OSTROWSKI QUOTIENTS FOR FINITE EXTENSIONS OF NUMBER FIELDS

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Abstract. For \( L/K \) a finite Galois extension of number fields, the relative Pólya group \( \text{Po}(L/K) \) coincides with the group of strongly ambiguous ideal classes in \( L/K \). In this paper, using a well known exact sequence related to \( \text{Po}(L/K) \), in the works of Brumer-Rosen and Zantema, we find short proofs for some classical results in the literature. Then we define the “Ostrowski quotient” \( \text{Ost}(L/K) \) as the cokernel of the capitulation map into \( \text{Po}(L/K) \), and generalize some known results for \( \text{Po}(L/\mathbb{Q}) \) to \( \text{Ost}(L/K) \).

Keywords: Pólya group, relative Pólya group, Ostrowski quotient, Galois cohomology, capitulation problem.

Notations. The following notations will be used throughout this article:

For a number field \( K \), the notations \( I(K) \), \( P(K) \), \( \text{Cl}(K) \), \( \mathcal{O}_K \), \( h_K \), \( U_K \), \( H(K) \), and \( \Gamma(K) \) denote the group of fractional ideals, group of principal fractional ideals, ideal class group, ring of integers, class number, group of units, Hilbert class field, and genus field of \( K \), respectively.

For a finite extension \( L/K \) of number fields, \( N_{L/K} : \text{Cl}(L) \to \text{Cl}(K) \) denotes the induced homomorphism by the ideal norm homomorphism \( N_{L/K} : I(L) \to I(K) \); and \( \epsilon_{L/K} : \text{Cl}(K) \to \text{Cl}(L) \) denotes the transfer of ideal classes induced by the homomorphism \( j_{L/K} : a \in I(K) \mapsto a\mathcal{O}_L \in I(L) \). Whenever \( L/K \) is Galois, for a prime ideal \( \mathfrak{P} \) of \( K \) denote by \( \epsilon_{\mathfrak{P}}(L/K) \) and \( f_{\mathfrak{P}(L/K)} \) the ramification index and residue class degree of \( \mathfrak{P} \) in \( L/K \), respectively.

1. Introduction

Let \( L \) be an algebraic number field with ring of integers \( \mathcal{O}_L \). For every prime number \( p \) and every integer \( f \geq 1 \), the Ostrowski ideal \( \Pi_{p^f}(L) \) of \( L \) is defined as follows [17]:

\[
\Pi_{p^f}(L) := \prod_{m \in \text{Max}(\mathcal{O}_L)} m, \quad m \in \text{Max}(\mathcal{O}_L), N_{L/\mathbb{Q}}(m) = p^f
\]

where by the convention, if \( L \) has no ideal with norm \( p^f \), then \( \Pi_{p^f}(L) = \mathcal{O}_L \). Following Zantema [24], \( L \) is called a Pólya field if all the Ostrowski ideals \( \Pi_{p^f}(L) \) of \( L \) for arbitrary prime powers \( p^f \) are principal. As an “obstruction measure” for \( L \) to be a Pólya field, the notion of Pólya-Ostrowski group or Pólya group was introduced in [1].

2010 Mathematics Subject Classification. Primary 11R29, 11R37.
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1 The third author’s research was supported by a grant from IPM.
Definition 1.1. The Pólya group of a number field $L$ is the subgroup $\text{Po}(L)$ of the class group $\text{Cl}(L)$ generated by the classes of the Ostrowski ideals $\Pi_q(L)$. Hence $L$ is Pólya iff its Pólya group is trivial. Obviously every number field with class number one is a Pólya field, but not conversely, see e.g. Proposition 1.2 below.

Now let $L$ be a Galois extension of $\mathbb{Q}$ with Galois group $G$. In this case, Pólyanness of $L$ is equivalent to principality of $\prod_{i=1}^g P_i$, where $P_1, P_2, \ldots, P_g$ are all distinct prime ideals of $L$ above a ramified prime $p$, see [24, Section 1]. On the other hand, one can show that the Ostrowski ideals $\Pi_q(L)$ freely generate the ambiguous ideals $I(L)^G$, see [1, Section 2]. Thus $\text{Po}(L)$ is the subgroup of $\text{Cl}(L)$ generated by the classes of ambiguous ideals. As the first result in this subject, Pólya groups of quadratic fields have been described by Hilbert [6]:

Proposition 1.2. [6, Theorem 105-106] Let $L$ be a quadratic field and denote the number of ramified primes in $L/\mathbb{Q}$ by $r_L$. Then:

$$\#\text{Po}(L) = \begin{cases} 2^{r_L-2} : \text{L is real with no unit of negative norm} \\ 2^{r_L-1} : \text{Otherwise} \end{cases}$$

The above theorem of Hilbert has been generalized by Zantema:

Proposition 1.3. [24, Section 3, page 9] Let $L/\mathbb{Q}$ be a Galois extension with Galois group $G$. Denote the ramification index of a prime $p$ in $L$ by $e_p$. Then the following sequence is exact:

$$(1.2) \quad \{0\} \rightarrow H^1(G, U_L) \rightarrow \bigoplus_{p \text{ prime}} \mathbb{Z}/e_p \mathbb{Z} \rightarrow \text{Po}(L) \rightarrow \{0\}.$$
We summarize some interesting consequences of Theorem 2.2 (for more results see [3, 15]):

**Corollary 2.3.** For $L/K$ a finite Galois extension of number fields with Galois group $G$, the following assertions hold:

(i) [3] Proposition 4.4] $\# \text{Po}(L/K) = \frac{h_K \cdot \prod_P \epsilon_{\text{Po}(L/K)}}{\# H^1(G, U_L)}$.

