Connections in holomorphic Lie algebroids

Alexandru IONESCU and Gheorghe MUNTEANU

Abstract

The main purpose of this note is the study of the total space of a holomorphic Lie algebroid $E$. The paper is structured in three parts.

In the first section we briefly introduce basic notions on holomorphic Lie algebroids. The local expressions are written and the complexified holomorphic bundle is introduced.

The second section is a little broader and includes two approaches to study the geometry of complex manifold $E$. The first part contains the study of the tangent bundle $T_C E = T'E \oplus T''E$ and its link, via tangent anchor map, with the complexified tangent bundle $T_C(T'M) = T''(T'M) \oplus T''(T'M)$. A holomorphic Lie algebroid structure has been emphasized on $T'E$. A special study is made for integral curves of a spray on $T'E$. Theorem 2.1 gives the coefficients of a spray, called canonical, according to a complex Lagrangian on $T'E$. In the second part of section two we study the prolongation $T'E$ of $E \times T'E$ algebroid structure.

In the third section we study how a complex Lagrange (Finsler) structure on $T'M$ induces a Lagrangian structure on $E$. Three particular cases are analyzed by the rank of anchor map, the dimensions of manifold $M$ and the fibre dimension. We obtain the correspondent on $E$ of the well-known (10) Chern-Lagrange nonlinear connection from $T'M$.

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Introduction

Lie algebroids are a generalization of Lie algebras and vector bundles. They are anchored vector bundles with a Lie bracket defined on the modules of sections induced from tangent bundle. Lie algebroids provide a natural setting in which one can develop the theory of differential operators such as the exterior derivative of forms and the Lie derivative with respect to a vector field. This setting is slightly more general than that of the tangent and cotangent bundles of a smooth manifold and their exterior powers.

Lie algebroids represent an active domain of research, with applications in many areas of mathematics and physics. A well-known example is the work of A. Weinstein [15] in the area of Mechanics, who developed a generalized theory of Lagrangians on Lie algebroids and obtained the Euler-Lagrange equations using the structure of the dual of Lie algebroids and Legendre transformations.
associated with a regular Lagrangian. On the other hand, E. Martinez developed another approach using the notion of prolongation of Lie algebroid, for that fundamental notions on tangent bundle geometry, such as spray theory and nonlinear connection, can be transferred to this prolongation. Many recently results are obtained on Lie algebroids (\cite{2} \cite{3} \cite{12} \cite{13} \cite{11} etc.).

In complex geometry, some properties of complex and holomorphic Lie algebroids have been studied in \cite{16} \cite{7} \cite{5}.

The present paper analyzes specific notions from real Lie algebroids theory in the case of holomorphic Lie algebroids. The paper is organized as follows. The first part gives basic definitions of a holomorphic anchor map, holomorphic Lie algebroid, Lie bracket on such an algebroid, which are also locally characterized. More details and linear connections on $E$ and $E_C$ are presented in \cite{6}.

In the geometry of the manifold holomorphic Lie algebroid $E$, two approaches are taken into account. One is of the tangent bundle $T'E$, which has in its turn a natural structure of Lie algebroid. The geometry of $T'E$ is "linearized" by using a nonlinear connection for which, in respect to its adapted frames, we study a distinguished complex linear connection. The classical notions of semisprays and sprays are defined in this case following the variational problem on an algebroid endowed with a Lagrangian structure. The main results is Theorem 2.1, which gives the coefficients of a spray, called canonical, from the variational problem.

The second approach concerns the prolongation $T'E$ of a holomorphic Lie algebroid. Using a complete lift we introduce the Liouville tensor and an almost tangent structure for defining a different type of nonlinear connection on the prolongation $T'E$. It is proved how the nonlinear connection on $T'E$ defines a nonlinear connection on $T'E$. Theorem 2.2 gives the procedure of deriving a nonlinear connection on $T'E$ from a spray on $T'E$. Corroborate with Theorem 2.1, we can say that we have solved the problem of determining of adapted frames, and so of "linearizing" of the geometry of a holomorphic algebroid endowed with a regular Lagrangian $L$.

In the last section we study the possibility of inducing Lagrange structures on holomorphic Lie algebroids from a Lagrangian structure on the tangent bundle $T'M$. Three particular cases are analyzed by the rank of anchor map and dimensions of manifold $M$ and fiber dimension. It is proved that a Lagrangian on $T'M$ and the well known Chern-Lagrange nonlinear connection on $T'M$ induces a Lagrangian structure on $T'E$ and consequently, by Theorems 2.1 and 2.2, a nonlinear connection on $T'E$.

### 1 Holomorphic Lie algebroids

Let $M$ be a complex $n$-dimensional manifold and $E$ a holomorphic vector bundle of rank $m$ over $M$. Denote by $\pi : E \to M$ the holomorphic bundle projection, by $\Gamma(E)$ the module of holomorphic sections of $\pi$ and let $T_2M = T'M \oplus T''M$ be the complexified tangent bundle of $M$, split into the holomorphic and antiholomorphic tangent bundles.

On a vector bundle $(E, \pi, M)$ the definition of a derivative law is $D : \chi(M) \times
\( \Gamma(E) \rightarrow \Gamma(E) \), \( D_X s \), such that \( D_{fX} s = fD_X s \) and \( D_X(fs) = fD_X s + X(f) \).

While these notions make sense on the fibers of \( E \), the Lie bracket \([s_1, s_2]|_f \), where \( s_1, s_2 \in \Gamma(E) \), has no mathematical meaning. Hence the notion of Lie algebroids.

**Definition 1.1.** The holomorphic vector bundle \( E \) over \( M \) is called anchored if there exists a holomorphic vector bundle morphism \( \rho : E \rightarrow T'M \), called anchor map.

Denote by \( \Gamma(T'M) \) the module of holomorphic sections of \( T'M \), that is, the holomorphic vector fields on \( M \), and by \( \mathcal{H}(M) \) the ring of holomorphic functions on \( M \).

Using the anchor map, we can define a Lie bracket on \( E \) from the Lie bracket on \( T'M \) by

\[
\rho_E([s_1, s_2]|_E) = [\rho_E(s_1), \rho_E(s_2)]|_{T'M}, \tag{1.1}
\]

for any \( s_1, s_2 \in \Gamma(E) \). For any \( f \in \mathcal{H}(M) \),

\[
\rho_E[s_1, fs_2]|_E = \rho_E(s_1), \rho_E(fs_2)]|_{T'M} = \rho_E(s_1), f\rho_E(s_2)]|_{T'M} = f[\rho_E(s_1), \rho_E(s_2)]|_{T'M} + \rho_E(s_1)(f)\rho_E(s_2).
\]

These considerations lead to the following definition (\[13\] \[7\] \[5\] \[8\]):

**Definition 1.2.** A holomorphic Lie algebroid over \( M \) is a triple \((E, [,], \rho_E)\), where \( E \) is a holomorphic vector bundle over \( M \), \([,]_E \) is a Lie bracket on \( \Gamma(E) \) and \( \rho_E : \Gamma(E) \rightarrow \Gamma(T'M) \) is the homomorphism of complex modules induced by the anchor map \( \rho \) such that

\[
[s_1, fs_2]|_E = f[s_1, s_2]|_E + \rho_E(s_1)(f)s_2 \tag{1.2}
\]

for all \( s_1, s_2 \in \Gamma(E) \) and all \( f \in \mathcal{H}(M) \).

Note that (1.1) means that \( \rho_E : (\Gamma(E), [,], \rho_E) \rightarrow (\Gamma(T'M), [,]) \) is a complex Lie algebra homomorphism.

Also, the Lie bracket \([,]_E \) satisfies the Jacobi identity

\[
[s_1, [s_2, s_3]|_E] + [s_2, [s_3, s_1]|_E] + [s_3, [s_1, s_2]|_E] = 0. \tag{1.3}
\]

The main examples of holomorphic Lie algebroids are, of course, offered by the holomorphic tangent bundle \( T'M \), or its cotangent bundle \( T'^*M \). Some other examples can be derived from those presented in \[5\], in the particular cases of complex manifolds (complex Poisson manifold, complete lift and prolongation of a Lie algebroid, direct product structure, etc.).

An example of interest is the projective bundle of a complex Finsler manifold. If \((M, F)\) is a complex Finsler manifold (\[10\]), then \( F : T'M \rightarrow R^+ \) is a real function of position \( z \in M \) and direction \( \eta \in T'M \). Consider homogeneous coordinates \([\eta]\) that determine \( P_zM \), the lines bundle in each \( z \in M \). The reunion of all these lines gives the projective bundle \( PM \cong T'M/C \eta \), which has a natural structure of holomorphic Lie algebroid by \( \rho : [\eta] \rightarrow \eta \). Here, things
are more subtle. $PM$ as a complex manifold is isometric with the indicatrix $IM = \bigcup_{z \in M} I_z M$, where $I_z M = \{ \eta \in T_z^1 M \mid F(z, \eta) = 1 \}$. If $g_{ij} (z, \eta)$ is the metric tensor of the complex Finsler space (see below in the paper the corresponding notations), then $\mathcal{G} = g_{ij} d\eta^i \otimes d\bar{\eta}^j$ is a metric structure on $IM$, and $h = g_{ij} dz^i \otimes d\bar{z}^j + (\log F^2)_{ij} d\eta^i \otimes d\bar{\eta}^j$ is a metric structure on $PM$ (see [4, 14]). Then a metric structure on $T' M$, descending from $h$ is $g = g_{ij} dz^i \otimes d\bar{z}^j + g_{ij} \delta \eta^i \otimes \delta \bar{\eta}^j$.

Local expressions

If $(z^k)_{k=1}^n$ is a local complex coordinate system on $U \subset M$ and $\{ e_a \}_{a=1}^m$ is a local frame of sections of $E$ on $U$, then $(\tilde{z}^k, \tilde{u}^\alpha)$ are local complex coordinates on $\pi^{-1}(U) \subset E$, where $e = u^a e_a(z), e \in E$.

