ON TRACES AND MODIFIED FREDHOLM DETERMINANTS
FOR HALF-LINE SCHRÖDINGER OPERATORS
WITH PURELY DISCRETE SPECTRA

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Abstract. After recalling a fundamental identity relating traces and modified
Fredholm determinants, we apply it to a class of half-line Schrödinger operators
\((-d^2/dx^2) + q\) on \((0, \infty)\) with purely discrete spectra. Roughly speaking, the
class considered is generated by potentials \(q\) that, for some fixed \(C_0 > 0, \, \varepsilon > 0, \, x_0 \in (0, \infty)\), diverge at infinity in the manner that \(q(x) \geq C_0 x^{(2/3)+\varepsilon_0}\) for all \(x \geq x_0\). We treat all self-adjoint boundary conditions at the left endpoint 0.

1. Introduction

To set the stage for describing the principal purpose of this note, we assume that
\(q\) satisfies \(q \in L^1_{\text{loc}}(\mathbb{R}_+; dx)\), \(q\) real-valued a.e. on \(\mathbb{R}_+\), and that for some \(\varepsilon_0 > 0, \, C_0 > 0, \, x_0 \in \mathbb{R}_+\), and sufficiently large \(x_0 > 0\),

\[ q(x) \geq C_0 x^{(2/3)+\varepsilon_0}, \quad x \in (x_0, \infty). \]  

(1.1)

Next, we introduce the half-line Schrödinger operator \(H_{+\alpha}\) in \(L^2(\mathbb{R}_+; dx)\) as the
\(L^2\)-realization of the differential expression \(\tau_+\) of the type

\[ \tau_+ = -\frac{d^2}{dx^2} + q(x) \quad \text{for a.e. } x \in \mathbb{R}_+ \]  

(1.2)

(here \(\mathbb{R}_+ = (0, \infty)\)), and a self-adjoint boundary condition of the form

\[ \sin(\alpha)g'(0) + \cos(\alpha)g(0) = 0, \quad \alpha \in [0, \pi] \]  

(1.3)

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for $g$ in the domain of $H_{+,\alpha}$. Then under appropriate additional technical assumptions on $q$ (cf. Hypothesis 3.1), we will prove in Theorem 3.3 that

$$\text{tr}_{L^2(\mathbb{R}_+;dx)}((H_{+,\alpha} - z I_+)^{-1} - (H_{+,\alpha} - z_0 I_+)^{-1}) = -\frac{d}{dz} \ln \left( \det_{2,L^2(\mathbb{R}_+,dx)}(I_+ - (z - z_0)(H_{+,\alpha} - z_0 I_+)^{-1}) \right)$$

$$= \frac{d}{dz} \ln \left( \sin(\alpha) f'_{+,1}(z,0,x_0) + \cos(\alpha) f_{+,1}(z,0,x_0) \right) \bigg|_{z=z_0}$$

$$- \frac{d}{dz} \ln \left( \sin(\alpha) f'_{+,1}(z,0,x_0) + \cos(\alpha) f_{+,1}(z,0,x_0) \right)$$

$$+ \frac{1}{2} I(z, z_0, x_0),$$

(1.4)

(with $I_+$ abbreviating the identity operator in $L^2(\mathbb{R}_+;dx)$) and

$$\det_{2,L^2(\mathbb{R}_+,dx)} \left( I_+ - (z - z_0)(H_{+,\alpha} - z_0 I_+)^{-1} \right)$$

$$= \left[ \begin{array}{c} \sin(\alpha) f'_{+,1}(z,0,x_0) + \cos(\alpha) f_{+,1}(z,0,x_0) \\ \sin(\alpha) f'_{+,1}(z,0,x_0) + \cos(\alpha) f_{+,1}(z,0,x_0) \end{array} \right]$$

$$\times \exp \left( - (z - z_0) \frac{\sin(\alpha) f'_{+,1}(z,0,x_0) + \cos(\alpha) f_{+,1}(z,0,x_0)}{\sin(\alpha) f'_{+,1}(z,0,x_0) + \cos(\alpha) f_{+,1}(z,0,x_0)} \right)$$

$$\times \exp \left( - \frac{1}{2} \int_{z_0}^{z} d\zeta I(\zeta, z_0, x_0) \right).$$

(1.5)

Here we abbreviated $t = d/dx$, $\cdot = d/dz$,

$$I(z, z_0, x_0) = \int_{z_0}^{\infty} dx \{ [q(x) - z]^{-1/2} - [q(x) - z_0]^{-1/2} \},$$

(1.6)

and $f_{+,1}(z,x,x_0)$ represents an analog of the Jost solution in the case where $q$ denotes a short-range potential (i.e., one that decays sufficiently fast as $x \to \infty$). Finally, $\det_{2}(\cdot)$ abbreviates the modified Fredholm determinant naturally associated with Hilbert–Schmidt operators.

Following the recent paper by Menon [22], which motivated us to write the present note, we then revisit the exactly solvable example $q(x) = x$, $x \in \mathbb{R}_+$, in Example 3.4.

In our final result, Theorem 3.5, we will also treat the case of different boundary condition parameters $\alpha_j \in [0, \pi]$, $j = 1, 2$, and derive the following extension of (1.4),

$$\text{tr}_{L^2(\mathbb{R}_+;dx)}((H_{+,\alpha_2} - z I_+)^{-1} - (H_{+,\alpha_1} - z_0 I_+)^{-1})$$

$$= -\frac{d}{dz} \ln \left( \sin(\alpha_2) f'_{+,1}(z,0,x_0) + \cos(\alpha_2) f_{+,1}(z,0,x_0) \right)$$

$$+ \frac{1}{2} I(z, z_0, x_0).$$

(1.7)

Our proofs of (1.4), (1.5), and (1.6) in Section 3 are based on fundamental connections between traces and modified Fredholm determinants briefly discussed in Section 2 in particular, we will employ the relation (with $I_H$ the identity operator
in \( \mathcal{H} \)
\[
\text{tr}_\mathcal{H} \left( (A - zI_\mathcal{H})^{-1} - (A - z_0I_\mathcal{H})^{-1} \right) = - (d/dz) \ln \left( \det_{\mathcal{H},2} \{ I_\mathcal{H} - (z - z_0)(A - z_0I_\mathcal{H})^{-1} \} \right),
\]
where \( A \) denotes a densely defined and closed operator in \( \mathcal{H} \) with \( \rho(A) \neq \emptyset \), and \( (A - zI_\mathcal{H})^{-1} \in \mathcal{B}_2(\mathcal{H}), z \in \rho(A) \).

