Preferred frame parameters in the tensor-vector-scalar theory of gravity and its generalization

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The Tensor-Vector-Scalar theory of gravity, which was designed as a relativistic implementation to the modified dynamics paradigm, has fared quite well as an alternative to dark matter, on both galactic and cosmological scales. However, its performance in the solar system, as embodied in the post-Newtonian formalism, has not yet been fully investigated. Tamaki has recently attempted to calculate the preferred frame parameters for TeVeS, but ignored the cosmological value of the scalar field, thus concluding that the Newtonian potential must be static in order to be consistent with the vector equation. We show that when the cosmological value of the scalar field is taken into account, there is no constraint on the Newtonian potential; however, the cosmological value of the scalar field is tightly linked to the vector field coupling constant $K$, preventing the former from evolving as predicted by its equation of motion. We then proceed to investigate the post-Newtonian limit of a generalized version of TeVeS, with Æther type vector action, and show that its $\beta, \gamma$ and $\xi$ parameters are as in GR, while solar system constraints on the preferred frame parameters $\alpha_1$ and $\alpha_2$ can be satisfied within a modest range of small values of the scalar and vector fields coupling parameters, and for values of the cosmological scalar field consistent with evolution within the framework of existing models.

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I. INTRODUCTION

As is known, General Relativity (GR) cannot explain the dynamics of our universe on large physical scales, since the amount of visible mass clearly lies below what would be expected from the observed gravitational effects. The usual remedy is to invoke a form of matter which does not couple to light, therefore being referred to as Dark Matter (DM). However, one can also take a different point of view and modify the law of gravity itself. Such a solution to the missing mass problem was first studied in great detail by Milgrom in his MOND paradigm. The modified Newtonian dynamics (MOND) paradigm \cite{1}, proposes that Newtonian gravity progressively fails as accelerations drop below a characteristic scale $a_0 \simeq 10^{-10}m/s^2$ which is typical of galaxy outskirts. MOND assumes that for accelerations of order $a_0$ or well below it, the Newtonian relation $a = -\nabla \Phi_N$ is replaced by

$$\tilde{\mu}(|a|/a_0) \, a = -\nabla \Phi_N,$$

where the function $\tilde{\mu}(x)$ smoothly interpolates between $\tilde{\mu}(x) = x$ at $x \ll 1$ and the Newtonian expectation $\tilde{\mu}(x) = 1$ at $x \gg 1$. This relation with a suitable standard choice of $\tilde{\mu}(x)$ in the intermediate range has proved successful not only in justifying the asymptotical flatness of galaxy rotation curves where acceleration scales are much below $a_0$, but also in explaining detailed shapes of rotation curves in the inner parts in terms of the directly seen mass, and in giving a precise account of the observed Tully-Fisher law which correlates luminosity of a disk galaxy with its asymptotic rotational velocity $v_\text{vir}$. This sharp relation, while obtained naturally in the framework of MOND, requires quite a fine tuning of dark halo parameters to be explained by the dark matter paradigm.

However, MOND alone is only a phenomenological prescription that does not fulfill the usual conservation laws, nor does it make clear if the departure from Newtonian physics is in the gravity or in the inertia side of the equation $F = ma$. Moreover, it is non relativistic, and as such it does not teach us how to handle gravitational lensing or cosmology in the weak acceleration regimes. To address these issues, Bekenstein designed TeVeS \cite{3}, a covariant field theory of gravity which has MOND as its low velocity, weak acceleration limit, while its non-relativistic strong acceleration limit is Newtonian and its relativistic limit is general relativity (GR). TeVeS sports two metrics, the “physical” metric on which all matter fields propagate, and the Einstein metric which interacts with the additional fields in the theory: a timelike dynamical vector field, $A$, and a scalar field, $\phi$. The theory also involves a free function $F$, a length scale $\ell$, and two positive dimensionless constants $k$ and $K$. The scalar field in TeVeS provides the additional gravitational potential for matter, whereas the vector field provides the desired light bending properties, in a fashion similar to the constant unit vector in Sanders’ stratified theory \cite{4}.

Many aspects of TeVeS have been investigated extensively, proving the theory to be faring quite well in view of the huge challenges it was designed to meet. Bekenstein showed that TeVeS’s weak acceleration limit reproduces MOND, and that it also has a Newtonian limit \cite{3}. Skordis \cite{5} formulated the cosmological equations for

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TeVeS, for both background and linear perturbations, and later extended his investigation to a version of TeVeS where the action of the vector field is of Einstein-Æther form [6]. The CMB spectrum and the matter power spectrum \( P(k) \) were calculated by Skordis, Mota, Ferreira and Boehm [7], who showed how TeVeS can reproduce the power spectrum in a manner similar to Dark Matter. Further inquiries into TeVeS cosmology have been made by Dodolsen and Liguori [8], who showed that perturbations in the TeVeS vector field can drive structure growth, and by Bourliot et al., [9] who considered a broad family of functions that lead to modified gravity and calculated the evolution of the field variables both numerically and analytically. Giannios [10] has found exact solutions of TeVeS for spherically symmetric systems, including Schwarzschild-like black holes, and Sagi and Bekenstein [11] expanded upon his work and found charged black hole solutions. Zhao and Famaey [12] have put a variety of constraints on the TeVeS free function from galaxy dynamics. Laski, Sotani and Giannios [13] investigated neutron stars in TeVeS, using them to place a lower bound on the allowed cosmological value of the scalar field, and Sotani [14] calculated the fundamental oscillation modes of neutron stars for the theory, showing how the imprint of the scalar field could be detected in gravitational waves.

TeVeS has also been tested against a multitude of data on gravitational lensing. Chiu, Ko and Tian have examined theoretical predictions of TeVeS for amplifications and time delays in strong gravitational lensing [15], while Zhao et al. have put TeVeS predictions for image splittings and amplifications to test against a large sample of lensed quasars [16]. And Chen and Zhao have compared the statistics of strong gravitational lensing by galaxies with TeVeS [17]. This work is admirably capped by Chen, who calculated the lensing probability with image separation larger than a given value \( \Delta \theta \) in an open, TeVeS cosmology, and showed that the predicted lensing probabilities with the simple interpolating function \( x/(1 + x) \) match the observational data quite well [18]. Angus et al. have criticized the claim that the colliding clusters of galaxies “the bullet” pose a threat to gravitational lensing a la TeVeS [19].

However, it is not yet clear where the theory stands with respect to solar system constraints, which are usually embodied in tight limits on the allowed values of post-Newtonian (PN) parameters. Any general metric theory of gravity can be fully characterized by ten parameters: \( \xi, \beta, \gamma, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7 \). Five of these parameters, \( \xi, \beta, \gamma, \alpha_1, \alpha_2 \), vanish identically for any semi-conservative theory, i.e. one derived, like TeVeS, from a covariant action principle. Two others, known as the Eddington–Robertson–Schiff parameters, \( \beta \) and \( \gamma \), characterize respectively the nonlinearity and the spatial curvature produced by gravity. Of the remaining three PPN parameters, two, \( \alpha_3 \) and \( \alpha_4 \), characterize preferred frame effects, and the third, \( \xi \), also known as the Whitehead parameter, characterizes a peculiar sort of three-body interaction. TeVeS’ PPN parameters have been calculated only under simplifying assumptions, such as spherical symmetry [3] or cosmological value of the scalar field set to zero [21]. These assumptions did not enable calculation of the preferred frame parameters \( \alpha_1 \) and \( \alpha_2 \). In a theory possessing a preferred Lorentz frame, the rest frame of the timelike vector field, one expects these parameters to be different from zero, whereas solar system experiments [22] constrain their measured value to be very close to zero, specifically \( \alpha_1 \lesssim 10^{-4} \) and \( \alpha_2 \lesssim 4 \times 10^{-7} \). The question then rises, is there a region of TeVeS parameter space within which the preferred frame parameters are zero as in GR, or is TeVeS ruled out or unreasonably constrained by solar system tests?

In this work, we start with an overview of TeVeS in Section II and elaborate on the PPN parameters in Section III, giving the full set of PPN parameters for TeVeS with no prior assumptions such as spherical symmetry or zero cosmological value of the scalar field. Although we find \( \beta \) and \( \gamma \) to be unity, and \( \xi = 0 \), as in GR, we encounter a difficulty with the vector equation, and find that to PN order, it links the cosmological value of the scalar field to the coupling parameter of the vector field \( K \) in such a way that the scalar field is not allowed to evolve with cosmological expansion. This is another hint to the fact that the simple Maxwell-like action of the vector field causes dynamical problems: it has previously been shown by Seifert [23] that static spherically symmetric solutions in this theory are unstable against spherically symmetric perturbations, and by Contaldi et al. [24] that the vector field runs into caustic singularities in rather generic situations. Therefore in Section IV we present results for the PPN parameters for a more general form of TeVeS, one with a full vector action of Æther type. We show that the preferred frame parameters can be set to zero within a modest range of small coupling parameters of the scalar and vector fields, and for reasonable values of the cosmological scalar field. We conclude that for particular ranges of the coupling parameters, TeVeS with generalized vector action is indiscernible from GR in the solar system. The full details of the calculations are given in the appendices.

