Some properties on $G$-evaluation and its applications to $G$-martingale decomposition

Yongsheng Song
Academy of Mathematics and Systems Science,
Chinese Academy of Sciences, Beijing, China;
yssong@amss.ac.cn

Abstract

In this article, a sublinear expectation induced by $G$-expectation is introduced, which is called $G$-evaluation for convenience. As an application, we prove that for any $\xi \in L^\beta_G(\Omega_T)$ with some $\beta > 1$ the decomposition theorem holds and that any $\beta > 1$ integrable symmetric $G$-martingale can be represented as an Itô integral w.r.t $G$-Brownian motion. As a byproduct, we prove a regular property for $G$-martingale: Any $G$-martingale $\{M_t\}$ has a quasi-continuous version.

Key words: $G$-expectation, $G$-evaluation, $G$-martingale, Decomposition theorem

MSC-classification: 60G07, 60G20, 60G44, 60G48, 60H05

1 Introduction

Recently, [P06], [P08] introduced the notion of sublinear expectation space, which is a generalization of probability space. One of the most important sublinear expectation space is $G$-expectation space. As the counterpart of Wiener space in the linear case, the notions of $G$-Brownian motion, $G$-martingale, and Itô integral w.r.t $G$-Brownian motion were also introduced. These notions have very rich and interesting new structures which nontrivially generalize the classical ones.

Because of the Sublinearity, the fact of $\{M_t\}$ being a $G$-martingale does not imply that $\{-M_t\}$ is a $G$-martingale. A surprising fact is that there
exist nontrivial processes which are continuous, decreasing and are also $G$-martingales. [P07] conjectured that for any $\xi \in L^1_G(\Omega_T)$, we have the following representation:

$$X_t := \hat{E}_t(\xi) = \hat{E}(\xi) + \int_0^t Z_s dB_s - K_t, \; t \in [0, T],$$

with $K_0 = 0$ and $\{K_t\}$ an increasing process.

[P07] proved the conjecture for cylindrical functions $Lip(\Omega_T)$ (see Theorem 2.18) by Itô’s formula in the setting of $G$-expectation space. So the left question is to extend the representation to the completion $L^1_G(\Omega_T)$ of $Lip(\Omega_T)$ under norm $\|\xi\|_{1,G} = \hat{E}(|\xi|)$.

[STZ09] make a progress in this direction. They define a much stronger norm $\|\xi\|_{L^2_G} = \{\hat{E}[\sup_{t \in [0,T]} \hat{E}_t(|\xi|^2)]\}^{1/2}$ on $Lip(\Omega_T)$ and generalized the above result to the completion $L^1_G(\Omega_T)$ of $Lip(\Omega_T)$. The shortcoming of this result is that no relations between the two norms are given. The space $L^2_G$ is just an abstract completion and we have no idea about the set $L^2_G(\Omega_T) \setminus L^0_G$.

The purpose of this article is to improve the decomposition theorem given in [P07] and [STZ09]. The main results of the article consist of:

We introduce a sublinear expectation called $G$-evaluation and investigate its properties. By presenting an estimate, which can be seen as the substitute of Doob’s maximal inequality, we proved that for any $\xi \in L^\beta_G(\Omega_T)$ with some $\beta > 1$ the decomposition theorem holds and that any $\beta > 1$ integrable symmetric $G$-martingale can be represented as an Itô integral w.r.t $G$-Brownian motion.

As a byproduct, we prove a regular property for $G$-martingale: Any $G$-martingale $\{M_t\}$ has a quasi-continuous version (see Definition 5.1). We also give several estimates for variables in the decomposition theorem, which may be useful in the follow-up work of $G$-martingale theory.

This article is organized as follows: In section 2, we recall some basic notions and results of $G$-expectation and the related space of random variables. In section 3, we introduce the notion of $G$-evaluation and present an estimate, which can be seen as the substitute of Doob’s maximal inequality. In section 4, we prove that for any $\beta > 1$ integrable $G$-martingale, the decomposition theorem holds and that any $\beta > 1$ integrable symmetric $G$-martingale can be represented as an Itô integral w.r.t $G$-Brownian motion. In section 5, we prove a regular property for $G$-martingale.
2 Preliminary

We present some preliminaries in the theory of sublinear expectations and the related $G$-Brownian motions. More details of this section can be found in [P07].

2.1 $G$-expectation

**Definition 2.1** Let $\Omega$ be a given set and let $\mathcal{H}$ be a linear space of real valued functions defined on $\Omega$ with $c \in \mathcal{H}$ for all constants $c$. $\mathcal{H}$ is considered as the space of random variables. A sublinear expectation $\hat{E}$ on $\mathcal{H}$ is a functional $\hat{E}: \mathcal{H} \rightarrow \mathbb{R}$ satisfying the following properties: for all $X, Y \in \mathcal{H}$, we have

(a) Monotonicity: If $X \geq Y$ then $\hat{E}(X) \geq \hat{E}(Y)$.

(b) Constant preserving: $\hat{E}(c) = c$.

(c) Sub-additivity: $\hat{E}(X) - \hat{E}(Y) \leq \hat{E}(X - Y)$.

(d) Positive homogeneity: $\hat{E}(\lambda X) = \lambda \hat{E}(X)$, $\lambda \geq 0$.

$(\Omega, \mathcal{H}, \hat{E})$ is called a sublinear expectation space.

**Definition 2.2** Let $X_1$ and $X_2$ be two $n$-dimensional random vectors defined respectively in sublinear expectation spaces $(\Omega_1, \mathcal{H}_1, \hat{E}_1)$ and $(\Omega_2, \mathcal{H}_2, \hat{E}_2)$. They are called identically distributed, denoted by $X_1 \sim X_2$, if $\hat{E}_1[\varphi(X_1)] = \hat{E}_2[\varphi(X_2)]$, $\forall \varphi \in \mathcal{C}_{l,lip}(\mathbb{R}^n)$, where $\mathcal{C}_{l,lip}(\mathbb{R}^n)$ is the space of real continuous functions defined on $\mathbb{R}^n$ such that

$$|\varphi(x) - \varphi(y)| \leq C(1 + |x|^k + |y|^k)|x - y|, \forall x, y \in \mathbb{R}^n,$$

where $k$ depends only on $\varphi$.

**Definition 2.3** In a sublinear expectation space $(\Omega, \mathcal{H}, \hat{E})$ a random vector $Y = (Y_1, \ldots, Y_n), Y_i \in \mathcal{H}$ is said to be independent to another random vector $X = (X_1, \ldots, X_m)$, $X_i \in \mathcal{H}$ under $\hat{E}(\cdot)$ if for each test function $\varphi \in \mathcal{C}_{l,lip}(\mathbb{R}^m \times \mathbb{R}^n)$ we have $\hat{E}[\varphi(X, Y)] = \hat{E}[\hat{E}[\varphi(x, y)]_{x=\hat{X}}]$.

**Definition 2.4** ($G$-normal distribution) A $d$-dimensional random vector $X = (X_1, \ldots, X_d)$ in a sublinear expectation space $(\Omega, \mathcal{H}, \hat{E})$ is called $G$-normal distributed if for each $a, b \in \mathbb{R}$ we have

$$aX + b\hat{X} \sim \sqrt{a^2 + b^2}X,$$

where $\hat{X}$ is an independent copy of $X$. Here the letter $G$ denotes the function

$$G(A) := \frac{1}{2} \hat{E}[(AX, X)] : S_d \rightarrow \mathbb{R},$$

where $S_d$ is the set of all $d$-dimensional symmetric matrices.
where $S_d$ denotes the collection of $d \times d$ symmetric matrices.

The function $G(\cdot): S_d \to R$ is a monotonic, sublinear mapping on $S_d$ and $G(A) = \frac{1}{2}E[(AX, X)] \leq \frac{1}{2}|A|E[X]^2] =: \frac{1}{2}|A|\sigma^2$ implies that there exists a bounded, convex and closed subset $\Gamma \subset S^2_d$ such that

$$G(A) = \frac{1}{2}\sup_{\gamma \in \Gamma} Tr(\gamma A).$$

If there exists some $\beta > 0$ such that $G(A) - G(B) \geq \beta Tr(A - B)$ for any $A \geq B$, we call the $G$-normal distribution is non-degenerate, which is the case we consider throughout this article.

