RECOVERY OF THE SINGULARITIES OF A POTENTIAL FROM BACKSCATTERING DATA IN GENERAL DIMENSION

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Abstract. We prove that in dimension $n \geq 2$ the main singularities of a complex potential $q$ having a certain a priori regularity are contained in the Born approximation $q_B$ constructed from backscattering data. This is archived using a new explicit formula for the multiple dispersion operators in the Fourier transform side. We also show that $q - q_B$ can be up to one derivative more regular than $q$ in the Sobolev scale. On the other hand, we construct counterexamples showing that in general it is not possible to have more than one derivative gain, sometimes even strictly less, depending on the a priori regularity of $q$.

1. Introduction and main theorems

The central problem in inverse scattering for the Schrödinger equation is to recover a potential $q(x)$, $x \in \mathbb{R}^n$, from the scattering data, the so called scattering amplitude $u_\infty$. The scattering amplitude measures the far field response of the Hamiltonian $H := -\Delta + q$ to incident plane waves. In backscattering, as the name suggest, only the far field response appearing in the opposite direction of the incoming wave is considered, or in other words, only the waves scattered in the opposite direction of the incident wave (the echoes). The usual reconstruction procedure is to construct the Born approximation $q_B$ of the potential, also an $\mathbb{R}^n$ function as $q$, from the backscattering data contained in $u_\infty$. This is the linear approximation to the inverse problem and it is widely used in applications.

From a mathematical point of view an important question that is not completely answered is to establish how much information does the Born approximation contain about the actual potential $q$. This problem can be approached in different ways. One is to look for uniqueness results, that is, if $q_B$ is enough to determine $q$ (this problem is still open, see next section for references). Motivated by the use of the Born approximation in applications, another approach is to ask how much and what kind of information about $q$ can be obtained just by looking at $q_B$, that is, in a very immediate way. In this sense, in [21] it was proposed that the Born approximation must contain the leading singularities of $q$. Since then, this approach has received great amount of attention in different scattering problems. In the case of backscattering we mention, among others, [12, 14] for recovery of conormal singularities, [18, 25] for recovery of singularities in 2 dimensions, [26, 29] in dimensions 2 and 3 and [6, 7] in odd dimension $n \geq 3$.

The main objective of this work is to quantify as exactly as possible how much more regular than $q$ can $q - q_B$ be in general, depending on the dimension $n$, and the a priori regularity of the potential $q$ measured in the Sobolev scale. The potential can be complex valued. We
provide positive and negative results, which answer this question almost completely, except for potentials in a certain range of the Sobolev scale where there is still a gap between them (see figure 1). To measure the regularity, we introduce the fractional derivative operator $< D > ^{\alpha}$, $\alpha \in \mathbb{R}$ given by the Fourier symbol $< \xi > ^{\alpha}$ with $< x > := (1 + |x|^2)^{1/2}$, and the weighted Sobolev space $W^{\alpha, p}_{\delta} (\mathbb{R}^n)$, $\delta \in \mathbb{R}$,

$$W^{\alpha, p}_{\delta} (\mathbb{R}^n) := \{ f \in \mathcal{S}'(\mathbb{R}^n) : \| < \cdot > ^{\delta} < D > ^{\alpha} f \|_{L^p(\mathbb{R}^n)} < \infty \}. $$

We usually use the notation $L^p_\delta(\mathbb{R}^n) := W^{0, p}_{\delta} (\mathbb{R}^n)$ and $W^{\alpha, p}_0(\mathbb{R}^n) := W^{\alpha, p}_{\delta} (\mathbb{R}^n)$, also we say that $f \in W^{\alpha, p}_0(\mathbb{R}^n)$ if $\phi f \in W^{\alpha, p}(\mathbb{R}^n)$ for every $\phi \in C^\infty_c(\mathbb{R}^n)$.

As we shall see in the next section, the Born approximations $q_B$ is related to the potential through the Born series expansion,

$$q_B \sim q + \sum_{j=2}^{\infty} Q_j(q),$$

where $Q_j(q)$ are certain multilinear operators describing the (multiple) dispersion of waves (we use the $\sim$ symbol to avoid claiming anything about convergence yet). We will call the $Q_2$ operator the double dispersion operator of backscattering. A key guiding principle is that in general $Q_j(q)$ is expected to be smoother as $j$ grows. We can introduce now the main theorems of this work.

**Theorem 1.1.** Let $n \geq 2$ and $\beta \geq 0$. Assume that $q - q_B \in W^{\alpha, 2}_{\text{loc}}(\mathbb{R}^n)$ for every $q \in W^{\beta, 2}(\mathbb{R}^n)$ compactly supported, radial, and real. Then $\alpha$ necessarily satisfies,

$$(1.1) \quad \alpha \leq \begin{cases} 2\beta - (n-4)/2, & \text{if } m \leq \beta < (n-2)/2, \\ \beta + 1, & \text{if } (n-2)/2 \leq \beta < \infty, \end{cases}$$

where $m = (n-4)/2 + 2/(n+1)$.

**Theorem 1.2** (Recovery of singularities). Let $n \geq 2$ and $\beta \geq 0$. Assume that $q \in W^{\beta, 2}(\mathbb{R}^n)$ is compactly supported. Then $q - q_B \in W^{\alpha, 2}(\mathbb{R}^n)$, modulo a $C^\infty$ function, if the following condition also holds

$$(1.3) \quad \alpha < \begin{cases} 2\beta - (n-3)/2, & \text{if } (n-3)/2 < \beta < (n-1)/2, \\ \beta + 1, & \text{if } (n-1)/2 \leq \beta < \infty. \end{cases}$$

See Figure 1 for a graphic representation of these results for $n = 2$ and $n = 4$.

Theorem 1.1 is the first result giving upper bounds for the maximum possible regularity that can be obtained from the Born approximation in backscattering. As we shall see, condition (1.1) is a consequence of upper bounds for the regularity of the $Q_2$ operator given by Theorem 1.4 below. The main reason we need $\beta \geq m$ and compact support is that the convergence of the (high frequency) Born series in a Sobolev space $W^{\alpha, 2}(\mathbb{R}^n)$ is known only under these assumptions (see Proposition 2.1 below). A remarkable consequence of condition (1.1) is that for $\beta < (n-2)/2$ and $n > 2$ it is not possible to reach the expected gain of one derivative over the regularity of $q$ (see figure 1 for the cases $n = 2, 4$). In fact, we reach the minimum value of $2/(n+1)$ in (1.1) for the upper bound of the derivative gain $\alpha - \beta$ when $\beta = m$, which approaches 0 as $n$ grows.

Theorem 1.2 is a consequence of new estimates of the $Q_j$ operators for $n \geq 2$ and $j \geq 2$ given in Theorem 1.3 below. As far as we know, these are the first results of recovery of singularities for every dimension $n$ in backscattering. We remark that Theorem 1.1 implies...
that a one derivative gain is the best possible result and so, the $1^-$ derivative gain in (1.3) is optimal except for the limiting case $\alpha = \beta + 1$. Observe this is the same range obtained in [7, Corollary 4.8] for odd dimension $n \geq 3$. Also, in [26] it was shown that $q - q_B$ is in $W^{\alpha, 2}(\mathbb{R}^n)$ (modulo a $C^\infty$ function) with $n = 2, 3$ and $\alpha < \beta + 1/2$. Therefore, in dimension 2 we improve the previous results for all $\beta \geq 0$ (see Figure 1), but in dimension 3 the result in [26] is still the best result for a low a priori regularity $0 \leq \beta < 1/2$.

Indeed, Theorems 1.1 and 1.2 leave a gap of up to $1/2$ derivative when $\max(m, 0) \leq \beta < (n - 1)/2$ between the positive and negative results. A similar situation is found in the fixed angle and full data scattering problems, where analogous results to Theorems 1.1 and 1.2 have been proved in [17] (see [1] for the positive results in the case of full data scattering). In backscattering, this gap has been partially closed in dimension 3 by the mentioned result in [26] and in dimension 2 in [2], where a uniform $1^-$ derivative gain has been obtained using a weaker regularity scale than the Sobolev scale $W^{\alpha, 2}$. We will make more observations about this problem in the final remarks.

We introduce now the Sobolev estimates of the $Q_j$ operators. Consider a constant $C_0 > 0$. Let $0 \leq \chi(\xi) \leq 1$, $\xi \in \mathbb{R}^n$ be a smooth cut-off function satisfying $\chi(\xi) = 1$ if $|\xi| > 2C_0$ and $\chi(\xi) = 0$ if $|\xi| < C_0$. We define the operator $\tilde{Q}_j$ by the relation

$$\tilde{Q}_j(q)(\xi) := \chi(\xi)\hat{Q}_j(q)(\xi),$$

so that $Q_j(q)$ differs from $\tilde{Q}_j(q)$ in a smooth function. $Q_j(q)$ will be introduced in the following section, see (2.8).

**Theorem 1.3.** Let $n \geq 2$ and $j \geq 2$. Assume that $0 \leq \beta \leq \infty$ and that the following condition also holds

$$\alpha < \begin{cases} 
\beta + (j - 1)(\beta - (n - 3)/2), & \text{if } (n - 3)/2 < \beta < (n - 1)/2, \\
\beta + (j - 1), & \text{if } (n - 1)/2 \leq \beta < \infty.
\end{cases}$$

Then for $q \in W^{\beta, 2}_2(\mathbb{R}^n)$ and $j = 2$ we have the estimate

$$\|\tilde{Q}_2(q)\|_{W^{\alpha, 2}} \leq C\|q\|_{W^{\beta, 2}}^2.$$

**Figure 1.** The (red) dashed line represents the limitation on the regularity gain given in Theorem 1.1 for $q - q_B$, and the solid (blue) line represents the positive results given in Theorem 1.2. When $n = 2$, the dot dashed line represents the previously known positive results of [26].
Otherwise if \( j \geq 3 \) and \( q \in W^{\beta,2}_\text{loc}(\mathbb{R}^n) \) we have that
\[
\|\tilde{Q}_j(q)\|_{W^{\alpha,2}} \leq C\|q\|_{W^{\beta,2}}^j.
\]

We estimate the \( \tilde{Q}_j \) operators in the Fourier transform side, as in [29]. The main advantage is that we can give an explicit formula for these operators as combinations of integral operators over spheres (spherical operators) and certain principal value operators (see Proposition 5.1 below). In the proof of Theorem 1.3 we use trace estimates to control the spherical integrals, and a new method to reduce the estimate of the principal value operators to estimates of the spherical operators. One advantage of these techniques is that with the same effort we can prove estimates for general dimension \( n \geq 2 \). In odd dimension, using very different techniques, similar estimates for the complete \( Q_j \) operator have been obtained in [6, Theorem 1.1], and in [7, Theorem 1.2] for \( Q_j, j \geq 2 \) and compactly supported potentials. We mention that the estimate of the \( \tilde{Q}_3 \) operator for \( n = 3 \) given in [26] is still the best estimate in the range \( 0 \leq \beta < 1/4 \). We give now the last theorem of this work, from which Theorem 1.1 will follow.

**Theorem 1.4.** Let \( 0 < \beta < \infty \) and assume that \( Q_2(q) \in W^{\alpha,2}_\text{loc}(\mathbb{R}^n) \) for every potential \( q \in W^{\beta,2}(\mathbb{R}^n) \) radial, real and compactly supported, then \( \alpha \) necessarily satisfies
\[
\alpha \leq \begin{cases} 
2\beta - (n - 4)/2, & \text{if } 0 \leq \beta < (n - 2)/2, \\
\beta + 1, & \text{if } (n - 2)/2 \leq \beta < \infty,
\end{cases}
\]

The paper is structured as follows. In section 2 we introduce with more detail the backscattering problem, and we show how to deduce Theorems 1.1 and 1.2 respectively from Theorems 1.4 and 1.3. Section 3 is dedicated to introducing the main result used for the estimate of the principal value operators and in section 4 we estimate the spherical part \( \tilde{Q}_2(q) \). In section 5 we study the general \( \tilde{Q}_j \) operators and we finish the proof of Theorem 1.3. In section 6 we give the estimates necessary to show the convergence of the Born series in Sobolev spaces, and section 7 is devoted to proving Theorem 1.4.

## 2. Convergence of the Born series in Sobolev spaces

Let us introduce the backscattering inverse problem more rigorously (see, for example, [11, chapter V] for an introduction to scattering theory from the point of view of PDEs). We follow a similar exposition to that of [17].

Consider a scattering solution \( u_s(k, \theta, x) \), \( k \in (0, \infty) \), \( \theta \in \mathbb{S}^{n-1} \), of the stationary Schrödinger equation satisfying
\[
\begin{aligned}
(-\Delta + q - k^2)u &= 0 \\
u(x) &= e^{ik\theta \cdot x} + u_s(k, \theta, x) \\
\lim_{|x| \to \infty} (\frac{\partial u_s}{\partial r} - iku_s)(x) &= o(|x|^{-(n-1)/2}),
\end{aligned}
\]

where the last line is the outgoing Sommerfeld radiation condition (necessary for uniqueness). If \( q \) is compactly supported, a solution \( u_s \) of (2.1) has the following asymptotic behavior when \( |x| \to \infty \)
\[
u_s(k, \theta, x) = C|x|^{-(n-1)/2}k^{(n+3)/2}e^{ik|x|}u_{\infty}(k, \theta, x/|x|) + o(|x|^{-(n-1)/2}),
\]

\( u_{\infty} \) is the solution of the stationary Schrödinger equation with potential \( q \) as in (2.1) with \( |x| = 1 \).
for a certain function \( u_\infty(k, \theta, \theta') \), \( k \in (0, \infty) \), \( \theta, \theta' \in \mathbb{S}^{n-1} \). As mentioned in the introduction, \( u_\infty \) is the so called scattering amplitude or far field pattern, and is given by the expression

\[
(2.2) \quad u_\infty(k, \theta, \theta') = \int_{\mathbb{R}^n} e^{-ik\theta' \cdot y} q(y) u(y) \, dy,
\]

where is important to notice that \( u \) depends also on \( k \) and \( \theta \) (for a proof of this fact when \( q \in C_c^\infty(\mathbb{R}^n) \) see for example [28, p. 53]).

Applying the outgoing resolvent of the Laplacian \( R_k \) in the first line of (2.1), where

\[
(2.3) \quad \hat{R}_k(f)(\xi) = (-|\xi|^2 + k^2 + i0)^{-1} \hat{f}(\xi),
\]

we obtain the Lippmann-Schwinger integral equation

\[
(2.4) \quad u_s = R_k(qe^{ik\theta \cdot \cdot}) + R_k(qu_s(k, \theta, \cdot)).
\]

The existence and uniqueness of scattering solutions of (2.1) follows from a priori estimates for the resolvent operator \( R_k \) and the previous integral equation (2.4). In the case of real potentials, this can be shown with the help of Fredholm theory for \( k > 0 \), see for example [28, pp. 79-82]. Otherwise, since the norm of the operator \( T(\xi) = R_k(qf) \) decays to zero as \( k \to \infty \) in appropriate function spaces, we can also use a Neumann series expansion in (2.4) which will be convergent for \( k > k_0 \) (in general \( k_0 \geq 0 \) will depend on some a priori bound of \( q \)). For our purposes it is enough to consider \( q \in L^r(\mathbb{R}^n) \), \( r > n/2 \) and compactly supported. Notice that by the Sobolev embedding this is satisfied if \( q \in W^{\beta,2}(\mathbb{R}^n) \) with \( \beta > (n-4)/2 \). See [1, p. 511] for more details and references.

