THE SHANNON-MCMILLAN-BREIMAN THEOREM
BEYOND AMENABLE GROUPS

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Abstract. We introduce a new isomorphism-invariant notion of entropy for measure preserving actions of arbitrary countable groups on probability spaces, which we call cocycle entropy. We develop methods to show that cocycle entropy satisfies many of the properties of classical amenable entropy theory, but applies in much greater generality to actions of non-amenable groups. One key ingredient in our approach is a proof of a subadditive convergence principle which is valid for measure-preserving amenable equivalence relations, going beyond the Ornstein-Weiss Lemma for amenable groups.

For a large class of countable groups, which may in fact include all of them, we prove the Shannon-McMillan-Breiman pointwise convergence theorem for cocycle entropy in their measure-preserving actions.

We also compare cocycle entropy to Rokhlin entropy, and using an important recent result of Seward we show that they coincide for free, ergodic actions of any countable group in the class. Finally, we use the example of the free group to demonstrate the geometric significance of the entropy equipartition property implied by the Shannon-McMillan-Breiman theorem.

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1. Introduction

Classical amenable entropy theory. The notion of entropy as an isomorphism invariant of dynamical systems originated in the work of Kolmogorov and Sinai, who defined entropy for probability measure preserving (p.m.p.) actions of $\mathbb{Z}$ and studied its basic properties. Entropy theory has been developed extensively and has played a major role in the theory of dynamical systems ever since. One of the main themes of development has been the program of extending entropy theory to the class of dynamical systems defined by p.m.p. actions of amenable groups. In particular, Owls settled the problem of classifying Bernoulli shifts for countable amenable groups by the value of the entropy of the underlying dynamical system. A central goal of classical entropy theory has been “entropy equipartition” results, established by Shannon, McMillan, and Breiman and culminating in the Shannon-McMillan-Breiman pointwise convergence theorem, a major achievement of the classical theory. In the more general context of amenable groups, the Shannon-McMillan mean convergence theorem for countable amenable groups has been established by Kieffer, and a subadditive ergodic theorem for entropy has been established by Ollagnier. Further, pointwise convergence for countable amenable groups admitting regular Følner sequences is due to Ornstein and Weiss. The Shannon-McMillan-Breiman pointwise pointwise convergence theorem for tempered Følner satisfying a mild growth condition was established by Lindenstrauss. This latter result was extended by Weiss to the case of non-ergodic actions and without a growth assumption on the sequences in question. The proof is based on previous joint work of Ornstein and Weiss, and avoids the random tiling and covering arguments used in.

Anticipating our discussion below, let us note that our approach to entropy theory has the significant advantage that it treats amenable and non-amenable groups on an equal footing, and thus allows us to make extensive use of the methods and ideas developed in the discussion of entropy theory of amenable groups, and apply them to the setting of non-amenable group.

Entropy for actions of non-amenable groups. The problem of developing entropy theory for p.m.p. actions of non-amenable countable groups remained open for nearly half a century, but has seen remarkable ground-breaking progress in recent years. The first major breakthrough was the introduction by Bowen of the concept of sofic entropy and the developments of its main properties as an isomorphism-invariant for p.m.p. actions of sofic groups. While this is certainly a very large class of groups, it is not yet known whether it includes all countable groups or not. Bowen’s notion of sofic entropy was developed further by Kerr and Li, in particular, sofic entropy constitutes a major extension of the classical entropy theory of p.m.p. group actions, since for amenable groups, sofic entropy coincides with the classical Kolmogorov-Sinai entropy.

Another important breakthrough in entropy theory, for actions of completely arbitrary groups, has recently been obtained by Seward, who introduced the concept of Rokhlin entropy. As for the case of sofic entropy, for amenable groups this notion coincides with the classical Kolmogorov-Sinai entropy as well, a result due to Rokhlin for the classical case of $\mathbb{Z}$-actions, and to Seward and Tucker-Drob in general. Furthermore, Seward proved several remarkable results about Rokhlin entropy, including the fact that every ergodic group action with finite Rokhlin entropy admits a finite generating partition. This greatly extends the classical finite generator theorems of Rokhlin and Krieger, to actions of general countable groups. This result is material for our work since the Shannon-McMillan-Breiman theorem proposed in Theorem below is valid in the context of finite partitions. Moreover, Seward implicitly deals with another notion of entropy which we will call finitary entropy. It follows from the fundamental inequality stated in that finitary entropy and Rokhlin entropy coincide for free, ergodic p.m.p. actions of arbitrary countable groups, a result that will be crucial in our considerations below.

Main results of the present paper. Let us note that while the three notions of entropy mentioned above - sofic, Rokhlin and finitary - generalize the classical entropy of amenable groups, none of them gives rise to an entropy equipartition theorem in the form of a Shannon-McMillan
mean convergence theorem or a Shannon-McMillan-Breiman pointwise convergence theorem. The present paper is devoted to solving this problem, as follows.

- We define a new isomorphism-invariant notion of entropy for probability measure preserving (p.m.p.) actions of a very extensive class of groups, which may in fact coincide with the class of all countable infinite groups. We establish the existence of this invariant, which we call cocycle entropy, via a new general subadditive convergence theorem for actions of non-amenable groups (see Theorem 2.1).

- We show that cocycle entropy is naturally bounded from below by finitary entropy and bounded from above by Rokhlin entropy (see Theorem 2.6). It follows that for free, ergodic p.m.p. countable group actions, cocycle entropy coincides with Rokhlin entropy and finitary entropy.

- We prove an analog of the Shannon-McMillan-Breiman pointwise convergence theorem for cocycle entropy, under a suitable ergodicity assumption (see Theorem 2.7). As a consequence, an analog of the Shannon-McMillan mean convergence theorem follows (see Corollary 2.8).

Let us add the following comments.

1) In the classical theory one defines the entropy of a partition as a limit of the normalized Shannon entropy of refinements of that partition. The crucial tool in this undertaking are abstract subadditive convergence theorems. The corresponding statement for Følner sequences in amenable groups is also called the Ornstein-Weiss Lemma. In the context of discrete, amenable structures, there are multiple versions of this result in the literature, cf. e.g. [OW87, Gr99, LW00, CCK14, Po14]. We will formulate a new subadditive convergence principle and use it to establish convergence in the mean of the normalized Shannon entropy of a sequence of successive refinements of a given partition of the underlying probability space, thus establishing the existence of cocycle entropy (see Theorem 2.1).

2) The subadditive convergence theorem we establish is the first subadditive convergence principle that we are aware of for actions of non-amenable groups. It is valid for general subadditive functions and its usefulness is not restricted to convergence of measure-theoretic information functions (see Theorem 4.2). In particular, it can be used to define a notion of topological entropy for certain actions of non-amenable groups on compact metric spaces, thus raising also the possibility of a variational principle for cocycle entropy.

The method of the proof: from amenable groups to amenable equivalence relations. Given a p.m.p. action of a group $\Gamma$ on $(X, \lambda)$, a main preoccupation in ergodic theory is to establish convergence properties for averages defined by a sequence of finite subsets $F_n \subset \Gamma$. Thus ergodic theorems concern the averages $\frac{1}{|F_n|} \sum_{\gamma \in F_n} f(\gamma^{-1} x)$ for a measurable function $f$ on $X$, and the Shannon-McMillan-Breiman theorem concerns the convergence of the normalized information functions, given in integrated form by

$$\frac{1}{|F_n|} H \left( \bigvee_{\gamma \in F_n} \gamma^{-1} \mathcal{P} \right),$$

where $\mathcal{P}$ is a (finite) partition of $X$ and $H$ denotes the Shannon entropy.

An mentioned briefly above, an elaborate and very useful set of techniques was developed when the group $\Gamma$ is amenable, and the sequences $F_n$ are asymptotically invariant (often satisfying some additional properties). These techniques allow for rather complete solutions to both convergence problems in the amenable case. Remarkably, it is possible to develop a point of view that treats amenable groups and non-amenable groups on an equal footing, thus raising the very attractive possibility of leveraging the arguments of classical amenable ergodic theory to deduce analogous results in the non-amenable case. The point view in question is based on the following heuristics. Suppose that $\mathcal{R}$ is an amenable probability-measure-preserving Borel equivalence relation on a space $Y$, with countable classes. Suppose that $\alpha : \mathcal{R} \to \Gamma$ is a measurable cocycle. Let there be given a sequence of finite subset functions $F_n$ with $F_n(y) \subset [y]_{\mathcal{R}}$, which are asymptotically invariant under the relation in a suitable sense. We then obtain a collection of finite subsets of $\Gamma$, ...
depending on the parameter $y \in Y$, given by $y \mapsto \{\alpha(y, z) : z \in \mathcal{F}_n(y)\}$. We can then consider the convergence of the averages $\frac{1}{|\mathcal{F}_n(y)|} \sum_{z \in \mathcal{F}_n(y)} f(\alpha(y, z)x)$, with $f$ being a measurable function on $X$, and the convergence of the information functions, given in integrated form by

$$
\frac{1}{|\mathcal{F}_n(y)|} \int H \left( \bigvee_{z \in \mathcal{F}_n(y)} \alpha(y, z) \mathcal{P} \right),
$$

(1.1)

where $\mathcal{P}$ is a finite partition of $X$. Given this set-up, it is to be expected that the amenability of the equivalence relation $\mathcal{R}$ and the asymptotic invariance of the subset functions $\mathcal{F}_n(y)$ can be utilized using the classical arguments of amenable ergodic theory to prove convergence.

This point of view was initiated and developed in [BN13a, BN13b, BN15a, BN15b], where it was applied to prove mean and pointwise convergence theorems for averages of functions on $X$. The present paper is concerned with the application of this method to the case of convergence of information functions. For this purpose, we consider a specific kind of asymptotically invariant subset functions $\mathcal{F}_n$, namely we choose them to be an increasing sequence of subequivalence relations $\mathcal{R}_n$ of $\mathcal{R}$ consisting of finite classes, such that $\mathcal{R} = \bigcup_{n \in \mathbb{N}} \mathcal{R}_n$ (a hyperfinite exhaustion). The generality of our approach is based on the fact that every amenable p.m.p. equivalence relation admits such a sequence, a result due to Connes, Feldman and Weiss [CFW81]. A natural question that arises here is to what extent can general asymptotically invariant sequences $\mathcal{F}_n$ (which can often be defined in geometric terms, e.g. via horospheres in hyperbolic groups) be approximated by hyperfinite sequences. We plan to address this issue and the issue of establishing amenability of equivalence relations in terms of combinatorial asymptotic invariance, in a separate paper.

It is a vindication of the point of view described above that our proof of the Shannon-McMillan-Breiman theorem proceeds by adapting the intricate overall strategy and some of the ingenious arguments developed in the amenable group case by E. Lindenstrauss in [Li01], to the setting of amenable equivalence relations. As in the situation of amenable groups, there are two major ingredients for the proof. The first is an abstract covering lemma for the averaging sequences under consideration. We prove a corresponding assertion in Lemma 5.2. Using the hyperfinite structure of the equivalence relation, we are able to avoid some difficult technicalities that appear in [Li01] such as the construction of random collection of tilings and establishing control of the overlapping of different tiles. The second ingredient is a suitable pointwise ergodic theorem. For the classical setting, it was observed in [OWS83] that the pointwise ergodic theorem is the crucial dynamical tool for the proof of the Shannon-McMillan-Breiman theorem. We will use the pointwise ergodic theorem proved in [BN13b], but note that in our present setting of hyperfinite equivalence relations, the required pointwise convergence also follows in a straightforward manner from the martingale convergence theorem.

Let us turn to comment briefly on precursors to our approach in the literature. The usefulness of cocycles for problems in ergodic theory, and in particular for entropy questions, has already been observed in [BW00], where Rudolph and Weiss show that p.m.p. actions of amenable groups which have the completely positive entropy property have strong mixing properties. For the proof of the main theorem, the authors employ orbit equivalence theory, where cocycles mapping from the orbit equivalence relation of one (amenable) group into another (amenable) group appear naturally. Another interesting precursor to our approach has been developed by Danilenko [Da01] and by Danilenko and Park [DP02]. In particular, [Da01] introduced the information function (1.1) for a given amenable relation $\mathcal{R}$ and a cocycle $\alpha : \mathcal{R} \to \Gamma$. Its properties were studied in [Da01] and [DP02] when $\Gamma$ is amenable. However, the notion of entropy considered by Danilenko and Park is not suitable for our purposes, since in all the examples we will consider in this paper it assumes the value $\infty$. In [Av10], Avni studies entropy defined via cocycles arising from cross sections in locally compact amenable groups. Among other interesting results, he proves a Shannon-McMillan type theorem for the underlying notion of entropy in this case.

We note, however, that considering cocycles taking values in non-amenable groups and establishing the validity of the corresponding subadditive convergence principle, as well as of the Shannon-McMillan-Breiman theorem, have no precedents in the literature that we are aware of.
Entropy equipartition and the geometric significance of cocycle entropy. One of the most important aspects of the classical Shannon-McMillan-Breiman theorem is that it establishes an "entropy equipartition" property of the dynamical system under consideration. This property asserts that the size of a typical atom in the partition $\mathcal{F}_n = \bigvee_{\gamma \in \mathcal{F}_n} \gamma^{-1}\mathcal{P}$ is comparable to $\exp(-h(\mathcal{P})|\mathcal{F}_n|)$, where $h(\mathcal{P})$ is the Shannon entropy of the partition $\mathcal{P}$. In particular, refining $\mathcal{P}$ by the action of the elements $\gamma \in \mathcal{F}_n$ in the space $X$ eventually produces refined partitions with most atoms of roughly the same size. The analog of the Shannon-McMillan-Breiman theorem that we prove for cocycle entropy implies the analogous equipartition property, upon refining the partition $\mathcal{P}$ of $X$ by the sets $\{\alpha(y,z); z \in \mathcal{R}_n(y)\} \subset \Gamma$, and thus expresses a property which is directly connected to the dynamics of the group action on $X$.

It is a very natural problem to study what are the sets of group elements that arise in the form $\{\alpha(y,z); z \in \mathcal{R}_n(y)\}$, namely as the cocycle images of hyperfinite exhaustions $\mathcal{R}_n(y)$. We emphasize that in many geometric situations, it is possible to give a very concrete and meaningful description of such sets, and this fact constitutes one of the main advantages of the definition of cocycle entropy. We will exemplify this statement in complete detail in §8 below for actions of finitely generated free groups $\mathbb{F}_r$. In that case, choosing $\mathcal{R}$ as the horospherical (=synchronous tail) relation on the boundary $\partial \mathbb{F}_r = Y$ of the free group, and $\alpha$ the canonical cocycle on it, the equivalence classes $[\omega]_{\mathcal{R}}$ (where $\omega \in \partial \mathbb{F}_r$) constitute a combinatorial construction of the unstable leaves of the horospherical foliation. Their images $\alpha(\omega, \xi) \in \mathbb{F}_r$ for $\xi \in \mathcal{R}_n(\omega)$ under the canonical cocycle coincide with the intersection of the word metric ball $B_{2n}(e)$ in $\mathbb{F}_r$ with the horoball based at $\omega$ and passing through $e$. Thus the Shannon-McMillan-Breiman theorem asserts in this case that refining a partition of $X$ along the sequence of (the inverses of) horospherical balls of increasing radii in the group has the entropy equipartition property almost surely. Namely, the information functions of the refined partitions converge to the cocycle entropy of the partition. We remark that a similar geometric interpretation holds in much greater generality, and serves as evidence for the existence of deep connections between the theory of p.m.p. actions of a non-amenable group, and the theory of its amenable actions, particularly its actions on its Poisson boundaries. We will briefly comment further on these topics in §7 and 8 below, and plan to give a more detailed exposition elsewhere.

Organization of the present paper. The paper is organized as follows. In Section 2 we present and discuss the main results of our work in detail. Section 3 is devoted to the discussion of amenability of measured equivalence relations and to hyperfinite exhaustions. Next, we prove a subadditive convergence theorem which amounts to an integrated Ornstein-Weiss type lemma in Section 4. We then establish pointwise covering and tiling lemmas for hyperfinite exhaustions in amenable equivalence relations in Section 5. Those statements will be the crucial tools in the proof of the Shannon-McMillan-Breiman theorem. This latter assertion is proven in Section 6 along with the Shannon-McMillan $L^1$-convergence theorem. In Section 7, we describe the generality of the framework in which the Shannon-McMillan-Breiman theorem holds true. We illustrate the convergence theorems by explicating the case of the free group in Section 8.

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2. Statement of main results

In the present section, we will briefly mention some key definitions, and then state our main results. A detailed discussion of all relevant concepts will appear in subsequent sections devoted to the proofs of the main results.

Throughout the paper, we consider standard Borel equivalence relations with countable classes $\mathcal{R} \subset Y \times Y$, where $(Y, \nu)$ is a probability space. $[y] = \mathcal{R}(y) \subset Y$ denotes the $\mathcal{R}$-class of $y \in Y$. We assume that the equivalence relation preserves the probability measure $\nu$. It is further assumed that $\mathcal{R}$ is hyperfinite, or equivalently, that $\mathcal{R}$ is amenable in the sense of [CFW81]. Thus $\mathcal{R}$ can
be written as an increasing union of equivalence subrelations \( R_n \subset R \), each with finite classes, i.e.

\[
R(y) := \bigcup_{n=1}^{\infty} R_n(y), \quad \text{for } \nu\text{-almost every } y \in Y.
\]

Such a representation of \( R \) will be called a \textit{hyperfinite exhaustion}. If for every \( n \), the relations \( R_n \) are bounded, namely the size of the equivalence classes \( R_n(y) \) is essentially bounded (with a bound depending on \( n \)), we will call the representation a \textit{bounded hyperfinite exhaustion}. Let us note that hyperfinite relations always admit a bounded hyperfinite exhaustion, as will be seen below. In fact it is possible to construct a hyperfinite exhaustion satisfying \( |R_n(y)| \leq n \) almost surely, as noted in [We84] in a more general context.

