EQUIVARIANT LOCAL INDEX

TAKAHIKO YOSHIDA

Abstract. This is an expository article on the equivariant local index developed by Fujita, Furuta, and the author in [7].

1. Background, motivation, and purpose

This is based on a joint work with Hajime Fujita and Mikio Furuta [5, 6, 7, 4]. In [5, 6] we developed an index theory for Dirac-type operators on possibly noncompact Riemannian manifolds and applied the index theory to the geometric quantization of Lagrangian fibrations. In [7] we refined the index theory in the case of torus actions. As an application we obtained a proof of the quantization conjecture, concerning the commutativity of the quantization and the symplectic reduction, in the case of torus actions. Other applications of the index theory will be described in [4, 8]. The purpose of this note is to explain the equivariant index theory developed in [7] in a simple symplectic case.

Let us recall the background and our motivation. One of our motivation comes from the geometric quantization. Let \((M, \omega)\) be a closed symplectic manifold. Suppose that \((M, \omega)\) is prequantizable, namely, the cohomology class represented by \(\omega\) is in the image of the natural map \(H^2(M; \mathbb{Z}) \to H^2(M; \mathbb{R})\). Then, there exists a Hermitian line bundle \(L \to M\) with Hermitian connection \(\nabla^L\) whose curvature form \(F_{\nabla^L}\) is equal to \(-2\pi\sqrt{-1}\omega\). \((L, \nabla^L)\) is called a prequantum line bundle.

It is well known that a symplectic manifold is equipped with an almost complex structure \(J\) compatible in the sense that \(g(u, v) := \omega(u, Jv)\) is a Riemannian metric. For example see [18]. We take and fix a compatible almost complex structure \(J\).

We extend \(J\) to \(TM \otimes \mathbb{C}\) complex linearly, and \(-\sqrt{-1}\) and \(-\sqrt{-1}\)-eigenspaces by \(T_{1,0}M\) and \(T_{0,1}M\), respectively. We put \(W := \wedge^0 \otimes T^*M \otimes L = \wedge^*(T^{0,1}M)^* \otimes L\).

The Levi-Civita connection with respect to the Riemannian metric \(g\) together with the Hermitian connection \(\nabla^L\) of \(L\) induces the canonical connection \(\nabla\) on \(W\). Moreover, the Clifford module structure \(c: \Gamma(T^*M) \to \text{End}(W)\) is defined as

\[c(u) := \sqrt{-2}(u^{0,1} \land \alpha - u^{0,1} \land \alpha)\]

for \(u \in T^*M\) and \(\alpha \in W\), where \(u^{0,1}\) is the \((0, 1)\)-factor of \(u \otimes 1 \in T^*M \otimes \mathbb{C} \cong (T^{1,0}M)^* \oplus (T^{0,1}M)^*\). Then, the \(\text{Spin}^c\) Dirac operator is defined to be the composition

\[D := c \circ \nabla: \Gamma(W) \to \Gamma(W)\]

It is well known that \(D\) is a first order, formally self-adjoint, elliptic differential operator of degree-one, and if \((M, \omega, J)\) is Kähler and \(L\) is holomorphic, then \(D\) is nothing but the Dolbeault operator with coefficients in \(L\) up to constant, namely, \(D = \sqrt{2}(\bar{\partial} \otimes L + \partial^* \otimes L)\).

2000 Mathematics Subject Classification. Primary 19K56; Secondary 53D50, 58J22.
Key words and phrases. equivariant local index, localization, quantization conjecture.
Partly supported by Grant-in-Aid for Young Scientists (B) 22740046.
Let $D^0$ and $D^1$ be the degree-zero and degree-one parts of $D$, namely,
\[ D^0 := D\mid_{\gamma \in \text{even} T^* M \otimes L}, \quad D^1 := D\mid_{\gamma \in \text{odd} T^* M \otimes L}, \]
respectively. Since $M$ is closed and $D$ is elliptic, $D$ is Fredholm, namely, both of the kernels of $D^0$ and $D^1$ are finite dimensional vector spaces. Then, the index of $D$ is defined by
\[ \text{ind } D := \dim \ker D^0 - \dim \ker D^1. \]
$\text{ind } D$ is called the Riemann-Roch index. Note that $\text{ind } D$ depends only on $\omega$ and does not depend on the choice of $J$ and $\nabla^L$ since the index is homotopy invariant and the space of compatible almost complex structures of $(M, \omega)$ is contractible. By the Atiyah-Singer index theorem, $\text{ind } D$ can be expressed as
\[ \text{ind } D = \int_M \omega \text{Td}(TM, J), \]
where $\text{Td}(TM, J)$ is the Todd class of the complex vector bundle $TM$ with complex structure $J$. Moreover, if $(M, \omega, J)$ is Kähler and $L$ is holomorphic, then $\text{ind } D$ is equal to the Euler-Poincaré characteristic
\[ \text{ind } D = \sum_{q \geq 0} (-1)^q \dim H^q(M, O_L). \]
For the Spin$^c$ Dirac operators see [16].

A Lagrangian fibration is a fiber bundle $\pi: (M, \omega) \to B$ from $(M, \omega)$ to a manifold $B$ whose fiber is a Lagrangian submanifold of $(M, \omega)$. Note that for a Lagrangian fibration $\pi: (M, \omega) \to B$, the restriction $(L, \nabla^L)\mid_{\pi^{-1}(b)}$ to each fiber $\pi^{-1}(b)$ is a flat line bundle since $F_{\nabla^L} = -2\pi \sqrt{-1} \omega$ and a fiber is Lagrangian. A fiber $\pi^{-1}(b)$ of a Lagrangian fibration $\pi: (M, \omega) \to B$ is said to be Bohr-Sommerfeld if $(L, \nabla^L)\mid_{\pi^{-1}(b)}$ has a non-trivial global parallel section. The Bohr-Sommerfeld condition is equivalent to that the degree zero cohomology $H^0(\pi^{-1}(b); (L, \nabla^L)\mid_{\pi^{-1}(b)})$ with coefficients in the local system $(L, \nabla^L)\mid_{\pi^{-1}(b)}$ is non-trivial. It is known that the Bohr-Sommerfeld fibers appear discretely. Then, in [1] Andersen showed that for a Lagrangian fibration $\pi: (M, \omega) \to B$ the Riemann-Roch index is equal to the number of Bohr-Sommerfeld fibers.

A completely integrable system can be thought of as a Lagrangian fibration with singular fibers. Similar results are known for several completely integrable systems, such as the polygon space [12], the Gelfand-Cetlin completely integrable system on the moduli space of flat $SU(2)$-bundles on a Riemann surface [13].

