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Multibody System with Elastic Connections for Dynamic Modeling of Compactor Vibratory Rollers

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Abstract: The dynamic model of the system of bodies with elastic connections substantiates the conceptual basis for evaluating the technological vibrations of the compactor roller as well as of the parameters of the vibrations transmitted from the vibration source to the remainder of the equipment components. In essence, the multi-body model with linear elastic connections consists of a body in vertical translational motion for vibrating roller with mass $m_1$, a body with composed motion of vertical translation and rotation around the transverse axis passing through its weight center for the chassis of the car with mass $m$ and the moment of mass inertia $J$ and a body of mass $m'$ representing the traction tire-wheel system located on the opposite side of the vibrating roller. The study analyzes the stationary motion of the system of bodies that are in vibrational regime as a result of the harmonic excitation of the $m$ mass body, with the force $F(t) = m_0 r^2 \omega^2 \sin \omega t$, generated by the inertial vibrator located inside the vibrating roller. The vibrator is characterized by the total unbalanced $m_0$ mass in rotational motion at distance $r$ from the axis of rotation and the angular velocity or circular frequency $\omega$.

Keywords: multibody; elastic bonds; vibrations; initial matrix; stiffness matrix

1. Introduction

The real-time assessment of the degree of compaction of the foundation soil both with stabilized natural soil as well as mixed with stone mineral aggregates or in the case of compaction of asphalt concrete layers, requires precision and high sensitivity of the dynamic response in amplitude of the compactor roller to the changes of soil rigidity as a result of the compaction process.

After each passage on the same compacted layer, the final rigidity of the soil has a new value, higher than the initial rigidity. In this case, after each passage, there can be estimated, through an appropriate instrumental system, the modified amplitude of vibration in correlation with the new state of compaction of the soil corresponding to modified rigidity.

Currently, there are several companies manufacturing vibration compactor machines that use instrumental and computer systems for capturing, treating, and processing the specific signal to the vibration of the vibrating roller. Usually, the dynamic calculation model used is reduced to that of the vibrating roller system with a single degree of freedom, without taking into account the effect of the other vibrating moving masses of the machine.

Frequently, for vibration regime at frequencies in the range of 40–50 Hz, the system ensures the degree of compaction in real time based on the change in rigidity with each passing on the same layer of land. In this case, the first two resonant frequencies are neglected, although they may be important in the work process.

At frequencies between 15 and 30 Hz, the automatic analysis of technological vibration systems produce errors 30% larger, which leads to major inconveniences. For these reasons, the current dynamic
study highlights the influence of the masses of the body assembly at various dynamic regimes for functional frequencies from 15 Hz to 80 Hz. According to the category of the compaction technology, that is, the change in final rigidity after each passage of the compacted layer, there are many scientific and technical approaches with case studies on technologically defined sites that require a more complete dynamic approach, highlighting the influences of the body system on the dynamic response and of the degree of compaction [1,2].

The numerical data used for the case study represent parametric values established on an experimental basis both in the laboratory and “in situ”. [2,3]

2. Multibody System Model

The dynamic multibody model of the vibrating roller is presented in Figure 1 [4–6], where the following notations are used:

- \( I_1 \)—elastic connection point of the vibrating roller with vertical translational movement;
- \( I_2 \)—connection point of the elastic system to the front side of the car chassis;
- \( I_3 \)—connection point between the rear of the car chassis to the traction unit consisting of tire-wheels;
- \( m' \)—mass of the vibrating roller;
- \( m \)—mass of the car chassis;
- \( J \) —moment of mass inertia in relation to the transverse axis \( z \) passing through the center of mass \( C \) of the car chassis;
- \( m_1 \)—mass of the traction group;
- \( k_1 \)—rigidity of the compacted material;
- \( k_2 \)—rigidity of the elastic connection system and dynamic insulation between the vibrating roller and the front chassis;
- \( k_3 \)—combined rigidity of the traction wheel tires in contact with the compacted material;
- \( a, b \)—distances of the \( C \) mass center in relation to the \( I_2 \) and \( I_3 \) ends of a chassis, so that \( a + b = l \), where \( l = I_2 I_3 \) is the equivalent length of the chassis;
- \( x, \varphi = \varphi_z \)—instantaneous displacements of the chassis; and
- \( x_1, x_2, x_3 \)—absolute instantaneous displacements relative to a fixed reference system.

