Analyticity Properties and Asymptotic Behavior of Scattering Amplitude in Higher Dimensional Theories

Jnanadeva Maharana\footnote{Adjunct Professor, NISER, Bhubaneswar}
E-mail maharana@iopb.res.in

Institute of Physics
Bhubaneswar - 751005, India

Abstract

The properties of the high energy behavior of the scattering amplitude of massive, neutral and spinless particles in higher dimensional field theories are investigated. The axiomatic formulation of Lehmann, Symanzik and Zimmermann is adopted. The analyticity properties of the causal, the retarded and the advanced functions associated with the four point elastic amplitudes are studied. The analog of the Lehmann-Jost-Dyson representation is obtained in higher dimensional field theories. The generalized J-L-D representation is utilized to derive the \( t \)-plane analyticity property of the amplitude. The existence of an ellipse analogous to the Lehmann ellipse is demonstrated. Thus a fixed-\( t \) dispersion relation can be written down with finite number of subtractions due to the temperedness of the amplitudes. The domain of analyticity of scattering amplitude in \( s \) and \( t \) variables is extended by imposing unitarity constraints. A generalized version of Martin’s theorem is derived to prove the existence of such a domain in D-dimensional field theories. It is shown that the amplitude can be expanded in a power series in \( t \) which converges for \(|t| < R\); \( R \) being \( s \)-independent. The positivity properties of absorptive amplitudes are derived to prove the \( t \)-plane analyticity of amplitude. In the extended analyticity domain dispersion relations are written with two subtractions. The bound on the total cross section is derived from LSZ axioms without any extra \textit{ad hoc} assumptions.

\footnotetext{Adjunct Professor, NISER, Bhubaneswar}
1. Introduction

The high energy scattering of hadrons is of paramount importance. The high energy accelerators measure total cross sections, elastic differential cross sections, the characteristics of the forward diffraction peaks to mentions a few observable quantities. On the other hand there are strong constraints on such measurable physical quantities based on general frame works of quantum field theories - the so called axiomatic field theoretic approach. One of the celebrated achievements of the so called axiomatic field theoretic formalism is that the total cross sections in hadronic reactions is bounded by square of the logarithm of the center of mass energy in the four spacetime dimensional theories. The experimental data for hadronic reactions respect this bound from energy of a few GeV to highest accelerator energies.

Heisenberg [1], based on his deep insight and intuitions, concluded that the total cross section for scattering of hadrons will grow as square of logarithm of the energy. He supported his arguments in a field theoretic model. The experimental data, at that time, limited to cosmic energy data in the MeV range lent supports to his theory. Froissart’s derivation of the bound on total cross sections, that bears his name [2], was based on the assumption that the amplitudes satisfy the Mandelstam representation. However, the Mandelstam representation has not been proved in the general frame work of field theory. The derivation of the bound, on total cross section, which is based on the rigorous results deduced from general field theory, is credited to Martin [3].

The purpose of this investigation is to study analyticity properties of four point scattering amplitudes of field theories in higher spacetime dimensions. Our intent is to derive rigorous results on scattering amplitude and total cross section based on the axiomatic frame work of Lehmann, Symanzik and Zimmermann [4]. The interests in theories of higher spacetime dimensions have been increasingly growing over a few decades. It is believed that supersymmetry will provide an understanding of questions related to consistencies of the standard model such as the gauge hierarchy problem. There is hope that supersymmetry might be discovered in the high energy collision experiments in near future. Supersymmetric theories have been generalized to higher spacetime dimensions subsequent to their discovery. The supergravity theories incorporate gravity and they possess interesting attributes when generalized to spacetime dimensions beyond four dimensions. Moreover, the five perturbatively consistent string theories are constructed in ten spacetime dimensions. The string theories hold the prospect of unifying the fundamental forces. There are intense research activities and keen interests to study diverse aspects of theories in higher dimensions. Moreover, there are proposals that, in certain scenario, the evidence for the existence of higher spacetime dimensions might be revealed in the experiments of Large Hadron Collider (LHC) [5, 6]. In view of such prospects it is necessary to examine what kind of precise theoretical results can be derived in a model indepen-

\footnote{see these two articles for reviews and extended references}
dent manner as far as possible. In recent years, several important issues related to high energy gravitational scatterings and the conceptual frame works for such processes have drawn attentions. A comprehensive review of this topic will be found in Erice lectures of Giddings [7]. The importance of analyticity and unitarity in higher dimensional field theories and their relevance in string theory has been investigated by Pius and Sen [8]. Moreover, Irrizary-Gelpi and Siegel have studied the properties of four point amplitude in higher dimensional relativistic theory in the JWKB approximation [9]. Therefore, there are strong motivations to study the analyticity properties of higher dimensional field theories and derive rigorous results which are not based on any specific model. Our strategy in deriving the analog of Froissart-Martin bound for higher dimensional field theories is to adopt the LSZ formulation of field theory in higher spacetime dimensions. As we shall deliberate on our approach in sequel, we shall be guided by the developments in the four dimensional theories \(^ 3\) [10, 11, 12, 13, 14, 15, 16, 17]; however, there are certain obstacles to be surmounted. We shall mention them at the appropriate places. We have adopted the axiomatic frame work of Lehmann, Symanzik and Zimmermann (LSZ). It might be possible to derive the results presented here from Wightman axioms [18, 19, 20] or from the more general structure proposed in theory of local observable [21, 22, 23]. However, we adhere to LSZ formulation and endeavor to deduce results for higher dimensional field theories. We consider scattering of massive, spinless, neutral particles of a single specie in D-dimensional flat spacetime with Lorentzian signature. We also assume that there are no bound states in this theory. Our approach to the higher dimensional theories will be clear as we proceed and we shall state our axioms in the next section. It is well known that there are host of exact results for collisions of strongly interacting particles which are stated as theorems. These theorems are derived under the general assumptions of quantum field theories without appealing to any model. Notable among them is the Froissart-Martin bound

\[ \sigma_t(s) < \frac{4\pi}{t_0 - \epsilon} \ln^2(s/s_0) \]

which restricts the growth of the total cross section, \( \sigma_t \), at high energies; \( s \) being the center of mass (c.m.) energy squared and \( t_0 \) is a known constant derived from field theory. This bound is derived from exact results which follow from axiomatic field theory. We shall elaborate on these aspects in Section 2. However, we present the essential ingredients that lead to the bound on total cross section:

(a1) Unitarity of S-matrix.

(a2) The amplitude is analytic in complex \( \cos \theta \) inside the Lehmann-Martin ellipse; \( \theta \) being the scattering angle in the center of mass frame. The focii of the ellipse lie at \((-1, +1)\) and its semi-major axis is \( \cos \theta_0 > 1 \). The partial wave expansion of the

\(^3\)We refer to several books, lecture notes and review articles here. These references provide a background on dispersion relations, derivation of analyticity domains and Froissart bound.
amplitude, $F(s, t)$, converges absolutely inside Lehmann-Martin ellipse,

$$F(s, t) = \frac{\sqrt{s}}{k} \sum_{l=0}^{\infty} (2l + 1) f_l(s) P_l(cos \theta)$$  \hspace{1cm} (2)

and $F(s, t)$ is analytic in this region, $k$ is the c.m. momentum. Unitarity bounds on the partial-wave amplitudes are

$$0 \leq |f_l(s)|^2 \leq \text{Im } f_l(s) \leq 1$$  \hspace{1cm} (3)

(a3). Polynomial boundedness of the amplitude [24]. For $0 \leq t < t_0$

$$|F(s, t)| < |s|^{-N}$$  \hspace{1cm} (4)

N being a positive integer.

(a4). $F(s, t)$ is analytic in the complex $s$-plane. There are cuts in the $s$-plane as a consequence of $s$-channel unitarity and crossing symmetry - crossing is a requirement and is proved for several reactions. There are other important bounds on the elastic differential cross sections, the width of the diffraction peak and slope of the diffraction peak to name a few. The statements (a1) - (a4) have been proved in the frame work of axiomatic field theory.

In the context of higher dimensional field theories, the analog of Froissart bound has been derived for scattering of massive spinless particles [25, 26]. There is only one scattering angle for spinless scattering although for the most general case, the amplitude is to expressed in terms of complete set of basis functions (see Section 4 for detailed discussions) for representation of $SO(D-1)$ rotation group. The bound is

$$\sigma_t \leq C_0 (\ln s)^{D-2}$$  \hspace{1cm} (5)

where $C_0$ is a constant, independent of $s$. This bound was derived under certain reasonable assumptions inspired by the proven results for $D = 4$ field theories. Note that for $D = 4$, one recovers the high energy bound i.e. $\sigma_t \leq \text{Const.}(\ln s)^2$.

Let us briefly discuss the essentials steps for derivation of (5) in [25, 26]. The amplitude for scattering of massive spinless particles in D-dimensional spacetime admits a partial wave amplitude expansion [27]

$$F^\lambda(s, t) = A_1 s^{-\lambda+1/2} \sum_{l=0}^{\infty} (l + \lambda) f_l^\lambda(s) C_l^\lambda(t)(1 + 2t/s)$$  \hspace{1cm} (6)

where $\lambda = \frac{1}{2}(D - 3)$ and $(s, t)$ are the usual Mandelstam variables. $A_1$ is a constant which is independent of $s$ and $t$; however it depends on $\lambda = \frac{1}{2}(D - 3)$ and contains some prefactor like powers of $\pi$ and other numerical constants. We shall display them in Section 4. Here the basis functions are the Gegenbauer polynomials, $C_l^\lambda(x)$, and they satisfy orthonormality properties with weight factor $(1 - x^2)^{D/2-2}$, $-1 \leq x \leq +1$. 

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The expansion (6) converges in the domain \(-1 \leq \cos \theta \leq +1\) [28]. The prefactor \(s^{-\lambda+1/2}\) is introduced on the right hand side of (6) in order that the partial wave amplitudes are dimensionless. Furthermore, the partial-wave amplitudes, \(\{f^\lambda_l(s)\}\), satisfy the unitarity constraint (3) [27]. The bound (5) was derived [25, 26] with certain extra assumptions which had not been derived from a field theoretic basis.

The two crucial assumptions were: (AI) The amplitude is polynomially bounded i.e. \(|F^\lambda(s, t)| < Cs^N\); C and N are undetermined constants, N is real positive number. (AII) The domain of convergence of the \(F^\lambda(s, t)\) is an extended ellipse with the semi-major axis \(1 + 2\bar{T}_0/s\) in the \(t\)-plane. The Gegenbauer polynomial is the basis set of functions for the case at hand and its domain of convergence is \(-1 \leq \cos \theta \leq +1\). Notice that \(\bar{T}_0\) is an undetermined constant and it is independent of \(s\). In other words the amplitude has a larger analyticity domain in the \(\cos \theta\)-plane than \(-1 \leq \cos \theta \leq +1\). Therefore, assuming the existence of the analog of the Lehmann ellipse looks quite reasonable. Similarly, the assumption (AI) of polynomial boundedness for the scattering amplitude is an acceptable proposition, although there existed no proof for it in D-dimensions. The authors [25, 26] derive the above mentioned bound (5) under assumptions (AI) and (AII). The constant \(C_0\), although is independent of \(s\), is expressed in terms of the dimensionality of spacetime, D, the constant, \(\bar{T}_0\) and \(N\); thus \(N\) decides the number of subtractions required in writing the dispersion relation for the scattering amplitude. To contrast with the Froissart-Martin bound (case of \(D = 4\)), the two parameters \(\bar{T}_0\) and \(N\) are determined from axiomatic field theoretic consideration and the analyticity domain of the amplitude is specified. In most of the cases (in \(D = 4\)), the dispersion relations for hadronic scattering have been proved. The bound (5), interesting as it is, lacks the rigorous basis which is known to have been proved for 4-dimensional case. Thus it is quite desirable to make an endeavors to derive (AI) and (AII) from a field theoretic frame work.

It is pertinent to mentioned that the consequence of assumptions (AI) and (AII) lead to certain important conclusions. First thing to note is that the partial wave amplitudes in the expansion for \(F^\lambda(s, t)\) fall off exponentially beyond a cut off \(L\). This is a very crucial feature and we shall dwell upon this aspect in sequel. Moreover, in the above mentioned work [25, 26], as is obvious from the bound on \(\sigma_\tau\), the growth is still a power of \(\ln s\). Furthermore, there are some interesting results where scattering amplitude and slope associated with the forward amplitude are constrained.

I examined [29] the scattering problem for \(D > 4\) and derived some new results in that one could obtain a upper bound for \(|F^\lambda(s, t)|\) in a certain domain in the complex \(t\)-plane. Furthermore, I presented a theorem on the distributions of zeros of the scattering amplitude in the complex \(t\)-plane. Similar results have been derived for four dimensional theories [30] and their importance is well known in that context. Moreover, with an additional assumption about the distribution of zeros in a small domain in \(t\)-plane which includes the physical \(t\)-region, I derived upper and lower bounds on absorptive part of the scattering amplitude [29]. I also proposed that it might be possible to get a glimpse of the presence of higher
dimensions [29] from the measurement of high energy total cross section data. All theories defined in higher spacetime dimensions have to adopt a compactification scheme in the sense that the radii of compactification of the extra compact dimensions are very small and therefore, such length scales might not be revealed by present high energy experiments. There are several compactification schemes. Moreover, the compactification schemes for the higher dimensional theories have revealed interesting symmetries of the theories dimensionally reduced to lower spacetime dimensions. The compactification scale is argued to be generally at higher energy scale beyond the reach of accelerator energies in the foreseeable future. However, there are concrete models where the scale of compactification is expected to be relatively low (i.e. 500 GeV to TeV) in the sense that the massive particle spectra arising from such compactification proposals could be observed in Large Hadron Collider (LHC) experiments. Therefore, it might be possible to observe the presence of extra dimensions at LHC. I have proposed [29] that the energy dependence of total cross sections might display such a feature that \( \sigma_t \) would seemingly violate the Froissart-Martin bound when very high energy total cross section data is fitted in the energy range beyond decompactification regime such as 500 GeV to 1 TeV. However, the high energy bounds in higher dimensions as alluded to above have different energy dependence as power of \( \ln s \). Thus a precision analysis of data might reveal departure from the Froissart bound and it can be interpreted as decompactification to extra dimensions. Indeed there are specific models which advocate that the scale of compactification could be as low as 500 GeV to 1 TeV and the data from the LHC does not completely rule out low scale compactification models [5, 6].

If we were to pursue the proposition presented above, look for (possible) evidence for the violation of Froissart-Martin bound in very high energy scatterings where extra spatial dimensions might have undergone decompactification then it is desirable that the constant \( \tilde{T}_0 \) is determined in terms of mass scale parameters of the theory under discussion i.e. mass scale of decompactification or mass parameter of the theory in question. It is obvious, this constant \( \tilde{T}_0 \) is not to be related to presently determines scale (i.e. \( t_0 = 4m^2 \)). Second issue that deserves attention is what is the value of \( N \) in (AI) which decides the number of subtractions required to write a dispersion relation for the amplitude (6). Our attention is focused to resolve questions alluded to in the preceding paragraph which are pertaining to the assumptions (AI) and (AII) stated above. We shall work in the frame work of LSZ formulation as mentioned earlier. Therefore, the sequence of our investigation is close to earlier formulations pursued in the the case of 4-dimensional field theory. There are certain difficulties when we attempt to address problems in D-dimensional field theory and we shall discuss them in each of the sections as we proceed. However, we may mention in passing some of these issues to illustrate the type of problems we have encountered. One of the important results that led Martin to extend the domain of analyticity of scattering amplitude in \( s \) and \( t \) is to use certain positivity properties of the absorptive amplitude and its
$t$-derivatives. This property was crucial to derive certain inequalities and eventually to prove analyticity. In order to arrive at the proof of the positivity properties, one has to resort to partial wave expansion. We have derived these positivity relations in Section 4. Another important problem which deserved attention is to show the existence of the analog of the Lehmann ellipses [31]. We have developed the required technical tools and presented the derivation of the analog-Lehmann ellipse.

The rest of the paper is organized as follows. In the next section, we present a short account of earlier known results of field theory which led to derivation of Froissart-Martin bound. This section outlines the prescriptions for derivation of the main results. We formulate the D-dimensional field theory following the axioms of LSZ formalism. There are no difficulties in defining $in$ and $out$ states for a D-dimensional theory. Moreover, the LSZ reduction procedure can be adopted for the D-dimensional case to derive the expressions for the amplitudes. We also derive expressions for the retarded function, the advanced function and the causal function (which are vacuum expectation values of distribution valued operators) and therefore the aforementioned functions are distributions. The goal is to derive the Jost-Lehmann-Dyson representation for these distributions. We adopt a specific coordinate frame for the D-dimensional case to obtain the Jost-Lehmann representation for the causal function. Note that the derived J-L-D representation is expressed as a function of Lorentz invariant variables and therefore, the results are true in any other frame. Subsequently, we obtain the representation for the retarded function. We also show how the elegant technique of Dyson can be generalized to the D-dimensional case. The generalized Dyson’s theorem and its proof utilizing Dyson’s technique is given in Appendix A in some details. Section 3 contains the derivation of the analog-Lehman ellipse for D-dimensions. This is quite important for the derivations of generalized Martin’s theorem in the next section. Section 4 is devoted to study the analyticity of the amplitude for D-dimensional case. We derive the generalized Martin’s theorem in Section 4. Finally, we derive the analog of Froissart-Martin bound in this section. It is argued that the scattering amplitude requires at most two subtractions in the extended domain of holomorphy. We determine the semimajor axis of the large Lehmann ellipse to be $1 + \frac{2R_s}{\sigma}$ where $R = 4m^2 - \epsilon$, $m$ being the mass of the scalar particle in the D-dimensional theory. Therefore, no ad hoc parameter appear in the expression for the bound on $\sigma_t$ in the D-dimensional theory. We summarise our results in Section 5. In proving Martin’s theorem, Martin had used two ‘tricks’ in his original paper and they are also presented in his books [10, 11]. We have presented these as lemmas in Appendix B. Appendix C contains collection of some useful formulas for the Gegebauer polynomials.
2. Review of Analyticity Properties of Scattering Amplitude and Bound on the Total Cross Section.

We review some of the important results of axiomatic field theory which are necessary to derive analyticity properties of the scattering amplitudes and to obtain asymptotic bound like the Froissart bound. Indeed, these techniques will be implemented for higher dimensional field theories. However, the results of four dimensional field theories are not automatically applicable in higher dimensions as we shall discuss in the next section. In the case of D-dimensional field theories, we encounter certain obstacles in our intent to (eventually) derive constraints on the growth properties of the amplitude at asymptotic energies. The resolution of these issues will be presented at the appropriate junctures in the text. In fact the existing results of 4-dimensional theories provide guidance for our investigations. Let us first envisage the axioms necessary to initiate the approach in the LSZ formulation. It is worth mentioning that the axioms stated below hold for D-dimensional theories and these are not special features of 4-dimensional theories.

The Axioms:

A1. The states of the system are represented in a Hilbert space, \( \mathcal{H} \). All the physical observables are self-adjoint operators in the Hilbert space.

A2. The theory is invariant under inhomogeneous Lorentz transformations.

A3. The energy-momentum of the states are defined. It follows from the requirements of the Lorentz invariance that we can construct a representation of the orthochronous Lorentz group. The representation corresponds to unitary operators, \( U(a, \Lambda) \), and the theory is invariant under these transformations. Thus there are hermitian operators corresponding to spacetime translations, denoted as \( P_\mu \) which have following properties:

\[
\left[ P_\mu, P_\nu \right] = 0 \tag{7}
\]

If \( \mathcal{F}(x) \) is any Heisenberg operator then its commutator with \( P_\mu \) is

\[
\left[ P_\mu, \mathcal{F}(x) \right] = i\partial_\mu \mathcal{F}(x) \tag{8}
\]

The operator does not depend explicitly on spacetime coordinates, \( x^\mu, \mu = 0, 1, \ldots, D-1 \). If one chooses a representation where the translation operators, \( P_\mu \), are diagonal and the basis vectors \( |p, \alpha > \) span the Hilbert space, \( \mathcal{H} \), such that

\[
P_\mu |p, \alpha > = p_\mu |p, \alpha > \tag{9}
\]

then we are in a position to make more precise statements:

- Existence of the vacuum: there is a unique invariant vacuum state \( |0 > \) which has the property

\[
U(a, \Lambda)|0 > = |0 > \tag{10}
\]
The vacuum is unique and Lorentz invariant.

- The eigenvalue of $P_\mu, p_\mu$, is light-like, with $p_0 > 0$. We are concerned only with massive stated in this discussion. If we implement infinitesimal Poincare transformation on the vacuum state then

$$P_\mu |0> = 0, \quad \text{and} \quad M_{\mu\nu} |0> = 0 \quad (11)$$

from above postulates. $M_{\mu\nu}$ are the generators of Lorentz transformations.

**A4.** The locality of theory implies that a (bosonic) local operator at spacetime point $x^\mu$ commutes with another (bosonic) local operator at $x'^\mu$ when their separation is spacelike i.e. if $(x - x')^2 < 0$. Our Minkowski metric convention is as follows: the inner product of two D-vectors is given by $x.y = x^0y^0 - x^1y^1 - ... - x^{D-1}y^{D-1}$. Since we are dealing with a neutral scalar field, for the field operator $\phi(x)$: $\phi(x)^\dagger = \phi(x)$ i.e. $\phi(x)$ is hermitian. By definition it transforms as a scalar under inhomogeneous Lorentz transformations as

$$U(a, \Lambda)\phi(x)U(a, \Lambda)^{-1} = \phi(\Lambda x + a) \quad (12)$$

The micro causality can be stated as

$$\left[\phi(x), \phi(x')\right] = 0, \quad \text{for} \quad (x - x')^2 < 0 \quad (13)$$

It is well known that in the LSZ formalism we are concerned with vacuum expectation values of time ordered products of operators as well as with the the retarded products. The requirements of the above listed axioms are realized as certain attributes of the T-products and R-products of operators. Furthermore, the axioms also establish important relationships for vacuum expectation values of time ordered products and similarly for the R-products. It is recognized that when we consider vacuum expectation values of retarded products of field operators (the so called r-functions) the implementation of the axioms, listed above, lead to certain linear relations among these functions [13, 14] as we shall derive later. Notice that when we impose unitarity constraints they yield nonlinear relations among these functions. It is worth emphasizing that separation into set of linear relations and nonlinear relations is a hallmark of the axiomatic approach.

If we contrast the LSZ formulation with the familiar Lagrangian formalism, the (free) linear theory is rendered trivial. Moreover, equations of motioned derived for the interacting theory are nonlinear in the Lagrangian approach in general. In the case of latter, it is not possible to state rigorously the attributes of the solutions to the field equations. Therefore, the computation of the S-matrix elements, in the Lagrangian formulation, is carried out through a chain of well defined and consistent prescriptions in the frame work of perturbation theory. On the other hand in the LSZ approach the linear relations have important consequences.

