UNIVERSITY IN OSCILLATION MODES OF SUPERFLUID NEUTRON STARS?
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ABSTRACT

It has been well established that the $f$-mode of relativistic ordinary fluid neutron stars displays a universal scaling behavior. Here, we study whether the "ordinary" $f_o$- and "superfluid" $f_s$-modes of superfluid neutron stars also show similar universal behavior. We first consider a simple case where the neutron superfluid and normal fluid are decoupled, and with each fluid modeled by a polytropic equation of state. We find that the $f_s$-mode obeys the same scaling laws as established for the $f$-mode of ordinary fluid stars. However, the oscillation frequency of the $f_s$-mode obeys a different scaling law, which can be derived analytically from a homogenous two-fluid stellar model in Newtonian gravity. Next the coupling effect between the two fluids is studied via a parameterized model of entrainment. We find that the coupling in general breaks the universal behavior seen in the case of decoupled fluids. Based on a relativistic variational principle, an approximated expression is derived for the first-order shift of the $f_s$-mode squared frequency due to the entrainment.

Key words: dense matter – equation of state – gravitational waves – stars: neutron – stars: oscillations

Online-only material: color figures

1. INTRODUCTION

Ground-based gravitational-wave detectors are either already operating or will soon be operating. For example, the Laser Interferometer Gravitational-Wave Observatory (LIGO) detectors have conducted five science runs since 2002. While the current detectors are still not sensitive enough to detect gravitational waves directly, interesting upper limits have been placed on the gravitational-wave strains from several potential astrophysical sources (e.g., isolated pulsars and stochastic background) (Abbott et al. 2007a, 2007b). It is very likely that the detection of gravitational waves will come within a decade or so after the upgrade of the detectors.

Pulsating neutron stars are one of the interesting potential sources of gravitational waves. They may also rotate rapidly enough for various kinds of rotational induced mode instabilities to develop and enhance the emitted gravitational waves (Andersson 2003). The pulsation modes of neutron stars are damped due to the emission of gravitational waves, and hence are called quasi-normal modes (see, e.g., Kokkotas & Schmidt 1999 and Ferrari & Gualtieri 2008 for reviews), instead of normal modes as in Newtonian theory. The study of quasi-normal modes of neutron stars has a long history dating back to the pioneering works of Thorne and his collaborators (Thorne & Campolattaro 1967; Price & Thorne 1969; Thorne 1969a, 1969b; Campolattaro & Thorne 1970). It is by now well established that the quasi-normal mode spectrum of a neutron star is tremendously rich and the gravitational waves emitted from a pulsating star carry important information about the internal structure of the star (e.g., Andersson & Kokkotas 1996, 1998; Benhar et al. 1999, 2004; Kokkotas et al. 2001; Tsui & Leung 2005a). The gravitational-wave signals from pulsating neutron stars, should they be detected by ongoing or future detectors, can thus in principle be used to constrain the supranuclear equation of state (EOS), which is still poorly understood.

While the quasi-normal modes of a pulsating neutron star in general depend sensitively on the stellar model and EOS, some empirical universal behaviors have also been observed with different EOSs. Andersson & Kokkotas (1998) noted that the frequencies and damping times of the leading gravitational-wave $w$-modes and fluid $f$-mode of nonrotating neutron stars can be approximated by some empirical relations which depend only on the mass and radius of the star for most EOSs (see also Benhar et al. 1999). In particular, Andersson & Kokkotas (1998) obtained the following formulae, respectively, for the real and imaginary parts of $f$-mode frequency:

$$\text{Re}(\omega M) = \alpha_1 M + \alpha_2 C^{3/2},$$

$$\text{Im}(\omega M) = C^4 (\beta_1 C + \beta_2),$$

where $\alpha_1$, $\alpha_2$, $\beta_1$, and $\beta_2$ are model-independent constants determined by curve fitting. $M$ is the mass of the star and the dimensionless parameter $C \equiv M/R$ (in units where $G = c = 1$) is the compactness of the star. The real part of $\omega$ is the mode oscillation frequency, while the imaginary part is inversely proportional to the damping time due to the emission of gravitational waves. Recently, the physical mechanism behind such universal behaviors has been investigated in detail by Tsui & Leung (2005b). In their notation, the (complex) frequency $\omega$ of the $w$-mode or $f$-mode can be approximated by

$$\omega M = aC^2 + bC + c,$$

where the complex parameters ($a$, $b$, $c$) are also model-independent constants determined by curve fitting. Exploring such universality for the $w$-modes, Tsui & Leung (2005a) have developed an inversion scheme to determine the mass, radius, density distribution, and the EOS of a neutron star model from the frequencies of the first few $w$-modes of the star.

Mature neutron stars are known to be very cold on the nuclear temperature scale ($10^8$ K). It is believed that a newborn neutron star can cool down rather quickly (on a timescale of a few weeks to months) to the transition temperature ($\sim 10^9$ K) for nucleon superfluidity and superconductivity to occur (see, e.g., Lombardo 1999; Lombardo & Schulze 2001; Andersson et al. 2005). For sufficiently high density, nuclear matter can transform to deconfined quark matter, which may also be in the so-called color-superconducting phase (see, e.g., Alford &

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Reddy 2003; Alford 2004). To date, there is still no direct evidence for the existence of nucleon superfluidity in neutron stars. However, the well-established pulsar glitch phenomenon is best explained by the pinning and unpinning of large numbers of superfluid vortices to the solid crust (Radhakrishnan & Manchester 1969; Lyne 1993). So, how would nucleon superfluidity affect the dynamics and hence the oscillation mode spectrum of neutron stars? For the simplest model, neutron stars consist of neutrons, protons, and electrons. When neutrons become superfluid, they dynamically decouple from protons and electrons. The protons can also be in a superconducting state, but they are coupled to the ordinary electron fluid via electromagnetic interaction on a very short timescale. In effect, the neutron star interior can be approximated by a two-fluid model: one fluid is the neutron superfluid; the other fluid is a conglomerate of all other charged constituents.

To a first approximation, the two interpenetrating fluids could be considered as independent. However, due to the strong interaction between neutrons and protons, the two fluids in general can couple via the so-called entrainment effect. Entrainment is a multifluid effect and it arises when the flow of one fluid induces a momentum in the other. In this work, we will study the parameterized entrainment model used previously by Andersson et al. (2002).

Building on the formalism developed by Carter and his collaborators (e.g., Carter 1989; Comer & Langlois 1993, 1994; Carter & Langlois 1995; Langlois et al. 1998); Comer et al. (1999) and Andersson et al. (2002) have calculated the quasi-normal modes of such a nonrotating two-fluid stellar model (see, e.g., Lindblom & Mendell 1994; Lee 1995; Prix & Rieutord 2002, for the corresponding study in Newtonian theory). These works show the existence of a new family of modes for a two-fluid star, the so-called superfluid modes. These modes have the distinguishing characteristic that the two fluids are essentially counter-moving, as a manifestation of an extra fluid degree of freedom which is missing in the study of pulsating ordinary, single-fluid neutron stars (Comer et al. 1999). Another characteristic of the superfluid modes is that they depend sensitively on the entrainment effects between the two fluids (Andersson et al. 2002). In a two-fluid star, there is essentially a doubling of the fluid modes. For example, the single \( f \)-mode which exists in an ordinary fluid star is split into an “ordinary” \( f \)-mode (denoted by \( f_o \)), where the two fluids tend to move together, and a superfluid counterpart (denoted by \( f_s \)) where the fluids are counter-moving (Comer et al. 1999). On the other hand, the \( w \)-modes do not show such kind of mode doubling effect (Comer et al. 1999). The two fluids of the \( w \)-modes always move in “lock step” due to the fact that \( w \)-modes are largely spacetime oscillations.

