K3-fibered Calabi-Yau threefolds I,
the twist map

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Introduction

A natural situation occurring in the general classification theory of algebraic varieties is that of algebraic fiber spaces $X \rightarrow Y$ such that the general fiber $F$ has Kodaira dimension $0$. This will be the case in particular if $F$ is Calabi-Yau (a regular variety with trivial canonical bundle), and if this is the case it is an interesting problem to find conditions characterizing the case in which the total space $X$ is itself Calabi-Yau. In this note we make a few remarks on this problem for three-dimensional $X$.

Much of what we do is in the context of weighted hypersurfaces. Let $X$ be a hypersurface of weight $d$ in $P_{(w_0, \ldots, w_n)}$. Then a sufficient condition for $X$ to be Calabi-Yau is $d = \sum w_i$. We show by example (due to I. Dolgachev) that this condition is not necessary. Given a variety $X$ with a fibration $\pi: X \rightarrow Y$, a necessary condition for $X$ to be Calabi-Yau is that $\pi^* K_X|_Y = K_Y^{-1}$, and we show by example (of a K3-fibered threefold) that this condition is not sufficient. Both examples are based on a particular K3 surface, which is the surface with the largest automorphism group which preserves the Picard lattice, namely $\mathbb{Z}/66\mathbb{Z}$.

Our interest in this paper is in particular with fibrations whose fibers have constant modulus. Such varieties are covered by products, and if both factors of that product are weighted hypersurfaces, this covering can be neatly described in terms of weighted projective spaces; this is our twist map. This map arose in [14] in the context of dualities of heterotic and type II superstrings, but its mathematical formulation gives a very convenient method for resolving singularities of such quotients, see Corollary [3]. Even in the case of conventional projective spaces, the twist map is interesting. For example, one corollary is the following perhaps already known but somewhat startling result.

**Theorem 1:** Let $f(x_1, \ldots, x_n) = x_1^d + \cdots + x_n^d$, $n > 3$ be a Fermat polynomial of degree $d$ in $n$ variables, and let $(n_1, \ldots, n_\lambda)$ be any partition of $n$ with $n_i \geq 2$, $x_{i,k}$, $k = 1, \ldots, n_i$, $i = 1, \ldots, \lambda$ the corresponding coordinates. Then the Fermat hypersurface $X = \{f = 0\}$ is birational to a quotient of the product $X_1 \times \cdots \times X_\lambda$ of $\lambda$ Fermat hypersurfaces $X_i = \{x_{i,0}^d + x_{i,1}^d + \cdots + x_{i,n_i}^d = 0\}$ of the same degree $d$ by $\mathbb{Z}/d\mathbb{Z}$, acting only on the coordinates $x_{i,0}$ by multiplication by a primitive $d$th root of unity.

As a particular case of this, consider the Fermat quartic surface $S$ in $P^3$, let $C = \{x_0^4 + x_1^4 + x_2^4 = 0\} \subset P^2$ (resp. $C' = \{y_0^4 - y_1^4 - y_2^4 = 0\}$) be the Fermat quartic curve in the projective plane. Then under the map

$$P^2 \times P^2 \rightarrow P^3$$

$$(x_0, x_1, x_2), (y_0, y_1, y_2) \mapsto (y_0 x_1, y_0 x_2, x_0 y_1, x_0 y_2)$$

the product $C \times C'$ maps onto the surface $S$, displaying the latter as a $\mathbb{Z}/4\mathbb{Z}$-quotient of the former. Recall also that this particular K3 surface is birational to the Kummer surface of the product of elliptic
curves $E_i \times E_{2i}$, where $E_i$ is the elliptic curve with modulus $\tau = i$ (the unique elliptic curve with an automorphism of order 4) and $E_{2i}$ is the elliptic curve with modulus $\tau = 2i$ (see \[16\], p. 546; it is interesting to note that Inose’s construction is a particularly simple case of our twist map). On the other hand, there is a natural morphism $\mathbb{P}^2 \to \mathbb{P}_{(1,1,2)}$ taking the curve $C$ onto the elliptic curve $E_i$, which is the weighted hypersurface $y_0^4 + y_1^4 + y_2^2 = 0 \subset \mathbb{P}_{(1,1,2)}$. It is then clear from construction that the $4-1$ cover of $S$ by $C \times C'$ factors:

\[
\begin{array}{ccc}
C \times C' & \to & S \\
\downarrow & & \downarrow \\
E_i \times E_{2i} & \to & S
\end{array}
\]

Other applications of our twist map are the construction of interesting examples. In particular, we prove the following existence theorems, the first of which has no analog for elliptic fibrations or fibrations of abelian surfaces.

**Theorem 2:** There exist Calabi-Yau threefolds (an example of which is given by a hypersurface of degree 12 in the weighted projective space $\mathbb{P}_{(1,1,2,2,6)}$) which have two different, constant modulus $K3$-fibrations.

**Theorem 3:** There are examples of Calabi-Yau threefolds with both extreme Euler-Poincaré characteristics 960 and $-960$, which are images of the twist map, hence have a constant modulus fibration. The first of these results Lemma 3.8, the second is discussed in section 3.4.1 in the text.

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1 General properties of fibered Calabi-Yau threefolds

1.1 Fiber spaces

Let $\pi : X \to Y$ be an algebraic fiber space, i.e., a proper morphism of algebraic varieties with connected fibers; in general we will be assuming the base $Y$ of the fibration is smooth and the fibers are generically smooth. The discriminant $\Delta \subset Y$ is the locus of $y \in Y$ such that the fiber $X_y$ over $y$ is not smooth, in other words the image of the set of points for which $d\pi$ fails to have maximal rank. Assume that $X$ is also smooth, and let $K_X$, $K_Y$ denote the canonical bundles. The relative canonical bundle is $K_{X|Y}$, defined as

\[
K_{X|Y} = \pi^*K_Y^{-1} \otimes K_X.
\]

It follows from (1) that $\pi_*(K_{X|Y}) = K_Y^{-1} \otimes \pi_*K_X$, so in particular

\[
K_X = \mathcal{O}_X \Rightarrow \pi_*(K_X) = \mathcal{O}_Y \Rightarrow \pi_*K_{X|Y} = K_Y^{-1}.
\]

This formula gives a necessary condition on a fiber space $\pi : X \to Y$ for $X$ to be Calabi-Yau. Note also that equation (3) is an equation for divisors on the base $Y$ of the fibration.

1.2 Fiber spaces with section

Let $\pi : X \to Y$ be an algebraic fiber space, and assume now that $\pi$ has a section $\sigma : Y \to X$. Let $\Sigma = \sigma(Y)$; we can identify $Y$ with the subvariety $\Sigma$ of $X$. As such $\Sigma$ has a normal bundle $N_X\Sigma$, and
we may apply adjunction, yielding (now as an equation of divisors on $\Sigma$, written additively)

$$K_{X|\Sigma} = K_\Sigma - c_1(N_X\Sigma),$$  \hspace{1cm} (3)

where $c_1$ denotes the first Chern class, an equivalence class of divisors. Thus the necessary condition (3) takes the form:

$$c_1(N_X\Sigma) = K_\Sigma.$$  \hspace{1cm} (4)

### 1.3 Elliptic fibrations

Assume now that $X \to Y$ is elliptic with a section $\sigma : Y \to X$ and let as above $\Sigma$ denote the image $\Sigma = \sigma(Y)$. It follows from fundamental results of Nakayama $[23]$, that there is a Weierstraß model $W(\mathcal{L}, g_2, g_3)$ over $Y$ and a proper birational map $\mu : X \to W(\mathcal{L}, g_2, g_3)$ over $Y$ such that $\mu(\Sigma)$ is the zero section of the Weierstraß model. A Weierstraß model over $Y$ consists of (a) a line bundle $\mathcal{L}$ over $Y$, and (b) sections $g_2 \in H^0(Y, \mathcal{L}^{\otimes 2})$, $g_3 \in H^0(Y, \mathcal{L}^{\otimes 3})$. The discriminant $\Delta := g_2^3 - 27g_3^2 \in H^0(Y, \mathcal{L}^{\otimes 12})$ is the discriminant of the Weierstraß model $W(\mathcal{L}, g_2, g_3)$, which is defined as follows. Fix meromorphic sections $x, y, z$ of $\mathcal{O}_P(1) \otimes \mathcal{L}^{-2}$, $\mathcal{O}_P(1) \otimes \mathcal{L}^{-3}$ and $\mathcal{O}_P(1)$, respectively, where $P := \text{Proj}(\mathcal{O}_Y \oplus \mathcal{L}^2 \oplus \mathcal{L}^3)$. Then $W(\mathcal{L}, g_2, g_3)$ is the divisor on $P$ defined by

$$y^2z = 4xz^3 + g_2xz^2 + g_3z^3.$$  \hspace{1cm} (5)

The zero section of this Weierstraß model is determined by dehomogenizing, i.e., choosing the inflection point at infinity of the Weierstraß cubic as the zero point of the curve. Note that, if $\Sigma$ denotes the image of a section as above, then $\mathcal{L} \cong N_X\Sigma$, where we view $\mathcal{L}$ as a line bundle on $\Sigma$. Then it follows that $\Delta = -12c_1(N_X\Sigma)$.

A Weierstraß model is minimal, if there is no prime divisor $D$ on $Y$ such that $\text{div}(g_2) \geq 4D$ and $\text{div}(g_3) \geq 6D$. Nakayama shows that if $Y$ is smooth and the discriminant $\Delta$ has normal crossings, then $W$ has only rational singularities, if and only if, $W(\mathcal{L}, g_2, g_3)$ is minimal.

**Lemma 1.1** Let $X \to Y$ be elliptic with section. The condition (3) is necessary and sufficient for $X$ to be Calabi-Yau.

**Proof:** We must show that $K_{X|\Sigma} = \mathcal{O}_\Sigma$ implies $K_X = \mathcal{O}_X$. This follows from Kodaira’s formula for the canonical bundle, in the higher-dimensional formulation as in $[17]$, which is

$$K_X = \pi^*(K_Y + \sum a_i[\Delta_i]),$$  \hspace{1cm} (6)

where $\Delta_i$ are the irreducible components of the discriminant $\Delta$ and the rational numbers $a_i$ are determined by the type of singular fiber over $\Delta_i$. For every $a_i$ we have $12a_i \in \mathbb{Z}$, and setting $\Delta = \sum 12a_i\Delta_i$, the divisor $\Delta$ is divisible by 12 with $\Delta = \sum 12a_i\Delta_i = -12c_1(N_X\Sigma) = -12c_1(\mathcal{L}) \Rightarrow c_1(\mathcal{L}) = -\sum a_i\Delta_i$. At any rate, (5), (6) and the assumption $K_{X|\Sigma} = \mathcal{O}_\Sigma$ together give $K_X = \pi^*(K_{X|\Sigma}) = \pi^*(\mathcal{O}_\Sigma) = \mathcal{O}_X$. \hfill \square

Finally, if $X \to Y$ is an elliptic fibration and $X$ is a smooth threefold with $K_X = \mathcal{O}_X$, then $X$ is birational to a minimal Weierstraß model over a surface $S$ which is one of the following:

1. a minimal surface with $\kappa(Y) = 0$, $g_2$, $g_3$ constant.

2. a minimal ruled surface over an elliptic curve.  

\footnote{In general there will be an additional error term which is present due to less accurate control over the birational geometry of these spaces in higher dimensions; by results of Grassi $[11]$ this term can be avoided by choosing our model correctly}
3. $S = \mathbb{P}^2$ or $S$ is one of the Hirzebruch surfaces $\mathbb{F}_n$, $0 \leq n \leq 12$.

(For $n > 12$, any Weierstraß model is necessarily non-minimal, hence by the above result has non-rational singularities and is consequently not Calabi-Yau).

1.4 Abelian surface fibrations

Next we assume that the general fiber of $X \rightarrow Y$ is an abelian surface, and let $\sigma : Y \rightarrow X$ be a section. Now the image $\Sigma$ has codimension two, being just a point in each fiber. The normal bundle $N_{X \Sigma}$ is a rank two bundle on $\Sigma$ (and on $Y$), and $c_1(N_{X \Sigma})$ is a divisor on $\Sigma$, so the formula (4) still makes sense. Here we also have a formula for the canonical bundle [1], 2.16, which is the following:

$$K_X = \pi^*(K_Y + \sum a_i[\Delta_i]) + \sum (e_j F_j + E_j),$$

where the first sum is over all singular fibers, the second sum over all multiple fibers and the third sum is over all components of the singular fibers which contain two or more components. It follows from this that we have

**Lemma 1.2** Let $X \rightarrow Y$ be a fiber space of abelian varieties over a curve, and suppose that

1. There are no multiple fibers.
2. All singular fibers are irreducible.

Then the condition (4) is necessary and sufficient for $X$ to be Calabi-Yau.

**Proof:** Using the formula (7), the proof is just as above, the important fact being that under the stated assumptions $K_X$ is the pullback of a class on the base. \(\square\)

1.5 K3-fibrations

In the case that $X$ is a K3-fibration, assuming $X$ is a Calabi-Yau threefold, the base $Y$ is necessarily $\mathbb{P}^1$. The necessary condition (4) becomes in this case $c_1(N_{X \Sigma}) = -2$. As we shall see below, however, this condition is not sufficient, not even under assumptions as in the previous lemma. We will give an example of a K3-fibration, for which (4) is satisfied, without multiple fibers, and for which all singular fibers are irreducible, but which is far from being Calabi-Yau. To discuss this example, we first discuss monodromy and how this relates to the coefficients $a_i$ which occur in (4) and (5).

2 Torsion monodromy and fibrations with constant modulus

2.1 Monodromy

Let $X \rightarrow Y$ be an algebraic fiber space, and let $Y_0 \subset Y$ denote the Zariski open subset of $Y$ over which $\pi$ is smooth, i.e., the complement of the discriminant $\Delta$. Let $\pi_0 : X_0 \rightarrow Y_0$ denote the restriction of $\pi$ to $X_0 = X - \pi^{-1}(Y - Y_0)$, so that $\pi_0$ is a locally trivial fibration in the sense of topology. The collection of the integral homology groups of the fibers, $H^k(F_y, \mathbb{Z})$, ($y \in Y_0$) form a sheaf $\mathcal{F}_0$ over $Y_0$, denoted $R^k(\pi_0)_* \mathbb{Z}$, and the monodromy around any $s \in \Delta := Y - Y_0$ can be viewed as a translation in the fiber over some fixed base point $* \in Y_0$, $(\mathcal{F}_0)_*$; sending $\gamma \in \pi_1(Y_0, *)$ to the matrix $T_\gamma \in \text{Aut}((\mathcal{F}_0)_*) = \text{Aut}(H^k(F_s, \mathbb{Z}))$ which describes this translation gives the monodromy representation

$$\rho : \pi_1(Y_0, *) \rightarrow \text{Aut}((\mathcal{F}_0)_*).$$
By the monodromy theorem, each monodromy matrix around an isolated bad fiber has eigenvalues which are roots of unity. Let $T$ be the monodromy matrix; if $(T - 1)$ is nilpotent, one says $T$ is unipotent. Suppose $T^k = 1$ for some $k$; in this case we will speak of torsion monodromy. The general monodromy matrix is quasi-unipotent, meaning that for some $N$ and $k$ we have $(T^k - 1)^N = 0$. In this case, $N$ is called the degree of unipotency, and it is at most the dimension of the fiber.

