AN INTRODUCTION TO QUANTUM ALGEBRAS
AND THEIR APPLICATIONS

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Abstract: A very elementary introduction to quantum algebras is presented and a few examples of their physical applications are mentioned.

I shall give a very elementary introduction to the topic of quantum algebras and mention a few physical applications. Quantum algebras, or quantum groups, extend the domain of classical group theory and constitute a new and growing field of mathematics with vast potential for applications in physics. In fact, the origins of quantum groups lie in physics: in the studies on the behaviour of integrable systems in quantum field theory and statistical mechanics, using the quantum inverse scattering method, by Sklyanin, Kulish, Reshetikhin, Takhtajan, and Faddeev in the 1980s. Mathematical abstraction from the observations on the common features of these systems led to the definition of the concept of a quantum group and the studies of Drinfeld, Jimbo, Manin, Woronowicz, Connes, Wess, Zumino, Macfarlane, Biedenharn, ..., revealed many aspects of quantum groups from different mathematical and physical points of view. I shall neither go into the history of the developments of the various significant concepts nor give any specific references to their origins. I shall cite at the end some books and articles which would give these details and lead to the vast literature on the subject of quantum groups and their applications.

Let us consider a two dimensional classical vector space whose elements can be written as

$$\begin{pmatrix} x \\ y \end{pmatrix},$$

where the coordinates $x$ and $y$ are real variables and commute with each other, i.e.,

$$xy = yx.$$

In the space of functions $\{f(x,y)\}$ defined in the above vector space the partial derivative operations $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ and the operations of multiplications by $x$ and $y$ satisfy the differential calculus

$$[x, y] = 0, \quad \left[ \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right] = 0, \quad \left[ \frac{\partial}{\partial x}, y \right] = 0, \quad \left[ \frac{\partial}{\partial y}, x \right] = 0,$$

$$\left[ \frac{\partial}{\partial x}, x \right] = 1, \quad \left[ \frac{\partial}{\partial y}, y \right] = 1,$$

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where the commutator bracket $[\ ,\ ]$ is defined by

$$[A, B] = AB - BA.$$  \hfill (4)

Let us make a linear transformation of the vector $\begin{pmatrix} x \\ y \end{pmatrix}$ as

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = M \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$ \hfill (5)

where the entries of the matrix $M$ are real and satisfy the condition

$$ad - bc = \det M = 1,$$ \hfill (6)

or in other words the transformation (5) is an element of the group $SL(2, R)$. Then, the new coordinates $x'$ and $y'$ and partial derivatives with respect to them, namely $\frac{\partial}{\partial x'}$ and $\frac{\partial}{\partial y'}$, also satisfy the same relations as (3). This is easily checked by noting

$$ \begin{pmatrix} \frac{\partial}{\partial x'} \\ \frac{\partial}{\partial y'} \end{pmatrix} = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} = \tilde{M}^{-1} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix},$$ \hfill (7)

where $\tilde{A}$ means the transpose of the matrix $A$. We say that the differential calculus on the two dimensional $(x, y)$-plane is covariant under the group $SL(2, R)$.

In nature all physical systems are quantum mechanical. But, the quantum mechanical behaviour is generally revealed only at the microscopic (molecular and deeper) level. At the macroscopic level of everyday experience quantum physics becomes classical physics as an approximation. So, only classical physics was discovered first and observations of failure of classical physics at the atomic level led to the discovery of quantum physics in the 20th century. This passage from classical physics to quantum physics can be mathematically described as a process of deformation of the classical physics wherein the commuting classical observables of a physical system are replaced by noncommuting hermitian operators. This process is characterized by a very small deformation parameter known as the Planck constant $\hbar$ and, roughly speaking, in the limit $\hbar \to 0$ classical physics is recovered from the structure of quantum physics.

In analogy with the process of quantizing the classical physics let us now quantize the classical vector space to get a quantum vector space by assuming that the coordinates do not commute with each other at any point. Let us model the noncommutativity of the coordinates $X$ and $Y$ of a two dimensional quantum plane by

$$XY = qYX,$$ \hfill (8)

where $q$ is the deformation parameter which we shall consider, in general, to be any nonzero complex number. Note that in the limit $q \to 1$ the noncommuting quantum coordinates $X$ and $Y$ become commuting classical coordinates. To be specific, let us choose

$$q = e^{i\theta}.$$ \hfill (9)
It is easy to find an example of such noncommuting variables. Let $T_\alpha$ and $G_{\theta/\alpha}$ be operators acting on functions of a single real variable $x$ such that

$$T_\alpha \psi(x) = \psi(x + \alpha), \quad G_{\theta/\alpha} \psi(x) = e^{i\theta x/\alpha} \psi(x).$$

