Generalized Broughton polynomials and characteristic varieties

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Abstract

We introduce a family of generalized Broughton polynomials and compute the characteristic varieties of complement of a curve arrangement defined by fibers of some generalized Broughton polynomials.

1 Introduction

In [Br] Broughton considered the polynomial

\[ f(x, y) = x(xy - 1). \]

The associated function \( f : \mathbb{C}^2 \to \mathbb{C} \) has no critical value, but the fiber \( f^{-1}(0) \) is not diffeomorphic to the generic one. This is explained by the existence of the so-called "critical value at infinity", see [HI], [Br], [DI].

In the paper [Z] Zahid introduced a family of polynomials:

\[ f_{p,q}(x, y) = x^p[xy(x + 2) \cdots (x + q) - 1], \]

which are called generalized Broughton polynomials, where \( p \geq 1 \) and \( q \geq 1 \) are integer number, with the convention

\[ f_{p,1} = x^p(xy - 1). \]

By computing the characteristic variety \( \mathcal{V}_1(M) \), where

\[ M = \mathbb{C}^2 \setminus (C_0 \cup C_1) \]

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is a complement of a curve arrangement defined by a component of the 0–fiber:

\[ C_0 = \begin{cases} \{ xy(x + 2) \cdots (x + q) - 1 = 0 \} & \text{if } q > 1 \\ \{ xy - 1 = 0 \} & \text{if } q = 1 \end{cases} \]

and the generic fiber of \( f_{p,q} \):

\[ C_1 = \{ f_{p,q}(x, y) = 1 \}, \]

the author obtained examples of characteristic varieties with an arbitrary number of translated components for complements of affine curve arrangements consisting of just two rational curves, see [Z].

The aim of this paper is to generalize the Zahid’s work in [Z]. More precisely, we introduce a family of generalized Broughton polynomials, which generalizes the Zahid’s one. Namely

\[ F(x, y) := p(x)(yq(x) - 1) \]

where \( p(x), q(x) \in \mathbb{C}[x] \).

Put \( f(x, y) := F(x, y) - 1 \) and \( g(x, y) := yq(x) - 1 \). We denote by \( M \) the complement

\[ M = \mathbb{C}^2 \setminus \{ f(x, y) = 0, g(x, y) = 0 \}. \]

The main result in this note shows how to compute the characteristic variety \( V_1(M) \), for all polynomials \( p(x), q(x) \) such that they have at least one common root and \( p(x) + 1, q(x) \) have not any common root.

In Section 2 we recall the definition and the basic properties of the characteristic and resonance varieties. In Section 3 we compute the characteristic variety \( V_1(M) \). In particular, we obtain examples of characteristic varieties with an arbitrary number of translated components (Theorem 3.6). This is an extension for Theorem 4.1 in [Z].

\section{Characteristic and Resonance varieties}

Let \( M \) be a smooth, irreducible, quasi-projective complex variety. The characteristic variety of \( M \) is defined by

\[ \mathbb{T}(M) := \text{Hom}(H_1(M), \mathbb{C}^*). \]

This is an algebraic group whose identity irreducible component \( \mathbb{T}(M)_1 \) is an algebraic torus \( (\mathbb{C}^*)^{b_1(M)} \). Consider the exponential mapping

\[ \exp : H^1(M, \mathbb{C}) \to H^1(M, \mathbb{C}^*) = \mathbb{T}(M) \]

induced by the usual exponential function \( \exp : \mathbb{C} \to \mathbb{C}^* \). Clearly

\[ \exp(H^1(M, \mathbb{C})) = \mathbb{T}(M)_1. \]
The characteristic varieties of $M$ are the jumping loci for the first cohomology of $M$, with coefficients in rank one local systems:

$$\mathcal{V}^1_k(M) = \{ \rho \in T(M) : \dim H^i(M, \mathcal{L}_\rho) \geq k \}.$$ 

When $i = 1$, we use the simpler notation $\mathcal{V}^1_k(M) = \mathcal{V}^1_k(M)$.

