Spontaneous Symmetry Breaking in Fermion-Gauge Systems:  
A Non Standard Approach

Vicente Azcoiti $^a$, Victor Lalena $^a$ and Xiang-Qian Luo $^b$

$^a$Departamento de Física Teórica, Facultad de Ciencias, Universidad de Zaragoza, 50009 Zaragoza, Spain

$^b$HLRZ, c/o KFA, D-52425 Jülich, Germany

ABSTRACT

We propose a new method for the study of the chiral properties of the ground state in QFT's based on the computation of the probability distribution function of the chiral condensate. It can be applied directly in the chiral limit and therefore no mass extrapolations are needed. Furthermore this approach allows to write up equations relating the chiral condensate with quantities computable by standard numerical methods, the functional form of these relations depending on the broken symmetry group. As a check, we report some results for the compact Schwinger model.
The study of the chiral properties of the ground state in Quantum Field Theories (QFT’s) is one of the main points of attention in recent developments of Lattice Gauge Theories (LGT’s). In \(QCD\), the gauge theory for the strong interacting sector of the Standard Model, the low energy physics could not be understood without a detailed analysis of the chiral properties of the vacuum. Furthermore there are strong evidences from numerical simulations of \(QCD\) at finite temperature, suggesting that a chiral transition appears at some critical temperature \(T_c\) separating the quark-gluon plasma phase from the hadronic phase.

Concerning the abelian model, recent numerical studies in the noncompact formulation \[4\] show the existence of a strongly coupled \(QED\) and open the possibility to get a non-trivial continuum limit for a non asymptotically free gauge theory in four dimensions. The strongly coupled phase of this model is characterized by a vacuum state not invariant under chiral transformations and therefore a mechanism for dynamical mass generation appears.

Last, the analysis of the chiral properties of the vacuum in 2+1 QFT’s and of their dependence on the flavor number has also become of great interest because of its possible relevance in the high \(T_c\) superconductivity phenomenon.

In this paper we propose a new method for the study of the chiral properties of the ground state in QFT’s which is based on the computation of the probability distribution function (p.d.f.) of the chiral condensate in the chiral limit. This is an standard procedure when analysing spontaneous symmetry breaking in spin systems or in QFT’s with bosonic degrees of freedom, since this kind of degrees of freedom can be simulated directly in a computer. However, in in the numerical simulations of a LGT with dynamical fermions, the Grassmann fields must be integrated analytically for obvious reasons. Then, even if the ground state is degenerate, in the analytical procedure we integrate over all possible vacuum states, the chiral order parameter being always zero for massless fermions. We will show here how despite of the fact that Grassmann variables cannot be simulated in a computer, an analysis of spontaneous symmetry breaking without a symmetry breaking external field, which is standard in the case of spin systems, can also be done in QFT’s with fermion degrees of freedom. Previous attempts to analyse the vacuum structure in fermion-gauge systems \[3\] were based on several approximations. We want to remark that our analysis is completely free from approximations.

The main advantage of this method, when compared with standard simulations, is that we can work directly in the chiral limit and therefore no mass extrapolations are needed. Furthermore this approach allows to write up equations relating the chiral condensate with quantities computable by stan-
standard numerical methods, the functional form of these relations depending on
the nature of the broken symmetry group.

Another interesting field of application of $p.d.f.$ of order parameters is
the quantitative description of interacting social groups in stochastic models
for the formation of Public Opinion. Previous work in this field [3] analyses
the time dependent $p.d.f.$ of public opinion for the simplest case of two kind
of opinions $(+, -)$, modeling it with an Ising ferromagnet, the dynamics of
which takes into account the influence on an individual of its neighbors.
Generalization of this work to the case of a continuous opinion (continuous
symmetry) could be another possible application of the formalism developed
here. The application of fermionic models to Public Opinion formation is
interesting not only for the continuous character of the symmetry but also
because of the long range interactions induced by the fermion dynamics,
which would give a more precise description of the high degree of long distance
communication in present world.

1. Theoretical Grounds

Our starting point is a $QFT$ describing a gauge field coupled to a fermion
matter field and regularized by means of a space-time lattice. Next suppose
the ground or equilibrium state of this model is degenerate, and be $\alpha$ the
index which characterize all possible vacuum states. The ensemble of all
equilibrium states is a partition of the Gibbs state and the probability $w_\alpha$
to get the vacuum state $\alpha$ when choosing randomly an equilibrium state
depends on the total free energy of the $\alpha$ state [4].

