Remarks on the Cauchy functional equation and variations of it

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The Cauchy functional equation
\[ f(x + y) = f(x) + f(y), \]

Fundamental in the theory of functional equations
Has many applications in various scientific domains
Has been extensively investigated by many people during the last 200 years (some are in the audience)

Remarks on the Cauchy equation
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Well-known background

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Background: Domain and range of \( f \) and the variables

Positive orthant

Subsets of \( \mathbb{R}^2 \)

Locally compact Polish groups

Banach spaces
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- More
Background: Regularity conditions on $f$
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Common consequence of regularity conditions

Frequently, if \( f \) satisfies a regularity condition, then \( f \) is linear.
Common consequence of regularity conditions

Phenomenon

**Frequently,** if $f$ satisfies a regularity condition, then $f$ is linear.
Stability

Definition

$\epsilon > 0$ is given

A function $f$ satisfies the following inequality

$$|f(x + y) - f(x) - f(y)| \leq \epsilon$$

is called an $\epsilon$-additive function.

Phenomenon

In various settings an $\epsilon$-additive function is an $\epsilon$-perturbation of a (unique) pure additive function $g$:

$$|f(x) - g(x)| \leq \epsilon, \forall x.$$
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\[ |f(x) - g(x)| \leq \epsilon, \quad \forall x. \]
Results: schematic description

Solvability and stability of the Cauchy equation relative to subsets of multi-dimensional Euclidean spaces and tori. New regularity conditions, e.g., $e^{if}$ is locally measurable.

New approach: initial value approach. The analysis is extended to related equations, e.g.,:
- Cauchy's equation on restricted domains
- The multiplicative Cauchy equation,
- The Jensen equation,
- The Pexider equation.

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Theorem

If $f : \mathbb{R}^n \to \mathbb{R}$ satisfies the Cauchy equation and $e$ is locally measurable, i.e., measurable on a hypercube, then there exists some $c \in \mathbb{R}^n$ such that $f(x) = c \cdot x$ for all $x \in \mathbb{R}^n$.

Remark

This regularity condition is strictly weaker than merely measurability of $f$.
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If \( f : \mathbb{R}^n \to \mathbb{R} \) satisfies the Cauchy equation and \( e^{if} \) is locally measurable, i.e., measurable on a hypercube, then there exists some \( c \in \mathbb{R}^n \) such that \( f(x) = c \cdot x \) for all \( x \in \mathbb{R}^n \).

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Proof: Approach 1

Theorems about automatic continuity from measurability and the description of measurable characters

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Remarks on the Cauchy equation

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1 Theorems about automatic continuity from measurability and the description of measurable characters
Proof 2: Initial value approach

Let $c \in \mathbb{R}^n$ be the unique solution to

$$c \cdot u_k = f(u_k), \quad k = 1, \ldots, n.$$ 

One solution to Cauchy's equation is $g_1 = f$, another is $g_2(x) = c \cdot x$.

Because Cauchy's equation is linear, to show that $f$ is linear it suffices to show that if $g$ is additive and $U$-periodic, i.e.,

$$g(x+y) = g(x) + g(y), \quad \forall x, y \in \mathbb{R}^n,$$

then $g(u_k) = 0, \quad k = 1, \ldots, n$,

then $g \equiv 0$.

Can be done by elementary but a bit technical arguments.
Proof 2: Initial value approach

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Corollaries of the initial value approach

Corollary

There exists \( 2^{\text{card}(\mathbb{R})} \) non-constant functions \( f : \mathbb{R} \to \mathbb{R} \), each of them has \( \text{card}(\mathbb{R}) \) linearly independent (over \( \mathbb{Q} \)) periods.

Corollary

If \( f : \mathbb{R}^n \to \mathbb{R} \) solves the Cauchy equation and satisfies an abstract regularity condition which significantly generalizes the condition that \( e^{ift} \) is locally measurable, then \( f \) is linear.