Finally, we briefly summarize some of the basic notation used in this paper. Let \( \mathcal{H} \) be a separable, complex Hilbert space, \( (\cdot, \cdot)_\mathcal{H} \) the scalar product in \( \mathcal{H} \) (linear in the second factor), and \( I_\mathcal{H} \) the identity operator in \( \mathcal{H} \). The domain and range of an operator \( T \) are denoted by \( \text{dom}(T) \) and \( \text{ran}(T) \), respectively. The kernel (null space) of \( T \) is denoted by \( \ker(T) \). The spectrum, point spectrum, and resolvent set of \( T \) (i.e., points in \( \sigma(T) \) which are isolated from the rest of \( \sigma(T) \), and which are eigenvalues of \( T \) of finite algebraic multiplicity) is abbreviated by \( \sigma_a(T) \).

The algebraic multiplicity \( m_a(z_0; T) \) of an eigenvalue \( z_0 \in \sigma_a(T) \) is the dimension of the range of the corresponding Riesz projection \( P(z_0; T) \),
\[
m_a(z_0; T) = \dim(\text{ran}(P(z_0; T))) = \text{tr}_\mathcal{H}(P(z_0; T)),
\]
where (with the symbol \( \hat{\cdot} \) denoting counterclockwise oriented contour integrals)
\[
P(z_0; T) = \frac{-1}{2\pi i} \oint_{C(z_0; \varepsilon)} d\zeta (T - \zeta I_\mathcal{H})^{-1},
\]
for \( 0 < \varepsilon < \varepsilon_0 \) and \( D(z_0; \varepsilon_0) \setminus \{ z_0 \} \subset \rho(T) \); here \( D(z_0; r_0) \subset \mathbb{C} \) is the open disk with center \( z_0 \) and radius \( r_0 > 0 \), and \( C(z_0; r_0) = \partial D(z_0; r_0) \) the corresponding circle.

The Banach spaces of bounded and compact linear operators in \( \mathcal{H} \) are denoted by \( \mathcal{B}(\mathcal{H}) \) and \( \mathcal{B}_\infty(\mathcal{H}) \), respectively. Similarly, the Schatten–von Neumann (trace) ideals will subsequently be denoted by \( \mathcal{B}_p(\mathcal{H}), p \in [1, \infty) \). In addition, \( \text{tr}_\mathcal{H}(T) \) denotes the trace of a trace class operator \( T \in \mathcal{B}_1(\mathcal{H}) \), \( \det_{\mathcal{H}}(I_\mathcal{H} - T) \) the Fredholm determinant of \( I_\mathcal{H} - T \), and for \( p \in \mathbb{N}, p \geq 2 \), \( \det_{\mathcal{H},p}(I_\mathcal{H} - T) \) abbreviates the \( p \)-th modified Fredholm determinant of \( I_\mathcal{H} - T \).

2. TRACES AND (MODIFIED) FREDHOLM DETERMINANTS OF OPERATORS

In this section we recall some well-known formulas relating traces and (modified) Fredholm determinants. For background on the material used in this section see, for instance, \([11, 12, 13, \text{Ch. XIII}], [14, \text{Ch. IV}], [21, \text{Ch. 17}], [25], [26, \text{Ch. 3}]\).

To set the stage we start with densely defined, closed, linear operators \( A \) in \( \mathcal{H} \) having a trace class resolvent, and hence introduce the following assumption:

**Hypothesis 2.1.** Suppose that \( A \) is a densely defined and closed in \( \mathcal{H} \) with \( \rho(A) \neq \emptyset \), and \( (A - zI_\mathcal{H})^{-1} \in \mathcal{B}_1(\mathcal{H}) \) for some (and hence for all) \( z \in \rho(A) \).

Given Hypothesis \( 2.1 \) and \( z_0 \in \rho(A) \), consider the bounded, entire family \( A(\cdot) \) defined by
\[
A(z) := I_\mathcal{H} - (A - zI_\mathcal{H})(A - z_0I_\mathcal{H})^{-1} = (z - z_0)(A - z_0I_\mathcal{H})^{-1}, \quad z \in \mathbb{C}.
\]
Employing the formula (cf. \([14, \text{Sect. IV.1}]\), see also \([28, \text{Sect. I.7}]\)),
\[
\text{tr}_\mathcal{H}(I_\mathcal{H} - (T(z))^{-1}T'(z)) = -(d/dz)\ln(\det_{\mathcal{H}}(I_\mathcal{H} - T(z))),
\]
\[1\text{One applies the resolvent equation for } A \text{ and the binomial theorem.}
valid for a trace class-valued analytic family $T(\cdot)$ on an open set $\Omega \subset \mathbb{C}$ such that $(I_H - T(\cdot))^{-1} \in \mathcal{B}(\mathcal{H})$, and applying it to the entire family $A(\cdot)$ then results in

$$\text{tr}_H((A - zI_H)^{-1}) = -(d/dz)\ln(\det_H((A - zI_H)(A - z_0I_H)^{-1})), \quad z \in \rho(A).$$

(2.3)

One notes that the left- and hence the right-hand side of (2.3) is independent of the choice of $z_0 \in \rho(A)$.

Next, following the proof of [26, Theorem 3.5(c)] step by step, and employing a Weierstrass-type product formula (see, e.g., [26, Theorem 3.7]), yields the subsequent result (see also [28, Sect. I.7]).

**Lemma 2.2.** Assume Hypothesis 2.1 and let $\lambda_k \in \sigma(A)$ then

$$\det_H(I_H - (z - z_0)(A - z_0I_H)^{-1}) = (\lambda_k - z)^{m_s(\lambda_k)}[C_k + O(\lambda_k - z)], \quad C_k \neq 0$$

(2.4)

as $z$ tends to $\lambda_k$, that is, the multiplicity of the zero of the Fredholm determinant $\det_H(I_H - (z - z_0)(A - z_0I_H)^{-1})$ at $z = \lambda_k$ equals the algebraic multiplicity of the eigenvalue $\lambda_k$ of $A$.

In addition, denote the spectrum of $A$ by $\sigma(A) = \{\lambda_k\}_{k \in \mathbb{N}}$, $\lambda_k \neq \lambda_{k'}$ for $k \neq k'$. Then

$$\det_H(I_H - (z - z_0)(A - z_0I_H)^{-1}) = \prod_{k \in \mathbb{N}} \left[1 - (z - z_0)(\lambda_k - z_0)^{-1}\right]^{m_s(\lambda_k)}$$

(2.5)

$$= \prod_{k \in \mathbb{N}} \left(\frac{\lambda_k - z}{\lambda_k - z_0}\right)^{m_s(\lambda_k)},$$

with absolutely convergent products in (2.5).

