An alternative derivation of the Germano identity as the residual of the LES equation

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1. Introduction

The Germano identity [1] and the resulting dynamic procedure [2, 3] to compute subgrid model coefficients have been among the most successful and popular developments in large eddy simulation (LES). The original rationale for the dynamic procedure was that the same subgrid model should be applicable with the same model coefficient at two different coarse-graining levels (or filter levels), which was later interpreted as an argument based on scale-invariance (cf. [4]), a property that is expected of turbulence in the inertial subrange. The rationale based on scale-invariance was first questioned by Jimenez and Moser [5] and later by Pope [6], partly based on the fact that the dynamic procedure works well at low Reynolds numbers (transitional flow, near-wall behavior, etc.) where scale-invariance does not hold. Jimenez and Moser [5] argued that the success of the dynamic procedure is probably due to the balance between the production of Leonard stresses and the dissipation rate resulting from the application of the dynamic procedure. Pope [6], on the other hand, argued that the reason for success is that the dynamic procedure minimizes the sensitivity of the total (resolved plus modeled) Reynolds stresses to the coarse-graining level. This general argument was later used by Meneveau [7] as well. The current consensus understanding of why the dynamic procedure works seems to be a combination of these arguments, the strength of which is that they require no specific assumptions about the characteristics of the flow (e.g., whether it satisfies the scale-similarity hypothesis) and to some degree about its nature (e.g., whether it is turbulent or not).

The objective of this Note is to present an alternative derivation of the Germano identity and its error which provides a subtly different argument for why the dynamic procedure works. While the previous arguments rest on recognizing the importance of the Reynolds stress or the dissipation rate, the present derivation instead follows the path of deriving the residual (in the sense of numerical analysis) of the LES equation. The residual is of central importance in the field of numerical analysis since it is the source of errors; therefore, the present derivation shows the connection between the error in the Germano identity and the source of error in LES based on the governing equation alone, with no physical insight required. The present derivation does not contradict the prior arguments by Jimenez and Moser [5] or by Pope [6] and Meneveau [7] in any way; rather, it is offered here as a complement to the prior explanations.

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2. LES equations

The Navier-Stokes equation for an incompressible and constant viscosity fluid is

\[ \frac{\partial U_i}{\partial t} + \frac{\partial U_i U_j}{\partial x_j} + \frac{1}{\rho} \frac{\partial P}{\partial x_i} - \nu \frac{\partial^2 U_i}{\partial x_j \partial x_j} = 0, \]

or in short notation \( \mathcal{N}(U_i) = 0 \), where \( \rho \) and \( \nu \) are the density and viscosity (both assumed constant here) and \( U_i \) and \( P \) are the exact velocity and pressure fields (corresponding to a perfect direct numerical simulation, DNS). When coarse-grained or implicitly filtered to a resolution length scale (or filter width) \( \Delta \) the equation becomes (assuming that filtering and differentiation commute)

\[ \frac{\partial \hat{U}_i}{\partial t} + \frac{\partial \hat{U}_i \hat{U}_j}{\partial x_j} + \frac{1}{\rho} \frac{\partial \hat{P}}{\partial x_i} - \nu \frac{\partial^2 \hat{U}_i}{\partial x_j \partial x_j} + \frac{\partial \tau_{ij}^{\text{exact}}}{\partial x_j} = 0, \tag{1} \]

or \( \mathcal{N}_\Delta^{\text{exact}}(U_i) = 0 \) where \( \hat{U}_i \) and \( \hat{P} \) are coarse-grained representations of the exact fields and \( \tau_{ij}^{\text{exact}} = U_i U_j - \hat{U}_i \hat{U}_j \) is the exact subgrid scale (SGS) stress tensor. An interesting property of \( \tau_{ij}^{\text{exact}} \) is that it satisfies the Germano identity \( \hat{\tau}_{ij}^{\text{exact}} - \tau_{ij}^{\text{exact}} = \hat{\tau}_{ij} - \hat{\tau}_{ij} \), \( \hat{\tau}_{ij} \) and \( \hat{\tau}_{ij} \) is the result of consecutive application of filters \( \Delta \) and \( \hat{\Delta} \), and satisfying this identity provides a “self-consistency condition” \( \hat{\tau}_{ij} \) that also applies to Eqn. (1) at filter levels \( \Delta \) and \( \hat{\Delta} \).

