Abstract

In this study, the properties of convex pentagons that can form rotationally symmetric edge-to-edge tilings are discussed. Because the rotationally symmetric tilings are formed by concave octagons that are generated by two convex pentagons connected through a line symmetry, they are considered to be equivalent to rotationally symmetric tilings with concave octagons. In addition, under certain circumstances, tiling-like patterns with a regular polygonal hole at the center can be formed using these convex pentagons.

Keywords: pentagon, octagon, tiling, rotational symmetry, monohedral, spiral

1 Introduction

In this study, as shown in Figure 1(a), let us label the vertices (interior angles) of the convex pentagon \( A, B, C, D, \) and \( E \), and its edges \( a, b, c, d, \) and \( e \) in a fixed manner. A convex pentagonal tile \( 1 \) that satisfies the conditions \( A + B + C = 360^\circ, C = 2D, a = b = c = d \) belongs to the Type 1 family \( 2 \) \([3, 9, 10, 14]\). The tile is called “C11-T1A” in \([13]\) and its geometric properties are shown. Because this convex pentagon has four equal-length edges, its interior can be divided into a triangle \( BDE \) and two isosceles triangles \( ABE \) and \( BCD \) as shown in Figure 1(b). Given the foregoing properties, the relational expression of the interior angle of each vertex of C11-T1A can be rewritten as follows:

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\(^1\) A tiling (or tessellation) of the plane is a collection of sets that are called tiles, which covers a plane without gaps and overlaps, except for the boundaries of the tiles. The term "tile" refers to a topological disk, whose boundary is a simple closed curve. If all the tiles in a tiling are of the same size and shape, then the tiling is monohedral \( 3, 14 \). In this study, a polygon that admits a monohedral tiling is called a polygonal tile \( 9, 11 \). Note that, in monohedral tiling, it admits the use of reflected tiles.

\(^2\) To date, fifteen families of convex pentagonal tiles, each of them referred to as a “Type,” are known \( 3, 10, 14 \). For example, if the sum of three consecutive angles in a convex pentagonal tile is \( 360^\circ \), the pentagonal tile belongs to the Type 1 family. Convex pentagonal tiles belonging to some families also exist. Known convex pentagonal tiles can form periodic tiling. In May 2017, Michaël Rao declared that the complete list of Types of convex pentagonal tiles had been obtained (i.e., they have only the known 15 families), but it does not seem to be fixed as of March 2020 \( 14 \).

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Figure. 1: Nomenclature for vertices and edges of a convex pentagon, three triangles in the convex pentagonal tile C11-T1a, and the Octa-unit

Figure. 2: Examples of variations of Type 1 tilings by C11-T1A
where $0^\circ < B < 180^\circ$. From the relationship between the five interior angles, the vertex concentrations that are always valid in tilings are \[A + B + C = 360^\circ, \quad 2E + C = 360^\circ, \quad 2D + A + B = 360^\circ, \quad 2D + 2E = 360^\circ.\] If $a = b = c = d = 1$, then the edge length of $e$ can be expressed as follows:

\[e = 2 \sin \left( \frac{B}{2} \right).\]

As shown in Figure 2, C11-T1A can form the representative tiling of Type 1 or variations of Type 1 tilings (i.e., tilings whose vertices are formed only by the relations of $A + B + C = 360^\circ$ and $D + E = 180^\circ$). The edge $e$ of C11-T1A is the only edge of different length. As shown in Figure 1(c), an equilateral concave octagon formed by two convex pentagons, connected through a line symmetry whose axis is edge $e$, is referred to as the Octa-unit. C11-T1A can form tilings that are not contained in the variations of Type 1 tilings, as shown in Figure 3, by using Octa-units with different directions in one direction [5,13].

In this study, we introduce the convex pentagon C11-T1A that can generate countless numbers of rotationally symmetric tilings. Because the Octa-unit of C11-T1A is used in these rotationally symmetric tilings, it is considered similar to the concave octagons that can generate countless rotationally symmetric tilings.

## 2 Rotationally symmetric tilings

In [5], there are figures of four-fold, five-fold, six-fold, and ten-fold rotationally symmetric edge-to-edge\(^3\) tilings using convex pentagonal tiles (or tiles that can be regarded as concave octagons). In [5], there are multiple types of edge-to-edge tilings with six-fold rotational symmetry by an equilateral convex pentagon, including Hirschhorn and Hunt’s types [4]. The convex pentagons forming these rotationally symmetric edge-to-edge tilings corresponded to C11-T1A.

