A SYSTEM OF INTEREST IN SPECTROSCOPY:
THE $qp$-ROTOR SYSTEM

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Abstract

A rotor system, having the symmetry afforded by the two-parameter quantum algebra $U_{qp}(u_2)$, is investigated in this communication. This system is useful in rotational spectroscopy of molecules and nuclei. In particular, it is shown to lead to a model (viz., the $qp$-rotor model) for describing (via an energy formula and a $qp$-deformation of $E2$ reduced transition probabilities) rotational bands of deformed and superdeformed nuclei.

1 Introduction

Quantum groups and quantum algebras, introduced at the beginning of the eightees,\textsuperscript{1–5} continue to attract much attention both in mathematics and physics. For the Physicist, a quantum algebra is commonly considered as a deformation ($q$-deformation) of a given Lie algebra. During the last four years, several works have been performed on two-parameter quantum algebras and quantum groups ($qp$-deformations).\textsuperscript{6–17}

Most of the physical applications, ranging from chemical physics to particle physics, have been mainly concerned up to now with one-parameter quantum algebras ($q$-deformations). In particular, in nuclear physics we may mention applications to rotational spectroscopy of deformed and superdeformed nuclei,\textsuperscript{18–26} to the interacting boson model,\textsuperscript{27,28} to the Moszkowski model,\textsuperscript{29,30} to the U(3) shell model\textsuperscript{31} and to the Lipkin-Meshkov-Glick model.\textsuperscript{32} There exist also applications to particle physics, as for example to quote a few, to hadron mass formulas\textsuperscript{33,34} and to Veneziano amplitudes.\textsuperscript{35,36} Among the just mentioned applications, only the ones in Refs. 25 and 36 rely on the use of two-parameter deformations.

The aim of the present communication is to describe a rotor system, with the $U_{qp}(u_2)$ symmetry, that constitutes a basic tool for the nonrigid rotor model briefly introduced in Ref. 25. The latter model, referred to as the $qp$-rotor model, is based on the two-parameter quantum algebra $U_{qp}(u_2)$ in contradistinction with the $q$-rotor models introduced by Iwao\textsuperscript{18} and Raychev, Roussev and Smirnov\textsuperscript{19} (see also Refs. 20-23 and 26) that are based on the one-parameter quantum algebra $U_q(su_2)$. 

The organization of this work is as follows. The \( qp \)-rotor model is introduced in Sec. 2. Subsection 2.1 deals with the mathematical ingredients of the model. The \( qp \)-rotor model itself is developed in Subsec. 2.2 (rotational energy formula) and in Subsec. 2.3 (\( E2 \) reduced transition probabilities). Finally, some concluding remarks are presented in Sec. 3.

## 2 A \( qp \)-Rotor System

### 2.1 The quantum algebra \( U_{qp}(u_2) \)

The quantum algebra \( U_{qp}(u_2) \) can be constructed from two pairs, say \( \{ \tilde{a}^+_+, \tilde{a}^+_- \} \) and \( \{ \tilde{a}^-+, \tilde{a}^-_- \} \), of \( qp \)-deformed (creation and annihilation) boson operators. The action of these \( qp \)-bosons on a nondeformed two-particle Fock space \( \{ |n_+, n_- \rangle : n_+ \in \mathbb{N}, n_- \in \mathbb{N} \} \) is controlled by

\[
\begin{align*}
\tilde{a}^+_+ |n_+, n_- \rangle &= \sqrt{[ [n_+ + \frac{1}{2} + \frac{1}{2}]_{qp} |n_++1, n_- \rangle}, \\
\tilde{a}^+_+ |n_+, n_- \rangle &= \sqrt{[ [n_+ + \frac{1}{2} - \frac{1}{2}]_{qp} |n_-1, n_- \rangle}, \\
\tilde{a}^-+ |n_+, n_- \rangle &= \sqrt{[ [n_- + \frac{1}{2} + \frac{1}{2}]_{qp} |n_+, n_- + 1 \rangle}, \\
\tilde{a}^-_- |n_+, n_- \rangle &= \sqrt{[ [n_- + \frac{1}{2} - \frac{1}{2}]_{qp} |n_+, n_- - 1 \rangle}.
\end{align*}
\]

