THE $R_\infty$ PROPERTY FOR NILPOTENT QUOTIENTS OF GENERALIZED SOLVABLE BAUMSLAG-SOLITAR GROUPS

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Abstract. We say a group $G$ has property $R_\infty$ if the number $R(\varphi)$ of twisted conjugacy classes is infinite for every automorphism $\varphi$ of $G$. For such groups, the $R_\infty$-nilpotency degree is the least integer $c$ such that $G/\gamma_{c+1}(G)$ has property $R_\infty$. In this work, we compute the $R_\infty$-nilpotency degree of all Generalized Solvable Baumslag-Solitar groups $\Gamma_n$. Moreover, we compute the lower central series of $\Gamma_n$, write the nilpotent quotients $\Gamma_{n,c} = \Gamma_n/\gamma_{c+1}(\Gamma_n)$ as semidirect products of finitely generated abelian groups and classify which integer invertible matrices can be extended to automorphisms of $\Gamma_{n,c}$.

1. Introduction

The main task of our paper is to show the following:

Theorem 1 (Theorem 4.4). Let an integer $n \geq 1$ have a prime decomposition with at least two prime numbers involved. Then the $R_\infty$-nilpotency degree of any Generalized Solvable Baumslag-Solitar group $\Gamma_n$ is infinite.

Let us first define property $R_\infty$ and the $R_\infty$-nilpotency degree and then problematize. An automorphism $\varphi$ of a group $G$ gives rise to a “twisted conjugacy” equivalence relation on $G$ given by $x \sim_\varphi y \iff \exists z \in G : zx\varphi(z)^{-1} = y$. The number of equivalence classes (or Reidemeister classes, or twisted conjugacy classes) is denoted by $R(\varphi)$, and we say $G$ has property $R_\infty$ if $R(\varphi)$ is infinite for every automorphism $\varphi$ of $G$. This twisted conjugacy relation first appeared in a work of K. Reidemeister [12] and has many connections with other areas of mathematics, in particular to fixed point theory (see [8]). We refer the introduction of the paper [3] to a good exhibition of those connections and a discussion of the historical context and development of $R_\infty$, including a list of families of groups with this property. The search of $R_\infty$-groups is still very active. For example, a first proof of $R_\infty$ for the pure Artin braid groups $P_n$, $n \geq 3$, was published in 2021 [5] and, even more recently, an alternative proof was obtained in [1].

For groups $G$ with property $R_\infty$, the $R_\infty$-nilpotency degree is the least integer $c$ such that the quotient $G/\gamma_{c+1}(G)$ has property $R_\infty$, where $\gamma_k(G)$ are the terms of the lower central series of $G$, that is, $\gamma_1(G) = G$ and $\gamma_{c+1}(G) = [\gamma_c(G),G]$, for any $c \geq 1$. If none of the quotients have $R_\infty$, we say the $R_\infty$-nilpotency degree of $G$ is infinite. This degree relates to the following problem: it is well known that if $G$ has a characteristic subgroup $N$ (for example, $N = \gamma_k(G)$, $k \geq 1$) such that $G/N$ has $R_\infty$, then $G$ has $R_\infty$. One can ask then...
whether the converse is true, that is, are there groups $G$ whose property $R_\infty$ is still valid for the quotients $G/\gamma_k(G)$? The answer to this question is positive and gives us the more specific motivation for our work, as we will discuss now.

Knowing that the free groups $F_n$ have $R_\infty$ (since they are hyperbolic [9]), the authors K. Dekimpe and D. L. Gonçalves studied property $R_\infty$ for infinitely generated free groups, free nilpotent groups and free solvable groups [2]. They generalized a result of [13], showing that the free nilpotent groups $F_{n,c} = F_n/\gamma_{c+1}(F_n)$, that is, nilpotent quotients of free groups $F_n$, have property $R_\infty$ if and only if $c \geq 2n$. So, the $R_\infty$-nilpotency degree of free groups $F_n$ is $2n$. This fact is actually what motivated the definition of $R_\infty$-nilpotency degree above, which was first given in [3]. In that work, also motivated by the fact that fundamental groups of hyperbolic surfaces have $R_\infty$ (again by [9]), the same authors studied nilpotent quotients of such surface groups, showing that the $R_\infty$-nilpotency degree $c$ of an orientable surface $S_g$ of genus $g \geq 2$ is $c = 4$ and that, for $N_g$ a connected sum of $g \geq 3$ projective planes, $c = 2(g-1)$. It is worth noticing that a consequence of their research was the discovery of new examples of nilmanifolds on which every self-homotopy equivalence can be deformed into a fixed point free map.

On the non-hyperbolic side, an important example of $R_\infty$ groups are the Baumslag-Solitar groups

$$BS(m,n) = \langle a, t \mid ta^mt^{-1} = a^n \rangle,$$

for $m, n$ integers (see [6]). These are important examples of combinatorial and geometric group theory. In [4], K. Dekimpe and D. L. Gonçalves determined the $R_\infty$-nilpotency degree of $BS(m,n)$:

**Theorem 2** (Theorem 5.4 in [4]). Let $0 < m \leq |n|$ with $m \neq n$ and take $d = \gcd(m,n)$. Let $p$ denote the largest integer such that $2^p | 2m/d + 2$. Then, the $R_\infty$-nilpotency degree $r$ of $BS(m,n)$ is given by

- In case $n < 0$ and $n \neq -m$, then $r = 2$.
- In case $n = -m$ then $r = \infty$.
- In case $n = m$ then $r = \infty$.
- In case $n - m = d$, then $r = \infty$.
- In case $n - m = 2d$, then $2 \leq r \leq p + 2$.
- In case $n - m \geq 3d$, then $r = 2$.

In the particular case of the solvable Baumslag-Solitar groups $BS(1,n)$, $n \geq 2$, we have $r = \infty$ for $n = 2$, $r = 4$ for $n = 3$ and $r = 2$ for $n \geq 4$. The investigation of the $R_\infty$-nilpotency degree for generalizations of these groups is therefore natural. The authors comment about the family of Generalized Baumslag-Solitar groups (or GBS groups), which are $R_\infty$ groups. In this paper, we add one more important family to that discussion, namely, the Generalized Solvable Baumslag-Solitar groups $\Gamma_n$. Let $n \geq 2$ be an integer with prime decomposition $n = p_1^{y_1} \cdots p_r^{y_r}$, the $p_i$ being pairwise distinct and $y_i > 0$. We consider the Generalized Solvable Baumslag-Solitar group

$$\Gamma_n = \langle a, t_1, \ldots, t_r \mid t_it_j = t_jt_i, \ i \neq j, \ t_iat_i^{-1} = a^{b_i^{y_i}}, \ i = 1, \ldots, r \rangle.$$

In particular, if $n = p^y$ involves only one prime number, then $\Gamma_n = BS(1,n)$ is a solvable Baumslag-Solitar group. We will focus, therefore, in the case of two or more prime numbers in the decomposition of $n$.