By using the relational expression of the interior angles of [1], the conditions of pentagonal tile C11-T1A that can form $n$-fold rotationally symmetric edge-to-edge tilings are expressed in [2].

\[^3\text{A tiling by convex polygons is edge-to-edge if any two convex polygons in a tiling are either disjoint or share one vertex or an entire edge in common. Then other case is non-edge-to-edge. [3,9,10].}\]
Figure 3: Examples of tilings with Octa-units of C11-T1A

\[
\begin{align*}
A &= 180^\circ - \frac{240^\circ}{n}, \\
B &= \frac{360^\circ}{n}, \\
C &= 180^\circ - \frac{120^\circ}{n}, \\
D &= 90^\circ - \frac{60^\circ}{n}, \\
E &= 90^\circ + \frac{60^\circ}{n}, \\
a &= b = c = d,
\end{align*}
\] (2)

where \(n\) is an integer greater than or equal to three, because \(0^\circ < B < 180^\circ\).

Table 1 presents some of the relationships between the interior angles of convex pentagons satisfying (2) that can form the \(n\)-fold rotationally symmetric edge-to-edge tilings. (For \(n = 3-13\), tilings with convex pentagonal tiles are drawn. For further details, Figures 5-15.) Given
Table 1: Interior angles of convex pentagons satisfying (2) that can form the n-fold rotationally symmetric edge-to-edge tilings

| n | Value of interior angle (degree) | Edge length of e | Figure number |
|---|----------------------------------|------------------|---------------|
| 3 | 100 120 140 70 110              | 1.732            | 5             |
| 4 | 120 90 150 75 105              | 1.414            | 6             |
| 5 | 132 72 156 78 102              | 1.176            | 7             |
| 6 | 140 60 160 80 100              | 1                | 8             |
| 7 | 145.71 51.43 162.86 81.43 98.57 | 0.868            | 9             |
| 8 | 150 45 165 82.5 97.5           | 0.765            | 10            |
| 9 | 153.33 40 166.67 83.33 96.67    | 0.684            | 11            |
| 10| 156 36 168 84 96              | 0.618            | 12            |
| 11| 158.18 32.73 169.09 84.55 95.45 | 0.563            | 13            |
| 12| 160 30 170 85 95              | 0.518            | 14            |
| 13| 161.54 27.69 170.77 85.38 94.62 | 0.479            | 15            |
| 14| 162.86 25.71 171.43 85.71 94.29 | 0.445            |               |
| 15| 164 24 172 86 94              | 0.416            |               |
| 16| 165 22.5 172.5 86.25 93.75    | 0.390            |               |
| 17| 165.88 21.18 172.94 86.47 93.53 | 0.367            |               |
| 18| 166.67 20 173.33 86.67 93.33 | 0.347             |               |
| ...| ... ... ... ... ...             | ...              |               |

that \( n = 6 \), the convex pentagonal tile that satisfies (2) is an equilateral convex pentagon, and can form several different six-fold rotationally symmetric edge-to-edge tilings \[4,5,13,14\]. The tiling in Figure 8 is one of the six-fold rotationally symmetric edge-to-edge tilings by the equilateral convex pentagon that Hirschhorn and Hunt presented \[4\]. Note that the n-fold rotationally symmetric edge-to-edge tilings by convex pentagonal tiles satisfying (2) have \( C_n \) symmetry \[4\] because they have rotational symmetry, but no axis of reflection symmetry.

Here, the formation of rotationally symmetric edge-to-edge tiling with convex pentagonal tiles is briefly explained. First, as shown in STEP 1 in Figure 4, create a unit connecting the Octa-units that are generated by convex pentagons satisfying (2) in one direction so that \( A + B + C = 360^\circ \). The Octa-units can then be assembled in such a way as to increase the number of pieces from one to two to three, and so on, in order. Then, copy the two units in STEP 1. In STEP 2, first, rotate one of the two copied units by \( \frac{120^\circ}{n} \), with respect to the original (see STEP 2 of Figure 4, wherein the axis of (2) has a rotation of \( \frac{120^\circ}{n} \) with respect to the axis of (1)), and connect it to the original to get the following results: \( 2E + C = 360^\circ \), \( 2D + A + B = 360^\circ \), and \( A + B + C = 360^\circ \), as expressed in STEP 2 of Figure 4. Next, rotate the remaining unit by \( \frac{240^\circ}{n} \), with respect to the original (see STEP 2 of Figure 4, wherein the axis of (3) has a rotation of \( \frac{240^\circ}{n} \) with respect to the axis of (1)), and connect it in the same way as the one previously made to make a unit like that in STEP 2 in Figure 4.

