Structured strong linearizations of structured rational matrices

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ABSTRACT
Structured rational matrices such as symmetric, skew-symmetric, Hamiltonian, skew-Hamiltonian, Hermitian, and para-Hermitian rational matrices arise in many applications. Linearizations of rational matrices have been introduced recently for computing poles, eigenvalues, eigenvectors, minimal bases and minimal indices of rational matrices. For structured rational matrices, it is desirable to construct structure-preserving linearizations so as to preserve the symmetry in the eigenvalues and poles of the rational matrices. With a view to constructing structure-preserving linearizations of structured rational matrices, we propose a family of Fiedler-like pencils and show that the family of Fiedler-like pencils is a rich source of structure-preserving strong linearizations of structured rational matrices. We construct symmetric, skew-symmetric, Hamiltonian, skew-Hamiltonian, Hermitian, skew-Hermitian, para-Hermitian and para-skew-Hermitian strong linearizations of a rational matrix $G(\lambda)$ when $G(\lambda)$ has the same structure. We also describe recovery of eigenvectors, minimal bases and minimal indices of $G(\lambda)$ from those of the linearizations of $G(\lambda)$ and show that the recovery is operation-free.

1. Introduction
Structured rational matrices such as symmetric, Hamiltonian, skew-symmetric, skew-Hamiltonian, Hermitian, skew-Hermitian, para-Hermitian and para-skew-Hermitian rational matrices arise in many applications, see [1–8] and the references therein. For example, the Hermitian rational eigenvalue problem

$$G(\lambda)u := \left( \lambda^2 M + K - \sum_{i=1}^{k} \frac{1}{1 + \lambda b_i} \Delta K_i \right) u = 0$$

arises in the study of damped vibration of a structure, where $M$ and $K$ are positive definite, $b_i$ is a relaxation parameter and $\Delta K_i$ is an assemblage of element stiffness matrices [6,7].

Also various structured rational matrices arise as transfer functions of linear time-invariant (LTI) systems, see [1,3–5,8,9].
Our main aim in this paper is to construct structure-preserving strong linearizations of structured rational matrices and to recover eigenvectors, minimal bases and minimal indices of rational matrices from those of the linearizations. Let \( G(\lambda) \) be an \( n \times n \) rational matrix, that is, the entries of \( G(\lambda) \) are scalar rational functions of the form \( p(\lambda)/q(\lambda) \), where \( p(\lambda) \) and \( q(\lambda) \) are scalar polynomials. We consider the following structures:

\[
\begin{align*}
\text{symmetric} : & \quad G(\lambda)^T = G(\lambda) \\
\text{skew-symmetric} : & \quad G(\lambda)^T = -G(\lambda) \\
\text{Hamiltonian} : & \quad G(\lambda)^T = G(-\lambda) \\
\text{skew-Hamiltonian} : & \quad G(\lambda)^T = -G(-\lambda)
\end{align*}
\]

where \( X^T \) (resp., \( X^* \)) denotes the transpose (resp., conjugate transpose) of a matrix \( X \) and \( \bar{\lambda} \) denotes the conjugate of \( \lambda \). For more on these structured rational matrices, we refer to [1–9] and the references therein.

We mention that there is a slight difference in the naming convention between some of the structured rational matrices and structured matrix polynomials. The Hamiltonian (resp., skew-Hamiltonian) structure for rational matrices is known as \( T \)-even (resp., \( T \)-odd) structure for matrix polynomials [10]. On the other hand, para-Hermitian (resp., para-skew-Hermitian) structure for rational matrices is known as \( * \)-even (resp., \( * \)-odd) structure for matrix polynomials [10]. We follow both the naming conventions in the rest of the paper without any bias.

Linearization of rational matrices is a relatively new concept and has been studied in [7,11–15]. However, barring symmetric linearizations [15,16], structure-preserving linearizations of structured rational matrices have not been constructed in the literature. The frameworks of Fielder pencils, generalized Fiedler pencils, and affine spaces of pencils for rational matrices presented in [11,12,15] are not adequate for construction of structure-preserving linearizations of structured rational matrices.

The main aim of this paper is to present a framework for construction of structure-preserving strong linearizations of structured rational matrices considered in (1). For this purpose, we propose a new family of Fiedler-like pencils of \( G(\lambda) \) which we refer to as generalized Fiedler pencils with repetition (GFPRs) of \( G(\lambda) \). We show that the family of GFPRs of \( G(\lambda) \) is a rich source of structure-preserving linearizations of \( G(\lambda) \) and utilize these pencils to construct structure-preserving Rosenbrock strong linearizations of \( G(\lambda) \). In fact, we show that structure-preserving linearizations of \( G(\lambda) \) can be constructed directly from structure-preserving linearizations of the polynomial part of \( G(\lambda) \) and the construction is operation-free. Also, we describe operation-free recovery of eigenvectors, minimal bases and minimal indices of \( G(\lambda) \) from those of the GFPRs of \( G(\lambda) \).

The rest of the paper is organized as follows. We collect some basic results in Section 2. We introduce GFPRs of \( G(\lambda) \) in Section 3. We construct structure-preserving Rosenbrock strong linearizations of structured rational matrices in Section 4. Finally, we describe recovery of eigenvectors, minimal bases and minimal indices of \( G(\lambda) \) from those of the Rosenbrock strong linearizations of \( G(\lambda) \) in Section 5.

**Notation.** We denote the \( j \)-th column of the \( n \times n \) identity matrix \( I_n \) by \( e_j \) and the transpose (resp., conjugate transpose) of an \( m \times n \) matrix \( A \) by \( A^T \) (resp., \( A^* \)). The right and left
null spaces of $A$ are given by $\mathcal{N}_r(A) := \{x \in \mathbb{C}^n : Ax = 0\}$ and $\mathcal{N}_f(A) := \{x \in \mathbb{C}^m : x^T A = 0\}$, respectively. We denote the Kronecker product of two matrices $A$ and $B$ by $A \otimes B$.

2. Preliminaries

Let $\mathbb{C}[\lambda]$ denote the ring of scalar polynomials with coefficients in $\mathbb{C}$ and $\mathbb{C}(\lambda)$ denote the field of rational functions of the form $p(\lambda)/q(\lambda)$, where $p(\lambda)$ and $q(\lambda)$ are polynomials in $\mathbb{C}[\lambda]$. We denote by $\mathbb{C}^{m \times n}, \mathbb{C}[\lambda]^{m \times n}$ and $\mathbb{C}(\lambda)^{m \times n}$, the sets of all $m \times n$ matrices with entries in $\mathbb{C}$, $\mathbb{C}[\lambda]$ and $\mathbb{C}(\lambda)$, respectively. The elements of $\mathbb{C}[\lambda]^{m \times n}$ are called matrix polynomials. If $P(\lambda) \in \mathbb{C}[\lambda]^{m \times n}$ then it can be written as

$$P(\lambda) = A_0 + \lambda A_1 + \cdots + \lambda^\ell A_\ell$$

for some $m \times n$ matrices $A_0, \ldots, A_\ell$ such that $A_\ell \neq 0$. The number $\ell$ is called the degree of $P(\lambda)$. We write $\ell = \deg(P(\lambda))$. The elements of $\mathbb{C}(\lambda)^{m \times n}$ are called rational matrices.

Zeros and poles. Let $G(\lambda) \in \mathbb{C}(\lambda)^{m \times n}$. The rank of $G(\lambda)$ over the field $\mathbb{C}(\lambda)$ is called the normal rank of $G(\lambda)$ and is denoted by $\text{nrank}(G)$. It is well known that $\text{nrank}(G) = \max\{\text{rank}(G(\lambda)) : \lambda \in \mathbb{C} \text{ is not a pole}\}$. If $\text{nrank}(G) = n = m$ then $G(\lambda)$ is said to be regular, otherwise $G(\lambda)$ is said to be singular. A complex number $\mu \in \mathbb{C}$ is said to be an eigenvalue of $G(\lambda)$ if $\text{rank}(G(\mu)) < \text{nrank}(G)$. We denote the set of eigenvalues of $G$ by $\text{eig}(G)$. Let

$$D(\lambda) := \text{diag} \left( \frac{\phi_1(\lambda)}{\psi_1(\lambda)}, \ldots, \frac{\phi_k(\lambda)}{\psi_k(\lambda)}, 0_{m-k,n-k} \right)$$

be the Smith–McMillan form [17,18] of $G(\lambda)$, where $k := \text{nrank}(G)$ and the scalar polynomials $\phi_i(\lambda)$ and $\psi_i(\lambda)$ are monic and pairwise coprime, $\phi_i(\lambda)$ divides $\phi_{i+1}(\lambda)$ and $\psi_{i+1}(\lambda)$ divides $\psi_i(\lambda)$, for $i = 1, 2, \ldots, k - 1$. Consider the zero polynomial $\phi_G(\lambda) := \prod_{j=1}^k \phi_j(\lambda)$ and the pole polynomial $\psi_G(\lambda) := \prod_{j=1}^k \psi_j(\lambda)$. Then $\mu \in \mathbb{C}$ is a pole of $G(\lambda)$ if $\phi_G(\mu) = 0$. A complex number $\mu$ is said to be a zero of $G(\lambda)$ if $\phi_G(\mu) = 0$. The spectrum of $G(\lambda)$ is given by $\text{Sp}(G) := \{\lambda \in \mathbb{C} : \phi_G(\lambda) = 0\}$ and consists of the finite zeros of $G(\lambda)$. Note that $\text{eig}(G) \subset \text{Sp}(G)$. See [11,17] for more on eigenvalues and zeros of $G(\lambda)$.

Minimal bases and minimal indices. Let $\mathbb{C}[\lambda]^n$ and $\mathbb{C}(\lambda)^n$ denote the sets of all column vectors of length $n$ with entries in $\mathbb{C}[\lambda]$ and $\mathbb{C}(\lambda)$, respectively. Then $\mathbb{C}[\lambda]^n$ is a vector space over $\mathbb{C}$ and the elements of $\mathbb{C}[\lambda]^n$ are called polynomial vectors. Note that $\mathbb{C}(\lambda)^n$ is a vector space over $\mathbb{C}(\lambda)$ and is called a rational vector space. Let $\mathcal{V}$ be a rational subspace of $\mathbb{C}(\lambda)^n$. Then it is well known [19] that $\mathcal{V}$ has a polynomial basis, that is, $\mathcal{V}$ has a basis consisting of polynomial vectors. Let $\mathcal{B} := (x_1(\lambda), \ldots, x_p(\lambda))$ be a polynomial basis of $\mathcal{V}$ ordered so that

$$\deg(x_1(\lambda)) \leq \cdots \leq \deg(x_p(\lambda)), \quad (2)$$

where $x_1(\lambda), \ldots, x_p(\lambda)$ are elements of $\mathbb{C}[\lambda]^n$. Then $\text{Ord}(\mathcal{B}) := \deg(x_1(\lambda)) + \cdots + \deg(x_p(\lambda))$ is called the order of the basis $\mathcal{B}$. A polynomial basis $\mathcal{B}$ of $\mathcal{V}$ is said to be a minimal basis of $\mathcal{V}$ if $\mathcal{B}$ has minimum order among all polynomial bases of $\mathcal{V}$, see [19]. In other words, $\mathcal{B}$ is a minimal basis of $\mathcal{V}$ provided that for any polynomial basis $\mathcal{E}$ of $\mathcal{V}$ we have $\text{Ord}(\mathcal{E}) \geq \text{Ord}(\mathcal{B})$. Although, minimal bases of $\mathcal{V}$ are not unique, however, for any minimal basis of $\mathcal{V}$ the ordered list of degrees as given in (2) is always the same. Thus, the
ordered list of degrees of a minimal basis of \( \mathcal{V} \) is uniquely determined by \( \mathcal{V} \). The ordered degrees \( \deg(x_1(\lambda)) \leq \cdots \leq \deg(x_p(\lambda)) \) of a minimal basis \( B := (x_1(\lambda), \ldots, x_p(\lambda)) \) of \( \mathcal{V} \) are called the minimal indices of \( \mathcal{V} \), see [19].

Let \( G(\lambda) \in \mathbb{C}(\lambda)^{m \times n} \) be a rational matrix. The right null-space \( \mathcal{N}_r(G) \) and the left null-space \( \mathcal{N}_l(G) \) of \( G(\lambda) \) are given by

\[
\mathcal{N}_r(G) := \{ x(\lambda) \in \mathbb{C}(\lambda)^n : G(\lambda)x(\lambda) = 0 \} \subset \mathbb{C}(\lambda)^n,
\]

\[
\mathcal{N}_l(G) := \{ y(\lambda) \in \mathbb{C}(\lambda)^m : y(\lambda)^T G(\lambda) = 0 \} \subset \mathbb{C}(\lambda)^m.
\]

Note that if \( G(\lambda) \) is singular then at least one of the null-spaces is non-trivial. If \( \mathcal{N}_r(G) \) is non-trivial then minimal bases and minimal indices of the rational subspace \( \mathcal{N}_r(G) \) are, respectively, called right minimal bases and right minimal indices of \( G(\lambda) \). Similarly, when \( \mathcal{N}_l(G) \) is non-trivial, minimal bases and minimal indices of the rational subspace \( \mathcal{N}_l(G) \) are, respectively, called left minimal bases and left minimal indices of \( G(\lambda) \). More on minimal bases and minimal indices of rational matrices and their applications can be found in [17,19].

Let \( Z(\lambda) \in \mathbb{C}[\lambda]^{m \times n} \) be a matrix polynomial. We say that \( Z(\lambda) \) is a minimal basis if the columns of \( Z(\lambda) \) form a minimal basis of the subspace of \( \mathbb{C}(\lambda)^m \) spanned (over the field \( \mathbb{C}(\lambda) \)) by the columns of \( Z(\lambda) \).

**Example 2.1:** Consider the \( 3 \times 4 \) rational matrix

\[
G(\lambda) := \begin{bmatrix}
1 & 0 & 0 & 0 \\
\lambda - 1 & 1 & \lambda^2 & 0 \\
0 & 1 & 0 & 1 & \lambda^3
\end{bmatrix}.
\]

Since \( \text{nrank}(G) = 3 \), we have \( \dim \mathcal{N}_l(G) = 0 \) and \( \dim \mathcal{N}_r(G) = 1 \). Hence \( G(\lambda) \) has a right minimal index but no left minimal indices. Note that \( x(\lambda) := [0 \lambda^5 - \lambda^3 1]^T \in \mathcal{N}_r(G) \) and hence \( x(\lambda) \) is a polynomial basis of \( \mathcal{N}_r(G) \). It is easily seen that if \( y(\lambda) \) is a polynomial basis of \( \mathcal{N}_r(G) \) then \( y(\lambda) = r(\lambda)x(\lambda) \) for some \( r(\lambda) \in \mathbb{C}[\lambda] \subset \mathbb{C}(\lambda) \). Consequently, \( x(\lambda) \) is a right minimal basis and \( \epsilon := \deg(x(\lambda)) = 5 \) is the right minimal index of \( G(\lambda) \).

Let \( G(\lambda) \in \mathbb{C}(\lambda)^{n \times n} \). We consider a realization of \( G(\lambda) \) of the form

\[
G(\lambda) = \sum_{j=0}^{m} A_j \lambda^j + C(\lambda E - A)^{-1}B =: P(\lambda) + C(\lambda E - A)^{-1}B,
\]

where \( \lambda E - A \) is an \( r \times r \) matrix pencil with \( E \) being non-singular, \( C \in \mathbb{C}^{n \times r} \) and \( B \in \mathbb{C}^{r \times n} \). The realization (3) is said to be minimal if the size of the pencil \( \lambda E - A \) is the smallest among all the realizations of \( G(\lambda) \), see [17]. The matrix polynomial

\[
S(\lambda) := \begin{bmatrix}
P(\lambda) \\
C/B
\end{bmatrix}
\]

is called the system matrix (or the Rosenbrock system matrix) of \( G(\lambda) \) associated with the realization (3). The system matrix \( S(\lambda) \) is said to be irreducible if the realization (3) is minimal. The system matrix \( S(\lambda) \) is irreducible if and only if \( \text{rank}([B \ A - \lambda E]) = r =
rank\([\{C^T (A - \lambda E)^T\}]\), see [17,18]. Observe that \(\text{eig}(G) \subset \text{eig}(S)\) and we have \(\text{eig}(S) = \text{Sp}(G)\) when \(S(\lambda)\) is irreducible, see [11,18].

An \(n \times n\) matrix polynomial \(U(\lambda)\) is said to be \textit{unimodular} if \(\det(U(\lambda))\) is a non-zero constant independent of \(\lambda\). A rational matrix \(G(\lambda)\) is said to be \textit{proper} if \(G(\lambda) \to D\) as \(\lambda \to \infty\), where \(D\) is a matrix. An \(n \times n\) rational matrix \(F(\lambda)\) is said to be \textit{biproper} if \(F(\lambda)\) is proper and \(F(\infty)\) is a non-singular matrix [20].

**Definition 2.2 ([15]):** Let \(\mathbb{L}(\lambda)\) be an \((mn + r) \times (mn + r)\) irreducible system matrix of the form

\[
\mathbb{L}(\lambda) := \begin{bmatrix}
\mathcal{X} - \lambda \mathcal{Y} & C \\
B & H - \lambda K
\end{bmatrix},
\]

where \(H - \lambda K\) is an \(r \times r\) pencil with \(K\) being non-singular. Then \(\mathbb{L}(\lambda)\) is said to be a Rosenbrock strong linearization of \(G(\lambda)\) if the following conditions hold.

(a) There exist \(mn \times mn\) unimodular matrix polynomials \(U(\lambda)\) and \(V(\lambda)\), and \(r \times r\) non-singular matrices \(U_0\) and \(V_0\) such that

\[
\begin{bmatrix}
U(\lambda) & 0 \\
0 & U_0
\end{bmatrix} \mathbb{L}(\lambda) \begin{bmatrix}
V(\lambda) & 0 \\
0 & V_0
\end{bmatrix} = \begin{bmatrix}
\mathcal{I}_{(m-1)n} & 0 \\
0 & S(\lambda)
\end{bmatrix}.
\]

(b) There exist \(mn \times mn\) biproper rational matrices \(O_L(\lambda)\) and \(O_R(\lambda)\) such that

\[
O_L(\lambda) \lambda^{-1} G(\lambda) O_R(\lambda) = \begin{bmatrix}
\mathcal{I}_{(m-1)n} & 0 \\
0 & \lambda^{-m} G(\lambda)
\end{bmatrix},
\]

where \(G(\lambda) := \mathcal{X} - \lambda \mathcal{Y} + C(\lambda K - H)^{-1} B\) is the transfer function of \(\mathbb{L}(\lambda)\).