**Definition 2.5** i) Let $\Omega_T = C_0([0, T]; R^d)$ with the supremum norm, $\mathcal{H}_T^0 := \{\varphi(B_{t_1}, ..., B_{t_n})| \forall n \geq 1, t_1, ..., t_n \in [0, T], \forall \varphi \in C_{lip}(R^{d \times n})\}$, $G$-expectation is a sublinear expectation defined by

$$\tilde{E}[\varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \cdots, B_{t_m} - B_{t_{m-1}})]$$

$$= \tilde{E}[\varphi(\sqrt{t_1 - t_0}{\xi_1}, \cdots, \sqrt{t_m - t_{m-1}}{\xi_m})],$$

for all $X = \varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \cdots, B_{t_m} - B_{t_{m-1}})$, where $\xi_1, \cdots, \xi_n$ are identically distributed $d$-dimensional $G$-normal distributed random vectors in a sublinear expectation space $(\Omega, \mathcal{H}, \tilde{E})$ such that $\xi_{i+1}$ is independent to $(\xi_1, \cdots, \xi_i)$ for each $i = 1, \cdots, m$. $(\Omega_T, \mathcal{H}_T^0, \tilde{E})$ is called a $G$-expectation space.

ii) For $t \in [0, T]$ and $\xi = \varphi(B_{t_1}, ..., B_{t_n}) \in \mathcal{H}_T^0$, the conditional expectation defined by (there is no loss of generality, we assume $t = t_i$)

$$\tilde{E}_{t_i}[\varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \cdots, B_{t_m} - B_{t_{m-1}})]$$

$$= \tilde{E}_{t_i}(\varphi(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \cdots, B_{t_i} - B_{t_{i-1}}),$$

where

$$\tilde{E}_{t_i}(x_1, \cdots, x_i) = \tilde{E}[\varphi(x_1, \cdots, x_i, B_{t_{i+1}} - B_{t_i}, \cdots, B_{t_m} - B_{t_{m-1}})].$$

Let $\|\xi\|_{p,G} = [\tilde{E}(|\xi|^p)]^{1/p}$ for $\xi \in \mathcal{H}_T^0$ and $p \geq 1$, then $\forall t \in [0, T]$, $\tilde{E}_t(\cdot)$ is a continuous mapping on $\mathcal{H}_T^0$, with norm $\|\cdot\|_{1,G}$ and therefore can be extended continuously to the completion $L^1_G(\Omega_T)$ of $\mathcal{H}_T^0$ under norm $\|\cdot\|_{1,G}$.

**Proposition 2.6** Conditional expectation defined above has the following properties: for $X, Y \in L^1_G(\Omega_T)$

i) If $X \geq Y$, then $\tilde{E}_t(X) \geq \tilde{E}_t(Y)$.

ii) $\tilde{E}_t(\eta) = \eta$, for $\eta \in L^1_G(\Omega_t)$.  

4
\[
\hat{E}(X) = \max_{P \in \mathcal{P}} E_P(X) \quad \text{for all } X \in \mathcal{H}_T^0.
\]

\(\mathcal{P}\) is called a set that represents \(\hat{E}\).

**Remark 2.8**

i) [HP09] gave a new proof to the above theorem. From the proof we know that any sublinear expectation \(E\) on \(\mathcal{H}_T^0\) satisfying

\[\mathcal{E}[(B_t - B_s)^2] \leq d_n(t - s)^n, \forall n \in \mathbb{N},\]

has the above representation.

ii) Let \(\mathcal{A}\) denotes the sets that represent \(\hat{E}\). \(\mathcal{P}^* = \{P \in \mathcal{M}_1(\Omega_T)|E_P(\xi) \leq \hat{E}(\xi), \forall \xi \in \mathcal{H}_T^0\}\) is obviously the maximal one, which is convex and weak compact. However, by Choquet capacitability Theorem, all capacities induced by weak compact sets of probabilities in \(\mathcal{A}\) are the same, i.e. \(c_P := \sup_{P \in \mathcal{P}} P = \sup_{P \in \mathcal{P}^*} P =: c_{\mathcal{P}^*}\) for any weak compact set \(\mathcal{P}, \mathcal{P}^* \in \mathcal{A}\). So we call it the capacity induced by \(\hat{E}\). In fact, By Choquet capacitability Theorem, it suffices to prove the compact sets case. For any compact set \(\hat{K} \subset \Omega_T\), there exists an decreasing sequence \(\{\varphi_n\} \subset C_b^\infty(\Omega_T)\) such that \(1_K \leq \varphi_n \leq 1\) and \(\varphi_n \downarrow 1_K\). Then by Theorem 28 in [DHP08],

\[c_P(K) = \lim_{n \to \infty} \sup_{P \in \mathcal{P}} E_P(\varphi_n) = \lim_{n \to \infty} \hat{E}(\varphi_n) = \lim_{n \to \infty} \sup_{P \in \mathcal{P}^*} E_P(\varphi_n) = c_{\mathcal{P}^*}(K).\]

iii) All capacities induced by sets of probabilities in \(\mathcal{A}\) are the same on open sets. In fact, for any \(\mathcal{P} \in \mathcal{A}\), let \(\hat{\mathcal{P}}\) be the weak closure of \(\mathcal{P}\). Since \(c_P = c_{\mathcal{P}}\) on open sets, we get the desired result by ii).

iv) Let \((\Omega^0, \{\mathcal{F}_t^0\}, \mathcal{F}, P^0)\) be a filtered probability space, and \(\{W_t\}\) be a d-dimensional Brownian motion under \(P^0\). [DHP08] proved that

\[\mathcal{P}'_M := \{P_0 \circ X^{-1}|X_t = \int_0^t h_s dW_s, h \in L_2^T([0, T]; \Gamma^1/2)\} \in \mathcal{A},\]

where \(\Gamma^1/2 := \{\gamma^{1/2}|\gamma \in \Gamma\}\) and \(\Gamma\) is the set in the representation of \(G(\cdot)\).

v) Let \(\mathcal{P}_M\) be the weak closure of \(\mathcal{P}'_M\). Then under each \(P \in \mathcal{P}_M\), the canonical process \(B_t(\omega) = \omega_t\) for \(\omega \in \Omega_T\) is a martingale. In fact, for any
$P \in \mathcal{P}_M$, there exists $\{P_n\} \subset \mathcal{P}'_M$ such that $P_n \to P$ weakly. For any $0 \leq s \leq t \leq T$ and $\varphi \in C_b(\Omega_s)$, $E_{P_n}[\varphi(B)B_s] = E_{P_n}[\varphi(B)B_t]$ since $\{B_t\}$ is a martingale under $P_n$. Then by the integrability of $B_t, B_s$ and the weak convergence of $\{P_n\}$ we have

$$E_P[\varphi(B)B_s] = \lim_n E_{P_n}[\varphi(B)B_s] = \lim_n E_{P_n}[\varphi(B)B_t] = E_P[\varphi(B)B_t].$$

Thus we get the desired result. □

Definition 2.9 i) Let $c$ be the capacity induced by $\hat{\mathcal{E}}$. A map $X$ on $\Omega_T$ with values in a topological space is said to be quasi-continuous w.r.t $c$ if

$$\forall \varepsilon > 0, \text{there exists an open set } O \text{ with } c(O) < \varepsilon \text{ such that } X|_O \text{ is continuous.}$$

ii) We say that $X : \Omega_T \to R$ has a quasi-continuous version if there exists a quasi-continuous function $Y : \Omega_T \to R$ with $X = Y$, c.q.s.. □

By the definition of quasi-continuity and iii) in Remark 2.8, we know that the collections of quasi-continuous functions w.r.t. capacities induced by any set(not necessary weak compact) that represents $\hat{\mathcal{E}}$ are the same.