Suppose $(M, \omega)$ is equipped with an effective Hamiltonian action of a compact Lie group $G$ which lifts to $L$ and preserves all the data. Then, $\ker D^0$ and $\ker D^1$ become $G$-representations. In this case the equivariant Riemann-Roch index is defined as
\[ \text{ind}_G D := \ker D^0 - \ker D^1 \in R(G), \]
where $R(G)$ is the representation ring of $G$. In the case where $G$ is a torus $(S^1)^n$, $M$ is a complex $n$-dimensional nonsingular projective toric variety, and $L$ is an ample line bundle, it is known by Danilov [2] that $\text{ind}_G D$ has the following irreducible decomposition
\[ \text{ind}_G D = \bigoplus_{\gamma \in \mu(M) \cap \mathbb{Z}^n} \mathbb{C}_\gamma^*, \]
where $\mu$ is the moment map associated to $M$ and $\mathbb{C}_\gamma^*$ is the irreducible representation with weight $\gamma^*$. The moment map $\mu$ can be thought of as a Lagrangian fibration with singular fibers. All singular fibers of $\mu$ are smooth tori. Hence, in this case, the notion of a Bohr-Sommerfeld fiber makes sense even for singular...
fibers. Moreover, elements of $\mu(M) \cap t^*_\mathbb{Z}$ correspond one-to-one to Bohr-Sommerfeld fibers. In particular, the Danilov formula can be thought of as a refinement of Andersen’s result. The Danilov formula was generalized to non-symplectic cases, such as, presymplectic toric manifolds [15], Spin$^c$ manifolds [9], and torus manifolds [17].

In the Geometric quantization the Riemann-Roch index and the number of Bohr-Sommerfeld fibers correspond one-to-one to Bohr-Sommerfeld fibers. In particular, the Danilov formula can be thought of as a refinement of Andersen’s result. The Danilov formula was generalized to non-symplectic cases, such as, presymplectic toric manifolds [15], Spin$^c$ manifolds [9], and torus manifolds [17].

In the Geometric quantization the Riemann-Roch index and the number of Bohr-Sommerfeld fibers correspond to the dimensions of the quantum Hilbert spaces obtained by the Spin$^c$ quantization and the geometric quantization using a real polarization, respectively. From the viewpoint of the geometric quantization it is fundamental to investigate the relationship between these two quantizations.

The above results are localization phenomena of the Riemann-Roch index to the Bohr-Sommerfeld fibers. So we have a natural question: We wonder whether all of these localization phenomena might be caused by the same mechanism. If it is true, make clear the mechanism of the phenomena.

For this question we gave a partial answer in [5, 6, 7]. Namely, in [5] we developed an index theory for Dirac-type operators on possibly noncompact Riemannian manifolds, which we call the local index, and improved the result obtained in [5] and obtained a product formula for the local index in [6]. In [7] we refined the local index to the case of torus actions and gave a proof of the quantization conjecture for the Hamiltonian torus actions. Some of the above results, such as the equality between the Riemann-Roch index and the number of Bohr-Sommerfeld fibers for a nonsingular Lagrangian fibration and the Danilov formula for a toric variety, were obtained as consequences of the excision property for the (equivariant) local index. See [5, 6, 7, 8].

Although the index theory developed in [5, 6, 7] is formulated for Riemannian manifolds it seems complicated for non experts. So, in this note, for simplicity, we will explain the equivariant version of the local index for the Hamiltonian $S^1$-actions. It is one of the simplest case.

This note is organized as follows. In Section 2 we will explain two versions of local indices for prequantizable symplectic manifolds with Hamiltonian $S^1$-actions. In particular, when the manifold has an open covering that satisfies certain conditions we can obtain localization formulas for local indices. In Section 3, we will give a way to take such an open covering and will consider the localization formulas for the open covering in details.

## 2. Equivariant local index

### 2.1. $\text{ind}_{S^1}(M, V; L)$

In this subsection let us recall the equivariant local index in the symplectic case. Let $(M, \omega)$ be a possibly non-compact symplectic manifold and $(L, \nabla^L) \to (M, \omega)$ a prequantum line bundle on it. Suppose $M$ is equipped with an effective Hamiltonian $S^1$-action which lifts to $L$ and preserves all the data. Note that each orbit $O$ is isotropic in the sense that $\omega|_O \equiv 0$. In particular, the restriction of $(L, \nabla^L)$ to each orbit is a flat line bundle because of $\nabla^L_{\vec{v}} = \omega$.

In order to define the equivariant local index we introduce the following notion.

**Definition 2.1.** An orbit $O$ is said to be $L$-acyclic if it satisfies the condition $H^0(O; (L, \nabla^L)|_O) = 0$.

Note that the non $L$-acyclic condition is a generalization of the Bohr-Sommerfeld condition for Lagrangian submanifolds.

The following lemma is one of the key points to define the equivariant local index.

**Lemma 2.2.** Let $O$ be an orbit of the $S^1$-action on $M$. Then, the following conditions are equivalent:

1. $O$ is $L$-acyclic.
2. $(L, \nabla^L)|_O$ admits no non-trivial global parallel section.
(3) \( H^\bullet (\mathcal{O}; (L, \nabla^L))|_\mathcal{O} = 0 \).

(4) The kernel of the de Rham operator of \( \mathcal{O} \) with coefficients in \( L \) vanishes.

**Remark 2.3.** An orbit consisting of a fixed point is not \( L \)-acyclic since on such an orbit \((L, \nabla^L)\) always has a non-trivial global parallel section.

**Proof of Lemma 2.29.** It is clear that the first two conditions are equivalent. Since an \( L \)-acyclic orbit \( \mathcal{O} \) is a circle the first and third conditions are equivalent. See [6, Lemma 2.29]. Moreover, by the Hodge theory, the third condition is also equivalent to the fourth condition. \( \square \)

**Example 2.4 (Non-\( L \)-acyclic orbits in \( \mathbb{C}P^1 \)).** Let \( k \) be a positive integer. Define \((M, \omega)\) and \((L, \nabla^L)\) to be the following quotient spaces by the equivalence relations

\[
\begin{align*}
(M, \omega) & := \left( S^3_k, \frac{\sqrt{-1}}{2\pi} \sum_{j=0}^1 dz_j \wedge d\bar{z}_j \right) / \{(z_0, z_1) \sim (h_{z_0}, h_{z_1}) \quad (h \in S^1) \}, \\
(L, \nabla^L) & := \left( S^3_k \times \mathbb{C}, d + \frac{1}{2} \sum_{j=0}^1 (z_j d\bar{z}_j - \bar{z}_j dz_j) \right) / \{(z_0, z_1, v) \sim (h_{z_0}, h_{z_1}, h^k v) \},
\end{align*}
\]

where \( S^3_k := \{ z = (z_0, z_1) \in \mathbb{C}^2 : \|z\|^2 = k \} \). Namely, \((M, \omega)\) is the one-dimensional complex projective space \( \mathbb{C}P^1 \) with \( k\omega_{FS} \), where \( \omega_{FS} \) is the Fubini-Study form which represents the generator of \( H^\bullet (\mathbb{C}P^1; \mathbb{Z}) \), and \( L \) is the \( k \)th tensor power of the hyperplane line bundle \( H^\otimes k \).

Take and fix an integer \( m \). Let us consider the toric \( S^1 \)-action on \( M \) and its lift on \( L \) which is defined by

\[
g[z_0 : z_1, v] := [z_0 : g z_1, g^m v]
\]

for \( g \in S^1 \) and \([z_0 : z_1, v] \in L \). In this example we have the following exactly \( k + 1 \) non-\( L \)-acyclic orbits

\[
\mathcal{O}_i := \{ [z_0 : z_1] \in M : |z_1|^2 = i \} \quad (i = 0, 1, \ldots, k).
\]

In fact, for an orbit \( \mathcal{O} \) take and fix an element \([z_0 : z_1]\) in \( \mathcal{O} \). Then, \( \mathcal{O} \) can be written as \( \mathcal{O} = \{ [z_0 : h z_1] : h \in S^1 \} \). Suppose \( s \in H^0(\mathcal{O}; (L, \nabla^L)) |_\mathcal{O} \) is a non-trivial global parallel section. Then, it is easy to show that \( s \) should be of the form

\[
s([z_0 : h z_1]) = [z_0 : h z_1, h |z_1|^2 s_0]
\]

for some complex number \( s_0 \in \mathbb{C} \). In particular, by (2.1), \( |z_1|^2 \) should be integer since \( s \) is a global section on \( \mathcal{O} \). Conversely, suppose \( |z_1|^2 \) is an integer. Then, (2.1) defines a non-trivial global parallel section on \( \mathcal{O} \).

