A disproof of the Riemann hypothesis on zeros of $\zeta$–function

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Abstract

Applying the known Beurling–Nyman criterion, it is disproved the Riemann hypothesis on zeros of $\zeta$–function.

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1 Introduction

In his famous presentation at the International Congress of Mathematicians held in Paris in 1900, David Hilbert included the Riemann Hypothesis as number 8 in his list of 23 challenging problems published later. After over 100 years, it is one of the few on that list that have not been solved. At present many mathematicians consider it the most important unsolved problem in mathematics.

Recall that, exactly one hundred years later, the Clay Mathematics Institute has published a list of 7 unsolved problems for the 21st century, including 6 unresolved problems from the Hilbert list, offering a reward of one million dollars for a solution to any of these problems.

One of them is the Riemann hypothesis, i.e. a conjecture that the so-called Riemann zeta function has as its zeros only complex numbers with real part $1/2$ in addition to its trivial zeros at the negative even integers. It was proposed by Bernhard Riemann in his 1859 paper [40]. The Riemann zeta function plays a great role in analytic number theory as well as in physics, probability theory and applied statistics.
The **Riemann zeta function** $\zeta(s)$ is a function of a complex variable $s$ that analytically continues the sum of the **Dirichlet series**

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \text{Re } s > 1. \quad (1.1)$$

As known, the series (1.1) is extended to the meromorphic function $\zeta(s)$ of the whole plane having only one simple pole at the point $s = 1$.

The Riemann zeta function satisfies the **Riemann functional equation**

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s) \quad \forall \ s \in \mathbb{C} \quad (1.2)$$

which is an equality of meromorphic functions where $\Gamma(s)$ is the **gamma function of Euler**, see [40], see also [43]. Recall that $\Gamma(s)$ is a meromorphic function on the whole complex plane $\mathbb{C}$ having no zeros and only simple poles at the points $s = 0, -1, -2, \ldots$. Equation (1.2) implies that $\zeta(s)$ has simple zeros at all even negative integers $s = -2n$, these are the **trivial zeros of $\zeta(s)$**.

Riemann has also found in [40] a symmetric form of the functional equation (1.2). One of such equivalent forms, see e.g. [32] and also [43], is the equation

$$\xi(s) = \xi(1-s) \quad \forall \ s \in \mathbb{C} \quad (1.3)$$

where

$$\xi(s) = \frac{1}{2} \pi^{-\frac{s}{2}} s(s-1) \Gamma\left(\frac{s}{2}\right) \zeta(s). \quad (1.4)$$

Note that by the previous items the function $\xi(s)$ is an entire function, i.e., an analytic function in the whole complex plane $\mathbb{C}$ without any poles, and, moreover, $\xi(s)$ has no above trivial zeros of $\zeta(s)$ but all their rest zeros coincide.

**Remark 1.** After the replacement $z = s - 1/2$, equation (1.3) can be written in the form

$$\xi\left(\frac{1}{2} + z\right) = \xi\left(\frac{1}{2} - z\right) \quad \forall \ z \in \mathbb{C} \quad (1.5)$$

meaning that the function $\xi(s)$ is symmetric with respect to the point $z_0 = 1/2$.

Thus, to verify the Riemann hypothesis it is sufficient to prove the absence of zeros of $\zeta(s)$ in the half-plane $\text{Re } s > 1/2$. Note also by the way that zeros of the Riemann zeta function are symmetric with respect to real axes because at all $\overline{\zeta(s)} = \zeta(\bar{s})$. 


Note that there exist fine monographs especially devoted to the theory of the Riemann zeta function, see e.g. [22], [27]–[31], [34] and the classic [43]. Moreover, it was even appeared the 3 volumes of equivalents of the Riemann hypothesis, see [15]–[17]. The great number of such equivalents makes possible, on the one hand, to attack the Riemann hypothesis from many positions and, on the other hand, to obtain many consequences in the case of its proof or its disproof. We prefer one of these equivalents.

