Clique dynamics of locally cyclic graphs

with $\delta \geq 6$

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Abstract

We prove that the clique graph operator $k$ is divergent on a locally cyclic graph $G$ (i.e. $N_G(v)$ is a circle) with minimum degree $\delta(G) = 6$ if and only if $G$ is 6-regular. The clique graph $kG$ of a graph $G$ has the maximal complete subgraphs of $G$ as vertices, and the edges are given by non-empty intersections. If all iterated clique graphs of $G$ are pairwise non-isomorphic, the graph $G$ is $k$-divergent; otherwise, it is $k$-convergent.

To prove our claim, we explicitly construct the iterated clique graphs of those infinite locally cyclic graphs with $\delta \geq 6$ which induce simply connected simplicial surfaces. These graphs are $k$-convergent if the size of triangular-shaped subgraphs of a specific type is bounded from above. We apply this criterion by using the universal cover of the triangular complex of an arbitrary finite locally cyclic graph with $\delta = 6$, which shows our divergence characterisation.

Keywords: Iterated clique graphs, clique convergence, clique dynamics, locally cyclic, hexagonal grid, covering graph

1. Introduction

Applied to a graph $G$, the clique graph operator constructs its clique graph $kG$. The vertices of $kG$ are the maximal complete subgraphs of $G$, called cliques. These cliques are adjacent in $kG$ if they intersect in $G$. In 1972, Hedetniemi and Slater first studied line graphs and triangle free graphs using the clique graph operator [2]. We are interested in locally cyclic graphs, which means that the set of vertices incident to a given vertex $v$ always induces a circle. Popular locally cyclic graphs are the octahedron, the icosahedron, and the hexagonal grid, which are displayed in Figure 1. For minimum degree $\delta$ of at least 4, they can be described as Whitney triangulations of surfaces, which were investigated for example in [5], [6], and [7]. In 1999, Larrión and Neumann-Lara showed that some 6-regular triangulations of the torus are $k$-divergent [3] and, in 2000, they generalised this result to every 6-regular locally cyclic graph [4]. Furthermore, Larrión, Neumann-Lara, and Pizaña [6] showed that graphs in which every open neighbourhood of a vertex has a girth of at least 7 are $k$-convergent. Thus,
locally cyclic graphs of minimum degree \( \delta \) of at least 7 are \( k \)-convergent. The question remains whether every non-regular locally cyclic graph with \( \delta = 6 \) is \( k \)-convergent. In this paper, we provide the affirmative answer by generalising the approach from [4] and using the theoretical background from [8].

In the remainder of this paper, a graph is not necessarily considered finite. Furthermore, we extend the incidence terminology of a graph to three-circles. Thus, the three vertices and the three edges of a three-circle are each incident to the three-circle itself. Throughout the paper, we use different kinds of neighbourhoods, which are defined in the appendix.

### 2. Overview and basic concepts

The focus of this paper consists in proving our main result:

**Theorem (Main result).** Let \( G \) be a finite, locally cyclic graph with minimum degree \( \delta = 6 \). The clique graph operator diverges on \( G \) if and only if \( G \) is 6-regular.

This paper is based on two core insights:

1. The explicit description of iterated clique graphs is massively simplified if we restrict ourselves to triangularly simply connected graphs (there, triangular substructures do not “self-overlap”). Subsection 2.1 gives a short refresher on the definitions. In Section 8, we extend the result to general locally cyclic graphs with \( \delta = 6 \) using universal covers.

2. Clique graphs grow only in regions with “many” vertices of degree 6. More precisely, these vertices have to form a triangular-shaped structure. We can also parametrise these regions by barycentric coordinates. The necessary formalism is given in Subsection 2.2. This way, we combinatorially encode the adjacencies in the iterated clique graphs in Section 4.

For simplification, we shrink the specifications of the centrally discussed object with a new definition.

**Definition 2.1.** *A pika* is a triangularly simply connected locally cyclic graph \( G \) with minimum degree \( \delta = 6 \).
For pikas, we head for the following theorem.

**Theorem (Main Theorem for Pikas).** Let $G$ be a pika. If there is an $m \geq 0$ such that the triangular-shaped graph of side length $m$ (see Definition 2.2) cannot be embedded into $G$, the clique operator is convergent on $G$.

We prove the main theorem for pikas by induction. For a pika $G$ and for every $n \in \mathbb{Z}_{\geq 0}$, we define a graph $G_n$, beginning with $G_0 = G$. We construct all the cliques of $G_n$ and their intersection, which yields $G_{n+1} \cong k(G_n)$ and, by induction, $k^nG = G_n$. Thus, $G$ is clique convergent if and only if the sequence $G_n$ converges.

Before diving into this line of arguments, in Section 3, we discuss some intricate topological arguments that are based on the discrete curvature of a locally cyclic graph. The results will be used to show that all cliques in $G_n$ are of one of the types we describe and that the adjacencies in $G_{n+1}$ correspond to the intersections of cliques of $G_n$.

In Section 4, we define the graph $G_n$ and construct two types of cliques in $G_n$. In Section 5, our goal is to ensure the existence of a combined parametrisation of intersecting triangular-shaped subgraphs in the general case and to handle the exceptional cases. The results of this discussion are used in Section 6 to prove that $G_n$ has no more cliques than the ones from Section 4. In Section 7, we finish the inductive proof for $G_n \cong k^nG$ by describing the clique intersections in $G_n$ through the vertex adjacencies in $G_{n+1}$ and prove the main theorem for pikas. In Section 8, we deduce a convergence criterion that does not rely on the simple connectivity of $G$. In the special case of finite $G$ we conclude the main result. Section 9 includes our conjecture about the infinite case and our suggestions for further research questions.

### 2.1. Review: Simple Connectivity

In this subsection, we review topological aspects of locally cyclic graphs. The definition of a path we use here originates from topological settings. In graph theoretic literature, those paths would be called walks. A path $p = x_0x_1\ldots x_k$ in a graph $G$ is a finite sequence of vertices such that $x_ix_{i+1} \in E$ for all $0 \leq i < k$. The length of a path is the number of contained edges. Let $G$ be a locally cyclic graph. Following [4], its triangular complex is the simplicial complex $\mathbb{K}(G)$, whose simplices are the vertices, edges, and three-circles of $G$. In this way, the three-circles of $G$ become the facets of $\mathbb{K}(G)$ and, from now on, we will call them the facets of $G$, too.

Like in [8], we call two paths $\alpha$ and $\beta$ in $G$ equivalent if we can reach one path from the other by applying a finite number of elementary moves consisting in replacing two consecutive path edges $uv$ and $vw$ by the edge $uw$ or the other way around whenever $\{u, v, w\}$ is a facet. The complex $\mathbb{K}(G)$ is called simply connected if $G$ is connected — i.e. for every pair of vertices, there is a path in $G$ connecting them —, and if every closed path $\alpha$ is equivalent to a path consisting of a single vertex (which is the origin and end of $\alpha$). We call a locally cyclic graph $G$ trianuglarly simply connected if $\mathbb{K}(G)$ is simply connected.
2.2. Hexagonal Grid

If a locally cyclic graph has an area of vertex degree 6, the graph locally looks like the hexagonal grid, a term we will define now.

**Definition 2.2.** We define the coordinate set

\[ \tilde{D}_0 := \{(1, -1, 0), (1, 0, -1), (-1, 1, 0), (0, 1, -1), (-1, 0, 1), (0, -1, 1)\} \]

For \( m \in \mathbb{Z} \), the **hexagonal grid of height** \( m \) is the graph \( \text{Hex}_m = (V_m, E_m) \) with

\[
V_m := \{(x_1, x_2, x_3) \in \mathbb{Z}^3 \mid x_1 + x_2 + x_3 = m\} \quad \text{and} \quad E_m := \\{\{x, y\} \subseteq V_m \mid x - y \in \tilde{D}_0\}.
\]

For \( m \geq 0 \), we denote the **triangular-shaped graph of side length** \( m \), which is defined as \( \text{Hex}_m[V_m \cap \mathbb{Z}^3_{\geq 0}] \), by \( \Delta_m \). Figure 2 shows the smallest five of those subgraphs.

![Figure 2: \( \Delta_m \) for \( m \in \{0, \ldots, 4\} \)](image)

For a locally cyclic graph \( G \), a **hexagonal chart** is a graph isomorphism \( \mu : H \to F \) (also written \( H \xrightarrow{\mu} F \)) with vertex-induced subgraphs \( H \subseteq \text{Hex}_m \) and \( F \subseteq G \). If \( F \cong \Delta_m \), we call it **standard chart**.

Since the symmetric group on three points acts on the hexagonal grid by coordinate permutations, every subgraph \( F \cong \Delta_m \) with \( m \geq 1 \) has six standard charts.

For \((t_1, t_2, t_3) \in \mathbb{Z}^3\), we define the **triangle inclusion map**:

\[ \Delta_{t_1,t_2,t_3} : \Delta_m \to \text{Hex}_{m+t_1+t_2+t_3}, \quad (a_1, a_2, a_3) \mapsto (a_1 + t_1, a_2 + t_2, a_3 + t_3). \]

3. Topology

We translate \( \Delta_m \)-shaped graphs into the setting of locally cyclic graphs. A **locally cyclic graph with boundary** is a simple graph \( G = (V, E) \) such that for every vertex \( v \in V \) the (open) neighbourhood \( N_G(v) \) is either a circle graph or a path graph. If \( N_G(v) \) is a circle, \( v \) is called an **inner vertex** of \( G \); otherwise, \( v \) is called a **boundary vertex**.
An edge $xy \in E$ is called an **inner edge** if its incident vertices $x$ and $y$ have two common neighbours, and a **boundary edge**, if not. The **boundary graph** $\partial G$ is the subgraph of $G$ consisting of the boundary vertices and the boundary edges. $G$ is called **locally cyclic** if $\partial G = \emptyset$.

The boundary graph $\partial G$ is well-defined, since for every inner vertex $x$ and every edge $xy$, the vertex $y$ lies in the cyclic neighbourhood $N_G(x)$ and has, therefore, two neighbours in $N_G(x)$. Thus, $xy$ is an inner edge. Conversely, there exist inner edges that are incident to only boundary vertices.

### 3.1. Straight Paths

A monomorphism of locally cyclic graphs with boundary preserves vertex degrees of inner vertices. Furthermore, the number of incident facets on either side of a path is preserved, too. We formalise this by the concept of **path degree**:

Let $p = x_0 x_1 \ldots x_k$ be a path in a locally cyclic graph $G$ and consider a vertex $x_i$ for $0 < i < k$.

- If $x_i$ is an inner vertex, $N_G(x_i)$ is a circle, say of length $L$, and marking $x_{i-1}$ and $x_{i+1}$ splits the circle into two paths of lengths $l_1$ and $l_2$, satisfying $l_1 + l_2 = L$. The **path degree** $\deg^p_G(x_i)$ is defined as $\{l_1, l_2\}$, as is visualized in Figure 3.

- If $x_i$ is a boundary vertex, $N_G(v)$ is a path graph containing a unique shortest path $q$ from $x_{i-1}$ to $x_{i+1}$ with length $l$. The **path degree** $\deg^p_G(x_i)$ is defined as $\{l\}$.

The concept of path degrees is illustrated in Figure 3. The path $p$ is called **straight** if 3 is contained in $\deg^p_G(x_i)$ for every $0 < i < k$.

As an important application, we construct the straight paths within $\Delta_m$.

**Remark 3.1.** Up to symmetry (see Subsection 2.2), the maximal straight paths with length at least $m - 2$ in $\Delta_m$ (with $m \geq 3$) are the following, depicted in Figure 4:

1. For length $m$, we have $\alpha : \{0, \ldots, m\} \to \mathbb{Z}^3$ with $t \mapsto (m - t, t, 0)$.

2. For length $m - 1$, we have $\beta : \{0, \ldots, m - 1\} \to \mathbb{Z}^3$ with $t \mapsto (m - 1 - t, t, 1)$.

3. For length $m - 2$, we have $\gamma : \{0, \ldots, m - 2\} \to \mathbb{Z}^3$ with $t \mapsto (m - 2 - t, t, 2)$. 

Figure 3: The path degrees of the inner vertex $v$ and the boundary vertex $w$. Since the path degree $\deg^p_G(w)$ does not contain 3, $p$ is not straight.
Proof. The boundary $\partial \Delta_m$ consists of six straight paths of length $m$, given by $\alpha$ and its images under coordinate permutations. Since any other straight path contains an inner vertex and inner vertices have degree 6, the value of one of the three coordinates is constant along the path (compare Subsection 3.1). Without loss of generality (see Subsection 2.2), let this be the third coordinate. This way, we receive $\beta$ and $\gamma$.

3.2. Topological consequences of $\delta = 6$

To understand the structure of a pika $G$ properly, we need to make sure that the boundary vertices of a $\Delta_m$-shaped subgraph are not connected by paths in an unexpected way.

