We consider a number of prior probability distributions of particular interest, all being defined on the three-dimensional convex set of two-level quantum systems. Each distribution is — following recent work of Petz and Sudar — taken to be proportional to the volume element of a monotone metric on that Riemannian manifold. We apply an entropy-based test (a variant of one recently developed by Clarke) to determine which of two priors is more noninformative in nature. This involves converting them to posterior probability distributions based on some set of hypothesized outcomes of measurements of the quantum system in question. It is, then, ascertained whether or not the relative entropy (Kullback-Leibler statistic) between a pair of priors increases or decreases when one of them is exchanged with its corresponding posterior. The findings lead us to assert that the maximal monotone metric yields the most noninformative prior distribution and the minimal monotone (that is, the Bures) metric, the least. Our conclusions both agree and disagree, in certain respects, with ones recently reached by Hall, who relied upon a less specific test criterion than our entropy-based one.

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I. INTRODUCTION

At the outset of this letter, let us note that subsequent to an earlier version of it (quant-ph/9703012), Hall communicated a study [1] having quite similar (though perhaps more clearly articulated) objectives. Nevertheless, in terms of methodologies and conclusions, the two studies appear to differ significantly. We shall indicate these interesting points of agreement and disagreement.

Let us reiterate the fundamental question motivating Hall, which he states at the beginning of his presentation: “what statistical ensemble corresponds to minimal prior knowledge about a quantum system? Such an ensemble may be identified as the most random ensemble of possible states of the system. It would provide, for example, a natural benchmark for assessing how ‘random’ a given evolution process is; a worst-case scenario for general schemes for extracting information about the system; and a natural unbiased measure over the set of possible states of the system (which would allow one to calculate, e.g., the average effectiveness of a general scheme for distinguishing between quantum states”).

In our work presented below, we have relied upon a specific entropy-based test (related to one recently developed by Clarke [2]) to address these issues Hall has raised. The conclusions of Hall themselves, on the other hand, rest on the less specific (and apparently, from our point of view, not determinative) proposition that “maximal randomness corresponds to an ensemble with maximal symmetry.” In fact, we will argue that Hall’s minimal-knowledge ensemble (based on the Bures metric [3]) is not truly minimal. However, we do agree with Hall that this particular ensemble is superior in information terms to that associated with a uniform distribution (as previously employed by Larson and Dukes [4]) over the convex set (Bloch sphere [5]) of two-level quantum systems.

Brody and Meister [6] recently studied the problem of deciding between two a priori possible two-level quantum mechanical pure states. They assert that “if prior knowledge is not available, one can still employ the Bayesian approach, using a noninformative prior. However, the analysis of such cases is beyond the scope of the present Letter.” In this study, we do consider the problem of determining an appropriate noninformative prior — for the more general situation, in which advance information regarding the degree of purity of the unknown system is lacking. (“In a real situation one can never design a preparator such that it produces an ensemble of identical pure states. What usually happens is that the ensemble consists of a set of pure states each of which is represented in the ensemble with a certain probability” [6].)

As noted, we adapt recent work of Clarke [2] to provide us with an operational criterion for deciding which of two priors is more noninformative in nature. We apply our test to a number of priors, all of which (with the exception...
of the uniform distribution of Larson and Dukes [4] are obtained by normalizing the volume elements of monotone metrics \[ 1 \leq H \leq f \text{symmetrization conditions}, \]

number of such metrics. Each corresponds to an operator monotone function that, in the quantum/noncommutative case, there is not a single monotone metric, but rather a nondenumerable

[8, p. 786]. Petz and Sudar [9] — building upon work of Morozova and Chentsov [18] — have recently shown

manifold formed by the family. "An infinitesimal statistical distance has to be monotone under stochastic mappings"

p — that the minimal monotone metric (of the symmetric logarithmic derivative) yields the least noninformative prior of those examined (other than the uniform one of Larson and Dukes [4]). It also appears that our (Bayesian) notion of noninformativity is equivalent to the (classical) one of Petz and Toth [11, p. 215] in their work comparing (Cramér-Rao) lower bounds for the variances/covariances of unbiased estimators of the parameters of quantum systems.

