Qualitative Analysis and Traveling wave Solutions for the Nonlinear Convection Equations with Absorption

Bashayir N. Abed¹  Salam J. Majeed²  Habeeb A. Aal-Rkhais³

¹²³Department of Mathematics, College of Computer Science and Mathematics, University of Thi-Qar, Iraq

ABSTRACT: We discuss qualitative behavior of the solutions for the nonlinear parabolic equation which modeling nonlinear convection equation with absorption. This model represents the movement of growing population that is ruled by convection process. In this paper, we concentrate on proving the existence of traveling wave solutions for the nonlinear convection-reaction equations. In addition, we consider the model when the speed of advective wave may breakdown and the problem has a shock wave solution. The mathematical interesting of the waves comes from the behaviors of singular differential equation and discussing the stability of the solution.

Keywords: traveling-waves, convection-reaction process, characteristic methods, stability.

1. Introduction

The traveling waves have played a very important role in many nonlinear parabolic equations modeling reaction-diffusion-convection processes. In this paper we are interested in solutions of nonlinear advection equation model

\[
\frac{\partial \phi}{\partial t} = a \frac{\partial \phi^\gamma}{\partial x} + b \phi^\beta(1 - \phi)
\]

(1)

where \(\beta > 0, \gamma > 1, \gamma + \beta > 2\); \(a\) and \(b\) are positive constants and \(\phi = \phi(x; t)\) is a nonnegative unknown density (concentration), with space \(x\) and time \(t\). The study and application of this model clearly appeared in many areas of science such as biological and physical models including shock waves and traveling waves of oscillatory chemical reactions. Existence and uniqueness of the local solution and traveling waves for reaction-diffusion-convection equation are introduced in [1, 2, 3, 7, 8, 12]. Several models of partial differential equations are represented as pattern formations, critical patch sizes, traveling waves, ecological invasion and many others in [5, 9, 10]. Combining population growth dynamics with models of movement has ecological interest. the Fisher model is one of the classical model of ecology that represents dispersion and population growth see [6]. In addition, we consider the standard nonlinear reaction-diffusion-convection equation in one dimension. Generally, this equation can show shock wave solutions [4, 5, 11].
In this paper, we considerate on the problem of a nonlinear advection-absorption process which has traveling wave solutions for special situations in one dimensional space. Particularly, we consider the population at a particular position which grows according to the diffusion process that is very weak with respect to the advection effects. It is interesting that we discuss the solutions in the form \( \varphi = \varphi(x \pm \lambda t) \) where \( \lambda > 0 \) represents the speed of the wave and it travels without changing shape.

2. Traveling Wave Solutions

In this section, we find travelling-wave solutions for the equation (1), and give the asymptotic behavior of these solutions of (1) and a description of a nonlinear convection process with a logistic population growth. The second part of (1) represents the nonlinear absorption term. The solution represents population density which is changed per unit time. In the spatially homogeneous status, the steady states of equation (1) for \( b > 0, \varphi = 0 \) and \( \varphi = 1 \) which are unstable and stable respectively. Before we discuss the existence of solutions, it is appropriate to change the variable \( \varphi = v^{1/(\gamma - 1)} \) in the equation (1) and it becomes

\[
\frac{\partial v}{\partial t} - a \gamma v \frac{\partial v}{\partial x} = b(\gamma - 1) v^\alpha \left(1 - \frac{1}{v^{\gamma - 1}}\right)
\tag{2}
\]

where \( \alpha = (\gamma + \beta - 2)/\gamma - 1 \).

**Theorem 1.** If \( \lambda > 0, \gamma > 1, \beta > 0, \gamma + \beta > 2; \) then the traveling wave solution \( v(x; t) = f(\xi), \xi = x + \lambda t \) of (2) is satisfied for \( 0 \leq f \leq 1 \), with the boundary conditions \( \lim_{\xi \to -\infty} f(\xi) = 0 \) and \( \lim_{\xi \to +\infty} f(\xi) = 1 \).

**Proof.** Let us use rescaling technique to equation (2) by writing new variables as \( \tau = bt \) and \( y = (b/a \gamma)x \). Then equation (2) becomes

\[
\frac{\partial v}{\partial \tau} - v \frac{\partial v}{\partial x} = (\gamma - 1) v^\alpha \left(1 - \frac{1}{v^{\gamma - 1}}\right),
\tag{3}
\]

where \( \gamma > 1 \). We consider nonnegative solution to (3) for \( v \leq 1 \) because the uniformly steady states of the solutions are only \( v = 0 \) and \( v = 1 \). We can formulate the traveling wave solution as

\[
v(x; t) = f(\xi), \quad \xi = x + \lambda t
\tag{4}
\]

