On the stability of two-dimensional extremal black holes

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Abstract

We discuss the stability of the extremal ground states of a two-dimensional (2D) charged black hole which carries both electric ($Q_E$) and magnetic ($Q_M$) charges. The method is first to find the physical field and then to derive the equation of the Schrödinger type. It is found that the presenting potential to an on-coming tachyon (as a spectator) takes a barrier-well type. This provides the bound state solution, which implies an exponentially growing mode with respect to time. The 2D extremal ground states all are classically unstable. We conclude that the 2D extremal charged black holes are not considered as the candidates for the stable endpoint of the Hawking evaporation.
Recently the extremal black holes have received much attention. Extremal black holes provide a simple laboratory in which to investigate the quantum aspects of black hole [1]. One of the crucial features is that the Hawking temperature vanishes. The black hole with $M > Q$ will tend to Hawking radiate down to its extremal $M = Q$ state. Thus the extremal black hole may play a role of the stable endpoint for the Hawking evaporation. It has been also proposed that although the extremal black hole has nonzero area, it has zero entropy [2]. This is because the extremal case is distinct topologically from the nonextremal one.

It is very important to enquire into the stability of the extremal black holes, which establishes their physical existence. It has been shown that the 4D extremal charged black holes including the Reissner-Nordström one are shown to be classically stable [3]. Since all potentials are positive definite outside the outer horizon, one can infer the stability of 4D extremal charged black holes using the same argument as employed by Chandrasekhar [4]. However, it was demonstrated that the 2D electrically charged extremal black hole is unstable [5].

In this letter, we shall perform a complete analysis of the stability for the 2D extremal black holes with electric and magnetic charges. One easy way of understanding a black hole is to find out how it reacts to external perturbations. We always visualize the black hole as presenting an effective potential barrier (or well) to the on-coming waves [4]. In deciding whether or not the the extremal black hole is stable, one starts with a physical perturbation which is regular everywhere in space at the initial time $t = 0$ [6]. And then see whether such a perturbation will grow with time. If there exists an exponentially growing mode, the extremal black hole is unstable. As a compact criterion, the extremal black hole is unstable if it has the potential well to the on-coming waves [7]. This is so because in the Schrödinger equation the potential well always allows the bound state as well as scattering states. The former shows up as an imaginary frequency mode, leading to an exponentially growing mode.

We start with the low energy action from heterotic string theory [8]
\[ S_{\text{I} - e} = \int d^2x \sqrt{-G} e^{-2\Phi} \left\{ R + 4(\nabla \Phi)^2 - \frac{1}{2} F^2 - \frac{1}{2} (\nabla T)^2 + V(\Phi, T) \right\} \] (1)

with the potential \( V(\Phi, T) = \alpha^2 - Q_M^2 e^{4\Phi} + T^2 \). Here are all string fields (metric \( G_{\mu\nu} \), dilaton \( \Phi \), Maxwell field \( F_{\mu\nu} \), and tachyon \( T \)). We introduce the tachyon as a spectator and the term \(-Q_M^2 e^{4\Phi}\) for a magnetically charged configuration. From the 4D magnetic Maxwell field \( F = Q_M \sin \theta d\theta \wedge d\varphi \), one finds the modification in two dimensions: \( V(0, T) \rightarrow V(\Phi, T) \). Also, this type of dilaton potential can be generated from the closed string loop corrections.

Setting \( \alpha^2 = 8 \) and after deriving equations, we take the transformation

\[ -2\Phi \rightarrow \Phi, \quad T \rightarrow \sqrt{2} T, \quad -R \rightarrow R. \] (2)

Then the equations of motion become

\[ R_{\mu\nu} + \nabla_{\mu} \nabla_{\nu} \Phi + \nabla_{\mu} T \nabla_{\nu} T + F_{\mu\rho} F_{\nu}^{\rho} - Q_M^2 e^{-2\Phi} G_{\mu\nu} = 0, \] (3)

\[ \nabla^2 \Phi + (\nabla \Phi)^2 - \frac{1}{2} F^2 + Q_M^2 e^{-2\Phi} - 2T^2 - 8 = 0, \] (4)

\[ \nabla_{\mu} F^{\mu\nu} + (\nabla_{\mu} \Phi) F^{\mu\nu} = 0, \] (5)

\[ \nabla^2 T + \nabla \Phi \nabla T + 2T = 0. \] (6)