These groups arise mainly in a geometric generalization of the solvable Baumslag-Solitar groups $BS(1,n)$. Consider the group $PSL_2(\mathbb{Z}[1/n])$, which acts on the product of the hyperbolic space $H_2$ with the product of the Bruhat-Tits trees for $PSL_2(\mathbb{Q}_{p_i})$. The stabilizer of a
point at infinity under this action is the upper triangular subgroup of $PSL_2(\mathbb{Z}[1/n])$, which we are denoting by $\Gamma_n$ (see [14]). In this paper, however, dealing with the above presentation was enough to establish our results. The $\Gamma_n$ are also known to be solvable and metabelian groups, fitting in a short exact sequence of the form

$$1 \rightarrow \mathbb{Z} \left[ \frac{1}{n} \right] \rightarrow \Gamma_n \rightarrow \mathbb{Z}^r \rightarrow 1.$$ 

More details about $\Gamma_n$ can be seen in the paper [14], where J. Taback and P. Wong show property for $\Gamma_n$ and for any group quasi-isometric to it.

We started dealing with Baumslag-Solitar groups in the first author’s doctoral dissertation. There, one realized that elementary techniques involving the BNS invariant $\Sigma^1$ of the groups $BS(1, n)$ could also be applied to $\Gamma_n$, leading to an elementary proof of $R_\infty$ to these groups. One can then ask whether other techniques - such as the investigation of nilpotent quotients of $BS(m, n)$ in [14] - can also be adapted to $\Gamma_n$. The main question was whether $\Gamma_n$ would behave more like $BS(1, 2)$ (with infinite $R_\infty$-nilpotency degree) or like $BS(1, n), n \geq 3$ (finite degree cases). As Theorem 1 shows, the groups $\Gamma_n$ fit in the infinite degree case. We believe this happens because of the large automorphism groups of its nilpotent quotients.

We will denote the nilpotent quotients of $\Gamma_n$ by

$$\Gamma_{n,c} = \frac{\Gamma_n}{\gamma_{c+1}(\Gamma_n)}$$

for any $c \geq 1$, where $\gamma_{c+1}$ is the $(c+1)^{th}$ term of the lower central series of $\Gamma_n$. We know that $\Gamma_{n,c}$ is a nilpotent group with nilpotency class $\leq c$, for $\gamma_{c+1}(\Gamma_{n,c}) = \gamma_{c+1}(\Gamma_n)(\Gamma_{n,c}) = \gamma_{c+1}(\Gamma_n) = 1$. In this work, the torsion subgroup of a nilpotent group $G$ will be denoted by 

$$\tau G = \{ g \in G \mid g^n = 1 \text{ for some } n \geq 1 \}.$$ 

This work is divided as follows: in section 2, we compute the torsion subgroup and the lower central series of $\Gamma_{n,c}$. Then, in section 3, we use these computations to create an useful isomorphism between $\Gamma_{n,c}$ and a semidirect product of the form $G_{n,c} = \mathbb{Z}_{m^c} \rtimes \mathbb{Z}^r$, for some $m \geq 1$. Finally, in section 4, we classify which matrices in $GL_r(\mathbb{Z})$ can be extended to automorphisms of $\Gamma_{n,c}$ and use the classification to find specific automorphisms with finite Reidemeister numbers in $\Gamma_{n,c}$.

2. TORSION AND LOWER CENTRAL SERIES OF $\Gamma_{n,c}$

We begin by showing the following

Lemma 2.1. Let $m = \gcd(p_1^{y_1} - 1, \ldots, p_r^{y_r} - 1)$. Then $a^{m^k} \in \gamma_{k+1}(\Gamma_n)$ for all $k \geq 1$.

Proof. Induction on $k$. First, $k = 1$. By using the group relations, note that, for any $1 \leq i \leq r$, $a^{p_{yi}^{r-1}} = t_{i}^{-1} a_{i}^{-1} a_{i}^{-1} = [t_{i}, a] \in \gamma_{2}(\Gamma_n)$. Since this is true for any $i$ and $m$ is an integer combination of the $p_r^{y_i} - 1$, we have $a^{m} \in \gamma_{2}(\Gamma_n)$. Now, suppose the lemma is true for some $k \geq 1$. Then

$$a^{(p_{yi}^{r-1})m^k} = a^{p_{yi}^{r-1}m^k} a^{-m^k} = t_{i} a^{m^k} a_{i}^{m^k} a^{-m^k} = [t_{i}, a^{m^k}] \in \gamma_{k+2}(\Gamma_n).$$

Again, since this is true for any $i$ and $m$ is an integer combination of the $p_r^{y_i} - 1$, we have $a^{m^k} \in \gamma_{k+2}(\Gamma_n)$, as desired. This completes the proof. □

To compute the torsion of the groups $\Gamma_{n,c}$, we need the following standard results:
Proposition 2.2 (see [11]). Let \( k, m, n \geq 1 \) and let \( x, y, z \in G \) be elements of a group \( G \) such that \( x \in \gamma_k(G) \), \( y \in \gamma_m(G) \) and \( z \in \gamma_n(G) \). Then:

a) \( xy = yx \mod \gamma_{k+m}(G) \);

b) \([x, yz] = [x, y][x, z] \mod \gamma_{k+m+n}(G)\);

c) \([xy, z] = [x, z][y, z] \mod \gamma_{k+m+n}(G)\).

Proposition 2.3. (Theorem 5.4) If \( G \) is finitely generated by elements \( x_1, \ldots, x_r \) then, for any \( k \geq 1 \), \( \gamma_k(G)/\gamma_{k+1}(G) \) is abelian and finitely generated by the cosets of the \( k \)-fold commutators \([x_1, \ldots, x_i]\), where \( 1 \leq i \leq r \).

