Generators of Internal Lorentz Transformations and of General Affine Coordinate Transformations in Teleparallel Theory of (2+1)-Dimensional Gravity — Cases with Static Circularly Symmetric Space-Times—

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Abstract

In a teleparallel theory of (2+1)-dimensional gravity developed in a previous paper, we examine generators of internal Lorentz transformations and of general affine coordinate transformations for static circularly symmetric exact solutions of gravitational field equation. The “spin” angular momentum, the energy-momentum and the “extended orbital angular momentum” are explicitly given for each solution. Also, we give a critical comment on Deser’s claim that neither momentum nor boosts are definable for finite energy solutions of three-dimensional Einstein gravity.

§ 1. Introduction

In a previous paper,¹ we have developed a teleparallel theory of (2+1)-dimensional gravity, in which three types of static circularly symmetric exact solutions of the gravitational
field equation are given, and horizons and singularities of space-times given by these solutions have been examined. Two of these solutions are also solutions of the three-dimensional Einstein equation.

In the Einstein theory of three-dimensional gravity\textsuperscript{2),3)} and in the theory of (2+1)-dimensional supergravity,\textsuperscript{4)} global charges such as energy-momentum, angular momentum and supercharge have been examined, and the following has been claimed: (1)Neither momentum nor boosts are definable for finite energy solutions in the three-dimensional Einstein theory.\textsuperscript{2)} (2)There is no way to define linear momentum and supercharge for a generic solution to (2+1)-dimensional supergravity.\textsuperscript{4)}

Thus, it would be significant to examine global charges in the teleparallel theory of (2+1)-dimensional gravity. The main purpose of this paper is to investigate the generators of the internal Lorentz transformations and of the general affine coordinate transformations for the exact solutions mentioned above.

§ 2. Basic Framework of the Theory and Generalities on Generators

For the convenience of the latter discussion, we briefly summarize the basic part of the teleparallel theory developed in Ref. 1).

The three-dimensional space-time is assumed to be a differentiable manifold endowed with a Lorentzian metric \( g_{\mu\nu} dx^\mu \otimes dx^\nu \) related to the fields \( e^k = e^k_\mu dx^\mu \) \((k = 0, 1, 2)\) through the relation \( g_{\mu\nu} = e^k_\mu \eta_{kl} e^l_\nu \) with \((\eta_{kl}) \triangleq \text{diag}(−1, 1, 1)\). Here, \( \{x^\mu ; \mu = 0, 1, 2\} \) is a local coordinate of the space-time. The fields \( e^k = e^\mu_k \partial / \partial x^\mu \), which are dual to \( e^k \), are the dreibein fields. The strength of \( e^k_\mu \) is given by \( T^k_\mu\nu = \partial_\mu e^k_\nu - \partial_\nu e^k_\mu \). The covariant derivative of the Lorentzian vector field \( V^k \) is defined by \( \nabla_l V^k \triangleq e^\mu_l \partial_\mu V^k \), and the covariant derivative of the world vector field \( V = V^\mu \partial / \partial x^\mu \) with respect to the affine connection \( \Gamma^\mu_\lambda_\nu \) is given by \( \nabla_\nu V^\mu = \partial_\nu V^\mu + \Gamma^\mu_\lambda_\nu V^\lambda \). The requirement \( \nabla_l V^k = e^\nu_l e^k_\mu \nabla_\nu V^\mu \) for \( V^\mu \triangleq e^\mu_k V^k \)
leads to

\[ T^k_{\mu\nu} \equiv e^k_\lambda T^\lambda_{\mu\nu}, \quad (2\cdot1) \]

\[ R^\mu_{\nu\lambda\rho} \overset{\text{def}}{=} \partial_\lambda \Gamma^\mu_{\nu\rho} - \partial_\rho \Gamma^\mu_{\nu\lambda} + \Gamma^\mu_{\tau\lambda} \Gamma^\tau_{\nu\rho} - \Gamma^\mu_{\tau\rho} \Gamma^\tau_{\nu\lambda} \equiv 0, \quad (2\cdot2) \]

\[ \nabla_\lambda g_{\mu\nu} \overset{\text{def}}{=} \partial_\lambda g_{\mu\nu} - \Gamma^\rho_{\mu\lambda} g_{\rho\nu} - \Gamma^\rho_{\nu\lambda} g_{\rho\mu} \equiv 0, \quad (2\cdot3) \]

where \( T^\lambda_{\mu\nu} \) is defined by \( T^\lambda_{\mu\nu} \overset{\text{def}}{=} \Gamma^\lambda_{\nu\mu} - \Gamma^\lambda_{\mu\nu} \). The components \( T^\lambda_{\mu\nu} \) and \( R^\mu_{\nu\lambda\rho} \) are those of the torsion tensor and of the curvature tensor, respectively. Equation (2·2) implies the teleparallelism. The field components \( e^k_\mu \) and \( e^\mu_k \) are used to convert Latin and Greek indices. Also, raising and lowering the indices \( k, l, m, \ldots \) are accomplished with the aid of \((\eta^{kl}) = (\eta_{kl})^{-1}\) and \((\eta_{kl})\), respectively.

