ON THE DYNAMICS OF A DEGENERATE PARABOLIC EQUATION:
GLOBAL BIFURCATION OF STATIONARY STATES AND CONVERGENCE

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Abstract. We study the dynamics of a degenerate parabolic equation with a variable, generally non-smooth diffusion coefficient, which may vanish at some points or be unbounded. We show the existence of a global branch of nonnegative stationary states, covering both the cases of a bounded and an unbounded domain. The global bifurcation of stationary states, implies-in conjunction with the definition of a gradient dynamical system in the natural phase space-that at least in the case of a bounded domain, any solution with nonnegative initial data tends to the trivial or the nonnegative equilibrium. Applications of the global bifurcation result to general degenerate semilinear as well as to quasilinear elliptic equations, are also discussed.

1. Introduction

The mathematical modelling of various physical processes, where spatial heterogeneity has a primary role, has usually as a result, the derivation of nonlinear evolution equations with variable diffusion, or dispersion. Applications are ranging from physics to biology. To name but a few, equations of such a type have been successfully applied to the heat propagation in heterogeneous materials [27, 42, 51, 52], the study of transport of electron temperature in a confined plasma [30], the propagation of varying amplitude waves in a nonlinear medium [70] (and [24] for linear Schrödinger equation), to the study of electromagnetic phenomena in nonhomogeneous superconductors [24, 49, 50] and the dynamics of Josephson junctions [36, 37], to epidemiology and the growth and control of brain tumors [61].

In this work we continue the study, initiated in [55], of the qualitative behavior of solutions of some degenerate evolution equations (involving degenerate coefficients). Work [55] concerns the asymptotic behavior of solutions, of a complex evolution equation of Ginzburg-Landau type. Here we study the following semilinear parabolic equation with variable, nonnegative diffusion coefficient, defined on an arbitrary domain (bounded or unbounded) $\Omega \subseteq \mathbb{R}^N$, $N \geq 2$,

$$
\partial_t \phi - \text{div}(\sigma(x) \nabla \phi) - \lambda \phi + |\phi|^{2\gamma} \phi = 0, \quad x \in \Omega, \quad t > 0,
$$

$$
\phi(x, 0) = \phi_0(x), \quad x \in \Omega,
$$

$$
\phi|_{\partial \Omega} = 0, \quad t > 0.
$$

Equation (1.1) can be derived as a simple model for neutron diffusion (feedback control of nuclear reactor) [27, 50]. In this case $\phi$ (which must be nonnegative) and $\sigma$ stand for the neutron flux and neutron diffusion respectively.

The degeneracy of problem (1.1) is considered in the sense that the measurable, nonnegative diffusion coefficient $\sigma$, is allowed to have at most a finite number of (essential) zeroes, at some points

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or even to be unbounded. The point of departure for the consideration of suitable assumptions on the diffusion coefficient is the work [18], where the degenerate elliptic problem is studied: we assume that the function \( \sigma : \Omega \to \mathbb{R} \) satisfies the following assumptions

\[(H_\alpha) \quad \sigma \in L^1_{\text{loc}}(\Omega) \text{ and for some } \alpha \in (0, 2), \liminf_{x \to z} |x - z|^{-\alpha} \sigma(x) > 0, \text{ for every } z \in \overline{\Omega}, \text{ when the domain } \Omega \text{ is bounded,}\]

\[(H_\infty^\alpha) \quad \sigma \text{ satisfies condition } (H_\alpha) \text{ and } \liminf_{|x| \to \infty} |x|^{-\beta} \sigma(x) > 0, \text{ for some } \beta > 2, \text{ when the domain } \Omega \text{ is unbounded.}\]

The assumptions \((H_\alpha)\) and \((H_\infty^\alpha)\) imply (see [18, Lemma 2.2]) that (i) the set of zeroes is finite, (ii) the function \( \sigma \) could be non smooth (cannot be of class \( C^2 \), if \( \alpha \in (0, 2) \) and it cannot have bounded derivatives if \( \alpha \in (0, 1) \)). Moreover, in the unbounded domain case the function \( \sigma \) has to be unbounded. The approach in [18], was based on Caffarelli-Kohn-Nirenberg type inequalities (see (2.1)). For some recent results concerning these inequalities and their applications to the study of elliptic equations, we refer to \([1, 20, 35]\).

The physical motivation of the assumption \((H_\alpha)\), is related to the modelling of reaction diffusion processes in composite materials, occupying a bounded domain \( \Omega \), which at some points they behave as perfect insulators. Following [27, pg. 79], when at some points the medium is perfectly insulating, it is natural to assume that \( \sigma(x) \) vanishes at these points. On the other hand, when condition \((H_\infty^\infty)\) is satisfied, it follows from [18, Lemma 2.2], that in addition, the diffusion coefficient has to be unbounded. Physically, this situation corresponds to a nonhomogeneous medium, occupying the unbounded domain \( \Omega \), which behaves as a perfect conductor in \( \Omega \setminus B_R(0) \) (see [27, pg.79]), and as a perfect insulator in a finite number of points in \( B_R(0) \). Note that when \( \partial \Omega = \emptyset \), the function \( \sigma(x) \), need not be locally bounded. These conditions arise in various simple transport models of electron temperature in a confined plasma. See [52] for a discussion concerning the one-dimensional case: the electron thermal diffusion is density dependent such that it vanishes with density, rendering the problem singular. Note that in various diffusion processes, the equations involve diffusion \( \sigma(x) \sim |x|^\alpha, \alpha < N \): We refer to \([30, 53]\) for equations describing heat propagation.

The main purpose of this work is to combine basic results from the theory of infinite dimensional dynamical systems and bifurcation theory, to give a description of the dynamics of \((1.1)\). We remark here the crucial role of the conditions \((H_\alpha)\) and \((H_\infty^\infty)\) on the “degeneracy exponents” \( \alpha, \beta \) which give rise to necessary compactness properties of various linear and nonlinear operators associated to the study of \((1.1)\) and its related stationary problem (a degenerate elliptic equation). We are restricted in the case \( N \geq 2 \) since the case \( N = 1 \), despite its similarities with the higher dimensional case with respect to the definition and properties of the appropriate functional setting, recovers also important differences. For the definition and properties of the related function spaces and detailed discussions on one dimensional versions of generalized Hardy and Caffarelli-Kohn-Nirenberg inequalities, we refer to \([19, 20]\).

More precisely, the first part of the present work is devoted to some results concerning the existence of a global attractor. While the result in [55], for the complex evolution equation, concerns the existence of a global attractor in \( L^2(\Omega) \), here it is verified that the dynamical system associated to \((1.1)\) is a gradient system, and that there exists a connected global attractor in the weighted Sobolev space \( D_0^{1,2}(\Omega; \sigma) \), the closure of \( C_0^\infty(\Omega) \) with respect to the norm \( ||\phi||^2_{D_0^{1,2}(\Omega; \sigma)} = \int_\Omega \sigma(x) |\nabla \phi|^2 \). This space appears to be the natural energy space for \((1.1)\). The main result of Section 3, can be stated by the following theorem.
Theorem 1.1. Let $\Omega \subseteq \mathbb{R}^N$, $N \geq 2$, be an arbitrary domain (bounded or unbounded). Assume that $\sigma$ satisfies condition $(H_\alpha)$ or $(H_\infty)$ and

$$0 < \gamma < \frac{2 - \alpha}{2(N - 2 + \alpha)} := \gamma^*.$$  

Equation (1.1) defines a semiflow

$$S(t) : D_0^{1,2}(\Omega, \sigma) \rightarrow D_0^{1,2}(\Omega, \sigma),$$

which possesses a global attractor $A$ in $D_0^{1,2}(\Omega, \sigma)$. Let $E$ denote the (bounded) set of equilibrium points of $S(t)$. For each positive orbit $\phi$ lying in $A$ the limit set $\omega(\phi)$ is a connected subset of $E$ on which

$$J : D_0^{1,2}(\Omega, \sigma) \rightarrow \mathbb{R},$$

$$J(\phi) := \frac{1}{2} \int_{\Omega} \sigma(x)|\nabla \phi|^2 \, dx \quad \text{and} \quad \frac{1}{2} \int_{\Omega} |\phi|^2 \, dx + \frac{1}{2} \int_{\Omega} |\phi|^{2\gamma+2} \, dx,$$

the Lyapunov functional associated to $S(t)$, is constant. If $E$ is totally disconnected (in particular if $E$ is countable), the limit

$$z_+ = \lim_{t \to +\infty} \phi(t),$$

exists and is an equilibrium point. Furthermore, any solution of (1.1), tends to an equilibrium point as $t \to +\infty$.

Further analysis is carried out, regarding the bifurcation of the corresponding steady states with respect to the parameter $\lambda \in \mathbb{R}$. More precisely, we prove the existence of a global branch of nonnegative solutions for the equation

$$- \text{div}(\sigma(x)\nabla u) = \lambda u - |u|^{2\gamma} u, \quad \text{in } \Omega,$$

$$u|_{\partial \Omega} = 0,$$

(1.2)

bifurcating from the trivial solution at $(\lambda_1, 0)$, where $\lambda_1$, is the positive principal eigenvalue of the corresponding linear problem

$$- \text{div}(\sigma(x)\nabla u) = \lambda u, \quad \text{in } \Omega,$$

$$u|_{\partial \Omega} = 0.$$

(1.3)

This is the main result of Section 4, described by the following Theorem.

Theorem 1.2. Let $\Omega \subseteq \mathbb{R}^N$, $N \geq 2$, be an arbitrary domain (bounded or unbounded). Assume that $\sigma$ satisfies condition $(H_\alpha)$ or $(H_\infty)$, and

$$0 < \gamma < \frac{2 - \alpha}{2(N - 2 + \alpha)}.$$  

Then, the principal eigenvalue $\lambda_1$ of (1.3) is a bifurcation point of the problem (1.2) and $C_{\lambda_1}$ is a global branch of nonnegative solutions, which "bends" to the right of $\lambda$. For any fixed $\lambda > \lambda_1$ these solutions are unique.

The technique leading to the global bifurcation result, is included in the general strategy of the approximation of solutions of a degenerate partial differential equation, by constructing an approximate sequence of solutions of nondegenerate problems. The approximation procedure has been successfully applied to evolution [31, 34], and to stationary problems [25, 26, 62], and in the context of bifurcation theory [5, 14, 32, 38].

One of the main difficulties arising, on the attempt to establish the global character of the branch of nonnegative solutions for (1.3), is that Harnack-type Inequalities are not valid in general (see [29, Remark 3.2]). This is a common fact for non-uniformly elliptic equations [39]. However,
we refer to $\Omega$ and the references therein for generalized Harnack-type inequalities, applied to degenerate elliptic equations. Distinguishing between the bounded and the unbounded domain case, we consider different families of approximate boundary value problems.

When $\Omega$ is bounded, $\Omega$ Lemma 2.2 and Remark 2.3 implies that under assumption $(H_\alpha)$, there exists a finite set $Z := \{z_1, \ldots, z_k\} \subset \overline{\Omega}$ and $r, \delta > 0$, such that the balls of center $z_i$ and radius $r$, $B_r(z_i), i = 1, \ldots, k$, are pairwise disjoint and

$$
(i) \quad \sigma(x) \geq |x - z_i|^{\alpha} \quad \text{for} \quad x \in B_r(z_i) \cap \Omega, \quad i = 1, \ldots, k,
(ii) \quad \sigma(x) \geq \delta, \quad \text{for} \quad x \in \Omega \setminus \bigcup_i B_r(z_i).
$$

Moreover if $\sigma$ satisfies $(H_\alpha)$, then $\sigma \geq 0$ in $\overline{\Omega}$, the set of zeroes of $\sigma Z := \{z \in \overline{\Omega} : \sigma(z) = 0\}$ is finite, and $Z_\sigma \subseteq Z$ (Remark 2.3). It is not a loss of a generality to assume that $Z_\sigma = Z$.

For convenience and simplicity, in the bounded domain case, we consider as a model for the diffusion coefficient, the function

$$
\sigma(x) = |x|^\alpha, \quad \alpha \in (0, 2),
$$

satisfying $(H_\alpha)$. Quite naturally, we construct a family of approximating nondegenerate problems as follows: Setting $\Omega_r := \Omega \setminus B_r(0)$, we consider the boundary value problems

$$(P)_r \quad \begin{cases} -\text{div}(\sigma(x) \nabla u) = \lambda u - |u|^{2\gamma} u, & \text{in} \ \Omega_r, \\
u|_{\partial \Omega_r} = 0. \end{cases}
$$

From the characterization [12], problems $(P)_r$ are non-degenerate, and it can be shown that for fixed $r > 0$, there exists a global branch of positive solutions (see Definition 2.6), by using Harnack type inequalities. The next step is to prove that the limit of the approximating family $(P)_r$, as $r \to 0$, preserves the same property, thus Theorem 1.2.

