ON THE COHERENCE CONDITIONS FOR
PSEUDO-DISTRIBUTIVE LAWS

NICOLA GAMBINO

Abstract. We survey the development of the formal theory of pseudo-monads, the analogue for pseudo-monads of the formal theory of monads. One of the main achievements of the theory is a satisfactory axiomatisation of the notion of a pseudo-distributive law between pseudo-monads.

1. Towards a formal theory of pseudo-monads

The formal theory of monads, originally introduced by Street in [22] and developed further by Lack and Street [17], provides a mathematically efficient treatment of several aspects of the theory of monads [1]. For example, it exhibits a universal property of the category of algebras for a monad and provides a clear explanation for Beck’s axioms for a distributive law [2]. Over the past few years, there has been substantial progress in the development of a formal theory of pseudo-monads [4, 15, 19, 20, 21, 22, 23, 24], with applications to pure mathematics [7, 8] and theoretical computer science [3, 5, 25]. Our aim here is to give a survey this development, both to facilitate further applications and to provide a reference for future work. We shall focus our attention on the formulation of coherence conditions, which has proved to be one of the most delicate aspects of the theory.

Just as the formal theory of monads is developed within two-dimensional category theory [14], the formal theory of pseudo-monads is developed within three-dimensional category theory [10]. Within this setting, it is convenient to work with Gray-categories, which are semistrict tricategories [10, Section 4.8]. Working with Gray-categories is easier than working with general tricategories, but does not lead to an essential loss of generality, since every tricategory is triequivalent to a Gray-category [10, Theorem 8.1].

The starting point of the formal theory of pseudo-monads is the definition, for a Gray-category $K$, of the Gray-category $\text{Psm}_K$ of pseudo-monads in $K$. As we will see, this is done following a different approach to the one taken in the formal theory of monads to define the 2-category of monads in a 2-category. The change of approach allows one to avoid building into the definition of $\text{Psm}_K$ the notions of a pseudo-monad morphism, pseudo-monad transformation, and pseudo-monad modification, which involve complex coherence conditions. These notions can be introduced at a later stage and then shown to lead to a tricategory that is triequivalent to $\text{Psm}_K$. 

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A byproduct of the formal theory of pseudo-monads that is of particular interest for applications is the definition of a satisfactory notion of pseudo-distributive law between pseudo-monads. The notion of a pseudo-distributive law between pseudo-monads should be understood as an analogue of the classical notion of a distributive law between monads introduced by Beck [2]. For this notion, the four diagrams that are required to commute in the definition of a distributive law are replaced by diagrams commuting up to invertible 3-cells, which are then required to satisfy appropriate coherence conditions. Extending the set of coherence conditions for semistrict pseudo-distributive laws between 2-monads introduced by Kelly [12], Marmolejo identified a set of nine coherence conditions for pseudo-distributive laws [20]. Later, Marmolejo and Wood [21] showed not only that an additional tenth coherence condition, introduced by Tanaka [23], can be derived from Marmolejo’s conditions, but also that one of the nine conditions originally introduced by Marmolejo is derivable from the others, thereby reducing the axiomatization of the coherence conditions for a pseudo-distributive law to eight axioms.

The first main contribution of the survey is to state precisely all the coherence conditions for pseudo-monads, pseudo-monad morphisms, pseudo-monad transformations, and pseudo-monad modification, and to prove that they give rise to a tricategory that is equivalent to the Gray-category $\mathbf{Psm}_{\mathbb{C}}$. The second main contribution is to give clearly all the coherence conditions for pseudo-distributive laws: the eight core conditions that are part of the definition, the ninth, derivable, condition originally introduced in [20] and the tenth, derivable, condition stated in [23]. We also provide an interpretation of these conditions in terms of the notions of pseudo-monad morphisms, pseudo-monad transformation, and pseudo-monad modification. As direct consequence of this interpretation, we obtain an analogue of Beck’s fundamental theorem relating distributive laws and liftings to categories of Eilenberg-Moore algebras [2].

We conclude these introductory remarks by recalling from [4] that the development of the formal theory of pseudo-monads in the Gray-categorical setting does not immediately account for the Kleisli construction. Indeed, the Kleisli construction for a pseudo-monad in the Gray-category $\mathbf{2-Cat}_{\mathbb{O}}$ of 2-categories, 2-functors, pseudo-natural transformations, and modifications, produces a bicategory, not a 2-category, thus leading outside $\mathbf{2-Cat}_{\mathbb{O}}$. It seems therefore that a tricategorical version of the formal theory of monads, based on the existing Gray-categorical theory, will eventually be needed.

2. Gray-categories

We begin by reviewing the notion of a Gray-category. Let us start with some preliminaries. We write $\mathbf{2-Cat}$ for the category of 2-categories and 2-functors. For 2-categories $X$ and $Y$, let $[X,Y]$ be the 2-category of 2-functors from $X$ to $Y$, pseudo-natural transformations, and modifications [14]. This definition equips the category $\mathbf{2-Cat}$ with the structure of a closed category [6]. The closed structure of $\mathbf{2-Cat}$ is part of symmetric monoidal structure, the tensor product of which is known as the Gray tensor product [10] Section 4.8. We will write $X \otimes Y$ for the Gray tensor product of 2-categories $X$ and $Y$. We
write $\mathbf{2\text{-Cat}}_\otimes$ to emphasise that we consider $\mathbf{2\text{-Cat}}$ as being equipped with the closed symmetric monoidal structure given by the Gray tensor product.