II. TEVES

TeVeS has MOND as its weak potential, low acceleration limit, while its weak potential, high acceleration limit is the usual Newtonian gravity. TeVeS is endowed with three dynamical gravitational fields: a scalar field \( \phi \), a timelike unit normalized vector field \( A^a \), and the Einstein metric \( g_{\alpha \beta} \) on which the gravitational fields of the theory propagate. The theory also employs a “physical” metric \( g_{\alpha \beta} \) on which gauge, spinor and Higgs fields...
propagate. It is related to $g_{\alpha\beta}$ by
\[ \tilde{g}_{\alpha\beta} = e^{-2\phi}g_{\alpha\beta} - 2A_\alpha A_\beta \sinh(2\phi). \] (2)
The index of $A_\alpha$ or of $\phi, \alpha$ is always raised with the metric $g^{\alpha\beta}$, the inverse of $g_{\alpha\beta}$.

The equations of motion for the fields in TeVeS derive from a five-term action depending on four parameters: the fundamental gravity constant $G$, two dimensionless parameters $k$ and $K$ and a fixed length scale $\ell$. The familiar Einstein-Hilbert action for the metric and the matter action for field variables collectively denoted $f$ have the form
\[ S_g = \frac{1}{16\pi G} \int g^{\alpha\beta} R_{\alpha\beta} \sqrt{-g} \, d^4x, \] (3)
\[ S_m = \int \mathcal{L} \left( \tilde{g}_{\mu\nu}, f_{\alpha\beta} \right) \sqrt{-\tilde{g}} \, d^4x. \] (4)

Next comes the vector field’s action, with $K$ a dimensionless positive parameter
\[ S_v = -\frac{K}{32\pi G} \int \left[ (g^{\alpha\beta} g^{\mu\nu} A_{[\alpha,\mu]} A_{[\beta,\nu]}) - \frac{2\lambda}{K} (g^{\mu\nu} A_\mu A_\nu + 1) \right] \sqrt{-g} \, d^4x, \] (5)

which includes a constraint that forces the vector field to be timelike (and unit normalized); $\lambda$ is the corresponding Lagrange multiplier. The presence of a nonzero $A^\alpha$ establishes a preferred Lorentz frame, thus breaking Lorentz symmetry in the gravitational sector. Finally, we have the scalar’s action ($k$ is a dimensionless positive parameter while $\ell$ is a constant with the dimensions of length, and $F$ a dimensionless free function)
\[ S_\phi = -\frac{1}{2k^2\ell^2G} \int F \left( k^2h^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta} \right) \sqrt{-g} \, d^4x, \] (6)

Above $h^{\alpha\beta} = g^{\alpha\beta} - A^\alpha A^\beta$ with $A^\alpha \equiv \tilde{g}^{\alpha\beta} A_\beta$.

Variation of the action with respect to $g^{\alpha\beta}$ yields the TeVeS Einstein equations for $g_{\alpha\beta}$
\[ G_{\alpha\beta} = 8\pi G \left( \tilde{T}_{\alpha\beta} + (1 - e^{-4\phi}) A^\mu \tilde{T}_{\mu(\alpha} A_{\beta)} + \tau_{\alpha\beta} \right) + \theta_{\alpha\beta}, \] (7)

where $v_{(\alpha} A_{\beta)} \equiv v_\alpha A_\beta + A_\alpha v_\beta$, etc. The sources here are the usual matter energy-momentum tensor $\tilde{T}_{\alpha\beta}$ (related to the variational derivative of $S_m$ with respect to $\tilde{g}^{\alpha\beta}$), as well as the energy-momentum tensors for the scalar and vector fields,
\[ \tau_{\alpha\beta} = -\frac{\mu(y)}{kG} \left( \phi_{,\alpha} \phi_{,\beta} - A^\mu \phi_{,\mu} A_{(\alpha} \phi_{,\beta)} \right) - \frac{\mathcal{F}(y)g_{\alpha\beta}}{2k^2\ell^2G}, \] (8)
\[ \theta_{\alpha\beta} = K \left( g^{\mu\nu} A_{[\mu,\alpha]} A_{[\nu,\beta]} - \frac{1}{4} g^{\tau\gamma} g^{\mu\nu} A_{[\mu,\tau]} A_{[\nu,\gamma]} g_{\alpha\beta} \right) - \lambda A_\alpha A_\beta \] (9)

where $v_{(\alpha} A_{\beta)} \equiv v_\alpha A_\beta - A_\alpha v_\beta$, etc., and
\[ \mu(y) \equiv \mathcal{F}'(y); \quad y \equiv k^2h^{\gamma\delta} \phi_{,\gamma} \phi_{,\delta}. \] (10)
Each choice of the function $\mathcal{F}(y)$ defines a separate TeVeS theory. Its derivative $\mu(y)$ functions somewhat like the $\mu$ function in MOND. For $y > 0$, $\mu(y) \approx 1$ corresponds to the high acceleration, i.e., Newtonian, limit, while the limit $0 < \mu(y) < 1$ corresponds to the deep MOND regime. We shall only consider functions such that $\mathcal{F} > 0$ and $\mu > 0$ for either positive or negative arguments.

The equations of motion for the vector and scalar fields are obtained by varying the action with respect to $\phi$ and $A_\alpha$, respectively. We have
\[ \left[ \mu(y) h^{\alpha\beta} \phi_{,\alpha} \right]_{,\beta} = kG \left[ g^{\alpha\beta} + (1 + e^{-4\phi}) A^\alpha A^\beta \right] \tilde{T}_{\alpha\beta}, \] (11)
for the scalar and
\[ KA^{[\alpha;\beta]}_{,\beta} + \lambda A^\alpha + \frac{8\pi}{k} \mu A_\alpha \phi_{,\beta} g^{\alpha\beta} = 8\pi G \left( 1 - e^{-4\phi} \right) g^{\mu\nu} A^\beta \tilde{T}_{\nu\beta} \] (12)
for the vector. Additionally, there is the normalization condition on the vector field
\[ A^\alpha A_\alpha = g_{\alpha\beta} A^\alpha A^\beta = -1. \] (13)
The $\lambda$ in Eq. (12), the lagrange multiplier charged with the enforcement of the normalization condition, can be calculated from the vector equation.

The three parameters, $k$, $K$, and $\ell$, all specific to TeVeS, are constant in the framework of the theory, as is $G$, the fundamental gravitational coupling constant, which does not coincide with Newton’s $G_N$.

### III. The PPN Parameters for TeVeS

In the weak field, slow motion limit, the next-to-Newtonian order gravitational effects of any metric gravitational theory can be described in terms of a set of functionals of the matter variables that answer certain criteria of ”reasonableness” and simplicity (see [20] for details), known as the Post-Newtonian potentials, and of ten parameters, $\gamma$, $\beta$, $\zeta_1$, $\zeta_2$, $\zeta_3$, $\zeta_4$, $\xi$, $\alpha_1$, $\alpha_2$, $\alpha_3$, known as the PPN parameters. Since TeVeS has two metrics, the Einstein metric on which the gravitational fields propagate, and the physical matter metric, serving as background for massive and massless matter particles, the potentials are of course to be expressed in terms of physical coordinates defined by the physical metric, and physical fluid variables, with indices raised and lowered by the physical metric.

The standard form of the Post Newtonian metric is
the dimensionless gravitational potential $G_{W}$ of Newtonian order have the following form scalar field, necessary order, and the form of the metric components cosmological boundary conditions. The fluid velocity is dependence on $g \chi$, where an index of $(1)$ denotes a term of first order in $g_{00}$ and $g_{ij}$, where $\chi$ is the "superpotential" defined as

$$\chi \equiv - \int \rho(x', t)|x - x'|d^3x'$$

Thus the coordinate frame is determined up to the necessary order, and the form of the metric components is unique.