For the matter field \( \varphi \) belonging to a representation of the three-dimensional Lorentz group, \( \mathcal{L}_M(\varphi, \nabla_k \varphi) \) with \( \nabla_k \varphi \overset{\text{def}}{=} e^\mu_k \partial_\mu \varphi \) is a Lagrangian density invariant under global Lorentz transformations and under general coordinate transformations, if \( \mathcal{L}_M(\varphi, \partial_k \varphi) \) is an invariant Lagrangian density on the three-dimensional Minkowski space-time. For the dreibein fields \( e_k \), we have employed\(^1\)

\[ L_G = c_1 t^{klm}_{klm} + c_2 v^k v_k + c_3 a^{klm}_{klm}, \quad (2\cdot4) \]

as the Lagrangian density. Here, \( t^{klm}_{klm} \), \( v^k \), and \( a^{klm}_{klm} \) are the irreducible components of \( T^{klm}_{klm} \), which are defined by

\[ t^{klm}_{klm} \overset{\text{def}}{=} \frac{1}{2}(T^{klm}_{klm} + T^{lkm}_{lkm}) + \frac{1}{4}(\eta^{mk}v_l + \eta^{ml}v_k) - \frac{1}{2}\eta^{kl}v_m, \quad (2\cdot5) \]

\[ v^k \overset{\text{def}}{=} T^{l}_{lk}, \quad (2\cdot6) \]

and

\[ a^{klm}_{klm} \overset{\text{def}}{=} \frac{1}{3}(T^{klm}_{klm} + T^{mkl}_{mkl} + T^{lmk}_{lmk}), \quad (2\cdot7) \]

respectively, and \( c_1, c_2 \) and \( c_3 \) are real constant parameters.\(^1\) Then,

\[ I \overset{\text{def}}{=} \frac{1}{c} \int L dx^0 dx^1 dx^2, \quad (2\cdot8) \]

\(^1\) The parameters \( c_1, c_2 \) and \( c_3 \) correspond to \( \alpha, \beta \) and \( \gamma \) in Ref. 1), respectively.
is the total action of the system, where $c$ is the light velocity in vacuum and $L$ is defined by

$$ L \equiv L_G + L_M \quad \text{with} \quad L_G \equiv \sqrt{-g} L_G, \quad L_M \equiv \sqrt{-g} L_M(\varphi, \nabla_k \varphi) \quad \text{and} \quad g \equiv \det(g_{\mu\nu}). $$

The gravitational field equation following from the action $I$ is

$$ -2\nabla^m F_{klm} + 2v^m F_{klm} + 2H_{kl} - \eta_{kl}L_G = T_{kl} \quad (2.9) $$

with

$$ F_{klm} \equiv c_1(t_{klm} - t_{kml}) + c_2(\eta_{kl}v_m - \eta_{km}v_l) + 2c_3a_{klm} = -F_{kml}, \quad (2.10) $$

$$ H_{kl} \equiv T_{mnk}F_{mn}^m - \frac{1}{2}T_{lmn}F_{k}^{mn} = H_{lk}, \quad (2.11) $$

$$ \nabla^m F_{klm} \equiv e^{\mu m} \partial_\mu F_{klm}. \quad (2.12) $$

Also, $T_{kl}$ is the energy-momentum density defined by

$$ \sqrt{-g}T_{kl} \equiv e_{\mu k} \frac{\delta L_M}{\delta e^k_\mu} = e_{\mu k} \left\{ \frac{L_M}{\partial e^k_\mu} - \partial_\nu \left( \frac{\partial L_M}{\partial (\partial_\nu e^k_\mu)} \right) \right\}. \quad (2.13) $$

The field equation (2.9) reduces to the three-dimensional Einstein equation, when the conditions

$$ 3c_1 = -4c_2 = -8c_3 = -\frac{1}{\kappa}, \quad T_{kl} = T_{lk}, \quad (2.14) $$

are satisfied. Here, $\kappa$ stands for the “Einstein gravitational constant” $\kappa = 8\pi G/c^4$ with $G$ being the “Newton gravitational constant.”

We consider the global internal Lorentz transformation

$$ e'^{k}_\mu = e^{k}_\mu + d^{k}_l e'^{l}_\mu, \quad \phi' = \phi + \frac{i}{2}d^{kl}M_{kl}\phi \quad (2.15) $$

with $d^{kl} = -d^{lk}$ and $\{M_{kl} = -M_{lk} \mid k, l = 0, 1, 2\}$ being an arbitrary infinitesimal constant and a set of representation matrices of the standard basis of the Lie algebra of the three-dimensional homogeneous Lorentz group, respectively. For the space-like surface $\sigma$, the generator $G(\sigma)$ of this transformation is given by

$$ G(\sigma) = -\frac{1}{2}d^{kl}S_{kl} \quad (2.16) $$

with the “spin” angular momentum$^*$)

$^*$) By “spin” we mean the quantum number associated with the three-dimensional Lorentz group.
\[ S_{kl} \overset{\text{def}}{=} \int_{\sigma} S_{kl}^{\mu} d\sigma_{\mu}. \]  

(2·17)

Here, \( S_{kl}^{\mu} \) is defined by\(^*)

\[ S_{kl}^{\mu} \overset{\text{def}}{=} -2 F_{[k}^{\nu\mu} e_{l]\nu} - i \frac{\partial L}{\partial \phi_{\lambda}} (M_{kl} \phi)^{\lambda} \]  

with

\[ F_{k}^{\nu\mu} \overset{\text{def}}{=} \frac{\partial L}{\partial e_{k\nu,\mu}} = \frac{\partial L_{G}}{\partial e_{k\nu,\mu}} = -F_{k}^{\mu\nu}. \]  

(2·19)

For the infinitesimal affine coordinate transformation

\[ x'_{\mu} = x_{\mu} + C_{\mu} + \Omega_{\nu}^{\mu} x'_{\nu} \]  

(2·20)

with \( C_{\mu} \) and \( \Omega_{\nu}^{\mu} \) being infinitesimal constants, the generator \( G^{\sharp}(\sigma) \) is given by

\[ G^{\sharp}(\sigma) = C_{\mu} P^{\mu} - \frac{1}{2} \Omega_{\nu}^{\mu} L_{\nu}^{\mu} \]  

(2·21)

with

\[ P^{\mu} \overset{\text{def}}{=} \int_{\sigma} \tilde{P}^{\mu}_{\nu} d\sigma_{\nu}, \quad L_{\nu}^{\mu} \overset{\text{def}}{=} \int_{\sigma} M^{\mu}_{\nu\lambda} d\sigma_{\lambda}. \]  

(2·22)

