Quantum Hermite–Hadamard-type inequalities for functions with convex absolute values of second $q^b$-derivatives

Muhammad Aamir Ali¹, Hüseyin Budak², Mujahid Abbas³ and Yu-Ming Chu⁴*

Abstract

In this paper, we obtain Hermite–Hadamard-type inequalities of convex functions by applying the notion of $q^b$-integral. We prove some new inequalities related with right-hand sides of $q^b$-Hermite–Hadamard inequalities for differentiable functions with convex absolute values of second derivatives. The results presented in this paper are a unification and generalization of the comparable results in the literature on Hermite–Hadamard inequalities.

Keywords: Hermite–Hadamard inequality; $q$-integral; Quantum calculus; Convex function

1 Introduction

The Hermite–Hadamard inequality introduced by Hermite and Hadamard (see also [1] and [2, p. 137]) is one of the most well-known inequalities in the theory of convex functional analysis. It has an interesting geometrical interpretation with several applications.

These inequalities state that if $f : I \rightarrow \mathbb{R}$ is a convex function on an interval $I$ of real numbers and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) \, dx \leq \frac{f(a) + f(b)}{2}.$$  (1.1)

Both inequalities hold in the reversed manner if $f$ is a concave function. Note that the Hermite–Hadamard inequalities may be viewed as a refinement of the concept of convexity and follows from Jensen’s inequality. Hermite–Hadamard inequalities for convex functions have received much attention in the recent years, and, consequently, a remarkable variety of refinements and generalizations have been obtained.

Many well-known integral inequalities such as the Hölder, Hermite–Hadamard, Ostrowski, Cauchy–Bunyakovsky–Schwarz, Gruss, Gruss-Chebyshev, and other integral inequalities have been studied in the setup of $q$-calculus using the concept of classical convexity. For more results in this direction, we refer to [3–18].
The purpose of this paper is to study Hermite–Hadamard-like inequalities for convex functions by applying the new concept of $q^b$-integral. We also discuss the relation of our results with comparable results existing in the literature.

The organization of this paper is as follows. In Sect. 2, we give a brief description of the concepts of $q$-calculus and some related works in this direction. In Sect. 3, we present the Hermite-Hadamard-type inequalities for the $q^b$-integrals. We also study the relation between the results presented herein and comparable results in the literature. Section 4 contains some conclusions and further directions for the future research. We believe that the study initiated in this paper may inspire new research in this area.

2 Preliminaries of $q$-calculus and some inequalities

In this section, we first present some known definitions and related inequalities in $q$-calculus. Set the following notation (see [19]):

$$[n]_q = \frac{1 - q^n}{1 - q} = 1 + q + q^2 + \cdots + q^{n-1}, \quad q \in (0, 1).$$

Jackson [20] defined the $q$-Jackson integral of a given function $f$ from 0 to $b$ as follows:

$$\int_0^b f(x) d_qx = (1 - q)b \sum_{n=0}^{\infty} q^n f(bq^n), \quad \text{where } 0 < q < 1,$$

provided that the sum converges absolutely.

Jackson [20] defined the $q$-Jackson integral of a given function over the interval $[a, b]$ as follows:

$$\int_a^b f(x) d_qx = \int_a^b f(x) d_qx - \int_0^a f(x) d_qx.$$

Definition 1 ([21]) Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then the $q_a$-derivative of $f$ at $x \in [a, b]$ is identified as

$$aD_qf(x) = \frac{f(x) - f(qx + (1 - q)a)}{(1 - q)(x - a)}, \quad x \neq a. \quad (2.2)$$