The pencil \(\mathbb{L}(\lambda)\) is also referred to as a Rosenbrock strong linearization of \(S(\lambda)\).

We refer to [15] for more on Rosenbrock strong linearizations of \(G(\lambda)\) and the relation between the structural indices of (finite and infinite) zeros and poles of \(G(\lambda)\) and \(\mathbb{L}(\lambda)\). Suffice it to say that the condition (a) ensures (see, [12, Theorem 3.4]) that \(U(\lambda)G(\lambda)V(\lambda) = \text{diag}(\mathcal{I}_{(m-1)n}, G(\lambda))\) which in turn ensures that \(G(\lambda)\) and \(G(\lambda)\) have the same finite zeros and poles. The irreducibility of \(\mathbb{L}(\lambda)\) guarantees that the finite zeros and poles of \(G(\lambda)\) are the same as the finite eigenvalues of \(\mathbb{L}(\lambda)\) and \(H - \lambda K\), respectively; see [15,17]. On the other hand, the condition (b) ensures that the structural indices of zeros and poles of \(G(\lambda)\) at infinity can be recovered from the structural indices of eigenvalues and poles of \(\mathbb{L}(\lambda)\) at infinity (see [15]). Thus the zeros and poles of \(G(\lambda)\) including their structural indices can be obtained by solving the eigenvalue problems \(\mathbb{L}(\lambda)v = 0\) and \((H - \lambda K)u = 0\); see [11,12,15,21]. As mentioned in [15], Definition 2.2 is equivalent to the definition of strong linearization of rational matrices presented in [13].
2.1. Fiedler matrices

For \( k, \ell \in \mathbb{Z} \), we use the following notation

\[
  k : \ell := \begin{cases} 
  k, k+1, \ldots, \ell & \text{if } k \leq \ell, \\
  \emptyset & \text{if } k > \ell.
  \end{cases}
\]

When \( k \leq \ell \), \((k : \ell)\) is called a string of integers from \( k \) to \( \ell \).

Assumption: For the rest of the paper, we assume that \( P(\lambda) := \sum_{i=0}^{m} \lambda^i A_i \) with \( A_m \neq 0 \) and the realization \( G(\lambda) = P(\lambda) + C(\lambda E - A)^{-1} B \) of \( G(\lambda) \) given by (3) is minimal. The system matrix \( S(\lambda) \) associated with \( G(\lambda) \) is given by (4).

For an arbitrary matrix \( X \in \mathbb{C}^{n \times n} \), we define the elementary matrices by [22]

\[
  M_0(X) := \begin{bmatrix} I_{(m-1)n} & X \end{bmatrix},
\]

\[
  M_i(X) := \begin{bmatrix} I_{(m-i-1)n} & X & I_n & I_n & I_{(i-1)n} \\
  X & I_{(m-1)n} & 0 & 
\end{bmatrix}
\quad \text{for } i = 1 : m - 1,
\]

\[
  M_{-m}(X) := \begin{bmatrix} X & 
\end{bmatrix},
\]

\[
  M_{-i}(X) := \begin{bmatrix} I_{(m-i-1)n} & X & I_n & I_n & I_{(i-1)n} \\
  0 & I_n & X & I_{(i-1)n} & 
\end{bmatrix}
\quad \text{for } i = 1 : m - 1.
\]

Note that, for \( i = 1 : m - 1 \), \( M_i(X) \) and \( M_{-i}(X) \) are invertible and \( (M_i(X))^{-1} = M_{-i}(-X) \) for any arbitrary matrix \( X \in \mathbb{C}^{n \times n} \). On the other hand, the matrices \( M_0(X) \) and \( M_{-m}(X) \) are invertible if and only if \( X \) is invertible. Further, \( M_i(X)M_j(Y) = M_{j}(Y)M_{i}(X) \) holds for any matrices \( X, Y \in \mathbb{C}^{n \times n} \) if \( ||i| - |j|| > 1 \), see [22]. For \( i \in \{-m : m - 1\} \), we define [22]

\[
  M_i^p := \begin{cases} 
  M_i(-A_i) & \text{if } i \geq 0, \\
  M_i(A_{-i}) & \text{if } i < 0.
  \end{cases}
\]

Then \( M_i^p, i \in \{-m : m - 1\} \), are the Fiedler matrices of \( P(\lambda) \) (see [23]).

For an arbitrary matrix \( X \in \mathbb{C}^{n \times n} \), we define \( (mn + r) \times (mn + r) \) elementary matrices \( M_i(X) \) by

\[
  \mathbb{M}_i(X) := \begin{bmatrix} M_i(X) & I_r \\
  I_r & 
\end{bmatrix}
\quad \text{for } i \in \{-m : m - 1\}.
\]

Note that \( \mathbb{M}_i(X) \) and \( \mathbb{M}_{-i}(X) \) are invertible and \( (\mathbb{M}_i(X))^{-1} = \mathbb{M}_{-i}(-X) \) for \( i = 1 : m - 1 \). On the other hand, the matrices \( \mathbb{M}_0(X) \) and \( \mathbb{M}_{-m}(X) \) are invertible if and only if \( X \) is invertible. For any arbitrary matrices \( X, Y \in \mathbb{C}^{n \times n} \), we have \( \mathbb{M}_i(X)\mathbb{M}_j(Y) = \mathbb{M}_j(Y)\mathbb{M}_i(X) \) if \( ||i| - |j|| > 1 \).
The \((mn + r) \times (mn + r)\) Fiedler matrices \(M_i^S\), \(i \in \{-m : m - 1\}\), associated with the system matrix \(G\) are defined by Alam and Behera [11,12]

\[
M_0^S := \begin{bmatrix} M_0^P & -e_m \otimes C \\ -e_m^T \otimes B & -A \end{bmatrix}, \quad M_{-m}^S := \begin{bmatrix} M_{-m}^P & 0 \\ 0 & -E \end{bmatrix}, \quad M_i^S := \begin{bmatrix} M_i^P & 0 \\ 0 & I_r \end{bmatrix},
\]

for \(i = 1 : m - 1\), and \(M_i^S := (M_i^S)^{-1}\) for \(i = 1 : m - 1\). The matrices \(M_i^S\) are also referred to as Fiedler matrices of \(G(\lambda)\). We have \(M_i^S M_j^S = M_{ij}^S M_{ji}^S\) for \(||i| - |j|| > 1\) except for \(||i| - |j|| = m\). For convenience in defining Fiedler-like pencils, we define

\[
M_i^P := \begin{bmatrix} M_i^P \\ I_r \end{bmatrix} \quad \text{for} \quad i \in \{-m : m - 1\}.
\]

**Remark 2.3:** Note that \(M_i^P = M_i^P\), for \(i = \pm 1, \ldots, \pm (m - 1)\), and \(M_0^S \neq M_0^P\) and \(M_{-m}^S \neq M_{-m}^P\). The utility of the notation \(M_i^P\) will be clear when we analyse Fiedler-like pencils.

**Definition 2.4 ([12]):** Let \(N\) be a finite set. A bijection \(\omega : N \to N\) is called a permutation of \(N\). \(\tau\) is said to be a sub-permutation of \(N\) if \(\tau\) is a permutation of a subset of \(N\).

For a sub-permutation \(\sigma := (i_1, i_2, \ldots, i_p)\) of \(-m : m - 1\), we define

\[
M_0^S := M_{i_1}^S M_{i_2}^S \cdots M_{i_p}^S.
\]

The Fiedler pencils of \(G(\lambda)\) are defined as follows.

**Definition 2.5 ([11], Fiedler pencil):** Let \(\sigma\) be a permutation of \(\{0 : m - 1\}\). Then \(L_\sigma(\lambda) := \lambda \overline{M}_\sigma^S - \overline{M}_\sigma^S\) is called a Fiedler pencil (FP) of \(G(\lambda)\) associated with \(\sigma\). The pencil \(L_\sigma(\lambda)\) is also referred to as a Fiedler pencil of \(S(\lambda)\).

**Example 2.6:** Let \(G(\lambda) := \sum_{i=0}^{4} \lambda^i A_i + C(\lambda E - A)^{-1} B\). Consider the permutations \(\sigma := (1, 2, 3, 0)\) and \(\tau := (2, 3, 0, 1)\) of \(\{0, 1, 2, 3\}\). Then

\[
L_\sigma(\lambda) = \lambda \overline{M}_{-4}^S - \overline{M}_{(1,2,3,0)}^S = \begin{bmatrix} \lambda A_4 + A_3 & -I_n & 0 & 0 & 0 \\
A_2 & \lambda I_n & -I_n & 0 & 0 \\
A_1 & 0 & \lambda I_n & A_0 & C \\
-I_n & 0 & 0 & \lambda I_n & 0 \\
0 & 0 & 0 & B & A - \lambda E \end{bmatrix}
\]

and

\[
L_\tau(\lambda) = \lambda \overline{M}_{-4}^S - \overline{M}_{(2,3,0,1)}^S = \begin{bmatrix} \lambda A_4 + A_3 & -I_n & 0 & 0 & 0 \\
A_2 & \lambda I_n & A_1 & -I_n & 0 \\
-I_n & 0 & \lambda I_n & 0 & 0 \\
0 & 0 & A_0 & \lambda I_n & C \\
0 & 0 & B & A - \lambda E & 0 \end{bmatrix}
\]

are Fiedler pencils of \(G(\lambda)\).

A drawback of Fiedler pencils of \(G(\lambda)\) is that they do not preserve structures of \(G(\lambda)\). Hence, with a view to constructing structure-preserving linearizations of \(G(\lambda)\), we introduce a family of Fielder-like pencils of \(G(\lambda)\) in Section 3.
2.2. Index tuple

We now briefly discuss index tuples and their properties, which will be needed for defining Fiedler-like pencils.

Definition 2.7 ([12]): An ordered tuple \( t := (t_1, t_2, \ldots, t_p) \) is said to be an index tuple containing indices from \( \mathbb{Z} \) if \( t_i \in \mathbb{Z} \) for \( i = 1 : p \). We define \( -t := (-t_1, -t_2, \ldots, -t_p) \), \( \text{rev}(t) := (t_p, t_{p-1}, \ldots, t_1) \) and \( t + k := (t_1 + k, t_2 + k, \ldots, t_p + k) \) for \( k \in \mathbb{Z} \). For any index tuples \( t := (t_1, \ldots, t_p) \) and \( s := (s_1, \ldots, s_q) \), we define \( t \cup s := (t_1, \ldots, t_p, s_1, \ldots, s_q) \).

Next, we define SIP, rsf and csf of an index tuple which will be used extensively.

Definition 2.8 ([24,25]): Let \( \sigma := (i_1, i_2, \ldots, i_t) \) be an index tuple containing indices from \( \{0, 1, \ldots, h\} \) for some non-negative integer \( h \). Then:

(a) \( \sigma \) is said to satisfy the Successor Infix Property (SIP) if for every pair of indices \( i_a, i_b \in \sigma \) with \( 1 \leq a < b \leq t \) satisfying \( i_a = i_b \), there exists at least one index \( i_c = i_a + 1 \) such that \( a < c < b \). Let \( \alpha \) be an index tuple containing indices from \( \{-h, -h + 1, \ldots, -1\} \). Then \( \alpha \) is said to satisfy the SIP if \( \alpha + h \) satisfies the SIP.

(b) \( \sigma \) is said to be in column standard form if

\[
\sigma = (a_s : b_s, a_{s-1} : b_{s-1}, \ldots, a_2 : b_2, a_1 : b_1),
\]

with \( 0 \leq b_1 < \cdots < b_{s-1} < b_s \leq h \) and \( 0 \leq a_j \leq b_j \), for all \( j = 1, \ldots, s \). We denote the column standard form of \( \sigma \) by csf(\( \sigma \)). Let \( \beta \) be an index tuple containing indices from \( \{-h, -h + 1, \ldots, -1\} \). Then \( \beta \) is said to be in column standard form if \( \beta + h \) is in column standard form.

Definition 2.9 ([22]): Let \( \alpha \) and \( \beta \) be two index tuples. Then \( \alpha \) is said to be a subtuple of \( \beta \) if \( \alpha = \beta \) or if \( \alpha \) can be obtained from \( \beta \) by deleting some indices in \( \beta \).

Example 2.10: Let \( \alpha = (1, 2, 0, 3, 0, 2) \) be an index tuple. Then \( (2, 3, 2) \) is a subtuple of \( \alpha \) but \( (2, 2, 3) \) is not a subtuple of \( \alpha \).

We now present the concept of consecutive consecutions and consecutive inversions of an index tuple which we will use extensively in the paper.

Definition 2.11 ([26], Consecutions and inversions): Let \( \alpha \) be an index tuple containing indices from \( \{0 : m\} \). Suppose that \( t \in \alpha \). Then we say that \( \alpha \) has \( p \) consecutive consecutions at \( t \) if \( (t, t + 1, \ldots, t + p) \) is a subtuple of \( \alpha \) and \( (t, t + 1, \ldots, t + p, t + p + 1) \) is not a subtuple of \( \alpha \). We denote the number of consecutive consecutions of \( \alpha \) at \( t \) by \( c_t(\alpha) \). Similarly, we say that \( \alpha \) has \( s \) consecutive inversions at \( t \) if \( (t + s, \ldots, t + 1, t) \) is a subtuple of \( \alpha \) and \( (t + s + 1, t + s, \ldots, t + 1, t) \) is not a subtuple of \( \alpha \). We denote the number of consecutive inversions of \( \alpha \) at \( t \) by \( i_t(\alpha) \). For any index \( k \in \{0 : m\} \), if \( k \notin \alpha \), we define \( c_k(\alpha) := -1 \) and \( i_k(\alpha) := -1 \).
**Example 2.12:** Let \( \alpha := (1, 0, 2, 1, 3, 2, 4, 1, 3, 2, 1) \) be an index tuple containing indices from \([0 : 6]\). Then \( c_0(\alpha) = 3 \) as \((0, 1, 2, 3)\) is a subtuple of \( \alpha \) and \((0, 1, 2, 3, 4)\) is not a subtuple of \( \alpha \).

**Remark 2.13 ([26]):** Let \( \alpha \) be a permutation of \([0 : m - 1]\). We denote the total number of consecutions and inversions of \( \alpha \) by \( c(\alpha) \) and \( i(\alpha) \), respectively. Note that \( c(\alpha) + i(\alpha) = m - 1 \).

3. Generalized Fiedler pencils with repetition

We now introduce a new family of Fiedler-like pencils for rational matrices which we refer to as generalized Fiedler pencils with repetition (GFPRs).

**Definition 3.1 ([22], Matrix assignments):** Let \( t := (t_1, t_2, \ldots, t_k) \) be an index tuple containing indices from \([-m : m - 1]\) and \( X := (X_1, X_2, \ldots, X_k) \) be a tuple of \( n \times n \) matrices. We define \( M_t(X) := M_{t_1}(X_1)M_{t_2}(X_2) \cdots M_{t_k}(X_k) \) and say that \( X \) is a matrix assignment for \( t \). Further, we say that the matrix \( X_j \) is assigned to the position \( j \) in \( t \). The matrix assignment \( X \) for \( t \) is said to be non-singular if the matrices assigned by \( X \) to the positions in \( t \) occupied by the 0 and \(-m\) indices are non-singular. Further, we define \( rev(X) := (X_k, \ldots, X_2, X_1) \).

Let \( t := (t_1, \ldots, t_k) \) be an index tuple containing indices from \([-m : m - 1]\) and \( X := (X_1, \ldots, X_k) \) be a matrix assignment for \( t \). Then we say that \( X \) is the trivial matrix assignment for the index tuple \( t \) associated with the matrix polynomial \( P(\lambda) \) if \( M_{t_j}(X_j) = M_{t_j}^P \) for \( j = 1 : k \). Further, we define \( M_t^P := M_{t_1}^P \cdots M_{t_k}^P \). Similarly, we define \( M_t^S := M_{t_1}^S \cdots M_{t_k}^S \), and \( M_t := M_{t_1}^S \cdots M_{t_k}^P \).

**Definition 3.2 (GFPR of \( G(\lambda) \)):** Let \( 0 \leq h \leq m - 1 \), and let \( \sigma \) and \( \tau \) be permutations of \([0 : h]\) and \([-m : -h - 1]\), respectively. Let \( \sigma_1 \) and \( \sigma_2 \) be index tuples containing indices from \([0 : h - 1]\) such that \((\sigma_1, \sigma, \sigma_2)\) satisfies the SIP. Similarly, let \( \tau_1 \) and \( \tau_2 \) be index tuples containing indices from \([-m : -h - 2]\) such that \((\tau_1, \tau, \tau_2)\) satisfies the SIP. Let \( X_1, X_2, Y_1 \) and \( Y_2 \) be any arbitrary matrix assignments for \( \sigma_1, \sigma_2, \tau_1 \) and \( \tau_2 \), respectively. Then the pencil

\[
\mathbb{L}(\lambda) := M_{\tau_1}(Y_1)M_{\sigma_1}(X_1)(\lambda M_{\tau}^S - M_{\sigma}^S)M_{\sigma_2}(X_2)M_{\tau_2}(Y_2)
\]  \hspace{1cm} (7)

is said to be a generalized Fiedler pencil with repetition (GFPR) of \( G(\lambda) \). We also refer to \( \mathbb{L}(\lambda) \) as a GFPR of \( \mathcal{S}(\lambda) \).

