Hierarchy of Supersymmetric Higher Spin Connections

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ABSTRACT

We focus on the geometrical reformulation of free higher spin supermultiplets in 4D, $\mathcal{N} = 1$ flat superspace. We find that there is a de Wit-Freedman like hierarchy of superconnections with simple gauge transformations. The requirement for sensible free equations of motion imposes constraints on the gauge parameter superfields. Unlike the non-supersymmetric case there is no unique way of doing that and thus generating many different but, on-shell equivalent, constrained descriptions of the same physical system. By lifting the constraints non-geometrically we find that all known descriptions of integer and half-integer supermultiplets are produced by the different ways of decoupling higher order superconnections. Also we find that there exist a consistent constrained description of half-integer supermultiplets which can not be lifted to an unconstrained formulation. In the constrained formulation, the various descriptions can be labeled as geometrical or non-geometrical if the equations of motion can be expressed only in terms of superconnections or not.
1 Introduction

The study of higher spin theories plays a special role in the search for underlying principles and symmetries of nature. Depending on your viewpoint, this can be understood either based on the important part they play in string theory [1–7] and the intriguing hypothesis that higher spin symmetry may control the high energy regime of a UV completion of gravity [8, 9] or based on the various no-go results [10–14] which under specific assumptions constrain the list of nontrivial interactions among particles.

Most of the progress done in higher-spin theories falls under two categories: (i) constructing consistent interactions involving higher spin gauge fields and (ii) the geometrical re-formulation of free higher spins on Minkowski and AdS backgrounds. The correlation between these two directions can be recognized in the case of gravity, where the geometrical formulation of the theory dictates its interactions. By analogy, a better understanding of the underlying geometrical structure of the higher spin theory, assuming one exist, may result to a deeper understanding of the higher spin interactions.

Non-trivial higher spin interactions have been constructed employing a variety of techniques such as Noether method [15–21], BRST [22–29], light cone [30–35], and frame-like formulation [36–51]. Although the frame-like formulation is successful in constructing consistent interactions and provides an economy of ideas, the metric-like description offers an economy of fields which makes the geometrical interpretation of the theory more direct. This was first demonstrated by de Wit and Freedman in [54] where it was found that for a bosonic spin $s$ the object replacing the usual connection of Riemannian geometry is a tower of $s-1$ connection-like objects, each being the derivative of the previous one and with the top connection to allow the definition of an invariant curvature tensor which is the $s$-th spacetime derivative of the higher spin gauge field. However, extracting Frønsdal’s second order equation of motion required imposing a traceless condition on the gauge parameters and thus reducing the symmetry group, in order to decouple the higher order connections. Later developments include an unconstrained, geometrical but non-local description (it has inverse powers of $\Box$) [57, 58] and an unconstrained, local but non-geometrical description (using compensators, fields unrelated to connections and curvature tensors) [59, 60].

For higher spin theories with manifest $\mathcal{N} = 1$ supersymmetry [61–70] there has been some recent progress in the direction of constructing consistent interactions [71–80]. Nevertheless, no steps have been taken towards the geometrical re-formulation of these theories. This paper is a first analysis in that direction.

We study the properties of a set of natural objects which define the notion of generalized higher spin superconnections and their corresponding supercurvature superfields. We find that these objects arrange into a hierarchy à la de Wit and Freedman [54]. The top member of this superspace

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4It resembles the structure of a non-abelian Yang-Mills theory for the group of isometries of the underlying manifold and its higher spin symmetry extension.

5The seemingly disconnected choices of approach (a) frame-like (gauging of the underlying symmetry group) or (b) metric-like (geometry) align coherently in Felix Klein’s view of geometry as the action of a Lie group $G$ on coset spaces $G/H$ [52] and Cartan’s generalization of it [53].

6For the case of gravity, $s = 1$, this tower collapses to one connection, the Christoffel symbol.

7Higher connection-like objects also appear in the frame-like description of higher spins [55, 56, 40]. These are called ‘auxiliary fields’ but they can be decoupled from the free theory by traceless conditions, precisely to get two derivative equations.
hierarchy is a proper superconnection in the sense that it allows the definition of an invariant supercurvature superfield which will match the known high derivative, higher superspin superfield strength.

The structure of these superconnection-like objects control the extraction of sensible free theory superspace equations of motion by generating appropriate set of constraints on the gauge parameters. For non-supersymmetric higher spin theories this corresponds to the known trace condition. However, in superspace there are many non-equivalent constraints that one can impose and all of them originate from the decoupling of higher order superconnections. In this process we reproduce all known formulations of lower and higher spin gauge theories. Furthermore, we discover that some of these theories have the property that in the constrained formulation their equations of motion can be written purely in terms of a superconnection and thus giving a sense of a geometrical origin of the theory. In addition, we find a new description of the half integer superspin supermultiplet which generalizes the known new minimal and new-new minimal descriptions of linearized supergravity. However, this description is possible only in the constrained formulation. For every other theory we are able to non-geometrically lift the constraints by introducing extra, compensating superfields.

The paper is organized as follows. In section 2, getting inspiration from super Yang-Mills theory, we define the notion of a generalized superconnection and its corresponding supercurvature tensor. In section 3, we focus on the \( (s + 1, s + 1/2) \) class of supermultiplets which are described by a bosonic gauge superfield and show that there is a hierarchy of \( (s + 1) \) superconnection-like objects. We demonstrate that at the level of components this hierarchy contains the known de Wit-Freedman connections. In section 4, we present the extraction of standard free equations of motions by constraining the gauge parameter and thus decouple higher superconnections. In section 5, we deviate from the geometrical approach mindset in order to have unconstrained formulation of the various supermultiplets. This is done via the introduction of compensators and we compare with known results. We find that all known formulations of lower and higher spin gauge supermultiplets correspond to the constraints generated by the superconnections, but one of the constrained descriptions can not have a compensator completion and exist only in the constrained formulation. In section 6, the analysis is repeated for the \( (s + 1/2, s) \) class of supermultiplets described by a fermionic gauge superfield. We show that there are two independent hierarchies, with \( s \) members and \( s + 1 \) members respectively and use them to generate all appropriate constraints in order to extract free equations of motion. Similarly to the previous case, all known descriptions of such supermultiplets correspond to one of these constraints. Last section contains the conclusions and discussion.

2 Superconnections

Gauge redundancy has been proven crucial in constructing manifestly supersymmetric field theories for higher spins and a particular set of their interactions. However, as it stands the formulation used in these constructions is not very geometrical. This is because the superspace actions \([61–63,65,66]\) for free integer and half-integer superspins have been determined by hand and there is no obvious way of rewriting them in terms of higher spin superfield strengths that involve higher derivatives. A step towards a more geometrical description would require a generalization of the notion of superconnection, in the context of higher spins.
Following de Wit and Freedman [54], the signal of a proper connection is its ability to allow the definition of a gauge invariant tensor in terms of its derivatives. To identify this signal in manifestly supersymmetric theories and set the stage for our later examinations, let us first recall the case of super Yang-Mills [81].

2.1 Superconnections in super Yang-Mills theory

Let’s consider a 4D, N = 1 chiral superfield \( \Phi \) (\( D_\alpha \Phi = 0 \)). The free action \( \int d^8z \bar{\Phi} \Phi \) has a global \( U(1) \) symmetry: \( \Phi \rightarrow e^{i\Lambda} \Phi, \ \bar{\Phi} \rightarrow \bar{\Phi} e^{-i\bar{\Lambda}} \), where \( \Lambda \) is a constant superfield. Once we gauge the symmetry, the transformation of the chiral and antichiral superfield becomes

\[
\Phi \rightarrow e^{i\Lambda} \Phi, \quad \bar{\Phi} \rightarrow \bar{\Phi} e^{-i\bar{\Lambda}}
\]

where \( \Lambda \) is a chiral superfield such that \( \Phi \) is mapped to a chiral superfield and \( \bar{\Lambda} \) is an antichiral superfield such that \( \bar{\Phi} \) is mapped to an antichiral superfield. The kinetic energy term, however is not invariant anymore:

\[
\bar{\Phi} \Phi \rightarrow \bar{\Phi} \Phi e^{-i\bar{\Lambda}} e^{i\Lambda} \Phi
\]

and just like in the non-supersymmetric theory, a compensating gauge superfield must be introduced to restore the invariance\(^8\). In this case for the \( U(1) \) gauge group\(^9\), we introduce a real scalar superfield \( V(z) \), we modify the kinetic energy term to the form:

\[
\bar{\Phi} e^{gV} \Phi = \bar{\Phi} \Phi + g \ \bar{\Phi} V \Phi + \frac{1}{2} g^2 \ \bar{\Phi} V^2 \Phi + \ldots
\]

where \( g \) is the coupling constant and can be used as a bookkeeping device for doing perturbation theory and we assign to \( V(z) \) the following gauge transformation:

\[
e^{gV} \rightarrow e^{i\Lambda} e^{gV} e^{-i\bar{\Lambda}} \quad \Rightarrow \quad \delta V = \frac{i}{g} (\bar{\Lambda} - \Lambda)
\]

\[
= \frac{1}{g} \left( \bar{D}^2 L + D^2 \bar{L} \right).
\]

where \( L \) is an arbitrary scalar superfield. One way of identifying the ‘curvature’ or field strength superfield we need to construct a set of covariant derivatives \( \nabla_A = \{ \nabla_\alpha, \nabla_{\dot{\alpha}}, \nabla_{\alpha\dot{\alpha}} \} \) and study their algebra. Let’s consider superfields that have the same gauge transformation as the chiral superfield. For such superfields, a consistent set of covariant derivatives must have the following gauge transformations:

\[
\nabla_A \rightarrow e^{i\Lambda} \nabla_A e^{-i\bar{\Lambda}}.
\]

A consistent and convenient choice of covariant derivatives is the following

\[
\nabla_\alpha = e^{-gV} D_\alpha e^{gV}, \quad \nabla_{\dot{\alpha}} = \bar{D}_{\dot{\alpha}}, \quad \nabla_{\alpha\dot{\alpha}} = -i \{ \nabla_\alpha, \nabla_{\dot{\alpha}} \}
\]

\(^8\)In this process, we find the minimal coupling of the gauge superfield with the chiral matter supermultiplet.

