A LOCALIZATION ALGORITHM FOR D-MODULES

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ABSTRACT. We present a method to compute the holonomic extension of a \( D \)-module from a Zariski open set in affine space to the whole space. A particular application is the localization of coherent \( D \)-modules which are holonomic on the complement of an affine variety.

Throughout this article let \( R = K[x_1, \ldots, x_n] \) be the ring of polynomials in \( n \) variables over the field \( K \) of characteristic zero and \( D = R(\partial_1, \ldots, \partial_n) \) the \( n \)-th Weyl algebra over \( K \). Here, \( \partial_i = \frac{\partial}{\partial x_i} \). Suppose \( f \in R \), let \( J \) be a left ideal of \( D \) and set \( M = D/J \). Let \( X = K^n, Y = V(f) \subseteq X, U = X \setminus Y \) and \( j : U \hookrightarrow X \) the natural inclusion.

In the landmark paper [K2] M. Kashiwara proved that \( j_* j^{-1} M \) is holonomic provided that \( M \) is holonomic on \( U \), but not necessarily on all of \( X \). In this note we make this important result algorithmic in the sense that we provide an algorithm that computes the module structure of \( j_* j^{-1} M \) over \( D \).

We remark that \( U \) is affine and \( X = K^n \), and so \( j_* j^{-1} M = R_f \otimes_R M \) as \( D \)-module where \( R_f = R[\partial^{-1}] = \Gamma(O_X, U) \). An algorithm to compute \( R_f \otimes_R M \) as a \( D \)-module has been given in [Oa3] under the assumption that \( M \) is holonomic on \( K^n \) and \( f \)-torsion free, and in [O-T2] under the assumption that \( M \) is holonomic on \( K^n \). The advantage of our new algorithm, besides removing the requirement of holonomicity on \( V(f) \), is that the natural map \( M \to R_f \otimes_R M \) is traced, which is an important property for other computations as local cohomology ([Oa3], [W1]) and de Rham cohomology ([O-T1], [W2]). The algorithm is illustrated in a section devoted to computational examples.

1. DESCRIPTION OF THE ALGORITHM

We shall start with a motivating construction which will also provide a skeleton of the proof of the correctness of our algorithm.

Assume that \( K \) is in fact the field \( \mathbb{C} \) of complex numbers. Let \( s, v \) be two new variables and let \( D_v = D(v, \partial_v) \).

Notation 1. Throughout we will denote \( \partial f/\partial x_i \) by \( f_i \). If \( P \in D_v \) is an operator interpreted as a polynomial in \( \partial_1, \ldots, \partial_n \), we will simply write \( P(\partial_x) \). Similarly we will abbreviate \( P(\partial_i - v^2 f_i \partial_v, \ldots, \partial_n - v^2 f_n \partial_v) \) by \( P(\partial_x - v^2 f_x \partial_v) \).

Consider the ring \( D_v \) and the \( \mathbb{C} \)-vectorspace \( R_f[s] \otimes f^{-s} \). Here, \( f^{-s} \) should be thought of as a symbol that behaves like the complex valued function \( f^{-s} \) under differentiation and the tensor is over \( \mathbb{C} \). We shall turn this vector space into a module over \( D_v \) by means of the action \( \bullet \) defined as follows. We require that all \( x_i \) act as multiplication on the left factor, while \( \partial_i \) acts by the “product rule”
\[ \partial_t \left( \frac{g(x,s)}{f} \otimes f^{-s} \right) = \partial_t \left( \frac{g(x,s)}{f} \right) \otimes f^{-s} + \frac{g(x,s)}{f} \otimes (-s)f_i \otimes f^{-s}. \]

Thus, we only need to define the action of \( v \) and \( \partial_v \). We set
\[ v \cdot \left( \frac{g(x,s)}{f} \otimes f^{-s} \right) = \frac{g(x,s + 1)}{f} \otimes f^{-s}, \]
\[ \partial_v \cdot \left( \frac{g(x,s)}{f} \otimes f^{-s} \right) = -(s - 2) \frac{f \cdot g(x, s - 1)}{f} \otimes f^{-s}. \]

We observe that \( v \) is simply shifting \( s \) up by one (in particular, the extra \( f \) in the denominator comes from shifting the exponent in \( f^{-s} \)) and \( \partial_v \) is shifting \( s \) down by one, together with the “differential” \( 2 - s \) of the shift operator (cf. the action defined in [M]).

