On symmetric Weierstraß semigroups and Poincaré series *

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Abstract

The aim of this paper is to introduce and investigate the Poincaré series associated with the Weierstraß semigroup of one and two rational points at a (not necessarily irreducible) non-singular projective algebraic curve defined over a finite field, as well as describe their functional equations in the case of affine complete intersection.

1 Introduction

The origin of the Weierstraß semigroup lies in the “Lückensatz” pointed out by Karl Weierstraß in some of his lectures in the 1860’s: for every point \( P \) on a compact Riemann surface \( X \) of genus \( g \), there are exactly \( g \) integers \( l_i(P) \) with

\[
1 = l_1(P) < \ldots < l_g(P) = 2g - 1
\]

so that there is no meromorphic function on \( X \) having a pole at \( P \) of multiplicity \( l_i(P) \) as its only singularity (see [W, III, pp.297-307]). The set \( G(P) := \{l_i(P)\} \) is called the gap set of \( P \). After some remarkable results of Brill and Noether ([B-N]), Hürwitz realized that, if \( \alpha, \beta \in \mathbb{N} \setminus G(P) \) and \( z_\alpha, z_\beta \) are two functions having a pole of order \( \alpha \) resp. \( \beta \) at \( P \) as their only singularities, then the function \( z_\alpha z_\beta \) has a pole of order \( \alpha + \beta \) at \( P \) as its only singularity. It implies that \( \mathbb{N} \setminus G(P) \) is a semigroup (cf. [H], p. 409 after equation (6)): the Weierstraß semigroup associated with the point \( P \).

Let \( X \) be a non-singular projective algebraic curve defined over a finite field \( \mathbb{F} \). Let \( P_1, \ldots, P_r \) be a set of rational points on \( X \) and consider the family of (finitely dimensional) \( \mathbb{F} \)-vector subspaces \( \Gamma(X, mP) := \Gamma(X, m_1P_1 + \ldots + m_rP_r) \) with \( m_i \in \mathbb{Z} \) for all \( i = 1, \ldots, r \) (cf. Subsection 2.2 below). This family gives rise to a \( \mathbb{Z}^r \)-multi-index filtration on the \( \mathbb{F} \)-algebra \( A := \bigcup_{m \in \mathbb{Z}^r} \Gamma(X, mP) \) of

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the affine curve $X \setminus \{P_1, \ldots, P_r\}$. This multi-index filtration is related to Weierstraß semigroups (with respect to several points in general, see \cite{D-2}) and, in the case of finite fields, to the methodology for trying to improve the Goppa estimation of the minimal distance of algebraic-geometrical codes (see \cite{Ca-To}). In \cite{C-D-GZ} is shown a connection of that filtration with global geometrical-topological aspects in a particular case. Furthermore, if we consider the dimensions of the spaces $\Gamma(X, mP)$, then we associate a suitable Poincaré series with them. Poincaré series are typically series for which the coefficients represent discrete information about the class of the study objects – the Weierstraß semigroup in this case.

Thus, a natural question, which this work is devoted to, is to define in a proper way the notion of Poincaré series and to describe it, in the case of a Weierstraß semigroup associated with one or two points, i.e., when $r \in \{1, 2\}$. The paper goes as follows. Section 2 is devoted to the definition and main properties of the Weierstraß semigroup for not necessarily irreducible curves using the notion of Manis valuation. In Section 3 we introduce and describe the Poincaré series of the Weierstraß semigroup associated with one and two points. We also consider symmetry properties in Section 4 (revising the work of Delgado \cite{D-2}), from which we deduce functional equations for the Poincaré series.

2 Weierstraß semigroup for reducible curves

2.1 Reducible curves and Manis valuations

We can extend the concept of valuation to start from a ring satisfying certain properties, instead of a field. More precisely, let $K$ be a ring with large Jacobson radical (i.e., a ring in which every prime ideal containing the Jacobson radical is maximal; for instance, a semilocal ring) and being its own total ring of fractions. A surjective map $w : K \rightarrow \mathbb{Z}_\infty := \mathbb{Z} \cup \{\infty\}$ such that

1. $w(1) = 0$,
2. $w(0) = \infty$,
3. $w(ab) = w(a) + w(b)$, for all $a, b \in K$,
4. $w(a + b) \geq \min\{w(a), w(b)\}$, for all $a, b \in K$,

is called a Manis valuation of the ring $K$ (cf. \cite{Ki-Vi} Chapter I, Definition 2.8, p. 9)).