\(^4\)\(C_n\) is based on the Schoenflies notation for symmetry in a two-dimensional point group \[15,16\]. \(C_n\) represents an n-fold rotation axis without reflection. The notation for symmetry is based on that presented in \[6\].
Next, copy the unit in STEP 2. Subsequently, take the unit from STEP 2 and rotate it by the value of the interior angle of vertex $B$. When the original unit and the rotated unit are arranged as shown in STEP 3 (Rotationally symmetric tiling) in Figure 3, $\frac{2}{n}$ parts in the $n$-fold rotationally symmetric tiling can be formed. Then, by repeating this process as many times as necessary, an $n$-fold rotationally symmetric edge-to-edge tiling with convex pentagonal tiles can be formed.

Klaassen [6] presented the convex pentagonal tiles that can generate countless rotationally symmetric edge-to-edge tilings; however, the pentagon tilings were non-edge-to-edge. On the other hand, it was shown that the convex pentagonal tiles satisfying (2) can generate countless
rotationally symmetric edge-to-edge tilings. Although the corresponding theorem and proof are not presented, similar to how it was in [3], it will still be understood that countless rotationally symmetric tilings can be generated by the foregoing explanations and methods.

3 Rotationally symmetric tilings (tiling-like patterns) with a regular polygonal hole at the center

In [5], there are figures of rotationally symmetric tiling-like patterns with a hole in the center of a regular heptagon, 10-gon, 12-gon, or 18-gon formed using convex pentagons (or elements that can be regarded as concave octagons). The regular 18-gonal hole can be filled with convex pentagons, but the other holes cannot be filled with convex pentagons. Note that the tiling-like patterns are not considered tilings due to the presence of a gap, but are simply called tilings in this study. The convex pentagons forming these rotationally symmetric tilings with a regular $m$-gonal hole corresponded to C11-T1A.

By using the relational expression of the interior angles of (1), the conditions of pentagonal tile C11-T1A that can form rotationally symmetric tilings with a regular $m$-gonal hole are expressed in (3).

$$\begin{align*}
A &= 180^\circ - \frac{720^\circ}{m}, \\
B &= \frac{1080^\circ}{m}, \\
C &= 180^\circ - \frac{360^\circ}{m}, \\
D &= 90^\circ - \frac{180^\circ}{m}, \\
E &= 90^\circ + \frac{180^\circ}{m}, \\
a &= b = c = d,
\end{align*}$$

where $m$ is an integer greater than or equal to seven, because $0^\circ < B < 180^\circ$. Note that, because “$180^\circ - \frac{360^\circ}{m}$” corresponds to one interior angle of a regular $m$-gon, the value of “$A + B$” in (3) is equal to the outer angle $(180^\circ + \frac{360^\circ}{m})$ of one vertex of a regular $m$-gon.

Table 2 presents some of the relationships between the interior angles of convex pentagons satisfying (3) that can form the rotationally symmetric tilings with a regular $m$-gonal hole at the center. (For $m = 7, 10, 12, 14, 15, 18, 21, 24, 27$, tilings with a regular $m$-gonal hole at the center formed by convex pentagons are drawn. For further details, Figures 18–26.) If these elements are considered to be convex pentagons, the connection is edge-to-edge. These tilings with a regular $m$-gonal hole with $D_m$ symmetry at the center have $C_m$ symmetry. If convex pentagons satisfying (3) have $m$ that is divisible by three, they are also convex pentagonal tiles that satisfy (2).

Table 2 presents some of the relationships between the interior angles of convex pentagons satisfying (3) that can form the rotationally symmetric tilings with a regular $m$-gonal hole at the center. (For $m = 7, 10, 12, 14, 15, 18, 21, 24, 27$, tilings with a regular $m$-gonal hole at the center formed by convex pentagons are drawn. For further details, Figures 18–26.) If these elements are considered to be convex pentagons, the connection is edge-to-edge. These tilings with a regular $m$-gonal hole with $D_m$ symmetry at the center have $C_m$ symmetry. If convex pentagons satisfying (3) have $m$ that is divisible by three, they are also convex pentagonal tiles that satisfy (2).