![Figure 1. Dynamic multibody model with linear elastic connections.](image)

Instantaneous displacements of points \( i = 1, 2, 3 \), can be determined with the following matrix relation [7,8]:

\[
\mathbf{u}_i = \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix} = \begin{bmatrix} 0 & 0 & -\varphi_z \\ 0 & 0 & -\varphi_z \\ -\varphi_y & \varphi_z & 0 \end{bmatrix} \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix}
\]

where \( x, y, z \) are the tri-orthogonal instantaneous linear coordinates of the mass center belonging to each rigid body \( I_1 \) and \( C \), respectively.

\( \varphi_x, \varphi_y, \varphi_z \)—the tri-orthogonal instantaneous angular coordinates relative to the competing \( x,y,z \) axes in the center of mass of each rigid body \( C_1 \) and \( C_2 \), respectively.
For the \( m_1 \) mass body with vertical translational motion and the null instantaneous angular coordinates, that is \( \varphi_x = \varphi_y = \varphi_z = 0 \), the displacement of the \( l_1 \equiv C_1 \) point is

\[
\mathbf{u}_1 = \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = x_1
\] (2)

For the mass body \( m \) and moment of inertia \( I_z = J \), with the instantaneous angular coordinates \( \varphi_x = \varphi_y = 0 \) and \( \varphi_z = \varphi \), it has a plane motion \((x, \varphi)\), so that the displacements of points \( l_2 \) and \( l_3 \) can be determined as follows:

\[
\mathbf{u}_2 = \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & -\varphi & 0 \\ \varphi & 0 & 0 \\ 0 & 0 & a \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ a \end{bmatrix} = x - \alpha \varphi
\] (3)

\[
\mathbf{u}_3 = \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & -\varphi & 0 \\ \varphi & 0 & 0 \\ 0 & 0 & -b \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -b \end{bmatrix} = x + b \varphi
\] (4)

2.1. Kinetic Energy of the Multibody System

Taking into account the motion of body \( C_1 \) of mass \( m_1 \) with translation coordinate \( x_1 \) and of the assembled body \( C_2C_3 \) with mass \( m + m' \), moment of mass inertia \( J + m'b^2 \), with coordinates \( x \), \( \varphi \) (vertical translation and rotation), the kinetic energy of the assembly of bodies is \([9,10]\)

\[
2E = \langle \dot{\mathbf{q}}, M\dot{\mathbf{q}} \rangle = \mathbf{q}^T M \dot{\mathbf{q}}.
\] (5)

where \( \dot{\mathbf{q}} \) is the column vector of the generalized velocity with \( \dot{\mathbf{q}} = [\dot{x}_1 \ \dot{x} \ \dot{\varphi}]^T \);

\( M \) — symmetric and positively defined inertia matrix; and

\( \langle \dot{\mathbf{q}}, M\dot{\mathbf{q}} \rangle \) — scalar product between vectors \( \dot{\mathbf{q}} \) and \( M\dot{\mathbf{q}} \).

Matrix \( M \) of the entire system of instantaneous moving bodies with generalized coordinates \( x_1, x, \) and \( \varphi \), consists of inertial elements of zero order \( m_1, m + m' \), one order \( m'b \) and two order \( J + m'b^2 \), placed on the main diagonal and symmetrically in relation to it, highlighting an inertial coupling due to a \( C_3 \) body eccentrically assembled on body \( C_2 \). In this case, matrix \( M \) can be written as follows:

\[
M = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m + m' & m'b \\ 0 & m' & J + m'b^2 \end{bmatrix}
\] (6)

The analytical expression of the kinetic energy, based on relations (5) and (6), can be developed in the form of

\[
2E = \bar{m}_1 x_1^2 + \bar{m}_2 x^2 + \bar{m}_3 \dot{\varphi}^2 + 2\bar{m}_2 \dot{x} \dot{\varphi}
\] (7)

where the following notations were used for the inertia coefficients \( m_2, m_3, \) and \( m_{23} \), so \( \bar{m}_2 = m + m' \); \( \bar{m}_3 = J + m'b^2 \); \( m_{23} = m'b \).