The case of trace class resolvent operators is tailor-made for a number of one-dimensional Sturm–Liouville operators (e.g., finite interval problems). But for applications to half-line problems with potentials behaving like $x$, or increasing slower than $x$ at $+\infty$, and similarly for partial differential operators, traces of higher-order powers of resolvents need to be involved which naturally lead to modified Fredholm determinants as follows.

**Hypothesis 2.3.** Let $p \in \mathbb{N}$, $p \geq 2$, and suppose that $A$ is densely defined and closed in $\mathcal{H}$ with $\rho(A) \neq \emptyset$, and $(A - zI_H)^{-1} \in \mathcal{B}_p(\mathcal{H})$ for some (and hence for all) $z \in \rho(A)$.

Applying the formula

$$\text{tr}_H((I_H - T(z))^{-1}T(z)^{p-1}T'(z)) = -(d/dz)\ln(\det_{H,p}(I_H - T(z))),$$

(2.6)

valid for a $\mathcal{B}_p(\mathcal{H})$-valued analytic family $T(\cdot)$ on an open set $\Omega \subset \mathbb{C}$ such that $(I_H - T(\cdot))^{-1} \in \mathcal{B}(\mathcal{H})$, [14, Sect. IV.2] (see also [28, Sect. I.7]) to the entire family $A(\cdot)$ in (2.1), assuming Hypothesis 2.3 then yields

$$\begin{align*}
(z - z_0)^{p-1}\text{tr}_H((A - zI_H)^{-1}(A - z_0I_H)^{-1-p}) \\
= -(d/dz)\ln(\det_{H,p}(I_H - (z - z_0)(A - z_0I_H)^{-1})), \quad z \in \rho(A).
\end{align*}$$

(2.7)
In the special case \( p = 2 \) this yields
\[
\text{tr}_H((A - zI_H)^{-1} - (A - z_0I_H)^{-1}) = -(d/dz)\ln(\det_H(I_H - (z - z_0)(A - z_0I_H)^{-1})).
\]
(2.8)

We refer to Section 3 for an application of (2.8) to half-line Schrödinger operators with potentials diverging at infinity. For additional background and applications of (modified) Fredholm determinants to ordinary differential operators we also refer to [2], [3], [5], [7], [8], [10], [16]–[21], [23], and the extensive literature cited therein.

3. Schrödinger Operators on a Half-Line

We now illustrate (2.8) with the help of self-adjoint Schrödinger operators \(-d^2/dx^2 + q\) on the half-line \(\mathbb{R}_+ = (0, \infty)\) in the particular case where the potential \(q\) diverges at \(\infty\) and hence gives rise to a purely discrete spectrum (i.e., the absence of essential spectrum).

To this end we introduce the following set of assumptions on \(q\):

**Hypothesis 3.1.** Suppose \(q\) satisfies
\[
q \in L^1_{\text{loc}}(\mathbb{R}_+; dx), \quad q \text{ is real-valued a.e. on } \mathbb{R}_+,
\]
and for some \(\varepsilon_0 > 0, C_0 > 0,\) and sufficiently large \(x_0 > 0,\)
\[
q, q' \in AC([x_0, R]) \text{ for all } R > x_0, \quad q(x) \geq C_0 x^{2/3 + \varepsilon_0}, \quad x \in (x_0, \infty),
\]
\[
q'/q = o(q^{1/2}), \quad \text{as } x \to \infty,
\]
\[
(q^{-3/2}q')' \in L^1((x_0, \infty); dx).
\]

Given Hypothesis 3.1, we take \(\tau_+\) to be the Schrödinger differential expression
\[
\tau_+ = -d^2/dx^2 + q(x) \text{ for a.e. } x \in \mathbb{R}_+,
\]
and note that \(\tau_+\) is regular at 0 and in the limit point case at \(+\infty\). The maximal operator \(H_{+,\text{max}}\) in \(L^2(\mathbb{R}_+; dx)\) associated with \(\tau_+\) is defined by
\[
H_{+,\text{max}} f = \tau_+ f,
\]
\[
f \in \text{dom}(H_{+,\text{max}}) = \{ g \in L^2(\mathbb{R}_+; dx) \mid g, g' \in AC([0, b]) \text{ for all } b > 0; \tau_+ g \in L^2(\mathbb{R}_+; dx) \},
\]
while the minimal operator \(H_{+,\text{min}}\) in \(L^2(\mathbb{R}_+; dx)\) associated with \(\tau_+\) is given by
\[
H_{+,\text{min}} f = \tau_+ f,
\]
\[
f \in \text{dom}(H_{+,\text{min}}) = \{ g \in L^2(\mathbb{R}_+; dx) \mid g, g' \in AC([0, b]) \text{ for all } b > 0; \}
\]
\[
g(0) = g'(0) = 0; \tau_+ g \in L^2(\mathbb{R}_+; dx) \}.
\]

One notes that the operator \(H_{+,\text{min}}\) is symmetric and that
\[
H_{+,\text{min}}^* = H_{+,\text{max}}, \quad H_{+,\text{max}}^* = H_{+,\text{min}}
\]
(3.9)
Moreover, all self-adjoint extensions of $H_{+, \text{min}}$ are given by the one-parameter family in $L^2(\mathbb{R}_+; dx)$

$$H_{+, \alpha}f = \tau_+ f,$$

$$f \in \text{dom}(H_{+, \alpha}) = \{g \in L^2(\mathbb{R}_+; dx) \mid g, g' \in AC([0, b]) \text{ for all } b > 0; \}
\sin(\alpha)g'(0) + \cos(\alpha)g(0) = 0; \tau_+ g \in L^2(\mathbb{R}_+; dx)\},$$

$$\alpha \in [0, \pi).$$

Next, we introduce the fundamental system of solutions $\phi_\alpha(z, \cdot)$ and $\theta_\alpha(z, \cdot)$, $\alpha \in [0, \pi)$, $z \in \mathbb{C}$, associated with $H_{+, \alpha}$ satisfying

$$\tau_+ \psi(z, \cdot)(x) = z \psi(z, x), \quad z \in \mathbb{C}, \quad x \in \mathbb{R}_+, \quad (3.11)$$

and the initial conditions

$$\phi_\alpha(z, 0) = -\sin(\alpha), \quad \phi'_\alpha(z, 0) = \cos(\alpha), \quad \theta_\alpha(z, 0) = \cos(\alpha), \quad \theta'_\alpha(z, 0) = \sin(\alpha). \quad (3.12)$$

