Topological phase transition at quantum criticality

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Recently topological states of matter have witnessed a new physical phenomenon where both gapless edge and bulk excitations coexist. This manifests in the existence of exponentially localized edge modes living at the certain criticalities with topological properties. The criticalities with topological and non-topological properties enable one to look into an unusual and interesting multicritical phenomenon: topological phase transition at criticality. We explore the existence of such topological transition and reconstruct various suitable theoretical frameworks to characterize them. The boundary solution of the Dirac equation and the winding number are constructed for the criticality to detect the multicritical points. We reframe the scaling theory of the curvature function and obtain the critical exponents to identify the topological transition between different critical phases separated by multicritical points. Finally, we discuss the experimental observabilities of our results in superconducting circuits and ultracold atoms.

Introduction: In the quest of classifying novel phases of quantum matter in the absence of local order parameters, the topology of electronic band structure plays a prime role [1–4]. It enables the distinction between gapped phases in terms of a quantized invariant number, which counts the number of localized edge modes present [5]. The transition between the distinct topological phases involves a bulk band gap closing at the point of topological phase transition. Across the transition the number of edge modes, protected by the bulk gap, changes [6]. In the gapped phases, the localization length of the edge modes diverges as the system drives towards the transition point or criticality [7].

Interestingly, this conventional knowledge is revised recently, realizing even criticalities can host the stable localized edge modes despite the vanishing bulk gap [8–19]. This results in the emergence of non-trivial criticalities with unique topological properties even in the presence of gapless bulk excitations. The non-trivial criticalities can be effectively characterized in terms of the zeros and poles of complex function associated with the Hamiltonian [8]. The localized edge modes at criticality are protected by novel phenomena such as kinetic inversion [11] (in fermionic models) and finite high energy charge gap [16] (in bosonic models). It has been shown that they also remain robust against interactions and disorders [8, 9]. This intriguing interplay between topology and criticality causes an unconventional topological transition between critical phases [11, 20].

In this letter, we investigate criticality in a generic two-band model and find that there exist two species of multicritical points with the quadratic and linear dispersions both corresponding to the topological transition between distinct critical phases. By constructing the Dirac equation for criticality we show localized edge mode solutions in the non-trivial critical phases. This is supported by our proposal of obtaining the non-zero integer winding number for non-trivial critical phases. To analyze the critical system further, we generalize curvature function which diverges on approaching the multicritical points. We extract the critical exponents for the transitions at the multicritical points. Based on the diverging property of the curvature function we develop a renormalization group scheme to distinguish the different topological critical phases. Finally, we discuss experiments that can test our results.

Model: We consider a one-dimensional lattice chain of spinless fermions in momentum space [21, 22] represented by a generic two-band Bloch Hamiltonian of the form

\[ H(k, \Gamma) = \chi \sigma = \chi_x \sigma_x + \chi_y \sigma_y, \]

where \( \Gamma = \{ \Gamma_0, \Gamma_1, \Gamma_2 \} \), \( \chi_x = \Gamma_0 + \Gamma_1 \cos k + \Gamma_2 \cos 2k \), and \( \chi_y = \Gamma_1 \sin k + \Gamma_2 \sin 2k \) and \( \sigma = (\sigma_x, \sigma_y) \) are the Pauli matrices. The model represents extended Su-Schrieffer-Heeger (SSH) [23] and extended Kitaev models [24] by uniquely defining the parameters (see Ref.[25] for details). The parameters, \( \Gamma_0, \Gamma_1 \) and \( \Gamma_2 \) describe the onsite potential, the nearest neighbor couplings and the next nearest neighbor couplings respectively. In general, the model in Eq.1, can support three distinct gapped phases (\( w = 0, 1 \) and 2) distinguished by the number of edge modes they possess [25]. Uniquely, the model supports the edge modes and topological transition at critical surfaces \( \Gamma_1 = \pm(\Gamma_0 + \Gamma_2) \) corresponding to \( k_0 = \pi, 0 \) respectively [25].

\[ \begin{align*}
\text{MC}_2: & \Gamma_2 = -\Gamma_0 \\
\text{MC}_1: & \Gamma_2 = \Gamma_0
\end{align*} \]

\[ \begin{align*}
\Gamma_2 < -\Gamma_0 & \quad \Gamma_0 > \Gamma_2 > -\Gamma_0 \\
\Gamma_2 > \Gamma_0 & \quad \Gamma_2 > \Gamma_0
\end{align*} \]

FIG. 1: Schematic representation of the criticalities \( \Gamma_1 = \pm(\Gamma_0 + \Gamma_2) \). Non-trivial critical phases: \( \Gamma_0 < \Gamma_2 \) and \( \Gamma_2 < -\Gamma_0 \) (solid lines), trivial critical phase: \( \Gamma_0 > \Gamma_2 > -\Gamma_0 \) (dashed line) are separated by the multicriticalities \( \text{MC}_{1,2} \) (\( \Gamma_2 = \pm \Gamma_0 \)).

To further explore these unique phenomena we propose a framework that works out for criticality without...
referring to any of the gapped phases of the model. Ideally driving the system to criticality involves \( k \to k_0 \) and \( \Gamma \to \Gamma_c \), where \( \Gamma_c \) is the critical point in the parameter space. To avoid the singularities involving the exact critical point, one can define the Hamiltonian critical only in the parameter space as \( \mathcal{H}(k, \Gamma_c) \) with \( k = k_0 + \Delta k \), where \( \Delta k << 2\pi \). This situation is hereafter referred as "criticality" in this letter. The model in Eq. 1 can be written at criticality using \( \chi_x = \Gamma_0 (1 \pm \cos k) + \Gamma_2 (\cos 2k \pm \cos k) \), and \( \chi_y = \Gamma_2 (\sin 2k \pm \sin k) + \Gamma_0 \sin k \) for the critical surfaces \( \Gamma_1 = \pm (\Gamma_0 + \Gamma_2) \). The topological and non-topological critical phases are separated by the multicritical lines \( \Gamma_2 = \Gamma_0 \) (\( MC_1 \)) and \( \Gamma_2 = -\Gamma_0 \) (\( MC_2 \)). Without loss of any generality, we assume \( \Gamma_0 = 1 \). Hence hereafter the critical surfaces and the multicritical lines will be called as the critical lines and the multicritical points respectively, as shown in Fig. 1. The multicriticals, \( MC_{1,2} \), are identified with quadratic and linear dispersion respectively. Moreover, \( MC_2 \) interestingly exhibits swapping of the values of \( k_{0_{MC}}^2 \) (which are \( k_0 = 0 \) and \( \pi \)) which emerges as a result of the intersection of both the critical lines. For all the details of \( MC_{1,2} \) see Ref.[25]. Although \( \mathcal{H}(k, \Gamma_c) \) can be regarded as the near-critical Hamiltonian as only parameters are exactly at critical values and momentum is not, we will show in the following that using this approach, one can capture the physics of topological transition at criticality and associated multicritical points.

**Bound state solution of the Dirac equation:** The presence of edge modes in topological insulators and superconductors is lucid from the bound state solution of Dirac equation [26]. We solve the model in Eq.1 for the bound state solution at criticality. Interestingly, as a consequence of the near-critical approach adopted here, the Dirac Hamiltonian at criticality naturally fixes the interface at a multicritical point. Dirac Hamiltonian at criticality up to third-order expansion around \( k_{0_{MC}}^2 \) for \( MC_1 \) is

\[
\mathcal{H}(k) \approx \epsilon_1 k^2 \sigma_x + (\epsilon_2 k - \epsilon_3 k^3)\sigma_y,
\]

where \( \epsilon_1 = (\Gamma_0 - 3\Gamma_2)/2, \epsilon_2 = (\Gamma_2 - \Gamma_0) \) and \( \epsilon_3 = (7\Gamma_2 - 3\Gamma_0)/6 \). We look for zero energy solution in real space (we set \( \hbar = 1 \) throughout the discussion), \( \mathcal{H}(\psi) = 0 \). Identifying the spinor of the wave-function \( \psi(x) = \rho_\theta \phi(x) \) is an eigenstate of \( \sigma_z \) and using \( \phi(x) \propto e^{-x/\xi} \), inverse of the non-zero decay length can be obtained as

\[
\xi^{-1} = (\eta \epsilon_1 \pm \sqrt{\epsilon_1^2 - 4\epsilon_2 \epsilon_3})/(-2\epsilon_3).
\]

For both roots to be positive, it requires \( \xi_+^2 + \xi_-^2 > 0 \), which implies \( \eta = \text{sign}([\epsilon_1]/\epsilon_3) \). The edge mode decay length (longer one of two) is \( \xi_+ \approx |\epsilon_1|/\epsilon_3 \) remains finite and positive for \( \epsilon_3 > 0 \) i.e., \( \Gamma_2 > \Gamma_0 \), which means the criticality in this region possesses edge modes and is the topological non-trivial phase. Note that, the term \( \epsilon_2 \) is the gap term at criticality, which mimics the role of mass. As \( \epsilon_2 \to 0 \) the decay length \( \xi_+ \to \infty \) indicating the edge mode delocalize into the bulk and topological transition takes place at \( MC_1 \) i.e., at \( \Gamma_2 = \Gamma_0 \). To visualize this phenomenon we write the bound state solution \( \psi(x) \propto (\eta \ 0)^T (e^{-x/\xi_+} - e^{-x/\xi_-}) \), which distributes dominantly near the boundary and decay exponentially as \( x \to \infty \), as shown in Fig.2(a).

**FIG. 2:** Bound state solutions of the edge modes at the non-trivial critical phases. (a) Plotted for \( \Gamma_2 = 3\Gamma_0 \) (with \( \Gamma_0 = 1 \)) at the critical phase \( \Gamma_2 > \Gamma_0 \). (b) Plotted for \( \Gamma_2 = -3\Gamma_0 \) (with \( \Gamma_0 = 1 \)) at the critical phase \( \Gamma_2 < -\Gamma_0 \).