Let $g_{UV} : U \cap V \to GL(m, \mathbb{C})$ be the holomorphic transition functions of $E$. In $z \in U \cap V, g_{UV}(z)$ is represented by the complex matrix of holomorphic functions $(M^a_{\beta}(z))$, such that, if $(\tilde{z}^k, \tilde{u}^\alpha)$ are local coordinates on $\pi^{-1}(V)$, then these change by the rules

$$ \tilde{z}^k = \tilde{z}^k (z), \quad \tilde{u}^\alpha = M^\alpha_{\beta}(z) u^\beta. \quad (1.4) $$

The Jacobi matrix of the transformation laws (1.4) is

$$ \left( \begin{array}{c} \frac{\partial \tilde{z}^k}{\partial z^\mu} \\ M^a_{\beta} \end{array} \right) \quad (1.5) $$

Let $(W^a_{\beta})$ be the inverse matrix of $(M^a_{\beta})$, and $\{ e_a \}$ a base of sections on $E$, that is, $u = u^a e_a$ for any $u \in \Gamma(E)$. Then these change by the rules

$$ \bar{e}_a = W^a_{\beta} e_\beta. $$

The action of the holomorphic anchor map $\rho_E$ can locally be described by

$$ \rho_E(e_a) = \rho^k_a \frac{\partial}{\partial z^k}, \quad (1.6) $$

while the Lie bracket $[\cdot, \cdot]_E$ is locally given by

$$ [e_a, e_\beta]_E = C_{a\beta}^\gamma e_\gamma. \quad (1.7) $$

The holomorphic functions $\rho^k_a(z)$ and $C_{a\beta}^\gamma(z)$ on $M$ are called the holomorphic structure functions of the Lie algebroid $E$. A change of local charts on $E$ implies

$$ \tilde{\rho}^k_a = W^a_{\beta} \rho^k_\beta \frac{\partial \tilde{z}^k}{\partial z^\beta}. \quad (1.8) $$

Since $E$ is a holomorphic vector bundle, it has the structure of a complex manifold, and the natural complex structure acts on its sections by $J_E(e_a) = ie_a$
and $J_E(\bar{e}_\alpha) = -ie_\alpha$. Hence, the complexified bundle $E_C$ of $E$ decomposes into $E_C = E' \oplus E''$. The sections of $E_C$ are given as usual by $\Gamma(E') = \{ s - iJ_E s \mid s \in \Gamma(E) \}$ and $\Gamma(E'') = \{ s + iJ_E s \mid s \in \Gamma(E) \}$, respectively. The local basis of sections of $E'$ is $\{ e_{\alpha} \}_{\alpha=1,m}$, while for $E''$, the basis is represented by their conjugates $\{ \bar{e}_\alpha := e_{\bar{\alpha}} \}_{\bar{\alpha}=1,m}$. Since $\rho_E : E \to T'M$ is a homomorphism of complex modules, it extends naturally to the complexified bundle by $\rho_{\bar{E}}(\bar{e}_\alpha) = \rho_E(e_{\bar{\alpha}})$ and $\rho''(e_{\bar{\alpha}}) = \rho_E(e_{\bar{\alpha}})$. Thus, the anchor map can be decomposed into $\rho_E = \rho' \oplus \rho''$ on the complexified bundle, and since $E$ is holomorphic, the functions $\rho(z)$ are holomorphic, hence $\rho_k^\beta = \rho_k^\beta = 0$ and $\rho_k^\alpha = \rho_k^\alpha$. Thus, the anchored bundles $(E', \rho', T'M)$ and $(E'', \rho'', T''M)$ are complex Lie algebroids $(\mathbb{H})$. The Lie brackets are defined as

$$\left[e_{\alpha}, e_{\beta}\right]' = [e_{\alpha}, e_{\beta}]_E = C_{\alpha\beta}^{\gamma}e_\gamma; \quad [e_{\alpha}, e_{\beta}]'' = [e_{\alpha}, e_{\beta}]_{E'} = C_{\alpha\beta}^{\gamma}e_\gamma,$$

where $C_{\alpha\beta}^{\gamma} = C_{\bar{\alpha}\bar{\beta}}^{\bar{\gamma}}$. On the complexified bundle $E_C$, we have to consider also the Lie brackets

$$\left[e_{\alpha}, e_{\beta}\right] = C_{\alpha\beta}^{\gamma}e_\gamma + C_{\alpha\beta}^{\gamma}e_\gamma; \quad [e_{\alpha}, e_{\beta}] = C_{\alpha\beta}^{\gamma}e_\gamma + C_{\alpha\beta}^{\gamma}e_\gamma.$$

It is obvious that $[e_{\alpha}, e_{\beta}] = [e_{\bar{\alpha}}, e_{\bar{\beta}}]$, hence $C_{\alpha\beta}^{\gamma} = C_{\bar{\alpha}\bar{\beta}}^{\bar{\gamma}}$ and $C_{\alpha\beta}^{\gamma} = C_{\bar{\alpha}\bar{\beta}}^{\bar{\gamma}}$.

**Proposition 1.1.** The structure functions of the complexified Lie algebroid $(E_C, [\cdot, \cdot], \rho_E)$ satisfy the identities:

$$\rho_\alpha^\gamma \frac{\partial \rho^\beta_{\bar{\gamma}}}{\partial z^\alpha} - \rho_\beta^\gamma \frac{\partial \rho^\alpha_{\bar{\gamma}}}{\partial z^\beta} = \rho_\gamma^\alpha C_{\alpha\bar{\beta}}^{\bar{\gamma}}, \quad \rho_\gamma^\beta C_{\alpha\beta}^{\gamma} = -\rho_\beta^\gamma \frac{\partial \rho^\alpha_{\bar{\gamma}}}{\partial z^\beta}, \quad \rho_\gamma^\gamma C_{\alpha\beta}^{\gamma} = \rho_\beta^\gamma \frac{\partial \rho^\alpha_{\bar{\gamma}}}{\partial z^\beta},$$

$$\rho_\alpha^\gamma \frac{\partial \rho^\beta_{\bar{\gamma}}}{\partial z^\alpha} - \rho_\beta^\gamma \frac{\partial \rho^\alpha_{\bar{\gamma}}}{\partial z^\beta} = \rho_\gamma^\alpha C_{\alpha\bar{\beta}}^{\bar{\gamma}}, \quad \rho_\gamma^\beta C_{\alpha\beta}^{\gamma} = -\rho_\beta^\gamma \frac{\partial \rho^\alpha_{\bar{\gamma}}}{\partial z^\beta}, \quad \rho_\gamma^\gamma C_{\alpha\beta}^{\gamma} = \rho_\beta^\gamma \frac{\partial \rho^\alpha_{\bar{\gamma}}}{\partial z^\beta}.$$

**Proof.** The identities follow by direct computations using (1.1), (1.6) and (1.7).

## 2 The geometry of the total space of $E$

Two approaches on the tangent bundle of a holomorphic Lie algebroid $E$ will be described in this section. The first is the classical study of the tangent bundle of $E$, while the second is that of the prolongation on $E$. The latter idea appeared from the need of introducing geometrical objects such as nonlinear connections or sprays which could be studied in a similar manner to the tangent bundle of a complex manifold.

### 2.1 The tangent bundle of a holomorphic Lie algebroid

Recall (13) that a complex Lagrange space is a pair $(M, L)$, where $L : T'M \to \mathbb{R}$ is a regular Lagrangian defined on the holomorphic tangent bundle
of a complex manifold. The geometrical objects acting on such a space are sections in the complexified tangent bundle $T_C(T'M) = T'(T'M) \oplus T''(T'M)$.

The holomorphic tangent bundle $T'M$ of $M$ is in its turn a complex manifold, and the changes of local coordinates $(z^h, \eta^h)$ to $(z^k, \eta^k)$ are

$$z^k = z^k(z), \quad \eta^k = \frac{\partial z^k}{\partial z^h} \eta^h.$$  \hspace{1cm} (2.1)

The natural frame \(\left\{ \frac{\partial}{\partial z^k}, \frac{\partial}{\partial \eta^k} \right\}\) of $T'_C(T'M)$ in a fixed point, changes from $(z^h, \eta^h)$ to $(z^k, \eta^k)$ by the rules

\[
\frac{\partial}{\partial z^h} = \frac{\partial z^k}{\partial z^h} \frac{\partial}{\partial z^k} + \frac{\partial^2 z^k}{\partial z^j \partial z^h} \eta^j \frac{\partial}{\partial \eta^k},
\]

\[
\frac{\partial}{\partial \eta^h} = \frac{\partial z^k}{\partial \eta^h} \frac{\partial}{\partial \eta^k}.
\]  \hspace{1cm} (2.2)

The generalization consists in introducing a Lagrange structure (in particular, Finsler) on a holomorphic vector bundle, a well-known idea from T. Aikou [1], and G. Munteanu [10]. The basis manifold of such a space is the complex manifold $E$ endowed with a regular Lagrangian $L: E \to \mathbb{R}$ and the geometry of the space obviously implies studying geometrical objects (vectors, metric structures, connections) which act on sections in the complexified tangent bundle $T_C E = T'E \oplus T''E$, where $T'E$ is the holomorphic tangent bundle and $T''E = \overline{T'E}$. In particular, a Lie algebroid is first of all a holomorphic vector bundle and its geometry must be studied.

On $T'E$, a natural frame of fields is \(\left\{ \frac{\partial}{\partial z^k}, \frac{\partial}{\partial u^\alpha} \right\}\), which, due to the (1.5) matrix, changes by the rules

\[
\frac{\partial}{\partial z^h} = \frac{\partial z^k}{\partial z^h} \frac{\partial}{\partial z^k} + \frac{\partial^2 z^k}{\partial z^j \partial z^h} u^j \frac{\partial}{\partial u^\alpha},
\]

\[
\frac{\partial}{\partial u^\alpha} = M^\beta_{\alpha} \frac{\partial}{\partial \eta^\beta}.
\]  \hspace{1cm} (2.3)

Since $E$ is a complex manifold, it follows that \(\left\{ \frac{\partial}{\partial z^k}, \frac{\partial}{\partial \eta^\alpha} \right\}\) is a local frame on $T''E = \overline{T'E}$ and its rules of change are deduced from (2.3) by conjugation.

Now, let us consider $E$ and $T'M$ as manifolds and we prove that the anchor map $\rho_E$ maps the local coordinates $(z^k, u^\alpha)$ on $E$, with changes (1.4), in a local map $(z^k, \eta^k)$ on $T'M$, with changes (2.1). Further, let us consider the same local charts on $M$ for $E$ and $T'M$, that is, we have the same changes $\tilde{z}^k(z) = z^k(z)$.

As a mapping between manifolds, the holomorphic anchor $\rho$ induced by $\rho_E$ maps $(z^k, u^\alpha)$ on $E$ to $(z^k, \eta^k)$ on $T'M$, where we define the directional coordinates by

$$\eta^k = u^\alpha \rho^k_\alpha(z).$$  \hspace{1cm} (2.4)
Let us prove that \([2.4]\) define a sistem of coordinates on \(T'M\). A change of local charts implies that \((\bar{z}^k, \bar{u}^\alpha)\) is mapped to \((z^k, \eta^k)\), where \(z^k = \bar{z}^k(z) = z^k(z)\) and
\[
\eta^k = \bar{u}^\alpha \rho^\alpha_k(\bar{z}) = M^\alpha_\beta u^\beta W^\gamma_\alpha \rho^\alpha_k \partial z^k = u^\beta \rho^\alpha_k \partial z^k = \eta^k \partial z^k,
\]
so that the changes \([2.1]\) are satisfied, and moreover we have:
\[
z^k = z^k(z), \quad \eta^k = u^\alpha \rho^\alpha_k \partial z^k.
\]
These transformation laws have the following Jacobi matrix:
\[
\begin{pmatrix}
\frac{\partial z^k}{\partial z^j} & 0 \\
\frac{\partial}{\partial z^j} \left( \rho^\alpha_k \frac{\partial z^k}{\partial z^h} \right) u^\gamma & \rho^\alpha_k \frac{\partial z^k}{\partial z^h}
\end{pmatrix}
\] (2.6)

Denote by \(\rho_* : T_C E \to T_C (T'M)\) the tangent mapping of the anchor \(\rho_E : \Gamma(E) \to \Gamma(T'M)\) and by \(J^*_{T'M} : T_C (T'M) \to T_C (T'M)\) the natural complex structure on \(T_C (T'M)\).

