Generalized Epsilon-Pseudospectra *

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Abstract

We generalize $\epsilon$-pseudospectra and the associated computational algorithms to the generalized eigenvalue problem. Rank one perturbations are used to determine the $\epsilon$-pseudospectra.

Keywords: epsilon-pseudospectra, generalized singular value decomposition, generalized eigenvalues, transient growth, operator theory

1 $\epsilon$-pseudospectra of the eigenvalue problem

Normal matrices have complete sets of orthonormal eigenvectors, and therefore the spectral decomposition is useful in studying the properties of normal operators. In contrast, the eigenvectors of non-normal matrices can be nearly linearly dependent, and the eigenvalue problem may be highly ill-conditioned. Thus additional concepts and analysis techniques are useful in examining non-normal operators. $\epsilon$-pseudospectra, introduced by L.N. Trefethen, have proven to be a powerful tool in the analysis of non-normal operators [10,11]. In examining the transient behavior of the Orr-Sommerfeld operator, Reddy et al. calculated the $\epsilon$-pseudospectra, the numerical range and the maximum transient growth [8]. The corresponding analysis for resistive magneto-hydrodynamics (MHD) generates a generalized eigenvalue problem [2,5,6,9]. In this note, we generalize $\epsilon$-pseudospectra to the generalized eigenvalue problem.

We begin by reviewing $\epsilon$-pseudospectra for the standard eigenvalue problem for the $n \times n$ matrix $A$. We use the standard 2-norm, $||v||^2 \equiv v^*v$. We denote the spectrum of $A$ by $\Lambda(A)$ and the resolvent set of $A$ by $\rho(A)$. 

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Definition 1.1 (Trefethen [10,11]). Let $\epsilon \geq 0$ be given. A complex number $z$ is in the $\epsilon$-pseudospectrum of $A$, which we denote by $\Lambda_\epsilon(A)$, if one of the following equivalent conditions is satisfied:

(i) the smallest singular value of $A - zI$ is less than or equal to $\epsilon$,
(ii) $\exists u \in \mathbb{C}^n$ such that $\|u\|^2 = 1$ and $\|(A - zI)u\|^2 \leq \epsilon^2$, 
(iii) $z \in \Lambda(A)$ or $z \in \rho(A)$ and $\exists u \in \mathbb{C}^n$ such that $\|u\|^2 = 1$ and $u^*(A - zI)^{-1}u \geq 1/\epsilon^2$,
(iv) $z$ is in the spectrum of $A + \epsilon E$: $(A + \epsilon E)u = zu$, where the matrix $E$ satisfies $\|E\| \leq 1$,
(v) $z$ is in the spectrum of $A + \epsilon v_2 v_1^*$, where $\|v_1\| \leq 1$ and $\|v_2\| \leq 1$,
(vi) $z$ is in the spectrum of $A - (Av - zv)v^*$ where $\|v\| \leq 1$ and $\|Av - zv\| \leq \epsilon$.

The equivalence of (i), (ii), (iii) and (iv) is given in Refs. [8,10,11]. Conditions (v) and (vi) are new, equivalent definitions of $\epsilon$-pseudospectra. (vi) $\Rightarrow$ (v) $\Rightarrow$ (iv) is trivial, as is (vi) $\iff$ (ii). The original proof that (ii) $\Rightarrow$ (iv) implicitly contained the proof that (ii) $\Rightarrow$ (vi) since the constructed $E$ matrix is given by (vi).

The stated definition is for finite dimensional matrices and needs to modified when $A$ is a closed linear operator on a sub-space of a Hilbert-space. In this case, we extend these definitions by replacing $u$ with a sequence of functions, $\{u_n\}$, in the domain of $A$, i.e. require that Definition 1 hold on the closure of the domain of $A$. Thus, condition (iii) becomes $\|(A - \lambda I)^{-1}\| \geq 1/\epsilon$.