In the present work, we use the notations

\[
[[X]]_{qp} := \frac{q^X - p^X}{q - p}
\]

and

\[
[X]_q := [[X]]_{qp^{-1}} = \frac{q^X - q^{-X}}{q - q^{-1}},
\]

where \( X \) may stand for an operator or a (real) number. For Hermitian conjugation requirements, the values of the parameters \( q \) and \( p \) must be restricted to some domains that can be classified as follows: (i) \( q \in \mathbb{R} \) and \( p \in \mathbb{R} \), (ii) \( q \in \mathbb{C} \) and \( p \in \mathbb{C} \) with \( p = q^* \) (the \( * \) indicates complex conjugation), and (iii) \( q = p^{-1} = e^{i\beta} \) with \( 0 \leq \beta < 2\pi \). The two pairs \( \{ \tilde{a}^+_+, \tilde{a}^+_- \} \) and \( \{ \tilde{a}^-+, \tilde{a}^-_- \} \) of \( qp \)-bosons commute and satisfy

\[
\tilde{a}^\pm \tilde{a}^\mp = [[N_\pm + 1]]_{qp}, \quad \tilde{a}^\mp \tilde{a}^\pm = [[N_\pm]]_{qp},
\]

where \( N_+ \) and \( N_- \) are the usual number operators with

\[
N_\pm |n_+, n_- \rangle = n_\pm |n_+, n_- \rangle.
\]
Of course, the $qp$-bosons $\tilde{a}^\pm$ and $\tilde{a}_\pm$ reduce to ordinary bosons (denoted as $a^\pm$ and $a_\pm$ in Refs. 37 and 38 and in Subsec. 2.3) in the limiting situation where $p = q^{-1} \to 1$.

The passage from the (harmonic oscillator) state vectors $|n_+, n_-\rangle$ to angular momentum state vectors $|I, M\rangle$ is achieved through the relations

$$I := \frac{1}{2}(n_+ + n_-), \quad M := \frac{1}{2}(n_+ - n_-)$$

and

$$|I, M\rangle \equiv |I + M, I - M\rangle = |n_+, n_-\rangle.$$  

Equations (1) may thus be rewritten as

$$\tilde{a}^\pm |I, M\rangle = \sqrt{[[I\pm M + \frac{1}{2} + \frac{1}{2}]/qp]} |I + \frac{1}{2}, M\pm \frac{1}{2}\rangle,$$

$$\tilde{a}_\pm |I, M\rangle = \sqrt{[[I\pm M + \frac{1}{2} - \frac{1}{2}]/qp]} |I - \frac{1}{2}, M\pm \frac{1}{2}\rangle,$$

so that the $qp$-bosons behave as ladder operators for the quantum numbers $I$ and $M$ (with $|M| \leq I$).

We are now in a position to introduce a $qp$-deformation of the Lie algebra $u_2$. A simple calculation shows that the four operators $J_\alpha$ ($\alpha = 0, 3, +, -$) given by

$$J_0 := \frac{1}{2}(N_+ + N_-), \quad J_3 := \frac{1}{2}(N_+ - N_-), \quad J_+ := \tilde{a}_+\tilde{a}_-, \quad J_- := \tilde{a}_+\tilde{a}_+$$

satisfy the following commutation relations

$$[J_3, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = (qp)^{J_0-J_3} [2J_3]_{qp}, \quad [J_0, J_\alpha] = 0.$$  

