A Partial Order on Bipartite Graphs with \( n \) Vertices

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ABSTRACT. The paper examines a partial order on bipartite graphs \((X_1, X_2, E)\) with \( n \) vertices, \( X_1 \cup X_2 = \{1, 2, \ldots, n\} \). The basis of such bipartite graph is \( X_1 = \{1, 2, \ldots, k\} \), \( 0 \leq k \leq n \). If \( U = (X_1, X_2, E(U)) \) and \( V = (Y_1, Y_2, E(V)) \) then \( U \leq V \) iff \( |X_1| \leq |Y_1| \) and \( \{ (i, j) \in E(U) : j > |Y_1| \} = \{ (i, j) \in E(V) : i \leq |X_1| \} \). This partial order is a natural partial order of subobjects of an object in a triangular category with bipartite graphs as morphisms.

KEYWORDS: Bipartite graph; partial order; triangular category.

1 The set \( B_n \) of bipartite graphs

We restrict attention to finite simple graph and use standard notations and definitions of graph theory. A graph \( G \) is a pair \((X, E)\), where \( X \) is a set \( \{x_1, x_2, \ldots, x_n\} \) of elements called vertices, and \( E \) is a set of pairs of vertices \((x_i, x_j) = (x_j, x_i)\). An element \((x_i, x_j)\) of \( E \) is called an edge of \( G=(X, E) \). Any two vertices, \( x_i \) and \( x_j \), are said to be adjacent if and only if the pair \((x_i, x_j)\) is an edge of \( G \). A graph \((X, E)\) is bipartite if its vertices can be partitioned into two sets \( X_1 \) and \( X_2 \) (\( X_1 \cup X_2 = X ; X_1 \cap X_2 = \emptyset \)) such that no two vertices in the same set are adjacent. One often writes \( U=(X_1, X_2, E(U)) \) to denote a bipartite graph, and we say that the first set \( X_1 \) is the basis of the bipartite graph \( U \).

An isomorphism of graphs \( G=(X, E) \) and \( G'=(X', E') \) is a bijection \( f : X \to X' \) such that any two vertices \( x_i, x_j \in X \) are adjacent in \( G \) if and only if \( f(x_i), f(x_j) \in X' \) are adjacent in \( G' \).
Now, we denote by $B_n$ the set of bipartite graphs $U=(X_1, X_2, E(U))$ with $n$ vertices such that the following laws hold:

1) The family $\{x_1, x_2, \ldots, x_n\}$ of the vertices of $U$ is denoted by its set of indices $\{1, 2, \ldots, n\}$ such that the first indices $(i=1, 2, \ldots, k)$ are in the same partite set namely in the basis $X_1$ of $U$ and $X_2=\{k+1, k+2, \ldots, n\}$.

2) If $i$ and $j$ are adjacent in $U$ such that $i \in X_1$ and $j \in X_2$ then $(i, j)$ denotes the corresponding edge of $U$. Thus $(i, j) \in E(U)$ implies $i < j$.

For instance, the following bipartite graph $U$:

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\end{array}
\]

with the set of edges $E=\{(1, 7), (2, 6), (4, 7), (5, 6)\}$ and $X_1=\{1, 2, 3, 4, 5\}$, $X_2=\{6, 7\}$ is an element of $B_7$. The following two elements of $B_7$:

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\phi & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\phi & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\end{array}
\]

are two distinct elements of $B_7$. The first has basis the empty set, and $\{1, 2, 3, 4, 5, 6, 7\}$ is the basis of the second bipartite graph.

It is straightforward to check that

\[|B_n| = \sum_{k=0}^{n} 2^{k(n-k)},\]

where $|B_n|$ is the cardinality of $B_n$.

2. A partial order on $B_n$

Let $U = (X_1, X_2, E(U))$ and $V = (Y_1, Y_2, E(V))$ be two elements of $B_n$.

**Proposition 1.** The relation defined by

$U \leq V \iff |X_1| \leq |Y_1|$ and $\{(i, j) \in E(U) \mid j > |Y_1|\} = \{(i, j) \in E(V) \mid i \leq |X_1|\}$

is a partial order on $B_n$. 
Proof. It is immediate that the relation defined above is reflexive and antisymmetric. To show that it is also transitive, let $U = (X_1, X_2, E(U)), V = (Y_1, Y_2, E(V)), W = (Z_1, Z_2, E(W)) \in B_n$ such that $U \leq V$ and $V \leq W$. It follows that

a) $|X_1| \leq |Y_1| \leq |Z_1|$,  

b) $(i_0, j_0) \in \{(i, j) \in E(U) | j >|Z_1| \}$  

$\Rightarrow (i_0, j_0) \in \{(i, j) \in E(U) | j >|Y_1| \} = \{(i, j) \in E(V) | i \leq |X_1| \}$  

$\Rightarrow (i_0, j_0) \in \{(i, j) \in E(V) | i \leq |X_1| \ and \ j >|Z_1| \} = \{(i, j) \in E(W) | i \leq |X_1| \}$  