When $\Omega$ is unbounded $\Omega$ Lemma 2.2 and Remark 2.3 implies that under $(H_\beta)$, in addition to (1.3), there exists $R > 0$, such that $B_r(z_i) \subset B_R(0)$ for every $i, \ldots, k$ and

$$
(iii) \quad \sigma(x) \geq \delta |x|^\beta, \quad \text{for} \quad x \in \Omega, |x| > R.
$$

In the unbounded domain case we consider as a model, the diffusion coefficient

$$
\sigma(x) = |x|^\alpha + |x|^\beta, \quad \alpha \in (0, 2), \quad \beta > 2,
$$

satisfying $(H_\beta)$. Note that since $\sigma$ is unbounded the Harnack inequality is still not applicable. To approximate [12] defined in the unbounded domain $(\Omega \subseteq \mathbb{R}^N)$, this time we consider the approximate family of boundary value problems in $\Omega_R := \Omega \cap B_R(0)$:

$$(P)_R \quad \begin{cases} -\text{div}(\sigma(x) \nabla u) = \lambda u - |u|^{2\gamma} u, & \text{in} \ \Omega_R, \\
u|_{\partial \Omega_R} = 0. \end{cases}
$$

Theorem 1.2 holds for $(P)_R$ and the claim is that as $R \to \infty$ the theorem remains valid at the limit.

To establish the properties of the principal eigenvalues corresponding to both of the approximating problems $(P)_r$ and $(P)_R$, we prefer an alternative proof, based on an appropriate adaptation of Picone’s Identity. This identity has been used in [2, 3, 4], where the author established certain properties of the principal eigenvalue of the p-Laplacian operator, and extends Sturm Theorems to degenerate elliptic equations.

Furthermore, we note that the presented method is applicable independently of the shape of $\Omega$. In general, the situation becomes more complicated for non-uniformly elliptic problems in terms of $u$. As an example of the appearance of local bifurcation, for such a type of equation, we refer to [60].
A general treatment of degenerate elliptic equations is provided by the monograph [29], focusing on the existence and properties of solutions (the issue of global bifurcation in the degenerate case is not addressed). Especially in the unbounded domain case, the problems are non-degenerate (at least in the sense of degeneracy, imposed by assumption (1.6)). In [32] a global bifurcation result is proved for a degenerate semilinear elliptic equation, with a degenerate diffusion coefficient of “critical exponent” (inducing non-compactness). Recent global bifurcation results for non-degenerate problems are included in the works [6, 32, 38, 40, 58, 66, 67]. For an overview, we also refer to the latest monographs [15, 54].

It is our intention to use Theorem 1.2 as a main tool, for a more detailed description of the asymptotic behavior of solutions of (1.1), at least for the case of a bounded domain. A consequence of Theorem 1.2 is that for any fixed \( \lambda > \lambda_1 \), the set \( E \) includes the trivial, the unique nonnegative solution of (1.1) and its (unique) nonpositive reflection. A combination of Theorems 1.1-1.2 could be used to design an intuitive picture for the dynamics of (1.1): It seems that the system undergoes through \( \lambda_1 \) a pitchfork bifurcation of supercritical type, where exchange of stability holds, i.e., the trivial solution is stable when \( \lambda < \lambda_1 \), while for \( \lambda_1 < \lambda \) the nonnegative (nonpositive) solution of the global branch become the stable stationary state. Section 5 is devoted to some remarks related to the rigorous verification of the bifurcation picture for (1.2). The fact that solutions of (1.1) with nonnegative initial data, remain nonnegative for all times (a “maximum principle” property), and the stability analysis of the unique nonnegative steady state, in conjunction with [13, Theorem 2.7], implies the following

**Corollary 1.3.** Assume that condition \((H_\alpha)\), holds. If \( \phi_0 \geq 0 \ a.e \ in \ \Omega \), any solution \( \phi(t) \) of (1.1), tends to either the trivial or the unique nonnegative equilibrium point, as \( t \to \infty \).

As it is expected, the nonnegative steady state is a global minimizer for the Lyapunov functional (Remark 5.3). A comment on the role of the “degeneracy exponent” \( \alpha \) and a discussion concerning some possible further developments with respect to the case of noncompactness, is given in Remarks 5.5, 5.6.

We conclude by mentioning the main results, on the convergence of globally defined and bounded solutions of evolution equations to rest points, as \( t \to \infty \).

For scalar parabolic equations we refer to [59, 60, 74] for convergence to a single equilibrium. In [57] the result is proved for a semilinear heat equation defined in a higher dimensional domain, assuming a special structure of the set of rest points (semistable solutions). In [44], convergence to a unique rest point, at least for the scalar case, is proved without the hypothesis that the set of rest points is totally disconnected. The same result is extended to semilinear parabolic and wave equations considered in multidimensional domains in [45, 46, 48], when the nonlinearity is analytic. For a scalar degenerate parabolic problem (porous medium equation) a positive answer is given in [34]. In the recent work [17], the result of convergence to a (single) equilibrium is extended to a semilinear parabolic equation in \( \mathbb{R}^N \): The main difficulty in the unbounded domain case is that even there exists a unique rest point \( z \) (radial with respect to 0), the \( \omega \)-limit set, may contains infinite many distinct translates of \( z \). The authors introduce a new method, by defining moments of energy, which can discriminate against different translates of a rest point. The work [17] provides also a brief but complete review of the existing results and methods. For a more detailed survey we refer to [64].

In the case of non-autonomous systems or in the case where uniqueness of solutions of the evolution equation is not expected, the question on the convergence of solutions to rest points, and generally, on the existence of a global attractor, is discussed through the framework of generalized processes and semiflows in [11, 12, 13]. Applications include nonautonomous semilinear wave
2. Preliminaries

Function spaces and formulation of the problem. We recall some of the basic results on functional spaces defined in [18]. Let $N \geq 2$, $\alpha \in (0, 2)$ and

\[ 2^*_\alpha := \begin{cases} \frac{4}{\alpha} \in (2, +\infty), & \text{if } \alpha \in (0, 2), \quad N = 2, \\ \frac{2N}{N-2+\alpha} \in (2, \frac{2N}{2-N}), & \text{if } \alpha \in (0, 2), \quad N \geq 3. \end{cases} \]

The exponent $2^*_\alpha$, has the role of the critical exponent in the classical Sobolev embeddings. The following Caffarelli-Kohn-Nirenberg inequality holds, for a constant $c$ depending only on $\beta, N$,

\begin{equation}
\left( \int_{\mathbb{R}^N} |\phi|^{2^*_\alpha} \, dx \right)^{\frac{1}{2^*_\alpha}} \leq c \int_{\mathbb{R}^N} |x|^\beta |\nabla \phi|^2 \, dx, \quad \text{for every } \phi \in C_0^\infty(\mathbb{R}^N).
\end{equation}

By using (2.1) and conditions $(\mathcal{H}_\alpha)$ and $(\mathcal{H}_{2^*_\alpha})$, it is proved in [18, Proposition 2.5], the following generalized version of (2.1),

\begin{equation}
\left( \int_{\Omega} |\phi|^{2^*_\alpha} \, dx \right)^{\frac{1}{2^*_\alpha}} \leq K \int_{\Omega} \sigma(x) |\nabla \phi|^2 \, dx, \quad \text{for every } \phi \in C_0^\infty(\Omega).
\end{equation}

As a consequence of (2.1) and (2.2), we have the following generalized version of Poincaré inequality ([18, Corollary 2.6-Proposition 3.5], see also [55, Section 5]).

Lemma 2.1. Let $\Omega$ be a bounded (unbounded) domain of $\mathbb{R}^N$, $N \geq 2$ and assume that condition $(\mathcal{H}_\alpha)$ ($(\mathcal{H}_{2^*_\alpha})$) is satisfied. Then there exists a constant $c > 0$, such that

\begin{equation}
\int_{\Omega} |\phi|^2 \, dx \leq c \int_{\Omega} \sigma(x) |\nabla \phi|^2 \, dx, \quad \text{for every } \phi \in C_0^\infty(\Omega).
\end{equation}

We emphasize that inequalities (2.1), (2.2) (and (2.3) in the case of a bounded domain), hold for some $\alpha \in [0, 2]$. However, the case $\alpha = 2$ can be considered as a “critical case” with respect to compactness of various embeddings, even in the bounded domain case. Moreover, condition $(\mathcal{H}_\alpha)$ is optimal in the following sense: For $\alpha > 2$ there exist functions such that (2.3) is not satisfied ([18]. Note also that in the case of an unbounded domain, (2.3) does not hold in general, if $\beta \leq 2$ in $(\mathcal{H}_{2^*_\alpha})$. We refer also to the examples of [1].

The natural energy space for the problems (1.1) and (1.2) involves the space $D_0^{1,2}(\Omega, \sigma)$, defined as the closure of $C_0^\infty(\Omega)$ with respect to the norm

\[ ||\phi||_{D_0^{1,2}(\Omega, \sigma)} := \left( \int_{\Omega} \sigma(x) |\nabla \phi|^2 \right)^{1/2}. \]

The space $D_0^{1,2}(\Omega, \sigma)$ is a Hilbert space with respect to the scalar product

\[ (\phi, \psi)_\sigma := \int_{\Omega} \sigma(x) \nabla \phi \nabla \psi \, dx, \quad \text{for every } \phi, \psi \in D_0^{1,2}(\Omega, \sigma). \]

The following two lemmas refer to the continuous and compact inclusions of $D_0^{1,2}(\Omega, \sigma)$ ([18, Propositions 3.3-3.5]).

Lemma 2.2. Assume that $\Omega$ is a bounded domain in $\mathbb{R}^N$, $N \geq 2$ and $\sigma$ satisfies $(\mathcal{H}_\alpha)$. Then the following embeddings hold:

i) $D_0^{2,\alpha}(\Omega, \sigma) \hookrightarrow L^{2^*_\alpha}(\Omega)$ continuously,

ii) $D_0^{1,2}(\Omega, \sigma) \hookrightarrow L^p(\Omega)$ compactly if $p \in [1, 2^*_\alpha).$
Lemma 2.3. Assume that $\Omega$ is an unbounded domain in $\mathbb{R}^N$, $N \geq 2$, and $\sigma$ satisfies $(\mathcal{H}_N^\infty)$. Then the following embeddings hold:

i) $D_0^{1,2}(\Omega, \sigma) \hookrightarrow L^p(\Omega)$ continuously for every $p \in [2^*_\beta, 2^*_\alpha]$.

ii) $D_0^{1,2}(\Omega, \sigma) \hookrightarrow L^p(\Omega)$ compactly if $p \in (2^*_\alpha, 2^*_\alpha)$.

Remark 2.4. It is crucial to note that as a special case, the embedding $D_0^{1,2}(\Omega, \sigma) \subset L^2(\Omega)$ is compact if either conditions $(\mathcal{H}_\alpha)$ or $(\mathcal{H}_N^\infty)$ hold: Observe that $\beta > 2$ implies $2^*_\beta = \frac{2N}{N-2} < 2$, i.e. $2 \in (2^*_\alpha, 2^*_\alpha)$. In the unbounded domain case, we need $\sigma$ to grow faster than quadratically at infinity, to ensure compactness. We also stress the fact, that since $\sigma$ is not in $L^\infty_{\text{loc}}(\Omega)$, there is not in general any inclusion relation between the space $D_0^{1,2}(\Omega, \sigma)$ and the standard Sobolev space $H_0^1(\Omega)$.

To justify the natural energy space for equation (1.1), we have applied in [55], Friedrich’s extension theory [21, pg. 28, 32], [73, pg. 126-135]: Assuming conditions $(\mathcal{H}_\alpha)$ or $(\mathcal{H}_N^\infty)$, the operator $T = -\text{div}(\sigma(x) \nabla \phi)$ is positive and self-adjoint, with domain of definition

$$D(T) = \left\{ \phi \in D_0^{1,2}(\Omega, \sigma), \ T\phi \in L^2(\Omega) \right\}.$$  

The space $D(T)$, is a Hilbert space endowed with the usual graph scalar product. Moreover, there exist a complete system of eigensolutions $\{e_j, \lambda_j\}$,

$$\left\{ -\text{div}(\sigma(x) \nabla e_j) = \lambda_j e_j, \quad e_j \in D_0^{1,2}(\Omega, \sigma), \quad \lambda_j \to \infty, \quad \text{as} \quad j \to \infty. \right.$$  

The fractional powers are defined as follows: For every $s > 0$, $T^s$ is an unbounded selfadjoint operator in $L^2(\Omega)$, with domain $D(T^s)$ to be a dense subset in $L^2(\Omega)$. The operator $T^s$ is strictly positive and injective. Also, $D(T^s)$ endowed with the scalar product $(\phi, \psi)_{D(T^s)} = (T^s \phi, T^s \psi)_{L^2}$, becomes a Hilbert space. We write as usual, $V_{2s} = D(T^s)$ and we have the following identifications $D(T^{-1/2}) = D_0^{1,2}(\Omega, \sigma)$ the dual of $D_0^{1,2}(\Omega, \sigma)$, $D(T^0) = L^2(\Omega)$ and $D(T^{1/2}) = D_0^{1,2}(\Omega, \sigma)$.

Moreover, the injection $V_{2s_1} \subset V_{2s_2}, \quad s_1, s_2 \in \mathbb{R}, \quad s_1 > s_2$, is compact and dense.

