Arnold Tongues in Area-Preserving Maps

Mark Levi & Jing Zhou

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Abstract

In the early 60’s J. B. Keller and D. Levy discovered a fundamental property: the instability tongues in Mathieu-type equations lose sharpness with the addition of higher-frequency harmonics in the Mathieu potentials. Twenty years later, V. Arnold discovered a similar phenomenon on the sharpness of Arnold tongues in circle maps (and rediscovered the result of Keller and Levy). In this paper we find a third class of object where a similar type of behavior takes place: area-preserving maps of the cylinder. loosely speaking, we show that periodic orbits of standard maps are extra fragile with respect to added drift (i.e. non-exactness) if the potential of the map is a trigonometric polynomial. That is, higher-frequency harmonics make periodic orbits more robust with respect to “drift”. This observation was motivated by the study of traveling waves in the discretized sine-Gordon equation which in turn models a wide variety of physical systems.

1. Introduction

Understanding invariant sets of area-preserving maps is one of the central problems of dynamics and one of the most studied—starting with Poincaré’s geometrical theorem [5,7,15], through KAM theory [2,3,14] and the Aubry-Mather theory [4,12,13]. All of the results require the exactness assumption.

Much less is known about area-preserving maps which are non-exact, such as the maps $\varphi$ of the cylinder $\mathbb{S} \times \mathbb{R}$ with a “drift”:

$$\int_{\varphi(\gamma)} v \, dx - \int_\gamma v \, dx = \delta \neq 0$$

(1)

Here $\gamma$ is a closed curve encircling the cylinder $\mathbb{R} \mod 1 \times \mathbb{R}$ once; see Fig. 1.

Such maps are ubiquitous in Hamiltonian dynamics, and arise in numerous settings. We mention four examples.
1. The Frenkel–Kontorova model of electrons in a crystal lattice [6,9]. The model consists of an infinite chain of particles on the line a periodic potential \( V(x) \) and with nearest neighbor coupling. The equilibria are the critical points of the total energy

\[
\sum_{i \in \mathbb{Z}} \frac{k}{2} (x_{i+1} - x_i)^2 + V(x_i);
\]

although the sum is divergent, the variational equation, i.e. the equilibrium condition

\[
x_{i+1} - 2x_i + x_{i-1} + k^{-1}V'(x_i),
\]

is well defined. This discrete analog of the Euler-Lagrange equation has a Hamiltonian counterpart obtained by the introduction of the discrete momentum \( v_i = x_i - x_{i-1} \):

\[
\begin{cases}
x_{i+1} = x_i + v_i - V'(x_i) \\
v_{i+1} = v_i - V'(x_i)
\end{cases}
\]

Used it an area-preserving map.

If \( V' \) is periodic, then (3) defines a cylinder map. However a periodic \( V' \) leaves the possibility that \( V \) itself may have a “tilt”, i.e. a linear part

\[
V(x) = a x + V_{\text{periodic}}(x),
\]

where \( V_{\text{periodic}}(x + T) = V_{\text{periodic}}(x) \) for some fixed \( T \).

The tilt causes the map (3) to be non-exact; (1) holds for this map with \( \delta = -aT \) (Fig. 2).

2. Chains of coupled pendula. In the special case of \( V(x) = c \sin x \), the Frenkel–Kontorova model has a mechanical interpretation as the chain of torsionally coupled pendula (Fig. 3); here \( x_i \) denotes the angle of the \( i \)th pendulum with the downward vertical. Now if each pendulum is subjected to a constant torque \( \tau \) then the potential acquires a linear part: \( V(x) = c \sin x + \tau x \), and the corresponding cylinder map becomes non-exact, with \( \delta = -2\pi \tau \).

3. Coupled Josephson junctions. A Josephson junction consists of two superconductors separated by a narrow gap of a few angstroms [8]. The junction can behave as a superconductor or as a resistor, depending on the initial conditions.
Fig. 2. A the Frenkel–Kontorova model; B the tilt added, leading to the non-exact cylinder map; C tilt interpreted as torque acting on coupled pendula

Fig. 3. Discretized sine-Gordon equation: pendula with nearest-neighbor torsional coupling

Fig. 4. Josephson junctions: single and coupled. Voltage across the junction is proportional to $\langle \dot{x} \rangle$

(there were some hopes in the 1970s to use this property as a memory device). This behavior reminds one of a pendulum with torque $\delta$ described by

$$\ddot{x} + c\dot{x} + \sin x = \delta$$

(4)

For the $|\delta| < 1$ there are two stable limiting regimes: either the stable equilibrium or the “running” periodic solution corresponding to the tumbling motion $x = \omega t + p(t)$ where $p$ is periodic. In fact the same equation (4) is satisfied by the jump $x = \arg \theta_2 - \arg \theta_1$ of the phase of the electron wave function across the gap if current $\delta$ is driven across the gap, Fig. 4. The voltage across the gap is proportional to the average $\langle \dot{x} \rangle$, or the average angular velocity in the pendulum interpretation. Thus the equilibrium solution with its average $\langle \dot{x} \rangle = 0$ corresponds to zero voltage and thus to the superconducting state, while the tumbling solution with the voltage $\langle \dot{x} \rangle \neq 0$ corresponds to the resistive state.
4. **Particle in \( \mathbb{R} \) subject to a force that varies periodically both in time and position.** The motion of such a particle is governed by

\[ \ddot{x} = \Phi(x, t), \]

where \( \Phi \) is periodic in both variables of period 1 (without loss of generality). The Poincaré map \( \varphi : (x, y)_{t=0} \mapsto (x, y)_{t=1} \), where \( y = \dot{x} \), is generally non-exact, satisfying (1) with the drift equal to the average of force \( \Phi \):

\[ \delta = \int_0^1 \int_0^1 \Phi(x, t) \, dx \, dt = \langle \Phi \rangle. \]

This completes our list of examples where non-exact area-preserving cylinder maps ares. In this paper we show that periodic orbits of the standard map (3) are extra sensitive to the added drift \( \delta \) if the potential has harmonics of only low frequencies. There are (at least) two known phenomena with a similar flavor: (i) the sharpness of Arnold tongues in circle maps [1]

\[ x \mapsto x + \omega + \varepsilon f(x), \]

where \( f \) is a trigonometric polynomial is related to the degree of \( f \), and (ii), the sharpness of resonance zones in Hill’s equations

\[ \ddot{x} + (\omega^2 + \varepsilon q(t))x = 0, \]

where \( q \) is a trigonometric polynomial is related to the degree of \( q \) [11]. The present paper adds one more example to this list. According to Arnold [1], Gelfand conjectured the existence of a general theorem which covers cases (i) and (ii); to this conjecture one can add the problem studied in the present paper.