Armed with this action on \( R_f[s] \otimes f^{-s} \) we can define an action of \( D_v \) on \( D/J \otimes_R (R_f[s] \otimes f^{-s}) \) as follows. \( v \) and \( \partial_v \) commute with the first factor in the tensor product, \( x_i \) acts by left multiplication on the first factor and \( \partial_i \) acts by the product rule.

We detect a certain set of trivial relations for this action. Namely, let us ask which operators on \( D_v \) are candidates for annihilating the element \( \bar{T} \otimes 1 \otimes f^{-s} \), the bar denoting cosets in \( D/J \). Independently of \( J \), \( 1 - vf \) is such an annihilator. Assume that \( J \) is generated by \( \{ P_j(\partial_x) \} \) where we think of \( P_j \) as a polynomial in \( \partial_x \) with left coefficients in \( R \). Let us define a ring map \( \phi \) from \( D \) to \( D_v \) by mapping \( P \in D \) to \( \phi(P_j(\partial_x)) = Q_j(\partial_x) := P_j(\partial_x - v^2 f \partial_x) \in D_v \) (recall \( f_i = \partial_i(f) \)). It is easy to see that this assignment preserves commutators and hence is an actual ring map. We remark that \( \phi \) represents the map \( U \rightarrow V(1 - vf) \subseteq \mathbb{C}^{n+1} \) given by \( x \rightarrow (1/f(x), x) \) on the level of differential operators, while the inclusion \( \phi(D) \subseteq D_v \) corresponds to \( V(1 - vf) \subseteq \mathbb{C}^{n+1} \).

We claim that \( Q_j(\partial_x) \) annihilates \( \bar{T} \otimes 1 \otimes f^{-s} \). To prove this, we observe that \( (\partial_i - v^2 f_i \partial_x) \cdot (\bar{T} \otimes 1 \otimes f^{-s}) \) equals \( \bar{T} \cdot F \otimes 1 \otimes f^{-s} \) for all \( P \in D \). Thus \( Q_j(\partial_x) \cdot (\bar{T} \otimes 1 \otimes f^{-s}) = P_j(\partial_x) \otimes 1 \otimes f^{-s} = 0 \). We point out that \( -v \partial_v + 1 \) acts as multiplication by \( s \) and that this construction is (formally) independent of the base field.

Our main statement is the following

**Theorem 2.** Let \( M = D/J \). Consider the left ideal \( D_v \cdot \{ 1 - xv, \phi(J) \} \) and the right ideal \( \{ \partial_v \} \cdot D_v \) in \( D_v \). The quotient \( D_v/(D_v \cdot \{ 1 - xv, \phi(J) \} + \{ \partial_v \} \cdot D_v) \) is isomorphic to \( M \otimes_R R_f \) as \( D \)-module. Moreover, the natural map \( M \rightarrow M \otimes_R R_f \) sends \( \bar{T} \in M \) to \( \bar{T} \otimes \frac{1}{T} \).

It is worth noticing that this is in fact a left \( D \)-module due to the fact that \( D \) and \( \partial_v \) commute.

In the general context in which the theorem is stated, the \( D \)-module \( D_v/(D_v \cdot \{ 1 - xv, \phi(J) \} + \{ \partial_v \} \cdot D_v) \) will not be a finitely generated \( D \)-module. If however \( M \) is holonomic on \( U = K^n \setminus V(f) \), then \( M \otimes_R R_f = D_v/(D_v \cdot \{ 1 - xv, \phi(J) \} + \{ \partial_v \} \cdot D_v) \) will be holonomic, and in particular finitely generated. In order to compute the structure of the localized module, we observe that it is the integration of the left \( D_v \)-module \( D_v/D_v \cdot \{ 1 - xv, \phi(J) \} \) with respect to \( \partial_v \) (cf. [Oa3, Theorem 5.7, O-Ti] Section 6).

