THE PARABOLIC INFINITE-LAPLACE EQUATION IN CARNOT GROUPS

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Abstract. By employing a Carnot parabolic maximum principle, we show existence-uniqueness of viscosity solutions to a class of equations modeled on the parabolic infinite Laplace equation in Carnot groups. We show stability of solutions within the class and examine the limit as $t$ goes to infinity.

1. Motivation

In Carnot groups, the following theorem has been established.

Theorem 1.1. [3, 16, 5] Let $\Omega$ be a bounded domain in a Carnot group and let $v : \partial \Omega \to \mathbb{R}$ be a continuous function. Then the Dirichlet problem

$$\begin{cases}
\Delta_\infty u = 0 & \text{in } \Omega \\
 u = v & \text{on } \partial \Omega
\end{cases}$$

has a unique viscosity solution $u_\infty$.

Our goal is to prove a parabolic version of Theorem 1.1 for a class of equations (defined in the next section), namely

Conjecture 1.2. Let $\Omega$ be a bounded domain in a Carnot group and let $T > 0$. Let $\psi \in C(\Omega)$ and let $g \in C(\Omega \times [0, T))$ Then the Cauchy-Dirichlet problem

$$\begin{cases}
 u_t - \Delta_\infty u = 0 & \text{in } \Omega \times (0, T), \\
u(x, 0) = \psi(x) & \text{on } \overline{\Omega}
\end{cases}$$

(1.1)

$$u(x, t) = g(x, t) & \text{on } \partial \Omega \times (0, T)$$

has a unique viscosity solution $u$.

In Sections 2 and 3, we review key properties of Carnot groups and parabolic viscosity solutions. In Section 4, we prove uniqueness and Section 5 covers existence.

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2. Calculus on Carnot Groups

We begin by denoting an arbitrary Carnot group in $\mathbb{R}^N$ by $G$ and its corresponding Lie Algebra by $g$. Recall that $g$ is nilpotent and stratified, resulting in the decomposition

$$g = V_1 \oplus V_2 \oplus \cdots \oplus V_l$$

for appropriate vector spaces that satisfy the Lie bracket relation $[V_i, V_j] = V_{i+j}$. The Lie Algebra $g$ is associated with the group $G$ via the exponential map $\exp : g \to G$. Since this map is a diffeomorphism, we can choose a basis for $g$ so that it is the identity map. Denote this basis by $X_1, X_2, \ldots, X_{n_1}, Y_1, Y_2, \ldots, Y_{n_2}, Z_1, Z_2, \ldots, Z_{n_3}$

so that

$$V_1 = \text{span}\{X_1, X_2, \ldots, X_{n_1}\}$$
$$V_2 = \text{span}\{Y_1, Y_2, \ldots, Y_{n_2}\}$$
$$V_3 \oplus V_4 \oplus \cdots \oplus V_l = \text{span}\{Z_1, Z_2, \ldots, Z_{n_3}\}.$$

We endow $g$ with an inner product $\langle \cdot, \cdot \rangle$ and related norm $\| \cdot \|$ so that this basis is orthonormal. Clearly, the Riemannian dimension of $g$ (and so $G$) is $N = n_1 + n_2 + n_3$. However, we will also consider the homogeneous dimension of $G$, denoted $Q$, which is given by

$$Q = \sum_{i=1}^{l} i \cdot \dim V_i.$$

Before proceeding with the calculus, we recall the group and metric space properties. Since the exponential map is the identity, the group law is the Campbell-Hausdorff formula (see, for example, [7]). For our purposes, this formula is given by

$$p \cdot q = p + q + \frac{1}{2}[p, q] + R(p, q)$$

where $R(p, q)$ are terms of order 3 or higher. The identity element of $G$ will be denoted by 0 and called the origin. There is also a natural metric on $G$, which is the Carnot-Carathéodory distance, defined for the points $p$ and $q$ as follows:

$$d_C(p, q) = \inf_{\Gamma} \int_0^1 \|\gamma'(t)\| dt$$

where the set $\Gamma$ is the set of all curves $\gamma$ such that $\gamma(0) = p, \gamma(1) = q$ and $\gamma'(t) \in V_1$. By Chow’s theorem (see, for example, [2]) any two points can be connected by such a curve, which means $d_C(p, q)$ is an honest metric. Define a Carnot-Carathéodory ball of radius $r$ centered at a point $p_0$ by

$$B(p_0, r) = \{ p \in G : d_C(p, p_0) < r \}.$$
In addition to the Carnot-Carathéodory metric, there is a smooth (off the origin) gauge. This gauge is defined for a point \( p = (\zeta_1, \zeta_2, \ldots, \zeta_l) \) with \( \zeta_i \in V_i \) by
\[
N(p) = \left( \sum_{i=1}^{l} \| \zeta_i \|^{\frac{2l}{l+1}} \right)^{\frac{1}{2l}}
\]
and it induces a metric \( d_N \) that is bi-Lipschitz equivalent to the Carnot-Carathéodory metric and is given by
\[
d_N(p, q) = N(p^{-1} \cdot q).
\]
We define a gauge ball of radius \( r \) centered at a point \( p_0 \) by
\[
B_N(p_0, r) = \{ p \in G : d_N(p, p_0) < r \}.
\]
In this environment, a smooth function \( u : G \to \mathbb{R} \) has the horizontal derivative given by
\[
\nabla_0 u = (X_1 u, X_2 u, \ldots, X_{n_1} u)
\]
and the symmetrized horizontal second derivative matrix, denoted by \((D^2 u)^*\), with entries
\[
((D^2 u)^*)_{ij} = \frac{1}{2} (X_i X_j u + X_j X_i u)
\]
for \( i, j = 1, 2, \ldots, n_1 \). We also consider the semi-horizontal derivative given by
\[
\nabla_1 u = (X_1 u, X_2 u, \ldots, X_{n_1} u, Y_1 u, Y_2 u, \ldots, Y_{n_2} u).
\]
Using the above derivatives, we define the \( h \)-homogeneous infinite Laplace operator for \( h \geq 1 \) by
\[
\Delta_h^\infty f = \| \nabla_0 f \|^{h-3} \sum_{i,j=1}^{n_1} X_i f X_j f X_i X_j f = \| \nabla_0 f \|^{h-3} \langle (D^2 f)^* \nabla_0 f, \nabla_0 f \rangle.
\]
Given \( T > 0 \) and a function \( u : G \times [0, T] \to \mathbb{R} \), we may define the analogous subparabolic infinite Laplace operator by
\[
u_t - \Delta_h^\infty u = 0.
\]
and we consider the corresponding equation
\[
(2.3)
\]
We note that when \( h \geq 3 \), this operator is continuous. When \( h = 3 \), we have the subparabolic infinite Laplace equation analogous to the infinite Laplace operator in \([5]\). The Euclidean analog for \( h = 1 \) has been explored in \([14]\) and the Euclidean analog for \( 1 < h < 3 \) in \([15]\).
We recall that for any open set \( \mathcal{O} \subset G \), the function \( f \) is in the horizontal Sobolev space \( W^{1,p}(\mathcal{O}) \) if \( f \) and \( X_i f \) are in \( L^p(\mathcal{O}) \) for \( i = 1, 2, \ldots, n_1 \). Replacing \( L^p(\mathcal{O}) \) by \( L^p_{\text{loc}}(\mathcal{O}) \), the space \( W^{1,p}_{\text{loc}}(\mathcal{O}) \) is defined similarly. The space \( W^{1,p}_{\text{loc}}(\mathcal{O}) \) is the closure in \( W^{1,p}(\mathcal{O}) \) of smooth functions with compact support. In addition, we recall a function \( u : G \to \mathbb{R} \) is \( C^2_{\text{sub}} \) if \( \nabla_1 u \) and \( X_i X_j u \) are continuous for all \( i, j = 1, 2, \ldots, n_1 \). Note that \( C^2_{\text{sub}} \) is not equivalent to (Euclidean) \( C^2 \). For spaces involving time, the space \( C(t_1, t_2; X) \) consists
of all continuous functions \( u : [t_1, t_2] \to X \) with \( \max_{t_1 \leq t \leq t_2} \| u(\cdot, t) \|_X < \infty \). A similar definition holds for \( L^p(t_1, t_2; X) \).

Given an open box \( O = (a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_N, b_N) \), we define the parabolic space \( O_{t_1, t_2} \) to be \( O \times [t_1, t_2] \). Its parabolic boundary is given by \( \partial_{\text{par}} O_{t_1, t_2} = (\overline{O} \times \{ t_1 \}) \cup (\partial O \times \{ t_1, t_2 \}) \).

