First Order Variational Calculation of Form Factor in a Scalar Nucleon–Meson Theory

A. W. Schreiber
TRIUMF, 4004 Wesbrook Mall, Vancouver B. C., Canada V6T 2A3

R. Rosenfelder
Paul Scherrer Institute, CH-5232 Villigen PSI, Switzerland

Abstract
We investigate a relativistic quantum field theory in the particle representation using a non-perturbative variational technique. The theory is that of two massive scalar particles, ‘nucleons’ and ‘mesons’, interacting via a Yukawa coupling. We calculate the general Euclidean Green function involving two external nucleons and an arbitrary number of external mesons in the quenched approximation for the nucleons. The non-perturbative renormalization and truncation is done in a consistent manner and results in the same variational functional independent of the number of external mesons. We check that the calculation agrees with one-loop perturbation theory for small couplings. As an illustration the special case of meson absorption on the nucleon is considered in detail. We derive the radius of the dressed particle and numerically investigate the vertex function after analytic continuation to Minkowski space.

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1 Introduction

There are many instances in relativistic quantum field theory where perturbation theory is inadequate, in particular when the coupling constant characterizing the interaction becomes too large. An alternative approach, which remains valid in the strong coupling regime, is the use of a variational principle. Although there exists a sizable body of literature on the use of variational methods in relativistic quantum field theories [1, 2, 3, 4, 5] it is fair to say that these methods are less widely used in this area of physics than the standard Rayleigh-Ritz variational method in non-relativistic quantum mechanics. This is largely due to the fact that one has to deal with a system with infinitely many degrees of freedom, the requirements of relativistic invariance and renormalizability of the theory.

The problem of the infinitely many degrees of freedom, however, is also present in non-relativistic field theories and here there exists a variational method which deals with this problem very successfully – Feynman’s variational treatment of the polaron [6, 7, 8, 9, 10, 11, 12]. A very early attempt to use this method in a relativistic setting was made in Ref. [13] and was recently reinvented and extended by us [14, 15, 16]. (The method has also previously been adapted for use in non-relativistic nuclear physics [17, 18, 19]). The particular Lagrangian which was considered in these papers was that of two scalar particles, one heavy (henceforth referred to as the ‘nucleon’) and one light (the ‘meson’), interacting via a Yukawa coupling. This variant of the Wick-Cutkosky model [20, 21] was chosen because of its simplicity, its similarity to several realistic Lagrangians (such as those of QED and nucleon-meson theories) and because it has traditionally been used for the study of non-perturbative techniques [22, 23, 24, 25, 26, 27, 28].

The quantities of interest in a relativistic field theory are the Green functions of the theory. The work in Refs. [14, 15, 16] concerned itself with the calculation of a subset of these Green functions in the quenched approximation for the nucleon. In particular, Green functions with 2 external (on-shell) nucleon legs and any number of external meson legs were calculated in Ref. [16].

Feynman’s variational method involves the use of the path integral formalism. As is well known, in general it is only possible to do gaussian path integrals analytically. Because of this he made use of a particular approximation – Jensen’s inequality which can be exploited for variational purposes. Equivalently, this corresponds to the first order term in a cumulant expansion (see, for example, Ref. [29]). In Refs. [14, 15], the variational calculation of the nucleon propagator was performed using the same approximation. The only quantity of interest there was the residue of this propagator, the position of the pole being defined as the (non-perturbatively) renormalized mass of the nucleon. (In addition, the ‘width’ of the nucleon was calculated – this quantity is not zero as the Wick-Cutkosky model is inherently unstable.) In Ref. [16] the general $(2 + n)$-point function was then written down to zeroth order in the cumulant expansion. Although already qualitatively interesting at this order (for example, even at zeroth order these Green functions contain parts of all Feynman diagrams to all orders in the coupling), these Green functions clearly had some unwanted numerical features. In particular, at this order all dependence on the external momenta is gaussian, resulting in, for example, a gaussian form factor. Furthermore, one of the strong points of Feynman’s treatment of the polaron was that agreement with perturbation theory was assured to the order in the cumulant expansion in which one calculates. Hence, the $(2 + n)$-point functions calculated in Ref. [16] only agree with perturbation theory at tree level.
In the present paper we calculate the \((2 + n)\)-point function to first order in the cumulant expansion, therefore greatly improving the numerical results. More importantly, it will be shown that the variational equations for any of these Green functions are in fact the same, i.e. once one has determined the numerical value of the variational parameters from one of these functions, say the propagator of the nucleon \((n = 0)\), the other ones may be calculated with the use of the same parameters. It will also be shown that it is possible to truncate the \((2 + n)\)-point function consistently; that is, the propagators on the external nucleon legs have poles at the physical nucleon mass, as they should.

An important ingredient of the method has been the use of the particle representation of a field theory \([7, 30, 31, 32, 33]\), which has undergone something of a revival in recent years \([22, 23, 34, 35, 36, 37, 38]\). We start in Section 2 by briefly outlining how one derives the expression for the quenched \((2 + n)\)-point function in the particle representation. Having done this, we perform the actual variational calculation in Section 3, resulting in the ‘master formula’ for the \((2 + n)\)-point function at first order in the cumulant expansion. An important check, that this formula agrees with perturbation theory to one-loop order, is relegated to the Appendix. It should be noted that throughout this derivation we work in Euclidean space-time. The Green functions can however be continued to Minkowski space. This is done, for the \((2+1)\)-point function, in Section 4 and numerical results are derived and discussed in Section 5. We conclude with some general observations in Section 6.

## 2 The \((2+n)\)-point function in the particle representation

Before we turn to the calculation of the general quenched Green function with two external nucleon and \(n\) external meson legs in first order in the cumulant expansion, we need to derive the particle representation of this function. For the propagator this was already done in Ref. \([14]\), so here we shall limit ourselves to briefly summarizing the relevant steps involved. We shall follow the notation of Ref. \([14]\), and the reader who is interested in the details is referred to this reference. The Wick–Cutkosky Lagrangian is given, in Euclidean space-time, by

\[
\mathcal{L} = \frac{1}{2}(\partial_{\mu}\Phi)^2 + \frac{1}{2}M_0^2\Phi^2 + \frac{1}{2}(\partial_{\mu}\varphi)^2 + \frac{1}{2}m^2\varphi^2 - g\Phi^2\varphi
\]

where \(M_0\) is the bare mass of the heavy nucleon, \(\varphi\) and \(\Phi\) are the fields of the meson and nucleon respectively and \(m\) is the meson mass. The latter does not get renormalized in the quenched approximation while the nucleon mass does. The corresponding generating functional for the general Green function is written in terms of the sources \(J\) and \(j\) of the external nucleons and mesons as

\[
Z[J, j] = \int \mathcal{D}\Phi \mathcal{D}\varphi \exp \left(-S[\Phi, \varphi] + (J, \Phi) + (j, \varphi) \right).
\]

Here the action is given by

\[
S[\Phi, \varphi] = \int d^4x \mathcal{L}(\Phi(x), \varphi(x))
\]

and we have defined

\[
(J, \Phi) \equiv \int d^4x J(x)\Phi(x) \quad \text{etc.}
\]

One implements the quenched approximation by integrating out the nucleon field and setting the resulting determinant equal to unity. After functionally differentiating twice with respect to
the nucleon source \( J(x) \) and then setting this source to zero we obtain the generating functional for the Green functions with two external nucleon legs and an arbitrary number of meson legs. It is given by

\[
Z' [j, x] \equiv \left. \frac{\delta^2 Z [J, j]}{\delta J(x) \delta J(0)} \right|_{J=0} = \int \mathcal{D}\varphi < x \mid -\Box + M_0^2 - 2g\varphi \mid 0 > \exp \left[ -\frac{1}{2} (\varphi, (-\Box + m^2) \varphi) + (j, \varphi) \right].
\]  

(5)

This generating functional is written in terms of the meson field \( \varphi \). In order to implement the particle representation of the theory in terms of the path of the nucleon we exponentiate the propagator and write the resulting relativistic matrix element in terms of a proper time path integral [39]:

\[
< x \mid \exp \left[ -\beta \left( \frac{p^2}{2} - g\varphi(x) \right) \right] \mid 0 > = \int_{x(0)=0}^{x(\beta)=x} \mathcal{D}x(\tau) \exp \left( -\int_0^\beta d\tau \left[ \frac{1}{2} x^2 - g\varphi(x(\tau)) \right] \right).
\]  