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\[ ^4\text{Itzykson and Zubber have discussed this aspect in their treatise [12]} \]
We proceed to state some of the salient features of the LSZ formulation. One of the most important requirements is the asymptotic condition. This, stated in nutshell, says that the field theory can be described in terms of asymptotic observables which correspond to particles of definite mass and charge. Note, however, that we are to deal with neutral massive particles. \( \phi(x) \) in represents a free field and it generates a Fock space. The dynamics is encoded in this formulation. The physical observables are expressible in terms of the field in a unique manner. LSZ also provide a method to relate the field \( \phi_{in}(x) \) with the interacting field \( \phi(x) \). According to their formulation, \( \phi_{in}(x) \) is to be defined in an appropriate limit of \( \phi(x) \). They invoke the concept of adiabatic switching off of interaction which is another ingredient in the LSZ approach. They introduce the postulate of an adiabatic cut off function so that this function controls the interactions. It is \( 1 \) at finite time and it has smooth limit of going to zero as \( |\text{time}| \to \infty \). Moreover, another postulate is that if we remove the adiabatic switching it will be possible to define all physical quantities. The relationship between \( \phi_{in}(x) \) and \( \phi(x) \) is given by

\[
x_0 \to -\infty \quad \phi(x) \to Z^{1/2} \phi_{in}(x) \tag{14}
\]

By the first postulate, \( \phi_{in}(x) \) creates free particle states. However, in general \( \phi(x) \) will create multi particle states besides the single particle one since it is the interacting field. Moreover, \( <1|\phi_{in}(x)|0> \) and \( <1|\phi(x)|0> \) carry same functional dependence in \( x \). If the factor of \( Z \) were not the scaling relation between the two fields (i.e. \( \phi_{in}(x) \) and \( \phi(x) \)) will be the same. Thus in the absence of \( Z \) the two theories will be identical. Moreover, the postulate of asymptotic condition states that in the remote future

\[
x_0 \to \infty \quad \phi(x) \to Z^{1/2} \phi_{out}(x). \tag{15}
\]

Furthermore, the vacuum is unique for \( \phi_{in}, \phi_{out} \) and \( \phi(x) \). The normalizable single particle states are the same i.e. \( \phi_{in}|0>=\phi_{out}|0> \). We do not display \( Z \) from now on. If at all any need arises, \( Z \) can be introduced in the relevant expressions. It is essential to define creation and annihilation operators for \( \phi_{in}, \phi_{out} \) and \( \phi \). We use the plane wave basis for simplicity; however, in a more formal approach, it is desirable to use wave packets [12]. Now

\[
\phi_{in}(x) = \frac{1}{(2\pi)^{(D-1)/2}} \int \frac{d^{D-1}k}{2|k_0|} [e^{-ik.x}a_{in}(k) + e^{+ik.x}a_{in}^\dagger(k)] \tag{16}
\]

and

\[
\phi_{out}(x) = \frac{1}{(2\pi)^{(D-1)/2}} \int \frac{d^{D-1}k}{2|k_0|} [e^{-ik.x}a_{out}(k) + e^{+ik.x}a_{out}^\dagger(k)] \tag{17}
\]

note that \( k \) is \( (D-1) \)-component spatial momentum vector of D-momentum, \( k \). The operators \( a_{in}(k) \) and \( a_{out}(k) \) and their hermitian conjugates are postulated to be weak coupling limits of \( a(k,x_0) \) and its hermitian conjugate in the asymptotic limits,
\(x_0 \rightarrow \pm \infty\) for 'in' and 'out' operators respectively i.e. \(a_{in}(k) = (\text{weak lim } x_0 \rightarrow -\infty) a(k, x_0)\). Similar definition is to be understood for creation operators for the 'in' case and corresponding limiting prescription is to be defined for 'out' operators. The mode expansion of the interacting field \(\phi(x)\) is defined below

\[
\phi(x) = \frac{1}{(2\pi)^{(D-1)/2}} \int \frac{d^{D-1}k}{2|k_0|} \left[ e^{-ik \cdot x} a(k, x_0) + e^{i\frac{k \cdot x}{2}} a^\dagger(k, x_0) \right]
\]

(18)

It is obvious from above discussions that \(\phi(x)\) interpolates between \(\phi_{in}(x)\) and \(\phi_{out}(x)\) and hence the nomenclature: interpolating field for \(\phi(x)\). Moreover, as it is an interacting field, the field equation is of the form

\[
(\Box x - m^2)\phi(x) = j(x)
\]

(19)

where \(\Box x\) is the D-dimensional d'Alembertian and \(j(x)\) is the source current operator; this is to be contrasted with the free field equations satisfied by \(\phi_{in}\) and \(\phi_{out}\). We are going to work in the Fourier (momentum) space quite often. The Fourier transform of the current is defined as

\[
j(x) = \frac{1}{(2\pi)^{(D-1)/2}} \int d^D k e^{-ik \cdot x} \tilde{j}(k)
\]

(20)

The solution for \(a(k, x_0)\) assumes form of an integral equation

\[
a(k, x_0) = a_{in}(k) + \int d^D k' \delta^{(D-1)}(k - k') \frac{e^{-i(k_0 - k'_0)x_0}}{k'_0 - k_0 + i\epsilon} \tilde{j}(k')
\]

(21)

Notice that the Fourier transformed \(\tilde{j}(k)\) is well defined on the mass shell i.e. \(k^2 = m^2\). We are in a position to define incoming and outgoing states using the corresponding creation operators.

\[
|k_1, k_2, ..., k_n \text{ in} > = a_{in}^\dagger(k_1) a_{in}^\dagger(k_2) ... a_{in}^\dagger(k_n) |0 >
\]

(22)

\[
|k_1, k_2, ..., k_n \text{ out} > = a_{out}^\dagger(k_1) a_{out}^\dagger(k_2) ... a_{out}^\dagger(k_n) |0 >
\]

(23)

An important comment is in order here. The generic matrix element \(\langle \alpha | \phi(x_1) \phi(x_2) ... | \beta >\) is not an ordinary function but a distribution. Thus it is to be always understood as smeared with a Schwarz type test function \(f \in S\). The test function is infinitely differentiable and it goes to zero along with all its derivatives faster than any power of its argument. We shall derive expressions for scattering amplitudes and the absorptive parts. It is to be understood that these are generalized functions and such matrix elements are properly defined with smeared out test functions. We envisage vacuum expectation values of product operators in LSZ formulation: either the time ordered products, the so called T-products or the retarded products, often denoted
as R-product. We shall be mostly concerned with the R-product throughout this investigation

\[ R \phi(x)\phi_1(x_1)\ldots\phi_n(x_n) = (-1)^n \sum_P \theta(x_0 - x_{10})\theta(x_{10} - x_{20})\ldots\theta(x_{n-10} - x_{n0}) \]

\[ \left([\ldots[\phi(x), \phi_{i_1}(x_{i_1})], \phi_{i_2}(x_{i_2})]\ldots], \phi_{i_n}(x_{i_n})\right) \]  

(24)

with \( R\phi(x) = \phi(x) \). Here \( P \) stands for all permutations \((i_1, \ldots, i_n)\) of \(1, 2, \ldots n\). The R-product is hermitial for hermitial fields \( \phi_i(x_i) \) and the product is symmetric under exchange of any fields \( \phi_1(x_1)\ldots\phi_n(x_n) \). Notice that the field \( \phi(x) \) is kept where it is located in its position. We list below some of the important properties for future use [13]:

(i) \( R \phi(x)\phi_1(x_1)\ldots\phi_n(x_n) \neq 0 \) only if \( x_0 > \max \{x_{10}, \ldots x_{n0}\} \).

(ii) An important property of the R-product is that

\[ R \phi(x)\phi_1(x_1)\ldots\phi_n(x_n) = 0 \]  

(25)

whenever the time component \( x_0 \), appearing in the argument of \( \phi(x) \) whose position is held fix, is less than time component of any of the four vectors \((x_1, \ldots x_n)\) appearing in the arguments of \( \phi(x_1)\ldots\phi(x_n) \).

(iii) We recall that

\[ \phi(x_i) \rightarrow \phi(\Lambda x_i) = U(\Lambda, 0)\phi(x_i)U(\Lambda, 0)^{-1} \]  

(26)

Under Lorentz transformation \( U(\Lambda, 0) \). Therefore,

\[ R \phi(\Lambda x)\phi(\Lambda x_1)\ldots\phi_n(\Lambda x_n) = U(\Lambda, 0)R \phi(x)\phi_1(x_1)\ldots\phi_n(x_n)U(\Lambda, 0)^{-1} \]  

(27)

And

\[ \phi_i(x_i) \rightarrow \phi_i(x_i + a) = e^{ia.P} \phi_i(x_i)e^{-ia.P} \]  

(28)

under spacetime translations. Consequently,

\[ R \phi(x + a)\phi(x + a)\ldots\phi_n(x + a) = e^{ia.P} R \phi(x)\phi_1(x_1)\ldots\phi_n(x_n)e^{-ia.P} \]  

(29)

We conclude, therefore, that the vacuum expectation value of the R-product dependents only on difference between pair of coordinates: in other words it depends on the following set of coordinate differences: \( \xi_1 = x_1 - x, \xi_2 = x_2 - x_1\ldots\xi_n = x_{n-1} - x_n \) as a consequence of translational invariance.

(iv) The retarded property of R-function and the asymptotic conditions lead to the following relations.

\[ R \left[ \phi(x)\phi_1(x_1)\ldots\phi_n(x_n), \phi_i^{\text{in}}(y_i) \right] = i \int d^D y_i \Delta(y_i - y_i')(\Box y' - m_i^2)R \phi(x)\phi_1(x_1)\ldots\phi_n(x_n)\phi_i(y_i)' \]  

(30)

where \( \Delta(y_i - y_i') \) admits the representation

\[ \Delta_i(y_i - y_i') = \frac{i}{(2\pi)^{(D-1)/2}} \int d^D k e^{-ik.i(y_i - y_i')}\delta(k^2 - m_i^2) \]  

(31)
Remarks: (i) We draw attention to the fact, the axioms (A1) - (A4) introduced in the beginning are not special to four dimensional spacetime. These axioms are true for theories living in arbitrary spacetime spacetime dimensions, \(D\).

(ii) The concept of asymptotic states and interpolating field (thus \(\phi_{\text{in}}\) and \(\phi_{\text{out}}\)) are valid in arbitrary \(D\). Moreover, we can construct the Fock space from these fields as described above.

(iii) The definition of the retarded operator, R-product, and other features holds in \(D\)-dimensions. Moreover, the properties of the vacuum expectation value of R-product hold good in \(D\)-dimensions and the consequences of the spacetime and Lorentz transformations are satisfied.

Therefore, adopting the LSZ reduction technique to study analyticity properties of scattering amplitude does not encounter any problem to fulfill the requirements of the axioms and fundamental formulation of LSZ formalism.

2.1 The Kinematics

We describe the kinematics for two body scattering. Although we consider scattering of neutral, scalar particles of equal mass, we shall continue to designate the four external particles with their momenta and denote mass by a label. We shall use equality of mass relation whenever we so desire. We focus attention only on \(2 \to 2\) elastic scattering. We mention passing that for scattering of scalars in \(D\)-dimensions the four point amplitude still depends on two Mandelstam variables \(s\) and \(t\) (note that for the case at hand the available variables are energy involved in scattering and scattering angle - see more discussions later). Our goal is to describe the essential steps used in deriving the analyticity properties of the four point elastic amplitude in \(D\)-dimensions and state important results.

The \(D\)-momenta of incoming particles are \(p_a\) are \(p_b\) and outgoing particles are \(p_c\) and \(p_d\). The mass shell condition is \(p_i^2 = m_i^2, i = a, b, c, d\). Our convention is that all the momenta are coming in and energy momentum conservation law is expressed as \(p_a + p_b + p_c + p_d = 0\). In this convention the matrix element for \(p_a + p_b \to p_c + p_d\) is \(\langle -p_d - p_c | p_a p_b \rangle\), i.e. \(D\)-momenta of \(c\) and \(d\) are denoted with negative sign from now on. The Mandelstam variables are

\[
s = (p_a + p_b)^2, \quad t = (p_a + p_d)^2, \quad u = (p_a + p_c)^2, \quad s + t + u = m_a^2 + m_b^2 + m_c^2 + m_d^2 = 4m^2 \quad (32)
\]

It is necessary to define some more 'mass' variables for subsequent discussions in the next section. The four 'mass' variables \(M_a, M_b, M_c, M_d\) are the lowest mass two or more particle states which have the same quantum numbers as particles \(a, b, c, d\) respectively. For the case at hand we define \(M_a = M_b = M_c = M_d = M\) since the particles carry no internal quantum numbers. We keep carrying these indices since
we keep the option open to consider (elastic) scattering of unequal mass particles in future and we might have to assign additive quantum numbers in those cases if we so desire. There are intermediate states in two or more particle states in various channels (i.e. \(s, t\) and \(u\)). Therefore, it is necessary to define the mass variables \(\mathcal{M}_{ab}, \mathcal{M}_{cd}\) \(\mathcal{M}_{ac}, \mathcal{M}_{bd}\) and \(\mathcal{M}_{ad}, \mathcal{M}_{bc}\). These masses correspond to two or more particle states carrying quantum numbers of particle pair \((ab, cd)\), \((ac, bd)\) and \((ad, bc)\). Moreover these masses could be different in a given channel since in the general case, the threshold for \((a, b)\) need not be same for \((c, d)\) although quantum numbers might be the same. The same logic holds for the other two pairs. In our case, there is only one such mass variable and we denote it by \(\mathcal{M}\). It also starts from the two particle intermediate state.

We assume that there are no bound states in the theory and consequently, there will be no anomalous thresholds. We define

\[
s_{\text{thr}} = 4m^2 = \mathcal{M}^2, \quad \text{and} \quad u_{\text{thr}} = \mathcal{M}^2
\]

and they also coincide with \(s_{\text{phys}}\) and \(u_{\text{phys}}\) respectively and we might use this definition interchangeably.

It is very convenient to go over to the center of mass (c.m.) system for two body scatterings. If we denote \(K_1(s)\) and \(K_2(s)\) as initial and final center of momenta for \((a, b)\) and \((c, d)\) respectively then (for equal mass cases)

\[
K_1(s)^2 = \frac{1}{4}(s - 4m^2), \quad K_2(s)^2 = \frac{1}{4}(s - 4m^2)
\]

when all masses are equal and \(K_1^2 = K_2^2 = K(s)^2\). The c.m. scattering angle is expressed as

\[
t = -2K(s)^2(1 - \cos\theta), \quad -1 \leq \cos\theta \leq +1, \quad -4K^2 \leq t \leq 0.
\]

We shall suppress \(s\) dependence of \(K\) from now on unless it is necessary. Now we proceed to discuss the general frame work which facilitates the investigations of analyticity properties of scattering amplitudes leading to derivation of the Froissart-Martin bound and other such rigorous bounds. One of the classic results is the proof of dispersion relations in the forward direction starting from LSZ formulation and to extend it to finite interval \(-T < t \leq 0, T\) positive, for physical \(t\). The scattering amplitude, \(F(s, t)\) is boundary value \(F(s, t) = \lim_{\epsilon \to 0} F(s + i\epsilon, t)\) of \(s\) and it is an analytic function in complex \(s\)-plane with a right hand cut from real \(s = s_{\text{thr}}\) and a left hand cut starting from \(u = u_{\text{thr}}\). Moreover, along the left hand cut \(\lim_{\epsilon \to 0} F(s - i\epsilon, t) = F_{\text{ad} \to \text{bc}}\). The corresponding c.m. energy squared is \(u = 4m^2 - s - t\). Note that for this \(u\)-channel process \(t\) represents c.m. momentum transfer squared too. Moreover, the discontinuity across the cuts are the absorptive parts of the \(s\)-channel and \(u\)-channel amplitudes respectively. We remark that the fixed-\(t\) analyticity properties are not sufficient to derive dispersion relations. It is necessary to know boundedness properties of these absorptive parts. If the absorptive amplitudes are polynomially bounded
then following dispersion relation may be written down

\[ F(s, t, u) = \frac{s^N}{\pi} \int_{s_{thr}}^{\infty} \frac{A_s(s', t)ds'}{s'^N(s' - s)} + \frac{u^N}{\pi} \int_{u_{thr}}^{\infty} \frac{A_u(u', t)du'}{u'^N(u' - u)} + \text{polynomials in } s \text{ and } u \]

(36)

The dispersion relation is expected to hold good for fixed \( t \) as stated earlier. There are subtleties for case of \( t < 0 \) and finite. Recall that \( t = -2K^2(1 - \cos \theta) \). If we desire \( t \) be finite and negative (in the physical region) then as \( s \rightarrow s_{thr} \), \( K \rightarrow 0 \), \( \cos \theta \) should be negative and much less than unity. Lehmann attempted and resolved this issue successfully in the frame work of LSZ formalism. He showed that \( F(s, t, u) \) is defined for physical \( s \) (even close to threshold) outside the interval \( -1 \leq \cos \theta \leq 1 \). This complex domain is known as small Lehmann ellipse (SLE). This result of Lehmann’s was not adequate to resolved the problem since even if \( \cos \theta \) lies inside the SLE; as \( K \rightarrow 0 \), still \( t \rightarrow 0 \sim K \) as \( s \) approaches the threshold value. Subsequently, Lehmann proved [31], in the LSZ formulation, that the absorptive part \( A_s(s, t) \) is analytic inside a larger ellipse, the large Lehmann ellipse (LLE), whose foci coincide with those of SLE but the semimajor axis is larger so that he resolved the problem alluded to in the context of \( s \rightarrow s_{thr} \). Indeed, the powerful theorem of Jost-Lehmann-Dyson was instrumental in proving the existence of SLE and LLE in the LSZ formulation. Therefore, fixed-\( t \) dispersion relations could be written down in \( s \) for the scattering amplitude. A further progress was made when it was demonstrated that the scattering amplitude is analytic in \( s \) and in \( t \). It is worth while to mention here that the results of Lehmann although very important could not be utilized to derive the Froissart bound as we know it today. The bound derived earlier was a weaker one.

Another important ingredient was incorporated by Martin to derive the Froissart bound as we know of now. He recognized the power of unitarity and used it; especially in the context of partial wave expansion of scattering amplitude. Note that in \( D = 4 \), Legendre polynomials \( (P_l(\cos \theta)) \) are the basis function for scattering of spinless massive particles. One of the crucial component in this advancement was use of positivity properties of the absorptive amplitude. This was proved elegantly through the partial wave expansion of \( A_s(s, t) \). Martin, through his celebrated theorem, proved the enlargement of domain of analyticity of the scattering amplitude. Furthermore, he concluded that the amplitude was analytic in \( s \) in the cut plane and it was also analytic in a domain in the \( t \)-plane denoted by \( D_t \). These advancements in identifying the domains of analyticity of scattering amplitude (in both \( s \) and \( t \)) paved the way to prove the Froissart-Martin bound for total cross section (1). The importance of this bound lies in the fact that there is no unknown constant in (1) except one. Notice that the prefactor in the right hand side of (1) is fixed in terms of the known parameters of strong interactions. However, in the logarithm squared of \( s \), one has to scale \( s \) with a dimensionful quantity: \( s_0 \). Recently, Martin and Roy [32] have argued and shown that this scale \( s_0 \) can be determined from considerations of \( \pi \pi \) scattering. They have put forward convincing arguments to determine \( s_0 \) in terms of mass of pion i.e. \( s_0^{-1} = 17\pi \sqrt{\pi/2m_\pi}^{-2} \) for \( \pi \pi \) scattering. The task ahead, keeping in mind
the preceding discussions, is to prove analyticity properties of scattering amplitude for scattering of massive, neutral particles in $D$-dimensions. We accept the axioms (A1) to (A4) stated in the beginning of this section. Our procedure is to follow the formalism of LSZ. We have argued that the amplitudes are tempered distributions in the $D$-dimensional case. The next result we need to derive is the good behavior of the amplitude as $s \rightarrow s_{th}, i.e. K \rightarrow 0$ for fixed negative $t$. This can be achieved if there exists analog of SLE and LLE in the LSZ formulation of the higher dimensional theory. As we shall see that in order to derive the existence of SLE and LLE it is required that equivalent generalization of Jost-Lehmann-Dyson theorem be proved in $D$-dimensional field theories. This is not a straight forward extension of the $D = 4$ result. We recall that the absorptive part of the amplitude appears in the dispersion relation. As we have mentioned already, the positivity properties of the absorptive part play a crucial role in deriving the analyticity of the amplitude in $s$ and $t$ variables. Moreover, to derive positivity properties of absorptive amplitude, one has to take the route of partial wave unitarity and their positivity relations as was utilized by Martin. As we shall show, in $D$-dimensions, the Gegenbauer polynomials are the basis functions for partial expansions. Thus the proof of positivity needs handling of these basis functions. Moreover, in order to derive the analog of Froissart-Martin bound for $\sigma_t$, some more efforts will be needed. As alluded to in the introduction, the earlier bound on $\sigma_t$, for the $D$-dimensional theory, contained unknown parameters: one them is $N$, that appeared on the polynomial boundedness property of the amplitude and a second unknown parameter is $\tilde{T}_0$ which was introduced to define the semimajor axis of the ellipse within which the partial wave amplitude (here basis is the Gegenbauer polynomial) converges. Then there is a third unknown parameter which scales the $\ln s$. This scale was present in the improved proof that Martin obtained (see remarks earlier) for the total cross section.

We shall systematically proceed to obtain the necessary result to derive analyticity and asymptotic behavior of scattering amplitude.
3. Analyticity Properties of Scattering Amplitude

We develop the necessary formalism to study analyticity properties of scattering of massive, neutral, spinless particles in D-dimensional spacetime in this section. We have adopted the LSZ formalism and we have stated all axioms and requisite definitions in the previous section. Thus we begin with LSZ reduction of the four point amplitude. Let us outline the relevant steps of reduction formula for two particle in 'in' state: \(| - p_d - p_c \text{ in } > \). If we reduce particle 'c' then

\[
| - p_d - p_c \text{ in } > = \delta^c_1(-p_c)| - p_d >
\]  

(37)

The state \(| - p_d - p_c \text{ out } > \) may be reduced following an analogous prescription. Our interest lies in evaluating the difference between the following two four-point functions by LSZ technique

\[
< -p_d - p_c \text{ out}|p_a p_b \text{ in } > - < -p_d - p_c \text{ in}|p_a p_b \text{ in } >
\]

\[
= lim_{x_0 \to \infty} < -p_d|a_b(-p_c, x_0)|p_a p_b \text{ in } > - lim_{x_0 \to -\infty} < -p_d|a_c(-p_c, x_0)|p_a p_b \text{ in } >
\]

\[
= \frac{i}{(2\pi)^{(D-1)/2}} \int d^Dx e^{-ip_c x} (\Box x - m_c^2) < -p_d|\phi_c(x)|p_a p_b \text{ in } >
\]

(38)

We retained the label 'c' in order to identify which particle was reduced and we have written \(m_c^2\) also for the same reason. We shall continue to follow this convention of labeling particles and their momenta which will serve useful purpose as will be clear soon. The above equation is a straight forward implementation of the reduction technique. We have two possibilities for the next step of reduction: (i) either we reduce the single particle state \(< -p_d >\) or (ii) one of the particle from \(|p_a p_b \text{ in } >\). We end up in getting vacuum expectation value of an R-product in either case.

\[
< -p_d - p_c \text{ out}|p_a p_b \text{ in } > - < -p_d - p_c \text{ in}|p_a p_b \text{ in } >
\]

\[
= \frac{i}{(2\pi)^{(D-1)/2}} \int d^Dx d^Dy e^{-ip_c x - ip_b y} (\Box x - m_c^2)(\Box y - m_b^2) < -p_d|R\phi_c(x)\phi_b^d(y)|p_a p_b >
\]

\[
= \frac{i}{(2\pi)^{(D-1)/2}} \int d^Dx d^Dy e^{-ip_c x - ip_d y} (\Box x - m_c^2)(\Box y - m_d^2)
\]

\[
< 0|R\phi_c(x)\phi_d(y)|p_a p_b \text{ in } >
\]

(39)

We have written \(\phi^d_b(y)\) deliberately to keep a tag on the field that it arises from reduction of 'b' in the 'in' state although we have only neutral scalar fields. The scattering amplitude is defined with the convention that

\[
< -p_d - p_c \text{ out}|p_a p_b \text{ in } > - < -p_d - p_c \text{ in}|p_a p_b \text{ in } >
\]

\[
= 2\pi \delta^D(p_a + p_b + p_c + p_d) F(p_a, p_b, p_c, p_d)
\]

(40)

As defined earlier the currents are \((\Box - \phi_l(x)) = j_l(x), \ l = a, b, c, d\). A few comments are in order here.

