Inverse spectral results for Schrödinger operators on the unit interval with potentials in $L^p$ spaces

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Abstract

We consider the Schrödinger operator on $[0, 1]$ with potential in $L^p$. We prove that two potentials already known on $[a, 1]$ ($a \in (0, 1/2]$) and having their difference in $L^p$ are equal if the number of their common eigenvalues is sufficiently large. The result here is to write down explicitly this number in terms of $p$ (and $a$) showing the role of $p$.

1. Introduction

This paper is concerned with the Schrödinger operator

$$A_{q,h,H} = -\frac{d^2}{dx^2} + q$$

on $[0, 1]$ associated with the boundary conditions

$$u'(0) + hu(0) = 0, \quad u'(1) + H u(1) = 0.$$ (2)

Here, the potential $q$ is a real-valued function belonging to $L^1([0, 1])$ and $h, H \in \mathbb{R}$. The corresponding spectrum is a sequence of simple eigenvalues. Let us denote by $(\lambda_j(q, h, H))_{j \in \mathbb{N} \cup \{0\}}$ the increasing sequence of its eigenvalues. Then, the asymptotic expansion is [LG]

$$\lambda_j(q, h, H) = j^2 \pi^2 + 2(H - h) + \int_0^1 q(x) \, dx + o(1) \quad \text{as} \quad j \to +\infty.$$ (3)

In 1978, Hochstadt and Lieberman [HL] proved that the whole spectra of $A_{q,h,H}$ determine uniquely $q$ when it is already known on $[\frac{1}{2}, 1]$. More precisely, if $q_1, q_2 \in L^1[0, 1]$, $q_1 = q_2$ a.e. on $[\frac{1}{2}, 1]$ and if the spectrum of $A_{q_1,h_1,H}$ is exactly the spectrum of $A_{q_2,h_2,H}$ then $q_1 = q_2$ and $h_1 = h_2$. In 2000, several extensions of this result are given by Gesztesy and Simon [GS] considering that the potentials belonging to $L^1([0, 1])$ are known on a larger interval $([a, 1]$ with $a \in (0, 1/2]$) and assuming that the common spectrum of $A_{q_1,h_1,H}$ and $A_{q_2,h_2,H}$ is...
sufficiently large (in terms of \( a \)). Another result in \([GS]\) is to assume that the potential belongs to \( C^3 \) so that the number of common eigenvalues is given in terms of \((a, k)\). Our aim here is to obtain a similar result for potentials in \( L^p \). Actually, only their difference needs to be in \( L^p \). We shall derive that two potentials already known on \([a, 1]\) and having their difference in \( L^p \) are equal if their common spectrum is sufficiently large (depending on \((a, p)\)) (see theorem 1.1).

Let us also mention at this point that our proof is different from the proof in \([GS]\).

For any \( a = (\alpha_j)_{j \in \mathbb{N}, |a_j| < t}, \alpha_j \in \mathbb{C} \), set

\[
\sigma (\alpha_j) = \{ j \in \mathbb{N} \cup \{0\} \mid |\alpha_j| < t \}, \quad \forall t \geq 0.
\]

Our purpose here is to prove the following result.

**Theorem 1.1.** Fix \( q_1, q_2 \in L^1([0, 1]) \) and \( h_1, h_2, H \in \mathbb{R} \). Consider an infinite set \( S \)

\[
S \subset \sigma (A_{q_1, h_1, H}) \cap \sigma (A_{q_2, h_2, H}).
\]  \((4)\)

Fix a \( a \in (0, \frac{1}{2}] \) and \( p \in [1, +\infty) \). Suppose that \( q_1 = q_2 \) on \([a, 1]\) and \( q_1 - q_2 \in L^p ([0, a]) \).

Assume that there exists a real number \( C \) such that

\[
2an_{\sigma(A)} (t) + C \geq n_{\sigma(A)} (t) \geq 2an_{\sigma(A)} (t) + \frac{1}{2p} - 2a, \quad t \in S, \quad t \text{ large enough,} \quad (H_1)
\]

where \( A \) denotes either \( A_{q_1, h_1, H} \) or \( A_{q_2, h_2, H} \). Then, \( h_1 = h_2 \) and \( q_1 = q_2 \).