(6)

We have given the particle the “mass” 1 which corresponds to the ‘proper time gauge’ [40]. Our variational results can be shown to be gauge-independent although some variational parameters are not [15]. One may now perform the functional integration over the meson fields to obtain the generating functional of connected Green functions

\[
Z'_{\text{conn}} [j, x] = \text{const} \int_0^\infty d\beta \exp \left( -\frac{\beta}{2} M_0^2 \right) \int_{x(0)=0}^{x(\beta)=x} \mathcal{D}x(\tau) \exp(-S_{\text{eff}} [x(\tau), j])
\]  

(7)

Here the effective action is given by

\[
S_{\text{eff}} [x(\tau), j] = S [x(\tau)] - g \int d^4 y \, j(y) \int_0^\beta d\tau < y \mid -\Box + m^2 \mid x(\tau) >
\]  

(8)

with

\[
S [x(\tau)] = S_0 [x(\tau)] + S_1 [x(\tau)]
\]  

(9)

\[
S_0 [x(\tau)] = \int_0^\beta d\tau \left[ \frac{1}{2} x^2 \right]
\]  

(10)

\[
S_1 [x(\tau)] = -\frac{g^2}{2} \int_0^\beta d\tau_1 \int_0^\beta d\tau_2 < x(\tau_1) \mid -\Box + m^2 \mid x(\tau_2) >
\]  

(11)

Finally, one may obtain the Green function with \( n \) external meson legs by differentiating \( n \) times with respect to the source \( j \) and then setting this source to zero. After Fourier transforming to momentum space and removing the external meson legs as well as the overall momentum conserving delta function one obtains

\[
G_{2,n}(p, p'; \{q\}) = \text{const.} \int_0^\infty d\beta \exp \left[ -\frac{\beta}{2} M_0^2 \right] \int d^4 x \, e^{-ip' \cdot x} \cdot \int_{x(0)=0}^{x(\beta)=x} \mathcal{D}x(\tau) \prod_{i=1}^n \left[ g \int_0^\beta d\tau_i e^{iq_i \cdot x(\tau_i)} \right] e^{-S[x(\tau)]}.
\]  

(12)
Here the definition of the meson momenta \{q\} and of the final nucleon momentum \(p'\) is such that the latter is in–going while the former are out–going. The integration parameters \(\tau_i\) have an obvious physical significance – they mark the (proper) times at which the external mesons interact with the (bare) nucleon.

Note that the external nucleon legs are still contained in this expression. As the nucleon mass needs to be renormalized their truncation is not as trivial as that of the meson legs. In particular, the nucleon mass is a dynamical quantity which needs to be calculated in terms of the bare nucleon mass. Of course, if one could calculate the Green functions of the theory exactly, the poles of the external nucleon legs would all be automatically at the same position as the pole of the propagator. In an approximate treatment, however, this is not automatically fulfilled. In fact, it is non-trivial to even isolate these poles, let alone to have them occur at the correct position. As will be seen in the next Section, it is indeed possible to perform the truncation of the general Green function in a consistent manner within the context of the variational calculation described here. The result (see (Eq. 49) ) will be a rather compact expression for the general truncated \((2+n)\) - point function, which encapsulates, as we shall see, information contained in any order of perturbation theory as well as genuinely non-perturbative effects.

3 Variational Calculation of the \((2+n)\)-point function to first order in the cumulant expansion

The variational treatment is based on the decomposition of the action into a trial action \(\tilde{S}_t\) plus a remainder

\[
\tilde{S} = \tilde{S}_t + \tilde{S} - \tilde{S}_t = \tilde{S}_t + \Delta \tilde{S}
\]

(13)

and uses the cumulant expansion to first order (which is related to Jensen’s inequality):

\[
< e^{-\Delta \tilde{S}} >_{\tilde{S}_t} \cong e^{-<\Delta \tilde{S}>_{\tilde{S}_t}}.
\]

(14)

Here the action \(\tilde{S}[x]\) is related to \(S[x]\) through

\[
\tilde{S}[x] = S[x(\tau)] + ip' \cdot x - i \sum_{i=1}^n q_i \cdot x(\tau_i)
\]

(15)

and thus contains all the dependence on the nucleon’s path in Eq. (12). The first order cumulant has the particular property that it is stationary (for real actions even minimal) under arbitrary variations of \(\tilde{S}_t\). We shall perform the variational calculation in momentum space so that the averages \(^1\) in Eq. (14) are defined by [14]

\[
< O >_{\tilde{S}_t} \equiv \frac{\int D\tilde{x} O \exp(-\tilde{S}_t)}{\int D\tilde{x} \exp(-\tilde{S}_t)}.
\]

(16)

\(^1\)These averages in momentum space were denoted by \(< < .. > >\) in Ref. [14] in order to distinguish them from averages in coordinate space which were indicated by a single bracket \(< .. >\). In the present paper we shall only deal with momentum space averaging, so as no confusion can arise we shall simply make use of the single brackets.
where
\[
\int D\hat{\mathbf{x}} = \int d^4x \int^{x(\beta)=x}_{x(0)=0} \mathcal{D}x
\] (17)

To this order in the cumulant expansion the \((2+n)\)-point function is given by
\[
G_{2,n}(p, p'; \{q\}) = \text{const.} \int_0^\infty d\beta \left[ \prod_{i=1}^n g \int_0^\beta d\tau_i \right] \exp \left[ -\frac{\beta}{2} M_0^2 \right] e^{-\langle \Delta S \rangle_{S_t}} \int D\hat{\mathbf{x}} e^{-\hat{S}^t[x(\tau)]} \] (18)

The actions \(\tilde{S}\) and \(\tilde{S}_t\) are expressed in terms of the absolute times \(\tau_i\). It will at times turn out to be more useful to define relative times \(\alpha_i\), as indicated in Fig. 1. It should be noted that Fig. 1 is not to be understood as a Feynman diagram - it only serves to label the interaction times of the external mesons with the nucleon. Furthermore, in this Figure we have ordered these interaction times such that \(\tau_1 \leq \tau_2 \ldots \leq \tau_n\). It is always possible to do this, provided we sum over all permutations of the \(q_i\)’s afterwards.

We shall do the path integrals which are involved by expanding the general path \(x(\tau)\) into its Fourier components:
\[
x(\tau) = x \frac{\tau}{\beta} + \sum_{k=1}^\infty \frac{2\sqrt{\beta}}{k\pi} b_k \sin \left( \frac{k\pi\tau}{\beta} \right).
\] (19)

The action \(\tilde{S}\) now becomes
\[
\tilde{S} = S[x(\tau)] + i \left[ \sqrt{\frac{2}{\beta}} b_0 \cdot \sum_{i=1}^{n+1} \alpha_i p_i - \sum_{k=1}^\infty Q^{(n)}_k \cdot b_k \right]
\] (20)
\[ Q_k^{(n)} = \frac{2\sqrt{\beta}}{\pi k} \sum_{i=1}^{n} \sin \left( \frac{k\pi \tau_i}{\beta} \right) q_i, \] (21)

\[ x = \sqrt{\frac{2\beta b_0}{\rho_k}} \] (22)

and the definition of the momenta \( p_i \) is indicated in Fig. [1].

The question now arises as to what the form of the trial action \( \tilde{S}_t \) should be. If one were to calculate all the corrections to Eq. (14) the functional form of this trial action would in principle be arbitrary, as would be the actual numerical values of the variational parameters. In practice, of course, one does not calculate the corrections, except the lowest order one which is needed to make the variational principle work. It is this fact that makes the form of the variational action assume importance. Indeed, for best results, the trial action should encapsulate as much of the information contained in the true action as possible. In particular, in order to reproduce the lowest non-trivial order of perturbation theory, the trial action should reduce to the free action in the small coupling limit [14]. On the other hand, it should be of such a form that the mathematical manipulations required become possible without further approximations, as otherwise one loses control over the variational properties. In the present case this simply means that the trial action must be quadratic in the Fourier coefficients \( b_k \) so that the path integrals may be performed analytically. One is therefore restricted to the following general form:

\[ \tilde{S}_t = S_t + i \left[ \sqrt{\frac{2}{\beta}} b_0 \cdot \sum_{i=1}^{n+1} \alpha_i p_i - \sum_{k=1}^{\infty} \rho_k Q_k^{(n)} \cdot b_k \right], \] (23)

where \( S_t \) is given by (as in Ref. [14]):

\[ S_t = \sum_{k=0}^{\infty} A_k b_k^2. \] (24)

Note that in the limit in which the coupling goes to zero the true action \( \tilde{S} \) becomes

\[ \tilde{S}_0 = \sum_{k=0}^{\infty} b_k^2 + i \left[ \sqrt{\frac{2}{\beta}} b_0 \cdot \sum_{i=1}^{n+1} \alpha_i p_i - \sum_{k=1}^{\infty} Q_k^{(n)} \cdot b_k \right], \] (25)

so that \( A_k \) and \( \rho_k \) must go to 1 in this limit.

