\[ \mathcal{N}=1 \] dynamics with \( T_N \) theory

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ABSTRACT: We study the dynamics of \( \mathcal{N}=1 \) supersymmetric systems consisting of the strongly-coupled superconformal theory \( T_N \), SU(\( N \)) gauge groups, and fundamental chiral multiplets. We demonstrate that such systems exhibit familiar phenomena such as deformation of the vacuum moduli space, appearance of the dynamical superpotential, and Coulomb branches with \( \mathcal{N}=1 \) Seiberg-Witten curves. The analysis requires a rather detailed knowledge of the chiral ring of the \( T_N \) theory, which will also be discussed at length.

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1 Introduction and summary

Since the seminal work [1, 2], people have extensively studied the dynamics of $\mathcal{N} = 1$ conventional supersymmetric QFTs, made of gauge multiplets and elementary chiral multiplets. It is becoming more and more apparent, however, that we also need to study the dynamics of non-conventional supersymmetric QFTs, made of gauge multiplets coupled to already strongly-coupled supersymmetric theories, in order to fully understand the duality. This is because the dual of a conventional theory can often be non-conventional [3–5].

So far the study of the non-conventional theories was restricted to the case when they flow to nontrivial superconformal theories in the infrared. In this paper we study non-conventional theories in a different regime, where they flow to almost free theories. The
basic ingredient is the $T_N$ theory introduced by [4, 6], which is an $\mathcal{N} = 2$ supersymmetric strongly-coupled theory with SU($N$)$^3$ flavor symmetry. If we gauge one of three SU($N$) symmetries, the contribution to the one-loop beta function is the same as that from $N$ flavors of chiral multiplets in the fundamental and the antifundamental representations. By giving a suitable vev, the $T_N$ theory flows to a system of free bifundamental chiral multiplets. We distinguish three SU($N$) symmetries by calling them SU($N$)$_X$ for $X = A, B, C$. The $T_N$ theory has dimension-2 chiral operators $\mu_X$, which transforms as adjoints of SU($N$)$_X$, for each $X = A, B, C$. They satisfy the constraints

$$\text{tr} \mu_A^k = \text{tr} \mu_B^k = \text{tr} \mu_C^k,$$

or equivalently

$$v_{A,k} = v_{B,k} = v_{C,k}, \quad \text{where} \quad \det(x - \mu_X) = x^N + v_X 2x^{N-2} + \cdots + v_{X,N}. \quad (1.2)$$

This theory also has operators $Q^{iAibc}_C$ and $Q^{iAibc}_C$ of dimension $N - 1$, transforming as trifundamentals under three SU($N$)$_X$ symmetries.

We will study the following three cases in detail:

- The $T_N$ theory coupled to one $\mathcal{N} = 1$ SU($N$) gauge multiplet. This leads to the quantum deformation of the vacuum moduli space of the $T_N$ theory. For definiteness let us gauge SU($N$)$_C$. Then the deformed moduli space is given by deforming the constraints (1.1) to

$$\text{tr} \mu_A^k - \text{tr} \mu_B^k = \Lambda 2^N \delta_{N,k}, \quad (1.3)$$

where $\Lambda$ is the dynamical scale of the gauge theory. We can give a suitable vev and make the system reproduce the deformed moduli space of SU($N$)$_X$ theory with $N$ flavors.

- The system above plus one flavor of fundamental and antifundamental chiral multiplets, $q^C_i$ and $\tilde{q}_iC$. Here, the IR description involves gauge-invariant composite fields behaving almost freely, with a dynamically-generated superpotential. In more detail, the list of the gauge invariant operators of this theory is

$$B^{iAB} = Q_{iAibc}q_{iC}, \quad \tilde{B}^{iAB} = Q_{iAibc}\tilde{q}_{iC}, \quad \mu_X \quad (X = A, B), \quad M_k = \tilde{q}_k^C q^k \quad (k = 0, 1, \cdots, N - 2). \quad (1.4)$$

Then the non-perturbative superpotential is

$$W = -\frac{1}{\Lambda^{2N-1}} \left[ \sum_{k=0}^{N-2} M_k f_{N-k}(\mu_A, \mu_B) - \text{tr}[(B\tilde{B})_A\mu_A] + \text{tr}[(\tilde{B}B)_B\mu_B] \right], \quad (1.5)$$

where $f_{N-k}(\mu_A, \mu_B)$ is given by

$$f_{N-k}(\mu_A, \mu_B) = \sum_{\ell=0}^{N-k-2} \frac{v_{A,\ell}}{N-k-\ell}(\text{tr} \mu_A^{N-k-\ell} - \text{tr} \mu_B^{N-k-\ell}). \quad (1.6)$$

Again, by giving a vev, the system reproduces the dynamics of SU($N$) theory with $N + 1$ flavors.
• The $T_N$ theory coupled to three SU($N$) gauge multiplets. The IR theory on a generic point on the vacuum moduli space is given by $2(N - 1)$ almost-free U(1) vector multiplets and $(N - 1)$ neutral chiral multiplets $v_{2,...,N}$. We determine the $\mathcal{N} = 1$ Seiberg-Witten curve describing the holomorphic coupling constant matrices. It has the form

$$w^N + v_2w^{N-2} + \cdots + v_{N-1}w + v_N - t\Lambda_{A}^{2N} - \frac{\Lambda_{B}^{2N}}{t} - \frac{\Lambda_{C}^{2N}}{t-1} = 0.$$ (1.7)

We also study the dynamics of a class of $\mathcal{N} = 2$ theories specified by a Riemann surface with several full punctures [4], when their flavor symmetries are gauged by $\mathcal{N} = 1$ vector multiplets. It is also possible to add some flavors of quarks to these gauge groups. We propose that the curve of these systems are given by

$$w^N + \sum_{k=2}^{N} V_k(t)w^{N-k} = 0,$$ (1.8)

where $V_k(t)$ are meromorphic functions on the Riemann surface with simple poles at the punctures. This is a generalization to $N > 2$ of the result of [7].

The techniques we use in the analysis are holomorphy, matching of ’t Hooft anomalies, the behavior under decoupling of the flavors, and so on. They are the same ones we use to analyze conventional $\mathcal{N} = 1$ supersymmetric gauge theories; the point is that these standard methods do not rely on the existence of a Lagrangian description with gauge multiplets and chiral multiplets only, and therefore are also applicable to non-conventional theories.

Since we use the $T_N$ theory as one of the ingredients of the system we study, a rather extensive knowledge of its properties is necessary. Those properties can be deduced from other known dualities and from the known superconformal index [8], which we will discuss in detail. The overall consistency of the properties of the $T_N$ theory thus obtained and the behavior of the $T_N$ theory coupled to SU($N$) gauge multiplets gives nontrivial checks of the whole procedure.

The rest of the paper is organized as follows. In section 2 we study the chiral ring of the $T_N$ theory which will be extensively used in the subsequent sections. In section 3 the low energy behavior of the $T_N$ theory coupled to the $\mathcal{N} = 1$ SU($N$) gauge group is proposed. We check this proposal by performing ’t Hooft anomaly matching and showing that it reduces to $\mathcal{N} = 1$ SU($N$) gauge theory with $N$ flavors by a suitable vev. The addition of one flavor to the previous case is explored in section 4. The similar checks are performed. In section 5 we see that the $T_N$ theory coupled to three $\mathcal{N} = 1$ SU($N$) gauge groups is in the abelian Coulomb phase and derive the $\mathcal{N} = 1$ Seiberg-Witten curve encoding the holomorphic coupling constant matrix of low energy U(1) gauge groups. We study the case where the gauge groups are also coupled to flavors. Finally we generalize this to a large class of $\mathcal{N} = 1$ theories which consists of a couple of $T_N$ theories and gauge groups. We conclude with brief discussions in section 6. In appendix A, we show the detailed calculation of the reduction to $\mathcal{N} = 1$ gauge theories with $N$ and $N + 1$ flavors discussed in sections 3 and 4.
2 The chiral ring of $T_N$ theory

2.1 Summary of chiral operators

The $T_N$ theory is an $\mathcal{N} = 2$ SCFT which has $\text{SU}(N)_A \times \text{SU}(N)_B \times \text{SU}(N)_C$ flavor symmetries. In terms of $\mathcal{N} = 1$ language, this theory has the following set of Higgs branch chiral operators; 

1. Operators $\mu_A, \mu_B$ and $\mu_C$ transforming in the adjoint representations of $\text{SU}(N)_A, \text{SU}(N)_B$ and $\text{SU}(N)_C$, respectively. They are the holomorphic moment maps associated to these flavor symmetries in the hyperkähler manifold. They have scaling dimension 2.

2. Operators $Q^{(k)} (k = 1, \cdots, N - 1)$ in the $(\wedge^k, \wedge^k, \wedge^k)$ representations of $\text{SU}(N)_A \times \text{SU}(N)_B \times \text{SU}(N)_C$, where $\wedge^k$ is the $k$-index anti-symmetric tensor representation (see [6] for $k = 1, N - 1$ and [5] for others). $Q^{(k)}$ has scaling dimension $k(N - k)$. In particular, the operators with $k = 1$ and $k = N - 1$ are denoted as $Q^{i_A i_B i_C}$ and $Q^{i_A i_B i_C}$ respectively, where $i_X = 1, \ldots, N$ for $X = A, B, C$ are the indices of $\text{SU}(N)_X$.

In this paper, we do not discuss the operators $Q^{(k)}$ with $k = 2, \cdots, N - 2$ in detail; their presence does not affect our analysis.

In addition to the above flavor symmetries, there are $\text{U}(1)_{R(N=2)} \times \text{SU}(2)_R$ $R$-symmetries of the $\mathcal{N} = 2$ superconformal algebra. All the lowest components of $\mu_{A,B,C}$ and $Q^{(k)}$ are neutral under the $\text{U}(1)_{R(N=2)}$ symmetry, while their charges under $\text{U}(1)_{2I_3}$ are equal to the scaling dimensions of these fields, where $\text{U}(1)_{2I_3}$ is twice the Cartan subalgebra of $\text{SU}(2)_R$. For example, $\mu_{A,B,C}$ have $\text{U}(1)_{2I_3}$ charge 2 and $Q^{(k)}$ has $k(N - k)$.

We also have Coulomb branch operators $u_{d,i}$ with dimension $d$, zero $\text{U}(1)_{2I_3}$ charge, and $\text{U}(1)_{R(N=2)}$ charge $2d$, for $d = 3, 4, \ldots, N$ and $i = 1, \ldots, d - 2$:

$$u_{3,1}; u_{4,1}, u_{4,2}; u_{5,1}, u_{5,2}, u_{5,3}; \cdots; u_{N,1}, u_{N,2}, \cdots, u_{N,N-2}. \quad (2.1)$$

We discuss the chiral ring relations among these operators in detail below. Those readers who are mainly interested in the dynamics of the $\mathcal{N} = 1$ systems with $T_N$ theories can skip to the next section, coming back to this section only when it is necessary.

2.2 Summary of chiral ring relations

First, there are no relations among the Coulomb branch operators $u_{d,i}$. This is an analogue of the statement that there are no relations among the operators $\text{tr} \phi^k$, $k = 2, \ldots, N$, parameterizing the Coulomb branch of an $\mathcal{N} = 2$ $\text{SU}(N)$ gauge theory. This feature, of the complete absence of the chiral ring relations among the Coulomb branch operators, is believed to be generic for any $\mathcal{N} = 2$ superconformal field theory in four dimensions.

Second, there are generic relations between Coulomb branch and Higgs branch operators of the form

$$u_X = 0 \quad (2.2)$$
where $X$ is a generic Higgs branch operator and $u$ is a generic Coulomb branch operator. It is possible that some of them satisfy more complicated relations, due to the possibility that on a sublocus on the Coulomb branch there can be a smaller Higgs branch, etc. In this paper we focus on a generic point of the Higgs branch, and hence all the Coulomb branch directions are lifted unless otherwise stated.

The rest of the discussions is devoted to the chiral ring relations among Higgs branch operators. Let us define the characteristic polynomial of $\mu_X$ ($X = A, B, C$) as

$$P_X(x) = \text{det}(x1 - \mu_X) = \sum_{k=0}^{N} v_{X,k} x^{N-k},$$

(2.3)

where $v_{X,0} = 1$ and $v_{X,1} = 0$. We claim that the chiral operators satisfy the following chiral ring relations:

$$\text{tr} \mu_A^k = \text{tr} \mu_B^k = \text{tr} \mu_C^k \quad (k = 2, \ldots, N),$$

(2.4)

$$(\mu_A)^{i_A}_{j_A} Q_{j_A B}^{i_A B i_C} = (\mu_B)^{i_B}_{j_B} Q_{j_B A}^{i_B A i_C} = (\mu_C)^{i_C}_{j_C} Q_{j_C A}^{i_C A i_B},$$

(2.5)

$$(\mu_A)^{i_A}_{j_A} Q_{j_A B}^{i_A B i_C} = (\mu_B)^{i_B}_{j_B} Q_{j_B A}^{i_B A i_C} = (\mu_C)^{i_C}_{j_C} Q_{j_C A}^{i_C A i_B},$$

(2.6)

$$Q_{A B}^{i_A B i_C} Q_{j_A j_B}^{i_A B i_C} = \sum_{l=0}^{N} \sum_{m=0}^{N-l-1} (\mu_A^{N-l-1-m})^{i_A}_{j_A} (\mu_B^{m})^{i_B}_{j_B},$$

(2.7)

$$= \frac{1}{(N-1)!} Q_{A B}^{i_A B i_C} Q_{A C}^{i_A C i_B} Q_{B C}^{i_B C i_A} \cdots Q_{A N}^{i_A N i_C} \epsilon_{i_{A,1} i_{A,2} \cdots i_{A,N}} \epsilon_{i_{B,1} i_{B,2} \cdots i_{B,N}} \epsilon_{i_{C,1} i_{C,2} \cdots i_{C,N}}.$$ 

(2.8)

Note that $\mu_X^k$ is a $k$-th power of the matrix $\mu_X$, and in particular $(\mu_X^0)^i_j = \delta^i_j$. There are also relations obtained by applying the cyclic permutation $A \rightarrow B \rightarrow C \rightarrow A$ to the above relations.

