More Communication with Less Entanglement

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We exhibit the intriguing phenomena of “Less is More” using a set of multipartite entangled states. We consider the quantum communication protocols for the exact teleportation, superdense coding, and quantum key distribution. We find that sometimes less entanglement is more useful. To understand this phenomena we obtain a condition that a resource state must satisfy to communicate a n-qubit pure state with m terms. We find that the an appropriate partition of the resource state should have a von-Neumann entropy of $\log_2 m$. Furthermore, it is shown that some states may be suitable for exact superdense coding, but not for exact teleportation.

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I. INTRODUCTION

The entanglement has been used as a quantum resource to carry out a number of communication tasks such as teleportation [1], superdense coding [2], teleportation, [2], secret sharing [3], quantum-key distribution (QKD) [4–7]. The entangled resource state may have bipartite or multipartite entanglement. A number of protocols which were first introduced in the context of a bipartite system can be extended to a multipartite system. However, in the case of multipartite systems, the nature of entanglement is still not fully understood [8]. Furthermore, there can be many variants and extensions of such protocols in this new setting. The multipartite entangled states can be classified according to various schemes [9–11]. Different classes exhibit different types of entanglement properties. The three-qubit states have been classified according to stochastic local operation and classical communication (SLOCC) into six categories. Two of these categories have genuine tripartite entanglement, viz. GHZ-states and W-states [12]. The utilities of these states have been explored in a number of papers [13, 14]. The four-qubit states have also been classified according to SLOCC [15]. There are nine categories. Some of them have genuine quadrupartite entanglement. We use these states to explore some of the variations of the bipartite protocols. Such studies may even allow a better understanding of multiparticle entanglement and classification of quantum states according to their ability to carry out a specific task. SLOCC classification is not a useful guide for the suitability of a state to be a resource state for a specific task [14].

In this paper, we are interested in the exact teleportation, superdense coding and quantum key distribution. These tasks may not be carried out maximally. A multipartite state allows many possibilities. A resource state is suitable for exact teleportation if it can be used to teleport a n-qubit state with m terms, for some m and n, with unit probability and unit fidelity. By exact superdense coding we mean the ability to communicate n + 1 or higher integral values of classical bits by transmitting n qubits. For the case of quantum key distribution, one should be able to generate a key using the resource state. We note that to carry out a task maximally, one would need a task-oriented maximally entangled state (TMES) [16]. Such states can be suitable for maximal teleportation - an n-qubit state would allow us to teleport an unknown arbitrary $2^n$-qubit state, when n is even and $(n-1)$-qubit state, when n is odd. For maximal superdense coding, one would need a TMES that would allow us to transmit n classical bits of information by sending $n/2$ qubits when n is even and $(n+1)/2$ qubits when n is odd. A prescription was given in [11] to construct such TMESs.

In this Brief Report, we study the protocols in the context of a specific set of genuinely quadripartite entangled states. We explore the possibility of teleporting an unknown one-qubit, two-qubit, and three-qubit states. We shall study only conventional teleportation. More interesting situations involving multi-measurements or more than two parties have also been discussed [17]. In exploring the usefulness of a state, we use the notion of von Neumann entropy of the subsystems. We find the value of entropy that a bipartite partition must have for the protocol to succeed. We also use these states to explore the possibility of exact superdense coding and quantum key distribution.

In the next section, we discuss the four-qubit states that we use to explore the viability of a protocol and illustrate some intriguing features. In section 3, we discuss the usefulness of the states for exact teleportation. We obtain a condition to determine the usefulness of a resource state. In section 4, we discuss exact superdense coding and show that some states that are suitable for exact superdense coding are not suitable for exact teleportation. We also see a connection between the success of the protocol and the von Neumann entropy of the subsystems. In section 5, we discuss the protocol of quantum key distribution. Finally we conclude in section 6.

II. FOUR-QUBIT ENTANGLED STATES AND THE VON NEUMANN ENTROPY

Some bipartite and the genuine tripartite entangled states can be used successfully to carry out the protocols of exact teleportation and superdense coding. We consider the following inequivalent quadripartite entangled states in terms of
SLOCC [15].

\[ |GHZ\rangle = \frac{1}{\sqrt{2}} (|0000\rangle + |1111\rangle), \]  
\[ |W\rangle = \frac{1}{2} (|0001\rangle + |0010\rangle + |0100\rangle + |1000\rangle), \]  
\[ |\Omega\rangle = \frac{1}{2} (|0000\rangle + |0110\rangle + |1001\rangle - |1111\rangle), \]  
\[ |S_1\rangle = \frac{1}{2} (|0000\rangle + |0101\rangle + |1000\rangle + |1110\rangle), \]  
\[ |S_2\rangle = \frac{1}{2} (|0000\rangle + |1011\rangle + |1101\rangle + |1110\rangle). \] (5)