2.2. Elastic Potential Energy

For the elastic elements, modeled as linear springs with rigidities \( k_1, k_2, k_3 \), the vector of the elastic deformations \( \mathbf{v} \), with \( \mathbf{v}^T = [v_1 \ v_2 \ v_3] \) has the following components \([7,11]\):

\[
v_1 = x_1
\]

\[
v_2 = u_2 - x_1 = x - \alpha \varphi - x_1
\] (8)
Thus, vector $v$ can be written as

$$v = \begin{cases} x_1 \\ x-a_1 \varphi - x_1 \\ x + b_1 \varphi \end{cases}$$ \hspace{1cm} (9)$$

The transition from the elastic deformations vector $v$ to the vector of instantaneous displacements $q$ with $q^T = \begin{bmatrix} x_1 & x & \varphi \end{bmatrix}$ can be done by the linear transformation of

$$v = Aq$$ \hspace{1cm} (10)$$

where $A$ is the matrix of the linear transformation as an operator of influence of the displacements on deformations.

Taking into account relations (9) and (10), matrix $A$ can be formulated as follows:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & -a \\ 0 & 1 & b \end{bmatrix}$$ \hspace{1cm} (11)$$

The potential elastic energy $2V$ can be formulated based on the use of the scalar product between vectors $v$ and $K_0 v$, where $K_0 = \text{diag}[k_1, k_2, k_3]$, as follows:

$$2V = \langle v, K_0 v \rangle$$ \hspace{1cm} (12)$$

Using the linear transformation (10) where $A$ has the property of a self-adjoint operator inside the scalar product, relation (12) becomes

$$2V = \langle Aq, K_0 Aq \rangle$$

or

$$2V = \langle q, A^T K_0 A q \rangle = \langle q, Kq \rangle$$ \hspace{1cm} (13)$$

where $K$ is the rigidity matrix of the multibody elastic system.

In this case, matrix $K = A^T K_0 A q$ can be written as

$$K = A^T K_0 A q = \begin{bmatrix} k_1 + k_2 & -k_2 & ak_2 \\ -k_2 & k_2 + k_3 & -ak_2 + bk_3 \\ ak_2 & -ak_2 + bk_3 & a^2 k_2 + b^2 k_3 \end{bmatrix}$$ \hspace{1cm} (14)$$

It is found that matrix $K$ is symmetrical and positively defined with elastic coupling elements symmetrically placed in relation to the main diagonal. In general form, matrix $K$ can be written as follows:

$$K = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix}$$ \hspace{1cm} (15)$$

where elements $k_{ij}$ are those in formulation (15), that is:

$$k_{11} = k_1 + k_2; \quad k_{12} = -k_2; \quad k_{13} = ak_2$$

$$k_{21} = -k_2; \quad k_{22} = k_2 + k_3; \quad k_{23} = ak_2 + bk_3$$

$$k_{31} = ak_2; \quad k_{32} = -ak_2 + bk_3; \quad k_{33} = a^2 k_2 + b^2 k_3$$
The potential elastic energy in analytical form, in this case, can be formulated in the form of 
\[ 2V = \Phi, \]  
follows:
\[ 2V = (k_1 + k_2)x_1^2 + (k_2 + k_3)x_2^2 + \left(a^2k_2 + b^2k_3\right)\varphi^2 - 2k_2x_1x + 2(-ak_2 + bk_3)x\varphi + 2ak_2x_1\varphi = \Phi \]  
(16)

Elastic force \(Q_j\), which corresponds to the generalized coordinate \(q_j\) can be written as follows:
\[ Q_j^V = -\frac{\partial V}{\partial q_j} \]  
(17)

