Variation and Rough Path Properties of Local Times of Lévy Processes

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Summary. In this paper, we will prove that the local time of a Lévy process is of finite p-variation in the space variable in the classical sense, a.s. for any $p > 2$, $t ≥ 0$, if the Lévy measure satisfies $\int_{\mathbb{R}} \{ |y|^2 \wedge 1 \} n(dy) < \infty$, and is a rough path of roughness $p$ a.s. for any $2 < p < 3$ under a slightly stronger condition for the Lévy measure. Then for any function $g$ of finite $q$-variation ($1 ≤ q < 3$), we establish the integral $\int_{-\infty}^{\infty} g(x) dL^x_t$ as a Young integral when $1 ≤ q < 2$ and a Lyons’ rough path integral when $2 ≤ q < 3$. We therefore apply these path integrals to extend the Tanaka-Meyer formula for a continuous function $f$ if $\nabla^\perp f$ exists and is of finite $q$-variation when $1 ≤ q < 3$, for both continuous semi-martingales and a class of Lévy processes.

Keywords: semimartingale local time; geometric rough path; finite $p$-variation; Young integral; rough path integral; Lévy processes.

1 Introduction

The variation of a stochastic process is a classical problem of fundamental importance in probability theory. There are two kinds of variations, namely in the sense of probability and in the classical sense. The quadratic variation of a Brownian motion (a martingale) in the sense of probability made it possible to define Itô’s stochastic integral of a square integrable progressive process with respect to the Brownian motion (the martingale) (Lévy [20], Itô [16], Kunita and Watanabe [18]). The classical $p$-variation of the Brownian motion and its Lévy area, $p > 2$, led to Lyons’ pathwise approach to stochastic differential equations (Lyons [21], [22]). Local time is an important and useful stochastic process. The investigation of its variation and integration has attracted attentions of many mathematicians. Similar to the case of the Brownian motion, the variation of the local time of a semimartingale in the spatial variable is also fundamental in the construction of an integral with respect to the local time. There have been many works on the quadratic or $p$-variations (in the case of stable processes) of local times in the sense of probability. Bouleau and Yor (23), Perkins (28), first proved that, for the Brownian local time, and a sequence of partitions $\{ D_n \}$ of an interval $[a, b]$, with the mesh $|D_n| \to 0$ when $n \to \infty$, $\lim_{n \to \infty} \sum_{D_n} (L^{x+1}_{a+1} - L^x_a)^2 = 4 \int_a^b L^x_t \, dx$ in probability. Subsequently, the process $x \to L^x_t$ can be regarded as a semimartingale (with appropriate filtration). This result allowed one to construct various stochastic integrals of the Brownian local time in the spatial variable. See also Rogers and Walsh [31]. Numerous important extensions on the variations, stochastic integrations of local times and Itô’s formula have been done, e.g. Marcus and Rosen [24], [25], Eisenbaum [5], [6], Eisenbaum and Kyprianou [7], Flandoli,
for every Borel function $g$ to a multiple of a constant). In case

$$\sigma \quad \text{respect to the Lebesgue measure on}$$

notion of local time defined as the Radon-Nikodym derivative of the occupation measure of $X, X$ is defined from the following formula (Meyer [26]):

$$a_q \quad \text{order to establish the continuity of the local time in the spa ce variable. The first main task of this}

considered by Boylan [4], Getoor and Kesten [14] and Barlow [2], using potential theory approach, in

define the path integral

literature. Here the Tanaka formula is directly used in our a pproach. As a direct application, one can

where the supremum is taken over all finite partition $D_{(-\infty, +\infty)} = \{ -\infty < x_0 < x_1 < x_2 < \cdots < x_n < \infty \}$. This allowed us to define the path integral $\int_{-\infty}^{\infty} g(x)d_x L_t^x$ as a Young integral, for any $g$ being of a finite $q$-variation for a number $q \in [1, 2)$. The main purpose of this paper is to solve the rough path part of the problem, for the local times of both continuous semi-martingales and a class of Lévy processes.

As a first step, we consider the classical $p$-variation for the local time of the Lévy process which is represented by the following Lévy-Itô decomposition

$$X_t = X_0 + \sigma B_t + bt + \int_0^t \int_{R \setminus \{0\}} y 1_{\{|y| \geq 1\}} N_p(ds, dy) + \int_0^t \int_{R \setminus \{0\}} y 1_{\{|y| < 1\}} \tilde{N}_p(ds, dy).$$

This is a non-trivial problem as the jumps, especially the small jumps, create a lot of difficulties in estimating the increment $L_t^{x_{i+1}} - L_t^{x_i}$. Recall that for a general semimartingale $X_t, L = \{L_t^x; x \in R\}$ is defined from the following formula (Meyer [26]):

$$\int_0^t g(X_s) d[X, X]^c_s = \int_{-\infty}^{\infty} g(x)L_t^x dx,$$

where $[X, X]^c$ is the continuous part of the quadratic variation process $[X, X]$. There is a different notion of local time defined as the Radon-Nikodym derivative of the occupation measure of $X$ with respect to the Lebesgue measure on $R$ i.e.

$$\int_0^t g(X_s) ds = \int_{-\infty}^{\infty} g(x)\gamma_t^x dx,$$

for every Borel function $g : R \rightarrow R^+$. For the Lévy process [2], if $\sigma \neq 0$, $L_t^x$ and $\gamma_t^x$ are the same (up to a multiple of a constant). In case $\sigma = 0$ e.g. for a stable process, there is no diffusion part so these two definitions are different. In fact, in this case $L_t^x = 0$. The increment of $\gamma_t$ for stable processes was considered by Boylan [1], Getoor and Kesten [14] and Barlow [2], using potential theory approach, in order to establish the continuity of the local time in the space variable. The first main task of this paper is to prove that when $\sigma \neq 0$, and [14] is satisfied, for any $p > 2$ and $t \geq 0$, the process $x \mapsto L_t^x$, is of finite $p$-variation in the classical sense almost surely. Both our result and our method are new in literature. Here the Tanaka formula is directly used in our approach. As a direct application, one can define the path integral $\int_{-\infty}^{\infty} g(x)dL_t^x$ as a Young integral for any $g$ being of bounded $q$-variation for a $q \in [1, 2)$. It is noted when $q \geq 2$, Young’s condition $\frac{1}{p} + \frac{1}{q} > 1$ is broken.

The main task of this paper is to construct a geometric rough path over the processes $Z(x) = (L_t^x, g(x))$, for a deterministic function $g$ being of finite $q$-variation when $q \in [2, 3)$. This implies
establishing the path integrals $\int_{-\infty}^{\infty} L_t^x d_x L_t^x$ and $\int_{-\infty}^{\infty} g(x) d_x L_t^x$. For these two integrals, all the classical integration theories of Riemann, Lebesgue and Young fail to work. To overcome the difficulty, we use the rough path theory pioneered by Lyons, see \[21, 22, 23\], also \[19\]. However, our $p$-variation result of the local time does not automatically make the desired rough path exist or the integral well defined, though it is a crucial step to study first. Actually further hard analysis is needed to establish an iterated path integration theory for $Z$. First we introduce a piecewise curve of bounded variation as a generalized Wong-Zakai approximation to the stochastic process $Z$. Then we define a smooth rough path by defining the iterated integrals of the piecewise bounded variation process. We need to prove the smooth rough path converges to a geometric rough path $Z = (1, Z^1, Z^2)$ when $1 \leq q < 3$. For this, an important step is to compute $E(L_t^{x_{i+1}} - L_t^{x_i})(L_t^{x_{i+1}} - L_t^{x_i})$, and obtain the correct order in terms of the increments $x_{i+1} - x_i$ and $x_{j+1} - x_j$, especially in disjoint intervals $[x_i, x_{i+1}]$ and $[x_j, x_{j+1}]$ when $i \neq j$. Actually, this is a very challenging task. In this analysis, one can see that the main difficulty is from dealing with jumps, especially the small jumps of the process. One can also see that \[14\] is not adequate to construct the geometric rough path, a slightly stronger condition \[38\] is needed here.

Using this key estimate, we can establish the geometric rough path $Z = (1, Z^1, Z^2)$. Then from Chen’s identity, we define the following two integrals

$$\int_a^b L_t^x dL_t^x = \lim_{m(D[a,b]) \to 0} \sum_{i=0}^{r-1} ((Z^2_{x_i, x_{i+1}}), 1, 1 + L_t(x_i)(L_t^{x_{i+1}} - L_t^{x_i}))$$

and

$$\int_a^b g(x) dL_t^x = \lim_{m(D[a,b]) \to 0} \sum_{i=0}^{r-1} ((Z^2_{x_i, x_{i+1}}), 2, 1 + g(x_i)(L_t^{x_{i+1}} - L_t^{x_i})).$$

Note that the Riemann sum $\sum_{i=0}^{r-1} L_t^{x_{i+1}}(L_t^{x_{i+1}} - L_t^{x_i})$ and $\sum_{i=0}^{r-1} g(x_i)(L_t^{x_{i+1}} - L_t^{x_i})$ themselves may not have limits as the mesh $m(D[a,b]) \to 0$. At least there are no integration theories, rather than Lyons’ rough path theory, to guarantee the convergence of the Riemann sums for almost all $\omega$. Here it is essential to add Lévy areas to the Riemann sum. Furthermore, we can prove if a sequence of smooth functions $g_j \to g$ as $j \to \infty$, then the Riemann integral $\int_a^b g_j(x) dL_t^x$ converges to the rough path integral $\int_a^b g(x) dL_t^x$ defined in \[3\]. It is also noted that to establish \[5\], one only needs \[39\]. This is true as long as the power of $|y|$ in the condition of Lévy measure is anything less (better) than $\frac{q}{2}$. It seems to us $\frac{q}{2}$ is a critical value here. Our conjecture is that when the Lévy measure satisfies $\int_{R \setminus \{0\}} (|y|^{\frac{q}{2}} \wedge 1) m(dy) = \infty$, the local time may be a rough path of roughness $p > 3$.

Having established the path integration of local time and the corresponding convergence results, as a simple application, we can easily prove a useful extension of Itô’s formula for the Lévy process when the function is less smooth: if $f: R \to R$ is an absolutely continuous function and has left derivative $\nabla^- f(x)$ being left continuous and of bounded $q$-variation, where $1 \leq q < 3$, then P-a.s.

$$f(X_t) = f(X_0) + \int_0^t \nabla^- f(X_s) dX_s - \int_{-\infty}^{\infty} \nabla^- f(x) d_x L_t^x$$

$$+ \sum_{0 \leq s \leq t} [f(X_s) - f(X_s^-) - \Delta X_s \nabla^- f(X_s^-)], \quad 0 \leq t < \infty.$$  

Here the path integral $\int_{-\infty}^{\infty} \nabla^- f(x) d_x L_t^x$ is a Lebesgue-Stieltjes integral when $q = 1$, a Young integral when $1 < q < 2$ and a Lyons’ rough path integral when $2 \leq q < 3$ respectively. Needless to say...
that Tanaka’s formula (32) and Meyer’s formula (26, 33) are very special cases of our formula when \( q = 1 \). The investigation of Itô’s formula to less smooth functions is crucial and useful in many problems e.g. studying partial differential equations with singularities, the free boundary problem in American options, and certain stochastic differential equations. Time dependent cases for a continuous semimartingale \( X_t \) were investigated recently by Elworthy, Truman and Zhao (8), Feng and Zhao (9), where two-parameter Lebesgue-Stieltjes integrals and two-parameter Young integral were used respectively. We would like to point out that a two-parameter rough path integration theory, which is important to the study of local times, and some other problems such as SPDEs, still remains open.

A part of the results about the rough path integral of local time for a continuous semimartingale was announced in Feng and Zhao [10]. In summary, we have obtained complete results of the variation and rough path of roughness \( p \) for any \( p \in (2, 3) \) of local times for any continuous semi-martingales and a class of Lévy processes satisfying (38) and \( \sigma \neq 0 \). There is no need to include a full proof of the rough path result for continuous semimartingales in this paper. We believe the reader can see easily that the proof is essentially included in this paper, noticing the idea of decomposing the local time to continuous and discontinuous parts in [9] and the key estimate (8) in [10].

2 The \( p \)-variation of the local time of Lévy processes

Let \( X_t \) be a one dimensional time homogeneous Lévy process, and \( (\mathcal{F}_t)_{t \geq 0} \) be generated by the sample paths \( X_t, \ p(\cdot) \) be a stationary \( (\mathcal{F}_t) \)-Poisson process on \( R \setminus \{0\} \). From the well-known Lévy-Itô decomposition theorem, we can write \( X_t \) as follows:

\[
X_t := X_0 + \sigma B_t + V_t + \tilde{M}_t, \tag{8}
\]

where

\[
V_t := b t + \int_0^{t+} \int_{R \setminus \{0\}} y 1_{\{|y| \geq 1\}} N_p(dsdy),
\]

\[
\tilde{M}_t := \int_0^{t+} \int_{R \setminus \{0\}} y 1_{\{|y| < 1\}} \tilde{N}_p(dsdy).
\]

Here, \( N_p \) is the Poisson random measure of \( p \), the compensator of \( p \) is of the form \( \hat{N}(dsdy) = dsn(dy) \), where \( n(dy) \) is the Lévy measure of process \( X \). The compensated random measure \( \tilde{N}_p(t, U) = N_p(t, U) - \hat{N}_p(t, U) \) is an \( (\mathcal{F}_t) \)-martingale. Before we give our main theorem of this section, we first give a \( p \)-moment estimate formula. This will be used in later proofs.