Let $\|\varphi\|_{p,G} = [\hat{\mathcal{E}}(|\varphi|^p)]^{1/p}$ for $\varphi \in C_b(\Omega_T)$, the completions of $C_b(\Omega_T)$, $\mathcal{H}_T^0$ and $L_{ip}(\Omega_T)$ under $\|\cdot\|_{p,G}$ are the same and denoted by $L_{ip}^p(\Omega_T)$, where

$$L_{ip}(\Omega_T) := \{\varphi(B_{t_1},...,B_{t_n})|\forall n \geq 1, t_1,...,t_n \in [0,T], \forall \varphi \in C_{b,Lip}(R^{d \times n})\}$$

and $C_{b,Lip}(R^{d \times n})$ denotes the set of bounded Lipschitz functions on $R^{d \times n}$. Theorem 2.10[DHP08] For $p \geq 1$ the completion $L_{ip}^p(\Omega_T)$ of $C_b(\Omega_T)$ is

$$L_{ip}^p(\Omega_T) = \{X \in L^0 : X \text{ has a q.c. version, } \lim_{n \to \infty} \hat{\mathcal{E}}[|X|^p1_{|X| > n}] = 0\},$$

where $L^0$ denotes the space of all R-valued measurable functions on $\Omega_T$.

2.2 Basic notions on stochastic calculus in sublinear expectation space

Now we shall introduce some basic notions on stochastic calculus in sublinear expectation space $(\Omega_T, L_{ip}^1, \hat{\mathcal{E}})$. The canonical process $B_t(\omega) = \omega_t$ for $\omega \in \Omega_T$ is called $G$-Brownian motion.

For convenience of description, we only give the definition of Itô integral with respect to 1-dimensional $G$-Brownian motion. However, all results in sections 3-5 of this article hold for the d-dimensional case.
For $p \geq 1$, let $M_{G}^{p, 0}(0, T)$ be the collection of processes in the following form: for a given partition $\{t_0, \cdots, t_N\} = \pi_T$ of $[0, T]$,

$$
\eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega) 1_{[t_j, t_{j+1})}(t),
$$

where $\xi_i \in L^p_G(\Omega_{t_i})$, $i = 0, 1, 2, \cdots, N - 1$. For each $\eta \in M_{G}^{p, 0}(0, T)$, let $
\|\eta\|_{M_{G}^{p}} = \{\hat{E} \int_0^T |\eta_s|^p ds\}^{1/p}
$ and denote $M_{G}^{p}(0, T)$ the completion of $M_{G}^{p, 0}(0, T)$ under norm $\|\cdot\|_{M_{G}^{p}}$.

**Definition 2.11** For each $\eta \in M_{G}^{2, 0}(0, T)$ with the form

$$
\eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega) 1_{[t_j, t_{j+1})}(t),
$$

we define

$$
I(\eta) = \int_0^T \eta(s) dB_s := \sum_{j=0}^{N-1} \xi_j(B_{t_{j+1}} - B_{t_j}).
$$

The mapping $I : M_{G}^{2, 0}(0, T) \rightarrow L_{G}^2(\Omega_T)$ is continuous and thus can be continuously extended to $M_{G}^{2}(0, T)$.

We denote for some $0 \leq s \leq t \leq T$, $\int_s^t \eta_u dB_u := \int_0^T 1_{[s, t]}(u) \eta_u dB_u$. We have the following properties:

Let $\eta, \theta \in M_{G}^{2}(0, T)$ and let $0 \leq s \leq r \leq t \leq T$. Then in $L_{G}^2(\Omega_T)$ we have

(i) $\int_s^t \eta_u dB_u = \int_s^r \eta_u dB_u + \int_r^t \eta_u dB_u$,

(ii) $\int_s^t (\alpha \eta_u + \theta_u) dB_u = \alpha \int_s^t \eta_u dB_u + \int_s^t \theta_u dB_u$, if $\alpha$ is bounded and in $L_{G}^1(\Omega_s)$,

(iii) $\hat{E}_t(X + \int_0^T \eta_s dB_s) = \hat{E}(X), \forall X \in L_{G}^1(\Omega_T)$ and $\eta \in M_{G}^{2}(0, T)$.

**Definition 2.12** Quadratic variation process of $G$-Brownian motion defined by

$$
\langle B \rangle_t = B_t^2 - 2 \int_0^t B_s dB_s
$$

is a continuous, nondecreasing process.

For $\eta \in M_{G}^{1, 0}(0, T)$, define $Q_{0,T}(\eta) = \int_0^T \eta(s) d\langle B \rangle_s := \sum_{j=0}^{N-1} \xi_j(\langle B \rangle_{t_{j+1}} - \langle B \rangle_{t_j}) : M_{G}^{1, 0}(0, T) \rightarrow L_{G}^1(\Omega_T)$. The mapping is continuous and can be extended to $M_{G}^{1}(0, T)$ uniquely.

**Definition 2.13** A process $\{M_t\}$ with values in $L_{G}^1(\Omega_T)$ is called a $G$-martingale if $\hat{E}_s(M_t) = M_s$ for any $s \leq t$. If $\{M_t\}$ and $\{-M_t\}$ are both $G$-martingale, we call $\{M_t\}$ a symmetric $G$-martingale.
**Definition 2.14** For two processes \( \{X_t\}, \{Y_t\} \) with values in \( L^1_G(\Omega_T) \), we say \( \{X_t\} \) is a version of \( \{Y_t\} \) if
\[
X_t = Y_t, \quad \text{q.s. \forall } t \in [0, T].
\]

By the same arguments as in the classical linear case, for which we refer to [HWY92] for instance, we have the following lemma.

**Lemma 2.15** Any symmetric \( G \)-martingale \( \{M_t\}_{t \in [0,T]} \) has a RCLL (right continuous with left limit) version. \( \square \)

In the rest of this article, we only consider the RCLL versions of symmetric \( G \)-martingales.

**Theorem 2.16 [P07]** For each \( x \in \mathbb{R}, Z \in M^2_G(0, T) \) and \( \eta \in M^1_G(0, T) \), the process
\[
M_t = x + \int_0^t Z_s dB_s + \int_0^t \eta_s d\langle B \rangle_s - \int_0^t 2G(\eta_s)ds, \quad t \in [0, T]
\]
is a martingale. \( \square \)

**Remark 2.17** Specially, \( -K_t = \int_0^t \eta_s d\langle B \rangle_s - \int_0^t 2G(\eta_s)ds \) is a \( G \)-martingale, which is a surprising result because \( -K_t \) is a continuous, non-increasing process. [P07] conjectured that any \( G \)-martingale has the above form and gave the following result. \( \square \)

**Theorem 2.18 [P07]** For all \( \xi = \varphi(B_{t_1} - B_{t_0}, \ldots, B_{t_n} - B_{t_{n-1}}) \in L_{ip}(\Omega_T) \), we have the following representation:
\[
\xi = \hat{E}_t(\varphi) + \int_0^T Z_t dB_t + \int_0^T \eta_t d\langle B \rangle_t - \int_0^T 2G(\eta_t)dt.
\]  \hspace{1cm} (2.2.1)

where \( Z \in M^2_G(0, T) \) and \( \eta \in M^1_G(0, T) \).

[STZ09] defined \( \|\xi\|_{L^2} = \{\hat{E}[\sup_{t \in [0,T]} |\hat{E}_t(\xi)|] \}^{1/2} \) on \( L_{ip}(\Omega_T) \) and generalized the above result to the completion \( L^0_2 \) of \( L_{ip}(\Omega_T) \) under \( \| \cdot \|_{L^2} \).

**Theorem 2.19 [STZ09]** For all \( \varphi \in L^0_2 \), there exists \( \{Z_t\}_{t \in [0,T]} \in M^2_G(0, T) \) and a continuous increasing process \( \{K_t\}_{t \in [0,T]} \) with \( K_0 = 0, K_T \in L^0_2(\Omega_T) \) and \( \{-K_t\}_{t \in [0,T]} \) a \( G \)-martingale such that
\[
X_t := \hat{E}_t(\varphi) = \hat{E}(\varphi) + \int_0^t Z_s dB_s - K_t =: M_t - K_t, \quad \text{q.s.} \hspace{1cm} (2.2.2)
\]
\( \square \)
3 G-evaluation

In this section, we introduce an sublinear expectation which is induced by G-expectation and investigate some of its properties.