Finally, recall that if \( G \) is a Carnot group with homogeneous dimension \( Q \), then \( G \times \mathbb{R} \) is again a Carnot group of homogeneous dimension \( Q + 1 \) where we have added an extra vector field \( \frac{\partial}{\partial t} \) to the first layer of the grading. This allows us to give meaning to notations such as \( W^{1,2}(O_{t_1, t_2}) \) and \( \mathcal{C}^{2}_{\text{sub}}(O_{t_1, t_2}) \) where we consider \( \nabla_0 u \) to be \((X_1 u, X_2 u, \ldots, X_n u, \frac{\partial u}{\partial t})\).

### 3. Parabolic Jets and Viscosity Solutions

#### 3.1. Parabolic Jets

In this subsection, we recall the definitions of the parabolic jets, as given in [4], but included here for completeness. We define the parabolic superjet of \( u(p, t) \) at the point \((p_0, t_0) \in O_{t_1, t_2}\), denoted \( P^{2,+}u(p_0, t_0) \), by using triples \((a, \eta, X) \in \mathbb{R} \times V_1 \oplus V_2 \times S^{n_1} \) so that \((a, \eta, X) \in P^{2,+}u(p_0, t_0) \) if

\[
\begin{align*}
    u(p, t) \leq u(p_0, t_0) + a(t - t_0) + \langle \eta, \frac{p_0^{-1} \cdot p}{|p_0^{-1} \cdot p|} \rangle + \frac{1}{2} \langle X_{\frac{p_0^{-1} \cdot p}{|p_0^{-1} \cdot p|}}, \frac{p_0^{-1} \cdot p}{|p_0^{-1} \cdot p|} \rangle + o(|t - t_0| + |p_0^{-1} \cdot p|^2) \quad \text{as} \quad (p, t) \to (p_0, t_0).
\end{align*}
\]

We recall that \( S^k \) is the set of \( k \times k \) symmetric matrices and \( n_i = \dim V_i \). We define \( p_0^{-1} \cdot p \) as the first \( n_1 \) coordinates of \( p_0^{-1} \cdot p \) and \( \frac{p_0^{-1} \cdot p}{|p_0^{-1} \cdot p|} \) as the first \( n_1 + n_2 \) coordinates of \( p_0^{-1} \cdot p \). This definition is an extension of the superjet definition for subparabolic equations in the Heisenberg group [4]. We define the subjet \( P^{2,-}u(p_0, t_0) \) by

\[
    P^{2,-}u(p_0, t_0) = -P^{2,+}(-u)(p_0, t_0).
\]

We define the set theoretic closure of the superjet, denoted \( \overline{P}^{2,+}u(p_0, t_0) \), by requiring \((a, \eta, X) \in \overline{P}^{2,+}u(p_0, t_0) \) exactly when there is a sequence \((a_n, p_n, t_n, u(p_n, t_n), \eta_n, X_n) \to (a, p_0, t_0, u(p_0, t_0), \eta, X) \) with the triple \((a_n, \eta_n, X_n) \in \overline{P}^{2,+}u(p_n, t_n) \). A similar definition holds for the closure of the subjet.

We may also define jets using appropriate test functions. Given a function \( u : O_{t_1, t_2} \to \mathbb{R} \) we consider the set \( \mathcal{A}u(p_0, t_0) \) given by

\[
    \mathcal{A}u(p_0, t_0) = \{ \phi \in \mathcal{C}^{2}_{\text{sub}}(O_{t_1, t_2}) : u(p, t) - \phi(p, t) \leq u(p_0, t_0) - \phi(p_0, t_0) = 0 \ \forall (p, t) \in O_{t_1, t_2} \},
\]

consisting of all test functions that touch \( u \) from above at \((p_0, t_0) \). We define the set of all test functions that touch from below, denoted \( \mathcal{B}u(p_0, t_0) \), similarly.

The following lemma relates the test functions to jets. The proof is identical to Lemma 3.1 in [4], but uses the (smooth) gauge \( \mathcal{N}(p) \) instead of Euclidean distance.

**Lemma 3.1.**

\[
    P^{2,+}u(p_0, t_0) = \{ (\phi_1(p_0, t_0), \nabla \phi(p_0, t_0), (D^2 \phi(p_0, t_0))^* : \phi \in \mathcal{A}u(p_0, t_0) \}.
\]
3.2. Jet Twisting. We recall that the set \( V_1 = \text{span}\{X_1, X_2, \ldots, X_{n_1}\} \) and notationally, we will always denote \( n_1 \) by \( n \). The vectors \( X_i \) at the point \( p \in G \) can be written as

\[ X_i(p) = \sum_{j=1}^{N} a_{ij}(p) \frac{\partial}{\partial x_j} \]

forming the \( n \times N \) matrix \( A \) with smooth entries \( A_{ij} = a_{ij}(p) \). By linear independence of the \( X_i \), \( A \) has rank \( n \). Similarly,

\[ Y_i(p) = \sum_{j=1}^{N} b_{ij}(p) \frac{\partial}{\partial x_j} \]

forming the \( n_2 \times N \) matrix \( B \) with smooth entries \( B_{ij} = b_{ij} \). The matrix \( B \) has rank \( n_2 \).

The following lemma differs from [5, Corollary 3.2] only in that there is now a parabolic term. This term however, does not need to be twisted. The proof is then identical, as only the space terms need twisting.

**Lemma 3.2.** Let \( (a, \eta, X) \in \overline{P}_{\text{eucl}}^{2,+} u(p, t) \). (Recall that \( (\eta, X) \in \mathbb{R}^N \times S^N \).) Then

\[ (a, A \cdot \eta \oplus B \cdot \eta, AXA^T + M) \in \overline{P}^{2,+} u(p, t) \]

Here the entries of the (symmetric) matrix \( M \) are given by

\[
M_{ij} = \begin{cases} 
\sum_{k=1}^{N} \sum_{l=1}^{N} \left( a_{il}(p) \frac{\partial}{\partial x_l} a_{kj}(p) + a_{jl}(p) \frac{\partial a_{ik}}{\partial x_l}(p) \right) \eta_k & \text{if } i \neq j, \\
\sum_{k=1}^{N} \sum_{l=1}^{N} a_{il}(p) \frac{\partial a_{ik}}{\partial x_l}(p) \eta_k & \text{if } i = j.
\end{cases}
\]

3.3. Viscosity Solutions. We consider parabolic equations of the form

\[ u_t + F(t, p, u, \nabla_1 u, (D^2 u)^*) = 0 \]

for continuous and proper \( F : [0, T] \times G \times \mathbb{R} \times g \times S^n \rightarrow \mathbb{R} \). We recall that \( S^n \) is the set of \( n \times n \) symmetric matrices (where \( \text{dim} \, V_1 = n \) and the derivaties \( \nabla_1 u \) and \( (D^2 u)^* \) are taken in the space variable \( p \). We then use the jets to define subsolutions and supersolutions to Equation (3.1) in the usual way.

**Definition 1.** Let \( (p_0, t_0) \in O_{t_1, t_2} \) be as above. The upper semicontinuous function \( u \) is a *parabolic viscosity subsolution* in \( O_{t_1, t_2} \) if for all \( (p_0, t_0) \in O_{t_1, t_2} \) we have \( (a, \eta, X) \in \overline{P}^{2,+} u(p_0, t_0) \) produces

\[ a + F(t_0, p_0, u(p_0, t_0), \eta, X) \leq 0. \]

A lower semicontinuous function \( u \) is a *parabolic viscosity supersolution* in \( O_{t_1, t_2} \) if for all \( (p_0, t_0) \in O_{t_1, t_2} \) we have \( (b, \nu, Y) \in \overline{P}^{2,-} u(p_0, t_0) \) produces

\[ b + F(t_0, p_0, u(p_0, t_0), \nu, Y) \geq 0. \]
A continuous function $u$ is a \textit{parabolic viscosity solution} in $\mathcal{O}_{t_1, t_2}$ if it is both a parabolic viscosity subsolution and parabolic viscosity supersolution.