**Definition 2.1.** A complex structure \(J^*_E\) on the complex tangent bundle \(T_C E\) is an endomorphism \(J^*_{E} : T_C E \to T_C E\) given by
\[
J^*_E \left( \frac{\partial}{\partial z^k} \right) = i \frac{\partial}{\partial z^k}, \quad J^*_E \left( \frac{\partial}{\partial \bar{z}^k} \right) = -i \frac{\partial}{\partial \bar{z}^k},
\] (2.7)
\[
J^*_E \left( \frac{\partial}{\partial u^\alpha} \right) = i \frac{\partial}{\partial u^\alpha}, \quad J^*_E \left( \frac{\partial}{\partial \bar{u}^\alpha} \right) = -i \frac{\partial}{\partial \bar{u}^\alpha}.
\]

The complex structure \(J^*_E\) satisfies the identities \(J^*_E{}^2 = - \text{Id}_{T_C E}\) and \(J^*_{T'M} \circ \rho_* = \rho_* \circ J^*_E\).

The splitting \(T_C E = T'E \oplus T''E\) of the complexified tangent bundle is due to the complex structure \(J^*_E\), the holomorphic and antiholomorphic tangent bundles of \(E\) corresponding to the eigenvalues \(\pm i\) of \(J^*_E\). Moreover, there are two modules of sections on \(T_C E\), \(\Gamma(T'_E s \mid s \in \Gamma(T'_E))\) and \(\Gamma(T''_E s \mid s \in \Gamma(T''_E))\).

Since the tangent anchor map \(\rho_*\) is holomorphic, if \(s \in \Gamma(T'E)\) is a holomorphic section on \(T_C E\), then \(\rho_*(s) \in \Gamma(T'(T'M))\). Similarly, for an antiholomorphic section \(\bar{s} \in \Gamma(T''E)\), \(\rho_*(\bar{s}) \in \Gamma(T''(T'M))\).

The anchor \(\rho\) maps the coordinates \((z^k, u^\alpha)\) from a local chart on the manifold \(E\) to the coordinates \((z^k, \eta^k = u^\alpha \rho^\alpha_k(z))\) in a local chart on \(\rho(E) \subset T'M\). The Jacobi matrix of the morphism \(\rho\) is
\[
\begin{pmatrix}
\delta^k_h & 0 \\
\rho^\alpha_k & \rho^\alpha_h \frac{\partial u^\alpha}{\partial z^h}
\end{pmatrix}
\]
Then \( \left\{ \frac{\partial}{\partial z^k}, \frac{\partial}{\partial \eta^k}, \frac{\partial}{\partial z'^k}, \frac{\partial}{\partial \eta'^k} \right\} \) is the natural frame field on \( T_{\mathbb{C}}(T'M) \) and the action of the tangent mapping \( \rho_* \) is locally described on \( \rho(E) \) by

\[
\rho_* \left( \frac{\partial}{\partial z^k} \right) =: \frac{\partial}{\partial z^k} + u^\alpha \frac{\partial \rho^k_h}{\partial \eta^h} \frac{\partial}{\partial \eta^h},
\]

(2.8)

\[
\rho_* \left( \frac{\partial}{\partial u^\alpha} \right) =: \frac{\partial}{\partial u^\alpha} = \rho^k_h \frac{\partial}{\partial \eta^h}
\]

and their conjugates. The dual basis of the natural frame \( \left\{ \frac{\partial}{\partial z^k}, \frac{\partial}{\partial \eta^k} \right\} \) induced by \( \rho_* \) on \( \rho(E) \) is

\[
d^* z^k = dz^k
\]

(2.9)

\[
d^* \eta^k = u^\alpha \frac{\partial \rho^k_h}{\partial z^h} dz^h + \rho^k_h du^\alpha
\]

For a change of coordinates on \( T_{\mathbb{C}}(T'M) \), the change laws on \( T_{\mathbb{C}}E \) are, due to the Jacobi matrix (2.6),

\[
\frac{\partial^*}{\partial z^j} = \frac{\partial z^k}{\partial z^j} \frac{\partial}{\partial z^k} + \frac{\partial \rho^k_h}{\partial z^h} \left( \frac{\partial z^j}{\partial z^k} \right) u^\gamma \frac{\partial}{\partial \eta^k},
\]

(2.10)

\[
\frac{\partial^*}{\partial u^\beta} = \rho^k_h \frac{\partial z^k}{\partial z^h} \frac{\partial}{\partial \eta^k}
\]

and the conjugates.

### 2.1.1 The Lie algebroid structure of \( T'E \)

We prove that \( T'E \) has a Lie algebroid structure over the basis manifold \( M \). Let \( p' : T'M \to M \) be the projection of the holomorphic tangent bundle of \( M \) and \( p'_* : T'T'M \to T'M \) its tangent map, acting at a point \( (z, \eta) \) by

\[
Z^k \frac{\partial}{\partial z^k} + V^k \frac{\partial}{\partial \eta^k} \xrightarrow{p'_*} Z^k \frac{\partial}{\partial z^k}.
\]

Then, the following diagram

\[
\begin{array}{ccc}
T'E & \xrightarrow{\pi_E} & T'T'M \\
\downarrow & & \downarrow \rho'_* \\
E & \xrightarrow{\rho} & T'M \\
\downarrow & & \downarrow p' \\
M & \xrightarrow{\text{Id}_M} & M
\end{array}
\]

(2.11)

suggests the definition of the map \( \Upsilon : T'E \to T'M \), \( \Upsilon = p'_* \circ \rho_* \), in order to introduce a holomorphic Lie algebroid structure on \( T'E \). Since \( T'M \) and \( T'E \) are holomorphic bundles, from the definition of \( \Upsilon \) follows that it is a vector bundle morphism.
Locally, we have $Z = Z^k \frac{\partial}{\partial z^k} + V^\alpha \frac{\partial}{\partial u^\alpha} \rho^\nu \rightarrow Z^* = Z^k \frac{\partial^*}{\partial z^k} + V^\alpha \frac{\partial^*}{\partial u^\alpha}$. The action \([2,8]\) and the definition of $\rho^\nu$ yield $\Upsilon(Z) = Z^k \frac{\partial}{\partial z^k}$.

Since $T' E$ is a vector bundle over $M$, taking the Lie bracket of two sections $[Z, W]_{T' E}$ and $f : M \rightarrow \mathbb{C}$, we obtain $\frac{\partial f}{\partial u^\alpha} = 0$, such that

$$[Z, fW]_{T' E} = f[Z, W]_{T' E} + \Upsilon(Z)f W. \quad (2.12)$$

This leads to the following

**Proposition 2.1.** The holomorphic tangent bundle $T' E$ has a structure of a Lie algebroid over the complex manifold $M$, with the anchor map $\Upsilon$.

Using the definition of $\Upsilon$, it follows that it is a homomorphism between the complex Lie algebras \( \Gamma(T' E), [\cdot, \cdot]_{T' E} \) and \( \Gamma(T' M), [\cdot, \cdot] \), that is,

$$\Upsilon[Z, W]_{T' E} = [\Upsilon(Z), \Upsilon(W)], \quad \forall Z, W \in \Gamma(T' E).$$

### 2.1.2 Nonlinear connections on $T' E$

It is obvious that the rules of change of the natural frame of fields on $T_C E$ are complicated. As in the case of Finsler geometry, the solution to this problem is the method of nonlinear connection. Consider $\pi_*$ the tangent mapping of the projection $\pi : E \rightarrow M$. Then the **vertical holomorphic tangent bundle** of $E$ can be defined by $V E = \ker \pi_*$. A local frame of fields on $V E$ is \( \left\{ \frac{\partial}{\partial u^\alpha} \right\}_{\alpha = 1, m} \) and if $\pi^*(T' M)$ is the pull-back bundle of the holomorphic tangent bundle of $M$, then the following fundamental sequence is obtained \([10]\):

$$0 \rightarrow V E \overset{\iota}{\rightarrow} T' E \overset{d\pi}{\rightarrow} \pi^*(T' M) \rightarrow 0. \quad (2.13)$$

As usual, a splitting $C : T' E \rightarrow V E$ in this sequence is called a **connection** on the vertical bundle and it determines the decomposition

$$T' E = V E \oplus H E \quad (2.14)$$

of the holomorphic tangent bundle of $E$, where $H E$ is the **horizontal distribution**, isomorphic to the pull back bundle $\pi^*(T' M)$ by the morphism $d\pi$ from the exact sequence \((2.13)\). This isomorphism is called **complex nonlinear connection** or **Ehresmann connection** (T. Aikou, \([1]\)) on the holomorphic vector bundle $E$.

The decomposition of the complexified tangent bundle $T_C E$ is obtained by conjugation:

$$T_C E = H E \oplus V E \oplus \overline{H E} \oplus \overline{V E}.$$ 

The **horizontal lift** $l^h : \pi^*(T' M) \rightarrow H E$ determined by the nonlinear connection is defined by

$$l^h \left( \frac{\partial}{\partial z^k} \right) = \frac{\delta}{\delta z^k} = \frac{\partial}{\partial z^k} - N^\alpha_k \frac{\partial}{\partial u^\alpha}, \quad (2.15)$$
(see [10]), where the functions $N^k_\alpha(z,u)$ are called the coefficients of the complex nonlinear connection on $E$. A change of local coordinates implies that $\frac{\delta}{\delta z^h}$ changes by the rule

$$\frac{\delta}{\delta z^h} = \frac{\partial z^k}{\partial z^h} \frac{\delta}{\delta z^k},$$

such that, using also (2.3), the laws of change for the functions $N^\alpha_k$ are obtained:

$$\frac{\partial z^k}{\partial z^h} N^\alpha_k = M^\alpha_\beta N^\beta_h - \frac{\partial M^\alpha_\beta}{\partial z^h} u^\beta.$$

(2.16)

A field of frames $\{\frac{\delta}{\delta z^k}, \frac{\partial}{\partial u^\alpha}\}$ on $T'E$ is obtained, called the adapted frame of the complex nonlinear connection. A simple computation using (2.12) and (2.15) leads to the following result.

**Proposition 2.2.** The Lie brackets of the adapted frame on $T'E$ are

$$\left[ \frac{\delta}{\delta z^k}, \frac{\delta}{\delta z^h} \right]_{T'E} = \left( \frac{\partial N^\beta_k}{\partial z^h} - \frac{\partial N^\beta_h}{\partial z^k} \right) \frac{\partial}{\partial u^\alpha};$$

$$\left[ \frac{\delta}{\delta z^k}, \frac{\partial}{\partial u^\beta} \right]_{T'E} = \left[ \frac{\partial}{\partial u^\alpha}, \frac{\partial}{\partial u^\beta} \right]_{T'E} = 0.$$

(2.17)

Let $\{\frac{\delta}{\delta z^k}, \frac{\partial}{\partial \eta^h}\}$ be the adapted frame of a complex nonlinear connection on $T'T'M$, where

$$\frac{\delta}{\delta z^k} = \frac{\partial}{\partial z^k} - N^h_k \frac{\partial}{\partial \eta^h}$$

(2.18)

and denote by $\{dz^k, d\eta^h\}$ the dual basis, with

$$d\eta^h = d\eta^h + N^h_k dz^k.$$

(2.19)

The coefficients of the complex nonlinear connection change by the rules (10)

$$N^i_j \frac{\partial z^j}{\partial z^k} = \frac{\partial z^i_j}{\partial z^k} - \frac{\partial^2 z^i_j}{\partial z^h \partial z^k} \eta^h.$$

(2.20)

Analogously, on $T'E$, the dual basis of the adapted frame is $\{dz^k, \delta u^\alpha\}$, where

$$d\alpha^\alpha = du^\alpha + N^h_k dz^h.$$

(2.21)

From now on, we will use the well-known abbreviations

$$\delta_k = \frac{\delta}{\delta z^k}, \quad \delta = \frac{\partial}{\partial z^k}, \quad \alpha = \frac{\partial}{\partial u^\alpha}. $$
2.1.3 Linear connections on $T'E$

Linear connections can be introduced on the holomorphic tangent bundle $T'E$ of the holomorphic Lie algebroid $E$ in a similar manner as in the case of the holomorphic tangent bundle $T'T'M$. Denote by $A'$ the set of complex $r$-forms over $M$ and $A'(E)$, the set of complex $r$-forms on $E$.