In (classical/nonquantum) Bayesian theory [13], in the absence of any information regarding the specific values of the parameters of a family of probability distributions, one uses a noninformative (Jeffreys) prior (cf. [16,17]). This is taken to be proportional to the volume element of the unique monotone (Fisher information) metric on the Riemannian manifold formed by the family. “An infinitesimal statistical distance has to be monotone under stochastic mappings” [8, p. 786]. Petz and Sudar [9] — building upon work of Morozova and Chentsov [18] — have recently shown

that, in the quantum/noncommutative case, not a single monotone metric, but rather a nondenumerable number of such metrics. Each corresponds to an operator monotone function \( f(t) \) satisfying the normalization and symmetrization conditions, \( f(1) = 1, f(t) = tf(t^{-1}) \). (A function \( f: \mathbb{R}^+ \to \mathbb{R} \) is called operator monotone if the relation \( 0 \leq K \leq H \), meaning that \( H - K \) is semipositive definite, implies \( 0 \leq f(K) \leq f(H) \), for any matrices \( K \) and \( H \) of arbitrary order [13].) “Therefore, more than one privileged metric shows up in quantum mechanics. The exact clarification of this point requires and is worth further studies” [9, p. 2672]. This Letter seeks to contribute to such a clarification (cf. [20]). We, thus, restrict our considerations to priors which are proportional to volume elements of monotone metrics. Although this still leaves a nondenumerable number of candidate priors, our results indicate that certain ones (in particular, that obtained from the maximal monotone metric) can be distinguished for their information-theoretic properties.

II. ANALYSIS

A. Two prior probability distributions

We begin our analysis by examining two monotone metrics of particular interest [9] — the Kubo-Mori metric (given by \( f(t) = (t - 1)/\log t \)) and the minimal monotone SLD (symmetric logarithmic derivative) “Bures-type” [21] metric (given by \( f(t) = (1 + t)/2 \)). In recent papers [22] (cf. [20,23]), the author has proposed and analyzed the use of a prior probability distribution,

\[
(1 - x^2 - y^2 - z^2)^{-1/2}/\pi^2.
\]

This is the normalized form of the volume element of the SLD-metric over the Bloch sphere [24] — the unit ball in three-space, comprised of the points \( x^2 + y^2 + z^2 \leq 1 \) — of \( 2 \times 2 \) density matrices,

\[
\frac{1}{2} \left( \begin{array}{cc} 1 + z & x - iy \\ x + iy & 1 - z \end{array} \right).
\]

In spherical coordinates \((x = r \cos \phi \sin \theta, y = r \sin \phi \sin \theta, z = r \cos \theta)\), [4] takes the form,

\[
p_{SLD}(r, \theta, \phi) = r^2(1 - r^2)^{-1/2} \sin \theta / \pi^2.
\]

On the other hand, use of the Kubo-Mori metric yields a prior probability distribution,

\[
p_{KM}(r, \theta, \phi) = r(1 - r^2)^{-1/2} \log((1 + r)/(1 - r)) \sin \theta / 4\pi^2.
\]

The distributions [2] and [3] are obtained by substituting the corresponding operator monotone functions given above into the formula [1, eq. 3.17],

\[
r^2(1 - r^2)^{-1/2} \log((1 + r)/(1 - r)) \sin \theta / f((1 - r)/(1 + r)),
\]

and, then, normalizing. Both \( p_{KM} \) and \( p_{SLD} \) are monotonically increasing with \( r \), with \( p_{KM} \) assigning greater probability to systems that are nearly pure \((r > .957504)\) and, compensatingly less, to relatively mixed systems \((r < .957504)\).
We, first, compare the suitabilities of \( p_{SLD} \) and \( p_{KM} \) as possible noninformative or “reference” priors \([13]\) for the quantum inference or estimation of an unknown two-level system. (Appropriate informative priors should, of course, be used if specific knowledge regarding the parameters of the system is available \([3]\).) We modify (as explained below) — in any case, doing so apparently, at least in our context, leads to no substantive differences — a general line of reasoning recently elaborated upon by Clarke \([2]\). We note, in this regard, that the relative entropy (Kullback-Leibler number or information gain or directed divergence) \([15]\) of \( p_{SLD} \) with respect to \( p_{KM} \), that is,

\[
D(p_{SLD} \parallel p_{KM}) = \int_0^1 \int_0^\pi \int_0^{2\pi} p_{SLD} \log \left( \frac{p_{SLD}}{p_{KM}} \right) d\phi d\theta dr
\]

(the natural logarithm is understood throughout this communication) is .0891523 “nats” (1 nat equals \( \frac{1}{\log 2} \approx 1.4227 \) bits), while \( D(p_{KM} \parallel p_{SLD}) \) is .0975976 nats. It proves possible to reduce the former statistic by incorporating certain information into our considerations, but not the latter. This leads us to the conclusion that \( p_{KM} \) is more noninformative than \( p_{SLD} \).

