N_2| = k_1$. Consider also all pairs $(L_1^i, L_2^i)$ of 4-tuples in $M_x$ with the same double ratio, such that $|L_1^i \cap L_2^i| = k_2$. For any $i$ there exists a point $z_i \in M_x$ such that $z_i \in L_1^i \cup L_2^i$, but $z_i \notin N_1 \cup N_2$, since $|L_1^i \cup L_2^i| = 8 - k_2 > 8 - k_1 = |N_1 \cup N_2|$. So for each point $x \in D_{k_1} \cap D_{k_2}$ and for each Zariski neighborhood $x \in U \subset D_{k_1}$ there exists a point $y \in U$ not belonging to $D_{k_2}$: just slightly move all the points $z_i$ in an independent way. This provides the transversality of the intersection $D_{k_1} \cap D_{k_2}$.

From lemma 4.3 we get that the set of hyperelliptic curves with auxiliary automorphisms is contained inside the closed subset

$$\bigcup_{0 \leq k_1 < k_2 < 4} (D_{k_1} \cap D_{k_2}),$$

which is of codimension 2, if $g \geq 3$, or $\rho$ is not conjugated to $(12)(34)(56)$ for $g = 2$.

Now consider the case, when $g = 2$ and $\rho$ is conjugated to $(12)(34)(56)$. The dimension of the moduli space of such hyperelliptic curves (i.e. hyperelliptic curves of genus 2 having an auxiliary automorphism with the action of type $(12)(34)(56)$ on the ramification points) is equal to 5=3+2: six ramification points are uniquely defined by three of them and also the involution in $PGL_2$ that is a two-dimensional space, since the involutions are parameterized by a couple of their fixed points. Thus we get the desired statement.
Remark 4.4. For the moduli space $M_g$ of curves of genus $g$ (with $g \geq 3$), the locus $M_g^{\text{ut}}$ of curves with non-trivial automorphisms is a closed subset of dimension $2g - 1$ and it has a unique irreducible component of maximal dimension made by hyperelliptic curves.

Note that $H_g$ is a normal variety since it is the quotient of a normal variety by the action of a group. We determine its smooth locus.

**Proposition 4.5.** If $g \geq 3$, then the smooth locus of $H_g$ is $H_g^0$. On the other hand, assuming $\text{char}(k) \neq 5$, $H_2$ has a unique singular point corresponding to the hyperelliptic curve $y^2 = x^6 - x$.

**Proof.** We will use a smoothness criterion of K. Lonsted ([Lon80]) for quotients of smooth varieties by finite groups, that we now briefly recall.

Let $X$ be a smooth $k$-variety, $\Gamma \subset \text{Aut}_k(X)$ a finite subgroup, and set $Y = X/\Gamma$. Let $P$ be a point in $X$ with image $Q$ in $Y$. Let’s denote with $\Gamma(P)$ the inertia group at $P$ and let $\Gamma'(P)$ be the subgroup of $\Gamma(P)$ generated by all the pseudoreflections, that is, the elements of $\Gamma(P)$ that leave a hypersurface through $P$ pointwise fixed. Then one has the following:

(i) If $Q$ is a smooth point, then $\Gamma(P) = \Gamma'(P)$.

(ii) Conversely, if $\Gamma(P) = \Gamma'(P)$ and the order of $\Gamma(P)$ is prime with $\text{char}(k)$, then $Q$ is a smooth.

Note that $H_g$ can be realized as a quotient of a smooth variety by a finite group in the following way. Given a $(2g + 2)$-tuple of ordered distinct points of $\mathbb{P}^1$, acting with an element of $\text{PGL}_2$ we can assume that the first three points of it are $0, \infty, 1$. Hence

$$H_g = \left( (\mathbb{P}^1 - \{0, \infty, 1\})^{2g-1} - \Delta \right)/\text{S}_{2g+2}$$

where $\Delta$ is the locus where at least two points coincide and the action of an element $\sigma$ in $\text{S}_{2g+2}$ on an ordered $(2g-1)$-tuple $\{x_1, \ldots, x_{2g-1}\}$ is obtained first letting $\sigma$ act in the natural way on the $(2g+2)$-tuple $\{0, \infty, 1, x_1, \ldots, x_{2g-1}\}$ and then applying the element of $\text{PGL}_2$ that sends the first three elements into $\{0, \infty, 1\}$ and taking the remaining $(2g - 1)$ points.

Apply the preceding smoothness criterion with $X = (\mathbb{P}^1 - \{0, \infty, 1\})^{2g-1} - \Delta$ and $\Gamma = \text{S}_{2g+2}$. If $g \geq 3$, proposition [4.1] implies that there aren’t non-trivial pseudoreflections. In fact such a non-trivial pseudoreflection would imply the existence of a hypersurface on $X$ made by points having non-trivial stabilizer and, passing to the quotient, this would give a codimension 1 locus of hyperelliptic curves with extra-automorphisms contradicting proposition [4.1]. Hence, by the criterion, a point on the quotient $H_g$ is non-singular if and only if it comes from a point above with trivial stabilizer, hence if and only if it belongs to $H_g^0$.

If $g = 2$, this argument fails because in that case the elements of $\text{S}_6$ conjugated to (12)(34)(56) are pseudoreflections (and by proposition [4.1] these are the only ones). In this case we can use, instead, an explicit description of Igusa (see [Igu60]) which showed that (under the hypothesis $\text{char}(k) \neq 5$):

$$M_2 = H_2 = \text{Spec}(k[z_1, z_2, z_3] \langle \zeta_5 \rangle) = \mathbb{A}_k^3/(\mathbb{Z}/5\mathbb{Z})$$

where the action of the 5-th root of unity $\zeta_5$ is given by $z_i \mapsto \zeta_5^iz_i$, $i = 1, 2, 3$ and the origin corresponds to the hyperelliptic curve defined by the equation $y^2 = x^6 - x$.

It is well-known that the origin in $\mathbb{A}_k^3$ is mapped to the singular point on $H_2$. Also we could get it applying the smoothness criterion to this quotient, since in this case there aren’t pseudoreflections, it follows that the only singularity of the quotient is the point corresponding to the curve $y^2 = x^6 - x$. \qed
Remark 4.6. Compare this result with the determination of the smooth locus of $M_g$ for $g \geq 3$. In this case it holds that $M_g^{smooth} = M_g^0$ for $g \geq 4$ (i.e. exactly when the locus of curves with automorphisms has codimension greater than 1) while $M_g^{smooth} = M_g^0 \cup H^0$ (see [Pau69] for an analytic proof over the complex numbers, [Pop69] for an algebraic proof in the case $g \geq 4$, [Oor74] for an algebraic proof in the case $g = 3$ and finally [Lon84] for an algebraic unified treatment of the cases $g = 3$ and $g \geq 4$ based on its smoothness criterion [Lon80]).

We want now to compute the Picard groups (i.e. the group of Cartier divisors modulo linear equivalence) and the divisor class groups (i.e. the group of Weyl divisors modulo linear equivalence) of $H_g$ and of $H^0_g$, away from some bad characteristic of the base field $k$. Note that since $H_g$ is a normal variety we have an inclusion $\text{Pic}(H_g) \hookrightarrow \text{Cl}(H_g)$; on the other hand, $H^0_g$ is smooth (see proposition 4.5) and hence $\text{Pic}(H^0_g) = \text{Cl}(H^0_g)$.

Theorem 4.7. Suppose that $\text{char}(k)$ doesn’t divide $2g + 2$. The Picard group of $H_g^0$ is equal to

$$\text{Pic}(H^0_g) = \begin{cases} \mathbb{Z}/(4g + 2)\mathbb{Z} & \text{if } g \geq 3 \\ \mathbb{Z}/5\mathbb{Z} & \text{if } g = 2. \end{cases}$$

Moreover, under the additional hypothesis that $\text{char}(k) \neq 5$ if $g = 2$, the natural restriction map $\text{Cl}(H_g) \to \text{Cl}(H^0_g) \cong \text{Pic}(H^0_g)$ is an isomorphism.

Proof. We will use the theory of equivariant Picard group (see [GIT 1.3] and also [EG98]) whose definition we now briefly recall. Given an action of an algebraic group $G$ on a algebraic variety $X$, $\sigma : G \times X \to X$, the equivariant Picard group $\text{Pic}^G(X)$ is defined as

$$\text{Pic}^G(X) = \{(\mathcal{L}, \phi) : \mathcal{L} \in \text{Pic}(X), \phi \text{ is a } G - \text{linearization}\}/\sim$$

where a $G$-linearization $\phi$ of a line bundle $\mathcal{L}$ is an isomorphism $\phi : \sigma^*(\mathcal{L}) \cong p_2^*(\mathcal{L})$ ($p_2$ is the projection $G \times X \to X$) satisfying the obvious cocycle condition (see [GIT, pag. 30]). We will apply this in our case with $G = PGL_2$ and $X = \mathbb{B}_{sm}(2, 2g + 2)$ (see 1.2).

In this case, since there aren’t non-trivial homomorphisms $PGL_2 \to \mathbb{G}_m$, we have an injection $\text{Pic}^{PGL_2}(\mathbb{B}_{sm}(2, 2g + 2)) \hookrightarrow \text{Pic}(\mathbb{B}_{sm}(2, 2g + 2))$ (see [GIT prop. 1.4]).

Moreover, since $\Delta = \mathbb{B}(2, 2g + 2) - \mathbb{B}_{sm}(2, 2g + 2)$ is an irreducible hypersurface $\Delta$ of degree $4g + 2$ (lemma 4.3, from the exact sequence (see [Har, II, 6.5])

$$\mathcal{Z} \cdot [\Delta] \to \text{Pic}(\mathbb{B}(2, 2g + 2)) \to \text{Pic}(\mathbb{B}_{sm}(2, 2g + 2)) \to 0$$

we get that $\text{Pic}(\mathbb{B}_{sm}(2, 2g + 2)) = \mathbb{Z}/(4g + 2)\mathbb{Z}$ generated by the hyperplane section $O(1) := O_{\mathbb{B}(2, 2g + 2)}(1)|_{\mathbb{B}_{sm}(2, 2g + 2)}$.

CLAIM : $O(1)$ admits a $PGL_2$-linearization.

In fact since the action of $\sigma : PGL_2 \times \mathbb{B}(2, 2g + 2) \to \mathbb{B}(2, 2g + 2)$ is linear in $\mathbb{B}(2, 2g + 2)$ and of degree $2g + 2$ in $PGL_2$, we have that

$$\sigma^*(O_{\mathbb{B}(2, 2g + 2)}(1)) = p_1^*(O_{PGL_2}(2g + 2)) \otimes p_2^*(O_{\mathbb{B}(2, 2g + 2)}(1)).$$

Moreover since $PGL_2 = \mathbb{P}^1 - \{\det = 0\}$ and $\det$ is of degree 2, $\text{Pic}(PGL_2) = \mathbb{Z}/2\mathbb{Z}$ and hence $O_{PGL_2}(2g + 2) \cong O_{PGL_2}$. From this, it follows that $\sigma^*(O_{\mathbb{B}(2, 2g + 2)}(1)) \cong p_2^*(O_{\mathbb{B}(2, 2g + 2)}(1))$ and hence the claim. So we reached the conclusion that

$$\text{Pic}^{PGL_2}(\mathbb{B}_{sm}(2, 2g + 2)) = \mathbb{Z}/(4g + 2)\mathbb{Z}$$

generated by $O_{\mathbb{B}(2, 2g + 2)}(1)$ (for every $g \geq 2$).

The last statement has another explanation: if an algebraic group $G$ acts on the projective space $\mathbb{P}^n = \mathbb{P}(V)$, then the sheaf $O_{\mathbb{P}^n}(1)$ admits a $G$-linearization if
and only if the initial action is induced from a representation of $G$ in the vector space $V$. It follows from the inclusion of the tautological bundle $O_{\mathbb{P}^n}(-1)$ into the product $\mathbb{P}^n \times V$ and the diagonal action of $G$ on $\mathbb{P}^n \times V$. In our case $\text{SL}_2$ does act on the vector space of binary forms of degree $2g + 2$ in two variables by the same formula as $\text{PGL}_2$ on the projective space. Moreover, $\{ \pm 1 \} = \text{Ker}(\text{SL}_2 \to \text{PGL}_2)$ acts trivially on the binary forms of even degree, so $\text{PGL}_2$ also acts on this vector space, and hence on $O_{\mathbb{P}(2, 2g+2)}(1)$. Explicitly the action of a class $[A]$ of $\text{PGL}_2$ on a binary form $f(x)$ is given by: $[A] \cdot f(x) = \det(A)^{g+1} f(A^{-1} \cdot x)$.

Now we are going to relate this equivariant Picard group with the divisor class group of the quotient variety $H_g = \mathbb{B}_{m(2, 2g + 2)}/\text{PGL}_2$ (note that a priori the equivariant Picard group is the Picard group of the quotient stack $[\mathbb{B}_{m(2, 2g + 2)}/\text{PGL}_2]$ (see [EG98, prop. 18]).