Among these five entangled states only \(|GHZ\rangle\) and \(|W\rangle\) states are symmetric with respect to the permutation of qubits; thus any quantum information task performed using these states are independent of distribution of particles among the parties. There are a number of ways to see that the above states have genuine multipartite entanglement. One way is to find out the states of the subsystems after tracing out one, two or three particles. If the state is mixed in each case, then this would be an indication of genuine quadrpartite entanglement. This is what we observe. The \(|\Omega\rangle\) state is the cluster state introduced by Briegel and Raussendorf [18]. This state is considered to have maximum connectedness and high persistence of entanglement and has been discussed extensively in the context of one-way quantum computation.

It will be useful to catalog the von Neumann entropy (henceforth, called entropy) of the subsystems obtained by all possible bipartite partitions of the four-qubit system. In a bipartite partition, both subsystems will have identical entropies. Therefore, if we split the system in particle 1 on one hand and particles 2, 3 and 4 on the other, then the entropy of the particle 1, \(S(p_1)\) will be the same as that of particle 234 subsystem, \(S(p_{234})\). These entropies are given in Table I.

| States | \(S(p_1)\) | \(S(p_2)\) | \(S(p_23)\) | \(S(p_3)\) | \(S(p_34)\) | \(S(p_{234})\) |
|--------|-------------|-------------|-------------|-------------|-------------|-------------|
| GHZ    | 1           | 1           | 1           | 1           | 1           | 1           |
| \(\Omega\) | 1           | 1           | 1           | 1           | 1           | 1           |
| \(W\)  | 0.81        | 0.81        | 0.81        | 0.81        | 1           | 1           |
| \(S_1\) | 0.81        | 1           | 0.81        | 1.5         | 1.22        | 1.22        |
| \(S_2\) | 0.81        | 1           | 1           | 1.5         | 1.5         | 1.5         |

Table I: Entropies of the subsystems

In our analysis of above four-qubit states, from the point of view of teleportation and superdense coding, these entropies will play an important role [16]. In fact, we will see that the possible success of various protocols depends on the numerical values of these entropies.

### III. TELEPORTATION AND ENTROPY

With a four-qubit resource state, one can look for teleporting one-qubit, two-qubit, or three-qubit unknown states with varying number of terms. As we shall see, usefulness of a resource state would depend on the entropies of its subsystems. For maximal teleportation, a subsystem must have maximal entropy. For non-maximal teleportation, smaller entanglement is enough.

#### A. Teleportation of a Single-Qubit State

In this scenario, Alice wishes to teleport an unknown qubit state \(|\psi\rangle_a = \alpha|0\rangle_a + \beta|1\rangle_a\) to Bob with a four-qubit entangled resource state. The particles of the states are distributed in such a way that Bob will have one qubit and Alice the rest. To find the usefulness of a resource state, one can look at the entropies of the subsystems. For any type of teleportation to succeed, the entropy of the Bob’s qubit needs to be one. In that case, any resource state can be written as \(|R\rangle = (|\psi_0\rangle|0\rangle + |\psi_1\rangle|1\rangle)/\sqrt{2}\). Here \(|\psi_i\rangle\) are orthonormal vectors. We note that Alice, with prior understanding, could encode in the cbits either the results of her measurements or the unitary operations that Bob should apply to his qubit.

Case-I. The \(|GHZ\rangle\) and \(|\Omega\rangle\) states: It is known that with the \(|GHZ\rangle\) state the teleportation protocol will work for any distribution of the particles. We can understand this success as all individual qubits of a \(|GHZ\rangle\) state has entropy as one. The \(|\Omega\rangle\) state is not symmetric under the permutations of qubits. Still, the teleportation would succeed regardless of which qubit Bob has because all individual qubits has entropy as one. For example, let Alice has the qubits 1, 2, 3 and Bob has the qubit 4. The combined state of the the five qubits can be rewritten as

\[ |\psi\rangle_a|\Omega\rangle_{1234} = [|\Omega^+_a\rangle_{a123}\sigma_0|\psi\rangle_4 + |\Omega^-_a\rangle_{a123}\sigma_3|\psi\rangle_4 + |\Omega^+_a\rangle_{a123}\sigma_1|\psi\rangle_4 + |\Omega^-_a\rangle_{a123}\sigma_2|\psi\rangle_4]/2, \]

where \(|\Omega^+_a\rangle = (|000\rangle|\varphi^+\rangle + |111\rangle|\varphi^-\rangle)/\sqrt{2}\), \(|\Omega^-_a\rangle = (|001\rangle|\varphi^-\rangle + |110\rangle|\varphi^+\rangle)/\sqrt{2}\) and \(|\varphi^\pm\rangle = (|00\rangle \pm |11\rangle)/\sqrt{2}\). With these measurement vectors, the teleportation protocol can be carried out with two cbits of classical communication [17].