Since $f : [a, b] \to \mathbb{R}$ is a continuous function, we can define

$$aD_qf(a) = \lim_{x \to a} aD_qf(x).$$

The function $f$ is said to be $q_a$-differentiable on $[a, b]$ if $aD_qf(x)$ exists for all $x \in [a, b]$. If we take $a = 0$ in (2.2), then we have $aD_qf(x) = D_qf(x)$, where $D_qf(x)$ is the $q$-derivative of $f$ at $x \in [0, b]$ (see [19]) given by

$$D_qf(x) = \frac{f(x) - f(qx)}{(1 - q)x}, \quad x \neq 0.$$

Definition 2 ([22]) Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then the $q^b$-derivative of $f$ at $x \in [a, b]$ is given by

$$bD_qf(x) = \frac{f(qx + (1 - q)b) - f(x)}{(1 - q)(b - x)}, \quad x \neq b.$$
Definition 3 Let \( f : [a, b] \rightarrow \mathbb{R} \) be a continuous function. Then the second \( q^b \)-derivative of \( f \) at \( x \in [a, b] \) is given by

\[
\frac{\partial^2 f}{\partial q^b} (x) = \frac{f(q^b a + (1 - t q^2) b) - (1 + q^b)(q^b a + (1 - q^b)b) + q^b (f(a) + f(b))}{(1 - q^b)^2 (b - a)^2 t^2}.
\]

Definition 4 ([21]) Let \( f : [a, b] \rightarrow \mathbb{R} \) be a continuous function. Then the \( q_a \)-definite integral on \([a, b]\) is defined by

\[
\int_a^b f(x) \, d_q x = (1 - q) (b - a) \sum_{n=0}^{\infty} q^n f(q^n a + (1 - q^n) b) = (b - a) \int_0^1 f((1 - t) a + t b) \, d_q t.
\]

Alp et al. [3] proved the following \( q_a \)-Hermite–Hadamard inequalities for convex functions in the setting of quantum calculus.

Theorem 1 If \( f : [a, b] \rightarrow \mathbb{R} \) is a convex differentiable function on \([a, b]\) and \( 0 < q < 1 \), then we have

\[
f\left(\frac{qa + b}{1 + q}\right) \leq \frac{1}{b - a} \int_a^b f(x) \, d_q x \leq \frac{q f(a) + f(b)}{1 + q}.
\]

In [3] and [23] authors established some bounds for the left- and right-hand sides of inequality (2.3).

On the other hand, Bermudo et al. [22] gave the following definition and obtained the related Hermite–Hadamard-type inequalities.

Definition 5 ([22]) Let \( f : [a, b] \rightarrow \mathbb{R} \) be a continuous function. Then the \( q^b \)-definite integral on \([a, b]\) is given by

\[
\int_a^b f(x) \, d_q x = (1 - q) (b - a) \sum_{n=0}^{\infty} q^n f(q^n a + (1 - q^n) b) = (b - a) \int_0^1 f((1 - t) a + t b) \, d_q t.
\]

Theorem 2 ([22]) If \( f : [a, b] \rightarrow \mathbb{R} \) is a convex differentiable function on \([a, b]\) and \( 0 < q < 1 \), then we have the following \( q \)-Hermite–Hadamard inequalities:

\[
f\left(\frac{a + qb}{1 + q}\right) \leq \frac{1}{b - a} \int_a^b f(x) \, d_q x \leq \frac{f(a) + q f(b)}{1 + q}.
\]

From Theorems 1 and 2 we obtain the following inequalities.

Corollary 1 [22] For any convex function \( f : [a, b] \rightarrow \mathbb{R} \) and \( 0 < q < 1 \), we have

\[
f\left(\frac{qa + b}{1 + q}\right) + f\left(\frac{a + qb}{1 + q}\right) \leq \frac{1}{b - a} \left\{ \int_a^b f(x) \, d_q x + \int_a^b f(x) \, d_q x \right\} \leq f(a) + f(b)
\]
and

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2(b-a)} \left\{ \int_a^b f(x) d_qx + \int_a^b f(x) d_qx \right\} \leq \frac{f(a) + f(b)}{2}. \quad (2.6)$$