Let $X$ be a reducible projective curve of arithmetical genus $g$ defined over a finite field $\mathbb{F}$, which is locally a complete intersection (or simply a curve from now on). Let $X = X_1 \cup \ldots \cup X_r$ be its decomposition into irreducible components. Each $X_i$ has associated a function field $K(X_i)$, and $K(X) := K(X_1) \times \ldots \times K(X_r)$ is the ring of rational functions over $X$. A point of $X$ is called regular
if it lies on just one irreducible component of $X$ and if it is a simple point of this component. Let us take $P$ a regular point of $X$. Then there exists some $i \in \{1, \ldots, r\}$ such that $P \in X_i$ and the local ring of $X$ at $P$ coincides with the local ring of $X_i$ at $P$, and therefore one can associate with $P$ a discrete valuation $v_P : K(X_i) \to \mathbb{Z}_\infty$. Let us consider $g = (g_1, \ldots, g_r) \in K(X)$. For every $1 \leq j \leq r$, we define

$$w'(g_j) = \begin{cases} v_P(g_j), & \text{if } j = i; \\
\infty, & \text{otherwise.}\end{cases}$$

One can easily prove that the map $w_P : K(X) \to \mathbb{Z}_\infty$ given by $w_P(g) = \min_{1 \leq j \leq r} w'_P(g_j)$ is a discrete Manis valuation of the ring $K(X)$.

Let $P := \{P_1, \ldots, P_r\}$ be a set of regular points on $X$, and set $w_i = w_{P_i}$ a discrete Manis valuation of $K(X)$ associated with each $P_i$, for all $1 \leq i \leq r$. If we consider the ring of rational functions $K(X')$ of the affine curve $X' := X \setminus P$, then we may define the Weierstraß semigroup of $X$ at $P$ as the following subsemigroup of $(\mathbb{Z}_\infty)^r$:

$$\Gamma_P := \{-(w_1, \ldots, w_r)(f) \mid f \in K(X')\}.$$

**Notations.** Let $\underline{n} = (n_1, \ldots, n_r) \in \mathbb{Z}^r$, $J = \{i_1, \ldots, i_d\} \subset I = \{1, \ldots, r\}$. It is useful to introduce the followings subsets of $\mathbb{Z}^r$:

$$\nabla_J(\underline{n}) = \{(m_1, \ldots, m_r) \in \mathbb{Z}^r \mid m_i = n_i \forall i \in J, \ m_j < n_j \forall j \not\in J\};$$

$$\nabla(\underline{n}) = \bigcup_{i=1}^r \nabla_i(\underline{n});$$

and the following subsets of $\Gamma_P$:

$$\nabla_J(\underline{n}) = \nabla_J(\underline{n}) \cap \Gamma_P;$$

$$\nabla(\underline{n}) = \nabla(\underline{n}) \cap \Gamma_P = \{(m_1, \ldots, m_r) \in \Gamma_P \mid m_j < n_j \forall j \text{ and } m_i = n_i \text{ for exactly one } i\};$$

$$\nabla^*_J(\underline{n}) = \bigcup_{i \in J} \nabla_J(\underline{n}) = \{(m_1, \ldots, m_r) \in \Gamma_P \mid m_i = n_i \text{ and } m_j \leq n_j \forall j \neq i\}.$$

Denote the standard basis of $\mathbb{Z}^r$ as $e_1, \ldots, e_r$, i.e., the vectors in $\mathbb{N}^r$ with 1 in the $i$-th position and 0 in the other ones.

**Definition 2.1** An element $\underline{n} = (n_1, \ldots, n_r) \in \mathbb{Z}^r$ is said to be an absolute maximal for $\Gamma_P$ if $\nabla(\underline{n}) = \emptyset$. If moreover, for any $J \subset I$ with $\sharp J \geq 2$ one has $\nabla_J(\underline{n}) \neq \emptyset$, then $\underline{n}$ is said to be relative maximal for $\Gamma_P$.

**Remark 2.2** For the case $r = 2$, the concepts of relative and absolute maximal coincide. Thus we will simply say “maximal” points of $\Gamma_{P_1,P_2}$. The set of maximal points of $\Gamma_{P_1,P_2}$ will be denoted by $\mathcal{M}_{P_1,P_2}$.
2.2 The Riemann-Roch theorem for reducible curves

Let $X$ be a curve. Let $\mathcal{O}_X$ be the sheaf of local rings on $X$. We can take, for each invertible sheaf $L$ on $X$, a divisor $D$ with support contained in the set of regular points of the curve such that $L = L(D)$, where $L(D)$ is the divisorial sheaf related to $D$. As $L(D)$ is coherent, the cohomology groups $H^q(X, L(D))$ are $\mathbb{F}$-vector spaces of finite dimension (if $q \geq 2$ its dimension is 0). Furthermore, since $L(D) \cap \mathcal{O}_X$ is a coherent sheaf too, we can define the degree of the divisor $D$ as

$$\deg(D) = \dim_\mathbb{F} \Gamma(X, L(D))/\mathcal{O}_X) - \dim_\mathbb{F} \Gamma(X, \mathcal{O}_X/(L(D) \cap \mathcal{O}_X)).$$

Such a definition extends in a natural way the classical one.

