Stability of the Poincaré Bundle

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30 May 1995

Abstract

Let $C$ be a nonsingular projective curve of genus $g \geq 2$ defined over the complex numbers, and let $M_\xi$ denote the moduli space of stable bundles of rank $n$ and determinant $\xi$ on $C$, where $\xi$ is a line bundle of degree $d$ on $C$ and $n$ and $d$ are coprime. It is shown that the universal bundle $U_\xi$ on $C \times M_\xi$ is stable with respect to any polarisation on $C \times M_\xi$. It is shown further that the connected component of the moduli space of $U_\xi$ containing $U_\xi$ is isomorphic to the Jacobian of $C$. 
Introduction

In the study of moduli spaces of stable bundles on an algebraic curve $C$, various bundles on the moduli space or on the product of the moduli space with $C$ arise in a natural way. An interesting question to ask about any such bundle is whether it is itself stable in some sense.

More precisely, let $C$ be a nonsingular projective curve of genus $g \geq 2$ defined over the complex numbers, and let $M = M_{n,d}$ denote the moduli space of stable bundles of rank $n$ and degree $d$ on $C$, where $n$ and $d$ are coprime. For any line bundle $\xi$ of deg $d$ on $C$, let $M_\xi$ denote the subvariety of $M$ corresponding to bundles with determinant $\xi$. There exists on $C \times M$ a universal (or Poincaré) bundle $U$ such that $U\big|_{C \times \{m\}}$ is the bundle on $C$ corresponding to $m$. Moreover the bundle $U$ is determined up to tensoring with a line bundle lifted from $M$.

The direct image of $U$ on $M$ is called the Picard sheaf of $U$; for $d > n(2g-2)$, this sheaf is a bundle. It was shown recently by Y. Li [Li] that, if $d > 2gn$, this bundle is stable with respect to the ample line bundle corresponding to the generalised theta divisor (cf. [DN]). (Recall here that, if $H$ is an ample divisor on a projective variety $X$, the degree $\deg E$ of a torsion-free sheaf $E$ on $X$ is defined to be the intersection number $[c_1(E) \cdot H^{\dim X-1}]$. $E$ is said to be stable with respect to $H$ (or $H$-stable) if, for every proper subsheaf $F$ of $E$,

$$\frac{\deg F}{\rank F} < \frac{\deg E}{\rank E}.$$  

The definition depends only on the polarisation defined by $H$.) This extends previously known results for the case $n = 1$ ([U, Ke1, EL]). We remark that the question of stability of the Picard sheaf of $U_\xi$ is still open (cf. [BV]).

In this paper, we investigate the stability of the Poincaré bundle $U$ and its restriction $U_\xi$ to $M_\xi$ using methods similar to those of [Li].
Our main results are

**Theorem 1.5.** $\mathcal{U}_\xi$ is stable with respect to any polarisation on $C \times M_\xi$.

**Theorem 1.6.** $\mathcal{U}$ is stable with respect to any polarisation of the form

$$a\alpha + b\Theta, \quad a, b > 0$$

where $\alpha$ is ample on $C$ and $\Theta$ is the generalised theta divisor on $M$. (Note that $C$ has a unique polarisation whereas $M$ does not.)

These results are proved in §1. In §2 we discuss some properties of the bundles $\text{End}\mathcal{U}_\xi$ and $\text{ad}\mathcal{U}_\xi$. It is reasonable to conjecture that $\text{ad}\mathcal{U}_\xi$ is also stable but we are able to prove this only in the case $n = 2$.

Finally, in §3 we consider the deformation theory of $\mathcal{U}_\xi$ using the results in §2. The main result we prove is that the only deformations of $\mathcal{U}_\xi$ are those of the form $\mathcal{U}_\xi \otimes p_C^* L$, where $L$ is a line bundle of degree 0 on $C$. More precisely, let $H$ be any ample divisor on $C \times M_\xi$ and let $M(\mathcal{U}_\xi)$ denote the moduli space of $H$-stable bundles with the same numerical invariants as $\mathcal{U}_\xi$ on $C \times M_\xi$; then

**Theorem 3.1.** The connected component $M(\mathcal{U}_\xi)_0$ of $M(\mathcal{U}_\xi)$ containing $\{\mathcal{U}_\xi\}$ is isomorphic to the Jacobian $J(C)$, the isomorphism $J(C) \to M(\mathcal{U}_\xi)_0$ being given by

$$L \mapsto \mathcal{U}_\xi \otimes p_C^* L,$$

where $p_C : C \times M_\xi \to C$ is the projection.