**Lemma 1** We have the \( p \)-moment estimate formula: for any \( p \geq 1 \),

\[
E \left( \sum_{0 \leq s \leq t} |f(s, p_s(\omega), \omega)| \right)^p \leq c_p \sum_{k=0}^m E \left( \int_0^t \int_R |f(s, y, \omega)|^{2^k} n(dy)ds \right)^{2^k} + c_p \left( E \int_0^t \int_R |f(s, y, \omega)|^{2^{m+1}} n(dy)ds \right)^{\frac{m}{2^{m+1}}}, \tag{9}
\]

for a constant \( c_p > 0 \). Here \( m \) is the smallest integer such that \( 2^{m+1} \geq p \).

**Proof:** From the definition of \( N_p, \tilde{N}_p \), the Burkholder-Davis-Gundy inequality and Jensen’s inequality, we can have the \( p \)-moment estimation:
the following alternative form is often convenient

\[
E\left( \sum_{0 \leq s \leq t} |f(s, p_s(\omega), \omega)|^p \right)
\]

\[
= E\left( \int_0^t \int_R |f(s, y, \omega)| N_p(dy)ds \right)^p
\]

\[
= E\left( \int_0^t \int_R |f(s, y, \omega)| n(dy)ds + \int_0^t \int_R |f(s, y, \omega)| (N_p(dy) - n(dy)ds) \right)^p
\]

\[
\leq pE\left( \int_0^t \int_R |f(s, y, \omega)| n(dy)ds \right)^p + pE\left( \int_0^t \int_R |f(s, y, \omega)| \tilde{N}_p(dy)ds \right)^p
\]

\[
\leq pE\left( \int_0^t \int_R |f(s, y, \omega)| n(dy)ds \right)^p + cpE\left( \int_0^t \int_R |f(s, y, \omega)|^2 n(dy)ds \right)^\frac{p}{2}
\]

\[
+ \cdots + cpE\left( \int_0^t \int_R |f(s, y, \omega)|^{2m} n(dy)ds \right)^\frac{p}{2m+1}
\]

\[
= pE\left( \int_0^t \int_R |f(s, y, \omega)| n(dy)ds \right)^p + cpE\left( \int_0^t \int_R |f(s, y, \omega)|^2 n(dy)ds \right)^\frac{p}{2}
\]

\[
+ \cdots + cpE\left( \int_0^t \int_R |f(s, y, \omega)|^{2m} n(dy)ds \right)^\frac{p}{2m+1}
\]

where \( m \) is the smallest integer such that \( 2^{m+1} \geq p \).

Recall the Tanaka formula for the Lévy process \( X_t \) (c.f. [1]), we have

\[
L_t^a = (X_t - a)^+ - (X_0 - a)^+ - \int_0^t 1_{\{X_s > a\}} dX_s
\]

\[
+ \sum_{0 \leq s \leq t} [(X_s - a)^+ - (X_s - a)^+] 1_{\{X_s > a\}} \Delta X_s.
\]

(10)

Since

\[
\sum_{0 \leq s \leq t} 1_{\{X_s > a\}} \Delta X_s 1_{\{\Delta X_s \geq 1\}} = \int_0^{t+} \int_{R \setminus \{0\}} 1_{\{X_s > a\}} y 1_{\{|y| \geq 1\}} N_p(dy)ds,
\]

the following alternative form is often convenient

\[
L_t^a = \varphi_t(a) - bI_t(a) - \sigma \tilde{B}^a_t + K_1(t, a) + K_2(t, a) + K_3(t, a),
\]

(11)

where

\[
\varphi_t(a) := (X_t - a)^+ - (X_0 - a)^+,
\]

\[
I_t(a) := \int_0^t 1_{\{X_s > a\}} ds,
\]

\[
\tilde{B}^a_t := \int_0^t 1_{\{X_s > a\}} dB_s,
\]

\[
K_1(t, a) := \sum_{0 \leq s \leq t} [(X_s - a)^+ - (X_s - a)^+] 1_{\{\Delta X_s \geq 1\}},
\]

\[
K_2(t, a) := \int_0^t \int_{R \setminus \{0\}} [(X_s - y - a)^+ - (X_s - a)^+] 1_{\{|y| < 1\}} \tilde{N}_p(dy)ds,
\]

\[
K_3(t, a) := \int_0^t \int_{R \setminus \{0\}} [(X_s - y - a)^+ - (X_s - a)^+] 1_{\{|y| \geq 1\}} \tilde{N}_p(dy)ds.
\]
the related to Lévy process, especially small jumps. The main idea to deal with the jump part is to use of this section. We need to consider the variation of each term in (11). As the terms related to the increment from

\[ K_3(t, a) := \int_0^t \int_R [(X_s - y - a)^+ - (X_s - a)^+ + 1_{\{X_s - y > a\}} y] 1_{\{|y| < 1\}} n(dy) ds. \]

For the convenience in what follows in later part, we denote

\[ J_1(s, a) := (X_{s^-} - a)^+ - (X_{s^-} - a)^+, \quad J_2(s, a) := 1_{\{X_{s^-} > a\}} \Delta X_s. \]

Note we have the following important decompositions that we will use often: for any \( a_i < a_{i+1} \),

\[ J_s(X_s, X_{s^-}, a_i, a_{i+1}) := J_1(s, a_{i+1}) - J_1(s, a_i) \]
\[ = -(X_s - a_i) 1_{\{X_s \leq a_i\}} 1_{\{a_i < X_{s^-} \leq a_{i+1}\}} - (a_{i+1} - a_i) 1_{\{X_s \leq a_i\}} 1_{\{X_{s^-} > a_{i+1}\}} \]
\[ + (X_s - a_i) 1_{\{a_i < X_{s^-} \leq a_{i+1}\}} 1_{\{X_{s^-} \leq a_i\}} - (a_{i+1} - a_i) 1_{\{a_i < X_{s^-} \leq a_{i+1}\}} 1_{\{X_{s^-} > a_{i+1}\}} \]
\[ + (a_{i+1} - a_i) 1_{\{X_s > a_{i+1}\}} 1_{\{X_{s^-} \leq a_i\}} + (a_{i+1} - a_i) 1_{\{X_s > a_{i+1}\}} 1_{\{a_i < X_{s^-} \leq a_{i+1}\}} \]
\[ + (X_s - X_{s^-}) 1_{\{a_i < X_{s^-} \leq a_{i+1}\}} 1_{\{a_i < X_{s^-} \leq a_{i+1}\}}, \quad (12) \]

and

\[ J^*(X_s, X_{s^-}, a_i, a_{i+1}) := [J_1(s, a_{i+1}) - J_1(s, a_i)] + [J_2(s, a_{i+1}) - J_2(s, a_i)] \]
\[ = -(X_s - a_i) 1_{\{X_s \leq a_i\}} 1_{\{a_i < X_{s^-} \leq a_{i+1}\}} - (a_{i+1} - a_i) 1_{\{X_s \leq a_i\}} 1_{\{X_{s^-} > a_{i+1}\}} \]
\[ + (X_s - a_i) 1_{\{a_i < X_{s^-} \leq a_{i+1}\}} 1_{\{X_{s^-} \leq a_i\}} - (a_{i+1} - a_i) 1_{\{a_i < X_{s^-} \leq a_{i+1}\}} 1_{\{X_{s^-} > a_{i+1}\}} \]
\[ + (a_{i+1} - a_i) 1_{\{X_s > a_{i+1}\}} 1_{\{X_{s^-} \leq a_i\}} + (a_{i+1} - a_i) 1_{\{X_s > a_{i+1}\}} 1_{\{a_i < X_{s^-} \leq a_{i+1}\}}. \quad (13) \]

The following theorem on the \( p \)-variation of local time in the spatial variable is the main result of this section. We need to consider the variation of each term in (11). As the terms related to the continuous part of the Lévy process were considered in [9], so the main difficulty is from the jumps related to Lévy process, especially small jumps. The main idea to deal with the jump part is to use the \( p \)-moment estimate formula for the jump part and change it to integration of \( X_s \). Then we can use occupation times formula, Jensen’s inequality and Fubini theorem to obtain the \( p \)-moment of the increment from \( a_i \) to \( a_{i+1} \). With the increment estimate, the \( p \)-variation can then be proved by the classical approach of Lévy.

**Theorem 2** If \( \sigma \neq 0 \) and the Lévy measure \( n(dy) \) satisfies

\[ \int_{R \setminus \{0\}} (|y|^{\frac{d}{2}} \wedge 1) n(dy) < \infty, \quad (14) \]

then the local time \( L_t^\alpha \) of time homogeneous Lévy process \( X_t \) given by (3) is of bounded \( p \)-variation in \( a \) for any \( t \geq 0 \), for any \( p > 2 \), almost surely, i.e.

\[ \sup_{D(-\infty, \infty)} \sum \left| L_t^{a_{i+1}} - L_t^a \right|^p < \infty \quad a.s., \]

where the supremum is taken over all finite partition on \( R \), \( D(-\infty, \infty) := \{-\infty < a_0 < a_1 < \cdots < a_n < \infty\} \).
Proof: Will estimate every term in (11). First note that the function \( \varphi_t(a) := (X_t - a)^+ - (X_0 - a)^+ \) is Lipschitz continuous in \( a \) with Lipschitz constant 2. This implies that for any \( p > 2 \) and \( a_i < a_{i+1} \),
\[
|\varphi_t(a_{i+1}) - \varphi_t(a_i)|^p \leq 2^p (a_{i+1} - a_i)^p.
\] (15)

Secondly, for the second term, by the occupation times formula, Jensen’s inequality and Fubini theorem,
\[
E[I_t(a_{i+1}) - I_t(a_i)]^p = E \left( \int_0^t 1\{a_i < X_s \leq a_{i+1}\} ds \right)^p
\]
\[
= \frac{1}{\sigma^2 c} (a_{i+1} - a_i)^p E \left( \frac{1}{a_{i+1} - a_i} \int_{a_i}^{a_{i+1}} L_t^2 dx \right)^p
\]
\[
\leq \frac{1}{\sigma^2 c} (a_{i+1} - a_i)^p \sup_x E(L_t^2)^p.
\] (16)

By (10) and noting that \( \sum_{0 \leq s \leq t} |(X_s - a)^+ - (X_s - a)^+ + 1\{X_s > a\} \Delta X_s| \) is a decreasing process in \( t \), we have
\[
L_t^a \leq (X_t - a)^+ - (X_0 - a)^+ - \int_0^t 1\{X_s > a\} dX_s.
\]

Now using the Burkholder-Davis-Gundy inequality again, we have
\[
E(L_t^a)^p \leq p \left[ E|X_t - X_0|^p + E \left| \int_0^t \sigma 1\{X_s > a\} dB_s \right|^p + E \left| \int_0^t 1\{X_s > a\} dV_s \right|^p + E \left| \int_0^t 1\{X_s > a\} d\tilde{M}_s \right|^p \right]
\]
\[
\leq c \sigma^p t^{\frac{p}{2}} + cE(\int_0^t |dV_s|)^p + c_p \sum_{k=1}^{m+1} \left[ \int_0^t \int_R |\gamma|^p x^k 1\{|x| < 1\} n(dy) ds \right]^{\frac{p}{2}}
\]
\[
\leq c(p, b, \sigma, t),
\]
where \( m > 0 \) is the smallest integer such that \( 2^{m+1} \geq p \), \( c(p, b, \sigma, t) \) is a universal constant depending on \( p, b, \sigma, \) and \( t \). By Jensen’s inequality, we also have
\[
E(L_t^a)^p \leq c(p, b, \sigma, t).
\] (17)

This inequality will be used later. So
\[
E[I_t(a_{i+1}) - I_t(a_i)]^p \leq c(p, b, \sigma, t)(a_{i+1} - a_i)^p.
\] (18)

Thirdly, for the term \( \hat{B}_t^a \),
\[
E[\hat{B}_t^{a_{i+1}} - \hat{B}_t^{a_i}]^p = E \left( \int_0^t 1\{a_i < X_s \leq a_{i+1}\} dB_s \right)^p
\]
\[
\leq c_p E \left( \int_0^t 1\{a_i < X_s \leq a_{i+1}\} ds \right)^{\frac{p}{2}}
\]
\[
\leq c(t, p, \sigma)(a_{i+1} - a_i)^{\frac{p}{2}}.
\] (19)

The last estimate can be obtained similarly to (16). About \( K_1(t, a) \), it is easy to see that
\[
|K_1(t, a_{i+1}) - K_1(t, a_i)|
\]
\[
\leq \sum_{0 \leq s \leq t} |(X_s - a_{i+1})^+ - (X_s - a_{i+1})^+ - (X_s - a_i)^+ - (X_s - a_i)^+ + 1\{|\Delta X_s| \geq 1\}|
\]
\[
\leq 2(a_{i+1} - a_i) \sum_{0 \leq s \leq t} 1\{|\Delta X_s| \geq 1\}.
\]
where $m$ is the smallest integer such that $2^{m+1} \geq p$. In the following we will often use the following type of method to estimate integrals with respect to the Lévy measure: let

$$Q := \int_{a_i - a_{i+1}}^{0} \int_{y+ai}^{a_i} (x - y - a_i)dx1_{\{|y| < 1\}}n(dy)$$

$$= \frac{1}{2} \int_{a_i - a_{i+1}}^{0} |y|^21_{\{|y| < 1\}}n(dy).$$

Then, $1_{\{|y| < 1\}}\frac{1}{Q}(x - y - a_i)dxn(dy)$ is a probability measure on $\{(x, y) : y + a_i \leq x \leq a_i, (-1) \wedge (a_i - a_{i+1}) \leq y \leq 0\}$. So by Jensen’s inequality, we have that

$$E\left(\int_{a_i - a_{i+1}}^{0} \int_{y+ai}^{a_i} L_i^2(a_{i+1} - a_i)(x - y - a_i)dx1_{\{|y| < 1\}}n(dy)\right)^\frac{p}{2}$$

$$\leq Q^\frac{p}{2} E\left(\frac{1}{Q} \int_{a_i - a_{i+1}}^{0} \int_{y+ai}^{a_i} (L_i^2)^{\frac{p}{2}}(a_{i+1} - a_i)^{\frac{p}{2}}(x - y - a_i)dx1_{\{|y| < 1\}}n(dy)\right)$$

$$\leq c(\sigma, t, p)|a_{i+1} - a_i|^{\frac{p}{2}} \sup_x E(L_i^x)^{\frac{p}{2}} \left(\int_{a_i - a_{i+1}}^{0} |y|^21_{\{|y| < 1\}}n(dy)\right)^{\frac{p}{2}}. \quad (22)$$