Case-II. The \(|W\rangle\) state: One cannot teleport the state \(|\psi\rangle_a\) using this resource state as none of the qubits has entropy as one. However, the \(|W_{mn}\rangle\) state of [14, 17] can be used for the teleportation, as one of its qubits has the entropy as one.

Case-III. The \(|S_1\rangle\) and \(|S_2\rangle\) states: These states are not symmetric under the permutations of the qubits; therefore only specific distributions of the qubits leads to the successful teleportation. For the \(|S_1\rangle\) state, only when Bob has the qubit 2, it works. This is because only second qubit has entropy as one. We can see this as [17]

\[ |\psi\rangle_a|S_1\rangle_{1234} = [|\rho^+_a\rangle_{a134}\sigma_0|\psi\rangle_2 + |\rho^-_a\rangle_{a134}\sigma_3|\psi\rangle_2 + |\rho^+_a\rangle_{a134}\sigma_1|\psi\rangle_2 + |\rho^-_a\rangle_{a134}\sigma_2|\psi\rangle_2]/2, \]

where, \(|\rho^+_a\rangle = (|0000\rangle + |0100\rangle) \pm (|1001\rangle + |1011\rangle)/2\), \(|\rho^-_a\rangle = (|1001\rangle + |0011\rangle) = (|1001\rangle + |1100\rangle)/2\). In the case of \(|S_2\rangle\) state Bob can have any qubit, except the qubit 1. This is because the qubit 1 has entropy less than one.
The protocol’s success can be seen as

\[ |\psi\rangle_a |S_2\rangle_{1234} = \frac{1}{\sqrt{2}}[|\psi_1^+\rangle_{a123\sigma_0}|\psi_1\rangle_4 + |\psi_2^+\rangle_{a123\sigma_3}|\psi_1\rangle_4 + |\psi_2^+\rangle_{a123\sigma_1}|\psi_1\rangle_4 + |\psi^-\rangle_{a123\sigma_2}|\psi_1\rangle_4]/2, \]

where, \[|\psi_1^+\rangle = [(|0000\rangle + |0111\rangle) \pm (|1011\rangle + |1100\rangle)]/\sqrt{2}, \]

\[|\psi_2^+\rangle = [(|0010\rangle + |0110\rangle) \pm (|1000\rangle + |1111\rangle)]/\sqrt{2}. \]

After making the measurement using the orthonormal set \{\{|\psi_1^+\rangle, |\psi_2^+\rangle\}\}, Alice needs to send two classical bits to Bob who then applies appropriate unitary transformations.

**B. Teleportation of a Two-Qubit State**

We now consider the possibility of teleporting an unknown arbitrary two-qubit state \(|\psi\rangle_{ab} = \alpha|00\rangle_{ab} + \beta|01\rangle_{ab} + \gamma|10\rangle_{ab} + \delta|11\rangle_{ab}\). As we shall see that it would be possible with only a few four-qubit entangled states. However, sometime one would be able to teleport subclasses of the general state. If one considers a product state of two Bell-states as a quantum resource, or the state \(|\chi\rangle\) of Ref [20], then one can teleport an arbitrary unknown two-qubit state. We will see that the cluster state \(|\Omega\rangle\) may also be used for an arbitrary two-qubit state. For any type of teleportation of an arbitrary and unknown two-qubit state to succeed, the entropy of the Bob’s two qubits should be two. For the teleportation of the restricted subclass, with two arbitrary parameters, the entropy needs to be one only. In such a situation, one can find suitable measurement vectors.

**Case-I. The \(|\text{GHZ}\rangle\) state:** The GHZ state is not a suitable resource state for the teleportation of an arbitrary unknown two-qubit state. However, an entangled two-qubit state \(|\psi_1\rangle_{ab} = \sigma_1\sigma_2(\alpha|00\rangle_{ab} + \beta|11\rangle_{ab})\) can be teleported if Alice uses the measurement vectors \(|\pi_1^\pm\rangle = \sigma_1\sigma_3(0000) \pm |1111\rangle)/\sqrt{2}\) and \(|\pi_2^\pm\rangle = \sigma_1\sigma_3(0011) \pm |1100\rangle)/\sqrt{2}\). One can teleport the state because the entropies of two-qubit subsystems of the GHZ-state is one. This state superficially looks like one-qubit state.