2 The Beurling–Nyman criteria

Let us recall the contents of the paper [14] of the known Swedish mathematician Arne Beurling. Denote by \( \{\tau\} \) the fractional part \( \tau - \lfloor \tau \rfloor \) of a real number \( \tau \) where \( \lfloor \tau \rfloor \) is the greatest integer that is less or equal to \( \tau \). Denote also by \( \mathcal{B} \) the collection of all functions \( \varphi : (0, 1] \to \mathbb{R} \) of the form

\[
\varphi(t) = \sum_{k=1}^{n} c_k \left\{ \frac{\theta_k}{t} \right\}, \quad c_k \in \mathbb{R}, \, \theta_k \in (0, 1], \, k = 1, \ldots, n, \quad (2.1)
\]

with the condition

\[
\sum_{k=1}^{n} c_k \theta_k = 0. \quad (2.2)
\]

Now, let \( \mathcal{B}_p \) be the closure of \( \mathcal{B} \) in \( L_p = L_p(0, 1) \), \( 1 < p < \infty \). It is shown in [14] that \( \mathcal{B}_p = L_p \) if and only if the function \( f(t) \equiv 1, \, t \in (0, 1) \), is in \( \mathcal{B}_p \). Moreover, it is shown in [14] that the Riemann zeta function has no zeros in \( \text{Re} s > 1/p, \, p \in (1, 2] \) if and only if \( \mathcal{B}_p = L_p \). Thus, by Remark 1 we have from this the following consequences.

**Theorem A.** The Riemann hypothesis is true if and only if the function \( f(t) \equiv 1, \, t \in (0, 1] \), can be approximated in \( L_2 \) by a sequence in the class \( \mathcal{B} \).

**Theorem B.** The Riemann zeta function has no zeros in the half–plane \( \text{Re} s > 1/p \) for \( p \in (1, 2] \) if and only if the function \( f(t) \equiv 1, \, t \in (0, 1] \), can be approximated in \( L_p \) by a sequence in the class \( \mathcal{B} \).

Theorem A was first proved in the thesis [39] of Bertil Nyman (1950). Recall also that Beurling was his advisor. The paper [14] first (1955) represented and
generalized this result as Theorem B. Later on, the Nyman–Beurling criterion was reproved and generalized in many different ways, as well as, the approach was applied for the research of the problem on the distribution of zeros of the Riemann zeta function, see e.g. [3]–[13], [18]–[20], [24], [33]–[38], [41] and [46]. It is impossible to list here hundreds of other papers devoted immediately or indirectly to the Riemann hypothesis.

Baez-Duarte in [6] proved Theorem 1 on a new version of criteria A in terms of class $\mathcal{D}$ of functions $\varphi$ in $L_2(0, \infty)$ of the form (2.1) with the special $\theta_k = 1/k$, $k \in \mathbb{N} := \{1, 2, \ldots \}$, generally speaking without the condition (2.2). Note by the way, (2.2) can be added by Lemma 1 and the proof of Proposition 1 in [8]. But we do not need it.

**Theorem C.** The Riemann hypothesis is true if and only if the characteristic function of interval $(0, 1]$ can be approximated in $L_2(0, \infty)$ by a sequence in class $\mathcal{D}$.

Theorem C admits wording more convenient for our purposes. Namely, let us denote by $\mathfrak{K}$ the collection of all functions $\varphi : (0, \infty) \to \mathbb{R}$ of the form

$$\varphi(t) = \sum_{k=1}^{n} h_k a_k(t), \quad h_k \in \mathbb{R}, \quad a_k(t) := \left\{ \frac{t}{k} \right\}, \quad k \text{ and } n \in \mathbb{N}, \quad (2.3)$$

by $L^*_2$ the Hilbert space $L_2((0, \infty); t^{-2}dt)$, by $\|f\|_*$ the norm of $f$ in $L^*_2$ and by $\langle f, g \rangle_*$ the scalar product of functions $f$ and $g$ in $L^*_2$,

$$\langle f, g \rangle_* := \int_{0}^{\infty} f(t) g(t) \frac{dt}{t^2}. \quad (2.4)$$

Applying the replacement $t \mapsto 1/t$, we come to the following equivalent formulation of the Baez-Duarte criterion that makes our study more visual.

**Theorem D.** The Riemann hypothesis is true if and only if the characteristic function $\chi$ of $[1, \infty)$ can be approximated by a sequence of the class $\mathfrak{K}$ in $L^*_2$.