**Acknowledgement.** Most of the work for this paper was carried out during a visit by the first two authors to Liverpool. They wish to acknowledge the generous hospitality of the University of Liverpool.
§1 Stability of $\mathcal{U}$

We begin with some lemmas which are probably well known, but which we could not find in the literature.

**Lemma 1.1.** Let $X$ and $Y$ be smooth projective varieties of the same dimension $m$. Let $f : X \dashrightarrow Y$ be a dominant rational map defined outside a subset $Z \subset X$ with $\operatorname{codim}_X Z \geq 2$. Suppose that $D_X$ and $D_Y$ are ample divisors on $X$ and $Y$ such that $f^* D_Y |_{X-Z} \simeq D_X |_{X-Z}$.

Let $E$ be a vector bundle on $Y$ such that $f^* E$ extends to a vector bundle $F$ on $X$. If $F$ is $D_X$-semi-stable (resp. stable) on $X$, then $E$ is $D_Y$-semi-stable (resp. stable) on $Y$.

**Proof.** The proof is fairly straightforward (cf. [Li, pp.548, 549]). Let rank $E = n$. Suppose that $V$ is a torsion-free quotient of $E$,

$$E \longrightarrow V \longrightarrow 0$$

with rank $V = r < n$. When $F$ is $D_X$-semi-stable, we need to check the inequality:

$$\frac{\deg V}{r} \geq \frac{\deg E}{n}. \quad (2)$$

From (1), we have

$$f^* E \longrightarrow f^* V \longrightarrow 0.$$ 

If $G = f^* V/(\text{torsion})$, then

$$\deg f^* V \geq \deg G. \quad (3)$$

(Note that $\deg f^* V = [c_1(f^* V) \cdot D_X^{m-1}]$, where $c_1(f^* V)$ makes sense since $f^* V$ is defined outside a subset of codimension 2.)

Moreover we have

$$0 \longrightarrow G^* \longrightarrow f^*(E)^*$$
on X-Z. Let \( \tilde{G} \) be a subsheaf of \( F^* \) extending \( G^* \). Since \( \text{codim}_X Z \geq 2 \), we have \( \deg \tilde{G} = \deg G^* = -\deg G \).

By semi-stability of \( F^* \),

\[
\frac{\deg \tilde{G}}{r} \leq \frac{\deg F^*}{n}. \tag{4}
\]

Note that

\[
\deg E = \left[ c_1(E).D_Y^{m-1} \right] = \left[ c_1(f^*E).D_X^{m-1} \right](\deg f)^{-1} = (\deg f^*E)(\deg f)^{-1}.
\]

The same applies to \( V \), so we have

\[
n \deg V - r \deg E = (n \deg f^*V - r \deg f^*E)(\deg f)^{-1} \\
\geq (n \deg G - r \deg F)(\deg f)^{-1} \quad \text{by (3)} \\
= (-n \deg \tilde{G} + r \deg F^*)(\deg f)^{-1} \\
\geq 0 \quad \text{by (4)}.
\]

This proves (2).

The proof in the stable case is similar.

**Lemma 1.2.** Let \( X^m \) and \( Y^n \) be smooth projective varieties, and \( D_X \) and \( D_Y \) ample divisors on \( X \) and \( Y \). Let \( \eta = aD_X + bD_Y \), \( a, b > 0 \). Suppose that \( E \) is a vector bundle on \( X \times Y \), such that for generic \( x \in X \), \( y \in Y \), \( E_x \simeq E|_{\{x\} \times Y} \) and \( E_y \simeq E|_{X \times \{y\}} \) are respectively \( D_Y \)-semi-stable and \( D_X \)-semi-stable. Then \( E \) is \( \eta \)-semi-stable.

