Nonlinearly coupled harmonic oscillators: high frequency oscillations yield synchronization

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Abstract
Synchronization of coupled harmonic oscillators is investigated. Coupling considered here is pairwise, unidirectional, and described by a nonlinear function (whose graph resides in the first and third quadrants) of some projection of the relative distance (between the states of the pair being coupled) vector. Under the assumption that the interconnection topology defines a connected graph, it is shown that the synchronization manifold is semiglobally practically asymptotically stable in the frequency of oscillations.

1 Introduction
Synchronization in coupled dynamical systems, due to the broad range of applications, has been a common ground of investigation for researchers from different disciplines. Most of the work in the area studies the case where the interconnection between individual systems is linear; see, for instance, [17, 9, 11, 16, 4, 7]. Nonlinear coupling is also of interest since certain phenomena cannot be meaningfully modelled by linear coupling. A particular system exemplifying nonlinear coupling that attracted much attention is Kuramoto model and its like [5, 10]. Among more general results allowing nonlinear coupling are [2, 13] where passivity theory is employed to obtain sufficient conditions for synchronization under certain symmetry or balancedness assumptions on the graph describing the interconnection topology.

In this paper we consider the basic equation of the theory of oscillations

\[ \ddot{q} = -\omega^2 q \]

i.e., the harmonic oscillator. One physical example is a unit mass attached to a spring. We interest ourselves with the following question. If we take a number of identical mass-spring systems and couple some pairs by unidirectional nonlinear dampers, will they eventually oscillate synchronously? To be precise, do the
solutions of the following array of coupled harmonic oscillators
\[ \ddot{q}_i = -\omega^2 q_i + \sum_{j \neq i} \gamma_{ij} (\dot{q}_j - \dot{q}_i), \quad i = 1, 2, \ldots, p \]
(where \( \gamma_{ij}(\cdot) \) is either identically zero or it is some function whose graph lies entirely in the first and third quadrants) synchronize? This question may be of direct importance for certain simple mechanical systems or electrical circuits, but our main interest in it is due to the possibility that it may serve as a lucky starting point for understanding a more general scenario.

Regarding the above question, our finding in the paper is roughly that synchronization occurs among mass-spring systems if the springs are stiff enough and there is at least one system that directly or indirectly effects all others. More formally, what we show is that, for a given set of functions \( \{\gamma_{ij}(\cdot)\} \) describing the coupling configuration, if the graph representing the interconnection topology is connected, then the solutions can be made to converge to an arbitrarily small neighborhood of the synchronization manifold, starting from initial conditions arbitrarily far from it by choosing large enough \( \omega \). In technical terms, what we establish is the semiglobal practical asymptotic stability (in \( \omega \)) of the synchronization manifold. Intuition and simulations tell us that global asymptotic synchronization should occur regardless of what \( \omega \) is. This however we have not been able to prove (nor disprove).

We reach our final result in three steps. Note that the solution of an uncoupled harmonic oscillator defines a rotating vector on the plane. Thanks to linearity of the system the speed of this rotation (\( \omega \)) is independent of the initial conditions. As a first step therefore we express the systems with respect to a rotating coordinate system. This change of variables yields coupled systems whose righthand sides are periodic in time (with period \( 2\pi/\omega \)). Our second step is to exploit this periodicity. We obtain the average systems and realize that they belong to a well-studied class of systems pertaining to consensus problems [6]. We then deduce that the solutions of average systems converge to a fixed point on the plane. As our final step we use the result of [14] to conclude that the global asymptotic synchronization of average systems implies the semiglobal practical asymptotic synchronization of coupled harmonic oscillators.