Similarly one can estimate

$$E\left(\int_{-\infty}^{a_i - a_{i+1}} \int_{y+ai}^{a_i} L_i^2 dx |y|^21_{\{|y| < 1\}}n(dy)\right)^\frac{p}{2}$$

$$\leq c(\sigma, t, p)|a_{i+1} - a_i|^{\frac{p}{2}} \sup_x E(L_i^x)^{\frac{p}{2}} \left(\int_{-\infty}^{a_i - a_{i+1}} |y|^21_{\{|y| < 1\}}n(dy)\right)^{\frac{p}{2}}. \quad (23)$$
Then we can estimate each term in (21). When \( k = 1 \), we change the orders of the integration and use Jensen’s inequality to have

\[
E \left( \int_{-\infty}^{a_{i}} L_{t}^{x} \int_{x-a_{i+1}}^{x-a_{i}} |x - y - a_{i}|^{2}1_{\{y\leq 1\}}n(dy)dx \right)^{\frac{p}{2}}
\]

\[
= E \left( \int_{-\infty}^{a_{i}} L_{t}^{x} \int_{y+a_{i}}^{y+a_{i+1}} L_{t}^{x}(x - y - a_{i})^{2}1_{\{y\leq 1\}}dxn(dy) \right)^{\frac{p}{2}}
\]

\[
+ \int_{a_{i} - a_{i+1}}^{0} \int_{y+a_{i}}^{a_{i}} L_{t}^{x}(x - y - a_{i})(x - y - a_{i})dx1_{\{y\leq 1\}}dxn(dy) \leq E \left( \int_{-\infty}^{a_{i}} L_{t}^{x} dx|y|^{2}1_{\{y\leq 1\}}n(dy) \right)^{\frac{p}{2}}
\]

\[
\leq c(\sigma, t, p)|a_{i+1} - a_{i}|^{\frac{p}{2}} \sup_{x} E(L_{t}^{x})^{\frac{p}{2}} \left( \int_{-\infty}^{a_{i} - a_{i+1}} |y|^{2}1_{\{y\leq 1\}}n(dy) \right)^{\frac{p}{2}}
\]

\[
+ c(\sigma, t, p)|a_{i+1} - a_{i}|^{\frac{p}{2}} \sup_{x} E(L_{t}^{x})^{\frac{p}{2}} \left( \int_{a_{i} - a_{i+1}}^{0} |y|^{2}1_{\{y\leq 1\}}n(dy) \right)^{\frac{p}{2}}
\]

\[
\leq c(\sigma, t, p)|a_{i+1} - a_{i}|^{\frac{p}{2}}.
\]

For the term when \( 2 \leq k \leq m_{i} \),

\[
E \left( \int_{-\infty}^{a_{i}} L_{t}^{x} \int_{x-a_{i+1}}^{x-a_{i}} |x - y - a_{i}|^{2k}1_{\{y\leq 1\}}n(dy)dx \right)^{\frac{p}{2k}}
\]

\[
\leq E \left( \int_{-\infty}^{a_{i}} L_{t}^{x} \int_{x-a_{i+1}}^{x-a_{i}} |x - y - a_{i}|^{2k-1}|y|^{2k-1}1_{\{y\leq 1\}}n(dy)dx \right)^{\frac{p}{2k}}
\]

\[
\leq (a_{i+1} - a_{i})^{\frac{p}{2k}}E \left( \int_{-\infty}^{a_{i}} L_{t}^{x} \int_{x-a_{i+1}}^{x-a_{i}} |y|^{2k-1}1_{\{y\leq 1\}}n(dy)dx \right)^{\frac{p}{2k}}
\]

\[
\leq c(t, p)|a_{i+1} - a_{i}|^{\frac{p}{2k}}.
\]

Actually we can see that \( \int_{-\infty}^{a_{i}} \int_{x-a_{i+1}}^{x-a_{i}} |y|^{2k-1}1_{\{y\leq 1\}}n(dy)dx \leq \infty \) by using the same method as in (24). For the last term in (21), similarly, we have

\[
\left( E \int_{-\infty}^{a_{i}} L_{t}^{x} \int_{x-a_{i+1}}^{x-a_{i}} |x - y - a_{i}|^{2^{m+1}}1_{\{y\leq 1\}}n(dy)dx \right)^{\frac{p}{2^{m+1}}}
\]

\[
\leq \left( E \int_{-\infty}^{a_{i}} L_{t}^{x} \int_{x-a_{i+1}}^{x-a_{i}} |a_{i+1} - a_{i}|^{2m}|y|^{2m}1_{\{y\leq 1\}}n(dy)dx \right)^{\frac{p}{2^{m+1}}}
\]

\[
\leq |a_{i+1} - a_{i}|^{\frac{p}{2m}} \left( E \int_{-\infty}^{a_{i}} L_{t}^{x} \int_{x-a_{i+1}}^{x-a_{i}} |y|^{2m}1_{\{y\leq 1\}}n(dy)dx \right)^{\frac{p}{2m+1}}
\]

\[
\leq c(t, p)|a_{i+1} - a_{i}|^{\frac{p}{2m}}.
\]
We can see that the key point is to estimate the term when \( k = 1 \) because the higher order term can always be dealt by the above method. We can use the similar method to deal with other terms and derive that

\[
E|K_2(t, a_{i+1}) - K_2(t, a_i)|^p \leq c(t, \sigma, p)|a_{i+1} - a_i|^\frac{p}{q},
\]

(25)

About \( K_3(t, a) \), with the decomposition \[13\], we will estimate the sum of each term for jumps \( |\Delta X_s| < 1 \). There are six such terms.

For the first term in \[13\], by the \( p \)-moment estimate formula and occupation times formula, we have

\[
E \left( \int_0^t \int_{-\infty}^{\infty} |X_s - a_i|1\{X_s \leq a_i\}1\{a_i < X_s - y \leq a_{i+1}\}1\{|y| < 1\}n(dy)ds \right)^p
= E \left( \int_0^t \int_{X_s - a_i}^{a_i} |X_s - a_i|1\{X_s \leq a_i\}1\{|y| < 1\}n(dy)ds \right)^p
= \frac{1}{\sigma^{2p}} E \left( \int_{-\infty}^{a_i} L_t^x \int_{x-a_i+1}^{x-a_i} |x - a_i|1\{|y| < 1\}n(dy)dx \right)^p.
\]

Now we change the orders of the integration and use Jensen’s inequality, we have

\[
E \left( \int_0^t \int_{-\infty}^{\infty} |X_s - a_i|1\{X_s \leq a_i\}1\{a_i < X_s - y \leq a_{i+1}\}1\{|y| < 1\}n(dy)ds \right)^p
\leq \frac{1}{\sigma^{2p}} E \left( \int_{-\infty}^{a_i-a_{i+1}} \int_{y+a_i}^{y+a_{i+1}} L_t^x (a_i - x)1\{|y| < 1\}dxn(dy)
+ \int_{a_i-a_{i+1}}^{0} \int_{y+a_i}^{a_i} L_t^x (a_i - x - x)\frac{1}{2}1\{|y| < 1\}dxn(dy) \right)^p
\leq \frac{1}{\sigma^{2p}} E \left( \int_{-\infty}^{a_i-a_{i+1}} \int_{y+a_i}^{y+a_{i+1}} L_t^dx |y|1\{|y| < 1\}n(dy)
+ \int_{a_i-a_{i+1}}^{0} \int_{y+a_i}^{a_i} L_t^x (a_{i+1} - a_i)\frac{1}{2}1\{|y| < 1\}n(dy) \right)^p
\leq c(\sigma)|a_{i+1} - a_i|^{\frac{p}{q}} sup_x E(L_t^x)^p \left( \int_{-\infty}^{a_i-a_{i+1}} |y|^{\frac{1}{q}+1}1\{|y| < 1\}n(dy) \right)^p
+ c(\sigma)|a_{i+1} - a_i|^{\frac{p}{q}} sup_x E(L_t^x)^p \left( \int_{a_i-a_{i+1}}^{0} |y|^{\frac{1}{q}}1\{|y| < 1\}n(dy) \right)^p
\leq c(t, \sigma, p)|a_{i+1} - a_i|^{\frac{p}{q}}.
\]

We can use the similar method to deal with other terms. In the following, we will only sketch the estimate without giving great details.

2) For the second term, we have

\[
E \left( \int_0^t \int_{-\infty}^{\infty} (a_{i+1} - a_i)1\{X_s \leq a_i\}1\{X_s - y > a_{i+1}\}1\{|y| < 1\}n(dy)ds \right)^p
= \frac{1}{\sigma^{2p}} (a_{i+1} - a_i)^p E \left( \int_{-\infty}^{a_i} L_t^x \int_{x-a_i+1}^{x-a_i} 1\{|y| < 1\}n(dy)dx \right)^p.
\]
= c(\sigma)(a_{i+1} - a_i)^p E \left( \int_{-\infty}^{a_{i+1} - a_i} \left( \int_{-\infty}^{a_{i+1}} L_t^x dx \right) \mathbb{1}_{\{ |y| < 1 \}} n(dy) \right)^p \\
\leq c(\sigma)|a_{i+1} - a_i|^\frac{1}{2} \sup_x E(L_t^x)^p \left( \int_{-\infty}^{a_{i+1} - a_i} |y|^\frac{1}{2} + 1 \mathbb{1}_{\{ |y| < 1 \}} n(dy) \right)^p \\
\leq c(t, \sigma, p)|a_{i+1} - a_i|^\frac{1}{2} p.

3) For the third term, we have

\[ E \left( \int_0^t \int_{-\infty}^{\infty} |X_s - a_i| 1_{\{ a_i < X_s \leq a_{i+1} \}} 1_{\{ X_s - y \leq a_i \}} 1_{\{ |y| < 1 \}} n(dy) ds \right)^p = \frac{1}{\sigma^{2p}} E \left( \int_{a_i}^{a_{i+1}} L_t^x \int_{-\infty}^{\infty} |x - a_i| 1_{\{ |y| < 1 \}} n(dy) dx \right)^p \\
\leq \frac{1}{\sigma^{2p}} E \left( \int_0^{a_{i+1} - a_i} \int_{a_i}^{a_i + y} L_t^x (a_{i+1} - a_i)^{\frac{1}{2}} (x - a_i)^{\frac{1}{2}} dx 1_{\{ |y| < 1 \}} n(dy) \\
+ \int_{a_{i+1} - a_i}^{\infty} \int_{a_i}^{a_{i+1}} L_t^x dx |y|^\frac{1}{2} 1_{\{ |y| < 1 \}} n(dy) \right)^p \\
\leq c(\sigma, p)|a_{i+1} - a_i|^\frac{1}{2} p \sup_x E(L_t^x)^p \left( \int_0^{\infty} |y|^\frac{1}{2} 1_{\{ |y| < 1 \}} n(dy) \right)^p \\
\leq c(t, \sigma, p)|a_{i+1} - a_i|^\frac{1}{2} p.

4) The fourth term is symmetric to the third term, so by a similar computation, we have

\[ E \left( \int_0^t \int_{-\infty}^{\infty} |a_{i+1} - X_s| 1_{\{ a_i < X_s \leq a_{i+1} \}} 1_{\{ X_s - y > a_{i+1} \}} 1_{\{ |y| < 1 \}} n(dy) ds \right)^p \leq c(t, \sigma, p)|a_{i+1} - a_i|^\frac{1}{2} p.

5) For the fifth term, as it is symmetric to the second term, so we can use a similar computation to have

\[ E \left( \int_0^t \int_{-\infty}^{\infty} (a_{i+1} - a_i) 1_{\{ a_i < X_s \leq a_{i+1} \}} 1_{\{ X_s - y > a_{i+1} \}} 1_{\{ |y| < 1 \}} n(dy) ds \right)^p \leq c(t, \sigma, p)|a_{i+1} - a_i|^\frac{1}{2} p.

6) The last term is symmetric to the first term, so by a similar computation, we have

\[ E \left( \int_0^t \int_{-\infty}^{\infty} |a_{i+1} - X_s| 1_{\{ a_i < X_s \leq a_{i+1} \}} 1_{\{ X_s - y \leq a_{i+1} \}} 1_{\{ |y| < 1 \}} n(dy) ds \right)^p \leq c(t, \sigma, p)|a_{i+1} - a_i|^\frac{1}{2} p.

