Topological sources of soliton mass and supersymmetry breaking

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Received 5 January 2018, revised 11 April 2018
Accepted for publication 12 April 2018
Published 9 May 2018

Abstract
We derive the Smarr formulae for two five-dimensional solutions of supergravity, which are asymptotically $\mathbb{R}^{1,3} \times S^1$; in particular, one has a magnetic ‘bolt’ in its center, and one is a two-center solution. We show for both spacetimes that supersymmetry—and so the BPS-bound—is broken by the holonomy and how each topological feature of a space-like hypersurface enters Smarr’s mass formula, with emphasis on the ones that give rise to the stated violation of the BPS-bound. In this light, we question if any violating extra-mass term in a spacetime with such asymptotics is only evident in the ADM mass while the Komar mass per se ‘tries’ to preserve BPS. Finally, we derive the cohomological fluxes for each situation and examine in a more general fashion how the breaking of supersymmetry—and so the BPS-bound violation—is associated with their topologies. In the second (and more complicated) scenario, we especially focus on the compact cycle linking the centers, and the contribution of non-vanishing bulk terms in the mass formula to the breaking of supersymmetry.

Keywords: supergravity, solitons, Smarr formula, Komar mass, cohomological fluxes, homological cycles, black holes

(Some figures may appear in colour only in the online journal)

1. Introduction

It has been shown that horizonless solitonic solutions of supergravity can indeed be constructed purely by means of nontrivial topology; in particular, homological cycles which smoothly embed into the solution and pave the structural ground for the supergravity limit of black hole microstate geometries within the fuzzball proposal of string theory [1]. Their dual cohomological fluxes counteract the cycles’ gravitational collapse and so represent the main ingredient for the solitonic mass.
The Smarr formula—the mass formula for a supergravity soliton based on the nontrivial topology—has been derived in multiple works by means of the Komar integral formalism over cohomology of degree two or higher in five or more spacetime dimensions [2–5], one important result being the role of Chern–Simons terms to only support the topological nature of the integral.

In this work, we consider two five-dimensional non-BPS solutions of supergravity which are topologically distinct. The idea in each case is to compute all contributions from topology and boundary that are flowing into the total mass formula, to see which pieces precisely cause the breaking of supersymmetry, by rendering \( M \neq \sum_{I=1}^{3} Q^I \), and which, in particular, make up \( \Delta M = M - \sum_{I=1}^{3} Q^I \).

In section 2, we construct a spacetime with a magnetic ‘bolt’ in its center and that is asymptotically \( \mathbb{R}^{1,3} \times S^1 \); in the fashion of [6, 7], we define a four-dimensional Ricci-flat base space which carries a Euclidean Schwarzschild metric and magnetic flux from a ‘floating brane’ ansatz [8] for the Maxwell fields.

Topologically, this spacetime can be described by entirely two homological cycles: the bolt 2-sphere and a non-compact cycle extending from the center to infinity. In this spirit, the main analysis will be done also in the framework of intersection homology.

The supersymmetry conditions require that the curvature tensor be either self-dual or anti-self-dual and tell how this duality has to be correlated with the one of the magnetic parts of the Maxwell-fields. Since the rotation group in four-dimensional space decomposes like \( SO(4) = SU(2)_{\text{self-dual}} \times SU(2)_{\text{anti-self-dual}} \), only one half of the Killing-spinors would ‘feel’ space’s holonomy and the other half flat space. In simple examples, this half-flatness, and the preservation or breaking of supersymmetry can be easily arranged by just changing a sign in the duality of the fields [9–11].

However, the curvature tensor for the Euclidean Schwarzschild bolt is neither self-dual nor anti-self-dual, so one essential BPS-condition is not fulfilled; but, because the Schwarzschild-geometry is Ricci-flat, the (almost-)BPS equations of motion are still satisfied [6, 8], and hence one speaks of an ‘almost-BPS’-solution. This provides a ground for more general solutions.

Before computing the fluxes and the Komar integral, we will briefly consider a vastly simplified scenario in which the angular momentum of the running bolt is set to zero and the five-dimensional warp-factor set to one. The reason for this is to demonstrate in a very clear and quick way that in a spacetime which is not asymptotically behaving like \( \mathbb{R}^{1,4} \), the theorem \( M_{\text{Komar}} = M_{\text{ADM}} \) does in general not hold anymore. In particular, the Komar mass vanishes under the simplified conditions, while the ADM mass equals the mass parameter of the Schwarzschild solution. The latter is responsible for the violation of the BPS-bound according to \( M = Q^1 + Q^2 + Q^3 + m \).

This raises a very important question: is the extra-mass term violating the BPS-bound always the difference between the ADM mass and the Komar mass, or, are there situations in which Komar also reflects terms breaking supersymmetry? The answer will be illuminated in this work along with the derivation of the Smarr formula based on the Komar-integral formalism in the sense of [2] for the present boundary conditions and for non-BPS solutions. Further analysis is done within intersection homology—in particular, it will be examined how the BPS-bound breaking extra-mass is composed by the period-integrals of the fields and fluxes in virtue of the homological cycles. This is to see which intersecting components exist in the present topology and how they contribute to the breaking of supersymmetry.

We will show that BPS-bound violating mass terms are, in the presence of compact dimensions, not solely ‘seen’ by the ADM mass but indeed also arise in the Komar mass from
space’s topology, and that this is due to the cohomology dual to the non-compact cycle; in particular, the latter’s intersection with the compact bolt cycle.

This raises the obvious question how supersymmetry-breaking emerges from the cohomological structure of more complex non-BPS solutions.

In section 3, we consider a more general non-supersymmetric solution of supergravity, this time with two intersecting pieces of compact homology: a non-extremal center in form of a charged bolt, constructed in a similar fashion as in section 2, and an extremal Gibbons–Hawking center. From previous works, there are known extremal results for BPS [12] and almost-BPS systems [9–11]—the situation considered here is a non-extremal generalization for a non-BPS system.

In particular, we have a running-bolt homology 2-sphere linked with a Gibbons–Hawking nut by a bubble carrying additional fluxes. This solution was derived in [13], first for an arbitrary number of extremal Gibbons–Hawking centers and then specialized to only one—so in total two independent pieces of intersecting compact homology and a non-compact cycle, much as in section 2. The interesting new feature is to look at the interaction between the bolt and the Gibbons–Hawking nut1.

The main part of the calculations in this section is the explicit examination of the cohomological fluxes coming from the homological 2-cycles that result from the bolt and the center-linking bubble. The focus then is on the analysis of the topological integrals over the harmonics in terms of intersection homology, in order to see how each cycle contributes to the mass, the charges, and the BPS-bound violating extra-mass. Although the intersection matrix will turn out to be rather trivial in section 2, a more complicated form is to be expected in section 3. Special emphasis is on the question if and how the supersymmetry-breaking extra-mass term results in part from space’s topology aside from pure boundary effects.

2. An almost-BPS spacetime with a magnetic ‘running bolt’

2.1. ADM versus Komar

Before we move on to the computation of the Komar mass, it is essential to remark its relation to the ADM mass and illustrate this by an easy and quick example.

The main difference between the two mass definitions is that, ADM follows from integrating the time-component of the energy-momentum tensor in asymptotically flat spacetimes and Komar from integrating over the Hodge-dual of a time-like Killing vector field in stationary spacetimes. The latter is precisely the Komar integral formalism, which will be outlined further below and used to derive Smarr formulae for the masses of the two present solitons.

On a first note, in case of space being asymptotically $\mathbb{R}^{1,d}$, that is, no compactified dimensions, it is known that ADM and Komar mass are equal in value. The presence of the $S^1$-direction in the spacetimes considered in this work though, does in fact create a difference between the two masses. To illustrate this in a quick way, we take a look at a strongly simplified version of the later stated Ricci-flat metrics:

$$ds^2 = -dt^2 + \left(1 - \frac{2m}{r}\right)dr^2 + \left(1 - \frac{2m}{r}\right)^{-1}dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2).$$  \hspace{1cm} (1)

1 Another interesting non-supersymmetric multi-center solution with more than one homological 2-cycle can be found in [14].
This metric is Ricci-flat and the simplest example of a time-fibration over a Euclidian Schwarzschild base space.

At first, it is important to outline that the ADM mass is in general the more authoritative measure for the gravitational mass of a system, since Komar, as stated earlier, requires stationarity; however, the latter allows for the desired consideration of solitonic mass being purely sourced by space’s topology which is represented by compact and non-compact cycles smoothly embedded into the present spacetime solutions.

For a time-like Killing vector, $K = \frac{\partial}{\partial t}$, with dual 1-form $K = g_{00}dt$, the Komar integral becomes

$$M \sim \int x^i \star_5 dK \sim \int x^i \frac{\partial}{\partial r} (g_{00}) \star_5 (dr \wedge dt),$$

and from $g_{00} = -1$ we can see directly that the Komar mass vanishes:

$$M_{\text{Komar}} = 0.$$  \hspace{1cm} (3)

To elaborate on the mass in more detail, we consider orbits in this simplified metric. The geodesic equations at $\theta = \frac{\pi}{2}$ are:

$$\frac{d}{d\lambda} \frac{d}{d\lambda} = E, \quad (1 - \frac{2m}{r}) \frac{dr}{d\lambda} = L_1, \quad r^2 \frac{d\phi}{d\lambda} = L_2,$$

where $E, L_1,$ and $L_2$ are conserved quantities.

The radial equation can be obtained through the normalization condition of the four-velocity, $u^\mu = \frac{dx^\mu}{d\lambda}$,

$$-1 = g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = -E^2 + \frac{L_2^2}{r^2} + \left(1 - \frac{2m}{r}\right)^{-1} \left(L_1^2 + \left(\frac{dr}{d\lambda}\right)^2\right),$$

and so, keeping only terms of up to first order in $\frac{1}{r}$ at infinity,

$$\left(\frac{dr}{d\lambda}\right)^2 = \left(1 - \frac{2m}{r}\right) (E^2 - 1) - L_1^2.$$  \hspace{1cm} (6)

The radial acceleration at infinity is now:

$$a^r = \frac{d^2 r}{d\lambda^2} = \frac{E^2 - 1}{\sqrt{(1 - \frac{2m}{r}) (E^2 - 1) - L_1^2}} \frac{m}{r^2} \to \frac{E^2 - 1}{E^2 - L_1^2} \frac{m}{r^2}.$$  \hspace{1cm} (7)

Setting off any rotation, $L_1 = L_2 = 0$, we read off the Keplarian mass seems to be

$$M_{\text{Kepler}} = \sqrt{E^2 - 1} m,$$

which carries a factor of $\sqrt{E^2 - 1}$. In particular, one has $a^r = 0$ for $E = 1$; thus this simple calculation does not directly ‘see’ the intrinsic mass of the background. To resolve this issue, we go to a $3+1$ description in terms of gravity in $3+1$ dimensions to look at the ADM mass.

Dimensionally reducing the metric along the $\tau$-direction, means to introduce a conformal scale factor, $\Omega$:

$$\text{d}s^2_5 = \left(1 - \frac{2m}{r}\right) \text{d}r^2 + \Omega \text{d}s^2_4.$$  \hspace{1cm} (9)

The goal is that $\text{d}s^2_5$ will be the metric apparent to observers in $3+1$ dimensions. As mentioned above, Komar and ADM mass are equal in value if asymptotics are $\mathbb{R}^{1,3}$. 