Here, \( \tilde{P}^{\mu}_{\nu} \) and \( M^{\mu}_{\nu\lambda} \) are defined by

\[ \tilde{P}^{\mu}_{\nu} \overset{\text{def}}{=} \tilde{t}^{\mu}_{\nu} + \tilde{T}^{\mu}_{\nu} \]  

with

\[ \tilde{t}^{\mu}_{\nu} \overset{\text{def}}{=} \delta^{\mu}_{\nu} L_{G} - F_{k}^{\lambda\nu} e_{k\lambda,\mu}, \quad \tilde{T}^{\mu}_{\nu} \overset{\text{def}}{=} \delta^{\mu}_{\nu} L_{M} - \frac{\partial L_{M}}{\partial \phi_{\lambda}} \phi_{\lambda}^{a}. \]  

(2·24)

and

\[ M^{\mu}_{\nu\lambda} = 2 (F_{k}^{\nu\lambda} e_{k\mu} - x^{\nu} \tilde{P}^{\lambda}_{\mu}) , \]  

(2·25)

respectively. The quantities \( P^{\mu} \) and \( L_{\nu}^{\mu} \) are the canonical energy-momentum and “the extended orbital angular momentum”;\(^5)\(^6)\) respectively, and \( \tilde{t}^{\mu}_{\nu} \) and \( \tilde{T}^{\mu}_{\nu} \) are energy-momentum densities of the gravitational field and of the matter field \( \varphi \), respectively. We have the relation

\[ \tilde{T}^{\mu}_{\nu} = \sqrt{-g} e_{k}^{\mu} e^{\nu} T_{kl}. \]  

(2·26)

The transformation rules of \( S_{kl}, P^{\mu} \) and \( L_{\nu}^{\mu} \) and the differential conservation laws for \( S_{kl}^{\mu}, \tilde{P}^{\mu}_{\nu} \) and \( M^{\mu}_{\nu\lambda} \) are obtainable in a usual way, if the total action \( I \) are invariant under

\(^*) \) The symbol \( [~] \) denotes the antisymmetrization: \( A_{...[k...l...]} \overset{\text{def}}{=} (A_{...k...l...} - A_{...l...k...})/2. \)
the corresponding transformations. In cases which we shall deal with in the following sections, however, the Lagrangian densities of the source matters are unknown, and the invariance of the total actions are not guaranteed. But these laws are obtainable on the basis of the field equation (2·9) and of the invariance of the gravitational action

\[ I_G \overset{\text{def}}{=} \frac{1}{c} \int L_G dx^0 dx^1 dx^2, \quad (2·27) \]

under the corresponding transformations. We need not explicitly assume the invariance of the total action \( I \).

Under the transformation (2·15), \( S_{kl}, P_\mu \) and \( L_\mu^\nu \) transform according as

\[
S'_{kl} = S_{kl} + d_k^m S_{ml} + d_l^m S_{km},
\]

(2·28)

\[
P'_\mu = P_\mu ,
\]

(2·29)

\[
L'_\mu^\nu = L_\mu^\nu ,
\]

(2·30)

respectively. Also, under the transformation (2·20), they transform according as

\[
S'_{kl} = S_{kl} ,
\]

(2·31)

\[
P'_\mu = P_\mu - \Omega^\nu_\mu P_\nu ,
\]

(2·32)

\[
L'_\mu^\nu = L_\mu^\nu - \Omega^\lambda_\mu L_\lambda^\nu + \Omega^\nu_\lambda L_\mu^\lambda - 2C^\nu P_\mu ,
\]

(2·33)

respectively. These rules are derivable by the use of Eqs. (2·9), (2·17), (2·18), (2·22), (2·23), (2·25) and (2·26).

Next, we consider the differential conservation laws. We first consider \( S_{kl}^\mu \). For the case with \( M_{kl} = 0 \), as are the cases discussed in the following sections, \( S_{kl}^\mu \) has the expression

\[
S_{kl}^\mu = -2F_{[k}^\nu_\mu e_{l]\nu} .
\]

(2·34)

From the invariance of the action \( I_G \) under the global Lorentz transformation (2·15), the identity

\[
\partial_\mu S_{kl}^\mu - 2 \frac{\delta L_G}{\delta e^k_\mu} e_{l\mu} \equiv 0
\]

(2·35)

follows, which, on the use of the field equation \( \delta L / \delta e^k_\mu = 0 \), leads to

\[
\partial_\mu S_{kl}^\mu = -2\sqrt{-g} T_{[kl]} .
\]

(2·36)
Thus, we have the differential conservation law

$$\partial_\mu S_{kl}^{\mu} = 0,$$

(2.37)

when $T_{[kl]} = 0$. Now, we consider $\tilde{P}_\mu^{\nu}$ and $M_\mu^{\nu\lambda}$. From the invariance of the action $I_G$ under the general coordinate transformations, the identity

$$\tilde{P}_\mu^{\nu} \equiv e^k_\mu \frac{\delta L}{\delta e^k_\nu} + \partial_\lambda (F_k^{\nu\lambda} e^k_\mu),$$

(2.38)

follows, from which we can get the differential conservation laws

$$\partial_\nu \tilde{P}_\mu^{\nu} = 0,$$

(2.39)

$$\partial_\lambda M_\mu^{\nu\lambda} = 0,$$

(2.40)

by using the field equation $\delta L/\delta e^k_\mu = 0$.

For the differential conservation laws, the following should be mentioned: For the dreibeins having singularities, it is highly desirable to confirm these conservation laws by explicit calculations in which singularities are treated in a proper way, because the above derivations are formal ones.

