Rewindable Quantum Computation and Its Equivalence to Cloning and Adaptive Postselection

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Abstract

We define rewinding operators that invert quantum measurements. Then, we define complexity classes RwBQP, CBQP, and AdPostBQP as sets of decision problems solvable by polynomial-size quantum circuits with a polynomial number of rewinding operators, cloning operators, and adaptive postselections, respectively. Our main result is that BPP^{PP} \subseteq RwBQP = CBQP = AdPostBQP \subseteq PSPACE. As a byproduct of this result, we show that any problem in PostBQP can be solved with only postselections of events that occur with probabilities polynomially close to one. Under the strongly believed assumption that BQP \nsubseteq SZK, or the shortest independent vectors problem cannot be efficiently solved with quantum computers, we also show that a single rewinding operator is sufficient to achieve tasks that are intractable for quantum computation. Finally, we show that rewindable Clifford circuits remain classically simulatable, but rewindable instantaneous quantum polynomial time circuits can solve any problem in PP.

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Related Version A shortened version of this paper appeared in Proceedings of the 18th Conference on the Theory of Quantum Computation, Communication and Cryptography (TQC 2023), pp. 9:1-9:23, 2023 [1]. As a difference from the conference paper, this paper includes the detail of rewindable Clifford and instantaneous quantum polynomial time circuits.
1 Introduction

1.1 Background

It is believed that universal quantum computers outperform their classical counterparts. There are two approaches to strengthening this belief. The first is to introduce tasks that seem intractable for classical computers but can be efficiently solved with quantum computers. For example, no known efficient classical algorithm can solve the integer factorization, but Shor’s quantum algorithm [2] can do it efficiently. The second approach is to consider what happens if classical computers can efficiently simulate the behaviors of quantum computers. So far, sampling tasks have often been considered in this approach [3]. It has been shown that if any probability distribution obtained from some classes of quantum circuits (e.g., instantaneous quantum polynomial time (IQP) circuits [4]) can be efficiently simulated with classical computers, then PH collapses to its second [5, 6] or third level [4, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18], or BQP is in the second level of PH [19]. Since the collapse of PH and the inclusion of BQP in PH are considered to be unlikely, these results imply quantum advantages.

1.2 Our Contribution

In this paper, we take the second approach. If efficient classical simulation of quantum measurements is possible, then the measurements become invertible because classical computation can be made reversible 5. From the analogy of the rewinding technique used in zero-knowledge (see e.g., [20, 21, 22]), we call such measurements rewindable measurements. They make quantum computation genuinely reversible and incredibly powerful 6. More formally, the following rewinding operator $R$ becomes possible. $R$ receives a post-measurement $n$-qubit quantum state $\langle z \rangle (z \otimes I^{\otimes n-1}) \langle \psi \rangle$ with $z \in \{0, 1\}$ and a succinct classical description $D$ of a pre-measurement quantum state $\langle \psi \rangle$ and outputs the quantum state $\langle \psi \rangle$:

$$R \left( \langle z \rangle (z \otimes I^{\otimes n-1}) \langle \psi \rangle \otimes |D\rangle \right) = \langle \psi \rangle,$$

where $I \equiv |0\rangle \langle 0| + |1\rangle \langle 1|$ is the two-dimensional identity operator. As an important point, $R$ requires the classical description $D$ as an input. If it requires only $\langle z \rangle (z \otimes I^{\otimes n-1}) \langle \psi \rangle$ as an input, the output state cannot be uniquely determined. For example, in the case of both $\langle \psi \rangle = |0\rangle + |1\rangle$ and $\langle (0)\rangle + |1\rangle \rangle / \sqrt{2}$, the post-measurement state is $|0\rangle |+\rangle$ for $z = 0$, where $|\pm\rangle \equiv (|0\rangle \pm |1\rangle) / \sqrt{2}$. To circumvent this problem, we require the classical description $D$ as information about $\langle \psi \rangle$. As a concrete example, the classical descriptions of $|0\rangle |+\rangle$ and $(|0\rangle |+\rangle + |1\rangle |-\rangle) / \sqrt{2}$ are $I \otimes H$ and CZ($H \otimes H$), respectively. Here, $H \equiv |+\rangle \langle 0| + |\rangle \langle -1|$ is the Hadamard gate, $CZ \equiv |0\rangle \langle 0| \otimes I + |1\rangle \langle 1| \otimes Z$ is the controlled-Z ($CZ$) gate, and $Z \equiv |0\rangle \langle 0| - |1\rangle \langle 1|$ is the Pauli-Z operator. These descriptions are proper because $|0\rangle |+\rangle$ and $(|0\rangle |+\rangle + |1\rangle |-\rangle) / \sqrt{2}$ can be prepared by applying $I \otimes H$ and CZ($H \otimes H$) on the fixed initial state $|00\rangle$, respectively. Furthermore, we define rewinding operators for only pure states, and hence their functionality is arbitrary for mixed states. Due to this restriction, we can avoid contradictions with an ordinary interpretation of mixed states (see Sec. 3) and the no-signaling principle (see Sec. 4.3).

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5 More concretely, if quantum measurements can be represented as logic circuits composed of AND and NOT, we can make them invertible by replacing each AND with the Toffoli gate and an ancillary bit.

6 This result would imply the hardness on the efficient classical simulation of the whole behavior of quantum computers but does not say anything about the separation between BQP and BPP.
It is strongly believed that the rewinding of measurements cannot be performed in ordinary quantum mechanics, i.e., the superposition is destroyed by measurements, and it cannot be recovered after measurements. One may think that if the rewinding were possible against this belief, it could add extra computation power to universal quantum computers. We show that this expectation is indeed correct. More formally, we define \( \text{RwBQP} \) (BQP with rewinding) as a set of decision problems solvable by polynomial-size quantum circuits with a polynomial number of rewinding operators and show \( \text{BQP} \subseteq \text{BPP} \subseteq \text{RwBQP} \).

The rewinding operator can be considered a probabilistic postselection. By just repeating measurements and rewinding operations until the target outcome is obtained, we can efficiently simulate the postselection with high probability if the output probability of the target outcome is at least the inverse of some polynomial. However, the original postselection enables us to deterministically obtain a target outcome even if the probability is exponentially small [23]. In this case, the above simple repeat-until-success approach requires an exponential number of rewinding operations on average. Surprisingly, we show that it is possible to exponentially mitigate probabilities of nontarget outcomes with a polynomial number of rewinding operators. By using this mitigation protocol, we can obtain the target outcome with high probability even if the output probability of the target outcome is exponentially small. In this sense, the rewinding and postselection are equivalent. More formally, we show that \( \text{RwBQP} \) is equivalent to the class \( \text{AdPostBQP} \) (BQP with adaptive postselection) of decision problems solvable by polynomial-size quantum circuits with a polynomial number of adaptive postselections. Here, an adaptive postselection is a projector \( |b\rangle\langle b| \) such that the value of \( b \in \{0, 1\} \) depends on previous measurement outcomes. From this equivalence, we also obtain \( \text{RwBQP} \subseteq \text{PSPACE} \).

The rewinding is also related to cloning. By strengthening our rewinding operator in Eq. (1), we define the cloning operator \( C \) as follows:
\[
C|D\rangle = |\psi\rangle.
\]
(2)

Unlike Eq. (1), this operator does not require the post-measurement state \( |z\rangle(z \otimes I^\otimes n^{-1})|\psi\rangle \). Since it is easy to duplicate the classical description \( D \), we can efficiently duplicate \( |\psi\rangle \), i.e., generate \( |\psi\rangle \otimes 2 \) by simply applying \( C \otimes C \) on \( |D\rangle \otimes 2 \). Although the ordinary cloning operator \( \tilde{C} \) is defined such that \( \tilde{C}(|\psi\rangle|0^n\rangle) = |\psi\rangle \otimes 2 \) for some \( m \in \mathbb{N} \) [24], we define \( C \) as an operator whose input is the classical description \( D \) of \( |\psi\rangle \) rather than \( |\psi\rangle \) itself. This makes sense because we can always obtain a classical description of \( |\psi\rangle \) in our setting. Note that it could be difficult to realize \( C \) in quantum polynomial time because \( |\psi\rangle \) might be prepared by using measurements. More precisely, it may be defined as a quantum state prepared when the measurement outcome is 0, e.g., \( |\psi\rangle = U_2(I \otimes |0\rangle \otimes I^\otimes n^{-2})U_1|0^n\rangle \) for some unitary operators \( U_1 \) and \( U_2 \). We show that \( \text{RwBQP} \) is also equivalent to the class \( \text{CBQP} \) (BQP with cloning) of decision problems solvable by polynomial-size quantum circuits with a polynomial number of cloning operators. That is, the difference between Eqs. (1) and (2) does not matter to the computation power. The following theorem summarizes our main results explained above:

**Result 1** (Theorem 17). \( \text{BPP} \subseteq \text{RwBQP} = \text{CBQP} = \text{AdPostBQP} \subseteq \text{PSPACE} \).

The computation power of the cloning has been addressed in [25] as an open problem. Result 1 gives lower and upper bounds on our class \( \text{CBQP} \), and it seems to be a reasonable approach.
to capturing the power of cloning. Note that more precisely speaking, in Corollary 21, we obtain the lower bound $BQP^{\text{classical}}$, which may be slightly tighter than $BPP^{\text{P}}$. Here, the subscript “classical” is a symbol used in [23] and means that only classical queries are allowed.

All the above results assume that rewinding operators can be utilized a polynomial number of times. Under the strongly believed assumption that the shortest independent vectors problem (SIVP) [26] cannot be efficiently solved with universal quantum computers, we show that a single rewinding operator is sufficient to achieve a task that is intractable for universal quantum computation:

► Result 2 (Informal Version of Theorem 24). Assume that there is no polynomial-time quantum algorithm that solves the SIVP. Then, there exists a problem such that it can be efficiently solved with a constant probability if a single rewinding operator is allowed for quantum computation, but the probability is super-polynomially small if it is not allowed.

We also show a superiority of a single rewinding operator under a different assumption:

► Result 3 (Informal Version of Corollary 27). Let $RwBQP(1)$ be $RwBQP$ with a single rewinding operator. Then, $RwBQP(1) \supset BQP$ unless $BQP \supset SZK$.

It is strongly believed that $BQP$ does not include $SZK$. At least, we can say that it is hard to show $BQP \supset SZK$ because there exists an oracle $A$ such that $BQP^A \not\supset SZK^A$ [27]. For example, by assuming that the decision version of SIVP, gapSIVP, is hard for universal quantum computation, Result 3 implies that a single rewinding operator is sufficient to achieve a task that is intractable for universal quantum computation. This is because the gapSIVP (with an appropriate parameter) is in $SZK$ [28]. As a difference from Result 2, Result 3 shows the superiority of a single rewinding operator for decision problems.

As simple observations, we also consider the effect of rewinding operators for restricted classes of quantum circuits. It has been shown that polynomial-size Clifford circuits are classically simulatable [29]. In Sec. 4.2, we show that such circuits with rewinding operators are still classically simulatable. It is also known that IQP circuits are neither universal nor classically simulatable under plausible complexity-theoretic assumptions [10]. In Sec. 4.3, we show that IQP circuits with rewinding operators can efficiently solve any problem in $PP$.

Our mitigation protocol used to show $AdPostBQP \subseteq RwBQP$ also has an application for $PostBQP$ [23], which is a class of decision problems solvable by polynomial-size quantum circuits with non-adaptive postselections. By slightly modifying our mitigation protocol and replacing rewinding operators with postselections, we obtain the following corollary:

► Result 4 (Corollary 22). For any polynomial function $P(|x|)$ in the size $|x|$ of an instance $x$, $PP = PostBQP$ holds even if only non-adaptive postselections of outputs whose probabilities are $1 - O(1/P(|x|))$ are allowed.

The equality $PP = PostBQP$ was originally shown in [23] by using postselections of outputs whose probabilities may be exponentially small. Result 4 shows that such postselections can be replaced with those of outputs whose probabilities are polynomially close to one. This result is optimal in the sense that polynomially many repetitions of non-adaptive postselections of outputs whose probabilities are $1 - 1/f(|x|)$ with a super-polynomial function $f(|x|)$ can be simulated in quantum polynomial time. It is worth mentioning that when the probabilities are at least some constant, the above replacement is obvious in $PostBPP$ (or $BPP_{\text{path}}$). This is because any behavior of a probabilistic Turing machine can be represented as a binary tree such that each path is chosen with probability $1/2$. However, in its quantum analogue $PostBQP$, it was open as to whether such replacement is possible even if the probabilities are at least some constant.
Related Work. As a hypothetical ability that enables us to perform the cloning, a closed timelike curve (CTC) has already been studied [30]. It also rewinds the time in a sense. Among several formulations of the CTC [31, 32], the Deutschian CTC [31] is the major one. Let $\mathcal{R}_{\text{CTC}}$ and $\mathcal{Q}$ be quantum registers that are input to and not input to the CTC, respectively, and $\text{Tr}_{\mathcal{R}[:]}$ be the partial trace over the system $\mathcal{R}$. Suppose that the register $\mathcal{Q}$ is initialized to $|0^m\rangle|0^m\rangle$ with $m \in \mathbb{N}$. The Deutschian CTC prepares a quantum state $^\otimes |0^m\rangle$ on $\mathcal{R}_{\text{CTC}}$ such that for any unitary operator $U$ acting on $\rho \otimes |0^m\rangle|0^m\rangle$,

\[
\text{Tr}_\mathcal{Q} \left[ U (\rho \otimes |0^m\rangle|0^m\rangle) U^\dagger \right] = \rho. \tag{3}
\]

Equation (3) means that the future state $\rho$ generated after applying $U$ is prepared before applying $U$ under the constraint of the causal consistency. Aaronson and Watrous have shown that classes $\text{P}_{\text{CTC}}$ and $\text{BQP}_{\text{CTC}}$ of decision problems solvable by polynomial-size logic and quantum circuits with the Deutschian CTC are equivalent and that they are also equivalent to $\text{PSPACE}$ [33]. From Result 1, the Deutschian CTC should be at least as powerful as the rewinding.

As another hypothetical ability, non-collapsing measurements, which allow us to obtain measurement outcomes without perturbing quantum states, have been considered in [25, 34]. For the class $\text{PDQP}$ of decision problems solvable by polynomial-size quantum circuits with non-collapsing measurements, Aaronson et al. have shown $\text{SZK} \subseteq \text{PDQP} \subseteq \text{BPP}_{\text{PP}}$ [25]. Therefore, Result 1 implies that the rewinding should be at least as powerful as the non-collapsing measurements.