An electrically and magnetically charged black hole solution to the above equations is given by

\[ \bar{\Phi} = 2 \sqrt{2} r, \quad \bar{F}_{tr} = Q_E e^{-2\sqrt{2} r}, \quad \bar{T} = 0, \quad \bar{G}_{\mu\nu} = \begin{pmatrix} -f & 0 \\ 0 & f^{-1} \end{pmatrix}, \] (7)

with

\[ f = 1 - \frac{2M}{2\sqrt{2}} e^{-2\sqrt{2} r} + \frac{Q^2}{8} e^{-4\sqrt{2} r}, \] (8)

where \( M \) and \( Q = \sqrt{Q_E^2 + Q_M^2} \) are the mass and total charge of the black hole, respectively. Here we take \( M = \sqrt{2} \) for convenience. For \( 0 < Q < M \), the double horizons \( (r_{\pm}) \) are given by

\[ r_{\pm} = \frac{1}{2\sqrt{2}} \log \left[ \frac{1 \pm \sqrt{1 - \frac{Q^2}{2}}}{2} \right], \] (9)
where $r_+(r_-)$ correspond to the event (Cauchy) horizons. This charged black hole may provide an ideal setting for studying the late stages of Hawking evaporation. For $Q = M$, two horizons coincide $r_+ = r_- \equiv r_o$. We are mainly interested in this extremal limit.

To study the propagation of string fields, we introduce small perturbation fields around the background solution as

\[
F_{tr} = \bar{F}_{tr} + \delta F_{tr} = \bar{F}_{tr}[1 - \frac{\mathcal{F}(r, t)}{Q_E}],
\]

(10)

\[
\Phi = \bar{\Phi} + \phi(r, t),
\]

(11)

\[
G_{\mu\nu} = \bar{G}_{\mu\nu} + h_{\mu\nu} = \bar{G}_{\mu\nu}[1 - h(r, t)],
\]

(12)

\[
T = \bar{T} + \bar{t} \equiv \exp(-\frac{\bar{\Phi}}{2})[0 + t(r, t)].
\]

(13)

One has to linearize (3)-(6) in order to obtain the equations governing the perturbations as

\[
\delta R_{\mu\nu}(h) + \bar{\nabla}_\mu \bar{\nabla}_\nu \bar{\Phi} - \delta \Gamma^\rho_{\mu\nu}(h) \bar{\nabla}_\rho \bar{\Phi} + 2 \bar{F}_{\mu\rho} \mathcal{F}_\nu^\rho \bar{\Phi} \bar{\Phi} h^{\rho\alpha} + Q^2 M e^{-4\sqrt{2}r} \bar{G}_{\mu\nu}(h + 2\phi) = 0,
\]

(14)

\[
\bar{\nabla}^2 \phi - h^{\mu\nu} \bar{\nabla}_\mu \bar{\nabla}_\nu \bar{\Phi} - \bar{G}^{\mu\nu} \delta \Gamma^\rho_{\mu\nu}(h) \partial_\rho \bar{\Phi} - h^{\mu\nu} \partial_\mu \bar{\Phi} \partial_\nu \bar{\Phi} + 2 \bar{G}^{\mu\nu} \partial_\mu \bar{\Phi} \partial_\nu \phi
\]

\[- \bar{F}_{\mu\nu} \mathcal{F}_{\mu\nu} + \bar{F}_{\mu\rho} \mathcal{F}_\rho^\nu h^{\mu\rho} - 2Q^2 M e^{-4\sqrt{2}r} \phi = 0,
\]

(15)

\[
(\bar{\nabla}_\mu + \partial_\mu \bar{\Phi})(\mathcal{F}_{\mu\nu} - \bar{F}_\alpha^\nu h^{\mu\alpha} - \bar{F}_\mu^\nu h^{\mu\nu}) + \bar{F}_{\mu\nu}(\delta \Gamma^\sigma_{\sigma\mu}(h) + (\partial_\mu \phi)) = 0,
\]