While in [55], the local in time solvability was discussed via compactness methods, for the purposes of the present work, it is more convenient to study the local in time solvability of equation (1.1) in $D_0^{1,2}(\Omega, \sigma)$, via the semigroup method approach: The discussion above clearly shows, that the operator $-T$ is the generator of a linear strongly continuous semigroup $T(t)$ ([10], [22], [55]).

Definition 2.5. For a given function $\phi_0 \in D_0^{1,2}(\Omega, \sigma)$, $0 < \gamma < \infty$ and $T > 0$, a solution for the problem (1.1) is a function

$$\phi(x, t) \in C([0, T]; D_0^{1,2}(\Omega, \sigma)) \cap C^1([0, T]; L^2(\Omega)),$$

satisfying the variation of constants formula

$$\phi(t) = T(t)\phi_0 + \int_0^t T(t-s)f(\phi(s))ds$$

where $f(s) = \lambda s - |s|^{2\gamma} s$.

Solutions of (1.1) satisfying Definition 2.5 and solutions satisfying [55] Definition 2.3 (weak solutions) are the same. This is an immediate consequence of (1.1).

We conclude this introductory section, by stating for the convenience of the reader, some basic definitions and results for our analysis. We state first a result on the existence of a branch of solutions of an operator equation (bifurcation in the sense of Rabinowitz [88], see also [28]).
**Theorem 2.6.** Let $X$ be a Banach space with norm $\| \cdot \|_X$ and consider the operators

$$G(\cdot, \cdot), L, H(\cdot, \cdot) : X \to X^*,$$

where $G(\cdot, \cdot) = \lambda L(\cdot) + H(\cdot, \cdot)$, $L$ is a compact linear operator and $H(\cdot, \cdot)$ is compact and satisfies

$$\lim_{\|u\|_X \to 0} \|H(\lambda, u)\|_{X^*} = 0.$$

If $\lambda$ is a simple eigenvalue of $L$ then the closure of the set

$$C = \{ (\lambda, u) \in \mathbb{R} \times X : (\lambda, u) \text{ solves } N(\lambda, u) := u - G(\lambda, u) = 0 \text{ in } X^*, \ u \neq 0 \},$$

possesses a maximal continuum (i.e. connected branch) of solutions, $C_\lambda$, such that $(\lambda, 0) \in C_\lambda$ and $C_\lambda$ either:

(i) meets infinity in $\mathbb{R} \times X$ or,

(ii) meets $(\lambda^*, 0)$, where $\lambda^* \neq \lambda$ is also an eigenvalue of $L$.

In the approximation procedure, we are making use of a generalized Harnack-type inequality (see [29, 39] and the references therein).

**Theorem 2.7.** (Harnack-type Inequality) Consider the equation

$$-\text{div}(a(x, u) |\nabla u|^{p-2} \nabla u) = f(x, u), \quad x \in \Omega,$$

where $\Omega \subseteq \mathbb{R}^N$, $1 < p < N$ and the functions $a$ and $f$ satisfy the following conditions:

(i) $a$ is a Carathéodory function, such that $a(x, s)$ is uniformly separated from zero and bounded for almost every $x \in \Omega$ and all $s \in \mathbb{R}$,

(ii) $f$ is a Carathéodory function and for any $M > 0$ there exists a constant $c_M > 0$, such that

$$|f(x, s)| \leq c_M |s|^{p^* - 1},$$

for almost every $x \in \Omega$ and all $s \in (-M, M)$, where $p^*$ is the critical Sobolev exponent $p^* = \frac{Np}{N-p}$.

Assume that $u \in D^{1,p}(\Omega) := \{ u \in L^{p^*}(\Omega) : \nabla u \in (L^p(\Omega))^N \}$ is a weak solution of (2.6) satisfying the weak formula

$$\int_\Omega a(x, u)|\nabla u|^{p-2}\nabla u\nabla \phi \, dx = \int_\Omega f(x, u) \phi \, dx,$$

holds for any $\phi \in C_c^\infty(\Omega)$. Then, for any cube $K = K(3\rho) \subset \Omega$ with $0 \leq \rho < M$ in $K$, we have that

$$\max_{x \in K, \rho} u(x) \leq C \min_{x \in K, \rho} u(x).$$

In particular, if the weak solution $u \neq 0$ of (2.6) satisfies $u \geq 0$ in $\Omega$ then it follows that $u$ is strictly positive in $\Omega$.

**Remark 2.8.** In the case where $a(x, s) \equiv a(x)$ satisfies condition (i) of Theorem 2.6, the norms of $D^{1,2}_{0}(\Omega, a)$ and $D^{1,2}(\Omega)$ are equivalent.

We also recall some basic definitions and results on semiflows (see [13] [14] and [11, 71]). Let $X$ be a complete metric space. For each $\phi_0 \in X$, via the correspondence $S(t)\phi_0 = \phi(t)$, a semiflow is a family of continuous maps $S(t) : X \to X$, $t \geq 0$, satisfying the semigroup identities (a) $S(0) = I$,

(b) $S(s + t) = S(s)S(t)$.

For $B \subset X$, and $t \geq 0$

$$S(t)B := \{ \phi(t) = S(t)\phi_0 \text{ with } \phi(0) = \phi_0 \in B \}.$$

The **positive orbit** of $\phi$ through $\phi_0$ is the set $\gamma^+ (\phi_0) = \{ \phi(t) = S(t)\phi_0, t \geq 0 \}$. If $B \subset X$ then the positive orbit of $B$ is the set

$$\gamma^+ (B) = \bigcup_{t \geq 0} S(t)B = \{ \gamma^+ (\phi) : \phi(0) = S(t)\phi_0 \text{ with } \phi(0) = \phi_0 \in B \}.$$
If \( t_0 \geq 0 \), \( \gamma^{t_0}(B) := \bigcup_{t \geq t_0} S(t)B = \gamma^+(S(t_0)B) \). The \( \omega \)-limit set of \( \phi_0 \in X \) is the set \( \omega(\phi_0) = \{ z \in X : \phi(t_j) = S(t_j)\phi_0 \rightarrow z \text{ for some sequence } t_j \rightarrow +\infty \} \). A complete orbit containing \( \phi_0 \in X \), is a function \( \phi : \mathbb{R} \rightarrow X \) such that \( \phi(0) = \phi_0 \) and for any \( s \in \mathbb{R}, S(t)\phi(s) = \phi(t+s) \) for \( t \geq 0 \). If \( \phi \) is a complete orbit containing \( \phi_0 \), then the \( \alpha \)-limit set of \( \phi_0 \) is the set
\[ \alpha(\phi_0) = \{ z \in X : \phi(t_j) \rightarrow z \text{ for some sequence } t_j \rightarrow -\infty \}. \]

The subset \( A \) attracts a set \( B \) if \( \text{dist}(S(t)B, A) \rightarrow 0 \) as \( t \rightarrow +\infty \). The set \( A \) is positively invariant if \( S(t)A \subset A \) for all \( t \geq 0 \) and invariant if \( S(t)A = A \) for all \( t \geq 0 \). The set \( A \) is a global attractor if it is compact, invariant, and attracts all bounded sets.

The semiflow \( S(t) \) is eventually bounded if given any bounded set \( B \subset X \), there exists \( t_0 \geq 0 \) such that the set \( \gamma^{t_0}(B) \) is bounded. The semiflow \( S(t) \) is said to be point dissipative if there is a bounded set \( B_0 \) that attracts each point of \( X \). It is called asymptotically compact if for any bounded sequence \( \phi_n \) in \( X \) and for any sequence \( t_n \rightarrow \infty \), the sequence \( S(t_n)\phi_n \) has a convergent subsequence. It is called asymptotically smooth if whenever \( B \) is nonempty, bounded and positively invariant, there exists a compact set \( \mathcal{K} \) which attracts \( B \).

A complete orbit is stationary if \( \phi(t) = z \) for all \( t \in \mathbb{R} \) for some \( z \in X \) and each such \( z \), is called an equilibrium point. We denote \( \mathcal{E} \) the set of stationary points.

The functional \( J : X \rightarrow \mathbb{R} \) is a Lyapunov functional for the semiflow \( S(t) \) if (i) \( J \) is continuous, (ii) \( J(S(t)\phi_0) \leq J(S(s)\phi_0) \) and \( t \geq s \geq 0 \), (iii) if \( J(\phi(t)) = \text{constant} \) for some complete orbit \( \phi \) and all \( t \in \mathbb{R} \), then \( \phi \) is stationary.

To derive the convergence result we shall use the following Theorem.

**Theorem 2.8.** ([13]) Let \( S(t) \) be an asymptotically compact semiflow and suppose that there exists a Lyapunov function \( J \). Suppose further that the set \( \mathcal{E} \) is bounded. Then \( S(t) \) is point dissipative, so that there exists a global attractor \( A \). For each complete orbit \( \phi \) containing \( \phi_0 \) lying in \( A \) the limit sets \( \alpha(\phi_0) \) and \( \omega(\phi_0) \) are connected subsets of \( \mathcal{E} \) on which \( J \) is constant. If \( \mathcal{E} \) is totally disconnected (in particular if \( \mathcal{E} \) is countable) the limits
\[ z_- = \lim_{t \rightarrow -\infty} \phi(t), \quad z_+ = \lim_{t \rightarrow +\infty} \phi(t) \]
exist and are equilibrium points. Furthermore any solution \( S(t)\phi_0 \) tends to an equilibrium point as \( t \rightarrow \infty \).

### 3. Global Attractor in \( D_{0.2}^{1,2}(\Omega, \sigma) \)

In this section we shall show, that the degenerate semilinear parabolic equation ([14]) defines a semiflow in the energy space \( D_{0.2}^{1,2}(\Omega, \sigma) \), possessing a global attractor. We state first an auxiliary lemma.

**Lemma 3.1.** Assume that either conditions \( (\mathcal{H}_\alpha) \) or \( (\mathcal{H}^\infty_\alpha) \) hold. The function \( f_1(s) := \lvert s \rvert^{2\gamma} s, s \in \mathbb{R}, \) defines a sequentially weakly continuous map \( f_1 : D_0^{1,2}(\Omega, \sigma) \rightarrow L^2(\Omega) \) if
\begin{equation}
0 < \gamma \leq \frac{2 - \alpha}{2(N - 2 + \alpha)} := \gamma_*.
\end{equation}
Furthermore, if \( F_1(\phi) := \int_0^\phi f_1(s)ds \), the functional \( E_1 : D_{0.2}^{1,2}(\Omega, \sigma) \rightarrow \mathbb{R} \) defined by \( E_1(\phi) = \int_\Omega F_1(\phi)dx \), is \( C^1(D_{0.2}^{1,2}(\Omega, \sigma), \mathbb{R}) \) and sequentially weakly continuous.

**Proof:** It can be easily checked that the functional \( f_1 \) is well defined, under the restriction ([8], by using Lemmas 2.2(i), 2.3(i)). Similarly, it follows that \( E_1 \) is well defined if
\begin{equation}
0 < \gamma \leq \frac{2 - \alpha}{N - 2 + \alpha} := \gamma_1.
\end{equation}
and note that $\gamma^* < \gamma_1$. To show that both functionals are sequentially weakly continuous, we use the compactness of the embeddings stated in Lemmas 2.2(ii)-2.3(ii), and repeat the lines of the proof of [13, Lemma 3.3, pg. 38 & Theorem 3.6, pg. 40]. To verify that $E_1$ is a $C^1$-functional, and its derivative is given by

\[(3.3)\]

\[E_1'(\phi)(z) = (f_1(\phi), z), \quad \text{for every } \phi \in D_0^{1,2}(\Omega, \sigma), \quad z \in D_0^{-1}(\Omega, \sigma),\]

we consider for $\phi, \psi \in D_0^{1,2}(\Omega, \sigma)$, the quantity

\[
\frac{E_1(\phi + s\psi) - E_1(\phi)}{s} = \frac{1}{s} \int_0^1 \frac{d}{d\theta} F_1(\phi + \theta s\psi) d\theta dx \\
= \int_0^1 f_1(\phi + s\theta \psi) d\theta dx.
\]

(3.4)

Setting $q = \frac{2N}{N+2-\alpha}, \quad q + 2_\alpha^* = 1$, we observe that

\[
\int f_1(\phi + \theta s\psi) \psi dx \leq c \left( \int (|\phi|^{2\gamma+1} + |\psi|^{2\gamma+1})^q dx \right)^{\frac{1}{q}} \left( \int |\psi|^2 dx \right)^{\frac{2}{q}}.
\]

(3.5)

Lemmas 2.2.4(i)-2.3(i) are applicable under the requirement $(2\gamma + 1)q \leq 2\gamma^*$ which justifies (3.3).

Using the dominated convergence theorem, we may let $s \to 0$, to obtain that $E$ is differentiable with the derivative (3.3).

