THE MAIN COMPONENT OF THE TORIC HILBERT SCHEME

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Abstract

Let $X$ be an affine toric variety with big torus $T \subset X$ and let $T' \subset T$ be a subtorus. The general $T$-orbit closures in $X$ and their flat limits are parametrized by the main component $H_0$ of the toric Hilbert scheme. Further, the quotient torus $T/T'$ acts on $H_0$ with a dense orbit. We describe the fan of this toric variety; this leads us to an integral analogue of the fiber polytope of Billera and Sturmfels. We also describe the relation of $H_0$ to the main component of the inverse limit of GIT quotients of $X$ by $T$.

1 Introduction

The multigraded Hilbert scheme parametrizes, in a technical sense specified below, all homogeneous ideals in a polynomial algebra (or, more generally, in an arbitrary finitely generated algebra) having a fixed Hilbert function with respect to a grading by an abelian group. In [8] it was shown that the multigraded Hilbert scheme always exists as a quasiprojective scheme.

We consider the following case. Let $X$ be an affine toric (not necessarily normal) variety with big torus $T \subset X$ and let $T' \subset T$ be a subtorus acting on $X$ by the restriction of the action of $T$. This defines a grading of the algebra of regular functions $k[X]$ by the group of characters of $T$. Denote by $H_{X,T}$ the toric Hilbert scheme, that is, the multigraded Hilbert scheme parametrizing those $T$-invariant ideals in $k[X]$ having the same Hilbert

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function as the toric $T$-variety $X = \overline{T \cdot x}$, where $x \in X$ lies in the open $T$-orbit [11]. There is a canonical irreducible component $H_0$ of $H_{X,T}$ parametrizing general $T$-orbit closures in $X$ and their flat limits (Proposition 3.6(2)). This component contains an open orbit for a natural action of $T/T$ on $H_{X,T}$. The main result of this work is a description of the fan of this toric variety (Theorem 4.5). Also, we compare the fan of $H_0$ with the fan of the toric Chow quotient.

The Chow quotient of a projective toric variety was considered in [10]. In particular, in this paper there is a description of its fan. Namely, recall that the fan of a projective toric variety is the normal fan of a convex polytope $P$ in the space generated by the lattice of characters of $T$. Let $Q$ be the projection of this polytope on the space $\mathcal{X}(T)_{\mathbb{R}}$ generated by the lattice of characters of the subtorus $T$. Then the fan of the Chow quotient is the normal fan to the fiber polytope $F(P,Q)$ [3], which, in a well-defined sense, is the average over all fibers of the projection of $P$ on $Q$. More generally, the fiber fan for a projection of an arbitrary polyhedron was defined in [4]. In this paper some results of [10] were generalized on the case of a variety that is projective over some affine variety.

In our affine setting, we show that the fan corresponding to the toric variety $H_0$ is the normal fan to the average over all “integral” fibers of the corresponding cone projection. Here by an integral fiber we mean the polyhedron generated by all integral points of a fiber of the projection. Thus this object can be regarded as an integral analogue of the fiber fan. If $X$ is a finite-dimensional $T$-module and the grading of $k[X]$ by the weights of $T$ is positive, then the fan of $H_0$ coincides with the normal fan to the state polytope of Sturmfels (see [12, Theorem 2.5]).

In the last section we consider the toric Chow morphism from the Hilbert scheme to the inverse limit of GIT quotients $X/_T$. This morphism was constructed in [3] Section 5] in the case when $X$ is a finite-dimensional $T$-module. We generalize this to the case of a normal affine toric $T$-variety $X$ (Theorem 5.4).

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2 Terminology and notation

We consider the category of schemes over an algebraically closed field $k$. A variety is a separated integral scheme of finite type. Recall that any scheme $Z$ is characterized by its functor of points from the category of $k$-algebras to the category of sets:

$$Z : k - Alg \rightarrow Set, \quad Z(R) := Mor(\text{Spec} \, R, Z),$$
where \( \text{Mor}(\text{Spec } R, Z) \) is the set of morphisms of schemes over \( k \) from \( \text{Spec } R \) to \( Z \) (we denote the functor of points of a scheme by the corresponding underlined letter). Our main reference on schemes is [5]. We denote by \( \mathcal{O}_Z \) the structure sheaf of \( Z \), and if \( Z \) is affine, then \( k[Z] \) denotes the algebra of sections of \( \mathcal{O}_Z \) over \( Z \). We denote by \( \mathbb{A}^n \) the affine space \( \text{Spec } k[x_1, \ldots, x_n] \).

An \( n \)-dimensional torus \( T \) is an algebraic group isomorphic to the direct product of \( n \) copies of the multiplicative group \( \mathbb{G}_m \) of the field \( k \). For the lattices of characters and one-parameter subgroups of \( T \), we use the notations \( \mathcal{X}(T) = \text{Hom}(T, \mathbb{G}_m) \) and \( \Lambda(T) = \text{Hom}(\mathbb{G}_m, T) \). We denote by \( \langle \cdot, \cdot \rangle \) the natural pairing between \( \mathcal{X}(T) \) and \( \Lambda(T) \). For a lattice \( \mathcal{X} \), let \( \mathcal{X}_\mathbb{R} = \mathcal{X} \otimes_\mathbb{Z} \mathbb{R} \). If \( \Sigma \subset \mathcal{X} \) is a monoid, then \( \text{cone}(\Sigma) \) denotes the cone in \( \mathcal{X}_\mathbb{R} \) generated by \( \Sigma \). For subsets \( D_1, D_2 \) of a vector space, we denote by \( D_1 + D_2 \) the Minkowski sum.

By a toric variety under a torus \( T \) we mean a variety \( X \) such that \( T \) is embedded as an open subset into \( X \), the action of \( T \) on itself by multiplication extends to an action on \( X \), and \( X \) admits an open covering by affine \( T \)-invariant charts. We do not require \( X \) to be normal.

We denote by \( \mathcal{C}_X \) the associated fan of a toric variety \( X \), so the cones of \( \mathcal{C}_X \) lie in \( \Lambda(T)_\mathbb{R} \) (see [7, Sec. 1.4]). The \( T \)-orbits on \( X \) are in order-reversing one-to-one correspondence with the cones of \( \mathcal{C}_X \). If \( \sigma(Y) \) is the cone in \( \mathcal{C}_X \) corresponding to a \( T \)-orbit \( Y \), then a one-parameter subgroup \( \lambda \in \Lambda(T) \) lies in the interior of \( \sigma(Y) \) if and only if \( \lim_{s \to 0} \lambda(s) \) exists and lies in \( Y \). A toric variety is determined by its fan up to normalization.

### 3 Definitions and background on multigraded Hilbert schemes

Let \( X \) be an affine variety over \( k \) with an action of a torus \( T \), so its algebra of regular functions \( S := k[X] \) is graded by the group \( \mathcal{X}(T) \) of characters of \( T \):

\[
S = \bigoplus_{\chi \in \mathcal{X}(T)} S_\chi,
\]

where \( S_\chi \) is the subspace of \( T \)-semiinvariant functions of weight \( \chi \). Let

\[
\Sigma := \{ \chi \in \mathcal{X}(T) \; ; \; S_\chi \neq 0 \}.
\]

This is a finitely generated monoid. Conversely, if \( S \) is a finitely generated commutative \( k \)-algebra without zero divisors graded by \( \mathcal{X}(T) \), then we have a \( T \)-action on the affine variety \( X = \text{Spec } S \).
DEFINITION 3.1. The grading of $S$ by $\mathcal{X}(T)$ is positive if $k[X]_0 = k$ and cone($\Sigma$) is strictly convex.

Notice that in the case of a positive grading there exists a unique minimal system of generators of $\Sigma$. The following definition was introduced in [8].

DEFINITION 3.2. Given a function $h : \mathcal{X}(T) \to \mathbb{N}$, the Hilbert functor is the covariant functor $H^h_{X,T}$ from the category of $k$-algebras to the category of sets assigning to any $k$-algebra $R$ the set of all $T$-invariant ideals $I \subseteq R \otimes_k S$ such that $(R \otimes_k S)/I_\chi$ is a locally free $R$-module of rank $h(\chi)$ for any $\chi \in \mathcal{X}(T)$.

Remark that we can also view $H^h_{X,T}(R)$ as a set of closed $T$-invariant subschemes $Y \subset \text{Spec } R \times X$ such that the projection $Y \to \text{Spec } R$ is flat.