Using the theory of descent, one can show that if the action is free and the quotient is a geometric quotient then the equivariant Picard group is the Picard group of the quotient stack $\mathbb{B}_{m(2, 2g + 2)}$ (see [GKZ, pag. 32]), so that in our case:

$$\text{Pic}(H_g^0) = \text{Pic}^{\text{PGL}_2}(\mathbb{B}_{m(2, 2g + 2)^0}).$$

Now, if $g \geq 3$, the proposition [4.4] says that $H_g - H_g^0$ has codimension greater or equal to 2. From this, it follows that $\text{Cl}(H_g) \cong \text{Pic}(H_g^0)$ (see [Har, II.6.5]), and $\text{Pic}^{\text{PGL}_2}(\mathbb{B}_{m(2, 2g + 2)}/\text{PGL}_2) = \text{Pic}^{\text{PGL}_2}(\mathbb{B}_{m(2, 2g + 2)^0})$ (see [EG98 sect. 2.4, lem. 2]) so that:

$$\text{Cl}(H_g) \cong \text{Pic}(H_g^0) = \text{Pic}^{\text{PGL}_2}(\mathbb{B}_{m(2, 2g + 2)})$$

which together with [4.4] gives the conclusion.

If $g = 2$, this argument fails because in this case $H_2 - H_2^0$ contains a divisor and removing it affects the divisor class group. We will compute $\text{Cl}(H_2)$ and $\text{Pic}(H_2^0)$ in two different ways obtaining that they are both isomorphic to $\mathbb{Z}/5\mathbb{Z}$.

First of all, to compute $\text{Cl}(H_2)$ we use the explicit description of Igusa under the hypothesis $\text{char}(k) \neq 5$ (see formula [4.4]). Since the action of $< \zeta_5 >$ is free outside the point $C_0 := \{ y^2 = x^5 - x \}$ (that has codimension 3), the same reasoning as before gives

$$\text{Cl}(H_2) = \text{Cl}(H_2 - [C_0]) = \text{Pic}(H_2 - [C_0]) = \text{Pic}^{\mathbb{Z}/5\mathbb{Z}}(\mathbb{A}_k^4 - 0) = \text{Pic}^{\mathbb{Z}/5\mathbb{Z}}(\mathbb{A}_k^3) \cong \mathbb{Z}/5\mathbb{Z}.$$  

By lemma [4.8], $\Delta$ is an irreducible hypersurface of degree 10 in $\mathbb{B}(2, 6) \cong \mathbb{P}^6$ and, by lemma [4.9] $\mathcal{D}$ is an irreducible hypersurface of degree 15, so that $\text{Pic}(\mathbb{B}(2, 6) - (\Delta \cup \mathcal{D})) \cong \mathbb{Z}/5\mathbb{Z}$ (use the usual exact sequence of [Har II.6.5]). Moreover from the claim above it follows that $\text{Pic}^{\text{PGL}_2}(\mathbb{B}(2, 6) - (\Delta \cup \mathcal{D})) = \text{Pic}(\mathbb{B}(2, 6) - (\Delta \cup \mathcal{D}))$ and hence the desired conclusion.

**Lemma 4.8.** Suppose that $\text{char}(k)$ doesn’t divide $2g + 2$. The closed subset $\Delta = \mathbb{B}(2, 2g + 2) - \mathbb{B}_{m(2, 2g + 2)}$ (given by the vanishing of the discriminant) is an irreducible hypersurface of degree $4g + 2$ in $\mathbb{B}(2, 2g + 2) = \mathbb{P}^{2g+2}$.

**Proof.** Let’s consider the polynomial $f$ of degree $n := 2g + 2$ associated to a binary form. Recall that the discriminant $\Delta(f)$ is the resultant $R(f, f')$ of the polynomial with its derivative divided by the leading coefficient (see [GKZ, pag. 104]). The resultant $R(f, f')$ is the determinant of a square matrix of size $n + n - 1 = 2n - 1$ whose entries are the coefficients of our polynomial and hence it will be a homogeneous polynomial in these coefficients of degree $2n - 1$. It follows that the discriminant
will be homogeneous of degree $2n - 2$ which in our case gives $4g + 2$ (for another proof see [Ran91]).

The irreducibility of the discriminant polynomial (under the hypothesis that \text{char}(k)$ doesn’t divide $2g + 2$) is proved in [AV03] pag. 658-659].

**Lemma 4.9.** Let $D$ be the unique irreducible component of codimension 1 of $B_{sm}(2,6) - B_{sm}(2,6)^0$ (see proposition [4.7] and let $\overline{D}$ be its closure in $\mathbb{B}(2,6)$. Then $\overline{D}$ is an irreducible hypersurface in $\mathbb{B}(2,6) = \mathbb{P}^6$ of degree 15.

**Proof.** Let us consider the map $\pi : (\mathbb{P}^1)^6 - \Delta \rightarrow \text{Sym}^6(\mathbb{P}^1) - \Delta$, where $\Delta$ indicates in both spaces (with an abuse of notation) the locus of 6-tuples of points with at least 2 coincident points.

We want to decompose the divisor $\pi^{-1}(D)$ in $(\mathbb{P}^1)^6 - \Delta$ or, more precisely, its closure $\overline{\pi^{-1}(D)}$ in $(\mathbb{P}^1)^6$. By proposition [4.4], an element of $\pi^{-1}(D)$ is a 6-tuple of distinct ordered points of $\mathbb{P}^1$ that has an automorphism of order 2, whose action on these six points is conjugated to $(12)(34)(56)$, or in other words such that there exists an element $A \in \text{PGL}_2$, inducing such permutation $\sigma$ of the 6-tuple. So we obtain a decomposition

$$\pi^{-1}(\overline{D}) = \bigcup_{\sigma \sim (12)(34)(56)} \overline{D}_\sigma$$

where the union is taken over the 15 elements of $S_6$ conjugated to $(12)(34)(56)$, and for each of them $\overline{D}_\sigma$ is an hypersurface.

Now we will compute the class of $\overline{D}_\sigma$ in the Picard group $\text{Pic}((\mathbb{P}^1)^6) \cong (\mathbb{Z})^6$ (without loss of generality we can consider $D_{(12)(34)(56)}$). Take a line $l = \{P_1\} \times \ldots \times \{P_3\} \times \mathbb{P}^1$ in $(\mathbb{P}^1)^6$ for general points $P_i \in \mathbb{P}^1$. Let $P = (P_1, \ldots, P_5, P_6) \in l \cap D_{(12)(34)(56)}$, and let $A \in \text{PGL}_2$ be an automorphism, inducing the corresponding permutation of $P_i$. We have the following conditions on $A$:

$$\begin{cases} A(P_1) = P_2, \\ A(P_3) = P_4, \\ A^2 = 1. \end{cases}$$

The point $P_6 = A(P_6)$ is uniquely determined by $A$, so we want to understand how many $A$ are satisfying the conditions above.

Choose the homogenous coordinates of $P_1, P_2, P_3$ and $P_4$ to be equal to $[1 : 0], [1 : 1], [0 : 1]$ and $[c : d]$ respectively (with $c \neq 0, d \neq 0, c \neq d$). Then, due to the first two conditions, the matrix $A$ should be equal in this basis to

$$A = \begin{pmatrix} 1 & \lambda c \\ 1 & \lambda d \end{pmatrix}$$

for some nonzero $\lambda$. The last condition $A^2 = 1$ gives $\lambda = -1/d$. Besides, since the $P_i$ are general, $A(P_3) \neq P_1, P_2, P_3, P_4, P_5$, so $l \cap D_{(12)(34)(56)}$ consists of one point, that is a transversal intersection.

Moreover, the five points from the intersection $l \cap \Delta$ cannot lie on $D_{(12)(34)(56)}$: if a point $Q = (P_1, \ldots, P_5, Q_6) \in l \cap \Delta$ is a limit of points $Q^t \in D_{(12)(34)(56)}$ then at each moment $t$ the point $Q^t \in \mathbb{P}^1$ is uniquely algebraically determined by $Q_5^t$ and $A^t \in \text{PGL}_2$, that is uniquely algebraically determined by $(Q_1^t, Q_2^t, Q_3^t, Q_4^t)$. Hence $Q_6$ must be equal to $P_6$, so $l \cap (D_{(12)(34)(56)} - D_{(12)(34)(56)})$ is empty.

Now due to the symmetry of $D_{(12)(34)(56)}$ the same is true for all other “coordinate” lines in $(\mathbb{P}^1)^6$, and so the class of $D_{(12)(34)(56)}$ in $\text{Pic}((\mathbb{P}^1)^6)$ is equal to $(1, 1, 1, 1, 1, 1)$. Thus, combining this result with the decomposition [4.7] and comparing it with the fact that $\pi^{-1}(Q_{\sigma^6}(1))$ is also of type $(1, 1, 1, 1, 1, 1)$, we obtain that the degree of $\overline{D}$ is equal to 15.
Theorem 4.10. Suppose that char(k) doesn’t divide 2g + 2 or 2g + 1. Then
\[ \text{Pic}(H_g) = 0. \]

Proof. Consider the following natural maps (see theorem 4.7):
\[ \text{Pic}(H_g) \to \text{Pic}^{PGL_2}(B_{sm}(2, 2g + 2)) \to \text{Pic}^{PGL_2}(B_{sm}(2, 2g + 2)^0) \cong \text{Cl}(H_g). \]
Since \( H_g \) is normal, the composition of the two maps is an injection, hence also the first map is an injection.

Recall (see formula 4.5) that \( \text{Pic}^{PGL_2}(B_{sm}(2, 2g + 2)) \) is a cyclic group of order \( 4g + 2 \) generated by the tautological line bundle \( \mathcal{O}_B_{sm}(2, 2g + 2)^{-1} \) with its natural \( PGL_2 \)-linearization that comes from its embedding inside \( B_{sm}(2, 2g + 2) \times \mathbb{A}_{sm}(2, 2g + 2) \), where \( PGL_2 \) acts diagonally.

We want to see which \( PGL_2 \)-linearized line bundles \( L \) on \( B_{sm}(2, 2g + 2) \) come from line bundles on \( H_g \). Clearly a necessary condition is that for each point \( x \in B_{sm}(2, 2g + 2) \) its stabilizer \( \text{Stab}_x \subset PGL_2 \) is acting trivially on the fiber \( L_x \).

Consider first the binary form \( f_1 := X^{2g+1}Y - Y^{2g+2} \) (which is in \( B_{sm}(2, 2g + 2) \) since char(k) doesn’t divide 2g + 1). Its stabilizer is the cyclic group of order 2g + 1:
\[ \text{Stab}_f_1 = C_{2g+1} = \left\{ \begin{bmatrix} \zeta_{2g+1} & 0 \\ 0 & 1 \end{bmatrix} \right\} \]
where \( \zeta_{2g+1} \) is a primitive \((2g + 1)\)-root of unity.

The fiber of the line bundle \( \mathcal{O}_{B_{sm}(2, 2g+2)}(-1) \) above \( f_1 \) is the 1-dimensional vector space of all scalar multiples of \( f_1 \) inside \( \mathbb{A}_{sm}(2, 2g + 2) \):
\[ \mathcal{O}_{B_{sm}(2, 2g+2)}(-1)_{f_1} = \{ \lambda \cdot (X^{2g+1}Y - Y^{2g+2}) : \lambda \in k \}. \]
Recall (from the proof of theorem 4.7) that \( PGL_2 \) acts on \( \mathbb{A}_{sm}(2, 2g + 2) \) by the formula: \( [A] \cdot f(x) = \det(A)^{g+1}f(A^{-1} \cdot x) \). So the generator of the stabilizer group acts on the fiber as multiplication by \( \zeta_{2g+1}^2 = \zeta_{2g+1} \). Hence only the multiples of \( \mathcal{O}_{B_{sm}(2, 2g+2)}(2g + 1) \) can come from line bundles on \( H_g \).

Next consider the binary form \( f_2 := X^{2g+2} - Y^{2g+2} \) (which is in \( B_{sm}(2, 2g + 2) \) since char(k) doesn’t divide 2g + 2). Its stabilizer is the dihedral group of order 4g + 4:
\[ \text{Stab}_f_2 = D_{2g+2} = \left\{ \begin{bmatrix} \zeta_{2g+2} & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\} \]
where \( \zeta_{2g+2} \) is a primitive \((2g + 2)\)-root of unity.

The fiber of the line bundle \( \mathcal{O}_{B_{sm}(2, 2g+2)}(-1) \) above \( f_2 \) is:
\[ \mathcal{O}_{B_{sm}(2, 2g+2)}(-1)_{f_2} = \{ \lambda \cdot (X^{2g+2} - Y^{2g+2}) : \lambda \in k \}. \]
The two generators of the stabilizer group act respectively as multiplication by \(-1\) and \((-1)^g\). Hence only the multiples of \( \mathcal{O}_{B_{sm}(2, 2g+2)}(2g + 1) \) can come from line bundles on \( H_g \).