Explicitly, one infers

$$\phi_\alpha(z, x) = \phi_\alpha^{(0)}(z, x) + \int_0^x dx' \frac{\sin(z^{1/2}(x - x'))}{z^{1/2}} q(x') \phi_\alpha(z, x'), \quad z \in \mathbb{C}, \quad \text{Im}(z^{1/2}) \geq 0, \quad x \geq 0,$$

with

$$\phi_\alpha^{(0)}(z, x) = \cos(\alpha) - \frac{\sin(z^{1/2}x)}{z^{1/2}} - \sin(\alpha) \cos(z^{1/2}x), \quad z \in \mathbb{C}, \quad \text{Im}(z^{1/2}) \geq 0, \quad x \geq 0,$$

and

$$\theta_\alpha(z, x) = \theta_\alpha^{(0)}(z, x) + \int_0^x dx' \frac{\sin(z^{1/2}(x - x'))}{z^{1/2}} q(x') \theta_\alpha(z, x'), \quad z \in \mathbb{C}, \quad \text{Im}(z^{1/2}) \geq 0, \quad x \geq 0,$$

with

$$\theta_\alpha^{(0)}(z, x) = \cos(\alpha) \cos(z^{1/2}x) + \sin(\alpha) \frac{\sin(z^{1/2}x)}{z^{1/2}}, \quad z \in \mathbb{C}, \quad \text{Im}(z^{1/2}) \geq 0, \quad x \geq 0. \quad (3.16)$$

The Weyl–Titchmarsh solution, $\psi_{+, \alpha}(z, \cdot)$, and Weyl–Titchmarsh $m$-function, $m_{+, \alpha}(z)$, corresponding to $H_{+, \alpha}$, $\alpha \in [0, \pi)$, are then related via,

$$\psi_{+, \alpha}(z, \cdot) = \theta_\alpha(z, \cdot) + m_{+, \alpha}(z) \phi_\alpha(z, \cdot), \quad z \in \rho(H_{+, \alpha}), \quad \alpha \in [0, \pi), \quad (3.17)$$

where

$$\psi_{+, \alpha}(z, \cdot) \in L^2(\mathbb{R}_+; dx), \quad z \in \rho(H_{+, \alpha}), \quad \alpha \in [0, \pi). \quad (3.18)$$

Let $I_+$ be the identity operator on $L^2(\mathbb{R}_+; dx)$. One then obtains for the Green’s function $G_{+, \alpha}$ of $H_{+, \alpha}$ expressed in terms of $\phi_\alpha$ and $\psi_{+, \alpha}$,

$$G_{+, \alpha}(z, x') = (H_{+, \alpha} - zI_+)^{-1}(x, x')$$

$$= \begin{cases} \phi_\alpha(z, x) \psi_{+, \alpha}(z, x'), & 0 \leq x \leq x' < \infty, \\ \phi_\alpha(z, x') \psi_{+, \alpha}(z, x), & 0 \leq x' \leq x < \infty, \end{cases}, \quad z \in \rho(H_{+, \alpha}), \quad \alpha \in [0, \pi), \quad (3.19)$$

utilizing

$$W(\theta_\alpha(z, \cdot), \phi_\alpha(z, \cdot)) = 1, \quad z \in \mathbb{C}, \quad \alpha \in [0, \pi), \quad (3.20)$$
implying $W(\psi_{+,\alpha}(z, \cdot), \phi_{\alpha}(z, \cdot)) = 1$, $z \in \rho(H_{+,\alpha})$.

By [6 Corollary 2.2.1], Hypothesis 3.1 implies the existence of two solutions $f_{+,j}(\lambda, \cdot, x_0)$, $j = 1, 2$, of $\tau_+ \psi(\lambda, \cdot) = \lambda \psi(\lambda, \cdot)$, $\lambda < 0$ sufficiently negative (and below $\inf(\sigma(H_{+,\alpha})))$, satisfying

$$f_{+,j}(\lambda, x, x_0) = 2^{-1/2} |q(x) - \lambda|^{-1/4} \exp \left( -1j \int_{x_0}^{x} \frac{dx'}{q(x') - \lambda} \right) \times [1 + o(1)],$$

$$f'_{+,j}(\lambda, x, x_0) = -2^{-1/2} |q(x) - \lambda|^{-1/4} \exp \left( -1j \int_{x_0}^{x} \frac{dx'}{q(x') - \lambda} \right) \times [1 + o(1)], \quad j = 1, 2,$$

with

$$W(f_{+,1}(\lambda, \cdot, x_0), f_{+,2}(\lambda, \cdot, x_0)) = 1.$$  \hfill (3.22)

(Here we explicitly introduced the $x_0$ dependence of $f_{+,j}$, implied by the choice of normalization in (3.21), as keeping track of it later on will become a necessity.) In particular, $f_{+,1}(\lambda, \cdot, x_0)$ now plays the analog of the Jost solution in the case of a short-range potential $q$ (i.e., $q \in \mathcal{L}^1(\mathbb{R}^3; (1 + x)dx)$, $q$ real-valued a.e. on $\mathbb{R}^3$.

By the limit point property of $\tau_+$ at $+\infty$ and the asymptotic behavior of $f_{+,1}$ in (3.21) one infers, in addition,

$$\psi_{+,\alpha}(\lambda, \cdot) = f_{+,1}(\lambda, \cdot, x_0) \frac{[\sin(\alpha) f_{+,1}(\lambda, 0, x_0) + \cos(\alpha) f_{+,1}(\lambda, 0, x_0)]}{[\sin(\alpha) f'_{+,1}(\lambda, 0, x_0) + \cos(\alpha) f'_{+,1}(\lambda, 0, x_0)]},$$

$$\phi_{\alpha}(\lambda, \cdot) = [\cos(\alpha) f_{+,2}(\lambda, 0, x_0) + \sin(\alpha) f_{+,2}(\lambda, 0, x_0)] f_{+,2}(\lambda, \cdot, x_0) - \left[ \cos(\alpha) f_{+,2}(\lambda, 0, x_0) + \sin(\alpha) f'_{+,2}(\lambda, 0, x_0) \right] f_{+,2}(\lambda, \cdot, x_0)$$

for $\lambda < 0$ sufficiently negative. Analytic continuation with respect to $\lambda$ in (3.21) then yields the existence of a unique Jost-type solution $f_{+,1}(z, \cdot, x_0)$ satisfying

$$\tau_+ f_{+,1}(z, \cdot, x_0) = z f_{+,1}(z, \cdot, x_0), \quad z \in \mathbb{C} \setminus \mathbb{R},$$

$$f_{+,1}(z, \cdot, x_0) \in \mathcal{L}^2(\mathbb{R}^3; dx), \quad z \in \mathbb{C} \setminus \mathbb{R},$$

coinciding with $f_{+,1}(\lambda, \cdot, x_0)$ for $z = \lambda < 0$ sufficiently negative. In addition one has

$$W(f_{+,1}(z, \cdot, x_0), \phi_{\alpha}(z, \cdot, x_0)) = \cos(\alpha) f_{+,1}(z, 0, x_0) + \sin(\alpha) f'_{+,1}(z, 0, x_0),$$

$$z \in \rho(H_{+,\alpha}),$$

which should be compared with the Jost function $f_+(z, 0)$ in the case where $q$ represents a short-range potential and $\alpha = 0$.