In [6, 7], we obtained the following theorem.

**Theorem 2.5 (6, 7).** Let \((M, \omega)\) be a possibly non-compact symplectic manifold with effective Hamiltonian \( S^1 \)-action and \((L, \nabla^L) \to (M, \omega)\) an \( S^1 \)-equivariant pre-quantum line bundle on it. Let \( V \subset M \) be an \( S^1 \)-invariant open set which contains only \( L \)-acyclic orbits and whose complement \( M \setminus V \) is compact. For these data, there exists an element \( \text{ind}_{S^1}(M, V; L) \in R(S^1) \) of the representation ring \( R(S^1) \) of \( S^1 \) that satisfies the following properties:

1. \( \text{ind}_{S^1}(M, V; L) \) is invariant under continuous deformation of the data.
2. If \( M \) is closed, then \( \text{ind}_{S^1}(M, V; L) \) is equal to the equivariant index of a \( \text{Spin}^c \) Dirac operator.
3. If \( V' \) is an \( S^1 \)-invariant open subset of \( V \) with complement \( M \setminus V' \) compact, then we have

\[
\text{ind}_{S^1}(M, V; L) = \text{ind}_{S^1}(M, V'; L).
\]
We call $M$ a compact neighborhood of $M \setminus V$. An orbifold version is available. It will be necessary in Section 3.

Remark 2.6. Let $\rho$ be an $S^1$-invariant de Rham operator of degree-one. For $t \geq 0$, the idea used here is the following infinite dimensional analog of the Witten deformation. Let $D_t : \Gamma(W) \to \Gamma(W)$ be the $S^1$-invariant Spin$^c$ Dirac operator. For $t \geq 0$ consider the following perturbation of $D$

$$D_t := D + t \rho D_{\text{fiber}},$$

where $\rho$ is an $S^1$-invariant cut-off function on $M$ with $\rho|_{M \setminus V} \equiv 0$ and $\rho \equiv 1$ outside a compact neighborhood of $M \setminus V$, and $D_{\text{fiber}}$ is an $S^1$-invariant de Rham operator on $V$ along orbits in the following sense, namely,

1. $D_{\text{fiber}} : \Gamma(W|_V) \to \Gamma(W|_V)$ is an order-one, formally self-adjoint $S^1$-invariant differential operator of degree-one.
2. $D_{\text{fiber}}$ contains only derivatives along orbits.
3. For each orbit $O$ in $V$ $D_{\text{fiber}}|_O$ is the de Rham operator of $O$ with coefficients in $(L, \nabla^L)|_O$.
4. For each orbit $O$ in $V$ let $u \in \Gamma(TV|_O)$ be an $S^1$-invariant section perpendicular to the orbit direction. $u$ acts on $\Gamma(W|_O)$ as the Clifford multiplication $c(u)$. Then, $D_{\text{fiber}}$ anti-commutes with $c(u)$.

It is possible to take such a $D_{\text{fiber}}$. Since $V$ contains only $L$-acyclic orbits the third condition for $D_{\text{fiber}}$ and Lemma 2.2 imply that the kernel of $D_{\text{fiber}}|_O$ is trivial for any orbit $O$ in $V$.

First we give a definition of $\text{ind}_{S^1}(M, V; L)$ for the special case where $M$ has a cylindrical end. In [5, 6, 7] we showed the following proposition.

Proposition 2.7. Under the assumption in Theorem 2.5 suppose that $M$ has a cylindrical end $V = N \times (0, \infty)$ and all the data are translationally invariant on the end. Then for a sufficiently large $t \gg 0$, the space of $L^2$-solutions of $D_t s = 0$ is finite dimensional and its super-dimension is independent of a sufficiently large $t \gg 0$ and any other continuous deformations of data.

Definition 2.8. In the case of Proposition 2.7 we define the $\text{ind}(M, V, W)$ to be the super-dimension of the space of $L^2$-solutions of $D_t s = 0$, namely,

$$\text{ind}_{S^1}(M, V; L) := \dim \ker D^0_t \cap L^2 - \dim \ker D^1_t \cap L^2$$

for a sufficiently large $t \gg 0$.

For the general end case, we replace $V$ by a cylindrical end so that all the data are translationally invariant on the end, and come down to the cylindrical end case. We can show that $\text{ind}_{S^1}(M, V, W)$ is well-defined, namely, it does not depend on various choices of the construction. See [5, 6, 7] for more details.

Remark 2.9. To obtain a product formula we need to formulate and define $\text{ind}_{S^1}(M, V; L)$ for a manifold whose end is the total space of a fiber bundle such that both of its base space and its fiber are manifolds with cylindrical end.
Let $(L, \nabla^L) \to (M, \omega)$ and $V$ be the data as in Theorem 2.3. Suppose that there exist finitely many mutually disjoint $S^1$-invariant open sets $V_1, \ldots, V_n$ of $M$ such that $V_1, \ldots, V_n$, and $V$ form an open covering of $M$, namely, $M = V \cup (\cup_{i=1}^n V_i)$. Then, for each $i = 1, \ldots, n$ the equivariant local index $\text{ind}^{S^1}_{V_i}(V_i, V_i \cap V; L|_{V_i}) \in R(S^1)$ is defined, and as a corollary of Theorem 2.3 we have the following localization formula for $\text{ind}^{S^1}(M, V; L)$.

**Corollary 2.10.** $\text{ind}^{S^1}(M, V; L)$ is written as the sum of $\text{ind}^{S^1}(V_i, V_i \cap V; L|_{V_i})$'s, namely,

$$\text{ind}^{S^1}(M, V; L) = \sum_{i=1}^n \text{ind}^{S^1}(V_i, V_i \cap V; L|_{V_i}).$$

Corollary 2.10 implies that $\text{ind}^{S^1}(M, V; L)$ can be described in terms of the data restricted to the neighborhood $V_i$ of non $L$-acyclic orbits.

**Proof of Corollary 2.10.** Since $\cup_{i=1}^n V_i$ is an $S^1$-invariant open neighborhood of $M \setminus V$ the excision property shows

$$\text{ind}^{S^1}(M, V; L) = \text{ind}^{S^1}(\cup_{i=1}^n V_i, \cup_{i=1}^n V_i \cap V; L|_{\cup_{i=1}^n V_i}).$$

Moreover, since $V_i$'s are mutually disjoint, by the sum formula, we obtain the equality in Corollary 2.10.

**Remark 2.11.** We can prove this theorem in the torus action case. In that case we need to construct an additional geometric structure named “strongly acyclic compatible system” on $V$. See [6] [7]. As an application of the theorem for the torus action we can obtain the Danilov formula for a nonsingular projective toric variety $M$. [2]. It will be explained in [8].