2. **Results**

We consider periodic potentials with a linear part added:

\[ V(x) = \delta x + \varepsilon F(x), \quad F(x + 2\pi) = F(x). \]

Thus the standard map (3) with such \( V \) takes form

\[
\begin{align*}
x_{i+1} &= x_i + v_i - \delta - \varepsilon f(x_i) \\
v_{i+1} &= v_i - \delta - \varepsilon f(x_i)
\end{align*}
\]

where \( f(x) = F'(x) \) is periodic of period \( 2\pi \).

For \( \delta = 0 \) the cylinder map (5) is exact, and it possesses a \( p/q \) periodic orbit for any integer \( p, \ q \neq 0 \), i.e. an orbit satisfying

\[ x_{i+q} = x_i + 2p\pi, \quad v_{i+q} = v_i; \]

this follows from Poincaré’s Last Geometric Theorem as generalized by Franks [7]. In his generalization Franks removed the requirement of invariant boundary circles.
of an annulus, replacing it with the assumption of exactness. Since the theorem no longer applies to the non-exact case $\delta \neq 0$, a natural question arises: for what range of drift $\delta$ do periodic orbits persist? We show that if $V$ is a trigonometric polynomial, then this range becomes narrower if the degree of the trigonometric polynomial $f$ becomes smaller; furthermore, the tightness of the range is exponentially small in terms of the period $q$ (Fig. 5). More precisely, one has the following:

**Theorem 1.** (Width of Arnold tongues) Let $f(x)$ in (5) be a trigonometric polynomial of degree $d$, and let $p \geq 0$, $q > 0$ be integers. There exist positive constants $\bar{c}$ and $\varepsilon$ depending only on $q$ and $f$, such that for any $0 < \varepsilon < \bar{c}$, all $p/q$ periodic orbits of (5) disappear if the drift $|\delta| > c \varepsilon \lfloor q/d \rfloor$; here $\lfloor \cdot \rfloor$ denotes the integer part.

The other observation of this paper is that the $p/q$ periodic points move on special curves as $\delta$ changes.

**Theorem 2.** Let $p > 0$, $q > 0$ be integers. If the perturbation term $f(x)$ in the cylinder map (5) is a $2\pi$-periodic analytic function (not necessarily a trigonometric polynomial), then there exists a positive constant $\bar{c}$ depending only on $q$ and $f$ such that for any $|\varepsilon| < \bar{c}$ and $|\delta| < \bar{c}$, any $p/q$ periodic orbit of the perturbed map (5), if it exists, lies on the graph of the function

$$v = \mu + v_1(x)\varepsilon + v_2(x)\varepsilon^2 + \cdots$$

where $\mu = 2\pi p/q$ and $v_n(x)$ is an $n$th degree polynomial in $f(x + k\mu)$ ($0 \leq k \leq q - 1$) and their derivatives up to order $n - 1$. In particular,

$$v_1(x) = -\frac{q}{2} \frac{1}{q} \sum_{k=0}^{q-1} f(x + k\mu),$$

$$v_2(x) = \frac{1}{q} \sum_{k=0}^{q-1} (q - k) f(x + k\mu).$$
Fig. 6. Iterates of two periodic points (a center and a saddle). The arrows show the direction of motion of the iterates as $\delta$ increases. The two orbits disappear in a saddle-node bifurcation. As $\delta$ changes, these points move with large speed (at least) $O(\varepsilon^{-[q/d]})$ (at bifurcation the speed becomes infinite).

Remark 1. If $f$ is a trigonometric polynomial, then $\overline{f}$ and $\overline{f}$ are trigonometric polynomials as well, of a degree not exceeding the degree $d$ of $f$. Moreover, if $d < q$, then $\overline{f} = 0$.

Remark 2. Birkhoff $p/q$ periodic orbits come in saddle-center pairs. As $\delta$ increases from 0 to a critical value, a pair disappears in a saddle-node bifurcation, Fig. 6. The first theorem therefore states that critical values of $\delta$ are $O(\varepsilon^{-[q/d]})$.

Remark 3. A special case of Theorem 1 for $q \leq 3$ was proven in [10] by a direct calculation. Unfortunately, as $q$ increases, the complexity of this calculation tends to infinity. We overcome this problem by extending Arnold’s approach [1] (that he used for circle maps) to the maps of the cylinder.

Remark 4. (An implication of Theorem 1 for traveling waves.) Consider an infinite periodic chain of pendula governed by the discretized sine-Gordon equation with damping:

$$\ddot{x}_k + \gamma \dot{x}_k + \varepsilon \sin x_k = (x_{k+1} - 2x_k + x_{k-1}) + \delta. \quad (6)$$

Fixing $q \in \mathbb{N}$ and the “twist” $p \in \mathbb{Z}$, consider space-periodic “twisted” solutions, i.e. the ones satisfying

$$x_{k+q}(t) = x_k(t) + 2\pi p \quad \text{for all } t. \quad (7)$$

According to [10] there exists a constant $\varepsilon_0 = \varepsilon_0(\gamma)$ depending only on the damping coefficient $\gamma > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ and for all $\delta$ every solution approaches asymptotically either to an equilibrium or to a traveling wave. This wave satisfies

$$x_k(t) = x_{k-1}(t + T/q),$$

(that is, each pendulum repeats what its neighbor is doing but with a delay). A $q$-fold application of this identity gives

$$x_k(t) = x_{k-q}(t + T) \equiv x_k(t + T) - 2\pi p, \quad (7)$$

i.e.

$$x_k(t + T) = x_k(t) + 2\pi p,$$
so that this solution is periodic modulo rotation $x_k \mapsto x_k + 2\pi p$ \((k = 1, \ldots, q)\) in $\mathbb{S} = \mathbb{R} \mod 2\pi$. Now, according to Theorem 1 the equilibria of (6) disappear if $\delta > c_0 \varepsilon^g$ (a threshold exponentially small in the number of pendula), and thus (6) has a globally attracting periodic traveling wave for all such $\delta$. This traveling wave appears as the result of a saddle-node bifurcation of the equilibria which exist for smaller $\delta$.