Roughly speaking, theorem 1.1 says that the potential given on \([a, 1]\) together with a sufficiently large part (depending on \( a \)) of its spectrum determine entirely the potential on \([0, 1]\).

In the particular case \( p = 1 \), a similar result (among many others) is proved in \([GS, \text{ theorem 1.3}]\) with the following modifications in \([GS]\):

(i) the lower bound in \((H_1)\) is \(2an_{\sigma(A)} (t) + \frac{1}{2} - a\) and \( t \in \mathbb{R}, t > 0 \) is large enough;

(ii) there is no upper bound in \((H_1)\).

Concerning (ii) our result is weaker and concerning (i) it is stronger for the following two reasons: 1. we have \(-2a\) instead of \(-a\); 2. the parameter \( t \) needs only to be in \( S \). These two points are involved in the proof of the following corollary.

The upper bound in \((H_1)\) imposes here that the given spectrum is in some sense regularly spaced. This is not required in \([GS]\). However, because of the points 1 and 2, our lower bound is well-adapted to results like the even spectrum and the potential given on \([\frac{1}{4}, 1]\) determine the potential on \([0, 1]\) (see corollary below) whereas in that case \([GS]\) needs slightly more than half of the spectrum (see the remark below theorem 1.3 in \([GS]\)).

**Corollary 1.2.** The even (resp. odd) spectrum \( \{\lambda_{2j}(q, h, H)\}_{j \geq 0} \) (resp. \( \{\lambda_{2j+1}(q, h, H)\}_{j \geq 0} \)) and \( q|_{[0, \frac{1}{4}]} \) determine \( q \) on \([0, 1]\).

**Proof.** For the even (resp. odd) case, apply theorem 1.1 with \( a = \frac{1}{2}, S = \{\lambda_{2j}\}_{j \geq 0} \) (resp. \( S = \{\lambda_{2j+1}\}_{j \geq 0} \)) and use \( n_{\sigma(A)} (\lambda_{2j}) = 2j + 1 \) and \( n_{\sigma(A)} (\lambda_{2j+1}) = 2j + 2 \) (resp. \( n_{\sigma(A)} (\lambda_{2j+1}) = 2j + 1 \)).

**Remark 1.3.** Similar results may be obtained also for the Dirichlet boundary conditions. Moreover, this method may be applied analogously to the AKNS systems. For AKNS systems, one may also refer to the work in \([DG]\) where the given spectrum is regularly spaced: \( S = \{\lambda_{jk}, j \geq 0\} \) with \( k \) being a fixed positive integer.

Whereas the proof of the results in \([GS]\) relies on the Weyl–Titmarsh functions, the starting point here is different and it is based on an idea taken in \([L]\) (which appears in a short proof that
two spectra determine the potential). Let us describe the main points. (1) An entire function $f$ depending on $q_1 - q_2$ restricted to $[0, a]$ is introduced having the property to vanish on the common eigenvalues (to be complete, we mention that $f(z)$ is the rhs of (14)). (2) The second step is to use the growth property of $f$ to derive that it is identically vanishing which directly follow from the maximum modulus principle. (3) The last step is to derive that $f \equiv 0$ implies that $q_1 \equiv q_2$.

Our contribution here is to modify the second step above in order to deal with potentials in $L^p$. The main fact is to replace the maximum modulus principle by a result of Levinson stated in Levin [L] (see step 4). Whereas the maximum modulus principle is applied to $f$, we shall apply Levinson’s result to the Fourier transform of $f$. More precisely, we rather use the Fourier transform of $f$ that we call $g$ for the following two reasons. The first one is that $f$ is actually roughly speaking close to the inverse Fourier transform of $q_1 - q_2$, so that $g$ is close in some sense to $q_1 - q_2$ and it is expected that this imply that the assumption $(q_1 - q_2) \in L^p$ is rewritten as $g \in L^p$ without any loss of information. The second one is (since the inverse Fourier transform of $g$ vanishes on the common eigenvalues) to remark a result given in [L] due to Levinson and essentially stating that the inverse Fourier transform of function being in $L^p$ is entirely vanishing if it has a sufficiently large number (depending on $p$) of zeros. Therefore, this shall replace in our proof the point 2 above.