2 Preliminaries

Let \( \mathbb{N} \) denote the set of nonnegative integers and \( \mathbb{R}_{\geq 0} \) the set of nonnegative real numbers. Let \( |\cdot| \) denote Euclidean norm. For \( x = [x_1^T, x_2^T, \ldots, x_p^T]^T \) with \( x_i \in \mathbb{R}^n \) we let \( \mathcal{A} := \{x \in \mathbb{R}^{np} : x_i = x_j \text{ for all } i, j\} \) be synchronization manifold. A function \( \alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \) is said to belong to class-\( K \) (\( \alpha \in \mathcal{K} \)) if it is continuous, zero at zero, and strictly increasing. A function \( \beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \) is said to belong to class-\( KL \) if, for each \( t \geq 0 \), \( \beta(\cdot, t) \) is nondecreasing and \( \lim_{s \to 0^+} \beta(s, t) = 0 \), and, for each \( s \geq 0 \), \( \beta(s, \cdot) \) is nonincreasing and \( \lim_{t \to \infty} \beta(s, t) = 0 \). Given a closed set \( S \subset \mathbb{R}^n \) and a point \( x \in \mathbb{R}^n \), \( |x|_S \) denotes the (Euclidean) distance from \( x \) to \( S \).
A (directed) graph is a pair \((\mathcal{N}, \mathcal{E})\) where \(\mathcal{N}\) is a nonempty finite set (of nodes) and \(\mathcal{E}\) is a finite collection of ordered pairs (edges) \((n_i, n_j)\) with \(n_i, n_j \in \mathcal{N}\). A directed path from \(n_1\) to \(n_\ell\) is a sequence of nodes \((n_1, n_2, \ldots, n_\ell)\) such that \((n_i, n_{i+1})\) is an edge for \(i \in \{1, 2, \ldots, \ell-1\}\). A graph is connected if it has a node to which there exists a directed path from every other node.\(^1\)

A set of functions \(\{\gamma_{ij} : \mathbb{R} \to \mathbb{R}\}\), where \(i, j = 1, 2, \ldots, p\) with \(i \neq j\), describes (is) an interconnection if the following hold for all \(i, j\) and all \(s \in \mathbb{R}\):

(i) \(\gamma_{ij}(0) = 0\) and \(s\gamma_{ij}(s) \geq 0\).

(ii) Either \(\gamma_{ij}(s) \equiv 0\) or there exists \(\alpha \in \mathbb{K}\) such that \(|\gamma_{ij}(s)| \geq \alpha(|s|)\).

To mean \(\gamma_{ij}(s) \equiv 0\) or there exists \(\alpha \in \mathbb{K}\) such that \(|\gamma_{ij}(s)| \geq \alpha(|s|)\).

To give an example, consider a set of functions \(C := \{\gamma_{ij} : i, j = 1, 2, \ldots, 4\}\).

Let \(\gamma_{13}\), \(\gamma_{23}\), \(\gamma_{24}\), \(\gamma_{32}\) be as in Fig. 1 while the remaining functions be zero. Note that each \(\gamma_{ij}\) satisfies conditions (i) and (ii). Therefore set \(C\) describes an interconnection. To determine whether \(C\) is connected or not we examine its graph, see Fig. 2. Since there exists a path to node \(n_4\) from every other node, we deduce that the graph (hence interconnection \(C\)) is connected.

### 3 Problem statement

We consider the following array of coupled harmonic oscillators

\[
\begin{align*}
\dot{q}_i &= \omega p_i, \quad (1a) \\
\dot{p}_i &= -\omega q_i + \sum_{j \neq i} \gamma_{ij} (p_j - p_i), \quad i = 1, 2, \ldots, p \quad (1b)
\end{align*}
\]

where \(\omega > 0\) and \(\{\gamma_{ij}\}\) is a connected interconnection. We assume throughout the paper that \(\gamma_{ij}\) are locally Lipschitz. Let \(\xi_i \in \mathbb{R}^2\) denote the state of \(i\)th oscillator, i.e., \(\xi_i = [q_i, p_i]^T\). When coupling functions \(\gamma_{ij}\) are linear, the oscillators are known to (exponentially) synchronize for all \(\omega\). That is, solutions \(\xi_i(\cdot)\) converge to a common (bounded) trajectory, see [12]. In this paper we investigate the behaviour of oscillators under nonlinear coupling. In particular, we aim to understand the effect of the frequency of oscillations \(\omega\) on synchronization for a given interconnection \(\{\gamma_{ij}\}\).