A remarkable fragility of equilibria is illustrated in Fig. 2(C), where shows an equilibrium of $q = 6$ coupled pendula with the same torque applied to each. With the choice of “gravity” $\varepsilon = \frac{1}{2}$ the pendula sag by a comparable amount and one might expect that the equilibrium could withstand the torque of a comparable magnitude. However, the equilibrium (and hence the corresponding periodic orbit of the map) disappears for $\delta > 0.0027$, about .05% of the “gravity” $\varepsilon$! Sinusoidal potentials are thus quite special: they are remarkably bad at pinning down the equilibria.

The effect of non-exactness on the dynamics of the map is of interest in itself; its understanding would also shed light on physical effects, such as the fragility of Frenkel–Kontorova equilibria in crystals to imposed voltages.

### 3. The Preliminaries

The cylinder map (5) with $\varepsilon = \delta = 0$ has an invariant circle $v = 2\pi p/q \overset{\text{def}}{=} \mu$ consisting of $p/q$ periodic points. This suggests introducing a shifted momentum

$$y := v - \mu$$

with which the cylinder map (5) takes a new form,

$$\begin{cases}
  x_{i+1} = x_i + y_i + \mu + g(x_i; \varepsilon, \delta) \\
  y_{i+1} = y_i + g(x_i; \varepsilon, \delta),
\end{cases} \tag{8}$$

where

$$g(x; \varepsilon, \delta) := -\delta - \varepsilon f(x),$$

and where $f(x)$ is a $2\pi$-periodic analytic function as in Theorem 2.

This map (8) restricted to a neighborhood of the circle $y = 0$ can be viewed as a perturbation of the shift by $\mu$ in the $x$ direction; the $n$th iterate will thus be a perturbation of the shift by $n\mu$ in the $x$-direction. This suggests writing the $n$-th iterate of (8) in the form

$$\begin{cases}
  x_n = x_0 + n\mu + nR(x_0, y_0, \varepsilon, \delta) \\
  y_n = y_0 + nS(x_0, y_0, \varepsilon, \delta).
\end{cases} \tag{9}$$

A quick calculation shows that

$$nR(x_0, y_0, \varepsilon, \delta) = n(y_0 + g(x_0)) + (n - 1)g(x_1) + \cdots + 2g(x_{n-2}) + g(x_{n-1}),$$

$$nS(x_0, y_0, \varepsilon, \delta) = g(x_0) + g(x_1) + \cdots + g(x_{n-2}) + g(x_{n-1}).$$
where \( \{x_0, x_1, \ldots, x_{n-1}\} \) are the \( x \)-coordinates of the iterates of \((x_0, y_0)\) under the cylinder map (8).

The cylinder map (8) possesses a \( p/q \) periodic orbit, \( x_q = x_0 + \frac{2\pi p}{q} \mu, y_q = y_0 \), if and only if for some \((x_0, y_0)\) the remainders of the \( q \)-th iterate vanish:

\[
qR(x_0, y_0, \varepsilon, \delta) = qS(x_0, y_0, \varepsilon, \delta) = 0.
\]

The overall plan of the proof of Theorem 1 (the main result) is as follows: The vanishing of the remainders (11) defines \( y_0 \) and \( \delta \) as functions of \( x_0 \) and \( \delta \):

\[
y_0 = Y(x_0, \varepsilon), \quad \delta = \Delta(x_0, \varepsilon)
\]

Proving the narrowness of the Arnold tongue (as specified in Theorem 1) amounts to showing that the range \( \Delta(\mathbb{R}, \varepsilon) \) is \( O(\varepsilon^r) \)--small, where \( r = [q/d] \). The proof of this narrow range statement goes as follows: expanding \( \Delta \) in powers of \( \varepsilon \) we consider the first \( x_0 \)-dependent term \( \Delta_r(x_0)\varepsilon^r \). Our goal is to show that \( r > [q/d] \). To that end, we prove (i) that \( \Delta_r(x_0) \) is periodic of period \( \mu \), and (ii), that \( \Delta_r(x_0) \) is a trigonometric polynomial of degree \( rd \). However a non-constant trigonometric polynomial periodic of period \( \mu = 2\pi p/q \) must have degree \( > q \), i.e. \( rd > q \), or \( r > [q/d] \), as desired.

### 4. Structure of the Remainders

In this section we examine the \( n \)-th iterate of the cylinder map (8) for any \( n \in \mathbb{N} \) and the associated remainders.

For brevity, we write \( g(x) = g(x; \varepsilon, \delta) \) and \( g'(x) = \frac{\partial}{\partial x} g(x; \varepsilon, \delta) = -\varepsilon f'(x) \) (recall that \( g(x; \varepsilon, \delta) := -\delta - \varepsilon f(x) \)), and also

\[
\begin{align*}
    g_k^{(r)} &:= g^{(r)}(x_0 + k\mu), \quad g_k^{(0)} = g(x_0 + k\mu) = g_k; \\
    f_k^{(r)} &:= f^{(r)}(x_0 + k\mu), \quad f_k^{(0)} = f(x_0 + k\mu) = f_k.
\end{align*}
\]

The followed lemma gives the structure of the remainders for any iterate of the cylinder map (8).