**Example 2.12** (Equivariant localization formula for $\mathbb{C}P^1$). Let us consider the case of Example 2.3. Recall that we have the exactly $k+1$ non $L$-acyclic orbits $O_0, \ldots, O_k$. For each $i = 0, 1, \ldots, k$ we take a sufficiently small positive real number $\varepsilon_i > 0$ and define $V_i$ by

$$V_i = \{[z_0 : z_1] \in M : i - \varepsilon_i < |z_1|^2 < i + \varepsilon_i\}.$$ 

We put

$$V := \{[z_0 : z_1] \in M : |z_1|^2 \notin \mathbb{Z}\}.$$ 

Then, for each $i = 0, 1, \ldots, k$ the local index $\text{ind}^{S^1}(V_i, V_i \cap V; L|_{V_i})$ is defined. Now we show the following formula

$$\text{ind}^{S^1}(V_i, V_i \cap V; L|_{V_i}) = C_{i-m}.$$ 

For each $i = 1, \ldots, k-1$, $(L, \nabla^L)|_{V_i} \to (V_i, \omega|_{V_i})$ is equivariantly isomorphic to the trivial line bundle on the cylinder $S^1 \times (i - \varepsilon_i, i + \varepsilon_i)$

$$(S^1 \times (i - \varepsilon_i, i + \varepsilon_i) \times \mathbb{C}, d - 2\pi \sqrt{-1} d\theta) \to (S^1 \times (i - \varepsilon_i, i + \varepsilon_i), dr \wedge d\theta)$$

with $S^1$-action

$$(g e^{2\pi \sqrt{-1} \theta}, r, v) := (g e^{2\pi \sqrt{-1} \theta}, r, g^m v)$$

where $m$ is the order of $V_i$.
for \((e^{2\pi\sqrt{-1}\theta}, r, v) \in S^1 \times (i - \varepsilon_i, i + \varepsilon_i) \times \mathbb{C}\). The isomorphism \(f_i : S^1 \times (i - \varepsilon_i, i + \varepsilon_i) \times \mathbb{C} \to (L, \nabla L)|_{V_i}\) is given as
\[
f_i(e^{2\pi\sqrt{-1}\theta}, r, v) := \left[\sqrt{k - r} : e^{2\pi\sqrt{-1}\theta}\sqrt{r}, v\right].
\]
For each \(i = 0, k, (L, \nabla L)|_{V_i} \to (V_i, \omega|_{V_i})\) is equivariantly isomorphic to the trivial line bundle on the disc \(D_{\varepsilon_i} := \{w \in \mathbb{C} : |w|^2 < \varepsilon_i\}\)
\[
(D_{\varepsilon_i} \times \mathbb{C}, d + \frac{1}{2}(wd\bar{w} - \bar{w}dw)) \to \left(D_{\varepsilon_i}, \sqrt{-1} \frac{1}{2\pi} dw \wedge d\bar{w}\right)
\]
with \(S^1\)-action
\[
g(w, v) := \begin{cases} (gw, g^m v) & \text{if } i = 0 \\ (g^{-1}w, g^{-k}v) & \text{if } i = k \\
\end{cases}
\]
for \((w, v) \in D_{\varepsilon_i} \times \mathbb{C}\). The isomorphism \(f_i : D_{\varepsilon_i} \times \mathbb{C} \to (L, \nabla L)|_{V_i}\) is given as
\[
f_i(w, v) := \begin{cases} [\sqrt{k - |w|^2} : w, v] & \text{if } i = 0 \\ [w : \sqrt{k - |w|^2}, v] & \text{if } i = k. \\
\end{cases}
\]
With the above identifications \(f_i\), we can compute \(\text{ind}_{S^1}(V_i, V_i \cap V : L|_{V_i})\). According to \([5, \text{Remark 6.10}]\) both of \(\text{dim ker} D^0_i \cap L^2\) and \(\text{dim ker} D^1_1 \cap L^2\) in Definition \(2.8\) for \(\text{ind}_{S^1}(V_i, V_i \cap V : L|_{V_i})\) are computed as
\[
\text{dim ker} D^0_i \cap L^2 = 1, \quad \text{dim ker} D^1_1 \cap L^2 = 0,
\]
and a generator of \(\ker D^0_i \cap L^2\) is given as
\[
s_i(e^{2\pi\sqrt{-1}\theta}, r) = \left(e^{2\pi\sqrt{-1}\theta}, r, a_i(r)e^{2\pi\sqrt{-1}\theta}\right)
\]
for \(i = 1, \ldots, k - 1\) and
\[
s_i(w) = (w, b_i(|w|))
\]
for \(i = 0, k\), where \(a_i(r)\) and \(b_i(|w|)\) are some functions on \(r\) and \(|w|\), respectively. See \([5, \text{Remark 6.10}]\) or \([25, \text{Section 5.3}]\) for more details. The \(S^1\) acts on \(\ker D^0_i \cap L^2\) by pull-back. For \(g \in S^1\) we denote by \(\varphi_g\) and \(\psi_g\) the \(S^1\)-action on \(M\) and \(L\), respectively. By using the explicit expressions for the \(S^1\)-actions \((2.3), (2.4)\) and the generators \((2.5), (2.6)\) the \(S^1\)-actions on the generators are written as
\[
(\psi_g^{-1} \circ s_i \circ \varphi_g)(e^{2\pi\sqrt{-1}\theta}, r) = g^{i-m}s_i(e^{2\pi\sqrt{-1}\theta}, r)
\]
for \(i = 1, \ldots, k - 1\), and
\[
(\psi_g^{-1} \circ s_i \circ \varphi_g)(w) = \begin{cases} g^{-m}(s_i(w)) & \text{if } i = 0 \\ g^{k-m}(s_i(w)) & \text{if } i = k.
\end{cases}
\]
Hence, we can obtain the formula \((2.2)\).

By the second property for \(\text{ind}_{S^1}(M, V : L)\) in Theorem \(2.8\) and the formula \((2.2)\) for \(\text{ind}_{S^1}(V_i, V_i \cap V : L|_{V_i})\), the equivariant localization formula in Corollary \(2.10\) is written as
\[
H^0(M; \mathcal{O}_L) = \text{ind}_{S^1} D
= \text{ind}_{S^1}(M, V : L)
= \bigoplus_{i=0}^k \text{ind}_{S^1}(V_i, V_i \cap V : L|_{V_i})
= \bigoplus_{i=0}^k C_{i-m}.\]
2.2. \( \text{ind}_{S^1}^\gamma (M, V; L) \). Let \( t^*_\mathbb{Z} \) be the weight lattice of \( S^1 \). For each \( \gamma^* \in t^*_\mathbb{Z} \) and an element \( U \in R(S^1) \) let us denote by \( U^\gamma \) the multiplicity of the irreducible representation with weight \( \gamma^* \) in \( U \). By taking the multiplicities of the irreducible representations with weight \( \gamma^* \) in the both sides of the equivariant localization formula in Corollary 2.10, we obtain the following localization formula for \( \text{ind}_{S^1}^\gamma (M, V; L) \)\(^\gamma \).

\[
\text{ind}_{S^1}^\gamma (M, V; L) = \bigoplus_{i=1}^{n} \text{ind}_{S^1}^\gamma (V_i, V_i \cap V; L|_{V_i}) \gamma^* .
\]

In this subsection, for each \( \gamma^* \in t^*_\mathbb{Z} \), we define an \((L, \gamma^*)\)-acyclic condition which is a milder condition than the \( L \)-acyclic condition. By using the \((L, \gamma^*)\)-acyclic condition we obtain a version of a local index, which is denoted by \( \text{ind}_{S^1}^\gamma (M, O; L) \), and its localization formula. In particular, (2.7) is obtained as a special case of the localization formula for \( \text{ind}_{S^1}^\gamma (M, O; L) \).