The regularity condition is related to a complex mean.
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Theorem

Given an additive homomorphism $f$ from a finite dimensional topological torus to $\mathbb{R}$, if $e$ is (Haar) measurable, then $f \equiv 0$. 

Remarks on the Cauchy equation

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Definition \((G, +)\) is a group, \(S \subseteq G\). \(S\) strongly generates \(G\) if for all \(x_1, x_2 \in G\) there exist \(s_1, s_2, t_1, t_2 \in S\) such that \(x_i = s_i - t_i\), \(i = 1, 2\) and such that \(s_1 + s_2 \in S, t_1 + t_2 \in S\).
Definition

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Examples of \( S \subseteq \mathbb{R}^n \) which strongly generate \( \mathbb{R}^n \) and containing hypercubes:

orthants (with or without the origin), halfspaces,

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- orthants (with or without the origin),
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- $S = \bigcup_{m=1}^{\infty} [10^m, 5 \cdot 10^m)^n$. 
Theorem

Suppose that $S \subseteq \mathbb{R}^n$ strongly generates $\mathbb{R}^n$ or $S$ convex with nonempty interior. Let $A \subseteq \mathbb{R}^n$ satisfy $S \cup (S + S) \subseteq A$. Let $f : A \to \mathbb{R}$ and assume that $f$ satisfies

$$f(x + y) = f(x) + f(y), \quad \forall (x, y) \in S^2.$$ 

If $S$ contains a hypercube $I$ on which $f$ is measurable, then there exists $c \in \mathbb{R}^n$ such that $f(x) = c \cdot x$ for each $x \in S$. 

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If \( S \) contains a hypercube \( I \) on which \( e^{if} \) is measurable, then there exists \( c \in \mathbb{R}^n \) such that \( f(x) = c \cdot x \) for each \( x \in S \).
Theorem

Let $S \subseteq \mathbb{R}^n$. Let $A \subseteq \mathbb{R}^n$ satisfy $S \cup (S + S) \subseteq A$. Suppose that $f : A \to \mathbb{R}$ is a positive function satisfying $f(x + y) = f(x)f(y) \forall (x, y) \in S^2$.

Assume also that either $S$ is a convex subset having a nonempty interior and $f$ is measurable on $S$, or $S$ strongly generates $\mathbb{R}^n$ and it contains a hypercube $I$ on which $f$ is measurable.

Then there exists $c \in \mathbb{R}^n$ such that $f(x) = e^{c \cdot x}$ for each $x \in S$. 

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Assume also that either

- $S$ is a convex subset having a nonempty interior and $f^i$ is measurable on $S$, or
- $S$ strongly generates $\mathbb{R}^n$ and it contains a hypercube $I$ on which $f^i$ is measurable.
Results: the multiplicative Cauchy’s equation

**Theorem**

Let $S \subseteq \mathbb{R}^n$. Let $A \subseteq \mathbb{R}^n$ satisfy $S \cup (S + S) \subseteq A$. Suppose that $f : A \to \mathbb{R}$ is a positive function satisfying

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Let $S$ be a convex subset of $\mathbb{R}^n$ and assume that its interior is nonempty. Suppose that $f : S \to \mathbb{R}$ satisfies
\[ f(x + y) = f(x) + f(y) \]
for all $(x, y) \in S^2$. If $f$ is measurable on $S$, then there exist $c \in \mathbb{R}^n$ and $b \in \mathbb{R}$ such that $f(x) = c \cdot x + b$ for each $x \in S$. 
Theorem

Let $S$ be a convex subset of $\mathbb{R}^n$ and assume that its interior is nonempty. Suppose that $f : S \to \mathbb{R}$ satisfies

$$f \left( \frac{x + y}{2} \right) = \frac{f(x) + f(y)}{2}$$

for all $(x, y) \in S^2$. 
Theorem

Let $S$ be a convex subset of $\mathbb{R}^n$ and assume that its interior is nonempty. Suppose that $f : S \rightarrow \mathbb{R}$ satisfies

$$f \left( \frac{x + y}{2} \right) = \frac{f(x) + f(y)}{2}$$

for all $(x, y) \in S^2$. If $e^{if}$ is measurable on $S$, then there exist $c \in \mathbb{R}^n$ and $b \in \mathbb{R}$ such that $f(x) = c \cdot x + b$ for each $x \in S$. 
Theorem