Putting together these two conditions plus the fact that \( \mathcal{O}_{B_{sm}(2, 2g+2)}(4g + 2) = 0 \) in \( \text{Pic}^{PGL_2}(B_{sm}(2, 2g + 2)) \), one concludes that \( \text{Pic}(H_g) = 0. \)

5. Stack of hyperelliptic curves and its Picard group

Recall that the moduli functor \( H_g \) of hyperelliptic curves is the contravariant functor
\[ H_g : \text{Sch}_k \to \text{Set} \]
which associates to every \( k \)-scheme \( S \) the set
\[ H_g(S) = \{ F \to S \text{ family of hyperelliptic smooth curves of genus } g \}_|_{/\text{et}}. \]
By the results of Lonsted-Kleiman (see theorem 3.1 and 3.2), a family $\pi : F \to S$ of hyperelliptic curves is a double cover of a family $p : C \to S$ of $\mathbb{P}^1$, namely we have the following situation

$$
\begin{array}{c}
W \subset F \\
\xrightarrow{f} \\
\pi \\
\downarrow \\
C \supset D \\
\xrightarrow{p^1} \\
S
\end{array}
$$

where the branch divisor $D$ and the ramification divisor $W$ (the Weierstrass subscheme) are relative Cartier divisor finite and étale of degree $2g + 2$ over the $S$. By the classical theory of double covers, the divisor $D$ is divisible by 2 in the Picard group of $C$, namely there exists an invertible sheaf $L$ in $\text{Pic}(C)$ such that

$$
(L^{-1})^\otimes 2 = \mathcal{O}_C(D).
$$

This invertible sheaf satisfies the following two relations

$$
f_*(\mathcal{O}_F) = \mathcal{O}_C \oplus L.
$$

Moreover the Hurwitz formula gives

$$
\omega_{F/S} = f^*(\omega_{C/S}) \otimes \mathcal{O}_F(W).
$$

Consider, inside the affine space $\mathbb{A}(2, 2g + 2)$ of linear forms in two variables of degree $2g + 2$, the open subset $\mathbb{A}_{sm}(2, 2g + 2)$ of smooth linear forms (i.e. forms having distinct roots) and an action of $GL_2$ by: $A \cdot f(x) = f(A^{-1} \cdot x)$. Let us remark that the projective space $\mathbb{P}(2, 2g + 2)$ is just the projectivization of $\mathbb{A}(2, 2g + 2)$ (the same is true for $\mathbb{P}_{sm}(2, 2g + 2)$ and $\mathbb{A}_{sm}(2, 2g + 2)$). Clearly the subgroup $\mu_{g+1}$, embedded diagonally in $GL_2$, acts trivially on $\mathbb{A}_{sm}(2, 2g + 2)$. The result is the following:

**Theorem 5.1.** (Arsie-Vistoli, [AV04] theo. 4.1) The stack of hyperelliptic curves of genus $g$ can be realized as

$$
\mathcal{H}_g = [\mathbb{A}_{sm}(2, 2g + 2)/(GL_2/\mu_{g+1})]
$$

with action given by $[A] \cdot f(x) = f(A^{-1} \cdot x)$.
Proof. (Sketch) Consider the auxiliary functor $\tilde{H}_g$ which associates to every $k$-schemes $S$ the set
\[
\tilde{H}_g(S) = \{ C \ra S, \mathcal{L}, \mathcal{L}^{\otimes 2} \ra \mathcal{O}_C, \phi : (C, \mathcal{L}) \cong (\mathbb{P}^1_S, \mathcal{O}_{\mathbb{P}^1_S}(-g-1)) \}
\]
where $p : C \ra S$, $\mathcal{L}$ and $i : \mathcal{L} \hookrightarrow \mathcal{O}_C$ are as before and the isomorphism $\phi$ consists of an isomorphisms of $S$-schemes $\phi_0 : C \cong \mathbb{P}^1_S$ plus an isomorphism of invertible sheaves $\phi_1 : \mathcal{L} \cong \mathcal{O}_{\mathbb{P}^1_S}(-g-1)$. Forgetting the isomorphism $\phi$, one gets a natural transformation of functors $\tilde{H}_g \ra H'_g \cong H_g$.

This rigidified functor $\tilde{H}_g$ is isomorphic to $\mathbb{A}_{sm}(2, 2g+2)$ (thought as the functor $\Hom(-, \mathbb{A}_{sm}(2, 2g+2))$). In fact for any object in $\tilde{H}_g(S)$ the isomorphism $\phi_1$ (precisely, its “tensor square”) and the inclusion $i : \mathcal{L}^{\otimes 2} \hookrightarrow \mathcal{O}_C$ provide a canonical inclusion $\mathcal{O}_{\mathbb{P}^1_S}(-2g-2) \hookrightarrow \mathcal{O}_{\mathbb{P}^1_S}$. This morphism of sheaves corresponds to a section of $\mathcal{O}_{\mathbb{P}^1_S}(2g+2)$, that is smooth on any geometric fiber of $\mathbb{P}^1_S \ra S$, and so defines an element of $\mathbb{A}_{sm}(2, 2g+2)(S)$. The inverse functor is obtained by sending an element of $\mathbb{A}_{sm}(2, 2g+2)(S)$, thought as a homomorphism $f : \mathcal{O}_{\mathbb{P}^1_S}(-2g-2) \ra \mathcal{O}_{\mathbb{P}^1_S}$, into the object $\{(\mathbb{P}^1_S \ra S, \mathcal{O}_{\mathbb{P}^1_S}(-g-1), f : \mathcal{O}_{\mathbb{P}^1_S}(-g-1)^{\otimes 2} \ra \mathcal{O}_{\mathbb{P}^1_S}, \text{id} : (\mathbb{P}^1_S, \mathcal{O}_{\mathbb{P}^1_S}(-g-1)) \ra (\mathbb{P}^1_S, \mathcal{O}_{\mathbb{P}^1_S}(-g-1))\}$ of $\tilde{H}_g(S)$.

Next consider the group $\text{Aut}(\mathbb{P}^1, \mathcal{O}(-g-1))$ consisting of automorphisms of $\mathbb{P}^1$ with a linearization of the sheaf $\mathcal{O}(-g-1)$. This group is canonically isomorphic to $GL_2/\mu_{g+1}$, and the corresponding group sheaf $\text{Aut}(\mathbb{P}^1, \mathcal{O}(-g-1))$ acts naturally on $\tilde{H}_g$ by composition with the isomorphism $\phi$. One can check that the corresponding action of $GL_2/\mu_{g+1}$ on $\mathbb{A}_{sm}(2, 2g+2)$ is the one given in the statement.

Finally, descent theory implies that the forgetful morphism $\tilde{H}_g \ra H_g$ makes $\tilde{H}_g$ into a principal bundle over the stack $H_g$ respect to the group sheaf $\text{Aut}(\mathbb{P}^1, \mathcal{O}(-g-1))$. From this, one gets the representation of $H_g \cong H'_g$ as the quotient stack $[\mathbb{A}_{sm}(2, 2g+2)/(GL_2/\mu_{g+1})]$. \hfill \Box

Note that also the bigger group $\mu_{2g+2}$ acts trivially on $\mathbb{A}_{sm}(2, 2g+2)$ the stack $[\mathbb{A}_{sm}(2, 2g+2)/(GL_2/\mu_{2g+2})]$ is not isomorphic to $[\mathbb{A}_{sm}(2, 2g+2)/(GL_2/\mu_{g+1})]$ (see remark ??).

One can give a more explicit description of the quotient group appearing in the preceding theorem as in the following:

**Lemma 5.2.** For the group $GL_2/\mu_{g+1}$ it holds:

(i) If $g$ is even then the homomorphism of algebraic groups
\[
GL_2/\mu_{g+1} \ra GL_2
\]

is an isomorphism. The group of characters of $GL_2/\mu_{g+1}$ is isomorphic to $\mathbb{Z}$ and is generated by $\det^{g+1}$.

(ii) If $g$ is odd then the homomorphism of algebraic groups
\[
GL_2/\mu_{g+1} \ra G_m \times PGL_2
\]

is an isomorphism. The group of characters of $GL_2/\mu_{g+1}$ is isomorphic to $\mathbb{Z}$ and is generated by $\det^{g+1}$.

**Proof.** (i) An inverse is given by the homomorphism $A \ra [\det(A)^{\frac{g+1}{2}}, A]$. 

(ii)
The second assertion follows from the fact that the group of characters of \( GL_2 \) is isomorphic to \( \mathbb{Z} \) and is generated by \( \det \).

(ii) An inverse is given by the homomorphism

\[(\alpha, [A]) \mapsto [\alpha^{-1} \det(A)^{-1/2} A].\]

The second assertion follows from the fact that the group of characters of \( \mathbb{G}_m \times P GL_2 \) is isomorphic to \( \mathbb{Z} \) and is generated by the projection onto the first factor.

\[\square\]

Using these isomorphisms, one can give another description of the moduli stack of hyperelliptic curves.

**Corollary 5.3.** ([AV04, cor. 4.7]) The stack \( \mathcal{H}_g \) of hyperelliptic curves of genus \( g \) can be represented by:

(i) If \( g \) is even

\[\mathcal{H}_g = [\mathbb{A}_{sm}(2,2g+2)/GL_2] \]

with action given by \( A \cdot f(x) = \det(A)^g f(A^{-1} x) \).

(ii) If \( g \) is odd

\[\mathcal{H}_g = [\mathbb{A}_{sm}(2,2g+2)/\mathbb{G}_m \times P GL_2] \]

with action given by \( (\alpha, [A]) \cdot f(x) = \alpha^{-2} \det(A)^{g+1} f(A^{-1} x) \).

Using this description of \( \mathcal{H}_g \) as a quotient, Arsie and Vistoli were able to compute the Picard group of it. For later reference, we include here their instructive proof.

First, recall the notion of a functorial Picard group of a stack, as defined by Mumford [Mum65], see also [EG98, pag. 624]. For an algebraic stack \( \mathcal{F} \), an element \( E \in \text{Pic}(\mathcal{F}) \) consists of two sets of data:

(i) An invertible sheaf \( E(\pi) \in \text{Pic}(S) \) for every morphism \( S \to \mathcal{F} \);

(ii) For each diagram

\[
\begin{array}{ccc}
S_1 & \xrightarrow{f} & S_2 \\
\downarrow{\pi_1} & & \downarrow{\pi_2} \\
\mathcal{F} & \xrightarrow{\pi} & \mathcal{F}
\end{array}
\]

a functorial isomorphism \( \mathcal{E}(f) : \mathcal{E}(\pi_1) \xrightarrow{\cong} f^*(\mathcal{E}(\pi_2)) \).

The product of two elements \( \mathcal{E} \) and \( \mathcal{E}' \) is the line bundle that associates to every morphism \( \pi : S \to \mathcal{F} \) the line bundle \( \mathcal{E}(\pi) \otimes \mathcal{E}'(\pi) \).

For a quotient stack \( [X/G] \), there is an isomorphism between the equivariant Picard group \( \text{Pic}^G(X) \) and the functorial Picard group \( \text{Pic}([X/G]) \) (see [EG98, prop. 18]). Explicitly, to \( E \in \text{Pic}^G(X) \) we associate the element \( \mathcal{E} \in \text{Pic}([X/G]) \) whose value on a morphism \( S \to [X/G] \), that is a principal \( G \)-bundle \( B \to S \) together with an equivariant map \( B \to X \) (this is the definition of the quotient stack, see [EG98, section 5.1]), is just the image of \( E \) under the maps \( \text{Pic}^G(X) \to \text{Pic}^G(B) \cong \text{Pic}(S) \). Equivalently, viewing an element of \( \text{Pic}^G(X) \) as a vector bundle \( E \to X \) of rank 1 together with a compatible action of \( G \), the element \( \mathcal{E} \in \text{Pic}([X/G]) \) is the functor represented by the line bundle stack \( \mathbb{E} := [E/G] \) over the stack \([X/G]\), i.e. we have
Theorem 5.4. (Arsie-Vistoli, [AV04, theo. 5.1]) Assume that char(k) doesn’t divide 2g + 2. The Picard group of \( \mathcal{H}_g \) is

\[
\text{Pic}(\mathcal{H}_g) = \begin{cases} 
\mathbb{Z}/(4g + 2)\mathbb{Z} & \text{if } g \text{ is even} \\
\mathbb{Z}/2(4g + 2)\mathbb{Z} & \text{if } g \text{ is odd}.
\end{cases}
\]

Proof. Recall that the equivariant Picard group of a \( k \)-linear representation \( V \) of a group \( G \) is equal to (see [EG98, lemma 2]):

\[
\text{Pic}^G(V) = \text{Pic}^G(\text{Spec}(k)) = G^*,
\]

where \( G^* = \text{Hom}(G, \mathbb{G}_m) \) (Hom is taken in the category of algebraic groups). Indeed, the group \( G \) acts trivially on the automorphism group \( H^0(V, O_V) = k^* \) of the trivial line bundle on \( V \), thus the \( G \)-linearizations of the trivial line bundle on \( V \) are elements of \( \text{Hom}(G, \mathbb{G}_m) \).