When taking into account the cosmological value of the scalar field, $\phi_0$, the Einstein metric and fields to post-Newtonian order have the following form

$$g_{00} = -1 + 2U - 2\beta U^2 - 2\zeta \Phi_{W} + (2\gamma + 2\alpha + 1 - 2\xi)\Phi_{1} + 2(3\gamma - 2\beta + 1 + 2\zeta)\Phi_{2} + 2(1 + \zeta)\Phi_{3} + 2(3\gamma + 3\zeta - 4\xi)\Phi_{4} - (1 + 2\xi)\Phi $$

$$g_{0j} = - \frac{1}{2}(4\gamma + 3 + \alpha + 2\zeta - 2\xi)V_j - \frac{1}{2}(1 + \alpha + 2\zeta - 2\xi)W_j$$

$$g_{jk} = (1 + 2\gamma U)\delta_{jk}$$

The potentials are all of the form

$$\Phi(x', t) = \int \rho(x', t)f(x', t)\frac{d^3x'}{|x - x'|}$$

where $f(x', t)$ is given for each potential as follows

$$U \gamma \Phi_{1} \gamma \Phi_{2} \gamma \Phi_{3} \gamma \Phi_{4} = \frac{p}{\rho}$$

$$V_{i} \gamma A_{i} \gamma W_{i} = \frac{v_{i}(x - x')_{i}}{|x - x'|^{2}}, W_{i} = \frac{v_{i}(x - x')_{i}(x - x')^{i}}{|x - x'|^{2}}.$$}

The standard post Newtonian gauge is such that all dependence on $\chi_{00}$ and $\chi_{ij}$ has been eliminated from $g_{00}$ and $g_{ij}$, where $\chi$ is the "superpotential" defined as

$$\chi = - \int \rho(x', t)|x - x'|d^3x'$$

The potentials must then be rescaled accordingly, for example $U = e^{-2\phi_0/2}U$, $V_i = e^{-\phi_0/2}V_i$, and $W_i = e^{-\phi_0/2}W_i$. Additionally, we will work in units such that $G_N \equiv 1$. In these units, the coupling constant $G$ can be expressed in terms of $K$, $k$

$$G = \left(1 - \frac{K}{2} + \frac{k}{4\pi}\right)^{-1}.$$}

In local quasi-Cartesian coordinates, and with $G_N \equiv 1$, the post-Newtonian metric for TeVeS is given by

$$\tilde{g}_{00} = -1 + 2U - 2U^2 + 4\Phi_{1} + 4\Phi_{2} + 2\Phi_{3} + 6\Phi_{4},$$

$$\tilde{g}_{ij} = \delta_{ij}(1 + 2U)$$

and

$$\tilde{g}_{0i} = - \frac{1}{2}(7 + \alpha_1 - \alpha_2)\tilde{V}_i - \frac{1}{2}(1 + \alpha_2)\tilde{W}_i$$

with

$$\alpha_1 = \frac{4G}{K}(2K - 1)e^{-4\phi_0} - e^{4\phi_0} + 8) - 8$$

and

$$\alpha_2 = \frac{2G}{(2 - K)^2}(3(2 - K) - (K + 4)e^{4\phi_0}) - 1$$

Here we omitted the bars over the potentials. The time-time and space-space components of the physical metric do not depend on the cosmological value of the scalar field in the standard post-Newtonian coordinate system, yielding $\beta = \gamma = 1$, $\xi = \zeta = 0$, with $\alpha = 1.4$, as for GR.

The result of $\beta = 1$ is apparently inconsistent with Giannios' [10], who obtained $\beta \neq 1$ for $A^r \neq 0$. However, Giannios performed the calculation assuming spherical symmetry, and took the radial component of the vector field to be of $O(1)$, and not $O(1.5)$ as dictated by the
Post-Newtonian formalism. While such a term is allowed by the field equations, the vector equation to $O(1)$ is

$$\nabla^2 A^i - A^i_{,ij} = 0.$$  \hfill (34)

This means that such a term in the vector field does not originate in sources or fields, and hence it is not clear how it would be incorporated in the PPN formalism. This was not noticed by Giannios since he performed the calculation in vacuum, so that the sources did not appear directly in the equations, but only as boundary conditions for integration. An $O(1)$ term in the vector field would require extra parameters to set the boundary conditions, and thus seems unnatural. An $A^r$ which is $O(0.5)$ could be acceptable, but it would have no effect on the calculation of the PPN parameters, since it could be eliminated by an appropriate Lorentz transformation.

However, the time-space component of the metric does depend on $\phi_0$, as do the preferred frame parameters, and factors of $e^{2\phi_0}$ cannot be "rescaled out" by a coordinate transformation. Therefore it is incorrect to perform the PPN parameters calculation without taking into account the cosmological value of the scalar field. In fact, its role in the vector equation is quite critical, as we shall soon see.

The covariant divergence of the vector equation, unlike the covariant divergence of the Einstein equation, is not automatically satisfied, but yields a constraint on the divergence of the vector field or on the coupling constants of the theory (see \cite{20}, Section 5.4 for details). In the case of TeVeS, we obtain to $O(1.5)$

$$\frac{K}{1-K/2} U_{,0} = -2(1-e^{-4\phi_0}) U_{,0} \hfill (35)$$

Here Tamaki’s calculation \cite{21} was in error; since he assumed $\phi_0 = 0$, his conclusion at this step should have been that necessarily $K = 0$, namely one cannot take into account the coupling of the vector field in this particular theory without the scalar field, because obviously one cannot demand $U_{,0} = 0$ always. When a nonzero cosmological value of the scalar field is taken into account, it is constrained by the divergence of the vector equation. From the requirement that $U_{,0} \neq 0$, we obtain

$$\frac{K}{1-K/2} = -2(1-e^{-4\phi_0}), \hfill (36)$$

which links the cosmological value of the scalar field to the coupling constant of the vector field as follows

$$\phi_0 = -\frac{1}{4} \ln \left( \frac{2}{2-K} \right) \hfill (37)$$

This contradicts the basic assumption of TeVeS that its coupling parameters are constant, namely that they do not change during the evolution of the universe. Or vice-versa, if the assumption remains valid, then $\phi_0$ is not allowed to evolve. Moreover, this relation would mean that $\phi_0$ is negative, since it has been shown \cite{11} that one must have $K < 2$. This would bring the problem of superluminal propagation of scalar waves.

When relation (37) holds, it can be shown that there is no ambiguity in the determination of the physical metric and the preferred frame parameters, although the spatial divergence of the vector field remains indeterminate. However, it looks like the constraint (37) is another sign of the existence of a dynamical problem with the vector action in TeVeS, adding up to the analysis of Contaldi et al. \cite{24} regarding the formation of caustic singularities in the evolution of the vector field. Therefore, we leave aside simple TeVeS, and proceed instead to calculate the PPN parameters of TeVeS with Æther \cite{20} type vector action.

\section{IV. \textsc{teves with Æther type vector action}}

This version of TeVeS has been proposed by Skordis \cite{9}, who investigated its cosmology, as a natural generalization, and Contaldi et al. \cite{24} adopted it as a resolution to the problem of vector caustic singularities. The vector action is taken to be of the most general form quadratic in derivatives of the vector fields, whereas its scalar and metric actions are unaltered, thus preserving the correct MOND and Newtonian limits. We take a vector action of the form

$$S_v = -\frac{1}{16\pi G} \int \sqrt{-g} d^4x \left( \frac{K}{2} F_{\alpha\beta} F^{\alpha\beta} + \frac{K_-}{2} S_{\alpha\beta} S^{\alpha\beta} + K_2 (\nabla A)^2 + K_4 A_\alpha A^\alpha - \lambda (A^\alpha A_\alpha + 1) \right) \hfill (38)$$

where $F_{\alpha\beta} = A_{\alpha;\beta} - A_{\beta;\alpha}$, $S_{\alpha\beta} = A_{\alpha;\beta} + A_{\beta;\alpha}$, and $A^\alpha = A^\beta A^\gamma_{,\beta}$. The relation between the coupling constants $K_i$ and the $c_i$ of Æther \cite{26} is $c_1 - c_3 = 2K, c_1 + c_3 = 2K, c_2 = K_2$ and $c_4 = -K_4$.

The equation of motion and the stress tensor for the vector field are then
and the scalar equation remains unaltered. To post-Newtonian order, the Einstein and physical metric and fields are as in Section III and the coordinate rescaling is as for simple TeVeS. The new gravitational coupling constant \( G_N \) is given by

\[
G_N = G \left( \frac{1 - K + K_+ - K_4}{2} \right)^{-1} + \frac{k}{4\pi},
\]

(41)

and for units in which \( G_N = 1 \),

\[
G = \left( \frac{1 - K + K_+ - K_4}{2} \right)^{-1} + \frac{k}{4\pi} \right)^{(-1)}
\]

\[
= \frac{4\pi(2 - (K + K_+) + K_4)}{8\pi + k(2 - (K + K_+) + K_4)}
\]

(42)

The PPN metric is

\[
\bar{g}_{00} = -1 + 2U - 2U^2 + 4\Phi_1 + 4\Phi_2 + 2\Phi_3 + 6\Phi_4,
\]

(43)

and

\[
\bar{g}_{ij} = \delta_{ij}(1 + 2U)
\]

(44)

with

\[
\alpha_1 = 8 \left( \frac{G \left( e^{2\phi_0} - (1 - 2K) e^{-2\phi_0} \sinh(2\phi_0) - (K + e^{4\phi_0} + K) \right)}{2KK_+ - (K + K_+)} - 1 \right)
\]

(46)

\[
\alpha_2 = \frac{\alpha_1}{2} + \frac{4G \left( (2 + 3K_2 + 2K_+) e^{2\phi_0} - (K + K_+ - K_4) e^{-2\phi_0} \sinh(2\phi_0) \right)}{(K_2 + 2K_+) (2 - (K + K_+ - K_4))} - \frac{2G \left( (K_4 - K + 3K_+)(2 + 3K_2 + 2K_+) e^{4\phi_0} + (3K_+ + 3K_2 - K_4)(2 - (K + K_+ - K_4)) \right)}{(K_2 + 2K_+) (2 - (K + K_+ - K_4))^2} + 3
\]

(47)

(48)

Hence TeVeS with Æther type vector kinetic term has \( \beta = \gamma = 1, \xi = \zeta = \alpha_3 = 0, \) with \( i = 1, 4, \) as for GR, and preferred frame parameters \( \alpha_1 \) and \( \alpha_2 \) given above.