As mentioned above, the frequencies of the \( f \)- (or \( w \))-modes of nonrotating ordinary fluid neutron stars can be approximated by universal scaling laws (Equations (1)–(3)) for different EOSs. It is thus natural to ask whether this universality also holds for two-fluid neutron star models. In this paper, we investigate whether the \( f_o \) and \( f_s \) modes of two-fluid stars establish any kind of universality. In order to provide the reader with a better understanding of the main results obtained in this paper, we will give a brief summary in the following.

In this work, we first study the simplest case of two decoupled fluids, each with a polytropic EOS. We vary the polytropic indices of the two fluids to mimic the effects of different EOSs. We find that the \( f_o \)-mode (i.e., the “ordinary” \( f \)-mode) can still be approximated very well by the same scaling laws (Equations (1)–(3)) as established for ordinary fluid neutron stars. For the superfluid \( f_s \)-mode, we find that the real part of the mode frequency satisfies a different universal scaling law (Equation (14)), while the imaginary part in general does not have any universal behavior. We are also able to derive the universal scaling curve for the real part of the \( f_s \)-mode frequency analytically based on a homogeneous two-fluid model in Newtonian gravity.

We then study the effect of coupling between the two fluids via a parameterized model of entrainment. We analyze both numerically and analytically how the entrainment affects the \( f_s \)-mode frequency. Based on a relativistic variational principle, we have derived a general integral formula to calculate the first-order shift in the mode (squared) frequency (Equation (53)). Furthermore, for the particular class of background EOS and entrainment models we considered in the study, we are able to approximate the integral formula by an algebraic expression (Equation (73)) which depends explicitly only on the model parameters.

The main results obtained in this work (Equations (14) and (17)) can be used by other researchers to obtain a good approximation to the \( f_s \)-mode frequency (without the need to construct their own numerical code) for the class of background EOS and entrainment models that have been used extensively to study superfluid neutron stars.

The plan of this paper is as follows. Section 2 briefly summarizes the general relativistic two-fluid formalism. In Section 3, we describe the EOS models we use in the study. Section 4 presents our numerical results. Section 5 presents our analytical analysis for a homogeneous two-fluid star model. The effects of entrainment on the superfluid mode frequency are studied analytically in Section 6. Finally, we summarize our results in Section 7. We use units where \( G = c = 1 \) unless otherwise noted.

2. THE TWO-FLUID FORMALISM

In this paper, we make use of the formalism and numerical code developed by Comer et al. (1999) to compute the oscillation modes of a general relativistic superfluid star. As discussed in Section 1, the superfluid neutron star interior is approximated by a two-fluid model. One of the fluids is composed of superfluid neutrons, and the other contains all other charged particles (proton, electrons, and crust nuclei, etc.), which will be called a proton fluid for simplicity. We focus our attention on a simplified model where the two fluids are assumed to coexist throughout the whole star as in Comer et al. (1999). The work of Comer et al. (1999) is based on Carter’s general relativistic superfluid formalism. Here, we will only summarize the formalism briefly (see Andersson & Comer 2007 for a recent review).

The central quantity of the two-fluid formalism is the master function \( \Lambda(n^a, p^a, x^2) \), which is formed by three scalars, \( n^2 = -n_a n^a, p^2 = -p_a p^a, \) and \( x^2 = -n_a p^a \). The four vectors \( n^a \) and \( p^a \) are, respectively, the conserved number density currents for the neutrons and protons. The master function is a two-fluid analog of the EOS. Here, we take \( -\Lambda \) to be the total thermodynamic energy density. Given a master function \( \Lambda \), the stress–energy tensor is found to be

\[
T_{\beta}^{\alpha} = \Psi \delta_{\beta}^{\alpha} + n^a \mu_\alpha + p^a \chi_\alpha, \tag{4}
\]

where \( \Psi \) is the generalized pressure

\[
\Psi = \Lambda - n^a \mu_a - p^a \chi_a. \tag{5}
\]
The chemical potential covectors $\mu_a$ and $\chi_a$, respectively, for neutrons and (conglomerate) protons are given by

$$\mu_a = B_n a + A p_a, \quad \chi_a = C n a + A n_a,$$  \hspace{1cm} (6)

where

$$A = - \frac{\partial \Lambda}{\partial x^2}, \quad B = - \frac{\partial \Lambda}{\partial n^2}, \quad C = - \frac{\partial \Lambda}{\partial p^2}. \hspace{1cm} (7)$$

It is noted that the so-called entrainment effect is described by the coefficient $A$. If $A \neq 0$, the mass current of one particle constituent will induce a momentum in the other constituent.

The equations of motion for the two-fluid system consist of two conservation equations,

$$\nabla_a n^a = 0, \quad \nabla_a p^a = 0, \hspace{1cm} (8)$$

and two Euler equations,

$$n^a \nabla_{(a} \mu_{b)} = 0, \quad p^a \nabla_{(a} \chi_{b)} = 0. \hspace{1cm} (9)$$

Solving the linearized superfluid and Einstein field equations, Comer et al. (1999) obtained a system of first-order differential equations to describe the (even parity or polar) nonradial oscillations of superfluid neutron stars. We refer the readers to Comer et al. (1999) for the detailed equations and numerical techniques.

3. EQUATION OF STATE

As discussed in the previous section, the master function $\Lambda(n^2, p^2, x^2)$ is a two-fluid analog of the EOS, replacing the role of $P = P(\rho)$ (with $P$ and $\rho$ being, respectively, the pressure and total energy density) for ordinary one-fluid stellar models. While the universality of the $f$-mode of ordinary fluid neutron stars is determined by many different EOSs, we cannot employ those EOSs directly for our two-fluid neutron stars. The reason is that realistic nuclear-matter EOSs are generally given in the form of pressure versus density. This is clearly insufficient for the two-fluid formalism used here in which the energy density $-\Lambda$ must be given as a function of the two number densities $(n, p)$, and hence standard tabulated EOSs are incomplete for our purposes. We refer the reader to Lin et al. (2008) for a detailed discussion on what is required from next-generation EOSs that allow for proper treatment of the dynamics of multifluid neutron stars.

In this paper, we will first study the oscillation modes of superfluid neutron stars without considering the entrainment. The master function $\Lambda$ is taken to be

$$\Lambda = \lambda_0 \equiv -m(n + p) - \sigma_a n^{h_a} - \sigma_p p^{h_p}, \hspace{1cm} (10)$$

where the two-fluid particles are assumed to have the same mass $m$ (i.e., the baryon mass). The parameters $\sigma_n$, $\sigma_p$, $\beta_n$, and $\beta_p$ are freely chosen. This master function has been used extensively to study superfluid neutron stars (Comer et al. 1999; Andersson & Comer 2001b; Andersson et al. 2002; Prix & Rieutord 2002; Yoshida & Lee 2003). For this EOS, each fluid behaves as a relativistic polytrope and can be regarded as decoupled (though it should be noted that the fluids still couple globally through gravity). In general, we say that the two fluids are decoupled if the master function is separable, in the sense that it can be decomposed into two contributions corresponding, respectively, to each fluid, i.e., $\Lambda(n^2, p^2) = \Lambda_n(n^2) + \Lambda_p(p^2)$.