In the following we want to illustrate how Hypothesis 2.3 and 2.7 apply to $H_{+,\alpha}$ in the case $p = 2$. For this purpose we first recall the following standard convergence property for trace ideals in $\mathcal{H}$:

**Lemma 3.2.** Let $q \in [1, \infty)$ and assume that $R, R_n, T, T_n \in \mathcal{B}(\mathcal{H})$, $n \in \mathbb{N}$, satisfy $s\text{-}\lim_{n \to \infty} R_n = R$ and $s\text{-}\lim_{n \to \infty} T_n = T$ and that $S, S_n \in \mathcal{B}(\mathcal{H})$, $n \in \mathbb{N}$, satisfy $s\text{-}\lim_{n \to \infty} \|S_n - \mathcal{T}_R S_n\|_{\mathcal{B}(\mathcal{H})} = 0$. Then \(\lim_{n \to \infty} \|R_n S_n T_n^* - R \mathcal{T}_R S_n T_n^*\|_{\mathcal{B}(\mathcal{H})} = 0\).

This follows, for instance, from [13 Theorem 1], [26, p. 28–29], or [28 Lemma 6.1.3] with a minor additional effort (taking adjoints, etc.).

Next, we introduce the family of self-adjoint projections $P_R$ in $\mathcal{L}^2(\mathbb{R}^3; dx)$ via

$$(P_R f)(x) = \chi_{[0,R]}(x) f(x), \quad f \in \mathcal{L}^2(\mathbb{R}^3; dx), \quad R > 0,$$  \hfill (3.28)
with \( \chi_{[0, R]}(\cdot) \) the characteristic function associated with the interval \([0, R], R > 0\). \((P_0)\) will play the role of \( R_n, T_n \) in our application of Lemma 3.2 in the proof of Theorem 3.3 below.

One then obtains the following results.

**Theorem 3.3.** Assume Hypothesis 3.3, \( z, z_0 \in \rho(H_{+, \alpha}), \) and \( \alpha \in [0, \pi) \). Then,

\[
[(H_{+, \alpha} - zI_+)^{-1} - (H_{+, \alpha} - z_0I_+)^{-1}] \in \mathcal{B}_1(L^2(\mathbb{R}_+; dx)),
\]

(3.29)

and

\[
\text{tr}_{L^2(\mathbb{R}_+; dx)}((H_{+, \alpha} - zI_+)^{-1} - (H_{+, \alpha} - z_0I_+)^{-1})
\]

\[
= -\frac{d}{dz}\ln\left( \det_{L^2(\mathbb{R}_+; dx)}(I_+ - (z - z_0)(H_{+, \alpha} - z_0I_+)^{-1}) \right)
\]

\[
= \frac{d}{dz}\ln\left( \sin(\alpha)f_{+,1}'(z, 0, x_0) + \cos(\alpha)f_{+,1}(z, 0, x_0) \right)_{|z=z_0}
\]

\[
- \frac{d}{dz}\ln\left( \sin(\alpha)f_{+,1}'(z, 0, x_0) + \cos(\alpha)f_{+,1}(z, 0, x_0) \right)
\]

\[
+ \frac{1}{2} I(z, z_0, x_0),
\]

(3.30)

as well as,

\[
\det_{L^2(\mathbb{R}_+; dx)}(I_+ - (z - z_0)(H_{+, \alpha} - z_0I_+)^{-1})
\]

\[
= \left[ \frac{\sin(\alpha)f_{+,1}'(z, 0, x_0) + \cos(\alpha)f_{+,1}(z, 0, x_0)}{\sin(\alpha)f_{+,1}'(z_0, 0, x_0) + \cos(\alpha)f_{+,1}(z_0, 0, x_0)} \right]
\]

\[
\times \exp\left( - (z - z_0)\frac{\sin(\alpha)f_{+,1}'(z, 0, x_0) + \cos(\alpha)f_{+,1}(z, 0, x_0)}{\sin(\alpha)f_{+,1}'(z_0, 0, x_0) + \cos(\alpha)f_{+,1}(z_0, 0, x_0)} \right)
\]

\[
\times \exp\left( - \frac{1}{2} \int_{z_0}^{z} d\zeta I(\zeta, z_0, x_0) \right),
\]

(3.31)

where we abbreviated \( \cdot = d/dz \) and

\[
I(z, z_0, x_0) = \int_{z_0}^{\infty} dx \{ [q(x) - z]^{-1/2} - [q(x) - z_0]^{-1/2} \}.
\]

(3.32)

**Proof.** Since the resolvents of \( H_{+, \alpha}, \alpha \in (0, \pi), \) and \( H_{+, 0} \) differ only by a rank-one operator, it suffices to choose \( \alpha = 0 \) when establishing (3.29). Employing the resolvent equation,

\[
(H_{+, 0} - zI_+)^{-1} - (H_{+, 0} - z_0I_+)^{-1} = (z - z_0)(H_{+, 0} - zI_+)^{-1}(H_{+, 0} - z_0I_+)^{-1},
\]

\[
z, z_0 \in \rho(H_{+, 0}),
\]

(3.33)

relation (3.29) follows upon establishing

\[
(H_{+, 0} - zI_+)^{-1} \in \mathcal{B}_2(L^2(\mathbb{R}_+; dx)), \quad z \in \rho(H_{+, 0}).
\]

(3.34)

To prove (3.34) in turn it suffices to establish the Hilbert–Schmidt property for some \( z = \lambda < 0 \) sufficiently negative. Given the Green’s function of \( H_{+, 0} \) in (3.19), it thus suffices to prove that

\[
\int_{\mathbb{R}_+} \int_{\mathbb{R}_+} dx\, dx' |\phi_0(\lambda, x) \psi_{+,0}(\lambda, x')|^2 < \infty.
\]

(3.35)
This can be verified, however, it is quicker to prove \( \text{(3.29)} \) directly, upon employing monotonicity of resolvents with respect to \( \lambda < 0 \) sufficiently negative, that is,

\[
(H_{+,0} - \lambda I_+)^{-1} \geq (H_{+,0} - \lambda_0 I_+)^{-1}, \quad \lambda_0 < \lambda < 0,
\]

with \( \lambda < 0 \) sufficiently negative, which will be assumed for the remainder of this proof.