As $X$ is locally a complete intersection, the dualizing sheaf of $X$ is an invertible sheaf on $X$ and we can take the corresponding canonical divisor $K$ such that supp($K$) is contained in the set of regular points on $X$. By setting

$$\ell(D) = \dim_\mathbb{F} H^0(X, L(D)) = \dim_\mathbb{F} \Gamma(X, L(D))$$
$$i(D) = \dim_\mathbb{F} H^1(X, L(D)) = \ell(D - K)$$
$$\gamma = \dim_\mathbb{F} H^0(X, \mathcal{O}_X) = \text{number of connected parts of } X,$$

the Riemann-Roch theorem for reducible curves can be proven (see [Oo, p. 103]):

**Theorem 2.3 (Riemann-Roch)** For every divisor $D$ on $X$, we have

$$\ell(D) - i(D) = \deg(D) - g + \gamma.$$

If $D = \{(U_i, f_i)\}_{i \in I}$ is a Cartier divisor in $X$, and $P$ is a regular rational point on $X$, we denote by $w_P(D)$ the value $w_P(f_i)$ if $P \in U_i$ (in the sense of Manis). Usual notations for effectivity of $D$ (namely $D > 0$) and linear equivalence between two divisors ($D \sim D'$) hold.

Let us consider $P_1, \ldots, P_r$ a set of regular rational points on $X$, $\underline{n} = (n_1, \ldots, n_r) \in \mathbb{Z}^r$ and the divisor $nP = \sum_{i=1}^r n_i P_i$. Denote the $\mathbb{F}$-vector space of global sections of the invertible sheaf $\mathcal{L}(nP)$ by $\Gamma(X, nP)$, its dimension by $\ell(nP)$ and $\ell(K - nP) := \dim_\mathbb{F} \Gamma(X, \mathcal{L}(K - nP))$. We have

$$\Gamma(X, nP) = \{f \in K(X)^* \mid f + nP > 0\}$$
$$= \{f \in K(X)^* \mid w_i(P) \geq -n_i, \ i = 1, \ldots, r\},$$

where $K(X)^* := K(X) \setminus \{0\}$. Notice that

$$\Gamma(X, nP) \supseteq \Gamma(X, mP)$$

for every $\underline{n}, \underline{m} \in \mathbb{Z}^r$ with $n_i \geq m_i$, for all $i \in \{1, \ldots, r\}$; hence the set $\{\Gamma(X, nP)\}_{\underline{n} \in \mathbb{Z}^r}$ defines a filtration given by multi-indices (or multi-index filtration) of the curve.
2.3 Structure and properties of the semigroup

We collect now some basic properties and definitions of the Weierstraß semigroup.

**Definition 2.4** An element \( m \in \mathbb{Z}^r \) is said to be a gap of \( \Gamma_P \) if \( m \notin \Gamma_P \). Otherwise \( m \) is said to be a non-gap of \( \Gamma_P \).

The proof of the following lemma was given in [D-2]; we reproduce it however for the reader’s convenience.

**Lemma 2.5** If \( m \in \mathbb{Z}^r \), then \( m \) is a non-gap of \( \Gamma_P \) if and only if \( \ell(mP) = \ell((m-e_i)P) + 1 \) for all \( i \in \{1, \ldots, r\} \).

**Proof.** The integer \( m \) belongs to the semigroup if and only if there exists a function \( f \in K(X)^* \) regular on \( X \setminus \{P_1, \ldots, P_r\} \) such that \( w_i(f) = m_i \) for every \( i \in \{1, \ldots, r\} \). Thus \( f \in \Gamma(X, mP) \) but \( f \notin \Gamma(X, (m-e_i)P) \) for every \( i \in \{1, \ldots, r\} \). Conversely, if there exists a function \( f_i \in \Gamma(X, mP) \setminus \Gamma(X, (m-e_i)P) \) for all \( i \in \{1, \ldots, r\} \), then, since \( g(\mathbb{F}) \geq r \), then one can take \( \mu_1, \ldots, \mu_r \in \mathbb{F} \) so that \( f = \sum_{i=1}^r \mu_i f_i \in K(X)^* \) is a regular function on \( X \setminus \{P_1, \ldots, P_r\} \) with \( f \in \Gamma(X mP) \setminus \Gamma(X, (m-e_i)P) \) for all \( i \in \{1, \ldots, r\} \), and therefore \( w_i(f) = m_i \) for all \( i \) and \( m \in \Gamma_P \). ◊

**Lemma 2.6** If \( X \) is a curve of genus \( g \) and \( m = (m_1, \ldots, m_r) \in \mathbb{Z}^r \) is a gap, then \( 0 \leq m_1 + \ldots + m_r < 2g \).

**Proof.** Let \( D_{2g, P} \) be a divisor on \( X \) of degree \( 2g \) and support \( P \) and let \( D_{2g-1, P} \) be a divisor on \( X \) of degree \( 2g-1 \) and support \( P \). If \( m_1 + \ldots + m_r \geq 2g - 1 \), then as a consequence of Theorem 2.3 one has that \( m_1 + \ldots + m_r \geq 0 \) and, for \( i \in \{1, \ldots, r\} \):

\[
\ell(D_{2g, P}) = 2g + 1 - g = g + 1 \neq g = 2g - 1 + 1 = \ell(D_{2g-1, P}),
\]

which implies that \( m \) is a non-gap. Hence, if \( m \) is a gap, then \( m_1 + \ldots + m_r < 2g \). ◊

**Remark 2.7** Notice that \( \Gamma_P \subset \Gamma_{P_1} \times \cdots \times \Gamma_{P_r} \).

**Lemma 2.8** Let \( X \) be a curve, \( P_1, \ldots, P_r \) rational points on \( X \) and \( \Gamma_P \) the Weierstraß semigroup of \( P_1, \ldots, P_r \). Then we have:

(i) There exists a number \( \vartheta \in \mathbb{Z} \setminus \{0\} \) so that \( \vartheta(P_1 - P_2) \) is a principal divisor.