\(^9\)Similarly, one can consider internal symmetries associated to a compact, non-abelian Lie group \( G \otimes U(1)^p \), where \( G \) is semi-simple.
and their algebra is\(^\text{10}\):\(^\text{10}\)

\[
\begin{align*}
\{\nabla_\alpha, \nabla_\beta\} &= 0 , \quad \{\nabla_\hat{\alpha}, \nabla_\hat{\beta}\} = 0 , \quad \{\nabla_\alpha, \nabla_\hat{\alpha}\} = i\nabla_{\alpha\hat{\alpha}} , \\
[\nabla_\hat{\alpha}, \nabla_\beta]\] &= iC_{\hat{\alpha}\hat{\beta}}W_\beta , \quad [\nabla_\alpha, \nabla_\beta]\] &= iC_{\alpha\beta}\bar{W}_\beta , \\
[\nabla_{\alpha\hat{\alpha}}, \nabla_\beta]\] &= C_{\alpha\hat{\beta}}\nabla_\hat{\alpha}\bar{W}_\beta + C_{\hat{\alpha}\hat{\beta}}\nabla_\alpha W_\beta
\end{align*}
\]

where \(W_\alpha = \bar{D}^2 (e^{-gV}D_\alpha e^{gV})\) and \(\bar{W}_\hat{\alpha} = e^{-gV} [D^2 (e^{gV}\bar{D}_\hat{\alpha}e^{-gV})] e^{gV} U(1) = \bar{D}^2 (e^{gV}\bar{D}_\hat{\alpha}e^{-gV})\).

Based on the above, one can immediately find the superconnections \(\Gamma_\alpha\) defined as the difference between the \(U(1)\) covariant derivatives \(\nabla_A\) and the supersymmetry covariant derivatives \(D_\alpha\): \(\nabla_A = D_\alpha + \Gamma_\alpha\)

\[
\Gamma_\alpha = e^{-gV}(D_\alpha e^{gV}) , \quad \Gamma_{\hat{\alpha}} = 0 , \quad \Gamma_{\alpha\hat{\alpha}} = -i\bar{D}_{\hat{\alpha}}\Gamma_\alpha
\]

and the field strength superfields \(W_\alpha = \bar{D}^2\Gamma_\alpha\). The superspace action for \(V(z)\) is proportional to \(\int d^8 z W_\alpha W_\alpha + \int d^8 \bar{z}\bar{W}_{\hat{\alpha}}\bar{W}_{\hat{\alpha}}\). For the linearized theory we get the following:

\[
\begin{align*}
\Gamma_\alpha &= gD_\alpha V , \\
\Gamma_{\hat{\alpha}} &= 0 , \\
\Gamma_{\alpha\hat{\alpha}} &= -ig\bar{D}_{\hat{\alpha}}D_\alpha V , \\
W_\alpha &= g\bar{D}^2D_\alpha V ,
\end{align*}
\]

One can also project\(^\text{11}\) these equation down to components to find (in the W.Z gauge):

\[
\Gamma_\alpha\big|_{W.Z} = 0 , \quad \Gamma_{\hat{\alpha}}\big|_{W.Z} = 0 , \quad \Gamma_{\alpha\hat{\alpha}}\big|_{W.Z} = iA_{\alpha\hat{\alpha}} \quad (\delta A_{\alpha\hat{\alpha}} = \partial_{\alpha\hat{\alpha}}\xi)
\]

which is exactly what is expected from the component theory, one real, vector, gauge field.

Based on the above review, we recognize the role of \(\Gamma_\alpha\) as a proper connection. The properties that gives it this characterization are \((i)\) its transformation has the structure \(D_\alpha\bar{D}^2 L\) and \((ii)\) it allows the definition of an invariant field strength \(W_\alpha\) by acting with \(\bar{D}^2\) on it. Getting inspiration from the above we define a notion of a generalized superconnection in the following way. In the context of a linearized theory we will call a superfield to be a \textit{superconnection} if it has a gauge transformation of the form

\[
\delta \Gamma_{\alpha\ldots} = D_\alpha \bar{D}^2 (\ldots)
\]

and therefore it allows the definition of an invariant \textit{supercurvature}

\[
W_{\alpha\ldots} = \bar{D}^2\Gamma_{\alpha\ldots} , \quad \delta W_{\alpha\ldots} = 0 .
\]

3 \textbf{Hierarchy of de Wit-Freedman superconnections for half integer superspins}

Let’s consider a supersymmetric system of massless higher spins, in 4D Minkowski spacetime. This system will include a bosonic and a fermionic higher spin gauge field, which are related by

\(^\text{10}\)We are using \textit{Superspace’s} [82] conventions.

\(^\text{11}\)Denoted by \(|\ldots|\) which is shorthand notation for evaluation at \(\theta = \bar{\theta} = 0\).
symmetrization of indices with weight one is denoted by \( H \). However, for that to happen it needs the appropriate gauge transformation. It is easy to verify that quantities \( (s+1) \gamma \) is a symmetric \( (s+1) \)-rank spacetime tensor which will play the role of the highest spin boson

\[
h_{\alpha(s+1)\dot{a}(s+1)} \propto \frac{1}{(s+1)!} [D_{\alpha s+1}\bar{D}_{\dot{a}s+1}] H_{\alpha(s)\dot{a}(s)} \right| .
\]  

(18)

However, for that to happen it needs the appropriate gauge transformation. It is easy to verify that the most general transformation of \( H_{\alpha(s)\dot{a}(s)} \) that gives \( h_{\alpha(s+1)\dot{a}(s+1)} \) the correct gauge transformation \( (\delta h_{\alpha(s+1)\dot{a}(s+1)} \propto \partial_{\alpha(s+1)\dot{a}(s+1)}(\zeta_{\alpha(s)}\dot{a}(s)) ) \) and is consistent with the reality of \( H \) is:

\[
\delta H_{\alpha(s)\dot{a}(s)} = \frac{1}{s!} D_{\alpha s} L_{(s-1)\dot{a}(s)} - \frac{1}{s!} \bar{D}_{\dot{a}s} L_{\alpha(s)\dot{a}(s-1))} .
\]  

(19)

This transformation is fixed by the reality of \( H_{\alpha(s)\dot{a}(s)} \) and the must have gauge transformation of

Looking back to the super Yang-Mills example of previous section, the goal is starting from the above \( H \)-superfield to construct a set of objects by the action of supersymmetric spinorial covariant derivatives with simple properties under transformation law (19). Consider the following quantities\(^{14}\):

\[
\Gamma_{\alpha(s)\dot{a}(s)} = D_{\beta} H_{\alpha(s)\dot{a}(s)} , \quad \delta \Gamma_{\alpha(s)\dot{a}(s)} = -\frac{1}{s!} C_{\beta s} D^2 L_{(s-1)\dot{a}(s)} - \frac{1}{s!} D_{\beta \dot{a}} L_{\alpha(s)\dot{a}(s-1))} ,
\]  

(20)

\[
\Gamma_{\beta(s)\dot{a}(s)} = \bar{D}_{\dot{\alpha}} H_{\alpha(s)\dot{a}(s)} , \quad \delta \Gamma_{\beta(s)\dot{a}(s)} = -\frac{1}{s!} C_{\dot{\beta} s} \bar{D}^2 D_{(s-1)\dot{a}(s)}
\]

\[
- \frac{i}{(s+1)!} \partial_{\beta(s)\dot{a}} L_{\alpha(s)\dot{a}(s-1))}
\]

\[
- \frac{1}{(s+1)!} C_{\dot{\beta}(s)\dot{a}} D_{\alpha s} D^2 L_{\alpha(s)\dot{a}(s-1))}
\]

\[
- \frac{1}{(s+1)!} C_{\dot{\beta}(s)\dot{a}} \dot{D} \beta s D_{\alpha(s)\dot{a}(s-1))}
\]

\[
+ \frac{s-1}{(s+1)!} C_{\dot{\beta}(s)\dot{a}} \dot{D} \beta s \bar{D}_{\dot{a}s-1} L_{\alpha(s)\dot{a}(s-2)).
\]  

\(^{12}\)We are using two component notation, where a single spacetime index is converted to a pair of spinorial indices. In 4D with Lorentzian signature, spinors can be split to Weyl spinors which have a definite helicity and carry undotted / dotted indices respectively that take two values. Irreducible \((p,q)\)-superfield tensors carry \( p \) undotted indices which are symmetrized (denoted as \( \alpha(p) \)) and \( q \) dotted indices which are also independently symmetrized (denoted as \( \dot{a}(q) \)). Symmetrization of indices with weight one is denoted by \( . \)

\(^{13}\)A detailed discussion regarding the components of a 4D, \( N = 1 \) superfield can be found in [83].