The scale factor is necessary to ensure that
\[
\frac{1}{G_5} \int d^5x \sqrt{-g^{(5)}} R^{(5)} = \frac{1}{G_4} \int d^4x \sqrt{-g^{(4)}} R^{(4)} + \text{‘derivatives of scale factors’}. \tag{10}
\]

With these scaling factors one has
\[
g^{(5)} \rightarrow \left(1 - \frac{2m}{r}\right) \Omega^4 g^{(4)} \text{ and } R^{(5)} \rightarrow \Omega^{-1} R^{(4)}, \tag{11}
\]
and hence
\[
\sqrt{-g^{(5)}} R^{(5)} \rightarrow \sqrt{1 - \frac{2m}{r} \Omega^{-1} \sqrt{-g^{(4)}} R^{(4)}}.
\]

Thus one must take
\[
\Omega = \left(1 - \frac{2m}{r}\right)^{-\frac{1}{2}}. \tag{12}
\]

Without the scale factor \(\Omega\), the four-dimensional Newton constant would gain radial dependence through multiplication by a power of \((1 - \frac{2m}{r})\). In general, those scale factors are necessary if one dimensionally reduces spacetime metrics to guarantee consistency of Newton’s constants in different dimensions.

Rewriting (1) in this sense,
\[
ds^2_5 = \left(1 - \frac{2m}{r}\right) d\tau^2 + \left(1 - \frac{2m}{r}\right)^{-\frac{1}{2}} \left\{- \left(1 - \frac{2m}{r}\right)^{\frac{1}{2}} dr^2 + \left(1 - \frac{2m}{r}\right)^{-\frac{1}{2}} \right\}
\times \left[dr^2 + \left(1 - \frac{2m}{r}\right) r^2 \left(d\theta^2 + \sin^2 \theta d\phi^2\right)\right],
\]
leads to the along the \(\tau\)-direction reduced four-dimensional metric:
\[
ds^2_4 = - \left(1 - \frac{2m}{r}\right)^{\frac{1}{2}} dr^2 + \left(1 - \frac{2m}{r}\right)^{-\frac{1}{2}} \left[dr^2 + \left(1 - \frac{2m}{r}\right) r^2 \left(d\theta^2 + \sin^2 \theta d\phi^2\right)\right], \tag{13}
\]
which is asymptotically \(\mathbb{R}^{1,3}\).

Reading off \(g^{(4)}_{00} = - \left(1 - \frac{2m}{r}\right)^{\frac{1}{2}}\) yields the stated expression for the ADM-mass:
\[
M_{\text{ADM}} = m. \tag{14}
\]

This result is the mass parameter of the Schwarzschild solution, which precisely accounts for the BPS-bound breaking extra-mass term of the solution within the current simplifications of zero charge.

One concludes that the Komar mass does not detect the breaking of BPS/super-symmetry while the ADM mass does.

After deriving the Komar mass and Maxwell-charges under the more general conditions in the following, we will examine the obvious question whether this is generally true for spacetimes asymptotically behaving like \(\mathbb{R}^{1,3} \times S^1\), or, possibly \(M_{\text{Komar}} = Q^1 + Q^2 + Q^3 + \Delta M\) for some non-vanishing \(\Delta M\).
2.2. Preliminaries

In five dimensions, the action is

\[
S = \int \left( \ast_5 R - Q_{IJ} dX^I \wedge \ast_5 dX^J - Q_{IJ} F^I \wedge \ast F^J - \frac{1}{6} C_{IJK} F^I \wedge F^J \wedge A^K \right),
\]

where \( C_{IJK} = |\epsilon_{IJK}| \) and \( X^I, I = 1, 2, 3 \), are scalar fields arising from reducing the eleven-dimensional metric,

\[
\begin{align*}
\text{d}x_{11}^2 &= \frac{Z_2 Z_3}{Z_1^2} \left( \text{d}x_5^2 + \text{d}x_6^2 \right) + \frac{Z_1 Z_3}{Z_2^2} \left( \text{d}x_7^2 + \text{d}x_8^2 \right) + \frac{Z_1 Z_2}{Z_3^2} \left( \text{d}x_9^2 + \text{d}x_{10}^2 \right),
\end{align*}
\]

with the reparametrization,

\[
X^1 = \left( \frac{Z_2 Z_3}{Z_1^2} \right)^{\frac{1}{3}}, \quad X^2 = \left( \frac{Z_1 Z_3}{Z_2^2} \right)^{\frac{1}{3}}, \quad X^3 = \left( \frac{Z_1 Z_2}{Z_3^2} \right)^{\frac{1}{3}},
\]

(16)

and fulfilling the constraint \( X^1 X^2 X^3 = 1 \).

Moreover, there is a metric for the kinetic terms,

\[
Q_{IJ} = \frac{1}{2} \text{diag} \left( \left( \frac{1}{X^1} \right)^2, \left( \frac{1}{X^2} \right)^2, \left( \frac{1}{X^3} \right)^2 \right).
\]

(18)

We do not allow for cohomology of degree one, so the equations of motion and symmetry equations correspond to those in [7], and we only give a brief summary of the mathematical background emerging the five-dimensional Komar integral including all the boundary terms.

From varying the action we receive the Einstein and the Maxwell equations,

\[
R_{\mu\nu} = Q_{IJ} \left( F^I_{\mu\rho} F^JJ_{\rho} - \frac{1}{6} Q_{IJ} F^I_{\mu\rho} F^J_{\rho\sigma} + \partial_{\mu} X^I \partial_{\nu} X^J \right),
\]

(19)

\[
J_{\mu}^{\text{CS}} = \nabla_{\mu} (Q_{IJ} F^I_{\mu \rho}),
\]

(20)

with the five-dimensional Chern–Simons 1-form current,

\[
J_{\mu}^{\text{CS}} = \frac{1}{16} C_{IJK} \epsilon_{\mu\rho\sigma\lambda} F^{I\rho\sigma} F^{J\kappa\lambda}.
\]

(21)

In differential form language, the Maxwell equations give the known identity for the dual field strengths,

\[
dG_I = \frac{1}{4} C_{IJK} F^J \wedge F^K,
\]

(22)

where

\[
G_I = Q_{IJ} \ast_5 F^J.
\]

(23)

Equation (19) can be rewritten such that the RHS is free of any trace terms,

\[
R_{\mu\nu} = Q_{IJ} \left( \frac{2}{3} F^I_{\mu\rho} F^I_{\rho\sigma} + \partial_{\mu} X^I \partial_{\nu} X^I \right) + \frac{1}{6} Q_{IJ} G_{\mu\rho\sigma} G_{J\nu}^{\rho\sigma}.
\]

(24)

especially since this form is much more helpful for the derivation of the Komar mass formula.
We assume the metric to have a time-like Killing vector, $K^\mu$, and can hence write the five-dimensional mass formula in terms of a Komar integral in five dimensions,

$$M = \frac{3}{32 \pi G_5} \int_{X^5} \star_3 dK,$$

(25)

where $X^3$ is the 3-boundary of the five-dimensional spacetime. Smoothness of spatial sections, $\Sigma_4$, allows in virtue of properties of the Killing vector to rewrite this formula as an integral over such by $X^3$ bound space-like hypersurfaces:

$$M = \frac{3}{32 \pi G_5} \int_{X^3} \star_3 dK = \frac{3}{16 \pi G_5} \int_{\Sigma_4} K^\mu R_{\mu \nu} d\Sigma^\nu. $$

(26)

Assuming again that the matter fields have the symmetries of the metric, means them to be invariant under the Lie-derivative along the Killing vector, $K$,

$$L_K F = 0 = L_K G,$$

(27)

from which follow the equations with Cartan’s magic formula,

$$0 = d (i_K F) \Leftrightarrow i_K F^I = d\lambda^I \text{ and } i_K G_I = d\Lambda_I - \frac{1}{2} C_{IJK} \lambda^J F^K + H_I^{(2)},$$

(28)

where $\lambda^I$ are magnetostatic potentials of the $G_I$ and electrostatic potentials of the $F^I$, respectively; $\Lambda_I$ are globally defined 1-forms and $H_I^{(2)} \in H^2 (M_5)$ closed but not exact 2-forms. These topological objects represent the cohomological fluxes along their dual homological cycles constituting the supergravity soliton and embedding smoothly into the considered space-time. They are precisely the source of the soliton’s mass in virtue of the below-stated Smarr formula.

With (28) the Einstein equations (24) become

$$K^\mu R_{\mu \nu} = \frac{1}{3} \nabla_\rho (2Q_{I\sigma} \lambda^{I\rho^\sigma} + Q^{I\sigma} \Lambda_\sigma G_{J^\nu \rho^\sigma}) + \frac{1}{6} Q^{I\sigma} H_I^{(2)\rho^\sigma} G_{J^\nu \rho^\sigma}. $$

(29)

The Smarr formula follows now from the Komar mass integral [2, 15–18] over the spatial hypersurface, $\Sigma_4$, including the boundary terms over $X^3$:

$$M = \frac{1}{16 \pi G_5} \left[ \int_{\Sigma_4} H_I^{(2)} \wedge F^I - \int_{X^3} (2\lambda^I G_I - \Lambda_I \wedge F^I) \right]. $$

(30)

Since in [2] the spacetime was assumed to be asymptotic to $\mathbb{R}^{1,4}$, the boundary integral was taken over $X^3 = S^3$. Throughout this work, the spacetimes will be asymptotic to $\mathbb{R}^{1,3} \times S^1$ and so we have $X^3 = S^3_\infty \times S^1$.

As we will see later, there is a gauge choice for which $\lambda^I$ can be made zero at infinity. This together with the fact that the $\Lambda_I$ are exact and vanishing at infinity, will prove the Komar mass to be an integral purely over cohomology, given by the first term in (30):

$$M = \frac{1}{16 \pi G_5} \int_{\Sigma_4} H_I^{(2)} \wedge F^I. $$

(31)

2The here used convention $\text{div} \ X = -\delta X \Rightarrow \delta dZ_I = -\nabla^2 Z_I$, where $\delta$ is the to $d$ adjoint exterior derivative, means for here: $\nabla^2 K_\mu = R_{\mu \nu} K^\nu$, so the opposite sign as in [2].
The Maxwell-charge is computed like:

\[ Q^I = -\frac{1}{\text{vol} (X^3)} \int_{X} G_I = -\frac{1}{32\pi^2 m} \int_{\Sigma^4} dG_I = -\frac{1}{128\pi^2 m} C_{IJK} \int_{\Sigma^4} F^I \wedge F^K. \]  

(32)

2.3. Metric and equations of motion

The five-dimensional metric, called the ‘running bolt’ [6, 7] is a time fibration over Euclidian Schwarzschild:

\[ ds^2_5 = -Z^{-2} \left( dt + k \right)^2 + Z ds^2_4 \]

\[ = -Z^{-2} \left( dt + k \right)^2 + Z \left[ \left( 1 - \frac{2m}{r} \right) d\tau^2 + \left( 1 - \frac{2m}{r} \right)^{-1} dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \right], \]

where \( k \) is the angular-momentum 1-form of the running bolt and \( Z \) the warp factor linking the five- and four-dimensional metrics, each following from the non-BPS equations, (37) and (38), stated below. The coordinate, \( \tau \), results from the Wick-rotation of the time-coordinate in the Euclidian Schwarzschild base manifold; it parameterizes the \( S^1 \) with periodicity \( \tau \sim \tau + 8\pi m \).

Note: In the limits \( k \to 0 \) and \( Z \to 1 \), the metric (33) becomes the metric (1) considered earlier.