After some initial numerical experimentation, we were led to assume that six spin measurements had been performed — two in the \( X^- \), two in the \( Y^- \), and two in the \( Z \)-direction — using six replicas of a spin-1/2 system and that for each of these pairs we obtained one “up” and one “down”. (It would be of interest to conduct a parallel series of analyses to that reported below, based on what have been found to be optimal sets of measurements \([4,5]\).) Then — applying Bayes’ Theorem \([13]\) — we converted \( p_{SLD} \) and \( p_{KM} \) to posterior distributions by multiplying them by the likelihood of such a set of six outcomes \([2,3]\),

\[
(1 - x^2)(1 - y^2)(1 - z^2)/64 = [(1 - x)/2][(1 + x)/2][(1 - y)/2][(1 + y)/2][(1 - z)/2][(1 + z)/2],
\]

and normalizing the resulting products over the Bloch sphere. (The normalization factors are \( 64 \times 192/71 \) in the \( SLD \)-case and \( 64 \times 19600/6047 \) in the \( KM \)-case.) The relative entropy of \( p_{SLD} \) with respect to the \( KM \)-posterior is, then, reduced, as a result of the added information, from .0891523 to .0720681. On the other hand, the relative entropy of \( p_{KM} \) with respect to the \( SLD \)-posterior is increased dramatically from .0975976 to .457259. Paraphrasing Clarke \([2, p. 173]\), “[\( p_{SLD} \)] is already more informative than [\( p_{KM} \)], so we cannot make it less informative by adding information”. However, if we were to replace the likelihood \([3] \) by its square — that is, in effect, assume twelve measurements, giving two “ups” and two “downs” in each of the three mutually orthogonal directions — then the relative entropy of \( p_{SLD} \) with respect to the corresponding revised or updated \( KM \)-posterior would not further decrease from .0720681, but would increase to .334699. Thus, the informativity of \( p_{SLD} \) with respect to \( p_{KM} \) is limited, in this manner. (In an approximate sense, then, the information contained in \( p_{SLD} \) can be described as that in \( p_{KM} \) with the addition of that gained by knowledge of the outcomes of the six measurements.)

If we had conformed strictly to the line of argument of Clarke \([2]\), we would have exchanged the positions of the priors and posteriors in the relative entropy statistics reported above. Nevertheless, it seems rather evident that — in the context of the present study — we would reach the same fundamental conclusions if we had done so. For example (again, based on the same six measurement outcomes), the relative entropy of the \( KM \)-posterior with respect to \( p_{SLD} \) is .0603743 (cf. .0720681) and that of the \( SLD \)-posterior (based on the same hypothetical six observations) with respect to \( p_{KM} \) itself is .151575, while the analogous \( SLD \)-result is less, \( \log 2 + \log \frac{1}{2^6} \approx .140862 \). Also, a single spin measurement yields an information gain of .157404 with respect to \( p_{KM} \), but less (\( \frac{1}{2} \) − log 2 \( \approx .140186 \) with respect to \( p_{SLD} \).
B. Two additional prior probability distributions

Using a recursively defined double sequence, Petz [10, eq. 21] has arrived at the operator monotone function,

\[ f(t) = \frac{2(t-1)^2}{(1+t)(\log t)^2}, \]  

(8)

the operator mean of which had been found by Morozova and Chentsov [3]. Normalizing, using numerical methods, the volume element (6) of the associated monotone metric, we obtain the following prior probability distribution,

\[ p_{MC}(r, \theta, \phi) = .00513299(1-r^2)^{-1/2}(\log[(1-r)/(1+r)])^2 \sin \theta. \]  

(9)