1.3 Overview of Techniques

To obtain Result 1, we show (i) $\text{RwBQP} \subseteq \text{CBQP}$; (ii) $\text{CBQP} \subseteq \text{AdPostBQP}$; (iii) $\text{AdPostBQP} \subseteq \text{RwBQP}$; (iv) $\text{BPP}_{\text{PP}} \subseteq \text{BQP}_{\text{PP}}$, which immediately means $\text{BPP}_{\text{PP}} \subseteq \text{RwBQP}$ because $\text{BPP}_{\text{PP}} \subseteq \text{BQP}_{\text{classical}}$; and (v) $\text{AdPostBQP} \subseteq \text{PSPACE}$. The first inclusion (i) is obvious from the definitions of the rewinding operator $R$ and cloning operator $C$ (see Eqs. (1) and (2)). The fifth inclusion (v) can also be easily shown by using the Feynman path integral that is used to show $\text{BQP} \subseteq \text{PSPACE}$ [35]. In $\text{BQP}$, measurements are only performed at the end of quantum circuits. On the other hand, in $\text{AdPostBQP}$, intermediate ordinary and postselection measurements are also allowed. However, this difference does not matter in showing the inclusion in $\text{PSPACE}$.

The second inclusion (ii) is a natural consequence from the simple observation that postselections can simulate the cloning operator $C$. On the other hand, the third inclusion (iii) is nontrivial because we have to efficiently simulate postselection by using only a polynomial number of rewinding operators. To this end, we give an efficient protocol to exponentially mitigate the amplitude of a nontarget state by using a polynomial number of rewinding operators. Let $|\psi\rangle = \sqrt{2^{-p(n)}}|\psi_t\rangle + \sqrt{1 - 2^{-p(n)}}|\psi^+_t\rangle$, where $p(n)$ is some polynomial in the size $n$ of a given $\text{AdPostBQP}$ problem, $|\psi_t\rangle$ is a target state that we would like to postselect, and $\langle \psi_t | \psi^+_t \rangle = 0$. By using our mitigation protocol, from $|\psi_t\rangle$, we can obtain $\sqrt{2^{-p(n)}}|\psi_t\rangle + \sqrt{2^{-p(n)}(1 - 2^{-p(n)})}|\psi^+_t\rangle$ up to a normalization factor. Since $2^{-p(n)}$ is larger than $2^{-p(n)}(1 - 2^{-p(n)})$, we now obtain $|\psi_t\rangle$ with probability of at least $1/2$. By repeating these procedures, we can simulate the postselection of $|\psi_t\rangle$ with high probability.

Our mitigation protocol is also useful in showing the fourth inclusion (iv). First, from $\text{PP} = \text{PostBQP}$ [23], we obtain $\text{PP} \subseteq \text{RwBQP}$ by using our mitigation protocol. Then, we show

\footnote{A fixed-point theorem guarantees the existence of such $\rho$ for any $U$ [31].}
that the completeness-soundness gap in RWBQP can be amplified to a value exponentially close to 1, and RWBQP is closed under composition if only classical queries are allowed. By combining these results, we obtain BQP_{classical}^{PP} \subseteq RWBQP^{RWBQP_{classical}} = RWBQP. Note that BQP_{classical}^{PP} \subseteq RWBQP_{classical}^{PP} is obvious from the definition of RWBQP (see Def. 9).

We show Result 2 as follows. Cojocaru et al. have shown that under the hardness of SIVP, there exists a family \( F = \{ f_K \}_{K \in \mathcal{K}} \) of functions that is collision resistant against quantum computers, i.e., no polynomial-time quantum algorithm can output a collision pair \((x, x')\) such that \( x \neq x' \) and \( f_K(x) = f_K(x') \) [36]. Here, \( \mathcal{K} \) is a finite set of parameters uniquely specifying each function (see Sec. 2.2 for details). We show that a collision pair can be output with a constant probability if only one rewinding operator is given. From the construction of \( F \), the last bits of collision pairs are different, i.e., there exist \( x_0 \) and \( x_1 \) such that \( x = (x_0, 0) \) and \( x' = (x_1, 1) \). Using the idea in [37], we can efficiently prepare

\[
\frac{|x_0\rangle|0\rangle + |x_1\rangle|1\rangle}{\sqrt{2}}
\]

for some output value \( y = f_K(x) = f_K(x') \). Note that since the preparation of Eq. (4) includes a measurement, if we perform it again, we will obtain a quantum state in Eq. (4) for a different output value \( y' \), and hence it is difficult to simultaneously obtain \( x \) and \( x' \) for the same \( y \). When we can use a rewinding operator, the situation changes. By measuring the state in Eq. (4), we can obtain \( x_0 \) or \( x_1 \). For simplicity, suppose that we obtain \( x_0 \). Then, by performing the rewinding operator \( R \) on \(|x_0\rangle|0\rangle\) and a classical description of Eq. (4), we can prepare the quantum state in Eq. (4) for the same \( y \). From this state, we can obtain \( x_1 \) with probability 1/2. As an important point, since the last bits of \( x \) and \( x' \) differ, a single rewinding operator (i.e., the rewinding of a single qubit) is sufficient to find a collision pair with a constant probability.

Finally, we show Result 3. To this end, we show that a SZK-complete problem is in RWBQP(1) by using a technique inspired by [34].

2 Preliminaries

In this section, we review some preliminaries that are necessary to understand our results. In Sec. 2.1, we introduce a complexity class PostBQP and explain the postselection. In Sec. 2.2, we introduce the SIVP and a collision-resistant and \( \delta - 2 \) regular family of functions.

2.1 Quantum Complexity Class

In this subsection, we review PostBQP and explain the postselection. Then, we clarify a difference between PostBQP and our class AdPostBQP (see Def. 11). Note that we assume that readers are familiar with classical complexity classes [38]. PostBQP is defined as follows:

**Definition 5 (PostBQP [23]).** A promise problem \( L = (L_{yes}, L_{no}) \subseteq \{0, 1\}^\ast \) is in PostBQP if and only if there exist polynomials \( n \) and \( q \) and a uniform family \( \{U_x\}_x \) of polynomial-size quantum circuits, such that

1. \( \Pr[p = 1] \geq 1/2^q \)
2. when \( x \in L_{yes}, \Pr[o = 1 \mid p = 1] \geq 2/3 \)
3. when \( x \in L_{no}, \Pr[o = 1 \mid p = 1] \leq 1/3 \).
Then, the family (i.e., the smallest $\chi$ and at most regular $\chi$ where $\gamma$)

\[ \Pr[p = z_2] = \langle 0^n|U_z^r (I \otimes |z_2\rangle \otimes I^{\otimes n-2}) U_x|0^n\rangle \]  

\[ \Pr[o = z_1 \mid p = z_2] = \langle 0^n|U_z^r (|z_1z_2\rangle \otimes I^{\otimes n-2}) U_x|0^n\rangle \Pr[p = z_2] \]  

In this definition, “polynomial” means the one in the length $|x|$ of the instance $x$.

From Def. 5, we notice that the postselection is to apply a projector. In PostBQP, it is allowed to apply the projector $|1\rangle\langle 1|$ to the qubit in the postselection register at the end of a quantum circuit. Therefore, PostBQP is a set of promise problems solvable by polynomial-size quantum circuits (in uniform families) with a single non-adaptive postselection. On the other hand, in AdPostBQP, we allow the application of a polynomial number of intermediate measurements and projectors. This means that the value $b \in \{0, 1\}$ of a projector $|b\rangle\langle b|$ can depend on previous measurement outcomes, while it is determined before executing a quantum circuit in PostBQP.

### 2.2 Shortest Independent Vectors Problem (SIVP)

The SIVP with approximation factor $\gamma$ (SIVP$_\gamma$) is defined as follows:

**Definition 6 (SIVP$_\gamma$).** Let $n$ be any natural number and $\gamma \geq 1$ be any real number. Given $n$ bases of a lattice $L$, output a set of $n$ linearly independent lattice vectors of length at most $\gamma \cdot \lambda_n(L)$. Here, $\gamma$ can depend on $n$, and $\lambda_n(L)$ is the $n$th successive minimum of $L$ (i.e., the smallest $r$ such that $L$ has $n$ linearly independent vectors of norm at most $r$).

Since there is no known polynomial-time quantum algorithm to solve SIVP$_\gamma$ for polynomial approximation factor, it is used as a basis of the security of lattice-based cryptography [26].

The hardness of the SIVP is also used to construct families of collision-resistant functions against universal quantum computers. From [36], we can immediately obtain the following theorem:

**Theorem 7 (adapted from [36]).** Let $n$ be any natural number, $q = 2^{5\lceil \log_2 n \rceil + 21}$, $m = 23n + 5n\lceil \log_2 n \rceil$, $\mu = 2mn\sqrt{23} + 5\log_2 n$, and $\mu' = \mu/m$, where $\lceil \cdot \rceil$ is the ceiling function. Let $K \equiv (A, As_0 + e_0) \in K$ being the multiset $\{(A, As_0 + e_0)\}_{A \in \mathbb{Z}_q^{n \times m}, s_0, e_0 \in \mathbb{C}, e_0 \in \chi^m}$, where $\mathbb{Z}_q^{n \times m}$ be the set of $n \times m$ matrices each of whose entry is chosen from $\mathbb{Z}_q \equiv \{0, 1, \ldots, q - 1\}$, and $\chi'$ is the set of integers bounded in absolute value by $\mu'$. Assume that there is no polynomial-time quantum algorithm that solves SIVP$_{p(n)}$ for some polynomial $p(n)$ in $n$. Then, the family $\mathcal{F} \equiv \{f_K : \mathbb{Z}_q^n \times \chi^m \times \{0, 1\} \rightarrow \mathbb{Z}_q^m\}_{K \in K}$ of functions

\[ f_K(s, c, e) \equiv As + e + c \cdot (As_0 + e_0) \pmod{q}, \]  

where $\chi$ is the set of integers bounded in absolute value by $\mu$, is collision resistant$^{10}$ and $\delta$-2 regular$^{11}$ for a constant $\delta$.

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$^9$ Note that a polynomial number of postselections are allowed if they can be unified as a single non-adaptive postselection.

$^{10}$ Let $\mathcal{F} \equiv \{f_K : \mathcal{D} \rightarrow \mathcal{R}\}_{K \in K}$ be a function family. We say that $\mathcal{F}$ is collision resistant if for any polynomial-time quantum algorithm $A$, which receives $K$ and outputs two bit strings $x, x' \in \mathcal{D}$, the probability $Pr[K[A(K)] = (x, x')]$ such that $x \neq x'$ and $f_K(x) = f_K(x')$ is super-polynomially small. Note that $K$ is chosen from $K$ uniformly at random, and the probability is also taken over the randomness in $A$.

$^{11}$ Let $\mathcal{F} \equiv \{f_K : \mathcal{D} \rightarrow \mathcal{R}\}_{K \in K}$ be a function family. For a fixed $K$, we say that $y \in \mathcal{R}$ has two
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From Eq. (7), the function $f_K$ has a collision pair\(^{12}\) $(s, e, 1)$ and $(s + s_0, e + e_0, 0)$, and Theorem 7 shows that it is difficult to find them simultaneously. This function family will be used to show that a single rewinding operator is sufficient to achieve a task that seems difficult for universal quantum computers.

Note that in [36], the matrix $A$ is constructed so that it has a trapdoor to efficiently invert $As + e$, and its distribution is statistically close to uniform over $Z_q^{n \times m}$. In Theorem 7, we consider a simplified variant of the function family of [36] in which the matrix $A$ is chosen uniformly at random.

### 3 Computational Complexity of Rewinding

In this section, we show Results 1 and 4.

#### 3.1 Our Complexity Classes

To this end, first, we define the rewinding operator $R$ and cloning operator $C$ as follows:

▶ **Definition 8** (Rewinding and Cloning Operators). Let $n$ be any natural number, $Q$ be any $n$-qubit linear operator composed of unitary operators and the $Z$-basis projective operators $\{|0\rangle, |1\rangle\}$, $D$ be a classical description of the linear operator $Q$, and $I$ be the single-qubit identity operator. The rewinding and cloning operators $R$ and $C$ are maps from a quantum state to a quantum state such that for any $s \in \{0, 1\}$, when $(|s\rangle \otimes I^{\otimes n-1}) Q |0^n\rangle \neq 0$,

$$\begin{align*}
R \left( \frac{|s\rangle \otimes I^{\otimes n-1}}{\sqrt{|0^n\rangle Q^\dagger |0^n\rangle}} \right) &= \frac{Q |0^n\rangle}{\sqrt{|0^n\rangle Q^\dagger |0^n\rangle}} \quad (8) \\
C |D\rangle &= \frac{Q |0^n\rangle}{\sqrt{|0^n\rangle Q^\dagger |0^n\rangle}} \quad (9)
\end{align*}$$

For other input states, the functionality of $R$ and $C$ is undefined, that is outputs are arbitrary $n$-qubit states. Particularly when it depends on a value of classical bits whether $R$ and $C$ are applied, we call them classically controlled rewinding and cloning operators, respectively.

Note that since the linear operator $Q$ may include projective operators (e.g., $Q = U_2 (I \otimes \{|0\rangle\langle 0| \otimes I^{\otimes (n-2)} U_1)$ for some $n$-qubit unitary operators $U_1$ and $U_2$), in general, $Q^\dagger Q \neq I^{\otimes n}$. From Def. 8, it is easily observed that the cloning operator $C$ can efficiently simulate the rewinding operator $R$.

An example of classically controlled rewinding operators is an operator such that if a measurement outcome is 0, the identity operator is applied to the post-measurement state, but if the outcome is 1, the rewinding operator $R$ is applied to it. Classically controlled rewinding and cloning operators play an important role in giving our main results. It is worth mentioning that $|D\rangle$ is consumed by applying a single rewinding or cloning operator. However, we can always retain $|D\rangle$ by duplicating it before applying the rewinding or cloning operator, and hence subsequent rewinding or cloning operators can also be applied. These remarks are explained more explicitly in Fig. 1.

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\(^{12}\) Since $q$ is larger than $\mu$, the second element $e + e_0$ of the second input may not be in the set $\chi^m$. Therefore, the probability of $f_K$ having a collision pair is not 1.
Simply speaking, the rewinding operator $R$ rewinds the state projected onto $|s\rangle$ to the state before the measurement. As an important point, Def. 8 implies that the rewinding operator $R$ only works for pure states. The following contradiction for an ordinary interpretation of mixed states occurs without the restriction to pure states. Suppose that we measure a maximally mixed state $I/2$ in the computational basis\textsuperscript{13}, and then obtain the measurement outcome 0. In this case, it is natural that even if we rewind this measurement and perform the same measurement again, the outcome is always 0. However, if we define the rewinding operator $R$ so that it also works for mixed states, then we can obtain $I/2$ from $|0\rangle$ with the rewinding operator, and the measurement on it may output 1. In other words, if the rewinding operator works for a mixed state $\rho$, we can measure $\rho$ again and again, and thus we obtain its information as much as we want without changing $\rho$. This situation contradicts with the natural interpretation that mixed states arise due to the lack of knowledge about them. Furthermore, the restriction to pure states would be useful in circumventing the contradiction with the no-signaling principle as explained in Sec. 4.3.