In [8,10,11], two methods for computing the $\epsilon$-pseudospectrum are given. First, using (i), we define $\epsilon_b(z)$ to be the smallest singular value of $A - zI$. The subscript $b$ on $\epsilon$ is used because $\epsilon_b(z)$ is the boundary of the $\epsilon$-pseudospectrum: $z \in \Lambda_\epsilon(A) \iff \epsilon_b(z) \leq \epsilon$.

$\epsilon_b(z)^2$ can also be determined by computing the smallest eigenvalue of $(A - zI)^*(A - zI)$, which is equivalent to minimizing $\|(A - zI)u\|^2$ over $u$ with $\|u\|^2 = 1$. However, this alternative is not as well conditioned as the singular value decomposition of $A - zI$ due to rounding error computing $(A - zI)^*(A - zI)$.

Second, for a fixed value of $\epsilon$, Trefethen and Reddy et al. approximately compute the $\epsilon$-pseudospectrum by generating random perturbing matrices, $E$, of unit norm, and then calculating the eigenvalues, $\{z(E, \epsilon)\}$, of $A + \epsilon E$. The eigenvalues, $\{z(E, \epsilon)\}$, are plotted in the complex $z$-plane for the ensemble of matrices, $E$. As the number of perturbing matrices increases, the scatterplot of the calculated eigenvalues densely fills the $\epsilon$-pseudospectrum (provided that the ensemble is representative of all possible perturbations).

We can modify this algorithm by considering only rank 1 perturbing matrices, $E$. This approach has several advantages. First, the space of all rank one perturbations has dimension $2(n - 1)$, while the space of all matrices with unit norm has dimension $n^2 - 1$. Thus, for a given number, $N$, of perturbing matrices, the distance, as measured in trace norm, between an arbitrary element, $E'$, and the closest matrix in the test ensemble is $O(N^{-1/d})$. 

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where \( d \) equals \( 2(n-1) \) and \( n^2-1 \) respectively. Thus the typical distance to a matrix in the test ensemble is significantly smaller in the rank one case. Using only rank 1 perturbations of the form, \((Av - zv)v^*\), further reduces the space of possible perturbations.

Second, for general perturbing matrices, \( E \), a singular value decomposition needs to be computed to determine \( \|E\| \). This step is unnecessary for rank one perturbations.

Third, if the matrix has a special structure, an arbitrary perturbation will destroy these characteristics. In contrast, rank one perturbations only modify the special structure. In certain cases, it may be possible to use this special structure in specialized algorithms to compute the eigenvalue decomposition.

Another important stability question is “How close is a matrix, \( A \), to being unstable?” In Ref. 13, Van Loan addressed this question. We now reformulate his results in the language of \( \epsilon \)-psuedospectra.

**Definition 1.2** Let \( \epsilon \geq 0 \) and \( A \in \mathbb{C}^{m \times m} \), \( A \) is \( \epsilon \)-asymptotically stable if and only if there exists no matrix, \( E \in \mathbb{C}^{m \times m} \) such that \( \|E\| < \epsilon \) and \( A + E \) has an eigenvalue, \( \lambda \), with \( \Re[\lambda] \geq 0 \).

**Theorem 1.3** \( A \) is \( \tau \)-asymptotically stable if and only if \( \inf_{y \in \mathbb{R}} \epsilon_b(iy) \geq \tau \).

Thus the minimum of \( \epsilon_b(z) \) on the imaginary axis defines the \( \epsilon \)-instability threshold.

### 2 Generalized singular value decomposition

Before examining generalized pseudospectra, we review the generalized singular value decomposition [1,7,12].

**Definition 2.1** (Van Loan). The \( B \)-singular values of a matrix \( A \) are elements of the set \( \mu(A,B) \) defined by

\[
\mu(A,B) \equiv \{ \mu | \mu \geq 0, \ det \ (A^*A - \mu^2B^*B) = 0 \}.
\]

where \( A \in \mathbb{C}^{m_a \times n} \), \( B \in \mathbb{C}^{m_b \times n} \) and \( m_a \geq n \).