Suppose that we have a local fiber space $X \to D$ over the unit disc $D$, and assume that all fibers $X_z$ are smooth for $z \in D^* = D - \{0\}$. Assume furthermore that the monodromy is torsion, say $T^k = 1$, and consider the base change $D \to D$, $z \mapsto z^k$. Pulling back the fibration $X$, we see that the monodromy is now $T^k = 1$, i.e., trivial. Again, if $X \to Y$ is an algebraic fiber space over a curve $Y$, and all monodromies are torsion, then we can find a (branched) cover $Y' \to Y$, such that the monodromy of the pull-back of $X$ is trivial. For this it is sufficient to have the branching of order $k_i$ at each point $z_i$ for which the local monodromy matrix $T_i$ has order $k_i$.

While the sheaf $\mathcal{F}_0$ is a sheaf of $\mathbb{Z}$-modules, by tensoring with $\mathcal{O}_Y$ we get a sheaf of $\mathcal{O}_Y$-modules, call it

$$\mathcal{E}_0 := \mathcal{F}_0 \otimes \mathcal{O}_Y.$$  

There is a very special extension of $\mathcal{E}_0$, called the canonical extension (Schmidt) $\mathcal{E}$ of $\mathcal{E}_0$ to $Y$, i.e., such that $\mathcal{E}|_Y = \mathcal{E}_0$. It is known how to describe holomorphic sections generating $\mathcal{E}$: let $r = rk \mathbb{Z}H^k(F, \mathbb{Z})$, and let $v_1, \ldots, v_r$ be a $\mathbb{Z}$-basis. For each $s \in \Delta$ let $T = T_s$ be the monodromy map defined by a small loop around $s$, and let $t = t_s$ be a local coordinate on the base near $s$ (i.e., $t = 0$ defines the point $s \in Y$). Then, for $j = 1, \ldots, r$, the expressions

$$\sigma_j = \exp \left( \frac{1}{2\pi i} \log T \log t \right) v_j$$

define holomorphic sections of $\mathcal{E}$ and in fact generate it. If this expression seems somewhat formidable\footnote{The first named author found it so; he is indebted to Donu Arapura for the following explanation}, consider the case that we only have torsion monodromy. Consider a Galois cover of $Y$ such that for each $s_i \in \Delta$, if $k_i$ denotes the order of $T_{s_i}$ ($T_{s_i}^k = 1$), then the cover $p : Y' \to Y$ is given locally by $t'_s \mapsto (t'_{s'})^{k_i} = t_s$ for any $s' \in p^{-1}(s_i)$, $s_i \in \Delta$. By the results above, the lift of $X \to Y$ to $Y'$,

$$\begin{align*}
X' & \xrightarrow{p} X \\
\pi' & \downarrow \downarrow \pi \\
Y' & \xrightarrow{p} Y
\end{align*}$$

will have trivial monodromy. The eigenvalues of $T$ are of the form $e^{2\pi i \alpha_j}$, $j = 1, \ldots, r$, where $\alpha_j = p_j/k_i$ are rational numbers with $k_i, p_j \in \mathbb{Z}$ (not necessarily in lowest terms). Let us suppose that $T$ is diagonalized; then the matrix $\log T$ can be calculated as follows:

$$\log T = \log \begin{pmatrix} e^{2\pi i \alpha_1} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & 0 \\ 0 & \cdots & e^{2\pi i \alpha_r} & 0 \end{pmatrix} = \begin{pmatrix} 2\pi i \alpha_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & 0 \\ \cdots & \ddots & 2\pi i \alpha_r & 0 \end{pmatrix},$$

so that the expression $\exp \left( \frac{1}{2\pi i} \log T \log t \right)$ can be calculated as

$$\exp \left( \frac{1}{2\pi i} \log T \log t \right) = \left( e^{\log t} \right)^{\frac{1}{2\pi i} \log T} = t^{\frac{1}{2\pi i} \log T} = \begin{pmatrix} e^{\alpha_1} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & 0 \\ \cdots & \ddots & e^{\alpha_r} & 0 \end{pmatrix},$$
and we have $\sigma_j = \begin{pmatrix} t^{\alpha_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & t^{\alpha_r} \end{pmatrix} v_j$ for $j = 1, \ldots, r$. So in this case we see how to calculate generators of $\mathcal{E}$ – they are just certain (rational) powers of the local coordinate times the locally constant section.

Now consider the sheaves $\pi_* K_{X|Y}$ and $\pi'_* K_{X'|Y'}$. This sheaf is related to the canonical extension $\mathcal{E}$ above by the following result of Kawamata (\cite{K}, Theorem 1):

$$\pi_* K_{X|Y} \cong i_* \mathcal{E}_0^{k,0} \cap \mathcal{E},$$

where $i : Y_0 \to Y$ is the inclusion and $\mathcal{E}_0^{k,0} \subset \mathcal{E}_0$ is the $H^{k,0}$-part of the Hodge decomposition of $H^k(F, \mathbb{C})$. Note this is an isomorphism of line bundles, and the monodromy group acts here with a single root of unity at each $\Delta_i \subset \Delta$; this root of unity $e^{2\pi i \alpha_i}$ gives the coefficient of $\Delta_i$ in the sheaf $\pi_* K_{X|Y}$ above. We now apply this to K3 fibrations $X \to \mathbb{P}^1$, the $\Delta_i$ are points on $\mathbb{P}^1$, $k = 2$, and the result above shows that:

the action of the monodromy on the holomorphic two-form determines the coefficient $r_i$ of $\Delta_i$ in $\pi_* K_{X|Y}$ by the rule:

$$\text{If } T_i(\omega) = e^{2\pi i \alpha_i} \omega \quad (\omega \text{ the holomorphic two-form}),$$

then $r_i = \alpha_i$, i.e., $\pi_* K_{X|Y} = \sum \alpha_i \Delta_i$. In particular, $\sum \alpha_i = 2$ is a necessary condition for $X$ to be Calabi-Yau.

### 2.2 Constant modulus

In general, given an algebraic fiber space $X \to Y$, the complex structure (modulus) of the fibers vary, in a holomorphic manner. However, it can also occur that the modulus is fixed; in this case we speak of constant modulus. Easy examples are given by elliptic surfaces for which the $J$-invariant is a constant; indeed, suppose $X \to Y$ is an elliptic fibration, and let $W \to Y$ be the Weierstraß model. Then $J = g_3^2/\Delta$, and $J$ is constant when $g_2$ or $g_3$ vanish. For example, suppose $g_2 = 0$, so that $J \equiv 0$. Then all singular fibers are of types $II, II^*, IV, IV^*$. More precisely, suppose we choose $Y = \mathbb{P}^1$, and let $\Delta$ consist of two points, with singular fibers of type $II$ at one and $II^*$ at the other. This is a rational elliptic surface, and clearly all elliptic fibers have $J = 0$ and so are copies of the elliptic curve with automorphism group $\mathbb{Z}/6\mathbb{Z}$. In this case by the result of the previous section, there is a cover $Y' \to \mathbb{P}^1$, such that the pull back of the fibration has trivial monodromy. Furthermore, the modulus is still constant. Under these circumstances, it is clear that the pull back of the fibration is a fibration without monodromy and with constant modulus. It follows that this pull back is a product. This fact holds more generally.

### 2.3 Uniformisation

Let $X \to Y$ be an algebraic fiber space satisfying the following conditions:

1. The dimension of $Y$ is one.
2. All monodromies around bad fibers are torsion.
3. The modulus of the fibers of $X$ is constant.
Lemma 2.1 Under the assumptions just made, there is a finite, branched cover $Y' \rightarrow Y$ such that the pull back of $X$ to $Y'$ is birational to a product.

Proof: For this it is sufficient to construct a cover $Y' \rightarrow Y$, which is branched at each of the base points $y_i \in Y$ of the bad fibers to degree $k_i$, where $T_i^{k_i} = 1$, $T_i$ the local monodromy matrix at the point $y_i$. Since the dimension of $Y$ is one, this can clearly be done. □

Remark: Under some mild assumptions of the moduli space of the fiber in question (as in, for example, [8]), the assumption 3. implies the assumption 2. above. If this is the case and the general fiber is smooth, the assumption 3. means that the period map has image a point which is in the interior of the moduli space. Since the corresponding moduli point is for a smooth variety, the only degeneration which can occur is a torsion one, gotten essentially by taking a finite quotient of a smooth fiber. Thus 3. ⇒ 2. □

3 A construction of quotients as weighted projective hypersurfaces

3.1 Notations

We will be working with weighted projective spaces, which are certain (singular) quotients of usual projective space. Alternatively, they may be described as quotients of $C^{n+1}$ by a $C^*$-action. We assume the weights $(w_0, \ldots, w_n)$ are given, let $\mu_{w_i}$ denote the group of $w_i$th roots of unity, and consider the action of $\mu := \mu_{w_1} \times \cdots \times \mu_{w_n}$ on $P^n$ as follows. Let $g = (g_0, \ldots, g_n) \in \mu$, and consider for $(z_0 : \ldots : z_n)$ homogenous coordinates on $P^n$ the action

$$(g, (z_0 : \ldots : z_n)) \mapsto (g_0z_0 : \ldots : g_nz_n).$$

Alternatively, consider the action of $C^*$ on $C^{n+1}$ given by

$$(t, (z_0, \ldots, z_n)) \mapsto (t^{w_0}z_0, \ldots, t^{w_n}z_n).$$

In both cases, the resulting quotient is the weighted projective space, which we will denote by $P_{(w_0, \ldots, w_n)}$. General references for weighted projective spaces are [2] and [8]. A weighted hypersurface is the zero locus of a weighted homogenous polynomial $p$. Such a hypersurface or the corresponding polynomial is called transversal, if the only singularities are the intersections with the singular locus of the ambient weighted projective space, and quasismooth, if the cone over the hypersurface is quasismooth, i.e., smooth outside of the vertex; for weighted hypersurfaces, these notions are equivalent (cf. [2], Propositions 6 and 8).

We will assume the weights are normalized in the sense that no $n$ of the $n+1$ weights have a common divisor $> 1$. Both for the weighted projective spaces as well as for the weighted hypersurfaces this assumption is no restriction (cf. [2] 1.3.1 and [8], pp. 185-186). We will write such isomorphisms in the sequel without further comment, for example $P_{(2,3,6)} \cong P_{(2,1,2)} \cong P_{(1,1,1)} = P^2$, where the first equality is because the last two weights are divisible by 3, the second while the first and last are divisible by 2.

We will use the notation $P_{(w_0, \ldots, w_n)}[d]$ to denote either a certain weighted hypersurface of degree $d$, or to denote the whole family of such (the context will make the usage clear). In the particular case that the weighted polynomial $p$ is of Fermat type, then there is a useful fact, corresponding to the above normalizations. For example, in $P_{(2,3,6)}$ consider the weighted hypersurface $x_0^3 + x_1^3 + x_2^3 = 0$. Then the isomorphism $P_{(2,3,6)} \cong P_{(2,1,2)}$ above is given by the introduction of new variables $(x_0') = x_3^3$, which is in spite of appearances a one to one coordinate transformation (because of admissible rescalings), and the Fermat polynomial becomes $(x_0')^2 + x_1^3 + x_2^3 = 0$. Again, the isomorphism $P_{(2,1,2)} \cong P_{(1,1,1)}$
is given by setting \(x_1' = x_1^2\), and the Fermat polynomial becomes \((x_0')^2 + (x_1')^2 + x_2^2 = 0\), which is a quadric in the projective plane. We denote this process by the symbolic expressions

\[
P_{(2,3,0)[12]} \cong P_{(2,1,2)[4]} \cong P_{(1,1,1)[2]}.
\]

### 3.2 The construction

We now introduce the twist map; this map will give an explicit form to the forming of quotients of products \(V_1 \times V_2\) of weighted hypersurfaces by an abelian group acting on the product.

#### 3.2.1 The twist map

Let \(V_1, V_2\) be weighted hypersurfaces defined as follows.

\[
V_1 = \{x_0^p + p(x_1, \ldots, x_n) = 0\} \subset P_{(w_0, w_1, \ldots, w_n)},
\]

\[
V_2 = \{y_0^q + q(y_1, \ldots, y_m) = 0\} \subset P_{(v_0, v_1, \ldots, v_m)},
\]

where we assume both \(p\) and \(q\) are quasi-smooth. The degrees of these hypersurfaces are

\[
\nu = \deg(V_1) = \ell \cdot w_0, \quad \mu = \deg(V_2) = \ell \cdot v_0.
\]

We then consider the hypersurface

\[
X := \{p(z_1, \ldots, z_n) - q(t_1, \ldots, t_m) = 0\} \subset P_{(w_0 w_1, \ldots, w_0 w_n, v_0 v_1, \ldots, v_0 v_m)}.
\]

Note that the degree of \(X\) is \(\nu \cdot \deg(p) = w_0 \cdot \deg(q) = v_0 w_0 \ell\).

**Lemma 3.1** The rational map

\[
\Phi : P_{(w_0, w_1, \ldots, w_n)} \times P_{(v_0, v_1, \ldots, v_m)} \rightarrow P_{(v_0 v_1, \ldots, v_0 v_n, w_0 v_1, \ldots, w_0 v_m)}
\]

\[
((x_0, \ldots, x_n), (y_0, \ldots, y_m)) \mapsto \left(\prod_{i=0}^{w_0} x_i, \ldots, y_0^{v_0} x_n, x_0^{v_1/v_0}, \ldots, y_m^{v_m/v_0}ight)
\]

restricts to \(V_1 \times V_2\) to give a rational generically finite map onto \(X\).