To identify the topological transition at \( MC_2 \) and the corresponding topological non-trivial phase one has to consider the swapping property of \( k_{0_{MC}}^2 \), which emerge as a result of intersection of critical lines at \( MC_2 \) [25]. In this case, after expanding around \( k_{0_{MC}}^2 \) and using the swapping property [25], the Dirac Hamiltonian can be obtained up to second order as

\[
\mathcal{H}(k) \approx (\epsilon_1 - \epsilon_3 k^2)\sigma_x + (\epsilon_2 k - \epsilon_3 k^3)\sigma_y,
\]

where \( \epsilon_1 = 2(\Gamma_0 + \Gamma_2), \epsilon_2 = (\Gamma_2 - 3\Gamma_0) \) and \( \epsilon_3 = (5\Gamma_2 + 3\Gamma_0)/2 \). With \( \eta = \text{sign}([\epsilon_2]/\epsilon_3) \), the edge mode decay length \( \xi_+ \approx -(\xi_2)/\epsilon_3 \) is obtained using \( \phi(x) \propto e^{x/\xi} \) and is positive if \( \epsilon_1 < 0 \). Therefore, in this case, the gap term is \( \epsilon_1 \) which vanish at the multicritical point \( MC_2 \), i.e. at \( \Gamma_2 = -\Gamma_0 \). This implies that the criticality \( \Gamma_2 < -\Gamma_0 \) is topological non-trivial phase and the topological transition occur at \( MC_2 \), i.e. \( \Gamma_2 = -\Gamma_0 \) as a consequence of the delocalization of edge mode into the bulk as \( \epsilon_1 \to 0 \). In this case the bound state solution \( \psi(x) \propto (0 \ \eta)^T (e^{x/\xi} - e^{-x/\xi}) \), distribute near the boundary and decay as \( x \to -\infty \) as shown in Fig.2(b).

**Winding number:** The hurdle in defining the winding number at criticality (\( w_c \)) is naturally avoided in the near-critical approach and allows one to calculate the winding number in its usual integral form as \( w_c = \frac{1}{2\pi} \lim_{\delta \to 0} \int_{|k - k_0| > \delta} F(k, \Gamma_c) dk \), where integrand is the curvature function at criticality. However, it yields fractional value [25] which does not account correctly for the number of edge modes present at criticalities.

The fractional values at criticality imply the presence of fractional winding of unit vector \( \mathbf{\hat{\chi}} = \mathbf{\chi}/|\mathbf{\chi}| \), in the auxiliary space over the Brillouin zone [11, 12, 27]. For non-trivial critical phases, one can identify integer winding (\( w_E \)) of the unit vector along with an extended fractional winding (\( w_F \)) in the Brillouin zone, as shown in Fig.3(a). For trivial criticalities, only fractional winding can be observed as in Fig.3(b). Based on this, we propose that the winding number at criticality should be approximated to only the integer values which effectively captures the number of edge modes at criticality.
Proposition: Winding number at criticality ($w_c$), which acquires fractional values ($w_c = w_c^I + w_c^F$), can be effectively approximated only to its integer part i.e. $w_c \approx w_c^I$, to quantify the number of edge modes present at criticality.

The proposal roots in the fact that the momentum zones can be divided into integer and fractional windings as $-\pi < k_I < \pi/\lambda$ and $\pi/\lambda < k_F < \pi$ respectively. The cut-off $\lambda$ differentiates the momentum zones responsible for integer and fractional windings of the unit vector, as shown in Fig.3(a). Therefore, the volume of the Brillouin zone is $2\pi = (\pi + \pi/\lambda) + (\pi - \pi/\lambda)$. To define the integer part over the full Brillouin zone we set $\lambda = 1$, which drives the fractional winding $w_c^F \to 0$. Therefore, the winding number at criticality can be written only in terms of integer winding

$$w_c^I = \frac{1}{2\pi} \lim_{\delta \to 0} \int_{|k_I - k_0| > \delta} F(k_I, \Gamma_c) dk_I. \quad (4)$$

The winding number in the non-trivial phases of the model can be found to have $w_c = 1.5$, for which the corresponding $w_c^I = 1$. Hence $w_c^I$ correctly accounts for the one edge mode living at the criticalities which we also find from the solution of the Dirac equation. For the trivial critical phase $w_c = 0.5$ and $w_c^I = 0$ implying no localized edge modes. The transition between the critical phases with $w_c^I = 0$ and $w_c^I = 1$ occur at the multicritical points (see Fig.3(b) of Ref.[25]). This clearly demonstrates the topological transition at criticality through multicritical points.

This proposal can be found consistent with the method used in Ref.[8], where the winding number is defined using number of zeros ($N_z$) and order of poles ($N_p$), $w = N_z - N_p$. The zeros and poles of a complex function is obtained from the Hamiltonian in momentum space with a substitution $e^{ik} = \zeta$, (where $\zeta$ is a complex number and interpreted on a unit circle.

We construct the complex function for the model with zeros $\zeta_1 = \pm 1$ and $\zeta_2 = \pm \Gamma_0/\Gamma_2$ and no poles respectively for the criticalities $\Gamma_1 = \pm (\Gamma_0 + \Gamma_2)$ [25]. One of the zeros ($\zeta_1$) always remain on the unit circle signalling the criticality and the other ($\zeta_2$) lie inside (outside) for non-trivial (trivial) critical phases with $w = 1(0)$, as shown in Fig.3(c,d). This also support the bound state solution of Dirac equation.

Curvature function at criticality: The information of the topological property of the system is embedded in the curvature function $F(k, \Gamma)$ [28–38]. As the system approaches critical point to undergo topological phase transition i.e. $\Gamma \to \Gamma_c$, curvature function diverges at $k_0$, with the diverging curve satisfying $F(k_0 + \delta k, \Gamma) = F(k_0 - \delta k, \Gamma)$. Interestingly, even at criticality, as one tunes the parameters $\Gamma_c$ towards a multicritical point $\Gamma_{mc}$, the curvature function diverges at $k_0^{mc}$ with $F(k_0^{mc} + \delta k, \Gamma_c) = F(k_0^{mc} - \delta k, \Gamma_c)$ [25]. Topological transition is signaled as the sign of the diverging peak flips [28] if the parameters are tuned across the multicritical points, $\lim_{\Gamma_c \to \Gamma_{mc}^{+}} F(k_0^{mc}, \Gamma_c) = - \lim_{\Gamma_c \to \Gamma_{mc}^{-}} F(k_0^{mc}, \Gamma_c) = \pm \infty$.

The curvature function on the critical lines $\Gamma_1 = \pm (\Gamma_0 + \Gamma_2)$ can be defined as

$$F(k, \Gamma_c) = \frac{\chi_x \partial_y \chi_y - \chi_y \partial_x \chi_x}{\chi_x^2 + \chi_y^2} = \frac{\Gamma_0^2 + 3\Gamma_2^2 \pm 4\Gamma_0 \Gamma_2 \cos k}{2(\Gamma_0^2 + \Gamma_2^2 \pm 2\Gamma_0 \Gamma_2 \cos k)} \quad (5)$$

The peak of this curvature function tends to diverge as the parameters approach $MC_{1,2}$ from both the sides and flips sign as $\Gamma_2 = \pm \Gamma_0$ is crossed, as shown in Fig.4(a). This evidently signals the topological transition across $MC_{1,2}$ at criticality. The diverging peak develops at $k_0^{mc}$ of the corresponding criticalities. Note that, for $MC_2$, one can observe the swapping of $k_0^{mc}$ values [25]. This interesting property is also observed and confirmed in the renormalization group and correlation function analysis.

Critical exponents: A proper choice of gauge allows
F(k, Γc) to take Ornstein-Zernike form around k_{mc}^0,
\[ F(k_{mc}^0 + \delta k, \Gamma_c) = \frac{F(k_{mc}^0, \Gamma_c)}{1 + \xi_2 \delta k^2}, \tag{6} \]
where \( \delta k = |k - k_{mc}^0|, \xi_2 \) is the characteristic length scale at criticality. As we approach multicritical point, one can also find the divergence in the characteristic length along with the curvature function, as \( F(k_{mc}^0, \Gamma_c) \propto |\Gamma_c - \Gamma_{mc}|^{-\gamma} \) and \( \xi_2 \propto |\Gamma_c - \Gamma_{mc}|^{-\nu}. \) The critical exponents \( \gamma \) and \( \nu \) define the universality class of the undergoing topological phase transition at criticality. We obtain the values of \( \gamma \) and \( \nu \) from a numerical fitting to the Ornstein-Zernike form. Fig. 4(b) shows the critical exponents on approaching from either sides (represented as \( \gamma_{+/-} \) and \( \nu_{+/-} \)) of the multicritical point MC1. The critical exponents \( \gamma_{+/-} = \gamma \approx 1 \) and \( \nu_{+/-} = \nu \approx 1 \) for both MC1 and MC2. 

The exponents can also be estimated analytically after expanding the \( \chi(k) \) around \( k_{mc}^0 \) up to third order. Then we recast the \( F(k, \Gamma_c) \) into Ornstein-Zernike form with \( F(k_{mc}^0, \Gamma_c) = A_0|\Gamma_c - \Gamma_{mc}|^{-1} \) and \( \xi_2 = A_0|\Gamma_c - \Gamma_{mc}|^{-1} \), where \( \delta \Gamma_c = |\Gamma_c - \Gamma_{mc}|. \) This implies \( \gamma = 1 \) and \( \nu = 1. \) For MC1, \( \delta \Gamma_c = (\Gamma_2 - \Gamma_0), A = (\Gamma_0 - 3\Gamma_2)/2 \) and for MC2, \( \delta \Gamma_c = 2(\Gamma_2 + \Gamma_0), A = (\Gamma_0 + 3\Gamma_2). \) Thus both the numerical and analytical methods are consistent with each other and yield the same values of critical exponents. These exponents obey a scaling law, imposed by the conservation of topological invariant, which reads \( \gamma = \nu \) for 1D systems [39]. Therefore, the scaling law is also valid for the topological transition at criticality. In addition to \( \gamma \) and \( \nu \), dynamical exponent \( z \) can also be calculated at MC1,2. It dictates the nature of the spectra near \( k_{mc}^0 \), i.e. \( E_k \propto k^z. \) The numerical evaluation of \( z \) yields \( z \approx 2 \) at MC1 and \( z \approx 1 \) at MC2 [25]. Therefore, topological transition at criticality occurs through multicritical points which belongs to distinct universality classes obtained from the set of three critical exponents \( (\gamma, \nu, z). \)

Moreover, correlation function in terms of Wannier-state representation [39] also identify the unique topological transition. The decay in the correlation function slow down as one approaches MC1,2 with the correlation length \( \xi_c \rightarrow \infty [25]. \) This quantity may be measured in higher dimensions [31, 40, 41].