In [8, Theorem 1.1] it was proved that there exists a quasiprojective scheme $H^h_{X,T}$ which represents this functor in the case when $X$ is a finite-dimensional $T$-module $V$. In the case of an arbitrary $X$ there exists a $T$-equivariant closed immersion $X \hookrightarrow V$, where $V$ is a finite-dimensional $T$-module. Then the Hilbert functor $H^h_{X,T}$ is represented by a closed subscheme of $H^h_{V,T}$ (see [1, Lemma 1.6]). Namely, for an algebra $R$ the subset $H^h_{X,T}(R) \subset H^h_{V,T}(R)$ consists of those ideals $I \subset R \otimes_k k[V]$ that $I \in H^h_{V,T}(R)$ and $R \otimes_k I_\chi \subset T$.

Recall that the universal family is the closed subscheme $W_{X,T}$ of $H_{X,T} \times X$ corresponding to the identity map $\{\text{Id} : H_{X,T} \to H_{X,T}\} \in H_{X,T}(H_{X,T}) := \text{Mor}(H_{X,T}, H_{X,T})$. For any $Y \in H_{X,T}(R)$ (so $Y$ is a closed subscheme in Spec $(R \otimes_k S)$) we have $Y = W_{X,T} \times_{H_{X,T}} \text{Spec } R$. In fact, the $k$-rational points of $W$ are those pairs $(y,Y)$, where $Y \in H_{X,T}(k)$ and $y \in Y(k)$.

If $V$ is a finite dimensional $T$-module such that $k[V]^T = k$, then $H^h_{V,T}$ is projective (see [8, Corollary 1.2]) The following lemma generalizes this statement.

LEMMA 3.3. Assume that $h(0) = 1$. Then the morphism

$$p : H^h_{X,T} \longrightarrow X//T := \text{Spec } k[X]^T$$

which assigns to any element $I \in H^h_{X,T}(R)$ the morphism $k[X]^T \to (R \otimes k[X]^T)/I^T \simeq R$, is projective.

PROOF. Since we know that $H^h_{X,T}$ quasi-projective, it is sufficient to check that the valuative criterion of properness for $p$ is satisfied. Let $S$ be the spectrum of a discrete valuation ring $R$ with generic point $\eta$ and closed point $s$. We have to show that any morphism $\phi_\eta : \eta \to H^h_{X,T}$ such that the composition $p \circ \phi : \eta \to X//T$ extends to a morphism $S \to X//T$, extends to a unique morphism $S \to H^h_{X,T}$. Consider $Y_\eta = \eta \times_{H^h_{X,T}} U^h_{X,T} \subset \eta \times X$. By [9, Prop. 9.7], a closed subscheme $Y \subset S \times X$ such that
$Y \times_S \eta = Y_{\eta}$, is flat over $S$ if and only if $Y$ is the closure of $Y_{\eta}$ in $S \times \mathbb{X}$. It follows that the desired extension $Y \in \overline{H^h_{\mathbb{X},T}(S)}$ is unique. For the existence, we consider $Y := \overline{Y_{\eta}} \subset S \times \mathbb{X}$. It remains to show that the fiber $Y_{\eta}$ is non-empty (then, by flatness, it has the Hilbert function $h$). Indeed, we have the following commutative diagram:

\[
\begin{array}{ccc}
Y = \overline{Y_{\eta}} & \subset & S \times \mathbb{X} \\
\downarrow & & \downarrow \\
S = \overline{\eta} & \subset & S \times \mathbb{X}/T,
\end{array}
\]

where the morphism $Y \to S$ is the quotient by $T$, so it is surjective.

We prove the following lemma to treat one particular case of the Hilbert scheme which we shall need later (see the corollary below).

**Lemma 3.4.** Let $P$ be an $\mathbb{N}$-graded algebra: $P = \bigoplus_{r \geq 0} P_r$, and

\[ (*) \quad \text{there exists } r_0 \text{ such that } P_{r+1} = P_1 P_r \text{ for any } r \geq r_0. \]

Consider the Hilbert scheme $H_P$ of the graded algebra $P$ for the Hilbert function

\[ h(r) := \begin{cases} 
1 & \text{if } r \geq 0, \\
0 & \text{otherwise}. 
\end{cases} \]

Let $R$ be an algebra and $Y = \text{Spec } (R \otimes_k P/I) \in H_P(R)$. Then the projection $Y \to \text{Spec } R$ is a locally trivial bundle with fiber $\mathbb{A}^1$.

**Proof.** (1) Consider the open subscheme in $\text{Spec } P$ that is the complement to the subscheme defined by the ideal $\bigoplus_{r \geq 0} P_r$:

\[ (\text{Spec } P)_0 = \{ p \in \text{Spec } P ; \ p \nmid (\bigoplus_{r > 0} P_r) \} \]

and the natural morphism

\[ \psi : (\text{Spec } P)_0 \to \text{Proj } P. \]

Locally $\psi$ is given by the embeddings of algebras $(P_f)_0 \subset P_f$, where $f \in P$ is homogeneous, $\deg f > 0$ (it is clear that the corresponding morphisms of affine schemes satisfy the compatibility conditions). Note that $\psi$ is a locally trivial bundle with fiber $\mathbb{G}_m$. Indeed, condition $(*)$ implies that $\text{Proj } P$ is covered by open affine subschemes $\text{Spec } (P_h)_0$, where $h \in P_1$, and for any $h \in P_1$ we have $P_h = (P_h)_0[h, h^{-1}]$.

(2) Consider $Y_0 = Y \cap (\text{Spec } R \times (\text{Spec } P)_0)$. We have the morphisms

\[ Y_0 \xrightarrow{\delta} \text{Proj } (R \otimes_k P/I) \xrightarrow{\delta} \text{Spec } R. \]
Since $R \otimes_k P/I$ satisfies condition (*), by (1), it follows that $\rho$ is a locally trivial bundle with fiber $\mathbb{G}_m$. So $\delta$ is an isomorphism. Consider the following morphism from $\text{Spec } R$ to $\text{Proj } P$:

$$\text{Spec } R \cong \text{Proj } (R \otimes_k P/I) \subset \text{Spec } R \times \text{Proj } P \xrightarrow{p} \text{Proj } P,$$

where $p$ is the projection.

(a) Note that $Y_0 = (\text{Spec } P)_0 \times_{\text{Proj } P} \text{Spec } R$. Indeed, locally we have

$$R \otimes_k P_f/I_f \cong P_f \otimes (P_f)_0/(I_f)_0,$$

where $f \in P$ is homogeneous of positive degree.

(b) Consider $Y' = Y_0 \times_{\mathbb{G}_m} \mathbb{A}^1$. Here $Y_0 \times_{\mathbb{G}_m} \mathbb{A}^1$ denotes the categorical quotient $(Y_0 \times \mathbb{A}^1)/\mathbb{G}_m$, where $\mathbb{G}_m$ acts on $\mathbb{A}^1$ as follows: $t \cdot s = t^{-1}s$, $t \in \mathbb{G}_m$, $s \in \mathbb{A}^1$. Then $Y'$ is a locally trivial bundle over $\text{Spec } R$ with fiber $\mathbb{A}^1$ and we have the natural morphism $\eta : Y' \to Y$, which is locally given by the homomorphisms

$$R \otimes_k P/I \to \bigoplus_{r \geq 0} (R \otimes_k P_f/I_f)_r,$$

where $f \in P$ is homogeneous of positive degree. So we have a commutative diagram:

$$\begin{array}{ccc}
Y' & \xrightarrow{\eta} & Y \\
\alpha \downarrow & & \alpha \downarrow \\
\text{Spec } R. & & \\
\end{array}$$

Note that for any $r \geq 0$, the corresponding homomorphism $\alpha_*(\mathcal{O}_Y)_r \to \alpha'_*(\mathcal{O}'_Y)_r$, is a surjective homomorphism of locally free sheaves of $R$-modules of rank 1 and, consequently, is an isomorphism. Thus $\eta$ is an isomorphism. □

The statement of the following corollary was given in [8, Section 5] with a proof for algebras generated by elements of degree 1.