In this case, deriving the relation (16) in the form of \(2V = \Phi\) in relation to coordinate \(q_j\), that is
\[ \frac{\partial(2V)}{\partial q_j} = \frac{\partial \Phi}{\partial q_j}, \]  
leads to
\[ \frac{\partial V}{\partial q_j} = \frac{1}{2} \frac{\partial \Phi}{\partial q_j}, \]  
and thus we obtain
\[ \begin{align*}
Q_1^V &= -\frac{\partial V}{\partial q_1} = -(k_1 + k_2)x_1 + k_2x - ak_2\varphi \\
Q_2^V &= -\frac{\partial V}{\partial q_2} = -(k_2 + k_3)x + k_2x_1 - (-ak_2 + bk_3)\varphi \\
Q_3^V &= -\frac{\partial V}{\partial q_3} = -\left(a^2k_2 + b^2k_3\right)\varphi - (-ak_2 + bk_3)x - ak_2x_1 
\end{align*} \]  
(18)

Taking into account function \(\Phi\) in relation (16) and the fact that \(q_1 = x_1, q_2 = x\) and \(q_3 = \varphi\), applying relations (18), we obtain
\[ \begin{align*}
Q_1^V &= -\frac{1}{2} \frac{\partial \Phi}{\partial q_1} = -(k_1 + k_2)x_1 + k_2x - ak_2\varphi \\
Q_2^V &= -\frac{1}{2} \frac{\partial \Phi}{\partial q_2} = -(k_2 + k_3)x + k_2x_1 - (-ak_2 + bk_3)\varphi \\
Q_3^V &= -\frac{1}{2} \frac{\partial \Phi}{\partial q_3} = -\left(a^2k_2 + b^2k_3\right)\varphi - (-ak_2 + bk_3)x - ak_2x_1 
\end{align*} \]  
(19)

2.3. Disruptive Force

The harmonic excitation is given by the disruptive force \(F(t) = F_0 \sin \omega t\), where the amplitude of the force is \(F_0 = m_0ra^2\). This is applied on body \(C_1\) in order to generate forced vibrations in the vertical direction so that the mass body \(m_1\) and coordinate \(x_1\) only have vertical translational movement.

In this case, the vector of disruptive forces is
\[ \mathbf{f}^F = \begin{bmatrix} F_0 \sin \omega t & 0 & 0 \end{bmatrix} \]

The generalized force corresponding to the disruptive force after the generalized coordinated \(q_j\) can be determined as follows:
\[ Q_j^F = \frac{\delta L_j}{\delta q_j} \]  
(20)

where \(\delta L_j\) is the virtual mechanical work of force \(F\);
\(\frac{\partial q_j}{\partial \varphi}\)—virtual variation of coordinate \(q_j\).

In this case, forces \(Q_1^F, Q_2^F, Q_3^F\) emerge as
\[ Q_1^F = \frac{F\delta x_1}{\delta x_1} = F = F_0 \sin \omega t \]  
(21)

and
\[ Q_2^F = Q_3^F = 0 \text{ because } \delta L_2 = \delta L_3 = 0. \]

3. Analysis of Forced Vibrations

The response of the multibody system with elastic connections is given by the excitation given by the harmonic force \(F(t) = F_0 \sin \omega t\). \(F_0 = m_0ra^2\) defines the inertial force of mass \(m_0\) in the rotational motion at distance \(r\) with the circular frequency \(\omega\) in relation to the axis of rotation of the vibrating device placed symmetrically inside the vibrating roller \([1,2,8]\).
For the multibody system, the Lagrange equations of the second kind can be applied as follows [5,11]:

$$\frac{d}{dt}\left( \frac{\partial E}{\partial \dot{q}_j} \right) - \frac{\partial E}{\partial q_j} = Q^V_j + Q^F_j, \quad j = 1, 2, 3$$  \hspace{1cm} (22)

where $E$ is the kinetic energy expressed by relation (7), and the generalized forces $Q^V_j$ and $Q^F_j$ are given by the relations (19) and (21), respectively.