By definition, a Gray-category is a $\mathbf{2\text{-Cat}}_\otimes$-enriched category [10, Section 5.1]. For a Gray-category $\mathcal{K}$, we write $\mathcal{K}$ also for its set of objects. Given $X, Y \in \mathcal{K}$, we write $\mathcal{K}(X, Y)$ for the hom-2-category of maps from $X$ to $Y$. We refer to the objects of $\mathcal{K}$ also as 0-cells, and to the $n$-cells of the hom-2-categories of $\mathcal{K}$ as the $n+1$-cells of $\mathcal{K}$. Following this idea, every Gray-category $\mathcal{K}$ can be viewed as a tricategory [10, Proposition 3.1]. Gray-categories are rather special tricategories, in that their only non-strict operation is horizontal composition of 2-cells [10, Section 5.2]. For an example of a Gray-category, recall that $\mathbf{2\text{-Cat}}_\otimes$ is a monoidal closed category and so it is enriched over itself. Therefore, it can be viewed as a Gray-category, as we will do from now on. More explicitly, $\mathbf{2\text{-Cat}}_\otimes$ is the Gray-category having 2-categories as 0-cells, 2-functors as 1-cells, pseudo-natural transformations as 2-cells, and modifications as 3-cells.

Let us briefly describe what the non-strictness of the horizontal composition of 2-cells in a Gray-category amounts to. Let $\mathcal{K}$ be a Gray-category. Let us consider 0-cells $I, X, Y$, 1-cells $A, B : I \to X$ and $H, K : X \to Y$. The non-strictness of $\mathcal{K}$ means that for every pair of 2-cells $f : A \to B$ and $p : H \to K$, we have invertible 3-cells

$$
\begin{array}{ccc}
HA & \xrightarrow{Hf} & HB \\
\downarrow^{pA} & \cong & \downarrow^{pB} \\
KA & \xrightarrow{Kf} & KB
\end{array}
$$

If, as usual in category theory [13, Section V.5], we think of 1-cells $A, B : I \to X$ as generalised elements of $X$, then these 3-cells are analogous to the 2-cells that are part of the structure of a pseudo-natural transformation, and indeed they satisfy very similar coherence conditions [20, Section 2]. In the following, these coherence conditions will often be used implicitly.

The notions of a Gray-functor and a Gray-natural transformation are instances of the general notions of enriched functor and enriched natural transformation [13, Section 1.2]. We will use the terminology of Gray-modification and Gray-perturbation to denote the strict counterparts of the corresponding tricategorical notions [10, Section 3.3]. When working with the Yoneda embedding for Gray-categories, which is just an instance of the Yoneda embedding for enriched categories [13, Section 2.4], we often identify an object of $\mathcal{K}$ with the representable Gray-functor associated to it. Analogous conventions will be used also for the $n$-cells of $\mathcal{K}$, where $n = 1, 2, 3$.

For further information on Gray-categories and tricategories, we invite the reader to refer to [9, 10, 11, 16].

3. The Gray-category of pseudo-monads

From now on, unless otherwise specified, we work with a fixed Gray-category $\mathcal{K}$.
Definition 3.1. A pseudo-monad \((X, S)\) in \(K\) consists of a 0-cell \(X\), a 1-cell \(S : X \to X\), 2-cells \(u : 1_X \to S, m : S^2 \to S\), and invertible 3-cells

\[
\begin{array}{cccc}
S^3 & \xrightarrow{\alpha} & S^2 & \\
\downarrow m & & \downarrow m & \\
S^2 & \xrightarrow{\lambda} & S & \\
\end{array}
\]

satisfying the coherence axioms in (1) and (2) below:

\[
\begin{array}{cccc}
S^4 & \xrightarrow{S^2m} & S^3 & \\
\downarrow mS^2 & & \downarrow S\alpha & \\
S^3 & \xrightarrow{S\alpha} & S^2 & \\
\downarrow mS & & \downarrow \alpha & \\
S^2 & \xrightarrow{\alpha} & S & \\
\end{array}
= \begin{array}{cccc}
S^4 & \xrightarrow{S^2m} & S^3 & \\
\downarrow mS^2 & & \downarrow S\alpha & \\
S^3 & \xrightarrow{S\alpha} & S^2 & \\
\downarrow mS & & \downarrow \alpha & \\
S^2 & \xrightarrow{\alpha} & S & \\
\end{array}
\]

Note that the notion of a pseudo-monad is self-dual, in the sense that a pseudo-monad in \(K\) is the same thing as a pseudo-monad in \(K^{op}\), where \(K^{op}\) is the Gray-category obtained from \(K\) by reversing the direction of the 1-cells, but not that of the 2-cells and 3-cells.

Proposition 3.2 (Marmolejo). Let \((X, S)\) be a pseudo-monad in \(K\). The following coherence conditions

\[
\begin{array}{cccc}
S^2 & \xrightarrow{m} & S & \\
\downarrow uS^2 & & \downarrow uS & \\
S^3 & \xrightarrow{Sm} & S^2 & \\
\downarrow mS & & \downarrow \alpha & \\
S^2 & \xrightarrow{\lambda} & S & \\
\end{array}
= \begin{array}{cccc}
S^2 & \xrightarrow{m} & S & \\
\downarrow uS^2 & & \downarrow uS & \\
S^3 & \xrightarrow{Sm} & S^2 & \\
\downarrow mS & & \downarrow \alpha & \\
S^2 & \xrightarrow{\lambda} & S & \\
\end{array}
\]
are derivable.

Proof. See [19, Proposition 8.1].