**Proof:** First we show that the map is well-defined outside of the locus\(^3\)

\[
\{y_0 = x_0 = 0\} \cup \{(1, 0, \ldots, 0), (1, 0, \ldots, 0)\} \subset P_w \times P_v,
\]

where we have used the abbreviations \(P_w\) and \(P_v\) for the projective spaces above; similarly we shall use the abbreviation \(P_{w,v}\) for the image projective space. That \(\Phi\) is well-defined as claimed holds because the variables \(z_1, \ldots, z_n\) (resp. \(t_1, \ldots, t_m\)) have weights all divisible by \(w_0\) (resp. by \(v_0\)), hence a change of the branch of the \(w_0^{th}\) roots of \(y_0\) (resp. of the \(v_0^{th}\) roots of \(x_0\)) just amounts to an admissible overall scaling of the coordinates. The locus \(\{y_0 = x_0 = 0\}\) is the locus where all image values are zero, hence this constitutes the locus where \(\Phi\) is not defined (put differently, where \(\Phi\) is not a morphism but only a rational map). If we restrict \(\Phi\) to \(V_1 \times V_2\), then \(x_0 = -p(x_1, \ldots, x_n)\) and \(y_0 = -q(y_1, \ldots, y_m)\). Hence

\[
p(z_1, \ldots, z_n) - q(t_1, \ldots, t_m) = p(y_0^{w_0} x_1, \ldots, y_0^{w_0} x_n) - q(x_0^{v_0} y_1, \ldots, x_0^{v_0} y_m)
\]

\[
= y_0^{\nu/w_0} p(x_1, \ldots, x_n) - x_0^{\mu/v_0} q(y_1, \ldots, y_m)
\]

\[
= y_0^{\ell} p(x_1, \ldots, x_n) - x_0^{\ell} q(y_1, \ldots, y_m)
\]

\[
= -q(y_1, \ldots, y_m) \cdot p(x_1, \ldots, x_n) + p(x_1, \ldots, x_n) \cdot q(y_1, \ldots, y_m) = 0.
\]

\(^3\) As the point \(\{(1, 0, \ldots, 0), (1, 0, \ldots, 0)\}\) is not contained on \(V_1 \times V_2\) we will disregard this in what follows.
Since $V_1 \times V_2$ and $X$ both have dimension $n + m - 2$, it is clear that $\Phi|_{V_1 \times V_2}$ is finite-to-one onto its image.

We call the rational map $\Phi$ the twist map.

Let $\mu_\ell$ denote the group of the $\ell$th roots of unity in $\mathbb{C}$.

**Corollary 3.2** Assume that $\gcd(w_0, v_0, \ell) = 1$. Then $\Phi$ maps $V_1 \times V_2$ to the quotient $V_1 \times V_2 / \mu_\ell$, where $\mu_\ell$ acts effectively on $V_1 \times V_2 \subset \mathbb{P}_{(w_0, w)} \times \mathbb{P}_{(v_0, v)}$ by $\gamma \in \mu_\ell$

$$(\gamma, (x_0 : \cdots : x_n), (y_0 : \cdots : y_m)) \mapsto ((\gamma x_0 : x_1 : \cdots : x_n), (\gamma y_0 : y_1 : \cdots : y_m)).$$

Hence the map $V_1 \times V_2 \rightarrow X$ is generically $\ell : 1$.

**Proof:** The product $V_1 \times V_2$ is given by the two equations $\{x_0^\ell + p(x_1, \ldots, x_n) = 0 = y_0^\ell + q(y_1, \ldots, y_m)\}$, hence it is invariant under the given action. Suppose $((x_0, \ldots, x_n), (y_0, \ldots, y_m))$ is a solution of the equations; then for any $\gamma \in \mu_\ell$, $((\gamma x_0, \ldots, x_n), (\gamma y_0, \ldots, y_m))$ is also a solution. Provided the weights $w_0$, $v_0$ and $\ell$ have no common divisor, none of the $\gamma$ act as an admissible overall scaling, hence the result.

### 3.2.2 Resolving quotients

It is well-known that the weighted projective spaces have only quotient singularities (see [1], 1.2.5 and 1.3.3) which can be resolved by the methods of torus embeddings. Furthermore, from our assumption that $p$ and $q$ are quasismooth, it follows that also $V$ has only quotient singularities (cf. [1], 3.1.6). This is then also true of the polynomial $p - q$ defining $X$, hence $X$ also has only quotient singularities, which can again be resolved by torus methods.

We have seen that there is an action of $\mu_\ell$ on $V$; we let $\tilde{V}$ be a resolution of $V$ to which the action of $\mu_\ell$ lifts. Let $\tilde{X}$ be a resolution of $X$. From these assumptions, the map $\Phi$ lifts to a map $\Phi : \tilde{V} \rightarrow \tilde{X}$, and we have the following commutative diagram

$$\begin{array}{ccc}
\tilde{V} & \xrightarrow{\Phi} & \tilde{X} \\
\downarrow & & \downarrow \\
V & \xrightarrow{\Phi} & X.
\end{array}$$

By assumption both $\tilde{V}$ and $\tilde{X}$ are smooth.

**Corollary 3.3** Under the above assumptions, $\tilde{X}$ is a resolution of the quotients $V_1 \times V_2 / \mu_\ell$ and $\tilde{V} / \mu_\ell$.

### 3.2.3 The fibration

If we project the quotient $V_1 \times V_2 / \mu_\ell$ onto the individual factors, we get two rational fibrations, $\tilde{X} \rightarrow V_1 / \mu_\ell$ and $\tilde{X} \rightarrow V_2 / \mu_\ell$. In each case, the generic smooth fibers are copies of resolutions of $V_2$ (resp. $V_1$). We are especially interested in the case that $\tilde{X}$ is Calabi-Yau.

**Lemma 3.4** Suppose $\tilde{X}$ is Calabi-Yau and $w_0 > 1$. Then the rational fibration $\tilde{X} \rightarrow V_1 / \mu_\ell$ induces a genuine fibration onto a resolution $Y$ of $V_1 / \mu_\ell$, if and only if $\tilde{V}_2$ is also Calabi-Yau.

**Proof:** “$\Rightarrow$” if $\tilde{X} \rightarrow Y$ is a fibration and $c_1(\tilde{X}) = 0$, then by adjunction $c_1(F) = c_1(\tilde{V}_2) = c_1(\tilde{X})|_F - c_1(N_{\tilde{X}|F}) = 0$. 

---

9
Suppose \( c_1(\widetilde{X}) = c_1(F) = c_1(\widetilde{V}_2) = 0 \). Since by adjunction this implies \( c_1(N_XF) = 0 \), it suffices to show that \( X \to V_1/\mu_\ell \) lifts to a morphism \( \widetilde{X} \to Y \) for a resolution \( Y \) of \( V_1/\mu_\ell \). The map \( X \to V_1/\mu_\ell \) is given by fixing \((x_1 : \cdots : x_n)\) and mapping all \((y_0 : \cdots : y_m)\) with \( \Phi(x, y) \in X \) to \((x_1 : \cdots : x_n)\). Thus the projection is well-defined unless \( y_0 = 0 \), a locus which is however blown up upon resolution of \( X \), provided \( w_0 > 1 \).

\[ \text{Corollary 3.5} \quad \text{Let } V_2 \text{ and } X \text{ fulfill the necessary conditions for being Calabi-Yau, } \sum_{j=0}^m v_j = \ell v_0 \text{ and } v_0 \sum_{i=1}^n w_i + w_0 \sum_{j=1}^m v_j = v_0 w_0 \ell. \text{ Then } X \text{ has a resolution of singularities } \widetilde{X} \text{ which is a fiber space over } Y \text{ with constant modulus.} \]

We now consider the possible singular fibers which can occur.

\[ \text{Lemma 3.6} \quad \text{The singular fibers of } \widetilde{X} \to Y \text{ occur at the (image in } V_1/\mu_\ell \text{ of the) set of fixed points of the } \mu_\ell \text{ action on } V_1. \]

**Proof:** This is well-known.

We must consider two situations. First, if \( x_0 = 0 \), then the entire locus \( \Delta = \{p(x_1, \ldots, x_n) = 0\} \subset V_1 \) is contained in the discriminant and maps to \( \Delta \subset V_1/\mu_\ell \), and we let \( \rho^*(\Delta) \) denote the total transform on \( Y \), where \( \rho : Y \to V_1/\mu_\ell \) is the resolution induced by that of \( P_{(y,w)} \). Secondly, it can happen that for \( x_0 \neq 0 \) we have further fixed points, a phenomenon which however only occurs in the case of weighted projective spaces (we will show examples below). It is easy to see that this can only occur along loci of the type \( x_i = 0 \) for some \( i \in \{1, \ldots, n\} \).

**Remark:** Finding the fibers of the fibration

In the physics literature these fibrations and the fibers are often found by the method of “eliminating coordinates”. As this is a source of confusion, we remark here on this. Sometimes it works, but often one gets misleading results. Consider the example of the K3 surface given by the equation

\[ X = \{z_1^{12} + z_2^6 + z_3^4 + z_4^2 = 0\} \subset P_{(1,2,3,6)}. \]

According to our twist map, this is the quotient of the product

\[ \{x_0^3 + x_1^{12} + x_2^6 = 0\} \times \{y_0^3 + y_1^4 + y_2^3 = 0\} \subset P_{(3,1,2)} \times P_{(1,1,2)}. \]

The latter curve is elliptic, and by our results above, the quotient of the product by \( Z/4Z \) has a fibration with constant fibers equal to that elliptic curve. For \( \lambda = (\lambda_1 : \lambda_2) \in P_{(1,2)} \) let \( D_\lambda = \{\lambda_2 z_2^2 - \lambda_1^2 z_2 = 0\} \subset P_{(1,2,3,6)}. \) Let \( X_\lambda = X \cap D_\lambda \). This actually defines the fibration we already derived above, over the weighted projective line \( P_{(1,2)} \) (which is of course isomorphic to the usual \( P^1 \)). The equation for the fiber can be derived by eliminating \( z_2 \), but upon eliminating \( z_1 \) (which occurs quadratically in the equation of \( D_\lambda \)), one gets the equation of a quotient of the fiber. These equations are:

\[ X_\lambda = \{(1 + \left(\frac{\lambda_1^2}{\lambda_2}\right)^6) z_2^6 + z_3^4 + z_4^2 = 0\} \subset P_{(2,3,6)} \cong \{(1 + \left(\frac{\lambda_2^2}{\lambda_1}\right)^6) (z_2')^2 + (z_3')^2 + z_4^2 = 0\} \subset P_{(1,1,1)}, \]

which describes a rational curve (a quotient of the actual fiber), and

\[ X_\lambda = \{(1 + \left(\frac{\lambda_2^2}{\lambda_1}\right)^6) z_1^{12} + z_3^4 + z_4^2 = 0\} \subset P_{(1,3,6)} \cong \{(1 + \left(\frac{\lambda_2^2}{\lambda_1}\right)^6) (z_1')^4 + (z_3')^4 + z_4^2 = 0\} \subset P_{(1,1,2)}, \]

which describes the elliptic curve which is the fiber. The rule to follow is then: if one of the weights of the eliminated variables is one, and the variable which is eliminated occurs linearly in the expression
of the divisor $D_\lambda$, then one gets the correct answer. To see that one must have one of the weights equal to unity, we consider one more example. Consider the product of two weighted hypersurfaces of degrees 6 and 12, respectively, in the product $\mathbb{P}^2 \times \mathbb{P}^2$, and let $\mu_3$ act as above on this product $V_1 \times V_2$. Clearly $V_2$ is a K3 surface, and we get a fibration over $\mathbb{P}^2$, whose fibers are isomorphic to $V_2$. By the twist map this maps to a hypersurface of degree 24 in $\mathbb{P}^4$, which is isomorphic to a hypersurface of degree 12 in $\mathbb{P}^2$. Note that in the equation of the corresponding divisor $D_\lambda$, both coordinates $z_1$ and $z_2$ occur linearly, but both have weight 2. Upon elimination of either, we get a hypersurface of degree 12 in $\mathbb{P}^2$, which is of general type and not K3. However, it is easy to see that it is a 2-cover of the K3 we started with. Hence the linearity of the coordinate in the equation of the divisor is not sufficient.

### 3.3 K3 surfaces

We now apply the results above to the construction of K3 surfaces as quotients of the type $C \times E$, where $C$ is a curve with an action of $\mathbb{Z}/\ell\mathbb{Z}$, and $E$ is an elliptic curve with the same automorphism. To apply the above, we must present both as weighted hypersurfaces.

#### 3.3.1 Elliptic curves

We consider the following elliptic curves.

\[
E_1 = \{ y_0^3 + y_1^3 + y_2^3 = 0 \} \subset \mathbb{P}^1 = \mathbb{P}^2.
\]

\[
E_2 = \{ y_0^4 + y_1^4 + y_2^2 = 0 \} \subset \mathbb{P}^1 = \mathbb{P}^2.
\]

\[
E_3 = \{ y_0^6 + y_1^3 + y_2^2 = 0 \} \subset \mathbb{P}^1 = \mathbb{P}^2.
\]

Both $E_1$ and $E_3$ are elliptic curves with modulus $\tau = \rho = e^{2\pi i/3}$, while $E_2$ has $\tau = i$.

#### 3.3.2 K3 surfaces

We let $C_{(w_0,w_1,w_2)}$ denote the following curve

\[
C_{(w_0,w_1,w_2)} = \{ x_0^\ell + p(x_1, x_2) = 0 \} \subset \mathbb{P}_{(w_0,w_1,w_2)}
\]

of degree $\ell \cdot w_0$. Then applying the map $\Phi$ of Lemma 3.1 we get a rational map of $C_{(w_0,w_1,w_2)} \times E_i$ (where $i = 1$ or 3 if $\ell = 3$ or 6, respectively, and $i = 2$ for $\ell = 4$) onto a hypersurface $X \subset \mathbb{P}_{(w_0,w_1,w_2,v_0,v_1,v_2)}$ of degree $d$, where $(v_0, v_1, v_2) = (1,1,1)$, (1,1,2) and (1,2,3) for the three elliptic curves. We list some of the possible hypersurfaces, all of which are of Fermat type, that one gets in this manner. Assuming that $d = \sum k_i$, where $(k_1, k_2, k_3, k_4) = (v_0 w_1, v_0 w_2, v_0 v_1, v_0 v_2)$, we necessarily have that $X$ is a (singular) K3 surface, and $\tilde{X}$ is its minimal desingularisation. The singular fibers we list are those occurring if the polynomial $p$ in the definition of the curve $C_{(w_0,w_1,w_2)}$ is of the form $p(x_1, x_2) = \ldots$
\[ x_1^{\ell \cdot w_0/w_1} + x_2^{\ell \cdot w_0/w_2} \]. In the last case the polynomials are not of Fermat type.

| #  | \( (w_0, w_1, w_2) \) | \( (v_0, v_1, v_2) \) | \( \ell \) | \( (k_1, k_2, k_3, k_4) \) | \( d \) | singular fibers |
|----|-----------------|-----------------|------|-----------------|------|---------------|
| 1  | (2, 1, 1)       | (1, 1, 1)       | 3    | (1, 1, 2, 2)    | 6    | \( 6 \times IV \) |
| 2  | (1, 1, 2)       | (1, 1, 2)       | 4    | (1, 1, 2, 4)    | 8    | \( 8 \times III \) |
| 3  | (1, 2, 3)       | (1, 1, 4, 6)    | 6    | (1, 1, 4)       | 12   | \( 12 \times II \) |
| 4  | (3, 1, 2)       | (1, 1, 2)       | 4    | (1, 2, 3, 6)    | 12   | \( 6 \times III, 1 \times I_0^6 \) |
| 5  | (1, 2, 3)       | (1, 2, 6, 9)    | 6    | (1, 2, 6)       | 18   | \( 9 \times II, 1 \times I_0^6 \) |
| 6  | (4, 1, 3)       | (1, 1, 1)       | 3    | (1, 3, 4, 4)    | 12   | \( 4 \times IV, 1 \times IV^* \) |
| 7  | (1, 2, 3)       | (1, 3, 8, 12)   | 6    | (1, 3, 8)       | 24   | \( 8 \times II, 1 \times IV^* \) |
| 8  | (5, 1, 4)       | (1, 1, 2)       | 4    | (1, 4, 5, 10)   | 20   | \( 5 \times III, 1 \times III^* \) |
| 9  | (7, 1, 6)       | (1, 2, 3)       | 6    | (1, 6, 14, 21)  | 42   | \( 7 \times II, 1 \times II^* \) |
| 10 | (5, 2, 3)       | (1, 2, 3)       | 6    | (2, 3, 10, 15)  | 30   | \( 5 \times II, 1 \times IV^*, 1 \times I_0^6 \) |
| 11 | (11, 5, 6)      | (1, 2, 3)       | 6    | (5, 6, 22, 33)  | 66   | \( 2 \times II, 2 \times II^* \) |

