On a class of bi–orthogonal polynomials on the unit circle

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Abstract

We consider the system of bi–orthogonal polynomials with respect to a complex valued measure supported on the unit circle and give a uniform compound asymptotic expansion formula consisting of the sum of two inverse factorial series, giving the explicit expression of the terms and including error bounds. This asymptotic expansion holds uniformly in compact subsets of \(C \setminus \{1\}\) and turns out to be convergent in compact subsets of \(|z| < |z−1| \cap \{1 < |z−1|\}\). We give also an explicit expression for the coefficients of the terms of an asymptotic formula given by Askey for this bi–orthogonal system. An electrostatic interpretation in the unit circle for the zeros of a class of para-orthogonal polynomials associated with the bi–orthogonal system is also considered.

1 Introduction and statement of the results

In this paper we present some results for a system of bi–orthogonal polynomials introduced by R. Askey \(^2\) Vol. 1\) in his discussions regarding the Szegő paper: Beiträge zur Theorie der Toeplitzschen Formen, 1921–1. Motivated by the fact that the families of Hermite, Jacobi, and Laguerre orthogonal polynomials are orthogonal with respect to the normal, beta and gamma distributions respectively, Askey found a complex measure with support on the unit circle that is of beta function type and pointed out that there is a family of bi–orthogonal polynomials for this measure. To be more precise, we are interested in the two–parameter system \(\{P_n, Q_n\}_{n \geq 0}\) of polynomials given by

\[
P_n(z; \alpha, \beta) = _2F_1(-n, \alpha + \beta + 1; 2\alpha + 1; 1 - z) \\
Q_n(z; \alpha, \beta) = P_n(z; \alpha, -\beta),
\]

which is bi-orthogonal with respect to the complex valued weight \(\omega(\theta) = (1 - e^{i\theta})^{\alpha + \beta}(1 - e^{-i\theta})^{\alpha - \beta} = (2 - 2 \cos \theta)^{\alpha}(e^{-i\theta})^{\beta}, \theta \in [-\pi, \pi], \Re(\alpha) > -\frac{1}{2}\), that is

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} P_n(e^{i\theta}; \alpha, \beta)Q_m(e^{-i\theta}; \alpha, \beta)\omega(\theta)d\theta = \frac{\Gamma(2\alpha + 1)}{\Gamma(\alpha + \beta + 1)\Gamma(\alpha - \beta + 1)} \frac{n!}{(2\alpha + 1)_n} \delta_{n,m}, \tag{2}
\]

where \(\Gamma\) denotes the Euler Gamma function.
The bi–orthogonality (2) was stated in [2] in a slightly different form and a formal proof was given in [3]. Here we have used the notation of [3] for (1). A proof different from Askey’s was first given by Greiner and Koornwinder in [11] (1.15) where it was pointed out that the polynomials $P_n$ can be expressed in terms of the Heisenberg polynomials, introduced by Greiner [10]. Another interpretation and proof of the bi–orthogonality was given in [13 a) after Remark (3.4)].

Asymptotic properties and electrostatic interpretation of the zeros of orthogonal polynomials are commonly studied themes in the theory of orthogonal polynomials and mathematical physics.

Asymptotic properties for Toeplitz and Hankel determinants, for a more general class of weights including $\omega$, i.e, weights with a fixed number of Fisher–Hartwig singularities [8] [16] [17] has been obtained by Basor, Tracy and also other authors in a series of papers. More recently, Deift, Its and Krasovsky in [5] by using the Riemann–Hilbert approach obtained the general non–degenerate asymptotic behavior for Toeplitz determinants for such weights, as conjectured by Basor and Tracy. They also obtained asymptotics expansions for Hankel and Toeplitz–Hankel determinants, see also please the references within this last paper. Basor’s classic work [4], as Askey has already pointed out in [2], suggests that the bi–orthogonal system (1) give the right analogue of the Jacobi polynomials on the unit circle rather than the polynomials given by Szegő in [26 (11.5.4)]. Without having an explicit formula for the bi–orthogonal system polynomials, and by using an indirect method of solving the strong Szegő limit problem for Toeplitz determinants, she obtained an asymptotic formula for the bi–orthogonal system (1) analogous to the existent for Jacobi polynomials in terms of Bessel functions, c.f. [26 (8.21.17)]. Having the explicit expression of the bi–orthogonal system, Askey obtains the formula

$$P_n(e^{i\theta}; \alpha, \beta) \sim _1F_1(\alpha + \beta + 1; 2\alpha + 1; n\theta), \quad \text{as} \quad n \to \infty, \quad (3)$$

which is analogous to the one for Jacobi polynomials $P_n^{(\alpha, \beta)}$\n
$$n^{-\alpha}P_n^{(\alpha, \beta)}(\cos \theta/n) \sim \left(\frac{\theta}{2}\right)^{-\alpha} J_\alpha(\theta), \quad \text{as} \quad n \to \infty.$$

Askey remarked that it is interesting to understand the effect of the zeros of the weight function on the asymptotic behavior of the orthogonal polynomials and this raises the question, which shall be referred to in the present manuscript as the Askey problem, of how to obtain the first term or preferably, more terms for the remainder in the asymptotic formula (3) as well as bounds for the remainder.

Progress in understanding the asymptotic behavior in compact subsets of the complex plane of orthogonal polynomials for weights having Fisher–Hartwig singularities has been obtained by Deift, Its and Krasovsky in [5] by using the Riemann–Hilbert approach and for positive weights with such singularities but without jumps by Martinez–Finkelshtein, McLaughlin and Saff in [19].

Temme in [27] found an infinite power series asymptotic expansion for the bi–orthogonal system (1). He proved that, for $z$ and $(\alpha, \beta)$ varying in compact subsets of $\mathbb{C} \setminus \{0\}$ and $\{ (\alpha, \beta) \in \mathbb{C}^2 : \Re(\alpha + \beta) > -1, \Re(\alpha – \beta) \geq 0 \}$ respectively, it holds that

$$P_n(z; \alpha, \beta) \sim \frac{\Gamma(2\alpha + 1)}{\Gamma(\alpha + \beta + 1)} z^{\alpha-\beta-1} \left(\frac{\ln z}{z - 1}\right)^{2\alpha} \times$$

$$\left(\sum_{k=0}^{\infty} \frac{A_k}{(n + 1)^k} + \sum_{k=0}^{\infty} \frac{B_k}{(n + 1)^{k+1}} + R_p\right), \quad \text{as} \quad n \to \infty,$$
where \( \varphi_0 = \frac{\Gamma(\alpha + \beta + 1)}{\Gamma(2\alpha + 1)} F_1(\alpha + \beta + 1, 2\alpha + 1; (n + 1) \ln z) \), \( \varphi_1 = \frac{\Gamma(\alpha + \beta + 2)}{\Gamma(2\alpha + 2)} F_1(\alpha + \beta + 2, 2\alpha + 2; (n + 1) \ln z) \) and \( A_k, B_k \) are coefficients defined by the recursion relations \([27, (2.13)]\). Moreover, a bound for the remainder \( R_p \) for this asymptotic expansion defined as

\[
P_n(z; \alpha, \beta) = \frac{\Gamma(2\alpha + 1)}{\Gamma(\alpha + \beta + 1)} z^{\alpha - \beta - 1} \left( \frac{\ln z}{z - 1} \right)^{2\alpha} \times \left( \varphi_0 \sum_{k=0}^{p-1} \frac{A_k}{(n + 1)^k} + \varphi_1 \sum_{k=0}^{p-1} \frac{B_k}{(n + 1)^k} + R_p \right), \quad n, p \in \mathbb{N},
\]

is given by

\[
|R_p| \leq \frac{M_p}{(n + 1)^p} \left| \frac{\Gamma(\alpha + \beta + 1)}{\Gamma(2\alpha + 1)} \left| F_1(\alpha + \beta + 1, 2\alpha + 1; (n + 1) \Re \ln z) \right| \right|,
\]

where \( M_p \) is some positive constant depending only on \( p \). Temme has remarked that the evaluation of the coefficients \( A_k, B_k \) is difficult, especially near or at unity.

The above asymptotic expansion gives, for \( z = e^{i\theta} \) as a particular case, an answer to the Askey problem, provided that \((\alpha, \beta)\) varies in compact subsets of \( \{(\alpha, \beta) \in \mathbb{C}^2 : \Re(\alpha + \beta) > -1, \Re(\alpha - \beta) \geq 0\}\).

In the present manuscript we give a new uniform asymptotic expansion for the bi–orthogonal system \([1]\) consisting of a sum of two inverse factorial series, for \( z \) and \((\alpha, \beta)\) varying in compact subsets of \( \mathbb{C} \setminus \{1\} \) and \( \{(\alpha, \beta) \in \mathbb{C}^2 : \Re(\alpha + \beta) > -1, \Re(\alpha - \beta) > 0\} \) respectively. We give the explicit expression of all the terms and bounds for the remainders as well. The expression for the remainders allows us to prove that the asymptotic expansion turn out to be convergent in compact subsets of \( \{|z| < |z - 1|\} \cap \{1 < |z - 1|\} \). In regard to Askey’s problem, we give a different solution from the one given by Temme for the explicit expression of the terms of the asymptotic formula \([3]\). We also consider bounds for the remainder for our expansion, which turns out to be convergent.

We show also that, as their closest relatives, the Jacobi Polynomials, the zeros of a class of para–orthogonal polynomials associated to the bi–orthogonal system \([1]\) have an electrostatic interpretation very much in the classical sense of Stieltjes. We recall that similar electrostatic models exists for the zeros of other families of orthogonal polynomials with respect to measures supported on the real line. Grünbaum in \([12]\) described an electrostatic model for the zeros of the Koornwinder polynomials, Ismail in \([14]\) gave another model for the zeros of orthogonal polynomials with respect to a measure satisfying certain integrability conditions with an absolutely continuous part and a finite discrete part.

We state our results in the next subsection below. The proof for the asymptotic expansion will be given in Section \([2]\). Section \([3]\) deals with Askey’s problem and in Section \([4]\) we give the proof for the electrostatic model of the zeros.

### 1.1 Statement of the results

In this subsection we state the results and introduce some notation as well.

For \( a, b \in \mathbb{C} \), we denote by \( \binom{a}{b} \) the generalized binomial coefficient, i.e.,

\[
\binom{a}{b} = \frac{\Gamma(a + 1)}{\Gamma(b + 1)\Gamma(a - b + 1)}.
\]
Let $B(a, b)$ denote the beta function

$$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a + b)}.$$  

For $a \in \mathbb{R}$, the symbol $\lfloor a \rfloor$ stands for the floor function of $a$. For $z, \alpha \in \mathbb{C}$, we choose $\arg z$ as the principal value of the argument. We define the functions $z^\alpha$ and $\ln z$ according to the branch of $\arg z$.

Let $D$ be the open unit disk $\{z : |z| < 1\}$ and $\Omega_0 = \{(\alpha, \beta) \in \mathbb{C}^2 : -1 < \Re(\alpha + \beta), 0 < \Re(\alpha - \beta)\}$.

If $K \subset \mathbb{C}$, then $\partial(K), K, \bar{K}$ denote the boundary, closure and interior of $K$ respectively.

We denote by using $B^{(n)}_n(x)$ the generalized Bernoulli polynomials, defined from the generating function \[38\] Sec. 2.8

$$\left( \frac{z}{e^z - 1} \right)^\alpha e^{xz} = \sum_{n=0}^{\infty} B^{(n)}_n(x) \frac{z^n}{n!}, |z| < 2\pi, \alpha \in \mathbb{C}.$$  

**A new asymptotic expansion**

The concept of asymptotic expansion goes back to Poincaré and Stieltjes, both introduced this concept in 1886, this notion embraces a wide class of useful divergent series. Special asymptotic expansion, however, had been discovered before and used in the eighteenth century by Stirling, MacLaurin and Euler. Asymptotic expansions are of great importance in various areas of mathematical physics, please see \[9\] and references within in. We will not give here a detailed discussion on this theory, for this we refer the reader to the monographs \[24\], \[31\], and \[30\]. We remember that

**Definition 1.** Let $f$ and $g_1, \ldots, g_N$ and $\{f^{(i)}_k\}_{k \in \mathbb{N}}, \ldots, \{f^{(N)}_k\}_{k \in \mathbb{N}}$ be functions in the variable $z$ defined in a common set $\Omega$. The relation,

$$f(z) = g_1(z) \left( \sum_{k=0}^{P_1} f^{(1)}_k(z) + \xi^{(1)}_{P_1}(z) \right) + \ldots + g_N(z) \left( \sum_{k=0}^{P_N} f^{(N)}_k(z) + \xi^{(N)}_{P_N}(z) \right),$$

is called a compound asymptotic expansion of Poincaré type as $z \to z_0$, if $\xi_{p_i}^{(i)}(z) = O(f^{(i)}_{p_i+1}(z))$, $z \to z_0$, $\forall i = 1, \ldots, N$ and $\forall p_1, \ldots, p_N \in \mathbb{N} \cup \{0\}$.

Sometimes the above relation in the definition is expressed equivalently

$$f(z) \sim g_1(z) \sum_{k=0}^{\infty} f^{(1)}_k(z) + \ldots + g_N(z) \sum_{k=0}^{\infty} f^{(N)}_k(z), \quad z \to z_0.$$  

The functions $\{\xi_{p_i}^{(i)}\}_{i=1}^{N}$ in the above definition are called the remainders of the expansion. Compound asymptotic expansions are characterized by having several remainders, none of which is absorbable in the others.

A series of inverse factorials, or a factorial series of the first kind, is a series involving Pochhammer symbols

$$a_0 z + \frac{a_1 1!}{z(z+1)} + \frac{a_2 2!}{z(z+1)(z+2)} + \ldots = \sum_{k=0}^{\infty} \frac{a_{\nu} \nu!}{(z)_{\nu+1}}.$$  

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They were used by Stirling in his classic book *Methodus Differentialis* (1730) and after by Schlömilch in 1863, Kluwer and Pincherle in 1902, please see [30] pp. 142 for details. Background information for this class of series may be found in the treatises [20, Ch. 10], [22, Ch. 6], and [30, pp. 142].

Factorial asymptotic expansions involving transformations of Gamma functions has been obtained by [23, Eq. 43], [29, Ch. V], for Bessel functions by [29, Ch. V]. In [6] it is found a factorial series expansions as the argument $z \to \infty$ for all of the standard Bessel and modified Bessel functions, such asymptotic expansions turns out to be convergent in the half plane $\Re(z) \geq \epsilon > 0$, with $\epsilon$ arbitrary. There exists a general theory for the asymptotic expansion in terms of such series, see [28], in the present manuscript we will follow this approach.