We recall that a bounded, nonnegative (hence self-adjoint) integral operator with continuous integral kernel in \( L^2((a,b); dx) \), \( [a,b] \subseteq \mathbb{R}_+ \) (specializing to the situation at hand), has a nonnegative integral kernel on the diagonal (cf., e.g., [4, Proposition 5.6.8]). Moreover, we will rely on Mercer’s theorem (see, e.g., [4, Proposition 5.6.9]), according to which a bounded, nonnegative integral operator in \( L^2((a,b); dx) \), with continuous integral kernel belongs to the trace class if and only if its integral kernel on the diagonal lies in \( L^1((a,b); dx) \).

Equations \( \text{(3.23)} \) and \( \text{(3.24)} \) yield for \( \alpha = 0 \),

\[
\phi_0(\lambda, \cdot) \psi_{+,0}(\lambda, \cdot) = f_{+,1}(\lambda, \cdot, x_0) f_{+,2}(\lambda, \cdot, x_0)
- f_{+,1}(\lambda, 0, x_0)^{-1} f_{+,2}(\lambda, 0, x_0) f_{+,1}(\lambda, \cdot, x_0)^2,
\]

and since by \( \text{(3.21)} \) for \( j = 1 \) integrability properties of \( \text{(3.37)} \) over \( \mathbb{R}_+ \) depend on those of \( f_{+,1}(\lambda, \cdot, x_0) f_{+,2}(\lambda, \cdot, x_0) \), we now investigate the latter on \( [x_0, \infty) \).

Employing \( \text{(3.21)} \) once more then yields

\[
0 \leq [\phi_0(\lambda, x) \psi_{+,0}(\lambda, x) - \phi_0(\lambda_0, x) \psi_{+,0}(\lambda_0, x)]
= 2^{-1} \left\{ [q(x) - \lambda]^{-1/2} - [q(x) - \lambda_0]^{-1/2} \right\} [1 + o(1)]
\xrightarrow{x \to \infty} 4^{-1} (\lambda - \lambda_0) q(x)^{-3/2} [1 + o(1)]
\xrightarrow{x \to \infty} 4^{-1} (\lambda - \lambda_0) C_0 x^{-1-3\varepsilon_0/2} [1 + o(1)],
\]

according to \( \text{(3.33)} \), proving integrability near \( +\infty \) and hence \( \text{(3.29)} \).

By \( \text{(3.7)} \) with \( p = 2 \) this proves the first equality in \( \text{(3.30)} \).

To prove the second equality in \( \text{(3.30)} \), we now apply Lemma \( \text{3.2} \) in the trace class case \( q = 1 \) and combine it with \( \text{(3.29)} \) to arrive at

\[
\text{tr}_{L^2([a,b]; dx)} ((H_{+,\alpha} - \lambda I_+)^{-1} - (H_{+,\alpha} - \lambda_0 I_+)^{-1})
= \lim_{R \to \infty} \text{tr}_{L^2([a,b]; dx)} \left( P_R [(H_{+,\alpha} - \lambda I_+)^{-1} - (H_{+,\alpha} - \lambda_0 I_+)^{-1}] P_R \right)
= \lim_{R \to \infty} \int_0^R d\lambda \left[ \phi_0(\lambda, x) \dot{\psi}_{+,\alpha}(\lambda, x) - \dot{\phi}_0(\lambda_0, x) \dot{\psi}_{+,\alpha}(\lambda_0, x) \right]
\]

\[
= \lim_{R \to \infty} \left[ W(\phi_0(\lambda_0, \cdot), \dot{\psi}_{+,\alpha}(\lambda_0, \cdot))(R) - W(\phi_0(\lambda_0, \cdot), \dot{\psi}_{+,\alpha}(\lambda_0, \cdot))(R) \right]
+ W(\phi_0(\lambda_0, \cdot), \dot{\psi}_{+,\alpha}(\lambda_0, \cdot))(0) - W(\phi_0(\lambda_0, \cdot), \dot{\psi}_{+,\alpha}(\lambda_0, \cdot))(0)
\]

\[
= \lim_{R \to \infty} \left[ W(\phi_0(\lambda_0, \cdot), \dot{\psi}_{+,\alpha}(\lambda_0, \cdot))(R) - W(\phi_0(\lambda_0, \cdot), \dot{\psi}_{+,\alpha}(\lambda_0, \cdot))(R) \right], \tag{3.39}
\]

since

\[
W(\phi_0(\lambda_0, \cdot), \dot{\psi}_{+,\alpha}(\lambda_0, \cdot))(0) = -\sin(\alpha) \dot{\psi}_{+,\alpha}(\lambda_0, 0) - \cos(\alpha) \dot{\psi}_{+,\alpha}(\lambda_0, 0)
= -\frac{d}{d\lambda} [\sin(\alpha) \psi_{+,\alpha}(\lambda, 0) + \cos(\alpha) \psi_{+,\alpha}(\lambda, 0)] = 0. \tag{3.40}
\]
It remains to analyze the right-hand side of (3.39). To this end we note that
\[ \tau_+ \hat{f}_{+1}(z, x, x_0) = z \hat{f}_{+1}(z, x, x_0) + f_{+1}(z, x, x_0), \] (3.41)
and hence
\[ \hat{f}_{+1}(z, x, x_0) = c_1(z) f_{+1}(z, x, x_0) + c_2(z) f_{+2}(z, x, x_0) + f_{+1}(z, x, x_0) \int_0^x dx' f_{+1}(z, x', x_0) f_{+2}(z, x', x_0) \] (3.42)
\[ - f_{+2}(z, x, x_0) \int_0^x dx' f_{+1}(z, x', x_0)^2, \]
\[ \hat{f}_{+1}(z, x, x_0) = c_1(z) f'_{+1}(z, x, x_0) + c_2(z) f'_{+2}(z, x, x_0) + f'_{+1}(z, x, x_0) \int_0^x dx' f_{+1}(z, x', x_0) f_{+2}(z, x', x_0) \] (3.43)
\[ - f'_{+2}(z, x, x_0) \int_0^x dx' f_{+1}(z, x', x_0)^2. \]
Next, we claim that
\[ c_2(z) = \int_0^\infty dx' f_{+1}(z, x', x_0)^2, \] (3.44)
and hence (3.42), (3.43) simplify to
\[ \hat{f}_{+1}(z, x, x_0) = c_1(z) f_{+1}(z, x, x_0) + f_{+1}(z, x, x_0) \int_0^x dx' f_{+1}(z, x', x_0) f_{+2}(z, x', x_0) \] (3.45)
\[ f_{+2}(z, x, x_0) \int_0^\infty dx' f_{+1}(z, x', x_0)^2, \]
\[ \hat{f'}_{+1}(z, x, x_0) = c_1(z) f'_{+1}(z, x, x_0) + f'_{+1}(z, x, x_0) \int_0^x dx' f_{+1}(z, x', x_0) f_{+2}(z, x', x_0) \] (3.46)
\[ f'_{+2}(z, x, x_0) \int_0^\infty dx' f_{+1}(z, x', x_0)^2. \]
To infer the necessity of (3.44) one can argue by contradiction as follows: If (3.44) does not hold, then integrating \( \hat{f}_{+1}(z, x) \) with respect to \( z \) from \( \lambda_0 \) to \( \lambda \) along the negative real axis on the left-hand side of (3.42) yields
\[ \int_{\lambda_0}^{\lambda} dz \hat{f}_{+1}(z, x, x_0) = f_{+1}(\lambda, x, x_0) - f_{+1}(\lambda_0, x, x_0) \xrightarrow{\lambda \to \infty} 0 \] (3.47)
by the first asymptotic relation in (3.21). However, with (3.44) violated, integrating the right-hand side of (3.42) with respect to \( z \) from \( \lambda_0 \) to \( \lambda \) along the negative real axis now yields several contributions vanishing as \( x \to \infty \) (again invoking (3.21)), but there will also be one integral of the type
\[ \int_{\lambda_0}^{\lambda} dz f_{+2}(z, x, x_0) A(z, x) \xrightarrow{x \to \infty} 0 \] (3.48)
where $A(z, \cdot)$ is bounded with a finite nonzero limit, $\lim_{x \to \infty} A(z, x) = A(z, \infty) \neq 0$. Relation (3.48) contradicts (3.47), proving (3.44).