We consider next a sequence $\{\phi_n\}_{n \in \mathbb{N}}$ of $D_0^{1,2}(\Omega, \sigma)$ such that $\phi_n \to \phi$ in $D_0^{1,2}(\Omega, \sigma)$ as $n \to \infty$. It holds that

\[(3.6)\]

\[\langle E_1'(\phi_n) - E_1'(\phi), z \rangle \leq ||f_1(\phi_n) - f_1(\phi)||_{L_2} ||z||_{L_2}^{2\gamma^*}.\]

Setting $p_1 = \frac{2_\alpha^*}{q}$ we observe that the requirement for $p_1 > 1$, justifies the restrictions on the exponent of degeneracy $\alpha$, imposed by $(\mathcal{H}_\alpha)$ or $(\mathcal{H}_\alpha^\infty)$. Setting now $p_2 = \frac{N+2-\alpha}{2(2-\alpha)}$, $p_2^{-1} + p_1^{-1} = 1$, we get

\[||f_1(\phi_n) - f_1(\phi)||_{L_2} \leq c \left( \int (|\phi_n|^{2\gamma} + |\phi|^{2\gamma})^q dx \right)^{\frac{1}{q}} \left( \int |\phi_m - \phi|^{2\alpha^*} dx \right)^{\frac{1}{p_1}} \equiv \Lambda(\phi_m, \phi).
\]

Let $p_3 = 2\gamma qp_2$. To apply Lemmas 2.2.4(i)-2.3(i) once again, we need $p_3 \leq 2\gamma^*$ or (3.3).

Under this condition we have that $\Lambda(\phi_n, \phi) \to 0$ as $n \to \infty$ and from (3.3), we get the continuity of $E'$. \dots

We consider the energy functional $J : D_0^{1,2}(\Omega, \sigma) \to \mathbb{R}$

\[(3.7)\]

\[J(\phi) := \frac{1}{2} \int \sigma(x)|\nabla \phi|^2 dx - \frac{\lambda}{2} \int |\phi|^2 dx + \frac{1}{2\gamma + 2} \int |\phi|^{2\gamma + 2} dx.
\]

**Proposition 3.2.** Let $\phi_0 \in D_0^{1,2}(\Omega, \sigma)$ and either conditions $(\mathcal{H}_\alpha)$ or $(\mathcal{H}_\alpha^\infty)$ be fulfilled, and assume that (3.7) holds. Then equation (3.7) has a unique, global in time (weak) solution $\phi$, such that

\[(3.8)\]

\[\phi \in C([0, \infty); D_0^{1,2}(\Omega, \sigma)) \cap C^1([0, \infty); L^2(\Omega)).\]

For each (weak solution) $J(\phi(\cdot)) \in C^1([0, \infty))$ and

\[(3.9)\]

\[\frac{d}{dt} J(\phi(t)) = - \int \partial_t \phi^2 dx.
\]
Lemma 2.3 (i), we have that for some \( \theta \) to derive (3.9).

(3.11) Now using the fact that \( J \) we get that (3.12)

\[
\phi(0) = \phi_0, \text{ defined on a maximal interval } [0, T_{max}], \text{ where } 0 < T_{max} \leq \infty \text{ [22].}
\]

We proceed by showing that \( T_{max} = \infty \). First note, that by Lemma 5.5, pg. 246-247 (see also [13, Theorem 3.6, pg. 41]) we get that (3.12).

We define \( \phi(0) = \phi_0 \), \( \phi_n \) \( \in D(T) \), such that

\[
\phi_n \rightarrow \phi, \text{ in } C([0, T]; D^{1,2}_0(\Omega; \sigma)),
\]

\[
\phi_{0n} \rightarrow \phi_0, \text{ in } D^{1,2}_0(\Omega; \sigma).
\]

We repeat the main lines of the proof, only for the shake of completeness: For all \( \phi \in D(T) \), then

\[
\langle -T\phi + f(\phi), J'(\phi) \rangle = -\int_\Omega |\nabla(\sigma(x)\nabla \phi)|^2 dx
\]

(3.10) \[
= -\int_\Omega |\partial_t \phi|^2 dx \leq 0.
\]

Setting \( g(t) = f(\phi(t)) \) we consider sequences \( g_n(t) \in C^1([0, T]; D^{1,2}_0(\Omega; \sigma)) \) and \( \phi_{0n} \in D(T) \) such that

\[
g_n \rightarrow g, \text{ in } C^1([0, T]; D^{1,2}_0(\Omega; \sigma)),
\]

\[
\phi_{0n} \rightarrow \phi_0, \text{ in } D^{1,2}_0(\Omega; \sigma).
\]

We define \( \phi_n(t) = T(t)\phi_{0n} + \int_0^t T(t-s)g_n(s)ds \), and it follows from [63, Corollary 2.5, p107] that \( \phi_n(t) \in D(T), \phi_n \in C^1([0, T]; D^{1,2}_0(\Omega; \sigma)) \) satisfying \( \frac{d}{dt}\phi_n(t) + T\phi + f(\phi) = 0 \). Also, from [11, Lemma 5.5, pg. 246-247] (see also [13, Theorem 3.6, pg. 41]) we get that

\[
\phi_n \rightarrow \phi, \text{ in } C([0, T]; D^{1,2}_0(\Omega; \sigma)).
\]

Now using the fact that \( J \) is \( C^1 \) and (3.10), we may pass to the limit to

\[
J(\phi_n(t)) - J(\phi_{0n}) = \int_0^t \langle J'(\phi_n(s)), -T\phi_n(s) + g_n(s) \rangle ds
\]

\[
= -\int_0^t |\partial_t \phi_n(s)|^2 ds + \int_0^t \langle J'(\phi_n(s)), g_n(s) - f(\phi_n(s)) \rangle ds
\]

to derive (3.9).

Multiplying (1.1) by \( \phi \), and integrating over \( \Omega \), we obtain the equation

\[
\frac{1}{2} \frac{d}{dt} ||\phi||^2_{L^2} + \int_\Omega \sigma(x)|\nabla \phi|^2 dx - \lambda ||\phi||^2_{L^2} + \int_\Omega |\phi|^{2\gamma+2} dx = 0.
\]

(3.11) We are focusing on the case where \( \lambda > \lambda_1 \) and the domain is unbounded. By interpolation and Lemma 2.3 (i), we have that for some \( \theta \in (0, 1) \),

\[
2\lambda ||\phi||^2_{L^2} \leq 2\lambda ||\phi||^2_{L^{2\gamma+2}} ||\phi||^{2(1-\theta)}_{L^{2\gamma+2}} \leq 2\lambda C_{1}^{2(1-\theta)} ||\phi||^2_{L^{2\gamma+2}} ||\phi||^{2(1-\theta)}_{D^{1,2}_0(\Omega, \sigma)} \leq \frac{1}{2} ||\phi||^2_{L^{1,2}_0(\Omega, \sigma)} + c_1 ||\phi||^2_{L^{2\gamma+2}} \leq \frac{1}{2} ||\phi||^2_{L^{1,2}_0(\Omega, \sigma)} + \frac{1}{2} ||\phi||^2_{L^{2\gamma+2}} + R_1,
\]

(3.12)
where $C_\beta$ is the constant of the embedding $D_0^{1,2}(\Omega, \sigma) \subset L^{2^\ast}$. Then

$$
\frac{d}{dt}\frac{1}{2}\|\phi\|_{L_2}^2 + \int_{\Omega} \sigma(x)|\nabla \phi|^2 dx - \frac{\lambda}{2} \int_{\Omega}\|\phi\|^2 dx \\
\geq \frac{1}{2} \int_{\Omega} \sigma(x)|\nabla \phi|^2 dx - \frac{\lambda}{2} \rho_1^2, \quad t \geq t_0.
$$

Hence, since $J(\phi(t))$ is nonincreasing in $t$, we conclude that

$$
\|\phi(t)\|_{L_2}^2 \leq 2J(\phi_0) + \lambda \rho_1^2, \quad t \geq t_0.
$$

Thus solutions are globally defined in $D_0^{1,2}(\Omega, \sigma)$.

**Proof of Theorem 1.1** It is not a loss of generality to assume that $\phi_0 \in B(0, R)$, a closed ball of $D_0^{1,2}(\Omega, \sigma)$, of center 0 and radius $R$. Then from Lemma 3.4 and 5.7, it follows that there exists a constant $c(R)$ such that $J(\phi_0) \leq c(R)$. Hence, 4.15 implies that $S(t)$ is eventually bounded. Since the resolvent of the operator $-T$ is compact, $S(t)$ is completely continuous for $t > 0$, thus asymptotically smooth. The equivalence criterion [13, Proposition 2.3, pg. 36], implies that $S(t)$ is asymptotically compact. The positive orbit $\gamma^+(\phi_0)$ is precompact, having a nonempty compact connected invariant $\omega$-limit set $\omega(\phi_0)$. From 3.9 and the continuity of $S(t)$ it follows that $\omega(\phi_0) \in E$.

It remains to show that $E$ is bounded, to conclude that $S(t)$ is point dissipative. An equilibrium point of $S(t)$, is an extreme value of the functional $J$ or equivalently, satisfies the weak formula

$$
\int_{\Omega} \sigma(x)|\nabla u|^2 dx - \lambda \int_{\Omega} u v dx + \int_{\Omega} |u|^{2^\ast} v dx, \quad \text{for every } v \in D_0^{1,2}(\Omega, \sigma).
$$

Setting $v = u$ in 3.19 and using inequality 3.12 we obtain

$$
\int_{\Omega} \sigma(x)|\nabla u|^2 dx \leq (2\lambda)^{\frac{2^\ast - 1}{2^\ast - 2}} R_2(\gamma, \theta),
$$

(see 3.18), which implies that for fixed $\lambda$ the set $E$ is bounded. 


4. Global Bifurcation of Stationary States

The validity of the continuous imbedding $D^{1,2}_0(\Omega, \sigma) \hookrightarrow L^2(\Omega)$ (Lemmas 2.2, 2.3 (i)), enables us to use the same arguments as in the proof [29, Theorem 4.1 (Step 2)] (see also [1, Lemma 2.8]), in order to prove $L^\infty$-estimates, for the weak solutions of (1.2) and (1.3).

**Lemma 4.1.** Assume that $\Omega$ is an arbitrary domain (bounded or unbounded) and the conditions $(H_\alpha)$ or $(H_{\alpha}^\gamma)$ are satisfied. Then any weak solution $u$ of (1.2) or (1.3) is uniformly bounded in $\Omega$, i.e. $\|u\|_{L^\infty(\Omega)} < C$, where $C$ is a positive constant depending on $\lambda, \gamma$ and $K$, where $K$ is the constant appearing in (2.3).

**A. The bounded domain case:** We assume that the diffusion coefficient is given by (1.5), and we consider the following problems:

\[
(P) \quad \begin{cases}
-\text{div}(\sigma(x)\nabla u) = \lambda u - |u|^{2\gamma} u, & \text{in } \Omega, \\
u|_{\partial \Omega} = 0,
\end{cases}
\]

\[
(P)_r \quad \begin{cases}
-\text{div}(\sigma(x)\nabla u) = \lambda u - |u|^{2\gamma} u, & \text{in } \Omega_r = \Omega \setminus B_r(0), \\
u|_{\partial \Omega_r} = 0,
\end{cases}
\]

for some $r > 0$ sufficiently small. Standard regularity results (cf. [39, Theorem 8.22]) imply that if $u$ is a weak solution of the problem $(P)$, $(P)_r$ then $u \in C^{1,\xi}_{loc}(\Omega \setminus \{0\})$, $(u \in C^{1,\xi}_{loc}(\Omega_r))$, for some $\xi \in (0,1)$.

For the linear eigenvalue problems

\[
(PL, ((PL)_r) \quad \begin{cases}
-\text{div}(\sigma(x)\nabla u) = \lambda u, & \text{in } \Omega (\Omega_r), \\
u|_{\partial \Omega} = 0, \quad (u|_{\partial \Omega_r} = 0),
\end{cases}
\]

we have the following lemma.

**Lemma 4.2.** Assume that $\sigma$ is given by (1.5). Problem $(PL)$, $(PL)_r)$ admits a positive principal eigenvalue $\lambda_1 (\lambda_{1,r})$, given by

\[
\lambda_1 (\lambda_{1,r}) = \inf_{\phi \in D^{1,2}_0(\Omega(\Omega_r), \sigma), \phi \neq 0} \frac{\int_{\Omega(\Omega_r)} \sigma(x) |\nabla \phi|^2 \, dx}{\int_{\Omega(\Omega_r)} |\phi|^2 \, dx},
\]

with the following properties: (i) $\lambda_1 (\lambda_{1,r})$, is simple with a nonnegative (positive) associated eigenfunction $u_1, (u_{1,r})$. (ii) $\lambda_1 (\lambda_{1,r})$, is the only eigenvalue of $(PL)$, $(PL)_r)$, with nonnegative (positive) associated eigenfunction.

**Proof:** The existence of $\lambda_1 (\lambda_{1,r})$ is a consequence of Lemma 2.2 (ii) (see also [24]). For the proof of (i), let us assume that $u_1 \geq 0 (u_{1,r} > 0)$ in $\Omega (\Omega_r)$ (since if $u (u_{1,r})$ is a minimizer of (4.1), then $|u_1| (|u_{1,r}|)$ must be also a minimizer-similar arguments may also be find in [39, Theorem 8.38]). The simplicity of $\lambda_1 (\lambda_{1,r})$, can be shown by an alternative argument, based on the so called Picone’s Identity [23, 38].