**Theorem 3** (Hölder’s inequality, [24, p. 604]) Suppose that \(x > 0, 0 < q < 1, p_1 > 1.\) If \(\frac{1}{p_1} + \frac{1}{q_1} = 1,\) then

$$\left( \int_0^x |f(x)|^{p_1} d_qx \right)^{\frac{1}{p_1}} \left( \int_0^x |g(x)|^{q_1} d_qx \right)^{\frac{1}{q_1}} \leq \left( \int_0^x |f(x)||g(x)| d_qx \right).$$

In this paper, we will also find some bounds for right-hand side of inequality (2.4).

### 3 New Hermite–Hadamard-type inequalities for quantum integrals

We now give some new Hermite–Hadamard-type inequalities for functions whose second \(q^b\)-derivatives in absolute value are convex.

We start with the following useful lemma.

**Lemma 1** If \(f : [a,b] \subset \mathbb{R} \rightarrow \mathbb{R}\) is a twice \(q^b\)-differentiable function on \((a,b)\) such that \(bD_q^2f\) is continuous and integrable on \([a,b]\), then we have:

$$\frac{f(a) + qf(b)}{1 + q} - \frac{1}{b-a} \int_a^b f(x) d_qx = \frac{q^2(b-a)^2}{1 + q} \int_0^1 t(1-qt) bD_q^2f(ta + (1-t)b) d_qt, \quad (3.1)$$

where \(0 < q < 1.\)

**Proof** From Definition 2 it follows that

$$bD_q^2f(ta + (1-t)b)$$

$$= bD_q \left( bD_q(f(ta + (1-t)b)) \right)$$

$$= bD_q \left( \frac{f(qta + (1-qt)b) - f(ta + (1-t)b)}{(1-q)(b-a)t} \right)$$

$$= \frac{1}{(1-q)(b-a)t} \left[ \frac{f(q^2ta + (1-tq^2)b) - f(qta + (1-qt)b)}{(1-q)q(b-a)t} \right]$$

$$- \frac{f(qta + (1-qt)b) - f(ta + (1-t)b)}{(1-q)(b-a)t}$$

$$\frac{f(qta + (1-qt)b) - f(ta + (1-t)b)}{(1-q)^2q(b-a)^2t^2}$$

$$- \frac{f(qta + (1-qt)b) - f(ta + (1-t)b)}{(1-q)^2(b-a)^2t^2}$$

$$= \frac{f(q^2ta + (1-tq^2)b) - f(qta + (1-qt)b)}{(1-q)^2q(b-a)^2t^2}.$$
Also,

\[
\begin{aligned}
\int_0^1 t(1-qt)^b D_1^2 f(ta + (1-t)b) \, dt \\
&= \int_0^1 \frac{f(q^2ta + (1-tq^2)b) - (1+q)f(qta + (1-qt)b) + qf(ta + (1-t)b)}{(1-q)^2q(b-a)^2} \, dt \\
&\quad - \int_0^1 q \left[ \frac{f(q^2ta + (1-tq^2)b) - (1+q)f(qta + (1-qt)b) + qf(ta + (1-t)b)}{(1-q)^2q(b-a)^2} \right] \, dt.
\end{aligned}
\]

\hfill (3.3)

By equality (2.1) we obtain that

\[
\begin{aligned}
\int_0^1 f(q^2ta + (1-tq^2)b) - (1+q)f(qta + (1-qt)b) + qf(ta + (1-t)b) \\
&\quad - \sum_{n=0}^{\infty} \frac{f(q^n a + (1-q^n)b)}{(1-q)^2q(b-a)^2} \\
\end{aligned}
\]

\hfill (3.4)