Let \((X, S)\) be a pseudo-monad in \(\mathcal{K}\). Let us recall the definition of the 2-category \(\text{Ps-}S\text{-Alg}(I)\) of \(I\)-indexed pseudo-\(S\)-algebras, pseudo-algebra morphisms, and pseudo-algebra 2-cells, where \(I \in \mathcal{K}\). An \(I\)-indexed pseudo-\(S\)-algebra consists of a 1-cell \(A : I \to X\), called the underlying 1-cell of the pseudo-algebra, a 2-cell \(a : SA \to A\), called the structure map of the pseudo-algebra, and invertible 3-cells

\[
\begin{array}{ccc}
S^2 A & \xrightarrow{S a} & SA \\
m_A & \Downarrow \tilde{a} & a \\
SA & \xrightarrow{\tilde{a}} & A \\
\end{array}
\]

\[
\begin{array}{ccc}
A & \xrightarrow{u_A} & SA \\
\tilde{a} & \Downarrow 1_A & a \\
A & \xrightarrow{1_A} & A
\end{array}
\]

called the associativity and unit of the pseudo-algebra, satisfying the coherence axioms (6) and (7) stated below.
Proposition 3.3 (Marmolejo). Let $A$ be a pseudo-algebra for a pseudo-monad $(X,S)$. The coherence condition

$$S^3A \xrightarrow{S^2a} S^2A \xrightarrow{m_A} S^2A \xrightarrow{ Sa } SA = S^3A \xrightarrow{S^2a} S^2A \xrightarrow{m_A} S^2A \xrightarrow{ Sa } SA = S^3A \xrightarrow{S^2a} S^2A \xrightarrow{m_A} S^2A \xrightarrow{ Sa } SA,$$

is derivable.

Proof. See [19, Lemma 9.1]. □

As usual, we refer to a pseudo-algebra by the name of its underlying 1-cell, leaving the rest of its data implicit. Given pseudo-algebras $A$ and $B$, a pseudo-algebra morphism $f : A \to B$ consists of a 2-cell $f : A \to B$ and an invertible 3-cell

$$SA \xrightarrow{sf} SB$$
satisfying the coherence conditions \((9)\) and \((10)\) stated below.

\[
\begin{align*}
\text{(9)} & \quad S^2A \xrightarrow{S^2f} S^2B \\
& \quad m_A \quad Sf \quad Sb \\
& \quad \downarrow a \quad \downarrow f \quad \downarrow b \\
& \quad A \quad f \quad B \\
& = \\
\text{(10)} & \quad S^2A \xrightarrow{S^2f} SB \\
& \quad m_A \quad Sf \quad Sb \\
& \quad \downarrow a \quad \downarrow f \quad \downarrow b \\
& \quad A \quad f \quad B \\
\end{align*}
\]

Given pseudo-algebra morphisms \(f : A \to B\) and \(g : A \to B\), a pseudo-algebra 2-cell consists of a 3-cell \(\alpha : f \to g\) satisfying the coherence condition \((11)\).

\[
\begin{align*}
\text{(11)} & \quad Sf \quad \downarrow S\alpha \quad SB \\
& \quad \quad \downarrow Sg \quad \quad \downarrow \beta \\
& \quad \downarrow y \quad \quad \downarrow g \\
& \quad A \quad \quad B \\
& = \\
& \quad Sf \quad \downarrow f \quad SB \\
& \quad \quad \downarrow \alpha \\
& \quad \downarrow y \quad \quad \downarrow g \\
& \quad A \quad \quad B \\
\end{align*}
\]

There is a forgetful 2-functor \(U_I : \text{Ps-S-Alg}(I) \to \mathcal{K}(I, X)\), defined by mapping a pseudo-S-algebra to its underlying 1-cell, which has a left pseudo-adjoint, defined by mapping a 1-cell \(A : I \to X\) to the free pseudo-algebra on it, given by the composite 1-cell \(SA : I \to X\).

The function mapping an object \(I \in \mathcal{K}\) to the 2-category \(\text{Ps-S-Alg}(I)\) extends to a Gray-functor \(\text{Ps-S-Alg} : \mathcal{K}^{\text{op}} \to \text{2-Cat}^{\otimes}\). We also have a Gray-transformation \(U : \text{Ps-S-Alg} \to X\), with components given by the forgetful 2-functors \(U_I : \text{Ps-S-Alg}(I) \to \mathcal{K}(I, X)\), for \(I \in \mathcal{K}\). Note that here we are using the notational conventions regarding representable Gray-functors introduced in Section 2. These conventions will be exploited repeatedly below.

We now recall from [20, Section 7] and [15, Section 6] the definition of the Gray-category \(\text{Psm}_\mathcal{K}\) of pseudo-monads in \(\mathcal{K}\). The 0-cells are pseudo-monads \((X, S)\) in \(\mathcal{K}\). For 0-cells \((X, S)\) and \((Y, T)\), a 1-cell \((H, \hat{H}) : (X, S) \to (Y, T)\)
consists of a 1-cell \( H : X \to Y \) in \( K \) and a Gray-transformation \( \hat{H} : \text{Ps-}S\text{-Alg} \to \text{Ps-}T\text{-Alg} \) making the following diagram commute

\[
\begin{array}{ccc}
\text{Ps-}S\text{-Alg} & \xrightarrow{\hat{H}} & \text{Ps-}T\text{-Alg} \\
U & & U \\
X & \xrightarrow{H} & Y
\end{array}
\]