**Scaling theory at criticality:** Based on the divergence of the curvature function at criticality, one can reframe the scaling theory [28] to incorporate the multicriticality and to efficiently identify the unique topological phase transition. This can be achieved by performing the deviation reduction mechanism at criticality where the deviation of the curvature function from its fixed point configuration can be reduced gradually. As a consequence of the same topology of the system at \( \Gamma_c \) and at fixed point \( \Gamma_c^d \), the curvature function can be written as \( F(k, \Gamma_c) = F_I(k, \Gamma_c^d) + F_D(k, \Gamma_c^d), \) where \( F_I(k, \Gamma_c^d) \) is the curvature function at fixed point and \( F_D(k, \Gamma_c^d) \) is deviation from the fixed point. For a given \( \Gamma_c \), one can find a new \( \Gamma_c' \) which satisfies \( F(k_{mc}^0, \Gamma_c') = F(k_{mc}^0 + \delta k, \Gamma_c). \) Iteratively performing this scaling procedure and solving \( \Gamma_c \) for every \( \delta k \), deviation of curvature function decreases and eventually \( F(k, \Gamma_c) \rightarrow F_I(k, \Gamma_c^d). \)

One can obtain a renormalization group (RG) equation for the coupling parameters using the scaling parameter \( \delta k^2 = dl \) and \( \Gamma_c' - \Gamma_c = d\Gamma_c \) as [25]
\[ \frac{d\Gamma_c}{dl} = \frac{1}{2} \partial^2 F(k, \Gamma_c)[k - k_{mc}^0] \quad \text{and} \quad \frac{d\Gamma_c}{dl} = -\frac{\Gamma_c(\Gamma_0 + \Gamma_2)}{2(\Gamma_0 + \Gamma_2)} \quad \left( \frac{d\Gamma_c}{dl} \right) \rightarrow 0, \]
\[ \frac{d\Gamma_c}{dl} \rightarrow \infty. \]

The RG flow distinguishes between distinct phases and clearly captures the topological phase transitions at criticality. The multicritical points and fixed points are easily identified by analyzing the RG flow lines. At the (i) multicritical point: \( \left| \frac{d\Gamma_c}{dl} \right| \rightarrow \infty \), flow directs away and (ii) stable (unstable) fixed point: \( \left| \frac{d\Gamma_c}{dl} \right| \rightarrow 0 \), flow directs into (away). We obtain the RG equations for MC1 at both the critical lines as
\[ \frac{d\Gamma_c}{dl} = \frac{\Gamma_0(\Gamma_0 + \Gamma_2)}{2(\Gamma_0 - \Gamma_2)} \quad \text{and} \quad \frac{d\Gamma_c}{dl} = -\frac{\Gamma_2(\Gamma_0 + \Gamma_2)}{2(\Gamma_0 + \Gamma_2)} \]

Therefore, in the RG flow diagram \( \Gamma_2 = \Gamma_0 \) (MC1) manifests as a multicritical line. Also we surprisingly obtain \( \Gamma_2 = -\Gamma_0 \) (MC2) as a line consisting of the unstable fixed points, as shown in Fig. 5(a). In order to realize the topological transition at criticality through MC2 one has to consider the swapping behavior of \( k_{mc}^0 \). This procedure yields the RG equations of the form
\[ \frac{d\Gamma_c}{dl} = \frac{\Gamma_0(\Gamma_0 - \Gamma_2)}{2(\Gamma_0 + \Gamma_2)} \quad \text{and} \quad \frac{d\Gamma_c}{dl} = -\frac{\Gamma_2(\Gamma_0 - \Gamma_2)}{2(\Gamma_0 + \Gamma_2)} \]

In this case the multicritical line appear at MC2 (\( \Gamma_2 = -\Gamma_0 \)) and unstable fixed points appear on the multicritical line MC1 (\( \Gamma_2 = \Gamma_0 \)). For details see Ref. [25].

![FIG. 5: RG flow diagrams. The multicritical lines are represented in solid lines and fixed lines are represented in dashed lines. (a) For MC1 where MC1 and MC2 appear as multicritical and unstable fixed lines respectively. (b) For MC2, where MC2 and MC1 appear as multicritical and unstable fixed lines respectively. The RG flow lines clearly demonstrates the topological phase transition at criticality.](image-url)

**Experimental observability:** The generic model presented in Eq.1 can be simulated using superconducting circuit with a single qubit driven by the tunable external microwave pulses as it has been done for the Kitaev version of the Eq. 1 [42, 43]. In Ref. [42], the energy
dispersion and the winding number have been measured by controlling the Rabi frequency of the pulses and by observing the evolution of the qubit respectively. In such experiments, by creating suitable components of the microwave field one can realize the parameters satisfying $\Gamma_1 = \pm(\Gamma_0 + \Gamma_2)$ at criticality. Hence the nature of energy dispersion which distinguishes between $MC_{1,2}$ and critical lines in general can be obtained in the experiments. Due to the straightforward mapping between the state of the qubit and the orientation of the pseudospin vector [42], we believe that the measurement on the state of the qubit yields results similar to the one depicted in Fig.3(a,b) [25].

Moreover, the ultracold atoms with a full control over the coupling parameters and onsite energy of the topological Hamiltonian provides a promising platform [44–48] and may be useful to test our results by tuning the parameters to the critical relation $\Gamma_1 = \pm(\Gamma_0 + \Gamma_2)$. Realization of Kitaev model with controlled NN and NNN couplings [46, 49, 50] may provide evidences of zero modes at criticality [25].

**Conclusion:** In this letter, we have shown that the extended models of topological insulators and superconductors have a generic property of exhibiting topological phase transition between distinct critical phases through multicritical points. This intriguing transition has been characterized by reframing the conventional theoretical frameworks to work at criticality.

Our proposed framework in general can be applied to the driven systems and in the systems with higher dimension. A unique advantage of having topological non-trivial criticalities is that the quantum information remains robust upon tuning the system towards it. By identifying the multicritical points one can choose a proper criticality to tune into and avoid the decoherence due to bulk gap closing and opening. Inclusion of disorder and interaction into the framework could eventually lead to the study of topological quantum criticality in real materials.

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Supplementary Material for “Topological phase transition at quantum criticality”

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I. PHYSICAL RELEVANCE OF MODEL HAMILTONIAN

The model considered in Eq.1 of main text is a generic two band model for spinless fermions in 1D lattice with nearest neighbor (NN) and next nearest neighbor (NNN) hopping amplitudes of electrons. It maps into extended Su–Schrieffer–Heeger (SSH) [1, 2] and Kitaev chains [3, 4] in momentum space, which are the simplest 1D models for topological insulators and superconductors respectively. The tight-binding Hamiltonians can be written as

\[ H_{SSH} = \alpha_0 \sum_i c_{i,a}^\dagger c_{i,b} + \alpha_1 \sum_{\langle ij \rangle} (c_{i,a}^\dagger c_{j,b} + h.c) + \alpha_2 \sum_{\langle\langle ij \rangle\rangle} (c_{i,a}^\dagger c_{j,b} + h.c), \]

\[ H_{Kitaev} = \beta_0 \sum_i 2(c_{i,a}^\dagger c_{i-1}^\dagger - 1) - \beta_1 \sum_{\langle ij \rangle} (c_{i}^\dagger c_{j} + c_{i}^\dagger c_{j}^\dagger + h.c) - \beta_2 \sum_{\langle\langle ij \rangle\rangle} (c_{i}^\dagger c_{j} + c_{i}^\dagger c_{j}^\dagger + h.c), \]

where \( c_{i,j}^\dagger \) and \( c_{i,j} \) are the fermionic creation and annihilation operators. In \( H_{SSH} \), the subscripts \( a,b \) denote the sub-lattices, with onsite potential \( \alpha_0 \) and NN (NNN) hopping amplitude \( \alpha_{1(2)} \). In \( H_{Kitaev} \), \( \beta_0 \) is onsite potential and \( \beta_{1(2)} \) is NN (NNN) pairing and hopping amplitudes.

The Hamiltonians can be readily diagonalised by Fourier transformation to obtain a generalized Bloch Hamiltonian in the basis of spinor \( \psi_k \)

\[ H_{SSH} = \sum_k \langle \psi_k | H_{SSH} | \psi_k \rangle \text{ with } \psi_k = (c_{a,k}^\dagger c_{b,k})^T \]

The Hamiltonian \( H_{SSH} = \chi_x \sigma_x + \chi_y \sigma_y \), where \( \chi_x = \alpha_0 + \alpha_1 \cos k + \alpha_2 \cos 2k \) and \( \chi_y = \alpha_1 \sin k + \alpha_2 \sin 2k \). The excitation spectra can be obtained as \( E_k = \pm \sqrt{\chi_x^2 + \chi_y^2} \). The gap closing points (i.e., \( E_k = 0 \)) for a specific \( k_0 \) defines critical surfaces or phase boundaries which separate topologically distinct gapped phases. The gapless edge excitations of these gapped phases are quantified in terms of winding number \( w \), which counts the number of edge modes present in the corresponding gapped phases. There are three critical surfaces for extended SSH model. Two of them are with high symmetry nature (i.e., \( k_0 = k_\pi \)), \( \alpha_1 = -(\alpha_0 + \alpha_2) \) and \( \alpha_2 = (\alpha_0 + \alpha_2) \) respectively for \( k_0 = 0 \) and \( \pi \). One with non-high symmetry nature (i.e., \( k_0 \neq k_\pi \)), \( \alpha_1 = \alpha_2 \) for \( k_0 = \cos^{-1}(-\alpha_1/2\alpha_2) \). Without loss of generality, we assume \( \alpha_0 = 1 \), hence critical surfaces and multicritical lines will be critical lines and multicritical points respectively on the \( \alpha_1 \) plane, as shown in Fig.1(a). The three critical lines distinguish the gapped phases with invariant number \( w = 0, 1, 2 \). There are three multicritical points named \( MC_1 \) and \( MC_2 \), with distinct nature, at which the critical lines meet.

The edge mode remains localized at the criticalities (critical lines) between the topological non-trivial gapped phases \((w = 1 \text{ and } w = 2)\), which give rise to the topological characteristics to the criticality. The same does not occur at the criticality between a trivial and non-trivial gapped phases \((w = 0 \text{ and } w = 1)\). This results in the criticality to get separated into two distinct critical phases with trivial and non-trivial topological properties. The multicritical points \( MC_{1,2} \), with quadratic (i.e., \( E_k \propto k^2 \)) and linear dispersions (i.e., \( E_k \propto k \)) respectively, facilitates the topological transition at criticality between trivial and non-trivial critical phases.