The Maxwell fields are set up by the ‘floating brane’ ansatz [8],

\[ A^I = -\varepsilon Z^{-1} \left( dt + k \right) + B^{(I)}, \]

(34)

where \( \varepsilon \) is set by the (anti-)self-duality of the fields. The magnetic field strengths are

\[ \Theta^{(I)} = dB^{(I)}. \]

(35)

The three forms, \( Z_I, \Theta^{(I)} \) and \( k \), are determined through the (non-)BPS equations [6, 9, 19, 20]:

\[ \Theta^{(I)} = \varepsilon \star_4 \Theta^{(I)}, \]

(36)

\[ \nabla^2 Z_I = \frac{1}{2} \varepsilon C_{IJK} \star_4 \left[ \Theta^{(J)} \wedge \Theta^{(K)} \right], \]

(37)

\[ dk + \varepsilon \star_4 dk = \varepsilon Z_I \Theta^{(I)}. \]

(38)

Note, that (36)–(38) are purely represented on the base manifold.

Following the choice of solution for the field strength made in [6],

\[ \Theta^{(I)} = q_I \left( \frac{1}{r^2} d\tau \wedge dr + \varepsilon d\Omega_2 \right), \]

(39)

we have also

\[ Z_I = 1 - \frac{1}{2m r} C_{IJK} q_I q_J q_K \]

(40)

\[ k = \mu \left( r \right) d\tau = \varepsilon \left( \frac{1}{r} - \frac{1}{2m} \right) \left[ \sum_{i=1}^{3} q_i - \frac{3}{2m} q_1 q_2 q_3 \left( \frac{1}{r} + \frac{1}{2m} \right) \right] d\tau, \]

(41)

where the \( q_I \) are M5-charges associated with the magnetic field strength component in (39).
It is important to note that exact terms proportional to \( d\tau \) have been chosen such that \( k \) vanishes on the bolt, which is essential for regularity and to remove closed timelike curves. With this choice, the asymptotic limit of the angular momentum does not vanish but has a finite value:

\[
\mu \to \infty \quad \gamma = -\frac{\varepsilon}{2m} \left( \sum_{i=1}^{3} q_i - \frac{3}{4m^2} q_1 q_2 q_3 \right).
\]

It is this finite limit which led to the name ‘running bolt’.

Transforming (42) leads to a formula for the magnetic charges:

\[
\sum_{i=1}^{3} q_i = -2\varepsilon m \gamma + \frac{3}{4m^2} q_1 q_2 q_3.
\]

2.4. Topological data

The spacetime at hand has entirely two cycles—the bolt and the non-compact cycle extending from \( r = 2m \) to infinity. Each of these carries an independent cohomological flux: \( d\Omega_2 \), which is carried by the bolt cycle \((B)\), and its dual, \( \frac{1}{r} dr \wedge d\tau \), which is carried by the non-compact cycle \((C)\).

The 2-form, \( d\Omega_2 \), is manifestly harmonic. Although its dual can be written as a total derivative, \( \frac{1}{r} dr \wedge d\tau = d\left( \frac{1}{2m} - \frac{1}{r} \right) d\tau \), it has a nonvanishing value, \( \frac{1}{2m} dr \), at infinity and thus is not exact.

From this, a basis for the cohomology can be directly inferred:

\[
\psi^B = \frac{1}{4\pi} d\Omega_2 \quad \text{(44)}
\]

\[
\psi^C = \frac{1}{4\pi r^2} dr \wedge d\tau \quad \text{(45)}
\]

It is immediately clear that \( \psi^B \) is cohomology. For \( \psi^C \), on the other hand, it is not so obvious; but as explained above, its potential, \( \frac{1}{2m} \left( \frac{1}{r} - \frac{1}{r_0} \right) d\tau \), is either singular at the bolt or non-vanishing at infinity, depending on \( r_0 \), and hence cohomological.

From the known results of the fields and the symmetry conditions, we will derive the cohomological 2-form fluxes. These and the fields will be used to compute the period-integrals—the topological ‘building blocks’—to particularly analyse the mass, charges, and BPS-bound breaking extra-mass within intersection homology.

2.4.1 Deriving the fluxes. In the following we derive expressions for the fields and fluxes from the RHS of (30) to understand their contributions to the mass formula.

From (34), (35) and (39) follows the Maxwell-field strength, \( F^l = dA^l \), which decomposes into an exact and a (on the base manifold) harmonic part,

\[
F^l = dA^l + \varepsilon q_l d\Omega_2 - \left( 2m \varepsilon \gamma + q_l \right) \frac{1}{r^2} dr \wedge d\tau,
\]

where

\[
\hat{A}^l = \varepsilon \left[ -Z^{-1} \mu + \gamma \left( 1 - \frac{2m}{r} \right) \right] d\tau + \text{terms proportional to } dr.
\]
Since the time-coordinate, \( t \), is not part of the base space, its 1-form, \( dt \), is irrelevant for the cohomology.

Note that the \( \lambda^I \) vanish at the bolt and at infinity and are thus globally smooth.

Choosing the Killing vector like \( K = \frac{\partial}{\partial t} \), (48)

we get from (28a):

\[
\lambda^I = \varepsilon Z_i^{-1} - \beta^I,
\]

where \( \beta^I \) are constants.

It is directly obvious that the choice \( \beta^I = \varepsilon \), (50)

causes the \( \lambda^I \) to vanish at infinity, where \( Z_i \to 1 \). This way the term \( 2\lambda^IG_I \) drops from the boundary integral of (30). Also, the below computed exact 1-forms, \( \Lambda^I \), from (29) have to fall off at infinity; and so the Komar mass is rendered purely topological.

With (34) and (23) we have

\[
G_I = \frac{1}{2} \left[ - \varepsilon r^2 \left( 1 - \frac{2m}{r} \right) Z_i^I + \varepsilon r^2 Z_i Z^{-3} \mu \mu' + q_i Z_i^2 Z^{-3} \mu \right] \, dt \wedge d\Omega_2
\]

\[
+ \frac{1}{2} Z_i Z^{-3} (\varepsilon r^2 \mu' + q_i Z_i) \, dr \wedge d\Omega_2 + \frac{\varepsilon}{2^r^2} q_i Z_i^2 Z^{-3} \, dr \wedge \, \varepsilon \, r \wedge d\tau,
\]

(51)

and find from this together with (49),

\[
i_k G_I + \frac{1}{2} C_{IK} \lambda^I F^K = - \frac{1}{2} C_{IK} \beta^I F^K - \frac{1}{2} \left( Z_i Z^{-3} (dt + \mu d\tau) \right),
\]

(52)

which is a manifestly closed expression.

Using (28b), we put all exact pieces of (52) into \( d\Lambda_I \) so that we have:

\[
\Lambda_I = - \frac{1}{2} \left[ Z_i Z^{-3} \mu - \gamma \left( 1 - \frac{2m}{r} \right) - \varepsilon C_{IK} \beta^I \left( Z_k^{-1} \mu - \gamma \left( 1 - \frac{2m}{r} \right) \right) \right] \, dt + \, dr',
\]

(53)

which is smooth at the bolt and vanishes at infinity.

The remaining terms in (52) sum up to the 2-form harmonic:

\[
H^{(2)}_I = - \frac{\varepsilon}{2} C_{IK} q_I \beta^K \, d\Omega_2 - \left[ m \gamma - \frac{1}{2} C_{IK} (q_I + 2m \varepsilon \gamma) \beta^K \right] \frac{1}{r} \, dr \wedge d\tau.
\]

(54)

As mentioned earlier, the 2-form, \( \frac{1}{r} \, dr \wedge d\tau \), has nonvanishing potential at infinity and thus is cohomological.

2.4.2. Intersection homology. The intersection technique relates an integral of two wedged 2-forms over the whole four-dimensional base space to integrals of the single 2-forms over the homological 2-cycles.

Applying the gauge choice (50), \( \beta^I = \varepsilon \), rendering the Komar mass purely topological, the period-integrals, forming the topological ‘building blocks’, ammount to:

It is instructive to introduce a canonical integer basis for the cohomology,
11

\[ \int CA v^\prime = \delta A^\prime A \quad \text{and} \quad \int \Sigma 4 v A \wedge v A^\prime = I A A^\prime, \]

(55)

with

\[ F^I = \sigma A^I v A \quad \text{and} \quad H^{I(2)} = \tilde{\sigma}_{IA} v A, \]

(56)

where \( \sigma A^I \) and \( \tilde{\sigma}_{IA} \) (\( A = B, C \)) are precisely the entries of the above tables (the ‘building blocks’), and \( I A A^\prime = I A A^\prime \) is the inverse intersection matrix.

The choice of the cohomological basis (44) and (45) manifestly fulfills the above-stated orthonormality condition.

The integrals for the mass and charge formulae become then

\[ \int_{\Sigma} H^{I(2)} \wedge F^I = 16\pi^2 \sum_{J=1}^{3} (C_{IJK} q^J + 3\varepsilon m \gamma q_I) = \tilde{\sigma}_{IA} \sigma A^I I A A^\prime \]

(57)

\[ C_{IJK} \int_{\Sigma} F^I \wedge F^K = -32\pi^2 \varepsilon C_{IJK} q^I (2\varepsilon \gamma + q_K) = C_{IJK} \sigma A^I \sigma A^K I A A^\prime. \]

(58)

In particular, this means that the integrals need to be reproduced by composing the products of the period-integrals, \( \sigma A^I \) and \( \tilde{\sigma}_{IA} \), by the integer coefficients of \( I A A^\prime \) in the sense of the last expressions, which can only be achieved by

\[ I A A^\prime = I A A^\prime = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \]

(59)

This is merely a trivial one-time intersection between the bolt and the non-compact cycle, proving that there is no self-intersecting homology at hand.

Note: If we take the building blocks from table 1 and compute \( \sigma A^I v A \), then we get

\[ F^I_{\text{harmonic}} = \varepsilon q_I d\Omega_2 - (2m \varepsilon \gamma + q_I) \frac{1}{r^2} dr \wedge d\tau, \]

(60)

but it is

\[ F^I - F^I_{\text{harmonic}} = d\hat{\Lambda}, \]

(61)

what is cohomologous to the total Maxwell-field strength, where \( \hat{\Lambda} \) is a global 1-form falling off at infinity.

In this spirit, it is now easy to compute the total Komar mass and Maxwell-charge.

### 2.5. Mass, charges, and the breaking of supersymmetry

In this section, we will evaluate the expressions for the Komar mass and the Maxwell-charges by means of the just introduced intersection homology method.

The fact that \( \Lambda_I \to 0 \) at infinity and the gauge choice,
\[ \beta^I = \varepsilon, \]  
(62)

for which the function \( \lambda^I \) goes zero according to (49), make the boundary integral drop out of (30), rendering the Komar mass an integral purely over cohomology (31):

\[ M = -\frac{1}{16\pi G_5} \int_{\Sigma_i} H_i^{(2)} \wedge F^I. \]
(63)

From (63) and (32) we get with (57) and (58) the mass and charges:

\[ M = -\frac{\pi}{G_5} \sum_{I=1}^3 (3m\varepsilon\gamma q_I + C_{JKI}q_J q_K) \]
(64)

\[ Q^I = \frac{\varepsilon}{4m} \sum_{K=1}^3 C_{JKI}q_J (2m\varepsilon\gamma + q_K). \]
(65)

From that follows

\[ M = -\frac{\varepsilon\pi m}{G_5} (4\sum_{I=1}^3 Q^I - \gamma \sum_{I=1}^3 q_I), \]
(66)

and hence the BPS-bound violating extra-mass

\[ \Delta M = M + 4\varepsilon\pi m \sum_{I=1}^3 q_I \]
(67)

\[ = -\frac{1}{16\pi G_5} \left( \int_{\Sigma_i} H_i^{(2)} \wedge F^I + \frac{\varepsilon}{2} \sum_{I=1}^3 C_{JKI} \int_{\Sigma_i} F^J \wedge F^K \right) \]
(68)

\[ = -\frac{1}{16\pi G_5} \sigma^K_A \left( \bar{\sigma}_{K,A} + \frac{\varepsilon}{2} \sum_{I=1}^3 C_{JKI} \sigma^I_A \right) I^A. \]
(69)

Interestingly, the breaking of supersymmetry is caused by the total M5-charges, \( q_I \).