**Corollary 3.5.** With the notation of the previous lemma, the Hilbert scheme $H_P$ is isomorphic to $\text{Proj } P$.

**Proof.** We shall show that $\text{Proj } P$ represents the Hilbert functor $H_P$. For this we prove that the tautological bundle over $\text{Proj } P$ is the universal family, i. e., we are going to prove the universal property for $E := (\text{Spec } P)_0 \times_{\mathbb{G}_m} \mathbb{A}^1$. Let $Y = \text{Spec } (R \otimes_k P/I) \in H_P(R)$. We have to show that $Y = E \times_{\text{Proj } P} \text{Spec } R$. Indeed, we have $Y = Y_0 \times_{\mathbb{G}_m} \mathbb{A}^1 = ((\text{Spec } P)_0 \times_{\text{Proj } P} \text{Spec } R) \times_{\mathbb{G}_m} \mathbb{A}^1 = E \times_{\text{Proj } P} \text{Spec } R$. □

Let us return to the case of an affine toric $\mathbb{T}$-variety $X$. We have

$$S = k[X] = \bigoplus_{\nu \in \Omega} S_{\nu},$$
where $\Omega \subset \mathcal{X}(\mathbb{T})$ is a finitely generated monoid and $S_\nu$ is the subspace of $\mathbb{T}$-semiinvariant functions of weight $\nu \ (\dim S_\nu = 1)$. Let $T \subset \mathbb{T}$ be a subtorus. We have a surjective linear map $\pi : \mathcal{X}(\mathbb{T}) \to \mathcal{X}(T)$ given by the restriction. The action of $T$ on $X$ arising from the action of $\mathbb{T}$ gives a grading
\[ S = \bigoplus_{\chi \in \Sigma} S_\chi, \]
where $\Sigma = \pi(\Omega)$. We shall consider the following Hilbert function:
\[ h(\chi) := \begin{cases} 1 & \text{if } \chi \in \Sigma, \\ 0 & \text{otherwise}. \end{cases} \]

Let $H_{X,T}$ be the corresponding Hilbert scheme (we shall also denote it by $H_{S,T}$). Note that all the ideals $I \in H_{S,T}(k)$ are binomial (see [6, Proposition 1.11]). If $x \in X$ lies in the open $T$-orbit, then we have the point $X := T \cdot x \in H_{X,T}(k)$.

The group $\mathbb{T}(R)$ acts on $H_{X,T}(R)$ in the natural way. Namely, we have an action of $\mathbb{T}(R)$ on $R \otimes_k S$ : for $f \in R \otimes_k S_\nu$, where $\nu \in \Omega$, and $t \in \mathbb{T}(R)$ let $t \cdot f = \nu(t)f$. Hence for $I \in H_{X,T}(R)$ let $t \cdot I = \{t \cdot f ; f \in I\}$. These actions commute with base extensions, thus we have an action of $\mathbb{T}$ on $H_{X,T}$. Since $T$ acts trivially, this yields an action of the torus $\mathbb{T}/T$. The universal family $\mathbb{W}_{X,T}$ is invariant under the diagonal action of $\mathbb{T}$ on $H_{X,T} \times X$.

Let $H_0$ be the toric orbit closure $\mathbb{T} \cdot X \subset H_{X,T}$, and denote by $\mathbb{W}_0$ its preimage under the projection
\[ p : \mathbb{W}_{X,T} \to H_{X,T} \]
(we consider $H_0$ and $\mathbb{W}_0$ with their structure of reduced schemes).

**Proposition 3.6.** (1) The stabilizer of $X$ under the action of $\mathbb{T}$ on $H_0$ is $T$. Moreover, $H_0$ is a toric variety under the torus $\mathbb{T}/T$.

(2) The orbit $\mathbb{T} \cdot X$ is open in $H_{X,T}$. Consequently, $H_0$ is an irreducible component of $H_{X,T}$.

(3) $\mathbb{W}_0$ is a toric variety under the torus $\mathbb{T}$ (and, consequently, $\mathbb{W}_0$ is an irreducible component of $\mathbb{W}_{X,T}$).

**Proof.** (1) If $t \cdot X = X$ for $t \in \mathbb{T}$, then $t \cdot x \in T \cdot x$ and $t \in T$. So we have only to show that $H_{X,T}$ admits an open covering by affine $T$-invariant charts. Indeed, let $\chi \in \Sigma$. Then for any $I \in H_{X,T}(R)$ the locally trivial $R$-module $(R \otimes k[\mathbb{X}]_\chi)/I_\chi$ defines a morphism from $\text{Spec } R$ to the projectivisation $\mathbb{P}(k[\mathbb{X}]_\chi^*) = \text{Proj } (\text{Sym}(k[\mathbb{X}]_\chi))$, where $\text{Sym}(k[\mathbb{X}]_\chi)$ denotes the symmetric algebra. These maps commute with base changes and, consequently, define a morphism $p_\chi : H_{X,T} \to \mathbb{P}(k[\mathbb{X}]_\chi^*)$. Note that $p_\chi$ is $\mathbb{T}$-equivariant (the action of $\mathbb{T}$ on $\mathbb{P}(k[\mathbb{X}]_\chi^*)$ is induced by the linear action of $\mathbb{T}$ on $k[\mathbb{X}]_\chi^*$). By [8, Proposition 3.2, Corollary...
3.4], it follows that there exists a finite set of characters \( \chi_1, \ldots, \chi_r \in \Sigma \) such that the morphism

\[
p \times p_{\chi_1} \times \ldots \times p_{\chi_r} : H_{X,T} \to \mathbb{X}/T \times \mathbb{P}(k[\mathbb{X}]_{\chi_1}^*) \times \ldots \times \mathbb{P}(k[\mathbb{X}]_{\chi_r}^*)
\]

is injective. Since the morphism \( p \) is projective (Lemma 3.3), it follows that \( p \times p_{\chi_1} \times \ldots \times p_{\chi_r} \) is a closed embedding. Since any \( \mathbb{P}(k[\mathbb{X}]_{\chi_i}^*) \) admits an open covering by \( T \)-invariant affine charts, it follows that \( H_{X,T} \) does.

(2) We shall prove that \( \mathbb{T} \cdot X \) is open in \( H_{X,T} \). Since the stabilizer of \( X \) in \( T \) is \( T \), it suffices to prove that \( \dim T_X H_{X,T} \leq \dim \mathbb{T} \cdot X = \dim \mathbb{T} - \dim T \), where \( T_X H_{X,T} \) denotes the tangent space to \( H_{X,T} \) at \( X \). By [8, Prop. 1.6], we have

\[
T_X H_{X,T} = \text{Hom}_{k[\mathbb{X}]}(I_X, k[X])_0.
\]

This vector space is isomorphic to

\[
\text{Hom}_{k[\mathbb{T}]}(I_T, k[T])_0 = \text{Hom}_{k[\mathbb{T}]}(I_T/I_T^2, k[T])_0,
\]

where \( I_T \) is the ideal of functions in \( k[\mathbb{T}] \) vanishing on \( T \). Indeed, since \( (I_X)_\chi \subset k[\mathbb{X}]_{\chi}(I_T)_0 \), for any \( \phi \in \text{Hom}_{k[\mathbb{T}]}(I_T, k[T])_0 \) we have \( \phi(I_X) \subset k[\mathbb{X}]\phi((I_T)_0) = k[\mathbb{X}] \). Conversely, \( I_T = k[\mathbb{T}]I_X \), so any \( \phi \in \text{Hom}_{k[\mathbb{X}]}(I_X, k[X])_0 \) can be extended to a homomorphism of \( k[\mathbb{T}] \)-modules from \( I_T \) to \( k[T] \).

Further, we can choose coordinates on \( \mathbb{T} \) such that

\[
k[\mathbb{T}] = k[t_1, t_1^{-1}, \ldots, t_m, t_m^{-1}, s_1, s_1^{-1}, \ldots, s_r, s_r^{-1}],
\]

where \( r = \dim \mathbb{T} - \dim T \), and the ideal \( I_T \) is generated by \( s_i - 1 \) for \( i = 1, \ldots, r \). The linear space \( I_T \) is spanned by the elements \( t_1^{a_1} \ldots t_m^{a_m} s_1^{b_1} \ldots s_m^{b_m} (s_i - 1) \), where \( a_i, b_j \in \mathbb{Z} \), and the projections of the elements \( t_1^{a_1} \ldots t_n^{a_n} (s_i - 1) \) span the linear space \( I_T/I_T^2 \) (since \( s_i(s_j - 1) = (s_j - 1) + (s_i - 1)(s_j - 1) \) and \( s_i^{-1}(s_j - 1) = (s_j - 1) - s_i^{-1}(s_i - 1)(s_j - 1) \)). Hence a homomorphism of \( k[\mathbb{T}] \)-modules from \( I_T \) to \( k[T] \) is uniquely determined by the images of \( s_i - 1 \). Thus the dimension of the vector space of such homomorphisms of degree zero is not greater than \( r \).