Lemma 3.2. For every induced subgraph $S \cong \Delta_m$ of $G$ the following statements hold:

1. Any edge incident to two boundary vertices of $S$ lies in $S$.

2. Let $v_0v_1v_2$ be a path with $v_0, v_2 \in \partial S$ and $v_1 \notin S$. Then, either $v_0 = v_2$ or $\{v_0, v_1, v_2\}$ is a facet (i.e. $v_0v_2$ is a boundary edge).

3. Let $v_0v_1v_2v_3$ be a non-repeating path with $v_0, v_3 \in \partial S$ and $v_1, v_2 \notin S$ such that neither $\{v_0, v_1, v_3\}$ nor $\{v_0, v_2, v_3\}$ are facets. Then, there exists a boundary vertex $v \in \partial S$ such that $\{v, v_1, v_2\}$ is a facet.

Proof. See Appendix B.

This helps proving two more auxiliary lemmas.

Lemma 3.3. Let $m \geq 1$ and $\Delta_m \cong S = S_1 \cup S_2 \cup S_3$ with $S_i \cong \Delta_{m-1}$. Then, $N_G[S_1] \cap N_G[S_2] \cap N_G[S_3] \subseteq S$.

Proof. See Appendix B.

Lemma 3.4. Let $S \cong \Delta_m$. Then, $N_G[S] \setminus S$ is a cycle and vertices in $N_G[S] \setminus S$ are incident to at most three faces in $N_G[S]$.

Proof. See Appendix B.
4. The graph $G_n$

We construct a graph sequence $G_n$ for every pika $G$ in a geometric way (see Def. 4.1). In Subsection 4.1, we prove some general statements concerning the relation between different triangular-shaped subgraphs. These will be helpful in all further analyses. In Subsection 4.2, we construct some cliques of $G_n$ explicitly. This is the first step on our way to prove $G_n \cong k^n G$ inductively.

If not otherwise stated, from now on $G$ will always refer to a pika and $G_n$ will be the geometric clique graph defined in Definition 4.1.

**Definition 4.1.** Let $G$ be a pika. For a non-negative integer $n$, the geometric clique graph $G_n$ has the following form:

- Its vertices are the subgraphs of $G$ isomorphic to triangle graphs $\Delta_m$ with $m \leq n$, where $m$ and $n$ have the same parity.

- Its edges are defined as follows:
  1. Two subgraphs (of $G$) $S_1 \cong \Delta_m$ and $S_2 \cong \Delta_m$ are adjacent (in $G_n$) if $S_1 \subseteq N_G[S_2]$ or $S_2 \subseteq N_G[S_1]$.
  2. Two subgraphs $S_1 \cong \Delta_m$ and $S_2 \cong \Delta_{m-2}$ (with $m \geq 2$) are adjacent if $S_2 \subseteq S_1$.
  3. Two subgraphs $S_1 \cong \Delta_m$ and $S_2 \cong \Delta_{m-4}$ (with $m \geq 4$) are adjacent if $S_2 \subseteq S_1$ and $S_2$ does not contain any vertex $\partial S_1$, i.e. $S_2 \cap \partial S_1 = \emptyset$.
  4. Two subgraphs $S_1 \cong \Delta_m$ and $S_2 \cong \Delta_{m-6}$ (with $m \geq 6$) are adjacent if $S_2 \subseteq S_1$ and $S_2$ does not contain any vertex with distance at most 1 from the boundary of $\partial S_1$, i.e. $S_2 \cap N_G[\partial S_1] = \emptyset$.

A subgraph $S \cong \Delta_m$ of $G$ is said to be of level $m$ in $G_n$.

Clearly, $G_0 = G$. Thus we can try to prove $G_n \cong k^n G$ by induction.

**Example 4.2.** The subgraph $\Delta_4$ of the pika Hex$_4$ is a vertex of every geometric clique graph (Hex$_4$)$_n$ with an even $n \geq 4$. The adjacent vertices of level 0 are the three $\Delta_0$ that are depicted in blue in Figure 5 and the adjacent vertices of level 2 are the “face-down” $\Delta_2$, which is depicted in red, and the six “face-up” $\Delta_2$, two of which are depicted in yellow.

4.1. Properties of Triangles

In this subsection, we collect some technical results about triangles and their relations.

**Remark 4.3.** Let $m \geq 3$. The vertices of $\Delta_m$ with distance at least 1 to the boundary induce the graph $\Delta_m \setminus \partial \Delta_m \cong \Delta_{m-3}$. Thus, for $m \geq 6$, the vertices with distance at least 2 induce $\Delta_m \setminus N_G[\partial \Delta_m] \cong \Delta_{m-6}$.

1 These two conditions are in fact equivalent. This is a direct consequence of Lemma 5.4 shown later.
Figure 5: The (types of) subgraphs of level 0 and 2 of Hex_4 that are adjacent to Δ_4 in any (Hex_4)_n with an even n ≥ 4

We define some graphs and vertex sets for future reference. The set
\[ \tilde{E} := \tilde{V}_1 \cap \mathbb{Z}_3^3 = \{(1,0,0), (0,1,0), (0,0,1)\} \]
is the canonical basis, the graph
\[ \nabla_1 := \text{Hex}_2[(1,1,0), (0,1,1), (1,0,1)] \]
is the downward triangle of side length 1 in the centre of Δ_2, the graphs
\[ \nabla_{\tilde{e}}^1 := \text{Hex}_3[(1,1,0) + \tilde{e}, (0,1,1) + \tilde{e}, (1,0,1) + \tilde{e}] \]
with \( \tilde{e} \in \tilde{E} \) are the downward triangles of side length 1 inside Δ_3, and the graph
\[ \nabla_2 := \text{Hex}_4[(2,2,0), (0,2,2), (2,0,2)] \]
is the downward triangle of side length 2 in the centre of Δ_4.

The following two auxiliary lemmas discuss small special cases.

**Lemma 4.4.** Let \( m \geq 1 \) and consider \( \Delta_m \subseteq \text{Hex}_m \). If \( \Delta_{m-1} \cong S \subseteq \Delta_m \), either
1. \( S \) is the image of \( \Delta_{\tilde{e}}^{m-1} \) with \( \tilde{e} \in \tilde{E} \), or
2. \( m = 2 \) and \( S = \nabla_1 \).

In particular, \( \Delta_m \subseteq N_G[S] \).

**Proof.** See Appendix C.

**Lemma 4.5.** Consider \( \Delta_m \subseteq \text{Hex}_m \) with \( m \geq 2 \). If \( \Delta_{m-2} \cong S \subseteq \Delta_m \), either
1. \( S \) is the image of \( \Delta_{\tilde{f}}^{m-2} \) with \( \tilde{f} \in \tilde{E} + \tilde{E} = \tilde{V}_2 \cap \mathbb{Z}_3^3 \),
2. \( m = 3 \) and \( S = \nabla_{\tilde{e}}^1 \) for some \( \tilde{e} \in \tilde{E} \), or
3. \( m = 4 \) and \( S = \nabla_2 \).

**Proof.** See Appendix C.
4.2. Clique construction of $G_n$

In this subsection, we construct different cliques of $G_n$. The constructed cliques fall into two classes; those that are formed from three $\Delta_m$ within one $\Delta_{m+1}$ (Lemma 4.6), and those that are formed by all $\Delta_1$ incident to a vertex (Lemma 4.7).

In the next lemma, we employ a shorthand: For a hexagonal chart $\mu : \Delta_{m+1} \rightarrow S$ and $(t_1, t_2, t_3) \in \mathbb{Z}^3$, we denote the image of $\mu \circ \Delta_{m+1-t_1-t_2-t_3}$ by $\mu_{t_1,t_2,t_3}$.

Lemma 4.6. Let $G$ be a pika and $\Delta_{m+1} \not\rightarrow S \subseteq G$ a hexagonal chart with $m \leq n$ and $m \equiv 2$ $n$. The common neighbourhood $N_{G_n}[\mu_{1,0,0}, \mu_{0,1,0}, \mu_{0,0,1}]$ forms a clique in $G_n$.

Proof. By Definition 4.1 the $\mu_\varepsilon$ with $\varepsilon \in \vec{E}$ are vertices of $G_n$. They are all contained in $S \subseteq N_G[\mu_\varepsilon]$ (by Lemma 4.4). Thus, by Definition 4.1 they are all adjacent to each other. The first step of the proof is finding all elements in the common neighbourhood. Let $T$ be in $N_{G_n}[\mu_{1,0,0}, \mu_{0,1,0}, \mu_{0,0,1}]$. We consider adjacency to smaller triangles: If $T$ is contained in $S \subseteq N_G[\mu_\varepsilon]$, then $\mu_\varepsilon$ has distance 1 to $\partial S$. By Remark 4.3, this uniquely defines $T$ with $N_G[S] \subseteq T$.

1. If $T \cong \Delta_{m-k}$ for $k \in \{2, 4, 6\}$ and $k \leq m$, by Definition 4.1 $T \subseteq \mu_{1,0,0} \cap \mu_{0,1,0} \cap \mu_{0,0,1}$. For $m \in \{0, 1\}$ we have $\mu_{1,0,0} \cap \mu_{0,1,0} \cap \mu_{0,0,1} = \emptyset$, thus this is a contradiction. For $m \geq 2$, we have $\mu_{1,0,0} \cap \mu_{0,1,0} \cap \mu_{0,0,1} = \mu_{1,1,1} \cong \Delta_{m-2}$. We distinguish between the possible values of $k$.
   a) $k = 2$: We conclude $T = \mu_{1,1,1}$.
   b) $k = 4$: We have $T \subseteq \mu_\varepsilon \setminus \partial \mu_\varepsilon$ for the three $\varepsilon \in \vec{E}$. Thus, by Remark 4.3 $\Delta_{m-4} \cong T \subseteq \mu_{1,1,1} \setminus \partial \mu_{1,1,1} \cong \Delta_{m-5}$, which is impossible.
   c) $k = 6$: We have $T \subseteq \mu_\varepsilon \setminus \partial N_G[\mu_\varepsilon]$ for the three $\varepsilon \in \vec{E}$. Thus, by Remark 4.3 $\Delta_{m-6} \cong T \subseteq \mu_{1,1,1} \setminus \partial N_G[\mu_{1,1,1}] \cong \Delta_{m-8}$, which is impossible.

We conclude $m \geq k = 2$ and $T = \mu_{1,0,0} \cap \mu_{0,1,0} \cap \mu_{0,0,1} = \mu_{1,1,1}$.

2. If $T \cong \Delta_m$, by Definition 4.1 $T \subseteq N_G[\mu_{1,0,0}] \cap N_G[\mu_{0,1,0}] \cap N_G[\mu_{0,0,1}]$. Since by Lemma 4.4 $S = N_G[\mu_{1,0,0}] \cap N_G[\mu_{0,1,0}] \cap N_G[\mu_{0,0,1}]$, Lemma 4.4 shows that $T$ can only appear if $m = 1$ (in which case it comes from a $\nabla_1$).

In particular, it is never part of the common neighbourhood if $\mu_{1,1,1}$ is.

3. If $T \cong \Delta_{m+k}$ for $k \in \{2, 4, 6\}$, we have $\Delta_{m+1} \cong S = \mu_{1,0,0} \cup \mu_{0,1,0} \cup \mu_{0,0,1} \subseteq T$. Again, we distinguish between the possible values of $k$.
   a) $k = 2$: We employ Lemma 4.4 to describe how $S$ can lie in $T$. By the same lemma and Definition 4.1 all of these $T$ are pairwise adjacent.

   Consider adjacency to smaller triangles: If $m = 1$, the additional $\Delta_m$ from Lemma 4.4 lies in $S$ and is thus adjacent to all $T$. If $m \geq 2$, the intersection $\mu_{1,1,1} \cong \Delta_{m-2}$ has distance 1 to $\partial S$. Thus, it also has distance 1 to $\partial T$ and it is therefore adjacent to all of them.
   b) $k = 4$: The subgraphs $\mu_\varepsilon$ need to have distance 1 to the boundary of $T$. Thus, $S$ also has this distance. By Remark 4.3 this uniquely defines $T$ with $N_G[S] \subseteq T$. 

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Since the additional $\Delta_{m+2}$ lie in $N_G[S]$ (Lemma 4.4), they are adjacent to $T$. For $m = 1$, the additional $\Delta_m$ lies in $S$ and has distance 1 from the boundary of $T$. For $m \geq 2$, the intersection $\mu^{1,1,1}$ has distance 1 from $\partial S$. Since $S$ has distance 1 from $\partial T$, the total distance between $\mu^{1,1,1}$ and $\partial T$ is 2, showing their adjacency.

c) $k = 6$: By Remark 4.3 there is only one embedding $\Delta_m \to \Delta_{m+6}$ with distance 2 to the boundary. Thus, there is no such element adjacent to all $\mu^{\vec{e}}$ simultaneously.

Finally, we conclude that $N_G^n[\mu^{1,0,0}, \mu^{0,1,0}, \mu^{0,0,1}]$ is a clique of $G_n$. □

After having covered the triangle case, we now cover the vertex case. The neighbours of a vertex $v$ form a circle $w_1 w_2 \ldots w_k$ for some $k \in \mathbb{N}$. The **umbrella** of $v$ is the set containing the $k$ facets $\{v, w_i, w_{i+1}\}$ for all $1 \leq i \leq k$, where we read indices modulo $k$.