In a sequel to this paper we will describe in detail how one gets the singular fibers listed. Let us just make a few remarks about the example 11. For the weights in this case, a Fermat hypersurface is not possible. We consider instead the following polynomial:

\[ \{ z_0^{12} z_1 + z_1^{11} + z_2^3 + z_3^2 = 0 \} \subset P_{(5,6,22,33)}. \]

We see without difficulty that this is the image under the twist map

\[ P_{(11,5,6)} \times P_{(1,2,3)} \rightarrow P_{(5,6,22,33)} \]

\[ ((x_0 : x_1 : x_2), (y_0 : y_1 : y_2)) \rightarrow (y_0^{5/11} x_1 : y_0^{6/11} x_2 : x_0^2 y_1 : x_0^3 y_2) \]

of the product \( \{ x_0^6 + x_1^{12} x_2 + x_2^{11} = 0 \} \times \{ y_0^6 + y_1^3 + y_2^2 = 0 \} \). First we note that since \( z_1 \) only occurs in the mixed monomial to first power, that the polynomial \( p(z_0 : z_1 : z_2 : z_3) := z_0^{12} z_1 + z_1^{11} + z_2^3 + z_3^2 \) is transversal. It follows that, since the degree of \( X := \{ p(z) = 0 \} \) is 66, which is also equal to the sum of the weights, that \( X \) is a K3 surface. Its proper transform, after resolving the singularities, is a smooth K3 surface, which we for convenience also denote by \( X \). From the fact that it is the image of the twist map above, it follows immediately that \( X \) has a constant-modulus elliptic fibration.

### 3.3.3 An exotic surface

We already mentioned that the condition \( d = \sum k_i \) is sufficient for \( X \) to be K3. It is, however, not necessary, which we show by example. We are indebted to Igor Dolgachev for explaining this example to us. Consider the weighted hypersurface

\[ x^2 + y^3 + z^{11} + w^{66} = 0 \]

in \( P_{(1,6,22,33)} \). Here \( d = 66 \) while \( \sum k_i = 62 \), hence the sufficient condition above is not met, and the hypersurface does not look like a K3. By general principles of weighted hypersurfaces, the geometric genus is given by the number of monomials of degree 66 – 62, of which there is only one, namely, \( w^4 \). It follows that a canonical divisor is given by the locus \( w = 0 \), which is a curve isomorphic to

\[ C = \{ z^{11} + y^3 + x^2 = 0 \} \subset P_{(6,22,33)}. \quad (12) \]

The resolution of such a weighted hypersurface is well-known, see \[25\]. One considers continued fractions of rational numbers derived from the equation of the hypersurface as follows. Let \( (k_1, k_2, k_3, k_4) \)
be the weights and for each \( i = 1, \ldots, 4 \), \( m_i \) the corresponding degrees of the monomial \( x_i^{m_i} \) occurring in the equation of the hypersurface. Then, to each singular point, with local stabilizer \( \mathbb{Z}_{\alpha_i} \), the rational number of which one considers the continued fraction is \( \frac{\alpha_i}{\beta_i} \), where \( \beta_i \) is a solution of \( k_i \cdot \beta_i \equiv 1 \pmod{\alpha_i} \). In our case, the \( \alpha_i \)'s are just the \( m_i \)'s, and the three fractions we need to expand are \( \frac{2}{1}, \frac{3}{1}, \frac{11}{2} \). The first two give rise to a single \(-2\) and \(-3\) curve, respectively, while the latter gives a \(-6\) curve meeting a \(-2\) curve. Together with the curve \( C \) given here by \( x^2 + y^3 + z^{11} = 0 \) we have the following resolution of the three singular points on the K3 surface:

![Diagram of curves and intersections]

The self-intersection number of the central curve \( C \) (the proper transform of the hyperplane section \( w = 0 \) above) can be determined as follows. Letting \( (\alpha_i, \beta_i) \) denote the pairs giving the continued fractions as above, the formula is (cf [23], 3.6.1, which do the case of an isolated singularity, but the proof is the same; note that the formula in loc. cit. should have a minus sign in front of the first term): the self-intersection number is \(-b\), where \( b \) is given by

\[
b = -\frac{1}{k_2 k_3 k_4} + \sum \frac{\beta_i}{\alpha_i}.
\]

This yields in our case

\[
b = -\frac{1}{66} + \frac{1}{2} + \frac{3}{3} + \frac{1}{11} = 1.
\]

Hence, the central curve \( C \) is exceptional of the first kind and can be blown down. The \(-2\) curve becomes a \(-1\) curve and can be blown down, then the \(-3\) curve of the original curve is now a \(-1\) curve and can be blown down. The result is depicted below.

![Diagram showing the resolution process and the resulting curves]

This leaves two curves \( F \) and \( D \), where \( F \) is a cuspidal genus 1 curve and \( D \) is a \(-2\) curve intersecting it transversally. The linear system \( |F| \) is an elliptic pencil and \( D \) its section. Letting \( S \) denote this smooth surface, it is clearly an elliptic K3 surface. It admits an action of \( \mathbb{Z}/66\mathbb{Z} \) which lifts the action \( (x, y, z, \zeta w) \) on the weighted hypersurface.

We can describe this fibration as follows. Consider the subspace \( \{z = w = 0\} \cong \mathbb{P}(22,33) \cong \mathbb{P}^1 \subset \mathbb{P}(1,6,22,33) \). The set of hypersurfaces of minimal degree containing this \( \mathbb{P}^1 \) are given by the equations \( D_{\lambda} = \{z - \lambda w^6 = 0\} \) (for \( \lambda \neq 0 \)). The intersection with the hypersurface \( X \) is then

\[
X_{\lambda} := D_{\lambda} \cap X = \{x^2 + y^3 + (1 + \lambda^{11})w^{66} = 0\} \cong \mathbb{P}_{(1,22,33)}[66]
\]

\[
\cong \{x^2 + y^3 + (1 + \lambda^{11})(w')^6 = 0\} = \mathbb{P}_{(1,2,3)}[6],
\]

which is the elliptic curve of degree 6 in \( \mathbb{P}_{(1,2,3)} \), which is isomorphic to the Fermat curve, which, as we pointed out above, is the elliptic curve \( E_\rho \) with modulus \( \rho = e^{2\pi i} \). The proper transforms of the
$X_{\lambda}$ are the fibers of our fibration. We need also the action of $\mathbb{Z}/66\mathbb{Z}$ on the holomorphic two-form on the K3 surface. The action on the fibers is:

$$w \mapsto \zeta_{66}w \Rightarrow \zeta_{66}^{11}w' = \zeta_{6}w',$$

which is the usual action of $\mathbb{Z}/6\mathbb{Z}$ on the sextic in $\mathbb{P}_{(1,2,3)}$. The action on the base of the fibration is the action on the parameter $\lambda$ (on some Zariski open set) above,

$$\lambda w^6 \mapsto \zeta_{11}\lambda w^6, \text{ i.e., } z \mapsto \zeta_{11}z, \lambda \mapsto \zeta_{11}\lambda,$$

and the action on the base is just multiplication by $\zeta_{11}$, which permutes the 11 singular fibers at the 11th roots of $-1$ and fixes the singular fiber at $\infty$. Since in terms of local coordinates $(z, t)$, where $z$ is a fiber coordinate, $t$ a coordinate on the base, the holomorphic two-form is given by $dz \wedge dt$, it follows that

$$dz \wedge dt = dw \wedge d\lambda \mapsto \zeta_{66}dw \wedge \zeta_{11}d\lambda = \zeta_{66}dz \wedge dt,$$

and the action is by multiplication by $\zeta_{66}$, i.e., if $h$ denotes the generator of $\mathbb{Z}/66\mathbb{Z}$ and $\omega$ denotes the holomorphic two-form, then

$$h^* (\omega) = \zeta_{66}\omega. \quad (13)$$

This surface is known from the work of Kondo. He shows in [19] that there is in fact a unique K3 surface which admits $\mathbb{Z}/66\mathbb{Z}$ as an automorphism group which preserves the Picard group of the K3 surface, or equivalently, acts non-trivially on the holomorphic two-form (see section 3.3.2 below). Kondo describes the same surface as an elliptic surface with 12 fibers of type $\text{II}$ at $t = 0$ and at the 11th roots of unity. The affine equation is

$$y^2 = x^3 + t \prod_{i=1}^{11} (t - \zeta_{11}^i),$$

and the automorphism is given by

$$g : (x, y, t) \mapsto (\zeta_{66}^2 x, \zeta_{66}^3 y, \zeta_{66}^6 t).$$

He shows that the Picard lattice of this surface is just a hyperbolic plane, while the transcendental lattice is of the form $U \oplus U \oplus E_8 \oplus E_8$, where $E_8$ denotes the negative definite copy of the root lattice of the exceptional group of type $E_8$. For the elliptic surface in Weierstraß form, the holomorphic two-form is given by $\frac{dx}{y} \wedge dt$, and it follows that $g^* (\omega) = \zeta_{66}^5 \omega$, which is a primitive 66th root of unity, showing that the action of the group of order 66 acts trivially on the Picard group. Comparing with (13), we see that $h = g^{55}$.

### 3.4 Calabi-Yau threefolds

In this section we apply the twist map to construct Calabi-Yau threefolds with K3-fibrations, which are quotients of either products $S \times E$, where $S$ is a surface and $E$ is an elliptic curve, or products $C \times K$, where $K$ is now a K3 surface. To apply the latter, we need some information on the automorphisms of K3 surfaces. This information can be found in [19] and [20]. We first recall this, then pass to the construction.
| $(w_0, w_1, w_2, w_3,)$ | $(v_0, v_1, v_2)$ | $\ell$ | $(k_1, k_2, k_3, k_4, k_5)$ | $d$ |
|-------------------------|-----------------|-------|-----------------------------|-----|
| (3, 1, 1, 1)            | (1, 1, 1)       | 3     | (1, 1, 1, 3, 3)             | 9   |
|                         | (1, 1, 2)       | 4     | (1, 1, 1, 3, 6)             | 12  |
|                         | (1, 2, 3)       | 6     | (1, 1, 1, 6, 9)             | 18  |
| (4, 1, 1, 2)            | (1, 1, 1)       | 3     | (1, 1, 2, 4, 4)             | 12  |
|                         | (1, 1, 2)       | 4     | (1, 1, 2, 4, 8)             | 16  |
|                         | (1, 2, 3)       | 6     | (1, 1, 2, 8, 12)            | 24  |
| (5, 1, 1, 3)            | (1, 1, 1)       | 3     | (1, 1, 3, 5, 5)             | 15  |
|                         | (1, 2, 3)       | 6     | (1, 1, 3, 10, 15)           | 30  |
| (5, 1, 1, 2)            | (1, 1, 2)       | 4     | (1, 2, 2, 5, 10)            | 20  |
|                         | (1, 2, 3)       | 6     | (1, 2, 2, 10, 15)           | 30  |
| (6, 1, 1, 4)            | (1, 1, 2)       | 4     | (1, 1, 4, 6, 12)            | 24  |
|                         | (1, 2, 3)       | 6     | (1, 1, 4, 12, 18)           | 36  |
| (6, 1, 2, 3)            | (1, 1, 1)       | 3     | (1, 2, 3, 6, 6)             | 18  |
|                         | (1, 1, 2)       | 4     | (1, 2, 3, 6, 12)            | 24  |
|                         | (1, 2, 3)       | 6     | (1, 2, 3, 12, 18)           | 36  |
| (7, 1, 2, 4)            | (1, 1, 2)       | 4     | (1, 2, 4, 7, 14)            | 28  |
| (7, 1, 3, 3)            | (1, 1, 1)       | 3     | (1, 3, 3, 7, 7)             | 21  |
|                         | (1, 2, 3)       | 6     | (1, 3, 3, 14, 21)           | 42  |
| (7, 2, 2, 3)            | (1, 2, 3)       | 6     | (2, 2, 3, 14, 21)           | 42  |
| (8, 1, 1, 6)            | (1, 1, 1)       | 3     | (1, 1, 6, 8, 8)             | 24  |
|                         | (1, 2, 3)       | 6     | (1, 1, 6, 16, 24)           | 48  |
| (8, 1, 3, 4)            | (1, 2, 3)       | 6     | (1, 3, 4, 16, 24)           | 48  |
| (8, 2, 3, 3)            | (1, 2, 3)       | 6     | (2, 3, 3, 16, 24)           | 48  |
| (9, 1, 2, 6)            | (1, 1, 2)       | 4     | (1, 2, 6, 9, 18)            | 36  |
|                         | (1, 2, 3)       | 6     | (1, 2, 6, 18, 27)           | 54  |
| (9, 1, 4, 4)            | (1, 1, 2)       | 4     | (1, 4, 4, 9, 18)            | 36  |
| (9, 2, 3, 4)            | (1, 1, 2)       | 4     | (2, 3, 4, 9, 18)            | 36  |
| (10, 1, 1, 8)           | (1, 1, 2)       | 4     | (1, 1, 8, 10, 20)           | 40  |
| (10, 2, 3, 5)           | (1, 1, 1)       | 3     | (2, 3, 5, 10, 10)           | 30  |
|                         | (1, 2, 3)       | 6     | (2, 3, 5, 20, 30)           | 60  |
| (10, 1, 3, 6)           | (1, 1, 1)       | 3     | (1, 3, 6, 10, 10)           | 30  |
|                         | (1, 2, 3)       | 6     | (1, 3, 6, 20, 30)           | 60  |
| (10, 3, 3, 4)           | (1, 2, 3)       | 6     | (3, 3, 4, 20, 30)           | 60  |
| (12, 1, 2, 9)           | (1, 1, 1)       | 3     | (1, 2, 9, 12, 12)           | 36  |
|                         | (1, 2, 3)       | 6     | (1, 2, 9, 24, 36)           | 72  |
| (13, 1, 6, 6)           | (1, 2, 3)       | 6     | (1, 6, 6, 26, 39)           | 78  |
| (14, 1, 1, 12)          | (1, 2, 3)       | 6     | (1, 1, 12, 28, 42)          | 84  |

Table 1: List of elliptic Calabi-Yau threefold weighted hypersurfaces of Fermat type and with constant fiber modulus
3.4.1 Elliptic fibrations

We do not attempt to give a classification, but instead describe a few examples. Lists of such elliptic weighted hypersurface Calabi-Yau threefolds have been compiled, and for many of these we can realize them as quotients. As before we use the three elliptic curves of section 3.2.1 but instead of the curve \( \text{(1)} \) now the variety \( V_1 \) is the surface

\[ S_{(w_0, w_1, w_2, w_3)} = \{ x_0^\ell + p(x_1, x_2, x_3) = 0 \} \tag{14} \]

of degree \( \ell w_0 \). Applying the map \( \Phi \) of Lemma 3.1, we get a rational map of \( S_{(w_0, \ldots, w_3)} \times E_i \) (where \( i = 1 \) or 3 if \( \ell = 3 \) or 6, respectively, and \( i = 2 \) for \( \ell = 4 \)) onto a threefold of degree \( d = \ell w_0 \). Again we list some of the possibilities which a brief manual search comes up with. All these cases have the following properties: they are of Fermat type (quite restrictive) and they have a fibration of constant modulus (it is not clear how restrictive this is). This list is given in Table 1.