(3) Consider the restriction \( p_0 \) of \( p \) to \( \mathbb{W}_0 \):

\[
p_0 : \mathbb{W}_0 \to H_0.
\]

This is a flat morphism. By Lemma 3.7 below and [9 Corollary 9.6], the dimension of any irreducible component \( Z \) of \( \mathbb{W}_0 \) is equal to \( \dim \mathbb{T} \). This implies that \( p_0(Z) = H_0 \) and \( Z \subset p^{-1}(\mathbb{T} \cdot X) \). Thus \( \mathbb{W}_0 = p^{-1}(\mathbb{T} \cdot X) = \overline{\mathbb{T} \cdot (x, X)} \) is irreducible and \( \mathbb{T} \cdot (x, X) \subset \mathbb{W}_0 \) is dense and, consequently, open. Since \( \mathbb{W}_0 \) is a closed subscheme in \( H_0 \times \mathbb{X} \), it follows that \( \mathbb{W}_0 \) admits an open covering by affine \( T \)-invariant charts. \( \square \)
Lemma 3.7. For any point $Y \in H_{X,T}$, the dimension of any irreducible component of its fiber $p^{-1}(Y)$ equals $\dim T$.

Proof. We denote by $k(Y)$ the residue field of $Y \in H_{X,T}$. Then we have

$$p^{-1}(Y) = \text{Spec } k(Y) \times_{H_{X,T}} \mathbb{W}_{X,T} = \text{Spec } L,$$

where $L$ is a coherent sheaf of $\Sigma$-graded $k(Y)$-algebras:

$$L = \bigoplus_{\chi \in \Sigma} L_{\chi},$$

and $L_{\chi} := k(Y) \otimes_{\mathcal{O}_{H_{X,T}}} (\mathcal{O}_{\mathbb{W}_{X,T}})_{\chi}$ is isomorphic to $k(Y)$. Let

$$\Sigma_{\text{red}} := \{ \chi \in \Sigma ; L_{\chi} \text{ is not nilpotent} \}.$$

Note that $\text{cone}(\Sigma_{\text{red}}) = \text{cone}(\Sigma)$. Every point $Y \in H_{X,T}$ gives us a subdivision of $\text{cone}(\Sigma)$ into subcones, namely two points $\chi, \chi' \in \Sigma_{\text{red}}$ lie in the same cone if and only if $L_{\chi}L_{\chi'} \neq 0$.

The irreducible components $Z$ of $p^{-1}(Y)$ correspond to the maximal cones $C$ of this subdivision:

$$Z = \text{Spec } \left( \bigoplus_{\chi \in \Sigma_{\text{red}} \cap C} L_{\chi} \right).$$

Note that $\Sigma_{\text{red}} \cap C$ is a monoid. It suffices to prove that the dimension of $Z$ is equal to $\dim T$. We can extend the action of $T$ on $Z$ to an action of the torus $T \times \text{Spec } k(Y)$ (over the field $k(Y)$). Thus $Z$ is a toric variety under the torus $T \times \text{Spec } k(Y)$ and $\dim Z = \dim C = \dim T$.

4 Fan of a toric Hilbert scheme

Our aim is to describe the fans of the toric varieties $H_0$ and $\mathbb{W}_0$.

Let us fix the notations. Recall that $X$ is an affine toric variety under an action of a torus $T$:

$$X = \overline{T \cdot x_0},$$

$T \subset T$ is a subtorus, and

$$\pi : \mathcal{X}(T) \rightarrow \mathcal{X}(T)$$

is the restriction map. Fix isomorphisms $T \simeq \mathbb{G}_m^n$, $T \simeq \mathbb{G}_m^r$, this gives us a basis in $k[T]$ and $k[T]$:

$$k[T] = \bigoplus_{\nu \in \mathcal{X}(T)} kt^\nu, \quad k[T] = \bigoplus_{\chi \in \mathcal{X}(T)} kt^\chi.$$
We denote by $X$ the $T$-orbit closure $T \cdot x_0$ and $I_X \subset k[X]$ denotes the corresponding ideal. Also, we have

$$S = k[X] = \bigoplus_{\nu \in \Omega} kt^\nu, \quad k[X] = \bigoplus_{\chi \in \Sigma} kt^\chi.$$ 

The restriction homomorphism $k[X] \to k[X]$ is given by $t^\nu \to t^{\pi(\nu)}$ and its kernel $I_X$ is generated by all the binomials of the form $t^\nu_1 - t^\nu_2$ such that $\pi(\nu_1) = \pi(\nu_2)$ (see [12, Lemma 4.1]).

Let us recall some definitions concerning convex polyhedra. They are taken from [12], which we shall use as a general reference on convex polyhedra.

**Definition 4.1.** Let $P$ be a convex polyhedron in a vector space $V$. For any face $F$ of $P$ the normal cone $N_F(P)$ is the following cone in the dual vector space $V^*$:

$$N_F(P) := \{ l \in V^* : l(v - v') \geq 0 \text{ for all } v \in P, v' \in F \}.$$ 

The normal fan $N(P)$ of $P$ is the fan whose cones are normal cones to the faces of $P$.

**Definition 4.2.** The recession cone of a polyhedron $P \subset V$ is the set of those vectors $v \in V$ such that $u + v \in P$ for any $u \in P$.

**Definition 4.3.** A fan $C_1$ is a refinement of a fan $C_2$ if any cone of $C_1$ is contained in some cone of $C_2$.

**Definition 4.4.** We say that two polyhedra $P_1, P_2 \subset X(T)_\mathbb{R}$ are equivalent if they have the same normal fan.

We fix an open $T$-equivariant embedding of $T/T$ (resp. of $T$) in $H_0$ (resp. in $W_0$) such that the image of $eT$ (resp. of $e$) is $X$ (resp. $(X, x_0)$), where $e$ is the unit in $T$.

**Theorem 4.5.** (1) The fan $C_{H_0} \subset \Lambda(T)_{\mathbb{R}}$ of the toric $T/T$-variety $H_0$ is the coarsest common refinement of the normal fans $C_X$ of the polyhedra

$$P_X := \text{conv}(\pi^{-1}(\chi) \cap \Omega) \subset X(T)_{\mathbb{R}},$$

where $\chi \in \Sigma$.

(2) The fan of $\mathbb{W}_0$ is the coarsest common refinement of the fans $C_{H_0}$ and $N(\text{cone}(\Omega))$.

**Remark 4.6.** We can consider fans in $\Lambda(T/T)_{\mathbb{R}}$ as fans in $\Lambda(T)_{\mathbb{R}}$ whose cones contain $\Lambda(T)_{\mathbb{R}}$. In particular, we view the fan of the toric $T/T$-variety $H_0$ as a fan in $\Lambda(T)_{\mathbb{R}}$.

**Proof.** Let us recall that a one-parameter subgroup $\lambda \in \Lambda(T)_{\mathbb{R}}$ belongs to the support of the fan $C_{H_0}$ if and only if there exists a limit of $X \in H_0$ under $\lambda$. Further, one-parameter
subgroups $\lambda, \lambda' \in \Lambda(\mathbb{T})_R$ lie in the interior of the same cone of $C_{H_0}$ if and only if they define the same limit of $X \in H_0$.

We shall calculate the limit of $X$ under a one-parameter subgroup $\lambda \in \Lambda(\mathbb{T})$. Consider the closed embedding

$$G_m \times X \subset G_m \times X,$$

$$(s, x) \rightarrow (s, \lambda(s) \cdot x).$$

Let $\Xi$ be the closure of the image of this embedding in $A^1 \times X$ (so $\Xi$ is a variety). Since the projection $p_{A^1} : \Xi \rightarrow A^1$ is a flat morphism, we have a morphism $A^1 \rightarrow H_{X,T}$ such that $\Xi = W_{X,T} \times_{H_{X,T}} A^1$. Thus the limit $X_\lambda$ of $X$ under $\lambda$ is equal to the fiber of $p_{A^1}$ over 0 if this fiber is non-empty and the limit does not exist otherwise. Consider the commutative diagram:

\[ \Xi \supset G_m \times X \cap A^1 \times X \supset G_m \times X. \]

We have the corresponding homomorphisms of algebras:

\[ k[\Xi] \rightarrow k[G_m \times X] \]

\[ k[\Lambda^1 \times X] \rightarrow k[G_m \times X], \]

where the vertical maps are surjective.