**Lemma 4.7.** Let $G$ be a pika and $v$ a vertex in $G$. For odd $n$, the common neighbourhood $N_G^n[T \cong \Delta_1 | v \subseteq \Delta_1]$ of all $\Delta_1$ containing $v$ forms a clique in $G_n$.

**Proof.** Clearly, all these $\Delta_1$ are pairwise adjacent in $G_n$, since they share $v$. Thus, they lie in a clique, which itself lies in the common neighbourhood $N_G^n[T \cong \Delta_1 | v \subseteq \Delta_1]$.

We consider all $\Delta_{1+k} \cong T$ which lie in this common neighbourhood (for $k \in \{0, 2, 4, 6\}$).

1. Case $k = 0$: If there was a $\Delta_1$ adjacent to all facets in the umbrella, all of its vertices would lie in $N_G(v)$ (each of its vertices can only lie in two facets and the number of facets is at least 6). Thus, $N_G(v)$ contains a three-circle, in contradiction to being at least a 6-cycle.

2. Case $k = 2$: Any $\Delta_3$ which is adjacent to all facets in the umbrella, would be the $\Delta_3$ containing $v$ as its middle vertex. Thus, $v$ has degree 6. In this case, there are two $\Delta_3$ with $v$ as their central vertex and these two are clearly adjacent.

3. Case $k = 4$: By Remark 4.3 every $\Delta_1$ adjacent to a given $\Delta_5$ has to lie within the central $\Delta_2$. By Lemma 4.4 there are four of these $\Delta_1$. Since $\deg(v) \geq 6$, no $\Delta_5$ can be adjacent to all the facets in the umbrella.

4. Case $k = 6$: By Remark 4.3 and Definition 4.1 a $\Delta_7$ is only adjacent to one $\Delta_1$. Since $\deg(v) \geq 6$, no $\Delta_7$ can be adjacent to all the facets in the umbrella.

Thus, all the elements in $N_G^n[T \cong \Delta_1 | v \subseteq \Delta_1]$ are pairwise adjacent, and we obtain a clique. □

Those two lemmas suggest a correspondence between the cliques of $G_n$ we constructed and the vertices of $G_{n+1}$.

**Remark 4.8.** For every pika $G$, and every $n \in \mathbb{Z}_{\geq 0}$ there is a map

$$C : V(G_{n+1}) \to \{\text{cliques of } G_n\},$$

$$S \mapsto \text{the clique from } \begin{cases} \text{Lemma 4.7 if } S \text{ is of level 0}, \\ \text{Lemma 4.6 otherwise.} \end{cases}$$
In Section 5, we discuss some theory that helps to show the bijection of this map in Section 6. In Section 7, we prove that it is a graph isomorphism between $G_{n+1}$ and $kG_n$.

5. Chart Extensions

In Section 4, we introduced the graph $G_n$ (of a pika $G$) and constructed several of its cliques in Subsection 4.2. We still need to show that $G_n$ has no more cliques than those.

To do so, we transfer local regions of $G_n$ to local regions of the hexagonal grid, where the calculations become simpler. This transfer is easy if we only consider “smaller” triangles within a “larger” hexagonal chart. However, Definition 4.1 also includes edges between triangles of the same size. In this case, we extend a hexagonal chart to a larger domain, containing all adjacent triangular-shaped subgraphs as well, if possible. The existence of such an extension is non-trivial and needs several intricate arguments about topology and straight paths. In this technical chapter, we show that such an extension of charts is always possible for $m \geq 3$ (Lemma 5.4).

For $m \geq 4$ and a $\Delta_m$-shaped subgraph $S$ of $\text{Hex}_m$, any neighbouring $\Delta_m$-shaped subgraph (with respect to the geometric clique graph $(\text{Hex}_m)_n$) can be constructed by adding vectors from $\vec{D}_0$ (see Def. 2.2) to $S$. For $m = 3$, one additional neighbour occurs, which is the rotation of $S$ by $\frac{\pi}{3}$. We want to prove that the same structure holds for subgraphs of $G$. We start by noting that each $\vec{d} \in \vec{D}_0$ can lead to an adjacent $\Delta_m$. (The complementary claim is proven in Lemma 5.2).

Remark 5.1. Let $G$ be a pika and let $\nu : H \to F$ be a hexagonal chart, with $\Delta_m \subseteq H \subseteq \text{Hex}_m$ for an $m \geq 3$. If for a $\vec{d} \in \vec{D}_0$, the image of $\Delta_m$ is contained in $H$, the image of $\nu \circ \Delta_m$ lies inside $N_{G_{\nu}(\Delta_m)}$.

We employ facet-paths to extend charts. For a locally cyclic graph with boundary $G = (V, E)$, these are finite sequences of facets $f_1 f_2 \ldots f_k$ such that $f_i \cap f_{i+1} \in E$ for all $1 \leq i < k$.

Given a monomorphism $\mu : H \to G$ and a facet-path $f_1 f_2 \ldots f_k$ in $H$, we have the following transfer: For each pair $f_i f_{i+1}$ with $1 \leq i < k$, the image $\mu(f_i \cap f_{i+1})$ is an inner edge and $\mu(f_i) \neq \mu(f_{i+1})$. In particular, the images of the vertices in $f_i$ uniquely determine the image of the vertex $f_{i+1} \setminus (f_i \cap f_{i+1})$. Inductively, the images of the vertices in $f_1$ determine those in $f_k$.

This allows a unique extension along a facet-path. Unfortunately, the extensions from different facet-paths are not compatible in general.

Given a $\Delta_m$-shaped subgraph $S \subseteq G$ with neighbouring $\Delta_m$-shaped subgraphs $T_k$, we want to show the existence of a chart containing all of them. We start by extending the chart from $S$ to one neighbouring triangular-shaped subgraph.

Lemma 5.2. Let $G$ be a pika and $\Delta_m \hookrightarrow S \subseteq G$ be a standard chart with $m \geq 3$. Let $\Delta_m \cong T \subseteq N_G[S]$. Then, there is a $H \subseteq \text{Hex}_m$ and a hexagonal chart $\hat{\mu} : H \to T$ such that $\mu^{-1}(x) = \hat{\mu}^{-1}(x)$ holds for every vertex $x$ in $S \cap T$. Furthermore, $H$ falls in one of these two cases:
• $H = \vec{d} + \Delta_m$ for some $\vec{d} \in \vec{D}_0$.
• $H = \nabla_3 := \{(a, b, c) \in \text{Hex}_3 \mid a \leq 2, b \leq 2, c \leq 2\}$, with corner vertices $(2, 2, -1)$, $(2, -1, 2)$, and $(-1, 2, 2)$.

Consequently, for $m \geq 4$ there are at most six of these $\Delta_m \cong T \subseteq N_G[S]$, and for $m = 3$ there are at most seven.

Proof. By Remark 3.1, the boundary of $\Delta_m$ consists of three straight paths of length $m$. Thus, to find $\Delta_m \cong T \subseteq N_G[S]$, we start by describing all straight paths with $m$ edges within $N_G[S]$. Each of those paths either completely lies in $N_G[S] \setminus S$ or it intersects $S$ in at least one vertex.

If all boundary paths of $T$ lie in $N_G[S] \setminus S$, Lemma 3.4 would imply $\partial T = N_G[S] \setminus S$ since both are cyclic graphs. However, this would imply $S \subseteq T$, contradicting $S \cong T$. Thus, at least one boundary path of $T$ intersects $S$. We can construct each of those paths by extending a straight path from $S$ into $N_G[S] \setminus S$. The first extended edge cannot be a boundary edge of $N_G[S]$.

By Lemma 3.4, vertices in $N_G[S] \setminus S$ are incident to at most three facets. Thus, any straight path in $S$ can only be extended by one edge into $N_G[S] \setminus S$ on each side (the path degree of any extension is at most 2), which can be seen in Figure 6.

Figure 6: Possible extension of a straight path into the neighbourhood of $\Delta_m$.

Consequently, we start with the straight paths in $S$ with at least $m - 2$ edges, extend them to straight paths of length $m$ and add the possible two other sides of the $\Delta_m$-shaped subgraph. Up to symmetry, the preimages of the paths of length $m - 2$ with respect to $\mu$ are described in Remark 3.1 whose notation ($\alpha, \beta,$ and $\gamma$) we employ. The images of $\alpha, \beta, \gamma \subseteq \Delta_m$ under $\mu$ are called $\alpha^\mu, \beta^\mu, \gamma^\mu \subseteq S$.

A straight path with $m$ edges can only be the boundary of at most two $T \cong \Delta_m$. However, if we start at a vertex $\mu(m - t - k, t, k) \in S$ and follow a straight path “down” (i.e. in the direction of smaller third coordinates), it can only go $k$ steps while staying in $S$. Conversely, if we follow a straight path in the direction of larger third coordinates (“up”), we can go at most $m - k$ steps within $S$. We conclude that from any vertex of

\[
\begin{pmatrix}
\alpha^\mu \\
\beta^\mu \\
\gamma^\mu 
\end{pmatrix}
\]

we can go \[\begin{pmatrix}
m + 1 \\
m \\
m - 1
\end{pmatrix}\] steps ‘up’ and \[\begin{pmatrix}
1 \\
2 \\
3
\end{pmatrix}\] steps ‘down’.
staying in \( N_G[S] \).

Since \( m \geq 3 \), beginning at a vertex of \( \alpha^\mu \) or \( \beta^\mu \) only in the ‘up’-direction we find a straight path of length \( m \). For \( m = 3 \), beginning at a vertex of \( \gamma^\mu \) only in the ‘down’-direction we have the space for a straight path of length \( m \) and for \( m \geq 4 \), no such straight path beginning at a vertex of \( \gamma^\mu \) exists, neither ‘upwards’ nor ‘downwards’.

Now, we discuss which parts of \( \alpha^\mu \), \( \beta^\mu \), and \( \gamma^\mu \) can be the straight boundary paths of a \( \Delta_m \cong T \subseteq N_G[S] \). Since \( \alpha^\mu \) lies in \( \partial S \) with \( S \cong \Delta_m \), it cannot be part of the boundary of \( T \). Thus, two possible extensions of \( \alpha \) and two possible extensions of \( \beta^\mu \) remain, as can be seen in Figure 7.

We extend \( \alpha^\mu_{\{0, \ldots, m-1\}} \) at 0. This path ends at \((1, m - 1, 0)\), where a rotation of \( \beta^\mu \) starts:

\[
\beta' : \{0, \ldots, m-1\} \to \mathbb{Z}^3, \quad t \mapsto (0, m - t, t).
\]

If these paths lie in the boundary of \( T \), the group action from Subsection 2.2 allows us to find a hexagonal chart \( \nu : \Delta_m \to T \) with

\[
\nu(0, m, 0) = \mu(1, m - 1, 0) \quad \text{and} \quad \nu(1, m - 1, 0) = \mu(2, m - 2, 0).
\]

A translation allows us to rewrite the hexagonal chart as

\[
\mu_{1, -1, 0} : \Delta_m + (1, -1, 0) \to T, \quad x \mapsto \nu(x + (-1, 1, 0)).
\]

In particular, all \( \mu(k, 0, m - k) \) with \( 2 \leq k \leq m - 1 \) have degree 6 if this chart exists.
Since the group action of Subsection 2.2 acts transitively on the extensions of $\alpha$ and $\beta$, respectively, there are exactly six triangular-shaped graphs $T$ that could be constructed in such a manner. These correspond to the elements of $\bar{D}_0$.

To complete the construction of $\Delta_m \cong T \subseteq N_G[S]$, we consider the path $\gamma$ for $m = 3$. It has to be extended in both directions. Similarly to our construction of $\mu_{1,-1,0}$, we extend the chart $\mu$ to incorporate the vertices $(2,2,-1)$, $(2,-1,2)$, and $(-1,2,2)$.

Next, we combine these different hexagonal charts. We show that they define compatible maps.

**Lemma 5.3.** Let $G$ be a pika and $\mu : \Delta_m \rightarrow S \subseteq G$ be a standard chart with $m \geq 3$. Let $\mu_1 : H_1 \rightarrow T_1$ and $\mu_2 : H_2 \rightarrow T_2$ be two hexagonal charts from Lemma 5.2. For any $x \in H_1 \cap H_2$, we have $\mu_1(x) = \mu_2(x)$.

**Proof.** Let $x \in H_1 \cap H_2$. If $x \in \Delta_m$, the claim follows directly from Lemma 5.2. Otherwise, we have to consider the extension construction along facet-paths.

If there is a facet path $f_1f_2$ in $H_1 \cap H_2$ with $f_1 \subseteq \Delta_m$ and $x \in f_2$, both $\mu_1$ and $\mu_2$ have to map $x$ to the same value. Thus, only the corner vertices of $\vec{d} + \Delta_m$ might be problematic.