For \( \xi \in \mathcal{H}_T^0 \), let \( \mathcal{E}(\xi) = \hat{E}[\sup_{u \in [0,T]} \hat{E}_u(\xi)] \) for all \( \xi \in \mathcal{H}_T^0 \) where \( \hat{E} \) is the G-expectation. For convenience, we call \( \mathcal{E} \) G-evaluation. First we give the following representation for G-evaluation, which is similar to that of G-expectation.

**Theorem 3.1** There exists a weak compact subset \( \mathcal{P}^\mathcal{E} \subset M_1(\Omega) \) such that

\[
\mathcal{E}(\xi) = \max_{P \in \mathcal{P}^\mathcal{E}} \mathcal{E}_P(\xi) \quad \text{for all} \quad \xi \in \mathcal{H}_T^0.
\]

**Proof.** 1. Obviously, \((\Omega, \mathcal{H}_T^0, \mathcal{E})\) is a sublinear expectation space. Then there exists a family of positive linear functionals \( \mathcal{I}_P \) on \( \mathcal{H}_T^0 \) such that

\[
\mathcal{E}(\xi) = \max_{I \in \mathcal{I}_P} I(\xi) \quad \text{for all} \quad \xi \in \mathcal{H}_T^0.
\]

2. In the following, we give some calculations.

For any \( 0 \leq s < t \leq T \) and \( u \in [0, T] \),

\[
\hat{E}_u|B_t - B_s|^{2n} \leq \begin{cases} |B_t - B_s|^{2n}, & \text{if } u \geq t, \\ c_n(t-s)^n, & \text{if } u \leq s, \\ 2^{2n-1}[c_n(t-u)^n + |B_u - B_s|^{2n}], & \text{if } s < u < t. \end{cases}
\]

Thus \( \hat{E}_u|B_t - B_s|^{2n} \leq 2^{2n-1}[c_n(t-s)^n + \sup_{u \in [s,t]} |B_u - B_s|^{2n}] \), and by B-D-G inequality

\[
\mathcal{E}|B_t - B_s|^{2n} \leq 2^{2n-1}[c_n(t-s)^n + b_n(t-s)^n] =: d_n(t-s)^n.
\]

3. Noting the discussion in Remark 2.8, we can prove the desired representation by just the same arguments as in [HP09]. \( \square \)

For \( p \geq 1 \) and \( \xi \in \mathcal{H}_T^0 \), define \( \|\xi\|_{p,\mathcal{E}} = [\mathcal{E}(|\xi|^p)]^{1/p} \) and denote \( L^p_{\mathcal{E}}(\Omega_T) \) the completion of \( \mathcal{H}_T^0 \) under \( \| \cdot \|_{p,\mathcal{E}} \).

We shall give an estimate between the two norms \( \| \cdot \|_{p,\mathcal{E}} \) and \( \| \cdot \|_{p,G} \). As the substitute of Doob’s maximal inequality, the estimate will play a critical role in the proof to the martingale decomposition theorem in the next section. First, we shall give a lemma.

For convenience, we say \( \xi \) is symmetric if \( \xi \in L^1_G \) with \( \hat{E}(\xi) + \hat{E}(-\xi) = 0 \).
Lemma 3.2 For $\xi \in L_{lip}(\Omega_T)$, there exists nonnegative $K_T \in L^1_G(\Omega_T)$ such that $\xi + K_T$ is symmetric. Moreover, for any $1 < \gamma < \beta$, $\gamma \leq 2$, $K_T \in L^2_G(\Omega_T)$ and

$$\|K_T\|^{\gamma}_{L^\gamma_G} \leq 14C_{\beta/\gamma}^{\gamma} \|\xi\|^{\beta}_{L^\beta_G},$$

where $C_{\beta/\gamma} = \sum_{i=1}^{\infty} i^{-\beta/\gamma}$.

**Proof.** Let $\xi^n = (\xi \wedge n) \lor (-n)$ and $\eta^n = \xi^{n+1} - \xi^n$ for $n \geq 0$. Then by Theorem 2.18, for each $n$, we have the following representation (2.2.1):

$$X^n_t := \hat{E}_t(\eta^n) = M^n_t - K^n_t,$$

where $\{M^n_t\}$ is a symmetric $G$-martingale with $M^n_T \in L^2_G(\Omega_T)$ and $\{K^n_t\}$ is a continuous increasing process with $K^n_0 = 0$, $K^n_T \in L^2_G(\Omega_T)$ and $\{-K^n_t\}$ a $G$-martingale. Fix $P \in \mathcal{P}_M$. By Itô’s formula

$$(\eta^n)^2 = 2 \int_0^T X^n_t dX^n_t + [M^n]_T, \ P - a.s.$$\)

Take expectation under $P$, we have

$$E_P[(M^n_T)^2] \leq E_P[(\eta^n)^2] + 2E_P(K^n_T).$$

Take supremum over $\mathcal{P}_M$, we have

$$\hat{E}[(M^n_T)^2] \leq \hat{E}[(\eta^n)^2] + 2\hat{E}(K^n_T) \leq 5\hat{E}(|\eta^n|).$$

Therefore,

$$\hat{E}[(K^n_T)^2] \leq 2(\hat{E}[(\eta^n)^2] + \hat{E}[(M^n_T)^2]) \leq 12\hat{E}(|\eta^n|).$$

Consequently, for any $1 < \gamma < \beta$ and $\gamma \leq 2$

$$\hat{E}[(K^n_T)^\gamma] \leq \hat{E}(K^n_T) + \hat{E}[(K^n_T)^2]) \leq 14\hat{E}(|\eta^n|).$$

$$\hat{E}[(\sum_{i=n+1}^{n+m} K^i_T)^\gamma] \leq \sum_{i=n+1}^{n+m} i^{-\beta/\gamma} \sum_{i=n+1}^{n+m} i^{\beta/\gamma} \hat{E}[(K^i_T)^\gamma] \leq 14C_{\beta/\gamma}^{\gamma-1}(n, m) \sum_{i=n+1}^{n+m} i^{\beta/\gamma} \hat{E}(|\eta^i|) \leq 14C_{\beta/\gamma}^{\gamma-1}(n, m) \sum_{i=n+1}^{n+m} i^{\beta/\gamma} c(|\xi| > i) \leq 14\hat{E}(|\xi|^\beta)C_{\beta/\gamma}^{\gamma}(n, m).$$
where \( C_{\beta/\gamma}(n, m) = \sum_{i=n+1}^{n+m} i^{-\beta/\gamma} \), \( \gamma^* = \gamma/(\gamma - 1) \).

So \( \{ \sum_{n=0}^{\infty} K_T^n \} \) is a Cauchy sequence in \( L_G^1(\Omega_T) \). Let \( K_T := \lim_{L_G^1(\Omega_T)} \sum_{n=0}^{\infty} K_T^n \), then \( \|K_T\|_{L_G^\gamma} \leq 14C_{\beta/\gamma}^\gamma \|\xi\|_{L_G^\beta}. \) Since \( \eta^n + K_T^n \) is symmetric for each \( n \geq 0 \), \( \xi^N + \sum_{n=0}^{N-1} K_T^n \) is symmetric for each \( N \geq 1 \). Consequently, \( \xi + K_T \) is symmetric. \( \square \)

**Theorem 3.3** For any \( \alpha \geq 1 \) and \( \delta > 0 \), \( L_{G}^{\alpha+\delta}(\Omega_T) \subset L_{E}^{\alpha}(\Omega_T) \). More precisely, for any \( 1 < \gamma < \beta := (\alpha + \delta)/\alpha, \gamma \leq 2 \), we have

\[
\|\xi\|_{\alpha, E} \leq \gamma^*\{\|\xi\|_{\alpha+\delta, G} + 14^{1/\gamma}C_{\beta/\gamma}\|\xi\|_{\alpha+\delta, G}^{(\alpha+\delta)/\gamma}\}, \forall \xi \in L_{ip}(\Omega_T),
\]

where \( C_{\beta/\gamma} = \sum_{i=1}^{\infty} i^{-\beta/\gamma} \), \( \gamma^* = \gamma/(\gamma - 1) \).