Rotational symmetry tilings with a regular $m$-gonal hole at the center can be made in almost the same way as the $n$-fold rotationally symmetric edge-to-edge tiling (i.e., the manner of creation follows the same steps up to STEP 2 of Figure 1). The part with three edges $AB$, at the center of the unit made in STEP 2 of Figure 4, corresponds to the contour of a regular $D_m$ is based on the Schoenflies notation for symmetry in a two-dimensional point group. “$D_m$” represents an $m$-fold rotation axis with $m$ reflection symmetry axes.
Table 2: Interior angles of convex pentagons satisfying (3) that can form the rotationally symmetric tilings with a regular \( m \)-gonal hole at the center

| \( n \) | Value of interior angle (degree) | Edge length of \( e \) | \( n \) of Table 1 | Figure number |
|-------|-------------------------------|----------------------|-----------------|---------------|
| 7     | 77.14 154.29 128.57 64.29 115.71 | 1.950                | 16              |               |
| 8     | 90 135 135 67.5 112.5         | 1.848                | 17              |               |
| 9     | 100 120 140 70 110          | 1.732                | 3               | 18            |
| 10    | 108 108 144 72 108          | 1.618                | 19              |               |
| 11    | 114.55 98.18 147.27 73.64 106.36 | 1.511                |                 |               |
| 12    | 120 90 150 75 105          | 1.414                | 4               | 20            |
| 13    | 124.62 83.08 152.31 76.15 103.85 | 1.326                |                 |               |
| 14    | 128.57 77.14 154.29 77.14 102.86 | 1.247                |                 | 21            |
| 15    | 132 72 156 78 102          | 1.176                | 5               | 22            |
| 16    | 135 67.5 157.5 78.75 101.25 | 1.111                |                 |               |
| 17    | 137.65 63.53 158.82 79.41 100.59 | 1.053                |                 |               |
| 18    | 140 60 160 80 100          | 1                  | 6               | 23            |
| 19    | 142.11 56.84 161.05 80.53 99.47 | 0.952                |                 |               |
| 20    | 144 54 162 81 99          | 0.908                |                 |               |
| 21    | 145.71 51.43 162.86 81.43 98.57 | 0.868                | 7               | 25            |
| 22    | 147.27 49.09 163.64 81.82 98.18 | 0.831                |                 |               |
| 23    | 148.70 46.96 164.35 82.17 97.83 | 0.797                |                 |               |
| 24    | 150 45 165 82.5 97.5     | 0.765                | 8               | 26            |
| 25    | 151.2 43.2 165.6 82.8 97.2 | 0.736                |                 |               |
| 26    | 152.31 41.54 166.15 83.08 96.92 | 0.709                |                 |               |
| 27    | 153.33 40 166.67 83.33 96.67 | 0.684                | 9               | 27            |
| ...   | ...  ...  ...  ...  ...   | ...  ...  ...  ...  ... | ...  ...  ...  ...  ... | ...  ...  ...  ...  ... |

\( m \)-gon. Therefore, copy the unit in STEP 2, take the unit from STEP 2 and rotate it by the value of the interior angle of vertex B. When the original unit and the rotated unit are arranged as shown in STEP 3 (Tiling with a regular \( m \)-gonal hole) in Figure 4, \( m \) parts of a regular \( m \)-gon can be formed. Then, by repeating this process as many times as necessary, a rotationally symmetric tiling with a regular \( m \)-gonal hole at the center can be formed.

As shown in Table 2 and Figure 23, the convex pentagonal tile that satisfies (3), where \( m = 18 \), is an equilateral convex pentagon. As such, the regular 18-gonal hole can be filled with equilateral convex pentagons. The pentagonal arrangement pattern in the regular 18-gon is unique; however, the regular 18-gon can be reversed; therefore, two patterns can be used in the hole (see Figure 23). The tiling in Figure 24 is one of the six-fold rotationally symmetric edge-to-edge tilings with equilateral convex pentagons that Hirschhorn and Hunt present in [4]. In addition, the tiling in Figure 23 is that of Hirschhorn [7]. In [5], there are figures of tilings corresponding to Figures [8] [23], and [24] with the one corresponding to Figure 24 labeled as Hirschhorn’s rosette. These tilings with a regular 18-gonal hole that is filled with equilateral convex pentagons have \( C_6 \) symmetry. In Pegg’s research [2], they stated there exists the tiling of Figure 8 using the equilateral convex pentagon, and that
there are also other interesting tilings, such as that with the “crystal-like” structure. Note that in Pegg’s research \(^2\), there are non-monohedral tilings created using convex and concave pentagons.