Consider the action of \( \mathbb{G}_m \) on \( \mathbb{A}(2, 2g + 2) \), given by \( \alpha \cdot f(x) = \alpha^{-2}f(x) \), and the usual exact sequence:

\[
\mathbb{Z}(\Delta) \rightarrow \text{Pic}^G(\mathbb{A}(2, 2g + 2)) \rightarrow \text{Pic}^G(\mathbb{A}_{sm}(2, 2g + 2)) \rightarrow 0,
\]

where \( \Delta = \mathbb{A}(2, 2g + 2) - \mathbb{A}_{sm}(2, 2g + 2) \) is the locus defined by the vanishing of the discriminant. More precisely, here and below \( \Delta \) denotes the generator of the subgroup of such linearizations of the trivial line bundle over \( \mathbb{A}(2, 2g + 2) \), that become isomorphic to the trivial linearization over \( \mathbb{A}(2, 2g + 2) \backslash \Delta \). Explicitly, since \( \Delta \) is an irreducible hypersurface of degree \( 4g + 2 \) (see lemma 4.8), its equation is a polynomial \( F \) of degree \( 4g + 2 \). Multiplication by \( F \) defines an automorphism of the trivial line bundle over \( \mathbb{A}(2, 2g + 2) \backslash \Delta \). Since \( F(\alpha \cdot f) = \alpha^{-2(4g + 2)}F(f) \) for \( \alpha \in \mathbb{G}_m \) and \( f \in \mathbb{A}(2, 2g + 2) \), we see that the latter automorphism sends a trivial linearization to the trivialization, which corresponds to the character \( -2(4g + 2) \) (see also [AV04]), and therefore:

\[
\text{Pic}^G(\mathbb{A}_{sm}(2, 2g + 2)) = \mathbb{Z}/2(4g + 2)\mathbb{Z}.
\]

When \( g \) is even, consider the two compatible actions:

\[
\mathbb{A}(2, 2g + 2) \xrightarrow{\text{id}} \mathbb{A}(2, 2g + 2)
\]

\[
\mathbb{G}_m \xrightarrow{\text{diagonal}} GL_2,
\]

where the first action is as before and the second one is that of corollary 5.8 \( A \cdot f(x) = \det(A)^y f(A^{-1}x) \). Since \( GL_2 \cong \mathbb{Z} \) generated by the determinant morphism, the diagonal inclusion \( \mathbb{G}_m \hookrightarrow GL_2 \) induces a map \( GL_2^* \rightarrow \mathbb{G}_m^* \) which is the multiplication by 2, i.e. it holds:

\[
\text{Pic}^{GL_2}(\mathbb{A}(2, 2g + 2)) \rightarrow 2 \cdot \text{Pic}^G(\mathbb{A}(2, 2g + 2)).
\]
From the two usual exact sequences
\[ \mathbb{Z} \langle \Delta \rangle \to \text{Pic}^{GL_2}(\mathbb{A}(2, 2g + 2)) \to \text{Pic}^{GL_2}(\mathbb{A}_{sm}(2, 2g + 2)) \to 0, \]
\[ \mathbb{Z} \langle \Delta \rangle \to \text{Pic}^{G_m}(\mathbb{A}(2, 2g + 2)) \to \text{Pic}^{G_m}(\mathbb{A}_{sm}(2, 2g + 2)) \to 0, \]
combined with formula (5.6) one deduces:
\[ \text{Pic}(\mathcal{H}_g) = \text{Pic}^{GL_2}(\mathbb{A}_{sm}(2, 2g + 2)) = \mathbb{Z}/(4g + 2)\mathbb{Z}, \]
as desired in the even case.

When \( g \) is odd, consider the two compatible actions:
\[ \mathbb{A}(2, 2g + 2) \xrightarrow{\text{id}} \mathbb{A}(2, 2g + 2) \]
\[ G_m \xrightarrow{\alpha} G_m \times PGL_2, \]
where the first action is as before while the second is (according to corollary 5.3):
\[ (\alpha, [A]) \cdot f(x) = \alpha^{-2} \det(A)^{g+1} f(A^{-1}x). \]
In this case, since \((G_m \times PGL_2)^* \cong G_m^*,\) we have an isomorphism
\[ (5.8) \quad \text{Pic}^{G_m \times PGL_2}(\mathbb{A}(2, 2g + 2)) \cong \text{Pic}^{G_m}(\mathbb{A}(2, 2g + 2)). \]
Hence from the exact sequences
\[ \mathbb{Z} \langle \Delta \rangle \to \text{Pic}^{G_m \times PGL_2}(\mathbb{A}(2, 2g + 2)) \to \text{Pic}^{G_m \times PGL_2}(\mathbb{A}_{sm}(2, 2g + 2)) \to 0, \]
combined with formula (5.6) one deduces:
\[ \text{Pic}(\mathcal{H}_g) = \text{Pic}^{G_m \times PGL_2}(\mathbb{A}_{sm}(2, 2g + 2)) = \mathbb{Z}/2(4g + 2)\mathbb{Z}, \]
as desired in the odd case. \( \square \)

**Remark 5.5.** This result still holds if one consider the stack \( \mathcal{H}_{1, 1} \) of families of elliptic curves with a section. In that case the Picard group was computed by Mumford in his legendary paper [Mum65], and it is isomorphic to \( \mathbb{Z}/12\mathbb{Z} \) (see also [AV04, remark 5.5]).

Observe that the stack \( \mathcal{H}_g^0 \) of hyperelliptic curves of genus \( g \) without extra-automorphisms is isomorphic to the quotient
\[ \mathcal{H}_g^0 = \mathbb{A}_{sm}(2, 2g + 2)^0/(GL_2/\mu_{g+1})], \]
being equal to the fiber product \( \mathcal{H}_g \times_{\mathcal{H}_g} \mathcal{H}_g^0 \). Here \( \mathbb{A}_{sm}(2, 2g + 2)^0 \) is the set of forms such that the corresponding \( 2g + 2 \) points on \( \mathbb{P}^1 \) have no automorphisms.

**Corollary 5.6.** Assume that \( \text{char}(k) \) doesn’t divide \( 2g + 2 \). The Picard group of \( \mathcal{H}_g^0 \) is
\[ \text{Pic}(\mathcal{H}_g^0) = \begin{cases} \mathbb{Z}/(4g + 2)\mathbb{Z} & \text{if } g \text{ is even and } g \neq 2, \\ \mathbb{Z}/5\mathbb{Z} & \text{if } g = 2, \\ \mathbb{Z}/2(4g + 2)\mathbb{Z} & \text{if } g \text{ is odd}. \end{cases} \]
Proof. For \( g > 2 \) it holds that \( \text{Pic}(\mathcal{H}_g) = \text{Pic}(\mathcal{H}_g^0) \) since the difference between \( \mathbb{A}_{\text{sm}}(2, 2g + 2) \) and \( \mathbb{A}_{\text{sm}}(2, 2g + 2)^0 \) is of codimension at least 2 by proposition 4.3. On the other hand, \( \text{Pic}(\mathcal{H}_g^0) \) is the quotient of \( \text{Pic}(\mathbb{H}_2) = \mathbb{Z}/10\mathbb{Z} \) over the subgroup generated by the divisorial component of \( \mathbb{A}_{\text{sm}}(2, 2g + 2) - \mathbb{A}_{\text{sm}}(2, 2g + 2)^0 \) which, in view of lemma 4.3 and the explicit calculations from the proof of theorem 5.4, is the subgroup generated by the residue of 5 in \( \mathbb{Z}/10\mathbb{Z} \).

Now we are going to give an explicit description of the generators of \( \text{Pic}(\mathcal{H}_g) \) using the functorial description of the Picard group.

**Theorem 5.7.** A generator of \( \text{Pic}(\mathcal{H}_g) \) is the element \( \mathcal{G} \) that associates to a family of hyperelliptic curves \( \pi : \mathcal{F} \to S \) with Weierstrass divisor \( W \) the line bundle on \( S \):

\[
\mathcal{G}(\pi) = \begin{cases} 
\pi^* \left( \omega_{\mathcal{F}/S}^{g+1} (-(g-1)W) \right) & \text{if } g \text{ is even,} \\
\pi^* \left( \omega_{\mathcal{F}/S}^g \left( -\frac{g-1}{2} W \right) \right) & \text{if } g \text{ is odd.}
\end{cases}
\]

**Proof.** From the proof of theorem 5.4, it follows that \( \text{Pic}^{GL_2/\mu_{g+1}}(\mathbb{A}_{\text{sm}}(2, 2g + 2)) \) is a cyclic group generated by the trivial line bundle \( \mathbb{A}_{\text{sm}}(2, 2g + 2) \times k \) on which \( GL_2/\mu_{g+1} \) acts via a generator of its group of characters. Let us choose as the generator the character \( \det^{-(g+1)} \) if \( g \) is even and \( \det^{\frac{g-1}{2}} \) if \( g \) is odd (see lemma 5.2). Note that this is true without any assumption on \( \text{char}(k) \) (apart from the usual \( \text{char}(k) \neq 2 \)), while the hypothesis that \( \text{char}(k) \) doesn’t divide \( 2g + 2 \) is necessary to compute the order of the Picard group.

We have to express this generator as an element of \( \text{Pic}(\mathcal{H}_g) \) from the point of view of Mumford’s functorial description. Consider the following diagram of \( (GL_2/\mu_{g+1}) \)-equivariant maps (the notation is that of theorem 5.4):

\[
\begin{array}{ccc}
\widetilde{\mathcal{H}}_g \times k & \xrightarrow{\cong} & \mathbb{A}_{\text{sm}}(2, 2g + 2) \times k \\
\downarrow & & \downarrow \\
\mathcal{H}_g & \cong & \mathbb{A}_{\text{sm}}(2, 2g + 2).
\end{array}
\]

The functor \( \widetilde{\mathcal{H}}_g \times k \) associates to a \( k \)-scheme \( S \) the set

\[
\left( \widetilde{\mathcal{H}}_g \times k \right)(S) = \left\{ \mathcal{C} \xrightarrow{p_S} S, \mathcal{L} : \mathcal{L} \cong O_C, \phi : (\mathcal{C}, \mathcal{L}) \cong (\mathbb{P}^1_S, O_{\mathbb{P}^1_S}(-g-1)), \mathcal{M} \right\},
\]

where \( \mathcal{M} = O_S \) is the structure sheaf, on which the action of \( (GL_2/\mu_{g+1})(S) \) is defined via multiplication by \( \det^{-(g+1)} \) if \( g \) is even and \( \det^{\frac{g-1}{2}} \) if \( g \) is odd.

Let \( \mathbb{P}^1_S = \mathbb{F}(V_S) \), where \( V \) is a two-dimensional vector space over the ground field \( k \). From the Euler exact sequence for the trivial family \( p_S : \mathbb{P}^1_S \to S \)

\[
0 \to O_{\mathbb{P}^1_S} \to p_S^*(\mathcal{V}_S^*)(1) \to \omega_{\mathbb{P}^1_S/S}^{-1} \to 0
\]

one deduces a \( (GL_2/\mu_{g+1})(S) \)-equivariant isomorphism

\[(5.9) \quad p_S^*((\det V_S)^{-1}) \otimes O_{\mathbb{P}^1_S}(2) \cong \omega_{\mathbb{P}^1_S/S}^{-1}, \]

where we consider the canonical actions of \( (GL_2/\mu_{g+1})(S) \) on \( \mathbb{P}^1_S \) and on the invertible sheaves involved. Using projection formula, the fact that \( (p_S)_*(O_{\mathbb{P}^1_S}) = O_S \) and the \( (GL_2/\mu_{g+1})(S) \)-equivariant identity \( (\det V_S)^{g+1} = \mathcal{M} \) we get \( (GL_2/\mu_{g+1})(S) \)-equivariant isomorphisms

\[
\left\{ \begin{array}{l}
\mathcal{M} \cong (p_S)_* \left( \omega_{\mathbb{P}^1_S/S}^{g+1} \otimes O_{\mathbb{P}^1_S}(2g+2) \right) & \text{if } g \text{ is even,} \\
\mathcal{M} \cong (p_S)_* \left( \omega_{\mathbb{P}^1_S/S}^{\frac{g-1}{2}} \otimes O_{\mathbb{P}^1_S}(g+1) \right) & \text{if } g \text{ is odd.}
\end{array} \right.
\]
Now remark that $\phi : (\mathcal{C}, \mathcal{L}) \cong (\mathbb{P}^1_S, \mathcal{O}_{\mathbb{P}^1_S}(-g - 1))$ induces a canonical isomorphism $\omega_{\mathcal{C}/S} \cong \omega_{\mathbb{P}^1_S}$ by the $\phi_0$-component. Hence the line bundle quotient $G = \left( \left( \widetilde{\mathcal{H}}_g \times k \right) / (GL_2/\mu_{g+1}) \right)$ over $\mathcal{H}_g'$ is isomorphic to

$$G(S) = \begin{cases} \mathcal{C} \overset{p}{\to} S, \mathcal{L}, \mathcal{L} \otimes \mathcal{L}^{\otimes 2} \overset{i}{\to} \mathcal{O}_C, p_* \left( \omega_{\mathcal{C}/S}^{g+1} \otimes \mathcal{L}^{-2} \right) & \text{if } g \text{ is even}, \\ \mathcal{C} \overset{p}{\to} S, \mathcal{L}, \mathcal{L} \otimes \mathcal{L}^{\otimes 2} \overset{i}{\to} \mathcal{O}_C, p_* \left( \omega_{\mathcal{C}/S}^{g+1} \otimes \mathcal{L}^{-1} \right) & \text{if } g \text{ is odd}. \end{cases}$$

To express the preceding line bundles as push-forward of line bundles on the hyperelliptic family $\pi : \mathcal{F} \to S$, we use formulas 5.2 and 5.3 together with the fact that the line bundles $\omega_{\mathcal{C}/S}^{g+1} \otimes \mathcal{L}^{\otimes (-2)}$ and $\omega_{\mathcal{C}/S}^{g+1} \otimes \mathcal{L}^{-1}$ for $g$ odd are trivial on each fiber of $p$, and we get

$$\begin{cases} f^* \left( \omega_{\mathcal{C}/S}^{g+1} \otimes \mathcal{L}^{-2} \right) = \omega_{\mathcal{F}/S}^{g+1} \left( -(g - 1)W \right) & \text{if } g \text{ is even}, \\ f^* \left( \omega_{\mathcal{C}/S}^{g+1} \otimes \mathcal{L}^{-1} \right) = \omega_{\mathcal{F}/S}^{g+1} \left( -\frac{g - 1}{2}W \right) & \text{if } g \text{ is odd}. \end{cases}$$

Hence the line bundle $G$ over $\mathcal{H}_g$ is equal to

$$G(S) = \begin{cases} \mathcal{F} \to S, \pi_* \left( \omega_{\mathcal{F}/S}^{g+1} \left( -(g - 1)W \right) \right) & \text{if } g \text{ is even}, \\ \mathcal{F} \to S, \pi_* \left( \omega_{\mathcal{F}/S}^{g+1} \left( -\frac{g - 1}{2}W \right) \right) & \text{if } g \text{ is odd}, \end{cases}$$

from which the conclusion follows.