Since there is one to one correspondence between the sets $(\text{tr} \mu_X^2, \ldots, \text{tr} \mu_X^N)$ and $(v_{X,2}, \ldots, v_{X,N})$, we also have $v_{A,k} = v_{B,k} = v_{C,k}$, or more concisely

$$P_A(x) = P_B(x) = P_C(x).$$

(2.10)

Because of this relation, we may just write $v_{X,k}$ and $P_X(x)$ as $v_k$ and $P(x)$ by dropping the subscripts $A, B$ or $C$. This is already used in writing (2.7).

\footnote{However, we do not claim that these are the complete list of chiral ring relations. At least there should be relations involving $Q_{(i)}$ for $k = 2, \ldots, N - 2$.}
2.3 Derivation of chiral ring relations

Here we explain how the chiral ring relations written down in the previous section are derived or guessed. Some of the results have been obtained in [9, 10].

The relations on \( \text{tr} \mu^k \). The relations (2.4) are derived in the same way as the derivation of \( \text{tr} \mu_A^2 = \text{tr} \mu_B^2 = \text{tr} \mu_C^2 \) in [10], which we now review. We consider the \( T_N \) theory coupled to the following quiver,

\[
SU(N-1) - SU(N-2) - \cdots - SU(2). \tag{2.11}
\]

A subgroup \( SU(N-1) \subset SU(N)_C \) is gauged by the above \( SU(N-1) \), there are bifundamental hypermultiplets between \( SU(k) \) and \( SU(k-1) \) \( (k = N-1, N-2, \cdots, 2) \), and there is one additional flavor for \( SU(2) \) to make it superconformal. This is dual [4] to the following linear quiver:

\[
SU(N)_A - SU(N)_2 - \cdots - SU(N)_{N-1} - SU(N)_B, \tag{2.12}
\]

where \( SU(N)_A \) and \( SU(N)_B \) are flavor symmetries and \( SU(N)_k \) \( k = 2, \cdots, N-1 \) are gauge symmetries. We denote \( SU(N)_1 = SU(N)_A \) and \( SU(N)_N = SU(N)_B \) for simplicity. There are bifundamentals \((\tilde{q}_k)^i_{i+1}_A\) and \((\tilde{q}_k)^i_{i+1}_N\) between \( SU(N)_k \) and \( SU(N)_{k+1} \). The identification of \( SU(N)_B \) with \( SU(N)_N \) is done as \( g \in SU(N)_N \mapsto \,^tg^{-1} \in SU(N)_B \) so that the fundamental and anti-fundamental representations are exchanged, e.g., \( (q_{N-1})^i_{i+1} = (q_{N-1})^{i+N-1}_{i+1} \).

The \( T_N \) operators \( \mu_A \) and \( \mu_B \) are identified with the following operators in the dual quiver,

\[
\mu_A = (\tilde{q}_1 \tilde{q}_1)_A, \quad \mu_B = (\tilde{q}_{N-1} q_{N-1})_B, \tag{2.13}
\]

where the subscript in a bilinear like \((q\tilde{q})_X\) means that we take the adjoint representation of \( SU(N)_X \) constructed from that bilinear, e.g., \((\tilde{q}_k q_k)_{k+1} = \tilde{q}_k q_k - \tfrac{1}{N} \text{tr} \tilde{q}_k q_k \). Taking the direction of \( SU(N)_C \) which is singlet under \( SU(N-1) \) as \( i_C = 1 \), we identify the operators \( Q^{i_A B_1} \) and \( Q_{i_A i_B} \) as

\[
Q^{i_A B_1} \propto (q_1 q_2 \cdots q_{N-1})^i_A, \quad Q_{i_A i_B} \propto (\tilde{q}_{N-1} \cdots \tilde{q}_2 \tilde{q}_1)^i_B, \tag{2.14}
\]

where we used the rules between the indices \( i_N \) and \( i_B \) discussed above.

The superpotential of the linear quiver (2.12) is given as

\[
\sum_{k=2}^{N-1} \text{tr} \Phi_k (-q_{k-1} q_{k-1}) + (q_k \tilde{q}_k)_k], \tag{2.15}
\]

where the minus sign in the first term comes from the fact that \( q_{k-1} \) is in the anti-fundamental representation of \( SU(N)_k \) and hence it couples to \( -^t \Phi_k \). Then the equations of motion give \((\tilde{q}_k - q_{k-1})_k = (q_k \tilde{q}_k)_k \). By using this and cyclicity of trace, we get

\[
\text{tr} \mu_A^k = \text{tr} \left[(q_1 \tilde{q}_1)_1^k \right] = \text{tr} \left[(\tilde{q}_1 q_1)_2^k \right] = \text{tr} \left[(q_2 \tilde{q}_2)_3^k \right] = \cdots = \text{tr} \left[(\tilde{q}_{N-1} q_{N-1})_{N-1}^k \right] = \text{tr} \left[^t \mu_B \right]^k. \tag{2.16}
\]

Taking the traceless part of matrices, e.g., \((q_1 \tilde{q}_1)_1 = q_1 \tilde{q}_1 - \frac{1}{N} \text{tr} q_1 \tilde{q}_1 \), does not spoil the trace cyclicity used here.
The relations on $\mu Q$. Next, let us consider relations (2.5), and (2.6). We use the same quiver dual. We start from the following relation:

\[
(q_1\tilde{q}_1)q_1q_2\cdots q_{N-1} = q_1(q_1\tilde{q}_1)q_2\cdots q_{N-1} = q_1(q_2\tilde{q}_2)q_2\cdots q_{N-1} = \cdots = q_1q_2\cdots q_{N-1}(\tilde{q}_{N-1} q_{N-1})^N.
\]  

(2.17)

This equation is equivalent to $(\mu_A)_{i_A}^j B Q^{i_A i_B} = (\mu_B)_{i_B}^j A Q^{i_A j_B}$. This relation should extend to (2.5) for general $i_C$. The relation (2.6) is obtained in the same way.

The relation on $QQ$. The relation (2.7) is determined by the following argument.\(^2\) Let us define $R_{i_A i_B}^{A i_B} := Q^{i_A i_B} Q_{j_A j_B}^{A i_C}$. Consider $N$ dimensional linear spaces $V_{A,B}$ on which $SU(N)_{A,B}$ acts respectively. $R$ is a linear operator on $V_A \otimes V_B$. The operators $\mu_A \otimes 1$ and $1 \otimes \mu_B$ also act on $V_A \otimes V_B$; we abbreviate them as just $\mu_A$ and $\mu_B$.

Now, using the relation (2.5) and (2.6), it is easy to show that $R$ commutes with $\mu_A$ and $\mu_B$. Therefore $R$ should be a polynomial in $\mu_A$ and $\mu_B$. The dimension of $Q$ says that this is a polynomial of degree $N - 1$. Again by (2.5) and (2.6), we get $(\mu_A - \mu_B)R = 0$. Such an $R$ can be constructed nicely using (2.10). From the property of the characteristic polynomial, we have $P(\mu_A) = P(\mu_B) = 0$. Thus we can suppose

\[
R \propto \frac{P(\mu_A) - P(\mu_B)}{\mu_A - \mu_B},
\]

(2.18)

where the right hand side is understood that we first calculate the polynomial $[P(x) - P(y)]/(x - y)$ with a later substitution of $x$ by $\mu_A$ and $y$ by $\mu_B$; as $\mu_A$ and $\mu_B$ commute as an operator acting on $V_A \otimes V_B$ there is no problem in this procedure. Writing (2.18) explicitly, we get (2.7). The coefficient of (2.18) is absorbed in the normalization of $Q^{i_A i_B} Q_{i_A i_B}$.

The relations relating $Q_{i_A i_B i_C}$ and $Q^{i_A i_B i_C}$. Finally we discuss the relations (2.8) and (2.9). Let us start with the case of $T_2$ theory, that is, the theory of a free single trifundamental chiral field $Q^{i_A i_B i_C}$. In this theory, we define $Q_{i_A i_B i_C} \equiv -\epsilon_{i_A i_B i_C} Q^{i_A i_B i_C} Q_{i_A i_B i_C}$, where the convention for the totally anti-symmetric tensor in this paper is $\epsilon^{12} = \epsilon_{12} = 1$. Then we have

\[
Q^{i_A j_B i_C} \epsilon_{i_B i_C} = Q_{j_A i_B i_C} \epsilon_{i_A i_C}, \quad Q_{i_A j_B i_C} \epsilon^{i_B i_C} \epsilon^{i_A i_C} = -Q^{i_A i_B i_C} \epsilon_{i_A i_C}.
\]

(2.19)

For the $T_3$ theory, all the chiral ring relations are worked out in [9]. There is a relation of the form\(^3\)

\[
Q^{i_A j_B j_C} Q^{i_A k_B k_C} \epsilon_{i_B j_B k_B} \epsilon_{i_C j_C k_C} \sim Q_{j_A i_B i_C} (\mu_A)^{i_A}_{i_B} (\mu_A)^{j_A}_{i_C} \epsilon_{i_A i_C},
\]

(2.20)

and a similar relation with upper and lower indices exchanged. We need a generalization of (2.19) and (2.20) to $N > 3$.

\(^2\)It was Davide Gaiotto who originally came up with the argument a few years ago. The authors thank him for kindly allowing them to reproduce it here.

\(^3\)The relations in [9] are written in the $SU(2) \times SU(6) \subset E_6$ covariant form. We need to rewrite them in $SU(3) \times SU(3) \times SU(3)$ covariant way to get (2.20).
On a generic point of the Higgs branch, the constraint (2.4) is solved up to complexified SU\((N)A \times SU(N)B \times SU(N)C\) transformations by the vevs
\[
\mu_A = \mu_B = \mu_C = \text{diag}(m_1, m_2, \cdots, m_N),
\]
with \(\sum_i m_i = 0\). Then, the relations (2.5) and (2.6) says that the only nonzero entries of \(Q^i_{AB}Q^i_{BC}\) and \(Q^i_{AIBC}\) are for \(i_A = i_B = i_C\), which we write as
\[
Q^{ii} = q^i, \quad Q_{ii} = q_i \quad (i = 1, \cdots, N),
\]
with other components zero. The relation (2.7) fixes the product \(q^i q_i\) for each \(i\) (note that no summation is involved):
\[
q^i q_i = \prod_{j \neq i} (m_i - m_j).
\]

In terms of these “gauge-fixed” variables, (2.19) gives us \(q^1 = q_2\), while (2.20) gives \(q^1 q^2 \propto q_3(m_1 - m_2)\) and its cyclic permutations. Using (2.23), they are written in more symmetric way as \(q^1 q^2 = (m_2 - m_1)\) and \(q^1 q^2 q^3 \propto (m_1 - m_2)(m_1 - m_3)(m_2 - m_3)\). From these, it is easy to guess a general relation of the form
\[
q^1 q^2 \cdots q^N \propto \prod_{1 \leq i < j \leq N} (m_i - m_j).
\]
This equation is consistent with the scaling dimensions of the operators and the Weyl group action. Dividing this by (2.23), we get
\[
q^1 q^2 \cdots q^{N-1} \propto q_N \prod_{i=1}^{N-1} \prod_{j=i+1}^{N-1} (m_i - m_j),
\]
and its cyclic permutation. We also have the version with \(q^i \leftrightarrow q_i\).

The relation (2.24) is crucial for reproducing the correct dimension of the Higgs branch of the \(T_N\) theory. The complex dimension of the Higgs branch is given by \((3N+2)(N-1)\), as can be shown by the S-duality discussed above or using the result of [11]. On the other hand, the \(\mu_{A,B,C}\) with the constraint (2.4) contributes to the dimension as \(3(N^2-1) - 2(N-1) = (3N+1)(N-1)\). The remaining \(N - 1\) directions of the Higgs branch are provided by the \(q^i\’s with the constraint (2.24).\(^4\)

One can check explicitly that the relations (2.8) and (2.9) reproduce (2.25) and its cyclic permutation and the version with \(q^i \leftrightarrow q_i\), up to coefficients. The coefficient of (2.8) is absorbed in the relative normalization of \(Q^i_{AB}Q^i_{BC}\) and \(Q^i_{ABC}\), while the coefficient of (2.9) is determined by the consistency with (2.8) and (2.23).

As one can see from the above discussion, the relations (2.8) and (2.9) are more speculative than others. However, they play an important role when we discuss reduction of the \(T_N\) theory to a bifundamental hypermultiplet in appendix A (see (A.13) and (A.14) where they are used), and that can be regarded as an overall consistency check of (2.8) and (2.9).

\(^4\)We believe that other \(Q^i_{(k)}\) for \(k = 2, \cdots, N - 1\) are determined by \(q^i\’s and m_i\’s in a similar way as the \(q^i\’s are determined by them in (2.23).\)
2.4 Chiral rings from superconformal index

In the subsequent sections we will study the $T_N$ theory coupled to an $N = 1$ vector multiplet by gauging, say, SU($N)_C$ flavor symmetry. To do that, we need the spectrum of the Higgs branch operators which are invariant under the gauged SU($N)_C$ symmetry. Here we will see they are generated by $\mu_A$ and $\mu_B$ only, with the constraints $tr \mu_A^k = tr \mu_B^k$ via the study of the Hall-Littlewood index. We also analyze the SU($N)_C$ invariant operators in the case with a generic puncture.

The Higgs branch of a class-$S$ theory can be characterized by the Hall-Littlewood index \[ I^{HL} = \text{Tr}_{HL} (-1)^F \tau^{2(E-I_3)}, \] (2.26)
where $\text{Tr}_{HL}$ denotes the trace restricted to states with $\delta_{1\pm} = E \pm 2j_1 - 2I_3 + \frac{1}{2} R_{N=2} = 0$ and $E$ is conformal dimension. Note that for the Higgs branch operators $2(E-I_3) = E$.