By an easy recursive argument, one shows the following:

Lemma 2.4. Let \( G \) be a nilpotent group of class \( \leq c \) and denote \( \gamma_i = \gamma_i(G) \). If the quotients \( \gamma_2/\gamma_3, \ldots, \gamma_c/\gamma_{c+1} \) are finite, then \( \gamma_2 \) is a torsion subgroup of \( G \). \( \square \)

Proposition 2.5. \( \tau \Gamma_{n,c} = \langle \overline{a}, \gamma_2(\Gamma_{n,c}) \rangle \), where \( \overline{a} = a\gamma_{c+1} = a\gamma_{c+1}(\Gamma_n) \in \Gamma_{n,c} \).

Proof. In the case \( c = 1 \) we have \( \Gamma_{n,1} \) is the abelianized group of \( \Gamma_n \), so

\[
\Gamma_{n,1} = \left\{ \overline{a}, \overline{t_1}, \ldots, \overline{t_r} \mid \overline{t_it_j} = \overline{t_jt_i}, \overline{t_it} = \overline{t_i}, \overline{a^{p_i+1}} = 1 \right\} \cong \mathbb{Z}_m \times \mathbb{Z}_r,
\]

where \( m = \gcd(p_1, p_2, \ldots, p_r) - 1 \). So \( \tau \Gamma_{n,1} = \langle \overline{a} \rangle = \langle \overline{a}, \gamma_2(\Gamma_{n,1}) \rangle \), since \( \gamma_2(\Gamma_{n,1}) = 1 \).

Now let us show the proposition in the case \( c \geq 2 \). For \( (\overline{a} \in \langle \overline{a} \rangle \), let \( x\gamma_{c+1} \in \tau \Gamma_{n,c} \). This means \( x^k \in \gamma_{c+1} \) for some \( k \geq 1 \). Since \( c \geq 2 \), we have \( x^k \in \gamma_{c+1} \subset \gamma_2 \), so \( x^k \in \tau \Gamma_{n,1} = \langle \overline{a} \rangle \).

Write then \( x = a'g_2 \) for \( l \in \mathbb{Z} \) and \( g_2 \in \gamma_2 = \gamma_2(\Gamma_n) \). This gives \( x\gamma_{c+1} = (a\gamma_{c+1})g_2 \gamma_{c+1} \in \langle \overline{a}, \gamma_2(\Gamma_{n,c}) \rangle \), as we wanted. To show \( (\gamma) \), we know that by Lemma 2.4 we get \( \overline{a}^{p_i+1} = 1 \) in \( \Gamma_{n,c} \), so \( \overline{a} \in \tau \Gamma_{n,c} \). So, we just need to show that \( \gamma_2(\Gamma_{n,c}) \) is a torsion subgroup of \( \Gamma_{n,c} \). To do this, we invoke Lemma 2.4, by which we know it is enough to show the quotients

\[
\frac{\gamma_2(\Gamma_{n,c})}{\gamma_3(\Gamma_{n,c})} = \ldots = \frac{\gamma_c(\Gamma_{n,c})}{\gamma_{c+1}(\Gamma_{n,c})}
\]

are all finite. But for every \( 2 \leq i \leq c \), by the known Isomorphism Theorem for quotients, we have

\[
\frac{\gamma_i(\Gamma_{n,c})}{\gamma_{i+1}(\Gamma_{n,c})} = \frac{\gamma_i(\Gamma_n)/\gamma_{i+1}(\Gamma_n)}{\gamma_{i+1}(\Gamma_n)/\gamma_{i+1}(\Gamma_n)} \simeq \frac{\gamma_i(\Gamma_n)}{\gamma_{i+1}(\Gamma_n)} = \gamma_i/\gamma_{i+1},
\]

so let us show that \( \gamma_2/\gamma_3, \ldots, \gamma_c/\gamma_{c+1} \) are finite by induction. By Proposition 2.5, we know they are abelian groups, generated by their \( i \)-fold commutator cosets. The group \( \gamma_2/\gamma_3 \) is generated by the elements \([t_i, a] \gamma_3, 1 \leq i \leq r \) and by \([t_i, t_j] \gamma_3 = 1 \gamma_3 = \gamma_3 \), which are trivial. By Proposition 2.2, we get \([t_i, a]^m \gamma_3 = [t_i, a^m] \gamma_3 = \gamma_3 \), since \( a^m \in \gamma_2 \) (Lemma 2.1). So all generators of \( \gamma_2/\gamma_3 \) have torsion. Since it is finitely generated and abelian, it must be a finite group. Finally, suppose by induction that \( \gamma_i/\gamma_{i+1} \) is finite for some \( i \geq 2 \). By Proposition 2.2, \( \gamma_{i+1}/\gamma_{i+2} \) is then generated by the elements of the form \([x, y] \gamma_{i+2} \) with \( x \in \gamma_i \) and \( y \in \Gamma_n \). Since \( \gamma_i/\gamma_{i+1} \) is finite, let \( k = k(x, y) \geq 1 \) such that \( x^k \in \gamma_{i+1} \). Then \([x, y] \gamma_{i+2} = [x^k, y] \gamma_{i+2} = \gamma_{i+2} \). By the same argument we just used, this implies \( \gamma_{i+1}/\gamma_{i+2} \) is finite and completes the proof. \( \square \)

Proposition 2.6. \( \gamma_k(\Gamma_{n,c}) = \left\langle \overline{a}^{p_i-1} \right\rangle \) for all \( k \geq 2 \) and \( c \geq 1 \).

Proof. First, we will show that

\[
\frac{\gamma_k(\Gamma_{n,c})}{\gamma_k(\Gamma_{n,c})} = \left\langle \overline{a}^{p_i-1} \right\rangle \gamma_{k+1}(\Gamma_{n,c}) \right\rangle.
\]
For \( k = 2 \), by Proposition 2.3 \( \frac{\gamma_2(\Gamma_{n,c})}{\gamma_3(\Gamma_{n,c})} \) is generated by the cosets \([\bar{t}_i, \bar{u}]\gamma_3(\Gamma_{n,c})\). Since \([\bar{t}, \bar{u}] = \bar{w}^{m-1}\), we have

\[
\frac{\gamma_2(\Gamma_{n,c})}{\gamma_3(\Gamma_{n,c})} = \left\langle \bar{w}^{m-1}\gamma_3(\Gamma_{n,c}), \ldots, \bar{w}^{m-1}\gamma_3(\Gamma_{n,c}) \right\rangle = \left\langle \bar{w}^{m-1}\gamma_3(\Gamma_{n,c}) \right\rangle
\]

(remember that \( m = \gcd(p_{\gamma i} - 1, \ldots, p_{\gamma r} - 1) \)). Suppose now (1) is true for some \( k \geq 2 \). We know \( \frac{\gamma_{k+1}(\Gamma_{n,c})}{\gamma_{k+2}(\Gamma_{n,c})} \) is generated by the cosets \([x, z]\gamma_{k+2}(\Gamma_{n,c}), \) where \( x \in \gamma_k(\Gamma_{n,c}) \) and \( z \in \Gamma_{n,c} \).