Further, if either \( E_x \) or \( E_y \) is stable, then \( E \) is stable.

**Proof.** Let \( F \subset E \) be a subsheaf. Since \( \text{Sing} F \) has codimension \( \geq 2 \), we can choose \( x \in X \) and \( y \in Y \) such that \( \text{Sing} F_x \) and \( \text{Sing} F_y \) also have codimension
\[ \geq 2. \] Thus any torsion in \( F_x \) or \( F_y \) is supported in codimension \( \geq 2 \) and does not contribute to \( c_1(F_x) \) or \( c_1(F_y) \). Let \( \text{rank}(E) = n \), \( \text{rank}(F) = r \). Then we need to show that

\[
\frac{\deg F}{r} \leq \frac{\deg E}{n} \tag{5}
\]

assuming \( E_x \) and \( E_y \) are semi-stable. Now,

\[
\deg E = c_1(E) \cdot [aD_X + bD_Y]^{m+n-1}
\]

\[
= c_1(E) [\lambda D_X^m \cdot D_Y^{n-1} + \mu D_X^{m-1} \cdot D_Y^n] \quad \text{for some} \quad \lambda, \mu > 0
\]

\[
= [c_1(E_x) + c_{1,1}(E) + c_1(E_y)] \cdot [\lambda D_X^m \cdot D_Y^{n-1} + \mu D_X^{m-1} \cdot D_Y^n]
\]

\[
= [c_1(E_x) \cdot D_Y^{n-1}] \cdot \lambda D_X^m + [c_1(E_y) \cdot D_X^{m-1}] \cdot \mu D_Y^n. \tag{6}
\]

We have a similar expression for \( \deg F \),

\[
\deg F = [c_1(F_x) \cdot D_Y^{n-1}] \cdot \lambda D_X^m + [c_1(F_y) \cdot D_X^{m-1}] \cdot \mu D_Y^n. \tag{7}
\]

(5) follows trivially by comparing the terms in (6) and (7) and using semi-stability of \( E_x \) and \( E_y \). The rest of the lemma follows in a similar fashion.

Before stating the next lemma, we recall very briefly some facts on spectral curves. For details see [BNR] and [Li].

Let \( K = K_C \) be the canonical bundle and let \( W = \oplus_{i=1}^n H^0(C, K^i) \). Let \( s = (s_1, \cdots, s_n) \in W \), and let \( C_s \) be the associated spectral curve. Then we have a morphism

\[
\pi : C_s \longrightarrow C
\]

of degree \( n \), such that for \( x \in C \), the fibre \( \pi^{-1}(x) \) is given by points \( y \in K_x \) which are zeros of the polynomial

\[
f(y) = y^n + s_1(x) \cdot y^{n-1} + \cdots + s_n(x).
\]

The condition that \( x \) be unramified is that the resultant \( R(f, f') \) of \( f \) and its derivative \( f' \) be non-zero at the point \( (s_1(x), \cdots, s_n(x)) \). Note that \( R(f, f') \) is a polynomial in the \( s_i(x), i = 1, \cdots, n \).
Lemma 1.3. Given $x \in C$, there exists a smooth spectral curve $C_s$ such that the covering map $\pi : C_s \rightarrow C$ is unramified at $x$.

Proof. Note first that, if $x \in C$, there exists $s = (s_1, \cdots, s_n) \in W$ such that

$$R(f, f')(s_1(x), \cdots, s_n(x)) \neq 0.$$ 

Indeed, since $|K^i|$ has no base points, given any $(\alpha_1, \cdots, \alpha_n) \in \bigoplus_{i=1}^n K^i_x$, there exist $s_i \in H^0(C, K^i)$ such that $s_i(x) = \alpha_i$, $i = 1, \cdots, n$.

Observe that this is clearly an open condition on $W$.

Further, the subset $\{s \in W \mid C_s \text{ is smooth} \}$ is a non-empty open subset of $W$ [BNR, Remark 3.5] and the lemma follows.

