Estimate of asymptotics of moments of arithmetic functions defined on an arithmetic progression and having a limit - normal distribution

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ABSTRACT We consider a method for estimating asymptotics of arithmetic functions on a geometric progression in the paper. This question is closely related with the distribution of primes. Problems often arise even with the definition of asymptotic of the mean value of arithmetic functions, and even more with the determination of asymptotics of moments of higher orders. Therefore, the paper proposes (a different from the traditional) probabilistic approach based on the limiting distribution of arithmetic functions on an arithmetic progression. We consider arithmetic functions with limiting normal distribution. Several assertions are proved about the estimation of asymptotics of moments of strongly additive arithmetic functions on a geometric progression, as well as additive functions of the class $H$ and arithmetic functions of the class $V$ having the same limit distribution.

Keywords: arithmetic function, additive arithmetic function, strongly additive arithmetic function, probability space, analogue of the law of large numbers, limit distribution, normal distribution, asymptotics of moments of arithmetic functions on a geometric progression, sequence of random variables, independence of random variables, Central limit theorem.
1. INTRODUCTION

A method for estimating asymptotics of arithmetic functions defined on an arithmetic progression is considered. This question is closely related to the distribution of primes.

This work considers arithmetic functions that have a limiting normal distribution. This question has already been studied in [1], [2], but there was investigated the case of arithmetic functions defined on a natural series.

An arithmetic function is a function defined on a subset of natural numbers and taking values on a set of complex numbers. The name arithmetic function is associated with the fact that this function expresses some arithmetic property of the natural series. An arithmetic progression is considered (in this case) as a subset of natural numbers.

If an arithmetic function satisfies the condition:

\[ f(m) = f(p_1^a_1 \ldots p_i^a_i) = f(p_1^a_1) + \ldots + f(p_i^a_i) = \sum_{p \mid m} f(p^n), \]

then this arithmetic function is called additive.

Problems often arise with the determination of the asymptotic of the mean value of arithmetic functions [3] and even more with the determination of asymptotics of moments of higher orders. Therefore, I propose to approach the solution of this problem from the other side.

There are the following prerequisites for this approach.

1. Let there are two sequences of random variables that have one limiting distribution. Naturally, in this case, these sequences have the same asymptotic behavior of the mean value and moments of higher orders.

2. It is known that an arithmetic function can be represented as a sequence of random variables. If this sequence converges to the limiting distribution function, which coincides with the limiting distribution function of another sequence of random variables, then the arithmetic function and the sequence of random variables have the same asymptotic behavior of the mean value and moments of higher orders. Then it is possible to construct a sequence of random variables for which the specified characteristics are determined more simply. On the other hand, a given sequence of random variables must converge to the same distribution function as the arithmetic function. Then, using asymptotics of characteristics of this sequence of random variables, one can more simply determine asymptotics of characteristics of an arithmetic function.
3. If two arithmetic functions have the same limiting distribution, then their asymptotics for all moments coincide.

Let us start with strongly additive arithmetic functions. Let me remind you that by definition a strongly additive arithmetic function is a function for which \( f(p^a) = f(p) \). Therefore, for an arbitrary natural number \( m = p_1^{a_1} \ldots p_t^{a_t} \) for a strongly additive arithmetic function, we have:

\[
f(m) = f(p_1^{a_1}) + \ldots + f(p_t^{a_t}) = \sum_{p|m} f(p).
\] (1.1)

Any initial segment of a natural series \( \{1, \ldots, n\} \) can be naturally transformed into a probability spaces \((\Omega_n, \mathcal{A}_n, \mathbb{P}_n)\) by taking \( \Omega_n = \{1, \ldots, n\} \), \( \mathcal{A}_n \) - all subsets \( \Omega_n \), \( \mathbb{P}_n(A) = \frac{\#A}{n} \). Then an arbitrary (real) function \( f \) of a natural argument \( m \) (or rather, its restriction to \( \Omega_n \)) can be considered as a random variable \( \xi_k \) on this probability space \((\Omega_k, \mathcal{A}_k, \mathbb{P}_k)\):

\[
\xi_k(m) = f(m), 1 \leq m \leq n.
\]

Therefore, we can write Chebyshev's inequality for an arithmetic function \( f(m), m = 1, \ldots, n \) on the probability space:

\[
P_n( \mid f(m) - A_n \mid \leq b\sigma_n ) \geq 1 - 1/b^2,
\] (1.2)

where the value \( b \geq 1 \), and \( A_n, \sigma_n \), accordingly, the mean and standard deviation \( f(m), m = 1, \ldots, n \).