Denote by $s$ the coordinate in $A^1$. Then the homomorphism $k[G_m \times X] \rightarrow k[G_m \times X]$ is given by $s \rightarrow s$ and $t^\nu \rightarrow s^{(\lambda, \nu)} t^{\pi(\nu)}$. Thus, the vector subspace $k[\Xi] \subset k[G_m \times X]$ is generated by the elements of the form $s^m t^{\pi(\nu)}$, where $m \geq \langle \lambda, \nu \rangle$, $\nu \in \Omega$. The fiber $p^{-1}_{A^1}(0)$ is empty if and only if the ideal $sk[\Xi]$ contains 1. Thus $\lambda$ belongs to the support of the fan $C_{H_0}$ if and only if $\langle \lambda, \nu \rangle \leq 0$ for any $\nu \in \pi^{-1}(0) \cap \Omega$. Since any $k[X_\chi]$ is a finitely generated $k[X]$-module, this is equivalent to say that $\lambda$ attains its minimum on $\pi^{-1}(\chi) \cap \Omega$ for any $\chi \in \Sigma$. In this case

$$k[X_\lambda] = \bigoplus_{\chi \in \Sigma} ks^{n_\lambda(\chi)} t^\chi,$$

where

$$n_\lambda(\chi) := \min_{\nu \in \pi^{-1}(\chi) \cap \Omega} \langle \lambda, \nu \rangle.$$

The product $s^{n_\lambda(\chi_1)} t^{\lambda_1} s^{n_\lambda(\chi_2)} t^{\chi_2} = s^{n_\lambda(\chi_1) + n_\lambda(\chi_2)} t^{\lambda_1 + \chi_2}$ equals zero if and only if $n_\lambda(\chi_1) + n_\lambda(\chi_2) > n_\lambda(\chi_1 + \chi_2)$. The embedding of $X_\lambda$ in $X$ is given by the homomorphism of algebras $k[X] \rightarrow k[X_\lambda]$, where $t^\nu, \nu \in \Omega$, maps to $s^{n_\lambda(\pi(\nu))} t^{\pi(\nu)}$ if $\langle \lambda, \nu \rangle = n_\lambda(\chi)$, and to 0 otherwise. We denote by $I_\lambda$ the kernel of this homomorphism. Hence we see that one-parameter subgroups $\lambda_1$ and $\lambda_2$ define the same limit if and only if $I_{\lambda_1} = I_{\lambda_2}$. This
holds if and only if $\lambda_1$ and $\lambda_2$ attain the minimum over $\pi^{-1}(\chi) \cap \Omega$ at the same point for any $\chi \in \Sigma$ or, equivalently, $\lambda_1$ and $\lambda_2$ lie in the interior of the same cone of $N(P_\chi)$ for any $\chi \in \Sigma$.

(2) Since $W_0 = T \cdot (X,x) \subset H_0 \times \mathbb{X}$, the second statement is evident.

The statement below follows directly from the description of the limit of $X$ under $\lambda$ in the proof of the theorem.

**Remark 4.7.** Let $\prec_\lambda$ be the preorder on $\mathcal{X}(\mathbb{T})$ such that $\nu_1 \prec_\lambda \nu_2$ if $\langle \lambda, \nu_1 \rangle \leq \langle \lambda, \nu_2 \rangle$. For any $f = \sum f_{\nu_i} \in k[\mathbb{X}]_{\nu_i}$, denote by $\text{in}_\lambda(f)$ the sum of $f_{\nu_i}$s where $\nu_i$ is maximal with respect to $\prec_\lambda$. Then the limit of $I_X \in H_0$ under $\lambda$ exists if and only if $\langle \lambda, \nu \rangle \geq 0$ for any $\nu \in \pi^{-1}(0) \cap \Omega$. In this case the limit is the ideal $\text{in}_\lambda(I_X)$ generated by all $\text{in}_\lambda(f), f \in I_X$.

**Example 4.8.** Let $\mathbb{X} = \mathbb{A}^n$, $T = G_m^n$ act on $\mathbb{A}^n$ by rescaling of coordinates, $T = G_m^n$, and let the $\mathcal{X}(T)$-grading of $k[x_1, \ldots, x_n]$ be positive.

(1) Consider the case $n = 3$. It was proved by Arnold, Korkina, Post, Roelfs (see, for example, [12, Theorem 10.2]), that any ideal $I \in H_{\mathbb{A}^n,T}(k)$ is of the form $t \cdot \text{in}_\lambda(I_X)$ for some $t \in G_m^n$ and $\lambda \in \Lambda(G_m^n)$. This means that in this case the toric Hilbert scheme is irreducible.

(2) Let $n = 4$ and $\chi_1 = 1, \chi_2 = 3, \chi_3 = 4, \chi_4 = 7$. Then the toric Hilbert scheme is reducible. Moreover, in $H_{\mathbb{A}^n,T}$ there are infinitely many orbits of $G_m^n$ (see [12, Theorem 10.4]).

**Proposition 4.9.** (1) The support of any $C_\chi$ is the cone generated by those one-parameter subgroups $\lambda$ that $\langle \lambda, \nu \rangle \geq 0$ for any $\nu \in \pi^{-1}(0) \cap \Omega$. In particular, the grading of $S$ by $\mathcal{X}(T)$ is positive if and only if this support is the whole space $\Lambda(T)_R$, i.e., any polyhedron $P_\chi$ is a polytope. This holds if and only if $H_0$ is projective.

(2) There are only finitely many non-equivalent polyhedra $P_\chi$ for $\chi \in \Sigma$. Hence $C_{H_0}$ is the normal fan of the Minkowski sum of representatives of the equivalence classes (we denote this sum by $P_{H_0}$).

**Proof.** (1) First note that $P_0$ is a cone and its normal cone $C_0$ is generated by those one-parameter subgroups $\lambda$ that $\langle \lambda, \nu \rangle \geq \langle \lambda, 0 \rangle = 0$ for any $\nu \in \pi^{-1}(0) \cap \Omega$. Further, note that the recession cone of any $P_\chi$ is $P_0$. Indeed, $S_\chi$ is a finitely generated $S_0$-module. Let $\mu_1, \ldots, \mu_d \in \mathcal{X}(\mathbb{T})$ be the weights of a set of $\mathbb{T}$-semiinvariant generators. Then

$$P_\chi = \text{conv}(\bigcup_{i=1}^d (\mu_i + P_0)) = \text{conv}(\mu_1, \ldots, \mu_d) + P_0.$$ 

It follows that the support of $C_\chi$ is $C_0$. 


If the support of $C_{H_0}$ is not $\Lambda(T)_\mathbb{R}$, then $H_0$ is not complete and, consequently, is not projective. Conversely, if the grading is positive, then the Hilbert scheme $H_{\mathbb{X},T}$ is projective, and $H_0$ is projective.

(2) There are only finitely many fans $\mathcal{C}$ such that $C_{H_0}$ is a refinement of $\mathcal{C}$ and the supports of $\mathcal{C}$ and $C_{H_0}$ coincide. \hfill \Box

**Remark 4.10.** By [12, Theorem 7.15], it follows that in the case when $\mathbb{X} = \mathbb{A}^n$, $T = G_m^n$ acts by rescaling of coordinates, and the $\mathcal{X}(T)$-grading of $k[\mathbb{X}]$ is positive, the polytope $P_{H_0}$ is equivalent to the Minkowski sum of $P_\chi$ corresponding to the weights $\chi$ of the elements of the universal Gröbner basis of $I_\mathbb{X}$.

Let $\mathbb{X}$ be normal. Now we are going to give a precise description of those characters $\chi \in \Sigma$ having equivalent polyhedra $P_\chi$. Recall that we have a homomorphism of lattices $\pi: \mathcal{X}(T) \to \mathcal{X}(T)$, a finitely generated monoid $\Omega \subset \mathcal{X}(T)$ such that $\Omega = \text{cone}(\Omega) \cap \mathcal{X}(T)$, and we put $\Sigma = \pi(\Omega)$. To any point $\chi \in \Sigma$ we associate the polyhedron

$$P_\chi = \text{conv}(\pi^{-1}(\chi) \cap \Omega) \subset \mathcal{X}(T)_\mathbb{R}.$$ 

Two points $\chi, \chi' \in \Sigma$ are said to be equivalent if the corresponding polyhedra $P_\chi$ and $P_{\chi'}$ are equivalent. The question is to describe equivalence classes constructively.