We can introduce now the inverse backscattering problem. If we insert (2.4) in (2.2), we can expand the Lippmann-Schwinger equation in a Neumann series, as we mentioned before. Then we obtain the Born series expansion relating the scattering amplitude and the potential in the Fourier transform side.

\[
(2.5) \quad u_\infty(k, \theta, \theta') = \hat{q}(\xi) + \sum_{j=2}^l \int_{\mathbb{R}^n} e^{-ik\theta' \cdot y}(qR_k)_{j-1}^{-1}(q(\cdot))e^{ik\theta \cdot \cdot}(y) \, dy
\]

where \( \xi = k(\theta' - \theta) \) and the last is the error term. Since we are considering complex potentials, \( u_\infty(k, \theta, \theta') \) is not defined for \( k \leq k_0 \) as we have seen. Therefore we also have to ask \( k > k_0 \) in (2.5).

The problem of determining \( q \) from the knowledge of the scattering amplitude is formally overdetermined in the sense that the data \( u_\infty(k, \theta, \theta') \) is described by \( 2n - 1 \) variables, while the unknown potential \( q(x) \) has only \( n \). We avoid the overdetermination by reducing to the backscattering data, assuming only knowledge of \( u_\infty(k, \theta, -\theta) \), for all \( k > k_0 \) and \( \theta \in \mathbb{S}^{n-1} \). For backscattering data the problem is formally well determined, and the Born approximation \( q_B \) is defined by the identity

\[
(2.6) \quad \hat{q}_B(\xi) := u_\infty(k, \theta, -\theta), \quad \text{where} \quad \xi = -2k\theta.
\]

Since \( u_\infty(k, \theta, -\theta) \) is not defined for \( k \leq k_0 \), from now on we consider that \( q_B(x) \) is defined modulo a \( C^\infty \) function.
By (2.6), the condition \( k > k_0 \) is equivalent to asking \(|\xi| > 2k_0\). Therefore, using the cut-off introduced before (1.4) with \( C_0 > 2k_0 \), and assuming convergence of the series, we can write (2.5) as follows

\[
(2.7) \quad \chi(\xi)\tilde{q}_B(\xi) = \chi(\xi)\tilde{q}(\xi) + \sum_{j=2}^{\infty} \tilde{Q}_j(q)(\xi),
\]

where \( \tilde{Q} \) was defined in (1.4) and

\[
(2.8) \quad \tilde{Q}_j(q)(\xi) = \int_{\mathbb{R}^n} e^{ik\theta y}(qR_k)^{j-1}(q(\cdot)e^{ik\theta(\cdot)}(y) dy,
\]

again with \( \xi = -2k\theta \).

We examine now the question of the convergence in Sobolev spaces of the series (2.7), an essential step in the proof of Theorems 1.1 and 1.2.

**Proposition 2.1.** Let \( n \geq 2, j \geq 2 \) and \( \max(0,m) \leq \beta < \infty \), where \( m \) was defined in (1.2). If \( q \in W^{\beta,2}(\mathbb{R}^n) \) is compactly supported in \( B_\rho \), the ball of radius \( \rho \), then \( \tilde{Q}_j(q) \in W^{\alpha,2}(\mathbb{R}^n) \) if \( \alpha < \alpha_j \), with

\[
(2.9) \quad \alpha_j = \beta + (j-1) - \frac{n}{2} - \frac{(n-1)}{2}(j-2) \max\left(0, \frac{1}{2} - \frac{\beta}{n}\right).
\]

Moreover, for every \( \alpha < \alpha_l \), \( l \geq 2 \) the series \( \sum_{j=l}^{\infty} \tilde{Q}_j(q) \), converges absolutely in \( W^{\alpha,2}(\mathbb{R}^n) \) provided we take \( C_0 = C||q||_{W^{\beta,2}}^{1/\varepsilon} \) in (1.4) for a large constant \( C = C(n,\alpha,\beta,\rho) \) and a certain \( \varepsilon = \varepsilon(n,\beta) > 0 \).

This proposition improves the original result of [29, Proposition 4.3] given for the range \( m \leq \beta \leq n/2 \) and later extended in [26] for \( \beta \geq n/2 \). We have used certain properties of the fractional Laplacian \((-\Delta)^s\) to improve the value of \( \alpha_j \) in dimension \( n \) (see section 6). It also improves the regularity gain given in [26] for the \( \tilde{Q}_4 \) operator with \( n = 3 \). This would allow to obtain the results of recovery of singularities in that paper without the very technical proof to estimate \( \tilde{Q}_4(q) \).

Restricting a bit the range of \( \beta \), a better result can be proved in odd dimension \( n \geq 3 \). In fact in [7, corollary 1.3] it is shown that, if \( (n-3)/2 \leq \beta < \infty \), the complete series \( \sum_{j=1}^{\infty} \tilde{Q}_j(q) \), converges in \( W^{\alpha,2}(\mathbb{R}^n) \) with \( \alpha \) in the same range given in theorem 1.2. This implies that the backscattering transform, the mapping which gives \( q_B \) from \( q \), is an entire analytic mapping of Sobolev spaces.

Using Proposition 2.1, we can reduce the proof of Theorem 1.2 to proving Theorem 1.3.

**Proof of Theorem 1.2.** Taking the inverse Fourier transform of (2.7), we can write modulo a \( C^\infty \) function

\[
(2.10) \quad q(x) - q_B(x) = -\sum_{j=2}^{\infty} \tilde{Q}_j(x).
\]

Consider \( \alpha \) and \( \beta \geq 0 \) satisfying condition (1.3). Observe that by Theorem 1.3, we have \( \tilde{Q}_j(q) \in W^{\alpha,2}(\mathbb{R}^n) \), so we just need to study the convergence of the series. But, since the value of \( \alpha_j \) in (2.9) grows linearly with \( j \), we can always find an integer \( l \) such that \( \alpha_l > \alpha \). As a consequence, if \( q \) has compact support, by Proposition 2.1, the series \( \sum_{j=l}^{\infty} \tilde{Q}_j(q)(x) \) converges in \( W^{\alpha,2}(\mathbb{R}^n) \). \( \square \)
Similarly, the proof of Theorem 1.1 can be reduced to Theorem 1.4 and Proposition 2.1.

**Proof of Theorem 1.1.** Take \( \alpha \geq 0 \) and assume that we have that \( q - q_B \in W^{\alpha,2}_{\text{loc}}(\mathbb{R}^n) \) for every compactly supported, real and radial potential \( q \in W^{\beta,2}(\mathbb{R}^n) \). We are going to prove that then necessarily \( Q_j(q) \in W^{\alpha,2}_{\text{loc}}(\mathbb{R}^n) \).

We denote by \( q_B(\lambda) \) the Born approximation of the potential \( q(\lambda) = \lambda q \), where \( \lambda \in (0, 1) \). By the multilinearity of the \( \tilde{Q}_j \) operators, the Born series \((2.10)\) for \( q(\lambda) \) becomes

\[
\lambda q - q_B(\lambda) = - \sum_{j=2}^{l-1} \lambda^j \tilde{Q}_j(q) - \sum_{j=1}^\infty \lambda^j \tilde{Q}_j(q),
\]

modulo a \( C^\infty \) function (which depends on \( \lambda \)).

By Proposition 2.1 we have that if \( m \leq \beta < \infty \), we can choose \( l \) in \((2.11)\) such that \( \alpha < \alpha_l \), and hence \( \sum_{j=1}^\infty \lambda^j \tilde{Q}_j(q) \) will converge absolutely in \( W^{\alpha,2}(\mathbb{R}^n) \). Now, let \( V := \mathcal{S}'/W^{\alpha,2}_{\text{loc}} \) be the quotient vector space of \( \mathcal{S}'(\mathbb{R}^n) \) with \( W^{\alpha,2}_{\text{loc}}(\mathbb{R}^n) \). We denote the elements of \( V \) by \([h]\) where \( h \in \mathcal{S}' \) is any member of the equivalence class. Since, by hypothesis we have that \( \lambda q - q_B(\lambda) \) is also a \( W^{\alpha,2}(\mathbb{R}^n) \) function, equation \((2.11)\) becomes in \( V \)

\[
\sum_{j=2}^{l-1} \lambda^j \tilde{Q}_j(q) = 0.
\]

But, we can always choose a \( \lambda_i \in (0, 1) \) for every \( 2 \leq i \leq l - 1 \) such that the \( l-2 \) vectors of \( \mathbb{R}^{l-2} \) with coordinates \( (\lambda_i^2, \ldots, \lambda_l^{l-1}) \) are linearly independent. This implies that \([\tilde{Q}_j(q)] = 0\) for all \( 2 \leq j \leq l - 1 \), and hence that \( Q_j(q) \in W^{\alpha,2}_{\text{loc}}(\mathbb{R}^n) \). But, by condition \((1.7)\) of Theorem 1.4, we know that this implies that \( \alpha \) must be in the range given in \((1.1)\). \(\square\)

As we have mentioned in the introduction, the question of uniqueness of the inverse scattering problem for backscattering data is still open. In [23] it has been proved for \( n = 3 \) that two potentials differing in a finite number of spherical harmonics with radial coefficients must be identical if they have the same backscattering data. The question of uniqueness for small potentials was studied in [22]. Generic uniqueness and uniqueness for small potentials has been obtained in [10, 32] for dimensions 2 and 3 and in [16] for \( n = 3 \). Similar results have been obtained in odd dimension \( n \geq 3 \) in [35], and in [7] for potentials in the Sobolev space \( W^{\beta,2}(\mathbb{R}^n) \) with \( \beta > (n - 3)/2 \). The case of even dimension has been treated in [36].

The recovery of singularities has been studied in other inverse scattering problems. The case of full data has been studied in [19–21] (real potentials) and [1,17] (complex potentials) and the case of fixed angle data in [30] in dimension 2, and [27] in dimension \( n \geq 2 \). The regularity gain has been improved recently in [17]. Analogous problems have been formulated in elasticity to study the recovery of singularities of the Lamé parameters, see [3,4].

Before going to the next section, we want to highlight the following property of Sobolev norms that we will use frequently in this work.

**Remark 2.2.** We have that \( W^{\beta,2}_\delta \subset W^{\beta',2}_{\delta'} \) if \( \beta \geq \beta' \) and \( \delta \geq \delta' \). This follows from the equivalence

\[
\|<\cdot>\beta \leq D >\beta f\|_{L^2(\mathbb{R}^n)} \sim \|<D>\beta <\cdot>\beta f\|_{L^2(\mathbb{R}^n)},
\]

and Plancherel theorem, see for example [6, p. 222].
3. From the spherical integral to the P.V. integral

As we have explained in the introduction, the $Q_j$ operators that appear in the Born series expansion of $q$ can be expressed as a sum of a spherical term and several principal value operators. The usual strategy is to estimate the spherical part and then try to extend this estimate to the other terms. This is generally a very long and technical process that must be repeated case by case if the dimension or the value of $j$ is changed (see [2, 25, 26, 29]).

We want to simplify the structure of the $Q_j$ estimates by applying recursive arguments that allow to obtain estimates of the principal value terms from estimates of the spherical terms.

First, we define the following distributions. Let $f \in C_c^\infty((0,\infty))$, we put

\begin{equation}
(3.1) \quad d(f) = \int_0^\infty \delta(1-r)f(r)\,dr, \quad \text{and} \quad P(f) = P.V. \int_0^\infty \frac{1}{1-r}f(r)\,dr,
\end{equation}

where $\delta$ denotes the Dirac delta distribution, as usual.

**Proposition 3.1.** Let $r \in (0,\infty)$ and consider the modified Ewald spheres defined by the equation

\begin{equation}
(3.2) \quad \Gamma_r(\eta) := \{\xi \in \mathbb{R}^n : |\xi - \eta|/2 = r|\eta|/2\},
\end{equation}

(see Figure 2 in section 7 below). Then we have that

\begin{equation}
(3.3) \quad \widehat{Q_2(q)}(\eta) = (i\pi d + P)S_r(q)(\eta),
\end{equation}

where, if we denote by $\sigma_{r,\eta}$ the Lebesgue measure of $\Gamma_r(\eta)$,

\begin{equation}
(3.4) \quad S_r(q)(\eta) := \frac{2}{|\eta|(1+r)} \int_{\Gamma_r(\eta)} \hat{q}(\xi)\hat{q}(\eta - \xi)\,d\sigma_{r,\eta}(\xi).
\end{equation}

We omit the proof since it is just the case $j = 2$ of Proposition 5.1 below.

Motivated by the previous proposition, we are ready to introduce the following definition. The interest of giving an abstract definition capturing the main properties of $S_r$ is that it will simplify a great amount of work necessary to study the $Q_j$ operators with $j > 2$.

**Definition 3.2** (admissible family of spherical operators). Consider the Schwartz class of functions $S = S(\mathbb{R}^n)$ and assume we have a family of operators $\{G_r\}_{r \in (0,\infty)}$ satisfying

\begin{equation}
G_r : S \longrightarrow L^1_{\text{loc}}(\mathbb{R}^n),
\end{equation}

for every $r \in (0,\infty)$, such that for every $f \in S$, $G_r(f)(\eta)$ is a function in $L^1_{\text{loc}}((0,\infty) \times \mathbb{R}^n)$ in the variables $(r, \eta)$. We call each $G_r$ a spherical operator.

Assume also that we have some $1 \leq p < \infty$ and a Banach space $X$ where $S$ is dense. Then we say that the family of spherical operators $\{G_r\}_{r \in (0,\infty)}$ is a (one parameter) admissible family of spherical operators in the Banach space $X$, if the following conditions are satisfied.

1. $G_r(f)(\eta)$, is a $C^1$ function in the $r$ variable for almost every $\eta \in \mathbb{R}^n$.
2. There is a $0 < \delta < 1$, a constant $C_1$ and an integer $1 \leq k < \infty$ such that it satisfies the a priori estimate

\begin{equation}
(3.5) \quad \|\partial_r G_r(f)\|_{L^r} \leq C_1 \|f\|_X^k,
\end{equation}

for some $r \in \mathbb{R}$, and for $r \in (1 - \delta, 1 + \delta)$.

In our case, to verify that a certain family of operators satisfies Definition 3.2, usually the only difficult point is to prove estimate (3.5). In Proposition 4.5 below we shall see that
a slight modification of \( S_r \) satisfies this definition for \( p = 2 \). From a formal point of view, the main motivation to introduce Definition 3.2 is that the following proposition holds.

**Proposition 3.3.** Let \( f \in S \) and assume that the family of operators \( \{ G_r \}_{r \in (0, \infty)} \) fulfills the conditions of Definition 3.2 for some \( 1 < p < \infty \) and some \( \tau \in \mathbb{R} \). If we also have the estimate

\[
\| G_r(f) \|_{L_p^\alpha} \leq C_1 (1 + r)^{-\gamma} \| f \|_{X}^k,
\]

for some \( \gamma > 0, \alpha \in \mathbb{R} \), then for every \( \alpha' < \alpha \) we have that

\[
\| (i \pi d + P) G_r(f) \|_{L_p^{\alpha'}} \leq C_2 C_1 \| f \|_{X}^k,
\]

where \( C_2 = C_2(\delta, \alpha, \alpha', \tau, p, \gamma) \).

Notice that the value of \( \tau \) in (3.5) does not have any influence on the value of \( \alpha' \) in (3.7).