Taking into account relations (7), (19), and (21), respectively, the Lagrange equations of the second kind given by relation (22), for each degree of freedom, can be written in the form

$$\begin{aligned}
&\frac{m_1 \ddot{x}_1 + (k_1 + k_2)x_1 - k_2x + ak_2 \phi}{m_2 \ddot{x} + m_{23} \ddot{\phi} + (k_2 + k_3)x - k_2x_1 + (-ak_2 + bk_3) \phi} = F_0 \sin \omega t \\
&\frac{\ddot{m}_2x + \ddot{m}_{23}\ddot{\phi} + (k_2 + k_3)x - k_2x_1 + (-ak_2 + bk_3) \phi}{\ddot{m}_{23}x + \ddot{m}_3\ddot{\phi} + (a^2k_2 + b^2k_3) \phi + (-ak_2 + bk_3)x + ak_2x_1} = 0
\end{aligned}$$  \hspace{1cm} (23)

In stationary forced mode, the dynamic response is given by the solutions of the system of linear differential Equation (23), as follows:

$$\begin{aligned}
&x_1 = A_1 \sin \omega t \\
&x = A_x \sin \omega t \\
&\phi = A_\phi \sin \omega t
\end{aligned}$$  \hspace{1cm} (24)

which introduced together with $\ddot{x}_1$, $\ddot{x}$ and $\ddot{\phi}$ in system (23) results in an algebraic system having as unknown amplitudes $A_1, A_x, \text{ and } A_\phi$, as

$$\begin{aligned}
&a_{11}A_1 + a_{12}A_x + a_{13}A_\phi = F_0 \\
&a_{21}A_1 + a_{22}A_x + a_{23}A_\phi = 0 \\
&a_{31}A_1 + a_{32}A_x + a_{33}A_\phi = 0
\end{aligned}$$  \hspace{1cm} (25)

Coefficients $a_{ij}$ $i, j = 1, 2, 3$ have the following expressions thus determined:

$$\begin{aligned}
&a_{11} = k_1 + k_2 - m_1 \omega^2 \\
&a_{22} = k_2 + k_3 - m_2 \omega^2 \\
&a_{33} = a^2k_2 + b^2k_3 - m_3 \omega^2 \\
&a_{12} = a_{21} = -k_2 \\
&a_{13} = a_{31} = -ak_2 \\
&a_{23} = a_{32} = -ak_2 + bk_3 - \ddot{m}_{23} \omega^2
\end{aligned}$$  \hspace{1cm} (26)

The determinant of the unknown coefficients based on relation (25) emerges as follows:

$$D = a_{11}a_{22}a_{33} + 2a_{12}a_{13}a_{23} - a_{11}a_{23}^2 - a_{22}a_{13}^2 - a_{33}a_{12}^2$$  \hspace{1cm} (27)

Condition $D = 0$ generates the pulse equation, from where there emerges three real values of $\omega$ that coincide with the three own pulses $\omega_{ij}, j = 1, 2, 3$.

Amplitudes $A_1, A_x, \text{ and } A_\phi$ are obtained by solving out the algebraic equation system (25) applying Cramer’s method, so that we have

$$A_1 = \left(a_{22}a_{33} - a_{23}^2\right)\frac{m_0r_\omega^2}{D}$$  \hspace{1cm} (28)

$$A_x = A_2 = \left(a_{13}a_{23} - a_{12}a_{33}\right)\frac{m_0r_\omega^2}{D}$$  \hspace{1cm} (29)

$$A_\phi = A_3 = \left(a_{12}a_{23} - a_{13}a_{22}\right)\frac{m_0r_\omega^2}{D}$$  \hspace{1cm} (30)
For a vibrating equipment modeled as a multibody system, the parametric values resulting from
the numerical evaluation are given as follows:

\[ m_1 = 2 \times 10^3 \text{ kg}; \quad \overline{m}_2 = 4.5 \times 10^3 \text{ kg}; \quad \overline{m}_3 = 32 \times 10^3 \text{ kgm}^2; \quad \overline{m}_{23} = 10^6 \text{ kgm}; \quad k_1 = (1; 2; 4; 6) \times 10^7 \text{ N/m}; \quad k_2 = 10^6 \text{ N/m}; \quad k_1 = 1.25 \times 10^6 \text{ N/m}; \quad m_0 r = 2 \text{ kgm}; \quad a = 1 \text{ m}; \quad b = 2 \text{ m}. \]