Although not apparent from the above formulation of the alphas, when the coupling constants of TeVeS, \( k \) and the \( K_i \), as well as the cosmological value of the scalar field are zero, then \( \alpha_1 = \alpha_2 = 0 \). Moreover, when the scalar field is decoupled from the theory by setting \( k = \phi_0 = 0 \), the preferred frame parameters acquire the values calculated for Æther theory, as given in \( \text{[27]} \)

\[
\alpha_1 = \frac{4 ((K - K)^2 - K_4(K + K_+))}{2KK_+ - (K + K_+)}
\]

(49)

\[
\alpha_2 = \frac{\alpha_1}{2} + \frac{(K - 3K - K_4)(K + 3K_+ + 3K_2 - K_4)}{(2K_2 + K_4)(2 - (K + K_+ - K_4))}
\]

(50)

This was to be expected, since when TeVeS is exempted of the scalar field, the physical and Einstein metric coincide, and the theory is identical to Æther.

Current data strongly constrains the preferred frame parameters to \( |\alpha_1| < 10^{-4} \) (from measurement of the earth-moon orbital polarization with lunar laser ranging and from pulse timing of binary pulsar PSR J2317+1439) and \( |\alpha_2| < 4 \times 10^{-7} \) (from solar alignment with the ecliptic) \( \text{[22, 28, 29]} \). It has been shown that in Æther theory the two alphas can be set to zero with two parameters
to spare [27]; it is obvious that in the extended version of TeVeS, which has five parameters, we can solve for \( \alpha_1, \alpha_2 = 0 \) with three parameters to spare. However, since the alphas depend on the cosmological value of the scalar field, their value is expected to change with cosmological evolution, and the question that arises is whether the present day cosmological value of the scalar field, as evolved from reasonable initial conditions, is consistent with the experimentally measured values of \( \alpha_1 \) and \( \alpha_2 \) today, and with small values of the coupling constants \( k, K_i \). Small values are required since in simple TeVeS, by allowing for a growing mode in the order of magnitude as \( k \). It then seems natural to assume that the extra kinetic terms added to the vector action should also have small couplings.

There is also the question of cosmological evolution of the alphas themselves; however, since all experimental data on the values of the preferred frame parameters originates nearby by cosmological standards (solar system or pulsars in our galaxy), we allow ourselves to assume that \( \phi_0 \) is constant for practical purposes.

## A. Allowed Ranges of Parameters

Solving \( \alpha_1 = 0 \) and \( \alpha_2 = 0 \), we obtained expressions for \( K_2 \) and \( k \) in terms of \( K_4, K, K_+ \) and \( \phi_0 \). The constraints on the alphas are satisfied for large ranges of the parameters \( K_4, K, K_+ \); however, when demanding that the parameters be small with respect to unity, the ranges narrow significantly. One can look at the allowed region in the \( K - K_+ \) parameter space, obtained from the overlap of the regions in which both \( k \) and \( K_2 \) are small, for specific values of the parameter \( K_4 \). We take \( \phi_0 = 0.003 \), as suggested by [33], and show, for example, the region in the \( K - K_+ \) plane for which \( \alpha_1, \alpha_2 = 0 \), \( K_2 \) and \( k \) are in the range \([0, 0.3]\), with \( K + K_+ - K_4 = c \), for \( c = 0.01 \) and \( c = 0.001 \), in Fig. [1] and Fig. [2], respectively. The allowed ranges for small parameters are quite narrow, especially for \( K_+ \), which is restricted to a small range near 0.01. The plots remain similar if instead of requiring \( \alpha_1 = 0 \) and \( \alpha_2 = 0 \), we only demand, for instance, \( \alpha_1 = 10^{-5} \) and \( \alpha_2 = 0 \), in compliance with current experimental constraints.

If we increase \( \phi_0 \), the shape of the allowed region in the \( K - K_+ \) plane remains similar, but it is shifted upwards, favoring larger values of \( K_+ \). For too large a value of \( \phi_0 \) (around 0.01), there would be no overlap between the regions in which both \( k \) and \( K_2 \) are small; the same happens when \( K + K_+ - K_4 \) is taken to be too large. When \( K + K_+ - K_4 = 0.1 \) there is no overlap between the two regions.

In the special case \( K + K_+ - K_4 = 0 \), the forms of \( K_2 \) and \( k \) become particularly simple

\[
k = \frac{2\pi ((2K - 1)e^{-4\phi_0} - (2K_+ - 1)e^{4\phi_0} - 2(2KK_+ - (K + K_+) + 1))}{2KK_+ - (K + K_+)}
\]

\[
K_2 = \frac{-2(K_+(2K - 1) + 2K)}{3(2K - 1)}
\]

Both \( k \) and \( K_2 \) no longer depend on \( K_4 \), and \( K_2 \) doesn’t depend on \( \phi_0 \). However, Skordis [6] has shown that cosmologically the combination \( K + K_+ - K_4 \) in TeVeS with \( \phi \text{Ether} \) type vector action plays the same role as \( K \) in simple TeVeS, by allowing for a growing mode in the vector field in order to source structure formation. For this same reason we have not considered negative values of \( K + K_+ - K_4 \).

## V. Conclusions

We calculated the preferred frame parameters for TeVeS, and in the process of calculation we found that the divergence of the vector equation constrains the cosmological value of the scalar field to be related to the coupling parameter of the vector field. Since such a link does not allow the scalar field to evolve cosmologically, we set aside simple TeVeS and proceeded to calculate the preferred frame parameters for a generalized version of TeVeS previously suggested in the literature to resolve possible dynamical problems [6, 24], TeVeS with \( \phi \text{Ether} \) like vector action. We obtained expressions for \( \alpha_1 \) and \( \alpha_2 \) in terms of the coupling parameters of TeVeS, \( k \) and \( K_i \), and the cosmological scalar field \( \phi_0 \). Since all existing experimental data on values of the alphas originates from within our galaxy, we assumed the cosmological value of the scalar field to be constant, and of modest value, and analyzed the allowed ranges of coupling parameters for which both preferred frame parameters are zero. We found that the conditions \( \alpha_1 = 0, \alpha_2 = 0 \) can be satisfied for small ranges of the coupling parameters, and in these
ranges TeVeS has its PPN metric identical to GR, making it indiscernible from the latter in the solar system. Future work on gravitational wave speeds and stability is expected to constrain the allowed values of parameters further.

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APPENDIX A: CALCULATION OF PPN PARAMETERS FOR TEVES

The PPN parameters are determined by solving the gravitational field equations with a perfect fluid source, in a standard coordinate gauge where the spatial part of the metric is diagonal and isotropic. The fluid variables are assigned orders of $U \sim v^2 \sim \rho \sim \Pi \sim p / \rho \sim O(1)$, where $U$ is the Newtonian gravitational potential. Taking the time derivative of a quantity raises its order by one half, since $\partial / \partial t \sim \nabla v$. In the Newtonian limit of any theory of gravity, the temporal component of the metric $g_{00}$ is required to first order in the potential, whereas the spatial components of the metric $g_{ij}$ are of zero order; the post-Newtonian limit calls for knowledge of $g_{00}$ to $O(2)$, $g_{ij}$ to $O(1)$, and $g_{0j}$, which must change sign under time reversal, is order $O(1.5)$. The standard form of the Post-Newtonian metric is given in Eqs. [14]-[16].

In the calculation we will use the following relation that holds for $\dot{U}$, $\Phi_{1,2,3,4}$, and $V_i$

$$\nabla^2 \Phi = -4\pi G \rho f. \quad (A1)$$

The superpotential Eq. [19] satisfies

$$\nabla^2 \chi = -2U \quad (A2)$$

$$\chi_{,00} = V_i - W_i \quad (A3)$$

$$V_i, i = -\dot{U}, 0 \quad (A4)$$

the last two relations follow from the continuity equation for the fluid, assumed to hold to $O(1.5)$

$$\rho, 0 + (\rho v^i), i = 0. \quad (A5)$$

The standard post Newtonian gauge is such that all dependence on $\chi_{,00}$ and $\chi_{ij}$ has been eliminated from $g_{00}$ and $g_{ij}$.

We will follow the procedure described in Will’s classic reference [20]. Although many details are similar to Tamaki’s computation [21], we will show the calculation again to make it clear where factors of $e^{\pm 2\phi_0}$ enter, and how they alter the results. In TeVeS, which is endowed with a scalar and vector field additional to the metric, they must be solved for to the order necessary to derive the metric as required; namely, $\phi$ to $O(2)$ in the Newtonian gravitational potential, $A^i$ to $O(2)$ in the Newtonian gravitational potential, and $A^i$ to $O(1.5)$. Moreover, the metric from which the PPN parameters are to be determined is the physical metric, $g_{\alpha \beta}$, since this is the metric on which matter fields propagate. For the calculation it is convenient to rearrange the Einstein equations (7) as follows

$$R_{\alpha \beta} = \left[ 8\pi G \left( \bar{T}_{\mu \nu} + (1 - e^{-4\phi}) A^\gamma A_{\mu \nu} + \tau_{\mu \nu} \right) \right. + \left. \theta_{\mu \nu} \right] \left( \delta^i_\alpha \delta^j_\beta - \frac{1}{2} g_{\alpha \beta} g^{ij} \right). \quad (A5)$$
where \( \tau_{\alpha\beta} \) and \( \theta_{\alpha\beta} \) are given by Eqs. (8)-(9) respectively. For a perfect fluid source, we take as customary

\[
T_{\alpha\beta} = (\rho + \rho \Pi + p) u_\alpha u_\beta + p g_{\alpha\beta} \quad (A6)
\]

In the Solar system, where accelerations are high compared to the MOND acceleration scale \( a_0 \), we can take \( \mu \approx 1 \), and, correspondingly, \( \mathcal{F}(y) = y \).