We consider the SCFTs obtained by compactifying $N$ M5-branes on a sphere with three punctures. A puncture is specified by a partition $\Lambda = (n_1, \ldots, n_\ell)$ with $n_1 \geq n_2 \geq \ldots \geq n_\ell$ and $\sum n_i = N$. The flavor symmetry of the puncture is $S[\prod_i U(r_i)]$ where $r_i = n_i^{T} - n_{i+1}^{T}$ and $\Lambda^T = (n_1^T, \ldots, n_\ell^T)$ is the dual partition to $\Lambda$. The Hall-Littlewood index of the theory associated with a sphere with three generic punctures is expressed in terms of the Hall-Littlewood polynomial $P_{HL}^\lambda(a|\tau)$ of partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{N-1})$, which are orthogonal polynomials with respect to the pairing

\[ \int \Delta_{Schur}(c) I^{HL}(a, b, c) = \delta_{\lambda\lambda'}, \] \[ \Delta_{Schur}(c) = (1 - \tau^2)^N \Delta_{Schur}(c) \hat{K}(a), \] (2.27)
where $\Delta_{Schur}(c)$ is the standard Haar measure. The Hall-Littlewood index is given by

\[ I^{HL}(a_A, a_B, a_C) = \frac{\prod_{j=2}^{N} (1 - \tau^2)}{(1 - \tau)^{N-2}} \prod X \hat{K}_X(a_X) \sum_{\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{N-1}} \frac{\prod_{X} P_{HL}^\lambda(a_X(\Lambda_X)|\tau)}{P_{HL}^\lambda(\tau^{N-1}, \ldots, \tau^{1-N}|\tau)}, \] (2.28)
where $a_X(\Lambda_X)$ are flavor fugacities of the puncture $\Lambda_X$ and

\[ \hat{K}_A(a) = \prod_{i=1}^{n_1} \prod_{j,k=1}^{n_i^T} \frac{1}{1 - \tau^{n_j+n_k-2i+2}a_j/a_k}. \] (2.29)

The flavor fugacities satisfy $\prod_{j=1}^{\ell} a_j^{n_j} = 1$. In what follows, let us choose the punctures $A$ and $C$ are maximal with SU($N)_C$ flavor symmetries, and define the following index by integrating out the fugacities corresponding to the SU($N)_C$ symmetry

\[ I^{sing}(a, b) = \int [dc] \Delta_{Schur}(c) I^{HL}(a, b, c), \] (2.30)
where we defined $a_A = a$, $a_B = b$ and $a_C = c$.

---

\(^5\)We changed the notation from [8] as $(R, r) \rightarrow (I_3, -\frac{1}{2} R_{N=2})$. 
Let us first consider the $T_N$ theory with three maximal punctures. The fugacities $a(\Lambda_{\text{max}})$ are just fugacities of SU($N$) flavor group $a_1, a_2, \cdots, a_N$ with constraint $\prod_i N a_i = 1$, so as $b$ and $c$. The $\hat{K}$ factor in this case is

$$\hat{K}(a) = \frac{1}{(1 - \tau^2)^N} \prod_{i \neq j} \frac{1}{1 - \tau^2 a_i / a_j}. \quad (2.31)$$

Using (2.27) the relevant part of the integral (2.30) is

$$(1 - \tau^2)^{-N} \int [dc] \Delta_{HL}(c) P_{HL}^\lambda_{\ldots, \lambda_{N-1}}(c | \tau). \quad (2.32)$$

The orthogonality of HL polynomial implies that the integral is zero except $\lambda_1 = \lambda_2 = \cdots = \lambda_{N-1} = 0$ in which case the HL polynomial is a constant. Therefore, the $I^{\text{sing}}$ index is given simply by

$$I^{\text{sing}} = \prod_{j=2}^N (1 - \tau^{2j}) (1 - \tau^2)^{-2} \hat{K}(a) \hat{K}(b). \quad (2.33)$$

This can be written in terms of the plethystic exponential as

$$I^{\text{sing}}(a, b) = PE \left[ \tau^2 (\chi_{\text{adj}}(a) + \chi_{\text{adj}}(b)) - \sum_{j=2}^N \tau^{2j} \right], \quad (2.34)$$

where we used $\hat{K}(a)(1 - \tau^2) = PE[\tau^2 \chi_{\text{adj}}(a)]$ and $\prod_{j=2}^N (1 - \tau^{2j}) = PE[- \sum_{j=2}^N \tau^{2j}]$. The first two terms $\chi_{\text{adj}}(a) + \chi_{\text{adj}}(b)$ inside the plethystic exponential correspond to polynomials generated by $\mu_A$ and $\mu_B$. Then the subtraction terms correspond to the relations $\text{tr} \mu_A^k = \text{tr} \mu_B^k$, $k = 2, \ldots, N$.

Let us then consider other examples where the puncture $A$ is not maximal while $B$ and $C$ kept intact. When the puncture $A$ is minimal associated with the Young diagram $\Lambda = (N-1, 1)$ with U(1)$_A$ flavor symmetry, the theory is just a hypermultplet in the bi-fundamental of SU($N$)$_B$ and SU($N$)$_C$ and charged under the U(1)$_A$. The resulting index is

$$I^{\text{sing}} = (1 - \tau^{2N}) PE \left[ \tau^2 \left( \sum_{i=1}^N b_i \right) \left( \sum_{j=1}^N \frac{1}{b_j} \right) \right] \left[ \tau^N (a^N + \frac{1}{a^N}) \right]. \quad (2.35)$$

The three factors represent, from right to left, the baryons $B$, the antibaryons $\tilde{B}$, the mesons $M$, and one relation $\det M - BB = 0$.

A possible generalization is the puncture $A$ associated with the partition $(N - k, 1, \cdots, 1)$ which corresponds to the flavor symmetry S[U(1) $\times$ U($k$)]. The corresponding index is

$$\prod_{j=N-k+1}^N (1 - \tau^{2j}) \left[ \tau^2 \chi_{\text{adj}}(b) \right] \left[ \tau^2 \chi_{\text{adj}}(a) \right] \left[ \tau^N (a^N + \frac{1}{a^N}) \right] \left[ \tau^{N-k+1} \sum_{i=1}^k \left( \frac{a}{a_i} + \frac{a_i}{a} \right) \right]. \quad (2.36)$$

where $a$ and $a_i$ are the fugacities of U(1) and U($k$) satisfying $a^{N-k} \prod_{i=1}^k a_i = 1$. This represents the following spectrum of the SU($N$)$_C$ invariants and $k$ constraints among them.
\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|}
\hline
 & $\Delta$ & $U(1)$ & SU($k$) & SU($N_B$) \\
\hline
$\mu_B$ & 2 & 0 & \text{adj} & \\
$\mu_0$ & 2 & 0 & - & \\
$\mu_{SU(k)}$ & 2 & 0 & \text{adj} & - \\
$B$ & $N - k + 1$ & $N/k$ & $\Box$ & - \\
$\bar{B}$ & $N - k + 1$ & $-N/k$ & $\Box$ & - \\
\hline
\end{tabular}
\caption{SU($N_C$) invariant operators.}
\end{table}

3 Deformed moduli space

3.1 Statement of the result

In this section, we consider $\mathcal{N} = 1$ SU($N$) supersymmetric gauge theory obtained by gauging an SU($N$) flavor symmetry of the $T_N$ theory. Let us denote the gauged symmetry by SU($N_C$) and the remaining flavor symmetry of the $T_N$ theory by SU($N_A$) $\times$ SU($N_B$). As is shown in section 2.4, the chiral operators which are singlets under the SU($N_C$) symmetry of the $T_N$ theory are $\mu_A$ and $\mu_B$ which satisfy the relation

$$\text{tr} \mu^k_A = \text{tr} \mu^k_B,$$

where $k = 2, \ldots, N$. This describes the original moduli space of vacua before coupling to the gauge fields.

Our proposal is that in the IR the theory is confined as in the case of SU($N$) supersymmetric QCD (SQCD) with $N_f = N$ flavors [1]: the effective theory consists of the gauge singlet fields $\mu_A$ and $\mu_B$. However the classical moduli space is deformed to the quantum one

$$\text{tr} \mu^k_A - \text{tr} \mu^k_B = N \Lambda^{2N} \delta^{kN},$$

where $\Lambda^{2N}$ is the holomorphic dynamical scale of SU($N_C$). They can be concisely summarized as

$$P_A(x) = P_B(x) - \Lambda^{2N},$$

where $P_{A,B}(x)$ are the characteristic polynomials defined in (2.3).

The global non-anomalous symmetry of the UV theory is SU($N_A$) $\times$ SU($N_B$) $\times$ U(1)$_R$ where U(1)$_R$ is the same as the U(1)$_{R(N=2)}$ symmetry of the $\mathcal{N} = 2$ $R$-symmetry of the $T_N$ theory. Another $R$-symmetry of the $T_N$ theory, U(1)$_{2I_3}$ is anomalous and the dynamical scale $\Lambda^{2N}$ has charge 2$N$ under it. The chiral operators $\mu_{A,B}$ have charges 0 and 2 under the U(1)$_R$ and U(1)$_{2I_3}$ symmetries respectively. Also the gaugino in the $\mathcal{N} = 1$ vector multiplet has charge 1 under each U(1)$_R$ and U(1)$_{2I_3}$.

Then, up to coefficient, (3.2) is the only possibility consistent with these symmetries. We assume that the coefficient of $\Lambda^{2N}$ is nonzero and absorb it in the definition of $\Lambda^{2N}$. Note that we cannot have a dynamical superpotential of $\mu_A$ and $\mu_B$ since they are neutral under the $R$-symmetry U(1)$_R$. Therefore the SUSY cannot be broken spontaneously. These
discussions are completely parallel to the SQCD case \cite{1}. The fact that the coefficient of $\Lambda^{2N}$ in (3.2) is nonzero will be supported by the fact that we can reproduce the result of SQCD from reduction of the $T_N$ theory as discussed in section 3.4, if and only if the coefficient is nonzero.

On a generic point of the quantum moduli space, the flavor symmetry is broken to $U(1)^{2(N-1)} \times U(1)_R$, where the $U(1)^{2(N-1)}$ come from the Cartan subalgebra of $SU(N)_{A,B}$.

3.2 Higgsing from $SU(N)$ to $SU(N-1)$

It is possible to reduce this theory to the $SU(N-1)$ theory coupled to the $T_{N-1}$ theory, as the $SU(N)$ SQCD with $N_f = N$ flavors of quarks can be reduce to the $SU(N-1)$ SQCD with $N-1$ flavors by giving vevs to quarks. We can consider a large vev

$$\langle \mu_A \rangle = \langle \mu_B \rangle = \langle \mu_C \rangle = m \text{diag}(1, \cdots, 1, -N + 1), \quad \langle Q_{NNN}^N Q_{NNN} \rangle = (-N m)^{N-1}, \quad (3.4)$$

where we used (2.23). By this vev, the gauge symmetry is broken to $SU(N-1)$, which is coupled to the $T_{N-1}$ theory.\footnote{The fact that the $T_{N-1}$ theory is obtained in this way from the $T_N$ theory may be seen as follows from the M-theory point of view. The $T_N$ theory is realized as $N$ coincident M5 branes wrapped on a Riemann sphere with three punctures. Then, the vev (3.4) for $\mu$’s is interpreted as separating one M5 brane from the other $N-1$ M5 branes in a certain direction. These $N-1$ M5 branes give the $T_{N-1}$ theory.}

We also have free Nambu-Goldstone multiplets $(\mu_X)_{i_X}^N$ and $(\mu_X)_{i_X}^N (X = A, B)$ for $i_X \leq N - 1$ and a modulus field $m$ which are decoupled from the strong dynamics.

The deformed moduli constraint for this $SU(N-1)$ theory can be obtained in the following way. In (3.3) we redefine $x' = x - m$, and take the limit $m \to \infty$ while $x'$ and $\Lambda^{2N}/m$ fixed. Then we get

$$P'_A(x') = P'_B(x') - \frac{\Lambda^{2N}}{Nm}, \quad (3.5)$$

where $P'_X$ is the characteristic polynomial for $(\mu'_X)_{i_X}^N \equiv (\mu_X - \langle \mu_X \rangle)_{i_X}^N$, where $1 \leq i_X, j_X \leq N - 1$. This is clearly the deformed moduli constraint for the $SU(N-1)$ theory with the low energy dynamical scale given as $\Lambda^{2N}/Nm$.

3.3 Check of the anomaly

As a check of the deformation of the moduli space (3.2), we check ’t Hooft anomaly matching here. To compare the anomaly coefficients of the UV and IR theories, we choose the vevs as $\langle tr \mu_A^N \rangle = N \Lambda^{2N}$ and $\langle \mu_B \rangle = 0$ where the $SU(N)_B$ flavor symmetry is unbroken, while the $SU(N)_A$ symmetry is broken to $U(1)^{N-1}$. The number of the massless Nambu-Goldstone fields from $\mu_A$ is $N^2 - N$.

Let us first compare the anomaly coefficient $tr T_B^a T_B^b R$, where $T_B^a$ and $R$ are the generators of the $SU(N)_B$ and $U(1)_R$. In the UV theory this anomaly is calculated from the flavor central charge of the $T_N$ theory

$$\text{tr} T_B^a T_B^b R = -N \delta^{ab}. \quad (3.6)$$

$$\text{tr} T_B^a T_B^b R = -N \delta^{ab}. \quad (3.6)$$
This agrees with the anomaly of the IR theory \(-t(\text{adj})\delta^{ab}\) to which only the \(\mu_B\) field contributes, where we used the normalization of the Dynkin index \(t\) such that \(t(\square) = \frac{1}{2}\) and \(t(\text{adj}) = N\).

Next let us consider the \(\text{tr} \, R^3\) anomaly. In the UV theory the contribution of the \(T_N\) theory can be computed by using \(\text{tr} \, R^3_{N=2} = 2(n_v - n_h) = -3N^2 + N + 2\) [6, 12]. Adding the contribution of the gauginos, we get

\[
\text{tr} \, R^3 = -2N^2 + N + 1. \tag{3.7}
\]

This agrees with that of the IR theory. Note that the reduction of the components of \(\mu_A\), due to the quantum constraints, is crucial in the matching. Also the anomaly \(\text{tr} \, R\) is the same as the above because \(\text{tr} \, R^3_{N=2} = \text{tr} \, R^3_{N=2}\), and the \(R\) charges of the gauginos and the fermionic partner of the low energy fields are 1 and \(-1\).