(ii) [15] Remark 2.3] If $\gcd(h_L, [L : K]) = 1$, then $\text{Po}(L/K) = \epsilon_{L/K}(\text{Cl}(K))$.

(iii) [15] Corollary 2.4] If $\gcd(h_K, [L : K]) = 1$ then $\text{Cl}(K) \hookrightarrow \text{Po}(L/K)$. In particular, $h_K | h_L$. Moreover, the following sequence is exact:

$$
\{0\} \rightarrow H^1(G, U_L) \rightarrow \bigoplus_{\mathfrak{p} \mid \text{disc}(L/K)} \mathbb{Z} / \mathfrak{p} \mathbb{Z} \rightarrow \text{Po}(L/K) / \text{Cl}(K) \rightarrow \{0\}.
$$

(iv) [15] Corollary 2.9] If every ideal class of $K$ extended to $L$ is principal and all finite places of $K$ are unramified in $L$, then $\text{Cl}(K) \simeq H^1(G, U_L)$ and $\text{Po}(L/K) = \{0\}$. In particular, $\text{Po}(H(K)/K)$ is trivial, for the Hilbert class field $H(K)$ of $K$.

**Remark 2.4.** Indeed classically, $\# \text{Po}(L/K)$ can be obtained also from Brumer-Rosen results in [11, Lemma 2.1 and Proposition 2.2].

**Lemma 2.5.** [15] Lemma 2.10] Let $F \subseteq K \subseteq L$ be a tower of finite extensions of number fields.

(i) If $K/F$ and $L/F$ are Galois extensions, then $\epsilon_{L/K}(\text{Po}(K/F)) \subseteq \text{Po}(L/F)$.

(ii) If $L/K$ is a Galois extension, then $\text{Po}(L/F) \subseteq \text{Po}(L/K)$.

**Remark 2.6.** Note that if either $K/F$ or $L/F$ is not Galois, then the containment in the part (i) in Lemma 2.3 might not hold. For instance, consider the pure cubic field $K = \mathbb{Q}(\sqrt[3]{19})$. One can show that the Galois closure $L$ of $K$ is a Polya field while $\text{Po}(K) = \text{Cl}(K) \simeq \mathbb{Z}/3\mathbb{Z}$, see [14, Example 2.9]. On the other hand, since $\gcd(h_K, [L : K]) = 1$, $\mathcal{N}_{L/K} \circ \epsilon_{L/K} : \tilde{a} \in \text{Cl}(K) \mapsto \tilde{a}^{[L : K]} \in \text{Cl}(K)$ is injective, hence so is $\epsilon_{L/K}$. Therefore $\epsilon_{L/K}(\text{Po}(K)) = \text{Po}(K) = \text{Cl}(K) \not\subseteq \text{Po}(L)$.

### 2.1. Some Applications.

In this section, using the exact sequence (BRZ) we find easier and shorter proofs for some results in the literature. We begin with the following result of Tannaka:

**Proposition 2.7.** [21] Theorem 8] For a number field $K$, one has

$$(2.1) \quad H^1(\text{Gal}(H(K)/K), U_{H(K)}) \simeq \text{Gal}(H(K)/K),$$

where $H(K)$ denotes the Hilbert class field of $K$.

**Proof.** By the principal ideal theorem, $\text{Ker}(\epsilon_{H(K)/K}) = \text{Cl}(K)$. Since $H(K)/K$ is an unramified extension, using the isomorphism induced by Artin reciprocity, and the exact sequence (BRZ), we find

$$\text{Gal}(H(K)/K) \simeq \text{Cl}(K) = \text{Ker}(\epsilon_{H(K)/K}) \simeq H^1(\text{Gal}(H(K)/K), U_{H(K)}).$$

\[\square\]

**Remark 2.8.** Isomorphism (2.1) also has been obtained independently by Khare and Prasad, see [9, §4, Corollary 3]. They mention that this isomorphism is equivalent to the principal ideal theorem, see [9, §4, Remark 3].
A similar statement holds for any finite extension of the Hilbert class field, due to Iwasawa:

**Proposition 2.9.** [7] For $L/K$ a finite Galois extension of number fields which is unramified at all finite places, if $H(K) \subseteq L$ then

$$H^1(\text{Gal}(L/K), U_L) \simeq \text{Cl}(K).$$

**Proof.** Since $\epsilon_{L/K} = \epsilon_{L/H(K)} \circ \epsilon_{H(K)/K}$, by the principal ideal theorem we have $\text{Ker}(\epsilon_{L/K}) = \text{Cl}(K)$. The exact sequence [BRZ] yields the claim. □

In a more general case (without considering the Hilbert class field), Iwasawa [7] and Khare-Prasad [9, § 4, Proposition 2] independently proved that:

**Proposition 2.10.** Let $L/K$ be a finite Galois extension of number fields that is unramified at all finite places. Then $\text{Ker}(\epsilon_{L/K}) \simeq H^1(\text{Gal}(L/K), U_L)$.