**Definition 2.2.** A complex linear connection on $T'E$ is a map

$$D : \Gamma(T'E) \times \Gamma(T'E) \to \Gamma(T'E), \quad (Z,W) \mapsto DZW,$$

such that

$$DZ(fW) = (\nabla(Z)f)W + fDZW, \quad \forall f \in A^0(E), \forall Z,W \in \Gamma(T'E). \quad (2.22)$$

Locally, a section $Z \in \Gamma(T'E)$ can be decomposed in the adapted frame $\{\delta_z^{\delta j}, \frac{\partial}{\partial u^\alpha}\}$ of a complex nonlinear connection as introduced in the previous section. Hence, the connection forms have vertical and horizontal components. If the linear connection preserves the distributions from (2.14), then it is called distinguished. A distinguished complex linear connection $D$ on $T'E$ has the following coefficients:

$$D_{\delta k}\delta_j = L^i_{jk}\delta_i, \quad D_{\delta j}\delta_k = L^i_{ij}\delta_i, \quad D_{\delta k}\partial_\beta = L^\alpha_{jk}\partial_\alpha, \quad D_{\delta j}\partial_\beta = C^\gamma_{\beta j}\partial_\alpha. \quad (2.23)$$

As usual, the next step is considering the torsion and curvature of such a connection.

As usual, the torsion of a distinguished complex linear connection on $T'E$ is

$$T(Z,W) = DZW - DWZ - [Z,W]. \quad (2.24)$$

Its coefficients are denoted by $T(\delta h, \delta k) = T^i_{hk}\delta_i + T^\alpha_{hk}\delta_\alpha$, etc. They are given by

$$T^i_{hk} = L^i_{hk} - L^i_{kh}, \quad T^\alpha_{hk} = \partial_k N^\alpha_h - \partial_h N^\alpha_k,$$

$$T^i_{\alpha h} = -L^i_{h\alpha}, \quad T^\beta_{\alpha h} = L^\beta_{\alpha h}, \quad T^\gamma_{\alpha \beta} = C^\gamma_{\alpha \beta} - T^\gamma_{\beta \alpha}. \quad (2.24)$$

The curvature of a distinguished complex linear connection on $T'E$ is defined by

$$R(Z,W) = DZW - DWZ - D[Z,W]. \quad (2.25)$$
In the adapted frame of fields, the coefficients of the curvature are
\[ R_{jkh}^i = \partial_k L_{jh}^i - \partial_h L_{jk}^i + L_{jk}^i L_{ih}^j - L_{jh}^i L_{ik}^j - (\partial_h N^\alpha_k - \partial_k N^\alpha_h) L_{ij}^h, \]
\[ R_{jkh}^\alpha = \partial_k L_{jh}^\alpha - \partial_h L_{jk}^\alpha + L_{jk}^\alpha L_{ih}^\gamma - L_{jh}^\alpha L_{ik}^\gamma - (\partial_h N^\gamma_k - \partial_k N^\gamma_h) C_{\gamma k}^\alpha, \]
\[ R_{\gamma k\beta}^\alpha = \partial_k C_{\gamma h}^\alpha + C_{\gamma \beta}^\alpha L_{\alpha k}^\gamma - L_{\gamma k}^\alpha C_{\alpha \beta}^\gamma, \]
\[ R_{i jh}^\alpha = \partial_h L_{ij}^\alpha + L_{ik}^\gamma L_{jh}^\gamma - L_{jk}^\gamma L_{ih}^\gamma, \]
\[ R_{i k\beta}^\alpha = L_{kj}^\gamma L_{ih}^\gamma - L_{jk}^\gamma L_{ih}^\gamma, \]
\[ R_{\sigma \alpha \beta}^\gamma = C_{\gamma j}^\beta C_{\alpha \tau}^\gamma - C_{\gamma \alpha}^\tau C_{\tau \beta}^\gamma. \]

2.1.4 Semisprays and sprays

The notion of semispray on a holomorphic Lie algebroid has been introduced in [6] following the steps from the real case ([2, 3]). Let \( \rho_E \) denote the anchor map and \( \pi^* \), the tangent map of the projection \( \pi \) and \( \tau_E : T'E \to E \).

**Definition 2.3.** A holomorphic section \( S : E \to T'E \) is called semispray if
i) \( \tau_E \circ S = \text{Id}_E \),
ii) \( \pi^* \circ S = \rho_E \).

Let \( c : I \to M, I \subset \mathbb{R} \) be a complex curve on \( M \), \( \tilde{c} : I \to E \) a complex curve on \( E \) such that \( \pi \circ \tilde{c} = c \) and denote by \( \dot{\tilde{c}} \) the tangent vector field to the curve \( \tilde{c} \).

**Definition 2.4.** The vector field \( \dot{\tilde{c}} \) is called admissible if
\[ \pi_* (\dot{\tilde{c}}) = \rho(\dot{\tilde{c}}). \] (2.26)

Locally, \( c(t) = (z^k(t)), \tilde{c} = (z^k(t), u^\alpha(t)) \) and \( \dot{\tilde{c}} = \frac{dz^k}{dt} \frac{\partial}{\partial z^k} + \frac{du^\alpha}{dt} \frac{\partial}{\partial u^\alpha}, t \in I. \)

Then, the curve \( \tilde{c} \) is admissible if and only if
\[ \frac{dz^k}{dt}(t) = \rho_k^\alpha(z(t)) u^\alpha(t), \forall t \in I. \]

If \( S = Z^k \frac{\partial}{\partial z^k} + U^\alpha \frac{\partial}{\partial u^\alpha} \), then, using the definition, it follows that \( S \) is a semispray if and only if
\[ Z^k(z, u) = \rho_k^\alpha(z) u^\alpha. \] (2.27)

The coefficients \( U^\alpha(z, u) \) are not determined, thus, for easier computations, let \( U^\alpha = -2G^\alpha \), such that
\[ S = \rho_k^\alpha u^\alpha \frac{\partial}{\partial z^k} - 2G^\alpha(z, u) \frac{\partial}{\partial u^\alpha}. \] (2.28)

The rules of change for the coordinates of \( S \) are obtained using the (1.5) matrix:
\[ Z^k = \frac{\partial z^k}{\partial z^h} Z^h \] (2.29)
\[ G^\alpha = M^\alpha_{\beta} G^\beta - \frac{1}{2} \frac{\partial M^\alpha_{\beta}}{\partial z^k} u^\beta \rho^k_\gamma u^\gamma. \] (2.30)

Moreover, due to (1.8), the coefficients \( Z^k(z,u) \) given by (2.27) verify the (2.29) laws of change, which leads to the following result.

**Proposition 2.3.** A vector field \( S = \rho^k_\alpha u^\alpha \frac{\partial}{\partial z^k} - 2G^\alpha \frac{\partial}{\partial u^\alpha} \in \Gamma(T'E) \) is a semispray if and only if the coefficients \( G^\alpha \) verify the (2.30) rules of transformation.

A curve \( c: t \mapsto (z^i(t),u^\alpha(t)) \) on \( E \) is an integral curve of the semispray \( S \) if it satisfies the system of differential equations

\[
\frac{dz^i}{dt} = \rho^i_\alpha(t) u^\alpha, \quad \frac{du^\alpha}{dt} + 2G^\alpha(z,u) = 0. \tag{2.31}
\]

A semispray can then be characterized also by

**Proposition 2.4.** A vector field on \( E \) is a semispray if and only if all its integral curves are admissible.

Let \( L = \frac{\partial L}{\partial u^\alpha} \) be the complex Liouville vector field on \( E \). Then, an even simpler formulation for the condition of spray can be obtained using Euler’s theorem for homogeneous functions:

\[ [L, S]_E = S. \] (2.34)

In [6], we have obtained a complex spray from the variational problem. Following the ideas of Weinstein ([15]), the Euler-Lagrange equations on \( E \) are

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial u^\alpha} \right) = \rho^k_\beta \frac{\partial L}{\partial z^k} + \rho^k_\beta \frac{\partial L}{\partial \bar{z}^k} + Q^\alpha_\beta \frac{\partial L}{\partial u^\alpha} + Q^\alpha_\beta \frac{\partial L}{\partial \bar{u}^\alpha}, \tag{2.35}
\]

where \( \rho^k_\beta = 0 \) since \( E \) is holomorphic and \( Q^\alpha_\beta \) and \( Q^\alpha_\beta \) must be determined.

**Theorem 2.1.** On a holomorphic Lie algebroid \( E \) endowed with a regular Lagrangian \( L(z,u) \) and a Hermitian metric tensor \( g_{\bar{\alpha}\beta} \) with \( \det(g_{\bar{\alpha}\beta}) \neq 0 \), a complex canonical spray is given by

\[
G^\alpha = \frac{1}{2} \left( g^\bar{\beta}_{\bar{\alpha}} \frac{\partial^2 L}{\partial z^k \partial \bar{u}^\beta} + \frac{1}{2} W^\epsilon_{\bar{\alpha}} \frac{\partial M^\epsilon_{\bar{\beta}}}{\partial z^k} u^\beta \right) \rho^k_\gamma u^\gamma. \tag{2.36}
\]

**Remark 2.1.** If the Lagrangian on \( E \) is complex homogeneous, then the spray is complex homogeneous of degree 2 in \( u \).
2.2 The prolongation of a holomorphic Lie algebroid

For the holomorphic Lie algebroid $E$ over a complex manifold $M$, its prolongation will be introduced using the tangent mapping $\pi_\ast : T'\!E \to T'M$ and the holomorphic anchor map $\rho_\!E : E \to T'M$. Define the subset $\mathcal{T}'\!E$ of $E \times T'\!E$ by $\mathcal{T}'\!E = \{(e, v) \in E \times T'\!E | \rho(e) = \pi_\ast(v)\}$ and the mapping $\pi_\!T : \mathcal{T}'\!E \to E$, given by $\pi_\!T(e, v) = \pi_\!E(v)$, where $\pi_\!E : T'\!E \to E$ is the tangent projection. Then $(\mathcal{T}'\!E, \pi_\!T, E)$ is a holomorphic vector bundle over $E$, of rank $2m$. Moreover, it is easy to verify that the projection onto the second factor $\rho_\!T : \mathcal{T}'\!E \to T'\!E$, $\rho_\!T(e, v) = v$, is the anchor of a new holomorphic Lie algebroid over the complex manifold $E$ (see [8,9,12] for details in the real case).