Finally, we consider the anomaly coefficients involving unbroken \(U(1)^{N-1}\) symmetries from \(SU(N)_A\). Let \((U_i)^k_i\) \((i = 1, \ldots, N - 1, \) and \(k, \ell = 1, \ldots, N)\) be generators of these \(U(1)\)'s such that the only non-zero components are \((U_i)^i_i = -(U_i)^{i+1}_{i+1} = 1\). In the IR theory, the components \((\mu_A)^i_{i+1}\) and \((\mu_A)^{i+1}_i\) have charge \(\pm 2\), and the components \((\mu_A)^k_i\) \((k \neq i, i + 1)\) have charge \(\pm 1\). Then the anomaly under \(U_i^2 R\) in IR is given as

\[
-2 \cdot (\pm 2)^2 - 4(N - 2) \cdot (\pm 1)^2 = -4N \tag{3.8}
\]

In the UV theory, this can be computed as

\[
\text{tr} \, U_i^2 R = -4N, \tag{3.9}
\]

where the factor 4 comes from the normalization of \(U_i\), \(\text{tr} \, U_i^2 = 2\), compared with \(\text{tr} \, T^a T^b = \frac{1}{2} \delta^{ab}\). Therefore the UV and IR anomalies match. The anomalies \(\text{tr} \, U(1)_i R^2\) and \(\text{tr} \, U(1)_i^3\) vanish. The only other nonzero anomalies are \(\text{tr} \, U_i U_{i+1} R\), which can be computed similarly with the result \(\text{tr} \, U_i U_{i+1} R = 2N\).

### 3.4 Higgsing to SQCD and others

It is known [12, 13] (see also [14] in a different context) that a puncture specified by a partition described in [4] can be obtained from a maximal puncture by giving a nilpotent vev to the corresponding moment map (say \(\mu_A\)). Let us consider an embedding \(\rho: SU(2) \to SU(N)\) given by \(\square \to n_1 + n_2 + \cdots + n_\ell\), where \(n_k\) \((k = 1, \ldots, \ell)\) are \(n_k\) dimensional (spin \((n_k - 1)/2\)) representations of SU(2). This embedding \(\rho\) is specified by the partition \(\Lambda = (n_1, \ldots, n_\ell)\) with \(n_1 \geq \cdots \geq n_\ell\), i.e., \(\rho\) has a Jordan block structure such that \(k\)-th block has the size \(n_k \times n_k\). Then, by giving a vev \(\langle \mu_A \rangle = \rho_\Lambda (\sigma^+)\), we get a puncture specified by the partition and some Nambu-Goldstone multiplets associated with the symmetry breaking of \(SU(N)_A\) by the vev \(\rho(\sigma^+)\). Under such embedding the adjoint representation of \(SU(N)\) decomposes as

\[
\text{adj} = \bigoplus_{\alpha=1}^{\ell} \bigoplus_{j=1}^{n_{\alpha-1}} V_j \oplus (\ell - 1)V_0 \oplus \bigoplus_{\alpha<\beta}^{\ell} \bigoplus_{k=1}^{n_{\alpha}} V_{n_{\beta} + n_{\alpha - 2k}} \equiv \bigoplus_j R_j \otimes V_j, \tag{3.10}
\]
where $V_j$ is the spin $j$ representation of SU(2) and $R_j$ is the flavor symmetry representation. The first two terms and the last term come from the diagonal and off-diagonal blocks, respectively.

In particular, the $T_N$ theory is reduced to a bifundamental hypermultiplet $Q^{ib,i'c}, \tilde{Q}^{i'b,i'c}$ together with Nambu-Goldstone multiplets by closing one puncture into the minimal (U(1)) puncture with a vev of $\mu_A$ as

$$\langle \mu_A \rangle = \rho_*(\sigma^+),$$

(3.11)

where $\rho_*$ is the embedding SU(2) $\rightarrow$ SU($N$) given by $\square \rightarrow (N - 1) + 1$. In this subsection we sketch the reduction of the deformed moduli constraints proposed in section 3.1 to that of SU($N$) SQCD with $N_f = N$ flavors by the above process. The detailed derivation is left to appendix A.

By the vev (3.11), the SU($N$)$_A$ flavor symmetry is broken to U(1) generated by $T_{U(1)} = \text{diag}(1, \cdots, 1, -(N - 1))$. Then $\mu_A$ is decomposed as

$$\mu_A = \rho_*(\sigma^+) + \sum_{j=1}^{N_f} T_{j,m,0} \rho_A^{j,m,0} + T_{U(1)} \mu_A^{0,0,0} + \sum_{m=-\frac{N_f-2}{2}}^{\frac{N_f-2}{2}} T_{\frac{N_f-2}{2},m,\pm N} \mu_A^{\frac{N_f-2}{2},m,\pm N},$$

(3.12)

where $T_{j,m,q}$, $T_{U(1)}$ and $\rho_*$ satisfy the following commutation relation

$$[\rho_*(\sigma^3), T_{j,m,q}] = m T_{j,m,q}, \quad [\rho_*(\sigma^\pm), T_{j,m,q}] \propto T_{j,m,\pm 1,q}, \quad [T_{U(1)}, T_{j,m,q}] = q T_{j,m,q}. \quad (3.13)$$

One can factor out the Nambu-Goldstone modes $\mu_A^{j,m,q}$ with $m > -j$ corresponding to the broken SU($N$)$_A$ by reparametrization $\mu_A = G_A \mu_A \Lambda_A^{-1}$, where $G_A$ is an element of the complexified SU($N$)$_A$ group containing the Nambu-Goldstone multiplets. All the holomorphic constraint equations are covariant under the complexified SU($N$)$_A$, and hence $G_A$ drops out from them. That is, the Nambu-Goldstone multiplets do not enter into the constraints. With these multiplets out, $\mu_A$ can be parametrized into the following matrix form

$$\mu_A \sim \begin{pmatrix}
\mu_A^{0,0,0} & (\rho_*(\sigma^+))_2^0 & 0 & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\mu_A^{N-3,-(N-3),0} & \cdots & (\rho_*(\sigma^+))_{N-1}^0 & 0 \\
\mu_A^{N-2,-(N-2),0} & \mu_A^{N-3,-(N-3),0} & \cdots & \mu_A^{0,0,0} & \mu_A^{(N-2),-(N-2),N} \\
\mu_A^{N-2,-(N-2),0} & \mu_A^{N-3,-(N-3),0} & \cdots & \mu_A^{0,0,0} & \mu_A^{(N-2),-(N-2),N} \\
\mu_A^{0,0,0} & \cdots & \cdots & \cdots & \mu_A^{0,0,0} \\
\end{pmatrix}, \quad (3.14)$$

where we have neglected order one coefficients of components containing $\mu_A^{k-j,0} (j \geq 1)$.

Now we impose the constraints $\text{tr} \mu_A^k - \text{tr} \mu_B^k = N \Lambda_A^{2N} \delta^{kN}$. Solving the constraints with $k = 2, \ldots, N - 1$ determines $\mu_A^{k-1,-(k-1),0}$ in terms of $\mu_A^{0,0,0}$ and $\mu_B$. The constraint with $k = N$ gives the relation

$$\det(-(N-1)\mu_A^{0,0,0} I - \mu_B) + C(\rho) \mu_A^{\frac{N_f-2}{2},-\frac{N_f-2}{2},N} \mu_A^{\frac{N_f-2}{2},-\frac{N_f-2}{2},-N} = \Lambda_A^{2N},$$

(3.15)
where $C(\rho) = \prod_{k=1}^{N-2}(\rho_+^k(\sigma^+)^k)_{k+1}$.

In SQCD, we have mesons $\mathcal{M}^{ij}_{B\beta} = Q^i{}^{i}c_c \tilde{Q}_{j\beta}^{c_c}$ and baryons $B_+ = \det \mathcal{Q}$ and $B_- = \det \tilde{\mathcal{Q}}$. The deformed moduli constraint is given by $\det \mathcal{M} - B_+ B_- = \Lambda^2 N$. Eq. (3.15) is the same as this SQCD constraint by the following identification:

$$\hat{\mathcal{M}} \equiv \mathcal{M} - \frac{1}{N} \text{tr} \mathcal{M} = \mu_B, \quad \text{tr} \mathcal{M} = N(N-1)\mu_A^{0,0,0}, \quad B_\pm = c_\pm \mu_A^{\frac{N-2}{2}, -\frac{N-2}{2}, \pm N}, \quad (3.16)$$

where $c_\pm$ are constants satisfying $c_+ c_- = (-1)^{N-1} C(\rho)$. This can be checked explicitly as demonstrated in appendix A. Here let us shortly see this is reasonable one. Before gauging $\text{SU}(N)_C$, the theory has $N = 2$ supersymmetry. The operators $\hat{\mathcal{M}}$ and $\text{tr} \mathcal{M}$ are the holomorphic moment maps of the $\text{SU}(N)_B$ and $\text{U}(1)$ symmetries respectively, so the first two identifications of (3.16) are expected from the mapping of these moment maps. Note that the operator $\text{tr} \mathcal{M}$ maps to $\text{tr}(\mu_A T_{U(1)}) = N(N-1)\mu_A^{0,0,0}$, which explains the coefficient $N(N-1)$ in the second equation of (3.16). Up to coefficients, the identification of the baryon operators is also easy to expect just from the $\text{U}(1)$ charges.

The reproduction of the SQCD result is highly nontrivial. Recall that the usual SQCD has $\text{SU}(N) \times \text{SU}(N) \times \text{U}(1)$ flavor symmetry which almost fixes the form of the constraint. However, only the subgroup $\text{SU}(N)_B \times \text{U}(1)$ of the SQCD symmetry group is manifest in the reduction of the $T_N$ theory to the SQCD, and other symmetries should be realized as accidental symmetries at low energies. The manifest symmetry $\text{SU}(N)_B \times \text{U}(1)$ is not enough at all to determine the form of the constraint (3.15) which is consistent with the SQCD.

It is also interesting to consider the closure of one puncture of the $T_N$ theory to other generic one. This is an $\text{SU}(N)$ gauge theory coupled to the SCFT associated with a sphere with two maximal punctures and one generic puncture. Here we consider the case with the next-to-maximal puncture $\Lambda = (N-k, 1, \cdots, 1)$ with the flavor symmetry $\text{U}(1) \times \text{SU}(k)$ and derive the constraints of the quantum moduli space of this theory. As in the previous section, the index indicates that the spectrum is given by (1) with $k$ relations. $\mu_A$ can be parametrized (after factoring out Nambu-Goldstone modes), in terms of the gauge invariants in (1), in a similar block diagonal form

$$\mu_A \sim \begin{pmatrix} \mu^0_0 & \rho^0_1 & 0 & 0 & 0 & 0 & \vdots \\ \mu^1_0 & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \mu^{N-k-1}_{N-k} & \cdots & 0 & 0 & 0 & \mu^0_{N-k} \\ \mu^{N-k}_{N-k} & \cdots & \mu^0_1 & 0 & 0 \\ B & 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ \mu^0 \\ -\frac{N-k}{k} \mu^0 I_{k \times k} + \mu_{\text{SU}(k)} \end{pmatrix},$$

(3.17)

where $B = (B_1, B_2, \cdots, B_k)$, $\tilde{B} = (\tilde{B}^1, \tilde{B}^2, \cdots, \tilde{B}^k)$, and we suppressed unnecessary indices. A similar argument as the previous case gives the constraints in matrix form. The $\text{SU}(k)$ flavor symmetry guarantees that there are only $k$ independent constraints. Here we present the expression of $k = 2$ case: the constraint $P_B(\mu_A) = \Lambda^{2N} I_{N \times N}$ gives

$$P_B \left( -\frac{N-2}{2} \mu^0 I_{2 \times 2} + \mu_{\text{SU}(2)} \right) + C(\rho) \left( \mu_{\text{SU}(2)} \tilde{B} B + B B \mu_{\text{SU}(2)} \right) = \Lambda^{2N} I_{2 \times 2},$$

(3.18)
Table 2. U(1) charges of fields.

|     | U(1)_B | U(1)_F | U(1)_R |
|-----|--------|--------|--------|
| q   | 1      | N      | 1      |
| \tilde{q} | -1    | N      | 1      |
| \mu_X | 0      | -2     | 0      |

where $C(\rho) = \prod_{k=1}^{N-3} \rho_{k+1}^k$.

With a similar procedure, for example, we should be able to derive the deformed moduli space of $\mathcal{N} = 1$ SU($N$) theory coupled to two fundamental flavors together with one flavor of two-index anti-symmetric tensor [15].

4 Adding one flavor

4.1 Statement of the result

We add one massless flavor $q$ and $\tilde{q}$ to the setup of the previous section. So we have one $T_N$ theory, an SU($N$) vector multiplet gauging one SU($N$) flavor symmetry of the $T_N$ theory, and this additional flavor $q$ and $\tilde{q}$.

The theory has the global symmetry SU($N$)$_A \times$ SU($N$)$_B \times$ U(1)$_F \times$ U(1)$_R$. Here, the two SU($N$)'s come from the flavor symmetry of the $T_N$ theory, and the U(1)$_B$ acts only on $q$ and $\tilde{q}$ with charge 1 and $-1$. The U(1)$_F$ is given by

$$F = R_{N=2} + 2I_3 + NA,$$

where $A$ is the generator of the U(1) symmetry under which both $q$ and $\tilde{q}$ have charge 1. Finally, the $R$-symmetry is given by

$$R = R_{N=2} + A.$$ (4.2)

All these are anomaly free. Charges of various fields are summarized in table 2.