**Proof.** Immediately follows from the exact sequence [BRZ]. □

**Remark 2.11.** Note that in our proofs, we have assumed a weaker assumption, i.e. finite places are unramified versus all places. For instance, for $L = \mathbb{Q}(\sqrt{-1}, \sqrt{3})$ and $K = \mathbb{Q}(\sqrt{3})$, ramification happens only for infinite places of $K$ in $L$; whereas by the exact sequence [BRZ] $H^1(\text{Gal}(L/K), U_L)$ is trivial, since $L/K$ is unramified in all finite places (Note that $h_K = 1$).

As another application of the exact sequence [BRZ] we find the following results on the “kernel” and “cokernel” of the capitulation map in cyclic unramified extensions:

**Theorem 2.12.** Let $L/K$ be a finite cyclic extension of number fields with Galois group $G$.

(i) If $L$ is unramified at all finite and infinite places of $K$, then

$$(2.2) \quad \# \text{Ker}(\epsilon_{L/K}) = \left( U_K : \text{Norm}_{L/K}(U_L) \right) [L : K].$$

(ii) If $L$ is unramified at all finite places of $K$, then

$$(2.3) \quad H^2(G, U_L) \hookrightarrow \text{Coker}(\epsilon_{L/K}).$$

**Proof.** (i) Since $L/K$ is unramified, by the exact sequence [BRZ] we have

$$(2.4) \quad \# H^1(G, U_L) = \# \text{Ker}(\epsilon_{L/K}).$$

On the one hand, since $L/K$ is cyclic, we can use the Herbrand quotient:

$$(2.5) \quad Q(G, U_L) = \frac{\# H^0(G, U_L)}{\# H^1(G, U_L)},$$

where

$$(2.6) \quad H^0(G, U_L) = \frac{U_L^G}{\text{Norm}_{L/K}(U_L)} = \frac{U_K}{\text{Norm}_{L/K}(U_L)}.$$ 

On the other hand,

$$Q(G, U_L) = \frac{2^s}{[L : K]}.$$
where \( s \) is the number of infinite places of \( K \) ramified in \( L \) [11, Chapter IX]. By the assumption \( L/K \) is unramified which implies that \( s = 0 \) i.e.

\[
(2.7) \quad Q(G,U_L) = \frac{1}{[L : K]}.
\]

Using relations (2.4)–(2.7) we obtain the desired equality.

(ii) Consider the following exact sequence

\[
\{0\} \to P(L) \to I(L) \to \text{Cl}(L) \to \{0\}.
\]

Taking Cohomology, we get the sequence

\[
\{0\} \to P(L)^G \to I(L)^G \to (\text{Cl}(L))^G \to H^1(G,P(L)) \to \{0\}
\]

is exact, since \( H^1(G,I(L)) = \{0\} \), see e.g. [10, proof of Theorem 1] or [23, Lemma 2.1]. Since \( L/K \) is Galois, \( P(L/K) = \frac{I(L)^G}{P(L)^G} \), see [15, § 2]. Hence the last exact sequence can be rewritten as

\[
\{0\} \to P(L/K) \to (\text{Cl}(L))^G \to H^1(G,P(L)) \to \{0\},
\]

which implies that

\[
(2.8) \quad \frac{(\text{Cl}(L))^G}{P(L/K)} \simeq H^1(G,P(L)).
\]

On the other hand, Kisilevsky [10, Lemma 1] proved that

\[
(2.9) \quad H^1(G,P(L)) \simeq H^2(G,U_L).
\]

Therefore

\[
(2.10) \quad \frac{(\text{Cl}(L))^G}{P(L/K)} \simeq H^2(G,U_L).
\]

Since all finite places of \( K \) are unramified in \( L \), the exact sequence [BRZ] yields

\[
(2.11) \quad P(L/K) = \epsilon_{L/K}(\text{Cl}(K)).
\]

Using equations (2.10) and (2.11) we have

\[
(2.12) \quad H^2(G,U_L) \simeq \frac{(\text{Cl}(L))^G}{\epsilon_{L/K}(\text{Cl}(K))}
\]

which is a subgroup of \( \text{Coker}(\epsilon_{L/K}) \). \( \square \)

