Quantum anchor : $U_q(sl(2))$ case

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Abstract

We introduce the tangent space $T(H_q)$ on the quantum hyperboloid ($A^0_{0,q}$) and equip it with an action on $A^0_{0,q}$ being a deformation of the action of vectors fields on functions. An embedding $sl(2)_q \to T(H_q)$ of $q$-deformed Lie algebra $sl(2)$ being an analogue of the anchor $sl(2) \to Vect(H)$ is called “quantum anchor”.

1 Introduction

Let us consider the sphere of radius $R > 0$

$$S^2 = \{(x, y, z) \in K^3; \quad x^2 + y^2 + z^2 = R^2\}$$

and the infinitesimal rotations

$$X = y\partial_z - z\partial_y, \quad Y = z\partial_x - x\partial_z, \quad Z = x\partial_y - y\partial_x.$$ (Hereafter the basic field $K$ is $\mathbb{R}$, but all results are still valid for its complexification. In this case we assume $K = \mathbb{C}$.)

It is known that these infinitesimal rotations generate the space $Vect(S^2)$ (or $Vect(A)$, where $A = Fun(S^2)$ is the space of functions on the sphere) and $Vect(S^2)$ is the space of all vectors fields on $S^2$. (In what follows all functions are assumed to be restrictions of polynomials onto the sphere in question.) This means that every element $\mathcal{X} \in Vect(S^2)$ is of the form

$$\mathcal{X} = \alpha X + \beta Y + \gamma Z; \quad \alpha, \beta, \gamma \in Fun(S^2).$$

However, the vectors fields $X, Y, Z$ are not free. They satisfy the following identity

$$xX + yY + zZ = 0.$$ So, the tangent space $T(S^2)$ on the sphere considered as a (left) $A$-module can be defined by this identity, i.e., it is the factor-module of the rank 3 free module

$$M = \{ \alpha X + \beta Y + \gamma Z; \quad \alpha, \beta, \gamma \in Fun(S^2) \}.$$
over its submodule

\[ M_1 = \{ f(xX + yY + zZ); \ f \in \text{Fun}(S^2) \}. \]

Let us fix \( k \in K \) and consider a hyperboloid

\[ H = \{(u, v, w) \in K^3; \ 2uw + vv + 2wu = k \}. \]

Similarly to the sphere \( S^2 \), the space of vectors fields on the hyperboloid \( \text{Vect}(H) \) (or \( \text{Vect}(A) \) where \( A = \text{Fun}(H) \)) is generated by three infinitesimal hyperbolic rotations \( U, V, W \) satisfying the identity

\[ 2uW + vV + 2wU = 0. \] (1)

Then, similarly to the previous case we can introduce the tangent space \( T(H) \) on the hyperboloid by means of the identity [I].

We remark that in both cases the tangent modules are equipped with a Lie algebra structure and that an action

\[ \beta : T(H) \otimes A \rightarrow A, \quad A = \text{Fun}(H) \]

is well defined: apply a vector field \( \mathcal{X} \) to a function \( f \):

\[ \mathcal{X} \otimes f \rightarrow \mathcal{X}f. \]

Thus, the tangent space \( T(H) \) is realized as that of vectors fields \( \text{Vect}(H) \) on \( H \) and, consequently, we have an embedding of the Lie algebra \( sl(2) \) generated by the elements \( u, v, w \) into the space \( \text{Vect}(H) \) generated by the elements \( U, V, W \). Traditionally such an embedding is called an anchor.

Consider a quantum hyperboloid. The explicit description below, let us already give three proprieties of the corresponding “quantum function” algebra:

1. this algebra is a flat deformation of its classical counterpart (see for example [DGK] for the definition of a flat deformation),
2. its product is \( U_q(sl(2)) \)-covariant (that is, it is a \( U_q(sl(2)) \)-morphism),
3. it is in some sens \( q \)-commutative (more precisely, it is a “\( q \)-commutative” algebra of a two parameters family of \( U_q(sl(2)) \)-covariant algebras considered below).