The above trial action ensures that upon expansion in the coupling constant, and without having to solve the variational equations, the one-loop perturbative result is obtained. However, as indicated in Section 2, we also demand that for arbitrary coupling one should be able to truncate the general Green function consistently. It is not obvious that this is possible in general. Indeed, we have not been able to perform this truncation with the simplest choice of \( \rho_k \) being a constant independent of \( k \). On the other hand, if one chooses \( \rho_k \) to be proportional to the ‘profile function’ \( A_k \)

\[ \rho_k = \lambda A_k, \] (26)

a consistent truncation becomes possible. Here \( \lambda \) is a variational parameter which will turn out to be precisely the same quantity which already appeared in the study of the propagator in Refs. [14, 13]. Actually for \( n = 0 \) Eq. (23) with the choice (26) is just the ansatz which was used there for ‘momentum averaging’ of the 2-point function. However, for this special
case it is more a convenience than a necessity to write the variational parameter $\rho_0$ as $\lambda A_0$
whereas for $n > 0$ the $k$-dependence chosen in Eq. (26) is essential. At the present stage we
have not investigated other parameterizations of the trial action in a systematic manner. As
the parameterization of the trial action is connected to the analytic structure of the resulting
Green functions it would clearly be a rather interesting exercise to determine what the most
general allowable $\rho_k$ is.

One may now calculate the various quantities required for the evaluation of the Green
functions in Eq. (18). Writing

$$\tilde{S} = \tilde{S}_0 + S_1$$

(27)

one obtains, up to irrelevant overall constants,

$$\int D\tilde{x} e^{-\tilde{S}_t} = \left[ \prod_{k=0}^{\infty} \frac{1}{A_k^2} \right] \exp \left\{ -\frac{\lambda^2}{2\beta} \left[ A_0 P^2 + \frac{\beta}{2} \sum_{k=1}^{\infty} A_k Q_k^{(n)}^2 \right] \right\}$$

(28)

and

$$\langle \tilde{S}_0 - \tilde{S}_t \rangle = \frac{P^2}{\beta} \left[ \lambda - \frac{\lambda^2}{2} (1 + A_0) \right] + 2 \sum_{k=0}^{\infty} \left( \frac{1}{A_k} - 1 \right) + \sum_{k=1}^{\infty} \frac{Q_k^{(n)}^2}{2} \left[ \lambda - \frac{\lambda^2}{2} (1 + A_k) \right].$$

(29)

For brevity we have defined the quantity $P$ to be

$$P = \sum_{i=1}^{n+1} \alpha_i p_i$$

(30)

Combining these terms yields

$$\exp \left[ -\langle \tilde{S}_0 - \tilde{S}_t \rangle \right] \int D\tilde{x} e^{-\tilde{S}_t} = \exp \left[ 2 \sum_{k=0}^{\infty} \left( \frac{1}{A_k} \right) \right] \sum_{i=1}^{n+1} \alpha_i p_i$$

(31)

The remaining quantity which needs to be calculated is $\langle S_1 \rangle \tilde{S}_t$. It is given by

$$\langle S_1 \rangle \tilde{S}_t = -\frac{g^2}{2} \int_0^{\beta} dt_1 dt_2 \int \frac{d^4 q'}{(2\pi)^4} \frac{1}{q'^2 + m^2} < \exp \left[ iq' \cdot (x(t_1) - x(t_2)) \right] > \tilde{S}_t.$$  

(32)

The path integrals in this expression can be performed analogously to the equivalent expression
for the propagator in Ref. [14]. Furthermore, the integral over $q'$ may also be done with the
help of the momentum space representation for the propagator

$$\frac{1}{q'^2 + m^2} = \frac{1}{2} \int_0^\infty du \exp \left[ -\frac{u}{2} (q'^2 + m^2) \right].$$

(33)

One obtains, after shifting the $u$ integration in the same way as in Ref. [14],

$$\langle S_1 \rangle \tilde{S}_t = -\frac{g^2}{16\pi^2} \int_0^{\beta} dt_1 dt_2 \frac{1}{\tilde{\mu}^2(\sigma, T)} \int_0^1 du \left( m\tilde{\mu}(\sigma, T), \frac{-i\lambda W^{(n)}(n)}{\tilde{\mu}(\sigma, T)}, u \right).$$

(34)
The quantity $\tilde{\mu}^2(\sigma, T)$ is defined in terms of the pseudotime $\mu^2(\sigma, T)$ as in Ref. [14], i.e.

$$\tilde{\mu}^2(\sigma, T) = \frac{\sigma^2}{A_0 \beta} + \mu^2(\sigma, T)$$

$$= \frac{\sigma^2}{A_0 \beta} + \beta \sum_{k=1}^{\infty} \frac{\lambda_k^2}{A_k}. \quad (35)$$

Here

$$\lambda_k = \frac{\sqrt{2}}{k \pi} \left[ \sin \left( \frac{k \pi t_1}{\beta} \right) - \sin \left( \frac{k \pi t_2}{\beta} \right) \right]. \quad (36)$$

and we have introduced the relative and total time $\sigma = t_1 - t_2$ and $T = (t_1 + t_2)/2$. Finally, the function $e(s, t, u)$ is defined as

$$e(s, t, u) = \exp \left( - \frac{s^2}{2} \frac{1 - u}{u} - \frac{t^2}{2} u \right). \quad (37)$$

It should be noted that the interaction term Eq. (34) looks identical to the interaction term written down for the propagator in Ref. [14], Eq. (97). Indeed, all the dependence on the number of external mesons is hidden in the (four-dimensional) quantity $W(n)$:

$$W^{(n)} = \frac{\sigma}{\beta} P - \sqrt{\beta} \sum_{k=1}^{\infty} \frac{\lambda_k Q_k^{(n)}}{2}. \quad (38)$$

For $n = 0$ this just reduces to $\sigma$ times the nucleon momentum $p$, while for general $n$ it contains the full complexity necessary for the $(2 + n)$-point function $G_{2,n}$. Indeed, using the standard formula for the summation of a series of cosines weighted by $1/k^2$ [11], we may evaluate the sums appearing in $W^{(n)}$ to obtain

$$W^{(n)} = \frac{\sigma}{2} (p_1 + p_{n+1}) + \frac{1}{2} \sum_{i=1}^{n} q_i \left( \left| \tau_i - t_1 \right| - \left| \tau_i - t_2 \right| \right). \quad (39)$$

Hence $W^{(n)}$ depends on the ordering of the proper times $\tau_i$ at which the nucleon interacts with the external mesons relative to the proper times $t_{1,2}$ at which the internal meson is emitted and absorbed. The form of Eqs. (38) and (39) bears some similarity to the “phase averaged” approximation for the Green function and corrections to it which have been investigated in Ref. [12].

It is instructive to eliminate the absolute values appearing in Eq. (39) by explicitly breaking up the sum into three contributions: one contribution ($i = 1$ to $b$, say) where the internal meson’s interaction times $t_{1,2}$ are both larger than all $\tau_i$, one contribution where one of them is smaller and the other one is larger than all $\tau_i$ ($i = b + 1$ to $a - 1$) and one contribution where they are both smaller than all $\tau_i$ ($i = a$ to $n$). This division of the sum is illustrated in Fig. 2. Similarly to Fig. 1, Fig. 2 should not be interpreted in the sense of a Feynman diagram, although it is closely related. It is a diagrammatic representation of the (in this case averaged) action (not an amplitude) in the particle representation. It serves to indicate the relative ordering of the proper times involved. Furthermore, it also defines the momenta: similarly to a Feynman diagram, four-momentum is conserved at each vertex, with the proviso that one
associates 0 four-momentum with the internal meson line (the momentum flowing through the loop has already been integrated over).