(PT): Assume that $u \geq 0, v > 0$ are almost everywhere differentiable functions in $\Omega$. Define

\[
L(u,v) := |\nabla u|^2 + \frac{u^2}{v^2} |\nabla v|^2 - 2 \frac{u}{v} \nabla u \cdot \nabla v,
\]

\[
R(u,v) := |\nabla u|^2 - \nabla \left( \frac{u^2}{v} \right) \cdot \nabla v.
\]
Then $L(u, v) = R(u, v)$, $L(u, v) \geq 0$, and $L(u, v) = 0$, if and only if $u = kv$ for some constant $k$, a.e. in $\Omega$.

Let $\Omega_0 \subset \Omega$ a compact subset of $\Omega$, and $0 \leq \phi \in C_0^\infty(\Omega)$. For $\lambda > 0$ we consider $u \in C^1_{loc}(\Omega)$, $\zeta \in (0, 1)$, a weak solution of $(P^L)$, such that $0 \leq u$ a.e in $\Omega$. Then, for any $\varepsilon > 0$, we have that

$$0 \leq \int_{\Omega_0} \sigma(x) L(\phi, u + \varepsilon) \, dx \leq \int_{\Omega} \sigma(x) L(\phi, u + \varepsilon) \, dx =$$

$$= \int_{\Omega} \sigma(x) R(\phi, u + \varepsilon) \, dx =$$

$$= \int_{\Omega} \sigma(x) |\nabla \phi|^2 \, dx - \int_{\Omega} \sigma(x) \nabla \left( \frac{\phi^2}{u + \varepsilon} \right) \cdot \nabla u \, dx =$$

$$= \int_{\Omega} \sigma(x) |\nabla \phi|^2 \, dx + \int_{\Omega} \left( \frac{\phi^2}{u + \varepsilon} \right) \nabla \sigma(x) \nabla u \, dx =$$

$$(4.2) \quad = \int_{\Omega} \sigma(x) |\nabla \phi|^2 \, dx - \lambda \int_{\Omega} \left( \frac{\phi^2}{u + \varepsilon} \right) u \, dx.$$

Assume now that $\lambda_1$ is not simple. Let $v \neq u_1$ be another associated eigenfunction, $v \in D^2_0(\Omega, \sigma)$ almost everywhere differentiable in $\Omega$, such that $v(x) \geq 0$ in some $\Omega^+ \subset \Omega$. Consider $\Omega_0 \subseteq \Omega^+$, $\lambda = \lambda_1$ and $u = u_1$. Letting $\phi \to v$ in $\Omega^+$ and $\varepsilon \to 0$, Fatou's Lemma and Lebesgue Dominated Convergence Theorem, imply that $L(v, u_1) = 0$ a.e. in $\Omega^+$. Hence from $(P^L)$ we get that $v = ku_1$, a.e. in $\Omega^+$, which implies the simplicity of $\lambda_1$. Property (i) is proved.

For the proof of (ii), we suppose that there exists another eigenvalue of $(PL)$, $\lambda^* > \lambda_1$, to which corresponds a nonnegative eigenfunction $u^*$. Consider $\Omega_0 \subseteq \Omega^+$, $\lambda = \lambda^*$ and $u = u_*$. Letting $\phi \to u_1$ in $\Omega$ and $\varepsilon \to 0$, we obtain that

$$0 \leq \int_{\Omega} \sigma(x) L(u_1, u^*) \, dx < 0,$$

which is a contradiction.

**Lemma 4.3.** Assume that $\sigma$ is given by (1.2). Let also $\lambda_1$, $\lambda_{1,r}$, be the positive principal eigenvalues of the problems $(PL)$, $(PL)_r$, respectively. Then, $u_{1,r} \to u_1$ in $D^{1,2}_0(\Omega, \sigma) \cap L^\infty_{loc}(\Omega \setminus \{0\})$, and $\lambda_{1,r} \downarrow \lambda_1$, as $r \downarrow 0$.

**Proof:** We extend $u_{1,r}$ on $\Omega$ as

$$\hat{u}_{1,r}(x) = \begin{cases} u_{1,r}(x), & x \in \Omega_r, \\ 0, & x \in B_r, \end{cases}$$

for any sufficiently small $r > 0$, but in the sequel, for convenience, we shall use the same notation $u_{1,r} \equiv \hat{u}_{1,r}$. Observe that

$$\lambda_{1,r} = \frac{\int_{\Omega_r} \sigma(x) |\nabla u_{1,r}|^2 \, dx}{\int_{\Omega_r} |u_{1,r}|^2 \, dx} \geq \lambda_1,$$

and $\lambda_{1,r}$ is an decreasing sequence, as $r \to 0$, since $\Omega_\rho \subset \Omega_\varrho$, for any $\rho > \varrho$. Clearly, $u_{1,r}$ forms a bounded sequence in $D^{1,2}_0(\Omega, \sigma)$. Lemma 2.2 (ii) and Lemma 4.2 imply the existence of a pair $(\lambda^*, u^*)$, and a subsequence of $u_{1,r}$ (not relabelled), such that

$$\lambda_{1,r} \int_{\Omega} |u_{1,r}|^2 \, dx \to \lambda^* \int_{\Omega} |u^*|^2 \, dx,$$
as $r \to 0$. Then (4.1) implies that $u_{1,r} \to u^*$ in $D^{1,2}_0(\Omega, \sigma)$ and $u^*$ satisfies

$$\int_\Omega \sigma(x) \|\nabla u^*\|^2 \, dx = \lambda^* \int_\Omega |u^*|^2 \, dx.$$  

From Lemma 4.2 (ii), we obtain that $(\lambda^*, u^*) \equiv (\lambda_1, u_1)$. We conclude by justifying the claim that $u_{1,r}$ is uniformly bounded in $L^\infty(\Omega)$. Note that $\lambda_{1,r} \in (\lambda_1, \lambda_1 + \epsilon)$, for some $\epsilon > 0$ and any $r$ small enough. Since $u_{1,r} \in D^{1,2}_0(\Omega, \sigma)$, it holds that $\|u_{1,r}\|_{L^\infty(\Omega)} < K \|u_{1,r}\|_{D^{1,2}_{0}(\Omega, \sigma)}$, $K$ is given in (2.2) and is independent of $r$. Hence, from Lemma 4.1 we have that $u_{1,r}$ is uniformly bounded in $L^\infty(\Omega)$. Then by a standard bootstrap argument we get that $u_{1,r} \to u_1$ in $L^\infty_{loc}(\Omega \backslash \{0\})$ and the proof is completed. \hfill $\Box$

**Proposition 4.4.** Assume that $\sigma$ is given by (1.2). The principal eigenvalues $\lambda_1, \lambda_{1,r}$ of the linear problems $(PL), (PL)_r$, are bifurcation points of the problems $(P), (P)_r$ respectively. Moreover, for any (sufficiently small) $r > 0$, the branch $C_{\lambda_{1,r}}$ is global, and any function which belongs to $C_{\lambda_{1,r}}$, is strictly positive.

**Proof:** The existence of branches bifurcating from $\lambda_1, \lambda_{1,r}$ follows by Theorem 2.6, since Lemma 2.2 (ii) and Lemmas 4.2, 4.3 are in hand. We outline the proof for the branch $C_{\lambda_1}$.

As in (10), we define a bilinear form in $C^0_\infty(\Omega)$ by

$$\langle u, v \rangle = \int_\Omega \sigma(x) \nabla u \nabla v \, dx - \frac{c-1}{2} \int_\Omega uv \, dx, \text{ for all } u, v \in C^0_\infty(\Omega).$$

($c$ is the constant in (2.3)) and we define $X$ to be the completion of $C^0_\infty(\Omega)$ with respect to the norm induced by (4.2), $\|u\|^2_X = \langle u, u \rangle$: from inequality 2.3 we get that

$$\frac{1}{2} \|u\|^2_{D^{1,2}_0(\Omega, \sigma)} \leq \|u\|^2_X \leq \frac{3}{2} \|u\|^2_{D^{1,2}_0(\Omega, \sigma)}, \text{ for all } u, v \in C^0_\infty(\Omega),$$

and by density it follows that $X = D^{1,2}_0(\Omega, \sigma)$. Henceforth we may suppose that the norm in $X$ coincides with the norm in $D^{1,2}_0(\Omega, \sigma)$ and that the inner product in $X$ is given by $\langle u, v \rangle = \langle u, v \rangle_X$ (moreover, we may assume that if $\langle \cdot, \cdot \rangle_X$ denotes the duality pairing on $X$, then $\langle \cdot, \cdot \rangle_{X, X^*} = \langle \cdot, \cdot \rangle$ [73, Identification Principle 21.18, pg. 254]). On the other hand, the bilinear form

$$a(u, v) = \int_\Omega uv \, dx, \text{ for all } u, v \in X,$$

is clearly continuous in $X$ as it follows from Lemma 2.2, and by the Riesz representation theorem we can define a bounded linear operator $L$ such that

$$a(u, v) = \langle Lu, v \rangle, \text{ for all } u, v \in X.$$  

The operator $L$ is self adjoint and by Lemma 2.2 (ii) is compact. The largest eigenvalue $\nu_1$ of $L$ is given by

$$\nu_1 = \sup_{u \in X} \frac{\langle Lu, u \rangle}{\langle u, u \rangle} = \sup_{u \in X} \frac{\int_\Omega u^2 \, dx}{\int_\Omega \sigma(x) |\nabla u|^2 \, dx}.$$  

It follows from Lemma 4.2 that the the positive eigenfunction $u_1$ of $(PL)$ corresponding to $\lambda_1$ is a positive eigenfunction of $L$ corresponding to $\nu_1 = 1/\lambda_1$. We consider now the nonlinear operator $N(\lambda, \cdot) : \mathbb{R} \times X \to X^*$ defined by

$$\langle N(\lambda, u), v \rangle = \int_\Omega \sigma(x) \nabla u \nabla v \, dx - \lambda \int_\Omega uv \, dx + \int_\Omega |u|^{2\gamma}uv \, dx, \text{ for all } v \in X.$$
Arguments very similar to those used for the proof of Lemma 3.1 can be used in order to verify that for fixed $u \in X$, the functional $S$ defined by

$$S(v) = \int_{\Omega} \sigma(x) \nabla u \nabla v \, dx - \lambda \int_{\Omega} uv \, dx + \int_{\Omega} |u|^{2\gamma} v \, dx, \quad v \in X,$$

is a bounded linear functional and thus $N(\lambda, u)$ is well defined from (4.8). Moreover by using the fact that $X = D^{1,2}_0(\Omega, \sigma)$ and relation (4.7), we can rewrite $N(\lambda, u)$ in the form $N(\lambda, u) = u - G(\lambda, u)$ where $G(\lambda, u) := \lambdaLu - H(u)$,

$$< H(u), v > = \int_{\Omega} |u|^{2\gamma} uv \, dx$$

for all $v \in X$.

The restriction (3.1) and Lemma 2.2 (ii) implies that $H$ is compact. Moreover we observe that

$$\frac{1}{||u|||X|} < H(u), v > | \leq \frac{1}{||u|||X|} ||u||^{2\gamma} ||u||_{L^{2\gamma+2}} ||v||_{L^{2\gamma+2}}$$

(4.6)

Therefore, we get from (4.6) that

$$\lim_{||u|||X| \to 0} \frac{||H(u)|||X|}{||u|||X|} = \lim_{||u|||X| \to 0, ||v|||X| \leq 1} \frac{1}{||u|||X|} < H(u), v > = 0.$$

To prove that $C_{\lambda_1, r}$ is global for sufficiently small $r > 0$, we proceed in two steps.

(a) We shall prove first that for all solutions $(\lambda, u) \in C_{\lambda_1, r}$ close to $(\lambda_1, r, 0)$ it holds that $u(x) > 0$, $x \in \Omega_r$. In other words, we have to show that there exists $\epsilon_0 > 0$, such that for any $(\lambda, u(x)) \in C_{\lambda_1, r} \cap B_{\epsilon_0}((\lambda_1, r, 0))$, it holds that $u(x) > 0$, for any $x \in \Omega_r$ (By $B_{\epsilon_0}((\lambda_1, r, 0))$, we denote the open ball of $C_{\lambda_1, r}$ of center $(\lambda_1, r, 0)$ and radius $\epsilon_0$).