Very important note: the factored term,

\[ \chi_{K,A} = \bar{\sigma}_{K,A} + \frac{\varepsilon}{2} \sum_{I=1}^3 C_{JKI} \sigma^I_A, \]
(70)

considered at each cycle separately,

\[ \chi_{K,B} = 0 \]
(71)

\[ \chi_{K,C} = -4\pi m\gamma, \]
(72)

vanishes identically for the bolt-cycle, \( A = B \), so all contribution to the breaking of supersymmetry comes from the non-compact cycle, \( A = C \):

\[ \Delta M = -\frac{1}{16\pi G_5} \sigma^K_B \left( \bar{\sigma}_{K,C} + \frac{\varepsilon}{2} \sum_{I=1}^3 C_{JKI} \sigma^I_C \right) I^B. \]
(73)

\[ = -\frac{1}{16\pi G_5} \sum_{K=1}^3 4\pi \varepsilon q_K (-4\pi m\gamma) \]
(74)

\[ = \frac{\varepsilon \pi m\gamma}{G_5} \sum_{K=1}^3 q_K. \]
(75)
3. Topological contributions to the BPS-bound violation in a 2-center solution

3.1. The 2-center solution

In the following we outline the main results of the non-supersymmetric 2-center solution given in [13].

The geometry of the spacetime and the three \( U(1) \)-gauge fields are a solution of the Einstein–Maxwell equations in the floating brane ansatz [8] within five-dimensional ungauged supergravity. The stationary spacetime carries a Killing vector, \( \mathbf{K} = \partial_t \), and has the metric:

\[
d s^2_5 = - \left( \frac{1}{2} LL_a L^a \right) \frac{1}{\sqrt{\Delta}} \left( dt + \hat{k} \right)^2 + \left( \frac{1}{2} LL_a L^a \right) d s^2_4, \tag{76}
\]

where the functions \( L \) and \( L_a \) and the angular momentum 1-form \( \hat{k} \) will be defined below.

The parameter \( a \) in (76) counts vector multiplets; like in [2, 6, 7] we choose to have two \((a = 2, 3)\) to have a total of three Maxwell-charges, \( A^I (I = 1, 2, 3) \). This enables non-trivial Chern–Simons interactions.

However, like [13] we use the STU truncation in which the fields and fluxes with index \( I = 1 \) are treated in a different way than \( I = a \). In detail, the raise of the latter index, \( K_a = \eta_{ab} K^b \), happens with an \( SO(1,1) \) metric following from the non-zero Chern–Simons coupling \( \mathbf{C}_{ab} = \eta_{ab} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \).

As for the running bolt solution from the last section and Gibbons–Hawking metrics, the four-dimensional base space is a \( U(1) \)-fibration over a 3-manifold which is asymptotically \( \mathbb{R}^3 \) at infinity, rendering the asymptotics of the whole spatial base \( S^1 \times \mathbb{R}^3 \) at infinity.

In particular, the 3-manifold is parameterized by \((r, \theta, \phi)\) and the fiber by \( \psi \). As in the last section, the latter defines a compact spatial dimension—the \( S^1 \)—, this time with periodicity \( \psi \sim \psi + 4\pi k \), where \( k \) is the scale parameter of the \( S^1 \).

The special topology considered here, is constituted by a non-extremal charged bolt at \( r = c \) and an extremal Gibbons–Hawking center at \( r = R > c \) and \( \theta = 0 \). This leads to a non-extremal and non-BPS generalization of known extremal results for BPS [12] and almost-BPS systems [9–11].

The most crucial aspect of this topology is that, the \( \psi \)-fiber pinches off at two locations—the bolt and the Gibbons–Hawking center. This \( \psi \)-fiber along an interval between \( r = c \) and \( r = R \) at \( \theta = 0 \) defines the new non-trivial compact homology cycle.

The metric of the four-dimensional base manifold is

\[
d s^2_4 = V^{-1} (d\psi + \omega_z) \left( d\psi + \omega_z \right) + V \left( dr^2 + \left( r^2 - c^2 \right) \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \right), \tag{77}
\]

where \( V \) has poles at \( r = c \) and \((r, \theta) = (R, 0)\). This causes the \( \psi \)-fiber to pinch off, but leaves the metric smooth.

This leads to the main difference to the running-bolt spacetime of the last section, which had only one pinch-off point for the circle-fibration.

As we will see in detail, the fact that \( \psi \) pinches off at two centers generates a new homological cycle defined by the fiber and the radial interval between the two pinch-off points.

In particular, if a periodic coordinate pinches off at only one existing center, like in the running-bolt solution, it can be fixed at the bolt at the cost of creating flux that does not vanish at infinity and hence giving rise to a non-compact cycle. Two pinch-off points, as considered here, give rise to both a further compact as well as the non-compact cycle.
There are three independent homological 2-cycles (figure 1): the bolt-sphere at $r = c$; the bubble-cycle being the $\psi$-fibered radial line along the positive $z$-axis, connecting the bolt’s north pole and the Gibbons–Hawking center at $(r, \theta) = (R, 0)$; the non-compact cycle, also running radially along the positive $z$-axis from the Gibbons–Hawking center, $z = R$, towards infinity, fibred by the $\psi$-circles.

The appendix gives more information on the functions used here along with their asymptotics—they are very complicated, which reinforces the interest in cohomology, but we will need some details here:

\[
V (r, \theta) = \frac{r + m_-}{2(r^2 - c^2)} \left( r + m_+ - \frac{2k}{R + m_-} \frac{Rr - c^2 \cos \theta}{\sqrt{r^2_1 - c^2 \sin^2 \theta}} \right)
\]

(78)

\[
\hat{k} = \omega - \frac{M}{V} \left( d\psi + \omega^0 \right)
\]

(79)

\[
\omega^0 (r, \theta) = -\frac{1}{2} \left[ (m_+ - m_-) \cos \theta + \frac{2k}{R + m_-} \frac{R^2 - m_- r - R (r - m_-) \cos \theta - c^2 \sin^2 \theta}{\sqrt{r^2_1 - c^2 \sin^2 \theta}} \right] d\phi
\]

(80)

\[
\omega (r, \theta) = -\frac{e_- R}{2(R + m_-)} u_a u^a \left[ \left( 1 - \frac{r + R}{r_1} \right) (1 - \cos \theta) + \frac{c^2}{R r_1} \sin^2 \theta \right] d\phi
\]

(81)

\[
L_a (r, \theta) = \frac{(r + m_-) (c^2 + m_- r)}{2m_- (r^2 - c^2)} \frac{L_a}{V} + u_a
\]

(82)

\[
L (r, \theta) = \frac{e_-}{2m_-^2} \frac{1}{V} L_a u^a - \frac{e_-}{c^2 (m_+ + r) (m_- + R)} u_a u^a
\]

(83)

3 The functions used here are taken from equations (2.25), (2.26), (3.55), (4.2)-(4.6) of [13].

4 In [13] the functions were equipped with an extra-parameter, $n_A$, that can be set equal to 1, which we do throughout this section.
\[ M(r, \theta) = -\frac{e_+}{2m_+} l_\alpha L^\alpha + \frac{e_-}{2(m_- + R)} \left[ \frac{R - r}{m_- + r} V + \frac{(c^2 + m_- r) (f_1 r + f_2)}{2c^2 (c + m_-)^2 (r^2 - c^2)} \right] u_\alpha u^\alpha, \] (84)

where \( r_1 = \sqrt{r^2 + R^2 - 2R r \cos \theta} \) is the distance measured from the Gibbons–Hawking center; \( m_-, m_+, e_-, l_\mu, u_\mu, f_1, \) and \( f_2 \) are parameters of the solution which are non-trivially inter-
related (see appendix A).

The function \( V \) has two singularities—one at the bolt, going like \( \frac{1}{r^4} \), and one at the
Gibbons–Hawking point, going like \( \frac{1}{r^2} \) at \( \theta = 0 \); in the metric (77) it poses a coordinate
singularity and so does not harm regularity at the centers.5

Like the geometry, the Maxwell-fields are solutions to the Einstein–Maxwell equations;
their potentials in the floating brane ansatz are:

\[ A^1 = \frac{1}{L} (U^1 + \hat{U}^1) - \frac{1}{2e_-} \left[ \frac{r + m_-}{V (r^2 - c^2)} \left( \frac{c^2 + m_- r}{V} \right) \phi \left( \phi + \omega \right) \right] \] \[ + \left( \cos \theta + \frac{2k}{R + m_-} \sqrt{r^2 - c^2 \sin^2 \theta} \right) \phi \right] \] (85)

where in both cases the first part represents the terms which are globally defined—yet not
exact, for they do not fall off at infinity.

From this follow their field-strengths, \( F^i = dA^i \),

\[ F^1 = d \left( \frac{1}{L} (U^1 + \hat{U}^1) \right) + Z_1 \left[ V \left( r^2 - c^2 \right) d\Omega_2 - dr \wedge (\phi + \omega) \right] + Z_2 \left[ d\theta \wedge (\phi + \omega) + V \sin \theta dr \wedge d\phi \right] \] (87)

\[ F^a = d \left( \frac{1}{L} (U^a + \hat{U}^a) \right) - \frac{e_+}{m_-} \left[ \frac{r + m_-}{V (r^2 - c^2)} \left( \frac{c^2 + m_- r}{V} \right) \phi \left( \phi + \omega \right) \right] \] \[ + \left( \cos \theta + \frac{2k}{R + m_-} \sqrt{r^2 - c^2 \sin^2 \theta} \right) \phi \right] \right] \] (88)

where the terms have been sorted according to the ones of the potentials.

Extracting the topological bits out of the globally defined terms, to make them exact, can
be done with help of the cohomological basis which we derive later. Adding them to the other
terms, however, would render them not harmonic anymore, so the present topology has both
self-dual and anti-self-dual parts.