Our first result deals with a compound asymptotic expansion of Poincaré type involving two series of inverse factorials, with an accurate estimation of the remainders, which is given in the following

**Theorem 1.** The following compound asymptotic expansion of Poincaré type holds,

$$P_n(z; \alpha, \beta) \sim \frac{\Gamma(2\alpha + 1)}{\Gamma(\alpha + \beta + 1)} \frac{\Gamma(n + 1)}{\Gamma(n + \alpha - \beta + 1)} z^{n+\alpha-\beta} (z-1)^{\beta-\alpha}$$

$$\times \left( \sum_{k=0}^{\infty} \binom{\alpha + \beta}{k} \left( \frac{z}{1-z} \right)^k \frac{\Gamma(k + \alpha - \beta)}{\Gamma(\alpha - \beta)} \frac{1}{(n + 1 + \alpha - \beta)_k} \right) + \frac{\Gamma(2\alpha + 1)}{\Gamma(\alpha - \beta)} \frac{\Gamma(n + 1)}{\Gamma(n + \alpha + \beta + 2)} (1-z)^{-\alpha-\beta-1}$$

$$\times \left( \sum_{k=0}^{\infty} \binom{\alpha - \beta - 1}{k} \frac{1}{(z-1)^k} \frac{\Gamma(k + \alpha + \beta + 1)}{\Gamma(\alpha + \beta + 1)} \frac{1}{(n + 2 + \alpha + \beta)_k} \right),$$

as $n \to \infty,$

uniformly in $z$ and $(\alpha, \beta)$ varying in compact subsets of $\mathbb{C} \setminus \{1\}$ and $\Omega_0$ respectively. For $n, p \in \mathbb{N} \cup \{0\}$, the remainders of the above asymptotic expansion defined by

$$P_n(z; \alpha, \beta) = \frac{\Gamma(2\alpha + 1)}{\Gamma(\alpha + \beta + 1)} \frac{\Gamma(n + 1)}{\Gamma(n + \alpha - \beta + 1)} z^{n+\alpha-\beta} (z-1)^{\beta-\alpha}$$

$$\times \left( \sum_{k=0}^{p_1} \binom{\alpha + \beta}{k} \left( \frac{z}{1-z} \right)^k \frac{\Gamma(k + \alpha - \beta)}{\Gamma(\alpha - \beta)} \frac{1}{(n + 1 + \alpha - \beta)_k} + \xi_{1,n,p_1}(z; \alpha, \beta) \right)$$

$$+ \frac{\Gamma(2\alpha + 1)}{\Gamma(\alpha - \beta)} \frac{\Gamma(n + 1)}{\Gamma(n + \alpha + \beta + 2)} (1-z)^{-\alpha-\beta-1}$$

$$\times \left( \sum_{k=0}^{p_2} \binom{\alpha - \beta - 1}{k} \frac{1}{(z-1)^k} \frac{\Gamma(k + \alpha + \beta + 1)}{\Gamma(\alpha + \beta + 1)} \frac{1}{(n + 2 + \alpha + \beta)_k} + \xi_{2,n,p_2}(z; \alpha, \beta) \right),$$

as $n \to \infty,$

uniformly in $z$ and $(\alpha, \beta)$ varying in compact subsets of $\mathbb{C} \setminus \{1\}$ and $\Omega_0$ respectively.
satisfy

\[
|\xi_{1,n,p_1}(z;\alpha,\beta)| \leq \frac{1}{(n+1+\Re(\alpha-\beta))_{p_1+1}} \frac{\Gamma(p_1+\Re(\alpha-\beta))}{|\Gamma(\alpha-\beta)|} |z|^{p_1+1} \times \\
\begin{cases}
  m_1 \left( \frac{z}{1-z}; \alpha+\beta \right) + p_1 e^{[\alpha+\beta]_2+\Re(\alpha+\beta)} + 1, & 0 \leq \Re(\alpha+\beta), \\
  m_2(z;p_1-2\beta,\alpha+\beta) \frac{\Gamma(p_1+1-2\Re(\beta))}{\Gamma(p_1+1+\Re(\alpha-\beta))} \frac{\Gamma(n+p_1+2+\Re(\alpha-\beta))}{\Gamma(n+p_1+2-2\Re(\beta))} + \\
  m_3(p_1,\alpha+\beta), & -1 < \Re(\alpha+\beta) < 0,
\end{cases}
\]

\[
|\xi_{2,n,p_2}(z;\alpha,\beta)| < \frac{1}{(n+2+\Re(\alpha+\beta))_{p_2+1}} \frac{\Gamma(p_2+2+\Re(\alpha+\beta))}{|\Gamma(\alpha+\beta+1)|} |z|^{p_2+1} \times \\
\begin{cases}
  m_1 \left( \frac{1}{z-1}; \alpha - \beta - 1 \right) + p_2 e^{[\alpha-\beta-1]_2+\Re(\alpha-\beta)-1} + 1, & 1 \leq \Re(\alpha-\beta), \\
  m_2(z^{-1};p_2+2\beta+2,\alpha-\beta-1) \frac{\Gamma(p_2+2\Re(\beta)+3)}{\Gamma(p_2+\Re(\alpha+\beta)+2)} \times \\
  \frac{\Gamma(n+p_2+3+\Re(\alpha+\beta))}{\Gamma(n+p_2+2\Re(\beta)+4)} + m_3(p_2,\alpha-\beta+1), & 0 < \Re(\alpha-\beta) < 1,
\end{cases}
\]

and

\[
m_1(z;\gamma) = e^{[\Im(\gamma)] \max(2\Re(\gamma), (1+|z|)^\Re(\gamma))}, \\
m_2(z;\gamma) = e^{[\Im(\gamma)] \left( \frac{|z|+1+\Re(\gamma)}{|z-1||z|^\Re(\gamma)} \left( 1+2 \left( \frac{q+1}{\gamma+1} \right) \right) \right) }, m_2(\infty;\gamma) = \lim_{z \to \infty} m_2(z;p,\gamma), \\
m_3(p;\gamma) = \max \left( e^{-[\Im(\gamma)][p-\gamma+1]} + 1 + p e^{[\Im(\gamma)]_2+\Re(\gamma)}, 1 + \frac{\sin \pi \gamma}{\pi (1+\Re(\gamma))} \right).
\]

**Remark 1.** By virtue of the second relation in (1) we have

\[ Q_n(z;\alpha,\beta) = P_n(z;\alpha,-\beta), \]

therefore, a similar expansion will holds for the polynomial \( Q_n \), with \( \beta \) replaced by \(-\beta\), provided that \( \Re(\alpha-\beta) > -1, \Re(\alpha+\beta) > 0 \), please see the beginning of section[2]

As a consequence of the above theorem we obtain

**Corollary 1.** Let \( \Omega_1 = \{ z \in \mathbb{C} : |z| < |z-1| \} \cap \{ z \in \mathbb{C} : |z-1| > 1 \} \) and \( n \in \mathbb{N} \cup \{0\} \) fixed. Then,

\[
P_n(z;\alpha,\beta) = \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+\beta+1)} \frac{\Gamma(n+1)}{\Gamma(n+1+\alpha-\beta)} e^{z^{n+\alpha-\beta} (z-1)^{\beta-\alpha}} \\
\times \sum_{k=0}^{\infty} \binom{\alpha+\beta}{k} \left( \frac{z}{1-z} \right)^k \frac{\Gamma(k+\alpha-\beta)}{\Gamma(\alpha-\beta)} \frac{1}{(n+1+\alpha-\beta)_k} \\
+ \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha-\beta)} \frac{\Gamma(n+1)}{\Gamma(n+2+\alpha+\beta)} (1-z)^{-\alpha-\beta-1} \\
\times \sum_{k=0}^{\infty} \binom{\alpha-\beta-1}{k} \frac{1}{(z-1)^k} \frac{\Gamma(k+\alpha+\beta+1)}{\Gamma(\alpha+\beta+1)} \frac{1}{(n+2+\alpha+\beta)_k}.
\]
for \((\alpha, \beta)\) and \(z\) varying in compact subsets of \(\Omega_0\) and \(\Omega_1\) respectively.

For Askey’s Problem, we prove that

**Theorem 2.** Assume that \((\alpha, \beta) \in \Omega_0\), then

\[
P_n \left( e^{in}; \alpha, \beta \right) = \frac{1}{\alpha} \binom{\alpha + \beta + 1}{2\alpha + 1} n! \sum_{j=1}^k \frac{B_{i_1}^{(-\alpha-\beta)}(\alpha - \beta)}{i_1!} \frac{B_{i_2}^{(-\alpha+\beta+1)}(0)}{i_2!} \frac{B_{i_3}^{(2\alpha)}(0)}{i_3!} \times
\]

\[
\frac{(\alpha + \beta + 1)i_1(\alpha - \beta)i_2}{(2\alpha + 1)i_1 + i_2} \binom{\alpha + \beta + 1}{\alpha + \beta + 1} \binom{\alpha + \beta + 1}{\alpha - \beta} X \left( \frac{i\theta}{n} \right)^j n! + R_{k, n}(\theta),
\]

where \(|i| = i_1 + i_2 + i_3, i_1, i_2, i_3 \in \mathbb{N} \cup \{0\}\) and

\[
|R_{k, n}(\theta)| \leq \frac{\Gamma(\alpha + \beta + 1)\Gamma(\alpha - \beta)}{\Gamma(\alpha + \beta + 1)\Gamma(\alpha - \beta)} \left| \frac{2\theta}{3\pi - 2\theta} \right| \left| \frac{2\theta}{3\pi} \right|^k \max_{|v| = \frac{\alpha}{\beta}} \frac{e^{v(\alpha - \beta)}v}{e^v - 1}.
\]

**An electrostatic model**

This subsection discusses an electrostatic model for the zeros for a class of para–orthogonal polynomials associated with the system.

It is well known that, for a positive definite functional, the zeros of the Szegő polynomials all lie in \(D\). In order to develop quadrature rules on the unit circle, it is useful to have orthogonal polynomials with respect to a linear functional whose zeros lie on \(\partial D\). Motivated by this fact Jones, Njastad, and Thron in [15, pp. 130] defined a sequence \(\{X_n\}_{n=0}^\infty\) of para–orthogonal polynomials with respect to a quasi-definite linear functional \(\mu\), if for each \(n \geq 0\), \(X_n\) is a polynomial of degree \(n\) satisfying

\[
\langle X_n, 1 \rangle \neq 0, \quad \langle X_n, z^m \rangle = 0 \text{ for } 1 \leq m \leq n - 1, \quad \text{and} \quad \langle X_n, z^n \rangle \neq 0,
\]

where \(\langle X, Y \rangle = \mu(X(z)\overline{Y}(1/z))\); \(X, Y \in \Lambda, \Lambda\) being the space of all Laurent polynomials. According to these authors, if \(\Phi_n\) is the \(n\)–th monic polynomial with respect to a linear functional \(\mu\), the polynomial

\[
B_n(z; c) = \Phi_n(z) + c\Phi^*(z), \quad |c| = 1,
\]

where \(\Phi^*\) is the reciprocal polynomial, is para-orthogonal polynomial of degree \(n\). From [15] Th. 6.2, if \(\mu\) is a positive definite functional, the \(n\) zeros of the para-orthogonal polynomials are simple and lie on \(\partial D\).

We found that the zeros of a class of para–orthogonal polynomials described in [25] associated to a positive definite functional defined using the weight function of the bi–orthogonal system [1] obey an electrostatic model.

Let us consider the moment functional

\[
\mu(X) = \frac{\Gamma(\alpha + \beta + 1)\Gamma(\alpha)}{2\pi\Gamma(2\alpha + 1)} \int_{-\pi}^\pi X(e^{i\theta})\omega(\theta)d\theta.
\]
Notice that $\mu$ is positive definite if and only if the weight $\omega$ is positive. From the expression for $\omega$, this happens when $\alpha \in \mathbb{R}, \alpha > -\frac{1}{2}$ and $\beta \in \mathbb{R}$. We shall assume in this section that $\alpha$ and $\beta$ satisfy these conditions. Notice that for this case $\omega(\theta) = 2^{2\alpha}e^{(\pi-\theta)\beta(\beta)}\sin^{2\alpha}(\frac{\theta}{2})$.

Ranga in [25] studied the sequence \( \left\{ \frac{(2\alpha + 1)_n}{(n + \beta)_n} P_n(z; \alpha, \beta) \right\}_{n=0}^{\infty} \) of monic orthogonal polynomials with respect to $\mu$, the author obtained that the polynomial

\[ B_n(z; (\alpha - \beta)_{n+1} \frac{(\alpha + \beta + 1)_n}{(\alpha + \beta)_{n+1}} \] is the par-orthogonal monic polynomial with respect to the positive definite linear functional $\mu$.

The electrostatic model for the zeros of $B_n$ can be formulated as the solution of the following problem:

**Problem 1.** Let $p, q$ be two given real numbers, $p > 0$. If $n$ unit masses, $n \geq 2$ at the variable points \( \{e^{i\theta_1}, \ldots, e^{i\theta_n}\} \) in the unit circumference, and one fixed mass point $p$ at $+1$ is considered, for what position of the points \( \{e^{i\theta_1}, \ldots, e^{i\theta_n}\} \) does the expression

\[ E(\theta_1, \ldots, \theta_n) = \sum_{k \neq j} \ln \frac{1}{|e^{i\theta_k} - e^{i\theta_j}|} + p \sum_{j=1}^{n} \ln \frac{1}{|1 - e^{i\theta_j}|} + q \sum_{j=1}^{n} \theta_j, \ \theta_j \in (0, 2\pi), \]

becomes a minimum?

The solution of the above problem is given in the following theorem.

**Theorem 3.** Let $p, q$ be two real numbers, $p > 0$, and let \( \{\theta_1, \ldots, \theta_n\} \) be a system of values for which $E(\theta_1, \ldots, \theta_n)$ is a minimum. Then, \( \{e^{i\theta_1}, \ldots, e^{i\theta_n}\} \) are the zeros of the para-orthogonal polynomial $B_n$ with parameters $\alpha = p$ and $\beta = 2q$.

## 2 Asymptotic expansion for $C \setminus \{1\}$

In this section we prove Theorem 1 and Corollary 1, we start with some general preliminary considerations and notations. From the Euler integral representation for hypergeometric functions [7, (10) pp. 59]

\[ P_n(z; \alpha, \beta) = \frac{\Gamma(2\alpha + 1)}{\Gamma(\alpha + \beta + 1)\Gamma(\alpha - \beta)} \int_0^1 t^{\alpha + \beta}(1-t)^{\alpha - \beta - 1}(1-t(1-z))^n dt. \] (4)

In order to ensure the convergence of the integral we will assume that the parameters $\alpha, \beta$ satisfy the conditions $\Re(\alpha + \beta) > -1, \Re(\alpha - \beta) > 0$.

It follows from the second relation in [3] that the polynomial $Q_n$ has a similar integral representation with $\beta$ replaced by $-\beta$, therefore, the asymptotic expansion given in Theorem 1 holds for $Q_n$ with $\beta$ replaced by $-\beta$, provided that $\alpha$ and $\beta$ satisfy conditions $\Re(\alpha - \beta) > -1, \Re(\alpha + \beta) > 0$.

For a given oriented arc $\delta$, we use $\theta^+$ to denote $\delta$ without self-intersection points or end points. The positive side $\delta^+$ is on the left while traversing $\delta$ according to its orientation, and the negative side $\delta^-$ is on the right.
**Definition 2.** Let $f$ be a function on $\mathbb{C} \setminus \delta$, and $s \in \delta^0$. We say that $f$ has continuous boundary values from $\delta^-$ or from $\delta^+$ if

$$\lim_{v \to s} f(v) = f(s) \quad \text{or} \quad \lim_{v \to s} f(v) = f(s),$$

respectively.

We sketch our proof as follows. In Subsection 2.1 we obtain a decomposition of (4) as a sum of two integrals. In Subsection 2.2, basically by using several identities for hypergeometric functions and the maximum modulus principle for analytic functions we obtain a bound for two terms, which, as will be seen in Subsection 2.3, are the remainders of the asymptotic expansion. In Subsection 2.3, by using the results of the previous subsections, we prove the asymptotic nature of the expansion.