The one-loop beta function is the same as that of the $\mathcal{N} = 1$ SQCD with $N_f = N + 1$. Thus we expect that the low energy theory exhibits the confinement and is described by the gauge invariant operators with dynamically generated superpotential as in [1]. Let us first list the gauge invariant operators of this theory:

$$B_{iA_1B} = Q_{iA_1B}^{ij} q_{iC}, \quad \tilde{B}^{iA_1B} = Q^{iA_1B}_{ijC} \tilde{q}_{iC},$$
$$\mu_X \quad (X = A, B), \quad M_k = \tilde{\mu}_k^C q \quad (k = 0, 1, \ldots , N - 2).$$ (4.3)

The U(1) charges of these gauge invariants are summarized in table 3.

Note that the meson $M_{N-1} = \tilde{\mu}_0^C q$ is excluded. This is because we can express this operator using other gauge invariant operators due to the relation (2.7).

From the global symmetry, the low energy superpotential is given as

$$W = -\frac{1}{A^{2N-1}} \left[ \sum_{k=0}^{N-2} M_k f_{N-k}(\mu_A, \mu_B) - tr[(B\tilde{B})_A\mu_A] + tr[(BB)_{B}\mu_B] \right],$$ (4.4)
where \( f_{N-k}(\mu_A, \mu_B) \) is an SU\((N)_A \times SU(N)_B \) invariant polynomial of degree \( N-k \) given below, and \( \Lambda \) is the dynamical scale. The coefficient of the term \( \text{tr}[(BB)_A \mu_A] \) can be absorbed in the definition of \( \Lambda^{2N-1} \), while that of \( \text{tr}[(B\bar{B})B\mu_B] \) is determined by later consistency.

The \( f_{N-k} \) is determined to be

\[
f_{N-k}(\mu_A, \mu_B) = \sum_{\ell=0}^{N-k-2} v_\ell(\mu_A, \mu_B) \frac{\text{tr} \mu_A^{N-k-\ell} - \text{tr} \mu_B^{N-k-\ell}}{N-k-\ell},
\]

(4.5)

where \( v_\ell(\mu_A, \mu_B) \) is a function of \( \mu_A \) and \( \mu_B \) of degree \( \ell \) which coincides with \( v_{A,\ell} \) and \( v_{B,\ell} \) when \( \mu_A^k = \mu_B^k \) (\( k = 2, \ldots, N \)). This \( f_{N-k} \) is determined such that the equations of motion give the “classical” constraints which are the chiral ring relations of the \( T_N \) theory. Indeed, the equations of motion with respect to \( M_k \) give the relation \( \text{tr} \mu_A^k = \text{tr} \mu_B^k \) (\( k = 2, 3, \ldots, N \)), which is (2.4). The equation of motion of \( \bar{B} \) gives

\[
0 = (\mu_A)^{ja}_i B_{jaiB} - (\mu_B)^{ja}_i B_{iAjb}.
\]

(4.6)

This is derived from (2.6) in the UV. The equations of motion with respect to \( \mu_A \) is

\[
0 = \sum_{k=0}^{N-2} \sum_{\ell=0}^k v_\ell M_{k-\ell} \left( \frac{1}{N} \text{tr} \mu_A^{N-k-1} - \mu_A^{N-k-1} \right)^{ia}_j B_{jaiB} \bar{B}_{iAjb} \delta^{ia}_j + \frac{\delta^{ia}_j}{N} B_{kakb} \bar{B}_k^{kab},
\]

(4.7)

where we have used \( \text{tr} \mu_A^k = \text{tr} \mu_B^k \). This can be obtained from the relation (2.7).

The consistency with the theory in the previous section is seen as follows: By adding a mass term of \( q \) and \( \bar{q} \), the IR superpotential is just (4.4) plus \( mM_0 \), which can be seen by using the global symmetries. Thus we can easily get \( \text{tr} \mu_A^N - \text{tr} \mu_B^N = Nm\Lambda^{2N-1} \) and \( \text{tr} \mu_A^k - \text{tr} \mu_B^k = 0 \) (\( k = 2, \ldots, N-1 \)). These are the quantum constraints of the previous case.

In the above analysis, we have not completely determined the function \( v_\ell(\mu_A, \mu_B) \). However, the ambiguity in this function can always be absorbed by redefining \( M_k \) as

\[
M_k \rightarrow M'_k = M_k + \sum_{\ell=1}^k h_\ell(\mu_A, \mu_B)M_{k-\ell},
\]

(4.8)
where $h_t(\mu_A, \mu_B)$ is an arbitrary function which vanishes under the condition $\text{tr} \, \mu_A^k = \text{tr} \, \mu_B^k$. This new $M'_k$ is equivalent to the original $M_k$ as an element of chiral rings.\footnote{If we add a tree level superpotential like $\lambda M_k$, we get $\text{tr} \, \mu_A^k - \text{tr} \, \mu_B^{k-N-1} \propto \Lambda^{2N-1}$. However, one should note that there is a general ambiguity in the definition of composite operators like $M_k = \tilde{q}_i \mu_i^k$. We need some regularization to define composite operators. (If we could consider them just as classical objects, we could have proved e.g. $\det M - B_i B_j = 0$ in SQCD by classical algebraic manipulation, which is not the case.) In the regularization, there is an ambiguity in the choice of (finite part of) counterterms. We require that these ambiguous terms are: (1) consistent with global symmetries and holomorphy, (2) going to zero when $\Lambda^{2N-1} \to 0$, since there is a canonical way of defining composite operators of chiral fields when we turn off gauge interactions (by using point splitting regularization). In these criteria, $M'_k$ is as good as $M_k$ as a definition of the ambiguous operator $\tilde{q}_i \mu_i^k$. There is no physical principle to select one of them.}

Therefore, the remaining ambiguity is not important as long as we do not try to determine the effective Kähler potential.

### 4.2 Check of the anomaly

At the origin of the moduli space, the full global symmetry is unbroken. We compare the anomaly coefficients at this point. In the UV theory, these are computed from the anomalies of the $T_N$ theory as in section 3.4, to the effective superpotential of SQCD with $N_f = N + 1$,

$$\Lambda^{2N-1} W_{\text{SQCD}} = B M \tilde{B} - \det M,$$

(4.10)

where the baryon and meson operators are written in terms of quark fields $Q^{I}_{iC}$ and $\tilde{Q}^{I}_{iC}$ ($I = 1, \ldots, N + 1$) as

$$B_I = \frac{1}{N!} Q^{I}_{iC,1} \ldots Q^{I}_{iC,N} \epsilon_{i_1 \ldots i_N} \epsilon_{i_{C,1} \ldots i_{C,N}},$$

$$\tilde{B}^I = \frac{1}{N!} \tilde{Q}^{I}_{jC,1} \ldots \tilde{Q}^{I}_{jC,N} \epsilon_{j_1 \ldots j_N} \epsilon_{j_{C,1} \ldots j_{C,N}},$$

(4.11)

$$\mathcal{M}^I_{ij} = Q^{I}_{iC} \tilde{Q}^{J}_{jC}.$$
Let us consider the first term of the superpotential (4.4). For \( k = 1, \ldots, N-2 \), by using (A.4) we can see that \( M_k \) and \( \mu_A^{N-k-1} - (N-k-1) \mu_B \) form a mass term in the superpotential and they can be integrated out. The equations of motion of these massive fields give \( \text{tr} \mu_A^{N-k} = \text{tr} \mu_B^{N-k} \) for \( k = 1, \ldots, N-2 \). Then, the remaining term in the first term of (4.4) is

\[
M_0 f_N = M_0 \sum_{l=0}^{N-2} \frac{v_l}{N-l}(\text{tr} \mu_A^{N-l} - \text{tr} \mu_B^{N-l}).
\]

(4.12)

Again the terms with \( l \neq 0 \) vanish by using the constraints. Under the condition \( \text{tr} \mu_A^k = \text{tr} \mu_B^k \) for \( k = 2, \ldots, N-1 \), we get \( \frac{1}{N}(\text{tr} \mu_A^N - \text{tr} \mu_B^N) = -P_A(x) + P_B(x) \), where \( x \) is arbitrary. Taking \( x = -(N-1)\mu_A^{0,0,0} \) as in (A.5), we obtain

\[
M_0 f_N = (-1)^N \mathcal{M}^{N+1}_{N+1} \left( \det \begin{pmatrix} \mathcal{M} - \mathcal{B}_N \tilde{\mathcal{B}}^N \end{pmatrix} \right),
\]

(4.13)

where we have used identification (3.16) and \( \mathcal{M}^{N+1}_{N+1} = M_0 \).

Next, let us consider the other two terms of (4.4). Because of the vev of \( \mu_A \), the fields \( B_{iA} (i_A = 1, \ldots, N-2) \) and \( \tilde{B}^{iA} (i_A = 2, \ldots, N-1) \) become massive. Therefore we need to integrate them out. The equations of motion with respect to these massive fields are

\[
\tilde{B}^{iA} (\mu_B)^i_B = \tilde{B}^{iA} (\mu_A)^i_A (i_A = 1, \ldots, N-2),
\]

(4.14)

\[
B_{iA} (\mu_B)^i_B = B_{iA} (\mu_A)^i_A (i_A = 2, \ldots, N-1).
\]

(4.15)

By using these and after some algebra which is detailed in appendix A, we can rewrite the last two terms of (4.4) as

\[
(-1)^{N-1} \mathcal{B}_{iB} \tilde{B}^{iB} \mathcal{M}^{iB}_{jB} + \mathcal{B}_{iB} \mathcal{M}^{iB}_{jB} \tilde{B}^{N+1} + \mathcal{M}^{N+1}_{iB} \tilde{B}^{iB} \mathcal{B}_{N+1} + \mathcal{M}^{N+1}_{iB} \det \mathcal{M} (\mathcal{M}^{-1})^{iB}_{jB} \mathcal{M}^{jB}_{N+1},
\]

(4.16)

where the identification of operators is as in (A.29). By combining (4.13) and (4.16), the superpotential (4.4) is thus reduced to

\[
\frac{(-1)^{N-1}}{\Lambda^{2N-1}} \left[ \det \begin{pmatrix} \mathcal{M} - \mathcal{B}_I \mathcal{M}^I \tilde{\mathcal{B}} \end{pmatrix} \right],
\]

(4.17)

which is the SQCD superpotential.

5 \( \mathcal{N} = 1 \) Coulomb branch

So far, we have considered the \( T_N \) theory where the SU(\( N \)) flavor symmetry is gauged. In this section we will see that further gauging of additional SU(\( N \)) flavor symmetries leads to abelian Coulomb phase, whose low energy holomorphic coupling matrix is encoded in an \( \mathcal{N} = 1 \) Seiberg-Witten curve [16]. After analyzing the case of the \( T_N \) theory coupled to \( \mathcal{N} = 1 \) SU(\( N \)) vector multiplets and with or without flavors, we generalize this to \( \mathcal{N} = 1 \) quiver gauge theory. Our analysis is a generalization of the previous studies [7, 16–22] of conventional theories in the Coulomb phase to non-conventional ones.
5.1 \( T_N \) theory coupled to SU(\(N\))^3

Let us consider the \( T_N \) theory coupled to \( \mathcal{N} = 1 \) SU(\(N\))^3 vector multiplets. Recall that in the theory considered in section 3, the independent operators are \( \mu_A \) and \( \mu_B \). Since the SU(\(N\))_A and SU(\(N\))_B symmetries are gauged, the gauge invariant operators are the \( v_k \) (\( k = 2, \ldots, N \)). There are another Higgs branch operators \( Q_{A}^{A}B_{C} \) and \( Q^{A}_{B}B_{C} \) of the \( T_N \) theory. As discussed in section 2.3, a generic point of the moduli space at the “classical” level is given by the diagonal vevs (2.21) and (2.22). Therefore, the gauge symmetry is broken to U(1)^2(\(N\)-1), represented in terms of the gauge fields \( A^{\alpha}_{X} \) of U(1)^2(\(N\))_X (\( \alpha = 1, \ldots, N - 1 \)) by

\[
A^{\alpha}_{A} + A^{\alpha}_{B} + A^{\alpha}_{C} = 0.
\] (5.1)

Thus the theory is in the Coulomb phase. Classically there are codimension one singular loci on the moduli space parameterized by \( v_k \) where some of the W-bosons become massless and the gauge symmetry is enhanced.

To consider quantum theory, let us turn on the dynamical scales \( \Lambda^{A,B,C} \). In the regime \( \Lambda_C \gg \Lambda_{A,B} \), we can first use the result of section 3 and think of the system as SU(\(N\))_A coupled to \( \mu_A \), together with SU(\(N\))_B coupled to \( \mu_B \), with the constraints (3.2). Thus, each ingredient is the \( \mathcal{N} = 2 \) SU(\(N\)) pure SYM theory. It is clear that there are 2(\(N\)-1) abelian vector multiplets on a generic point on the moduli space. The classical singularities are split into singularities where massless charged particles appear as in [23–25]. The constraint \( \text{tr} \mu^{N}_{A} - \text{tr} \mu^{N}_{B} = N \Lambda^{2N}_{C} \) says that the singular loci for SU(\(N\))_A and SU(\(N\))_B are generically separated at the quantum level.

Naively, in the limit \( \Lambda_C \gg \Lambda_{A,B} \), we just have two separate SU(\(N\)) gauge groups with an adjoint field for each gauge group. However, the deformed moduli space of section 3 has a Wess-Zumino-Witten term.\(^8\) As discussed in detail in [7] for the case of SU(2), this makes the dynamics much more complicated and interesting.

To avoid the complication due to the Wess-Zumino-Witten term, we follow the procedure in [7]. We first add one flavor to the SU(\(N\))_C gauge group with mass \( m \), and consider the regime \( \Lambda_C \gg \Lambda_{A,B} \). At the energy scale around \( \Lambda_C \) the SU(\(N\))_C theory is confined with the superpotential as in section 4. This theory does not have a Wess-Zumino-Witten term since the topology of the moduli space is trivial (in the limit \( m \to 0 \)). We can regard \( \mu_A \) and \( \mu_B \) as adjoint chiral fields of SU(\(N\))_A and SU(\(N\))_B, respectively, and \( B, \tilde{B} \) as bifundamental fields of SU(\(N\))_A \times SU(\(N\))_B. Furthermore, the superpotential (4.4) is similar to the one in an \( \mathcal{N} = 2 \) theory except for the first term. The curve of this \( \mathcal{N} = 2 \) theory is written as [27]

\[
\Lambda^{2N}_{A} t^3 - P_A(w) t^2 + P_B(w) t - \Lambda^{2N}_{B} = 0,
\] (5.2)

where \( P_{A,B}(w) = \det(w \cdot 1 - \mu_{A,B}) \). The equations of motion with respect to \( M_k \) give the relation \( P_A = P_B - \Lambda^{2N}_{C} \), where the low energy dynamical scale \( \Lambda^{2N}_{C} \) is defined as \( m \Lambda^{2N-1}_{C} \).

