Non-monotonic spontaneous magnetization in a Sznajd-like Consensus Model

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Abstract. Ising or Potts models \cite{1} of ferromagnetism have been widely used to describe locally interacting social or economic systems\cite{2}\cite{3}\cite{4}. We consider a related model, introduced by Sznajd\cite{5}\cite{6} to describe the evolution of consensus in a society. In this model, the opinion or state of any spins can only be changed through the influence of neighbouring pairs of similarly aligned spins. Such pairs can polarize their neighbours. We show that, assuming the global dynamics evolve in a synchronous manner, the two-state Sznajd model exhibits a non-monotonically decreasing overall orientation that has a maximum value when the system is subject to a finite value of noise. Reinterpreting the model in terms of opinions within a society we predict that consensus can be increased by the addition of an appropriate amount of random noise. These features are explained by the presence of islands of complete orientation that are stable in the absence of noise but removed via the presence of added noise.

Key words: Statistical Mechanics, cellular automata, phase transitions, opinion dynamics

Introduction

Simple cellular automaton based models, such as the Ising or Potts models, are useful not only to understand physical phenomena such as ferromagnetism \cite{1}, but also to study the effect of interactions between human beings in a society. In this way we may cast light on the way opinions or behaviours spread throughout society.

We examine a cellular automaton consensus model first introduced by Sznajd. The model is based on the idea that, within human societies, it is generally easier to change someone’s opinion by acting within from within a group than by acting alone. To quote Abraham Lincoln: 'United we stand, divided we fall'.
The simplest version of the Sznajd model is implemented on a two-dimensional lattice. Each site carries a spin, \( S \) that may be either up or down. This represents either a positive or negative opinions on any question. Two neighbouring parallel spins, representing for instance two neighbouring people sharing the same opinion, in the model, are able to convince their neighbours of this opinion. If these two neighbours are not parallel or in step, then they have no influence on their neighbours. Allowing such a system to evolve from one time step to another via a random sequential updating mechanism, always leads after a sufficiently long time to complete orientation of spins which for the social systems is analogous to complete consensus providing that the initial net orientation of spins is greater than zero.

However, if the random sequential updating is replaced by a synchronous updating mechanism the possibility of reaching total alignment is reduced quite dramatically. Now, updating is performed by going systematically through the lattice to find the first member of the pair, then choosing randomly the second member of the pair within the neighbourhood of the first. Having in this way completed the assembly of pairs, each spin then orients itself according to its neighbours at time step \( t \). Parallel pairs of spins will induce their neighbours to turn to their same state. However a single spin may often belong, simultaneously, to the neighbourhood of more than one couple (of like-oriented spins). In this case, if the neighbouring pairs have different orientations, the state of the individual spin does not change. It is also possible to assume that each spin has a memory of its past orientations. The introduction of such a mechanism clearly helps overcome the frustration of individual spins [9] however we shall not consider this aspect in this short note.

With synchronous updating only if the initial net orientation of spins is above a critical value (that in turn depends on the lattice size \( L \) [8] and memory length \( T \) [9]) does the system reach complete orientation or complete consensus. Below the critical value, the system evolves to a partial orientation or consensus.

This phenomenon has, thus far, been studied at zero temperature or the absence of noise. In this note we examine the influence of noise in the two dimensional Sznajd model where updating is synchronous.

**Synchronous updating in the Sznajd Model with added noise**

We define the Magnetization in the following way
\[ M = \frac{|(N_{\text{up}}) - N_{\text{down}}|}{(N_{\text{up}}) + N_{\text{down}}} \]

where \( N_{\text{up}} \) is the number of up spins and \( N_{\text{down}} \) is the number of down spins and, as we have already remarked, neglect memory effects. During the evolution of the system from the initial random state into a stable configuration, we allow each spin to flip randomly with probability \( q \) where \( 0 < q < 0.5 \).

Figure 1 shows the variation on a log-log scale of \( 1 - Mc \) with lattice size, \( L \), in the absence of noise or zero temperature. When the initial net magnetisation, \( M(0) > Mc \) the system evolves to complete consensus \( (M(\infty) = 1) \). When the initial net magnetisation, \( M(0) < Mc \) the system never reaches complete consensus \( (M(\infty) < 1) \).