We refer to $U_{qp}(u_2)$ the (quantum) algebra described by (10). To endow $U_{qp}(u_2)$ with a Hopf algebraic structure, it is necessary to introduce a co-product $\Delta_{qp}$. The latter co-product is such that:

$$\Delta_{qp}(J_0) = J_0 \otimes 1 + 1 \otimes J_0,$$

$$\Delta_{qp}(J_3) = J_3 \otimes 1 + 1 \otimes J_3,$$

$$\Delta_{qp}(J_\pm) = J_\pm \otimes (qp)^{\frac{1}{2}J_0} (qp^{-1})^{\frac{1}{2}J_3} + (qp)^{\frac{1}{2}J_0} (qp^{-1})^{-\frac{1}{2}J_3},$$

and is clearly seen to depend on the two parameters $q$ and $p$. [Note that with the constraint $p = q^*$, to be used in Subsec. 2.2, the co-product satisfies the Hermitean conjugation property]
\[
(\Delta_{qp}(J_\pm))^\dagger = \Delta_{pq}(J_\mp)
\]
and is compatible with the commutation relations for the four operators \(\Delta_{qp}(J_\alpha)\) (with \(\alpha = 0, 3, +, -\).] The universal \(R\)-matrix (for the coupling of two angular momenta \(I = \frac{1}{2}\)) associated to the co-product \(\Delta_{qp}\) reads
\[
R_{pq} = \begin{pmatrix}
p & 0 & 0 & 0 \\
0 & \sqrt{pq} & 0 & 0 \\
0 & p - q & \sqrt{pq} & 0 \\
0 & 0 & 0 & p
\end{pmatrix},
\]
and it can be proved that \(R_{pq}\) verifies the so-called Yang-Baxter equation.

The operator defined by
\[
C_2(U_{qp}(u_2)) := \frac{1}{2}(J_+ J_- + J_- J_+) + \frac{1}{2} [[2]]_{qp} (qp)^{J_0 - J_3} ([J_3]_{qp})^2
\]
is an invariant of the quantum algebra \(U_{qp}(u_2)\). It depends truly on the two parameters \(q\) and \(p\). The invariant \(C_2(U_{qp}(u_2))\) will be one of the main mathematical ingredients for the \(qp\)-rotor model to be developed below. Hence, it is worth to examine its structure more precisely, especially its dependence on two independent parameters. Equation (11) suggests the following change of parameters
\[
Q := (qp^{-1})^{\frac{1}{2}}, \quad P := (qp)^{\frac{1}{2}}.
\]
Then, by introducing the generators \(A_\alpha\) (\(\alpha = 0, 3, +, -\))
\[
A_0 := J_0, \quad A_3 := J_3, \quad A_\pm := (qp)^{-\frac{1}{2}(J_0 - \frac{1}{2})} J_\pm,
\]
it can be shown that the two-parameter quantum algebra \(U_{qp}(u_2)\) is isomorphic to the central extension
\[
U_{qp}(u_2) = u_1 \otimes U_Q(su_2),
\]
where \(u_1\) is spanned by the operator \(A_0\) and \(U_Q(su_2)\) by the set \(\{A_3, A_+, A_-\}\). The \(Q\)-deformation \(U_Q(su_2)\) (a one-parameter deformation!) of the Lie algebra \(su_2\) corresponds to the usual commutation relations
\[
[A_3, A_\pm] = \pm A_\pm, \quad [A_+, A_-] = [2A_3]_Q.
\]
Furthermore, the co-product relations (11) leads to
\[
\Delta_{qp}(J_\pm) = P^{\Delta_Q(A_0)-\frac{1}{2}} \Delta_Q(A_\pm),
\]
where the co-product \(\Delta_Q\) is given via
\[
\Delta_Q(A_0) = A_0 \otimes 1 + 1 \otimes A_0, \quad \Delta_Q(A_3) = A_3 \otimes 1 + 1 \otimes A_3, \quad 
\Delta_Q(A_\pm) = A_\pm \otimes Q^{+A_3} + Q^{-A_3} \otimes A_\pm.
\]
Equations (17) involve only one parameter, i.e., the parameter $Q$. However, two parameters ($Q$ and $P$) occur in (18) as well as in the invariant $C_2(U_{qp}(u_2))$ transcribed in terms of $Q$ and $P$. As a matter of fact, (13) can be rewritten as