With the Einstein metric and fields of the form (20)-(24), the physical metric to the required order is given by

\[
\tilde{g}_{00} = e^{2\phi_0} \left[ -1 + \left( \tilde{h}^{(1)}_{00} - 2\phi^{(1)} \right) + \left( \tilde{h}^{(2)}_{00} - 2\phi^{(2)} - 2(\phi^{(1)})^2 + 2\phi^{(1)}\tilde{h}^{(1)}_{00} \right) \right] \quad (A7)
\]

\[
\tilde{g}_{ij} = e^{-2\phi_0} \left[ 1 + \left( \tilde{h}_{ij} - 2\phi^{(1)} \right) \right] \quad (A8)
\]

\[
\tilde{g}_0 = e^{2\phi_0} h_0 + 2A^i \sinh 2\phi_0 \quad (A9)
\]

where different orders were separated by brackets.

In our calculation, we will impose the following gauge conditions

\[
h_{kj,k} = -\frac{1}{2} \left( \tilde{h}^{(1)}_{00,j} - \tilde{h}_{kk,j} \right) \quad (A10)
\]

\[
h_{0k,k} = \frac{1}{2 - K} \tilde{h}_{kk,0} \quad (A11)
\]

the first condition is frequently used in GR, whereas the second condition is required to bring \( \tilde{g}_{00} \) to the standard post-Newtonian gauge. Here and throughout the calculation we assume the Einstein summation convention to relevant order, meaning that repeated spatial indices are summed over. Indices on the vector field are raised and lowered with the Einstein metric, and indices for the matter fields and fluid velocity are raised and lowered with the physical metric.

**1. Calculation of \( A^i \) and \( v^i \) from normalization**

We are now ready to proceed with the calculation. First, we find the temporal components of the vector field and of the fluid velocity from the normalization conditions Eqs. (13), (26), respectively

\[
A^{(1)} = \frac{1}{2} \tilde{h}^{(1)}_{00} \quad (A12)
\]

\[
A^{(2)} = \frac{1}{2} \tilde{h}^{(2)}_{00} + \frac{3}{8} (\tilde{h}^{(1)}_{00})^2 \quad (A13)
\]

\[
v^i = \frac{1}{2} \left( \tilde{h}^{(1)}_{00} - 2\phi^{(1)} + e^{-4\phi_0} v^2 \right) \quad (A14)
\]

**2. \( \tilde{g}_{00} \) to \( O(1) \)**

Next, we solve for \( \tilde{g}_{00} \) to \( O(1) \). To this end, we must find \( h^{(1)}_{00} \) and \( \phi^{(1)} \). The temporal component of the Einstein equations (A5) to \( O(1) \) is

\[
\frac{1}{2} (1 - K/2) \nabla^2 h^{(1)}_{00} = -4\pi G e^{-2\phi_0} \rho \quad (A15)
\]

yielding for \( h^{(1)}_{00} \)

\[
h^{(1)}_{00} = \frac{2G e^{-2\phi_0}}{1 - K/2} \int \frac{\rho}{|x - x'|} d^3x' \quad (A16)
\]

For convenience, we define

\[
G_K = \frac{G}{1 - K/2} \quad (A17)
\]

since this is a recurring quantity in the calculation.

\( \phi^{(1)} \) is determined from Eq. (11) to \( O(1) \)

\[
\nabla^2 \phi^{(1)} = kG e^{-2\phi_0} \rho \quad (A18)
\]

giving

\[
\phi^{(1)} = -\frac{kGe^{-2\phi_0}}{4\pi} \int \frac{\rho}{|x - x'|} d^3x' \quad (A19)
\]

Combining Eqs. (B2), (A19) and (A7), we have

\[
\tilde{g}_{00} = Ge^{-2\phi_0} \left( \frac{1}{1 - K/2} + \frac{k}{4\pi} \right) \int \frac{\rho}{|x - x'|} d^3x' \quad (A20)
\]

We now put the physical metric in standard Newtonian and post-Newtonian form in local quasi-Cartesian coordinates, as described in Ref. 20, through the following coordinate transformation

\[
x^0 = e^{\phi_0} x^0, x^i = e^{-\phi_0} x^i \quad (A21)
\]

In the new coordinates, the physical metric to Newtonian order is

\[
\tilde{g}_{00} = -1 + 2G_N U, \tilde{g}_{ij} = \delta_{ij}, \tilde{g}_{0i} = 0 \quad (A22)
\]

with

\[
G_N = G \left( \frac{1}{1 - K/2} + \frac{k}{4\pi} \right) \quad (A23)
\]

Here the potential \( U \) has been redefined in accordance with the coordinate transformation \( U = e^{-2\phi_0} U \). In the asymptotically Minkowskian coordinates, we have

\[
h^{(1)}_{00} = 2G_K U \quad (A24)
\]

and

\[
\phi_1 = -\frac{kG}{4\pi} U \quad (A25)
\]
In order to compare our result with (14), we must select units such that \( G_N \equiv 1 \). The coupling constant \( G \) is then given in terms of \( K, k \)
\[
G = \left( \frac{1}{1 - K/2} + \frac{k}{4\pi} \right)^{-1}
\]
so that finally
\[
g_{00} = -1 + 2\bar{U}, \quad \bar{g}_{ij} = \delta_{ij}, \quad \bar{g}_{0i} = 0 \tag{A27}
\]

3. \( \bar{g}_{ij} \) to \( O(1) \)

The next step will be to solve for \( \bar{g}_{ij} \) to \( O(1) \). From the space-space component of Eq. (A5) we have for \( h_{ij} \)
\[
\nabla^2 h_{ij} - h_{00,ij} + h_{kk,ij} + \bar{h}_{ki,jk} - \bar{h}_{kj,ik} = -8\pi G K e^{-2\phi_0} \rho \delta_{ij},
\]
making use of the gauge Eq. (A10), this becomes
\[
\nabla^2 h_{ij} = -8\pi G K e^{-2\phi_0} \rho \delta_{ij},
\]
whose solution is
\[
h_{ij} = \delta_{ij} 2G K \int \frac{pe^{-2\phi_0}}{|x'| - x|} d^3x',
\]
or
\[
h_{ij} = 2G K e^{-2\phi_0} U \delta_{ij}
\]
Combining this result with Eq. (A19) we obtain for \( \bar{g}_{ij} \)
\[
\bar{g}_{ij} = e^{-2\phi_0} \delta_{ij} \left( 1 + 2e^{-2\phi_0} G_N U \right)
\]
We now transform the coordinate system according to (A21), set units such that \( G_N \equiv 1 \), and obtain
\[
\bar{g}_{ij} = \delta_{ij} \left( 1 + 2\bar{U} \right)
\]
Comparing this with Eq. (16) we see that for TeVeS the PPN parameter \( \gamma = 1 \). This is in accordance with previous work [3 11 21], and agrees with current solar system data.

4. \( \bar{g}_{0j} \) to \( O(1.5) \)