One obtains

\[ W(n) = \text{sign}(\sigma) \left\{ \left[ \tau_{b+1} - \min(t_1, t_2) \right] p_{b+1} + \sum_{i = b+2}^{a-1} \alpha_i p_i + \left[ \max(t_1, t_2) - \tau_{a-1} \right] p_a \right\} \]

\[ a > b + 1 \]

\[ = \sigma p_a \]

\[ a = b + 1 . \] (40)

(For the case \( a = b + 2 \), the sum appearing in Eq. (39) is defined to be empty.) In other words, \( W(n) \) is equal to (up to a sign, which is not relevant as only the square of \( W(n) \) enters) the integral of the proper time multiplied by the nucleon’s four-momentum for the duration of the exchange of the internal meson. The reason for the appearance of this quantity will become clear once we make the connection to perturbation theory. Particular cases of interest, for the purpose of the eventual truncation of the Green function, are those where \( t_{1,2} \) are both smaller or both larger than all \( \tau_i \), in which case \( W(n) = \sigma p_1 \) and \( \sigma p_{n+1} \), respectively.

As was shown in Ref. [14], if one is interested in the Green function for the case where the external nucleon legs (not the internal ones) are on-shell, then one should consider the above expression in the limit where the proper times corresponding to the external legs (i.e. \( \alpha_1 \) and \( \alpha_{n+1} \), and hence \( \beta \)) tend towards infinity. To implement this limit, the following equality derivable from the Poisson summation formula [43, 44] is extremely useful: If \( F(k) \) is only a function of \( k \pi / \beta \) and if it is even, then up to exponentially small terms in \( \beta \)

\[ \sum_{k=1}^{\infty} F(k) \simeq \frac{\beta}{\pi} \int_{0}^{\infty} dE F \left( \frac{E \beta}{\pi} \right) - \frac{1}{2} F(0). \] (41)
As was shown in Ref. [14], $A_k$ satisfies these criteria, which results in the following simplifications:

$$
\lim_{\beta \to \infty} \tilde{\mu}^2(\sigma, T) \equiv \mu^2(\sigma) = \frac{4}{\pi} \int_0^\infty dE \frac{1}{A(E)} \frac{\sin^2(E\sigma/2)}{E^2}.
$$

(42)

and

$$
2 \lim_{\beta \to \infty} \sum_{k=0}^\infty \left[ \log A_k + \frac{1}{A_k} - 1 \right] = \frac{2\beta}{\pi} \int_0^\infty dE \left[ \log A(E) + \frac{1}{A(E)} - 1 \right]
$$

$$
+ \log A(0) + \frac{1}{A(0)} - 1
$$

$$
\equiv \log A(0) + \frac{1}{A(0)} - 1 + \Omega \sum_{i=1}^{n+1} \alpha_i.
$$

(43)

Here $\Omega$ is identical to the kinetic term defined in Ref. [14].

Using the above simplifications for the case of on-shell external nucleon legs, as well as the fact that the integrand appearing in the interaction term $< S_1 > \tilde{S}_t$ is even in $\sigma$, one can now write down the expression for the (untruncated) $(2 + n)$-point function:

$$
G_{2,n}(p, p'; \{q\}) = \frac{N_0}{2g} \sum_{P_{\{q\}}} \left\{ \prod_{i=1}^{n+1} g \int_0^\infty d\alpha_i \exp \left[ -\frac{\alpha_i}{2} \left( M_0^2 + 2\Omega + p_i^2[1 - (1 - \lambda)^2] \right) \right] \right\}
$$

$$
\cdot \exp \left[ \frac{g^2}{8\pi^2} \int_0^\beta dt_1 \int_0^{t_1} dt_2 \frac{1}{\mu^2(\sigma)} \int_0^1 du e \left( m\mu(\sigma), \frac{-i\lambda W^{(n)}(\sigma)}{\mu(\sigma)}, u \right) \right].
$$

Here $N_0$ is defined to be

$$
N_0 = \frac{1}{A(0)} \exp \left( 1 - \frac{1}{A(0)} \right)
$$

(45)

and it should be noted that $G_{2,n}(p, p'; \{q\})$ is normalized in such a way that the correct tree level result is obtained in the limit in which the coupling vanishes:

$$
G_{2,n}^{\text{tree}}(p, p'; \{q\}) = \frac{1}{2g} \sum_{P_{\{q\}}} \prod_{i=1}^{n+1} \frac{2g}{p_i^2 + M^2_{\text{phys}}}
$$

(46)

Furthermore, for non-zero couplings Eq. [14] needs to be renormalized in order for it to become meaningful. This is because $\mu^2(\sigma) \to \sigma$ for small $\sigma$ and hence the integral in Eq. [14] diverges logarithmically. Indeed, the mass renormalization was already performed in Ref. [14] by demanding that the pole of the nucleon propagator occurs at the physical nucleon mass. Actually, the nucleon propagator is just the special case of the Green function in Eq. [14] with $n = 0$. It is therefore not surprising, but nevertheless crucial for the internal consistency of the variational approach described here, that with the same renormalization on the poles of the external legs we arrive at the same relationship between the unrenormalized nucleon mass $M_0$ and the physical nucleon mass $M$ as one did in Ref. [14]:

$$
M_0^2 + 2\Omega = M^2[1 - (1 - \lambda)^2] + \frac{g^2}{4\pi^2} \int_0^\infty \frac{d\sigma}{\mu^2(\sigma)} \int_0^1 du e \left( m\mu(\sigma), \frac{\lambda\sigma M}{\mu(\sigma)}, u \right).
$$

(47)
Figure 3: The $t_1 - t_2$ integration region. Regions (A) and (B) correspond to dressing of the external legs and give rise to the external propagators and residues.

It should be understood, of course, that in principle one should regularize both Eqs. (44) and (47). Only after substitution of the latter into the former does everything become finite and therefore meaningful with the regulator removed.

Because the meaning of $W^{(n)}$ depends on the relative ordering of the proper times $t_{1,2}$ with respect to the interaction times $\tau_i$, the $t_1 - t_2$ integration region in Eq. (44) needs to be broken up in order to do explicit calculations. In particular, in order to isolate those parts of the expression which contribute to the dressing of the external legs one needs to separate out the parts of the integration region for which $t_{1,2}$ are either both smaller than all $\tau_i$ (Region (A) in Fig. 3) or both larger than all $\tau_i$ (Region (B) in Fig. 3). Explicitly, in the limit in which $\alpha_1$ and $\alpha_{n+1}$ become very large, the relevant integrals for these two regions become

$$
\int_{(A)} dt_1 dt_2 \rightarrow \int_0^\infty d\sigma (\alpha_1 - \sigma) \\
\int_{(B)} dt_1 dt_2 \rightarrow \int_0^\infty d\sigma (\alpha_{n+1} - \sigma),
$$

respectively. Here we have made use of the fact that in regions (A) and (B) the integrand does not depend on the overall time $T$.

It is important to note that there is no further dependence on $\alpha_1$ and $\alpha_{n+1}$ dependence coming from the integration region (C). This allows one to perform the integrals over these two proper times. Comparing with the results for the propagator in Ref. [14], one sees that
the result of these two integrations just yield the external nucleon propagators, including their residues:

\[ \int_0^\infty d\alpha_i \exp \left\{ -\frac{\alpha_i}{2} \left[ (M^2 + p_i^2)(1 - (1 - \lambda)^2) \right] \right\} \]

\[ + \frac{g^2}{4\pi^2} \int_0^\infty \frac{d\sigma}{\mu^2(\sigma)} \int_0^1 du \left[ e \left( m_\mu(\sigma), \frac{\lambda_\sigma M_\mu(\sigma)}{\mu(\sigma)}, u \right) - e \left( m_\mu(\sigma), \frac{-i\lambda_\sigma p_i}{\mu(\sigma)}, u \right) \right] \}

\[ \cdot \exp \left\{ -\frac{g^2}{8\pi^2} \int_0^\infty \frac{d\sigma}{\mu^2(\sigma)} \int_0^1 du e \left( m_\mu(\sigma), \frac{-i\lambda_\sigma p_i}{\mu(\sigma)}, u \right) \right\} \]

\[ = \frac{2Z/N_0}{p_i^2 + M^2} \]

Here \( i \) is 1 or \( n + 1 \), respectively, and \( Z \) is the same expression for the nucleon propagator’s residue as derived in Ref. [13].