**Proof.** For \( \xi \in L_{ip}(\Omega_T), |\xi|^\alpha \in L_{ip}(\Omega_T) \). By Lemma 3.2, there exists \( K_T \in L_{G}^1(\Omega_T) \) such that for any \( 1 < \gamma < \beta, \gamma \leq 2 \) \( K_T \in L_{G}^\gamma(\Omega_T) \) and \( M_T := |\xi|^\alpha + K_T \) is symmetric. Let \( M_t = \hat{E}_t(M_T) \). Then

\[
\|\xi\|_{\alpha, E}^\alpha = \hat{E}[\sup_{t \in [0,T]} \hat{E}_t(|\xi|^\alpha)]
\leq \hat{E}(\sup_{t \in [0,T]} M_t)
\leq [\hat{E}(\sup_{t \in [0,T]} M_t^\gamma)]^{1/\gamma}
\leq \gamma^*\|M_T\|_{\gamma, G}
\leq \gamma^*\{\|\xi\|_{\alpha+\delta, G} + \|K_T\|_{\gamma, G}\}
\leq \gamma^*\{\|\xi\|_{\alpha+\delta, G} + 14^{1/\gamma}C_{\beta/\gamma}\|\xi\|_{\alpha+\delta, G}^{(\alpha+\delta)/\gamma}\}.
\]

\( \square \)

Let \( \mathcal{P}_E \) be weak compact subsets of \( M_1(\Omega_T) \) which represent \( E \). Define capacity \( c_{\mathcal{E}}(A) = \sup_{P \in \mathcal{P}_E} (A) \). We call \( c_{\mathcal{E}} \) the capacity induced by \( \mathcal{E} \).

By the above estimates, we can get the following equivalence between the Choquet capacities induced by \( \hat{E} \) and \( \mathcal{E} \).

**Corollary 3.4** There exists \( C > 0 \) such that for any set \( A \in \mathcal{B}(\Omega_T), c(A)^2 \leq c_{\mathcal{E}}(A)^2 \leq Cc(A) \).

**Proof.** By Choquet capacitability Theorem, it suffices to prove the compact sets case. For any compact set \( K \subset \Omega_T \), there exists an decreasing sequence \( \{\varphi_n\} \subset C_b^+(\Omega_T) \) such that \( 1_K \leq \varphi_n \leq 1 \) and \( \varphi_n \downarrow 1_K \). Let \( \alpha = \delta = 1 \) in the above Theorem 3.3, there exists \( 1 < \gamma < 2 \) and \( C > 0 \), such that

\[
[\mathcal{E}(\varphi_n)]^2 \leq C\hat{E}(\varphi_n).
\]

11
Then by Theorem 28 in [DHP08],
\[ c_{\mathcal{E}}(K)^2 \leq Cc(K). \]
□

**Corollary 3.5** The collections of quasi-continuous functions on \( \Omega_T \) w.r.t \( c \) and \( c_{\mathcal{E}} \) are the same. □

4 Applications to \( G \)-martingale decomposition

4.1 Generalized It\( \hat{o} \) integral

Let \( H^0_G(0,T) \) be the collection of processes in the following form: for a given partition \( \{t_0, \cdots, t_N\} = \pi_T \) of \([0,T] \),
\[ \eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega)1_{[t_j, t_{j+1})}(t), \]
where \( \xi_i \in L_{ip}(\Omega_{t_i}), i = 0, 1, 2, \cdots, N - 1 \). For each \( \eta \in H^0_G(0,T) \) and \( p \geq 1 \), let \( \|\eta\|_{H^p_G} = \{\hat{E}(\int_0^T |\eta_s|^2 ds)^{p/2}\}^{1/p} \) and denote \( H^p_G(0,T) \) the completion of \( H^0_G(0,T) \) under norm \( \| \cdot \|_{H^p_G} \). It’s easy to prove that \( H^2_G(0,T) = M^2_G(0,T) \).

**Definition 4.1** For each \( \eta \in H^0_G(0,T) \) with the form
\[ \eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega)1_{[t_j, t_{j+1})}(t), \]
we define
\[ I(\eta) = \int_0^T \eta(s)dB_s := \sum_{j=0}^{N-1} \xi_j(B_{t_{j+1}} - B_{t_j}). \]

By B-D-G inequality, the mapping \( I : H^0_G(0,T) \to L^p_G(\Omega_T) \) is continuous under \( \| \cdot \|_{H^p_G} \) and thus can be continuously extended to \( H^p_G(0,T) \).

4.2 \( G \)-martingale decomposition

Let \( \mathcal{B}_t = \sigma\{B_s|s \leq t\}, \mathcal{F}_t = \bigcap_{r \geq t} \mathcal{B}_r \) and \( \mathbb{F} = \{\mathcal{F}_t\}_{t \in [0,T]} \). \( \tau : \Omega_T \to [0,T] \) is called a \( \mathbb{F} \) stopping time if \( [\tau \leq t] \in \mathcal{F}_t, \forall t \in [0,T] \).
In order to prove the more general $G$-martingale decomposition decomposition theorem, we first introduce a famous lemma, for which we refer to [RY94].

**Definition 4.3.** A positive, adapted right-continuous process $X$ is dominated by an increasing process $A$ with $A_0 \geq 0$ if

$$E[X_\tau] \leq E[A_\tau]$$

for any bounded stopping time $\tau$.

**Lemma 4.4.** If $X$ is dominated by $A$ and $A$ is continuous, for any $k \in (0, 1)$

$$E[(X^*_T)^k] \leq \frac{2 - k}{1 - k} E[A^k_T],$$

where $X^*_T = \sup_{t \in [0, T]} X_t$.

**Theorem 4.5.** For $\xi \in L^\beta_G(\Omega_T)$ with some $\beta > 1$, $X_t = \hat{E}_t(\xi), t \in [0, T]$ has the following decomposition:

$$X_t = X_0 + \int_0^t Z_s dB_s - K_t, \text{ q.s.}$$

where $\{Z_t\} \in H^1_G(0, T)$ and $\{K_t\}$ is a continuous increasing process with $K_0 = 0$ and $\{-K_t\}_{t \in [0, T]}$ a $G$-martingale. Furthermore, the above decomposition is unique and $\{Z_t\} \in H^\alpha_G(0, T)$ and $\{K_t\} \in L^\alpha_G(\Omega_T)$ for any $1 \leq \alpha < \beta$.

**Proof.** For $\xi \in L^\beta_G(\Omega_T)$, there exists a sequence $\{\xi^n\} \subset L^\beta_{ip}(\Omega_T)$ such that $\|\xi^n - \xi\|_{\beta, G} \to 0$. By Theorem 2.18, we have the following decomposition

$$X^n_t := \hat{E}_t(\xi^n) = X_0 + \int_0^t Z^n_s dB_s - K^n_t := M^n_t - K^n_t, \text{ q.s.}$$

where $\{Z^n_t\} \in H^2_G(0, T)$ and $\{K^n_t\}$ is a continuous increasing process with $K^n_0 = 0$ and $\{-K^n_t\}_{t \in [0, T]}$ a $G$-martingale.

Fix $P \in \mathcal{P}_M$, by Itô’s formula,

$$(X^n_\tau)^2 = 2 \int_0^\tau X^n_s dX^n_s + [M^n]_\tau, \forall \text{ stopping time } \tau.$$

Take expectation under $P$, we have

$$E_P[(M^n_\tau)^2] = E_P[(X^n_\tau)^2] + 2E_P(\int_0^\tau X^n_s dK^n_s) \leq E_P[(X^n_\tau)^2] + 2E_P(\int_0^\tau X^{n*}_s dK^n_s),$$

13
where $X_n^* = \sup_{0 < s \leq t} |X_n^s|$.

In the following, $C_\alpha$ will always designate a universal constant, which may vary from line to line.

$\beta \leq 2$ case.

Consequently, for any $1 < \alpha < \beta$, by Lemma 4.4

$$
E_P[(M_T^{n*})^{\alpha}] \leq C_\alpha \{E_P[(X_T^{n*})^{\alpha}] + E_P[(X_T^{n*})^{\alpha/2}(K_T^{n*})^{\alpha/2}]\}
$$

$$
\leq C_\alpha \{E_P[(X_T^{n*})^{\alpha}] + \{E_P[(X_T^{n*})^{\alpha/2}]\}^{1/2} \{E_P[(K_T^{n*})^{\alpha}]\}^{1/2}\},
$$

where $M_T^{n*} = \sup_{0 < s \leq T} |M_s^n|$.