**Remark 2.** Note that the functions $\varphi \in \mathfrak{K}$ are linear with the slope $\sum h_k k$ on all open intervals of length 1 that appeared in $(0, \infty)$ after removing all natural numbers. Moreover, functions $\varphi \in \mathfrak{K}$ are continuous from the right at
each point $k \in \mathbb{N}$ and has the jump $-h_k$ at each point $km, k, m \in \mathbb{N}$, where the corresponding jumps summarized if $k_1m_1 = k_2m_2$ for some indexes $m_1 \neq m_2$ and $k_1 \neq k_2$. Consequently, the functions $a_k, k = 1, 2, \ldots, n$, are linearly independent as it is clear from the behavior of any finite linear combination $\sum h_k a_k$ at a neighborhood of the minimal $k_0 \in \mathbb{N}$ where $h_{k_0} \neq 0$.

3 Some more preliminary remarks

Let us start from the following general statement on the Hilbert spaces.

**Lemma 1.** Let $\mathcal{H}$ be a Hilbert space and let $v_1, v_2, \ldots$ be a sequence of linearly independent vectors in $\mathcal{H}$. Then each vector $v \in \mathcal{V} := \text{sp} \{v_k\}_{k=1}^\infty$ can be represented as a series $\sum_{k=1}^\infty \alpha_k v_k$ that is convergent to $v$ weakly in $\mathcal{H}$.

Furthermore, if $e_1, e_2, \ldots$ is an orthonormal sequence in $\mathcal{H}$, then it is a basis by Schauder of its Hilbert subspace $\mathcal{V} := \text{sp} \{e_k\}_{k=1}^\infty$, i.e., each $v \in \mathcal{V}$ can be represented as a unique series $\sum_{k=1}^\infty \alpha_k e_k$ that is convergent to $v$ with respect to the norm of $\mathcal{H}$.

It is evident that the latter conclusion of Lemma 1 remains true for any orthogonal sequences but not for arbitrary linearly independent sequences of vectors $v_1, v_2, \ldots$ in a Hilbert space $\mathcal{H}$ as the corresponding examples show. Thus, the uniqueness request cannot be added in the former conclusion of Lemma 1 because a weak basis is a basis in the strong sense by Schauder. The latter fact goes back to Banach S. for the Hilbert spaces and to Mazur S. for the Banach spaces and had many extensions to more general spaces by many authors, see e.g. [12] and [42], Theorem 13.1 and page 209 for brief history.

**Proof.** Let us start from the proof of the latter conclusion of the lemma. So, let $v \in \mathcal{V} := \text{sp} \{e_k\}_{k=1}^\infty$, where $e_1, e_2, \ldots$ is an orthonormal sequence in a Hilbert space $\mathcal{H}$. Then $\|v - v^{(n)}\| \to 0$ as $n \to \infty$ for some vectors $v^{(n)} \in \text{sp} \{e_k\}_{k=1}^\infty$, i.e., $v^{(n)} = \sum \alpha_k^{(n)} e_k$, where only a finite number of coefficients $\alpha_k^{(n)}$ is different from zero for each $n \in \mathbb{N}$. Note that the convergent sequence $v^{(n)}, n = 1, 2, \ldots$ is fundamental with respect to the norm of $\mathcal{H}$, see e.g. Lemma I.6.6 in [21], i.e.,
\|v^{(n)} - v^{(m)}\| \to 0 \text{ as } n, m \to \infty. \text{ Thus, by Lemma IV.4.9 in [21], we have that}
\[
\lim_{n,m \to \infty} \sum_{k=1}^{\infty} \left| \alpha_k^{(n)} - \alpha_k^{(m)} \right|^2 = 0, \tag{3.1}
\]
i.e., the sequence \((\alpha_1^{(n)}, \alpha_2^{(n)}, \ldots), n = 1, 2, \ldots,\) is fundamental in the Hilbert space \(l_2,\) see e.g. Lemma IV.4.19 in [21], which is a Banach space, see Theorem IV.4.1. Consequently, \(\{\alpha_k^{(n)}\}\) converges as \(n \to \infty\) to some \(\{\alpha_k\}\) in \(l_2\) with respect to the norm in \(l_2.\) Hence \(v\) can be really represented as a series \(\sum \alpha_k e_k\)
that is convergent to \(v\) with respect to the norm of \(\mathcal{H},\) see again Lemma IV.4.9 in [21].