Then, for any $\psi(x)$,

$$T_\alpha G_{\theta/\alpha} \psi(x) = e^{i\theta(x+\alpha)/\alpha} \psi(x + \alpha) = e^{i\theta} G_{\theta/\alpha} T_\alpha \psi(x),$$

for a given fixed value of $\theta$. Thus, with the variation of $\alpha$, $T_\alpha$ and $G_{\theta/\alpha}$ become noncommuting variables obeying the relation

$$T_\alpha G_{\theta/\alpha} = e^{i\theta} G_{\theta/\alpha} T_\alpha,$$

with fixed value for $\theta$.

Now the interesting questions are:

- Is it possible to define a differential calculus on the two dimensional $(X,Y)$-plane?
- If so, will it be covariant under some generalization of the classical group $SL(2)$?

The answers are yes, yes! First let us understand how one can give a meaning to partial derivatives with respect to $X$ and $Y$. These have to operate on the space of functions $\{f(X,Y)\}$ which we shall consider to be polynomials in $X$ and $Y$. We can write $f(X,Y) = \sum_{m,n} f_{mn} X^m Y^n$ since any polynomial in $X$ and $Y$, with coefficients commuting with $X$ and $Y$, can be brought to this form using the commutation relation (8). Then, if we take formally

$$\frac{\partial}{\partial X} X^m = m X^{m-1}, \quad \frac{\partial}{\partial Y} Y^n = n Y^{n-1},$$

we would have a differential calculus in the $(X,Y)$-plane, as desired, once we prescribe consistently the remaining commutation relations between $X$, $Y$, $\frac{\partial}{\partial X}$ and $\frac{\partial}{\partial Y}$.

Without going into any further details, let me state the complete set of commutation relations defining the differential calculus in the $(X,Y)$-plane (with $q = e^{i\theta}$):

$$XY = qYX, \quad \frac{\partial}{\partial X} \frac{\partial}{\partial Y} = q^{-1} \frac{\partial}{\partial Y} \frac{\partial}{\partial X}, \quad \frac{\partial}{\partial Y} Y = qY \frac{\partial}{\partial X}, \quad \frac{\partial}{\partial Y} X = qX \frac{\partial}{\partial Y},$$

$$\frac{\partial}{\partial X} X - q^2 X \frac{\partial}{\partial X} = 1 + (q^2 - 1)Y \frac{\partial}{\partial Y}, \quad \frac{\partial}{\partial Y} Y - q^2 Y \frac{\partial}{\partial Y} = 1.$$  

This noncommutative differential calculus on the two-dimensional quantum plane is seen to be covariant under the transformations

$$\begin{pmatrix} X' \\ Y' \end{pmatrix} = T \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix},$$

$$\begin{pmatrix} \frac{\partial}{\partial X} \\ \frac{\partial}{\partial Y} \end{pmatrix} = \tilde{T}^{-1} \begin{pmatrix} \frac{\partial}{\partial X} \\ \frac{\partial}{\partial Y} \end{pmatrix} = \begin{pmatrix} D & -qC \\ -q^{-1}B & A \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial X} \\ \frac{\partial}{\partial Y} \end{pmatrix},$$
provided
\[ A, B, C, \text{ and } D \text{ commute with } X \text{ and } Y , \]
\[ AB = qBA, \quad CD = qDC, \quad AC = qCA, \quad BD = qDB, \]
\[ BC = CB, \quad AD - DA = (q - q^{-1}) BC , \]
and
\[ AD - qBC = \det_q T = 1 . \]  

In other words \( X', Y', \frac{\partial}{\partial X}, \) and \( \frac{\partial}{\partial Y} \) defined by (15) satisfy the relations obtained from (14) by just replacing \( X \) and \( Y \) by \( X' \) and \( Y' \) respectively. Note that \( \det_q T \) defined in (17) commutes with all the matrix elements of \( T \). Verify that
\[ T^{-1} = \begin{pmatrix} D & -q^{-1}B \\ -qC & A \end{pmatrix} \]
is such that
\[ TT^{-1} = T^{-1}T = \mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} . \]

A matrix \( T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \) is called a 2 \( \times \) 2 quantum matrix if its matrix elements \{\( A, B, C, D \)\} satisfy the commutation relations in (17). In the limit \( q \to 1 \) a quantum matrix \( T \) becomes a classical matrix with commuting elements. Note that the identity matrix \( \mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) is a quantum matrix.