**Proof.** By (3.1), we clearly have that

\[
\| d(G_r(f)) \|_{L_p^\alpha} \leq 2^{-\gamma} C_1 \| f \|_{X}^k
\]

follows directly by putting \( r = 1 \) in (3.6). Therefore it remains to estimate the principal value operator

\[
P_G(f)(\eta) := P.V. \int_{0}^{\infty} \frac{1}{1-r} G_r(f)(\eta) \, dr.
\]

We have that

\[
P_G(f)(\eta) = P.V. \int_{|1-r|<\delta} \frac{G_r(f)(\eta)}{1-r} \, dr + \int_{\delta <|1-r|} \frac{G_r(f)(\eta)}{1-r} \, dr
\]

\[
= P_G,\delta(f)(\eta) + P_G,L(f)(\eta),
\]

where \( \delta \) is as in Definition 3.2. Applying Minkowski’s integral inequality and estimate (3.6), we obtain that

\[
\| P_G,L(f) \|_{L_p^p} \leq \int_{\delta <|1-r|} \frac{\| G_r(f) \|_{L_p^\alpha}}{|1-r|} \, dr \leq C(\delta, \gamma) C_1 \| f \|_{X}^k.
\]

To estimate \( P_G,\delta \) we need a finer decomposition in regions. Set

\[
\delta_\eta := \delta < \eta >^{-s},
\]

for some \( s \geq 0 \) that will be chosen later, and to simplify notation define the region

\[
B := \{ r \in (0, \infty) : \delta_\eta \leq |1-r| \leq \delta \}.
\]

Using that \( P.V. \int_{|1-r|<\delta} \frac{dr}{1-r} = 0 \) for any \( a > 0 \), we have

\[
P_G,\delta(f)(\eta) = \int_{|1-r|<\delta_\eta} \frac{G_r(f)(\eta) - G_1(f)(\eta)}{1-r} \, dr + \int_{B} \frac{G_r(f)(\eta)}{1-r} \, dr
\]

\[
:= P_G,s(f)(\eta) + P_G,B(f)(\eta),
\]

where in (3.10) the \( P.V. \) is not necessary any more, since we can cancel the singularity in the denominator with the difference in the numerator thanks to condition 1) in the definition.
Now, since \( p > 1 \), by Hölder’s inequality in the \( r \) variable (denoting by \( q \) the conjugate exponent), and the fundamental theorem of calculus we have

\[
\|P_{G,s}(f)\|_{L^p_{\alpha}} = \int_{\mathbb{R}^n} \left| \nabla \int_{|r|<\delta} \frac{G_r(f)(\eta) - G_1(f)(\eta)}{1 - r} \, dr \right|^p \, d\eta \leq \int_{\mathbb{R}^n} \left| \nabla \int_{|r|<\delta} \frac{G_r(f)(\eta) - G_1(f)(\eta)}{1 - r} \, dr \right|^{p/q} \, d\eta \leq \left( \int_{|r|<\delta} \left| \nabla \int_{0}^{1} \partial_r G_{\tilde{r}(t)}(\eta) \, dt \right|^p \, dr \right)^{p/q}
\]

where \( \tilde{r}(t) = (r-1)t + 1 \) and we have used that

\[
\left( \int_{|r|<\delta} \left| \nabla \int_{0}^{1} \partial_r G_{\tilde{r}(t)}(\eta) \, dt \right|^p \, dr \right)^{p/q} = (2\delta)^{p/q} = \frac{(2\delta)^{p/q}}{\eta > s/p/q},
\]

by (3.9). We have two cases. If in (3.5) and (3.6) we have \( \alpha \leq \tau \) we can choose \( s = 0 \), otherwise, if \( \alpha > \tau \), we choose \( s \) such that \( \alpha - s/q = \tau \). In both cases applying Minkowski’s inequality and (3.5) we obtain

\[
\|P_{G,s}(f)\|_{L^p_{\alpha}} \leq (2\delta)^{1/q} \left( \int_{|r|<\delta} \left( \int_{0}^{1} \left| \partial_r G_{\tilde{r}(t)}(\eta) \right| \, dt \right)^p \, dr \right)^{1/p}
\]

(3.11)

To finish we need estimate \( P_{G,B} \) which is non-zero when \( s > 0 \), that is when \( \alpha > \tau \). We set \( N(\eta) = -\log_2(\delta < \eta > s) \), and consider the next dyadic decomposition,

\[
P_{G,B}(f)(\eta) := \int_{B} \frac{G_r(f)(\eta)}{1 - r} \, dr = \sum_{0 \leq j < N(\eta)} \int_{|2^{-j+1} < |r| < 2^{-j}} \chi_{B(r)} \frac{G_r(f)(\eta)}{1 - r} \, dr.
\]

If \( j = 0, 1, \ldots, N(\eta) \), for \( \eta \) fixed, the definition of \( N(\eta) \) implies that \( 2^j \leq \eta > s/\delta \), therefore

\[
|P_{G,B}(f)(\eta)| \leq \sum_{j=0}^{N(\eta)} 2^{j+1} \chi_{(\delta,1,\infty)}(\eta > s) \int_{|1-r|<2^{-j}} |G_r(f)(\eta)| \, dr.
\]

(3.12)

But observe that in the last line we have an operator of the kind

\[
P^\lambda(f)(\eta) := \chi_{(\delta,1,\infty)}(\eta > s) \int_{|1-r|<\lambda} |G_r(f)(\eta)| \, dr,
\]

with \( 0 < \lambda \leq 1 \). Computing its \( L^p_{\alpha-\varepsilon} \) norm when \( \varepsilon > 0 \) and applying Minkowski’s integral inequality we obtain

\[
\|P^\lambda(f)\|_{L^p_{\alpha-\varepsilon}} \leq \chi_{\varepsilon/s} \int_{|1-r|<\lambda} \|G_r(f)\|_{L^p_{\alpha}} \, dr \leq \lambda^{1+\varepsilon/s} C_1 \|f\|_{L^p_{\alpha}},
\]

(3.13)

where we have used estimate (3.6), and that in the region where the characteristic function does not vanish we have that \( \eta > s/\delta \leq \delta^{-\varepsilon/s} \lambda^{\varepsilon/s} \). Hence, taking the \( L^p_{\alpha'} \) norm of (3.12)
and applying estimate (3.13) with $\varepsilon = \alpha - \alpha'$,

$$\|P_{G,B}(f)\|_{L^p_{\alpha'}} \leq 2 \sum_{j=0}^{\infty} 2^j \|P^{2^j}(f)\|_{L^p_{\alpha'}} \leq 2^{j+\varepsilon/s} C_1 \|f\|_X \sum_{j=0}^{\infty} 2^{-j\varepsilon/s}$$

$$\leq C(\delta, \alpha, \alpha', \tau, p) C_1 \|f\|_X.$$  (3.14)

Observe that this is the first time we need the strict inequality $\alpha' < \alpha$ in the statement of the theorem. Therefore since $P_G = P_{G,L} + P_{G,s} + P_{G,B}$ we conclude the proof putting together estimates (3.8), (3.11) and (3.14). \(\square\)

**Remark 3.4.** In the previous proposition, we have excluded the case $p = 1$ to estimate $P_{G,s}$, but this restriction is not necessary to prove (3.8) and (3.14). Therefore, if the assumptions in the statement of Proposition 3.3 hold for $p = 1$, then for $\alpha' < \alpha$ we have the estimate

$$\int_{|\eta| \geq \delta_n} G_r(f)(\eta) \, d\eta \leq C \|f\|_X^2.$$  \(\square\)

### 4. Sobolev estimates for the double dispersion operator

In this section, we study in detail the spherical operator $S_r$ of the double dispersion operator $\tilde{Q}_2$ in order to prove Theorem 1.3 for $j = 2$. This section will serve to illustrate the approach used to obtain in section 5 the main estimates of the spherical operators related to the $\tilde{Q}_j$ operators.

For notational convenience we define the operator

$$\tilde{S}_r(q)(\eta) := \chi(\eta) S_r(q)(\eta).$$

Then, multiplying both sides of equation (3.3) by the smooth cut-off $\chi(\eta)$ we get

(4.1) $$\tilde{Q}_2(q)(\eta) = (i\pi d + P)\tilde{S}_r(q)(\eta).$$

Hence, the main idea to estimate the $\tilde{Q}_2$ operator is to apply Proposition 3.3 to the particular case $G_r = \tilde{S}_r$. We begin with the necessary estimates for $\tilde{S}_r$.

**Lemma 4.1.** Let $n \geq 2$ and $q \in W_1^{\beta,2}(\mathbb{R}^n)$ with $\beta \geq 0$. Then the estimate

$$\|\tilde{S}_r(q)\|_{L^2_{\alpha'}} \leq C(1 + r)^{-\gamma} \|q\|_{W_1^{\beta,2}}^2,$$

holds when

(4.2) $$\begin{cases}
\alpha \leq \beta + (\beta - (n - 3)/2), & \text{if } (n - 3)/2 < \beta < (n - 1)/2, \\
\alpha < \beta + 1, & \text{if } (n - 1)/2 \leq \beta < \infty,
\end{cases}$$

for some real number $\gamma > 0$ (possibly depending on $\beta$ and $\alpha$).

To simplify later computations we define the operator

$$\tilde{K}_r(g_1, g_2)(\eta) = \chi(\eta) K_r(g_1, g_2)(\eta),$$

where

(4.3) $$K_r(g_1, g_2)(\eta) := \frac{1}{|\eta|} \int_{r(\eta)} |g_1(\xi)||g_2(\eta - \xi)| \, d\sigma_{r\eta}(\xi).$$

Then we have that

$$\left|\tilde{S}_r(q)(\eta)\right| \leq \frac{2}{1 + r} \tilde{K}_r(\tilde{q}, \tilde{q})(\eta).$$
and therefore the proof of Lemma 4.1 is an immediate consequence of the following lemma taking $\gamma = 1 - \lambda$.

**Lemma 4.2.** Let $n \geq 2$ and $f_1, f_2 \in W^{\beta,2}_1(\mathbb{R}^n)$ with $\beta \geq 0$. Then the estimate

$$\|K_r(\hat{f}_1, \hat{f}_2)\|_{L^2_\alpha} \leq C r^\lambda \|f_1\|_{W^{\beta,2}_1} \|f_2\|_{W^{\beta,2}_1},$$

holds when condition (4.2) is also satisfied, for some real number $0 < \lambda < 1$ (possibly depending on $\beta$ and $\alpha$).

In the proof we are going to use the following result.

**Lemma 4.3** (Lemma 3.4). Let $S_\rho \subset \mathbb{R}^n$ be any sphere of radius $\rho$ and let $d\sigma_\rho$ be its Lebesgue measure. Then for any $0 < \lambda \leq (n-1)/2$, we have that

$$\frac{1}{|x-y|(n-1)-2\lambda} d\sigma_\rho(y) \leq C_\lambda \rho^{2\lambda},$$

for any $x \in \mathbb{R}^n$, and for a constant $C_\lambda$ that only depends on $\lambda$.

This can be proved by direct computation (for a detailed proof see [17, Appendix]).

**Proof of Lemma 4.2.** First, using that $|\eta|^{-1} \leq C < \eta >^{-1}$ in the region where $\chi(\eta)$ does not vanish, we have

$$\|K_r(\hat{f}_1, \hat{f}_2)\|^2_{L^2_\alpha} \leq C \int_{\mathbb{R}^n} < \eta >^{2\alpha-2} \left( \int_{\Gamma_r(\eta)} |\hat{f}_1(\xi)| |\hat{f}_2(\eta - \xi)| d\sigma_r(\xi) \right)^2 d\eta.$$

Now, $\eta = (\eta - \xi) + \xi$, so if we choose any $0 < c < 1/2$ at least one of the conditions $|\xi| > c|\eta|$ and $|\eta - \xi| > c|\eta|$ must hold. But observe now that the change of variables $\xi' = \eta - \xi$ leaves invariant $\Gamma_r(\eta)$ and $K_r(\hat{f}_1, \hat{f}_2)$, except for the fact that interchanges the roles of $\hat{f}_1$ and $\hat{f}_2$. Therefore is enough to study only the case of $|\xi| > c|\eta|$ since then the other follows applying the change of variables. We want to estimate

$$I := \int_{\mathbb{R}^n} < \eta >^{2\alpha-2} \left( \int_{\Gamma_r^+(\eta)} |\hat{f}_1(\xi)| |\hat{f}_2(\eta - \xi)| d\sigma_r(\xi) \right)^2 d\eta,$$

where $\Gamma_r^+(\eta) := \{ \xi \in \Gamma_r(\eta) : |\xi| > c|\eta| \}.$

We introduce a real parameter $0 < \lambda \leq (n-1)/2$. Then by Cauchy-Schwarz’s inequality we have

$$I \leq C \int_{\mathbb{R}^n} < \eta >^{2\alpha-2} \int_{\Gamma_r^+(\eta)} |\hat{f}_1(\xi)|^2 |\hat{f}_2(\eta - \xi)|^2 |\eta - \xi|^{n-1-2\lambda} d\sigma_r(\xi) \times \ldots$$

$$\ldots \times \int_{\Gamma_r^+(\eta)} \frac{1}{|\eta - \xi|^{n-1-2\lambda}} d\sigma_r(\xi) d\eta.$$

Then, since $\Gamma_r(\eta)$ has radius $r|\eta|/2$, using Lemma 4.3 to bound the second integral we obtain

$$I \leq C r^{2\lambda} \int_{\mathbb{R}^n} < \eta >^{2\alpha-2} |\eta|^{2\lambda} \times \ldots$$

$$\ldots \times \int_{\Gamma_r^+(\eta)} |\hat{f}_1(\xi)|^2 |\hat{f}_2(\eta - \xi)|^2 < \eta - \xi >^{n-1-2\lambda} d\sigma_r(\xi) d\eta$$

$$\leq C r^{2\lambda} \int_{\mathbb{R}^n} \int_{\Gamma_r(\eta)} \frac{1}{|\hat{f}_1(\xi)|^2 |\hat{f}_2(\eta - \xi)|^2 < \eta - \xi >^{n-1-2\lambda} d\sigma_r(\xi) d\eta,$$

(4.5)
Therefore changing the order of integration and using that by Plancherel theorem we have

\[ \alpha \]

hence, the condition \( \alpha \) of parameters given in the statement.

We are going to use the Trace theorem to bound the \( L^2(\Gamma, \eta) \) norm given by the second integral in \( (4.5) \). The fundamental point is that when the surface in which we apply the Trace theorem is a sphere, we can choose a constant for the estimate independent of its radius (actually \( C = 1 \)), see \( (A.1) \) below. As a consequence, the constant \( C \) in the following estimate does not depend on \( \eta \) or \( r \),

\[
I \leq C r^{2\lambda} \int_{\mathbb{R}^n} |\hat{f}_1(\xi)|^2 < \xi >^{2\alpha-2+2\lambda} |\hat{f}_2(\eta-\xi)|^2 < \eta - \xi >^{(n-1)-2\lambda} \, d\xi \, d\eta \\
+ C r^{2\lambda} \int_{\mathbb{R}^n} \left| \nabla \left( \hat{f}_1(\xi) < \xi >^{\alpha-1+\lambda} \right) \right|^2 |\hat{f}_2(\eta-\xi)|^2 < \eta - \xi >^{(n-1)-2\lambda} \, d\xi \, d\eta \\
+ C r^{2\lambda} \int_{\mathbb{R}^n} |\hat{f}_1(\xi)|^2 < \xi >^{2\alpha-2+2\lambda} \left| \nabla \left( \hat{f}_2(\eta-\xi) < \eta - \xi >^{(n-1)/2-\lambda} \right) \right|^2 \, d\xi \, d\eta.
\]

Therefore changing the order of integration and using that by Plancherel theorem we have

\[
\int_{\mathbb{R}^n} \left| \nabla (\hat{f}(\xi) < \xi >^{\lambda}) \right|^2 \, d\xi \leq C \|f\|_{W^{1,2}_t}^2,
\]

we obtain

\[
I \leq C r^{2\lambda} \|f_1\|_{W^{\alpha-1+\lambda,2}_t}^2 \|f_2\|_{W^{(n-1)/2-\lambda,2}_x}^2.
\]

As we have explained before, in the case \( |\eta - \xi| > c|\eta| \) we obtain the same estimate but interchanging the roles of \( f_1 \) and \( f_2 \). Putting both estimates together we get

\[
\|K_x(f_1, f_2)\|_{L^2_x} \leq C r^\lambda \left( \|f_1\|_{W^{\alpha-1+\lambda,2}_t} \|f_2\|_{W^{(n-1)/2-\lambda,2}_x} + \|f_2\|_{W^{\alpha-1+\lambda,2}_t} \|f_1\|_{W^{(n-1)/2-\lambda,2}_x} \right).
\]

We also add the extra restriction \( \lambda < 1 \), this is necessary to have a negative value for \( \gamma \) in Lemma 4.1. Now, fix

\[
(4.6) \quad \beta = \alpha - 1 + \lambda,
\]

hence, the condition \( \alpha - 1 + \lambda \geq 0 \) used in the proof implies we must have \( \beta \geq 0 \). As a consequence of \( (4.6) \), equation \( (4.4) \) follows directly in the range \( \beta \geq (n-1)/2 \) (we are using remark 2.2). But, by the conditions imposed in the proof we have to take into account the restrictions

\[
(4.7) \quad \begin{cases} 0 < \lambda < 1 \\ 0 < \lambda \leq \frac{n-1}{2} \end{cases} \quad \iff \begin{cases} \beta < \alpha < \beta + 1 \\ \beta + 1 - \frac{n-1}{2} \leq \alpha < \beta + 1. \end{cases}
\]

We can discard the lower bounds for \( \alpha \) using that \( \|f\|_{L^2_\alpha} \leq \|f\|_{L^2_{\alpha'}} \) always holds if \( \alpha \leq \alpha' \). Therefore only the restriction \( \alpha < \beta + 1 \) remains.