Let $M_\xi$ and $U_\xi$ be as in the introduction, and let $\Theta_\xi$ denote the restriction of the generalized theta divisor to $M_\xi$.

Proposition 1.4. Let $U_\xi$ be the Poincaré bundle on $C \times M_\xi$ and $x \in C$. Then the bundle

$$U_{\xi,x} \cong U_\xi|_{\{x\} \times M_\xi}$$

is $\Theta_\xi$-semi-stable on $M_\xi$.

Proof. For the point $x \in C$ above, choose a spectral curve $C_s$ by Lemma 1.3, so that

$$\pi : C_s \rightarrow C$$

is unramified at $x$. Let $\pi^{-1}(x) = \{y_1, \cdots, y_n\}$, $y_i$ being distinct points in $C_s$.

Let $J^\delta(C_s)$ denote the variety of line bundles of degree

$$\delta = d - \deg \pi_*(\mathcal{O}_{C_s})$$

on $C_s$, and let $P_s$ denote the subvariety of $J^\delta(C_s)$ consisting of those line bundles $L$ for which the vector bundle $\pi_* L$ has determinant $\xi$. ($P_s$ is a
translate of the Prym variety of $\pi$.) Let $\mathcal{L}$ denote the restriction of the Poincaré bundle on $C_s \times J^d(C_s)$ to $C_s \times P_s$. Then [BNR, proof of Proposition 5.7], we have a dominant rational map defined on an open subset $T_s$ of $P_s$ such that $\text{codim}(P_s - T_s) \geq 2$, and

$$\phi : T_s \longrightarrow M_\xi$$

is generically finite. The morphism $\phi$ on $T_s$ is defined by the family $(\pi \times 1)_* \mathcal{L}$ on $C_s \times T_s$; so, by the universal property of $\mathcal{U}_\xi$, we have

$$(\pi \times 1)_* \mathcal{L} \simeq (1 \times \phi)^* \mathcal{U}_\xi \otimes p_T^* L_0$$

for some line bundle $L_0$ on $T_s$. (Here $p_T : C_s \times T_s \longrightarrow T_s$ is the projection.) By (8) we have

$$\phi^* \mathcal{U}_{\xi,x} \simeq [(\pi \times 1)_* \mathcal{L}]_x \otimes L_0^{-1} \text{ on } T_s.$$  

But $[(\pi \times 1)_* \mathcal{L}]_x \simeq \bigoplus_{i=1}^n \mathcal{L}_{y_i}$ on $P_s$. Hence

$$\phi^* \mathcal{U}_{\xi,x} \simeq \bigoplus_{i=1}^n (\mathcal{L}_{y_i} \otimes L_0^{-1}) \text{ on } T_s.$$  

We observe that the $\mathcal{L}_{y_i} \otimes L_0^{-1}$ are the restrictions to $T_s$ of algebraically equivalent line bundles on $P_s$. Further one knows [Li, Theorem 4.3] that $\phi^* \Theta_\xi$ is a multiple of the restriction of the usual theta divisor on $J(C_s)$ to $T_s$.

Now we are in the setting of Lemma 1.1 and we conclude that $\mathcal{U}_{\xi,x}$ is semi-stable with respect to $\Theta_\xi$ on $M_\xi$.

**Theorem 1.5.** The Poincaré bundle $\mathcal{U}_\xi$ on $C \times M_\xi$ is stable with respect to any polarisation.

**Proof.** Since $\text{Pic } M_\xi = \mathbb{Z}$, $\text{Pic } (C \times M_\xi) = \text{Pic } C \oplus \text{Pic } M_\xi$. Thus, any polarisation $\eta$ on $C \times M_\xi$ can be expressed in the form

$$\eta = a\alpha + b\Theta_\xi, \quad a, b > 0.$$
for some ample divisor $\alpha$ on $C$.