In the present note we discuss two problems: what is the “tangent space” on the quantum hyperboloid? Do we have an analogue of the above mentioned anchor?

First of all, we should find quantum analogues of the vectors fields \( U, V, W \). Usually, one considers the generators \( X, Y, H \) of the quantum group (QG) \( U_q(sl(2)) \) (see Section 1) as such quantum analogues. However, contrary to the classical case, those generators do not satisfy any \( q \)-analogue of identity [II]. So, if we introduce the tangent space on the quantum hyperboloid as the familly of all linear combinations of these operators (with coefficients in the \( q \)-analogue of \( \text{Fun}(H) \), called quantum hyperboloid in what follows), then we do not have any flat deformation of the classical tangent space \( T(H) \).
Here we propose other candidates to play the role of $q$-analogues of the operators $U, V, W$, and satisfying an identity that can be viewed as a $q$-analogue of (1). Thus, the tangent space $T(H_q)$ on the quantum hyperboloid $H_q$ defined as the set of all linear combinations of these quantum vectors fields modulo this identity will be a deformation with respect to its classical counterpart.

Once the tangent space is defined, the following natural problem arises: does there exist an action

$$T(H_q) \otimes A \to A$$

which is a deformation of the classical anchor? Here $A$ designs the quantum analogue of the algebra $\text{Fun}(H)$. We construct such an action and give an interpretation of the elements of $T(H_q)$ as \textit{braided vectors fields}. We will obtain an embedding of $\mathfrak{sl}(2)_q$ into $T(H_q)$, called \textit{quantum anchor}. Here $\mathfrak{sl}(2)_q$ is the “braided” analogue of $\mathfrak{sl}(2)$ and $T(H_q)$ is generated by braided analogues of the vectors fields $U, V, W$.

The organization of this note is the following. In the Section 1, we recall the construction of a quantum hyperboloid. In section 2 we define the tangent space on a quantum hyperboloid and equip it with an action on all “function” on the quantum hyperboloid. Finally, we defined a \textit{quantum anchor}.

Throughout the note, the parameter $q \in K$ is assumed to be generic.

## 2 Quantum hyperboloid

In order to introduce a quantum hyperboloid, let us consider the QG $U_q(\mathfrak{sl}(2))$ defined, as usual, by the generators $X, Y, H$ subject to the well known relations (cf. [CP]). Let us fix a coproduct in $U_q(\mathfrak{sl}(2))$ and the corresponding antipode and consider a spin 1 $U_q(\mathfrak{sl}(2))$-module $V = V'$.

We only need the fact that the fusion ring of finite dimensional $U_q(\mathfrak{sl}(2))$-modules is exactly the same as in the classical case (we consider only the finite dimensional $U_q(\mathfrak{sl}(2))$-modules which are deformations of the classical $\mathfrak{sl}(2)$-modules). Thus, if $V_i$ is a spin $i$ $U_q(\mathfrak{sl}(2))$-module, the classical formula

$$V_i \otimes V_j = \bigoplus_{k=|i-j|} V_k$$

is still valid although the Clebsch-Gordan coefficients (which depend on the choice of a base) are $q$-deformed.

In particular, we have:

$$V \otimes^2 = V_0 \oplus V_1 \oplus V_2.$$
Let us denote $\mathcal{A}_{c,h,q}$ the algebra defined by the quotient algebra
\[ T(V) / \{I_h\} \]
where $\{I_h\}$ is the ideal generated by the elements of the left hand side of (2) and all the derived elements.

The algebra $\mathcal{A}_{c,h,q}$ is multiplicity free. More precisely, each integer spin module appears once in its decomposition into a direct sum of irreducible $U_q(sl(2))$-modules. Moreover, each element of $\mathcal{A}_{c,h,q}$ can be represented in a unique way as a sum of homogeneous elements such that each of them belongs to one of the highest components of $V^\otimes i$ (a proof of this fact is given in [A]).