We put \( b = b(n) \) in (1.2), where \( b(n) \) is an unboundedly increasing function when the value \( n \to \infty \):

\[
P_n( \mid f(m) - A_n \mid \leq b(n)\sigma_n ) \geq 1 - 1/b^2(n).
\] (1.3)

The limit \( P_n \) in (1.3) exists when the value \( n \to \infty \):

\[
P_n( \mid f(m) - A_n \mid \leq b(n)\sigma_n ) \to 1, n \to \infty.
\] (1.4)

Expression (1.4) is an analogue of the law of large numbers for arithmetic functions [4].
Turan proved [5] that if an arithmetic function \( f(m), m = 1, \ldots, n \) is strongly additive and for all primes \( p \) satisfies the condition: \( 0 \leq f(p) < c \) and when the value \( n \to \infty \), then the following form of the analogue of the law of large numbers holds for it:

\[
P_n( |f(m) - A_n| \leq b(n) \sqrt{A_n} ) \to 1, n \to \infty.
\]  

(1.5)

Any arithmetic function \( f(m), m = 1, \ldots, n \) can be associated with a sequence of random variables \( \xi_n \) (living in different probability spaces). Each such random variable has a distribution function, so it is natural to consider the convergence of the sequence of distribution functions corresponding to the arithmetic function to the limiting distribution function.

In 1939 Erdős and Katz proved [6] that for an arithmetic function \( \omega(m), m \leq n \) (the number of prime divisors of a number \( m \)) the limiting distribution function is normal, that is, for any \( x \in \mathbb{R} \) is performed:

\[
P_n\left( \frac{\omega(m) - \ln \ln n}{\sqrt{\ln \ln n}} \leq x \right) \to \Phi(x), n \to \infty.
\]  

(1.6)

where \( \Phi(x) \) is the standard normal distribution function.

Erdős and Katz also generalized (1.6) and proved that if \( f(m) \) is a strongly additive arithmetic function and it is satisfied \( |f(p)| \leq 1 \) and \( \sigma_n \to \infty, n \to \infty \) for all primes \( p \), then

\[
P_n\left( \frac{f(m) - A_n}{\sigma_n} \leq x \right) \to \Phi(x), n \to \infty.
\]  

(1.7)

In this case, we define a random variable \( f^{(p)}(m) = f(p) \) for each prime number \( p(p \leq n) \), if \( p \mid m \) and \( f^{(p)}(m) = 0 \) otherwise.

Then \( f^{(p)}(m) = f(p) \) with probability \( \frac{1}{n \left\lfloor \frac{n}{p} \right\rfloor} \) and \( f^{(p)}(m) = 0 \) with probability \( 1 - \frac{1}{n \left\lfloor \frac{n}{p} \right\rfloor} \).

Therefore, the average value \( f^{(p)}(m) \) over the interval \([1, n]\) is:

\[
E[f^{(p)}, n] = \frac{f(p)}{n} \left\lfloor \frac{n}{p} \right\rfloor.
\]

It is executed for a strongly additive arithmetic function \( f(m) : \)
\[ f(m) = \sum_{p \leq n} f^{(p)}(m). \]

Therefore, the average value \( f(m) \) over the interval \([1, n]\) is:

\[ A_n = \sum_{p \leq n} \frac{f(p)}{p^{\frac{1}{b}}}. \]

This implies the following asymptotic behavior of the mean value of a strongly additive arithmetic function \( f(m) \):

\[ A_n \to \sum_{p \leq n} \frac{f(p)}{p^{\frac{1}{b}}}, n \to \infty. \]  

(1.8)

Therefore, based on (1.5), (1.7) and (1.8), it is true for a strongly additive arithmetic function \( f(m), m = 1, \ldots, n \) satisfying the conditions: \( 0 \leq f(p) \leq 1 \) and \( A_n \to \infty \) when the value \( n \to \infty \):

\[ P_n\left(\frac{f(m) - A_n}{\sqrt{A_n}} \leq x\right) \to \Phi(x), n \to \infty, \]  

(1.9)

where \( A_n \to \sum_{p \leq n} \frac{f(p)}{p}, n \to \infty \).