**Case-II. The \(|\Omega\rangle\) state:** This state can be used to teleport an arbitrary two-qubit unknown state for all appropriate distributions of qubits, except the partition of (1,4) and (2,3) qubits. We can see the protocol by rewriting the six-qubit state as [17]

\[|\psi\rangle_{ab}|\Omega\rangle_{1234} = \sum_{i=1}^{2} [\Omega_{i+1}\rangle_{ab12} U^i_{+1} |\psi\rangle_4 + \Omega_{i-1}\rangle_{ab12} U^i_{-1} |\psi\rangle_4 + \Omega_{i+2}\rangle_{ab12} U^{i+2}_{+1} |\psi\rangle_4 + \Omega_{i+2}\rangle_{ab12} U^{i+2}_{-1} |\psi\rangle_4 + \Omega_{i+4}\rangle_{ab12} U^{i+4}_{+1} |\psi\rangle_4 + \Omega_{i+4}\rangle_{ab12} U^{i+4}_{-1} |\psi\rangle_4 + \Omega_{i+6}\rangle_{ab12} U^{i+6}_{+1} |\psi\rangle_4 + \Omega_{i+6}\rangle_{ab12} U^{i+6}_{-1} |\psi\rangle_4]/4(6) \]

where,

\[\Omega_{i+1} = \langle 0|\pi_{i+1}\rangle (0 \pm 1)|\pi_{i+1}\rangle (1)/\sqrt{2} \]

\[\Omega_{i+2} = \langle 0|\pi_{i+2}\rangle (0 \pm 1)|\pi_{i+2}\rangle (1)/\sqrt{2} \]

\[\Omega_{i+4} = \langle 0|\pi_{i+4}\rangle (0 \pm 1)|\pi_{i+4}\rangle (1)/\sqrt{2} \]

\[\Omega_{i+6} = \langle 0|\pi_{i+6}\rangle (0 \pm 1)|\pi_{i+6}\rangle (1)/\sqrt{2} \]

Here \(i = 1, 2\) and \(|\eta_1^\pm\rangle = |\varphi^\pm\rangle, |\eta_2^\pm\rangle = |\psi^\pm\rangle\) and \(|\psi^\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle_1 \pm |1\rangle_2)\). \(U^i_{\pm1}\) are appropriate unitary operators.

**Case-III. The \(|W\rangle, |S_1\rangle\) and \(|S_2\rangle\):** A general unknown two-qubit state cannot be teleported using these states. However, in the case of the \(|W\rangle\)-state the two-qubit subsystems have entropy one, so one can teleport some subclasses of arbitrary two-qubit states. For example, one can teleport the state \(|\psi_2\rangle_{ab} = \alpha|00\rangle + \beta|01\rangle_{ab}\). Bob has to apply joint-unitary operations \((I \otimes I, |00\rangle_2 - |\psi^\pm\rangle_2 |\psi^\pm\rangle_2 - |\psi^-\rangle_2 |\psi^-\rangle_2 + |1\rangle_2)\). One can teleport the state because the entropies of two-qubit subsystems of the general state \(|\varphi\rangle\) is one. This state superficially looks like one-qubit state.

**C. Teleportation of a Three-Qubit State**

For the teleportation of a general three-qubit state, one would need an entangled state of six qubits [17]. From the Table I, we see that the maximum entropy of three-qubit subsystems for the states under consideration is one. Therefore one can teleport at most a state with two terms only. We note that although one can use \(|\Omega\rangle\) state to teleport a two-qubit state with four unknown parameters, one can teleport a three-qubit state with only two unknown parameters. This is because, while the two-qubit subsystems can have entropy two, three-qubit subsystems can have entropy at most one.

**IV. CONDITION FOR TELEPORTATION OF A PURE MULTI-PARTITE STATE**

In the preceding section, we have studied the teleportation of a single qubit, two-qubit and three-qubit states. We saw that the entropy of the reduced state at the receiver end of the given resource entangled state plays an important role. For the teleportation of an arbitrary two-qubit state with four terms, we need the entropy as two for the reduced state at the receiver end, while entropy of one bit is needed to teleport a two-qubit state with two terms. Now we can ask a question that if we are given a \(n\)-qubit state with \(m\) terms to teleport, what kind of resource state is needed? In a reverse way, the question can be posed as: given a resource state, what are the states that can be teleported using this resource? The answer to this question tells us that why sometimes a more entangled state is less suitable.