Figures 2 shows the effect of temperature or noise on the final magnetisation, \( M(\infty) \) for lattice size of 50. Figures 3 and 4 are similar results for lattice sizes of 100 and 250 respectively. When \( M(0) > Mc \) the final orientation, \( M(\infty) \) decreases monotonically as \( q \) increases. When \( M(0) < Mc \) the final orientation \( M(\infty) \) displays non-monotonic behaviour as a function of \( q \). For very small values of \( q \) (typically below 0.005), \( M \) increases and may reach values close (but still below) 1, for intermediate values of \( q \) (roughly between 0.006 and 0.06) \( M(\infty) \) may decrease slowly or even display oscillations depending on the value of the initial magnetization. For larger values of \( q \), \( M(\infty) \) quickly decays to zero. The results, captured in Figures 2, 3 and 4 for different values of lattice size \( L \), have been obtained using Monte Carlo simulations and averaging over ensembles of 500 realizations for each set of the three parameters \( \{L, M(0), q\} \).

**Microstructure**

To gain further insight into this non-monotonic behaviour, we began with a randomly selected configuration of spins on the lattice, having a fraction \( p \) up and \( 1-p \) down (say \( p=0.4 \), i.e. initial magnetization \( M(0)=0.2 \)), and no added random noise \( (q=0) \). After only a few hundred iterations, the system evolved into an archipelago-like structure. Spins could be observed aligned in the manner of islands located in a sea of opposite orientation. For values of \( p \) close to 0.5, the 'landscape' is irregular and jagged. The geometrical structure of these small islands is now crucial. Spins surrounded by four others of the same sign (so forming cross-shaped islands) form a stable state since, at any time, they are prevented, at zero temperature, from flipping by frustration. Islands made up of one or more copies (even overlapping) of such a cross-shaped structure are destined to last and no total orientation
can be reached in the system. However now the addition of random noise can flip a fraction $q$ of spins, at each time step. The occasional flipping of one spin within a cross-shaped island now removes the frustration and breaks the previous stability of the island structure. Equally some new cross-shaped islands may also be randomly created. $M$ thus is now dependent on the balance between creation and destruction of cross-shaped islands. The effect is illustrated by figures 5 and 6 which represent the microstructure at two consecutive time steps. The area located by the arrow show how an island has become destabilised by the noise.

**Conclusions**

A non-monotonic dependence of magnetization on random noise (temperature) arises in a square lattice model where spins interact via a synchronous Sznajd fashion. This is due to the presence of stable 'cross-shaped' islands of parallel spins. Random fluctuations can destabilise these islands. So, returning to our social analogue, one might expect, if the model applies, that consensus may actually increase when a small amount of noise is present. These results suggest that a certain degree of individual freedom and independence of thought may actually increase the degree of consensus within a society.

A more detailed analysis of these effects will be subject of further study.

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**References**

[1] See for example, Huang, K. Statistical Mechanics. John Wiley

[2] See for example, Vaga, T., Profiting from Chaos. McGraw Hill New York 1994 ISBN 0 07 066786 1
Fig.1.
The straight line (estimated slope $-0.39$) displays in log-log scale the quantity $1 - Mc$ as a function of $L$, at zero temperature. $Mc$ is the initial net magnetization at the phase transition point from the state without consensus to the state with consensus. $L$ is the lattice linear dimension.

Fig.2
Magnetization at equilibrium as a function of $q$, for $L=50$ and initial magnetization values 0.9,0.2,0.02 and zero.

Fig.3
Magnetization at equilibrium as a function of $q$, for $L=100$ and initial magnetization values 0.9,0.2,0.02 and zero.

Fig.4
Magnetization at equilibrium as a function of $q$, for $L=250$ and initial magnetization values 0.9,0.2,0.02 and zero.

Fig.5
This shows the microscopic detail of a calculation for a lattice of linear dimension $L=50$, a probability of random flipping $q=0.001$ and an initial magnetization $M(0)=0.2$. The arrow points to the 'cross-shaped' island (positive spins, black) centred in the site 33,17 and surrounded by the sea (negative spins, white) that has formed after 396 time-steps.

Fig.6

This shows the microscopic detail for the calculation shown in figure 5 at the subsequent time step. The random flipping of spin 33, 16 (see the arrow) has now eased the frustration, it would otherwise exhibit at zero temperature, destabilising the 'cross shaped' island. In this particular case it took just one time-step for the other four positive spins forming the island to be flipped via interaction with the surrounding negative spins. In general that may take up to tens of steps.
1 - (critical M(initial) value)
Initial Magnetization value (before the thermalization)

- □ M(initial)=0.9
- ○ M(initial)=0.2
- ▲ M(initial)=0.02
- ⧓ M(initial)=0
Initial Magnetization value
(before the thermalization)

- □ - $M(\text{initial})=0.9$
- ◯ - $M(\text{initial})=0.2$
- ▲ - $M(\text{initial})=0.02$
- ∗ - $M(\text{initial})=0$
Initial Magnetization value (before the thermalization)

- $M(\text{initial})=0.9$
- $M(\text{initial})=0.2$
- $M(\text{initial})=0.02$
- $M(\text{initial})=0$