### 2.1 Decomposing the integral representation

The aim of this subsection is to decompose the integral (4) as a sum of two integrals, as is shown in the following lemma.

**Lemma 1.** Let $(\alpha, \beta) \in \Omega_0$, $n \in \mathbb{N} \cup \{0\}$ and $z \in \mathbb{C} \setminus \{1\}$. Then,

$$P_n(z; \alpha, \beta) = \frac{\Gamma(2\alpha + 1)}{\Gamma(\alpha - \beta)\Gamma(\alpha + \beta + 1)} \left( z^{n+\alpha-\beta} (z-1)^{\beta-\alpha} \int_0^1 t^n (1-t)^{\alpha-\beta-1} \left( \frac{1-z}{1-t} \right)^{\alpha+\beta} \, dt \right).$$

**Proof.** By using the transformation $u = 1 - t(1 - z)$, for $z \in (-\infty, 0)$ in (4) and splitting the integral as a sum we obtain

$$\int_0^1 t^n (1-t)\alpha+\beta \left( \frac{1-t}{1-z} \right)^{\alpha-\beta-1} \, dt = \int_0^1 u^n (1-u)^{\alpha+\beta} \left( \frac{1-u}{1-z} \right)^{\alpha-\beta-1} \, du.$$

(5)

The substitution $w = \frac{u}{z}$ in the integral $\int_0^1$ of the right hand side of (5) gives

$$\int_0^1 t^{\alpha+\beta} (1-t)^{\alpha-\beta-1} (1-t(1-z))^n \, dt = (1-z)^{-\alpha-\beta-1} \int_0^1 w^n (1-w)^{\alpha+\beta} \left( \frac{w}{1-z} \right)^{\alpha-\beta-1} \, dw$$

Let us define the functions

$$w^n \left( \frac{w}{1-z} \right)^{\alpha-\beta} \int_0^1 w^n (1-w)^{\alpha-\beta-1} \left( \frac{1-wz}{1-z} \right)^{\alpha+\beta} \, dw. \quad (6)$$
\begin{align*}
  h_1(z) &= (1 - z)^{-\alpha - \beta - 1} \int_0^1 \lambda^{\alpha} (1 - \lambda)^{\alpha + \beta} \left( \frac{z - \lambda}{z - 1} \right)^{\alpha - \beta - 1} d\lambda, \\
  h_2(z) &= z^{\alpha} \left( \frac{z}{z - 1} \right)^{-\beta} \int_0^1 \lambda^{\alpha} (1 - \lambda)^{-\alpha - \beta - 1} \left( \frac{1 - \lambda z}{1 - z} \right)^{\alpha + \beta} d\lambda,
\end{align*}

which are analytic for \( z \in \mathbb{C} \setminus (-\infty, 0) \). From (6) we have
\[
\int_0^1 t^{\alpha + \beta} (1 - t)^{\alpha - \beta - 1} (1 - t(1 - z))^{\alpha} d\lambda = h_1(z) + h_2(z), z \in (-\infty, 0). \tag{7}
\]

Notice that the left hand side of (7) has continuous boundary values by the upper and lower half planes, thus by analytic continuation we deduce that (7) holds for \( \mathbb{C} \setminus \{1\} \) which together with (1) and (4) proves the lemma.

\begin{remark}
The preceding lemma can be extended to the case in which \( n \) is not necessarily a natural number. The proof is somewhat more complicated than the current given here. For this case we will have that the relation in the lemma holds for \( z \in \mathbb{C} \setminus (-\infty, 0) \).
\end{remark}

Lemma 1 is fundamental for obtaining the asymptotic expansion. By developing the functions
\[
  \left( \frac{1 - zt}{1 - z} \right)^{\alpha + \beta} \quad \text{and} \quad \left( \frac{z - t}{z - 1} \right)^{-\alpha - \beta - 1}
\]
inside the integrals in terms of a binomial series, commuting the integral with summa symbols and performing some algebraic manipulation, the factorial series that define the remaining terms of the asymptotic expansion will appear, as will be seen in the proof of Theorem 1. The accurate estimation of the remainders will be done in the next subsection.

### 2.2 Estimation of the remainders

The goal of this subsection is the proof of Proposition 1. As will be seen in the proof Theorem 1 (next subsection), this proposition plays a fundamental role in the estimation of the remainders of the asymptotic expansion.

Through this subsection we define the set \( \tau_\psi = \{ u \in \mathbb{C} : \arg(1 + u\psi) = \pi \} \), if \( \psi \in \mathbb{C} \setminus \{0\} \) fixed. Consider an orientation in \( \tau_\psi \) by traversing this segment from \( -\psi^{-1} \) to \( e^{i\arg(-\psi^{-1})} \).

Before we prove the proposition we need to prove some preliminary lemmas.

**Lemma 2.** Assume that \( \gamma \in \mathbb{C} \) and \( k \in \mathbb{N} \cup \{0\} \). Then,
\[
  a) \quad \left| \left( \frac{\gamma}{k} \right) \right| < \frac{e^{\gamma^2 + \Re(\gamma)}}{k^{1 + \Re(\gamma)}}, \gamma \neq 0, k \neq 0,
  \\
  b) \quad \sup_{u \in \tau_\psi} \left| e^{-\pi(k-\gamma)} \left( \frac{1 + \psi u}{\psi u} \right) \gamma + \frac{(1 + \psi u)^\gamma}{(\psi u)^{k+1}} \right| < e^{-\pi\Im(\gamma)}|k + 1 - \gamma|, -1 < \Re(\gamma) < 0.
\]

**Proof.** a) Using the inequalities of arithmetic and geometric means we have
\[
  \left| \left( \frac{\gamma}{k} \right) \right|^2 = \prod_{j=1}^k \left| 1 - \frac{1 + \gamma}{j} \right|^2 \leq \left( \frac{1}{k} \sum_{j=1}^k \left| 1 - \frac{1 + \gamma}{j} \right|^2 \right)^k. \tag{8}
\]
From the equality
\[ \left| 1 - \frac{1 + \gamma}{j} \right|^2 = 1 - \frac{2}{j} \Re(1 + \gamma) + \frac{|1 + \gamma|^2}{j^2}, \]
we obtain
\[ \frac{1}{k} \sum_{j=1}^{k} \left| 1 - \frac{1 + \gamma}{j} \right|^2 = 1 + \frac{1}{k} \left( -2(1 + \Re(\gamma)) \sum_{j=1}^{k} \frac{1}{j} + |1 + \gamma|^2 \sum_{j=1}^{k} \frac{1}{j^2} \right). \]

Substituting the last equality in (8) follows
\[ \left| (\gamma_k) \right|^2 \leq \left( 1 + \frac{1}{k} \left( -2(1 + \Re(\gamma)) \sum_{j=1}^{k} \frac{1}{j} + |1 + \gamma|^2 \sum_{j=1}^{k} \frac{1}{j^2} \right) \right)^k. \]

Using the elementary inequalities
\[ \sum_{j=1}^{k} \frac{1}{j} \leq 1 + \ln k, \quad \sum_{j=1}^{k} \frac{1}{j^2} < 2, \quad k \geq 1, \]
we obtain
\[ \left| (\gamma_k) \right|^2 < e^{-2(1+\Re(\gamma))(1+\ln k)+2|1+\gamma|^2} = \frac{e^{|\gamma|^2+\Re(\gamma)}}{k^2(1+\Re(\gamma))}, \]
and this proves a).

b) Notice that \( \tau_\psi = \{ u : u = x\psi^{-1}, x < -1 \} \), therefore, from the statement of the lemma, we have to find an upper bound for
\[ \sup_{x \in (-\infty, -1)} \left| \left( \frac{\gamma}{k} \right) \right| \leq \frac{1}{k} \left( -2(1 + \Re(\gamma)) \sum_{j=1}^{k} \frac{1}{j} + |1 + \gamma|^2 \sum_{j=1}^{k} \frac{1}{j^2} \right) \left( 1 + \frac{x}{k} \right)^k. \]

By using \( x = \frac{1}{t-1} \), relation (9) transforms to
\[ \sup_{t \in (0, 1)} \left| t^\gamma \left( 1 - e^{-i\pi(\gamma)}(1-t)^{k-\gamma+1} \right) \right|. \]

Let us define the function
\[ g(t) = e^{-i\pi(k-\gamma)}(1-t)^{k-\gamma+1}, t \in [0, 1]. \]

We have that \( g(t) \) can be expressed as
\[ \sup_{t \in (0, 1)} |t^\gamma (g(t) - g(0))|. \]

Notice that
\[ |g(t) - g(0)| \leq \int_{t}^{e} |g'(\epsilon)| d\epsilon \leq te^{-\pi\Im(\gamma)}|k - \gamma + 1|, \]
hence
\[ |t^\gamma (g(t) - g(0))| \leq e^{-\pi \Im(\gamma)}|k - \gamma + 1|, \forall t \in (0, 1), \]
and this proves the lemma.

**Lemma 3.** Let \( p \in \mathbb{N} \cup \{0\} \) and \( \psi, \gamma \in \mathbb{C} \setminus \{0\} \) be fixed parameters, then
\[
    r_p(u; \psi, \gamma) = \frac{(1 + \psi u)^\gamma - \sum_{k=0}^{p} \binom{\gamma}{k} (\psi u)^k}{(\psi u)^{p+1}}, \quad u \neq 0,
\]
and
\[
    r_p(0; \psi, \gamma) = \lim_{u \to 0} r_p(u; \psi, \gamma),
\]
is an analytic function in \( \mathbb{C} \setminus \tau_\psi \), with continuous boundary values from \( \tau_{-\psi} \).

**Proof.** Analyticity of \( r_p(u; \psi, \gamma) \) for \( u \psi \in \mathbb{D} \) follows from Taylor’s theorem applied to the function \( (1 + z)^\gamma, z \in \mathbb{D} \). Noticing that \( (1 + z)^\gamma \) has continuous boundary values in \((-\infty, -1]\) by the upper half plane, we obtain what we claim by analytic continuation.

The function \( r_p \) in the preceding lemma will play a fundamental role in the definition of the remainders. For \( \psi = 0 \) or \( \gamma = 0 \) we define \( r_p = 0 \).

The following lemma gives an integral representation for the function \( r_p \), which will be used conveniently to obtain bounds on \( |r_p| \), for some values of the parameter \( \gamma \).

**Lemma 4.** Let us have \( \gamma \in \mathbb{C} \setminus \{0\}, \Re(\gamma) > -1 \) and \( p \in \mathbb{N} \cup \{0\}, p \geq \lfloor \Re(\gamma) \rfloor \). Then,
\[
    r_p(u; \psi, \gamma) = \sin \frac{\pi (\gamma - p)}{\pi} \int_0^1 \frac{t^{\gamma - 1 - t} \gamma \, dt}{1 + t u \psi}, \quad u \in \mathbb{C} \setminus \tau_\psi.
\]

**Proof.** Expanding the function \( \frac{1}{1 + t u \psi} \) in
\[
    \int_0^1 \frac{t^{\gamma - 1 - t} \gamma \, dt}{1 + t u \psi},
\]
for \( u \psi \in \mathbb{D} \) as a binomial series, identifying the remaining integral as a Beta function and applying Euler’s reflection formula we have (13) for \( u \psi \in \mathbb{D} \). We obtain the lemma for the specified domain by analytic continuation.

**Remark 3.** The formula provided in the preceding lemma, with a slight modification, can be extended to the set \( \tau_\psi \), by defining the integral in the principal value sense, giving an integral representation for the remainder of a binomial series in the whole complex plane. Since this is not necessary in our discussion, we shall not dwell on this.

**Lemma 5.** Let \( p \in \mathbb{N} \cup \{0\}, \gamma \in \mathbb{C} \setminus \{0\}, \Re(\gamma) \geq 0, \psi \in \mathbb{C} \setminus \{0\} \) and \( r_p \) be as in (12). Then,
\[
    |r_p(u; \psi, \gamma)| < m^*_1(\psi; \gamma) + pe^{\psi^2 + \Re(\gamma) + 1}, \quad u \in (0, 1) \setminus \{-\psi^{-1}\},
\]
where \( m^*_1(\psi; \gamma) = e^{\pi \Im(\gamma)} \max(2^{\Re(\gamma)}, 1 + |\psi|) \Re(\gamma) \).

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Proof. We shall prove the lemma by analyzing two cases: \( \psi \in \mathbb{C} \setminus \mathbb{D} \) and \( \psi \in \mathbb{D} \). Let’s first consider \( \psi \in \mathbb{C} \setminus \mathbb{D} \), \( \psi \) fixed.

For \( R \) sufficiently small, we define the open set \( \Omega_R \) as the region enclosed by the closed contour \( ABCDEA \), as is shown in figure 1.

Denote by \( \Gamma_{BCD} \) the subset of the boundary of \( \Omega_R \) defined by a semicircle of radius \( R \) with center at \(-\psi^{-1}\). Let us denote by \( \Gamma_{AB}, \Gamma_{DE} \subset \partial \Omega_R \) the closed segments with end points \( \{A, B\} \) and \( \{E, D\} \) respectively, and parallel to the segment \( \tau_{\psi:D} = D \cap \tau_{\psi} \). Define by \( \Gamma_{EA} \) the part of the boundary lying at the boundary of the unit circle and joining the points \( A \) and \( E \).

From Lemma 3 we have that \( r_p \) has continuous boundary values from \( \tau_{\psi:D} \), therefore the following relation holds

\[
\sup_{u \in \mathbb{D} \setminus \{-\psi^{-1}\}} |r_p(u; \psi, \gamma)| = \lim_{R \to 0} \sup_{u \in \partial \Omega_R} |r_p(u; \psi, \gamma)|. \tag{14}
\]

Also from Lemma 3, \( r_p \) is analytic in \( \Omega_R \) and continuous in \( \overline{\Omega_R} \), for all \( R \) sufficiently small. Hence, from the maximum principle for analytic functions [1, Thm. 12’ pp. 134], we have

\[
\sup_{u \in \partial \Omega_R} |r_p(u; \psi, \gamma)| = \sup_{u \in \partial \Omega_R} |r_p(u; \psi, \gamma)|. \tag{15}
\]

From (14) and (15) we obtain

\[
\sup_{u \in \mathbb{D} \setminus \{-\psi^{-1}\}} |r_p(u; \psi, \gamma)| = \lim_{R \to 0} \sup_{u \in \partial \Omega_R} |r_p(u; \psi, \gamma)|, \tag{16}
\]

therefore, in order to prove the lemma for \( \psi \in \mathbb{C} \setminus \mathbb{D} \), we need to estimate the right hand side of (16).