We will treat the algebra $\mathcal{A}_{0,q}$ as a $q$-analogue of the commutative algebra Fun (H) and call it quantum hyperboloid if $c \neq 0$ (being equipped with an involution it can be treated as the Podles quantum sphere ([P]) and quantum cone if $c = 0$. This is motivated by an analogy with the classical case. Since the $sl(2)$-module $sl(2)_{\otimes 2}$ is multiplicity free we can introduce $q$-analogues $I_{\pm}$ of symmetric and skew symmetric subspaces of $sl(2)_{\otimes 2}$ by setting, as in the classical case
\[ I_+ = V_0 \oplus V_2 \quad I_- = V_1. \]

Let us emphasize that the corresponding algebras $A_{\pm} = T(sl(2))/\{I_{\pm}\}$ are flat deformations of their classical counterparts. Since the algebra $\mathcal{A}_{0,q}$ is a factor of the “$q$-symmetric” algebra $A_+$ we consider it also as “$q$-symmetric”. It is a particular case of the family of two parameters algebras $\mathcal{A}_{c,h,q}$ (where $c$ is assumed to be fixed).

**Remark 1** In the case $n \geq 3$, the $sl(n)$-module $sl(n)_{\otimes 2}$ is not multiplicity free. Consequently, it is not obvious what $q$-analogues of the symmetric and skew symmetric algebras of the space $sl(n)$ should be. However, as shown in [3], there exists a flat deformation of the algebra $Sym(sl(n))$. A way to construct these algebras in a more explicit way is suggested in [AC].

For any simple Lie algebra $g$ different from $sl(n)$ its tensor square $g_{\otimes 2}$ is multiplicity free. This implies that there exists natural $q$-analogues $I_{\pm}^q$ of the subspaces $I_{\pm} \subset g_{\otimes 2}$. However, the $q$-algebras $A_{\pm}^q = T(V)/\{I_{\pm}^q\}$ are not flat deformations of their classical counterparts (cf. [G]).

### 3 Quantum anchor on quantum hyperboloid

In the present section we introduce the tangent space on the quantum hyperboloid and we equip it with a quantum anchor.

In the classical case, we can restate the identity (II) in a symbolic way as
\[ (V \otimes V')_0 = 0 \]
where $V = Span (u, v, w)$, $V' = Span (U, V, W)$ and $(V \otimes V')_i$ designs the spin $i$ $U(sl(2))$-module in the tensor product $V \otimes V'$.

In order to define the tangent space on the quantum hyperboloid, we should consider the
identity (3) in the $U_q(sl(2))$-module category. Denote $V'^q$ the $q$-analogue of $V'$. Let $(V^q \otimes V'^q)_0$ be the spin 0 $U_q(sl(2))$-module and $\{(V^q \otimes V'^q)_0\}$ be the left $A_{\bar{h},q}$-module generated by $(V^q \otimes V'^q)_0$.

**Definition 1** The “left tangent space” on the quantum hyperboloid consider as a left $A_{\bar{h},q}$-module is defined as follows:

$$\mathcal{T}(H_q)_l = (A_{\bar{h},q} \otimes V'^q)/\{(V^q \otimes V'^q)_0\}.$$  \hspace{1cm} (4)

In a similar way one can define the right tangent space $\mathcal{T}(H_q)_r$ of the quantum hyperboloid considered as a right $A_{\bar{h},q}$-module.

**Proposition 1** ([DG], [AC])

The $A_{\bar{h},q}$-module $\mathcal{T}(H_q)$ is a flat deformation of its classical counterpart.

Now we discuss the problem of a suitable definition of a “quantum anchor” on the quantum hyperboloid.

For this, we need the notion of a braided Lie algebra. Following [DG], we define a braided Lie bracket

$$[,]_q : V^\otimes 2 \rightarrow V$$

as a no trivial $U_q(sl(2))$-morphism (here $V$ is spin 1 $U_q(sl(2))$-module). By this request, the bracket is defined uniquely up to a factor. The product table of such a bracket is given for example in [DG]. Let us note that this construction is generalized for the $sl(n)$ $n > 2$ case, in [LS].