So we have

\[ E|K_3(t, a_{i+1}) - K_3(t, a_i)|^p \leq c(t, \sigma, p)|a_{i+1} - a_i|^\frac{1}{2} p. \tag{28} \]

Now we use Proposition 4.1.1 in [23] (i = 1, \gamma > p - 1), for any finite partition \{a_k\} of [a, b]

\[ \sup \left\{ \sum_i \left| \hat{B}_t^{a_{i+1}} - \hat{B}_t^{a_i} \right|^p \leq c(p, \gamma) \sum \sum_{k=1}^{2n} \left| \hat{B}_t^{a_{k+1}} - \hat{B}_t^{a_{k-1}} \right|^p, \right\} \]

where

\[ a_k^n = a + \frac{k}{2^n}(b - a), \quad k = 0, 1, \ldots, 2^n. \]
The key point here is that the right hand side does not depend on the partition $D$. We take the expectation and use (19), it follows that
\[
E \sum_{n=1}^{\infty} n^\gamma \sum_{k=1}^{2^n} |\hat{B}_{t_k}^n - \hat{B}_{t_{k-1}}^n|^p = \sum_{n=1}^{\infty} n^\gamma \sum_{k=1}^{2^n} E|\hat{B}_{t_k}^n - \hat{B}_{t_{k-1}}^n|^p \\
\leq c \sum_{n=1}^{\infty} n^\gamma \left( \frac{b-a}{2^n} \right)^p < \infty,
\]
as $p > 2$. Therefore
\[
\sum_{n=1}^{\infty} n^\gamma \sum_{k=1}^{2^n} |\hat{B}_{t_k}^n - \hat{B}_{t_{k-1}}^n|^p < \infty \text{ a.s.}
\]
It turns out that for any interval $[a, b] \subset \mathbb{R}$
\[
\sup_D \sum_i |\hat{B}_{t_{i+1}} - \hat{B}_{t_i}|^p < \infty \text{ a.s.} \quad (29)
\]
But we know $L_b^a$ has a compact support $[-K, K]$ in $a$. So for the partition $D := D_{-K,K} = \{-K = a_0 < a_1 < \cdots < a_r = K\}$, we obtain
\[
\sup_D \sum_i |\hat{B}_{t_{i+1}} - \hat{B}_{t_i}|^p < \infty \text{ a.s.} \quad (30)
\]
In the same way, from (18), (20), (25), (28) we can prove that
\[
\sup_D \sum_i |I_{t_{i+1}} - I_{t_i}|^p < \infty \text{ a.s.} \quad (31)
\]
\[
\sup_D \sum_i |K_1(t, a_{i+1}) - K_1(t, a_i)|^p < \infty \text{ a.s.} \quad (32)
\]
\[
\sup_D \sum_i |K_2(t, a_{i+1}) - K_2(t, a_i)|^p < \infty \text{ a.s.} \quad (33)
\]
\[
\sup_D \sum_i |K_3(t, a_{i+1}) - K_3(t, a_i)|^p < \infty \text{ a.s.} \quad (34)
\]
On the other hand, it is easy to see from (15) that
\[
\sum_i |\varphi_t(a_{i+1}) - \varphi_t(a_i)|^p \leq 2^p \sum_i (a_{i+1} - a_i)^p \leq 2^p |\sum_i (a_{i+1} - a_i)|^p = 2^p (b-a)^p.
\]
Then from (18), (30), (31), (32), (33), (34), and (35), it turns out that
\[
\sup_D \sum_i |L_{t_{i+1}} - L_{t_i}|^p < \infty \quad \text{ a.s.}
\]

**Remark 3** From (20) and (27), we can see easily that if we require the following slightly stronger condition on the Lévy measure
\[
\int_{R \setminus \{0\}} (|y|^{\frac{1}{2} - \xi} \wedge 1)n(dy) < \infty,
\]
for a $\xi \in (0, \frac{1}{2}]$, then for any $p \geq 1$,
\[
E|K_3(t, a_{i+1}) - K_3(t, a_i)|^p \leq c(t, \sigma, p)|a_{i+1} - a_i|^{\frac{1}{2} + \xi}p.
\]
This estimate will be used in the construction of the geometric rough path where (28) is not adequate.
3 The local time rough path

The p-variation ($p > 2$) result of the local time enables one to use Young’s integration theory to define $\int_{-\infty}^{\infty} g(x)dx L_t^\delta$ for $g$ being of bounded $q$-variation when $1 \leq q < 2$. This is because in this case, for any $q \in [1, 2)$, one can always find a constant $p > 2$ such that the condition $\frac{1}{p} + \frac{1}{q} > 1$ for the existence of the Young integral is satisfied. However, when $q \geq 2$, Young integral is no longer well defined. We have to use a new integration theory. Lyons’ integration of rough path provides a way to push the result further. But from [23], generally, we cannot expect to have an integration theory to define integrals such as $\int_{-\infty}^{\infty} g(x)dx L_t^\delta$. However, inspired by the method in Chapter 6 in [23], we can treat $Z_x := (L_t^\delta, g(x))$ as a process of variable $x$ in $R^2$. Then it’s easy to know that $Z_x$ is of bounded $\hat{q}$-variation in $x$, where $\hat{q} = q$, if $q > 2$, and $\hat{q} > 2$ can be taken any number when $q = 2$. In the following, we only consider the case that $2 \leq q < 3$. We obtain the existence of the geometric rough path $Z = (1, Z^1, Z^2)$ associated to $Z$.

We assume a slightly stronger condition than (14) for the Lévy measure: there exists a constant $\varepsilon > 0$ such that

$$\int_{R \setminus \{0\}} (|y|^{\frac{1}{4}-\varepsilon} \wedge 1) n(dy) < \infty. \tag{38}$$

We will prove with this condition, the desired geometric rough path $Z = (1, Z^1, Z^2$) is well defined. We need to point out that in the following when we consider the control function and the convergence of the first level path, condition (14) is still adequate. But we need (38) in the convergence of the second level path. Denote $\delta = \frac{1}{q} - (3-q)\varepsilon$. Note when $q = 2$, $\delta = \frac{1}{q} - \varepsilon$. So condition (38) becomes: there exists $\varepsilon > 0$ such that

$$\int_{R \setminus \{0\}} (|y|^{\frac{1}{4}-\varepsilon} \wedge 1) n(dy) < \infty. \tag{39}$$

Later in this section, we will see under this condition, the integral $\int_{-\infty}^{\infty} L_t^\delta dL_t^\delta$ can be well-defined as a rough path integral. Also note $\inf_{2 \leq q < 3} \delta(q, \varepsilon) = \frac{1}{q}$. So under the condition

$$\int_{R \setminus \{0\}} (|y|^{\frac{1}{2}} \wedge 1) n(dy) < \infty, \tag{40}$$

(38) is satisfied for any $2 \leq q < 3$. In this case, our results imply that we can construct the geometric rough path for any $g$ being of finite $q$-variation, where $2 \leq q < 3$ can be arbitrary.

Recall the $\theta$-variation metric $d_\theta$ on $C_{0, \theta}([0, \theta], (R^2))$ defined in [23],

$$d_\theta(Z, Y) = \max_{1 \leq i \leq |\theta|} d_i, \theta(Z^i, Y^i) = \max_{1 \leq i \leq |\theta|} \sup_{1 \leq i \leq |\theta|} \left( \sum_i |Z^i_{x_{i-1} - x_i} - Y^i_{x_{i-1} - x_i}| \right)^{\frac{2}{\theta}}. \tag{41}$$

Assume condition (14) through to Proposition 5. Let $[x', x'']$ be any interval in $R$. From the proof of Theorem 2 for any $p \geq 2$, we know there exists a constant $c > 0$ such that

$$E|L_t^b - L_t^a|^p \leq c|b - a|^{\frac{p}{\theta}}, \tag{41}$$

i.e. $L_t^\delta$ satisfies Hölder condition in [23] with exponent $\frac{1}{\theta}$. First we consider the case when $g$ is continuous. Recall in [23], a control $w$ is a continuous super-additive function on $\Delta := \{(a, b) : x' \leq a < b \leq x''\}$ with values in $[0, \infty)$ such that $w(a, a) = 0$. Therefore
If \( g(x) \) is of bounded \( q \)-variation, we can find a control \( w \) s.t.

\[
|g(b) - g(a)|^q \leq w(a, b),
\]

for any \((a, b) \in \Delta := \{(a, b) : x' \leq a < b \leq x''\}\). It is obvious that \( w_1(a, b) := w(a, b) + (b - a) \) is also a control of \( g \). Set \( h = \frac{1}{q} \), it is trivial to see for any \( \theta > q \) (so \( h\theta > 1 \)) we have,

\[
|g(b) - g(a)|^\theta \leq w_1(a, b)^{h\theta}, \text{ for any } (a, b) \in \Delta.
\]

Considering (41), we can see \( Z_x \) satisfies, for such \( h = \frac{1}{q} \), and any \( \theta > q \), there exists a constant \( c \) such that

\[
E|Z_b - Z_a|^\theta \leq cw_1(a, b)^{h\theta}, \text{ for any } (a, b) \in \Delta.
\]

For any \( m \in \mathbb{N} \), define a continuous and bounded variation path \( Z(m) \) by

\[
Z(m)_x := Z_{x_{m+1}}^x + \frac{w_1(x) - w_1(x_{m+1})}{w_1(x_{m+1}) - w_1(x_{m-1})} \Delta_{m}^1 Z,
\]

where \( w_1(x) := w_1(x', x) \). It is obvious that \( x_{m+1}^m - x_{m+1}^{m-1} \leq \frac{1}{2m} w_1(x', x) \) and by the superadditivity of the control function \( w_1 \),

\[
w_1(x_{m+1}^m, x_{m+1}^{m-1}) \leq w_1(x_{m+1}^m) - w_1(x_{m+1}^{m-1}) = \frac{1}{2m} w_1(x', x).
\]

The corresponding smooth rough path \( Z(m) \) is built by taking its iterated path integrals, i.e. for any \((a, b) \in \Delta\),

\[
Z(m)_{a, b}^j = \int_{a < x_1 < \cdots < x_j < b} dZ(m)_{x_1} \otimes \cdots \otimes dZ(m)_{x_j}.
\]

In the following, we will prove \( \{Z(m)\}_{m \in \mathbb{N}} \) converges to a geometric rough path \( Z \) in the \( \theta \)-variation topology when \( 2 \leq q < 3 \). We call \( Z \) the canonical geometric rough path associated to \( Z \).

**Remark 4** The bounded variation process \( Z(m)_x \) is a generalized Wong-Zakai approximation to the process \( Z \) of bounded \( q \)-variation. Here we divide \([x', x'']\) by equally partitioning the range of \( w_1 \). We then use (47) to form the piecewise curved approximation to \( Z \). Note here Wong-Zakai’s standard piecewise linear approximation does not work immediately.

Let’s first look at the first level path \( Z(m)_{a, b}^1 \). The method is similar to Chapter 4 in [23] for Brownian motion. Similar to Proposition 4.2.1 in [23], we can prove that for all \( n \in \mathbb{N} \), \( m \mapsto \sum_{k=1}^n |Z(m)_{x_{k-1}, x_k}^1|^\theta \) is increasing and for \( m \geq n \),

\[
Z(m)_{x_{k-1}, x_k}^1 = Z(m+1)_{x_{k-1}, x_k}^1 = Z_{x_k^m}^n - Z_{x_{k-1}^m}^n.
\]
Let $Z_{a,b}^1 = Z_b - Z_a$. Then (53) implies $E|Z_{a,b}^1|^\theta \leq cw_1(a,b)^{h\theta}$. For such points $\{x^n_k\}, k = 1, \ldots, 2^n$, $n = 1, 2, \ldots$, defined above we still have the inequality in Proposition 4.1.1 in [23],

$$E \sup_D \sum_{l} |Z_{x_{l-1}, x_l}^1|^\theta \leq C(\theta, \gamma)E \sup_m \sum_{n=1}^{\infty} n^\gamma \sum_{k=1}^{2^n} |Z_{x_{k-1}^n, x_k^n}^1|^\theta \leq C_1 \sum_{n=1}^{\infty} n^\gamma \left(\frac{1}{2^n}\right)^{h\theta-1} w_1(x', x'')^{h\theta}$$

(47)

for constant $C_1 = C(\theta, \gamma)c$. Since $h\theta - 1 > 0$, the series on the right-hand side of (47) is convergent, so that $\sup D \sum_l |Z_{x_{l-1}, x_l}^1|^\theta < \infty$ almost surely. This shows that $Z^1$ has finite $\theta$-variation almost surely. Moreover, for any $\gamma > \theta - 1$, there exists a constant $C_1(\theta, \gamma, c) > 0$ such that

$$E \sup_m \sup_D \sum_{l} |Z(m)_{x_{l-1}, x_l}^1|^\theta \leq C(\theta, \gamma)E \sup_m \sum_{n=1}^{\infty} n^\gamma \sum_{k=1}^{2^n} |Z(m)_{x_{k-1}^n, x_k^n}^1|^\theta \leq C(\theta, \gamma, c) \sum_{n=1}^{\infty} n^\gamma \left(\frac{1}{2^n}\right)^{h\theta-1} w_1(x', x'')^{h\theta}$$

(48)

< \infty.