From (2.1) and Definition 5 it follows that

\[
\begin{aligned}
\int_0^1 q \left[ \frac{f(q^2ta + (1-tq^2)b) - (1+q)f(qta + (1-qt)b) + qf(ta + (1-t)b)}{(1-q)^2q(b-a)^2} \right] \, dt \\
&= q \left[ (1-q)(b-a) \sum_{n=0}^{\infty} \frac{q^{n+2}f(q^{n+2}a + (1-q^{n+2})b)}{(1-q)^2q^3(b-a)^3} \\
&\quad - (1-q)(1+q)(b-a) \sum_{n=0}^{\infty} \frac{q^{n+1}f(q^{n+1}a + (1-q^{n+1})b)}{(1-q)^2q^3(b-a)^3} \\
&\quad + q(1-q)(b-a) \sum_{n=0}^{\infty} \frac{q^n f(q^n a + (1-q^n)b)}{(1-q)^2q^3(b-a)^3} \right] \\
&= q \left[ \frac{1}{(1-q)^2q^3(b-a)^3} \right. \\
&\quad \times \left( \int_a^b f(x)^b d_q x - (1-q)(b-a)f(a) - (1-q)(b-a)qf(qa + (1-q)b) \right) \\
&\quad \left. - \frac{1+q}{(1-q)^2q^3(b-a)^3} \left( \int_a^b f(x)^b d_q x - (1-q)(1+q)(b-a)f(a) \right) \right].
\end{aligned}
\]
Theorem 4

If \( f \) is continuous and integrable on \( \mathbb{R} \), we complete the proof of Lemma 1.

Remark 1

Multiplying both sides of (3.6) by \( \frac{qf}{1-q} \), we obtain the required identity (3.1) and hence we complete the proof of Lemma 1.

\[ f(a) + qf(b) = \frac{1}{1+q} \int_a^b f(x) \, dqx \]

as given in [25].

Theorem 4

If \( f : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R} \) is a twice \( q^2 \)-differentiable function on \( (a, b) \) such that \( bD_q^2 f \) is continuous and integrable on \( [a, b] \), then we have the following inequality, provided that \( |bD_q^2 f| \) is convex on \( [a, b] \):

\[ |f(a) + qf(b)| \leq \frac{1+q}{b-a} \int_a^b f(x) \, dqx \]

\[ \leq \frac{q^2(b-a)^2}{(1+q)(q^2+q+1)(q^2+q+2+1)} \left[ |bD_q^2 f(a)| + q^2 |bD_q^2 f(b)| \right]. \]

where \( 0 < q < 1 \).

Proof

Taking the modulus in Lemma 1 and applying the convexity of \( |bD_q^2 f| \), we obtain

\[ |f(a) + qf(b)| \leq \frac{1+q}{b-a} \int_a^b f(x) \, dqx \]

\[ \leq \frac{q^2(b-a)^2}{1+q} \int_0^1 (t(1-qt)) \left| bD_q^2 f(ta + (1-t)b) \right| dq t \]

\[ \leq \frac{q^2(b-a)^2}{1+q} \int_0^1 (t(1-qt)) \left[ t |bD_q^2 f(a)| + (1-t) |bD_q^2 f(b)| \right] dq t \]
Suppose that \( f \) is continuous and integrable on \( [a, b] \) and \( |bD_q^2f|^{p_1}, p_1 > 1 \), is convex on \( [a, b] \), then we have the following inequality:

\[
\frac{f(a) + qf(b)}{1 + q} - \frac{1}{b - a} \int_a^b f(x) \, dq_x
\leq \frac{q^2(b - a)^2}{(1 + q)^2} \left( \frac{1}{q^2 + q^2 + 1} \right) \frac{1}{p_1} \left( |bD_q^2f(a)|^{p_1} + q^2 |bD_q^2f(b)|^{p_1} \right)^{\frac{1}{p_1}},
\]

where \( 0 < q < 1 \).