We analyze now the contribution of each arc of the boundary. Notice that

\[
\lim_{R \to 0} \sup_{u \in \Gamma_{EA}} |r_p(u; \psi, \gamma)| \leq \sup_{u \in \partial \mathbb{D}} e^{2\pi|\theta(\gamma)|}/0 + |r_p(u; \psi, \gamma)| \leq \sup_{u \in \partial \mathbb{D}} \frac{e^{2\pi|\theta(\gamma)|}/0 + \sum_{k=0}^{p} |(\gamma_k)| |\psi|^k}{|\psi|^{p+1}}, \tag{17}
\]

\[
\lim_{R \to 0} \sup_{u \in \Gamma_{AB}} |r_p(u; \psi, \gamma)| \leq \sup_{u \in \tau_{\psi:D}} g_0(u; \psi, \gamma), \tag{18}
\]
where \( g_0(u; \psi, \gamma) = \frac{e^{\pi|3(\gamma)|} |1 + \psi u|^{|R(\gamma)}| + \sum_{k=0}^{p} \left(\frac{\gamma}{k}\right)}{|\psi u|^{p+1}} \)

\[
\lim_{R \to 0} \sup_{\psi, \gamma \in \Gamma_{DE}} |r_p(u; \psi, \gamma)| = \sup_{u \in \tau_D} \left| e^{-2i\pi\gamma(1 + \psi u)\gamma + \sum_{k=0}^{p} \left(\frac{\gamma}{k}\right) (\psi u)^{k}} \right| \leq \sup_{u \in \tau_D} g_0(u; \psi, \gamma).
\]

From (21) and (22) we deduce that

\[
\sup_{u \in (0,1) \setminus \{-1\}} |r_p(u; \psi, \gamma)| \leq \sup_{u \in D \setminus \{-1\}} |r_p(u; \psi, \gamma)| \leq \max \left( \sup_{u \in \partial D, \gamma \in \tau_D} g_0(u; \psi, \gamma), \lim_{R \to 0} \sup_{u \in \Gamma_{BCD}} |r_p(u; \psi, \gamma)| \right).
\]

Let’s estimate the right hand side of inequality (20). We have that

\[
\sup_{u \in \partial D} g_0(u; \psi, \gamma) \leq e^{\pi|3(\gamma)|} M_\gamma + \sum_{k=0}^{p} \left(\frac{\gamma}{k}\right),
\]

where \( M_\gamma = \sup_{u \in \partial D} |1 + \psi u|^{|R(\gamma)}| = |1 + \psi||^{|R(\gamma)}| \).

From a) of Lemma 2 we have

\[
\sum_{k=0}^{p} \left(\frac{\gamma}{k}\right) \leq pe^{2|\gamma|^2 + \Re(\gamma)} + 1.
\]

From (21) and (22) we deduce

\[
\sup_{u \in \partial D} g_0(u; \psi, \gamma) \leq e^{\pi|3(\gamma)|} M_\gamma + pe^{2|\gamma|^2 + \Re(\gamma)} + 1.
\]

In a similar way, for \( u \in \tau_D \)

\[
\sup_{u \in \tau_D} g_0(u; \psi, \gamma) \leq \sup_{u \in \tau_D} e^{\pi|3(\gamma)|} |1 + \psi u|^{|R(\gamma)}| + \sup_{u \in \tau_D} \sum_{k=0}^{p} \left(\frac{\gamma}{k}\right) |\psi u|^{k+p-1}
\]

\[
\leq e^{\pi|3(\gamma)|} \sup_{u \in \tau_D} |1 + \psi u|^{|R(\gamma)}| + \sum_{k=0}^{p} \left(\frac{\gamma}{k}\right)
\]

\[
< e^{\pi|3(\gamma)|} \sup_{u \in \tau_D} |1 + \psi u|^{|R(\gamma)}| + pe^{\gamma^2 + \Re(\gamma)} + 1.
\]

Notice that \( \sup_{u \in \tau_D} |1 + \psi u|^{|R(\gamma)}| \) occurs at \( u = e^{\arg(-\psi^{-1})} \), hence

\[
\sup_{u \in \tau_D} |1 + \psi u|^{|R(\gamma)}| \leq \sup_{u \in \partial D} |1 + \psi u|^{|R(\gamma)}| = M_\gamma,
\]

therefore, from (24) and (25), we obtain

\[
\sup_{u \in \tau_D} g_0(u; \psi, \gamma) \leq e^{\pi|3(\gamma)|} M_\gamma + pe^{\gamma^2 + \Re(\gamma)} + 1 < e^{\pi|3(\gamma)|} |1 + \psi||^{|R(\gamma)}| + pe^{\gamma^2 + \Re(\gamma)} + 1.
\]

(26)
From (23) and (26), we deduce that
\[
\sup_{u \in \partial D \cup \tau \psi} g_0(u; \psi, \gamma) < e^{\pi |\Im(\gamma)||1 + |\psi||^R(\gamma) + p e^{|\gamma|^2 + R(\gamma) + 1}}. \tag{27}
\]

Let us consider now the contribution of the semicircle $\Gamma_{BCD}$. We have that
\[
\lim_{R \to 0} \sup_{u \in \Gamma_{BCD}} |r_p(u; \psi, \gamma)| \leq \sup_{u \in \Gamma_{BCD}} g_0(u; \psi, \gamma) \leq e^{\pi |\Im(\gamma)| + \sum_{k=0}^{P} \left| \binom{\gamma}{k} \right|} < e^{\pi |\Im(\gamma)| + p e^{|\gamma|^2 + R(\gamma) + 1}}. \tag{28}
\]

From (20), (27) and (28) we deduce that
\[
\sup_{u \in (0, 1) \setminus \{-\psi^{-1}\}} |r_p(u; \psi, \gamma)| < e^{\pi |\Im(\gamma)|} + |\psi||^R(\gamma) + p e^{|\gamma|^2 + R(\gamma) + 1}, \tag{29}
\]
which gives an upper bound for the case $\psi \in \mathbb{C} \setminus \bar{D}$.

Let’s consider now the case $\psi \in \bar{D}$. For $R$ sufficiently small we denote by $\Delta_R$ the interior of the region enclosed by the closed contour $ABC$, as shown in figure 2.

![Figure 2: Region $\Theta_R$](image)

Let us denote by $\Gamma_{ABC}$ the subset of the boundary of $\Delta_R$ defined by a semicircle of radius $R$ with center at $-\psi^{-1}$. Let $\Gamma_{CA}$ be the subset of the boundary of $\Delta_R$ which joins the points $A$ and $C$.

Notice that
\[
\sup_{u \in \bar{D}} |r_p(u; \psi, \gamma)| = \lim_{R \to 0} \sup_{u \in \Delta_R} |r_p(u; \psi, \gamma)|. \tag{30}
\]

From Lemma 3, $r_p$ is analytic in $\Delta_R$ and continuous in $\bar{\Delta}_R$. According to the maximum principle for analytic functions \[1\] Thm. 12’ pp. 134\] we have
\[
\sup_{u \in \Delta_R} |r_p(u; \psi, \gamma)| = \sup_{u \in \partial \Delta_R} |r_p(u; \psi, \gamma)|, \tag{31}
\]
therefore, from (30) and (31) we obtain
\[
\sup_{u \in \bar{D}} |r_p(u; \psi, \gamma)| = \lim_{R \to 0} \sup_{u \in \partial \Delta_R} |r_p(u; \psi, \gamma)|. \tag{32}
\]
Notice that
\[
\lim_{R \to 0} \sup_{u \in \Gamma_{R}} |r_p(u; \psi, \gamma)| = \sup_{|u| = \psi^{-1}, u \neq \psi^{-1}} |r_p(u; \psi, \gamma)|, \tag{33}
\]
therefore, from (32) and (33) we have
\[
\sup_{u \in (0, 1)} |r_p(u; \psi, \gamma)| \leq \max \left( \sup_{|u| = \psi^{-1}, u \neq \psi^{-1}} |r_p(u; \psi, \gamma)|, \lim_{R \to 0} \sup_{u \in \Gamma_{ABC}} |r_p(u; \psi, \gamma)| \right). \tag{34}
\]
We estimate now the right hand side of (32) by analyzing the contribution of each term. Notice that
\[
\sup_{|u| = \psi^{-1}, u \neq \psi^{-1}} |r_p(u; \psi, \gamma)| \leq \sup_{|u| = \psi^{-1}, u \neq \psi^{-1}} e^{\pi |\Im(\gamma)|} \left| 1 + u \right|^{\Re(\gamma)} + \sum_{k=0}^{p} \left| \binom{\gamma}{k} \right| \tag{35}
\]
\[
< 2^{\Re(\gamma)} e^{\pi |\Im(\gamma)|} + pe^{\gamma^2 + \Re(\gamma)} + 1.
\]
Consider now the set $\Gamma_{ABC}$. We have
\[
\lim_{R \to 0} \sup_{u \in \Gamma_{ABC}} |r_p(u; \psi, \gamma)| \leq \lim_{R \to 0} \sup_{u \in \Gamma_{ABC}} \left| \frac{1 + u}{u^{p+1}} \right| + \left| \sum_{k=0}^{p} \binom{\gamma}{k} u^k \right| \tag{36}
\]
\[
\leq 1 + \sum_{k=0}^{p} \left| \binom{\gamma}{k} \right| < 2 + pe^{\gamma^2 + \Re(\gamma)}.
\]
From (34), (35) and (36), we obtain
\[
\sup_{u \in (0, 1)} |r_p(u; \psi, \gamma)| \leq \max \left( \sup_{u \in \partial \Delta_R} |r_p(u; \psi, \gamma)|, \lim_{R \to 0} \sup_{u \in \Gamma_{ABC}} |r_p(u; \psi, \gamma)| \right) < 2^{\Re(\gamma)} e^{\pi |\Im(\gamma)|} + pe^{\gamma^2 + \Re(\gamma)} + 1, \tag{37}
\]
and from (29) and (37) we obtain the lemma.

**Lemma 6.** Let $\gamma \in \mathbb{C}$, $-1 < \Re(\gamma) < 0$, $\psi \in \mathbb{C} \setminus \{0\}$, $p \in \mathbb{N} \cup \{0\}$ and $r_p$ be as in (12). Then,
\[
r_p(u; \psi, \gamma) = -e^{-\pi(p-\gamma)} \left( \frac{1 + \psi u}{\psi u} \right)^{\gamma} + f_p(u; \psi, \gamma), \quad u \in \mathbb{C} \setminus \{-\psi^{-1}\},
\]
where $f_p$ satisfies
\[
|f_p(u; \psi, \gamma)| < m_3(p, \gamma), \quad u \in (0, 1),
\]
\[
m_3(p, \gamma) = \max \left( e^{-\pi \Im(\gamma)} |p - \gamma + 1| + 1 + pe^{\gamma^2 + \Re(\gamma)} , 1 + \left| \frac{\sin \pi \gamma}{\pi (1 + \Re(\gamma))} \right| \right).
\]
**Proof.** For $R$ sufficiently large, define the open set $\Theta_R$ as the region enclosed by the contour $ABCDEFGHJA$ as shown in figure 3.

Denote by $\Gamma_{EF}, \Gamma_{JA} \subset \partial \Theta_R$ the arcs with end points $\{E, F\}$ and $\{A, J\}$ respectively, lying at the boundary of a circle of radius $R$ and center at the origin. Let’s $\Gamma_{BCD}, \Gamma_{GHI} \subset \partial \Theta_R$ denote semicircles of radius $R^{-1}$, with center at $-\psi^{-1}$ and 0 respectively. Denote by $\Gamma_{DE}, \Gamma_{AB}, \Gamma_{FG}$ and $\Gamma_{IJ}$ the closed segments with end points $\{E, D\}, \{A, B\}, \{G, F\}$ and $\{I, J\}$ respectively and parallel to the line segments $\tau_\psi$ or $\tau_1 = \{u \in \mathbb{C} : \arg(\psi u) = 0\}$.

From Lemma 3, $r_p$ can be expressed as

$$r_p(u; \psi, \gamma) = -e^{-i\pi(p-\gamma)} \left(\frac{1 + \psi u}{\psi u}\right)^\gamma + f_p(u; \psi, \gamma),$$

where

$$f_p(u; \psi, \gamma) = e^{-i\pi(p-\gamma)} \left(\frac{1 + \psi u}{\psi u}\right)^\gamma + (1 + \psi)^\gamma \frac{(1 + \psi)^{p+1}}{(\psi u)^{p+1}} - \sum_{k=0}^{p-1} \binom{\gamma}{k} (\psi u)^{k-p-1}.$$  \hspace{1cm} (38)

Since the following limit exists

$$\lim_{u \to -\psi^{-1}} e^{-i\pi(p-\gamma)} \left(\frac{1 + u\psi}{u\psi}\right)^\gamma + (1 + u\psi)^\gamma (u\psi)^{p+1} = 0,$$  \hspace{1cm} (39)

we define $f_p(-\psi^{-1}; \psi, \gamma)$ as

$$f_p(-\psi^{-1}; \psi, \gamma) = \sum_{k=0}^{p} \binom{\gamma}{k} (-1)^{k-p-1}.$$  

From Lemma 3, $r_p$ is continuous at $u = 0$. Hence, if we define $f_p(0; \psi, \gamma) = r_p(0; \psi, \gamma)$, we have that $f_p$ is continuous at the points $\{-\psi^{-1}, 0\}$.

Notice that $f_p$ is the sum of the functions $-e^{-i\pi(p-\gamma)} \left(\frac{1 + \psi u}{\psi u}\right)^\gamma$, which is analytic in $u \in \Theta_R$ (by virtue of the fact that it is possible to separate a uniform branch of the root in this region), and continuous in $\Theta_R$ and the function $r_p$, which by Lemma 3 is analytic in $\Theta_R$ and continuous in $\Theta_R$. We note that these statements hold for all $R$ sufficiently large. Hence, from the maximum principle for analytic functions [1, Thm. 12’ pp. 134] we have

$$\sup_{u \in \Theta_R} |f_p(u; \psi, \gamma)| = \sup_{u \in \partial \Theta_R} |f_p(u; \psi, \gamma)|.$$  \hspace{1cm} (40)
Consider an orientation in the segment $\tau_1$ by traversing it starting from the point 0 in the direction of $e^{i \arg(-\psi^{-1})}_\infty$. Since $f_p$ has continuous boundary values from $\tau^-$ and $\tau_1^-$ and is continuous at the points $\{-\psi^{-1}, 0\}$, then it is not difficult to see that the following relation holds

$$
\sup_{u \in \mathcal{D}} |f_p(u; \psi, \gamma)| \leq \lim_{R \to \infty} \sup_{u \in \partial \Theta_R} |f_p(u; \psi, \gamma)|.
$$

From (40) and (41) we deduce that

$$
\sup_{u \in \mathcal{D}} |f_p(u; \psi, \gamma)| \leq \lim_{R \to \infty} \sup_{u \in \partial \Theta_R} |f_p(u; \gamma)|.
$$

Let’s find an upper bound for $\lim_{R \to \infty} \sup_{u \in \partial \Theta_R} |f_p(u; \psi, \gamma)|$ by analyzing the contributions of the arcs that form the boundary. For the arcs $\Gamma_{EF}, \Gamma_{JA}, \Gamma_{AB}$ and $\Gamma_{DE}$ we have

$$
\lim_{R \to \infty} \sup_{u \in \Gamma_{EF \cup \Gamma_{JA}}} |f_p(u; \psi, \gamma)|
\leq \lim_{u \to \infty} \left| e^{-i\pi(p-\gamma)} \left(\frac{1 + \psi u}{\psi u}\right)^\gamma + \frac{(1 + \psi u)^\gamma}{(\psi u)^{p+1}} + \sum_{k=0}^p \left(\frac{\gamma}{k}\right) |\psi u|^{k-p-1}\right| = \sup_{u \in \tau_\psi} |f_p(u; \psi, \gamma)| = 1,
$$

where $g_1(u; \psi, \gamma) = \left| e^{-i\pi(p-\gamma)} \left(\frac{1 + \psi u}{\psi u}\right)^\gamma + \frac{(1 + \psi u)^\gamma}{(\psi u)^{p+1}} + \sum_{k=0}^p \left(\frac{\gamma}{k}\right) |\psi u|^{k-p-1}\right|$.