Entries of a quantum matrix \( T \) are noncommuting variables satisfying the commutation relations (17). Let \( T_1 = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \) and \( T_2 = \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix} \) be any two quantum matrices; i.e., \{\( A_1, B_1, C_1, D_1 \)\} obey the relations (17), and \{\( A_2, B_2, C_2, D_2 \)\} also obey the relations (17). The matrix elements of \( T_1 \) and \( T_2 \) may be ordinary classical matrices satisfying the required relations (17). Define the product
\[ \Delta_{12}(T) = T_1 \hat{\otimes} T_2 = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \hat{\otimes} \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix} = \begin{pmatrix} A_1 \otimes A_2 + B_1 \otimes C_2 & A_1 \otimes B_2 + B_1 \otimes D_2 \\ C_1 \otimes A_2 + D_1 \otimes C_2 & C_1 \otimes B_2 + D_1 \otimes D_2 \end{pmatrix} = \begin{pmatrix} \Delta_{12}(A) & \Delta_{12}(B) \\ \Delta_{12}(C) & \Delta_{12}(D) \end{pmatrix} \]
where \( \otimes \) denotes the direct product with the property \((P \otimes R)(Q \otimes S) = PQ \otimes RS\). Then one finds that the matrix elements of \( \Delta_{12}(T) \), namely,
\[ \Delta_{12}(A) = A_1 \otimes A_2 + B_1 \otimes C_2 , \quad \Delta_{12}(B) = A_1 \otimes B_2 + B_1 \otimes D_2 , \]
\[ \Delta_{12}(C) = C_1 \otimes A_2 + D_1 \otimes C_2 , \quad \Delta_{12}(D) = C_1 \otimes B_2 + D_1 \otimes D_2 , \]
also satisfy the commutation relations [17]. In other words, $\Delta_{12}(T)$ is also a quantum matrix. This product, $\Delta_{12}(T) = T_1 \hat{\otimes} T_2$, is called the coproduct or comultiplication. Note that there is no inverse for this coproduct. Under this coproduct the $2 \times 2$ quantum matrices $\{T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\}$ form a pseudomatrix group, commonly called a quantum group, denoted by $SL_q(2)$. The algebra of functions over $SL_q(2)$, or the algebra of polynomials in $\{A,B,C,D\}$, is denoted by $Fun_q(SL(2))$. The coproduct operation $(\Delta)$ defined by (22), is symbolically written as

$$\Delta(A) = A \otimes A + B \otimes C, \quad \Delta(B) = A \otimes B + B \otimes D,$$

$$\Delta(C) = C \otimes A + D \otimes C, \quad \Delta(D) = C \otimes B + D \otimes D. \quad (23)$$

For any $f(A,B,C,D) \in Fun_q(SL(2))$ the definition of $\Delta$ is extended as

$$\Delta f(A,B,C,D) = f(\Delta(A), \Delta(B), \Delta(C), \Delta(D)). \quad (24)$$

Now, for example, look at $T^{(1)} = \begin{pmatrix} T_{11} & T_{12} & \ldots & T_{1n} \\ T_{21} & T_{22} & \ldots & T_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ T_{n1} & T_{n2} & \ldots & T_{nn} \end{pmatrix}$, is said to be an $n$-dimensional representation of $SL_q(2)$ if its matrix elements $(T_{ij})$ are polynomials in $\{A,B,C,D\}$, or in other words elements of $Fun_q(SL(2))$, and satisfy the property

$$(T \hat{\otimes} T)_{ij} = \sum_{l=1}^{n} T_{il} \otimes T_{lj} = \Delta(T_{ij})$$

$$= T_{ij}(\Delta(A), \Delta(B), \Delta(C), \Delta(D)), \quad \forall \ i,j = 1,2,\ldots,n. \quad (25)$$

It can be verified that $T^{(1)}$ is the 3-dimensional representation of $SL_q(2)$. For instance, see that

$$(T^{(1)} \hat{\otimes} T^{(1)})_{11} = \sum_{l=1}^{3} T^{(1)}_{1l} \otimes T^{(1)}_{1l}$$
\[
\begin{align*}
  & = A^2 \otimes A^2 + (1 + q^{-2}) AB \otimes AC + B^2 \otimes C^2 \\
  & = A^2 \otimes A^2 + AB \otimes AC + BA \otimes CA + B^2 \otimes C^2 \\
  & = (A \otimes A + B \otimes C)^2 = (\Delta(A))^2 = \Delta \left( T^{(1)}_{11} \right), \tag{27}
\end{align*}
\]

as required. Similarly, for other matrix elements of \( T^{(1)} \) one can verify the property (23), namely,

\[
\left( T \otimes T \right)_{ij} = \Delta \left( T_{ij} \right). \tag{28}
\]