(16)

\[
\nabla^2 \bar{t} + \nabla_\mu \bar{\Phi} \nabla^\mu \bar{t} + 2 \bar{t} = 0,
\]

(17)

where

\[
\delta R_{\mu\nu}(h) = \frac{1}{2} \bar{\nabla}_\mu h_{\rho}^{\rho} + \frac{1}{2} \bar{\nabla}_\rho h_{\mu\nu} - \frac{1}{2} \bar{\nabla}_\rho h_{\rho\mu} - \frac{1}{2} \bar{\nabla}_\mu \bar{\nabla}_\rho h_{\nu\rho},
\]

(18)

\[
\delta \Gamma^\rho_{\mu\nu}(h) = \frac{1}{2} \bar{G}^{\rho\sigma}(\bar{\nabla}_\nu h_{\mu\sigma} + \bar{\nabla}_\mu h_{\nu\sigma} - \bar{\nabla}_\sigma h_{\mu\nu}).
\]

(19)

From (16) one can express $\mathcal{F}$ in terms of $\phi$ and $h$ as

\[
\mathcal{F} = -Q_E(\phi + h).
\]

(20)
This means that $\mathcal{F}$ is no longer an independent mode. Also from the diagonal element of (14), we have

\begin{align}
\bar{\nabla}^2h - 2\bar{\nabla}^2\phi - 2\sqrt{2} \bar{G}_{rr} \partial_r h - 2Q^2 e^{-4\sqrt{2}r} (h + 2\phi) &= 0, \\
\bar{\nabla}^2h - 2\bar{\nabla}^2\phi + 2\sqrt{2} \bar{G}_{rr} \partial_r h - 2Q^2 e^{-4\sqrt{2}r} (h + 2\phi) &= 0.
\end{align}

Adding the above two equations leads to

\[ \bar{\nabla}^2(h - \phi) - 2Q^2 e^{-4\sqrt{2}r} (h + 2\phi) = 0. \] (23)

Also the dilaton equation (15) leads to

\[ \bar{\nabla}^2\phi + 4\sqrt{2} f \partial_r \phi + 2\sqrt{2} (\partial_r f + 2\sqrt{2} f) h - 2Q^2 e^{-4\sqrt{2}r} \phi = 0. \] (24)

And the off-diagonal element of (14) takes the form

\[ \partial_t \{(\partial_r - \Gamma^t_{tr})\phi + \sqrt{2}h\} = 0, \] (25)

which provides us the relation between $\phi$ and $h$ as

\[ \partial_r \phi = -\sqrt{2}h + \frac{1}{2} \frac{\partial_r f}{f} \phi + U(r). \] (26)

Here $U(r)$ is the residual gauge degrees of freedom and thus we set $U(r) = 0$ for simplicity. Substituting (26) into (24), we have

\[ \bar{\nabla}^2\phi + 2\sqrt{2} \partial_r f (h + \phi) - 2Q^2 e^{-4\sqrt{2}r} \phi = 0. \] (27)

Calculating (23) $+ 2 \times$ (27), one finds the other equation

\[ \bar{\nabla}^2(h + \phi) + 4\sqrt{2} \partial_r f (h + \phi) - 2Q^2 e^{-4\sqrt{2}r} (h + 4\phi) = 0. \] (28)

Although (23) and (27) look like the very complicated forms, these reduce to

\[ \bar{\nabla}^2(h - \phi) = 0, \] (29)

\[ \bar{\nabla}^2(h + \phi) + 4\sqrt{2} \partial_r f (h + \phi) = 0 \] (30)
in the asymptotically flat region ($r \to \infty$). This suggests that one obtain two graviton-dilaton modes. However, it is important to check whether the graviton ($h$), dilaton ($\phi$), Maxwell mode ($F$) and tachyon ($t$) are physically propagating modes in the 2D charged black hole background. We review the conventional counting of degrees of freedom. The number of degrees of freedom for the gravitational field ($h_{\mu \nu}$) in $D$-dimensions is $(1/2)D(D-3)$. For a Schwarzschild black hole, we obtain two degrees of freedom. These correspond to the Regge-Wheeler mode for odd-parity perturbation and Zerilli mode for even-parity perturbation [4].