This work is concerned with $L^p$ spaces and a work involving other spaces is in progress. In the following section, we establish theorem 1.1. Its proof is split into five steps. The first step is to define properly the function $g$ and to give some of the properties that shall be used in the following. The second point is to recall that the inverse Fourier transform of $g$ is vanishing on the common spectra of $A_{q_1, h_1, H}$ and $A_{q_2, h_2, H}$. In the third step, we introduce an auxiliary property $(H_2)$ derived from $(H_1)$. The fourth step consists of proving that if $(H_2)$ is satisfied then $g$ is vanishing. It is at this point that we use Levinson’s result. In the fifth step, we give a short proof of the already known fact: $g$ equals zero implies $q_1 = q_2$ and $h_1 = h_2$.

2. Proof of theorem 1.1

2.1. Step 1: definition of $g$

- **Definition of $\psi$.** For $z \in \mathbb{C}$, let $\psi(\cdot, z, q, h)$ defined on $[0, 1]$ be the solution to $(-\frac{d^2}{dx^2} + q)\psi = z\psi, \psi(0) = 1, \psi'(0) = -h$. It is known that $\psi(x, \cdot, q, h)$ is an entire function and [LG]

$$\psi(x, z, q, h) = \cos \sqrt{z}x + O \left( \frac{q \sqrt{z}^x}{\sqrt{|z|}} \right), \quad \text{as } |z| \to +\infty, \quad (5)$$

uniformly in $x \in [0, 1]$.

- **Definition of $r$.** Fix $h_1, h_2 \in \mathbb{R}$ and $q_1, q_2 \in L^1([0, 1])$. For $x \in [0, 1], z \in \mathbb{C}$, let $r(z, x) = -\psi(x, z^2, q_1, h_1)\psi(x, z^2, q_2, h_2) + \frac{1}{2}(1 + \cos 2\pi x)$. Clearly,

$$r(z, x) = O \left( \frac{e^{2|z|x}}{|z|} \right), \quad \text{as } |z| \to +\infty, \quad (6)$$

uniformly in $x \in [0, 1]$.

In order to apply Levinson’s result on $[-2a, 2a]$ in step 4, we introduce below a scaling and extension by parity operator $\mathcal{E}$ which shall be always applied to $r$ and $q_1 - q_2$ in the following. In particular, it allows us to define below $s$ with the usual Fourier transform instead of the cosine Fourier transform.
Remark 2.1. It may be also natural to consider for \( z \in \mathbb{C}, f(z) = \int_0^\alpha (-1 + 2\psi(x, z^2, q_1, h_1)\psi(x, z^2, q_2, h_2))(q_1(x) - q_2(x)) \, dx \) (see (14)) and define \( g \) as the Fourier transform of \( f \). However, it is not directly clear that \( g \) would be in \( L^p \) when \( q_1 - q_2 \) is \( L^p \), in particular \( g(y) \) may not be written as \( \mathcal{F}^{-1} g = f \). Moreover, \( f \) itself is not involved in the main step (step 4). Therefore, we choose to define \( g \) by (11), verify (12), (13) and check that \( \mathcal{F}^{-1} g = f \) (cf (14)).
2.2. Step 2: \( z^2 \in S \Rightarrow \int_{-2a}^{2a} e^{izy} g(y) \, dy = 0 \)

Since \( g \in L^1([-2a, 2a]) \) then (11) implies that \( \int_{-2a}^{2a} e^{izy} g(y) \, dy = \mathcal{F}^{-1}(\mathcal{E}(q_1 - q_2))(z) - 2 \int_{-2a}^{2a} e^{izy} s(y, x) \mathcal{E}(q_1 - q_2)(x) \, dx \, dy \).