We now solve for \( \bar{g}_{0j} \) to \( O(1.5) \), which requires calculating \( A^i \) and \( h_{0j} \) to the same order. First, we find the Lagrange multiplier \( \lambda \) to \( O(1) \) from the temporal component of the vector equation Eq. (12)
\[
\lambda = \frac{1}{2} K \nabla^2 h_{00}^{(1)} - 16\pi G \rho \sinh 2\phi_0
\]
The time-space component of Eq. (A5) to \( O(1.5) \) is then
\[
-\frac{1}{2} \left( \nabla^2 h_{0j} - h_{0k,kj} + h_{kk,0j} - h_{kj,0k} \right) = -8\pi G e^{-6\phi_0} \rho U^j.
\]
Making use of the gauge condition Eq. (A11), written as \( h_{0k,k} = \frac{1}{2} G \frac{\nabla^2 k}{k} \), this becomes
\[
-\frac{1}{2} \left( \nabla^2 h_{0j} - \frac{1}{2} G \frac{\nabla^2 k}{k} h_{0k,0j} + h_{kk,0j} - h_{kj,0k} \right) = -8\pi G e^{-6\phi_0} \rho U^j.
\]
We note that \( h_{kk} = 6G_K e^{-2\phi_0} U \) and \( h_{kk,0j} - h_{kj,0k} = \frac{2}{5} h_{kk,0j} \), obtaining
\[
-\frac{1}{2} \left( \nabla^2 h_{0j} + 3G K \left( \frac{2}{3} - \frac{1}{2} G_k \right) e^{-2\phi_0} U_0 \right) = -8\pi G e^{-6\phi_0} \rho U^j.
\]
Using the definition of the potential \( V_j \) and the relation Eq. (A22), we obtain
\[
-\frac{1}{2} \left( \nabla^2 h_{0j} - 3G K \left( \frac{2}{3} - \frac{1}{2} G_k \right) e^{-2\phi_0} U^j \right) = 2G K e^{-6\phi_0} \nabla^2 V_j.
\]
Finally, with the aid of Eq. (A3), we rearrange and solve for \( h_{0j} \)
\[
h_{0j} = 3e^{-2\phi_0} G K \left( \frac{2}{3} - \frac{1}{2} G_k \right) (V_j - W_j) - 4e^{-6\phi_0} G V_j.
\]
Now to the spatial part of the vector. Eq. (12) to \( O(1.5) \) is
\[
K \left( \nabla^2 A^i - A^i_{,ji} + \nabla^2 h_{0i} - h_{0j,ji} + \frac{1}{2} h_{00}^{(1)} \right) = -8\pi G e^{-2\phi_0} \rho v^i (1 - e^{-4\phi_0})
\]
Taking the covariant divergence of the vector equation, we obtain to \( O(1.5) \)
\[
K \frac{h_{00}^{(1)}}{1 - K/2} = -8\pi G e^{-2\phi_0} (\rho v^i)_i (1 - e^{-4\phi_0})
\]
using relations (A24 A11) this can be written as
\[
K G_K e^{-2\phi_0} \nabla^2 U_{,0} = 2G e^{-2\phi_0} \nabla^2 V_{,i} (1 - e^{-4\phi_0}),
\]
or, with Eq. (A4) and canceling some terms on both sides,
\[
\frac{K}{1 - K/2} U_{,0} = -2(1 - e^{-4\phi_0}) U_{,0}
\]
From the requirement that \( U_{,0} \neq 0 \), we obtain
\[
\frac{K}{1 - K/2} = -2(1 - e^{-4\phi_0}),
\]
which links the cosmological value of the scalar field to the coupling constant of the vector field as follows
\[
\phi_0 = -\frac{1}{4} \ln \left( \frac{2}{2 - K} \right)
\]
If the constraint Eq. (37) holds, it is possible to solve for the vector field to $O(1.5)$ and obtain the PPN metric and the preferred frame coefficients, and this we proceed to demonstrate. Since the covariant divergence of the vector equation does not constrain $A^i$, we are free to set its value as we please. We can regard it as a $U(1)$ gauge transformation, with $A^a \to A^a + \varphi^a$. Although the action is not fully $U(1)$ gauge invariant because of the lagrange multiplier term, to $O(1.5)$, the relevant order for the preferred frame PPN parameters, the vector equation is gauge invariant.

We thus set, for convenience

$$A^i = 0,$$

simplifying the vector equation

$$K \left( \nabla^2 A^i + \nabla^2 h_{0i} - h_{0j,ji} + \frac{1}{2} h_{00,0i} \right) = -8\pi G e^{-2\phi_0} \rho V^i (1 - e^{-4\phi_0})$$

from Eq. (A35) we have

$$\nabla^2 h_{0i} - h_{0j,ji} = 16\pi G e^{-6\phi_0} \rho V^i \frac{2}{3} h_{jj,0i},$$

yielding

$$K \left( \nabla^2 A^i + 16\pi G e^{-6\phi_0} \rho V^i \frac{2}{3} h_{jj,0i} + \frac{1}{2} h_{00,0i} \right) = -8\pi G \rho V^i e^{-2\phi_0} (1 - e^{-4\phi_0}).$$

Since $h_{00,0i}^{(1)} = \frac{1}{2} h_{jj,0i}$, we have

$$K \left( \nabla^2 A^i + 16\pi G e^{-6\phi_0} \rho V^i - \frac{1}{2} h_{jj,0i} \right) = -8\pi G e^{-2\phi_0} \rho V^i (1 - e^{-4\phi_0}).$$

With the definition of the potential $V_i$ and

$$h_{jj,0i} = -3G_K e^{-2\phi_0} \nabla^2 \chi_{,0i} = -3e^{-2\phi_0} G_K \nabla^2 (V_i - W_i),$$

we get

$$K \left( A^i - 4e^{-6\phi_0} G V_i + \frac{3}{2} e^{-2\phi_0} G_K (V_i - W_i) \right) = 2e^{-2\phi_0} G V_i (1 - e^{-4\phi_0})$$

The spatial part of the vector field is then given by

$$A^i = e^{-2\phi_0} \left( 2G e^{-4\phi_0} \left( \frac{1 - e^{-4\phi_0}}{K} \right) V_i - \frac{3}{2} G_K (V_i - W_i) \right),$$

or, with Eq. (A17)

$$A^i = 2G e^{-2\phi_0} \left( \left( e^{-4\phi_0} + \frac{1 - e^{-4\phi_0}}{K} \right) V_i - \frac{3}{2 - K} (V_i - W_i) \right).$$

This result satisfies $A^i = 0$ if relation (37) holds.

We now combine Eqs. (A39), (A52) and (A9) to obtain

$$\tilde{g}_{0i} = 2G \left( \frac{1 - 2K}{(2 - K)^2} (V_i - W_i) - 2V_i + e^{4\phi_0} V_i - \frac{3}{2 - K} (V_i - W_i) \right)$$

This remains to be converted to asymptotically Minkowskian coordinates using prescription (A21), with the potentials rescaled accordingly $V_i = e^{4\phi_0} V_i$ and $W_i = e^{-4\phi_0} W_i$, yielding

$$\tilde{g}_{0i} = 2G \left( \frac{e^{4\phi_0} (1 - 2K)}{(2 - K)^2} (V_i - W_i) - 2V_i + (e^{4\phi_0} - 1) \left( \frac{1 + (2K - 1)e^{-4\phi_0}}{K} V_i - \frac{3}{2 - K} (V_i - W_i) \right) \right)$$

would diverge.

5. $\tilde{g}_{00}$ to $O(2)$

We have now arrived at the most complicated step—calculation of $\tilde{g}_{00}$ to $O(2)$. To this end, we should solve for $h_{00}^{(2)}$ from the temporal part of the Einstein equations Eqs. (A5), and for $\phi^{(2)}$ from the scalar equation Eq. (11) to $O(2)$. We start with $h_{00}^{(2)}$
\[ \left( 1 - \frac{K}{2} \right) \left( -\frac{1}{2} \nabla^2 h_{00}^{(2)} + \frac{1}{2} h_{00,jj} \left( h_{jk,k} - \frac{1}{2} h_{kk,j} \right) - \frac{1}{4} \left| \nabla h_{00}^{(1)} \right|^2 + \frac{1}{2} h_{jk} h_{00,jk} \right) \]

\[ - \frac{1}{2} K A^j_{j0} - \frac{1}{2} \left( h_{jj,00} - 2 \left( 1 - \frac{K}{2} \right) h_{j0,j0} \right) \]

\[ = -4\pi G e^{-2\phi_0} \rho \left( h_{00}^{(1)} + 2\phi^{(1)} \right) + 4\pi G e^{-2\phi_0} \rho \left( \Pi + 3\frac{B}{\rho} + 2e^{-4\phi_0}\nu^2 \right) \quad (A56) \]

All the terms in the second line disappear by virtue of the gauge conditions Eqs. (A11), (A40), leaving us with

\[ \left( 1 - \frac{K}{2} \right) \left( -\frac{1}{2} \nabla^2 h_{00}^{(2)} + \frac{1}{2} h_{00,jj} \left( h_{jk,k} - \frac{1}{2} h_{kk,j} \right) - \frac{1}{4} \left| \nabla h_{00}^{(1)} \right|^2 + \frac{1}{2} h_{jk} h_{00,jk} \right) \]

\[ = -4\pi G e^{-2\phi_0} \rho \left( h_{00}^{(1)} + 2\phi^{(1)} \right) + 4\pi G e^{-2\phi_0} \rho \left( \Pi + 3\frac{B}{\rho} + 2e^{-4\phi_0}\nu^2 \right) \quad (A57) \]

The left hand side of the equation yields, similarly to GR

\[ - \frac{G}{2G_K} \left( \nabla^2 h_{00}^{(2)} + 2 (G_K)^2 e^{-4\phi_0} \nabla^2 U - 8 (G_K)^2 e^{-4\phi_0} \nabla^2 \Phi_2 \right) \]

For the right hand side we use the definitions of the potentials \( \Phi_1, \Phi_2, \Phi_3, \Phi_4 \), together with Eqs. (A24), (A25) to obtain

\[ - Ge^{-2\phi_0} \nabla^2 \left( 2e^{-4\phi_0} \Phi_1 \right) \]
\[ - 2e^{-2\phi_0} \left( G_K - \frac{kG}{4\pi} \right) \Phi_2 + \Phi_3 + 3\Phi_4 \]