$$
\lim_{R \to \infty} \sup_{u \in \Gamma_{DE}} |f_p(u; \psi, \gamma)|
\leq \sup_{u \in \tau_\psi} \left| e^{-i\pi(p-\gamma)} \left(\frac{1 + \psi u}{\psi u}\right)^\gamma + \sum_{k=0}^p \left(\frac{\gamma}{k}\right) |\psi u|^{k-p-1}\right| = \sup_{u \in \tau_\psi} g_1(u; \psi, \gamma).
$$

For the segments $\Gamma_{FG}, \Gamma_{JJ}$, it is more convenient to use another representation for $f_p$. From Lemma 4 we deduce that

$$
f_p(u; \psi, \gamma) = e^{-i\pi(p-\gamma)} \left(\frac{1 + \psi u}{\psi u}\right)^\gamma + \frac{\sin \pi(\gamma - p)}{\pi} \int_0^1 \frac{t^{p-\gamma}(1-t)^{\gamma-1}}{1+tu\psi} dt, u \in \mathbb{C} \setminus \tau_\psi, \ p \geq 0.
$$

From (46) we find that

$$
\lim_{R \to \infty} \sup_{u \in \Gamma_{FG}} |f_p(u; \psi, \gamma)| \leq \sup_{u \in \tau_1} |f_p(u; \psi, \gamma)|.
$$

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where \( g_2(u; \psi, \gamma) = e^{\pi |\Im(\gamma)|} \left| \frac{1 + \psi u}{\psi u} \right|^{\Re(\gamma)} + \left| \frac{\sin \pi (\gamma - p)}{\pi} \right| \left| \int_0^1 t^{p-\gamma}(1-t)^\gamma dt \right| \).

\[
\lim_{R \to \infty} \sup_{u \in \Gamma_{B,C,D}} |f_p(u; \psi, \gamma)| \leq \sup_{u \in \Gamma_{B,C,D}} \left| \int_0^1 t^{p-\gamma}(1-t)^\gamma dt \right| = \sup_{u \in \Gamma_{B,C,D}} g_2(u; \psi, \gamma). (51)
\]

From (39) we deduce, that for the contour \( \Gamma_{B,C,D} \), we have
\[
\lim_{R \to \infty} \sup_{u \in \Gamma_{B,C,D}} |f_p(u; \psi, \gamma)| = |f_p(-\psi^{-1}; \psi, \gamma)|. (50)
\]

In a similar way, from (46), we deduce that, for the contour \( \Gamma_{G,H,I} \), it holds that
\[
\lim_{R \to \infty} \sup_{u \in \Gamma_{G,H,I}} |f_p(u; \psi, \gamma)| = |f_p(0; \psi, \gamma)|. (50)
\]

From (42) up to (45); and from (47) up to (50) we deduce
\[
\sup_{u \in \mathbb{D}} |f_p(u; \psi, \gamma)| \leq \lim_{R \to \infty} \sup_{u \in \partial \Theta_R} |f_p(u; \psi, \gamma)| \leq \max \left(1, \sup_{u \in \Gamma_{B,C,D}} g_1(u; \psi, \gamma), \sup_{u \in \Gamma_{G,H,I}} g_2(u; \psi, \gamma)\right). (51)
\]

By using inequalities (44), (45), (47) and (48), we estimate the right hand side of the second inequality in (51). From a) and b) of Lemma 2 we have
\[
\sup_{u \in \Gamma_{B,C,D}} g_1(u; \psi, \gamma) < e^{-\pi |\Im(\gamma)|} |p - \gamma + 1| + 1 + pe^{\gamma |\gamma|^2 + \Re(\gamma)}. (52)
\]

We also have
\[
\sup_{u \in \Gamma_{G,H,I}} g_2(u; \psi, \gamma) = \sup_{u \in \Gamma_{B,C,D}} \left| \frac{1 + \psi u}{\psi u} \right|^\gamma + \left| \frac{\sin \pi (\gamma - p)}{\pi} \right| \left| \int_0^1 t^{p-\gamma}(1-t)^\gamma dt \right| \leq 1 + \frac{\sin \pi \gamma}{\pi (1 + \Re(\gamma))} \leq 1 + \frac{\sin \pi \gamma}{\pi (1 + \Re(\gamma))}. (53)
\]

From (52) and (53), we obtain that (51) gives
\[
\sup_{u \in (0,1)} |f_p(u; \psi, \gamma)| \leq \max \left(1, e^{-\pi |\Im(\gamma)|} |p - \gamma + 1| + 1 + pe^{\gamma |\gamma|^2 + \Re(\gamma)}, 1 + \frac{\sin \pi \gamma}{\pi (1 + \Re(\gamma))}\right) =
\]
\[
\max \left(e^{-\pi |\Im(\gamma)|} |p - \gamma + 1| + 1 + pe^{\gamma |\gamma|^2 + \Re(\gamma)}, 1 + \frac{\sin \pi \gamma}{\pi (1 + \Re(\gamma))}\right),
\]
and this proves the lemma. \( \square \)
For the next lemma, we define the following function as

\[
m_2(z; q, \gamma) = e^{\pi|\Im(\gamma)|}\left(\frac{|z| + 1}{|z - 1||z|^{|\Re(\gamma)|}}\left(1 + 2\frac{q + 1}{|\gamma + 1|}\right)\right),
\]

\(z \in \mathbb{C} \setminus \{1\}, \gamma \in \mathbb{C}; -1 < \Re(\gamma) < 0, \) \hspace{1cm} (54)

Notice that \(m_2\) is an analytic function in a reduced neighborhood of \(z = \infty\). We define \(m_2(\infty; q, \gamma) = \lim_{z \to \infty} m_2(z; q, \gamma)\).

**Lemma 7.** Let \(\gamma \in \mathbb{C}; -1 < \Re(\gamma) < 0, q \in \mathbb{C}; \Re(q) > 0\) and \(z \in \mathbb{C} \setminus \{1\}\). Then,

\[a) \left| \int_0^1 t^n(1-t)^q (1-zt)^\gamma dt \right| \leq m_2(z; q, \gamma) B(n + 1, \Re(q) + 1),\]

\[b) \left| \int_0^1 t^n(1-t)^q (z-t)^\gamma dt \right| \leq m_2(z^{-1}; q, \gamma) B(n + 1, \Re(q) + 1),\]

where \(m_2\) as in (54).

**Proof.** a) From relation [7] Eq. (33) pp. 103 and from Euler’s formula for hypergeometric functions [7] Eq. (10) pp. 59 with parameters \(a = -\gamma - 1, b = n + 1, \) and \(c = n + q + 2\), we have

\[ (\gamma + 1)(z - 1) \int_0^1 t^n(1-t)^q(1-zt)^\gamma dt =\]

\[ - (\gamma + 2 + q) \int_0^1 t^n(1-t)^q(1-zt)^\gamma + 1 dt + n \int_0^1 t^{n-1}(1-t)^{q+1}(1-zt)^\gamma + 1 dt,\]

hence

\[ \left| \int_0^1 t^n(1-t)^q \left(\frac{1-zt}{z}\right)^\gamma dt \right| \leq \]

\[ \frac{|\gamma + 2 + q|}{|\gamma + 1|} \frac{1}{|z - 1||z|^{|\Re(\gamma)|}} \int_0^1 t^n(1-t)^{|\Re(q)|} |1-zt|^{|\Re(\gamma)| + 1} e^{\Im(\gamma) \arg\left(\frac{1}{1-zt}\right)} dt\]

\[ + \frac{n}{|\gamma + 1||z - 1||z|^{|\Re(\gamma)|}} \int_0^1 t^{n-1}(1-t)^{|\Re(q)| + 1} |1-zt|^{|\Re(\gamma)| + 1} e^{\Im(\gamma) \arg\left(\frac{1}{1-zt}\right)} dt.\] \hspace{1cm} (55)

Notice that

\[ \int_0^1 t^n(1-t)^{|\Re(q)|} |1-zt|^{|\Re(\gamma)| + 1} e^{\Im(\gamma) \arg\left(\frac{1}{1-zt}\right)} dt \leq e^{\pi|\Im(\gamma)|}(|z| + 1)^{|\Re(\gamma)| + 1} B(n + 1, \Re(q) + 1), \] \hspace{1cm} (56)

\[ \int_0^1 t^{n-1}(1-t)^{|\Re(q)| + 1} |1-zt|^{|\Re(\gamma)| + 1} e^{\Im(\gamma) \arg\left(\frac{1}{1-zt}\right)} dt \leq e^{\pi|\Im(\gamma)|}(|z| + 1)^{|\Re(\gamma)| + 1} B(n, \Re(q) + 2) \leq e^{\pi|\Im(\gamma)|}(|z| + 1)^{|\Re(\gamma)| + 1} B(n + 1, \Re(q) + 1). \] \hspace{1cm} (57)
From inequalities (55), (56) and (57), we obtain
\[
\left| \int_0^1 t^n (1-t)^q (1-zt)^\gamma \, dt \right| \leq \frac{e^{\gamma|\Im(\gamma)|}(|z| + 1)^{\Re(\gamma)+1}}{|z-1||z|^{\Re(\gamma)}} \left( 1 + 2 \left| \frac{q + 1}{\gamma + 1} \right| B(n + 1, \Re(q) + 1) \right),
\]
which is a).

b) Follows by performing the transformation \( z \mapsto z^{-1} \) in inequality a), for \( z \neq 0 \). Since \( m_2 \) is analytic at \( \infty \) we deduce that b) is also valid at \( z = 0 \).

Before we prove the next result, we define the following functions, for \( n, p_1, p_2 \in \mathbb{N} \cup \{0\} \) and \( z \in \mathbb{C} \setminus \{1\} \)
\[
\xi_{1,n,p_1}(z; \alpha, \beta) = \frac{1}{B(n + 1, \alpha - \beta)} \left( \frac{z}{1 - z} \right)^{p_1+1} \int_0^1 t^n (1-t)^{p_1+\alpha-\beta} r_{1,p_1} \left( 1 - t; \frac{z}{1 - z}, \alpha + \beta \right) \, dt, \quad (58)
\]
and
\[
\xi_{2,n,p_2}(z; \alpha, \beta) = \frac{1}{B(n + 1, \alpha + \beta + 1)} (z - 1)^{p_2+1} \times \int_0^1 t^n (1-t)^{p_2+\alpha+\beta} r_{2,p_2} \left( 1 - t; \frac{1}{z - 1}, \alpha - \beta - 1 \right) \, dt, \quad (59)
\]
where \( r_{1,p_1} \) and \( r_{2,p_2} \) are as in (12). The functions (58) and (59), as will be seen in the proof of Theorem 1, define the remainders of the asymptotic expansion.

As a consequence of the preceding lemmas, we obtain the following proposition which is fundamental in our asymptotic formulas

**Proposition 1.** Let \((\alpha, \beta) \in \Omega_0, z \in \mathbb{C} \setminus \{1\} \) and \( n, p_1, p_2 \in \mathbb{N} \cup \{0\} \). Then,
\[
\begin{align*}
&\quad a) \ |\xi_{1,n,p_1}(z; \alpha, \beta)| \leq \frac{\Gamma(p_1 + \Re(\alpha - \beta))}{|\Gamma(\alpha - \beta)|} \left( \frac{z}{1 - z} \right)^{p_1+1} \frac{c_{1,n,p_1}(z; \alpha, \beta)}{(n + \Re(\alpha - \beta) + 1)^{p_1+1}}, \\
&\quad b) \ |\xi_{2,n,p_2}(z; \alpha, \beta)| < \frac{\Gamma(p_2 + 2 + \Re(\alpha + \beta))}{|\Gamma(\alpha + \beta + 1)|} \frac{c_{2,n,p_2}(z; \alpha, \beta)}{(n + 2 + \Re(\alpha + \beta) + 1)^{p_2+1}},
\end{align*}
\]
where
\[
c_{1,n,p_1}(z; \alpha, \beta) = \begin{cases} 
m_1 \left( \frac{z}{1 - z}; \alpha + \beta \right) + p_1 e^{\alpha + \beta + \Re(\alpha + \beta)} + 1, & 0 \leq \Re(\alpha + \beta), \\
m_2(z; p_1 - 2\beta, \alpha + \beta) \frac{\Gamma(p_1 + 1 - 2\Re(\beta))}{\Gamma(n + 1 + \Re(\alpha - \beta))} \frac{\Gamma(n + 1 + \Re(\alpha - \beta))}{\Gamma(n + p_1 + 2 - 2\Re(\beta))} + \\
m_3(p_1, \alpha + \beta), & -1 < \Re(\alpha + \beta) < 0,
\end{cases}
\]

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\[ c_{2,n,p_{2}}(z;\alpha,\beta) = \begin{cases} 
    m_{1}\left(\frac{1}{z-1};\alpha - \beta - 1\right) + p_{2}e^{(\alpha - \beta - 1)^{2} + \Re(\alpha - \beta) - 1} + 1, & 1 \leq \Re(\alpha - \beta), \\
    m_{2}(z^{-1};p_{2} + 2\beta + 2,\alpha - \beta - 1) \frac{\Gamma(p_{2} + 2\Re(\beta) + 3)}{\Gamma(p_{2} + \Re(\alpha + \beta) + 2)} \times \\
    \frac{\Gamma(n + p_{2} + 3 + \Re(\alpha + \beta))}{\Gamma(n + p_{2} + 2\Re(\beta) + 4)} + m_{3}(p_{2},\alpha - \beta + 1), & 0 < \Re(\alpha - \beta) < 1,
\end{cases} \]

and
\[
m_{1}(z;\gamma) = e^{\pi \Im(\gamma)} \max(2\Re(\gamma), (1 + |z|)^{\Re(\gamma)}), \\
m_{2}(z;\gamma) = e^{\pi \Im(\gamma)} \left(\frac{|z| + 1 + \Re(\gamma)}{|z - 1||z|^{\Re(\gamma)}} \left(1 + 2 \left|\frac{q + 1}{\gamma + 1}\right\right)\right), \\
m_{3}(p,\gamma) = \max(e^{-\pi \Im(\gamma)}|p - \gamma + 1| + 1 + pe^{\gamma + \Re(\gamma)} + 1, \left|\frac{\sin \pi \gamma}{\pi(1 + \Re(\gamma))}\right|) .
\]