Investigating the leading asymptotic behavior (3.21), then shows that to obtain the leading relations (3.21) with respect to $\lambda$, finally implies $\psi_\alpha f'_{+,1}(\lambda, x, x_0)$, $\hat{f}_{+,2}(\lambda, x)$ $(3.41)$ with respect to $\lambda$, where $\hat{f}_{+,2}(\lambda, x)$ $(3.41)$ with respect to $\lambda$, $\hat{f}_{+,2}(\lambda, x)$ $(3.41)$ with respect to $\lambda$, $\hat{f}_{+,2}(\lambda, x)$ $(3.41)$ with respect to $\lambda$, $\hat{f}_{+,2}(\lambda, x)$ $(3.41)$ with respect to $\lambda$, $\hat{f}_{+,2}W_{\alpha}(\lambda, \cdot), \psi_{+,\alpha}(\lambda, \cdot)) = f_{+,2}(\lambda, R, x_0) \hat{f}_{+,1}(\lambda, R, x_0)
- f_{+,2}(\lambda, R, x_0) f'_{+,1}(\lambda, R, x_0) \frac{\sin(\alpha) \hat{f}_{+,1}(\lambda, 0, x_0) + \cos(\alpha) \hat{f}_{+,1}(\lambda, 0, x_0)}{\sin(\alpha) \hat{f}_{+,1}(\lambda, 0, x_0) + \cos(\alpha) \hat{f}_{+,1}(\lambda, 0, x_0)}
- f_{+,2}(\lambda, R, x_0) f_{+,1}(\lambda, R, x_0)
+ f_{+,2}(\lambda, R, x_0) f_{+,1}(\lambda, R, x_0) \sin(\alpha) f_{+,1}(\lambda, 0, x_0) + \cos(\alpha) f_{+,1}(\lambda, 0, x_0)
+ f_{+,1}(\lambda, R, x_0) f_{+,2}(\lambda, R, x_0) - f_{+,1}(\lambda, R, x_0) f_{+,2}(\lambda, R, x_0)
+ \frac{\sin(\alpha) f_{+,1}(\lambda, 0, x_0) + \cos(\alpha) f_{+,1}(\lambda, 0, x_0)}{\sin(\alpha) f_{+,1}(\lambda, 0, x_0) + \cos(\alpha) f_{+,1}(\lambda, 0, x_0)}.

(3.50)

for $\lambda < 0$ sufficiently negative. Insertion of (3.29) and (3.30) into (3.50) finally implies

$$W(\phi_\alpha(\lambda, \cdot), \psi_{+,\alpha}(\lambda, \cdot), \psi_{+,\alpha}(\lambda, \cdot)) = \frac{\sin(\alpha) f_{+,1}(\lambda, 0, x_0) + \cos(\alpha) f_{+,1}(\lambda, 0, x_0)}{\sin(\alpha) f_{+,1}(\lambda, 0, x_0) + \cos(\alpha) f_{+,1}(\lambda, 0, x_0)}
- 2^{-1} \left( \int_{x_0}^{R} dx [q(x) - \lambda]^{-1/2} \right) [1 + o(1)].$$

(3.51)
Returning to (3.39) this yields

\[
\begin{align*}
\text{tr}_{L^2(\mathbb{R}^+; dx)}((H_{+,\alpha} - \lambda I_{+})^{-1} - (H_{+,\alpha} - \lambda_0 I_{+})^{-1}) \\
= \lim_{{R \to \infty}} \left[ W(\phi_{\alpha}(\lambda_0, \cdot), \psi_{+,\alpha}(\lambda_0, \cdot))(R) - W(\phi_{\alpha}(\lambda, \cdot), \psi_{+,\alpha}(\lambda, \cdot))(R) \right], \\
= \frac{\sin(\alpha)f_{+,1}(\lambda_0, 0, x_0) + \cos(\alpha)f_{+,1}(\lambda_0, 0, x_0)}{\sin(\alpha)f_{+,1}(\lambda_0, 0, x_0) + \cos(\alpha)f_{+,1}(\lambda_0, 0, x_0)} \\
- \frac{\sin(\alpha)f'_{+,1}(\lambda_0, 0, x_0) + \cos(\alpha)f'_{+,1}(\lambda_0, 0, x_0)}{\sin(\alpha)f'_{+,1}(\lambda_0, 0, x_0) + \cos(\alpha)f'_{+,1}(\lambda_0, 0, x_0)} \\
+ 2^{-1}\left( \int_0^R dx \left\{ [q(x) - \lambda]^{-1/2} - [q(x) - \lambda_0]^{-1/2} \right\}[1 + o(1)] \right)
\end{align*}
\]

(3.52)

and hence (3.30) for \( z = \lambda < 0 \), \( z_0 = \lambda_0 < 0 \), both sufficiently negative. In this context one observes that for \( x_0 > 0 \) sufficiently large,

\[
2^{-1}\left( \int_0^R dx \left\{ [q(x) - \lambda]^{-1/2} - [q(x) - \lambda_0]^{-1/2} \right\} \right) = \frac{1}{4}(\lambda - \lambda_0) \left( \int_0^R dx q(x)^{-3/2} \right)[1 + o(1)]
\]

(3.53)

with \( q^{-3/2} \in L^1([x_0, \infty); dx) \) by Hypothesis (3.3).