Let $S \subseteq \mathbb{R}^n$ be a semigroup satisfying $0 \in S$. Assume that $f: S \to \mathbb{R}$, $g: S \to \mathbb{R}$, $h: S \to \mathbb{R}$ satisfy

$$f(x + y) = g(x) + h(y), \quad \forall (x, y) \in S^2.$$

Suppose that $S$ generates $\mathbb{R}^n$ and it contains a hypercube $I$ on which a complex exponent of one of the given functions is measurable. Then there exist $c \in \mathbb{R}^n$ and constants $a, b \in \mathbb{R}$ such that

$$f(x) = c \cdot x + a,$$

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Theorem

Let $S \subseteq \mathbb{R}^n$ be a semigroup which satisfies $\mathbb{R}^n = S - S$ and containing a hypercube $I$. Suppose that $f : S \to \mathbb{R}$ is an $\epsilon$-additive function, and $x \mapsto e^{(mx)/m}$ is Lebesgue measurable on $I$ for infinitely many positive integers $m$.

Then there exists $c \in \mathbb{R}^n$ such that $|f(x) - c \cdot x| \leq \epsilon$, $\forall x \in S$. 
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Let $S \subseteq \mathbb{R}^n$ be a semigroup which satisfies $\mathbb{R}^n = S - S$ and containing a hypercube $I$. Suppose that

- $f : S \rightarrow \mathbb{R}$ is an $\varepsilon$-additive function, and
- $x^{m} / m$ is Lebesgue measurable on $I$ for infinitely many positive integers $m$. 

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The End
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The slideshow and the paper can be found online
Appendix: abstract regularity condition

A is a set of real functions defined on $\mathbb{R}^n$. $B$ is a set of complex functions containing $\{e^{ig} : g \in A\}$. There exist a functional $F : B \rightarrow \mathbb{C}$ such that several conditions hold. Under the above assumptions, if $f \in A$ and it satisfies the Cauchy equation, then $f(x) = c \cdot x$ for some $c \in \mathbb{R}^n$. 
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Appendix: abstract regularity condition

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Appendix (Cont.): the conditions on $A, B, F$

1. For all $\beta \in C$, $|\beta| = 1$ and $h \in B$ we have $\beta h \in B$ and $F(\beta h) = \beta F(h)$.

2. $A$ contains all the affine functions from $\mathbb{R}^n$ to $\mathbb{R}$.

3. $A$ is closed under addition and under multiplication by positive rationals.

4. There exists a basis $\{u_1, \ldots, u_n\}$ in $\mathbb{R}^n$ such that for all $U$-periodic $g \in A$ (i.e., satisfying the relation $g(x + u_k) = g(x)$ for all $x \in \mathbb{R}^n$ and $k \in \{1, \ldots, n\}$), the functions $gy(x) := g(x + y)$, $x \in \mathbb{R}^n$, are in $A$ for each $y \in \mathbb{R}^n$, and $F(e^{igy}) = F(e^{igy})$.

5. For each $g \in A$ there exists $\alpha > 0$ rational such that $F(e^{i\alpha g}) \neq 0$. 

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Example

\[ A := \{ f : \mathbb{R}^n \to \mathbb{R} : \text{if is measurable} \} \]

\[ B = \{ e \in g : g \in A \} \]

\[ F(\mathbf{u}) = \int_I u(x) \, dx \]  
(Condition (6) is not immediate)

Problem

Any "essentially different" examples of \((A, B, F)\)?

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Any example of \((A, B, F)\) in an infinite dimensional setting?

If yes, then this generalizes many of the previous results to such a setting.
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