Fundamental results on the structure of the cohomology support loci for local systems on quasi-projective algebraic varieties were obtained by Beauville [Be], Green and Lazarsfeld [GL], Simpson [S] (for the proper case), and Arapura [A] (for the quasi-projective case and first characteristic varieties $\mathcal{V}^1_1(M)$).

**Theorem 2.1.** The strictly positive dimensional irreducible components of the first characteristic variety $\mathcal{V}^1_1(M)$ are translated subtori in $T(M)$ by elements of finite order. When $M$ is proper, all the components of $\mathcal{V}^i_k(M)$ are translated subtori in $T(M)$ by elements of finite order.

The strictly positive dimensional irreducible components of the first characteristic variety $\mathcal{V}^1_1(M)$ are described as follows.

**Theorem 2.2.** ([Be], [A]) Let $W$ be a $d$-dimensional irreducible component of $\mathcal{V}^1_1(M)$, $d > 0$. Then, there is a regular morphism $f : M \to S$ onto a smooth curve $S$ with $h_1(S) = d$ such that the generic fiber $F$ of $f$ is connected, and a torsion character $\rho \in T(M)$ such that the composition

$$\pi_1(F) \to \pi_1(M) \to \mathbb{C}^*$$

is trivial and $W = \rho \cdot f^*(\mathbb{T}(S))$.

**Remark 2.3.** When $M$ is a hypersurface complement in $\mathbb{P}^n$, the curve $S$ in Theorem 2.2 above is obtained from $\mathbb{C}$ by deleting $d$ points, see [D3], Theorem 1.11.

If we fix a regular mapping $f : M \to S$ as above, the number of irreducible components $W = \rho \cdot f^*(\mathbb{T}(S))$ obtained by varying the torsion character $\rho$ is given by the following.

**Theorem 2.4.** ([D3]) For a given regular map $f : M \to S$ as above, the associated irreducible components $W = \rho \cdot f^*(\mathbb{T}(S))$ are parametrized by the Pontrjagin dual $\hat{T}(f) = \text{Hom}(T(f); \mathbb{C}^*)$ of the finite abelian group

$$T(f) = \frac{\text{ker}\{f^* : H_1(M) \to H_1(S)\}}{\text{im}\{\iota^* : H_1(F) \to H_1(M)\}}$$

if $\chi(S) < 0$ and by the non-trivial elements of this Pontrjagin dual $\hat{T}(f)$ if $\chi(S) = 0$.

The group $T(f)$ is determined as follows.

**Theorem 2.5.** ([D3]) Let $S$ be a non-proper smooth curve and $f : M \to S$ be a regular function. Then the group $T(f)$ is computed by the following

$$T(f) = \oplus_{c \in \mathcal{C}(h)} \mathbb{Z}/m_c \mathbb{Z},$$

where $m_c$ is the multiplicity of the divisor $f^{-1}(c)$ and $\mathcal{C}(f)$ is the set of bifurcation values of $f$. 3
The (first) resonance varieties of $M$ are the jumping loci for the first cohomology of the complex $H^*(H^*(M, \mathbb{C}), \alpha \wedge)$, namely

$$R_k(M) = \{ \alpha \in H^1(M, \mathbb{C}) : \dim H^1(H^*(M, \mathbb{C}), \alpha \wedge) \geq k \}.$$ 

The relation between the resonance and characteristic varieties can be summarized as follows, see [D5].

**Theorem 2.6.** Assume that $M$ is any hypersurface complement in $\mathbb{P}^n$. Then the irreducible components $E$ of the resonance variety $R_1(M)$ are linear subspaces in $H^1(M, \mathbb{C})$ and the exponential mapping (1) sends these irreducible components $E$ onto the irreducible components $W$ of $R_1(M)$ with $1 \in W$.