The class of strongly additive functions is expanded in [4] and the class H of arithmetic functions satisfying the following conditions is investigated.

Let \( f(m) \in H \) and \( q^b \) is a sequence of positive integer primes such that: \( \sum_{b} 1/q^b \) converges.

Let us define another additive function \( f^*(m) \), setting \( f^*(p^a) = f^*(p^b) \) for all \( p^a \) other than \( q^b \). Let \( f^*(q^b) \) - run through any values \( q^b \). Then it was proved that the limiting laws:

\[ P_n\left(\frac{f(m) - A_n}{D_n} < x\right) \text{ and } P_n\left(\frac{f^*(m) - A_n}{D_n} < x\right) \text{ exist and coincide.} \]

In particular, as \( f^*(m) \) one can take a strongly additive arithmetic function related with \( f(m) \) the ratio \( f^*(p^a) = f(p)(a = 1, 2, \ldots) \) for all prime \( p \) when the value \( n \to \infty \).
Therefore, additive arithmetic functions \( f(m) \) which have in the class H with strongly additive arithmetic functions \( f^*(m) \) (having a limit normal distribution) also have a limit normal distribution.

It was shown in [4] that strongly additive arithmetic functions \( f^*(m) \) defined on an arithmetic progression \( m = k_1 + t, (t = 0, 1, 2, \ldots), (k_1, 1) \) can act as additive arithmetic functions \( f(m) \).

2. ESTIMATE OF THE ASYMPTOTICS OF THE MOMENTS OF ARITHMETIC FUNCTIONS DEFINED ON THE AN ARITHMETIC PROGRESSION AND HAVING A LIMIT - NORMAL DISTRIBUTION

Using the results obtained in [7], we consider various cases of the sum of prime numbers:

\[
\sum_{p \leq x, \text{primes}} \frac{f(p)}{p},
\]  

(2.1)

where \( |f(p)| \leq 1 \) and \( \sum_{p \leq x, \text{primes}} \frac{f(p)}{p} \rightarrow \infty, x \rightarrow \infty \).

First, we consider the convergence of some series of a prime argument:

\[
\sum_{p \leq x, \text{primes}} \frac{1}{p^2} \leq \sum_{n \leq x, \text{numbers}} \frac{1}{n^2} = \frac{C}{\varphi(k)},
\]

where \( C \) is a constant, \( \varphi(k) \) is the Euler function at the point \( k \).

Therefore, the series \( \sum_{p \leq x, \text{primes}} \frac{1}{p^2} \) converges.

\[
\sum_{p \leq x, \text{primes}} \frac{1}{p \ln^2(p)} \leq \sum_{n \leq x, \text{numbers}} \frac{1}{n \ln^2(n)} = \frac{C_1}{\varphi(k)}
\]

where \( C_1 \) is a constant, \( \varphi(k) \) is the Euler function at the point \( k \).

Therefore, the series \( \sum_{n \leq x, \text{numbers}} \frac{1}{p \ln \ln(p)} \) also converges.

Now consider a sum of the form (2.1), where \( f(p) = 1/\ln(p) \).
If \( p = 3>e, \ln(p) > 1, 1/\ln(p) < 1 \), hence the condition \( |f(p)| \leq 1 \) is met.

Therefore, in this case, the sum looks like:

\[
\sum_{3 \leq p \leq x, p \neq km+1} \frac{1}{p \ln(p)}. \tag{2.2}
\]

We check the fulfillment of the necessary conditions for using the formulas of this paper to the sum (2.2). Let’s start with the 3rd condition.

Denote \( g(t) = \frac{1}{t \ln(t)} \), then the 3rd condition will be written in the form:

\[
\lim_{n \to \infty} \frac{\int_3^n g'(t) dt}{\ln(t)} = \infty(-\infty). \tag{2.3}
\]

Let us find \( g'(t) = -\frac{1}{t^2} \left( \frac{1}{\ln(t)} - \frac{1}{\ln^2(t)} \right) \), substitute in (2.3) and get:

\[
-\lim_{n \to \infty} \left( \int_3^n \frac{d \ln(t)}{\ln^2(t)} + \int_3^n \frac{d \ln(t)}{\ln^3(t)} \right) = C_2,
\]

where \( C_2 \) is a constant, i.e. 3rd condition is not met.