By using the rewinding and cloning operators and postselections, we define three complexity classes—RwBQP (BQP with rewinding), CBQP (BQP with cloning), and AdPostBQP (BQP with adaptive postselection)—as follows:

\begin{itemize}
  \item **Definition 9 (RwBQP and CBQP).** Let $n$ and $k$ be any natural number, $\ell$ be a polynomial in $n$, and $0 \leq s < c \leq 1$. A promise problem $L = (L_{\text{yes}}, L_{\text{no}}) \subseteq \{0, 1\}^*$ is in RwBQP($c, s$)($k$) if and only if there exists a polynomial-time deterministic Turing machine that receives $1^n$ as an input and generates a $\ell$-bit description $\mathcal{D}$ of an operator $Q_n$ such that it consists of a polynomial number of elementary gates in a universal gate set, single-qubit measurements in the computational basis, and $k$ (classically controlled) rewinding operators $R$ defined in Def. 8 and satisfies, for the instance $x \in \{0, 1\}^n$ and a polynomial $m$, that

$$
\begin{align*}
  &\text{if } x \in L_{\text{yes}}, \sum_{z \in A} \left| \langle 1 | (1 \otimes I^{\otimes n+m+\ell-1}) Q^{(z)}_n (|z\rangle |0^m\rangle |\mathcal{D}\rangle) \right|^2 \geq c \\
  &\text{if } x \in L_{\text{no}}, \sum_{z \in A} \left| \langle 1 | (1 \otimes I^{\otimes n+m+\ell-1}) Q^{(z)}_n (|z\rangle |0^m\rangle |\mathcal{D}\rangle) \right|^2 \leq s,
\end{align*}
$$

\end{itemize}

where $A$ is the set of possible outcomes of intermediate measurements, $Q^{(z)}_n$ is the same as $Q_n$ except for that the $i$th measurement is replaced with $|z_i\rangle \langle z_i|$ for all $i$, and $z_i$ is the $i$th bit of $z$. Here, $||v|| = \sqrt{\langle v | v \rangle}$ for any vector $|v\rangle$, and “polynomial” is the abbreviation of “polynomial in $n$.” Particularly, for the set poly($n$) of all polynomial functions, we denote $\bigcup_{k \in \text{poly}(n)}$ RwBQP($c, s$)($k$) and $\bigcup_{k \in \text{poly}(n)}$ RwBQP($2/3, 1/3$)($k$) as RwBQP($c, s$) and RwBQP, respectively.

By replacing $R$ with the cloning operator $C$ defined in Def. 8, CBQP($c, s$)($k$), CBQP($c, s$), and CBQP are defined in a similar way.

To perform a rewinding operator $R$ to recover an intermediate state $|\psi\rangle$, a classical description $\mathcal{D}$ of $|\psi\rangle$ is necessary. It can always be generated from $\mathcal{D}$ and measurement outcomes obtained before preparing $|\psi\rangle$. As in the case of BQP, computations performed to solve RwBQP problems can be written as uniform families of quantum circuits.

Due to the addition of rewinding operators, it may be difficult to imagine quantum circuits used in RwBQP. To clarify them, as an example, we give a concrete circuit diagram for the following RwBQP computation. Suppose that we would like to prepare a two qubit state $(|0\rangle \otimes I) U |00\rangle$ (up to normalization) for a two-qubit unitary operator $U$. To this end, we use at most two classically controlled rewinding operators. More precisely, the rewinding

\textsuperscript{13}We sometimes call the $Z$ basis the computational basis.
operator \( R \) is applied if and only if the measurement outcome is 1. This computation can be depicted as a fixed quantum circuit in Fig. 1.

From Def. 9, we immediately obtain the following lemma:

\begin{itemize}
  \item \textbf{Lemma 10.} \( \text{RwBQP} \subseteq \text{CBQP} \).
\end{itemize}

\textbf{Proof.} The only difference between \( \text{RwBQP} \) and \( \text{CBQP} \) is whether the rewinding or cloning operator is allowed. Since the cloning operator \( C \) can exactly simulate the rewinding operator \( R \), this lemma holds.

\begin{itemize}
  \item \textbf{Definition 11 (AdPostBQP).} Let \( n \) be any natural number, \( \ell \) be a polynomial in \( n \), and \( 0 < s < c \). A promise problem \( L = (L_{\text{yes}}, L_{\text{no}}) \subseteq \{0,1\}^* \) is in \( \text{AdPostBQP}(c, s) \) if and only if there exists a polynomial-time deterministic Turing machine that receives \( 1^n \) as an input and generates a \( \ell \)-bit description \( \tilde{D} \) of an operator \( Q_n \) such that it consists of a polynomial number of elementary gates in a universal gate set, single-qubit measurements in the computational basis, and single-qubit projectors \( |1\rangle\langle 1| \) and satisfies, for the instance \( x \in \{0,1\}^n \) and a polynomial \( m \), that

  \begin{align*}
    \quad & \quad \text{if } x \in L_{\text{yes}}, \sum_{z \in A} q_z \left\| \left( |1\rangle\langle 1| \otimes I^{\otimes n+m+\ell-1} \right) \mathcal{N}[Q_n^{(z)}(|x\rangle\langle 0|^n)\tilde{D})] \right\|^2 \geq c \\
    \quad & \quad \text{if } x \in L_{\text{no}}, \sum_{z \in A} q_z \left\| \left( |1\rangle\langle 1| \otimes I^{\otimes n+m+\ell-1} \right) \mathcal{N}[Q_n^{(z)}(|x\rangle\langle 0|^n)\tilde{D})] \right\|^2 \leq s,
  \end{align*}

  \end{itemize}

where \( A \) is the set of possible outcomes of intermediate measurements, \( Q_n^{(z)} \) is the same as \( Q_n \) except for that the \( i \)th measurement is replaced with \( |z_i\rangle\langle z_i| \) for all \( i \), \( z_i \) is the \( i \)th bit of \( z \), \( q_z \) is the probability of obtaining \( z \), and \( \mathcal{N}[\cdot] \) denotes the normalization of the vector in the square brackets. Here, “polynomial” is the abbreviation of “polynomial in \( n \).” Note that for
1 ≤ i ≤ n, a projector \(|1⟩⟨1|\) on the ith qubit can be applied only when a quantum state |ψ⟩ to be applied satisfies
\[
||(|1⟩⟨1|) |ψ⟩||^2 ≥ 2^{-p(n)}
\]  
for a polynomial p(n) in n. Particularly, we denote \(\text{AdPostBQP}(2/3, 1/3)\) as \(\text{AdPostBQP}\).

Equation (10) can be automatically satisfied by using standard gate sets whose elementary gates involve only square roots of rational numbers. From Defs. 9 and 11, we notice that the main difference between \(\text{RwBQP}\), \(\text{CBQP}\), and \(\text{AdPostBQP}\) is whether the rewinding or cloning operators or projectors are allowed. We also mention a difference between \(\text{AdPostBQP}\) and \(\text{PostBQP}\) as the following remark:

\[\text{Remark 12.} \quad Q_n \text{ can include adaptive postselections because depending on previous measurement outcomes, we can decide whether } \ket{1} = \ket{0} + \ket{1} \text{ is applied before and after applying } \ket{1⟩⟨1|}. \]

Here, \(X ≡ |1⟩⟨0| + |0⟩⟨1|\) is the Pauli-X operator. It is worth mentioning that it is unknown whether the adaptive postselection can be efficiently done in \(\text{PostBQP}\) as discussed in [23]. Indeed, if it is possible, \(\text{SZK} \subseteq \text{PP}\) should be immediately obtained from the argument in [25, 34], while it is a long-standing problem. The difference between \(\text{AdPostBQP}\) and \(\text{PostBQP}\) should arise from intermediate measurements allowed in \(\text{AdPostBQP}\).

The following three corollaries hold:

\[\text{Corollary 13.} \quad \text{RwBQP}, \text{CBQP}, \text{and AdPostBQP are closed under complement.} \]

\[\text{Corollary 14.} \quad \text{RwBQP} = \text{RwBQP}(1 - 2^{-p(n)}, 2^{-p(n)}), \text{CBQP} = \text{CBQP}(1 - 2^{-p(n)}, 2^{-p(n)}), \text{and AdPostBQP} = \text{AdPostBQP}(1 - 2^{-p(n)}, 2^{-p(n)}) \text{ for any polynomial function } p(n) \text{ in the size } n \text{ of a given instance } x. \]

\[\text{Corollary 15.} \quad \text{When only classical queries are allowed, RwBQP, CBQP, and AdPostBQP are closed under composition. In other words, } \text{RwBQP}^{\text{classical}} = \text{RwBQP}, \text{CBQP}^{\text{classical}} = \text{CBQP}, \text{and AdPostBQP}^{\text{classical}} = \text{AdPostBQP.} \]

Since they are obvious from Defs. 9 and 11 and can be shown by using standard techniques, proofs are given in Appendix A.

3.2 Relations with Classical Complexity Classes

From Def. 11, we immediately obtain the following lemma:

\[\text{Lemma 16.} \quad \text{AdPostBQP} \subseteq \text{PSPACE.} \]

**Proof.** The proof is essentially the same as that of \(\text{BQP} \subseteq \text{PSPACE}\) [35]. The details are given in Appendix B.

In the rest of this subsection, we consider a relation between the rewinding, cloning, and postselection (i.e., \(\text{RwBQP}\), \(\text{CBQP}\), and \(\text{AdPostBQP}\)), and also obtain lower and upper bounds on them. More formally, we show the following theorem:

\[\text{Theorem 17.} \quad \text{BPP}^{\text{PP}} \subseteq \text{RwBQP} = \text{CBQP} = \text{AdPostBQP} \subseteq \text{PSPACE.} \]

**Proof.** This theorem can be obtained by showing (i) \(\text{RwBQP} \subseteq \text{CBQP}; \) (ii) \(\text{CBQP} \subseteq \text{AdPostBQP}; \) (iii) \(\text{AdPostBQP} \subseteq \text{RwBQP}; \) (iv) \(\text{BPP}^{\text{PP}} \subseteq \text{RwBQP}, \) which immediately means \(\text{BPP}^{\text{PP}} \subseteq \text{RwBQP}\) because \(\text{BPP}^{\text{PP}} \subseteq \text{BQP}^{\text{classical}}\); and (v) \(\text{AdPostBQP} \subseteq \text{PSPACE.}\) The inclusions (i) and (v) are already shown in Lemmas 10 and 16, respectively. The remaining inclusions (ii), (iii), and (iv) will be shown in Lemma 18 and Corollary 21.
To simplify our argument in proofs of Lemma 18 and Theorem 19, we particularly consider the universal gate set \( \{X, CH, CCZ \} \cup \{H_k \mid k \in \mathbb{Z}, -p(|x|) \leq k \leq p(|x|) \} \) with a polynomial \( p(|x|) \) in the instance size \(|x|\) of a given problem. Here, \( CH \equiv |0\rangle \langle 0| \otimes I + |1\rangle \langle 1| \otimes H \) is the controlled-Hadamard gate, \( CCZ \equiv |0\rangle \langle 0| \otimes I^{\otimes 2} + |1\rangle \langle 1| \otimes CZ \) is the controlled-controlled-Z (CCZ) gate, and \( H_k \) is the generalized Hadamard gate such that \( H_k|0\rangle = (|0\rangle + 2^k|1\rangle)/\sqrt{1 + 4^k} \) and \( H_k|1\rangle = (2^k|0\rangle - |1\rangle)/\sqrt{1 + 4^k} \). Therefore, \( H_0 \) is the ordinary Hadamard gate \( H \), and hence, from [39], our gate set is universal. By using our universal gate set, we can make output probabilities of any Pauli-Z measurement in any polynomial-size quantum circuit 0 or at least \( 2^{-q(|x|)} \) for some polynomial \( q(|x|) \). Due to this property, we can postselect any outcome for any polynomial-size quantum circuit [see Eq. (10)], which simplifies a proof of Lemma 18. Furthermore, by using this gate set, we can perform all quantum operations required in a proof of Theorem 19 without any approximation. Note that our argument can also be applied to other universal gate sets such as \( \{H, T, CZ\} \) with \( T \equiv |0\rangle \langle 0| + e^{i\pi/4}|1\rangle \langle 1| \) by using the Solovay-Kitaev algorithm [40].

We show the second inclusion (ii):

**Lemma 18.** CBQP \( \subseteq \text{AdPostBQP} \).

**Proof.** To obtain this lemma, it is sufficient to show that for any polynomial-size linear operator \( Q \) and its classical description \( D \), the cloning operator \( C \) can be simulated in quantum polynomial time by using the adaptive postselection. That is, our purpose is to perform the cloning operator \( C \) on the input state \( |D\rangle \). Let \( m \) be the number of \( Z \)-basis projective operators included in \( Q \). By using \( n \)-qubit unitary operators \( \{|U(i)\rangle\}_{i=1}^{m+1} \) and \( Z \)-basis projective operators \( \{P(i)\}_{i=1}^{m} \), \( Q = U^{(m+1)} \prod_{i=1}^{m} (P(i)U(i)) \). We can obtain the classical description \( D \) of \( Q \) by measuring the state \( |D\rangle \) in the Pauli-Z basis. The description \( D \) informs us about whether \( P(i) \) is \( |0\rangle \langle 0| \) or \( |1\rangle \langle 1| \) and how to construct \( U^{(i)} \) from \( \{X, H_k, CH, CCZ\} \) for all \( i \). Therefore, by using the postselection, we can prepare \( Q|0^m\rangle \) (up to normalization) in quantum polynomial time. When we would like to apply \( U^{(i)} \), we just apply it. On the other hand, when we apply \( P(i) \), we use the postselection. Since we assume the universal gate set \( \{X, H_k, CH, CCZ\} \), the postselection is possible in any case. These efficient procedures simulate the non classically-controlled cloning operator \( C \).