2.1. Probability measure preserving actions of groups and cocycle extensions. Throughout the paper, \( \Gamma \) denotes a countable group. The collection of all finite subsets of \( \Gamma \) is denoted by \( \text{Fin}(\Gamma) \). We consider a probability measure preserving (p.m.p.) ergodic action \( \Gamma \curvearrowright (X, \lambda) \), and aim to define a notion of entropy for the action.

To that end, assume that there is an amenable p.m.p. equivalence relation \( R \) over \( (Y, \nu) \), admitting a measurable cocycle \( \alpha : R \to \Gamma \). Thus for \( \nu\text{-almost every } y \in Y \) and every \( w, u, z \in [y] = R(y) \), the cocycle identity holds:

\[
\alpha(z, u) = \alpha(z, w) \cdot \alpha(w, u).
\]

A crucial construction in our discussion is the equivalence relation, denoted \( R^X \), which is the extension of the equivalence relation \( R \) by the cocycle \( \alpha \) and the \( \Gamma \)-action on \( X \). The \textit{extended equivalence relation} \( R^X \) over \( (X \times Y, \lambda \times \nu) \) is defined by the condition

\[
((x, y), (x', y')) \in R^X \iff yRy' \text{ and } x = \alpha(y, y')x'.
\]

Assuming that the measure \( \nu \) is \( R \)-invariant, it follows that \( \lambda \times \nu \) is \( R^X \)-invariant, since the \( \Gamma \)-action on \( X \) preserves \( \lambda \). The projection map \( \pi : R^X \to R \) given by \((x, y) \to y\) is \textit{class injective}, namely injective on almost every \( R^X \)-equivalence class.

Further, it is well known that an extension of an amenable action is amenable, and thus in particular if \( R \) is amenable, so is \( R^X \). Thus, when \( R \) is hyperfinite, so is \( R^X \). But since the extension is class-injective, in fact every hyperfinite exhaustion \( (R_n^X) \) of \( R^X \), via \( R_n((x, y)) = \{(\alpha(z, y)x, z) : z \in R_n(y)\} \).

Note that if \((R_n)\) is a bounded hyperfinite exhaustion, then so is \((R_n^X)\), with the same bounds on the equivalence classes.

The cocycle \( \alpha \) is called \textit{class injective} if for \( \nu\text{-almost every } y \in Y \), we have that \( \alpha(y, z) \neq \alpha(y, w) \) whenever \( w \neq z \). In order to avoid degenerate cases, we will assume in the sequel that the cocycle under consideration is class injective.

2.2. Definition of cocycle entropy. In order to define the measure theoretic entropy of the p.m.p. \( \Gamma \)-action on \( X \), recall first the following. For a countable measurable partition \( \mathcal{P} = \{A_i : i \in I\} \) of \( X \) the Shannon entropy \( H(\mathcal{P}) \) is defined as

\[
H(\mathcal{P}) := - \sum_{A \in \mathcal{P}} \lambda(A) \log \lambda(A),
\]

where we use the convention that \( 0 \cdot \log 0 = 0 \). For two countable partitions \( \mathcal{P} \) and \( \mathcal{Q} \), their common refinement is \( \mathcal{P} \vee \mathcal{Q} := \{ P \cap Q : P \in \mathcal{P}, Q \in \mathcal{Q} \} \). For a finite set \( F \in \text{Fin}(\Gamma) \), we set

\[
\mathcal{P}^F := \bigvee_{g \in F} g^{-1}\mathcal{P},
\]

where \( g^{-1}\mathcal{P} = \{ g^{-1}A_i : i \in I \} \). If \( F \) is the empty set, then we define \( \mathcal{P}^F \) to be the trivial partition, which of course has Shannon entropy zero. Given two partitions \( \mathcal{P} \) and \( \mathcal{Q} \), \( \mathcal{Q} \) is called \textit{finer than} \( \mathcal{P} \) or a \textit{refinement of} \( \mathcal{P} \), denoted \( \mathcal{Q} \succeq \mathcal{P} \), if for every \( Q \in \mathcal{Q} \), there is some \( P \in \mathcal{P} \) such that \( Q \subseteq P \) up to \( \lambda \)-measure zero.
Now for a subset function $F : Y \to \text{Fin}(Y)$ (as always, with $F(y) \subseteq R(y)$ a.e.) and a countable partition $\mathcal{P}$ of $X$ with $H(\mathcal{P}) < \infty$, we consider the entropy function
\[
h^\mathcal{P}(F) : Y \to [0, \infty) : h^\mathcal{P}(F)(y) := H \left( \bigvee_{z \in F(y)} \alpha(z,y)^{-1} \mathcal{P} \right).
\]

We can now state our first main theorem which shows that we can attach a notion of cocycle entropy to every partition $\mathcal{P}$ with finite Shannon entropy.

**Theorem 2.1.** Given a class injective cocycle on $R$, for every countable partition $\mathcal{P}$ of $X$ with $H(\mathcal{P}) < \infty$, there is a number $h^*_R(\alpha)$ such that for every bounded hyperfinite exhaustion $(R_n)$,
\[
h^*_R(\alpha) := \lim_{n \to \infty} \int_Y h^\mathcal{P}(R_n)(y) d\nu(y).
\]

Let us highlight the crucial fact that it is part of the conclusion of Theorem 2.1 that the limit is independent of the choice of the bounded hyperfinite exhaustion.

**Remark 2.2.** As can be seen from the proof of Theorem 2.1, the assumption on the cocycle being class injective is not necessary for this convergence result. But we dispense with this additional generality in order to make sure that the values $h^*_R(\alpha)$ accurately reflect entropy theoretic information regarding the action of $\Gamma$ on $X$.

In the following, the number $h^*_R(\alpha)$ shall be called the cocycle entropy of the partition $\mathcal{P}$ for the action $\Gamma \curvearrowright (X, \lambda)$. Let us proceed to use it to define a notion of measure-theoretic entropy for the dynamical system $\Gamma \curvearrowright (X, \lambda)$ itself. For this, we must restrict ourselves to generating partitions, i.e. countable partitions $\mathcal{P}$ for $X$ such that (modulo null sets)
\[
\bigvee_{y \in \Gamma} g \mathcal{P} = B.
\]

**Definition 2.3 (Cocycle entropy).** Let $(X, \lambda)$ be a p.m.p. ergodic action of $\Gamma$. Let $\alpha : R \to \Gamma$ be a class injective cocycle defined on a hyperfinite relation. The number
\[
h^C(\Gamma \curvearrowright X) := \inf \left\{ h^*_R(\alpha) \mid \mathcal{P} \text{ countable, generating partition} \right\}
\]
is called the cocycle entropy for the action $\Gamma \curvearrowright (X, \lambda)$ and for the injective cocycle $\alpha : R \to \Gamma$.

As we shall see presently, the cocycle entropy of a free ergodic action is an intrinsic invariant, independent of the p.m.p. hyperfinite relation $R$, the bounded hyperfinite exhaustion $R_n$, and the class injective cocycle $\alpha$ used to define it. This remarkable fact is ultimately based on the important recent result established by Seward relating two completely different notions of entropy, which we now proceed to describe.

### 2.3. Cocycle entropy, finitary entropy and Rokhlin entropy

The notion of Rokhlin entropy has its origin in Rokhlin’s studies of entropy for a single transformation. It has recently been greatly generalized and studied intensively in the context of general group actions by Seward in the series of papers [Se14, Se15, Se16]. Is particular, Seward defines

**Definition 2.4 (Rokhlin entropy).** Let $\Gamma \curvearrowright (X, \lambda)$ be a p.m.p. ergodic group action. Then, the number
\[
h^\text{Rok}(\Gamma \curvearrowright X) := \inf \left\{ H(\mathcal{P}) \mid \mathcal{P} \text{ countable, generating partition} \right\}
\]
is called the Rokhlin entropy of the group action.

Seward and Tucker-Drob showed in [ST12] that for free ergodic actions of amenable groups, Rokhlin entropy coincides with the classical Kolmogorov-Sinai entropy. In a recent important breakthrough Seward [Se16] established the following upper bound for the Rokhlin entropy for every countable, generating partition $\mathcal{P}$, assuming the $\Gamma$ action on $X$ is essentially free.

\[
h^\text{Rok}(\Gamma \curvearrowright X) \leq \inf_{T \in \text{Fin}(\Gamma)} \frac{1}{|T|} H \left( \bigvee_{g \in T} g^{-1} \mathcal{P} \right).
\]
Proof. Let bounded hyperfinite exhaustion for \( R \) for every amenable p.m.p. equivalence relation. Shannon-McMillan-Breiman theorem concerns mean or pointwise almost everywhere convergence of a sequence of natural information functions set almost surely, we obtain 2.4.

Knowledge, the question of determining Rokhlin entropy for Bernoulli shifts over general countable groups is still open.

Entropy is bounded from below by sofic entropy, we obtain what we claimed. To the best of our knowledge, the question of determining Rokhlin entropy for Bernoulli shifts over general countable groups is still open.

**Theorem 2.6.** Assume that \( \Gamma \curvearrowright (X, \lambda) \) is an ergodic essentially free p.m.p. action. Then, for every amenable p.m.p. equivalence relation \( R \) over \( (Y, \nu) \) and every class injective cocycle \( \alpha : R \to \Gamma \), we obtain

\[
h^\text{fin}(\Gamma \curvearrowright X) = h^C(\Gamma \curvearrowright X, \alpha) = h^\text{Rok}(\Gamma \curvearrowright X).
\]

**Proof.** Let \( \mathcal{P} \) be a countable partition with finite Shannon entropy. Assume that \((\mathcal{R}_n)\) is any bounded hyperfinite exhaustion for \( \mathcal{R} \). Since the cocycle \( \alpha \) is class injective, and \( \mathcal{R}_n(y) \) is a finite set almost surely, we obtain

\[
\inf_{T \in Fin(\mathcal{R})} \frac{1}{|T|} H \left( \bigvee_{g \in T} g^{-1} \mathcal{P} \right) \leq \frac{h^\mathcal{P}(\mathcal{R}_n)(y)}{|\mathcal{R}_n(y)|} \leq H(\mathcal{P}),
\]

for each \( n \in \mathbb{N} \) and for \( \nu \)-almost every \( y \in Y \). Note that these inequalities remain valid even if \( H(\mathcal{P}) = \infty \). In the case that there exist generating partitions with finite Shannon entropy, we integrate over \( Y \) and pass to the limit as \( n \to \infty \). By Theorem 2.4, we conclude that \( h^\text{fin}(\Gamma \curvearrowright X) \leq h^\mathcal{P}(\alpha) \leq H(\mathcal{P}) \) independently of the cocycle \( \alpha \). Now taking the infimum over all generating partitions, since by the discussion above \( h^\text{Rok}(\Gamma \curvearrowright X) = h^\text{Rok}(\Gamma \curvearrowright X) \), the above inequality immediately implies equality of all three notions of entropy.

To conclude this subsection, let us comment briefly on the relation of the concept of entropy described above to Bowen’s sofic entropy introduced in [Bo10c]. First, we note the fact that for every sofic approximation \( \Sigma \) of a sofic group \( \Gamma \), the corresponding sofic entropy \( h^\text{sof}(\Gamma \curvearrowright X) \) is bounded above by Rohklin entropy, cf. [Bo10c, Ke13]. It follows from Theorem 2.4 that for essentially free actions, the same is true if we replace Rohklin entropy by cocycle entropy. Consequently, we can easily conclude that for sofic groups \( \Gamma \) and a Bernoulli shift \( \Gamma \curvearrowright (A^\Gamma, \otimes, p) \) with \( A \) being a countable alphabet and \( p = (p_a) \) denoting a probability vector with \( H(p) := -\sum_{a \in A} p_a \log p_a < \infty \), the cocycle entropy is equal to \( H(p) \). Indeed, the cocycle entropy is less than or equal to \( H(p) \) since the sets \( \{ [a] = i, i \in A \} \) form a generating partition of \( A^\Gamma \) with Shannon entropy \( H(p) \). Bowen shown that the sofic entropy for Bernoulli shifts is equal to \( H(p) \), cf. Theorem 8.1 in [Bo10c] (see also the work of Kerr [Ke13], Theorem 4.2). Hence, since cocycle entropy is bounded from below by sofic entropy, we obtain what we claimed. To the best of our knowledge, the question of determining Rohklin entropy for Bernoulli shifts over general countable groups is still open.

**2.4. The Shannon-McMillan-Breiman theorem.** The generalization that we propose of the Shannon-McMillan-Breiman theorem concerns mean or pointwise almost everywhere convergence of a sequence of natural information functions on \( X \). For a given countable partition \( \mathcal{P} \) with \( H(\mathcal{P}) < \infty \), we set

\[
\mathcal{J}(\mathcal{P})(x) := -\log \lambda(\mathcal{P}(x)),
\]

(In fact, Theorem 1.5 in [Bo10b] shows a stronger statement involving entropy conditioned on \( \Gamma \)-invariant \( \sigma \)-algebras.)
where we define $\mathcal{P}(x)$ to be the unique set $A \in \mathcal{P}$ containing the point $x$ (namely the $\mathcal{P}$-name of $x$). Note that by definition, we have

$$H(\mathcal{P}) = \int_X J(\mathcal{P})(x) \, d\lambda(x).$$

Given an ergodic p.m.p. group action $\Gamma \curvearrowright (X, \lambda)$, and a class-injective cocycle $\alpha : \mathcal{R} \to \Gamma$ consider the extended relation $\mathcal{R}^X$ over $X \times Y$. For the proof of the Shannon-McMillan-Breiman theorem, we will assume that the extended relation $\mathcal{R}^X$ is ergodic. Though being a non-trivial assumption on the equivalence relation and the action under consideration, ergodicity of the extension is satisfied in many situations. One important example is that of arbitrary ergodic actions of irreducible lattices in connected semisimple Lie groups with finite center. In general, a sufficient condition in order to guarantee ergodicity of $\mathcal{R}^X$ is weak mixing of the relation $\mathcal{R}$ over $(Y, \nu)$, as defined in [BN13a]. For a more detailed elaboration of these issues, we refer the reader to Section 7.

We are now able to state the Shannon-McMillan-Breiman pointwise convergence theorem in our context.

**Theorem 2.7.** Let $\Gamma \curvearrowright (X, \lambda)$ be an ergodic essentially free p.m.p. action. Assume that $\mathcal{R}$ is an amenable, p.m.p. equivalence relation, and $\alpha : \mathcal{R} \to \Gamma$ is a class injective cocycle, such that the extended relation $\mathcal{R}^X$ is ergodic. Then, for every bounded hyperfinite exhaustion $(\mathcal{R}_n)$ satisfying the growth condition

$$\lim_{n \to \infty} \inf_y |\mathcal{R}_n(y)|/\log n = \infty$$

the information functions satisfy the following convergence property. Given a finite partition $\mathcal{P}$ of $X$, for $(\lambda \times \nu)$-almost every $(x, y) \in X \times Y$,

$$\lim_{n \to \infty} \frac{J(\mathcal{P}^{\mathcal{R}_n}(y))(x)}{|\mathcal{R}_n(y)|} = h^*_p(\alpha),$$

where $h^*_p(\alpha)$ is the cocycle entropy of the partition $\mathcal{P}$.

Let us point out that the growth condition on the $(\mathcal{F}_n)$ is very mild, and in fact one can always find bounded hyperfinite exhaustions which satisfy it. To see this, recall that any two ergodic p.m.p. actions of amenable groups are orbit equivalent. This fact goes back to Dye [D59] for a pair of (ergodic) $\mathbb{Z}$-actions and was stated in full generality in [OW80]. For a survey on the topic of orbit-equivalence, see also [Ga00]. Now in [CFW81], it was shown that any amenable equivalence relation is hyperfinite and any hyperfinite relation is generated by the action of a single transformation. This action is orbit equivalent to the standard odometer action, and we can transfer the hyperfinite exhaustion of the odometer to the underlying equivalence relation (using the orbit equivalence). It follows that bounded hyperfinite exhaustions satisfying the growth condition required in Theorem 2.7 always exist.

We note that the analogous growth condition for (tempered) Folner sequences appears also in E. Lindenstrauss’ proof of the Shannon-McMillan-Breiman theorem for amenable groups, see Theorem 1.3 in [L01]. In the survey paper [W03] Weiss gives a deterministic combinatorial proof of the Shannon-McMillan-Breiman theorem based on previous joint work of Ornstein and Weiss. This proof is valid for general tempered Folner sequences even without an additional growth condition and also for non-ergodic actions, see Theorem 6.2 in [W03].

As a corollary of Theorem 2.7, we obtain the corresponding Shannon-McMillan theorem, asserting convergence of the information functions in $L^1$.

**Corollary 2.8.** Convergence in Theorem 2.7 holds for $\nu$-almost every $y \in Y$ in the $L^1(X, \lambda)$-norm, and also in the $L^1(X \times Y, \lambda \times \nu)$-norm.