Otherwise, if \( \beta \) is in the range \( 0 \leq \beta < (n-1)/2 \), estimate \( (4.4) \) will follow if we add the extra condition

\[
(4.8) \quad (n-1)/2 - \lambda \leq \beta.
\]

Then, since \( \lambda < 1 \), we must have \( \beta > (n-3)/2 \) (the other conditions on \( \lambda \) don’t add new restrictions). Also \( (4.6) \) and \( (4.8) \) imply together that \( \alpha \leq 2\beta - (n-3)/2 \), which is a stronger condition than \( \alpha < \beta + 1 \) since we have \( \beta < (n-1)/2 \). Hence, we have obtained the ranges of parameters given in the statement. \( \square \)
Lemma 4.4. Let \( q \in \mathcal{S}(\mathbb{R}^n) \). We have the following pointwise inequality
\[
|\partial_r S_r(q)(\eta)| \leq C K_r(\hat{q}, \hat{q})(\eta) + C|\eta| \sum_{i=1}^{n} K_r(\hat{x}_i q, \hat{q}).
\]

In general, the constant \( C \) in the estimate might depend on \( \delta \), but this is harmless.

Proof. Assume that \( q \in \mathcal{S}(\mathbb{R}^n) \). We centre the Ewald sphere in (3.4) at the origin with the change \( \xi = \eta/2 + r|\eta/2|\theta \), where \( \theta \in S^{n-1} \), to obtain
\[
S_r(q)(\eta) = \frac{r^{n-1}|\eta|^{n-2}}{2^{n-2}(1+r)} \int_{S^{n-1}} \hat{q} \left( r \frac{|\eta|}{2} \theta + \frac{\eta}{2} \right) \hat{q} \left( -r \frac{|\eta|}{2} \theta + \frac{\eta}{2} \right) d\sigma(\theta).
\]

Now we can compute derivatives in the \( r \) variable. Consider \( \eta \) fixed, then
\[
\partial_r S_r(q)(\eta) = \frac{r^{n-1}|\eta|^{n-2}}{2^{n-2}(1+r)^2} \int_{S^{n-1}} q \left( r \frac{|\eta|}{2} \theta + \frac{\eta}{2} \right) \hat{q} \left( -r \frac{|\eta|}{2} \theta + \frac{\eta}{2} \right) d\sigma(\theta)
\]
\[
+ \frac{r^{n-1}|\eta|^{n-1}}{2^{n-2}(1+r)} \int_{S^{n-1}} \theta \cdot \nabla \hat{q} \left( r \frac{|\eta|}{2} \theta + \frac{\eta}{2} \right) \hat{q} \left( -r \frac{|\eta|}{2} \theta + \frac{\eta}{2} \right) d\sigma(\theta)
\]
\[
- \frac{r^{n-1}|\eta|^{n-1}}{2^{n-2}(1+r)} \int_{S^{n-1}} \hat{q} \left( r \frac{|\eta|}{2} \theta + \frac{\eta}{2} \right) \theta \cdot \nabla \hat{q} \left( -r \frac{|\eta|}{2} \theta + \frac{\eta}{2} \right) d\sigma(\theta).
\]

We have passed the derivative inside the integral since we are integrating in finite measure and \( \hat{q} \) is smooth. This implies that \( S_r(q)(\eta) \) is \( C^1 \) in the \( r \) variable (in fact smooth) for every \( \eta \neq 0 \), so \( S_r \) and \( \tilde{S} \) satisfy condition 1 of Definition 3.2. Observe also that the two last terms are identical (this can be verified with the change \( \omega = -\theta \)). Hence, if we undo the change to spherical coordinates we get
\[
\partial_r S_r(q)(\eta) = \frac{(n-2)r + (n-1)}{r(1+r)^2} \frac{2}{|\eta|} \int_{r=r(\eta)} \hat{q}(\xi) \hat{q}(\eta - \xi) d\sigma_{r\eta}(\xi)
\]
\[
+ \frac{2}{(1+r)} \frac{\xi - \eta/2}{|\xi - \eta/2|} \cdot \nabla \hat{q}(\xi) \hat{q}(\eta - \xi) d\sigma_{r\eta}(\xi).
\]

Therefore by (4.3), if we fix some \( 0 < \delta < 1 \), for \( r \in (1 - \delta, 1 + \delta) \) we obtain
\[
|\partial_r S_r(q)(\eta)| \leq C K_r(\hat{q}, \hat{q})(\eta) + C|\eta| K_r(|\nabla \hat{q}|, \hat{q})(\eta).
\]

The estimate follows then using that
\[
K_r(|\nabla \hat{q}|, \hat{q}) = \sum_{i=1}^{n} K_r(\hat{x}_i \hat{q}, \hat{q}) = C \sum_{i=1}^{n} K_r(\hat{x}_i \hat{q}, \hat{q}).
\]

From Lemmas 4.1 and 4.4 we get the following proposition.

Proposition 4.5. Let \( n \geq 2 \) and fix some \( 0 < \delta < 1 \). Then for every \( r \in (1 - \delta, 1 + \delta) \), and \( q \in \mathcal{S}(\mathbb{R}^n) \) we have that
\[
\|\partial_r \tilde{S}_r(q)\|_{L^2_{\omega, 2}} \leq C \|q\|_{W^{\beta, 2}_2}^2,
\]
holds when \( \alpha \) and \( \beta \geq 0 \) satisfy condition (4.2). As a consequence we obtain that \( \{\tilde{S}_r\}_{0, \infty} \) is an admissible family of spherical operators as defined in Definition 3.2 for \( p = 2 \), \( r = \alpha - 1 \), \( k = 2 \) and \( X = W^{\beta, 2}_{1}(\mathbb{R}^n) \) with \( (n-3)/2 < \beta < \infty \).
Proposition 5.1

Proof. Multiplying (4.9) by \(\chi(\eta)\) we get

\[
\|\partial_r \tilde{S}_r(q)\|_{L^2_{\alpha-1}} \leq C \|\tilde{K}_r(\tilde{q}, \tilde{q})\|_{L^2_{\alpha-1}} + C \sum_{i=1}^{n} \|\tilde{K}_r(\tilde{q}_i, \tilde{q})\|_{L^2_{\alpha-1}}.
\]

Notice that we get the \(L^2_{\alpha}\) norm in the last term due to the extra \(|\eta|\) factor appearing in (4.9). Then, by Lemma 4.2 we obtain the desired estimate using that

\[
\|\tilde{K}_r(\tilde{q}_i, \tilde{q})\|_{L^2_{\alpha}} \leq C \|x_i q\|_{W^{2,2}_{1,2}} \|q\|_{W^{2,2}_{2,2}} \leq C \|q\|_{W^{2,2}_{2,2}}^2.
\]

The estimate \(\|x_i q\|_{W^{2,2}_{1,2}} \leq C \|q\|_{W^{2,2}_{2,2}}\) can be verified for integer \(\beta\) and extended by interpolation to the general case.

By Lemma 4.1 and 4.5 we can apply Proposition 3.3 to estimate the \(\tilde{Q}_2\) operator, but we leave this for the next section.

5. SOBOLEV ESTIMATES FOR THE GENERAL \(\tilde{Q}_j\) OPERATOR

In this section we prove Theorem 1.3.

Let \(\ell \geq 1\), and assume we have \(r \in (0, \infty)^{\ell}\), \(r = (r_1, \ldots, r_\ell)\) and \(f \in C^\infty_c((0, \infty)^{\ell})\). We define the operators,

\[
P_i, d_i : C^\infty_c((0, \infty)^{\ell}) \rightarrow C^\infty_c((0, \infty)^{\ell-1}),
\]

following the notation introduced in (3.1),

\[
d_i(f)(r_1, \ldots, \hat{r}_i, \ldots, r_\ell) := \int_0^\infty \delta(r_i - 1)f(r) \, dr_i,
\]

\[
P_i(f)(r_1, \ldots, \hat{r}_i, \ldots, r_\ell) := P. V. \int_0^\infty \frac{1}{1 - r_i} f(r) \, dr_i,
\]

where \(\hat{r}_i\) indicates that this coordinate is deleted in the list. Hence, if \(\ell = 1\), \(d_i(f)\) and \(P_i(f)\) are just scalars. Also, if \(r \in (0, \infty)^{\ell}\) we define the manifold,

\[
\Gamma_r(\eta) = \Gamma_{r_1}(\eta) \times \cdots \times \Gamma_{r_\ell}(\eta),
\]

and we denote by \(\sigma_r\) its Lebesgue measure (product of the measures of the spheres \(\Gamma_{r_i}(\eta)\)),

\[
d\sigma_r(\xi_1, \ldots, \xi_\ell) = d\sigma_{r_1}(\xi_1) \times \cdots \times d\sigma_{r_\ell}(\xi_\ell).
\]

Proposition 5.1 (\(Q_j(q)\) structure). Let \(n \geq 2\) and \(j \geq 2\). Then we have that

\[
\tilde{Q}_j(q)(\eta) = \prod_{i=1}^{j-1} (i\pi d_i \pm P_i) S_{j,r}(q)(\eta),
\]

where \(r = (r_1, \ldots, r_{j-1})\), and

\[
S_{j,r}(q)(\eta) := \left(\prod_{i=1}^{j-1} \frac{2}{1 + r_i}\right) \times 
\]

\[
\frac{1}{|\eta|^{j-1}} \int_{\Gamma_r(\eta)} \tilde{q}(\eta - \xi_1) \left(\prod_{i=1}^{j-2} \tilde{q}(\xi_i - \xi_{i+1})\right) \tilde{q}(\xi_{j-1}) \, d\sigma_r(\xi_1, \ldots, \xi_{j-1}).
\]
Proposition 5.1 implies that the higher order operators $Q_j$ have a similar structure to the $Q_2$ operator (we consider $\prod_{k}^{n} = 1$ if $k > m$, as it is usual). In fact, when $j = 2$, (5.1) is equivalent to equation (3.3) since with the new notation we have $S_r = S_{2,r}$ (in this case we have $r = r$, since there is only one parameter).

Proof. Let $k \in (0, \infty)$, we are going need the identity

\begin{equation}
R_k(f)(x) = i \frac{\pi}{2} k^{n-2} \int_{S^{n-1}} \hat{f}(\omega) e^{ikx \cdot \omega} d\sigma(\omega) + P.V. \int_{\mathbb{R}^n} e^{ix \cdot \zeta} \frac{\hat{f}(\zeta)}{|\zeta|^2 + k^2} d\zeta,
\end{equation}

for the resolvent of the Laplacian, used in [29] (it follows from computing explicitly the limit in (2.3) in the sense of distributions, see for example chapter 5 of [28] and [13, pp. 209-236] for more details). We take spherical coordinates in the principal value integral, denoting by $t$ the radial variable and use the change of variables $\xi = rk$ in the radial integral,

\begin{align*}
P.V. \int_{\mathbb{R}^n} e^{ix \cdot \zeta} \frac{\hat{f}(\zeta)}{|\zeta|^2 + k^2} d\zeta &= P.V. \int_{0}^{\infty} \frac{1}{(k-t)(k+t)} \int_{S^{n-1}} e^{ix \cdot \omega} \hat{f}(t \omega) t^{n-1} d\sigma(\omega) dt \\
&= P.V. \int_{0}^{\infty} \frac{1}{k} \int_{0}^{\infty} \frac{1}{(1-r)(1+r)} \int_{S^{n-1}} e^{ix \cdot \xi} \hat{f}(\xi) r^{n-1} d\sigma(\omega) dr \\
&= P.V. \int_{0}^{\infty} \frac{1}{k} \int_{0}^{\infty} \frac{1}{(1-r)(1+r)} \int_{\Gamma_r(\eta)} e^{i(-\xi - k\theta) \cdot x} \hat{f}(-\xi - k\theta) d\sigma(r \eta(\xi)) d\xi,
\end{align*}

where to obtain the integral over the Ewald sphere in the last line we have used the change of variables $r k \omega = -\xi - k\theta$ in the spherical integral, and that $\Gamma_r(\eta) = \{ \xi \in \mathbb{R}^n : |\xi + k\theta| = rk \}$ if $\eta = -2k\theta$ (see (3.2)). Hence, using the analogous change of variables $k\omega = -\xi - k\theta$ in the first integral in (5.3), we finally obtain

\begin{align*}
R_k(f)(x) &= i \frac{\pi}{|\eta|} \int_{\Gamma_r(\eta)} e^{i(-\xi - k\theta) \cdot x} \hat{f}(-\xi - k\theta) d\sigma(\eta) \\
&+ P.V. \int_{0}^{\infty} \frac{2}{|\eta|(1+r)} \int_{\Gamma_r(\eta)} e^{i(-\xi - k\theta) \cdot x} \hat{f}(-\xi - k\theta) d\sigma(r \eta(\xi)) d\xi \\
&= (i\pi d + P) \left( \frac{2}{|\eta|(1+r)} \int_{\Gamma_r(\eta)} e^{i(-\xi - k\theta) \cdot x} \hat{f}(-\xi - k\theta) d\sigma(r \eta(\xi)) \right).
\end{align*}

We recall that by (2.8), we have

\begin{equation}
\widehat{Q_2(\eta)}(-2k\theta) = \int_{\mathbb{R}^n} e^{ik\theta \cdot y} (q R_k)^{-1}(q(\cdot) e^{ik\theta \cdot (\cdot)})(y) dy.
\end{equation}

Let $m \in \mathbb{N}$, we define

\begin{equation}
f_m(x) := R_k((q R_k)^{m}(q(\cdot) e^{ik\theta \cdot (\cdot)}))(x).
\end{equation}

We claim that

\begin{equation}
f_m(x) = \left( \prod_{i=1}^{m} (i \pi d_i + P_i) \right) \left( \prod_{i=1}^{m} \frac{2}{1 + r_i} \right) \frac{1}{|\eta|^m} \times \ldots
\end{equation}

\begin{equation}
\int_{\Gamma_m(\eta)} \ldots \int_{\Gamma_1(\eta)} e^{i(-\xi_m - k\theta) \cdot x} \hat{q}(\eta - \xi_1) \left( \prod_{i=1}^{m-1} \hat{q}(\eta - \xi_{i+1}) \right) d\sigma(\xi_1, \ldots, \xi_m).
\end{equation}
We prove the claim by induction. The case $m = 1$ follows directly from (5.4) using that $q e^{i k \theta} \cdot (\xi) = \tilde{q}(\xi - k \theta)$ and that $\eta = -2k\theta$.