So

$$\sup_m \sup_D \sum_{l} |Z(m)_{x_{l-1}, x_l}^1|^\theta < \infty \ a.s. \quad (49)$$

This means that $Z(m)_{a,b}^1$ have finite $\theta$-variation uniformly in $m$. And furthermore, from (46) and some standard arguments,

$$E \sum_{n=1}^{\infty} n^\gamma \sum_{k=1}^{2^n} |Z(m)_{x_{k-1}^n, x_k^n}^1 - Z_{x_{k-1}^n, x_k^n}^1|^\theta \leq C \left(\frac{1}{2^n}\right)^{\frac{h\theta-1}{2}},$$

(50)

where $C$ depends on $\theta$, $h$, $w_1(x', x'')$, and $c$ in [43]. By Proposition 4.1.2 in [23], Jensen’s inequality and [40],

$$E \sum_{m=1}^{\infty} \sup_D \left( \sum_{l} \left| Z(m)_{x_{l-1}, x_l}^1 - Z_{x_{l-1}, x_l}^1 \right|^\theta \right)^{\frac{1}{\theta}} \leq E \sum_{m=1}^{\infty} \left( \sum_{n=1}^{\infty} n^\gamma \sum_{k=1}^{2^n} \left| Z(m)_{x_{k-1}^n, x_k^n}^1 - Z_{x_{k-1}^n, x_k^n}^1 \right|^\theta \right)^{\frac{1}{\theta}} \leq C \sum_{m=1}^{\infty} \left( \frac{1}{2^n}\right)^{\frac{h\theta-1}{2\theta}} \leq \infty,$$

(51)

for $h\theta > 1$. So we obtain
Theorem 5 Let $L_t^\xi$ be the local time of the time homogeneous Lévy process $X_t$ given by \((3)\), and $q$ be a continuous function of bounded $q$-variation. Assume $q \geq 1$, $\sigma \neq 0$ and the Lévy measure $\mu(dy)$ satisfies \((14)\). Then for any $\theta > q$, the continuous process $Z_t = (L_t^\xi, g(x))$ satisfying \((15)\), we have

$$\sum_{m=1}^{\infty}\sup_D\left(\sum_{l} |Z(m)_{x_{l-1},x_l}^{1} - Z_{x_{l-1},x_l}^{1}|^\theta \right)^{\frac{1}{\theta}} \leq \mathbb{E}Z(m)_{x_{m-1},x_m}^{2} \mathbb{E}\mu(dy) \leq \mathbb{E}Z(m)_{x_{m-1},x_m}^{2},$$

(52)

In particular, $(Z(m)_{a,b}^{1})$ converges to $(Z_{a,b}^{1})$ in the $\theta$-variation distance a.s. for any $(a, b) \in \Delta$.

We next consider the second level path $Z(m)_{a,b}^{2}$. As in \((5)\), we can also see that if $m \leq n$,

$$Z(m)_{x_{m-1},x_m}^{2} = \frac{1}{2} \Delta_{k}^{m} \mu(dy) \leq \mathbb{E}Z(m)_{x_{m-1},x_m}^{2} \mathbb{E}\mu(dy) \leq \mathbb{E}Z(m)_{x_{m-1},x_m}^{2}$$

(53)

where $l$ is chosen such that $x_{n-1}^m \leq x_{k-1}^m < x_k^m \leq x_l^m$; if $m > n$,

$$Z(m)_{x_{m-1},x_m}^{2} = \frac{1}{2} \Delta_{k}^{m} \mu(dy) \leq \mathbb{E}Z(m)_{x_{m-1},x_m}^{2} \mathbb{E}\mu(dy) \leq \mathbb{E}Z(m)_{x_{m-1},x_m}^{2}$$

(54)

$k = 1, \ldots, 2^n$. Similar to the proof of Proposition 4.3.3 in \((5)\), we have

Proposition 6 Assume $g$ is a continuous function of finite $q$-variation with a real number $q \geq 2$, and the Lévy measure satisfies \((15)\). Let $\theta > q$. Then for $m \leq n$,

$$\sum_{k=1}^{2^n} \mathbb{E}|Z(m + 1)_{x_{k-1},x_k}^{2} - Z(m)_{x_{k-1},x_k}^{2}|^\theta \leq C\left(\frac{1}{2^n + m}\right)^{\frac{2n+1}{2}},$$

(55)

where $C$ depends on $\theta$, $h(\varepsilon) := \frac{1}{q}$, $w_{1}(x', x'')$, and $c$ in \((5)\).

The main step to establish the geometric rough path integral over $Z$ is the following estimate. The proof is rather complicated. We will use Lemma 8 about the correlation of $K_2(t, a_{i+1}) - K_2(t, a_i)$ and $K_2(t, a_{j+1}) - K_2(t, a_j)$, and \((3)\) for the term $K_3$, for $\xi = \frac{2-2\gamma}{2\theta} + (3 - q)\varepsilon$.

Proposition 7 Assume $g$ is a continuous function of finite $q$-variation with a real number $q \in [2, 3)$, and the Lévy measure satisfies \((5)\). Let $q < \theta < 3$. Then for $m > n$,

$$\mathbb{E}|Z(m + 1)_{x_{k-1},x_k}^{2} - Z(m)_{x_{k-1},x_k}^{2}|^\theta \leq C\left(\frac{1}{2^n + m}\right)^{\frac{3}{2}} + \left(\frac{1}{2^n + m}\right)^{\frac{1}{2}} + \left(\frac{1}{2^n + m}\right)^{\frac{3}{2}},$$

(56)

where $C$ is a generic constant and also depends on $\theta$, $h(\varepsilon) := \frac{1}{q}$, $w_{1}(x', x'')$, and $c$ in \((5)\).

Lemma 8 Assume the Lévy measure satisfies \((5)\) with $0 \leq \xi \leq \frac{1}{\theta}$, then for any $a_0 < a_1 < \cdots < a_m$,

$$\left|\mathbb{E}K_2(t, a_j) - \mathbb{E}K_2(t, a_{j+1})\right| \leq \left\{\begin{array}{ll}
c(t, \sigma)(a_{i+1} - a_i), & \text{when } 0 \leq i = j \leq m, \\
c(t, \sigma)(a_{j+1} - a_j) + (a_{j+1} - a_j)^{\frac{3}{2}}, & \text{when } 0 \leq i \neq j \leq m.
\end{array}\right.$$
Proof: When \( i = j \), (57) follows from (25) directly. Now we consider the case when \( i \neq j \). Without losing generality, we assume that \( i < j \). From (12), it is easy to see that
\[
E(K_2(t, a_{i+1}) - K_2(t, a_i)) - (K_2(t, a_{j+1}) - K_2(t, a_j))
\]
\[
= E\left(\int_0^{t+} \int_R J_s(X_s, X_s - y, a_i, a_{i+1}) \tilde{N}_p(dyds) \cdot \int_0^{t+} \int_R J_s(X_s, X_s - y, a_j, a_{j+1}) \tilde{N}_p(dyds)\right)
\]
\[
= E\int_0^{t} \int_R J_s(X_s, X_s - y, a_i, a_{i+1}) J_s^*(X_s, X_s - y, a_j, a_{j+1}) n(dyds).
\]
But from (12), we know
\[
\int_0^{t} \int_R J_s(X_s, X_s - y, a_i, a_{i+1}) J_s^*(X_s, X_s - y, a_j, a_{j+1}) n(dyds) = A_1 + A_2 + A_3 + A_4,
\]
where
\[
A_1 := \int_0^{t} \int_R \left[-(a_{i+1} - a_i)(a_j - X_{s-})\right] 1_{\{X_s \leq a_i\}} 1_{\{a_j < X_s \leq a_{j+1}\}} 1_{\{|y| < 1\}} n(dy)ds
\]
\[
+ \int_0^{t} \int_R \left[(a_{i+1} - a_i)(a_{j+1} - a_j)\right] 1_{\{X_s \leq a_i\}} 1_{\{a_j < X_s \leq a_{j+1}\}} 1_{\{|y| < 1\}} n(dy)ds,
\]
\[
A_2 := \int_0^{t} \int_R \left[-(a_{i+1} - X_s)(a_j - X_{s-})\right] 1_{\{a_i < X_s \leq a_{i+1}\}} 1_{\{a_j < X_s \leq a_{j+1}\}} 1_{\{|y| < 1\}} n(dy)ds
\]
\[
+ \int_0^{t} \int_R \left[(a_{i+1} - X_s)(a_{j+1} - a_j)\right] 1_{\{a_i < X_s \leq a_{i+1}\}} 1_{\{a_j < X_s \leq a_{j+1}\}} 1_{\{|y| < 1\}} n(dy)ds,
\]
\[
A_3 := \int_0^{t} \int_R \left[(a_{i+1} - a_i)(X_s - a_j)\right] 1_{\{X_s \leq a_i\}} 1_{\{a_j < X_s \leq a_{j+1}\}} 1_{\{|y| < 1\}} n(dy)ds
\]
\[
+ \int_0^{t} \int_R \left[(a_{i+1} - a_i)(a_{j+1} - a_j)\right] 1_{\{X_s \leq a_i\}} 1_{\{a_j < X_s \leq a_{j+1}\}} 1_{\{|y| < 1\}} n(dy)ds,
\]
\[
A_4 := \int_0^{t} \int_R \left[-(X_s - a_{i+1})(X_s - a_j)\right] 1_{\{a_i < X_s \leq a_{i+1}\}} 1_{\{a_j < X_s \leq a_{j+1}\}} 1_{\{|y| < 1\}} n(dy)ds
\]
\[
+ \int_0^{t} \int_R \left[-(X_s - a_{i+1})(a_{j+1} - a_j)\right] 1_{\{a_i < X_s \leq a_{i+1}\}} 1_{\{a_j < X_s \leq a_{j+1}\}} 1_{\{|y| < 1\}} n(dy)ds.
\]
To estimate \(|EA_1|\), we notice that
\[
|EA_1| \leq E\int_0^{t} \int_{-\infty}^{\infty} (a_{i+1} - a_i)(a_j - X_{s-}) 1_{\{X_s \leq a_i\}} 1_{\{a_j < X_s - y \leq a_{j+1}\}} 1_{\{|y| < 1\}} n(dy)ds
\]
\[
+ E\int_0^{t} \int_{-\infty}^{\infty} (a_{i+1} - a_i)(a_{j+1} - a_j) 1_{\{X_s \leq a_i\}} 1_{\{a_j < X_s - y \leq a_{j+1}\}} 1_{\{|y| < 1\}} n(dy)ds
\]
\[
:= A_{11} + A_{12}.
\]
Let’s estimate every term on the righthand side of the above inequality. By the occupation times formula, Fubini theorem, Jensen’s inequality, similar as before, we have:
\[
A_{11} = \frac{1}{\sigma^2} E\int_{-\infty}^{a_{i+1}} L_t^{\alpha_i} \int_{x - a_{j+1}}^{x - a_j} (a_{i+1} - a_i)(x - a_j - y) 1_{\{|y| < 1\}} n(dy)dx
\]
\[
\leq \frac{1}{\sigma^2} (a_{i+1} - a_i)^{\frac{1}{2} + \xi} E\int_{-\infty}^{a_{i+1} - a_{j+1}} |y|^\frac{1}{2} - \xi \int_{y + a_j}^{y + a_{j+1}} L_t^{\alpha_i} |y| 1_{\{|y| < 1\}} dx n(dy)
\]
Therefore we proved (57).

So when

$$c(t, \sigma)(a_{i+1} - a_i)^{\frac{1}{2} + \xi}(a_{j+1} - a_j).$$

In the same way, we can have

$$A_{12} \leq \frac{1}{\sigma^2}(a_{i+1} - a_i)^{\frac{1}{2} + \xi}(a_{j+1} - a_j) \sup_x E(L_t^x) \int_{-\infty}^{a_{i+1} - a_j} |y|^{\frac{3}{2} - \xi} |1_{\{|y|<1\}}| n(dy)

\leq c(t, \sigma)(a_{i+1} - a_i)^{\frac{1}{2} + \xi}(a_{j+1} - a_j).$$

Therefore, we get

$$|EA_1| \leq c(t, \sigma)((a_{i+1} - a_i)^{1+2\xi} + (a_{j+1} - a_j)^{1+2\xi}).$$

Using the same method, we can have a similar estimation in the other cases:

$$|EA_2| \leq E\int_0^t \int_{-\infty}^{\infty} (a_{i+1} - X_s)(X_s - y - a_j)1_{\{a_i < X_s \leq a_{i+1}\}}1_{\{a_j < X_s - y \leq a_{j+1}\}}1_{\{|y|<1\}} n(dy) ds

+ E\int_0^t \int_{-\infty}^{\infty} (a_{i+1} - X_s)(a_{j+1} - a_j)1_{\{a_i < X_s \leq a_{i+1}\}}1_{\{X_s - y > a_{j+1}\}}1_{\{|y|<1\}} n(dy) ds

\leq c(t, \sigma)((a_{i+1} - a_i)^{1+2\xi} + (a_{j+1} - a_j)^{1+2\xi});$$

and

$$|EA_3| \leq E\int_0^t \int_{-\infty}^{\infty} (a_{i+1} - a_i)(X_s - a_j)1_{\{X_s - y \leq a_i\}}1_{\{a_j < X_s \leq a_{j+1}\}}1_{\{|y|<1\}} n(dy) ds

+ E\int_0^t \int_{-\infty}^{\infty} (a_{i+1} - a_i)(a_{j+1} - a_j)1_{\{X_s - y \leq a_i\}}1_{\{X_s > a_{j+1}\}}1_{\{|y|<1\}} n(dy) ds

\leq c(t, \sigma)((a_{i+1} - a_i)^{1+2\xi} + (a_{j+1} - a_j)^{1+2\xi});$$

and

$$|EA_4| \leq E\int_0^t \int_{-\infty}^{\infty} (a_{i+1} - X_s + y)(X_s - a_j)1_{\{a_i < X_s - y \leq a_{i+1}\}}1_{\{a_j < X_s \leq a_{j+1}\}}1_{\{|y|<1\}} n(dy) ds

+ E\int_0^t \int_{-\infty}^{\infty} (a_{i+1} - X_s + y)(a_{j+1} - a_j)1_{\{a_i < X_s - y \leq a_{i+1}\}}1_{\{X_s > a_{j+1}\}}1_{\{|y|<1\}} n(dy) ds

\leq c(t, \sigma)((a_{i+1} - a_i)^{1+2\xi} + (a_{j+1} - a_j)^{1+2\xi}).$$

So when $i \neq j$,

$$|E(K_2(t, a_{i+1}) - K_2(t, a_i))(K_2(t, a_{j+1}) - K_2(t, a_j))| \leq c(t, \sigma)((a_{i+1} - a_i)^{1+2\xi} + (a_{j+1} - a_j)^{1+2\xi}).$$

Therefore we proved (57).