By removing the external legs one may now arrive at the main result of this paper, the truncated \((2 + n)\)-point function \( G_{2,n}^{tr}(p, p';\{q\}) \), valid for on-shell external nucleon momenta:

\[ G_{2,n}^{tr}(p, p';\{q\}) = \frac{2g}{N_0} \sum_{\mathcal{P}(\alpha_i)} \left\{ \prod_{i=2}^n g \int_0^\infty d\alpha_i \exp \left[ -\frac{\alpha_i}{2} \left( (M^2 + p_i^2)(1 - (1 - \lambda)^2) \right) \right] \right\} \]

\[ \cdot \exp \left\{ -\frac{g^2}{8\pi^2} \int_0^1 du \left[ \int_0^\infty \frac{d\sigma}{\mu^2(\sigma)} \sum_{i=2}^n \alpha_i e \left( m_\mu(\sigma), \lambda_\sigma M_\mu(\sigma), u \right) \right] \]

\[ - \int_{(C)} \frac{dt_1 dt_2}{\mu^2(\sigma)} e \left( m_\mu(\sigma), \frac{-i\lambda W^{(n)}(\sigma)}{\mu(\sigma)}, u \right) \right\}. \quad (49) \]

In order to appreciate the information contained in Eq. (49) it is very instructive to expand the exponential in the second and third lines in something akin to a perturbative expansion (i.e. expand in \( g^2 \), disregarding the dependence of \( N_0, \mu \) and \( \lambda \) on \( g \)). The leading term contains the tree diagram and for this one the integrations over the proper times \( \alpha_i \) may actually be carried out, resulting in the appropriate propagators for the internal nucleon legs. The next term contains diagrams occurring at one loop order. Because the proper times \( t_1 \) and \( t_2 \) at which the internal meson couples to the internal nucleon line are integrated over, they occur in all possible permutations with respect to the external times \( \tau_i \). These permutations give rise to all the individual Feynman diagrams at this order. Similar reasoning applies to the higher order terms – at order \( g^4 \) four integrations over the (two) internal mesons’ interaction times must be carried out, each ordering with respect to the times \( \tau_i \) corresponding to a piece contained in a particular Feynman diagram, etc. Of course, there is no reason for a particular one of these terms to reproduce a given Feynman diagram exactly. This is only guaranteed for the one-loop result and is explicitly checked in the Appendix.

Using this procedure, it becomes possible to address the question whether or not Eq. (49) explicitly contains all possible Feynman diagrams of any order. (In addition, it contains additional information implicitly because, after all, \( N_0, \mu \) and \( \lambda \) do depend on the coupling. A diagram by diagram identification of this implicit dependence on the coupling constant does not seem possible.) In fact, it is easy to convince oneself that Eq. (49) does contain, in an approximate way, almost every diagram occurring at any order in perturbation theory, except...
for an important exception. Because of the restriction of the \( t_1 - t_2 \) integration to Region \((C)\) in Fig. 3, diagrams of fourth order or higher in which \( t_1 \) and \( t_2 \) are both smaller than \( \tau_1 \) or both larger than \( \tau_n \) (excluding, of course, the one-particle reducible diagrams taken away in the truncation) are not contained explicitly. It would be very interesting to see to what extent this feature depends on the form of the variational function \( \rho_k \) in Eq. (23).

Although Eq. (49) is a very compact expression for all the Green functions in the one nucleon sector of the theory, it is in general not possible to continue without specifying the number of external meson legs \( n \). The simplest case (apart from the nucleon propagator, which has already been worked out in Refs. \([14, 15]\)) is the vertex function, i.e. \( n = 1 \). We shall discuss this function in the next Section. The rather more complicated case of the \((2+2)\)-point function (‘meson – nucleon’, or ‘Compton’ scattering) will be discussed in a future paper.

4 The vertex function

Setting \( n = 1 \) in the expression for the general truncated Green function \( G_{2,n}^\nu(p,p';\{q\}) \) in Eq. (49) yields the vertex function. For this function, the only possible values for \( a \) and \( b \), as defined in Fig. 2, are 2 and 0, respectively. Hence \( W^{(n)} \) takes on the form, for positive \( \sigma \),

\[
W^{(1)} = (\alpha_1 - t_2)p_1 + (t_1 - \alpha_1)p_2 \\
= \sigma/2 (p_1 + p_2) + (T - \alpha_1)q_1.
\]

(50)

It is important to note that although \( W^{(1)} \) depends on \( \alpha_1 \), the integral in the last line in Eq. (49) does not. The \( \alpha_1 \) dependence disappears by a shift of variable: the integration region \((C)\) for the vertex function is given by, in the limit of \( \alpha_1 \to \infty \) appropriate for on-shell external nucleons,

\[
\int_{(C)} dt_1 dt_2 = \int_0^\infty d\sigma \int_{\alpha_1 - \sigma/2}^{\alpha_1 + \sigma/2} dT.
\]

(51)

Letting \( T \to T - \alpha_1 \) eliminates both the \( \alpha_1 \) dependence in the integral limits as well as in \( W^{(1)} \). This is absolutely necessary – after all, the variable \( \alpha_1 \) has already been integrated over, resulting in one of the external propagators of the untruncated Green function. With this shift, the scattering amplitude is given by

\[
A_{2,1}(p,p';q) = \frac{2gZ}{N_0} \exp \left\{ \frac{g^2}{8\pi^2} \int_0^1 du \int_0^\infty d\sigma \frac{\mu^2(\sigma)}{\mu^2(\sigma)} \int_{-\sigma/2}^{\sigma/2} dT \right. \\
\left. \cdot \exp \left[ -\frac{1}{2} \left( m^2 \mu^2(\sigma) \frac{1-u}{u} - \frac{\lambda^2}{\mu^2(\sigma)} (p_1 + p_2 + T q_1)^2 \right) \right] \right\}.
\]

(52)

Note the extra factor of \( Z \) relating the Green function in Eq. (49) and the corresponding amplitude.

As it stands, the integrand in Eq. (24) looks as if it may diverge for small \( \sigma \) due to the factor \( 1/\mu^2(\sigma) \). It is easy to see that this is actually not the case by scaling the \( T \) integral by \( \sigma \). That we do not need an additional (coupling constant) renormalization is, of course, a special feature of the Wick-Cutkosky model in the quenched approximation which is a superrenormalizable
theory. By also making use of the fact that \( p_1^2 = p_2^2 = -M^2 \), one obtains the form factor as a function of the momentum transfer \( q^2 = -q_1^2 \) in Minkowski space

\[
A_{2,1}(q^2) = \frac{2}{N_0} g Z \exp \left\{ \frac{g^2}{8\pi^2} \int_0^1 du \int_0^\infty ds \frac{s}{\mu^2(is)} \int_0^1 dt \right. \\
\left. \quad \cdot \exp \left[ -\frac{1}{2} \left( m^2 \mu^2(is) \frac{1-u}{u} + \frac{s^2 \lambda^2}{\mu^2(is)} \left( M^2 - (1-t^2)\frac{q^2}{4} \right) u \right) \right] \right\}
\]

Note that, in contrast to perturbation theory, for example, the change from Euclidean space to Minkowski space was trivial here as there are no integrations over momenta to be performed. Of course, Eq. (53) only defines the amplitude in the region in which the integrals converge. This is the case for spacelike momentum transfers, as well as (unphysical) timelike momentum transfers as long as \( q^2 < 4M^2 \). In these regions the amplitude is purely real. For timelike momentum transfers larger than \( 4M^2 \), however, one needs to analytically continue Eq. (53) before obtaining numerical results. It is possible to do this if the analytic structure of \( \mu^2(\sigma) \) is known. In particular, if \( \mu^2(\sigma) \) has no cuts or poles for \( \text{Re}(\sigma) > 0 \), then one may deform the contour of the \( \sigma \) integration in Eq. (53) to the imaginary axis. If one gives \( q^2 \) a small positive imaginary part, one must perform the deformation in the positive half-plane so that one does not get a contribution from the contour at infinity. Setting \( \sigma = is \), one obtains

\[
A_{2,1}(q^2) = \frac{2}{N_0} g Z \exp \left\{ \frac{g^2}{8\pi^2} \int_0^1 du \int_0^\infty ds \frac{s}{\mu^2(is)} \int_0^1 dt \right. \\
\left. \quad \cdot \exp \left[ -\frac{1}{2} \left( m^2 \mu^2(is) \frac{1-u}{u} + \frac{s^2 \lambda^2}{\mu^2(is)} \left( (1-t^2)\frac{q^2}{4} - M^2 \right) u \right) \right] \right\}.
\]

It is clear that for \( q^2 > 4M^2 \) the amplitude is no longer purely real. Indeed, it develops a cut, corresponding to the fact that this is the physical region for nucleon pair-production. It should of course be remembered that the calculation has been performed in the quenched approximation; hence the numerical results in this region certainly do not correspond to what one would obtain in the full theory.