Without loss of generality (Subsection 2.2), let $H_1 = (1,-1,0) + \Delta_m$. The corner vertex $(m+1,-1,0)$ does not lie in any $H_2$, so it can be ignored. The corner $x = (1,-1,m)$ also lies in $H_2 = (0,-1,1) + \Delta_m$. We can define $x$ by a facet path in $H_1$ with three facets. Since $m \geq 3$, this facet path also lies in $H_2$. Thus, the charts $\mu_1$ and $\mu_2$ cannot conflict, as can be seen in Figure 8.

**Figure 8:** The edge-face-paths that can be used for extending the standard chart to a neighbouring $\Delta_m$

Finally, we put all of the pieces together.

**Lemma 5.4.** Let $G$ be a pika and $\mu : \Delta_m \rightarrow S \subseteq G$ be a standard chart with $m \geq 3$. There is a hexagonal chart $\hat{\mu} : E \rightarrow \hat{S}$, with $\Delta_m \subseteq E$ and $S \subseteq \hat{S} \subseteq G$, such that $\hat{\mu}|_{\Delta_m} = \mu$ and such that any $T \cong \Delta_m$ with $T \subseteq N_G[S]$ is either

- the image of $\hat{\mu} \circ \Delta^\vec{t}_m$ for some $\vec{t} \in \bar{D}_0$ or
the image of $\hat{\mu}(\nabla_3)$ from Lemma 5.2 if $m = 3$.

Proof. For any $\Delta_m \cong T \subseteq N_G[S]$, we construct the hexagonal chart $\mu_T : H_T \to T$ from Lemma 5.2. We define $E$ as the union of $\Delta_m$ with all those $H_T$.

For $m > 3$, this graph is

\[ E := \Delta_m + \bigcup_{\mathbf{d} \in \mathcal{D}_0} (\mathbf{d} + \Delta_m), \]

in which $\mathcal{D}$ is the set of all applicable translation vectors. For $m = 3$, the graph also contains $\nabla_3$. By Lemma 5.2 and Lemma 5.3, we can define $\hat{\mu}$ as follows:

\[ \hat{\mu} : E \to G \quad x \mapsto \begin{cases} \mu(x), & x \in \Delta_m, \\ \mu_T(x), & x \in H_T \text{ for } \Delta_m \cong T \subseteq N_G[S]. \end{cases} \]

It remains to show that $\hat{\mu}$ is injective. Since $\mu$ is injective on $S$, we only need to consider pairs of vertices, in which at least one is from $N_G[S] \setminus S$.

1. Let $x \in E \setminus \Delta_m$ and $y \in \Delta_m$ such that $\hat{\mu}(x) = \hat{\mu}(y) \in S$. Then, there is a $\mathbf{d} \in \mathcal{D}_0$ such that $x \in \mathbf{d} + \Delta_m$. By construction of $\mu_T$, the point $x$ is mapped to a point in $N_G[S]$, which is different from $S$ by Lemma 3.2(1), in contradiction to our assumption.

2. Let $x, y \in E \setminus \Delta_m$ such that $\hat{\mu}(x) = \hat{\mu}(y)$. Except the translates of the triangle tips, every vertex in $E \setminus \Delta_m$ in adjacent to two different boundary vertices of $\Delta_m$. Thus, we first consider the case where $x$ and $y$ are adjacent to different boundary vertices $a$ and $b$, respectively. Thus, we have a path of length 2 from $\hat{\mu}(a) \in \partial S$ over $\hat{\mu}(x) \in N_G[S] \setminus S$ to $\hat{\mu}(b) \in \partial S$. Then, Lemma 3.2[1] implies that $\{\hat{\mu}(a), \hat{\mu}(b), \hat{\mu}(x)\}$ is a facet. By our proof of well-definedness, this is only possible if $x = y$.

It remains to show the claim if $x$ and $y$ both are adjacent to the same triangle tip, say $x = (0, -1, m + 1)$ and $y = (-1, 0, m + 1)$. In this case, $\hat{\mu}(x) = \hat{\mu}(y)$ would imply $\deg(\hat{\mu}(0, 0, m)) = 5$, in contradiction to $\delta = 6$.

The condition $m \geq 3$ in Lemma 5.4 is necessary. Extending a chart $\mu : \Delta_2 \to S$ simultaneously to all subgraphs $T \cong \Delta_2$ of $G$ is only possible if the vertices $(1, 1, 0), (1, 0, 1)$, and $(0, 1, 1)$ are mapped to vertices of degree 6.

6. Full Clique Description

In this section, we show that the cliques from Subsection 4.2 are all cliques of the geometric clique graph $G_n$. This section culminates in a full description of all cliques of $G_n$ (Corollary 6.9) and the correspondence to the vertices of $G_{n+1}$ (Theorem 6.8).

For $m \geq 3$, we employ the chart extensions from Section 5. The smaller cases have to be argued differently.
6.1. Exceptional (Small) Cases

In this subsection, we discuss the cliques which only contain elements of levels smaller than 3.

**Lemma 6.1.** Let $C$ be a clique of $G_n$, in which every vertex is of level 0 or 2. Then, $C$ is one of the cliques described in Lemma 4.6.

**Proof.** We start with the case where all vertices of $C$ are of level 0, i.e. they are isomorphic to $\Delta_0$. In this case, they form a clique of $G$, i.e. a triangle $S \cong \Delta_1$. So, $C$ is constructed from $S$ by Lemma 4.6.

For the remainder, we assume that $C$ contains a vertex of level 2, i.e. a subgraph $S \cong \Delta_2$ of $G$. Thus, $C$ lies in the closed neighbourhood $N_G[S]$. We visualise the neighbourhood in Figure 9. Remark 5.1 shows that all the depicted $\Delta_2$-shaped subgraphs exist. We label the subgraphs which are isomorphic to $\Delta_0$ with their preimage under a standard chart of $S$. Since it is not necessarily possible to extend this chart to all the $\Delta_2$-shaped subgraphs in the neighbourhood, we label those in a new labelling scheme.

We place every label inside the central facet of the subgraph. Two different $\Delta_2$-shaped subgraphs are adjacent if and only if their central facets have facet-distance at most 2. We describe all the cliques of $N_G[S]$ which contain $S$ using the labels in Figure 9.

1. If a corner-vertex of $S$, like $(2, 0, 0)$, is contained in the clique, the common neighbourhood of this vertex and $S$ is a clique, which by Lemma 4.6 is constructed from the $\Delta_1$ in $S$ containing the corner-vertex.

2. Assume no corner-vertex of $S$ is contained in the clique. For all three corner-vertices, there must be an element in the clique which is not adjacent to it; otherwise, the clique would not be maximal. From the remaining elements in $N_G[S]$, the three middle vertices are each adjacent to exactly two corner-vertices, the other elements are each adjacent to exactly one corner-vertex. Thus, to exclude the corner-vertices, either the three middle vertices are in $C$ or at least one other element is in $C$.

   a) In the first case, the clique is constructed from the three middle-vertices using Lemma 4.6.

   b) In the second case, there is an element not adjacent to two of the corner vertices. Without loss of generality, these corner-vertices are $(0, 2, 0)$ and $(0, 0, 2)$ and the element not adjacent to them is $\wedge_2^{-1,0,1}$ or $\vee_2^{1,0,0}$, which will be called $S_1$. Additionally, in $C$ there needs to be an element not adjacent to $(2, 0, 0)$ called $S_2$ which is adjacent to $S_1$ since they both lie in $C$.

   Thus, $S_2$ can be neither $\wedge_2^{-1,1,0}$ nor $\wedge_2^{-1,0,1}$ nor $\wedge_2^{0,-1,1}$ since those are not adjacent to each of the possible $S_1$. Picking $S_2$ to be $\vee_2^{0,1,0}$ is only possible if $S_1$ is $\vee_2^{1,0,0}$. In this case, $N_G[S, S_1, S_2]$ is a clique containing the middle-vertices and we are in case 2a. The same happens if we choose $S_2$ to be $\vee_2^{0,0,1}$. 

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Figure 9: Neighbourhood of an $S \cong \Delta_2$ in $G_n$, subgraphs isomorphic to $\Delta_2$ are labelled in their middle face with the symbols $\wedge$ for 'upward' and $\vee$ for 'downward' facing.
If the degree of \((1, 1, 0)\) is at least 7, there is no other possibility for \(S_2\), but if the degree of \((1, 1, 0)\) is 6, the vertices \(\land_{1}^{1,0,-1}\) and \(\land_{2}^{1,0,-1}\) are adjacent, and if \(S_1 = \land_{2}^{1,0,-1}\) we can choose \(S_2\) to be \(\land_{2}^{1,0,-1}\). In this case, \(S, S_1,\) and \(S_2\) are contained in a common \(T \cong \Delta_3\), from which \(C\) is constructed by Lemma 4.6.

**Lemma 6.2.** Let \(C\) be a clique of \(G_n\), in which every vertex is of level 1. Then, \(C\) is one of the cliques described in Lemma 4.6 or in Lemma 4.7.

**Proof.** If \(C\) is not given as the common neighbourhood of the set of facets incident to a given vertex like in Lemma 4.7, the intersection of the elements of \(C\) is empty and \(C\) has at least three elements. Furthermore, there are two elements of \(C\) that do not intersect in an edge: otherwise, for any three element subset of \(C\) there would be a vertex \(v\) in the intersection of the three elements and the neighbourhood of \(v\) would contain a three-circle.

Thus, we choose two elements \(S\) and \(T\) from \(C\) which intersect in a vertex \(v\) but not in an edge. Since the common intersection of all elements of \(C\) is empty, there must be an element \(U \in C\) not containing \(v\), but intersecting \(S\) and \(T\) in at least one vertex each, which we will call \(s\) and \(t\). Those two vertices are distinct since \(S, T,\) and \(U\) do not have a common vertex, and they are connected by an edge from \(U\). As \(s\) and \(t\) also lie in the neighbourhood of \(v\), the edge \(st\) also lies in this neighbourhood. Since, by assumption, the third vertex of \(U\) is not \(v\), it is the other common neighbour of \(s\) and \(t\). This way, we proved that \(C\) is constructed from the union of \(S, T,\) and \(U\), which is \(\Delta_2\)-shaped, using Lemma 4.6 as it is depicted in Figure 10.

![Figure 10: The clique of \(G_n\) containing \(S, T,\) and \(U\) is constructed from their union \(R \cong \Delta_2\) using Lemma 4.6](image)

**6.2. The Generic (Large) Case**

Up to now we only investigated cliques lying in the lower levels of \(G_n\). The cliques left to discuss are those containing a \(\Delta_m\) with \(m \geq 3\). In this generic case, we describe the neighbourhood \(N_{G_n}[S]\) of a \(S \cong \Delta_m\) explicitly by using triangle inclusion maps. Then, we classify the cliques there explicitly.
We can describe the adjacency conditions of Definition \ref{def:adjacency} combinatorially with triangle inclusion maps. Additional to the aforementioned set
\[
\vec{D}_0 = \{(1, -1, 0), (1, 0, -1), (-1, 1, 0), (0, 1, -1), (-1, 0, 1), (0, -1, 1)\},
\]
we define the following sets of coordinates:
\[
\vec{D}_{-2} := \{(2, 0, 0), (1, 1, 0), (0, 2, 0), (0, 1, 1), (0, 0, 2), (1, 0, 1)\},
\vec{D}_{-4} := \{(2, 1, 1), (1, 2, 1), (1, 1, 2)\}, \text{ and}
\vec{D}_{-6} := \{(2, 2, 2)\}.
\]

**Lemma 6.3.** Let \( \mu : H \to F \subseteq G \) be a hexagonal chart of the pika \( G \). Let \( \vec{s}, \vec{t} \in \mathbb{Z}^3 \) and \( k \in \{0, 2, 4, 6\} \) and \( m \geq k \), be such that the images of \( \Delta^\vec{s}_m \) and \( \Delta^\vec{t}_{m-k} \) are subsets of \( H \). Further, let \( S \subseteq F \) be the image of \( \mu \circ \Delta^\vec{s}_m \) and \( T \subseteq F \) the image of \( \mu \circ \Delta^\vec{t}_{m-k} \). Then, \( S \) and \( T \) are adjacent in the clique graph \( G_n \) for all \( n \geq m \) with \( n \equiv_2 m \) if and only if \( \vec{t} - \vec{s} \in \vec{D}_{-k} \).

**Proof.** Since \( \mu \) is an isomorphism, \( S \) and \( T \) are adjacent in \( G_n \) if and only if the images of \( \Delta^\vec{s}_m \) and \( \Delta^\vec{t}_{m-k} \) are connected by an edge of the \( n \)-th iterated geometric clique graph \( (\text{Hex}_{m+|\vec{s}|})_n \) of the hexagonal grid. Therefore, it is sufficient to prove the claim for \( G = \text{Hex}_m \). Since
\[
\text{Hex}_{m+|\vec{s}|} \to \text{Hex}_m, \quad a \mapsto a - \vec{s}
\]
is an isomorphism between hexagonal grids, we can assume without loss of generality, that \( S = \Delta_m \) and \( T \) is the image of \( \Delta^\vec{t}_{m-k} \) with corners \( (m-k, 0, 0) + \vec{t} - \vec{s}, (0, m-k, 0) + \vec{t} - \vec{s}, \text{ and } (0, 0, m-k) + \vec{t} - \vec{s} \). Now, we distinguish with respect to \( k \):

1. \( k = 0 \): \( T \) is adjacent to \( S \) if the corners of \( T \) lie in the neighbourhood \( N_G[S] \). A vertex \( (v_1, v_2, v_3) \in \text{Hex}_m \) lies in \( N_G[\Delta_m] \) if and only if \(-1 \leq v_i \leq m + 1 \). Since the components of \( \vec{t} - \vec{s} \) sum to 0, this is equivalent to \( \vec{t} - \vec{s} \in \vec{D}_0 \).