Since we are interested in the analysis of the chiral properties of the
ground state, let us choose as order parameter the chiral condensate $\bar{\psi}\psi$
and characterize each vacuum state $\alpha$ by the expectation value $c_\alpha$ of the order
parameter in the $\alpha$ state, i.e.

$$c_\alpha = \frac{1}{N} \sum_x <\bar{\psi}(x)\psi(x)>_\alpha$$

(1)

where $N$ is the total number of lattice sites and the sum is over all lattice
points. The $p.d.f.$ $P(c)$ of the chiral order parameter $c$ will be given by

$$P(c) = \sum_\alpha w_\alpha \delta(c - c_\alpha).$$

(2)
The function $P(c)$ tells us what is the probability that choosing randomly a vacuum state, we get the value $c$ for the chiral order parameter. If the vacuum state is invariant under chiral transformations, i.e., if it is unique as concerning the chiral symmetry, $P(c)$ will be a single $\delta$ function $\delta(c)$. Otherwise $P(c)$ will be a more complex function, sum of $\delta$ functions in the case of a discrete symmetry group or a continuous function in the other cases.

The next step now is to relate $P(c)$ with magnitudes which can be computed by numerical simulations. To this end, let us write

$$P(c) = \lim_{N\to\infty} <\delta\left(\frac{1}{N} \sum_x \bar{\psi}(x)\psi(x) - c\right)>$$

where the expectation value in (3) is computed in the Gibbs state and the integration measure is that associated to the partition function

$$Z = \int d\bar{\psi}d\psi dU e^{-S_G(U) + \bar{\psi}\Delta\psi}.$$  

$S_G$ in (4) is the pure gauge action and $\Delta$ the fermionic matrix.

To check that equation (3) gives us indeed the p.d.f. of the chiral condensate, it is enough to verify that all the moments of the p.d.f. defined in (2) agree with those of (3). The verification for the first moment is trivial whereas for the higher moments it is enough to take into account that in the thermodynamical limit intensive quantities do not fluctuate in any equilibrium state $\alpha$, i.e.

$$\frac{1}{N^p} <\sum_{x_1,x_2,...,x_p} \bar{\psi}(x_1)\psi(x_1)\bar{\psi}(x_2)\psi(x_2)......\bar{\psi}(x_p)\psi(x_p)>_\alpha$$

$$= \frac{1}{N^p} \sum_{x_1,x_2,...,x_p} <\bar{\psi}(x_1)\psi(x_1)>_\alpha <\bar{\psi}(x_2)\psi(x_2)>_\alpha ...... <\bar{\psi}(x_p)\psi(x_p)>_\alpha.$$  

Expression (3) is not suitable for numerical computation. However we can define its Fourier transformed $P(q)$

$$P(q) = \int e^{iqc} P(c) dc$$

which, as will be shown, can be numerically computed.
Since we are interested in the analysis of the chiral symmetry on the lattice, we will use the staggered fermions regularization. The fermion matrix $\Delta$ can be written as

$$\Delta = m + i\Lambda$$

where $m$ is the fermion mass and $\Lambda$ a hermitian matrix which depends on the gauge field configuration. The eigenvalues of $\Lambda$ are real and symmetric. Taking into account all these properties of $\Delta$, the following expression for $P(q)$ can be derived from (6)

$$P(q) = \langle \prod_j (1 - q^2 - i2mqN^2/(m^2 + \lambda_j^2)) \rangle$$

and in the chiral limit

$$P(q) = \langle \prod_j (1 - q^2/N^2\lambda_j^2) \rangle$$

where the product in (8) and (9) runs over all positive eigenvalues $\lambda_j$ and the mean values are computed with the probability distribution function of the effective gauge theory obtained after integrating out the fermion fields. In order to get meaningful results for (9), special care must be taken with the boundary conditions for the fermion field to avoid zero modes.