### 3 The Characteristic varieties $V_1(M)$

Consider from now on the complement $M = \mathbb{C}^2 \setminus C$, where $C = C_0 \cup C_1$, $C_0 = \{ g(x, y) = 0 \}$ and $C_1 = \{ f(x, y) = 0 \}$.

By the same argument as in Section 3 in [Z] we can prove the following.

**Theorem 3.1.** The integral (co)homology of the surface $M$ is torsion free and

$$b_1(M) = 2, b_2(M) = s + t,$$

where $s$ and $t$ are the numbers of roots of $q(x)$ and $p(x)q(x)$, respectively. Moreover, the cup-product

$$\cup : H^1(M) \times H^1(M) \to H^2(M)$$

is non-trivial.

Using the definition of the resonance varieties we get the following.

**Corollary 3.2.** The resonance varieties of $M$ are trivial, i.e. $R_k(M) = 0$ for any $k > 0$.

Since the resonance varieties are trivial, and $M$ is a hypersurface complement, it follows from Theorem 2.6 that the characteristic varieties $V_1(M)$ can contain only isolated points and 1-dimensional translated components. In this section we determine the latter ones.

In view of Theorem 2.2 and Remark 2.3 any such component comes from a mapping $h : M \to \mathbb{C}^*$. If we regard $h$ as a regular function on the affine variety $M$, it follows that $h$ should have the form

$$h = \frac{P(x, y)}{f^m g^n}$$

for some polynomial $P$ and some positive integers $m, n$. If $P$ is not in the multiplicative system spanned by $f$ and $g$, then $P$ vanishes at some point of $M$ and this is a contradiction. It follows that we may assume that

$$h = f^m g^n$$
for some (positive or negative) integers \( m, n \). Now, we are looking for all such mappings such that they have multiple fibers and connected generic fiber.

**Lemma 3.3.** For all integer numbers \( m > 1, n > 1 \) and \( c \in \mathbb{C} \setminus \{0\} \), then the generic fiber of the polynomial \( f^m(x, y) + cg^n(x, y) \) is connected.

We need the following fact.

**Lemma 3.4.** (D3) For any polynomial map \( P : \mathbb{C}^n \to \mathbb{C} \) the followings are equivalent:

1. The generic fiber of \( P \) is connected;
2. There do not exist polynomials \( H : \mathbb{C} \to \mathbb{C} \) and \( Q : \mathbb{C}^n \to \mathbb{C} \) such that \( \deg(H) > 1 \) and \( P = H(Q) \).

**Proof of Lemma 3.4.** Let \( \Phi : \mathbb{C}^2 \to \mathbb{C}^2 \) be given by \( \Phi(x, y) = (x, g(x, y)) \). We have

\[
f^m(x, y) + cg^n(x, y) = h \circ \Phi,
\]

where \( h(u, v) := (p(u)v - 1)^m + cv^n \).

It is easy to see that the restriction of \( \Phi \) on \( \mathbb{C}^2 \setminus A \) is a homeomorphism, where \( A = \{(u, y) : q(u) = 0, y \in \mathbb{C}\} \). Then, the generic fiber of \( f^m(x, y) + cg^n(x, y) \) is connected if and only if the generic fiber of \( h(u, v) \) is connected.

Now, we assume by contradiction that the generic fiber of \( h(u, v) \) is not connected. According to Lemma 3.4 there are polynomials \( H : \mathbb{C} \to \mathbb{C} \) and \( Q : \mathbb{C}^2 \to \mathbb{C} \) such that \( \deg(H) > 1 \) and

\[
(p(u)v - 1)^m + cv^n = H(Q(u, v)).
\]

We consider the singular locus of the polynomials in the above equality. Since \( \deg(H) > 1 \) then the singular locus of \( H(Q(u, v)) \) has dimension at least one. In particular, there are infinitely many points. However, singular points of \( h(u, v) \) are roots of the following systems.

\[
\begin{align*}
& p'(u) = 0, \\
& mp(u)(p(u)v - 1)^{m-1} + cv^{n-1} = 0
\end{align*}
\]

or

\[
\begin{align*}
& p(u) = 0, \\
& v = 0
\end{align*}
\]

It is easy to see that the above systems have only finitely many points. Contradiction.