\(^{14}\)Because of the reality of \( H \) the objects generated by the exchange of \( D \) and \( \bar{D} \) are not independent.
\[ \Gamma_{\gamma\beta\alpha(s)}h\dot{\alpha}(s) = D_\gamma \bar{D}_\beta D_\beta H_{\alpha(s)}\dot{\alpha}(s) \ , \ \delta \Gamma_{\gamma\beta\alpha(s)}h\dot{\alpha}(s) = -\frac{i}{s!} C_{\beta(\alpha_s} \partial_{\gamma\beta} D^2L_{\alpha(s-1)}\dot{\alpha}(s) - \frac{i}{(s+1)!} \partial_{\beta(\dot{\alpha}} D_\gamma \bar{D}_\delta \dot{\alpha}(s)\dot{\alpha}(s-1)) + \frac{1}{(s+1)!} C_{\beta(\dot{\alpha}} C_{\gamma\beta} D^2D_\delta \dot{\alpha}(s-1)) + \frac{s-1}{(s+1)!} C_{\beta(\dot{\alpha}} D_\gamma \bar{D}_\delta \bar{D}_\beta \bar{D}_{\dot{\alpha}(s-1)}L_{\alpha(s)|\gamma}\dot{\alpha}(s-2)) - \frac{1}{(s+1)!} C_{\beta(\dot{\alpha}} D_\gamma \bar{D}_\delta \bar{D}_\beta L_{\alpha(s)}\dot{\alpha}(s-1)) \] (22)

Observe that by imposing various (anti)symmetrizations of indices we can simplify the above transformations. Also notice that the last term in (22) has the characteristic structure of the transformation of a superconnection (16). So let’s consider the following quantity:

\[ \Gamma_{\alpha(s+2)}h\dot{\alpha}(s-1) = \frac{1}{(s+2)!} D_{(\alpha_{s+2}} \partial_{\alpha_{s+1}} \dot{\alpha}^s H_{\alpha(s)}\dot{\alpha}(s) \ , \] (23a)

\[ \delta \Gamma_{\alpha(s+2)}h\dot{\alpha}(s-1) = -\frac{i}{s} \frac{1}{(s+2)!} D_{(\alpha_{s+2}} \bar{D}^2D_{\alpha_{s+1}} L_{\alpha(s)}\dot{\alpha}(s-1)) \] (23b)

\[ - \frac{1}{s} \frac{1}{(s+2)!(s-1)!} D_{(\alpha_{s+2}} \bar{D}_{(\dot{\alpha}_{s-1}}} \partial_{\alpha_{s+1}} \dot{\gamma} L_{\alpha(s)}|\gamma|\dot{\alpha}(s-2)) \ . \]

Because of the presence of the second term in (23b) this quantity is not quite yet a superconnection. However, for the special case of \( s = 1 \) (linearized supergravity: (2, 3/2)-supermultiplet) this term drops and \( \Gamma_{\alpha\beta\gamma} = \frac{1}{3} D_{(\alpha} \partial_{\beta} \dot{\gamma} H_{\alpha(s)}|\gamma) \dot{\alpha}(s) \gamma) \) is the superconnection for linearized supergravity. One can confirm that it’s \( \bar{\theta} \)-component is the linearized Christoffel symbol

\[ \bar{D}_\alpha \Gamma_{\alpha\beta\gamma} \bigg|_{\text{W.Z.}} \propto \partial_{(\alpha} h_{\beta\gamma)} \gamma \dot{\alpha} \ . \] (24)

As expected by (17), it allows the definition of a supercurvature tensor \( W_{\alpha\beta\gamma} = \bar{D}^2 \Gamma_{\alpha\beta\gamma} \propto \bar{D}^2 D_{(\alpha} \partial_{\beta)} \dot{\gamma} H_{\alpha(s)}|\gamma) \dot{\alpha}(s) \gamma) \) which is exactly the known invariant superfield strength [84, 85]. It includes the bosonic and fermionic linearized curvature tensors.

It is now straightforward to define generalized higher spin superconnections by recursive application of superspace derivatives:

\[ \Gamma^{(t)}_{\alpha(s+t+1)\dot{\alpha}(s-t)} = \frac{1}{(s + t + 1)!} D_{(\alpha_{s+t+1}} \partial_{\alpha_{s+t}} \dot{\gamma} l \ldots \partial_{\alpha_{s+1}} \dot{\gamma} l H_{\alpha(s)|\gamma(t)\dot{\alpha}(s-t))} \] (25a)

\[ \delta \Gamma^{(t)}_{\alpha(s+t+1)\dot{\alpha}(s-t)} = -i \frac{t}{s} \frac{1}{(s + t + 1)!} D_{(\alpha_{s+t+1}} \bar{D}^2 D_{\alpha_{s+t}} \partial_{\alpha_{s-t-1}} \dot{\gamma} l \ldots \partial_{\alpha_{s+1}} \dot{\gamma} l L_{\alpha(s)}|\gamma(t-1)\dot{\alpha}(s-t))} \] (25b)

\[ - \frac{s-t}{s} \frac{1}{(s + t + 1)!(s-t)}! D_{(\alpha_{s+t+1}} \bar{D}_{(\dot{\alpha}_{s-t}} \partial_{\alpha_{s+t}} \dot{\gamma} l \ldots \partial_{\alpha_{s+1}} \dot{\gamma} l L_{\alpha(s)}|\gamma(t)\dot{\alpha}(s-t-1))} \ . \]

This is a hierarchy of \( (s+1) \) superconnection-like objects à la de Wit and Freedman [54], parametrized by the values of \( t \) \( (t = 0, 1, 2, \ldots, s) \). Each of which is defined in terms of superspace derivatives of
the previous one, via the recursive relation
\[
\Gamma^{(t)}_{\alpha(s+t+1)\dot{\alpha}(s-t)} = \frac{1}{(s + t + 1)!} D_{(\alpha_{s+t+1})} \bar{D}^{\gamma_t} \Gamma^{(t-1)}_{\alpha(s+t+1)\gamma_t\dot{\alpha}(s-t)}. \tag{26}
\]
Only the top one \((t = s)\) is a proper superconnection in the sense that
\[
\delta \Gamma^{(s)}_{\alpha(2s+1)} = -i \frac{1}{(2s + 1)!} D_{(\alpha_{2s+1})} \bar{D}^2 D_{\alpha_{2s}} \partial_{\alpha_{2s-1}} \gamma_{s-1} \ldots \partial_{\alpha_{s+1}} \gamma_{s-1} L_{\alpha(s)} \bar{\gamma}_{(s-1)}. \tag{27}
\]
Consequently, it allows the definition of the invariant higher spin superfield strength
\[
W_{\alpha(2s+1)} = \bar{D}^2 \Gamma^{(s)}_{\alpha(2s+1)}. \tag{28}
\]
This is the exactly the invariant higher spin superfield strength constructed in [61] and later found in [65] by studying the transition between irreducible, massive higher superspin representation and irreducible, massless massless higher superspin representations.

Therefore, our analysis shows, that there actually exist a geometrical structure for higher spin gauge superfields which naturally extends the known super Yang-Mills and Supergravity cases. This hierarchy of higher spin superconnections provides the supersymmetric extension of the de Wit - Freedman higher spin connections. One can check that the \(\bar{\theta}\) and \(\theta^2\) components of \(\Gamma^{(t)}_{\alpha(s+t+1)\dot{\alpha}(s-t)}\) are the bosonic and fermionic de Wit - Freedman higher spin connections, respectively:
\[
\bar{D}_\beta \Gamma^{(t)}_{\alpha(s+t+1)\dot{\alpha}(s-t)} \left|_{W,Z.} \right. \propto \frac{1}{(s + t + 1)!} \partial_{(\alpha_{s+t+1})} \dot{\alpha}_s \ldots \partial_{\alpha_{s-1}} \dot{\alpha}_{s-t+1} h_{\alpha(s+1)} \beta \dot{\alpha}(s), \tag{29a}
\]
\[
\bar{D}^2 \Gamma^{(t)}_{\alpha(s+t+1)\dot{\alpha}(s-t)} \left|_{W,Z.} \right. \propto \frac{1}{(s + t + 1)!} \partial_{(\alpha_{s+t+1})} \dot{\alpha}_s \ldots \partial_{\alpha_{s+1}} \dot{\alpha}_{s-t+1} \psi_{\alpha(s+1)} \dot{\alpha}(s). \tag{29b}
\]

4 Extracting free equations of motion

Ordinary free field theory requires a second (first) order equation for bosons (fermions) which translates to four (two) spinorial superspace derivatives. However, the previously discussed geometrical approach to higher spin supermultiplets indicates that the only gauge invariant quantities involve higher derivatives. Therefore, it is not clear how one can obtain reasonable free superfield equations. The answer\(^{15}\) is that only if appropriate constraints are imposed on the gauge parameter, and thus reducing the underlying symmetry group, this can be achieved. This behavior is homologous to non-supersymmetric higher spins were a traceless condition must be imposed and the fields are restricted to \(SO(D)\) irreducible tensors instead of \(GL(D)\) tensors and thus forcing the constrained formulation.