Van Loan’s original work decomposed real matrices with orthogonal matrices. We state the analogous result for complex matrices and unitary transformations.

**Theorem 2.2** (Van Loan) (The \( B \)-singular value decomposition (BSV).) Suppose \( A \in \mathbb{C}^{m_a \times n} \), \( B \in \mathbb{C}^{m_b \times n} \) and \( m_a \geq n \). There exist unitary matrices, \( U \ (m_a \times m_a) \) and \( V \ (m_b \times m_b) \), and a nonsingular \( n \times n \) matrix, \( X \), such that

\[
U^*AX = D_A = \text{diag} \ (\alpha_1, \ldots, \alpha_n), \quad \alpha_i \geq 0,
\]

\[
V^*BX = D_B = \text{diag} \ (\beta_1, \ldots, \beta_n), \quad \beta_i \geq 0,
\]
where \( q = \min \{m, n\} \), \( r = \text{rank}(B) \) and \( \beta_1 \geq \ldots \geq \beta_r > \beta_{r+1} = \ldots = \beta_q = 0 \). If \( \alpha_j = 0 \) for any \( j \), \( r + 1 \leq j \leq n \), then \( \mu(A, B) = \{\mu|\mu \geq 0\} \). Otherwise, \( \mu(A, B) = \{\alpha_i/\beta_i|i = 1, \ldots, r\} \).

In [7], Paige and Saunders generalized this definition to relax the requirement that \( m_a \geq n \). Van Loan also gave a second generalization of the singular value decomposition, by using two different norms in the variational formulation.

**Definition 2.3** (Van Loan). Let \( P \in C^{n \times n} \) be positive definite. A matrix \( Q \in C^{n \times n} \) is unitary if \( Q^*PQ = I_n \).

**Definition 2.4** (Van Loan). Let \( S \) and \( T \) be positive definite matrices of orders \( m \) and \( n \), respectively, with \( m \geq n \). The \((S, T)\)-singular values of \( A \in C^{m \times n} \) are elements of the set \( \mu(A, S, T) \) defined by

\[
\mu(A, S, T) = \{\mu|\mu \geq 0, \mu^2 \text{ is a stationary value of} \frac{x^*Ax}{x^*Tx}\}.
\]

**Theorem 2.5** (Van Loan) (The \((S, T)\)-singular value decomposition). Let \( A, S \) and \( T \) be in \( C^{m \times n}, C^{m \times m}, C^{n \times n} \), respectively, with \( S \) and \( T \) positive definite \((m \geq n)\). There exists an \( S \) unitary \( U \in C^{m \times m} \) and a \( T \) unitary \( V \in C^{n \times n} \) such that

\[
U^{-1}AV = D = \text{diag} (\mu_1, \ldots, \mu_n).
\]

We denote the \((S, T)\)-singular values of \( A \) by \( \mu(A, S, T) \). The proof is based on the singular value decomposition of \( L^*AK^{-1} \), where \( K \) and \( L \) are the Cholesky factorizations of \( T \) and \( S \): \( T = KK^* \) and \( S = LL^* \). We relate the two generalizations of the singular value decomposition by the following corollary.

**Corollary 2.1** (The \((S, T)\)-singular value decompositions corresponding to the \(B\)-singular value decomposition). Let \( A \) and \( B \) be in \( C^{m \times m} \), with \( B \) positive definite, self-adjoint. Then

\[
a) \ \mu(A, B) = \mu(A, I_m, B^*B) \\
b) \ \mu(A, B) = \mu(A, B^{-1}B^{-1}, I_m).
\]

The corollary follows from determinant identities.