2. \( k = 2 \): \( T \subseteq S \) if and only if the corners of \( T \) lie in \( S \). Equivalently, all components of \( \vec{t} - \vec{s} \) have to be non-negative. Since the components sum to 2, this is equivalent to \( \vec{t} - \vec{s} \in \vec{D}_{-2} \).

3. \( k = 4 \): The corners of \( T \) do not lie on the boundary if and only if all components of \( \vec{t} - \vec{s} \) are at least 1. Since the components sum to 4, this is equivalent to \( \vec{t} - \vec{s} \in \vec{D}_{-4} \).

4. \( k = 6 \): The corners of \( T \) have distance 2 from the boundary of \( S \) if and only if all components of \( \vec{t} - \vec{s} \) are at least 2. Since the components sum to 6, this is equivalent to \( \vec{t} - \vec{s} \in \vec{D}_{-6} \). \( \square \)

From every clique we can choose an element \( S \) of maximal level \( m \). Then, we describe the clique as a clique of the lower-level neighbourhood \( N_{G_n}[S] \cap V(G_m) \). To describe \( N_{G_n}[S] \) combinatorially, we introduce the **local hexagonal graph**: Its vertices are
\[
\mathcal{V}_{\text{LHG}} := \left\{ v_0^{0,0,0} \right\} \cup \left\{ v_r^{\vec{d}} \mid r \in \{0, -2, -4, -6\}, \vec{d} \in \vec{D}_r \right\}
\]

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and its edges are given by

\[ E_{LHG} := \left\{(v^x_r, v^y_{r-k}) \mid \vec{y} - \vec{x} \in \vec{D}_k \text{ for a } k \in \{0, 2, 4, 6\}\right\}. \]

For a set \( Q \) of vertices of \( G_n \) of a given level \( m \), the **lower level neighbourhood** of \( Q \) is defined as the set \( N_{G_m}^Q \subseteq G_n \), which consists of all the common neighbours of the elements in \( Q \) that have a level of at most \( m \).

**Lemma 6.4.** Let \( S \cong \Delta_m \) be a vertex in \( G_n \) with \( m \geq 3 \). The lower-level-neighbourhood of \( S \) in \( G_n \) is isomorphic to an induced subgraph of the local hexagonal graph.

**Proof.** We give a graph monomorphism \( \varphi : N_{G_m}[S] \cap G_m \rightarrow LHG \) that maps non-edges to non-edges. We start with the generic case \( m \geq 6 \) and a standard chart \( \Delta_m \rightarrow S \). By Lemma 5.4, we can extend it to a hexagonal chart \( \mu : E \rightarrow G \) such that all adjacent \( T \cong \Delta_m \) are contained. We have the following adjacencies of smaller level:

1. The inclusions of \( \Delta_{m-2} \) into \( \Delta_m = S \) are all described by triangle inclusion maps since \( m > 4 \) (Lemma 4.5).
2. The inclusions of \( \Delta_{m-4} \) into \( \Delta_{m-3} \) (compare Remark 4.3) are all described by triangle inclusion maps since \( m - 3 > 2 \) (Lemma 4.4).
3. The inclusion of \( \Delta_{m-6} \) into \( \Delta_{m-6} \) (compare Remark 4.3) is unique and also given by a triangle inclusion map.

Thus, all adjacent triangles of smaller level are given by triangle inclusion maps. Therefore, by Lemma 6.4, \( \varphi(\mu(\Delta^e_{m-k})) = v^e_{r-k} \) is a monomorphism of the required property, but it is not necessarily an isomorphism since not all of the \( \vec{D}_0 \)-translated neighbours of \( S \) need to be present.

We continue with the case \( m = 5 \), illustrated in Figure 11. Since \( 5 > 4 \), all neighbours of level \( m - 2 \) are given by triangle inclusion maps (Lemma 4.5). For level \( m - 4 = 1 \), we need to consider inclusions of \( \Delta_1 \) into \( \Delta_2 \) (Remark 4.3). By Lemma 4.4, one exceptional case occurs: a graph \( T \cong \Delta_1 \) with vertices \((2, 1, 2), (2, 2, 1), \) and \((1, 2, 2)\). However, there is no neighbour of level \( m - 6 \) since \( m - 6 = -1 \). Thus, we define \( \varphi \) as in the generic case, but we map \( T \) to \( v^{2,2,2}_r \). Since Lemma 6.3 shows the correct edge correspondence for all neighbours given by triangle inclusion maps, it remains to show that the edges of the local hexagonal graph correctly describe the adjacencies of \( v^{2,2,2}_r \).

- By definition, the middle \( \Delta_1 \) is adjacent to \( \Delta_5 \) in \( G_n \) as well as \( v^{0,0,0}_0 \) and \( v^{2,2,2}_{-6} \) are adjacent in \( LHG \).
- Furthermore, \( T \) is adjacent to the three triangles \( \Delta^{0,1,1}_3, \Delta^{1,0,1}_3, \) and \( \Delta^{1,1,0}_3 \) (as an exceptional case in Lemma 4.5), exactly as in the local hexagonal graph.
- Since \( T \) is adjacent to all three relevant \( \Delta_1 \), the description of the local hexagonal graph is correct again.
Next, we move on to $m = 4$. Again, the only difference to the generic case is the designated preimage of $v_{-6}^{2,2,2}$, which is a $\nabla_2$ with corners $(2,2,0)$, $(0,2,2)$, and $(2,0,2)$. As can be seen in Figure 12, we check the adjacencies to the levels 0 and 2. Both are satisfied again.

Finally, we deal with $m = 3$, with several differences to the generic case:

1. There is another adjacent $\Delta_3$ adjacent to $S$ which is “facing down”. We denote it by $T$ and set $\varphi(T) = v_{-4}^{2,2,2}$. 

2. There are the three subgraphs isomorphic to $\Delta_1$ from Lemma 4.5 adjacent to $S$, called $T_{1,0,0}, T_{0,1,0}$, and $T_{0,0,1}$, and we map them by $\varphi(T_{1,0,0}) = v_{-4}^{1,2,1}$, $\varphi(T_{0,1,0}) = v_{-4}^{1,2,1}$, and $\varphi(T_{0,0,1}) = v_{-4}^{1,1,2}$.

Figure 13 shows that the local hexagonal graph describes the adjacencies correctly.

We describe the cliques of the local hexagonal graph.

**Lemma 6.5.** Let $C$ be a clique in the local hexagonal graph with $v_0^{0,0,0} \in C$. Then, one
Figure 12: Adjacencies for $m = 4$

Figure 13: Adjacencies for $m = 3$
of the following three cases holds:

1. \( C = C_{-6} := \{ v_0^{0,0}, v_2^{-1,0}, v_0^{-1,1}, v_0^{1,0}, v_0, v_2^{-2,1}, v_4^{-1,2}, v_0^{2,2} \} \)

   \[ N_{\text{LHG}} \left[ v_0^{-2,1}, v_4^{-1,2}, v_0^{2,2} \right] = N_{\text{LHG}} \left[ v_0^{-2,2} \right], \]

2. \( C = C_{-4}^\varepsilon := \{ v_0^{(1,0)+\varepsilon}, v_0^{(0,0)+\varepsilon}, v_0^{(0,1)+\varepsilon}, v_0^{(1,1)+\varepsilon} \} \)

   \[ N_{\text{LHG}} \left[ v_0^{(1,0)+\varepsilon}, v_0^{(0,1)+\varepsilon}, v_0^{(0,0)+\varepsilon}, v_0^{(1,1)+\varepsilon} \right] = N_{\text{LHG}} \left[ v_0^{2\varepsilon} \right] \text{ for an } \varepsilon \in \vec{E}, \]

3. \( C = C_{-2}^\varepsilon := \{ v_0^{(1,0)-\varepsilon}, v_0^{(0,1)-\varepsilon}, v_0^{(0,0)-\varepsilon}, v_0^{(1,1)-\varepsilon} \} \)

   \[ N_{\text{LHG}} \left[ v_0^{(1,0)-\varepsilon}, v_0^{(0,1)-\varepsilon}, v_0^{(0,0)-\varepsilon}, v_0^{(1,1)-\varepsilon} \right] \text{ for an } \varepsilon \in \vec{E}. \]

**Proof.** By the definition of the local hexagonal graph, the given sets form complete subgraphs. Furthermore, they are represented as common neighbourhoods of triangles or as closed neighbourhoods of vertices in the claimed way. Thus, they are also maximal.

It remains to show that there cannot be any other cliques.

If \( v_{-2,2}^{2,2} \in C \), we get \( C = N_{\text{LHG}} \left[ v_{-2,2}^{2,2} \right] \) since this neighbourhood already forms a clique. Thus, the first case of the lemma holds.

Otherwise, \( C \) contains an element not incident to \( v_{-2,2}^{2,2} \). Those elements are either given by \( v_{-2}^{2\varepsilon} \) for an \( \varepsilon \in \vec{E} \) or by \( v_0^{(2,2)-e_i} \) for \( e_1, e_2 \in \vec{E} \) with \( e_1 \neq e_2 \).

If \( v_{-2}^{2\varepsilon} \in C \), we get \( C = N_{\text{LHG}} \left[ v_{-2}^{2\varepsilon} \right] \) since this neighbourhood already forms a clique. Thus, the second case of the lemma holds. This neighbourhood is a clique.

Finally, we assume \( v_0^{(2,2)-e_1} \in C \), but \( v_{-2}^{2\varepsilon} \notin C \) (the other two vertices \( v_{-2}^{2\varepsilon} \) with \( e \in \vec{E} \) are not adjacent to \( v_0^{(2,2)-e_1} \in C \), anyway). For reasons of symmetry, we can choose \( e_1 = (1,0,0) \) and \( e_2 = (0,1,0) \). Thus, we have \( v_0^{(1,1,0)} \in C \), but \( v_0^{(0,2,0)} \notin C \). The set of neighbours of \( v_0^{(1,1,0)} \) is

\[ \{ v_0^{0,0}, v_0^{0,1,-1}, v_0^{0,1,0}, v_0^{0,1,1}, v_0^{0,2,0}, v_0^{0,2,1} \}. \]

Only \( v_0^{0,1,0} \) is not adjacent to \( v_0^{0,2,0} \), so it has to lie in \( C \) since \( C \) is maximal. Then, \( C = N_{\text{LHG}} \left[ v_0^{(1,0,0)}, v_0^{(0,1,1)}, v_0^{(0,0,0)} \right] = C_{-2}^\varepsilon \), as described in the third case of the lemma.

The following lemma describes how we can find the cliques of an induced subgraph using the cliques of the surrounding graph.

**Lemma 6.6.** For a graph \( G \) and an induced subgraph \( H \), every clique of \( H \) is given as the intersections of a (not necessarily unique) clique of \( G \) with \( H \).

**Proof.** Let \( C \) be a clique of \( H \). Then, \( C \) is a complete subgraph of \( G \). Therefore, there is at least one clique \( C_G \) of \( G \) containing \( C \). Obviously, \( C \subseteq C_G \cap H \). If there was an \( x \in C_G \cap H \setminus C \), the union \( C \cup \{ x \} \) were a complete subgraph of \( H \) since \( H \) is an induced subgraph in contradiction to \( C \) being chosen maximal. \( \square \)
We apply this to the image of a lower-level-neighbourhood under the embedding given in \[6.3\] This way, we can classify all the cliques of $G_n$.

**Theorem 6.7.** If $C$ is a clique of $G_n$ containing a vertex $\Delta_m$ with $m \geq 3$, $C$ is given by the construction in \[4.6\] or \[4.7\].

**Proof.** Let $C$ be a clique of $G_n$ and let $S \cong \Delta_m$ be a vertex with maximal $m \geq 3$ of $C$. Thus, $C$ is contained in the lower level neighbourhood of $S$. The lower level neighbourhood of $S$ is isomorphic to an induced subgraph $H$ of the local hexagonal graph containing $v_0^{0,0,0}$ and the isomorphism $\mu$ maps $S$ to $v_0^{0,0,0}$. Thus, by Lemma \[6.6\] $C$ is isomorphic to the intersection of $H$ with a clique $C_{\text{LHG}}$ of the local hexagonal graph.

Thus, $C_{\text{LHG}}$ is one of the cliques given in Lemma \[4.6\]. For reasons of symmetry, we can restrict our investigation to the cliques $C_{-6}, C_{-4}^{1,0,0}$, and $C_{-2}^{1,0,0}$.