The series \( \sum_{3 \leq p \leq x, p \neq km+1} \frac{1}{p \ln(p)} \) - converges, so the necessary condition is not met.

Now we consider another sum of the form (2.1), where \( f(p) = 1/\ln(\ln(p)) \).

If \( p = 11 > e', \ln(\ln(p)) > 1, 1/\ln(\ln(p)) < 1 \), then the condition \( |f(p)| \leq 1 \) is satisfied.

The necessary and sufficient conditions using formulas are satisfied in this case.

We use a more precise formula, assuming the fulfillment of the Riemann hypothesis to find the asymptotic \( \sum_{11 \leq p \leq x, p \neq km+1} \frac{1}{p \ln(p)} \):
\[
\sum_{p \leq x, \rho \neq m+1} g(p) = \frac{1}{\phi(k)} \int_{1}^{x} g(t) dt + O\left(\left| g(x) \right| x^{1/2} \ln(x) \right) + O\left(\int_{1}^{x} |g'(t)| t^{1/2} \ln(t) dt \right). \tag{2.4}
\]

Having in mind (2.4) we get:

\[
\sum_{p \leq x, \rho \neq m+1} \frac{1}{p \ln(p)} = \frac{1}{\phi(k)} \int_{1}^{x} \frac{d(\ln \ln(t))}{\ln \ln(t)} + O\left(\frac{\ln(x)}{x^{1/2} \ln(x)} \right) = \frac{1}{\phi(k)} \ln \ln(x) + O(1). \tag{2.5}
\]

Now we consider \( f(p) = 1 - 1/p < 1 \).

The asymptotic of the sum is:

\[
\sum_{p \leq x, \rho \neq m+1} \frac{1-1/p}{p} = \sum_{p \leq x, \rho \neq m+1} \frac{1}{p} - \sum_{p \leq x, \rho \neq m+1} \frac{1}{p^2} = \frac{1}{\phi(k)} \ln \ln(x) + O(1),
\]

as the series \( \sum_{p \leq x, \rho \neq m+1} \frac{1}{p^2} \) converges.

Now let us consider \( f(p) = 1 - 1/\ln(p) < 1 \).

The asymptotic of the sum is:

\[
\sum_{p \leq x, \rho \neq m+1} \frac{1-1/\ln(p)}{p} = \sum_{p \leq x, \rho \neq m+1} \frac{1}{p} - \sum_{p \leq x, \rho \neq m+1} \frac{1}{p \ln(p)} = \frac{1}{\phi(k)} \ln \ln(x) + O(1),
\]

since the series \( \sum_{p \leq x, \rho \neq m+1} \frac{1}{p \ln(p)} \) converges.

Let us draw the following conclusions:

1. If \( f(p) = C \), where \( 0 \leq C \leq 1 \), then the asymptotic of the sum:

\[
\sum_{p \leq x, \rho \neq m+1} \frac{f(p)}{p} \sim \frac{C}{\phi(k)} \ln \ln(x).
\]

2. If \( f(p) \) increases monotonically and \( \lim_{p \to \infty} f(p) = C \), then the asymptotic of the sum:

\[
\sum_{p \leq x, \rho \neq m+1} \frac{f(p)}{p} \sim \frac{C}{\phi(k)} \ln \ln(x).
\]
3. If \( f(p) \) monotonically decreases, as \( \frac{C}{\ln \ln(p)} \) or slower, then the asymptotic of the sum:

\[
\sum_{p \leq x, p \neq m} \frac{f(p)}{p} \sim \frac{C}{\phi(k)} \ln \ln(x) \text{ or growing more slowly.}
\]

4. If \( f(p) \) monotonically decreases, as \( \frac{C}{\ln(p)} \) or faster, then the series:

\[
\sum_{p_0 \leq p \leq x, p \neq m} \frac{f(p)}{p} (p_o \geq 2) \text{ - converges.}
\]

Therefore the conditions \( |f(p)| \leq 1 \) and \( \sum_{p \leq x, p \neq m} \frac{f(p)}{p} \to \infty, x \to \infty \) satisfy only cases 1-3.

Now based on [7], we present an analysis of the asymptotic of the expression \( \sum_{p \leq n} \frac{g(p)}{p} \) under the condition that \(-1 \leq g(p) < 0\).