We refer to \( \hat{H} \) as a lifting of \( H \) to pseudo-algebras. Analogous terminology will be used for the notions introduced below. Given 1-cells \((H, \hat{H}) : (X, S) \to (Y, T)\) and \((K, \hat{K}) : (X, S) \to (Y, T)\), a 2-cell \((p, \hat{p}) : (H, \hat{H}) \to (K, \hat{K})\) consists of a 2-cell \( p : H \to K \) in \( K \) and a Gray-modification \( \hat{p} : \hat{H} \to \hat{K} \) such that the following diagram commutes

\[
\begin{array}{ccc}
U\hat{H} & \xrightarrow{U\hat{p}} & U\hat{K} \\
\| & & \| \\
HU & \xrightarrow{pU} & KU
\end{array}
\]

The vertical arrows are the identities given by the assumption that \( \hat{H} \) and \( \hat{K} \) are liftings of \( H \) and \( K \), respectively. Finally, for 2-cells \((p, \hat{p})\) and \((q, \hat{q})\), a 3-cell \( \alpha : (p, \hat{p}) \to (q, \hat{q}) \) consists of a 3-cell and \( \alpha : p \to q \) and a Gray-perturbation \( \hat{\alpha} : \hat{p} \to \hat{q} \) making the following diagram commute

\[
\begin{array}{ccc}
U\hat{p} & \xrightarrow{U\hat{\alpha}} & U\hat{q} \\
\| & & \| \\
pU & \xrightarrow{\alpha U} & qU
\end{array}
\]

As before, the vertical arrows are the identities that are part of the assumption that \( \hat{p} \) and \( \hat{q} \) are liftings of \( p \) and \( q \), respectively. Composition and identities of \( \text{Psm}_K \) are defined in the evident way, using those of \( K \) and \( \text{2-Cat} \).

A Gray-category \( K \) is said to admit the construction of pseudo-algebras if for every pseudo-monad \((X, S)\) in \( K \), the Gray-functor \( \text{Ps-}S\text{-Alg} : K^{op} \to \text{2-Cat} \) is representable. When this is the case, the Yoneda lemma for Gray-categories implies that the notions of liftings given above can be given an evident alternative equivalent description, expressed purely in terms of the structure of \( K \). Let us also recall that \( \text{2-Cat} \) admits the construction of pseudo-algebras: the representing object for the Gray-functor \( \text{Ps-}S\text{-Alg} \) associated to a pseudo-monad \((X, S)\) in \( \text{2-Cat} \) is the 2-category of pseudo-algebras, pseudo-algebra morphism, and algebra 2-cells [4, 19].

4. Coherence axioms for the Gray-category of pseudo-monads

We provide an alternative description of the Gray-category \( \text{Psm}_K \), closer in spirit to the definition of the 2-categories of monads in a 2-category given in [22] and formulated without reference to the notion of pseudo-algebra. The material in this section is essentially an account of [23, Chapter 5] and [21, Section 3],
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except for some changes in terminology, that we explain later. Theorem 4.5, however, does not seem to appear in the form given here in the existing literature, even if it is closely related to [23, Theorem 5.23] and [21, Theorem 3.5]. Corollary 4.6, which is inspired by [15, Section 6], seems also to be new.

**Definition 4.1.** Let \((X, S)\) and \((Y, T)\) be pseudo-monads in \(K\). A pseudo-monad morphism \((H, h) : (X, S) \to (Y, T)\) consists of a 1-cell \(H : X \to Y\), a 2-cell \(h : TH \to HS\), and invertible 3-cells

\[
\begin{array}{c}
T^2 H \\ \uparrow T H \\
\downarrow n H \\
H S \\
\end{array}
\]

\[
\begin{array}{c}
T^2 H \\ \uparrow T H \\
\downarrow n H \\
H S \\
\end{array}
\]

These data are required to satisfy the coherence axioms in (12) and (13).

\[
\begin{array}{c}
T^3 H \\ \uparrow T^2 H \\
\downarrow n T H \\
T^2 H \\
\end{array}
\]

\[
\begin{array}{c}
T^3 H \\ \uparrow T^2 H \\
\downarrow n H \\
H S \\
\end{array}
\]

(12)
By a pseudo-monad op-morphism we mean a pseudo-monad morphism in $K^{\text{op}}$. Let $(X, S)$ and $(Y, T)$ be pseudo-monads in $K^{\text{op}}$. For a 1-cell $H : X \to Y$, the data of a 2-cell $h : TH \to HS$ and invertible 3-cells as in Definition 4.1 is referred to as a transition from $(X, S)$ to $(Y, T)$ along $H$ in [21]. In [23], a pseudo-monad morphism of the form $(H, h) : (X, S) \to (X, S)$, for a pseudo-monad $(X, S)$, is referred to as a pseudo-distributive law of $S$ over $H$.

**Proposition 4.2** (Marmolejo and Wood). Let $(H, h) : (X, S) \to (Y, T)$ be a pseudo-monad morphism. The coherence condition

\[
\begin{align*}
TH & \xrightarrow{h} HS \\
T^2H & \xrightarrow{Th} TSH \\
& \xrightarrow{nH} THS \\
& \xrightarrow{Hm} HS
\end{align*}
\]

is derivable.