Denote by $\pi_\mathbb{R}$ the linear map induced by $\pi$:

$$\pi_\mathbb{R}: \mathcal{X}(T)_\mathbb{R} \to \mathcal{X}(T)_\mathbb{R}.$$ 

Let $C_\chi^\mathbb{R}$ denote the normal fan to the polyhedron

$$P_\chi^\mathbb{R} := \pi_\mathbb{R}^{-1}(\chi) \cap \text{cone}(\Omega).$$

**Definition 4.11.** (see [4]) The cell decomposition of cone($\Sigma$) induced by $\pi_\mathbb{R}$ is the subdivision of cone($\Sigma$) into the following set of cones: the characters $\chi$ and $\chi'$ lie in the interior of the same cone of this decomposition if and only if the set of those faces of $\Omega_{R,+}$ whose images under $\pi_\mathbb{R}$ contain $\chi$ coincides with the set of such faces for $\chi'$.

**Remark 4.12.** Note that the cell decomposition of cone($\Sigma$) induced by $\pi_\mathbb{R}$ coincides with the subdivision by GIT-cones ([2, Section 2]).

Note that if $\chi$ lies in the interior of a cone $\sigma$ of the cell decomposition and $\chi' \in \sigma$, then $C_\chi^\mathbb{R}$ refines $C_{\chi'}^\mathbb{R}$. In particular, the polyhedra $P_\chi^\mathbb{R}$ corresponding to interior points $\chi$ of $\sigma$ are equivalent. Let $P_\mathbb{R}$ denote the Minkowski sum of $P_\chi^\mathbb{R}$ for representatives of interior points for all cones of the cell decomposition and let $C_\mathbb{R}$ denote the normal fan to $P_\mathbb{R}$ (note that in the Minkowski sum it suffices to take representatives of interior points for the maximal cones of the cell decomposition).
Remark 4.13. In [4] the fan $C_R$ is called the fiber fan by analogy with the normal fan of the fiber polytope for a linear projection of polytopes (see [3]).

Definition 4.14. (See [8] Definition 5.4.) A character $\chi \in \Sigma$ is integral if the inclusion of the convex polyhedra $P_\chi \subseteq P^R_\chi$ is an equality.

We shall denote by $\Sigma^\text{int}_\chi$ the set of integral characters. The following proposition gives us an algorithm for computing the fan of $H_0$.

Proposition 4.15. For any cone $\sigma$ of the cell decomposition of cone$(\Sigma)$ induced by $\pi$ let $\mu_1, \ldots, \mu_r$ be generators of the monoid $\sigma \cap \Sigma$ and let $c_1, \ldots, c_r \in \mathbb{N}$ be such that $c_i \mu_i$ are integral, $i = 1, \ldots, r$. Then, the polyhedra $P_\chi$, where $\chi = \sum_{i=1}^r d_i \mu_i$ and $0 < d_i < l(\sigma) c_i$, form representatives of all equivalence classes of points in $\sigma$ up to Minkowski sum with $P_R$. Here $l(\sigma)$ is the number of vertices of $P^R_\chi$ for $\chi$ lying in the interior of $\sigma$.

Hence $P_{H_0}$ is the Minkowski sum of such representatives for all (maximal) cones $\sigma$ of the cell decomposition of cone$(\Sigma)$ induced by $\pi_R$.

Proof. Consider a point $\chi$ lying in the interior of $\sigma$ and the corresponding polyhedron $P^R_\chi$. For any vertex $v$ of $P^R_\chi$ there exists a unique minimal face $F$ of cone$(\Omega)$ such that $F \cap P^R_\chi = \{v\}$ (indeed, since $P^R_\chi$ is the intersection of cone$(\Omega)$ with the affine subspace $\pi^{-1}_R(\chi)$, it follows that any face of $P^R_\chi$ is the intersection of $\pi^{-1}_R(\chi)$ with some face of cone$(\Omega)$). Let $v^\chi_1, \ldots, v^\chi_{l(\sigma)} \in \mathcal{X}(\mathbb{T})_R$ be the vertices of $P^R_\chi$ and let $F^\sigma_1, \ldots, F^\sigma_{l(\sigma)}$ be the corresponding faces (the set of such faces does not depend on a point $\chi$ in the interior of $\sigma$). Note also that the intersection $F_i \cap P^R_\chi$ is a vertex of $P^R_\chi$ for any $\chi \in \sigma$. Just as above, we denote this vertex by $v^\chi_i$. For two vectors $u, u' \in \mathcal{X}(\mathbb{T})_R$ we say $u \prec u'$ if $u' - u \in \text{cone}(\Omega)$.

Let us show that if $\chi = \sum_{i=1}^r d_i \mu_i$ lies in the interior of $\sigma$ and there exists $i$ such that $d_i \geq c_i l(\sigma)$, then

\[ P_\chi = P_{\chi - c_i \mu_i} + P_{c_i \mu_i}. \]

Indeed, the inclusion $P_{\chi - c_i \mu_i} + P_{c_i \mu_i} \subseteq P_\chi$ is evident. For the converse, it is sufficient to show that $P_\chi \cap \Omega \subseteq P_{\chi - c_i \mu_i} + P_{c_i \mu_i}$. Note that $P_\chi \cap \Omega = P^R_\chi \cap \Omega$. Denote by $D_\chi$ the convex hull of the $v^\chi_i$, $i = 1, \ldots, l(\sigma)$. By Proposition 4.9 (1), $P^R_\chi = D_\chi + P_0$. Then for any $v \in P_\chi \cap \Omega$ we have $v = u + v_0$ for some $v_0 \in P_0, u \in D_\chi$ and $u = \sum_{j=1}^{l(\sigma)} q_j v^\chi_j$ for some $q_j \geq 0$ such that $\sum_{j=1}^{l(\sigma)} q_j = 1$. There exists $j$ such that $q_j \geq 1/l(\sigma)$. Hence $v \succ q_j v^\chi_j = q_j (v^\chi_j - c_i \mu_i + v^\chi_j c_i \mu_i) \succ v^\chi_j c_i \mu_i$. Thus $v - v^\chi_j c_i \mu_i \in \text{cone}(\Omega) \cap \mathcal{X}(\mathbb{T}) = \Omega$.

In particular, this implies that for any $\chi$ in the interior of $\sigma$ the polyhedron $P_\chi + P_R$ is equivalent to $P_{\chi'} + P_R$ for some $\chi' = \sum_{i=1}^r d_i \mu_i$ such that $d_i \prec c_i l(\sigma)$ for any $i$. The second statement of the proposition is evident. \qed
**Corollary 4.16.** With the preceding notation, if \( \chi = \sum_{i=1}^{r} d_i \mu_i \) lies in the interior of \( \sigma \) and there exists \( i \) such that \( d_i \geq c_i l(\sigma) \), then \( P_\chi \) is equivalent to \( P_{\chi+c_i \mu_i} \).

**Proof.** By \((*)\), it follows that \( P_{\chi+c_i \mu_i} = P_{\chi-c_i \mu_i} + 2P_{c_i \mu_i} \) is equivalent to \( P_\chi \).

**Example 4.17.** Let \( X = \mathbb{A}^n, T = \mathbb{G}^n_m \) act be rescaling of coordinates, and let \( T = \mathbb{G}^n_m \) act on \( \mathbb{A}^n \) with characters \( \chi_1, \ldots, \chi_n \in \mathbb{Z} \). Then \( \Omega \subset \mathbb{Z}^n \) is the set of vectors with integral non-positive coordinates, and \( \Sigma \subset \mathbb{Z} \) is the monoid generated by \( -\chi_i \). Moreover, \( \Sigma = (\Sigma \cap \mathbb{Z}_+) \cup (\Sigma \cap \mathbb{Z}_-) \) is the subdivision of \( \Sigma \) induced by \( \pi \). Let \( n_+ \) and \( n_- \) be the numbers of positive and negative \( \chi_i \) respectively. A number \( \chi \in \mathbb{Z}_+ \) (resp. \( \mathbb{Z}_- \)) is integral (in the sense of Definition 4.14) if and only if \( \chi \) is divisible by any \( \chi_i < 0 \) (resp. \( > 0 \)). Let \( \chi_+ \) (resp. \( \chi_- \)) be the least common (positive) multiple of all positive (resp. negative) \( \chi_i \). Then \( P_{H_0} \) is the Minkowski sum of polyhedra \( P_\chi \) for \( -n_+ \chi_+ < \chi < n_- \chi_- \).