1. If \( g(p) = C \), where \(-1 \leq C < 0\), then the asymptotic of the sum will be equal to:

\[
\sum_{p \leq x, p \neq m} \frac{g(p)}{p} \sim \frac{C}{\phi(k)} \ln \ln(x) .
\]

2. If \( g(p) \) monotonically decreases and \( \lim_{p \to \infty} g(p) = C \), then the asymptotic of the sum also is equal to:

\[
\sum_{p \leq x, p \neq m} \frac{g(p)}{p} \sim \frac{C}{\phi(k)} \ln \ln(x) .
\]

3. If \( g(p) \) increases monotonically, as \( \frac{C}{\ln \ln(p)} \) or more slowly, then the asymptotic of the sum is equal to:

\[
\sum_{p \leq x, p \neq m} \frac{g(p)}{p} \sim \frac{C}{\phi(k)} \ln \ln \ln(x)
\]
or decreases more slowly.

4. If \( g(p) \) monotonically increases, as \( \frac{C}{\ln(p)} \) or faster, then the series:

\[
\sum_{p_0 \leq p \leq x, p \neq k} \frac{g(p)}{p} (p, \geq 2) - 
\]

converges.

Therefore, only cases 1-3 satisfy the conditions \( |g(p)| \leq 1 \) and

\[
- \sum_{p \leq x, p \neq k} \frac{g(p)}{p} \rightarrow \infty, x \rightarrow \infty .
\]

Assertion 1

Let \( f(m) \) is strongly additive arithmetic function, \( |f(p)| \leq 1 \) and \( \sum_{p \neq n, p \neq k, p+l} \frac{f(p)}{p} \rightarrow \infty \) when

the value \( n \rightarrow \infty \). Then:

1. It is always possible to construct a random variable having a normal distribution, mean
value and variance equal to the asymptotics of the mean value and variance \( f(m) \), where
\( m = kt + l, (t = 0, 1, 2, \ldots), (k, l = 1) \).

2. The asymptotic of the central moment of \( u \) -th order \( f(m) \) is equal to:

\[
\sum_{p \neq n, p \neq k, p+l} \frac{f^u(p)}{p} .
\]

Proof

Based on [4], asymptotics of the mean value and variance of a strongly additive \( f(m) \) at
value \( m = l, l+k, l+2k, \ldots, n \) and \( n \rightarrow \infty \) are equal, respectively:

\[
\sum_{p \neq n, p \neq k, p+l} \frac{f(p)}{p} \quad \text{and} \quad \sum_{p \neq n, p \neq k, p+l} \frac{f^2(p)}{p} .
\]
Let us construct a random variable $X_p$ that takes two values: $X_p = f(p)$ with probability $1/p$ and $X_p = 0$ with probability $1-1/p$. Then the mean value $X_p$ will be $E[X_p] = \frac{f(p)}{p}$, and the variance $X_p$ will be $D[X_p] = \frac{f^2(p)}{p}$.

Let us consider a random variable $S_n = \sum_{p\in S_n, p\geq m-l} X_p$, where $X_p$ are independent random variables. Then the mean value $S_n$ is equal to $E[S,n] = \sum_{p\in S_n, p\geq m-l} \frac{f(p)}{p}$, and the variance $S_n$ is equal to $D[S,n] = \sum_{p\in S_n, p\geq m-l} \frac{f^2(p)}{p}$, which corresponds to the asymptotics of the mean value and variance $f(m)$ for the value $m = l, l+k, l+2k, ..., n$ and $n \to \infty$.

Based on the Central Limit Theorem, the random variable $S_n$ has a normal distribution. Thus, we have proved the first part of the assertion.

Note that if $|f(p)| \leq 1$ and $D_n \to \infty$ at the value $n \to \infty$, then the arithmetic function $f(m)$ (where $m = kt + l, (t = 0, 1, 2, ...)$) also has a limiting normal distribution (see Section 1) with similar asymptotics of the mean value and variance. Consequently, the limit distributions also coincide $S_n$ and $f(m)$ for $m = l, l+k, l+2k, ..., n$, $n \to \infty$, and, consequently, all central moments of the $u$-th order are also coincide. Now we will calculate them.