We do not know whether one can also expect convergence in $L^1(Y, \nu)$ for $\lambda$-almost every $x \in X$.

3. Amenable equivalence relations

In this section, we discuss measured Borel equivalence relations which are amenable in the sense of Connes, Feldman and Weiss [CFW81]. This condition was shown in that paper to be equivalent to hyperfiniteness.
3.1. Measurable equivalence relations. Consider a Borel measurable equivalence relation $\mathcal{R}$ defined over a standard Borel probability space $(Y, \mathcal{B}(Y), \nu)$, namely $\mathcal{R}$ is a Borel measurable subset of $Y \times Y$ with the properties

- $(y, y) \in \mathcal{R}$ for all $y \in Y$,
- if $(y, z) \in \mathcal{R}$, then $(z, y) \in \mathcal{R}$ for all $y, z \in Y$,
- if $(y, z) \in \mathcal{R}$ and $(z, w) \in \mathcal{R}$, then $(y, w) \in \mathcal{R}$ for all $y, z, w \in Y$.

Two points $y$ and $z$ are called $\mathcal{R}$-equivalent points if $(y, z) \in \mathcal{R}$, and the equivalence class is denoted $\mathcal{R}(y) = [y] = [y]_\mathcal{R}$. Each $y \in Y$ determines a left and a right fiber in $\mathcal{R}$, given by $\mathcal{R}_y = \{(y, z) ; z \mathcal{R} y\} \subset \mathcal{R}$ and $\mathcal{R}_y = \{(z, y) ; z \mathcal{R} y\} \subset \mathcal{R}$. We will always assume that for almost every $y$, the fiber $\mathcal{R}^y$ (and hence also $\mathcal{R}_y$) is countable. $\mathcal{c}^y$ will denote the counting measure on $\mathcal{R}^y$, and $c_y$ the counting measure on $\mathcal{R}_y$. Integrating the counting measures over $Y$, we obtain two $\sigma$-finite (but in general not finite) measures on $\mathcal{R}$, namely $\tilde{\nu}_i = \int_y c^y dv(y)$ and $\tilde{\nu}_r = \int_y c^y dv(y)$. The measure $\nu$ on $Y$ is called $\mathcal{R}$-non-singular if these two measures are equivalent. Note that $(\mathcal{R}, \tilde{\nu}_i)$ is a standard Borel space, and $\pi_t : \mathcal{R} \rightarrow Y$ given by $\pi_t(y, z) = y$ is a measurable factor map, and similarly for $\tilde{\nu}_r$ and $\pi_r$. Note that under the coordinate projection $\pi_t : \mathcal{R} \rightarrow Y$, the integral above expresses the measure disintegration of $\tilde{\nu}$ with respect to $\nu$. If $\tilde{\nu}_i = \tilde{\nu}_r$, then we denote it by $\tilde{\nu}$, and then $\tilde{\nu}$ as well as $\nu$ are called $\mathcal{R}$-invariant, and $\mathcal{R}$ is called a probability-measure-preserving (p.m.p.) equivalence relation. This is the only case we will consider below.

An inner automorphism of the relation $\mathcal{R}$ is a measurable mapping $\phi : Y \rightarrow Y$ which is almost surely bijective with measurable inverse and with its graph $\text{gr}(\phi)$ being contained in $\mathcal{R}$. The collection of all inner automorphisms gives rise to a group $\text{Aut}(\mathcal{R}) = [\mathcal{R}]$, called the full group of $\mathcal{R}$. A countable subset $\Phi_0 \in \text{Aut}(\mathcal{R})$ is said to be generating (for $\mathcal{R}$) if for $\tilde{\nu}$-almost all $(y, z) \in \mathcal{R}$ there is some $\phi \in \Phi_0$ such that $z = \phi(y)$. $\Phi_0$ of course generates a countable subgroup of $\text{Aut}(\mathcal{R})$, denoted $\Phi$.

For measurable subsets $A, B \subseteq Y$ of positive measure, we say that $\psi$ is a partial transformation if $\psi : A \rightarrow B$ is measurable, essentially bijective with measurable inverse and again, $\text{gr}(\psi) \subseteq \mathcal{R}$. The space $\text{Fin}(Y)$ of finite subsets of a Borel space $Y$ is a Borel space in a natural way, using the obvious Borel structure on $\mathcal{U}_{n \in \mathbb{N}} Y^n / \text{Sym}(n)$. Measurable mappings of the form $F : Y \rightarrow \text{Fin}(Y)$ satisfying that for almost every $y$, the set $F(y)$ consists of finitely many points equivalent to $y$, are called subset functions of $\mathcal{R}$. The possibility that $F(y) = \phi$ is the empty set is allowed. We consider two subset functions $\mathcal{T}, S$ to be equal if the set $\{y \mid S(y) \neq \mathcal{T}(y)\}$ has zero measure. We write $S \subseteq T$ if $S(y) \subseteq T(y)$ for $\nu$-almost every $y$. Subset functions can be composed with each other, inverted, and subtracted from each other. We refer to [BN13a] and [BN13b] for a full discussion, and recall here the following definitions.

\[
S \circ \mathcal{T}(y) := \bigcup_{z \in \mathcal{T}(y)} S(z),
\]

\[
\mathcal{T}^{-1}(y) := \{z \in [y] \mid y \in \mathcal{T}(z)\},
\]

\[
(S \setminus \mathcal{T})(y) := S(y) \setminus \mathcal{T}(y).
\]

A finite non-empty set $D \subset \text{Aut}(\mathcal{R})$ gives rise to the subset function $D(y) = \{\phi(y) : \phi \in D\}$. Given a subset function $\mathcal{T}$ and a finite set $D \subset \text{Aut}(\mathcal{R})$, $D \circ \mathcal{T}$ is defined as above, and is given by

\[
D \circ \mathcal{T}(y) := \bigcup_{\phi \in D} \phi(\mathcal{T}(y)).
\]

We will also use the notation $D \circ \mathcal{T}$ for this expression. A subset $\mathcal{K} \subseteq \mathcal{R}$ is said to be bounded if

\[
\|\mathcal{K}\| := \text{ess-sup}_y \max \{|\mathcal{K}_y|, |\mathcal{K}^y|\} < \infty,
\]

where $\mathcal{K}_y := \{z \in [y] \mid (z, y) \in \mathcal{K}\}$ and $\mathcal{K}^y := \{z \in [y] \mid (y, z) \in \mathcal{K}\}$. Analogously, we say that a subset function $\mathcal{T}$ is bounded, if $\|\mathcal{T}\| := \text{ess-sup}_y \max \{|\mathcal{T}(y)|, |\mathcal{T}^{-1}(y)|\}$ is finite.
3.2. Hyperfiniteness and amenability. The relation $\mathcal{R}$ is called hyperfinite if there exists a sequence $(\mathcal{R}_n)$ consisting of subrelations $\mathcal{R}_n \subseteq \mathcal{R}$, where each $\mathcal{R}_n$ has finite classes, satisfying

$$\mathcal{R}_n \subseteq \mathcal{R}_{n+1}, \quad \mathcal{R} = \bigcup_{n=1}^{\infty} \mathcal{R}_n.$$ 

We refer to such a sequence as a hyperfinite exhaustion of $\mathcal{R}$. Note that each $\mathcal{R}_n$ is a subset function as defined above. If each $\mathcal{R}_n$ is a bounded subset function we will call the hyperfinite exhaustion a bounded hyperfinite exhaustion. Recall that it was proved by Connes, Feldman and Weiss [CFWS1 Thm. 10] that $\mathcal{R}$ being amenable is equivalent to $\mathcal{R}$ being hyperfinite.

Let us note that hyperfinite exhaustions are asymptotically invariant under inner automorphisms of finite rank, in the following sense (see also [BN13a, BN13b]):

**Proposition 3.1.** If $(\mathcal{R}_n)$ is a bounded hyperfinite exhaustion of the relation $\mathcal{R}$, then there exists an increasing sequence of finite subgroups $\Phi_n \subseteq \text{Aut}(\mathcal{R})$, $n \geq 1$, such that $\Phi := \bigcup_{n \geq 1} \Phi_n$ is generating for $\mathcal{R}$ and for all $n \in \mathbb{N}$ and $\phi \in \Phi_n$, the graph $\text{gr}(\phi)$ is contained in $\mathcal{R}_n$.

In particular, for every $\phi \in \Phi$, there is an $n_0 \in \mathbb{N}$ such that $\mathcal{R}_n(y) \triangle \phi(\mathcal{R}_n(y)) = \emptyset$ for $\nu$-almost every $y \in Y$ and every $n \geq n_0$.

**Proof.** Let $\mathcal{T}$ be any equivalence subrelation of $\mathcal{R}$ with finite classes of bounded size. We can divide $Y$ to finitely many measurable $\mathcal{T}$-invariant sets where the size of $\mathcal{T}(y)$ is fixed, and we can restrict $\mathcal{T}$ to one of them. Without loss of generality we can thus assume that $\mathcal{T}$ has classes of fixed size $N$ in $Y$. As is well known (see e.g. [EM77 §4], or [CFWS1 Lem. 3b]) the factor space $Y/\mathcal{T}$ consisting of equivalence classes of $\mathcal{T}$ is a standard Borel space and admits measurable sections $J_1, \ldots, J_N : Y/\mathcal{T} \to Y$ such that $\{J_i(\mathcal{T}(y)) : 1 \leq i \leq N\} = \{y\} \mathcal{T} = \mathcal{T}(y)$ for almost every $y$. Define the cyclic permutation $\sigma_{\mathcal{T}(y)}$ given by $J_1(\mathcal{T}(y)) \mapsto J_2(\mathcal{T}(y)) \mapsto \cdots J_N(\mathcal{T}(y)) \mapsto J_1(\mathcal{T}(y))$ in each class, whose cycle length is $N$. Denote by $\Phi_\mathcal{T}$ the map on $Y$ which coincides with $\sigma_{\mathcal{T}(y)}$ on each class $\mathcal{T}(y)$, and denote the cyclic group generated by $\phi_\mathcal{T}$ by $\Phi_\mathcal{T}$. Then $\phi_\mathcal{T}$ is measurable and constitutes an inner automorphism of $\mathcal{R}$ which leaves invariant almost every class of the relation $\mathcal{T}$, and the group $\Phi_\mathcal{T}$ generates the relation $\mathcal{R}$.

Applying this procedure to each of the finite bounded relations $\mathcal{R}_n$ and taking the union of the corresponding groups, the stated result follows.

We proceed to state and prove another lemma which will be useful in our considerations below. The statement of part (i) is very close to [BN13b Lem. 2.8, 2.9], but we give full details in order to eliminate the assumption of uniformity of the sets $\mathcal{R}_n$ that appears there. Part (ii) is close to [Da01 Cor. 2.3] but we will use the form stated below so again give full details.

**Lemma 3.2.** Let $\mathcal{R}$ be hyperfinite and $(\mathcal{R}_n)$ a bounded hyperfinite exhaustion.

(i) Assume that the subset function $\mathcal{D}$ is bounded. Then

$$\lim_{n \to \infty} \int_Y \frac{\lvert \mathcal{R}_n(y) \triangle \mathcal{D} \circ \mathcal{R}_n(y) \rvert}{\lvert \mathcal{R}_n(y) \rvert} \, d\nu(y) = 0.$$ 

(ii) Assume that $\mathcal{T} \subseteq \mathcal{R}$ is a bounded sub-equivalence relation. For every $\varepsilon > 0$, and for all large enough $n \in \mathbb{N}$ (depending on $\varepsilon$) there is a set $Y_n \subseteq Y$ with $\nu(Y_n) \geq 1 - \varepsilon$ such that for all $y \in Y_n$

$$\frac{\lvert \{z \in \mathcal{R}_n(y) \mid \mathcal{T}(z) \subseteq \mathcal{R}_n(y)\} \rvert}{\lvert \mathcal{R}_n(y) \rvert} > 1 - \varepsilon \cdot \lVert \mathcal{T} \rVert. \quad (3.1)$$

**Proof.** (i). First let us assume that $\mathcal{D}$ is defined by a finite set of inner automorphisms $D = \{\psi_i, 1 \leq i \leq N\} \subseteq [\mathcal{R}]$. It suffices to show that for each $1 \leq i \leq N$

$$\lim_{n \to \infty} \int_Y \frac{\lvert \mathcal{R}_n(y) \triangle \psi_i(\mathcal{R}_n(y)) \rvert}{\lvert \mathcal{R}_n(y) \rvert} \, d\nu(y) = 0.$$
Using the invariance of the measure $\nu$ and $\mathcal{R}_n(y) = \mathcal{R}_n(z)$ for $z \in \mathcal{R}_n(y)$, we compute for $\psi = \psi_i$
\[
\int_Y \frac{1}{|\mathcal{R}_n(y)|} \sum_{z \in \mathcal{R}_n(y)} |\{\psi(z)\} \setminus \mathcal{R}_n(y)| \, d\nu(y)
\]
\[
= \int_Y \sum_{z \in \mathcal{R}_n(z)} |\mathcal{R}_n(z)|^{-1} |\{\psi^{-1}(z)\} \setminus \mathcal{R}_n(z)| \, d\nu(z)
\]
\[
= \int_Y \sum_{z \in \mathcal{R}_n(z)} |\mathcal{R}_n(z)|^{-1} |\{\psi^{-1}(z)\} \setminus \mathcal{R}_n(z)| \, d\nu(z)
\]
Now since the union of the $\mathcal{R}_n$ is the full relation $\mathcal{R}$, we obtain that the integrand tends to zero pointwise almost surely as $n \to \infty$. Since all integrands are bounded by 1, we can use the dominated convergence theorem to deduce that the above integrals converges to zero as well. In the general case, for every bounded subset function $\mathcal{U}$, there exists a finite set of inner automorphisms $D$ such that $\mathcal{U}(y) \subset D(y)$ for a.e. $y$, as shown in the proof of Lemma 2.9 in [BN13a]. This establishes the validity of the assertion (i).
(ii). Fix $\varepsilon > 0$. By (i), we can choose $n(\varepsilon) \in \mathbb{N}$ such that for $n \geq n(\varepsilon)$
\[
\int_Y \frac{|\mathcal{T} \circ \mathcal{R}_n(y) \setminus \mathcal{R}_n(y)|}{|\mathcal{R}_n(y)|} \, d\nu(y) < \varepsilon^2.
\]
It is a consequence of Markov’s inequality that we can find a set $Y_n \subset Y$ with $\nu(Y_n) \geq 1 - \varepsilon$ and such that for all $y \in Y_n$
\[
\frac{|\mathcal{T} \circ \mathcal{R}_n(y) \setminus \mathcal{R}_n(y)|}{|\mathcal{R}_n(y)|} < \varepsilon.
\]
Now suppose $y \in Y_n$ and (3.1) fails to hold for some $\varepsilon > 0$. Then there are at least $\varepsilon \|\mathcal{T}\| |\mathcal{R}_n(y)|$ many elements $z \in \mathcal{R}_n(y)$ with $\mathcal{T}(z) \setminus \mathcal{R}_n(y)$ containing at least one element $\varepsilon_z$. $\mathcal{T}$ is a sub equivalence relation consisting of disjoint cells, and each cell $\mathcal{T}(z)$ contains at most $\|\mathcal{T}\|$ points. Thus there are at least $\varepsilon |\mathcal{R}_n(y)|$ many distinct such elements $\varepsilon_z$. However, this contradicts the inequality (3.2).

4. **Subadditive convergence and entropy**

The notion of subadditivity and the convergence of subadditive functions along a suitable family of sets are two key concepts in the development of ergodic theory of amenable groups. In the present section we will extend the scope of this concept and develop a natural notion of subadditivity for p.m.p. amenable equivalence relations, along bounded hyperfinite exhaustions.

This will be the crucial tool in the proof of Theorem 2.1, establishing the existence of cocycle subadditivity for p.m.p. amenable equivalence relations, along bounded hyperfinite exhaustions.

In the present section we will extend the scope of this concept and develop a natural notion of subadditive functions in the sense proposed below. Applying the subadditive convergence lemma below immediately gives the validity of Theorem 2.1.

Let us turn to state the definition of subadditive functions. Recall that a measurable subset function is a measurable map $F : Y \to \mathcal{F}(Y)$, satisfying that $F(y)$ is a finite subset of $\mathcal{R}(y)$ for almost every $y$. $F$ is bounded if $|F(y)|$ and $|F^{-1}(y)|$ are essentially bounded. Let $\mathcal{BSF}(\mathcal{R})$ denote the space of measurable bounded subset functions on the equivalence relation $\mathcal{R}$.

**Definition 4.1** (Subadditive functions). A mapping $\mathcal{F} : \mathcal{BSF}(\mathcal{R}) \to \text{Map} (Y, [0, \infty))$ is called subadditive if

- for every bounded subset function $\mathcal{A}$, the function $\mathcal{F}(\mathcal{A}) : Y \to [0, \infty)$ is measurable;
\[ \mathcal{H}(A)(y) \leq \sum_{i=1}^{m} \mathcal{H}(A_i)(z_i), \]

where \( \{z_1(y), \ldots, z_m(y)\} \subseteq \mathcal{A}(y) \) is a set of points depending measurably on \( y \).

We now prove the subadditive convergence theorem.