Note that if all the matrix assignments \( X_1, X_2, Y_1 \) and \( Y_2 \) in Definition 3.2 are the trivial matrix assignments then \( \mathbb{L}(\lambda) = M_{\tau_1}^P M_{\sigma_1}^P(\lambda M_{\tau}^S - M_{\sigma}^S)M_{\sigma_2}^P M_{\tau_2}^P \) is called a Fiedler pencil with repetition (FPR) of \( G(\lambda) \) [14,27]. Hence the family of FPRs of \( G(\lambda) \) is a subclass of the family of GFPRs of \( G(\lambda) \).
Example 3.3: Let \( G(\lambda) := \sum_{i=0}^{4} \lambda^i A_i + C(\lambda E - A)^{-1} B. \) Consider \( \sigma := (1, 2, 3, 0), \tau := (-4), \sigma_2 := (2, 1) \) and \( \sigma_1 = \tau_1 = \tau_2 = 0. \) Then

\[
(\lambda M^S_{-4} - M^S_{(1,2,3,0)}) M_{(2,1)}(X, Y) = \begin{bmatrix}
\lambda A_4 + A_3 & -X & -Y & -I_n \\
A_2 & \lambda X - I_n & \lambda Y & \lambda I_n \\
A_1 & \lambda I_n & A_0 & 0 \\
-I_n & 0 & \lambda I_n & 0 \\
0 & 0 & B & 0
\end{bmatrix}
A - \lambda E
\]

is a GFPR of \( G(\lambda), \) where \((X, Y)\) is an arbitrary matrix assignment for \( \sigma_2. \)

Remark 3.4: The pencil \( L(\lambda) := M_{\tau_1}(Y_1)M_{\sigma_1}(X_1)(\lambda M^P - M^P_{\sigma_2}(X_2)M_{\tau_2}(Y_2) \) is called a generalized Fiedler pencil with repetition (GFPR) of \( P(\lambda) \) \([22], \) where \( \sigma, \tau, \sigma_j \) and \( \tau_j, j = 1, 2, \) are as given in Definition 3.2. In particular, if \( X_1, X_2, Y_1 \) and \( Y_2 \) are the trivial matrix assignments then \( L(\lambda) := M^P_{\tau_1} M^P_{\sigma_1}(\lambda M^P - M^P_{\sigma_2})M^P_{\sigma_2}M^P_{\tau_2} \) is called a Fiedler pencil with repetition (FPR) of \( P(\lambda) \) \([24,25]. \)

We now show that a GFPR of \( G(\lambda) \) can be constructed directly from a GFPR of \( P(\lambda) \) without performing any arithmetic operations. For this purpose we need the following result which is given in \([26, \text{Lemma 3.10}]. \)

Lemma 3.5 ([26]): Let \( L(\lambda) := M_{(\tau_1,\sigma_1)}(Y_1, X_1)(\lambda M^P_{\tau_1} - M^P_{\sigma_2})M_{(\sigma_2,\tau_2)}(X_2, Y_2) \) be a GFPR of \( P(\lambda) \). Then we have \( (e^T_{m-c_0(\sigma,\sigma_2)} \otimes I_n)M_{(\sigma_2,\tau_2)}(X_2, Y_2) = e^T_{m-c_0(\sigma,\sigma_2)} \otimes I_n \) and \( M_{(\tau_1,\sigma_1)}(Y_1, X_1)(e_{m-i_0(\sigma,\sigma_2)} \otimes I_n) = e_{m-i_0(\sigma,\sigma_2)} \otimes I_n. \)

Theorem 3.6: Let \( \mathbb{L}(\lambda) := \mathbb{M}_{(\tau_1,\sigma_1)}(Y_1, X_1)(\lambda M^S - M^S_{\sigma_2})\mathbb{M}_{(\sigma_2,\tau_2)}(X_2, Y_2) \) and \( L(\lambda) := M_{(\tau_1,\sigma_1)}(Y_1, X_1)(\lambda M^P - M^P_{\sigma_2})M_{(\sigma_2,\tau_2)}(X_2, Y_2) \) be GFPRs of \( G(\lambda) \) and \( P(\lambda), \) respectively. Then

\[
\mathbb{L}(\lambda) = \begin{bmatrix}
L(\lambda) & e_{m-i_0(\sigma,\sigma_2)} \otimes C \\
e^T_{m-c_0(\sigma,\sigma_2)} \otimes B & A - \lambda E
\end{bmatrix}
\]

Thus, the map

\[
\text{GFPR}(P) \to \text{GFPR}(G), \quad L(\lambda) \mapsto \begin{bmatrix}
L(\lambda) & e_{m-i_0(\sigma,\sigma_2)} \otimes C \\
e^T_{m-c_0(\sigma,\sigma_2)} \otimes B & A - \lambda E
\end{bmatrix}
\]

is a bijection, where \( \text{GFPR}(P) \) and \( \text{GFPR}(G) \) denote the set of GFPRs of \( P(\lambda) \) and \( G(\lambda), \) respectively.

Proof: Let \( \sigma \) be given by \( \sigma = (\delta_1, 0, \delta_2). \) A straight forward calculation shows that

\[
\mathbb{L}(\lambda) = \mathbb{M}_{(\tau_1,\sigma_1)}(Y_1, X_1) \left( \lambda \begin{bmatrix}
M^P_{\delta_1} & 0 \\
0 & -E
\end{bmatrix} - \begin{bmatrix}
M^P_{\delta_1} M^P_{\tau_0} M^P_{\delta_2} M_{\delta_1} \left(-e_m \otimes C\right) \\
(-e_m \otimes B) M^P_{\delta_2} & -A
\end{bmatrix} \right)
\times \mathbb{M}_{(\sigma_2,\tau_2)}(X_2, Y_2)
\]

\[= \begin{bmatrix}
L(\lambda) & M_{(\tau_1,\sigma_1)}(Y_1, X_1) M^P_{\delta_1} (e_m \otimes C) \\
e^T_{m-c_0(\sigma,\sigma_2)} \otimes B M^P_{\delta_2} M_{(\sigma_2,\tau_2)}(X_2, Y_2) & A - \lambda E
\end{bmatrix}
\]
It is shown in the proof of [27, Theorem 5.12] that $M_{\delta_1}^P(e_m \otimes I_n) = e_{m-i_0(\sigma)} \otimes I_n$ and $(e_m^T \otimes I_n)M_{\delta_2}^P = e_{m-c_0(\sigma)}^T \otimes I_n$. Consequently, by Lemma 3.5, we have $(e_m^T \otimes I_n)M_{\delta_2}^P M_{(\sigma_2, \tau_2)}(X_2, Y_2) = e_{m-c_0(\sigma, \tau_2)}^T \otimes I_n$ and $M_{(\tau_1, \sigma_1)}(Y_1, X_1)M_{\delta_1}^P(e_m \otimes I_n) = e_{m-i_0(\sigma_1, \tau_1)} \otimes I_n$. Hence the desired form of $\mathbb{L}(\lambda)$ follows from (8).

**Remark 3.7:** We mention that FPRs and GFPRs of matrix polynomials can be generated by automatic algorithms without performing any arithmetic operations (see, Algorithms 1, 2, 3 and 4, in [28, p.49–52]). Thus, in view of Theorem 3.6, GFPRs of rational matrices can be generated by an operation-free automatic algorithm.

Fiedler pencils (FPs) and generalized Fiedler pencils (GFPs) of rational matrices have been studied in [11,12]. The GFPs of $G(\lambda)$ are defined as follows.

**Definition 3.8 ([12], GFP):** Let $\omega := (\omega_0, \omega_1)$ be a permutation of $\{0 : m\}$. Then the pencil $T_\omega(\lambda) := \lambda M_{-\omega}^S - M_{\omega}^S$ is said to be a generalized Fiedler pencil (GFP) of $G(\lambda)$ associated with the permutation $\omega$.

FPs, GFPs and GFPRs of $G(\lambda)$ are in fact Rosenbrock strong linearizations of $G(\lambda)$. Indeed, the fact that FPs of $G(\lambda)$ are Rosenbrock strong linearizations of $G(\lambda)$ can be proved by following a very similar proof as given in [23] for FPs of matrix polynomials. Since the proof is long, technical and is not germane to the present paper, we refer the reader to [29] for a proof. We mention that an alternative proof can also be found in a recent paper [30, Section 8]. That GFPs and GFPRs are Rosenbrock strong linearizations can be deduced from the fact that FPs of $G(\lambda)$ are Rosenbrock strong linearizations. For completeness, we briefly state the results.

**Theorem 3.9 ([29]):** Let $\mathbb{L}_\sigma(\lambda) := \lambda M_{-\sigma}^S - \lambda M_{\sigma}^S$ be the Fiedler pencil of $G(\lambda)$ associated with a permutation $\sigma$ of $\{0 : m - 1\}$. Then $\mathbb{L}_\sigma(\lambda)$ is a Rosenbrock strong linearization of $G(\lambda)$.

The following result, which can be deduced from [16, Lemma 2.7], yields that GFPRs and GFPs are strong linearizations. For completeness, we provide a proof.

**Proposition 3.10:** Let $T(\lambda) := \lambda M_{-\sigma}^S - \lambda M_{\sigma}^S$ be the Fiedler pencil of $G(\lambda)$ associated with a permutation $\sigma$ of $\{0 : m - 1\}$. Let $\mathbb{L}(\lambda)$ be a pencil given by $\mathbb{L}(\lambda) := \text{diag}(\mathcal{X}, X_0)T(\lambda)\text{diag}(\mathcal{Y}, Y_0)$, where $\mathcal{X}, \mathcal{Y} \in \mathbb{C}^{m \times m}$ and $X_0, Y_0 \in \mathbb{C}^{r \times r}$ are non-singular matrices. Then $\mathbb{L}(\lambda)$ is a Rosenbrock strong linearization of $G(\lambda)$.

**Proof:** Since $T(\lambda)$ is a Fiedler pencil of $G(\lambda)$, by Theorem 3.9, $T(\lambda)$ is a Rosenbrock strong linearization of $G(\lambda)$. Hence there exist $mn \times mn$ unimodular matrix polynomials $U(\lambda)$ and $V(\lambda)$, and $r \times r$ non-singular matrices $U_0$ and $V_0$ such that

$$\text{diag}(I_{(m-1)n}, S(\lambda)) = \text{diag}(U(\lambda), U_0)T(\lambda)\text{diag}(V(\lambda), V_0)$$

$$= \text{diag}(U(\lambda)X^{-1}, U_0X_0^{-1})\mathbb{L}(\lambda)\text{diag}(Y^{-1}V(\lambda), Y_0^{-1}V_0).$$  

(9)
By Theorem 3.6, we have

\[
\mathbb{T}(\lambda) = \begin{bmatrix}
L(\lambda) & e_m - i_\alpha(\alpha) \otimes C \\
e^T_m - c_0(\alpha) \otimes B & A - \lambda E
\end{bmatrix},
\]

where \( L(\lambda) = \lambda M_P^P - M_P^P \) is the Fiedler pencil of \( P(\lambda) \) associated with \( \alpha \). Then

\[
\mathbb{L}(\lambda) = \begin{bmatrix}
\mathcal{X}L(\lambda)Y & \mathcal{X}(e_m - i_\alpha(\alpha) \otimes C)Y_0 \\
X_0(e^T_m - c_0(\alpha) \otimes B)Y & X_0(A - \lambda E)Y_0
\end{bmatrix},
\]

and \( \mathbb{G}_L(\lambda) := \mathcal{X}L(\lambda)Y + \mathcal{X}(e_m - i_\alpha(\alpha) \otimes C)(\lambda E - A)^{-1}(e^T_m - c_0(\alpha) \otimes B)Y \) is the transfer function of \( \mathbb{L}(\lambda) \). Since \( \mathbb{T}(\lambda) \) is a Rosenbrock strong linearization of \( \mathbb{G}(\lambda) \), there exist biproper rational matrices \( \mathcal{O}_\ell(\lambda) \) and \( \mathcal{O}_r(\lambda) \) such that

\[
\mathcal{O}_\ell(\lambda) \mathbb{T}(\lambda)^{-1} \mathbb{G}_T(\lambda) \mathcal{O}_r(\lambda) = \begin{bmatrix}
I_{(m-1)n} & 0 \\
0 & \lambda^{-m}G(\lambda)
\end{bmatrix},
\]

(10)

where \( \mathbb{G}_T(\lambda) = L(\lambda) + (e_m - i_\alpha(\alpha) \otimes C)(\lambda E - A)^{-1}(e^T_m - c_0(\alpha) \otimes B) \) is the transfer function of \( \mathbb{T}(\lambda) \). Since \( \mathcal{X}^{-1} \mathbb{G}_L(\lambda)Y^{-1} = \mathbb{G}_T(\lambda) \), it follows from (10) that

\[
\mathcal{O}_\ell(\lambda) \mathcal{X}^{-1} \mathbb{G}_L(\lambda)Y^{-1} \mathcal{O}_r(\lambda) = \begin{bmatrix}
I_{(m-1)n} & 0 \\
0 & \lambda^{-m}G(\lambda)
\end{bmatrix}.
\]

(11)

Note that \( \mathcal{O}_\ell(\lambda) \mathcal{X}^{-1} \) and \( Y^{-1} \mathcal{O}_r(\lambda) \) are biproper rational matrices. Hence it follows from (9) and (11) that \( \mathbb{L}(\lambda) \) is a Rosenbrock strong linearization of \( \mathbb{G}(\lambda) \).

As a consequence, we have the following result.

**Theorem 3.11:** Let

\[
\mathbb{L}(\lambda) := \mathbb{M}_{\tau_1}(Y_1) \mathbb{M}_{\sigma_1}(X_1)(\lambda \mathbb{M}^S_{\tau} - \mathbb{M}^S_{\sigma_1}) \mathbb{M}_{\sigma_2}(X_2) \mathbb{M}_{\tau_2}(Y_2)
\]

be a GFPR of \( \mathbb{G}(\lambda) \) as given in Definition 3.2, where all the matrix assignments \( X_j \) and \( Y_j \), \( j = 1, 2 \), are non-singular. Then \( \mathbb{L}(\lambda) \) is a Rosenbrock strong linearization of \( \mathbb{G}(\lambda) \).

**Proof:** Let \( \tau \) be given by \( \tau = (\beta, -m, \gamma) \). Define \( \alpha := (-\text{rev}(\beta), \sigma, -\text{rev}(\gamma)) \) and \( \mathbb{T}(\lambda) := \lambda \mathbb{M}^S_{-m} - \mathbb{M}^S_{\alpha} \). Then \( \mathbb{T}(\lambda) \) is a Fiedler pencil of \( \mathbb{G}(\lambda) \) associated with the permutation \( \alpha \) of \( \{0 : m - 1\} \). It is easily seen that \( \mathbb{L}(\lambda) = \mathbb{A} \mathbb{T}(\lambda) \mathbb{B} \), where

\[
\mathbb{A} = \mathbb{M}_{(\tau_1, \sigma_1)}(Y_1, X_1) \mathbb{M}^S_{\beta} = \begin{bmatrix}
M_{(\tau_1, \sigma_1)}(Y_1, X_1)M^P_{\beta} \\
I_r
\end{bmatrix},
\]

and

\[
\mathbb{B} = \mathbb{M}^S_{\gamma} \mathbb{M}_{(\sigma_2, \tau_2)}(X_2, Y_2) = \begin{bmatrix}
M^P_{\gamma}M_{(\sigma_2, \tau_2)}(X_2, Y_2) \\
I_r
\end{bmatrix}.
\]

Since \( X_j \) and \( Y_j, j = 1, 2 \), are non-singular matrix assignments, the matrices \( M_{(\tau_1, \sigma_1)}(Y_1, X_1) \) and \( M_{(\sigma_2, \tau_2)}(X_2, Y_2) \) are non-singular. Hence by Proposition 3.10, \( \mathbb{L}(\lambda) \) is a Rosenbrock strong linearization of \( \mathbb{G}(\lambda) \).
The next result shows that the GFPs of $G(\lambda)$ are also Rosenbrock strong linearizations.

**Theorem 3.12:** Let $T_\omega(\lambda) := \lambda M_\omega(\lambda) - M_\omega(\lambda)$ be a GFP of $G(\lambda)$, where $0 \in \omega$. Then $T_\omega(\lambda)$ is a Rosenbrock strong linearization of $G(\lambda)$.

**Proof:** It is shown in [12, Theorem 2.13] that $T_\omega(\lambda) = \text{diag}(X_\omega, X_0) F(\lambda) \text{diag}(Y_\omega, Y_0)$ for some non-singular matrices $X_\omega, Y_\omega, X_0, Y_0 \in \mathbb{C}^{mn \times mn}$ and $F(\lambda)$ is a Fiedler pencil of $G(\lambda)$. Hence by Proposition 3.10, $T(\lambda)$ is a Rosenbrock strong linearization of $G(\lambda)$. ■

**4. Structure-preserving strong linearizations**

This section is devoted to the construction of structure-preserving strong linearizations of structured rational matrices. We consider only symmetric, skew-symmetric, Hamiltonian and skew-Hamiltonian rational matrices and construct their structure-preserving strong linearizations. The construction of structure-preserving strong linearizations of Hermitian, skew-Hermitian, para-Hermitian and para-skew-Hermitian rational matrices is similar. We show that the family of GFPRs of $G(\lambda)$ is a rich source of structure-preserving strong linearizations of $G(\lambda)$. Recall that $G(\lambda) = P(\lambda) + G_{sp}(\lambda)$, where $P(\lambda) := \sum_{j=0}^{m} A_j \lambda^j$ with $A_m \neq 0$ and $G_{sp}(\lambda)$ is strictly proper, that is, $G_{sp}(\lambda) \rightarrow 0$ as $\lambda \rightarrow \infty$.

**4.1. Symmetric GFPRs**

Suppose that $G(\lambda)$ is symmetric, that is, $G(\lambda)^T = G(\lambda)$. Since $G(\lambda) = P(\lambda) + G_{sp}(\lambda)$, it follows that both $P(\lambda)$ and $G_{sp}(\lambda)$ are symmetric. As $G_{sp}(\lambda)$ is strictly proper and symmetric, there exists a minimal symmetric realization of $G(\lambda)$ given by $G_{sp}(\lambda) = B^T (\lambda I_r - A)^{-1} B$, where $A$ is a symmetric matrix [1,3,4]. Hence $G(\lambda) = P(\lambda) + B^T (\lambda I_r - A)^{-1} B$ is a minimal symmetric realization of $G(\lambda)$. The system matrix

$$S(\lambda) := \begin{bmatrix} P(\lambda) & B^T \\ B & A - \lambda I_r \end{bmatrix}$$

is then symmetric and irreducible. Also, there exists a minimal symmetric realization of $G(\lambda)$ of the form $G(\lambda) = P(\lambda) + B^T (\lambda E - A)^{-1} B$, where $A$ and $E$ are symmetric matrices with $E$ being non-singular [16]. The system matrix

$$S(\lambda) := \begin{bmatrix} P(\lambda) & B^T \\ B & A - \lambda E \end{bmatrix}$$

(12)

is obviously symmetric and irreducible.

The block transpose of an $m \times n$ block matrix $\mathcal{H} = [H_{ij}]$ is an $n \times m$ block matrix $\mathcal{H}^B$ given by $\mathcal{H}^B = [H_{ji}]$. A block matrix $\mathcal{H}$ is said to be block-symmetric provided that $\mathcal{H}^B = \mathcal{H}$, see [23]. The block-transpose of a system matrix is defined as follows.
Definition 4.1 ([12,14]): Let
\[
\mathcal{A} := \begin{bmatrix}
A & u \otimes X \\
v^T \otimes Y & Z
\end{bmatrix} \in \mathbb{C}^{(mn+r) \times (mn+r)},
\]
where \( A = [A_{ij}] \) is an \( m \times m \) block matrix with \( A_{ij} \in \mathbb{C}^{n \times n} \), \( u, v \in \mathbb{C}^m \), \( X \in \mathbb{C}^{n \times r} \), \( Y \in \mathbb{C}^{r \times n} \) and \( Z \in \mathbb{C}^{r \times r} \). Define the block transpose of \( \mathcal{A} \) by
\[
\mathcal{A}^B := \begin{bmatrix}
A^B & v \otimes X \\
u^T \otimes Y & Z
\end{bmatrix}.
\]
Observe that \( \mathcal{A} \) is block-symmetric if and only if \( \mathcal{A} \) is block-symmetric and \( u = v \).