First, we define the central moment of the $u$-th order for the random variable $X_p$:

$$E[(X_p - \frac{f(p)}{p})^u] = E[X_p - uE[X_p] + \frac{u(u-1)}{2}E[X_p^2 - \frac{f(p)}{p}] + ... + (-1)^u \frac{u!}{2} \frac{f^{(u)}(p)}{p} + \frac{u!(u-1)!}{2!} \frac{f^{(u)}(p) f^{(u)}(p) f^{(u)}(p) + ... + (-1)^u f^{(u)}(p) f^{(u)}(p) f^{(u)}(p)}{p^u}$$

Thus, the central moment of the $u$-th order for a random variable $X_p$ is equal to:

$$E[(X_p - \frac{f(p)}{p})^u] = \frac{f^{(u)}(p)}{p} + O\left(\frac{f^{(u)}(p)}{p^2}\right).$$

Having in mind that $|f(p)| \leq 1$ we get that asymptotic of the central moment of the $u$-th order for a random variable $S_n$ is equal to the value:
Therefore, we have proved the second part of the assertion.

Note that if the value \( |f(p)| \leq 1 \) in assertion 1, then \(- |f''(p)| \leq 1\), therefore, the study of the sum of the form \( \sum_{p \leq n, p \neq km \pm l} \frac{g(p)}{p} \), if \( |g(p)| \leq 1 \), can be also used in this case.

Let us look at an example for assertion 1.

Let we have a strongly additive arithmetic function \( f(m) \) that for prime numbers takes the value \( f(p) = 1/\ln \ln(p) \).

It is required to find the asymptotics of the moments of all orders for the function \( f(m) \) with the value \( m = l, l+k, l+2k, \ldots, n \) and \( n \to \infty \).

First, we find the asymptotic of the mean value \( f(m) \) for the value \( m = l, l+k, l+2k, \ldots, n \) and \( n \to \infty \):

\[
\sum_{p \leq n, p \neq km \pm l} \frac{f(p)}{p} = \sum_{p \leq n, p \neq km \pm l} \frac{1}{\ln \ln(p)}.
\]

Let us use [7] to determine \( \sum_{p \leq n, p \neq km \pm l} \frac{1}{\ln \ln(p)} \):

\[
\sum_{p \leq n, p \neq km \pm l} g(p) = \frac{1}{\varphi(k)} \int_{2}^{n} \frac{g(t)dt}{\ln(t)} + O(|g(n)|n^{1/2} \ln(n)) + O\left( \int_{2}^{n} |g(t)| t^{1/2} \ln(t)dt \right).
\]

Based on this formula, we get:

\[
\sum_{p \leq n, p \neq km \pm l} \frac{1}{\ln \ln(p)} = \frac{1}{2\varphi(k)} \int_{1}^{n} \frac{dt}{t \ln(t) \ln \ln(t)} + O\left( \frac{\ln(n)n^{1/2}}{n \ln \ln(n)} \right) = \frac{1}{2\varphi(k)} \ln \ln(n) + O(1).
\]

Compare with (2.5).

Now let us find the asymptotic of the variance \( f(m) \) for the value \( m = l, l+k, l+2k, \ldots, n \) and \( n \to \infty \):
\[ \sum_{p \in S_n, p \neq km+1} \frac{f^2(p)}{p} + O(1) = \sum_{p \in S_n, p \neq km+1} \frac{1}{p(\ln p)^2} + O(1). \]

Based on the above, the series \( \sum_{p \in S_n, p \neq km+1} \frac{1}{p(\ln p)^2} \) converges.

Therefore, the asymptotic of the variance \( f(m) \) for \( m = l, l+k, l+2k, \ldots, n \) and \( n \to \infty \) is equal to:

\[ \sum_{p \in S_n, p \neq km+1} \frac{f^2(p)}{p} + O(1) = O(1). \]

Next, we find the asymptotic of the \( u \)-th central moment \( f(m) \) for the value \( m = l, l+k, l+2k, \ldots, n \) and \( n \to \infty \):

\[ \sum_{p \in S_n, p \neq km+1} \frac{f^u(p)}{p} + O(1) = \sum_{p \in S_n, p \neq km+1} \frac{1}{p(\ln p)^u} + O(1). \]

Having in mind that for \( k > 1 \) the series \( \sum_{p \in S_n, p \neq km+1} \frac{1}{p(\ln p)^u} \) converges, the asymptotic of the \( u \)-th central moment of the function \( f(m) \) for the value \( m = l, l+k, l+2k, \ldots, n \) and \( n \to \infty \), is equal to:

\[ \sum_{p \in S_n, p \neq km+1} \frac{f^u(p)}{p} + O(1) = O(1). \]

So far, we have considered strongly additive arithmetic functions \( f(m) \) that have a limiting normal distribution at the value \( m = l, l+k, l+2k, \ldots, n \) and \( n \to \infty \), for which the condition \( |f(p)| \leq 1 \) is satisfied.