**Theorem 4.2.** Let \( \mathcal{R} \) be p.m.p. and hyperfinite and let \( \mathcal{H} : BSF(\mathcal{R}) \to \text{Map}(Y, [0, \infty)) \) be subadditive. Then, for all bounded hyperfinite exhaustions \( (\mathcal{R}_n) \), the following holds:

\[
\mathcal{H}^* := \inf_{\mathcal{T} \in \mathcal{R}} \int_Y \frac{\mathcal{H}(\mathcal{T})(y)}{|\mathcal{T}(y)|} \, d\nu(y) = \lim_{n \to \infty} \int_Y \frac{\mathcal{H}(\mathcal{R}_n)(y)}{|\mathcal{R}_n(y)|} \, d\nu(y),
\]

where the infimum is taken with respect to all non-trivial, bounded sub-equivalence relations \( \mathcal{T} \subseteq \mathcal{R} \).

**Proof.** Let \( (\mathcal{R}_n) \) be a non-trivial, bounded hyperfinite exhaustion and denote by \( \mathcal{T} \) an arbitrary sub-equivalence relation of \( \mathcal{R} \). It is enough to show that

\[
\limsup_{n \to \infty} \int_Y \frac{\mathcal{H}(\mathcal{R}_n)(y)}{|\mathcal{R}_n(y)|} \, d\nu(y) \leq \int_Y \frac{\mathcal{H}(\mathcal{T})(y)}{|\mathcal{T}(y)|} \, d\nu(y). \tag{4.1}
\]

To this end, fix \( \varepsilon > 0 \), as well as \( m \in \mathbb{N} \). We apply Lemma 3.2 (ii) to \( \mathcal{T} \). Hence, for large enough \( n \in \mathbb{N} \), there is \( Y_n \subseteq Y \) with \( \nu(Y_n) > 1 - \varepsilon \) and for all \( y \in Y_n \), we have \( |\mathcal{R}_n(y)| \geq (1 - \varepsilon \|\mathcal{T}\|)|\mathcal{R}_n(y)| \), where \( \mathcal{R}_n(y) := \{z \in \mathcal{R}_n(y) \mid \mathcal{T}(z) \subseteq \mathcal{R}_n(y)\} \).

\( \mathcal{R}_n(y) \) is of course a bounded subset function (possibly assuming the empty set as a value), and consider now the disjoint decomposition \( \mathcal{R}_n(y) = \mathcal{R}_n(y) \bigcup \bigcup_{i=1}^{\infty} (\mathcal{R}_n(y) \setminus \mathcal{R}_n(y)) \). Applying the subadditivity of \( \mathcal{H} \) to this decomposition, this yields for a.e. \( y \in Y \)

\[
\mathcal{H}(\mathcal{R}_n)(y) \leq \mathcal{H}(\mathcal{R}_n)(y) + \mathcal{H}(\mathcal{R}_n \setminus \mathcal{R}_n)(y). \tag{4.2}
\]

Now note that if \( z \in \mathcal{R}_n(y) \) and \( w \in \mathcal{R}_n(y) \) as well, so that \( \mathcal{R}_n(y) \) is a union of disjoint \( \mathcal{T} \)-classes. Hence, we can choose points \( z_1, \ldots, z_m \) in each of the resulting cells such that the choice of representatives is measurable in \( y \) (since an equivalence relation with finite classes admits measurable sections). Hence, \( \mathcal{R}_n(y) = \bigcup_{i=1}^{n} \mathcal{T}(z_i) \), and using the subadditivity property of \( \mathcal{H} \), along with the invariance property with respect to \( \mathcal{T} \), we get

\[
\mathcal{H}(\mathcal{R}_n)(y) \leq \sum_{z \in \mathcal{R}_n(y)} \frac{\mathcal{H}(\mathcal{T})(z)}{|\mathcal{T}(z)|}. \]

Dividing inequality 4.2 by \( |\mathcal{R}_n(y)| \), and using the boundedness of \( \mathcal{H} \) applied to the second summand, we conclude that for a.e. \( y \in Y_n \):

\[
\mathcal{H}^*_n(y) \leq \sum_{z \in \mathcal{R}_n(y)} \frac{\mathcal{H}^*_n(z)}{|\mathcal{R}_n(y)|} + C \varepsilon \|\mathcal{T}\|,
\]

where, for brevity, we have set \( \mathcal{H}^*_n(y) := \mathcal{H}(\mathcal{T})(y)/|\mathcal{T}(y)| \). By the boundedness of \( \mathcal{H} \), clearly \( \mathcal{H}^*_n(y) \leq C \) for all \( y \in Y \) and since \( \mathcal{H}^*_n \geq 0 \), as \( \nu(Y_n) < \varepsilon \), we integrate over \( Y = Y_n \bigcup (Y \setminus Y_n) \) in order to arrive at

\[
\int_Y \mathcal{H}^*_n(y) \, d\nu(y) \leq \int_Y \sum_{z \in \mathcal{R}_n(y)} \frac{\mathcal{H}^*_n(z)}{|\mathcal{R}_n(y)|} \, d\nu(y) + C (1 + \|\mathcal{T}\|) \varepsilon.
\]
Now using the \( R \)-invariance of the measure \( \nu \), along with the fact that \( R_n(y) = R_n(z) \) for \( zR_n y \), we obtain
\[
\int_Y \sum_{z \in R_n(y)} \frac{\mathcal{H}_T^+(z)}{|R_n(y)|} \, d\nu(y) = \int_Y \sum_{y \in R_n(z)} \frac{\mathcal{H}_T^+(z)}{|R_n(z)|} \, d\nu(z) = \int_Y \mathcal{H}_T^+(z) \, d\nu(z).
\]
Consequently, one gets
\[
\limsup_{n \to \infty} \int_Y \mathcal{H}_T^+(y) \, d\nu(y) \leq \int_Y \mathcal{H}_T^+(z) \, d\nu(z) + C (1 + \|T\|) \varepsilon.
\]
Letting \( \varepsilon \to 0 \), we see that the inequality (1.1) is verified.

Before proceeding with the proof of Theorem 2.1, let us recall the following standard properties of the Shannon entropy of countable partitions.

**Proposition 4.3.** Let \( (X, \lambda) \) be a p.m.p. action of \( \Gamma \). Fix \( F \in Fin(\Gamma) \) and \( g \in \Gamma \). Further, let two countable partitions \( \mathcal{P} \) and \( \mathcal{Q} \) of \( X \) be given. Then,
\[
\begin{align*}
(i) \quad & H(\mathcal{P} \lor \mathcal{Q}) \leq H(\mathcal{P}) + H(\mathcal{Q}); \\
(ii) \quad & H(\bigvee_{g \in \mathcal{F}} g^{-1} \mathcal{P}) \leq H(\mathcal{P})|\mathcal{F}|; \\
(iii) \quad & H(\mathcal{P}) \leq H(\mathcal{Q}) \text{ if } \mathcal{Q} \supseteq \mathcal{P}; \\
(iv) \quad & H(g \mathcal{P}) = H(\mathcal{P}).
\end{align*}
\]

**Proof.** See e.g. [CFS, Ch. 10, §6].

The following proposition shows that for a countable partition \( \mathcal{P} \) with \( H(\mathcal{P}) < \infty \), choosing for \( \mathcal{H} \) the Shannon entropy map \( h^P : BS\mathcal{F}(R) \to \text{Map}(\mathcal{Y}, [0, \infty)) \) as defined in Section 2, gives rise to a subadditive function.

**Proposition 4.4.** Fix a countable partition \( \mathcal{P} \) of \( (X, \lambda) \) with finite Shannon entropy \( H(\mathcal{P}) \). Then the mapping \( h^P : BS\mathcal{F}(R) \to \text{Map}(\mathcal{Y}, [0, \infty)) \) is subadditive according to Definition 4.1.

**Proof.** Recall that we have defined
\[
h^P(\mathcal{F}) : Y \to [0, \infty) \text{ via } h^P(\mathcal{F})(y) := H\left( \bigvee_{w \in \mathcal{F}(y)} \alpha(w, y)^{-1} \mathcal{P} \right).
\]

The boundedness and the subadditivity property for \( h^P \) are easily verified by properties (i) and (ii) listed in Proposition 4.3. Concerning the invariance property, fix an arbitrary bounded sub-equivalence relation \( T \subseteq R \). Let \( y, z \in Y \) be such that \( yTz \). We need to show that \( h^P(\mathcal{T})(z) = h^P(\mathcal{T})(y) \). To this end, note that by the cocycle identity, the set of group elements \( \{\alpha(w, z) \mid w \in T(z)\} \) is equal to the set \( \{\alpha(w, y) \mid w \in T(y)\} \cdot \alpha(y, z) \). Hence, the desired invariance follows from part (iv) of Proposition 4.3. It remains to show the measurability statement. To this end, let \( \mathcal{A} \) be any measurable bounded subset function. Note first that given a partition \( \mathcal{P} \), the function \( h^P(\mathcal{A})(\cdot) \) takes at most countably many different values, since \( Fin(\Gamma) \) is countable. More precisely, those values only depend on the finite collection of group elements of the form \( \{\alpha(w, y) \mid w \in \mathcal{A}(y)\} \in Fin(\Gamma) \) for \( y \in Y \). Since \( \mathcal{A} \) and the cocycle \( \alpha \) are measurable by assumption, the function \( \mathcal{H}(\mathcal{A}) : R \to [0, \infty), \) defined by \( \mathcal{H}(\mathcal{A})(y, z) := H\left( \bigvee_{w \in \mathcal{A}(y)} \alpha(w, z)^{-1} \mathcal{P} \right) \) is measurable. Using the cocycle identity \( \alpha(z, w)\alpha(w, y) = \alpha(z, y) \), we observe that \( \mathcal{H}(\mathcal{A})(y, z) = \mathcal{H}(\mathcal{A})(y, y) \) for all \( z \in [y] \), by property (iv) in Proposition 4.3. Consequently, we can write \( h^P(\mathcal{A}) := \mathcal{H}(\mathcal{A}) \circ \ell_\Delta \), where \( \ell_\Delta : Y \to R \) is the lift \( y \mapsto (y, y) \) of \( Y \) to the diagonal. Since the latter function is measurable, so is \( h(\mathcal{A}) \).

We can now complete the proof of Theorem 2.1.

**Proof of Theorem 2.1.** By Proposition 4.4, the mapping \( h^P : BS\mathcal{F}(R) \to \text{Map}(\mathcal{Y}, [0, \infty)) \) is subadditive. Hence, the convergence claimed in the theorem, as well as the fact that the limit does not depend on the hyperfinite exhaustion, both follow from Theorem 4.2 and this concludes the proof of Theorem 2.1.
As noted in the introduction, Theorem 4.2 establishes a general subadditive convergence principle, valid for all functions in the large class of subadditive set functions over equivalence relations, as described in Definition 4.1. In particular, it can be applied to prove an analog of Theorem 2.1 for a natural notion of topological entropy in the present context.

5. POINTWISE COVERING LEMMAS

We now establish pointwise decomposition results for subset functions which will provide the central tool for proving the Shannon-McMillan-Breiman theorem in the next section. Our main lemma builds on the techniques developed by Lindenstrauss for the proof of the corresponding covering lemma for tempered Følner sequences in amenable groups (see [Li01, Lem. 2.4]). However, working with hyperfinite exhaustions, we are able to avoid two difficult technicalities:

- we do not need to use an auxiliary random parameter in order to be able to choose the desired coverings with high probability;
- we are able to produce strictly disjoint coverings, and the discussion of $\delta$-disjointness (see [Li01, Lem. 2.6, 2.7]) becomes unnecessary.

Instead, we exploit the disjointness properties inherent in a sequence of nested equivalence relations.

As usual, $\mathcal{R}$ will denote a p.m.p amenable equivalence relation over $(Y, \nu)$ with $\bar{\nu}$ denoting the invariant measure on $\mathcal{R}$, and $(\mathcal{R}_n)$ will denote a bounded hyperfinite exhaustion for $\mathcal{R}$.

We start with the following elementary covering (and disjointification) lemma.

**Proposition 5.1.** Let $N, L \in \mathbb{N}_{\geq 1}$, with $N < L$ and consider an arbitrary finite sequence of subset functions $\mathcal{B}_j \subseteq \mathcal{R}_L$, $1 \leq j \leq N$. Further, for $y \in Y$, consider a collection of classes (of the relations $\mathcal{R}_n$ where $1 \leq n \leq N$) given by

$$\mathcal{S}(y) := \{ \mathcal{R}_{n(j)}(w) | w \in \mathcal{B}_j(y), 1 \leq j \leq N \}.$$

Then, for a.e. $y \in Y$, we can extract from $\mathcal{S}(y)$ a disjoint subcollection $\mathcal{G}(y)$ of classes such that

$$\prod_{C \in \mathcal{G}(y)} C \supseteq \bigcup_{j=1}^{N} \mathcal{B}_j(y), \text{ and so } \sum_{C \in \mathcal{G}(y)} |C| \geq \left| \bigcup_{j=1}^{N} \mathcal{B}_j(y) \right|.$$ 

**Proof.** Fix $y \in Y$. We note first that for $z_1, z_2 \in \mathcal{R}_L(y)$ and $1 \leq j_1, j_2 \leq N$, there are three possibilities for the inclusion relation between $\mathcal{R}_{j_1}(z_1)$ and $\mathcal{R}_{j_2}(z_2)$:

- $\mathcal{R}_{j_1}(z_1) \cap \mathcal{R}_{j_2}(z_2) = \emptyset$,
- $\mathcal{R}_{j_1}(z_1) \subseteq \mathcal{R}_{j_2}(z_2)$, or
- $\mathcal{R}_{j_1}(z_1) \supseteq \mathcal{R}_{j_2}(z_2)$.

Any collection $\mathcal{S}(y)$ of sets with the latter property has the property that the union of its constituents has a unique representation as a disjoint union of some of the constituents. To find this representation explicitly, namely to choose the subcollection $\mathcal{G}(y)$, enumerate all the elements in $\mathcal{S}(y)$ and give them distinct labels collected in an index set $\mathcal{I}$. Then run the following checking algorithm.

1. Set $\mathcal{S}^*(y) = \emptyset$, $\mathcal{I}^* = \mathcal{I}$.
2. Check an arbitrary class $C \in \mathcal{S}(y)$ with its corresponding label being contained in $\mathcal{I}^*$.
3. Given $C$, there are two possibilities:
   - (A) Either for all $1 \leq j \leq N$ and $z \in \mathcal{B}_j(y)$ such that $\mathcal{R}_{n(j)}(z) \cap C \neq \emptyset$, we have $C \supseteq \mathcal{R}_{n(j)}(z)$,
   - (B) or there is some $1 \leq j_1 \leq N$, $z_1 \in \mathcal{B}_{j_1}(y)$ such that $C \subseteq \mathcal{R}_{n(j_1)}(z_1)$ and $C \neq \mathcal{R}_{n(j_1)}(z_1)$.
4. Only in case of (A), add $C$ to the subcollection $\mathcal{S}^*(y)$. Then remove from $\mathcal{I}^*$ all labels corresponding to classes $\mathcal{R}_{n(j)}(z) \in \mathcal{S}(y)$ being contained in $C$. If the new set $\mathcal{I}^* = \emptyset$, jump to step (5), otherwise return to step (2).
5. We have $\mathcal{I}^* = \emptyset$ (meaning all classes have been checked) and we set $\mathcal{G}(y) = \mathcal{S}^*(y)$. This is the collection we aim for.
By construction, for a.e. $Y$, the elements $C \in \mathcal{S}(y)$ are pairwise disjoint. Also, we made sure that for every $1 \leq j \leq N$, every single $b \in B_j(y)$ is contained in some class $C$ taken into $\mathcal{S}(y)$. This proves the above inequality. ■

We now prove the main covering lemma, motivated by [Li01, Lemma 2.1].

**Lemma 5.2.** Fix $0 < \delta < 1$ and fix an (arbitrary) finite non-empty set $D \subset \Phi$. Then, for sufficiently large $M \in \mathbb{N}$, depending only on $D$ and $\delta$, the following property holds.

Let $T_{i,j} \subseteq R_L$ $(1 \leq i \leq M, 1 \leq j \leq N_i)$ be an array of subset functions, such that for a.e. $y$, $T_{i,j}(y) = R_n(i,j)(y)$, where $n(i,j) \leq L$. Assume that for $2 \leq i \leq M$ and every $1 \leq j \leq N_i$

$$\left| \bigcup_{k<i} D \circ (T_{k,i}^{-1} T_{i,j}) \right| \leq (1 + \delta) |T_{i,j}|$$  \hspace{2cm} (5.1)

almost surely, where $T_{k,i} := \bigcup_{j=1}^{N_k} T_{k,j}$ for $1 \leq k \leq M$.