By Proposition 1.4, $\mathcal{U}_{\xi,x}$ is semi-stable with respect to $\Theta_{\xi}$ for all $x \in C$ and by definition $\mathcal{U}_{\xi}|_{C \times \{m\}}$ is stable with respect to any polarisation on $C$. Hence by Lemma 1.2, $\mathcal{U}_{\xi}$ is stable with respect to $\eta$ on $C \times M_{\xi}$.

Note that Proposition 1.4 remains true if we replace $M_{\xi}$, $\mathcal{U}_{\xi}$ and $\Theta_{\xi}$ by $M$, $\mathcal{U}$ and $\Theta$. (The key point is that [Li, Theorem 4.3] is valid in this context).

We deduce at once

**Theorem 1.6.** $\mathcal{U}$ is stable with respect to any polarisation of the form

$$a\alpha + b\Theta, \quad a, b > 0,$$

where $\alpha$ is ample on $C$ and $\Theta$ is the generalized theta divisor on $M$.

**Remark 1.7.** Since $C \times M$ is a Kähler manifold, then by a theorem of Donaldson-Uhlenbeck-Yau, $\mathcal{U}$ admits an Hermitian-Einstein metric. One can expect that the restriction of this metric to each factor is precisely the metric on the factor. It would be interesting to know an explicit description of the metric on $\mathcal{U}$. Note that [Ke2] contains such a description for the Picard sheaf in the case $g = 1$, $n = 1$.

§2 Some properties of $\text{End} \mathcal{U}_{\xi}$

Our first object in this section is to calculate the dimension of some of the cohomology spaces of $\text{End} \mathcal{U}_{\xi}$. We denote by $p : C \times M_{\xi} \rightarrow M_{\xi}$ and $p_C : C \times M_{\xi} \rightarrow C$ the projections.

**Proposition 2.1.** Let $h^i(\text{End} \mathcal{U}_{\xi}) = \dim H^i(\text{End} \mathcal{U}_{\xi})$. Then,

$$h^0(\text{End} \mathcal{U}_{\xi}) = 1, \quad h^1(\text{End} \mathcal{U}_{\xi}) = g, \quad h^2(\text{End} \mathcal{U}_{\xi}) = 3g - 3.$$
Proof. We can write $\text{End} U_\xi \cong \mathcal{O} \oplus \text{ad} U_\xi$; hence

$$H^i(C \times M_\xi, \text{End} U_\xi) = H^i(C \times M_\xi, \mathcal{O}) \oplus H^i(C \times M_\xi, \text{ad} U_\xi).$$

Since $U_\xi \big|_{C \times \{m\}}$ is always stable, we have $R^0_p(\text{ad} U_\xi) = 0$. So by the Leray spectral sequence and the fact that $R^1_p(\text{ad} U_\xi)$ is the tangent bundle $TM_\xi$ of $M_\xi$, we have

$$H^i(C \times M_\xi, \text{ad} U_\xi) \cong H^{i-1}(M_\xi, R^1_p(\text{ad} U_\xi))$$

$$\cong H^{i-1}(M_\xi, TM_\xi)$$

By [NR, Theorem 1], this space is 0 if $i \neq 2$, and has dimension $3g - 3$ if $i = 2$.

On the other hand, since $M_\xi$ is unirational, it follows from the Künneth formula that

$$H^i(C \times M_\xi, \mathcal{O}) = H^i(C, \mathcal{O}_C).$$

This space has dimension 1 if $i = 0$, $g$ if $i = 1$ and 0 otherwise. The proposition follows.

Remark 2.2. In fact the proof shows that

$$h^i(\text{End} U_\xi) = 0 \text{ if } i > 2.$$ 

Lemma 2.3. Let $L \in J(C)$ and suppose that $E \cong E \otimes L$ for all $E \in M_\xi$. Then $U_\xi \cong U_\xi \otimes p_C^*L$.