1. If $C_{\text{LHG}} = C_{-6}$, and $m \geq 4$, the preimages of $v_{-4}^{2,1,1}, v_{-4}^{1,2,1}$, and $v_{-4}^{1,1,2}$ are subgraphs of $S$ isomorphic to $\Delta_{m-4}$. Thus, they do exist and $C$ is given by the construction of Lemma \[4.6\]. If $m = 3$, the preimages of $v_{-4}^{2,1,1}, v_{-4}^{1,2,1}$, and $v_{-4}^{1,1,2}$ do exist, but they are not contained in a common $\Delta_3$ and we cannot apply Lemma \[4.6\]. Therefore, we look at the preimages of $v_{-2}^{1,1,0}, v_{-2}^{0,1,1}, v_{-2}^{1,0,1}, v_{-2}^{2,1,1}, v_{-4}^{1,1,2}$, and $v_{-4}^{1,1,2}$ which do exist, since they are induced subgraphs of $S$ isomorphic to $\Delta_1$. Furthermore, those preimages are the subgraphs isomorphic to $\Delta_1$ containing the middle vertex of $S$. Thus $C$ is constructed from this vertex by Lemma \[4.7\].

2. If $C_{\text{LHG}} = C_{-4}^{1,0,0}$, the preimages of $v_{-2}^{(1,0,0)+\vec{e}}, v_{-2}^{(0,1,0)+\vec{e}}$, and $v_{-2}^{(0,0,1)+\vec{e}}$ are subgraphs of $S$ isomorphic to $\Delta_{m-2}$. Thus, they exist and $C$ is given by the construction of Lemma \[4.6\].

3. If $C_{\text{LHG}} = C_{-2}^{1,0,0}$, either the preimages of $v_0^{(1,0,0)-\vec{e}}, v_0^{(0,1,0)-\vec{e}}, v_0^{(0,0,1)-\vec{e}}$ exist and $C$ is their common neighbourhood, or one of them does not exist. In the second case, without loss of generality, $\vec{e} = (1,0,0)$. Thus $v_0^{(1,0,0)-\vec{e}} = v_0^{0,0,0}$ and there is no preimage of $v_0^{(0,0,1)-\vec{e}} = v_0^{1,0,-1}$. The remaining elements of $C_2^{\vec{e}}$ are at most $v_0^{0,0,0}, v_{-1,1,0}$, and $v_{0,1,1}$ which also lie in $C_{-4}^{0,1,0}$ Hence, we can also see $C$ as the intersection of LHG with $C_{-4}^{0,1,0}$ and, by applying the second case, it is given by the construction of Lemma \[4.6\].

We finally managed to prove the surjectivity of the map from Remark \[4.8\] between the vertices of $G_{n+1}$ and the cliques of $G_n$. We continue proving the injectivity.

**Theorem 6.8.** The map

$$C: V(G_{n+1}) \to \{\text{cliques of } G_n\},$$

$$S \mapsto \text{the clique from }$$

\[\begin{cases} \text{Lemma } 4.6 \text{ if } S \text{ is of level } 0, \\ \text{Lemma } 4.7 \text{ otherwise, } \end{cases}\]

is bijective.
\[ C(S) = \bigcup_{|\cdot|=3} \bigcup_{|\cdot|=0, \text{ if } n=m, \text{ or } n=m+2} \left\{ \emptyset, \begin{cases} \bar{\mu} \Delta_1^{(1,1)} (\nabla_2), & \text{if } m=1 \text{ and } n \leq 1, \\ \{\mu(\nabla_1)\}, & \text{if } m=2, \\ \{S \setminus \partial S\}, & \text{if } m \geq 3. \end{cases} \right\} \]

- \( M_{m-1} \) consists of the elements \( \Delta_{m-1} \cong \mu \vec{e} \) for \( \vec{e} \in \vec{E} \).
- \( M_{m+1} \) consists of the elements \( \Delta_{m+1} \nrightarrow T \) fulfilling \( \mu = \nu \vec{e} \) for an \( \vec{e} \in \vec{E} \).
- \( M_{m+3} \) consists of the element \( \Delta_{m+3} \cong T \) enclosing \( S \) with distance 1, i.e. \( S = T \setminus \partial T \).

2. For \( \Delta_0 \cong S \in V(G_{n+1}) \), if we denote the vertex of \( S \) by \( v \), an explicit description of \( C(S) \) is given through
\[ C(S) = \left\{ T \in V(G_n) \mid T \cong \Delta_1, \ S \subseteq T \right\} \cup \left\{ T \in V(G_n) \mid T \cong \Delta_3, \ S \subseteq T \setminus \partial T \right\} \]
\[ [|\cdot|=0, \text{ if } \deg_G(v) \geq 7 \text{ or } n \leq 2, \]
\[ |\cdot|=2, \text{ if } \deg_G(v) = 6 \text{ and } n \geq 5. \]

7. **Clique intersections of \( G_n \)**

After having constructed all cliques of the geometric clique graph \( G_n \) (and proven that these cliques correspond to vertices of \( G_{n+1} \)), we need to show that two cliques \( C(S_1) \) and \( C(S_2) \) intersect if and only if the corresponding vertices \( S_1 \) and \( S_2 \) in \( G_{n+1} \) are connected by an edge. From now on, we assume that \( S_1 \cong \Delta_m \) and \( S_2 \cong \Delta_{m+k} \) for \( k \in \{0, 2, 4, 6\} \) and \( m \geq 0 \).
7.1. Case: $S_2 \cong \Delta_m$

**Lemma 7.1.** For $S_1, S_2 \in V(G_{n+1})$ with $S_1 \cong S_2 \cong \Delta_m$ for some $m \geq 0$, the cliques $C(S_1)$ and $C(S_2)$ do not intersect in a vertex $T \cong \Delta_{m+3}$. Furthermore, if $m \geq 4$, they do not intersect in an element $T \cong \Delta_{m-3}$.

*Proof.* From Corollary 6.9 we see that if a clique $C(S)$ contains an element $T \cong \Delta_{m+3}$, the clique is uniquely defined by this element since $S = T \setminus \partial T$. Furthermore, for $m \geq 4$ the clique is also uniquely defined by an element $T \cong \Delta_{m-3}$ since then $T = S \setminus \partial S$, which has only one solution $S$. In either way, the vertex $T$ cannot lie in two distinct cliques of $G_n$. \hfill \Box

**Lemma 7.2.** For $S_1, S_2 \in V(G_{n+1})$ with $S_1 \cong S_2 \cong \Delta_0$, the cliques $C(S_1)$ and $C(S_2)$ intersect non-trivially if and only if $S_1$ and $S_2$ are adjacent in $G_{n+1}$, i.e. they are adjacent in $G$.

*Proof.* At first, we suppose that $S_1$ and $S_2$ are adjacent in $G$. Since $G$ is locally cyclic, they have two common neighbours, there is a $\Delta_1 \cong T \subseteq G$ with $S_1 \subseteq T$ and $S_2 \subseteq T$. Thus, $T$ lies in both $C(S_1)$ and $C(S_2)$. Conversely, suppose there is a $T \in C(S_1) \cap C(S_2)$. By Lemma 7.1, $T \cong \Delta_1$. Furthermore, by Corollary 6.9 $S_1$ and $S_2$ are both vertices of $T$ and thus adjacent. \hfill \Box

**Lemma 7.3.** For $S_1, S_2 \in V(G_{n+1})$ with $S_1 \cong S_2 \cong \Delta_m$ for some $m \geq 1$, if the cliques $C(S_1)$ and $C(S_2)$ intersect in a $T_1 \cong \Delta_{m+1}$, they also intersect in a $T_2 \cong \Delta_{m-1}$.

*Proof.* If $T \in C(S_1) \cap C(S_2)$ for a $\Delta_{m+1} \subseteq T$, both $S_1 \subseteq T$ and $S_2 \subseteq T$. If $m + 1 \geq 3$, it follows that $S_1 = \nu e$ and $S_2 = \nu f$ for $e, f \in \vec{E}$ and $e \neq f$, implying $S_1 \cap S_2 \cong \Delta_{m-1}$.

If $m + 1 = 2$, either the situation is the same as in the foregoing case or, without loss of generality, $S_1 = \mu \vec{e}$ for an $\vec{e} \in \vec{E}$ and $S_2 = \mu (\nabla 1)$. Even in this case, $S_1$ and $S_2$ intersect in a vertex. \hfill \Box

Consequently, if $m \geq 1$, we only have to investigate whether two cliques intersect in a $\Delta_{m-1}$-shaped vertex of $G_n$.

**Lemma 7.4.** For $S_1, S_2 \in V(G_{n+1})$ with $S_1 \cong S_2 \cong \Delta_m$ for an $m \geq 1$, the cliques $C(S_1)$ and $C(S_2)$ intersect non-trivially if and only if $S_1$ and $S_2$ are adjacent in $G_{n+1}$, i.e. $S_1 \subseteq N_G [S_2]$.

*Proof.* If there is a $T \in C(S_1) \cap C(S_2)$, by Lemma 7.1 and Lemma 7.3 we can choose $T \cong \Delta_{m-1}$. Thus, we have $S_1 \subseteq N_G [T] \subseteq N_G [S_2]$, where $S_1 \subseteq N_G [T]$ follows from Lemma 7.3. Conversely, suppose $S_1 \subseteq N_G [S_2]$. We distinguish between the values of $m$.

- If $m = 1$, $S_1$ is one of the additional faces in $N_G [S_2]$. Thus $S_1$ and $S_2$ intersect in at least one vertex, which lies in both $C(S_1)$ and $C(S_2)$.
- If $m = 2$, there is a $\Delta_1 \cong T \subseteq S_1 \cap S_2$. Thus, $T \in C(S_1) \cap C(S_2)$. 

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• If $m \geq 4$, let $\mu: \Delta_m \to S_1$ be a standard chart. By Lemma 4.4, there is an extension $\hat{\mu}: E \to \hat{S}$ such that $S_2$ is the image of $\hat{\mu} \circ \Delta^{\hat{\mu}}$ for a $\hat{\mu} \in \mathcal{D}_0$. Therefore, $S_1 \cap S_2 \cong \hat{\mu}^{-1}(S_1 \cap S_2) = \Delta_m \cap \Delta^{\hat{\mu}}(\Delta_m) \cong \Delta_{m-1}$. Thus, by definition of the cliques, $S_1 \cap S_2 \subseteq C(S_1) \cap C(S_2)$, so they intersect non-trivially.

• If $m = 3$, by Lemma 4.3, we are either in the same situation as for $m \geq 4$, which proves the claim, or $S_2$ lies twisted in the middle of $N_G[S_1]$. In this case, $C(S_1)$ and $C(S_2)$ share the vertex equivalent to the midpoint of both $S_1$ and $S_2$.

Thus, for $S_1 \cong S_2 \in V(G_{n+1})$, intersection in $G_n$ and adjacency in $G_{n+1}$ are equivalent.

7.2. Case: $S_2 \cong \Delta_{m+k}$ for $k \in \{2, 4, 6\}$

Lemma 7.5. For $S_1, S_2 \in V(G_{n+1})$ with $S_1 \cong \Delta_m$ and $S_2 \cong \Delta_{m+2}$ for an $m \geq 0$, the cliques $C(S_1)$ and $C(S_2)$ intersect non-trivially if and only if $S_1$ and $S_2$ are adjacent in $G_{n+1}$, i.e., $S_1 \subseteq S_2$.

Proof. At first, we suppose that $S_1 \subseteq S_2$. By Lemma 4.3 if $m \neq 2$ and by choosing a chart $\Delta_{m+2} \nsubseteq S_2$, we get $S_1 = \mu^{\text{ref}}\tilde{f}$ for some $\tilde{c}, \tilde{f} \in \tilde{E}$ and $T := \mu^{\text{ref}}$ fulfills $S_1 \subseteq T \subseteq S_2$. By Corollary 6.9, $T$ lies in both cliques $C(S_1)$ and $C(S_2)$.

If $m = 2$, the triangle $S_1$ either lies inside a $T \subseteq S_2$ which is isomorphic to $\Delta_3$ or $S_1$ contains the unique $S \cong \Delta_1$, which has distance 1 to $\partial S_2$. In both cases, $S$ or $T$ respectively lies in both $C(S_1)$ and $C(S_2)$.

Conversely, we now suppose that $C(S_1)$ and $C(S_2)$ intersect non-trivially. We distinguish between the possibilities for the element in the intersection. Any element $T \in C(S_1) \cap C(S_2)$ is isomorphic to $\Delta_{m-1}$, $\Delta_{m+1}$, or $\Delta_{m+3}$.

• If $T \cong \Delta_{m-1}$ (i.e., $m \geq 1$), $T$ has distance 1 to the boundary of $S_2$; thus $T = S_2 \setminus \partial S_2$. All the graphs isomorphic to $\Delta_m$, which contain $T$, are subgraphs of $S_2$; thus $S_1 \subseteq S_2$.

• If $T \cong \Delta_{m+1}$, by Corollary 6.9, this means $S_1 \subseteq T \subseteq S_2$, which proves the claim.