In the theory of classical Lie groups we know that an element of a Lie group \( G \), say \( g \), can be written as

\[
g = e^{\epsilon_1 L_1} e^{\epsilon_2 L_2} \cdots e^{\epsilon_n L_n}, \tag{29}
\]

where the parameters \( \{\epsilon_i\} \) characterize the group element \( g \) and \( \{L_i\} \) are constant generators of the group \( G \) satisfying a Lie algebra

\[
[L_i, L_j] = \sum_{k=1}^{n} C_{ij}^k L_k, \quad i, j = 1, 2, \ldots, n, \tag{30}
\]

with \( \{C_{ij}^k\} \) as the structure constants. When the group element \( g \) is close to the identity element \( (I) \) of the group, the parameters \( \{\epsilon_i\} \) are infinitesimals and one can write

\[
g \approx I + \sum_{i=1}^{n} \epsilon_i L_i. \tag{31}
\]

Now, the interesting question is

- Is there an analogue of the Lie algebra in the case of a quantum group?

The answer is yes! To this end, first we have to recall some basic notions of the theory of \( q \)-series.

One defines the \( q \)-shifted factorial by

\[
(x; q)_n = \begin{cases} 
1, & n = 0, \\
(1 - x)(1 - xq)(1 - xq^2) \cdots (1 - xq^{n-1}), & n = 1, 2, \ldots.
\end{cases} \tag{32}
\]

Then, with the notation

\[
(x_1, x_2, \ldots, x_m; q)_n = (x_1; q)_n (x_2; q)_n \cdots (x_m; q)_n, \tag{33}
\]

an \( r, s \)-basic hypergeometric series, or a general \( q \)-hypergeometric series, is given by

\[
\begin{align*}
  r, s \Phi_s (a_1, a_2, \ldots, a_r; b_1, b_2, \ldots, b_s; q, z) \\
  = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_r; q)_n}{(b_1, b_2, \ldots, b_s; q)_n (q; q)_n} \left( (-1)^n q^{(n-1)/2} \right)^{1+r-s} z^n,
\end{align*}
\]

\( r, s = 0, 1, 2, \ldots. \tag{34} \)

Consider

\[
1 \Phi_0 (0; -; q, (1 - q)z) = e^z_q = \sum_{n=0}^{\infty} \frac{z^n}{[n]_q}!, \tag{35}
\]
where
\[ [n]_q = \frac{1 - q^n}{1 - q}, \]
\[ [n]_q! = [n]_q[n - 1]_q[n - 2]_q \ldots [2]_q[1]_q, \quad n = 1, 2, \ldots, \quad [0]_q! = 1. \quad (36) \]
The \( q \)-number \([n]_q\), or the so-called basic number, was defined by Heine (1846). Much of Ramanujan's work is related to these \( q \)-series. Note that
\[ [n]_q q^{-1} \to n, \quad e^z q^{-1} \to e^z. \quad (37) \]
Thus, \( e^z \) defined by (35) is a \( q \)-generalization of the exponential function, called a \( q \)-exponential function; note that there can be several generalizations of the exponential function satisfying the condition that in the limit \( q \to 1 \) it should become the standard exponential function. In the theory of quantum groups a new definition of the \( q \)-number is often useful. It is
\[ [[n]_q] = \frac{q^n - q^{-n}}{q - q^{-1}}. \quad (38) \]
Note that \([n]_q\) also becomes \( n \) in the limit \( q \to 1 \), and \([n]_q\) is symmetric with respect to the interchange of \( q \) and \( q^{-1} \) unlike Heine's \([n]_q\).