We are going to prove (5.7) for $m + 1$ assuming that it is true for $m$. On the one hand, by (5.6) and (5.4) we have

$$ f_{m+1}(x) = R_k(q f_m)(x) = (i \pi d_{m+1} + P_{m+1}) \ldots $$

(5.8) $$\left( \frac{2}{(1 + r_{m+1})|\eta|} \int_{\Gamma_{r_{m+1}}(\eta)} e^{i(-\xi_{m+1} - k \theta)x} (q f_m)(-\xi_{m+1} - k \theta) d\sigma_{r_{m+1}+\eta}(\xi_{m+1}) \right).$$

On the other hand, by (5.7), changing the order of integration we have

$$ (q f_m)(\zeta) = \left( \prod_{i=1}^m (i \pi d_i + P_i) \right) \left( \prod_{i=1}^{m+1} \frac{2}{(1 + r_i)} \right) \frac{1}{|\eta|^{m+1}} \int_{\Gamma_{r_1}(\eta)} \ldots \int_{\Gamma_{r_m}(\eta)} \tilde{q}(\eta - \xi_1) \times \ldots $$

(5.9) $$\left( \prod_{i=1}^m \tilde{q}(\xi_i - \xi_{i+1}) \right) \tilde{q}(k \theta + \zeta + \xi_m) d\sigma_\xi(\xi_1, \ldots, \xi_m).$$

Thus, putting $\zeta = -\xi_{m+1} - k \theta$ in the previous equality and using (5.8), we get

$$ f_{m+1}(x) = \left( \prod_{i=1}^{m+1} (i \pi d_i + P_i) \right) \left( \prod_{i=1}^{m+1} \frac{2}{(1 + r_i)} \right) \frac{1}{|\eta|^{m+1}} \times \ldots $$

$$\int_{\Gamma_{r_{m+1}}(\eta)} \ldots \int_{\Gamma_{r_1}(\eta)} e^{i(-\xi_{m+1} - k \theta)x} \tilde{q}(\eta - \xi_1) \left( \prod_{i=1}^m \tilde{q}(\xi_i - \xi_{i+1}) \right) d\sigma_\xi(\xi_1, \ldots, \xi_{m+1}),$$

which proves the claim.

By (5.5) we have that $\tilde{Q}_j(-2k \theta) = q f_{j-1}(-k \theta)$, and hence, in order to obtain (5.1), is enough to put $\zeta = -k \theta$ in (5.9),

$$ \tilde{Q}_j(-2k \theta) = \left( \prod_{i=1}^{j-1} (i \pi d_i + P_i) \right) \left( \prod_{i=1}^{j-1} \frac{2}{(1 + r_i)} \right) \frac{1}{|\eta|^{j-1}} \times \ldots $$

$$\int_{\Gamma_{r_{j-1}}(\eta)} \ldots \int_{\Gamma_{r_1}(\eta)} \tilde{q}(\eta - \xi_1) \left( \prod_{i=1}^{j-2} \tilde{q}(\xi_i - \xi_{i+1}) \right) \tilde{q}(\xi_{j-1}) d\sigma_\xi(\xi_1, \ldots, \xi_{j-1}).$$

We introduce now the $\tilde{S}_{j,x}$ spherical operators,

$$ \tilde{S}_{j,x}(q)(\eta) := \chi(\eta) S_{j,x}(q)(\eta), $$

as we did for $j = 2$. The following proposition generalizes the results of Lemma 4.1 and Proposition 4.5 for $j \geq 2$. Its proof will be given later on.

**Proposition 5.2.** Let $q \in S(\mathbb{R}^n)$, $n \geq 2$, $j \geq 3$ and $0 < \delta < 1$. Consider all the multi-indices $a = (a_1, \ldots, a_{j-1})$ with $a_i, 1 \leq i \leq j-1$, either $0$ or $1$. Then the estimate

(5.10) $$ \|\partial_{\eta}^a \tilde{S}_{j,x}(q)\|_{L^2_{\eta} \cdot [\cdot]} \leq C \left( \prod_{i=1}^{j-1} \frac{1}{(1 + r_i)^\gamma} \right) \|q\|_{W^{\beta,2}}. $$
holds for $\beta \geq 0$, a certain $\gamma > 0$ (possibly dependent on $\beta$), and some constant $C = C(n,j,\alpha,\beta)$, if the following conditions also hold

$$r_i \in (1 - \delta, 1 + \delta) \text{ if } a_i = 1 \text{ and } r_i \in (0, \infty) \text{ if } a_i = 0.$$  

(5.11)

$$
\begin{cases}
\alpha \leq \beta + (j - 1)(\beta - (n - 3)/2), & \text{if } (n - 3)/2 < \beta < (n - 1)/2, \\
\alpha < \beta + (j - 1), & \text{if } (n - 1)/2 < \beta < \infty.
\end{cases}
$$

(5.12)

With this proposition we can prove finally Theorem 1.3, with the help of the following density argument.

**Lemma 5.3.** Assume that the operator $\tilde{Q}_j$ satisfies an a priori estimate

$$||\tilde{Q}_j(q)||_{W^{\alpha,2}} \leq C||q||_{W^{\beta,p}}^j,$$

(5.13)

for every $q \in C^\infty_c(\mathbb{R}^n)$. Then there is a unique continuous extension $\tilde{Q}_j : W^{\beta,p}_0(\mathbb{R}^n) \rightarrow W^{\alpha,2}(\mathbb{R}^n)$ of the operator, and estimate (5.13) holds also for $q \in W^{\beta,p}(\mathbb{R}^n)$.

This lemma is just a trick to extend estimates for $\tilde{Q}_j(q)$ without having to give an estimate for the multilinear operator $Q_j(f_1, \ldots, f_j)$ (this operator is defined by putting $f_i$ instead of $q$ in (2.8) following the order of appearance of each $q$ in the formula). It holds for more general spaces and operators, but for the sake of brevity we have stated it for the specific operators $\tilde{Q}_j$ and the functional spaces we need. The advantage of having $f_i = q$ in most of the estimates in this work is a question of (great) notational simplicity, but it is not an essential restriction in any of them. Nonetheless, we give a detailed proof of the previous lemma in the Appendix. The key idea is to symmetrize $Q_j(f_1, f_2, \ldots, f_j)$ and use a polarization identity for multilinear operators.

**Proof of Theorem 1.3.** We begin with the case $j = 2$. By Proposition 4.5 we have that $\{\tilde{S}_r\}_{r \in (0, \infty)}$ satisfies Definition 3.2 for $X = W^{\beta,2}_2(\mathbb{R}^n)$ with $(n - 3)/2 < \beta < \infty$ and $p = 2$. Therefore using (4.1) we can put together Lemma 4.1, Proposition 3.3 and Plancherel theorem to obtain an a priori bound for $\tilde{Q}_2(q)$, $q \in \mathcal{S}$, in the $W^{\alpha',2}(\mathbb{R}^n)$ norm with $\alpha' < \alpha$ and $\alpha$ in the range (4.2). By Lemma 5.3 we can extend by density these estimates for $q \in W^{\beta,2}_2(\mathbb{R}^n)$. This is enough to prove estimate (1.5).

Now, let's study the case $j \geq 3$. Consider $f \in \mathcal{S}$. We introduce the following operators,

$$T_1(r_1, \ldots, r_{j-1})(f) := \tilde{S}_{j,r}(f),$$

(5.14)

$$T_k(r_{k-1}, r_k, \ldots, r_{j-1})(f) := (i\pi d_{k-1} + P_{k-1})T_{k-1}(r_{k-1}, \ldots, r_{j-1})(f)$$

$$= \prod_{i=1}^{k-1} (i\pi d_i + P_i)\tilde{S}_{j,r}(f),$$

(5.15)

for $2 \leq k \leq j$. $T_k(r_k, \ldots, r_{j-1})(f)(x)$ is a well defined function, smooth in the variables $r_k, \ldots, r_{j-1}$ and $x$ (see Proposition A.2 in the Appendix for more details). As we are going to see, the proof can be reduced to proving the following claim.

**Claim.** Let $1 \leq k \leq j$, and let $a = (a_1, \ldots, a_{j-1})$ with $a_i = 0$ if $1 \leq i \leq k - 1$, and $a_i = 0, 1$ if $k \leq i \leq j - 1$. Then the estimate

$$||\partial_x^\alpha T_k(r_k, \ldots, r_{j-1})(f)||_{L_{\alpha'}^2} \leq c_k ||f||_{W^{\beta,2}}^j,$$

(5.16)
holds for \( \alpha' < (\alpha - |a|) \) if conditions (5.11) and (5.12) are satisfied, with

\[
(5.17) \quad c_k = CC_2^{k-1} \prod_{i=k}^{j-1} \frac{1}{(1 + r_i)^\gamma},
\]

where \( C \) and \( C_2 \) are the constants introduced respectively in Propositions 5.2 and 3.3.

Since by (5.15) we have

\[
T_j(f) = \hat{Q}_j(f),
\]

by Plancherel theorem, for \( q_\xi = \frac{1}{1 + r_k} \) (5.17)

Therefore, for \( \alpha \) since condition (3.5) is given by (5.19) and condition (3.6) by (5.18) with \( C \)

This is why we have to include the estimate of the derivatives \( \partial_x^\alpha T_k(r_k, \ldots, r_{j-1})(f) \) in the

We prove now the claim by induction. By (5.14), the case \( k = 1 \) of estimate (5.16) is equivalent to Proposition 5.2. To prove that (5.16) holds true for \( 2 \leq k \leq j \), in each induction step we are going to use Proposition 3.3 and (5.15). This means that we need to show that each \( T_k(r_k, \ldots, r_{j-1})(f) \) is an admissible family of spherical operators in the parameter \( r_k \).

This is why we have to include the estimate of the derivatives \( \partial_x^\alpha T_k(r_k, \ldots, r_{j-1})(f) \) in the

Let’s assume that the claim holds for a certain \( k, 1 \leq k < j - 1 \), then we are going to prove it for \( k + 1 \). Let \( \alpha' = (a'_1, \ldots, a'_{k-1}) \) with \( a'_i = 0 \) if \( 1 \leq i \leq k \), and \( a'_i = 1 \) if \( k + 1 \leq i \leq j - 1 \). We are going to apply Proposition 3.3 with

\[
G_{r_k}(f)(x) := \partial_x^\alpha T_k(r_k, \ldots, r_{j-1})(f)(x).
\]

By the induction hypothesis (5.16) with \( a = \alpha' \), and (5.17) we have

\[
(5.18) \quad \|G_{r_k}(f)\|_{L^2_{\alpha}} \leq \frac{c_k}{(1 + r_k)^\gamma} \|f\|_{W^{\beta, 2}_4},
\]

for \( \alpha' < (\alpha - |a'|) \) and \( r_k \in (0, \infty) \). Moreover, taking now \( a \) with \( a_i = a'_i \) for \( i \neq k \), and \( a_k = 1 \), we get also from (5.16) the estimate

\[
(5.19) \quad \|\partial_x^\alpha G_{r_k}(f)\|_{L^2_{\alpha-\delta}} \leq \frac{c_k}{(1 + r_k)^\gamma} \|f\|_{W^{\beta, 2}_4},
\]

with \( \alpha' < (\alpha - |a'| - 1) \) and \( r_k \in (1 - \delta, 1 + \delta) \). Then we can apply Proposition 3.3 since condition (3.5) is given by (5.19) and condition (3.6) by (5.18) with \( C_1 = c_k + 1 C_2^{-1} \).

Therefore, for \( \alpha' < (\alpha - |a'|) \), we obtain that

\[
\|\partial_x^\alpha T_{k+1}(r_k, \ldots, r_{j-1})(f)\|_{L^2_{\alpha}}, = \quad \|(i\pi d_k + P_k)\partial_x^\alpha T_k(r_k, \ldots, r_{j-1})(f)\|_{L^2_{\alpha}}, \leq c_k \|f\|_{W^{\beta, 2}_4},
\]

where the first equality is true by Proposition A.2 in the Appendix. This concludes the proof of the claim. \(\square\)

We devote the remaining part of this section to prove Proposition 5.2. We define the operator

\[
K_{j,r}(g_1, \ldots, g_j)(\eta) = K_{j,r}(g_i)(\eta) := \frac{1}{|\eta|^{j-1}} \int_{\Gamma_r(\eta)} |g_1(\eta - \xi_1)| \prod_{i=1}^{j-2} |g_{i+1}(\xi_i - \xi_{i+1})| |g(\xi_{j-1})| \, d\sigma_r(\xi_1, \ldots, \xi_{j-1}),
\]
and \( \tilde{K}_{j,r}(g_1, \ldots, g_j)(\eta) := \chi(\eta)K_{j,r}(g_1, \ldots, g_j)(\eta) \). Hence we have that

\[
|\tilde{S}_{j,r}(q)(\eta)| \leq \left( \prod_{i=1}^{j-1} \frac{2}{1 + r_i} \right) \tilde{K}_{j,r}(q, \ldots, q)(\eta).
\]

The main tool to prove Proposition 5.2 is the following Lemma.

**Lemma 5.4.** Let \( n \geq 2 \) and \( j \geq 3 \), and consider \( f_i \in W^{\beta,2}_{2}(\mathbb{R}^n) \), \( 1 \leq i \leq j \) with \( \beta \geq 0 \). Then the estimate

\[
\|\tilde{K}_{j,r}(\hat{f}_1, \ldots, \hat{f}_j)\|_{L^2} \leq C \left( \prod_{i=1}^{j-1} (1 + r_i)^{\lambda} \right) \prod_{i=1}^{j-2} \|f_i\|_{W^{\beta,2}},\]

holds when \( \alpha \) is in the range given in (5.12) for some real number \( 0 < \lambda < 1 \).

**Proof.** Since

\[
\eta = (\eta - \xi_1) + \sum_{i=1}^{j-2} (\xi_i - \xi_{i+1}) + \xi_{j-1},
\]

if we fix \( c < 1/j \), one of the conditions \(|\eta - \xi_1| > c|\eta|, |\xi_i - \xi_{i+1}| > c|\eta|\) for some \( 1 \leq i \leq j - 2 \), or \( |\xi_{j-1}| > c|\eta| \) must hold. Hence, the sets

\[
A^1_r(\eta) = \{(\xi_1, \ldots, \xi_{j-1}) \in \Gamma_r(\eta) : |\eta - \xi_1| > c|\eta|\},
\]

\[
A^i_r(\eta) = \{(\xi_1, \ldots, \xi_{j-1}) \in \Gamma_r(\eta) : |\xi_i - \xi_1| > c|\eta|\}, \quad i = 2, \ldots, j - 1,
\]

\[
A^j_r(\eta) = \{(\xi_1, \ldots, \xi_{j-1}) \in \Gamma_r(\eta) : |\xi_{j-1}| > c|\eta|\},
\]

satisfy \( \Gamma_r(\eta) = \bigcup_{k=1}^{j} A^k_r(\eta) \). As a consequence

\[
\|\tilde{K}_{j,r}(\hat{f}_1, \ldots, \hat{f}_j)\|_{L^2} \leq \sum_{k=1}^{j} \|\tilde{K}^k_{j,r}(\hat{f}_1, \ldots, \hat{f}_j)\|_{L^2},
\]

where \( \tilde{K}^k_{j,r} \) is defined as \( \tilde{K}_{j,r} \) but integrating over \( A^k_r(\eta) \) instead of \( \Gamma_r(\eta) \).