Definition 4.2 ([22]):
(a) Let \( h \geq 0 \) be an integer. We say that \( w \) is an admissible tuple of \( \{0 : h\} \) if \( w \) is a permutation of \( \{0 : h\} \) and
\[
\text{csf}(w) = (h - 1 : h, h - 3 : h - 2, \ldots, p + 1 : p + 2, 0 : p)
\]
for some \( 0 \leq p \leq h \). We call \( p \) the index of \( w \) and denote it by \( Ind(w) \).
(b) Let \( h \geq 0 \) be an integer and let \( w \) be an admissible tuple of \( \{0 : h\} \) with index \( p \). Then the symmetric complement of \( w \), denoted by \( c_w \), is defined by
\[
c_w := \begin{cases}
(h - 1, h - 3, \ldots, p + 3, p + 1, (0 : p)_{revc}) & \text{if } p \geq 1, \\
(h - 1, h - 3, \ldots, 1) & \text{if } p = 0 \text{ and } h > 0, \\
\emptyset & \text{if } h = 0,
\end{cases}
\]
where \( (0 : p)_{revc} := (0 : p - 1, 0 : p - 2, \ldots, 0 : 1, 0) \).

For simplicity, we always consider an admissible tuple of the form (13). Clearly, for an integer \( h \geq 0 \), there exists a unique admissible tuple of \( \{0 : h\} \) with index 0 or 1 (see [22]).

Definition 4.3: An admissible tuple \( w \) of \( \{0 : h\}, h \geq 0 \), is said to be the simple admissible tuple if \( Ind(w) = 0 \) or \( Ind(w) = 1 \).

Note that for the simple admissible tuple \( w \) of \( \{0 : h\} \), we have \( Ind(w) = 0 \) (resp., \( Ind(w) = 1 \)) if \( h \) is even (resp., odd).

Remark 4.4: Let \( v \) be an admissible tuple of \( \{0 : k\}, k \geq 0 \), and let \( c_v \) be the symmetric complement of \( v \). Then it follows from Definition 4.2 that \( 0 \in c_v \) if and only if \( Ind(v) \geq 1 \). In particular, for the simple admissible tuple \( w \) of \( \{0 : h\} \), we have \( 0 \in c_w \) (resp., \( 0 \notin c_w \)) if \( h \) is odd (resp., even), where \( c_w \) is the symmetric complement of \( w \).

Definition 4.5 ([22]): Given \( h \geq 0 \), we say that an index tuple \( t \) is in canonical form for \( h \) if \( t \) is of the form
\[
(a_1 : h - 2, a_2 : h - 4, \ldots, a_{\lceil \frac{h}{2} \rceil} : h - 2\lfloor \frac{h}{2} \rfloor)
\]
with \( a_i \geq 0, i = 1 : \lfloor \frac{h}{2} \rfloor \), where \( \lfloor \cdot \rfloor \) stands for the greatest integer function.
Note that an index tuple in canonical form for \( h \) is necessarily empty for \( h = 0, 1 \).

The following result characterizes all symmetric GFPRs of a matrix polynomial.

**Theorem 4.6 ([22], Theorem 6.11):** Let \( 0 \leq h < m \). Let \( w_h \) and \( v_h + m \) be the simple admissible tuples of \( \{0 : h\} \) and \( \{0 : m - h - 1\} \), respectively. Let \( t_{w_h} \) and \( t_{v_h} + m \) be index tuples in canonical form for \( h \) and \( m - h - 1 \), respectively. Let \( X \) and \( Y \) be non-singular matrix assignments for \( t_{w_h} \) and \( t_{v_h} \), respectively. Then

\[
L(\lambda) := M(t_{v_h}, t_{w_h}) (Y, X) (\lambda M_{v_h}^P - M_{w_h}^P) M_{(c_{w_h}, c_{v_h})} (\lambda (rev(t_{w_h}), rev(t_{v_h}))) (rev(X), rev(Y)),
\]

is a block symmetric GFPR of \( P(\lambda) \), where \( c_{w_h} \) and \( c_{v_h} + m \) are the symmetric complements of \( w_h \) and \( v_h + m \), respectively. Moreover, any block symmetric GFPR of \( P(\lambda) \) is of the form (14). Furthermore, if all the matrices in the matrix assignments \( X \) and \( Y \) are symmetric, then \( L(\lambda) \) is symmetric when \( P(\lambda) \) is symmetric.

The pencil in (14) is denoted by \( L_p(h, t_{w_h}, t_{v_h}, X, Y) \) and is uniquely determined by \( h, t_{w_h}, t_{v_h}, X \) and \( Y \), see [22].

**Definition 4.7:** Let \( h, w_h, c_{w_h}, t_{w_h}, v_h, c_{v_h}, t_{v_h}, X \) and \( Y \) be as given in Theorem 4.6. Then we define

\[
\mathbb{L}(\lambda) := \mathbb{M}(t_{v_h}, t_{w_h}) (Y, X) (\lambda M_{v_h}^S - M_{w_h}^S) M_{(c_{w_h}, c_{v_h})} (\lambda (rev(t_{w_h}), rev(t_{v_h}))) (rev(X), rev(Y)).
\]

The pencil \( \mathbb{L}(\lambda) \) in (15) is uniquely determined by \( h, t_{w_h}, t_{v_h}, X \) and \( Y \). We denote \( \mathbb{L}(\lambda) \) by \( \mathbb{L}_S(h, t_{w_h}, t_{v_h}, X, Y) \).

The following result characterizes all block-symmetric GFPRs of \( G(\lambda) \).

**Theorem 4.8:** Let \( S(\lambda) \) be given in (4). Let \( 0 \leq h \leq m - 1 \) be even. Consider the GFPR

\[
\mathbb{L}(\lambda) := \mathbb{L}_S(h, t_{w_h}, t_{v_h}, X, Y) \text{ associated with } S(\lambda). \text{ Then}
\]

\[
\mathbb{L}(\lambda) = \begin{bmatrix}
L_p(h, t_{w_h}, t_{v_h}, X, Y) & \mathbb{C}_{m-\ell_0(t_{w_h}, w_h)} \otimes B \\
\mathbb{C}_{m-\ell_0(w_h, c_{w_h}, rev(t_{w_h}))} \otimes B & A - \lambda E
\end{bmatrix}
\]

(16)

Further, we have the following:

(a) \( \mathbb{L}(\lambda) \) is a block symmetric GFPR of \( S(\lambda) \). Further, any block symmetric GFPR of \( S(\lambda) \) must be of the form \( \mathbb{L}_S(h, t_{w_h}, t_{v_h}, X, Y) \) for some even \( 0 \leq h \leq m - 1 \).

(b) If \( m \) is odd then \( \mathbb{L}(\lambda) \) is a Rosenbrock strong linearization of \( S(\lambda) \). If \( m \) is even then \( \mathbb{L}(\lambda) \) is a Rosenbrock strong linearization of \( S(\lambda) \) when the leading coefficient of \( P(\lambda) \) is non-singular.

**Proof:** By substituting \( \sigma = w_h, \sigma_1 = t_{w_h}, \sigma_2 = (c_{w_h}, rev(t_{w_h})) \), \( \tau = v_h, \tau_1 = t_{v_h} \) and \( \tau_2 = (c_{v_h}, rev(t_{v_h})) \) in Theorem 3.6, we obtain (16).
By Theorem 4.6, \( L_P(h, t_w h, t_y h, \mathcal{X}, \mathcal{Y}) \) is a block symmetric pencil. Hence it follows that \( \mathbb{L}(\lambda) \) is block symmetric if and only if \( c_0(w_h, c_{w_h}, rev(t_{w_h})) = i_0(t_{w_h}, w_h) \). Next, we show that \( c_0(w_{f}, c_{w_{f}}, rev(t_{w_{f}})) = i_0(t_{w_{f}}, w_{f}) \).

Case-I: Suppose that \( h = 0 \). Then \( w_h = (0) \) and \( c_{w_h} = \emptyset = t_{w_h} \). Hence \( i_0(t_{w_h}, w_h) = 0 = c_0(w_{f}, c_{w_{f}}, rev(t_{w_{f}})) \).

Case-II: Suppose that \( h > 0 \). Since \( h \) is even and \( w_h \) is the simple admissible tuple of \( \{0 : h\} \), we have \( w_h = (h - 1 : h, h - 3 : h - 2, \ldots , 1 : 2, 0) \) and \( c_{w_h} = (h - 1, h - 3, \ldots , 3, 1) \). Thus \( c_0(w_{f}, c_{w_{f}}, rev(t_{w_{f}})) = 2 + c_2(rev(t_{w_{f}})) \) and \( i_0(t_{w_{f}}, w_{f}) = 2 + i_2(t_{w_{f}}) \). (Recall that for any index tuple \( \beta \) and for any index \( t \), if \( t \notin \beta \) then \( c_1(\beta) = -1 = i_1(\beta) \)). Hence \( \mathbb{L}(\lambda) \) is block-symmetric since \( i_1(\beta) = c_1(rev(\beta)) \) for any index tuple \( \beta \) and any index \( t \). This proves the first part of (a).

Next we prove that, if \( h \) is odd, then \( c_0(w_{h}, c_{w_{h}}, rev(t_{w_{h}})) \neq i_0(t_{w_{h}}, w_{h}) \). Then it follows from (16) that \( \mathbb{L}(\lambda) \) is not a block symmetric GFPR of \( S(\lambda) \). This will prove the second part of (a).

Let \( h \geq 0 \) be odd. If \( h = 1 \) then \( w_h = (0, 1) \), \( c_{w_h} = (0) \) and \( t_{w_h} = \emptyset \). Thus \( c_0(w_{h}, c_{w_{h}}, rev(t_{w_{h}})) = 1 = i_0(t_{w_{h}}, w_{h}) = 0 \). Hence \( \mathbb{L}(\lambda) \) is not block symmetric.

Next, suppose that \( h > 1 \). Then \( w_h = (h - 1 : h, h - 3 : h - 2, \ldots , 2 : 3, 0 : 1) \) and \( c_{w_h} = (h - 1, h - 3, \ldots , 2, 0) \). Thus \( c_0(w_{h}, c_{w_{h}}, rev(t_{w_{h}})) = 3 + c_3(rev(t_{w_{h}})) = 3 + i_3(t_{w_{h}}) \) and \( i_0(t_{w_{h}}, w_{h}) = 1 + i_1(t_{w_{h}}) \). We show that \( 3 + i_3(t_{w_{h}}) \neq 1 + i_1(t_{w_{h}}) \). Let \( i_1(t_{w_{h}}) = p \). If \( p = -1 \) or \( p = 0 \) then \( 1 + i_1(t_{w_{h}}) < 2 \leq 3 + i_3(t_{w_{h}}) \) and hence the desired result follows.

Suppose that \( p \geq 1 \). Note that \( t_{w_h} \) is in canonical form for \( h > 1 \) is odd, i.e.

\[
\begin{aligned}
t_{w_h} &= \left( a_1 : h - 2, a_2 : h - 4, \ldots , a_{h-1} : 3, a_{h-1} : 1 \right).
\end{aligned}
\]

We call \( (a_j : h - 2j) \), \( j = 1, 2, \ldots , \frac{h-1}{2} \), as the strings of \( t_{w_h} \) and \( h - 2j \) as the right end point of the string \( (a_j : h - 2j) \). Since \( i_1(t_{w_h}) = p \), \( (p + 1, p, \ldots , 3, 2, 1) \) is a subtuple of \( t_{w_h} \) and \( (p + 2, p + 1, p, \ldots , 2, 1) \) is not a subtuple of \( t_{w_h} \). It is clear from (17) that each index of the subtuple \( (p + 1, p, \ldots , 2, 1) \) of \( t_{w_h} \) belongs to distinct string of \( t_{w_h} \). By collecting all those strings we have a subtuple

\[
\left( (p + 1 : b_{p+1}), (p : b_{p}), \ldots , (3 : b_3), (2 : b_2), (1 : b_1) \right)
\]

of \( t_{w_h} \), where \( b_j \)'s are the right end points of the collected strings. Hence \( b_j \in \{1, 3, 5, \ldots , h - 4, h - 2\} \) for \( j = 1 : p + 1 \) is such that \( b_{p+1} > b_p > \cdots > b_3 > b_2 > b_1 \). This implies that \( b_2 \geq 3 \) and hence \( 3 \in (2 : b_2) \), \( b_3 \geq 5 \) and hence \( 4 \in (3 : b_3) \), and so on \( p + 1 \in (p : b_p) \) and \( p + 2 \in (p + 1 : b_{p+2}) \). Consequently, \( (p + 2, p + 1, p, \ldots , 4, 3) \) is a subtuple of \( t_{w_h} \) and \( i_3(t_{w_h}) \geq p - 1 \). So \( 3 + i_3(t_{w_h}) \geq p + 2 > p + 1 = 1 + i_1(t_{w_h}) \). Hence \( c_0(w_{h}, c_{w_{h}}, rev(t_{w_{h}})) \neq i_0(t_{w_{h}}, w_{h}) \) and \( \mathbb{L}(\lambda) \) is not a block symmetric GFPR of \( S(\lambda) \). This completes the proof of the second part of (a).

(b) Since \( h \) is even, by Remark 4.4 we have \( 0 \notin c_{w_h} \). This implies that the matrix assignment for \( c_{w_h} \) is non-singular. Further, it is given that \( \mathcal{X} \) and \( \mathcal{Y} \) are non-singular matrix assignments for \( t_{w_h} \) and \( t_{y_h} \), respectively. Consequently, by taking \( \sigma := w_{h}, \tau := v_{h}, \sigma_1 := t_{w_{h}}, \sigma_2 := (c_{w_{h}}, rev(t_{w_{h}})), \tau_1 := t_{y_{h}} \) and \( \tau_2 := (c_{y_{h}}, rev(t_{y_{h}})) \), it follows from Theorem 3.11 that \( \mathbb{L}(\lambda) \) is a Rosenbrock strong linearization of \( S(\lambda) \) if the matrix assignment for \( c_{y_h} \) is non-singular. Suppose that \( m \) is odd. Then \( m - h - 1 \) is even (since \( h \) is even) and by Remark 4.4, it follows that \( 0 \notin c_{y_{h}} + m \implies -m \notin c_{y_{h}} \). Hence the matrix assignment for
Corollary 4.9: Let G(\lambda) be symmetric and S(\lambda) be given in (12). Let 0 \leq h \leq m - 1 be even. Consider the GFPR
\[
\mathbb{L}(\lambda) := \mathbb{L}_S(h, t_{w_h}, t_{v_h}, \mathcal{X}, \mathcal{Y}) = \begin{bmatrix}
L_p(h, t_{w_h}, t_{v_h}, \mathcal{X}, \mathcal{Y}) & e_{m-\alpha} \otimes B^T \\
\frac{e_{m-\alpha}^T}{\lambda - \alpha E} & A - \lambda E
\end{bmatrix}
\]
associated with S(\lambda), where \alpha := i_0(t_{w_h}, w_h), \mathcal{X} and \mathcal{Y} are non-singular matrix assignments and all the matrices in \mathcal{X} and \mathcal{Y} are symmetric. If m is odd then \mathbb{L}(\lambda) is a symmetric Rosenbrock strong linearization of G(\lambda). If m is even then \mathbb{L}(\lambda) is a symmetric Rosenbrock strong linearization of G(\lambda) when the leading coefficient of P(\lambda) is non-singular. Also the transfer function \mathcal{G}(\lambda) := L(\lambda) + (e_{m-\alpha} \otimes B^T)(\lambda E - A)^{-1}(e_{m-\alpha}^T \otimes B) of \mathbb{L}(\lambda) is symmetric, where L(\lambda) := L_p(h, t_{w_h}, t_{v_h}, \mathcal{X}, \mathcal{Y})).

Proof: By considering C = B^T it follows from the proof of Theorem 4.8 that
\[
\mathbb{L}(\lambda) = \begin{bmatrix}
L_p(h, t_{w_h}, t_{v_h}, \mathcal{X}, \mathcal{Y}) & e_{m-\alpha} \otimes B^T \\
\frac{e_{m-\alpha}^T}{\lambda - \alpha E} & A - \lambda E
\end{bmatrix}
\]
is a block symmetric Rosenbrock strong linearization of S(\lambda), where \alpha := i_0(t_{w_h}, w_h). Since P(\lambda) is symmetric and all the matrices in the matrix assignments \mathcal{X} and \mathcal{Y} are symmetric, by Theorem 4.6, we have L_p(h, t_{w_h}, t_{v_h}, \mathcal{X}, \mathcal{Y}) is symmetric. Further, since A and E are symmetric, it follows from (18) that \mathbb{L}(\lambda) and \mathcal{G}(\lambda) are symmetric.

Example 4.10: Let G(\lambda) = \sum_{i=0}^5 \lambda^i A_i + B^T (\lambda E - A)^{-1} B be symmetric. Consider h = 2, t_{w_h} = 0 and t_{v_h} = -5. Let X and Y be any arbitrary non-singular symmetric matrices. Then the GFPR
\[
\mathbb{L}_S(h, t_{w_h}, t_{v_h}, \mathcal{X}, \mathcal{Y}) = \begin{bmatrix}
0 & -Y & \lambda Y & 0 & 0 & 0 \\
-Y & \lambda A_5 - A_4 & \lambda A_4 & 0 & 0 & 0 \\
\lambda Y & \lambda A_4 & \lambda A_3 + A_2 & A_1 & -X & 0 \\
0 & 0 & A_1 & -\lambda A_1 + A_0 & \lambda X & 0 \\
0 & 0 & -X & \lambda X & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & A - \lambda E
\end{bmatrix}
\]
is a symmetric Rosenbrock strong linearization of S(\lambda). Note that \mathbb{L}_S(h, t_{w_h}, t_{v_h}, \mathcal{X}, \mathcal{Y}) is a block penta-diagonal pencil.