where \( \lambda > 0 \) is the wave speed. Then the wave fronts of the solutions move to the left in the \( \xi \)-plane. We substitute the function (4) in the equation (3), then \( f(\xi) \) satisfies
\[ \frac{\partial f}{\partial \xi} = (\gamma - 1)(\lambda - f) f^\alpha \left( 1 - f^{\frac{1}{\gamma - 1}} \right) \] (5)

where differentiation is satisfied according to the variable \( \xi \). A singularity of the solution happens at \( f(\xi) = \lambda \). We can get the wave front solution \( f(\xi) \) to have limiting values. The problem is to govern the traveling wave solution with respect to \( \lambda \) where the solution of (5) is nonnegative and exists. It satisfies \( f'(\xi) > 0 \) and,

\[ \lim_{\xi \to -\infty} f(\xi) = 0 \text{ and } \lim_{\xi \to +\infty} f(\xi) = 1, \] (6)

which are steady states and also \( f(\xi) \) can be monotonically increasing. Where equation (5) has steady states at \( f(\xi) = 0 \) and \( f(\xi) = 1 \) and stability of them relies too much on value of \( \lambda \). Linearity of the equation (5) displays that the solution \( f(\xi) = 0 \) is unstable for \( \lambda > 0 \), and \( f(\xi) = 1 \) is stable for \( \lambda > 0 \). Also, it is generally unstable for \( 0 < \lambda < 1 \). If \( \lambda = 1 \), we can reduce equation (5) into \( f'(\xi) = f(\xi) \) provided that \( f(\xi) \neq 1 \). Definitely, \( f(\xi) = 1 \) is a singularity of (5), and \( f(\xi) \) is exponentially increasing. \( \blacksquare \)

Next, we introduce in particular case the traveling wave of (5) with \( \gamma = 2 \) and \( \alpha = \beta \), for \( \lambda > 1 \). Let us consider the equation (2) which becomes the following equation

\[ \frac{\partial v}{\partial t} - 2av \frac{\partial v}{\partial x} = bv^\beta (1 - v), \] (7)

and after rescaling equation (7) by assuming \( \tau = bt \) and \( y = (b/2a)x \), we get

\[ \frac{\partial v}{\partial \tau} - v \frac{\partial v}{\partial y} = v^\beta (1 - v). \] (8)

Then the similar way in Theorem 1, the traveling wave solution \( f(\xi) \) satisfies

\[ \frac{\partial f}{\partial \xi} = f^\beta (\lambda - f)^{-1}(1 - f) \] (9)

Then for \( 0 < f(\xi) < 1 \), we have three cases to get the solution of the ODEs. First, if \( \beta = 1/2 \), then the solution of (9) is

\[ \ln \left( \frac{1 + \sqrt{f}}{1 - \sqrt{f}} \right)^{\lambda - 1} = \xi - 2\sqrt{f} + \xi_1. \] (10)

If the parameter \( \beta = 1 \) , then the solution of (9) is in the following form
\[ \ln \left( \frac{f^\lambda}{(1-f)^{\lambda-1}} \right) = \xi + C_2. \]  

(11)

Finally, let us choose \( \beta = 2 \), then the solution of (9) has the following form

\[ \ln \left( \frac{f}{1-f} \right)^{\lambda-1} = \frac{\lambda}{f} + (x - C_3), \]  

(12)

where \( C_i, i = 1, 2, 3 \); are constants of integration. The solutions (10)-(12) of the equation (9) for \( \beta = 1/2, 1, 2 \); respectively are satisfied with the initial condition \( f(0) = 1/2, \) for all \( \lambda > 0 \) with the boundary conditions (6) at \(-\infty\) for any constants \( C_1 = (1 + (\lambda - 1) \ln 3)/2, C_2 = -\ln 2, C_3 = (2\lambda + 2(\lambda - 1) \ln 2) \).

Also, the boundary conditions (5) are satisfied at \(+\infty\) for \( \lambda > 1 \) but they are not satisfied for \( \lambda < 1 \). The solution \( f \) is exponentially increasing and satisfied travelling wave solutions for \( \lambda = 1 \). Because the travelling wave solutions are invariant, the equation (9) is unchanged if \( \xi \rightarrow \xi + c \), where \( c \) is any constant. Let us take \( \xi = 0 \) to be the origin point so the behavior of solutions is invariant to any shifting from the origin.

**Fig.1:** Traveling wave solution \( f(\xi) \) where \( \beta = 1/2, \gamma = 2, \alpha = 1/2 \)
Fig.2: Traveling wave solution $f(\xi)$ where $\beta = 1, \gamma = 2, \alpha = 1$

Therefore, traveling wave solution $f = f(\xi)$ of (9) with $\beta = 1/2, 1, 2$ are shown in Figs.1-3; respectively. These matching to values $\lambda = 1.5, \lambda = 2, \lambda = 3$ and $\lambda = 4$. We observe that the derivative of the solution at $\xi = 0$ explains the steepness of the traveling waves is decreasing but the wave speed is increasing.