Next, we show that the above procedures can also be applied to simulate a classically controlled cloning operator. Suppose that when \( a \in \{0, 1\} \) is 1, we would like to apply the cloning operator \( C \). On the other hand, when \( a = 0 \), we do not apply \( C \). Note that without loss of generality, we can assume that \( C \) is controlled by a single bit \( a \) because \( C \) is either applied or not. Only when \( a = 1 \), we must apply \( P(i) \) to simulate the classically controlled cloning operator, which means that a classically controlled postselection seems to be necessary. This problem can be resolved in the following way. Let \( P(i) = |b\rangle \langle b| \) for \( b \in \{0, 1\} \). Such classically controlled \( P(i) \) can be simulated by adding an ancillary qubit \( |b\rangle \) and applying the classically controlled SWAP gate as shown in Fig. 2. Classically controlled quantum gates are allowed in \( \text{AdPostBQP} \) computation because any classically controlled quantum gate can be realized by combining elementary quantum gates in a universal gate set.

In conclusion, we obtain CBQP \( \subseteq \text{AdPostBQP} \).

As the first step to obtain inclusions (iii) and (iv), we show the following theorem:

**Theorem 19.** RwBQP \( \supseteq \text{PP} \).

**Proof.** We show this theorem by replacing the postselection used in the proof of PP \( \subseteq \text{PostBQP} \) in [23] with a polynomial number of rewinding operators. To this end, we consider
the following PP-complete problem [23]: let $f : \{0, 1\}^n \rightarrow \{0, 1\}$ be a Boolean function computable in classical polynomial time, and $s \equiv \sum_{x \in \{0, 1\}^n} f(x)$. Decide $0 < s < 2^{n-1}$ or $s \geq 2^{n-1}$. Note that it is promised that one of them is definitely satisfied.

To solve this problem with an exponentially small error probability using rewinding operators, first, we prepare

$$\sqrt{p}\ket{\psi_+} + \sqrt{1-p}\ket{\psi_-},$$

where

$$p \equiv \frac{2\alpha^2(2^n-s)^2 + \beta^24^n}{2[(2^n-s)^2 + s^2]}$$

(12)

$$\ket{\psi_+} \equiv \frac{\sqrt{2}\alpha(2^n-s)\ket{0} + \beta2^n\ket{1}}{\sqrt{2\alpha^2(2^n-s)^2 + \beta^24^n}} \otimes \ket{0}$$

(13)

$$\ket{\psi_-} \equiv \frac{\sqrt{2}\alpha s\ket{0} + \beta(2^n-2s)\ket{1}}{\sqrt{2\alpha^2s^2 + \beta^2(2^n-2s)^2}} \otimes \ket{1}$$

(14)

for positive real numbers $\alpha$ and $\beta$ such that $\alpha^2 + \beta^2 = 1$ and $\beta/\alpha = 2^k$, where $k$ is an integer whose absolute value is upper bounded by $n$. This preparation can be efficiently done without the postselection and rewinding operators by using a protocol in [23]. First, we prepare

$$\frac{1}{\sqrt{2^n}} \sum_{x \in \{0, 1\}^n} (H^{\otimes n}\ket{x})\ket{f(x)}$$

(15)

in quantum polynomial time. Then, we measure all $n$ qubits in the first register in the Pauli-Z basis. We repeat these procedures until we obtain the outcome $0^n$ or the repetition number reaches $n$. In previous calculations [23, 41], the probability of $0^n$ being output in each repetition is lower bounded by $1/4$. However, we tighten the lower bound by calculating as follows:

$$\frac{1}{\sqrt{2^n}} \sum_{y \in \{0, 1\}^n} (\bra{y}H^{\otimes n}\bra{f(y)}) \ket{(0^n)(0^n) \otimes I} \left(\frac{1}{\sqrt{2^n}} \sum_{x \in \{0, 1\}^n} (H^{\otimes n}\ket{x})\bra{f(x)}\right)$$

$$= \frac{1}{2^n} \sum_{x,y \in \{0, 1\}^n} \frac{1}{2^n} (f(y)\ket{f(x)})$$

(16)

$$= \frac{M_0^2 + M_1^2}{4^n} = \frac{M_0^2 + (2^n - M_0)^2}{4^n} \geq \frac{1}{2},$$

(17)
where $M_b \equiv \{|x \in \{0,1\}^n : f(x) = b\}$ for any $b \in \{0,1\}$, and we have used $M_0 + M_1 = 2^n$. Note that $M_1$ is equal to $s$. Therefore, we can obtain at least one $0^n$ with probability of at least
\[ 1 - 1/2^n. \tag{18} \]

When the measurement outcome is $0^n$, we obtain
\[ |\psi \rangle = \frac{(2^n - s)|0\rangle + s|1\rangle}{\sqrt{(2^n - s)^2 + s^2}}. \tag{19} \]

From this state, for any positive real numbers $\alpha$ and $\beta$ such that $\alpha^2 + \beta^2 = 1$ and $\beta/\alpha = 2^k$, we can prepare
\[ CH[(\alpha|0\rangle + \beta|1\rangle)|\psi\rangle] = \alpha|0\rangle|\psi\rangle + \beta|1\rangle H|\psi\rangle, \tag{20} \]

which is identical to the state in Eq. (11), in quantum polynomial time\(^\dagger\).

If the second qubit in Eq. (11) is projected onto $|1\rangle$, we can obtain
\[ |\phi_{\beta/\alpha}\rangle \equiv \frac{\sqrt{2}\alpha s|0\rangle + \beta(2^n - 2s)|1\rangle}{\sqrt{2\alpha^2 s^2 + \beta^2(2^n - 2s)^2}}. \tag{21} \]

Aaronson has shown that when $0 < s < 2^{n-1}$, there exists an integer $k$ such that $|\langle + |\phi_{\beta/\alpha}\rangle| \geq (1 + \sqrt{2})/\sqrt{3}$. On the other hand, if $s \geq 2^{n-1}$, then $|\langle + |\phi_{\beta/\alpha}\rangle| \leq 1/\sqrt{2}$ holds for all $-n \leq k \leq n$. Therefore, by simply measuring $n$ copies of $|\phi_{\beta/\alpha}\rangle$ in the Pauli-X basis for all $k$, we can decide whether $0 < s < 2^{n-1}$ or $s \geq 2^{n-1}$ with an exponentially small error probability $p_{err}$ in quantum polynomial time [23].

However, since $p$ may be exponentially close to 1, the efficient preparation of Eq. (21) is difficult without postselection. We resolve this problem by using rewinding operators. Our idea is to amplify the probability of $|1\rangle$ being observed by mitigating the probability of $|0\rangle$ being observed. We propose the mitigation protocol in Algorithm 1 (see also Fig. 3).

\(^\dagger\)For clarity, we here explicitly write the procedure of Aaronson’s state-preparation protocol in [23]. First, the state in Eq. (15) is prepared, and then the state in Eq. (19) is generated from it. Finally, it is transformed to the target state in Eq. (20).
Algorithm 1 Mitigation protocol

1. Set $i = 0$ and $c = 0$.
2. By using the state in Eq. (11), prepare
\[ \sqrt{p_0} |\psi_t^-\rangle + \sqrt{1-p_0} |\psi_t^+\rangle, \]
where $p_0 = p$, and measure the last register in the Pauli-Z basis. Let $z$ be the measurement outcome. Furthermore, replace $c$ with $c + 1$.
3. Depending on the values of $z$, $i$, and $c$, perform one of following steps:
   (a) When $z = 0$, replace $i$ with $i + 1$, reset $c$ to 0, and obtain
\[ \sqrt{p_{i+1}} |\psi_t^-\rangle + \sqrt{1-p_{i+1}} |\psi_t^+\rangle, \]
where
\[ p_{i+1} = \frac{p_i}{2 - p_i}. \]
   If $i + 1 < 2n + 3$, do step 2 by using the state in Eq. (23). On the other hand, if $i + 1 = 2n + 3$, output the state in Eq. (23) and halt the mitigation protocol.
   (b) When $z = 1$ and $c < 3n$, apply the rewinding operator $R$ and do step 2 again for the same $i$.
   (c) When $z = 1$ and $c = 3n$, answer $0 < s < 2^{n-1}$ or $s \geq 2^{n-1}$ uniformly at random, and halt the mitigation protocol.

In this protocol, $i$ and $c$ count how many times the mitigation succeeds and how many times the mitigation fails for a single $i$, respectively. From Eq. (24),
\[ \frac{1 - p_{i+1}}{p_{i+1}} = 2 \frac{1 - p_i}{p_i}, \]
and hence we succeed in mitigating the amplitude of the nontarget state $|\psi_t^\perp\rangle$. Note that Eq. (24) is derived from Eq. (44) and the fact that the normalization factor of Eq. (44) is $\sqrt{1-p_i/2}$.

To evaluate our mitigation protocol, we show the following Claim in Appendix C:

> Claim 20. The mitigation protocol succeeds, i.e., the counter $i$ becomes $2n + 3$ with probability of at least
\[ 1 - \frac{5n}{8^n}. \]
In this case, the output state satisfies
\[ \frac{1 - p_{2n+3}}{p_{2n+3}} \geq 1. \]
From Eq. (26),
\[ \sqrt{p_{2n+3}} |\psi_t^-\rangle + \sqrt{1-p_{2n+3}} |\psi_t^+\rangle \]
is output with probability of at least $1 - 5n/8^n$. From Eq. (27), we can obtain the outcome 1 with probability of at least $1/2$ by measuring the second qubit in Eq. (28). If we obtain 0, we do the same measurement again by using the rewinding operator. Therefore, by repeating this procedure $n$ times, we obtain the outcome 1 with probability of at least
\[ 1 - 1/2^n. \]
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In total, from Eqs. (18), (26), and (29) and the fact that the mitigation protocol is used for all \(-n \leq k \leq n\), with probability of at least

\[ p_{\text{suc}} \equiv \left(1 - \frac{1}{2^n}\right)^2 \left(1 - \frac{5n}{8^n}\right)^{n(2n+1)} , \tag{30}\]

we obtain \(n\) copies of \(|\phi_{\beta/\alpha}\rangle\) for all \(k\). As a result, we can correctly decide whether \(0 < s < 2^{n-1}\) or \(s \geq 2^{n-1}\) in polynomial time with probability of at least \(p_{\text{suc}}(1 - p_{\text{err}})\) that is exponentially close to 1.

From Theorem 19, we obtain the following corollary:

**Corollary 21.** \(\text{BQP}^{\text{PP}_{\text{classical}}} \subseteq \text{AdPostBQP} \subseteq \text{RwBQP}\)

**Proof.** From Lemmas 10 and 18, it is sufficient to show \(\text{BQP}^{\text{PP}_{\text{classical}}} \subseteq \text{RwBQP}\) and \(\text{AdPostBQP} \subseteq \text{RwBQP}\) to obtain this corollary. First, we show the former inclusion. From Def. 9, it is obvious that any process in \(\text{BQP}\) can be simulated by a process in \(\text{RwBQP}\) in polynomial time. Furthermore, from Corollary 14 and Theorem 19, the \(\text{PP}\) oracle can be replaced with the \(\text{RwBQP}\) oracle. Therefore, from Corollary 15, \(\text{BQP}^{\text{PP}_{\text{classical}}} \subseteq \text{RwBQP}^{\text{RwBQP}_{\text{classical}}} = \text{RwBQP}\).

To obtain the latter inclusion, we show that each postselection can be simulated by rewinding operators. From Def. 11, each postselection (i.e., each projector) acts on a single qubit. Therefore, we can write a quantum state immediately before a postselection as

\[ \alpha|0\rangle_p|\psi_0\rangle + \beta|1\rangle_p|\psi_1\rangle \]

for some quantum states \(|\psi_0\rangle\) and \(|\psi_1\rangle\) and complex numbers \(\alpha\) and \(\beta\) such that \(|\alpha|^2 + |\beta|^2 = 1\). Here, the subscript \(p\) denotes the postselection register. Although \(|\beta|^2\) may be exponentially small, the postselection onto \(|1\rangle\) can be simulated by using our mitigation protocol.

As the most important difference between the proof of Theorem 19 and that of \(\text{PostBQP} \supseteq \text{RwBQP}\) in [23], we have not used postselections of outputs whose probabilities are exponentially small by proposing the mitigation protocol. To state the difference more explicitly on the technical level, we show the following corollary:

**Corollary 22.** For any polynomial function \(P(|x|)\) in the size \(|x|\) of an instance \(x\), \(\text{PP} = \text{PostBQP}\) holds even if only non-adaptive postselections of outputs whose probabilities are at least \(q \equiv 1 - 1/P(|x|)\) are allowed.

**Proof.** We can obtain this corollary by slightly modifying our mitigation protocol and showing that it can be realized with non-adaptive postselections of outputs whose probabilities are at least \(q \equiv 1 - 1/P(|x|)\). For some natural number \(m\), we prepare

\[ \sqrt{p}|\psi_1^m\rangle \left( \sqrt{q}|0\rangle + \sqrt{1-q}|1\rangle \right)^\otimes m + \sqrt{1-p}|\psi_t\rangle|0\rangle^\otimes m \]

instead of the state in Eq. (22). By postselecting \(m\) qubits in the second register onto \(|0\rangle\) one by one, we obtain

\[ \sqrt{pq^m}|\psi_1^m\rangle + \sqrt{1-p}|\psi_t\rangle \]

\[ \sqrt{1-p} + pq^m \]

These \(m\) postselections are non-adaptive ones of outputs whose probabilities are at least \(q\). If the amplitude of \(|\psi_t\rangle\) in Eq. (32) is at least \(\sqrt{1/2}\), we can obtain \(|\psi_t\rangle\) with at least a constant probability, and hence we can solve the PP-complete problem. Such the amplitude is realized by setting \(m \geq \log_2 [p/(1-p)] / \log_2 (1/q)\). Since \(\log_2 [p/(1-p)] / \log_2 (1/q) \leq 2(n+1)/\log_2 (1/q) \leq 2(n+1)/\log_2 (1/q) \leq 2(n+1)/O(P(|x|))\) from Eq. (54) in Appendix C, where \(n\) is at most a polynomial function in \(|x|\), a polynomial number of postselections are sufficient in the above argument.
Restricted Rewindable Quantum Computation

4.1 Power of Single Rewinding Operator

In Sec. 3, a polynomial number of rewinding operators was available. If the number is restricted to a constant, the question is: how is the rewinding useful for universal quantum computation? We show that a single rewinding operator is sufficient to solve the following problem with a constant probability, which seems hard for universal quantum computation:

Definition 23 (Collision-finding Problem). Given the function family \( F \equiv \{f_K\}_{K \in \mathcal{K}} \) in Theorem 7 and a parameter \( K \) for \( F \), output a pair \((x, x')\) with \( x, x' \in \mathbb{Z}_n^m \times \chi^m \times \{0, 1\} \) such that (i) \( x \neq x' \) and (ii) \( f_K(x) = f_K(x') \).