The vertical subbundle of the prolongation is defined using the projection onto the first factor $\tau_\!_1 : \mathcal{T}'\!E \to E$, $\tau_\!_1(e, v) = e$, by

$$V\mathcal{T}'\!E = \ker \tau_\!_1 = \{(e, v) \in \mathcal{T}'\!E | \tau_\!_1(e, v) = 0\}.$$ 

From the construction above, it follows that any element of $V\mathcal{T}'\!E$ has the form $(0, v) \in E \times \mathcal{T}'\!E$, with $\pi_\ast(v) = 0$. Then, vertical elements $(0, v) \in V\mathcal{T}'\!E$ have the property $v \in \ker \pi_\ast$ and $v$ is a vertical vector on $E$.

The local coordinates on $\mathcal{T}'\!E$ are $(z^k, u^\alpha, v^\alpha, w^\alpha)$, obtained from the local coordinates $(z^k, u^\alpha)$ of $e$ by using the identity $\rho(e) = \pi_\ast(v)$, which yields the vector $v$ in the form

$$v = \rho_\!_\alpha = v^\alpha \frac{\partial}{\partial z^k} + w^\alpha \frac{\partial}{\partial u^\alpha}.$$ 

The local basis of holomorphic sections in $\Gamma(\mathcal{T}'\!E)$ is $\{Z_\alpha, V_\alpha, \}$, defined by

$$Z_\alpha(e) = \left(s_\alpha(\pi(e)), \rho_\!_\alpha \frac{\partial}{\partial z^k}\right), \quad V_\alpha(e) = \left(0, \frac{\partial}{\partial u^\alpha}\right),$$

where $\left\{\frac{\partial}{\partial z^k}, \frac{\partial}{\partial u^\alpha}\right\}$ is the natural frame on $T'\!E$.

If $W$ is a holomorphic section of $\mathcal{T}'\!E$, then its decomposition in the basis $\{Z_\alpha, V_\alpha, \}$ is

$$W = Z_\alpha Z_\alpha + V_\alpha V_\alpha,$$

where $Z_\alpha$ and $V_\alpha$ are holomorphic functions of $z$ and $u$.

Also, the holomorphic vector field $\rho_\!T(W) \in \Gamma(\mathcal{T}'\!E)$ can be written as

$$\rho_\!T(W) = \rho_\!_\alpha Z_\alpha(z, u) \frac{\partial}{\partial z^k}(z, u) + V_\alpha(z, u) \frac{\partial}{\partial u^\alpha}(z, u).$$

A section $Z \in \Gamma(E)$ can be lifted to sections of the prolongation $\mathcal{T}'\!E$ by considering its vertical and complete lifts $Z^v$ and $Z^c$, which will be defined in the following (see [11]). The vertical lift of a section $Z \in \Gamma(E)$, $Z = Z_\alpha s_\alpha$, is a vector field on $E$ given by

$$Z^v(z, u) = Z_\alpha(z) \frac{\partial}{\partial u^\alpha}.$$ (2.37)
In particular, \( s^v_\alpha = \frac{\partial}{\partial u^\alpha} \).

The complete lift \( Z^c \) of a section \( Z \in \Gamma(E) \) is a vector field on \( E \) defined by

\[
Z^c(z,u) = Z^\alpha \rho^k \frac{\partial}{\partial z^k} + \left( \rho^k \frac{\partial Z^\alpha}{\partial z^k} - Z^\gamma C^\alpha_{\gamma \beta} \right) u^\beta \frac{\partial}{\partial u^\alpha}.
\] (2.38)

In particular, \( s^c_\alpha = \rho^k \frac{\partial}{\partial z^k} - C^\alpha_{\gamma \beta} u^\beta \frac{\partial}{\partial u^\gamma} \).

The lifts on \( T^E \) are defined as

\[
Z^V(e) = (0, Z^v(e)), \quad Z^C(e) = (Z(\pi(e)), Z^c(e)), \quad e \in E.
\]

In local coordinates, if \( Z = Z^\alpha s^\alpha \), then the expressions of \( Z^V \) and \( Z^C \) are

\[
Z^V = Z^\alpha V^\alpha, \quad Z^C = Z^\alpha z^\alpha + \left( \rho^k \frac{\partial Z^\alpha}{\partial z^k} - Z^\gamma C^\alpha_{\gamma \beta} \right) u^\beta V^\alpha.
\]

In particular, \( s^V_\alpha = V_\alpha \) and \( s^C_\alpha = Z_\alpha - C^\beta_{\alpha \gamma} u^\gamma V^\beta \).

The Lie bracket \([\cdot, \cdot]_T\) on \( T^E \) satisfies the identities

\[
[Z^V, W^V]_T = 0, \quad [Z^V, W^C]_T = [Z, W]^V_E, \quad [Z^C, W^C]_T = [Z, W]^C_E
\]

for \( Z, W \in \Gamma(E) \). The structure of a holomorphic Lie algebroid on the vector bundle \((T^E, \pi_T, E)\) is therefore given by \([\cdot, \cdot]_T, \rho_T\). The action of the anchor \( \rho_T \) on \( T^E \) is locally described by

\[
\rho_T(z^\alpha) = \rho^k \frac{\partial}{\partial z^k}, \quad \rho_T(V^\alpha) = \frac{\partial}{\partial u^\alpha}.
\]

**Lemma 2.1.** The Lie brackets of the basis \( \{z_\alpha, v_\alpha\} \) are:

\[
[z_\alpha, z_\beta]_T = C^\gamma_{\alpha \beta} z_\gamma, \quad [z_\alpha, v_\beta]_T = 0, \quad [v_\alpha, v_\beta]_T = 0.
\]

As in the real case ([8]), a differential \( \partial_T \) can be defined on \( T^E \). Denoting by \( \{z^\alpha, v^\alpha\} \) the dual base of \( \{z_\alpha, v_\alpha\} \), then

\[
\partial_T z^k = \rho^k z^\alpha, \quad \partial_T u^\alpha = v^\alpha
\]

and

\[
\partial_T z^\alpha = -\frac{1}{2} C^\beta_{\alpha \gamma} z^\beta \wedge z^\gamma, \quad \partial_T v^\alpha = 0.
\]

As announced in the previous section, the notion of semispray for a Lie algebroid can be introduced in another manner as well ([8] [12]). More precisely, a semispray can also be considered on the prolongation \( T^E \) of the holomorphic Lie algebroid \( E \). Let \( \mathcal{L} \) be the complex Liouville section on \( T^E \), defined by

\[
\mathcal{L}(e) = (0, e^V_v), \quad e \in E.
\] (2.39)

The coordinate expression of \( \mathcal{L} \) is

\[
\mathcal{L} = u^\alpha v^\alpha.
\] (2.40)
Also, let \( T \) be the tangent structure (or vertical endomorphism) defined on \( \mathcal{T}'E \) by
\[
T(Z^C) = Z^V, \quad T(Z^V) = 0.
\]
(2.41)
In local coordinates,
\[
T = Z^\alpha \otimes V_\alpha,
\]
(2.42)
which yields
\[
T(Z_\alpha) = V_\alpha, \quad T(V_\alpha) = 0.
\]
(2.43)

**Lemma 2.2.** If \( T \) is the complex tangent structure on \( \mathcal{T}'E \) and \( C \) is the complex Liouville section, then
\[
T^2 = 0, \quad \text{Im} \, T = \ker T = V\mathcal{T}'E, \quad [\mathcal{L}, T]_T = -T.
\]
(2.44)

These two canonical complex objects on \( \mathcal{T}'E \) can now be used for defining the notion of complex semispray on the prolongation of \( E \).

**Definition 2.5.** A section \( S \) of the holomorphic Lie algebroid \( \mathcal{T}'E \) is called complex semispray on \( E \) if
\[
T(S) = \mathcal{L}.
\]

In order to describe locally a complex semispray on \( \mathcal{T}'E \), let \( S = A^\alpha Z_\alpha + B^\alpha V_\alpha \). Then, (2.40) and (2.43) yield \( A^\alpha = u^\alpha \), and for convenience let \( B^\alpha = -2G^\alpha \). Therefore, the local expression of a semispray on \( \mathcal{T}'E \) is
\[
S = u^\alpha Z_\alpha - 2G^\alpha(z, u)V_\alpha.
\]
If \([\mathcal{L}, S]_T = S\), then \( S \) is called spray and \( G^\alpha \) are homogeneous functions of degree 2.

### 2.2.1 Nonlinear connections on \( \mathcal{T}'E \)

The method of nonlinear connection discussed in Section 2.1.2 will be applied here for the prolongation \( \mathcal{T}'E \) of the holomorphic Lie algebroid \( E \). A complex nonlinear connection on \( \mathcal{T}'E \) is given by a complex vector subbundle \( H\mathcal{T}'E \) of \( \mathcal{T}'E \) such that \( \mathcal{T}'E = H\mathcal{T}'E \oplus V\mathcal{T}'E \). If \( l^h \) is the horizontal lift, then similar considerations as in the real case ([12]) lead to the following local expression of
\[
l^h(Z_\alpha) = Z_\alpha - N_\alpha^\beta V_\beta, \quad l^h(V_\alpha) = 0,
\]
where \( N_\alpha^\beta = N_\alpha^\beta(z, u) \) are functions defined on \( E \), called the coefficients of the complex nonlinear connection on \( \mathcal{T}'E \).

Denote by
\[
\delta_\alpha = Z_\alpha - N_\alpha^\beta V_\beta \quad (2.45)
\]
in order to obtain a local frame \( \{\delta_\alpha, V_\alpha\} \) on \( \mathcal{T}'E \), called the *adapted frame* with respect to the complex nonlinear connection \( N \) on \( \mathcal{T}'E \). Then
\[
\rho_T(\delta_\alpha) = \rho^k_\alpha \frac{\partial}{\partial z^k} - N_\alpha^\beta \frac{\partial}{\partial u^\beta}, \quad \rho_T(V_\alpha) = \frac{\partial}{\partial u^\alpha}.
\]
(2.46)
The dual of the adapted frame of fields is \( \{ \mathcal{Z}^\alpha, \delta \mathcal{V}^\alpha \} \), where
\[
\delta \mathcal{V}^\alpha = \mathcal{V}^\alpha + N^\beta_\alpha \mathcal{Z}^\beta
\]
and \( \{ \mathcal{Z}^\alpha, \mathcal{V}^\alpha \} \) is the dual basis of \( \{ \mathcal{Z}_\alpha, \mathcal{V}_\alpha \} \).