From now on we fix

\[
\beta := \alpha - (j - 1)(1 - \lambda).
\]

In the region where \( \chi(\eta) \) does not vanish, \(|\eta| \sim < \eta >\), and hence

\[
\|\tilde{K}^k_{j,r}(\hat{f}_1, \ldots, \hat{f}_j)\|_{L^2}^2 \leq \int_{\mathbb{R}^n} \eta > 2^\beta |\eta|^{-2(j-1)\lambda} \left( \int_{A^k_r(\eta)} |\hat{f}_1(\eta - \xi_1)| \left( \prod_{i=1}^{j-2} |\hat{f}_i \xi_{i+1} - (\xi_{i+1})| \right) |\hat{f}_j(\xi_{j-1})| \right) d\sigma_r \ d\eta,
\]

where \( \lambda \) is a real parameter satisfying \( 0 < \lambda \leq (n - 1)/2 \), and \( d\sigma_r = d\sigma_r(\xi_1, \ldots, \xi_{j-1}) \).

The analysis is exactly the same for each \( \tilde{K}^k_{j,r} \), \( k = 1, \ldots, j \), so we only show one explicitly, for example the case \( k = j \).

If \( \beta > 0 \) we can use that \( \eta > \beta < C < \xi_{j-1} > \beta \) in \( A^k_r(\eta) \). Hence multiplying and dividing by \(|\eta - \xi_1|^{n-1-2\gamma} \prod_{i=1}^{j-2} |\xi_i - \xi_{i+1}|^{n-1-2\gamma} \) and applying Cauchy-Schwarz inequality, we get
the following point-wise estimate for the integrand of (5.23),

\[
< \eta >^{2\beta} |\eta|^{-2(j-1)\lambda} \left( \int_{\mathcal{A}_j'(\eta)} |\hat{f}_1(\eta - \xi_1)| \left( \prod_{i=1}^{j-2} |\hat{f}_{i+1}(\xi_i - \xi_{i+1})| \right) |\hat{f}_j(\xi_{j-1})| \, d\sigma_r \right)^2 \\
\leq |\eta|^{-2(j-1)\lambda} \int_{\Gamma_r(\eta)} |\hat{f}_1(\eta - \xi_1)|^2 |\eta - \xi_1|^{n-1-2\lambda} \left( \prod_{i=1}^{j-2} |\hat{f}_{i+1}(\xi_i - \xi_{i+1})|^2 |\xi_i - \xi_{i+1}|^{n-1-2\lambda} \right) \\
(5.24) \ldots \times |\hat{f}_j(\xi_{j-1})|^2 < \xi_{j-1} >^{2\beta} \, d\sigma_r \int_{\Gamma_r(\eta)} \frac{1}{|\eta - \xi_1|^{n-1-2\gamma}} \left( \prod_{i=1}^{j-2} |\xi_i - \xi_{i+1}|^{n-1-2\gamma} \right) d\sigma_r.
\]

Now, by Lemma 4.3 we have that

\[
|\eta|^{-2(j-1)\lambda} \int_{\Gamma_r(\eta)} \frac{1}{|\eta - \xi_1|^{n-1-2\gamma}} \left( \prod_{i=1}^{j-2} |\xi_i - \xi_{i+1}|^{n-1-2\gamma} \right) d\sigma_r \leq C \prod_{i=1}^{j-1} r_i^{2\lambda},
\]

where \( C \) is some constant independent of \( \eta \) (to see this, always compute first the integral in the variable \( \xi_i \) that only appears in one factor, in this case \( \xi_{j-1} \)). Hence, using this in (5.24) and integrating in the \( \eta \) variable we get the estimate

\[
\| \tilde{K}^j_{j,r}(\hat{f}_1, \ldots, \hat{f}_j) \|^2_{L^2} \leq C \prod_{i=1}^{j-1} r_i^{2\lambda} \left( \int_{\mathcal{R}_r(\eta)} |F(\xi_1, \ldots, \xi_{j-1}, \eta)|^2 \, d\sigma_r \, d\eta, \right.
\]

where

\[
F(\xi_1, \ldots, \xi_{j-1}, \eta) := \hat{f}_1(\eta - \xi_1) < \eta - \xi_1 >^{(n-1)/2-\lambda} \times \\
\left( \prod_{i=1}^{j-2} \hat{f}_{i+1}(\xi_i - \xi_{i+1}) < \xi_i - \xi_{i+1} >^{(n-1)/2-\lambda} \right) \hat{f}_j(\xi_{j-1}) < \xi_{j-1} >^{\beta}.
\]

Therefore, as in Lemma 4.2, we apply the Trace theorem to the integrals in \( \Gamma_r(\eta) \), to obtain

\[
\| \tilde{K}^j_{j,r}(\hat{f}_1, \ldots, \hat{f}_j) \|^2_{L^2} \leq C \left( \prod_{i=1}^{j-1} r_i^{2\lambda} \right) \times \ldots \]

\[
(5.26) \sum_{0 \leq |\alpha_1|, \ldots, |\alpha_{j-1}| \leq 1} \int_{\mathcal{R}_n} \ldots \int_{\mathcal{R}_n} |\partial_{\xi_1}^{\alpha_1} \ldots \partial_{\xi_{j-1}}^{\alpha_{j-1}} F(\xi_1, \ldots, \xi_{j-1}, \eta)|^2 \, d\xi_1 \ldots d\xi_{j-1} \, d\eta,
\]

where the \( \alpha_i \) are multi-indices related to the corresponding the \( \mathcal{R}_n \) variables \( \xi_i \), see Lemma A.1 in the Appendix for a more detailed formulation. Now, using the Leibniz rule in (5.26) we can put the derivative operators on the functions \( \hat{f}_i < \cdot >^\alpha \). In the worst case we are going to get terms of the kind

\[
\partial_{\xi_1}^{\alpha_1} \partial_{\xi_{i+1}}^{\alpha_{i+1}} \left( \hat{f}_{i+1}(\xi_i - \xi_{i+1}) < \xi_i - \xi_{i+1} >^\alpha \right),
\]

with at most two derivative operators having \( |\alpha_i| = |\alpha_{i+1}| = 1 \). Therefore, if we integrate each summand first in \( \eta \) and then in \( \xi_1, \xi_2, \ldots, \xi_{j-1} \), we obtain

\[
(5.27) \| \tilde{K}^j_{j,r}(\hat{f}_1, \ldots, \hat{f}_j) \|^2_{L^2} \leq C \left( \prod_{i=1}^{j-1} r_i^{2\lambda} \right) \| f_j \|^2_{W^{2,2}} \prod_{i=1}^{j-1} \| f_i \|^2_{W^{2(n-1)/2-\lambda,2}}.
\]
Putting together (5.27) and the analogous estimates coming from the analysis of the other cases in (5.21), we obtain

\[
\|\tilde{K}_{\delta,\gamma}(\tilde{f})\|_{W^\mu,2} \leq C \left( \prod_{j=1}^{n-1} r_j^{2\lambda} \right) \sum_{i=1}^{j} \|f_i\|_{W^\beta,2} \prod_{1 \leq \ell, j \leq n \atop \ell \neq i} \|f_j\|_{W^\gamma,2}.
\]

We reason as in Lemma 4.2, first we impose the extra condition \( \lambda < 1 \).

As a consequence of Remark 2.2, equation (5.21) follows directly in the range \( \beta \geq (n - 1)/2 \). The restrictions on \( \lambda \) and (5.22) give us the following restrictions on \( \alpha \),

\[
\begin{align*}
0 < \alpha < 1 \\
0 < \alpha \leq \frac{s-1}{2}
\end{align*}
\]

\[
\iff
\begin{align*}
\beta + (j - 2) < \alpha < \beta + (j - 1) \\
\beta + (j - 1) - \frac{s-1}{2} \leq \alpha < \beta + (j - 1).
\end{align*}
\]

We discard the lower bounds for \( \alpha \) as in Lemma 4.2.

Otherwise, if \( \beta \) is in the range \( \beta < (n - 1)/2 \), estimate (5.21) holds if we add the extra condition \( (n - 1)/2 - \lambda \leq \beta \). Then, since \( \lambda < 1 \), we must have \( \beta > (n - 3)/2 \) as in Lemma 4.2. Also, (5.22) together with \( (n - 1)/2 - \lambda \leq \beta \) imply that \( \alpha \leq \beta + (j - 1)(\beta - (n - 3)/2) \) which is a more restrictive condition than \( \alpha < \beta + (j - 1) \) since we have \( \beta < (n - 1)/2 \). Hence, we have obtained the ranges of parameters given in the statement.

\[\square\]

**Proof of Proposition 5.2.** By (5.20), estimate (5.10) follows directly if \( a = 0 \). Therefore, we consider the case \( a \neq 0 \).

Let \( \mathbf{r} = (r_1, \ldots, r_k), 1 \leq k < \infty \). Doing the same computation we did to obtain \( \partial_r S_r(\eta) \) in (4.11) from (4.10), we have that for a general function \( F(\xi_1, \ldots, \xi_k, \eta) \), \( C^1 \) in the first \( k \) variables,

\[
\partial_r \left( \frac{1}{1 + r_i} \int_{\Gamma_r(\eta)} F(\xi_1, \ldots, \xi_k, \eta) \, d\sigma_r(\xi_1, \ldots, \xi_k) \right) = \frac{(n - 2)r_i + (n - 1)}{r_i(1 + r_i)^2} \int_{\Gamma_r(\eta)} F(\xi_1, \ldots, \xi_k, \eta) \, d\sigma_r
\]

\[
+ |\eta| \frac{1}{1 + r_i} \int_{\Gamma_r(\eta)} \theta_i \cdot \nabla \xi_i F(\xi_1, \ldots, \xi_k, \eta) \, d\sigma_r,
\]

where \( \theta_i = \frac{\xi_i - \eta/|\xi_i - \eta/2|} {r_i} \) is a unitary vector. Observe that the coefficients before the integrals are functions of \( r_i \) which are bounded for \( r_i \in (1 - \delta, 1 + \delta) \) for any \( 0 < \delta < 1 \) fixed. Hence if we take a derivative \( \partial^a_r \) with \( a = (a_1, \ldots, a_k) \) and \( a_i = 0, 1 \), we have

\[
\left| \partial^a_r \left( \prod_{i=1}^{n-1} \frac{1}{1 + r_i} \int_{\Gamma_r(\eta)} F(\xi_1, \ldots, \xi_k, \eta) \, d\sigma_r \right) \right| \leq C \left( \prod_{i=1}^{n-1} \frac{1}{1 + r_i} \right) \sum_{0 \leq |\alpha_1| + \cdots + |\alpha_k|} |\eta|^{\alpha_1} \cdots |\eta|^{\alpha_k} \int_{\Gamma_r(\eta)} |\partial_{\xi_1}^{\alpha_1} \cdots \partial_{\xi_k}^{\alpha_k} F(\xi_1, \ldots, \xi_k, \eta)| \, d\sigma_r,
\]

where \( \alpha_i \) are multi-indices associated to derivatives in \( \mathbb{R}^a \), and we have imposed \( r_i \in (1 - \delta, 1 + \delta) \) if \( a_i = 1 \) (to bound the coefficients dependent on \( r_i \) as we did before). Notice that \( |\alpha_1| + \cdots + |\alpha_k| \) can take all the integer values from 0 to \( |a| \). We are interested in computing
Lemma 6.2. The proof can be found in [15] and for the compactly supported case in [24, pp. 182-183].

We need also a theorem of Zolesio on the product of functions in the Sobolev spaces (a proof can be found in [15] and for the compactly supported case in [24, pp. 182-183]).

Lemma 6.1. Let \( s \geq 0 \) and let \( r \) and \( t \) be such that \( 0 \leq 1/t - 1/2 \leq 1/(n+1) \) and \( 0 \leq 1/2 - 1/r \leq 1/(n+1) \). There exist \( \delta, \delta' > 0 \) and \( C \) (independent of \( k \)) such that

\[
\| R_0(q) \|_{W^{\delta, r}_{t}} \leq C k^{-1+(1/t-1/r)(n-1)/2} \| q \|_{W^{\delta', t}}.
\]

Lemma 6.2 (Zolesio). Let \( s_1, s_2, s \geq 0 \), \( s \leq s_1 \), \( s \leq s_2 \), and let \( r, t \) and \( p \) be such that \( t < \min(p, r) \) and

\[
s_1 + s_2 - s \geq n \left( \frac{1}{p} + \frac{1}{r} - \frac{1}{t} \right).
\]

Then

\[
\| qf \|_{W^{s, t}} \leq C \| q \|_{W^{s_1, p}} \| f \|_{W^{s_2, r}}.
\]
Moreover, if \( q \) is compactly supported and \( \delta, \delta' \in \mathbb{R} \), then
\[
\| qf \|_{W^{-\delta',r}} \leq C(\text{supp } q, \delta, \delta') \| q \|_{W^{\delta,r}} \| f \|_{W^{\delta',r}}.
\]

Proof of Proposition 2.1. For brevity, we will omit the dependency of the constants on the dimension \( n \). Without loss of generality assume \( q \in C_c^\infty(B_\rho) \), where \( B_\rho \) denotes the ball of radius \( \rho \). In terms of \( R_\theta \), defined in (6.1), the expression of \( Q_j \) given in (2.8) becomes
\[
Q_j(q)(\xi) = \int_{\mathbb{R}^n} e^{i2k\theta \cdot y}(qR_\theta)^{j-1}(q)(y) \, dy,
\]
with \( \xi = -2k\theta \). In spherical coordinates we can write
\[
\| Q_j(q) \|_{W^{\alpha,2}}^2 \leq \int_{C_0}^{\infty} k^{n-1+2\alpha-2\beta} \int_{S^{n-1}} \int_{\mathbb{R}^n} (-\Delta)^{\beta/2}(e^{i2k\theta \cdot y}) (qR_\theta)^{j-1}(q)(y) \, dy \, d\sigma(\theta) \, dk.
\]
Now, if \( f \) is a \( C^\infty_c(\mathbb{R}^n) \) function and \( \beta \geq 0 \), the fractional Laplacian (see for example [9, Section 3]) can be defined by the identity
\[
(\mathcal{F} \Delta^{\beta/2} f)(\xi) = |\xi|^{\beta} \hat{f}(\xi),
\]
and we have that in the sense of distributions
\[
(\mathcal{F} \Delta^{\beta/2} e^{i2k\theta \cdot x} = (2k)^\beta e^{i2k\theta \cdot x},
\]
see [31, chapter 2] for a rigorous extension to distributions of the fractional Laplacian. Hence applying this to the previous inequality, since \( (qR_\theta)^{j-1}(q) \in C_c^\infty(\mathbb{R}^n) \) we obtain
\[
\| Q_j(q) \|_{W^{\alpha,2}}^2 \leq C(\beta) \int_{C_0}^{\infty} k^{n-1+2\alpha-2\beta} \int_{S^{n-1}} \int_{\mathbb{R}^n} (-\Delta)^{\beta/2}(e^{i2k\theta \cdot y}) (qR_\theta)^{j-1}(q)(y) \, dy \, d\sigma(\theta) \, dk.
\]
Now, applying Lemma A.3 in the Appendix we have
\[
\| (-\Delta)^{\beta/2} ((qR_\theta)^{j-1}(q)) \|_{L^1} \leq C(\beta) \| q \|_{W^{\delta,2}} \| R_\theta ((qR_\theta)^{j-2}(q)) \|_{W^{\delta',2}}
\]
using that \( q \) is compactly supported. Now, choose \( \delta \) in the previous equation as in Lemma 6.1. The idea to deal with the norm in the right hand side is to iterate lemmas 6.1 and 6.2 following the diagram,
\[
\begin{align*}
W^{\beta,t_j-1} & \xrightarrow{R_\theta} W^{\beta,r_{j-1}} \xrightarrow{q} W^{\beta,r_{j-2}} \ldots \xrightarrow{q} W^{\beta,t_1} \xrightarrow{R_\theta} W^{\beta,r_1} \\
q & \xrightarrow{R_\theta(q)} qR_\theta(q) \ldots \xrightarrow{(qR_\theta)^{j-2}(q)} R_\theta((qR_\theta)^{j-2}(q))
\end{align*}
\]
where \( r_1 = 2 \) and \( t_{j-1} = 2 \) and \( r_\ell \) and \( t_\ell \), \( \ell = 1, \ldots, j-2 \) have to satisfy the conditions.
$$0 \leq \frac{1}{t_\ell} - \frac{1}{2} \leq \frac{1}{n+1}$$

and

$$0 \leq \frac{1}{2} - \frac{1}{r_{\ell+1}} \leq \frac{1}{n+1}.$$ 

Hence we obtain

$$\|R_0((qR_0)^{j-2}(q))\|_{W^{\beta,2}} \leq C^j(\beta, \rho) k^{\gamma_j} \|q\|_{W^{\beta,2}}^{j-1},$$

in (6.2) the constant depends on the support $B_\rho$ of $q$ where

$$\gamma_j = -(j-1) + \frac{(n-1)}{2} \sum_{\ell=1}^{j-1} \left( \frac{1}{t_\ell} - \frac{1}{r_\ell} \right)$$

$$= -(j-1) + \frac{(n-1)}{2} \sum_{\ell=1}^{j-2} \left( \frac{1}{t_\ell} - \frac{1}{r_{\ell+1}} \right).$$