### 3 Generalized \( \epsilon \)-pseudospectrum

We now consider the generalized eigenvalue problem \( Ae = \lambda Me \), where \( M \) is self-adjoint and positive definite. We could transform the problem into a standard eigenvalue problem: \( A'e' = \lambda e' \) where \( F^*F = M \), \( e' = Fe \), and \( A_F \equiv F^{-1}AF^{-1} \), and then examine the \( \epsilon \)-pseudospectrum of the standard linear problem.
In Appendix A of [8], Reddy et al. consider the \( \epsilon \)-pseudospectra of an operator, \( \hat{A} \), where both the domain and the range of \( \hat{A} \) have a metric \( M = F^*F \). In this case, the \( \epsilon \)-pseudospectrum is defined as the set of \( z \) values such that there exists a vector \( u \) with \( u^* \hat{A} \hat{M} u \leq \epsilon^2 \) and \( u^* Mu = 1 \). Reddy et al. show that the \( \epsilon \)-pseudospectrum for the \( F^*F \) norm is equivalent to the standard \( \epsilon \)-pseudospectrum for \( F \hat{A} F^{-1} \).

We modify Reddy et al.’s approach in two ways. First, we consider the generalized eigenvalue problem with \( A e = \lambda M e \) with \( A \equiv F^*F \hat{A} \) and \( M = F^*F \). This type of generalized eigenvalue problem tends to occur when differential equations are discretized using a variational formulation. For this formulation, the \( \epsilon \)-pseudospectrum of the generalized eigenvalue problem is equivalent to the standard definitions of \( \epsilon \)-pseudospectra for \( F^{-1}A F^{-1} \).

Second, we prefer not to transform the generalized \( \epsilon \)-pseudospectrum problem into an equivalent standard problem. The transformation \( A \rightarrow F^{-1}A F^{-1} \) disguises the effect of the matrix norm and can result in the loss of accuracy due to roundoff error. For similar problems such as the generalized singular value decomposition and the generalized eigenvalue decomposition, the best numerical algorithms do not transform the generalized problem, but solve it directly. Therefore, we state all of the equivalent definitions of \( \epsilon \)-pseudospectra for the generalized case. We do this for our formulation, with \( A \equiv F^*F \hat{A} \).

The Reddy et al. formulation is given by transforming the results in definition 3.1.

We restrict our consideration to the finite dimensional case. We denote the spectrum of the generalized eigenvalue problem, \( A e = \lambda M e \), by \( \Lambda(A,M) \) and the resolvent set by \( \rho(A,M) \).

**Definition 3.1** Let \( M \) be a positive self-adjoint matrix and let \( \epsilon \geq 0 \) be given. A complex number \( z \) is in the \( \epsilon \)-pseudospectrum of \( (A,M) \), which we denote by \( \Lambda_\epsilon(A,M) \), if any of the following equivalent conditions is satisfied:

1. \( z \) is in the \( \epsilon \)-pseudospectrum of \( F^{-1}AF^{-1} \), where \( F^*F = M \).
2. The smallest generalized \( (M^{-1}, M) \) singular value of \( A - zM \) is less than or equal to \( \epsilon \), i.e. \( \epsilon \geq \min\{\mu(A - zM, M^{-1}, M)\} \).
3. \( \exists u \in \mathbb{C}^n \) such that \( u^*Mu = 1 \) and \( u^*(A - zM)^{-1}(A - zM)u \leq \epsilon^2 \),
4. \( \exists z \in \Lambda(A,M) \) or \( \exists z \in \rho(A,M) \) and \( \exists w \in \mathbb{C}^n \) such that \( w^*M^{-1}w = 1 \) and \( w^*(A - zM)^{-1}w \geq 1/\epsilon^2 \),
5. \( z \) is in the generalized spectrum of \( A + \epsilon F^*EF : (A + \epsilon F^*EF)u = zMu \), where \( F^*F = M \) and the matrix \( E \) satisfies \( \|E\| \leq 1 \),
6. \( \exists a n \times n \) matrix, \( H \), such that \( z \) is in the generalized spectrum of \( A + \epsilon H : (A + \epsilon H)u = zMu \), where the matrix \( H \) satisfies

\[
\max_{u \in \mathbb{C}^n} \frac{u^*H^*M^{-1}Hu}{u^*Mu} \leq 1,
\]