**Proposition 2.5.** The Lie brackets of the adapted frame \( \{ \delta \alpha, \mathcal{V}_\alpha \} \) are
\[
\begin{align*}
[\delta \alpha, \delta \beta]_T &= C^\gamma_{\alpha \beta} \delta \gamma + \mathcal{R}^\gamma_{\alpha \beta} \mathcal{V}_\gamma, \\
[\delta \alpha, \mathcal{V}_\beta]_T &= \frac{\partial N^\gamma_\beta}{\partial u^\alpha} \mathcal{V}_\gamma, \\
[\mathcal{V}_\alpha, \mathcal{V}_\beta]_T &= 0,
\end{align*}
\]
where
\[
\mathcal{R}^\gamma_{\alpha \beta} = C^\gamma_{\alpha \beta} N^\gamma_\epsilon + \rho^k_{\beta} \frac{\partial N^\gamma_\alpha}{\partial z^k} - \rho^k_{\alpha} \frac{\partial N^\gamma_\beta}{\partial z^k} - N^\gamma_\beta \frac{\partial N^\gamma_\alpha}{\partial u^\epsilon} + N^\gamma_\alpha \frac{\partial N^\gamma_\beta}{\partial u^\epsilon}.
\]

Now, consider the complex nonlinear connection on \( T' E \) introduced in Section 2.1.2. Its coefficients, \( N^\beta_\alpha(z,u) \), change by the rules (2.16) and the adapted frame of fields is \( \{ \delta \delta z^k, \frac{\partial}{\partial u^\alpha} \} \), where
\[
\frac{\delta}{\delta z^k} = \frac{\partial}{\partial z^k} - N^\rho_\alpha \frac{\partial}{\partial u^\rho}.
\]

It is interesting to study the relation between the two nonlinear connections on \( T' E \) and \( T'E' \), respectively. The first relation in (2.46) suggests considering another adapted frame on \( T'E' \), defined by
\[
\delta \alpha = \rho^k_\alpha \frac{\partial}{\partial z^k} - N^\beta_\alpha \frac{\partial}{\partial u^\beta}, \tag{2.47}
\]
and imposing that it changes by the rules
\[
\delta \alpha = M^\beta_\alpha \delta \beta \tag{2.48}
\]
or, using (2.16) and (2.3),
\[
M^\beta_\alpha \tilde{N}^{\tilde{\gamma}}_\beta = M^\gamma_\beta N^\gamma_\alpha - \rho^k_\alpha \frac{\partial M^\gamma_\beta}{\partial z^k} u^\beta. \tag{2.49}
\]
Note that these rules of change can be obtained from (2.16) by contracting with \( \rho^k_\alpha \) and by denoting
\[
N^\beta_\alpha = \rho^k_\alpha N^\beta_k. \tag{2.50}
\]
It follows that

**Proposition 2.6.** A complex nonlinear connection on \( T' E \) with the coefficients \( N^\beta_\alpha \) induces a complex nonlinear connection on \( T'E' \) with the coefficients \( N^\beta_\alpha \) given by (2.50). Moreover, the relations between the adapted frames on \( T'E' \) and \( T'E' \) are
\[
\rho_T(\delta \alpha) = \rho^k_\alpha \delta \delta z^k. \tag{2.51}
\]
Proof. We first have to prove that $N_\alpha^\beta$ from (2.50) are the coefficients of a complex nonlinear connection on $T'E$. Let $\rho_\gamma^k$ be the dual map of $\rho_\gamma$. Then $\rho_\gamma^k dz^k = \rho_\alpha^k z_\alpha$ and $\rho_\gamma^k (du^\alpha) = V_\alpha$, such that $\rho_\gamma^k (du^\alpha) = \rho_\gamma^k (du^\alpha + N_\delta^k dz^k) = V_\alpha + N_\delta^k \rho_\beta^k z_\beta$. On the other hand, on $(T'E)^*$, the dual adapted frame is $\delta V_\alpha = V_\alpha + N_\beta^\gamma z_\beta$ and $\rho_\gamma^k |(V'T'E)^* : (V'T'E)^* \to (V'T'E)^*$ is an isomorphism, that is, $\rho_\gamma^k (du^\alpha) = \delta V_\alpha$, which yields $N_\delta^k \rho_\beta^k = N_\alpha^\beta$. Also, $$\rho_\gamma (\delta_\alpha) = \rho_\gamma (z_\alpha - N_\alpha^\beta V_\beta) = \rho_\alpha^k \frac{\partial}{\partial z^k} - N_\alpha^\beta \frac{\partial}{\partial u^\beta} = \rho_\alpha^k \left( \frac{\partial}{\partial z^k} - N_\beta^\gamma \frac{\partial}{\partial u^\beta} \right) = \rho_\alpha^k \delta_\alpha^\beta.$$  

This result shows that the adapted frame (2.47) can very well be interpreted as defining a new complex nonlinear connection on $E$. An interesting property of this connection is that it can be derived from a spray.

**Theorem 2.2.** If $G_\alpha$ are the coefficients of a complex spray on $T'E$, as defined in (2.30), then the functions

$$N_\alpha^\beta = \frac{\partial G_\alpha}{\partial u^\beta} + P_\alpha^\beta$$

(2.52)

define a complex nonlinear connection on $E$, where

$$P_\alpha^\beta = \frac{1}{4} W_\gamma^\alpha \left( \rho_\alpha^k \frac{\partial M_\gamma^\beta}{\partial z^k} u^\delta - \frac{\partial M_\alpha^\beta}{\partial z^k} \rho_\delta^k u^\delta \right).$$

(2.53)

**Proof.** Deriving in (2.30) with respect to $u^\gamma$ and taking into account the (1.8) rules of change yields

$$M_\gamma^\alpha \frac{\partial \tilde{G}_\alpha}{\partial u^\gamma} = M_\beta^\alpha \frac{\partial G_\beta}{\partial u^\gamma} - \frac{1}{2} \frac{\partial M_\alpha^\beta}{\partial z^k} \rho_\delta^k u^\delta - \frac{1}{2} \frac{\partial M_\alpha^\beta}{\partial z^k} \rho_\delta^k u^\delta.$$  

Replacing $\frac{\partial G_\beta}{\partial u^\gamma} = N_\gamma^\beta - P_\gamma^\beta$ gives

$$M_\gamma^\delta N_\delta^\beta - M_\gamma^\delta P_\delta^\beta = M_\beta^\alpha N_\alpha^\beta - M_\beta^\alpha P_\gamma^\beta - \rho_\delta^k \frac{\partial M_\alpha^\beta}{\partial z^k} u^\delta + \frac{1}{2} \rho_\delta^k \frac{\partial M_\alpha^\beta}{\partial z^k} u^\delta - \frac{1}{2} \frac{\partial M_\alpha^\beta}{\partial z^k} \rho_\delta^k u^\delta.$$  

Comparing this with (2.49) means that we have to show that

$$M_\beta^\alpha P_\gamma^\beta - M_\gamma^\delta P_\delta^\beta = \frac{1}{2} \frac{\partial M_\alpha^\beta}{\partial z^k} u^\delta - \frac{1}{2} \frac{\partial M_\alpha^\beta}{\partial z^k} \rho_\delta^k u^\delta.$$  

(2.54)

First, from (2.53), it follows that

$$M_\beta^\alpha P_\gamma^\beta = \frac{1}{4} \left( \rho_\delta^k \frac{\partial M_\alpha^\beta}{\partial z^k} u^\delta - \frac{\partial M_\alpha^\beta}{\partial z^k} \rho_\delta^k u^\delta \right).$$  

(2.55)
Then, again using (2.53),
\[
\tilde{P}^\alpha_8 = \frac{1}{4} M^\alpha_8 \left( \rho^k_8 \frac{\partial W^\beta_8}{\partial z^k} u^\gamma - \frac{\partial W^\beta_8}{\partial z^k} \rho^h_8 u^\gamma \right),
\]
which, multiplied by \(- M^\delta_8\) and using (1.8) and
\[
M^\alpha_8 \frac{\partial W^\beta_8}{\partial z^k} = - \frac{\partial M^\alpha_8}{\partial z^k} W^\beta_8,
\]
gives after some basic computations
\[
- M^\delta_k \tilde{P}^\alpha_8 = \frac{1}{4} \left( \rho^k_8 \frac{\partial M^\alpha_8}{\partial z^k} - \frac{\partial M^\alpha_8}{\partial z^k} \rho^k_8 \right). \tag{2.56}
\]
Adding (2.55) and (2.56) yields (2.54).

3 Induced Lagrange structures on holomorphic Lie algebroids

It is interesting to study the conditions under which two nonlinear connections on \(E\) and \(T'M\), respectively, are linked.

The geometry of the bundle \(T'M\) endowed with a complex Lagrangian \(L(z, \eta)\), where
\[
g_{ij}(z, \eta) = \frac{\partial^2 L}{\partial \eta^i \partial \bar{\eta}^j}, \tag{3.1}
\]
defines a nondegenerate metric, is well-known. The pair \((M, L)\) is called complex Lagrange space ([10]). In order to distinguish it from \(E\), we will further denote it \((T'M, L)\).

A remarkable complex nonlinear connection on \(T'M\) is
\[
N^i_k = g^{ji} \frac{\partial^2 L}{\partial z^k \partial \bar{\eta}^j}, \tag{3.2}
\]
called the Chern-Lagrange nonlinear connection. If \(L\) if homogeneous in \(\eta\), i.e. \(L(z, \lambda \eta) = \lambda L(z, \eta)\) for all \(\lambda \in \mathbb{C}\), then \((M, L)\) is called complex Finsler space and in particular (3.2) defines a complex nonlinear connection on \(T'M\), called Chern-Finsler nonlinear connection.

The problem of introducing a Lagrange (in particular, Finsler), on \(E\) by a Lagrange (Finsler) complex space \((T'M, L)\) is relatively simple if we impose some additional conditions on the fibers of \(E\) and on the anchor map \(\rho\). We will now analyze the three possible cases, depending on the relation between the dimensions of \(M\), \(E\) and the rank of \(\rho\).

I) The case when \(m = n = \text{rank} \rho\).

Recall that on the manifold \(E\) we have local coordinates in a map \((z^k, u^\alpha)\), while on \(T'M\) we have \((z^k, \eta^k)\), where \(\eta^k = u^\alpha \rho^k_\alpha(z)\), as stated in (2.4), and \(\alpha, \beta, ..., i, j, ...\) all range over \(1, n\).
Since $n = \text{rank} \rho$, it follows that $\rho$ is a diffeomorphism with $\rho^{-1} = [\rho^a_k]$, such that $\rho^a_k \rho^k_i = \delta^a_i$ and $u^a = \rho^a_k \eta^k$.

If $(T'M, L)$ is a complex Lagrange (Finsler) complex space and $L(z, \eta)$ is the Lagrange function, then, according to (2.4), it induces on $E$ another Lagrange function, $L^*(z, u) = L(z, \eta(u))$, where the metric tensor is

$$g_{\alpha \beta}(z, u) := \frac{\partial^2 L^*}{\partial u^\alpha \partial u^\beta} = \rho^k_\alpha \rho^\beta_k g_{ij}.$$ 

Since $\rho$ is a diffeomorphism, rank $g_{\alpha \beta} = n = m$ and $L^*$ can be interpreted as a function on $E$.

The pair $(E, L^*)$ is called Lagrange structure on the Lie algebroid $E$. Notice that if $L$ is homogeneous, then $L^*(z, \lambda u) = \lambda \lambda L^*(z, u)$, in which case a Finsler structure is induced on $E$.