Next, let G(\lambda) := \sum_{i=0}^6 \lambda^i A_i + B^T (\lambda E - A)^{-1} B be symmetric. Consider h = 0, t_{w_h} = 0 and t_{v_h} = (-6 : -3, -6 : -5). Then the GFPR \mathbb{L}_S(h, t_{w_h}, t_{v_h}, \mathcal{X}, \mathcal{Y})
\[
= \begin{bmatrix}
0 & 0 & 0 & 0 & -A_6 & \lambda A_6 & 0 \\
0 & 0 & 0 & -A_6 & \lambda A_6 - A_5 & \lambda A_5 & 0 \\
0 & 0 & -A_6 & \lambda A_6 - A_5 & \lambda A_5 - A_4 & \lambda A_4 & 0 \\
0 & -A_6 & \lambda A_6 - A_5 & \lambda A_5 - A_4 & \lambda A_4 - A_3 & \lambda A_3 & 0 \\
\lambda A_6 & \lambda A_5 & \lambda A_4 & \lambda A_3 & \lambda A_2 & \lambda A_1 + A_0 & B^T \\
0 & 0 & 0 & 0 & 0 & B & A - \lambda E
\end{bmatrix}
\]
is a block symmetric Rosenbrock strong linearization of S(\lambda).
is a symmetric Rosenbrock strong linearization of $G(\lambda)$ when $A_0$ is non-singular, where $X$ and $Y$ are the trivial matrix assignments.

We now show that the transfer function of a real symmetric strong linearization preserves the Cauchy-Maslov index of a real symmetric rational matrix.

**Definition 4.11:** [31] The Cauchy-Maslov index of a real symmetric rational matrix $G(\lambda)$ is defined by

$$\text{Ind}_{\text{CM}}(G) := (\# \text{ eigenvalues of } G(\lambda) \text{ which jump from } -\infty \text{ to } +\infty) - (\# \text{ eigenvalues of } G(\lambda) \text{ which jump from } +\infty \text{ to } -\infty) \text{ as the real parameter } \lambda \text{ traverses from } -\infty \text{ to } +\infty.$$

The Cauchy–Maslov index of a real symmetric rational matrix plays an important role in many applications such as in networks of linear systems, see [31–34] and the references therein. It is therefore desirable to construct real symmetric linearizations of $G(\lambda)$ whose transfer functions preserve the Cauchy-Maslov index of $G(\lambda)$.

**Theorem 4.12:** Let $G(\lambda)$ be real symmetric and $S(\lambda)$ be as given in (12). Let $L(\lambda) := L_S(h, t_w, t_h, X, Y)$ be a symmetric Rosenbrock strong linearization of $G(\lambda)$ as given in Corollary 4.9. Let $G(\lambda)$ be the associated transfer function of $L(\lambda)$. Then $G(\lambda)$ is real and symmetric and has the same Cauchy–Maslov index as $G(\lambda)$, that is, $\text{Ind}_{\text{CM}}(G) = \text{Ind}_{\text{CM}}(G)$.

**Proof:** By Corollary 4.9 we have $G(\lambda) = L(\lambda) + (e_m - \alpha \otimes B^T)(\lambda E - A)^{-1}(e_m - \alpha \otimes B)$ is symmetric, where $\alpha := i_0(t_w, w_h)$ and $L(\lambda)$ are as given in Corollary 4.9.

Next, we show that $\text{Ind}_{\text{CM}}(G) = \text{Ind}_{\text{CM}}(G)$. Set $G_{sp}(\lambda) := B^T(\lambda E - A)^{-1}B$. Then we have $G(\lambda) = P(\lambda) + G_{sp}(\lambda)$ and

$$G(\lambda) = L(\lambda) + (e_m - \alpha \otimes B^T)(\lambda E - A)^{-1}(e_m - \alpha \otimes B)$$

$$= L(\lambda) + \text{diag}(0, \ldots, 0, G_{sp}(\lambda), 0, \ldots, 0).$$

Hence $\text{Ind}_{\text{CM}}(G) = \text{Ind}_{\text{CM}}(G_{sp}) = \text{Ind}_{\text{CM}}(G)$, where we have used the fact that the Cauchy–Maslov index is invariant under perturbation by a matrix polynomial.

**Remark 4.13:** Although the Cauchy–Maslov index is defined for real symmetric rational matrices, it can be extended to Hermitian rational matrices.

**Remark 4.14:** Let $G(\lambda) = P(\lambda) + B^T(\lambda E - A)^{-1}B$ be symmetric, where $P(\lambda) = \sum_{j=0}^{m} A_j \lambda^j$ and $m > 1$. Then the construction given in [15, Theorem 5.3] generates only one symmetric linearization of $G(\lambda)$ which is explicitly given by

$$T(\lambda) := \lambda^m + A_m$$

$$\begin{bmatrix}
  \cdots \\
  \cdot \\
  \cdot \\
  A_m & A_{m-1} & \cdots & A_1 \\
  \vdots & \vdots & \ddots & \vdots \\
  \cdots \\
  -E
\end{bmatrix}.$$
Further, the pencil $T(\lambda)$ is a Rosenbrock strong linearization of $G(\lambda)$ if and only if $A_m$ is non-singular [15, Theorem 5.3]. By contrast, the family of GFPRs enables us to construct an infinite number of symmetric strong linearizations of $G(\lambda)$. In fact, by considering $h = 0$, $t_{wh} = \emptyset$ and $t_{vh} = -m + (0 : m - 3, 0 : m - 5, \ldots)$, we have $L_S(h, t_{wh}, t_{vh}, X, Y) = T(\lambda)$, where $X$ and $Y$ are the trivial matrix assignments.

### 4.2. Hamiltonian linearizations

Recall that a rational matrix $G(\lambda)$ is said to be Hamiltonian (i.e. $T$-even) if $G(-\lambda)^T = G(\lambda)$. Since $G(\lambda) = P(\lambda) + G_{sp}(\lambda)$, it follows that if $G(\lambda)$ is $T$-even then both $P(\lambda)$ and $G_{sp}(\lambda)$ are $T$-even. We now construct $T$-even Rosenbrock strong linearizations of $G(\lambda)$. We proceed as follows.

For the rest of the paper, we define $J := \begin{bmatrix} 0 & I_r \\ -I_r & 0 \end{bmatrix}$ when $r = 2\ell$. Note that $J^T = J^{-1} = -J$.

Further, we define $J_{k,r} := \text{diag}(I_k, J)$ for any integer $k \geq 1$ when $r = 2\ell$.

**Definition 4.15 ([4])**: A matrix $X \in \mathbb{C}^r$ with $r := 2\ell$ is said to be Hamiltonian (resp., skew-Hamiltonian) if $JX$ is symmetric (resp., $JX$ is skew-symmetric), that is, $(JX)^T = JX$ (resp., $(JX)^T = -JX$).

If $X$ is Hamiltonian then $(JX)^T = JX \Rightarrow (XJ)^T = XJ$. Similarly, if $X$ is skew-Hamiltonian then we have $(XJ)^T = -XJ$.

**Definition 4.16**: Let $G(\lambda)$ be a Hamiltonian (i.e. $T$-even) rational matrix.

(a) A realization of $G(\lambda)$ of the form $G(\lambda) = P(\lambda) + C(\lambda I_r - A)^{-1}B$ is said to be a Hamiltonian realization of $G(\lambda)$ if $P(\lambda)$ is $T$-even, $A$ is Hamiltonian with $r = 2\ell$ and $JB = C^T$.

(b) A system matrix $S(\lambda)$ of the form

$$S(\lambda) := \begin{bmatrix} P(\lambda) & C \\ B & A - \lambda I_r \end{bmatrix}$$

is said to be a Hamiltonian system matrix if $r = 2\ell$ and $J_{n,r} S(\lambda)$ is $T$-even, that is, if $(J_{n,r} S(-\lambda))^T = J_{n,r} S(\lambda)$, where $J_{n,r} := \text{diag}(I_n, J)$.

(c) A realization of $G(\lambda)$ of the form $G(\lambda) = P(\lambda) + C(\lambda E - A)^{-1}B$ with $E$ being non-singular is said to be a $T$-even realization of $G(\lambda)$ if $C = B^T$ and both $P(\lambda)$ and $\lambda E - A$ are $T$-even.

See [2] for a para-Hermitian realization of $G(\lambda)$. Note that the system matrix $S(\lambda)$ associated with a $T$-even realization of $G(\lambda)$ is $T$-even, that is, $S(-\lambda)^T = S(\lambda)$. 

Remark 4.17: Observe that \( G(\lambda) = P(\lambda) + C(\lambda I_r - A)^{-1}B \) is a Hamiltonian realization of \( G(\lambda) \) if and only if
\[
S(\lambda) := \begin{bmatrix} P(\lambda) & C \\ B & A - \lambda I_r \end{bmatrix}
\]
is a Hamiltonian system matrix of \( G(\lambda) \). On the other hand, \( G(\lambda) = P(\lambda) + C(\lambda E - A)^{-1}B \) is a \( T \)-even realization of \( G(\lambda) \) if and only if
\[
S(\lambda) := \begin{bmatrix} P(\lambda) & C \\ B & A - \lambda E \end{bmatrix}
\]
is a \( T \)-even system matrix of \( G(\lambda) \).

For convenience, we often refer to \( S(\lambda) \) as a \( T \)-even (resp., Hamiltonian) realization of \( G(\lambda) \) when \( S(\lambda) \) is \( T \)-even (resp., Hamiltonian).

Proposition 4.18: Suppose that \( G(\lambda) \) is Hamiltonian (i.e. \( T \)-even). Then we have the following:

(a) There exists a minimal Hamiltonian realization of \( G(\lambda) \) of the form \( G(\lambda) = P(\lambda) + C(\lambda I_r - A)^{-1}B \) with \( r = 2\ell \) and \( JB = C^T \). Thus the associated system matrix
\[
S(\lambda) := \begin{bmatrix} P(\lambda) & B^T J^T \\ B & A - \lambda I_r \end{bmatrix}
\]
is Hamiltonian.

(b) There exists a minimal \( T \)-even realization of \( G(\lambda) \) of the form \( G(\lambda) = P(\lambda) + B^T(\lambda E - A)^{-1}B \). Thus the system matrix
\[
S(\lambda) := \begin{bmatrix} P(\lambda) & B^T \\ B & A - \lambda E \end{bmatrix}
\]
is \( T \)-even.

Proof: Since \( G(\lambda) = P(\lambda) + G_{sp}(\lambda) \) is \( T \)-even, we have \( P(\lambda) \) and \( G_{sp}(\lambda) \) are \( T \)-even. Also since \( G_{sp}(\lambda) \) is strictly proper and \( T \)-even, there exists a minimal Hamiltonian realization of \( G_{sp}(\lambda) \) of the form \( G_{sp}(\lambda) = C(\lambda I_r - A)^{-1}B \) with \( r = 2\ell \) and \( JB = C^T \); see [4]. Hence \( G(\lambda) = P(\lambda) + C(\lambda I_r - A)^{-1}B \) is a minimal Hamiltonian realization of \( G(\lambda) \). Obviously the system matrix \( S(\lambda) \) is Hamiltonian, that is, \((J^n_r S(-\lambda))^T = J^n_r S(\lambda)\), where \( J^n_r := \text{diag}(I_{n,r}, J) \). This proves (a).

The results in (b) follow from (a). Indeed, by part (a) we have \( G(\lambda) = P(\lambda) + B^T J^T (\lambda I_r - A)^{-1}B = P(\lambda) + B^T (\lambda J - AJ)^{-1}B \). Since \( A \) is Hamiltonian, it follows that \( \lambda J - AJ \) is \( T \)-even. Hence setting \( E := J \) and redefining \( A := AJ \), it follows that \( G(\lambda) := P(\lambda) + B^T (\lambda E - A)^{-1}B \) is a minimal \( T \)-even realization of \( G(\lambda) \). Evidently, the system matrix \( S(\lambda) \) is \( T \)-even, that is, \( S(-\lambda)^T = S(\lambda) \). This proves (b).

We construct \( T \)-even (resp., Hamiltonian) linearizations of \( G(\lambda) \) corresponding to a \( T \)-even (resp., Hamiltonian) realization of \( G(\lambda) \). We proceed as follows.
Definition 4.19 ([35]): A matrix $Q \in \mathbb{C}^{mn \times mn}$ is said to be a quasi-identity matrix if $Q = \epsilon_1I_n \oplus \cdots \oplus \epsilon_mI_n$, where $\epsilon_i \in \{\pm 1\}$ for $i = 1 : m$. We refer to $\epsilon_j$, $j = 1:m$, as the $j$-th parameter of $Q$.

We need the following result which is a particular case of [35, Theorem 4.15].

Theorem 4.20 ([35], Theorem 4.15): Let $0 \leq h \leq m - 1$ be even. Let $w$ be the simple admissible tuple of $\{0 : h\}$ and $c_w$ be the symmetric complement of $w$. Let $z + m$ be any admissible tuple of $\{0 : m - h - 1\}$ and $c_z + m$ be the symmetric complement of $z + m$. Let $L(\lambda) := (\lambda M_z^B - M_w^c)M_w^c M_z^c$. Then, up to multiplication by $-1$, there exists a unique quasi-identity matrix $Q$ such that $QL(\lambda)$ is $T$-even (resp., $T$-odd) when $P(\lambda)$ is $T$-even (resp., $T$-odd).

We refer to [35, Algorithm 4.14] for more on the construction of the quasi-identity matrix $Q$. The next result provides $T$-even linearizations of $G(\lambda)$.

Theorem 4.21: Let $G(\lambda)$ be $T$-even and $S(\lambda)$ be a $T$-even realization of $G(\lambda)$ as given in Proposition 4.18(b). Let $h, w, c_w, z$ and $c_z$ be as in Theorem 4.20. Consider the GFPR $L(\lambda) := (\lambda M_z^S - M_w^c)M_w^c M_z^c$ associated with $S(\lambda)$. Then there exists a unique quasi-identity matrix $Q := \text{diag}(s Q, I_r)$ such that

$$QL(\lambda) = \begin{bmatrix}
    s QL(\lambda) & e_{m-i_0(w)} \otimes B^T \\
    e_{m-i_0(w)}^T \otimes B & A - \lambda E
\end{bmatrix}$$

is $T$-even, where $Q$ and $L(\lambda)$ are as in Theorem 4.20 and $s$ is the $(m - i_0(w))$-th parameter of $Q$.

Assume that $\text{Ind}(z + m) = 0$ when the leading coefficient of $P(\lambda)$ is singular. Then $QL(\lambda)$ is a Rosenbrock strong linearization of $G(\lambda)$. The transfer function $G(\lambda) := s QL(\lambda) + (e_{m-i_0(w)} \otimes B^T)(A - \lambda E)^{-1}(e_{m-i_0(w)} \otimes B)$ of $QL(\lambda)$ is $T$-even.

Proof: By Theorem 3.6, we have

$$L(\lambda) = \begin{bmatrix}
    L(\lambda) & e_{m-i_0(w)} \otimes B^T \\
    e_{m-c_0(w, c_w)}^T \otimes B & A - \lambda E
\end{bmatrix},$$

where $L(\lambda)$ is as given in Theorem 4.20. Since $h$ is even and $w$ is the simple admissible tuple of $\{0 : h\}$, we have $w = (h - 1 : h, \ldots, 3 : 4, 1 : 2, 0)$ and $c_w = (h - 1, h - 3, \ldots, 1)$. This implies that $i_0(w) = c_0(w, c_w) = 0$ if $h = 0$, and $i_0(w) = c_0(w, c_w) = 1$ if $h > 0$. By Theorem 4.20, $s QL(\lambda)$ is $T$-even. Set $\alpha := i_0(w)$. Then $Q(e_{m-\alpha} \otimes I_n) = s(e_{m-\alpha} \otimes I_n)$. Note that $ss = 1$. Consequently, we have

$$QL(\lambda) = \begin{bmatrix}
    s QL(\lambda) & s Q(e_{m-\alpha} \otimes B^T) \\
    e_{m-\alpha}^T \otimes B & A - \lambda E
\end{bmatrix} = \begin{bmatrix}
    s QL(\lambda) & e_{m-\alpha} \otimes B^T \\
    e_{m-\alpha}^T \otimes B & A - \lambda E
\end{bmatrix}. \quad (20)$$

Since $s QL(\lambda)$ and $A - \lambda E$ are $T$-even, it follows from (20) that $QL(\lambda)$ is $T$-even.

Since $h$ is even, by Remark 4.4 we have $0 \notin c_w$. This implies that the matrix assignment for $c_w$ is non-singular. Hence by taking $\sigma := w, \tau := z, \sigma_1 := \emptyset, \sigma_2 := c_w, \tau_1 := \emptyset$ and $\tau_2 := c_z$, it follows from Theorem 3.11 that $L(\lambda)$ is a Rosenbrock strong linearization
of $G(\lambda)$ if the matrix assignment for $c_z$ is non-singular. If the leading coefficient of $P(\lambda)$ is non-singular then the matrix assignment for $c_z$ is non-singular. On the other hand, if the leading coefficient of $P(\lambda)$ is singular and $\text{Ind}(z + m) = 0$, then by Remark 4.4, we have $0 \notin c_z + m \Rightarrow -m \notin c_z$. Hence the matrix assignment for $c_z$ is non-singular. Thus, $QL(\lambda)$ is a $T$-even Rosenbrock strong linearization of $G(\lambda)$. Obviously the transfer function $G(\lambda)$ is $T$-even.

**Remark 4.22:** Note that if $m$ is even then $\text{Ind}(z + m) > 0$ because $h$ is always even. This implies that $-m \in c_z$. Hence if the leading coefficient of $P(\lambda)$ is singular then $QL(\lambda)$ in Theorem 4.21 is not a linearization of $G(\lambda)$ as $M^P_{c_z}$ is singular.

**Example 4.23:** Let $G(\lambda) := \sum_{i=0}^{5} \lambda^i A_i + B^T(\lambda E - A)^{-1}B$ be a $T$-even realization of $G(\lambda)$ and $S(\lambda)$ be as in Proposition 4.18(b). Consider the GFPR $QL(\lambda) = \lambda M^S_I(-4: -3, -5) - M^P_{(1:2, 0)} M^P_{(-4: -3, -4)}$ and $Q = \text{diag}(I_n, I_n, -I_n, -I_n, I_r)$. Then

$$QL(\lambda) = \begin{bmatrix}
0 & -I_n & \lambda I_n & 0 & 0 & 0 \\
-I_n & \lambda A_5 - A_4 & \lambda A_4 & 0 & 0 & 0 \\
-\lambda I_n & -\lambda A_4 & -\lambda A_3 - A_2 & -A_1 & I_n & 0 \\
0 & 0 & A_1 & -\lambda A_1 + A_0 & \lambda I_n & B^T \\
0 & 0 & I_n & -\lambda I_n & 0 & 0 \\
0 & 0 & 0 & B & 0 & 0
\end{bmatrix}
$$

is a $T$-even Rosenbrock strong linearization of $G(\lambda)$. Observe that $QL(\lambda)$ is a block pentadiagonal pencil.