3.1. Entrainment

After studying the decoupled case, we will also study the entrainment between the fluids. As already mentioned, entrainment describes essentially the effect where the momentum of one species depends on the relative motion between the two species. In the relativistic framework, this effect is specified by the coefficient $A$, which is in turn determined by the dependence of the master function $\Lambda$ on $x^2$. Note that since the background is assumed static, $x^2 - np$ should remain small even when the star oscillates. We will follow Andersson et al. (2002) and approximate the master function by

$$\Lambda = \lambda_0 + \lambda_1 (x^2 - np), \hspace{1cm} (11)$$

where the first term $\lambda_0$ is given by Equation (10). The second term is used to describe the entrainment and $\lambda_1$ is given by

$$\lambda_1(n^2, p^2) = \epsilon \frac{m}{p + \epsilon(n + p)}. \hspace{1cm} (12)$$

The positive constant $\epsilon$ is used to parameterize the strength of the entrainment. As discussed in Andersson et al. (2002), we will consider the physically reasonable range to be $0 \leq \epsilon \leq 0.2$. Note that the second term of the master function $\Lambda$ vanishes in the background (where $x^2 = np$). Hence, the background quantities of the star are determined solely by $\lambda_0$. We give the expressions for some of the thermodynamic coefficients on the background which will be used in later sections:

$$A = - \lambda_1(n^2, p^2), \quad B = \frac{1}{n} \frac{\partial \lambda_0}{\partial n} - \frac{p}{n} A, \quad C = - \frac{1}{p} \frac{\partial \lambda_0}{\partial p} - \frac{n}{p} A. \hspace{1cm} (13)$$

4. NUMERICAL RESULTS

4.1. Decoupled Case

In order to study whether the $f_e$- and $f_p$-modes of superfluid neutron stars establish any kind of universality, we have varied the values of $\beta_n$ and $\beta_p$ for the master function (Equation (10)) to mimic the effects of different EOSs. The effects of $\sigma_n$ and $\sigma_p$ on the mode frequencies are relatively small and hence we will fix them to be $\sigma_n = \sigma_p = 0.5m$. Table 1 summarizes the different EOS models we will study in this section. In Table 1, we have defined two parameters $\beta = (\beta_n + \beta_p)/2$ and $\Delta \beta = \beta_n - \beta_p$. The parameter $\beta$ can be considered as a parameter that controls the bulk fluid motion, while $\Delta \beta$ represents the asymmetry between the two fluids. A larger value of $\Delta \beta$ would imply that the effects of the two-fluid dynamics become more important. On the other hand, the case $\Delta \beta = 0$ represents the one-fluid limit in which case the two-fluid components are indistinguishable and the stellar models reduce to ordinary one-fluid neutron stars.

In Figures 1(a) and (b), we plot the real and imaginary parts of the $f_e$-mode frequencies, respectively, against the compactness $C = M/R$ for the EOS models $\Lambda_i$ (with $i = 1$–5). These models have the same “bulk” EOS parameter $\beta = 1.9$, but different “asymmetry” parameters $\Delta \beta$. In particular, Model $A_1$ (with $\Delta \beta = 0$) corresponds to the one-fluid limit for this series, while Model $A_5$ (with $\Delta \beta = 0.2$) represents the most “asymmetric” one. For comparison, we also plot the universal curves (Equations (1)–(3)) for the $f$-mode of ordinary fluid
Figure 1. (a) Real and (b) imaginary parts of $\omega_M$ of the $f_o$-mode plotted against the compactness $C$ for neutron stars described by the EOS models $A_i$ ($i = 1–5$). The solid line in (a) represents Equation (1) for star models with $M$ and $C$ obtained by EOS A1, while the solid line in (b) represents Equation (2). The dashed lines in both figures represent Equation (3). (A color version of this figure is available in the online journal.)

Figure 2. Similar to Figure 1 but for the EOS models $B_i$ ($i = 1–5$). The solid line in (a) represents Equation (1) for the star models with the masses and compactnesses obtained by EOS B1, while the solid line in (b) represents Equation (2). The dashed lines in both figures represent Equation (3). (A color version of this figure is available in the online journal.)

Table 1
Models for the “Polytropic” EOS Defined in Equation (10)

| Models | $\beta_n$ | $\beta_p$ | $\bar{\beta}$ | $\Delta \beta$ |
|--------|-----------|-----------|----------------|----------------|
| A1     | 1.9       | 1.9       | 1.9            | 0.0            |
| A2     | 1.905     | 1.895     | 1.9            | 0.01           |
| A3     | 1.925     | 1.875     | 1.9            | 0.05           |
| A4     | 1.95      | 1.85      | 1.9            | 0.1            |
| A5     | 2.0       | 1.8       | 1.9            | 0.2            |
| B1     | 2.0       | 2.0       | 2.0            | 0.0            |
| B2     | 2.005     | 1.995     | 2.0            | 0.01           |
| B3     | 2.025     | 1.975     | 2.0            | 0.05           |
| B4     | 2.05      | 1.95      | 2.0            | 0.1            |
| B5     | 2.1       | 1.9       | 2.0            | 0.2            |
| C1     | 2.1       | 2.1       | 2.1            | 0.0            |
| C2     | 2.105     | 2.095     | 2.1            | 0.01           |
| C3     | 2.125     | 2.075     | 2.1            | 0.05           |
| C4     | 2.15      | 2.05      | 2.1            | 0.1            |
| C5     | 2.2       | 2.0       | 2.1            | 0.2            |

Notes.

- $\sigma_n = \sigma_p = 0.5m$ are fixed in all models.
- $\bar{\beta} = (\beta_n + \beta_p)/2$.
- $\Delta \beta = \beta_n - \beta_p$.

A universal behavior. For a given compactness $C$, the real and imaginary parts of the mode frequency do not depend sensitively on the EOS. Furthermore, the mode frequency can still be approximated very well by the universal curves (Equations (1)–(3)) satisfied by the $f$-mode of ordinary fluid neutron stars.

The corresponding results for the EOS models $B_i$ and $C_i$ (with $i = 1–5$) are plotted in Figures 2 and 3. It is seen clearly from Figures 1–3 that the $f_o$-modes of superfluid neutron stars satisfy the same universal scaling as their ordinary fluid counterpart. This result might not be so surprising as the fluid motions of the $f_o$-mode are such that the two fluids move in “lock step” and reach essentially the same maximum displacements at the stellar surface: a feature that is similar to the situation of an ordinary fluid neutron star in which the (single) fluid displacement reaches a maximum at the surface (see Figure 9 of Comer et al. 1999).

Now we consider the superfluid $f_s$-mode, which does not have an ordinary fluid counterpart. For this class of modes, there is no a priori reason to expect that they would establish a similar universal behavior. However, as shown in Figures 4–6, the $f_s$-modes do indeed follow another universal scaling law. In these figures, we plot the real parts of the $f_s$-mode frequencies against the compactness $C$ for the EOS models $A_i$, $B_i$, and $C_i$ (with $i = 2–5$). In particular, the data can be approximated very well by the curve (dashed lines in the figures)

$$\text{Re}(\omega_M) = (lC^3)^{1/2},$$

where $l$ is the spherical harmonics index. Note that the $f_s$-mode does not exist in the Models A1, B1, and C1 since they correspond to the one-fluid limit $\Delta \beta = 0$. 

neutron stars in the figures. In Figure 1(a), the solid line represents Equation (1) for the star models with the masses and compactnesses obtained by EOS A1, while the dashed line represents Equation (3). Similarly, in Figures 1(b), the solid and dashed lines represent Equations (2) and (3), respectively. It is seen that the $f_o$-modes of superfluid neutron stars still exhibit
while the solid line in (b) represents Equation (2). The dashed lines in both figures represent Equation (3). (Andersson & Kokkotas 1998; Tsui & Leung 2005b).

In Figure 7, we focus on Model B5 and plot the real part of the $f_s$-mode frequency against the compactness for different values of the spherical harmonics index $l = 2$–4. It is seen that the data can be approximated by Equation (14) very well in general. The discrepancy between the numerical data and Equation (14) becomes significant only for the $l = 4$ modes in the region of high compactness ($C \gtrsim 0.2$). In Section 5, we will derive this scaling relation in the framework of Newtonian gravity and under the assumptions that the star is composed of two homogeneous and decoupled fluids. It should be noted that Equation (14) does not contain any free parameters, while the scaling Equations (1)–(3) for the $f$-modes of ordinary fluid stars depend on parameters that are determined by curve fitting (Andersson & Kokkotas 1998; Tsui & Leung 2005b).