Let $N^h_k(z, \eta)$ be the coefficients of a nonlinear connection on $E$ and $\delta = \frac{\partial}{\partial \delta^k} - N^h_k \frac{\partial}{\partial u^k}$ the corresponding adapted frame of fields. It is mapped by $\rho_k$, using (2.8), to $\delta^\ast = \frac{\partial}{\partial \delta^k} - N^h_k \frac{\partial}{\partial \eta^k}$ and we obtain

$$N^h_k(z, \eta) = \rho^h_k N^\alpha_k(z, u) - \frac{\partial \rho^h_k}{\partial \delta^k} u^\alpha (3.3)$$

where $\eta^k = u^\alpha \rho^h_\alpha(z)$. From $\frac{\partial}{\partial \delta^k} = \frac{\partial \delta^k}{\partial \delta^k} \delta^k$, it follows that for a change of local maps on $E$ we have $\delta^\ast = \frac{\partial}{\partial \delta^k} \delta^\ast$, hence $N^h_k$ defines a nonlinear connection, induced on $T'M$ by $N^\alpha_k$.

Conversely, let $N^\alpha_k(z, u)$ be a complex nonlinear connection on $T'M$. We search for a nonlinear connection $N^\ast_k$ induced on $E$ by $N^h_k$. Obviously, we will need that $N^h_k(z, \eta) = \rho^h_k N^\alpha_k(z, u) - \frac{\partial \rho^h_k}{\partial \delta^k} u^\alpha$, which, contracted with $\rho^\beta_k$ gives $N^\beta_k(z, u) = \rho^\beta_k N^h_k(z, \eta) + \rho^\beta_k \frac{\partial \rho^h_k}{\partial \delta^k} u^\alpha$. By deriving the condition $\rho^\beta_k \rho^\alpha_k = \delta^\beta_k$ and substituting in the above we get the following nonlinear connection induced on $E$:

$$N^\ast_k(z, u) = \rho^h_k N^\alpha_k(z, \eta) - \frac{\partial \rho^h_k}{\partial \delta^k} \eta^k (3.4)$$

To conclude, in the case when $m = n = \text{rank} \rho$, the diffeomorphism $\rho_k$ maps the decomposition $T'E = VE \oplus HE$ in $T'T'M = VT'M \oplus HT'M$ preserving the distributions, and $\rho^{-1}$ has the converse role.

**Proposition 3.1.** If $(T'M, L)$ is a complex Lagrange (Finsler) space and $(3.2)$ is an associated complex nonlinear connection, then

$$N^\ast_k = g^{\beta \alpha} \frac{\partial^2 L^*}{\partial z^\alpha \partial u^\beta} (3.5)$$

is the nonlinear connection induced on $E$, called the Chern-Lagrange connection of the Lie algebroid.
II) The case when \( \text{rank } \rho = m < n \).

Note that, in this case, the morphism \( \rho \) maps \( E \) in \( \rho(E) \) which is an immersed submanifold of \( T'M \).

As in the first case, we will introduce on \( E \) Lagrange structures induced by a Lagrange structure \((T'M, L)\).

Let \( g_{i\bar{j}}(z, \eta) = \frac{\partial^2 L}{\partial \eta^i \partial \bar{\eta}^j} \) the metric tensor defined by the regular Lagrangian \( L : T'M \to R \). As in the first case, we consider the Lagrangian induced on \( \rho(E) \) given by \( L^* (z, u) = L(z, \eta(u)) \), with the metric tensor

\[
g_{\alpha \bar{\beta}}(z, u) := \frac{\partial^2 L^*}{\partial u^\alpha \partial \bar{u}^\beta} = \rho_{\alpha \beta} \bar{g}_{ij}.
\]

Since \( \text{rank } \rho = m \) and \( \text{rank}[g_{i\bar{j}}] = n > m \), it follows that \( \text{rank}[g_{\alpha \bar{\beta}}] = m \).

Let \( X_\alpha = \rho_{\alpha k}^i \frac{\partial}{\partial z^k} \in \chi(M) \), \( \alpha = \overline{1,m} \), be fields on the basis manifold \( M \).

From \( \text{rank } \rho = m \) we read that \( \{X_\alpha\} \) are linear independent and can be lifted to \( \rho(E) \) by \( \{X'_\alpha := \frac{\partial}{\partial \eta^\alpha} = \rho_{\alpha \beta}^k \frac{\partial}{\partial \eta^k}\}_{\alpha = \overline{1,m}} \), which defines an \( m \)-dimensional subdistribution \( V \rho(E) \) of the \( n \)-dimensional distribution \( VT'M \).

Let us fix a complex nonlinear connection \( N_k^h(z, \eta) \) on \( T'M \), in particular the Chern-Lagrange connection, and let us consider its adapted basis and cobasis.

We search for a nonlinear connection \( N_k^h(z, \eta(u)) \) induced by \( N_k^h(z, \eta) \) as the image through \( \rho \) of a nonlinear connection \( N_k^h(z, u) \) on \( E \).

Denote by \( G = g_{i\bar{j}} \delta \eta^i \otimes \delta \bar{\eta}^j \) the metric structure on \( VT'M \). We complete \( \{X'_\alpha\}_{\alpha = \overline{1,m}} \) with \( \{Y_a\}_{a = \overline{1,n-m}} \), vector fields normal to \( V \rho(E) \) with respect to \( G \).

Moreover, we assume these vectors to be orthonormal. By writing \( Y_a = Y_a^k \frac{\partial}{\partial \eta^k} \), the orthogonality conditions read

\[
g_{i\bar{j}} Y_a^i \rho_{\alpha \beta}^a Y_a^\bar{j} = 0 \tag{3.6}
\]

and the normality ones give \( g_{i\bar{j}} Y_a^i Y_a^\bar{j} = \delta_{ab} \).

We have thus obtained \( \mathcal{R}^* = \{X'_\alpha, Y_a\} \), a frame on \( VT'M \) with the matrix \( R = [\rho_{\alpha k}^i ; Y_a^k] \) of change from the natural frame \( \left\{ \frac{\partial}{\partial \eta^k} \right\} \). Let \( R^{-1} = [\rho_{i \alpha}^a ; Y_i^a] \) its inverse matrix, such that

\[
\rho_{i \alpha}^a \rho_{\alpha \beta}^a = \delta_{i \beta} \quad ; \quad \rho_{i \alpha}^a Y_a^i = 0 \quad ; \quad Y_i^a Y_i^\beta = 0 \quad ; \quad \rho_{i \alpha}^a + Y_i^a Y_i^\alpha = \delta_i^\beta. \tag{3.7}
\]

The technique used in the following is known from holomorphic subspaces, \cite{[10]}. From (3.7) we get

\[
g_{i\bar{j}} Y_a^i \rho_{\alpha \beta}^a \bar{Y}_a^\bar{j} = 0 \quad ; \quad g_{i\bar{j}} \rho_{\alpha \beta}^a \bar{Y}_a^\bar{j} = 0 \tag{3.8}
\]

Now, let \( N_k^h(z, \eta) \) be a complex nonlinear connection on \( T'M \) and \( N_k^h(z, u) \) a connection on \( E \), with the adapted cobases \( \{dz^k, \delta u^\alpha = du^\alpha + N_h^a dz^k \} \) and...
\{dz^k, \delta \eta^k = dy^k + N_h^k dz^h\}, respectively. The identities in (2.9) suggest considering the cobasis \delta \eta^k as a frame of forms \delta^* \eta^k = d^* \eta^k + N_h^k d^* z^h induced on \rho(E), where \( N_h^k = Y_h^k(z, \eta) \) with \eta^k = \rho_a^k u^\alpha. Among these, only \( m \) are linear independent. In the following, for the simplicity of writing, we will identify \( N_h^k = N_h^k \).

**Definition 3.1.** \( N_h^\alpha \) is called an induced connection on \( E \) by \( N_h^k \) from \( T'M \) if

\[
\delta^\alpha u^\alpha = \rho_h^\alpha \delta^* \eta^k.
\]  

(3.9)

In matrix form, the right-hand side yields \( \text{rank}[\delta^\alpha u^\alpha] = m \), thus \( \{\delta^\alpha u^\alpha\} \) can define a cobasis on \( E \). Substituting from (2.9), we obtain

\[
du^\alpha + N_h^\alpha dz^h = \rho_h^\alpha \left\{ u^\beta \partial \rho_h^k \partial z^h + \rho_h^k du^\beta + N_h^k dz^h \right\},
\]

such that \( \rho_h^\alpha \rho_h^k = \delta^\alpha_h \) yields \( du^\alpha + N_h^\alpha dz^h = du^\alpha + \rho_h^\alpha \left\{ N_h^k + u^\beta \partial \rho_h^k \partial z^h \right\} dz^h \). This means that the connections are linked by

\[
N_h^\alpha(z, u) = \rho_h^\alpha \left\{ Y_h^k(z, \eta) + u^\beta \partial \rho_h^k \partial z^h \right\}, \quad \text{where} \quad \eta^k = \rho_a^k u^\alpha.
\]  

(3.10)

Thus, given a nonlinear connection \( N_h^k(z, \eta) \) on \( T'M \), we have obtained a nonlinear connection \( N_h^\alpha(z, u) \) on \( E \). The relations between the two connections are given in the following

**Proposition 3.2.** If \( N_h^\alpha(z, u) \) is the nonlinear connection induced on \( E \) by the nonlinear connection \( N_h^k(z, \eta) \) from \( T'M \), then

i) \( dz^k = d^* z^k \); \( \delta^* \eta^k = \rho_h^k \delta^\alpha u^\alpha + Y_h^k Y_a^t H_h^l dz^h \), where \( H_h^l = N_h^l + u^\beta \partial \rho_h^l \partial z^h \).

ii) \( \frac{\delta^*}{\delta z^k} = \frac{\delta}{\delta z^k} + Y_h^k Y_a^t H_h^l \frac{\partial}{\partial \eta^l} ; \quad \frac{\partial}{\partial \eta^l} = \rho_h^\alpha \frac{\partial}{\partial \eta^k} \).

**Proof.** The first point is already proven, thus we only have to prove ii):

\[
\frac{\delta^*}{\delta z^k} = \frac{\partial}{\partial z^k} \frac{\partial^*}{\partial u^\alpha} - N_h^\alpha \frac{\partial}{\partial u^\alpha} = \frac{\partial}{\partial z^k} \frac{\partial}{\partial u^\alpha} + u^\alpha \frac{\partial \rho_h^k}{\partial z^k} \frac{\partial}{\partial \eta^l} = \frac{\partial}{\partial z^k} + N_h^k \frac{\partial}{\partial \eta^l} + u^\alpha \frac{\partial \rho_h^k}{\partial z^k} \frac{\partial}{\partial \eta^l} - \rho_h^\alpha \rho_h^k \left\{ N_h^l + u^\beta \partial \rho_h^l \partial z^h \right\} \frac{\partial}{\partial \eta^l}.
\]

Using \( \rho_h^\alpha \rho_h^k = \delta_h^k - Y_h^k Y_a^t \) yields \( \frac{\delta^*}{\delta z^k} = \frac{\delta}{\delta z^k} + Y_h^k Y_a^t H_h^l \frac{\partial}{\partial \eta^l} \).