As an explicit example consider the case \((2k, 1, 1, 2(k - 1))\), \((1, 2, 3)\) for which the Calabi-Yau is the hypersurface of degree \( 12k \) in \( \mathbb{P}(1,1,2(k-1),4k,6k) \). The elliptic fibration is onto \( \mathbb{P}(1,1,2(k-1)) \), whose desingularisation is the Hirzebruch surface \( F_{2(k-1)} \). By Lemma 3.6, the discriminant is the total transform of \( \Delta = \{ p(x_1, x_2, x_3) = 0 \} \subset \mathbb{P}(1,1,2(k-1)) \). The projection \( \mathbb{P}(1,1,2(k-1)) \rightarrow \mathbb{P}(1,1) \) lifts to the projection \( F_{2(k-1)} \rightarrow \mathbb{P}^1 \). Let \( C_\infty \) be the section of negative self-intersection \(-2(k-1)\) (the exceptional curve of the resolution \( F_{2(k-1)} \rightarrow \mathbb{P}(1,1,2(k-1)) \)), and let \( C_0 \) denote the class of a positive section, \( F \) the class of a fiber. The curve \( \Delta \) is reducible; its total transform on \( F_{2(k-1)} \) consists of an irreducible curve \( C \) of class \( C \sim aC_0 + bF \) (\( a, b > 0 \)) and a multiple of \( C_\infty \), \( \Delta \sim aC_0 + bF + \nu C_\infty \). Assume that the exceptional point \((0,0,1)\) of \( \mathbb{P}(1,1,2(k-1)) \) is not contained in \( \Delta \) (for example \( p \) of Fermat type). Then \( C \) and \( C_\infty \) are disjoint. Hence given the explicit form of \( p \) one can easily determine exactly the degeneracy locus of the smooth elliptic Calabi-Yau.

In the cases \( k = 2, 3, 4, 7 \) and \( \ell = 6 \) or \( k = 5 \) and \( \ell = 4 \) one can take \( p \) to be of Fermat type. In these cases, the class of \( C \) can be calculated as follows: it is a curve in \( \mathbb{P}(1,1,2(k-1)) \) of degree \( 12k \), which maps \( k \) to one onto a \( \mathbb{P}^1 \), totally branched at \( 12k \) points, hence the Euler number (which is the first Chern number) is equal to \( k \cdot (2 - 2k) + 2k = 2k(2 - k) \). Then applying adjunction on the resolution \( F_{2(k-1)} \), one gets \( a = k \) and \( b = -2k(k - 1) \). The fiber over \( C_\infty \) is of type IV, \( \text{I}_v^* \), \( \text{IV}^* \) and \( \text{II}^* \) in the \( \ell = 6 \) cases and \( \text{III}^* \) in the \( k = 5, \ell = 4 \) case, and this determines \( \nu \) to be 3, 6, 8, 10 and 9 in the respective cases. These have been studied in more detail in \( \text{[22]} \).

Lemma 3.7 Let \( X \) be a Calabi-Yau threefold with both an elliptic and a K3 fibration, and suppose these are compatible (i.e., that the elliptic fibration of \( X \) restricts to an elliptic fibration of the smooth fibers), and assume moreover that the degenerate fibers of the K3 fibration are irreducible. Then the base of the elliptic fibration is a rational ruled surface \( F_n \).

Proof: From the first assumption, we clearly have on the base of the elliptic fibration, for each K3 fiber, a \( \mathbb{P}^1 \), the base of the elliptic fibration of that K3 fiber. Thus the surface fibers over the original \( \mathbb{P}^1 \), and is thus the blow up of one of the Hirzebruch surfaces. Now we use the assumption that each of the degenerate fibers is irreducible; being so, each of these can fiber over at most a single \( \mathbb{P}^1 \), and it follows that all fibers of the base surface are just \( \mathbb{P}^1 \), in other words that it is a rational ruled surface, as claimed. \( \square \)

In general the base of such a fibration can be some blow-up of one of the \( F_n \).

We now turn to some examples of elliptic fibrations which are not K3 fibrations. The examples given up to now were all in the geography region where the Euler number is negative. We now give some examples where the Euler number is positive, which are listed in Table 2. We will discuss one
exceptional curves resolving the point: $w$ in more detail. These examples are all images of the twist map 

$$P_{(w_0,w_1,w_2,w_3)} \times P_{(1,2,3)} \rightarrow P_{(w_1,w_2,w_3,2-w_0,3-w_0)}.$$ 

Hence they all have constant elliptic fibrations with fiber the sextic in $P_{(1,2,3)}$. We now consider the first example in some more detail.

The variety $V_1$ is a weighted hypersurface in $P_{(581,41,42,498)}$ of degree 3486, given by the equation

$$x_0^6 + p(x_1, x_2, x_3) = x_0^6 + x_1^{13} x_2 + x_2^{13} + x_3^7 = 0$$

and is a 6 − 1 cover of the weighted $P_{(41,42,498)} \cong P_{(41,7,83)}$, branched over the locus $p = 0$. Then the base of the elliptic fibration will be the resolution of $P_{(41,7,83)}$. This is standard toric geometry. Take the lattice spanned by the three vectors

$$v_0 = \frac{1}{41} \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \quad v_1 = \frac{1}{7} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad v_2 = \frac{1}{83} \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

and in each of the cones spanned by two of the $v_i$ one must determine all lattice points. Then taking the convex hull of these, one gets a simplicial decomposition of the cones into subcones all of which have unit area. Then one counts the number of new vertices, and each corresponds to an exceptional divisor. There are three singular points, and our curve $p = 0$ passes through only $(1,0,0)$, so this is the only relevant singularity. It is easy to see that we are looking for integral solutions $(\alpha, \beta, \gamma)$ of the inequalities

$$\alpha(w_1 + w_2) - w_0(\beta + \gamma) + w_0 \leq 0, \quad w_0 \beta > w_1 \alpha, \quad w_0 \gamma > w_2 \alpha,$$

where $(w_0, w_1, w_2) = (41, 7, 83)$, of which there are the following 20, in other words, there are 20 exceptional curves resolving the point:

$$(\alpha, \beta, \gamma) = (11,2,23),(16,3,33),(17,3,35),(21,4,43),(22,4,45),(23,4,47),(26,5,53),(27,5,55),(28,5,57),(29,5,59),(31,6,63),(32,6,65),(33,6,67),(34,6,69),(35,6,71),(36,7,73),(37,7,75),(38,7,77),(39,7,79),(40,7,81).$$

| $(w_0, w_1, w_2, w_3)$ | $(k_1, k_2, k_3, k_4, k_5)$ | $d$ | $\chi$ | $h^{1,2}$ |
|-------------------------|-------------------------|-----|------|--------|
| $(581,41,42,498)$       | $(41,42,498,1162,1743)$ | 3486| 960  | 491    |
| $(498,36,41,421)$       | $(36,41,421,996,1494)$  | 2988| 960  | 491    |
| $(539,36,41,462)$       | $(36,41,462,1078,1617)$ | 3234| 900  | 462    |
| $(469,31,42,396)$       | $(31,42,396,938,1407)$  | 2814| 900  | 462    |
| $(463,31,41,391)$       | $(31,41,391,926,1389)$  | 2778| 900  | 462    |
| $(433,31,36,366)$       | $(31,36,366,866,1299)$  | 2598| 840  | 433    |
| $(483,28,41,414)$       | $(28,41,414,966,1449)$  | 2898| 804  | 416    |
| $(414,24,41,349)$       | $(24,41,349,828,1242)$  | 2484| 804  | 416    |
| $(385,28,31,326)$       | $(28,31,326,770,1155)$  | 2310| 744  | 387    |
| $(434,21,41,372)$       | $(21,41,372,868,1302)$  | 2604| 720  | 377    |
| $(372,18,41,313)$       | $(18,41,313,744,1116)$  | 2232| 720  | 377    |
It follows that along the proper transform of \( p = 0 \), we have singularities of type II, while along the 20 exceptional curves we have singularities of type II\(^*\). The morphism onto the base is not yet flat; to achieve this one could take Miranda’s small resolutions, which would amount to blowing up the intersection points of the 20 exceptional curves.

### 3.4.2 Automorphisms of K3 surfaces

This section is preparatory and recalls some facts about the automorphism of K3 surfaces, which will be applied in the same way the automorphisms of orders 2, 3, 4 and 6 were for elliptic curves. Let \( X \) be a K3 surface; the integral homology \( H^2(X, \mathbb{Z}) \) forms a lattice in \( H^2(X, \mathbb{R}) \cong \mathbb{R}^{22} \) which is a copy of \( U \oplus U \oplus E_8 \oplus E_8 \), where \( E_8 \) denotes here the negative definite even unimodular lattice in \( \mathbb{R}^8 \) and \( U \) is the hyperbolic lattice. We consider in this note polarized K3 surfaces, letting \( \omega \in H^{1,1}(X) \) denote the Kähler form; the primitive cohomology \( H^2(X, \mathbb{Z}) \) is injective (cf. [20], section 4) so it is convenient to consider this as a representation of \( \text{Aut}(\mathbb{Z}^{22}) \). Letting \( \text{Aut}(\mathbb{Z}^{22}) \) be applied in the same way the automorphisms of orders 2, 3, 4 and 6 were for elliptic curves. Let \( S \) be a K3 surface; the integral homology \( H^2(X, \mathbb{Z}) \) is of finite index in \( H^2(X, \mathbb{Z}) \), and the map \( * \) actually maps into \( \text{Aut}(S) \cong \mathbb{R} \times \mathbb{Z}^2 \) (cf. [19], Proposition 1.1), so we can faithfully consider the action of \( \text{Aut}(X) \) on the Picard and transcendental lattices. Note that if \( T \) (and hence \( S \)) is unimodular, then

\[
S_X \oplus T_X = H^2(X, \mathbb{Z}) = U^{\oplus 2} \oplus E_8^{\oplus 2},
\]

so both of \( S_X \) and \( T_X \) are themselves direct sums with summands \( U \) and \( E_8 \).

Every element \( g \in \text{Aut}(X) \) acts on this decomposition, and letting \( \alpha_g \in \mathbb{C}^* \) denote the factor such that \( g^*(\Omega) = \alpha_g \Omega \), where \( \Omega \) denotes the holomorphic two form, the map \( g \mapsto \alpha_g \) gives rise to an exact sequence

\[
1 \rightarrow G_X \rightarrow \text{Aut}(X) \xrightarrow{\alpha} H_X \rightarrow 1.
\]

Elements of \( G_X \) are called symplectic, as they preserve the symplectic form \( \Omega \). If \( X \) is algebraic, then \( H_X = \mathbb{Z}_k \) for some \( k \). The following facts were proved by Nikulin in [23]:

1. \( G_X = \text{Ker}(\text{Aut}(X) \rightarrow \text{Aut}(T_X)) \) and \( H_X = \text{Ker}(\text{Aut}(X) \rightarrow \text{Aut}(S_X)). \)

2. The representation \( \mathbb{Z}_k \rightarrow \text{Aut}(T_X) \) is a direct sum of a number \( K \) of irreducible representations, each of rank \( \phi(k) \) (\( \phi \) denotes the Euler phi-function), so \( \sum_1^K \phi(k) = \text{rank}(T_X). \)

3. The representation \( G_X \rightarrow \text{Aut}(S_X) \) was described by Nikulin for abelian \( G_X \); the possible subgroups occurring are

\[
(Z_2)^m, \quad 0 \leq m \leq 4; \quad Z_4; \quad Z_2 \times Z_4; \quad (Z_4)^2; \quad Z_8; \quad Z_3; \quad (Z_3)^2; \quad Z_5; \quad Z_7; \quad Z_6; \quad Z_2 \times Z_6.
\]

From the fact that \( \phi(k) \leq 22 \) one deduces \( k \leq 66 \), and \( k = 66 \) does occur. This was considered by Kondo in [19] and by Vorontsov in [27], who proved the following results:
1. If \( T_X \) is unimodular, then \( k \) is a divisor of 66, 44, 42, 36, 28 or 12.

2. Suppose \( \phi(k) = \text{rank}(T_X) \). Then \( k=66, 44, 42, 36, 28 \) or 12, and in these cases there exists a unique (up to isomorphism) K3 surface with given \( k \).

3. If \( T_X \) is not unimodular, then \( k = 2^r \) (1 \( \leq \) \( r \) \( \leq \) 4), 3\( \nu \) (1 \( \leq \) \( r \) \( \leq \) 3), 5\( \nu \) (\( r = 1, 2 \)), 7, 11, 13, 17 or 19.

In the case that \( T_X \) is unimodular, it is a direct sum of factors \( U \) and \( E_8 \). For the cases above, the lattices are listed in the following table.

| \( k \) | \( S_X \) | \( T_X \) |
|--------|--------|--------|
| 66     | \( U \) | \( U \oplus U \oplus E_8 \oplus E_8 \) |
| 44     | \( U \) | \( U \oplus U \oplus E_8 \oplus E_8 \) |
| 42     | \( U \oplus E_8 \) | \( U \oplus U \oplus E_8 \) |
| 36     | \( U \oplus E_8 \) | \( U \oplus U \oplus E_8 \) |
| 28     | \( U \oplus E_8 \) | \( U \oplus U \oplus E_8 \) |
| 12     | \( U \oplus E_8 \oplus E_8 \) | \( U \oplus U \) |

### 3.4.3 Calabi-Yau threefolds with K3 fibrations

Again we just list some examples. As above we consider three K3’s of the above list.

\[
K_1 = \{ y_0^6 + y_1^6 + y_2^3 + y_3^3 = 0 \} \subset P_{(1,1,2,2)}
\]

\[
K_2 = \{ y_0^{12} + y_1^6 + y_2^4 + y_3^2 = 0 \} \subset P_{(1,2,3,6)}
\]

\[
K_3 = \{ y_0^{42} + y_1^7 + y_2^3 + y_3^2 = 0 \} \subset P_{(1,6,14,21)}
\]

All three K3’s are elliptic fibrations; the elliptic fibers are \( E_1, E_2 \) and \( E_3 \), respectively, and \( \ell = 6, 12, 42 \).