We find that the constraints needed for having sensible higher superspin equations of motion, are dictated by the higher spin superconnections (25a) and are such that the higher derivative members of the hierarchy decouple. We also find that there are more than one inequivalent set of constraints that allow such a decoupling to take place and they correspond to the known minimal and non-minimal description of the \((s + 1, s + 1/2)\) supermultiplet. In the constrained formulation,\(^{15}\)Under the requirement of having a local theory.
the free theory equations of motion corresponding to the non-minimal description have a geometrical interpretation because they can be written only in terms of the superconnections.

Motivated from (28), one can define the \( t \)-generalization of it:

\[
W^{(t)}_{\alpha(s+1)\dot{\alpha}(s-t)} = \bar{D}^2 \Gamma^{(t)}_{\alpha(s+1)\dot{\alpha}(s-t)} .
\]

This is an interesting, secondary hierarchy because it’s top is the invariant superfield strength, but the members of it are not superconnection-like objects because their gauge transformation is:

\[
\delta W^{(t)}_{\alpha(s+1)\dot{\alpha}(s-t)} = -i \frac{s - t}{s} \frac{1}{(s + t + 1)!(s - t)!} \bar{D}^2 \partial_{\alpha s+t+1}(\alpha_{s-t} \partial_{\alpha s+t} \gamma_1 \ldots \partial_{\alpha s+1} \gamma_t \bar{L}(s))|\gamma(t)|\dot{\alpha}(s-t-1) .
\]

In [65] it was shown that the higher spin superfield strength has knowledge of the free equation of motion in the sense that the quantity \( D^{\alpha(s+1)}W_{\alpha(2s+1)} \) is expressed as the sum of higher derivative operators acting on the free equations of motion. It is natural to attempt this for the entire secondary hierarchy. One can show that:

\[
D^{\alpha(s+1)}W^{(t)}_{\alpha(s+1)\dot{\alpha}(s-t)} = \frac{1}{(s+1)!} \partial_{\alpha s+t} \gamma_1 \ldots \partial_{\alpha s+1} \gamma_t \left\{ (s + t + 1) D^\beta \bar{D}^2 D^\beta H_{\alpha(s)}|\gamma(t)|\dot{\alpha}(s-t) \right. \\
+ s D_{\alpha s} \bar{D}^2 D^\beta H_{[\beta|\alpha(s)|]|\gamma(t)|\dot{\alpha}(s-t)} \bigg\} \\
- i \frac{s - t}{(s + t + 1)!} D_{\alpha s+t} \partial_{\alpha s+t-1} \gamma_1 \ldots \partial_{\alpha s+1} \gamma_t \bar{D}^2 \bar{D}^\beta \bar{D}^\beta H_{\alpha(s)}|\gamma(t)|\dot{\alpha}(s-t) .
\]

The answer is similar, we get the sum of higher derivative operators acting on terms that have the correct characteristics (engineering dimensions and index structures) to appear in the equations of motions. Therefore, the \( t = 0 \) level seems the appropriate one to focus on in order to have quantities without higher derivatives. Under the gauge transformation we find that:

\[
\delta \left( D^{\alpha(s+1)}W^{(t)}_{\alpha(s+1)\dot{\alpha}(s-t)} \right) = \frac{(s-0)(s+1)}{s(s+1)!} \frac{1}{(s-t)!} \bar{D}_{(\dot{\alpha} s+t)} \bar{D}^2 \partial_{\alpha s+t} \gamma_1 \ldots \partial_{\alpha s+1} \gamma_t \bar{L}(s)|\gamma(t)|\dot{\alpha}(s-t-1) \\
+ \frac{(s-t)!}{s(s+1)!} \frac{1}{(s-t)!} \bar{D}_{(\dot{\alpha} s+t)} \bar{D}^2 \partial_{\alpha s+t} \gamma_1 \ldots \partial_{\alpha s+1} \gamma_t \bar{L}(s)|\gamma(t)|\dot{\alpha}(s-t-1) \\
- \frac{s - t}{s + t + 1} \frac{1}{(s-t)!} D_{(\dot{\alpha} s+t)} \bar{D}_{(\dot{\alpha} s-t)} \bar{D}^2 \partial_{\alpha s+t} \gamma_1 \ldots \partial_{\alpha s+1} \gamma_t \bar{L}(s)|\gamma(t)|\dot{\alpha}(s-t-1) .
\]

and for our case of interest \( (t = 0) \) we have the simpler transformation law

\[
\delta \left( D^{\alpha(s+1)}W^{(0)}_{\alpha(s+1)\dot{\alpha}(s)} \right) = \frac{1}{s!} \bar{D}_{(\dot{\alpha} s+1)} \bar{D}^2 \bar{L}(s)|\dot{\alpha}(s-1)\dot{\alpha}(s-1) - \frac{s}{s+1} \frac{1}{s!} D_{(\alpha s)} \bar{D}_{(\dot{\alpha} s+1)} \bar{D}^2 \bar{L}(s)|\gamma(s-1)|\dot{\alpha}(s-1) .
\]

4.1 Non-minimal Constraints

Based on the above transformation law, the quantity \( D^{\alpha(s+1)}W^{(0)}_{\alpha(s+1)\dot{\alpha}(s)} \) is invariant if we constraint the gauge parameter \( L_{\alpha(s)\dot{\alpha}(s-1)} \) as follows:

\[
D^2 L_{\alpha(s)}(s-1) + D^{\alpha(s+1)}L_{\alpha(s+1)\dot{\alpha}(s-1)} = 0
\]
where $\Lambda_{\alpha(s+1)\dot{\alpha}(s-1)}$ is an arbitrary superfield consistent with the condition $\bar{D}_\beta D^{\alpha(s+1)} \Lambda_{\alpha(s+1)\dot{\alpha}(s-1)} = 0$. It is immediately evident that, in this constraint formulation, the gauge invariant equation

$$\mathcal{E}_{\alpha(s)\dot{\alpha}(s)} \propto D^{\alpha(s+1)} \bar{D}^2 \Gamma_{\alpha(s)\dot{\alpha}(s)}^{(0)} = 0$$

(36)

should be considered the free equation of motion for the $(s+1, s+1/2)$ supermultiplet. This equation is geometrical in nature because it involves only superconnection $\Gamma^{(0)}$. Using (32), equation (36) can be decomposed to the following two equations:

$$D^\beta \bar{D}^2 D_\beta H_{\alpha(s)\dot{\alpha}(s)} = 0, \quad \bar{D}^2 D^\beta H_{\beta\alpha(s-1)\dot{\alpha}(s)} = 0$$

(37)

which are both gauge invariant due to (35). The constraint (35) is more restrictive than necessary, meaning that one can impose softer constraints. Moreover, it will be shown that it generates the known non-minimal description of $(s + 1, s + 1/2)$ supermultiplets.

4.2 Minimal Constraints

The transformation (34) can be written in a different way by realizing that (19) allows the decomposition of it to the sum of appropriate $\delta H$-dependent terms and the remainder. The $\delta H$-terms can then be absorbed to the left hand side of the equation. By isolating as many as possible of the $\delta H$-terms, the remainder will provide an alternative structure of constraints. With this in mind, the first term of (34) can be written as:

$$\frac{1}{s!} \bar{D}_{(\dot{\alpha}s)} D^2 \bar{D}^2 L_{\alpha(s)\dot{\alpha}(s-1))} = -\frac{s(s+1)}{2s+1} \frac{1}{s!} \bar{D}_{(\dot{\alpha}s)} D^2 \bar{D}^\rho \delta H_{\alpha(s)\dot{\alpha}(s-1))}$$

$$+ \frac{s^2}{2s+1} \frac{1}{s!} \bar{D}_{(\dot{\alpha}s)} D_{(\alpha s)} \bar{D}^\rho \delta H_{|\rho|\alpha(s-1))|\rho|\dot{\alpha}(s-1))}$$

$$- \frac{s^2}{2s+1} \frac{1}{s!} \bar{D}_{(\dot{\alpha}s)} D_{(\alpha s)} \bar{D}^2 D^\rho L_{|\rho|\alpha(s-1))|\rho|\dot{\alpha}(s-1))}$$

$$+ \frac{s(s-1)}{2s+1} \frac{1}{s!} \bar{D}_{(\dot{\alpha}s)} D_{(\alpha s)} \bar{D}_{(\dot{\alpha}s-1)} D^\rho \bar{D}^\rho L_{|\rho|\alpha(s-1))|\rho|\dot{\alpha}(s-2))}$$