Let us consider a \(n\)-qubit state with \(m\)-terms that Alice wishes to teleport to Bob,
We need to find a condition on a suitable measurement basis. In state to be useful, we should be able to write it as, 
\[ |\eta_k\rangle = \delta_{kl}, \quad \sum_{k=1}^{m} |\alpha_k|^2 = 1. \]

Let the resource state be a $N$-qubit state. For this resource state to be useful, we should be able to write it as,
\[ |\Psi_n\rangle = \frac{1}{\sqrt{m}} \sum_{i=1}^{m} |\chi_i\rangle_{N-n} |\eta_l\rangle_n. \]

Here the states $|\chi_i\rangle_{N-n}$ may not be orthonormal. The combined state can be written as
\[ |\Psi_n\rangle |R_N\rangle = \frac{1}{\sqrt{m}} \sum_{i=1}^{m} \sum_{k=1}^{m} \alpha_i |\eta_l\rangle_n |\chi_k\rangle_{N-n} |\eta_k\rangle_n \]
\[ = \frac{1}{\sqrt{m}} \sum_{i=1}^{m} \sum_{k=1}^{m} |\eta_l\rangle_n |\chi_k\rangle_{N-n} |\alpha_i\rangle |\eta_k\rangle_n. \]

Alice will make a measurement in an orthonormal basis $|\theta_l\rangle_p$. Therefore, we should be able to write
\[ |\eta_l\rangle_n |\chi_k\rangle_{N-n} = \frac{1}{\sqrt{m}} \sum_{i=1}^{m} C_{ik,l} |\theta_l\rangle_N. \]

$C_{ik,l}$ is an interesting object. For each $l$, it is a $m \times m$ matrix in $ik$ space. It is also a $m^2 \times m^2$ matrix with row label as $ik$. We need to find a condition on $C_{ik,l}$ such that we indeed have a suitable measurement basis.
\[ |\Psi_n\rangle |R_N\rangle = \frac{1}{m} \sum_{i=1}^{m} \sum_{k=1}^{m} C_{ik,l} |\theta_l\rangle_N |\alpha_i\rangle |\eta_k\rangle_n \]
\[ = \frac{1}{m} \sum_{i=1}^{m} |\theta_l\rangle_N \sum_{k=1}^{m} \sum_{l=1}^{m} C_{ik,l} |\alpha_i\rangle |\eta_k\rangle_n. \]

For the teleportation to succeed, we should have,
\[ \sum_{i=1}^{m} \sum_{k=1}^{m} C_{ik,l} |\alpha_i\rangle |\eta_k\rangle_n = V_l \sum_{n=1}^{m} |\alpha_n\rangle |\eta_l\rangle_n. \]

For each $l$,
\[ \sum_{i=1}^{m} \sum_{i'=1}^{m} (C_{ik,l}^* C_{ik,l'}^*) |\alpha_i\rangle |\alpha_{i'}\rangle = 1. \]

For this equation to be satisfied,
\[ (C_{ik,l}^* C_{ik,l'}) = \delta_{i'i}. \]

This suggests that $C$ is unitary in $ik$ space for each $l$ for teleportation to succeed. Let us now see what it means for the resource state.

Taking the adjoint and the scalar product,
\[ \langle \eta_l^l | \eta_l^l \rangle |\chi_k^k\rangle = \frac{1}{m} \sum_{l=1}^{m} \sum_{i=1}^{m} (C_{ik,l}^* C_{ik,l'}^*) |\theta_l\rangle |\theta_{l'}\rangle, \]
\[ \delta_{i'i'} |\chi_k^k\rangle = \frac{1}{m} \sum_{l=1}^{m} C_{ik,l} C_{ik,l'} |\theta_l\rangle |\theta_{l'}\rangle. \]

Multiplying by $\delta_{i'i'}$ and summing over $i'$ and $i$, we get,
\[ |\chi_k^k\rangle = \frac{1}{m^2} \sum_{l=1}^{m} (C_{ik,l}^* C_{ik,l'}) |\theta_l\rangle |\theta_{l'}\rangle. \]

Since $C$ is unitary in the subspace, we get
\[ |\chi_k^k\rangle = \delta_{kk'}. \]

Therefore $|\chi_k^k\rangle$ should be orthonormal for the exact teleportation. Thus the resource state should have the form
\[ |R_N\rangle = \frac{1}{\sqrt{m}} \sum_{i=1}^{m} |\chi_i\rangle_{N-n} |\eta_l\rangle_n, \]
with both $|\chi_k^k\rangle$ and $|\eta_l\rangle$ being orthonormal. So, the reduced density matrix of the Bob’s qubits would be $\log_2 m$.