We argue by contradiction: Let $(\lambda_n, u_n)$ be a sequence of solutions of $(P)_r$, such that $(\lambda_n, u_n) \to (\lambda_1, r, 0)$ and assume that $u_n$ are changing sign in $\Omega_r$. Let $u_n^- := \min\{0, u_n\}$ and $\mathcal{U}_n^- := \{ x \in \Omega_r : u_n(x) < 0 \}$. Since $u_n = u_n^+ - u_n^-$ is a solution of the problem $(P)_r$, it can be easily seen that $u_n^-$, satisfies (in the weak sense) the equation

$$-\text{div} \left( \sigma(x) \nabla u_n^- \right) - \lambda_n u_n^- + |u_n|^{2\gamma} u_n^- = 0,$$

$$|u_n^-|_{\partial \Omega_r} = 0.$$

Then, multiplying (4.7) with $u_n^-$ and integrating over $\Omega_r$ we have that

$$\int_{\mathcal{U}_n^-} |\nabla u_n^-|^2 \, dx - \lambda_n \int_{\mathcal{U}_n^-} |u_n^-|^2 \, dx + \int_{\mathcal{U}_n^-} |u_n|^2 |u_n^-|^2 \, dx = 0.$$

Since $\lambda_n$ is a bounded sequence, it follows from (4.8), Hölder’s inequality and relation (4.8) that

$$||u_n^-||_{D^{1,2}_0(\mathcal{U}_n^-, \sigma)} \leq \lambda_n \int_{\mathcal{U}_n^-} |u_n^-|^2 \, dx \leq C |\mathcal{U}_n^-|^{\frac{2-\alpha}{2}} \left( \int_{\mathcal{U}_n^-} |u_n^-|^{2\alpha} \right)^{\frac{2}{2\alpha}} \leq C |\mathcal{U}_n^-|^{\frac{2-\alpha}{2}} ||u_n^-||_{D^{1,2}_0(\mathcal{U}_n^-, \sigma)}^2,$$

or, equivalently

$$M \leq \mathcal{U}_n^- \quad \text{for all } n,$$

where the constant $M$ is independent of $n$. We denote now by $\tilde{u}_n = u_n/||u_n||$ the normalization of $u_n$. Then there exists a subsequence of $\tilde{u}_n$ (not relabelled) converging weakly in $D^{1,2}_0(\Omega_r, \sigma)$ to
some function $\tilde{u}_0$. By following the lines of the proof of Lemma 4.3 it can be seen that $\tilde{u}_0 = u_{1,r}$. Moreover, $u_n \to u_{1,r} > 0$ in $L^2(\Omega)$. Passing to a further subsequence if necessary, by Egorov’s Theorem, $\tilde{u}_n \to u_{1,r}$ uniformly on $\Omega_r$ with the exception of a set of arbitrary small measure. This contradicts (4.9) and we conclude the functions $u_n$ cannot change sign (for a similar argument, we refer to [28, 29, 69]).

(b) Suppose now that for some solution $(\lambda, u) \in C_{\lambda_1,r}$, there exists a point $x_0 \in \Omega_r$, such that $u(x_0) < 0$. Using (a), the fact that the continuum $C_{\lambda_1,r}$ is connected (see Theorem 2.6 and the $C^1_{\text{loc}}(\Omega_r)$- regularity of solutions, we get that there exists $(\lambda_0, u_0) \in C_{\lambda_1,r}$, such that $u_0(x) \geq 0$, for all $x \in \Omega_r$, except possibly some point $x_0 \in \Omega_r$, such that $u_0(x_0) = 0$. Then Theorem 2.7 implies that $u_0 \equiv 0$ on $\Omega_r$. Thus, we may construct a sequence $\{\lambda_n, u_n\} \subseteq C_{\lambda_1,r}$, such that $u_n(x) > 0$, for all $n$ and $x \in \Omega_r$, $u_n \to 0$ in $D^2_{\text{loc}}(\Omega_r, \sigma)$, and $\lambda_n \to \lambda_0$. However, this is true only for $\lambda_0 = \lambda_1$. As a consequence, we have that $C_{\lambda_1,r}$ cannot cross $(\lambda, 0)$ for some $\lambda \neq \lambda_1$, and every function which belongs to $C_{\lambda_1,r}$ is strictly positive.

Theorem 4.5. Assume that $\sigma$ is given by (1.3). Then, $C_{\lambda_1}$ is a global branch of nonnegative solutions for the problem (P).

Proof. It suffices to prove that $C_{\lambda_1,r} \to C_{\lambda_1}$, as $r \to 0$. The global character of $C_{\lambda_1,r}$ implies that for any fixed positive number $R$, and any $r$ sufficiently small, the set $C_{\lambda_1,r} \cap B_R(\lambda_1,r,0)$ is not empty. By using the properties of $\lambda_1$ established in Lemma 4.2 and the compactness arguments of Lemma 1.3 we can show that

$$\lim_{r \to 0} C_{\lambda_1,r} \cap B_R(\lambda_1,r,0) = C_{\lambda_1} \cap B_R(\lambda_1,0),$$

for every $R > 0$, which implies that $C_{\lambda_1,r} \to C_{\lambda_1}$, as $r \to 0$. Alternatively, one may use Whyburn’s Theorem 5 [4, 32, 48].

Proof of Theorem 1.2 in the case of (H.o): One has to extend Theorem 1.3 in the case of a diffusion coefficient satisfying (H.o). Since the set of zeroes of $\sigma$, $Z_{\sigma}$ is finite, we may use (1.2) and consider approximating problems similar to (P), defined this time in the domain $\Omega_r = \Omega \setminus \bigcup B_r(z_i)$. The finiteness of $Z_{\sigma}$ allows to repeat the proofs of Lemmas 1.2, 1.3 and Proposition 1.4 without additional complications.

B. The unbounded domain case We assume that the diffusion coefficient is given by (1.7) and we consider the following problem:

$$(P)_\infty \quad \begin{cases} -\text{div}(\sigma(x)\nabla u) = \lambda u - |u|^{2\gamma} u, & \text{in } \Omega, \\ u|_{\partial \Omega} = 0. \end{cases}$$

where $\Omega \subseteq \mathbb{R}^N$, $N \geq 2$, is an unbounded domain containing the origin. The regularity results of [39, Theorem 8.22], imply once again that if $u$ is a weak solution of the problem $(P)_\infty$, then $u \in C^1_{\text{loc}}(\Omega \setminus \{0\})$, for some $\zeta \in (0, 1)$. This time, we consider the approximating problem,

$$(P)_R \quad \begin{cases} -\text{div}(\sigma(x)\nabla u) = \lambda u - |u|^{2\gamma} u, & \text{in } \Omega_R = \Omega \cap B_R(0), \\ u|_{\partial \Omega R} = 0. \end{cases}$$

We consider the linear eigenvalue problems

$$(PL)_\infty, (PL)_R \quad \begin{cases} -\text{div}(\sigma(x)\nabla u) = \lambda u, & \text{in } \Omega \setminus \{0\}, \\ u|_{\partial \Omega} = 0, & (u|_{\partial \Omega R} = 0). \end{cases}$$

A result similar to Lemma 1.2 holds.
Lemma 4.6. Assume that $\sigma$ is given by (1.4). Problem $(PL)_\infty ((PL)_R)$, admits a positive principal eigenvalue $\lambda_1 (\lambda_{1,R})$, given by

$$\lambda_1 (\lambda_{1,R}) = \inf_{\phi \in D_0^{1,2}(\Omega (\Omega_R), \sigma)} \frac{\int_{\Omega (\Omega_R)} \sigma(x) |\nabla \phi|^2 \, dx}{\int_{\Omega (\Omega_R)} |\phi|^2 \, dx}. \tag{4.10}$$

with the following properties: (i) $\lambda_1 (\lambda_{1,R})$, is simple with a nonnegative (positive) associated eigenfunction $u_1, (u_{1,R})$. (ii) $\lambda_1 (\lambda_{1,R})$, is the only eigenvalue of $(PL)_\infty ((PL)_R)$, with nonnegative (positive) associated eigenfunction.

To prove a similar to Lemma 4.3 result, we shall use the extension

$$\hat{u}_{1,R}(x) = \begin{cases} u_{1,R}(x), & x \in \Omega_R, \\ 0, & x \in \Omega \setminus \Omega_R, \end{cases}$$

and use for convenience the notation $\hat{u}_{1,R} \equiv u_{1,R}$.

Lemma 4.7. Let $\lambda_1 (\lambda_{1,R})$ be the positive principal eigenvalues of the problems $(PL)_\infty ((PL)_R)$.

Then, $\hat{u}_{1,R} \to u_1, \text{ in } D_0^{1,2}(\Omega, \sigma) \cap L_\infty (\Omega \setminus \{0\})$ and $\lambda_{1,R} \downarrow \lambda_1$, as $R \to \infty$.

We remark that for each $R > 0$, Theorem 4.6 is applicable for $(P)_R$: There exists a global branch, $C_{\lambda_{1,R}}$, of nonnegative solutions, bifurcating from $\lambda_{1,R}$. This suffices for a repetition of arguments similar to those used for the proof of Theorem 4.4 to show that $C_{\lambda_{1,R}} \to C_{\lambda_1}$, as $R \to \infty$.

Theorem 4.8. Assume that $\sigma$ is given by (1.4). Then, $\lambda_1$ is a bifurcating point of the problem $(P_\infty)$ and $C_{\lambda_1}$ is a global branch of nonnegative solutions.

Proof of Theorem 1.2 in the case of $(H^\infty)$: One has to consider approximating problems similar to $(P)_R$, defined in the domain $\Omega_R = \Omega \cap B_R(0)$. The conclusion follows from Theorem 1.2 in the case of $(H_\alpha)$, repeating the proofs of Lemmas 4.6 4.7 and the arguments of the proof of Theorem 1.6.

C. Properties of the global branches In the remaining part of this section, we state some further properties of the global branch $C_{\lambda_1}$, both in the bounded and the unbounded domain case. For similar properties possessed by solutions of nondegenerate elliptic equations, we refer to 45 52 38.

Lemma 4.9. Assume that $\Omega$ is a bounded domain and the condition $(H_\alpha)$ is fulfilled, $\lambda > \lambda_1$ is fixed and $u_{\lambda,r} \in C_{\lambda_{1,r}}$ and $u_\lambda \in C_{\lambda_1}$. Then, we have that

$$u_{\lambda,r}(x) \leq u_\lambda(x), \text{ for any } x \in \bar{\Omega}_r, \text{ and any } r \to 0,$$

and

$$u_{\lambda,r} \to u_\lambda \text{ in } L_\infty (\Omega \setminus \{0\}), \text{ as } r \to 0.$$

Proof: The solution $u_{\lambda,r}$ satisfies $(P)_r$, while $u_\lambda$ satisfies

$$\begin{cases} -\text{div}(\sigma(x) \nabla u) = \mu u - |u|^2 \gamma u, \\ u|_{\partial \Omega} = 0, \quad u|_{\partial B_r} > 0. \end{cases}$$

Having in mind, that both $u_{\lambda,r}$ and $u_\lambda$ are sufficiently smooth and positive functions on $\bar{\Omega}_r$, from the Comparison Principle [35 Theorem 10.5], we conclude the first assertion of Lemma.

Next, we proceed as in Lemma 4.3. Since $u_{\lambda,r} \in D_0^{1,2}(\Omega, \sigma)$, it holds that $||u_{\lambda,r}||_{L_\infty (\Omega)} < K ||u_{\lambda,r}||_{D_0^{1,2}(\Omega, \sigma)}$, where $K$ is given in 22 and is independent of $r$. Hence, from Lemma 4.11 we
have that \( u_{\lambda,r} \) is uniformly bounded in \( L^\infty(\Omega) \). Consider \( \psi = u - u_{\lambda,r} \). Then, from [39] Theorem 8.8 we obtain that

\[
||\psi||_{W^{2,2}_{\text{loc}}(\Omega \setminus \{0\})} \leq c ||\psi||_{W^{1,2}(\Omega)} + O(r), \quad \text{as } r \to 0,
\]

for some positive constant \( c \) independent from \( r \). Then, by a standard bootstrap argument, we conclude that \( u_{\lambda,r} \to u_\lambda \) in \( L^\infty_{\text{loc}}(\Omega \setminus \{0\}) \) and the proof is completed.

Similar results may be obtained for the unbounded domain case.

**Lemma 4.10.** Assume that \( \Omega \) is an unbounded domain and the condition \((\mathcal{H}^\infty_\beta)\) is fulfilled, \( \lambda > \lambda_1 \) is fixed, \( u_{\lambda,R} \in C_{\lambda,1,R} \) and \( u_\lambda \in C_{\lambda,1} \). Then, we have that

\[
u_{\lambda,R}(x) < u_\lambda(x), \quad \text{for any } x \in \Omega_R,
\]

and

\[
u_{\lambda,R} \to u_\lambda \in L^\infty_{\text{loc}}(\Omega \setminus \{0\}), \quad \text{as } R \to \infty.
\]

For both, the bounded and the unbounded domain case, we have the following

**Proposition 4.11.** Assume that conditions \((\mathcal{H}_\alpha)\) or \((\mathcal{H}^\infty_\beta)\) hold. Then,

1. The global branch \( C_{\lambda,1} \) bends to the right of \( \lambda_1 \) (supercritical bifurcation) and it is bounded for \( \lambda \) bounded.