\[
\int_{-2a}^{2a} e^{izy} g(y) \, dy = \mathcal{F}^{-1}(\mathcal{E}(q_1 - q_2))(z) - 2 \int_{-2a}^{2a} e^{izy} \mathcal{F}_1 \mathcal{E}_2(r)(y, x) \mathcal{E}(q_1 - q_2)(x) \, dx \, dy
\]

\[
= \int_{-2a}^{2a} (e^{izy} - 2 \mathcal{E}_2(r)(z, x)) \mathcal{E}(q_1 - q_2)(x) \, dx
\]

\[
= 4 \int_{0}^{\pi} (\cos 2zx - 2r(z, x)) (q_1 - q_2)(x) \, dx
\]

\[
= 4 \int_{0}^{\pi} (-1 + 2 \psi(x, z^2, q_1, h_1) \psi(x, z^2, q_2, h_2))(q_1(x) - q_2(x)) \, dx. \quad (14)
\]

This is known to vanish for \( z \in S \). Indeed, multiply \( (-\frac{d^2}{dx^2} + q_1(x) - z^2) \psi(x, z^2, q_1, h_1) = 0 \) by \( \psi(x, z^2, q_2, h_2) \), multiply \( (-\frac{d^2}{dx^2} + q_2(x) - z^2) \psi(x, z^2, q_1, h_1) = 0 \) by \( \psi(x, z^2, q_2, h_2) \) and integrate their difference on \([0, 1] \) to obtain that the term in (14) equals \( 2(h_1 - h_2) + \int_{0}^{\pi} q_2(x) - q_1(x) \, dx \). This term is zero from (3) since it is assumed that \( S \) contains an infinite number of points.

**Remark 2.2.** Since the translation on the potential \( q \) acts as a translation on the spectrum of \( A_{q,h,H} \), it is assumed without loss of generality that the \( \lambda_j(q_1, h_1, H) \) and \( \lambda_j(q_2, h_2, H) \) are positive real numbers.

Let us write \( S = (s_j)_{j \geq 0} \), where \( (s_j) \) is an increasing sequence and consider the following property: there exists \( D \in \mathbb{R} \) such that

\[
\frac{\pi}{2a} j + D \leq \sqrt{s_j} \leq \frac{\pi}{2a} j + C + \frac{1}{j} \quad \text{as} \quad j \to \infty. \quad (H_2)
\]

2.3. Step 3: \( (H_1) \Rightarrow (H_2) \)

Since \( s_j \in (\lambda_n(q_1, h_1, H))_{n \in \mathbb{N}\cup\{0\}} \) then \( s_j = \lambda_{m_j}(q_1, h_1, H) \) for some increasing sequence of integer numbers \( (m_j)_{n \in \mathbb{N}\cup\{0\}} \). In particular, \( n_{\sigma(A_{q_1, h_1})}(s_j) = m_j + 1 \) and following (3) we have \( n_{\sigma(A_{q_1, h_1})}(s_j) = \mathbb{Z}[n \geq 0 \mid \lambda_n(q_1, h_1, H) \leq s_j] \)

\[
= \mathbb{Z} \left\{ n \geq 0 \mid n \pi + O \left( \frac{1}{n} \right) \leq m_j \pi + O \left( \frac{1}{m_j} \right) \right\}. \quad (15)
\]

We also have

\[
n_{\mathcal{E}}(s_j) = j + 1. \quad (16)
\]

Let us check that \( (H_1), (15), (16) \) imply \( (H_2) \).