## 5 Numerical Results and Discussion

Our numerical results for the vertex function will be presented as a function of the dimensionless coupling constant

\[
\alpha = \frac{g^2}{4\pi M^2}.
\]

We only consider \( \alpha \) below the critical coupling \( \alpha_c \approx 0.82 \). For couplings larger than \( \alpha_c \) the variational equations do not have real solutions due to the well-known instability of the Wick-Cutkosky model.[14][15]

We first investigate the vertex function for low values of the momentum transfer. Writing Eq. (53) as

\[
A_{2,1}(q^2) = A_{2,1}(0) F(q^2)
\]

one may easily read off the effective coupling at zero momentum transfer and the elastic form factor for scattering of mesons from the dressed particle. The effective coupling takes on a
particularly simple form as it may be simplified through the use of the variational equation for
the parameter $\lambda$ (see Ref. [15], Sec. 5.1). One obtains

$$2g_{\text{eff}} = A_{2,1}(0) = \frac{2g}{\lambda}.$$  \hfill (57)

Like the famous anomalous magnetic moment of the electron in QED the effective coupling is
enhanced because the ‘velocity parameter’ $\lambda$ is always less than unity. It is interesting to note
that in zeroth variational order we had obtained the enhancement $g_{\text{eff}} = g A_0$ [16] which can be
shown to be less than the one obtained in first order. Indeed, solving the variational equations
perturbatively for $m = 0$ we obtain

$$\frac{g_{\text{eff}} - g}{g} = \left\{ \frac{\alpha}{\pi} + \ldots \quad \text{(zeroth order)} \right\} \frac{\alpha}{\pi} + \ldots \quad \text{(first order)}.$$  \hfill (58)

In addition $A_0$ varies much more among the different parametrizations in Ref. [15] of the re-
tardation function than does the parameter $\lambda$ which is essential for determining the critical
coupling.

The elastic form factor is given by

$$F(q^2) = \exp \left\{ -\frac{g^2}{8\pi^2} \int_0^1 du \int_0^\infty d\sigma \frac{\sigma}{\mu^2(\sigma)} \exp \left[ -\frac{1}{2} m^2 \mu^2(\sigma) \frac{1-u}{u} - \frac{\lambda^2 M^2 \sigma^2}{2\mu^2(\sigma)} \right] \right\} \int_0^1 dt \left[ 1 - \exp \left( \frac{\lambda^2 \sigma^2}{8\mu^2(\sigma)} (1-t^2) q^2 u \right) \right].$$  \hfill (59)

From its low-$q$ expansion we deduce the mean square radius of the dressed particle to be

$$\langle r^2 \rangle = \frac{g^2}{16\pi^2} \lambda^2 \int_0^1 du \int_0^\infty d\sigma \frac{\sigma^3}{\mu^4(\sigma)} \exp \left[ -\frac{1}{2} \left( m^2 \mu^2(\sigma) \frac{1-u}{u} + \frac{\lambda^2 M^2 \sigma^2}{\mu^2(\sigma)} u \right) \right].$$  \hfill (60)

In the limit $\lambda \to 1$, $\mu^2(\sigma) \to \sigma$ we can perform the $\sigma$-integration and recover the perturbative result

$$\langle r^2 \rangle_{\text{pert}} = \frac{g^2}{4\pi^2} \int_0^1 du \frac{u^3}{[1-u]m^2 + u^2 M^2]^2}. \hfill (61)$$

For large spacelike momenta $q^2 \ll -M^2$ the elastic form factor (59) approaches the constant value

$$F(q^2) \to \exp \left\{ -\frac{g^2}{8\pi^2} \int_0^1 du \int_0^\infty d\sigma \frac{\sigma}{\mu^2(\sigma)} \exp \left[ -\frac{1}{2} \left( m^2 \mu^2(\sigma) \frac{1-u}{u} + \frac{\lambda^2 M^2 \sigma^2}{\mu^2(\sigma)} u \right) \right] \right\} \exp \left[ \frac{\alpha}{\pi} + \ldots \right].$$  \hfill (62)

which is identical with the constant $N_1 = \lambda Z/N_0$ encountered in the calculation of the residue
[15]. Although the large $q$-limit certainly is outside the region of applicability of the quenched
model the result still is meaningful: for asymptotically large momentum transfers the meson
“sees” the bare particle inside the Wick-Cutkosky polaron.

The effective coupling constant and the radius of the dressed particle are plotted as a func-
tion of the dimensionless coupling in Figs. 4 and 5. We have employed several parametrizations
Table 1: First-order root-mean-square radius (in fm) of the dressed particle from Eq. (60) for various parametrizations of the retardation function. The heading ‘Feynman’ gives the result in the Feynman parametrization whereas ‘improved’ refers to the ‘improved’ parametrization. In both cases the parameters have been determined from the variational calculation for the nucleon self-energy in Ref. ([15]). The ‘extended’ parametrization was introduced in Ref. ([16]) for additional use in meson-nucleon scattering. The radius calculated with the solution of the variational equations is denoted by ‘variational’. For comparison the perturbative result is also given.

| α   | ‘Feynman’ | ‘improved’ | ‘extended’ | ‘variational’ | perturbative |
|-----|-----------|------------|------------|---------------|-------------|
| 0.1 | 0.0480    | 0.0481     | 0.0481     | 0.0481        | 0.0470      |
| 0.2 | 0.0697    | 0.0697     | 0.0697     | 0.0697        | 0.0664      |
| 0.3 | 0.0878    | 0.0879     | 0.0879     | 0.0879        | 0.0813      |
| 0.4 | 0.1047    | 0.1049     | 0.1049     | 0.1049        | 0.0939      |
| 0.5 | 0.1216    | 0.1220     | 0.1220     | 0.1220        | 0.1050      |
| 0.6 | 0.1397    | 0.1404     | 0.1404     | 0.1404        | 0.1150      |
| 0.7 | 0.1610    | 0.1622     | 0.1622     | 0.1623        | 0.1242      |
| 0.8 | 0.1937    | 0.1980     | 0.1979     | 0.1985        | 0.1328      |

Table 1: First-order root-mean-square radius (in fm) of the dressed particle from Eq. (60) for various parametrizations of the retardation function. The heading ‘Feynman’ gives the result in the Feynman parametrization whereas ‘improved’ refers to the ‘improved’ parametrization. In both cases the parameters have been determined from the variational calculation for the nucleon self-energy in Ref. ([15]). The ‘extended’ parametrization was introduced in Ref. ([16]) for additional use in meson-nucleon scattering. The radius calculated with the solution of the variational equations is denoted by ‘variational’. For comparison the perturbative result is also given.

of the retardation function in the quadratic trial action $\mathcal{A}$. The nucleon mass is taken to be 939 MeV, while the meson mass is fixed at 140 MeV. For comparison, the perturbative and zeroth order results are also shown. Table 1 demonstrates quantitatively that the results of the variational calculation are practically independent of which ansatz for the trial action is chosen. Indeed, if one leaves the form of the profile function free and lets the variational principle determine it (corresponding to the ‘variational’ trial action in Ref. [14]), the results for the effective coupling and radius are nearly indistinguishable from those of the ‘improved’ or the ‘extended’ parametrization. Furthermore, the results agree with perturbation theory for small couplings while they deviate quite strongly for large couplings.