$\Rightarrow \{(i, j) \in E(U) | j >|Z_1| \} \subseteq \{(i, j) \in E(W) | i \leq |X_1| \}$

c) $(i_0, j_0) \in \{(i, j) \in E(W) | i \leq |X_1| \}$  

$\Rightarrow (i_0, j_0) \in \{(i, j) \in E(W) | i \leq |Y_1| \} = \{(i, j) \in E(V) | j >|Z_1| \}$  

$\Rightarrow (i_0, j_0) \in \{(i, j) \in E(V) | i \leq |X_1| \ and \ j >|Z_1| \} = \{(i, j) \in E(U) | j >|Z_1| \}$  

$\Rightarrow \{(i, j) \in E(W) | i \leq |X_1| \} \subseteq \{(i, j) \in E(U) | j >|Z_1| \}$

a), b) and c) implies that $U \leq W$.

**Proposition 2.** If the bases of two elements $U, V \in B_n, U \neq V,$ are equal then $U$ and $V$ are incomparable.

**Proof.** The equality

$\{(i, j) \in E(U) | j >|Y_1| \} = \{(i, j) \in E(V) | i \leq |X_1| \}$

where $X_1 = Y_1$, implies that $U = V$.

Now, let $n=3$. Then the Hasse diagram of the partially ordered set $(B_3, \leq)$ is the following one:
Without specifying the bipartite graphs, the Hasse diagram of 
\((B_3, \leq)\) is given by:

The case \(n > 3\) is somewhat laborious. For example, the Hasse diagram of 
the partial ordered set \((B_4, \leq)\) is the following one:
3. Connection with a triangular category

Möbius inversion for categories was considered for the first time by Leroux [Ler75]. A Möbius category in the sense of Leroux is a decomposition finite category $C$ (i.e. a small category where each morphism $\alpha$ has only finitely many nontrivial factorizations) such that an incidence function $f : MorC \rightarrow \mathbb{R}$ has a convolution inverse if and only if $f(1_A) \neq 0$ for any identity morphism $1_A$ of $C$.

The convolution $f \ast g$ of two incidence functions $f$ and $g$ is defined by

$$(f \ast g)(\alpha) = \sum_{\alpha = \beta \gamma} f(\beta) \cdot g(\gamma) \quad (\alpha \in MorC).$$

Möbius categories have also been characterized as decomposition-finite categories in which

1. each identity morphism is indecomposable, i.e., $1_A = \beta \gamma$ implies $\beta = 1_A = \gamma$;
2. $\beta \gamma = \gamma$ implies that $\beta$ is identity morphism.

Now, it is straightforward to see that a special class of categories (called triangular categories by Leroux [Ler80]) in which the set of objects is the set of nonnegative integers $\mathbb{N}$ and the family of numbers $|\text{Hom}(k,n)|$
(where $|\text{Hom}(k,n)|$ is the number of morphisms from $k$ to $n$) constitute a triangular family of numbers, that is:

$$|\text{Hom}(n,n)| = 1 \text{ for all } n \in N; \text{ and } |\text{Hom}(m,n)| = 0 \text{ if } m > n.$$  

The prime example of a triangular category (denoted $\Delta$ in [Ler80]) is that for which $0 \in N$ is the initial object and $\text{Hom}_\Delta(k,n)=$"the set of all injective and isotone maps from $\{1,2,\ldots,k\}$ to $\{1,2,\ldots,n\}". The corresponding triangular family of numbers is the following one:

$$|\text{Hom}_\Delta(k,n)| = \binom{n}{k} \quad (k \leq n)$$

More combinatorial triangular families of numbers can be represented by triangular categories (see [Ler80], [Ler90]).

Let $C$ be a triangular category. The set $S(n)$ of subobjects of $n \in N$ (or, rather, monomorphisms into $n$) is an ordered set:

$$\alpha \leq \beta \iff \exists \gamma : \beta \gamma = \alpha$$

This relation is called the natural partial order on the set of subobjects of $n$.

**Proposition 3.** Since $|\text{Hom}(k,k)| = 1$ for every $k \in N$, two monomorphisms $\alpha, \beta$ into $n, \alpha \neq \beta$, with the same domain $k$ are incomparable.

A triangular category is called lattice-triangular if $(S(n), \leq)$ is a lattice for every $n \in N$. A triangular category $C$ is called monomorphic-triangular if any morphism of $C$ is a monomorphism. We have:

**Theorem 4.** ([Sch03]) Let $C$ be a monomorphic-triangular category. Then $C$ is lattice-triangular if and only if $C$ has pullbacks.

The triangular category $\Delta$ is a lattice triangular category. It is straightforward to check that in $\Delta$ the lattice $(S(n), \leq)$ is isomorphic to the Boolean algebra of all subsets of the set $\{1,2,\ldots,n\}$. This category is not a category with pushout and therefore in Theorem 1, the “pullback” cannot be replaced by “pushout”.