(38)

and similarly the second term:

$$\frac{1}{s!s!} D_{(\alpha s)} \bar{D}_{(\dot{\alpha}s)} D^\beta \bar{D}^2 L_{|\beta|\alpha(s-1))|\dot{\alpha}(s-1))} = -\frac{s^2}{2s+1} \frac{1}{s!} D_{(\alpha s)} \bar{D}^2 D^\rho \delta H_{|\rho|\alpha(s-1))|\dot{\alpha}(s-1))}$$

$$- \frac{s(s+1)}{2s+1} \frac{1}{s!} D_{(\alpha s)} \bar{D}_{(\dot{\alpha}s)} \bar{D}^\rho \delta H_{|\rho|\alpha(s-1))|\rho|\dot{\alpha}(s-1))}$$

$$- \frac{s(s+1)}{2s+1} \frac{1}{s!} D_{(\alpha s)} \bar{D}_{(\dot{\alpha}s-1)} \bar{D}^\rho \bar{D}^\rho L_{|\rho|\alpha(s-1))|\rho|\dot{\alpha}(s-1))}$$

$$+ \frac{(s+1)(s-1)}{2s+1} \frac{1}{s!} D_{(\alpha s)} \bar{D}_{(\dot{\alpha}s-1)} \bar{D}^\rho \bar{D}^\rho \bar{D}^\rho L_{|\rho|\alpha(s-2))|\rho|\dot{\alpha}(s-1))}.$$

(39)
Therefore, it is obvious that if one deforms the left hand side of (34), by considering the quantity

\[ I_{\alpha(s)\dot{a}(s)} = D^{\alpha s+1}W^{(0)}_{\alpha(s+1)\dot{a}(s)} + \frac{s(s + 1)}{2s + 1} \frac{1}{s!} \bar{D}(\dot{a}_s)D^2\bar{D}^\dot{a}_sH_{\alpha(s)|\dot{a}(s-1)} + \frac{s^3}{(2s + 1)(s + 1)} \frac{1}{s!} D_{(\alpha s}\bar{D}^2\bar{D}^\dot{a}_sH_{|\rho(\alpha(s-1))|\dot{a}(s)\rho(\dot{a}(s-1))} - \frac{s^2}{2s + 1} \frac{1}{s!} \bar{D}(\dot{a}_s)D(\alpha s)\bar{D}^2\bar{D}^\dot{a}_sH_{|\rho(\alpha(s-1))|\dot{a}(s-1)\rho(\dot{a}(s-1))} \]  

(40)

then we have the following transformation law:

\[ \delta I_{\alpha(s)\dot{a}(s)} = -s^2 2s + 1 \frac{1}{s!} \bar{D}(\dot{a}_s)D_{(\alpha s} \left[ D^2\bar{D}^\gamma L_{|\gamma(\alpha(s-1))|\dot{a}(s-1)} - \frac{s - 1}{s} D_{\alpha s-1} D^\gamma\bar{D}^\dot{a}_s L_{|\gamma|\alpha(s-1)|\dot{a}(s-1)|} \right] \]  

(41)

\[ + \frac{s^2}{2s + 1} \frac{1}{s!} D_{(\alpha s} \bar{D}\dot{a}_s \left[ D^2\bar{D}^\gamma \bar{L}_{\alpha(s-1)|\gamma|\dot{a}(s-1)} - \frac{s - 1}{s} D_{\alpha s-1} D^\gamma\bar{D}^\dot{a}_s \bar{L}_{|\gamma|\alpha(s-2)|\dot{a}(s-1)|} \right] \]

Hence, gauge invariance is achieved if we constrained the gauge parameter \( L_{\alpha(s)\dot{a}(s-1)} \) in the following way:

\[ s > 1 : \quad D^\gamma\bar{D}^\dot{a}_s L_{\gamma\alpha(s-1)|\gamma|\dot{a}(s-2)} = \frac{s}{s - 1} D^\gamma\bar{D}^\dot{a}_s L_{\gamma\alpha(s-1)|\gamma|\dot{a}(s-2)} + \frac{s}{(s - 1)!} \bar{D}(\dot{a}_s) J_{\alpha(s-1)\dot{a}(s-3)} = 0 \]  

(42a)

\[ s = 1 : \quad \bar{D}^2\bar{D}^\gamma L_{\gamma\dot{a}(s-1)} = 0 \]  

(42b)

where \( J_{\alpha(s-1)\dot{a}(s-3)} \) is an arbitrary superfield. These are weaker constraints that will be later shown to generate the known minimal description of the half-integer superspin supermultiplet. In this constraint formulation the gauge invariant equation

\[ \mathcal{E}_{\alpha(s)\dot{a}(s)} \propto I_{\alpha(s)\dot{a}(s)} = 0 \]  

(43)

should be considered the free equations of motion. This equation of motion yields the following two equations:

\[ D^\gamma\bar{D}^\dot{a}_s L_{\gamma\alpha(s-1)|\gamma|\dot{a}(s-2)} + \frac{s(s + 1)}{2s + 1} \frac{1}{s!} \left[ D_{(\alpha s} D^2\bar{D}^\gamma H_{|\gamma|\alpha(s-1)|\dot{a}(s-1)|} + \text{c.c.} \right] \]  

(44)

\[ -\frac{s^2}{2s + 1} \frac{1}{s!} \left[ D_{(\alpha s} \bar{D}(\dot{a}_s) D^\gamma\bar{D}^\dot{a}_s L_{|\gamma|\alpha(s-1)|\dot{a}(s-1)|} + \text{c.c.} \right] = 0 \]

(45)

An interesting observation is that this minimally constraint formulation, unlike the previous non-minimally constraint formulation, does not have a geometrical origin, in the sense that the equation of motion (43) cannot be written purely in terms of the superconnection \( \Gamma_{\alpha(s+1)\dot{a}(s)}^{(0)} \), but additional terms depending on the gauge superfield had to be added.
4.3 More Minimal Constraints

Moreover, there is yet another approach to constraining the gauge parameter in order to reach
gauge invariant second order equations. Using the approach of extracting $\delta H$-terms, we can re-write
(34) alternatively as follows:

$$
\delta \left( D_{\alpha + 1}^{\alpha} W^{(0)}_{\alpha(s+1)\dot{\alpha}(s)} \right) = \frac{s}{s + 1} \frac{sc^*}{s(c^* + 1) + 1} \frac{1}{s!} D_{(\alpha_s} D^2 D^\rho \delta H_{|\rho|\alpha(s-1))\dot{\alpha}(s)}
$$

$$
- \frac{s}{s(c + 1) + 1} \frac{1}{s!} \bar{D}_{(\dot{\alpha}_s} D^2 \bar{D}^\delta \delta H_{\alpha(s)|\bar{\rho}|\dot{\alpha}(s-1))}
$$

$$
+ \frac{s}{s(c + 1) + 1} \frac{1}{s!} \bar{D}_{(\dot{\alpha}_s} D \left( \bar{D}^\delta \bar{D}^2 L_{\alpha(s-1))\dot{\alpha}(s-1)) |\bar{\rho}|\dot{\alpha}(s-1)) \right) - c^* \bar{D} \bar{D}^2 L_{\alpha(s-1))\dot{\alpha}(s-1)) |\bar{\rho}|\dot{\alpha}(s-1))
$$

$$
- \frac{s}{s(c^* + 1) + 1} \frac{1}{s!} \bar{D}_{(\alpha_s} \bar{D}(\dot{\alpha}_s} \left( \bar{D}^\rho \bar{D}^2 L_{\alpha(s-1))\dot{\alpha}(s-1)) |\bar{\rho}|\dot{\alpha}(s-1)) \right) - c^* \bar{D} \bar{D}^2 \bar{L}_{\alpha(s-1))|\bar{\rho}|\dot{\alpha}(s-1))
$$

for an arbitrary complex number $c$. However if $c$ is chosen to be a phase\(^\text{16}\), then we can impose the constraint

$$
D^\rho \bar{D}^2 L_{\rho\alpha(s-1)\dot{\alpha}(s-1)} + \bar{D}^\delta \bar{D}^2 L_{\alpha(s-1)\dot{\alpha}(s-1)} = 0.
$$

(47)

This is another minimal constraint. Under the assumption of this constraint, we conclude that the equation

$$
\mathcal{E}_{\alpha(s)\dot{\alpha}(s)} = D^{\alpha+1} W^{(0)}_{\alpha(s+1)\dot{\alpha}(s)} - \frac{s}{s + 1} \frac{sc}{s(c + 1) + 1} \frac{1}{s!} D_{(\alpha_s} \bar{D}^2 D^\rho H_{|\rho|\alpha(s-1))\dot{\alpha}(s)}
$$

$$
+ \frac{s}{s(c + 1) + 1} \frac{1}{s!} \bar{D}_{(\dot{\alpha}_s} D^2 \bar{D}^\delta H_{\alpha(s)|\bar{\rho}|\dot{\alpha}(s-1))} = 0
$$