\textbf{Proof.} a) Let’s consider \(0 \leq \Re(\alpha + \beta)\). From the definitions of \(\xi_{1,n,p_{1}}\), \(r_{1,p_{1}}\) and from Lemma 5 we have
\[
\begin{align*}
    r_{1,p_{1}} \left(1 - t; \frac{z}{1 - z},\alpha + \beta\right) < \\
m_{1}\left(\frac{z}{1 - z};\alpha + \beta\right) + p_{1}e^{(\alpha + \beta)^{2} + \Re(\alpha + \beta) + 1}, t \in (0,1), t \neq 1/z, if z \neq 0; or t \in (0,1), if z = 0,
\end{align*}
\]
hence, for every \(z \in \mathbb{C} \setminus \{1\}\) fixed
\[
|\xi_{1,n,p_{1}}(z;\alpha,\beta)| \leq \frac{\Gamma(n + 1 + \alpha - \beta)}{\Gamma(n + 1)|\Gamma(\alpha - \beta)|} \left|\frac{z}{1 - z}\right|^{p_{1} + 1} \times \\
\int_{0}^{1} t^{n}(1 - t)^{p_{1} + \Re(\alpha - \beta)} \left| r_{1,p_{1}} \left(1 - t; \frac{z}{1 - z},\alpha + \beta\right) \right| dt < \\
\frac{1}{|\Gamma(\alpha - \beta)|} \left|\frac{z}{1 - z}\right|^{p_{1} + 1} \left(m_{1}\left(\frac{z}{1 - z};\alpha + \beta\right) + p_{1}e^{(\alpha + \beta)^{2} + \Re(\alpha + \beta) + 1}\right) \times \\
B(n + 1, p_{1} + 1 + \Re(\alpha - \beta)) \frac{\Gamma(n + 1 + \alpha - \beta)}{\Gamma(n + 1)}. \tag{60}
\]
Notice that, for \((\alpha,\beta) \in \mathbb{C}^{2}\) in the set \([-1 < \Re(\alpha + \beta), 0 < \Re(\alpha - \beta)\]}, it holds that
\[
B(n + 1, p_{1} + 1 + \Re(\alpha - \beta)) \frac{\Gamma(n + 1 + \alpha - \beta)}{\Gamma(n + 1)} = \frac{\Gamma(p_{1} + \Re(\alpha - \beta))\Gamma(n + 1 + \alpha - \beta)}{\Gamma(n + 2 + p_{1} + \Re(\alpha - \beta))} \leq \\
\frac{\Gamma(p_{1} + \Re(\alpha - \beta))}{(n + \Re(\alpha - \beta) + 1)p_{1} + 1}. \tag{61}
\]
From the last inequality in \((60)\) and inequality \((61)\), we deduce that

\[
|\xi_{1,n,p_1}(z;\alpha,\beta)| < \frac{1}{|\Gamma(\alpha-\beta)|} \left| \frac{z}{1-z} \right|^{p_1+1} \times \left( m_1 \left( \frac{z}{1-z}; \alpha + \beta \right) + p_1e^{(\alpha+\beta)^2+\Re(\alpha+\beta)} + 1 \right) \frac{\Gamma(p_1+\Re(\alpha-\beta))}{(\Re(\alpha-\beta)+1)p_1+1},
\]

which is item a) for \(0 \leq \Re(\alpha+\beta)\).

Consider now the case \(-1 < \Re(\alpha+\beta) < 0\). From Lemma \([8]\) the function \(r_{1,p_1}\) can be expressed as

\[
r_{1,p_1} \left( 1-t; \frac{z}{1-z}, \alpha + \beta \right) = -e^{-i\pi(\alpha+\beta)} \left( \frac{1-zt}{z(1-t)} \right)^{\alpha+\beta} + f_{1,p_1} \left( 1-t; \frac{z}{1-z}, \alpha + \beta \right), \ t \in (0,1), \ t \neq 1/z,
\]

where \(f_{1,p_1}\) satisfies

\[
|f_{1,p_1} \left( 1-t; \frac{z}{1-z}, \alpha + \beta \right)| < m_3(p_1, \alpha + \beta).
\]

Hence, for every \(z \in \mathbb{C} \setminus \{0,1\}\) fixed

\[
\int_0^1 t^n(1-t)^{p_1+\alpha-\beta} r_{1,p_1} \left( 1-t; \frac{z}{1-z}, \alpha + \beta \right) dt =
\]

\[
e^{-i\pi(\alpha+\beta)} \int_0^1 t^n(1-t)^{p_1+2\beta} \left( \frac{1-zt}{z} \right)^{\alpha+\beta} dt + \int_0^1 t^n(1-t)^{p_1+\alpha-\beta} f_{1,p_1} \left( 1-t; \frac{z}{1-z}, \alpha + \beta \right) dt. \ (62)
\]

From \((62)\), a) of Lemma \([7]\) and the definition of \(r_{1,p_1}\) we have

\[
\left| \int_0^1 t^n(1-t)^{p_1+\alpha-\beta} r_{1,p_1} \left( 1-t; \frac{z}{1-z}, \alpha + \beta \right) dt \right| \leq m_2(z;p_1-2\beta, \alpha + \beta) B(n+1, p_1+1-2\Re(\beta)) + m_3(p_1, \alpha + \beta) \times
\]

\[
B(n+1, p_1+1+\Re(\alpha-\beta)),
\]

for every \(z \in \mathbb{C} \setminus \{1\}\), therefore

\[
|\xi_{1,n,p_1}(z;\alpha,\beta)| \leq \frac{1}{|\Gamma(\alpha-\beta)|} \left| \frac{z}{1-z} \right|^{p_1+1} \frac{(m_2(z;p_1-2\beta, \alpha + \beta) B(n+1, p_1+1-2\Re(\beta)) + m_3(p_1, \alpha + \beta)) B(n+1, p_1+1+\Re(\alpha-\beta))}{\Gamma(n+1)} \frac{\Gamma(n+1+\alpha-\beta)}{\Gamma(n+1+\alpha-\beta)\Gamma(n+1)}. \ (63)
\]

Notice that

\[
\frac{B(n+1, p_1+1-2\Re(\beta))}{B(n+1, p_1+1+\Re(\alpha-\beta))} = \frac{\Gamma(p_1+1-2\Re(\beta))}{\Gamma(p_1+1+\Re(\alpha-\beta))} \frac{\Gamma(n+1+2+\Re(\alpha-\beta))}{\Gamma(n+p_1+2-2\Re(\beta))}. \ (64)
\]
From (63), (64) and from relation (61) (which is valid in the set \{-1 < \Re(\alpha + \beta), 0 < \Re(\alpha - \beta)\}), we have

\[
|\xi_{1,n,p_1}(z; \alpha, \beta)| \leq \frac{\Gamma(p_1 + \Re(\alpha - \beta))}{\Gamma(\alpha - \beta)} \left| \frac{z}{1 - z} \right|^{p_1 + 1} \times \\
(m_2(z; p_1 - 2\beta, \alpha + \beta) \left| \frac{\Gamma(p_1 + 1 - 2\Re(\beta))}{\Gamma(p_1 + 1 + \Re(\alpha - \beta))} \frac{\Gamma(n + p_1 + 2 + \Re(\alpha - \beta))}{\Gamma(n + p_1 + 2 - 2\Re(\beta))} + m_3(p_1, \alpha + \beta) \right| \frac{1}{(n + \Re(\alpha - \beta) + 1)p_1 + 1},
\]

which is item a) for \(-1 < \Re(\alpha + \beta) < 0\).

b) The proof of b) is similar to a), however we will include it for if the reader is interested in following the calculations. Assume that \(1 \leq \Re(\alpha - \beta)\). From Lemma 5 and from the definition of \(\xi_{2,n,p_2}\), we have

\[
\left| r_{2,p_2} \left( 1 - t; \frac{1}{z - 1}, \alpha - \beta - 1 \right) \right| < \\
m_1 \left( \frac{1}{1 - z}; \alpha - \beta - 1 \right) + p_2 e^{(\alpha - \beta - 1)^2 + \Re(\alpha - \beta) - 1} + 1, \ t \in (0, 1), \ t \neq z,
\]

therefore, if \(z \in C \setminus \{1\}\) is fixed

\[
|\xi_{2,n,p_2}(z; \alpha, \beta)| \leq \frac{\Gamma(n + 2 + \alpha + \beta)}{\Gamma(n + 1)|\Gamma(\alpha + \beta + 1)|} \left| \frac{1}{z - 1} \right|^{p_2 + 1} \times \\
\int_0^1 t^n (1 - t)^{p_2 + \Re(\alpha + \beta) + 1} \left| r_{2,p_2} \left( 1 - t; \frac{1}{z - 1}, \alpha - \beta - 1 \right) \right| dt < \\
\frac{\Gamma(n + 2 + \alpha + \beta)}{\Gamma(n + 1)|\Gamma(\alpha + \beta + 1)|} \left| \frac{1}{z - 1} \right|^{p_2 + 1} \left( m_1 \left( \frac{1}{1 - z}; \alpha - \beta - 1 \right) + p_2 e^{(\alpha - \beta - 1)^2 + \Re(\alpha - \beta) - 1} + 1 \right) \times \\
B(n + 1, p_2 + \Re(\alpha + \beta) + 2). \quad (65)
\]

Notice that if \((\alpha, \beta)\) satisfies \(-1 < \Re(\alpha + \beta), 0 < \Re(\alpha - \beta)\), then

\[
B(n + 1, p_2 + \Re(\alpha + \beta) + 2) \frac{\Gamma(n + 2 + \alpha + \beta)}{\Gamma(n + 1)} = \frac{\Gamma(p_2 + \Re(\alpha + \beta) + 2)}{\Gamma(n + 3 + p_2 + \Re(\alpha + \beta))} \leq \frac{\Gamma(p_2 + 2 + \Re(\alpha + \beta))}{\Gamma(n + 2 \Re(\alpha + \beta))_{p_2 + 1}}. \quad (66)
\]

From (66) and the last inequality in (65) we obtain

\[
|\xi_{2,n,p_2}(z; \alpha, \beta)| < \frac{\Gamma(p_2 + 2 + \Re(\alpha + \beta))}{\Gamma(\alpha + \beta + 1)} \left| \frac{1}{z - 1} \right|^{p_2 + 1} \times \\
m_1 \left( \frac{1}{z - 1}; \alpha - \beta - 1 \right) + p_2 e^{(\alpha - \beta - 1)^2 + \Re(\alpha - \beta) - 1} + 1 \left( n + 2 + \Re(\alpha + \beta) \right)_{p_2 + 1}. \]

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which is item b) for \(-1 \leq \Re(\alpha - \beta)\).

Consider now the case \(0 < \Re(\alpha - \beta) < 1\). From Lemma 6, \(r_{p_2}\) can be expressed as

\[
\begin{align*}
r_{2,p_2}\left(1 - t; \frac{1}{z - 1}, \alpha - \beta - 1\right) &= -e^{-\pi (p_2 - (\alpha - \beta) + 1)} \left(z - t \right)^{\alpha - \beta - 1} + \int_0^1 t^n (1 - t)^{p_2 + \alpha + \beta + 1} f_{2,p_2}\left(1 - t; \frac{1}{z - 1}, \alpha - \beta - 1\right) dt, \quad t \in (0, 1), \ t \neq z,
\end{align*}
\]

where

\[
\left|f_{2,p_2}\left(1 - t; \frac{1}{z - 1}, \alpha - \beta - 1\right)\right| < m_3(p_2, \alpha - \beta + 1),
\]

hence

\[
\begin{align*}
\int_0^1 t^n (1 - t)^{p_2 + \alpha + \beta + 1} r_{2,p_2}\left(1 - t; \frac{1}{z - 1}, \alpha - \beta - 1\right) dt &= -e^{-\pi (p_2 - \alpha + \beta + 1)} \int_0^1 t^n (1 - t)^{p_2 + 2\beta + 2} (z - t)^{\alpha - \beta - 1} dt + \int_0^1 t^n (1 - t)^{p_2 + \alpha + \beta + 1} f_{2,p_2}\left(1 - t; \frac{1}{z - 1}, \alpha - \beta - 1\right) dt, \quad (67)
\end{align*}
\]

for \(z \in C \setminus \{1\}\) fixed. From (67) and b) of Lemma 7, we have

\[
\begin{align*}
\left|\int_0^1 t^n (1 - t)^{p_2 + 2\beta + 2} r_{2,p_2}\left(1 - t; \frac{1}{z - 1}, \alpha - \beta - 1\right) dt\right| &< m_2(z^{-1}; p_2 + 2\beta + 2, \alpha - \beta - 1) B(n + 1, p_2 + 2\Re(\beta) + 3) + m_3(p_2, \alpha - \beta + 1) B(n + 1, \Re(\alpha + \beta) + p_2 + 2).
\end{align*}
\]

Therefore,

\[
\begin{align*}
|\xi_{2,n,p_2}(z; \alpha, \beta)| &< \frac{|\Gamma(n + 2 + \alpha + \beta)|}{\Gamma(n + 1)|\Gamma(\alpha + \beta + 1)|} \left|\frac{1}{\left(z - 1\right)^{p_2 + 1}}\right| \times m_2(z^{-1}; p_2 + 2\beta + 2, \alpha - \beta - 1) B(n + 1, p_2 + 3 + 2\Re(\beta)) + m_3(p_2, \alpha - \beta + 1) B(n + 1, p_2 + 2 + \Re(\alpha + \beta)) \frac{|\Gamma(n + 2 + \alpha + \beta)|}{\Gamma(n + 1)}. \quad (68)
\end{align*}
\]

Notice that

\[
\begin{align*}
\frac{B(n + 1, p_2 + 2\Re(\beta) + 3)}{B(n + 1, p_2 + \Re(\alpha + \beta) + 2)} &= \frac{\Gamma(p_2 + 2\Re(\beta) + 3)}{\Gamma(p_2 + \Re(\alpha + \beta) + 2)} \frac{\Gamma(n + p_2 + 3 + \Re(\alpha + \beta))}{\Gamma(n + p_2 + 4 + 2\Re(\beta))}. \quad (69)
\end{align*}
\]

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From (66), (68) and (69), we obtain

\[
|\xi_{2,n,p_2}(z;\alpha,\beta)| < \frac{\Gamma(p_2 + 2 + \Re(\alpha + \beta))}{\Gamma(\alpha + 1)} \left| z - 1 \right|^{p_2+1} \times \left( m_2(z^{-1}; p_2 + 2\beta + 2, \alpha - \beta - 1) \right) \left( \frac{\Gamma(p_2 + 2\Re(\beta) + 3)}{\Gamma(p_2 + 2\Re(\beta) + 2)} \frac{\Gamma(n + p_2 + 3 + \Re(\alpha + \beta))}{\Gamma(n + p_2 + 2\Re(\beta) + 4)} + \frac{1}{(n + \Re(\alpha + \beta) + 2)p_2+1} \right)
\]

and this proves item b) for \(0 < \Re(\alpha - \beta) < 1\).

\[\Box\]

### 2.3 The Asymptotic expansion

As a consequence of the lemmas of previous subsections, we obtain

**Proof. (Of Theorem 1)**

From Lemma 1 we have

\[
P_n(z;\alpha,\beta) = \frac{\Gamma(2\alpha + 1)}{\Gamma(\alpha - \beta)\Gamma(\alpha + 1)} \left( z^{n+\alpha-\beta}(z-1)^{\beta-\alpha} \left( \int_0^1 t^n(1-t)^{\alpha-\beta-1} \left( \frac{1-zt}{1-z} \right)^{\alpha+\beta} dt \right) + (1-z)^{-\alpha-\beta-1} \left( \int_0^1 t^n(1-t)^{\alpha+\beta} \left( \frac{zt}{z-1} \right)^{\alpha-\beta-1} dt \right) \right), \quad z \in \mathbb{C} \setminus \{1\},\]

or equivalently

\[
P_n(z;\alpha,\beta) = \frac{\Gamma(2\alpha + 1)}{\Gamma(\alpha - \beta)\Gamma(\alpha + 1)} \left( I_n^{(1)}(z;\alpha,\beta) + I_n^{(2)}(z;\alpha,\beta) \right), \quad z \in \mathbb{C} \setminus \{1\},\]

where \(I_n^{(1)}\) and \(I_n^{(2)}\) denote the first and second summand respectively inside the parenthesis in the right hand side of (70).