**Remark 2.13.** Part (i) of Theorem 2.12 can be thought of as a quantitative version of “Hilbert’s Theorem 94” [6].

### 3. Ostrowski Quotient

By exact sequence (1.2), there exists a surjective map from \( \bigoplus_p \frac{\mathbb{Z}}{\mathbb{Z}(L/Q)} \) onto \( P(L) \). Whereas switching to the relative case, namely \( P(L/K) \) for \( L/K \) a finite Galois extension of number fields, this statement does not hold, in general. There are finite Galois extensions \( L/K \) such that the only map from \( \bigoplus_p \frac{\mathbb{Z}}{\mathbb{Z}(L/Q)} \) to \( P(L/K) \) is the zero map while \( P(L/K) \) is non trivial:
Example 3.1. Let $K = \mathbb{Q}(\zeta_p)$ and $L = \mathbb{Q}(\zeta_{p^n})$ where $n \geq 1$ and $p$ is a regular prime, i.e. $p \nmid h_K$, and $\zeta_p$ (resp. $\zeta_{p^n}$) denote the $p$-th (resp. $p^n$-th) primitive root of unity. By a result of Iwasawa [8], since $h_K$ is not divisible by $p$, so is not $h_L$. In particular $p \nmid #\text{Po}(L/K)$. The only prime of $K$ that ramifies in $L$ is $\mathfrak{p} = < 1 - \zeta_p >$ whose ramification index is $[L : K] = p^{n-1}$, i.e. $\mathfrak{p}$ is totally ramified.

Therefore the only map from $\bigoplus_{\mathfrak{p}} \mathbb{Z}_{\mathfrak{p}(L/K)^{\mathfrak{p}}} \to \text{Po}(L/K)$ is the zero map. Masley and Montgomery [16] proved that $h_K \neq 1$ for $p > 19$. Since $\gcd(h_K, [L : K]) = 1$, one has $\text{Cl}(K) \leftrightarrow \text{Po}(L/K)$, see [15] Corollary 2.4. Hence for $p > 19$, $\text{Po}(L/K)$ is nontrivial. Note that

$$P_L = (1 - \zeta_p)O_L = (1 - \zeta_{p^n})^{p^{n-1}}$$

is a principal ideal of $L$, hence is trivial in $\text{Po}(L/K)$.

The above example shows that the controllability of the Pólya group by ramification, cannot transfer to the relative Pólya group. This motivates us to modify the notion of relative Pólya group to arrive at a notion directly governed by ramification.

Definition 3.2. [19] For a finite extension $L/K$ of number fields, the Ostrowski quotient $\text{Ost}(L/K)$ is defined as

$$\text{Ost}(L/K) := \frac{\text{Po}(L/K)}{\text{Po}(L/K) \cap \epsilon_{L/K}(\text{Cl}(K))}.$$  

In particular, $\text{Ost}(L/\mathbb{Q}) = \text{Po}(L/\mathbb{Q}) = \text{Po}(L)$ and $\text{Ost}(L/L) = \{0\}$. The extension $L/K$ is called “Ostrowski” (or $L$ is called $K$-Ostrowski) if $\text{Ost}(L/K)$ is trivial.

Remark 3.3. If $L/K$ is a Galois extension with Galois group $G$, then $\epsilon_{L/K}(\text{Cl}(K)) \subseteq \text{Po}(L/K)$ [15 § 2], and the exact sequence (BRZ) can be rewritten as follows

$$(3.2) \quad \{0\} \xrightarrow{\theta_{L/K}} \text{Ker}(\epsilon_{L/K}) \xrightarrow{H^1(G,U_L)} \bigoplus_{\mathfrak{p}} \mathbb{Z}_{\epsilon_{\mathfrak{p}(L/K)^{\mathfrak{p}}}} \xrightarrow{\Phi} \text{Ost}(L/K) \xrightarrow{\partial} \{0\}.$$  

Hence in this case we have

$$\# \text{Ost}(L/K) = \frac{\# \text{Ker}(\epsilon_{L/K}) \cdot \prod_{\mathfrak{p} \mid \text{disc}L/K} \epsilon_{\mathfrak{p}(L/K)^{\mathfrak{p}}}}{\# H^1(G,U_L)},$$

which relates the Ostrowski quotient of $L/K$ to the “capitulation kernel”.

Using Corollary [23] we immediately find:

Theorem 3.4. Let $L/K$ be a finite Galois extension of number fields with Galois group $G$.

(i) If $\gcd(h_K, [L : K]) = 1$, then the following sequence is exact:

$$\{0\} \xrightarrow{\theta_{L/K}} H^1(G,U_L) \xrightarrow{\bigoplus_{\mathfrak{p}} \mathbb{Z}_{\epsilon_{\mathfrak{p}(L/K)^{\mathfrak{p}}}}} \text{Ost}(L/K) \xrightarrow{\partial} \{0\}.$$  

(ii) If either $\gcd(h_K, [L : K]) = 1$ or all finite places of $K$ are unramified in $L$, then $L/K$ is Ostrowski. In particular, the extensions $H(K)/K$ and $\Gamma(K)/K$ are Ostrowski, where $H(K)$ and $\Gamma(K)$ denote the Hilbert class field and genus field of $K$, respectively.