### 4 Change of coordinates

We define \(S(\omega) \in \mathbb{R}^{2 \times 2}\) and \(H \in \mathbb{R}^{1 \times 2}\) as

\[
S(\omega) := \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}, \quad H := [0 \ 1].
\]

\(^1\)This is another way of saying that the graph contains a spanning tree.
Figure 1: Some examples of coupling functions.

Then we rewrite (1) as

\[ \dot{\xi}_i = S(\omega)\xi_i + H^T \sum_{j \neq i} \gamma_{ij}(H(\xi_j - \xi_i)) . \]  

(2)

Let us recall the geometric meanings of the terms in (2). The first term \( S(\omega)\xi_i \) defines a rotation (with period \( 2\pi/\omega \)) since \( S(\omega) \) is a skew-symmetric matrix. The second term \( H^T \sum_{j \neq i} \gamma_{ij}(H(\xi_j - \xi_i)) \) induces a contraction in the direction specified by vector \( H^T \) (so to speak, along the vertical axis) such that the projections of states \( \xi_i \) on the vertical axis tend to approach each other. The combined effect of these two terms is relatively more difficult to visualize. One trick to partially overcome this difficulty is to look at the system from the point of view of the observer that sits on a rotating frame of reference.

Under change of coordinates \( x_i(t) := e^{-S(\omega)t} \xi_i(t) \) we can by (2) write

\[
\dot{x}_i = e^{S(\omega)^T t} H^T \sum_{j \neq i} \gamma_{ij}(He^{S(\omega)t}(x_j - x_i)) \\
= \left[ -\sin \omega t \cos \omega t \right] \sum_{j \neq i} \gamma_{ij} \left[ -\sin \omega t \cos \omega t \right] (x_j - x_i).
\]

(3)

Since the change of coordinates is realized via rotation matrix \( e^{-S(\omega)t} \), the
relative distances are preserved, that is \(|x_i(t)−x_j(t)| = |ξ_i(t)−ξ_j(t)|\) for all \(t\) and all \(i, j\). This means from the synchronization point of view that the behaviour of array (2) will be inherited by array (3). However, exact analysis of (3) seems still far from yielding. Therefore we attempt to understand this system via its approximation.

5 Average systems

Observe that the righthand side of (3) is periodic in time. Time average functions \(\bar{\gamma}_{ij} : \mathbb{R}^2 \to \mathbb{R}^2\) are given by

\[
\bar{\gamma}_{ij}(x) := \frac{1}{2\pi} \int_0^{2\pi} \left[ -\sin \varphi \cos \varphi \right] \gamma_{ij}([-\sin \varphi \cos \varphi]x) d\varphi.
\]

Then the \textit{average} array dynamics read

\[
\dot{\eta}_i = \sum_{j \neq i} \bar{\gamma}_{ij}(\eta_j − \eta_i).
\] (4)

Theory of perturbations [3, Ch. 4 § 17] tells us that, starting from close initial conditions, the solution of a system with a periodic righthand side and the solution of the time-average approximate system stay close for a long time provided that the period is small enough. Therefore (4) should tell us a great deal about the behaviour of (3) when \(\omega \gg 1\).

Understanding (4) requires understanding average function \(\bar{\gamma}_{ij}\). The following lemma is helpful from that respect.

\textbf{Lemma 1} We have \(\bar{\gamma}_{ij}(x) = \rho_{ij}(|x|) \frac{x}{|x|}\) where \(\rho_{ij} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}\) is

\[
\rho_{ij}(r) := \frac{1}{2\pi} \int_0^{2\pi} \gamma_{ij}(r \sin \varphi) \sin \varphi d\varphi.
\]