We have $-1$ for $D = 2$. This means that in two dimensions the contribution of the graviton is equal and opposite to that of a spinless particle (dilaton). The graviton-dilaton modes ($h + \phi$, $h - \phi$) are gauge degrees of freedom and thus turn out to be nonpropagating modes[6]. In addition, the Maxwell field has $D - 2$ physical degrees of freedom. The Maxwell field has no physical degrees of freedom for $D = 2$. Actually from (20) it turns out to be a redundant one. Since these all are nonpropagating modes, it is necessary to consider the remaining one (17). The tachyon as a spectator is a physically propagating mode. This is used to illustrate many of the qualitative results about the 2D charged black hole in a simpler context. The stability should be based on the physical degrees of freedom. Its linearized equation is

$$f^2 t'' + f f't' - [\sqrt{2} ff' - 2f(1-f)]t - \partial^2_t t = 0, \quad (31)$$

where the prime ($'$) denotes the derivative with respect to $r$. To study the stability, the above equation should be transformed into one-dimensional Schrödinger equation. Introducing a tortoise coordinate

$$r \to r^* \equiv g(r),$$

(31) can be rewritten as

$$f^2 g'^2 \frac{\partial^2}{\partial r^*^2} t + f\{fg'' + f'g'\} \frac{\partial}{\partial r^*} t - [\sqrt{2} ff' - 2f(1-f)]t - \frac{\partial^2}{\partial t^2} t = 0. \quad (32)$$

Requiring that the coefficient of the linear derivative vanish, one finds the relation

$$g' = \frac{1}{f}. \quad (33)$$
Assuming \( t(r^*, t) \sim \tilde{t}(r^*) e^{i\omega t} \), one can cast (32) into the Schrödinger equation

\[
\left\{ \frac{d^2}{d r^*^2} + \omega^2 - V(r) \right\} \tilde{t} = 0,
\]

(34)

where the effective potential \( V(r) \) is given by

\[
V(r) = f(\sqrt{2} f' - 2(1 - f)).
\]

(35)

Fig. 1 shows the graphs of potentials for \( Q = 0.1, 1, \) and \( \sqrt{2} \). When \( M(= \sqrt{2}) > Q \), the potentials outside the event horizon are simple barriers. However a barrier-well potential appears outside the event horizon when the nonextremal black hole (a simple barrier) approaches the extremal one. This takes the form

\[
V_{\text{extr}}(r) = 2e^{-2\sqrt{2}r}(1 - \frac{1}{2} e^{-2\sqrt{2}r})^2(1 - \frac{3}{4} e^{-2\sqrt{2}r}).
\]

(36)

The event horizon is located at \( r_o = -0.245 \). Now let us translate the potential \( V_{\text{extr}}(r) \) into \( V_{\text{extr}}(r^*) \). With \( f = (1 - \frac{1}{2} \exp(-2\sqrt{2}r))^2 \), one obtains the explicit form of \( r^* \)

\[
r^* = r - \frac{1}{2\sqrt{2}(1 - \frac{1}{2} e^{-2\sqrt{2}r})} + \frac{1}{2\sqrt{2}} \log |1 - \frac{1}{2} e^{-2\sqrt{2}r}|.
\]

(37)

Since both the forms of \( V_{\text{extr}}(r) \) and \( r^* \) are very complicated, we are far from obtaining the exact form of \( V_{\text{extr}}(r^*) \). Instead we can find an approximate form. From (37), in the asymptotically flat region one finds that \( r^* \sim r \). (36) takes the asymptotic form

\[
V_{r^*\to\infty} \simeq 2 \exp(-2\sqrt{2}r^*).
\]

(38)