Methods for the algorithmic computation of 0-th integration worked out in [Oa3, Theorem 5.7] (i.e., the restriction to \( v = 0 \) of the Fourier transform of the module,) together with our theorem give the following algorithm. Let \( N \) denote \( D_v/(\{ \partial_v \} \cdot D_v + D_v \cdot \{ 1 - fv, \phi(J) \}) \).
Algorithm 3. Input: \( f \in R; \{ P_1, \ldots, P_r \} \subseteq D \) generating \( J, D/J \) holonomic on \( U = K^n \setminus V(f) \).
Output: \( k \in \mathbb{N}; \{ Q_1, \ldots, Q_t \} \subseteq D \) such that \( D/D \cdot \{ Q_1, \ldots, Q_t \} \) is isomorphic to \( D/J \otimes_R R_f \) generated by \( T \otimes f^{-1} \).

Begin.

1. For \( P_i, i = 1, \ldots, r \) compute \( \phi(P_i) \in D_v \) defined by replacing \( \partial_x \) by \( \partial_x - v^2 f_x \partial_v \).
2. Compute the \( b \)-function \( b(s) \) for integration of \( D_v/D_v \cdot \{ 1 - f v, \phi(J) \} \) with respect to \( \partial_v \). That is, find \( K[v\partial_v] \cap \cap w(D_v \cdot \{ 1 - f v, \phi(J) \}) \) where \( w \) is the weight assigning 1 to \( v \), \(-1\) to \( \partial_v \) and 0 to all other variables. Replace \( v \partial_v \) by \( -s - 1 \).
3. Let \( k \) be the largest non-negative integer root of \( b(s) \). If there is no such root, then output \( I = D \), else continue. The integral \( N \) is generated by the coset of \( v^k \) in \( N \).
4. Compute the annihilator \( I \) over \( D \) of \( v^k \) in \( N \).
5. \( D/J \otimes_R R_f \) is generated by \( T \otimes f^{-1}(k + 2) \) and isomorphic to \( D/I \).
6. Return \( k \) and \( I \).

End.

An algorithm to perform step 2 of the algorithm is given in [On2, Algorithm 4.5].

The steps 3 and 4 are nothing but an algorithm to get the 0-th integral of \( D \)-modules. Here is a more precise description of steps 3 and 4.

1. Let \( G = \{ g_1, \ldots, g_m \} \) be a Gröbner basis of \( D_v \cdot \{ 1 - v f, \phi(J) \} \) with respect to \( w \) (see, e.g., [SST], section 1.1).
2. Let \( G_k \) be the set
   \[ \{ \text{normalForm}(v^i g_j, \{ \partial_v \} \cdot D_v) | j = 1, \ldots, m, 0 \leq i \leq k - \text{ord}_w(g_j) \} \].
3. Regard \( D \cdot \{ G_k \} \) as a left submodule of the free module
   \[ D v^{0} + D v^{1} + \cdots + D v^{k} = D^{k+1} \]
   and find generators of \( D \cdot \{ G_k \} \cap D v^{k} \). The last intersection can be computed by using an order to eliminate \( v^{i}, \ldots, v^{k-1} \), i.e., by using an order \( > \) such that \( a_i v^{i} > a_k v^{k} \) for all \( i = 0, \ldots, k - 1 \) and \( a_i, a_k \neq 0 \) in \( D \).

Here, we put
\[ \text{ord}_w(g) := \max \left\{ w \cdot (\alpha, \beta) | g = \sum a_{\alpha, \beta} v^{\alpha} \partial_v^{\beta} \right\} \]
following the notation of [SST], \( \text{normalForm}(g, \{ \partial_v \} \cdot D_v) \) means taking the normal form of \( g \) with respect to the right ideal \( \{ \partial_v \} \cdot D_v \). For example,
\[ \text{normalForm}(v^2 \partial_v^2, \{ \partial_v \} \cdot D_v) = \text{normalForm}(\partial_v^2 v^2 - 4 \partial_v v + 2, \{ \partial_v \} \cdot D_v) = 2. \]