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\(^8\)This is because we can reduce this theory to the SU(2) theory coupled to \( T_2 \) (i.e., a trifundamental chiral multiplet) by the process described in section 3, and this SU(2) theory contains the Wess-Zumino-Witten term [26].
However, this relation is valid only in the limit $\Lambda_C \gg \Lambda_{A,B}$. We propose that the exact relation between $P_A$ and $P_B$ is given as

$$P_A = P + \Lambda_A^2 N, \quad P_B = P + \Lambda_B^2 N + \Lambda_C^2 N,$$

where $P = w^N + \sum_{k=2}^N v_k w^{N-k}$ is a polynomial of $w$ of degree $N$. Then, the curve is written as

$$F(w,t) \equiv w^N + \sum_{k=2}^N V_k(t)w^{N-k} = 0,$$

(5.4)

where $V_N(t) = v_N - t\Lambda_A^2 N - \frac{\Lambda_B^2 N}{t} - \frac{\Lambda_C^2 N}{t-1}$. (5.5)

The exact form (5.3) has been obtained so that $V_N$ becomes a meromorphic function on a Riemann sphere with simple poles at $t = 0, 1$ and $\infty$. This should be the case because there is an obvious symmetry under the exchange of $A,B$ and $C$. The $N=1$ curve is the $N$-sheeted cover of the base sphere with three punctures.

As mentioned above, the holomorphic gauge coupling matrix is identified with the period matrix of this curve. Let us see the genus of the curve. There are two types of branch points on the base Riemann sphere. One is at the poles of $V_N(t)$. The other is at the points where $F(w,t) = 0$ has double roots, that is, $F(w,t) = 0$ and $\frac{\partial F(w,t)}{\partial w} = 0$. We can find $3(N-1)$ points of this type. The branching number of each simple pole is $N-1$, i.e. $N$ sheets meet at these points. Meanwhile that of each double root point is 1. By using the Riemann-Hurwitz relation, the genus of the curve $g'$ is

$$g' = (g-1) + 1 + \frac{B}{2} = 2(N-1),$$

(5.6)

where $g$ is the genus of the base curve, which is zero in this case, and $B$ is the total branching number. This coincides with the number of massless U(1) fields.

Let us study what happens when we decouple some of the gauge groups. By decoupling the SU($N_A$) vector multiplet by setting $\Lambda_A = 0$, the theory is SU($N_B$) $\times$ SU($N_C$) theory coupled to the $T_N$ theory. The theory is still in the Coulomb phase with U(1)$^{N-1}$ gauge groups. The curve is simply (5.5) with $\Lambda_A = 0$, and this is the same curve as the pure $\mathcal{N} = 2$ SYM theory. Now, further decoupling the SU($N_B$) vector multiplet by $\Lambda_B = 0$, the theory goes back to the one considered in section 3. Indeed, the curve degenerates to a genus zero curve, indicating that all massless photons decouple.

### 5.2 $T_N$ theory coupled to SU($N_X$)$^3$ with a number of additional flavors

We can further generalize the above theory by introducing some more fundamental flavors of quarks. Let us add $N_X (< N)$ flavors of quarks $q^X_I, \bar{q}^X_I$ ($I = 1, \ldots, N_X$) which are in the (anti)-fundamental representation of SU($N_X$), where $X = A, B$. The superpotential for these flavors is taken as

$$W = \sum_{X=A,B} \sum_{I=1}^{N_X} (m^X_I \bar{q}^X_I q^X_I + \lambda^X_I \bar{q}^X_I \mu_X q^X_I),$$

(5.7)
We will soon give a motivation for this form of the superpotential. Again, the theory is similar to one studied in \[27\] in the limit \(\Lambda_C \gg \Lambda_{A,B}\). The curve is

\[
\Lambda_A^{2N-N_A} Q_A(w)t^3 - P_A(w)t^2 + P_B(w)t - \Lambda_B^{2N-N_B} Q_B(w) = 0, \tag{5.8}
\]

where \(Q_X(w) = \prod_{I=1}^{N_X}(\lambda_I^X w + m_I^X)\). The dependence of the curve on \(m_I^{A,B}\) and \(\lambda_I^{A,B}\) is determined by using spurious flavor symmetries such as a U(1) spurious symmetry

\[
q_I^X \to e^{i\theta_I,X} q_I^X, \quad m_I^X \to e^{-i\theta_I,X} m_I^X, \quad \lambda_I^X \to e^{-i\theta_I,X} \lambda_I^X, \quad \Lambda_X^{2N-N_X} \to e^{i\theta_I,X} \Lambda_X^{2N-N_X}. \tag{5.9}
\]

We neglect possible order one coefficients. As in the previous case, we assume the relation between \(P_A\) and \(P_B\) is given as \(P_A = P + \Lambda_A^{2N-N_A} Q_A, P_B = P + \Lambda_B^{2N} + \Lambda_B^{2N-N_B} Q_B\). The curve is then

\[
P(w) = t\Lambda_A^{2N-N_A} Q_A(w) + \frac{\Lambda_B^{2N-N_B} Q_B(w)}{t} + \frac{\Lambda_C^{2N}}{t-1}. \tag{5.10}
\]

This is again written as the \(N\)-sheeted cover of the sphere with three punctures

\[
w^N + \sum_k V_k(t)w^{N-k} = 0, \tag{5.11}
\]

but in the present case \(V_k(t)\) are as follows: at \(t = 0\), \(V_k\) with \(k \geq N - N_B\) has a simple pole, at \(t = 1\), only \(V_N\) has a simple pole and at \(t = \infty\), \(V_k\) with \(k \geq N - N_A\) has a simple pole.

Notice that when \(\lambda_I^X = 0\), we recover the curve without flavors by identifying \(\Lambda_X^{2N} = \Lambda_X^{2N-N_X} \prod_{I=1}^{N_X} m_I^X\). This means that in the theory without the cubic term in (5.7) the addition of the flavors does not change the form of the \(\mathcal{N} = 1\) curve. Note also that in the limit where all the flavors are massless the \(V_k\) functions behave such that at \(t = \infty\) only \(V_{N-N_A}\) has a simple pole and at \(t = 0\) only \(V_{N-N_B}\) has a simple pole.

So far we have considered the theories as purely \(\mathcal{N} = 1\) theories. However, we can obtain them from \(\mathcal{N} = 2\) theories by mass deformations. Let us take the gauge multiplets as \(\mathcal{N} = 2\) vector multiplets and introduce the superpotential,

\[
W = \sum_{X=A,B,C} (M_X \text{ tr } \Phi_X^2 + \text{ tr } \Phi_X \mu_X) + \sum_X \sum_{I=1}^{N_X} (m_I^X \hat{q}_I^X q_I^X + \hat{q}_I^X \Phi_X q_I^X), \tag{5.12}
\]

where \(\Phi_X\) are the adjoint chiral multiplets. The \(\mathcal{N} = 2\) supersymmetry is broken only by the adjoint mass terms. Assuming that the mass parameters \(M_X\) are large enough such that the classical analysis is valid, we integrate \(\Phi_X\) out and get terms like \(\text{ tr } \mu_X^2, (\hat{q}_I^X q_I^X)^2\) and \(\hat{q}_I^X \mu_X q_I^X\) with coupling constant \(1/M_X\). Since there is the relation \(\text{ tr } \mu_A^2 = \text{ tr } \mu_B^2 = \text{ tr } \mu_C^2\), the first kind of terms is cancelled by setting \(\sum_X 1/M_X = 0\), which is necessary for the moduli space not to be lifted. The second kind of terms can be shown to be irrelevant for the curve (5.8) by spurious symmetry argument. The third kind of terms is precisely the ones in (5.7) with \(\lambda_I^X \to 1/M_X\).
5.3 Generalization

We consider the following class of $N = 1$ gauge theories in the Coulomb phase as a generalization of the previous model. Instead of a single copy of $T_N$ theory, we use a class of theories specified by a Riemann surface with several maximal punctures, and we gauge the flavor symmetries associated with the punctures by $N = 1$ vector multiplets.

The class is specified by a graph consisting of trivalent vertices connecting circles or boxes as in figure 1. The $\mathcal{N} = 1$ theory can be read off from a given graph as follows:

- Label vertices by $v$ and circles by $i$ or $e$.
- To each trivalent vertex $v$, we introduce a copy of $T_N$ theory denoted as $T^{(v)}_N$. If it is connected to circles $i, j$ and $k$, it has operators $\mu_{v,i}$, $\mu_{v,j}$ and $\mu_{v,k}$, etc.
- To each circle $i$ connected to two vertices $v$ and $v'$, we associate an $\mathcal{N} = 2$ SU($N_i$) vector multiplet. The superpotential is taken as
  \[
  W = \text{tr}(\mu_{v,i} - t\mu_{v',i})\Phi_i. \tag{5.13}
  \]
  Note that the second term is $-t\mu_{v',i}$ instead of just $\mu_{v',i}$. This comes from our choice of the embedding $g \in \text{SU}(N_i) \mapsto (g, t^{-1}g^{-1}) \in \text{SU}(N_{v,i}) \times \text{SU}(N_{v',i})$.
- To each box with SU($N$) written inside it, we associate a flavor SU($N$) symmetry.
- To each box with $N_e$ inside it, we introduce $N_e$ flavors of quarks $q_{e,l}$ and $\bar{q}_{e,l}$ with mass $m_{e,l}$ ($I = 1, \ldots, N_e$). We only consider the case $N_e < N$.
- To each circle $e$ connected to only one vertex $v$ and possibly to a box with $N_e$, we introduce a mass-deformed $\mathcal{N} = 2$ SU($N_e$) vector multiplet. The superpotential is
  \[
  W = M_e \text{tr} \Phi_e^2 + \text{tr} \mu_{v,e} \Phi_e + \sum_{l=1}^{N_e} [\bar{q}_{e,l} \Phi e q_{e,l} + m_{e,l} \bar{q}_{e,l} q_{e,l}]. \tag{5.14}
  \]
  We require the mass parameters $M_e$ satisfy
  \[
  \sum_e \frac{1}{M_e} = 0. \tag{5.15}
  \]

Suppose that there are $g$ loops in the graph and $n$ SU($N_e$) gauge fields which are coupled to only one trivalent vertex. Our proposal is that the $\mathcal{N} = 1$ curve of this class of models which encode the holomorphic coupling matrix of low energy U(1) fields is given by
  \[
  w^N + \sum_k V_k(t) w^{N-k} = 0, \tag{5.16}
  \]
  where $V_k$ are meromorphic functions on a genus $g$ Riemann surface with $n$ punctures $c_{g,n}$. At each puncture, $V_k$ with $k \geq N - N_e$ have simple poles, and the residues of these poles are given in terms of dynamical scales of SU($N_e$), $\Lambda^{2N-N_e}_e$, and the masses of quarks, $m_{e,l}$.
This is a generalization of the result in [7] for SU(2) with trifundamentals to SU(N) with copies of $T_N$ theory and some flavors of quarks.\footnote{Aside from the introduction of flavors of quarks, there is a few minor changes from [7]. First, the adjoint fields $\Phi_i$ connecting two vertices was given mass terms in [7]. In the case $g \geq 1$, there is actually a flat direction of $\Phi_i$. However, we can show that the vevs of $\Phi_i$ do not enter in the curve because they are charged under the $R_{N=2}$ symmetry while the curve is neutral under this symmetry. Another difference is that the massive adjoint was not introduced for gauge groups SU($N$)$_e$ connected to one vertex. If there are no flavors of quarks as in [7], this difference does not matter as long as (5.15) is satisfied. However, massive adjoint fields are important to get the couplings $\tilde{q}_{e,I} \mu_{e,e} q_{e,I}$ as explained in the previous subsection.} Note the similarity to the family of curves Gaiotto has written down for $\mathcal{N}=2$ theories [4]

$$x^N + \sum_k \phi_k x^{N-k} = 0,$$

(5.17)

where $\phi_k$ are $k$-th differentials on a Riemann surface. In our case, they are replaced by functions $V_k$ on a Riemann surface.

Now we perform checks of our proposal. Some of the checks of [7] are extended straightforwardly to the present case, so we only study a few basic properties here.

**Number of U(1) gauge groups and moduli.** First let us study the moduli space and massless U(1) fields at the ‘classical’ level, i.e. before discussing the quantum effects from the gauge fields. If a vertex $v$ is connected to circles labeled as e.g., $i$, $j$ and $e$, we have the $T_N^{(v)}$ constraints

$$\text{tr} \mu^k_{v,i} = \text{tr} \mu^k_{v,j} = \text{tr} \mu^k_{v,e}.$$  

(5.18)

We also have the equations of motion of $\Phi_i$, $\mu_{v,i} = \mu_{v,i}'$. Up to gauge and flavor transformations, these equations are solved by giving the same diagonal vevs to all the $\mu$ operators as

$$\mu_{v,i} = \mu_{v,e} = \text{diag}(m_1, \cdots, m_N),$$

(5.19)
where $\sum_k m_k = 0$. Also, as discussed in section 2.3, the operators $Q_{(v)}$ should also have diagonal vevs as in (2.22). Then the massless gauge fields are given by $A_t^a$ of $U(1)^a_t \subset SU(N)$, $(\alpha = 1, \ldots, N - 1)$ with the constraints at each vertex $v$,

$$
(1 - h(v,i))A_t^a + (-1)^{(h(v,j))}A_j^a + (-1)^{(h(v,e))}A_e^a = 0,
$$

(5.20)

where $h(v,i) = 0$ or 1 if $SU(N)_i$ is embedded in $SU(N)_{v,i}$ as $g \rightarrow g \text{ or } g \rightarrow t g^{-1}$, respectively. This has the form of current conservation in an electric circuit at each vertex.