Similar qualitative behavior can also be observed in the extended Kitaev model due to the striking similarity in the phase diagram with SSH model. For the Kitaev model one can obtain

\[ H_{Kitaev} = \sum_k \langle \psi_k | H_{Kitaev} | \psi_k \rangle \text{ with } \psi_k = (c_k^\dagger c_{-k})^T \]

The Hamiltonian \( H_{Kitaev} = \chi_x \sigma_x + \chi_y \sigma_y \), where \( \chi_x = 2\beta_0 - 2\beta_1 \cos k - 2\beta_2 \cos 2k \) and \( \chi_y = 2\beta_1 \sin k + 2\beta_2 \sin 2k \), after a rotation along \( \sigma_y \). The gap closing critical surfaces for this case are \( \beta_1 = -(\beta_0 - \beta_2) \), \( \beta_1 = (\beta_0 - \beta_2) \) and \( \beta_0 = -\beta_2 \) respectively for \( k_0 = 0 \), \( k_0 = \pi \) and \( k_0 = \cos^{-1}(-\beta_1/2\beta_2) \). These phase boundaries separate the gapped phases with invariant numbers \( w = 0, 1, 2 \) as shown in Fig.1(b) (for \( \beta_0 = 0.5 \)). Localized edge modes living at the
criticalities between the non-trivial topological gapped phases can be observed here as well which defines trivial and non-trivial critical phases with distinct topological properties. The multicritical points \( MC_{1,2} \) mediate the topological transition at criticality between critical phases with distinct topological nature and share the same properties as in the case of SSH model.

To study the unusual topological transition at criticalities we consider a generic model which essentially summarize both SSH and Kitaev model, thereby giving one platform to study both topological insulator and superconductor models in one dimension. We define a generalized Bloch Hamiltonian for two band model by setting \( \alpha_0 = 2\beta_0 = \Gamma_0 \), \( \alpha_1 = -2\beta_1 = \Gamma_1 \), and \( \alpha_2 = -2\beta_2 = \Gamma_2 \). This model captures the physics of both SSH and Kitaev models, especially the phenomenon of multicriticality and the corresponding topological transition. The energy dispersion of the generalized model can be written to be

\[
E_k = \pm \sqrt{\chi_x^2 + \chi_y^2} = \pm \sqrt{(\Gamma_0 + \Gamma_1 \cos k + \Gamma_2 \cos 2k)^2 + (\Gamma_1 \sin k + \Gamma_2 \sin 2k)^2}
\]  

(1.5)

The model can support three distinct gapped phases \( (w = 0, 1, 2) \) distinguished by the number of edge modes they possess. These phases can be pinned with the winding numbers \( w = 0, 1, 2 \), which quantify the edge modes. The model undergoes transition between these phases with necessarily involving the gap closing, \( E_k = 0 \), at the phase boundaries. The criticalities, where the bulk gap closes, occurs for the momentum \( k_0 = 0, \pm \pi, \cos^{-1}(\Gamma_1/2\Gamma_2) \), which respectively gives the critical surfaces \( \Gamma_1 = -(\Gamma_0 + \Gamma_2) \), \( \Gamma_1 = (\Gamma_0 + \Gamma_2) \) and \( \Gamma_0 = \Gamma_2 \). The model possesses three multicritical lines \( (MC_{1,2}) \), at which two critical surfaces intersect. One of them shows linear dispersion around the gap closing point and the other two are identified with quadratic dispersion and share identical properties.

**Model at criticality:** To obtain the model at criticality we use the critical surface relation which modifies the components into \( \chi_x = \Gamma_0(1 + \cos k) + \Gamma_2(\cos 2k + \cos k) \), and \( \chi_y = \Gamma_2(\sin 2k + \sin k) + \Gamma_0 \sin k \) for \( \Gamma_1 = (\Gamma_0 + \Gamma_2) \) and \( \chi_x = \Gamma_0(1 - \cos k) + \Gamma_2(\cos 2k - \cos k) \), and \( \chi_y = \Gamma_2(\sin 2k - \sin k) - \Gamma_0 \sin k \) for \( \Gamma_1 = -(\Gamma_0 + \Gamma_2) \). The possible topological trivial and non-trivial critical phases are separated by the phase boundaries at which the dispersion \( E_k^c = \pm \sqrt{\chi_x^2 + \chi_y^2} = 0 \). These phase boundaries turns out to be the multicriticalities \( \Gamma_2 = \Gamma_0 \) \( (MC_1) \) and \( \Gamma_2 = -\Gamma_0 \) \( (MC_2) \). They can be obtained for the following \( k_0^{mc} \).
For $\Gamma_2 = \Gamma_0$ ($MC_1$):

\[
k_0^{mc} = \cos^{-1}\left(-\frac{\Gamma_2 + \Gamma_0}{2\Gamma_2}\right) \quad \text{at the criticality} \quad \Gamma_1 = (\Gamma_0 + \Gamma_2) \tag{I.6}
\]

\[
k_0^{mc} = \cos^{-1}\left(\frac{\Gamma_2 + \Gamma_0}{2\Gamma_2}\right) \quad \text{at the criticality} \quad \Gamma_1 = -(\Gamma_0 + \Gamma_2) \tag{I.7}
\]

For $\Gamma_2 = -\Gamma_0$ ($MC_2$):

\[
k_0^{mc} = 0 \quad \text{at the criticality} \quad \Gamma_1 = (\Gamma_0 + \Gamma_2) \tag{I.8}
\]

\[
k_0^{mc} = \pi \quad \text{at the criticality} \quad \Gamma_1 = -(\Gamma_0 + \Gamma_2) \tag{I.9}
\]

Interestingly, one can observe swapping of $k_0^{mc}$ at $MC_2$, i.e. $k_0^{mc} = 0$ for $\Gamma_1 = (\Gamma_0 + \Gamma_2)$ which was obtained for $k_0 = \pi$ and $k_0^{mc} = \pi$ for $\Gamma_1 = -(\Gamma_0 + \Gamma_2)$ which was obtained for $k_0 = 0$. This property is observed at $MC_2$ since this multicriticality emerge as a result of intersection of two criticalities. This swapping of $k_0^{mc}$ is observed in all the analysis carried out in this work.

**II. NUMERICAL ANALYSIS OF EDGE MODES AND TOPOLOGICAL TRANSITION AT CRITICALITY**

At first, we discuss the behavior of parameter space of the spin-vectors to identify the trivial and non-trivial criticalities. The characteristic feature of the parameter space curve at criticality is that it intersects the origin while tracing closed curve. Non-trivial critical phases can be identified with the emergence of secondary loops in the parameter space curve. One among them intersects the origin while the other encircle it indicating a finite winding number or edge modes at criticality. In trivial critical phase parameter space curve are always seen to be intersecting the origin and represent no encircling loops, thus no edge modes at criticality. To demonstrate this explicitly we have considered a schematic representation of criticality with multicritical points $MC_{1,2}$ in Fig.2. Among the parameter space curves shown the Figures A.1 and A.3 represents non-trivial critical phases and Figure A.2 represents trivial critical phase. Fixing $\Gamma_0 = 1$, Figures A.1, A.2 and A.3 are respectively for $\Gamma_2 = 2, 0, -2$.

The edge modes at criticalities can also be identified numerically using probability and eigenvalue distributions under open boundary condition. The BdG Hamiltonian of the model can be written as

\[
H_{BdG} = \begin{pmatrix}
\chi_x & \chi_y \\
\chi_y & -\chi_x
\end{pmatrix}
\]

where the submatrices of the Hamiltonian are

\[
\chi_{x,(i,j)}(k) = -\Gamma_0 \delta_{j,i} - \frac{\Gamma_1}{2} (\delta_{j,i+1} + \delta_{j,i-1}) - \frac{\Gamma_2}{2} (\delta_{j,i+2} + \delta_{j,i-2})
\]

\[
\chi_{y,(i,j)}(k) = -\frac{\Gamma_1}{2} (\delta_{j,i+1} - \delta_{j,i-1}) - \frac{\Gamma_2}{2} (\delta_{j,i+2} - \delta_{j,i-2})
\]

Numerical diagonalization of the BdG Hamiltonian reveals that for the non-trivial phases the probability of wave function significantly distributes at the edge of the finite open chain representing the stable localized edge modes. In case of the trivial phase, probability distribution can be found in the bulk than at the edge, as a consequence of delocalization of the edge modes into the bulk states. The localization and de-localization property of the edge modes can be found to switch across the multicritical points $MC_{1,2}$ which thus differentiates between trivial and non-trivial critical phases. Correspondingly, the eigenvalue distribution shows two of the eigenvalues trapped at the zero energy even if there is no bulk gap in the non-trivial phase. Trivial phase is identified with no eigenvalues explicitly living at zero energy among the continuous distribution of the eigenvalues. For the model Hamiltonian these behaviors of probability and energy eigenvalue distributions are shown in Fig.2.B.1, B.2, B.3, C.1, C.2 and C.3 for the system size $N = 40$. Edge modes are observed at the criticalities with the parameter values $\Gamma_2 = \pm 2$ and trivial phase is shown with $\Gamma_2 = 0.5$. This clearly shows that the stable localized edge modes living at criticalities and the trivial and non-trivial critical phases separated by the multicritical points. To further understand the nature of topological transition at the multicriticalities, one can study the discontinuity in the derivatives of ground state energy density $E_0$ with respect to $\Gamma_2$.

\[
E_0 = -\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\partial^2}{\partial \Gamma_2^2} \left(\sqrt{\chi_x^2 + \chi_y^2}\right) dk \tag{II.3}
\]
FIG. 2: Numerical results for edge mode and topological phase transition at criticality. Criticalities $\Gamma_1 = \pm (\Gamma_0 + \Gamma_2)$ is shown schematically with two multicritical points $MC_{1,2}$, shown as Magenta and Purple dots respectively. Fixing $\Gamma_0 = 1$, the multicritical points $MC_{1,2}$ are obtained respectively at $\Gamma_2 = \pm 1$. Therefore, critical phases $\Gamma_2 > 1$ and $\Gamma_2 < -1$ are non-trivial with stable localized edge modes and $-1 < \Gamma_2 < 1$ is trivial. The topological transition between these critical phases occur at the multicritical points $\Gamma_2 = \pm 1$. Figures (A.1,A.2,A.3) parameter space curves, (B.1,B.2,B.3) probability distribution, (C.1,C.2,C.3) eigenvalue distribution, are obtained for the system size $N = 40$ for the parameter space $\Gamma_2 = 2$, $\Gamma_2 = 0.5$ and $\Gamma_2 = -2$ respectively. (D.1,D.3) Second and (D.2,D.4) third order derivatives of ground state energy density with respect to the parameter $\Gamma_2$, around multicritical points $MC_{1,2}$.