On one side, the second inequality in \( (H_1) \) together with (15), (16) reads

\[
j + 1 \geq 2a \left( \frac{\sqrt{s_j}}{\pi} + 1 + O \left( \frac{1}{j} \right) \right) + \frac{1}{2p} \geq 2a,
\]

which is the second inequality in \( (H_2) \). On the other side, the first inequality in \( (H_1) \) together with (15), (16) gives \( j + 1 \leq 2a \left( \frac{\sqrt{s_j}}{\pi} + 1 + O \left( \frac{1}{j} \right) \right) + C \), that is to say,

\[
\sqrt{s_j} \geq \pi \left( \frac{j + 1 - C}{2a} - 1 \right) + O \left( \frac{1}{j} \right),
\]

which imply the first inequality in \( (H_2) \).
2.4. Step 4: \((H_2) \Rightarrow g = 0\)

Let \(\alpha = (\alpha_j)\) be a sequence of complex numbers and define \(N_\alpha(R) = \int_0^R n_\alpha(t) \, dt.\) Let us recall the following result [L, app. III, section 2, theorem 3].

Let \(q \in ]1, +\infty[, \alpha = (\alpha_j)\) a sequence of complex numbers satisfying

\[
\limsup_{R \to \infty} N_\alpha(R) = 2R + \frac{1}{q} \ln R > -\infty.
\]

Then the family \(\{y \mapsto e^{\alpha_j y}\}\) is complete in \(L^q([-\pi, \pi])\) (if \(q = \infty\) then \(\{y \mapsto e^{\alpha_j y}\}\) is complete in \(C^0([-\pi, \pi])\)).

Set (see remark 2.2)

\[
S_2^1 = \{ \pm \sqrt{j}, \, j \geq 0 \}, \quad S_2^{1*} = \{ \sqrt{j}, \, j \geq 0 \}.
\]

In particular, replacing \(\pi\) by \(2a\) and setting \(a = S_2^1\), this result reads as: if

\[
\limsup_{R \to \infty} N_{S_2^1}((R) - \frac{4a}{\pi} R + \frac{1}{q} \ln R > -\infty \quad (H_L)
\]

then \(\{e^{\alpha_j y}\}\) is complete in \(L^q([-2a, 2a])\) (resp. \(C^0([-2a, 2a])\) if \(q \in ]1, +\infty[\) (resp. if \(q = +\infty\)). To see this, one may either reproduce the proof taken in [L] while replacing \(\pi\) with \(2a\) or use the Riesz representation theorem to write \(L \in (L^q([-2a, 2a]))'\) as \(L(h) = \int_{-2a}^{2a} l(y) h(y) \, dy\) for some \(l \in L^q([-2a, 2a])\) and make the change of variable \(y' = \frac{\pi}{2a} y\), then use \(n_\alpha(t) = n_{2a}(\lambda t)\) and \(N_\alpha(t) = N_{2a}(\lambda t)\) for all \(\lambda > 0\) and \(t > 0\).

Let us check that

\[
(H_2) \implies (H_L) \quad \text{if} \quad \frac{1}{p} + \frac{1}{q} = 1.
\]

Since \(s_j > 0\) for all \(j \geq 0\), \(n_{S_2^1}(t) = 2n_{S_2^{1*}}(t)\) for all \(t \geq 0\). Moreover, \(n_{S_2^{1*}}(t) = j + 1, \forall t \in [\sqrt{j}, \sqrt{j+1}], \forall j \geq 0\). Therefore,

\[
N_{S_2^1}(\sqrt{j}) = 2 \sum_{k=0}^{j-1} \int_{\sqrt{k}}^{\sqrt{k+1}} \frac{1}{\sqrt{t}} \, dt \quad \text{which gives}
\]

\[
N_{S_2^1}(\sqrt{j}) = 2 \left( j \ln \sqrt{j} - \sum_{k=0}^{j-1} \ln \sqrt{k} \right), \quad \forall j \geq 0. \quad (18)
\]

Following \((H_2)\) (left inequality),

\[
j \ln \sqrt{j} \geq j \left( \ln j + \ln \frac{\pi}{2a} \right) + O(1), \quad \text{as} \quad j \to +\infty. \quad (19)
\]