§ 3. Generators for Static Circularly Symmetric Solutions of Gravitational Field Equation

We consider static circularly symmetric gravitational fields produced by point-like static circular bodies located at the origin $\vec{r} \equiv (x^1, x^2) = (0, 0)$, assuming that the “spin” of constituent particles of the bodies can be completely neglected: $M_{kl} = 0$. For these cases, $(e^k_\mu)$ can be assumed, without loss of generality, to have a diagonal form:\(^1\)

$$ (e^k_\mu) = \begin{bmatrix} A(r) & 0 & 0 \\ 0 & B(r) & 0 \\ 0 & 0 & B(r) \end{bmatrix} $$

(3.1)
with $A$ and $B$ being functions of $r \equiv |\vec{r}|$, which leads to $a_{klm} \equiv 0$.

The right hand sides of the field equation (2.9) can be expressed in terms of $A$ and of $B$,\(^1\) and there are the relations $T_{[kl]} = 0\), $T_{(0)(1)} = 0 = T_{(0)(2)}$, where, to avoid the confusion, Latin indices in $T_{kl}$ are enclosed in the parentheses. Hence, the differential conservation law (2.37) and the relation $\tilde{T}_0^\alpha = 0 = \tilde{T}_\alpha^0$ hold.

For the dreibeins having the form (3.1), the quantities $F_k^{\mu \nu}$ have the expression

$$\begin{align*}
F_{(0)^{0\alpha}} &= -F_{(0)^{0\alpha}} = C(r)x^\alpha, \quad \alpha = 1, 2, \\
F_{(1)^{12}} &= -F_{(1)^{21}} = D(r)x^2, \\
F_{(2)^{12}} &= -F_{(2)^{21}} = -D(r)x^1, \\
F_{(0)^{\alpha \beta}} &= 0, \quad F_{k^{0\alpha}} = -F_{k^{\alpha 0}} = 0,
\end{align*}$$

(3.2)

with

$$\begin{align*}
C(r) &\equiv \frac{1}{2r} \left( 3c_1 \frac{d}{dr} \ln \frac{A}{B} + 4c_2 \frac{d}{dr} \ln AB \right), \\
D(r) &\equiv -\frac{A}{2rB} \left( 3c_1 \frac{d}{dr} \ln \frac{A}{B} - 4c_2 \frac{d}{dr} \ln AB \right).
\end{align*}$$

(3.3)

The “spin” angular momentum density $S_{kl}^{\mu \nu}$ can be expressed in terms of $B(r)$ and of $F_k^{\mu \nu}$, and it gives the vanishing “spin” angular momentum: $S_{kl} = 0$. The energy-momentum density $\tilde{t}_{\mu \nu}$ is expressed as

$$\begin{align*}
\tilde{t}_{0^\nu} &= \delta_{0^\nu} A \left\{ \frac{3c_1}{4} \left( \frac{d}{dr} \ln \frac{A}{B} \right)^2 + c_2 \left( \frac{d}{dr} \ln AB \right)^2 \right\}, \quad \tilde{t}_{0^0} = 0, \\
\tilde{t}_{\alpha^\beta} &= \delta_{\alpha^\beta} \left\{ \frac{3c_1}{4} \left( \frac{d}{dr} \ln \frac{A}{B} \right)^2 + c_2 \left( \frac{d}{dr} \ln AB \right)^2 \right\} \\
&\quad - \frac{x^\alpha x^\beta A}{2r^2} \left[ (3c_1 + 4c_2) \left( \frac{d}{dr} \ln A \right)^2 + \left( \frac{d}{dr} \ln B \right)^2 \right] \\
&\quad - 2(3c_1 - 4c_2) \left( \frac{d}{dr} \ln A \right) \left( \frac{d}{dr} \ln B \right). \quad (3.4)
\end{align*}$$

The density $M_\mu^{\nu \lambda}$ takes the form

$$\begin{align*}
M_0^{00} &= -2x^0 \tilde{P}_0^0, \quad M_0^{0\alpha} = 2AF_{(0)^{0\alpha}}, \\
M_0^{\alpha 0} &= -2AF_{(0)^{0\alpha}} - 2x^\alpha \tilde{P}_0^0, \quad M_0^{\alpha \beta} = 0, \\
M_\alpha^{\beta \gamma} &= 2BF_{(0)^{\beta \gamma}} - 2x^\beta \tilde{P}_\alpha^\gamma, \quad M_\alpha^{00} = 0, \\
M_\alpha^{0\beta} &= -2x^0 \tilde{P}_\alpha^\beta, \quad M_\alpha^{\beta 0} = 0. \quad (3.5)
\end{align*}$$
In Ref. 1), exact solutions of the field equation (2.9) with point-like sources have been given, which are normalized as $A(r_0) = 1 = B(r_0)$ for a radius $r = r_0$ and classified into three cases:

**Case 1.** $3c_1 + 4c_2 \neq 0$:

$$A(r) = X(r)Y(r), \quad B(r) = \left[X(r)\right]^{(1+\Lambda)/(1-\Lambda)}\left[Y(r)\right]^{(1-\Lambda)/(1+\Lambda)},$$  \hspace{1cm} (3.6)

where $\Lambda \equiv \sqrt{-4c_2/3c_1} \neq 1$, and

$$X(r) \equiv 1 + \frac{\Lambda - 1}{4\Lambda}K_1 \ln \left( \frac{r}{r_0} \right), \quad Y(r) \equiv 1 + \frac{\Lambda + 1}{4\Lambda}K_2 \ln \left( \frac{r}{r_0} \right).$$  \hspace{1cm} (3.7)

Here, $K_1$ and $K_2$ are complex constants satisfying the relation $K_1K_2 - (1 - \Lambda)K_1 - (1 + \Lambda)K_2 = 0$.

**Case 2A.** $3c_1 + 4c_2 = 0$:

$$A(r) = 1 + a \ln \left( \frac{r}{r_0} \right), \quad B(r) = \frac{r_0}{r}$$  \hspace{1cm} (3.8)

with $a$ being a real constant.

**Case 2B.** $3c_1 + 4c_2 = 0$:

$$A(r) = 1, \quad B(r) = \left( \frac{r}{r_0} \right)^b$$  \hspace{1cm} (3.9)

with $b$ being a real constant.