\textbf{Proof of Proposition 7.} For $m > n$, it is easy to see that
\[ E[Z(m+1)^2 x_{k-1} x_k - Z(m)^2 x_{k-1} x_k]^2 \]

\[ = \frac{1}{4} E \left| \sum_{l=2^{m-n}(k-1)+1}^{2^{m-n}k} (\Delta_{2l-1}^{m+1} Z \otimes \Delta_{2l}^{m+1} Z - \Delta_{2l}^{m+1} Z \otimes \Delta_{2l-1}^{m+1} Z)^2 \right| \]

\[ = \frac{1}{4} \sum_{i,j=1 \atop i \neq j}^{2^{m-n}k} E \left( \Delta_{2l-1}^{m+1} Z_i \Delta_{2l}^{m+1} Z_j - \Delta_{2l}^{m+1} Z_i \Delta_{2l-1}^{m+1} Z_j \right)^2 \]

\[ = \frac{1}{4} \sum_{l,r} E \left( \Delta_{2l-1}^{m+1} L r \Delta_{2l}^{m+1} L r \right) (\Delta_{2l}^{m+1} g(x) \Delta_{2l}^{m+1} g(x)) \]

\[ + \left( \Delta_{2l-1}^{m+1} g(x) \Delta_{2l}^{m+1} g(x) \right) (\Delta_{2l}^{m+1} L r \Delta_{2l}^{m+1} L r) \]

\[ \frac{1}{4} \sum_{l,r} E \left( \Delta_{2l-1}^{m+1} L r \Delta_{2l}^{m+1} L r \right) (\Delta_{2l}^{m+1} g(x) \Delta_{2l}^{m+1} g(x)) \]

\[ + \left( \Delta_{2l-1}^{m+1} g(x) \Delta_{2l}^{m+1} g(x) \right) (\Delta_{2l}^{m+1} L r \Delta_{2l}^{m+1} L r) \]

\[ - \frac{1}{4} \sum_{l,r} E \left( \Delta_{2l-1}^{m+1} L r \Delta_{2l}^{m+1} L r \right) (\Delta_{2l}^{m+1} g(x) \Delta_{2l}^{m+1} g(x)) \]

\[ + \left( \Delta_{2l-1}^{m+1} g(x) \Delta_{2l}^{m+1} g(x) \right) (\Delta_{2l}^{m+1} L r \Delta_{2l}^{m+1} L r) \]

\[ \frac{1}{4} \sum_{l,r} E \left( \Delta_{2l-1}^{m+1} L r \Delta_{2l}^{m+1} L r \right) (\Delta_{2l}^{m+1} g(x) \Delta_{2l}^{m+1} g(x)) \]

\[ + \left( \Delta_{2l-1}^{m+1} g(x) \Delta_{2l}^{m+1} g(x) \right) (\Delta_{2l}^{m+1} L r \Delta_{2l}^{m+1} L r) \].

The main difficulty is to estimate the following expectation which can be derived from Tanaka’s formula:

\[ E \left[ \Delta_{2l-1}^{m+1} L r \Delta_{2l}^{m+1} L r \right] \]

\[ = E \left[ (L_r(x_{2l-1}^{m+1}) - L_r(x_{2l-2}^{m+1})) (L_r(x_{2l-1}^{m+1}) - L_r(x_{2l-2}^{m+1})) \right] \]

\[ = E \left[ \varphi_r(x_{2l-1}^{m+1}) - \varphi_r(x_{2l-2}^{m+1}) - b \int_0^t 1_{(x_{2l-2}^{m+1} < x_r - x_{2l-1}^{m+1})} ds \right] \]

\[ - \sigma \int_0^t 1_{(x_{2l-2}^{m+1} < x_r - x_{2l-1}^{m+1})} dB_s + (K_1(t, x_{2l-1}^{m+1}) - K_1(t, x_{2l-2}^{m+1})) \]

\[ + (K_2(t, x_{2l-1}^{m+1}) - K_2(t, x_{2l-2}^{m+1})) + (K_3(t, x_{2l-1}^{m+1}) - K_3(t, x_{2l-2}^{m+1})) \]

\[ + [\varphi_r(x_{2l-1}^{m+1}) - \varphi_r(x_{2l-2}^{m+1}) - b \int_0^t 1_{(x_{2l-2}^{m+1} < x_r - x_{2l-1}^{m+1})} ds \]

\[ - \sigma \int_0^t 1_{(x_{2l-2}^{m+1} < x_r - x_{2l-1}^{m+1})} dB_s + (K_1(t, x_{2l-1}^{m+1}) - K_1(t, x_{2l-2}^{m+1})) \]

\[ + (K_2(t, x_{2l-1}^{m+1}) - K_2(t, x_{2l-2}^{m+1})) + (K_3(t, x_{2l-1}^{m+1}) - K_3(t, x_{2l-2}^{m+1})) \].

Firstly, from [15, 13, 10], the Cauchy-Schwarz inequality and the quadratic variation of stochastic integrals, we have

\[ \left| E(\varphi_r(x_{2l-1}^{m+1}) - \varphi_r(x_{2l-2}^{m+1}) - b \int_0^t 1_{(x_{2l-2}^{m+1} < x_r - x_{2l-1}^{m+1})} ds - \sigma \int_0^t 1_{(x_{2l-2}^{m+1} < x_r - x_{2l-1}^{m+1})} dB_s \right) \]
\[ \begin{align*}
& \left( \varphi_t(x_{2i-1}^{m+1}) - \varphi_t(x_{2i-2}^{m+1}) - b \int_0^t 1_{\{x_{2i-2}^{m+1} < X_s \leq x_{2i-1}^{m+1}\}} ds - \sigma \int_0^t 1_{\{x_{2i-2}^{m+1} < X_s \leq x_{2i-1}^{m+1}\}} dB_s \right) \\
& \quad \leq c(t) \left[ (1 + 2b + b^2)(x_{2i-1}^{m+1} - x_{2i-2}^{m+1})(x_{2i-1}^{m+1} - x_{2i-2}^{m+1}) + \sigma(x_{2i-1}^{m+1} - x_{2i-2}^{m+1})(x_{2i-1}^{m+1} - x_{2i-2}^{m+1}) \right] \\
& \quad + \sigma^2 E \left[ \int_0^t 1_{\{x_{2i-1}^{m+1} < X_s \leq x_{2i-2}^{m+1}\}} 1_{\{x_{2i-1}^{m+1} \leq X_s \leq x_{2i-2}^{m+1}\}} ds \right] \\
& \quad \leq C \left[ \left( \frac{1}{2m+n} \right)^2 w_1(x', x'') + \left( \frac{1}{2m+n} \right)^2 w_1(x', x'') \right] \\
& \quad + \sigma^2 E \left[ \int_0^t 1_{\{x_{2i-1}^{m+1} < X_s \leq x_{2i-2}^{m+1}\}} 1_{\{x_{2i-1}^{m+1} \leq X_s \leq x_{2i-2}^{m+1}\}} ds \right] \\
& \quad \leq \begin{cases} 
C \left( \frac{1}{2m+n} \right)^2, & \text{if } r \neq l, \\
C \left( \frac{1}{2m+n} \right)^2, & \text{if } r = l.
\end{cases}
\end{align*} \]

(59)

Here \( C \) is a generic constant and also depends on \( t, b, \sigma, w_1(x', x'') \). Secondly, recall the fact that \( E(P.M.) = 0 \), if \( P \) is a process of bounded variation and \( M \) is a martingale with mean 0 and at least one of \( M \) and \( P \) is continuous. Note here \( K_1 \) is a process of bounded variation. Recall also that the cross-variation of \( \int_0^t 1_{\{a_i < X_s \leq a_{i+1} \}} dB_s \) and the jump parts such as \( (K_2(t, a_{j+1}) - K_2(t, a_j)) \) are zero. So we have

\[ \begin{align*}
E \int_0^t 1_{\{a_i < X_s \leq a_{i+1} \}} dB_s \cdot (K_1(t, a_{j+1}) - K_1(t, a_j)) &= 0, \\
E \int_0^t 1_{\{a_i < X_s \leq a_{i+1} \}} dB_s \cdot (K_2(t, a_{j+1}) - K_2(t, a_j)) &= 0, \\
E \int_0^t 1_{\{a_i < X_s \leq a_{i+1} \}} ds \cdot (K_2(t, a_{j+1}) - K_2(t, a_j)) &= 0, \\
E \int_0^t 1_{\{a_i < X_s \leq a_{i+1} \}} dB_s \cdot (K_3(t, a_{j+1}) - K_3(t, a_j)) &= 0, \\
E(K_2(t, a_{i+1}) - K_2(t, a_i)) \cdot (K_3(t, a_{j+1}) - K_3(t, a_j)) &= 0.
\end{align*} \]

Thirdly, by Lemma 8 we can see that when \( 0 \leq \xi \leq \frac{b}{2} \),

\[ \begin{align*}
E(K_2(t, a_{i+1}) - K_2(t, a_i))(K_2(t, a_{j+1}) - K_2(t, a_j)) \leq c(t, \sigma)[(a_{i+1} - a_i)^{1+2\xi} + (a_{j+1} - a_j)^{1+2\xi}].
\end{align*} \]

For other terms, by the Cauchy-Schwarz inequality, it is easy to see that

\[ \begin{align*}
& |E(\varphi_t(a_{i+1}) - \varphi_t(a_i))(K_1(t, a_{j+1}) - K_1(t, a_j))| \leq c(t)[(a_{i+1} - a_i)(a_{j+1} - a_j)]; \\
& |E(\varphi_t(a_{i+1}) - \varphi_t(a_i))(K_2(t, a_{j+1}) - K_2(t, a_j))| \leq c(t, \sigma)[(a_{i+1} - a_i)(a_{j+1} - a_j)^\frac{1}{2}]; \\
& |E(\varphi_t(a_{i+1}) - \varphi_t(a_i))(K_3(t, a_{j+1}) - K_3(t, a_j))| \leq c(t)[(a_{i+1} - a_i)(a_{j+1} - a_j)^\frac{1}{2} + \xi]; \\
& |E \int_0^t 1_{\{a_i < X_s \leq a_{i+1} \}} ds (K_1(t, a_{j+1}) - K_1(t, a_j))| \leq c(t, \sigma)[(a_{i+1} - a_i)(a_{j+1} - a_j)]; \\
& |E \int_0^t 1_{\{a_i < X_s \leq a_{i+1} \}} ds (K_3(t, a_{j+1}) - K_3(t, a_j))| \leq c(t, \sigma)[(a_{i+1} - a_i)(a_{j+1} - a_j)^\frac{1}{2} + \xi]; \\
& |E(K_1(t, a_{i+1}) - K_1(t, a_i))(K_1(t, a_{j+1}) - K_1(t, a_j))| \leq c(t, \sigma)[(a_{i+1} - a_i)(a_{j+1} - a_j)];
\end{align*} \]
\[
E(K_1(t, a_{i+1}) - K_1(t, a_i))(K_2(t, a_{j+1}) - K_2(t, a_j)) \leq c(t, \sigma)[(a_{i+1} - a_i)(a_{j+1} - a_j)^\frac{1}{\gamma}];
\]
\[
E(K_1(t, a_{i+1}) - K_1(t, a_i))(K_3(t, a_{j+1}) - K_3(t, a_j)) \leq c(t, \sigma)[(a_{i+1} - a_i)(a_{j+1} - a_j)^{\frac{1}{\gamma} + \xi}];
\]
\[
E(K_3(t, a_{i+1}) - K_3(t, a_i))(K_3(t, a_{j+1}) - K_3(t, a_j)) \leq c(t, \sigma)[(a_{i+1} - a_i)^{\frac{1}{\gamma} + \xi}(a_{j+1} - a_j)^{\frac{1}{\gamma} + \xi}].
\]

Thus
\[
E \left[ \Delta_{2^n-1} L_i^2 \Delta_{2^n-1} L_i^2 \right] \leq \begin{cases} 
C \left( \frac{1}{2m+1} \right)^{1+2\xi}, & \text{if } r \neq l, \\
C \left( \frac{1}{2m+1} \right), & \text{if } r = l.
\end{cases}
\] (60)

The other terms in (68) can be treated similarly. Therefore
\[
E |Z(m+1)^2_{x_{k-1} \cdots x_k} - Z(m)^2_{x_{k-1} \cdots x_k}|^2 \leq C \left[ 2^{m-n} \left( \frac{1}{2m+1} \right)^{1+2h} + 2^{2(m-n)} \left( \frac{1}{2m+1} \right)^{2+2h} \right].
\]

Hence, for \(2 \leq \theta < 3\), by Jensen’s inequality,
\[
E |Z(m+1)^2_{x_{k-1} \cdots x_k} - Z(m)^2_{x_{k-1} \cdots x_k}|^2 \leq \left( E |Z(m+1)^2_{x_{k-1} \cdots x_k} - Z(m)^2_{x_{k-1} \cdots x_k}|^2 \right)^{\frac{\theta}{2}} \leq \left( 2^{m-n} \left( \frac{1}{2m+1} \right)^{1+2h} + 2^{2(m-n)} \left( \frac{1}{2m+1} \right)^{2+2h} \right)^{\frac{\theta}{2}} \leq \left( \frac{1}{2m} \right)^{\frac{\theta}{2}} \left[ \frac{1}{2m} \right]^{\frac{1}{2}h \theta} \leq \left( \frac{1}{2m} \right)^{\frac{\theta}{2}} \left[ \frac{1}{2m} \right]^{\frac{1}{2}h \theta} \leq C \left[ \frac{1}{2m} \right]^{\frac{\theta}{2}} \left[ \frac{1}{2m} \right]^{\frac{1}{2}h \theta} \leq \left( \frac{1}{2m} \right)^{\frac{\theta}{2}} \left( \frac{1}{2m} \right)^{\frac{1}{2}h \theta} \leq \left( \frac{1}{2m} \right)^{\frac{\theta}{2}} \left( \frac{1}{2m} \right)^{\frac{1}{2}h \theta} \leq C \left[ \frac{1}{2m} \right]^{\frac{\theta}{2}} \left[ \frac{1}{2m} \right]^{\frac{1}{2}h \theta} \leq C \left[ \frac{1}{2m} \right]^{\frac{\theta}{2}} \left[ \frac{1}{2m} \right]^{\frac{1}{2}h \theta} \leq C \left[ \frac{1}{2m} \right]^{\frac{\theta}{2}} \left[ \frac{1}{2m} \right]^{\frac{1}{2}h \theta} \leq C \left[ \frac{1}{2m} \right]^{\frac{\theta}{2}} \left[ \frac{1}{2m} \right]^{\frac{1}{2}h \theta} \leq C \left[ \frac{1}{2m} \right]^{\frac{\theta}{2}} \left[ \frac{1}{2m} \right]^{\frac{1}{2}h \theta}.
\]

where \(C\) is a generic constant and also depends on \(\theta, h, w_1(x', x''),\) and \(c.\) Note \(\xi = \frac{q-2}{2q} + (3-q)\varepsilon,\) so we get (66).