$$C_2(U_{qp}(u_2)) = P^{2A_0-1} C_2(U_Q(u_2)),$$

(20)

where

$$C_2(U_Q(u_2)) := \frac{1}{2} (A_+A_- + A_-A_+) + \frac{1}{2} [2]_Q \ ([A_3]_Q)^2$$

(21)

is an invariant of $U_Q(u_2)$ [compare Eqs. (13) and (21)]. As a consequence, of central importance for the $qp$-rotor model of Subsec. 2.2, the invariant $C_2(U_{qp}(u_2))$, in either the form (13) or the form (20), depends on two parameters. Finally, it should be noted that $C_2(U_{qp}(u_2))$ can be identified to the invariant of $U_q(u_2)$ and to the Casimir of $su_2$ when $p = q^{-1}$ and $p = q^{-1} \to 1$, respectively. In this sense, the $U_{qp}(u_2)$ symmetry encompasses the $U_q(u_2)$ and $su_2$ symmetries.

To close this section, let us say a few words on the representation theory of $U_{qp}(u_2)$ in the case where neither $q$ nor $p$ are roots of unity. An irreducible representation of this quantum algebra is described by a Young pattern $[\varphi_1; \varphi_2]$ with $\varphi_1 - \varphi_2 = 2I$, where $2I$ is a nonnegative integer ($I$ will represent a spin angular momentum in what follows). We note $[[\varphi_1; \varphi_2], M]$ (with $M = -I, -I+1, \ldots, +I$) the basis vectors for the representation $[\varphi_1; \varphi_2]$.

We are now ready to develop a $qp$-rotor model for describing energy levels and reduced transition probabilities for rotation spectra of molecules and nuclei.

### 2.2 Energy levels

We want to construct a nonrigid rotor model. As a first basic hypothesis (Hypothesis 1), we take a rigid rotor with $U_{qp}(u_2)$ symmetry, thus introducing the nonrigidity through the $qp$-deformation of the Lie algebra $u_2$. More precisely, we assume that the $qp$-rotor Hamiltonian $H$ is a linear function of the invariant $C_2(U_{qp}(u_2))$:

$$H = \frac{1}{2I} C_2(U_{qp}(u_2)) + E_0,$$

(22)

where $I$ denotes the moment of inertia of the rotor and $E_0$ the bandhead energy. As a second hypothesis (Hypothesis 2), we take $\varphi_1 = 2I$ and $\varphi_2 = 0$. This means that we work with state vectors of the type $|I, M\rangle \equiv |2I; 0\rangle, M\rangle$. Therefore, the eigenvalues of $H$ are obtained by the action of $H$ on the physical subspace $\{|I, M\rangle : M = -I, -I+1, \ldots, +I\}$ of constant angular momentum $I$ corresponding to the irreducible representation $|2I; 0\rangle$ of $U_{qp}(u_2)$. The two preceding hypotheses lead to the energy formula

$$E(I)_{qp} = \frac{1}{2I} [[I]_{qp} [[I+1]_{qp} + E_0$$

(23)
for the \( qp \)-deformed rotational level of angular momentum \( I \).