• If $T \cong \Delta_{m+3}$, $S_1$ is the unique subgraph of $T$ with distance 1 from the boundary, i.e., $S_1 = T \setminus \partial T$. This subgraph is contained in every subgraph of $T$ isomorphic to $\Delta_{m+2}$; thus $S_1 \subseteq S_2$.

Lemma 7.6. For $S_1, S_2 \in V(G_{n+1})$ with $S_1 \cong \Delta_m$ and $S_2 \cong \Delta_{m+4}$ for an $m \geq 0$, the cliques $C(S_1)$ and $C(S_2)$ intersect non-trivially if and only if $S_1$ and $S_2$ are adjacent in $G_{n+1}$, i.e., $S_1 \subseteq S_2 \setminus \partial S_2$.

Proof. If $S_1 \subseteq S_2 \setminus \partial S_2$, the element $S_2 \setminus \partial S_2 \cong \Delta_{m+1}$ lies in both cliques.

Conversely, if the cliques intersect, they intersect in a $T$ fulfilling $S_1 \subseteq T \subseteq S_2$ with $T \cong \Delta_{m+1}$. Further, the distance of $T$ and $\partial S_2$ is 1 or $T \cong \Delta_{m+3}$ and the distance of $S_1$ and $\partial T$ is 1. Thus, the distance between $S_1$ and $\partial S_2$ is 1 and $S_1 \subseteq S_2 \setminus \partial S_2$. 

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Lemma 7.7. For \( S_1, S_2 \in V(G_{n+1}) \) with \( S_1 \cong \Delta_m \) and \( S_2 \cong \Delta_{m+6} \) for an \( m \geq 0 \), the cliques \( C(S_1) \) and \( C(S_2) \) intersect non-trivially if and only if \( S_1 \) and \( S_2 \) are adjacent in \( G_{n+1} \), i.e. \( S_1 \subseteq S_2 \setminus N_G[\partial S_2] \).

Proof. If \( S_1 \subseteq S_2 \setminus N_G[\partial S_2] \), the only \( T \cong \Delta_{m+3} \) such that \( S_1 \subseteq T \setminus \partial T \) and \( T \subseteq S_2 \setminus \partial S_2 \) is \( T = S_2 \setminus \partial S_2 \), by Corollary 6.9. This subgraph \( T \) lies in both \( C(S_1) \) and \( C(S_2) \).

Conversely, if the cliques intersect in a \( T \cong \Delta_{m+3} \), this subgraph \( T \) has distance 1 to both the boundaries of \( S_1 \) and \( S_2 \); therefore, the boundaries of \( S_1 \) and \( S_2 \) have distance 2.

The preceding lemmata can be summarised in the following way:

Corollary 7.8. For \( S_1, S_2 \in V(G_{n+1}) \), the cliques \( C(S_1) \) and \( C(S_2) \) intersect non-trivially if and only if \( S_1 \) and \( S_2 \) are adjacent in \( G_{n+1} \). Furthermore, for every \( n \in \mathbb{Z}_{\geq 0} \), \( G_n \cong k^n G \).

Now we can finally prove our main theorem for pikas.

Theorem 7.9. Let \( G \) be a triangulary simply connected locally cyclic graph with minimum degree at least 6. If there is an \( m \geq 0 \) such that \( \Delta_m \) cannot be embedded into \( G \), the clique operator is convergent on \( G \).

Proof. If \( \Delta_m \) cannot be embedded into \( G \), this means \( m \geq 2 \) and \( G_{m-2} = G_m \) since the graphs \( G_m \) and \( G_{m-2} \) can only differ in vertices isomorphic to \( \Delta_m \), which would be subgraphs of \( G \). But by Corollary 7.8 this means \( k^m G \cong k^{m-2} G \), which is the definition of the clique operator being convergent on \( G \).

8. Coverings

Up to this point, we only considered triangularly simply connected locally cyclic graphs. Fortunately, any other locally cyclic graph is covered by a simply connected one, to which we will apply Theorem 7.9.

For the generalisation of the theory, we need results from [4] and [8], whose ways of notation look incompatible at first glance. Instead of repeating and re-deriving large parts of both, we show how to fit the definitions of [4] into the setting of [8].

While one of the sources talks about simple graphs with edges and triangles ([4, Section 1, p. 160]), the other one describes complexes with 1-simplices and 2-simplices ([8, Section 1, p. 642]). We can transition from graphs to complexes by constructing the triangular complex \( \mathbb{K}(G) \), which is defined in [4], as we did before.

Lemma 8.1. Let \( G, G' \) be simple graphs. A vertex map \( V(G) \to V(G') \) defines a homomorphism \( G \to G' \) in the sense of [4, Section 2, p. 161] if and only if it defines a map \( \mathbb{K}(G) \to \mathbb{K}(G') \) in the sense of [8, Section 1, p. 642].

Proof. Let \( f : G \to G' \) be a homomorphism in the sense of [4] and \( \{u,v,w\} \) a triangle in \( G \), i.e. a 2-simplex in \( \mathbb{K}(G) \). By assumption, \( \{f(u), f(v)\}, \{f(u), f(w)\}, \{f(v), f(w)\} \) are edges in \( G' \). Thus, \( \{f(u), f(v), f(w)\} \) is a triangle in \( G' \), i.e. a 2-simplex in \( \mathbb{K}(G') \). The other implication is trivial.
We will continue calling these maps graph homomorphisms. We also take the next definition from [4, Section 2, p. 162].

**Definition 8.2.** Let $G, \tilde{G}$ be connected, simple graphs. A graph homomorphism $p : \tilde{G} \to G$ is called a **triangular covering map** if it fulfills the triangle lifting property: For each triangle $\{u, v, w\} \in G$ and each preimage $\tilde{u}$ of $u$, there exists a (unique) triangle $\{\tilde{u}, \tilde{v}, \tilde{w}\}$ in $\tilde{G}$ which is mapped to $\{u, v, w\}$ by $p$.

**Lemma 8.3.** Let $G, \tilde{G}$ be connected, simple graphs and $p : \tilde{G} \to G$ be a homomorphism. Then, $p$ is a triangular covering map if and only if $(\mathbb{K}(\tilde{G}), p)$ is a covering complex ([8, Section 2, p. 650]).

**Proof.** We only need to show that the lifting properties are equivalent. For the first part, assume that $(\mathbb{K}(\tilde{G}), p)$ is a covering complex. By Theorem 2.1 ([8, p. 651]), $p$ has the unique path lifting property. It remains to show that $p$ has the triangle lifting property. Let $\{u, v, w\}$ be a triangle in $G$, i.e. a 2-simplex in $\mathbb{K}(G)$. Since $p^{-1}(\{u, v, w\})$ is the union of 2-simplices, every $\tilde{u} \in p^{-1}(u)$ lies in some triangle of $\tilde{G}$.

For the second part, assume that $p$ is a triangular covering map. Consider a 1-simplex $\{u, v\}$ in $\mathbb{K}(G)$ and let $\tilde{u} \in p^{-1}(u)$. By the unique edge lifting property of $p$ ([4, Section 2, p. 161]), there is a unique $\tilde{v} \in p^{-1}(v)$ adjacent to $\tilde{u}$. By the same argument, $\tilde{u}$ is unique with respect to $\tilde{v}$. Thus, $p^{-1}(\{u, v\})$ splits into pairwise disjoint 1-simplices.

Next, consider a 2-simplex $\{u, v, w\}$ in $\mathbb{K}(G)$. By the triangle lifting property ([4, Section 2, p. 162]), $p^{-1}(\{u, v, w\})$ is the union of triangles in $\tilde{G}$. If two different triangles would intersect, the unique edge lifting property would be violated.

We take the following definition from [8 Section 3, p. 663].

**Definition 8.4.** A **universal covering complex** of $K$ is a covering complex $p : \tilde{K} \to K$ such that, for every covering complex $q : \tilde{J} \to K$ there exists a unique map $h : \tilde{K} \to \tilde{J}$ making the following diagram commute:

![Diagram showing the commuting properties of universal covering complexes](attachment:universal_covering_diagram.png)

Universal covering complexes are unique up to isomorphism, and every connected complex has a universal covering complex ([8 Section 3, p. 663]). We apply this to our graph setting.

**Definition 8.5.** Let $G, \tilde{G}$ be connected, simple graphs and $p : \tilde{G} \to G$ be a triangular covering map. Then, $\tilde{G}$ is the **universal cover** of $G$ if $(\mathbb{K}(\tilde{G}), p)$ is the universal covering complex of $\mathbb{K}(G)$.
We would like to apply Proposition 3.2 from [4] to the universal cover:

**Proposition 8.6.** Let $p: \tilde{G} \to G$ be Galois with group $\Gamma$. Then, $p_k: k\tilde{G} \to kG$ is also Galois with group $\Gamma_k \cong \Gamma$.

To apply this proposition to the universal cover of an arbitrary locally cyclic graph of minimum degree at least 6, we need to show that the universal cover always defines a *Galois triangular map* ([4, Section 3, p. 165]).

**Lemma 8.7.** Let $G, \tilde{G}$ be connected (simple) graphs such that $K(\tilde{G})$ is the universal cover of $K(G)$. Then, the associated covering map $p: K(\tilde{G}) \to K(G)$ is Galois.

**Proof.** Define $\Gamma := \{ \gamma \in \text{Aut}(\tilde{G}) \mid p \circ \gamma = p \}$ (the deck transformations from [8, Section 3, p. 665]). We need to show that $\Gamma$ acts transitively on each fibre of $p$. This is proven in Corollary 3.11 ([8, p. 667]) if $p$ is regular. Since $p$ is a covering map from the universal cover, the regularity follows from the remark at the top of p. 666 in [8].

Now, we can conclude that clique convergence of the universal cover transfers to the original graph.

**Lemma 8.8.** Let $G$ be a locally cyclic graph with $\delta(G) = 6$, whose universal cover is $k$-convergent. Then, $G$ is $k$-convergent as well.

**Proof.** Let $p: \tilde{G} \to G$ be the triangular covering map. By Lemma 8.7, $p$ is a Galois covering map with group $\Gamma$, the group of deck transformations. We define $p_{k^n} : k^n\tilde{G} \to k^nG$ by $p_{k^n} = p_k$ and $p_{k^n}(Q) = \{ p_{k^{n-1}}(v) \mid v \in Q \}$ for every $n \geq 1$. By Proposition 8.6, the maps $p_{k^n}$ are Galois with groups $\Gamma_{k^n} \cong \Gamma$. Since the universal cover is $k$-convergent, there are $n, l \in \mathbb{N}$ such that $k^n\tilde{G} \cong k^{n+l}\tilde{G}$. Thus, $p_{k^n}$ and $p_{k^{n+l}}$ are Galois covering maps with a group isomorphic to $\Gamma$. Therefore, $k^nG \cong k^n(\tilde{G})/\Gamma \cong k^{n+l}(\tilde{G})/\Gamma \cong k^{n+l}G$, and $G$ is $k$-convergent as well.

Combining Lemma 8.8 with Theorem 7.9, we get a general criterion for convergence.

**Theorem 8.9.** If $G$ is a locally cyclic graph with $\delta(G) = 6$ and there is an $m \geq 0$ such that $\Delta_m$ cannot be embedded into the universal cover of $G$, then $G$ is $k$-convergent.

If the graph $G$ is finite, we also have a criterion for divergence.

**Lemma 8.10.** Let $G$ be a finite locally cyclic graph with $\delta(G) = 6$ whose universal cover is $k$-divergent. Then, $G$ is 6-regular.

**Proof.** Let $\tilde{G}$ be the universal cover and $p: \tilde{G} \to G$ the universal covering map. Since the universal cover diverges, there is a $\Delta_m \subseteq \tilde{G}$ for any $m \geq 1$ which is mapped to $G$ via $p$. Since $G$ is finite, the maximal length of a facet-path between any two vertices can be bounded by a finite number $d$.  

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Now, consider $\Delta_{3d+3} \subseteq \tilde{G}$. For any vertex $x \in G$, there is a facet-path between $p(d+1, d+1, d+1)$ and $x$ with length at most $d$. Since $p$ is a covering map (compare Definition 8.2), this facet-path lifts to a facet-path in $\tilde{G}$. All vertices in $\Delta_{3d+3}$ with distance at most $d$ from $(d+1, d+1, d+1)$ are inner vertices of $\Delta_{3d+3}$; thus, the vertex $x$ has the same degree as such an inner vertex, namely, 6.

Since by [H Theorem 1.1], a locally cyclic graph which is 6-regular is $k$-divergent, we state our main theorem.

**Corollary 8.11** (Main result). Let $G$ be a locally cyclic graph with minimum degree at least 6.

1. For a finite graph $G$, the clique graph operator diverges on $G$ if and only if $G$ has only vertices of degree 6.
2. For an infinite graph $G$, if there exists an $m \geq 0$, such that $\Delta_m$ cannot be embedded into the universal cover of $G$, the clique operator is convergent on $G$.

Whether an infinite locally cyclic $G$ can be convergent, even though its universal cover diverges, is still an open question.