The solutions (3.8) and (3.9) are also solutions of the three-dimensional Einstein equation, when $3c_1 = -1/\kappa$.

We examine the generator $G^\sharp(\sigma)$ for each of these solutions. These solutions have singularities at the origin $\vec{r} = 0$, which will be regularized by replacing $r$ by $\sqrt{r^2 + \varepsilon^2}$ with $\varepsilon$ being an infinitesimal real constant. The following should be understood: (1)Any $r$ in expressions of $X, Y, A$ and $B$ is replaced with $\sqrt{r^2 + \varepsilon^2}$. (2)The limit $\varepsilon \to 0$ is taken at final stages of calculations.

**Case 1.** The energy-momentum density $\tilde{t}_\mu^\nu$ for this case is

$$\begin{cases} 
\tilde{t}_0^\nu &= \frac{3c_1}{4} \delta_0^\nu \frac{K_1K_2 r^2}{(r^2 + \varepsilon^2)^2}, \quad \nu = 0, 1, 2, \\
\tilde{t}_0^0 &= 0, \\
\tilde{t}_\alpha^\beta &= \frac{3c_1}{4} \delta_\alpha^\beta \frac{K_1K_2 r^2}{(r^2 + \varepsilon^2)^2} - \frac{3c_1}{2} x^\alpha x^\beta \frac{K_1K_2}{(r^2 + \varepsilon^2)^2}, \quad \alpha, \beta = 1, 2.
\end{cases}$$  \hspace{1cm} (3.10)
Also, from Eqs. (2-9), (2-26) and (3-6), we can get

\[
\begin{align*}
\tilde{T}_0^0 &= \frac{3c_1\varepsilon^2}{2(r^2 + \varepsilon^2)^2}\{(1 + \Lambda)K_1 Y + (1 - \Lambda)K_2 X\}, \\
\tilde{T}_0^\alpha &= 0 = \tilde{T}_0^\alpha, \\
\tilde{T}_\alpha^\beta &= -\frac{3c_1}{4}K_1K_2\varepsilon^2\delta_\alpha^\beta. 
\end{align*}
\]

The first of Eq. (2·22), Eqs. (2·23), (3·10) and (3·11) give

\[
\begin{align*}
P_0 &= \frac{3}{2}\pi c_1 K_1 K_2 \int_{0}^{\infty} \frac{r^3dr}{(r^2 + \varepsilon^2)^2} \\
&\quad +3\pi c_1\varepsilon^2 \int_{0}^{\infty} \frac{\{(1 + \Lambda)K_1 Y + (1 - \Lambda)K_2 X\}rdr}{(r^2 + \varepsilon^2)^2}, \\
P_\alpha &= 0, \\
\end{align*}
\]

and

\[
\int_C \tilde{P}_\mu^\nu d\sigma_\nu = 0,
\]

where \( C \) is the cylinder defined in Appendix A. Equations (2·39), (3·12) and (3·13) satisfy Eq. (A·1) and they are consistent each other, although \( P_0 \) is diverging.

The density \( M_{\mu}^{\nu}\lambda \) is evaluated by the use of Eqs. (2·23), (3·2), (3·5), (3·10) and (3·11), and we obtain the following:

\[
L_\mu^\nu = -2\delta_\mu^0\delta_\nu^0 x^0 P_0, \\
\int_C M_{\mu}^{\nu}\lambda d\sigma_\lambda = \frac{3}{2}c_1\delta_\mu^0\delta_\nu^0 \int_C \{(1 + \Lambda)K_1 Y + (1 - \Lambda)K_2 X\} \frac{x^\alpha d\sigma_\alpha}{r^2}.
\]

Here, \( t \overset{\text{def}}{=} x^0/c \) stands for the time coordinate of the space-like surface \( \sigma \) on which \( L_0^0 \) is defined. The component \( L_0^0 \) is not conserved, although \( \partial_\lambda M_{0}^{0\lambda} = 0 \). This is not a contradiction, as is shown below. There is a relation

\[
L_0^0(\sigma_2^R) - L_0^0(\sigma_1^R) = \int_{V^R} \partial_\lambda M_{0}^{0\lambda} dx^0 dx^1 dx^2 - \int_{C^R} M_{0}^{0\lambda} d\sigma_\lambda.
\]

Here, \( C^R \) is a cylinder with the spatial radius \( R \) between the the space-like surfaces \( \sigma_1^R, \sigma_2^R \), and the \( V^R \) is the domain enclosed by the surfaces \( C^R, \sigma_1^R, \sigma_2^R \). The limit \( R \to \infty \) of Eq. (3·16), although both sides of which diverge, corresponds to Eq. (A·1) applied to the generator \( L_0^0 \). We can show

\[
\lim_{R \to \infty} \frac{1}{\ln(R/r_0)} \{ \text{the left hand side of Eq. (3·16)} \} = \lim_{R \to \infty} \frac{1}{\ln(R/r_0)} \{ \text{the right hand side of Eq. (3·16)} \} = \text{finite}.
\]

Thus, we have a consistency in this sense.
**Case 2A.** The energy-momentum density $\tilde{t}_{\mu}^{\nu}$ for this case is

$$
\begin{align*}
\tilde{t}_0^{\nu} &= \frac{3ac_1 r^2}{(r^2 + \varepsilon^2)^2} \delta_0^{\nu} , \\
\tilde{t}_\alpha^{\nu} &= 0 , \\
\tilde{t}_\alpha^{\beta} &= \frac{3ac_1 r^2}{(r^2 + \varepsilon^2)^2} \delta_\alpha^{\beta} - \frac{6ac_1 x^\alpha x^\beta}{(r^2 + \varepsilon^2)^2} .
\end{align*}
$$