(i) There is a subtlety involved in the operation

\[
(\Box x - m_c^2)(\Box y - m_d^2)(R\phi_c(x)\phi_d(y)) = R(j_c(x)j_d(y)
\]  

(41)
which will be used in sequel. When we let \((\Box_x - m_c^2) (\Box_y - m_d^2)\) pass over \((R\phi_c(x)\phi_d(y))\) we eventually get \(R(j_c(x)j_d(y))\); it is to be understood that in writing this equality, in general, there will be extra terms containing \(\delta\)-functions and the derivatives of \(\delta\)-functions in such operations in addition to the term \((R\phi_c(x)\phi_d(y))\). It has been argued by Symanzik [37, 13, 14] that in a local quantum field theory only finite number of derivatives of delta functions can appear. Therefore, when we Fourier transform an amplitude into functions of momentum space variables (in fact functions of Lorentz invariant variables such as \(s\) and \(t\) in case of four point functions), these \(\delta\)-function derivatives will appear as powers of momenta. Therefore, these will be only finite number of terms with powers of momenta i.e. the amplitude will be at most polynomials in momenta [37]. Indeed, the N-subtracted dispersion relation we displayed in the previous section is justified on these grounds when we follow the LSZ formulation. In nutshell, we see that these amplitudes are polynomially bounded. (ii) We may use the translation properties of the fields to simplify the above expressions. For example, consider the product of operators \(A(x)B(x')\) and use the translation operation on the matrix element say \(M(x, x') = <\alpha | [A(x), B(x')]|\beta >\) where \(\alpha\) and \(\beta\) designate the momenta. Now use translation shift by \('a'\). Then \(M(x, x') = e^{-i(\beta - \alpha)\cdot a}M(x + a, x' + a)\). Choose \(a = -(x + x')/2\) and \(M(x, x')\) depends on \(x - x'\) as expected. Thus in host of cases, we shall see that matrix elements depend on difference of coordinates. Therefore, we write \(Rj_l(x)j_m(x') = Rj_l(z/2)j_m(-z/2)\) where indices \(l, m\) stand for \(a, b, c, d\). Thus, the scattering amplitude (40) expressed as [13, 14]

\[
F(p_a, \ldots p_d) = -\int d^Dz e^{iP\cdot z} < -p_d | Rj_c(z)j_b(\frac{-z}{2}) | p_a >
\]

where \(P = (p_b - p_c)/2\). In deriving (42) we have reduced \(c\) and \(b\). If we reduce \(c\) and \(d\) the amplitude is expressed as

\[
F(p_a, \ldots p_d) = -\int d^Dz e^{-iQ\cdot z} < 0 | Rj_c(z)j_d(\frac{-z}{2}) | p_a \ p_b \ in >
\]

Now \(Q\) is difference of momenta \(p_c\) and \(p_d\) with factor 2 dividing. We could reduce all states of the matrix element \(< -p_d - p_c \ out | p_a \ p_b \ in >\) and we shall get vacuum expectation of the R-products of four corresponding currents. This reduction is not very useful for our investigation at the moment. The above expressions (42) and (43) are quite useful. The two equations derived above for the amplitude, \(F\), are special cases of a generic retarded function

\[
F_R(q) = \int d^Dz e^{iQ_0} \theta(z_0) < Q_f | [j_l(\frac{z}{2}), j_m(-\frac{z}{2})] | Q_i >
\]

\(j_l\) and \(j_m\) are two generic currents and indices take values \(a, b, c, d\). The two states \(|Q_f\rangle\) and \(|Q_i\rangle\) carry D-dimensional momenta \(Q_f\) and \(Q_i\) respectively and these momenta are held fixed. Thus the argument of \(F_R\) does not display \(Q_f\) and \(Q_i\) and we treat them as parameters for the discussions to follow. We define two more functions

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for our later conveniences

$$F_A = - \int d^D z e^{iq \cdot z} \theta(-z_0) < Q_f | j_i(\frac{z}{2}), j_m(-\frac{z}{2})|Q_i >$$  \hspace{1cm} (45)$$

and

$$F_C(q) = \int d^D z e^{iq \cdot z} < Q_f | j_i(\frac{z}{2}), j_m(-\frac{z}{2})|Q_i >$$  \hspace{1cm} (46)$$

From above definitions, it follows that

$$F_C(q) = F_R(q) - F_A(q)$$  \hspace{1cm} (47)$$

Since $F_C$ is commutator of two currents, we explicitly write the commutator in terms of products of currents $j_i(\frac{z}{2})j_m(-\frac{z}{2})$. Let us introduce two complete set of physical states: $\sum_n |p_n\alpha_n > < p_n\alpha_n| = 1$ and $\sum_{n'} |p_{n'}\beta_{n'} > < p_{n'}\beta_{n'}| = 1$. Here $\{\alpha_n, \beta_{n'}\}$ stand for quantum numbers that are permitted for the intermediate states. Now eq. (47) can be expresses as

$$\int d^D z e^{iq \cdot z} \left[ \sum_n \left( \int d^D p_n < Q_f | j_i(\frac{z}{2})|p_n\alpha_n > < p_n\alpha_n|j_m(-\frac{z}{2})|Q_i > \right) 
- \sum_{n'} \left( \int d^D p_{n'} < Q_f | j_m(-\frac{z}{2})|p_{n'}\beta_{n'} > < p_{n'}\beta_{n'}|j_i(\frac{z}{2})|Q_i > \right) \right]$$  \hspace{1cm} (48)$$

We may use spacetime translations on the above matrix elements of each term to bring the arguments of the currents to $z = 0$. Consequently, the intermediate states satisfy energy momentum conservation conditions requiring $p_n = (\frac{Q_i + Q_f}{2}) - q$ and $p_{n'} = (\frac{Q_i + Q_f}{2}) + q$ and therefore,

$$F_C(q) = \sum_n \left( < Q_f | j_i(0)|p_n = \frac{(Q_i + Q_f)}{2} - q, \alpha_n > < \alpha_n, p_n = \frac{(Q_i + Q_f)}{2} - q|j_m(0)|Q_i > \right) 
- \sum_{n'} \left( < Q_f | j_m(0)|p_{n'} = \frac{(Q_i + Q_f)}{2} + q, \alpha_{n'} > < \alpha_{n'}, p_{n'} = \frac{(Q_i + Q_f)}{2} + q|j_i(0)|Q_i > \right)$$  \hspace{1cm} (49)$$

The matrix element $F_C$ vanishes, only when each term on the right hand side of the above equation vanishes at the same time. Therefore,

$$2A_s(q) = \sum_{n'} \left( < Q_f | j_i(0)|p_{n'} = \frac{(Q_i + Q_f)}{2} + q, \alpha_{n'} > \times < \alpha_{n'}, p_{n'} = \frac{(Q_i + Q_f)}{2} + q|j_i(0)|Q_i > \right) = 0$$  \hspace{1cm} (50)$$

Similarly

$$2A_u = \sum_n \left( < Q_f | j_m(0)|p_n = \frac{(Q_i + Q_f)}{2} - q, \alpha_n > \times < \alpha_n, p_n = \frac{(Q_i + Q_f)}{2} - q|j_i(0)|Q_i > \right) = 0$$  \hspace{1cm} (51)$$
Thus the expressions for $2A_a$ and $2A_u$ given above must vanish simultaneously if we desire $F_C = 0$. The intermediate states inserted in the expressions of equations (50) and (51) are the physical states i.e. their D-momenta must lie in the forward light cone, $V^+$. These requirements translate to
\[
\frac{Q_i + Q_f}{2} + q \geq 0, \quad \frac{Q_i + Q_f}{2} + q_0 \geq 0 \tag{52}
\]
and
\[
\frac{Q_i + Q_f}{2} - q \geq 0, \quad \frac{Q_i + Q_f}{2} - q_0 \geq 0 \tag{53}
\]
Thus we should have minimum mass parameters in each of the cases which satisfy the requirements: (i) $(\frac{Q_i + Q_f}{2} + q)^2 \geq M_+^2$ and (ii) $(\frac{Q_i + Q_f}{2} - q)^2 \geq M_-^2$. The matrix elements for $A_s(q)$ and $A_u(q)$ will not vanish and if the two conditions stated above, pertinent to each of them, are fulfilled. If we define $\tilde{F}_C(z)$ to be the Fourier transform of $F_C(q)$
\[
\tilde{F}_C(z) = \frac{1}{(2\pi)^D} \int d^Dq e^{-iqSZ}F_C(q) = \langle Q_f || j_m(\frac{z}{2}), j_l(-\frac{z}{2}) || Q_i \rangle > \tag{54}
\]
It follows from axiom of micro causality that the current commutator vanishes outside the light cone i.e. $\tilde{F}_C(z) = 0$ for $z^2 < 0$. Thus to repeat, $F_C(q) \neq 0$ if one of the two conditions stated in equations (52) and (53) are satisfied.

We emphasize that the retardedness property of $F_R(q)$ and similar feature of $F_A(q)$ are crucial ingredients in order to deduce analyticity properties of scattering amplitudes. A very important observation is, when $F_C(q)$ is zero i.e. $F_C(q) = F_R(q) - F_A(q) = 0$. Thus $F_R(q) = F_A(q)$ for those values of $q$. This information is immensely useful to identify the analytic functions $F_R$ and $F_A$ from the generalization of reflection principle of Schwarz. The study of the causal function, $F_C(q)$ and its analyticity properties enables construction of $F_R(q)$ or $F_A(q)$ from the fact that $F_R(q) = F_A(q)$ whenever $F_C(q) = 0$ over certain values of $q$. We may represent the retarded function as [14]
\[
F_R(q) = \frac{1}{2\pi i} \int d^Dq' \delta^{D-1}(q' - q) \frac{1}{(q_0' - q_0)} F_C(q'), \quad \text{Im } q_0 > 0 \tag{55}
\]
In fact the above relationship is more transparent if we go over to the coordinate space through a Fourier transform and note
\[
\tilde{F}_R(z) = \int d^Dq e^{-iqSZ}F_R(q) = \theta(z_0) < Q_f || j_l(\frac{z}{2}), j_m(-\frac{z}{2}) || Q_i >= \theta(z_0)\tilde{F}_C(z) \tag{56}
\]
Let us consider a specific case where we identify $|Q_i > = |p_a >$ and $|Q_f > = |-p_d >$. Therefore, we have reduced 'b' and 'c' and the associated currents in the R-product matrix elements are: $j_l(\frac{z}{2}) = j_z(\frac{z}{2})$ and $j_m(-\frac{z}{2}) = j_l(-\frac{z}{2})$ (we continue to write $j^+$). A few remarks are called for at this stage.
(i) We have noted how the matrix element $\tilde{F}_C$ vanishes outside the light cone.
(ii) We observed that for certain values of \(q\), \(F_C(q)\) vanishes and consequently, \(F_R(q)\) and \(F_A(q)\) coincide there. We recall that in the context of \(D = 4\), the edge-of-the-wedge theorem plays a powerful role in the study of the four point function (with four momenta \((p_a, p_b, p_c, p_d)\)). The amplitude is uniquely represented by analytic function of these complexified momenta. The amplitude is an analytic function on the manifold \(p_a + p_b + p_c + p_d = 0\). The method to find the domain of holomorphy is termed as linear problem since unitarity condition is not invoked. Bremermann, Oehme and Taylor [38] proved the edge-of-the-wedge theorem for 4-point function in the LSZ framework.

(iii) We have not furnished detail proof of edge-of-the-wedge theorem for the massive scalar theory in D-dimensions. However, it is quite conceivable that the proof of Bremermann, Oehme and Taylor [38] is likely to go through. It seems there are no serious obstacles in generalizing the theorem to D-dimensions. Let us recapitulate the essential arguments of Bremermann, Oehme and Taylor [38]. The Appendix of their paper proves Lemma 1 and Lemma 2 prior to proving the edge-of-the-wedge theorem. We briefly outline the content of Lemma 1 of [38]; we refer the reader to the appendix of the paper for details. \(f(z_0, z_1)\) is a function of two complex variables \((z_0, z_1)\) and is given as a Fourier transform of two tempered distributions. The function is analytic in the "wedge" \(W\) defined below:

\[
W = \{(x_0, z_1) : |y_1| < |y_0|, \ |x_0| < \infty, \ |x_1| < \infty\} \quad (57)
\]

where \((x_0, x_1)\) and \((y_0, y_1)\) are real and imaginary parts of \((z_0, z_1)\) respectively. Let \(E\) be a given domain in \((x_0, x_1)\) plane. The authors define the "E-limiting sequence" for a pair of complex numbers \((z_{0n}, z_{1n})\) if they satisfy the following conditions: (a) \(\lim_n y_{0n} = \lim_n y_{1n} = 0\). (b) \(\lim_n (x_{0n}, x_{1n}) \in E\). (c) There is a number \(c > 1\), independent of \(n\), so that for all \(n\), \(|y_{0n}| > |y_{1n}|\). It is then assumed that \(f(z_0, z_1)\) has the limiting property that for any E-limiting sequence; the limit \(\lim_n f(z_{0n}, z_{1n})\) exists. It is independent of the particular sequence and depends on the limit point. Then BOT [38] proved that if \(f(z_0, z_1)\) is analytic in some neighborhood \(N\) of the set \(S = \{(z_0, z_1) : y_0 = y_1 = 0, \ (x_0, x_1) \in E\}\). There was a choice of a coordinate system such that \((x_0, x_1) = (0, 0)\) to be a particular point of \(E\) and analyticity was proven at \(z_0 = z_1 = 0\). The authors assumed that \(f(z_0, z_1)\) is analytic in the neighborhood of \(z_0 = z_1 = 0\) and therefore, the power series expansion (like \(f(z_0, z_1) = \sum_{m=0}^{\infty} a_{mn} z_0^m z_1^n\)) exists for some \(r\), \(|z_0| < r, \ |z_1| < r\) and the expansion uniformly convergent. One can define an analytic plane \(\pi_\alpha : z_0 = \alpha_0 \lambda, \ z_1 = \alpha_1 \lambda; \ (\alpha_0, \alpha_1)\) real and \(\frac{\alpha_0}{\alpha_1} < 1\). Eventually, BOT proved that power series expansions on analytic planes \(\pi_\alpha\) (these planes can be suitably defined) can be joined together to give a power series expansion (this is defined in specific domains for \((z_0\) and \(z_1))\ which equals the separate power series expansions and therefore, is equal to \(f(z_0, z_1)\) from where one started with. Thereby, the analyticity of \(f(z_0, z_1)\) is proved in the neighborhood of \(z_0 = z_1 = 0\). This Lemma is subsequently used to prove edge-of-the-wedge theorem for the four point amplitude of equal mass scattering of scalars in four dimensions. In the Lemma 2, they consider a
function $f(z)$ which is a function of four complex variables i.e. $(z = z_0, z_1, z_2, z_3 = z_0, z)$. Thus by definition $z$ has three components. Now define a wedge $W$ for the system of four complex variables.

$$W = \left\{ (z_0, z) : |y_0| > |y|, |x_0| < \infty, |x| < \infty \right\}$$

Next, they extend the proof of analyticity of properties $f(z)$ as a function of four complex variables along the technique utilized in the proof of Lemma 1. In order to prove the edge-of-the-wedge theorem for the 4-dimensional case, they define a function $f(z, z')$ as a function of 8 complex variables, $z = (z_0, z_1, z_2, z_3)$ and $z' = (z'_0, z'_1, z'_2, z'_3)$ and define two wedges associated with each of the four complex variables. It is assumed that $f(z, z')$ is analytic in the double wedge $W \otimes W$. Furthermore, this function is Fourier transform of tempered distributions. Subsequently, these authors [38] prove that this function (of eight complex variables) can be analytically continued and they arrive at the proof of the edge-of-the-wedge theorem. In four dimensions, there are two momentum variables for the problem (each momentum has four components) and when complexified it gives rise to eight complex variables. Therefore, intuitively, it looks plausible that the proof might go through when we deal with amplitude in D-dimensional field theory. Therefore, existence of the proof can be taken as a well judged conclusion.

(iv) Bros, Epstein and Glaser [39] studied analyticity domain of the four point amplitude in complex four momentum space. They adopted a geometrical technique for analytic completion\(^5\). They derived analyticity in both the variables $s$ and $t$ for the on mass shell amplitude. As noted earlier, we deal with the Fourier transforms of the vacuum expectation values corresponding to retarded and advanced products of field operators. The Fourier transformed functions are analytically continued (the functions in the $x$-space are tempered distributions). Furthermore, the functions defined in the momentum space coincide in the coincidence region. It follows from the edge-of-the-wedge theorem that amplitudes for various of four particle reactions are represented by boundary values of unique analytic functions depending on three independent complex variables. The $2 \to 2$ amplitudes are functions $\mathcal{F}_I(p_a, p_b, p_c, p_d)$ with $\sum p_l = 0, l = a, b, c, d$; therefore, the amplitude is effectively a function of three complexified variables. There are many more functions besides $F_R, F_A$ and $F_c$. We have discussed the support properties of these three functions already. $\mathcal{F}_I(p_a, p_b, p_c, p_d)$ is the boundary value of a function $\mathcal{F}_I(k_a, k_b, k_c, k_d), k_l = p_l + iq_l, l = a, b, c, d$. $\mathcal{F}_I(k_a, \ldots)$ is analytic in the tube $\mathcal{T}_A = \{k, q \in V^+_A\}$; such that $\sum k_l = 0$. Here $V^+_A$ is the forward light cone. Moreover, due to the spectrum conditions, these functions $\mathcal{F}_I(k_a, k_b, k_c, k_d)$ coincide in some real regions of the momentum space ($\text{Im } k_l = 0, l = a, b, c, d$). Therefore, by the edge-of-the-wedge theorem, there exists a function $H(k_a, k_b, k_c, k_d)$ which is their analytic continuations. It is natural to ask whether the formalism of [39] is

\(^5\)see Martin [11] for lucid exposition to technique of analytic completion and for illustrative examples.
specifically applicable to four dimensional theories if we restrict the analysis to four point amplitude. We argue that the BEG [39] procedure may be applied to the four point amplitude in D-dimensional field theories. In their more general formulation, BEG were also working within the LSZ frame work. It is not very clear if the crucial fact that they were working in \(D = 4\) is essential for their proof. In the case of D-dimensional spacetime, the BEG theorem [39] might be proved for the four point amplitude. In the case of four point function, let us choose a coordinate system where particles \(a, b, c, d\) are assigned following momenta:

\[
\begin{align*}
    p_a &= (p_{a0}, p_{a1}, p_{a2}, p_{a3}, 0) \\
    p_b &= (p_{b0}, p_{b1}, p_{b2}, p_{b3}, 0) \\
    p_c &= (p_{c0}, p_{c1}, p_{c2}, p_{c3}, 0) \\
    p_d &= (p_{d0}, p_{d1}, p_{d2}, p_{d3}, 0)
\end{align*}
\]

where 0 stands for the \(D - 3\) dimensional spatial components of the \(D - 1\) spatial vectors. We can always choose the \(D\)-momentum vectors of the four particles in this manner. The BEG [39] proof of the edge-of-the-wedge theorem will go through as long as we consider the four point amplitude.\(^6\) However, in \(D\) – dimensions the arguments of BEG cannot be extended for an arbitrary \(n\)-point amplitudes in this manner in general. Moreover, the four point amplitude depends only on the Lorentz invariant variables \(s\) and \(t\). Thus if we obtain the BEG proof of the edge-of-the-wedge theorem in this special choice of (momentum) frame, then it will be valid in any Lorentz frame. However, when BEG argument is adopted in this frame, we are to implement the analytic continuation in this four dimensional subspace from one domain to another domain following the arguments used for \(D = 4\) theories [39, 41]. Let us consider the simple case of \(n\)-point amplitude where \(4 < n < D\). Here we are unable to restrict the \(D\)-momenta of \(n\)-particles to a four dimensional subspace to apply BEG theorem (proved for \(D = 4\) theories). We cannot offer arguments for the proof of the edge-of-the-wedge theorem for higher point amplitudes in \(D\)-dimensions as we have advanced for the four point amplitude.

\(^{(v)}\) It is worth while to point out the approach of Bogoliubov and collaborators [23, 40]. In this formalism, the proof of non-forward dispersion relation does not make use of the theory of several complex variables as was adopted by [38]. The formalism of this group [23, 40] is to employ certain parametrizations and the technique of distributions. Their formalism is mathematically rigorous. In their treatise on axiomatic field theory [23], they present the formalism to prove edge-of-the-wedge theorem. They develop a theory for study of the analyticity properties of a function of \(n\)-complex variable systematically. Then prove analog of the edge-of-the-wedge theorem before resorting to usual 4-dimensional spacetime case. It is quite possible that their technique might be useful to prove the edge-of-the-wedge theorem in D-

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\(^6\)I thank H. Epstein for very illuminating discussions on this point and in advancing these arguments.
dimensions.

(vi) Let us focus attention on $F_R(q)$. It is a function of D-component complex vector $q$. We remind that, in coordinate space expression, it vanishes outside the light cone. Moreover, the function is defined in the future light cone: $V^+ = \{ z \mid z^2 > 0, \, z_0 > 0 \}$. The matrix element is tempered. The expression will converge (thus $F_R(q)$ will be analytic in $q$) if in the Fourier transform $e^{iq \cdot z}$ falls off exponentially for $z \to \infty$ in all directions such that $z$ is in the support of the integrand. It is evident that we must have $z.\text{Im} \ q > 0$ and at the same time all $z \in V^+$. Moreover, we should have $\text{Im} \ q \in V^+$.

This is the definition of the forward tube. Thus $F_R(q)$ is holomorphic in $q$ for $q \in T^+$. We may go through the same arguments and conclude that $F_A(q)$ is also holomorphic in $q$ for $q \in T^- = -T^+$. The region where $F_R(q) = F_A(q)$ i.e. where $F_C(q) = 0$ corresponds to the domain where $q$ is real ($\text{Im} \ q = 0$). If we invoke the preceding arguments, we may conclude that $F_R(q)$ and $F_A(q)$ are the same analytic functions.