For “abelian” number fields, Zantema proved:
Proposition 3.5. [24, Proposition 2.5] If $K/Q$ is an abelian extension ramifying at only one prime, then $K$ is a Pólya field.

Using some notions in class field theory for finite abelian extensions, namely the “conductor” and the “ray class field” [11, Chapter X, §3], we can find an analogous statement to Proposition 3.5 for the Ostrowski quotient:

**Theorem 3.6.** Let $K/F$ be a finite abelian extension of number fields such that only one prime of $F$ is ramified in $K$. Let $L$ be the ray class field of $F$ for the modulus $c(K/F)$, where $c(K/F)$ denotes the conductor of $K$ over $F$. If $L/F$ is Ostrowski, then so is $K/F$.

**Proof.** Let $p$ be the only prime of $F$ ramified in $K$ and

$$pO_K = \left(\mathfrak{a}, \mathfrak{b}, \ldots, \mathfrak{g}\right)^{e_p(K/F)} = \left(\Pi_{p}^{K/F} (K/F)\right)^{e_p(K/F)},$$

(3.4)

$$pO_L = \left(\mathfrak{c}, \mathfrak{c}_2, \ldots, \mathfrak{c}_t\right)^{e_p(L/F)} = \left(\Pi_{p}^{L/F} (L/F)\right)^{e_p(L/F)}.$$  

Since $L/F$ is Ostrowski, $\Pi_{p}^{L/F} (L/F) \in \epsilon_{L/F}(Cl(F))$. Equivalently $\Pi_{p}^{L/F} (L/F) = \epsilon_{L/F}(\epsilon_{K/F}([\mathfrak{a}]))$ for some $[\mathfrak{a}] \in Cl(F)$. Hence

$$\mathcal{N}_{L/K} \left(\Pi_{p}^{L/F} (L/F)\right) = \mathcal{N}_{L/K} \left(\epsilon_{L/F}(\epsilon_{K/F}([\mathfrak{a}]))\right) = \left(\epsilon_{K/F}([\mathfrak{a}])\right)^{[L:K]}.$$  

On the other hand,

$$\mathcal{N}_{L/K} (pO_L) = \left(pO_K\right)^{[L:K]} = \left(\Pi_{p}^{K/F} (K/F)\right)^{[L:K]e_p(K/F)}.$$  

By relations (3.4), (3.5) and (3.6) we get

$$\mathcal{N}_{L/K} (pO_L) = \left(pO_K\right)^{[L:K]} = \left(\Pi_{p}^{K/F} (K/F)\right)^{[L:K]e_p(K/F)}.$$  

Finally, since $K/F$ is a Galois extension, for each prime $p' \neq p$ of $F$ we have $\Pi_{p'}^{F/K} (K/F) \in \epsilon_{K/F}(Cl(F))$, see part (i) of Lemma 12 below. Therefore $\text{Ost}(K/F) = \{0\}$ as claimed. □

**Remark 3.7.** For $F = Q$, triviality of $\text{Ost}(L/F)$ is a classical result for cyclotomic fields [24, Proposition 2.6].

Leriche proved that the Hilbert class field $H(K)$ of $K$ is a Pólya field [12, Corollary 3.2], which recently has been generalized to the triviality of $\text{Po}(H(K)/K)$, see [15, Corollary 2.9]. Using the same method as in [15, §2] we have:

**Theorem 3.8.** For a finite extension $L/K$ of number fields, the extension $H(L)/K$ is Ostrowski.

**Proof.** Using the principal ideal theorem, we have

$$\epsilon_{H(L)/K}(\text{Cl}(K)) = \epsilon_{H(L)/L} (\epsilon_{L/K}(\text{Cl}(K))) \subseteq \epsilon_{H(L)/L} (\text{Cl}(L)) = \{0\}.$$  

Therefore

$$\text{Ost}(H(L)/K) = \text{Po}(H(L)/K) \subseteq \text{Po}(H(L)/L) = \{0\}.$$  

Indeed, the first equality follows from Definition 3.2, the middle containment follows from part (ii) of Lemma 2.5 and the last equality follows from part (iii) of Corollary 2.6. □
Remark 3.9. There is another proof for Theorem 3.8: One can relativize Leriche’s method in [12, proof of Proposition 3.1] to show that all relative Ostrowski ideals \( \Pi_{f}(H(L)/K) \) belong to \( \epsilon_{H(L)/K}(\text{Cl}(K)) \). Further, in the case that \( L/K \) is a Galois extension, using equality (3.3) one can see that the result of Brumer-Rosen in [1, Proposition 2.4] coincides with Theorem 3.8.

Definition 3.10. [12, Definition 2.3] An extension \( L/K \) is said to be a Pólya extension if \( \epsilon_{L/K}(\text{Po}(K)) = \{0\} \).