Following \((H_2)\) (right inequality) and \(\ln j! = (j + \frac{1}{2}) \ln j - j + O(1)\) as \(j \to +\infty\),

\[
\sum_{k=0}^{j-1} \ln \sqrt{k} \leq j \ln \frac{\pi}{2a} + \sum_{k=0}^{j-1} \ln(k + 1) + \ln \left( 1 - \frac{1}{2p} + O \left( \frac{1}{k} \right) \right)
\]

\[
\leq j \ln \frac{\pi}{2a} + \ln j! - \sum_{k=0}^{j-1} \frac{1}{2p} + O \left( \frac{1}{k} \right) + O \left( \frac{1}{k^2} \right)
\]

\[
\leq j \ln \frac{\pi}{2a} + \left( j + \frac{1}{2} \right) \ln j - j - \frac{1}{2p} (\ln j + \gamma) + O(1), \quad (20)
\]

where \(\gamma\) is the Euler constant and \(j \to +\infty\). In particular, combining (18) with (19), (20) and again using \((H_2)\) give

\[
N_{S_2^1}(\sqrt{j}) \geq \left( \frac{1}{p} - 1 \right) \ln j + 2j + O(1)
\]

\[
\geq \left( \frac{1}{p} - 1 \right) \ln \sqrt{j} + \frac{4a}{\pi} \sqrt{j} + O(1),
\]
as $j \to \infty$. This proves (17) and $[e^{\pm i \sqrt{T_j^3}}]$ is complete in $L^q([-2a, 2a])$ (resp. $C^0([-2a, 2a])$) if $q \in [1], +\infty]$ (resp. if $q = +\infty$). For $p \in [1], +\infty]$ (resp. $p = 1$), define $L \in (L^q([-2a, 2a]))^\prime$ (resp. $L \in (C^0([-2a, 2a]))^\prime$) by

$$L : \left\{ \begin{array}{ll} L^q([-2a, 2a]) & \to \mathbb{C} \\ h & \mapsto \int_{-2a}^{2a} h(y) g(y) \, dy \end{array} \right.$$

According to step 1, $L$ is well defined. Following step 2, $L(y \mapsto e^{\pm i \sqrt{T_j^3}}) = 0$ for all $j$. The completeness property of $\{ y \mapsto e^{\pm i \sqrt{T_j^3}} \}$ implies that $L$ is vanishing identically. This proves that $g \equiv 0$.

2.5. Step 5: $g = 0 \Rightarrow (h_1, q_1) = (h_2, q_2)$

The fact that $h_1 = h_2$ shall follow the asymptotic expansions of the eigenvalues and the fact that $q_1 = q_2$ is actually proved in [L] since $g = 0$ implies $\int_0^1 \left( -\frac{1}{x} + \psi(x, z, q_1, h_1) \psi(x, z, q_2, h_2)(q_1 - q_2)(x) \right) \, dx = 0, \forall z \in \mathbb{R}$. For sake of completeness, let us give a shorter proof involving only the function $s$ and its properties. Indeed, (8), (9) and the definition of $T_s$, Fubini’s theorem yield for any $u \in L^1([-2a, 2a])$ and for any $n \geq 1$

$$\left\| T_s^n u \right\|_{L^1([-2a, 2a])} \leq 2^n \int_{|t_{n+1}| \leq \cdots \leq |t_2| \leq |t_1| \leq 2a} |s(t_{n+1}, t_n) \cdots s(t_2, t_1) u(t_1)| \, dt_1 \cdots dt_{n+1}.$$

Consequently,

$$\left\| T_s^n \right\|_{L^1([-2a, 2a])} \leq \frac{(4 \sqrt{a} \sup_{s \in [-2a, 2a]} \|s(\cdot, x)\|_{L^2([-2a, 2a])})^n}{\sqrt{n!}}.$$

In particular, this proves that $T_s^n$ is a contracting map in $L^1([-2a, 2a])$ for $n$ large enough. It follows that 0 is the unique fixed point of $T_s$ in $L^1([-2a, 2a])$. By (13), this proves that $q_1 = q_2$ and the fact that $h_1 = h_2$ follows the asymptotic expansion of the eigenvalues (3).

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