Therefore, when $n \geq 1$, the number of massless photons are given as

$$
N_{U(1)} = (N - 1)(n + g - 1).
$$

(5.21)

The products $(Q_{(v)})^{kkk}(Q_{(v)})_{kkk} (k = 1, \cdots, N)$ and $\prod_{k=1}^N (Q_{(v)})^{kkk}$ are determined by $m_k$ as in (2.23) and (2.24), while other $N - 1$ unconstrained fields $(Q_{(v)})^{kkk}$ are absorbed by the linear combination of gauge fields appearing in (5.20) to form massive vector fields. Therefore the moduli space is $N - 1$ dimensional which is spanned by $m_k$ or equivalently $v_k$ ($k = 2, \cdots, N$).

Let us compare the above field theory results to the properties of the curve. When all the residues of poles of $V_k$ are fixed, the only remaining freedom is to change the constant parts of $V_k \sim v_k$. These $N - 1$ constants represent the moduli fields, consistent with the field theory analysis.

The number of photons should be compared with the genus of the curve. The genus can be calculated as follows. The equation (5.16) has $N$ solutions as a polynomial equation of $w$, which we denote as $w_k(t)$ ($k = 1, \cdots, N$). At the puncture corresponding to $SU(N)_{v,e}$, $V_k$ with $k \geq N - N_e$ have simple poles, and hence the behavior of $w_k$ near this pole is given as

$$
w_k(t) \sim \begin{cases} 
\text{finite} & (1 \leq k \leq N_e) \\
(t - t_e)^{-1/(N - N_e)} (N_e + 1 \leq k \leq N) 
\end{cases}
$$

(5.22)

where $t_e$ is the position of the puncture. Therefore $N - N_e$ sheets meet at this point. Next, let us determine the number of points at which (5.16) has double roots. Such points are the zeros of the discriminant,

$$
D(t) = \prod_{k < \ell} (w_k(t) - w_{\ell}(t))^2.
$$

(5.23)

This discriminant $D(t)$ is a meromorphic function on the Riemann surface. At the singular point, the behavior of $D(t)$ is determined by using (5.22) as $D(t) \sim t^{-(N + N_e - 1)}$. Since the total of degrees of poles of a meromorphic function is the same as the total of degrees of zeros, $D(t)$ has generically $\sum_e (N + N_e - 1)$ zero points. Therefore, the genus of the curve $g'$ is determined by the Riemann-Hurwitz relation as

$$
g' = N(g - 1) + 1 + \frac{1}{2} \sum_e [(N - N_e - 1) + (N + N_e - 1)] = N_{U(1)} + g.
$$

(5.24)

This is larger than (5.21) by $g$. This excess was already present in e.g., [4, 7, 27]. The cycles pulled back from the base Riemann surface has a trivial monodromy when we change the moduli $v_k$. They decouple from the rest of the cycles, and the remaining $g' - g = N_{U(1)}$ cycles give us nontrivial monodromy and the coupling matrix of massless photons in the field theory.
Conditions on dynamical scales. Let us consider the linear quiver theory as in figure 2. Here we focus on the case with $N_e = 0$ or 1. We consider the limit where all the dynamical scales $\Lambda^{2N}_e$ are large such that the SU($N_e)$ gauge theories ($e = 1, \ldots, n$) confine. When $N_e = 0$ the low energy theory is described by

$$tr \mu^k_{e,n+e} - tr \mu^k_{e,n+e+1} = N\Lambda^{2N}_e \delta^{kN}. \quad (5.25)$$

When $N_e = 1$ the relevant superpotential is given by

$$W = -\frac{1}{\Lambda^{2N-1}_e} \left( \sum_k M_e,k f_{e,N-k} - tr B_e \tilde{B}_e \mu_{e,n+e} + tr \tilde{B}_e B_e \mu_{e,n+e+1} \right) + m_e M_e,0 + \frac{1}{M_e} M_e,1. \quad (5.26)$$

where $M_e,k$, $B_e$ and $\tilde{B}_e$ are the SU($N_e)$-singlet operators and $f_{e,N-k}$ are the functions of $\mu_{e,n+e}$ and $\mu_{e,n+e+1}$ like (4.5). The equations of motion with respect to $M_e,k$ give

$$tr \mu^k_{e,n+e} - tr \mu^k_{e,n+e+1} = Nm_e \Lambda^{2N-1}_e \delta^{kN} + \frac{N - 1}{M_e} \Lambda^{2N-1}_e \delta^{k,N-1}. \quad (5.27)$$

Also, from the $\Phi_i = n+e$ equation of motion, we get $\mu_{e-1,n+e} = \mu_{e,n+e}$.

Let us gauge the diagonal subgroup of SU($N_e)$, $n+1$ and SU($N_e)$, $2n+1$. Effectively, this is an SU($N$) gauge theory coupled to the three adjoints $\Phi$, $A := \mu_{1,n+1}$ and $B := \mu_{n,2n+1}$ with the superpotential

$$W = \sum_{k=2}^N X_k \left( tr A^k - tr B^k - Nm \delta^{kN} - (N - 1)\lambda \delta^{k,N-1} \right) + tr \Phi (A - B), \quad (5.28)$$

where $m = \sum e m_e \Lambda^{2N-1}_e$, $\lambda = \sum e \frac{\Lambda^{2N-1}_e}{M_e}$ and $X_k$ are Lagrange multipliers. The F-term equation with respect to $\Phi$ gives $A = B$, while the F-term equations with respect to $X_k$ restrict the terms in the parenthesis to vanish for each $k$. Thus the supersymmetry is broken unless $m = \lambda = 0$. 

![Figure 2](image-url)
This can be seen as conditions on residues of the functions $V_N$ and $V_{N-1}$ in the curve. In general, for each holomorphic one-form $\lambda_i$ ($i = 1, \cdots, g$) on the base Riemann surface $C_{g,n}$, we can define meromorphic one-forms $V_k\lambda_i$. The sum of residues of these meromorphic one-forms must vanish in any compact Riemann surface. The above field theory constraints on parameters corresponds to the constraints on the residues of $V_k$ in the curve.

**Pinching of the curve.** Let us consider the graph with $g$ loops and $n$ SU($N_e$) gauge groups with $N_e$ flavors ($e = 1, \cdots, n$). By the decoupling of an $\mathcal{N} = 2$ vector multiplet, in some cases the graph is split into two graphs, one with $g_1$ loops and $n_1$ SU($N_{e_1}$) gauge groups with $N_{e_1}$ flavors ($e_1 = 1, \cdots, n_1$), and one with $g_2$ loops and $n_2$ SU($N_{e_2}$) gauge groups with $N_{e_2}$ flavors ($e_2 = 1, \cdots, n_2$), where $g_1 + g_2 = g$ and $n_1 + n_2 = n$. In the other cases the graph is simply reduced to the one with $g - 1$ loops and the same SU($N_e$) gauge groups.

This can be easily seen from the curve. Each SU($N_e$) gauge group corresponds to a point $p_e$ on the genus $g$ Riemann surface. The function $V_k$ has a simple pole at the point $p_e$ if $N_e - k \leq N$. In the first cases, the Riemann surface is split into two Riemann surfaces: one with genus $g_1$ and points $p_{e_1}$ ($e_1 = 1, \cdots, n_1$) and one with genus $g_2$ and points $p_{e_2}$ ($e_2 = 1, \cdots, n_2$). Accordingly, $V_k$ becomes two functions: $V_{1k}$ on the former surface with simple poles at $p_{e_1}$ if $N_e - N_{e_1} \leq k \leq N$, and $V_{2k}$ on the latter surface with simple poles at $p_{e_2}$ if $N_e - N_{e_2} \leq k \leq N$. In the second cases, $V_k$ becomes a meromorphic function on the pinched genus $g - 1$ surface whose singularity at $p_e$ is the same as the original $V_k$. It is important that $V_k$ are meromorphic functions in this analysis.

### 6 Conclusion and discussions

We have studied the low energy dynamics of the $\mathcal{N} = 1$ system consisting of copies of the $T_N$ theory and $\mathcal{N} = 1$ and $\mathcal{N} = 2$ vector multiplets. The $T_N$ theory coupled to an $\mathcal{N} = 1$ vector multiplet (with the additional flavor) displays confinement in the IR where the effective theory is described by the gauge invariant operators with quantum deformed constraints (with the dynamically-generated superpotential). The $T_N$ theory coupled to two or three $\mathcal{N} = 1$ vector multiplets is in the Coulomb phase with moduli space of vacua, whose low energy holomorphic coupling matrix is identified with the period matrix of an $\mathcal{N} = 1$ Seiberg-Witten curve. We have also generalized it to $\mathcal{N} = 1$ systems denoted by generalized quiver graphs and determined their curves. It should be emphasized that while the system is not a conventional one, we can confirm these results by using the methods widely used in the literature: holomorphy and symmetry, 't Hooft anomaly matching, and so on.

The curve of the $\mathcal{N} = 1$ systems discussed in the previous section is reminiscent of the Seiberg-Witten curve of $\mathcal{N} = 2$ gauge theory which is understood widely from the M-theoretical point of view [4, 27, 28]. So, it would be interesting to construct the $\mathcal{N} = 1$ system by using M5-branes or if one prefers, just the 6d $\mathcal{N} = (2,0)$ theory. Let us briefly discuss this here, leaving details to future work. Recall that the $T_N$ theory is obtained by putting $N$ M5-branes on a sphere with three maximal punctures. A maximal puncture is the intersections of $N$ M5-branes with a particular choice of other branes. Many other types
of punctures are known, which correspond to the intersections with other combinations of branes. The punctures are divided into two main classes, the tame ones and the wild ones. Each intersection locally preserves half of the supersymmetry. If we decide to use all the punctures to preserve the same half of the supersymmetry, we can realize $\mathcal{N} = 2$ supersymmetric theory which can be either tame [4, 15] or wild [28–34]. We can also make some of the punctures to preserve different half of the supersymmetry, realizing $\mathcal{N} = 1$ supersymmetric theory in the end. The tame cases have been analyzed before [5, 10, 35–37] and correspond to $\mathcal{N} = 1$ theories which flow to IR superconformal fixed points. The $\mathcal{N} = 1$ system analyzed above is the wild version. With the brane construction, it would be possible to derive the results in the previous sections geometrically.

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A Details of Higgsing to the SQCD results

When one puncture is closed to the minimal puncture by giving a vev to $\mu_A$ as $\langle \mu_A \rangle = \rho_\ast (\sigma^+)$, the $T_N$ theory is reduced to a bifundamental hyper multiplet $Q_i^B, \tilde{Q}_i^B$ and some Nambu-Goldstone multiplets. In this appendix we provide all the details needed to show that the deformed moduli constraints in section 3.1 and the effective superpotential 4.1 reduce to those of SU($N$) SUSY QCD with $N_f = N$ and $N_f = N + 1$ flavors respectively.

As in (3.10), the adjoint representation of SU($N$) decomposes as a representation of $SU(2) \times U(1)$ as

$$\text{adj} = \bigoplus_{j=1}^{N-2} V_{j,0} \oplus V_{0,0} \oplus V_{N-2,0, N} \oplus V_{N-2,-N},$$

(A.1)

where $V_{j,q}$ is a spin $j$ representation of SU(2) with U(1) charge $q$. Then the $N^2 - 1$ generators of SU($N$)$_A$ decomposes into $T_{j,m,0}$ ($m = -j, -j+1, \cdots, j$), $T_{U(1)}^0$ and $T_{\frac{N-2}{2},m,\pm N}$ ($m = -(N-2)/2, \cdots, (N-2)/2$), which satisfy the commutation relations (3.13). The argument below (3.13) leads to

$$\mu_A = \rho_\ast (\sigma^+) + \sum_{j=1}^{N-2} T_{j,-j,0} \mu_A^{j,-j,0} + T_{U(1)}^{0,0,0} + T_{\frac{N-2}{2},-\frac{N-2}{2},\pm N} \mu_A^{\frac{N-2}{2},-\frac{N-2}{2},\pm N}. \quad \text{(A.2)}$$

We take the matrix form of $\rho_\ast$ such that the only nonzero components are $(\rho_\ast (\sigma^\pm))_k^{k+1} \neq 0$ ($k = 1, \cdots, N-2$) and $(\rho_\ast (\sigma^3))_k^k = (N-2k)/2$ ($k = 1, \cdots, N-1$). Then
the only nonzero components of the matrix $T_{j,-j,0}$ are $(T_{j,-j,0})_{k-j}^k$ ($k = j + 1, \ldots, N - 1$).
For the matrices $T_{N-2,-N-2,\pm N}$, we take $(T_{N-2,-N-2,\pm N})^{N-1}_{N} = (T_{N-2,-N-2})^{N-1}_N = 1$ and other components are zero. Finally, $T_{U(1)} = \text{diag}(1, \ldots, 1, -(N-1))$. By using these, a matrix representation of (A.2) is given as

$$
\mu_A \sim \begin{pmatrix}
\mu_A^{0,0}, & \ldots, & (\rho_*^1)^1_2 & 0 & 0 & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
\mu_A^{N-3,-(N-3),0} & \ldots & (\rho_*^N)^N_{N-1} & 0 & 0 & 0 \\
\mu_A^{N-2,-(N-2),0} & \mu_A^{N-3,-(N-3),0} & \ldots & \mu_A^{0,0} & \mu_A^{N-2,-N-2,N} & \mu_A^{N-2,-N-2,N} \\
\mu_A^{N-2,-N-2,N} & 0 & \ldots & 0 & -(N-1) & 0
\end{pmatrix},
$$

(A.3)

where we have neglected order one coefficients of components containing $\mu_A^{j-j,0}$ ($j \geq 1$).