\textbf{Remark 3.3.} \textit{In the special case when $F(t, p, u, \nabla_1 u, (D^2 u)^*) = F^h_\infty(\nabla_0 u, (D^2 u)^*) = -\Delta_\infty^h u$, for $h \geq 3$, we use the terms “parabolic viscosity $h$-infinite supersolution”, etc.}

In the case when $1 \leq h < 3$, the definition above is insufficient due to the singularity occurring when the horizontal gradient vanishes. Therefore, following \cite{14} and \cite{15}, we define viscosity solutions to Equation (2.3) when $1 \leq h < 3$ as follows:

\textbf{Definition 2.} Let $\mathcal{O}_{t_1, t_2}$ be as above. A lower semicontinuous function $v : \mathcal{O}_{t_1, t_2} \to \mathbb{R}$ is a \textit{parabolic viscosity $h$-infinite subsolution} of $u_t - \Delta_\infty^h u = 0$ if whenever $(p_0, t_0) \in \mathcal{O}_{t_1, t_2}$ and $\phi \in \mathcal{B}_u(p_0, t_0)$, we have

$$
\begin{align*}
\phi_t(p_0, t_0) - \Delta_\infty^h \phi(p_0, t_0) &\geq 0 & \text{when } \nabla_0 \phi(p_0, t_0) \neq 0 \\
\phi_t(p_0, t_0) - \min_{\|\eta\|=1} \langle (D^2 \phi)^*(p_0, t_0) \eta, \eta \rangle &\geq 0 & \text{when } \nabla_0 \phi(p_0, t_0) = 0 \text{ and } h = 1 \\
\phi_t(p_0, t_0) &\geq 0 & \text{when } \nabla_0 \phi(p_0, t_0) = 0 \text{ and } 1 < h < 3
\end{align*}
$$

An upper semicontinuous function $u : \mathcal{O}_{t_1, t_2} \to \mathbb{R}$ is a \textit{parabolic viscosity $h$-infinite supersolution} of $u_t - \Delta_\infty^h u = 0$ if whenever $(p_0, t_0) \in \mathcal{O}_{t_1, t_2}$ and $\phi \in \mathcal{A}_u(p_0, t_0)$, we have

$$
\begin{align*}
\phi_t(p_0, t_0) - \Delta_\infty^h \phi(p_0, t_0) &\leq 0 & \text{when } \nabla_0 \phi(p_0, t_0) \neq 0 \\
\phi_t(p_0, t_0) - \max_{\|\eta\|=1} \langle (D^2 \phi)^*(p_0, t_0) \eta, \eta \rangle &\leq 0 & \text{when } \nabla_0 \phi(p_0, t_0) = 0 \text{ and } h = 1 \\
\phi_t(p_0, t_0) &\leq 0 & \text{when } \nabla_0 \phi(p_0, t_0) = 0 \text{ and } 1 < h < 3
\end{align*}
$$

A continuous function is a \textit{parabolic viscosity $h$-infinite solution} if it is both a parabolic viscosity $h$-infinite subsolution and parabolic viscosity $h$-infinite supersolution.

\textbf{Remark 3.4.} \textit{When $1 < h < 3$, we can actually consider the continuous operator}

$$
(3.2) F^h_\infty(\nabla_0 u, (D^2 u)^*) = \begin{cases} -\|\nabla_0 u\|^h - 3 \langle (D^2 u)^*(\nabla_0 u, \nabla_0 u) = -\Delta_\infty^h u & \nabla_0 u \neq 0 \\
0 & \nabla_0 u = 0 \end{cases}
$$

\textit{Definitions 1 and 2 would then agree. (cf. \cite{15})}

We also wish to define what \cite{12} refers to as parabolic viscosity solutions. We first need to consider the set

$$
\mathcal{A}^- u(p_0, t_0) = \{ \phi \in C^2(\mathcal{O}_{t_1, t_2}) : u(p, t) - \phi(p, t) \leq u(p_0, t_0) - \phi(p_0, t_0) = 0 \text{ for } p \neq p_0, t < t_0 \}
$$

consisting of all functions that touch from above only when $t < t_0$. Note that this set is larger than $\mathcal{A} u$ and corresponds physically to the past alone playing a role in determining the present. We define $\mathcal{B}^- u(p_0, t_0)$ similarly. We then have the following definition.

\textbf{Definition 3.} An upper semicontinuous function $u$ on $\mathcal{O}_{t_1, t_2}$ is a \textit{past parabolic viscosity subsolution} in $\mathcal{O}_{t_1, t_2}$ if $\phi \in \mathcal{A}^- u(p_0, t_0)$ produces

$$
\phi_t(p_0, t_0) + F(t_0, p_0, u(p_0, t_0), \nabla_1 \phi(p_0, t_0), (D^2 \phi(p_0, t_0))^*) \leq 0.
$$
An lower semicontinuous function \( u \) on \( O_{t_1,t_2} \) is a past parabolic viscosity supersolution in \( O_{t_1,t_2} \) if \( \phi \in \mathcal{B}^{-}u(p_0,t_0) \) produces
\[
\phi_t(p_0,t_0) + F(t_0,p_0,u(p_0,t_0),\nabla_1 \phi(p_0,t_0), (D^2 \phi(p_0,t_0))^{*}) \geq 0.
\]
A continuous function is a past parabolic viscosity solution if it is both a past parabolic viscosity supersolution and subsolution.

We have the following proposition whose proof is obvious.

**Proposition 3.5.** Past parabolic viscosity sub(super-)solutions are parabolic viscosity sub(super-)solutions. In particular, past parabolic viscosity h-infinite sub(super-)solutions are parabolic viscosity h-infinite sub(sub(super-)solutions for \( h \geq 1 \).

### 3.4. The Carnot Parabolic Maximum Principle

In this subsection, we recall the Carnot Parabolic Maximum Principle and key corollaries, as proved in [6].

**Lemma 3.6 (Carnot Parabolic Maximum Principle).** Let \( u \) be a viscosity subsolution to Equation (3.1) and \( v \) be a viscosity supersolution to Equation (3.1) in the bounded parabolic set \( \Omega \times (0,T) \) where \( \Omega \) is a (bounded) domain and let \( \tau \) be a positive real parameter. Let \( \phi(p,q,t) = \varphi(p \cdot q^{-1},t) \) be a \( C^2 \) function in the space variables \( p \) and \( q \) and a \( C^1 \) function in \( t \). Suppose the local maximum
\[
M_\tau \equiv \max_{\Omega \times [0,T]} \{ u(p,t) - v(q,t) - \tau \phi(p,q,t) \}
\]
occurs at the interior point \( (p_\tau, q_\tau, t_\tau) \) of the parabolic set \( \Omega \times (0,T) \). Define the \( n \times n \) matrix \( W \) by
\[
W_{ij} = X_i(p)X_j(q)\phi(p_\tau, q_\tau, t_\tau).
\]
Let the \( 2n \times 2n \) matrix \( \mathcal{W} \) be given by
\[
\mathcal{W} = \begin{pmatrix} 0 & \frac{1}{2}(W - W^T) \\ \frac{1}{2}(W^T - W) & 0 \end{pmatrix}
\]
and let the matrix \( \mathcal{W} \in S^{2N} \) be given by
\[
\mathcal{W} = \begin{pmatrix} D_{pp}^2 \phi(p_\tau, q_\tau, t_\tau) & D_{pq}^2 \phi(p_\tau, q_\tau, t_\tau) \\ D_{qp}^2 \phi(p_\tau, q_\tau, t_\tau) & D_{qq}^2 \phi(p_\tau, q_\tau, t_\tau) \end{pmatrix}
\]
Suppose
\[
\lim_{\tau \to \infty} \tau \phi(p_\tau, q_\tau, t_\tau) = 0.
\]
Then for each \( \tau > 0 \), there exists real numbers \( a_1 \) and \( a_2 \), symmetric matrices \( X_\tau \) and \( Y_\tau \), and vector \( \Upsilon_\tau \in V_1 \oplus V_2 \), namely \( \Upsilon_\tau = \nabla_1 \varphi(p_\tau, q_\tau, t_\tau) \), so that the following hold:

A) \( (a_1, \tau \Upsilon_\tau, X_\tau) \in \mathcal{P}^{2,+} v(p_\tau, t_\tau) \) and \( (a_2, \tau \Upsilon_\tau, Y_\tau) \in \mathcal{P}^{2,-} v(q_\tau, t_\tau) \).
B) \( a_1 - a_2 = \phi_t(p_\tau, q_\tau, t_\tau) \).
Remark 4.2. The functions \( \psi \) and \( g \) may be replaced by one function \( g \in C(\overline{\Omega_T}) \). This combines conditions (E) and (BC) into one condition
\[
(4.2) \quad u(p, t) = g(p, t), \quad (p, t) \in \partial_{par} \Omega_T \quad (IBC)
\]
Theorem 4.3. Let \( \Omega \) be a bounded domain in \( G \) and let \( h \geq 1 \). If \( u \) is a parabolic viscosity subsolution and \( v \) a parabolic viscosity supersolution to Problem (4.1) then \( u \leq v \) on \( \Omega_T \equiv \Omega \times [0, T) \).