Proof. Since $E \cong E \otimes L$ and $U_\xi \otimes p_C^*L$ is a family of stable bundles, there is a line bundle $L_1$ over $M_\xi$ such that

$$U_\xi \cong U_\xi \otimes p_C^*L \otimes p^*L_1.$$
Fix \( x \in C \), then \( \mathcal{U}_{\xi,x} \cong \mathcal{U}_{\xi,x} \otimes L_1 \) over \( M_\xi \). Hence \( c_1(\mathcal{U}_{\xi,x}) = c_1(\mathcal{U}_{\xi,x}) + nc_1(L_1) \) so \( nc_1(L_1) = 0 \). But \( \text{Pic} \mathcal{M}_\xi \cong \mathbb{Z} \) (see [DN]); so \( c_1(L_1) = 0 \), which implies that \( L_1 \) is the trivial bundle.

The next lemma will also be required in §3.

**Lemma 2.4.** If \( \mathcal{U}_\xi \cong \mathcal{U}_\xi \otimes p^*_C L \) then \( L \cong \mathcal{O}_C \).

**Proof.** If \( \mathcal{U}_\xi \cong \mathcal{U}_\xi \otimes p^*_C L \) then

\[
\mathcal{O} \oplus \text{ad} \mathcal{U}_\xi \cong \text{End} \mathcal{U}_\xi \\
\cong \text{End} \mathcal{U}_\xi \otimes p^*_C L \\
\cong p^*_C L \oplus \text{ad} \mathcal{U}_\xi \otimes p^*_C L.
\]

Hence \( H^0(C \times M_\xi, p^*_C L) \) and \( H^0(C \times M_\xi, \text{ad} \mathcal{U}_\xi \otimes p^*_C L) \) cannot both be zero. Suppose there is a non-zero section \( \phi : \mathcal{O} \longrightarrow \text{ad} \mathcal{U}_\xi \otimes p^*_C L \). For some \( x \in C \), the restriction of \( \phi \) to \( \{x\} \times M_\xi \) will define a non-zero section of \( \text{ad} \mathcal{U}_{\xi,x} \), which is a contradiction since \( H^0(M_\xi, \text{ad} \mathcal{U}_{\xi,x}) = 0 \) (see [NR, Theorem 2]). Hence \( H^0(C \times M_\xi, \text{ad} \mathcal{U}_\xi \otimes p^*_C L) = 0 \).

Therefore \( H^0(C \times M_\xi, p^*_C L) \neq 0 \). Since \( \deg L = 0 \), this implies \( L \cong \mathcal{O}_C \).

**Remark 2.5.** The proof of Lemma 2.4 fails when \( g = 1 \) since [NR, Theorem 2] is not then valid. In fact, Lemma 2.4 and the remaining results of this section are false for \( g = 1 \).

We show next that a general stable bundle \( E \) is not isomorphic to \( E \otimes L \) unless \( L \cong \mathcal{O}_C \).

**Proposition 2.6.** There exists a proper closed subvariety \( S \) of \( M_\xi \) such that, if \( E \not\in S \), then

\[
E \cong E \otimes L \implies L \cong \mathcal{O}_C.
\]
Proof. For any \( L \), the subset \( S_L = \{ E \in M_\xi | E \cong E \otimes L \} \) is a closed subvariety of \( M_\xi \). If \( L \not\cong O_C \), then, by Lemmas 2.3 and 2.4, \( S_L \) is a proper subvariety. On the other hand, \( S_L \) can only be non-empty if \( L^n \cong O_C \); so only finitely many of the \( S_L \) are non-empty. Since \( M_\xi \) is irreducible, the union \( S = \bigcup \{ S_L | L \not\cong O_C \} \) is a proper subvariety of \( M_\xi \) as required.

Remark 2.7. It follows at once from Proposition 2.6 that the action of \( J(C) \) on \( M \) defined by \( E \mapsto E \otimes L \) is faithful. Another proof of this fact has been given in [Li, Theorem 1.2 and Proposition 1.6]. As the following proposition shows, our set \( S \) is analogous to the set \( S \) of [Li, Theorem 1.2].

Proposition 2.8. Let \( S \) be as above and \( E \in M_\xi \). Then \( E \in S \) if and only if \( \text{ad} \, E \) has a line sub-bundle of degree zero.

Proof. The trivial bundle cannot be a subbundle of \( \text{ad} \, E \).