2. Proof of correctness of the algorithm

Let us first provide the

Proof of the Theorem: Given an operator \( P \in D_v \) we shall define its normal form in \( D_v \). The goal is a presentation of \( P \) where \( \partial_i - v^2 f_i \partial_v \) take the position of \( \partial_i \). In other words, we are aiming for a sum of the form \( P = \sum \partial_i v^k p_{ab}(x) q_{ab}(\partial_x - v^2 f_x \partial_v) \) where \( p_{ab} \) and \( q_{ab} \) are polynomials in \( n \) variables over \( K \). To this end, write first \( P \) as \( \sum \partial_i v^k p_{ab}(x) q_{ab}(\partial_x) \). The operator \( P^1 := \sum \partial_i v^k p_{ab}(x) \partial_x \) has the property that \( P - P^1 \) will have lower degree in \( \partial_x \) than \( P \). If \( P - P^1 = 0 \), quit. Otherwise write \( P - P^1 \) as \( \sum \partial_i v^k p_{ab}^1(x) q_{ab}^1(\partial_x) \) and set \( P^2 := \partial_i v^k p_{ab}^1(x) q_{ab}^1(\partial_x) - \)
$v^2 f_v \partial_v$). Repeat this procedure until we arrive at $P^l = 0$. Then $P = P^1 + \cdots + P^l$ and this sum is the desired normal form in $D_v$. We shall write $\bar{P}$ for the normal form of $P$ in $D_v$.

The normal form in $D_v$ of an operator induces a normal form in $D_v/\{\partial_v\} \cdot D_v$ by removing all terms in $\{\partial_v\} \cdot D_v$ from the normal form of $P$ in $D_v$. We denote the normal form of $P \in D_v$ in $D_v/\{\partial_v\} \cdot D_v$ by $\bar{P}$. Of course, $P + \{\partial_v\} \cdot D_v = \bar{P} + \{\partial_v\} \cdot D_v$ as cosets. We notice that both normal forms are unique. As an example, consider the representative of a coset.

Remark 4. The theorem and its proof generalize nearly verbatim to the situation where $M$ is any finitely generated $D$-module.

Now let us consider the situation in which $M = D/J$ is holonomic on $U$. Then $D_v/\{\partial_v\} \cdot D_v + D_v \cdot \{1 - f v, \varphi(J)\}$ is a finite $D$-module as it is isomorphic to the
module $M \otimes_R R_f$ which is holonomic by theorem 1.3 of [K2]. We can find a generator by computing the $b$-function for integration along $\partial_v$ for $D_v/D_v \cdot (1-vf, \phi(J))$ as in step 2 of algorithm 3. If $k$ is the largest root of the $b$-function then $v^k$ is a generator for $D_v/(\{\partial_v\} \cdot D_v + D_v \cdot (1-fv, \phi(J))) = M \otimes_R R_f$ (compare [9-T2], algorithm 5.4). Thus in order to represent the localization $M \otimes_R R_f$ as a quotient of $D$ all one needs to do is to find the annihilator of $v^k$ over $D$. This shows the correctness of our algorithm.

**Remark 5.** Again, the algorithm generalizes to the non-cyclic situation. Let $M = D^n/J$, $D^n = \otimes^n D e_j$. The modifications are as follows. Compute $m$ separate $b$-functions $b_j(s)$ to the integration of $D_v(e_j + \phi(J))/(D_v \cdot \{\phi(J), (1-vf)e_j\})$ along $\partial_v$. $M \otimes_R R_f$ is generated by the cosets of $v^{k_j} e_j$ in $N$ where $k_j$ is the largest integer root of $b_j(s)$.

### 3. Categorical explanation of the algorithm

Let us now give a more categorical explanation of the validity of our algorithm. Decompose $j: K^n \setminus V(f) \to K^n$ as $j = p \circ \iota \circ \phi$;

$$
\begin{array}{ccc}
W & \xrightarrow{\iota} & K^{n+1} \\
\phi & & p \\
K^n \setminus V(f) & \xrightarrow{j} & K^n
\end{array}
$$

where $\phi : K^n \setminus V(f) \to W = \{(v, x) \mid v \cdot f(x) = 1\}$ is defined by $x \mapsto (1/f(x), x)$; $\iota : W \hookrightarrow K^{n+1}$ is the closed embedding, and $p : K^{n+1} \to K^n$ is the natural projection. Then we have

$$M \otimes_R R_f = \int_j j^{-1} M = \int_p \int_{\phi} j^{-1} M$$

by the chain rule of integration functors (see, e.g., [B] p.251, 6.4 Proposition, [HT], 1.5.1). Note that the chain rule holds in the derived category in general, but, in this case, $\int_j$ is an exact functor since $D_{X \setminus U} = D_X[1/f]$ is flat over $D_U$ and $j$ is an affine morphism. $\int_p \int_{\phi}$ is also an exact functor. Hence, we have only to compute $0$-th integrals in each step of the computation of the integral functors. $\int_{\phi} j^{-1} M$ is obtained by the coordinate transformation represented by our ring map $\phi$; consider the map