Next, let $G(\lambda) := \sum_{i=0}^{4} \lambda^i A_i + B^T(\lambda E - A)^{-1}B$ be a $T$-even realization. Consider $L(\lambda) = (\lambda M^S_{(-4: -1)} - M^P_{(-4: -2, -4: -3, -4)})$ and $Q := \text{diag}(I_n, -I_n, I_n, -I_n, I_r)$. Then

$$QL(\lambda) = \begin{bmatrix}
0 & 0 & -A_4 & \lambda A_4 & 0 \\
0 & A_4 & -\lambda A_4 + A_3 & -\lambda A_3 & 0 \\
-A_4 & \lambda A_4 - A_3 & \lambda A_3 - A_2 & \lambda A_2 & 0 \\
-\lambda A_4 & -\lambda A_3 & -\lambda A_2 & -\lambda A_1 - A_0 & B^T \\
0 & 0 & 0 & 0 & B
\end{bmatrix}
$$

is a $T$-even Rosenbrock strong linearization of $G(\lambda)$ when $A_4$ is non-singular.

Next, we consider a Hamiltonian realization of $G(\lambda)$ and construct a Hamiltonian strong linearization of $G(\lambda)$.

**Theorem 4.24:** Let $G(\lambda)$ be Hamiltonian and $S(\lambda)$ be a Hamiltonian realization of $G(\lambda)$ as given in Proposition 4.18(a). Assume that $\text{Ind}(z + m) = 0$ when the leading coefficient of $P(\lambda)$ is singular, where $z$ is as given in Theorem 4.21. Then

$$\mathbb{T}(\lambda) := \left[ \begin{array}{c|c}
\psi QL(\lambda) \\
\psi^T m_i(w) \otimes B \\
& \psi m_i(w) \otimes B \end{array} \right]$$

is Hamiltonian and is a Rosenbrock strong linearization of $G(\lambda)$, where $w$ and $\psi QL(\lambda)$ are as given in Theorem 4.21.
The transfer function $G(\lambda) := sQL(\lambda) + (e_{m-\delta_0(w)} \otimes B^T J^T)(\lambda I_r - A)^{-1}(e_{m-\delta_0(w)} \otimes B)$ of $T(\lambda)$ is Hamiltonian.

**Proof:** Define

$$\hat{S}(\lambda) := J_{n,r} S(\lambda) = \begin{bmatrix} P(\lambda) & B^T J^T \\ JB & JA - \lambda J \end{bmatrix}.$$ 

Since $A$ is Hamiltonian, we have $JA - \lambda J$ is $T$-even. This shows that $\hat{S}(\lambda)$ is a $T$-even realization of $G(\lambda)$. Hence by Theorem 4.21,

$$\hat{L}(\lambda) := \begin{bmatrix} sQL(\lambda) & e_{m-\delta_0(w)} \otimes B^T J^T \\ e^T_{m-\delta_0(w)} \otimes JB & JA - \lambda J \end{bmatrix}$$ (21)

is a $T$-even Rosenbrock strong linearizations of $\hat{S}(\lambda)$. Note that $\hat{L}(\lambda) = J_{mn,r} \hat{T}(\lambda)$, where $J_{mn,r} := \text{diag}(I_{mn}, J)$. Since $\hat{L}(\lambda)$ is $T$-even, it follows that $\hat{T}(\lambda)$ is Hamiltonian, that is, $(J_{mn,r} \hat{T}(\lambda)) = J_{mn,r} \hat{T}(\lambda)$. Further, since $\hat{L}(\lambda)$ is a Rosenbrock strong linearization of $\hat{S}(\lambda)$ and $\hat{S}(\lambda) = J_{n,r} S(\lambda)$, it follows that $\hat{T}(\lambda)$ is a Rosenbrock strong linearization of $S(\lambda)$. Obviously the transfer function $G(\lambda)$ is Hamiltonian.

**4.3. Skew-Hamiltonian linearizations**

Recall that a rational matrix $G(\lambda)$ is said to be skew-Hamiltonian (i.e. $T$-odd) if $G(-\lambda)^T = -G(\lambda)$.

**Proposition 4.25:** Let $G(\lambda)$ be $T$-odd. Then there exists a minimal $T$-odd realization of $G(\lambda)$ of the form $G(\lambda) := P(\lambda) + B^T (\lambda I_r - A)^{-1} B$, where $P(\lambda)$ and $\lambda I_r - A$ are $T$-odd. Thus the system matrix

$$S(\lambda) := \begin{bmatrix} P(\lambda) & -B^T \\ B & \lambda I_r - A \end{bmatrix}$$

is $T$-odd.

**Proof:** Since $G(\lambda) = P(\lambda) + G_{sp}(\lambda)$ is $T$-odd, it follows that both $P(\lambda)$ and $G_{sp}(\lambda)$ are $T$-odd. Since $G_{sp}(\lambda)$ is $T$-odd and strictly proper, there exists a minimal $T$-odd realization of $G_{sp}(\lambda)$ of the form $G_{sp}(\lambda) = B^T (\lambda I_r - A)^{-1} B$, where $A$ is skew-symmetric; see [4]. Since $A$ is skew-symmetric, we have $\lambda I_r - A$ is $T$-odd. This shows that $G(\lambda) = P(\lambda) + B^T (\lambda I_r - A)^{-1} B$ is a minimal $T$-odd realization of $G(\lambda)$ and that the system matrix $S(\lambda)$ is $T$-odd.

The next result gives $T$-odd Rosenbrock strong linearizations of $G(\lambda)$.

**Theorem 4.26:** Let $G(\lambda)$ be $T$-odd and $S(\lambda)$ be as given in Proposition 4.25. Let $h, w, c_w, z$ and $c_z$ be as in Theorem 4.20. Consider the GFPR $L(\lambda) := (\lambda M_z^S - M_w^S)M_{c_w}^P M_{c_z}^P$ associated with $S(\lambda)$. Then there exists a unique quasi-identity matrix $Q := \text{diag} (sQ, I_r)$ such that

$$QL(\lambda) = \begin{bmatrix} sQL(\lambda) & -e_{m-\delta_0(w)} \otimes B^T \\ e^T_{m-\delta_0(w)} \otimes B & \lambda I_r - A \end{bmatrix}$$

is $T$-odd, where $Q$ and $L(\lambda)$ are as in Theorem 4.20 and $s$ is the $(m - \delta_0(w))$-th parameter of $Q$. 

Assume that Ind(z + m) = 0 when leading coefficient of P(λ) is singular. ThenQL(λ) is a T-odd Rosenbrock strong linearization of G(λ). The transfer function G(λ) := sQL(λ) + (em−i0(w) ⊗ B)T(λIr − A)−1(eTm−i0(w) ⊗ B) ofQL(λ) is T-odd.

**Proof:** By Theorem 3.6, we have

\[ L(λ) = \begin{bmatrix}
    \lambda - λ_{r} & e_{m−i0}(w) & (−B^{T})
\end{bmatrix}, \]

where L(λ) is as given in the proof of Theorem 4.20. It is shown in the proof of Theorem 4.21 that i0(w) = c0(w, c_w). Set α := i0(w). Then Q(λ−α ⊗ In) = s(e−α ⊗ In). Note that ss = 1. Consequently, we have

\[ QL(λ) = \begin{bmatrix}
    sQL(λ) & \frac{e_{m−α} ⊗ (−B^{T})}{λ_{r} − A}
\end{bmatrix} = \begin{bmatrix}
    QL(λ) & \frac{e_{m−α} ⊗ (−B^{T})}{λ_{r} − A}
\end{bmatrix}. \quad (22) \]

By Theorem 4.20, sQL(λ) is T-odd. Since λIr − A is T-odd, it follows from (22) thatQL(λ) is T-odd.

By the same arguments as given in the proof of Theorem 4.21, it follows thatQL(λ) is a Rosenbrock strong linearization of G(λ). Obviously, the transfer function G(λ) is T-odd.

**Example 4.27:** Let \( G(λ) = \sum_{i=0}^{5} λ^{i}A_{i} + B^{T}(λIr − A)^{-1}B \) be a T-odd realization of G(λ) and S(λ) be as given in Proposition 4.25. Set \( Q := \text{diag}(I_{n}, −I_{n}, I_{n}, −I_{n}, −I_{n}, I_{r}) \) and consider the GFPR \( L(λ) := (λM^{S}_{(−4:−3:−5)} − M^{S}_{(1:2:0)})M^{P}_{1}M^{P}_{4} \). Then

\[ QL(λ) = \begin{bmatrix}
    0 & −I_{n} & λI_{n} & 0 & 0 & 0 \\
    I_{n} & −λA_{5} + A_{4} & −λA_{4} & 0 & 0 & 0 \\
    λA_{5} + A_{4} & I_{n} & −λA_{5} + A_{2} & A_{1} & −I_{n} & 0 \\
    0 & 0 & A_{1} & −λA_{1} − A_{0} & −λI_{n} & −B^{T} \\
    0 & 0 & 0 & −λI_{n} & B & 0 \\
    0 & 0 & 0 & 0 & 0 & λI_{r} − A
\end{bmatrix} \]

is a T-odd Rosenbrock strong linearization of G(λ). Notice thatQL(λ) is a block penta-diagonal pencil.

Next, let \( G(λ) = \sum_{i=0}^{5} λ^{i}A_{i} + B^{T}(λIr − A)^{-1}B \) be a T-odd realization. Consider \( L(λ) := (λ^{S}_{(−4:−1)} − M^{S}_{0})M^{P}_{(−4:−2:−4:−3:−4)} \) and \( Q := \text{diag}(I_{n}, −I_{n}, I_{n}, −I_{n}, −I_{n}, I_{r}) \). Then

\[ QL(λ) = \begin{bmatrix}
    0 & 0 & −A_{4} & λA_{4} & 0 \\
    0 & A_{4} & −λA_{4} + A_{3} & −λA_{3} & 0 \\
    −A_{4} & λA_{4} − A_{3} & λA_{3} − A_{2} & λA_{2} & 0 \\
    −λA_{4} & −λA_{3} & −λA_{2} & −λA_{1} − A_{0} & −B^{T} \\
    0 & 0 & 0 & 0 & λI_{r} − A
\end{bmatrix} \]

is a T-odd Rosenbrock strong linearization of G(λ) when A_4 is non-singular.
4.4. Skew-symmetric linearizations

Suppose that \( G(\lambda) \) is skew-symmetric, that is, \( G(\lambda)^T = -G(\lambda) \). Since \( G(\lambda) = P(\lambda) + G_{sp}(\lambda) \), it follows that \( P(\lambda) \) and \( G_{sp}(\lambda) \) are skew-symmetric.

**Definition 4.28:** Suppose that \( G(\lambda) \) is skew-symmetric.

(a) A realization of \( G(\lambda) \) of the form \( G(\lambda) = P(\lambda) + C(\lambda I_r - A)^{-1}B \) is said to be a skew-Hamiltonian realization of \( G(\lambda) \) if \( P(\lambda) \) is skew-symmetric, \( A \) is skew-Hamiltonian with \( r = 2\ell \) and \( C^T = JB \).

(b) A system matrix \( S(\lambda) \) of the form

\[
S(\lambda) := \begin{bmatrix} P(\lambda) & -C \\ B & \lambda I_r - A \end{bmatrix}
\]

is said to be a skew-Hamiltonian system matrix if \( r = 2\ell \) and \( (\mathbb{J}_{n,r} S(\lambda))^T = -\mathbb{J}_{n,r} S(\lambda) \), where \( \mathbb{J}_{n,r} := \text{diag}(I_n, J) \).

(c) A realization of \( G(\lambda) \) of the form \( G(\lambda) = P(\lambda) + C(\lambda E - A)^{-1}B \) with \( E \) being nonsingular is said to be a skew-symmetric realization of \( G(\lambda) \) if \( C = B^T \) and both \( P(\lambda) \) and \( \lambda E - A \) are skew-symmetric.

**Remark 4.29:** Observe that \( G(\lambda) = P(\lambda) + C(\lambda I_r - A)^{-1}B \) is a skew-Hamiltonian realization of \( G(\lambda) \) if and only if

\[
S(\lambda) := \begin{bmatrix} P(\lambda) & -C \\ B & \lambda I_r - A \end{bmatrix}
\]

is a skew-Hamiltonian system matrix of \( G(\lambda) \). On the other hand, \( G(\lambda) = P(\lambda) + C(\lambda E - A)^{-1}B \) is a skew-symmetric realization of \( G(\lambda) \) if and only if

\[
S(\lambda) := \begin{bmatrix} P(\lambda) & -C \\ B & \lambda E - A \end{bmatrix}
\]

is a skew-symmetric system matrix of \( G(\lambda) \).

For convenience, we often refer to \( S(\lambda) \) as a skew-symmetric (resp., skew-Hamiltonian) realization of \( G(\lambda) \) when \( S(\lambda) \) is skew-symmetric (resp., skew-Hamiltonian).

**Proposition 4.30:** Suppose that \( G(\lambda) \) is skew-symmetric. Then we have the following:

(a) There exists a minimal skew-Hamiltonian realization of \( G(\lambda) \) of the form \( G(\lambda) = P(\lambda) + C(\lambda I_r - A)^{-1}B \) with \( r = 2\ell \) and \( JB = C^T \). Thus the system matrix

\[
S(\lambda) := \begin{bmatrix} P(\lambda) & -B^T J^T \\ B & \lambda I_r - A \end{bmatrix}
\]

associated with \( G(\lambda) \) is skew-Hamiltonian.
(b) There exists a minimal skew-symmetric realization of $G(\lambda)$ of the form $G(\lambda) = P(\lambda) + B^T(\lambda E - A)^{-1}B$. Thus the system matrix

$$S(\lambda) = \begin{bmatrix} P(\lambda) & -B^T \\ B & \lambda E - A \end{bmatrix}$$

associated with $G(\lambda)$ is skew-symmetric.

**Proof:** Since $G(\lambda) = P(\lambda) + G_{sp}(\lambda)$ is skew-symmetric, we have both $P(\lambda)$ and $G_{sp}(\lambda)$ are skew-symmetric. Also since $G_{sp}(\lambda)$ is strictly proper and skew-symmetric, there exists a minimal skew-Hamiltonian realization of $G_{sp}(\lambda)$ of the form $G_{sp}(\lambda) = C(\lambda I_r - A)^{-1}B$ with $r = 2\ell$ and $JB = C^T$; see [4]. Hence $G(\lambda) = P(\lambda) + C(\lambda I_r - A)^{-1}B$ is a minimal skew-Hamiltonian realization of $G(\lambda)$. Obviously the system matrix $S(\lambda)$ is skew-Hamiltonian, that is, $\langle P_{n,r}S(\lambda) \rangle = -\langle P_{n,r}S(\lambda) \rangle$, where $P_{n,r} := \text{diag}(I_n, J)$. This proves (a).

By part (a), $G(\lambda) = P(\lambda) + B^TJ^T(\lambda I_r - A)^{-1}B = P(\lambda) + B^T(\lambda J - AJ)^{-1}B$. Since $A$ is skew-Hamiltonian, it follows that $\lambda J - AJ$ is skew-symmetric. Hence setting $E := J$ and redefining $A := AJ$, it follows that $G(\lambda) = P(\lambda) + B^T(\lambda E - A)^{-1}B$ is a minimal skew-symmetric realization of $G(\lambda)$. Evidently, the system matrix $S(\lambda)$ is skew-symmetric, that is, $S(\lambda)^T = -S(\lambda)$. This proves (b). \hfill $\blacksquare$

Let $\alpha$ be a permutation of $\{0 : k\}$ for $k \geq 0$ with $\text{csf}(\alpha)$ being the column standard form of $\alpha$. Then an index $s \in \{0 : k - 1\}$ is said to be a right index of type-1 relative to $\alpha$ if there is a string $(s : t)$ in the $\text{csf}(\alpha)$ such that $s < t$, see [35].

**Definition 4.31 ([35], Associated simple tuple):** Let $\alpha$ be a permutation of $\{0 : k\}$ for some $k \geq 0$. Suppose that $\text{csf}(\alpha) = (b_d, b_{d-1}, \ldots, b_1)$, where $b_i = (a_{i-1} + 1 : a_i)$ for $i = 2 : d$ and $b_1 = (0 : a_1)$. If $s$ is a right index of type-1 relative to $\alpha$ then the simple tuple associated with $(\alpha, s)$ is denoted by $z_r(\alpha, s)$ and is given by

- $z_r(\alpha, s) := (b_d, b_{d-1}, \ldots, b_{h+1}, \tilde{b}_h, \tilde{b}_{h-1}, b_{h-2}, \ldots, b_1)$ if $s = a_{h-1} + 1 \neq 0$, where $\tilde{b}_h = (a_{h-1} + 2 : a_h)$ and $\tilde{b}_{h-1} = (a_{h-2} + 1 : a_{h-1} + 1)$.
- $z_r(\alpha, s) := (b_d, b_{d-1}, \ldots, b_2, \tilde{b}_1, b_0)$ if $s = 0$, where $\tilde{b}_1 = (1 : a_1)$ and $\tilde{b}_0 = (0)$.

**Definition 4.32 ([35], Type-1 index tuple):** Let $\alpha$ be a permutation of $\{0 : k\}$, $k \geq 0$, and let $\beta := (s_1, \ldots, s_r)$ be an index tuple containing indices from $\{0 : k - 1\}$. Then $\beta$ is said to be a right index tuple of type-1 relative to $\alpha$ if, for $i = 1: r$, $s_i$ is a right index of type-1 relative to $z_r(\alpha, (s_1, \ldots, s_{i-1}))$, where $z_r(\alpha, (s_1, \ldots, s_{i-1})) := z_r(z_r(\alpha, (s_1, \ldots, s_{i-2})), s_{i-1})$ for $i > 2$.

We need the following result which is a particular case of [35, Theorem 3.15].

**Theorem 4.33 ([35]):** Let $P(\lambda)$ be skew symmetric and let $0 \leq h \leq m - 1$ be even. Let $w$ be the simple admissible tuple of $\{0 : h\}$ and $c_w$ be the symmetric complement of $w$. Let $z + m$ be any admissible tuple of $\{0 : m - h - 1\}$. Let $c_z + m$ be the symmetric complement of $z + m$. Let $t_w$ containing indices from $\{0 : h - 1\}$ and $t_z + m$ containing indices from $\{0 : m - h -$
2) be right index tuples of type-1 relative to $\text{rev}(w)$ and $\text{rev}(z + m)$, respectively. Consider

$$L(\lambda) := M^P_{\text{rev}(t_z)} M^P_{\text{rev}(t_w)} (\lambda M^P_z - M^P_w) M^P_c M^P_{t_w} M^P_c M^P_{t_z}.$$ 

Then, up to multiplication by $-1$, there exists a unique quasi-identity matrix $Q$ such that $\text{QL}(\lambda)$ is skew-symmetric.

We now construct skew-symmetric Rosenbrock strong linearizations of $G(\lambda)$.