\textbf{Corollary 9} Under the same assumption as in Proposition 7, we have
\[
\sup_m \sum_{n=1}^{\infty} n^2 \sum_{k=1}^{2^n} |Z(m)^2_{x_{k-1} \cdots x_k}|^\frac{\theta}{2} < \infty \quad \text{a.s.}
\]

\textbf{Proof:} From the Minkowski inequality,
\[
\left( \sum_{k=1}^{2^n} |Z(m)^2_{x_{k-1} \cdots x_k}|^\frac{\theta}{2} \right)^{\frac{2}{\theta}} \leq \left( \sum_{k=1}^{2^n} |Z(m)^2_{x_{k-1} \cdots x_k} - Z(m-1)^2_{x_{k-1} \cdots x_k}|^\frac{\theta}{2} \right)^{\frac{2}{\theta}} + \left( \sum_{k=1}^{2^n} |Z(m-1)^2_{x_{k-1} \cdots x_k} - Z(m-2)^2_{x_{k-1} \cdots x_k}|^\frac{\theta}{2} \right)^{\frac{2}{\theta}} + \cdots + \left( \sum_{k=1}^{2^n} |Z(1)^2_{x_{k-1} \cdots x_k} - Z(0)^2_{x_{k-1} \cdots x_k}|^\frac{\theta}{2} \right)^{\frac{2}{\theta}} + \left( \sum_{k=1}^{2^n} |Z(0)^2_{x_{k-1} \cdots x_k}|^\frac{\theta}{2} \right)^{\frac{2}{\theta}}.
\]

Then it is easy to see from (61), Jensen’s inequality, (65), (66) and (68)
\[
E \sup_m \sum_{n=1}^{\infty} n^\frac{\theta}{2} \left( \sum_{k=1}^{2^n} |Z(m)^2_{x_{k-1} \cdots x_k}|^\frac{\theta}{2} \right)^{\frac{2}{\theta}}
\]
There exists a unique $Z$ as $2 < \theta < 3$, where $C$ is a generic constant and also depends on $\theta$, $h$, $w(x', x'')$, and $c$. Therefore,

$$\sup_m \sum_{n=1}^{\infty} \sum_{k=1}^{2^n} \left( Z(m)^2_{x_{k-1}, x_k} \right)^{\frac{\theta}{2}} < \infty \text{ a.s.}$$

However, it is easy to see as $\theta > 2$,

$$\left( \sup_m \sum_{n=1}^{\infty} \sum_{k=1}^{2^n} \left( Z(m)^2_{x_{k-1}, x_k} \right)^{\frac{\theta}{2}} \right)^{\frac{\theta}{\theta}} \leq \sup_m \sum_{n=1}^{\infty} \sum_{k=1}^{2^n} \left( Z(m)^2_{x_{k-1}, x_k} \right)^{\frac{\theta}{2}} < \infty \text{ a.s.}$$

So the claim follows.

**Theorem 10** Let $L^2_t$ be the local time of the time homogeneous Lévy process $X_t$ given by (2). Assume $g$ is a continuous function of finite $q$-variation with a real number $2 \leq q < 3$, and the Lévy measure $n(dy)$ satisfies (3), $\sigma \neq 0$. Then for any $\theta \in (q, 3)$, the continuous process $Z_x = (L^2_t, g(x))$ satisfying (3), there exists a unique $Z^1$ on $\Delta$ taking values in $(R^2)^{\Delta}$ ($i = 1, 2$) such that

$$\sum_{i=1}^{2} \sup_P \left( \sum_{l}^{l} \left( Z(m)^l_{x_{i-1}, x_i} - Z^i_{x_{i-1}, x_i} \right)^{\theta} \right)^{\frac{\theta}{\theta}} \rightarrow 0,$$

both almost surely and in $L^1(\Omega, F, P)$ as $m \rightarrow \infty$. In particular, $Z = (1, Z^1, Z^2)$ is the canonical geometric rough path associated to $Z_\cdot$, and $Z^1_{n,b} = Z_b - Z_a$.

**Proof:** The convergence of $Z(m)^l$ to $Z^l$ is actually the result of Theorem 5. In the following we will prove $Z(m)^2_{a,b}$ converges in the $\theta$-variation distance. By Proposition 4.1.2 in [23].
$E \sup_{D} \sum_{i} |Z(m + 1)_{x_{i-1}, x_i}^2 - Z(m)_{x_{i-1}, x_i}^2|^\frac{q}{2}$

$\leq C(\theta, \gamma) E \left( \sum_{n=1}^{\infty} n^2 \sum_{k=1}^{2^n} |Z(m + 1)_{x_{k-1}, x_k}^1 - Z(m)_{x_{k-1}, x_k}^1|^\theta \right)^{\frac{q}{2}}$

$\cdot \left( \sum_{n=1}^{\infty} n^2 \sum_{k=1}^{2^n} \left( |Z(m + 1)_{x_{k-1}, x_k}^1|^\theta + |Z(m)_{x_{k-1}, x_k}^1|^\theta \right) \right)^{\frac{q}{2}}$

$+ C(\theta, \gamma) E \sum_{n=1}^{\infty} n^2 \sum_{k=1}^{2^n} |Z(m + 1)_{x_{k-1}, x_k}^2 - Z(m)_{x_{k-1}, x_k}^2|^\theta$

$:= A + B.$

We will estimate part A, B respectively. First from the Cauchy-Schwarz inequality, \[48\] and \[50\], we know

$A \leq C \left( E \sum_{n=1}^{\infty} n^2 \sum_{k=1}^{2^n} \left( |Z(m + 1)_{x_{k-1}, x_k}^1 - Z(m)_{x_{k-1}, x_k}^1|^\theta + |Z(m)_{x_{k-1}, x_k}^1 - Z_{x_{k-1}, x_k}^1|^\theta \right) \right)^{\frac{q}{2}}$

$\cdot \left( E \sum_{n=1}^{\infty} n^2 \sum_{k=1}^{2^n} \left( |Z(m + 1)_{x_{k-1}, x_k}^1|^\theta + |Z(m)_{x_{k-1}, x_k}^1|^\theta \right) \right)^{\frac{q}{2}}$

$\leq C \left( \frac{1}{2^m} \right)^{\frac{q}{2} - 1} \left( \sum_{n=1}^{\infty} n^2 \left( \frac{1}{2^n} \right)^{\frac{q}{2} - 1} \right)^{\frac{q}{2}}.$

Secondly from Proposition \[6\] and Proposition \[7\] we know

$B \leq C \sum_{n=1}^{\infty} n^2 \left( \frac{1}{2^{m+n}} \right)^{\frac{h\theta}{2} - 1} C \sum_{n=1}^{m} n^2 \left( \frac{1}{2^m} \right)^{\frac{3\theta}{2} - 1} \left( \frac{1}{2m} \right)^{\frac{3\theta}{2} - 1} \left( \frac{1}{2m} \right)^{\frac{3\theta}{2} - 1}$

$\leq C \left[ \left( \frac{1}{2^m} \right)^{\frac{h\theta}{2} - 1} + \left( \frac{1}{2^m} \right)^{\frac{3\theta}{2} - 1} \right].$

as $q < \theta < 3$, and $h\theta > 1$. So

$E \sup_{D} \sum_{i} |Z(m + 1)_{x_{i-1}, x_i}^2 - Z(m)_{x_{i-1}, x_i}^2|^\frac{q}{2} \leq C \left[ \left( \frac{1}{2^m} \right)^{\frac{h\theta}{2} - 1} + \left( \frac{1}{2^m} \right)^{\frac{3\theta}{2} - 1} \right].$

Similar to the proof of Theorem \[5\], we can easily deduce that $(Z(m)^2)_{m \in N}$ is a Cauchy sequence in the $\theta$-variation distance. So when $m \to \infty$, it has a limit, denote it by $Z^2$. And from the completeness under the $\theta$-variation distance (Lemma 3.3.3 in [23]), $Z^2$ is also of finite $\theta$-variation. The theorem is proved.

\[\diamond\]

**Remark 11** We would like to point out that the above method does not seem to work for two arbitrary functions $f$ of $p$-variation and $g$ of $q$-variation ($2 < p, q < 3$) to define a rough path $Z_x = (f(x), g(x))$. However the special property (\[64\]) of local times makes our analysis work. A similar method was used in [23] for fractional Brownian motion with the help of long-time memory. Here \[64\] serves a similar role of the long-time memory as in [23].

As local time $L^x_t$ has a compact support in $x$ for each $\omega$ and $t$, so we can define integral of local time directly in $R$. For this, we take $[x', x'']$ covering the support of $L^x_t$. From Chen’s identity, it’s easy to know that for any $(a, b) \in \Delta$. 

\[\text{Local time rough path 23}\]
Let \( Z_{a,b}^2 = \lim_{m(D_{a,b}) \to 0} \sum_{i=0}^{r-1} (Z_{x_i,x_{i+1}}^2 + Z_{x_i,x_{i+1}}^1 \otimes Z_{x_i,x_{i+1}}^1) \).

In particular,

\[
(Z_{a,b}^2)_{2,1} = \lim_{m(D_{a,b}) \to 0} \sum_{i=0}^{r-1} (Z_{x_i,x_{i+1}}^2)_{2,1} + (Z_{a,x_i}^1 \otimes Z_{x_i,x_{i+1}}^1)_{2,1}
\]

exists. Here \((Z_{x_i,x_{i+1}}^2)_{2,1}\) means lower-left element of the \(2 \times 2\) matrix \(Z_{x_i,x_{i+1}}^2\). It turns out that

\[
\lim_{m(D_{a,b}) \to 0} \sum_{i=0}^{r-1} (Z_{x_i,x_{i+1}}^2)_{2,1} + (g(x_i)(L_t^{x_{i+1}} - L_t^{x_i}))
\]

exists. Denote this limit by \( \int_a^b g(x) dL_t^x \). Similarly, we can define \( \int_a^b L_t^x dL_t^x \). To verify the latter integral is well defined, we only need to consider the case \( q = 2 \). Then it is easy to see under condition \( (39) \), \( \int_a^b L_t^x dL_t^x \) is defined as a rough path integral. Therefore we have the following corollary.

**Corollary 12** Assume all conditions of Theorem 11 but the Lévy measure satisfies \( (39) \). Then the local time \( L_t^x \) is a geometrical rough path of roughness \( p \) in \( x \) for any \( t \geq 0 \) a.s. for any \( p > 2 \), and \( (a,b) \in \Delta \),

\[
\int_a^b L_t^x dL_t^x = \lim_{m(D_{a,b}) \to 0} \sum_{i=0}^{r-1} \langle Z_{x_i,x_{i+1}}^2 \rangle_{1,1} + L(x_i)(L_t^{x_{i+1}} - L_t^{x_i})
\]

Moreover, if \( g \) is a continuous function with bounded \( q \)-variation, \( 2 \leq q < 3 \), and the Lévy measure satisfies \( (38) \), then the integral \( \int_a^b g(x) dL_t^x \) is defined by

\[
\int_a^b g(x) dL_t^x = \lim_{m(D_{a,b}) \to 0} \sum_{i=0}^{r-1} (Z_{x_i,x_{i+1}}^2)_{2,1} + g(x_i)(L_t^{x_{i+1}} - L_t^{x_i})
\]  \( (62) \)

### 4 Continuity of the rough path integrals and applications to extensions of Itô’s formula

In this section we will apply the Young integral and rough path integral of local time defined in sections 2 and 3 to prove a useful extension to Itô’s formula. First we consider some convergence result of the rough path integrals.