**Lemma 3.5.** Assume that the map \( h = f^m g^n : M \to \mathbb{C}^* \) has connected generic fiber and a multiple fiber. Then \( n = 0 \) and \( m = \pm 1 \).
Proof. If \( n = 0 \) then \( m = \pm 1 \), because \( h \) has connected generic fiber. Similarly, if \( m = 0 \) then \( n = \pm 1 \). However, since \( \deg(f) > \deg(g) \), it is easy to show that the function \( g : \mathbb{C}^2 \setminus \{ fg = 0 \} \to \mathbb{C}^* \) has not any multiple fiber.

Now, we assume that \( mn \neq 0 \). Since \( M = \mathbb{C}^2 \setminus \{ fg = 0 \} \) and \( f, g \) are two irreducible polynomial then the map \( h : M \to \mathbb{C}^* \) has multiple fiber if and only if, there exist \( c \in \mathbb{C}^*, h_1 \in \mathbb{C}[x, y], h_1 \not\parallel f, h_1 \not\parallel g \) and integer numbers \( s, l, k, |s| > 1 \), such that
\[
f^m g^n = c + h_1^s f^l g^k. \tag{2}
\]
Since \( f, g, h_1 \) are pairwise relatively prime then \( ml \geq 0 \) and \( nk \geq 0 \).

There are four cases.

a) \( m, l, n, k \geq 0 \): This implies that \( l = k = 0 \) and the generic fiber of \( h \) has at least \( |s| > 1 \) connected components. Contradiction.

b) \( m, l, n, k \leq 0 \): By dividing two sides of the equality (2) by the lowest powers of \( f \) and \( g \), one can prove that \( m = l \) and \( n = k \). It means
\[
(f^m g^n)^{-1} = \frac{1}{c}(1 - h_1^s).
\]
So the generic fiber of \( f^m g^n \) is not connected.

c) \( m, l \geq 0 \) and \( n, k \leq 0 \): Similarly, we get \( l = 0 \) and \( n = k \). Hence \( f^m = cg^{-n} + h_1^s \). But, this implies that the generic fiber of the polynomial \( f^m - cg^{-n} \) is not connected, contradicts to Lemma 3.3.

d) \( m, l \leq 0 \) and \( n, k \geq 0 \): By the same argument, we also obtain the contradiction. \( \square \)

The main result in this paper is the following.

**Theorem 3.6.** Let \( p(x) \) and \( q(x) \in \mathbb{C}[x] \) be two polynomials such that they have at least one common root and \( p(x) + 1, q(x) \) have no common root. Then, if there exist an integer number \( s > 1 \) and a polynomial \( p_1 \in \mathbb{C}[x] \) such that
\[
p(x) = p_1(x)^s,
\]
the strictly positive dimensional components of \( V_1(M) \) are the translated 1-dimensional sub-tori
\[
W_j = \epsilon_j \times \mathbb{C}^*,
\]
where \( d \) is the maximum of the exponent \( s \) above and \( \epsilon_j = \exp(2\pi ij/d) \) for \( j = 1, 2, \ldots, d - 1 \). Moreover, for a local system \( \mathcal{L} \in W_j \) one has \( \dim H^1(M, \mathcal{L}) \geq 1 \) and equality holds with finitely many exceptions.

Otherwise, there do not exist strictly positive dimensional components of \( V_1(M) \).

**Proof.** According to Theorem 2.2 and Remark 2.3 any translated positive dimensional component of \( V_1(M) \) comes from a map \( h : M \to \mathbb{C}^* \) which has connected generic fiber.