**SQCD with $N_f = N$.** Now let us study the constraints of SQCD with $N_f = N$: \tr \mu_A^k = \tr \mu_B^k$ ($k = 2, \ldots, N - 1$) and $\tr \mu_A^N - \tr \mu_B^N = N\Lambda^{2N}$ under the above Higgsing. One can see that

$$
\tr \mu_A^k \propto \mu_A^{k-1,-(k-1),0} + \cdots, \quad (k = 2, \ldots, N - 1)
$$

(A.4)

where the ellipsis denotes a sum of products of $\mu_A^{j-j,0}$s with $0 \leq j < k - 1$. Thus \tr $\mu_A^k = \tr \mu_B^k$ ($k = 2, \cdots, N - 1$) just determines $\mu_A^{k-1,-(k-1),0}$ in terms of $\mu_A^{0,0}$ and $\mu_B$. We do not try to determine the explicit form of them.

Next let us consider the equation

$$
P_A(-(N-1)\mu_A^{0,0,0}) - P_A(-(N-1)\mu_A^{0,0,0}) = \Lambda^{2N}.
$$

(A.5)

Relations $\tr \mu_A^k = \tr \mu_B^k$ ($k = 2, \cdots, N - 1$) guarantee that this equation is equivalent to the constraint $\tr \mu_A^N - \tr \mu_B^N = N\Lambda^{2N}$. Directly from the definition $P_A(-(N-1)\mu_A^{0,0,0}) = \det[-(N-1)\mu_A^{0,0,0} 1 - \mu_A]$ and (A.3), we get

$$
P_A(-(N-1)\mu_A^{0,0,0}) = -C(\rho)\mu_A^{N-2,-N-2,N} \mu_A^{N-2,-N-2,N} = \Lambda^{2N},
$$

(A.6)

where $C(\rho)$ is a product of components of $\rho_*^1$ defined as

$$
C(\rho) = \prod_{k=1}^{N-2} (\rho_*^1)^k_{k+1}.
$$

(A.7)

Then we obtain

$$
\det(-(N-1)\mu_A^{0,0,0} 1 - \mu_B) + C(\rho)\mu_A^{N-2,-N-2,N} \mu_A^{N-2,-N-2,N} = \Lambda^{2N}.
$$

(A.8)

Now we see that equation (A.8) is indeed the deformed moduli constraint in SQCD $\det M - B_+ B_- = \Lambda^{2N}$ (up to a normalization of $\Lambda^{2N}$) if we assume the following identification;

$$
\hat{M} = M - \frac{1}{N} \tr M = \mu_B, \quad \tr M = N(N-1)\mu_A^{0,0,0}, \quad B_\pm = \pm \mu_A^{N-2,-N-2,\pm N},
$$

(A.9)
where \( c_\pm \) are constants satisfying \( c_+ c_- = (-1)^{N-1} C(\rho) \).

Let us see this is the case by considering the operators \( Q^{i_1 i_2 i_3 i_4} \) and \( \tilde{Q}^{i_1 i_2} \). Under the above Higgsing, we expect the following identification of the bifundamental fields \( Q^{i_1 i_2 i_3} \) and \( \tilde{Q}^{i_1 i_2} \):

\[
Q^{i_1 i_2 i_3} = c'_+ Q^{i_1 i_2 i_3}, \quad \tilde{Q}^{i_1 i_2} = c'_- Q_{N-1,i_1 i_2},
\]

(A.10)

where \( c'_\pm \) are constants. The other components of \( Q^{i_1 i_2 i_3 i_4} \) and \( \tilde{Q}^{i_1 i_2} \) are constants. The other components of \( Q^{i_1 i_2 i_3 i_4} \) and \( \tilde{Q}^{i_1 i_2} \) are obtained as follows. Using (2.5), we obtain

\[
(\mu_B)^{i_1 i_2} Q^{i_1 i_2} = c'_+ (\mu_A)^{i_1 i_2} Q^{i_1 i_2} \propto Q^{i_1 i_2} + \cdots \quad (k = 1, \ldots, N-2).
\]

(A.11)

This equation gives \( Q^{k,i_1 i_2} \) \( (k = 2, \ldots, N-1) \) in terms of \( \mu_B \) and \( Q \). \( Q^{k,i_1 i_2} \) \( (k = 1, \ldots, N-2) \) are determined in a similar way. We will discuss how to obtain \( Q^{N,i_1 i_2} \) and \( Q_{N,i_1 i_2} \) later.

The meson is given as \( M^{i_1 i_2}_{j_1 j_2} \). Using (2.7), we get

\[
M^{i_1 i_2}_{j_1 j_2} = c'_+ c'_- \sum_{l=0}^{N-1} \sum_{m=0}^{N-1-l} v_1(\mu_A^{N-l-1-m}) N-1_k (\mu_B^{m})^{i_1 i_2}_{j_1 j_2} = c'_+ c'_- C(\rho) \left( \mu_B + (N-1) \mu_A^{0,0,0,1} \right)^{i_1 i_2}_{j_1 j_2},
\]

(A.12)

where we have used \( (\mu_A^{k})^{N-1} = 0 \) for \( (k = 1, \ldots, N-3) \), \( (\mu_A^{N-2})^{N-1} = C(\rho) \) and \( \sum_{l=0}^{N-1} (\mu_B^{m})^{N-1} = (N-1) C(\rho) \mu_A^{0,0,0,1} \). By setting \( c'_+ c'_- = C(\rho)^{-1} \), we precisely get the first two equations of (A.9).

The baryons are given as \( B_+ = \text{det} Q \) and \( B_- = \text{det} \tilde{Q} \). Before considering them, we first determine \( Q^{N,i_1 i_2} \) and \( Q_{N,i_1 i_2} \), which has U(1) charges \(- (N-1)\) and \( N-1 \) respectively. We use the constraints (2.8) and (2.9) for \( i_{A,1} = \cdots = i_{A,N-1} = 1 \) and \( i_{A,1} = \cdots = i_{A,N-1} = N-1 \) respectively. We get

\[
\frac{1}{(N-1)!} Q^{i_1 i_2 i_3 i_4} Q^{i_5 i_6 i_7 i_8} \cdots Q^{i_{N-i} i_{N-1} i_{N-2} i_{N-3}} = (c'_+)^{-1} Q^{i_1 i_2 i_3} Q^{i_4 i_5 i_6} \cdots Q^{i_{N-1} i_{N-2} i_{N-3}} = C_+ (\rho) Q^{N,i_1 i_2}
\]

(A.13)

and similarly

\[
\frac{1}{(N-1)!} \tilde{Q}^{i_1 i_2 i_3 i_4} \tilde{Q}^{i_5 i_6 i_7 i_8} \cdots \tilde{Q}^{i_{N-i} i_{N-1} i_{N-2} i_{N-3}} = (-1)^{N-1} (c'_-)^{-1} C_- (\rho) Q^{N,i_1 i_2}
\]

(A.14)

where we have defined

\[
C_+ (\rho) = \prod_{k=1}^{N-2} \left( \rho_+(\sigma^+)^{k} \right)^{N-1-k}, \quad C_- (\rho) = \prod_{k=1}^{N-2} \left( \rho_-(\sigma^-)^{k} \right)^{k}.
\]

(A.15)
These equations determine $Q^{N_b i B i C}$ and $Q_{N_b i B i C}$ in terms of $Q^{i B i C}$ and $\hat{Q}_{i B i C}$. The baryons can be obtained by multiplying $Q^{i B i C}$ to (A.13) and $\hat{Q}_{i B i C}$ to (A.14). Using (2.7), we obtain

$$Q^{1, i B i C} Q_{N, i B i C} = NC(\rho) \mu_A^{N - \frac{N - 2}{2} - \frac{N - 2}{2}},$$

(A.16)

$$Q^{N_b, i B i C} Q_{N_b - 1, i B i C} = NC(\rho) \mu_A^{N - \frac{N - 2}{2} - \frac{N - 2}{2} - N}.$$  

(A.17)

Then, the baryons are given as

$$B_+ = (c'_+)^N C_+ (\rho) C(\rho) \mu_A^{N - \frac{N - 2}{2} - N},$$

(A.18)

$$B_- = (-1)^{(N - 1)} (c'_-)^N C_- (\rho) C(\rho) \mu_A^{N - \frac{N - 2}{2} - N}.$$  

(A.19)

This is consistent with (A.9) if the coefficients satisfy the relation

$$(1)^{(N - 1)} (c'_+ c'_-) (C_+(\rho) C_- (\rho)) C(\rho)^2 = (-1)^{(N - 1)} C(\rho).$$

(A.20)

Using $c'_+ c'_- = C(\rho)^{-1}$ and $C_+(\rho) C_- (\rho) = C(\rho)^{N - 1}$, we can see that this is really the case. 

**SQCD with $N_f = N + 1$.** Here let us see the derivation of (4.16) which is needed to get the effective superpotential of SQCD with $N_f = N + 1$. By using the equations of motion of massive fields (4.15), the last two terms of the superpotential (4.4) is

$$B_{i_A i_B} \tilde{B}^{i_A i_B} (\mu_B)_{i_B}^{i_B} - B_{i_A i_B} \tilde{B}^{i_A i_B} (\mu_A)_{j_A}^{i_A}$$

$$= B_{N_i B} \tilde{B}^{N_i B} (\mu_B)_{i_B}^{i_B} + B_{1_{i_B}} \tilde{B}_{1_{i_B}} (\mu_B)_{i_B}^{i_B} - B_{1_{i_B}} \tilde{B}^{N_i B} (\mu_A)_{j_A}^{j_A} - B_{1_{i_B}} \tilde{B}^{1_{i_B}} (\mu_A)_{i_A}^{i_A}$$

$$= B_{N_i B} \tilde{B}^{N_i B} M^{i_B}_{j_B} - B_{N_i B} \tilde{B}^{1_{i_B}} (\mu_B)_{i_B}^{i_B} - B_{N_i B} \tilde{B}^{N_i B} (\mu_A)_{j_A}^{j_A} - B_{N_i B} \tilde{B}^{1_{i_B}} (\mu_A)_{i_A}^{i_A}$$

$$+ \left( B_{1_{i_B}} M^{i_B}_{j_B} - \sum_{i_A = 1}^{N - 1} B_{i_A j_B} \tilde{M}^{i_A}_{j_A} \right) \tilde{B}_{j_B},$$

(A.21)

where we have defined $\tilde{\mu} = \mu_A + (N - 1) \mu_A^{0,0,0}$. 

Let us determine the last term of (A.21) in terms of $B_{N - 1, i_B}$. (4.15) gives us

$$B_{i_A} = (\rho_{i_A})^{i_A} \sum_{i_A < j_A < N - 1} B_{i_A j_A} (\delta_{i_A j_A}^{i_A + 1} M - \tilde{\mu}_{i_A + 1}^{i_A + 1})$$

(A.22)

where we have omitted the indices of SU($N$)$_B$ by considering $M$ and $B_{i_A}$ as a matrix and a vector respectively, and we have used abbreviation $\rho = \rho(\sigma^+)$.

Repeatedly using this equation, we obtain

$$\sum_{0 < i_A \leq N - 1} \sum_{i_A < j_A < N - 1} B_{i_A} (\delta_{i_A}^{i_A} M - \tilde{\mu}_i^{i_A})$$

$$= B_{N - 1} \sum_{a = 0}^{N - 2} \sum_{0 < i_A < \ldots < N - 1} (\delta_{i_A + 1}^{N - 1} M - \tilde{\mu}_{i_A + 1}^{N - 1}) \cdot (\rho_{i_A + 1}^{i_A + 1} - 1)$$

$$\cdot (\delta_{i_A + 1}^{i_A + 1} M - \tilde{\mu}_{i_A + 1}^{i_A + 1}) \cdot (\rho_{i_A + 1}^{i_A + 1} - 1) \cdot \ldots \cdot (\rho_{i_{N - 1} + 1}^{i_{N - 1} + 1} - 1) \cdot (\delta_{i_{N - 1} + 1}^{i_{N - 1} + 1} M - \tilde{\mu}_{i_{N - 1} + 1}^{i_{N - 1} + 1}).$$

(A.23)
By a careful inspection of the matrix (A.3), we can see that (A.23) is summarized as

\[ C(\rho)^{-1} B_{N-1} P_{\mu'}(M), \quad (A.24) \]

where \( \mu' \) is the \( N-1 \times N-1 \) matrix \( \mu' \), with \( 1 \leq i, j \leq N-1 \), and \( P_{\mu'}(x) = \det(xI - \mu') \) is the characteristic polynomial of \( \mu' \). This is related to \( P_{\mu}(x) = \det(xI - \mu) \) as

\[ P_{\mu}(x) = x P_{\mu'}(x) - C(\rho) \mu_A^{\frac{N-2}{2}}, \quad (A.25) \]

Because of the relation \( \text{tr} \mu^k_A = \text{tr} \mu^k_B \) \( (k = 2, \ldots, N-1) \), we have

\[ P_{\mu'}(x) = P_M(x) + (x \text{ independent term}) \quad (A.26) \]

where \( P_M(x) = \det(xI - M) \). Combining the above equations, we get

\[ P_{\mu'}(x) = x^{-1}(P_M(x) - (-1)^N \det M), \quad (A.27) \]

Therefore, (A.23) is finally reduced to

\[ (-1)^{N-1} C(\rho)^{-1} B_{N-1} M^{-1} \det M, \quad (A.28) \]

where we have used \( P_M(M) = 0 \). We identify

\[ B_{i_B} = -(c'_+)^{N-1} C_+(\rho) B_{Ni_B}, \quad \bar{B}_{i_B} = (-1)^N (c'_-)^{N-1} C_-(\rho) B_{Ni_B}, \]

\[ \mathcal{M}^{i_B}_{N+1} = c'_+ \bar{B}_{i_B}, \quad \mathcal{M}^{N+1}_{i_B} = c'_- B_{Ni_B}. \quad (A.29) \]

Using \( c_+ = (c'_+)^N C_+(\rho) C(\rho) \) and \( c_- = (-1)^{N-1}(c'_-)^N C_-(\rho) C(\rho) \), the superpotential terms (A.21) is now rewritten as (4.16).

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