Remark 3.11. There is another notion of Pólya extension due to Spickermann [20]: a finite Galois extension \( L/K \) of number fields with Galois group \( G \) is said to be a Pólya extension, in the sense of Spickermann, if \( I(L)G \subseteq I(K).P(L) \). Indeed, this definition is equivalent to \( \text{Po}(L/K) = \epsilon_{L/K}(\text{Cl}(K)) \), see [3, page 11]. In other words, for Galois extensions the notion of Pólya extension in Spickermann’s sense coincides with \( \text{Ost}(L/K) = \{0\} \).

Convention. Throughout this article, by a Pólya extension we mean the same notion as in Definition 3.10, and not in the sense of Spickermann.

The following relation between Pólya fields and Pólya extensions has been found by Leriche:

Proposition 3.12. [12, Proposition 3.4] Let \( L/K \) be a Pólya extension of Galois number fields. If all finite places are unramified in the extension \( L/K \), then \( L \) is a Pólya field.

We aim to generalize the above result of Leriche. First we relativize the notion of Pólya extension:

Definition 3.13. For \( F \subseteq K \subseteq L \) a tower of finite extensions of number fields, the extension \( L/K \) is called an \( F \)-relative Pólya extension whenever \( \epsilon_{L/K}(\text{Po}(K/F)) = \{0\} \). Note that \( \mathbb{Q} \)-relative Pólya extension is the same notion as in Definition 3.10.

Theorem 3.14. Let \( F \subseteq K \subseteq L \) be a tower of finite extensions of number fields. If both \( K/F \) and \( L/F \) are Galois extensions, then the following sequence is exact:

\[
\{0\} \to \text{Ker}(\psi) \to \text{Coker}(\gamma) \to \bigoplus_{\mathfrak{p}} \mathbb{Z}/\mathfrak{p}(\mathbb{Z}) \to \text{Ost}(L/F) / \psi(\text{Ost}(K/F)) \to \{0\},
\]

where the maps \( \gamma \) and \( \psi \) are defined as follows (\( \theta_{K/F} \) and \( \theta_{L/F} \) are as in exact sequence (3.2) and “Inf” denotes the inflation map):

\[
\gamma : \frac{H^1(Gal(K/F), U_K)}{\theta_{K/F}(\text{Ker}(\epsilon_{K/F}))} \to \frac{H^1(Gal(L/F), U_L)}{\theta_{L/F}(\text{Ker}(\epsilon_{L/F}))}
\]

\[
[\sigma] \pmod{\theta_{K/F}(\text{Ker}(\epsilon_{K/F}))} \mapsto \text{Inf}([\sigma]) \pmod{\theta_{L/F}(\text{Ker}(\epsilon_{L/F}))}
\]

and

\[
\psi : \text{Ost}(K/F) \to \text{Ost}(L/F)
\]

\[
[a] \pmod{\epsilon_{K/F}(\text{Cl}(F))} \mapsto \epsilon_{L/K}([a]) \pmod{\epsilon_{L/F}(\text{Cl}(F))}
\]
Proof. Using exact sequence (3.2) we find the following commutative diagram with exact rows of abelian groups:

\[
\begin{array}{ccccccccc}
0 & \rightarrow & \ker(\epsilon_{K/F}) & \rightarrow & H^1(Gal(K/F), U_K) & \rightarrow & \bigoplus_{\mathfrak{p}} \mathbb{Z}/\mathfrak{p}(K/F) & \rightarrow & \text{Ost}(K/F) & \rightarrow & 0 \\
& & \| & & \| & & \| & & \| & & \\
0 & \rightarrow & \ker(\epsilon_{L/F}) & \rightarrow & H^1(Gal(L/F), U_L) & \rightarrow & \bigoplus_{\mathfrak{p}} \mathbb{Z}/\mathfrak{p}(L/F) & \rightarrow & \text{Ost}(L/F) & \rightarrow & 0 \\
\end{array}
\]

Note that \(\psi\) is well-defined:

For \([a_1], [a_2] \in \text{Po}(K/F)\), if \([a_1] = [a_2] \mod \epsilon_{K/F}(\text{Cl}(F))\), then \([a_1], [a_2]^{-1} = \epsilon_{K/F}([b])\) for some \([b] \in \text{Cl}(F)\). Hence

\[
\epsilon_{L/K}([a_1], [a_2]^{-1}) = \epsilon_{L/K}(\epsilon_{K/F}([b])) = \epsilon_{L/F}([b]) \in \epsilon_{L/F}(\text{Cl}(F)).
\]

Equivalently, the following diagram is commutative:

\[
\begin{array}{ccccccccc}
0 & \rightarrow & H^1(Gal(K/F), U_K) & \rightarrow & \bigoplus_{\mathfrak{p}} \mathbb{Z}/\mathfrak{p}(K/F) & \rightarrow & \text{Ost}(K/F) & \rightarrow & 0 \\
& & \| & & \| & & \| & & \| & & \\
0 & \rightarrow & H^1(Gal(L/F), U_L) & \rightarrow & \bigoplus_{\mathfrak{p}} \mathbb{Z}/\mathfrak{p}(L/F) & \rightarrow & \text{Ost}(L/F) & \rightarrow & 0 \\
\end{array}
\]