Now, let \(v_1, v_2, \ldots\) be a sequence of linearly independent vectors in \(\mathcal{H}.\) Without loss of generality and as necessary at the end of the proof, we assume further that the sequence \(v_1, v_2, \ldots\) has a special normalization, namely, that \(\|v_k\|^2 = 2^{-k}\) for all \(k = 1, 2, \ldots,\) \(...\). Note that the set of the vectors \(v_1, \ldots, v_n\) in \(\mathcal{H}\) for each \(n \in \mathbb{N}\) generates the vector space \(\mathcal{V}^n\) as its basis which is isomorphic to \(\mathbb{R}^n,\) see e.g. Section 8 in [25]. Since we have the scalar product in \(\mathcal{V}^n\) as in \(\mathcal{H},\) we may apply here the Gram–Schmidt process, see e.g. Section 48 in [25], to obtain in \(\mathcal{V}^n\) first its orthogonal basis
\[
b_1 := v_1, \quad b_2 := v_2 - \text{proj}_{b_1} v_2, \ldots, \quad b_n := v_n - \text{proj}_{b_1} v_n - \ldots - \text{proj}_{b_{n-1}} v_n, \tag{3.2}
\]
where \(\text{proj}_b v\) denotes the orthogonal projection of a vector \(v\) on a vector \(b \neq 0,\)
\[
\text{proj}_b v := \frac{\langle v, b \rangle}{\langle b, b \rangle} b, \tag{3.3}
\]
and then its orthonormal basis
\[
e_1 := b_1/\|b_1\|, \quad e_2 := b_2/\|b_2\|, \ldots, \quad e_n := b_n/\|b_n\|. \tag{3.4}
\]
It is clear by construction that \(\text{sp}\{v_k\}_{k=1}^{n} = \text{sp}\{e_k\}_{k=1}^{n}\) for each \(n = 1, 2, \ldots\)
and that \(\overline{\text{sp}}\{v_k\}_{k=1}^{\infty} = \overline{\text{sp}}\{e_k\}_{k=1}^{\infty}.\)

Consequently, \(\mathcal{V} = \overline{\text{sp}}\{e_k\}_{k=1}^{\infty}\), and by the first part of the proof each \(v \in \mathcal{V}\)
can be represented as a unique series \(\sum_{k=1}^{\infty} \alpha_k e_k\) that is convergent to \(v\) with respect to the norm of \(\mathcal{H}.\) Thus, \(\mathcal{V}\) is a Hilbert subspace of \(\mathcal{H},\) which is in the natural way isometric to \(l_2,\) see Lemma IV.4.19 in [21].
As usual, let $l_1$ denote the Banach space of all sequences $s = \{\alpha_k\}$ of real numbers $\alpha_k$, $k \in \mathbb{N}$, with the norm $\|s\|_1 = \sum_{k=1}^{\infty} |\alpha_k|$. Note that $l_1$ is a Hilbert subspace of $l_2$ because $(\sum |\alpha_k|)^2 \geq \sum |\alpha_k|^2$, i.e., $\|s\|_1 \geq \|s\|_2$ for each $s \in l_1$.

Let us consider the continuous linear operator $O(\{\alpha_k\}) = \sum_{k=1}^{\infty} \alpha_k v_k$ from $l_1$ into the vector space $\mathcal{V}$. Note that its subspace $\mathcal{V}_1 := O(l_1)$ consists of elements of $\mathcal{V}$ represented by the absolutely convergent series with respect to the norm in $\mathcal{H}$, because $\|\sum_{k=1}^{\infty} \alpha_k v_k\| \leq \sum_{k=1}^{\infty} |\alpha_k|$. Next, setting $Z := O^{-1}(0)$, we see that $Z$ is a closed linear subspace of $l_1$.

Let $l_1^\perp := l_1/Z$ be the factor space in $l_1$ associated with $\mathcal{V}$ through $Z$. Let us show that each coset of $Z$ in $l_1$, i.e., each subset of $l_1$ of the form $s + Z$, $s \in l_1$, has the single element $s_0$ such that $\|s_0\|_2 = m := \inf_{s' \in s + Z} \|s'\|_2$. The existence of such elements $s_0$ follows from the closeness of $Z$ in $l_1$. Let us assume that there exist at least two such elements $s_1$ and $s_2$ in $s + Z$. Then all the points $s_t := ts_1 + (1 - t)s_2 = s_2 + t(s_1 - s_2)$, $t \in (0, 1)$, also belong to the hyperplane $s + Z$ by the convexity of $Z$, and, moreover, by the triangle inequality $\|s_t\|_2 = m$ for all $t \in (0, 1)$. However, by differentiating the function $\|s_t\|_2^2 = \langle s_t, s_t \rangle = \|s\|_2^2 + 2t\langle s_2, s_1 - s_2 \rangle + t^2\|s_1 - s_2\|_2^2$ twice on the variable $t$ in the identity $\|s_t\|_2^2 \equiv m^2$, we have that it should be the equality $s_1 = s_2$. It is hereafter important that the extreme $s_0$ is in $l_1$.