Now consider the 2-dimensional \( T \)-matrix parametrized as
\[ T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} e^\alpha & e^{\alpha \beta} \\ e^{-\gamma} e^\alpha & e^{-\alpha} + e^{\alpha \beta} \end{pmatrix}, \quad (39) \]
which requires the variable parameters \( \{\alpha, \beta, \gamma\} \) to satisfy a Lie algebra
\[ [\alpha, \beta] = (\log q) \beta, \quad [\alpha, \gamma] = (\log q) \gamma, \quad [\beta, \gamma] = 0, \quad (40) \]
so that \( \{A, B, C, D\} \) obey the algebra (17). Then, one can write
\[ T = e^{\gamma \mathcal{X}_0^{(1/2)}} e^{2\alpha \mathcal{X}_0^{(1/2)}} e^{\beta \mathcal{X}_+^{(1/2)}} e^{q^2 \mathcal{X}_+^{(1/2)}}, \quad (41) \]
with
\[ \mathcal{X}_0^{(1/2)} = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathcal{X}_-^{(1/2)} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \mathcal{X}_+^{(1/2)} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \quad (42) \]
Of course, in this case, \( e^{\gamma \mathcal{X}_0^{(1/2)}} \) and \( e^{\beta \mathcal{X}_+^{(1/2)}} \) are trivially the same as \( e^{\gamma \mathcal{X}_-^{(1/2)}} \) and \( e^{\beta \mathcal{X}_+^{(1/2)}} \), respectively, since \( (\mathcal{X}_+^{(1/2)})^2 = 0 \).

Actually, equation (11) is the special case of a universal formula for the representations of the \( T \)-matrices of \( SL_q(2) \) and corresponds to the fundamental representation. The generic form of the \( T \)-matrix is given by
\[ T = e^{\gamma \mathcal{X}_-} e^{2\alpha \mathcal{X}_0} e^{\beta \mathcal{X}_+}, \quad (43) \]
called the universal \( T \)-matrix, where \( \{X_0, X_+, X_-\} \) obey the algebra

\[
[X_0, X_\pm] = \pm X_\pm, \quad [X_+, X_-] = \frac{q^{2X_0} - q^{-2X_0}}{q - q^{-1}} = [2X_0]_q,
\]

called the quantum algebra \( sl_q(2) \). In the limit \( q \to 1 \), \( \{X\} \to \{X\} \) which generate \( sl(2) \),

\[
[X_0, X_\pm] = \pm X_\pm, \quad [X_+, X_-] = 2X_0,
\]

the Lie algebra of \( SL(2) \). The matrices \( \{X_0^{(1/2)}, X_+^{(1/2)}, X_-^{(1/2)}\} \) in (42) provide the fundamental 2-dimensional irreducible representation of the algebra (44) (actually, they also provide the fundamental representation of the generators of \( sl_q(2) \) algebra (45)). When a higher dimensional representation of (44) is plugged in the formula (43) \( T \) becomes a higher dimensional representation of the \( T \)-matrix. For example, the three dimensional representation (43) is obtained by substituting in (43)

\[
X_0^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},
\]

\[
X_-^{(1)} = \sqrt{2q} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1/\sqrt{q} & 0 \\ 0 & 0 & \sqrt{q} \end{pmatrix},
\]

\[
X_+^{(1)} = \sqrt{2q} \begin{pmatrix} 0 & 1/\sqrt{q} & 0 \\ 0 & 0 & \sqrt{q} \\ 0 & 0 & 0 \end{pmatrix},
\]

which provide the three dimensional irreducible representation of the \( sl_q(2) \) algebra (44), and using the parametrization of \( \{A, B, C, D\} \) in terms of \( \{\alpha, \beta, \gamma\} \) as given by (39). Note that in the limit \( q \to 1 \) these matrices (43) obey the \( sl(2) \) algebra (45). As seen from (39), (43) and (44), one can say that for a quantum group the group-parameter space is noncommutative.

The algebra \( sl_q(2) \) is also often called a quantum group. Actually, the relations in (44) define the generators of the \( q \)-deformation of the universal enveloping algebra of \( sl(2) \). Hence, the relations (44) are also referred to, more properly, as \( U_q(sl(2)) \), the \( q \)-deformed universal enveloping algebra of \( sl(2) \). The algebra \( U_q(sl(2)) \), generated by polynomials in \( \{X_0, X_+, X_-\} \) obeying the relations (44), is also a Hopf algebra. We shall consider only the coproduct(s) for \( U_q(sl(2)) \). A coproduct for \( sl_q(2) \) is

\[
\Delta_q(X_0) = X_0 \otimes 1 + 1 \otimes X_0, \quad \Delta_q(X_\pm) = X_\pm \otimes q^{X_0} + q^{-X_0} \otimes X_\pm.
\]

It can be easily verified that this comultiplication rule is an algebra isomorphism for \( sl_q(2) \):

\[
[\Delta_q(X_0), \Delta_q(X_\pm)] = \pm \Delta_q(X_\pm), \quad [\Delta_q(X_+), \Delta_q(X_-)] = [2\Delta_q(X_0)]_q.
\]