By induction, we can write \( x = \bar{w}^{m k-1} w_{k+1} \) for some \( w_{k+1} \in \gamma_{k+1}(\Gamma_{n,c}) \) and \( \alpha \in \mathbb{Z} \). Then, by using Proposition 2.2 we get

\[
[x, z]\gamma_{k+2}(\Gamma_{n,c}) = [\bar{w}^{m k-1} w_{k+1}, z]\gamma_{k+2}(\Gamma_{n,c})
\]

so the quotient \( \frac{\gamma_{k+1}(\Gamma_{n,c})}{\gamma_{k+2}(\Gamma_{n,c})} \) is actually generated only by the cosets \([\bar{w}^{m k-1}, z]\gamma_{k+2}(\Gamma_{n,c})\). Since \([\bar{u}, \bar{w}]\) is obviously trivial, the quotient group is generated only by the generators \([\bar{u}, \bar{w}]\gamma_{k+2}(\Gamma_{n,c})\).

Since \([\bar{u}, \bar{w}] = \bar{w}^{m k-1} \), we obtain

\[
\frac{\gamma_{k+1}(\Gamma_{n,c})}{\gamma_{k+2}(\Gamma_{n,c})} = \left\langle \bar{w}^{m k-1} w_{k+1}, \ldots, \bar{w}^{m k-1} w_{k+2}, \right\rangle = \left\langle \bar{w}^{m k-1} \gamma_{k+2}(\Gamma_{n,c}) \right\rangle,
\]

where

\[
\beta = \gcd((p_{\gamma i} - 1) m k-1, \ldots, (p_{\gamma r} - 1) m k-1) = m k-1 \gcd(p_{\gamma i} - 1, \ldots, p_{\gamma r} - 1) = m^k
\]

and this shows (1). Now, let us show the proposition. The (\(\supset\)) part is a direct consequence of Lemma 2.1. Let us show (\(\subset\)). In the case \( c < k \), we have \( \gamma_k(\Gamma_{n,c}) = 1 \) \textcolor{red}{
(2)} \(\subset\) \textcolor{red}{
\left\langle \bar{w}^{m k-1} \right\rangle \). Suppose then \( c \geq k \) and let \( x \in \gamma_k(\Gamma_{n,c}) \). Since \( x \gamma_{k+1}(\Gamma_{n,c}) \in \left\langle \bar{w}^{m k-1} \gamma_{k+1}(\Gamma_{n,c}) \right\rangle \) (by (1)), write \( x = \bar{w}^{j k m k-1} x_{k+1} \) for \( j_k \in \mathbb{Z} \) and \( x_{k+1} \in \gamma_{k+1}(\Gamma_{n,c}). \) By using (1) again, we write

\[
x_{k+1} = \bar{w}^{m k m k-1} x_{k+2} \text{ for } j_k \in \mathbb{Z} \text{ and } x_{k+2} \in \gamma_{k+2}(\Gamma_{n,c}).
\]

We can do this recursively to obtain

\[
x = \bar{w}^{j k m k-1} \bar{w}^{j k m k-1+1} \ldots \bar{w}^{j k m k-1+k} x_{k+1}
\]

and the proof is complete.

By Lemma 2.1 and the two propositions above, we get

\textbf{Corollary 2.7.} \( \tau \Gamma_{n,c} = \langle \bar{u} \rangle \) and \( \text{card}(\tau \Gamma_{n,c}) \leq m^c \).

\section{An Isomorphism for \( \Gamma_{n,c} \)}

The next step is to find a presentation to \( \Gamma_{n,c} \), so we will find an isomorphism between \( \Gamma_{n,c} \) and a more known group. We will use the notations from the previous section and will also denote \( \mathbb{Z} m^c = \langle x \mid x^{m^c} = 1 \rangle \) and \( \mathbb{Z}^r = \langle s_1, \ldots, s_r \mid s_i s_j = s_j s_i \rangle \). We define the group

\[
G_{n,c} = \mathbb{Z} m^c \rtimes \mathbb{Z}^r
\]

where the action of \( \mathbb{Z}^r \) on \( \mathbb{Z} m^c \) is given by \( s_i x s_i^{-1} = x^{p_{\gamma i}} \), \( 1 \leq i \leq r \).
Observation 3.1. Note first that the actions defined above are all automorphisms of \( \mathbb{Z}_{m^c} \), since \( \gcd(p_i^k, m) = 1 \) (and so \( \gcd(p_i^k, m^c) = 1 \) for any \( c \geq 1 \)). Second, all such automorphisms commute, for \( \mathbb{Z}_{m^c} \) is cyclic. These facts show that there is a well defined homomorphism \( \mathbb{Z}' \to \text{Aut}(\mathbb{Z}_{m^c}) \), so this semidirect product is well defined.

We will show that \( \Gamma_{n,c} \simeq G_{n,c} \). To do this, we need:

**Lemma 3.2.** \( G_{n,c} \) is nilpotent of class \( \leq c \).

**Proof.** Since \([s_i, x] = x^{p_i - 1} \in \langle x^m \rangle\) for every \( i \), we have \( \gamma_2(G_{n,c}) \subset \langle x^m \rangle \). Similarly, since \([s_i, x^m] = x^{(p_i^k - 1)m} \in \langle x^m \rangle \) for every \( i \), in particular we have \([s_i, z] \in \langle x^{m^2} \rangle\) for every \( z \in \gamma_2(G_{n,c}) \), so it is easy to see that \( \gamma_3(G_{n,c}) \subset \langle x^{m^2} \rangle \). Recursively, we can show that \( \gamma_k(G_{n,c}) \subset \langle x^{m^{k-1}} \rangle\) for every \( k \geq 2 \). In particular, \( \gamma_{c+1}(G_{n,c}) \subset \langle x^{m^c} \rangle = 1 \), since \( x^{m^c} = 1 \) in \( \mathbb{Z}_{m^c} \). This shows the lemma. \( \square \)

**Corollary 3.3.** \( \tau G_{n,c} \) is a subgroup of \( G_{n,c} \). Moreover, \( \tau G_{n,c} = \mathbb{Z}_{m^c} = \langle x \rangle \) and so \( \text{card}(\tau G_{n,c}) = m^c \).