\textbf{Proof.} Given \(x \in \mathbb{R}^2\), let \(r = |x|\) and \(\theta \in [0, 2\pi)\) be such that \(r[−\cos \theta \sin \theta]^T =\)
We focus on the second term in (5). Observe that
\[
\int_{0}^{\pi} \gamma_{ij}(r \sin \varphi) \cos \varphi \, d\varphi = \int_{-\pi/2}^{\pi/2} \gamma_{ij}(r \sin(\varphi + \frac{\pi}{2})) \cos(\varphi + \frac{\pi}{2}) \, d\varphi = 0
\]
since the integrand is an odd function on the interval of integration. Likewise,
\[
\int_{-\pi}^{\pi} \gamma_{ij}(r \sin \varphi) \cos \varphi \, d\varphi = \int_{-\pi/2}^{\pi/2} \gamma_{ij}(r \sin(\varphi + \frac{3\pi}{2})) \cos(\varphi + \frac{3\pi}{2}) \, d\varphi = 0.
\]
Therefore
\[
\int_{0}^{2\pi} \gamma_{ij}(r \sin \varphi) \cos \varphi \, d\varphi
= \int_{0}^{\pi} \gamma_{ij}(r \sin \varphi) \cos \varphi \, d\varphi + \int_{\pi}^{2\pi} \gamma_{ij}(r \sin \varphi) \cos \varphi \, d\varphi
= 0.
\]
Combining (5) and (6) we obtain
\[
\bar{\gamma}_{ij}(x) = \left( \frac{1}{2\pi} \int_{0}^{2\pi} \gamma_{ij}(r \sin \varphi) \, d\varphi \right) \left[ -\cos \theta \right] \sin \theta
= \rho_{ij}(|x|) \frac{x}{|x|}.
\]
Hence the result.
Lemma 1 tells us that, given any vector \( x \) on the plane, vector \( \bar{\gamma}_{ij}(x) \) (if it has nonzero magnitude) is in the same direction as \( x \). In the light of this if we now look at systems (4) we can roughly visualize the evolution of the trajectories. Take the \( i \)th system. Let \( j_1, \ldots, j_\ell \) be all the indices such that \( j \in \{j_1, \ldots, j_\ell\} \) implies \( \gamma_{ij} \neq 0 \). These indices are sometimes called the indices of neighbors of system \( i \). Then we can write \( \dot{\eta}_i = v_1 + \ldots + v_\ell \) where each \( v_k \) is a vector pointing from \( \eta_i \) to the state of \( k \)th neighbor system. Therefore the net velocity vector \( \sum v_k \) points to some “weighted mean” of the neighbor systems’ states. Synchronization of such systems, where the velocity vector of a system always points to some weighted mean of the positions of its neighbors, have been studied under the names consensus and state agreement; see, for instance, [8, 1, 6]. The finding of those works is roughly that if the interconnection is connected, then the solutions of systems converge to a common fixed point in space. We now give the formal application of this result to our case.

Let us stack the individual states \( \eta_i \) to form \( \eta := [\eta_1^T \eta_2^T \ldots \eta_p^T]^T \). Define

\[
\gamma_{av}(\eta) := \begin{bmatrix}
\sum \bar{\gamma}_{1j}(\eta_j - \eta_1) \\
\vdots \\
\sum \bar{\gamma}_{pj}(\eta_j - \eta_p)
\end{bmatrix}
\]

We then reexpress (4) as

\[
\dot{\eta} = \gamma_{av}(\eta) .
\]

We now have the following result.

**Theorem 1** Consider system (7). Synchronization manifold \( \mathcal{A} \) is globally asymptotically stable, i.e., there exists \( \beta \in KL \) such that \( \|\eta(t)\|_{\mathcal{A}} \leq \beta(\|\eta(0)\|_{\mathcal{A}}, t) \) for all \( t \geq 0 \).

**Proof.** By Lemma 1 we can write

\[
\dot{\eta}_i = \sum_{j \neq i} \rho_{ij}(|\eta_j - \eta_i|) \frac{\eta_j - \eta_i}{|\eta_j - \eta_i|} =: f_i(\eta)
\]

for \( i = 1, 2, \ldots, p \). Let \( \mathcal{G} \) denote the graph of interconnection \( \{\gamma_{ij}\} \). Note that \( \mathcal{G} \) is connected by assumption. We make the following simple observations. Function \( \rho_{ij} \) is continuous and zero at zero. If there is no edge of \( \mathcal{G} \) from node \( i \) to node \( j \) then \( \rho_{ij}(r) \equiv 0 \). If there is an edge from node \( i \) to node \( j \) then \( \sigma_{ij}(r) > 0 \) for \( r > 0 \).