**Proof.** See [21, Theorem 2.3].

Let $(H, h) : (X, S) \to (Y, T)$ be a pseudo-monad morphism. We show that we can define a lifting $\hat{H} : \text{Ps-S-Alg} \to \text{Ps-T-Alg}$ of $H : X \to Y$. Let us consider a fixed $I \in K$. First, let us observe that if $A$ is an $I$-indexed pseudo-$S$-algebra, then $HA$ is naturally an $I$-indexed pseudo-$T$-algebra, with structure map given by the composite

\[
THA \xrightarrow{h_A} HSA \xrightarrow{Ha} HA
\]
and associativity and unit 3-cells provided by the pasting diagrams

The coherence condition (6) for $H A$ follows by an application of the coherence condition (12) for $H$ and the coherence condition (6) for $A$. The coherence condition (7) for $H A$ follows by applying the coherence condition (13) for $H$ and the coherence condition (7) for $A$. Secondly, we observe that if $f : A \to B$ is a pseudo-$S$-algebra morphism, then $H f : HA \to HB$ is naturally a pseudo-$T$-algebra morphism, as we have the following pasting diagram:

The coherence conditions (9) and (10) follow immediately by the axioms for a Gray-category. Finally, if $\alpha : f \to g$ is a pseudo-$S$-algebra 2-cell, the required pseudo-$T$-algebra 2-cell is given by $H \alpha : H f \to H g$. We have thus defined the components of a Gray-natural transformation $\hat{H} : Ps-S-Alg \to Ps-T-Alg$, which is clearly a lifting of $H : X \to Y$.

We continue our analysis of liftings by describing what structure on a 2-cell allows us to define a lifting for it.

**Definition 4.3.** Let $(H, h) : (X, S) \to (Y, T)$ and $(K, k) : (X, S) \to (Y, T)$ be pseudo-monad morphisms. A pseudo-monad transformation $(p, \tilde{p}) : (H, h) \to (K, k)$ consists of a 2-cell $p : H \to K$ and an invertible 3-cell
satisfying the coherence conditions in (14) and (15) below:

\[
\begin{align*}
\text{(14)} & \quad T^2H \xrightarrow{T^2p} T^2K \\
& \quad \downarrow nH \quad \downarrow Th \quad \downarrow T\bar{p} \quad \downarrow Tk \\
& \quad TH \quad THS \xrightarrow{T\bar{p}S} TKS \\
& \quad \downarrow \bar{h} \quad \downarrow hS \quad \downarrow \bar{p}S \quad \downarrow kS \\
& \quad HS^2 \xrightarrow{p^2} KS^2 \\
& \quad \downarrow Hm \quad \downarrow \bar{p}m \quad \downarrow Km \\
& \quad HS \xrightarrow{pS} KS \\
\end{align*}
\]

\[
\begin{align*}
\text{(15)} & \quad H \xrightarrow{p} K \\
& \quad \downarrow vH \quad \downarrow Hu \quad \downarrow \bar{u}^{-1} \quad \downarrow Ku \\
& \quad TH \xrightarrow{T\bar{p}} TK \\
& \quad \downarrow \bar{h} \quad \downarrow \bar{h}u \quad \downarrow \bar{p}u \quad \downarrow \bar{k}u \\
& \quad HS \xrightarrow{pS} KS \\
\end{align*}
\]

Let \((p, \bar{p}) : (H, h) \to (K, k)\) be a pseudo-monad transformation. We show that we can define a lifting \(\hat{p} : \hat{H} \to \hat{K}\) of \(p : H \to K\), where \(\hat{H}\) and \(\hat{K}\) are the liftings of \(H\) and \(K\) associated to the pseudo-monad morphisms \((H, h)\) and \((K, k)\), respectively. Let \(I \in \mathcal{K}\). We need to define a pseudo-natural transformation \(\hat{p} : \hat{H}_I \to \hat{K}_I\). We define the component of \(\hat{p}\) associated to an \(I\)-indexed pseudo-\(S\)-algebra \(A\) to be the \(I\)-indexed pseudo-\(T\)-algebra morphism given by \(p_A : HA \to KA\) and the 2-cell

\[
\begin{align*}
T^2H & \xrightarrow{T^2p} T^2K \\
\downarrow nH & \quad \downarrow Th \quad \downarrow T\bar{p} \quad \downarrow Tk \\
TH & \quad THS \xrightarrow{T\bar{p}S} TKS \\
\downarrow \bar{h} & \quad \downarrow hS \quad \downarrow \bar{p}S \quad \downarrow kS \\
HS^2 & \xrightarrow{p^2} KS^2 \\
\downarrow Hm & \quad \downarrow \bar{p}m \quad \downarrow Km \\
HS & \xrightarrow{pS} KS \\
\end{align*}
\]

To prove the condition (9) for the pseudo-algebra morphism \(p_A\), we apply the axioms for a Gray-category and then condition (14) for the pseudo-monad transformation \(p\). To establish condition (10), it is sufficient to apply the coherence condition (15) for the pseudo-monad transformation \(p\), and then the axioms for a Gray-category. Clearly, \(\hat{p}\) is a lifting of \(p\) as required.
Finally, we describe what property a 3-cell has to satisfy in order to admit a lifting.

Definition 4.4. Let \((p, \tilde{p}) : (H, \tilde{H}) \to (K, \tilde{K}), (q, \tilde{q}) : (H, \tilde{H}) \to (K, \tilde{K})\) be pseudo-monad transformations. A pseudo-monad modification \(\alpha : (p, \tilde{p}) \to (q, \tilde{q})\) is a 3-cell \(\alpha : p \to q\) satisfying the coherence condition (16).

\[
\begin{array}{c}
TH \\ \downarrow Tp \\
\leftarrow T\alpha \\
\downarrow Tq \\
\leftarrow k \\
\hfill \tilde{q} \\
\downarrow qS \\
HS \\
\downarrow qS \\
\end{array}
= 
\begin{array}{c}
TH \\ \downarrow Tp \\
\leftarrow pS \\
\downarrow k \\
\hfill \tilde{q} \\
\downarrow qS \\
HS \\
\downarrow qS \\
\end{array}
\]
\]

Given a pseudo-monad modification \(\alpha : (p, \tilde{p}) \to (q, \tilde{q})\) we can define a lifting \(\hat{\alpha} : \hat{p} \to \hat{q}\) of \(\alpha\) as the Gray-perturbation with components given by the 3-cells \(\alpha_A : p_A \to q_A\), for a pseudo-\(S\)-algebra \(A\). It suffices to check that, these 3-cells are a pseudo-\(T\)-algebra 2-cells. To prove this, apply the axioms for a Gray-category and the coherence axiom (16).