\( \square \)
Let us consider on $T'M$ the Chern-Lagrange connection, $N^i_k = g^{ji} \frac{\partial^2 L}{\partial z^k \partial \bar{\eta}^i}$, while on $\rho(E)$ we consider a nonlinear connection of Chern-Lagrange type,

\[ N^\alpha_k = g^{\beta \alpha} \frac{\partial^2 L^*}{\partial^* z^k \partial^* \bar{u}^\beta}. \]  

Then

\[ N^\alpha_k = \rho^i \rho^j g^{ji} \frac{\partial^* \partial L}{\partial \bar{z}^k \partial \bar{\eta}^\alpha} = \rho^i \rho^j g^{ji} \left( \frac{\partial}{\partial \bar{z}^k} + u^\gamma \frac{\partial \rho^\gamma}{\partial \bar{z}^k} \right) \left( \rho^h \frac{\partial L}{\partial \bar{\eta}^h} \right) \]

and since $\rho$ is holomorphic, hence $\frac{\partial}{\partial \bar{z}^k} (\rho^h) = 0$, we get

\[ N^\alpha_k = \rho^i \rho^j g^{ji} \left( \frac{\partial^2 L}{\partial z^k \partial \bar{\eta}^\alpha} + u^\gamma \frac{\partial \rho^\gamma}{\partial z^k} \right) = \rho^i \rho^j g^{ji} \left( \frac{\partial^2 L}{\partial z^k \partial \bar{\eta}^h} + u^\gamma \frac{\partial \rho^\gamma}{\partial z^k} \right). \]

With $\rho^\gamma g^{ji} = \delta^i_j - Y^a_i Y^h_a$, it follows that

\[
N^\alpha_k = \rho^i g^{ki} \frac{\partial^2 L}{\partial z^k \partial \bar{\eta}^\alpha} + \rho^i u^\gamma \frac{\partial \rho^\gamma}{\partial z^k} - \rho^\alpha (Y^a_i Y^h_a g^{ji} \frac{\partial L}{\partial z^k \partial \bar{\eta}^h} + u^\gamma \frac{\partial \rho^\gamma}{\partial z^k \partial \bar{\eta}^h}).
\]

Now, using (3.8), we obtain $N^\alpha_k = \rho^i N^i_k = \rho^i \left( N^i_k + u^\gamma \frac{\partial \rho^\gamma}{\partial z^k} \right)$, which, compared to (3.10), gives the following

**Proposition 3.3.** The nonlinear connection induced on $E$ by the Chern-Lagrange connection given by (3.2) from $T'M$, coincides with (3.11).

**III)** The case when $\text{rank} \rho = n < m$.

In this case, $\rho$ is a submersion and $\rho(E)$ can be identified with $T'M$. We will introduce a Lagrange structure on $T'M$ induced by a Lagrange structure on $E$.

Let $L(z, u)$ be a regular Lagrangian on $E$ with the metric tensor $g_{\alpha \beta}(z, u) = \frac{\partial^2 L}{\partial u^\alpha \partial u^\beta}$, with $\det [g_{\alpha \beta}] \neq 0$, such that $\text{rank} g_{\alpha \beta} = m$. Let $g^{\bar{\alpha} \bar{\beta}}$ be the inverse of the metric tensor, that is, $g^{\bar{\alpha} \bar{\beta}} g_{\alpha \beta} = \delta^\alpha_\gamma$.

On $T'M$ we have the induced frame (2.8) and its coframe (2.9). The Lagrangian $L(z, u)$ will be mapped by $\rho$ in $L^*(z, \eta^k = \rho^k_i u^\alpha)$ and using (2.8) we compute

\[
g^{*}_{\alpha \beta} = \frac{\partial^2 L^*}{\partial u^\alpha \partial u^\beta} = \rho^i_\alpha \rho^j_\beta g_{ij}, \quad \text{where} \quad g_{ij}(z, \eta) = \frac{\partial^2 L}{\partial \eta^i \partial \eta^j}.
\]

We obtain $\text{rank}[g_{ij}] = n$, since $\text{rank} \rho = n$.  

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On $E$, we define the following $n$ vertical, linear independent forms

$$dv^k = \rho^k_\alpha du^\alpha.$$  

They form an $n$-dimensional distribution, denoted by $\rho^1 T^* M$, of the $m$-dimensional distribution $V^* E$, and, according to (2.8), are linked with their image through $\rho_*$ on $T^* M$ by

$$dz^k = dz^k; \quad dv^k = d^* \eta^k - u^\alpha \frac{\partial \rho^k_\alpha}{\partial z^h} dz^h,$$  \hspace{1cm} (3.12)

where $\eta^k = \rho^k_\alpha u^\alpha$.

We now complete $\{dv^k\}_{k=1,n}$ to a cobasis of $V^* E$ with $\{dy^a = Y^a_\alpha du^\alpha\}_{a=1,m-n}$. We obtain the matrix of this frame, $R = [\rho^k_\alpha : Y^a_\alpha]^t$. If $G_E = g_{\alpha \beta} du^\alpha \otimes d\bar{u}^\beta$ is a metric in the vertical bundle, using the isomorphism of the tangent and cotangent spaces, let $G^{-1}_E = g^{\bar{eta} \alpha} \frac{\partial}{\partial u^\alpha} \otimes \frac{\partial}{\partial \bar{u}^\beta}$ be its action on $V^* E$.

Using the same idea as in the second case, we impose that $\{dy^a\}$ are normal forms with respect to $G^{-1}_E$ on $\rho^1 T^* M$ and also that they are orthonormal, that is,

$$g^{\bar{eta} \alpha} Y^a_\alpha \bar{Y}^k_\beta = g^{\bar{eta} \alpha} \rho_\alpha^k Y^\beta_\alpha = 0 \quad \text{and} \quad g^{\bar{eta} \alpha} Y^a_\alpha \bar{Y}^\beta_\alpha = \delta^\beta_\alpha.$$  

We have thus obtained a different coframe from the natural one, $\{dv^k, dy^a\}$ on $V^* E$. Let $R = [\rho^k_\alpha : Y^a_\alpha]$ be the inverse matrix of $R$,

$$\rho^a_\alpha \rho^h_\beta = \delta^h_\beta; \quad \rho^a_\alpha Y^a_\alpha = 0; \quad Y^k_\alpha Y^a_\alpha = 0; \quad \rho^a_\alpha \rho^\beta_\alpha + Y^a_\alpha Y^\beta_\alpha = \delta^\beta_\alpha.$$  

The first identity yields that $\left\{ \frac{\partial}{\partial z^k}, \frac{\partial}{\partial v^k} := \rho^k_\alpha \frac{\partial}{\partial u^\alpha} \right\}$ is the dual frame of $\{dz^k, dv^k\}$ on $\rho^1 T^* M$, since $dz^k \left( \frac{\partial}{\partial z^k} \right) = \delta^k_k$. Indeed, using (3.12), we have

$$dz^k \left( \frac{\partial}{\partial z^h} \right) = 0,$$

since $d^* \eta^k \left( \frac{\partial}{\partial z^h} \right) = u^\alpha \frac{\partial \rho^k_\alpha}{\partial z^h}$. Also, $dv^k \left( \frac{\partial}{\partial v^k} \right) = \rho^k_\alpha \rho^h_\alpha = \delta^k_h$.

From (2.8), we have

$$\rho_* \left( \frac{\partial}{\partial z^k} \right) =: \frac{\partial^*}{\partial z^k} = \frac{\partial^*}{\partial z^k} + u^\alpha \frac{\partial \rho^k_\alpha}{\partial z^h} \frac{\partial}{\partial \eta^h},$$  

$$\rho_* \left( \frac{\partial}{\partial v^k} \right) =: \frac{\partial^*}{\partial v^k} = \rho^k_\alpha \frac{\partial^*}{\partial u^\alpha} = \rho^k_\alpha \rho^h_\alpha \frac{\partial}{\partial \eta^h} = \frac{\partial}{\partial \eta^k}.$$  

Thus,

$$g_{ij}(z, \eta) = \frac{\partial^2 L^{*}}{\partial \eta^i \partial \eta^j} = \rho^i_\alpha \rho^j_\beta \frac{\partial^*}{\partial u^\alpha \partial \bar{u}^\beta} = \frac{\partial^*}{\partial v^i \partial \bar{v}^j} := \left( g_{ij}(z, u) \right)^*$$  \hspace{1cm} (3.13)

where $\eta^k = \rho^k_\alpha u^\alpha$.  

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Let $N^k(\alpha,k)(z,u)$ be a nonlinear connection on $E$ and $N^h(z,\eta)$ a nonlinear connection on $T'M$, with the adapted cobases $\delta u^\alpha = du^\alpha + N^k z^h$ and $\delta \eta^k = d\eta^k + N^h z^h$, respectively. Let $(z^k, \eta^k = \rho^k u^\alpha)$ be the coordinates induced on $T'M$ by $\delta$ and $\delta^* \eta^k = d^* \eta^k + N^k(z, \rho^k u^\alpha) dz^h$.

**Definition 3.2.** The nonlinear connection $N^h(z,\eta)$ is called induced on $\rho(E) \equiv T'M$ if

$$\delta^* \eta^k = \rho^k u^\alpha. \quad (3.14)$$

Using (3.12), we obtain

$$dv^k + u^\alpha \partial \rho^k \alpha \partial z^h + N^k(z, \rho^k u^\alpha) dz^h = \rho^k (du^\alpha + N^h z^h).$$

With $dv^k = \rho^k u^\alpha$, this yields

**Proposition 3.4.** The nonlinear connection $N^h(z,\eta)$ is induced on $\rho(E)$ by the nonlinear connection $N^k(z,u)$ from $E$ if and only if

$$N^h(z,\eta) = \rho^k N^k(z,u) - u^\alpha \partial \rho^k \alpha \partial z^h, \quad (3.15)$$

where $\eta^k = \rho^k u^\alpha$.

We now compute

$$\frac{\delta^*}{\delta z^k} = \partial^* + N^k z^h \frac{\partial}{\partial z^h} + u^\alpha \partial \rho^h \partial \eta^h - N^k \rho^h \partial \partial \eta^h,$$

and replacing $N^k \rho^h$ from (3.15) gives $\delta^* \eta^k = \frac{\delta}{\delta z^k}$ on $T'M$.

Also, using (2.9), $\delta \eta^k = d\eta^k + N^h z^h = d^* \eta^k + \left( \rho^k (N^k(z,u) - u^\alpha \partial \rho^k \partial z^h) \right) dz^h = \rho^k \delta u^\alpha = \delta \eta^k$.

**Proposition 3.5.** The adapted basis and cobasis of the nonlinear connection $N^h(z,\eta)$ induced by $N^k(z,u)$ define the adapted frame and coframe on $T'M$ given by

$$\frac{\delta}{\delta z^k} = \frac{\delta^*}{\delta z^k}; \quad \frac{\partial}{\partial \eta^h} = \rho^k \frac{\partial}{\partial z^h} \frac{\partial^*}{\partial \eta^h} = \frac{\partial^*}{\partial \eta^h} ; \quad \frac{\delta \eta^k}{\delta z^h} = d^* \delta z^h.$$ \(\delta \eta^k = \rho^k \delta u^\alpha = \delta^* \eta^k.

**Proposition 3.6.** The Chern-Lagrange nonlinear connection from $E$ induces the Chern-Lagrange connection on $T'M$.

Proof. It follows in a similar manner an in the second case.

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