3.2 The Jost-Lehmann-Dyson Representation

Jost and Lehmann [42] obtained an integral presentation for the matrix element of the causal commutator. This is a very powerful result. The analyticity properties of the scattering amplitude, as a function of $s$ and $t$, can be investigated. These are of interests to us and we shall generalize the known result to the D-dimensional case. Dyson [43] introduced a very powerful technique and derived more general results in that the Jost-Lehmann representation is valid for the case of equal mass particles and had some limitations. On the other hand Dyson’s method is applicable for the case of unequal masses in a more general setting. We have generalized Dyson’s theorem for the case of D-dimensional spacetime and derived the corresponding representation for $F_C(q)$ and then we can also derive representation for $F_R(q)$. The constraint of micro causality on $\hat{F}_C(z)$ plays an important role in this process.

$$\hat{F}_C(z)$$ plays an important role in this process.
as stated earlier $V^+$ is the future light cone. The Jost-Lehmann representation for $F_C(q)$ is such that it is nonvanishing in the region $\bar{R}$ given by (61) and the Fourier transform vanishes outside light cone

$$F_C(q) = \int_S d^D u \int_0^\infty d\chi^2 \epsilon(q_0 - u_0)\delta((q - u)^2 - \chi^2)\Phi(u, Q, \chi^2) \tag{62}$$

Note that $u$ is also a D-dimensional vector (no relations with Mandelstam variable $u$).

The domain of integration of $u$ is the region $S$ specified below

$$S : \left\{ Q + u \in V^+, Q - u \in V^+, \text{Max} [0, M_+ - \sqrt{(Q + u)^2}, M_- - \sqrt{(Q - u)^2}] \leq \chi \right\} \tag{63}$$

and $\Phi(u, Q, \chi^2)$ arbitrary. Here $\chi^2$ is like a mass parameter. Notice that the assumptions about the features of the causal function stated above are the properties we have listed earlier if we identify $Q = \frac{(Q_t + Q_i)}{2}$. In order to obtain a representation for the retarded function, we recall that $F_R(q)$ and $F_C(q)$ are related by [42]

$$F_R(q) = \frac{i}{2\pi} \int d^D q' \delta^{D-1}(q' - q) \frac{1}{q_0' - q_0} F_C(q'), \text{Im } q_0 > 0 \tag{64}$$

Therefore, the Jost-Lehmann representation for $F_R(q)$ reads [42]

$$F_R(q) = \frac{i}{2\pi} \int_S d^D u \int_0^\infty d\chi^2 \frac{\Phi(u, Q, \chi^2)}{(q - u)^2 - \chi^2} \tag{65}$$

This integral representation is valid provided the integral converges. We have noted earlier in derivation of the expression for $\tilde{F}_R(z)$ that it is defined with the understanding that there could be additional terms corresponding to $\delta$-functions and their finite number of derivatives. The support for this argument is that in a local field theory only finite number of such derivatives could occur [37]. Therefore, the above integrand (it is defined now in the momentum space) might have at most polynomials in momentum which can be taken care of by appealing to the subtraction prescription. However, the analyticity properties are unaffected by subtractions.

An important point to note that the singularities are in the complex plane as is obvious from (65). These points are solution to the equation

$$(q_0 - u_0)^2 - (q_1 - u_1)^2 - (q_2 - u_2)^2 - ... - (q_{D-1} - u_{D-1})^2 = \chi^2 \tag{66}$$

This implies that the points of singularities lie on the hyperboloids. The points $u_0, u_1, ... u_{D-1}, \chi^2$ lie in the domain $S$. The hyperboloids where the parameters belong to $S$ are called admissible. We have defined a domain $\bar{R}$, eq.(61) where $F_C(q)$ is nonvanishing. Now define a set $\tilde{R}$ such that it is compliment to the real elements of $\bar{R}$. Therefore, we conclude from the definition of $\tilde{R}$ that $F_C(q) = 0$ for every point which lies in $\tilde{R}$ and is real. Moreover, $F_R(q) = F_A(q)$ in this domain. This is the coincidence region. If we examine the definition of domain $\bar{R}$ (61), it is bordered by
the upper branch of the hyperbola \((Q + q)^2 = M^2\) and the other branch is border of another hyperbola, \((Q - q)^2 = M^2\). We arrive at the conclusion that the region between these two hyperbolas can be identified as the coincidence region. We recall that the set \(S\) is defined by the range of values \(u\) and \(\chi^2\) take in the admissible parabolas. This set of values is a subset of \((u, \chi^2)\) of all hyperbolas [42, 15]. We remind the reader that the present discussion is for the case of equal masses. The more general scenario follows from Dyson’s analysis.

We are in a position to explore the analyticity properties of \(F_R(q)\) since we have defined various domain for our purpose. As stated earlier singularities are in the admissible parabola defined by \((u - q)^2 = \chi^2\). It is better to, eventually, explore this feature in terms of invariants since scattering amplitudes are expressed in terms of invariants. We focus on the case of \(Q \in V^+\). Now we choose a frame where \(Q = (Q_0, 0)\) where 0 stands for \((D - 1)\)-dimensional spatial components of the D-vector \(Q\) in this frame. Next we choose D-vector \(q\) to find out the location of the singularities\(^7\). In order to simplify the calculations and bring out the essence of Jost-Lehmann formalism, we make choice about \(q\). We single out one spatial component of \(q\) and treat as a variable to locate singularities and treat \(q_0\) and and rest of the spatial components (now \(D - 2\) spatial coordinates are fixed) as fixed parameters. The general case can be treated more elegantly in Dyson’s approach. The above mentioned choice would lead us to find singularities in a simple way. To be specific let us choose \(q_1\) to be the variable of the \(D - 1\) spatial vectors; in other words \(q_0, q_2, \ldots, q_{D-1}\) are treated as parameter and held fixed. If we examine the Jost-Lehmann representation (65) then we note that we have only Lorentz invariant objects appearing in the right hand side of the equation. Therefore, with the present choice the of \(q^2\), the study of location of singularities is reduced to concentrating on \((q_1)^2\) as a variable. Thus we are required to explore the location of the singularities in the \(q_1\)-variable. These points are

\[
q_1 = u_1 \pm i\sqrt{\chi_{min}^2(u) - (q_0 - u_0)^2 + (q_2 - u_2)^2 + \ldots (q_{D-1} - u_{D-1})^2 + \rho}, \rho > 0 \tag{67}
\]

Note that the set of points \(\{u_0, u_1, \ldots, u_{D-1}; \chi_{min} = min \chi^2\}\) lie in the domain \(S\). We are able to identify the domain where the singularities might reside with this choice for the variables \(Q\) and \(u\). Another feature is that the solution (67) is symmetric with respect to the real axis. In general, the case is not so when masses are unequal; the original derivation of Jost and Lehmann was applicable to equal mass case only. This region contains all points in the region Re \(q_1 + i\text{Im } q_1\), Im \(q_1 > 0\) which satisfy Re \(q_1 + i\text{Im } q_1 + \rho', \rho' > 0\). In order to illustrate the result in a very transparent manner, consider a very special case. Let us set \(q_0 = q_2 = \ldots = q_{D-1} = 0, u_1 \neq 0\). Now the singular points are

\[
q_1 = u_1 \pm i\sqrt{\beta(u_1)^2 + \rho}, \rho > 0 \tag{68}
\]

\(^7\)This treatment is analogous to that of [42, 15] which is generalized to D-dimensional spacetime.
\[ \beta^2(u_1) = \text{Min} \left[ \chi_{\text{min}}^2 - u_0^2 + u_2^2 + \ldots + u_{D-1}^2 \right] \] (69)

The minimization is taken over the variables \( \{u_0, u_2, \ldots, u_{D-1}\} \). Now we examine a still simpler case where the coincidence region is bounded by two branches of hyperboloids (so that \( M_+^2 = M_-^2 = M^2 \)). For this choice

\[ (Q + q)^2 = M^2, \quad (Q - q)^2 = M^2 \] (70)

and we have \( \beta(u_1) = M - \sqrt{Q^2 - u_1^2} \).

We briefly remark about Dyson’s [43] formulations of the problem to derive representation for \( F_C(q) \). This formalism is most suitable to obtain representations for causal function and the retarded functions in any spacetime dimensions. We present the details in the Appendix A. In order to obtain Jost-Lehmann representation, in \( D = 4 \), Dyson [43] enlarged the spacetime to six dimension with Minkowski signature such that there is one time coordinate and five spatial coordinates. Therefore, there will be six momentum variables as well. He constructed a six dimensional wave equation in the momentum space and obtained solutions. Thus a causal function can be defined in six dimensional spacetime and in terms of momentum space variables. Then he chooses special set of coordinates and boundary conditions to relate his solution to the function \( F_C(q) \). Dyson made a very important observation in relating solution to his six dimensional wave equation with the four dimensional causal function. He chose an arbitrary spacelike surface, \( \Sigma \), in the enlarged space. He used the fact that the solution can be expressed in terms of its value and its normal derivative on an arbitrary spacelike surface. Moreover, he demonstrated uniqueness of his solution. We have generalized the technique of Dyson to arbitrary spacetime dimension, \( D \) and derived the representation for the causal function, \( F_C(q) \). A more general and mathematically rigorous derivation of Dyson’s theorem is presented in the book of Bogolibov et. al. [23]. Their starting point is to consider generalized functions in an \( n \)-manifold. Subsequently, they set up the conditions for deriving the results of Dyson in a formal mathematical framework. It is possible that their formulation might be useful and more powerful in derivation of Dyson’s theorem for general \( D \)-dimensions. The analyticity properties of the 2-particle scattering amplitude and that it is polynomially bounded have been rigorously derived by Hepp [44] in a more general setting without appealing to the LSZ formalism.

### 3.3 D-dimensional Derivation of Lehmann’s Ellipses

Our goal is to analyze the analyticity properties of the scattering amplitude, \( F(s, t) \) and write fixed-\( t \) dispersion relations. We have already argued in the preceding section that for fixed negative \( t \), as \( s \to s_{\text{thr}} \), \( \cos \theta \) exceeds its limit when \( K^2 \to 0 \) in the above limit. We had mentioned that Lehmann’s investigation of analyticity of amplitude in \( t \) played an important role. Indeed, the Jost-Lehmann-Dyson representation is very crucial for deriving Lehmann’s result. We remind the reader the two...
equations representing $F(s,t)$, (42) and (43). Since we consider scattering of equal mass identical particles, the kinematics is simplified. We shall invoke the partial wave expansion for D-dimensional case in due course (see Section 4). The following argument will convince that there is only one scattering angle. Let us choose the c.m. frame system. Then D-momenta of particles, $a$ and $b$ are given by \( p_a = (E_a, K) \) and \( p_b = (E_b, -K) \) and \( E_a = \sqrt{m^2 + K^2} = E_b \). Thus \( K \) defines a direction. For the out going particles, c.m. momenta \(|K'| = |K|\) in the equal mass scatterings. Thus \( p_c = (E_c, K') \) and \( p_d = (E_d, -K') \), \( E_c = E_d \). The two vectors \( K \) and \( K' \) (of equal magnitude) define a plane and \( \cos \theta \) is the cosine of angle between vectors \( K \) and \( K' \).

If we considered particles with spin, as it happens in the 4-dimensional case, there will be more complications and the scattering amplitudes with helicity states are to be defined. There will be analogous complications in the case of D-dimensions when we consider scattering of particles with spin. In fact we can choose a coordinate frame to define the momenta of the four particles, now,

\[
\begin{align*}
p_a &= \left( \frac{\sqrt{s}}{2}, +K, 0, 0 \right), \quad p_b = \left( \frac{\sqrt{s}}{2}, -K, 0, 0 \right) \\
p_c &= \left( -\frac{\sqrt{s}}{2}, +K\cos \theta, +K\sin \theta, 0 \right), \quad p_d = \left( -\frac{\sqrt{s}}{2}, -K\cos \theta, -K\sin \theta, 0 \right)
\end{align*}
\] (71)

We have adopted a coordinate system where \( K \) lies along spatial direction '1' for \( p_a \) and the same is true for \( p_b \), except that sign is opposite; all other components are zero. For the out going particles, the c.m. momentum (magnitude \( K \)) lies along the \( 1 - 2 \) plane. Thus \( 0 \) represents a \( D - 3 \) dimensional vector all whose components are zero. Moreover, according to our conventions for scattering the vectors \( p_c \) and \( p_d \) have appropriate signs. Furthermore, \( s = (p_a + p_b)^2 = (-p_c - p_d)^2 \geq 4m^2 \), \( p_a + p_b + p_c + p_d = 0 \). Now we identify

\[
q = \frac{1}{2}(p_d - p_c) = (0, -K\cos \theta, -K\sin \theta, 0), \quad P = \frac{1}{2}(p_a + p_b) = \left( \frac{\sqrt{s}}{2}, 0, 0, 0 \right)
\] (72)

The next step is to identify the coincidence region to proceed further. We define (i) \( p_c^2 = (P + q)^2 < M_c^2 \) and (ii) \( p_d^2 = (P - q)^2 < M_d^2 \). Where \( M_c \) is the lowest threshold for two or more particle states which carry the same quantum number as \( c \). Similarly, \( M_d \) is defined for particle \( d \). We keep these identities for book keeping. However, for the present case \( M_c^2 = M_d^2 = 4m^2 \). We also need to use threshold constraints in various channels. The relevant conditions are: (ia) \( (p_a - p_c)^2 = (P - p_a + q)^2 < M_{ac}^2 \) and (iib) \( (p_a + p_d)^2 = (P - p_a - q)^2 < M_{ad}^2 \). Notice that \( M_{ac} \) and \( M_{ad} \) correspond to threshold for two or more particles carrying quantum numbers of \( a-c \) and \( a-d \) channels respectively. Note, again that \( M_{ac}^2 = M_{ad}^2 = 4m^2 \) here as well. In general, the problems arise when quantum number considerations forbid lowest two particle state in a given channel. In the present case, the center of the hyperboloid, \( 4m^2 \) in the region \( P + q \in V^+ \) and same is true for the other one, i.e. \( P - q \in V^+ \). We intend to find the location of the singularities. These are in the region specified by
the admissible hyperboloids i.e. \((q-u)^2 = \chi_{\text{min}}^2 + \rho, \rho > 0\) and \(\left(\frac{p_a + p_b}{2}\right) \pm u \in V^+\). We determine \(\chi_{\text{min}}^2\) from

\[
\chi_{\text{min}}^2 = \max \left\{0, M - \sqrt{\left(\frac{p_a + p_b}{2} + u\right)^2}, M - \sqrt{\left(\frac{p_a + p_b}{2} - u\right)^2}\right\}
\]

(73)

We have an integral over \(u\) in the Jost-Lehmann representation. We choose

\[
u = (u_0, |u|cos\alpha, |u|sin\alpha, 0)
\]

(74)

and compute \((q-u)^2\) with the constraint \((q-u)^2 = \chi_{\text{min}}^2 + \rho, \rho > 0\). This constraint leads to the equation

\[
u_0^2 - K^2 - u^2 - 2|K||u|cos(\theta - \alpha) = \chi_{\text{min}}^2 + \rho, \rho > 0
\]

(75)

Therefore, defining \(cos(\theta - \alpha) = z(u)\),

\[
z(u) = \frac{1}{2|K||u|}\left(K^2 + u^2 + \chi_{\text{min}}^2 + \rho - u_0^2\right)
\]

(76)

Noting that \(-1 \leq cos(\theta - \alpha) \leq +1\). The singularity occurs for \(z_{\text{min}} = Min_{u,\chi^2} z(u)\). It is found to be (for equal mass case)

\[
z_{\text{min}}(s) = \left(1 + \frac{9m^4}{sK^2}\right)^{1/2}
\]

(77)

Note that the amplitude has singularities for \(cos(\theta - \alpha) > z_{\text{min}}\). Moreover, the amplitude is holomorphic in the interior of an ellipse in the \(cos\theta\) plane which has its focii located at \(cos\theta = \pm 1\), with\(^8\)

\[
\cos\theta = z_{\text{min}}cos\alpha + i\sqrt{z_{\text{min}}^2 - 1} sin\alpha, \quad 0 \leq \alpha \leq 2\pi
\]

(78)

This is known as the small Lehmann ellipse (SLE) [31]. If we consider \(t\)-variable, \(t = 2K^2 cos\theta - 2K^2\). The domain of analyticity in the \(t\)-plane is

\[-4K^2 \leq t \leq 0\]

(79)

Thus it is quite satisfying that the analyticity in the SLE is derived for D-dimensional field theories from the axioms of LSZ.

We have remarked in the previous section that extension of the analyticity domain to SLE is not quite adequate since as \(K \to 0\), \(t \to \sim K\) and thus tends to zero softly [10] compared to the earlier case, before Lehmann proved existence of SLE.

On that occasion, the amplitude behave as \(K^2\) near \(s \to s_{\text{thr}}\) when it was assumed that the amplitude is well defined in the region, \(-1 \leq cos\theta \leq +1\). However, with\(^8\)

\(^8\)see book of Itzykson and Zuber [12] for another method to derive the Lehmann ellipse
the existence of SLE, we are able to go beyond. An important step was taken by Lehmann, in the $D = 4$ theories, when he proved that the absorptive part of the scattering amplitude is analytic inside a larger ellipse - the large Lehmann ellipse (LLE). This is accomplished by reducing the expression (42) once more. In other words two particles in state $\langle -p_a - p_c, out \rangle$ were reduced and we were left with two particles in the state $|p_a p_b, in \rangle$. If we reduce the latter

$$A_s(p_a...p_d) = 2\pi \sum_n \int d^D z' e^{-i(p_a - p_d)z'/2} < 0|Rj_c(z'/2)j_d(-\frac{z'}{2})|p_a + p_b, n > \times$$

$$\int d^D ze^{-i(p_a - p_b)z} < p_a + p_b, n |Rb_i(\frac{z}{2})j_a(-\frac{z}{2})|0 >$$

(80)

Notice that there is sum over physical intermediate states; however, the energy momentum of the intermediate states is constrained to be $p_a + p_b$. We may apply J-L-D construction to each of the matrix elements of the R-products. The intermediate physical state is $|p_a = |p_a + p_b, n >$. The first matrix element will be defined with a momentum and there will be an angle between this vector and the momentum vector $P_n$. Similarly, $P_n$ will subtend an angle with the momentum vector associated with second matrix element and we can choose this to be the c.m. frame momentum vector $K$ of $(p_a, p_b)$ system. We can implement J-L-D construction to each of the two systems: (i) one system corresponds the initial state $(p_a, p_b)$ and the other one is final state $(p_c, p_d)$ with c.m. momentum $K'$ (note $|K| = |K'|$). Moreover, the c.m. scattering angle is inner product of the unit vectors along two directions; $\cos\theta = \hat{K}.\hat{K'}$. Recall that in going through the steps to derive SLE, we introduced an angle $\alpha$ in defining the vector $u$. Now it requires two additional angular variables. Moreover, if we denote the angle between $K$ and $P_n$ as $\phi_1$ then the angle between $P_n$ and $K'$ is $\theta - \phi_1$. It is easy to see that we need to introduce two J-L-D functions, one for each matrix element: $\Phi_1^a$ and $\Phi_2^a$. When we go through the algebraic steps, interestingly enough Lehmann’s method exactly goes through in this case also. The region of analyticity of $A_s$ in $\cos\theta$ plane is found to be an enlarged ellipse with foci at $\cos\theta = \pm 1$. The semimajor axis is $(2\cos\theta^2_0 - 1)$ which is $2s_{min}^2 - 1$; thus it is substantially larger compared to the semimajor axis of the SLE.

Remarks: Now the amplitude is well defined for fixed $t$ as $s \rightarrow s_{thr}$. As noted earlier, $A_s$ and $A_u$ are discontinuities across the cuts $s > s_{thr} = 4m^2$ and $u = 4m^2 - s - t > M_{Of}^2 = 4m^2$. Thus $A_s$ and $A_u$ are analytic in LLE. Therefore, we can write dispersion relations in $s$ and $u$ variables for $A_s$ as well as for $A_u$ respectively. Our goal in this investigation is not to prove dispersion relations; however, we require knowledge of analyticity in $s$ and $t$ variables. We mention in passing that all these results were obtained without utilizing the powers of unitarity.

We proceed to investigate positivity conditions on the absorptive part $A_s$. According to axiom (A1) the field operators act on a Hilbert space, $\mathcal{H}$, to create states with positive norms. Let us envisage elastic scattering and focus attention on $A_s$. We consider the scenario where the two particle 'out' state and the two particle 'in' have
been reduced in the two step process as we followed in deriving the LLE. We express (80) in a slightly different form

$$A_s(p_a, p_b, p_c, p_d) = \mathcal{C}^2 \int d^D x_a d^D x_b d^D x_c d^D x_d e^{-i p_a \cdot x_a - i p_b \cdot x_b - i p_c \cdot x_c - i p_d \cdot x_d} \delta^D(p_a + p_b + p_c + p_d)$$

$$<0 | R(j_c(x_c)j_d(x_d))R(j_b(x_b)^\dagger j_a(x_a)^\dagger)0 >$$  \hspace{1cm} (81)

It is understood that the total energy momentum conserving $\delta$-function $\delta^D(p_a + p_b + p_c + p_d)$ multiplies $A_s$ on the left hand side of the above equation and we have suppressed it. Here $\mathcal{C}$ is a real constant. We are aware that the above matrix element is a distribution. Therefore, it is necessary to introduce suitable test functions in order that the right hand side is properly defined. We choose $C(x, y)$ to be such a test function which is rapidly decreasing and is infinitely differentiable; the variables $x$ and $y$ belong to $\mathbb{R}^D$. The Fourier transform of $C(x, y)$ is denoted as $\tilde{C}(p, q)$. Now considered the smeared state

$$\int d^D x_a d^D x_b C(x_a, x_b)R(j_b^\dagger(x_b)j_a^\dagger(x_a))0 >$$  \hspace{1cm} (82)

This state has a positive norm and therefore,

$$0 < || \int d^D x_a d^D x_b C(x_a, x_b)R(j_b^\dagger(x_b)j_a^\dagger(x_a))0 > ||^2$$

$$= <0 | \int d^D x_c d^D x_d C^\ast(x_c, x_d)R((j_c(x_c)j_d(x_d)) \times$$

$$\int d^D x_a d^D x_b C(x_a, x_b)R(j_b^\dagger(x_b)j_a^\dagger(x_a))0 >$$  \hspace{1cm} (83)

Let us take the Fourier transform of (82)

$$0 < \mathcal{C}^2 \int d^D p_a d^D p_b d^D p_c d^D p_d \tilde{C}(p_a, p_b)\tilde{C}^\ast(p_c, p_d) \delta^D(p_a + p_b + p_c + p_d)A_s(p_a, p_b, p_c, p_d)$$  \hspace{1cm} (84)

This integral is positive as defined and $\mathcal{C}'$ is a real constant. Now define

$$P = \frac{(p_a + p_b)}{2}, \quad q = -\frac{(p_a - p_b)}{2}, \quad q' = \frac{(p_c - p_d)}{2}$$  \hspace{1cm} (85)

Thus we get the relations (i) $p_a = P + q$, $p_b = P - q$ and (ii) $p_c = -P + q'$, $p_d = -P - q'$; using $\sum_p t_l = 0$, $l = a, b, c, d$. Moreover, Mandelstam variables $s$ and $t$ can be expressed in terms of $P, q, q'$ also. We choose $\tilde{C}(p_a, p_b) = \tilde{C}(P + q, P - q) = \tilde{f}(P)\tilde{g}(q)$

The functions, $\tilde{f}$ and $\tilde{g}$, defined in the momentum space are in $R^D$. The positivity condition (84) is expressed as

$$\int d^D P |\tilde{f}(P)|^2 \int d^D q d^D q' A_s(-P - q, -P + q, -P - q', -P + q')\tilde{g}(q)\tilde{g}^\ast(q') > 0$$  \hspace{1cm} (86)

\(^9\text{See discussion on this point by Martin in the ETH Lecture notes }[10]\)
We note that $A_s$ is a positive measure in $P$. The expression for $A_s$ in (81) which is defined in the coordinate space is product of two retarded functions. One is $R_{j_c}(x_c)j_d(x_d)$ and other is $R_{j_b}^\dagger(x_b)j_d^\dagger(x_a)$ Thus each of the R-product satisfies retardedness properties in pair of coordinates: the former in $(x_c, x_d)$ and the latter in $(x_a, x_b)$. We argue that if $P$ is held fixed, then $A_s$ is a function of $q$ and $q'$ and it possesses analytic properties in the two variables. We arrive at the conclusion that the domain of holomorphy (for $q$) lies in a domain, $\Sigma \in C^D$ (the space of complex coordinates) and the other variable ($q'$) is also in the same domain. We remind the reader the steps we followed in the context of deriving J-L-D representation for the scattering amplitude. First we obtained the domain of analyticity for $F(s, t)$ itself and, in the next step, we derived the larger analyticity domain for the absorptive part of the amplitude, $A_s$. This was the route taken to arrive at LLE. In the present context, let us take $P$ to be real and hold it fixed. Consequently, the positivity conditions

$$\int d\mu(q) d\mu(q') A_s(-P-q, -P+q, -P-q', -P+q') \sigma(q)\sigma(q') \geq 0$$

(87)

The measures are: $d\mu(q) = dq \wedge dq^*$ and $d\mu(q') = dq' \wedge dq'^*$. Moreover, $q, q'$ are in $C^D$ and $\sigma(q)$ is a suitably defined function in $L_2$. Now on we suppress the presence of test functions since we know that the distributions are defined with them; we might explicitly invoke their presence if necessary. We choose the following assignments for various D-vectors which are useful for our kinematical analysis and to study consequences of positivity.

$$P = (\sqrt{s}, 0, 0, 0), \quad q = (\sqrt{K^2 + m^2 - \sqrt{s}}, K\cos\phi, K\sin\phi, 0)$$

$$q' = (\sqrt{K^2 + m^2 - \sqrt{s}}, K\cos\phi', K\sin\phi', 0),$$

(88)

Note that as before, $0$ is the $D - 3$ dimensional vector of the spatial vector with $D - 1$ components. We can get expressions for $s$ and $t$ with the above assignments for $P, q, q'$. Of special interests to us is the vector $(q - q')$ since $t = (q - q')^2$. Note that

$$(q - q') = (0, K(\cos\phi - \cos\phi'), K(\sin\phi - \sin\phi'), 0)$$

(89)

Thus $t = -2K^2[1 - \cos(\phi - \phi')]$. Therefore, we may identify $\phi - \phi' = \theta$ and $\cos(\phi - \phi') = \cos\theta$. Now the scattering amplitude is a function of $s$ and $\cos(\phi - \phi') = \cos\theta$. In the case of $D = 4$ theories it was convenient to choose $\tilde{g}(q) = e^{i\phi} \tilde{g}^*(q') = e^{-i\phi'}$ then the positivity property of $A_s^{D=4}(s, t)$ could be proved since the amplitude is expanded in the Legendre polynomial basis [10, 11].