If \( L \in J(C) \) is a subbundle of \( \text{ad} \, E \), so is it of \( \text{End} \, E \), therefore

\[
H^0(C, \text{End} \, E \otimes L^*) \neq 0.
\]

Hence, there is a non-zero map \( \phi : E \otimes L \longrightarrow E \), which is an isomorphism since \( E \otimes L \) and \( E \) are stable bundles of the same slope. Hence \( E \in S \).

Conversely, suppose \( E \cong E \otimes L \) with \( L \not\cong O_C \). The isomorphism \( O_C \oplus \text{ad} \, E \cong L \oplus \text{ad} \, E \otimes L \) implies that \( \text{ad} \, E \otimes L \) has a section, i.e. there is a non-zero map \( \phi : L^* \longrightarrow \text{ad} \, E \). Since \( L^* \) and \( \text{ad} \, E \) have the same slope and \( \text{ad} \, E \) is semi-stable (being a subbundle of a semi-stable bundle \( \text{End} \, E \) with the same slope), \( \phi \) is an inclusion.

Corollary 2.9. If \( E \) is a general stable bundle of rank 2 and determinant \( \xi \), then \( \text{ad} \, E \) is stable.
Proof. Note that \( \text{ad} \ E \) has rank 3 and degree 0 and is semi-stable. By the Proposition, \( \text{ad} \ E \) has no line subbundle of degree 0. On the other hand \( \text{ad} \ E \) is self-dual, so it cannot have a quotient line bundle of degree 0.

Remark 2.10. It would be interesting to know if \( \text{ad} \ E \) is stable for a general stable bundle \( E \) of rank greater than 2. It is certainly true that \( \text{ad} \ E \) is semistable and also that it is stable as an orthogonal bundle [R].

Theorem 2.11. If \( n = 2 \), \( \text{ad} \ U_\xi \) is stable with respect to any polarisation on \( C \times M_\xi \).

Proof. In view of Corollary 2.9 and Lemma 1.2, we need only prove that \( \text{ad} \ U_{\xi,x} \) is semi-stable for some \( x \in C \). The argument is the same as for Proposition 1.4; indeed (9) shows at once that \( \phi^* \text{ad} \ U_{\xi,x} \) can be expressed as a direct sum of restrictions to \( T_s \) of algebraically equivalent line bundles on \( P_s \).

For \( n > 2 \), we can show similarly that \( \text{ad} \ U_\xi \) is semi-stable.

§3 Deformations

As in the introduction, let \( H \) be any ample divisor on \( C \times M_\xi \), let \( M(\mathcal{U}_\xi) \) denote the moduli space of \( H \)-stable bundles with the same numerical invariants as \( \mathcal{U}_\xi \) on \( C \times M_\xi \), and let \( M(\mathcal{U}_\xi)_0 \) denote the connected component of \( M(\mathcal{U}_\xi) \) which contains \( \mathcal{U}_\xi \). One can define a morphism

\[
\beta : J(C) \rightarrow M(\mathcal{U}_\xi)_0
\]

by

\[
\beta(L) = \mathcal{U}_\xi \otimes p^*_C L.
\]

Our object in this section is to prove
Theorem 3.1. β is an isomorphism.

Remark 3.2. Note that this implies in particular that $M(\mathcal{U}_\xi)_0$ is independent of the choice of $H$, and is a smooth projective variety of dimension $g$. Since $h^2(\text{End}\mathcal{U}_\xi) \neq 0$, there is no a priori reason why this should be so.

Proof of Theorem 3.1. By Lemma 2.4, β is injective. Moreover the Zariski tangent space to $M(\mathcal{U}_\xi)_0$ at $\mathcal{U}_\xi \otimes p_\xi^*L$ can be identified with $H^1(C \times M_\xi, \text{End}\mathcal{U}_\xi)$, which has dimension $g$ by Proposition 2.1. It follows that, at any point of $\text{Im } \beta$, $M(\mathcal{U}_\xi)_0$ has dimension precisely $g$ and is smooth. Hence by Zariski’s Main Theorem, β is an open immersion. Since $J(C)$ is complete, it follows that β is an isomorphism.

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