Then, given another array $\tilde{B}_{i,j} \subseteq R_L$ for $1 \leq i \leq M$ and $1 \leq j \leq N_i$, for $\nu$-almost every $y \in Y$ there are disjoint subcollections

$$\mathcal{S}(y) \subseteq \{ T_{i,j}(w) \mid w \in B_{i,j}(y), 1 \leq i \leq M, 1 \leq j \leq N_i \}$$

such that

$$\sum_{C \in \mathcal{S}(y)} |C| \geq \left( 1 - \delta \right) \min_{1 \leq i \leq M} \left| D \circ \bigcup_{j=1}^{N_i} B_{i,j}(y) \right|.$$  \hspace{2cm} (5.2)

**Proof.** Fix a conull set $Y_0 \subseteq Y$ such that for all $y \in Y_0$ the inequality (5.1) is fulfilled. For $i = M$, apply Proposition 5.1 to the subset functions defined as $B_j \equiv B_{M,j}$, where $1 \leq j \leq N_M$. This way, by passing to another conull subset of $Y_0$, for $\nu$-a.e. $y \in Y_0$, we obtain a disjoint subcollection

$$\mathcal{S}_M(y) \subseteq \{ T_{M,j}(w) \mid 1 \leq j \leq N_M, w \in B_{M,j}(y) \},$$

with

$$\sum_{C \in \mathcal{S}_M(y)} |C| \geq \left| \bigcup_{j=1}^{N_M} B_{M,j}(y) \right|.$$  \hspace{2cm} (5.2)

Proceeding iteratively, for $i < M$, we set

$$\tilde{B}_{i,j}(y) := B_{i,j}(y) \setminus \bigcup_{k>1} T_{k,i}^{-1} \circ (\bigcup \mathcal{S}_k)(y),$$

where $1 \leq j \leq N_i$ and $\bigcup \mathcal{S}_k$ denotes the union over all sets in $\mathcal{S}_k$. Applying Proposition 5.1 again, this time to the subset functions defined as $B_j := \tilde{B}_{i,j}$, $1 \leq j \leq N_i$ gives a subcollection

$$\mathcal{S}_i(y) \subseteq \{ T_{i,j}(w) \mid 1 \leq j \leq N_i, w \in \tilde{B}_{i,j}(y) \}$$

with the corresponding covering property, namely

$$\sum_{C \in \mathcal{S}_i(y)} |C| \geq \left| \bigcup_{j=1}^{N_i} \tilde{B}_{i,j}(y) \right|.$$  \hspace{2cm} (5.2)

Having constructed $\mathcal{S}_i$ for all $1 \leq i \leq M$, we finally set $\mathcal{S} := \bigcup_{i=1}^{M} \mathcal{S}_i$. By construction, $\mathcal{S}$ is a disjoint collection of sets. We are going to show

$$\sum_{i=m}^{M} \sum_{C \in \mathcal{S}_i(y)} |C| \geq \left( 1 - \delta, (M - m + 1) \frac{\delta^2}{|D|} \right) \min_{m \leq i \leq M} \left| D \circ \bigcup_{j=1}^{N_i} B_{i,j}(y) \right|$$  \hspace{2cm} (5.2)

for all $1 \leq m \leq M$ and $\nu$-almost every $y \in Y$. The proof is by induction on $m = M, \ldots, 1$, starting with $M$. By the first case in our argument above,

$$\sum_{C \in \mathcal{S}_M(y)} |C| \geq \left| \bigcup_{j=1}^{N_M} B_{M,j}(y) \right| \geq \frac{1}{|D|} \left| D \circ \bigcup_{j=1}^{N_M} B_{M,j}(y) \right|,$$
which shows the validity of an inequality which is in fact stronger than (5.2). In order to show the claim for \( m < M \), we assume that for the element \( y \in Y \) under consideration,

\[
\sum_{l \geq m} \sum_{C \in \mathcal{F}_l(y)} |C| < (1 - \delta) \min_{m \leq i \leq M} \left| D \circ \bigcup_{j=1}^{N_i} B_{i,j}(y) \right|,
\]

since, otherwise, there is nothing to prove. By construction of the collection \( \mathcal{G}_m(y) \), we have

\[
\sum_{C \in \mathcal{G}_m(y)} |C| \geq \left| \bigcup_{j=1}^{N_m} \tilde{B}_{m,j}(y) \right|.
\]

It follows from

\[
\tilde{B}_{m,j}(y) = B_{m,j}(y) \setminus \bigcup_{l > m} \bigcup_{C \in \mathcal{F}_l(y)} T_{m,j}^{-1} C
\]

that

\[
(D \circ \bigcup_{j=1}^{N_m} \tilde{B}_{m,j})(y) \supseteq \left( D \circ \bigcup_{j=1}^{N_m} B_{m,j} \right)(y) \setminus \bigcup_{l > m} \bigcup_{C \in \mathcal{F}_l(y)} D \circ \left( \bigcup_{j=1}^{N_m} T_{m,j}^{-1} C \right).
\]

Recall that by assumption (5.1), we have \( y \in \mathcal{A} \) for a.e. \( y \in Y \), which shows the validity of an inequality which is in fact stronger than (5.2). In order to show the claim for \( m < M \), which is in fact stronger than (5.2).

With (5.4) we obtain by taking into account (5.3)

\[
\sum_{C \in \mathcal{G}_m} |C| \geq \frac{1}{|D|} \left| D \circ \bigcup_{j=1}^{N_m} B_{m,j}(y) \right| - \frac{1 + \delta}{|D|} \sum_{l > m} \sum_{C \in \mathcal{F}_l} |C|
\]

\[
\geq (1 - (1 + \delta)(1 - \delta)) \frac{1}{|D|} \min_{m \leq i \leq M} \left| D \circ \bigcup_{j=1}^{N_i} B_{i,j}(y) \right|
\]

\[
= \frac{\delta^2}{|D|} \min_{m \leq i \leq M} \left| D \circ \bigcup_{j=1}^{N_i} B_{i,j}(y) \right|.
\]

Consequently,

\[
\sum_{l \geq m} \sum_{C \in \mathcal{F}_l} |C| \geq \sum_{l > m} \sum_{C \in \mathcal{F}_l} |C| + \frac{\delta^2}{|D|} \min_{m \leq i \leq M} \left| D \circ \bigcup_{j=1}^{N_i} B_{i,j}(y) \right|
\]

\[
\geq \left( (M - (m + 1)) \frac{\delta^2}{|D|} + \frac{\delta^2}{|D|} \right) \min_{m \leq i \leq M} \left| D \circ \bigcup_{j=1}^{N_i} B_{i,j}(y) \right|.
\]

Hence, the proof of the inequality (5.2) is complete. Finally, choosing \( M \) large enough such that

\[
\frac{(M - 1)\delta^2}{|D|} \geq (1 - \delta)
\]

concludes the proof of the lemma. \( \blacksquare \)
The goal of this section is to prove Theorem 6.1, as well as Corollary 6.2. To this end, we adapt the overall strategy given in [Li01, Section 4] to the situation of amenable equivalence relations. As usual, $R$ is p.m.p. and hyperfinite, $\Gamma$ is countable and $\alpha: R \to \Gamma$ a class injective measurable cocycle. Recall that for a p.m.p. group action $\Gamma \curvearrowright (X, \lambda)$, the extended equivalence relation $R^X$ over $(X \times Y, \lambda \times \nu)$ is defined by the condition

$$(x, y, (x', y')) \in R^X \iff y R y' \text{ and } x = \alpha(y, y')x',$$

When the measure $\nu$ is $R$-invariant, it follows that $\lambda \times \nu$ is $R^X$-invariant, since the $\Gamma$-action on $X$ preserves $\lambda$. The projection map $\pi: R^X \to R$ given by $(x, y) \to y$ is injective when restricted to the $R^X$-equivalence class of $(x, y)$, for almost all $(x, y) \in R^X$. Further, it is well known that an extension of an amenable action is amenable, and thus in particular if $R$ is amenable, so is $R^X$. But since the extension is class-injective, in fact every hyperfinite exhaustion $(R_n)$ of $R$ can be canonically lifted to a hyperfinite exhaustion $(R_n^X)$ of $R^X$, via $R_n^X((x, y)) = \{(\alpha(z, y)x, z) : z \in R_n(y)\}$. Note that if $(R_n)$ is a bounded hyperfinite exhaustion, then so is $(R_n^X)$, with the same bounds on the equivalence classes.

For a given hyperfinite exhaustion $(R_n)$ for $R$, we define a set $\Phi \subseteq \text{Aut}(R)$ satisfying the conclusions of Proposition 3.1. Then, every $\phi \in \Phi$ can be extended naturally to an inner automorphism $\phi^X \in \text{Aut}(R^X)$ by setting

$$\phi^X((x, y)) := (\alpha(\phi(y), y)x, \phi(y)).$$

For a subset $D \subseteq \Phi$, we write $D^X$ for the set $\{\phi^X : \phi \in D\}$. Note that by definition, the set $\Phi^X := \{\phi^X : \phi \in \Phi\}$ is generating for the relation $R^X$. Clearly, if $\phi$ preserves the classes of $R_n$ almost surely, then $\phi^X$ preserves the classes of $R_n^X$ almost surely.

Let us now recall the concept of ergodicity for a measure preserving equivalence relation. Let $Z$ be such a relation over a probability space $(Z, \eta)$. A subset $A \subseteq Z$ will be called $Z$-invariant if

$$(A \times Z) \cap Z = A \times A.$$ 

The relation $Z$ is ergodic if every $Z$-invariant set $A \subseteq Z$ satisfies $\eta(A) \in \{0, 1\}$. From now on, we will always assume that the relation $R^X$ is ergodic. For sufficient conditions guaranteeing this property, we refer to the discussion in [17] below.

One crucial ingredient for the proofs of Theorem 6.1 and Corollary 6.2 is the pointwise ergodic theorem. A general form of it being valid for suitable asymptotically invariant sequences of subset functions in an amenable equivalence relation was established in [13]. Since we restrict our discussion here to hyperfinite exhaustions, we will state a less general but easily accessible special case which is sufficient for our purposes. Indeed, the following fact is an immediate consequence of the martingale convergence theorem.

**Theorem 6.1.** Let $Z$ be an ergodic, p.m.p. equivalence relation over a probability space $(Z, \eta)$. Let $(Z_n)$ be a bounded hyperfinite exhaustion for $Z$. Then for all $f \in L^1(Z, \eta)$, we have

$$\lim_{n \to \infty} |Z_n(z)|^{-1} \sum_{w \in Z_n(z)} f(w) = \int_Z f(w) d\eta(w)$$

for $\eta$-almost every $z \in Z$.

We now turn to establish some preliminary lemmas that will be used below. The first step is a straightforward consequence of the ergodicity of the relation.

**Lemma 6.2.** Let $Z$ be an ergodic p.m.p. equivalence relation over a probability space $(Z, \eta)$ along with a countable set $\Phi$ of inner automorphisms generating $Z$. Then, for every $\delta > 0$, and every set $A \subseteq Z$ with $\eta(A) > 0$, there is a finite set $D \subseteq \Phi$ such that

$$\eta(D \circ A) \geq 1 - \delta.$$
Proof. Let \( A \subseteq Z \) be measurable with \( \eta(A) > 0 \). Assume that there is some \( \delta_0 > 0 \) such that for all finite collections \( D \subseteq \Phi \), we have \( \eta(D \circ A) < 1 - \delta_0 \). We define \( \overline{A} := \Phi \circ A = \bigcup_{\phi \in \Phi} \phi(A) \). This set is invariant under the relation \( Z \). To see this, consider \( z \in Z \), \( \phi \in \Phi \) and \( a \in A \) such that \( (\phi(a), z) \in Z \). Since \( \Phi \) is generating, there is a further inner automorphism \( \phi' \in \Phi \) such that \( z = \phi' \phi(a) \). We conclude that \( z \in \Phi \circ A \) and thus,
\[
(\overline{A} \times Z) \cap Z = \overline{A} \times Z.
\]
Since \( \Phi \) is countable, it follows \( \eta(\overline{A}) \leq 1 - \delta_0 \) by our assumption. Now \( \overline{A} \) is an invariant set, hence by ergodicity of the relation, we obtain \( \eta(\overline{A}) = 0 \). However, since every inner automorphism preserves the measure, we have \( \eta(\overline{A}) \geq \eta(A) > 0 \). This is a contradiction. 

We need the following combinatorial lemma which is the measured equivalence relation analog of the version for Følner sequences, see Lemma 1 in [OW83] and Lemma 4.2 in [Li01].

**Lemma 6.3.** Let \( \mathcal{R} \) be a p.m.p. equivalence relation over \((Y, \nu)\), and let \((\mathcal{R}_n)\) be a bounded hyperfinite exhaustion satisfying \( \lim_{n \to \infty} \inf_y |\mathcal{R}_n(y)| = \infty \). Then, for every \( \eta > 0 \), there is some \( n \in \mathbb{N} \) such that for all \( n \geq n_0 \), the number of possible disjoint subcollections \( \mathcal{G}(y) \) of the form
\[
\mathcal{G}(y) = \left\{ \mathcal{R}_{k_i}(c) \mid c \in \mathcal{R}_n(y), 1 \leq i \leq r(y) \right\}
\]
is at most \( 2^{n|\mathcal{R}_n(y)|} \).

**Proof.** Let \( \eta > 0 \). Consider \( \ell \in \mathbb{N} \), \( y \in Y \) and an increasing sequence \( (k_i) \), \( 1 \leq i \leq r(y) \) of integers (depending on \( y \)) with \( |\mathcal{R}_{k_i}(z)| \geq \ell \) for almost every \( z \in Y \). Then, there is \( n_0 \in \mathbb{N} \) such that for all \( n \geq n_0 \), the number of possible disjoint subcollections \( \mathcal{G}(y) \) of the form
\[
\mathcal{G}(y) = \left\{ \mathcal{R}_{k_i}(c) \mid c \in C(y) \right\}
\]
with \( 1 \leq t(c) \leq r(y) \).

Given \( \mathcal{G}(y) \) (and hence \( C(y) \)), we define a set of points \( \mathcal{N}(y) \) as follows. For \( c \in C(y) \), define \( n(c) := i \) as the maximal \( 1 \leq i \leq t(c) \) such that \( \mathcal{R}_{k_i}(c) \setminus \mathcal{R}_{k_{i-1}}(c) \neq \emptyset \), where \( \mathcal{R}_0 = \emptyset \) by convention. Then add to the set \( \mathcal{N}(y) \) an arbitrary point \( p(c) \in \mathcal{R}_{k_i}(c) \setminus \mathcal{R}_{k_{i-1}}(c) \). By processing in this way for all \( c \in C(y) \), we obtain a set \( \mathcal{N}(y) \) with cardinality at most \( |C(y)| \). We claim that we can uniquely recover \( \mathcal{G}(y) \) from knowing the elements in both sets \( C(y) \) and \( \mathcal{N}(y) \). Indeed, given \( C(y) \) and \( \mathcal{N}(y) \), by construction, for every \( c \in C(y) \), there is a minimal (and thus unique) \( 1 \leq t(c) \leq r(y) \) such that \( \mathcal{R}_{k_{t(c)}}(c) \cap \mathcal{N}(y) \neq \emptyset \). Hence, the number of possible disjoint subcollections must be bounded by the number of choices for the two sets \( C(y) \) and \( \mathcal{N}(y) \). Since the sizes of all classes involved are uniformly bounded from below by \( \ell \), the cardinality of every \( C(y) \) and \( \mathcal{N}(y) \) is bounded from above by \( |C(y)|/\ell \). In light of that, we need to bound the expression
\[
\left( \frac{|C(y)|}{\ell} + 1 \right)^2 \left( \frac{|\mathcal{R}_n(y)|}{|C(y)|} \right)^2.
\]
To do so, we use the entropy formula for the Stirling approximation. For this purpose, we define
\[
E(\ell) := \frac{1}{\ell} \log \ell + \left( 1 - \frac{1}{\ell} \right) \log \left( 1 - \frac{1}{\ell} \right)^{-1}.
\]
By Stirling’s approximation (see e.g. [FS08], Example VIII.10), for large enough \( n \in \mathbb{N} \), we get
\[
\left( \frac{|\mathcal{R}_n(y)|}{|C(y)|} \right)^2 \leq \left( \exp \left( E(\ell/2) |\mathcal{R}_n(y)| \right) \right)^2 \leq 2^{4E(\ell/2)|\mathcal{R}_n(y)|}.
\]
Now increasing \( n \) if necessary, we can make sure that
\[
\left( \frac{|\mathcal{R}_n(y)|}{\ell} + 1 \right)^2 \left( \frac{|\mathcal{R}_n(y)|}{|\mathcal{R}_n(y)|/\ell} \right)^2 \leq 2^{5E(\ell/2)|\mathcal{R}_n(y)|}.
\]
Since \( E(\ell) \to 0 \) as \( \ell \to \infty \), we can find some \( \ell \) such that \( E(\ell/2) < \eta/5 \), and this completes the proof of the claim. 

We are ready to prove the main lemma of this section. It is motivated by Lemma 4.3 in [5.1] and provides an analog of it for hyperfinite exhaustions.

**Lemma 6.4.** Let \((B_k)\) be a sequence of measurable sets in \(X \times Y\) such that
\[
\lambda \times \nu \left( \bigcap_{k=1}^{\infty} \bigcup_{j \geq k} B_j \right) > 0.
\]
Then, for every \(\delta > 0\) and \(\lambda \times \nu\text{-a.e.} (x, y) \in X \times Y\), there is \(n(x, y) \in \mathbb{N}\) for which the following holds true: for each \(n \geq n(x, y)\), there is a disjoint collection of subsets of \(\mathcal{R}_n(y)\)
\[
\mathcal{G} = \{ \mathcal{R}_k(b_i) \mid 1 \leq i \leq r \}
\]
with \(b_i \in \mathcal{R}_n(y)\) and \(1 \leq k_i < n(x, y)\), such that
(i) \((\alpha(b_i) x, b_i) \in B_{k_i}\) for all \(i\),
(ii) \(\sum_{C \in \mathcal{G}} |C| \geq (1 - \delta) |\mathcal{R}_n(y)|\).