7. \( \exists u_1 \) and \( u_2 \in \mathbb{C}^n \) such that \( z \) is in the generalized spectrum of \( A + \epsilon w_2 w_1^* \) w.r.t. \( M \), where \( w_1 = Mu_1, w_2 = Mu_2, u_1^*Mu_1 \leq 1 \) and \( u_2^*Mu_2 \leq 1 \).
(vi) \( \exists \mathbf{u} \in \mathbb{C}^n \) such that \( z \) is in the generalized spectrum of \( \mathbf{A} - (\mathbf{A} \mathbf{u} - z \mathbf{M} \mathbf{u}) \mathbf{w}^* \) w.r.t. \( \mathbf{M} \), where \( \mathbf{w} = \mathbf{M} \mathbf{u} \), \( \mathbf{u}^{*} \mathbf{M} \mathbf{u} \leq 1 \) and \( \mathbf{u}^{*} (\mathbf{A} - z \mathbf{M})^* \mathbf{M}^{-1} (\mathbf{A} - z \mathbf{M}) \mathbf{u} \leq \epsilon^2 \).

The equivalence may be proved directly or by simply transforming each of the properties from definition 1.1 as applied to \( \mathbf{A}_F \). Properties (i) and (v) are used in practice. Property (ii) corresponds to part b) of the corollary.

We now present a different generalization of \( \epsilon \)-pseudospectra for the generalized eigenvalue problem, \( \mathbf{A} \mathbf{u} = \lambda \mathbf{M} \mathbf{u} \). Our \( \mathbf{M} \)-weighted \( \epsilon \)-pseudospectrum has the advantage that definitions (i)-(vi) are simpler than in Def. 3.1. However, the \( \mathbf{M} \)-weighted \( \epsilon \)-pseudospectrum is not related to the standard \( \epsilon \)-pseudospectrum of Def. 1.1 through a change of variables, and therefore Def. 3.2 has no analog of (0) in Def. 3.1.

**Definition 3.2 (\( \mathbf{M} \)-weighted \( \epsilon \)-pseudospectrum).** Let \( \mathbf{M} \) be a positive self-adjoint matrix and let \( \epsilon \geq 0 \) be given. Define \( \tau \equiv \epsilon \left( \frac{||\mathbf{M}||}{||\mathbf{M}^{-1}||} \right)^{\frac{1}{2}} \). A complex number \( z \) is in the \( \mathbf{M} \)-weighted \( \epsilon \)-pseudospectrum of \( \mathbf{A} \), which we denote by \( \Lambda_{\epsilon}(\mathbf{A}|\mathbf{M}) \), if one of the following equivalent conditions is satisfied:

(i) the smallest singular value of \( \mathbf{A} - z \mathbf{M} \) is less than or equal to \( \tau \).

(ii) \( \exists \mathbf{u} \in \mathbb{C}^n \) such that \( \|\mathbf{u}\|^2 = 1 \) and \( \|(\mathbf{A} - z \mathbf{M})\mathbf{u}\|^2 \leq \tau^2 \),

(iii) \( z \in \Lambda(\mathbf{A}) \) or \( z \in \rho(\mathbf{A}) \) and \( \exists \mathbf{u} \in \mathbb{C}^n \) such that \( \|\mathbf{u}\|^2 = 1 \) and \( \mathbf{u}^{*}(\mathbf{A} - z \mathbf{M})^{-1}\mathbf{u} \geq 1/\tau^2 \),

(iv) \( z \) is in the generalized spectrum of \( \mathbf{A} + \tau \mathbf{E} \) w.r.t. \( \mathbf{M} \): \( (\mathbf{A} + \tau \mathbf{E})\mathbf{u} = z \mathbf{M} \mathbf{u} \), where the matrix \( \mathbf{E} \) satisfies \( \| \mathbf{E} \| \leq 1 \).