Theorem 24. Assume that a rewinding operator can be applied in one step, and there is no polynomial-time quantum algorithm solving \( \text{SIVP}_{p(n)} \) for some polynomial \( p(n) \) in \( n \). Then, the problem in Def. 23 can be solved with probability of at least \( \delta/2(1-\omega(1)) \) by uniformly generated polynomial-size quantum circuits with a single rewinding operator, but it cannot be achieved without rewinding operators. Here, the probability is taken over the uniform distribution on \( K \) and the randomness used in a quantum circuit to solve the problem.

The sketch of the proof is explained in Sec. 1.3, and the rigorous proof is given in Appendix D. Here, we explain the implication of Theorem 24. In Sec. 3, we have shown the equivalence between the postselection and rewinding. In contrast, Theorem 24 may represent their difference. A possible approach to solving the problem in Def. 23 is to generate two copies of the state in Eq. (4) (more precisely, Eq. (58) in Appendix D) by using the postselection. As a straightforward way, this can be achieved by postselecting the second register in the state \( \sum_x |x\rangle|f_K(x)\rangle \) (more precisely, Eq. (56) in Appendix D) onto the same \( f_K(x) \). However, it requires the postselection of a polynomial number of qubits (or the postselection of states whose amplitudes are exponentially small), while a single qubit is sufficient for the rewinding. Furthermore, since we do not know which \( f_K(x) \) has exactly two different preimages, the postselection applied to the second copy of \( \sum_x |x\rangle|f_K(x)\rangle \) needs to be adaptive, i.e., it depends on \( f_K(x) \) obtained from the first copy. On the other hand, a non-adaptive (i.e., non-classically-controlled) rewinding operator is sufficient for solving the problem (with a constant probability). Although there may be other ways to solve the problem by using a non-adaptive postselection of a single qubit, the above discussion may imply that the rewinding is superior to the postselection in some situations where the number of qubits to be rewound or postselected is restricted, and copies (i.e., \( \sum_x |x\rangle|f_K(x)\rangle \otimes 2 \)) are not processed collectively.

We also show a superiority of a single rewinding operator under a different assumption. To this end, we use the statistical difference (SD) problem, which is \( \text{SZK} \)-complete [42], and show the following theorem:

Definition 25 (Statistical Difference Problem [42]). Given classical descriptions of two Boolean circuits \( C_0, C_1 : \{0, 1\}^n \rightarrow \{0, 1\}^m \) with natural numbers \( n \) and \( m \), let \( P_0 \) and \( P_1 \) be distributions of \( C_0(x) \) and \( C_1(x) \) with uniformly random inputs \( x \in \{0, 1\}^n \), respectively. Decide whether \( \text{TV}(P_0, P_1) < 2^{-O(n^c)} \) or \( \text{TV}(P_0, P_1) > 1 - 2^{-O(n^c)} \) for some positive constant \( c \), where \( \text{TV}(\cdot, \cdot) \) is the total variation distance.

Theorem 26. The SD problem defined in Def. 25 is in \( \text{RwBQP}(1/2 - 2^{-O(n^c)}, 2 - 2^{-O(n^c)})(1) \), where \( n \) and \( c \) are the problem size and some positive constant as defined in Def. 25.

The proof is given in Appendix E. From Theorem 26, we obtain the following corollary:
Corollary 27. Let \( \text{RwBQP}(1) \equiv \bigcup_{1/(c-s) \in \text{poly}(|x|)} \text{RwBQP}(c, s)(1) \) for the set \( \text{poly}(|x|) \) of all polynomial functions in the size \( |x| \) of an instance \( x \). Then, \( \text{RwBQP}(1) \subseteq \text{BQP} \) unless \( \text{BQP} \supseteq \text{SZK} \).

Proof. From Theorem 26, if \( \text{RwBQP}(1) \subseteq \text{BQP} \), then \( \text{SZK} \subseteq \text{BQP} \).

For example, by assuming that the decision version of SIVP, gapSIVP, is hard for universal quantum computation, Corollary 27 implies that a single rewinding operator is sufficient to achieve a task that is intractable for universal quantum computation. This is because the gapSIVP (with an appropriate parameter) is in \( \text{SZK} \). Therefore, Corollary 27 shows the superior superiority of a single rewinding operator for promise problems, while Theorem 24 shows it for the search problem.

As a simple observation, a single cloning operator should also be sufficient to exceed universal quantum computation. This is because any problem in \( \text{PostBQP} \) is efficiently solvable with a single cloning operator. By applying the cloning operator \( C \) on a classical description of the output state

\[
\frac{(I \otimes |1\rangle \otimes I^\otimes n-2) \ U_x |0^n\rangle}{\sqrt{\Pr[p = 1]}}
\]

of a PostBQP computation (see Def. 5), the state in Eq. (33) is prepared in one step. Then, by measuring its first qubit in the computational basis, we can solve the PostBQP problem.

### 4.2 Rewindable Clifford Circuits

As another observation, we consider how useful rewinding operators are for restricted classes of quantum computation. Particularly, in this subsection, we focus on Clifford circuits defined as follows:

Definition 28 (Clifford Circuits). Let \( n \) be any natural number. An \( n \)-qubit Clifford circuit \( C_n \) is a quantum circuit such that

1. The initial state is \( |0^n\rangle \).
2. An applied \( n \)-qubit unitary operator \( U \) consists of elementary quantum gates chosen from \( \{H, S, CZ\} \), where \( H \) is the Hadamard gate, \( S \equiv |0\rangle \langle 0| + i|1\rangle \langle 1| \), and \( CZ \) is the CZ gate.
3. All or a part of qubits in the final state \( U |0^n\rangle \) are measured in the Pauli-Z basis \( \{|0\}, |1\}\). Particularly when \( U \) consists of a polynomial number of elementary quantum gates in \( n \), we say that the size of \( C_n \) is polynomial.

In Definition 28, the unitary operator \( U \) is in the Clifford group that is defined as the group of unitary operators normalizing the Pauli group. Here, \( n \)-qubit Pauli group \( \mathbf{P}_n \) is defined as \( \mathbf{P}_n \equiv \{e^{ik\pi/2} \otimes_{i=1}^n \sigma^{(j_i)} \mid k, j_i \in \{0, 1, 2, 3\} \text{ for } 1 \leq i \leq n\} \), where \( \sigma^{(0)} = I \), \( \sigma^{(1)} = X \), \( \sigma^{(2)} = Z \), and \( \sigma^{(3)} = iXZ \). Therefore, we call \( C_n \) a Clifford circuit.

It is well known that Clifford circuits are classically simulatable. More formally, the Gottesman-Knill theorem [29] states as follows:

Theorem 29. Let \( n \) be any natural number and \( P_S \) be any rank-one Z-basis projector applied on qubits in the subset \( S \) of \( n \) qubits. Any \( n \)-qubit polynomial-size Clifford circuit \( C_n \) can be strongly simulated in classical polynomial time in \( n \). That is, for any \( S \) and Clifford unitary \( U \) in \( C_n \), the output probability \( \langle 0^n | U^\dagger P_S U |0^n \rangle \) can be exactly computed in classical polynomial time in \( n \).

Simply speaking, Theorem 29 has been shown by using the stabilizer formalism. The initial state \( |0^n\rangle \) is a unique common +1 eigenstate of \( \{|Z_i\}_{i=1}^n\} \), where \( Z_i \) is the Pauli-Z operator.
applied on the $i$th qubit. Here, $\langle \rangle$ means that $\{Z_i\}_{i=1}^n$ are $n$ independent generators of a stabilizer group. The time evolution from $|0^n\rangle$ to $u_1|0^n\rangle$ with an elementary gate $u_1$ can be simulated by replacing $\{(Z_i)_{i=1}^n\}$ with $\{(u_1Z_iu_1^\dagger)_{i=1}^n\}$. Since stabilizer groups are abelian subgroups of the Pauli group and $u_1$ is in the Clifford group, this replacement can be done in classical polynomial time. By repeating a similar procedure for other elementary gates, we can uniquely represent $U|0^n\rangle$ by $\{(UZ_iU^\dagger)_{i=1}^n\}$. When the outcome of the Pauli-$Z$ measurement on the $i$th qubit is $o_i \in \{0, 1\}$, the measurement can be simulated by replacing a generator that anticommutes with $Z_i$ with $(-1)^{o_i}Z_i$. This replacement can also be done in classical polynomial time. Furthermore, it can be easily shown that the probability of obtaining $o_i$ is 1/2 for any $o_i \in \{0, 1\}$. Note that if all generators commute with $Z_i$, the measurement does not change the generators. In this case, the stabilizer group certainly contains $Z_i$ or $-Z_i$, and hence the measurement outcome is definitely 0 or 1, respectively. It is possible to check which, $Z_i$ or $-Z_i$, is contained in the stabilizer group in classical polynomial time. In short, since $n$ generators of any stabilizer group can be represented as a polynomial-size bit string $y$, Clifford circuits can be efficiently simulated by classically processing $y$.

We show that this classical simulatability holds even if rewinding operators are supplied:

**Theorem 30.** Let $n$ be any natural number, $P_S$ be any rank-one $Z$-basis projector applied on qubits in the subset $S$ of $n$ qubits, and $Q_{\text{Clifford}}$ be any $n$-qubit operator composed of a polynomial number of $\{H, S, CZ\}$, single-qubit measurements in the computational basis, and the (classically controlled) rewinding operator $R$. Let $Q_{\text{Clifford}}^{(z)}$ be the same as $Q_{\text{Clifford}}$ except for that the $i$th measurement is replaced with $|z_i\rangle\langle z_i|$ for all $i$, and $z_i$ is the $i$th bit of $z$. For any $z$ and $S$, the output probability $\langle 0^n|Q_{\text{Clifford}}^{(z)}P_SP_{\text{Clifford}}Q_{\text{Clifford}}|0^n\rangle$ can be exactly computed in classical polynomial time in $n$.

**Proof.** From the stabilizer formalism explained above, we can simulate Clifford unitary operators and Pauli-$Z$ measurements with outcomes $\{z_i\}_i$ in classical polynomial time by processing polynomial-length bit strings. Therefore, the remaining task is to show that the rewinding operator applied on a quantum state generated by Clifford unitary operators and Pauli-$Z$ measurements can also be simulated in classical polynomial time.

Suppose that we now have an $n$-qubit state $Q|0^n\rangle$ (up to normalization), where $Q$ is an $n$-qubit linear operator composed of a polynomial number of Clifford unitary operators and Pauli-$Z$ measurements. Then, we measure its first qubit in the $Z$ basis and obtain the post-measurement state $\langle z|\langle z| \otimes I^{0^n-1})Q|0^n\rangle$ (up to normalization) with $z \in \{0, 1\}$. Note that since the SWAP gate is a Clifford unitary, without loss of generality, we can assume that the first qubit is measured. Let $a \in \{0, 1\}$. Finally, if $z = a$, we apply the rewinding operator $R$ on the post-measurement state. Otherwise, we do nothing. For any $z, a \in \{0, 1\}$, the above procedures can be classically simulated in the following way:

1. By using the stabilizer formalism, represent $Q|0^n\rangle$ by a polynomial-length bit string $y$.
2. From $y$, make a duplicate $y'$.
3. Simulate the Pauli-$Z$ measurement on $Q|0^n\rangle$ by replacing $y$ with another polynomial-length bit string $y_z$.
4. When $z = a$, replace $y_z$ with the duplicate $y'$. Otherwise, do nothing.

The above argument can be used even if the rewinding operator is conditioned on a number of bits or is not classically controlled.

---

\[15\] Since the stabilizer group is a subgroup of the Pauli group, each generator commutes or anticommutes with $Z_i$. Furthermore, if there exist two or more generators that anticommute with $Z_i$, we can reduce the number to one by multiplying them.
Since Clifford circuits are classically simulatable even under postselection, Theorem 30 also implies an equivalence between the postselection and rewinding.

### 4.3 Rewindable IQP Circuits

In previous sections, we have considered the effect of the rewinding on universal quantum computation and classically simulatable quantum computation. How about sub-universal quantum computing models that are (believed to be) neither universal nor classically simulatable? Particularly, we consider IQP circuits [4] as a sub-universal quantum computing model. They are defined by replacing Clifford unitary operators $U$ in Clifford circuits with unitary operators $E$ diagonalizable in the Pauli-$X$ basis $\{|+, -\rangle\}$:

**Definition 31 (IQP Circuits [4]).** Let $n$ be any natural number. $n$-qubit IQP circuits are quantum circuits such that

1. The initial state is $|0^n\rangle$.
2. An applied $n$-qubit unitary operator is $E = H^\otimes n DH^\otimes n$, where $D$ is a unitary operator diagonalizable in the Pauli-$Z$ basis $\{|0\rangle, |1\rangle\}$.
3. All or some of the qubits in the final state $E|0^n\rangle$ are measured in the Pauli-$Z$ basis.

In particular, we consider the case that $D$ consists of a polynomial number of elementary gates chosen from $\{R_z(\theta), CZ \mid 0 \leq \theta < 2\pi\}$, where $R_z(\theta) \equiv |0\rangle\langle 0| + e^{i\theta}|1\rangle\langle 1|$. Since all elementary gates commute with each other, we can represent $D$ by $(\bigotimes_{i=1}^n R_z(\theta_i))D_{CZ}$, where $D_{CZ}$ is a unitary operator composed of only $CZ$'s.

When $D_{CZ}$ satisfies some property mentioned later, the IQP circuits can be regarded as measurement-based quantum computation (MBQC) [43]. To define MBQC, we first introduce the brickwork state as follows:

**Definition 32 (Brickwork State [44]).** Let $n$ and $m$ be natural numbers. A brickwork state $|B\rangle$ is an $nm$-qubit entangled state with $m \equiv 5$ (mod $8$) that is constructed as follows:

1. Prepare $|+\rangle^\otimes nm$ and assign an index $(i, j)$ to each qubit, where $|+\rangle \equiv (|0\rangle + |1\rangle)/\sqrt{2}$, $1 \leq i \leq n$, and $1 \leq j \leq m$.
2. For any $1 \leq i \leq n$ and $1 \leq j \leq m-1$, apply $CZ$ to two qubits labeled by $(i, j)$ and $(i, j+1)$.
3. For any odd $i$ and $j \equiv 3$ (mod $8$), apply $CZ$ to two qubits labeled by $(i, j)$ and $(i+1, j)$ or $(i, j+2)$ and $(i+1, j+2)$.
4. For any even $i$ and $j \equiv 7$ (mod $8$), apply $CZ$ to two qubits labeled by $(i, j)$ and $(i+1, j)$ or $(i, j+2)$ and $(i+1, j+2)$.