Therefore \( f_i \) is continuous; and vector \( f_i(\eta) \) always points to the (relative) interior of the convex hull of the set \( \{\eta_i\} \cup \{\eta_j : \text{there is an edge of } \mathcal{G} \text{ from node } i \text{ to node } j\} \). These two conditions together with connectedness of \( \mathcal{G} \) yield by [6, Corollary 3.9] that system (7) has the globally asymptotic state agreement property, see [6, Definition 3.4]. Another property of the system is invariance with respect to translations. That is, \( \gamma_{av}(\eta + \xi) = \gamma_{av}(\eta) \) for \( \xi \in \mathcal{A} \). These properties let us write the following.
(a) There exists a class-$\mathcal{K}$ function $\alpha$ such that $|\eta(t)|_A \leq \alpha(|\eta(0)|_A)$ for all $t \geq 0$.

(b) For each $r > 0$ and $\varepsilon > 0$, there exists $T > 0$ such that $|\eta(0)|_A \leq r$ implies $|\eta(t)|_A \leq \varepsilon$ for all $t \geq T$.

Finally, (a) and (b) give us the result by [15, Proposition 1].

Let us now go back to our discussion in the beginning of Section 4. There we talked about two actions that shape the dynamics of systems (2), namely, rotation and vertical contraction. The combined effect of those actions on synchronization of the systems was not initially apparent. However, by applying first a change of coordinates (3) and then averaging (4) we see that it is likely that two actions will result in synchronization, at least when the rotation is rapid enough. Vaguely speaking, rotation rescues contraction from being confined only to vertical direction and sort of "smears" it uniformly to all directions, which should bring synchronization. In the next section we formalize our observation.

6 Semiglobal practical asymptotic synchronization

Consider systems (3). Stack states $x_i$ to form $x := [x_1^T \ x_2^T \ \ldots \ x_p^T]^T$. Define

$$
\gamma(x, \omega t) := \begin{bmatrix}
-\sin \omega t \\
\cos \omega t \\
-\sin \omega t \\
\cos \omega t \\
\vdots \\
-\sin \omega t \\
\cos \omega t
\end{bmatrix} \sum_{j=1}^{p} \gamma_{ij} \left([-\sin \omega t \ \cos \omega t] (x_j - x_1)\right) \\
\sum_{j=1}^{p} \gamma_{pj} \left([-\sin \omega t \ \cos \omega t] (x_j - x_p)\right)
$$

Now reexpress (3) as

$$\dot{x} = \gamma(x, \omega t). \tag{8}$$

The following definition is borrowed with slight modification from [14].

**Definition 1** Consider system $\dot{x} = f(x, \omega t)$. Closed set $S$ is said to be semiglobally practically asymptotically stable if for each pair $(\Delta, \delta)$ of positive numbers, there exists $\omega^* > 0$ such that for each $\omega \geq \omega^*$ the following hold.

(a) For each $r > \delta$ there exists $\varepsilon > 0$ such that

$$|x(0)|_S \leq \varepsilon \implies |x(t)|_S \leq r \ \forall t \geq 0.$$  

(b) For each $\varepsilon < \Delta$ there exists $r > 0$ such that

$$|x(0)|_S \leq \varepsilon \implies |x(t)|_S \leq r \ \forall t \geq 0.$$
(c) For each $r < \Delta$ and $\varepsilon > \delta$ there exists $T > 0$ such that

$$|x(0)|_S \leq r \implies |x(t)|_S \leq \varepsilon \quad \forall t \geq T.$$ 

Below we establish the semiglobal practical asymptotic stability of synchronization manifold. To do that we use [14, Thm. 2], which says that the origin of system $\dot{x} = f(x, \omega t)$ (where $f$ is periodic in time) is semiglobally practically asymptotically stable if the origin of $\dot{x} = f_{av}(x)$ (where $f_{av}$ is the time average of $f$) is globally asymptotically stable.