We have that

\[
\left( \frac{1-zt}{1-z} \right)^{\alpha+\beta} = \left( 1 + \frac{z(1-t)}{1-z} \right)^{\alpha+\beta} = \sum_{k=0}^{\alpha+1} \binom{\alpha + \beta}{k} \left( \frac{z}{1-z} \right)^k (1-t)^k \]

\[
\left( \frac{z-t}{z-1} \right)^{\alpha-\beta-1} = \left( 1 + \frac{1-t}{z-1} \right)^{\alpha-\beta-1} = \sum_{k=0}^{\alpha-\beta-1} \binom{\alpha - \beta - 1}{k} \left( \frac{1-t}{z-1} \right)^k \]

where \(r_{1,p_1}\) and \(r_{2,p_2}\) are as in Lemma 3. From the expression that defines \(I_n^{(1)}\) and from (72) we have
\[
I_n^{(1)}(z; \alpha, \beta) = B(n+1, \alpha - \beta)z^{n+\alpha-\beta}(z-1)^{\beta-\alpha} \times \left( \sum_{k=0}^{p_1} \binom{\alpha + \beta}{k} \left( \frac{z}{1-z} \right)^k \frac{\Gamma(k + \alpha - \beta)}{\Gamma(\alpha - \beta)} \frac{1}{(n+1+\alpha-\beta)_k} + \xi_{1,n,p_1}(z; \alpha, \beta) \right),
\]

where \( \xi_{1,n,p_1} \) is given by \([58]\).

In a similar way,

\[
I_n^{(2)}(z; \alpha, \beta) = B(n+1, \alpha + \beta + 1)(1-z)^{-\alpha-\beta-1} \times \left( \sum_{k=0}^{p_2} \binom{\alpha - \beta - 1}{k} \left( \frac{z}{1-z} \right)^k \frac{\Gamma(k + \alpha + \beta + 1)}{\Gamma(\alpha + \beta + 1)} \frac{1}{(n+2+\alpha+\beta)_k} + \xi_{2,n,p_2}(z; \alpha, \beta) \right),
\]

where \( \xi_{2,n,p_2} \) is given by \([59]\).

From the preceding discussion, relation \((70)\) expanded as a formal infinite series gives

\[
P_n(z; \alpha, \beta) \sim \frac{\Gamma(2\alpha + 1)}{\Gamma(\alpha + \beta + 1) \Gamma(n + \alpha - \beta + 1)}z^{n+\alpha-\beta}(z-1)^{\beta-\alpha} \times \left( \sum_{k=0}^{\infty} \binom{\alpha + \beta}{k} \left( \frac{z}{1-z} \right)^k \frac{\Gamma(k + \alpha - \beta)}{\Gamma(\alpha - \beta)} \frac{1}{(n+1+\alpha-\beta)_k} \right. \\
\left. + \frac{\Gamma(2\alpha + 1)}{\Gamma(\alpha - \beta) \Gamma(n + \alpha + \beta + 2)}(1-z)^{-\alpha-\beta-1} \times \left( \sum_{k=0}^{\infty} \binom{\alpha - \beta - 1}{k} \left( \frac{z}{1-z} \right)^k \frac{\Gamma(k + \alpha + \beta + 1)}{\Gamma(\alpha + \beta + 1)} \frac{1}{(n+2+\alpha+\beta)_k} \right) \right).
\]

We analyze now the uniform asymptotic nature of the expansion as \( n \to \infty \). Let’s consider two compact subsets \( K_0 \subset \Omega_0, K_1 \subset \mathbb{C} \setminus \{1\} \). From \([58]\), item \( a) \) of Proposition \([1]\) and \([7] (4) \) pp. 47] we have that

\[
\xi_{1,n,p_1}(z; \alpha, \beta) = O(a_{p_1+1}(n; z, \alpha, \beta)) \quad \text{as} \quad n \to \infty, \quad z \in K_0, (\alpha, \beta) \in K_1,
\]

where

\[
a_{p_1}(n; z, \alpha, \beta) = \binom{\alpha + \beta}{p_1} \left( \frac{z}{1-z} \right)^{p_1} \frac{\Gamma(p_1 + \alpha - \beta)}{\Gamma(\alpha - \beta)} \frac{1}{(n+1+\alpha-\beta)_{p_1}}.
\]

In a similar way, from \([59]\), item \( b) \) of Proposition \([1]\) and \([7] (4) \) pp. 47] we have that

\[
\xi_{2,n,p_2}(z; \alpha, \beta) = O(a_{p_2+1}(n; z, \alpha, \beta)) \quad \text{as} \quad n \to \infty, \quad (\alpha, \beta) \in K_0, z \in K_1,
\]

where

\[
a_{p_2}(n; z, \alpha, \beta) = \binom{\alpha - \beta - 1}{p_2} \left( \frac{1-z}{1-z} \right)^{p_2} \frac{\Gamma(p_2 + \alpha + \beta + 1)}{\Gamma(\alpha + \beta + 1)} \frac{1}{(n+2+\alpha+\beta)_{p_2}}.
\]

From \((76)\) and \((77)\) we obtain the uniform asymptotic nature of the expansion, cf. \([31]\) pp. 356].
As a consequence of the preceding theorem, we obtain

**Proof. (Of Corollary 1)**

From (71) of Theorem 1 we have that

\[ P_n = \frac{\Gamma(2\alpha + 1)}{\Gamma(\alpha - \beta)\Gamma(\alpha + \beta + 1)} (I_n^{(1)} + I_n^{(2)}). \]

We will analyze the convergence for the expansions of \( I_n^{(1)} \) and \( I_n^{(2)} \).

Consider the remainder \( \xi_{1,n,p_1}(z;\alpha,\beta) \) of the expansion for \( I_n^{(1)} \). From [7, (4) pp. 47] and from the expression for \( \xi_{1,n,p_1}(z;\alpha,\beta) \), we have that, for every \( n \in \mathbb{N} \cup \{0\} \) fixed

\[ \xi_{1,n,p_1}(z;\alpha,\beta) = O\left(\frac{1}{p_1^{n+1}}\right), \quad p_1 \to \infty. \] (78)

Therefore, if \( |z/(1-z)| < 1 \), the series expansion for \( I_n^{(1)} \) converges.

In a similar way, from [7, (4) pp. 47] and from the expression of \( \xi_{2,n,p_2}(z;\alpha,\beta) \) for \( I_n^{(2)} \), we have that, for every \( n \in \mathbb{N} \cup \{0\} \) fixed

\[ \xi_{2,n,p_2}(z;\alpha,\beta) = O\left(\frac{1}{p_2^{n+1}}\right), \quad p_2 \to \infty. \] (79)

Therefore, the series expansion for \( I_n^{(2)} \) converges if \( \frac{1}{|z-1|} \leq 1 \).

From relation \( P_n = \frac{\Gamma(2\alpha + 1)}{\Gamma(\alpha - \beta)\Gamma(\alpha + \beta + 1)} (I_n^{(1)} + I_n^{(2)}) \) and the analysis of convergence done for \( I_n^{(1)} \) and \( I_n^{(2)} \), we deduce that the expansion for \( P_n \) converges if \( |z| < |z-1| \) and \( 1 < |z-1| \). \( \square \)

### 3 Askey’s problem

For the proof of Theorem 2 we use the following integral representation for \( P_n(z;\alpha,\beta) \), with \((\alpha,\beta) \in \Omega_0 \) and \( z \in \mathbb{C} \setminus \{0\} \), which is straightforward from [27 Sect. 2 Eqs. (2.1)–(2.4)],

\[ P_n(z;\alpha,\beta) = \frac{\Gamma(2\alpha + 1)}{\Gamma(\alpha + \beta + 1)\Gamma(\alpha - \beta)} \int_0^1 \left( \frac{z^u - 1}{z - 1} \right)^{\alpha+\beta} \left( \frac{z - z^u}{z - 1} \right)^{\alpha-\beta-1} e^{czu\ln z} e^{czu} e^{czu\ln z} du. \] (80)

We prove now the theorem.

**Proof. (Of Theorem 2)**

From (80) we have that

\[ P_n(e^{i\theta}z;\alpha,\beta) = \frac{\Gamma(2\alpha + 1)}{\Gamma(\alpha + \beta + 1)\Gamma(\alpha - \beta)} \times \int_0^1 \left( \frac{e^{i\theta} - 1}{e^{i\theta} - 1} \right)^{\alpha+\beta} \left( \frac{e^{i\theta} - e^{i\theta}z}{e^{i\theta} - 1} \right)^{\alpha-\beta-1} e^{i\theta z} e^{i\theta z} e^{i\theta z} du. \] (81)

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Notice that if $0 \leq x, y \leq \pi$ then
\[ 0 \leq \arg \left( \frac{e^{ix} - 1}{ix} \right) \leq \frac{\pi}{2}, \quad -\frac{\pi}{2} \leq \arg \left( \frac{iy}{e^{iy} - 1} \right) \leq 0. \] (82)

In a similar way, if $-\pi < x, y \leq 0$ then
\[ -\frac{\pi}{2} < \arg \left( \frac{e^{ix} - 1}{ix} \right) \leq 0, \quad 0 \leq \arg \left( \frac{iy}{e^{iy} - 1} \right) < \frac{\pi}{2}. \] (83)

Relations (82) and (83) give, for $0 \leq x, y \leq \pi$ or $-\pi < x, y \leq 0$,
\[ -\frac{\pi}{2} \leq \arg \left( \frac{e^{ix} - 1}{ix} \right) + \arg \left( \frac{iy}{e^{iy} - 1} \right) \leq \frac{\pi}{2}. \] (84)

It is well known that, if $z_1, z_2, \gamma \in \mathbb{C}$, then
\[ (z_1 z_2)^\gamma = e^{2k\pi \gamma z_1 z_2}, \] (85)

where $k = \left\lfloor \frac{\arg(z_1 z_2) - \arg(z_1) - \arg(z_2)}{2\pi} \right\rfloor$.

From (84) and (85), if $\theta_n = \frac{\theta}{n}$, $\theta \in [-\pi, \pi)$, $n \in \mathbb{N}$ and $u \in [0, 1]$
\[ \left( \frac{e^{\theta_n u} - 1}{e^{\theta_n} - 1} \right)^{\alpha+\beta} = u^{\alpha+\beta} \left( \frac{e^{\theta_n u} - 1}{\theta_n u} \right)^{\alpha+\beta} \left( \frac{\theta_n}{e^{\theta_n} - 1} \right)^{\alpha+\beta}, \] (86)

\[ \left( \frac{e^{\theta_n} - 1}{e^{\theta_n} - 1} \right)^{\alpha-\beta-1} = (1-u)^{\alpha-\beta-1} e^{u \theta_n (\alpha-\beta-1)} \left( \frac{e^{(1-u)\theta_n} - 1}{(1-u)\theta_n} \right)^{\alpha-\beta-1} \left( \frac{\theta_n}{e^{\theta_n} - 1} \right)^{\alpha-\beta-1}. \] (87)

Substituting (86) and (87) in (81)
\[ P_n(e^{\theta_n}; \alpha, \beta) = \frac{\Gamma(2\alpha + 1)}{\Gamma(\alpha + \beta + 1) \Gamma(\alpha - \beta)} \int_0^1 u^{\alpha+\beta}(1-u)^{\alpha-\beta-1} \times e^{u \theta_n (\alpha-\beta)} \left( \frac{u \theta_n}{e^{u \theta_n} - 1} \right)^{-\alpha-\beta} \left( \frac{\theta_n (1-u)}{e^{\theta_n (1-u)} - 1} \right)^{-\alpha+\beta+1} \left( \frac{\theta_n}{e^{\theta_n} - 1} \right)^{2\alpha} e^{n \theta_n u} du. \] (88)

From the generating functions for the generalized Bernoulli polynomials [18], we have
\[ e^{u \theta_n (\alpha-\beta)} \left( \frac{u \theta_n}{e^{u \theta_n} - 1} \right)^{-\alpha-\beta} = \sum_{j=0}^\infty B_j^{(-\alpha-\beta)}(\alpha-\beta) \frac{(u \theta_n)^j}{j!}, \]
\[ \left( \frac{\theta_n (1-u)}{e^{\theta_n (1-u)} - 1} \right)^{-\alpha+\beta+1} = \sum_{j=0}^\infty B_j^{(-\alpha+\beta+1)}(0) \frac{(1-u)^j \theta_n^j}{j!}, \]
\[ \left( \frac{\theta_n}{e^{\theta_n} - 1} \right)^{2\alpha} = \sum_{j=0}^\infty B_j^{(2\alpha)}(0) \frac{\theta_n^j}{j!}. \]
Substituting these relations in \(88\),

\[
P_n(e^{\theta u} \alpha, \beta) = \frac{\Gamma(2\alpha + 1)}{\Gamma(\alpha + \beta + 1)\Gamma(\alpha - \beta)} \int_0^1 u^{\alpha + \beta}(1 - u)^{\alpha - \beta - 1} \sum_{j=0}^{\infty} B_j(-\alpha+\beta)(\alpha - \beta) \frac{u^j \theta_n^j}{j!} \times
\]

\[
\sum_{j=0}^{\infty} B_j(-\alpha+\beta+1)(0) \frac{(1 - u)^j \theta_n^j}{j!} \sum_{j=0}^{\infty} B_j(2\alpha)(0) \frac{\theta_n^j}{j!} e^{\theta_n u} du
\]

\[
= \frac{\Gamma(2\alpha + 1)}{\Gamma(\alpha + \beta + 1)\Gamma(\alpha - \beta)} \int_0^1 u^{\alpha + \beta}(1 - u)^{\alpha - \beta - 1} \sum_{j=0}^{\infty} b_j(u) \theta_n^j e^{\theta_n u} du, \quad (89)
\]

where \(b_j(u) = \sum_{i_1 + i_2 + i_3 = j} \frac{B_{i_1}(-\alpha+\beta)(\alpha - \beta)}{i_1!} \frac{B_{i_2}(-\alpha+\beta+1)(0)}{i_2!} \frac{B_{i_3}(2\alpha)(0)}{i_3!} u^{i_1}(1 - u)^{i_2}\) are the coefficients of the Taylor development about \(v = 0\) of the function

\[
e^{uv(\alpha - \beta)} \left( \frac{uv}{e^{uv} - 1} \right)^{-\alpha - \beta} \left( \frac{v(1 - u)}{e^{v(1 - u)} - 1} \right)^{-\alpha + \beta + 1} \left( \frac{v}{e^v - 1} \right)^{2\alpha}.
\]