What we have shown is that if we wish to teleport a $n$-qubit state with $m$ terms, then we should be able to distribute resource states qubits in such a way such that Bob’s $n$ qubits have entropy $\log_2 m$. Given a resource state, we can compute entropy of all the partitions. If there is a partition where Bob’s $n$ qubits have entropy as $\log_2 m$, then the state with $m$ terms can be teleported with that partition.

### V. SUPERDENSE CODING AND ENTROPY

For a successful superdense coding protocol, Alice should be able to transmit more than $n$ classical bits by sending $n$ qubits to Bob. In this sense, any pure state can be used for superdense coding. However, our interest is in exact superdense coding, i.e., transmission of positive integer number of classical bits. We will consider the protocol to be successful, even if, a state is not suitable for maximal superdense coding [14].
So, for example, by transmitting two qubits, one may be able to transmit only 3 cbits, but not 4 cbits.

We now discuss the superdense coding capacity of various entangled states under discussion. There are three possible scenarios. In the scenario 1 (S1), Alice has one qubit; in the scenario 2 (S2), Alice has two qubits; in the scenario 3 (S3), Alice has three qubits. In each case, Bob has the rest of the qubits. For all such distributions, they follow the standard superdense coding protocol to transmit a classical message. In this protocol, Alice applies unitary operations \( \sigma_0, \sigma_1, i\sigma_2, \sigma_3 \) with equal probabilities on her qubits and sends them to Bob. Bob performs a joint measurement on all the four qubits to retrieve the original message. Since some states are asymmetric with respect to the permutation of qubits, the distribution may affect the superdense coding capacity of the states. The capacity mainly depends on how many orthogonal states are obtained using unitary transformations by Alice. This is because orthogonal states can be perfectly distinguished. Therefore, the task is to find out the number of orthogonal states that can be obtained by unitary operations on the particles possessed by the sender. In the S2 and S3 scenarios, Alice can also perform joint operations on more than one qubit.

The superdense coding capacity also appear to depend on the entropy of the subsystem on which Alice applies the unitary operations. This is specially true for maximal superdense coding. It also seems to be the case for non-maximal superdense coding, as we will see below.

### Case-I. The \(|\text{GHZ}\rangle\) state:
This state is symmetric under the permutation of particles, therefore the distribution of qubits would not affect the capacity. In the scenario S1, Alice applies unitary operations on her qubit producing four orthogonal states. Bob can perform an appropriate joint four-qubit von Neumann measurement to acquire two bits of classical information. This is possible because the entropy of the Alice’s qubit is one. In the scenario S2, Alice applies unitary operations on her two qubits. It gives rise to sixteen states out of which only eight are orthogonal. It allows Bob to access only three cbits. This happens because the entropy of Alice’s subsystem is one. In the scenario S3, Alice applies unitary operations on her three qubits yielding sixteen orthogonal states. It leads to the transmission of four cbits of information. In general using \(n\)-qubit GHZ-state one may be able to send \(n\) bits of classical information by sending \(n - 1\) qubits. We note that this state is not suitable for maximal superdense coding.

### Case-II. The \(|\text{W}\rangle\) state:
For the W-state, the classical capacities will also be independent of the distribution of the particles. In the scenario S1 and S3, this state is not suitable for superdense coding because the entropy of Alice’s subsystem is less than one. However, in S2 scenario, Bob can access three cbits, because the two-qubit subsystems have the entropy as one.

### Case-III. The \(|\Omega\rangle\) state:
This is the best quantum resource from the point of view of superdense coding. The classical capacity in the S1 scenario is two cbits irrespective of distribution of the qubits, as the entropy of each individual qubit is one. In scenario S2, Alice can transmit four cbits by transmitting two qubits with specific distributions of qubits. The entropy for the subsystems (1,2) or (1,3) is two, while it is one for the subsystem (1,4). Therefore, the classical capacity is 4 cbits when Alice has subsystems (1,2) or (3,4) or (1,3) or (2,4), but it is three cbits when Alice has the subsystem (1,4) or (2,3). To illustrate this let us consider the case when Alice has particles 1 and 2. On applying unitary transformations, \(\sigma_k \otimes \sigma_\ell \otimes \sigma_0 \otimes \sigma_0 \) \((k, \ell = 0, 1, 2, 3)\), the sixteen orthogonal states are obtained. (when \(k, \ell = 2\), we have \(i\sigma_2\)), Therefore, clearly the classical capacity is four cbits which is the maximum possible.