The function $P(q)$ can be computed numerically and then, by inverse Fourier transform we get $P(c)$. We can go deeper in the investigation of the form of the p.d.f. $P(c)$. In the lattice regularized action of a gauge theory with Kogut-Susskind fermions, only a $U(1)$ subgroup of the continuous chiral symmetry is preserved. Then if this continuous residual symmetry is spontaneously broken, we will get a continuum of equilibrium states characterized by an angle $\alpha$ with $\alpha$ taking values between $-\pi$ and $\pi$. The v.e.v. of the chiral condensate at each vacuum will be given by $c_0 \cos(2\alpha)$, $c_0$ being the value corresponding to the $\alpha$-vacuum selected when switching-on an external “magnetic” field. Therefore, the function $P(c)$ can be computed as

$$P(c) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\alpha \delta(c - c_0 \cos(2\alpha))$$
which gives for $P(c)$ the value $1/((\pi (c_0^2 - c^2)^{1/2})$ for $-c_0 \leq c \leq c_0$, $P(c) = 0$ otherwise (see Fig. 1). Its Fourier transformed

$$P(q) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta e^{iqc_0 \cos \theta}$$

(11)

is the well known zeroth order Bessel function of the first kind $J_0(qc_0)$.

Several relations between the chiral condensate and the eigenvalues of the fermionic matrix can be derived from (11). For example, by doing the second derivative of the function $P(q)$ we get the second moment of the distribution $P(c)$ and then the following relation holds

$$c_0^2 = \langle \bar{\psi} \psi \rangle^2 = \langle \frac{4}{N^2} \sum_j \frac{1}{\lambda_j^2} \rangle.$$  

(12)

In the symmetric phase, the right hand side of (12) is the longitudinal susceptibility normalized by the lattice volume $N$. Furthermore the function $P(q)$ is proportional to the partition function $Z$ evaluated in the imaginary axis of the complex mass plane and therefore a relation between the zeroes of the partition function in the complex mass plane and the value of the chiral condensate follows from this relation.

Using (11) the following relations between the chiral condensate $c_0$, the zeroes $q_i$ of $P(q)$ and the zeroes $a_i$ of the zeroth order Bessel function of first kind can be derived,

$$c_0(i) = \frac{a_i}{q_i}.$$  

(13)

In the thermodynamical limit the value of $c_0$ will be independent of the order $i$ of the zero in (13). At finite volume $c_0$ depends on $i$. However we have observed that a plateau appears in the plot of $c_0$ as a function of $i$ when chiral symmetry is spontaneously broken. The extent of this plateau increases with the lattice size becoming eventually infinite for infinite lattice volumes. These things are very well illustrated in Fig. 2 where we plot some preliminary results on the lattice chiral condensate $c_0(i)$ against $i$ for the one-flavor compact Schwinger model, obtained from MFA simulations [3]. The continuous line in this figure stands for the continuum analytical result (in units of charge) times $\beta^{-1/2}$. Of course the Schwinger model is not very interesting from a physical point of view but, as well known, it is a very
good laboratory to check new proposals for two reasons: exact analytical results are available and it shares many interesting properties with other more relevant physical models. This is the reason why we believe the results reported in Fig. 2 are very encouraging.

2. **Discrete versus continuous p.d.f.**

The analysis of p.d.f. of order parameters in Statistical Mechanics is not new. It has been developed for ferromagnetic Ising-like systems \[4\] and also for spin-glasses \[4\], the last with a much more complicated vacuum structure than magnets. However, this is the first time to our knowledge that this kind of analysis is applied, without any approximation, to QFT’s with dynamical fermion fields.

There are several important differences between p.d.f.’s in spin systems and fermionic systems and we would like to point out them here. First and from a practical point of view, the p.d.f. of the density of magnetization in a spin system can be directly measured by computer simulations whereas in a fermionic system this is not possible since Grassmann variables cannot be simulated in a computer. This is the reason why in our case we work with the Fourier transform \(P(q)\) \[9\] rather than with \(P(c)\) \[3\]. Furthermore and as a consequence of the anticommuting character of fermionic fields, all the moments of the p.d.f. \(P(c)\) of order larger than the lattice volume \(N\) vanish since they are correlation functions of anticommuting variables with some repeated index. In fact the function \(P(q)\) given by equation \[9\] is a polynomial of degree the lattice volume \(N\) and therefore its inverse Fourier transform does not exists as a regular function if \(P(q)\) is defined in an unbounded momentum space.