Let us describe more constructively the extreme $s_0$ from the previous item. Denoting by $\tilde{Z}$ the $l_2$–closure of $Z$ in $l_1$, we see that $\inf_{s' \in s + Z} \|s'\|_2 = m = \inf_{s' \in \tilde{Z}} \|s'\|_2$ for each $s \in l_1$, and the corresponding extreme in $\tilde{Z}$ is unique, see e.g. Theorem 12.2.1 in [29], and hence it coincides with $s_0$. Since the hyperplane $s + \tilde{Z}$ is parallel to $\tilde{Z}$, the distance from 0 to $s + \tilde{Z}$ is equal to the distance from $s$ to $\tilde{Z}$.

Note that the $l_2$–closed linear subspace $\tilde{Z}$ of the Hilbert space $l_1$ has an orthonormal basis $\mathcal{E}$, see Theorem IV.4.12 in [21], and the best approximation of $s$ by the vectors in $\tilde{Z}$ is given by the orthogonal projection of $s$ into $\tilde{Z}$: $P_{\tilde{Z}}(s) := \sum_{e \in \mathcal{E}} \langle s, e \rangle e$, see Theorem IV.4.10 in [21], which is a linear operator, and $s_0 = P_{\tilde{Z}}(s) := s - P_{\tilde{Z}}(s)$ is a linear operator (orthogonal projector of $s$ into $\tilde{Z}^\perp$), too, see e.g. Theorem 8.8 in [23]. Here $\tilde{Z}^\perp$ denotes the orthogonal
complement of \( \bar{Z} \) in \( l_1 \).

Remark also that \( \bar{Z} \) is a linear subspace of \( l_1 \), which is closed with respect to \( l_2 \)-norm, see e.g. Lemma IV.4.4 in [21]. Consequently, \( \bar{Z} \) is a Banach subspace (even a Hilbert subspace) of \( l_1 \subset l_2 \). Moreover, by construction the restriction \( O_\ast \) of the operator \( O \) to \( \bar{Z} \) is a bijective bounded linear operator of \( \bar{Z} \) onto \( \mathcal{V}_1 \). Then by the Banach theorem, see e.g. Lemma VII.1.4.1 in [30], there is its inverse bounded operator \( O_\ast^{-1} \) of \( \mathcal{V}_1 \) onto \( \bar{Z} \subset l_2 \), see also Lemma II.3.4 in [21]. Note that \( \mathcal{V}_1 \) is dense in \( \mathcal{V} \) because \( \text{sp}\{v_k\}_{k=1}^\infty \subset \mathcal{V}_1 \) is dense. Consequently, the operator \( O_\ast^{-1} \) can be extended by continuity to the bounded operator \( O^\ast \) of \( \mathcal{V} \) into \( l_2 \), see e.g. Theorem V.8.2 in [30].

Let \( v \) be a vector in \( \mathcal{V} \) and \( s = \{\alpha_k\} \) be its image in \( l_2 \) under \( O^\ast \). Note that the series \( \sum \alpha_k v_k \) generally speaking is nonconvergent to \( v \) with respect to the norm in \( \mathcal{H} \). However, let us show that \( \sum \alpha_k v_k \) is convergent to \( v \) weakly in \( \mathcal{H} \).