**Lemma 1.** The remainders \( nR \) and \( nS \) are a convergent series

\[
\begin{align*}
    nR &= nR_1 + nR_2 + \cdots \\
    nS &= nS_1 + nS_2 + \cdots,
\end{align*}
\]

where \( nR_m \) and \( nS_m \) are homogeneous polynomials of degree \( m \) in terms of the items in the list \( \{y_0 + g_0, \ g_k^{(l)} \ (0 \leq k \leq n - 1, \ 0 \leq l \leq m - 1)\} \). The two series converge for all \( x_0, y_0, \varepsilon, \delta \). Moreover, each term of degree \( m \geq 2 \) contains at least one derivative of \( g \) (so that, with \( g = -\delta - \varepsilon f(x) \) these terms are \( O(\varepsilon) \)). In particular,

\[
\begin{align*}
    nR_1 &= n(y_0 + g_0) + (n - 1)g_1 + (n - 2)g_2 + \cdots + g_{n-1}, \\
    nS_1 &= g_0 + g_1 + \cdots + g_{n-1}. \tag{12}
\end{align*}
\]
Proof. goes by induction on $n$. The statement is trivially true for $n = 1$, and we show that if the remainders $nR$ and $nS$ are of the form claimed in the Lemma then the same is true for $n + 1$. Indeed,

$$x_{n+1} = x_n + y_n + \mu + g(x_n) \overset{(9)}{=} x_0 + (n + 1)\mu + \frac{nR}{n+1} + \frac{nS}{n+1} + g(x_n).$$

Thanks to the inductive assumption, it only remains to show that $g(x_n)$ is of the form claimed. Taylor expansion yields

$$g(x_n) = g(x_0 + n\mu + nR) = g_n + \sum_{k \geq 1} \frac{1}{k!} g_n^{(k)} \cdot (nR)^k = g_n + \sum_{k \geq 1} \frac{1}{k!} g_n^{(k)} \cdot \left(\sum_{j \geq 1} nR_j \right)^k. \tag{13}$$

Now $m$th degree part of (13) is a linear combination (with constant coefficients) of products

$$g_n^{(k)} nR_{j_1} \cdots nR_{j_r},$$

with $j_1 + \cdots + j_r = m - 1$, and is thus a homogeneous polynomial of degree $m$ as claimed. Furthermore, every term with $m \geq 2$ comes from the series in (13) and thus contains a derivative of $g$.

The claim about $n+1S$ is proven similarly:

$$y_{n+1} = y_n + g(x_n) \overset{(9)}{=} y_0 + \frac{nS}{n+1} + g(x_n).$$

The rest of the proof is identical to the one above. This completes the proof of Lemma 1. \qed

Remark 5. As an illustration of the lemma, a short computation gives an explicit form of the degree-2 terms:

$$nR_2 = (n - 2)g_1' \cdot (y_0 + g_0) + (n - 3)g_2' \cdot (2(y_0 + g_0) + g_1) + \cdots + g_{n-1}' \cdot ((n - 1)(y_0 + g_0) + (n - 2)g_1 + \cdots + g_{n-2})$$

$$nS_2 = g_1' \cdot (y_0 + g_0) + g_2' \cdot (2(y_0 + g_0) + g_1) + \cdots + g_{n-1}' \cdot ((n - 1)(y_0 + g_0) + (n - 2)g_1 + \cdots + g_{n-2}).$$

Each term in $nR_2, nS_2$ contains the first derivative of $g$ at some shift $x_0 + k\mu$. 
5. The Existence of Periodic Orbits

In this section we discuss the existence of the $p/q$ periodic orbits of the cylinder map (5) using Implicit Function Theorem and then we prove Theorem 2. We begin by showing that the equations

$$qR(x, y, \varepsilon, \delta) = qS(x, y, \varepsilon, \delta) = 0$$

uniquely determine $\delta$ and $y$ as functions of $x$ and $\varepsilon$ for all $x \in \mathbb{R}$ and for all $|\varepsilon| < \bar{\varepsilon}$ for some positive $\bar{\varepsilon}$,

$$\begin{cases}
\delta = \Delta(x, \varepsilon) \\
y = Y(x, \varepsilon),
\end{cases}
$$

(14)

such that $qR$ and $qS$ vanish identically if (14) hold.

Wishing to apply the implicit function theorem, we note that $qR(x_0, 0, 0, 0) = qS(x_0, 0, 0, 0) = 0$ for all $x \in \mathbb{R}$ and that, using Lemma 1,

$$\left. \frac{\partial(qR, qS)}{\partial(\delta, y_0)} \right|_{y = \varepsilon = \delta = 0} = \begin{pmatrix}
-q(q + 1) \\
2q \\
-q \\
0
\end{pmatrix}$$

for all $x$. Since the determinant is $q^2 \neq 0$, the implicit function theorem applies: for any $x_0$ there exists an open disk $D_{x_0}$ centered at the point $(x_0, 0)$ in the $(x, \varepsilon)$-plane such that the implicit function $(x, \varepsilon) \mapsto (\delta, y)$ is well defined by the equations $qR = qS = 0$ on the disk $D_{x_0}$. The segment $[0, 2\pi] \times \{0\}$ in the $(x, \varepsilon)$-plane is covered by open disks and thus has a finite subcover; but then this finite union contains a strip nonzero width $\bar{\varepsilon} > 0$ around the $x$–axis. Moreover, the functions corresponding to the overlapping disks coincide. Thus the implicit function $(x, \varepsilon) \mapsto (\Delta, Y)$ is defined for all $x$ and for all $|\varepsilon| < \bar{\varepsilon}$.

**Proof of Theorem 2.** The proof proceeds by induction. If $(x_0, y_0)$ is a $p/q$-periodic point of the map (5) for some $\delta$, then (14) holds Proving the theorem thus amounts to showing that the coefficients in the expansion of $Y(x_0, \varepsilon)$ in powers of $\varepsilon$ are polynomials in $f_k$ and its derivatives up to order $k - 1$. We fix $\varepsilon < \bar{\varepsilon}$ so that $\Delta$ and $Y$ are well-defined. As the first step in induction we show that $\Delta_1$ and $Y_1$ in the expansions

$$\Delta(x_0) = \Delta_1(x_0)\varepsilon + o(\varepsilon),$$

$$Y(x_0) = Y_1(x_0)\varepsilon + o(\varepsilon)$$

are polynomials of degree 1 (i.e. linear) in $f_k$, where $0 \leq k \leq q - 1$. Indeed, by Lemma 1,