Now we consider a more general requirement for strongly additive arithmetic functions under which they have a limiting normal distribution. This is some analogue of the Lindeberg condition of the Central Limit Theorem.

The theorem was proved in [4].
Let \( f(m) \) is a strongly additive arithmetic function, and \( A(n) \) its mean value. If the variance \( f(m) \) is \( D(n) = \sum_{p \leq n} \frac{f^2(p)}{p} \to \infty, n \to \infty \) and for any fixed \( \epsilon > 0 \) condition is satisfied:

\[
\frac{1}{D(n)} \sum_{p \leq n, f(p) > D(n)} \frac{f^2(p)}{p} \to 0, n \to \infty,
\]

then the distribution: \( P_n \{ \frac{f(m) - A(n)}{\sqrt{D(n)}} < x \} \) is normal when the value \( n \to \infty \).

Corollary of the theorem.

Let \( f(m) \) is a strongly additive arithmetic function for which

\[
D(n) = \sum_{p \leq n, p \pm k, \pm l} \frac{f(p)}{p} \to \infty \text{ and } \max_{p \leq n, p \pm k, \pm l} f(p) = o(\sqrt{D(n)}) \text{ if the value } n \to \infty, \text{ then the conditions of this theorem are satisfied.}
\]

Assertion 2

Let \( f(m) \) is a strongly additive arithmetic function with \( D(n) = \sum_{p \leq n, p \pm k, \pm l} \frac{f(p)}{p} \to \infty \) and \( \max_{p \leq n, p \pm k, \pm l} f(p) = o(\sqrt{D(n)}) \) if the value \( n \to \infty \), then

1. It is always possible to construct a random variable having a normal distribution with the mean value and variance equal to the asymptotics of the mean value and variance \( f(m) \), where \( m = kt + l, (t = 0, 1, 2, \ldots), (k, l = 1) \).

2. The asymptotic of the central moment of \( u \)-th order for \( f(m) \) is equal to:

\[
\sum_{p \leq n, p \pm k, \pm l} \frac{f^u(p)}{p}.
\]

Assertion 2 is proved similarly to Assertion 1, taking into account the indicated corollary.

Let us define the asymptotic of the central moment of the \( u \)-th order for a strongly additive arithmetic function \( f(m) = \sum_{p \leq n} \sqrt{\ln \ln(p)} \) \( (f(p) = \sqrt{\ln \ln(p)}) \), which satisfies the condition of Assertion 2.
Therefore, based on Assertion 2 and [7], we obtain that the asymptotic of the central moment of the $u$ -th order for this function:

$$\sum_{p \leq n, pk \leq m + 1} \frac{f''(p)}{p} = \frac{1}{\varphi(k)} \sum_{p \leq n, pk \leq m + 1} \frac{(\ln \ln(p))^{u/2}}{p} \sim \frac{1}{\varphi(k)} \int_{t \leq n} (\ln \ln(t))^{u/2} dt - \frac{1}{\varphi(k)} \frac{(\ln \ln(n))^{u/2+1}}{u/2+1}.$$  

We have shown (in Assertions 1 and 2) that, under certain conditions, it is possible to construct a random variable that has a limiting normal distribution coinciding with the limiting distribution of strongly additive arithmetic functions defined on an arithmetic progression.

It is known that additive arithmetic functions belonging to the class $H$, like strongly additive arithmetic functions, have the same limit distribution as strongly additive arithmetic functions.

Now we will consider a wider class of real arithmetic functions - $V$, which includes not only real additive arithmetic functions, but any real arithmetic functions that have the same limit distribution. Thus, real arithmetic functions $f(m)$ and $f^*(m)$ belong to the class $V$ if their limit distributions $P_n\left(\frac{f(m) - A_n}{D_n} < x\right)$ and $P_n\left(\frac{f^*(m) - A_n}{D_n} < x\right)$ coincide.

Naturally, the class $V$ includes the class $H$ of real additive arithmetic functions.