(48)

with $c = -1, 1$ is suitable to play the role of the equation of motion. As in the previous case, this
is not a geometrical equation because of the explicit $H$-terms that can be expressed in terms of the
superconnection $\Gamma_{\alpha(s+1)\dot{\alpha}(s)}^{(0)}$. The above equation yields only one gauge invariant second order
equation of motion for $H$

$$
\bar{D}^\delta \bar{D}^2 \bar{D}^\rho H_{\alpha(s)\dot{\alpha}(s)} + \frac{s}{s(c + 1) + 1} \frac{1}{s!} D_{(\alpha_s} \bar{D}^2 D^\gamma H_{|\gamma|\alpha(s-1))\dot{\alpha}(s)}
$$

$$
+ \frac{s}{s(c + 1) + 1} \frac{1}{s!} \bar{D}_{(\dot{\alpha}_s} \bar{D}^2 \bar{D}^\gamma H_{\alpha(s)|\bar{\gamma}|\dot{\alpha}(s-1))} = 0.
$$

(49)

The $s = 1$ case is special. For $s = 1$, the constraint (47) has a superspace solution

$$
L_{\alpha} = \begin{cases}
 iD_{\alpha} L + \bar{D}^\dot{\alpha} \Lambda_{\alpha\dot{\alpha}} , & \text{for } c = -1 \\
 D_{\alpha} L + \bar{D}^\dot{\alpha} \Lambda_{\alpha\dot{\alpha}} , & \text{for } c = +1
\end{cases}
$$

(50)

\(^\text{16}\)After redefinitions we consider only the case of $c = -1, 1$.\]
where \( L \) is an arbitrary real scalar \((L = \bar{L})\) and \( \Lambda_{\dot{\alpha}} \) is an arbitrary vector superfield. Therefore, the \( s = 1 \) version of (49) remains valid for the unconstrained gauge transformation

\[
\delta H_{\dot{\alpha}a} = \begin{cases} 
\frac{1}{2} \partial_{\dot{\alpha}} L + \frac{1}{2} \left( D^2 \Lambda_{\dot{\alpha}} + \bar{D}^2 \bar{\Lambda}_{\dot{\alpha}} \right), & \text{for } c = -1 \\
\frac{1}{2} [D_{\dot{\alpha}}, \bar{D}_{\dot{\alpha}}] L + \frac{1}{2} \left( D^2 \Lambda_{\dot{\alpha}} + \bar{D}^2 \bar{\Lambda}_{\dot{\alpha}} \right), & \text{for } c = +1
\end{cases}
\]

(51)

The first one corresponds to the new-minimal [86–89] description of linearized supergravity supermultiplet and the second one to the new-new-minimal [90–92] description of linearized supergravity. However, for general \( s \), equation (49) is only valid under the assumption of the constraint (47) and corresponds to the higher spin version of the new minimal and new-new minimal descriptions.

It is important to emphasize that all the different constrained formulations presented above based on minimal or non-minimal constraints describe the same physical degrees of freedom. All of them have the same invariant superfield strength \( W_{\alpha(2s+1)} \) and their equations of motion are such that when substituted in (32) give the same on-shell condition \( D^{\alpha(s+1)} W_{\alpha(s+1)} = 0 \).

5 Non-Geometrical Unconstrained Formulation: Compensators

Similarly to the de Wit and Freedman story, we have shown that the extraction of sensible gauge invariant free equations of motion for the higher spin supermultiplets required the constraining of the gauge parameter and the low order members of the hierarchy generate the appropriate constraints. The difference for manifestly supersymmetric theories is that because of the non-trivial algebra of the supersymmetric covariant derivatives there are many ways of generating such constraints and they lead to different formulations of the same physical theory.

It would be desirable if we could have an unconstrained formulation. This can be easily achieved via introducing so called compensator superfields, which have a transformation law proportional to the constraint, which now can be lifted. The compensators will modify the equations of motion found in the constrained description such that they remain invariant under the full group of transformations. Unfortunately, this type of unconstrained formulation is not geometrical anymore because the compensators are unrelated to the superconnections or the field strength.

In particular, one can lift constraint (35) by introducing a fermionic compensator \( \chi_{\alpha(s)\dot{\alpha}(s-1)} \) equipped with the transformation law

\[
\delta \chi_{\alpha(s)\dot{\alpha}(s-1)} \propto \bar{D}^2 L_{\alpha(s)\dot{\alpha}(s-1)} + D^{\alpha(s+1)} \Lambda_{\alpha(s+1)\dot{\alpha}(s-1)} .
\]

(52)

This compensator will modify the right hand side of equations (37) according to (34) so they remain invariant for arbitrary gauge parameter \( L_{\alpha(s)\dot{\alpha}(s-1)} \). This process will give the known non-minimal description of \((s + 1, s + 1/2)\) supermultiplet [61, 65]. Likewise for (42), introduce compensators
\( \chi_{\alpha(s-1)\dot{\alpha}(s-2)} \) for \( s > 1 \) and chiral \( \Phi \) for \( s = 1 \) with transformations

\[
\delta \chi_{\alpha(s-1)\dot{\alpha}(s-2)} \propto \bar{D}^\gamma D^\gamma L_{\gamma \alpha(s-1)\dot{\gamma}(s-2)} + \frac{s-1}{s} D^\gamma \bar{D}^\gamma L_{\gamma \alpha(s-1)\dot{\gamma}(s-2)} + \frac{1}{(s-2)!} \bar{D}_{(\dot{\alpha}_{s-2})} J_{\alpha(s-1)\dot{\alpha}(s-3)} ,
\]

\[
\delta \Phi \propto \bar{D}^2 D^\gamma L
\]

which will modify equations (44), (45) according to (41). The result will be identical to the minimal description of \( (s + 1, s + 1/2) \) supermultiplet \([61, 65]\) and for the \( s = 1 \) case this will give the old-minimal description of linearized supergravity. Finally, constraint (47) requires the introduction of a real (imaginary) linear compensator \( U_{\alpha(s-1)\dot{\alpha}(s-1)} \) with the following transformation law

\[
\delta U_{\alpha(s-1)\dot{\alpha}(s-1)} = D^\rho \bar{D}^2 L_{\rho \alpha(s-1)\dot{\alpha}(s-1)} \pm \bar{D}^\rho D^2 \bar{L}_{\alpha(s-1)\dot{\rho}(s-1)}
\]

however such a transformation completely eliminates the compensator\(^{17}\) forcing back on us the constraint (47). Therefore, the non-geometrical method of compensators cannot provide an unconstrained formulation for this case. However at the constrained formulation, this is a new and consistent description. Of course, as we previously mentioned, the exception is the \( s = 1 \) case where the constraint can be explicitly solved in superspace introducing new unconstrained gauge parameters and thus effectively making the formulation unconstrained without the need of introducing a compensator.

6 Double hierarchy of de Wit-Freedman superconnections for integer superspins

In this section we consider integer superspin supermultiplets \( (s + 1/2, s) \) where the highest propagating spin is a fermion. The appropriate superfield for the description of this supermultiplet is a fermionic \( (s, s-1) \)-superfield tensor \( \Psi_{\alpha(s)\dot{\alpha}(s-1)} \) with a transformation\(^{18}\)

\[
\delta \Psi_{\alpha(s)\dot{\alpha}(s-1)} = \frac{1}{s!} D_\alpha K_{\alpha(s-1)\dot{\alpha}(s-1)} + \frac{1}{(s-1)!} \bar{D}_{(\dot{\alpha}_{s-1})} \Lambda_{\alpha(s)\dot{\alpha}(s-2)} .
\]

One can verify that the highest rank component of this superfield is a symmetric rank \( s \) spinor tensor

\[
\psi_{\alpha(s+1)\dot{\alpha}(s)} \propto \frac{1}{(s+1)!} \left[ D_{(\alpha_{s+1})} \bar{D}_{(\dot{\alpha}_s)} \Psi_{\alpha(s)\dot{\alpha}(s-1)} \right]
\]

and the transformation law (55) is the most general which will give the proper gauge transformation to the above gauge field \( \delta \psi_{\alpha(s+1)\dot{\alpha}(s)} \propto \partial_{(\alpha_{s+1})} (\lambda(s)\dot{\alpha}(s-1)) \).

Immediately, one can observe a very important qualitative difference with the previous half-integer case. There are two independent symmetries. One is parametrized by gauge parameters

\(^{17}\)Any real (imaginary) linear superfield \( U_{\alpha(s-1)\dot{\alpha}(s-1)} \) can be expressed in terms of an unconstrained superfield (prepotential) \( \dot{\psi}_{\alpha(s)\dot{\alpha}(s-1)} \). Based on (54) the transformation of \( \psi \) is algebraic \( \delta \psi_{\alpha(s)\dot{\alpha}(s-1)} = L_{\alpha(s)\dot{\alpha}(s-1)} + \bar{D}^\gamma \hat{\Xi}_{\beta\gamma\alpha(s)\dot{\alpha}(s-1)} \) and thus it can be set to zero immediately.