### Case-IV. The \(|S_1\rangle\) state:
For this state, the success of the protocol depends on the distribution of the particles between the parties. In the S1 scenario, the protocol succeeds only when Alice has the qubit 2. This is because the entropy of this qubit is one. Other qubits have individual entropy 0.81 which is less than 1. In the scenario S2, if Alice has qubits (1,2) and she applies unitary operations on her qubits, she gets eight orthogonal states. Therefore the capacity with this distribution is three cbits. If the qubits are distributed such that Alice has the (1,3) or (1,4), then unitary transformations yield at most four orthogonal states. So there is no enhancement in the classical capacity. We note that the entropy of the subsystem (1,2) is 1.5, while it is 1.22 for the subsystems (1,3) and (1,4). Here, we observe that there is an enhancement in the capacity for the entropy 1 (as earlier) and 1.5, but not for all values greater than 1. Surprisingly, the capacity of superdense coding is two cbits in spite of the entropy of two qubit subsystem being greater than one. It should be noted a similar observation is made in case of entangled qudits [21]. In the S3 scenario, Alice will be able to transmit four cbits by sending three qubits only when she has qubits 1, 3 and 4. This is, as earlier, because only qubit 2 has entropy as one.

We observe that when Alice has qubits (1,2), she can do superdense coding, but she would not be able to teleport a state. So we have a situation where a state is suitable for exact superdense coding but not for exact teleportation.

### Case-V. The \(|S_2\rangle\) state:
As in the case of the \(|S_1\rangle\) state, the distribution of the qubits is important. The protocol can be implemented in the S1 scenario for all possible distribution of qubits except when Alice has the qubit 1. This is because the qubit 1 has the entropy of 0.81, while others have the entropy one. In the S2 scenario, like \(|S_1\rangle\) state, we can use the state for superdense coding for all distributions. This is because now all such partitions have entropy 1.5. The channel capacity, in each case is 3 cbits. It appears that after 1, the entropy value of 1.5 has special significance. This state is not suitable for exact teleportation, but can be used for non-maximal superdense coding.

### VI. QUANTUM KEY DISTRIBUTION WITH A FOUR-QUBIT ENTANGLING STATE

In the previous sections, we have shown that there are some quantum information processing tasks like teleportation and dense coding which can be deterministically achieved with lesser entangled states compared to the higher entangled states. In this section, we will show that there exist another important quantum information processing task such as
quantum key distribution (QKD) which works well with lesser amount of shared entanglement.

To verify our claim we consider 2 four-qubit states – \(|GHZ\rangle\) and \(|S_1\rangle\), which were defined earlier. Let us suppose that the two distant partners Alice and Bob possess two qubits each. In this partition, we know from Table I that the two-qubit reduced subsystem of \(|S_1\rangle\) state has more entropy than the two-qubit reduced subsystem of the \(|GHZ\rangle\) state. Therefore, in this partition, the \(|S_1\rangle\) state has more entanglement than the \(|GHZ\rangle\) state. Although the amount of entanglement in the \(|S_1\rangle\) state is larger than the \(|GHZ\rangle\) state, we will see that \(|S_1\rangle\) state cannot be used in a variant of BB84 QKD scheme while \(|GHZ\rangle\) state can be used.

The variant of BB84 QKD protocol goes like this. Let us suppose that Alice and Bob share a four qubit \(|GHZ\rangle_{1234}\) state,

\[|GHZ\rangle_{1234} = \frac{1}{\sqrt{2}}(|00\rangle_{13}|00\rangle_{24} + |11\rangle_{13}|11\rangle_{24}). \tag{22}\]

Here Alice holds the qubits 1 and 3 and the remaining two qubits 2 and 4 are with Bob. In the Bell basis \(|\Phi^\pm, \Phi^-, \Psi^+, \Psi^-\rangle\), \(|GHZ\rangle_{1234}\) state can be re-written as

\[|GHZ\rangle_{1234} = \frac{1}{\sqrt{2}}(|\Phi^+\rangle_{13}|\Phi^+\rangle_{24} + |\Phi^-\rangle_{13}|\Phi^-\rangle_{24}). \tag{23}\]

Here, \(|\Phi^\pm\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle\) and \(|\Psi^\pm\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle\). In the next step, Alice randomly performs measurements on her particles in either \(|00\rangle, |11\rangle\) basis or in the basis \(|\Phi^+, \Phi^-\rangle\). The information is encoded by using the two binary digits 0 and 1. If the measurement outcome is \(|00\rangle\) then 0 is encoded and if the measurement outcome is \(|11\rangle\) then 1 is encoded. After Alice’s measurement, Bob also randomly chooses either the basis \(|00\rangle, |11\rangle\) or the basis \(|\Phi^+, \Phi^-\rangle\) and then performs measurement on his particles in that basis. In the next stage, Alice publicly announces the basis in which she had measured the state of the particles but does not declare the measurement outcome. If Bob finds that his measurement basis matches with the Alice’s basis, then he informs Alice to keep the data, otherwise the data is thrown out. In this way, quantum key can be distributed between Alice and Bob. Therefore, \(|GHZ\rangle_{1234}\) state can be used in generating the quantum key.