**9. Further Research**

In our research, we were able to decide which finite locally cyclic graphs with minimum degree $\delta = 6$ are $k$-convergent and which are $k$-divergent. But we are not able to decide this for infinite graphs, not even if they are triangularly simply connected. To prove in an analogous way that every pik\(^2\) which contains a subgraph isomorphic to $\Delta_m$, for every $m$, is $k$-divergent, it would be necessary to show that $k^nG \nsubseteq k^{n+1}G$ implies $k^nG \neq k^{n+1}G$. Even if this was proven, our classification of $k$-convergence would not be finished, since an infinite graph with a $k$-divergent universal cover can itself be $k$-convergent.

Our work shows that explicit consideration of the clique dynamics can be fruitful. It would be interesting to know whether this approach gives feasible results for smaller minimum degrees.

\[^2\text{i.e. a triangularly simply connected locally cyclic graphs with minimum degree } \delta = 6\]
A. Definitions

Definition A.1. For a graph $G = (V_G, E_G)$, the closed neighbourhood of $M \subseteq V_G$ in $G$ is given by the induced subgraph

$$N_G[M] := G[y \in V_G \mid y \in M \text{ or } \exists x \in M : xy \in E_G],$$

and the common neighbourhood of $M$ is

$$N^0_G[M] := G[y \in V_G \mid \forall x \in M : xy \in E_G].$$

For a subgraph $H$ of $G$, $N_G[H] := N_G[V_H]$ and $N^0_G[H] := N^0_G[V_H]$. Furthermore, for a vertex $v \in V_G$, the closed neighbourhood is given by $N_G[v] := N_G[\{v\}]$ and the open neighbourhood is given by $N_G(v) := G[y \in V_G \mid vy \in E_G]$.

For the two graphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$, the graph $G \setminus H$ is defined as $(V_G \setminus V_H, \{xy \in E_G \setminus E_H \mid x \notin V_H \land y \notin V_H\})$.

Definition A.2. A graph homomorphism $\varphi : G \rightarrow H$ is any adjacency-preserving vertex map, i.e. $uv \in E(G) \Rightarrow f(u)f(v) \in E(H)$, see [4, Section 2, p. 161]. Injective homomorphisms are called monomorphisms. An isomorphism is a bijective homomorphism whose inverse is also a homomorphism.

B. Proofs from Topology

Proof of Lemma 3.2. In all cases, we start with a path $v_0v_1 \ldots v_k$ with $v_0, v_k \in \partial S$ and $v_1, \ldots, v_{k-1} \notin S$ such that none of the edges $v_iv_{i+1}$, for $0 \leq i < k$, lies in $S$. In the first case, we aim for a contradiction, in the second and third we show the claims directly.

Since $G$ is simply connected, both the path and $S$ lie in a common planar subgraph $U$ of $G$, such that every bounded face is a triangle. Henceforth, we consider all paths as paths in the plane.

There are two paths along $\partial S$ that connect $v_k$ to $v_0$. By [9, Corollary 1.2], one of those together with $v_0v_1 \ldots v_k$ bound a disc containing the other one. These two paths cannot be the paths along $\partial S$, since $v_0 \ldots v_k$ does not lie in $S$. The path which lies “inside” will be denoted by $v_k = s_0s_1 \ldots s_m = v_0$, the other one by $v_kv_{k+1} \ldots v_r = v_0$.

We define $\alpha_i$ as the inner facet-degree at $v_i$ of the path $v_0 \ldots v_k s_1 \ldots s_m$ for every $0 \leq i \leq k$ (thus, $\alpha_0$ and $\alpha_k$ are the facet-degrees between the path and $\partial S$). To prove the lemma, we focus on the path $v_0 \ldots v_kv_{k+1} \ldots v_r$ in more detail. We denote the inner facet-degree at $v_j$ by $\beta_j$. This situation is displayed in Figure [3].

Since this path bounds a disc in the plane, Lemma 4.1.5 from [11] is applicable and gives

$$6 = \sum_{v \text{ inner vertex}} (6 - \deg(v)) + \sum_{j=0}^{r-1} (3 - \beta_j).$$

\[3\]Which counts the number of facets and not the number of edges
Since $\deg(v) = 6$ for all vertices in $G$, we obtain

$$6 \leq \sum_{j=0}^{r-1} (3 - \beta_j) = (3 - \beta_0) + \sum_{i=1}^{k-1} (3 - \alpha_i) + (3 - \beta_k) + \sum_{j=k+1}^{r-1} (3 - \beta_j).$$

If a vertex $v_j$ with $k < j < r$ lies in a corner of $S$, we have $\beta_j = 1$; otherwise, $\beta_j = 3$. Thus, it remains to analyse $\beta_0$ and $\beta_k$.

If $v_0$ is a corner vertex, we have $\beta_0 = 1 + \alpha_0$; otherwise, we have $\beta_0 = 3 + \alpha_0$. Whichever case applies, we can rewrite the inequality into

$$6 \leq \sum_{i=0}^{k} (3 - \alpha_i) - 6 + 2c,$$

where $c \in \{0, 1, 2, 3\}$ is the number of corner vertices in $\{v_k, v_{k+1}, \ldots, v_r\}$ (note the inclusion of $v_k$ and $v_r$ in this set). With this inequality, we proceed through the three cases of the lemma. Recall that $\alpha_0 \geq 1$ and $\alpha_k \geq 1$ have to hold (otherwise the edge $v_0v_1$ or the edge $v_{k-1}v_k$ lies in $S$).

1. For the path $v_0v_1$ we obtain $6 \leq 2c - \alpha_0 - \alpha_1$, which has no solutions. Thus, there can be no edge between these vertices that does not already lie in $S$.

2. For the path $v_0v_1v_2$, we obtain $3 \leq 2c - \alpha_0 - \alpha_1 - \alpha_2$. If $\alpha_1 = 0$, we obtain $v_0 = v_2$. Otherwise, the only possible solution is $c = 3$ and $\alpha_0 = \alpha_1 = \alpha_2 = 1$. This already implies that $\{v_0, v_1, v_2\}$ is a facet of $G$.

3. For the path $v_0v_1v_2v_3$, we obtain $2c \geq \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3$. Since the path is non-repeating, we have $\alpha_1 \geq 1$ and $\alpha_2 \geq 1$; thus $c \in \{2, 3\}$.

Now, $\alpha_1 = 1$ implies the facet $\{v_0, v_1, v_2\}$. In particular, $v_0v_2v_3$ is a path. With part (2) of this lemma, we conclude that $\{v_0, v_2, v_3\}$ is a facet, in contradiction to our assumption. The same argument applies if $\alpha_2 = 1$. Thus, both of them have to be at least 2.

Then, the only solution is $c = 3$ with $\alpha_0 = \alpha_3 = 1$ and $\alpha_1 = \alpha_2 = 2$. Since $\alpha_0 = 1$, the triple $\{v_0, s_{m-1}, v_1\}$ forms a facet. Since $\alpha_1 = 2$, the triple $\{v_1, s_{m-1}, v_2\}$ also has to be a facet. For $v = s_{m-1}$, this was the claim that needed to be shown. \qed
Proof of Lemma 3.3. Assume to the contrary that there is an \( x \in (N_G [S_1] \cap N_G [S_2] \cap N_G [S_3]) \setminus S \). Since \( S_i \subseteq S \), we conclude \( x \in N_G [S] \setminus S \). Without loss of generality, \( x \) is adjacent to \( (t, m - t, 0) \) for some \( 0 \leq t < m \). We permute the coordinates, such that \( t \) is maximal among all edges.

1. Case: \( t > 0 \):
   Since \( (t, m - t, 0) \notin \Delta^0_{m-1} \), the vertex \( x \) has to be adjacent to a boundary vertex of \( \Delta^0_{m-1} \) as well, say \( (s, 0, m - s) \) for some \( 0 \leq s < m \) (see Figure 15).

   \[ \text{Figure 15: Illustration of common neighbour } x. \]

   By Lemma 3.2(2), the vertices \( (t, m - t, 0) \) and \( (s, 0, m - s) \) have to be adjacent. This is only possible for \( t = s = m - 1 \). But both facets incident to the edge \( \{(m - 1, 1, 0), (m - 1, 0, 1)\} \) already lie in \( S \) (for \( m > 1 \)), contradicting \( x \notin S \).

2. Case: \( t = 0 \):
   Since \( x \) is only adjacent to corner vertices (otherwise we would be in the case \( t > 0 \)), it has to be adjacent to \( (0, m, 0) \) and \( (0, 0, m) \) as well (to lie in each \( N_G [S_i] \)). By Lemma 3.2(2), this implies the adjacency of the corner vertices, i.e. \( m = 1 \). But then, the neighbourhood of \( x \) contains a circle of length 3, in contradiction to neighbourhoods being circles of length at least 6.

Proof of Lemma 3.4. If \( m = 0 \), this is the definition of locally cyclic. For \( m \geq 1 \), we enumerate the boundary vertices of \( S \) in cyclic order by \( b_1 b_2 \ldots b_m \). For each adjacent pair \( (b_i, b_{i+1}) \), there is exactly one face containing \( \{b_i, b_{i+1}\} \) and not lying in \( S \). Call the final corner of this face \( n_{i,i+1} \). If \( n_{i,i+1} \in S \), Lemma 3.2 (1) would imply that the full face lied in \( S \).

With this notation, \( N_G (b_i) \setminus S \) is the path \( n_{i-1,i} x_1 x_2 \ldots x_k n_{i,i+1} \) (there are no further edges between these vertices since the neighbourhood of \( b_i \) is a cycle). None of the \( x_i \) lies in \( S \), since Lemma 3.2 (1) would imply further boundary edges of \( S \), in contradiction to our assumption. This situation is illustrated in Figure 16.

By Lemma 3.2(3), any edge between vertices in \( N_G [S] \setminus S \) already lies in one \( N_G (b_i) \setminus S \). Combining these paths gives the desired cycle.

C. Proofs from Properties of Triangles

Proof of Lemma 4.4. Let \( \tilde{\Delta}^m_{m-1} \to \text{Hex}_m \) be a triangle inclusion map whose image lies in \( \Delta_m \). Thus, each component of \( \tilde{t} \) has to be non-negative, leaving \( \tilde{t} \in V_1 \cap \mathbb{Z}_{\geq 0}^3 = \tilde{E} \).  

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Figure 16: Part of a local neighbourhood

It remains to consider those $\Delta_{m-1} \cong S \subseteq \Delta_m$ that are not the image of a triangle inclusion map. The boundary of such an $S$ consists of three straight paths of length $m-1$. By Remark 3.1 there are six such paths along the boundary of $\Delta_m$ and three such paths in the interior (each given by all the vertices for which one fixed has value 1). The boundary paths can only lie in one $\Delta_{m-1} \cong S \subseteq \Delta_m$. Since the triangle inclusion maps “use” two boundary paths and one interior path each, the only remaining possibility is combining the three interior paths into a $\Delta_{m-1}$. But this is only possible if the paths meet at the boundary (where one component is 0). Thus, the component sum $m$ has to be 2.

Proof of Lemma 4.5. Let $\Delta^f_{m-2} : \Delta_{m-2} \rightarrow \text{Hex}_m$ be a triangle inclusion map whose image lies in $\Delta_m$. Thus, each component of $f$ has to be non-negative and thus $f \in V_2 \cap \mathbb{Z}_{\geq 0}^3 = \hat{E} + \hat{E}$. For $m = 2$, all $\Delta_0 \cong S \subseteq \Delta_m$ are possible images.

It remains to consider those $\Delta_{m-2} \cong S \subseteq \Delta_m$ that are not the image of a triangle inclusion map. In this case $m \geq 3$. The boundary of such an $S$ consists of three straight paths of length $m-2$. Remark 3.1 describes all paths of this kind inside $\Delta_m$. Up to the group action (Subsection 2.2), we only need to look at four of these paths:

1. The path $\alpha_{\{0,\ldots,m-2\}}$ can only be the boundary of one $\Delta_{m-2}$ and it already is the boundary of $\Delta_{m-2}^{(2,0,0)}$.

2. The path $\alpha_{\{1,\ldots,m-1\}}$ can only be the boundary of one $\Delta_{m-2}$ and it already is the boundary of $\Delta_{m-2}^{(1,1,0)}$.

3. The path $\beta_{\{0,\ldots,m-2\}}$ is the boundary of $\Delta_{m-2}^{(1,0,1)}$ (whose third component is higher). There can only be a $\Delta_{m-2}$ with lower third component if $m - 2 \leq 1$, implying $m = 3$. The triangle has corner vertices $(2,0,1)$, $(1,1,1)$, and $(2,1,0)$.

4. The path $\gamma$ is the boundary of $\Delta_{m-2}^{(0,0,2)}$ (whose third component is higher). There can only be a $\Delta_{m-2}$ with lower third component if $m - 2 \leq 2$. The case $m = 3$ gives a triangle which is brought to the triangle of (3) by the group action. The case $m = 4$ gives vertices $(2,0,2)$, $(0,2,2)$, and $(2,2,0)$.
Applying the group action to these $\Delta_{m-2}$ gives the desired results.

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