Now, we shall consider the category $B$ of bipartite graphs (see $Bipis$ in [Ler80]) defined by:
- \( \text{Ob}B = \mathbb{N} \);
- \( \text{Hom}_B(k,n) = \begin{cases} \{ U \in B_n \mid \{1,2,\ldots,k\} \text{ is the basis of } U \} & \text{if } k \leq n \\ \emptyset & \text{if } k > n \end{cases} \)

The composition of two morphisms: if \( U \in \text{Hom}_B(m,k) \) and \( V \in \text{Hom}_B(k,n) \) then the composition \( V \circ U \) is the bipartite graph with \( n \) vertices, 1,2,\ldots,\( n \), having the basis \( \{1,2,\ldots,m\} \); and the set of edges being the union of the set of edges of \( U \) and the set of those edges of \( V \) which have an endpoint in the basis of \( U \).

For example, if \( U \in \text{Hom}_B(2,5) \) is given by

\[
\begin{array}{ccc}
1 & 2 & \\
3 & 4 & 5
\end{array}
\]

and if \( V \in \text{Hom}_B(5,7) \) is given by

\[
\begin{array}{ccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
& & & & & &
\end{array}
\]

then

\[
\begin{array}{ccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
& & & & & &
\end{array}
\]

that is,

\[
E(V \circ U) = \{(1,4);(2,3);(2,5);(1,7);(2,6)\}.
\]
Hence,

\[ E(V \bullet U) = E(U) \cup \{(i, j) \in E(V) : i \leq m\}. \]

The identity morphism from \( n \) to \( n \) is the bipartite graph with \( n \) vertices and with basis \( \{1, 2, \ldots, n\} \); the set of edges being the empty set.

**Proposition 5.** The category \( B \) is a triangular category and the corresponding triangular family of numbers is the following one:

\[ |\text{Hom}_B(k, n)| = 2^{k(n-k)} \quad (k \leq n) \]

**Proposition 6.** The category \( B \) is monomorphic but it is not epimorphic.

**Proof.** Let \( V \in \text{Hom}_B(m, n) \) and \( U, U' \in \text{Hom}_B(k, m) \) be such that

\[ V \bullet U = V \bullet U'. \]

Then,

\[ U \cup \{(i, j) \in V : i \leq k\} = U' \cup \{(i, j) \in V : i \leq k\} \]

and therefore \( U = U' \).

Now, if \( V' \in \text{Hom}_B(m, n) \) such that

\[ V \bullet U = V' \bullet U, \]

then,

\[ \{(i, j) \in V : i \leq k\} = \{(i, j) \in V' : i \leq k\}. \]

But this does not imply that \( V = V' \).

**Proposition 7.** The partial order on \( B_n \) is the natural partial order on the set of subobjects of \( n \) in the triangular category \( B \).

**Proof.** Let \( U = (X_1, X_2, E(U)) \) and \( V = (Y_1, Y_2, E(V)) \) be two elements of \( B_n \) such that

\[ |X_1| \leq |Y_1| \text{ and } \{(i, j) \in E(U) : j > |Y_1|\} = \{(i, j) \in E(V) : i \leq |X_1|\}. \]

These two bipartite graphs \( U \) and \( V \) are two morphisms of the category \( B \) having the same codomain \( n \). Consider the morphism

\[ W = (X_1, Y_1 - X_1, E(W)) \in \text{Hom}_B(|X_1|, |Y_1|), \]

where

\[ E(W) = \{(i, j) \in E(U) : j \leq |Y_1|\} \]

and we obtain:

\[ E(V \bullet W) = E(W) \cup \{(i, j) \in E(V) : i \leq |X_1|\} = \{(i, j) \in E(U) : j \leq |Y_1|\} \cup \{(i, j) \in E(U) : j > |Y_1|\} = E(U). \]

It follows that the diagram
is commutative in $B$.

Conversely, if the above diagram is commutative in $B$, where $U = (X_1, x_2, E(U)), V = (Y_1, Y_2, E(V))$ and $W = (X_1, y_1, X_2, E(W))$, then

$$|X_1| \leq |Y_1|$$

and

$$E(W) = \{(i, j) \in E(U) : j \leq |Y_1|\}$$

$$E(U) = E(V \bullet W) = E(W) \cup \{(i, j) \in E(V) : i \leq |X_1|\}$$

$$\Rightarrow \quad E(U) = \{(i, j) \in E(U) : j \leq |Y_1|\} \cup \{(i, j) \in E(V) : i \leq |X_1|\}$$

$$\Rightarrow \quad \{(i, j) \in E(U) : j > |Y_1|\} = \{(i, j) \in E(V) : i > |X_1|\}$$

as required.

Taking into account the Hasse diagram of $(B_3, \leq)$ and Proposition 7, it follows:

**Proposition 8.** The monomorhic-triangular category $B$ is not a lattice-triangular category.

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