Proof. Taking the modulus in Lemma 1 and applying the well-known power mean inequality, we have

\[
\frac{f(a) + qf(b)}{1 + q} - \frac{1}{b - a} \int_a^b f(x) \, dq_x
\leq \frac{q^2(b - a)^2}{1 + q} \int_0^1 (t(1 - qt))^{\frac{1}{p_1}} |bD_q^2f(ta + (1 - t)b)| \, dq_t
\leq \frac{q^2(b - a)^2}{1 + q} \left( \int_0^1 (t(1 - qt)) \, dq_t \right)^{1 - \frac{1}{p_1}}
\times \left( \int_0^1 (t(1 - qt)) \frac{1}{p_1} \right)^{\frac{1}{p_1}}
\times \left( \int_0^1 (t(1 - qt)) \frac{1}{p_1} \right)^{\frac{1}{p_1}}.
\]

By the convexity of \( |bD_q^2f|^{p_1} \) we have

\[
\frac{f(a) + qf(b)}{1 + q} - \frac{1}{b - a} \int_a^b f(x) \, dq_x
\leq \frac{q^2(b - a)^2}{1 + q} \left( \int_0^1 (t(1 - qt)) \frac{1}{p_1} \right)^{1 - \frac{1}{p_1}}
\times \left( \int_0^1 (t(1 - qt)) \frac{1}{p_1} \right)^{\frac{1}{p_1}}
\times \left( \int_0^1 (t(1 - qt)) \frac{1}{p_1} \right)^{\frac{1}{p_1}}.
\]
\begin{align*}
&= \frac{q^2(b-a)^2}{1+q} \left( \int_0^1 (t(1-qt)) \, dq \right)^{1-\frac{1}{p_1}} \\
&\quad \times \left( \left| b D_q^2 f(a) \right|^{p_1} \int_0^1 t(1-qt) \, dq + \left| b D_q^2 f(b) \right|^{p_1} \int_0^1 1-t(1-qt) \, dq \right)^{\frac{1}{p_1}} \\
&= \frac{q^2(b-a)^2}{1+q} \left( \frac{1}{(1+q)(1+q+q^2)} \right)^{1-\frac{1}{p_1}} \\
&\quad \times \left( \left( \frac{|b D_q^2 f(a)|^{p_1}}{(q^2+q+1)(q^2+q^2+q+1)} + \frac{q^2|b D_q^2 f(b)|^{p_1}}{(q^2+q+1)(q^2+q^2+q+1)} \right)^{\frac{1}{p_1}} \right),
\end{align*}

which completes the proof. \hfill \square

**Remark 3** If we take the limit as $q \to 1^-$ in Theorem 5, then we have

\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{(b-a)^2}{12.2 p_1^2} \left( [f''(a)]^{p_1} + [f''(b)]^{p_1} \right)^{\frac{1}{p_1}}.
\]

**Theorem 6** Suppose that $f : [a, b] \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is a twice $q^b$-differentiable function on $(a, b)$ and $b D_q^2 f$ is continuous and integrable on $[a, b]$. If $|b D_q^2 f|^{p_1}$ is convex on $[a, b]$ for some $p_1 > 1$ and $\frac{1}{q_1} + \frac{1}{p_1} = 1$, then we have

\[
\left| \frac{f(a) + qf(b)}{1 + q} - \frac{1}{b-a} \int_a^b f(x) \, dq \right| \\
\leq \frac{q^2(b-a)^2}{1+q} \left( u_1 \right)^{\frac{1}{p_1}} \left( \frac{|b D_q^2 f(a)|^{p_1} + q|b D_q^2 f(b)|^{p_1}}{q+1} \right)^{\frac{1}{p_1}},
\tag{3.7}
\]

where $u_1 = (1-q)\sum_{n=0}^{\infty} (q^n)^{r_1} (1-q^{n+1})^{r_1}$ and $0 < q < 1$.