**Theorem 3.4.** \( \Gamma_{n,c} \simeq G_{n,c} \).

**Proof.** Let \( f : \Gamma_n \to G_{n,c} \) be the map \( f(a) = x = f(t_i) = s_i \). Since \( f(t_i)f(a)f(t_i)^{-1} = s_i x s_i^{-1} = x^{p_i^k} = f(a)^{p_i^k} \), \( f \) is a well defined group homomorphism. Since \( f(\gamma_i(\Gamma_n)) \subset \gamma_i(G_{n,c}) \), \( f \) induces the morphism

\[
f : \Gamma_{n,c} = \frac{\Gamma_n}{\gamma_{c+1}(\Gamma_n)} \to \frac{G_{n,c}}{\gamma_{c+1}(G_{n,c})} = G_{n,c}
\]

given by \( f(\overline{a}) = x \) and \( f(\overline{t_i}) = s_i \). It is obviously surjective. We are just left to show that \( \ker(f) = 1 \), and to do that we will make use of the torsion subgroups. Since \( f(\tau \Gamma_{n,c}) \subset \tau G_{n,c} \) (this is true for any homomorphisms between nilpotent groups), there is the restriction morphism \( f_r : \tau \Gamma_{n,c} \to \tau G_{n,c} \). By Corollaries 2.7 and 3.3 we can actually write \( f_r : \langle \overline{a} \rangle \to \langle x \rangle \). Since \( f_r(\overline{a}) = x \), it is clearly surjective. Now, \( f_r \) is a surjective map from a finite set of \( \leq m^c \) elements (Corollary 2.7) to a finite set with exactly \( m^c \) elements (Corollary 3.3), so we must have \( \text{card}(\langle \overline{a} \rangle) = m^c \) and \( f_r \) an isomorphism. In particular, \( \ker(f_r) = 1 \). We claim that \( \ker(f) \subset \tau \Gamma_{n,c} \). In fact, let \( z \in \ker(f) \). By using the relations in \( \Gamma_n \), we can write

\[
z = \overline{l_1^k_1 \cdots l_r^k_r \alpha_1} \overline{\alpha_1} \overline{\alpha_r} \cdots \overline{\alpha_1}
\]

for \( k_i, l \in \mathbb{Z} \) and \( \alpha_i \geq 0 \). So

\[
1 = f(z) = s_1^{k_1} \cdots s_r^{k_r} s_1^{-\alpha_1} \cdots s_r^{-\alpha_r} x^{l_1} s_1^{\alpha_1} \cdots s_r^{\alpha_r}
\]

Since \( x \in \tau G_{n,c} \subset \Gamma_{n,c} \) we have \( s_1^{-\alpha_1} \cdots s_r^{-\alpha_r} x^{l_1} s_1^{\alpha_1} \cdots s_r^{\alpha_r} \in \tau G_{n,c} = \langle x \rangle \), so \( 1 = f(z) = s_1^{k_1} \cdots s_r^{k_r} x^{l' \alpha} \) for some \( l' \in \mathbb{Z} \). By projecting this equality under the natural homomorphism \( G_{n,c} \to \mathbb{Z}' \) we get \( 1 = s_1^{k_1} \cdots s_r^{k_r} \), which implies \( k_i = 0 \) for every \( i \). Therefore \( z = \overline{l_1^k_1 \cdots l_r^k_r \alpha_1} \overline{\alpha_1} \overline{\alpha_r} \cdots \overline{\alpha_1} \in \tau G_{n,c} \), since \( \overline{\alpha} \in \tau \Gamma_{n,c} < \Gamma_{n,c} \), which shows the claim. Finally, this gives \( \ker(f) = \ker(f_r) \cap \tau \Gamma_{n,c} = \ker(f_r) = 1 \) and the theorem is proved. \( \square \)

**Corollary 3.5.** For any \( c \geq 1 \), the nilpotent quotient \( \Gamma_{n,c} \) has the following presentation:

\[
\Gamma_{n,c} = \langle x, s_1, \ldots, s_r \ | \ x^{m^c} = 1, s_is_j = s_js_i, s_is_i^{-1} = x^{p_i^k} \rangle.
\]
4. Reidemeister numbers

Because of the theorem above, from now on we will make the following identifications
\[ \Gamma_{n,c} = G_{n,c} = \mathbb{Z}_{m_c} \rtimes \mathbb{Z}^r = \langle x \rangle \rtimes \langle s_1, \ldots, s_r \rangle. \]

It’s also worth remembering that we will restrict us to investigate Reidemeister numbers of \( \Gamma_{n,c} \) only in the case \( r \geq 2 \), for, if \( r = 1 \), then \( \Gamma_n \) is by definition a Baumslag-Solitar group \( BS(1,n) \) and its Reidemeister numbers were studied in \([7]\). Let \( \varphi \in Aut(\Gamma_{n,c}) \). Since \( \varphi(\tau \Gamma_{n,c}) \subset \tau \Gamma_{n,c} \), we have an induced automorphism
\[ \overline{\varphi} : \frac{\Gamma_{n,c}}{\tau \Gamma_{n,c}} = \mathbb{Z}^r \to \mathbb{Z}^r = \frac{\Gamma_{n,c}}{\tau \Gamma_{n,c}}. \]

From now on, we will use the usual identification \( Aut(\mathbb{Z}^r) = GL_r(\mathbb{Z}) \) which sees an automorphism of \( \mathbb{Z}^r \) as its (integer invertible) matrix with respect to the coordinates \( s_i \). So, if \( \overline{\varphi}(s_i) = s_1^{a_{i1}} \cdots s_r^{a_{ir}} \), we will identify
\[ \overline{\varphi} = (a_{ij})_{ij} = \begin{bmatrix} a_{11} & \cdots & a_{1r} \\ \vdots & \ddots & \vdots \\ a_{r1} & \cdots & a_{rr} \end{bmatrix} = [A_1 \cdots A_r], \quad \text{where} \quad A_i = \begin{bmatrix} a_{ii} \\ \vdots \\ a_{ri} \end{bmatrix} \in \mathbb{Z}^r. \]

**Proposition 4.1.** If \( \varphi \in Aut(\Gamma_{n,c}) \), the following are equivalent:

1. \( R(\varphi) = \infty \);
2. \( R(\overline{\varphi}) = \infty \);
3. \( \det(\overline{\varphi} - Id) = 0 \);
4. \( \overline{\varphi} \) has 1 as an eigenvalue.