In the case of scattering in $D$-dimensions, the basis functions are the Gegenbauer polynomials as has been remarked earlier. In this case the positivity property holds also. We shall show positivity of $A_s$ when we study the partial wave expansion and the problem of enlarging the domain of analyticity. The reader may consult the Appendix C where we have collected some useful formulas relevant in our work where the Gegenbauer polynomials appear.
3.4 Fixed $t$ Dispersion Relations

Let us discuss the fixed-$t$ dispersion relations in $s$ for the scattering amplitudes. We postulated that there are no bound states and therefore, $s_{\text{thr}} = s_{\text{phys}} = 4m^2$. The elastic scattering amplitude, for fixed $t$ in the region, $-T, T > 0$, admits a dispersion relation in $s$. If the integrand in the dispersion relation does not have a good convergence property then an unsubtracted dispersion relation may be substituted by a subtracted dispersion relation with $N$ subtractions. We know from our earlier discussions that in the LSZ formulation $N$ is finite. Moreover, $F(s, t)$ is also an analytic function in both $s$ and $t$ in some neighborhood of any $\tilde{s}$ where $\tilde{s}$ lies in some interval below $s_{\text{phys}}$ such that $4m^2 - \delta < \tilde{s}$, $4m^2$ and it also lies in some neighborhood $|t| < \bar{R}(\tilde{s})$ of $t = 0$. This result has been proved for the case of $D = 4$ in [39]. Lehmann [46] (in $D = 4$) approached the problem of writing fixed-$t$ dispersion relation from a different viewpoint. He considered scattering amplitude for fixed $s > s_{\text{thr}}$, for values of $t$ that lie within SLE and thus $t$-analyticity is valid. This is to be contrasted with results of [39]. As we shall see later, Martin exploited the analyticity domain ordained in [39] to prove his theorem. We have not rigorously proved existence of such a domain of analyticity for D-dimensional theories. This analyticity property was derived by [39] in the LSZ formulation, with micro causality. We have shown that there is analyticity in $t$ in the Lehmann ellipse for D-dimensional theories and polynomial bounded (in $s$) in such higher dimensional theories. Therefore, we strongly believe that BEG result is also valid for D-dimensional theories (see discussion (iv) after (56)). The absorptive parts $A_s$ and $A_u$ defined on the right hand and left hand cuts respectively, for $s' > s_{\text{thr}}$ and $u' > u_{\text{thr}}$ are holomorphic in the LLE. Thus, assuming no subtractions necessary

$$F(s, t) = \frac{1}{\pi} \int_{4m^2}^{\infty} \frac{ds' A_s(s', t)}{s' - s} + \frac{1}{\pi} \int_{4m^2}^{\infty} \frac{du' A_u(u', t)}{u' + s - 4m^2 - t}$$

(90)

We have argued that the integrands satisfy polynomial boundedness properties and therefore, if required, we might need subtractions.
4. Analyticity in $s$ and $t$ and the Asymptotic Behavior of Scattering Amplitude in D-dimensions

We shall further study the analyticity properties of the scattering amplitude in this section. Let us recall the main results of the previous sections. We have derived the expressions for the absorptive parts of the scattering amplitude $A_s$ and $A_u$ from LSZ formulation and we have argued that the amplitude is polynomially bounded in $s$ when we write a dispersion relation. In Section 3, we devoted our investigations to the analyticity properties in that the representations for $F_C(q)$ and $F_A(q)$ were obtained from the Jost-Lehmann-Dyson theorem. We also showed that in D-dimensions, the analog of Lehmann ellipses exist. However, it was mentioned that the results of generalized Lehmann ellipses were not adequate to derive the Froissart bound. In order to derive the higher dimensional Froissart-Martin bound we have to go through a few more important steps. This will be focus of this section. A crucial result in the derivation of the Froissart-Martin bound, as is known in the present form, relies on a theorem due to Martin [47]. We shall present generalized Martin’s theorem. However, we shall summarize below the essence of the theorem, as was derived in the four dimensional case and provide remarks on our way to generalize the theorem.

Statement of Martin’s Theorem for $D = 4$: If following requirements are satisfied
I. $F(s, t)$ satisfies fixed-$t$ dispersion relation in $s$ with finite number of subtractions ($-T \leq t \leq 0$).

Remark: This property is true for the case of D-dimensional theory in LSZ formulation as has been argued by us.

II. $F(s, t)$ is an analytic function of the two Mandelstam variables, $s$ and $t$, in a neighborhood of $\bar{s}$ in an interval below the threshold, $4m^2 - \rho < \bar{s} < 4m^2$ and also in some neighborhood of $t = 0$, $|t| < R(\bar{s})$. This statement hold due to the work of Bros, Epstein and Glaser [39, 45].

Remark: We have not proved the BEG theorem for the four point amplitude in the D-dimensional case. However, there seems to be no serious obstacles to generalize it to D-dimension in the LSZ formalism. Thus if we follow the arguments presented following equation (56) (see the remark (iv)) the BEG results hold for the four point amplitude.

III. Holomorphicity of $A_s(s', t)$ and $A_u(u', t)$: The absorptive parts of $F(s, t)$ on the right hand and left hand cuts with $s' > 4m^2$ and $u' > 4m^2$ are holomorthic in the LLE.

Remark: We have shown that for D-dimensional field theories, there exist SLE and LLE (see Sec 3.3). This result was derived on the basis of the generalized J-L-D construction in the D-dimensional case.

IV. The absorptive parts $A_s(s', t)$ and $A_u(u', t)$, for $s' > 4m^2$ and $u' > 4m^2$ satisfy
the following positivity properties

\[ \left| \left( \frac{\partial}{\partial t} A_s(s', t) \right)^n \right| \leq \left. \left( \frac{\partial}{\partial t} \right)^n A_s(s', t) \right|_{t=0}, \quad -4K^2 \leq t \leq 0 \] (91)

and

\[ \left| \left( \frac{\partial}{\partial t} A_u(u', t) \right)^n \right| \leq \left. \left( \frac{\partial}{\partial t} \right)^n A_u(u', t) \right|_{t=0}, \quad -4K^2 \leq t \leq 0 \] (92)

Remarks: The above positivity properties were proved, in $D = 4$ case, using properties of the Legendre polynomials in the partial wave expansion for the scattering amplitude with unitarity constraints on the partial wave amplitudes. For the $D$-dimensional case, the basis functions are the Gegenbauer polynomials. We shall prove similar inequalities for absorptive parts in the case of $D$-dimensions in the latter part of this section. The inequalities (91) and (92) indeed hold.

The Martin’s theorem aims at deriving the domain of analyticity of the scattering amplitude in the complex planes of $s$ and $t$ variables. This can be proved if the scattering amplitude can be expanded in a power series

\[ F(s, t) = \sum_{n=0}^{\infty} t^n \frac{d^n}{dt^n} F(s, 0), \] (93)

Then appealing to the Hartog’s theorem\(^1\), $F(s, t)$ will be analytic in the quasi-product of topological domains $D_s \otimes D_t$ if for every $s \in D_s$ the series is uniformly convergent for $t \in D_t$. Therefore, the first step is to define and derive a bound on $\left. (\frac{d}{dt})^n F(s, t) \right|_{t=0}$ and then analyze the convergence properties of (93). Let us first consider the dispersion relation for $F(s, t)$ only in the presence of the right hand cut and momentarily ignore the presence of the left hand cut. We shall account for the presence of the left hand cut later. Moreover, we also assume that there is no subtraction.

\[ F(s, t) = \frac{1}{\pi} \int_{s_{thr}}^{\infty} ds' \frac{A_s(s', t)}{s' - s} \] (94)

Next, we use the BEG [39, 45] result that for each point of $s$ in the cut plane and fixed $t$, it is possible to write a dispersion relation. More specifically, if one chooses $s = 4m^2$, there is an analyticity neighborhood in $s$ and in $t$ such that $s = 4m^2 - \epsilon$ and $t$ in the analyticity neighborhood. One can choose an $\epsilon$ which is small enough to fulfill these requirements. Thus when we have chosen an $\bar{s}$ which is sufficiently close to $s_{thr}$, we can find an $R$ such that $F(\bar{s}, t)$ is analytic in $t$ in the region $|t| < R$.

\(^1\)Hartog’s theorem for functions of several complex variables $f(z_1, z_2, \ldots z_n)$ may stated as follows. Let $f(z_1, z_2, \ldots z_n)$ be defined in an $n$-complex domain $D$ and let $f(z'_1, z'_2, z'_k, z'^{k+1}_n)$ for all $k = 1, 2, \ldots n$ be holomorphic in $|z_k - z'_k| < \epsilon$ as a function of $z_k$ for all $f(z'_1, z'_2, z'_k, z'^{k+1}_n) \in D$. Then $f(z_1, z_2, \ldots z_n)$ is holomorphic in $D$ simultaneously. See [10] for examples to illustrate the technique of analytic continuations and how the domain of holomorphicity is enlarged.
We need to choose \( \bar{s} \) below the threshold to avoid the divergence difficulties of the integral above (94). Therefore, we rewrite the above relation as

\[
F(\bar{s}, t) = \frac{1}{\pi} \int_{s_{\text{thr}}}^{\infty} ds' \frac{A(s', t)}{s' - \bar{s}} \quad (95)
\]

Notice that \( F(\bar{s}, t) \) is analytic in the domain \( |t| < R \). Consequently, all derivatives of the function exists at \( t = 0 \). Now if we apply Cauchy’s inequality [48] for an analytic function inside a domain \( |t| < R - \epsilon \) then its derivatives at \( t = 0 \) are bounded i.e.

\[
|F(\bar{s}, t)| \leq M, \quad \text{and,} \quad |(\frac{d}{dt})^n F(\bar{s}, t)| < \frac{n!M}{(R - \epsilon)^n} \quad (96)
\]

We follow Martin’s argument and endeavor to show that, if we adopt his procedures, there is a method to analytically continue the inequalities (96) to the complex cut \( s \)-plane. This is accomplished by utilizing the powerful result of fixed-\( t \) dispersion relations. The above result can be proved if one is permitted to interchange differentiation with respect to \( t \) with the integration over \( s \) in the dispersion relation. The Lemma 1, given in the Appendix B can be readily utilized for this purpose. Thus

\[
\frac{d}{dt} F(\bar{s}, 0) = \lim_{\tau \to 0, \tau > 0} \frac{F(\bar{s}, 0) - F(\bar{s}, -\tau)}{\tau} = \frac{1}{\pi} \lim_{\tau \to 0, \tau > 0} \int_{s_{\text{thr}}}^{\infty} \frac{A_s(\bar{s}, 0) - A_s(\bar{s} - \tau)}{\tau(s' - \bar{s})} ds' \quad (97)
\]

The absorptive part \( A_s(\bar{s}, t) \) and its \( t \)-derivatives satisfy the inequalities

\[
A_s(s, t = 0) \geq \left| A_s(s, t) \right|_{-4k^2 \leq t \leq 0} \\
\left| (\frac{d}{dt})^n A_s(s, t) \right|_{t=0} \geq \left| (\frac{d}{dt})^n A_s(s, t) \right|_{-4k^2 \leq t \leq 0} \quad (98)
\]

The first inequality follows from optical theorem applied to elastic amplitude using positivity property of the imaginary part of the partial wave amplitudes. As noted, for D-dimensional case, the basis functions are the Gegenbauer polynomials and the first inequality is easily derived. The other inequalities need some careful analysis to use relations among the Gegenbauer polynomials. At this moment, we would like the reader to accept (98) and see the subsequent results in this section. We use the arguments of Lemma 1 and decompose the right hand side of the integral eq. (97) as \( \int_{s_{\text{thr}}}^{X} + \int_{X}^{\infty} \) then we argue as in Lemma1 that

\[
\lim_{\tau \to 0} \frac{F(\bar{s}, 0) - F(\bar{s}, -\tau)}{\tau} \geq \lim_{\tau \to 0} \int_{s_{\text{thr}}}^{X} \frac{A_s(\bar{s}, 0) - A_s(\bar{s} - \tau)}{\tau(s' - \bar{s})} \quad (99)
\]

as is obvious. We are free to choose \( \tau \) to be small enough so that for all \( 4m^2 \leq s' \leq X \) the interval \( -\tau \leq t \leq 0 \) inside all the Lehmann ellipses for a given \( X \). We remind
the reader that the size of the Lehmann ellipses depend on the value of $s'$ we choose since the semimajor axis depends on $s'$. Thus

$$\lim_{\tau \to 0} \frac{A_s(s',0) - A_s(s',-\tau)}{\tau} = \frac{dA_s(s',0)}{dt}$$  \quad (100)$$

and using Lemma 1

$$\frac{dF(s,0)}{dt} = \frac{1}{\pi} \lim_{\tau \to 0} \frac{1}{s' - \bar{s}} \int_{s' \tau}^{\infty} \frac{dA_s(s',-\tau'(s'))}{dt} ds'$$  \quad (101)$$

Our aim is to show that $F(s,t)$ can be expanded in a power series. Thus we should prove the inequality other way around for $t$-derivative of the amplitude. To this end, let us begin with

$$\frac{dF(s,0)}{dt} = \frac{1}{\pi} \lim_{\tau \to 0} \frac{1}{s' - \bar{s}} \int_{s' \tau}^{\infty} \frac{dA_s(s',-\tau'(s'))}{dt} ds'$$  \quad (102)$$

Now choose $-\tau < -\tau'(s') < 0$ (it corresponds to a physical region). Moreover, if we have $s' > s_{thr} + \rho$, $\rho > 0$, $t = -\tau'(s')$ is in physical region. We can utilize the positivity property and the inequality associated with the absorptive part

$$\frac{dF(s,0)}{dt} \leq \lim_{\tau \to 0} \frac{1}{\pi} \int_{s_{thr} + \rho}^{\infty} ds' \frac{dA_s(s',-\tau'(s'))}{dt} ds'$$  \quad (103)$$

The point to notice that in the above equation the first integral is taken over a compact interval. Moreover, the integrand is positive and regular. Therefore, we are permitted to choose $\rho$ as small as we desire. As a consequence,

$$\frac{dF(s,0)}{dt} \leq \lim_{\tau \to 0} \frac{1}{\pi} \int_{s_{thr}}^{\infty} ds' \frac{dA_s(s',0)}{dt} \frac{1}{s' - \bar{s}}$$  \quad (104)$$

Thus combining (101) and (104) we come to conclusion that that

$$\left. \frac{dF(s,0)}{dt} \right|_{t=0} = \frac{1}{\pi} \int_{s_{thr}}^{\infty} \frac{dA_s(s',0)}{dt} ds'$$  \quad (105)$$

Therefore, using the above result for the $t$-derivative of $F(s,0)$ we can derive the relation

$$\left( \frac{d}{dt} \right)^n F(s,t) = \frac{1}{\pi} \int_{s_{thr}}^{\infty} \left( \frac{d}{dt} \right)^n A_s(s',0) \frac{ds'}{s' - \bar{s}} \leq \frac{Mn!}{(R - \epsilon)^n}$$  \quad (106)$$

Note that the positivity property i.e. $\left( \frac{d}{dt} \right)^n A_s(s',0) \geq 0$ has been utilized. We shall demonstrate it later in this section. Thus the absorptive part is also bounded. The last inequality is a consequence of the Cauchy’s inequality. We can also define

$$\left( \frac{d}{dt} \right)^n F(s,0) = \frac{1}{\pi} \int_{s_{thr}}^{\infty} \frac{\left( \frac{d}{dt} \right)^n A_s(s',0)}{s' - s} ds'$$  \quad (107)$$
The function as defined in (106) is defined in a finite segment of the real $s$-axis. We are allowed to move $\bar{s}$ and the argument does not change. If the integral has uniform convergence, we can continue to it arbitrary complex plane. Thus $\left(\frac{d}{dt}\right)^n F(s, 0)$ can be continued to arbitrary complex $s$-plane since there exist an analyticity neighborhoods. Since $\left.\left(\frac{d}{dt}\right)^n A_s(s, t)\right|_{t=0} \geq \left.\left(\frac{d}{dt}\right)^n A_s(s, t)\right|_{-4K^2 \leq t \leq 0}$, we use

$$\left|\left(\frac{d}{dt}\right)^n A_s(s', 0)\right| - \left|\left(\frac{d}{dt}\right)^n A_s(s', 0)\right|_{s'-s} < \mu(s, \bar{s}) (108)$$

and

$$\mu(s, \bar{s}) = \sup_{4m^2 < s' < \infty} \frac{|s' - \bar{s}|}{|s' - s|} (109)$$

note that this remains finite as long we are outside the cut. Moreover, for $\Re s > s_{thr}$ as $\Im s \to 0$, $\mu(s, \bar{s})$ diverges. We utilize the earlier inequality to argue that

$$\left|\left(\frac{d}{dt}\right)^n F(s, 0)\right| < \mu(s, \bar{s}) \frac{Mn!}{(R - \epsilon)^n} (110)$$

Thus the expansion of $F(s, t)$ in a power series in $t$, (93), converges for $|t| < R$. As a consequence we conclude that, for fixed $s$, $F(s, t)$ is an analytic function of $t$. Moreover, for any fixed, $t$, $|t| < R$, if we remain in a compact region of complex $s$-plane where $\mu(s, \bar{s})$ is bounded; it converges. Note that each term in the power series expansion of $F(s, t)$ is analytic in $s$. It follows from Hartog’s [49, 10] theorem that the amplitude is analytic in topological product of the domains $D_s \otimes D_t$. This is defined by: $|t| < R$ and $s$ outside the cut $s_{thr} + \lambda = 4m^2 + \lambda, \lambda > 0$.

The importance of this result is recognized if we recall BEG theorem [39]. It was shown that in neighborhood of any point $s_0, t_0$, $-T < t_0 \leq 0$, $s_0$ outside the cuts, there is analyticity in $s$ and $t$ in a region

$$|s - s_0| < \eta(s_0, t_0), \quad |t - t_0| < \eta(s_0, t_0) (111)$$

The size of this analyticity neighborhood can vary as we vary $s_0$ and $t_0$. Moreover, as $s \to 0$, $\eta(s)$ can shrink to zero. Martin’s theorem proves that there is a lower bound to $\eta(s)$ such that $\eta(s) \geq R$ and $R$ is independent of $s$. Consequently, we can argue that in the region $|t| < R$, $F(s, t)$ satisfies unsubtracted dispersion relation; that is where we started with. Notice that if the amplitude satisfies polynomial boundedness, then with subtractions, the analyticity properties are not affected.

We remind the reader that the above result was derived in the presence of only the right hand cut. There is a problem, when we include the left hand cut part, $A_u(u', t)$. Let us recall

$$F(\bar{s}, t) = \frac{1}{\pi} \int_{s_{thr}}^\infty ds' A_s(s', t) + \frac{1}{\pi} \int_{u_{thr}}^\infty A_u(u', t) (112)$$

and $\bar{u} = 4m^2 - \bar{s} - t$. The following problem crops up when we take into account the presence of the left hand cut in the dispersion relation. If we take derivative of
$F(\bar{s}, t)$ at $t = 0$, then right hand side of (112) is not a sum of two positive terms (and positivity played a key role in all preceding arguments in this context). The positivity property is spoiled due to the presence of $\bar{u} = 4m^2 - \bar{s} - t$ in the denominator of the second integral and its presence makes the previous procedure inadequate. The Lemma 2 is very useful to resolve this issue (see Martin’s lecture notes for details)\textsuperscript{11}. The analyticity properties of $F(\bar{s}, t)$, in both $s$ and $t$ was proved in the presence of only the right hand cut. If we consider a function
\[
\frac{F(\bar{s}, t)}{\bar{s} - R - t}
\]
it is also analytic in $|t| < R$. The analyticity property of newly defined function (113) is unaffected as long as $R < \bar{s} < s_{\text{thr}}$. In order to facilitate application of Lemma 2, we identify
\[\begin{align*}
G_1(s', t) &= A_s(s', t), & G_2(s', t) &= \frac{1}{(s' - \bar{s}) (s' - R - t)}, & \beta = s_{\text{thr}}
\end{align*}\]
for the right hand cut contribution to dispersion integral. For the left hand cut now our identifications are
\[\begin{align*}
G_1(u', t) &= A_u(u', t) \\
G_2(u', t) &= \frac{1}{(u' - \bar{s} - 4m^2 + t) (s' - R - t)}, & z = -s' + 4m^2 - t, & \beta = u_{\text{thr}}
\end{align*}\]
The functions identified with $G_1$ and $G_2$ defined in Lemma 2 satisfying the requirements (b) and (c) laid down in proving Lemma 1 and Lemma 2. Therefore, in going through Martin’s arguments for the function in complex $s$-plane. The new bound is
\[
|F(s, t)| < \left[ \sup_{s_{\text{thr}} \leq s' \leq \infty} \left| \frac{s' - \bar{s}}{s' - \bar{s}} \right| + \sup_{u_{\text{thr}} \leq u' \leq u_{\text{thr}}} \left| \frac{u' - \bar{s} - 4m^2}{u' + s - 4m^2 + t} \right| \right] \times \\
(\bar{s} - R) \sum_{n=0}^{\infty} \frac{|s|^n}{R^n} \text{Max}_{|t| < R} \left( \frac{F(\bar{s}, t)}{\bar{s} - R - t} \right) \]

\textit{Determinaton of $R$:} We remark that in derivation of the generalized Martin’s theorem, it was crucial to to use the fact that the power series expansion of the scattering amplitude, $F(s, t)$ converges in the domain $|t| < R$. The second important point to note is that the BEG function, $\eta(s) > R$, not only it is bounded from below but $R$ is independent of $s$. Therefore, it is essential to determine $R$. We also recall that in the preceding discussion, we have used the fact that $F(s, t)$ in analytic in the SLE whose existence was derived in Section 3. Moreover, the analyticity of $A_s$ and $A_u$ inside the LLE is also a very important ingredient in writing the dispersion relation. Note that

\begin{footnote}
\textsuperscript{11}See the works of Martin [10] and also see [15, 50]
\end{footnote}
the right hand extremity of SLE is \( r_{sLE} = 2K^2(z_{\text{min}} - 1) \) whereas that of the LLE is \( r_{1LE} = 4K^2(z_{\text{min}}^2 - 1) \).