The most important property of this coproduct is its noncommutativity. Note that the algebra (14) is invariant under the interchange \( q \leftrightarrow q^{-1} \) since \( [2X_0]_q = [2X_0]_{q^{-1}} \).
However, the comultiplication (47) is not invariant under such an interchange. This means that the comultiplication obtained from (47) by an interchange $q \leftrightarrow q^{-1}$ should also be an equally good comultiplication. It can be verified that the coproduct so obtained, namely,

$$\Delta_{q^{-1}}(X_0) = X_0 \otimes 1 + 1 \otimes X_0, \quad \Delta_{q^{-1}}(X_\pm) = X_\pm \otimes q^{-X_0} + q^{X_0} \otimes X_\pm. \quad (49)$$

is indeed an algebra isomorphism for $sl_q(2)$. This coproduct (49), $\Delta_{q^{-1}}$, is called the opposite coproduct in view of the relation

$$\Delta_{q^{-1}}(X) = \tau(\Delta_q(X)),$$

where $\tau(u \otimes v) = v \otimes u$. \quad (50)

Since

$$\Delta_{q^{-1}} \neq \Delta_q, \quad \text{or} \quad \tau(\Delta) \neq \Delta, \quad (51)$$

the comultiplications $\Delta_q$ and $\Delta_{q^{-1}}$ of $sl_q(2)$ are noncommutative. In the limit $q \to 1$, the classical $sl(2)$ has only a single comultiplication, $\Delta(X) = X \otimes 1 + 1 \otimes X$, which is commutative (i.e., $\tau(\Delta) = \Delta$).

One can show that the two comultiplications of $sl_2(q)$, namely $\Delta_q$ and $\Delta_{q^{-1}}$, are related to each other by an equivalence relation such that there exists an $R \in U_q(sl(2)) \otimes U_q(sl(2))$, called the universal $R$-matrix, satisfying the relation

$$\Delta_{q^{-1}}(X) = R\Delta_q(X)R^{-1}.$$

This universal $R$-matrix is the central object of the quantum group theory. In this case it can be shown that

$$R = q^{2(X_0 \otimes X_0)} \sum_{n=0}^{\infty} \frac{(1 - q^2)^n}{[n]_q!} \left(q^{X_0} X_+ \otimes q^{-X_0} X_-\right)^n.$$

If we insert the matrix representations of $\{X\}$ in this expression for $R$ we get numerical $R$-matrices. For example, substituting in (53) the $2 \times 2$ representation of $\{X\}$, given in (42), we get the fundamental 4-dimensional $R$-matrix

$$R = \frac{1}{\sqrt{q}} \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & (q - q^{-1}) & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (54)$$

Let us now write any $R$ in the form

$$R = \sum_i a_i u_i \otimes v_i. \quad (55)$$

It is clear from (53) that this can be done. Now, define

$$R_{12} = R \otimes 1, \quad R_{13} = \sum_i a_i u_i \otimes 1 \otimes v_i, \quad R_{23} = 1 \otimes R. \quad (56)$$

Then, these satisfy the remarkable relation

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}. \quad (57)$$
known as the quantum Yang-Baxter equation, or simply the Yang-Baxter equation (YBE).

We have considered only the simplest example of a quantum group, namely $SL_q(2)$, associated with the classical group $SL(2)$. There exists a systematic theory of deformation of any classical group. It is also possible, in certain cases, to obtain deformations with several $q$-parameters. Actually, the study of quantum groups sheds more light on the structure of the classical group theory. I shall not go further into the details of the formalism of quantum group theory.

Now, I am in a position to mention a few applications of quantum groups and algebras. First, let us see how these things started. Define

$$T_1 = T \otimes \mathbf{1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$T_2 = \mathbf{1} \otimes T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \quad (58)$$

Note that

$$T_1 T_2 = \begin{pmatrix} A^2 & AB & BA & B^2 \\ AC & AD & BC & BD \\ CA & CB & DA & DB \\ C^2 & CD & DC & D^2 \end{pmatrix} \neq T_2 T_1 = \begin{pmatrix} A^2 & BA & AB & B^2 \\ CA & DA & CB & DB \\ AC & BC & AD & BD \\ C^2 & DC & CD & D^2 \end{pmatrix}, \quad (59)$$

because $\{A, B, C, D\}$ are noncommutative. The relation between $T_1 T_2$ and $T_2 T_1$ turns out to be

$$RT_1 T_2 = T_2 T_1 R. \quad (60)$$

This type of relation is commonly encountered in the quantum inverse scattering method approach to integrable models in quantum field theory and statistical mechanics. Substituting in (60) $R$ from (54), and $T_1$ and $T_2$ from (58), it is found that equation (60) is a compact way of stating the commutation relations (17) defining the fundamental $T$-matrix of $SL_q(2)$. What about the commutation relations (44) defining $sl_q(2)$? Define