It is also instructive to compare the scaling law (Equation (1)) for the $f$-mode of ordinary fluid stars with Equation (14) since they both contain the factor $C^{3/2}$. As discussed by Andersson & Kokkotas (1998), since the characteristic timescale is related to the mean density of the star $\bar{\rho}$, it is expected that the $f$-mode frequency scales with $\bar{\rho}^{1/2} \sim (M/R^3)^{1/2}$. Fitting to the numerical data for the $f$-mode frequencies of ordinary fluid stars, Andersson & Kokkotas (1998) obtained the scaling law (Equation (1)). While Equations (1) and (14) both contain the scale factor $C^{3/2}$, we note two important differences: (1) our scaling law (Equation (14)) for the $f_s$-modes does not contain any free parameter; (2) it also does not contain a term that scales only with $M$ as in Equation (1). As we will show in Section 5, the scaling law (Equation (14)) is not a trivial generalization of Equation (1) to the case of two-fluid stars. It is also not the well-known Kelvin mode for an incompressible stellar model (Chandrasekhar 1981), whose oscillation frequency is given by

$$\omega M = \left[ \frac{2(l - 1)}{(2l + 1)} C^3 \right]^{1/2}. \quad (15)$$

However, Equations (14) and (15) agree with each other in the limit $l \gg 1$.

Now we turn to the imaginary part of the $f_s$-mode. We see that in general it does not follow any universal scaling law. For a given compactness, its value depends very sensitively on the EOS models. In Figure 8, we plot $\text{Im}(\omega M)$ against $C$ for models $B_i (i = 2$–5$)$. These models have the same “bulk motion” parameter $\beta = 2$, but different “asymmetry” parameter $\Delta \beta$. It is seen that $\text{Im}(\omega M)$ decreases significantly with decreasing $\Delta \beta$. In fact, for the particular form of the master function Equation (10), it is known that the superfluid modes become non-radiating, and hence $\text{Im}(\omega M) \to 0$, in the limit $\Delta \beta \to 0$ (Andersson et al. 2002).

While $\text{Im}(\omega M)$ does not follow any universal law, it is interesting to note that $\text{Im}(\omega M)$ can be rescaled in such a way that the data from models with the same value of $\beta$ lie on the same curve. In Figure 9, we plot $\text{Im}(\omega M)/\Delta \beta^2$ versus $C$ for the models $A_i$, $B_i$, and $C_i (i = 2$–5$)$. It is seen clearly that, for a given $\beta$ (e.g., the models $A_i$) and compactness $C$, the imaginary parts of the mode frequencies (normalized by the asymmetry parameter $\Delta \beta^2$) are rather insensitive to $\Delta \beta$.

The above results suggest that, for the polytropic EOS model (Equation (10)), the imaginary parts of the $f_s$-modes are given...
given by Equation (10), we believe that the universal behavior of \( \sigma \) depends essentially only on the compactness \( C \).

While we have only focused on a class of EOSs with \( \Delta \beta \) approximately equal to \( 0 \) when \( \Delta \beta \leq 0 \), we thus expect that the leading dependence of \( \omega \) on \( \Delta \beta \) must be quadratic. In fact, it can also be shown that the leading dependence is also quadratic even if \( \sigma_n \neq \sigma_p \) (Wong 2008). We will present in the Appendix a qualitative argument to explain why the damping time of the \( s \)-mode oscillations is then likely to provide clues to the coupling between neutron and proton fluids in neutron stars, which is not yet thoroughly understood.

4.2. Effects of Entrainment

In the above we have demonstrated that the real part of the \( f \)-mode frequency as a function of \( \epsilon \) (normalized by its value when \( \epsilon = 0 \)) for Model B2. In the figure, the circle and square data points correspond to the compactness \( C = 0.1 \) and \( C = 0.2 \), respectively. The corresponding results for Model B5 are shown in Figure 11. As found by Andersson et al. (2002), we see that the mode frequency increases with the entrainment parameter \( \epsilon \). It is also seen that the rate of increase of the mode frequency depends on the compactness of the stellar models.

In Figure 10, we plot the squared \( f \)-mode frequency \( \omega \) (normalized by its value when \( \epsilon = 0 \)) for Model B2. In the figure, the circle and square data points correspond to the compactness \( C = 0.1 \) and \( C = 0.2 \), respectively. The corresponding results for Model B5 are shown in Figure 11. As found by Andersson et al. (2002), we see that the mode frequency increases with the entrainment parameter \( \epsilon \). It is also seen that the rate of increase of the mode frequency depends on the compactness of the stellar models.
for the real parts of the superfluid Equation (17) occurs only for higher values of numerical data very well. Significant deviation of the data from the figures represent Equation (17), respectively, for Models B2 and B5. Note that the value $\omega(0)$ is the same for both models. Nevertheless, we will show in Section 6.2 that the shift in the frequency for a given value of $\epsilon$, namely, the ratio $\omega(\epsilon)^2/\omega(0)^2$, can be obtained analytically from the following equation (see Equation (73)) which involves only the parameters of the background EOS model and the compactness:

$$\frac{\omega(\epsilon)^2}{\omega(0)^2} \approx 1 + \epsilon \frac{1 + \frac{(\partial \rho)_{\rho_p}}{(\partial \rho)_{\rho_p}} (g(0.2) + \pi G)}{[1 + g(0.2)]},$$

(17)

where the function $g(x)$ is defined in Equation (72). The excellent agreement between the analytic result and numerical data can be seen in Figures 10 and 11. The solid and dashed lines in the figures represent Equation (17), respectively, for $C = 0.1$ and $C = 0.2$. It is seen clearly that, for the physically reasonable values of $\epsilon$ considered by us, Equation (17) agrees with the numerical data very well. Significant deviation of the data from Equation (17) occurs only for higher values of $\epsilon$.

5. SUPERFLUID MODE FOR A HOMOGENEOUS TWO-FLUID STAR

In this section, we will derive the scaling law (Equation (14)) for the real parts of the superfluid $f_s$-modes using the Newtonian two-fluid hydrodynamics equations. Our analysis is based on a model in which the densities of the two fluids are constant throughout the interior of the star. The general set of two-fluid equations in Newtonian gravity can be found in Andersson & Comer (2001a):

$$0 = \partial_t \rho_n + \partial_i (\rho_n v_i^p),$$

(18)

$$0 = \partial_t \rho_p + \partial_i (\rho_p v_i^p),$$

(19)

$$0 = \partial_t \left[ v_i^p + \frac{2\alpha}{\rho_n} (v_i - v_i^p) \right] + v_i^p \partial_j \left[ v_i^p + \frac{2\alpha}{\rho_n} (v_i - v_i^p) \right]$$

$$+ \delta^{ij} \partial_j (\Phi + \mu_n) + \frac{2\alpha}{\rho_n} \delta^{ij} \delta_{kl} v_k^p \partial_j (v_l - v_l^p),$$

(20)

where $\rho_n$, $v_i^p$, and $\mu_n$ (x = n, p) are the mass density, velocity, and chemical potential per unit mass of species x, respectively. $\Phi$ is the gravitational potential. $\alpha$ is a function to describe the entrainment and is defined by

$$dU(\rho_n, \rho_p, \Delta^2) = \mu_n d\rho_n + \mu_p d\rho_p + \alpha d\Delta^2,$$

(23)

where $U$ is the internal energy density and $\Delta^2 = |v_n - v_p|^2$. Note that the relativistic coefficient $A$ defined in Equation (7) is proportional to $\alpha$ in the Newtonian limit (Andersson & Comer 2001a).