(v) \( z \) is in generalized spectrum of \( \mathbf{A} + \tau \mathbf{v}_2 \mathbf{v}_1^{*} \) w.r.t. \( \mathbf{M} \), where \( \| \mathbf{v}_1 \| \leq 1 \) and \( \| \mathbf{v}_2 \| \leq 1 \).

(vi) \( z \) is in the generalized spectrum of \( \mathbf{A} - (\mathbf{A} \mathbf{v} - z \mathbf{M} \mathbf{v}) \mathbf{v}^{*} \) w.r.t. \( \mathbf{M} \), where \( \| \mathbf{v} \| \leq 1 \) and \( \| \mathbf{A} \mathbf{v} - z \mathbf{M} \mathbf{v} \| \leq \tau \).

The normalization, \( \tau \equiv \epsilon \left( ||\mathbf{M}||/||\mathbf{M}^{-1}|| \right)^{\frac{1}{2}} \), allows Def. 3.2 to reduce to Def. 1.1 when \( \mathbf{M} \) is a multiple of the identity matrix. When \( ||\mathbf{M}^{-1}|| \) is infinite, we can replace this normalization with the normalization: \( \tau \equiv \epsilon ||\mathbf{M}|| \). When \( \mathbf{M} \) is an unbounded operator with finite \( ||\mathbf{M}^{-1}|| \), we can replace this definition with the normalization: \( \tau \equiv \epsilon/||\mathbf{M}^{-1}|| \).

Since definition 3.1 is a transformed version of Def. 1.1, we believe that Def. 3.1 is preferable to the simpler, but coordinate dependent Def. 3.2. In particular, definition 3.1 is useful in the analysis of differential operators which have variational formulations.

In definition 3.1, \( \mathbf{M} \) is the metric of the domain of \( \mathbf{A} \) and \( \mathbf{M}^{-1} \) is the metric of the range of \( \mathbf{A} \). In the corresponding Reddy et al. formulation, \( \mathbf{M} \) is the metric of both the domain and the range of \( \hat{\mathbf{A}} \). We give a more general formulation of \( \epsilon \)-pseudospectra which incorporates both previous cases. We say that \( z \) is in the \( \epsilon \)-pseudospectrum of \( \langle \mathbf{A}, \mathbf{M} \rangle \) with respect to the operator norms \( \| \cdot \|_1 \) and \( \| \cdot \|_2 \) if and only if there exist matrices \( \mathbf{E}_1 \) and \( \mathbf{E}_2 \) such that \( \| \mathbf{E}_1 \|_1^2 + \| \mathbf{E}_2 \|_2^2 \leq \epsilon^2 \) and \( z \) is a generalized eigenvalue of \( \langle \mathbf{A} + \mathbf{E}_1, \mathbf{M} + \mathbf{E}_2 \rangle \). This
extended definition corresponds to part (iv) of Defs. 1.1 & 3.1. Definition 3.1 is equivalent to restricting to $E_2 \equiv 0$ and using the matrix norm: $\| E \|_{M,M^{-1}} \equiv \max_{u \in \mathbb{C}^n} \frac{u^* E M^{-1} E u}{u^* M u}$.

Restricting to $E_2 \equiv 0$ is natural when $M$ is known to higher precision than $A$. When $\| \cdot \|_1 \equiv \| \cdot \|_2$, then allowing for both $E_1$ and $E_2$ simply transforms the definitions: $E_1 = E_1 + |z|^2$ and $E_2 = -z E_1 + |z|^2$, where $E$ is the optimal perturbation. The resulting perturbation has norm: $\| E_1 \|^2 + \| E_2 \|^2 = \frac{1}{1 + |z|^2} \| E \|^2$. Thus by allowing an $E_2$ perturbation, we replace the critical value of $\epsilon$, $\epsilon_b(z)$, with $\frac{1}{\sqrt{1 + |z|^2}} \epsilon_b(z)$. J. Demmel points out that a modification of the proof of Lemma 5 of [3] shows the equivalence of (i) and (iv) in Def. 3.2.