We recall that the generalized Martin’s theorem was derived for \( s < s_{\text{thr}} \), it lies just below the threshold although it lies in the analyticity region as per of BEG results. As has been pointed out earlier, BEG formalism is not adequate to determine \( R \). Therefore, it is important to determine \( R \) and to employ the analyticity in \( t \) to arbitrary \( s \), not just below the threshold. Thus, it is worth while to utilize Lehmann’s analysis which implies that if \( s > s_{\text{thr}} \), then \( F(s_1,t) \) is analytic within the domain

\[
|t| < 2K^2(s_1)\left[\left(1 + \frac{(M_a^2 - m_a^2)(M_b^2 - m_b^2)}{K^2(s_1)|s_1 - (M_a - M_b)^2|}\right)^{1/2} - 1\right]
\]

A few comments are necessary: (i) Note that c.m. momentum \( K = K(s_1) \) and \( s_1 \) is above the physical threshold. (ii) We have deliberately retained \( M_a, M_b, m_a, m_b \) and remind that \( (M_a, M_b) \) correspond to two or more particle states carrying the quantum number of ‘a’ and ‘b’. In our case these are equal and \( (M_a = M_b = 2m) \) since \( m_a = m_b = m \). In case of \( \pi \pi \) scattering it starts with \( 3\pi \) state due to \( G \)-parity considerations. (iii) \( s_1 \) is on the physical cut. The analytic continuation mentioned earlier is applicable in the range \( 0 < s = \bar{s} < 4m^2 \). Sommer [50] provided a general method to remove the cut \( 4m^2 < s < s_1 \) which permits to obtain Martin’s result beyond \( \bar{s} \). This procedure led to derivation of the value of \( R \).\(^\text{12}\) Martin first applied the technique, described below, to obtain value of \( R \) for \( \pi^0\pi^0 \) scattering. Martin’s original arguments for \( \pi^0\pi^0 \) scattering is applicable for the case at hand since the essential logical sequences are the same. We consider scattering of equal mass neutral spinless particles. Therefore, all the three channels, \( s, t \) and \( u \), are identical and consequently, Martin’s procedure goes through. Then following the BEG [45] argument the dispersion relation for fixed \( t (-4K^2 < t \leq 0) \) can be written down. Consequently, all the points inside the region (a triangle): (a) \(-4m^2 < t \leq 0, s < 4m^2, 4m^2, u < 4m^2 \) lie in the domain of analyticity for the scattering amplitude. Note that \( t \) can take a much lower value; however, we have chosen a value which is required for what follows in sequel. We can also consider the other two channels: the one choice of kinematical region is (b) \(-4m^2 < s \leq 0, t < 4m^2, s < 4m^2 \) and the other is (c) \(-4m^2 < u \leq 0, t < 4m^2, s < 4m^2 \).

As in the case of region (a), for the two other regions, (b) and (c), to each of the points in (b) as well as in (c), we can attach a neighborhood in \( s \) and \( t \) where the scattering amplitude is analytic. Note that if we choose \( s = s_1 \), there is an analyticity domain \(|t| < R \). Moreover, the same analyticity argument goes through for \( s_1 < s < 4m^2 \).

Now fix \( s, 4m^2 - R + \epsilon < s < 4m^2 \); following [45], for each \(-s < t_0 < 4m^2 \) there is a neighborhood \(|t - t_0| < \eta(s, t_0) \) of analyticity in \( t \). Note that we have analyticity in this compact region. Then invoke the Heine-Borel-Lebesgue theorem to argue that the interval \(-s + \epsilon \leq t \leq 4m^2 - \epsilon \) can be covered by finite number of (such) compact

\(^{12}\)Martin [47] had derived \( R = 4m^2_\pi \) for \( \pi\pi \) scattering.
intervals. Moreover, the fixed-s amplitude is analytic in t in a region which also contains the real domain $-s + \epsilon \leq t < 4m^2 - \epsilon$. First consider the case when an unsubtracted dispersion relation can be written for $F(s, t)$ for $0 < s < 4m^2$. If we consider a function

$$\frac{F(s, t)}{s - t}$$

in the interval $0 < s < 4m^2$, then the function has all its $t$-derivatives positive at $t = 0$. The power series expansion of this function acquires a singularity at $|t| = R$. Thus, the strip $-s + \epsilon < t < s + \epsilon$ is singularity free. Therefore, one concludes $R = s$. Now $s$ can be taken as much closer to $4m^2$ as we desire. Therefore, the analytic domain is $|t| < 4m^2$.

Remarks: (i) If the amplitude needs subtractions to write dispersion relation then it allows Martin’s arguments to go through. We can construct suitable amplitude in this case and implement Martin’s prescription to determine $R$. Note that the the dispersion relation still holds for the newly defined amplitude [47]. (ii) The above arguments of Martin goes through for the elastic amplitude for scattering of massive neutral scalar particles in the D-dimensional theories as has been argued above. The crucial point to underline here is that all three channels are identical. Therefore, the amplitude $F(s, t)$ satisfies all the criterion we require to determine $R$.

(iii) The contrasting point is that in the case of hadronic scattering (i.e. ππ scattering) the of Martin’s triangle is determined through the introduction of $m_\pi$ which is experimentally determined to have a numerical value i.e. $m_\pi = 140$ MeV. This value is not determined from the theory but it is experimentally measurable as a number. However, what will be $R$ (that it is $4m_\pi^2 - \epsilon$) is derived from the theory. What is important is that value of $R$ is expressed in terms of the mass parameter in the theory. In the case of scattering in D-dimensional theory, $R$ is also determined in terms of the mass parameter of the theory i.e. $R$ is not an arbitrary parameter to be introduced ad hoc. The important point to note, from Martin’s work, is that in the t-plane, in the intersection of the regions of SLE and LLE, there is a circle of radius $R$ which is independent of $s$. Thus the analyticity domain of $F^X$ is contained in this region. Therefore, the amplitude is analytic in the domain: $|t| < R \otimes$ cut s-plane.

Martin proved the theorem for $s$ below the threshold from the BEG [39, 45] results. The analytic continuation is applicable for $0 < s, s_{thr} = 4m^2$. In order to proceed further, so that we can investigate the analyticity properties (and hence the growth properties) of the amplitude, it is necessary to get information about the amplitude beyond this rage of $s$. Sommer [50] provided the resolution. He removed the cut $4m^2 < s < s_1$ by defining a function

$$F^X(s, t) = F(s, t) - \frac{1}{\pi} \int_{s_{thr}}^{X} ds' \frac{A_s(s', t)}{s' - s}$$

in the range $-T \leq t \leq 0$. Now $F^X(s, t)$ has a right hand cut starting at $s = X$ (earlier the cut of $F(s, t)$ started from $s_{thr} = 4m^2$). The positivity property of $F^X(s, t)$, so
crucial to us, remains the same as that of $F(s,t)$. Let us consider $s$ in the following region: $4m^2 < s_1 < X$. Notice that $F^X(s_1,t)$ is analytic in the domain which lies in the intersection of the two functions

$$F(s_1,t) \quad \text{and} \quad -\frac{1}{\pi} \int_{s_{th}}^{X} ds' \frac{A_s(s',t)}{s' - s}$$

(120)

Thus $F(s_1,t)$ is analytic in the Lehmann region given by (117). Moreover, for complex $s$

$$-\frac{1}{\pi} \int_{s_{th}}^{X} ds' \frac{A_s(s',t)}{s' - s}$$

(121)

is analytic in the large Lehmann ellipse associated with $4m^2 < s' < X$. We also know that, in the $\cos \theta$-plane the semimajor axis of large Lehmann ellipse shrinks with energy. What is important for us is that

$$-\frac{1}{\pi} \int_{s_{th}}^{X} ds' \frac{A_s(s',t)}{s' - s}$$

(122)

has still the same analyticity domain in $t$ [50] and it is a distribution in $s$. Moreover, as has been proved [50, 11] for the case of $D = 4$, the domain of convergence in the $t$-plane is circle of radius $R$ and it is $s$-independent. Sommer’s arguments are also valid for the amplitude of $D$-dimensional theory i.e. it is analytic inside a circle in the $t$-plane and the radius is $s$-independent.

4.2 The Partial Wave Expansion and Asymptotic Behavior of Amplitude

We present the partial wave expansion for the scattering amplitude in this subsection. Soldate [27] considered graviton-graviton scattering in arbitrary dimensions without accounting for their spins. He noted that the amplitude admits a partial wave expansion with the Gegebauer polynomial as the basis function. This function is also known as ultraspherical Jacobi polynomial and its domain of convergence lies in the interval $-1 \leq \cos \theta \leq +1$. We have argued earlier that for elastic scattering of equal mass spinless particle there are only two admissible kinematic variables (also holds for elastic scattering of unequal mass particles). The easiest way to arrive at this conclusion is to note that the momenta of the four particles can be chosen to lie in a four dimensional subspace of the $D$-dimensional space without loss of generality as was argued in the preceding section (see remark (iv) after (56)). Once we impose mass shell conditions on scattering particles we are left with only two Lorentz invariant variables $s$ and $t$ which correspond to the c.m. energy squared and the momentum transfer squared. If we consider, say, the case of $D = 10$, then we have to consider the rotation group $SO(9)$, (in D-dimensions it is $SO(D-1)$). The general cases have been studied in mathematical literature [51]. These authors have considered the problem of representations in a series of paper. For 9-dimensional spherical harmonics, it will depend on angular momenta denoted by $l_2, \ldots l_5$, magnetic quantum numbers
\( m_1, \ldots m_4, \) correspondingly there will be the angles \( \theta_2, \ldots \theta_5 \) and four azimuthal angles \( \phi_1, \ldots \phi_4 \). The 'spherical harmonics' will be given by

\[
Y_{l_2, l_5}^{l_1 \ldots l_4} (\theta_2, \ldots \theta_5 : \phi_1, \ldots \phi_4)
\]

(123)

We know that there is only one scattering angle and therefore, all other angles can be integrated out. We are left with the Gegenbauer polynomial denoted by: \( C_1^\lambda (\cos \theta) \)

The partial wave expansion is [27]

\[
F^\lambda = A_1 s^{-\lambda + 1/2} \sum_{l=0}^{\infty} (l + \lambda) f_l^\lambda C_1^\lambda (\cos \theta)
\]

(124)

The amplitude \( F^\lambda (s, t) = F(s, t) \). Thus with this identification, the analyticity properties of \( F^\lambda (s, t) \) have been studied in the previous sections. We introduce the index \( \lambda \), where \( \lambda = \frac{1}{2}(D - 3) \), to keep track of the spacetime dimension we are dealing with. Note that for \( D = 4 \), \( \lambda = \frac{1}{2} \) and in this case the Gegenbauer polynomial is the Legendre polynomial.

\[
A_1 = 2^{4\lambda + 3}\pi^\lambda \Gamma(\lambda), \text{ independent of } s \text{ and } t.
\]

The factor \( s^{-\lambda + 1/2} \) on right hand side of (124) begs explanation. On the dimensional ground, we need this factor if we want the partial wave amplitudes \( \{ f_l^\lambda \} \) to be dimensionless in order to facilitate the partial wave unitarity relation in the conventional form

\[
0 \leq |f^\lambda|^2 \leq \text{Im} f^\lambda \leq 1
\]

(125)

Moreover, eventually, when one derives the bound [25, 26] on \( \sigma_t \), the factor \( s^{-\lambda + 1/2} \) disappears in the expression for the bound. \( C_1^\lambda (x) \) are the Gegenbauer polynomials satisfying orthogonality conditions with weight factor \( (1 - x^2)^{\lambda - 1/2}, -1 \leq x \leq +1 \) [28]. The partial wave unitarity (125) can be derived through the standard procedure; using the orthogonality relations of the Gegenbauer polynomials. Let us discuss, the positivity properties of the absorptive amplitude. Recall that [53](see p184)

\[
C_1^\lambda (1) = \frac{\Gamma(l + 2\lambda)}{l!\Gamma(2\lambda)}
\]

(126)

Again from [54] (see p206)

\[
\text{Max}_{-1 \leq x \leq +1} |C_1^\lambda (x)| = C_1^\lambda (1)
\]

(127)

\( C_1^\lambda (1) \) is positive, therefore,

\[
|\text{Im} F^\lambda (s, t)| = A_1 s^{-\lambda + 1/2} \sum_{l=0}^{\infty} (\lambda + l) \text{Im} f_l^\lambda C_1^\lambda (x)
\]

\[
\leq A_1 s^{-\lambda + 1/2} \sum_{l=0}^{\infty} (\lambda + l) \text{Im} f_l^\lambda |C_1^\lambda (x)|
\]

\[
\leq A_1 s^{-\lambda + 1/2} \sum_{l=0}^{\infty} (\lambda + l) \text{Im} f_l^\lambda C_1^\lambda (1)
\]

\[
= A_1^\lambda (s, t = 0)
\]

(128)
Here the notations are $x = \cos \theta$ and $A_s^\lambda(s,t) = A_s(s,t)$ as defined earlier and $0 \leq \text{Im} f^\ell_k \leq 1$. Thus we have proved the first inequality of positivity for $A_s(s,t = 0) \geq |A_s(s,t)|$ as promised. Now we need to show $\frac{d}{dt}A_s(s,t)|_{t=0} \geq \frac{d}{dt}A_s(s,t)$. This is achieved from the derivative relation of the Gegenbauer polynomial $[54]$

$$\frac{d}{dx}C^\lambda_l(x) = 2\lambda C^{\lambda+1}_{l-1}(x), \quad (129)$$

The above relation and eq. (127) is used to prove $\frac{d}{dt}A_s(s,t)|_{t=0} \geq \frac{d}{dt}A_s(s,t)$. For the $n^{th}$ $t$-derivative of the absorptive part, we may use the chain of relation (the first one) in the above equation repeatedly as follows. First start with the expression for $\frac{d}{dt}A^\lambda(s,t)$ which will involve first derivative of the Gegenbauer polynomial and use (129) to convert derivative of $C^\lambda_l$ to another Gegenbauer polynomial. Then take the $t$-derivative of this expression to get second derivative of $A_s(s,t)$. We would derive the positivity property of second $t$-derivative of $A_s(s,t)$. We then continue this chain of arguments to derive positivity property of the $n^{th}$ $t$-derivative of the absorptive amplitude. Thus, the positivity properties of the absorptive amplitudes that were so useful to prove power series expansion in case of $D = 4$ theories, are also valid in arbitrary dimension, $D > 4$. Therefore, the generalized Martin theorem goes through as was shown already in this section.

Now we shall very briefly recapitulate the derivation of Froissart-Martin bound for D-dimensional theories [25, 26]. We would like to remind that the earlier result was derived under two assumptions: (AI) polynomial boundedness of the scattering amplitude and (AII) convergence of partial wave amplitude inside an extended ellipse with semimajor axis $1 + \frac{2\tilde{T}_0}{s}$, we use notation $\tilde{T}_0$ to distinguish it from $T$ we have introduced already. These assumption crucially used in [25, 26]. Here, they have been derived $ab\ initio$.

In the present investigation, we adopted LSZ formalism to discuss scattering of massive, spinless particle in D-dimensions. In this approach, the amplitude is a tempered distribution. Moreover, within the frame work of LSZ approach, the Fourier transformed amplitude is polynomially bounded in momentum variables. Moreover, we know how to write down dispersion relation for such a case i.e. we might have to write subtracted dispersion relations.

We have shown the existence of SLE and LLE within the LSZ frame work. Moreover, since we are dealing with only a single type of particle (these are their antiparticles too) the direct channel and the two crossed channels are the same. In such a crossing symmetric theory, we have argued, following Martin, that the the radius of the circle in the $t$-plane, $R$, is $4m^2$. To remind, the scattering amplitude $F(s,t)$ is analytic in $t$ in quasi topological product $\{|t| < R = 4m^2\} \otimes$ cut $s$-plane. Thus we have provided proof of the two assumptions used in [25, 26] in the present investigation.

We have not determined how many subtractions are required in the dispersion relation i.e. what is the integer $N$ that appears in the dispersion relation. In other words, can we write an unsubtracted dispersion relation or we need to write a subtracted
dispersion relation? If the answer to second question is in affirmative, the next question is how many subtractions we need?

In order to answer this question, we are required to determine the asymptotic growth properties of the scattering amplitude, especially in the forward direction. We begin by utilizing the property of polynomial boundedness of the scattering amplitude.

4.3 The High Energy Behavior of Scattering Amplitude

We investigate the behavior of scattering amplitude at asymptotic energies which is based on the results we have derived until now. (i) The scattering amplitude is polynomially bounded in $s$ in the sense that the dispersion integral is written with $N$ subtractions. We may take $N$ to be even without loss of generality. (ii) The analyticity property of the amplitude in the domain $|t| < R$. We recall the results, in the context of Froissart-Martin bound [25, 26], which were derived with (i) and (ii) as ad hoc assumptions; presently (i) and (ii) are not so. Now on we shall take $R = 4m^2$. Therefore, in the region $t = \bar{R}$ with $\bar{R} = R - \epsilon$ we can write a dispersion relation since

$$A^\lambda_s(s, t = \bar{R}) < \bar{C}s^N, \quad \bar{C} = \text{Constant}$$

Note that $A^\lambda_s(s, t) = A_s(s, t)$, we use this definition since there will be some $\lambda$-dependent constants i.e. $D$-dependent constants as we proceed. We recall that $A^\lambda_s(s, t = \bar{R})$ is analytic in this region in $t$-plane. Thus the partial wave expansion

$$A^\lambda_s(s, t = \bar{R}) = A_1 s^{-\lambda+1/2} \sum_{l=0}^{\infty} (l + \lambda) \text{Im} \ f^\lambda_l(s) C^\lambda_l(1 + \frac{\bar{R}}{2K^2})$$

converges since $A^\lambda_s(s, t = \bar{R})$ is analytic in the domain $|t| < R \otimes \text{cut } s$-plane. We also know the large-$s$ behavior of $A^\lambda_s(s, t = \bar{R})$ inside Lehmann ellipse. Note that the overall factor $\frac{\sqrt{s}}{K}$ which usually appears in definition of the amplitude has been dropped since this ratio is equal to 1 in the large $s$ limit. Moreover, in the forward direction

$$A^\lambda_s(s, t = 0) = A_1 s^{-\lambda+1/2} \sum_{l=0}^{\infty} (l + \lambda) \text{Im} \ f^\lambda_l(s) C^\lambda_l(1)$$

There is a constant positive factor $C^\lambda_l(1)$ appearing in the above equation. This is the starting point to prove the Froissart bound which Martin improved by exactly determining certain constants. We shall not go through all the steps since this has been undertaken by [25, 26]. They adopt the same maximization program proposed by Martin [11]; however, for general $D$-dimensional case there are departures which we shall point out in sequel. The extremization is achieved by resorting to Martin’s method. (i) Choose $\text{Im} \ f^\lambda_l(s) = 1$ for $0 \leq l \leq L$ (ii) $\text{Im} \ f^\lambda_l(s) = \epsilon < 1$ for $l = L + 1$. (iii) And $\text{Im} \ f^\lambda_l(s) = 0$ for $l > L + 2$. Here we consider the case when the ratio $\frac{L}{\sqrt{s}} \to \infty$ for large $s$ which eventually leads to Froissart-like bound i.e. the total cross section is bounded by power of $\ln s$. The other situation where the ratio $\frac{L}{\sqrt{s}}$ goes to a
constant would make total cross section bound by a constant. As is well known, the polynomial boundedness and the partial wave expansion (132) are crucial ingredients to choose the cut-off value of $L$. In contrast to $D = 4$ case, where one dealt with Legendre polynomials, there are some departures to determine the cut off, $L$. The large $l$ behavior of $C^\lambda_l(1 + \frac{\bar{R}}{s})$ is

$$C^\lambda_l(1 + \frac{\bar{R}}{s}) \sim e^{2l\sqrt{\bar{R}/s}}(\frac{s}{\bar{R}})^{\frac{\lambda - 1}{2}}G(\lambda)$$

where $G(\lambda)$ is a function which depends only on $\lambda$; we shall display it whenever necessary. Noting the polynomial boundedness property (130) and the large $l$ behavior of the Gegengauer polynomials for the argument greater than 1, we get

$$A^\lambda_s(\bar{R}) = \frac{2^{4\lambda - 3}}{\pi^{\lambda/2}}\Gamma(2\lambda)s^{\frac{\lambda}{2} + 1}e^{2L\sqrt{R}}L^\lambda \leq \tilde{C}s^\lambda$$

Thus we find that the cut-off value, $L$, is

$$L = \frac{1}{2}\sqrt{\frac{s}{\bar{R}}}lns + \text{terms nonleading in } lns$$

A remarkable feature is the energy dependence of the cut-off $L \sim \sqrt{slns}$ [25, 26]; there is no power of $\lambda$ in energy dependence. This $s$-dependence is the same as in the 4-dimensional theory. The bound on $A^\lambda_s(s, t = 0)$ now follows [25, 26]

$$A^\lambda_s(s, t = 0) = \sum_{0}^{L}(l + \lambda)C^\lambda_l(1) \leq B(\lambda)\Psi(N, \bar{R})s(lns)^{D-2}$$

where $\bar{R} = 4m^2 - \epsilon$. The total cross section is bounded from the above as

$$\sigma_{\text{total}} \leq B(\lambda)\Psi(N, \bar{R})(lns)^{D-2}$$

One interesting feature of the bound is its energy dependence i.e. is a power of $lns$. Thus for the four dimensional case one recovers the the Froissart-Martin bound, $(lns)^2$. It is worth while to mention that the upper bound (138) contains an unknown parameter. I do not consider $B(\lambda)$ as an unknown function since it gets fixed once we decide the dimensionality of spacetime we work in. However, $\Psi$ depends on $N$; the number of necessary subtraction is not determined so far.