**Theorem 4.34:** Let $G(\lambda)$ be skew-symmetric and $S(\lambda)$ be a skew-symmetric realization of $G(\lambda)$ as in Proposition 4.30(b). Let $h, w, c_w, t_w, z, c_z$ and $t_z$ be as in Theorem 4.33. Consider the GFPR

$$\mathbb{L}(\lambda) := \begin{bmatrix} L(\lambda) & e_{m-\alpha} \otimes B \\ e^{T}_{m-\alpha} \otimes B & \lambda E - A \end{bmatrix},$$

associated with $S(\lambda)$. Then there exists a unique quasi-identity matrix $Q := \text{diag}(s Q, I_r)$ such that

$$\text{QL}(\lambda) := \begin{bmatrix} s Q L(\lambda) & -e_{m-\alpha} \otimes B^T \\ e^{T}_{m-\alpha} \otimes B & \lambda E - A \end{bmatrix},$$

is skew-symmetric, where $Q$ and $L(\lambda)$ are as in Theorem 4.33 and $s$ is the $(m - \alpha)$-th parameter of $Q$ with $\alpha := c_0(w, c_w, t_w)$.

Assume that $\text{Ind}(z + m) = 0$ when the leading coefficient $A_m$ of $P(\lambda)$ is singular. Further, suppose that $0 \notin t_w$ (resp., $-m \notin t_z$) when $A_0$ (resp., $A_m$) is singular. Then $\text{QL}(\lambda)$ is a skew-symmetric Rosenbrock strong linearization of $G(\lambda)$. The transfer function $\mathbb{G}(\lambda) := s QL(\lambda) + (e_{m-\alpha} \otimes B^T)(\lambda E - A)^{-1}(e_{m-\alpha} \otimes B)$ of $\text{QL}(\lambda)$ is skew-symmetric.

**Proof:** By Theorem 3.6, we have

$$\mathbb{L}(\lambda) = \begin{bmatrix} L(\lambda) & e_{m-\alpha} \otimes \text{rev}(t_z) \otimes B \\ e^{T}_{m-\alpha} \otimes \text{rev}(t_w, c_w, t_w) & \lambda E - A \end{bmatrix},$$

where $L(\lambda)$ is as in Theorem 4.33. Next, we show that $i_0(\text{rev}(t_w), w) = c_0(w, c_w, t_w)$. If $h = 0$ then $w = (0)$ and $c_w = \emptyset = t_w$. Thus $i_0(\text{rev}(t_w), w) = 0 = c_0(w, c_w, t_w)$. Next, suppose that $h > 0$. Then we have $w = (h - 1 : h, h - 3 : h - 2, \ldots, 1 : 2, 0)$ and $c_w = (h - 1, h - 3, \ldots, 1, 0)$. This implies that $c_0(w, c_w, t_w) = 2 + c_2(t_w)$ and $i_0(\text{rev}(t_w), w) = 2 + i_2(\text{rev}(t_w)) = 2 + c_2(t_w)$. Hence $i_0(\text{rev}(t_w), w) = c_0(w, c_w, t_w)$.

By Theorem 4.33, we have $s QL(\lambda)$ is skew-symmetric. Note that $Q(e_{m-\alpha} \otimes I_n) = s (e_{m-\alpha} \otimes I_n)$ and $ss = 1$. Consequently, we have

$$\text{QL}(\lambda) = \begin{bmatrix} s QL(\lambda) & s Q(e_{m-\alpha} \otimes (-B^T)) \\ e^{T}_{m-\alpha} \otimes B & \lambda E - A \end{bmatrix} \begin{bmatrix} s QL(\lambda) & e_{m-\alpha} \otimes (-B^T) \\ e^{T}_{m-\alpha} \otimes B & \lambda E - A \end{bmatrix} = \begin{bmatrix} s QL(\lambda) & e_{m-\alpha} \otimes (-B^T) \\ e^{T}_{m-\alpha} \otimes B & \lambda E - A \end{bmatrix}.$$ (23)

Since $s QL(\lambda)$ and $\lambda E - A$ are skew-symmetric, it follows from (23) that $\text{QL}(\lambda)$ is skew-symmetric.

Since $0 \notin t_w$ (resp., $-m \notin t_z$) when $A_0$ (resp., $A_m$) is singular, the matrix assignments of $t_w, \text{rev}(t_w), t_z$ and $\text{rev}(t_z)$ are non-singular. Hence by taking $\sigma := w, \tau := z, \sigma_1 := \text{rev}(t_w), \sigma_2 := (c_w, t_w), \tau_1 := \text{rev}(t_z)$ and $\tau_2 := (c_z, t_z)$, it follows from Theorem 3.11 that
\( \mathbb{L}(\lambda) \) is a Rosenbrock strong linearization of \( S(\lambda) \) if the matrix assignments for \( c_w \) and \( c_z \) are non-singular. By the similar arguments as given in the proof Theorem 4.21, it follows that the matrix assignments for \( c_w \) and \( c_z \) are non-singular.

Example 4.35: Let \( G(\lambda) = \sum_{i=0}^{5} \lambda^i A_i + B^T(\lambda E - A)^{-1} B \) be skew-symmetric and \( S(\lambda) \) be as in Proposition 4.30(b). Define \( \mathbb{L}(\lambda) := (\lambda M^S_{(4,-3,-5)} - M^S_{(1;2,0)}) M^P_{-4} \) and \( Q := \text{diag}(I_n, -I_n, -I_n, -I_n, I_n, I_r) \). Then

\[
\begin{bmatrix}
0 & -I_n & \lambda I_n & 0 & 0 & 0 \\
I_n & -\lambda A_5 + A_4 & -\lambda A_4 & 0 & 0 & 0 \\
-\lambda I_n & -\lambda A_4 & -\lambda A_3 - A_2 & -A_1 & I_n & 0 \\
0 & 0 & -A_1 & \lambda A_1 - A_0 & -\lambda I_n & -B^T \\
0 & 0 & -I_n & \lambda I_n & 0 & 0 \\
0 & 0 & 0 & B & 0 & \lambda E - A \\
\end{bmatrix}
\]

is a skew-symmetric Rosenbrock strong linearization of \( G(\lambda) \). Observe that \( \mathbb{Q}\mathbb{L}(\lambda) \) is a block penta-diagonal pencil.

Next, let \( G(\lambda) = \sum_{i=0}^{4} \lambda^i A_i + B^T(\lambda E - A)^{-1} B \) be skew-symmetric. Consider the GFPR \( \mathbb{L}(\lambda) := (\lambda M^S_{(4,-3,0)} - M^S_{(1;2,0)}) M^P_{-4} \) and \( Q := \text{diag}(I_n, I_n, I_n, -I_n, I_r) \). Then

\[
\begin{bmatrix}
-A_4 & \lambda A_4 & 0 & 0 & 0 \\
\lambda A_4 & \lambda A_3 + A_2 & A_1 & -I_n & 0 \\
0 & A_1 & -\lambda A_1 + A_0 & \lambda I_n & -B^T \\
0 & I_n & -\lambda I_n & 0 & 0 \\
0 & 0 & B & 0 & \lambda E - A \\
\end{bmatrix}
\]

is a skew-symmetric Rosenbrock strong linearization of \( G(\lambda) \) when \( A_4 \) is non-singular.

Next, we construct skew-Hamiltonian strong linearizations of \( G(\lambda) \).

Theorem 4.36: Let \( G(\lambda) \) be skew-symmetric and \( S(\lambda) \) be a skew-Hamiltonian realization of \( G(\lambda) \) as in Proposition 4.30(a). Let \( \mathbf{w}, c_w, \mathbf{t}_w, \mathbf{z}, c_z \) and \( \mathbf{t}_z \) be as in Theorem 4.34. Suppose that \( 0 \notin \mathbf{t}_w \) (resp., \( -m \notin \mathbf{t}_z \)) when \( A_0 \) (resp., \( A_m \)) is singular. Assume that \( \text{Ind}(z + m) = 0 \) when \( A_m \) is singular. Then

\[
\mathbb{T}(\lambda) := \begin{bmatrix}
\mathbf{s} \mathbb{Q}\mathbb{L}(\lambda) & -e_{m-a} \otimes B^T J^T \\
e_{m-a}^T \otimes B & \lambda I_r - A \\
\end{bmatrix}
\]

is a skew-Hamiltonian Rosenbrock strong linearization of \( G(\lambda) \), where \( \alpha \) and \( \mathbf{s} \mathbb{Q}\mathbb{L}(\lambda) \) are as in Theorem 4.34. The transfer function \( \mathbb{G}(\lambda) := \mathbf{s} \mathbb{Q}\mathbb{L}(\lambda) + (e_{m-a} \otimes B^T J^T)(\lambda I_r - A)^{-1}(e_{m-a}^T \otimes B) \) of \( \mathbb{T}(\lambda) \) is skew-symmetric.
Proof: Define
\[
\hat{S}(\lambda) := J_{n,r}S(\lambda) = \begin{bmatrix}
P(\lambda) & -B^TJ^T \\
JB & \lambda J - JA
\end{bmatrix}.
\]
Since A is skew-Hamiltonian, we have \(\lambda J - JA\) is skew-symmetric. Hence \(\hat{S}(\lambda)\) is skew-symmetric as \(P(\lambda)\) and \(\lambda J - JA\) are skew-symmetric. Now by Theorem 4.34,
\[
\hat{L}(\lambda) := \begin{bmatrix}
sQL(\lambda) & -e_{m-\alpha} \otimes B^TJ^T \\
e_{m-\alpha} \otimes JB & \lambda J - JA
\end{bmatrix}
\]
is a skew-symmetric Rosenbrock strong linearizations of \(\hat{S}(\lambda)\), where \(\alpha\) and \(s\) \(QL(\lambda)\) are as in Theorem 4.34. Note that \(\hat{L}(\lambda) = J_{mn,r}T(\lambda)\). Since \(\hat{L}(\lambda)\) is skew-symmetric, it follows that \(T(\lambda)\) is skew-Hamiltonian, that is, \((J_{mn,r}T(\lambda))^T = -J_{mn,r}T(\lambda)\). Further, since \(\hat{L}(\lambda)\) is a Rosenbrock strong linearization of \(\hat{S}(\lambda)\) and \(\hat{S}(\lambda) = J_{n,r}S(\lambda)\), it follows that \(T(\lambda)\) is a Rosenbrock strong linearization of \(S(\lambda)\). Obviously \(\mathcal{G}(\lambda)\) is skew-symmetric and is the transfer function of \(T(\lambda)\).

5. Recovery of eigenvectors and minimal bases

We now describe the recovery of eigenvectors, minimal bases and minimal indices of \(G(\lambda)\) from those of the GFPRs of \(G(\lambda)\). We need the following result.

Theorem 5.1 ([15,36]): Let \(G(\lambda)\) and \(S(\lambda)\) be as in (3) and (4), respectively.

(I) Suppose that \(G(\lambda)\) is singular. Let \(Z(\lambda) := \begin{bmatrix} Z_n(\lambda) \\ Z_r(\lambda) \end{bmatrix}\) be a matrix polynomial, where \(Z_n(\lambda)\) has \(n\) rows and \(Z_r(\lambda)\) has \(r\) rows. If \(Z(\lambda)\) is a right (resp., left) minimal basis of \(S(\lambda)\) then \(Z_n(\lambda)\) is a right (resp., left) minimal basis of \(G(\lambda)\). Further, the right (resp., left) minimal indices of \(G(\lambda)\) and \(S(\lambda)\) are the same.

(II) Suppose that \(G(\lambda)\) is regular and \(\mu \in \mathbb{C}\) is an eigenvalue of \(G(\lambda)\). Let \(Z := \begin{bmatrix} Z_n \\ Z_r \end{bmatrix}\) be an \((n + r) \times p\) matrix such that \(\text{rank}(Z) = p\), where \(Z_n\) has \(n\) rows and \(Z_r\) has \(r\) rows. If \(Z\) is a basis of \(N_r(S(\mu))\) (resp., \(N_l(S(\mu))\)) then \(Z_n\) is a basis of \(N_r(G(\mu))\) (resp., \(N_l(G(\mu))\)).

Thus, in view of Theorem 5.1, we only need to describe the recovery of eigenvectors, minimal bases and minimal indices of \(S(\lambda)\) from those of the GFPRs of \(G(\lambda)\). To that end, we need the following result.

Theorem 5.2 ([12,27]): Consider the GF pencil \(T_\omega(\lambda) := \lambda M_{\omega}S - M_{\omega_0}S\) of \(G(\lambda)\) associated with a permutation \(\omega := (\omega_0, \omega_1)\) of \(\{0 : m\}\), where \(0 \in \omega_0\) and \(m \in \omega_1\). Then we have the following:

(I) Minimal bases. Suppose that \(S(\lambda)\) is singular. Then the maps

\[
\mathbb{F}_\omega(S) : N_r(T_\omega) \to N_r(S), \quad \begin{bmatrix} u(\lambda) \\ v(\lambda) \end{bmatrix} \mapsto \begin{bmatrix} (e_{m-c_0}\otimes I_n)u(\lambda) \\ v(\lambda) \end{bmatrix},
\]

\[
\mathbb{K}_\omega(S) : N_l(T_\omega) \to N_l(S), \quad \begin{bmatrix} u(\lambda) \\ v(\lambda) \end{bmatrix} \mapsto \begin{bmatrix} (e_{m-c_0}\otimes I_n)u(\lambda) \\ v(\lambda) \end{bmatrix},
\]

where \(c_0\) is the multiplicity of \(\omega_0\) in \(\lambda = \omega(\mu)\).
are linear isomorphisms, where $u(\lambda) \in \mathbb{C}(\lambda)^{mn}$ and $v(\lambda) \in \mathbb{C}(\lambda)^{\tau}$. Further, $\mathbb{F}^{\tau \omega}(S)$ (resp., $\mathbb{K}^{\tau \omega}(S)$) maps a minimal basis of $\mathcal{N}_r(\mathbb{T}_\omega)$ (resp., $\mathcal{N}_l(\mathbb{T}_\omega)$) to a minimal basis of $\mathcal{N}_r(S)$ (resp., $\mathcal{N}_l(S)$).

Let $\omega_1$ be given by $\omega_1 := (\omega_0^j, m, \omega_0^i)$. Set $\alpha := (\text{rev}(\omega_0^j), \omega_0, \text{rev}(\omega_0^i))$. Let $c(\alpha)$ and $i(\alpha)$ be the total number of consecutations and inversions of the permutation $\alpha$, respectively. If $\varepsilon_1 \leq \cdots \leq \varepsilon_p$ are the right (resp., left) minimal indices of $\mathbb{T}_\omega(\lambda)$ then $\varepsilon_1 - i(\alpha) \leq \cdots \leq \varepsilon_p - i(\alpha)$ (resp., $\varepsilon_1 - c(\alpha) \leq \cdots \leq \varepsilon_p - c(\alpha)$) are the right (resp., left) minimal indices of $S(\lambda)$.

(II) Eigenvectors. Suppose that $S(\lambda)$ is regular and $\mu \in \mathbb{C}$ is an eigenvalue of $S(\lambda)$. Let $Z := \begin{bmatrix} Z_{mn} & Z_r \end{bmatrix}$ be an $(mn + r) \times p$ matrix such that $\text{rank}(Z) = p$, where $Z_{mn}$ has $mn$ rows and $Z_r$ has $r$ rows. If $Z$ is a basis of $\mathcal{N}_r(\mathbb{T}_\omega(\mu))$ (resp., $\mathcal{N}_l(\mathbb{T}_\omega(\mu))$) then

$$
\begin{bmatrix}
\begin{pmatrix} e^{T}_{m-c(\alpha)} \otimes I_n \end{pmatrix} Z_{mn} \\
\begin{pmatrix} e^{T}_{m-i(\alpha)} \otimes I_n \end{pmatrix} Z_r
\end{bmatrix}
$$

is a basis of $\mathcal{N}_r(S(\mu))$ (resp., $\mathcal{N}_l(S(\mu))$).

The pencil $\mathbb{T}_\omega(\lambda)$ in Theorem 5.2 is referred to as a PGF (proper generalized Fiedler) pencil of $G(\lambda)$ (also refer to as a PGF pencil of $S(\lambda)$).

For the rest of the paper, we only consider GFPRs with non-singular matrix assignments. Thus, if $L(\lambda) := \mathbb{M}_{(t_1, \sigma_1)}(Y_1, X_1)(\lambda \mathbb{M}_{\tau}^S - \mathbb{M}_{\sigma}^S)\mathbb{M}_{(t_2, \tau_2)}(X_2, Y_2)$ is a GFPR of $S(\lambda)$ then we assume that $X_j$ and $Y_j$, $j = 1, 2$, are non-singular matrix assignments.

**Theorem 5.3:** Let $\mathbb{L}(\lambda) := \mathbb{M}_{(t_1, \sigma_1)}(Y_1, X_1)(\lambda \mathbb{M}_{\tau}^S - \mathbb{M}_{\sigma}^S)\mathbb{M}_{(t_2, \tau_2)}(X_2, Y_2)$ be a GFPR of $S(\lambda)$. Let $Z(\lambda) := \begin{bmatrix} Z_{mn}(\lambda) & Z_r(\lambda) \end{bmatrix}$ be an $(mn + r) \times p$ matrix polynomial, where $Z_{mn}(\lambda)$ has $mn$ rows and $Z_r(\lambda)$ has $r$ rows.

(a) If $Z(\lambda)$ is a right (resp., left) minimal basis of $\mathbb{L}(\lambda)$ then

$$
\begin{bmatrix}
\begin{pmatrix} e^{T}_{m-c(\sigma, \tau)} \otimes I_n \end{pmatrix} Z_{mn}(\lambda) \\
\begin{pmatrix} e^{T}_{m-i(\sigma, \tau)} \otimes I_n \end{pmatrix} Z_r(\lambda)
\end{bmatrix}
$$

is a right (resp., left) minimal basis of $S(\lambda)$.

(b) Let $\tau$ be given by $\tau := (t_1, -m, t_2)$. Set $\alpha := (-\text{rev}(t_1), \sigma, -\text{rev}(t_2))$. Let $c(\alpha)$ and $i(\alpha)$ be the total number of consecutations and inversions of the permutation $\alpha$. If $\varepsilon_1 \leq \cdots \leq \varepsilon_p$ are the right (resp., left) minimal indices of $\mathbb{L}(\lambda)$ then $\varepsilon_1 - i(\alpha) \leq \cdots \leq \varepsilon_p - i(\alpha)$ (resp., $\varepsilon_1 - c(\alpha) \leq \cdots \leq \varepsilon_p - c(\alpha)$) are the right (resp., left) minimal indices of $S(\lambda)$.