Now, \( D(p_{KM} \parallel p_{MC}) = .112421 \) and \( D(p_{MC} \parallel p_{KM}) = .117982 \). These statistics are transformed to .106655 and .482023, respectively, when they are computed with respect to the corresponding \( MC \) and \( KM \)-posteriors, both based on the previously hypothesized set of six outcomes. So, we can assert — by our rule — that \( p_{MC} \) is itself more noninformative than \( p_{KM} \). (Strict adherence to the scheme of Clarke yields the corresponding statistics, .0910048 and .452794, leading to the very same conclusion. It appears, however, that our procedure yields somewhat larger statistics than does Clarke’s.) This is consistent with the earlier result, in the sense that \( p_{MC} \) assigns greater probability to some set of nearly pure systems \((r > .9846)\) than does \( p_{KM} \). Assuming that noninformativity is a transitive relation, one would expect to find that \( p_{MC} \) is also more noninformative than \( p_{SLD} \). This proves to be the case, as \( D(p_{SLD} \parallel p_{MC}) = .388323 \) and \( D(p_{MC} \parallel p_{SLD}) = .445981 \). The posterior version of these statistics are, .186964 and .991175, respectively, with \( p_{MC} \) assigning greater probability than \( p_{SLD} \) to systems with \( r > .973932 \).

Larson and Dukes [4] have utilized a uniform prior,

\[ p_{LD}(r, \theta, \phi) = 3r^2 \sin \theta/4\pi, \]  

(10)

over the Bloch sphere of two-level quantum systems. “The simplest prior which does not confine itself to pure states assigns equal probability to equal infinitesimal volumes within the unit sphere of the geometrical parameterization. (This is a physically reasonable prior, since the geometrical parameterization is metrically faithful)” [4]. Hall [1] obtains the uniform distribution by two distinct arguments, one based on randomly correlated ensembles, and the other on the Hilbert-Schmidt metric.

Although \( p_{LD} \) can be written as proportional to the volume element (5), using \( f(t) = (1+t)^2/\sqrt{t} \), this particular function is {not} operator monotone, so \( p_{LD} \) does not, in fact, correspond to a monotone metric. We can, nevertheless, examine its informative properties. We have that \( D(p_{LD} \parallel p_{MC}) = 1.07895 \) and \( D(p_{MC} \parallel p_{LD}) = 1.98719 \). These statistics are transformed to .559829 and 2.79851, respectively, if one replaces the second probability distribution in each formula with its posterior counterpart based on the set of six outcomes previously hypothesized (6). These results are, then, consistent with the earlier patterns, since \( p_{MC} \) assigns greater probability than \( p_{LD} \) to systems with \( r > .948724 \). By assuming a doubling or repeating of the set of six measurements — involving the squaring of the likelihood (6) — we are able to reduce the statistic .559829 still further, to .310686. While a tripling (corresponding to eighteen measurements) yields less still, that is .307632. However, use of a posterior based on twenty-four such measurements results in .529577. So, it might be asserted that \( p_{LD} \) is considerably more informative than \( p_{MC} \).

The variance of \( z \), that is \( \langle z^2 \rangle \), is .301762 for \( p_{MC} \), .277778 for \( p_{KM} \), .25 for \( p_{SLD} \) and .2 for \( p_{LD} \), thus, agreeing in order with the relative noninformativities of these priors. In the relative ranking of \( p_{SLD} \) and \( p_{LD} \), we are in full agreement with Hall [1]. Hall’s argument that \( p_{SLD} \) generates the maximally random ensemble is that “the Bures metric for a two-dimensional system corresponds to the surface of a unit 4-ball, i. e., to the maximally symmetric 3-dimensional space of positive curvature . . . This space is homogenous and isotropic, and hence the Bures metric does not distinguish a preferred location or direction in the space of density operators.” Although we can not disagree with these statements, they do not appear to be determinative in judging the relative noninformativity of prior distributions over the two-level quantum systems.