In order to derive what value $N$ takes, let us consider the modulus of the forward scattering amplitude and expand it in partial waves. First of all, we can cut off the partial wave sum at $L$. Thus

$$|F^\lambda(s, t = 0)| \leq \sum_{0}^{L}(l + \lambda)C^\lambda_l(cos\theta = 1)|f^\lambda_l(s)| + \text{terms with sum starting } l > L + 1$$
where \( L = \sqrt{\frac{1}{R^2}} (N - 1) \ln s \). Thus the remainder of the sum starting \( L + 1 \) can be made as small as we desire. It is understood that the right hand side of the above equation might have constant prefactors; however, their presence will not affect the ensuing discussions. Using partial wave inequality, we conclude from (136) that

\[
|F^\lambda(s, t = 0)| < \text{Constant } s \ln s^{D-2}
\]  

(140)

Remark: This bound is generalization of Jin and Martin [52] bound to D-dimensions. As we have argued elsewhere, crossing symmetry is valid for the case under study and invoking crossing, we conclude that the modulus of the forward scattering amplitude \(|F(s, t = 0)| < |s| \ln s^{D-2}\). The bound holds on the right hand cut as well as on the left hand cut. Thus \( F^\lambda(s, t = 0) \) is polynomially bounded in the complex \( s \)-plane. Now invoke Phragman-Lindelof theorem [48]: \(|F^\lambda(s, t = 0)|\) is bounded by Constant \( s \ln s^{D-2} \) in the entire complex \( s \)-plane. Therefore, we need at most two subtractions, i.e. \( N = 2 \), not only in the forward direction, \( t = 0 \) but for \( -T \leq t \leq 0 \). Moreover, for \(|t| < R \) the number of subtractions, \( N = 2 \) (even) is conserved, this is true in the complex \( s \)-plane. We have now fixed \( N = 2 \). Therefore, our work can be summarized as :

**Theorem:** For a massive neutral scalar field theory which satisfies axioms of Lehmann, Symanzik and Zimmermann formalism, the upper bound on the total cross section, \( \sigma_t \), is

\[
\sigma_{\text{total}} \leq B(\lambda) \left( \frac{1}{2\sqrt{4m^2 - \epsilon}} \right)^{D-2} (\ln s)^{D-2}
\]  

(141)

where \( B(\lambda) = \frac{2\lambda^\Gamma(\lambda)\Gamma(\lambda+1/2)}{\pi^{3/2}\Gamma^2(2\lambda)} \).

5. Summary and Discussions

We summarize our results and discuss the consequences. We began with an intent to derive the high energy behavior of scattering amplitude in D-dimensional massive field theories. Our principal goal was to remove certain arbitrariness in the derivation of the bound on total cross section which was obtained earlier [25, 26]. Essentially, there were two assumptions which were not proven in the field theoretic framework. We have proven that these two assumptions can be derived from LSZ formalism. In order to arrive at our goal, we needed the edge-of-the-wedge theorem. We have argued that the theorem is likely to hold so long as we consider the four point scattering amplitude. We have argued that the proof of Bremermann, Oehme and Taylor [38] will also be valid in D-dimensional theories. We have presented the supportive arguments in the preceding section. In fact, by choosing suitable coordinate frame, in case of four particle amplitude, we can confine to a four dimensional subspace of the D-dimensional momentum space. Subsequently, the BEG wedge-of-the-edge theorem will be proved as has been argued in Section 3. The second important result used by us is the analog of BEG [45] theorem. We have argued regarding existence of analyticity domain in the neighborhood of \( s \) and \( t \) just below \( s_{\text{thr}} \) to prove Martin’s theorem.
Once again, in case of four point amplitude, if we confine ourselves to a 4-dimensional momentum subspace, as alluded to above, this theorem will also be valid. We have not presented explicit proofs of these two results for the D-dimensional theory. However, we feel that the arguments are adequate to utilize the results of these theorems for our purpose. It is worth while to point out that we adopted the LSZ formulation to achieve this goal without resorting to any specific model. It is assumed that there are no bound states in this theory.

The strategy adopted to derive the asymptotic behavior of scattering amplitude is as follows. As a first step, it was necessary to establish that a fixed-\( t \) dispersion relation can be written for the scattering amplitude. In order to reach this goal, the essential step was to prove that the absorptive part of the amplitude is well behaved for fixed physical \( t \) as \( s \to s_{\text{thr}} \). We showed that there are ellipses in the \( t \)-plane where the amplitudes have desired behavior. In particular, starting from the LSZ reduction technique we showed the existence of a large ellipse in D-dimensional theory which is analogous to the large Lehmann ellipse. In order to prove the existence of the ellipses, we needed to prove the existence of Jost-Lehmann-Dyson representation for the retarded function. We proved the generalized Dyson theorem to achieve our goal. We have accomplished the target of establishing the dispersion relations in \( s \) for fixed \( t \). We needed to prove a generalized version of Martin’s theorem to derive constraints on the growth properties of scattering amplitude as a function of \( s \). It was shown that, indeed there is a circle inside the domain of analyticity in the \( t \)-plane inside which the scattering amplitude, \( F^\lambda(s, t) \), can be expanded in a power series in \( t \) and the power series converges absolutely. We also proved positivity properties of the absorptive part of the amplitude and its \( t \)-derivatives for the D-dimensional case by exploiting some of the properties of the Gegenbauer polynomials. This is achieved, after we expanded the amplitude in the basis of the Gegenbauer polynomial. Thus the generalized version of Martin’s theorem could be proved for the D-dimensional field theories.

The asymptotic growth properties of the amplitude had been investigated in [25, 26] under the assumptions (AI) and (AII) as stated in Section 1. These assumptions played central role in derivation of the bounds in [25, 26]. We recall that these authors had assumed the existence of an ‘analog’ Lehmann ellipse whose semimajor axis is characterized by a constant \( \tilde{T}_0 \) which is independent of \( s \). In the present work we have proved existence of such a domain of analyticity i.e. the Large Lehmann Ellipse (LLE). We may remind that such a parameter, \( t_0 \), also appears in four dimensional theories; however, it is determined from the first principles. In most of the hadronic processes it turns out to be \( 4m_\pi^2 \) [47]. Moreover, Sommer [50] has given a prescription to determine \( t_0 \). The second assumption [25, 26] is the polynomial boundedness of the scattering amplitude, \( |F^\lambda(s, t)| < s^N \) inside a certain ellipse [25, 26]. We have proved, within LSZ axioms, that the amplitude is polynomially bounded (due to the temperedness) and the number of required subtractions is \( N = 2 \). In nutshell, the work reported in [25, 26] left two important questions to be answered: (I) what is
value of $R$?, in our notation and (II) what is value of $N$?

Our long investigation has provided definite answers to these questions as was presented in Section 4. We showed that $R = 4m^2$. Moreover, the value is determined from LSZ formalism together with Martin’s analysis. We demonstrated that $N = 2$. Again, having proved the asymptotic growth properties of the absorptive amplitude in $s$ in a domain $|t| < R$, one can show how the forward scattering amplitude is bounded i.e. the asymptotic behavior of $F^\lambda(s, t = 0)$. Furthermore, from the bound on $A^\lambda_\lambda(s, |t| < R)$, we can impose a constraint on the scattering amplitude in the same $t$ domain. Finally, as we have shown, the scattering amplitude needs at most two subtractions i.e. $N = 2$. Therefore, the bound we have derived now has no free parameters. This statement is to be made with a qualifying remark that there is the unknown energy scale which is necessary to scale $s$ i.e. $(\ln s)^{D-2} \rightarrow [\ln(s_1)]^{D-2}$ in the bound which is not fixed from first principles. Note that corresponding scale for four dimensional theories is also not determined from first principles of quantum field theory. We feel that it is quite satisfying that both the unknown parameters are now determined in the framework LSZ formalism.

Furthermore, the upper bound [29] on $|F^\lambda(s, t)|$ for $|t| < R$, which was deduced for large $s$ is now established from the results proved here. No additional assumption is required. It is also important to mention that the upper bound and lower bound derived by me for the absorptive amplitude (the second theorem) [29] now needs no extra assumption. Indeed, the results derived in the present work removes what was termed as an extra assumption for derivation of the two bounds for the absorptive amplitude in a small $|t|$ region including the physical domain. Now the theorems of [29] can be utilized to derive new bounds on elastic differential cross sections and bounds on slope of diffraction peak. Moreover, since the differential cross sections have been measured at the LHC energies the scaling behavior of differential cross sections might be explored.

We recognize that it might be possible to derive the results presented here through more formal approach to axiomatic field theories. In such framework works some of the assumptions such as the field operators being operator values distributions are not invoked. In other words the temperedness property of the amplitude is not required in some of these formulations. It might be possible to prove the polynomial boundedness of the scattering amplitude as derived by Epstein, Glaser and Martin [24].

Now we present some arguments and the phenomenological scenario in the context of the present investigation. Let us envisage the scenario of low scale compactification. In this proposal, the scale of compactification could be as low as 500 GeV or 1 TeV. In other words, the extra dimensions decompactify at this energy scale. Consequently, the decompactification effects could manifest in very high energy accelerator experiments. The lowest mass particles will have mass-value same as this scale. Therefore, one could argue that $s_1 \sim s_{comp}$. This is a plausible proposal. We may ask: if the decompactification scale is so low can we get some hints of this low scale? The phenomenology of this scenario has already been worked out in some details [5, 6].
had proposed another scenario to experimentally investigate existence of low scale compactification proposals. Consider high energy collisions in an energy scale above decompactification scale. Then there is a possibility that effect of higher dimensions, $D > 4$, might manifest in high energy scatterings. In particular such an effect might show certain high energy behaviors unfamiliar to us. For example the data for total cross sections might seemingly violate the Froissart bound derived for $D = 4$ theories i.e. $\sigma_t \sim (\ln s)^2$; in fact the data fits this behaviors over wide energy range i.e. $\sigma_t$ exhibits a $(\ln s)^2$ behavior in very high energy processes. In order to explore the possible signal for decompactification at low energy scale (500 GeV to 1 TeV range) one should examine the energy dependence of total cross sections in collision energies above $\sqrt{s} > 500$ GeV and try to fit with a phenomenological formula for $\sigma_t$. Thus $\sigma_t$ might assume a form [55]

$$\sigma_t = \sigma_0 + C_1 \left( \frac{\ln s}{s_0} \right)^2 + C_2 \left( \frac{\ln s}{s_1} \right)^\beta, \quad \beta > 2$$

(142)

Here $\sigma_0$ corresponds to a constant, independent of $s$, the so called Pomeranchuk term. The next term is the Froissart bound-like energy dependence. The last term encodes effect of higher dimension decompactification. If we find a fit with such a parametrization, it might provide an indirect evidence for low scale compactification. It is fair to take $s_0$ in the range of decompactification scale. We tried to obtain a qualitative fit to total cross sections of old LHC data at 6 TeV and 7 TeV together with very high energy cosmic ray data. We found [55] that with $\beta \sim 2.3$ we get a fit with reasonable $\chi^2$. However, the cosmic ray data are reported with large error bars and therefore, the available set of data to fit cross sections is not large enough to conclude that $\beta > 2$. We feel that a more careful procedure to fit very high energy experimental data with the forthcoming results from the LHC might be a promising endeavor to explore the hypothesis of low scale decompactification at accelerator energies.

Indeed, it will be quite interesting to study analyticity properties of scattering amplitude in a higher dimensional theory where some of the spatial coordinates are compactified. It was pointed out, in the context of potential scattering [56, 57], that for nonrelativistic theory, the analyticity properties of scattering amplitudes are different from those of a nonrelativistic theory which has no compactified spatial coordinates. It is to be noted that momenta associated with compact directions are discrete. Moreover, the deviation from usual dispersion relations for certain potential models (with compact coordinates) raised the question that for field theories with compactified coordinates might not satisfy the known dispersion relations [56]. It was argued that these effects might be observed in high energy scatterings at LHC energies. Therefore, it is worth while to pursue these issues for a $D$-dimensional field theory where certain spatial coordinates are compact. The present investigation can be utilized towards this end.

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Appendix A: Proof of Dyson’s Theorem: Generalized to D-dimensions

We have derived the Jost-Lehmann representation for causal function $F_C(q)$ and for $F_R(q)$ in the D-dimensional theory. This was achieved in the LSZ formulation. Moreover, we analyzed the location of the singularities of $F_R(q)$ generalizing the approach of Jost and Lehmann. It is worth mentioning that Jost-Lehmann representation is valid for the case of equal mass particle.

Dyson [43] used an indigenous technique to derive the representation for the case of unequal mass in a more elegant mathematical framework. We have generalized Dyson’s formalism for theories in arbitrary dimensions which satisfy LSZ axioms. We mention in passing that Dyson’s derivation was also based on the LSZ formulation of field theories. To recapitulate, we have derived the expressions for $F_R(q)$, $F_A(q)$ and $F_C(q)$ already for the case of D-dimensional theories in Section 3. Furthermore, the support properties of these functions in their Fourier transformed coordinate space have been alluded to in that section. We consider D-dimensional Lorentzian space time manifold and supplement it with two extra spatial signature coordinates in order to generalize Dyson’s formalism. The coordinates of the $(D+2)$-dimensional spacetime are

$$\tilde{z} = \{\tilde{z}_0 = x_0, \tilde{z}_1 = x_1, ... \tilde{z}_{D-1} = x_{D-1}, \tilde{z}_D = y_1, \tilde{z}_{D+1} = y_2\} \quad (143)$$

The $(D+2)$-dimensional momenta are defined as

$$\tilde{r} = \{\tilde{r}_0 = q_0, \tilde{r}_1 = q_1, ... \tilde{r}_{D-1} = q_{D-1}, \tilde{r}_D = p_1, \tilde{r}_{D+1} = p_2\} \quad (144)$$

The metric is: $\text{diag} (+1, -1, ... -1)$ and

$$\tilde{z}^2 = \tilde{x}^2 - y^2 = x_0^2 - x_1^2 - ... - x_{D-1}^2 - y_1^2 - y_2^2 \quad (145)$$

We recall $\tilde{F}_C(x)$ is the Fourier transform of $F_C(q)$. Now we define $\tilde{F}_C(\tilde{z})$ in $(D+2)$-dimensions from the given D-dimensional function $\tilde{F}_C(x)$

$$\tilde{F}_C(\tilde{z}) = 4\pi \tilde{F}_C(x) \delta(x^2 - y^2) = 4\pi \tilde{F}_C(x) \delta(\tilde{z}^2) \quad (146)$$

We note that $\tilde{F}_C(\tilde{z})$ is defined on the light cone of the $(D+2)$-dimensional $\tilde{z}$-space.

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dy_1 dy_2 \tilde{F}_C(\tilde{z}) = 4\pi^2 \tilde{F}_C(x) , \text{for } x^2 \geq 0$$

$$= 0 , \text{for } x^2 < 0 \quad (147)$$

We have constructed $\tilde{F}_C(\tilde{z})$ in $(D+2)$-dimensions. Notice that by construction $\tilde{F}_C(\tilde{z})$ and $\tilde{F}_C(x)$ ($\tilde{F}_C(x) = 0$, for $x^2 < 0$) are equivalent in the sense that we may recover
$\tilde{F}_C(x)$ from $\tilde{F}_C(\tilde{z})$ by integrating over $d^2y$ (see eq. (147)). Now we choose a special $(D + 2)$-dimensional momentum vector:

$$\hat{q} = (q_0, q_1, ... q_{D-1}, 0, 0)$$  \hspace{1cm} (148)

We have set last two components of $\tilde{r}$ to zero with this choice. The Fourier transform of $\tilde{F}_C(\tilde{z})$, defined as $\tilde{F}_C(\tilde{r})$, is given by

$$\tilde{F}_C(\tilde{r}) = \frac{1}{(2\pi)^{D+2}} \int e^{i\tilde{r} \cdot \tilde{z}} \tilde{F}_C(\tilde{z}) d^{D+2} \tilde{z}$$  \hspace{1cm} (149)

Let us insert expression for $\tilde{F}_C(\tilde{z})$, (146), into the Fourier transform (149)

$$\tilde{F}_C(\tilde{r}) = \frac{4\pi}{(2\pi)^{D+2}} \int d^{D+2} \tilde{z} d^D q e^{i\hat{q} \cdot \tilde{z}} \tilde{F}_C(q)$$

$$= \int D^{(1)}(\tilde{r} - \hat{q}) \tilde{F}_C(q) d^D q$$  \hspace{1cm} (150)

where

$$D^{(1)}(\tilde{r}) = \frac{2}{(2\pi)^{D+1}} \int e^{-i\tilde{r} \cdot \tilde{z}} \delta(\tilde{z}^2) d^{D+2} \tilde{z}$$

$$= \frac{2}{(2\pi)^{D+1}} P \frac{1}{(\tilde{r})^{D/2}}$$  \hspace{1cm} (151)

$P$ stands for the principal value. From now on, I shall drop the prefactors like $(2\pi)^{D+2}$, $(2\pi)^{D+1}$ etc. which come from taking Fourier transforms. As the next equation will show, we derive an expression for the $\tilde{F}_C(\tilde{r})$ which will display the singularity structure and location of singularities in the $q$-plane. Now use the expression for $D^{(1)}(\tilde{r})$ in the above equation

$$\tilde{F}_C(\tilde{r}) = \int d^D q \frac{\tilde{F}_C(q)}{(\tilde{r} - \hat{q})^{D/2}}$$

$$= \int d^D q \frac{\tilde{F}_C(q)}{[(u - q)^2 - \bar{s}]^{D/2}}$$  \hspace{1cm} (152)

where $\bar{s} = p_1^2 + p_2^2$. ($\{p_1, p_2\}$ are momenta along extra directions). It is important to remember that $\tilde{F}_C(\tilde{z}) = \tilde{F}_C(x)\delta(\tilde{z}^2)$ whose support is on the light cone of the $(D + 2)$-dimensional spacetime. Moreover, the Fourier transformed $\tilde{F}_C(\tilde{r})$ is rotationally invariant on the $\tilde{r}_D - \tilde{r}_{D+1}$ plane since it depends on $\bar{s} = p_1^2 + p_2^2$. A crucial observation, originally due to Dyson [43], is that $D^{(1)}(\tilde{r})$ satisfies a $(D + 2)$-dimensional wave equation in the momentum space

$$\Box_{D+2} D^{(1)}(\tilde{r}) = 0$$

where $\Box_{D+2} = \frac{\partial^2}{\partial \tilde{r}^2} - \sum_{k=1}^{D+1} \frac{\partial^2}{\partial \tilde{r}_k^2}$  \hspace{1cm} (153)
Furthermore, \( \tilde{F}_C(\tilde{r}) \) also satisfies the \((D+2)\)-dimensional wave equation: \( \Box_{D+2} \tilde{F}_C(\tilde{r}) = 0 \). The argument of Dyson can invoked: if \( \tilde{F}_C(x) \) vanishes for \( x_2 < 0 \), then \( F_C(q) \) is the boundary value of \( \tilde{F}_C(q) \) on \( \bar{s} = 0 \) plane. In other words, \( \tilde{F}_C(\hat{q}) = F_C(q) \), \( \hat{q} = (q_0, ...q_{D-1}, 0, 0) \). Moreover,

\[
F_C(\hat{q}) = \int d^{D+2}z e^{i\hat{q} \cdot \bar{z}} 4\pi \delta(x^2 - y^2) \tilde{F}_C(x) \\
= \int d^{D}x e^{i\hat{q} \cdot x} 4\pi \theta(x^2) \tilde{F}_C(x) 
\]

This is achieved after integrating \( \int dy_1 dy_2 \) and setting \( \hat{q}, \bar{z} = q, x \). Now \( F_C(q) = \int d^Dq \frac{F_C(q')}{|q-q'|^{D/2}} \). Recall that there is a class of solution of \( F_C(q) \) whose Fourier transform, \( \tilde{F}_C(x) \), vanishes for \( x_2 < 0 \). Let this class be denoted by \( \mathcal{C} \). Dyson’s arguments can be generalized for \( D \)-dimensional case as follows: a necessary condition for \( \tilde{F}_C(x) \) is that it satisfies the micro causality property. \( \tilde{F}_C(q) \) should be the boundary value on \( \bar{s} = 0 \) plane of a solution \( \tilde{F}_C(q, \bar{s}) \), where \( \tilde{F}_C(q, \bar{s}) \) is a solution to the \((D + 2)\)-dimensional wave equation in the momentum space. We have observed that this class of solutions has to be rotationally symmetric in the plane \( \tilde{r}_D = \tilde{r}_{D+1} \). Note that \( \bar{s} = 0 \) is a time-like surface. Moreover, the boundary value of the hyperbolic equation \( \Box_{D+2} F_C(\tilde{r}) = 0 \) is not arbitrary on this surface. Alternatively, a more general approach is to consider a function which satisfies \((D + 2)\)-dimensional wave equation in the \( \tilde{r} \)-space and is rotationally invariant on the \((D) - (D + 1)\) plane. Its Fourier transform being

\[
\tilde{F}(\bar{z}) = \int d^{D+2}r e^{-i\bar{z} \cdot \bar{r}} \tilde{F}(\bar{r}) 
\]

The Fourier transformed \( \tilde{F}(\bar{z}) \) is endowed with the following features: since \( \Box_{D+2} \tilde{F}(\bar{r}) = 0 \); therefore, \( \tilde{F}(\bar{z}) = \delta(\bar{z}^2) \tilde{G}(\bar{z}) \) and it is noteworthy that \( \tilde{F}(\bar{z}) \) has its support on the light cone of the \( \bar{z} \)-spacetime. Moreover, \( \tilde{F}(\bar{r}) \) has rotational symmetry on a plane as noted earlier. Therefore,

\[
\tilde{F}(\bar{z}) = \int d^{D+2}r e^{-i\bar{z} \cdot \bar{r}} \tilde{F}(u, |p|) \\
= \int d^D u e^{-iu.x} \int_0^\infty dp \int_0^{2\pi} e^{i|y|\cos \theta} \tilde{F}(u, |p|) 
\]

in the polar decomposition of \((p_1, p_2)\) with \( \bar{s} = p_1^2 + p_2^2 \). Consequently,

\[
\tilde{F}(\bar{z}) = 2\pi \int d^D u e^{-iu.x} \int_0^{\infty} d\bar{s} J_0(\sqrt{\bar{s}|y|}) \tilde{F}(u, \bar{s}) 
\]

The Bessel function admits a power series expansion \( J_0(\sqrt{\bar{s}|y|}) = \sum_0^\infty \frac{(\bar{s}|y|)^n}{n!} \); it already shows the rotational invariance in the \( y \)-plane since it depends on \( y^2 \). Moreover, from its structure \( \tilde{F}(\bar{z}) = \delta(\bar{z}^2) \tilde{G}(x, y^2) = \delta(x^2 - y^2) \tilde{G}(x, y^2) \); we may conclude

\[
\tilde{F}(\bar{z}) = \delta(x^2 - y^2) \tilde{f}(x) 
\]

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With the above developments, \( \tilde{f}(x) \) need not vanish for \( x^2 < 0 \). If we want to relate it to causal function, we have to impose the additional condition: \( \tilde{f}(x) = 0 \), for \( x^2 < 0 \) from outside and then we can identify \( \tilde{F}_C(q) = F_C(q) \). Under this constraint we arrive at \( \tilde{F}_C(\tilde{z}) = F_C(\tilde{z}) \). We can now present the generalized version of Dyson’s condition: the necessary and sufficient condition for a function to vanish outside the light cone of D-dimensional spacetime, i.e. \( x^2 < 0 \) is that \( F_C(q) \) be the boundary value on the surface \( \tilde{s} = 0 \) of a solution to a \((D + 2)\)-dimensional wave equation \( \Box_{D+2} \tilde{F} = 0 \). The solution is required to be rotationally invariant in \( p_1-p_2 \) plane.