This peculiarity of the p.d.f. for fermionic systems complicates the finite size scaling analysis of it since in order to do such a kind of analysis, we need to put an arbitrary cut-off in momentum space which could induce extra finite-volume effects. However, we can by-pass this difficulty by applying the finite size scaling hypothesis, instead to the p.d.f. \(p(c)\) as in Binder’s work on Ising systems \[6\], to its Fourier transform \(P(q)\). Then the finite size scaling hypothesis for \(P(q)\) reads

\[
P_L(q) = f(qL^{-\beta}, \frac{\xi}{L}).
\] (14)

where \(L\) is the linear dimension of the lattice and \(f\) is a universal scaling function. Equation (14) should work for \(L\) large and near the critical point.
The next step now should be to estimate the explicit form of the universal function \( f \) in both the symmetric and broken phases. Since this kind of analysis is too long to be reported here, we will stress only some important differences which appear when comparing with Binder’s analysis of Ising-like systems. The physical origin of the different behavior of the universal function \( f \) for Ising and fermion-gauge systems is the existence of a continuum of equilibrium states in the broken phase of fermion-gauge systems in contrast to the two ground states of the low temperature phase of the Ising model.

The symmetric phase of both models is characterized by a non-degenerate vacuum and Binder’s analysis \([3]\), which is based on the assumption that in this case \( P_L(c) \) (equivalently \( P_L(q) \)) is well approximated by a gaussian function centered at the origin, should work also for fermionic systems, at least for not too large values of \( q \). The situation however changes drastically in the broken phase where Binder’s analysis, based on the two gaussian approximation of the \( p.d.f. \) at finite lattice size, fails completely to describe the spontaneous breaking of a continuous symmetry. In fact the infinite volume limit of \( P_L(q) \) (see eq. (11)) is a Bessel function in contrast with the cosine function which appears in the Ising case. Taking into account this result and writing the universal function \( f \) in the broken phase as a function of the chiral condensate

\[
f(qL^{\nu}, \xi) = F(qL^{\nu}, \frac{c_0}{L}).
\]

(15)

the following expression for the universal function \( F(z, z') \) holds

\[
F(z, z') = J_0(\frac{z}{z'}) + \ldots \ldots \ldots \ldots
\]

(16)

where \( J_0 \) is the Bessel function and the dots in (16) stand for size dependent terms which vanish in the infinite volume limit. There are many ways to parameterize the first corrections in (16), the simplest one being a term proportional to \( q^2 \) as follows from the fact that the universal function \( F \) and the Bessel function \( J_0 \) differ in an analytical function of \( q^2 \), which vanish at the origin in momentum space. A detailed analysis in this direction for some specific models will be reported in a separate paper. There are however some interesting features concerning finite volume effects which can be derived from the results reported in Fig. 2, and which we would like to point out.
Our analysis of the Schwinger model shows that, as expected, finite volume effects are more or less relevant depending on the quantity computed. By looking at Fig. 2 we can understand why finite size effects are relatively large when computing moments of $P(c)$, like equation (12). In fact in this case all the zeroes of $P(q)$, or equivalently all the eigenvalues of the fermionic matrix, give contribution to the moments of $P(c)$ and our results show clearly that higher order zeroes of $P(q)$ suffer from stronger finite size effects. What we find certainly surprising in the results of Fig. 2 is the fact that an accurate estimation of the chiral condensate in the Schwinger model can be obtained from rather small lattices, by analyzing the first zeroes of $P(q)$. They show an asymptotic scaling behavior which follows from the fact that the first eigenvalues of the dominant gauge field configurations scale with the inverse lattice volume, the window for this scaling increasing with the lattice size. These results should stimulate people working in this field to apply this formalism to more interesting physical systems, like QCD.

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Figure captions

**Figure 1.** Standard form of $P(c)$ (eq. (1.11)) in the broken phase.
**Figure 2.** Chiral order parameter in the compact Schwinger model in $32^2$ and $64^2$ lattices at $\beta = 6.344$. The solid line corresponds to its continuum analytical value.
Continuum value

Figure 2

\[ C(i) \]

\[ i \]

\[ 32^2 \]

\[ 64^2 \]

Continuum value