Given two pseudo-monads \((X, S)\) and \((Y, T)\) in \(\mathcal{K}\), we define \(\tilde{\mathcal{K}}((X, S), (Y, T))\) to be the 2-category having pseudo-monad morphisms from \((X, S)\) to \((Y, T)\) as 0-cells, pseudo-monad transformations as 1-cells, and pseudo-monad modifications as 2-cells. The development in this section shows that we have a 2-functor

\[ F_{(X,S),(Y,T)} : \tilde{\mathcal{K}}((X, S), (Y, T)) \to \text{Psm}_{\mathcal{K}}((X, S), (Y, T)). \]

Theorem 4.5 below can be read as saying that the coherence axioms for pseudo-monad morphisms, pseudo-monad transformations, and pseudo-monad modifications are not only sufficient, but also necessary in order to obtain liftings.

Theorem 4.5. For every pair of pseudo-monads \((X, S)\) and \((Y, T)\) in \(\mathcal{K}\), the 2-functor \(F_{(X,S),(Y,T)} : \tilde{\mathcal{K}}((X, S), (Y, T)) \to \text{Psm}_{\mathcal{K}}((X, S), (Y, T))\) is a pseudo-equivalence.

Proof. Let us begin by considering a lifting \(\hat{H} : \text{Ps-Alg} \to \text{Ps-T-Alg}\) of a 1-cell \(H : X \to Y\). By the definition of a lifting, the following diagram of 2-categories and 2-functors commutes:

\[
\begin{array}{ccc}
\text{Ps-S-Alg}(X) & \xrightarrow{\hat{H}_X} & \text{Ps-T-Alg}(X) \\
\downarrow U_X & & \downarrow U_X \\
\mathcal{K}(X, X) & \xrightarrow{\kappa(X, H)} & \mathcal{K}(X, Y)
\end{array}
\]

Let us now observe that \(S : X \to X\) can be regarded as an \(X\)-indexed pseudo-\(S\)-algebra, with structure map given by the 2-cell \(m : S^2 \to S\). By the commutativity of the diagram above, this pseudo-\(S\)-algebra is mapped by the 2-functor \(\hat{H}_X\) into a pseudo-\(T\)-algebra with underlying 1-cell \(HS : X \to Y\), with
structure map a 2-cell of the form $h' : THS \to HS$, and invertible 3-cells fitting in the diagrams

\[
\begin{array}{cccc}
T^2HS & \xrightarrow{Th'} & THS & \\
\downarrow{nH} & \searrow{k'} & \downarrow{h'} & \\
THS & \xrightarrow{h'} & HS & \\
\end{array}
\quad
\begin{array}{cccc}
HS & \xrightarrow{vHS} & THS & \\
\downarrow{1HS} & \searrow{k'} & \downarrow{h'} & \\
HS & \xrightarrow{} & HS & \\
\end{array}
\]

The desired pseudo-monad morphism $(H, h) : (X, S) \to (Y, T)$ is then obtained by letting $h : TH \to HS$ be the composite

\[TH \xrightarrow{THu} THS \xrightarrow{h'} HS\]

The appropriate 3-cells are provided by the following pasting diagrams

\[
\begin{array}{cccc}
T^2H & \xrightarrow{T^2Hu} & T^2HS & \xrightarrow{Th'} THS & \xrightarrow{THuS} THS^2 & \\
\downarrow{nH} & \searrow{nHu} & \downarrow{nHS} & \searrow{k'} & \downarrow{Hm} & \\
TH & \xrightarrow{THu} THS & \xrightarrow{h'} HS & \xrightarrow{} HS & \\
\end{array}
\quad
\begin{array}{cccc}
H & \xrightarrow{vH} & TH & \\
\downarrow{Hu} & \searrow{vHS} & \downarrow{THu} & \\
HS & \xrightarrow{1HS} & HS & \\
\end{array}
\]

where $\gamma$ is the inverse to the 2-cell obtained from the following pasting of invertible 2-cells:

\[
\begin{array}{cccc}
T^2H & \xrightarrow{T^2Hu} & T^2HS & \xrightarrow{Th'} THS & \xrightarrow{THuS} THS^2 & \\
\downarrow{nH} & \searrow{nHu} & \downarrow{nHS} & \searrow{k'} & \downarrow{Hm} & \\
TH & \xrightarrow{THu} THS & \xrightarrow{h'} HS & \xrightarrow{} HS & \\
\end{array}
\quad
\begin{array}{cccc}
H & \xrightarrow{vH} & TH & \\
\downarrow{Hu} & \searrow{vHK} & \downarrow{THu} & \\
HS & \xrightarrow{1HS} & HS & \\
\end{array}
\]