In accordance with Dyson’s prescription, in general, a solution to the wave equation (of the type being discussed) can be expressed in terms of its value and its normal derivative on an arbitrary spacelike surface. Thus, for the D-dimensional case, we also introduce a singular function and denote it as \( \tilde{D}(\tilde{r}) \). It also satisfies the homogeneous wave equation

\[
\Box_{D+2} \tilde{D}(\tilde{r}) = 0 \tag{159}
\]

The initial conditions are

\[
\tilde{D}(\tilde{r}_0, 0, \tilde{r}_1, ..., \tilde{r}_{D+1}) = 0, \quad \text{and} \quad \frac{\partial \tilde{D}(\tilde{r})}{\partial \tilde{r}_0} \bigg|_{\tilde{r}_0 = 0} = \Pi_{i=1}^{D+1} \delta(\tilde{r}_i) \tag{160}
\]

We can write (160) explicitly as

\[
\tilde{D}(\tilde{r}) = \int d^{D+2} z e^{-i\tilde{r}.\tilde{z}} \epsilon(\tilde{z}) \delta(\tilde{z}^2) \tag{161}
\]

Now we can choose a spacelike surface, \( \Sigma \), and prescribe initial data on it. If \( \tilde{F}(\tilde{r}) \) is solution to the wave equation. Let it assume the value \( \tilde{F}(\tilde{r}') \) and \( (\frac{\partial \tilde{F}(\tilde{r}')}{\partial \tilde{r}'_\alpha}) n^\alpha(\tilde{r}') \) on \( \Sigma \) (where \( n^\alpha \) is normal to the surface). Then

\[
\tilde{F}(\tilde{r}) = \int_{\Sigma} d\Sigma_\alpha \left[ \tilde{F}(\tilde{r}'), \frac{\partial}{\partial \tilde{r}'_\alpha} \tilde{D}(\tilde{r}' - \tilde{r}) \right] \tag{162}
\]

We define

\[
\left[ \tilde{F}(\tilde{r}'), \frac{\partial}{\partial \tilde{r}'_\alpha} \tilde{D}(\tilde{r}' - \tilde{r}) \right] = \tilde{F}(\tilde{r}') \frac{\partial}{\partial \tilde{r}'_\alpha} \tilde{D}(\tilde{r}') - \frac{\partial \tilde{F}(\tilde{r}')}{\partial \tilde{r}'_\alpha} \tilde{D}(\tilde{r}') \tag{163}
\]

Here \( d\Sigma_\alpha \) is the surface element and it is a \((D + 2)\)-dimensional vector normal to the spacelike surface.

We can derive the solution to the wave equation with assigned symmetry properties by choosing the surface appropriately and with desired boundary values for the solution (163).

Our original goal is to derive a representation for \( F_C(q) \). Therefore, we set \( \tilde{F}_C(q) = F_C(q) \). The integral equation for the latter is

\[
F_C(q) = \int_{\Sigma} d\Sigma' \left[ \tilde{F}(\tilde{r}'), \frac{\partial}{\partial \tilde{r}'_\alpha} \tilde{D}(\tilde{r}' - \tilde{q}) \right] = \int_{\Sigma} d\Sigma_\alpha \left[ \tilde{F}(\tilde{r}'), \frac{\partial}{\partial \tilde{r}'_\alpha} \left\{ \epsilon(u_0 - q_0) \delta'\left((u - q)^2 - \tilde{s}\right) \right\} \right] \tag{164}
\]

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This representation is unique, as has been argued by Dyson. We have a function $F_C(q)$ (for which $F_C(x)$ has the desired support property) and we have a given surface $\Sigma$ and $F_C(q)$ admits the representation (164) with any function $F_C(\tilde{r}) = F_C(u, \bar{s})$ so that it depends on the invariant $\bar{s} = p_1^2 + p_2^2$. Moreover $F_C(\tilde{r})$ satisfies the desired wave equation in $\tilde{r}$-space then $F_C(\tilde{r})$ (is identical to $F_C(\tilde{r})$) defined by

$$\bar{F}(\tilde{r}) = \int d^D q \frac{F_C(q)}{|(\tilde{r} - q)|^{D/2}} = \int \frac{d^D q F_C(q)}{|(u - q)^2 - \bar{s}|^{D/2}}$$

and

$$\hat{F}(\tilde{r}) = \int \hat{D}(1)(\tilde{r}, \tilde{q}) \bar{F}(\tilde{q}) d^D q$$

I have suppressed the factors of $(2\pi)^D$ etc. coming from Fourier transforms as before. We are in a position, now, to put forward an argument that there is one-to-one correspondence between the class of functions, $F_C(q)$, (this is in the class $C$) and solution of the wave equation in the $\tilde{r}$-space, $\hat{F}(\tilde{r})$ which is endowed with a rotational symmetry in the $p_1$-$p_2$ plane. Notice the representation of $\hat{F}(\tilde{r})$ is expressed in terms of $F_C(q)$. Our desired goal is to choose a suitable $\tilde{F}(u, \bar{s})$ and choose the surface, $\Sigma$ to achieve a representation for $F_C(q)$ with special support in the momentum space (which is obtained from the support properties of $\hat{F}(x)$). Our aim is to identify the analog of coincidence region (where $F_C(q) = 0$). Following Dyson, we define a region $\mathcal{R}$, in the momentum space ($q$-space) which is bounded by two spacelike surfaces $\sigma_1$ and $\sigma_2$. To be specific choose $\mathcal{R}$ as follows:

$$\mathcal{R} : \quad \bar{s}_1(q) < q_0 < \bar{s}_2(q)$$

Inside this domain $F_C(q) = 0$. Moreover, the two surfaces are chosen in such a way that

$$|\bar{s}_1(q) - \bar{s}_1(q')| < |q - q'|$$
$$|\bar{s}_2(q) - \bar{s}_2(q')| < |q - q'|$$

Here $q$ is the $(D - 1)$ component vector along spatial directions of the $D$-vector $q$ and the same definition holds for $q'$. Thus we have defined two spacelike surfaces with $q^0 = \bar{s}_1(q)$ and $q^0 = \bar{s}_2(q)$. Now define $C_R$ to be class of functions such that $F_C(x) = 0$ for $x^2 < 0$ and such that $F_C(q) = 0$ for any $q \in \mathcal{R}$. Notice that the hyperboloid $(q - u)^2 - \bar{s} = 0$ is $q$-space admissible. This property is valid if the upper sheet does not come below $\sigma_2$ and the lower sheet is above $\sigma_1$. In the $(D + 2)$-dimensional space, the hyperboloid in question corresponds to points $\tilde{r} = (u_0, u_1, ..., u_{D-1}, p_1, p_2)$, $\bar{s} = p_1^2 + p_2^2$ lying in a certain region $\mathcal{S}$ of $\tilde{r}$-space (recall $\mathcal{R}$ is defined in $q$-space and $C_R$ in coordinate space). Our intent is to derive a representation for $F_C(q)$. Now for every $\tilde{r}$ in $\mathcal{S}$ but $q$ in the region $\mathcal{R}$, $\hat{D}(\tilde{r} - \tilde{q})$ vanishes. The following expression is a prospective representation for $F_C(q)$ in $C_R$

$$F_C(q) = \int d\Sigma_\alpha \left[ \hat{F}(\tilde{r}), \frac{\partial}{\partial \tilde{r}_\alpha} \delta(u_0 - q_0) \delta'(\tilde{r} - q) \right]$$

55
Note that the points of $\tilde{r}$ are constrained to be in $S$. Furthermore, every point of $\tilde{r}$ and $\Sigma$ in $S$ are required to belong to $C_R$ which follows from the conditions stated above. It is important to point out that $F_C(q)$ has a representation using only the admissible hyperboloid i.e. every $F_C(q)$ we intend to construct must have variables belonging to the admissible hyperboloid: $(q - u)^2 - \bar{s} = 0$. Another constraint is that this must not cross the surface defined by $q_0 = \bar{s}_1(q)$ and $q_0 = \bar{s}_2(q)$ (see (168)). Let us focus attention at the upper sheet of the hyperboloid and it corresponds to the branch

$$q_0 = u_0 + \sqrt{(q - u)^2 + \bar{s}}$$  \hspace{1cm} (170)

This will cross $\sigma_2$ if

$$u_0 + \sqrt{(q - u)^2 + \bar{s}} \geq \bar{s}_2(q)$$ \hspace{1cm} (171)

for $q$ held fixed. We could rephrase the above constraint as

$$u_0 \geq \text{Max}_q \{\bar{s}_2(q) - \sqrt{(q - u)^2 + \bar{s}}\} = m(u, \bar{s})$$ \hspace{1cm} (172)

We can repeat the same steps for the lower sheet and obtain

$$u_0 \leq \text{Min}_q \{\bar{s}_1(q) + \sqrt{(q - u)^2 + \bar{s}}\} = M(u, \bar{s})$$ \hspace{1cm} (173)

We have closely followed Dyson’s notation and convention. The principal reason is that unlike Jost-Lehmann representation which was derived for the case of equal masses, the advantage of Dyson’s formulation lies in the fact that the case of unequal mass is treated elegantly and the approach is quite general. The connection with Jost-Lehmann formulation will be clear later. We have mentioned earlier that the points on the hyperboloid correspond to region $S$ (see the remark preceding (169) where we define region $S$ in $\tilde{r}$-space). For the present consideration the region $S$, in the $\tilde{r}$-space, can be identified to be

$$m(u, \bar{s}) \leq u_0 \leq M(u, \bar{s})$$ \hspace{1cm} (174)

and it is bounded by two surfaces $\Sigma_1$ and $\Sigma_2$ in the $\tilde{r}$-space. We mention in passing that these surfaces are envelopes of two families of hyperboloids and these two are also spacelike. Now define $T$: complements of $S$ i.e. it contains the set of points in the $\tilde{r}$-space such that

$$M(u, \bar{s}) \leq u_0 \leq m(u, \bar{s})$$ \hspace{1cm} (175)

The purpose is to impose a constraint on $\hat{F}(\tilde{r})$ in order that representation for $F_C(q)$ gives an $F_C(q)$ such that its Fourier transform belongs to a class which is in $C_R$. In order that this condition is fulfilled $F_C(\tilde{r})$ must vanish for each $\tilde{r}$ in $T$. Now, for equation (162), choose a spacelike surface, $\Sigma$, such that it lies between the two spacelike surfaces $\Sigma_1$ and $\Sigma_2$. This surface is identified to be

$$u_0 = \frac{1}{2}[m(u, \bar{s}) + M(u, \bar{s})]$$ \hspace{1cm} (176)
We have already constrained \( u_0 \) to lie in the regions given by (172) and (173) and it also chosen to be (176). Therefore, every point of the chosen spacelike surface, \( \Sigma \), is either in the domain \( \mathcal{S} \) or it lies in its complement \( \mathcal{T} \). According to stipulation \( \hat{F}(\hat{r}) \) is required to vanish for every \( \hat{r} \in \mathcal{T} \). A function \( F_C(q) \) belongs to \( C_R \) (the Fourier transform is meant to be in \( C_R \)) if and only if it admits a unique representation

\[
F_C(q) = \int_{\Sigma} d\Sigma_\alpha \left[ \hat{F}(\hat{r}), \frac{\partial}{\partial \hat{r}_\alpha} D(\hat{r} - \hat{q}) \right]
\]

(177)

where \( \Sigma \in \mathcal{S} \), in other words this integral extends only those points of \( \hat{r} \) of the spacelike surface \( \Sigma \) which belong to \( \mathcal{S} \). We recall that the set of points in domain \( \mathcal{S} \) are given by (172) and (173) and \( \mathcal{S} : m(u, \bar{s}) \leq q_0 \leq M(u, \bar{s}) \). Thus generalized version, for \( D \)-dimensional case, is

**Theorem:** For a function \( F_C(q) \) to vanish in the region \( \tilde{s}_1(q) < q_0 < \tilde{s}_2(q) \) and to have a Fourier transform, \( \tilde{f}(x) \) such that \( \tilde{f}(x) = 0 \) for \( x^2 < 0 \), it is necessary and sufficient to have a representation

\[
F_C(q) = \int d^D u \int_0^\infty \epsilon(q_0 - u_0) \delta[(q - u)^2 - \bar{s}] \Phi(u, \bar{s})
\]

(178)

\( \Phi(u, \bar{s}) \) vanishes outside the regions \( u_0 \geq \text{Max}_q \{\tilde{s}_2(q) - \sqrt{(q - u)^2 + \bar{s}}\} \) and \( u_0 \leq \text{Min}_q \{\tilde{s}_1(q) + \sqrt{(q - u)^2 + \bar{s}}\} \) and already noted earlier) and \( \mathcal{S} : m(u, \bar{s}) \leq u_0 \leq M(u, \bar{s}) \), but arbitrary otherwise. Note that \( \Phi(u, \bar{s}) \), appearing in (178), depends on \( q \)'s determined by \( (u - q)^2 = \bar{s} \) which lie entirely in \( \mathbb{R} \). It reproduces the function, \( F_C(q) \) on the left hand side of (178) with the requisite support properties in \( q \)-space and the support properties of \( \hat{F}_C(x) \) are satisfied. Thus we can write

\[
\hat{F}_C(x) = \int_0^\infty d\bar{s} \Delta(x; \bar{s}) \Phi(x, \bar{s})
\]

(179)

where \( \Phi(x, \bar{s}) \) is the Fourier transform of \( \Phi(u, \bar{s}) \) with respect to \( u \) and is the well known invariant function (now defined in \( D \)-dimensions) with mass \( \sqrt{\bar{s}} \). Thus the causality properties of \( \hat{F}_C(x) \), as desired by us, is satisfied.

Let us consider a specific situation to make connections with our derivation of the Jost-Lehmann representation in Section 3. Following Dyson, we choose the two surfaces \( \sigma_1 \) and \( \sigma_2 \) to be

\[
\begin{align*}
\tilde{s}_1(q) &= a - \sqrt{q^2 + m_2^2} \\
\tilde{s}_2(q) &= -a + \sqrt{q^2 + m_1^2}
\end{align*}
\]

(180)

We keep the masses \( m_1 \) and \( m_2 \) unequal. The region \( \mathcal{S} \) is identified to be

\[
\begin{align*}
m(u, \bar{s}) &= \text{Max}_q \{\sqrt{q^2 + m_1^2}, -a - \sqrt{(q - u)^2 + \bar{s}}\} \\
M(u, \bar{s}) &= \text{Max}_q \{+\sqrt{q^2 + m_2^2}, -a - \sqrt{(q - u)^2 + \bar{s}}\}
\end{align*}
\]

(181)
The extremum of \( m(u, \bar{s}) \) and \( M(u, \bar{s}) \) is derived by taking their gradients with respect to \( q \) and set each of the gradient to zero and derive the locations of maxima. In order to establish connections with the Jost-Lehmann representation, we identify \( \left( \frac{1}{2}(Q_i + Q_f) \right) \) and \( m_1 = m_2 = m \) then domain \( S \) is \( (Q + q) \in V^+, (Q - q) \in V^+, \bar{s} = \chi^2 \). Note, \( \bar{s} = p_1^2 + p_2^2 \), defined in terms of the momenta along extra directions and there is integration over \( d\bar{s} \) in the expression for \( F_C(q) \), (179). Indeed in the case of equal mass scattering we get back the result derived by the techniques of Jost and Lehmann. We know how to derive the presentation for the retarded function \( F_R(q) \) since the two are simply related in their coordinate space definition: \( \bar{F}_C(x) = \theta(x_0) \bar{F}_C(x) \). The power of the mathematical approach of Dyson is quite evident and its generalization to D-dimensions is achieved in a very elegant manner.

Appendix B: Martin’s Lemma

In this appendix we prove two lemmas which are very useful to prove Martin’s theorem. We mention that, essentially they deal with interchange of differentiation and integration in order to prove certain analyticity properties of functions which depend on two variables. Therefore, the proof of these lemmas are not dependent on the dimensionality of spacetime. As will be obvious, these functions, in nutshell, depend on two variables, and these are to be identified with the Mandelstam variables \( s \) and \( t \) eventually when the Martin’s theorem is discussed.

Lemma 1. Suppose \( F(t) = \int_0^\infty ds G(s, t) \) and it fulfills following requirements:
(a) \( F(t) \) is analytic in the neighborhood of \(-\alpha \leq t \leq 0\).
(b) For all \( s \in [\beta, \infty) \), \( G(s, t) \) in analytic in \( t \) in the neighborhood of \(-\alpha \leq t \leq 0\). The \( t \)-derivatives of \( G(s, t) \) satisfy the condition that \( (\frac{\partial}{\partial t})^n G(s, t) \) are bounded by some functions \( G_n(s) \).
(c) For all \( s \in [\beta, \infty) \) : \( |(\frac{\partial}{\partial t})^n G(s, t)| \leq (\frac{\partial}{\partial t})^n G(s, t)|_{t=0} \) in the interval \(-\alpha \leq t \leq 0\)

Then
\[
(\frac{\partial}{\partial t})^n F(t) = \int_\beta^\infty (\frac{\partial}{\partial t})^n G(s, t) ds \tag{182}
\]
for \( n = 0, 1, 2 \ldots \) and for \( t \) in the interval \(-\alpha \leq t \leq 0\). What is the purpose of this lemma? It is to prove, under what conditions, the operations of differentiation in variable \( t \) and the integration in variable \( s \) can the interchanged as we shall see.

Proof: In the interval for \(-\alpha \leq t \leq 0\), assumption (a) implies that
\[
\frac{\partial}{\partial t} F(t) = \frac{\partial}{\partial t} \int_\beta^\infty ds G(s, t)
= \lim_{\epsilon \to 0} \int_\beta^\infty ds \frac{G(s, t) - G(s, t - \epsilon)}{\epsilon} \tag{183}
\]
exists (moreover, it is analytic in the closed interval \(-\alpha \leq t \leq 0\)). Now we appeal to the assumption (b) regarding the analyticity property of \( G(s, t) \). We are required to show that the limit \( \epsilon \to 0 \) and the integral can be interchanged. In order to
accomplish this goal, we write
\[
\frac{\partial}{\partial t} F(t) = \int_{\beta}^{s_1} \frac{\partial}{\partial t} G(s, t) + \lim_{\epsilon \to 0} \int_{S_1}^{\infty} \frac{G(s, t) - G(s, t - \epsilon)}{\epsilon} \tag{184}
\]
We are permitted to carry out this operation due to the following reasons: the integral has a limit and it is bounded by finite function of \( s \). Therefore, the (Lebesgue) integral over a finite interval converges. Thus, what remains is to be demonstrated is that for the second integral of (184), i.e \( \int_{S_1}^{\infty} \), the limit and integration can be interchanged.

We argue, invoking (c), to achieve this
\[
0 \leq \left| \int_{S_1}^{\infty} \frac{G(s, t) - G(s, t - \epsilon)}{\epsilon} \right| = \left| \int_{S_1}^{\infty} \frac{\partial}{\partial t} G(s, t - \epsilon') \right| \leq \int_{S_1}^{\infty} \left| \frac{\partial}{\partial t} G(s, t - \epsilon') \right| \leq \int_{S_1}^{\infty} \frac{\partial}{\partial t} G(s, t = 0) \tag{185}
\]
When we write the last term, it is already understood that the limit \( \epsilon \to 0 \) has been taken at the appropriate stage.

Lemma 2: Consider two functions \( G_1(s, t) \) and \( G_2(s, t) \) which fulfill the requirements (b) and (c) of Lemma 1 then the product also have the same properties.

Proof: Define
\[
G(t) = \int_{\beta}^{\infty} ds G_1(s, t) G_2(s, t) \tag{186}
\]
and it is analytic in \(|t| < R\). Then for all complex \( z \notin [\beta, \infty) \) and it is analytic in \(|t| < R\). Consequently, for all complex \( z \notin [\beta, \infty) \)
\[
\left| \frac{(\frac{\partial}{\partial t})^n G_1(s, 0)}{s - z} \right| \leq \sup_{\beta \leq s \leq \infty} \left| \frac{1}{(s - z) G_2(s, 0)} \right| \int_{\beta}^{\infty} ds \left[ (\frac{\partial}{\partial t})^n G_1(s, t) \right] \bigg|_{t=0} \leq \sup_{\beta \leq s \leq \infty} \left| \frac{1}{(s - z) G_2(s, 0)} \right| \int_{\beta}^{\infty} \left[ (\frac{\partial}{\partial t})^n \left( G_1(s, t) G_2(s, t) \right) \right] \bigg|_{t=0} \leq \frac{n!}{R^n} \sup_{\beta \leq s \leq \infty} \left| \frac{1}{(s - z) G_2(s, 0)} \right| \max_{|t| < R} G(t) \tag{187}
\]
The first inequality obviously follows from the properties of function. The next one is a consequence of the requirement (c) stated in Lemma 1. The last inequality is due to application of the Cauchy’s inequality for the function \( G(t) \) defined above.

Appendix C: Useful formulas used for the Gegenbauer Polynomial

We compile some of the useful formulas used in this article. We give volume and page number of the Batesman manuscript - the exact reference is given in the reference.
section. In our case $\lambda = \frac{1}{2}(D-3)$ where D is number of spacetime dimensions. We are dealing with higher spacetime dimensions i.e. $D >$. Therefore, $\lambda \geq 1$

Orthogonal polynomials.

$$ (\phi(x), \phi_2(x)) = \int_a^b W(x)\phi_1(x)\phi_2(x)dx $$

(188)

For Gegebauer polynomials $C^\lambda_l$, $a = -1$, $b = +1$, $W(x) = (1 - x^2)^{\lambda-1/2}$

The formulas from Vol I.

p175

$$ C^\lambda_n(z) = \sum_{l=0}^{n} \frac{(-1)^l\Gamma(\lambda+l)\Gamma(n+2\lambda+l)}{l!(n-l)!\Gamma(\lambda)\Gamma(2l+\lambda)} (\frac{1}{2} - \frac{1}{2} z^l) $$

(189)

p176

$$ \left(\frac{d}{dz}\right)^n[C^\lambda_n(z)] = 2^n \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} $$

(190)

p176

$$ \frac{1}{2} (\Gamma(\lambda))^2 C^\lambda_n(cos\phi) \sum_{m=0}^{\leq n/2} \frac{\Gamma(m+\lambda)\Gamma(n-m+\lambda)cos[(n-2m)\phi]}{m!(n-m)!} $$

(191)

$$ C^\lambda_n(x) = \frac{1}{\sqrt{\pi}} \frac{\Gamma(n+2\lambda)\Gamma(\lambda+1/2)}{\Gamma(\lambda)\Gamma(2\lambda)\Gamma(n+1)} \int_0^\pi \left[ x + \sqrt{(x^2 - 1)cos\phi} \right]^\lambda (cos\phi)^{2\lambda-1} d\phi $$

(192)

p178

$$ \frac{d}{dz}C^\lambda_n(z) = 2\lambda C^\lambda_{n-1}(z) $$

(193)

Vol II.

Inequality: p206

$$ Max_{-1 \leq z \leq +1} |C^\lambda_n(z)| = C^\lambda_n(1) > 0 $$

(194)
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