2. Every solution \( u \in C_{\lambda,1} \), is the unique nonnegative solution for the problem (1.2).

**Proof:**

1. Assume that \( C_{\lambda,1} \) bends to the left of \( \lambda_1 \). Then there exists a pair \( (\lambda,u) \in \mathbb{R} \times D^1_{0,2}(\Omega,\sigma), \ 0 < \lambda < \lambda_1 \), such that

\[
\int_{\Omega} \sigma(x)|\nabla u|^2 \, dx = \lambda \int_{\Omega} |u|^2 \, dx - \int_{\Omega} |u|^{2\gamma + 2} \, dx,
\]

The last equality implies that

\[
||u||_{D^1_{0,2}(\Omega,\sigma)}^2 \leq \lambda ||u||_{L^2(\Omega)}^2, \quad \text{with } \lambda < \lambda_1,
\]

which contradicts the variational characterization (1.1) of \( \lambda_1 \). Thus, \( C_{\lambda,1} \) must bend to the right of \( \lambda_1 \). To show that \( C_{\lambda,1} \) is bounded for \( \lambda \) bounded, we proceed exactly as for the derivation of the estimate (3.20).

2. Let \( u \in C_{\lambda,1} \), and suppose that \( v \) is a nonnegative solution of (1.2) with \( u \not\equiv v \). We claim that \( u(x) \leq v(x) \), for any \( x \in \Omega \setminus \{0\} \). This is a consequence of Lemma 4.9 since by the Comparison Principle we have that

\[
u_{\lambda,r}(x) \quad (\text{or } u_{\lambda,R}(x)) \leq \min_{x \in \Omega \setminus \{0\}} \{u(x), \ v(x)\},
\]

and of the \( L^\infty_{\text{loc}} \) convergence of \( u_{\lambda,r} \) (or \( u_{\lambda,R}(x) \)) to \( u \). Then, from (4.11) we must have that

\[
\int_{\Omega} (|u|^{2\gamma} v - |v|^{2\gamma} u) \, dx = 0,
\]

which is a contradiction, unless \( u \equiv v \). \( \diamond \)

We emphasize that uniqueness results in the case of semilinear elliptic equations, have been treated by many authors. We refer to the discussion in [58, Theorem 2.4]. For an approach using variational methods we refer to [32, Theorem 4.1].
5. Convergence to the Nonnegative Equilibrium, in the Case of a Bounded Domain.

Theorem 1.1 establishes for any \( \lambda > \lambda_1 \), the existence of a unique nonnegative equilibrium point for the semiflow \( S(t) \). In the light of Theorem 1.2, in order to prove convergence of solutions of (1.1) to the nonnegative equilibrium, it remains to verify (a) that solutions of (1.1) remain positive for all times and (b) the asymptotic stability of the nonnegative equilibrium.

Proposition 5.1. Assume that condition \((\mathcal{H}_a)\) or \((\mathcal{H}_\beta^\infty)\) holds. The set

\[
D_+ := \left\{ \phi \in D_0^{1,2}(\Omega, \sigma) : \phi(x) \geq 0 \text{ on } \Omega \right\},
\]

is a positively invariant set for the semiflow \( S(t) \).

Proof: From Proposition 5.2 we have that solutions are globally defined in time. It suffices to show that a kind of maximum principle holds, that is, solutions of (1.1) corresponding to nonnegative initial data, remain positive. We adapt an argument from [22, Proposition 5.3.1]. Let \( \phi_0 \in D_0^{1,2}(\Omega, \sigma) \), \( \phi_0 \geq 0 \) a.e. in \( \Omega \), and \( \phi \in C([0, +\infty); D_0^{1,2}(\Omega, \sigma)) \cap C^1([0, +\infty); L^2(\Omega)) \) the global in time solution of (1.1), with initial condition \( \phi_0 \). We consider \( \phi^+ := \max\{\phi, 0\} \), \( \phi^- := -\min\{\phi, 0\} \).

Both \( \phi^+ \) and \( \phi^- \) are nonnegative, \( \phi^+, \phi^- \in C([0, +\infty); D_0^{1,2}(\Omega, \sigma)) \cap C^1([0, +\infty); L^2(\Omega)) \), and we set \( \phi = \phi^+ - \phi^- \). We get from (1.1), that \( \phi^- \) satisfies the equation

\[
\partial_t \phi^- - \text{div}(\sigma(x) \nabla \phi^-) - \lambda \phi^- + |\phi|^{2\gamma} \phi^- = 0.
\]

Multiplying (5.1) by \( \phi^- \) and integrating over \( \Omega \) we obtain

\[
\frac{1}{2} \frac{d}{dt} ||\phi^-||^2_{L^2} + \int_{\Omega} \sigma(x) |\nabla \phi^-|^2 dx - \lambda ||\phi^-||^2_{L^2} + \int_{\Omega} |\phi|^{2\gamma} |\phi^-|^2 dx = 0,
\]

which implies that

\[
\frac{1}{2} \frac{d}{dt} ||\phi^-||^2_{L^2} \leq c ||\phi^-||^2_{L^2}.
\]

Thus, by Gronwall’s Lemma we obtain

\[
||\phi^- (t)||^2_{L^2} \leq e^{ct} ||\phi^-||^2_{L^2} = 0, \quad \text{for every } t \in [0, +\infty),
\]

hence \( \phi \geq 0 \) for all \( t \in (0, +\infty) \), a.e. in \( \Omega \). \( \phi \)

Lemma 5.2. Let condition \((\mathcal{H}_a)\) be fulfilled. The unique nonnegative equilibrium point which exists for \( \lambda > \lambda_1 \) is uniformly asymptotically stable.

We discuss first the stability properties of the zero solution. The linearization about the zero solution which is an equilibrium point for any \( \lambda \) is

\[
\partial_t \psi - \text{div}(\sigma(x) \nabla \psi) - \lambda \psi = 0, \quad x \in \Omega,
\]

\[
\psi|_{\partial \Omega} = 0.
\]

It follows from (2.2), that \( \phi = 0 \) is asymptotically stable in \( D_0^{1,2}(\Omega, \sigma) \) if \( \lambda < \lambda_1 \), and unstable in \( D_0^{1,2}(\Omega, \sigma) \) if \( \lambda > \lambda_1 \).

The linearization around the nonnegative equilibrium point \( u \) of (1.1), is given by

\[
- \text{div}(\sigma(x) \nabla \psi) - \lambda \psi + (2\gamma + 1)|u|^{2\gamma} \psi = 0,
\]

\[
\psi|_{\partial \Omega} = 0,
\]

and we shall see that for the corresponding eigenvalue problem

\[
- \text{div}(\sigma(x) \nabla \psi) - \lambda \psi + (2\gamma + 1)|u|^{2\gamma} \psi = \mu \psi,
\]

\[
\psi|_{\partial \Omega} = 0,
\]
zero is not an eigenvalue. The weak formulation of (5.5) is

$$A(\psi, \omega) := \int_\Omega \sigma(x) \nabla \psi \nabla \omega dx - \lambda \int_\Omega \psi \omega dx$$

(5.6)

$$+ (2\gamma + 1) \int_\Omega |u|^{2\gamma} \psi \omega dx = \mu \int_\Omega \psi \omega dx,$$

for every $\omega \in D_0^{1,2}(\Omega, \sigma)$. The symmetric bilinear form $A_\sigma : D_0^{1,2}(\Omega, \sigma) \times D_0^{1,2}(\Omega, \sigma) \to \mathbb{R}$ defines a Garding form [73, pg. 366], since

$$A_\sigma(\psi, \psi) \geq ||\psi||_{D_0^{1,2}(\Omega, \sigma)}^2 - \lambda ||\psi||_{L^2(\Omega)}^2.$$

Hence, Garding’s inequality is satisfied. Then it follows from Lemmas 2.2, 2.3 and [73, Theorem 22G pg. 369-370], that the problem (5.5) has infinitely many eigenvalues of finite multiplicity, and if we count the eigenvalues according to their multiplicity, then

$$\lambda < \mu_1 \leq \mu_2 \leq \cdots$$

and $\mu_j \to \infty$ as $j \to \infty$.

The smallest eigenvalue can be characterized by the minimization problem

$$\mu_1 = \min_{\psi \in D_0^{1,2}(\Omega, \sigma)} A_\sigma(\psi, \psi), \quad ||\psi||_{L^2} = 1.$$

(5.8)

The $j$-th eigenvalue, can be characterized by the minimum-maximum principle

$$\mu_j = \min_{M \in \mathcal{L}_j} \max_{\psi \in M} A_\sigma(\psi, \psi).$$

(5.9)

where $M = \{ \psi \in D_0^{1,2}(\Omega, \sigma) : ||\psi||_{L^2} = 1 \}$ and $\mathcal{L}_j$ denotes the class of all sets $M \cap L$ with $L$ an arbitrary $j$-dimensional linear subspace of $D_0^{1,2}(\Omega, \sigma)$.

By using similar arguments as for the proof of Lemmas 4.2-4.6, we may see that for (5.5), the (nontrivial) eigenfunction corresponding to the principal eigenvalue $\mu_1$ is nonnegative, i.e $\psi_1 \geq 0$ a.e. on $\Omega$. Since $\mu_1, \psi_1$ satisfy (5.6) we get by setting $\omega = u$ that

$$\int_\Omega \sigma(x) \nabla \psi_1 \nabla u dx - \lambda \int_\Omega \psi_1 u dx + (2\gamma + 1) \int_\Omega |u|^{2\gamma} \psi_1 u dx = \mu_1 \int_\Omega \psi_1 u.$$

(5.10)

On the other hand, by setting $v = \psi_1$ to the weak formula (5.6) we get

$$\int_\Omega \sigma(x) \nabla \psi_1 \nabla v dx - \lambda \int_\Omega \psi_1 v dx + \int_\Omega |u|^{2\gamma} \psi_1 v dx = 0.$$

Subtracting these equations, we obtain that

$$2\gamma \int_\Omega |u|^{2\gamma} \psi_1 dx = \mu_1 \int_\Omega u \psi_1 dx,$$

which implies that $\mu_1 > 0$. \hfill \Box

**Proof of Corollary 1.3** The positivity property of Proposition 5.1 and Theorem 1.1 imply that the solution $\phi(\cdot, t)$ converges towards the set of nonnegative solutions of (1.2) as $t \to \infty$, in $D_0^{1,2}(\Omega, \sigma)$. In fact, it follows from Lemma 5.2 that in the case of $\lambda > \lambda_1$, for any nonnegative initial condition $\phi_0$, $\omega(\phi_0) = \{u\}$. On the other hand it is not hard to check, by following the computations leading to (5.11-5.14), that in the case $\lambda < \lambda_1$, dist$({\mathcal{S}(t)\mathcal{B}, \{0\}}) \to 0$ as $t \to \infty$, for every bounded set $\mathcal{B} \subset D_0^{1,2}(\Omega, \sigma)$. In this case, the global attractor $\mathcal{A}$ is reduced to $\{0\}$. \hfill \Box

**Remark 5.3.** (Minimization of the Lyapunov Function) We expect naturally, that the nonnegative equilibrium points, minimize the Lyapunov function. (5.11) \hfill \Box

Assume that condition (H_0) holds.
It is not hard to check that $J$ is a bounded from below functional on $D_0^{1,2}(\Omega, \sigma)$. Assume further that $\lambda_0 < \lambda_1$. Then, the variational characterization of $\lambda_1$ (4.12) - (4.17), implies that

$$J(\phi) \geq \frac{1}{2\gamma + 2}||\phi||_{L^{2\gamma+2}(\Omega)}^2,$$

for every $\phi \in D_0^{1,2}(\Omega, \sigma)$. Hence the trivial solution is the global minimizer of the functional $J$. However, for $\lambda > \lambda_1$ the origin is no longer the global minimizer of the functional: consider the function $tu_1$, where $u_1$ is the normalized nonnegative eigenfunction associated to $\lambda_1$ and $t > 0$ is small enough. Then from (3.7), we have that

$$J(tu_1) = \frac{t^2}{2}||u_1||_{L^{2\gamma+2}(\Omega, \sigma)}^2 - \frac{\lambda_1 t^2}{2}||u_1||_{L^2(\Omega)}^2 - \frac{(\lambda - \lambda_1) t^2}{2}||u_1||_{L^2(\Omega)}^2$$

$$+ \frac{t^{2\gamma+2}}{2\gamma + 2}||u_1||_{L^{2\gamma+2}(\Omega)}^2 < 0.$$

The justification of the Palais-Smale condition follows from Lemma 2.2 (ii). Then, Ekeland’s variational principle implies the existence of nontrivial minimizers for $J$. These minimizers are the solutions which belong to the branch $C_{\lambda_1}$.

Actually $C_{\lambda_1}$ is a pitchfork bifurcature of supercritical type, where Principle of Exchange of Stability holds.