FIG. 3: Energy spectra at criticality with respect to the parameter $\Gamma_2$, for system size $N = 50$. The multicritical points are represented as black dots. Zero energy states are present non-trivial critical phases i.e., for $\Gamma_2 < -1$ and $\Gamma_2 > 1$ with $\Gamma_0 = 1$. Trivial critical phase i.e., $-1 < \Gamma_2 < 1$ can be identified with the absence of zero energy states.

As shown in Fig.2, D.1, D.2, D.3 and D.4 the second derivative features a cusp at the multicritical points $MC_{1,2}$ respectively. Corresponding discontinuity can be observed in the third derivative at the parameter values $\Gamma_2 = \pm 1$.

The topological transition among the non-trivial and trivial critical phases can be identified in the energy spectrum with the system parameters. The presence and absence of zero energy states dictates the triviality and non-triviality
with respect to the system parameter, as shown in Fig.3. The spectrum is obtained for \( N = 50 \). Note that, there is no bulk gap in the spectrum since the system is at criticality. The zero energy states represents localized stable edge modes living at the critical phases for \( \Gamma_2 > 1 \) and \( \Gamma_2 < -1 \) (with \( \Gamma_0 = 1 \)). The transition among the trivial and non-trivial phases can be seen at the multicritical points. On approaching to trivial phase from non-trivial phase, the multicritical points are located at \( \Gamma_2 = 1 \) and \( \Gamma_2 = -1 \), at which the zero energy states delocalize into the bulk states. Therefore, this analysis clearly depicts the picture of topological phase transition at criticality between distinct critical phases.

### III. BOUND STATE SOLUTION OF DIRAC EQUATION

The presence of edge modes in topological insulators and superconductors can be understood from the bound state solution of Dirac equation [6, 7]. Jackiw and Rebbi obtained soliton solution at the interface \((x = 0)\), away from which the wavefunction distribution decays exponentially [8]. A general solution of zero energy is obtained from the mass distribution of the Dirac Hamiltonian which changes from negative to positive across the interface [9].

The model Hamiltonian can be recasted in the form of Dirac Hamiltonian in 1D, which represents the topological insulator and superconductor models. The Dirac Hamiltonian of the model can be obtained by the second order expansion of \( \chi \) around the gap closing momenta \( k_0 \)

\[
\mathcal{H}(k) \approx (m - \epsilon_1 k^2)\sigma_x + \epsilon_2 k\sigma_y. \tag{III.1}
\]

For \( k_0 = 0 \) we have \( m = (\Gamma_0 + \Gamma_1 + \Gamma_2), \epsilon_1 = (\Gamma_1 + 4\Gamma_2)/2 \) and \( \epsilon_2 = (\Gamma_1 + 2\Gamma_2). \) For \( k_0 = \pi, m = (\Gamma_0 - \Gamma_1 + \Gamma_2), \epsilon_1 = (4\Gamma_2 - \Gamma_1)/2 \) and \( \epsilon_2 = (2\Gamma_2 - \Gamma_1). \) The continuum version of the model reads (with \( \hbar = 1 \))

\[
\mathcal{H}(-i\partial_x) \approx (m + \epsilon_1 \partial_x^2)\sigma_x + (-i\epsilon_2 \partial_x)\sigma_y. \tag{III.2}
\]

To obtain zero energy solution \( \mathcal{H}\psi(x) = 0 \), we multiply \( \psi(y) \) from right-hand side. This implies the wavefunction \( \psi(x) = \rho_\eta\phi(x) \), is an eigenstate of \( \sigma_z\chi_\eta = \eta\chi_\eta \). The resulting second order differential equation can be written as

\[
\partial_x^2 \phi(x) = \frac{-\epsilon_2 \partial_x + \eta m}{\eta \epsilon_1} \phi(x). \tag{III.3}
\]

We set the trial wavefunction \( \phi(x) \propto e^{x/\xi} \) to obtain the secular equation, which yields the inverse of the decay length

\[
\xi^+ \approx -\frac{m}{|\epsilon_2|}. \tag{III.4}
\]

The decay length is positive if \( m < 0 \) which identifies the gapped topological non-trivial phase with \( w = 1 \). Similarly, topological phase with \( w = 2 \) can also be identified by using the ansatz \( \phi(x) \propto e^{-x/\xi}, \) which under the condition \( m\epsilon_1 > 0 \) yields the decay length \( \xi_- \approx |\epsilon_2|/m \). The decay length is positive if \( m > 0 \). Even though, the topological trivial phase with \( w = 0 \) is also identified with \( m > 0 \), it does not host any zero energy solution since the region \( m > 0 \) for trivial phase satisfies the relation \( m\epsilon_1 < 0 \). If the parameter \( m\epsilon_1 < 0 \), spin distribution of the ground state does not show anti-parallel spin orientation in momentum space [10]. If \( m\epsilon_1 > 0 \) is satisfied, spin orientation align in the opposite directions with the increasing momentum. Thus the gapped phases \( w = 2 \) and \( w = 0 \) are identified with the condition \( m\epsilon_1 \lesssim 0 \) respectively. The wave-function for zero energy solution can be derived to be

\[
\psi(x) \propto \psi(0)\left(e^{x/\xi^+} - e^{x/\xi^-}\right), \tag{III.5}
\]

up to normalization constant. The solution is exponentially localized near the boundary, as shown in Fig.4 for different gapped phases.

To work at criticality we make use of the near-critical approach and write Dirac Hamiltonian at criticality which naturally fix the interface at the multicritical point. We expect a bound state solution at the multicritical point which distinguish topological trivial and non-trivial phases at criticality. Dirac Hamiltonian at criticality can be written as

\[
\mathcal{H}(k) \approx \epsilon_1 k^2\sigma_x + \epsilon_2 k\sigma_y. \tag{III.6}
\]

We look for zero energy solution in real space

\[
\mathcal{H}(-i\partial_x)\psi(x) = [(-\epsilon_1 \partial_x^2)\sigma_x + (-i\epsilon_2 \partial_x)\sigma_y]\psi(x) = 0. \tag{III.7}
\]
Multiplying by $\sigma_y$ from the right-hand side, one can identify the wave-function is an eigenstate of $\sigma_z \psi(x \lesssim 0) = \eta \psi(x \lesssim 0)$. For $x > 0$, we use the ansatz $\psi(x) \propto e^{-x/\xi_{x>0}}$:

$$
(\eta \epsilon_1 \xi_{x>0}^{-2}) e^{-x/\xi_{x>0}} + (-\epsilon_2 \xi_{x>0}^{-1}) e^{-x/\xi_{x>0}} = 0.
$$

(III.8)

Therefore, the decay length of an edge mode can be obtained as

$$
\xi_{x>0} = \frac{\eta \epsilon_1}{\epsilon_2}.
$$

(III.9)

Similarly, for $x < 0$ we use the ansatz $\psi(x) \propto e^{x/\xi_{x<0}}$, which yields the decay length

$$
\xi_{x<0} = \frac{-\eta \epsilon_1}{\epsilon_2}.
$$

(III.10)

where $\epsilon_1 = (\Gamma_0 - 3\Gamma_2)/2$, $\epsilon_2 = (\Gamma_2 - \Gamma_0)$ and $\eta = \text{sign}(\epsilon_1)$. Note that, the term $\epsilon_2$ is the gap term, which mimic the role of mass, at criticality. The gap term is positive for $x > 0$ and negative for $x < 0$. Therefore, the interface of a topological trivial and non-trivial critical phases indeed represent the multicritical point since the term $\epsilon_2 = 0$ only when $\Gamma_2 = \Gamma_0$, which is a multicritical point $MC_1$. The solution can be obtained as

$$
\psi(x > 0) \propto \begin{pmatrix} \eta \\ 0 \end{pmatrix} e^{-x/\xi_{x>0}}, \quad \text{and} \quad \psi(x < 0) \propto \begin{pmatrix} 0 \\ \eta \end{pmatrix} e^{x/\xi_{x<0}},
$$

(III.11)

The solution distribution is dominant at the interface and decay exponentially as $x \to \pm \infty$, which provides a convincing evidence of the topological distinct nature of the criticalities separated by the multicritical point $MC_1$.

To capture the bound state solution at the interface set by the multicritical point $MC_2$, one has to consider the swapping of $k_{0mc}^{\infty}$. As a result of intersection of critical lines at $MC_2$, the critical properties swap between the $k_0 = 0$ and $\pi$. The Dirac Hamiltonian in this case can be obtained as

$$
\mathcal{H}(-i\partial_x) = \epsilon_1 \sigma_x + (-i\epsilon_2 \partial_x) \sigma_y
$$

(III.12)

Therefore, the wavefunction is an eigenstate of $\sigma_z$. The zero energy solution for $x > 0$ is obtained as $\xi_{x>0} = \eta \epsilon_2/\epsilon_1$, whilst for $x < 0$ is $\xi_{x<0} = \eta \epsilon_2/\epsilon_1$, where $\epsilon_1 = 2(\Gamma_0 + \Gamma_2)$, $\epsilon_2 = (\Gamma_0 + 3\Gamma_2)$ and $\eta = \text{sign}(\epsilon_2)$. In this case the gap term is $\epsilon_1$ which vanish at the multicritical point $MC_2$, i.e, at $\Gamma_2 = -\Gamma_0$. Therefore, there exists a bound state solution at the interface with exponential decay as $x \to \pm \infty$.

The identification of the topological trivial and non-trivial phases at criticality is not possible only from the sign of gap term. However, higher order correction to the gap term enables the identification of topological non-trivial phase at criticality.

IV. TOPOLOGICAL INVARIANT NUMBER

The edge excitations of the gapped phases can be quantified in terms of winding number ($w$) $[11]$

$$
w = \frac{1}{2\pi} \int_{BZ} F(k, \Gamma) dk,
$$

(IV.1)
where $F(k, \Gamma) = i \langle u_k | \partial_{k^*} | u_k \rangle$ is the Berry connection or curvature function of Bloch wavefunction $\psi_k(r) = u_k(r)e^{ikr}$.