Having in mind what has been said, we formulate the following assertion.

Assertion 3

1. Let a strongly additive arithmetic function $f^*(m)$ have a limiting normal distribution, and let $f(m)$ is a real arithmetic function, both belonging to the same class $V$. Then the asymptotics of the moments $f^*(m)$ and $f(m)$ are equal, and Assertions 1–2 hold for the function $f(m)$.

2. Let a strongly additive arithmetic function $f^*(m)$ have a limiting normal distribution, and let $f(m)$ is a real additive arithmetic function, both belonging to the same class $H$. Then the asymptotics of the moments $f^*(m)$ and $f(m)$ are equal, and Assertions 1 and 2 hold for the function $f(m)$.
Proof

Since a strongly additive arithmetic function $f^*(m)$ has a limiting normal distribution on the natural series, it has a limiting normal distribution on a geometric progression $m = kt + l, (t = 0, 1, 2, ...), (k, l = 1)$ (see Introduction).

Based on the definition of the classes, $V$ and $H$ arithmetic functions $f^*(m)$ and $f(m)$ have the same limiting normal distribution on a geometric progression $m = kt + l, (t = 0, 1, 2, ...), (k, l = 1)$, so these arithmetic functions have the same moments of all orders.

On the other hand, since the arithmetic function $f(m)$ has the same limiting normal distribution on the arithmetic progression $m = kt + l, (t = 0, 1, 2, ...), (k, l = 1)$ as the strongly additive arithmetic function $f^*(m)$, then assertions 1 and 2 hold for the function $f(m)$.

Let’s look at examples of real additive functions of class $H$.

The arithmetic function of the number of divisors of a number $m$ without taking into account the multiplicity $\omega(m)$ is a strongly additive arithmetic function $f^*(m) = \omega(m)$ and has a limiting normal distribution. Therefore, it also has a limiting - normal distribution on an arithmetic progression $kt + l, (k, l = 1)$ and, based on Assertion 1, has asymptotics of the mean value and variance equal to

$$1 \ln \ln(n) + O(1).$$

The arithmetic function of the number of divisors of a number, taking into account the multiplicity - $f(m) = \Omega(m)$ for all prime values $p$ coincides with a strongly additive function $f^*(m) = \omega(m)$, i.e. $\Omega(p) = \omega(p)$, both belong to the same class $H$ and have limiting normal distribution. Therefore, based on Assertion 3, asymptotics of the mean value and variance $\Omega(m)$ with $\omega(m)$ coincide on the arithmetic progression $kt + l, (k, l = 1)$

$$- \frac{1}{\varphi(k)} \ln \ln(n).$$

Another example of arithmetic functions of the class $H$ are the functions $\Omega(m), m = 1, 2, ..., n$ and $\omega(m)$, which do not coincide for a finite number of prime values $q_1, q_2, ..., q_k$, but for other prime values - $\Omega(p) = \omega(p)$. These functions also have the
same limiting normal distribution and therefore asymptotics of all moments on the arithmetic progression \( kt + l, (k, l = 1) \) coincide - \( \frac{1}{\varphi(k)} \ln \ln(n) \).

Let us take an example of class \( V \).

Let \( \omega_k(m) \) - the number of prime divisors of a natural number \( m \), without taking into account their multiplicity; belong to the sequence of natural numbers \( 4k + 1 \). Based on Assertion 1, asymptotics of the mean value and variance of this function is equal to - \( \frac{1}{\varphi(4)} \ln \ln(n) = \frac{1}{2} \ln \ln(n) \).

The arithmetic function \( f(m) = 0.5 \omega(m) \) has the same limiting normal distribution as \( \omega_k(m) \), i.e. both of these functions belong to the class \( V \). Therefore, based on Assertion 3, it has the same asymptotics of the mean value and variance as \( \omega_k(m) - \frac{1}{2} \ln \ln(n) \).

However, these two arithmetic functions are not equal for any prime values \( p \), since \( 0.5 \omega(p) = 0.5 \), but \( \omega_k(p) \) takes only values 1 or 0. Therefore, they do not belong to the class \( H \).

Naturally, as examples of the class \( V \), you can take all the examples of arithmetic functions of the class \( H \) given before.

3. CONCLUSION AND SUGGESTIONS FOR FURTHER WORK

The next article will continue to study the asymptotic behavior of some arithmetic functions.

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