We can now look at other natural elements of $\text{Pic}(\mathcal{H}_g)$ and express them in term of the generator found above. Recall that given a family $\pi : \mathcal{F} \to S$ of hyperelliptic curves, there are two natural line bundles over $\mathcal{F}$: the relative canonical sheaf $\omega_{\mathcal{F}/S}$ and the line bundle associated to the Weierstrass divisor $W = \mathcal{W}_{\mathcal{F}/S}$. Hence we can consider a linear combination of them $\omega_{\mathcal{F}/S}^{g} \otimes \mathcal{O}_{\mathcal{F}/S}(bW)$ and note that it restricts on every fiber $F$ of the family to

$$\omega_{\mathcal{F}/S}^{g} \otimes \mathcal{O}_{\mathcal{F}/S}(bW)|_F = aK_F + bW_F = a(g - 1)g_1 + b(g + 1)g_2 = [(a + b)g + (b - a)]g_2,$$

where we used that on a hyperelliptic curve $F$ the canonical class $K_F$ is $(g - 1)$-times the unique $g_2$ while the Weierstrass divisor $W_F$ is $(g + 1)$-times the $g_2$. Let’s call $m(a, b) := [(a + b)g + (b - a)]$ and let’s consider only those integers $a$ and $b$ for which $m(a, b) \geq 0$.

Moreover, since on a hyperelliptic curve $F$ it holds that $h^0(F, \mathcal{O}_{\mathcal{F}/S}(kg_2)) = k + 1$ for $k \geq 0$, the push-forward $\pi_* \left( \omega_{\mathcal{F}/S}^{g} \otimes \mathcal{O}_{\mathcal{F}/S}(bW) \right)$ is a vector bundle of rank $m(a, b) + 1$ on the base $S$ (see [Har cor. 12.9]). Hence we can define an element $\mathcal{T}_{a,b}$ of $\text{Pic}(\mathcal{H}_g)$ by

$$\mathcal{T}_{a,b}(\pi) = \det \left( \pi_* \left( \omega_{\mathcal{F}/S}^{g} \otimes \mathcal{O}_{\mathcal{F}/S}(bW) \right) \right) \in \text{Pic}(S).$$

**Theorem 5.8.** In terms of the generator $\mathcal{G}$ of $\text{Pic}(\mathcal{H}_g)$ (see theorem 5.7), if $0 \leq m(a, b) < g + 1$ the element $\mathcal{T}_{a,b}$ is equal to

$$\mathcal{T}_{a,b} = \begin{cases} \mathcal{G}^{\frac{(a+b)(m(a, b)+1)}{2}} & \text{if } g \text{ is even}, \\ \mathcal{G}^{(a+b)(m(a, b)+1)} & \text{if } g \text{ is odd}, \end{cases}$$

and if $m(a, b) \geq g + 1$ the element $\mathcal{T}_{a,b}$ is equal to

$$\mathcal{T}_{a,b} = \begin{cases} \mathcal{G}^{\frac{(a+b-1)(m(a, b)+g)}{2}} & \text{if } g \text{ is even}, \\ \mathcal{G}^{(a+b-1)(m(a, b)+g)} & \text{if } g \text{ is odd}. \end{cases}$$
leads to the following.

First of all, we want to express $\mathcal{T}_{a,b}$ as an element $\mathcal{T}'_{a,b}$ of the Picard group of $\mathcal{H}'_g \cong \mathcal{H}_g$. Since, by formulas \ref{eq:5.1} and \ref{eq:5.4} it holds

$$f^* \left( \omega_{C/S}^a \otimes \mathcal{L}^{-(a+b)} \right) = \omega_{F/S}^a \otimes \mathcal{O}_{F/S}(bW),$$

the element $\mathcal{T}'_{a,b}$ in $\text{Pic}(\mathcal{H}'_g)$ will associate to $\{ p : C \to S, \mathcal{L}, i : \mathcal{L} \to \mathcal{O}_C \} \in \mathcal{H}'_g(S)$ the element

\begin{equation}
(5.10) \quad \mathcal{T}'_{a,b}(S) = \text{det} \ p_*(\omega_{C/S}^a \otimes \mathcal{L}^{-(a+b)}) \otimes \text{det} \ p_*(\omega_{C/S}^a \otimes \mathcal{L}^{-(a+b)+1}) \in \text{Pic}(S).
\end{equation}

Here we used that $f_*(\mathcal{O}_F) = \mathcal{O}_C \oplus \mathcal{L}$. In order to compute the pull-back $\widetilde{\mathcal{T}}_{a,b}$ of $\mathcal{T}'_{a,b}$ we use the isomorphism $\phi : (\mathcal{L}, \mathcal{L}) \cong (\mathbb{P}_S^1, \mathcal{O}(-g-1))$ and the Euler formula \ref{eq:5.9} which give

$$\omega_{C/S}^a \otimes \mathcal{L}^{-(a+b)} \cong p_S^*((\text{det}\mathcal{V}_S)^a) \otimes \mathcal{O}_{\mathbb{P}_S^1}((-2a) \otimes \mathcal{O}_{\mathbb{P}_S^1}((a + b)(g + 1)) =
\quad \quad = p_S^*((\text{det}\mathcal{V}_S)^a) \otimes \mathcal{O}_{\mathbb{P}_S^1}(m(a,b)),$$

and, analogously,

$$\omega_{C/S}^a \otimes \mathcal{L}^{-(a+b)+1} \cong p_S^*((\text{det}\mathcal{V}_S)^a) \otimes \mathcal{O}_{\mathbb{P}_S^1}(m(a,b) - (g + 1)),$$

where $\mathbb{P}_S^1 = P(V_S)$.

Now we take the push-forward through the map $p_S$ and take the determinant, obtaining

$$\text{det} \ (p_S)_* \left( p_S^*((\text{det}\mathcal{V}_S)^a) \otimes \mathcal{O}_{\mathbb{P}_S^1}(m(a,b)) \right) = \text{det} \ (\text{det}\mathcal{V}_S)^a \otimes \text{Sym}^{m(a,b)}(V_S) =
\quad \quad = (\text{det}\mathcal{V}_S)^a(\text{det}\mathcal{V}_S)^{n(a+b) + 1},$$

where we used the relation $\text{det}(\text{Sym}^n(V_S)) = (\text{det}\mathcal{V}_S)^{n(n+1)}$. As for the second sheaf, the push forward is zero if $m(a,b) < g + 1$. Otherwise the analogous computation leads to the following

$$\text{det} \ (p_S)_* \left( p_S^*((\text{det}\mathcal{V}_S)^a) \otimes \mathcal{O}_{\mathbb{P}_S^1}(m(a,b) - (g + 1)) \right) = (\text{det}\mathcal{V}_S)^{((a+b-1)(g+1) - \frac{m(a,b) - g}{2}}.$$

Now we conclude recalling from theorem \ref{thm:5.7} that the pull-back $\widetilde{\mathcal{G}}$ of the generator $\mathcal{G}$ to the Picard group of $\mathcal{H}_g$ is $(\text{det}\mathcal{V}_S)^{g+1}$ for $g$ even and $(\text{det}\mathcal{V}_S)^{\frac{4g+3}{2}}$ for $g$ odd.

\begin{proof}
We will also show another way to find the expression in terms of the canonical generators, which is more explicit and doesn’t involve the stack description of theorem \ref{thm:5.1}.

We use the same notations as above. In addition, let $\tau$ denote the invertible “generator” sheaf $\pi_* \left( \omega_{F/S}^{g+1}(- (g-1)W) \right)$ on the base $S$, and let $\epsilon$ denote the invertible “generator” sheaf $\pi_* \left( \omega_{F/S}^{g+1}(- \frac{g-1}{2}W) \right)$ for the case $g$ odd, so that $\epsilon^2 = \tau$.

The idea is to express the sheaves $\omega_{C/S}$ and $\mathcal{L}$ in terms of $p^*\tau$ (or $p^*\epsilon$ for $g$ odd) and a certain invertible sheaf $\mathcal{E}$ on $C$, whose determinant of the direct image via $p$ can be expressed in terms of $\tau$ (or $\epsilon$ for $g$ odd). Then one concludes by projection formula, using the relation \ref{eq:5.10}.

Suppose $g$ is odd. We claim that in this case $\mathcal{L} \cong p^*(\epsilon^{-1}) \otimes \omega_{C/S}^{\frac{g+1}{2}}$. Indeed, by \ref{eq:5.11} $f_*(\omega_{F/S}^{g+1}(- \frac{g-1}{2}W) = \omega_{C/S}^{g+1} \otimes \mathcal{O}_C \oplus \mathcal{L}^{-1}).$ Thus $\epsilon = p_*(\omega_{C/S}^{g+1} \otimes \mathcal{L}^{-1})$, and we get the desired statement, since $\omega_{C/S}^{g+1} \otimes \mathcal{L}^{-1}$ is isomorphic to the structure sheaf.
on each fiber of \( p \). Moreover, in lemma 6.4 after the proof of theorem 6.3, we will prove that \( \det p_*(\omega_{C/S}^m) \) is a trivial line bundle on the base \( S \) for any \( m \in \mathbb{Z} \). Thus we are done in the case \( g \) odd, taking \( E = \omega_{C/S}^1 \).

Now we treat the case \( g \) even. In this case \( C \) is in fact the projectivization of a two-dimensional vector bundle \( p_*(\mathcal{M}) \) (see theorem 3.5 (ii)), where \( \mathcal{M} = \omega_{g/2} \mathcal{M}/S \otimes L^{-1} \).

Hence from Euler exact sequence on each fiber of the family \( p : C \to S \) we get an exact sequence

\[
0 \to \mathcal{O}_C \to p^*(\mathcal{M})^* \otimes \mathcal{M} \to \omega_{C/S}^{-1} \to 0.
\]

Thus we get an isomorphism

\[
\det(p^*(\mathcal{M})^*)) \cong \omega_{C/S}^{-1} \otimes \mathcal{M}^{-2},
\]

so

\[
\det(p_*(\mathcal{M})) \cong \tau.
\]

Moreover, there are expressions

\[
\omega_{C/S} \cong p^* \tau \otimes \mathcal{M}^{-2},
\]

\[
\mathcal{L} \cong (p^* \tau)^{g/2} \otimes \mathcal{M}^{-(g+1)}.
\]

Therefore we can take \( E = \mathcal{M} \) for \( g \) even.

Among the elements \( T_{a,b} \) one is of particular interest, namely the Hodge line bundle that in our notation is \( T_{1,0}^* \). It is known that, over the complex numbers, the Hodge line bundle generate the Picard group of \( \mathcal{M}_g \) (see [AC87]). For hyperelliptic curves we have the following

**Corollary 5.9.** In terms of the generator \( G \) of \( \text{Pic}(\mathcal{H}_g) \), the Hodge line bundle is equal to

\[
\det \pi_*(\omega_{F/S}) = \begin{cases} 
G^{g/2} & \text{if } g \text{ is even}, \\
G^g & \text{if } g \text{ is odd}.
\end{cases}
\]

In particular it generates \( \text{Pic}(\mathcal{H}_g) \) if \( g \) is not divisible by 4 while otherwise it generates a subgroup of index 2.

For \( g = 2 \), this was proved by Vistoli in [Vis98] (he computed the Chow ring of \( H_2 = \mathcal{M}_2 \) proving that it’s generated by the Chern classes of the Hodge bundle).

Note also that for \( g \) even there is another interesting generator of the Picard group of \( \mathcal{H}_g \), that is \( T_{2,1-\frac{a}{2}} \) (for which it holds that \( a + b = 1 \) and \( m(a,b) = 1 \)).

The interest of it is that it is the determinant of the push-forward of a globally defined \( g_2^1 \) (which is very far from being unique!) on the family \( F \to S \), that in fact, as we know from section 3, exists in general only for \( g \) even.

### 6. Comparison between stack and coarse moduli space of hyperelliptic curves

Now we want to compare the stack \( \mathcal{H}_g \) with its coarse moduli scheme \( H_g \) (as well as the open substack \( \mathcal{H}^0 \) with the open subvariety \( H^0 \)).

We introduce a new moduli functor that is "intermediate" between \( \mathcal{H}_g \) and \( H_g \).

**Definition 6.1.** The moduli functor \( D_{2g+2} \) is the contravariant functor

\[
D_{2g+2} : \text{Sch}/k \to \text{Set}
\]

which associates to every \( k \)-scheme \( S \) the set

\[
D_{2g+2}(S) = \left\{ \mathcal{C} \to S \text{ family of } \mathbb{P}^1 \text{ and } D \subset C \text{ an effective Cartier divisor finite and étale over } S \text{ of degree } 2g + 2 \right\}/\cong
\]
and $\mathcal{D}_{2g+2}$ is the subfunctor of families of effective divisors on $\mathbb{P}^1$ without automorphisms.