**Proof.** Items (2), (3) and (4) are well known to be all equivalent. Also, we have an obvious commutative diagram involving the automorphisms \( \varphi, \overline{\varphi} \) and the projection \( \pi : \Gamma_{n,c} \to \mathbb{Z}^r \). Thus, by Lemma 1.1 of \([7]\), it follows that (2) implies (1). So we only have to prove that (1) implies (2).

To simplify the computation, let us use the following notation in this proof: given \( y = (y_1, \ldots, y_r) \in \mathbb{Z}^r \) (either a row or a column vector), we will denote the element \( s_1^{y_1} \cdots s_r^{y_r} \in \Gamma_{n,c} \) by \( S^y \), and the scalar product of \( k \in \mathbb{Z} \) by \( y \) is denoted by \( ky \). With this notation, it turns out that any element of \( \Gamma_{n,c} \) is of the form \( S^y x^\beta \) for some \( y \in \mathbb{Z}^r \) and \( \beta \in \mathbb{Z} \). Suppose then that \( R(\overline{\varphi}) = d < \infty \) and write \( \mathcal{R}(\overline{\varphi}) = \{[v_1], \ldots, [v_d]\} \) for \( v_i \in \mathbb{Z}^r \) or, equivalently, \( \overline{\mathcal{R}(\overline{\varphi})} = \{\overline{v_1}, \ldots, \overline{v_d}\} \) (where \( \overline{v_i} = v_i + \im(\overline{\varphi} - Id) \)). Write \( \varphi(x) = x^\mu \) (for some \( \mu \in \mathbb{Z} \) with \( \gcd(\mu, m^c) = 1 \)) and \( \varphi(s_i) = S^{k_i} x^{\beta_i}, \beta_i \in \mathbb{Z} \). Given that the \( s_i \)-coordinates behave well in the \( \Gamma_{n,c} \), for any \( k = (k_1, \ldots, k_r) \in \mathbb{Z}^r \) and \( l \in \mathbb{Z} \) we have
\[ \varphi(S^k x^l) = S^{\overline{\varphi(k)}} x^\theta, \]
for some \( \theta \in \mathbb{Z} \). This implies that, for any \( j \in \mathbb{Z} \) and \( y \in \mathbb{Z}^r \),
\[ (S^k_x^l)(S^y x^j) \varphi(S^k_x^l)^{-1} = S^{y+(Id-\overline{\varphi}(k))} x^\theta, \]
for some \( \theta \in \mathbb{Z} \). This means that, if two vectors \( y, y' \in \mathbb{Z}^r \) are such that \( \overline{y} = \overline{y}' \in \im(\overline{\varphi} - Id) \), then every element \( S^y x^j \) is \( \varphi \)-conjugated to some element \( S^{y'} x^\theta \) for some \( 0 \leq \theta < m^c \). Since \( \overline{\mathcal{R}(\overline{\varphi})} = \{\overline{v_1}, \ldots, \overline{v_d}\} \), every element \( S^y x^j \) is \( \varphi \)-conjugated to some \( S^{v_i} x^\theta \), \( 1 \leq i \leq d, 0 \leq \theta < m^c \), so \( R(\varphi) \leq dm^c < \infty \) and the proposition is proved. \( \square \)
In the rest of the work we will use the following notation: we know that \( \gcd(p_i^{y_i}, m^c) = 1 \). This means that \( p_i^{y_i} \) is an invertible element in the commutative ring \( \mathbb{Z}_{m^c} \) (now thought in the abelian notation \( \mathbb{Z}_{m^c} = \{0, 1, \ldots, m^c - 1\} \)). So, we will naturally denote by \( p_i^{-y_i} \) the inverse element \( (p_i^{y_i})^{-1} \in \mathbb{Z}_{m^c} \) and, similarly, we define \( p_i^{-k y_i} \) as \( (p_i^{ky_i})^{-1} \) for any \( k \geq 0 \), so it makes sense to write \( p_i^{ky_i} \) for any \( k \in \mathbb{Z} \), thinking of it as an invertible element of the ring \( \mathbb{Z}_{m^c} \). We are saying this to avoid a possible misinterpretation of \( p_i^{-y_i} \) as \( \frac{1}{p_i^{y_i}} \in \mathbb{Q} \), for example. With this notation, it is clear that \( s_i^k x s_i^{-k} = x p_i^{ky_i} \) for any \( k \in \mathbb{Z} \).

**Proposition 4.2.** \( \Gamma_{n,c} \) has not property \( R_\infty \) if and only if there is \( M = (a_{ij})_{ij} \in \text{GL}_r(\mathbb{Z}) \) such that

- \( \det(M - Id) \neq 0 \);
- for any \( 1 \leq i \leq r \),
  \[
  p_1^{a_{1i}y_1} p_2^{a_{2i}y_2} \cdots p_r^{a_{ri}y_r} \equiv p_i^{y_i} \mod m^c. \quad (M, c, i)
  \]

**Proof.** Suppose first that \( \Gamma_{n,c} \) has not property \( R_\infty \). Let \( \varphi \in \text{Aut}(\Gamma_{n,c}) \) such that \( R(\varphi) < \infty \). Let \( M = \varphi \in \text{GL}_r(\mathbb{Z}) \), and write \( M = (a_{ij})_{ij} \). By Proposition 4.1 we have \( \det(M - Id) \neq 0 \). Since \( \varphi(\tau \Gamma_{n,c}) \subset \tau \Gamma_{n,c} \), we have \( \varphi(x) = x^\mu \) for some \( \mu \in \mathbb{Z} \) such that \( \gcd(\mu, m^c) = 1 \). Let us show that for any \( 1 \leq i \leq r \) the equation \( (M, c, i) \) holds. For any such \( i \), since \( \varphi \) is a homomorphism of \( \Gamma_{n,c} \) it must satisfy \( \varphi(s_i) \varphi(x) \varphi(s_i)^{-1} = \varphi(x)p_i^{y_i} \), so

\[
  s_1^{a_{1i}} \cdots s_r^{a_{ri}} x^\mu s_r^{-a_{ri}} \cdots s_1^{-a_{1i}} = x^{\mu p_i^{y_i}}
  \]
or, equivalently,

\[
  x^{\mu p_i^{a_{1i}y_1}} \cdots x^{\mu p_i^{a_{ri}y_r}} = x^{\mu p_i^{y_i}}.
  \]