For a nonrotating static background ($v_i^p = 0$), the equilibrium equations are given by

$$\partial_t (\Phi + \mu_n) = 0,$$

(24)

$$\partial_i \partial_j \Phi = 4\pi G(\rho_n + \rho_p).$$

(25)

In particular, we will assume that the background star is homogeneous in which $\rho_n$ and $\rho_p$ are constants. The solutions for the background equations are then given by

$$\Phi = \begin{cases} 
\frac{4\pi G}{6} (\rho_n + \rho_p) (3R^2 - r^2), & r < R, \\
\frac{4\pi G}{3r} (\rho_n + \rho_p) R^3, & r \geq R,
\end{cases}$$

(26)

$$\mu_n = \begin{cases} 
\frac{4\pi G}{6} (\rho_n + \rho_p) (R^2 - r^2), & r < R, \\
0, & r \geq R.
\end{cases}$$

(27)

The linearized versions of Equations (18)–(22) have been studied previously (e.g., Lindblom & Mendell 1994; Andersson & Comer 2001a; Prix & Rieutord 2002). In particular, the...
linearized hydrodynamics equations for a nonrotating static background are

$$\partial_t^2 \xi_n^i + \frac{2\alpha}{\rho_n} (\xi_n^i - \xi_n^0) = - \partial_i (\delta \Phi + \delta \mu_n),$$  
(28)

$$\partial_t^2 \xi_p^i + \frac{2\alpha}{\rho_p} (\xi_p^i - \xi_p^0) = - \partial_i (\delta \Phi + \delta \mu_p),$$  
(29)

$$\delta \rho_n + \partial_i (\rho_n \xi_n^i) = 0,$$  
(30)

$$\delta \rho_p + \partial_i (\rho_p \xi_p^i) = 0,$$  
(31)

$$\partial_i \partial_j \delta \Phi = 4\pi G (\delta \rho_n + \delta \rho_p),$$  
(32)

where \( \xi_n^i \) is the Lagrangian displacements for species \( x \) and \( \delta \) is used to denote Eulerian perturbations. For simplicity, we will set \( \alpha = 0 \) in the following analysis. The effect of entrainment will be studied in Section 6.

For our homogeneous background model, together with the assumption that the two fluids are decoupled, it can be shown from the linearized hydrodynamics equations that the superfluid modes are governed by the following equations (Andersson & Comer 2001a):

$$\frac{\partial^2 \xi}{\partial t^2} + \nabla \delta \beta = 0,$$  
(33)

$$\nabla \cdot \xi = 0,$$  
(34)

where \( \partial \xi / \partial t \equiv \delta \nu_n - \delta v_n \) and \( \delta \beta \equiv \delta \mu_n - \delta \mu_n \). A nonzero relative velocity \( \xi \) is a characteristic of the superfluid modes. For our simplified stellar model, we note that the equations for the superfluid modes are completely decoupled from those for the ordinary fluid modes (Andersson & Comer 2001a; Prix & Rieutord 2002). Furthermore, the superfluid modes are completely decoupled from the perturbation of the gravitational potential \( \delta \Phi \). The counter motion between the two fluids implies that the total density variation \( \delta \rho = \delta \rho_n + \delta \rho_p \), and hence \( \delta \Phi \), nearly vanish. Combining Equations (33) and (34), we obtain

$$\nabla^2 \delta \beta = 0.$$  
(35)

In solving Equation (35), boundary conditions should be imposed at the center and the stellar surface: (1) the solution should be regular at \( r = 0 \); (2) the Lagrangian variation of \( \beta \equiv \mu_p - \mu_n \) should vanish at \( r = R \).

The background Equation (24) implies that \( \mu_n' = \mu_p' \), where a “prime” denotes radial derivative. The boundary condition at the surface can thus be written as

$$\delta \beta + (\xi_n^i - \xi_n^0) \mu_n' = 0,$$  
(36)

where \( \xi_n^i \) is the radial component of the Lagrangian displacement of the species \( x \). Equations (33) and (36) imply that \( \delta \beta \) satisfies the following equation at the surface:

$$\frac{\partial^2 \delta \beta}{\partial t^2} - \mu_n' \delta \beta' = 0.$$  
(37)

Now let us decompose \( \delta \beta \) in the form \( \delta \beta = \delta \beta_{lm}(r) Y_{lm}(\theta, \phi)e^{i\omega t} \). Equation (35) and the regularity condition at \( r = 0 \) imply that

$$\delta \beta_{lm}(r) = Ar^l,$$  
(38)

where \( A \) is some constant. Using the radial background profile for \( \mu_n \) as given in Equation (27), the boundary condition (Equation (37)) implies that the oscillation frequency \( \omega \) of the superfluid modes is given by

$$\omega^2 = \frac{l GM}{R^3}.$$  
(39)

This is the scaling law (Equation (14)) presented in Section 4.1, except that we have restored the gravitational constant \( G \). As we have seen in Section 4.1, Equation (14) can describe the data very well as long as \( C \sim 0.2 \). Relativistic effects become important for higher compactness. As can be seen from Figures 5 and 6, the deviation of the numerical data from Equation (39) becomes noticeable for \( C \gtrsim 0.2 \). To further illustrate the relativistic corrections, we plot in Figure 12 the relative difference between the numerical data and the analytic result (Equation (39)) for the \( f \)-mode frequency of Model C5. We see that the relative difference increases linearly with the compactness of the star. The relativistic correction rises up to about 8% for the maximum mass configuration.

It is worth pointing out that Equation (39) in fact also the result for the \( f \)-mode of an incompressible single-fluid star in the Cowling approximation (i.e., assuming \( \delta \Phi = 0 \)). For such a single-fluid model, it can be shown that the perturbed one-fluid equations governing the \( f \)-mode reduce to the same Equation (35) with \( \delta \beta \) being replaced by the perturbed enthalpy \( \delta h = \delta P / \rho \):

$$\nabla^2 \delta h = 0.$$  
(40)

The equivalence between the two governing Equations (35) and (40) comes from the fact that the counter motion of a superfluid mode leads to a nearly vanishing \( \delta \Phi \) as discussed above. However, it should be noted that Equation (39) does not describe the numerical data of ordinary \( f \)-modes as good as it does for the superfluid \( f \)-modes. This suggests that incompressibility and the Cowling approximation are not good approximations for the ordinary \( f \)-modes (see Andersson et al. 2008 for the analysis of the ordinary \( f \)-mode with the compressibility taken into account).

Finally, if we also assume that the entrainment term \( \alpha \) is a constant, Equation (39) can be easily generalized to include the
effect of entrainment and becomes
\[ \omega^2 = \frac{GM}{R^3} \left( 1 + \epsilon \frac{p_n}{\rho_p} \right), \tag{41} \]
where the two parameters \( \alpha \) and \( \epsilon \) are related by \( \alpha = \epsilon \rho_p / \rho_p [2(\epsilon \rho_p + (1 + \epsilon) \rho_p)] \). At first sight, one might expect that this generalized result could be used directly to compare with the numerical data. However, it should be noted that Equation (41) involves the mass densities of the two-fluid components. This makes the comparison impractical since the numerical data are obtained from stellar models where \( \rho_n \) and \( \rho_p \) are varying throughout the stars. In the next section, we will use a variational principle to study the effect of entrainment perturbatively.

### 6. Perturbative Analysis of the Entrainment

#### 6.1. General Integral Formula

In this section, we will study the effect of entrainment on the superfluid mode frequency for a two-fluid star without rotation. Our approach is to treat the entrainment as a perturbation and employ a variational principle to calculate the first-order shift in the mode frequency. We have generalized the variational principle for polar oscillation modes of ordinary fluid relativistic stars developed by Detweiler & Ipser (1973) to the case of two-fluid stars. The derivation of the general relativistic variational principle is somewhat lengthy and we will not present the details in this paper (see Wong 2008). However, in order to illustrate the basic idea, we will first derive the corresponding variational principle in the Newtonian framework and then simply quote the relativistic result.