The functions \( Z_{1,2} \), have the form:

\[ Z_1 = \frac{(c^2 + m_- r) (r + m_-)}{2e_- V (r^2 - c^2)} \left( \frac{r + m_-}{2V (r^2 - c^2)} - \frac{m_-}{c^2 + m_- r} + \left( 1 - \frac{(r + m_-) (r + m_+)}{2V (r^2 - c^2)} \right) \right) \] \[ \times \left( \frac{R}{R r - c^2 \cos \theta} + \frac{R \cos \theta - r}{r_1^2 - c^2 \sin^2 \theta} \right) \] \[ (89) \]

\[ Z_2 = -\frac{(c^2 + m_- r) (r + m_-)}{2e_- V} \left( 1 - \frac{(r + m_-) (r + m_+)}{2V (r^2 - c^2)} \right) \left( \frac{R}{R r - c^2 \cos \theta} + \frac{R \cos \theta - r}{r_1^2 - c^2 \sin^2 \theta} \right) \sin \theta \] \[ \left( \frac{R}{R r - c^2 \cos \theta} + \frac{R \cos \theta - r}{r_1^2 - c^2 \sin^2 \theta} \right) \] \[ (90) \]

5 For a much more detailed outline of the regularity analysis, see [13].
The dual field strengths have to be computed by
\[ G_1 = \frac{1}{2} L^2 \left( \frac{1}{2} L L' \right)^{-\frac{i}{3}} \ast_5 F^1 \]
(91)
\[ G_a = \frac{1}{2} L^{-\frac{i}{2}} \left( \frac{1}{2} L L' \right)^{\frac{i}{3}} (L^a)^{-\frac{2}{3}} \ast_5 F^a, \]
(92)
where the prefactors correspond to the \( Q_{ij} \) from (23), composed in the same manner as in (18) of the scalars of the solution:
\[ X^1 = L^{-\frac{i}{2}} \left( \frac{1}{2} L_a L^a \right)^{\frac{i}{3}} \]
and \( X^a = L^{\frac{i}{2}} \left( \frac{1}{2} L_b L^b \right)^{-\frac{2}{3}} L^a. \)

The derivation of the Komar mass formula was done in the last section in (30); the three \( U(1) \)-charges relevant for this section, are given by the properly normalized formula:
\[ Q^l = -\frac{1}{\text{vol}(X^1)} \int_{X^1} G_l = -\frac{1}{16\pi^2 k} \int_{\Sigma_a} dG_l = -\frac{1}{64\pi^2 k} C_{ljk} \int_{\Sigma_4} F^l \wedge F^k, \]
(93)
where the last step follows from the equations of motion for the Maxwell-fields (22).

### 3.2. Deriving the fluxes

In the following, we derive expressions for the fields and fluxes to understand their contributions to the mass formula. As above, we write the timelike Killing vector near infinity like \( K = \frac{\partial}{\partial t} \).

From (87)–(88) we get
\[ i_k F^1 = -d \left( \frac{1}{L} \right) \]
(94)
\[ i_k F^a = -d \left( \frac{1}{L_a} \right), \]
(95)
and so with (28a):
\[ \lambda^1 = \beta^1 = -\frac{1}{L}, \]
(96)
\[ \lambda^a = \beta^a = \frac{1}{L_a}, \]
(97)
where the \( \beta^I \) are freely-choosable constants. We will later fix them though, so that \( \lambda^I \to 0 \) at infinity.

From (87)–(92) and (96)–(97) we find, decomposing (28b) according to the STU truncation:
\[ \Gamma_1 = \frac{e}{(r + m)^2} \left( u_a \beta^a - 2 \right) \left[ V (r^2 - c^2) \, d\Omega_2 - dr \wedge (d\psi + \omega^0) \right] \]
\[ + d \left[ \frac{1}{L_a L'} \left( \beta^a L_a - 1 \right) \left( dt + \hat{k} \right) - \frac{e}{2m} V u_a \beta^a (d\psi + \omega^0) \right] \]
(98)
\[
\Gamma_a = \left( \frac{e^a}{(r + m_-)^2} u_a \partial^1 + \frac{1}{2} \eta_{ab} \partial^b Z_1 \right) \left[ V (r^2 - c^2) \, d\Omega_2 - d\varphi \wedge (d\psi + \omega^0) \right] \\
+ \frac{1}{2} \eta_{ab} \partial^b Z_2 \left[ V \sin \theta \, d\varphi \wedge d\theta + d\theta \wedge (d\psi + \omega^0) \right] \\
+ d \left[ \frac{1}{2} \left( \frac{L}{L^a} \partial^1 + \frac{1}{2} \eta_{ab} \partial^b - \frac{1}{LL^a} \right) \left( d\varphi + \hat{k} \right) - \frac{e^a}{2m_- V} I_a \partial^1 (d\psi + \omega^0) \right],
\]

where we wrote for short,
\[
\Gamma_1 = i_k G_1 + \frac{1}{2} \eta_{ab} \lambda^a F^b
\]
\[
\Gamma_a = i_k G_a + \frac{1}{2} \eta_{ab} \left( \lambda^1 F^b + \lambda^b F^1 \right).
\]

These formulae suggest obvious choices for the analogues of \((53)\) and \((54)\):
\[
\tilde{H}_1^{(2)} = \frac{e^a}{(r + m_-)^2} (u_a \partial^a - 2) \left[ V (r^2 - c^2) \, d\Omega_2 - d\varphi \wedge (d\psi + \omega^0) \right]
\]
\[
\tilde{H}_a^{(2)} = \left[ \frac{e^a}{(r + m_-)^2} u_a \partial^1 + \frac{1}{2} \eta_{ab} \partial^b Z_1 \right] \left[ V (r^2 - c^2) \, d\Omega_2 - d\varphi \wedge (d\psi + \omega^0) \right]
\]
\[
+ \frac{1}{2} \eta_{ab} \partial^b Z_2 \left[ V \sin \theta \, d\varphi \wedge d\theta + d\theta \wedge (d\psi + \omega^0) \right]
\]
\[
\tilde{\Lambda}_1 = \frac{1}{L_a L^c} (\beta^a L_a - 1) \left( d\varphi + \hat{k} \right) - \frac{e^a}{2m_- V} I_a \partial^a (d\psi + \omega^0)
\]
\[
\tilde{\Lambda}_a = \frac{1}{2} \left( \frac{L^a}{L^2} \partial^1 + \frac{1}{2} \eta_{ab} \partial^b - \frac{1}{LL^a} \right) \left( d\varphi + \hat{k} \right) - \frac{e^a}{2m_- V} I_a \partial^1 (d\psi + \omega^0).
\]

Note: The tilde indicates that the \(d\tilde{\Lambda}_1\) are not exact, since the \(\tilde{\Lambda}_1\) do not vanish at infinity and are thus singular there; exactness can be achieved, however, by extracting the cohomological bits from the \(d\tilde{\Lambda}_1\) by means of the cohomological basis, which we will derive later, and shift them into the \(\tilde{H}_1^{(2)}\).

The \(\tilde{H}_1^{(2)}\) are manifestly self-dual, and one can check that \(d\tilde{H}_1^{(2)} = 0\). They are thus harmonic and can locally be written as \(\tilde{H}_1^{(2)} = \tilde{B}_1\), where the \(\tilde{B}_1\) are not globally defined since they do not vanish at the pinch-off points of the \(\psi\)-coordinate:
\[
\tilde{B}_1 = e_- (u_a \partial^a - 2) \left[ \frac{1}{r + m_-} (d\psi + \omega^0) - \left( \frac{\cos \theta}{2} + \frac{k}{r + m_-} \frac{r - R \cos \theta}{\sqrt{r_1^2 - c^2 \sin^2 \theta}} \right) \, d\phi \right]
\]
\[
(106)
\]
\[ \tilde{B}_a = e_- u_\alpha \beta^1 \left[ \frac{1}{r + m_-} (d\psi + \omega^0) - \left( \frac{\cos \theta}{2} + \frac{k}{R + m_-} \frac{r - R \cos \theta}{\sqrt{r^2 - c^2 \sin^2 \theta}} \right) d\phi \right] \]

\[ - \frac{1}{4e_-} \eta_{ab} \beta^b \left[ \frac{(r + m_-) (c^2 + m_- r)}{V (r^2 - c^2)} (d\psi + \omega^0) + (c^2 - m_-^2) \cos \theta d\phi \right]. \quad (107) \]

Note that, because \( V \) has a singularity at each center, \( V^{-1} \) goes zero there and ensures finite norms—except for the terms with the additional factor \( (r - c)^{-1} \) which cancels the zero at the bolt and hence the ‘good’ behavior.

As one can clearly see from the analysis, the factors in the volume integral of (30), \( H_I^{(2)} \) and \( F^I \), are each equipped with two dual flux terms, \( V (r^2 - c^2) d\Omega_2 \) and \( dr \wedge (d\psi + \omega^0) \), wedge-multiplying to space’s volume form. Hence, homology allows for the existence of purely topological terms in the mass formula, which cannot be converted into boundary terms, and even has self-intersection, as we will see in the following.

The topological objects described in (102) and (103) are the cohomological fluxes for this solution, the duals of which form the homological cycles constituting the supergravity soliton at hand.

### 3.3. Topological data and intersection homology

In this subsection, we explicitly derive the topological ingredients flowing into the formula for the mass and charges. The goal is to see if and how each of the 2-cycle’s contribution goes into either, so that we can precisely make out their sources of topology in virtue of intersection homology—with special emphasis on the breaking of supersymmetry.

As mentioned earlier, the cohomological fluxes (102)–(103) are still missing cohomology from the \( d\tilde{\Lambda}_I \). Since the \( \tilde{\Lambda}_I \) are only irregular at infinity, these cohomological pieces just concern the non-compact cycle, \( C_C \):

\[ H_I^{(2)} = \bar{H}_I^{(2)} + \left( \int_{C_C} d\tilde{\Lambda}_I \right) v^C \]

\[ = \bar{H}_I^{(2)} + \pi k \left[ \left( 2\beta^a - \frac{1}{l_a + u_a} \right) \frac{\gamma}{l_a + u_a} - \frac{4 e_-}{m_-} l_a \beta^a \right] v^C \]

\[ H_a^{(2)} = \bar{H}_a^{(2)} + \left( \int_{C_C} d\tilde{\Lambda}_a \right) v^C \]

\[ = \bar{H}_a^{(2)} + 2 \pi k \left[ \left( \frac{\gamma}{l^a + u^a} - \frac{2 e_-}{m_-} l_a \right) \beta^a + \frac{\gamma}{L} \eta_{ab} \beta^b - \frac{4 e_-}{L} \eta_{ab} \beta^b \right] v^C, \quad (108) \]

where \( v^C \) is the cohomological basis vector for the non-compact cycle derived below in (124).

The Komar mass and charges are not dependent on any choice of gauge, but it is very convenient to choose the \( \beta^I \) such that the boundary integral term \( 2 \lambda^I G_I \) in (112) vanishes; with (96) and (97) this is achieved by

\[ \beta^i = \lim_{r \to \infty} \frac{1}{L} = \left[ \frac{e_-}{m_-} l_a l^a + \frac{e_-}{c^2 (c + m_-)} \left( \frac{m_-}{R + m_-} - \frac{4 e_- k}{(c + m_-)^2} \right) u_a u^a \right]^{-1} \]

(110)
$19 \beta a = \lim_{r \to \infty} 1/L_a = \frac{1}{L_a + u_a}$.  \hfill (111)

With this choice the equations (30) and (93) then become completely topological expressions:

$$M = -\frac{1}{16\pi G_5} \int_{\Sigma_4} H_{1}^{(2)} \wedge F^4$$ \hfill (112)

$$Q' = -\frac{1}{64\pi^2 k} C_{ijk} \int_{\Sigma_4} F^j \wedge F^k.$$ \hfill (113)

We now compute the topological ‘building blocks’ of the fields and fluxes by which (112)-(113) shall be represented in the framework of intersection homology. Like in the last section, we derive the period-integrals (see tables 2 and 3) by doing the integrals of the 2-form fluxes over all topologically relevant 2-cycles (see figure 2):

The parameter $\alpha$ hereby represents the bolt’s NUT-charge.

It is instructive again to introduce a canonical integer basis for the cohomology,
\[ \int_{C_A} v^A = \delta^A_A \text{ and } \int_{\Sigma_A} v^A \wedge v^{A'} = F^{A'}. \] (114)

with

\[ F^l = \sigma^l_A v^A \text{ and } H^{(2)}_I = \tilde{\sigma}_I A v^A, \] (115)

where \( \sigma^l_A \) and \( \tilde{\sigma}_I A \) \((A = B, \Delta, C)\) represent the entries of the above tables, and \( F^{A'} = F^A A' \) is the inverse intersection matrix.