Being without automorphisms means for an effective divisor on $\mathbb{P}^1$ that there are no projective transformation of $\mathbb{P}^1$ that preserves the divisor. By the results of Lonstead-Kleiman (see theorem 3.1 and 3.2) it follows that there is a natural transformation of functors $\Psi : \mathcal{H}_g \to \mathcal{D}_{2g+2}$. Moreover, since over an algebraically closed field a hyperelliptic curve is uniquely determined (up to isomorphism) by the $2g + 2$ points on $\mathbb{P}^1$ (up to isomorphism) over which the double cover of $\mathbb{P}^1$ is ramified, it follows that both these moduli functors have $\mathcal{H}_g$ as a coarse moduli scheme. We end up with the following diagram:

$$
\begin{array}{ccc}
\mathcal{H}_g & \xrightarrow{\Phi} & \mathcal{D}_{2g+2} \\
\Phi_\mathcal{H} & & \Phi_\mathcal{D} \\
\text{Hom}(-, \mathcal{H}_g). & & \\
\end{array}
$$

Now we want to prove that $\mathcal{D}_{2g+2}$ is an algebraic stack providing a description of it as a quotient stack.

**Theorem 6.2.** $\mathcal{D}_{2g+2}$ is an algebraic stack isomorphic to the quotient stack $[\mathbb{B}_{sm}(2, 2g + 2)/\text{PGL}_2]$, where the action is given by $[\mathcal{A}] \cdot [f(x)] = [f(A^{-1} \cdot x)]$. Moreover it holds the following isomorphism of stacks $[\mathbb{B}_{sm}(2, 2g + 2)/\text{PGL}_2] \cong [\mathbb{A}_{sm}(2, 2g + 2)/(\text{GL}_2/\mu_{2g+2})]$, with the same action as before.

**Proof.** We prove the first part of the theorem with a strategy analogous to that of theorem 3.1 of Arsie and Vistoli, namely first rigidifying the functor so that it becomes a scheme and then viewing this rigidified functor as a principal bundle over the original one for the action of a suitable group.

Here the rigidified functor is the functor $\mathcal{D}_{2g+2}$ which associates to a $k$-scheme $S$ the set

$$
\mathcal{D}_{2g+2}(S) = \{ \mathcal{C} \to S, D, \phi : \mathcal{C} \cong \mathbb{P}^1 \}
$$

where $\mathcal{C} \to S$ is a family of $\mathbb{P}^1$, $D$ is an effective Cartier divisor as the one in definition 6.1 and $\phi$ is an isomorphism between the family $\mathcal{C} \to S$ and the trivial family $\mathbb{P}^1_S = S \times \mathbb{P}^1$.

This rigidified functor is isomorphic to $\mathbb{B}_{sm}(2, 2g + 2)$ (thought as the functor $\text{Hom}(-, \mathbb{B}_{sm}(2, 2g + 2))$). In fact an effective smooth divisor of degree $2g + 2$ on $\mathbb{P}^1$ is an element of $\mathbb{B}_{sm}(2, 2g + 2)$ and hence a divisor $D$ on $\mathcal{C} \cong \mathbb{P}^1_S$ as above can be identified with an element of $\mathbb{B}_{sm}(2, 2g + 2)(S)$.

The group sheaf $\text{Aut}(\mathbb{P}^1) \cong \text{PGL}_2$ acts on $\mathcal{D}_{2g+2}$ by composing with the isomorphism $\phi$ and it’s easy to see that the corresponding action of $\text{PGL}_2$ on $\mathbb{B}_{sm}(2, 2g + 2)$ is the one given in the statement.

Finally, descent theory implies that the forgetful morphism $\mathcal{D}_{2g+2} \to \mathcal{D}_{2g+2}$ makes $\mathcal{D}_{2g+2}$ into a $\text{Aut}(\mathbb{P}^1)$-principal bundle over $\mathcal{D}_{2g+2}$, from which one gets the description of $\mathcal{D}_{2g+2}$ as a quotient stack $[\mathbb{B}_{sm}(2, 2g + 2)/\text{PGL}_2]$.

To prove the second part of the theorem, observe that, applying lemma 5.2(ii) with $g+1$ replaced by $2g+2$, one deduces an isomorphism $\text{GL}_2/\mu_{2g+2} \cong \mathbb{G}_m \times \text{PGL}_2$. Moreover one can check that, under this isomorphism, the corresponding action of $\text{GL}_2/\mu_{2g+2} \cong \mathbb{G}_m \times \text{PGL}_2$ on $\mathcal{A}_{sm}(2, 2g + 2)$ is given by $(\alpha, [A]) \cdot f(x) = \alpha^{-1} \cdot (\det A)^{g+1} f(A^{-1} \cdot x)$. Hence the stack quotient of $\mathcal{A}_{sm}(2, 2g + 2)$ by $\text{GL}_2/\mu_{2g+2} \cong \mathbb{G}_m \times \text{PGL}_2$ can be taken in two steps: first take the quotient over the subgroup $\mathbb{G}_m/\mu_{2g+2} \cong \mathbb{G}_m$ which is isomorphic to $\mathbb{B}_{sm}(2, 2g + 2)$ since the action is free, and then take the quotient over $\text{GL}_2/\mathbb{G}_m \cong \text{PGL}_2$ with the usual action. $\square$
In the next theorem we compute the Picard group of \(D_{2g+2}\) and compare it with the Picard group of \(H_g\). We use the first description of \(D_{2g+2}\) in the preceding theorem, although everything can be proved also using the second description in a spirit similar to theorems 6.3 and 5.7.

**Theorem 6.3.** Assume that \(\text{char}(k)\) doesn’t divide \(2g + 2\). Then \(\text{Pic}(D_{2g+2}) = \mathbb{Z}/(4g + 2)\mathbb{Z}\) generated by the element \(\mathcal{G}_{\text{div}}\) that associates to a family \(p : C \to S\) of \(F^1\) together with a Cartier divisor \(D\) (as in the definition (6.1) the line bundle on \(S\))

\[
\mathcal{G}_{\text{div}}(p) = p_* \left( \omega_{C/S}^{g+1}(D) \right).
\]

Moreover, the natural map \(\text{Pic}(D_{2g+2}) \to \text{Pic}(H_g)\) is injective and hence it’s an isomorphism if \(g\) is even while it’s an inclusion of index 2 if \(g\) is odd.

**Proof.** In view of the explicit description of theorem 6.2 it holds that \(\text{Pic}(D_{2g+2}) = \text{Pic}_{PGL_2}(\mathbb{B}_{sm}(2, 2g + 2))\) and we already proved (see formula (4.5)) that this is a cyclic group of order \(4g + 2\) generated by \(\mathcal{O}_{\mathbb{B}_{sm}(2, 2g + 2)}(1)\) with its natural \(PGL_2\)-linearization.

To prove the functoriality of \(\mathcal{G}_{\text{div}}\), we can pull-back this element to \(\text{Pic}(D_{2g+2})\) (see theorem 6.2) and hence we reduce to show the isomorphism of the corresponding \(PGL_2\)-equivariant invertible sheaves for the case when \(S = \mathbb{B}_{sm}(2, 2g + 2), \ C = \mathbb{B}_{sm}(2, 2g + 2) \times F^1\), and \(D\) is the incidence divisor. As in the proof of theorem 5.7 from Euler formula applied to the given family one deduces a \(PGL_2\)-equivariant isomorphism

\[
p^*((\det E)^{-g+1}) \otimes \mathcal{O}_{\mathbb{P}(E)}(2g + 2) \cong \omega^{-g-1},
\]

where \(\omega\) denotes the (trivial) relative canonical sheaf for the morphism \(p : F^1 \times \mathbb{B}_{sm}(2, 2g + 2) \to \mathbb{B}_{sm}(2, 2g + 2)\) and \(E = V \times \mathbb{B}_{sm}(2, 2g + 2)\) is a trivial two-dimensional vector bundle on \(\mathbb{B}_{sm}(2, 2g + 2)\) such that \(\mathcal{C} = \mathbb{P}(E)\). So after taking push-forwards we get

\[
\text{Sym}^{2g+2}(E) \cong p_*((\omega^{-g+1}) \otimes (\det E)^{g+1}).
\]

Remark that the group \(PGL_2\) acts trivially on \(\det E\), thus we get an exact sequence of \(PGL_2\)-equivariant sheaves on \(F^1 \times \mathbb{B}_{sm}(2, 2g + 2)\)

\[
0 \to p^*\mathcal{O}_{\mathbb{B}(2, 2g + 2)}(-1) \to \omega^{-(g+1)} \to \omega^{-(g+1)}|_D \to 0.
\]

So on the base \(\mathbb{B}_{sm}(2, 2g + 2)\) there is an equality of \(PGL_2\)-equivariant sheaves \(\mathcal{O}_{\mathbb{B}(2, 2g + 2)}(-1) \cong p_*((\omega^{-(g+1)}(-D))\) (we use the fact that the restriction of \(\omega^{-(g+1)}(-D)\) on each fiber is trivial), and we get the desired statement.

Finally to study the map \(\text{Pic}(D_{2g+2}) \to \text{Pic}(H_g)\), let us first remark that

\[
\text{Pic}_{PGL_2/\mu_{g+1}}(\mathbb{A}(2, 2g + 2)) = \text{Pic}_{PGL_2/\mu_{g+1}}(\mathbb{A}(2, 2g + 2) \setminus \{0\}),
\]

since the origin in \(\mathbb{A}(2, 2g + 2)\) is of codimension \(\geq 2\) (see sect. 2.4, lemma 2). Now consider the compatible diagram:

\[
\begin{array}{ccc}
\mathbb{A}(2, 2g + 2) \setminus \{0\} & \longrightarrow & \mathbb{B}(2, 2g + 2) \\
\bigcup_{\mathbb{G}_m} & & \bigcup_{\mathbb{G}_m}
\end{array}
\]

where the action on the left is given by \(\alpha \cdot f(x) = \alpha^{-2} f(x)\) while on the right is the trivial one. There is an isomorphism \(\text{Pic}(\mathbb{G}_m(\mathbb{B}(2, 2g + 2))) \cong \mathbb{Z} \oplus \mathbb{Z}\), where the first component is generated by \(\mathcal{O}_{\mathbb{B}(2, 2g + 2)}(1)\) with the trivial action of \(\mathbb{G}_m\), and the second component is just \(\mathbb{G}_m^*\). The trivialization of the pull-back of \(\mathcal{O}_{\mathbb{B}(2, 2g + 2)}(1)\) on \(\mathbb{A}(2, 2g + 2) \setminus \{0\}\) is given by the section \(f(x) \to (f(x), f(x)) \in \mathbb{A}(2, 2g + 2) \setminus \{0\} \times \mathbb{A}(2, 2g + 2)\), so the pull back of the trivial \(\mathbb{G}_m\)-linearization of \(\mathcal{O}_{\mathbb{B}(2, 2g + 2)}(1)\) corresponds to the character \(-2 \in \mathbb{Z} = \mathbb{G}_m\), because of the action \(\alpha \cdot f(x) = \alpha^{-2} f(x)\). Thus we see, that the composition \(\mathbb{Z} = \text{Pic}_{PGL_2}(\mathbb{B}(2, 2g + 2)) \to
Pic\textsuperscript{G}\textsuperscript{m}(\mathbb{H}(2g + 2)) \to Pic\textsuperscript{G}\textsuperscript{m}(\mathbb{A}(2, 2g + 2)\setminus\{0\}) = \mathbb{Z} is equal to multiplication by 4.

We can complete the diagram \ref{diagram:Pic} from above as follows:

\[
\begin{array}{ccc}
GL_{2}/\mu_{g+1} & \xrightarrow{\quad} & PGL_{2} \\
\downarrow & & \downarrow \\
\mathbb{A}(2, 2g + 2)\setminus\{0\} & \xrightarrow{id} & \mathbb{B}(2, 2g + 2) \\
\downarrow & & \downarrow \\
\mathbb{A}(2, 2g + 2)\setminus\{0\} & \xrightarrow{id} & \mathbb{B}(2, 2g + 2).
\end{array}
\]

So, in the case \(g\) even we see from \ref{lemma:Pic} that the morphism \(\text{Pic}^{PGL_{2}}(\mathbb{B}(2, 2g + 2)) \to \text{Pic}^{GL_{2}/\mu_{g+1}}(\mathbb{A}(2, 2g + 2)\setminus\{0\}) = \text{Pic}^{GL_{2}}(\mathbb{A}(2, 2g + 2)\setminus\{0\})\) is an isomorphism \(\mathbb{Z} \xrightarrow{-1} \mathbb{Z}\), while for \(g\) odd we see from \ref{lemma:Pic} that \(\text{Pic}^{PGL_{2}}(\mathbb{B}(2, 2g + 2)) \to \text{Pic}^{GL_{2}/\mu_{g+1}}(\mathbb{A}(2, 2g + 2)\setminus\{0\})\) is equal to multiplication by 2.

Now the conclusion follows since \(\text{Pic}(\mathcal{D}_{2g+2})\) is the quotient of \(\text{Pic}^{PGL_{2}}(\mathbb{B}(2, 2g + 2)) = \mathbb{Z}\) of order 4\(g+2\) while \(\text{Pic}(\mathcal{H}_{g})\) is the quotient of \(\text{Pic}^{GL_{2}}(\mathbb{A}(2, 2g+2)\setminus\{0\}) = \mathbb{Z}\) of order 4\(g+2\) if \(g\) is even and order 2(4\(g+2\)) if \(g\) is odd (see theorem \ref{theorem:Pic}). \(\square\)

With the same technique of above, we can prove the following lemma that was used in the second proof of theorem \ref{theorem:Pic}.