Proof. Our proof follows that of [8, Thm. 8.2] and so we discuss only the main parts. For \( \varepsilon > 0 \), we substitute \( \tilde{u} = u - \frac{\varepsilon}{T^2} \) for \( u \) and prove the theorem for

\[
(4.3) \quad u_t + F_h^\infty(\nabla_0 u, (D^2 u)^*) \leq -\frac{\varepsilon}{T^2} < 0
\]

\[
(4.4) \quad \lim_{t \uparrow T} u(p, t) = -\infty \text{ uniformly on } \overline{\Omega}
\]

and take limits to obtain the desired result. Assume the maximum occurs at \((p_0, t_0) \in \Omega \times (0, T)\) with

\[ u(p_0, t_0) - v(p_0, t_0) = \delta > 0. \]

Case 1: \( h > 1 \).
Let \( H \geq h + 3 \) be an even number. As in Equation (3.8), we let

\[
\phi(p, q) = \frac{1}{H} \sum_{i=1}^{N} ((p \cdot q^{-1})_i)^H
\]

where \((p \cdot q^{-1})_i\) is the \(i\)-th component of the Carnot group multiplication group law. Let

\[
M_\tau = u(p_\tau, t_\tau) - v(q_\tau, t_\tau) - \tau \phi(p_\tau, q_\tau)
\]

with \((p_\tau, q_\tau, t_\tau)\) the maximum point in \( \overline{\Omega} \times \overline{\Omega} \times [0, T) \) of \( u(p, t) - v(q, t) - \tau \phi(p, q) \).

If \( t_\tau = 0 \), we have

\[
0 < \delta \leq M_\tau \leq \sup_{\Omega \times \Omega} (\psi(p) - \psi(q) - \tau \phi(p, q))
\]

leading to a contradiction for large \( \tau \). We therefore conclude \( t_\tau > 0 \) for large \( \tau \). Since \( u \leq v \) on \( \partial \Omega \times [0, T) \) by Equation (BC) of Problem (4.1), we conclude that for large \( \tau \), we have \((p_\tau, q_\tau, t_\tau)\) is an interior point. That is, \((p_\tau, q_\tau, t_\tau) \in \Omega \times \Omega \times (0, T)\). Using Corollary 3.7 Property A, we obtain

\[
(a, \tau \Upsilon(p_\tau, q_\tau), \mathcal{A}_\tau) \in \mathcal{P}_{2,+}^2 u(p_\tau, t_\tau)
\]

\[
(a, \tau \Upsilon(p_\tau, q_\tau), \mathcal{Y}_\tau) \in \mathcal{P}_{2,-}^2 v(q_\tau, t_\tau)
\]

satisfying the equations

\[
a + F_h^\infty(\tau \Upsilon(p_\tau, q_\tau), \mathcal{A}_\tau) \leq -\frac{\varepsilon}{T^2}
\]

\[
a + F_h^\infty(\tau \Upsilon(p_\tau, q_\tau), \mathcal{Y}_\tau) \geq 0.
\]
We then have

\[\begin{align*}
0 < \frac{\varepsilon}{T^2} \leq (\tau \Upsilon(p_r, q_r))^2 & \left( (\mathcal{X}_r \Upsilon(p_r, q_r), \Upsilon(p_r, q_r)) - (\mathcal{Y}_r \Upsilon(p_r, q_r), \Upsilon(p_r, q_r)) \right) \\
\lesssim \tau^h((\varphi(p_r, q_r) - \frac{H^2 - 1}{H})^{2} & \varphi(p_r, q_r) - \frac{H^2 - 3}{H}) = (\tau \varphi(p_r, q_r))^h \varphi(p_r, q_r) \frac{H^2 - 3}{H}.
\end{align*}\]

Because \(H > h + 3\), we arrive at a contradiction as \(\tau \to \infty\).

If we have \(p_r = q_r\), we arrive at a contradiction since

\[F^h_\infty(\tau \Upsilon(p_r, q_r), \mathcal{X}_r) = F^h_\infty(\tau \Upsilon(p_r, q_r), \mathcal{Y}_r) = 0.\]

**Case 2: \(h = 1\).**

We follow the proof of Theorem 3.1 in [14]. We let

\[\varphi(p, q, t, s) = \frac{1}{4} \sum_{i=1}^{N} ((p \cdot q^{-1})_i)^4 + \frac{1}{2} (t - s)^2\]

and let \((p_r, q_r, t_r, s_r)\) be the maximum of

\[u(p, t) - v(q, s) - \tau \varphi(p, q, t, s)\]

Again, for large \(\tau\), this point is an interior point. If we have a sequence where \(p_r \neq q_r\), then Lemma 3.2 yields

\[\begin{align*}
(\tau(t_r - s_r), \tau \Upsilon(p_r, q_r), \mathcal{X}_r) & \in \mathcal{P}^2_{+} u(p_r, t_r) \\
(\tau(t_r - s_r), \tau \Upsilon(p_r, q_r), \mathcal{Y}_r) & \in \mathcal{P}^2_{-} v(q_r, s_r)
\end{align*}\]

satisfying the equations

\[\begin{align*}
\tau(t_r - s_r) + F^h_\infty(\tau \Upsilon(p_r, q_r), \mathcal{X}_r) & \leq -\frac{\varepsilon}{T^2} \\
\tau(t_r - s_r) + F^h_\infty(\tau \Upsilon(p_r, q_r), \mathcal{Y}_r) & \geq 0.
\end{align*}\]

As in the first case, we subtract to obtain

\[\begin{align*}
0 < \frac{\varepsilon}{T^2} \leq (\tau \Upsilon(p_r, q_r))^{-2} & \tau^2 \left( (\mathcal{X}_r \Upsilon(p_r, q_r), \Upsilon(p_r, q_r)) - (\mathcal{Y}_r \Upsilon(p_r, q_r), \Upsilon(p_r, q_r)) \right) \\
\lesssim \varphi(p_r, q_r)^{-\frac{4}{3}} (\tau \varphi(p_r, q_r) \varphi(p_r, q_r)^{-\frac{4}{3}}) = \tau \varphi(p_r, q_r).
\end{align*}\]

We arrive at a contradiction as \(\tau \to \infty\).

If \(p_r = q_r\), then \(v(q, s) - \beta^\nu(q, s)\) has a local minimum at \((q_r, s_r)\) where

\[\beta^\nu(q, s) = -\frac{T}{4} \sum_{i=1}^{N} ((p_r \cdot q^{-1})_i)^4 - \frac{T}{2} (t_r - s)^2.\]

We then have

\[0 < \varepsilon(T - s_r)^{-2} \leq \beta^\nu_s(q_r, s_r) - \min_{\|\eta\| = 1} ((D^2 \beta^\nu)^*(q_r, s_r) \eta, \eta).\]
Similarly, \( u(p,t) - \beta u(p,t) \) has a local maximum at \((p_\tau, t_\tau)\) where
\[
\beta u(p,t) = \tau \sum_{i=1}^{N} ((p \cdot q^{-1})_i)^4 + \frac{\tau}{2} (t - s_\tau)^2.
\]

We then have
\[
0 \geq \beta_i u(p_\tau, t_\tau) - \max_{\|\eta\|=1} \langle (D^2 \beta u)(p_\tau, t_\tau) \eta, \eta \rangle
\]
and subtraction gives us
\[
0 < \varepsilon (T - s_\tau)^{-2} \leq \max_{\|\eta\|=1} \langle (D^2 \beta u)(p_\tau, t_\tau) \eta, \eta \rangle - \min_{\|\eta\|=1} \langle (D^2 \beta u)(q_\tau, s_\tau) \eta, \eta \rangle \\
+ \beta_s(q_\tau, s_\tau) - \beta_i u(p_\tau, t_\tau) \\
= \tau \max_{\|\eta\|=1} \langle (D^2 p \varphi(p \cdot q^{-1}))^*(p_\tau, t_\tau) \eta, \eta \rangle \\
- \tau \min_{\|\eta\|=1} \langle (D^2 q \varphi(p_\tau \cdot q^{-1}))^*(q_\tau, s_\tau) \eta, \eta \rangle \\
+ \tau (t_\tau - s_\tau) - \tau(t_\tau - s_\tau) = 0.
\]

Here, the last equality comes from the fact that \( p_\tau = q_\tau \) and the definition of \( \varphi(p \cdot q^{-1}) \).