C. Two truncated prior probability distributions

We now proceed to find two priors more noninformative than \( p_{MC} \), but only by imposing a restriction on the \textit{a priori} possible two-level systems — that is, we must eliminate the possibility that the unknown system under examination is either in a pure or “nearly” pure state. We consider the three operator monotone functions [10,11],

\[ f(t) = \frac{2t^{2-2n}}{(1+t)(\log t)^n}, \quad n = 0, 1, 2 \]  

(11)
For $n = 0$ (corresponding to the maximal monotone metric [of the left logarithmic derivative] and $n = 1$, the volume elements are improper, that is not normalizable over the Bloch sphere, while for $n = 2$, the corresponding volume element is proper or normalizable, corresponding to the operator monotone function and probability distribution $p_{MC}$, that is. To directly compare the three metrics based on $R$, we choose to normalize their volume elements over a three-dimensional ball of radius $R = 1 - 10^{-10}$. We, consequently, obtain the three probability distributions ($p_n$) over the so-truncated convex set (not containing the pure $[r = 1]$ and nearly pure $[1 > r > R]$ states),

\[ p_0 = .00000112542r^2(1 - r^2)^{-3/2} \sin \theta, \]
\[ p_1 = .000569121r(1 - r^2)^{-1} \log[(1 + r)/(1 - r)] \sin \theta, \]
\[ p_2 = .00513611(1 - r^2)^{-1/2} \log[(1 - r)/(1 + r)]^2 \sin \theta. \]

We have that the relative entropy or information gain of $p_0$ with respect to $p_1$, that is [cf. $R$],

\[ D(p_0 \parallel p_1) = \int_0^R \int_0^{2\pi} \int_0^{\pi} p_0 \log[p_0/p_1] d\phi d\theta dr \]

equals .867442 nats. Also, $D(p_0 \parallel p_2) = 5.76086$, $D(p_1 \parallel p_0) = 1.654$, $D(p_1 \parallel p_2) = 2.37198$, $D(p_2 \parallel p_0) = 7.06816$, and $D(p_2 \parallel p_1) = 1.52109$.

We assume now — again following our general methodology — that we have performed a two-level measurement on each of six replicas of a two-level quantum system, two measurements in each of three orthogonal $(X,Y,Z)$ directions, and obtained a single “up” and a single “down” in each direction. Multiplying the likelihood of such an occurrence by $p_0, p_1$ and $p_2$, in turn, and normalizing the products over the truncated Bloch sphere (of radius $R = 1 - 10^{-10}$) with normalization factors, 335.987, 327.546 and 249.378, respectively — we obtain (in accordance with Bayes’ Theorem) three posterior distributions, which we will designate as $P_0, P_1$ and $P_2$, in the obvious fashion. Now, we have that $D(p_0 \parallel P_1) = 1.07576$, $D(p_0 \parallel P_2) = 6.24184$, $D(p_1 \parallel P_0) = 1.53564$, $D(p_1 \parallel P_2) = 2.55172$, $D(p_2 \parallel P_0) = 6.94979$, and $D(p_2 \parallel P_1) = 1.42817$.

Our variation of the criterion elaborated upon by Clarke (in which we have exchanged the positions of priors and posteriors in the relative entropy statistics), then leads us to conclude that $p_0$ is more noninformative than both $p_1$ and $p_2$ and that $p_1$ is more noninformative than $p_2$. According to our rule, $p_1$ is more noninformative than $p_j$ if both $D(p_i \parallel P_j) > D(p_i \parallel P_j)$ and $D(p_j \parallel P_i) < D(p_j \parallel P_i)$. For instance, for $i = 0$ and $j = 1$, we have that $D(p_0 \parallel P_1) = 1.07576 > D(p_0 \parallel P_1) = .867442$ and $D(p_1 \parallel P_0) = 1.53564 < D(p_1 \parallel P_0) = 1.654$. Thus, by adding information to $p_0$ — in the form of the six hypothesized measurement results — we are able to more closely approximate $p_1$, but apparently not vice versa. It would be of interest to study the changes in the statistics given above as $R \to 1$. The ratio of $p_0$ to $p_1$ at $r = R = 1 - 10^{-10}$ is 5.89521, while that of $p_0$ to $p_2$ is 1947.41, and that of $p_1$ to $p_2$ is 330.338. These results are in accordance with those above, in that more noninformative priors were also found there to assign greater probability to more nearly pure states.

III. DISCUSSION

Petz and Toth [11, p. 215] found that the lower bound on the covariance matrix of unbiased estimators of parameters provided by use of the symmetric logarithmic derivative or minimal montone metric was “more informative” (that is, tighter) than the bound furnished by the Kubo-Mori (Bogoliubov) monotone metric. This bound, in turn, was tighter than that obtained from the maximal monotone metric. Their notion of informativity appears to be fully consistent with that derived here through an apparently quite different line of argument. We assert this since we have found that the noninformativity of $p_{SLD}$ is less than that of $p_{KM}$, and that of $p_{KM}$ is less than that of $p_{MC}$, while finally, the noninformativity of the truncated form of $p_{MC}$, that is $p_2$, is less than that of $p_0$ (based on the maximal monotone metric).