We show \(\gamma\) is also well-defined:

For \([\sigma_1], [\sigma_2] \in H^1(Gal(K/F), U_K)\), if \([\sigma_1] = [\sigma_2] \mod \theta_{K/F}(\ker(\epsilon_{K/F}))\), then \([\sigma_1], [\sigma_2]^{-1} = \theta_{K/F}([a])\) for some \([a] \in \ker(\epsilon_{K/F})\). Using diagram (3.3) we have

\[
\text{Inf}([\sigma_1], [\sigma_2]^{-1}) = \text{Inf}(\theta_{K/F}([a])) = \theta_{L/F}([a]) \in \theta_{L/F}(\ker(\epsilon_{L/F})).
\]

(Note that \(\ker(\epsilon_{K/F}) \subseteq \ker(\epsilon_{L/F})\)). Using the snake lemma, we obtain the desired exact sequence. (Note that the snake lemma also yields \(\gamma\) is one-to-one.). \(\square\)

Corollary 3.15. With the same notations and assumptions in Theorem 3.14 the following assertions hold:

(i) If all finite places of \(K\) are unramified in \(L\), then \(\text{Ost}(L/F) = \psi(\text{Ost}(K/F))\).

(ii) If \(L/K\) is an \(F\)-relative Pólya extension, then the following sequence is exact:

\[
\{0\} \rightarrow \text{Ost}(K/F) \rightarrow \text{Coker}(\gamma) \rightarrow \bigoplus_{\mathfrak{p}} \mathbb{Z}/\mathfrak{p}(L/K) \rightarrow \text{Ost}(L/F) \rightarrow \{0\}.
\]

(iii) If \(\epsilon_{K/F}(\text{Cl}(F)) = \{0\}\), then the following sequence is exact:

\[
\{0\} \rightarrow \ker(\epsilon_{L/K} |_{\text{Po}(K/F)}) \rightarrow \ker(\text{inf}) \rightarrow \bigoplus_{\mathfrak{p}} \mathbb{Z}/\mathfrak{p}(L/K) \rightarrow \text{Po}(L/F) \rightarrow \{0\}.
\]

Proof. Immediately follows from Theorem 3.14. For proving part (ii) note that since \(L/K\) is an \(F\)-relative Pólya extension, \(\psi\) is the zero map. \(\square\)

Remark 3.16. Combining the first two parts or using part (iii) of Corollary 3.15 we find a generalization of Leriche’s result in Proposition 3.12.
We recall that for a finite extension $K/F$ of number fields, the relative genus field of $K$ over $F$, denoted by $\Gamma(K/F)$, is the maximal abelian unramified extension of $K$ of the form $K\alpha$ for some abelian extension $F\alpha/F$ [5]. By the genus field of $K$, denoted by $\Gamma(K)$, we mean the absolute genus field of $K$ (i.e. $F = \mathbb{Q}$). As an application of Proposition 3.12, Leriche proved that:

**Proposition 3.17.** [12, Proposition 3.4, Theorem 3.8] If $K$ is an abelian number field then the genus field $\Gamma(K)$ of $K$ is Pólya.

Unlike the Hilbert class field, the relative Pólya group of the genus field, even for abelian number fields, is not necessarily trivial. For instance, one can show that for $K = \mathbb{Q}(\sqrt{-23})$, we have $\Gamma(K)/K = \mathbb{Q}/Z/3Z$, see [15, Example 2.14]. Whereas using the following principal ideal theorem of Terada [22], one can generalize Proposition 3.17 for Ostrowski quotients in “cyclic extensions”:

**Proposition 3.18.** [22] Let $K/F$ be a finite cyclic extension of number fields. Then every ambiguous ideal class in $K/F$ will be principal in $\Gamma(K/F)$.

**Theorem 3.19.** For $K/F$ a finite cyclic extension of number fields, the extension $\Gamma(K/F)/F$ is Ostrowski.

**Proof.** By Proposition 3.18, $\epsilon_{\Gamma(K/F)/K}(\text{Po}(K/F)) = \{0\}$, i.e. $\Gamma(K/F)/K$ is an $F$-relative Pólya extension. Since all finite places of $K$ are unramified in $\Gamma(K/F)$, part (ii) of Corollary 3.15 proves that $\text{Ost}(\Gamma(K/F)/F) = \{0\}$. □

4. Relativization of the of pre-Pólya group

As stated before, for a Galois number field $L$ all the Ostrowski ideals above unramified primes in $L$ are automatically principal, whereas in the non-Galois case this is not true, in general. Based on this, Zantema introduced the notion of the pre-Pólya field and gave an interesting group theoretical description of the pre-Pólya condition for number fields [24, §6]. Recently, the pre-Pólya condition has been generalized to the notion of the pre-Pólya group $\text{Po}(\cdot)$ by Chabert and Halberstadt [4]. Relativizing these notions, one can define:

**Definition 4.1.** The pre-Ostrowski quotient $\text{Ost}(L/K)_{nr}$ for a finite extension $L/K$ of number fields, is the subgroup of $\text{Ost}(L/K)$ defined as follows:

\[
\text{Ost}(L/K)_{nr} = \frac{\text{Po}(L/K)_{nr}}{\text{Po}(L/K)_{nr} \cap \epsilon_{L/K}(\text{Cl}(K))},
\]

where

\[
\text{Po}(L/K)_{nr} = \left\{ \left[ \prod_{f} (L/K) \right] : \text{f is a prime in } K \text{ unramified in } L \right\}.
\]

In particular $\text{Ost}(L/\mathbb{Q})_{nr}$ coincides with the “pre-Pólya group” $\text{Po}(L)_{nr}$ defined in [4] §1.

**Lemma 4.2.** For $L/K$ a finite extension of number fields,

(i) $\text{Ost}(L/K)_{nr}$ is trivial if and only if $\text{Po}(L/K)_{nr} \subseteq \epsilon_{L/K}(\text{Cl}(K))$. In particular if $L/K$ is a Galois extension, then $\text{Ost}(L/K)_{nr} = \{0\}$.

(ii) If $\epsilon_{L/K}(\text{Cl}(K)) = \{0\}$, then $\text{Ost}(L/K)_{nr} = \text{Po}(L/K)_{nr}$;

(iii) If $L/K$ is Galois and all finite places of $K$ are unramified in $L$, then $\text{Ost}(L/K)_{nr}$ is trivial and

$\text{Po}(L/K)_{nr} = \text{Po}(L/K) = \epsilon_{L/K}(\text{Cl}(K))$. 


Proof. (i) The first assertion is obvious by Definition 4.1. If \( L/K \) is Galois, then for a prime \( p \) of \( K \) unramified in \( L \), all primes of \( L \) above \( p \) have the same residue class degree, say \( f_p \), which implies that

\[
\Pi_{p/K}(L/K) = pO_L \in \epsilon_{L/K}(Cl(K)).
\]

(Note that for the special case \( K = \mathbb{Q} \) every Galois number field \( L \) is a “pre-Pólya” field, see [24, § 6]).

(ii) Immediately follows from Definition 4.1.

(iii) Since \( L/K \) is Galois and unramified, exact sequence (3.2) yields \( \text{Ost}(L/K) \) is trivial or equivalently

\[
\text{Po}(L/K) = \epsilon_{L/K}(Cl(K)).
\]

The equality \( \text{Po}(L/K)_{nr} = \text{Po}(L/K) \) holds since \( L/K \) is unramified. \( \square \)

**Example 4.3.** Let \( K = \mathbb{Q}(\sqrt{-4027}) \) and \( \alpha \) be a root of \( f(x) = x^3 + 10x + 1 \). One can show that \( L = K(\alpha) \) is a cyclic unramified extension of \( K \), see [14, Example 2.14]. By Lemma 4.2, \( \text{Ost}(L/K)_{nr} \) is trivial and

\[
\text{Po}(L/K)_{nr} = \text{Po}(L/K) = \epsilon_{L/K}(Cl(K)).
\]

Indeed, since \( L/K \) is unramified, by exact sequence (5.2) we find

\[
\# \ker(\epsilon_{L/K}) = \# H^1(\text{Gal}(L/K), U_L) = \frac{\# H^0(\text{Gal}(L/K), U_L)}{Q(\text{Gal}(L/K), U_L)},
\]

where \( Q(\text{Gal}(L/K), U_L) \) denotes the the Herbrand quotient of \( U_L \) (as \( \text{Gal}(L/K) \)-module). One can easily show that

\[
Q(\text{Gal}(L/K), U_L) = 1/3, \quad \# H^0(\text{Gal}(L/K), U_L) = 1,
\]

since \( U_K = \{\pm 1\} \) and \( N_{L/K}(-1) = -1 \) (for more details see proof of Theorem 2.12). Therefore the capitulation kernel of \( L/K \) has order three. On the other hand, since \( h_K = 9 \), we have

\[
\# \text{Po}(L/K)_{nr} = \# \text{Po}(L/K) = \# \epsilon_{L/K}(Cl(K)) = 3.
\]

Hence \( \text{Po}(L/K) \) is a nontrivial proper subgroup of \( Cl(L) \), since \( h_L = 108 \).

**Theorem 4.4.** For a finite Galois extension \( L/K \) of number fields, we have

\[
\text{Ost}(L/K) \subseteq \frac{\text{Po}(L/K)}{\text{Po}(L/K)_{nr}}.
\]

Proof. Since \( L/K \) is Galois, the containsments

\[
(4.3) \quad \text{Po}(L/K)_{nr} \subseteq \epsilon_{L/K}(Cl(K)) \subseteq \text{Po}(L/K)
\]

prove the assertions. \( \square \)

**Acknowledgment**

The authors would like to thank the anonymous referees for their valuable comments and carefully reading the first draft.
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