\(^{18}\)For the special case of \( s = 1 \) the transformation takes the form \( \delta \Psi_\alpha = D_\alpha K + \bar{D}^2 \Lambda_\alpha \).
This is identical to the invariant superfield strength constructed in \([55]\). For the half integer superspin case, the reality condition of superfield \(H_{\alpha(s)\dot{\alpha}(s)}\) forced a relation between the two parameters by complex conjugation and thus collapsed the two symmetries into one. The implications of this are that we can construct two types of superconnections: \(K\)-superconnections and \(\Lambda\)-superconnections. Each type will have its own hierarchy, and field strength.

### 6.1 \(\Lambda\)-superconnections

Gauge parameter \(\Lambda_{\alpha(s)\dot{\alpha}(s-2)}\) appears in (55) exactly the same way as \(L_{\alpha(s)\dot{\alpha}(s-1)}\) appears in (19). Therefore, we can immediately inherit the results of section 3:

There is a hierarchy of \(s\) \(\Lambda\)-superconnection-like objects parametrized by \(t\) \((t = 0, 1, 2, \ldots, s-1)\)

\[
\Gamma^{(t)}_{\alpha(s+t+1)\dot{\alpha}(s-t-1)} = \frac{1}{(s+t+1)!} D_{(\alpha_{s+t+1}} \partial_{\alpha_{s+t}} \dot{\gamma}^1 \ldots \partial_{\alpha_{s+1}} \dot{\gamma}_s \Psi_{\alpha(s))\dot{\gamma}(t)\dot{\alpha}(s-t-1)}
\]  

(57)

which satisfy the recursive relation

\[
\Gamma^{(t)}_{\alpha(s+t+1)\dot{\alpha}(s-t-1)} = \frac{1}{(s+t+1)!} D_{(\alpha_{s+t+1}} \bar{D} \dot{\gamma}^1 \Gamma^{(t-1)}_{\alpha(s+t))\dot{\gamma}(s-t-1)}
\]  

(58)

and have the following gauge transformation law

\[
\delta \Gamma^{(t)}_{\alpha(s+t+1)\dot{\alpha}(s-t-1)} = i \frac{t}{s-1} \frac{1}{(s+t+1)!} D_{(\alpha_{s+t+1}} \bar{D} \partial_{\alpha_{s+t}} \dot{\gamma}^1 \ldots \partial_{\alpha_{s+1}} \dot{\gamma}_s \Lambda_{\alpha(s))\dot{\gamma}(t)\dot{\alpha}(s-t-1)} + \frac{s-t-1}{s} \frac{1}{(s+t+1)!(s-t-1)!} D_{(\alpha_{s+t+1}} \bar{D} \partial_{\alpha_{s-t-1}} \dot{\gamma}^1 \ldots \partial_{\alpha_{s+1}} \dot{\gamma}_s \Lambda_{\alpha(s))\dot{\gamma}(t)\dot{\alpha}(s-t-2)}
\]  

(59)

The top one \((t = s-1)\) \(\Gamma^{(s-1)}_{\alpha(2s)}\) is a proper superconnection and as such allows the definition of an invariant superfield strength

\[
W_{\alpha(2s)} = \bar{D} \Gamma^{(s-1)}_{\alpha(2s)}
\]  

(60)

This is identical to the invariant superfield strength constructed in [62] and later in [65].

### 6.2 \(K\)-superconnections

Gauge parameter \(K_{\alpha(s-1)\dot{\alpha}(s-1)}\) appears in (55) exactly the same way as \(\bar{L}_{\alpha(s-1)\dot{\alpha}(s)}\) in (19). Hence, if we use \(\bar{\Psi}_{\alpha(s-1)\dot{\alpha}(s)}\) instead of \(\Psi_{\alpha(s)\dot{\alpha}(s-1)}\) in the construction of superconnections we can also use the results of section 3

There is a hierarchy of \((s+1)\) \(K\)-superconnection like objects

\[
\Delta^{(t)}_{\alpha(s+t)\dot{\alpha}(s-t)} = \frac{1}{(s+t)!} D_{(\alpha_{s+t}} \partial_{\alpha_{s+t-1}} \dot{\gamma}^1 \ldots \partial_{\alpha_2} \dot{\gamma}_s \bar{\Psi}_{\alpha(s))\dot{\gamma}(t)\dot{\alpha}(s-t)}
\]  

(61)

which is not related to the hierarchy in (57) by complex conjugation and must be studied indepen-
dently. Their gauge transformation is
\[
\delta \Delta^{(t)}_{\alpha(s+t)\dot{\alpha}(s-t)} = i \frac{t}{s} \frac{1}{(s+t)!} D_{(\alpha_s+t} \bar{D}^2 D_{\alpha_{s+t-1}} \partial_{\alpha_{s+t-2}} \dot{\gamma}_1 \cdots \partial_{\alpha_s} \dot{\gamma}_{t-1} \bar{K}_{\alpha(s-1)\dot{\alpha}(t-1)} \dot{\alpha}(s-t) + \frac{s-t}{s} \frac{1}{(s+t)!} (s-t)! D_{(\alpha_s+t} \bar{D}_{(\dot{\alpha}_s-t-1)} \partial_{\alpha_{s+t-2}} \dot{\gamma}_1 \cdots \partial_{\alpha_s} \dot{\gamma}_{t-1} \bar{K}_{\alpha(s-1)\dot{\alpha}(t-1)} .
\]

The top member of this hierarchy, \( \Delta^{(s)}_{\alpha(2s)} \) is a superconnection and it defines the following gauge invariant superfield strength
\[
Z_{\alpha(2s)} = \bar{D}^2 \Delta^{(s)}_{\alpha(2s)} .
\]

### 6.3 Free equations of motion

Once again the invariant tensors involve higher derivatives. Hence, the extraction of gauge invariant two derivative equations of motion for the \((s + 1/2, s)\) supermultiplet must rely on some appropriate set of constraints on the gauge parameters that will decouple the low derivative members in at least one of the available hierarchies. The relevant quantities to investigate are: \( D^\alpha_s \Delta^{(0)}_{\alpha(s)\dot{\alpha}(s)} \), \( D^\dot{\alpha}_s \Delta^{(0)}_{\alpha(s)\dot{\alpha}(s)} \) and \( D^\alpha_s \Gamma^{(0)}_{\alpha(s+1)\dot{\alpha}(s-1)} \). For, \( s > 1 \), we find:
\[
\delta \left( \bar{D}^\dot{\alpha}_s \Delta^{(0)}_{\alpha(s)\dot{\alpha}(s)} \right) = -\frac{1}{s} D_{(\alpha_s} \bar{D}^2 \bar{K}_{\alpha(s-1)\dot{\alpha}(s-1)} - \frac{1}{s!} D^2 D_{(\alpha_s} \bar{K}_{\alpha(s-1)\dot{\alpha}(s-1)} \right) + \frac{s-1}{s!} \bar{D}_{(\dot{\alpha}_s-1)\alpha(s)} \bar{D}_{(\dot{\alpha}) \dot{\alpha}} K_{\alpha(s-1)\dot{\alpha}(s-2)} ,
\]
\[
\delta \left( \bar{D}^\dot{\alpha}_s \Delta^{(0)}_{\alpha(s)\dot{\alpha}(s)} \right) = -\frac{1}{s} D^2 D_{(\alpha_s} \bar{K}_{\alpha(s-1)\dot{\alpha}(s-1)} \dot{\alpha} ,
\]
\[
\delta \left( D^{\alpha_s+1} \Gamma^{(0)}_{\alpha(s+1)\dot{\alpha}(s-1)} \right) = \frac{s+1}{s+1} \frac{1}{(s-1)!} D^2 \bar{D}_{(\dot{\alpha}_s-1)\alpha(s)\dot{\alpha}(s-2)}
\]

hence by constraining gauge parameter \( K_{\alpha(s-1)\dot{\alpha}(s-1)} \) in the following way
\[
D^\dot{\alpha}_s K_{\alpha(s-2)\dot{\alpha}(s-1)} = 0 \quad \Rightarrow \quad K_{\alpha(s-1)\dot{\alpha}(s-1)} = D^\alpha_s L_{\alpha(s)\dot{\alpha}(s-1)} ,
\]
\[
\bar{K}_{\alpha(s-1)\dot{\alpha}(s-1)} \pm K_{\alpha(s-1)\dot{\alpha}(s-1)} = 0 \quad \Rightarrow \quad D^\alpha_s L_{\alpha(s)\dot{\alpha}(s-1)} \pm \bar{D}^\dot{\alpha}_s \bar{L}_{\alpha(s-1)\dot{\alpha}(s)} = 0
\]

we get the following gauge invariant equation of motion
\[
E_{\alpha(s)\dot{\alpha}(s-1)} \propto \bar{D}^{\dot{\alpha}_s+1} \left( \frac{s+1}{s} \Delta^{(0)}_{\alpha(s)\dot{\alpha}(s)} \pm \bar{\Delta}^{(0)}_{\alpha(s)\dot{\alpha}(s)} \right)
\]

The equation of motion is expressed purely in terms of superconnection \( \Delta^{(0)} \) and in the constrained formulation it produces the following two gauge invariant equations for superfield \( \Psi_{\alpha(s)\dot{\alpha}(s-1)} \)
\[
\frac{1}{s} \bar{D}^{\dot{\alpha}_s} D_{(\alpha_s} \bar{D}_{\alpha(s-1)\dot{\alpha}(s-1)} = 0 \quad D^\alpha_s D^2 \Psi_{\alpha(s)\dot{\alpha}(s-1)} = 0 .
\]
As described previously one can lift constraint (65b) via a compensator in this case a real (imaginary) bosonic superfield \( V_{\alpha(s-1)\dot{\alpha}(s-1)} \) is required, with a transformation law

\[
\delta V_{\alpha(s-1)\dot{\alpha}(s-1)} \propto D^{\alpha s} L_{\alpha(s)\dot{\alpha}(s-1)} \pm \bar{D}^{\dot{\alpha} s} \bar{L}_{\dot{\alpha}(s-1)\alpha(s)} .
\]  

(68)

The compensator will modify the right hand side of (67) accordingly so they remain invariant under the full symmetry without constraint (65b). This non-geometric unconstrained formulation gives precisely the integer superspin description of [62, 65].