$$qS(x_0, Y, \varepsilon, \Delta) = \sum_{m \geq 1} qS_m(x_0, Y, \varepsilon, \Delta),$$
where, by (12), the linear term is

\[ qS_1(x_0, Y, \varepsilon, \Delta) = -q\Delta - \varepsilon q\overline{f}(x_0) \quad \text{with} \quad \overline{f}(x_0) = \frac{1}{q} \sum_{k=0}^{q-1} f_k, \]

while each higher-degree term \( qS_m \) \((m \geq 2)\) is a homogeneous polynomial of degree \( m \) in the items from the list \( \{ Y - \Delta - \varepsilon f_0, -\Delta - \varepsilon f_k, \varepsilon f_k^{(l)} \} \) with \( 0 \leq k \leq q - 1, \ 0 \leq l \leq m - 1 \) and thus is of order \( \mathcal{O}(\varepsilon^2) \) since \( \Delta, Y \sim \mathcal{O}(\varepsilon) \), so that

\[ qS(x_0, Y, \varepsilon, \Delta) = -q\Delta - \varepsilon q\overline{f}(x_0) + o(\varepsilon). \]

Since the above expression vanishes by the definition of \( Y, \Delta \), we conclude that

\[ \Delta = -\varepsilon \overline{f}(x_0) + o(\varepsilon), \quad (15) \]

so that the leading coefficient of \( \varepsilon \) is

\[ \Delta_1(x_0) = -\overline{f}(x_0). \]

Similarly, by Lemma 1, we have

\[ qR(x_0, Y, \varepsilon, \Delta) = \sum_{m \geq 1} qR_m(x_0, Y, \varepsilon, \Delta), \]

where

\[ qR_1(x_0, Y, \varepsilon, \Delta) \overset{(12)}{=} qY - \frac{q(q + 1)}{2} \Delta - \varepsilon \sum_{k=0}^{q-1} (q - k) f_k, \]

while each higher-degree term \( qR_m \) \((m \geq 2)\) is a degree-\( m \) homogeneous polynomial in the items from the list \( \{ Y - \Delta - \varepsilon f_0, -\Delta - \varepsilon f_k, \varepsilon f_k^{(l)} \} \), where \( 0 \leq k \leq q - 1, \ 0 \leq l \leq m - 1 \); and since \( Y, \Delta \sim \mathcal{O}(\varepsilon) \) all \( qR_m \) with \( m \geq 2 \) are at most \( \mathcal{O}(\varepsilon^2) \), so that

\[ qR(x_0, Y, \varepsilon, \Delta) = qY - \frac{q(q + 1)}{2} \Delta - \varepsilon \sum_{k=0}^{q-1} (q - k) f_k + o(\varepsilon). \]

Substituting into this we obtain that \( Y(x_0, \varepsilon) = Y_1(x_0)\varepsilon + o(\varepsilon) \), and (15) results in

\[ Y_1(x_0) = -\frac{q + 1}{2} \overline{f}(x_0) + \overline{f}(x_0), \]

where \( \overline{f}(x_0) = \frac{1}{q} \sum_{k=0}^{q-1} (q - k) f_k. \)

This completes the first step of induction. To carry out the \( n \)th inductive step, let \( n > 1 \) and assume that in the expansion

\[ \Delta(x_0) = \Delta_1(x_0)\varepsilon + \cdots + \Delta_n(x_0)\varepsilon^n + \cdots, \]
\[ Y(x_0) = Y_1(x_0)\varepsilon + \cdots + Y_n(x_0)\varepsilon^n + \cdots, \tag{16} \]
each $\Delta_m(x_0)$ and $Y_m(x_0)$ with $m \leq n$ is a polynomial of degree $m$ in $f_k^{(l)}$ with $0 \leq k \leq q - 1$, $0 \leq l \leq m - 1$. Our goal is show that then the coefficients $\Delta_{n+1}$ and $Y_{n+1}$ are polynomials of degree $n + 1$ in $f_k^{(l)}$ with $0 \leq k \leq q - 1$, $0 \leq l \leq n$.

Just as in the first step, we obtain $\Delta_{n+1}$ by extracting the coefficient of $\epsilon^{n+1}$ in $qS$:

$$0 = qS(x_0, Y, \epsilon, \Delta) = qS_1(x_0, Y, \epsilon, \Delta) + \sum_{m \geq 2} qS_m(x_0, Y, \epsilon, \Delta) + \sum_{m > n+1} qS_m(x_0, Y, \epsilon, \Delta).$$

For $m > n + 1$, $qS_m(x_0, Y, \epsilon, \Delta) \sim o(\epsilon^{n+1})$, and hence the last sum does not contribute to the coefficient of $\epsilon^{n+1}$. On the other hand, since

$$qS_1(x_0, Y, \epsilon, \Delta) = -q\Delta - \epsilon q\bar{f}(x_0),$$

$qS_1$ contributes $-q\Delta_{n+1}(x_0)\epsilon^{n+1}$, a constant multiple of $\Delta_{n+1}$. It thus suffices to show that the terms in the middle sum are polynomials of degree up to $n + 1$ in terms of $f_k^{(l)}$ with $0 \leq k \leq q - 1$, $0 \leq l \leq n$.

For each $m \in [2, n + 1]$, $qS_m(x_0, Y, \epsilon, \Delta)$ is a degree-$m$ homogeneous polynomial in the items from the list \(\{ Y - \Delta - \epsilon f_0, -\Delta - \epsilon f_k, \epsilon f_k^{(l)} \}\) with $0 \leq k \leq q - 1$, $0 \leq l \leq m - 1$. Thus by the inductive assumption, the coefficient of $\epsilon^{n+1}$ in $qS_m$ ($2 \leq m \leq n + 1$) is a linear combination of the terms

$$(Y_i \epsilon^i)^{m_i} (\Delta_j \epsilon^j)^{m_j} (f_k^{(l)})^{m_l},$$

with $im_i + jm_j + ms = n + 1$, $0 \leq l \leq m - 1$.