From Eqs. (2.9), (2.26) and (3.8), we can get

$$
\begin{align*}
\tilde{T}_0^{\nu} &= \frac{6c_1 A \varepsilon^2}{(r^2 + \varepsilon^2)^2} , \\
\tilde{T}_\alpha^{\nu} &= 0 = \tilde{T}_0^{\alpha} , \\
\tilde{T}_\alpha^{\beta} &= -3ac_1 \varepsilon \delta_\alpha^{\beta} - \frac{6ac_1 x^\alpha x^\beta}{(r^2 + \varepsilon^2)^2} .
\end{align*}
$$

We have

$$
\begin{align*}
P_0 &= 6\pi ac_1 \int_0^\infty \frac{r^3 dr}{(r^2 + \varepsilon^2)^2} + 12\pi c_1 \varepsilon^2 \int_0^\infty \frac{Ar}{(r^2 + \varepsilon^2)^2} dr , \\
P_\alpha &= 0 ,
\end{align*}
$$

and

$$
\int_C \tilde{P}_\mu^{\nu} d\sigma_\nu = 0 .
$$

Equations (2.39), (3.20) and (3.21) satisfy Eq. (A.1), although $P_0$ is not finite.

The density $M_{\mu}^{\nu\lambda}$ is evaluated by the use of Eqs. (2.23), (3.2), (3.5), (3.8), (3.18) and (3.19), and we obtain

$$
L_\mu^{\nu} = -2x^0 \delta_\mu^{0} \delta_0^{\nu} P_0 ,
$$

$$
\int_C M_{\mu}^{\nu\lambda} d\sigma_\lambda = 6c_1 \delta_\mu^{0} \delta_0^{\nu} \int_C \frac{Ax^\alpha}{r^2 + \varepsilon^2} d\sigma_\alpha .
$$

The component $L_0^{0}$ is diverging and not conserved. Both sides of Eq. (A.1) applied to the generator $L_0^{0}$ diverge, but we have a consistency in the same sense as for **Case 1.**

**Case 2B.** The energy-momentum density $\tilde{t}_{\mu}^{\nu}$ is vanishing for this case:

$$
\tilde{t}_{\mu}^{\nu} = 0 .
$$

Also, from Eqs. (2.9), (2.26) and (3.9), we can get

$$
\tilde{T}_\mu^{\nu} = -6bc_1 \delta_\mu^{0} \delta_0^{\nu} \frac{\varepsilon^2}{(r^2 + \varepsilon^2)^2} .
$$
and we have
\[ P_0 = -6\pi bc_1, \quad P_\alpha = 0, \quad (3.26) \]
and
\[ \int_C \tilde{P}_\mu^\nu d\sigma_\nu = 0. \quad (3.27) \]
Equations (2.39), (3.26) and (3.27) satisfy Eq. (A.1).
The density \( M_\mu^{\nu\lambda} \) is evaluated by the use of Eqs. (2.23), (3.2), (3.5), (3.9), (3.24) and (3.25), which leads to
\[ L_\mu^\nu = 12\pi bc_1 x^0 \delta_\mu^0 \delta_\nu^0, \quad (3.28) \]
\[ \int_C M_\mu^{\nu\lambda} d\sigma_\lambda = -6bc_1 \delta_\mu^0 \delta_\nu^0 \int_C \frac{x^\alpha}{r^2 + \varepsilon^2} d\sigma_\alpha. \quad (3.29) \]
The component \( L_0^0 \) is not conserved, although the differential conservation law Eq. (2.40) holds. This is not a contradiction, because we have Eq. (3.29) and Eq. (A.1) applied to \( L_0^0 \) is satisfied.

It is worth adding the following: For every of Case 1., Case 2A. and Case 2B., we have the following:

(a) That all of the differential conservation laws (2.37), (2.39) and (2.40) are actually satisfied is confirmed by explicit calculations in which the singularities at \( \vec{r} = 0 \) are treated in a proper way by the regularization procedure employed above.

(b) The relation
\[ \int_C S_{kl}^\mu d\sigma_\mu = 0 \quad (3.30) \]
holds, and Eqs. (2.37), (3.30) and \( S_{kl} = 0 \) satisfy Eq. (A.1).

(c) The orbital angular momentum \( L_{[\mu\nu]}^\text{def} (\eta_{\nu\lambda} L_\mu^\lambda - \eta_{\mu\lambda} L_\nu^\lambda)/2 \) is conserved and vanishes, although the \( L_0^0 \) is not conserved.
§ 4. Comment on Momentum and Boosts in Three-Dimensional Einstein Gravity

In Ref. 2), Deser has claimed that neither momentum nor boosts are definable for finite energy solution of the three-dimensional Einstein gravity. But, this claim is wrong, as is shown below.

He has considered the metric tensor having the expression

\[ g_{00} = -1, \ g_{0\alpha} = 0, \ g_{\alpha\beta} = \phi \delta_{\alpha\beta}, \ \alpha, \ \beta = 1, 2 \]  

(4·1)

with

\[ \phi = \prod_{i=1}^{n} |\vec{r} - \vec{r}_i|^{-\alpha_i}, \ \alpha_i = 8Gm_i/c^2, \]

(4·2)

which describes the gravitational field outside of particles with masses \( m_i \) located at \( \vec{r}_i \).

Our solutions (3·8) and (3·9) are also solutions of the three-dimensional Einstein equation, when the relation \( 3c_1 = -1/\kappa \), which we shall assume to hold from now on, is satisfied. The solution (3·9) gives a metric which agrees with the metric (4·1), when \( n = 1, \vec{r}_1 = 0, b = -4Gm_1/c^2, r_0 = 1 \). From the discussion for Case 2B., we see that the energy-momentum and affine coordinate transformation including also boosts are all definable, which excludes Deser’s claim mentioned above.