We define MBQC by using the brickwork state $|B\rangle$ as follows:

**Definition 33 (MBQC).** Let $m \equiv 5n$ (mod $8$) for any natural number $n$. MBQC is a quantum computing model proceeding as follows:

1. Prepare the $m$-qubit brickwork state $|B\rangle$.
2. Measure each qubit in $|B\rangle$ one by one in the basis $\{|+\theta\rangle, |+\theta+\pi\rangle\}$, where $|+\theta\rangle \equiv (|0\rangle + e^{i\theta}|1\rangle)/\sqrt{2}$ for any $\theta \in \mathbb{R}$. Each measurement basis (i.e., the value of $\theta$) is calculated from all the previous measurement outcomes in classical polynomial time. The universality of this computing model has been shown in [44].

From Defs. 31 and 33, we notice that when $D_{CZ}(|+\rangle^\otimes n) = |B\rangle$, the measurement basis for the $i$th qubit is $\{|+\theta_i\rangle, |+\theta_i+\pi\rangle\}$, and all measurement bases do not depend on previous measurement outcomes, MBQC becomes an IQP circuit. In short, (important subclasses of) IQP circuits are MBQC with non-adaptive single-qubit measurements.

Suppose that we would like to perform quantum computation by measuring all qubits in some $n$-qubit state $|\psi\rangle$ in the Pauli-$Z$ basis. To simulate this quantum computation by
MBQC, we prepare the $m \geq n$-qubit brickwork state $|B\rangle$ whose qubits can be divided into two sets, $M$ and $O$, each of which includes $(m - n)$ and $n$ qubits, respectively. By measuring all qubits in the set $M$ in appropriate bases, the state of $n$ qubits in the set $O$ becomes $H^\otimes n|\psi\rangle$ (up to a byproduct operator\textsuperscript{16}). Therefore, by measuring all qubits in $O$ in the Pauli-$X$ basis, we can simulate the target quantum computation. As an important point, for any single-qubit measurement on any qubit in the set $M$, the probability of the outcome being $0$ (corresponding to $|+\theta\rangle\langle+\theta|$) is exactly $1/2$, while it may not be the case for qubits in the set $O$. Furthermore, when all outcomes of qubits in the set $M$ are 0, adaptive measurements are not necessary. These properties are immediately obtained from the gate teleportation (see e.g., Fig. 1 in [45]) and will be used to show that if rewinding operators are given, then universal quantum computation is possible by MBQC with non-adaptive single-qubit measurements.

We show that polynomial-size IQP circuits can solve any PP problem if a polynomial number of rewinding operators are supplied. To formally state this, we define rewritable IQP circuits as follows:

\textbf{Definition 34 (Rewritable IQP Circuits).} Let $n$ be any natural number. $n$-qubit rewritable IQP circuits are quantum circuits such that

1. The initial state is $|0^n\rangle$.
2. An applied $n$-qubit unitary operator is $E \equiv H^\otimes nDH^\otimes n$, where $D$ is a unitary operator diagonalizable in the Pauli-$Z$ basis \{\ket{0}, \ket{1}\}.
3. All or some of the qubits in the final state $E|0^n\rangle$ are measured in the Pauli-$Z$ basis. For each qubit, if the outcome\textsuperscript{17} is 1, it is allowed to apply a rewinding operator and then perform the measurement again and again.

Particularly when $D$ consists of a polynomial number of elementary gates diagonalizable in the Pauli-$Z$ basis and the number of uses of rewinding operators grows at most polynomially with $n$, we say that $n$-qubit rewritable IQP circuits are polynomial size.

By using Def. 34, our result is formalized as follows:

\textbf{Theorem 35.} Let $L = (L_{\text{yes}}, L_{\text{no}}) \subseteq \{0, 1\}^*$ be a promise problem in PP. Then, there exists a uniform family of polynomial-size rewritable IQP circuits that decides $x \in L_{\text{yes}}$ or $x \in L_{\text{no}}$ with an exponentially small error probability for a given instance $x$.

\textbf{Proof.} First, we show that polynomial-size rewritable IQP circuits can prepare output states of polynomial-size quantum circuits in any uniform family. To this end, we use the relation between IQP circuits and MBQC with non-adaptive single-qubit measurements explained above. Due to the relation, our purpose is to show that if a polynomial number of rewinding operators are given, then we can perform universal quantum computation (with an exponentially small error) by MBQC with non-adaptive single-qubit measurements. Let $|\psi\rangle$ be any $n$-qubit quantum state that can be generated in quantum polynomial time. From [44], $|\psi\rangle$ can also be generated in quantum polynomial time by using the $m \geq n$-qubit brickwork state (see Def. 33) with $m$ being a polynomial in $n$. As explained above, $m$ qubits can be divided into a set $M$ of $(m - n)$ qubits and a set $O$ of the remaining $n$ qubits. When all measurement outcomes in the set $M$ are 0, the $n$-qubit state in the set $O$ exactly becomes

\textsuperscript{16}Since byproduct operators are tensor products of Pauli operators, they can be efficiently removed by classical postprocessing.

\textsuperscript{17}Without loss of generality, we can assume that an undesirable outcome is 1 because the bit-flip operation can be performed in IQP circuits.
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From Def. 33, we consider only single-qubit measurements in the set $M$ 0's with probability exponentially close to 1 by using a polynomial number of rewinding operators. Due to the property that the probability of 0 being output is exactly 1/2 for the measurements\(^{18}\) of all qubits in $M$, we can make it $1 - 2^{-(m-n)}$ in the following way:

1. Let $c = 0$.
2. Suppose that we now have $|\psi\rangle$. Measure its first qubit (in $M$) in the basis
   \[ \{|+\rangle\langle+|, |+\theta\rangle\langle+\theta|, |+\theta+\pi\rangle\langle+\theta+\pi|\} \]
   with $\theta \in \mathbb{R}$.
3. Depending on the measurement outcome and the value of $c$, perform one of following steps:
   
   (a) If the measurement outcome is 0 (corresponding to $|+\rangle\langle+|$), then halt the protocol.
   
   (b) If it is 1 (corresponding to $|+\theta\rangle\langle+\theta|$) and $c < m - n$, replace $c$ with $c + 1$, apply the rewinding operator $R$ on the post-measurement state, and perform step 2 by replacing $\{\{|+\rangle\langle+|, |+\theta\rangle\langle+\theta|, |+\theta+\pi\rangle\langle+\theta+\pi|\}$ with the $Z$ basis.
   
   (c) Otherwise, halt the protocol.

The reason we replace the measurement basis in step 2 with the $Z$ basis in step 3(b) comes from Def. 8. Since our rewinding operator works for only Pauli-$Z$ measurements, when we choose the measurement basis $\{\{|+\rangle\langle+|, |+\theta\rangle\langle+\theta|, |+\theta+\pi\rangle\langle+\theta+\pi|\}$ in step 2, the state after the rewinding is

\[ H_1 R_{z,1}(-\theta)|\psi\rangle, \tag{35} \]

where $R_z(\phi) \equiv |0\rangle\langle 0| + e^{i\phi}|1\rangle\langle 1|$ for any $\phi \in \mathbb{R}$, and the subscript number represents the qubit on which the gate is applied. Therefore, the measurement of $|\psi\rangle$ in the basis $\{\{|+\rangle\langle+|, |+\theta\rangle\langle+\theta|, |+\theta+\pi\rangle\langle+\theta+\pi|\}$ can be simulated by measuring the state in Eq. (35) in the $Z$ basis.

In the case of step 3(a), it is obvious that our purpose is achieved, i.e., the above protocol succeeds. On the other hand, in the case of step 3(c), the protocol fails to obtain the outcome 0, but it happens with the exponentially small probability $2^{-(m-n)}$. Therefore, the probability of the above protocol succeeding for all qubits in $M$ is $(1 - 2^{-(m-n)})^{m-n}$, which is exponentially close to one.

Second, by using the above argument, for any instance $x$ of the PostBQP problem, we can prepare

\[ \left( \prod_{j=1}^{q} CH_{2,n+j} \right) \left[ U_x|0^n\rangle \otimes \left( \bigotimes_{j=n+1}^{n+q} |+\rangle \right) \right] \tag{36} \]

as an output state of the polynomial-size rewindable IQP circuit. Here, $CH_{2,n+j}$ is the controlled-Hadamard gate whose control and target qubits are the second and $(n+j)$th ones, respectively, for $1 \leq j \leq q$. For the definitions of $U_x$, $n$, and $q$, see Def. 5. For the convenience, let $U_x|0^n\rangle = \alpha|0\rangle|0_1\rangle|\psi_0\rangle + \beta|1\rangle|1_2\rangle|\psi_1\rangle$ for some $(n-1)$-qubit states $|\psi_0\rangle$ and $|\psi_1\rangle$ and complex numbers $\alpha$ and $\beta$ such that $|\alpha|^2 + |\beta|^2 = 1$. According to our mitigation protocol, when the last $q$ qubits in Eq. (36) are measured in the computational basis, and the measurement outcome is $0^q$, we obtain $|\phi\rangle = \alpha'|0\rangle|0_1\rangle + \beta'|1\rangle|2\rangle$ such that $\beta'/\alpha' = \sqrt{2}\beta/\alpha$ (see also Fig. 4). From Appendix C, we can efficiently obtain the outcome $0^q$ with probability exponentially close to one by using a polynomial number of rewinding operators. From

\(^{18}\) From Def. 33, we consider only single-qubit measurements in the x-y plane of the Bloch sphere.
Figure 4 Quantum circuit diagram of our mitigation protocol. $R$ represents the rewinding operator, and $|\phi\rangle = \alpha'|0\rangle_2|\psi_0\rangle + \beta'|1\rangle_2|\psi_1\rangle$. The red wire represents the second qubit of $U_x|0\rangle_n$ that will be postselected.

$|\alpha|^2 + |\beta|^2 = 1$ and Def. 5, $|\beta'/\alpha'| \geq 1$, and hence the postselection onto $|1\rangle_2$ succeeds with probability of at least $1/2$. By using the rewriting operators again, this postselection probability is increased to $1 - o(1)$. Finally, the PostBQP problem is solved by measuring the first qubit of $|\psi_1\rangle$ in the computational basis. The proof is completed from PostBQP = PP and the fact that the probability amplification is possible in PostBQP [23].

It is worth mentioning why it should be hard or unlikely to show that $\text{RwBQP}$ problems are solvable with rewindable IQP circuits. In $\text{RwBQP}$ computations, measurements and rewinding are allowed to be performed during the computations. On the other hand, they are performed at the end of the computations in rewindable IQP circuits. This implies that the principle of deferred measurement [46] is necessary to be held even when the rewinding operators are supplied.

Here, we would like to emphasize that our rewinding operator seems to not violate the no-signaling principle. Although one may think that Theorem 35 contradicts the no-signaling principal from the result in [47], this is not the case. [47] shows that if MBQC can be realized without byproduct operators on an entangled state shared by Alice and Bob, then Alice can send her message to Bob without any signal. To apply their argument to our rewindable IQP circuits (i.e., MBQC with the rewinding), Alice has to rewind her measurements without Bob’s qubits. However, this is impossible because our rewinding operator defined in Def. 8 only works for pure states.

From [4], IQP circuits can solve any PP problem in quantum polynomial time under postselection. Therefore, Theorem 35 may imply an equivalence between the postselection and rewinding. On the other hand, our argument on rewinding operators seems to be not applicable to DQC1 [8], which can also solve any PP problem under postselection. This is because the input state of DQC1 includes maximally mixed states. This simple example may imply that the rewinding is weaker than the postselection in some situations.

## 5 Conclusion

We have defined $\text{RwBQP}$ and $\text{CBQP}$ by using rewriting and cloning operators, respectively, and $\text{AdPostBQP}$ by using postselections and have shown $\text{BPP}^{\text{PP}} \subseteq \text{RwBQP} = \text{CBQP} = \text{AdPostBQP} \subseteq \text{PSPACE}$. To this end, we have proposed a mitigation protocol that exponentially reduces amplitudes of nontarget states. To tighten the lower bound on $\text{RwBQP}$, it may be useful to consider whether $\text{PP}^{\text{PP}} \not\subseteq \text{RwBQP}_{\text{classical}}$. This inclusion cannot be...
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straightforwardly shown from Theorem 19 because this theorem does not necessarily mean that any process in the PP machine can be efficiently simulated with the RwBQP machine.

Our mitigation protocol also implies that a polynomial number of non-adaptive postselections of outputs whose probabilities are polynomially close to one are sufficient for solving any PostBQP problem. Then, we have considered how much rewinding operators are useful for restricted classes of quantum computation. Particularly, we have shown that Clifford circuits are still classically simulatable even if a polynomial number of rewinding operators are supplied. On the other hand, the rewinding promotes IQP circuits, which are believed to be neither universal nor classically simulatable, so that they can efficiently solve any PP problem. To this end, we have shown that if the rewinding is possible, then universal quantum computation can be efficiently performed by measurement-based quantum computation (MBQC) [43] without adaptive measurements. We have also shown that a single rewinding operator is sufficient to achieve tasks that are intractable for quantum computation under the post-quantum cryptographic assumption and BQP $\not\subseteq$ SZK.

Some of our results imply an equivalence between the postselection and rewinding for uniform families of quantum circuits with pure input states. How about non-uniform families and quantum interactive proof systems? So far, several advice complexity classes with PostBQP and quantum non-interactive proof systems with a PostBQP verifier have been explored. For example, the following results are already known:

1. $\text{BQP}/\text{qpoly} \subseteq \text{PostBQP}/\text{poly} \neq \text{ALL}$ [48]
2. $\text{PostBQP}/\text{qpoly} = \text{ALL}$ [49]
3. $\text{PostQMA} = \text{PSPACE}$ [50]
4. $\text{PostQCMA} = \text{NP}^\text{PP}$ [50]
5. $\text{PostQMA}(2) = \text{NEXP}$ [51, 52]

Here, PostQMA, PostQCMA, and PostQMA(2) are defined by replacing a BQP verifier with a PostBQP one in QMA, QCMA, and QMA(2), respectively. It would be interesting to consider what happens if we replace PostBQP with RwBQP or AdPostBQP in above complexity classes. Since some classical description is necessary to do our rewinding, while it is not for the postselection, a difference between the postselection and rewinding may be observed in the settings of 2, 3, and 5. As another direction, it would be interesting to precisely characterize RwBQP(1) and investigate the computational capability of rewindable boson sampling [7, 9, 14] by defining a bosonic rewinding operator that inverts the Fock-basis measurement in a single mode. It is also still open whether RwBQP is closed under composition even if quantum queries are allowed.