On the other hand,

$$(K_T^n)^2 \leq 2[(X_T^n)^2 + (M_T^n)^2], \forall \tau.$$

So

$$
E_P[(K_T^n)^2] \leq 2E_P[(X_T^n)^2 + (M_T^n)^2]
$$

$$
\leq 2E_P[(X_T^{n*})^2 + (M_T^{n*})^2].
$$

By this, we have

$$
E_P[(K_T^n)^\alpha] \leq C_\alpha E_P[(X_T^{n*})^\alpha + (M_T^{n*})^\alpha].
$$

So

$$
E_P[(M_T^{n*})^\alpha] \\
\leq C_\alpha E_P[(X_T^{n*})^\alpha] + C_\alpha \{E_P[(X_T^{n*})^{\alpha}]\}^{1/2} \{E_P[(M_T^{n*})^\alpha]\}^{1/2} \\
\leq 1/2C_\alpha E_P[(X_T^{n*})^\alpha] + 1/2E_P[(M_T^{n*})^\alpha].
$$

Now, we have $E_P(|M_T^{n*}|^\alpha) \leq C_\alpha E_P[(X_T^{n*})^\alpha]$ and $E_P(|K_T^n|^\alpha) \leq C_\alpha E_P[(X_T^{n*})^\alpha]$.

Let $\hat{X}_t := X_t^n - X_t^m$, $\hat{M}_t := M_t^n - M_t^m$, $\hat{K}_t := K_t^n - K_t^m$ and $\tilde{K}_t := K_t^n + K_t^m$. By Itô’s formula,

$$
\hat{X}_t^2 = 2 \int_0^\tau \hat{X}_s d\hat{X}_s + [\hat{M}]_\tau, \forall \tau.
$$

Take expectation under $P$, we have

$$
E_P[(\hat{M}_\tau)^2] = E_P[(\hat{X}_\tau)^2] + 2E_P(\int_0^\tau \hat{X}_s d\hat{K}_s)
$$

$$
\leq E_P[(\hat{X}_\tau)^2] + 2E_P(\int_0^\tau \hat{X}_s^* d\tilde{K}_s),
$$
where $\hat{X}_t^* = \sup_{0 < s \leq t} |\hat{X}_s|$.

By the same arguments as above,

$$E_F[(\hat{M}_T^*)^\alpha] \leq C_\alpha \{ E_F[(\hat{X}_T^*)^\alpha] + E_F[(\hat{X}_T^*)^{\alpha/2}(\hat{K}_T)^{\alpha/2}]\}$$

$$\leq C_\alpha \{ E_F[(\hat{X}_T^*)^\alpha] + \{ E_F[(\hat{X}_T^*)^{\alpha/2}]\}^{1/2} \{ E_F[(\hat{K}_T)^{\alpha}]\}^{1/2}\},$$

where $\hat{M}_T^* = \sup_{0 < s \leq T} |\hat{M}_s|$.

Take supremum over $\mathcal{P}_M$, we get

$$\hat{E}(\sup_{t \in [0,T]} |K_t^n - K_t^m|^\alpha) \to 0$$

as $n$ goes to infinity. Then there exists symmetric $G$-martingale $\{M_t\}$ and a process $\{K_t\}$ valued in $L_G^\alpha(\Omega_T)$ such that

$$\hat{E}(\sup_{t \in [0,T]} |M_t^n - M_t|^\alpha) \to 0$$

and

$$\hat{E}(\sup_{t \in [0,T]} |K_t^n - K_t|^\alpha) \to 0$$

as $n$ goes to infinity.

So by B-D-G inequality, there exists $\{Z_t\} \in H_G^\alpha(0,T)$ such that $\|Z - Z_n\|_{H_G^\alpha} \to 0$.

Consequently,

$$X_t = \lim_{L_G^n, n \to \infty} X_t^n = \lim_{L_G^n, n \to \infty} \int_0^t Z^n_s dB_s - \lim_{L_G^n, n \to \infty} K_t^n = \int_0^t Z_s dB_s - K_t.$$ 

$\beta > 2$ case.

For $2 < \alpha < \beta$,

$$[M_T^n]^{\alpha/2} \leq C_\alpha (|X_T^n|^{\alpha} + \int_0^T X^n_s dK^n_s)^{\alpha/2} + \int_0^T X^n_s dM^n_s)^{\alpha/2}. $$
So
\[
E_P([M^n]^{\alpha/2}_T) \\
\leq C_\alpha [E_P(|X^n_T|^\alpha) + E_P(\int_0^T X^n_s d\hat{K}^n_s)^{\alpha/2})] + E_P(\int_0^T X^n_s dM^n_s)^{\alpha/2})
\]
\[
\leq C_\alpha \{E_P(|X^n_T|^\alpha) + \{E_P([X^n_T]^{\alpha/2})\}^{1/2}\{E_P([\hat{K}^n_T]^{\alpha/2})\}^{1/2}\}^{1/2} \{E_P([M^n]^{\alpha/2}_T)\}^{1/2}.
\]

On the other hand
\[
E_P([K^n_T]^{\alpha}) \leq C_\alpha [E_P(|X^n_T|^\alpha) + E_P([M^n]^{\alpha/2}_T)]
\]
\[
\leq C_\alpha \{E_P([X^n_T]^{\alpha/2})\}^{1/2} \{E_P([\hat{K}^n_T]^{\alpha/2})\}^{1/2}.
\]

Therefore,
\[
E_P([M^n]^{\alpha/2}_T) \leq C_\alpha E_P([X^n_T]^{\alpha})
\]
and
\[
E_P([K^n_T]^{\alpha}) \leq C_\alpha E_P([X^n_T]^{\alpha}).
\]

By the same arguments, we get
\[
E_P([\hat{M}^{\alpha/2}_T]) \leq C_\alpha \{E_P([\hat{X}^{\alpha/2}_T]) + \{E_P([\hat{X}^{\alpha/2}_T])\}^{1/2}\{E_P([\hat{K}^{\alpha/2}_T])\}^{1/2}\}^{1/2}.
\]

The rest of the proof is just similar to the \(\beta \leq 2\) case.

\(\square\)

**Theorem 4.6** Let \(\xi \in L_G^\beta(\Omega_T)\) for some \(\beta > 1\) with \(\hat{E}(\xi) + \hat{E}(-\xi) = 0\), then there exists \(\{Z_t\}_{t \in [0,T]} \in H_G^\alpha(0,T)\) such that
\[
\xi = \hat{E}(\xi) + \int_0^T Z_s dB_s.
\]

Furthermore, the above representation is unique and \(\{Z_t\} \in H_G^\alpha(0,T)\) for any \(1 \leq \alpha < \beta\).

**Proof.** By Theorem 4.5, for \(\xi \in L_G^\beta(\Omega_T)\) with some \(\beta > 1\), \(X_t = \hat{E}_t(\xi), t \in [0,T]\) has the following decomposition:
\[
\xi = \hat{E}(\xi) + \int_0^T Z_s dB_s - K_T, \text{ q.s.}
\]

where \(\{Z_t\} \in H_G^\alpha(0,T)\) for any \(1 \leq \alpha < \beta\) and \(\{K_t\}\) is a continuous increasing process with \(K_0 = 0\) and \(\{-K_t\}_{t \in [0,T]}\) a \(G\)-martingale. If \(\xi\) is symmetric in addition, then
\[
\hat{E}(K_T) = \hat{E}(\xi) + \hat{E}(-\xi) = 0.
\]
So $K_T = 0, q.s.$ and
\[
\xi = \hat{E}(\xi) + \int_0^T Z_s dB_s.
\]
□

Remark 4.7 Since $\xi \in L_G^\beta(\Omega_T)$, we have by B-D-G and Doob’s maximal inequality
\[
\left\{ \hat{E}\left( \int_0^T |Z_s|^\beta ds \right)^{\beta/2} \right\}^{1/\beta} < \infty.
\]
But we still can’t say that $\{Z_t\}_{t \in [0, T]} \in H_G^\beta(0, T)$. Fortunately, by a stopping time technique, Corollary 5.2 in [S10] shows that $\{Z_t\}_{t \in [0, T]}$ does belong to $H_G^\beta(0, T)$ for $\xi \in L_G^\beta(\Omega_T)$. Here we will give a direct proof for $\beta = 2$ case.