The discussion in Ref. 2) is based on the assumption that the energy-momentum is given by

\[ P_0 = -\frac{1}{2\kappa} \int \sqrt{(2)g} \ (2)R(\{} \) dx\, dx^2, \]  

(4·3)

\[ P_\alpha = \int \partial_\beta \pi_{\alpha\beta} dx\, dx^2, \ \alpha, \ \beta = 1, 2 \]  

(4·4)

for the coordinate system \( \{x^0, x^1, x^2\} \) for which the metric takes the form (4·1) and by the corresponding primed expression for the primed coordinate system \( \{x'^0, x'^1, x'^2\} \) boosted from \( \{x^0, x^1, x^2\} \). Here, we have defined \( (2)g \overset{\text{def}}{=} \det(g_{\alpha\beta}) \), and \( (2)R(\{}) \) is the two-dimensional Riemann-Christoffel scalar curvature defined by
\[ (2) R(\{\}) \overset{\text{def}}{=} (2) g^{\alpha\beta} \left( \partial_\gamma \{\begin{array}{c} \alpha \\ \beta \end{array}\} - \partial_\beta \{\begin{array}{c} \gamma \\ \alpha \end{array}\} \right) + \{\begin{array}{c} \gamma \\ \delta \end{array}\} \{\begin{array}{c} \delta \\ \alpha \end{array}\} - \{\begin{array}{c} \delta \\ \beta \end{array}\} \{\begin{array}{c} \gamma \\ \alpha \end{array}\} \right) \] (4.5)

with the two-dimensional Christoffel symbol,

\[ \{\begin{array}{c} \gamma \\ \alpha \beta \end{array}\} \overset{\text{def}}{=} \frac{1}{2} (2) g^{\gamma\delta} (\partial_\alpha g_{\delta\beta} + \partial_\beta g_{\delta\alpha} - \partial_\delta g_{\alpha\beta} ) , \] (4.6)

where \((2) g^{\alpha\beta}\) \(\overset{\text{def}}{=} (g_{\alpha\beta})^{-1}\). Also, \(\pi_{\alpha\beta} \overset{\text{def}}{=} g_{\alpha\gamma} \pi_{\gamma\beta}\) with \(\pi_{\gamma\beta}\) being the momentum conjugate to \(g_{\gamma\beta}\). This momentum has the expression

\[ \pi_{\gamma\beta} = \frac{\sqrt{(2) g}}{2\kappa} (K_{\gamma\beta} - (2) g^{\gamma\beta} K) \] (4.7)

with

\[ K_{\gamma\beta} \overset{\text{def}}{=} (2) g^{\gamma\delta} (2) g^{\beta\epsilon} K_{\delta\epsilon} \overset{\text{def}}{=} (2) g^{\gamma\delta} (2) g^{\beta\epsilon} \left( \frac{\partial_\delta g_{\epsilon\lambda} - \nabla_\delta \lambda_\epsilon - \nabla_\epsilon \lambda_\delta}{2\sqrt{N}} \right) , \] (4.8)

\[ K \overset{\text{def}}{=} (2) g^{\alpha\beta} K_{\alpha\beta} , \] (4.9)

\[ \nabla_\alpha \lambda_\beta \overset{\text{def}}{=} \partial_\alpha \lambda_\beta - \{\begin{array}{c} \gamma \\ \alpha \beta \end{array}\} \lambda_\gamma , \] (4.10)

\[ N \overset{\text{def}}{=} -\frac{1}{g^{00}} , \quad \lambda_\alpha \overset{\text{def}}{=} g_{0\alpha} . \] (4.11)

Here, in Eqs. (4.5)\(\sim\)(4.11), the indices \(\alpha, \beta, \gamma, \delta\) and \(\epsilon\) take the values 1 and 2.

This assumption is groundless. For momentum of an asymptotically flat generic solution of four-dimensional Einstein equation, a representation corresponding to Eq. (4.4) holds for any coordinate system related by Lorentz transformations to an asymptotically Cartesian coordinate system. But, this does not justify the above assumption for the three-dimensional case.

In the coordinate system \(\{x^0, x^1, x^2\}\), the energy-momentum \(P_\mu\), defined as the generator of the space-time translations, can be expressed as

\[ P_0 = \int \tilde{P}_0^0 dx^1 dx^2 = -\frac{1}{2\kappa} \int \sqrt{(2) g} (2) R(\{\}) dx^1 dx^2 = \frac{1}{\kappa} \int \Delta \ln B dx^1 dx^2 , \] (4.12)

\[ P_\alpha = \int \tilde{P}_\alpha^0 dx^1 dx^2 = 0 = \int \partial_\beta \pi_\alpha^\beta dx^1 dx^2 , \] (4.13)
for the solution (3.9), where we have defined \( \Delta \equiv (\partial_1)^2 + (\partial_2)^2 \). In the coordinate system \( \{x^0, x^1, x^2\} \), however, we have

\[
P'_{\alpha} \overset{\text{def}}{=} \int \tilde{P}'_{\alpha} \, dx^1 \, dx^2 \neq \int \partial'_{\beta} \pi'_{\alpha} \beta \, dx^1 \, dx^2. \tag{4.14}
\]

Thus, the discussion in Ref. 2) is wrong in this respect.

The energy-momentum and general affine coordinate transformations including also boosts are definable for the solution (4.1) of the three-dimensional Einstein equation.

§ 5. Summary and Remarks

In the above, we have examined “spin” angular momentum, energy-momentum and the “extended orbital angular momentum” in a teleparallel theory of (2+1)-dimensional gravity. Also, we have given a critical comment on the discussion in Ref. 2) claiming that neither momentum nor boosts are definable for finite energy solution of the three-dimensional Einstein gravity. The results can be summarized as follows:

(1) Under the global Lorentz transformation (2.15), the “spin” angular momentum \( S_{kl} \), the energy-momentum \( P'_\mu \) and the “extended angular momentum” \( L_{\mu''} \) transform according as Eqs. (2.28), (2.29) and (2.30), respectively. Also, they transform according as Eqs. (2.31), (2.32) and (2.33), under the infinitesimal affine coordinate transformation Eqs. (2.20).