We again consider the curve \( \{ \} \), and the weighted hypersurfaces are then birational to quotients of \( C_{(w_0, w_1, w_2)} \times K_1 \) by \( \mathbb{Z}/\ell \mathbb{Z} \). We list some examples found after a brief manual search in Table 3. Of course most of these also occur in Table 1.

We now show by example that a Calabi-Yau threefold can have two different K3-fibrations with constant modulus.

**Lemma 3.8** An appropriately chosen weighted hypersurface of degree 12 in the weighted projective space \( P_{(2,2,1,1,6)} \) has two K3-fibrations with constant modulus; the two different fibers are \( P_{(1,1,4,6)}[12] \) and \( P_{(1,1,1,3)}[6] \).

**Proof:** Both of these fibrations can be constructed with the twist map. For the first, we let \( \mu_3 \) act on the product \( P_{(2,1,1)}[6] \times P_{(4,1,1,0)}[12] \); the twist map is onto a hypersurface of degree 24 in \( P_{(4,4,2,2,12)} \), which is the same thing (as all weights are divisible by 2) as a hypersurface of degree 12 in \( P_{(2,2,1,1,6)} \). By Corollary 3.7, we get a constant modulus fibration with fiber \( P_{(4,1,1,0)}[12] \). For the second, we let \( \mu_6 \) act on the product \( P_{(2,1,1)}[12] \times P_{(1,1,1,3)}[6] \). The image under the twist map is a hypersurface of degree 12 in \( P_{(2,2,1,1,6)} \), and by choosing equations of the K3 surface and auxiliary curves appropriately, these two hypersurfaces coincide. In particular it is true of the Fermat hypersurface. \( \square \)

### 3.4.4 A curious example

Consider the ten K3 surfaces of Fermat type in the table in section 3.3.2. For each, of weights \( (k_1, k_2, k_3, k_4) \) and degree \( d = \sum k_i \), we can form the image under the twist map with \( P_{(2,1,1)} \),

\[
P_{(2,1,1)}[2d] \times P_{(k_1, k_2, k_3, k_4)}[d] \longrightarrow P_{(1,1,2k_2, 2k_3, 2k_4)}[2d].
\]
Table 3: K3-fibered Calabi-Yau weighted hypersurfaces which are also elliptic fibered, have constant modulus and are of Fermat type

| $(w_0, w_1, w_2)$ | $(v_0, v_1, v_2, v_3)$ | $\ell$ | $(k_1, k_2, k_3, k_4, k_5)$ | $d$ | $\chi$ |
|-----------------|------------------|------|-----------------|-----|------|
| $(2, 1, 1)$     | $(1, 1, 2, 2)$   | 6    | $(1, 1, 2, 4)$  | 12  | −192 |
|                 | $(1, 2, 3, 6)$   | 12   | $(1, 1, 4, 6, 12)$ | 24  | −312 |
|                 | $(1, 6, 14, 21)$ | 42   | $(1, 1, 12, 28, 42)$ | 84  | −960 |
| $(3, 1, 2)$     | $(1, 1, 2, 2)$   | 6    | $(1, 2, 3, 6, 6)$ | 18  | −144 |
|                 | $(1, 2, 3, 6)$   | 12   | $(1, 2, 6, 9, 18)$ | 36  | −228 |
|                 | $(1, 6, 14, 21)$ | 42   | $(1, 2, 18, 42, 63)$ | 126 | −720 |
| $(4, 1, 3)$     | $(1, 1, 2, 2)$   | 6    | $(1, 3, 4, 8, 8)$ | 24  | −120 |
|                 | $(1, 2, 3, 6)$   | 12   | $(1, 3, 8, 12, 24)$ | 48  | −192 |
|                 | $(1, 6, 14, 21)$ | 42   | $(1, 3, 24, 56, 84)$ | 168 | −624 |
| $(5, 1, 4)$     | $(1, 2, 3, 6)$   | 12   | $(1, 4, 10, 15, 30)$ | 60  | −168 |
| $(7, 1, 6)$     | $(1, 2, 3, 6)$   | 12   | $(1, 6, 14, 21, 42)$ | 84  | −132 |
|                 | $(1, 6, 14, 21)$ | 42   | $(1, 6, 42, 98, 147)$ | 294 | −480 |
| $(5, 2, 3)$     | $(1, 1, 2, 2)$   | 6    | $(2, 3, 5, 10, 10)$ | 30  | −72  |
|                 | $(1, 2, 3, 6)$   | 12   | $(2, 3, 10, 15, 30)$ | 60  | −108 |
|                 | $(1, 6, 14, 21)$ | 42   | $(2, 3, 30, 70, 105)$ | 210 | −384 |

For the cases 1, 4, and 9 in that list, the resulting Calabi-Yau threefolds are the first three entries in Table 3. The analysis described there applies also to these cases, and the K3 fibration over $\mathbb{P}^1$ has $2d$ singular fibers, which are the affine K3 singularities listed in Table 2 of part II of this paper. Let $X$ be the type of bad fiber, $e(X)$ its Euler number. Then we have the formula for the Euler number of the Calabi-Yau

$$e(X) = (2 - 2d) \cdot 24 + 2d \cdot e(X) = 48 + 2d \cdot (e(X) - 24).$$

Now since $0 < e(X) < 24$, we have $-24 < e(X) - 24 < 0$ which yields the inequality for the Euler number of $X$:

$$-48d < e(X) - 48 = 2d(e(X) - 24) < 0.$$

Observe that $\mathbb{Z}/d\mathbb{Z}$ is an automorphism group of the K3 $V_2$, so by the results described above we have $d \leq 66$. Realizing that the case with minimal known Euler number $-960$ is realized in this manner by taking $V_2$ to be the K3 with $d = 42$ (this is the third example in Table 3), while there are cases $d = 44, 66$, one could imagine constructing a similar example with the $d = 66$ case.

Next observe that if $g$ denotes the generator of $\text{Aut}(V_2)$, fixing $x \in \Delta \subset \mathbb{P}^1$ in the discriminant of the fibration, formula (7) shows the contribution of the singular fiber $X_x$ at $x \in \Delta$ is determined by the action of $g$ on the holomorphic two-form. In each case of the fibrations described above, for each $x_i \in \Delta$, $a_i = \frac{1}{d}$, and we have the equality

$$\sum_{i=1}^{2d} a_i = 2d \frac{1}{d} = 2 = c_1(\mathbb{P}^1),$$

which is the necessary condition described at the beginning of this paper. For the case $d = 66$ this would require 132 singular fibers, each giving a contribution of $\frac{1}{66}$. If so, we can calculate what its Euler number would be. Since it has 132 singular fibers which are the affine surface $t_1^4 + t_2^3 + t_3^2 = 0$.
which has Milnor number $\mu = (11 - 1)(3 - 1)(2 - 1) = 20$, each singular fiber has Euler number 4, and our formula for the Euler number yields:
\[
e(X) = (2 - 132) \cdot 24 + 132 \cdot 4 = -2592.
\]

Is it possible to construct such an example?

Consider the image of the twist map
\[
\mathbb{P}_{(2,1,1)}[132] \times \mathbb{P}_{(1,6,22,33)}[66] \longrightarrow \mathbb{P}_{(1,1,12,44,66)}[132].
\]
If we take $V_1, V_2$ as Fermat hypersurfaces, then the image is the weighted threefold given by the equation
\[
X = \{z_1^{132} + z_2^{132} + t_1^{11} + t_2^3 + t_4^2 = 0\} \subset \mathbb{P}_{(1,1,12,44,66)}.
\]

Suppose this threefold did allow a fibration with fibers the resolved K3 surfaces discussed in section 3.3.3. It is easy to see that we have singular fibers at the 132 points which are the zeroes of $z_1^{132} + z_2^{132}$, and that the fibers $\{z_1 = 0\} \cap X$ and $\{z_2 = 0\} \cap X$ are both smooth, so that it looks as if we could in fact construct such a fibration. However, as $132 > 124 = \sum k_i$, the threefold $X$ does not satisfy the sufficient condition to be Calabi-Yau, and our arguments above (in particular Lemma 3.4) do not apply directly. In fact, assuming the necessary condition (4) is also sufficient, such a fibration does not exist, which can be seen as follows. There is no birational model of $X$ which is Calabi-Yau: the geometric genus $h^{3,0}$ is 9, not 1 as would be the case if $X$ were Calabi-Yau. Indeed, since the geometric genus is a birational invariant, this is just the number of sections of $O(132 - 124) = O(8)$, and this is the number of monomials in the first two variables of weight 8, of which there are 9. However, using the method of the proof of Lemma 3.4, the reader may verify that in fact such a fibration does exist (first the fiber is the non-Calabi-Yau surface $\{x^2 + y^3 + z^{11} + w^{66} = 0\} \subset \mathbb{P}_{(1,6,22,33)}$ of section 3.3.3, which is birationally modified as in that section over the base of the fibration). Consequently, condition (4) is not sufficient.

We note that we can also display $X$ as an elliptic fibration, and for these we have the necessary and sufficient condition (6), which utilizing the Weierstraß form says that
\[
-12K_Y = \Delta,
\]
where $\Delta$ is the discriminant locus (counted with appropriate multiplicities). The variety $X$ is the image of the twist map
\[
\Phi : \mathbb{P}_{(11,1,1,12)}[132] \times \mathbb{P}_{(1,2,3)}[6] \longrightarrow \mathbb{P}_{(1,1,12,44,66)}[132].
\]
The discriminant is the total transform of
\[
\Sigma = \{x_1^{132} + x_2^{132} + x_3^{11} = 0\} \subset \mathbb{P}_{(1,1,12)};
\]
projecting onto $\mathbb{P}_{(1,1)}$ displays the proper transform of this, which we denote also by $\Sigma$, as an 11 to 1 cover, totally branched at the 132rd roots of $-1$, so it has Euler number
\[
e(\Sigma) = 11(2 - 132) + 132 = -1298.
\]
We can determine its class in $H^2(\mathbb{F}_{12}, \mathbb{Z})$ as follows: from the fact that it is an 11 to 1 cover, it intersects a fiber in 11 points, so $\Sigma \sim 11C_0 + bF$, where $C_0$ denotes the (class of) a section of positive self-intersection and $F$ denotes the class of a fiber. On the other hand, the first Chern class of the resolution $\mathbb{F}_{12}$ of $\mathbb{P}_{(1,1,12)}$ is
\[
c_1(\mathbb{F}_{12}) = 2C_0 - 10F,
\]
and applying adjuction to $\Sigma$ on $F_{12}$, which is a smooth curve with Euler number we just calculated, one can determine its class in $H^2(F_{12}, \mathbb{Z})$. The result is

$$\Sigma \sim 11C_0 \sim 11(C_\infty + 12F),$$

while for the exceptional curve $C_\infty$, we have fiber types II again, so

$$\Delta = 2\Sigma + 2C_\infty = 24C_\infty + 264F,$$

and one sees that the sufficient condition (II) is not satisfied, as $-12K_{F_{12}} = 24C_\infty + 168F$.

It remains an open problem whether it is possible to construct such a K3 fibration; if possible, it would enlarge the range of possible Euler numbers of Calabi-Yau threefolds (and is therefore quite unlikely).

### 3.4.5 Birational fibrations

It is, however, possible to construct K3 fibrations which have the K3 surface of section [3.3.3] as fiber, which we now explain. There are examples of weighted hypersurfaces which, after resolution of singularities, are not fibrations, but still are, in the class of Calabi-Yau threefolds, birational to such a Calabi-Yau. These examples come from the exotic surface example of section [3.3.3]. Indeed, suppose we want a twist map

$$\Phi : \mathbb{P}(w_0, w_1, w_2)[66 \cdot w_0] \times \mathbb{P}(1,6,22,33)[66] \to \mathbb{P}(w_1, w_2, w_0, 6, w_0 \cdot 22, w_0 \cdot 33)[66 \cdot w_0]$$

to have an image which satisfies the sufficient condition $d = \sum k_i$ for it to be a Calabi-Yau threefold. Then writing down what the weights are, we get an equation $w_1 + w_2 + w_0(6 + 22 + 33) = 66w_0$, which means that we require $w_1 + w_2 = 5w_0$. The easiest solution to this is: $w_0 = 1, (w_1, w_2) = (2, 3)$ or $(1, 4)$. The image projective spaces is then $\mathbb{P}(2,3,6,22,33)$, and there are indeed Calabi-Yaus (of degree 66) in this space, as well as in $\mathbb{P}(1,4,6,22,33)$. They have Euler numbers $-240$ and $-300$, respectively. They do not possess a fibration a priori, because the base locus of the projection (which one gets by setting the first two coordinates $=0$) is not part of the singular locus (recall that in Lemma [3.4] we required $w_0 > 1$ to get a fibration). However, one can blow it up (upon which the surface is temporarily no longer C-Y) and then in each fiber do the blowing down process we have described in section [3.3.3]. After this is done, we do get a C-Y with fibration by those exotic surfaces.

### 4 Applications in Physics

In this section we briefly describe some of the applications of the twist map to physical dualities, which was the original source of motivation for the present investigation. It is intended for the non-expert, and we just try to explain the physical interpretation of the geometry, without going into any details.

#### 4.1 The physical theories and their moduli spaces

We are interested in superstring theories. These are theories about how a string (a smooth image of the circle) moves in Minkowski space; its vibrations give rise to all elementary particles. More precisely, consider a string moving in some Minkowski space $M^{1,d}$; it traces out with time a world sheet, and this is then an embedded Riemann surface in $M^{1,d}$. The physical theory one is interested in is a superconformal field theory on that Riemann surface; the adjective superconformal refers to the symmetry group of the equations of motion. It is well-known that for this symmetry to hold, the dimension is restricted to $d = 9$ (ten-dimensional Minkowski space), and it is also known that
there are five consistent theories of this type: Types I, IIA, IIB, Het\(_{E_8 \times E_8}\), Het\(_{SO(32)}\). The first is the theory which contains open strings and has only \(N = 1\) supersymmetry, while the others are theories of closed strings and have \(N = 2\) supersymmetry (on the world sheet). That is, one has a consistent supersymmetric quantum field theory on the world sheet of the string. (There is also an issue of space-time supersymmetry, but we neglect that here). The type II theories have at most abelian gauge groups (at least in ten dimensions, see [12], Chapter 14 for a discussion of this issue upon compactification), which is why they seemed completely uninteresting for many years, while the heterotic strings, which have only left-moving supersymmetries, have the two gauge groups of rank 16 (which arises from the fact that the consistent dimension for the bosonic, or non-supersymmetric string, is 26, and the 16 = 26 − 10 remaining dimensions are compactified to the maximal torus of a Lie group, the gauge group) listed above.