Then \( \mu p_i^{a_{1i}y_1} \cdots p_r^{a_{ri}y_r} \equiv \mu p_i^{y_i} \mod m^c \), and since \( \gcd(\mu, m^c) = 1 \), we have \( p_1^{a_{1i}y_1} \cdots p_r^{a_{ri}y_r} \equiv p_i^{y_i} \mod m^c \), which is exactly \( (M, c, i) \). This shows the “if” part. Suppose now that there is such a matrix \( M = (a_{ij})_{ij} \) and let us show \( \Gamma_{n,c} \) has not \( R_\infty \). Define \( \varphi : \Gamma_{n,c} \to \Gamma_{n,c} \) by \( \varphi(x) = x \) and \( \varphi(s_i) = s_1^{a_{1i}} s_2^{a_{2i}} \cdots s_r^{a_{ri}} \). Let us check that \( \varphi \) is a well defined homomorphism:

\[
  \varphi(s_i) \varphi(x) \varphi(s_i)^{-1} = s_1^{a_{1i}} s_2^{a_{2i}} \cdots s_r^{a_{ri}} x^{\mu \cdot a_{ri}} \cdots s_r^{a_{ri}} s_1^{a_{1i}} = x^{\mu p_i^{a_{1i}y_1}} \cdots x^{\mu p_i^{a_{ri}y_r}} = \varphi(x)p_i^{a_{1i}y_1} \cdots p_i^{a_{ri}y_r} = \varphi(x)p_i^{y_i},
  \]

the last equality being true by \( (M, c, i) \). Also, since the \( s_i \) commute, we obviously have

\[
  \varphi(s_i) \varphi(s_j) = s_1^{a_{1i}} \cdots s_r^{a_{ri}} s_1^{a_{1j}} \cdots s_r^{a_{rij}} = s_1^{a_{1j}} \cdots s_r^{a_{rij}} s_1^{a_{1i}} \cdots s_r^{a_{ri}} = \varphi(s_j) \varphi(s_i).
  \]

Finally,

\[
  \varphi(x)^m = x^{m^c} = 1,
  \]

so \( \varphi \) is in fact a homomorphism. Let us now construct an inverse homomorphism. Let \( N = M^{-1} \in \text{GL}_r(\mathbb{Z}) \) and write \( N = (b_{ij})_{ij} \). Let us show that, for any \( 1 \leq i \leq r \), \( N \) satisfies the equation \( (N, c, i) \), that is \( p_1^{b_{1i}y_1} p_2^{b_{2i}y_2} \cdots p_r^{b_{ri}y_r} = p_i^{y_i} \mod m^c \). Since \( MN = Id \), for any \( 1 \leq i, j \leq r \) we have

\[
  \sum_{k=1}^r a_{ik} b_{kj} = (MN)_{ij} \equiv Id_{ij} = \delta_{ij},
  \]

where \( \delta_{ij} \) is the Kronecker delta. Fix \( i \). We do the following: for each fixed \( 1 \leq j \leq r \), we raise both sides of equation \( (M, c, j) \) to the power of \( b_{ji} \) and obtain

\[
  p_1^{a_{1j}b_{1ji}y_1} p_2^{a_{2j}b_{2ji}y_2} \cdots p_r^{a_{rij}b_{rji}y_r} \equiv p_j^{b_{rji}y_j} \mod m^c.
  \]
Now, if we do the product of all the \( r \) equations above (on both sides, of course) and rearrange the left side according to the primes we get

\[
p_{1}^{(a_{11}b_{11}+\cdots+a_{1r}b_{1r})y_{1}}p_{2}^{(a_{21}b_{11}+\cdots+a_{2r}b_{1r})y_{2}}\cdots p_{r}^{(a_{r1}b_{11}+\cdots+a_{r1}b_{1r})y_{r}} \equiv p_{1}^{b_{11}y_{1}}p_{2}^{b_{21}y_{2}}\cdots p_{r}^{b_{r1}y_{r}} \mod m^{c},
\]
or

\[
p_{1}^{(\sum_{k}a_{1k}b_{1k})y_{1}}p_{2}^{(\sum_{k}a_{2k}b_{1k})y_{2}}\cdots p_{r}^{(\sum_{k}a_{rk}b_{1k})y_{r}} \equiv p_{1}^{b_{11}y_{1}}p_{2}^{b_{21}y_{2}}\cdots p_{r}^{b_{r1}y_{r}} \mod m^{c},
\]
or even

\[
\prod_{i=1}^{r}p_{1}^{s_{i1}y_{1}}p_{2}^{s_{21}y_{2}}\cdots p_{r}^{s_{r1}y_{r}} \equiv p_{1}^{b_{11}y_{1}}p_{2}^{b_{21}y_{2}}\cdots p_{r}^{b_{r1}y_{r}} \mod m^{c},
\]

which results in

\[
p_{1}^{y_{1}} \equiv p_{1}^{b_{11}y_{1}}p_{2}^{b_{21}y_{2}}\cdots p_{r}^{b_{r1}y_{r}} \mod m^{c},
\]

which is exactly \((N, c, i)\), as we wanted. Now define \( \psi : \Gamma_{n,c} \to \Gamma_{n,c} \) by \( \psi(x) = x \) and \( \psi(s_{i}) = s_{1}^{b_{i1}}s_{2}^{a_{i2}}\cdots s_{r}^{b_{ri}}. \) As we did with \( \varphi \), the fact that \( N \) satisfies \((N, c, i)\) for all \( i \) gives us that \( \psi \) is a group homomorphism. Of course we have \( \varphi(\psi(x)) = x. \) Also, by the fact that \( MN = Id \), straightforward calculations show that \( \varphi(\psi(s_{i})) = s_{i}. \) Similarly, we show that \( \psi \varphi = Id \) by using that \( NM = Id, \) so \( \varphi \in Aut(\Gamma_{n,c}). \) Since \( \overline{\varphi} = M \) we have \( \det(\overline{\varphi} - Id) = \det(M - Id) \neq 0 \) by hypothesis, so \( R(\varphi) < \infty \) by Proposition 4.1. This completes the proof.