For the variation of \( \omega \) in the range of values \((0 \div 400) \text{ rad/s}\), the response curves of amplitudes \( A_1(\omega) \), \( A_2(\omega) \), and \( A_3(\omega) \) were drawn and represented in Figures 2–4 for four discrete values of rigidity \( k_1 \). Thus, three own pulses emerge of which the first two at the values \( \omega_{n1} = 12.23 \text{ rad/s}, \quad \omega_{n2} = 22.24 \text{ rad/s} \), are common and constant for the four values of rigidity \( k_1 = (1; 2; 4; 6) \times 10^7 \text{ N/m}; \) the last value of the own pulse \( \omega_{n3} \) is different according to rigidity \( k_1 \). In this case, for \( k_j, j = 1, 2, 3, \) we have \( k_1 = 10^7 \text{ N/m}, \quad \omega_{n3}^{(1)} = 74.73 \text{ rad/s}, \quad k_1 = 2 \times 10^7 \text{ N/m}, \quad \omega_{n3}^{(2)} = 102.1 \text{ rad/s}, \quad k_1 = 4 \times 10^7 \text{ N/m}, \quad \omega_{n3}^{(3)} = 142.6 \text{ rad/s}, \) and \( k_1 = 6 \times 10^7 \text{ N/m}, \quad \omega_{n3}^{(4)} = 174.1 \text{ rad/s}. \) It can be found that in the post-resonance regime for \( \omega > \omega_{nj} \), amplitude \( A_1 \) tends asymptotically toward a constant value and stable motion at the value \( A_1 = 1.245 \text{ mm}, \) and amplitudes \( A_2 \) and \( A_3 \) tend toward very small values, of the order \( 1.87 \times 10^{-3} \text{ mm}, \) respectively, \( 3 \times 10^{-7} \text{ rad}. \)

![Figure 2. Family of curves for amplitude \( A_1 \). (a) Normal representation. (b) Enlarged scale representation.](image-url)
stable motion at the value $A_1 = 1.245$ mm, and amplitudes $A_2$ and $A_3$ tend toward very small values, of the order $1.87 \times 10^{-3}$ mm, respectively, $3 \times 10^{-7}$ rad.

In order to determine the resonance pulses to ensure a post-resonance regime, only the significant linear elastic case was considered, obviously with the neglect of the viscous effects.

**Figure 2.** Family of curves for amplitude $A_1$. (a) Normal representation. (b) Enlarged scale representation.

**Figure 3.** Family of curves for amplitude $A_2$. (a) Normal representation. (b) Enlarged scale representation.

**Figure 4.** Cont.
Thus, the resonance pulses were measured for each case, with an accuracy of ±5 Hz compared to the numerically obtained value. A Bosch hydrostatic control system and a Bruel & Kjaer vibration measurement system were used.

4. Conclusions

The structural assembly of a vibrating roller can be modeled as a system of two rigid bodies with linear elastic connections so that two contradictory desiderata can be achieved simultaneously, namely: achieving technological vibrations for body $C_1$ (vibrating roller) and the significant reduction of the vibrations transmitted to body $C_2$ (machine chassis) in the control cabin was assembled with the working space for the operating mechanic and the drive unit.

In this context, the modeling of the multi-body system was conducted taking into account the inertial characteristics in direct correlation with the possible and significant movements of the two rigid bodies. Thus, the vertical translational motion of body $C_1$ of mass $m_1$ is characterized by a coordinate or a single dynamic degree of freedom that describes the vertical instantaneous displacement.