In this spirit, the bulk integral (112) becomes

\[ \int_{\Sigma_A} H^{(2)}_I \wedge F^l = \tilde{\sigma}_I A \sigma^l_A F^{A'}, \] (116)

and analogously for (113). Reproducing the integrals by composing the products of the period-integrals, \( \sigma^l_A \) and \( \tilde{\sigma}_I A \), by the integer coefficients of \( F^{A'} \) in the sense of (116), can only be achieved by

\[ I^{A'} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & -n & -n \\ 1 & -n & -n - 1 \end{pmatrix} \Leftrightarrow I_{AA'} = \begin{pmatrix} n & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix}. \] (117)

It is obvious that in the present spacetime the homological structure is significantly more complex than in the one of the last section.

The off-diagonal 1’s in \( I_{AA'} \) are directly clear: the bolt intersects the bubble at the former’s north pole; the bubble intersects the non-compact cycle in the Gibbons–Hawking point. Less intuitive are the self-intersections of each cycle—it is \( n \)-fold for the bolt for NUT-charge of \( n \); the self-intersections of the bubble and the non-compact cycle arise, like the intersection between them, from the topology of the nut linking these cycles.

Since we want to study the explicit topological contribution of each intersecting part later, it is important to see in how far the elements of \( I_{AA'} \), representing the intersections, are reflected in \( F^{A'} \), representing the composition of the building blocks. To shed light on this, we leave the intersection numbers in \( I_{AA'} \) more general:

\[ I_{AA'} = \begin{pmatrix} I_{BB} & I_{B\Delta} & 0 \\ I_{B\Delta} & I_{\Delta\Delta} & I_{\Delta C} \\ 0 & I_{\Delta C} & I_{C C} \end{pmatrix}. \] (118)

After inversion, we receive

\[ I^{A'} = \frac{1}{\det(I_{AA'})} \begin{pmatrix} I_{\Delta\Delta} I_{C C} - I^2_{\Delta C} & -I_{B\Delta} I_{C C} & I_{B\Delta} I_{\Delta C} \\ -I_{B\Delta} I_{C C} & I_{B B} I_{C C} & -I_{B B} I_{\Delta C} \\ -I_{B B} I_{\Delta C} & I_{B B} I_{\Delta C} & I_{B B} I_{\Delta C} - I^2_{B\Delta} \end{pmatrix}. \] (119)

Note, that this form is merely a schematic ‘tracking device’ to qualitatively illustrate how the contributions of intersection from \( I_{AA'} \) distribute among the non-zero entries of \( F^{A'} \); hence, the determinant, \( \det(I_{AA'}) = 1 \), and the upper left entry, \( I_{\Delta\Delta} I_{C C} - I^2_{\Delta C} = 0 \) (not contributing to topology), are not of interest in this regard. So, the matrix can be written as:

\[ I^{A'} = \begin{pmatrix} 0 & -I_{B\Delta} I_{C C} & I_{B\Delta} I_{\Delta C} \\ -I_{B\Delta} I_{C C} & I_{B B} I_{C C} & -I_{B B} I_{\Delta C} \\ I_{B\Delta} I_{\Delta C} & -I_{B B} I_{\Delta C} & I_{B B} I_{\Delta C} - I^2_{B\Delta} \end{pmatrix}. \] (120)
\[ = I_{B\Delta} \begin{pmatrix} 0 & -I_{CC} & I_{\Delta C} \\ -I_{CC} & 0 & 0 \\ I_{\Delta C} & 0 & -I_{B\Delta} \end{pmatrix} + I_{BB} \begin{pmatrix} 0 & 0 & 0 \\ 0 & I_{CC} & -I_{\Delta C} \\ 0 & -I_{\Delta C} & I_{B\Delta} \end{pmatrix}. \quad (121) \]

The last expression indeed gives a clue about how to disentangle homology.

First, \( M^A \) decomposes with respect to the bolt-intersections \((I_{BB} \text{ and } I_{B\Delta})\). The resulting constituents are predominantly defined by the homology of the subsystem of the bubble and the non-compact cycle, that is, the topology of the nut; and if this was turned off, then there would still be a contribution \(-I_{B\Delta}^2 = -1\) from the bolt-bubble intersection.

This gives rise to the conclusion that there might be a possible topological hierarchy between the bolt and the nut.

In any case, these insights will prove to be very helpful in spectralizing the topological contributions later.

From (87)–(90) and (115)–(117) follows the cohomological basis:

\[ \nu^B = \frac{1}{4\pi} d\Omega_2 \quad (122) \]

\[ \nu^\Delta = \left( \Xi^\Delta_1 Z_1 + \frac{\Xi^\Delta_2}{(r + m)^2} \right) \left[ V \left( r^2 - c^2 \right) d\Omega_2 - dr \wedge (d\psi + \omega^0) \right] \]

\[ + \Xi^\Delta_3 Z_2 \left[ d\theta \wedge (d\psi + \omega^0) + V \sin \theta dr \wedge d\phi \right] + \Xi^\Delta_2 d\Omega_2, \quad (123) \]

\[ \nu^C = \left( \Xi^C_1 Z_1 + \frac{\Xi^C_2}{(r + m)^2} \right) \left[ V \left( r^2 - c^2 \right) d\Omega_2 - dr \wedge (d\psi + \omega^0) \right] \]

\[ + \Xi^C_3 Z_2 \left[ d\theta \wedge (d\psi + \omega^0) + V \sin \theta dr \wedge d\phi \right] + \Xi^C_2 d\Omega_2, \quad (124) \]

with the constant coefficients,

\[ \Xi^\Delta_1 = \frac{e_+}{\pi} \frac{e^+ + m_+}{(R + m_+)(e + m)} \]

\[ \Xi^\Delta_2 = -\frac{m_-}{\pi} \frac{e^+ + m_+}{(R + m_+)(e + m)} \]

\[ \Xi^\Delta_3 = -\frac{4\pi}{e_+} \frac{4km_+ (R + m_+)(e + m)}{4 \pi (c_+ + m_+)(c_- + m_-) - 4km_+ (c_+ - R)} \]

\[ \Xi^C_1 = \frac{e_+}{\pi} \frac{e^+ - R}{(R + m_+)(e + m)} \]

\[ \Xi^C_2 = -\frac{m_-}{\pi} \frac{e^+ - R}{(R + m_+)(e + m)} \]

\[ \Xi^C_3 = \frac{4\pi}{e_+} \frac{4km_+ (R - c_+)(c + m_-) + 2(e_+ - c)(3R - 2e + m_-)}{4 \pi (c_+ + m_+)(c_- + m_-) - 4km_+ (c_+ - R)}. \quad (125) \]

Note: \( \nu^\Delta \) and \( \nu^C \) are each self-dual and hence harmonic, up to a \( d\Omega_2 \)-term.

The orthonormality condition (114) is obviously fulfilled by \( \nu^B \). However, to show the same for \( \nu^\Delta \) and \( \nu^C \), one has to evaluate the integrals of the coefficient functions over the cycles with help of

\[ \int_{S_{\theta = \pi}} V \left( r^2 - c^2 \right) Z_1 d\Omega_2 = \frac{2\pi (e^2 - m^2)}{e_-} + \frac{\pi (e + m_-)^2}{e_-} n \]

\[ \int_{c}^{R} Z_1 |_{\theta = 0} dr = \frac{(e + m_-)^2}{4ke_-} \]

\[ \int_{R}^{\infty} Z_1 |_{\theta = 0} dr = \frac{m_-}{e_-}. \]

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Now, we can easily write the fluxes in terms of the cohomology basis:

\[ H^{(2)}_1 = \tilde{\sigma}_A \tilde{A}^A = \left[ \frac{8\pi e_+}{(c + m_-)^2} l_a u^A + \frac{4\pi e_-(c - R)}{(c + m_-)(R + m_-)} l_a u^A - \left( \frac{\gamma}{\nu + \nu'} - \frac{e_-}{m_-} \frac{4R}{R + m_-} \right) \right] \frac{\pi k}{l_a + u_a} \]

\[ H^{(2)}_a = \tilde{\sigma}_{aA} \tilde{A}^A = \left[ \frac{8\pi e_-}{(c + m_-)^2} l_a u^A + \frac{4\pi e_-(c - R)}{(c + m_-)(R + m_-)} l_a u^A + \frac{\pi e_-}{2e_-} \frac{1}{\nu + \nu'} \right] \nu^A \]

\[ \Lambda_1 = \tilde{\Lambda}_1 - \left( \oint_{C^C} d\tilde{\Lambda}_1 \right) \omega^C = \frac{1}{L_a L^A} \left( \frac{1}{l_a + u_a - 1} (dt + k) - \frac{e_-}{2m_- V} \frac{1}{l_a + u_a} (d\psi + \omega^0) - \frac{\pi k}{l_a + u_a} \gamma_{aA} \omega^C \right) \]

\[ \Lambda_a = \tilde{\Lambda}_a - \left( \oint_{C^C} d\tilde{\Lambda}_a \right) \omega^C = \frac{1}{2} \left( \frac{1}{L^A} \beta^b + \frac{1}{L} (\tau_{ab} \beta^b - \frac{1}{LL^A} (dt + k)) - \frac{e_-}{2m_- V} \frac{1}{l_a + u_a} (d\psi + \omega^0) - \frac{2\pi k}{L} \gamma_{aA} \omega^C \right), \]

where

\[ \gamma_a = \frac{\gamma}{\nu + \nu'} - \frac{2e_-}{m_-} \frac{l_a}{l_a + u_a} \]

\[ \tilde{\gamma}_a = \frac{\gamma}{\nu + \nu'} - \frac{4e_-}{m_-} \frac{l_a}{l_a + u_a} \]

and \( \omega^C \) is the potential for \( \nu^C = d\omega^C \).

\[ \omega^C = \left[ \frac{1}{r + m_-} - \frac{(r + m_-)(c^2 + m_- r)}{2e_- V (r^2 - c^2)} \Xi^C \Xi^C \right] (d\psi + \omega^0) \]

\[ = \left[ \cos \theta \left( \frac{c^2 - m^2}{e_-} \Xi^C + \Xi^C + 2\Xi^C \right) + \frac{k}{R - m_-} \frac{r - R \cos \theta}{\sqrt{r_1^2 - c^2 \sin^2 \theta}} \right] d\phi. \]

Analogously it holds for the fields,

\[ F^1 = \sigma^1_{A} \tilde{A}^A \]

\[ = \left( \frac{2\pi (c^2 - m^2)}{e_-} + \frac{\pi (c + m_-) n}{e_-} \right) \nu^A + \frac{\pi (c + m_-)^2}{e_-} \nu^A + 4\pi k \left( \frac{\gamma}{L} - \frac{2m_-}{e_-} \right) \nu^C \]
\[ F^a = \sigma^a \nu^A \]  
\[ = \frac{8\pi ke_\pm}{c + m_\pm} \left( \frac{2c}{c + m_\pm} - \nu^b + \frac{c - R}{R + m_\pm} \right) + 4\pi k \left[ \frac{\gamma}{l_a + u_a} - \frac{2e_\pm}{m_\pm} \left( \nu^b + \frac{m_\pm - u^a}{R + m_\pm} \right) \right] \nu^c, \]  
and the potentials of the \( H_1^{(2)} \),

\[ B_1 = \tilde{B}_1 + \left( \int_{c\subset} d\tilde{\Lambda}_1 \right) \omega^c = \tilde{B}_1 + \frac{\pi k}{l_a + u_a} \tilde{\gamma}_a \omega^c \]  
\[ B_a = \tilde{B}_a + \left( \int_{c\subset} d\tilde{\Lambda}_a \right) \omega^c = \tilde{B}_a + \frac{2\pi k}{L} \tilde{\gamma}_a \omega^c. \]  

### 3.4. Mass, charges, and BPS-bound breaking from cohomology

We now have the data we need to compute the total mass—the Smarr formula of the 2-center supergravity soliton—and charges in the framework of the present topology.