## 5 Toric Chow morphism

We are going to describe the toric Chow morphism from the Hilbert scheme to the inverse limit of GIT quotients \( X/\chi T \). In [8, Section 5] the toric Chow morphism was constructed in the case when \( X = \mathbb{A}^n \) is a \( T \)-module. We generalize this to the case of a normal affine toric \( T \)-variety \( X \).

In this section we fix a \( T \)-equivariant closed embedding \( X \hookrightarrow V \), where \( V \) is a finite-dimensional \( T \)-module such that \( X \) is not contained in a proper \( T \)-submodule. We use the notations of the previous sections. Let

\[
S^{(\chi)} := \bigoplus_{r=0}^{\infty} S_{r\chi},
\]

and let

\[
X/\chi T := \text{Proj} S^{(\chi)}
\]

be the GIT quotient. In particular, \( X/\chi T = X/\chi T = \text{Spec} \ S_0 \). Notice also that \( X/\chi T = X^{ss}/\chi T \), where

\[
X^{ss} := \{ x \in X; f(x) \neq 0 \text{ for some homogeneous } f \in S^{(\chi)} \}.
\]

If \( \chi \) lies in the interior of \( \text{cone}(\Sigma) \), then \( X/\chi T \) is a normal toric \( T/T \)-variety whose fan is \( C_\chi^R \), the normal fan to the polyhedron \( P_\chi^R \).

It is easy to see that for any \( \chi_1, \chi_2 \in \Sigma \), the inclusion \( X^{ss}_{\chi_1} \subset X^{ss}_{\chi_2} \) holds if and only if \( \chi_1 \) belongs to the cone of the cell decomposition of \( \text{cone}(\Sigma) \) induced by \( \pi \) (see Definition 4.11) containing \( \chi_2 \) in its interior. We consider the morphisms between GIT-quotients \( X/\chi T \)
induced by inclusions between $X_{\chi}^T$, where $\chi \in \Sigma$. So the GIT-quotients $X_{\chi}^T$ form a finite inverse system with $X_{\chi}^T$ sitting at the end. Consider the inverse limit

$$X_{\chi}^T := \lim_{\leftarrow} \{ X_{\chi}^T ; \chi \text{ lies in the interior of } \Sigma \}.$$ 

It is a closed subscheme in the product $X_{\chi_1}^T \times \ldots \times X_{\chi_r}^T$, where $\chi_1, \ldots, \chi_r$ are representatives of interior points of all maximal cones of the cell decomposition of cone$(\Sigma)$ induced by $\pi$. Note also that $X_{\chi}^T$ is a closed subscheme in $V_{\chi}^T$.

**Definition 5.1.** The main component $(X_{\chi}^T)^0$ of the inverse limit $X_{\chi}^T$ is the closure of the image of the map $T \to X_{\chi}^T$ induced by the maps $T \to X_{\chi}^T$, where $\chi$ lies in the interior of $\Sigma$. By [4, Proposition 3.8], it follows that the main component $(X_{\chi}^T)^0$ is an irreducible component of $X_{\chi}^T$ which satisfies the following universal property: given a $T/T$-variety $Y$ containing an irreducible component $Y^0$ such that $Y^0$ is a toric $T/T$-variety, and given $T/T$-equivariant morphisms $\phi_\chi : Y \to X_{\chi}^T$, where $\chi$ lies in the interior of $\Sigma$, such that the $\phi_\chi$ induce birational morphisms $Y^0 \to X_{\chi}^T$ and the $\phi_\chi$ are compatible with the morphisms of the inverse system (so the $\phi_\chi$ give a morphism $\phi : Y \to X_{\chi}^T$); then restricting the morphism $\phi$ to $Y^0$ we have a birational morphism of toric $T/T$-varieties $Y^0 \to (X_{\chi}^T)^0$.

**Remark 5.2.** By [4, Proposition 3.10], it follows that the fan of $(X_{\chi}^T)^0$ is $C_\mathbb{R}$, the maximal common refinement of all the normal fans to the polyhedra $P_{\chi}^\mathbb{R}$, $\chi \in \Sigma$. Since every character $\chi \in \Sigma$ has some integral positive multiple $c_\chi \in \Sigma^\mathbb{N}_{\chi}$ ($c \in \mathbb{N}$), the fan $C_{H^0}$ is a refinement of the fan $C_\mathbb{R}$.

The following example shows that $C_{H^0}$ and $C_\mathbb{R}$ do not always coincide.

**Example 5.3.** Let $X = \mathbb{A}^3$, $T = \mathbb{G}_m^3$ act by rescaling of coordinates, and let $T = \mathbb{G}_m$ act by $t(x_1, x_2, x_3) = (tx_1, tx_2, t^2 x_3)$.

$$\mathcal{X}(T) \xrightarrow{\pi} \mathcal{X}(T)$$

The Hilbert scheme $H_{\mathbb{A}^3,T}$ is the closed subscheme in $\mathbb{P}^1 \times \mathbb{P}^3$ defined by the equations $z_1 w_3 - z_2 w_1 = 0$ and $z_1 w_2 - z_2 w_3 = 0$ (where $z_1, z_2$ and $w_1, w_2, w_3, w_4$ are homogeneous coordinates in $\mathbb{P}^1$ and $\mathbb{P}^3$ respectively). The integral (in the sense of Definition 4.14) degrees are even. The fan $C_{H^0}$ consists of the following cones:

$$\mathbb{R}_+ (e_1 + e_2) + \mathbb{R}_+ e_2,$$
\[ R_+ (e_1 + e_2) + R_+ (-e_2), \]
\[ R_+ (e_2 - e_1) + R_+ e_2, \]
\[ R_+ (e_2 - e_1) + R_+ (-e_2), \]

where \( e_1 = \nu_1^* + \nu_2^* \), \( e_2 = -\nu_3^* \) is a basis of \( \Lambda(T/T) \). The inverse limit of GIT-quotients is \( A^3/C^T = \text{Proj} \ k[x_1, x_2, x_3] \) (where \( k[x_1, x_2, x_3] \) is graded by the weights of \( T \)), and its fan \( C_R \) consists of the following cones:

\[ \mathbb{R}_+ (e_1 + e_2) + \mathbb{R}_+ (-e_2), \]
\[ \mathbb{R}_+ (e_2 - e_1) + \mathbb{R}_+ (-e_2), \]
\[ \mathbb{R}_+ (e_1 + e_2) + \mathbb{R}_+ (e_2 - e_1). \]

By the statement \((*)\) from the proof of Proposition 4.15 it follows that if a character \( \chi \in \Sigma \) is integral, then there exists \( r_0 \) such that \( S_{(r+1)\chi} = S_\chi S_{r\chi} \) for all \( r \geq r_0 \). Thus Corollary 3.5 implies that

\[ H_{S(\chi), T} = \text{Proj} \ S^{(\chi)} = X/\chi T, \]

for any \( \chi \in \Sigma_{\text{int}}^\chi \).

For any subset \( D \subset \Sigma \) we can consider the restriction of the Hilbert scheme \( H_{X, T} \) on degrees \( D \), that is, the quasiprojective scheme \( H^D_{X, T} \) representing the covariant functor

\[ \underline{H^D_{X, T}} : k - \text{Alg} \to \text{Set} \]

such that \( H^D_{X, T}(R) \) is the set of families \( \{L_\chi\}_{\chi \in D} \), where \( L_\chi \subset R \otimes_k S_\chi \) is an \( R \)-submodule, such that \( (R \otimes_k S_\chi)/L_\chi \) is a locally free \( R \)-module of rank 1 and \( fL_{\chi_1} \subset L_{\chi_2} \) for any \( \chi_1, \chi_2 \in D \) and any \( f \in S_{\chi_1 - \chi_2} \) (see \[8\] Section 2]). In particular, \( H^D_{\chi, T} = H_{\chi, T} \) and \( H_{S(\chi), T} = H^{D_X}_{X, T} \), where \( D_X := \{c\chi : c \in \mathbb{Z}_+\} \). Note also that \( H^D_{\chi, T} \) is a closed subscheme of \( H^D_{V, T} \). For any \( D \subset \Sigma \) we have a degree restriction morphism \( H_{X, T} \to H^D_{X, T} \). In particular, we have canonical morphisms

\[ \phi^\chi_X : H_{X, T} \to X/\chi T. \]