**Proof.** Let \(\delta > 0\). We set
\[
B^* := \bigcap_{k=1}^{\infty} \bigcup_{j \geq k} B_j.
\]
By assumption, \((\lambda \times \nu)(B^*) > 0\). Given the hyperfinite exhaustion \((\mathcal{R}_n)\), we fix \(\Phi^X \subseteq \text{Aut}(\mathcal{R}^X)\) satisfying the conclusions of Proposition 5.1. Since the extended relation \(\mathcal{R}^X\) is ergodic, and since the set \(\Phi^X\) is generating for \(\mathcal{R}^X\), we can use Lemma 6.2 in order to find a finite set \(D \subset \Phi\) such that the lifted automorphisms \(D^X \subseteq \Phi^X\) satisfy
\[
\lambda \times \nu(D^X \circ B^*) \geq 1 - \delta/10.
\]
(6.2)
Thus clearly \(\lambda \times \nu(B^*) \geq (1 - \delta/10)/|D|\). Further, choose \(M \in \mathbb{N}\) such that Lemma 5.2 holds true for the fixed parameters \(\delta\) and \(D\) for the relation \(\mathcal{R}\).

The next step is to construct two finite increasing integer sequences \((m_i), (N_i)\), \(1 \leq i \leq M\), according to the following algorithm.

1. Define \(m_1 := 1\).
2. If \(m_i\) has been chosen, then determine \(N_i\) large enough such that
\[
\lambda \times \nu \left( B^* \setminus \bigcup_{j=m_i}^{m_i+N_i} B_j \right) < \frac{\delta \cdot \lambda \times \nu(B^*)}{10 |D|}.
\]
3. Further, if \(N_i\) has been chosen, choose \(m_{i+1}\) large enough such that for every \(l \geq m_{i+1}\), we get
\[
\bigcup_{j < m_{i+1}}^{} D \circ \mathcal{R}^{-1}_j \mathcal{R}_l \subseteq \mathcal{R}_l.
\]
This is possible by Proposition 5.3 and since \(D \subset \Phi\). In fact \(D \circ \mathcal{R} = \mathcal{R}\) for all sufficiently large \(n\). So it suffices that \(m_{i+1} > m_i + N_i\) and also that \(\mathcal{R}_{m_{i+1}}\) (and hence each \(\mathcal{R}_{m_{i+1}}\)) is invariant under \(D\).

With the sequences \((m_i), (N_i)\) at our disposal, we define
\[
\tilde{B}^* := \bigcap_{i=1}^{M} \bigcup_{j=m_i}^{m_i+N_i} B_j.
\]
Clearly
\[
\lambda \times \nu(B^* \setminus \tilde{B}^*) \leq \sum_{i=1}^{M} \lambda \times \nu \left( B^* \setminus \bigcup_{j=m_i}^{m_i+N_i} B_j \right) < \frac{\delta \cdot \lambda \times \nu(B^*)}{10 |D|},
\]
and so
\[
\lambda \times \nu(D^X \circ B^* \setminus D^X \circ \tilde{B}^*) < \frac{\delta \cdot \lambda \times \nu(B^*)}{10}.
\]
Consequently,

\[ \lambda \times \nu(D^X \circ \bar{B}^*) \geq \nu(D^X \circ B^*) - \frac{\delta \cdot \lambda \times \nu(B^*)}{10} \geq 1 - \frac{\delta}{10} - \frac{\delta \cdot \lambda \times \nu(B^*)}{10} \geq 1 - \frac{\delta}{5}. \]

Now by the pointwise ergodic theorem (Theorem 5.1), for \( \lambda \times \nu \)-a.e. \((x, y)\), we can define \( n(x, y) \) as the smallest integer value greater than \( m_M + N_M \) such that for all \( n \geq n(x, y) \), we have

\[ A(n, 1_{D \times \sigma \bar{B}^*})(x, y) := \left| \mathcal{R}_n(y) \right|^{-1} \sum_{z \in \mathcal{R}_n(y)} 1_{D \times \sigma \bar{B}^*}((\alpha(z, y)x, z)) > 1 - \frac{\delta}{4}. \] (6.3)

We fix some pair \((x, y)\) satisfying this condition, as well as \( n \geq n(x, y) \). For \( 1 \leq i \leq M \) and \( 1 \leq j \leq N_i \), set \( T_{i, j} := \mathcal{R}_{m_i + j - 1} \) and

\[ A_{i, j}(y) := \{ b \in \mathcal{R}_n(y) \mid (\alpha(b, y)x, b) \in B_{m_i + j - 1} \}. \]

By step (3) of the algorithm and since \( n > m_M + N_M \), we have \( D \circ \mathcal{R}_n \subseteq \mathcal{R}_n \). In particular, each \( \phi \in D \) gives rise to bijections \( \phi : \mathcal{R}_n(y) \rightarrow \mathcal{R}_n(y) \), for a.e. \( y \in Y \).

Applying the transformations in the set \( D \) to the sets \( A_{i, j}(y) \), we then obtain

\[
D \circ \bigcup_{j=1}^{N_i} A_{i, j}(y) = \bigcup_{\phi \in D} \{ \phi(b) \mid b \in \mathcal{R}_n(y), (\alpha(b, y)x, b) \in \bigcup_{j=1}^{N_i} B_{m_i + j - 1} \}
\]

\[
= \{ b \in \mathcal{R}_n(y) \mid \exists \phi \in D, b' \in \mathcal{R}_n(y) : b = \phi(b'), (\alpha(b', y)x, b') \in \bigcup_{j=1}^{N_i} B_{m_i + j - 1} \}. \]

Together with (6.3), this yields for all \( 1 \leq i \leq M \):

\[
\left| \bigcup_{j=1}^{N_i} D \circ A_{i, j}(y) \right| = \left| \{ b \in \mathcal{R}_n(y) \mid \exists \phi \in D : b' = \phi^{-1}(b), (\alpha(b', y)x, b') \in \bigcup_{j=1}^{N_i} B_{m_i + j - 1} \} \right|
\]

(using (5.1))

\[
= \left| \{ b \in \mathcal{R}_n(y) \mid (\alpha(b, y)x, b) \in D^X \circ \bigcup_{j=1}^{N_i} B_{m_i + j - 1} \} \right|
\]

(since \( \bar{B}^* \) is an intersection)

\[
\geq \left| \mathcal{R}_n(y) \right| \cdot A(n, 1_{D \times \sigma \bar{B}^*})(x, y)
\]

(by the pointwise ergodic theorem)

\[
\geq \left( 1 - \frac{\delta}{4} \right) \left| \mathcal{R}_n(y) \right|.
\]

We finally apply Lemma 5.2 with \( \delta/4 \) instead of \( \delta \) to the arrays \( T_{i, j} \) and \( A_{i, j}(y) \), where \( 1 \leq i \leq M \) and \( 1 \leq j \leq N_i \). The assumption of the Lemma is indeed satisfied, namely \( T_{i, j} = \mathcal{R}_{m_i + j - 1} \) satisfy (5.1) by the construction of \( m_2 \), which guarantees \( D \circ \mathcal{R}_m = \mathcal{R}_m \) for all \( m \geq m_2 \). Furthermore, we have just shown that

\[
\min_{1 \leq i \leq M} \left| \bigcup_{j=1}^{N_i} D \circ A_{i, j}(y) \right| \geq \left( 1 - \frac{\delta}{4} \right) \left| \mathcal{R}_n(y) \right|
\]

and hence Lemma 5.2 implies that there is a disjoint subcollection

\[
\mathcal{G}(y) \subseteq \{ T_{i, j}(b) \mid b \in A_{i, j}(y) \}
\]

with

\[
\sum_{C \in \mathcal{G}(y)} |C| \geq \left( 1 - \delta \right) \left| \mathcal{R}_n(y) \right|,
\]

as desired. \( \blacksquare \)

The following is an immediate consequence from the previous lemma.
Lemma 6.5. Let \((\mathcal{R}_n)\) be a bounded hyperfinite exhaustion, and keep the assumptions of the previous lemma. Then, for the number

\[
h := \text{ess-inf}_{x,y} \liminf_{n \to \infty} \frac{\mathcal{J}(\mathcal{P}^{\mathcal{R}_n(y)}(x))}{|\mathcal{R}_n(y)|},
\]

the following holds true. For every \(\delta > 0\), each \(N \in \mathbb{N}\) and \(\lambda \times \nu\)-a.e. \((x, y)\), there is a number \(n(x, y) \in \mathbb{N}\) such that for all \(n \geq n(x, y)\), we can find a disjoint collection (depending on both \(x\) and \(y\))

\[
\mathcal{G} = \{\mathcal{R}_i(b_i) | 1 \leq i \leq r\}
\]
such that all \(b_i \in \mathcal{R}_n(y)\) and \(N \leq k_i < n(x, y)\) and further,

(i) for all \(i\),

\[
\frac{\mathcal{J}(\mathcal{P}^{\mathcal{R}_n(b_i)}(\alpha(b_i, y)x))}{|\mathcal{R}_i(b_i)|} \leq h + \delta,
\]

(ii) \(\sum_{C \in \mathcal{G}} |C| \geq (1 - \delta) |\mathcal{R}_n(y)|\).

Proof. We apply the previous Lemma 6.4 to the sequence \((\mathcal{R}_n)_{n \geq N}\) and the sets

\[
B_k := \{(x, y) \in X \times Y | \frac{\mathcal{J}(\mathcal{P}^{\mathcal{R}_{n+k}}(x))}{|\mathcal{R}_{n+k}(y)|} \leq h + \delta \}.
\]

We are now in position to prove the Shannon-McMillan-Breiman Theorem. To do so, we combine the previous results of this section, motivated by the proof of Theorem 1.3 in [Li01].

Proof of Theorem 2.7 As before, we set

\[
h := \text{ess-inf}_{x,y} \liminf_{n \to \infty} \frac{\mathcal{J}(\mathcal{P}^{\mathcal{R}_n(y)}(x))}{|\mathcal{R}_n(y)|}.
\]

If \(h = \infty\), then there is nothing left to show. So assume that \(h < \infty\). We show that for a.e. \((x, y)\),

\[
\limsup_{n \to \infty} \frac{\mathcal{J}(\mathcal{P}^{\mathcal{R}_n(y)}(x))}{|\mathcal{R}_n(y)|} \leq h.
\] (6.4)

To this end, fix \(\delta > 0\). We find \(N \in \mathbb{N}\) large enough such that Lemma 6.3 holds for \(\eta = \delta\) and \(\ell = N\). By the growth assumption on the \((\mathcal{R}_n)\), we find \(N_1 \in \mathbb{N}\) such that for a full measure set of \(y\)'s, we have \(|\mathcal{R}_k(z)| \geq N\) for all \(k \geq N_1\) and each \(z \in [y]\). We have seen in Lemma 6.3 that for \(\lambda \times \nu\)-almost every \((x, y)\), there is \(n(x, y) \in \mathbb{N}\), \(n(x, y) > N_1\), such that for \(n > n(x, y)\), we obtain a special subcollection \(\mathcal{G} = \mathcal{G}(x, y)\) of subsets of \(\mathcal{R}_n(y)\). Namely, this collection

- is disjoint,
- satisfies \(\sum_{C \in \mathcal{G}} |C| \geq (1 - \delta) |\mathcal{R}_n(y)|\),
- each \(C \in \mathcal{G}\) is of the form \(\mathcal{R}_k(b)\) with \(N_1 \leq k < n(x, y)\) and \(b \in \mathcal{R}_n(y)\), which by the choice of \(N_1\) implies that \(|\mathcal{R}_k(b)| \geq N\) for all \(k \geq N_1\),
- writing \(C = \mathcal{R}_k(b) \in \mathcal{G}\), the following holds

\[
\frac{\mathcal{J}(\mathcal{P}^{\mathcal{R}_k(b)}(\alpha(b, y)x))}{|\mathcal{R}_k(b)|} \leq h + \delta.
\] (6.5)

By increasing \(n\) if necessary, due to \(|\mathcal{R}_n(y)| \to \infty\), we can assume that

\[
|\mathcal{R}_n(y)| \geq 8 \delta^{-1} (\log n).
\] (6.6)

Let us now fix \(x, y, n = n(x, y)\) and \(\mathcal{G}(x, y) = \mathcal{G}\) satisfying all the conditions stated above.

We first note that since \(\mathcal{R}_n(y)\) is the disjoint union of the sets \(C = \mathcal{R}_k(b) \in \mathcal{G}\) and \(\mathcal{G}_n(y) := \mathcal{R}_n(y) \setminus \bigcup_{C \in \mathcal{G}} \mathcal{R}_k(b)\), it follows that the partition \(\mathcal{P}^{\mathcal{R}_n(y)}\) is given by \(\bigvee_{C \in \mathcal{G}} \mathcal{P}^{\mathcal{R}_k(b)} \cup \mathcal{P}^{\mathcal{G}_n(y)}\).

Therefore for any point \(x \in X\), its \(\mathcal{P}^{\mathcal{R}_n(y)}\)-name arises as the intersection of the \(\mathcal{P}^{\mathcal{R}_k(b)}\)-name of \(\alpha(b, y)x\) where \(C = \mathcal{R}_k(b) \in \mathcal{G}\), and of the \(\mathcal{P}\)-name of \(\alpha(b, y)x\) for every \(b \in \mathcal{G}_n(y) := \mathcal{R}_n(y) \setminus \bigcup_{C \in \mathcal{G}} \mathcal{R}_k(b)\).
Consider the partition $\mathcal{P}^{R_n(y)}$ (which is finer than each partition $\mathcal{P}^{R_k(b)}$ when $b \in R_n(y)$) and define a set of atoms in it which we denote by $K_n(x, y)$. Namely, for our fixed $x$, we consider the disjoint collection $\mathcal{S}(x, y)$ of subsets of $R_n(y)$, and we put an atom of $\mathcal{P}^{R_n(y)}$ in $K_n(x, y)$ provided that $n(x,y) \in R_n(y)$.

Now, for every $\mathcal{R}_k(b) = C \in \mathcal{S}(x, y)$, inequality (6.3) gives a lower bound on the measure of some of the atoms in the partition, and hence an upper bound on their number. It follows that for each such $C$, there are at most $2^{(h + \delta)|\mathcal{R}_k(b)|}$ atoms of $\mathcal{P}^{\mathcal{R}_k(b)}$ (namely $\mathcal{R}^{\mathcal{R}_k(b)}$-names) for which the inequality (6.5) can hold true.

Consequently, the number of atoms of $\mathcal{P}^{R_n(y)}$ which appear as elements in $K_n(x, y)$ is bounded by

$$\prod_{C \in \mathcal{S}} 2^{(h + \delta)|\mathcal{R}_k(b)|} \cdot |\mathcal{P}|^{\mathcal{R}_n(y)} \leq \prod_{C \in \mathcal{S}} 2^{(h + \delta)|C|} \cdot |\mathcal{P}|^{\mathcal{R}_n(y)} \cdot \mathcal{R}_n(y).$$

Since $\mathcal{R}_n(y)$ is disjointly $(1 - \delta)$-covered by the elements in $\mathcal{S}$, we conclude that for a fixed $x \in X$, there are at most

$$2^{(h + \delta)|\mathcal{R}_n(y)|} \cdot |\mathcal{P}|^{\delta|\mathcal{R}_n(y)|}$$

many elements in $K_n(x, y)$.

$K_n(x, y)$ depends on both $x$ and $y$, having been constructed using the collection $\mathcal{S}(x, y)$ of subsets on $\mathcal{R}_n(y)$. Now, we define another collection of atoms of $\mathcal{P}^{\mathcal{R}_n(y)}$ which we denote by $K_m(y)$. It consists of all the atoms in the sets $K_n(x, y)$ as $x$ varies in $X$, provided that $n(x, y)$ satisfies the conditions stated above and in addition $n(x, y) \leq m$. By Lemma (6.3) (and the choices for $N$ and $N_1$), for all $m \geq m(y)$ the number of possibilities for $\mathcal{S}(x,y)$ is bounded by $2^{2^{\delta}|\mathcal{R}_m(y)|}$. Hence, we obtain

$$|K_m(y)| \leq 2^{(h + 2\delta + \delta \log |\mathcal{P}|)|\mathcal{R}_m(y)|}.$$

We now consider the sets

$$X_m(y) := \left\{ x \in X \mid \frac{\mathcal{J}(\mathcal{P}^{\mathcal{R}_m(y)}(x))}{|\mathcal{R}_m(y)|} > h + 3\delta + \delta \log |\mathcal{P}| \right\}.$$

Consider $\bigcup K_m(y)$, the union of all $\mathcal{P}^{\mathcal{R}_m(y)}$-atoms which belong to $K_m(y)$. Then,

$$\lambda(X_m(y) \cap \bigcup K_m(y)) \leq |K_m(y)| \cdot 2^{-(h + 3\delta + \delta \log |\mathcal{P}|)|\mathcal{R}_m(y)|} \leq 2^{-\delta|\mathcal{R}_m(y)|}.$$ 

It follows from the growth condition (6.3) stated above that

$$2^{-\delta|\mathcal{R}_m(y)|} \leq 2^{\log m^{-n}} \leq \frac{1}{m^2}$$

for large enough $m$. This implies that $\sum_{m=1}^{\infty} 2^{-\delta|\mathcal{R}_m(y)|} < \infty$. Thus, for $\nu$-almost-every $y \in Y$, we can apply the Borel-Cantelli lemma and obtain that for $\lambda \times \nu$-almost every $(x,y)$, $x \notin X_m(y) \cap \bigcup K_m(y)$ if $m$ is large enough. On the other hand, we deduce from Lemma (6.5) that for large enough $m$ (depending on $x$ and $y$), $x \in \bigcup K_m(y)$. This implies that for $\lambda \times \nu$-a.e. $(x,y)$, and $m$ large enough, we must have $x \notin X_m(y)$, which means by definition of $X_m(y)$ that

$$\limsup_{m \to \infty} \frac{\mathcal{J}(\mathcal{P}^{\mathcal{R}_m(y)}(x))}{|\mathcal{R}_m(y)|} \leq h + 3\delta + \delta \log |\mathcal{P}|.$$ 

Letting $\delta \to 0$ yields (6.3). This establishes almost everywhere convergence, as stated. Since the integrals of $\mathcal{J}(\mathcal{P}^{\mathcal{R}_m(y)}(x))/|\mathcal{R}_m(y)|$ over $X \times Y$ converge to the cocycle entropy by Theorem 2.1 we have $h = h'_\mathcal{P}(\alpha)$, as claimed.