**Theorem 2** Consider system (8). Synchronization manifold $A$ is semiglobally practically asymptotically stable.

**Proof.** Consider systems (3). Observe that the righthand side depends only on the relative distances $x_j - x_i$. Let us define

$$Y := \begin{bmatrix} x_2 - x_1 \\ x_3 - x_1 \\ \vdots \\ x_p - x_1 \end{bmatrix}$$

Note that $\dot{y} = f(y, \omega t)$ for some $f : \mathbb{R}^{2p-2} \times \mathbb{R}_{\geq 0} \to \mathbb{R}^{2p-2}$. Since functions $\gamma_{ij}$ are assumed to be locally Lipschitz, $f$ is locally Lipschitz in $y$ uniformly in $t$. Also, $f$ is periodic in time by (3). Now consider systems (4). Again the righthand side depends only on the relative distances $\eta_j - \eta_i$. Define

$$Z := \begin{bmatrix} \eta_2 - \eta_1 \\ \eta_3 - \eta_1 \\ \vdots \\ \eta_p - \eta_1 \end{bmatrix}$$

Then $\dot{z} = f_{av}(z)$ where $f_{av}$ is the time average of $f$ and locally Lipschitz both due to that $\bar{\gamma}_{ij}$ is the time average of $\gamma_{ij}$.

Theorem [14] implies that the origin of $\dot{z} = f_{av}(z)$ is globally asymptotically stable. Then [14, Thm. 2] tells us that the origin of $\dot{y} = f(y, \omega t)$ is semiglobally practically asymptotically stable. All there is left to complete the proof is to realize that semiglobal practical asymptotic stability of the origin of $\dot{y} = f(y, \omega t)$ is equivalent to semiglobal practical asymptotic stability of synchronization manifold $A$ of system (8). 

Theorem [2] can be recast into the following form.

**Theorem 3** Consider coupled harmonic oscillators [11]. For each pair $(\Delta, \delta)$ of positive numbers, there exists $\omega^* > 0$ such that for each $\omega \geq \omega^*$ the following hold.
(a) For each $r > \delta$ there exists $\varepsilon > 0$ such that
\[
\max_{i,j} |\xi_i(0) - \xi_j(0)| \leq \varepsilon \implies \max_{i,j} |\xi_i(t) - \xi_j(t)| \leq r \quad \forall t \geq 0.
\]

(b) For each $\varepsilon < \Delta$ there exists $r > 0$ such that
\[
\max_{i,j} |\xi_i(0) - \xi_j(0)| \leq \varepsilon \implies \max_{i,j} |\xi_i(t) - \xi_j(t)| \leq r \quad \forall t \geq 0.
\]

(c) For each $r < \Delta$ and $\varepsilon > \delta$ there exists $T > 0$ such that
\[
\max_{i,j} |\xi_i(0) - \xi_j(0)| \leq r \implies \max_{i,j} |\xi_i(t) - \xi_j(t)| \leq \varepsilon \quad \forall t \geq T.
\]

7 Conclusion

For nonlinearly coupled harmonic oscillators we have shown that synchronization manifold is semiglobally practically asymptotically stable in the frequency of oscillations. Our assumption on each coupling function is that it is locally Lipschitz and, if nonzero, its graph lies in the first and third quadrants and does not get arbitrarily close to the horizontal axis when far from the origin. Our assumption on the interconnection graph is the minimum; that is, it is connected.

One last remark we want to make is the following. If we look at (2) we realize that $H = [0 \ 1]$ is not necessary for the rest of the analysis. In fact any nonzero $H \in \mathbb{R}^{1 \times 2}$ is no worse than $[0 \ 1]$. For instance, for $H = [1 \ 0]$ coupled harmonic oscillators would be represented by
\[
\dot{q}_i = \omega p_i + \sum_{j \neq i} \gamma_{ij} (q_j - q_i)
\]
\[
\dot{p}_i = -\omega q_i
\]
for which Theorem 3 is valid.

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