$$L^{(+)} = \begin{pmatrix} q^{-x_0} & -\sqrt{q}(q-q^{-1})X_- \\ 0 & q^{x_0} \end{pmatrix},$$

$$L^{(-)} = \begin{pmatrix} q^{x_0} & 0 \\ q^{-1/2}(q-q^{-1})X_+ & q^{-x_0} \end{pmatrix},$$

$$L_1^{(\pm)} = L^{(\pm)} \otimes \mathbf{1}, \quad L_2^{(\pm)} = \mathbf{1} \otimes L^{(\pm)}. \quad (61)$$

Then, the commutation relations (44), defining the generators of $sl_q(2)$, can be stated elegantly as

$$R^{-1} L_1^{(\pm)} L_2^{(\pm)} = L_2^{(\pm)} L_1^{(\pm)} R^{-1}, \quad R^{-1} L_1^{(+)} L_2^{(-)} = L_2^{(-)} L_1^{(+)} R^{-1}. \quad (62)$$

Note that the $L^{(\pm)}$-matrices are special realizations of the $T$-matrices, i.e., the elements of $L^{(\pm)}$-matrices obey the commutation relations required of the $T$-matrix elements.
If we define for the $R$-matrix in (54),

$$S_1 = \hat{R} \otimes \mathbb{1}, \quad S_2 = \mathbb{1} \otimes \hat{R},$$

(63)

where

$$\hat{R} = PR, \quad P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

(64)

then, it is found that

$$S_1 S_2 S_1 = S_2 S_1 S_2,$$

(65)

which is an alternative form of the YBE (57). For any general $R$-matrix the YBE (57) can be put in this form (65). This relation (65) represents a property of the generators of a braid group which is a generalization of the well known symmetric group $S_n$. The symmetric group $S_n$ is the group of all permutations of $n$ objects. An element of the braid group $B_n$ can be depicted as a system of $n$ strings joining two sets of $n$ points, each set located on a line, the two lines, say top and bottom, being parallel, with over-crossings or under-crossings of the strings. The over-crossings and the under-crossings of the strings make $B_n$ an infinite group which will otherwise reduce to $S_n$. If $i$ and $i+1$ are two consecutive points on the top and bottom lines, the string starting at $i$ on the top line can reach $i+1$ on the bottom line by either under-crossing or over-crossing the string starting at $i+1$ on the top line and reaching $i$ on the bottom line. The corresponding elements of the braid group are usually denoted by $\sigma_i$ and $\sigma_i^{-1}$, respectively. The elements $\{\sigma_i | i = 1, 2, \cdots n - 1\}$, generating the braid group $B_n$, satisfy two relations,

$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{for} \quad |i - j| > 1,$$

(66)

and

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_{i} \sigma_{i+1}.$$

(67)

Now, comparing the relations (65) and (67) it is obvious that the solutions of the YBE ($R$-matrices), or the quantum algebras, should play a central role in the theory of representations of braid groups. Braid groups have many applications. In mathematics they are useful in the study of complex functions of hypergeometric type having several variables. In physics they appear in knot theory, statistical mechanics, two-dimensional conformal field theory, and so on.

In quantum mechanics the notion of continuous space-time with commutative coordinates is taken over from classical mechanics. This is an assumption. What will happen if at some deeper microscopic level the space-time coordinates themselves are noncommutative? It is clear that to deal with such a situation one will have to use a noncommutative differential calculus and the theory of quantum groups provides the necessary framework as we have seen above. For example, consider the motion of a quantum particle in a two dimensional noncommutative plane with $X$ and $Y$ as the coordinates. If we take the corresponding conjugate momenta to be proportional to $\frac{\partial}{\partial X}$ and $\frac{\partial}{\partial Y}$, respectively, then the relations (64) indicate how the two-dimensional
quantum mechanical phase-space would be deformed at that level. Thus, the theory of quantum groups would provide the mathematical framework for the future of quantum physics if it turns out that at some deeper microscopic level space-time manifold is noncommutative. Hence there has been a lot of interest in studying the fundamental modifications that would occur in the framework quantum mechanics, relativity theory, Poincaré group, ..., etc., if the space-time manifold happens to be noncommutative.