4 Transient growth for the generalized system

For completeness, we now describe algorithms for computing the numerical range and maximum transient growth for the generalized eigenvalue problem.

**Definition 4.1** Let $A$ be a linear operator and $M$ be a self-adjoint, positive definite operator. The numerical range of $A$ with respect to $M$ is defined by

$$R(A, M) \equiv \{ z \mid \exists u \text{ with } u^* Au = z \text{ and } u^* Mu = 1 \}.$$ 

The numerical range is convex, and its boundary can be computed by maximizing $Re(e^{-i\theta} \lambda_\theta)$ for all $\theta$, where $\lambda_\theta$ is a generalized eigenvalue. Thus for each value of $\theta \in [-\pi, \pi]$, the largest eigenvalue, $\lambda_\theta$, of the self-adjoint eigenvalue problem is computed,

$$(e^{-i\theta} A + e^{i\theta} A^*) e_\theta = 2 \lambda_\theta M e_\theta.$$ 

We note that the $\epsilon$-pseudo-spectrum is contained within $\epsilon$ of the numerical range [7,9,10].

Finally, we consider transient growth problems for evolutionary systems of partial differential equations. In [8], Reddy et al. studied the transient growth of solutions of the Orr-Sommerfeld equation. The corresponding problem in magneto-hydrodynamics again requires a generalized eigenvalue problem. We consider the initial value problem:

$$M \frac{\partial u}{\partial t} = Au.$$ 

We wish to maximize the energy at time $t$, $E_M(t) \equiv u(t)^* M u(t)$, subject to $E_M(t = 0) = 1$. We compute the eigenvalues and eigenvectors, $\{(\lambda_k, e_k)\}$ of $A e_k = \lambda_k M e_k$. We represent $u(t) = \sum_k a_k e_k e^{\lambda_k t}$. We define $Q_{k,\ell}(t) = e_k^* M e_\ell \exp(i(\lambda_k + \lambda_\ell) t)$. The energy at time, $E_M(t)$, is $a^* Q(t) a$, where $a$ is the $n$ vector of coefficients. Thus the maximum transient growth is computed by maximizing $E_M(t) - \lambda(E_M(t = 0) - 1)$ with respect to $a$. The resulting self-adjoint eigenvalue problem is

$$Q(t) a(t) = \lambda(t) Q(t = 0) a(t),$$
where \(a(t)\) are the coefficients which maximize the transient growth, and \(\lambda\) is the maximum energy growth. Transient growth depends on the norm which measures the energy. Replacing \(E_M(t)\) with \(E_N(t) \equiv u(t)^* N u(t)\), where \(N\) is an arbitrary positive definite matrix, can greatly alter the magnitude of the transient growth. This maximum transient growth problem can be recasted as a generalized singular value problem for \(\mu(A, B)\) where 
\[
F^* F = M, \quad B \cdot \ell \equiv F e_\ell, \quad A \cdot \ell \equiv e^{\lambda(t)} F e_\ell, \quad \text{and} \quad \mu^2(t) = \lambda(t).
\]
This generalized singular value decomposition formulation reduces the roundoff error in the computation.

In conclusion, \(\epsilon\)-pseudospectra, the numerical range, and the maximum transient growth rate have been useful in analyzing certain problems in fluid dynamics. For evolutionary systems, the \(\epsilon\)-pseudospectrum describes the norm of the Green's function to fixed frequency forcing. To treat magneto-hydrodynamics, the corresponding definitions and algorithms for the generalized eigenvalue problem are required. Parts (v) and (vi) of definitions 1.1 and 3.1 simplify the calculation of \(\epsilon\)-pseudospectra.

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