From (112)–(117) follow the expressions:

\[ M = -\frac{1}{16\pi G_5} \int_{\Sigma_4} H_1^{(2)} \wedge F^I = -\frac{1}{16\pi G_5} \tilde{\sigma}^I \tilde{\sigma}^I \hat{A}^{A'} \]  
\[ = \frac{\pi k}{2G_5} \left\{ \frac{4ke^2}{L} \left[ \frac{1}{(R + m_\pm)^2} + \frac{4c + n (c + m_\pm)}{(c + m_\pm)^3} \right] u_a u^a \right. \]  
\[ - \left. \frac{1}{l_a + u_a} \left[ \frac{\gamma}{2} \gamma_a - \frac{e_\pm}{2m_\pm} l_a \right] \left( \frac{e_\pm R + 2m_\pm}{L} u_a u^a - \frac{c^2}{e_\pm} \right) + \frac{e_\pm - \gamma}{2L} (u_a - l_a) \right\} \]  
\[ Q^1 = -\frac{1}{64\pi^2 k} \eta_{ab} \int_{\Sigma_4} F^a \wedge F^b = -\frac{1}{64\pi^2 k} \eta_{ab} \sigma^a \sigma^b I^{A'} \]  
\[ = \frac{ke^2}{L} \left[ \frac{1}{(R + m_\pm)^2} + \frac{4c + n (c + m_\pm)}{(c + m_\pm)^3} \right] u_a u^a + \frac{1}{4} \gamma^a (m_\pm - \gamma_a - 2e_\pm u_a) \]  
\[ Q^a = -\frac{1}{32\pi^2 k} \eta_{ab} \int_{\Sigma_4} F^a \wedge F^b = -\frac{1}{32\pi^2 k} \eta_{ab} \sigma^a \sigma^b I^{A'}, \]  
\[ = -\frac{1}{4} \left\{ \frac{2m_\pm - \gamma}{L} \left( e_\pm u_a - \gamma_a \right) + \frac{m_\pm^2 c^2}{e_\pm^2 \gamma_a} \right\}. \]  

The core piece of this work is to examine the topological origin of the BPS-bound breaking extra-mass, for which purpose we will sequence the foregoing results with respect to intersecting homology.

Comparing (143) with (145)–(147), the relation between the mass and the total charge,

\[ M = \frac{2\pi k}{G_5} (Q^1 + Q^2 + Q^3) + \Delta M, \]
gives the sought extra-mass term,
\[ \Delta M = M - \frac{2\pi k}{G_5} \sum_{i=1}^{\Sigma} Q^l \]

\[
= - \frac{1}{16\pi G_5} \left[ 2 \sum_{i=1}^{\Sigma} H_i^2 \wedge F^l - \frac{1}{2} \sum_{i=1}^{\Sigma} C_{IK} \int_{\Sigma_i} F^l \wedge F^K \right] 
\]

\[
= - \frac{1}{16\pi G_5} \sigma^k_A \left( \tilde{\sigma}_{kA} - \frac{1}{2} \sum_{i=1}^{\Sigma} C_{IK} \sigma^l_A \right) f^M. 
\]

With (143)–(147) it ammounts to
\[
\Delta M = \frac{\pi k}{2G_5} \sum_{a=2}^{\Sigma \gamma^3} \left( 4ke^2 - \frac{1}{L} - 1 \right) \left[ 1 - \frac{4(c + n(c + m_-))}{(c + m_-)^2} \right] u_u a^a 
\]

\[
= - \frac{1}{\mu + u^a} \left( \frac{3}{4} - \frac{c}{2m_-} \right) \left( e_- R + 2m_- u_u u^a - \frac{c^2}{e_-} \right) - e_- \gamma \left[ \frac{1}{\mu + u^a} - 1 \right] 
\]

\[
= \frac{\gamma}{2L} \left( 2e_- u_u - m_- \gamma_a \right) + \left( \frac{m_- - c^2}{e_-} \right) \gamma_a. 
\]

Like in the previous section, it is insightful to investigate the from (151) factored term,
\[
\chi_{kA} = \tilde{\sigma}_{kA} - \frac{1}{2} \sum_{i=1}^{\Sigma} C_{IK} \sigma^l_A, 
\]

in more detail for every cycle:
\[
\chi_{1,B} = - \frac{8\pi c e_-}{(c + m_-)^2} \left[ 2 - \sum_{a=2}^{\Sigma \gamma^3} u_u \left( \frac{1}{\mu + u^a} - 1 \right) \right] 
\]

\[
= \frac{8\pi c e_-}{(c + m_-)^2} \left[ \frac{1}{L} - 1 \right] u_u + \left( \pi \left( \frac{c_2^2 - m_-^2}{e_-} \right) + \frac{\pi (c + m_-)^2}{2e_-} \right) \left( \frac{1}{\mu + u^a} - 1 \right) 
\]

\[
\chi_{1,A} = - \frac{4\pi c e_- (c - R)}{(c + m_-)(R + m_-)} \left[ 2 - \sum_{a=2}^{\Sigma \gamma^3} u_u \left( \frac{1}{\mu + u^a} - 1 \right) \right] 
\]

\[
= \frac{4\pi c e_- (c - R)}{(c + m_-)(R + m_-)} \left[ \frac{1}{L} - 1 \right] u_u + \left( \frac{c + m_-}{2e_-} \right) \left( \frac{1}{\mu + u^a} - 1 \right) 
\]

\[
\chi_{1,C} = \pi k \sum_{a=2}^{\Sigma \gamma^3} \left[ \frac{\gamma}{\mu + u^a} - \frac{4e_-}{m_-} \left( l_a + \frac{m_-}{R + m_-} \right) \right] \left( \frac{1}{\mu + u^a} - 1 \right) - \gamma \left( \frac{1}{\mu + u^a} - 1 \right) 
\]

\[
\chi_{a,C} = 2\pi k \left( \frac{\gamma}{L} - \frac{2m_-}{\mu + u^a} - \frac{2e_-}{m_-} \left( l_a + \frac{m_-}{R + m_-} \right) \left( \frac{1}{L} - 1 \right) - \frac{\gamma}{\mu + u^a} \right). 
\]

In the previous section, all contribution from the bolt canceled out identically, so that only the non-compact cycle turned out to be responsible for the breaking of supersymmetry. Here we can clearly see that every cycle contributes, if one keeps the parameters general.
In the following, we consider a special choice of parameters, in order to both simplify the foregoing results and parallel the procedure even more with the one from the previous section.

In section 2, the warp factor of the five-dimensional metric, $Z$, goes to 1 at infinity. For the warp factor, $\frac{1}{2} \mathcal{L}_a \mathcal{L}^a$, of the present spacetime, however, this is not so obvious, since the asymptotics of $\mathcal{L}_a$ and $\mathcal{L}$ are composed of the very strictly bound parameters (see appendix B). Following the regularity discussion at the end of [13], one condition outlined (equation (4.17)) is

$$V L > 0 \text{ and } V L_a > 0 \text{ everywhere},$$

and so, with $V \to \frac{1}{2}$ at infinity, we learn that both $L$ and $L_a$ must have positive asymptotics. Hence, the choice

$$\hat{L} = 1 \text{ and } l_a + u_a = l^a + u^a = l + u = 1,$$

is in agreement with that, and doing so we would have

$$\frac{1}{2} \mathcal{L}_a \mathcal{L}^a \to \frac{1}{2} L (l_a + u_a) (l^a + u^a) = \hat{L} (l + u)^2 = 1,$$

as desired.

This way (152) simplifies significantly to

$$\Delta M = \frac{\pi ke^{-\gamma}}{2 G_5} \left( 2\ell + 5 - \frac{3 m - \gamma}{e_-} - \frac{c^2 - m^2}{2 e_-^2} \right),$$

where redefining the (restricted) degree of freedom $l > 0$ like $2\ell + 5 = \frac{2m}{e_-} \alpha$ yields

$$\Delta M = \frac{\pi km \gamma}{G_5} \left( \alpha - \frac{3 \gamma}{2} - \frac{c^2 - m^2}{4m e_-} \right).$$

This result compares nicely to (67) from the running-bolt spacetime, especially if one does

$$m_- \to m_{\text{Schwarzschild}} \text{ and } k \left( \alpha - \frac{3 \gamma}{2} - \frac{c^2 - m^2}{4m e_-} \right) \to \varepsilon \Sigma^3 l,$$

so clearly every cycle goes at least in part into the extra-mass and so contributes to the violation of the BPS-bound, where, like in the previous section, the non-compact cycle yields the strongest contribution.

To see the cycles’ contribution more particularly in this light, we decompose $\Delta M \propto \sigma^k \chi_{K'A'} \mu^{kl}$ into a part where the compact cycles only ‘talk’ to one another (c-c) ($A, A' = B, \Delta$), and one where the compact cycles correspond with the non-compact cycle (c-n):
\[ \Delta M_{\text{ex}} = -\frac{1}{16\pi G_5} \left[ (\sigma^K_m - n\sigma^K_\Delta - (n + 1)\sigma^K_\Delta) \chi_{K;C} + \sigma^K_\Delta (\chi_{1,R} - n\chi_{1,\Delta}) \right] \]

\[ = \frac{\pi k}{G_5} \left[ \beta \left( \frac{2\alpha - 3\gamma}{2} - \frac{c^2 - m^2}{4\epsilon_-} \right) - \frac{c^2 + Rm_-}{R + m_-} \right]. \quad (168) \]

So, there is a non-vanishing contribution from the compact cycles only.

Now, we take a closer look at how these results can be interpreted in the language of intersection homology. The period-integrals compose the extra-mass integral (151) through the inverse matrix, \( I^{\text{AI}} \); from (121) we see which cycles intersect in each component.

One finds from \( \Delta M \propto \chi_{K,\Delta} \sigma^K_\Delta I^{\text{AI}} \) and (121) that the vanishing of the components, \( \chi_{\alpha,B} \) and \( \chi_{\alpha,\Delta} \), weakens the contributions from the intersection terms: \( I_{B\Delta}I_{C;C}, I_{B\Delta}I_{\Delta;C}, I_{BB}I_{\Delta;C}, \) and \( I_{BB}I_{\Delta;C} \); so, all intersections are affected equally many times—except for the self-intersection of the bubble, \( I_{\Delta;\Delta} \). On the other hand, the above-mentioned dominance of the non-compact cycle means with (121) a stronger representation of the intersection terms: \( I_{B\Delta}I_{C;C}, I_{B\Delta}I_{C;C}, I_{BB}I_{\Delta;C}, \) and \( I_{BB}I_{\Delta;\Delta} \); this time being all intersections except for the self-intersection of the non-compact cycle, \( I_{C;C} \). So, under the bottom line one can make a qualitative estimation of the order of contribution strength for the intersections (starting with the strongest):

\[ I_{B\Delta} \rightarrow (I_{BB}, I_{\Delta;C}) \rightarrow (I_{\Delta;\Delta}, I_{C;C}). \quad (169) \]

Since in the previous section the only contributing (and existing) intersection was \( I_{B;C} \), one may rise the question whether the intersection of the bolt with its adjacent cycle might be generally the dominant one.