As described above, the tiling of \( m = 10 \) in Table 2 (see Figure 19) is listed in Iliev’s research \(^5\), however, the value of the interior angle of the convex pentagon was erroneous, as well as in Smith’s research \(^8\). It is shown that the convex pentagon of \( m = 10 \) in Table 2 is the same as the convex pentagonal tile belonging to both the Type 1 and Type 2 families, and that it can form various tilings \(^7\). In \(^1\) and \(^8\), there are also tilings with a regular pentagonal hole at the center using this convex pentagon of \( m = 10 \) in Table 2. Moreover, it is interesting to note that the ring-shaped layer forming the periphery can be reversed in \(^8\).

### 4 Spiral tilings

Klaassen \(^6\) also showed the viewpoint of rotationally symmetric tilings with convex pentagons as a spiral structure. A similar spiral structure can be found in the rotationally symmetric edge-to-edge tilings with convex pentagons of C11-T1A. Figure 28 pertains to the case where \( n = 5 \), which makes the spiral structure easier to understand. From this spiral structure, the difference between the rotationally symmetric tilings with convex pentagons in \(^6\) and the rotationally symmetric edge-to-edge tilings with C11-T1A other than the edge-to-edge property will be determinable.

Here, let us pay attention to the convex pentagon of \( m = 8 \) in Table 2. This convex pentagon has the property of \( B = C = 135^\circ \). From this property, the convex pentagon of \( m = 8 \) in Table 2 can form spiral tilings \(^6\) with two-fold rotational symmetry, as shown in Figure 29(a). This convex pentagonal tile belongs to both the Type 1 and Type 7 families \(^3, 9–11, 14\).

After finding the tiling in Figure 29(a), we found that, in “(6) Central block ‘wraparound’ ” of \(^1\), Bailey presented the spiral tilings with two-fold rotational symmetry by the convex pentagon corresponding to the case of \( m = 10 \) in Table 2. The spiral tiling is the same as that in Figure 30(a). Furthermore, from \(^1\), we found that it is possible to form tilings that keep the spiral structure and extend in one direction, as shown in Figures 30(b), 30(c), and 30(d). Then, we determined that the convex pentagon of \( m = 8 \) in Table 2 can also form tilings that keep the spiral structure and extend in one direction (see Figures 29(b), 29(c), and 29(d)).

The convex pentagon of \( m = 14 \) in Table 2 has \( B = D = \frac{3\pi}{7} \approx 77.14^\circ \), and belongs to both the Type 1 and Type 2 families. Similar to the convex pentagons of \( m = 8, 10 \) in Table 2 the convex pentagon of \( m = 14 \) in Table 2 can also form the spiral tiling with two-fold rotational symmetry (see Figure 31(a)) and tilings that keep the spiral structure and extend in one direction (see Figures 31(b), 31(c), and 31(d)). For all convex pentagons of \( m = 8, 10, 14 \) in Table 2, it is also possible to remove one spiral structure and to extend the belts wherein the Octa-units are arranged (see Figure 32).

It should be noted that the spiral tiling with a regular pentagonal hole at the center is shown in \(^8\) using the convex pentagon corresponding to \( m = 10 \) in Table 2.

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\(^6\)It is difficult to determine what the spiral tiling is. There are some discussions, such as those in \(^7\). The tiling shown in Figure 29 can be regarded as a spiral, however, it has a radial structure extending from the center. Thus, we feel that it is more straightforward to regard it as a rotationally symmetric tiling. On the other hand, because the tiling of Figure 29 does not have a radial structure extending from the center or singularity point, we feel that it is a spiral rather than a rotationally symmetric tiling.
5 Conclusions

The existence of convex pentagonal tiles that can generate countless number of rotationally symmetric edge-to-edge tilings is not well-known. As mentioned above, Klaassen presented a case where tilings are not edge-to-edge. From [5], we determined that the convex pentagon called C11-T1A in [13] had this property. In [5], Iliev presented the tilings corresponding to $n = 4, 5, 6, 10$ in Table 1 and $m = 7, 10, 12, 18$ in Table 2. It may have been inferred by Iliev that there are countless rotationally symmetric tilings similar to those; however, there seems to be no clarification on the matter. In particular, there is no description of the concrete properties (tile conditions), such as the convex pentagon of C11-T1A, as shown in this study.