Approximating the exact SGS stress tensor using a model leads to the LES equation in differential form (i.e., without numerical errors), where we intentionally exclude the numerical errors in order to isolate the effect of modeling errors in the equation, and to be faithful to many of the developments in the LES literature. The LES equations at two different filter levels \( \Delta \) (say, original) and \( \hat{\Delta} \) (test filtered) are

\[ \frac{\partial \overline{u}_i}{\partial t} + \frac{\partial \overline{u}_i \overline{u}_j}{\partial x_j} + \frac{1}{\rho} \frac{\partial \overline{P}}{\partial x_i} - \nu \frac{\partial^2 \overline{u}_i}{\partial x_j \partial x_j} + \frac{\partial \tau_{ij}^{\text{model}}}{\partial x_j} = 0, \tag{3} \]

\[ \frac{\partial \hat{\overline{u}}_i}{\partial t} + \frac{\partial \hat{\overline{u}}_i \hat{\overline{u}}_j}{\partial x_j} + \frac{1}{\rho} \frac{\partial \hat{\overline{P}}}{\partial x_i} - \nu \frac{\partial^2 \hat{\overline{u}}_i}{\partial x_j \partial x_j} + \frac{\partial \tau_{ij}^{\text{model}}}{\partial x_j} = 0, \tag{4} \]

where \((\overline{u}_i, \overline{P})\) and \((\hat{\overline{u}}_i, \hat{\overline{P}})\) are the solutions at the respective filter levels. These equations are referred to as \( \mathcal{N}^{\text{model}}(\overline{u}_i) = 0 \) and \( \mathcal{N}^{\text{model}}(\hat{\overline{u}}_i) = 0 \), respectively.

The principle of the dynamic procedure \( \mathcal{L} \) is that any approximate model should satisfy, as well as possible, the Germano identity. It therefore aims to minimize the error

\[ \mathcal{G}_{ij} = \mathcal{L}_{ij} - \left[ \tau_{ij}^{\text{model}}(\overline{u}_k) - \tau_{ij}^{\text{model}}(\hat{\overline{u}}_k) \right], \tag{5} \]

in a least squares sense \( \mathcal{M}_{ij} \). Here, \( \mathcal{G}_{ij} \) is the Germano identity error (GIE), \( \mathcal{L}_{ij} \) is the Leonard or resolved stress, and \( \mathcal{M}_{ij} \) is the modeled stress [cf. 8].
3. Residual due to modeling and its connection to the modeling error

The residual of an inexact equation $N_{\text{approx}}(u_{\text{approx}}) = 0$ is the misfit when evaluating the inexact equation for the exact solution, i.e., $N_{\text{approx}}(u_{\text{exact}})$ in this example. The importance of the residual is made clear by the Taylor expansion

$$N_{\text{approx}}(u_{\text{exact}}) \approx N_{\text{approx}}(u_{\text{approx}}) + \frac{\partial N_{\text{approx}}}{\partial u} (u_{\text{approx}} - u_{\text{exact}}),$$

which shows how the residual is the source of error in the solution for linearized dynamics. We therefore want to find the residual of the LES equation (3). This residual is $N_{\text{model}}$ for filter level $\Delta$ and $N_{\text{model}}^{\hat{\Delta}}$ for filter level $\hat{\Delta}$, where we must use the coarse-grained representations of the exact fields (clearly the full field $U_i$ is consistent only with the DNS equation, not the coarse-grained ones containing $\tau_{ij}$). Therefore, we can write

$$R_{\Delta} \equiv N_{\text{model}}(\hat{U}_i) = \partial \hat{U}_i / \partial t + \partial \hat{U}_i \hat{u}_j / \partial x_j + \frac{1}{\rho} \partial \hat{p} / \partial x_i - \nu \partial^2 \hat{u}_i / \partial x_j \partial x_j + \frac{\partial \tau_{\text{model}}}{\partial \hat{u}_j / \partial x_j} \hat{\Delta} \hat{U}_k \partial x_j.$$

(6)

where Eqn. (1) is used to replace the terms by the divergence of $\tau_{ij}^{\text{exact}}$. Similarly, we have

$$R_{\hat{\Delta}} \equiv N_{\text{model}}^{\hat{\Delta}}(\hat{U}_i) = \partial \hat{U}_i / \partial x_j \left[ \tau_{\text{model}}^{\hat{\Delta}}(\hat{U}_k) - \tau_{ij}^{\text{exact}} \right].$$

The exact solution is unknown and must therefore be approximated. In the area of numerical analysis, the residual is often approximated by evaluating the numerical operators on a finer representation of the solution [cf. 9]. This approach, however, does not work for LES equations with modeling of the discarded scales, because obtaining an exact (or sufficiently more accurate) LES equation requires a more accurate $\tau_{ij}$, which in turn requires the estimation of $\tau_{ij}$ from only the LES solution $\hat{U}_i$, which is impossible due to the limited spectral content of the filtered solution and the projection errors [cf. 10]. The solution is to use the $N_{\text{model}}$ operator (i.e., the direct approach, as in Eqn. 6, to avoid estimation of $\tau_{ij}$) and to compute the residual at a coarser filter level, specifically the test filter level $\hat{\Delta}$, such that the test filtered solution $\hat{U}_i$ can be used in place of $\hat{U}_i$ to compute the approximate residual. We should note that approximating $\hat{U}_i$ by $\hat{U}_i$ is only done for the purpose of estimating the residual (very similar to the use of the numerical solution for estimating the truncation errors in numerical analysis), and is assumed to be a much weaker approximation than saying that $\hat{U}_i$ is an accurate representation of $\hat{U}_i$ in general.