**Proof** Taking the modulus in Lemma 1 and applying well-known Hölder’s inequality, we obtain

\[
\left| \frac{f(a) + qf(b)}{1 + q} - \frac{1}{b-a} \int_a^b f(x) \, dq \right| \\
\leq \frac{q^2(b-a)^2}{1+q} \left( \int_0^1 (t(1-qt))^{\frac{1}{p_1}} |b D_q^2 f(ta + (1-t)b)| \, dq \right) \\
\leq \frac{q^2(b-a)^2}{1+q} \left( \int_0^1 (t(1-qt))^{r_1} \, dq \right)^{\frac{1}{p_1}} \left( \int_0^1 |b D_q^2 f(ta + (1-t)b)|^{p_1} \, dq \right)^{\frac{1}{p_1}}.
\]

Since $|b D_q^2 f|^{p_1}$ is convex, we have

\[
\left| \frac{f(a) + qf(b)}{1 + q} - \frac{1}{b-a} \int_a^b f(x) \, dq \right| \\
\leq \frac{q^2(b-a)^2}{1+q} \left( \int_0^1 (t(1-qt))^{r_1} \, dq \right)^{\frac{1}{p_1}}
\]
Thus
\[
\begin{align*}
\int_0^1 (t(1-qt))^{r_1} \, dt = (1-q) \sum_{n=0}^{\infty} (q^n)^{r_1+1} (1-q^{n+1})^{r_1},
\end{align*}
\]
which completes the proof. \(\square\)

**Remark 4** If we take the limit as \(q \to 1^-\) in Theorem 6, then we have
\[
\int_0^1 (t(1-t))^{r_1} \, dt = B(r_1 + 1, r_1 + 1),
\]
where \(B(x,y)\) is the Euler beta function. Moreover, inequality (3.7) reduces to
\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx \right| 
\leq \frac{(b-a)^2}{2} \left( B(r_1 + 1, r_1 + 1) \right)^{\frac{1}{r_1} \left( \frac{1}{r_1 + 1} \right)^{\frac{1}{r_1}}}. \]

We obtain another Hermite–Hadamard-type inequality for powers in terms of the second quantum derivatives.

**Theorem 7** With assumptions of Theorem 6, we have the inequality
\[
\left| \frac{f(a) + qf(b)}{1 + q} - \frac{1}{b-a} \int_a^b f(x)^b \, dx \right| 
\leq \frac{q^2(b-a)^2}{1 + q} \left( \frac{1}{[r_1 + 1]_q} \right)^{\frac{1}{r_1}} \left( u_2 \left| bD_q^2f(a) \right|^{p_1} + u_3 \left| bD_q^2f(b) \right|^{p_1} \right)^{\frac{1}{p_1}},
\]
where
\[
u_2 = (1-q) \sum_{n=0}^{\infty} q^n (1-q^{n+1})^{p_1} \quad \text{and} \quad u_3 = (1-q) \sum_{n=0}^{\infty} q^n (1-q^n)(1-q^{n+1})^{p_1}.
\]

**Proof** Taking the modulus of the right-hand side of the equality in Lemma 1 and applying well-known Hölder’s inequality, we obtain
\[
\left| \frac{f(a) + qf(b)}{1 + q} - \frac{1}{b-a} \int_a^b f(x)^b \, dx \right| 
\leq \frac{q^2(b-a)^2}{1 + q} \int_0^1 (t(1-qt))^{r_1} \frac{d}{dt} \left| bD_q^2f(ta + (1-t)b) \right| \, dt 
\leq \frac{q^2(b-a)^2}{1 + q} \left( \int_0^1 t^{r_1} \, dt \right)^{\frac{1}{r_1}} \left( \int_0^1 (1-qt)^{p_1} \left| bD_q^2f(ta + (1-t)b) \right|^{p_1} \, dt \right)^{\frac{1}{p_1}}.
\]
Since $|b^2f'|^{p_1}$ is convex, we have

$$\left| \frac{f(a) + qf(b)}{1 + q} - \frac{1}{b-a} \int_a^b f(x) b_q^x \right|$$