Since the \( S^1 \)-action preserves all the data, for each orbit \( O \), \( S^1 \) acts on \( H^0(O; (L, \nabla^L)|_O) \gamma^* \gamma^* \) by pull-back.

**Definition 2.13.** For each \( \gamma^* \in t^*_\mathbb{Z} \) an orbit \( O \) is said to be \((L, \gamma^*)\)-acyclic if \( O \) does not consist of a fixed point and satisfies the condition \( H^0(O; (L, \nabla^L)|_O) \gamma^* \gamma^* = 0 \).

**Remark 2.14.** By definition, any \( L \)-acyclic orbit is \((L, \gamma^*)\)-acyclic orbit.

The following lemma is a version of Lemma 2.2 for \((L, \gamma^*)\)-acyclic orbits.

**Lemma 2.15.** Let \( O \) be an orbit of the \( S^1 \)-action on \( M \). Then, the following conditions are equivalent:

1. \( O \) is \((L, \gamma^*)\)-acyclic.
2. \( H^*(O; (L, \nabla^L)|_O) \gamma^* = 0 \).
3. The irreducible component with weight \( \gamma^* \) of the kernel of the de Rham operator of \( O \) with coefficients in \( L \) vanishes.

The proof is similar to that of Lemma 2.2.

**Example 2.16 (Non \((L, \gamma^*)\)-acyclic orbits in \( \mathbb{C}P^1 \)).** Let us find non \((L, \gamma^*)\)-acyclic orbits for Example 2.3. By definition, non \((L, \gamma^*)\)-acyclic orbits are orbits consisting of a fixed point, or orbits with \( H^0(O; (L, \nabla^L)|_O) \gamma^* \gamma^* \neq 0 \). The orbits of the former type are

\[
O_0 = \{ [z_0 : 0] \}, \quad O_k = \{ [0 : z_1] \}.
\]

We can show that there exists an orbit of the latter type if and only if \( 0 \leq m + \gamma^* \leq k \), and in that case we have the unique orbit of the latter type which is

\[
O_{m+\gamma^*} = \{ [z_0 : z_1] \in M : |z_1|^2 = m + \gamma^* \}.
\]

Recall that \( O_i \)s are the only orbits which satisfy \( H^0(O_i; (L, \nabla^L)|_O) \neq 0 \), and in that case an element \( s \in H^0(O_i; (L, \nabla^L)|_O) \) has the form (2.1). \( S^1 \) acts on \( H^0(O_i; (L, \nabla^L)|_O) \) by pull-back. For \( g \in S^1 \) and \( s([z_0 : h z_1]) = [z_0 : h z_1, h^{|z_1|^2 - m} h^{|z_1|^2} s_0] \in H^0(O_i; (L, \nabla^L)|_O) \), the \( S^1 \)-action can be written as

\[
(\psi_{g^{-1} \circ s \circ \varphi_g})([z_0 : h z_1]) = [z_0 : h z_1, g^{|z_1|^2 - m} g^{|z_1|^2} s_0].
\]

Thus, \( (\psi_{g^{-1} \circ s \circ \varphi_g}) = g^\gamma s \) if and only if \( |z_1|^2 - m = \gamma^* \).

Now we have a version of Theorem 2.3. See [7] in case of \( \gamma^* = 0 \).

**Theorem 2.17.** Let \( (M, \omega) \) be a possibly non-compact symplectic manifold with effective Hamiltonian \( S^1 \)-action and \((L, \nabla^L) \) \( \rightarrow (M, \omega) \) an \( S^1 \)-equivariant prequantum line bundle on it. Let \( O \subset M \) be an \( S^1 \)-invariant open set which contains only \((L, \gamma^*)\)-acyclic orbits and whose complement \( M \setminus O \) is compact. For these data,
there exists an integer \( \text{ind}_{S^1}^{\gamma^*}(M, O; L) \in \mathbb{Z} \) that satisfies the same properties as in Theorem 2.16.

**Remark 2.18.** In order to define \( \text{ind}_{S^1}^{\gamma^*}(M, O; L) \in \mathbb{Z} \) we replace the \( L \)-acyclic condition by the \((L, \gamma^*)\)-acyclic condition in the construction of the equivariant local index, and consider the multiplicity of the irreducible representation with weight \( \gamma^* \) in \( \text{ker} D_1^0 \cap L^2 - \text{ker} D_1^1 \cap L^2 \) for the perturbed Spin\(^c\) Dirac operator \( D_t \) instead of \( \text{ker} D_1^0 \cap L^2 - \text{ker} D_1^1 \cap L^2 \) itself. In particular, since \( L \)-acyclic orbits are \((L, \gamma^*)\)-acyclic, \( V \) in Theorem 2.16 can be taken as \( O \) in Theorem 2.17. In this case, by definition, \( \text{ind}_{S^1}^{\gamma^*}(M, O; L) \) is equal to \( \text{ind}_{S^1}^{\gamma^*}(M, V; L) \).

Let \((L, \nabla_L) \rightarrow (M, \omega)\) and \( O \) be the data as in Theorem 2.17. Suppose that there exist finitely many mutually disjoint \( S^1 \)-invariant open sets \( O_1, \ldots, O_l \) of \( M \) such that \( O_1, \ldots, O_l \), and \( O \) form an open covering of \( M \), namely, \( M = O \cup (\cup_{i=1}^l O_i) \). Then, for each \( i = 1, \ldots, l \), \( \text{ind}_{S^1}^{\gamma^*}(O_i, O_i \cap O; L|_{O_i}) \in \mathbb{Z} \) is well defined, and we have the following localization formula for \( \text{ind}_{S^1}^{\gamma^*}(M, O; L) \).

**Corollary 2.19.** \( \text{ind}_{S^1}^{\gamma^*}(M, O; L) \) is written as the sum of \( \text{ind}_{S^1}^{\gamma^*}(O_i, O_i \cap O; L|_{O_i}) \)'s, namely,

\[
\text{ind}_{S^1}^{\gamma^*}(M, O; L) = \sum_{i=1}^l \text{ind}_{S^1}^{\gamma^*}(O_i, O_i \cap O; L|_{O_i}).
\]

This formula implies that \( \text{ind}_{S^1}^{\gamma^*}(M, O; L) \) can be described in terms of the data restricted to a sufficiently neighborhood of the fixed point set and orbits with \( H^*(O; (L, \nabla_L)|_O)^{\gamma^*} \neq 0 \).

**Remark 2.20.** By Remark 2.18 if we take \( V \) and \( V_i \)'s in Corollary 2.10 as \( O \) and \( O_i \)'s in Corollary 2.19, respectively, then (2.2) is obtained by Corollary 2.19.