Analytic continuation in \( z \) of both sides in (3.52) extends the latter to \( z \in \rho(H_{+,\alpha}) \). Similarly, analytic continuation in \( z_0 \) of both sides in (3.52) extends the latter to \( z_0 \in \rho(H_{+,\alpha}) \), completing the proof of (3.30).

Relation (3.31) then follows from integrating (3.30) with respect to the energy variable from \( z_0 \) to \( z \).

Next, we apply Theorem (3.3) to the following explicitly solvable example concerning the linear potential and denote by \( \text{Ai}(\cdot), \text{Bi}(\cdot) \) the Airy functions as discussed, for instance, in [1 Sect. 10.4].
Example 3.4. Consider the special case $q(x) = x$, $x \in \mathbb{R}_+$, and $\alpha = 0$. Then, for $x \in \mathbb{R}_+$, $z, z_0 \in \rho(H_{+,0})$,

\begin{align*}
  f_{+,1}(z, x, x_0) &= (2\pi)^{1/2}e^{(2/3)(x_0 - z)^{3/2}}Ai(x - z), \\
  f_{+,2}(z, x, x_0) &= (\pi/2)^{1/2}e^{-(2/3)(x_0 - z)^{3/2}}Bi(x - z), \\
  W(f_{+,1}(z, \cdot, x_0), f_{+,2}(z, \cdot, x_0)) &= 1, \\
  \phi_0(z, x) &= \pi[ Ai(-z)Bi(x - z) - Bi(-z)Ai(x - z) ], \\
  \psi_{+,0}(z, x) &= Ai(x - z)/Ai(-z), \\
  W(\phi_0(z, \cdot), \psi_{+,0}(z, \cdot))(x) &= \pi[ Ai'(x - z)Bi'(x - z) - (x - z)Ai(x - z)Bi(x - z) ] - [ Ai'(-z)/Ai(-z) ], \\
  I(z, z_0, x_0) &= \int_{x_0}^{\infty} dx \{ [x - z]^{-1/2} - [x - z_0]^{-1/2} \} \\
  &= 2((x_0 - z_0)^{1/2} - (x_0 - z)^{1/2}), \\
  \text{tr}_{L^2(\mathbb{R}_+:dx)}((H_{+,0} - zI_+)^{-1} - (H_{+,0} - z_0I_+)^{-1}) \\
  &= \psi'_{+,0}(z, 0) - \psi'_{+,0}(z_0, 0) = [ Ai'(z)/Ai(z) ] - [ Ai'(-z)/Ai(-z) ], \\
  \det_{L^2(\mathbb{R}_+:dx)}(I_+ - (z - z_0)(H_{+,0} - z_0I_+)^{-1}) \\
  &= [ Ai(-z)/Ai(-z_0) ] \exp \left( (z - z_0)[ Ai'(-z_0)/Ai(-z_0) ] \right).
\end{align*}

We note that (3.62) was recently considered in [22], but the exponential factor in (3.62) was missed in [22].

Finally, we generalize Theorem 3.3 to the following setting.

Theorem 3.5. Assume Hypothesis 3.1. $z \in \rho(H_{+,\alpha_2})$, $z_0 \in \rho(H_{+,\alpha_1})$, and $\alpha_1, \alpha_2 \in [0, \pi)$. Then,

\begin{align*}
  [(H_{+,\alpha_2} - zI_+)^{-1} - (H_{+,\alpha_1} - z_0I_+)^{-1}] \in B_1(L^2(\mathbb{R}_+:dx)),
\end{align*}

and (cf. 3.32)

\begin{align*}
  \text{tr}_{L^2(\mathbb{R}_+:dx)}((H_{+,\alpha_2} - zI_+)^{-1} - (H_{+,\alpha_1} - z_0I_+)^{-1}) = -\frac{d}{dz} \ln \left( \frac{\sin(\alpha_2)f_{+,1}^{(1)}(z, 0, x_0) + \cos(\alpha_2)f_{+,1}(z, 0, x_0)}{\sin(\alpha_1)f_{+,1}^{(1)}(z, 0, x_0) + \cos(\alpha_1)f_{+,1}(z, 0, x_0)} \right), \\
  + \frac{1}{2}I(z, z_0, x_0).
\end{align*}

Proof. Eq. (3.63) is established exactly as in the proof of Theorem 3.3. Furthermore, as argued there one has

\begin{align*}
  \text{tr}_{L^2(\mathbb{R}_+:dx)}((H_{+,\alpha_2} - \lambda I_+)^{-1} - (H_{+,\alpha_1} - \lambda_0 I_+)^{-1}) \\
  = \lim_{R \to \infty} \left[ W(\phi_{\alpha_1}(\lambda_0, \cdot), \psi_{+,\alpha_1}(\lambda_0, \cdot))(R) - W(\phi_{\alpha_2}(\lambda, \cdot), \psi_{+,\alpha_2}(\lambda, \cdot))(R) \right].
\end{align*}

Using eq. (3.61) then immediately implies (3.64).
Setting $z = z_0$, we obtain in particular
\[
\begin{align*}
\text{tr}_{L^2(\mathbb{R}^+; dx)}(H_{+,\alpha_2} - zI_+)^{-1} - (H_{+,\alpha_1} - zI_+)^{-1} = & -\frac{d}{dz} \ln \left( \frac{\sin(\alpha_1)f'_{+,1}(z, 0, x_0) + \cos(\alpha_1)f_{+,1}(z, 0, x_0)}{\sin(\alpha_2)f_{+,1}(z, 0, x_0) + \cos(\alpha_2)f'_{+,1}(z, 0, x_0)} \right). \\
\end{align*}
\]
(3.66)

Remark 3.6 In order to prove Theorem 3.5 one could instead have proven the slightly simpler result (3.66) and then note that
\[
\begin{align*}
\text{tr}_{L^2(\mathbb{R}^+; dx)} (H_{+,\alpha_2} - zI_+)^{-1} - (H_{+,\alpha_1} - z_0I_+)^{-1} = & \text{tr}_{L^2(\mathbb{R}^+; dx)} (H_{+,\alpha_2} - zI_+)^{-1} - (H_{+,\alpha_1} - zI_+)^{-1} \\
+ & \text{tr}_{L^2(\mathbb{R}^+; dx)} (H_{+,\alpha_1} - zI_+)^{-1} - (H_{+,\alpha_1} - z_0I_+)^{-1}, \\
\end{align*}
\]
(3.67)

which, using (3.66) together with Theorem 3.3 implies Theorem 3.5.

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