The relevant equations for the Newtonian analysis are given by Equations (28)–(32). We will treat this system of equations as an eigenvalue system with the operator containing the entrainment term \( \alpha \) as the perturbing Hamiltonian. Consider a normal-mode solution with the time dependence of the form \( \delta f(r, t) \equiv \delta f(r)e^{i\omega t} \), where \( \delta f \) is a perturbed quantity. Equations (28) and (29) then become
\[
\omega^2 \left[ \frac{\xi_n^i}{\rho_n} \left( \frac{\xi_p^i}{\rho_p} - \xi_n^i \right) \right] = \partial_i (\delta \Phi + \delta \tilde{\mu}_n), \tag{42}
\]
\[
\omega^2 \left[ \frac{\xi_n^i}{\rho_p} \left( \frac{\xi_n^i}{\rho_n} - \xi_p^i \right) \right] = \partial_i (\delta \Phi + \delta \tilde{\mu}_p). \tag{43}
\]

We want to formally express the above equations (to first-order of smallness in \( \alpha \)) in the form
\[
(\hat{\mathcal{H}}_0 + \hat{\mathcal{H}}_1) Y = \omega^2 Y, \tag{44}
\]
where \( \hat{\mathcal{H}}_i \) is an operator \( i \)th order in \( \alpha \) and \( Y \equiv (\xi_n, \xi_p) \) is an abstract eigenvector with the two Lagrangian displacement vectors as its components. The system is solved with the "constraint" Equations (30)–(32) which do not contain time derivatives.

First, we define the "unperturbed Hamiltonian" \( \hat{\mathcal{H}}_0 \) by
\[
\hat{\mathcal{H}}_0 \left( \begin{array}{c} \xi_n \\ \xi_p \end{array} \right) = \left( \begin{array}{c} \partial_i (\delta \Phi + \delta \tilde{\mu}_n) \\ \partial_i (\delta \Phi + \delta \tilde{\mu}_p) \end{array} \right), \tag{45}
\]

In order to apply standard perturbation theory as in quantum mechanics, we first need to define an inner product such that \( \hat{\mathcal{H}}_0 \) is symmetric. To this end, we define the inner product
\[
(\eta_n, \eta_p) \cdot (\xi_n, \xi_p) = \int dV (\rho_n \eta_n \cdot \xi_n + \rho_p \eta_p \cdot \xi_p). \tag{46}
\]

We shall show that, under this inner product, \( \hat{\mathcal{H}}_0 \) is symmetric. In other words, \( (\eta_n, \eta_p) \cdot \hat{\mathcal{H}}_0 (\xi_n, \xi_p) \) is symmetric in \( (\eta_n, \eta_p) \) and \( (\xi_n, \xi_p) \):
\[
(\eta_n, \eta_p) \cdot \hat{\mathcal{H}}_0 (\xi_n, \xi_p) = \int dV \left[ \rho_n \eta_n \cdot \partial_i (\delta \Phi + \delta \tilde{\mu}_n) + \rho_p \eta_p \cdot \partial_i (\delta \Phi + \delta \tilde{\mu}_p) \right]
= \int dS \delta f \left[ \rho_n \eta_n \cdot (\delta \Phi + \delta \tilde{\mu}_n) + \rho_p \eta_p \cdot (\delta \Phi + \delta \tilde{\mu}_p) \right]
+ \int dV (\delta \rho_n (\delta \Phi + \delta \tilde{\mu}_n) + \delta \rho_p (\delta \Phi + \delta \tilde{\mu}_p)), \tag{47}
\]
where the first term is a surface integral which is carried out on the two-sphere \( r = R \) and \( \hat{f} \) is the outward unit normal to the sphere. To arrive at the second row, we have performed an integration by parts and used Equations (30) and (31). \( \delta f \) and \( \delta \tilde{f} \) refer to the Eulerian variation of the quantity \( f \) associated with the displacement \( \xi_n \) and \( \eta_p \) respectively.

Note that the surface integral in Equation (47) vanishes since the densities tend to zero at the surface for our stellar models. The remaining volume integral can be shown to be symmetric in \( (\eta_n, \eta_p) \) and \( (\xi_n, \xi_p) \). To show this, we first note that
\[
\tilde{\mu}_n = \frac{\partial U(\rho_n, \rho_p, \Delta^2)}{\partial \rho_n}, \tag{48}
\]
and thus
\[
\delta \tilde{\mu}_n = \frac{\partial^2 U}{\partial \rho_n^2} \delta \rho_n + \frac{\partial U}{\partial \rho_n} \frac{\partial \rho_n}{\partial \rho_p} \delta \rho_p. \tag{49}
\]

It should be mentioned that a term associated with \( d \Delta^2 \) vanishes because in the background \( \Delta = 0 \). This implies that
\[
\delta \rho_n \delta \tilde{\mu}_n + \delta \rho_p \delta \tilde{\mu}_p = \delta \rho_n \delta \rho_n + \delta \rho_p \delta \rho_p \frac{\partial^2 U}{\partial \rho_n^2} + \frac{\partial U}{\partial \rho_n} \frac{\partial \rho_n}{\partial \rho_p} \delta \rho_p \frac{\partial U}{\partial \rho_n} \frac{\partial \rho_n}{\partial \rho_p} \delta \rho_p, \tag{50}
\]
which is symmetric in \( (\delta \rho_n, \delta \rho_p) \) and \( (\delta \rho_p, \delta \rho_n) \). It can also be shown easily that \( \int dV (\delta \rho_n + \delta \rho_p) \delta \Phi \) is symmetric. Thus, the volume integral in Equation (47), and hence \( \hat{\mathcal{H}}_0 \), is symmetric.

We can now apply standard perturbation theory as in quantum mechanics with the perturbing potential \( \hat{\mathcal{H}}_1 \) defined by
\[
\hat{\mathcal{H}}_1 = 2\alpha \omega^2 \left( \frac{1}{\rho_n} - \frac{1}{\rho_p} \right). \tag{51}
\]
where $\omega^2_{(2)}$ denotes the squared frequency of the mode in the absence of entrainment (i.e., $\alpha = 0$). The first-order shift in the squared frequency is given by

$$
\omega^2_{(1)} = \frac{\left( \xi_\alpha, \xi_\beta \right) \cdot \hat{H}_1 \left( \xi_n, \xi_p \right) \left( \xi_n, \xi_p \right)}{\left[ \left( \xi_n, \xi_p \right) \cdot \left( \xi_n, \xi_p \right) \right]^2} = \frac{2\omega^2_0 \int dV \left( \alpha \xi_n - \xi_p \right)^2}{\int dV \left( \rho_n \xi_n, \xi_p + \rho_p \xi_p \right)}.
$$

(52)

where $\xi_n$ and $\xi_p$ are the Lagrangian displacement vectors associated with the “zeroth-order” mode solution with squared frequency $\omega^2_0$.

For a given static background of two-fluid stellar model, the effect of a small entrainment on the oscillation mode frequency is to change the squared frequency $\omega^2_0$ to $\omega^2 = \omega^2_0 + \omega^2_{(1)}$. It can also be seen qualitatively from Equation (52) that the ordinary fluid modes in general do not depend on entrainment (as has been discussed in Andersson & Comer 2001a; Andersson et al. 2002; Prix & Rieutord 2002). The first-order shift depends on the relative displacement $|\xi_n - \xi_p|$, which tends to zero for ordinary fluid modes.

Equation (52) readily be used to compute the change in the mode frequency once the entrainment function $\alpha$ and the zeroth-order solution $|\xi_n, \xi_p|$ are given. However, we cannot employ the equation directly since it is derived in the Newtonian framework, while our zeroth-order solution $|\xi_n, \xi_p|$ is computed using a relativistic numerical code. Nevertheless, with the use of Equation (52) as an illustration of the basic idea involved, we have derived the corresponding result based on a two-fluid formalism extension of the work of Detweiler & Ipser (1973). The derivation of the relativistic case is somewhat tedious and lengthy (Wong 2008). Here, we will only present the final result of the first-order shift in $\omega^2$:

$$
\omega^2_{(1)} = \frac{\omega^2_0 \int dV \xi e^{-\nu/2} (Anp |\xi_n - \xi_p|^2)}{\int dV e^{-\nu/2} \left[ \frac{\partial n}{\partial \xi} \text{sgn}(\xi_n, \xi_p) + \frac{\partial p}{\partial \xi} \left( \xi_n, \xi_p \right) \right]}. \tag{53}
$$

where $dV = e^{\kappa/3} r^2 \sin \theta d\rho d\phi \delta \phi$ is the proper volume element for the spatial three-geometry of the background spacetime metric

$$
d^2s = -e^{\nu(r)} dr^2 + e^{\lambda(r)} d\theta^2 + r^2 d\theta^2 + \sin^2 \theta d\phi^2. \tag{54}
$$

The three-vectors $\xi_n$ and $\xi_p$ are still defined as the Lagrangian displacement vectors associated with the zeroth-order solution as before. But the scalar product between any two three-vectors $U$ and $V$ is now given by

$$
U \cdot V = e^\lambda U^r V^r + r^2 U^\theta V^\theta + r^2 \sin^2 \theta U^\phi V^\phi,
$$

where $U^r$ and $V^r$ are the coordinate components of the vectors. The entrainment is now described by the function $\Lambda$ as discussed in Section 2. It can also be shown easily that Equation (53) reduces to Equation (52) in the Newtonian limit.