The conventional definition of winding number to explain the bulk-boundary correspondence, fails at criticalities. This is due to the non-analyticity of the curvature function (integrand in Eq. IV.1) at criticalities. This constraint is naturally avoided in the near-critical approach adopted here, which allows one to calculate the winding number in its usual integral form even at criticality.

$$w_c = \frac{1}{2\pi} \lim_{\delta \to 0} \oint_{|k-k_0|>\delta} F(k, \Gamma_c)dk$$  \hspace{1cm} (IV.2)

However, it yields fractional values, as shown in Fig.5(a), which does not account correctly the number of edge modes present at criticalities.

Alternatively, one can refer to the auxiliary space and differentiate between NN and NNN loops and consider only one among them which gives integer contribution and accounts for the edge modes at criticality [12]. However, this method is dependent on the choice of parameter space and also the auxiliary space loops gets complicated as the NN couplings increases [13].

Therefore, we have proposed that the winding number at criticality can be effectively approximated to its integer part to quantify the number of edge modes at criticality, i.e. $w_c \approx w_c^I$.

$$w_c^I = \frac{1}{2\pi} \lim_{\delta \to 0} \oint_{|k-k_0|>\delta} F(kI, \Gamma_c)dk_l.$$  \hspace{1cm} (IV.3)

The number of edge modes can now be counted correctly using $w_c^I$ and the topological transition between them can be observed at multicritical points $MC_{1,2}$. The integer winding number $w_c^I$ can be found consistent with winding number defined using number of zeros and order of the poles, i.e, $w = N_z - N_p$ in Ref.[14]. Writing the fermionic creation and annihilation operators in terms of Majorana operators followed by Fourier transform and substitution $e^{i\beta} = \zeta$, where $\zeta$ is a complex number and $e^{i\beta}$ goes around the unit circle in the complex plane as $k$ varies over the Brillouin zone, yields the complex function $f(\zeta)$ living on the unit circle in the complex plane

$$f(\zeta) = \sum_{\mu=-\infty}^{\infty} t_\mu \zeta^\mu.$$  \hspace{1cm} (IV.4)

For extended Kitaev model it reads $f(\zeta) = \sum_{\mu=0}^{2} t_\mu \zeta^\mu$ (with no poles) where $t_{0,1,2}$ are respectively $-\beta_0, \beta_1, \beta_2$. Using the mapping $2\beta_0 = \Gamma_0, -2\beta_1 = \Gamma_1$, and $-2\beta_2 = \Gamma_2$ one can write the complex function for the generic model

$$f(\zeta) = \frac{\Gamma_0}{2} - \frac{\Gamma_1}{2} \zeta - \frac{\Gamma_2}{2} \zeta^2.$$  \hspace{1cm} (IV.5)

The solutions are $\zeta_{\pm} = (\Gamma_1 \pm \sqrt{\Gamma_1^2 - 4\Gamma_0\Gamma_2})/2\Gamma_2$. To characterize the topological trivial and non-trivial critical phases we write the solution at criticalities. For $\Gamma_1 = -(\Gamma_0 + \Gamma_2)$ we get $\zeta_1 = 1, \zeta_2 = \Gamma_0/\Gamma_2$ and for $\Gamma_1 = (\Gamma_0 + \Gamma_2)$ we get $\zeta_1 = -1, \zeta_2 = -\Gamma_0/\Gamma_2$. It is evident that one of the zero lie on the unit circle since the system is critical and other
zero falls inside (outside) the unit circle for topological non-trivial (trivial) critical phase, as shown in Fig.5. Winding number is determined by the number of zeros falls inside the unit circle, whose value can be found in consistent with \(w_c^f\). For non-trivial critical phases as shown in Fig.5(b,c,d,e) \(w = w_c^f = 1\) and for trivial critical phases as shown in Fig.5(f,g) \(w = w_c^e = 0\).

V. CURVATURE FUNCTION

Topological phase transition can be induced by changing the underlying topology of the system upon tuning the parameters \(\Gamma\) appropriately. The information of the topological property of the system is embedded in the curvature function \(F(k, \Gamma)\) defined at momentum \(k\) [15–26]. The topological quantum phase transition can be identified from the quantized jump of topological invariant number as the parameter tuned across the critical point \(\Gamma_c\). As the system approaches critical point to undergo topological phase transition i.e, \(\Gamma \to \Gamma_c\), curvature function diverges at \(k_0\), with the diverging curve satisfying \(F(k_0 + \delta k, \Gamma) = F(k_0 - \delta k, \Gamma)\). The sign of the diverging peak flips across the critical point as

\[
\lim_{r \to r'_c} F(k_0, \Gamma) = -\lim_{r \to r'_c} F(k_0, \Gamma) = \pm \infty. \tag{V.1}
\]

Interestingly, even at criticality, the qualitative behavior of the curvature function remains the same with the fact that, now the critical point is a multicriticality which governs the topological transition between critical phases. As one tunes the parameters at criticality \(\Gamma_c\) towards a multicritical point \(\Gamma_{mc}\), the curvature function diverges at \(k_{0 mc}\) with the symmetric nature \(F(k_{0 mc} + \delta k, \Gamma_c) = F(k_{0 mc} - \delta k, \Gamma_c)\), as shown in Fig.6a.

Topological transition is signalled as the sign of the diverging peak flips if the parameters tuned across the multicritical point.

\[
\lim_{\Gamma_c \to \Gamma_{mc}^+} F(k_{0 mc}, \Gamma_c) = -\lim_{\Gamma_c \to \Gamma_{mc}^-} F(k_{0 mc}, \Gamma_c) = \pm \infty. \tag{V.2}
\]

This is the characteristic feature of topological transition at criticality through both the multicritical points \(MC_{1,2}\). The curvature function of the generic model at criticality can be written using the critical line relations of the parameters. The pseudo-spin vectors on the two critical lines, \(\Gamma_1 = (\pm (\Gamma_0 + \Gamma_2))\), of the model are \(\chi_x(k) = \Gamma_0(1 \pm \cos k) + \Gamma_2(\cos 2k \pm \cos k)\), and \(\chi_y(k) = \Gamma_2(\sin 2k \pm \sin k) \pm \Gamma_0 \sin k\). This defines curvature function on the critical lines \(F(k, \Gamma_c) = F(k, \Gamma_{\Gamma_1 = \pm (\Gamma_0 + \Gamma_2)})\),

\[
F(k, \Gamma_{\Gamma_1 = \pm (\Gamma_0 + \Gamma_2)}) = \frac{\chi_x \partial k \chi_y - \chi_y \partial k \chi_x}{\chi_x^2 + \chi_y^2} = \frac{\Gamma_2^2 + 3 \Gamma_0^2 + 4 \Gamma_0 \Gamma_2 \cos k}{2(\Gamma_0^2 + \Gamma_2^2 \pm 2 \Gamma_0 \Gamma_2 \cos k)}. \tag{V.3}
\]

The property in Eq.V.2 can be observed to be obeyed by \(F(k, \Gamma_{\Gamma_1 = \pm (\Gamma_0 + \Gamma_2)})\) as shown in the Fig.6. It shows the critical behavior of curvature function around the multicritical points \(MC_{1,2}\), which distinguish between distinct critical phases. The peak of the curvature function tends to diverge as the parameters approach \(MC_{1,2}\) from both sides at criticality. Both the criticalities exhibit the universal nature of curvature function around the multicritical points.

The scenario around \(MC_1\) on the critical line \(\Gamma_1 = -(\Gamma_0 + \Gamma_2)\) shows the divergence in curvature function at the \(k_{0 mc}^0 = 0\), as shown in Fig.6b. As the parameter \(\Gamma_2\) is tuned towards its multicritical value (i.e \(MC_1\)) on both sides, the diverging peak of curvature function increases leading to a complete divergence at \(MC_1\) and flips sign as the critical value is crossed. This signals the topological transition across \(MC_1\) at criticality. Similar behavior of curvature function can be observed around \(MC_1\) on the critical line \(\Gamma_1 = (\Gamma_0 + \Gamma_2)\), for which the divergence occurs at \(k_{0 mc}^1 = \pi\), as shown in Fig.6c.

The nature of curvature function around \(MC_2\) at both the criticalities share the same property of divergence and flipping of sign as shown in Fig.6d and 6e. Note that, the \(k_{0 mc}^1\), at which the diverging peak increases on approaching the multicritical value, is \(k_{0 mc}^1 = \pi\) instead of \(k_{0 mc}^0 = 0\) for \(\Gamma_1 = -(\Gamma_0 + \Gamma_2)\) (and \(k_{0 mc}^0 = 0\) instead of \(k_{0 mc}^1 = \pi\) for \(\Gamma_1 = (\Gamma_0 + \Gamma_2)\)). This swapping of \(k_{0 mc}^1\) occur as a consequence of the intersection of critical lines. Typically the multicritical point \(MC_2\) is the same point for both the critical lines \(\Gamma_1 = \pm (\Gamma_0 + \Gamma_2)\) in parameter space. These critical lines intersect each other at \(MC_2\), which results in the swapping of respective \(k_{0 mc}^0\) values.
VI. NUMERICAL AND ANALYTICAL EVALUATION OF CRITICAL EXPONENTS

The condition in Eq.V.1 for curvature function allows one to choose the proper gauge for which \( F(k, \Gamma) \) can be written in Ornstein-Zernike form around the \( k_0 \) [17],

\[
F(k_0 + \delta k, \Gamma) = \frac{F(k_0, \Gamma)}{1 + \xi^2 \delta k^2},
\]

where \( \delta k \) is small deviation from \( k_0 \), \( F(k_0, \Gamma) \) is the height of the peak and \( \xi \) is characteristic length scale or the width of the peak. As we approach critical point, one can also find the divergence in the characteristic length \( \xi \) along with the curvature function. The divergences in both \( F(k_0, \Gamma) \) and \( \xi \) give rise to the critical exponents

\[
F(k_0, \Gamma) \propto |\Gamma - \Gamma_c|^{-\gamma}, \quad \xi \propto |\Gamma - \Gamma_c|^{-\nu},
\]

where \( \gamma \) and \( \nu \) are the critical exponents which define the universality class of the undergoing topological phase transition. These exponents obeys a scaling law, imposed by the conservation of topological invariant, which reads \( \gamma = \nu \) for 1D systems [18].