$$y_i = x_i, \ (i = 1, \ldots, n), \ y_{n+1} = 1/f(x).$$

Then, $\partial x_i = \partial y_i - \frac{\partial y_i}{\partial y_{n+1}} \partial y_{n+1} = \partial y_i - f_v^i \partial y_i$ modulo $1-fv = 0$, which commute with each other and define our ring map $\phi$. $\int_j$ is nothing but Kashiwara equivalence corresponding to $\phi(D) \hookrightarrow D_v$. It follows that $\int_\phi j^{-1} M = D_v/(D_v \cdot \{\phi(J), 1-fv\})$, compare also Proposition A.1 (p. 596) of [1]. Integration under the projection $p$ corresponds then to our last step accomplished by forming the quotient $(\int_p \int_{\phi} j^{-1} M)/\{\partial_v\} \cdot (\int_j \int_{\phi} j^{-1} M)$.

Since $j^{-1} M$ is holonomic, so is $\int_j \int_{\phi} j^{-1} M$ and hence the $b$-function for integration is nonzero, thus guaranteeing termination of the search for generators of $\int_j j^{-1} M$. 


4. Examples

Example 6. For our first example we take \( n = 1, x = x_1, K = \mathbb{C}, J = x\partial_x + \lambda, \lambda \in \mathbb{C}, f(x) = x \).

In this scenario, we have to compute the integral of the module \( D_v/(1-xv, x(\partial_x - v^2\partial_v)+\lambda) \) along \( \partial_v \). As \( D_v\{1-xv, x(\partial_x - v^2\partial_v)+\lambda\} = D_v\{\partial_x x - \partial_v v + 1 + \lambda, 1-xv\} \), one sees that the \( b \)-function is \( s(s + 1 + \lambda) \). Thus the largest integer root is either 0 or \(-1 - \lambda\), depending on whether \( \lambda \) is a negative integer or not. So \( M \otimes_R R_x \) is generated by \( 1 \otimes x - (\lambda + 2) \) in the former and \( 1 \otimes x - \lambda \) in the latter case.

If for example \( \lambda = -7 \) we compute a \( b \)-function of \( s(s - 6) \) indicating that \( M \otimes_R R_x \) is generated by \( 1 \otimes x - 8 \). Since in this case \( M = R \) generated by \( x \), we conclude that \( M \otimes_R R_f \) is in fact generated by \( x^{-8} \cdot x \), as it should.

If on the other hand \( \lambda = 1/2 \) then \( b(s) = s(s + 3/2) \) and hence the largest integer root is 0. Thus, \( M \) is generated by the germ of \( x^{-\lambda} \) and \( M \otimes_R R_x \) is generated by \( v^0 \) corresponding to \( 1/x^{2+\lambda} \). \( M \) is already isomorphic to \( M \otimes_R R_x \).

Example 7. In this example we consider the left ideal \( J \) generated by

\[
\partial_x(x^2 - y^3), \\
\partial_y(x^2 - y^3).
\]

These are annihilators of the function \( 1/f \), where \( f = x^2 - y^3 \), but they do not generate the annihilating ideal (see, e.g., [Oa2]). The left ideal \( J \) is not holonomic since the characteristic variety of \( J \) is \( V(x^3 - y^2) \cup V(\xi_x, \xi_y) \), whose first component has dimension 3 in \( \mathbb{C}^4 = \{(x, y; \xi_x, \xi_y)\} \). As to an algorithmic method to get the characteristic variety, see [Oa1]. We give now the output of a computer session using the computer algebra system Kan/sm1 ([T]) interspersed with comments.

We remark that in this case \( M \) restricted to \( U \) is an \( \mathcal{O}_U \)-coherent free module of rank one where \( U = \mathbb{C}^2 \setminus V(f) \).

\[ /ff \]
\[ (((x^2-y^3)*(Dx - v^-2*2*x*Dx) + 2*x) \\
((x^2-y^3)*(Dy + v^-2*3*y^2*Dy) -3*y^2) \\
(v*(x^2-y^3)-1) \]
\[ ]
\[ def \]
\[ (this is \phi(J)) \]

\[ sm1>ff [(v)] intbfm :: \]
\[ [ \begin{array}{l} 216*s^4+1296*s^3+2586*s^2+1716*s $ \end{array} ] \]
\[ [[6,1],[s^2,1],[6*s+11,1],[6*s+13,1],[s,1]] \]
\[ (these are the factors of the \( b \)-function) \]

\[ sm1>ff [(v)] -2 0 1 install_s ; \]
\[ Completed. \]
\[ (computing the integration) \]