### 6.2. Decoupled Polytropes

Equation (53) is a general result and could be used to obtain the frequency shift due to a small entrainment once the zeroth-order mode solution (i.e., without the entrainment) is given.

| Models | C   | $\eta$       |
|--------|-----|--------------|
| B2     | 0.10| $1.89 \times 10^{-5}$ |
|        | 0.15| $2.24 \times 10^{-5}$ |
|        | 0.20| $2.83 \times 10^{-5}$ |
| B5     | 0.10| $6.95 \times 10^{-3}$ |
|        | 0.15| $8.39 \times 10^{-3}$ |
|        | 0.20| $1.11 \times 10^{-2}$ |

Table 2

Values of the Ratio $\eta$ as Defined in Equation (61) for the $f_s$-Modes of Models B2 and B5 (see Table 1) at Various Values of Compactness C

In deriving Equation (53), we have not made any assumption about the master function $\Lambda$. In this subsection, we focus on the special case where the master function $\Lambda$ is given by Equation (11). We will show that the integral formula (Equation (53)) can be approximated very well by an algebraic relation which involves only the parameters of the background EOS (i.e., the part $\lambda_0$ in Equation (11)), the entrainment parameter $\epsilon$, and the compactness of the star $C$.

To obtain an approximation to Equation (53), we first note that the chemical potentials of the neutrons and the (conglomerate) protons evaluated on the static background are, respectively (Comer et al. 1999)

$$
\mu = Bn + Ap = \mu_\infty e^{-\nu/2}, \quad \chi = Cn + A = \chi_\infty e^{-\nu/2}, \tag{56}
$$

where $\mu_\infty$ and $\chi_\infty$ are constants. The condition of chemical equilibrium, with Equation (13) for the thermodynamic coefficients, implies that

$$
\frac{\partial \lambda_0}{\partial n} \frac{\partial \lambda_0}{\partial p} = \mu_\infty e^{-\nu/2}. \tag{57}
$$

Using this relation, Equation (53) can now be written as

$$
\omega^2_{(1)} = \frac{1}{\mu_\infty} \int dV \xi e^{-\nu/2} (Anp |D|^2) \tag{58}
$$

where the three-vectors $U$ and $D$ are defined by

$$
U \equiv \frac{n \xi_n + p \xi_p}{n + p}, \quad D \equiv \xi_n - \xi_p. \tag{59}
$$

We note that for the superfluid modes where the two fluids are dominated by counter-moving motion (i.e., when $|D|$ is large), Equation (58) can be approximated by

$$
\omega^2_{(1)} \approx \frac{1}{\mu_\infty} \int dV \xi e^{-\nu/2} (\frac{np}{\pi + p} |D|^2). \tag{60}
$$

To show that this is in general a good approximation, we define the ratio

$$
\eta \equiv \frac{\int dV \xi e^{-\nu/2} (n + p) |U|^2}{\int dV e^{-\nu/2} (\frac{np}{\pi + p} |D|^2)}. \tag{61}
$$

and present its value for the $f_s$-modes of a few typical stellar models in Table 2. It is seen that $\eta$ is in general much smaller than unity. Hence, it is a good approximation to neglect the first integral in the denominator of Equation (58).
In the relativistic perturbative formalism (Comer et al. 1999), the Lagrangian displacements are decomposed into
\[ \xi^r_s = e^{-\lambda/2} r^{l-1} W_s(r) P_l(\cos \theta), \]
\[ \xi^\theta_s = -r^{l-2} V_s(r) \frac{\partial}{\partial \theta} P_l(\cos \theta), \] (62)
where \( P_l(x) \) is the Lengendre polynomial. Hence, Equation (60) can be put into the following form
\[ \frac{\omega^2_{(l)}}{\omega^2_{(0)}} \approx -1 \int drr^2 e^{\frac{x}{2}} \frac{\sigma}{m} \tilde{D}(r), \] (63)
where
\[ \tilde{D}(r) \equiv e^{-\lambda(r)} \left[ W_n(r) - W_p(r) \right]^2 + l(l + 1) \left[ V_n(r) - V_p(r) \right]^2. \] (64)

We also observe that the two functions (which are the two integrands in Equation (63))
\[ f_1(r) \equiv r^{2l} e^{\frac{x}{2}} \frac{\sigma}{m} \tilde{D}(r), \]
\[ f_2(r) \equiv r^{2l} e^{\frac{x}{2}} \frac{np}{n + p} \tilde{D}(r), \] (65)
have very similar shapes and both reach a maximum value at the position \( r \approx 0.8R \) (see Figure 13). We thus further approximate Equation (63) by
\[ \frac{\omega^2_{(l)}}{\omega^2_{(0)}} \approx -1 \int drr^2 e^{\frac{x}{2}} \frac{\sigma}{m} \tilde{D}(r) \approx \left[ \frac{1}{\mu \epsilon_0 e^{-\frac{x}{2}} (n + p) N} \right]_{r=0.8R}. \] (66)

For the decoupled polytropic model \( \lambda_0 \) (see Equation (10)) we use for the background, we have
\[ \mu = \mu_\infty e^{-\frac{x}{2}} = m + \sigma_0 p_0 n^{\beta_0 - 1}, \] (67)
\[ \chi = \mu_\infty e^{-\frac{x}{2}} = m + \sigma_0 p_0 p^{\beta_0 - 1}. \] (68)

Since \( n(R) = p(R) = 0 \) and \( \epsilon = (1 - 2C) \), we deduce that
\[ \frac{\mu_\infty}{m} = \sqrt{1 - 2C}. \] (69)

Furthermore, it can be shown that
\[ n(r) = \left[ \frac{1}{\sigma_0 p_0} \left( \frac{\mu_\infty}{m} e^{-\frac{x}{2}} - 1 \right) \right]^{1/(\beta_0 - 1)}, \] (70)
\[ p(r) = \left[ \frac{1}{\sigma_0 p_0} \left( \frac{\mu_\infty}{m} e^{-\frac{x}{2}} - 1 \right) \right]^{1/(\beta_0 - 1)}, \] (71)
where \( \sigma_0 = \sigma/m \), and
\[ \frac{\mu_\infty}{m} e^{-\frac{x}{2}} \approx 1 + \frac{C}{(1 - 2C)^2} x + \frac{C (2 - C)}{2(1 - 2C)^2} x^2 \equiv 1 + g(x), \] (72)

where \( x \equiv 1 - r/R \).

Using the above relations, we obtain the final expression
\[ \frac{\omega^2_{(l)}}{\omega^2_{(0)}} \approx \frac{\epsilon \left[ \frac{1}{\mu_\infty e^{-\frac{x}{2}} (n + p) N} g(0.2) \right]^{\frac{1}{2}}}{1 + g(0.2)}. \] (73)

It should be noted that this final formula only depends on the parameters of the master function and the compactness of the model. In particular, in contrast to the “exact” integral formula (Equation (53)), it does not depend on the zeroth-order mode parameters of the master function and the compactness of the model. In particular, in contrast to the “exact” integral formula (Equation (53)), it does not depend on the zeroth-order mode functions \( \xi^{(0)}_n \), \( \xi^{(0)}_p \), which need to be determined numerically.