For this purpose, let us first show that the sequence \( V_n := \sum_{k=1}^n \alpha_k v_k, \ n \in \mathbb{N} \), is weakly fundamental in \( \mathcal{H} \). In this connection, recall that each \( h^\ast \in H^\ast \) uniquely determines \( h \in H \) such that \( h^\ast g = \langle g, h \rangle \) and the map \( \sigma : h^\ast \to h \) is a one-to-one map of \( H^\ast \) onto \( H \), see Theorem IV.4.5 in [21]. For every \( h \in H \) and \( n, m \in \mathbb{N}, \ m > n \), we have by the triangle and Schwartz inequalities that
\[
\Delta_{n,m} := \langle V_n - V_m, h \rangle^2 = \left( \sum_{k=m+1}^n \alpha_k \langle v_k, h \rangle \right)^2 \leq \|h\|^2 \left( \sum_{k=m+1}^n |\alpha_k| \cdot \|v_k\| \right)^2.
\]
To take it further, let us use the elementary inequality \((a + b)^2 \leq 2(a^2 + b^2)\) to get by induction a more general inequality \((\sum a_m)^2 \leq 2a_1^2 + 2^2a_2^2 + 2^3a_3^2 + \ldots\).

Consequently, taking into account the special normalization \( \|v_k\|^2 = 2^{-k} \), we see that \( \Delta_{n,m} \leq \|h\|^2 \cdot \sum_{k=m+1}^n |\alpha_k|^2 \to 0 \) as \( n \) and \( m \to \infty \) for each \( h \in H \), i.e., indeed, \( V_n := \sum_{k=1}^n \alpha_k v_k, \ n \in \mathbb{N} \), is weakly fundamental in \( \mathcal{H} \).

As is well known, every Hilbert space is weakly complete, see e.g. Corollary IV.4.7 in [21]. Hence the last item implies that the series \( \sum \alpha_k v_k \) is convergent to a vector \( v_0 \) weakly in \( \mathcal{H} \). It remains to show that \( v_0 = v \). Indeed, by construction there is a sequence \( v^{(m)} = \sum \alpha_k^{(m)} v_k \) in \( \mathcal{V}_1 \) with \( s_m := \{\alpha_k^{(m)}\} \) in \( \bar{Z} \subset l_1 \subset l_2, \ m = 1, 2, \ldots \), such that \( v^{(m)} \to v \) as \( m \to \infty \) with respect to the norm in \( \mathcal{H} \). Then \( O^*(v^{(m)}) = s_m \) and \( \|s - s_m\|_2 \to 0 \) as \( m \to \infty \) by continuity of the operator \( O^* : \mathcal{V} \to l_2 \). Finally, arguing similarly to the last item, we
obtain that $\langle v_0 - v^{(m)}, h \rangle^2 \leq \|h\|^2 \cdot \|s - s_m\|_2 \to 0$ as $m \to \infty$ for each $h \in \mathcal{H}$, i.e., $v^{(m)} \to v_0$ as $m \to \infty$ weakly in $\mathcal{H}$. On the other hand, $v^{(m)} \to v$ as $m \to \infty$ weakly in $\mathcal{H}$ because $v^{(m)} \to v$ as $m \to \infty$ with respect to the norm in $\mathcal{H}$. Thus, $v_0 = v$ in view of the uniqueness of a weak limit, see e.g. Lemma II.3.26 in \[21\]. The proof is complete. \(\square\)

Recall also that the Möbius function $\mu(n)$ is a function of the natural parameter $n = 1, 2, \ldots$ defined as follows:

(i) $\mu(1) = 1$, 
(ii) $\mu(n) = 0$ if $n$ is divisible by a square of a prime $p > 1$, 
(iii) $\mu(n) = (-1)^k$ if $n$ is the product of $k$ distinct primes.

The function $\mu(n)$ occurs implicitly in the work of Euler as early as 1748, but Möbius, in 1832, was the first to investigate its properties systematically, see e.g. \[32\], p. 567–587 and 901.

Its characteristic property, see e.g. Theorem 263 in \[26\], see also Theorem 2.1 in \[1\], p. 24, Theorem 7.2 in \[2\], p. 103, and \[32\], p. 575, is the following:

$$\sum_{d | n} \mu(d) = 0 \quad \forall \ n > 1.$$  \hspace{1cm} (3.5)

**Remark 3.** Arguing by induction, it is easy to see that if some arithmetical function $\alpha(n)$, $n \in \mathbb{N}$, satisfies this property and $\alpha(1) = 1$, then $\alpha(n) \equiv \mu(n)$, because we have by (3.5) that

$$\alpha(n + 1) = - \sum_{d | n+1, \ d < n+1} \alpha(d) \quad \forall \ n \in \mathbb{N} ,$$  \hspace{1cm} (3.6)

i.e., we have a guarantee for the next inductive step, and, consequently, such a function is uniquely determined.