**Example 2.21** (Localization formula for multiplicities in \( \mathbb{C}P^1 \)). In Example 2.10 we showed that for each \( \gamma^* \in \mathfrak{t}_G^\ast \) with \( 0 < m + \gamma^* < k \) there are exactly three \((L, \gamma^*)\)-acyclic orbits \( O_0, O_k \), and \( O_{m+\gamma^*} \), otherwise we have exactly two non-\((L, \gamma^*)\)-acyclic orbits \( O_0 \) and \( O_k \). We put

\[
O_0 := V_0, \ O_k := V_k, \ O_{m+\gamma^*} := V_{m+\gamma^*}, \ \text{and} \ O := \{[z_0 : z_1] \in M : |z_1|^2 \neq 0, k, m+\gamma^* \}.
\]

Then, for each \( i \) \( \text{ind}_{S^1}^{\gamma^*}(O_i, O_i \cap O; L|_{O_i}) \) is defined. By definition, \((O_i, O_i \cap O)\) is equal to \((V_i, V_i \cap V)\). Hence, by Remark 2.18 and the formula (2.2) we obtain

\[
\text{ind}_{S^1}^{\gamma^*}(O_i, O_i \cap O; L|_{O_i}) = \text{ind}_{S^1}^{\gamma^*}(V_i, V_i \cap V; L|_{V_i}) = \begin{cases} 1 & \text{if } 0 \leq m + \gamma^* \leq k \text{ and } i = m + \gamma^* \\ 0 & \text{otherwise} \end{cases}
\]

3. A special case

Let \((L, \nabla_L) \rightarrow (M, \omega)\) be as above. For \( g \in S^1 \) we denote by \( \varphi_g \) and \( \psi_g \) the \( S^1 \)-action on \( M \) and \( L \), respectively. It is well known that corresponding to the infinitesimal lift of the \( S^1 \)-action on \( L \), the moment map \( \mu : M \rightarrow \mathfrak{t}^\ast \) is determined uniquely by the following Kostant formula

\[
\frac{d}{dt}|_{t=0} \psi_{e^{-\mu t} \circ \varphi_{e^{\xi t}}} = \nabla X_\xi s + 2\sqrt{-1}\pi (\mu, \xi) s
\]

for \( \xi \in \mathfrak{t} \) and \( s \in \Gamma(L) \), where \( t \) is the Lie algebra of \( S^1 \), \( (\ , \ ) \) is the natural pairing between \( \mathfrak{t}^\ast \) and \( \mathfrak{t} \), and \( X_\xi \in \Gamma(TM) \) is the infinitesimal action of \( \xi \). We have the
following relationship between non \( L \)-acyclic orbits, \((L, \gamma^*)\)-acyclic orbits, and the values of \( \mu \).

**Lemma 3.1.** (1) Non \( L \)-acyclic orbits are contained in \( \mu^{-1}(t^*_2) \). In particular, fixed points are contained in \( \mu^{-1}(t^*_2) \).

(2) Orbits with \( H^* \mathcal{(O; (L, \nabla^L)|_\mathcal{O})} \gamma^* \neq 0 \) are contained in \( \mu^{-1}(\gamma^*) \).

**Proof.** Let \( \mathcal{O} \) be a non \( L \)-acyclic orbit with \( \mu(\mathcal{O}) = \eta^* \in t^* \). Then, by definition, there exists a non-trivial global parallel section \( s \in H^0 \mathcal{(O; (L, \nabla^L)|_\mathcal{O})} \). For any element \( \xi \) in the integral lattice \( t_2 \) we put

\[
\mathcal{s}_t := \psi_{e-\xi} \circ s \circ \varphi_{e\xi}.
\]

By (3.1), we have

\[
d \frac{ds_t}{dt}(x) = 2\sqrt{-1} \pi \langle \eta^*, \xi \rangle s_t(x)
\]

for \( x \in \mathcal{O} \). Then, \( s_t \) has the form

\[
s_t = e^{2\sqrt{-1} \pi \langle \eta^*, \xi \rangle t} s.
\]

Since \( \xi \in t_2 \), by putting \( t = 1 \),

\[
s = s_1 = e^{2\sqrt{-1} \pi \langle \eta^*, \xi \rangle} s.
\]

Thus, \( \langle \eta^*, \xi \rangle \) should be an integer for arbitrary \( \xi \in t_2 \). This implies the first part.

Let \( \mathcal{O} \) be an orbit with \( H^* \mathcal{(O; (L, \nabla^L)|_\mathcal{O})} \gamma^* \neq 0 \). Then, there exists a non-trivial global parallel section \( s \in H^0 \mathcal{(O; (L, \nabla^L)|_\mathcal{O})} \gamma^* \). For any element \( \xi \in t \), by (3.1), we have

\[
2\pi\sqrt{-1} \langle \gamma^*, \xi \rangle s(x) = \frac{d}{dt} \bigg|_{t=0} \psi_{e-\xi} \circ s \circ \varphi_{e\xi}(x)
\]

for \( x \in \mathcal{O} \). Since \( s \) is non-trivial this implies the second part. \( \square \)

In the rest of this section we assume that \( \mu \) is proper and the cardinality of \( \mu(M) \cap t^*_2 \) is finite. For each \( \gamma^* \in \mu(M) \cap t^*_2 \) let \( \mathcal{V}_{\gamma^*} \) be a sufficiently small \( S^1 \)-invariant neighborhood of \( \mu^{-1}(\gamma^*) \) so that \( \{ \mathcal{V}_{\gamma^*} \}_{\gamma^* \in \mu(M) \cap t^*_2} \) are mutually disjoint. Let \( \mathcal{V} \) be the complement of \( \mu^{-1}(t^*_2) \), namely, \( \mathcal{V} := M \setminus \mu^{-1}(t^*_2) \). By Lemma 3.1 \( \mathcal{V} \) contains only \( L \)-acyclic orbits. Moreover, by assumption, \( \mathcal{V}_{\gamma^*} \setminus \mathcal{V} \) is compact for each \( \gamma^* \in \mu(M) \cap t^*_2 \). Hence, for each \( \gamma^* \in \mu(M) \cap t^*_2 \) the equivariant local index \( \text{ind}_{S^1} \mathcal{(V}_{\gamma^*}, \mathcal{V}_{\gamma^*} \cap \mathcal{V} ; L|_{\mathcal{V}_{\gamma^*}}) \in R(S^3) \) is defined. By applying Corollary 2.10 to this case we have the following localization formula for \( \text{ind}_{S^1} \mathcal{(M, V; L)} \).

\[
(3.2) \quad \text{ind}_{S^1} \mathcal{(M, V; L)} = \bigoplus_{\gamma^* \in \mu(M) \cap t^*_2} \text{ind}_{S^1} \mathcal{(V}_{\gamma^*}, \mathcal{V}_{\gamma^*} \cap \mathcal{V} ; L|_{\mathcal{V}_{\gamma^*}}).
\]

We show the following theorem.