Surprisingly, these scaling behavior of curvature function also works well at criticality. Approaching multicritical points \( MC_{1,2} \), curvature function acquires Ornstein-Zernike form around \( k_0^{mc} \).

\[
F(k_0^{mc} + \delta k, \Gamma_c) = \frac{F(k_0^{mc}, \Gamma_c)}{1 + \xi_c^2 \delta k^2},
\]

where \( \delta k = |k - k_0^{mc}| \), \( \xi_c \) is the characteristic length scale at criticality and it represents the width of the curvature function that develops around \( k_0^{mc} \) as the parameters \( \Gamma_c \rightarrow \Gamma_0^{mc} \). The critical behavior of curvature function around the multicritical points \( MC_{1,2} \) can be captured by the same exponents \( \gamma \) and \( \nu \) defined by

\[
F(k_0^{mc}, \Gamma_c) \propto |\Gamma_c - \Gamma_0^{mc}|^{-\gamma}, \quad \xi_c \propto |\Gamma_c - \Gamma_0^{mc}|^{-\nu}.
\]

One can calculate these critical exponents and quantify the scaling properties, numerically, through fitting the diverging peak of curvature function with the Ornstein-Zernike form in Eq.VI.3, as shown in Fig.6(f). The data points...
collected for \( F(k_0^{mc}, \Gamma_c) \) and \( \xi_c \) can then be fitted again with the equation of the form in Eq.VI.4, to extract the exponents \( \gamma \) and \( \nu \) at the multicritical points. Fig.7(a) shows the acquired values of exponents for \( MC_2 \) on approaching from either sides (for \( MC_1 \) see Fig.4(b) of main text). The critical exponents are found to be, \( \gamma_{+/-} = \gamma \approx 1 \) and \( \nu_{+/-} = \nu \approx 1 \) for both multicritical points \( MC_{1,2} \), where \( \gamma_{+/-} \) and \( \nu_{+/-} \) represents the scaling behavior of curvature function with positive (negative) peaks around the multicritical points on both the criticalities.

The exponents can also be estimated analytically by writing the curvature function in Eq.V.3 in Ornstein-Zernike form. It can be achieved by expanding the pseudo-spin vector \( \chi(k) \) around \( k_0^{mc} \) up to third order.

\[
\chi(k)|_{k=k_0^{mc}} \approx \chi(k_0^{mc}) + \partial_k \chi(k_0^{mc}) \delta k + \frac{\partial^2 \chi(k_0^{mc})}{2} \delta k^2 + \frac{\partial^3 \chi(k_0^{mc})}{6} \delta k^3.
\]

Expansion of the individual components of the vectors \( \chi_x(k)|_{k=k_0^{mc}} = \chi_x^{(0)}(1) + \chi_x^{(2)}(2) \) and \( \chi_y(k)|_{k=k_0^{mc}} = \chi_y^{(0)}(1) + \chi_y^{(2)}(2) \) for both the criticalities of the model yields

For \( MC_1 \):

\[
\chi_x(k)|_{k=k_0^{mc}} \approx \frac{(\gamma_0 - 3\Gamma_2)}{2} \delta k^2.
\]

\[
\chi_y(k)|_{k=k_0^{mc}} \approx \frac{\Gamma_0 - 7\Gamma_2}{6} \delta k^3.
\]

For \( MC_2 \):

\[
\chi_x(k)|_{k=k_0^{mc}} \approx \frac{(\Gamma_0 + 3\Gamma_2)\delta k}{2}.
\]

\[
\chi_y(k)|_{k=k_0^{mc}} \approx 2(\Gamma_2 + \Gamma_0) + \frac{\Gamma_0 + 5\Gamma_2}{2} \delta k^2.
\]

The expression for \( MC_2 \) is obtained after considering the swapping of \( k_0^{mc} \). The Ornstein-Zernike form of the curvature function for \( MC_1 \) can be obtained as

\[
F(k, \delta \Gamma_c) = \frac{\chi_y \partial_k \chi_x - \chi_x \partial_k \chi_y}{\chi_x^2 + \chi_y^2} = \frac{(\delta \Gamma_c \delta k + B \delta k^3 - 2A \delta k^2)}{(2A \delta k) - (A \delta k^2)}
\]

\[
F(k, \delta \Gamma_c) = \frac{A\delta \Gamma_c - AB \delta k^2}{A^2 + 2AB \delta k + B^2}.
\]

where \( \delta \Gamma_c = |\Gamma_c - \Gamma_{mc}| = (\Gamma_2 - \Gamma_0), A = (\Gamma_0 - 3\Gamma_2)/2 \) and \( B = (\Gamma_0 - 7\Gamma_2)/6 \). Similarly, for \( MC_2 \) it reads

\[
F(k, \delta \Gamma_c) = \frac{\chi_y \partial_k \chi_x - \chi_x \partial_k \chi_y}{\chi_x^2 + \chi_y^2} = \frac{(\delta \Gamma_c + B \delta k^2) A - (A \delta k^2) (2B \delta k)}{(A \delta k^2) + (2B \delta k^2)^2}
\]

\[
F(k, \delta \Gamma_c) = \frac{A\delta \Gamma_c - AB \delta k^2}{A^2 + 2AB \delta k + B^2}.
\]

where \( \delta \Gamma_c = 2(\Gamma_2 + \Gamma_0), A = (\Gamma_0 + 3\Gamma_2) \) and \( B = (\Gamma_0 + 5\Gamma_2)/2 \). Now the critical exponents can be obtained using Eq.VI.4. The exponent \( \gamma \) is given by

\[
F(k_0^{mc}, \delta \Gamma_c) = A \delta \Gamma_c^{-1} \implies \gamma = 1.
\]

Exponent \( \nu \) can be obtained by identifying the dominant term among the coefficients of \( \delta k^2 \) and \( \delta k^4 \). It can be easily seen that approaching multicritical points \( MC_{1,2} \) on both the criticalities yields \( A > \sqrt{2B}, \sqrt{B} \). This implies

\[
\xi_c = A \delta \Gamma_c^{-1} \implies \nu = 1.
\]

Thus both the numerical and analytical methods yield the same values of critical exponents for topological transition through multicritical points at criticality.

The exponents calculated obey certain scaling laws and defines universality class of the multicriticals. For topological transition occurring through both the multicritical point \( MC_{1,2} \) the exponents are found to have \( \gamma = \nu = 1 \) both numerically and analytically. The scaling law \( \gamma = \nu \) for 1D systems [18] is thus true for the critical behavior of the multicritical points governing the topological transition at criticality. In addition, the dynamical exponent \( z \) dictates the nature of the spectra near the gap closing momenta \( k_0^{mc} \), i.e. \( E_k \propto k^z \). It can be calculated numerically using curve fitting method similar to the previous case. We first collect the data points of \( E_k \) very close to the gap.
FIG. 7: Critical exponents. (a) Exponents of curvature function ($\gamma$ and $\nu$) for $MC_2$. $\gamma_{+/\pm}$ and $\nu_{+/\pm}$ represents the exponents on approaching the point from either sides. (b) Dynamical exponent for $MC_1$ represents quadratic dispersion. (c) Dynamical exponent for $MC_2$: represents linear dispersion. Red and Blue in (b,c) corresponds to the criticalities $\Gamma_1 = \mp (\Gamma_0 + \Gamma_2)$ respectively.

closing point and fit the data points collected to the scaling equation. This procedure results in the Fig.7(b) and (c), where the data points around gap closing momenta $k_{0mc}$ at the multicritical points $MC_{1,2}$ are shown. The spectra is quadratic at $MC_1$ and linear at $MC_2$. The quadratic spectra results in the dynamical critical exponent $z \approx 2$, whilst for linear spectra $z = 1$. This behavior is true for both the criticalities. Therefore, the multicriticalities with both $z = 1$ and $z = 2$ favour the topological transition at criticality. The universality class for the topological transition at criticality through both $MC_{1,2}$ can now be obtained using the set of three critical exponents ($\gamma, \nu, z$), which captures the scaling behavior around the multicritical points with distinct nature. The universality class of the multicriticality at $MC_1 = (1,1,2)$ and for $MC_2$ it reads $(1,1,1)$. Therefore, it is clear that the topological transition at quantum criticality occurs through two distinct multicriticalities which belongs to different universality classes. The scaling laws are also obeyed at criticalities for the topological transition through multicritical points.

VII. CURVATURE FUNCTION RENORMALIZATION GROUP AND WANNIER STATE CORRELATION FUNCTION

Based on the divergence of the curvature function, a scaling theory has been developed [17, 19–27]. This is achieved by the deviation reduction mechanism where the deviation of the curvature function from its fixed point configuration can be reduced gradually. In the curvature function $F(k, \Gamma)$, for a given $\Gamma$ in the parameter space, one can find a new $\Gamma'$ which satisfies

$$F(k_0, \Gamma') = F(k_0 + \delta k, \Gamma'),$$  \hspace{1cm} (VII.1)

where $\delta k$ is small deviation away from the $k_0$, satisfying $F(k_0 + \delta k, \Gamma) = F(k_0 - \delta k, \Gamma)$. As a consequence of the same topology of the system at $\Gamma$ and at fixed point $\Gamma_f$, the curvature function can be written as $F(k, \Gamma) = F_f(k, \Gamma_f) + F_d(k, \Gamma_d)$, where $F_f(k, \Gamma_f)$ is curvature function at fixed point and $F_d(k, \Gamma_d)$ is deviation from the fixed point. The scaling procedure drives the deviation part of curvature function $|F_d(k_0, \Gamma_d)| \rightarrow 0$. The fixed point configuration is invariant under the scaling operation i.e., $F_f(k_0, \Gamma_f) = F_f(k_0 + \delta k, \Gamma_f)$.

Performing the scaling procedure in Eq.VII.1 iteratively and solving $\Gamma$ for every deviation $\delta k$, one can obtain a renormalization group (RG) equation for the coupling parameters. Expanding Eq.VII.1 in leading order and writing $\Gamma' = \Gamma + d\Gamma$ and $(\delta k)^2 = dl$, one can obtain a generic RG equation

$$\frac{d\Gamma}{dl} = \frac{1}{2} \frac{\partial^2 F(k, \Gamma)|_{k=k_0}}{\partial^2 F(k_0, \Gamma)}.$$ \hspace{1cm} (VII.2)

Since the curvature function diverges at $\Gamma_c$, the scaling procedure gradually drives the system away from $\Gamma_c$ towards $\Gamma_f$ without changing the topological invariant. Thus, eventually, the RG flow distinguishes between distinct gapped phases and correctly captures the topological phase transitions between the gapped phases in the system.