With this approximation, the residual at the test-filter level $\hat{\Delta}$ is

$$R_{\hat{\Delta}} \approx \frac{\partial \hat{U}_i}{\partial t} + \frac{\partial \hat{U}_i \hat{u}_j}{\partial x_j} + \frac{1}{\rho} \partial \hat{p} / \partial x_i - \nu \frac{\partial^2 \hat{U}_i}{\partial x_j \partial x_j} + \frac{\partial \tau_{\text{model}}}{\partial \hat{U}_j / \partial x_j} \hat{\Delta} \hat{U}_k \partial x_j.$$

(7)
This equation can be directly computed to estimate $R_\Delta^*$; however, quite interestingly, it can be simplified by test-filtering the LES equation \(3\) and subtracting it from Eqn. \(7\), which yields (assuming that filtering and differentiation commute)

\[
R_\Delta^* \approx \frac{\partial}{\partial x_j} \left[ \hat{\tau}_{ij,\Delta} - \hat{\tau}_{ij,\Delta} \right] = -\frac{\partial}{\partial x_j} \left[ L_{ij} - M_{ij} \right],
\]

where $L_{ij}$ and $M_{ij}$ are the familiar Leonard (resolved) and modeled stress terms from Eqn. \(5\). In other words, we have

\[
-\frac{\partial}{\partial x_j} \left[ L_{ij} - M_{ij} \right] \approx R_\Delta^* = \frac{\partial}{\partial x_j} \left[ \hat{\tau}_{model,ij} - \hat{\tau}_{exact,ij} \right].
\]

Therefore, the residual $R_\Delta^*$ of the LES equation at the test-filter level is approximately equal to the divergence of the error in the Germano identity $L_{ij} - M_{ij}$, and the tensor $L_{ij} - M_{ij}$ directly estimates the modeling error $\tau_{model,ij} - \tau_{exact,ij}$. Minimizing this Germano identity error (GIE) thus directly minimizes the modeling errors and the residual that is the source of errors in the (test-filtered) LES equation.

4. Concluding remarks

This Note illustrates the close connection between the residual of the test-filtered LES evolution equation and the error in the Germano identity (the GIE, generally written as $L_{ij} - M_{ij}$ in most texts, \[cf. 8\]). Equation \(9\) also shows that the GIE approximates the difference between the modeled SGS stress tensor and the exact one given the exact solution. This can explain why the dynamic procedure is successful at distinguishing between laminar, transitional, and turbulent flows, and why it is capable of recovering the correct near-wall behavior of the eddy viscosity at the vicinity of solid walls: the exact SGS stress tensor $\tau_{exact,ij}$ has all these properties built in \[cf. 11, 12\], and by minimizing the difference between $\tau_{exact,ij}$ and $\tau_{model,ij}$, the SGS model should inherit (to the largest degree possible given the chosen model form) those characteristics.

The main purpose of this Note is to complement prior interpretations of why the dynamic procedure works, and to serve as a connection between the fields of LES and numerical analysis. There is a great body of work in the numerical analysis literature that utilizes the residual to, for example, produce error estimates and to optimally adapt the computational grid. The connection between the GIE and the residual suggests that the dynamic procedure in a sense uses the same residual to improve the solution by optimally choosing the model parameter(s). The implication of such a connection is that many of the more advanced techniques that are currently used in residual minimization (weighting the residual by the adjoints, for instance) could (and maybe should?) be used in the dynamic procedure as well. Furthermore, the present derivation implies that one should be minimizing the volume integral of the residual (i.e., the GIE) as the more meaningful and more optimal approach of reducing the errors (optimally the GIE should be weighted by some adjoint field), as done by Ghosal \textit{et al.} \[13\].
and shows clearly that we should indeed be minimizing the divergence of the GIE rather than error itself, as done by Morinishi and Vasilyev.\footnote{14}

Finally, the implications of the findings of this Note extend to uncertainty quantification (UQ) and output-based grid/filter adaptation in LES, both of which require an estimate of the residual in the equation. In that sense, this Note also complements our prior work\footnote{15} on grid adaptation for LES, that used the same quantity (the divergence of the GIE) as its error indicator, but motivated its use from a different point-of-view of solution sensitivity.

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