In Section 4.2, we have seen how well Equation (73) agrees to the numerical data. Here, in Figure 14 we demonstrate the good agreement again by plotting the numerical data, the “exact” integral formula (Equation (53)), and the approximated formula (Equation (73)) for the \( f_s \)-modes of EOS Model B2 at compactness \( C = 0.15 \). In the figure, we plot the (normalized) squared \( f_s \)-mode frequency \( \epsilon \) against the entrainment parameter \( \epsilon \) (see Equation (12)). The circle data points are obtained directly from the relativistic numerical code. The solid and dashed lines are obtained, respectively, from Equations (53) and (73). It is seen clearly that there is excellent agreement among the three results for the physical range of \( \epsilon \) we consider.

7. CONCLUSIONS

In this work, we have studied whether the \( f_s \) and \( f_o \) modes of superfluid neutron stars exhibit any kind of universal scaling laws as seen in the \( f \)-mode of ordinary fluid neutron star models. We first focus on the simplified case of two decoupled fluids,
each with a polytropic EOS. We vary the polytropic indices to mimic the effects of different EOSs. Our numerical results show that the $f_s$-mode, where the two fluids move in “lock step,” obeys the same universal scaling laws as the $f$-mode of ordinary fluid stars.

On the other hand, we find that the oscillation frequency of the $f_s$-mode, which corresponds essentially to counter motion between the two fluids, obeys a different scaling law (Equation (14)). We have also derived the scaling law analytically based on a homogeneous two-fluid stellar model in Newtonian gravity. However, the damping time of the $f_s$-mode in general does not exhibit any kind of universality. While we have only used a generalized polytropic EOS in our study, we believe that our conclusion holds in general for superfluid neutron star models in which the two fluids exist throughout the whole star and are decoupled in the sense that the master function $\Lambda$ (i.e., the EOS) can be decomposed into two contributions corresponding to each fluid, i.e., $\Lambda(n^2, p^2) = \Lambda_n(n^2) + \Lambda_p(p^2)$.

The inclusion of a coupling term in the master function will in general break the universal behavior. To illustrate the effect of coupling, we have studied the entrainment between the two fluids using a parameterized entrainment model. We show numerically how the $f_s$-mode frequency increases with the strength of the entrainment. Furthermore, based on a relativistic variational principle, we have carried out a perturbative analysis and have derived an expression for the first-order shift of the frequency due to the entrainment. If the superfluid $f_s$-modes could be detected by future gravitational-wave detectors, then the deviation of the observed mode frequencies from the universal scaling curve for the decoupled fluids could then be a useful probe for the coupling effects between the neutron superfluid and normal fluids inside neutron stars. In summary, our main results (Equations (14) and (17)) can be used to obtain a good approximation to the oscillation frequency of the $f_s$-modes for the generalized polytropic EOS and entrainment models that have been used extensively to study superfluid neutron stars (Comer et al. 1999; Andersson & Comer 2001b; Andersson et al. 2002; Prix & Rieutord 2002; Yoshida & Lee 2003).

While we have not yet detected the gravitational waves emitted by neutron stars, it is worthy to mention that the first gravitational-wave search sensitive to the $f$-modes has recently been carried out by the LIGO detectors and interesting upper limits on the wave strain (within the predicted range of some theoretical models) have also been placed (Abbott et al. 2008). In view of the fact that the advanced LIGO detectors will have more than a factor of 10 improvement on the sensitivity of the wave strain, gravitational-wave astroseismology may soon become a reality.

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APPENDIX

Gravitational-Wave Emission of Superfluid Modes

To gain some insight into the reason why the imaginary part of the $f_s$-mode fails to follow a universal scaling which depends solely on the mass and radius of the star, we provide here a qualitative understanding based on the multipole formulae of the gravitational-wave luminosity for a single oscillation mode with frequency $\omega$ (Thorne 1980):

$$\frac{dE}{dt} = \sum_{l=2}^{\infty} N_l \omega_l^{2l+2} (|\delta D_{lm}|^2 + |\delta J_{lm}|^2),$$

(A1)

where $N_l$ is some constant depending on the spherical harmonic index $l$. $\delta D_{lm}$ and $\delta J_{lm}$ are, respectively, the mass and current multipoles associated with the oscillation modes. For a nonrotating two-fluid star with weak internal gravity (i.e., a Newtonian source), the multipoles are given by (Andersson et al. 2008)

$$\delta D_{lm} = \int (\delta \rho_n + \delta \rho_p) r^l Y_{lm}^* dV,$$

$$\delta J_{lm} = \frac{2}{c} \sqrt{\frac{l}{l+1}} \int r^l (\rho_n \delta v_n + \rho_p \delta v_p) \cdot \hat{r} \times \nabla Y_{lm} dV,$$

(A2)

where $Y_{lm} = [(l+1)^{-1/2} \hat{r} \times \nabla Y_{lm}$ is the magnetic-type vector spherical harmonics.

The ordinary $f_n$-mode of a two-fluid star is characterized by the fact that the two fluids are comoving, which implies a large total density variation $\delta \rho = \delta \rho_n + \delta \rho_p$. Hence, similar to the $f$-mode of a single-fluid star, it is expected that the gravitational-wave emission of a $f_n$-mode is dominated by the mass multipole $\delta D_{lm}$. The damping time of the $l = 2$ $f$-mode can be estimated by $\tau \sim E_{\text{m}}/\sigma n |\delta D_{22}|^2$. The mode energy $E_{\text{m}}$ can be calculated by giving the eigenfunction of the mode. For a homogeneous single-fluid model, it can be shown that the damping timescale depends only on the mass and radius of the star (Detweiler 1975). In fact, the proposal for the leading scaling term $C^4$ in Equation (2) was based on this rough estimation. Since the $f_n$-mode of a two-fluid star is essentially the same as the standard $f$-mode, it is thus not surprising that the $f_n$-mode also follows the same universal scaling law as we have seen in the numerical data.

Now let us turn to the superfluid $f_s$-mode. This mode is characterized by the counter-moving motion of the two fluids in such a way that the total density variation, and hence the mass multipole, nearly vanish. It is thus conceivable that the current multipole $\delta J_{lm}$ could provide the main radiation mechanism for the $f_s$-mode. It is also interesting to note that, for the case of two nearly symmetric fluids (e.g., $\sigma_n \approx \sigma_p$ and $\beta_n \approx \beta_p$ in the master function (Equation (10))), the superfluid modes become...
nonradiating as the current multipole nearly vanishes. This explains why the imaginary part of the mode complex frequency \(\text{Im}(\omega_M) \to 0\) in the limit \(\Delta \beta \to 0\) as shown in Figure 8. In the general situation, however, the mass current \(\rho_v \delta v_n + \rho_p \delta v_p\) would depend on the mass fractions of the two components, which are determined by the condition of chemical equilibrium. The equilibrium condition in turn depends sensitively on the underlying EOS models and thus the damping timescale of the \(f_s\)-mode would not have a simple scaling relation with the compactness of the star. Stellar models with the same global parameters (e.g., the compactness) could have vastly different fractions of the two particle species. The counter-moving character of the superfluid \(f_s\)-mode would then lead to different amount of gravitational-wave emission among the stellar models.

Finally, we note that it might be possible to perform a more detailed quantitative analysis by expanding the current multipole and extract the leading dependence of the damping time of the \(f_s\)-mode on the local thermodynamics quantities. Such an analysis might explain the relation (Equation (16)) seen in the numerical data. A natural starting point would be the extension of the recent work of Andersson et al. (2008), in which the dependence of the damping time of the ordinary \(f_o\)-mode due to the so-called mutual friction on the thermodynamics quantities has been studied.

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