By introducing \( s = \ln q \) and \( r = \ln p \), Eq. (23) yields

\[
E(I)_{qp} = \frac{1}{2\mathcal{L}} \ e^{(2I-1)\beta \cos \gamma} \ \frac{\sinh(I \beta \sin \gamma) \ \sinh[(I + 1) \beta \sin \gamma]}{\sinh^2(\beta \sin \gamma)} + E_0.
\]  

Preliminary studies have lead us to the conclusion that a good agreement between theory and experiment cannot be always obtained by varying the parameters \( s \) and \( r \) (or \( q \) and \( p \)) on the real line \( \mathbb{R} \), a fact that confirms a similar conclusion reached in Ref. 20 for \( p = q^{-1} \in \mathbb{R} \). In addition, if we want that our \( qp \)-rotor model reduces to the \( q \)-rotor model developed by Raychev, Roussev and Smirnov\(^{19} \) when \( p = q^{-1} \) (or equivalently \( r = -s \)), we are naturally left to impose that \( (s + r) \) and \( (s - r)/i \) should be real numbers. [Observe that the two constraints \( (s + r) \in \mathbb{R} \) and \( (s + r)/i \in \mathbb{R} \) ensure that the energy \( E(I)_{qp} \) is real as it should be.] Furthermore, work in progress in collaboration with J. Meyer (in Lyon) shows that for certain SD bands, a good agreement between theory and experiment requires that the parameters \( s \) and \( r \) vary on the real line \( \mathbb{R} \). Thus, we shall consider the two possible parametrizations:

\[
\begin{align*}
\text{(a)} \quad \frac{s + r}{2} &= \beta \cos \gamma \in \mathbb{R}, \quad \frac{s - r}{2i} = \beta \sin \gamma \in \mathbb{R}, \\
\text{(b)} \quad \frac{s + r}{2} &= \beta \cos \gamma \in \mathbb{R}, \quad \frac{s - r}{2i} = \beta \sin \gamma / i \in i\mathbb{R},
\end{align*}
\]  

so that the parameters \( q \) and \( p \) read

\[
\begin{align*}
\text{(a)} \quad q &= e^{\beta \cos \gamma} e^{+i\beta \sin \gamma}, \\
&\quad p = q^* = e^{\beta \cos \gamma} e^{-i\beta \sin \gamma}, \\
\text{(b)} \quad q &= e^{\beta \cos \gamma} e^{+\beta \sin \gamma}, \\
&\quad p = e^{\beta \cos \gamma} e^{-\beta \sin \gamma}.
\end{align*}
\]

Thus, our \( qp \)-rotor model involves two independent real parameters \( \beta \) and \( \gamma \) corresponding either to (a) the two complex parameters \( q \) and \( p \) subjected to the constraint \( p = q^* \) or to (b) the two real parameters \( q \) and \( p \). In terms of the parameters \( \beta \) and \( \gamma \), the energy formula (24) takes the form

\[
\begin{align*}
E(I)_{qq^*} &= \frac{1}{2\mathcal{L}} \ e^{(2I-1)\beta \cos \gamma} \ \frac{\sin(I \beta \sin \gamma) \ \sin[(I + 1)\beta \sin \gamma]}{\sin^2(\beta \sin \gamma)} + E_0 \\
\text{or} \\
E(I)_{qp} &= \frac{1}{2\mathcal{L}} \ e^{(2I-1)\beta \cos \gamma} \ \frac{\sinh(I \beta \sin \gamma) \ \sinh[(I + 1)\beta \sin \gamma]}{\sinh^2(\beta \sin \gamma)} + E_0
\end{align*}
\]

in the parametrizations of type (a) or (b), respectively.
In the (a)-parametrization, to better understand the connection between our \(qp\)-rotor model and the \(q\)-rotor model of Ref. 19, we can perform a series analysis of Eq. (27a). A straightforward calculation allows us to rewrite Eq. (27a) as

\[
E(I)_{qq^{-1}} = \frac{1}{2L\beta\gamma} \left( \sum_{n=0}^{\infty} d_n(\beta, \gamma) [I(I+1)]^n + (2I + 1) \sum_{n=0}^{\infty} c_n(\beta, \gamma) [I(I+1)]^n \right) + E_0,
\]

where the expansion coefficients \(d_n(\beta, \gamma)\) and \(c_n(\beta, \gamma)\) are given in turn by the series