On the other hand, near the horizon (\( r = r_o \)) one has

\[
r^* \simeq -\frac{1}{2\sqrt{2}(1 - \frac{1}{2} e^{-2\sqrt{2}r})}.
\]

(39)

Approaching the horizon (\( r \to r_o, r^* \to -\infty \)), the potential takes the form

\[
V_{r^*\to-\infty} \simeq \frac{1}{4r^{*2}}.
\]

(40)

Using (38) and (40) one can construct the approximate form \( V_{\text{app}}(r^*) \) (Fig. 2). This is also a barrier-well which is localized at the origin of \( r^* \). Our stability analysis is based on the equation
\[
\left\{ \frac{d^2}{dr^*2} + \omega^2 - V_{\text{app}}(r^*) \right\} \tilde{t} = 0. \tag{41}
\]

As is well known, two kinds of solutions to the Schrödinger equation correspond to the bound and scattering states. In our case \( V_{\text{app}}(r^*) \) admits two solutions depending on the signs of the energy.

(i) For \( E > 0 (\omega = \text{real}) \), the asymptotic solution for \( \tilde{t} \) is given by

\[
\tilde{t}_{\infty} = \exp(i\omega r^*) + R \exp(-i\omega r^*) \quad (r^* \to \infty), \tag{42}
\]

\[
\tilde{t}_{EH} = T \exp(i\omega r^*) \quad (r^* \to -\infty), \tag{43}
\]

where \( R \) and \( T \) are the scattering amplitudes of two waves which are reflected and transmitted by the potential \( V_{\text{app}}(r^*) \), when a tachyonic wave of unit amplitude with the frequency \( \omega \) is incident on the black hole from infinity.

(ii) For \( E < 0 (\omega = -i\alpha, \alpha \text{ is positive and real}) \), we have the bound state. Eq. (41) and possible asymptotic solutions are given by

\[
\frac{d^2}{dr^*2} \tilde{t} = (\alpha^2 + V_{\text{app}}(r^*)) \tilde{t}, \tag{44}
\]

\[
\tilde{t}_{\infty} \sim \exp(\pm \alpha r^*) \quad (r^* \to \infty) \tag{45}
\]

\[
\tilde{t}_{EH} \sim \exp(\pm \alpha r^*) \quad (r^* \to -\infty). \tag{46}
\]

To ensure that the perturbation falls off to zero for large \( r^* \), we choose \( \tilde{t}_{\infty} \sim \exp(-\alpha r^*) \). In the case of \( \tilde{t}_{EH} \), the solution \( \exp(\alpha r^*) \) goes to zero as \( r^* \to -\infty \). Now let us observe whether or not \( \tilde{t}_{EH} \sim \exp(\alpha r^*) \) can be matched to \( \tilde{t}_{\infty} \sim \exp(-\alpha r^*) \). Assuming \( \tilde{t} \) to be positive, the sign of \( d^2\tilde{t}/dr^*2 \) can be changed from + to − as \( r^* \) goes from \( \infty \) to \( -\infty \). If we are to connect \( \tilde{t}_{EH} \) at one end to a decreasing solution \( \tilde{t}_{\infty} \) at the other, there must be a point \( (d^2\tilde{t}/dr^*2 < 0, d\tilde{t}/dr^* = 0) \) at which the signs of \( \tilde{t} \) and \( d^2\tilde{t}/dr^*2 \) are opposite: this is compatible with the shape of \( V_{\text{app}}(r^*) \) in Fig. 2. It thus is possible for \( \tilde{t}_{EH} \) to be connected to \( \tilde{t}_{\infty} \) smoothly. Therefore a bound state solution is given by

\[
\tilde{t}_{\infty} \sim \exp(-\alpha r^*) \quad (r^* \to \infty) \tag{47}
\]

\[
\tilde{t}_{EH} \sim \exp(\alpha r^*) \quad (r^* \to -\infty). \tag{48}
\]
This is a regular solution everywhere in space at the initial time $t = 0$. It confirms this solution from the quantum mechanics which states that the bound state solution is always allowed if a potential well exists. However, on later $\omega = -i\alpha$ implies

$$t_\infty(r^*, t) = \tilde{t}_\infty(r^*) \exp(-i\omega t) \sim \exp(-\alpha r^*) \exp(\alpha t)$$

and

$$t_{EH}(r^*, t) = \tilde{t}_{EH}(r^*) \exp(-i\omega t) \sim \exp(\alpha r^*) \exp(\alpha t).$$