Now, for small $\varepsilon > 0$, when $\beta \geq m = (n-4)/2 + 2/(n+1)$ we can choose $r_\ell$ and $t_\ell$ satisfying all the previous conditions and

$$1/t_\ell - 1/r_{\ell+1} = \max(1/2 - \beta/n, \varepsilon),$$

for all $1 \leq \ell \leq j-2$, and so we obtain

$$\gamma_j = -(j-1) + \frac{(n-1)}{2} (j-2) \max(1/2 - \beta/n, \varepsilon).$$

Putting all the previous estimates together in (6.4) we obtain (6.5)

$$\|\tilde{Q}_j(q)\|_{W^{\alpha,2}} \leq C^{2j}(\beta, \rho) \|q\|_{W^{\beta,2}}^j \int_{C_0}^{\infty} k^{\beta - 2} \|q\|_{W^{\beta,2}}^{j-1} dk = C^{2j}(\beta, \rho) C_0^{-2(\alpha_j - \alpha)} \|q\|_{W^{\beta,2}}^{j+2},$$

with $\alpha < \alpha_j$ and $\alpha_j = \beta + (j-1) - \frac{n}{2} - \frac{(n-1)}{2} (j-2) \max(0, 1/2 - \beta/n)$. By density, we can extend estimate (6.5) for $q \in W^{\beta,2}(\mathbb{R}^n)$ compactly supported in $B_\rho$. This follows from Lemma 5.3, with minor changes to take into account the restriction in the support.

Hence we can consider now $q \in W^{\beta,2}(\mathbb{R}^n)$ compactly supported in $B_\rho$. Choose some $\alpha > 0$. Now, for $\beta \geq m$, $\alpha_j$ grows linearly with $j$. Then, for any integer $l > 0$ such that $\alpha_l > \alpha$ we have by the previous estimates that

$$\left\| \sum_{j=l}^{\infty} \tilde{Q}_j(q) \right\|_{W^{\alpha,2}} \leq \sum_{j=l}^{\infty} \|\tilde{Q}_j(q)\|_{W^{\alpha,2}} \leq \sum_{j=l}^{\infty} C_0^{-(\alpha_j - \alpha)} C^j(\alpha, \beta, \rho) \|q\|_{W^{\beta,2}}^j,$$

Using the linear growth of $\alpha_j$ we can choose some $\varepsilon(\beta) = \varepsilon > 0$ such that for every $j \geq l$, $(\alpha_j - \alpha) \geq j \varepsilon$. Therefore we obtain that

$$\left\| \sum_{j=l}^{\infty} \tilde{Q}_j(q) \right\|_{W^{\alpha,2}} \leq \sum_{j=l}^{\infty} C_0^{-\varepsilon j} C^j(\alpha, \beta, \rho) \|q\|_{W^{\beta,2}}^j,$$

and the right hand side converges taking $C_0 > (C(\alpha, \beta, \rho) \|q\|_{W^{\beta,2}})^{1/\varepsilon}$.

$\square$
7. Some limitations on the regularity of the double dispersion operator

In this section we use a certain family of compactly supported radial and real functions, to obtain the upper bounds to the maximum regularity of the $Q_2$ operator given by Theorem 1.4. This family of functions was constructed in [17] to illustrate an analogous phenomenon in the fixed angle and full data scattering problems.

See also [8, pp. 20] for an explicit radial counterexample for the the case $\beta = (1/2)^-$ and $n = 3$.

Lemma 7.1 (Proposition 5.3 of [17]). For every $0 < \beta < \infty$ there is a radial, real and compactly supported function $g_\beta$ such that $\hat{g}_\beta$ is non negative, $\hat{g}_\beta(0) > 0$, and for some $c > 0$ we have that

\begin{equation}
\hat{g}_\beta(\xi) \sim \xi^{-n/2-\beta} \quad \text{if } |\xi| > c.
\end{equation}

Notice that $g_\beta \in W^{\gamma,2}$ if and only if $\gamma < \beta$. The construction of these functions is not difficult, the idea is to define

\begin{equation}
g_\beta(x) := (\phi * \phi)(x)G_\beta(x),
\end{equation}

where $\phi$ is any real and radial $C_c^\infty(\mathbb{R}^n)$ function and the $G_\beta$ functions are, up to normalizing factors, kernels of Bessel potential operators. Indeed, if we have that

\[ \hat{G}_\beta(\xi) := \xi^{-n/2-\beta}, \]

then $G_\beta(x)$ is a smooth and exponentially decaying function outside the origin (see, for example [33, Chapter V]). Hence, multiplying $G_\beta$ as in (7.2) by a $C_c^\infty(\mathbb{R}^n)$ cut off function, non vanishing at the origin, we get the desired asymptotic behaviour of $\hat{g}_\beta(\xi)$. The choice of the cut off $\phi * \phi$ guarantees the positivity of $\hat{g}_\beta(\xi)$ (see [17] for more details).

The key idea behind the proof of Theorem 1.4 is to study the asymptotic behavior of $|Q_2(\hat{g}_\beta)(\eta)|$ when $|\eta| \to \infty$. This is greatly simplified by the fact we have the explicit formula (3.3). Now, $g_\beta$ has a real Fourier transform $\hat{g}_\beta(\xi)$ by construction, so $Q_2(\hat{g}_\beta)$ has a real part given by the principal value term in (3.3) and an imaginary part given by $\pi S_{r=1}(g_\beta)$. As there is no possible cancellation between the real an imaginary parts, we are going to study only the asymptotic behavior of the spherical integral, which has the advantage of having a positive integrand. To simplify notation we put $S(q) := S_{r=1}(q)$ and $\Gamma(\eta) := \Gamma_{r=1}(\eta)$. The main estimate is the following one.

Lemma 7.2. Let $\beta > -n/2$ and assume that $q_\beta \in S'(\mathbb{R}^n)$ satisfies the following conditions,

i) Its Fourier transform $\hat{q}_\beta(\xi)$ is real and non negative function in all $\mathbb{R}^n$.

ii) There is a constant $c > 0$ such that if $|\xi| > c$, then $\hat{q}_\beta(\xi) \geq C < \xi >^{-n/2-\beta}$.

iii) $\hat{q}_\beta(\xi)$ is continuous and satisfies $\hat{q}_\beta(0) > 0$.

Then we have that if $|\eta| > 4c$, there is a constant $C$ independent of $\eta$ such that

\[ S(q_\beta)(\eta) \geq C \max \left( < \eta >^{-n/2-1}, < \eta >^{-2\beta-2} \right). \]

Proof. Since $\hat{q}_\beta$ is non negative, we have that

\begin{equation}
S(q_\beta)(\eta) \geq \frac{1}{|\eta|} \int_{A(\eta)} \hat{q}_\beta(\xi)\hat{q}_\beta(\eta - \xi) d\sigma_\eta(\xi),
\end{equation}

where $A(\eta) := \{ \xi \in \mathbb{R}^n : |\xi - \eta| < 2c \}$. Since $\hat{q}_\beta(\xi)$ is non negative and $\hat{q}_\beta(0) > 0$, the integral in the right hand side of (7.3) is non negative, and hence

\[ S(q_\beta)(\eta) \geq \frac{1}{|\eta|} \int_{A(\eta)} \hat{q}_\beta(\xi)\hat{q}_\beta(\eta - \xi) d\sigma_\eta(\xi), \]

where $A(\eta) := \{ \xi \in \mathbb{R}^n : |\xi - \eta| < 2c \}$. Since $\hat{q}_\beta(\xi)$ is non negative and $\hat{q}_\beta(0) > 0$, the integral in the right hand side of (7.3) is non negative, and hence

\[ S(q_\beta)(\eta) \geq \frac{1}{|\eta|} \int_{A(\eta)} \hat{q}_\beta(\xi)\hat{q}_\beta(\eta - \xi) d\sigma_\eta(\xi), \]

where $A(\eta) := \{ \xi \in \mathbb{R}^n : |\xi - \eta| < 2c \}$.
where, if we write $\eta = |\eta|\theta$ with $\theta$ a unitary vector, $A(\eta) \subset \Gamma(\eta)$ is defined as follows

$$A(\eta) := \{ \xi \in \Gamma(\eta) : |(\xi - \eta/2) \cdot \theta| \leq |\eta|/4 \}.$$

That is, $A(\eta)$ is a band around the equator orthogonal to $\eta$ of width proportional to $|\eta|$. (see figure 2). Observe that we have that $\xi \in A(\eta)$ if and only if $\eta - \xi \in A(\eta)$, and that in this region $|\xi| \geq |\eta|/4$. Hence, if we consider $|\eta| > 4c$ (where $c$ is given in the statement) and $\xi \in A(\eta)$, we have that $|\xi| > c$ and $|\eta - \xi| > c$, so from (7.3) we get

$$S(q_\beta)(\eta) \geq C \frac{1}{|\eta|} \int_{A(\eta)} \eta - \xi > -\beta - n/2 \eta - \xi > -\beta - n/2 \ d\sigma_\eta(\xi)$$

$$\geq C < \eta > -2\beta - n |\eta|^{n-2} \eta > -2\beta - 2,$$

where to get the last line we have used that the measure of $A(\eta)$ is proportional to $|\eta|^{n-1}$, and that $|\xi| \leq |\eta|$ and $|\eta - \xi| \leq |\eta|$ always hold in $\Gamma(\eta)$.

Now, if $\tilde{q}_\beta$ is continuous and $\tilde{q}_\beta(0) > 0$, we can take a ball $B_\varepsilon$ around the origin of radius $0 < \varepsilon < c$ such that $\tilde{q}_\beta(\xi)$ is positive in its closure. Then if $|\eta| > 2c$, $\xi \in B_\varepsilon \cap \Gamma(\eta)$ implies $|\eta - \xi| > c$, so

$$S(q_\beta)(\eta) \geq C \frac{1}{|\eta|} \int_{B_\varepsilon \cap \Gamma(\eta)} \tilde{q}_\beta(\xi) \tilde{q}_\beta(\eta - \xi) \ d\sigma_\eta(\xi)$$

$$\geq C \frac{1}{|\eta|} \int_{B_\varepsilon \cap \Gamma(\eta)} < \eta - \xi > -\beta - n/2 \ d\sigma_\eta(\xi) \geq C < \eta > -\beta - n/2 - 1,$$

(7.4)

using that $|\eta - \xi| \leq |\eta|$ always, and that the measure $|B_\varepsilon \cap \Gamma(\eta)|$ is bounded below by a positive constant independent of $\eta$ (this is because the region $B_\varepsilon \cap \Gamma(\eta)$ approaches a flat disc of radius $\varepsilon$ for $\eta$ large). To finish we have just to put together (7.3) and (7.4). □

The proof of Theorem 1.4 follows from the previous lemma and the following simple result.

\[\text{Figure 2. The largest sphere is the Ewald sphere } \Gamma(\eta) := \Gamma_1(\eta), \text{ and the small one represents the Ewald sphere } \Gamma_r(\eta) \text{ for some } r < 1. \text{ The dashed region is the set } A(\eta) \subset \Gamma(\eta).\]
Lemma 7.3. Let \( f \in S'(\mathbb{R}^n) \) be such that \( \hat{f} \) is a non negative measurable function. Assume also that for some \( c > 0, \gamma \in \mathbb{R} \) and \( |\eta| > c \) we have \( \hat{f}(\eta) \geq C < \eta >^{n/2 - \gamma} \). Then we have that \( f \notin W^{\alpha, 2}_{\text{loc}}(\mathbb{R}^n) \) if \( \alpha \geq \gamma \).

Proof. We can always take a function \( \psi \in C_0^\infty(\mathbb{R}^n) \) such that \( \hat{\psi}(\xi) \geq 0 \) in \( \mathbb{R}^n \) and \( \hat{\psi}(0) > 0 \) (for example, is enough to choose \( \psi = \phi * \phi \) with \( \phi \in C_0^\infty(\mathbb{R}^n) \) radial and real, as in the definition of \( g_\beta \)). Then we can take an \( 0 < \varepsilon < c/2 \) small such that \( \hat{\psi}(\xi) \) is bounded below by a positive constant in \( B_\varepsilon \). Hence if \( |\eta| \geq 2c \),

\[
\hat{\psi} \hat{f}(\eta) = \int_{\mathbb{R}^n} \hat{\psi}(\xi) \hat{f}(\eta - \xi) d\xi \\
\geq \int_{B_\varepsilon} \hat{\psi}(\xi) \hat{f}(\eta - \xi) d\xi \geq C < \eta >^{n/2 - \gamma}.
\]

As a consequence we have that \( \psi \hat{f} \notin W^{\alpha, 2}(\mathbb{R}^n) \) for \( \alpha \geq \gamma \), which implies that \( f \notin W^{\alpha, 2}_{\text{loc}}(\mathbb{R}^n) \) by definition of the local Sobolev spaces. \( \square \)

Proof of Theorem 1.4. By Lemma 7.1 the function \( g_\beta \) satisfies all the conditions necessary to apply Lemma 7.2, so for \( \eta \) large we have

\[
S(g_\beta)(\eta) \geq C \max \left( < \eta >^{-\beta - n/2 - 1}, < \eta >^{-2\beta - 2} \right).
\]

By (3.3), we have that

\[
Q_2(g_\beta)(\eta) = P(S_r(g_\beta))(\eta) + i\pi S(g_\beta)(\eta),
\]

and \( \tilde{g}_\beta \) is real, so \( P(S_r(g_\beta)) \) and \( S(g_\beta) \) are real functions of \( \eta \) also. This means that if we assume \( Q_2(g_\beta) \in W^{\alpha, 2}_{\text{loc}}(\mathbb{R}^n) \), we must have \( \mathcal{F}^{-1}(S(g_\beta)) \in W^{\alpha, 2}_{\text{loc}}(\mathbb{R}^n) \), since there are no possible cancellations between the real and imaginary parts in (7.6).

As a consequence of (7.5), applying Lemma 7.3 with \( f = \mathcal{F}^{-1}(S(g_\beta)) \) we obtain that \( \alpha \) must satisfy simultaneously \( \alpha < \beta + 1 \) and \( \alpha < 2\beta + (n - 4)/2 \).