The following theorem was proved in \[8\] Theorem 5.6] for the case when \( X \) is a finite-dimensional \( T \)-module.
Theorem 5.4. Let \( H_{X,T}^{\text{int}} := H_{X,T}^{\Sigma_X} \) be the toric Hilbert scheme restricted to the set of integral degrees. Then there is a canonical morphism

\[
\phi_X^{\text{int}} : H_{X,T}^{\text{int}} \to \mathbb{X}/C T
\]

which induces an isomorphism of the corresponding reduced schemes. In particular, composing \( \phi_X^{\text{int}} \) with the degree restriction morphism, we obtain a canonical Chow morphism from the toric Hilbert scheme to the inverse limit of the GIT quotients

\[
\phi_X : H_{X,T} \to \mathbb{X}/C T.
\]

Proof. As in \cite[Lemma 5.7]{S}, we see that the morphisms \( \phi_X^{\chi} \) satisfy the compatibility conditions for \( \chi \in \Sigma_X^{\text{int}} \) and, consequently, give a canonical morphism

\[
H_{X,T} \to H_{X,T}^{\text{int}} \xrightarrow{\phi_X^{\text{int}}} \mathbb{X}/C T.
\]

Further, note that for any algebra \( R \) the morphism

\[
\phi_X^{\text{int}}(R) : H_{X,T}^{\text{int}}(R) \to \mathbb{X}/C T(R)
\]

is injective (since \( H_{S(\chi),T} = \mathbb{X}/T \), we view any element of \( \mathbb{X}/C T(R) \) as a family of \( R \)-submodules \( \{I_\chi \in R \otimes S_\chi \}_{\chi \in \Sigma_X^{\text{int}}} \) such that \( (R \otimes S_\chi)/I_\chi \) is a locally free \( R \)-module of rank 1, \( I_\chi := \bigoplus_{n \geq 0} I_{n \chi} \) is an ideal in \( R \otimes S(\chi) \), so it defines a point of \( H_{S(\chi),T}(R) \), and these points satisfy the compatibility conditions of the direct system). Hence to prove that \( \phi_X^{\text{int}} \) induces an isomorphism of the reduced schemes, it suffices to show that \( \phi_X^{\text{int}}(R) \) is surjective for any reduced \( R \).

Note that \( \phi_X^{\chi} \) coincides with the restriction of \( \phi_X^{\chi} \) to \( H_{X,T} \subset H_{V,T} \) for any \( \chi \in \Sigma_V \subset \Sigma_X^{\text{int}} \). By \cite[Theorem 5.6]{S}, the map \( \phi_X^{\text{int}}(R) \) is surjective for any reduced \( R \), and it follows that any element \( \{I_\chi \}_{\chi \in \Sigma_X^{\text{int}}} \) in \( \mathbb{X}/C T(R) \subset V_{C T}(R) \) gives an element \( \{I_\chi \}_{\chi \in \Sigma_V^{\text{int}}} \) in \( H_{V,T}^{\text{int}}(R) \), i.e., \( f I_{\chi_2} \subset I_{\chi_1} \) for any \( \chi_1, \chi_2 \in \Sigma_V^{\text{int}} \) and any \( f \in S_{\chi_1 - \chi_2} \). We have to prove that this condition holds for any \( \chi_1, \chi_2 \in \Sigma_X^{\text{int}} \). There exists \( c \in \mathbb{N} \) such that \( c\chi_1, c\chi_2 \in \Sigma_V^{\text{int}} \). For any \( f' \in I_{\chi_2} \) we have \( f^c(f')^c \in I_{c\chi_1} \). By Lemma 3.4, we see that the projection of \( \text{Spec}((R \otimes k S^{(\chi_1)})/I^{(\chi_1)}) \to \text{Spec} R \) is a locally trivial bundle with fiber \( A^1 \). Consequently, \( (R \otimes k S^{(\chi_1)})/I^{(\chi_1)} \) is reduced and \( f f' \in I^{(\chi_1)} \).

Remark 5.5. Note that restricting \( \phi_X \) to the main component \( H_0 \), we obtain a birational morphism of toric \( T/T \)-varieties from \( H_0 \) to \( (\mathbb{X}/C T)_0 \).

Example 5.6. Let \( V = A^3 \) where \( G_m^3 \) and \( T = G_m \) act as in Example 5.3 and \( \mathbb{T} = G_m^2 \) be embedded in \( G_m^3 \) by \( (t_1, t_2) \to (t_1, t_1, t_2) \). Consider the variety \( \mathbb{X} = \)}
\( T \cdot (1, 1, 1) = \text{Spec} \ S \), where \( S = k[x_1, x_2, x_3]/I_X \) and \( I_X = (x_1 - x_2) \). So \( H_{X,T} \) is defined in \( H_{A^3,T} \) by the equation \( z_1 = z_2 \). We have the homomorphisms of groups of characters

\[
\mathbb{Z}^3 = \mathcal{X}(\mathbb{G}_m^3) \xrightarrow{\pi} \mathbb{Z}^2 = \mathcal{X}(\mathbb{T}) \xrightarrow{\pi} \mathbb{Z} = \mathcal{X}(T)
\]

and of monoids

\[
\Omega_{A^3} \to \Omega \to \Sigma,
\]

where \( \Omega_{A^3} \) is the monoid in \( \mathcal{X}(\mathbb{G}_m^3) \) generated by characters with negative coordinates.

Note that \( \Sigma_{\mathbb{X}}^\text{int} = \Sigma_{A^3}^\text{int} \) is the set of even numbers. The scheme \( H_{X,T}^\text{int} \) is the closed subscheme in \( \mathbb{P}^3 \) defined by the equation \( w_3^2 = w_1 w_2 \), and \( H_{X,T}^\text{int} \) is defined by the equations \( w_1 = w_2 = w_3 \). The isomorphism

\[
\phi_{X,T}^\text{int} : H_{X,T}^\text{int} \to X/\mathbb{C}T = \text{Proj} S
\]

is the restriction of the isomorphism

\[
\phi_{A^3}^\text{int} : H_{A^3,T}^\text{int} \to A^3/\mathbb{C}T = \text{Proj} k[x_1, x_2, x_3],
\]

where the inverse isomorphism is given by

\[
(\phi_{A^3}^\text{int})^{-1}(x_1 : x_2 : x_3) = (x_1^2 : x_2^2 : x_1 x_2 : x_3).
\]

Concerning the morphism \( \phi_{A^3} : H_{A^3,T} \to A^3/\mathbb{C}T \), note that \( \phi_{A^3}^{-1}(X/\mathbb{C}T) \) is not contained in \( H_{X,T} \). Indeed, consider the ideal \( I = (x_1, x_2^2) \in H_{A^3,T}(k) \). We have \( (I_X)_{\mathbb{C}} \subset I_{\mathbb{C}} \) for any even \( r \), so \( \phi_{A^3}(I) \notin X/\mathbb{C}T(k) \), but \( I \notin H_{X,T}(k) \).

**References**

1. V. Alexeev and M. Brion, Moduli of affine schemes with reductive group action, J. Algebraic Geom. 14 (2005), 83–117.

2. F. Berchtold and J. Hausen, GIT-equivalence beyond the ample cone, Michigan Math. J. 54 (2006), 483-515.

3. L. J. Billera and B. Sturmfels, Fiber polytopes, Ann. of Math. (2) 135 (1992), 527–549.
[4] A. Craw and D. Maclagan, Fiber fans and toric quotients, Discrete Comput. Geom., (2) 37 (2007), 251-266.

[5] D. Eisenbud and J. Harris, The geometry of schemes, Grad. Texts in Math. 197, Springer-Verlage, New York, 2000.

[6] D. Eisenbud and B. Sturmfels, Binomial ideals, Duke Math. J. 84 (1996), 1–45.

[7] W. Fulton, Introduction to toric varieties, Ann. of Math. Stud. 131, Princeton Univ. Press, N.J., 1993.

[8] M. Haiman and B. Sturmfels, Multigraded Hilbert schemes, J. Algebraic Geom. 13 (2004), 725–769.

[9] R. Hartshorne, Algebraic geometry, Grad. Texts in Math. 52, Springer-Verlag, New York-Heidelberg, 1977.

[10] M. Kapranov, B. Sturmfels and A. Zelevinsky, Quotients of toric varieties, Math. Ann. 290 (1991), 644–655.

[11] I. Peeva and M. Stillman, Toric Hilbert schemes, Duke Math. J. 111 (2002), 419–449.

[12] B. Sturmfels, Gröbner bases and convex polytopes. Univ. Lecture Ser. 8, American Mathematical Society, Providence, R.I., 1996.

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