**Observation 4.3:** Implicit in the proof of Proposition 4.2 above is the classification of all matrices in \( GL_{r}(\mathbb{Z}) \) which can be extended to automorphisms of \( \Gamma_{n,c}. \) In other words, given a matrix \( M = (a_{ij})_{ij} \in GL_{r}(\mathbb{Z}), \) there is an automorphism \( \varphi \) of \( \Gamma_{n,c} \) such that \( \overline{\varphi} = M \) if and only if all equations \((M, c, i)\) are satisfied.

To proceed, we need the following lemma, which can be easily shown by elementary number theory and induction on \( k: \)

**Lemma 4.4.** Let \( x, m \geq 2. \) If \( x = 1 \mod m, \) then \( x^{m^{k}} = 1 \mod m^{k+1} \) for any \( k \geq 0. \)

**Theorem 4.5.** Let \( n \geq 2 \) have prime decomposition \( n = p_{1}^{y_{1}}\cdots p_{r}^{y_{r}}, \) the \( p_{i} \) being pairwise distinct and \( y_{i} > 0. \) Suppose \( r \geq 2, \) that is, there are at least two primes involved. Then the nilpotent quotient group \( \Gamma_{n,c} = \Gamma_{n}/\gamma_{c+1}(\Gamma_{n}) \) does not have property \( R_{\infty} \) for any \( c \geq 1. \) In other words, the \( R_{\infty} \)-nilpotency degree of \( \Gamma_{n} \) is infinite.

**Proof.** Let \( m = \gcd(p_{1}^{y_{1}} - 1, \ldots, p_{r}^{y_{r}} - 1), \) as we have done in this work. If \( m = 1, \) then none of the groups \( \Gamma_{n,c} \) have property \( R_{\infty}. \) This is because \( \Gamma_{n,c} \simeq \mathbb{Z}^{r} \) for any \( c \) in this case (see Theorem 3.4), and we know \( \mathbb{Z}^{r} \) has not \( R_{\infty}. \) So, from now on, suppose \( m \geq 2. \) Of course \( \Gamma_{n,1} \) does not have property \( R_{\infty}, \) for it is a finitely generated abelian group. Now, for any fixed \( c \geq 2, \) we will use Proposition 4.2 that is, for any \( r \geq 2, \) we will find a matrix \( M = (a_{ij})_{ij} \in GL_{r}(\mathbb{Z}) \) with \( \det(M - Id) \neq 0 \) and satisfying equations \((M, c, i)\) for \( 1 \leq i \leq r. \) We will look for a particular family of matrices \( M, \) that is,

\[
M = m^{k}N + Id.
\]

Here, \( k \) will be some suitable positive number, \( N = (j_{a\beta})_{a\beta} \) will be some integer \( r \times r \) matrix with determinant 1 and \( m^{k}N = (m^{k}j_{a\beta})_{a\beta} \) is the natural scalar product of a number by a matrix. The first thing to observe is that any such matrix \( M \) satisfies all the equations \((M, c, i)\) for some big enough \( k \geq 1. \) Let us see that. It is easy to see that, for such \( M, \) the equations \((M, c, i)\) become exactly

\[
(p_{1}^{y_{1}m}p_{2}^{y_{2}}\cdots p_{r}^{y_{r}})m^{k} \equiv 1 \mod m^{c}. \quad (M, c, i)
\]

For us to use the previous lemma, the term inside the parenthesis in the above equation must be congruent to 1 modulo \( m, \) so we claim this is true. Since \( m \) divides each number
of the determinant of
with $d$
Now let $k = c - 1$. By the above lemma we have $(p_1^{i_1}p_2^{i_2} \ldots p_r^{i_r})^m = 1 \mod m^c$, so for every $i$, equation $(M, c, i)$ is satisfied for such $M$.

It is then enough for us to find, for any $r \geq 2$, an integer matrix $N$ which makes $\det(M) = 1$ and $\det(M - Id) \neq 0$. Since $M = m^k N + Id$, we have

$$\det(M - Id) = \det(m^k N) = m^r \det(N),$$

so for $\det(M - Id)$ to be non-zero it suffices us to have $\det(N) \neq 0$. We claim therefore that, for any $r \geq 2$, there is a matrix $N$, such that $\det(N, r) = 1$ and $\det(M, r) = \det(m^k N, r + Id) = 1$.

For any $r \geq 2$, let

$$N_r = \begin{bmatrix}
1 & -(m^k + 2) & m^k + 1 & -(m^k + 1) & \ldots & (-1)^{r-4}(m^k + 1) & (-1)^{r-3}(m^k + 1) \\
1 & -(m^k + 1) & m^k & -m^k & \ldots & (-1)^{r-4}m^k & (-1)^{r-3}m^k \\
0 & 1 & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & 1 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & 1 & 0
\end{bmatrix}.$$ 

By developing the determinant of $N_r$, using the last column, we get that $\det(N, r) = 1$, since the two submatrices that appear are upper triangular with diagonal entries equal 1. Now, our task is to prove that $\det(M, r) = 1$, where

$$M_r = \begin{bmatrix}
d & -m^k(m^k + 2) & m^k d & -m^k d & \ldots & (-1)^{r-4}m^k d & (-1)^{r-3}m^k d \\
m^k & -m^k d + 1 & m^{2k} & -m^{2k} & \ldots & (-1)^{r-4}m^{2k} & (-1)^{r-3}m^{2k} \\
0 & m^k & 1 & 0 & \ldots & 0 & 0 \\
0 & 0 & m^k & 1 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & 1 & 0 \\
0 & 0 & 0 & 0 & \ldots & m^k & 1
\end{bmatrix}$$

with $d = m^k + 1$. We will prove this by induction. The case $r = 2$ is verified by the calculation of the determinant of

$$M_2 = \begin{bmatrix}
m^k + 1 & -m^k(m^k + 2) \\
m^k & -m^k(m^k + 1) + 1
\end{bmatrix}.$$ 

Now, for $r > 2$, developing the determinant of $M_r$ by the last column gives us:

$$\det(M_r) = (-1)^{r+1}(-1)^{r-3}m^k d m^{k(r-1)} + (-1)^{r+2}(-1)^{r-3}m^{2k} d m^{k(r-2)} + (-1)^{2r} \det(M_{r-1}) = m^{kr} d - m^{kr} d + 1 = 1.$$ 

This completes the induction step and finishes our proof. □

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