(ii) For any \( \lambda \in \mathbb{Z} \), then \( (m_1, \ldots, m_r) \in \Gamma_P \) if and only if \( (m_1, \ldots, m_r) + \lambda(\vartheta, -\vartheta, 0, \ldots, 0) \in \Gamma_P \).
Proof. (i) Let Pic(X) be the set of isomorphism classes of invertible sheaves on X, and Pic^0(X) the subgroup of Pic(X) of those classes of degree 0. The Jacobian variety of X is isomorphic to Pic^0(X) (cf. [L], Chapter 7, Theorem 4.39). Since the \( \mathbb{F} \)-points of \( J \) are a finite group, then the divisor \( P_1 - P_2 \in \text{Pic}^0(X) \) is an element of finite order. (ii) is a consequence of (i).

**Definition 2.9** Assume \( r = 2 \) and \( \sharp(\mathbb{F}) > 2 \). The smallest natural number \( \vartheta \neq 0 \) such that \( \vartheta(P_1 - P_2) \) is a principal divisor is called the period of the semigroup \( \Gamma_{P_1,P_2} \).

Consider the discrete plane \( \mathbb{Z}^2 \). The period of the semigroup defines an important area on it: the parallelogram in \( \Gamma_{P_1,P_2} \cap \mathbb{Z}^2 \) determined by the vertices \((0,0),(0,2g),(\vartheta,2g-\vartheta)\) and \((\vartheta,-\vartheta)\) not including the line joining \((0,0)\) and \((0,2g)\) is called the fundamental corner of \( \Gamma_{P_1,P_2} \). We denote it by \( C_{P_1,P_2} = \mathcal{C} \).

By Lemma 2.8 (ii), any translation of \( C_{P_1,P_2} \) of vector \( \lambda(\vartheta,-\vartheta) \) for every \( \lambda \in \mathbb{Z} \setminus \{0\} \), reproduces the same distribution of the points in \( \Gamma_{P_1,P_2} \). Therefore, the points in \( \mathcal{C} \) will play an essential role in the semigroup’s knowledge, as we will see in the next sections, especially the set of maximal points of \( \mathcal{C} \), which will be denoted by \( \mathcal{M}_C \).

### 3 Poincaré series of the Weierstrass semigroups associated with one and two points

This section is devoted to introduce the notion of Poincaré series associated with the multi-index filtration given by \( \Gamma(X,mP) \) and to show its behaviour in the case of one and two points. For every \( m \in \mathbb{Z}^r \), set

\[
\text{d}(m) := \dim_{\mathbb{F}} \frac{\Gamma(X,mP)}{\Gamma(X,(m-1)P)}.
\]

**Definition 3.1** The Poincaré series of the Weierstrass semigroup \( \Gamma_P \) is defined to be

\[
P_{\Gamma_P}(t) := \sum_{m \in \mathbb{Z}^r} \text{d}(m)t^m,
\]

where \( t^m := t_1^{m_1} \cdots t_r^{m_r} \).

The series \( P_{\Gamma_P}(t) \) contains negative powers of variables \( t_i \), however it does not contain monomials \( t^m \) with purely negative \( m \) (i.e., all components of which are negative). It is convenient to consider \( P_{\Gamma_P}(t) \) as an element of the set of formal Laurent series in the variables \( t_1, \ldots, t_r \) with integer coefficients without purely negative exponents; i.e., of expressions of the form \( \sum_{m \in \mathbb{Z}^r \setminus \mathbb{Z}_{\leq -1}^r} \text{d}(m)t^m \).

**Remark 3.2** Notice also that \( \text{d}(m) = 0 \) if and only if \( m \notin \Gamma_P \). If \( r = 1 \), then \( \text{d}(m) = 1 \) if and only if \( n \in \Gamma_P \) and so \( P_{\Gamma_P}(t) = \sum_{n \in \Gamma_P} t^n \).
3.1 Poincaré series associated with one point

Let $C$ be a plane model for $X$ having a unique branch at infinity (i.e., such that there exists a birational morphism $X \to C \subseteq \mathbb{P}^2$ and a line $L \subseteq \mathbb{P}^2$ defined over $\mathbb{F}$ such that $L \cap C$ consists of only one point $Q$ and $X$ has only one branch at $Q$. Hence there is only one point of $X$ over $Q$, which will be denoted by $P$). Set $C' = C \setminus \{Q\}$. We can define the set

$$S_P := \{-w_P(f) \mid f \in K(C') \setminus \{0\}\},$$

which is actually a subsemigroup of $\mathbb{N}$. Notice that $\Gamma_P = S_P$ whenever the curve $C$ is non-singular in the affine part. The Abhyankar-Moh theorem (see e.g. [A]) shows that $S_P$ is a strictly generated semigroup with generators \{r_0, \ldots, r_h\}. By setting $	heta_i := \gcd(r_0, \ldots, r_i-1), 1 \leq i \leq h+1$ and $d_i := \theta_i/\theta_{i+1}$ for $1 \leq i \leq h$, every element $n \in S_P$ can be written uniquely as

$$n = \lambda_0 r_0 + \lambda_1 r_1 + \ldots + \lambda_h r_h,$$

with $\lambda_0 \geq 0$ and $0 \leq \lambda_i < d_i$, for $1 \leq i \leq h$. We also know that the set $\Gamma_P \setminus S_P$ is finite (cf. [F, Lemma 3.3]):