Finally, let us recall one more result of Baez-Duarte, see Theorem 2.2 in \[4\], that after the replacement $t \mapsto 1/t$ can be formulated in the following way.

**Theorem D*. The sequence of the functions $\varphi_n := - \sum_{k=1}^{n} \mu(k) a_k$ cannot be convergent to $\chi$ with respect to the norm in $L^*_2$.**

In light of Lemma 1, we need the following strengthening of Theorem D*.

**Lemma 2. The sequence of the functions $\varphi_n := - \sum_{k=1}^{n} \mu(k) a_k$ cannot be convergent to $\chi$ even weakly in $L^*_2$.**
4 The main result

**Theorem 1.** The Riemann hypothesis is not true, i.e., the critical strip $0 < \text{Re } z < 1$ contains zeros of the $\zeta-$function outside the line $\text{Re } z = \frac{1}{2}$.

**Proof.** Let us assume that the Riemann hypothesis is true. Then first by Theorem D we have that $\chi \in \overline{\mathbb{R}} = \overline{\text{sp}} \{a_k\}_{k=1}^{\infty}$, where $a_k := \{t/k\}$, $k = 1, 2, \ldots$. Consequently, by Lemma 1 together with Remark 2, $\chi$ can be represented as a series $\sum_{k=1}^{\infty} \alpha_k a_k$, which is convergent to $\chi$ weakly in $L^*_2$, i.e., for each $g \in L^*_2$,

$$\langle \chi - \varphi_n , g \rangle_2 \to 0 \quad \text{as } n \to \infty,$$

where $\varphi_n(t) := \sum_{k=1}^{n} \alpha_k a_k(t)$. (4.1)

In this case, by Remark 2 the slope of $\varphi_n$ on each interval $(l-1, l)$, $l = 1, 2, \ldots$,

$$s_n = \sum_{k=1}^{n} k^{-1} \alpha_k \to 0 \quad \text{as } n \to \infty$$

(4.2)

because then it should be for $\delta \in (0, 1)$ that

$$\lim_{n \to \infty} \int_{\delta}^{1} \varphi_n(t) \frac{dt}{t^2} = 0.$$ (4.3)

Then we have that $\alpha_1 = -1 = -\mu(1)$, because by Remark 2

$$\lim_{n \to \infty} \int_{1}^{2} (1 - \varphi_n(t)) \frac{dt}{t^2} = \lim_{n \to \infty} \int_{1}^{2} (1 - (s_n t - \alpha_1)) \frac{dt}{t^2} = 0,$$

and, similarly, $\alpha_1 + \alpha_2 = 0$ because

$$\lim_{n \to \infty} \int_{2}^{3} (1 - \varphi_n(t)) \frac{dt}{t^2} = \lim_{n \to \infty} \int_{2}^{3} (1 - (s_n t - \alpha_1 - (\alpha_1 + \alpha_2))) \frac{dt}{t^2} = 0$$

and, arguing by induction, we have that $\sum_{d|k} \alpha_d = 0$ for all $k \in \mathbb{N}$ because

$$\lim_{n \to \infty} \int_{k}^{k+1} (1 - \varphi_n(t)) \frac{dt}{t^2} = \lim_{n \to \infty} \int_{k}^{k+1} (1 - (s_n t - \alpha_1 - (\alpha_1 + \alpha_2))) \frac{dt}{t^2} = 0.$$
\[
\lim_{n \to \infty} \int_{k}^{k+1} \left( 1 - \left( (s_n t - \alpha_1) - \ldots - \sum_{d \mid (k-1)} \alpha_d \right) - \sum_{d \mid k} \alpha_d \right) \frac{dt}{t^2} = 0 .
\]

Thus, by Remark 3 we have that \( \chi = -\sum_{k=1}^{\infty} \mu(k) a_k \), i.e., it should be that
\[
\varphi_n := -\sum_{k=1}^{n} \mu(k) a_k
\]
is convergent to \( \chi \) weakly in \( L_2^* \) that directly contradicts the conclusion of Lemma 2. The obtained contradiction disproves the above assumption, i.e., the Riemann hypothesis is not true. \( \square \)

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