Since, by Lemma 1, each higher-degree term $qS_m$ ($m \geq 2$) has at least one derivative of $g$, it follows that $m_s \geq 1$ and consequently $i, j \leq n$, so that the inductive assumption applies to $Y_i$ and $\Delta_j$ above. Thus $Y_i$ is a polynomial of degree $i$ in $f_k^{(l)}$ with $0 \leq k \leq q - 1$ with $0 \leq l \leq i - 1$, and similarly, $\Delta_j$ is a polynomial of degree $j$ with $0 \leq k \leq q - 1$, $0 \leq l \leq j - 1$. Therefore $(Y_i)^{m_i} (\Delta_j)^{m_j} (f_k^{(l)})^{m_l}$ is a polynomial of degree $n + 1$ in $f_k^{(l)}$ with $0 \leq k \leq q - 1$, $0 \leq l \leq n$. This completes the inductive step for $\Delta_{n+1}$. The step for $Y_{n+1}$ is carried out in the same way, and we have

$$0 = qR(x_0, Y, \epsilon, \Delta) = qR_1(x_0, Y, \epsilon, \Delta) + \sum_{m \geq 2} qR_m(x_0, Y, \epsilon, \Delta) + \sum_{m > n+1} qR_m(x_0, Y, \epsilon, \Delta).$$

Just as before, the last sum does not contribute to the coefficient of $\epsilon^{n+1}$. The first term

$$qR_1(x_0, Y, \epsilon, \Delta) = qY - \frac{q(q + 1)}{2} \Delta - \epsilon q\bar{f}(x_0).$$
contributes
\[ (qY_{n+1} - \frac{q(q + 1)}{2} \Delta_{n+1}) \varepsilon^{n+1}. \] (17)

The coefficients of $\varepsilon^{n+1}$ in the middle sum are polynomials of degree at most $n + 1$ in terms of $f$, its shifts and its derivatives up to order $n$, precisely as we proved before when treating $\Delta_{n+1}$. This shows that the coefficient in (17) is a polynomial of degree at most $n + 1$ in terms of $f$, its shifts and its derivatives up to order $n$. Since the same is true for $\Delta_{n+1}$, this holds for $Y_{n+1}$ as well, thus completing the induction step and the proof of Theorem 2. \(\Box\)

6. The Periodicity Lemma

In this section we show that the leading $x$-dependent coefficient in the $\varepsilon$-expansion of $\Delta(x, \varepsilon)$ is periodic of period $\mu$. This fact plays a key role in the proof of Theorem 1. Before proving this periodicity we show that this leading coefficient is also the leading term up to a constant factor in $qR$ and $qS$ as well, where is another crucial fact.

Let $r \geq 1$ be the smallest power of $\varepsilon$ where $x$ first appears in the coefficient of the expansion of $\Delta$ in powers of $\varepsilon$ so that
\[ \Delta(x_0, \varepsilon) = A(\varepsilon) + \Delta_r(x_0)\varepsilon^r + o(\varepsilon^r), \] (18)
where $A(\varepsilon)$ is a polynomial in $\varepsilon$ of degree at most $r - 1$ with constant coefficients. We claim that replacing $\Delta$ with its constant part $A(\varepsilon)$ in $qR(x_0, Y, \varepsilon, \Delta) = 0$ and $qS(x_0, Y, \varepsilon, \Delta) = 0$ changes these from 0 by the amount proportional to $\Delta_r(x_0)$ in the leading order:
\[ qR(x_0, Y, \varepsilon, A(\varepsilon)) = \frac{q(q + 1)}{2} \Delta_r(x_0)\varepsilon^r + o(\varepsilon^r) \] (19)
and
\[ qS(x_0, Y, \varepsilon, A(\varepsilon)) = q(\Delta_r(x_0))\varepsilon^r + o(\varepsilon^r). \] (20)
Here $Y = Y(x_0, \varepsilon)$. Indeed, by Lemma 1, we have
\[ qR(x_0, Y, \varepsilon, \Delta) - qR(x_0, Y, \varepsilon, A(\varepsilon)) = \sum_{m \geq 1} qR_m(x_0, Y, \varepsilon, \Delta) - qR_m(x_0, Y, \varepsilon, A(\varepsilon)). \]

Starting with $m = 1$, the terms $qR_1$ and $qS_1$ are linear in $\delta$ with constant coefficients, according to (12) More precisely,
\[ qR_1(x_0, Y, \varepsilon, \Delta) - qR_1(x_0, Y, \varepsilon, A(\varepsilon)) = q(Y - \Delta - \varepsilon f_0) + \sum_{k=1}^{q-1} (q - k)(-\Delta - \varepsilon f_k) \]
\[- q(y_0 - A(\epsilon) - \epsilon \bar{f}_0) - \sum_{k=1}^{q-1} (q - k)(-A(\epsilon) - \epsilon \bar{f}_k) \]
\[= -q \Delta_r(x_0) \epsilon^r - \sum_{k=1}^{q-1} (q - k) \Delta_r(x_0) \epsilon^r + o(\epsilon^r) \]
\[= - \frac{q(q + 1)}{2} \Delta_r(x_0) \epsilon^r + o(\epsilon^r). \]

To complete the proof of (19) it suffices to show that
\[q R_m(x_0, Y, \epsilon, \Delta) - q R_m(x_0, Y, \epsilon, A(\epsilon)) = \mathcal{O}(\epsilon^{r+1}) \text{ for } m \geq 2. \tag{21}\]

By Lemma 1, \(q R_m(x_0, Y, \epsilon, \delta)\) is a homogeneous polynomial of degree \(m\) in the items from the list \(\{Y + g_0, g_1, \ldots, g_{q-1}, g^{(l)}_k\}\) with \(1 \leq l \leq m-1, 0 \leq k \leq q-1\) and it contains at least one derivative \(g^{(l)}_k\) for some \(1 \leq l \leq m-1, 0 \leq k \leq q-1,\) thus contributing an extra factor of \(\epsilon\). Since \(g(x_0; \epsilon, \delta) = -\delta \cdot \epsilon \bar{f}(x_0)\), replacing \(\delta = \Delta = A + \Delta r \epsilon^r + o(\epsilon^r)\) by \(A(\epsilon)\) changes \(q R_m\) by \(\mathcal{O}(\epsilon^r) \cdot \epsilon = \mathcal{O}(\epsilon^{r+1})\), thus completing the proof of (19). The proof of (20) is identical and therefore omitted.