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\[\text{PostQMA and PostQCMA are denoted by } \text{QMA}_{\text{PostBQP}} \text{ and } \text{QCMA}_{\text{PostBQP}}, \text{ respectively. However, we use the terminology given in } [53]\]
Number JPMJMS2061, and the Grant-in-Aid for Scientific Research (A) No.JP22H00522 of JSPS.

\section*{A \hfill Proofs of Corollaries 13, 14, and 15}

The proof of Corollary 13 is as follows:

\textbf{Proof.} Since proofs are essentially the same for all three classes, we only write a concrete proof for $\text{RwBQP}$. Let $L$ be the complement of $L$. From Def. 9, when $x \in L_\text{yes}$ (i.e., $x \in L_\text{no}$),
\[
\sum_{z \in A} \left\vert \left\langle 1 \right\rangle \otimes I^{\otimes n+m+\ell-1} (X \otimes I^{\otimes n+m+\ell-1}) Q_n^{(z)} (|x \rangle |0^m \rangle |\tilde{D} \rangle) \right\vert^2 \geq 2/3. \tag{37}
\]
On the other hand, when $x \in L_\text{no}$ (i.e., $x \in L_\text{yes}$),
\[
\sum_{z \in A} \left\vert \left\langle 1 \right\rangle \otimes I^{\otimes n+m+\ell-1} (X \otimes I^{\otimes n+m+\ell-1}) Q_n^{(z)} (|x \rangle |0^m \rangle |\tilde{D} \rangle) \right\vert^2 \leq 1/3. \tag{38}
\]
Therefore, $\text{coRwBQP} \subseteq \text{RwBQP}$. By using the same argument, we can also show $\text{coRwBQP} \supseteq \text{RwBQP}$ and thus $\text{RwBQP} = \text{coRwBQP}$. $\blacksquare$

The proof of Corollary 14 is as follows:

\textbf{Proof.} Since proofs are essentially the same for all three classes, we only write a concrete proof for $\text{RwBQP}$. By repeating the same $\text{RwBQP}$ computation $m$ times and taking the majority vote on the outcomes, due to the Chernoff bound \cite{38}, the error probability is improved from $1/3$ to $2^{-q(m)}$ for a positive polynomial function $q(m)$ in $m$. Therefore, by setting $m$ so that $q(m) \geq p(n)$, we obtain this corollary. $\blacksquare$

The proof of Corollary 15 is as follows:

\textbf{Proof.} Since proofs are essentially the same for all three classes, we only write a concrete proof for $\text{RwBQP}$. From Def. 9, when a polynomial-time algorithm calls another polynomial-time algorithm as a subroutine, the resultant algorithm can still be realized in polynomial time. Since the $\text{RwBQP}$ computation has some error probability, a remaining concern is that errors may accumulate every time polynomial-time algorithms are called. However, the accumulation of errors is negligible from Corollary 14. As a result, we obtain $\text{RwBQP}_{\text{classic}} = \text{RwBQP}$. $\blacksquare$

\section*{B \hfill Proof of Lemma 16}

We give a proof of Lemma 16.

\textbf{Proof.} To show this lemma, it is sufficient to show that the acceptance probability
\[
p_{\text{acc}} = \sum_{z \in A} q_z \left\vert \left\langle 1 \right\rangle \otimes I^{\otimes n+m+\ell-1} \mathcal{N} Q_n^{(z)} (|x \rangle |0^m \rangle |\tilde{D} \rangle) \right\vert^2 \tag{39}
\]
can be computed in polynomial space. To this end, we use the Feynman path integral. Note that we only concretely show that
\[
\left\vert \left\langle 1 \right\rangle \otimes I^{\otimes n+m+\ell-1} \mathcal{N} Q_n^{(z)} (|x \rangle |0^m \rangle |\tilde{D} \rangle) \right\vert^2 \tag{40}
\]
can be computed in polynomial space because the derivation of $q_z$ in polynomial space can be done in a similar way. Let $k$ be some polynomial in $n$. By using $q_i^{(z)}$ that is an elementary gate in a universal gate set or a single-qubit postselection onto $|1 \rangle \langle 1|$ or $|0 \rangle \langle 0|$ for $1 \leq i \leq k$,
we can decompose $\mathcal{N}[Q_n^{(z)}(\cdot)]$ as $\mathcal{N}[Q_n^{(z)}(\cdot)] = \prod_{i=1}^{k} q_i^{(z)}(\cdot)$ for any $z \in A$. Let $N \equiv n + m + \ell$. Therefore,

$$\langle 1 \rangle \langle 1 \rangle \otimes I^{\otimes N-1} \mathcal{N}[Q_n^{(z)}( |x \rangle |0^m\rangle |\tilde{D}|)]$$

$$= \left[ \langle 1 \rangle \langle 1 \rangle \left( \sum_{d \in \{0,1\}^{N-1}} |d\rangle \langle d| \right) \right] q_k^{(z)} \left( \prod_{i=1}^{k-1} \left( \sum_{s(i) \in \{0,1\}^N} |s(i)\rangle \langle s(i)| \right) \right) q_i^{(z)} |x\rangle |0^m\rangle |\tilde{D}|. \tag{41}$$

Let $s$ be a shorthand notation of a $(k-1)N$-bit string $s(1)s(2)\ldots s(k-1)$. By defining

$$g(s,d,z) \equiv \langle 1 | |d\rangle \left[ q_k^{(z)} \left( \prod_{i=1}^{k-1} |s(i)\rangle \langle s(i)| q_i^{(z)} \right) \right] |x\rangle |0^m\rangle |\tilde{D}|, \tag{42}$$

$p_{\text{acc}}$ can be written as

$$\sum_{z \in A} q_z \sum_{s,d,z} g(s,d,z)g^* (\tilde{s}, \tilde{d}, z). \tag{43}$$

Since $q_i^{(z)}$ is just a constant-size matrix, each term $g(s,d,z)g^* (\tilde{s}, \tilde{d}, z)$ can be computed in polynomial space. Therefore, Eq. (43) can also be computed in polynomial space. \hfill \Box

## C Success Probability of Our Mitigation Protocol

We show that the probability that the state in Eq. (28) is output in our mitigation protocol is at least $1 - 5n/8^n$ and that $(1 - p_{2n+3})/p_{2n+3} \geq 1$. For clarity, we show a schematic diagram of our mitigation protocol in Fig. 3.

When the outcome 0 is obtained by measuring the last register of Eq. (22) in the $Z$ basis, the amplitude of the nontarget state $|\psi^+_i\rangle$ is mitigated with the factor $1/\sqrt{2}$ (up to normalization) because

$$\left( I^{\otimes 2} \otimes \langle 0 | \right) \left( \sqrt{p_i} |\psi^+_i\rangle |+\rangle + \sqrt{1-p_i} |\psi_i\rangle |0\rangle \right) = \sqrt{p_i/2} |\psi^+_i\rangle + \sqrt{1-p_i} |\psi_i\rangle. \tag{44}$$

Therefore, for any $i$, the probability $q_i$ that the outcome 0 is obtained by measuring the last register of Eq. (22) in the $Z$ basis is

$$q_i = \frac{p2^{-(i+1)} + (1 - p)}{1 - (1 - 2^{-i})p}, \tag{45}$$

where we have used $p_0 = p$. Therefore, for any $i$, the probability that we obtain 0 by measuring the last register of Eq. (22) in the $Z$ basis before or at $c = 3n$ is

$$1 - (1 - q_c)^{3n} \geq 1 - (1 - q_0)^{3n} = 1 - \left(\frac{p}{2}\right)^{3n}. \tag{46}$$

Our purpose is to sufficiently mitigate the amplitude of $|\psi^+_i\rangle$ so that we obtain the outcome 1 by measuring the second register of Eq. (23) in the $Z$ basis with probability of at least $1/2$. To this end, it is sufficient to run our mitigation protocol until $i = N$ such that

$$\frac{1 - p_N}{p_N} \geq 1. \tag{47}$$

\[20\] It may not be able to be computed in polynomial time, because $q_i^{(z)}$ may be a postselection.
From Eq. (25), this condition can be satisfied by setting

\[
N = \left\lfloor \log_2 \left( \frac{p}{1-p} \right) \right\rfloor. \tag{48}
\]

By combining Eqs. (46) and (48), the probability that the state in Eq. (28) is output in our mitigation protocol (i.e., the probability of our mitigation protocol reaching to Success in Fig. 3) is at least

\[
\left[ 1 - \left( \frac{p}{2} \right)^{3n} \right]^N \geq 1 - \left[ \log_2 \left( \frac{p}{1-p} \right) + 1 \right] \left( \frac{p}{2} \right)^{3n} \geq 1 - \left[ \log_2 \left( \frac{p}{1-p} \right) + 1 \right] \left( \frac{1}{2} \right)^{3n}. \tag{49}
\]

(50)

Since \( \log_2 \left[ p/(1-p) \right] \) is a monotonically increasing function of \( p \) in the range of \( 0 < p < 1 \), the remaining task is to upper bound \( p \). (Recall that our goal in this appendix is to show that Eq. (50) is lower bounded by \( 1 - 5n/8n \) and \( N \leq 2n + 3 \).)

From the simple observation that \( 2[(2^n - s)^2 + s^2] \), which is the denominator of \( p \) in Eq. (12), is a symmetric convex downward function that becomes minimum at \( s = 2^{n-1} \), and that \( (2^n - s)^2 \) in the numerator of \( p \) is a monotonically decreasing function in the range of \( 0 < s \leq 2^n \), the value of \( s \) maximizing \( p \) (for any \( \alpha, \beta \), and \( n \)) is between 1 and \( 2^{n-1} \). To upper bound \( p \), we separately consider three cases: (i) \( 1 \leq s \leq (1 - 1/\sqrt{2}) 2^n \), (ii) \( (1 - 1/\sqrt{2}) 2^n < s \leq 2^{n-1} - 1 \), and (iii) \( s = 2^{n-1} \).

(i) When \( 1 \leq s \leq (1 - 1/\sqrt{2}) 2^n \), the inequality \( 2(2^n - s)^2 \geq 4^n \) holds. From Eq. (12),

\[
p = \frac{\alpha^2(2^n - s)^2 + \beta^2 4^n}{2[(2^n - s)^2 + s^2]}.
\]

Since \( \alpha^2 + \beta^2 = 1 \), the numerator of \( p \) is upper bounded by \( 2\alpha^2(2^n - s)^2 + 2\beta^2(2^n - s)^2 = 2(2^n - s)^2 \). Therefore,

\[
p \leq \frac{(2^n - s)^2}{(2^n - s)^2 + s^2} \leq \frac{(2^n - 1)^2}{(2^n - 1)^2 + 1} = 1 - \frac{1}{(2^n - 1)^2 + 1}. \tag{51}
\]

(ii) When \((1 - 1/\sqrt{2}) 2^n < s \leq 2^{n-1} - 1\), the inequality \( 2(2^n - s)^2 < 4^n \) holds, and hence

\[
p \leq \frac{4^n}{2[(2^n - s)^2 + s^2]} \leq \frac{4^n}{2[(2^{n-1} + 1)^2 + (2^{n-1} - 1)^2]} = 1 - \frac{1}{4^{n-1} + 1}. \tag{52}
\]

The first inequality can be derived in a similar way as in (i).

(iii) When \( s = 2^{n-1} \),

\[
p = \frac{\alpha^2}{2} + \beta^2 \leq 1 - \frac{1}{2(4^n + 1)}, \tag{53}
\]

where we have used \( 2^{-n} \leq \beta/\alpha \leq 2^n \) and \( \alpha^2 + \beta^2 = 1 \).

From Eqs. (51), (52), and (53), \( p \leq 1 - 1/[2(4^n + 1)] \equiv p_{\text{max}} \), and hence

\[
N \leq \log_2 \left( \frac{p}{1-p} \right) + 1 \leq \log_2 \left( \frac{p_{\text{max}}}{1-p_{\text{max}}} \right) + 1 \leq 2n + 3. \tag{54}
\]

This implies that Eq. (50) is lower bounded by

\[
1 - \left[ \log_2 \left( \frac{p_{\text{max}}}{1-p_{\text{max}}} \right) + 1 \right] \left( \frac{1}{2} \right)^{3n} \geq 1 - \frac{2n + 3}{2^{3n}} \geq 1 - \frac{5n}{8n}. \tag{55}
\]
D Proof of Theorem 24

The proof of Theorem 24 is as follows:

**Proof.** To solve the problem in Def. 23 with a constant probability, we use the idea used in [37]. We prepare the state

\[
\sum_{s \in \mathbb{Z}_q^n \times \mathbb{Z}_q^m \times \{0,1\}} |s, e, d| f_K(s, e, d) \frac{1}{\sqrt{2^m(2\mu + 1)^m}},
\]

(56)

where \(1/\sqrt{2^m(2\mu + 1)^m}\) is the normalization factor (see Theorem 7). When there exists a natural number \(N\) satisfying \(2^N = 2^m(2\mu + 1)^m\), this preparation is trivially possible in quantum polynomial time with unit probability. If this is not the case, we prepare

\[
\sum_{(s, e, d) \in \mathbb{Z}_q^n \times \mathbb{Z}_q^m \times \{0,1\}} |s, e, d| f_K(s, e, d) |1\rangle + \sum_{(s, e, d) \notin \mathbb{Z}_q^n \times \mathbb{Z}_q^m \times \{0,1\}} |s, e, d| 0^n \mu_{\log_2 q} |0\rangle
\]

(57)

with unit probability, where \(\tilde{N}\) is the smallest natural number satisfying \(2^{\tilde{N}} \geq 2^m(2\mu + 1)^m\).

If we obtain the outcome 1 by measuring the third register in the computational basis, we can prepare the state in Eq. (56). From \(2^m(2\mu + 1)^m > 2^{\tilde{N} - 1}\), the probability of 1 being observed is larger than 1/2. Therefore, by repeating these procedures, we can obtain the outcome 1 at least once with probability of at least \(1 - o(1)\).

By measuring the second register in Eq. (56), we obtain a value of \(f_K(s, e, d)\). From the \(\delta\)-2 regularity of \(\mathcal{F}\), the obtained output \(f_K(s, e, d)\) has exactly two different preimages with probability of at least \(\delta\). When \(f_K(s, e, d)\) has exactly two different preimages, the state of the first register becomes

\[
|s, e, 1\rangle + |s + s_0, e + e_0, 0\rangle \frac{1}{\sqrt{2}},
\]

(58)

where \(f_K(s, e, 1) = f_K(s + s_0, e + e_0, 0)\). Then, we measure the state in Eq. (58) and obtain the values of \(s, e, 1\) or \(s + s_0, e + e_0, 0\).