**Lemma 6.4.** For a \(\mathbb{P}^{1}\)-family \(p: \mathcal{C} \to S\) with an effective Cartier divisor \(D \subset \mathcal{C}\) etale and finite of degree 2\(g+2\) over \(S\), the line bundle \(\det(p_{*}\omega_{\mathcal{C}/S}^{m})\) is trivial for any \(m \in \mathbb{Z}\).

**Proof.** By pulling-back to \(\mathcal{D}_{2g+2}\) (see theorem \ref{theorem:Pic}), one reduce to consider the \(PGL_{2}\)-equivariant line bundle \(\det(p_{*}\omega^{m})\) for the trivial family \(p: \mathbb{P}(E) \to \mathbb{B}_{sm}(2, 2g + 2)\) together with the incidence divisor \(D\), where \(E = V \times \mathbb{B}_{sm}(2, 2g + 2)\) is a trivial two-dimensional vector bundle. Using Euler formula \ref{equation:Euler} one expresses \(\det(p_{*}\omega^{m})\) as a power of \(\det(E)\) and hence \(PGL_{2}\) acts trivially on it. \(\square\)

Using theorem \ref{theorem:Pic} it is possible to proof a weaker form of theorem \ref{lemma:Pic} without computations for stacks. Namely, it is possible to proof the statement of theorem \ref{lemma:Pic} for \(g\) even and only up to 2-torsion (which is isomorphic to \(\mathbb{Z}/2\mathbb{Z}\)) for \(g\) odd.

In notations of theorem \ref{lemma:Pic} the generator \(G\) of the Picard group \(\text{Pic}(\mathcal{H}_{g})\) corresponds to the residue class of \(-1\) in Arsie-Vistoli description as a cyclic group (see theorem \ref{theorem:Pic}). Moreover, from the proof of theorem \ref{lemma:Pic} it follows that the map of cyclic groups \(\text{Pic}(\mathcal{D}_{2g+2}) \to \text{Pic}(\mathcal{H}_{g})\) is multiplication by \(-1\) for \(g\) even and by \(-2\) for \(g\) odd. Thus we have just to reinterpret the generator sheaf \(G_{\text{div}}\) from theorem \ref{lemma:Pic} in terms of the family of hyperelliptic curves and to take the square root in the case \(g\) odd. As in the proof of theorem \ref{lemma:Pic} one obtains that

\[f^{*}(\omega_{\mathcal{D}_{2g+2}/S}^{2g+1}(D)) \cong \omega_{\mathcal{F}/S}^{2g+1}(-(g+1)W) \otimes \mathcal{O}_{\mathcal{F}}(2W) = \omega_{\mathcal{F}/S}^{2g+1}(-(g-1)W),\]

where \(f: \mathcal{F} \to \mathcal{C}\) is the quotient over the hyperelliptic involution, \(W \subset \mathcal{F}\) is the Weierstrass divisor and \(D \subset \mathcal{C}\) is the branch divisor.

Now we want to study how much the stacks \(\mathcal{H}_{g}\) and \(\mathcal{D}_{2g+2}\) are far to be finely represented by their coarse moduli scheme \(\mathcal{H}_{g}\). Since the existence of automorphisms is always one of the most serious obstruction to the finess of moduli scheme, it’s very natural to restrict to the open subset \(\mathcal{H}_{g}^{0}\) of hyperelliptic curves without
extra-automorphisms as well as to the corresponding stacks $\mathcal{H}_g^0$ and $\mathcal{D}^0_{2g+2}$. In fact we get a positive answer for the stack $\mathcal{D}^0_{2g+2}$.

**Theorem 6.5.** $H^0_g$ is a fine moduli scheme for the functor $\mathcal{D}^0_{2g+2}$, i.e. the natural transformation of functors $$\Phi_D : \mathcal{D}^0_{2g+2} \xrightarrow{\cong} \text{Hom}(\mathcal{D}, H^0_g)$$ is an isomorphism.

**Proof.** We have to construct a family of $\mathbb{P}^1$ over $H^0_g$ plus an effective Cartier divisor (finite and étale of degree $2g + 2$ over $H^0_g$) that is universal for the functor $\mathcal{D}^0_{2g+2}$. To do this, we consider over $(\text{Sym}^{2g+2}(\mathbb{P}^1) - \Delta)^0$ (the open subset of $(2g+2)$-tuples without automorphisms) a trivial family of $\mathbb{P}^1$ together with the tautological divisor above it:

$$(\text{Sym}^{2g+2}(\mathbb{P}^1) - \Delta)^0 \times \mathbb{P}^1 \xrightarrow{(D, x)} \mathcal{D}_{2g+2} = \{(D, x) : x \in D\}$$

Now $PGL_2$ acts (naturally) on $(\text{Sym}^{2g+2}(\mathbb{P}^1) - \Delta)^0$ and diagonally on $(\text{Sym}^{2g+2}(\mathbb{P}^1) - \Delta)^0 \times \mathbb{P}^1$ and this action clearly preserves the tautological divisor $\mathcal{D}_{2g+2}$. Moreover, since we restrict over $(\text{Sym}^{2g+2}(\mathbb{P}^1) - \Delta)^0$, the action of $PGL_2$ is free on the family of $\mathbb{P}^1$ as well on the divisor $\mathcal{D}_{2g+2}$. Hence everything passes to the quotient giving the required universal family of $\mathbb{P}^1$ plus the divisor:

$$((\text{Sym}^{2g+2}(\mathbb{P}^1) - \Delta)^0 \times \mathbb{P}^1)/PGL_2 = C_g \xrightarrow{(D, x)} \mathcal{D}_{2g+2}/PGL_2 = \mathcal{D}_{2g+2}/PGL_2$$

Now since the parameterized objects are really without automorphisms, the existence of a tautological family shows that in fact it’s a universal one. □

In view of this result, we can reinterpret the last assertion in theorem 6.3 as follows:

**Corollary 6.6.** Assume that char($k$) doesn’t divide $2g + 2$. The natural map $\text{Pic}(H^0_g) \to \text{Pic}(H^0_g)$ is injective. Hence it’s an isomorphism for $g$ even, while it’s an inclusion of index 2 for $g$ odd.

**Remark 6.7.** Compare these results with the analogous ones for the moduli spaces of curves of genus $g \geq 3$ (results that up to now are known only over the complex numbers). In that case there is an inclusion

$$\text{Pic}(\mathcal{M}_g) \hookrightarrow \text{Pic}(\mathcal{M}_g) \cong \text{Pic}(\mathcal{M}_g^0) \cong \text{Cl}(\mathcal{M}_g^0) \cong \text{Cl}(\mathcal{M}_g).$$

It is known that $\text{Pic}(\mathcal{M}_g) \cong \mathbb{Z}$ generated by the Hodge class (see [Har] and [AC87]) but it’s still unknown the index of the first group into the second (see [AC87] section 4).

**Remark 6.8.** Using theorem 6.3 we could prove lemma 6.4 for families of divisors on $\mathbb{P}^1$ of degree $2g + 2$ without automorphisms and also find the generator of $\text{Pic}(H^0_g)$ repeating the proof of theorem 6.3 saying nothing about stacks. Thus without stack theory we could prove a weaker form of theorem 6.7 as in the discussion after lemma 6.4 and also theorem 6.8 for families of hyperelliptic curves without extra-automorphisms. In fact, this makes difference only for $g = 2$ by proposition 4.4.
Now we can study the natural transformation $\Phi : \mathcal{H}_g^0 \to \text{Hom}(-, H_0^0)$, or in other words we study how many families of hyperelliptic curves (without extra-automorphisms) can have the same modular map. We generalize Mumford’s arguments from the case of elliptic curves to the case of hyperelliptic curves without extra-automorphisms (see [Mum65, pag. 49-53, pag. 60-61]).

**Theorem 6.9.** Given a map $\phi : S \to H_g^0$, the set of families of hyperelliptic curves having $\phi$ as a modular map, if non empty, is a principal homogeneous space for $H^1_{\acute{e}t}(S, \mathbb{Z}/2\mathbb{Z})$.

**Proof.** Fix a map $\phi : S \to H_g^0$ and suppose that it is a modular map for some family of hyperelliptic curves over $S$. Let’s denote by $H_g^0(S)_{\phi}$ the (non-empty) set of families of hyperelliptic curves over $S$ having $\phi$ as modular map.

We are going to define an action of $H^1_{\acute{e}t}(S, \mathbb{Z}/2\mathbb{Z})$ on $H_g^0(S)_{\phi}$ as follows: for a family $\pi : \mathcal{F} \to S$ in $H_g(S)_{\phi}$ and an element of $H^1_{\acute{e}t}(S, \mathbb{Z}/2\mathbb{Z})$ (i.e. a double étale cover $f : S' \to S$), we define a new family $f \cdot \pi : \mathcal{F}' \to S$ of $H_g^0(S)_{\phi}$ by mean of the following diagram:

![Diagram](image)

where $j$ is the involution on $S'$ that exchanges the two sheets of the covering $f$, $i$ is the global hyperelliptic involution (see [5.1]) and $j \times i$ is the involution on the fiber product. So the new family $\mathcal{F}' \to S$ is obtained first by doing the pull-back of the family $\mathcal{F} \to S$ to $S'$ and then by taking the quotient with respect to $i \times j$.

Note that also the first part of the diagram is cartesian and that the original family $\mathcal{F} \to S$ can be re-obtained by taking the quotient of $S' \times S \mathcal{F}'$ with respect to the involution $j \times \text{id}$.

By construction, over a geometric point $s \in S$ the fibers of $\pi : \mathcal{F} \to S$ and $f \cdot \pi : \mathcal{F}' \to S$ are the same so that the new family is an element of $H_g(S)_{\phi}$ and the definition is well-posed.

We have to show that this action is simply transitive, namely that given two families $\pi_1 : \mathcal{F}_1 \to S$ and $\pi_2 : \mathcal{F}_2 \to S$ of hyperelliptic curves in $H_g(S)_{\phi}$ there exists a unique étale double cover of $S$ which realizes the construction in diagram 6.2. By general results of Grothendieck (see [Gro61]), there exists a scheme $\text{Isom}(\pi_1, \pi_2)$ over $S$ whose fiber over the geometric point $s \in S$ is

$$\text{Isom}(\pi_1, \pi_2)_s = \text{Isom}(\pi^{-1}_1(s), \pi^{-1}_2(s))$$

and hence, since the fibers of our families are hyperelliptic curves without extra-automorphisms, this is a double étale cover of $S$. Moreover the two families become
isomorphic above $\text{Isom}(\pi_1, \pi_2)$ and the corresponding diagram:

![Diagram](image)

satisfies exactly the property of diagram $6.2$ (see [Mum65, pag. 61]). Moreover this is the unique double cover with that property (see [Mum65, pag. 61]).

**Remark 6.10.** This proof is in fact an explicit form of a general principle and could be formulated shorter in the following way. Any two hyperelliptic families with the same modular map to $H^0_S$ are locally isomorphic in the étale topology since in a suitable étale neighborhood both families have a section. Moreover, the automorphism group of hyperelliptic families is $\mathbb{Z}$, branched along $\Phi$.

The non-uniqueness of a family with a given modular map may be also seen explicitly from the construction of double covers. By theorem 6.5, the action of $G$ acts on the scheme $X$ and maps $f_\alpha : U_\alpha \to X$, $g_{\alpha \beta} : U_\alpha \cap U_\beta \to G$ such that on the intersection $U_\alpha \cap U_\beta$ there is $f_\alpha = g_{\alpha \beta} f_\beta$ for all $\alpha, \beta$. Moreover, an equivalence (i.e. an isomorphisms of the initial groupoid) between $\{U_\alpha, f_\alpha, g_{\alpha \beta}\}$ and $\{U'_\alpha, f'_\alpha, g'_{\alpha \beta}\}$ is
given by a common subcovering $V_\gamma$ and maps $g_\gamma : V_\gamma \rightarrow G$ such that after suitable restriction $f_\gamma = g_\gamma f_\gamma'$ and $g_\gamma\delta = g_\gamma^{-1}g_\gamma g_\delta$ in evident notations. Thus the last map is just projection to $g_{\alpha\beta}$.

Now suppose we are given a central extension of group schemes
\[ 1 \rightarrow K \rightarrow G \rightarrow H \rightarrow 1 \]
and a free action of $H$ on the scheme $X$. Then for each scheme $S$ the group $H^1_{et}(S, K)$ acts naturally on $[X/G](S)$ by formula $\{U_\alpha, f_\alpha, g_{\alpha\beta}\} \mapsto \{U_\alpha, f_\alpha, g_{\alpha\beta} k_{\alpha\beta}\}$, where $k_{\alpha\beta}$ is a 1-cocycle.