We conclude this section with the proof of Corollary 2.8. The proof follows along the lines of the $L^1$-convergence case proved in [401, Thm. 4.1].
7. Amenable relations, injective cocycles and ergodic extensions

7.1. Groups admitting injective cocycles. As usual, let \((Y, \nu)\) be a probability space, and let \(\mathcal{R} \subset Y \times Y\) be a p.m.p. Borel equivalence relation with \(\mathcal{R}\)-invariant probability measure \(\tilde{\nu}\), such that \(\mathcal{R} = \bigcup_{n \in \mathbb{N}} \mathcal{R}_n\) is hyperfinite, or equivalently, \(\mathcal{R}\) is amenable in the sense of [CFW81]. Let \((X, \lambda)\) be a p.m.p. action of a countable group \(\Gamma\) and let \(\alpha : \mathcal{R} \to \Gamma\) be a measurable cocycle. Let \(\mathcal{R}^X\) denote the extended relation on \((X \times Y, \lambda \times \nu)\).

As noted in the proof of Theorem 2.4, given the ingredients just listed, the following limit exists

\[
h^*_\mathcal{R}(\alpha) := \lim_{n \to \infty} \int_Y \frac{H^{\mathcal{R}}(\mathcal{R}_n)(y)}{|\mathcal{R}_n(y)|} \, d\nu(y).
\]

However, in this generality we cannot say too much about its properties. A meaningful entropy invariant arises when we assume that the hyperfinite exhaustion is by bounded finite relations, and the cocycle is injective. The first condition can always be satisfied, and thus cocycle entropy exists as an invariant with the properties stated for all countable groups \(\Gamma\) admitting an injective cocycle defined on a hyperfinite relation. In particular, cocycle entropy then coincides with Rokhlin

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Proof of Corollary 2.8. By Theorem 2.7, the normalized information function converges pointwise almost surely (w.r.t. the measure \(\lambda \times \nu\)) to \(h^*_\mathcal{R} = h^*_P(\alpha)\). Fix \(\varepsilon > 0\), and for \(n \in \mathbb{N}\) and fixed \(y \in Y\), define

\[
C_n(y) := \{ x : h^*_\mathcal{R} - \varepsilon \leq \frac{\mathcal{J}(P(y)(x))}{\mathcal{R}_n(y)} \leq h^*_\mathcal{R} + \varepsilon \}.
\]

Integration over \(X\) gives

\[
(h^*_\mathcal{R} - \varepsilon) \lambda(C_n(y)) \leq \int_{C_n(y)} \frac{\mathcal{J}(P(y)(x))}{\mathcal{R}_n(y)} \, d\lambda(x) \leq \int_X \frac{\mathcal{J}(P(y)(x))}{\mathcal{R}_n(y)} \, d\lambda(x) \leq (h^*_\mathcal{R} + \varepsilon) \lambda(C_n(y)) + \int_{X \setminus C_n(y)} \frac{\mathcal{J}(P(y)(x))}{\mathcal{R}_n(y)} \, d\lambda(x).
\]

We define a new measure \(\lambda^*\) on \(X \setminus C_n(y)\) by setting

\[
\lambda^*(A) := \frac{\lambda(A)}{\lambda(X \setminus C_n(y))}.
\]

Then, since \(C_n(y)\) is a disjoint union of atoms of the partition \(P(y)\) of \(X\),

\[
\int_{X \setminus C_n(y)} \frac{\mathcal{J}(P(y)(x))}{\mathcal{R}_n(y)} \, d\lambda(x) = \lambda(X \setminus C_n(y)) \int_{X \setminus C_n(y)} \frac{\mathcal{J}(P(y)(x))}{\mathcal{R}_n(y)} \, d\lambda^*(x) - \lambda(X \setminus C_n(y)) \log \lambda(X \setminus C_n(y)) \log (\lambda(X \setminus C_n(y))).
\]

where \(\mathcal{J}^*\) denotes the information function with respect to \(\lambda^*\). Now since the integral on the right hand side in the inequality above is just the Shannon entropy of the partition \(P(y)\) with respect to the measure \(\lambda^*\), we arrive at

\[
\int_{X \setminus C_n(y)} \frac{\mathcal{J}(P(y)(x))}{\mathcal{R}_n(y)} \, d\lambda(x) \leq \lambda(X \setminus C_n(y)) \log |P| - \lambda(X \setminus C_n(y)) \log (\lambda(X \setminus C_n(y))).
\]

Clearly, by Theorem 2.7, for \(\nu\)-almost every \(y \in Y\), the latter expression tends to zero as \(n \to \infty\). Now sending \(\varepsilon \to 0\) yields the first assertion of the claimed statement. For the second statement note that due to the dominated convergence theorem (with dominating function \(g(y) := 2 |P|\)) we have

\[
\lim_{n \to \infty} \int_Y \int_X \frac{\mathcal{J}(P(y)(x))}{\mathcal{R}_n(y)} \, d\lambda(x) \, d\nu(y) = 0.
\]

This concludes the proof of the corollary. 

■
entropy and finitary entropy provided that the $\Gamma$-action on $X$ is ergodic and essentially free. It may be the case that every countable group admits an injective cocycle defined on an ergodic hyperfinite relation, but this remains to be seen.

To indicate briefly that the class of groups in question is extensive indeed, let us note the following constructions of cocycles on amenable actions.

1) Amenable groups.

Let $\Gamma$ be amenable, let $(Y, \nu)$ be any p.m.p. action of $\Gamma$, and assume that the action is essentially free. For example, we can take $Y$ to the Bernoulli action of $\Gamma$ on $\{0,1\}^\Gamma$. Let $\mathcal{R} = O\Gamma$ be the orbit relation of $\Gamma$ on $Y$, and then there is a cocycle $\alpha : \mathcal{R} \to \Gamma$ given by $\alpha(\gamma y, y) = \gamma$. This cocycle is indeed an injective cocycle on a p.m.p. amenable equivalence relation $\mathcal{R}$, by amenability of $\Gamma$ and freeness of the action. Given a p.m.p. action of $\Gamma$ on a space $(X, \lambda)$, clearly $\Gamma$ acts on the product $(X \times Y, \lambda \times \nu)$, and the orbit relation of $\Gamma$ in the product coincides with the extended relation $\mathcal{R}^X$ on $X \times Y$ that we have used throughout the paper. Thus $\alpha : \mathcal{R} \to \Gamma$ is an injective cocycle defined on a p.m.p. amenable relation.

2) The Maharam extension of the Poisson boundary. For every countable infinite group $\Gamma$, and every generating probability measure $\mu$ on $\Gamma$, the Poisson boundary $B = B(\Gamma, \mu)$ is an amenable action of $\Gamma$, in the sense defined by Zimmer [Zi78], or equivalently, in the sense of [CFW81]. Let $\eta$ denote the stationary measure on $B$, and $r_\eta(\gamma, b) = \frac{d\eta \circ \gamma}{d\eta}(b)$ the Radon-Nikodym derivative cocycle of $\eta$, so that $r_\eta : \Gamma \times B \to \mathbb{R}^*_+$. The Maharam extension of $\Gamma$ by the cocycle $r_\eta$ is the $\Gamma$-action on $B \times \mathbb{R}$ given by $\gamma(b, t) = (\gamma b, t - \log r_\eta(\gamma, b))$, and this action preserves the measure $\eta \times \theta$, where $d\theta(t) = e^t dt$ and $dt$ is Lebesgue measure on $\mathbb{R}$. This action is again an amenable action of $\Gamma$, being an extension of an amenable action. Let us define $Y = B \times (-\infty, 0)$, and let $\nu$ be the restriction of $\eta \times \theta$ to $Y$, a finite measure which we normalize to be a probability measure. Then $(Y, \nu)$ is a probability space, and we define the relation $\mathcal{R}$ on it to be the restriction of the orbit relation $O\Gamma$ defined by $\Gamma$ on $B \times \mathbb{R}$ to the subset $Y$. Thus $\mathcal{R}$ is a p.m.p. Borel equivalence relation with countable classes, since $\Gamma$ is countable and preserves the measure $\eta \times \theta$. The orbit relation $O\Gamma$ is an amenable relation, hence it is hyperfinite, and as a result so is its restriction $\mathcal{R}$ to the subset $Y$. Finally, if the action of $\Gamma$ on its Poisson boundary $B$ is essentially free, then we can define a cocycle $\alpha : O\Gamma \to \Gamma$, by the formula $\alpha(\gamma(b, t), (b, t)) = \gamma$. This cocycle is well-defined since the elements in a $\Gamma$-orbit are in bijective correspondence to the elements of $\Gamma$, by essential freeness. Restricting $\alpha$ to $\mathcal{R}$ we obtain a cocycle from an amenable p.m.p. equivalence relation $\mathcal{R}$ to $\Gamma$. Furthermore, this cocycle is injective in this case, since $\gamma(b, t) = (\gamma b, t - \log r_\eta(\gamma, b))$, so that if $\gamma \neq \gamma'$ then $\alpha((b, t), \gamma(b, t)) \neq \alpha((b, t), \gamma'(b, t))$.

Clearly, the class of groups admitting a random walk such that the action on the associated Poisson boundary is essentially free is extensive indeed. In fact typically many different random walks on a given non-amenable group $\Gamma$ give rise to Poisson boundaries admitting an essentially free action. It is a remarkable feature of the construction of cocycle entropy that it gives one and the same value for the entropy of the $\Gamma$-action on $X$, provided only that this action is ergodic and essentially free, and the value is independent of which cocycle $\alpha : \mathcal{R} \to \Gamma$ as above was chosen to calculate it.

7.2. Ergodicity of cocycle extensions. The Shannon-McMillan-Breiman theorem stated in Theorem 2.7, being a pointwise convergence result for cocycle entropy, requires an additional ergodicity assumption for its validity, beyond those sufficient to guarantee the existence of cocycle entropy itself. We note that an ergodicity assumption also plays a role in the Shannon-McMillan-Breiman theorem for amenable groups, see [Li01].

Let us recall that in [BN15a, Def. 2.2] a notion of weak-mixing for a cocycle $\alpha$ on a p.m.p. relation $\mathcal{R}$ on $(Y, \nu)$ was defined, as follows. A cocycle $\alpha : \mathcal{R} \to \Gamma$ is weak-mixing if for every p.m.p. ergodic action of $\Gamma$ on a space $(X, \lambda)$, the extended relation $\mathcal{R}^X$ on $X \times Y$ is ergodic with respect to the product measure $\lambda \times \nu$. In particular, the relation $\mathcal{R}$ itself must be ergodic. This definition is a natural extension of the notion of weak-mixing for group actions, where the action of $\Gamma$ on a space $(B, \eta)$ (with $\eta$ not necessarily invariant), is called weak-mixing if given any p.m.p. ergodic action of $\Gamma$ on a space $(X, \lambda)$, the product action of $\Gamma$ on $(X \times B, \lambda \times \eta)$ is still ergodic.
1) Amenable groups. Let us first consider the case where $\Gamma$ is amenable. Referring to the relation $\mathcal{R}$ and the cocycle $\alpha$ defined in §7.1, if the $\Gamma$-action on $(Y, \nu)$ is weak-mixing (in the usual sense for group actions), then the $\Gamma$-action on $(X \times Y, \lambda \times \nu)$ is ergodic for every ergodic p.m.p. action of $\Gamma$. Thus, referring to §7.1, the cocycle $\alpha : \mathcal{R} \to \Gamma$ defined there is a weak-mixing cocycle and so here the extended relation $\mathcal{R}^\times$ is ergodic.

2) Poisson boundaries. When $\Gamma$ is non-amenable, the most important source (but not always the only one) of weak-mixing actions of a countable group $\Gamma$ is the set of its actions on Poisson boundaries $B = B(\Gamma, \mu)$. These actions are amenable as noted above, and satisfy a stronger condition than weak-mixing, namely double ergodicity with coefficients, see [Ka03]. If $\Gamma$ is a non-amenable group, then the ergodic action on a Poisson boundary $(B, \eta)$ is not measure-preserving, and thus of type III. The action of $\Gamma$ on the Maharam extension $(B \times \mathbb{R}, \eta \times L)$ is in fact measure-preserving, on a $\sigma$-finite (but not finite) measure space, but the Maharam extension is not necessarily an ergodic action of $\Gamma$. The possibilities for it are determined by the type of the $\Gamma$-action on $(B, \eta)$, and in particular, if it is $III_1$, then the Maharam extension is ergodic. In general, this does not imply that for any ergodic action of $\Gamma$ on $(X, \lambda)$, the Maharam extension of $(X \times B, \lambda \times \eta)$ is ergodic. If indeed this is the case for every p.m.p. action of $\Gamma$, then the action of $\Gamma$ on $(B, \eta)$ is defined in [BN13b] to have stable type $III_1$.

Assume that the $\Gamma$-action on $(B, \eta)$ is essentially free, and let $(Y, \nu)$, $\mathcal{R}$ and $\alpha$ be as defined in §7.1. Then the cocycle $\alpha : \mathcal{R} \to \Gamma$ is injective, defined on a p.m.p. amenable relation, and if the type of the $\Gamma$-action on $(B, \eta)$ is $III_1$, it is ergodic. If, furthermore, the $\Gamma$-action on $(B, \eta)$ has stable type $III_1$, then the cocycle $\alpha$ is weak-mixing, and hence $\mathcal{R}^\times$ is ergodic for every p.m.p. action of $\Gamma$. Thus all the assumptions in the Shannon-McMillan-Breiman theorem are verified in this case, for which examples will be provided below.

3) Non-trivial stable type. If the type of the $\Gamma$-action on $(B, \eta)$ is $III_\tau$ for some $\tau > 0$ then the action of $\Gamma$ on the Maharam extension has a set of ergodic components admitting a free transitive action of the circle group $\mathbb{R}/\mathbb{Z} \cdot \log \tau$, which acts on the Maharam extension and which commutes with the $\Gamma$-action. If this is also the situation for the Maharam extension of all the spaces $(X \times B, \lambda \times \eta)$ for every ergodic p.m.p. action of $\Gamma$, then the $\Gamma$-action on $(B, \eta)$ is defined in [BN13b] to have stable type $\tau$. In that case, it is also possible to prove a version of the Shannon-McMillan-Breiman pointwise convergence theorem, which applies, rather than to the information functions we defined, to a further average of them. In particular, this provides a proof of the Shannon-McMillan mean-convergence theorem in our context. In the interest of brevity, however, we shall provide the details elsewhere. We refer to [BN13b] and [BN15a] for a detailed discussion of type, stable type and Maharam extensions in the context of pointwise ergodic theorems for group actions.

4) Ergodicity and mixing conditions. Given an injective cocycle $\alpha : \mathcal{R} \to \Gamma$ on an ergodic p.m.p. amenable relation and a p.m.p. action of $\Gamma$ on $(X, \lambda)$, it is possible to develop criteria to show that the extended relation $\mathcal{R}^\times$ is ergodic, provided that the $\Gamma$-action on $X$ satisfies additional ergodicity or mixing conditions. This implies that the Shannon-McMillan-Breiman theorem is valid for $\Gamma$-actions on a suitable class of p.m.p. actions on $(X, \lambda)$. A simple example of this phenomenon arises for the p.m.p. actions of finitely generated non-abelian free groups $\mathbb{F}_r$. Here, taking the boundary $(\partial \mathbb{F}_r, \nu)$ with the uniform measure $\nu$, we construct a cocycle $\alpha : \mathcal{R} \to \mathbb{F}_r$, where $\mathcal{R}$ is an amenable p.m.p. relation on $\partial \mathbb{F}_r$, which is not ergodic, but in fact has exactly two ergodic components. If $(X, \lambda)$ is a p.m.p. ergodic action of $\mathbb{F}_r$ for which the index 2 subgroup of even length words is ergodic, then the extended relation $\mathcal{R}^\times$ is an ergodic relation, see [BN13b] for a detailed exposition. Hence these actions of $\mathbb{F}_r$ satisfy the Shannon-McMillan-Breiman theorem. We will give a detailed exposition of this case, which will also demonstrate the geometric significance of the theorem, in the next section.

To conclude this section let us note the following results concerning type and stable type, which are relevant to the foregoing discussion.

**Examples 7.1**. (1) Let $\Gamma$ be an irreducible lattice in a connected semisimple Lie group with finite center and without compact factors. Then the action of $\Gamma$ on the maximal boundary

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(G/P, m), where P is a minimal parabolic subgroup and m is the Lebesgue measure class, is amenable and has stable type III₁, see [BN13a].