**Proof:** We have $\mathbb{L}(\lambda) = U \mathbb{T}_\omega(\lambda) V$, where $\mathbb{T}_\omega(\lambda) := \lambda \mathbb{M}_{\tau}^S - \mathbb{M}_{\sigma}^S$ is a PGF pencil of $G(\lambda)$ associated with the permutation $\omega := (\sigma, -\tau)$ of $\{0 : m\}$, and $U := \mathbb{M}_{(t_1, \sigma_1)}(Y_1, X_1)$ and $V := \mathbb{M}_{(t_2, \tau_2)}(X_2, Y_2)$. Since $V$ is a non-singular matrix, it is easily seen that the map $V : \mathcal{N}_r(\mathbb{L}) \rightarrow \mathcal{N}_r(\mathbb{T}_\omega), z(\lambda) \mapsto V z(\lambda)$, is an isomorphism and maps a minimal basis of $\mathcal{N}_r(\mathbb{L})$ to a minimal basis of $\mathcal{N}_r(\mathbb{T}_\omega)$. On the other hand, by Theorem 5.2,

$$
\mathbb{F}^{\tau \omega}(S) : \mathcal{N}_r(\mathbb{T}_\omega) \rightarrow \mathcal{N}_r(S), \begin{bmatrix} x(\lambda) \\
y(\lambda)
\end{bmatrix} \mapsto \begin{bmatrix} (e^{T}_{m-c(\sigma)} \otimes I_n) x(\lambda) \\
y(\lambda)
\end{bmatrix},
$$

is an isomorphism and maps a minimal basis of $\mathcal{N}_r(\mathbb{T}_\omega)$ to a minimal basis of $\mathcal{N}_r(S)$, where $x(\lambda) \in \mathbb{C}(\lambda)^{mn}$ and $y(\lambda) \in \mathbb{C}(\lambda)^{\tau}$. Consequently, $\mathbb{F}^{\tau \omega}(S) V : \mathcal{N}_r(\mathbb{L}) \rightarrow \mathcal{N}_r(S), z(\lambda) \mapsto \mathbb{F}^{\tau \omega}(S) V z(\lambda)$, is an isomorphism and maps a minimal basis of $\mathcal{N}_r(\mathbb{L})$ to a minimal basis
of \( \mathcal{N}_r(S) \). Now, by Lemma 3.5, we have 
\[
\text{Fr} \omega(S)V = \text{Fr} \omega(S)M_{(\sigma_2, \tau_2)}(X_2, Y_2) = \left[ \begin{array}{c} (e^T_{m-c_0(\sigma) \otimes I_n}) M_{(\sigma_2, \tau_2)}(X_2, Y_2) \\ I_r \end{array} \right] = \left[ \begin{array}{c} (e^T_{m-c_0(\sigma, \sigma_2) \otimes I_n}) I_r \end{array} \right],
\]
and hence the desired result for the recovery of right minimal bases follows.

Now we describe the recovery of left minimal bases. Since \( U \) is a non-singular matrix, it is easily seen that the map \( U^T : \mathcal{N}_l(\mathbb{L}) \to \mathcal{N}_l(\mathbb{T}_\omega) \), \( z(\lambda) \mapsto U^T z(\lambda) \), is an isomorphism and maps a minimal basis of \( \mathcal{N}_l(\mathbb{L}) \) to a minimal basis of \( \mathcal{N}_l(\mathbb{T}_\omega) \). On the other hand, by Theorem 5.2,
\[
\text{Fr} \omega(S) : \mathcal{N}_l(\mathbb{T}_\omega) \to \mathcal{N}_l(S), \left[ \begin{array}{c} x(\lambda) \\ y(\lambda) \end{array} \right] \mapsto \left[ \begin{array}{c} (e^T_{m-i_0(\sigma) \otimes I_n}) x(\lambda) \\ y(\lambda) \end{array} \right],
\]
is an isomorphism and maps a minimal basis of \( \mathcal{N}_l(\mathbb{T}_\omega) \) to a minimal basis of \( \mathcal{N}_l(S) \), where \( x(\lambda) \in \mathbb{C}(\lambda)^{mn} \) and \( y(\lambda) \in \mathbb{C}(\lambda)^{r} \). Consequently, \( \text{Fr} \omega(S) U^T : \mathcal{N}_l(\mathbb{L}) \to \mathcal{N}_l(S) \), \( z(\lambda) \mapsto \text{Fr} \omega(S) U^T z(\lambda) \), is an isomorphism and maps a minimal basis of \( \mathcal{N}_l(\mathbb{L}) \) to a minimal basis of \( \mathcal{N}_l(S) \). Now \( \text{Fr} \omega(S) U^T = \text{Fr} \omega(S)(M_{(\tau_1, \sigma_1)}(Y_1, X_1))^T = \left[ \begin{array}{c} (e^T_{m-i_0(\sigma) \otimes I_n}) (M_{(\tau_1, \sigma_1)}(Y_1, X_1) (e^T_{m-i_0(\sigma) \otimes I_n}) I_r) \\ I_r \end{array} \right] \).

By Lemma 3.5, we have \( M_{(\tau_1, \sigma_1)}(Y_1, X_1) (e^T_{m-i_0(\sigma) \otimes I_n}) I_n = e^T_{m-i_0(\sigma, \sigma_1) \otimes I_n} \). Hence the desired result for recovery of left minimal bases follows.

Finally, let \( \varepsilon_1 \leq \cdots \leq \varepsilon_p \) be the right (resp., left) minimal indices of \( \mathbb{L}(\lambda) \). Since the PGF pencil \( T_\omega(\lambda) \) is strictly equivalent to \( \mathbb{L}(\lambda) \), \( \varepsilon_1 \leq \cdots \leq \varepsilon_p \) are also the right (resp., left) minimal indices of \( T_\omega(\lambda) \). Hence by Theorem 5.2, \( \varepsilon_1 - i(\alpha) \leq \cdots \leq \varepsilon_p - i(\alpha) \) (resp., \( \varepsilon_1 - c(\alpha) \leq \cdots \leq \varepsilon_p - c(\alpha) \)) are the right (resp., left) minimal indices of \( S(\lambda) \).

The next result describes the recovery of eigenvectors of \( S(\lambda) \) from those of the GFPRs of \( S(\lambda) \) when \( S(\lambda) \) is regular.

**Theorem 5.4:** Let
\[
\mathbb{L}(\lambda) := M_{(\tau_1, \sigma_1)}(Y_1, X_1)(\lambda M_{(\sigma_2, \tau_2)}(X_2, Y_2) - M_{\sigma_1}(\sigma_1)),
\]
be a GFPR of \( S(\lambda) \). Suppose that \( S(\lambda) \) is regular and \( \mu \in \mathbb{C} \) is an eigenvalue of \( S(\lambda) \). Let \( Z := \left[ \begin{array}{c} Z_{mn} \\ Z_r \end{array} \right] \) be an \( (mn + r) \times p \) matrix such that \( \text{rank}(Z) = p \), where \( Z_{mn} \) has \( mn \) rows and \( Z_r \) has \( r \) rows. If \( Z \) is a basis of \( \mathcal{N}_r(\mathbb{L}(\mu)) \) (resp., \( \mathcal{N}_l(\mathbb{L}(\mu)) \)) then \( \left[ (e^T_{m-c_0(\sigma, \sigma_2) \otimes I_n}) Z_{mn} \right] \) (resp., \( \left[ (e^T_{m-i_0(\sigma, \sigma_1) \otimes I_n}) Z_{mn} \right] \)) is a basis of \( \mathcal{N}_r(S(\mu)) \) (resp., \( \mathcal{N}_l(S(\mu)) \)).

**Proof:** A verbatim proof of Theorem 5.3 together with part (II) of Theorem 5.2 yields the desired results.

Next, we briefly describe the recovery of eigenvectors, minimal bases and minimal indices of a structured \( G(\lambda) \) from those of the structured linearizations discussed in Section 4.
Table 1. $c_0(\sigma, \sigma_2)$.

| Structure  | symmetric | $T$-even/odd | skew-symmetric |
|------------|-----------|--------------|----------------|
| $c_0(\sigma, \sigma_2)$ | $2 + l_2(t_{m_a})$ | 1 | $2 + c_2(t_w)$ |

Note that if $G(\lambda)$ is singular then the left (resp., right) minimal indices of $G(\lambda)$ and $XG(\lambda)Y$ are the same for any non-singular matrices $X$ and $Y$. Hence it follows that if $G(\lambda)$ is symmetric (resp., skew-symmetric, Hamiltonian, skew-Hamiltonian) then the left minimal indices of $G(\lambda)$ are the same as the right minimal indices of $G(\lambda)$. Consequently, if $\mathbb{L}(\lambda)$ is a structure-preserving linearization of $G(\lambda)$ considered in Section 4 then the left minimal indices of $\mathbb{L}(\lambda)$ are the same as the right minimal indices of $\mathbb{L}(\lambda)$. Since $\mathbb{L}(\lambda)$ is strictly equivalent to a GFPR $T(\lambda) := \mathbb{M}_{(\tau_1, \tau_2)}(Y_1, X_1)(\lambda, \mathbb{M}_r^S - \mathbb{M}_r^S\mathbb{M}_{(\sigma_2, \tau_2)}(X_2, Y_2)$ of $G(\lambda)$, the left and right minimal indices of $T(\lambda)$ are the same. Let $\tau$ be given by $\tau = (\tau_\ell, -m, \tau_r)$. Define $\alpha := (\text{rev}(\tau_\ell), \sigma, -\text{rev}(\tau_r)).$ Then $\alpha$ is a permutation of $\{0 : m - 1\}$.

Let $c(\alpha)$ and $i(\alpha)$, respectively, be the total number of consecutions and inversions of $\alpha$. Let $\varepsilon_1 \leq \cdots \leq \varepsilon_k$ be the minimal (left and right) indices of $T(\lambda)$. Then by Theorem 5.3, $\varepsilon_1 - i(\alpha) \leq \cdots \leq \varepsilon_k - i(\alpha)$ and $\varepsilon_1 - c(\alpha) \leq \cdots \leq \varepsilon_k - c(\alpha)$, respectively, are the right and left minimal indices of $G(\lambda)$. Since the left and right minimal indices of $G(\lambda)$ are the same, we must have $i(\alpha) = c(\alpha)$. But $i(\alpha) + c(\alpha) = m - 1$. Consequently, we have $i(\alpha) = (m - 1)/2 = c(\alpha)$ which shows that $\varepsilon_1 - (m - 1)/2 \leq \cdots \leq \varepsilon_k - (m - 1)/2$ are the minimal (left and right) indices of $G(\lambda)$. Recall that $\mathbb{L}(\lambda)$ is not a linearization of $G(\lambda)$ if $m$ is even.

Thus, if $\mathbb{L}(\lambda)$ is a structure-preserving linearization of $G(\lambda)$ considered in Section 4 then the left minimal indices of $\mathbb{L}(\lambda)$ are the same as the right minimal indices of $\mathbb{L}(\lambda)$. Moreover, if $\varepsilon_1 \leq \cdots \leq \varepsilon_k$ are the minimal (left and right) indices of $\mathbb{L}(\lambda)$ then $\varepsilon_1 - (m - 1)/2 \leq \cdots \leq \varepsilon_k - (m - 1)/2$ are the minimal (left and right) indices of $G(\lambda)$. Hence we only need to comment on the recovery of eigenvectors and minimal bases of $G(\lambda)$ from those of the $\mathbb{L}(\lambda)$.

Note that the left minimal bases of $G(\lambda)$ are the same as the right minimal bases of $G(\lambda)$ when $G(\lambda)$ is symmetric (resp., Hamiltonian, skew-Hamiltonian, skew-symmetric). Hence if $\mathbb{L}(\lambda)$ is a structure-preserving linearization of $G(\lambda)$ considered in Section 4 then the left minimal bases of $\mathbb{L}(\lambda)$ are the same as the right minimal bases of $\mathbb{L}(\lambda)$. Consequently, minimal bases and eigenvectors of $G(\lambda)$ can be recovered from those of $\mathbb{L}(\lambda)$ as special cases of Theorems 5.3 and 5.4. Indeed, for structure-preserving linearizations, we have $c_0(\sigma, \sigma_2) = 0$ when $h = 0$ and $c_0(\sigma, \sigma_2)$ is given in the Table 1 when $h > 0$.

6. Conclusion

We have made three main contributions in this paper. First, we have generalized GFPRs of a matrix polynomial $P(\lambda)$ to the case of a rational matrix $G(\lambda)$. Moreover, we have shown that the transition from GFPRs of matrix polynomials to GFPRs of rational matrices is operation-free (Theorem 3.6). Second, and most importantly, we have utilized GFPRs of $G(\lambda)$ to construct structure-preserving Rosenbrock strong linearizations of a structured (symmetric, Hermitian, skew-symmetric, even, odd, etc.) rational matrix $G(\lambda)$. Third, we
have described automatic recovery rules for eigenvectors, minimal bases and minimal indices of \( G(\lambda) \) from those of the linearizations of \( G(\lambda) \).

**Disclosure statement**

No potential conflict of interest was reported by the author(s).

**References**

[1] Fuhrmann PA. On symmetric rational transfer functions. Linear Algebra Appl. 1983;50:167–250.
[2] Genin Y, Hachez Y, Nesterov Y, et al. Positivity and linear matrix inequalities. Eur J Control. 2002;8:275–298.
[3] Helmke U, Rosenthal J, Wang XA. Output feedback pole assignment for transfer functions with symmetries. SIAM J Control Optim. 2006;45:1898–1914.
[4] Hillar CJ, Sottile F. Complex static skew-symmetric output feedback control. SIAM J Control Optim. 2013;51:3011–3026.
[5] Kouhi Y, Bajcinca N, Raisch J, et al. On the quadratic stability of switched linear systems associated with symmetric transfer function matrices. Autom J IFAC. 2014;50:2872–2879.
[6] Mehrmann V, Voss H. Nonlinear eigenvalue problems: a challenge for modern eigenvalue methods. GAMM Mitt Ges Angew Math Mech. 2004;27:121–152.
[7] Su Y, Bai Z. Solving rational eigenvalue problems via linearization. SIAM J Matrix Anal Appl. 2011;32:201–216.
[8] Willems JC. Realization of systems with internal passivity and symmetry constraints. J Franklin Inst. 1976;301:605–621.
[9] Ran ACM. Necessary and sufficient conditions for existence of \( J \)-spectral factorization for para-Hermitian rational matrix functions. Automatica. 2003;39:1935–1939.
[10] Mackey DS, Mackey N, Mehl C, et al. Structured polynomial eigenvalue problems: good vibrations from good linearizations. SIAM J Matrix Anal Appl. 2006;28:1029–1051.
[11] Alam R, Behera N. Linearizations for rational matrix functions and Rosenbrock system polynomials. SIAM J Matrix Anal Appl. 2016;37:354–380.
[12] Alam R, Behera N. Generalized fiedler pencils for rational matrix functions. SIAM J Matrix Anal Appl. 2018;39:587–610.
[13] Amparan A, Dopico FM, Marcaida S, et al. Strong linearizations of rational matrices. SIAM J Matrix Anal Appl. 2018;39:1670–1700.
[14] Behera N. Fiedler linearizations for LTI state-space systems and for rational eigenvalue problems [dissertation]. Guwahati: Indian Institute of Technology; 2014.
[15] Das RK, Alam R. Affine spaces of strong linearizations for rational matrices and the recovery of eigenvectors and minimal bases. Linear Algebra Appl. 2019;569:335–368.
[16] Dopico FM, Marcanda S, Quintana MC. Strong linearizations of rational matrices with polynomial part expressed in an orthogonal basis. Linear Algebra Appl. 2019;570:1–45.
[17] Kailath T. Linear systems. Englewood Cliffs: Prentice-Hall; 1980.
[18] Rosenbrock HH. State-space and multivariable theory. New York: John Wiley & Sons, Inc.; 1970.
[19] Forney Jr. GD. Minimal bases of rational vector spaces, with applications to multivariable linear systems. SIAM J Control. 1975;13:493–520.
[20] Vardulakis AIG. Linear multivariable control. Chichester: John Wiley & Sons Ltd.; 1991.
[21] Alam R, Behera N. Recovery of eigenvectors of rational matrix functions from Fiedler-like linearizations. Linear Algebra Appl. 2016;510:373–394.
[22] Bueno MI, Dopico FM, Furtado S, et al. Large vector spaces of block-symmetric strong linearizations of matrix polynomials. Linear Algebra Appl. 2015;477:165–210.
[23] De Terán F, Dopico FM, Mackey DS. Fiedler companion linearizations and the recovery of minimal indices. SIAM J Matrix Anal Appl. 2010;31:2181–2204.
[24] Bueno MI, De Terán F. Eigenvectors and minimal bases for some families of Fiedler-like linearizations. Linear Multilinear Algebra. 2014;62:39–62.
[25] Vologiannidis S, Antoniou EN. A permuted factors approach for the linearization of polynomials. Math Control Signals Syst. 2011;22:317–342.
[26] Das RK, Alam R. Automatic recovery of eigenvectors and minimal bases of matrix polynomials from generalized Fiedler pencils with repetition. Linear Algebra Appl. 2019;569:78–112.
[27] Das RK, Alam R. Recovery of minimal bases and minimal indices of rational matrices from Fiedler-like pencils. Linear Algebra Appl. 2019;566:34–60.
[28] Das RK. Strong linearizations of polynomial and rational matrices and recovery of spectral data [dissertation]. Guwahati: Indian Institute of Technology; 2019.
[29] Das RK, Alam R. Structured strong linearizations of structured rational matrices. Extended version available as arXiv:2008.00427 [math.NA].
[30] Amparan A, Dopico FM, Marcaida S, et al. On minimal bases and indices of rational matrices and their linearizations. arXiv:1912.12293v2 [math.NA].
[31] Bitmead RR, Anderson BDO. The matrix cauchy index: properties and applications. SIAM J Appl Math. 1977;33:655–672.
[32] Byrnes CI. On a theorem of Hermite and Hutwitz. Linear Algebra Appl. 1983;50:61–101.
[33] Byrnes CI, Duncan TE. On certain topological invariants arising in system theory. In: Hilton P, Young G, editors. New directions in applied mathematics. New York: Springer-Verlag; 1981. p. 29–71.
[34] Hughes TH, Smith MC. Algebraic criteria for circuit realisations. In: Mathematical system theory – Festschrift in honor of Uwe Helmke. CreateSpace; 2013. p. 211–228.
[35] Bueno MI, Furtado S. Structured strong linearizations from Fiedler pencils with repetition II. Linear Algebra Appl. 2014;463:282–321.
[36] Verghese G, Van Dooren P, Kailath T. Properties of the system matrix of a generalized state-space system. Int J Control. 1979;30:235–243.