However, for the special case of \( s = 1 \) there is an alternative constraint that one can impose. In that case, we find:

\[
\delta \left( \bar{D}^{\dot{\alpha}} \Delta^{(0)}_{\alpha\dot{\alpha}} \right) = -D_{\alpha} D^2 \bar{K} - D^2 D_{\alpha} K
\]

\[
= -\frac{1}{2} D_{\alpha} \bar{D}^{\dot{\alpha}} \delta \Psi_{\dot{\alpha}} + \frac{1}{2} D^2 \bar{D}^{\dot{\alpha}} D_{\alpha} \bar{\Lambda}_{\dot{\alpha}} - D^2 D_{\alpha} K ,
\]  

(69a)

\[
\delta \left( \bar{D}^{\dot{\alpha}} \Delta^{(0)}_{\alpha\dot{\alpha}} \right) = -2 \bar{D}^2 D_{\alpha} K ,
\]

(69b)

\[
\delta \left( D^3 \Gamma^{(0)}_{\beta\alpha} \right) = \frac{3}{2} D^2 \bar{D}^2 \Lambda_{\alpha} .
\]  

(69c)

Therefore, if we consider the quantity

\[
\delta \left( \bar{D}^{\dot{\alpha}} \Delta^{(0)}_{\alpha\dot{\alpha}} \pm \frac{1}{2} \bar{D}^{\dot{\alpha}} \Delta^{(0)}_{a\dot{\alpha}} \pm \frac{1}{3} D^3 \Gamma^{(0)}_{\beta\alpha} \right) = -\frac{1}{2} D_{\alpha} \bar{D}^{\dot{\alpha}} \delta \Psi_{\dot{\alpha}} - D^2 D_{\alpha} \left\{ \bar{K} \pm K \right\}
\]

\[
+ \frac{1}{2} \left\{ D^2 \bar{D}^{\dot{\alpha}} D_{\alpha} \bar{\Lambda}_{\dot{\alpha}} \pm D^2 D^2 \Lambda_{\alpha} \right\}
\]  

(70)

then under the constraints

\[
\bar{K} \pm K = 0 \Rightarrow K = \begin{cases} i L , & \text{for } + \\ L , & \text{for } - \end{cases} , \quad L = \bar{L} ,
\]  

(71a)

\[
D^2 \bar{D}^{\dot{\alpha}} D_{\alpha} \bar{\Lambda}_{\dot{\alpha}} \pm D^2 D^2 \Lambda_{\alpha} = 0 \Rightarrow \Lambda_{\alpha} = \begin{cases} D_{\alpha} \Lambda , & \text{for } + \\ i D_{\alpha} \Lambda , & \text{for } - \end{cases} , \quad \Lambda = \bar{\Lambda}
\]  

(71b)

we get the invariant equation of motion:

\[
\mathcal{E}_{\alpha(s)\dot{\alpha}(s-1)} \propto \bar{D}^{\dot{\alpha}} \Delta^{(0)}_{\alpha\dot{\alpha}} \pm \frac{1}{2} \bar{D}^{\dot{\alpha}} \Delta^{(0)}_{a\dot{\alpha}} \pm \frac{1}{3} D^3 \Gamma^{(0)}_{\beta\alpha} + \frac{1}{2} D_{\alpha} \bar{D}^{\dot{\alpha}} \bar{\Psi}_{\dot{\alpha}} .
\]  

(72)

This gives an alternative free equation of motion for \( \Psi_{\alpha} \)

\[
\bar{D}^{\dot{\alpha}} D_{\alpha} \bar{\Psi}_{\dot{\alpha}} \pm \bar{D}^2 \Psi_{\alpha} + \frac{1}{2} D_{\alpha} \bar{D}^{\dot{\alpha}} \bar{\Psi}_{\dot{\alpha}} \pm \frac{1}{2} D^2 \Psi_{\alpha} = 0 .
\]  

(73)

These constraints (71) as indicated can be solved explicitly in superspace in terms of new uncon-
strained parameters $L, \Lambda$ thus making this description automatically unconstrained without the need of a compensator. The equation of motion for superfield $\Psi_\alpha$ corresponds to the description in [93].

7 Conclusions and Discussion

The description of free manifestly supersymmetric higher spin supermultiplets and some of their interactions have been constructed in literature in a not manifestly geometrical manner. This is because the superspace action principle or equations of motion have been determined on the base of some ansatz and gauge invariance had to be checked. Also, there is no obvious way of rewriting them in terms of geometrical objects like connections and curvatures.

In this work we focus towards a geometrical reformulation of these theories. We find that there exist an underlying geometrical structure which allows a more geometrical description of free higher spin supermultiplets. Specifically, we show that there is a notion of a higher spin superconnection and we find it seating on top of a hierarchy of superconnection-like objects that can be defined recursively by the action of supersymmetric covariant derivatives. This structure is the superspace manifestation of the de Wit - Freedman hierarchy, since the components of the superspace hierarchy include the de Wit Freedman higher spin connections. For half-integer superspin supermultiplets, we find a $(s+1)$-hierarchy of superconnections but for integer superspin supermultiplets, where the highest propagating spin is a fermion, we find two independent hierarchies, one has $s$ members and the other one has $s + 1$ members.

The top superconnection allows the definition of a corresponding higher spin supercurvature tensor which involves higher derivatives and matches the known higher spin superfield strength. This is the only gauge invariant object, therefore the extraction of the standard free superspace equations of motion is unclear. A simple answer is enforcing appropriate constraints on the various gauge parameters in order to decouple the higher derivative members of the hierarchy. These constraints are generated by the structure of the superconnections and unlike the non-supersymmetric case, we find many different classes of constraints that lead to different but on-shell equivalent descriptions of the same physical system.

In the constrained formulation, we find that in a few cases due to the nature of the constraint the equations of motion are expressed purely in terms of the superconnections, whereas in other cases this is not possible and terms that depend on the bare gauge superfield had to be added. In this sense we can label theories as geometrical and non-geometrical. We find that for all possible descriptions of $(s+1, s+1/2)$ supermultiplet only one of them is geometrical and the same holds true for the descriptions of $(s+1, s+1/2)$ supermultiplet.

Finding an unconstrained formulation would be desirable. In this work and in order to make contact with known results we use the non-geometrical method of compensators. These are gauge superfields unrelated to the superconnections or the supercurvature tensors, introduced by hand with a transformation law proportional to the constraints, allowing us to lift the constraints. By exploring all different possibilities of constraints as generated by the superconnections and introducing the corresponding compensators, we reproduce all known descriptions of lower and higher spin gauge theories in 4D Minkowski. However, we find that one of the constrained description of the $(s+1, s+1/2)$ supermultiplet can not have an unconstrained description, at least based on the compensator.
approach, but at the constrained formulation it is consistent. This is a new description which generalize the known new-minimal and new-new minimal descriptions of linearized supergravity to higher spins.

We hope that this geometrical structure suggested by free higher spin supermultiplets can play a role in describing consistent and non-trivial interactions in superspace. For example for non-supersymmetric theories the de Wit-Freedman connections can be related to the extra, auxiliary higher spin connections appearing in the frame-like description. These fields are required in the construction of several types of interactions. It is also possible that for an interacting theory such superconnections acquire their own dynamics and only through the enforcement of several ‘torsion’-like constraints are expressed in terms of the derivatives of the gauge superfield as we have done here. In addition we would like to investigate whether alternative and more geometrical unconstrained formulations can exist in superspace. For non-supersymmetric theories such unconstrained formulations exist by relaxing locality or exploiting Poincaré lemma. It would be interesting to find if and how such formulations can exist in superspace. Also we would like to investigate whether this geometrical structure holds in AdS superspace. For non-supersymmetric theories this has been demonstrated in [94].

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