As the series of the last equality in \(89\) converges uniformly in \([0, 1]\), we have that

\[
P_n(e^{\theta u} \alpha, \beta) = \frac{\Gamma(2\alpha + 1)}{\Gamma(\alpha + \beta + 1)\Gamma(\alpha - \beta)} \sum_{j=0}^{\infty} \left( \sum_{|i| = j} \frac{B_{i_1}(-\alpha+\beta)(\alpha - \beta)}{i_1!} \frac{B_{i_2}(-\alpha+\beta+1)(0)}{i_2!} \frac{B_{i_3}(2\alpha)(0)}{i_3!} \times
\]

\[
\int_0^1 u^{\alpha + \beta + i_1 + i_2} (1 - u)^{\alpha - \beta - 1 + i_2} e^{\theta u} du \left( \frac{\theta}{n} \right)^j = \sum_{j=0}^{\infty} \sum_{|i| = j} \frac{B_{i_1}(-\alpha+\beta)(\alpha - \beta)}{i_1!} \frac{B_{i_2}(-\alpha+\beta+1)(0)}{i_2!} \frac{B_{i_3}(2\alpha)(0)}{i_3!} \times
\]

\[
\frac{(\alpha + \beta + 1)i_1 (\alpha - \beta)i_2}{(2\alpha + 1)i_1 + i_2} F_1(1 + \alpha + \beta + i_1; 1 + 2\alpha + i_1 + i_2; \theta) \left( \frac{\theta}{n} \right)^j, \quad (90)
\]

where \(|i| = i_1 + i_2 + i_3, i_1, i_2, i_3 \in \mathbb{N} \cup \{0\}\). Consider now the remainder \(R_{k,n}\) defined as

\[
P_n(e^{\theta u} \alpha, \beta) = \sum_{j=0}^{k} \sum_{|i| = j} \frac{B_{i_1}(-\alpha+\beta)(\alpha - \beta)}{i_1!} \frac{B_{i_2}(-\alpha+\beta+1)(0)}{i_2!} \frac{B_{i_3}(2\alpha)(0)}{i_3!} \times
\]

\[
\frac{(\alpha + \beta + 1)i_1 (\alpha - \beta)i_2}{(2\alpha + 1)i_1 + i_2} F_1(1 + \alpha + \beta + i_1; 1 + 2\alpha + i_1 + i_2; \theta) \left( \frac{\theta}{n} \right)^j + R_{k,n}(\theta). \quad (91)
\]

From \(89\) we have
\[ P_n(e^{i\theta};\alpha,\beta) = \frac{\Gamma(2\alpha + 1)}{\Gamma(\alpha + \beta + 1)\Gamma(\alpha - \beta)} \times \left( \sum_{j=0}^{k} + \sum_{j=k+1}^{\infty} \right) \left( \int_0^1 u^{\alpha+\beta}(1-u)^{\alpha-\beta-1}b_j(u)e^{i\theta u}du \right) \left( \frac{i\theta}{n} \right)^j. \] (92)

From (91) and (92) we deduce
\[ R_{k,n}(\theta) = \frac{\Gamma(2\alpha + 1)}{\Gamma(\alpha + \beta + 1)\Gamma(\alpha - \beta)} \sum_{j=k+1}^{\infty} \left( \int_0^1 u^{\alpha+\beta}(1-u)^{\alpha-\beta-1}b_j(u)e^{i\theta u}du \right) \left( \frac{i\theta}{n} \right)^j. \] (93)

From Cauchy’s estimate [1, (25) pp. 122], we have that
\[ |b_j(u)| \leq M(r)r^{-j}, \quad 0 < r < 2\pi, \] (94)
where
\[ M(r) = \max_{|v| = \frac{3\pi}{2}} \left| e^{v(\alpha-\beta)} \left( \frac{uv}{e^{uv} - 1} \right)^{-\alpha-\beta} \left( \frac{v(1-u)}{e^{v(1-u)} - 1} \right)^{-\alpha+\beta+1} \left( \frac{v}{e^v - 1} \right)^{2\alpha} \right|. \]
Since \( u \in [0, 1] \), we have
\[ M \left( \frac{3\pi}{2} \right) \leq \max_{|v| = \frac{3\pi}{2}} \left| e^{v(\alpha-\beta)} \right|, \]
therefore, from (94)
\[ |b_j(u)| \leq \left( \frac{2}{3\pi} \right)^j \max_{|v| = \frac{3\pi}{2}} \left| e^{v(\alpha-\beta)} \right|. \] (95)

From the expression for the remainder (93) and from (95), we obtain
\[ |R_{k,n}(\theta)| \leq \frac{\Gamma(\Re(\alpha + \beta + 1))}{\Gamma(\alpha + \beta + 1)\Gamma(\alpha - \beta)} \max_{|v| = \frac{3\pi}{2}} \left| e^{v(\alpha-\beta)} \right| \sum_{j=k+1}^{\infty} \left| \frac{2\theta}{3n\pi} \right|^j \leq \frac{\Gamma(\Re(\alpha + \beta + 1))\Gamma(\Re(\alpha - \beta))}{\Gamma(\alpha + \beta + 1)\Gamma(\alpha - \beta)} \max_{|v| = \frac{3\pi}{2}} \left| e^{v(\alpha-\beta)} \right| \left| \frac{2\theta}{3n\pi - 2\theta} \right| \left| \frac{2\theta}{3n\pi} \right|^k. \]

\section{Electrostatic model for zeros of a class of para-orthogonal polynomials}

\textbf{Proof.} (Of Theorem 3)
Notice that as \((\theta_1, \ldots, \theta_n)\) approaches to the boundary of the set \(\Theta_0 = \{(\theta_1, \ldots, \theta_n) \in [0, 2\pi]^n : \theta_1 < \ldots < \theta_j < \ldots < \theta_n\}\) we have that \(E \to +\infty\), therefore the solution set \(\Theta_1\) of the problem

\[
\min_{(\theta_1, \ldots, \theta_n) \in \Theta_0} E(\theta_1, \ldots, \theta_n),
\]

belongs to the interior of the set \(\Theta_0\). It follows from the theory of constrained optimization \[21, pp. 327–328\] that the first–order optimality conditions (or Karush–Kuhn–Tucker conditions, more precisely) reduce to the equation \(\nabla \theta E = 0\).

From the relation

\[
\frac{\partial}{\partial \theta_j} \ln \frac{1}{|e^{i\theta_k} - e^{i\theta_j}|} = \Im \left( \frac{e^{i\theta_j}}{e^{i\theta_j} - e^{i\theta_k}} \right),
\]

we deduce that the partial derivatives of \(E\) can be expressed as

\[
\frac{\partial E}{\partial \theta_j} = \sum_{k \neq j} \Im \left( \frac{e^{i\theta_j}}{e^{i\theta_j} - e^{i\theta_k}} \right) - \Im \left( \frac{p + \Omega - 1}{1 - e^{i\theta_j} + e^{i\theta_j} - 1} \right) e^{i\theta_j}.
\]

Notice that we have introduced in the above expression the auxiliary term \(\frac{n + p - 1}{2} \). This term, as will be seen below, completes the expression for the differential equation that defines the para–orthogonal polynomial.

Introducing the polynomial \(f(z) = \prod_{j=1}^{n} (z - e^{i\theta_j})\), we obtain that the extremum conditions give

\[
\frac{\partial E}{\partial \theta_j} = \Im \left( z_j \frac{1}{2} f''(z_j) - \left( \frac{p}{1 - z_j} + \frac{n + p - 1}{2} \right) z_j \right) = \Im \left( z_j (1 - z_j) f''(z_j) - \frac{n + \alpha - 1 - 2 \Omega - (n - p - 1 - 2 \Omega) z_j f'(z_j)}{2 f'(z_j)(1 - z_j)} \right) = 0,
\]

where \(z_j = e^{i\theta_j}\). If \(p = \alpha = 2 \Omega = \beta\), we have that this last relation gives

\[
\frac{\partial E}{\partial \theta_j} = \Im \left( \frac{z_j (1 - z_j) f''(z_j) - (n + \alpha - 1 - \beta - (n - \alpha - \beta - 1) z_j f'(z_j)}{f'(z_j)(1 - z_j)} \right) = 0. \tag{97}
\]

Let’s write \([97]\) as

\[
\frac{\partial E}{\partial \theta_j} = \frac{1}{|g_n(z_j)|^2} \Im \left( \Pi_n(z_j) g_n(z_j) \right) = 0,
\]

where \(\Pi_n(z) = z (1 - z) f''(z) - (n + \alpha - 1 - \beta - (n - \alpha - \beta - 1) z) f'(z)\) and \(g_n(z) = f'(z)(1 - z)\). Relation \([97]\) gives that

\[
\Im \left( \Pi_n(z) g_n(z) \right) = \sum_{k=-n}^{n} a_k z^k, \tag{98}
\]

is a trigonometric polynomial of degree \(n\) vanishing at the \(2n\) points

\[
\{e^{i\theta_1}, \ldots, e^{i\theta_n}, \rho_1 e^{i\psi_1}, \ldots, \rho_{n-1} e^{i\psi_{n-1}}, 1\}, \quad 0 < \rho_i < 1,
\]

is a trigonometric polynomial of degree \(n\) vanishing at the \(2n\) points
counting the multiplicities of $g_n$, where \( \{ \rho_j e^{i\psi_j} \}_{j=1}^{n-1} \cup \{1\} \) are the set of zeros of the polynomial $g_n$. This condition determines $\Pi_n$ uniquely, except up to a multiplicative real constant $c_n$. Notice that the choice $\Pi_n(e^{i\theta}) = \kappa_n f(e^{i\theta})$, for some adequate constant $\kappa_n \in \mathbb{C}$, satisfies (97). By comparing the coefficient $z^n$ in

\[
\kappa_n f(z) = z(1-z)f''(z) - (n+\alpha-1-\beta-(n-\alpha-\beta-1)z)f'(z),
\]

we deduce that $\kappa_n = -n(\alpha+\beta)$ and from the fact that the hypergeometric differential equation

\[
z(1-z)y'' - (\alpha+n-\beta-1-(n-\alpha-\beta-1)z)y' + n(\alpha+\beta)y = 0,
\]

has the unique monic polynomial solution

\[
B_n \left( z; \frac{(\alpha-\beta)_{n+1}}{(\alpha+\beta)_{n+1}} \right) = \frac{(2\alpha)_n}{(\alpha+\beta)_n} \binom{2}{1}_F \left( -n, \alpha+\beta; 2\alpha; 1-z \right),
\]

we deduce that the zeros of the para–orthogonal polynomial $B_n$ define the local extremum of $E$.

In order to see if the local extremum is a global minimum one can check that the Hessian matrix is indeed positive definite or just apply the following argument.

Let us consider a compact subset $K \subset \Theta_0$ such that $E(\theta_1,\ldots,\theta_n) > M, \forall (\theta_1,\ldots,\theta_n) \in \partial K$, for $M > 0$ sufficiently large (such $M$ exists since $\lim_{(\theta_1,\ldots,\theta_n) \to \partial \Theta_0} E(\theta_1,\ldots,\theta_n) = +\infty$) and $(\theta_{1}^{*},\ldots,\theta_{n}^{*})$ being the local extremum. Now, $(\theta_{1}^{*},\ldots,\theta_{n}^{*})$ is the global minimum, since $E$ is continuous in $K$.

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References

[1] Ahlfors, L.V.: Complex analysis, McGraw–Hill Book Company, 3th ed., 1953.

[2] Askey, R.(editor): Gabor Szegő: Collected papers, 3 vols., Contemp. Math., Amer. Math. Soc., Providence, RI, 1982.

[3] Askey, R.: Some open problems about special functions and computations, Rend. Sem. Mat. Univ. Politec. Torino, Special Volume, 1–22 (1985).

[4] Basor, E.: Asymptotic formulas for Toeplitz determinants, Trans. Amer. Math. Soc. 239, 33–65 (1978).

[5] Deift, P., Its, A. and Krasovsky, I.: Asymptotic of Toeplitz, Hankel, and Toeplitz+Hankel determinants with Fisher–Hartwig singularities, Ann. of Math. 174, 1243–1299 (2011).

[6] Dunster, T.M. and Lutz, D.A.: Convergent factorial series expansions for Bessel functions, SIAM J. Math. Anal. 22 (4), 1156–1172 (1991).
[7] Erdélyi, A.: *Higher transcendental functions*, McGraw-Hill: New York, Vol. 1, 1953.

[8] Fisher, M.E. and Hartwig R.E.: Toeplitz determinants: Some applications, theorems, and conjectures, Adv. Chem. Phys. 15, 333–353 (1968).

[9] Friedrichs, K.O.: Asymptotic phenomena in mathematical physics, Bulletin of the American Mathematical Society, 61(6), 485–504 (1955).

[10] Gasper, G.: Orthogonality of certain functions with respect to complex valued weight functions, Canad. J. Math. 33, 1261–1270 (1981).

[11] Greiner, P.C. and Koornwinder, T.H.: *Variations on the Heisenberg spherical harmonics*, Report ZW 186/83, Mathematisch Centrum, Amsterdam, 1983.

[12] Grünbaum, F. A.: Variations on a theme of Heine and Stieltjes: an electrostatic interpretation of the zeros of certain polynomials, J. Comp. Appl. Math. 99 (1), 189–194, (1998).

[13] Hendriksen, E. and Rossum, H.V.: Orthogonal Laurent polynomials, Nederl. Akad. Wetensch. Indag. Math. 48, 17–36 (1986).

[14] Ismail, M. E.: More on electrostatic models for zeros of orthogonal polynomials, Numerical functional analysis and optimization, 21 (1–2), 191–204, (2000).

[15] Jones, W.B., Njastad, O. and Thron, W. J.: Moment theory, orthogonal polynomials, quadrature, and continued fractions associated with the unit circle, Bull. London Math. Soc. 21, 113–152 (1989).

[16] Lenard, A.: Moment distribution in the ground state of the one–dimensional system of impenetrable bosons, J. Math. Phys. 5, 930–943 (1964).

[17] Lenard, A.: Some remarks on large Toeplitz determinants, Pacific J. Math. 42, 137–145 (1972).

[18] Luke, Y. L.: *The Special Functions and their approximations*, Academic Press, New York, Vol. I, 1969.

[19] Martínez–Finkelshtein, A., McLaughlin, K.T.-R. and Saff E.B.: Asymptotics for orthogonal polynomials with respect to an analytic weight with algebraic singularities on the unit circle, Int. Math. Res. Not. Art. ID. 91426, 43pp. (2006).

[20] Milne–Thompson L. M.: *The calculus of finite differences*, MacMillan and Co., London, 1951.

[21] Wright, S. J.; Nocedal, J.: *Numerical optimization*, New York: Springer, 1999.

[22] Nörlund, N.E.: *Leçons sur les series dinterpolation*, Gauthiers-Villars et Cie, Paris, 1926.

[23] Nörlund, N.E.: Sur les valeurs asymptotiques des nombres et des polynômes de Bernoulli, Rend. Circ. Mat. Palermo 10 (2), 27–44 (1961).

[24] Olver, F. W. J.: *Asymptotics and Special Functions*, Academic, New York, 1974.

[25] Ranga, A. Sri: Szegö polynomials from hypergeometric functions, Proc. Amer. Math. Soc. 138 (12), 4259–4270 (2010).
[26] Szegő, G.: Orthogonal Polynomials, Amer. Math. Soc. Colloq. Publ. Vol. 23, Providence, RI, Amer. Math. Soc., 4th ed., 1975

[27] Temme, N.M.: Uniform asymptotic expansion for a class of polynomials biorthogonal on the unit circle, Constr. Approx. 2, 369–376 (1986).

[28] Wasow, W.: Asymptotic Expansions for Ordinary Differential Equations, Academic Press, New York, 1965.

[29] Watson, G.N.: The transformation of an asymptotic series into a convergent series of inverse factorials, Rend. Circ. Mat. Palermo 34 (1), 41–88 (1912).

[30] Whittaker, E.T. and Watson, G.N.: A course of modern analysis, Cambridge University Press, 1927.

[31] Wong, R.: Asymptotic approximations of integrals, Vol. 34. SIAM, 2001.