In order to capture the topological transition at criticality one can modify the same scaling scheme to incorporate the multicriticality. This is possible since the qualitative behavior of the curvature function defined at criticality exhibits the same diverging nature near multicritical points with the property $F(k_{0mc}, \Gamma'_c) = F(k_{0mc} + \delta k, \Gamma'_c)$ (here $\delta k$ is small deviation from $k_{0mc}$). As the parameters at criticality $\Gamma'_c \rightarrow \Gamma_{mc}$, the topology of the critical phase changes implying a topological transition at multicritical point.
Iteratively performing the scaling procedure reduces the deviation part of curvature function at criticality. This yields the RG equation

$$\frac{d\Gamma_c}{dl} = \frac{1}{2} \frac{\partial^2 F(k, \Gamma_c)}{\partial \Gamma_c^2} |_{k = k_{mc}}.$$  \hspace{1cm} (VII.3)

The distinct critical phases with different topological characters can be distinguished from the RG flow of Eq. VII.3. The multicritical points and fixed points are then easily captured by analyzing the RG flow lines.

- Multicritical point: \(\frac{d\Gamma_c}{dl} \rightarrow \infty\), flow directs away.
- Stable (unstable) fixed point: \(\frac{d\Gamma_c}{dl} \rightarrow 0\), flow directs into (away). \hspace{1cm} (VII.4)

Performing the RG scheme to the model at criticality, we obtain the RG equations for \(MC_1\) as

$$\frac{d\Gamma_0}{dl} = \frac{\Gamma_0(\Gamma_0 + \Gamma_2)}{2(\Gamma_0 - \Gamma_2)} \quad \text{and} \quad \frac{d\Gamma_2}{dl} = -\frac{\Gamma_2(\Gamma_0 + \Gamma_2)}{2(\Gamma_0 - \Gamma_2)}.$$  \hspace{1cm} (VII.5)

Both the critical lines \(\Gamma_1 = \pm(\Gamma_0 + \Gamma_2)\), yield the same RG equations. The multicritical point \(MC_1\) is manifested as a line \(\Gamma_0 = \Gamma_2\) with all flow lines flowing away (see Fig. 5(a) of main text). The condition in Eq. VII.4 for multicritical points is satisfied as the flow rate diverges at \(MC_1\), which also indicate that it is the topological phase transition point at criticality. Surprisingly, \(\Gamma_0 = -\Gamma_2\) (\(MC_2\)) is obtained as a line of unstable fixed points at which flow rate vanishes with all the flow lines are flowing away.

In order to realize the topological transition at criticality through \(MC_2\) one has to consider the swapping of \(k_{mc}\). The RG equation for the critical line \(\Gamma_1 = (\Gamma_0 + \Gamma_2)\), has to be derived with \(k_{mc} = 0\) and vice versa. This procedure yield the RG equations of the form

$$\frac{d\Gamma_0}{dl} = \frac{\Gamma_0(\Gamma_0 - \Gamma_2)}{2(\Gamma_0 + \Gamma_2)} \quad \text{and} \quad \frac{d\Gamma_2}{dl} = -\frac{\Gamma_2(\Gamma_0 - \Gamma_2)}{2(\Gamma_0 + \Gamma_2)}.$$  \hspace{1cm} (VII.6)

In this case, \(\Gamma_0 = -\Gamma_2\) (\(MC_2\)) is obtained to be the topological transition point at criticality, with the diverging flow rate and flow lines directing away (see Fig. 5(b) of main text). The unstable fixed point appear at \(\Gamma_0 = \Gamma_2\) (\(MC_1\)) with vanishing flow rate and flow lines flowing away.

Along with the RG scheme, a correlation function in terms of Wannier-state representation is proposed to characterize the topological phase transition [18]. It is the filled-band contribution to the charge-polarization correlation between Wannier states at different positions, and can be obtained after the Fourier transform of the curvature function. For the two-band model considered here with only the lower band occupied the Wannier state at a distance \(R\),

$$|R\rangle = \int dk e^{ik\hat{r} - R} |u_k\rangle.$$  \hspace{1cm} (VII.7)

with position operator \(\hat{r}\), defines Wannier state correlation function as the overlap of the states \(|0\rangle\) at the origin and at a distance \(|R\rangle\), as [18]

$$\lambda_R = \langle R|\hat{r}|0\rangle = \int dk e^{ikR} \langle u_k|\hat{r}|u_k\rangle.$$  \hspace{1cm} (VII.8)

\[\text{FIG. 8: Wannier state correlation function at criticality. (a) For } MC_1, \text{ (b) For } MC_2. \text{ Approaching the multicritical points } \Gamma_2 = \pm \Gamma_0 \text{ (with } \Gamma_0 = 1)\text{, the decay in the correlation function gets slower on either sides of } MC_{1,2}.\]
Meanwhile, the substitution of the Ornstein-Zernike form of curvature function (Eq. VI.1) yields the Wannier state correlation function \( \lambda_R \), to be
\[
\lambda_R = \int \frac{dk}{2\pi} e^{ikR} F(k, \Gamma) = e^{ik_0 R} \frac{F(k_0, \Gamma)}{2\xi} e^{-\frac{R}{\xi}}. 
\]
(VII.9)

where \( \xi \) can be treated as correlation length of topological phase transition. The correlation function \( \lambda_R \) decays exponentially on either sides of the critical point. The decay gets slower as the parameter is tuned towards criticality. Surprisingly, this notion of correlation function holds true even at criticality and identify the unique topological phase transition at criticality. The behavior of correlation function evidently show that the topological phase transition occurs at the multicritical points \( MC_{1,2} \) at both the criticalities. The Wannier state correlation function can be calculated at criticality as
\[
\lambda_{Rc} = e^{ik_{mc} R} \frac{F(k_{mc}), \Gamma_c)}{2\xi c} e^{-R/\xi_c}. 
\]
(VII.10)

where \( \xi_c = F(k_{mc}, \Gamma_c) = (\Gamma_0 - 3\Gamma_2)/2(\Gamma_2 - \Gamma_0) \) for \( MC_1 \). The correlation function decays faster away from the the multicritical point \( MC_1 \) and the decay slow down as one approaches \( MC_1 \) with the correlation length \( \xi_c \rightarrow \infty \), as shown in Fig.8(a). Both the criticalities shows same behavior of correlation function near this multicritical point on both sides indicating that the multicriticality is indeed a topological phase transition point at criticality. Note that, the only difference between the criticalities for \( k_0 = 0 \) and \( \pi \) is the oscillatory behavior of \( \lambda_{Rc} \) originating from the term \( e^{ik_0 R} \). To obtain the critical nature of \( MC_2 \) one has to consider the swapping of \( k_{mc} \) (Eq.I.8 and Eq.I.9), which yields \( \xi_c = F(k_{mc} R), \Gamma_c) = (\Gamma_0 + 3\Gamma_2)/2(\Gamma_0 + \Gamma_2) \). This captures the critical nature of \( MC_2 \), where the decay gets slower as one approaches this point from both the sides, as shown in Fig.8(b). Therefore, the behavior of the correlation function evidently shows that the topological phase transition occurs at the multicritical points. For both the criticalities, the correlation length \( \xi_c \) coinsides with the decay length of the edge modes at criticality studied earlier.

VIII. EXPERIMENTAL OBSERVABILITY

A. Using Superconducting Circuits

The generic model presented in Eq.1 of main text can be simulated using superconducting circuit with a single qubit driven by the tunable external microwave pulses [28, 29]. The connection established between momentum \( k \) and evolution time \( t \), mimic the adiabatic evolution in momentum space, under which the Anderson pseudospin vector \( \chi \) straight away maps to the external driving \( B \). The evolution of the qubit takes place in the \( xy \) plane of the Bloch sphere and the expectation values of the corresponding spin operators \( \langle \sigma_x \rangle \) and \( \langle \sigma_y \rangle \) yield the state of the qubit which in turn gives the winding number of a topological phase [28]. Microwave pulses created by choosing the critical values of the parameters with \( \Gamma_1 = \pm(\Gamma_0 + \Gamma_2) \), enable one to realize the external driving \( B \) at criticality. The measurement on the state of the qubit yields the results similar to the winding of unit vector, as shown in Fig.3(a,b) of main text. This similarity is due to the fact that there is a straightforward mapping between the state of the qubit and the orientation of the pseudospin vector: upward orientation \( \rightarrow |0 \rangle \) and downward orientation \( \rightarrow |1 \rangle \) (all other orientation represents the linear superposition of these two states) [28]. The measurements will be able to show that \(-\Gamma_0 < \Gamma_2 < \Gamma_0 \) represents a topological trivial critical phase with \( w_c = 1/2 \) and \( \Gamma_2 > \Gamma_0 \) and \( \Gamma_2 < -\Gamma_0 \) represents topological non-trivial critical phases with winding number \( w_c = 3/2 \). The experimental and theoretical winding numbers can be compared to confirm the results. The topological quantum phase transition between these distinct critical phases can be observed to be at multicritical points \( MC_{1,2} \) \( (\Gamma_2 = \pm \Gamma_0) \). The linear and quadratic nature of energy dispersion which distinguishes between these multicritical points can be obtained from the Rabi frequency of the microwave pulses.

B. Using Ultracold Atoms

The results discussed in this work can also be experimentally simulated using ultracold atoms in optical lattices [30]. The model used in this study represents topological insulators and superconductors with NN and NNN couplings. For the realization of the model for topological insulators, the Bose-Einstein condensates (BEC) of Rb-87 atoms provides a suitable platform to realise the critical point of the extended SSH models [31–33], with full control over the tuning parameters [34]. The critical phenomena discussed in the current work can be observed by tuning the parameters to
obey the criteria for criticality i.e., $\Gamma_1 = \pm (\Gamma_0 + \Gamma_2)$. Under this setting one can perform the measurement of mean chiral displacement [31, 33] which provides a measure of winding number and zero modes through the imaging of topological mid-gap states [32] at criticality. This would eventually be able to distinguish the topological trivial and non-trivial critical phases and the transition among them across the multicritical points. Topological superconducting model with real-time control over both hopping and pairing parameters can also be effectively simulated in optically trapped fermionic atoms [35, 36] to detect the Majorana zero modes using various techniques [36] at criticality.

C. Scanning Tunneling Microscopy

The topological and non-topological nature of criticalities and the corresponding topological transition among them can be observed from the profile of the local density of states [9]. At criticality where the bulk density of states is zero, using Scanning Tunneling Microscopy (STM) one would observe a raise in the local boundary density of states at the multicritical point through which the topological transition at criticality takes place. Further one may distinguish between two multicritical points by looking at the bulk local density of states [9]. At criticality where the bulk density of states is linearly vanishing, using Scanning Tunneling Microscopy one would observe a rise in the local boundary density of states [9].

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