All in all, \( h_{00}^{(2)} \) is given by

\[ h_{00}^{(2)} = - 2e^{-4\phi_0} (G_K)^2 U^2 + 4G_K e^{-6\phi_0} \Phi_1 + \left( 8 (G_K)^2 - 4G_K \left( G_K - \frac{kG}{4\pi} \right) e^{-4\phi_0} \Phi_2 \right) \]
\[ + 2G_K e^{-2\phi_0} \Phi_4 + 6G_K e^{-2\phi_0} \Phi_4 \]

which simplifies to

\[ h_{00}^{(2)} = 2G_K e^{-2\phi_0} \left( -G_K e^{-2\phi_0} U^2 + 2e^{-4\phi_0} \Phi_1 \right) \]
\[ + 2e^{-2\phi_0} G_N \Phi_2 + \Phi_3 + 3\Phi_4 \quad (A58) \]

Lastly, the scalar equation to \( O(2) \) is

\[ \nabla^2 \phi^{(2)} - h_{kk} \phi^{(1)}_{jk} + 2kGe^{-2\phi_0} \rho \phi^{(1)} = kG e^{-2\phi_0} \left( 2\rho e^{-4\phi_0} \nu^2 + \rho \Pi + 3p \right) \quad (A59) \]

Using Eqs. (A31), (A25) we see that

\[ h_{jk} \phi_{jk}^{(1)} - 2kGe^{-2\phi_0} \rho \phi^{(1)} = 2kGe^{-4\phi_0} \left( G_K + \frac{kG}{4\pi} \right) U \rho, \quad (A60) \]

and with Eq. (A23) this is

\[ h_{jk} \phi_{jk}^{(1)} - 2kGe^{-2\phi_0} \rho \phi^{(1)} = 2kGe^{-4\phi_0} G_N U \rho. \quad (A61) \]

We now easily recognize the potentials \( \Phi_1, \Phi_2, \Phi_3, \Phi_4 \) and can write

\[ \phi^{(2)} = - \frac{kG e^{-2\phi_0}}{4\pi} \left( 2e^{-4\phi_0} \phi_1 + 2e^{-2\phi_0} G_N \Phi_2 + \Phi_3 + 3\Phi_4 \right) \quad (A62) \]

We now have all that is required to find \( \tilde{g}_{00} \) to \( O(2) \).

We first write the contribution of the \( O(2) \) term

\[ h_{00}^{(2)} - 2\phi^{(2)} - 2(\phi^{(1)})^2 + 2\phi^{(1)} h_{00}^{(1)} = 2G_N e^{-2\phi_0} \left( 2e^{-4\phi_0} \phi_1 + 2e^{-2\phi_0} G_N \Phi_2 + \Phi_3 + 3\Phi_4 - e^{-2\phi_0} G_N U^2 \right) \quad (A63) \]

Combining all orders of \( \tilde{g}_{00} \), we get
Transforming coordinates according to [A21], rescaling the potentials to the new coordinates and choosing units such that \( G_N = 1 \), we finally obtain
\[
\tilde{g}_{00} = -1 + 2 \tilde{U} - 2 \tilde{U}^2 + 4 \tilde{\Phi}_1 + 4 \tilde{\Phi}_2 + 2 \tilde{\Phi}_3 + 6 \tilde{\Phi}_4, \tag{A65}
\]
exactly as in GR. From here we can instantly read that \( \alpha_3 = \zeta_1 = \zeta_2 = \zeta_3 = \zeta_4 = 0 \), as expected for equations of motion derived from a lagrangian. We also see that \( \beta = 1 \) and \( \xi = 0 \).

APPENDIX B: PPN PARAMETERS FOR TEVES WITH \Æ\Æ\ERO TYPE ACTION

All the setup for the calculation of the PPN parameters remains as in Section [A1] except for the gauge condition for the time-space component of the Einstein metric, which we shall recalculate. \( A' \) and \( v' \) are as in Section [A1]. We follow the same steps as in the calculation for regular TeVeS, and for brevity we only show the relevant equations and altered results. The main difference will be in the calculation of \( \tilde{g}_{0i} \), since the divergence of the vector equation will now constrain the spatial divergence of the vector field, and not the coupling parameters of the theory (see Ref. [20], Section 5.4 for details). This will enable us to calculate \( \tilde{g}_{0i} \) unambiguously, and obtain the preferred frame parameters.

1. \( \tilde{g}_{00} \) to \( O(1) \)

The temporal component of the Einstein equations (A5) with the altered stress tensor Eq. (B1), is to \( O(1) \)
\[
\frac{1}{2} \left( 1 - \frac{K + K_+ + K_4}{2} \right) \nabla^2 h_{00}^{(1)} = -4 \pi e^{-2\phi_0} G \rho \tag{B1}
\]
yielding for \( h_{00}^{(1)} \)
\[
h_{00}^{(1)} = 2 e^{-2\phi_0} G \left( 1 - \frac{K + K_+ + K_4}{2} \right)^{-1} \int \frac{\rho}{|X - X'|} d^3 x' \tag{B2}
\]
For convenience, we define
\[
G_v \equiv G \left( 1 - \frac{K + K_+ + K_4}{2} \right)^{-1} \tag{B3}
\]
In fact, this is the quantity that replaces \( G_K \) in the calculation. With the scalar equation unaltered, we have, as in [A2]
\[
h_{00}^{(1)} = 2 G_v \tilde{U} \tag{B4}
\]
and
\[
\phi_1 = -\frac{kG}{4\pi} \tilde{U}, \tag{B5}
\]
with \( G_v \) replacing \( G_K \). The physical metric in asymptotically Minkowski coordinates is
\[
\tilde{g}_{00} = -1 + 2 G_N \tilde{U} , \tilde{g}_{ij} = \delta_{ij} , \tilde{g}_{0i} = 0 \tag{B6}
\]
with Newton’s constant redefined as
\[
G_N = G \left( 1 - \frac{K + K_+ + K_4}{2} \right)^{-1} + \frac{k}{4\pi} \tag{B7}
\]

2. \( \tilde{g}_{ij} \) to \( O(1) \)

The result here is again the same as in Section [A3] with \( G_v \) replacing \( G_K \)
\[
h_{ij} = 2 G_v e^{-2\phi_0} U \delta_{ij}, \tag{B8}
\]
and in asymptotically Minkowski coordinates, with \( G_N \equiv 1 \)
\[
\tilde{g}_{ij} = \delta_{ij} \left( 1 + 2 \tilde{U} \right). \tag{B9}
\]
Comparing this with Eq. (16), we see that for TeVeS with \Æ\Æ\ERO type action, the PPN parameter \( \gamma \) is still unity. This is a promising result for the modified theory, since solar system data [22] strongly constrains the value of \( \gamma \) to be near unity.

3. \( \tilde{g}_{0j} \) to \( O(1.5) \)

The Lagrange multiplier \( \lambda \) to \( O(1) \) is now given by
\[
\lambda = \frac{1}{2} (K - K_+ ) \nabla^2 h_{00}^{(1)} + 16 \pi G \rho \sinh 2\phi_0 \tag{B10}
\]
The time-space component of Eq. (A5) with the new stress tensor Eq. (B4) to (O(1.5) is then
\[
-\frac{1}{2} \left( \nabla^2 h_{0j} - h_{0k,kj} + h_{k,kj} - h_{k,0j} \right) - \frac{K_2}{2} h_{kk,0j} = -\frac{1}{2} (K_2 + 2 K_+ ) - \frac{1}{2} (K_2 + K_+ ) A^k_{kj} - K_+ A^j_{kk} = -8 \pi G e^{-2\phi_0} \rho v_j , \tag{B11}
\]
and the vector equation to (O(1.5) is
\[
\frac{1}{2} \left( K + K_+ - K_4 \right) h_{00,0j}^ { (1) } + K (h_{0j,kk} - h_{0k,kj} ) + \frac{1}{2} K_2 h_{kk,0j} + K_+ h_{jk,k0} + (K_+ + K_2 - K ) A^k_{kj} + (K + K_+ ) A^j_{kk} = -8 \pi G e^{-2\phi_0} (1 - e^{-4\phi_0} ) \rho v_j \tag{B12}
\]
These two equations should be solved to yield \( h_{0j} \) and \( A^j \). We still need two ingredients to be able to solve the equations \( A^j \), which will be determined from the divergence of the vector equation, and \( h_{0k,k} \), whose value should be set from the demand that \( \tilde{g}_{00} \) be in the standard PPN gauge. In the meanwhile, we write

\[
h_{0k,k} = e^{-2\phi_0} G_v (3 + Q) U_{0,0}, \tag{B13}
\]

and we will solve for \( Q \) in the next step of the calculation. This is inspired by the standard GR gauge condition \( h_{0k,k} = 3U_{0,0} \), since in the GR limit, \( G_v \to G \equiv 1 \) and \( \phi_0 = 0 \). The divergence of the vector equation is

\[
\frac{1}{2} \left( K + K_+ - K_4 \right) h_{00,0,jj}^{(1)} + \frac{1}{2} K_2 h_{kk,jj,0} + K_+ h_{jk,k,j0} + (K_+ + K_2) A^k_{kkj} + K_+ A^k_{kjk} = -8\pi G e^{-2\phi_0} (1 - e^{-4\phi_0}) (\rho_0^j)_{,j} \tag{B14}
\]