**Lemma 3.3** Let $A$ be the affine $\mathbb{F}$-algebra of $C$, and $\overline{A}$ its normalization. Then we have

$$\sharp(\Gamma_P \setminus S_P) = \dim_{\mathbb{F}}(\overline{A}/A)$$

Thus, if $\{f_1, \ldots, f_l\}$ is a basis of the $\mathbb{F}$-vector space $\overline{A}/A$ and we put $s_j := -w_P(f_j)$, for $1 \leq j \leq l$, then every element $m \in \Gamma_P \setminus S_P$ can be written uniquely as

$$m = \lambda'_1 s_1 + \ldots + \lambda'_l s_l,$$

with $\lambda'_j \geq 0$ for $1 \leq j \leq l$. This leads to the following description of $P_{\Gamma_P}(t)$:

**Proposition 3.4** Let $\Gamma_P$ be the Weierstraß semigroup at $P$. Then:

$$P_{\Gamma_P}(t) = \frac{1}{1 - t r_0} \cdot \prod_{i=1}^{h} \frac{1 - t_{n_i} r_i}{1 - t r_i} + \prod_{j=1}^{l} \frac{1}{1 - t s_j},$$

In particular, the Poincaré series $P_{\Gamma_P}(t)$ is a rational function.

**Proof.** By equations $(\ast)$ and $(\ast\ast)$ we have

$$\sum_{n \in S_P} t^n = \sum_{\lambda_0 \geq 0 \atop 0 \leq \lambda_i < n_i \atop 1 \leq i \leq h} t^{\lambda_0 r_0 + \ldots + \lambda_h r_h} = \left(\sum_{\lambda_0 \geq 0} t^{\lambda_0 r_0}\right) \cdot \left(\sum_{0 \leq \lambda_1 < n_1} t^{\lambda_1 r_1}\right) \ldots \left(\sum_{0 \leq \lambda_h < n_h} t^{\lambda_h r_h}\right) = \frac{1}{1 - t r_0} \cdot \frac{1 - t_{n_1} r_1}{1 - t r_1} \ldots \frac{1 - t_{r_h} r_h}{1 - t r_h},$$

Using the above expressions, we can simplify the Poincaré series $P_{\Gamma_P}(t)$ to

$$P_{\Gamma_P}(t) = \frac{1}{1 - t r_0} \cdot \prod_{i=1}^{h} \frac{1 - t_{n_i} r_i}{1 - t r_i} + \prod_{j=1}^{l} \frac{1}{1 - t s_j},$$

which is a rational function.
and
\[
\sum_{m \in \Gamma \setminus S_p} t^m = \sum_{\substack{\lambda_j \geq 0 \\ 1 \leq j \leq l}} t^{s_1 + \ldots + s_l} = \frac{1}{1 - t^{s_1}} \ldots \frac{1}{1 - t^{s_l}}.
\]

\[\diamond\]

If the curve $C$ is non-singular in the affine part, then $\Gamma_P = S_P$ and we obtain the formula before Theorem 1 of [C-D-GZ]:

**Corollary 3.5** Let $C$ be a non-singular curve in the affine part. Then we have

\[
P_{\Gamma_P}(t) = \frac{1}{1 - t_0} \cdot \prod_{i=1}^{b} \frac{1 - t^{n_i}}{1 - t^{r_i}}.
\]

The semigroup $\Gamma_P$ is numerical, i.e., it has a conductor $c = c(\Gamma_P)$ (see for instance [D-2]).

**Proposition 3.6** We have

\[
P_{\Gamma_P}(t) = \frac{L_{\Gamma_P}(t)}{1 - t},
\]

with $L_{\Gamma_P}(t) = 1 - t + t^c + (1 - t) \sum_{n \in \Gamma \subset C} t^n \in \mathbb{Z}[t]$. In particular, if $X$ is an affine complete intersection, then we have

\[
P_{\Gamma_P} = \frac{1 - t + t^{2g}}{1 - t}.
\]

**Proof.** The first equality follows from the definition of conductor. The second one is a consequence of the fact that, if the $\Gamma_P$ is an affine complete intersection, then $c = 2g$ (cf. [Sa]).\[\diamond\]

### 3.2 Poincaré series associated with two points

Here are some elementary properties of the dimensions $d(m)$ for $m = (m_1, m_2) \in \mathbb{Z}^2$:

**Lemma 3.7** Let $\Gamma_{P_1, P_2}$ be the Weierstraß semigroup associated with two rational points $P_1, P_2$ on $X$. Then we have:

1. $m \in \Gamma_{P_1, P_2}$ if and only if $d(m) \geq 1$.
2. Let $m \in \Gamma_{P_1, P_2}$. Then $d(m) = 1$ if and only if $\nabla(m) = \emptyset$.
3. $d(m) = 2$ if and only if $m \in \Gamma_{P_1, P_2}$ and $\nabla(m) \neq \emptyset$. 
4. If $m_1 \in \Gamma_{P_1}$ and $m_2 \in \Gamma_{P_2}$, then $d(m) = 2$.