Theorem 4.8 Let $\xi \in L_G^2(\Omega_T)$ with $\hat{E}(\xi) + \hat{E}(-\xi) = 0$, then there exists $\{Z_t\}_{t \in [0, T]} \in M_G^2(0, T)$ such that
\[
\xi = \hat{E}(\xi) + \int_0^T Z_s dB_s.
\]

Proof. Let $\xi^n = (\xi \vee n) \wedge (-n)$, then $\hat{E}[(\xi - \xi^n)^2] \to 0$. Let $M^n_t = \hat{E}_t(\xi^n)$ and $-\tilde{M}^n_t = \hat{E}_t(-\xi^n)$. By Theorem 4.5, there exist $\{Z^n_t\}, \{\tilde{Z}^n_t\} \in M_G^2(0, T)$ and continuous increasing processes $\{K^n_t\}_{t \in [0, T]}$, $\{-\tilde{K}^n_t\}_{t \in [0, T]}$ with $K^n_0 = -\tilde{K}^n_0 = 0$ and $\{K^n_t\}_{t \in [0, T]}$, $\{-\tilde{K}^n_t\}_{t \in [0, T]}$ $G$-martingales such that
\[
M^n_t = M^n_0 + \int_0^t Z^n_s dB_s, \quad K^n_t = -\tilde{K}^n_t,
\]
\[
\tilde{M}^n_t = \tilde{M}^n_0 - \int_0^t Z^n_s dB_s, \quad \tilde{K}^n_t = \tilde{K}^n_t.
\]
Let
\[
\hat{M}^n_t := M^n_t - \tilde{M}^n_t = N^n_t - \tilde{N}^n_t, \quad (\hat{K}^n_t + K^n_t) =: \tilde{N}^n_t - \tilde{\hat{K}}^n_t.
\]

Fix $P \in \mathcal{P}_M$. By Itô’s formula,
\[
0 = (\hat{M}^n_T)^2 = 2 \int_0^T \hat{M}^n_s d\hat{M}^n_s + [\hat{N}^n]_T, \quad P-a.s. \forall t \in [0, T].
\]

Take expectation in the above equation,
\[
E_P((\hat{N}^n_T)^2) = 2E_P(\int_0^T \hat{M}^n_s d\hat{M}^n_s) \leq 4nE_P[\tilde{K}^n_T].
\]
So $\hat{E}[(\hat{N}_T^n)^2] \leq 4n\hat{E}[\hat{K}_T^n]$. Noting that

\[
\begin{align*}
\hat{E}[\hat{K}_T^n] &= \hat{E}[\hat{N}_T^n] + \hat{E}[-\hat{M}_T^n] \\
&= \hat{E}[\xi^n] + \hat{E}[-\xi^n] \\
&= \hat{E}[\xi^n - \xi] + \hat{E}[-(\xi^n - \xi)] - \hat{E}[\xi] \\
&\leq 2\hat{E}[|\xi^n - \xi|] \\
&\leq 2\hat{E}[|\xi|1_{|\xi|>n}],
\end{align*}
\]

we get

\[
\hat{E}[(\hat{N}_T^n)^2] \leq 8n\hat{E}[|\xi|1_{|\xi|>n}] \leq 8\hat{E}[|\xi|^21_{|\xi|>n}] \to 0.
\]

So

\[
\hat{E}[(\hat{K}_T^n)^2] \leq \hat{E}[(\hat{K}_T^n)^2] = \hat{E}[(\hat{N}_T^n)^2] \to 0.
\]

Let $X^n := \xi^n - \xi = N_T^n - \xi - K_T^n$. Then

\[
\hat{E}[(N_T^n - \xi)^2] \leq 2\{\hat{E}[(X^n)^2] + \hat{E}[(K_T^n)^2]\} = 2\{\hat{E}[(\xi^n - \xi)^2] + \hat{E}[(K_T^n)^2]\} \to 0.
\]

Since $\{\eta \in L^2_G(\Omega_T) \mid \eta = \int_0^T Z_s dB_s \text{ for some } Z \in M^2_G(0, T)\}$ is closed in $L^2_G(\Omega_T)$, we proved the desired result. □

By the proof in Theorem 4.5, we can get the following estimates, which may be useful in the follow-up work of $G$-martingale theory.

**Corollary 4.9** For $\xi, \xi' \in L^2_G(\Omega_T)$ with some $\beta > 1$, let $\xi = M_T - K_T$ and $\xi' = M'_T - K'_T$ be the decomposition in Theorem 4.5. Then for any $1 < \alpha < \beta$, $1 < \gamma < \beta/\alpha, \gamma \leq 2$, there exists $C_{\alpha, \beta, \gamma}$ such that

\[
\|K_T\|_{\alpha,G}^\alpha \leq C_{\alpha,\beta,\gamma}\{\|\xi\|_{\beta,G}^\alpha + \|\xi\|_{\beta,G}^{\beta/\gamma}\}
\]

and

\[
\|M_T - M'_T\|_{\alpha,G}^\alpha \leq C_{\alpha,\beta,\gamma}\{\|\xi - \xi'\|_{\beta,G}^\alpha + \|\xi - \xi'\|_{\beta,G}^{\beta/\gamma}\} + C_{\alpha,\beta,\gamma}\{\|\xi - \xi'\|_{\beta,G}^\alpha + \|\xi - \xi'\|_{\beta,G}^{\beta/\gamma}\}^{1/2}\{1 + \|\xi\|_{\beta,G}^{\beta/\gamma} + \|\xi'\|_{\beta,G}^{\beta/\gamma}\}^{1/2}.
\]

## 5 Regular properties for $G$-martingale

**Definition 5.1** We say that a process $\{M_t\}$ with values in $L^1_G(\Omega_T)$ is quasi-continuous if
∀ε > 0, there exists open set $G$ with $c(G) < ε$ such that $M(\cdot)$ is continuous on $G^c \times [0, T]$.

**Corollary 5.2** Any $G$-martingale $\{M_t\}$ with $M_T \in L^\beta_G(\Omega_T)$ for some $\beta > 1$ has a quasi-continuous version.

**Proof.** For $ξ \in L_{ip}(\Omega_T)$, $M_t = \hat{E}_t(ξ)$ is continuous on $[0, T] \times \Omega_T$. In fact, for $ξ = \varphi(B_{t_1} - B_{t_0}, \ldots, B_{t_n} - B_{t_{n-1}})$ with $\varphi \in C_{b, ip}(\mathbb{R}^n)$, $M_t(\cdot)$ is obvious continuous on $\Omega_T$ for fixed $t \in [0, T]$. On the other hand, fix $ω \in \Omega_T$

$$|M_{t_n}(\omega) - M_{t_{n-1}}(\omega)|$$

$$\leq |\varphi(x_1, \ldots, x_{n-1}, x_n) - \hat{E}[\varphi(x_1, \ldots, x_{n-1}, B_{t_n} - B_{t_{n-1}})]|$$

$$\leq L\hat{E}(|B_{t_n} - B_{t_{n-1}} - x_n|)$$

$$\leq L(\sigma(t_n - t_{n-1})^{1/2} + |ω_{t_n} - ω_{t_{n-1}}|),$$

where $L$ is the Lipschitz constant of $\varphi$ and $x_i = B_{t_i}(ω) - B_{t_{i-1}}(ω)$. In fact, the above estimate holds for any $s, t \in [t_i, t_{i-1}]$ for some $1 \leq i \leq n$. Then for any $s_k \to s$ and $ω^k \to ω$,

$$|M_{s_k}(ω^k) - M_s(ω)|$$

$$\leq |M_{s_k}(ω^k) - M_s(ω^k)| + |M_s(ω^k) - M_s(ω)|$$

$$\leq L(\sigma |s_k - s|^{1/2} + |ω_{s_k} - ω^k_s| + |M_{s_k}(ω^k) - M_s(ω)|) \to 0.$$