0-th cohomology: [ 1 , \\
[ -3*x*Dx-2*y*Dy-18 , \\
3*y^2*Dx+2*x*Dy , \\
-2*y^3*Dy+2*x^2*Dy-18*y^2 ] ]

-1-th cohomology: [ 0 , [ ] ]
The integration $\int j^{-1}M$ is not $\mathcal{O}_X$-coherent, although of course it is still coherent over the sheaf of differential operators $\mathcal{D}_X$ on $X = \mathbb{C}^2$. Since localization is an exact functor, the first cohomology group (corresponding to the first higher order integration) was known to be zero. The 0-th cohomology above coincides with the annihilating ideal of the function $f^{-3}$.

**Example 8.** Let $n = 3$, and consider the ideal $J$ generated by the system

\[
(x^3 - y^2z^2)^2\partial_x + 3x^2, \\
(x^3 - y^2z^2)^2\partial_y - 2yz^2, \\
(x^3 - y^2z^2)^2\partial_z - 2y^2z.
\]

These operators are annihilators of the exponential function $e^{1/f}$ where $f(x, y, z) = x^3 - y^2z^2$. The characteristic variety of $M = D/J$ has six components, defined by the prime ideals $(y, x)$, $(z, x)$, $(\xi_x, \xi_y, \xi_z)$, $(\xi_y, z, x)$, $(\xi_z, y, x)$ and the ideal generated by

\[
y\xi_y - z\xi_z, \\
2x\xi_x + 3z\xi_z, \\
8z\xi_x + 27\xi_y^2\xi_z, \\
8y\xi_y^3 + 27\xi_y^2\xi_z, \\
-4z^2\xi_x^2 + 9x\xi_y^2, \\
-4y\xi_x^2 + 9x\xi_y\xi_z, \\
-4y\xi_x^2 + 9x\xi_y^2, \\
2z^3\xi_x\xi_z - 3x^2\xi_y^2, \\
2y^2z\xi_x + 3x^2\xi_z, \\
yz^3\xi_z + x^3\xi_y, \\
yz^2 - x^3\xi_y.
\]

All but the first two are of dimension three. This implies that the non-holonomic locus of $M$ is contained in the hypersurface $x = 0$. Here, we used Asir to obtain the primary ideal decomposition.

Hence we may apply our algorithm to compute $M \otimes_R R_x$. We remark that contrary to the previous example in this case $j^{-1}M$ is holonomic but not coherent as $\mathcal{O}_U$-module on $U = X \setminus V(x)$. Using Kan/sm1 again one obtains the eight operators

\[
-3y\partial_y + 3z\partial_z, \\
-2xyz^2\partial_x - 3x^3\partial_y - 4yz^2, \\
-2y^2z\partial_x - 3x^3\partial_z - 4y^2z, \\
6xz^3\partial_x + 9x^3\partial_y + 6x^2z\partial_x + 6yz^2\partial_y + 6z^3\partial_z + 12z^2, \\
-6yz^3\partial_x + 4x^4\partial_x + 12x^3z\partial_x + 8x^3 + 12, \\
6yz^4\partial_z - 4x^4\partial_x - 12z^3\partial_y - 8yz^3\partial_y - 8y^3\partial_y - 12y, \\
8x^3\partial_y^2 + 24x^4z\partial_x + 18x^3z^2\partial_z^2 + 64x^4\partial_x + 102x^3z\partial_z + 80x^3 + 24xz\partial_x + 48, \\
-6z^5\partial_z^3 + 4x^4\partial_x\partial_y^2 + 12x^3z\partial_y^2\partial_x - 36z^4\partial_y^2 + 8x^3\partial_y^2 - 36z^3\partial_z + 12\partial_y^2
\]

which annihilate the function $x^{-2}e^{1/f}$. The characteristic variety of this holonomic left ideal of $D$ has the same last four components as the characteristic variety of $M$ while the first two components (of dimension 4) are replaced by $(\xi_z, \xi_y, x)$ and $(z, y, x)$ (of dimension 3).

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