The comparison principle has the following consequences concerning properties of solutions:

**Corollary 4.4.** Let \( h \geq 1 \). The past parabolic viscosity \( h \)-infinite solutions are exactly the parabolic viscosity \( h \)-infinite solutions.

**Proof.** By Proposition 3.5, past parabolic viscosity \( h \)-infinite sub(super-)solutions are parabolic viscosity \( h \)-infinite sub(super-)solutions. To prove the converse, we will follow the proof of the subsolution case found in [12], highlighting the main details. Assume that \( u \) is not a past parabolic viscosity \( h \)-infinite subsolution. Let \( \phi \in A^{-u}(p_0, t_0) \) have the property that
\[
\phi_t(p_0, t_0) - \Delta^h \phi(p_0, t_0) \geq \epsilon > 0
\]
for a small parameter \( \epsilon \). We may assume \( p_0 \) is the origin. Let \( r > 0 \) and define \( S_r = B_N(r) \times (t_0 - r, t_0) \) and let \( \partial S_r \) be its parabolic boundary. Then the function
\[
\bar{\phi}_r(p,t) = \phi(p,t) + (t_0 - t)^{8!} - r^{8!} + (N(p))^{8!}
\]
is a classical supersolution for sufficiently small \( r \). We then observe that \( u \leq \bar{\phi}_r \) on \( \partial S_r \), but \( u(0, t_0) > \bar{\phi}(0, t_0) \). Thus, the comparison principle, Theorem 4.3, does not hold. Thus, \( u \) is not a parabolic viscosity \( h \)-infinite subsolution. The supersolution case is identical and omitted. \( \square \)

The following corollary has a proof similar to [14, Lemma 3.2].
Corollary 4.5. Let $u : \Omega_T \to \mathbb{R}$ be upper semicontinuous. Let $\phi \in Au(p_0, t_0)$. If

\begin{equation}
\phi_t(p_0, t_0) - \frac{1}{2} \Delta \phi(p_0, t_0) \leq 0 \quad \text{when} \quad \nabla \phi(p_0, t_0) \neq 0
\end{equation}

\begin{equation}
\phi_t(p_0, t_0) \leq 0 \quad \text{when} \quad \nabla \phi(p_0, t_0) = 0, (D^2 \phi)^*(p_0, t_0) = 0
\end{equation}

then $u$ is a viscosity supersolution to ($E$) of Problem 4.1.

Proof. The function $u$ satisfies Equation 4.1 with boundary data $\phi$.

Corollary 4.6. Let $h \geq 1$. Let $g_1, g_2 \in C(\Omega_T)$ and $u_1, u_2$ be parabolic viscosity solutions to Equation 4.1 with boundary data $g_1$ and $g_2$, respectively. Then

\[ \sup_{(p, t) \in \Omega_T} |u_1(p, t) - u_2(p, t)| \leq \sup_{(p, t) \in \partial_{par} \Omega_T} |g_1(p, t) - g_2(p, t)|. \]

Proof. The function $u^+(p, t) = u_2(p, t) + \sup_{(p, t) \in \partial_{par} \Omega_T} |g_1(p, t) - g_2(p, t)|$ is a parabolic viscosity supersolution with boundary data $g_1$ and the function $u^-(p, t) = u_2(p, t) - \sup_{(p, t) \in \partial_{par} \Omega_T} |g_1(p, t) - g_2(p, t)|$ is a parabolic viscosity subsolution with boundary data $g_1$. Moreover, $u^- \leq u_1 \leq u^+$ on $\partial_{par} \Omega_T$ and by Theorem 4.3 $u^- \leq u_1 \leq u^+$ in $\Omega_T$. \qed

Corollary 4.7. Let $h \geq 1$. Let $g \in C(\Omega_T)$. Then every parabolic viscosity solution to Problem 4.1 satisfies

\[ \sup_{(p, t) \in \Omega_T} |u(p, t)| \leq \sup_{(p, t) \in \partial_{par} \Omega_T} |g(p, t)| \]

Proof. The proof is similar to the previous corollary, but using the functions $u^\pm(p, t) = \pm \sup_{(p, t) \in \partial_{par} \Omega_T} |g(p, t)|$ instead. \qed

5. Existence of Viscosity Solutions

5.1. Parabolic Viscosity Infinite Solutions: The Continuity Case. As above, we will focus on the equations of the form (3.1) for continuous and proper $F : [0, T] \times G \times \mathbb{R} \times g \times S_{\eta_1}^n \to \mathbb{R}$ that possess a comparison principle such as Theorem 4.3 or [6, Thm. 3.6].

We will use Perron’s method combined with the Carnot Parabolic Maximum Principle to yield the desired existence theorem. In particular, the following proofs are similar to those found in [10, Chapter 2] except that the Euclidean derivatives have been replaced with horizontal derivatives and the Euclidean norms have been replaced with the gauge norm.

Lemma 5.1. Let $\mathcal{L}$ be a collection of parabolic viscosity supersolutions to (3.1) and let $u(p, t) = \inf \{v(p, t) : v \in \mathcal{L}\}$. If $u$ is finite in a dense subset of $\Omega_T = \Omega \times [0, T)$ then $u$ is a parabolic viscosity supersolution to (3.1).

Proof. First note that $u$ is lower semicontinuous since every $v \in \mathcal{L}$ is. Let $(p_0, t_0) \in \Omega_T$ and $\phi \in Au(p_0, t_0)$. Now let

\[ \psi(p, t) = \phi(p, t) - (d_N(p_0, p))^{2l} - |t - t_0|^2 \]
and notice that $\psi \in \mathcal{A}u(p_0, t_0)$. Then
\[
(u - \psi)(p, t) - (d_N(p_0, p))^{2n} - |t - t_0|^2 = (u - \phi)(p, t)
\geq (u - \phi)(p_0, t_0)
= (u - \psi)(p_0, t_0)
= 0
\]
yields
\[
(5.1) \quad (u - \psi)(p, t) \geq (d_N(p_0, p))^{2n} + |t - t_0|^2.
\]
Since $u$ is lower semicontinuous, there exists a sequence $\{ (p_k, t_k) \}$ with $t_k < t_0$ converging to $(p_0, t_0)$ as $k \to \infty$ such that
\[
(u - \psi)(p_k, t_k) \to (u - \psi)(p_0, t_0) = 0.
\]
Since $u(p, t) = \inf \{v(p, t) : v \in \mathcal{L}\}$, there exists a sequence $\{ v_k \} \subset \mathcal{L}$ such that $v_k(p_k, t_k) < u(p_k, t_k) + 1/k$ for $k = 1, 2, \ldots$. Since $v_k \geq u$, (5.1) gives us
\[
(5.2) \quad (v_k - \psi)(p, t) \geq (u - \psi)(p, t) \geq (d_N(p_0, p))^{2n} + |t - t_0|^2.
\]
Let $B \subset \Omega$ denote a compact neighborhood of $(p_0, t_0)$. Since $v_k - \psi$ is lower semicontinuous, it attains a minimum in $B$ at a point $(q_k, s_k) \in B$. Then by (5.1) and (5.2) we have
\[
(u - \psi)(p_k, t_k) + 1/k > (v_k - \psi)(p_k, t_k) \geq (v_k - \psi)(q_k, s_k) \geq (d_N(p_0, q_k))^{2n} + |s_k - t_0|^2 \geq 0
\]
for sufficiently large $k$ such that $(p_k, t_k) \in B$. By the squeeze theorem, $(q_k, s_k) \to (p_0, t_0)$ as $k \to \infty$. Let $\eta = \psi - (d_N(q_k, p))^{2n} - |s_k - t|^2$. Then $\eta \in \mathcal{A}v_k(q_k, s_k)$ and we have that
\[
\eta(q_k, s_k) + F(s_k, q_k, v_k(q_k, s_k), \nabla_1 \psi(q_k, s_k), (D^2 \psi(q_k, s_k))^*) \geq 0.
\]
This implies
\[
\psi_t(q_k, s_k) + F(s_k, q_k, v_k(q_k, s_k), \nabla_1 \psi(q_k, s_k), (D^2 \psi(q_k, s_k))^*) \geq 0.
\]
Letting $k \to \infty$ yields
\[
\phi_t(p_0, t_0) + F(t_0, p_0, u(p_0, t_0)\nabla_1 \phi(p_0, t_0), (D^2 \phi(p_0, t_0))^*) \geq 0.
\]
and that $u$ is a parabolic viscosity supersolution as desired. $\square$

A similar argument yields the following.