Hence, we have shown that for every \( 0 < \beta < \infty \) there is a radial, real and compactly supported function \( g_\beta \) such that \( g_\beta \in W^{\alpha, 2} \) if and only if \( \gamma < \beta \), but we have that \( Q_2(g_\beta) \in W^{\alpha, 2}_{\text{loc}}(\mathbb{R}^n) \) only if \( \alpha < \min(\beta + 1, 2\beta - (n - 4)/2) \). This enough to conclude the proof. \( \square \)

Appendix A. Some technical results

In this section we give the proofs of some technical results used throughout this work.

Lemma A.1 (Trace theorem). Let \( \xi_1, \ldots, \xi_k \in \mathbb{R}^n \), and \( \mathbf{r} \in (0, \infty)^k \). Assume that for every \( \eta \in \mathbb{R}^n \) the function \( F(\xi_1, \ldots, \xi_k, \eta) \) is \( C^1 \) in the first \( k \) variables. Then if \( \alpha_1, \ldots, \alpha_k \) are multi indices corresponding to the variables \( \xi_1, \ldots, \xi_k \) we have that

\[
\int_{\Gamma_\mathbf{r}(\eta)} |F(\xi_1, \ldots, \xi_k, \eta)|^2 \, d\sigma_\mathbf{r}(\xi_1, \ldots, \xi_k) \\
\leq C \sum_{0 \leq |\alpha_1|, \ldots, |\alpha_k| \leq 1} \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} |\partial_{\xi_1}^{\alpha_1} \cdots \partial_{\xi_k}^{\alpha_k} F(\xi_1, \ldots, \xi_k, \eta)|^2 \, d\xi_1 \cdots d\xi_k,
\]

where the constant \( C \) does not depend on \( \eta \), or \( \mathbf{r} \).
Proof. The general case follows inductively using that, as mentioned in the proof of Lemma 4.2, the estimate

\[
(A.1) \quad \int_{S_\rho} |f(x)|^2 \, d\sigma_\rho(x) \leq \int_{\mathbb{R}^n} |f(x)|^2 \, dx + \int_{\mathbb{R}^n} |\nabla f(x)|^2 \, dx,
\]

holds for any sphere $S_\rho \subset \mathbb{R}^n$ of radius $\rho$ and (Lebesgue) measure $\sigma_\rho$, see [17, Proposition A.1] for an elementary proof of this result. \hfill \square

We give now the proof of Lemma 5.3, introduced in section 5.

**Proof of Lemma 5.3.** Let $\{q_i\}_{i \in \mathbb{N}}$, $q_i \in C^\infty_c(\mathbb{R}^n)$ for every $i \in \mathbb{N}$, be a Cauchy sequence of functions in the $W_\delta^{2,p}(\mathbb{R}^n)$ norm. To prove the proposition is enough to show that then $\{\tilde{Q}_{ij}(q_i)\}_{i \in \mathbb{N}}$ is also a Cauchy sequence in $W^{\alpha,2}(\mathbb{R}^n)$, since this implies that there must be a unique continuous extension of $\tilde{Q}_{ij}$ to the whole space $W_\delta^{2,p}(\mathbb{R}^n)$.

First we introduce the symmetric part of $\tilde{Q}_{ij}$, defined as usual by the formula

\[
\tilde{Q}_{j,S}(f_1, \ldots, f_j) := \frac{1}{j!} \sum_{\sigma} \tilde{Q}_{j}(f_{\sigma(1)}, \ldots, f_{\sigma(j)}),
\]

where the sum is over all the permutations $\sigma$ of $j$ elements. We remind that $\tilde{Q}_{j}(f_1, \ldots, f_j)$ is defined by putting $f_i$ instead of $q$ in (2.8), following the order of appearance of the potentials in the formula. By definition we have that $\tilde{Q}_{j,S}(q) = \tilde{Q}_{j}(q)$, and hence

\[
(A.2) \quad \tilde{Q}_{j}(q_k) - \tilde{Q}_{j}(q_l) = \tilde{Q}_{j,S}(q_k) - \tilde{Q}_{j,S}(q_l)
\]

\[= \tilde{Q}_{j,S}(q_k - q_l, q_k, \ldots, q_k) + \tilde{Q}_{j,S}(q_l, q_k - q_l, q_k, \ldots, q_k) + \cdots + \tilde{Q}_{j,S}(q_l, \ldots, q_l, q_k - q_l).
\]

Since $\tilde{Q}_{j,S}$ is symmetric, we can use a polarization identity to express each of the previous terms as combinations of diagonal terms. See [34] for the explicit derivation of the following polarization identity:

\[
j! \tilde{Q}_{j,S}(f_1, \ldots, f_k) = \sum_{m=1}^{j} (-1)^{j-m} \sum_{J, |J|=m} \tilde{Q}_{j}(S_J),
\]

where the inner sum in the right hand side is over all distinct subsets $J \subset \{1, 2, \ldots, j\}$ of $m$ elements, and $S_J = \sum_{i \in J} f_i$. Since each term in the last line of (A.2) can be treated in the same way, we illustrate only one case. Let $h > 0$ be a (small) constant that we will choose later. Then the polarization identity can be written in the following way

\[
\tilde{Q}_{j,S}(q_l, q_k - q_l, q_k, \ldots, q_k) = \tilde{Q}_{j,S}(hq_l, h^{-(j-1)}(q_k - q_l), hq_k, \ldots, hq_k)
\]

\[= \frac{1}{j!} \sum_{0 \leq a+b+c \leq j} (-1)^{j-1-(a+b+c)} N(a, b, c) \tilde{Q}_{j}(ah^{-(j-1)}(q_l - q_k) + h(bq_l + cq_k)),
\]

where $a, b, c$ are integers satisfying $0 \leq a, b \leq 1$, since $q_l$ and $q_k - q_l$ appear only once in the term we have chosen, and $0 \leq c \leq j - 2$ since $q_k$ appears $j - 2$ times (the integer coefficient $N(a, b, c)$ is just to account for repetitions).
Proposition A.2. Let\(\{q_i\}\) be a Cauchy sequence, it is bounded, so \(\|q_i\|_{W^{\alpha, 2}} \leq M\) for every \(i \in \mathbb{N}\) and some constant \(M > 0\). Hence, taking the \(W^{\alpha, 2}(\mathbb{R}^n)\) norm and using estimate (5.13) we obtain
\[
\| \tilde{Q}_j(q_i q_k - q_i q_k, \ldots, q_k) \|_{W^{\alpha, 2}} \leq \frac{1}{j!} \sum_{0 < a + b + c \leq j} N(a, b, c) \| \tilde{Q}_j(ah^{-j-1}(q_i - q_k) + h(bq_i + cq_k)) \|_{W^{\alpha, 2}}
\]
(A.3)
\[
\leq C(j) \left( h^{-j-1}\|q_i - q_k\|_{W^{\alpha, 2}}^j + h^j M^j \right) < \varepsilon / j,
\]
for the following choices (\(C(j)\) is some constant dependent only on \(j\)),
\[
h^j < \frac{\varepsilon}{2jC(j)M^j}, \text{ and } \|q_i - q_k\|_{W^{\alpha, 2}} < \frac{\varepsilon}{2jC(j)M^j-1}.
\]
So, using (A.3) for each term in (A.2) we finally obtain
\[
\| \tilde{Q}_j(q_i) - \tilde{Q}_j(q_k) \|_{W^{\alpha, 2}} < \varepsilon,
\]
which shows that \(\{\tilde{Q}_j(q_i)\}_{i \in \mathbb{N}}\) is a Cauchy sequence in \(W^{\alpha, 2}\).

Proposition A.2. Let \(f \in S\) and \(1 \leq k \leq j\). Then we have that
\[
T_k(f)(\eta) = \prod_{i=1}^{k-1} (i\pi d_i + P_i) \tilde{S}_{j, r}(f)(\eta),
\]
is a well defined function in \(S(\mathbb{R})\) (that is, the case \(k = 0\)). First any derivative in the \(\eta\) or \(\xi\) variables can be computed as in (4.10) (see also (5.29)). The \(|\eta|\) factors appearing in the expression of \(\tilde{S}_{j, r}\) and its derivatives are non-smooth for \(\eta = 0\), but this is not a problem since we have the smooth cut-off \(\chi(\eta)\) which vanishes in the origin. Essentially the estimate of each Schwartz class seminorm can be reduced to the basic case
\[
< \eta >^\gamma \int_{\Gamma_r(\eta)} |f(\eta - \xi_i)| \left( \prod_{i=1}^{j-2} |f(\xi_i - \xi_{i+1})| \right) |f(\xi_{j-1})| \, d\sigma_r \leq \frac{C}{|\eta|^{n-1}} \ldots
\]
\[
\int_{\Gamma_r(\eta)} |f(\eta - \xi_i)| < \eta - \xi_i >^{\gamma'} \left( \prod_{i=1}^{j-2} |f(\xi_i - \xi_{i+1})| < \xi_i - \xi_{i+1} >^{\gamma'} \right) |f(\xi_{j-1})| < \xi_{j-1} >^{\gamma'} \, d\sigma_r
\]
\[
\leq \| f < \cdot >^{\gamma'} \|_{\infty},
\]
where \(\gamma' = \gamma + n - 1\). If instead of a weight \(< \eta >^\gamma\) we have \(< r >^\gamma\), an analogous procedure can be followed using that \(r_i = 2|\xi_i - \eta/2|/|\eta| \leq C(1 + |\xi_i|)\) for \(|\eta| > C_0\), that is, where the cut-off \(\chi(\eta)\) does not vanish.

We give the following indications to prove (A.4) and that \(T_k(f) \in S(\mathbb{R})\) for \(k > 0\). Let \(g \in S((0, \infty)^k)\) be a function of the variable \(r \in (0, \infty)^k\). It is not very difficult to bound the principal value operators in the Schwartz class since we have the estimate
\[
\| P_i(g) \| \leq C(\| \partial_r g \|_{\infty} + \| r_i >^{\varepsilon} \|_{\infty}),
\]
for $\varepsilon > 0$. This implies that $\partial_{\tau_n} P_1(q) = P_1(\partial_{\tau_n} g)$ since the limit that defines the derivative $\partial_{\tau_n}$ is continuous in the norms of the right hand side (this is a consequence of the mean value theorem together with the fact that $g \in S((0, \infty)^k)$ which means that all the derivatives are uniformly bounded). The same reasoning can be applied to control the partial derivatives in $\eta$ of $T_k(f)$. \hfill \square

We state now Lemma A.3, used in the proof of Proposition 2.1.

**Lemma A.3.** Let $f, g \in C_c^\infty(\mathbb{R}^n)$, then if $\beta \geq 0$ we have that

$$\|(-\Delta)^{\beta/2}(fg)\|_{L^1} \leq C(\beta)\|f\|_{W^{\beta,2}}\|g\|_{W^{\beta,2}}.$$ 

**Proof.** Assume first that $0 < \beta < 2$ (the case of $\beta = 0$ is trivial). Then we have the pointwise relation (it can be computed by hand using the principal value formula of the fractional Laplacian see for example [5, p. 636])

$$(-\Delta)^{\beta/2}(fg)(x) = f(x)(-\Delta)^{\beta/2}(g)(x) + g(x)(-\Delta)^{\beta/2}(f)(x) + \int_{\mathbb{R}^n} \frac{(f(x) - f(y))(g(x) - g(y))}{|x - y|^{n+\beta}} \, dy,$$

where we need $0 < \beta < 2$ so that the singularity in the last integral can be controlled. Since by (6.3) we have that $\|(-\Delta)^{\beta/2}u\|_{L^2} \leq \|u\|_{W^{\beta,2}}$, taking the $L^1(\mathbb{R}^n)$ norm and applying Cauchy-Schwarz inequality to the first two terms we obtain

$$\|(-\Delta)^{\beta/2}(fg)\|_{L^1} \leq 2\|f\|_{W^{\beta,2}}\|g\|_{W^{\beta,2}} + \ldots$$

$$\ldots \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \frac{(f(x) - f(y))(g(x) - g(y))}{|x - y|^{n+\beta}} \, dy \right| \, dx.$$ 

But the last can be bounded using Cauchy Schwarz and that

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^2}{|x - y|^{n+\beta}} \, dy \, dx \leq C\|f\|_{W^{\beta/2,2}}^2,$$

(in fact, the left hand side is an equivalent norm for the homogeneous $W^{\beta/2,2}$ when $0 < \beta < 2$, see [9, Proposition 3.4]).

If we assume now that $\beta \geq 2$, define $k := [\beta/2]$, that is the integer part, and $\tilde{\beta} = \beta - 2k$ so we have now $\tilde{\beta} \in [0, 2)$, and

$$(-\Delta)^{\beta/2}(fg) = (-\Delta)^{\tilde{\beta}/2}(-\Delta)^k(fg).$$

An integer power of the Laplacian is an homogeneous constant coefficient differential operator of order $2k$, and therefore if $a, b \in \mathbb{N}^n$ we have

$$(-\Delta)^k(fg) = \sum_{|a| + |b| = 2k} c_{a,b} \partial^a \partial^b g,$$

where we are not interested in the particular values of the constants $c_{a,b}$. Then, to bound the $L^1(\mathbb{R}^n)$ norm of (A.5) we can apply the same arguments as before so we obtain

$$\|(-\Delta)^{\beta/2}(fg)\|_{L^1} \leq \sum_{|a| + |b| = 2k} |c_{a,b}|\|\partial^a f\|_{W^{\beta,2}}\|\partial^b g\|_{W^{\beta,2}} \leq C\|f\|_{W^{\beta,2}}\|g\|_{W^{\beta,2}},$$

using that $\|\partial^a f\|_{W^{\beta,2}} \leq \|f\|_{W^{\beta + |a|,2}} \leq \|f\|_{W^{\beta,2}}$ since $|a| \leq 2k$. \hfill \square
Further Remarks

In the introduction we have seen that there is a gap between the negative and positive results of recovery of singularities given in Theorems 1.1 and 1.2. This is a consequence of Theorems 1.3 and 1.4, where essentially the same gap is manifested in the results concerning the $Q_2$ operator. It appears in the range $(n-4)/2 < \beta < (n-2)/2$ (see for example figure 1 for the case $n = 4$). What happens in this range is not known except for some partial results in dimension 2 and 3. In [29, Proposition 3.1] a $1/2$ derivative gain for $\beta < 1/2$ is given in dimension 3 using finer properties of the structure of the Ewald spheres. In this work, thanks to the Trace theorem, we have not used any special properties of the spheres $\Gamma_r(\eta)$ in the estimates of the $Q_2$ operator. This suggest that there is an opportunity to improve the positive results in order to narrow this gap. Another possible strategy that we have already mentioned, is to choose a weaker scale for measuring the regularity of the $Q_2(q)$ operator. This is the approach of [2] in dimension 2, where they show that, if $\Lambda^\alpha(\mathbb{R}^2)$ denotes the H"older class and $q \in W^{\beta, 2}(\mathbb{R}^2)$, $\beta \geq 0$, then, modulo a $C^\infty$ function, $q - q_B \in \Lambda^\alpha(\mathbb{R}^2)$ for every $\alpha < \beta$. This is a $1^{-}$ derivative gain in the sense of integrability. A similar result holds also in dimension $n \geq 3$ and this will be the subject of a forthcoming work.

A similar problem is what happens in the limiting case $\alpha = \beta + 1$, when $\beta \geq (n-1)/2$. It is not difficult to show, modifying slightly the proof of lemma 4.2, that there is a whole $1$ derivative gain when $\beta > (n-1)/2$ for the spherical operator $S_r$. Unfortunately, it is not possible to say the same about the principal value operator, since in the estimate of the $P_{G, B}$ term in Proposition 3.3 is necessary to sacrifice an $\varepsilon$ of the regularity of the spherical operator (hence the final estimate for $\alpha' < \alpha$). Also, since this term involves cancellations, it is difficult to determine if this is a limitation of the techniques, or if it is possible to construct a counterexample.

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