Let us now consider a lifting $(p, \tilde{p}) : (H, \tilde{H}) \to (K, \tilde{K})$ of a 2-cell $p : H \to K$. We can define a pseudo-monad transformation $p : (H, h) \to (K, k)$ by considering the following pasting diagram:

\[
\begin{array}{cccc}
TH & \xrightarrow{Tp} & TK & \\
\downarrow{THu} & \searrow{Tpu^{-1}} & \downarrow{TKu} & \\
TSH & \xrightarrow{TpS} TKS & \xrightarrow{k'} KS & \\
\downarrow{h'} & \searrow{pS} & \downarrow{pS} & \\
HS & \xrightarrow{} KS & \\
\end{array}
\]


in which the bottom 3-cell is part of the structure making \( pS : HS \to KS \) into a pseudo-algebra morphism. Finally, if \((\alpha, \hat{\alpha}) : (p, \hat{p}) \to (q, \hat{q})\) is a lifting of a 3-cell \( \alpha : p \to q \), then \( \alpha : p \to q \) is a pseudo-monad modification. Lengthy calculations show that these definitions determine a 2-functor

\[
G_{(X,S),(Y,T)} : \text{PsSm}_K((X,S),(Y,T)) \to \tilde{K}((X,S),(Y,T))
\]

which provides the required quasi-inverse to \( F \). We omit the construction of the required invertible pseudo-natural transformations \( \eta : 1 \to GF \) and \( \varepsilon : FG \to 1 \), since this is not difficult.

**Corollary 4.6.** For every Gray-category \( \mathcal{K} \), there exist a tricategory \( \tilde{\mathcal{K}} \) having pseudo-monads in \( \mathcal{K} \) as 0-cells, pseudo-monad morphisms as 1-cells, pseudo-monad transformations as 2-cells, and pseudo-monad modifications as 3-cells, and a triequivalence \( F : \tilde{\mathcal{K}} \to \text{PsSm}_K \).

**Proof.** Theorem \([\text{4.5}]\) allows us to apply the lemma on transport of structure in \([\text{10}, \text{Section 3.6}]\). \(\square\)

It would be of interest to define a tricategory \( \tilde{\mathcal{K}} \) as in Corollary \([\text{4.6}]\) without reference to the Gray-category \( \text{PsSm}_K \) and to verify whether this is indeed only a tricategory, and not a Gray-category, as anticipated in \([\text{15}, \text{Section 6}]\).

**Remark.** For a pseudo-monad \((X,S)\) in \( \mathcal{K} \), there is a Gray-natural family of isomorphisms of 2-categories

\[
\text{Ps-S-Alg}(I) \cong \tilde{\mathcal{K}}((I,1_I),(X,S))
\]

for \( I \in \mathcal{K} \), where \((I,1_I)\) denotes the identity pseudo-monad on \( I \). Hence, \( \mathcal{K} \) admits the construction of pseudo-algebras if and only if for every pseudo-monad \((X,S)\) in \( \mathcal{K} \) there exists an object \( X^S \in \mathcal{K} \) and a pseudo-monad morphism

\[
\varepsilon_S : (X^S,1_{X^S}) \to (X,S)
\]

such that, for every \( I \in \mathcal{K} \), the pseudo-functor

\[
\mathcal{K}(I,X^S) \to \tilde{\mathcal{K}}((I,1_I),(X,S))
\]

defined by composition with \( \varepsilon_S \) is an isomorphism. More explicitly, to give a pseudo-monad morphism \( \varepsilon_S \) as above is to give a morphism \( U : X^S \to X \), a transformation \( u : SU \to U \), and invertible modifications

\[
\begin{array}{ccc}
S^2U & \xrightarrow{Su} & SU \\
\downarrow mU & & \downarrow u \\
SU & \xrightarrow{u} & U
\end{array}
\quad
\begin{array}{ccc}
U & \xrightarrow{uU} & SU \\
\downarrow \overline{u} & & \downarrow u \\
U & \xrightarrow{1_U} & U
\end{array}
\]

satisfying the coherence conditions for a pseudo-monad morphism.
5. Pseudo-distributive laws

**Definition 5.1.** Let \((X, S)\) and \((X, T)\) be pseudo-monads in \(K\). A *pseudo-distributive law* of \(T\) over \(S\) consists of a 2-cell \(d : ST \to TS\) and invertible 3-cells

\[
\begin{align*}
  S^2T & \xrightarrow{Sd} STS \\
  ST^2 & \xrightarrow{Sn} ST \\
  T^2S & \xrightarrow{nS} TS
\end{align*}
\]

satisfying the coherence conditions (D1)-(D8) stated in Appendix A.

**Proposition 5.2** (Marmolejo and Wood). Let \(d : ST \to TS\) be a pseudo-distributive law. The coherence conditions (D9) and (D10), as stated in Appendix A, are derivable.

**Proof.** See [21, Proposition 5.1] □

The development in Section 4 allows us to give a clear explanation for the coherence conditions for pseudo-distributive laws and for Proposition 5.2. The axioms (C1) and (C2) express that \((T, d) : (X, S) \to (X, S)\) is a pseudo-monad morphism. Hence, it clear that they imply (C9), since this is a special case of Proposition 4.2. Dually, (C7) and (C8) express that \((S, d) : (X, T) \to (X, T)\) is a pseudo-monad op-morphisms. Hence, by a dual of Proposition 5.2, as stated in [21, Proposition 4.2], they imply (C10). Let us also note that the axioms (C3) and (C4) express that \((n, \bar{n}) : (T, d)^2 \to (T, d)\) is a pseudo-monad transformation; the axioms (C5) and (C6) express that \((v, \bar{v}) : (X, 1_X) \to (T, d)\) is a pseudo-monad transformation; and finally the axioms (C7), (C8), and (C10), express that \(\alpha, \rho,\) and \(\lambda\), and are pseudo-monad modifications, respectively. It is then clear that giving a pseudo-distributive law of \(T\) over \(S\) is equivalent to giving a lifting of \(T\) to Ps-S-Alg, by which we mean a lifting of all the data that is part of the pseudo-monad \(T\), thus obtaining an analogue of Beck’s fundamental result on the equivalence between giving a distributive law of a monad \(T\) over a monad \(S\) and a lifting of the monad \(T\) to the category of Eilenberg-Moore algebras for \(S\).
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Appendix A. Coherence conditions for pseudo-distributive laws

We limit ourselves to drawing the boundaries of these diagrams and explain in text which 3-cells should be inserted in them, except from the 3-cells coming from the structure of a Gray-category of $\mathcal{K}$.