**Remark 5.4.** (Degeneracy exponent) Condition (4.1) can be written as a restriction on the “degeneracy” exponent

$$0 < \alpha \leq \frac{2(1 - \gamma(N - 2))}{2\gamma + 1} := \alpha^*.$$ 

This is a restriction on the “rate” of decrease of the diffusion coefficient $\sigma$ near every point $z \in \sigma^{-1}(0)$. Unique (since the nonlinearity defines a Lipschitz map) and global in time solutions of (1.1) exist, converging towards a global attractor, if $\sigma(x)$ decreases more slowly than $|x - z|^\delta$, $\delta \in (0, \alpha^*)$, near every point $z \in \sigma^{-1}(0)$. As an example we mention the case $N = 2$ and $\gamma = 1$ (cubic nonlinearity) where $\alpha^* = 2/3$. However, as it follows from the discussion in [13], (4.11) (or (5.11)) does not possibly define a critical exponent, concerning the existence of global attractor. In the case $\alpha > \alpha^*$, the dynamics related to (1.1), could be investigated through the theory of generalised semiflows [12, 13]. As in the case of the damped semilinear wave equation examined in [14], where uniqueness of solutions is not assumed, one could possibly prove the existence of a global attractor in the case $\alpha > \alpha^*$, under the hypothesis that weak solutions of (1.1) satisfy the corresponding energy equation.

**Remark 5.5.** (Lack of compactness) Our approach concerning convergence to the equilibrium, which combines the characterization of the global attractor of and global bifurcation theory depends heavily on $(H_0, \sigma)$ and $(\mathcal{H}_\infty^\delta)$, ensuring compactness of the linear and nonlinear operators involved to our study, either in the bounded or unbounded domain case. Thus it is natural to ask if a relaxation
of the aforementioned conditions which may give rise to noncompactness, could allow for a generalization of the results of Sections 3-5.

A starting point, could be the generalization of the result concerning the existence of the global attractor. One could assume conditions \((H_\alpha)\) and \((H^\infty_\alpha)\) for some \(\alpha \in [0, +\infty)\). As it is noted in [13, Remark 2.1], if \(\sigma \in L^1_{loc}(\Omega), \Omega \subseteq \mathbb{R}^N, N \geq 2\), satisfies \((H_\alpha)\) then it satisfies \((H_\beta)\) for any \(\beta \geq \alpha\) and if \((H^\infty_\alpha)\) holds, then \((H^\infty_\delta)\) is valid for any \(\delta \in [0, \alpha]\).

For example, in the case where \(\Omega = \mathbb{R}^N\) one can also consider as an energy space, the space \(H^1_0(\mathbb{R}^N, \sigma)\), defined as the closure of \(C^\infty_0(\mathbb{R}^N)\) with respect to the norm

\[
||\phi||^2_{H,\sigma} = \int_{\mathbb{R}^N} \sigma(x)|\nabla \phi|^2 \, dx + \int_{\mathbb{R}^N} |\phi|^2 \, dx.
\]

The embedding \(H^1_0(\mathbb{R}^N, \sigma) \subseteq L^2(\mathbb{R}^N)\) although obviously continuous, is not compact, in the case where \((H_\alpha)\) and \((H^\infty_\alpha)\), hold for some \(\alpha \in (0, 2]\). Recall also from Remark 2.4, that \(D^{1,2}_0(\mathbb{R}^N, \sigma)\) is not compactly embedded in \(L^2(\mathbb{R}^N)\), if \(\sigma\) grows less or equal than quadratically at infinity, even in the case where \((H_\alpha)\) is satisfied for some \(\alpha \in (0, 2]\).

For the existence of global attractors for reaction diffusion equations in unbounded domains, representative references include [8, 33] (for semilinear and degenerate (porous medium) parabolic equations considered on weighted Sobolev spaces) and [12]. The latter provides with an effective remedy for the lack of compactness of the Sobolev imbeddings, with respect to the existence of the global attractor for partial differential equations considered in unbounded domains and in the natural phase space. The idea of [12] is based on the approximation of \(\mathbb{R}^N\) by a bounded domain and on the derivation of suitable estimates for the approximation error of the norm of solutions, showing that this approximation error is arbitrary small. These estimates allow for the application of the method developed in [13] which makes use of the energy functional associated to the evolution equation (in [12] a reaction diffusion equation): Asymptotic compactness is shown by passing to the limit of the nonlinear term of the energy functional as the error tends to zero, establishing the existence of a global attractor in \(L^2(\mathbb{R}^N)\).

It would be possibly interesting to attempt to apply this method, to the degenerate equation of the form [14] and investigate if new restrictions could arise between degeneracy, nonlinearity and the parameters involved, through the process of the derivation of an appropriate energy functional, and the estimation of the relevant estimation errors of the generalised Sobolev norms.

On the other hand, as it is already mentioned in the introduction with respect to the convergence to equilibrium, in the unbounded domain case, one has to deal in general, not only with the lack of compactness but also with the possible appearance of infinite distinct translates of a unique rest point. Thus, it could be also interesting to investigate if the analysis of [17], could provide a framework for the generalization of the convergence result for [14], in the unbounded domain case.

6. APPLICATIONS OF THE GLOBAL BIFURCATION RESULT TO GENERAL ELLIPTIC EQUATIONS

We conclude, by mentioning some other examples of degenerate elliptic equations for which, extensions of the results of Section 4, could be investigated.

A. Semilinear Equations We consider the semilinear problem

\[
-\nabla (\sigma(x) \nabla u) = \lambda f(x) u - g(\lambda, x, u),
\]

\(u|_{\partial \Omega} = 0\).

and the corresponding linear eigenvalue problem

\[
-\nabla (\sigma(x) \nabla u) = \lambda f(x) u,
\]

\(u|_{\partial \Omega} = 0\),

(6.2)
where $\Omega \subseteq \mathbb{R}^N$, $N \geq 2$, is an arbitrary domain (bounded or unbounded). In this case, the coefficient functions satisfy:

$$(F)^{\alpha} f$$ is a smooth function, at least $C^{0,\zeta}_{loc}(\Omega)$, for some $\zeta \in (0, 1)$, such that $f \in L^{2N/(N-2)}(\Omega)$, and there exists $\Omega_f^+ \subset \Omega$, $|\Omega_f^+| > 0$, such that $f(x) > 0$ for all $x \in \Omega_f^+$.

$$(G)^{\alpha} g$$ is a Carathéodory function, i.e., $g(\lambda, x, \cdot)$ is a continuous function for a.e. $x \in \Omega$ and $g(\lambda, \cdot, u)$ is measurable for all $(\lambda, u) \in \mathbb{R}^2$. Moreover, there exist nonnegative functions $c(\lambda) \in C(\mathbb{R})$ and $\rho(x) \in L^\infty(\Omega) \cap L^{2N/(N-2)}(\Omega)$, such that $|g(\lambda, x, u)| \leq c(\lambda) \rho(x) |u|^\gamma$, for all $(\lambda, u) \in \mathbb{R}^2$ and almost every $x \in \mathbb{R}^N$.

Depending on the particular properties of the coefficient functions, the properties of the global branch could be represented by those of the corresponding approximating problems. For some applications, we refer to [5, 6, 40, 58].

**B. Quasilinear Equations** We consider quasilinear degenerate elliptic equations of the form

$$(6.3) \quad -\nabla(a(x) |\nabla u|^{p-2} \nabla u) = \lambda b(x) |u|^{p-2} u + f(x) |u|^{\gamma-1} u$$

and

$$(6.4) \quad -\nabla(a(x, u) |\nabla u|^{p-2} \nabla u) = \lambda b(x) |u|^{p-2} u + f(x) |u|^{\gamma-1} u,$$

where $\Omega$ is a bounded domain of $\mathbb{R}^N$, $N \geq 2$ and $1 < p < N$. A possible treatment could be based on the results of [29]. Assume that there exists a function $\nu(x) \geq 0$ in $\Omega$ satisfying

(N1) $\nu(x) = 0$ or $\nu(x) = \infty$ in a finite subset $Z \subset \Omega$,

(N2) $\nu \in L^1_{loc}(\Omega)$, $\nu^{-\frac{p}{p-1}} \in L^1_{loc}(\Omega)$ and $\nu^{-s} \in L^1(\Omega)$, for some $s \in \left(\max\left\{\frac{N}{p}, \frac{1}{p-1}\right\}, \infty\right)$.

The coefficient functions $a$, $b$, $f$ satisfy:

(A) $a$ is a smooth function at least $C^{0,\zeta}_{loc}$, for some $\zeta \in (0, 1)$ a.e. in $\Omega$, such that

$$\frac{\nu(x)}{c} \leq a(x) \leq c \nu(x), \quad \text{for some } c > 0.$$

(B) $b$ is a nonnegative and smooth function, at least $C^{0,\zeta}_{loc}(\Omega)$, for some $\zeta \in (0, 1)$, such that $b \in L^\infty(\Omega)$,

(F) $f$ is a smooth function, at least $C^{0,\zeta}_{loc}(\Omega)$, for some $\zeta \in (0, 1)$, such that $f \in L^{2N/(N-2)}(\Omega)$, where

$$p_s^* := \frac{Np_s}{N(s+1)-ps}$$

and $p < \gamma + 1 < p_s^*$.

We consider the weighted Sobolev space $D^{1,p}_0(\Omega, \nu)$ endowed with the norm

$$\|u\|_{D^{1,p}_0(\Omega, \nu)} := \left(\int_\Omega \nu(x) |\nabla u|^p \right)^{1/p} < \infty.$$

The space $D^{1,p}_0(\Omega, \nu)$ is a reflexive Banach space, enjoying the following embeddings:

i) $D^{1,p}_0(\Omega, \nu) \hookrightarrow L^{p_s^*}(\Omega)$ continuously for $1 < p_s^* < N$,

ii) $D^{1,p}_0(\Omega, \nu) \hookrightarrow L^r(\Omega)$ compactly for any $r \in [1, p_s^*)$.

For further properties of these spaces, we refer to [29], as well as for the proof of the following results.
Lemma 6.1. Assume that conditions (N2), (A), (B) and (F) hold. Then, the corresponding to eigenvalue problem

\[ -\nabla (a(x)|\nabla u|^{p-2}\nabla u) = \lambda b(x)|u|^{p-2}u, \]

admits a positive principal eigenvalue \( \lambda_1 \), given by

\[ \lambda_1 = \inf_{\int_{\Omega} b(x)|\phi|^p \, dx = 1} \int_{\Omega} a(x)|\nabla \phi|^p \, dx. \]

Moreover, \( \lambda_1 \) is simple with a nonnegative associated eigenfunction \( u_1 \). In addition, \( \lambda_1 \) is the only eigenvalue with nonnegative associated eigenfunction.

Lemma 6.2. Assume that conditions (N2), (A), (B) and (F) hold. Then, any weak solution \( u \in D_0^{1,p}(\Omega, \nu) \) of (6.3) belongs to \( L^\infty(\Omega) \). Moreover, \( u \in C_0^{0,\zeta} \) for some \( \zeta \in (0, 1) \), a.e. in \( \Omega \).

Proposition 6.3. Assume that conditions (N2), (A), (B) and (F) hold. Then the principal eigenvalue \( \lambda_1 \) of (6.3), is a bifurcation point of the problem (6.3).

Proposition 6.3 could be extended to a global bifurcation result as follows: Assuming that \( \nu \) satisfies in addition, condition (N1), the principal eigenvalue \( \lambda_1 \) is a bifurcating point of a global branch. We may adapt the same procedure described in Sections 2-4, by considering similar approximating problems. It is interesting to note that in this case, Picone’s identity is still applicable.

Theorem 6.4. Assume that the conditions (N1), (N2), (A), (B) and (F) hold. Then the branch bifurcating from the principal eigenvalue \( \lambda_1 \) of (6.3), is a global branch of solutions of the problem (6.3). Moreover, any solution which belongs in this branch, is nonnegative.

Concerning problem (6.3), we assume that \( a(x, u), b(x, u) \), are sufficiently smooth functions satisfying the following conditions:

\[ (A_S) \frac{g(x)}{c} \leq a(x, u) \leq c g(|u|) \nu(x), \quad 0 \leq b(x, u) \leq b(x) \quad \text{and} \quad \lim_{s \to 0} a(x, s) = a(x), \lim_{s \to 0} b(x, s) = b(x) \]

uniformly for a.e. \( x \in \Omega \).

Here \( c > 0 \), \( g \) is a nondecreasing bounded function and \( a, b \) satisfy conditions (A) and (B), respectively.

Based again on [29], and the analysis of Sections 2-4, we may prove

Theorem 6.5. Assume that condition \( (A_S) \) holds. Then the principal eigenvalue \( \lambda_1 \) of (6.3), is a bifurcation point of the problem (6.3). Moreover, the corresponding branch is global, and any solution which belongs to this branch, is nonnegative.

The quasilinear problems,

\[ -\nabla ((k |x|^\alpha + m |x|^\beta) |\nabla u|^{p-2}\nabla u) = \lambda |u|^{p-2}u + f(x) |u|^{\gamma-1}u \]

and

\[ -\nabla ((k |x|^\alpha + m |x|^\beta) (1 + e^{-1/u^2}) |\nabla u|^{p-2}\nabla u) = \lambda e^{-u^2} |u|^{p-2}u + f(x) |u|^{\gamma-1}u, \]

for some \( 0 < \alpha < \min\{p, N(p-1)\}, \quad -N < \beta < 0 \) and \( k, m \) nonnegative constants, could serve as examples.

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