**Lemma 5.2.** Let $\mathcal{L}$ be a collection of parabolic viscosity subsolutions to (3.1) and let $u(p, t) = \sup \{v(p, t) : v \in \mathcal{L}\}$. If $u$ is finite in a dense subset of $\Omega_T$ then $u$ is a parabolic viscosity subsolution to (3.1).

For the following lemmas, we need to recall the following definition.
Definition 4. The upper and lower semi-continuous envelopes of a function $u$ are given by

$$u^*(p, t) := \limsup_{r \searrow 0} \{ u(q, s) : |q^{-1}p|_g + |s - t| \leq r \}$$

and

$$u_*(p, t) := \liminf_{r \searrow 0} \{ u(q, s) : |q^{-1}p|_g + |s - t| \leq r \},$$

respectively.

Lemma 5.3. Let $h$ be a parabolic viscosity supersolution to (3.1) in $\Omega_T$. Let $S$ be the collection of all parabolic viscosity subsolutions $v$ of (3.1) satisfying $v \leq h$. If for $\hat{v} \in S$, $\hat{v}_*$ is not a parabolic viscosity supersolution of (3.1) then there is a function $w \in S$ and a point $(p_0, t_0)$ such that $\hat{v}(p_0, t_0) < w(p_0, t_0)$.

Proof. Let $\hat{v} \in S$ such that $\hat{v}_*$ is not a parabolic viscosity supersolution of (3.1). Then there exists $(\hat{p}, \hat{t}) \in \Omega_T$ and $\phi \in A\hat{v}_*(\hat{p}, \hat{t})$ such that

$$(\hat{v}_* - \phi)(p, t, \hat{v}_*(p, t), \nabla_1 \phi(p, t), (D^2 \phi(p, t))^*) > 0.$$ (5.3)

Let

$$\psi(p, t) = \phi(p, t) - (d_N(\hat{p}, p))^{2n} - |t - \hat{t}|^2$$

and notice that $\psi \in A\hat{v}_*(\hat{p}, \hat{t})$. As in Lemma 5.1.

$$(\hat{v}_* - \psi)(p, t) \geq (d_N(\hat{p}, p))^{2n} + |t - \hat{t}|^2.$$ (5.4)

Let $B$ denote a compact neighborhood of $(\hat{p}, \hat{t})$ and let

$$B_{\epsilon} = B \cap \{(p, t) : (d_N(\hat{p}, p))^{2n} \leq k\epsilon \text{ and } |t - \hat{t}|^2 \leq k\epsilon \}.$$ 

Since $\hat{v} \in S$, we have that $\hat{v} \leq h$ and thus $\psi(\hat{p}, \hat{t}) = \hat{v}_*(\hat{p}, \hat{t}) \leq \hat{v}(\hat{p}, \hat{t}) \leq h(\hat{p}, \hat{t})$. However, if $\psi(\hat{p}, \hat{t}) = h(\hat{p}, \hat{t})$, then $\psi \in Ah(\hat{p}, \hat{t})$ and inequality (5.3) would be contradictory. Thus,

$$\psi(\hat{p}, \hat{t}) < h(\hat{p}, \hat{t}).$$

Since $\psi$ is continuous and $h$ is lower semicontinuous, there exists $\epsilon > 0$ such that

$$\psi(p, t) + 4\epsilon \leq h(p, t)$$

for $(p, t) \in B_{2\epsilon}$. Notice that $\psi + 4\epsilon$ is a subsolution of (3.1) on the interior of $B_{2\epsilon}$. Further, by (5.4)

$$(\hat{v}(p, t) \geq \hat{v}_*(p, t) \geq \psi(p, t) + 4\epsilon \text{ for } (p, t) \in B_{2\epsilon} \setminus B_{\epsilon}.$$ (5.5)

We now define $\omega$ by

$$\omega = \begin{cases} \max\{\psi(p, t) + 4\epsilon, \hat{v}(p, t)\} & (p, t) \in B_{\epsilon} \\ \hat{v}(p, t) & (p, t) \in \Omega_T \setminus B_{\epsilon} \end{cases}$$

But by (5.5)

$$\omega(p, t) = \max\{\psi(p, t) + 4\epsilon, \hat{v}(p, t)\} \text{ for } (p, t) \in B_{2\epsilon},$$
not just for \((p, t) \in B_\epsilon\). Then by Lemma 5.2, \(\omega\) is a subsolution in the interior of \(B_{2\epsilon}\) and thus a subsolution in \(\Omega_T\). Therefore, \(\omega \in \mathcal{S}\). Since

\[
0 = (\hat{v} - \psi)(\hat{p}, \hat{t}) = \lim_{r \to 0} \inf_{(p, t) \in B_r} \{(\hat{v} - \psi)(p, t) : (p, t) \in B_r\}
\]

there is a point \((p_0, t_0) \in B_\epsilon\) that satisfies

\[
\hat{v}(p_0, t_0) - \psi(p_0, t_0) < 4\epsilon
\]

which yields

\[
\hat{v}(p_0, t_0) < \psi(p_0, t_0) + 4\epsilon = \omega(p_0, t_0).
\]

Thus, we have constructed \(\omega \in \mathcal{S}\) that satisfies \(\hat{v}(p_0, t_0) < \omega(p_0, t_0)\).

We then have the following existence theorem concerning parabolic viscosity solutions.

**Theorem 5.4.** Let \(f\) be a parabolic viscosity subsolution to \((3.1)\) and \(g\) be a parabolic viscosity supersolution to \((3.1)\) satisfying \(f \leq g\) on \(\Omega_T\) and \(f_* = g^*\) on \(\partial_{\text{par}} O_{0,T}\). Then there is a parabolic viscosity solution \(u\) to \((3.1)\) satisfying \(u \in C(\Omega_T)\). Explicitly, there exists a unique parabolic viscosity infinite solution to Problem 4.1 when \(h > 1\).

**Proof.** Let

\[
S = \{\nu : \nu \text{ is a parabolic viscosity subsolution to } (3.1) \text{ in } \Omega_T \text{ with } \nu \leq g \text{ in } \Omega_T\}
\]

and

\[
u(p, t) = \sup \{\nu(p, t) : \nu \in S\}.
\]

Since \(f \leq g\), the set \(S\) is nonempty. Notice that \(f \leq u \leq g\) by construction. By Lemma 5.2, \(u\) is a parabolic viscosity subsolution. Suppose \(u_*\) is not a parabolic viscosity supersolution. Then by Lemma 5.3 there exists a function \(w \in S\) and a point \((p_0, t_0) \in \Omega_T\) such that \(u(p_0, t_0) < w(p_0, t_0)\). But this contradicts the definition of \(u\) at \((p_0, t_0)\). Thus \(u_*\) is a parabolic viscosity supersolution. By our assumptions on \(f\) and \(g\) on \(\partial_{\text{par}} O_{0,T}\),

\[
u = u_* \leq g^* = f_* \leq u_*
\]

on \(\partial_{\text{par}} O_{0,T}\). Then by the (assumed) comparison principle, \(u \leq u_*\) on \(\Omega_T\). Thus we have \(u\) is a parabolic viscosity solution such that \(u \in C(\Omega_T)\).

\[
\square
\]

5.2. **The \(h = 1\) case.** We begin by recalling the definition of upper and lower relaxed limit of a function. \([8][10]\).
**Definition 5.** For \( \varepsilon > 0 \), consider the function \( h_\varepsilon : O_T \subset G \to \mathbb{R} \). The upper relaxed limit \( \overline{h}(p, t) \) and the lower relaxed limit \( \underline{h}(p, t) \) are given by

\[
\overline{h}(p, t) = \lim_{\hat{p} \to p, \hat{t} \to t, \varepsilon \to 0} h_\varepsilon(\hat{p}, \hat{t}) = \lim_{\varepsilon \to 0} \sup_{0 < \delta < \varepsilon} \{ h_\delta(\hat{p}, \hat{t}) : O_T \cap B_\varepsilon(\hat{p}, \hat{t}) \},
\]

and

\[
\underline{h}(p, t) = \lim_{\hat{p} \to p, \hat{t} \to t, \varepsilon \to 0} h_\varepsilon(\hat{p}, \hat{t}) = \lim_{\varepsilon \to 0} \inf_{0 < \delta < \varepsilon} \{ h_\delta(\hat{p}, \hat{t}) : O_T \cap B_\varepsilon(\hat{p}, \hat{t}) \}.
\]