\[
d_n(\beta, \gamma) = \frac{2^{2n-1}}{\sin^2(\beta \sin \gamma)} \times \sum_{k=0}^{\infty} \left\{ (\cos \gamma)^{2k+2n} \cos(\beta \sin \gamma) - \cos[(2k + 2n)\gamma] \right\} \frac{\beta^{2k+2n}}{(2k + 2n)!} \frac{(k + n)!}{k! n!},
\]

\[
c_n(\beta, \gamma) = \frac{2^{2n-1}}{\sin^2(\beta \sin \gamma)} \times \sum_{k=0}^{\infty} \left\{ (\cos \gamma)^{2k+2n+1} \cos(\beta \sin \gamma) - \cos[(2k + 2n + 1)\gamma] \right\} \frac{\beta^{2k+2n+1}}{(2k + 2n + 1)!} \frac{(k + n)!}{k! n!}.
\]

In Eq. (28), we have introduced the deformed moment of inertia:

\[
I_{\beta\gamma} = L \, e^{2\beta \cos \gamma},
\]

which gives back the ordinary moment of inertia when \(\gamma = \pi/2\) (i.e., \(q = p^{-1} = e^{i\beta}\)). In the limiting situation where \(\gamma = \pi/2\), the coefficients \(c_n(\beta, \gamma)\) vanish and the energy formula (28) simplifies to

\[
E(I)_{qq^{-1}} = \frac{1}{2L} \frac{\beta^2}{\sin^2 \beta} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{2^{n-1}}{n!} \beta^{n-1} j_{n-1}(\beta) [I(I+1)]^n + E_0,
\]

where \(j_{n-1}\) denotes a spherical Bessel function of the first kind. Equation (31) was derived by Bonatsos et al.\textsuperscript{20} for the \(q\)-rotor model with \(q = e^{i\beta}\) in order to prove the mathematical parentage between the \(q\)-rotor model and the variable moment of inertia (VMI) model.\textsuperscript{40} The series (31) corresponds indeed to the compact expression

\[
E(I)_{qq^{-1}} = \frac{1}{2L} [I]_q[I + 1]_q + E_0,
\]

to be compared with Eq. (23). Note that Eq. (32) corresponds also to the (b)-parametrization with \(\gamma = \pi/2\). The transition from Eq. (23) to Eq. (32) illustrates the descent from the \(U_{qp}(u_2)\) symmetry of the \(qp\)-rotor to the \(U_q(su_2)\) symmetry of the \(q\)-rotor. A further descent in symmetry is obtained when \(\beta \to 0\) (i.e., \(q = p^{-1} \to 1\)): in this case \([I]_q[I + 1]_q \to I(I + 1)\) and we get [from Eq. (32)] the usual energy formula corresponding to the rigid rotor with \(su_2\) symmetry.
2.3 E2 reduced transition probabilities

We now examine the implication of the $U_{q\mu}(u_2)$ symmetry on the calculation of the (electric quadrupole) reduced transition probability $B(E2; I + 2 \rightarrow I)$. Let us start with the ordinary expression of the reduced transition probability, namely,

$$B(E2; I + 2 \rightarrow I) = \frac{5}{16\pi} Q_0^2 \left| (I + 2, M, 2, 0 | I + 2, I, M) \right|^2$$

for an E2 transition. In Eq. (33), $Q_0$ is the intrinsic electric quadrupole moment in the body-fixed frame. The coefficient of type $(j, m, k, \mu | j, k, j', m')$ in the right-hand side of Eq. (33) is a usual Clebsch-Gordan coefficient for the group SU(2). Our goal is to find a $qp$-analog of $B(E2; I + 2 \rightarrow I)$. The strategy for obtaining a $qp$-analog of Eq. (33) is the following:

(i) We first rewrite the SU(2) Clebsch-Gordan coefficient of Eq. (33) in terms of a matrix realization of ordinary boson operators. In this respect, we may consider the so-called van der Waerden realization of $t_{k\mu\alpha}$. There are several ways to $qp$-deform the operator $t_{k\mu\alpha}$. Here, we choose to define a $qp$-deformation $t_{k\mu\alpha}(qp)$ by replacing, in the van der Waerden realization of $t_{k\mu\alpha}$, the ordinary bosons $\{a_+^-, a_+\}$ by $qp$-deformed bosons $\{\tilde{a}_+^-, \tilde{a}_+\}$ and the ordinary factorials $!$ by $qp$-deformed factorials $[[x]]_{qp}! = [[x]]_{qp} [[x - 1]]_{qp} \cdots [[1]]_{qp}$ for $x \in \mathbb{N}$. We thus obtain

$$t_{k\mu\alpha}(qp) = (-1)^{k+\alpha} \left( \frac{[[k+\mu]]_{qp}! [[k-\mu]]_{qp}! [[k+\alpha]]_{qp}! [[k-\alpha]]_{qp}! [[2k-j+k+\alpha]]_{qp}!}{[[2j+k+\alpha+1]]_{qp}!} \right)^{\frac{1}{2}} \times \sum_z (-1)^z \frac{((\tilde{a}_+)^{k-\mu-z}(\tilde{a}_-)^{k-\alpha-z}(\tilde{a}_+)^{\mu+\alpha+z}(\tilde{a}_+)^z)}{[[k-\mu-z]]_{qp}! [[k-\alpha-z]]_{qp}! [[\mu+\alpha+z]]_{qp}! [[z]]_{qp}!}.$$  

(ii) We now examine the implication of the $SU(2)$ Clebsch-Gordan coefficient of Eq. (33) in terms of a matrix realization of ordinary boson operators. In this respect, we may consider the so-called van der Waerden realization of $t_{k\mu\alpha}$. There are several ways to $qp$-deform the operator $t_{k\mu\alpha}$.

In particular, the $qp$-deformed operator $t_{20-2}(qp)$ connecting the state vector $|I + 2, M\rangle$, with $j \equiv I + 2$, to the state vector $|I, M\rangle$, with $j' \equiv I$, reads

$$t_{20-2}(qp) = \left( \frac{[[3]]_{qp} [[4]]_{qp} [[2I]]_{qp}!}{[[2]]_{qp} [[2I+5]]_{qp}!} \right)^{\frac{1}{2}} (\tilde{a}_+)^2 (\tilde{a}_-)^2.$$

$$B(E2; I + 2 \rightarrow I) = \frac{5}{16\pi} Q_0^2 \left| (I + 2, M, 2, 0 | I + 2, I, M) \right|^2$$

which shows that the irreducible tensor operator $t_{k\mu\alpha}$ produces the (angular momentum) state vector $|j + \alpha, m + \mu\rangle$ when acting upon the state vector $|j, m\rangle$. Then, Eq. (33) is amenable to the form

$$B(E2; I + 2 \rightarrow I) = \frac{5}{16\pi} Q_0^2 (2I + 1) \left| (I, M | t_{20-2} | I + 2, M) \right|^2$$

by making use of Eq. (34).
an expression of direct interest for deriving the \( qp \)-analog of \( B(E2; I + 2 \rightarrow I) \).

(iii) We assume that the \( qp \)-analog \( B(E2; I + 2 \rightarrow I)_{qp} \) of \( B(E2; I + 2 \rightarrow I) \) is simply

\[
B(E2; I + 2 \rightarrow I)_{qp} := \frac{5}{16\pi} Q_0^2 \left| \langle I, M | t_{20-2}(qp) | I + 2, M \rangle \right|^2.
\]  

(Equation (38) constitutes the third and last hypothesis (Hypothesis 3) for our \( qp \)-rotor model.)