For $ξ = M_T \in L^\beta_G(\Omega_T)$, by Theorem 3.3, there exists $1 < α < β$ and $\{ξ^n\} \subset L_{ip}(\Omega_T)$ such that $\|ξ^n - ξ\|_{ε, α} \to 0$. Let $M^n_t := \hat{E}_t(ξ^n)$. Then

$$\sup_{m > n} \hat{E}[\sup_{t \in [0, T]} |M_t^m - M_t^n|] \leq \sup_{m > n} \|ξ^n - ξ^m\|_{ε, α} \to 0$$

as $n$ goes to infinity. So there exists a subsequence $\{n_k\}$ such that

$$\sup_{m > n_k} \hat{E}[\sup_{t \in [0, T]} |M_t^m - M_t^{n_k}|] < 1/4^k.$$

Consequently,

$$\hat{E}\left[\sum_{k=1}^{∞} \sup_{t \in [0, T]} |M_t^{n_k+1} - M_t^{n_k}|\right] \leq \sum_{k=1}^{∞} \hat{E}[\sup_{t \in [0, T]} |M_t^{n_k+1} - M_t^{n_k}|] < ∞.$$

Then there exists $\{\tilde{M}_t\}$ such that

$$\sup_{t \in [0, T]} |M_t^{n_k} - \tilde{M}_t| \leq \sum_{i=k}^{∞} \sup_{t \in [0, T]} |M_t^{n_i+1} - M_t^{n_i}|, \forall k \geq 1.$$
For any \( \varepsilon > 0 \), let \( O^\varepsilon_k := \mathop{\sup}_{t \in [0,T]} (|M_{t+1}^{n_k} - M_t^{n_k}|) > 1/(2^k \varepsilon) \) and \( O^\varepsilon = \bigcup_{k=1}^\infty O^\varepsilon_k \). Then \( c(O^\varepsilon) \leq \sum_{k=1}^\infty c(O^\varepsilon_k) < \varepsilon \) and on \( (O^\varepsilon)^c \)

\[
\sup_{t \in [0,T]} (|M_{t+1}^{n_k} - \tilde{M}_t|) \leq \sum_{i=1}^\infty \sup_{t \in [0,T]} (|M_{t+i}^{n_k} - M_t^{n_k}|) \leq 1/(2^{k-1} \varepsilon), \quad \forall k \geq 1.
\]

So

\[
\sup_{\omega \in (O^\varepsilon)^c} \sup_{t \in [0,T]} (|M_{t+1}^{n_k} - \tilde{M}_t|) \to 0
\]

and \( \{\tilde{M}_t\} \) is a quasi-continuous version of \( \{M_t\} \). \( \square \)

**Theorem 5.3** Any \( G \)-martingale \( \{M_t\} \) has a quasi-continuous version.

**Proof.** Let \( \xi := M_T \) and \( \xi^n = (\xi \land n) \lor (-n) \). For \( m > n \), let \( \{X_t^m\}, \{X_t^n\}, \{X_t^{m,n}\} \) be the quasi-continuous versions of \( \{\hat{E}_t(\xi^n)\}, \{\hat{E}_t(\xi^m)\}, \{\hat{E}_t(\xi^n - \xi^m)\} \) respectively.

We claim that

\[
\hat{E}[\sup_{m>n} \sup_{t \in [0,T]} (|X_t^m - X_t^n| \land 1)] \leq \hat{E}[\sup_{m>n} \sup_{t \in [0,T]} (X_t^{m,n} \land 1)] \downarrow 0.
\]

Otherwise, there exists \( \varepsilon > 0 \) such that \( \hat{E}[\sup_{m>n} \sup_{t \in [0,T]} (X_t^{m,n} \land 1)] > \varepsilon \) for all \( n \in \mathbb{N} \). Consequently, for each \( n \in \mathbb{N} \), there exists \( m(n) > n \) such that \( \hat{E}[\sup_{t \in [0,T]} (X_t^{n,m(n)} \land 1)] > \varepsilon \).

Noting that

\[
\varepsilon < \hat{E}\left[ \sup_{t \in [0,T]} (X_t^{n,m(n)} \land 1) \right] \leq c\left( \sup_{t \in [0,T]} X_t^{n,m(n)} > \varepsilon/2 \right) + \varepsilon/2,
\]

we have \( c(\sup_{t \in [0,T]} X_t^{n,m(n)} > \varepsilon/2) > \varepsilon/2 \). Since

\[
\left[ \sup_{t \in [0,T]} X_t^{n,m(n)} > \varepsilon/2 \right] = \pi(\{(\omega,t) | X_t^{n,m(n)}(\omega) > \varepsilon/2\}),
\]

the projection of \( \{(\omega,t) | X_t^{n,m(n)}(\omega) > \varepsilon/2\} \) on \( \Omega \), we have stopping time \( \tau_n \leq T \) such that \( \hat{E}(X_t^{n,m(n)} > \varepsilon^2/4 \text{ by section theorem}) \).

On the other hand, by Theorem 4.5, \( \{X_t^{n,m(n)}\} \) has the following decomposition

\[
X_t^{n,m(n)} = M_t^n - K_t^n,
\]

where \( \{M_t^n\} \) is a symmetric \( G \)-martingale and \( \{-K_t^n\} \) a negative \( G \)-martingale with \( M_T^n, K_T^n \in L^1_G(\Omega_T) \). By Theorem 4.5 and Corollary 5.2, \( \{M_t^n\}, \{-K_t^n\} \) can be taken to be quasi-continuous.
So we have as $n \to \infty$

\[ \hat{E}(X_{n,m}^{n,m(n)}) \]
\[ \leq \hat{E}(M_{n}^{n}) + \hat{E}(-K_{n}^{n}) \]
\[ \leq \hat{E}(M_{T}^{n}) \]
\[ \leq \hat{E}(X_{T}^{n,m(n)}) + \hat{E}(K_{T}^{n}) \]
\[ \leq 2\hat{E}(X_{T}^{n,m(n)}) \to 0. \]

This is a contradiction.

Therefore, by the same arguments as in Corollary 5.2, $\{M_{t}\}$ has a quasi-continuous version. □

References

[DHP08] Denis, L., Hu, M. and Peng S. Function spaces and capacity related to a sublinear expectation: application to $G$-Brownian motion paths. arXiv:0802.1240v1 [math.PR] 9 Feb, 2008

[HP09] Hu, Mingshang and Peng, Shige (2009) On representation theorem of $G$-expectations and paths of $G$-Brownian motion. Acta Math Appl Sinica English Series, 25(3): 1-8.

[HWY92] He, S., Wang, J., Yan, J. Semimartingales and Stochastic Calculus Science Press, Beijing.

[P06] Peng, S. (2006) $G$-expectation, $G$-Brownian Motion and Related Stochastic Calculus of Itô type, preprint (pdf-file available in arXiv:math.PR/0601035v1 3Jan 2006), to appear in Proceedings of the 2005 Abel Symposium.

[P08] Peng, S. (2008) Multi-Dimensional $G$-Brownian Motion and Related Stochastic Calculus under $G$-Expectation, in Stochastic Processes and their Applications, 118(12), 2223-2253.

[P07] Peng, S. (2007) $G$-Brownian Motion and Dynamic Risk Measure under Volatility Uncertainty, Preprint: arXiv:0711.2834v1 [math.PR] 19 Nov 2007.

[P09] Peng, S. (2009) Survey on normal distributions, central limit theorem, Brownian motion and the related stochastic calculus under sublinear expectations, Science in China Series A: Mathematics, 52, No.7, 1391-1411, (www.springerlink.com).
[RY94] Revuz, D. and Yor, M. (1994) *Continuous Martingale and Brownian Motion*, Springer Verlag, Berlin-Heidelberg-New York.

[S10] Song, Y. (2010) *Properties of hitting times for G-martingale*. Preprint. arXiv:1001.4907v1 [math.PR] 27 Jan 2010.

[STZ09] Soner, M., Touzi, N., Zhang, J. (2009) *Martingale Representation Theorem under G-expectation*. Preprint.

[XZ09] Xu J, Zhang B. (2009) *Martingale characterization of G-Brownian motion*. Stochastic Processes Appl., 119(1): 232-248.

[Yan98] Yan, J.A. (1998) *Lecture Note on Measure Theory*, Science Press, Beijing, Chinese version.