The superstring theories are all \emph{perturbative} in essence, which means (contrary to general relativity) they are small perturbations of certain vacuum solutions. These theories are consistent, but phenomenologically uninteresting, as in reality one does not observe ten flat dimensions. To alleviate this, one can compactify six of the dimensions, and making the size of the compact factor small enough, it will be invisible to us yielding a phenomenologically more acceptable theory. The space time in which the world sheet resides is then \(M^{1,3} \times X\) for some compact manifold \(X\). In order to assure superconformal symmetry here also, it is required that \(X\) be Calabi-Yau\(^4\). Such a manifold has two types of moduli: complex structures and Kähler forms. These moduli turn out to be moduli of the physical theory also, that is, any \(X\) (whatever its moduli) gives a consistent compactification, and in this way, moduli spaces enter also in the physical theory. These correspond in a sense to “flat directions of the potential”, in other words change only the explicit form of the Lagrangian, not the equations of motion. The “physical moduli space” will in general consist of these geometric moduli in addition to others. In order to better understand these compactifications, one often compactifies fewer than six dimensions, an example of which we discuss next.

### 4.2 Toy model: IIA(K3) \(\longleftrightarrow\) Het\(_{E_8 \times E_8}(T^4)\)

The notation of the title is meant to indicate that one starts with one of the five theories above and compactifies on the Calabi-Yau manifold in parenthesis, in this case of dimension four, leaving a six-dimensional Minkowski space left as “space-time”. The arrow indicates the so-called \emph{duality}; this means that the non-perturbative theory, of which the superstring theory is an approximation (perhaps it would be better to say the superstring theory is a certain \emph{limit}), is \emph{the same} on both sides. Think of this as meaning there is an underlying theory, with certain moduli, and at certain moduli points one can approximate the theory by the two different superstring theories. It is quite involved to list what this implies physically, but one thing is certain: if this is the case, then both sides must necessarily have the same moduli space, and this can be verified and even understood.

The moduli space in question is given by

\[
\mathcal{M} = \text{SO}(4, 20; \mathbb{Z}) \backslash \text{SO}(4, 20)/\text{SO}(4) \times \text{SO}(20)
\]

and the heterotic/type II duality reduces in this context to giving two different interpretations to this moduli space. The group \(\text{SO}(4, 20; \mathbb{Z})\) is the discrete group preserving a particular lattice in \(\mathbb{R}^{24}\), which is described below.

The symmetric space on the right-hand side of (15) can be described as the space of 4-dimensional subspaces \(V\) of the space \(\mathbb{R}^{(4,20)}\), by which we mean \(\mathbb{R}^{24}\) endowed with a metric of signature \((4,20)\), on which the metric is positiv definite. The space \(\mathbb{R}^{(4,20)}\) contains a (unique) selfdual even integral

\[^4\]the super- invariance implies Kähler, the conformal invariance implies Ricci flatness
lattice, which we denote by $\Gamma^{(4,20)} \subset \mathbb{R}^{(4,20)}$, and which has the same signature (in the above notations, $\Gamma^{(4,20)} \cong U^{4,4} \oplus E_8^{(2)}$). The moduli space $\mathcal{M}$ then can be described as the space of all $V$ up to automorphisms of the lattice $\Gamma^{(4,20)}$.

Now, in the heterotic string interpretation of this space the subspace $V$ and its orthogonal complement $V^\perp$ are associated to gauge fields describing the Yang-Mills structure of the string. Generically, the non-abelian gauge group $E_8 \times E_8$ of the heterotic string is broken to the abelian group $U(1)^{16}$ (see for example [4], section 3.5 and page 468), i.e., none of the non-abelian gauge fields persist upon compactification. Of special significance are those points in the moduli space for which $V^\perp$ contains nonzero points of the lattice $\Gamma^{(4,20)}$. If $V^\perp$ contains such nonzero points for which the normalized length $d_P = -\frac{1}{2} < P, P >$ equals unity then physical considerations lead to an enhancement of the rank of the Yang-Mills group beyond the rank encountered at a generic point of the moduli space. So what is varying here in the moduli space is the surviving non-abelian gauge group.

The second, type II string, interpretation of the moduli space $(15)$ is obtained by viewing $\mathbb{R}^{(4,20)}$ as the real cohomology $H^*$ of a K3 surface $S$. It is known that type IIA compactified on a K3 surface is equivalent to a superconformal non-linear sigma models on that K3 surface. Its moduli space can be described as follows, see [4] §3 for details on this aspect. From the decomposition

$$\frac{SO(4, 20)}{S(O(4) \times O(20))} \cong \frac{SO(3, 19)}{S(O(3) \times O(19))} \times \mathbb{R}^{22} \times \mathbb{R}_+,$$

this has the following interpretation: the first factor is the space of Einstein metrics on a K3 surface, the second factor is the moduli space of the so-called $B$-field and the final factor is the size. It remains to determine the meaning of the classes of norm $-2$ occurring above. But it is well-known how this occurs. Referring back to section 3.4.2 (and letting our K3 surface be denoted by $X$ as there), we have the Picard lattice $S_X$ and the transcendental lattice $T_X$. The Picard lattice is spanned by the Kähler class and by rational $-2$ curves, which correspond to a generic K3 surface acquiring ordinary double points and then resolving these to yield a smooth K3 surface. In this respect it is important to recall that the number of moduli preserving such a Picard lattice is $20 - \rho$, see for example [15], Proposition 2.3.2. For a generic algebraic surface, there are 19 moduli and the Picard number $\rho$ (the rank of the Picard group) is 1, while for a K3 surface with $\rho > 1$, there are only $20 - \rho$ moduli of K3 surfaces which are deformations of the given one and which have isomorphic Picard lattice. The relation in this case to the heterotic theory was observed by Witten in [28]. Namely, if $S$ acquires a singularity of any of the A,D, or E types, a configuration of rational curves whose dual graph is the Dynkin diagram of one of the groups of type A, D or E, collapses to a point (these are the vanishing cycles). Such rational curves have selfintersection $-2$ and thus correspond precisely to the vectors encountered above in the heterotic description.

### 4.3 String dualities in $D = 4$

Next we consider the compactification of Type IIA on a Calabi-Yau threefold (down to $D = 4$) and the compactification of the heterotic string on a product $S \times T^2$, where $S$ is a K3 surface. Let us think of this in the complex category and make the simplifying assumption that $S$ is elliptically fibered (this is an 18-dimensional family, as compared with the 19-dimensional family of K3 surfaces with some fixed polarization) and that the Calabi-Yau threefold $X$ is fibered by K3 surfaces. Then, we have two fibrations:

$$S \rightarrow \mathbb{P}^1, \quad X \rightarrow \mathbb{P}^1,$$

and combining the first with $T^2$, we have two fibrations, one of $X$ and one of $S \times T^2$, both onto $\mathbb{P}^1$. It is natural to think of the above duality for a fixed $t \in \mathbb{P}^1$, and applying the duality fiberwise. There is a physical argument for this, the so-called adiabatic limit. Note however, that if $S$ is one of our K3
surfaces and $X$ one of our Calabi-Yau threefolds, for which the modulus of the fiber is constant, then there is no argument at all necessary: applying the duality above fiberwise implies that the gauge group of the heterotic theory is just the group whose weight lattice is the Picard group of the K3 fiber on $X$.

For these $D = 4$ theories, the heterotic side is quite complicated; there are vector bundles (background fields) on the K3 surface involved, and a Higgsing process. Nonetheless, our constructions gives us a good idea of what the gauge group of a heterotic dual theory would be. In [2], there are chains of such dual theories. We give four examples, three of which are in [4], the other of which is considered in [3]. We start with the K3 surfaces listed as # 5, 7, 8 and 9 in section 3.3.2. We then form the product with the curve $C$ with weights $(2, 1, 1)$ and $\ell = 18, 24, 20$ and 42 in the respective cases. The image of the twist map is in these four cases a weighted hypersurface with weights $(1, 1, 4, 12, 18), (1, 1, 6, 16, 24), (1, 1, 8, 10, 20), (1, 1, 12, 28, 42)$ of degrees 36, 48, 40, 84. Looking at the table in section 3.3.2, we see that the singular fibers $I^*_0, IV^*_0, III^*$ and $II^*$ of the elliptic fibrations of the K3 surfaces (the fibers of the fibration) have dual graphs which are the extended Dynkin diagrams of the types $D_4, E_6, E_7$ and $E_8$, respectively, yielding as gauge groups of the heterotic duals $SO(8)$, $E_6$, $E_7$ and $E_8$, respectively. For the three cases which occur in [2], these are (up to abelian factors) indeed the gauge groups of the heterotic dual theories, and for the remaining case this is verified in [3], p. 133.

As explained in [5], the moduli of the two physical theories are described in more detail as follows. The type IIA string compactified on a Calabi-Yau threefold $X$ has the following moduli:

1. The dilaton (which governs the string coupling constant) and the axion which together form a complex scalar $\Phi^{IIA}$;
2. A metric which is determined by the complex structure of $X$ and the cohomology class of a Kähler form;
3. A skew field $B \in H^2(X, \mathbb{R})/H^2(X, \mathbb{Z})$;
4. So-called Ramond-Ramond fields $R \in H^{odd}(X, \mathbb{R})/H^{odd}(X, \mathbb{Z})$.

These moduli split into two types, the so-called vector moduli and the hypermultiplet moduli, as follows:

V The Kähler form and the $B$-field together form the well-known “complexified Kähler form” used in mirror symmetry; these moduli together form a moduli space $\mathcal{M}_V$ which is a special Kähler manifold.

H The complex scalar $\Phi^{IIA}$, the Ramond-Ramond fields $R$ and the complex structure of $X$ form the moduli space $\mathcal{M}_H$ of hypermultiplet moduli; this is a quaternionic Kähler manifold.

The above descriptions are not valid in the complete quantum theory, but rather only for certain approximations: $\mathcal{M}_V$ is valid only in the “large radius limit” of $X$, $\mathcal{M}_H$ is valid only near the weakly-coupled limit $\Phi^{IIA} \rightarrow -\infty$.

The heterotic string compactified on a product of a K3 surface $S_H$ and an elliptic curve $E_H$, $S_H \times E_H$ (which is Kähler Ricci-flat) has the following moduli:

1. The dilaton (which governs the string coupling) and the axion, which together form a complex scalar $\Phi^{Het}$.

2. A Ricci-flat metric on the product $S_H \times E_H$. 

25
3. A skew-field $B \in \mathbb{H}^2(S_H \times E_H, \mathbb{R})/\mathbb{H}^2(S_H \times E_H, \mathbb{Z})$.

4. A $G$-bundle on $S_H \times E_H$ with a connection satisfying the Yang-Mills equations, where $G$ is the (unbroken) gauge group of the heterotic string, either Spin$(32)/\mathbb{Z}_2$ or $E_8 \times E_8$.

Once again, these moduli split into two types, vector and hypersmultiplet. Assume that the $G$ bundle is the product of a $G_S$-bundle over $S_H$ and a $G_E$-bundle over $E_H$, where $G_S \times G_E \subset G$ is a subgroup. Then these types can be described as follows:

- **V** The scalar $\Phi_{\text{Het}}$, the moduli of the $G_E$-bundle over $E_H$, the metric on $E_H$ and the $B$-field on $E_H$ form the vector multiplet space $\mathcal{M}_V$.

- **H** The moduli of the $G_S$-bundle on $S_H$, the metric on $S_H$ and the $B$-field on $S_H$ form the hypersmultiplet moduli space $\mathcal{M}_H$.

Again, these descriptions are only valid in certain limits: $\mathcal{M}_V$ when $\Phi_{\text{Het}} \to -\infty$ and the area of $E_H$ is large, and $\mathcal{M}_H$ when the volume of $S_H$ is large. Duality here means essentially matching these moduli spaces in the two cases. The match of vector moduli is described above, and the match of hypersmultiplet moduli leads to quite interesting mathematical constructs, for example intermediate Jacobians, Prym varieties and Deligne cohomology, see [7].

Perhaps the most fascinating aspect of this is determining the K3 surface explicitly in terms of the Calabi-Yau threefold $X$; for this one considers a stable degeneration of $X$ into the union of two generalized Fano-threefolds $X_1 \cup X_2$, and the K3 surface for the heterotic compactification is the intersection $X_1 \cap X_2$.

### 4.4 Conifold transitions

A second problem which is illuminated by our construction is the heterotic structure of the so-called conifold transition between Calabi-Yau manifolds. Such transitions are given by the following construction. Allow a smooth Calabi-Yau threefold $X_t$ depending on a parameter $t$ to acquire a certain number of ordinary double points at $t = 0$; let $X^* (= X_0)$ denote the singular space. Each of the ordinary double points can be resolved by a small resolution; we assume that there is at least one of these which is projective (i.e., Kähler), and let $X^s$ denote such a resolution. Schematically we have the following situation:

$$X_t \leftrightarrow X^* \leftrightarrow X^s.$$ 

The Hodge numbers of $X_t$ ($t \neq 0$) and $X^s$ are related as follows:

$$h^{2,1}(X^s) = h^{2,1}(X_t) - (P - R), \quad h^{1,1}(X^s) = h^{1,1}(X_t) + R,$$

where $P$ denotes the number of nodes and $R$ denotes the number of relations between the corresponding vanishing cycles. Although this transition passes through a singular space $X^*$, Strominger has shown [8] that the physics remains smooth. What happens is something quite similar to what occurs in the work of Seiberg and Witten: a massive particle (in this case a black hole) gets massless at the moduli point of $X^*$ (in the physical theory one always makes a low-energy approximation, and all massive particles are so heavy that they do not influence the physics; accordingly they are integrated out of the Lagrangian), and passing to $X^s$ amounts to the new theory with an additional massless particle.
4.4.1 Splittings

One way of describing such transitions is by means of splittings, which are described in terms of complete intersections in products of weighted projective spaces. As an example of this, consider first a transversal weighted hypersurface $\mathbb{P}(k_1,k_2,k_3,k_4)[d]$, where $d = 2k_1 + k_2 + k_3 + k_4$, and consider the following threefold in the product $\mathbb{P}(1,1) \times \mathbb{P}(k_1,k_2,k_3,k_4)$:

$$X_0 := \left\{ \begin{array}{l}
p_1(u,y) = u_0Q(y) + u_1R(y) = 0 \\
p_2(u,y) = u_0S(y) + u_1T(y) = 0
\end{array} \right\} \subset \mathbb{P}(1,1) \times \mathbb{P}(k_1,k_2,k_3,k_4).$$