$$\leq \frac{q^2(b-a)^2}{1 + q} \left( \int_0^1 t^{1/p_1} dt \right)^{1/p_1}$$

$$\times \left( \left| \frac{b^2f(a)}{p_1} \right| \int_0^1 (1 - qt)^{p_1} t d_q t + \left| \frac{b^2f(b)}{p_1} \right| \int_0^1 (1 - qt)^{p_1} (1 - t) d_q t \right)^{1/p_1}$$

$$= \frac{q^2(b-a)^2}{1 + q} \left( \frac{1}{[r_1 + 1]} \right)^{1/p_1} \left( u_2 \left| \frac{b^2f(a)}{p_1} \right| + u_3 \left| \frac{b^2f(b)}{p_1} \right| \right)^{1/p_1}.$$

We can easily see that

$$u_2 = \int_0^1 (1 - qt)^{p_1} t d_q t = (1 - q) \sum_{n=0}^\infty q^{2n}(1 - q^{n+1})^{p_1}$$

and

$$u_3 = \int_0^1 (1 - qt)^{p_1} (1 - t) d_q t = (1 - q) \sum_{n=0}^\infty q^n(1 - q^n)(1 - q^{n+1})^{p_1}.$$

This completes the proof. \[\square\]

**Remark 5** If we take the limit as $q \to 1^-$ in Theorem 7, then we have

$$u_2 = \int_0^1 (1 - t)^{p_1} t d_q t = \frac{1}{(p_1 + 1)(p_1 + 2)}$$

and

$$u_3 = \int_0^1 (1 - t)^{p_1} (1 - t) dt = \frac{1}{p_1 + 2}. $$

Moreover, inequality (3.8) reduces to

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right|$$

$$\leq \frac{(b-a)^2}{2} \left( \frac{1}{r_1 + 1} \right)^{1/p_1} \left( \frac{1}{(p_1 + 1)(p_1 + 2)} \right)^{1/p_1} \left( (p_1 + 2)|f''(a)|^{p_1} + |f''(b)|^{p_1} \right)^{1/p_1}.$$

**4 Conclusions**

In this paper, we obtained Hermite–Hadamard-type inequalities for convex functions by applying the newly defined $q^b$-integral. The results proved in this paper are a potential generalization of the existing comparable results in the literature. As future directions, we can find similar inequalities through different types of convexities.
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Authors’ contributions
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Author details
1 Jiangsu Key Laboratory for NSLSCS, School of Mathematical Sciences, Nanjing Normal University, Nanjing, China. 2 Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce, Turkey. 3 Department of Mathematics, Government College University, Lahore, 54000, Pakistan. 4 Department of Mathematics, Huzhou University, Huzhou, China.