It was previously recognized that the convex pentagon belonging to both the Type 1 and Type 7 families (i.e., convex pentagon corresponding to $m = 8$ in Table 2) can generate various albeit limited tilings. However, in reality, we noticed that it is possible to generate more tilings. Among them are other rotationally symmetric tilings and spiral tilings. We will introduce them in a different article [12].

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Figure. 5: Three-fold rotationally symmetric edge-to-edge tiling by a convex pentagon

Figure. 6: Four-fold rotationally symmetric edge-to-edge tiling by a convex pentagon
Figure. 7: Five-fold rotationally symmetric edge-to-edge tiling by a convex pentagon

Figure. 8: Six-fold rotationally symmetric edge-to-edge tiling by a convex pentagon
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Figure. 9: Seven-fold rotationally symmetric edge-to-edge tiling by a convex pentagon

Figure. 10: Eight-fold rotationally symmetric edge-to-edge tiling by a convex pentagon
Figure. 11: Nine-fold rotationally symmetric edge-to-edge tiling by a convex pentagon

Figure. 12: 10-fold rotationally symmetric edge-to-edge tiling by a convex pentagon
Figure. 13: 11-fold rotationally symmetric edge-to-edge tiling by a convex pentagon

Figure. 14: 12-fold rotationally symmetric edge-to-edge tiling by a convex pentagon
Figure. 15: 13-fold rotationally symmetric edge-to-edge tiling by a convex pentagon

Figure. 16: Rotationally symmetric tiling with $C_7$ symmetry with a regular convex heptagonal hole at the center by a convex pentagon
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Figure 17: Rotationally symmetric tiling with $C_8$ symmetry with a regular convex octagonal hole at the center by a convex pentagon

Figure 18: Rotationally symmetric tiling with $C_9$ symmetry with a regular convex nonagonal hole at the center by a convex pentagon
Figure. 19: Rotationally symmetric tiling with $C_{10}$ symmetry with a regular convex 10-gonal hole at the center by a convex pentagon

Figure. 20: Rotationally symmetric tiling with $C_{12}$ symmetry with a regular convex 12-gonal hole at the center by a convex pentagon
Figure. 21: Rotationally symmetric tiling with $C_{14}$ symmetry with a regular convex 14-gonal hole at the center by a convex pentagon

Figure. 22: Rotationally symmetric tiling with $C_{15}$ symmetry with a regular convex 15-gonal hole at the center by a convex pentagon
Figure 23: Rotationally symmetric tiling with a regular convex 18-gon at the center by an equilateral convex pentagon
Figure. 24: Hirschhorn and Hunt’s rotationally symmetric tiling with a regular convex 18-gon at the center by an equilateral convex pentagon
Figure. 25: Rotationally symmetric tiling with $C_{21}$ symmetry with a regular convex 21-gonal hole at the center by a convex pentagon

Figure. 26: Rotationally symmetric tiling with $C_{24}$ symmetry with a regular convex 24-gonal hole at the center by a convex pentagon
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Figure. 27: Rotationally symmetric tiling with $C_{27}$ symmetry with a regular convex 27-gonal hole at the center by a convex pentagon

Figure. 28: Spiral structure of five-fold rotationally symmetric edge-to-edge tiling by a convex pentagon
Figure 29: Spiral tilings with two-fold rotational symmetry by a convex pentagon of $m = 8$ in Table 2.
Figure 30: Spiral tilings with two-fold rotational symmetry by a convex pentagon of $m = 10$ in Table 2.
Figure 31: Spiral tilings with two-fold rotational symmetry by a convex pentagon of $m = 14$ in Table 2.
Figure 32: Tilings that removed one spiral structure and extend the belts of Octa-units using convex pentagons of \( m = 8, 10, 14 \) in Table 2.