Derka, Buzek and Adam (cf. [20]) approach the problem of using Bayesian reasoning to reconstruct a two-level quantum system that is possibly impure, by assuming that the unknown system is coupled to a second system (which they, for convenience, take to be two-level in nature), and that the pair of coupled systems is in a pure state. They utilize the invariant integration measure on all possible such (four-dimensional) pure states. Hall [12] adopts a related approach in the first part of his paper. In contrast, the analysis here and in [22,23] does not posit any such coupling of the unknown system, regarding it, in effect, as independent or autonomous.
A most interesting question that remains to be formally addressed is whether or not it is possible to find two different sets of measurements, one of which leads to a conclusion that a prior \( p \) is more noninformative than another prior \( q \), while the other set leads to a conclusion that \( q \) is more noninformative than \( p \). Of course, if such hypothetical sets were to exist, which we presume they do not, the validity of the general line of argument presented in this Letter would be called into question.

Another line of investigation that would be of interest to pursue would be the determination of the particular set of measurements that minimizes the relative entropy between the corresponding posterior form \( (P) \) of \( p \) and \( q \) itself (cf. [12]). Such a set of minimizing measurements could be said to best express the additional information contained in \( q \), above and beyond that in \( p \). (Clarke [2] argues that one can, in general, find a data set to minimize the relative entropy of the corresponding posterior with respect to the “informative” prior in question.)

In summary, based on our analysis here, we would recommend for the Bayesian inference of the parameters of an unknown two-level quantum system the use of \( p_{MC} \) as a prior or, preferably, some version (possibly modifying our choice of \( R \)) of \( p_0 \), if one has a priori knowledge that the unknown system is described by a polarization vector of length \( \sqrt{x^2 + y^2 + z^2} \leq R < 1 \). If we view \( p_0 \) as a function of \( R \), reexpress it in terms of Cartesian coordinates, integrate over \( z \), say, and take the limit \( R \to 1 \), we obtain the bivariate marginal probability distribution over the unit disk,

\[
(1 - x^2 - y^2)^{-1/2}/2\pi. \tag{16}
\]

(The bivariate marginal distribution of \( p_{SLD} \), on the other hand, is the uniform one — \( 1/\pi \) — over the unit disk. In [23] the distribution (16) was obtained from the Jeffreys prior for the family of trivariate normal distributions, with null mean vectors, having the density matrices (2) as their covariance matrices.) Thus, if one is content to estimate only two of the three parameters determining a two-level system, one can employ (16) as a prior and avoid having to rule out the possibility of a pure or nearly pure state \( p_{SLD} \). The distribution (16) is precisely the standard (classical/nonquantum) noninformative (Jeffreys) prior for the two-parameter family of trinomial distributions with probabilities \( x^2, y^2 \) and \( 1 - x^2 - y^2 \) [13,23]. The conditional distribution over [-1,1] of \( x \) in (16) — given that \( y = 0 \) — is the “cosine distribution” [27]

\[
(1 - x^2)^{-1/2}/\pi. \tag{17}
\]

(In the statistical literature this is termed the arc-sine distribution.) It is the noninformative (Jeffreys) prior for the one-parameter family of binomial distributions with probabilities \( x^2 \) and \( 1 - x^2 \) [13,23]. The distributions (16) and (17) can also be obtained as conditional distributions of \( p_{SLD} \) — which itself is the Jeffreys prior for the family of quadrinomial distributions with probabilities \( x^2, y^2, z^2 \) and \( 1 - x^2 - y^2 - z^2 \). (This corresponds to the geometry of a three-dimensional hemisphere, as pointed out in [1,3,24] (cf. [28]).)

For additional quantum applications of other (classical) work — besides [2] — of Clarke (jointly with Barron) [29,30], having relevance to comparative properties of the volume elements of the monotone metrics for the two-level quantum systems (cf. [3]), we refer the reader to [20]. For a further application — this one to spin-1 systems — in which the comparative properties of priors based on the minimal and maximal monotone metrics are assessed (with the maximal one again displaying a certain superiority — greater computational tractability), see [31].

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