(C1)
In (C1), the left-hand side pasting is obtained using $S\bar{n}$, $\bar{m}$, and the associativity of the pseudo-monad $S$; the right-hand side pasting is obtained using the associativity of the pseudo-monad $S$ and $\bar{m}$.

In (C2), the left-hand side pasting is obtained using $S\bar{u}$, $\bar{m}$, and the left unit of the pseudo-monad $S$; the right-hand side pasting is obtained using the left unit of the pseudo-monad $S$.

For (C3), the left-hand side pasting is obtained using $S\bar{v}$, $\bar{v}S$; the right-hand side is obtained using $\bar{m}$ and $\bar{m}$.
For (C4), the left-hand side pasting is obtained using $\bar{u}$ and $\bar{v}$.
For (C5), the left-hand side pasting is obtained using $S\bar{n}$, $\bar{n}S$ and $\bar{m}T$; the right-hand side pasting is obtained using $\bar{m}$ and $\mathit{mut}$.

\begin{equation}
\begin{tikzpicture}
\node (A) at (0,0) {$ST^2$}; \node (B) at (3,0) {$ST$}; \node (C) at (0,-3) {$T^2S$}; \node (D) at (3,-3) {$TS$}; \node (E) at (1.5,-1.5) {$ST^2$}; \node (F) at (2.5,-1.5) {$ST$}; \node (G) at (1.5,-2.5) {$T^2S$}; \node (H) at (2.5,-2.5) {$TS$};
\draw[->] (A) -- node[above] {$n$} (B);
\draw[->] (A) -- node[below] {$uT^2$} (C);
\draw[->] (B) -- node[above] {$Sn$} (E);
\draw[->] (B) -- node[below] {$dT$} (F);
\draw[->] (C) -- node[above] {$nS$} (D);
\draw[->] (C) -- node[below] {$Td$} (G);
\draw[->] (E) -- node[above] {$ST^2$} (F);
\draw[->] (E) -- node[below] {$TST$} (G);
\draw[->] (F) -- node[above] {$ST^2$} (D);
\draw[->] (F) -- node[below] {$T^2S$} (H);
\draw[->] (G) -- node[above] {$nS$} (H);
\draw[->] (G) -- node[below] {$T^2S$} (D);
\end{tikzpicture}
\end{equation}

In (C6), the left-hand side pasting is obtained using $\bar{u}$, and $\bar{n}$; the right-hand side pasting is obtained using $\bar{u}T$.

To state the coherence conditions (D8), (D9) and (D10), let $\alpha$, $\lambda$, and $\rho$ be the associativity, left unit, and right unit for the pseudo-monad $T$.

\begin{equation}
\begin{tikzpicture}
\node (A) at (0,0) {$ST^3$}; \node (B) at (3,0) {$ST$}; \node (C) at (0,-3) {$T^2S$}; \node (D) at (3,-3) {$TS$}; \node (E) at (1.5,-1.5) {$ST^3$}; \node (F) at (2.5,-1.5) {$ST$}; \node (G) at (1.5,-2.5) {$T^2S$}; \node (H) at (2.5,-2.5) {$TS$};
\draw[->] (A) -- node[above] {$STn$} (E);
\draw[->] (A) -- node[below] {$dT^2$} (G);
\draw[->] (B) -- node[above] {$ST^2$} (F);
\draw[->] (B) -- node[below] {$dT$} (H);
\draw[->] (C) -- node[above] {$ST^3$} (E);
\draw[->] (C) -- node[below] {$T^2S$} (G);
\draw[->] (D) -- node[above] {$ST^2$} (F);
\draw[->] (D) -- node[below] {$T^2S$} (H);
\draw[->] (E) -- node[above] {$STn$} (F);
\draw[->] (E) -- node[below] {$T^2S$} (G);
\draw[->] (F) -- node[above] {$ST^2$} (D);
\draw[->] (F) -- node[below] {$T^2S$} (H);
\draw[->] (G) -- node[above] {$ST^3$} (H);
\draw[->] (G) -- node[below] {$T^2S$} (D);
\end{tikzpicture}
\end{equation}

For (C7), the left-hand side pasting is obtained using $S\alpha$, $\bar{n}$, $\bar{n}T$; the right-hand side pasting is obtained using $\bar{n}$ and $\alpha S$.
For (C8), the left-hand side pasting is obtained using $S\rho$ and $\bar{u}, \bar{v}S$; the right-hand side pasting is $\rho S$.

For (C9), the left-hand side pasting is obtained using the right unit of the pseudo-monad $S$, $\bar{u}$; the right-hand side pasting is obtained using the right unit of the pseudo-monad $S$. 
For (C10), the left-hand side pasting uses $S\lambda$. The right-hand side pasting is obtained using $\bar{n}$ and $\lambda S$.

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University of Palermo & University of Manchester

E-mail address: nicola.gambino@gmail.com