This means that there exists an exponentially growing mode with time. Therefore, the 2D extremal ground state is classically unstable. The origin of this instability comes from a barrier-well. This potential appears when the nonextremal black hole approaches the extremal limit. As is discussed in Ref. [9], the quantum stress tensor of a scalar field (instead of the tachyon) in the extremal black hole diverges at the horizon.

This means that the 2D extremal black hole is also quantum-mechanically unstable. This divergence can be better understood by the regarding an extremal black hole as the limit of a nonextremal one. A nonextremal black hole has an outer (event) and an inner (Cauchy) horizon, and these come together in the extremal limit. In this case, we find that if we adjust the quantum state of the scalar field so that the stress tensor is finite at the outer horizon, it always diverges at the inner horizon. Thus it is not so surprising that in the extremal limit (when the two horizons come together) the divergence persists, although it has a softened form. By the similar way, it conjectures that the classical instability originates from the instability (blueshift) of the inner horizon. The potential of the nonextremal ($Q = 1$) black hole takes a barrier-well between the inner and outer horizons, while it takes a simple barrier outside the outer horizon. It confirmed that the inner horizon is unstable, whereas the outer one is stable [10]. When these coalesce, a barrier-well type potential appears outside the event horizon ($r > r_o$). This induces the instability of the extremal black holes.

The 2D black hole solution is symmetric in $Q_E$ and $Q_M$. The extremal cases include three:

$$Q_E = \sqrt{2}, Q_M = 0; Q_E = 0, Q_M = \sqrt{2}; Q_E = Q_M = 1/\sqrt{2}.\) This implies that there is no distinction between the electrically and magnetically charged extremal black holes [11]. It has already shown that the 2D electrically extremal charged black hole is unstable [5]. This is easily recognized from our case of $Q_E = \sqrt{2}, Q_M = 0$. Since the potential $V_{extr}(r)$ is left invariant under the transformation $(Q_E = \sqrt{2}, Q_M = 0) \rightarrow (Q_E = 0, Q_M = \sqrt{2})$, the mag-
netically charged extremal black hole is unstable. Furthermore a symmetric combination of $Q_E = Q_M = 1/\sqrt{2}$ is also unstable.

On the other hand, the Hawking temperature of a static black hole can be calculated from (18)

$$T_H^Q = \frac{\sqrt{2}}{\pi} \frac{\sqrt{M^2 - Q^2}}{M + \sqrt{M^2 - Q^2}}.$$  \hspace{1cm} (49)

The Hawking temperature vanishes when $M \rightarrow Q$.

In conclusion, although the 2D extremal charged black holes all have zero Hawking temperature, they cannot be the candidates for the stable endpoint of the Hawking evaporation.

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Fig. 1: Three graphs of generally charged black hole for $Q = 0.1$ (dashed line : \(\cdash\cdash\cdash\)), 1 (dotted line : \(-\cdash\cdash\)), and $\sqrt{2}$ (solid line : \(-\)). The corresponding event horizons are located at $r_+ = -0.004, -0.056, \text{and} -0.245$ respectively. When $M(= \sqrt{2}) > Q(Q = 0.1, 1)$, the potentials outside the event horizon are simple barriers. However a barrier-well ($V_{\text{ext}}(r)$) appears when the nonextremal black hole (a simple barrier) approaches the extremal one ($M = Q$).

Fig.2: The approximate form ($V_{\text{app}}(r^*)$) of extremal black holes outside the event horizon ($r^* = -\infty$). The asymptotically flat region is at $r^* = \infty$. This also takes a barrier-well type. This is localized at $r^* = 0$, falls to zero exponentially as $r^* \rightarrow \infty$ and inverse-squarely as $r^* \rightarrow -\infty$ (solid lines). The dotted line is used to connect two boundaries.