According to Lemma 3.5, the only morphisms associated to strictly positive dimensional components of \( V_1(M) \) are \( f : M \to \mathbb{C}^* \) and \( f^{-1} : M \to \mathbb{C}^* \).
\( \mathbb{C}^*, z \mapsto f(z)^{-1} \), but they give the same associated component of \( V_1(M) \). Thus all translated positive dimensional components of \( V_1(M) \) are associated to the map \( f : M \rightarrow \mathbb{C}^* \).

On the other hand, it is easy to see that the only possibly multiple fiber of \( f \) is \( f^{-1}(-1) \). Hence, according to Theorem 2.5, if \( p(x) \) is not a power of a polynomial then \( T(f) = 0 \) and there does not exist strictly positive dimensional components of \( V_1(M) \); unless \( T(f) = \mathbb{Z}/d\mathbb{Z} \), where

\[ d = \max\{ s \in \mathbb{N} : p(x) = p_1(x)^s, p_1 \in \mathbb{C}[x] \}. \]

We now consider the later case. It is deduced from Theorem 2.6 that there are exactly \( d - 1 \) associated 1-dimensional translated components. If we identify \( T(M) = \mathbb{C}^* \) by associating to a local system \( L \in T(M) \) the two monodromies \((\lambda_0, \lambda_1)\) about the curves \( C_0 \) and \( C_1 \), and in a similar way \( T(\mathbb{C}^*) = \mathbb{C}^* \), then the induced morphism

\[ f^* : T(\mathbb{C}^*) \rightarrow T(M) \]

is just \( \lambda \mapsto (1, \lambda) \).

With these identifications, the above \( d - 1 \) associated 1-dimensional translated components of \( V_1(M) \) are given by \( W_j = \epsilon_j \times \mathbb{C}^* \), where \( \epsilon_j = \exp(2\pi ij/d) \) for \( j = 1, 2, \ldots, d - 1 \).

The inequality on dimension of cohomology group of \( M \) is the direct consequence of Corollary 5.9 in [D3].

**References**

[A] Arapura, D., *Geometry of cohomology support loci for local systems. I*, J. Algebraic Geom., 6 (1997), no. 3, 563-597.

[Be] Beauville, A., *Annulation du \( H^1 \) pour les fibrés en droites plats*, in: Complex algebraic varieties (Bayreuth, 1990), 1-15, Lecture Notes in Math., vol. 1507, Springer, Berlin, 1992.

[Br] Broughton, S. A., *Milnor numbers and the topology of polynomial hypersurfaces*, Invent. Math., 92 (1988), no. 2, 217-241.

[D1] Dimca, A., *Singularities and Topology of Hypersurfaces*, Universitext, Springer, 1992.

[D2] Dimca, A., *Sheaves in Topology*, Universitext, Springer-Verlag, 2004.

[D3] Dimca, A., *Characteristic varieties and constructible sheaves*, Rend. Lincei Mat. Appl., 18 (2007), no. 4, 365-389.

[D4] Dimca, A., *On the irreducible components of characteristic varieties*, An. Stiin. Univ. "Ovidius" Constanta Ser. Mat. 15 (2007), no. 1, 6773.

[D5] Dimca, A., Papadima, S. and Suciu, A., *Topology and geometry of cohomology jump loci*, Duke Math. J., 148 (2009), no. 3, 405-457.
[GL] Green, M. and Lazarsfeld, R., *Higher obstructions to deforming cohomology groups of line bundles*, J. Amer. Math. Soc., 4 (1991), no. 1, 87-103.

[HL] Hà, H. V. and Lê, D. T., *Sur la topologie des polynomes complexes*, Acta Math. Vietnamica, 9 (1984), 21-32.

[S] Simpson, C., *Subspaces of moduli spaces of rank one local systems*, Ann. Sci. Ecole Norm. Sup. 26(1993), no. 3, 361401.

[Z] Zahid, R., *Broughton polynomials and characteristic varieties*, Studia Sci. Math. Hungarica, 47 (2010), no.2, 214-222.