**Proof.** See [Ca-To, Section 2]. ♦

Let $\Gamma_{P_1, P_2}$ be the Weierstraß semigroup associated with two rational points $P_1, P_2$ on $X$. Let us define the series

$$L_{\Gamma_{P_1, P_2}}(t) := (1-t_1)(1-t_2)P_{\Gamma_{P_1, P_2}}(t).$$

This expression can be rewritten as

$$L_{\Gamma_{P_1, P_2}}(t) = \sum_{m \in \mathbb{Z}^2} c(m)t^m,$$

where $c(m) := d(m) - d(m - e_1) - d(m - e_2) + d(m - 1)$. We may describe the series $L_{\Gamma_{P_1, P_2}}(t)$ in terms of the set $\mathcal{M}_{P_1, P_2}$ of maximal points of $\Gamma_{P_1, P_2}$ using the following result:

**Proposition 3.8** Let $m \in \mathbb{Z}^2$. Then we have

1. $c(m) = -1$ if and only if $m - 1$ is maximal.
2. $c(m) = 1$ if and only if $m$ is maximal.

**Proof.** Taking into account the main properties of the Weierstraß semigroup of Lemma 3.7 and since $c(m) = d(m) - d(m - e_1) - d(m - e_2) + d(m - 1)$, we consider the different values the coefficients $d(m)$ can take. So, if $d(m) = 2$, then the following statements hold:

- if $d(m - e_1) = d(m - e_2) = 1$, then the only possible case is $d(m - 1) = 0$ and $c(m) = 2 - 1 - 1 + 0 = 0$;

- if $d(m - e_1) = 1$ and $d(m - e_2) = 2$, then $d(m - 1) = 1$ and $c(m) = 2 - 1 - 2 + 1 = 0$;

- if $d(m - e_1) = 2$ and $d(m - e_2) = 1$, as in the previous case one has $d(m - 1) = 1$ and $c(m) = 2 - 2 - 1 + 1 = 0$;

- if $d(m - e_1) = d(m - e_2) = 2$, one has $d(m - 1) \geq 1$. If $d(m - 1) = 2$, then $c(m) = 2 - 2 - 2 + 2 = 0$; on the other hand, if $d(m - 1) = 1$, then $c(m) = -1$ and $\nabla(m - 1) = \emptyset$ by Lemma 3.7 (2). Thus the point $m - 1$ is maximal.

In the case $d(m) = 1$, we have to distinguish whether the point $m$ is maximal. If $m$ is maximal, then:

- if $d(m - e_1) = d(m - e_2) = 1$, one has either $d(m - 1) = 1$ and so $c(m) = 1 - 1 - 1 + 1 = 0$ and $m - 1$ is maximal, or $d(m - 1) = 2$ and therefore $c(m) = 1 - 1 - 1 + 2 = 1$ and $m - 1$ is not maximal by Lemma 3.7 (3);
\(-\text{ if } d(m-e_1) = 1 \text{ and } d(m-e_2) = 0, \text{ then one has only } d(m-1) = 1 \text{ and } c(m) = 1 - 1 - 0 + 1 = 1 \text{ with } m-1 \text{ non-maximal; }\)

\(-\text{ if } d(m-e_1) = 0 \text{ and } d(m-e_2) = 1, \text{ then one has again } d(m-1) = 1 \text{ and } c(m) = 1 \text{ with } m-1 \text{ non-maximal; }\)

\(-\text{ if } d(m-e_1) = 0 = d(m-e_2) = 0, \text{ then } d(m-1) = 0 \text{ and } c(m) = 1.\)

On the other hand, if \(m\) is non-maximal, it holds that:

\(-\text{ if } d(m-e_1) = 0 \text{ and } d(m-e_2) = 1, \text{ then } d(m-1) = 0 \text{ and } c(m) = 0; \)

\(-\text{ if } d(m-e_1) = 0 \text{ and } d(m-e_2) = 2, \text{ then we have only } d(m-1) = 1 \text{ and } c(m) = 1 - 1 - 1 + 0 = 0; \)

\(-\text{ if } d(m-e_1) = 1 = d(m-e_2) = 1, \text{ then again } d(m-1) = 1 \text{ and } c(m) = 1 - 1 - 1 + 1 = 0; \)

\(-\text{ if } d(m-e_1) = 1 \text{ and } d(m-e_2) = 2, \text{ then one has either } d(m-1) = 1 \text{ and } c(m) = 1 - 1 - 2 + 1 = -1 \text{ and } m-1 \text{ is maximal by Lemma 3.7(2), or } d(m-1) = 2 \text{ and } c(m) = 1 - 1 - 2 + 2 = 0.\)

The remaining case is \(d(m) = 0\), in which the following possibilities hold:

\(-\text{ if } d(m-e_1) = d(m-e_2) = 0, \text{ then } d(m-1) = 0 \text{ and } c(m) = 0; \)

\(-\text{ if } d(m-e_1) = 0 \text{ and } d(m-e_2) = 1, \text{ then } d(m-1) = 0 \text{ and } c(m) = 0; \)

\(-\text{ if } d(m-e_1) = 1 \text{ and } d(m-e_2) = 0 \text{ one has again } d(m-1) = 0 \text{ and } c(m) = 0; \)

\(-\text{ if } d(m-e_1) = d(m-e_2) = 1, \text{ then } d(m-1) = 2 \text{ and } c(m) = 0. \)

\textbf{Corollary 3.9}

\[L_{\Gamma_{P_1,P_2}}(t) = \left(1 - t_1 \cdot t_2\right) \sum_{m \in M_{P_1,P_2}} t^m.\]

\textit{Proof.} The only point to realize is that

\[L_{\Gamma_{P_1,P_2}}(t) = \sum_{m \in M_{P_1,P_2}} t^m + \sum_{m-1 \in M_{P_1,P_2}} t^m - \sum_{m \in M_{P_1,P_2}} t^{m+1}.\]

\[\diamondsuit\]

\textbf{Theorem 3.10} \ Let \(\mathcal{C} := \mathcal{C}_{P_1,P_2}\) be the fundamental corner of the semigroup \(\Gamma_{P_1,P_2}\) and \(\mathcal{M}_C := M_{P_1,P_2} \cap \mathcal{C}\). Then we have

\[L_{\Gamma_{P_1,P_2}}(t) = \left(1 - t_1 \cdot t_2\right) \sum_{m \in \mathcal{M}_C} t^m.\]
Proof. We denote \( \vartheta' := (\vartheta, -\vartheta) \). The series \( L_{\Gamma P_1, P_2}(t) \) can be written as

\[
L_{\Gamma P_1, P_2}(t) = (1 - t_1 \cdot t_2) \cdot \sum_{m \in M_{P_1, P_2}} t^m = (1 - t_1 \cdot t_2) \left( \sum_{m' \in M_c} t^{m'} + \sum_{m' + \lambda \vartheta' : \lambda \in \mathbb{Z}_{\geq 0} \setminus \{0\}} t^{m' + \lambda \vartheta'} + \sum_{m' - \lambda \vartheta' : \lambda \in \mathbb{Z}_{\geq 0} \setminus \{0\}} t^{m' - \lambda \vartheta'} \right).
\]

Set \( T := t^{\vartheta'} \). The proof is completed by showing that

\[
\sum_{m' + \lambda \vartheta' : \lambda \in \mathbb{Z} \setminus \{0\}} t^{m' + \lambda \vartheta'} = \sum_{m' \in M_c} t^{m'} \left( \sum_{\lambda=1}^{\infty} \left( T^{-(\lambda+1)} + T^{\lambda} \right) \right) = \left( \sum_{m' \in M_c} t^{m'} \right) \left( \sum_{\lambda=1}^{\infty} T^{-\lambda+1} + \sum_{\lambda=1}^{\infty} T^{\lambda} \right) = \left( \sum_{m' \in M_c} t^{m'} \right) \left( T \frac{T^{-1}}{1-T^{-1}} + \frac{T}{1-T} \right) = \left( \sum_{m' \in M_c} t^{m'} \right) \left( \frac{T}{T-1} + \frac{T}{1-T} \right) = 0.
\]

\( \diamond \)

Corollary 3.11 We have

\[
P_{\Gamma P_1, P_2}(t) = \frac{L_{\Gamma P_1, P_2}(t)}{(1 - t_1)(1 - t_2)} = \frac{1 - t_1 t_2}{(1 - t_1)(1 - t_2)} \sum_{m \in M_c} t^m.
\]

4 Symmetry and functional equations

4.1 The symmetry of the semigroup and affine embeddings

Some important considerations about the symmetry of the Weierstraß semigroup at several points were introduced by Delgado in [12], being the most remarkable the following result.

Theorem 4.1 Let \( X \) be a reducible projective space curve of arithmetical genus \( g \) and \( X' = X \setminus \{P_1, \ldots, P_r\} \), where \( P_i \) are smooth points for all \( i = 1, \ldots, r \). Then, the following statements are equivalent:
1. $X'$ is an affine complete intersection.

2. There exists a canonical divisor $K$ such that $\text{supp}(K) \subset \{P_1, \ldots, P_r\}$.

3. There exists a relative maximal \( \tau = (\tau_1, \ldots, \tau_r) \) in $P$ such that $\sum_{i=1}^r \tau_i = 2g - 2 + r$.

4. There exists $\sigma = (\sigma_1, \ldots, \sigma_r) \in \Gamma_P$ such that $\sum_{i=1}^r \sigma_i = 2g - 2 + r$ and $\Gamma_P$ is symmetrical with respect to $\sigma$.

Proof. The equivalences (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) go straight as in [D-2]. The rest of the subsection is devoted to present the suitable results to reach the equivalence (1) $\Leftrightarrow$ (2). ♦

Then we present now in a more clearly self-contained way the proof “(1) $\Leftrightarrow$ (2)” of [D-2] p. 630.