**Lemma 2.** (Periodicity) For all sufficiently small \(\epsilon\) the leading \(x\)-dependent coefficient \(\Delta_r\) in the expansion (18) of \(\Delta\) is periodic in \(\mu\), and for any \(x\) we have
\[\Delta_r(x + \mu) = \Delta_r(x). \]

**Proof.** We fix an initial point \((x_0, y_0 = Y(x_0, \epsilon))\) and set \(\delta = \Delta(x_0, \epsilon)\) in the cylinder map (8) (and consequently \(g(x_0; \epsilon, \delta) = -\Delta(x_0) - \epsilon \bar{f}(x)\)).

For future use we observe (dropping \(\epsilon\) from the notation for the sake of brevity) that
\[Y(x_1) = y_1, \quad \Delta(x_1) = \Delta(x_0) \tag{22}\]
for all sufficiently small \(\epsilon\). Indeed, the orbit \((x_0, y_0 = Y(x_0)), (x_1, y_1), (x_2, y_2), \ldots\) is \(q\)-periodic under the map with \(\delta = \Delta(x_0, \epsilon)\) by the definition of \(Y\) and \(\Delta\); thus \(q R\) and \(q S\) vanish at any point of this orbit, and, in particular, at \((x_1, y_1),\)
\[q R(x_1, y_1, \epsilon, \Delta(x_0)) = q S(x_1, y_1, \epsilon, \Delta(x_0)) = 0. \tag{23}\]

On the other hand, by the definition of \(Y\) and \(\Delta,\)
\[q R(x_1, Y(x_1), \epsilon, \Delta(x_1)) = q S(x_1, Y(x_1), \epsilon, \Delta(x_1)) = 0. \tag{24} \]

Provided that the conditions of the implicit function theorem in Section 5 are satisfied, the solution is unique, and thus comparison of (23) and (24) implies (22).

The conditions of the implicit function theorem are satisfied if \(\epsilon\) is restricted to be sufficiently small, and more precisely, that \(|y_1| < \tilde{\epsilon}\). To thus end we note that
\[|y_1| = |Y(x_0) - \Delta(x_0) - \epsilon \bar{f}(x_0)| \leq c \epsilon, \]
where \(c\) is a constant depending only on \(q\) and on \(\max |f|\). It thus suffices to restrict \(\epsilon\) to \(\epsilon < \tilde{\epsilon} := \min(\tilde{\epsilon}/c, \bar{\epsilon}),\) which we do from now on.
We now proceed with the rest of the proof. Recalling that \( y_0 = Y(x_0) \) we have

\[
qS(x, Y(x), \varepsilon, A(\varepsilon)) \bigg|_{x=x_0}^{x=x_1} = q(\Delta_r(x_1) - \Delta_r(x_0))\varepsilon' + o(\varepsilon'). \tag{25}
\]

Since \( x_1 = x_0 + \mu + y_0 - \Delta(x_0) - \varepsilon f(x_0) = x_0 + \mu + O(\varepsilon) \), this implies

\[
qS(x_1, Y(x_1), \varepsilon, A(\varepsilon)) - qS(x_0, y_0, \varepsilon, A(\varepsilon)) = q(\Delta_r(x_0 + \mu) - \Delta_r(x_0))\varepsilon' + o(\varepsilon'). \tag{26}
\]

We now show that the left-hand side is \( o(\varepsilon') \) (thus completing the proof of the lemma). Consider the orbit \((\tilde{x}_k, \tilde{y}_k)\) of the same initial point \((x_0, y_0)\) but under the map with \( \delta = A(\varepsilon) \) (instead of \( \delta = \Delta(x_0) \)). We will show that

\[
qS(\tilde{x}_1, \tilde{y}_1, \varepsilon, A(\varepsilon)) - qS(x_0, y_0, \varepsilon, A(\varepsilon)) = o(\varepsilon') \tag{27}
\]

and

\[
qS(x_1, Y(x_1), \varepsilon, A(\varepsilon)) - qS(\tilde{x}_1, \tilde{y}_1, \varepsilon, A(\varepsilon)) = o(\varepsilon'), \tag{28}
\]

thus implying that the left-hand side of (25) is \( o(\varepsilon') \). **Proof of (27).** By (10) we have

\[
qS(\tilde{x}_1, \tilde{y}_1, \varepsilon, A(\varepsilon)) - qS(x_0, y_0, \varepsilon, A(\varepsilon)) = -qA(\varepsilon) - \varepsilon \sum_{k=0}^{q-1} f(\tilde{x}_k) + qA(\varepsilon) + \varepsilon \sum_{k=1}^{q} f(\tilde{x}_k)
\]

\[
= -\varepsilon(f(\tilde{x}_q) - f(x_0)).
\]

However

\[
\tilde{x}_q \equiv x_0 + 2p\pi + qR(x_0, y_0, \varepsilon, A(\varepsilon)),
\]

which together with (19), shows that the last difference in parentheses is \( O(\varepsilon') \), thus implying 27. **Proof of (28).** By Lemma 1,

\[
qS(\tilde{x}_1, \tilde{y}_1, \varepsilon, A(\varepsilon)) = \sum_{m \geq 1} qS_m(\tilde{x}_1, \tilde{y}_1, \varepsilon, A(\varepsilon))
\]

and

\[
qS(x_1, y_1, \varepsilon, A(\varepsilon)) = \sum_{m \geq 1} qS_m(x_1, y_1, \varepsilon, A(\varepsilon)),
\]

where \( qS_m(x, y, \varepsilon, A(\varepsilon)) \) is a degree-\( m \) homogeneous polynomial in the items from the list \( \{y - A(\varepsilon) - \varepsilon f(x), -A(\varepsilon) - \varepsilon f(x + k\mu), \varepsilon f^{(l)}(x + k\mu)\} \) with \( 0 \leq k \leq q - 1, 1 \leq l \leq m - 1 \). We will show that the corresponding terms in each sum differ by \( o(\varepsilon') \). We have

\[
x_1 = x_0 + y_0 + \mu - \Delta(x_0) - \varepsilon f(x_0)
\]

\[
y_1 = y_0 - \Delta(x_0) - \varepsilon f(x_0);
\]
and

\[ \tilde{x}_1 = x_0 + y_0 + \mu - A(\varepsilon) - \varepsilon f(x_0) \]
\[ \tilde{y}_1 = y_0 - A(\varepsilon) - \varepsilon f(x_0). \]