Now the question arises whether the state \(|S_1\rangle\) is also suitable for this QKD scheme? To answer this question, we have to write the four-qubit state \(|S_1\rangle\) in the computational basis as well as in the Bell basis.

In the computational basis, the state \(|S_1\rangle\) can be expressed as

\[|S_1\rangle_{1234} = \frac{1}{2}(|00\rangle_{13}|00\rangle_{24} + |00\rangle_{13}|11\rangle_{24} + |10\rangle_{13}|00\rangle_{24} + |11\rangle_{13}|10\rangle_{24}). \tag{24}\]

In the Bell basis, this state can be re-expressed as

\[|S_1\rangle_{1234} = \frac{1}{4}(|2|\Phi^+\rangle_{13} + |2|\Phi^-\rangle_{13} + |\Psi^+\rangle_{13} + |\Psi^-\rangle_{13}\]

\[= |\Phi^+\rangle_{24} + (|\Psi^+\rangle_{13} + |\Psi^-\rangle_{13})|\Phi^-\rangle_{24} + (|\Psi^+\rangle_{13} - |\Psi^-\rangle_{13})|\Psi^+\rangle_{24} - (|\Psi^+\rangle_{13} - |\Psi^-\rangle_{13})|\Psi^-\rangle_{24}). \tag{25}\]

From eq. (24) and eq. (25), it is clear that one cannot generate key even if their basis matches. Therefore, we have shown that although the four-qubit state \(|S_1\rangle\) has more entanglement than \(|GHZ\rangle\) state but the former cannot be used in the QKD protocol while the latter can. Similarly we can also see that the state \(|S_2\rangle\) has more entanglement in bipartite partitions, but it is not useful for the QKD protocol. It would appear that, like teleportation, only for specific values of the entropy, our QKD protocol will succeed.

We note that our protocol may look like BB84 protocol but there are significant differences between the two protocols. In BB84 protocol, single particle (or single qubit) is used but in our protocol, we used four particle (or four qubit) shared entangled state between two parties. Secondly, In BB84 protocol, the sender have to send the particle to the receiver but in our case we don’t need to send any particle.

VII. CONCLUSION

Multipartite states allow many variations of the communication protocols that were introduced for bipartite states. We have considered a number of different genuine quadrupartite entangled states as quantum resources for exact teleportation, superdense coding and quantum key distribution protocols. With a multipartite state as a resource, one can consider possibilities of teleporting multiple qubit states with different number of terms. We find that in such scenarios the phenomenon of “less is more” may occur. It means that a less entangled state is sometimes more suitable for the teleportation of an unknown state. To understand this we have obtained a condition that a resource state must satisfy for the protocol to succeed. In particular, we find that to teleport a \(m\)-term \(n\)-qubit state, a subsystem of the resource state must have entropy \(\log_2 m\). So to be able to teleport a two-term two-qubit state, a subsystem need to have the entropy as one, but for a most general two-qubit teleportation, the required entropy is two. Therefore, a four-qubit GHZ state can be used to teleport a two-term two-qubit state but not the most general two-qubit state.

For the exact superdense coding, the capacity also seems to depend on the entropy of the subsystem on which Alice applies unitary transformations to encode the classical information. We also have a number of situations where one can encode more classical information by applying unitary transformations on a subsystem with smaller entropy. This again suggests that sometime “less is more”. Furthermore, for some specific distribution of qubits, sometimes one can carry out superdense coding, but not the teleportation. It also appears that if Alice’s subsystem has certain entropy then the superdense coding is possible. In particular for the maximal superdense
coding, the Alice’s subsystem should have maximal entropy, allowed for her number of qubits. We also find the concept of “less is more” holds for the quantum key distribution schemes. The amount of entanglement in $|GHZ\rangle$ state is less than the amount of entanglement in the state $|S_1\rangle$ but quantum key can be distributed with the $|GHZ\rangle$ state but not with $|S_1\rangle$ state. In general, it appears that the phenomenon of “less is more” is normal in the case of communications protocols involving multipartite state.

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