Apart from applications of fundamental nature, such as those mentioned above, there have been many phenomenological applications of quantum algebras in nuclear physics, condensed matter physics, molecular physics, quantum optics, and elementary particle physics. In these applications either an existing model is identified with a quantum algebraic structure, or a standard model is deformed to have an underlying quantum algebraic structure and studied to reveal the new features emerging. To give an idea of such applications, let me mention, as the final example, the \( q \)-deformation of the quantum mechanical harmonic oscillator algebra, also known as the boson algebra. The algebraic treatment of the quantum mechanical harmonic oscillator involves a creation operator \( (a^\dagger) \), an annihilation operator \( (a) \), and a number operator \( (N) \), obeying the commutation relations

\[
[a, a^\dagger] = 1, \quad [N, a^\dagger] = a^\dagger,
\]

where \( N \) is a hermitian operator and \( a^\dagger \) is the hermitian conjugate of \( a \). The energy spectrum of the harmonic oscillator is given by the eigenvalues of the Hamiltonian operator

\[
H = \frac{1}{2} \left( a a^\dagger + a^\dagger a \right),
\]

in the appropriate units. Taking two such sets of oscillator operators, \( \{a_1, a_1^\dagger, N_1\} \) and \( \{a_2, a_2^\dagger, N_2\} \), which are assumed to commute with each other, and defining

\[
J_0 = \frac{1}{2} (N_1 - N_2), \quad J_+ = a_1^\dagger a_2, \quad J_- = a_2^\dagger a_1,
\]

it is found that

\[
J_0^\dagger = J_0, \quad J_+^\dagger = J_-,
\]

and

\[
[J_0, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = 2J_0.
\]

This Lie algebra (72) is seen to be the same as the \( sl(2) \) algebra (45) subject to the hermiticity conditions (71) and is known as \( su(2) \) algebra, the Lie algebra of the group \( SU(2) \). The \( su(2) \) algebra is the algebra of three dimensional rotations, or the rigid rotator, with \( \{J_0, J_\pm\} \) representing the angular momentum operators. The coproduct rule

\[
\Delta (J_0) = J_0 \otimes 1 + 1 \otimes J_0, \quad \Delta (J_\pm) = J_\pm \otimes 1 + 1 \otimes J_\pm,
\]
for the algebra (72), obtained by setting $q = 1$ in (47) (or (49)), represents the rule for addition of angular momenta. Correspondingly, the relations (44) rewritten as

\[ [J_0, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = [2J_0]_q, \tag{74} \]

with the hermiticity conditions

\[ J_0^\dagger = J_0, \quad J_+^\dagger = J_- \tag{75} \]

represent the $su_q(2)$ (or $U_q(su(2))$) algebra or the $q$-deformed version of the $su(2)$ algebra (72). One can say that $su_q(2)$ is the algebra of the $q$-rotator. For the $q$-angular momentum operators there are two possible addition rules,

\[ \Delta_q^{\pm 1}(J_0) = J_0 \otimes 1 + 1 \otimes J_0, \quad \Delta_q^{\pm 1}(J_\pm) = J_\pm \otimes q^{\pm J_0} + q^{\mp J_0} \otimes J_\pm, \tag{76} \]

as seen from (47) and (49). Now the interesting fact is that one has a realization of $su_q(2)$ generators given by

\[ J_0 = \frac{1}{2} (N_1 - N_2), \quad J_+ = A_1^\dagger A_2, \quad J_- = A_2^\dagger A_1 \tag{77} \]

exactly analogous to the $su(2)$ case (70), where the two sets of operators $\{A_1, A_1^\dagger, N_1\}$ and $\{A_2, A_2^\dagger, N_2\}$ commute with each other and obey, within each set, the algebra

\[ AA^\dagger - qA^\dagger A = q^{-N}, \quad [N, A^\dagger] = A^\dagger. \tag{78} \]

Further, $N$ is hermitian and $\{A, A^\dagger\}$ is a hermitian conjugate pair. The $q$-deformed oscillator algebra (78) is known as the $q$-oscillator or the $q$-boson algebra. When $q \rightarrow 1$ the $q$-oscillator algebra (78) reduces to the canonical oscillator algebra (58).

As is easy to guess, phenomenological applications of quantum algebras in nuclear and molecular spectroscopy involve the substitution of harmonic oscillator model by the $q$-oscillator model and the rigid rotator model by the $q$-rotator model. Such applications lead to impressive results showing that the actual vibrational-rotational spectra of nuclei and molecules can be fit into schemes in which the number of phenomenological $q$-parameters required are very much fewer than the number of traditional phenomenological parameters required to fit the same spectral data. Somehow such $q$-deformed models seem to take into account more efficiently the anharmonicity of vibrations and the nonrigidity of rotations in nuclear and molecular systems.

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