Using results from previous steps, this can be written as

\[
-\frac{1}{2} (3K_+ + K + 3K_2 - K_4) e^{-2\phi_0} G_v \chi_{,0jjk} + (2K_+ + K_2) A^j_{,jkk} = 2G e^{-2\phi_0} (1 - e^{-4\phi_0}) V_{j,jkk} \tag{B15}
\]

which simplifies to

\[
A^j = \frac{Ge^{-2\phi_0}}{2K_+ + K_2} \left( 2(1 - e^{-4\phi_0})V_{j,j} + \frac{3K_+ + K + 3K_2 - K_4}{2 - (K + K_+ - K_4)} \chi_{,0j} \right) \tag{B16}
\]

and finally, with \( V_{j,j} = -U_{0,0} = \frac{1}{2} \chi_{,0jj} \), we have for the spatial divergence of the vector field

\[
A^j = \frac{Ge^{-2\phi_0}}{2K_+ + K_2} \left( 1 - e^{-4\phi_0} + \frac{3K_+ + K + 3K_2 - K_4}{2 - (K + K_+ - K_4)} \right) \chi_{,0jj}. \tag{B17}
\]

The addition of extra kinetic terms in the vector action has removed the \( U(1) \) gauge invariance, and enabled us to determine unambiguously the spatial divergence of the vector field.

We may now substitute the vector divergence Eq. (B17) and the metric gauge Eq. (B13) into Eqs. (B12, B11) and solve for \( A^i \) and \( h_{0i} \). Doing so, we obtain

\[
\begin{align*}
\chi_{,0i} &= \frac{4Ge^{-2\phi_0} (Ke^{-4\phi_0} + K_+)}{2KK_+ - (K + K_+)} V_i - \frac{Ge^{-2\phi_0} (2K(2 - (K + K_+ - K_4)) e^{-4\phi_0} + (2KK_+ - (K + K_+))Q + K_+ - 3K + 2K_+(K_4 - K_+ + 2K)) \chi_{,0i}}{(2 - (K + K_+ - K_4))(2KK_+ - (K + K_+))} \tag{B18} \\
A^i &= 2Ge^{-2\phi_0} ((1 - 2K) e^{-4\phi_0} - 1) V_i + Ge^{-2\phi_0} \left( \frac{(K + (2K - 1)(K_+ + K_2))e^{-4\phi_0}}{(K_2 + 2K_+)(2KK_+ - (K + K_+))} + \frac{4(1 - K)K_2^2 - 2(K_4 + 2K_2)(3K - 2) + 1)K_+ + 2(1 + 2K_2)K - (K_4 + 2)K_2}{(K_2 + 2K_+)(2KK_+ - (K + K_+))(2 - (K + K_+ - K_4))} \chi_{,0i} \right) \tag{B19}
\end{align*}
\]

We still need to find \( Q \); to this end, we proceed to the last step in the calculation.

4. \( \tilde{g}_{00} \to O(2) \)

The equation for \( h_{00}^{(2)} \) is
\[
(1 - \frac{K + K_+ - K_4}{2}) \left( -\frac{1}{2} \nabla^2 h_{00}^{(2)} + \frac{1}{2} h_{00,j} \left( h_{jk,k} - \frac{1}{2} h_{kk,j} \right) - \frac{1}{4} \nabla h_{00}^{(1)} \nabla h_{00}^{(1)} - \frac{1}{2} h_{j;j\,0} \right) - \frac{1}{2} (K + 3K_2 + 3K_+ - K_4) A_j^{(1)} - \frac{1}{2} \left( 1 + \frac{3}{2} K_2 + K_+ \right) h_{jj,0} + \frac{1}{2} \left( 2 - (K + K_+ - K_4) \right) h_{j0,j0}
\]

\[
= -4\pi e^{-2\phi_0} G\rho \left( h_{00}^{(1)} + 2\phi_0^{(1)} \right) + 4\pi e^{-2\phi_0} G\rho \left( \Pi + 3\frac{\rho}{\rho} + 2e^{-4\phi_0} \phi^2 \right)
\]  

(B20)

All terms on the second line are proportional to $\chi_{00}$; therefore the standard PPN gauge is obtained from the requirement

\[
- \frac{1}{2} (K + 3K_2 + 3K_+ - K_4) A_j^{(1)} - \frac{1}{2} \left( 1 + \frac{3}{2} K_2 + K_+ \right) h_{jj,0} + \frac{1}{2} \left( 2 - (K + K_+ - K_4) \right) h_{j0,j0} = 0
\]  

(B21)

or

\[
h_{j0,j} = \left( \frac{1 + \frac{3}{2} K_2 + K_+}{2 - (K + K_+ - K_4)} \right) h_{jj,0} + \frac{Ge^{-2\phi_0} (3K_+ + K + 3K_2 - K_4)}{2K_+ + K_2} \left( 1 - e^{-4\phi_0} + \frac{3K_+ + K + 3K_2 - K_4}{2 - (K + K_+ - K_4)} \right) \chi_{0jj}
\]  

(B22)

Combining this with (B13), we obtain for $Q$

\[
Q = \frac{3K_+ + K + 3K_2 - K_4}{2K_+ + K_2} \left( e^{-4\phi_0} + \frac{2 - 4K_+}{2 - (K + K_+ - K_4)} \right)
\]  

(B23)

\[
\tilde{g}_{0i} = \left( \frac{2 \left( (2 + 3K_2 + 2K_+) e^{2\phi_0} - (2 - (K + K_+ - K_4)) e^{-2\phi_0} \right) \sinh (2\phi_0)}{(K_2 + 2K_+) (2 - (K + K_+ - K_4))} - 2 \left( (e^{2\phi_0} - (1 - 2K)) e^{-2\phi_0} \sinh (2\phi_0) - (K_+ e^{4\phi_0} + K) \right) \right) \frac{2KK_+ - (K + K_+)}{2K + K_+}
\]

\[
- \frac{1}{2} \left( (K_4 - K + 3K_+)(2 + 3K_2 + 2K_+) e^{4\phi_0} + (K_2 + 3K_2 + K - K_4)(2 - (K + K_+ - K_4)) \right) \frac{2KK_+ - (K + K_+)}{(K_2 + 2K_+) (2 - (K + K_+ - K_4))^2}
\]

\[
+ \left( \frac{2 \left( (K_4 - K + 3K_+)(2 + 3K_2 + 2K_+) e^{4\phi_0} + (K_2 + 3K_2 + K - K_4)(2 - (K + K_+ - K_4)) \right) \sinh (2\phi_0)}{(K_2 + 2K_+) (2 - (K + K_+ - K_4))^2} - 2 \left( (2 + 3K_2 + 2K_+) e^{2\phi_0} + (2 - (K + K_+ - K_4)) e^{-2\phi_0} \sinh (2\phi_0) \right) \right) \frac{2KK_+ - (K + K_+)}{2K + K_+}
\]

(B24)

In the standard PPN gauge, for TeVeS with Æther type vector kinetic term, the equation for $h_{00}^{(2)}$ is the
same as for simple TeVeS, Eq. (A56), except for the replacement of $G_K$ by $G_v$. Since the scalar equation remains unchanged, we can immediately write the result for $\tilde{g}_{00}$, after coordinate redefinition and appropriate rescaling of the potentials, and this time with

$$G_N = G \left( \left( 1 - \frac{K + K_+ - K_1}{2} \right)^{-1} + \frac{k}{4\pi} \right) \equiv 1$$

$$\tilde{g}_{00} = -1 + 2\tilde{U} - 2\tilde{U}^2 + 4\tilde{\Phi}_1 + 4\tilde{\Phi}_2 + 2\tilde{\Phi}_3 + 6\tilde{\Phi}_4, \quad (B25)$$

Once again we have $\alpha_3 = \zeta_1 = \zeta_2 = \zeta_3 = \zeta_4 = 0$, as expected for equations of motion derived from a lagrangian, and also $\beta = 1$ and $\xi = 0$, as for simple TeVeS. We can now compare Eq. (B24) to the standard form Eq. (15), and extract the preferred frame parameters

$$\alpha_1 = 8 \left( G \left( (e^{2\phi_0} - (1 - 2K)e^{-2\phi_0}) \sinh (2\phi_0) - (K_+ e^{4\phi_0} + K) \right) - 1 \right) \quad (B26)$$

$$\alpha_2 = \frac{\alpha_1}{2} + 4G \left( (2 + 3K_2 + 2K_+)^2 \sinh (2\phi_0) - (K_2 + 2K_+) (2 - (K + K_+ - K_4)) \right) \quad (B27)$$

$$\alpha_3 = 8 \left( G \left( (e^{2\phi_0} - (1 - 2K)e^{-2\phi_0}) \sinh (2\phi_0) - (K_+ e^{4\phi_0} + K) \right) - 1 \right) \quad (B28)$$

In the above,

$$G = \frac{4\pi(2 - (K + K_+) + K_4)}{8\pi + k(2 - (K + K_+) + K_4)}.$$

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