Even though there are large numerical differences between the perturbative and variational calculation of the effective coupling and radius, one does not observe a qualitative difference. The situation is dramatically different for the form factor away from zero momentum transfer. The real part is plotted, for three different couplings, as a function of the meson’s momentum transfer squared in Fig. 6. For $q^2 \leq 4M^2$ the representation of Eq. (53) was used, while for $q^2 \geq 4M^2$ we used the analytically continued representation of Eq. (54). The physical region for meson absorption is $q^2 \leq 0$, while $q^2 \geq 4M^2$ corresponds to pair-production. (The unphysical region $0 < q^2 < 4M^2$ is also shown because the matching of the two representations at $q^2 = 4M^2$ provides a useful check on the accuracy of the numerical integrations.)

In general, the perturbative and variational calculations tend to approach each other at

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2See Refs. [14, 15] for more details about these trial actions; the ‘Feynman’ trial action is of the same form as that used in the variational calculation for the polaron [6] while the ‘improved’ trial action has the same ultraviolet behavior as the true action. The ‘extended’ parametrization gives in addition the correct threshold behaviour in zeroth order meson-nucleon scattering [16] and leads to the lowest value of the variational functional among the three parametrizations.
Figure 4: The effective coupling strength as a function of the dimensionless coupling $\alpha$: (a) in zeroth order, (b) in first order of the variational calculation. The solid (dashed) line corresponds to the use of the ‘improved’ (‘Feynman’) trial actions, while the dash-dotted line is the result from the ‘extended’ parametrization. The dotted line denotes the perturbative result.
Figure 5: The mean square radius as a function of the dimensionless coupling $\alpha$: (a) in zeroth order, (b) in first order of the variational calculation. The various curves have the same meaning as those in Fig. [4].
Figure 6: The real part of the amplitude as a function of the momentum transfer squared, for from top to bottom, $\alpha = 0.2$ (a), 0.4 (b) and 0.8 (c). The solid line corresponds to the use of the ‘improved’ trial action, while the dotted line is the perturbative result.
large values of $|q^2|$. In the area just above threshold, however, major differences arise as the coupling is increased. Whereas the perturbative amplitude increases smoothly in magnitude as the coupling is increased from $\alpha = 0.2$ to $\alpha = 0.8$, the variational amplitude decreases from $\alpha = 0.2$ to $\alpha = 0.4$ and actually becomes negative for $\alpha = 0.8$. As the amplitude has to be positive at threshold (Eq. (53) is always positive) this necessitates a rapid oscillation very close to threshold. Furthermore, near threshold the variational amplitude is numerically greatly enhanced (at $\alpha = 0.8$ by orders of magnitude) as compared to the perturbative result.

A somewhat similar behavior is exhibited by the imaginary part of the amplitude, shown in Fig. 7. This is, of course, only non-zero above threshold. Again the variational calculation and perturbation theory approach each other for large $q^2$ and the two calculations are qualitatively similar for the relatively small coupling of $\alpha = 0.2$ (Numerically, though, there is already a factor of two between the two calculations at the peak.) In contrast to the real part of the amplitude, the imaginary part of the variational amplitude is still increasing at modest couplings ($\alpha = 0.4$), but again at $\alpha = 0.8$ large oscillations show up.

The pronounced quantitative and qualitative differences between the leading order perturbative calculation and the variational one are due to the fact that the latter has not been expanded in powers of the coupling constant. Let us write Eq. (54) as

$$A_{2,1}(q^2) = \text{const.} \exp \left\{ \frac{\alpha}{2\pi} \left[ \Re \Xi(q^2, \alpha) + i \Im \Xi(q^2, \alpha) \right] \right\},$$

where the dependence of $\Xi(q^2, \alpha)$ on the coupling arises from the dependence of the variational parameters on the coupling. The perturbative amplitude up to one-loop order is just given by

$$A_{2,1}^{\text{pert.}}(q^2) = \text{const.'} \left\{ 1 + \frac{\alpha}{2\pi} \left[ \Re \Xi(q^2, 0) + i \Im \Xi(q^2, 0) \right] \right\}.$$

In the function $\Xi(q^2, \alpha)$, as opposed to the complete amplitude, there are in fact only quantitative, but not qualitative, differences between the perturbative and the variational calculations. For example, the imaginary parts of $\Im \Xi(q^2, 0)$ and $\Im \Xi(q^2, \alpha = 0.2)$ both rise rapidly from 0 at threshold to less than 20 just above threshold and then slowly decrease. At $\alpha = 0.8$ the same behavior takes place, except this time the maximum of the variational calculation is a little bit more than twice as large. The origin of the qualitative differences between Eqs. (53) and (54) are due to the fact that even though the coefficient $\alpha/2\pi$ is not particularly large (for example, it is only 0.11 for $\alpha = 0.8$) – so one would expect leading order perturbation theory to be quite good – the numerical value of the integral $\Xi(q^2, \alpha)$ more than compensates for it and makes the perturbative result meaningless. In particular, it is clear from Eq. (53) that the real (imaginary) parts of the amplitude will have zeros when $\frac{\alpha}{2\pi} \Im \Xi(q^2, \alpha)$ is $\pi/2$, $3\pi/2$, ... (0, $\pi$, ...) while the perturbative result in Eq. (54) will not. Keeping higher terms in the perturbative expansion may in fact also give rise to zeros, but it is clear that, with $\frac{\alpha}{2\pi} \Im \Xi(q^2, \alpha)$ numerically approaching 5 at the peak for $\alpha = 0.8$, to obtain any reasonable numerical agreement between the perturbative and the variational results one would have to expand the perturbation theory to a very high order indeed.

6 Conclusion and Outlook

The main result of this paper is a compact expression for the truncated Green function of an on-shell scalar nucleon interacting with an arbitrary number of mesons (Eq. (49)), calculated
Figure 7: The imaginary part of the amplitude as a function of the momentum transfer squared, for from top to bottom, $\alpha = 0.2$ (a), 0.4 (b) and 0.8 (c). The solid line corresponds to the use of the ‘improved’ trial action, while the dotted line is the perturbative result.
at lowest non-trivial order in the variational approach of Ref. [14]. A welcome feature of this non-perturbative approach to the relativistic field theoretic strong coupling problem is that the renormalization and the truncation of this general Green function may be done analytically and in a self-consistent manner. Furthermore, by construction the expansion of the Green function in the coupling constant agrees exactly with the result of renormalized 1-loop perturbation theory. Clearly, this assures that the small coupling limit of the calculation is correct numerically. This was an important characteristic in the original application of this variational technique – numerical agreement with both the weak and strong coupling limits was the reason why the famous application to the polaron problem was so successful. Furthermore, in the present case it has the additional consequence that the analytical structure of variational Green functions is at least as rich as that of 1-loop perturbation theory: cuts due to particle production thresholds opening up as well as poles due to particle exchange in the intermediate state are both contained in $G_{2,n}$. Also, of course, the technique is fully covariant; the Green functions may be used for any process, be it scattering or particle production, and indeed are even valid outside the physical regions.

Dependence on the coupling constant of the theory enters the Green function both explicitly, as well as implicitly through the variational parameters. The explicit dependence allows one to identify contributions to individual Feynman diagrams. Indeed, one finds that the expression for the Green function sums up, in an approximate way, Feynman diagrams up to infinite order in perturbation theory (the untruncated Green function in fact contains parts of all possible Feynman diagrams).

Most of the features mentioned above are independent of the functional form of the trial action. The analytic structure of the Green function $G_{2,n}$, however, does seem to depend on this functional form. For the present we have been satisfied with a particular (but still very general) choice for the trial action (Eqs. (23) and (26)) which does allow the non-trivial truncation of the external dressed nucleon propagators to be made. Indeed, it results in the same variational equations which have already been found in Refs. [14, 15] for the propagator, independent of the number of external mesons. This has the great practical advantage in that it means that the determination of the variational parameters only has to be done once. It would be interesting to see if this is a general feature or whether it is specific to the class of trial actions considered here.

In the second part of this paper we investigated the case of meson absorption or emission and nucleon pair-production, i.e. $n = 1$, numerically. The much richer case of particle scattering, $n = 2$, is considered in a following paper. After a trivial rotation to Minkowski space, the general expression for $G_{2,n}$ for $n = 1$ immediately yields the amplitude in the physical region for meson absorption/emission (i.e. $q^2 < 0$) as well as in the unphysical region $0 < q^2 < 4M^2$. The representation is not defined for $q^2 > 4M^2$, but may be analytically continued into this region. One finds a cut starting at the pair production threshold. At this point it should be remembered that we are working in the quenched approximation for the nucleons, so the numerical results in this region will differ to the unquenched ones. Comparison to other quenched calculations, such as perturbation theory, do of course remain meaningful.