The motion of the $C_2$ body is characterized by two degrees of dynamic freedom defined by the $x$ and $\varphi$ coordinates. They describe the instantaneous vertical translational motion and respectively, the instantaneous angular rotational motion around the horizontal axis passing through the center of gravity of body $C_2$. In this case, the multibody system is characterized by three degrees of dynamic freedom noted with $x_1$, $x$, and $\varphi$.

As a result of the dynamic study developed in the paper, based on the numerical analysis and the experimental results obtained on five categories of equipment, the presented model faithfully describes the dynamic behavior of the tested equipment. In this context, the following conclusions can be drawn.

(a) The dynamic model of the multibody system with elastic connections is characterized by the inertia matrix $M$ and by the rigidity matrix $K$, both symmetrical in relation to the main diagonal;

(b) The elements of inertial coupling $m_{23} = m'l'$ and of elastic coupling $-k_2$, $ak_2$ and $-ak_2 + bk_3$ are found in the differential equations of motion (23) with significant effects on the equation of own pulses (27) and of amplitudes $A_1$, $A_2$, $A_3$ as a dynamic response to the harmonic excitation $F(t) = m_0\varrho\omega^2 \sin \omega t$.

Figure 4. Family of curves for amplitude $A_3$. (a) Normal representation. (b) Enlarged scale representation.
(c) The numerical and experimental analysis on a vibrating roller equipment, with mass, elastic and excitation data, for the evaluated case study, provides the following conclusions:

- the first two own pulses with relatively low values $\omega_{n1} = 12.23$ rad/s and $\omega_{n2} = 22.24$ rad/s were influenced by the fact that the inertial effect is large enough and rigidity $k_2$ of the elastic connection system between bodies $C_1$ and $C_2$ is low enough for good post-resonance vibration isolation at $\omega > \omega_{n3}$;

- the last own pulse $\omega_{n3}$, is mainly influenced by rigidity $k_1$ of the compaction soil. Thus, for four distinct values of $k_1$, which correspond to successive passages on the same layer of road structure, in the compaction process, there emerged four distinct values of the own pulses (resonance) $\omega_{n3}$, in ascending order, as follows: 74.73 rad/s, 102.1 rad/s, 142.6 rad/s, 174.1 rad/s [12].

(d) The family of curves for amplitudes $A_1$, $A_2$, and $A_3$ represented in Figures 2–4 highlights the fact that in the post-resonance regime for $\omega > \omega_{n3}$, amplitude $A_1$ of the technological vibrations is constant and stable for $\omega \in (300\ldots400)$ rad/s, and amplitudes $A_2$ and $A_3$ tend toward small values, assuring the favorable effect of dynamic insulation for body $C_2$.

(e) The analytical relations (26), (27), and (28) can be used for the parametric optimization of the dynamic response, as follows:

- amplitude $A_1$ of the technological vibrations, which must be constant and stable at the excitation pulse $\omega$, must meet the post-resonance operating condition $\omega > 1.5 \omega_{n3}$. Practically, it is recommended that $\omega = 2\omega_{n3}$ to achieve the technological requirements of efficient compaction;

- amplitudes $A_2$ and $A_3$ of body $C_2$ must have low values so that the degree of isolation of the vibrations transmitted from the body $C_1$ to be $I_v \geq 95\%$; and

- the first two own pulses or resonance circular frequencies must be within the range $(10 \div 60)$ rad/s, so that the influence of the two resonance zones for $\omega = \omega_{n1}$ and $\omega = \omega_{n2}$ becomes negligible for stable optimal operation [13].

Given the above, the analytical approach of the dynamic behavior of multibody systems with effective applications for vibratory rollers for compacting road structures can be useful in the stage of establishing technical design solutions as well as in the parametric optimization stage.

This can be achieved by adjustments and tuning that can be made during the working process such as the discrete change in steps, the static moment $m_0$, and/or the continuous modification of the excitation pulse $\omega$ that can be achieved with the hydrostatic actuation of the vibrator [14].

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