(2) We have formally derived the differential conservation laws (2.37), (2.39) and (2.40), without assuming explicitly the invariance of the total action \( I \).

(3) For the static circularly symmetric solutions (3.6), (3.8) and (3.9), we have obtained the following:
(A) All the differential conservation laws (2·37), (2·39) and (2·40) are actually satisfied by every of the solutions (3·6), (3·8) and (3·9), although the dreibeins are singular at the origin \( \vec{r} = 0 \).

(B) All the “spin” angular momentum \( S_{kl} \), the momentum \( P_{\alpha} \) and the orbital angular momentum \( L_{[\mu\nu]} \) vanish for static circularly symmetric solutions.

(C) For each of the solutions (3·6) and (3·8), the energy \(-P_0\) is conserved and divergent. The component \( L_0^0 \) is diverging and not conserved. But, there arises no inconsistency.

(D) For the solution (3·9), the energy \(-P_0\) and the component \( L_0^0 \) of the “extended orbital angular momentum” have the finite values \( 6\pi bc_1 \) and \( 12\pi bc_1 x^0 \), respectively. There is no inconsistency, although \( L_0^0 \) is not conserved.

(4) Both of the energy-momentum and general affine coordinate transformations can be defined for the solution (4·1), which invalidates Deser’s claim\(^2\) that neither momentum nor boosts are definable for finite energy solutions of three-dimensional Einstein gravity.

Gravitating bodies for Case 1., Case 2.A. and Case 2.B. are all localized at the origin \( \vec{r} = 0 \), which is known from the expressions (3·11), (3·19) and (3·25) for the energy-momentum densities \( \tilde{T}_{\mu\nu} \) of source bodies by noting the relation

\[
\lim_{\varepsilon \to 0} \frac{\varepsilon^2}{(r^2 + \varepsilon^2)^2} = \pi \delta(\vec{r}) .
\]

The source body for Case 2B. is a mass point with the mass \( 6\pi bc_1/c^2 \). We can give tentative interpretations to the sources for Case 1. and for Case 2A.. When the source of gravity is fluid having the four-velocity field \( u^\mu(x) \), the mass density \( \rho(x) \) and the pressure \( p(x) \), the density \( \tilde{T}_{\mu\nu} \) is given by

\[
\tilde{T}_{\mu\nu} = \sqrt{-g} \left[ \rho(x) u_\mu(x) u_\nu(x) + p(x) \left\{ \frac{1}{c^2} u_\mu(x) u_\nu(x) + \delta_\mu^\nu \right\} \right] ,
\]

which reduces to

\[
\begin{align*}
\tilde{T}_0^0 &= -\rho(x) c^2 \sqrt{-g} , \\
\tilde{T}_\alpha^0 &= 0 = \tilde{T}_0^\alpha , \\
\tilde{T}_\alpha^\beta &= p(x) \sqrt{-g} \delta_\alpha^\beta ,
\end{align*}
\]
when the fluid is at rest: \( \sqrt{-g_{00}(x)} u^0(x) = -u_0(x)/\sqrt{-g_{00}(x)} = c, u^\alpha(x) = 0, \alpha = 1, 2. \)

Thus, if we *make bold to regard* the gravitating bodies for **Case 1.** and for **Case 2A.** as fluid, we get the following mass density and pressure for each case:

**Case 1.**

\[
\begin{aligned}
\rho(x) &= -\frac{3c_1 \varepsilon^2}{2c^2 (r^2 + \varepsilon^2)^2} \left\{ (1 + \Lambda) K_1 Y + (1 - \Lambda) K_2 X \right\} \\
&\quad \times [X(r)]^{(\Lambda+3)/(\Lambda-1)} [Y(r)]^{(\Lambda-3)/(\Lambda+1)}, \\
p(x) &= -\frac{3c_1 K_1 K_2 \varepsilon^2}{4 (r^2 + \varepsilon^2)^2} [X(r)]^{(\Lambda+3)/(\Lambda-1)} [Y(r)]^{(\Lambda-3)/(\Lambda+1)}.
\end{aligned}
\]

**(5.4)**

**Case 2A.**

\[
\begin{aligned}
\rho(x) &= -\frac{6c_1 \varepsilon^2}{c^2 r_0^2 (r^2 + \varepsilon^2)} , \\
p(x) &= -\frac{3ac_1 \varepsilon^2}{r_0^2 A (r^2 + \varepsilon^2)}.
\end{aligned}
\]

**(5.5)**

For **Case 2A.,** the mass density and the pressure both vanish in the limit \( \varepsilon \to 0. \) But, this does not mean that there is no source of gravity, because \( \bar{T}_{00} \) gives non-trivial (diverging) contribution to \( P_0, \) as is seen from the first of Eq. (3.20).

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**Appendix A**

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**Relation between Current and Charge---**

In general, we have the relation

\[
Q_a(\sigma_2) - Q_a(\sigma_1) = \int_{\Gamma} \partial_\mu j^\mu_a dx^0 dx^1 dx^2 - \int_{C} j^\mu_a d\sigma_\mu ,
\]

**(A.1)**

between the current \( j^\mu_a (a = 1, 2, ..., N) \) and the charges

\[
Q_a(\sigma_i) = \int_{\sigma_i} j^\mu_a d\sigma_\mu , \ i = 1, 2.
\]

**(A.2)**
Here $\sigma_1$ and $\sigma_2$ are space-like surfaces, $C$ is the cylinder between these surfaces at spatial infinity, and $V$ is the domain of the space-time enclosed by $\sigma_1, \sigma_2$ and $C$. Thus, we have the conservation law $Q_a(\sigma_2) = Q_a(\sigma_1)$, if

$$\partial_\mu j^\mu_a = 0, \quad \int_C j^\mu_a d\sigma_\mu = 0.$$  \hspace{1cm} (A.3)

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