(2) Let Γ be a discrete non-elementary subgroup of isometries of real hyperbolic space. By [Su78, Su82], the type of the action of Γ on the boundary of hyperbolic space with respect to the Lebesgue measure class is III₁. In [Sp87] this result was proved for the action of fundamental groups of compact connected negatively curved manifolds acting on the visual boundary with the manifold measure class.

(3) Let Γ be a finitely generated free group. By [RR07], the action of Γ has non-trivial type with respect to harmonic measures, namely stationary measures of suitable random walks. For Γ being a word hyperbolic group, it was proved in [INO08] that the action of Γ on the Poisson boundary associated with a generating measure of finite support has non-trivial type.

(4) In [Bo14] it was proved that for word hyperbolic groups (with an additional technical condition) the action on the Gromov boundary with respect to a quasi-conformal measure (and in particular the Patterson-Sullivan measure) measure has non-trivial stable type.

8. The free group

8.1. The boundary of the free group. We briefly describe the ingredients we need for our analysis, following the exposition of [BN13a].

Let $\mathbb{F} = \langle a_1, \ldots, a_r \rangle$ be the free group of rank $r \geq 2$, with $S = \{a_i^{\pm 1} : 1 \leq i \leq r\}$ a free set of generators. The (unique) reduced form of an element $g \in \mathbb{F}$ is the expression $g = s_1 \cdots s_n$ with $s_i \in S$ and $s_{i+1} \neq s_i^{-1}$ for all $i$. Define $|g| = n$, the length of the reduced form of $g$. The distance function on $\mathbb{F}$ is defined by $d(g_1, g_2) := |g_1^{-1} g_2|$. The Cayley graph associated with the generating set $S$ is a regular tree of valency 2, and $d$ coincides with its edge-path distance.

The boundary of $\mathbb{F}$ is the set of all sequences $\xi = (\xi_1, \xi_2, \ldots) \in S^\mathbb{N}$ such that $\xi_{i+1} \neq \xi_i^{-1}$ for all $i \geq 1$. We denote it by $\partial \mathbb{F}$. A metric $d_\mathbb{F}$ on $\partial \mathbb{F}$ is defined by $d_\mathbb{F}(g_1, g_2) := \min \{k \in \mathbb{N} \cup \{1\} : g_1 \cdot \cdots \cdot s_k = g_2 \cdot \cdots \cdot s_k\}$ where $n$ is the largest natural number such that $\xi_i = t_i$ for all $i < n$. If $\{g_i\}_{i=1}^{\infty}$ is any sequence of elements in $\mathbb{F}$ and $t_i := t_{1,i} \cdots t_{i,n}$ is the reduced form of $g_i$ then $\lim_i g_i = (\xi_1, \xi_2, \ldots) \in \partial \mathbb{F}$ if $t_{i,j}$ is eventually equal to $\xi_j$ for all $j$. If $\xi \in \partial \mathbb{F}$ then we will write $\xi_i \in S$ for $i$-th element in the sequence $\xi = (\xi_1, \xi_2, \xi_3, \ldots)$.

We define a probability measure $\nu$ on $\partial \mathbb{F}$, by the requirement that every finite sequence $t_1, \ldots, t_n$ with $t_{i+1} \neq t_i^{-1}$ for $1 \leq i < n$, the following holds:

$$\nu\left(\{\xi_1, \xi_2, \ldots \in \partial \mathbb{F} : \xi_i = t_i \quad 1 \leq i \leq n\}\right) := |S_n(e)|^{-1} = (2r - 1)^{-n+1}(2r)^{-1}.$$ 

There is a natural action of $\mathbb{F}$ on $\partial \mathbb{F}$ by

$$(t_1 \cdots t_n)\xi := (t_1, \ldots, t_{n-k}, \xi_{k+1}, \xi_{k+2}, \ldots)$$

where $t_1, \ldots, t_n \in S$, $t_1 \cdots t_n$ is in reduced form and $k$ is the largest number $\leq n$ such that $\xi_i^{-1} = t_{n+1-i}$ for all $i \leq k$. Observe that if $g = t_1 \cdots t_n$ then the Radon-Nikodym derivative satisfies

$$\frac{d\nu \circ g}{d\nu}(\xi) = (2r - 1)^{2k-n}.$$ 

8.2. The horospherical relation and the fundamental cocycle. Let $\mathcal{R}$ be the equivalence relation on $\partial \mathbb{F}$ given by $(\xi, \eta) \in \mathcal{R}$ if and only when writing $\xi = (\xi_1, \xi_2, \ldots)$ and $\eta = (\eta_1, \eta_2, \ldots)$, there exists $n$ such that $\eta_i = \xi_i$ for all $i > n$. Thus $\eta \mathcal{R} \xi$ if and only if $\eta$ and $\xi$ have the same synchronous tail, if and only if they differ by finitely many coordinates only.

Let $\mathcal{R}_n$ be the equivalence relation given by $(\xi, \eta) \in \mathcal{R}_n$ if and only if $\xi_i = \eta_i \forall i > n$. Then $\mathcal{R}$ is the increasing union of the finite subequivalence relations $\mathcal{R}_n$. Thus $\mathcal{R}$ is a hyperfinite relation.

Consider the relation $\mathcal{R}'$ on $\partial \mathbb{F}$ such that $\eta \mathcal{R}' \xi$ if and only if there is a $g \in \mathbb{F}$ such that $g \xi = \eta$ and $\frac{d\nu \circ g}{d\nu}(\xi) = 1$. Note that the level set of the Radon-Nikodym derivative, namely $\left\{g \in \mathbb{F} : \frac{d\nu \circ g}{d\nu}(\xi) = 1\right\}$ is the horosphere in the Cayley tree based at $\xi$ and passing through the identity in $\mathbb{F}$. Note that in that case $\xi = g^{-1} \eta$, and $\frac{d\nu \circ g^{-1}}{d\nu}(\eta) = \left(\frac{d\nu \circ g}{d\nu}(\xi)\right)^{-1} = 1$, so that the
relation is indeed symmetric. The transitivity of the horospherical relation $\mathcal{R}'$ follows from the cocycle identity which the Radon-Nikodym derivative satisfies. Thus $\mathcal{R}'$ is an equivalence relation, and by definition, the measure $\nu$ is $\mathcal{R}'$-invariant.

The relation $\mathcal{R}'$ on $\partial F_r$ can also be defined more concretely by the condition that $(\xi, \eta) \in \mathcal{R}'$ iff there exists $k$ s.t. $\eta = g\xi$ and $g = \eta_1 \cdots \eta_k \cdot \xi_{k+1}^{-1} \cdots \xi_1^{-1}$. It follows that $\eta = g\xi$ has the same synchronous tail as $\xi$ from the $k+1$-th letter onwards. Equivalently, $g^{-1}$ belongs to the horosphere based at $\xi$ and passing through the identity in $F_r$, namely the geodesic from $g^{-1}$ to $\xi$ and the geodesic from $e$ to $\xi$ meet at a point which is equidistant from $e$ and $g^{-1}$. Thus it is natural to call $\mathcal{R}'$ the horospherical relation and the equivalence class of $\xi$ under $\mathcal{R}_n$ the horospherical ball of radius $n$ based at $\xi$. Since $\xi$ and $g\xi$ have the same synchronous tail, $\mathcal{R}'$ coincides with the synchronous tail relation $\mathcal{R}$.

The fundamental cocycle of the tail relation is the measurable map $\alpha : \mathcal{R} \to F_r$, given for $\eta = (\eta_1, \ldots, \eta_k, \ldots)$ and $\xi = (\xi_1, \ldots, \xi_k, \ldots)$ (with $(\eta, \xi) \in \mathcal{R}_k$), by

\[ \alpha(\eta, \xi) = \eta_1 \cdots \eta_k \cdot \xi_{k+1}^{-1} \cdots \xi_1^{-1} \]

so that $\alpha(\eta, \xi)\xi = \eta$. The cocycle takes values in the subgroup $F^r_\eta$ consisting of words of even length, and more precisely for each $k$ the set of values $\alpha(\eta, \xi)$ for fixed $\eta$ and $\xi \in \mathcal{R}_k(\eta)$ coincides with the intersection of the word metric ball $B_{2k}(e)$ with the horoball based at $\eta$ and passing through $e$. This set is called the horospherical ball of radius $2k$ determined by $\eta$ and denoted by $B^\eta_{2k}$.

Note further that the set of cocycle values $\alpha(\eta, \xi)$ for $\xi \mathcal{R}_k \eta$ with $\xi$ fixed, namely the set $(B^\eta_{2k})^{-1}$, is given by

\[ \alpha(\xi, \eta) = \xi_1 \cdots \xi_k \cdot \eta_{k+1}^{-1} \cdots \eta_1^{-1} \]

and this set contains words of length at most $2k$ whose first $(k - 1)$ letters can be specified arbitrarily.

Finally, consider the set of values of the cocycles $\alpha(\eta, \xi)$ for all points $\eta \neq \xi$ which are $\mathcal{R}_k$ equivalent to another, i.e. as we go over all pairs of distinct points in some equivalence class of the form $\mathcal{R}_k(\xi)$. This set of values clearly contains all the even words in a ball of radius $2k - 2$, i.e. $B_{2k-2}(e) \cap F^r_\eta$.

The finite order automorphisms of $\mathcal{R}$ is the subgroup $\Phi$ of $\cup_{n \in \mathbb{N}}[\mathcal{R}_n]$ generated by the transformations defined as follows. Let $\pi_n : \partial F_r \to S^n$ be the projection given by $\pi_n(s_1, s_2, \ldots, s_n) = (s_1, s_2, \ldots, s_n)$. We say that a map $\psi : \partial F_r \to \partial F_r$ has order $n$ if $\psi(\xi) = \psi(\xi')$ for any two boundary points $\xi, \xi' \in \partial F_r$ with $\pi_n(\xi) = \pi_n(\xi')$.

For any $(\xi, \xi') \in \mathcal{R}$ there exists a map $\phi \in \Phi$ such that $\phi(\xi) = \xi'$ and $\phi$ has order $n$ for some $n \in \mathbb{N}$, see $\mathbf{BM}$ [29]. Thus the group of finite order automorphisms clearly generates the synchronous tail (i.e. the horospherical) relation.

The extended horospherical relation. Let $\mathcal{F}_r$ act by measure-preserving transformations on a probability space $(X, \lambda)$. Let $\mathcal{R}_X$ be the equivalence relation on $X \times \partial F_r$ defined by $((x, \xi), (x', \xi')) \in \mathcal{R}_X$ if and only if there exists $g \in F$ with $(gx, g\xi) = (x', \xi')$ and $(\xi, \xi') \in \mathcal{R}_n$ (i.e., if $\xi = (\xi_1, \ldots, \xi_n) \in S^n$ and $\xi' = (\xi'_1, \ldots) \in S^n$ then $\xi_i = \xi'_i$ for all $i \geq n$).

Inspecting the definitions, we see that the extended horospherical relation on $X \times \partial F_r$ coincides with the extension of the horospherical (i.e. synchronous tail) relation $\mathcal{R}$ on $\partial F_r$ via the fundamental cocycle $\alpha : \mathcal{R} \to F_r$, defined above.

It is easy to see that for any $(y, \xi) \in X \times \partial F_r$,

\[ |\mathcal{R}_X(y, \xi)| = |\mathcal{R}_n(\xi)| = (2n - 1)^n. \]

The relation $\mathcal{R}_X = \bigcup_{n \geq 1} \mathcal{R}_X^n$ is thus a hyperfinite measurable equivalence relation, it preserves the measure $\nu \times \lambda$, and it is uniform, namely for each $n \geq 1$ almost every equivalence class of the relation $\mathcal{R}_X^n$ has the same cardinality.

8.3. Shannon-McMillan-Breiman theorem for the free groups. Let $\mathcal{R}_X$ be the equivalence relation on $X \times \partial F_r$. We may assume the action of $\mathcal{F}_r$ on $(X, \lambda)$ is ergodic, and we will use the following.
Theorem 8.1. If $F^e_r$ acts on $(X, \lambda)$ ergodically, then the diagonal action $F^e_r \cap X \times \partial F^e_r$ is ergodic.

The type of the boundary action. The type of the action $F^e_r \cap (\partial F^e_r, \nu)$ is $III_2$ where $\tau = (2r - 1)^{-1}$. It is follows from [BN13a] Theorem 4.1 that the stable type of $F^e_r \cap (\partial F^e_r, \nu)$ is $III_{2r}$. In fact, if $F^e_2$ denotes the index 2 subgroup of $F$ consisting of all elements of even word length then $F^e_r \cap (\partial F^e_r, \nu)$ is of type $III_{2r}$ and stable type $III_{2r}$. It is also weakly mixing. Indeed, $(\partial F^e_r, \nu)$ is naturally identified with $B(F^e_r, \mu)$, the Poisson boundary of the random walk generated by the measure $\mu$ that is distributed uniformly on $S$. By [AL05], the action of any countable group $\Gamma$ on the Poisson boundary $B(\Gamma, \kappa)$ is weakly mixing whenever the measure $\kappa$ is adapted. This shows that $F^e_r \cap (\partial F^e_r, \nu)$ is weakly mixing. Moreover, if we denote $S^2 = \{s': s, t \in S\}$, then $(\partial F^e_r, \nu)$ is naturally identified with the Poisson boundary $B(\partial F^e_r, \mu_2)$ where $\mu_2$ is the uniform probability measure on $S^2$. So the action $F^e_r \cap (\partial F^e_r, \nu)$ is also weakly mixing.

Let $R$ act on $\partial F^e_r \times \mathbb{Z}$ by $g(b, t) = (gb, t - \log r,(g, b))$, which gives the discrete Maharam extension in this case. Let $\mathcal{R}$ be the orbit-equivalence relation restricted to $\partial F^e_r \times \{0\}$, which we may, for convenience, identify with $\partial F^e_r$. In other words, $b R b'$ if and only if there is an element $g \in F^e_r$ such that $gb = b'$ and $\frac{\partial g}{\partial b}(b) = 1$. As noted above, this is the same as the (synchronous) tail-equivalence relation on $F^e_r$.

Let $\alpha : \mathcal{R} \to F^e_r$ be the cocycle $\alpha(gb, b) = g$ for $g \in F^e_r, b \in \partial F^e_r$. This is well-defined almost everywhere because the action of $F^e_r$ is essentially free. Because $F^e_r \cap (\partial F^e_r, \nu)$ has type $III_{2r}$ and stable type $III_{2r}$, this cocycle is weakly mixing for $F^e_r$. In other words, if $F^e_r \cap (X, \mu)$ is any ergodic p.m.p. action, then the equivalence relation $\mathcal{R}^X$ defined on $X \times \partial F^e_r$ by the cocycle extension is ergodic. Since the relation is hyperfinite and the cocycle is injective, we conclude that the action of $F^e_r$ on any ergodic p.m.p. space satisfies the Shannon-McMillan-Breiman theorem, provided only that $\partial F^e_r$ acts ergodically on $X$.

References

[AL05] Aaronson, J., and Lemanczyk, M. Exactness of Rokhlin endomorphisms and weak mixing of Poisson boundaries. Algebraic and topological dynamics. Contemp. Math. 385, Amer. Math. Soc., Providence, RI (2005), pp. 77–87.

[ADR] Anantharaman-Delaroche, C., and Renault, J. Amenable groupoids. L’Enseignement Mathématique, vol. 36, 2000.

[Av10] Avni, N., Entropy Theory for Cross Sections, Geom. Func. Ana., vol. 19 (2010), pp. 1515-1538.

[Br57] Breiman, L., The ergodic and topological theorem of information theory, Ann. Math. Stat. 28 (1957), pp. 809–911. Correction, ibid. 31 (1960), pp. 809–910.

[Bo10a] Bowen, L., Invariant measures on the space of horofunctions of a word-hyperbolic group. Ergodic Theory and Dynamical Systems, 30, no. 1 (2010), pp. 97–129.

[Bo10b] Bowen, L. The ergodic theory of free group actions: entropy and the $f$-invariant. Groups Geom. Dyn. 4, no. 3 (2010), pp. 419-432.

[Bo10c] Bowen, L. Measure conjugacy invariants for actions of countable sofic groups. J. Amer. Math. Soc. 23 (2010), pp. 217–245.

[Bo12] Bowen, L. Sofic entropy and amenable groups. Ergodic Theory Dynam. Systems 32, no. 2 (2012), pp. 427–466.

[Bo14] Bowen, L. The type and stable type of the boundary of a Gromov hyperbolic group. Geometriae Dedicata 172 (2014), pp. 363–386.

[BN13a] Bowen, L. and Nevo, A. Geometric covering arguments and ergodic theorems for free groups. L’Enseignement Mathématique, 59 (2013), pp. 133–164.

[BN13b] Bowen, L. and Nevo, A. Pointwise ergodic theorems beyond amenable groups. Ergod. Th. and Dynam. Sys. 33 (2013), pp. 777–820.

[BN15a] Bowen, L. and Nevo, A. Amenable equivalence relations and the construction of ergodic averages for group actions. To appear in Journal d’Analyse Mathématique, (2015).

[BN15b] Bowen, L. and Nevo, A. von Neumann and Birkhoff ergodic theorems for negatively curved groups. Ann. Sci. Ec. Norm. Supér. 48 (2015), pp. 1113–1147.

[CKK14] Ceccherini-Silberstein, T., Coornaert, M. and Krieger, F. An analogue of Fekete’s lemma for subadditive functions on cancellative amenable semigroups. J. Ana. Math. 124 (2014), pp. 59–81.