3. Methods of Characteristics

Let us consider in this section the stability of the traveling wave solution. If we impose a small perturbation on the wavefront at initial time such as $t = 0$, then it decays away. Also, the behavior of the initial conditions effects on the speed of propagation of the wave. Development of traveling wave solutions the partial differential equation (3) with the initial condition $v(x, 0) = v_0(x)$ are satisfied. Now, we use the characteristic methods to solve the initial value problem of characteristic equations

$$\frac{dx}{d\tau} = -v, \quad \frac{dt}{d\tau} = 1, \quad \frac{dv}{d\tau} = v^\alpha(1 - v).$$

With the initial conditions that can be parameterized in the following forms

$$x(s, 0) = s, \quad t(s, 0) = 0, \quad v(s, 0) = v_0(s)$$

Integrating the equation for $t$ yields $t = \tau$. For $v$, after substituting $t$ for $\tau$, we consider particular case when $\alpha = 1$, $\beta = 1$, $\gamma > 1$; explicit solution

$$v(s, t) = e^{t}v_0(s) \left[ (e^{t} - 1)v_0(s) + 1 \right]^{-1} \quad (13)$$

Also, we obtain the characteristic curves as follow
\[ x = s - \ln\left((e^s - 1)v_0 + 1\right). \]  

(14)

On the other hand, if we suppose that such \( \alpha = 2, \beta = \gamma, \gamma > 1; \) we get implicit solution which has a complicated form and is not easy to consider its characteristic curves and behavior. The solution of equation (13) evolves along the characteristic curve (14) at \((s, 0), s \in \mathbb{R}\). We can assume initial guess of initial conditions

\[ v_0(x) = 1 \text{ if } x \leq a \text{ and } v_0(x) = 0 \text{ if } x \geq b, \]

where \( a < b \) and \( v_0(x) \) is continuous in \( a \leq x \leq b \). Let us begin with the above initial condition, and because the slope equals to the origin point (zero) for all \( x \) when \( v_0(x) = 0 \) for \( x \leq a \), the characteristic curves will intersect. Also, the derivative of \( v_0(x) \) is nonnegative and will move up to be shocks. Depending on the nature of traveling wave and the observation of the above initial data, we should consider the initial condition with the following inequality

\[ 0 \leq v'_0(x) \leq v_0(x) \leq 1, \quad \forall x \in \mathbb{R}, \]

should be satisfied. This restriction is significant because if \( v'_0(x) > v_0(x) \), then \( \partial v / \partial x \) may blow up at some \( t > 0 \).

Therefore, we observe that if initial conditions are smooth, then the curves may steepen into shocks-like solutions. Thus, from the above analysis, we shall assume the form of the initial data as

\[ v(x, 0) = \begin{cases} C e^{\mu(x-50)}, & x \leq 50, \\ C(2 - e^{-\mu(x-50)}), & x > 50, \end{cases} \]

(15)

with the nonnegative constant \( C \) and \( C \leq 0.5 \) and \( 0 < \mu < 1 \) we consider the traveling wave in the form (4) with the initial condition (15) and boundary conditions (6). Then, for \( \mu > 1 \), the derivative of the solution \( \partial v / \partial x \) with the initial condition (15) for \( x \leq x_0 \) will be blows up for some \( t > 0 \). Also for \( \mu = 1 \), then \( \partial v / \partial x \) is unbounded and the solution \( v \) does not represent the traveling wave. We observe numerically that the traveling wave solutions for equation (3) with the initial condition (15) for \( 0 < \mu < 1 \) are satisfied with the wave speed \( \lambda \) depends on the value of \( \mu \) and is inversely proportional to \( \mu \).

In Fig.4, numerical development of the traveling wave solutions is shown for \( \mu = 0.2, 0.4 \) and \( 0.8 \) with wave speeds \( \lambda = 1.5, 2, 3 \) and 4. For more motivation, the wave speed depending on the parameter \( \mu \) has a fundamental analysis in [10].
Fig.4: Development of solutions of (3) starting from initial condition (16) with $C = 0.3$. Top: $\mu = 0.2$, middle: $\mu = 0.4$, bottom: $\mu = 0.8$

4. Stability of Traveling Wave Solutions

In this section, we try to investigate the stability of traveling wave solutions in particular cases where $\gamma = 2$, $\alpha = \beta$ and for $\lambda > 1$. Let us write the equation (8) by assuming $\nu(x,t) = \psi(y,t)$ where $y = x + \lambda t$, and we get

$$\frac{\partial \psi}{\partial t} + (\lambda - \psi) \frac{\partial \psi}{\partial y} = \psi^\beta(1 - \psi). \tag{16}$$