**Theorem 3.2.** For each \( \gamma^* \in \mu(M) \cap t^*_2 \) and \( \sigma^* \in t^*_2 \) with \( \gamma^* \neq \sigma^* \)

\[
\text{ind}_{S^1} \mathcal{(V}_{\gamma^*}, \mathcal{V}_{\gamma^*} \cap \mathcal{V} ; L|_{\mathcal{V}_{\gamma^*}}) \sigma^* = 0.
\]

**Proof.** Since \( L \)-acyclic orbits are \( (L, \sigma^*) \)-acyclic \( \text{ind}_{S^1} \mathcal{(V}_{\gamma^*}, \mathcal{V}_{\gamma^*} \cap \mathcal{V} ; L|_{\mathcal{V}_{\gamma^*}}) \sigma^* \) is also defined, and by Remark 2.18 it is equal to \( \text{ind}_{S^1} \mathcal{(V}_{\gamma^*}, \mathcal{V}_{\gamma^*} \cap \mathcal{V} ; L|_{\mathcal{V}_{\gamma^*}}) \sigma^* \). \( \text{ind}_{S^1} \mathcal{(V}_{\gamma^*}, \mathcal{V}_{\gamma^*} \cap \mathcal{V} ; L|_{\mathcal{V}_{\gamma^*}}) \) is described in terms of the data restricted to a sufficiently neighborhood of the fixed point set and orbits with \( H^* \mathcal{(O; (L, \nabla^L)|_\mathcal{O})} \sigma^* \neq 0 \). By Lemma 3.1 and the definition of \( \mathcal{V}_{\gamma^*} \), if \( \gamma^* \neq \sigma^* \), then \( \mathcal{V}_{\gamma^*} \) contains no orbits of the latter type.
Suppose $V_\gamma$ contains fixed points. By Lemma 3.1 the fixed point set $(V_\gamma)^{S^1}$ is contained in $\mu^{-1}(\gamma^*)$. In particular, $(V_\gamma)^{S^1}$ is compact since $\mu$ is proper. Suppose $(V_\gamma)^{S^1}$ has the exactly $l$ connected components $(V_\gamma)^{S^1}_1, \ldots, (V_\gamma)^{S^1}_l$. For each $i = 1, \ldots, l$ we take a sufficiently small $S^1$-invariant neighborhood $O_i$ of $(V_\gamma)^{S^1}_i$, and also put $O := V_\gamma \setminus (V_\gamma)^{S^1}$. Then, for each $i = 1, \ldots, l$ $ind^{S^1}_{G^\gamma}(O_i, O_i \cap O; L|_{O_i})$ is defined, and by the third property in Theorem 2.11 and Corollary 2.10 for $ind^{S^1}_{G^\gamma}(V_\gamma, O; L|_{V_\gamma})$ we have

$$ind^{S^1}_{G^\gamma}(V_\gamma, V_\gamma \cap V; L|_{V_\gamma}) = ind^{S^1}_{G^\gamma}(V_\gamma, O; L|_{V_\gamma}) = \sum_{i=1}^l ind^{S^1}_{G^\gamma}(O_i, O_i \cap O; L|_{O_i}).$$

Now, for each fixed point $x_0 \in V_\gamma$, the fiber $L_{x_0}$ becomes a representation of $S^1$. By the Kostant formula (3.1), for $\xi \in t$ and $s \in \Gamma(L)$ we have

$$\frac{d}{dt}|_{t=0} \psi_{e^{-t\xi}} (s(x_0)) = \frac{d}{dt}|_{t=0} \psi_{e^{-t\xi}} \circ s \circ \varphi_{e^t}(x_0)$$

$$= (\nabla_{X_\xi} s)(x_0) + 2\sqrt{-1} \pi (\mu(x_0), \xi) s(x_0)$$

$$= 2\sqrt{-1} \pi (\gamma^*, \xi) s(x_0).$$

This implies $\psi_{\gamma}(v) = g^{-\gamma^*} v$ for $g \in S^1$ and $v \in L_{x_0}$. By definition it is easy to see that

$$ind^{S^1}_{G^\gamma}(O_i, O_i \cap O; L|_{O_i}) = ind^{S^1}_{G^\gamma}(O_i, O_i \cap O; L \oplus \mathbb{C}_{\sigma^*}|_{O_i}).$$

Thus, by [7] Theorem 4.1, for $\sigma^* \neq \gamma^*$ we obtain

$$ind^{S^1}_{G^\gamma}(O_i, O_i \cap O; L|_{O_i}) = 0$$

for each $i = 1, \ldots, l$. This proves the theorem. \hfill \Box

As a corollary we have the following formula.

**Corollary 3.3.**

$$ind^{S^1}_{G^\gamma}(M, V; L)^{\gamma^*} = \begin{cases} ind^{S^1}_{G^\gamma}(V_\gamma, V_\gamma \cap V; L|_{V_\gamma}) & \text{if } \gamma^* \in \mu(M) \cap t_2^c \\ 0 & \text{otherwise} \end{cases}$$

So, we have a natural question:

**Question 3.4.** How to compute $ind^{S^1}_{G^\gamma}(V_\gamma, V_\gamma \cap V; L|_{V_\gamma})$?

We give a partial answer of this question. First let us consider the case where $\gamma^* = 0$. Suppose $0 \in t_2^c$ is a regular value of $\mu$. Then, a new symplectic orbifold $(M_0, \omega_0)$ with prequantum line bundle $(L_0, \nabla_{L_0})$ is obtained by the symplectic reduction, namely,

$$(M_0, \omega_0) := (\mu^{-1}(0), \omega|_{\mu^{-1}(0)}) / S^1, \quad (L_0, \nabla_{L_0}) := (((L, \nabla_L)|_{\mu^{-1}(0)}) / S^1.$$

Since $\mu$ is proper $M_0$ is compact. Let $D_0$ be the Spin$^c$ Dirac operator on $(M_0, \omega_0)$ with coefficients in $L_0$. Then, in [7] Section 5.2 we showed the following formula.

**Theorem 3.5 ([7]).** Let $\gamma^* = 0 \in t_2^c$ be a regular value of $\mu$. Then, $ind^{S^1}_{G^\gamma}(V_0, V_0 \cap V; L|_{V_0})$ is equal to the index of $D_0$.

**Remark 3.6.** Note that if $0$ is a regular value of $\mu$, $V_0$ contains no fixed points. In fact, by Lemma 3.1 and the definition of $V_0$, if fixed points exist in $V_0$, they should be contained in $\mu^{-1}(0)$. But, by assumption, $0$ is a regular value of $\mu$. Thus $V_0$ contains no fixed point. In particular, $ind^{S^1}_{G^\gamma}(V_0, V_0 \cap V; L|_{V_0})$ is described
in terms of the data restricted to a sufficiently neighborhood of the orbits with \( H^\bullet \left( \mathcal{O}; \langle L, \nabla^L \rangle |_\mathcal{O} \right)^0 \neq 0 \).

Next let us consider the general case. For a general regular value \( \gamma^* \in t_\ast^2 \) of \( \mu \) we use the shifting trick. By tensoring \( \mathbb{C}_{\gamma^*} \) with \( \langle L, \nabla^L \rangle \) we obtain the prequantum line bundle \( \langle L, \nabla^L \rangle \otimes \mathbb{C}_{\gamma^*} \) on \( (M, \omega) \) with shifted \( S^1 \)-action. Then, the moment map associated to the shifted \( S^1 \)-action, which we denote by \( \mu_{\gamma^*} \), is written as

\[
(3.3) \quad \mu_{\gamma^*} = \mu - \gamma^*.
\]

Since \( \gamma^* \) is a regular value of \( \mu \), \( 0 \) is a regular value of \( \mu_{\gamma^*} \). Hence, by the symplectic reduction for the shifted \( S^1 \)-action, a new compact symplectic orbifold \( (M_{\gamma^*}, \omega_{\gamma^*}) \) with prequantum line bundle \( \langle L_{\gamma^*}, \nabla^{L_{\gamma^*}} \rangle \) is obtained as

\[
(M_{\gamma^*}, \omega_{\gamma^*}) := \left( \mu_{\gamma^*}^{-1}(0), \omega \big|_{\mu_{\gamma^*}^{-1}(0)} \right) / S^1, \quad \langle L_{\gamma^*}, \nabla^{L_{\gamma^*}} \rangle := \left( \langle L, \nabla^L \rangle \otimes \mathbb{C}_{\gamma^*} \big|_{\mu_{\gamma^*}^{-1}(0)} \right) / S^1.
\]

Let \( D_{\gamma^*} \) be the Spin\(^c\) Dirac operator on \( (M_{\gamma^*}, \omega_{\gamma^*}) \) with coefficients in \( \langle L_{\gamma^*} \rangle \). Then, as a corollary of Theorem 3.5 we obtain the following formula.