In any case, it is striking that the breaking of supersymmetry has for this solution of supergravity non-vanishing topological contributions from all cycle’s intersections.

4. Conclusion

In the long line of efforts in deriving and applying the Smarr formula in a huge and manifold framework of physical situations, the so-called ‘no-go theorems’, which prohibit the existence of massive supergravity solitons in the absence of singularities and horizons, were shown to be circumvented in recent works by allowing non-trivial topology to spacetimes of dimensions five or higher.

In this work, further accomplishments on exploring the scope of applicability and implications of Smarr’s formula have been demonstrated and discussed within the cases of two geometrically and topologically distinct spacetimes.

Since the known non-BPS solutions are very specialized, the assumption stood to reason that the breaking of supersymmetry might be in general a boundary phenomenon rather than an intrinsic part of the core solution. In this work, it could be shown that the breaking of supersymmetry gets indeed contributions from the core topology in addition to the boundary.

First, we have derived the Komar mass for an non-BPS solution of supergravity in a five-dimensional stationary spacetime where we gave space a magnetically charged ‘bolt’ at the center and made it asymptotically \( S^1 \times \mathbb{R}^3 \).

The very goal was to determine explicitly how each mass component follows from topology and especially how cohomology accounts particularly for the extra-mass causing the violation of the BPS-bound.

One essential question addressed was, whether the extra-mass term violating the BPS-bound in spacetimes asymptotically behaving like \( \mathbb{R}^{1,3} \times S^1 \), only appears in the ADM mass while the Komar mass preserves the BPS formula.
It was shown that for a vast simplification of the running bolt solution, in which the magnetic charges got turned off, this still holds; but in the more general situation, the Komar mass contains terms breaking supersymmetry as well.

The mass formula is all topology. This is due to the fact that the present spacetime allows for two harmonic fluxes—the volume form of the bolt and its dual flux on the non-compact cycle. Nonetheless, the topology of the Euclidian Schwarzschild base space does not inhabit any self-intersecting homology as opposed to the Gibbons–Hawking base and the spacetime in second two.

In any case, it is the fluxes on the non-compact cycle that render the supersymmetry-breaking extra-mass term non-zero, which raised the question whether the latter is in general a boundary term.

A whole different picture arose when the cohomological fluxes of a 2-center solution of five-dimensional supergravity were derived, which consists of a non-extremal magnetic bolt and an extremal Gibbons–Hawking center, both linked by a bubble 2-cycle.

A first striking consequence from the in this light computed 2-form harmonics is that the 2-center situation exhibits additional topological flux through the bubble’s pinching off at two centers and thus generating a kind of ‘interaction flux’, which is dual to the bolt-flux. As opposed to the 1-center running-bolt solution, the homology of this solution turned out to be self-intersecting and the purely topological contributions to the asymptotic mass and Maxwell-charges much more various. From the running-bolt solution it is already known that Komar does indeed reflect the breaking of supersymmetry in a spacetime with the given asymptotics; the main question, however, addressed at this point is, in which explicit manner it is caused by the various given features of space’s topology.

From the derived harmonic fluxes, the formulae for the mass and charges were computed and the topological pieces of the BPS-bound breaking extra-mass term explicitly analyzed in virtue of intersection homology. As a striking result, the extra-mass and hence the breaking of supersymmetry are indeed supported by all existing intersections of the homological cycles—dominated by the bolt-bubble intersection—, so in part even by the compact cycles and hence the core solution alone.

Acknowledgments

My gratitude goes to Nicholas Warner for his great work as my doctoral adviser. Further thanks shall be directed toward the High Energy Group of USC and the relaxing, work-supportive atmosphere of the University of Southern California. Last but not least, I would like to thank DOE grant DE-SC0011687 by whom this work was in part supported.

Appendix A. Functions and constants

In section 3, a more general solution of supergravity has been considered. It contains several degrees of freedom, that are non-trivially interrelated, and some fairly bulky functions. Although the latter were already written out in section 3, they shall be briefly listed here again for completeness.

The functions used here are taken from equations (2.25), (2.26), (3.55), (4.2)–(4.6) of [13]:

\[
V(r, \theta) = \frac{r + m_+}{2(r^2 - c^2)} \left( r + m_+ - \frac{2k}{R + m_-} \sqrt{\frac{Rr - c^2 \cos \theta}{r^2 + R^2 - 2Rr \cos \theta - c^2 \sin^2 \theta}} \right)
\]  

(A.1)
\[ \omega^0 (r, \theta) = - \frac{1}{2} \left[ (m_+ - m_-) \cos \theta + \frac{2k}{R + m_-} \frac{R^2 - m_- r - R (r - m_-) \cos \theta - c^2 \sin^2 \theta}{\sqrt{r^2 + R^2 - 2rR \cos \theta - c^2 \sin^2 \theta}} \right] d\phi \]

(A.2)

\[ \omega (r, \theta) = - \frac{e_- R}{2 (R + m_-)} u_a u^a \left[ (1 - \frac{r + R}{\sqrt{r^2 - 2rR \cos \theta + R^2}}) (1 - \cos \theta) + \frac{c^2}{R \sqrt{r^2 - 2rR \cos \theta + R^2}} \sin^2 \theta \right] d\phi \]

(A.3)

\[ \dot{k} (r, \theta) = - \frac{M}{V} (d\psi + \omega^0) \]

(A.4)

\[ L_a (r, \theta) = \frac{(r + m_-) (c^2 + m_- r)}{2m_- (r^2 - c^2)} \frac{l_a}{V} + u_a \]

(A.5)

\[ L (r, \theta) = \frac{e_-}{2m_-} \frac{1}{V} u_a u^a - \frac{e_-}{c^2 (c + m_-)^2} \frac{f_1 r + f_2}{(m_- + r) (m_- + R)} u_a u^a \]

(A.6)

\[ M (r, \theta) = - \frac{e_-}{2m_-} l_a L^a + \frac{e_-}{2 (m_- + R)} \left[ \frac{R - r}{m_- + R} + \frac{(c^2 + m_- r) (f_1 r + f_2)}{2c^2 (c + m_-)^2 (r^2 - c^2)} \right] u_a u^a. \]

(A.7)

The constants \( e_\pm, m_\pm, r, u^a, k, f_1, f_2, \) and the NUT-charge, \( n, \) are connected by the relations:

\[ c^2 = m_+ m_- - 2e_+ e_- \]  

(A.8)

\[ m_+ = c \left( -1 + \frac{4k}{c + m_-} + \frac{2k}{R + m_-} \right) = \frac{2kc}{c + m_-} \left( \frac{2m_-}{c + m_-} - n \right) \]  

(A.9)

\[ R = -m_- + \frac{2k (c + m_-)^2}{(c + m_-)^2 - 4ck - 2k (c + m_-) n} \]  

(A.10)

\[ - (n + 1) k = \frac{m_+ - m_-}{2} - \frac{kR}{R + m_-} \]  

(A.11)

\[ f_1 = \frac{m_+ (c + m)^2 (c - R) + 4c^2 k (R + m)}{c + m} \]  

(A.12)

\[ f_2 = \frac{cm_+ (c + m)^2 (R - c) + 4c^2 k (c^2 - 2Rc - Rm)}{c + m} \]  

(A.13)

Appendix B. Asymptotic limits

B.1. Functions

The asymptotic limits of the functions at the bolt and the Gibbons–Hawking center are (stated in table B1):
\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|}
\hline
$r \to c$ & \multicolumn{2}{c|}{$\theta = 0, r \to R$} \\
\hline
$V$ & $1/2$ & \\
$\omega^0$ & $-\frac{(m_+ - m_-)\cos \theta}{2}$ & $\left(-\frac{(R - c\cos \theta)^2 + (R\cos \theta - c)(c + m_-)}{R + m_-}\right) k$ \\
$\omega$ & $\frac{e_- R (1 - \cos \theta)}{2(R + m_-)^2} u_a u^a d\phi$ & $0$ \\
$\mathring{k}$ & $\frac{e_- R (1 - \cos \theta)}{2(R + m_-)^2} u_a u^a d\phi$ & $0$ \\
$L_a$ & $\left[\frac{1}{c + m_-} + u_a\right] u_a$ & $0$ \\
$L$ & $\frac{4k}{c + m_-} u_a u^a$ & $\frac{e_-^2 (R - c)^2 (c + m_-)^2 - 4c k u_a u^a}{c^6 (R + m_-)^2}$ \\
$\frac{M}{V}$ & $0$ & $0$ \\
\hline
\end{tabular}
\end{table}

Note: For some of the functions, the limits towards the Gibbons–Hawking point are direction dependent; we chose the approach along the positive $z$-axis ($\theta = 0$ and $r \to R$) since that is how the adjacent cycles run.

At infinity we have the limits, to leading orders:

\begin{align}
V & \to \frac{1}{2} \quad (B.1) \\
\omega^0 & \to \left[-\frac{(m_+ - m_-)\cos \theta}{2} + k \frac{R\cos \theta + m_-}{R + m_-}\right] d\phi \quad (B.2) \\
\omega & \to e_- \frac{R^2 - c^2}{2(R + m_-)^2} u_a u^a \frac{\sin^2 \theta}{r} d\phi \quad (B.3) \\
\mathring{k} & \to \gamma \left[d\psi - \left(-\frac{(m_+ - m_-)\cos \theta}{2} - k \frac{R\cos \theta + m_-}{R + m_-}\right) d\phi\right] \quad (B.4) \\
L_a & \to l_a + u_a \quad (B.5) \\
L & \to \frac{e_-^2}{m_-} l_a l^a + \frac{c^2}{c^2 (c + m_-)} \left(\frac{m_+ (R - c)}{R + m_-} - \frac{4c k}{(c + m_-)^2}\right) u_a u^a \quad (B.6) \\
-\frac{M}{V} & \to -\frac{e_-}{m_-} (l^a + u^a) = -\frac{e_-}{2} \left[\frac{(c + m_-)c^2 - m_- m_+ (R - c)}{c^2 (R + m_-) (c + m_-)} + \frac{4k m_-}{(c + m_-)^3}\right] u_a u^a = \gamma. \quad (B.7)
\end{align}

B.2. Fields and fluxes

The fields and fluxes yield at infinity:

\begin{align}
F^l & \to \left(\frac{m_-^2 - c^2}{2e_-} - \frac{m_- \gamma}{L}\right) d\Omega_2 \quad (B.8)
\end{align}
\begin{align}
F^\nu &\to \left[e_-(2p^\nu + \omega^\nu) - \frac{m-\gamma}{L_a+u_a}\right] dt^a \\
\mathcal{H}^{(2)} &\to \frac{e}{2} (\omega_{a b}^2 - 2) dt^2 + \frac{1}{2} \left(2b^\nu - \frac{1}{L_a+u_a}\right) \left(\frac{p^\nu + \omega^\nu}{p^\nu + \omega^\nu} - \frac{e}{m-\gamma} L_a^b\right) (d\psi + \omega^b) + 'dt' \\
\mathcal{H}^{(2)} &\to \left(\frac{e}{2} u_a b^1 - \frac{e^2 - m^2}{4 e^2} \eta_{a b}^2 b^1\right) dt^2 + \frac{1}{2} \left(\frac{\gamma}{p^\nu + \omega^\nu} - \frac{2e}{m-\gamma} L_a^b\right) b^1 + \frac{\gamma}{L_a} \eta_{a b} b^1 - \frac{\gamma}{L_a} (p^\nu + \omega^\nu) (d\psi + \omega^b) + 'dt' .
\end{align}

(B.9)

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