Let \( Z_j(x) := (L_t^x, g_j(x)) \), where \( g_j(\cdot) \) is of bounded \( q \)-variation uniformly in \( j \) for \( 2 \leq q < 3 \), and when \( j \to \infty \), \( g_j(x) \to g(x) \) for all \( x \in \mathbb{R} \). Assume the Lévy measure satisfies \( (38) \). Repeating the above argument, for each \( j \), we can find the canonical geometric rough path \( Z_j = (1, Z_j^1, Z_j^2) \) associated to \( Z_j \), and the smooth rough path \( Z_j^1(m) = (1, Z_j^1(m)^1, Z_j^1(m)^2) \). Actually, \( Z_j^1_{a,b} \to Z_{a,b}^1 \) in the sense of the uniform topology, and also in the sense of the \( \theta \)-variation topology. As for \( Z_j^2_{a,b} \), we can easily see that
From Theorem 10, we know that \( d_{2,\theta}(Z(m)^2, Z^2) \to 0 \) as \( m \to \infty \). Moreover, it is not difficult to see from the proofs of Propositions 6, 7, and Theorem 10, \( d_{2,\theta}((Z_j)^2, (Z_j(m))^2) \to 0 \) as \( m \to \infty \) uniformly in \( j \). So for any given \( \varepsilon > 0 \), there exists an \( m_0 \) such that when \( m \geq m_0 \), \( d_{2,\theta}(Z(m)^2, Z^2) < \frac{\varepsilon}{2} \), \( d_{2,\theta}((Z_j)^2, (Z_j(m))^2) < \frac{\varepsilon}{2} \) for all \( j \). In particular, \( d_{2,\theta}(Z(m)^2, Z^2) < \frac{\varepsilon}{2} \) for all \( j \). It’s easy to prove for such \( m_0 \), \( d_{2,\theta}((Z_j(m))^2, Z(m)^2) < \frac{\varepsilon}{2} \) for sufficiently large \( j \). Replacing \( m \) by \( m_0 \) in (63), we can get \( d_{2,\theta}((Z_j)^2, Z^2) < \varepsilon \) for sufficiently large \( j \). Then by (62) and the definition of \( \int_a^b g_j(x) dL_t^x \), we know that \( \int_a^b g_j(x) dL_t^x \to \int_a^b g(x) dL_t^x \) as \( j \to \infty \). Similarly, we can see from the last section, when we consider \( Z_t(m) = (1, Z_t(m))^1, Z_t(m)^2), d_{2,\theta}((Z_t)^2, (Z_t(m))^2) \to 0 \), as \( m \to \infty \) uniformly in \( t \in [0, T] \), for any \( T > 0 \). Therefore we can also conclude that \( Z_t^2 \) is continuous in \( t \) in the \( d_{2,\theta} \) topology. Note now that the local time \( L_t^x \) has a compact support in \( x \) a.s. So it is easy to see from taking \([x', x'']\) covering the support of \( L_t^x \) that the above construction of the integrals and the convergence can work for the integrals on \( R \). Therefore we have

**Proposition 13** Let \( Z_j(x) := (L_t^x, g_j(x)) \), \( Z_t(x) := (L_t^x, g(x)) \), where \( g_j(\cdot), g(\cdot) \) are continuous and of bounded \( q \)-variation uniformly in \( j \), \( 2 \leq q < 3 \), and the Lévy measure \( n(dy) \) satisfies [58], \( \sigma \neq 0 \). Assume \( g_j(x) \to g(x) \) as \( j \to \infty \) for all \( x \in R \). Then as \( j \to \infty \), \( Z_j(\cdot) \to Z(\cdot) \) a.s. in the \( \theta \)-variation distance. In particular, as \( j \to \infty \), \( \int_\infty^\infty g_j(x) dL_t^x \to \int_\infty^\infty g(x) dL_t^x \) a.s. Similarly, \( Z_t(\cdot) \) is continuous in \( t \) in the \( \theta \)-topology. In particular, \( \int_\infty^\infty g(x) dL_t^x \) is a continuous function of \( t \) a.s.

Now for any \( g \) being continuous and of bounded \( q \)-variation \((2 \leq q < 3)\), define

\[
g_j(x) = \int_\infty^\infty k_j^q(x-y)g(y)dy,
\]

where \( k_j^q \) is the mollifier given by

\[
k_j^q(x) = \begin{cases} \frac{c_j e^{-(x-j)^q}}{(q-1)j^{q-1}}, & \text{if } x \in (0, j), \\ 0, & \text{otherwise.} \end{cases}
\]

Here \( c \) is a constant such that \( \int_0^j k_j^q(x)dx = 1 \). It is well known that \( g_j \) is a smooth function and \( g_j(x) \to g(x) \) as \( j \to \infty \) for each \( x \). So the integral \( \int_\infty^\infty g_j(x) dL_t^x \) is a Riemann integral for the smooth function \( g_j(x) \). Moreover, Proposition 13 guarantees that \( \int_\infty^\infty g_j(x) dL_t^x \to \int_\infty^\infty g(x) dL_t^x \) a.s.

In the following, we will show that Proposition 13 is true for \( g \) being of bounded \( q \)-variation \((2 \leq q < 3)\) without assuming \( g \) being continuous. Note that a function with bounded \( q \)-variation \((q \geq 1)\) may have at most countable discontinuities. Using the method in [54], we will define the rough path integral \( \int_{x'}^{x''} L_t^x dg(x) \). Here we assume \( g(x) \) is càdlàg in \( x \).

First we can define a map

\[
\tau_\delta(\cdot) : [x', x''] \to [x', x'' + \delta \sum_{n=1}^\infty |j(x_n)|^q],
\]

in the following way:

\[
\tau_\delta(x) = x + \delta \sum_{n=1}^\infty |j(x_n)|^q 1_{\{x_n \leq x\}}(x),
\]

where \( j(x_i) := G(x_i) - G(x_i-), \{x_i\}_{i=1}^\infty \) are the discontinuous points of \( G \) inside \([x', x'']\), \( \delta > 0 \). The map \( \tau_\delta(\cdot) : [x', x''] \to [x', \tau_\delta(x'')] \) extends the space interval into the one where we can define the continuous path \( G_\delta(y) \) from a càdlàg path \( G \) by:
Take $G$ to be $g$ and $L_t$, we can define $g_\delta$ and $L_{t, \delta}$  respectively. As $L_t^\delta$ is continuous, we can easily see that $L_{t, \delta}(y) := L_{t, \delta}(\tau_\delta(x)) = L_t^\delta$.

**Theorem 14** Let $g(x)$ be a càdlàg path with bounded $q$-variation ($2 \leq q < 3$), and the Lévy measure $n(dy)$ satisfies (65), $\sigma \neq 0$. Then

$$\int_{x'}^{x^*} L_t^\delta dg(x) = \int_{x'}^{\tau_\delta(x^*)} L_{t, \delta}(y) dg_\delta(y).$$

**Proof:** First it is easy to see that the integral $\int_{x'}^{\tau_\delta(x^*)} L_{t, \delta}(y) dg_\delta(y)$ is a rough path integral that can be defined by the method of last section. Now note that at any discontinuous point $x_r$,

$$\int_{x_r}^{x_r^*} L_t^\delta dg(x) = L_t(x_r)(g(x_r) - g(x_r^-))$$

and

$$\sum_r \left( (Z_\delta)^2_{(\tau_\delta(x_r^-), \tau_\delta(x_r))} \right)_{2,1} = \sum_r \int_{\tau_\delta(x_r^-)}^{\tau_\delta(x_r)} (L_{t, \delta}(y) - L_{t, \delta}(\tau_\delta(x_r^-))) dg_\delta(y) = 0,$$

where $Z_\delta := (L_{t, \delta}(y), g_\delta(y))$. Thus

$$\sum_r L_t^\delta (g(x_r) - g(x_r^-))$$

$$= \sum_r L_{t, \delta}(\tau_\delta(x_r^-))(g_\delta(\tau_\delta(x_r)) - g_\delta(\tau_\delta(x_r^-)))$$

$$= \sum_r \left[ L_{t, \delta}(\tau_\delta(x_r^-))(g_\delta(\tau_\delta(x_r)) - g_\delta(\tau_\delta(x_r^-))) + ((Z_\delta)^2_{(\tau_\delta(x_r^-), \tau_\delta(x_r))})_{2,1} \right]$$

$$< \infty,$$

so

$$\int_{\tau_\delta(x_r^-)}^{\tau_\delta(x_r)} L_{t, \delta}(y) dg_\delta(y) = L_{t, \delta}(\tau_\delta(x_r^-))(g_\delta(\tau_\delta(x_r)) - g_\delta(\tau_\delta(x_r^-)))$$

$$= L_t(x_r)(g(x_r) - g(x_r^-)),$$

Thus

$$\int_{x_r}^{x_r^*} L_t^\delta dg(x) = \int_{\tau_\delta(x_r^-)}^{\tau_\delta(x_r)} L_{t, \delta}(y) dg_\delta(y).$$

Now define $g(x) = \tilde{g}(x) + h(x) = g(x_r^-) + h(x_r^*)$. Then $\tilde{g}$ is the continuous part of $g$ and $h$ is the jump part of $g$. Moreover, $\tilde{g}$ satisfies the $q$-variation condition. So $\int_{x'}^{x^*} L_t(x) d\tilde{g}(x)$ can be well defined as in the last section. For $h$, we can define $h_\delta$ by taking $G = h$ in (65). So the integral $\int_{x'}^{x^*} L_t^\delta dh(x)$ can be well defined by the followings:

$$\int_{x'}^{\tau_\delta(x^*)} L_{t, \delta}(y) dh_\delta(y) = \sum_r \int_{\tau_\delta(x_r^-)}^{\tau_\delta(x_r)} L_{t, \delta}(y) dh_\delta(y) = \sum_r L_t(x_r)(h(x_r) - h(x_r^-))$$

$$= \sum_r L_t(x_r)(g(x_r) - g(x_r^-)) = \sum_r \int_{x_r}^{x_r^*} L_t^\delta dh(x) = \int_{x'}^{x^*} L_t^\delta dh(x).$$
Therefore
\[
\int_{x'}^{x''} L^x_t dg(x) = \int_{x'}^{x''} L^x_t d\tilde{g}(x) + \int_{x'}^{x''} L^x_t dh(x)
\]
\[
= \int_{x'}^{\tau_3(x'')} L_{t,\beta}(y)d\tilde{g}_\delta(y) + \int_{x'}^{\tau_3(x'')} L_{t,\beta}(y)dh_\delta(y)
\]
\[
= \int_{x'}^{\tau_3(x'')} L_{t,\beta}(y)dg_\delta(y).
\]

Similarly to Proposition 13 we have

**Proposition 15** Under the condition of Proposition 13 as \( j \to \infty \), \( \int_{-\infty}^{\infty} L_t^x dg_j(x) \to \int_{-\infty}^{\infty} L_t^x dg(x) \) a.s. for such \( g \) with bounded \( q \)-variation \((2 \leq q < 3)\).

**Proof:** Define \( F_j(x) := (g_j - g)(x) \), so \( F_j(x) \to 0 \) as \( j \to \infty \), for all \( x \). It’s easy to see that \( F_j,\beta(x) \to 0 \) as \( j \to \infty \), for all \( x \). From the above theorem and Proposition 13 we have
\[
\int_{x'}^{x''} L^x_t d(g_j - g)(x) = \int_{x'}^{x''} L^x_t dF_j(x) = \int_{x'}^{\tau_3(x'')} L_{t,\beta}dF_j,\beta(y) \to 0, \quad \text{as } j \to \infty.
\]
Then the proposition follows easily.

 Applying the standard smoothing procedure on \( f(x) \), we can get \( f_n(x) \) which is defined in the same way as \( g_j(x) \) in (64). And by Itô’s formula (c.f. [29]), we have
\[
f_n(X_t) = f_n(X_0) + \int_0^t \nabla f_n(X_s)dX_s + A^n_t, \quad 0 \leq t < \infty.
\]

where
\[
A^n_t = \frac{1}{2} \int_0^t f''_n(X_{s-})d[X,X]_s + \sum_{0 \leq s \leq t} \left[ f_n(X_s) - f_n(X_{s-}) - \nabla f_n(X_{s-})\Delta X_s \right].
\]

From the occupation times formula, the definition of the integral of local time and the convergence results of the integrals, we have
\[
\lim_{n \to \infty} \frac{1}{2} \int_0^t f''_n(X_{s-})d[X,X]_s = \lim_{n \to \infty} \frac{1}{2} \int_{-\infty}^{\infty} L^x_t d\nabla^- f_n(x) = -\frac{1}{2} \int_{-\infty}^{\infty} \nabla^- f(x)dxL^x_t,
\]
and the rough path integral \( \int_{-\infty}^{\infty} \nabla^- f(x)dxL^x_t \) is continuous in \( t \) from Proposition 13. For the convergence of jump part in (68), we can conclude from the proof of Theorem 3 in Eisenbaum and Kyrianou [7], if the following assumption:

**Condition (A):** \( \int_{\{|y|<1\}} |f(x+y) - f(x) - \nabla^- f(x)y|n(dy) \) is well defined and locally bounded in \( x \), holds, then
\[
\lim_{n \to \infty} \sum_{0 \leq s \leq t} \left[ f_n(X_s) - f_n(X_{s-}) - \nabla f_n(X_{s-})\Delta X_s \right] = \sum_{0 \leq s \leq t} \left[ f(X_s) - f(X_{s-}) - \nabla^- f(X_{s-})\Delta X_s \right],
\]
in \( L^2(dP) \). Therefore we have:
Theorem 16 Let \( f : \mathbb{R} \to \mathbb{R} \) be an absolutely continuous function and have left derivative \( \nabla^- f(x) \) being left continuous and locally bounded, \( \nabla^- f(x) \) be of bounded \( q \)-variation, where \( 1 \leq q < 3 \). Then for \( X = (X_t)_{t \geq 0} \), a time homogeneous Lévy process with \( \sigma \neq 0 \) and Lévy measure \( \nu(dy) \) satisfying Condition (A), and (14) when \( 1 \leq q < 2 \), (38) when \( 2 \leq q < 3 \), we have P-a.s.

\[
f(X_t) = f(X_0) + \int_0^t \nabla^- f(X_s) dX_s - \frac{1}{2} \int_{-\infty}^{\infty} \nabla^- f(x) d_x L^x_t + \sum_{0 \leq s \leq t} [f(X_s) - f(X_{s-}) - \nabla^- f(X_{s-}) \Delta X_s], \quad 0 \leq t < \infty.
\]

(69)

Here the integral \( \int_{-\infty}^{\infty} \nabla^- f(x) d_x L^x_t \) is a Lebesgue-Stieltjes integral when \( q = 1 \), a Young integral when \( 1 < q < 2 \) and a Lyons’ rough path integral when \( 2 \leq q < 3 \) respectively. In particular, under the condition (40), (69) holds for any \( 2 \leq q < 3 \).

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