By using Eqs. (37) and (8), the relevant matrix element of the operator \( t_{20-2}(qp) \) is easily found to be

\[
\langle I, M | t_{20-2}(qp) | I + 2, M \rangle = \left( \frac{[[3]_q [4]_q [2I]_q ! [[I + M + 1]_q [I - M + 1]_q [I + M + 2]_q [I - M + 2]_q}{[[2]_q [2I + 5]_q !} \right)^{1/2}.
\]  

(39)

Then, the introduction of Eq. (39) into Eq. (38) yields

\[
B(E2; I + 2 \rightarrow I)_{qp} = \frac{5}{16\pi} Q_0^2 \frac{[[3]_q [4]_q [2I]_q ! [[I + M + 1]_q [I - M + 1]_q [I + M + 2]_q [I - M + 2]_q}{[[2]_q [2I + 5]_q !}
\]  

(40)

in the case of the \( K \equiv M = 0 \) bands.

At this stage, a contact with the formula \( B(E2; I + 2 \rightarrow I)_q \) derived by Raychev, Roussev and Smirnov\(^{19}\) is in order. First, by taking \( p = q^{-1} \) the right-hand side of (40) may be specialized to the expression of \( B(E2; I + 2 \rightarrow I)_q \) obtained in Ref. 19. Hence, our \( qp \)-rotor model for the \( E2 \) reduced transition probability admits as a particular case the corresponding \( q \)-rotor model worked out in Ref. 19. Second, it can be shown that

\[
B(E2; I + 2 \rightarrow I)_{qp} = P^{-4(I+1)} B(E2; I + 2 \rightarrow I)_Q,
\]  

(41)

where \( P \) and \( Q \) are given by (14).

Let us close with a remark. Should we have chosen to find a \( qp \)-analog of the Clebsch-Gordan coefficient in (33), we would have obtained\(^{39}\)

\[
\langle I + 2, M, 2, 0 | I + 2, 2, I, M \rangle_{qp} = \langle I + 2, M, 2, 0 | I + 2, 2, I, M \rangle_Q
\]  

(42)

and, consequently

\[
B(E2; I + 2 \rightarrow I)_{qp} = B(E2; I + 2 \rightarrow I)_Q.
\]  

(43)

We prefer to use (41) rather than (38) because the factorization in (41) parallels the one in (20).
3 Conclusions

In this communication, we concentrated on a \(qp\)-rotor system [in the framework of a revisiting of the two-parameter quantum algebra \(U_{qp}(u_2)\)] that leads to a nonrigid rotor model (the \(qp\)-rotor model) based on three hypotheses. The two facets of this model consist of a three-parameter energy level formula and a \(qp\)-deformed \(E_2\) reduced transition probability formula. As limiting cases, the \(qp\)-rotor model gives back the \(q\)-rotor model\(^{19}\) (when \(p = q^{-1}\)) based on the quantum algebra \(U_q(\text{su}_2)\) and the rigid rotor model (when \(p = q^{-1} \rightarrow 1\)) based on the Lie algebra \(\text{su}_2\).

A work to be published elsewhere\(^{42}\) shows that the \(qp\)-rotor model is appropriate for describing the collective phenomenon of distortion occurring in the rotation of the nucleus (increase or decrease of the dynamical moment of inertia with the spin). The net difference between the \(q\)- and \(qp\)-rotor models comes from the “quantum algebra”-type parameter \(a\) that tends to smooth the (spherical or hyperbolical) sine term in the energy and thus accentuates or moderates the distortion phenomenon of the nucleus.

To close this work, let us mention that Hypothesis 2 (i.e., \(\varphi_1 = 2I\) and \(\varphi_2 = 0\)) of our model might be abandoned. This would lead to a \(à \ la\) Dunham formulation for describing more complicated rotational spectra of deformed and superdeformed nuclei or rovibrational spectra of diatomic molecules. As a further extension, it would be also interesting to combine our model with one of Ref. 24 (based on the \(q\)-Poincaré symmetry) in the case of heavy nuclei. Work in these directions is in progress.

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