Schematically this is abbreviated with the following notation:

$$X_0 \in \mathbb{P}(1,1)_{(k_1,k_2,k_3,k_4)} \left[ \begin{array}{cc} 1 & 1 \\
\text{a} \cdot \text{k}_1 & \text{d} - \text{a} \cdot \text{k}_1 \end{array} \right],$$

where $a \cdot k_1 = \deg(Q) = \deg(R)$ and $d - a \cdot k_1 = \deg(S) = \deg(T)$. On the other hand, consider the determinantal variety

$$X^* := \{ (Q(y)T(y) - R(y)S(y) = 0 \} \subset \mathbb{P}(k_1,k_2,k_3,k_4).$$

Clearly $X^*$ is singular for generic choices of $Q, R, S$ and $T$ when $Q = R = S = T = 0$, which means that $X^*$ generically has isolated singularities, which one can check are ordinary double points. Furthermore, mapping $\mathbb{P}(k_1,k_2,k_3,k_4)$ into the product $\mathbb{P}(1,1) \times \mathbb{P}(k_1,k_2,k_3,k_4)$ in the obvious way, it is clear that $X^*$ maps to $X_0$: write the equation defining $X_0$ as

$$P(u,y) = (u_0,u_1) \left( \begin{array}{cc} Q(y) & R(y) \\
S(y) & T(y) \end{array} \right) = (u_0,u_1)\Pi(y) = 0,$$

with a $2 \times 2$ matrix $\Pi$. Then $y \in X^* \iff \det(\Pi(y)) = 0 \iff \exists (u_0,u_1)$ with $(u_0,u_1)$ is in the kernel of $\Pi(y) \iff P(u,y) = 0 \iff (u,y) \in X_0$. The singular $X^*$ can be deformed by adding some multiple of a transversal polynomial, i.e., by setting

$$X_t := \{ t_0(Q(y)T(y) - R(y)S(y)) + t_1p_{\text{trans}}(y) = 0 \} \subset \mathbb{P}(k_1,k_2,k_3,k_4) \quad (t = (t_0,t_1) \in \mathbb{P}^1),$$

and we are in the situation mentioned above: the smooth $X_t$ acquires singularities of the desired type (ordinary double points) at $t_1 = 0$, and this singular $X_0$ can be given a small resolution $X^*$. Once again, we schematically describe this process by the symbols

$$\mathbb{P}(k_1,k_2,k_3,k_4)[d] \longleftrightarrow \mathbb{P}(1,1)_{(k_1,k_2,k_3,k_4)} \left[ \begin{array}{cc} 1 & 1 \\
\text{a} \cdot \text{k}_1 & \text{d} - \text{a} \cdot \text{k}_1 \end{array} \right].$$

4.4.2 K3 fibrations

It was shown in [21] that such transitions can be constructed between K3-fibered Calabi-Yau manifolds, an example being provided by the transition

$$\mathbb{P}(1,1,2,4,4)[12]^{(5,101)} \longleftrightarrow \mathbb{P}(1,1)_{(4,4,1,1,2)} \left[ \begin{array}{cc} 1 & 1 \\
4 & 8 \end{array} \right]^{(6,70)},$$

where the notation on the right denotes a complete intersection manifold of codimension two defined by two polynomials of bi-degree $(1,4)$ and $(1,8)$ respectively and the superscripts indicate the Hodge numbers $(h^{1,1}, h^{2,1})$. Note that the smooth hypersurface on the left hand side is the first example.
in Table 3, so that the Fermat hypersurface is in fact the image of an appropriate twist map. In this case we have \( \deg(Q) = \deg(R) = 4 \) and \( \deg(S) = \deg(T) = 8 \), hence \( Q = R = S = T \) has (after rescalings) 32 solutions. Hence the singular \( X^* \) above has \( P = 32 \) and \( R = 1 \), which explains the change in Hodge numbers explicitly from this point of view. To see the K3-fibration on the right hand side, one considers the sections \( \lambda y_0 - \lambda_1 y_1 = 0 \); an easy calculation shows that these are K3 surfaces in the family of complete intersections in \( \mathbb{P}(1,1) \times \mathbb{P}(1,1,2,2) \) of degrees \( (1,2) \) and \( (1,4) \).

### 4.4.3 A generalized twist map

To see that the right hand side above can also be realized as a constant modulus fibration, we can generalize the twist map to this situation. Define the following rational map:

\[
\Phi : \mathbb{P}(w_0, \ldots, w_n) \times \mathbb{P}(v_0, \ldots, v_m) \rightarrow \mathbb{P}(w_1, \ldots, w_n, v_0v_1, \ldots, v_0v_m)
\]

\[
((x_0, \ldots, x_n), (u_0, u_1), (y_0, \ldots, y_m)) \mapsto ((u_0, u_1), (y_0^{w_1/v_0}x_1, \ldots, y_0^{v_0v_m}x_n, x_0^{v_0/v_1}y_1, \ldots, x_0^{v_0v_m}y_m)).
\]

Let the subvarieties \( V_1, V_2 \) be defined as follows: \( V_1 = \{ x_0^n + p(x_1, \ldots, x_n) = 0 \} \subset \mathbb{P}(w_0, \ldots, w_n) \);

\[
V_2 = \left\{ \begin{array}{l}
p_1(u, y) = u_0y_1 + u_1 \cdot p_{11}(y_1, \ldots, y_m) = 0 \\
p_2(u, y) = u_0(y_0^{p_{20}} + p_{20}(y_1, \ldots, y_m)) + u_1y_1^{-1} = 0 \end{array} \right\} \subset \mathbb{P}(1,1) \times \mathbb{P}(v_0, \ldots, v_m).
\]

This complete intersection has bidegrees \( \begin{bmatrix} 1 & 1 \\
v_0 & d - v_1 \end{bmatrix} \), where \( d = \sum_{i=0}^{m} v_i; p_{11} \) has degree \( v_1 \), and the degree of \( p_{20} \) is \( \deg(p_{20}) = (\deg(y_1))(\nu - 1) = v_1(\nu - 1) = \deg(y_0) \cdot \mu = v_0 \cdot \mu = d - v_1 \), hence we have the relation among the various weights:

\[
\mu = \frac{v_1(\nu - 1)}{v_0} = \frac{d - v_1}{v_0}.
\]

Clearly \( V_1 \) is invariant under the obvious action of \( \mathbb{Z}_\mu \). We claim that \( V_2 \) is invariant under the following action of \( \mathbb{Z}_\mu \):

\[
(u, y) \mapsto ((\eta^{\mu-1}u_0, u_1), (\eta y_0, y_1, \ldots, y_m))
\]

for a generator \( \eta \) of \( \mathbb{Z}_\mu \). Indeed, the first equation defining \( V_2 \) is invariant, while the second gets multiplied by a factor of \( \eta^{\mu-1} \), hence the zero locus is invariant. Set \( \ell := \gcd(\mu, \nu) \), then we get an action of \( \mathbb{Z}_\ell \) on the product space as follows:

\[
(\zeta, x, u, y) \mapsto ((\zeta^{\nu-1}x_0, x_1, \ldots, x_n), (\zeta^{\nu}(\nu-1)u_0, u_1), (\zeta\mu y_0, y_1, \ldots, y_m)).
\]

Finally, let \( X \) be defined in the product \( \mathbb{P}(1,1) \times \mathbb{P}(w_0, \ldots, w_n, v_0, \ldots, v_m) \) with coordinates \( ((u_0, u_1), (z_1, \ldots, z_n, t_1, \ldots, t_m)) \) by

\[
X = \left\{ \begin{array}{l}
u_0 \cdot t_1 + u_1 p_{11}(t_1, \ldots, t_m) = 0 \\
u_0 (p_{20}(t_1, \ldots, t_m) - p(z_1, \ldots, z_n)) + u_1 \cdot t_1^{-1} = 0 \end{array} \right.. \]

(19)

Then a calculation shows that \( \Phi(V_1 \times V_2) \subset X \), and that this displays \( X \) rationally as a quotient of \( V_1 \times V_2 \) by a \( \mu_\ell \)-operation. Indeed, the first equation defining \( X \), in terms of the coordinates \( x, y \), is

\[
u_0 \cdot x_0^{v_1/v_0} \cdot y_1 + u_1 \cdot (x_0^{v_1/v_0})^{\deg(p_{11})} \cdot p_{11}(y_1, \ldots, y_m) = x_0^{v_1/v_0}(u_0 y_1 + u_1 p_{11}(y_1, \ldots, y_m)),
\]

which clearly vanishes for \( (u, y) \in V_2 \). A similar calculation for the second equation is

\[
u_0 \left( \left( x_0^{v_1/v_0} \right)^{\mu v_0} \cdot p_{20}(y_1, \ldots, y_m) - \left( y_0^{1/v_0} \right)^{\mu v_0} \cdot p(x_1, \ldots, x_n) \right) + u_1 \left( x_0^{v_1/v_0} \right)^{-1} \cdot y_1^{1-},
\]

which again clearly vanishes for \( (x, u, y) \in V_1 \times V_2 \).
4.4.4 Gauge groups

Let us apply this to determine the gauge groups on both sides of \( \mathbb{P}(1,1,1) \times \mathbb{P}(1,1,2,2) \). For convenience we stick to Fermat polynomials. On the left hand side we have a constant modulus K3-fibration with fiber the K3 surface occurring in the first line of the table in section 3.3.2, with six singular fibers of type IV. Since each of these corresponds to a \( A_2 \), we get for the Picard group of the K3 a lattice of type \( A_2^0 \oplus H \), where \( H \) is a sum of hyperbolic and abelian factors. As we have already explained, since the Calabi-Yau threefold is the quotient of the product by a group, and the action of the group on the K3-surface is of non-Nikulin type, it preserves the Picard lattice, hence this is, up to abelian factors, the gauge group of the theory compactified on that Calabi-Yau. To do the same for the right hand side, we must first study the K3 fiber, which is the complete intersection of type \( (1,2) \), \( (1,4) \) in \( \mathbb{P}(1,1,1) \times \mathbb{P}(1,1,2,2) \). Recall that we can determine the Picard group from the elliptic fibration; in this case we first describe this K3 as the image of a twist map. This is given by the above construction, setting \( (v_0, v_1, v_2) = (1,1,1) \), \( (w_0, w_1, w_2) = (2,1,1) \), \( p(x_1, x_2) = x_1^4 + x_2^4 \), \( x_1 y_1 = y_2 \) and \( p_20(y_1, y_2) = y_1^2 \). In other words, let \( V_1 = \{ x_0^4 + x_1^4 + x_2^4 = 0 \} \subset \mathbb{P}(2,1,1) \) (so \( \mu = 2 \)), and let \( V_2 \) be the elliptic curve defined as follows:

\[
V_2 = \left\{ \begin{array}{l}
 u_0 \cdot y_1 + u_1 \cdot y_2 = 0 \\
 u_0(y_2^2 + y_1^2) + u_1 \cdot y_1^2 = 0
\end{array} \right\} \subset \mathbb{P}(1,1) \times \mathbb{P}(1,1,1) \cdot
\]

The image of \( V_1 \times V_2 \) under the twist map is the complete intersection \( W \) defined as follows:

\[
W = \left\{ \begin{array}{l}
 u_0 t_1 + u_1 t_2 = 0 \\
 u_0(t_1^2 - (z_1^2 + z_2^4)) + u_1 \cdot t_1^2 = 0
\end{array} \right\} \subset \mathbb{P}(1,1) \times \mathbb{P}(1,1,2,2) \cdot
\]

In this case, \( \nu = 3 \) and \( \ell = 6 \), so we have a group of order six acting on the product, with a subgroup of order 3 acting on the elliptic curve. The singular fibers of the fibration are hence of types \( IV \) or \( IV^* \). It is easy to see that at the four zeros of \( p(x_1, x_2) \), we have singular fibers of type \( IV \). The reader may check that there is, in addition, a singular fiber of type \( IV^* \). Hence, in this case the Picard group is \( A_2^0 \oplus E_6 \).

As we have described above, this is the same as a determinental hypersurface in \( \mathbb{P}(1,1,2,2) \), which acquires \( \deg(Q)\cdot\deg(R)\cdot\deg(S)\cdot\deg(T) = 4 \) ordinary double points. However, these four ordinary double points all lie on certain rational curves, and after resolution of singularities, they are components of the \( IV^* \) fiber. Since we have a constant modulus fibration, fiberwise duality implies that \( A_2^0 \oplus E_6 \) is indeed the gauge group of the heterotic string after the conifold transition. This kind of information is new and exciting from the point of view of physics.

4.5 Elliptic fibrations

More recently not only K3-fibrations have become of importance in string theory, but also elliptic fibrations in various dimensions. In the framework of F-theory a more general type of compactification of the ten-dimensional string (of type IIB) is considered in which the dilaton field of the string is not assumed to be constant, as in conventional compactifications, but instead can vary. In this solution of the string equations this dilaton is assumed to take values in an elliptic curve, thus leading to description of these vacua as 12-dimensional elliptic fibrations. Interesting theories then can be described by elliptically fibered Calabi-Yau manifolds of complex dimensions two, three and four.

These new vacua of the type IIB string are conjectured to be dual to the most general compactification possible for the heterotic string, based on stable vector bundles with vanishing first Chern class. Of particular interest in this context are Calabi-Yau fourfolds which not only are elliptic but also K3-fibered. Such spaces lead to a four-dimensional compactification of F-theories which are expected to be dual to stable vector bundles over Calabi-Yau threefolds which are elliptic. Clearly our twist construction provides a systematic method for building and analyzing such varieties.
In this general framework the question of phase transitions rises again in the context of the possible connectedness of these fourdimensional string theory ground states. This problem has been addressed in [6] in the framework of 4D F-theory. It was shown there that there indeed exists a generalization of the conifold transition for Calabi-Yau fourfolds. In the context of these higher dimensional varieties however the singularities are no longer nodes but the manifolds degenerate at curves of in general high genus.

A simple example of such a transition between CY
3
fibered fourfolds is provided by [6]

\[ \mathbb{P}(8,8,4,2,1,1)[24] \leftrightarrow \mathbb{P}(1,1,1,1) \oplus \mathbb{P}(8,8,4,2,1,1) \]

In this transition the singular locus is given by the smooth curve \( \Sigma = \mathbb{P}(16,16) \) of genus \( g = 385 \). At the singular configuration of this space the CY
3
fiber degenerates into a conifold configuration with 32 nodes. We can obtain this singular fiber by either deforming the generic hypersurface fiber \( \mathbb{P}(4,4,2,1,1)[12] \) which is obtained from the twist map

\[ \mathbb{P}(2,1,1)[12] \times \mathbb{P}(2,1,1)[6] \rightarrow \mathbb{P}(4,4,2,1,1)[12] \]

i.e. by collapsing 32 three-cycles, or by collapsing 32 two-cycles of the generic quasismooth codimension two complete intersection CY which leads to the codim\( \mathbb{C} = 2 \) fourfold via the twist map

\[ \mathbb{P}(2,1,1)[8] \times \mathbb{P}(1,1,1) \oplus \mathbb{P}(4,2,1,1) \[1 \times 4 8 \] \rightarrow \mathbb{P}(1,1,1) \oplus \mathbb{P}(8,8,4,2,1,1) \[1 \times 8 16 \] \]

From the analysis above we expect that via our twist map many of the aspects of the conifold transitions have important implications for the transitions among Calabi-Yau fourfolds and, via the conjectured duality between F-theory and the heterotic string compactified on stable vector bundles \( V \rightarrow \text{CY}_3 \) also for transitions between stable vector bundles.

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