Taking the relaxed limits as \( h \to 1^+ \) of the operator \( F^h_\infty(\nabla_0 u, (D^2 u)^*) \) in Equation 3.2, we have via the continuity of the operator

\[
F^1_\infty(\nabla_0 u, (D^2 u)^*) = F^1_\infty(\nabla_0 u, (D^2 u)^*) = \left\{ \begin{array}{ll}
-\|\nabla_0 u\|^2((D^2 u)^* \nabla_0 u, \nabla_0 u) & \nabla_0 u \neq 0 \\
0 & \nabla_0 u = 0.
\end{array} \right.
\]

We give this operator the label \( F(\nabla_0 u, (D^2 u)^*) \). Consider the relaxed limits \( \overline{u}(p, t) \) and \( \underline{u}(p, t) \) of the sequence of unique (continuous) viscosity solutions to Problem 4.1 \( \{ u_h(p, t) \} \) as \( h \to 1^+ \). By [10, Thm 2.2.1], we have \( \overline{u}(p, t) \) is a viscosity subsolution and \( \underline{u}(p, t) \) is a viscosity supersolution to

\[
u_t + F(\nabla_0 u, (D^2 u)^*) = 0.
\]

We have the following comparison principle, whose proof is similar to Theorem 4.3 in the case to \( h = 1 \) and is omitted.

**Lemma 5.5.** Let \( \Omega \) be a bounded domain in \( G \). If \( u \) is a parabolic viscosity subsolution and \( v \) a parabolic viscosity supersolution to

\[
u_t + F(\nabla_0 u, (D^2 u)^*) = 0,
\]

then \( u \leq v \) on \( \Omega_T = \Omega \times [0, T) \).

**Corollary 5.6.** \( \overline{u}(p, t) = \underline{u}(p, t) \).

**Proof.** By construction, \( \underline{u}(p, t) \leq \overline{u}(p, t) \). By the Lemma, \( \underline{u}(p, t) \geq \overline{u}(p, t) \).

Using the corollary, we will call this common relaxed limit \( u^1(p, t) \). By [10, Chapter 2] and [8, Section 6], it is continuous and the sequence \( \{ u_h(p, t) \} \) converges locally uniformly to \( u^1(p, t) \) as \( h \to 1^+ \).

We then have the following theorem.

**Theorem 5.7.** There exists a unique parabolic viscosity infinite solution to Problem 4.1 when \( h = 1 \).

**Proof.** Let \( \{ u_h(p, t) \} \) and \( u^1(p, t) \) be as above. Let \( \{ h_j \} \) be a subsequence with \( h_j \to 1^+ \) where \( u_h(p, t) \to u^1(p, t) \) uniformly. We may assume \( h_j < 3 \).
Let \( \phi \in \mathcal{A}u_1(p_0, t_0) \). Using the uniform convergence, there is a sequence \( \{p_j, t_j\} \to (p_0, t_0) \) so that \( \phi \in \mathcal{A}u_{h_j}(p_j, t_j) \). If \( \nabla_0 \phi(p_0, t_0) \neq 0 \), we have \( \nabla_0 \phi(p_j, t_j) \neq 0 \) for sufficiently large \( j \). We then have
\[
\phi_t(p_j, t_j) - \Delta_{\infty}^{h_j} \phi(p_j, t_j) \leq 0
\]
and letting \( j \to \infty \) yields
\[
\phi_t(p_0, t_0) - \Delta_{\infty}^1 \phi(p_0, t_0) \leq 0.
\]
Suppose \( \nabla_0 \phi(p_0, t_0) = 0 \). By Corollary 6.2 we may assume \( (D^2 \phi)^*(p_0, t_0) = 0 \). Suppose passing to a subsequence if needed, we have \( \nabla_0 \phi(p_j, t_j) \neq 0 \). Then
\[
\phi_t(p_j, t_j) - \max_{||\eta||=1} \langle (D^2 \phi)^*(p_j, t_j) \eta, \eta \rangle \leq \phi_t(p_j, t_j) - \Delta_{\infty}^{h_j} \phi(p_j, t_j) \leq 0.
\]
Letting \( j \to \infty \) yields
\[
\phi_t(p_0, t_0) = \phi_t(p_j, t_j) - \max_{||\eta||=1} \langle (D^2 \phi)^*(p_0, t_0) \eta, \eta \rangle \leq 0.
\]
In the case \( \nabla_0 \phi(p_j, t_j) = 0 \), since \( h_j < 3 \), we have \( \phi_t(p_j, t_j) \leq 0 \) and letting \( j \to \infty \) yields \( \phi_t(p_0, t_0) \leq 0 \). We conclude that \( u_1 \) is a parabolic viscosity \( h \)-infinite subsolution. Similarly, \( u_1 \) is a parabolic viscosity \( h \)-infinite supersolution. \( \square \)

6. The limit as \( t \to \infty \).

We now focus our attention on the asymptotic limits of the parabolic viscosity \( h \)-infinite solutions. We wish to show that for \( 1 \leq h \), we have the (unique) viscosity solution to
\[
u_t - \Delta_{\infty}^h \nu = 0
\]
approaches the viscosity solution of \( -\Delta_{\infty}^h \nu = 0 \) as \( t \to \infty \). Our goal is the following theorem:

**Theorem 6.1.** Let \( h > 1 \) and \( u \in C(\overline{\Omega} \times [0, \infty)) \) be a viscosity solution of
\[
\begin{align*}
\left\{ \begin{array}{l}
u_t - \Delta_{\infty}^h \nu = 0 \quad \text{in } \Omega \times (0, \infty), \\
u(p, t) = g(p) \quad \text{on } \partial_{\text{par}}(\Omega \times (0, \infty))
\end{array} \right.
\end{align*}
\]
with \( g : \overline{\Omega} \to \mathbb{R} \) continuous and assuming that \( \partial \Omega \) satisfies the property of positive geometric density (see [12] pg. 2909). Then \( u(p, t) \to U(p) \) uniformly in \( \Omega \) as \( t \to \infty \) where \( U(p) \) is the unique viscosity solution of \( -\Delta_{\infty}^h U = 0 \) with the Dirichlet boundary condition \( \lim_{q \to p} U(q) = g(p) \) for all \( p \in \partial \Omega \).

We first must establish the uniqueness of viscosity solutions to the limit equation. Note that for future reference, we include the case \( h = 1 \).

**Theorem 6.2.** Let \( 1 \leq h < \infty \) and let \( \Omega \) be a bounded domain. Let \( u \) be a viscosity subsolution to \( \Delta_{\infty}^h u = 0 \) and let \( v \) be a viscosity supersolution to \( -\Delta_{\infty}^h u = 0 \). Then,
\[
\sup_{p \in \Omega} (u(p) - v(p)) = \sup_{p \in \partial \Omega} (u(p) - v(p)).
\]
Proof. Let $u$ be a viscosity subsolution to $-\Delta^h_\infty u = 0$. Then choose $\phi \in C^2_{\text{sub}}(\Omega)$ such that $0 = \phi(p_0) - u(p_0) < \phi(p) - u(p)$ for $p \in \Omega$, $p \neq p_0$. If $\|\nabla_0 \phi(p_0)\| = 0$, then $-\langle (D^2\phi)\ast(p_0)\nabla_0 \phi(p_0), \nabla_0 \phi(p_0) \rangle = 0 \leq 0$. If $\|\nabla_0 \phi(p_0)\| \neq 0$, we then have

$$-\Delta^h_\infty \phi(p_0) = -\|\nabla_0 \phi(p_0)\|^h \langle (D^2\phi)\ast(p_0)\nabla_0 \phi(p_0), \nabla_0 \phi(p_0) \rangle \leq 0.$$  

Dividing, we have $-\langle (D^2\phi)\ast(p_0)\nabla_0 \phi(p_0), \nabla_0 \phi(p_0) \rangle \leq 0$. In either case, $u$ is a viscosity subsolution to $-\Delta^h_\infty u = 0$. Similarly, $v$ is a viscosity supersolution to $-\Delta^3_\infty u = 0$. The theorem follows from the corresponding result for $-\Delta^3_\infty u = 0$ in [5, 3, 16]. \hfill $\square$

We state some obvious corollaries:

**Corollary 6.3.** Let $1 \leq h < \infty$ and let $g : \partial \Omega \to \mathbb{R}$ be continuous. Then there exists exactly one solution to

$$\begin{cases} -\Delta^h_\infty u = 0 & \text{in } \Omega \\ u = g & \text{on } \partial \Omega. \end{cases}$$

**Corollary 6.4.** Let $1 \leq h < \infty$ and let $g : \partial \Omega \to \mathbb{R}$ be continuous. The unique viscosity solution to

$$\begin{cases} -\Delta^h_\infty u = 0 & \text{in } \Omega \\ u = g & \text{on } \partial \Omega \end{cases}$$

is the unique viscosity solution to

$$\begin{cases} -\Delta^3_\infty u = 0 & \text{in } \Omega \\ u = g & \text{on } \partial \Omega. \end{cases}$$

Our method of proof for Theorem 6.1 follows that of [12, Theorem 2], the core of which hinges on the construction of a parabolic test function from an elliptic one. In order to construct such a parabolic test function, we need to examine the homogeneity of Equation (6.1). A quick calculation shows that for a fixed $h > 1$, $k^{1-h}u(x, kt)$ is a $C^2_{\text{sub}}$ solution to Equation (6.1) if $u(x, t)$ is a $C^2_{\text{sub}}$ solution. A routine calculation then shows parabolic viscosity $h$-infinite solutions share this homogeneity. We use this property in the following lemma, the proof of which can be found in [9, pg. 170]. (Also, cf. [6, Lemma 6.2] and [12].)