We have numerically investigated the vertex function as a function of the momentum transfer and the strength of the coupling. A strong coupling limit of the Wick-Cutkosky model does not exist because of its instability, so a numerical appraisal of the variational method for very large $\alpha$ has to await its application to a different field theory. Nevertheless certain non-
perturbative properties can already be addressed in the range of coupling constants available in the variational treatment of the Wick-Cutkosky model. Among these are the critical coupling and the width of scalar nucleon for even larger coupling constants which have been extracted from the propagator in Ref. [15]. The vertex function evaluated in the present paper provides another example of large nonperturbative effects: even for the relatively ‘small’ couplings allowed in the Wick-Cutkosky model the results differ markedly from what one would obtain in perturbation theory, both quantitatively and qualitatively. The origin of this disagreement may be traced back to the fact that although the coupling is indeed relatively small, the integrals multiplying the coupling are in fact very large, in particular near threshold – both in perturbation theory as well as in the variational calculation. The lesson to be learned, well known of course, is that smallness of the coupling is not a good criterion to determine whether or not to trust a perturbative result. In the present case, (1-loop) perturbation theory is inadequate even for values of the dimensionless coupling constant as small as 0.4.

In conclusion, we think that the variational method described here is a rather promising technique for systematically calculating Green functions in problems where perturbation theory is inadequate. We have shown that it is possible to truncate Green functions with one nucleon and an arbitrary number of external mesons in a manner which is consistent with the variational ansatz. It remains an interesting problem to extend this treatment to several external nucleons and to study bound-state problems. In addition, it seems worthwhile to find out how this approach will fare in realistic field theories. Attempts to apply it in QED are currently under way.

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Appendix : Perturbative Limit

It is a straightforward, but nevertheless useful, exercise to verify that the truncated \((2 + n)\)-point function in Eq. \((49)\) has the correct perturbative limit. We shall sketch the derivation of the perturbative result in this appendix. It suffices to consider a single perturbative diagram \(G_{a,b}^{0,n}(p, p'; \{q_i\})\) of the form shown in Fig. 2. The complete amplitude is then obtained by a summation over all \(a\) and \(b\) as well as a summation over all permutations of the external meson momenta \(q_i\). To start off with, let us consider the case where \(a > b + 1\), i.e. a diagram which has at least one external meson coupling to the nucleon while the internal meson is being exchanged. For this type of diagram one may replace, at this order, the unrenormalized nucleon mass \(M_0\) by the physical mass \(M\). The perturbative truncated Green function corresponding to this situation is given by

\[
G_{a,b}^{0,n}(p, p'; \{q_i\}) = \frac{1}{2} \prod_{i=2}^{b+1} \frac{2g}{p_i^2 + M^2} \prod_{i=a}^{n} \frac{2g}{p_i^2 + M^2} \cdot (2g)^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + m^2} \prod_{i=b+1}^{a} \frac{2g}{(p_i - k)^2 + M^2} \tag{A.1}
\]

It is customary at this stage to rewrite the denominator in terms of integrals involving Feynman parameters. For the purposes of comparison to the variational calculation this is, however, not very useful as the variational calculation gives rise to an expression in terms of integrals over proper time. Although in general there will be a relation between these proper times and the Feynman parameters, it will be non-trivial.

Rather, it is easier to proceed by exponentiating all denominators in the usual way:

\[
\frac{1}{p^2 + M^2} = \frac{1}{2} \int_0^\infty dx e^{-x(p^2 + M^2)/2} \tag{A.2}
\]

and then performing the gaussian integration over the internal meson momentum \(k\). If one exponentiates all propagators in this way one obtains a product of \(n + 2\) integrals, the same number which one obtains if one expands the exponential appearing in the truncated Green’s function in Eq. \((49)\). Indeed, the \(n + 2\) integration parameters of these two expressions may be identified individually if one chooses them in the following way: A nucleon propagator involving \(p_i^2\) or \((p_i - k)^2\) between external mesons should just be exponentiated with the parameter \(\alpha_i\).

The two pairs of nucleon propagators between an external meson and the internal meson need to be exponentiated a little bit differently - one needs to recombine the integration parameters in the following way:

\[
\frac{2}{p_a^2 + M^2} \frac{2}{(p_a - k)^2 + M^2} = \int_0^\infty dx dy \exp \left[-\frac{x}{2}(p_a^2 + M^2)\right] \exp \left[-\frac{y}{2}((p_a - k)^2 + M^2)\right] = \int_0^\infty d\alpha_a \int_{\tau_{a-1}}^{\tau_a} dt_1 \exp \left[-\frac{t_1 - \tau_a}{2}((p_a - k)^2 - p_a^2)\right] \cdot \exp \left[-\frac{\alpha_a}{2}((p_a - k)^2 + M^2)\right] \tag{A.3}
\]
and similarly for the two propagators involving $p_{b+1}$. Here we have used the transformations $\alpha_a = x + y$ and $t_1 = \tau_a - x$, the time $\tau_a$ being defined to be the same sum of $\alpha'_s$ as used in the main text. The reason for this particular transformation should be clear from Fig. 4.

Finally, the integration parameter associated with the internal meson should be transformed in the same way as the parameter $u$ in the main text.

Using these definitions, one readily obtains

$$G_{2,n}^{(a,b)}(p, p'; \{q\}) = 2g \left\{ \prod_{i=2}^{n} g \int_{0}^{\infty} d\alpha_i \exp \left\{ -\frac{\alpha_i}{2}(M^2 + p_i^2) \right\} \right\} \cdot \left\{ 1 \right. \right.$$  

$$- \frac{g^2}{8\pi^2} \alpha_a \int_{0}^{\infty} \frac{d\sigma}{\sigma} \int_{0}^{1} du \exp \left( m\sqrt{\sigma}, M\sqrt{\sigma}, u \right)$$  

$$+ \frac{g^2}{8\pi^2} \int_{\tau_{a-1}}^{\tau_a} \frac{dt_1}{\sigma} \int_{0}^{1} du \exp \left( m\sqrt{\sigma}, \frac{-i\lambda W^{(n)}}{\sqrt{\sigma}}, u \right) \right\}.$$  

Finally then, we obtain the complete perturbative result for the truncated $(2+n)$-point function by summing over all permutations of the external momenta as well as over $a$ and $b$:

$$G_{2,n}^{\text{pert}}(p, p'; \{q\}) = \sum_{a=2}^{n} G_{2,n}^{(a)}(p, p'; \{q\}) + \sum_{b=0}^{n-1} \sum_{a=b+2}^{n+1} G_{2,n}^{(a,b)}(p, p'; \{q\})$$  

$$= 2g \left\{ \prod_{i=2}^{n} g \int_{0}^{\infty} d\alpha_i \exp \left\{ -\frac{\alpha_i}{2}(M^2 + p_i^2) \right\} \right\} \cdot \left\{ 1 \right. \right.$$  

$$- \frac{g^2}{8\pi^2} \int_{0}^{\infty} \frac{d\sigma}{\sigma} \int_{0}^{1} du \sum_{i=2}^{n} \alpha_i \exp \left( m\sqrt{\sigma}, M\sqrt{\sigma}, u \right)$$  

$$+ \frac{g^2}{8\pi^2} \int_{\tau_1}^{\tau_n} \frac{dt_1}{\sigma} \int_{0}^{1} du \exp \left( m\sqrt{\sigma}, \frac{-iW^{(n)}}{\sqrt{\sigma}}, u \right)$$  

$$+ \frac{g^2}{8\pi^2} \int_{\tau_{a-1}}^{\tau_a} \frac{dt_1}{\sigma} \int_{0}^{1} du \exp \left( m\sqrt{\sigma}, \frac{-iW^{(n)}}{\sqrt{\sigma}}, u \right) \right\}.$$  

Remembering that $A(E)$ and $\lambda$ have perturbative expansions of the form $1 + \mathcal{O}(g^2)$ (and hence $\mu^2(\sigma) = \sigma + \mathcal{O}(g^2)$), it is easily checked that this agrees with the corresponding result of the variational calculation given in Eq. (19).
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