To obtain the other one with probability 1/2, we would like to obtain the state in Eq. (58) again. It is possible by applying the rewinding operator \(R\) on \(|s, e, 1\rangle\) or \(|s + s_0, e + e_0, 0\rangle\) and a classical description\(^{21}\) of the state in Eq. (58). As an important point, since the state in Eq. (58) becomes \(|s, e, 1\rangle\) or \(|s + s_0, e + e_0, 0\rangle\) by measuring only the last single qubit in the \(Z\) basis, a single rewinding operator is sufficient to rewind it.

On the other hand, if rewinding operator is not allowed, the probability of the problem being solved is super polynomially small from the collision resistance of the function family \(\mathcal{F}\).

\[^{21}\]More precisely, the classical description means a transcript of how to prepare the state in Eq. (58) from a tensor product of \(|0\rangle\)'s. Let \(V\) be an operator that prepares the state in Eq. (56) from \(|0\rangle\)'s and \(f\) be the number of qubits required in the first register in Eq. (56). Then, the classical description is \((1^{\otimes f} \oplus |f_K(s, e, d)/f_K(s, e, d)|V\). Note that \(V\) can be decomposed into a polynomial number of elementary gates in a universal gate set and the postselection onto \(|1\rangle\langle 1|\).
E Proof of Theorem 26

In this appendix, we show that the RwBQP(1) machine can solve the SD problem with probabilities at least 1/2 − 2−O(α′) and 1 − 2−2−O(α′) when \( D_{TV}(P_0, P_1) < 2−O(α) \) and \( D_{TV}(P_0, P_1) > 1−2−O(α) \), respectively. To this end, we use an argument inspired by [34]22.

First, the RwBQP(1) machine prepares

\[
\frac{1}{\sqrt{2^{n+1}}} \sum_{b \in \{0,1\}, x \in \{0,1\}^n} |b⟩⟨x|C_b(x)⟩.
\]

By measuring the last register in the computational basis, it obtains the outcome \( y \in \{0,1\}^m \) and

\[
|0⟩\left(\sum_{x : C_b(x) = y} |x⟩\right) + |1⟩\left(\sum_{x : C_b(x) = \bar{y}} |x⟩\right)
\]

where for \( b \in \{0,1\} \), \( P_b(y) = \frac{1}{2^n} \sum_{x} \) is the probability of \( C_b \) outputting \( y \) for uniformly random inputs \( x \in \{0,1\}^n \). This event occurs with probability \( (P_0(y) + P_1(y))/2 \). Then, it measures the first register in Eq. (60) in the computational basis and obtain the outcome \( b_1 \in \{0,1\} \). By using a single rewinding operator, it can perform the same measurement again and obtain another outcome \( b_2 \in \{0,1\} \). Finally, it outputs 1 if \( b_1 \neq b_2 \). Otherwise, it outputs 0.

We now calculate error probabilities, i.e., probabilities of the machine outputting 0 and 1 when \( D_{TV}(P_0, P_1) < 2−O(α) \) and \( D_{TV}(P_0, P_1) > 1−2−O(α) \), respectively. First, we consider the case of \( D_{TV}(P_0, P_1) < 2−O(α) \). The probability \( p_{err} \) of the machine outputting 0, i.e., that of \( b_1 = b_2 \) is

\[
p_{err} = \frac{1}{2} \left( 1 + \sum_{y \in \{0,1\}^m} \delta(y) \frac{P_{min}(y) + \delta(y)}{2P_{min}(y) + \delta(y)} \right) \leq \frac{1}{2} \left( 1 + \sum_{y \in \{0,1\}^m} \delta(y) \right) < \frac{1}{2} + 2−O(α),
\]

where we have used \( \sum_{y \in \{0,1\}^m} \delta(y) = 2D_{TV}(P_0, P_1) \) in the last inequality.

Then, we consider the case of \( D_{TV}(P_0, P_1) > 1−2−O(α) \). The probability \( p'_{err} \) of the machine outputting 1, i.e., that of \( b_1 \neq b_2 \) is

\[
p'_{err} = \frac{1}{2} \left( 1 + \sum_{y \in \{0,1\}^m} \delta(y) \frac{P_{min}(y) + \delta(y)}{2P_{min}(y) + \delta(y)} \right) \leq \frac{1}{2} \left( 1 + \sum_{y \in \{0,1\}^m} \delta(y) \right) < \frac{1}{2} + 2−O(α),
\]

where we have used \( \sum_{y \in \{0,1\}^m} \delta(y) = 2D_{TV}(P_0, P_1) \) in the last inequality.

22 As a difference between their argument in [34] and ours, we replace their non-collapsing measurement with a single rewinding operator and an ordinary (i.e., a collapsing) measurement. Furthermore, although they use three non-collapsing measurements, we can perform the rewinding operator only once.
References

1. R. Hiromasa, A. Mizutani, Y. Takeuchi, and S. Tani, Rewindable Quantum Computation and Its Equivalence to Cloning and Adaptive Postselection, in Proceedings of the 18th Conference on the Theory of Quantum Computation, Communication and Cryptography (LIPIcs, Aveiro, 2023), p. 9:1.
2. P. W. Shor, Polynomial-time algorithms for prime factorization and discrete logarithms on a quantum computer, SIAM J. Comput. 26, 1484 (1997).
3. A. W. Harrow and A. Montanaro, Quantum computational supremacy, Nature (London) 549, 203 (2017).
4. M. J. Bremner, R. Jozsa, and D. J. Shepherd, Classical simulation of commuting quantum computations implies collapse of the polynomial hierarchy, Proc. R. Soc. A 467, 459 (2011).
5. K. Fujii, H. Kobayashi, T. Morimae, H. Nishimura, S. Tamate, and S. Tani, Impossibility of Classically Simulating One-Clean-Qubit Model with Multiplicative Error, Phys. Rev. Lett. 120, 200502 (2018).
6. T. Morimae, Y. Takeuchi, and H. Nishimura, Merlin-Arthur with efficient quantum Merlin and quantum supremacy for the second level of the Fourier hierarchy, Quantum 2, 106 (2018).
7. S. Aaronson and A. Arkhipov, The computational complexity of linear optics, in Proceedings of the 43rd Symposium on Theory of Computing (ACM Press, San Jose, 2011), p. 333.
8. T. Morimae, K. Fujii, and J. F. Fitzsimons, Hardness of Classically Simulating the One-Clean-Qubit Model, Phys. Rev. Lett. 112, 130502 (2014).
9. A. P. Lund, A. Laing, S. Rahimi-Keshari, T. Rudolph, J. L. O’Brien, and T. C. Ralph, Boson Sampling from a Gaussian State, Phys. Rev. Lett. 113, 100502 (2014).
10. M. J. Bremner, A. Montanaro, and D. J. Shepherd, Average-Case Complexity Versus Approximate Simulation of Commuting Quantum Computations, Phys. Rev. Lett. 117, 080501 (2016).
11. Y. Takahashi, S. Tani, T. Yamazaki, and K. Tanaka, Commuting quantum circuits with few outputs are unlikely to be classically simulatable, Quantum Inf. Comput. 16, 251 (2016).
12. Y. Takeuchi and Y. Takahashi, Ancilla-driven instantaneous quantum polynomial time circuit for quantum supremacy, Phys. Rev. A 94, 062336 (2016).
13. X. Gao, S.-T. Wang, and L.-M. Duan, Quantum Supremacy for Simulating a Translation-Invariant Ising Spin Model, Phys. Rev. Lett. 118, 040502 (2017).
14. C. S. Hamilton, R. Kruse, L. Sansoni, S. Barkhofen, C. Silberhorn, and I. Jex, Gaussian Boson Sampling, Phys. Rev. Lett. 119, 170501 (2017).
15. J. Miller, S. Sanders, and A. Miyake, Quantum supremacy in constant-time measurement-based computation: A unified architecture for sampling and verification, Phys. Rev. A 96, 062320 (2017).
16. S. Boixo, S. V. Isakov, V. N. Smelyanskiy, R. Babbush, N. Ding, Z. Jiang, M. J. Bremner, J. M. Martinis, and H. Neven, Characterizing quantum supremacy in near-term devices, Nat. Phys. 14, 595 (2018).
17. D. Hangleiter, J. Bermejo-Vega, M. Schwarz, and J. Eisert, Anticoncentration theorems for schemes showing a quantum speedup, Quantum 2, 65 (2018).
18. A. Bouland, B. Fefferman, C. Nirkhe, and U. Vazirani, On the complexity and verification of quantum random circuit sampling, Nat. Phys. 15, 159 (2019).
19. T. Morimae, Y. Takeuchi, and S. Tani, Sampling of globally depolarized random quantum circuit, arXiv:1911.02220.
20. J. Watrous, Zero-knowledge against quantum attacks, SIAM J. Comput. 39, 25 (2009).
21. A. Ambainis, A. Rosmanis, and D. Unruh, Quantum Attacks on Classical Proof Systems: The Hardness of Quantum Rewinding, in Proc. of the 55th Annual Symposium on Foundations of Computer Science (IEEE, Philadelphia, 2014), p. 474.
22. D. Unruh, Computationally Binding Quantum Commitments, in Proc. of the 35th Annual International Conference on the Theory and Applications of Cryptographic Techniques (Springer, Vienna, 2016), p. 497.
23 S. Aaronson, Quantum computing, postselection, and probabilistic polynomial-time, Proc. R. Soc. A 461, 3473 (2005).
24 W. K. Wootters and W. H. Zurek, A single quantum cannot be cloned, Nature (London) 299, 802 (1982).
25 S. Aaronson, A. Bouland, J. Fitzsimons, and M. Lee, The space “just above” BQP, arXiv:1412.6507.
26 O. Regev, On lattices, learning with errors, random linear codes, and cryptography, in Proc. of the 37th Annual Symposium on Theory of Computing (ACM, Baltimore, 2005), p. 84.
27 S. Aaronson, Quantum lower bound for the collision problem, in Proc. of the 34th Annual Symposium on Theory of Computing (ACM, Montréal, 2002), p. 635.
28 C. Peikert and V. Vaikuntanathan, Noninteractive Statistical Zero-Knowledge Proofs for Lattice Problems, in Proc. of the 28th International Cryptology Conference (Springer, Santa Barbara, 2008), p. 536.
29 D. Gottesman, The Heisenberg representation of quantum computers, in Group22: Proc. of the XXII International Colloquium on Group Theoretical Methods in Physics (International Press, Cambridge, 1999), p. 32.
30 T. A. Brun, M. M. Wilde, and A. Winter, Quantum State Cloning Using Deutschian Closed Timelike Curves, Phys. Rev. Lett. 111, 190401 (2013).
31 D. Deutsch, Quantum mechanics near closed timelike lines, Phys. Rev. D 44, 3197 (1991).
32 S. Lloyd, L. Maccone, R. García-Patron, V. Giovannetti, Y. Shikano, S. Pirandola, L. A. Rozema, A. Darabi, Y. Soudagar, L. K. Shalm, and A. M. Steinberg, Closed Timelike Curves via Postselection: Theory and Experimental Test of Consistency, Phys. Rev. Lett. 106, 040403 (2011).
33 S. Aaronson and J. Watrous, Closed timelike curves make quantum and classical computing equivalent, Proc. R. Soc. A 465, 631 (2009).
34 S. Aaronson, A. Bouland, J. Fitzsimons, and M. Lee, The Space “Just Above” BQP, in Proc. of the 7th ACM Conference on Innovations in Theoretical Computer Science (ACM, Cambridge, 2016), p. 271.
35 E. Bernstein and U. Vazirani, Quantum Complexity Theory, SIAM J. Comput. 26, 1411 (1997).
36 A. Cojocaru, L. Colisson, E. Kashefi, and P. Wallden, On the Possibility of Classical Client Blind Quantum Computing, Cryptography 5, 3 (2021).
37 A. Cojocaru, L. Colisson, E. Kashefi, and P. Wallden, QFactory: Classically-Instructed Remote Secret Qubits Preparation, in Proc. of the 25th Annual International Conference on the Theory and Application of Cryptology and Information Security (Springer, Kobe, 2019), p. 615.
38 S. Arora and B. Barak, Computational Complexity: A Modern Approach (Cambridge University Press, Cambridge, 2009).
39 Y. Shi, Toffoli and Controlled-NOT need little help to do universal quantum computation, Preprint at arXiv:quant-ph/0205115.
40 C. M. Dawson and M. A. Nielsen, The Solovay-Kitaev algorithm, Quantum Inf. Comput. 6, 81 (2006).
41 D. S. Abrams and S. Lloyd, Nonlinear Quantum Mechanics Implies Polynomial-Time Solution for NP-Complete and \# P Problems, Phys. Rev. Lett. 81, 3992 (1998).
42 A. Sahai and S. Vadhan, A complete problem for statistical zero knowledge, J. ACM 50, 196 (2003).
43 R. Raussendorf and H. J. Briegel, A One-Way Quantum Computer, Phys. Rev. Lett. 86, 5188 (2001).
44 A. Broadbent, J. Fitzsimons, and E. Kashefi, Universal Blind Quantum Computation, in Proc. of the 50th Annual Symposium on Foundations of Computer Science (IEEE, Atlanta, 2009), p. 517.
45 A. Mantri, T. F. Demarie, and J. F. Fitzsimons, Universality of quantum computation with cluster states and (X, Y)-plane measurements, Sci. Rep. 7, 42861 (2017).
32

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46 M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information 10th Anniversary Edition* (Cambridge University Press, Cambridge, 2010).

47 T. Morimae, Measurement-based quantum computation cannot avoid byproducts, International Journal of Quantum Information **12**, 1450026 (2014).

48 S. Aaronson, Limitations of Quantum Advice and One-Way Communication, in *Proc. of the 19th IEEE Annual Conference on Computational Complexity* (IEEE, Amherst, 2004), p. 320.

49 S. Aaronson, QMA/qpoly $\subseteq$ PSPACE/poly: De-Merlinizing Quantum Protocols, in *Proc. of the 21st Annual IEEE Conference on Computational Complexity* (IEEE, Prague, 2006), p. 261.

50 T. Morimae and H. Nishimura, Merlinization of complexity classes above BQP, Quantum Inf. Comput. **17**, 959 (2017).

51 S. A. Beikidezfuli, Quantum proof systems and entanglement theory, Ph.D. Thesis, Massachusetts Institute of Technology, 2009.

52 Y. Kinoshita, QMA(2) with postselection equals to NEXP, arXiv:1806.09732.

53 N. Usher, M. J. Hoban, and D. E. Browne, Nonunitary quantum computation in the ground space of local Hamiltonians, Phys. Rev. A **96**, 032321 (2017).