**Lemma 6.5.** Let $u$ be as in Theorem 6.1 and $h > 1$. Then for every $(x, t) \in \Omega \times (0, \infty)$ and for $0 < T < t$, we have

$$|u(x, t-T) - u(x, t)| \leq \frac{2 \|g\|_{\infty, \Omega}}{h-1} \left(1 - \frac{T}{t}\right) \frac{h}{t} T.$$  

Proof. [Theorem 6.1] Fix $h > 1$. Let $u$ be a viscosity solution of (6.1). The results of [9, Chapter III] imply that the family $\{u(\cdot, t) : t \in (0, \infty)\}$ is equicontinuous. Since it is uniformly bounded due to the boundedness of $g$, Arzela-Ascoli’s theorem yields that there exists a sequence $t_j \to \infty$ such that $u(\cdot, t_j)$ converge uniformly in $\Omega$ to a function $U \in C(\Omega)$ for which $U(p) = g(p)$ for all $p \in \partial \Omega$. Since it is known from [5, Lemma 5.5] that the Dirichlet problem for the subelliptic $p$-Laplace equation possesses a unique solution, it is enough to show that $U$ is a viscosity $p$-subsolution to $-\Delta_p U = 0$ on $\Omega$. With
that in mind, let \( p_0 \in \Omega \) and choose \( \phi \in C^2_{\text{sub}}(\Omega) \) such that \( 0 = \phi(p_0) - U(p_0) < \phi(p) - U(p) \) for \( p \in \Omega, p \neq p_0 \). Using the uniform convergence, we can find a sequence \( p_j \to p_0 \) such that \( u(t, p_j) - \phi \) has a local maximum at \( p_j \). Now define

\[
\phi_j(p, t) = \phi(p) + C \left( \frac{t}{t_j} \right)^{\frac{h}{t_j}} - \phi(p) - C \left( \frac{t}{t_j} \right)^{\frac{h}{t_j}} \]

where \( C = 2||g||_{\infty, \Omega}/(h - 1) \). Note that \( \phi_j(p, t) \in C^2_{\text{sub}}(\Omega \times (0, \infty)) \). Then using Lemma 6.5,

\[
u(p_j, t_j) - \phi(p_j, t_j) 
\geq u(p, t) - \phi(p) - C \left( \frac{t}{t_j} \right)^{\frac{h}{t_j}} - \phi(p) \]

for any \( p \in \Omega \) and \( 0 < t < t_j \). Thus we have that \( \phi_j \) is an admissible test function at \( (p_j, t_j) \) on \( \Omega \times [0, T] \). Therefore,

\[
(\phi_j)_t(p_j, t_j) - \Delta^h_{\infty} \phi_j(p_j, t_j) \leq 0.
\]

This yields

\[
-\Delta^h_{\infty} \phi(p_j) \leq \frac{C}{t_j}.
\]

The theorem follows by letting \( j \to \infty \).

Combining the results of the previous sections, we have the following theorem:

**Theorem 6.6.** The following diagram commutes:

\[
u(h, t) - \Delta^h_{\infty} u(h, t) = 0 \quad \xrightarrow{h \to 1} \quad u^1_{(t)} - \Delta^1_{\infty} u^1_{(t)} = 0
\]

\[
\begin{array}{ccc}
\downarrow & & \downarrow \\
\lim_{h \to 1^{+}} & & \lim_{t \to \infty}
\end{array}
\]

\[
-\Delta^h_{\infty} u^{h, \infty} = 0 \quad \xrightarrow{h \to 1^{+}} \quad -\Delta^1_{\infty} u^{1, \infty} = 0
\]

**Proof.** By Theorem 6.1, Corollary 6.4, and Theorem 5.7, the top, bottom and left limits exist, with the left limit being a uniform limit. By results of iterated limits (see, for example, [1]), we have the fourth limit exists, as does the full limit. In particular,

\[
\lim_{h \to 1^{+}} \lim_{t \to \infty} u^{h, t} = \lim_{h \to 1^{+}} \lim_{t \to \infty} u^{h, t} = \lim_{t \to \infty} \lim_{h \to 1^{+}} u^{h, t} = u^{1, \infty}
\]

\[
\Box
\]

**References**

[1] Bartle, Robert G. *The Elements of Real Analysis*; Second Edition, John Wiley & Sons: Hoboken, NJ, 1976.

[2] Bellaïche, André. The Tangent Space in Sub-Riemannian Geometry. In *Sub-Riemannian Geometry*; Bellaïche, André., Risler, Jean-Jacques., Eds.; Progress in Mathematics; Birkhäuser: Basel, Switzerland.1996; Vol. 144, 1–78.
[3] Bieske, Thomas. On Infinite Harmonic Functions on the Heisenberg Group. Comm. in PDE. 2002, 27 (3&4), 727–762.
[4] Bieske, Thomas. Comparison principle for parabolic equations in the Heisenberg Group. Electron. J. Diff. Equns. 2005, 2005 (95), 1–11.
[5] Bieske, Thomas. A Sub-Riemannian Maximum Principle and its application to the p-Laplacian in Carnot Groups. Ann. Acad. Sci. Fenn. 2012, 37, 119–134.
[6] Bieske, Thomas; Martin, Erin. The parabolic p-Laplace equation in Carnot groups. Ann. Acad. Sci. Fenn. 2014, 39, 605–623.
[7] Bourbaki, Nicolas, Lie Groups and Lie Algebras, Chapters 1–3, Elements of Mathematics, Springer-Verlag, 1989.
[8] Crandall, Michael.; Ishii, Hitoshi.; Lions, Pierre-Louis. User’s Guide to Viscosity Solutions of Second Order Partial Differential Equations. Bull. of Amer. Math. Soc. 1992, 27 (1), 1–67.
[9] DiBenedetto, Emmanuele. Degenerate Parabolic Equations; Springer-Verlag: New York, 1993.
[10] Giga, Yoshikazu. Surface Evolution Equations: A Level Set Approach; Monographs in Mathematics (99); Birkhäuser Verlag: Basel, Switzerland, 2006.
[11] Haller, E. Comparison Principles for Fully Nonlinear Parabolic Equations and Regularity Theory for Weak Solutions of Parabolic Systems in Carnot Groups. Ph.D. Dissertation, University of Arkansas (2008).
[12] Juutinen, Petri. On the Definition of Viscosity Solutions for Parabolic Equations. Proc. Amer. Math. Soc. 2001, 129 (10), 2907–2911.
[13] Juutinen, Petri.; Lindqvist, Peter.; Manfredi, Juan. On the Equivalence of Viscosity Solutions and Weak Solutions for a Quasi-linear Equation. Siam. J. Math. Anal. 2001, 33 (3), 699–717.
[14] Juutinen, Petri.; Kawohl, Bernd. On the Evolution governed by the Infinite Laplacian. Math. Ann. 2006, 335 (4), 819–851.
[15] Portilheiro, Manuel; Vázquez, Juan Luis. Degenerate homogeneous parabolic equations associated with the infinity-Laplacian. Calc. Var. Partial Differential Equations 2013, 46, (3 & 4), 705–724.
[16] Wang, C.Y. The Aronsson equation for gradient minimizers of $L^\infty$-functionals associated with vector fields satisfying Hörmander’s condition. Trans. Amer. Math. Soc. 2007, 359, 91–113.