Since \( \Delta(x_0) = A(\varepsilon) + \Delta_r(x_0)\varepsilon^r + o(\varepsilon^r) \), this implies

\[ \tilde{x}_1 - x_1 = \Delta_r(x_0)\varepsilon^r + o(\varepsilon^r) \]
\[ \tilde{y}_1 - y_1 = \Delta_r(x_0)\varepsilon^r + o(\varepsilon^r) \]

\[ f^{(l)}(\tilde{x}_1 + k\mu) - f^{(l)}(x_1 + k\mu) = f^{(l+1)}(x_1 + k\mu)\Delta_r(x_0)\varepsilon^r + o(\varepsilon^r). \]  \hspace{1cm} (29)

This \( O(\varepsilon^r) \) difference in \( x \) and \( y \) results in the \( o(\varepsilon^r) \) difference in the terms \( qS_m \) as we now show. Starting with \( m = 1 \) we have

\[ qS_1(\tilde{x}_1, \tilde{y}_1, \varepsilon, A(\varepsilon)) - qS_1(x_1, y_1, \varepsilon, A(\varepsilon)) \]
\[ \overset{(12)}{=} -\varepsilon \sum_{k=0}^{q-1} (f(\tilde{x}_1 + k\mu) - f(x_1 + k\mu)) \]
\[ = -\varepsilon \sum_{k=0}^{q-1} f'(x_1 + k\mu)\Delta_r(x_0)\varepsilon^r + o(\varepsilon^r) = o(\varepsilon^r). \]

For \( m \geq 2 \), according to Lemma 1, each term in \( qS_m \) contains at least one derivative of \( g(x; \varepsilon, A(\varepsilon)) = -A(\varepsilon) - \varepsilon f(x) \) for both \( qS_m(x_1, y_1, \varepsilon, A(\varepsilon)) \) and \( qS_m(\tilde{x}_1, \tilde{y}_1, \varepsilon, A(\varepsilon)) \), which contributes a factor of \( \varepsilon \). This, together with (29), implies

\[ qS_m(\tilde{x}_1, \tilde{y}_1, \varepsilon, A(\varepsilon)) - qS_m(x_1, y_1, \varepsilon, A(\varepsilon)) = o(\varepsilon^r). \]

We showed that \( qS(x_1, y_1, \varepsilon, A(\varepsilon)) - qS(\tilde{x}_1, \tilde{y}_1, \varepsilon, A(\varepsilon)) = o(\varepsilon^r) \). Since \( y_1 = Y(x_1) \) this proves (28), thus completing the proof of the lemma. \( \square \)

7. End of Proof of the Main Theorem

In this last section we complete the proof of Theorem 1 using the results of the previous sections. The main idea, similar to [1], is to observe that if \( f \) is a trigonometric polynomial of degree \( d \) then \( \Delta_r \) is a trigonometric polynomial of degree \( rd \). Since \( \Delta_r \) is nonconstant (by the definition) and periodic of period \( 2\pi p/q \), one must have \( rd > q \), so that \( r > \lfloor q/d \rfloor \). This would complete the proof of the theorem, since \( \Delta(x, \varepsilon) = A(\varepsilon) + \Delta_r(x)\varepsilon^r + o(\varepsilon^r) \) implies that the range of \( \delta \) for which \( p/q \)-periodic points exist is at most \( O(\varepsilon^r) \) with \( r > \lfloor q/d \rfloor \).

It remains therefore to show that \( \Delta_r \) is indeed a trigonometric polynomial of degree at most \( rd \). According to (20),

\[ \varepsilon^r \Delta_r(x) = q^{-1} \sum_{m \geq 1} qS_m(x, Y(x, \varepsilon), \varepsilon, A(\varepsilon)) + o(\varepsilon^r), \]
and we must show that the coefficient of $\varepsilon^r$ in the above sum is a trigonometric polynomial of degree at most $rd$. According to Lemma 1, $q S_m(x, Y(x), \varepsilon, A(\varepsilon))$ is homogeneous polynomial of degree $m$ in the items from the list

\begin{equation}
\{ Y(x) - A(\varepsilon) - \varepsilon f_0, -A(\varepsilon) - \varepsilon f_k, \varepsilon f_k^{(l)} \},
\end{equation}

with $0 \leq k \leq q - 1, 1 \leq l \leq m - 1$, and thus only finitely many terms - namely the ones with $m \leq r$ - contribute to the coefficient of $\varepsilon^r$. According to Theorem 2 the coefficients $Y_n$ in the expansion

$$Y(x) = Y_1(x)\varepsilon + Y_2(x)\varepsilon^2 + \cdots$$

are $n$th degree polynomials in $f$, its shifts by $\mu$, and its derivatives. The key point here is that the power of $\varepsilon$ in $Y(x)$ is also the degree of the polynomial $Y_k$ in $f$, its derivatives and shifts. In addition, $\varepsilon$ enters with power 1 in the list (30) as a factor of every $f$ and its derivatives and shifts (while $A(\varepsilon)$ has constant coefficients). This shows that the coefficient of $\varepsilon^r$ is a polynomial of degree at most $r$ in $f$, its derivatives and shifts. Moreover, this coefficient is a finite sum, since only finitely many terms ($m \leq r$) contribute to it. Finally, since $f$ is a trigonometric polynomial of degree $d$, it follows that the coefficient of $\varepsilon^r$ is a trigonometric polynomial of degree at most $rd$, as claimed.

This completes the proof of Theorem 1.

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Mark Levi · Jing Zhou
Department of Mathematics,
Penn State University,
State College
PA
USA.
e-mail: mxl48@psu.edu
e-mail: jingzhou@psu.edu

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