**Corollary 3.7.** For a regular value \( \gamma^* \in t_\ast^2 \) of \( \mu \) \( \text{ind}_D^\gamma \langle V_{\gamma^*}, V_{\gamma^*} \cap V; L |_{V_{\gamma^*}} \rangle \) is equal to the index of \( D_{\gamma^*} \).

**Proof.** By 3.3 \( V_{\gamma^*} \) is a sufficiently small \( S^1 \)-invariant neighborhood of \( \mu_{\gamma^*}^{-1}(0) \). Thus, by Theorem 3.5 for the prequantum line bundle \( \langle L, \nabla^L \rangle \otimes \mathbb{C}_{\gamma^*} \) on \( (M, \omega) \) with shifted \( S^1 \)-action we obtain

\[
\text{ind}_D^\ast \langle V_{\gamma^*}, V_{\gamma^*} \cap V; L \otimes \mathbb{C}_{\gamma^*} |_{V_{\gamma^*}} \rangle = \text{ind} D_{\gamma^*}.
\]

On the other hand, it is easy to see that

\[
\text{ind}_D^\gamma \langle V_{\gamma^*}, V_{\gamma^*} \cap V; L |_{V_{\gamma^*}} \rangle = \text{ind}_D^\gamma \langle V_{\gamma^*}, V_{\gamma^*} \cap V; L \otimes \mathbb{C}_{\gamma^*} |_{V_{\gamma^*}} \rangle.
\]

This proves the corollary. \( \square \)

In particular, for a closed \( M \), we obtain the the quantization conjecture for the \( S^1 \)-action.

**Corollary 3.8** (11, 11, 13, 19, 23, 24, 20, 22, etc.). Let \( \langle L, \nabla^L \rangle \to (M, \omega) \) be as above. Assume \( M \) is closed. If \( \gamma^* \in t_\ast^2 \) is a regular value of \( \mu \), then

\[
(\text{ind}_D^\gamma)^\gamma = \text{ind} D_{\gamma^*}.
\]

**Proof.** From the second property in Theorem 2.7 \( \text{ind}_D^\gamma (M, V; L) \) is equal to the equivariant index \( \text{ind}_D^\gamma D \) for the Spin\(^c\) Dirac operator \( D \) on \( M \) with coefficients in \( L \). Then, this is a consequence of Corollaries 3.3 and 5.7 \( \square \)

If \( \gamma^* \) is a critical value of \( \mu \), the reduced space \( (M_{\gamma^*}, \omega_{\gamma^*}) \) has more complicated singularities than orbifold singularities in general. Even in this case Meinrenken and Sjamaar defined the Riemann-Roch index for \( (M_{\gamma^*}, \omega_{\gamma^*}) \), and showed that the quantization conjecture still holds for a closed \( M \) [21].

We conclude this note with the following question:

**Question 3.9.** For a critical value \( \gamma^* \), is \( \text{ind}_D^\gamma \langle V_{\gamma^*}, V_{\gamma^*} \cap V; L |_{V_{\gamma^*}} \rangle \) equal to Meinrenken and Sjamaar’s Riemann-Roch index for \( (M_{\gamma^*}, \omega_{\gamma^*}) \)?
REFERENCES

1. J. E. Andersen, Geometric quantization of symplectic manifolds with respect to reducible non-negative polarizations, Comm. Math. Phys. 183 (1997), no. 2, 401–421.
2. V. Danilov, The geometry of toric varieties (Russian), Uspekhi Mat. Nauk 33 (1978), no. 2, 85–134. English translation: Russian Math. Surveys 33 (1978), no. 2, 97–154.
3. H. Duistermaat, V. Guillemin, E. Meinrenken, and S. Wu, Symplectic reduction and Riemann-Roch for circle actions, Math. Res. Lett. 2 (1995), no. 3, 259–266.
4. H. Fujita, M. Furuta, and T. Yoshida, Geodesic flows on the sphere and Riemann-Roch numbers, in preparation.
5. , Torus fibrations and localization of index I, J. Math. Sci. Univ. Tokyo 17 (2010), no. 1, 1–26.
6. , Torus fibrations and localization of index II, UTMS Preprint Series 2009-21, 64 pages. Also available at arXiv:0910.0358, 2009.
7. , Torus fibrations and localization of index III, UTMS Preprint Series 2010-11, 25 pages. Also available at arXiv:1008.5007, 2010.
8. H. Fujita and T. Yoshida, in preparation.
9. M. D. Grossberg and Y. Karshon, Equivariant index and the moment map for completely integrable torus actions, Adv. Math. 133 (1998), no. 2, 185–223.
10. V. Guillemin, Reduced phase spaces and Riemann-Roch, Lie theory and geometry, Progr. Math., vol. 123, Birkhäuser Boston, Boston, MA, 1994, pp. 305–334.
11. V. Guillemin and S. Sternberg, Geometric quantization and multiplicities of group representations, Invent. Math. 67 (1982), no. 3, 515–538.
12. , The Gelfand-Cetlin system and quantization of the complex flag manifolds, J. Funct. Anal. 52 (1983), no. 1, 106–128.
13. L. Jeffrey and J. Weitsman, Bohr-Sommerfeld orbits in the moduli space of flat connections and the Verlinde dimension formula, Comm. Math. Phys. 150 (1992), no. 3, 593–630.
14. Y. Kamiyama, The cohomology of spatial polygon spaces with anticanonical sheaf, Int. J. Math. 3 (2000), no. 3, 339–343.
15. Y. Karshon and S. Tolman, The moment map and line bundles over presymplectic toric manifolds, J. Differential Geom. 38 (1993), no. 3, 465–484.
16. H. B. Lawson and M. L. Michelsohn, Spin geometry, Princeton Mathematical Series, vol. 38, Princeton University Press, Princeton, NJ, 1989.
17. M. Masuda, Unitary toric manifolds, multi-fans and equivariant index, Tohoku Math. J. (2) 51 (1999), no. 2, 237–265.
18. D. McDuff and D. Salamon, Introduction to symplectic topology, second ed., Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 1998.
19. E. Meinrenken, On Riemann-Roch formulas for multiplicities, J. Amer. Math. Soc. 9 (1996), no. 2, 373–389.
20. , Symplectic surgery and the Spin$^c$-Dirac operator, Adv. Math. 134 (1998), no. 2, 240–277.
21. E. Meinrenken and R. Sjamaar, Singular reduction and quantization, Topology 38 (1999), 699–762.
22. Y. Tian and W. Zhang, An analytic proof of the geometric quantization conjecture of Guillemin-Sternberg, Invent. Math. 132 (1998), no. 2, 229–259.
23. M. Vergne, Multiplicities formula for geometric quantization. I, Duke Math. J. 82 (1996), no. 1, 143–170.
24. , Multiplicities formula for geometric quantization. II, Duke Math. J. 82 (1996), no. 1, 181–194.
25. T. Yoshida, RR = #BS via localization of index, Trends in Math. 12 (2010), no. 1, 1–41.