The space $V$ equipped with such bracket will be denoted by:

$$sl(2)_q := (V, [, ]_q)$$

and will be called a braided Lie algebra. The enveloping algebra of this braided Lie algebra can be defined similarly to that of $A_{\bar{h},q}$ but with omitting the first relation of (3). This enveloping algebra will be denoted $A_{\bar{h},q}$. We disregard here the problem of suitable relation between $\bar{h}$ and the factor coming in the definition of $sl(2)_q$, cf. [LS].

Let us consider (left) $q$-adjoint operators associated to the elements of the space $V$. For example the operator $U^q$ associated to the generator $u$ is defined by

$$U^q : V \rightarrow \text{V} \quad \quad z \mapsto U^q z = ad^q u(z) = [u, z]_q, \quad U^q 1 = 0.$$ 

In a similar way the operators $V^q, W^q$ associated to the generators $v, w$ respectively are well defined on the space $V$.

**Proposition 2** The following holds

$$(q^3 + q) u W^q + v V^q + (q + q^{-1}) w U^q = 0. \quad \quad (5)$$

Here all components are treated as operators acting from the space $V$ to $A_{\bar{h},q}$.
Throughout, we can assumed that the operators $U^q, V^q, W^q$ realize a representation of the braided Lie algebra $sl(2)_q$ (i.e. they satisfy the defining relation of its enveloping algebra). If it is not the case we can get such operators by a proper rescaling

$$U^q \rightarrow \lambda U^q, \quad V^q \rightarrow \lambda V^q, \quad W^q \rightarrow \lambda W^q \quad \lambda \in K.$$ 

Let us remark that such a rescaling does not breack the identity (5).

**Definition 2** We say that the $\mathcal{A}_{0,q}^c$-module $T(H_q)$ is equipped with a structure of “left quantum anchor” if there exists an action

$$\beta : T(H_q) \otimes \mathcal{A}_{0,q}^c \longrightarrow \mathcal{A}_{0,q}^c$$

such that the operators corresponding to $U^q, V^q, W^q$ realize a representation of the braided Lie algebra $sl(2)_q$ and the diagram

$$\begin{array}{c}
\mathcal{A}_{0,q}^c \otimes T(H_q) \otimes \mathcal{A}_{0,q}^c \\
\downarrow \\
\mathcal{A}_{0,q}^c \otimes \mathcal{A}_{0,q}^c \\
\downarrow \\
\mathcal{A}_{0,q}^c
\end{array}$$

is commutative. Here we suppose that the elements of $\mathcal{A}_{0,q}^c$ act on $\mathcal{A}_{0,q}^c$ (the low arrow) by the usual product. The vertical arrows are defined by means of $\beta$ and the top one makes use of the $\mathcal{A}_{0,q}^c$-module structure of $T(H_q)$.

In a similar way a notion of “right quantum anchor” can be defined.

Now we want to describe a quantum anchor in terms of the operators $U^q, V^q, W^q$. However, up to now the action of the operators $U^q, V^q, W^q$ is well defined on the space $V$. To complete the construction of quantum anchor we should extend their action on the whole algebra $\mathcal{A}_{0,q}^c$ with properties from definition 2. In the classical case, such an extension can be done via the Leibniz rule.

In the case under consideration such an extension is realize in $[A]$ (without any Leibniz rule). We represent here only the final result.

**Theorem 1** ([A]) A quantum anchor exists and, moreover, it is unique if we impose the additional condition that the extented operators $U^q, V^q, W^q$ send the highest component $(V_i)$ of $V^\otimes i$ into itself.

To finish this note, let us remark that the tangent space $T(H_q)$ is realize as an operator algebra on $\mathcal{A}_{0,q}^c$ and, moreover, the braided Lie algebra $sl(2)_q$ is embeded in it. This is the motivation of our definition of quantum anchor, inspite of the fact that we do not equip $T(H_q)$ with any structure fo Lie algebra (deformed or “braided”).

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