**Proposition 6.11.** The set $(X/H)(S)$ is a quotient of $[X/G](S)$ under the action of $H^1_{et}(S, K)$.

**Proof.** Suppose $\{U_\alpha, f_\alpha, g_{\alpha\beta}\}$ and $\{U'_\alpha, f'_\alpha, g'_{\alpha\beta}\}$ are equivalent in $(X/H)(S)$. By definition there exists a smaller subcovering $V_\gamma$ and $h_\gamma : V_\gamma \rightarrow H$ such that $f_\gamma = h_\gamma f'_\gamma$. Taking, if necessary, a smaller subcovering, we may suppose that for each $\gamma$ there is $g_\gamma : V_\gamma \rightarrow G$ such that it naturally maps to $h_\gamma$. Thus multiplying by $g_\gamma$ we see that we could suppose from the very beginning that $f_\alpha = f'_\alpha$. Hence we obtain $f_\alpha = g_{\alpha\beta} f_\beta = g'_{\alpha\beta} f'_\beta$. This means that $g_{\alpha\beta} = k_{\alpha\beta} g'_{\alpha\beta}$ for a certain 1-cocycle $k_{\alpha\beta} : U_\alpha \rightarrow K$ since the action of $H$ on $X$ is free. \[ \square \]

Besides, the map $[X/G](S) \rightarrow (X/H)(S)$ is not surjective. The obvious cohomological obstruction is provided by the image of a given element from $(X/H)(S)$ under the composition $(X/H)(S) \rightarrow H^1(S, H) \rightarrow H^2(S, K)$.

In our case $G = GL_2/\mu_{2g+2}$, $H = GL_2/\mu_{2g+2}$, $K = \mathbb{Z}/2\mathbb{Z}$, $X = \mathbb{A}_{sm}(2, 2g + 2)^0$ and proposition 6.11 becomes theorem 6.9. The cohomological obstruction takes values in $H^2_{et}(S, \mathbb{Z}/2\mathbb{Z})$ which is a reinterpretation of theorem 3.3. Indeed, using the isomorphism from lemma 6.2 we see that for $g$ odd the exact sequence of groups in question is
\[ 0 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow GL_2^{(\det(\cdot), [\cdot])} \mathbb{G}_m \times PGL_2 \rightarrow 1 \]
while for $g$ even this is
\[ 0 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow \mathbb{G}_m \times PGL_2^{(2,1)} \mathbb{G}_m \times PGL_2 \rightarrow 1, \]

Thus the exact sequence
\[ 0 \rightarrow \text{Pic}(S)/2\text{Pic}(S) \rightarrow H^2_{et}(S, \mathbb{Z}/2\mathbb{Z}) \rightarrow \text{Br}(S)_2 \rightarrow 0. \]

shows that the triviality of the cohomological obstruction means for $g$ odd that a certain divisor should be divisible by two, and for $g$ even, in addition, that the $\text{Pic}^1$ family should be Zariski locally trivial.

**Example.** Let $S = \text{Spec}(k)$. Then $H^1_{et}(S, \mathbb{Z}/2\mathbb{Z}) = k^*/(k^*)^2$. If we fix a divisor $D \subset \mathbb{P}^1_k$ over $k$ of degree $2g + 2$ then the set of all hyperelliptic curves over $k$, corresponding to the pair $(\mathbb{P}^1, D)$, may be described as follows: any such hyperelliptic curve is locally given by the equation
\[ ay^2 = P(x) \]
where $P(x)$ is some fixed equation of the divisor $D$ on $\mathbb{A}^1 \subset \mathbb{P}^1$ and $a$ corresponds to a class from $k^*/(k^*)^2$.

In the last part of this section, we are going to investigate the existence of a tautological family of hyperelliptic curves over an open subset of $H_g$ (compare [HMSS exercise 2.3] after having replaced universal with tautological).

**Theorem 6.12.** There exists a tautological family of hyperelliptic curves over an open subset of $H_g$ if and only if $g$ is odd.
Proof. Clearly it’s enough to restrict to $H^0_g$. We proved in theorem 3.5 that over $H^0_g$ there exists a family $C_g$ of $\mathbb{P}^1$ plus a divisor $D_{2g+2}$ (finite and étale over $H^0_g$ of degree $2g+2$) that are universal. Hence if a tautological family of hyperelliptic curves exists over an open subset $U \subset H^0_g$, then it has to be a double cover of $C_g|_U$ branched along $D_{2g+2}$.

Now if $g$ is odd, theorem 3.5 (iii) gives the existence of a tautological family over an open subset of $H^0_g$.

On the other hand for $g$ even, the non-existence of a tautological family over any open subset of $H^0_g$ will follow from theorem 3.5 (ii) once we will prove that the family $C_g \rightarrow H^0_g$ is not Zariski locally trivial.

Let’s consider again the situation of theorem 4.5:

First of all from [Har II ex. 6.1] one gets:

$$\text{Pic}((\text{Sym}^{2g+2}(\mathbb{P}^1) - \Delta)^0 \times \mathbb{P}^1) = (\mathbb{Z} \cdot p_1^*(\mathcal{O}(1)))/(4g+2)\mathbb{Z} \oplus \mathbb{Z} \cdot p_2^*(\mathcal{O}(1)).$$

where $p_1$ and $p_2$ are the projections on the first and on the second factor.

To compute the Picard group of $C_g$, we use again the theory of equivariant Picard group of Mumford ([GIT]). Note that in this case the action is free so that actually

$$\text{Pic}(C_g) = \text{Pic}^{PGL_2}((\text{Sym}^{2g+2}(\mathbb{P}^1) - \Delta)^0 \times \mathbb{P}^1).$$

Since the action of $PGL_2$ is diagonal, it holds

$$\text{Pic}^{PGL_2}((\text{Sym}^{2g+2}(\mathbb{P}^1) - \Delta)^0 \times \mathbb{P}^1) = \text{Pic}^{PGL_2}((\text{Sym}^{2g+2}(\mathbb{P}^1) - \Delta)^0) \times \text{Pic}^{PGL_2}(\mathbb{P}^1).$$

We already proved (see 4.5 4.6 and the last part of the proof of theorem 4.7) that:

$$\text{Pic}^{PGL_2}((\text{Sym}^{2g+2}(\mathbb{P}^1) - \Delta)^0) = \begin{cases} 
\mathbb{Z}/(4g+2)\mathbb{Z} & \text{if } g \geq 3, \\
\mathbb{Z}/5\mathbb{Z} & \text{if } g = 2.
\end{cases}$$

generated by the hyperplane section. As for the action $\sigma : PGL_2 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$, we have that $\sigma^*(\mathcal{O}_{\mathbb{P}^1}(1)) = p_1^*(\mathcal{O}_{PGL_2}(1)) \otimes p_2^*(\mathcal{O}_{\mathbb{P}^1}(1))$ and, since $\mathcal{O}_{PGL_2}(1)$ is of $2$-torsion in $\text{Pic}(PGL_2)$, it follows that only $\mathcal{O}_{\mathbb{P}^1}(2)$ admits a $PGL_2$-linearization or in other words:

$$\text{Pic}^{PGL_2}(\mathbb{P}^1) = \mathbb{Z} \cdot \mathcal{O}_{\mathbb{P}^1}(2).$$

Therefore

$$\text{Pic}(C_g) = \begin{cases} 
(\mathbb{Z} \cdot p_1^*(\mathcal{O}(1)))/(4g+2)\mathbb{Z} \oplus \mathbb{Z} \cdot p_2^*(\mathcal{O}(2)) & \text{if } g \geq 3, \\
(\mathbb{Z} \cdot p_1^*(\mathcal{O}(1)))/5\mathbb{Z} \oplus \mathbb{Z} \cdot p_2^*(\mathcal{O}(2)) & \text{if } g = 2.
\end{cases}$$

Hence, since there doesn’t exist a line bundle of vertical degree $1$, by proposition 2.1 (iv) the family $C_g \rightarrow H^0_g$ is not Zariski locally trivial.

The non existence of a line bundle of vertical degree $1$ on the family $C_g$ over $H^0_g$ may be also deduced from the universality of this family and from the existence of any family of divisors without automorphisms of degree $2g+2$ on $\mathbb{P}^1$ which has no divisor of horizontal degree $1$. For example, as such family we could take an open subset of the set of all conics in $\mathbb{P}^2$, on which the divisor of degree $2g+2$ is defined by the intersection with an irreducible curve of degree $g+1$ in $\mathbb{P}^2$. □

In the preceding theorem 6.12 we really need to take an open subset of $H_g$ for $g$ odd. Namely, the following is true...
Proposition 6.13. There doesn’t exist a tautological family over all \( H_g \) (and neither over \( H_g^0 \)) for \( g \) odd.

Proof. Clearly it’s enough to restrict to \( H_g^0 \). There are two ways to prove this fact. In fact they are rather just two different ways of looking at the same situation.

The first way is to suppose that such family exists and consider the corresponding morphism \( H_g^0 \to H_g^0 \). It would imply the existence of a projection for the inclusion \( \text{Pic}(H_g^0) \hookrightarrow \text{Pic}(H_g^0) \) but this is impossible since the first group is isomorphic to \( \mathbb{Z}/(4g+2) \) by theorem 4.7 and the second is isomorphic to \( \mathbb{Z}/2(4g+2) \mathbb{Z} \) by the corollary 3.3. Thus we get a contradiction.

The second way is to compute explicitly the class of \( D_{2g+2} \) in the Picard group \( \text{Pic}(C_g) \) (recall that \( D_{2g+2} \) is the universal divisor in the family \( C_g \to H_g^0 \)). Since the fiber of \( D_{2g+2} \) over a point \( x \in \mathbb{P}^1 \) is a hyperplane in \( \text{Sym}^{2g+2}(\mathbb{P}^1) \) and over a point \( D \in (\text{Sym}^{2g+2}(\mathbb{P}^1) - \Delta) \) consists of \( 2g+2 \) points of \( \mathbb{P}^1 \), the class of \( D_{2g+2} \) in the Picard group \( \text{Pic}(\text{Sym}^{2g+2}(\mathbb{P}^1) - \Delta) \) is still equal to \( \mathbb{T} \in \mathbb{Z}/(4g+2) \mathbb{Z} \) so it is undivisible by 2 in the Picard group. Hence by remark 6.13 there isn’t any tautological family over \( H_g^0 \).

Note that the situation is different for \( g = 1 \) as the following remark shows (in this case \( \text{Pic}(H_1^0) = 0 \)).

Remark 6.14. There exists a tautological family over \( H_1^0 \cong \mathbb{A}^1 - \{0, 1728\} \) (the isomorphism is given by associating to every elliptic curve its \( j \)-function). The following is an explicitly example (it’s not unique!) of such a family (see Mum65 page 58):

\[
y^2 = x^3 + \frac{27}{4} \cdot \frac{1278 - j}{j} (x + 1).
\]

Remark 6.15. If one considers the moduli space of “framed” hyperelliptic curves (i.e. hyperelliptic curve \( C \) plus a fixed double cover \( C \to \mathbb{P}^1 \), which is just \( \text{Sym}^{2g+2}(\mathbb{P}^1) - \Delta \) without taking the quotient for \( \text{PGL}_2 \), then one can prove that there doesn’t exist a universal (neither a tautological!) family above it (see Ran91). Nevertheless such a tautological family exists over an open subset: for example, if we remove the hyperplane consisting of tuples containing the point at infinity then the usual equation \( y^2 = P(x) \), with \( P(x) \) a monic polynomial of degree \( 2g+2 \) with distinct roots, defines a tautological hyperelliptic curve (see Ran91).

7. Application

There is an interesting application of the theory developed above. Consider a family \( F \to S \) of smooth hyperelliptic curves of genus \( g \) with automorphism group \( \mathbb{Z}/2 \mathbb{Z} \) over a regular irreducible base \( S \). Let us make two assumptions:

- the corresponding modular map \( S \to H_g^0 \) is dominant and generically finite,
- the family \( F \) satisfies the conditions of proposition 6.14.

By its universal property the nontrivial element \( \alpha \) in \( \text{Br}(k(H_g^0)) \), which corresponds the restriction of the family \( C_g \) on the generic point, becomes trivial in \( \text{Br}(k(S)) \) (see the cohomological interpretation of proposition 2.11. Recall the following well-known fact (see Ser page 12)):
Lemma 7.1. If a profinite group $H$ is a subgroup of order $n$ inside a profinite group $G$, then for any $G$–module $M$ and $i \geq 1$ the composition of restriction and corestriction maps $H^i(G, M) \xrightarrow{\text{res}} H^i(H, M) \xrightarrow{\text{cor}} H^i(G, M)$ is equal to the multiplication by $n$.

This lemma implies that $[k(S) : k(H^0_g)]$ must be even since $\alpha$ is of order 2 being a class of a conic over $k(H^0_g)$.

Remark 7.2. We cannot give a more precise statement about the divisibility of $[k(S) : k(H^0_g)]$ even if we replace the second assumption by a stronger one: $F$ has a rational section. Indeed, we may take any double cover $S$ of $H^0_g$ over which the pull-back of $C_g$ is Zariski locally trivial and then use the explicit construction from the last example to define a desired family of hyperelliptic curves over a Zariski open subset in $S$.

Remark 7.3. If we replace smooth hyperelliptic curves of genus $g$ with automorphism group $\mathbb{Z}/2\mathbb{Z}$ by smooth complex curves of genus $g$ without automorphisms and assume the analogous hypothesis about a family of such curves (namely that the modular map is generically finite and dominant and the family has a rational section), then the answer will be that the degree of a modular map should be always divisible by $2g - 2$ (Caporaso lemma 5). It follows from a deep statement that says that, over the complex numbers, the relative Picard group of the universal family over $M^0_g$ — the fine moduli space of smooth curves of genus $g$ without automorphisms — is generated by the relative canonical sheaf (this was first claimed by Franchetta but a correct proof is due to Harer and Arbarello-Cornalba). There is an analogous statement about families of trigonal curves (in arbitrary characteristic), which is also proved by considering the relative Picard group of the universal family (see [GV]). However, the proof for hyperelliptic case, as presented here, has a rather different spirit.

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