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References
1. Dragomir, S.S., Pearce, C.: Selected topics on Hermite–Hadamard inequalities and applications. Mathematics Preprint Archive 2003(3), 463–817 (2003)
2. Pečarić, J.E., Tong, Y.L.: Convex Functions, Partial Orderings, and Statistical Applications. Academic Press, Boston (1992)
3. Alp, N., Sankaya, M.Z., Kurt, M., İşcan, İ.: \( q \)-Hermite Hadamard inequalities and quantum estimates for midpoint type inequalities via convex and quasi-convex functions. J. King Saud Univ., Sci. 30(2), 193–203 (2018)
4. Yang, X.-Z., Farid, G., Nazeer, W., Yusuf, M., Chu, M.-C., Dong, C.-F.: Fractional generalized Hadamard and Fejér–Hadamard inequalities for \( m \)-convex functions. AIMS Math. 5(6), 6325–6340 (2020)
5. Budak, H., Ali, M.A., Tarhanaci, M.: Some new quantum Hermite–Hadamard-like inequalities for coordinated convex functions. J. Optim. Theory Appl. 186(3), 899–910 (2020)
6. Guo, S.-Y., Chu, Y.-M., Farid, G., Mehmoond, S., Nazeer, W.: Fractional Hadamard and Fejér–Hadamard inequalities associated with exponentially \( (s, m) \)-convex functions. J. Funct. Spaces 2020, Article ID 2410385 (2020)
7. Ernst, T.: A Comprehensive Treatment of \( Q \)-calculus. Springer, Berlin (2012)
8. Ernst, T.: The History of \( Q \)-calculus and a New Method. Department of Mathematics, Uppsala University, Sweden (2000)
9. Nwaeze, E.R., Tameru, A.M.: New parameterized quantum integral inequalities via \( \eta \)-quasiconvexity. Adv. Differ. Equ. 2019(1), 425 (2019)
10. Gauchman, H.: Integral inequalities in \( q \)-calculus. Comput. Math. Appl. 47(2–3), 281–300 (2004)
11. Jhanthamah, S., Tariboon, J., Ntouyas, S.K., Nonlaopon, K.: On \( q \)-Hermite–Hadamard inequalities for differentiable convex functions. Mathematics 7(7), 632 (2019)
12. Khan, M.A., Mohammad, N., Nwaeeze, E.R., Chu, Y.-M.: Quantum Hermite–Hadamard inequality by means of a Green function. Adv. Differ. Equ. 2020(1), 1 (2020)
13. Liu, W.-J., Zhuang, H.-F.: Some quantum estimates of Hermite–Hadamard inequalities for convex functions. J. Appl. Anal. Comput. 7(2), 501–512 (2017)
14. Noor, M.A., Noor, K.I., Awan, M.U.: Some quantum integral inequalities via preinvex functions. Appl. Math. Comput. 269, 243–261 (2015)
15. Noor, M.A., Awan, M.U., Noor, K.I.: Quantum Ostrowski inequalities for \( q \)-differentiable convex functions. J. Math. Inequal. 10(4), 1013–1018 (2016)
16. Sudsutad, W., Ntouyas, S.K., Tariboon, J.: Quantum integral inequalities for convex functions. J. Math. Inequal. 9(3), 781–793 (2015)
17. Vivas-Cortez, M., Aamir Ali, M., Kashuri, A., Bashir Sial, J., Zhang, Z.: Some new Newton’s type integral inequalities for co-ordinated convex functions in quantum calculus. Symmetry 12(9), 1476 (2020)
18. Zhuang, H., Liu, W., Park, J.: Some quantum estimates of Hermite–Hadamard inequalities for quasi-convex functions. Mathematics 7(2), 152 (2019)
19. Kac, V., Cheung, P.: Quantum Calculus. Springer, Berlin (2001)
20. Jackson, F.H.: On \( q \)-definite integrals. Q. J. Pure Appl. Math. 41, 193–203 (1910)
21. Tariboon, J., Ntouyas, S.K.: Quantum calculus on finite intervals and applications to impulsive difference equations. Adv. Differ. Equ. 2013(1), 282 (2013)
22. Bermudo, S., Körpus, P., Valdés, J.N.: On \( q \)-Hermite–Hadamard inequalities for general convex functions. Acta Math. Hung. 1–11 (2020)
23. Noor, M.A., Noor, K.I., Awan, M.U.: Some quantum estimates for Hermite–Hadamard inequalities. Appl. Math. Comput. 251, 673–679 (2015)
24. Anastassiou, G.A.: Intelligent Mathematics: Computational Analysis. Springer, New York (2011)
25. Alomari, M.W., Darus, M., Dragomir, S.S.: New inequalities of Hermite–Hadamard type for functions whose second
derivatives absolute values are quasi-convex. Tamkang J. Math. 41(4), 353–359 (2010)
26. Sarıkaya, M.Z., Aktan, N.: On the generalization of some integral inequalities and their applications. Math. Comput.
Model. 54(9–10), 2175–2182 (2011)