Suppose that $f(y)$ is a traveling wave solution of the equation (9) which is defined for $\lambda > 0$. Let us consider the equation (14) that has a solution in the form
\( \psi(y,t) = f(y) + P(y,t) \) \hspace{1cm} (17)

where \( P(y,t) \) is a small perturbation of \( f(y) \). Thus, for some \( x_0 \in \mathbb{R} \), we suppose that \( P(y,t) = 0 \) for \( y < x_0 \), which means that the perturbation can be vanished on the interfaces of the waves. Let us substitute the form (17) in the equation (16), then we obtain a partial differential equation of the perturbation \( P(y,t) \) as

\[
\frac{\partial P}{\partial t} + (\lambda - f(y)) \frac{\partial P}{\partial y} + (2f(y) - 1 - f'(y))P - P \frac{\partial P}{\partial y} + P^2 = 0, \quad (18)
\]

By vanishing the last two terms since \( P \) is too small and \( \frac{\partial P}{\partial y} \) is very small at low density (if it is advection). Also if \( \beta = 2 \), we use the similar calculation thus (18) becomes

\[
\frac{\partial P}{\partial t} + (\lambda - f(y)) \frac{\partial P}{\partial y} + (2f(y) - 1 - f'(y))P = 0. \quad (19)
\]

Since \( \lim_{t \to \infty} P(y,t) \approx 0 \) for any fixed \( y \), we shall apply the same technique that introduced in [11], for \( \lambda > 1 \) to investigate the stability of the traveling wave solution \( f(y) \) of (19) to small perturbations \( P(y,t) \).

### 5. Conclusion

Existence and uniqueness of the solutions for the nonlinear parabolic equation which modeling nonlinear convection equation with absorption have introduced in several studies in [1, 2, 7, 8, 12]. Proving the existence of traveling wave solutions for the nonlinear convection-reaction equations in some cases was discussed. Also, Shock wave solutions happens in some restrictions of the parameters where the speed of traveling wave may breakdown. Qualitative techniques displayed the traveling wave depends on the behavior of the initial conditions particularly at the edges of the waves. The equation (1) with \( \gamma = \beta = 1 \) has speed of the traveling wave that depends on the initial conditions at infinity. We satisfy that traveling wave solution which has a compact support cannot grow from the initial data.

### References

[1] Aal-Rkhais, H. A. 2018 On the Qualitative Theory of the Nonlinear Degenerate Second Order Parabolic Equations Modeling Reaction-Diffusion-Convection Processes (Florida, USA: FIT).
[2] Habeeb A. Aal Rkhais, Ayed E. Hashoosh; Asymptotic Behavior of Solutions to the Nonlinear Fokker-Planck Equation with Absorption; Jour of Adv Research in Dynamical & Control Systems, Vol. 10, 10-Special Issue, 2018
[3] Abdulla, Ugur G., and Habeeb A. Aal-Rkhais. "Development of the Interfaces for the Nonlinear Reaction-Diffusion equation with Convection." IOP Conference Series: Materials Science and Engineering. Vol. 571. No. 1. IOP Publishing, 2019.
[4] Dai, J. L. H. (2007). On the study of singular nonlinear traveling wave equations: Dynamical system approach.
[5] Fife, P. C. Mathematical Aspects of Reacting and Diffusing Systems Lecture Notes in Biomathematics, Vol 28, Springer-Verlag, Berlin, 1979.

[6] Fisher, R. A. The wave of advance of advantageous. Ann. Eugenics, 7: 335-369, 1937.

[7] Huang, J., Lu, G., & Ruan, S. (2003). Existence of traveling wave solutions in a diffusive predator-prey model. Journal of Mathematical Biology, 46(2), 132-152.

[8] Li, J., & Liu, Z. (2002). Traveling wave solutions for a class of nonlinear dispersive equations. Chinese Annals of Mathematics, 23(03), 397-418.

[9] Malchow, H. Flow- and locomotion-induced pattern formation in nonlinear population dynamics. Ecological Modelling, 82: 257-265, 1995

[10] Mollison, D. Spatial contact models for ecological and epidemic spread. J. Roy. Stat. Soc., B39: 283-326, 1977.

[11] Smoller, J. Shock Waves and Reaction-Diffusion Equations. Springer-Verlag, Berlin, 2nd edition, 1994.

[12] Zhang, Z. Y., Liu, Z. H., Miao, X. J., & Chen, Y. Z. (2011). Qualitative analysis and traveling wave solutions for the perturbed nonlinear Schrödinger's equation with Kerrlaw nonlinearity. Physics Letters A, 375(10), 1275-1280.