We need to collect now some important classical results.

**Lemma 4.2** Let $R$ be a ring. Assume that every projective $R$-module of ranks 1 and 2 are free. Let $I$ be a nonzero ideal of $R$ with projective dimension less than or equal to 1. The following statements are equivalent:

1. $I$ may be generated by two elements.

2. $\text{Ext}^1_R(I, R)$ is a cyclic $R$-module.

Proof. See [Se, §2.4., Corollaire]. ♦

Let $V$ be a non-singular variety of dimension $r$, $O_V$ be the affine coordinate ring of $V$ and $\Omega_V$ be the sheaf of differential forms of degree $r$ over $V$. It is a locally free sheaf of rank 1. Let $W$ be a subvariety of $V$ of codimension $h$.

**Definition 4.3** The module of differential forms on $W$ is defined to be

$$\Omega_W := \text{Ext}^h_{O_V}(O_W, \Omega_V).$$

**Remark 4.4** For any ring $R$, there is a correspondence between free finitely generated projective $R$-modules of finite rank and trivial vector bundles on Spec($R$) of the same rank.

Next result was proven by Serre (cf. [Se, §2.7., Prop. 6]). By the sack of completeness, we rewrite his proof adding further details.

**Proposition 4.5** Let $V$ be a non-singular affine variety over which every vector bundle of rank 1 is trivial. Let $W \subset V$ be a subvariety Cohen-Macaulay of $V$ of codimension 2. Then, if $W$ is a complete intersection on $V$, then $\Omega_W \cong O_W$. Conversely, assuming moreover that every vector bundle of rank 2 on $V$ is trivial, we have that $W$ is a complete intersection on $V$ if $\Omega_W \cong O_W$. 
Proof. The sheaf $\Omega_V$ is locally free of rank 1. Every algebraic vector bundle of rank 1 is trivial by hypothesis, then $\Omega_V \cong \mathcal{O}_V$. Furthermore, if $W$ is a complete intersection, we have the exact sequence

$$0 \to A_W \to \mathcal{O}_V \to \mathcal{O}_W \to 0,$$

where $A_W$ denotes the coherent ideal sheaf defining $W$. Applying the functor $\operatorname{Ext}_{\mathcal{O}_V}(-, \mathcal{O}_V)$ to this exact sequence, the necessity is proven.

Conversely, let $\mathcal{O}_V$ be the affine coordinate ring of $V$, and $A_W$ the ideal defining $W$. Then the sheaf associated with $\operatorname{Ext}^2_{\mathcal{O}_V}(\mathcal{O}_W, \mathcal{O}_V) \cong \operatorname{Ext}^2_{\mathcal{O}_V}(\mathcal{O}_W, \Omega_V)$ is isomorphic to $\Omega_W$, and $\Omega_W \cong \mathcal{O}_W$, thus $\operatorname{Ext}^2_{\mathcal{O}_V}(\mathcal{O}_W, \mathcal{O}_V) \cong \mathcal{O}_W$ and so it is cyclic. By the other hand, since $\mathcal{O}_W$ is locally Cohen-Macaulay, then the homological dimension of the ideal defining $W$ is at most 1. By Remark 4.4 we can apply Lemma 4.2 and see that this ideal may be generated by two elements, i.e., $W$ is a complete intersection on $V$. ♦

Remark 4.6 Let $k$ be a principal ideal domain. Then every finitely generated projective $k[x_1, \ldots, x_n]$-module is free (see [Q, Theorem 4, p. 169]).

Next result is indeed the statement $(1) \iff (2)$ of Theorem 4.1.

Corollary 4.7 Every non-singular affine curve is complete intersection if and only if there exists a canonical divisor with support totally contained in the places at infinity of the curve.

Proof. The existence of a canonical divisor totally supported in places at infinity for the curve is equivalent to the existence of a trivial canonical line bundle. Let $\mathbb{A}^n = \operatorname{Spec}(\mathbb{F}[x_1, \ldots, x_n])$. As $\mathbb{F}$ is a field, every finitely generated projective $\mathbb{F}[x_1, \ldots, x_n]$-module is free by Remark 4.6, and that means by Remark 4.4 that every vector bundle on $\operatorname{Spec}(\mathbb{F}[x_1, \ldots, x_n]) = \mathbb{A}^n$ is trivial. Let $W$ be a smooth affine curve, then it is Cohen-Macaulay. Now, from Proposition 4.5 taking $V = \mathbb{A}^n$ and $W$ the smooth affine curve, that is the case if and only if $W$ is a complete intersection. ♦

4.2 Functional equations

In local contexts of Poincaré series associated with semigroups of values of a curve singularity, for instance, or zeta functions associated with local singular rings, one usually has functional equations when the base ring is Gorenstein (see [D-1] and [SI]). In our global context the same holds for complete intersections.

Definition 4.8 The point $\sigma$ of the last assertion of Theorem 4.1 is called the symmetry point of $\Gamma_P$.

Notice that, if $r = 1$, then the symmetry point of $\Gamma_P$ is the conductor, which is equal to $c = 2g$ (cf. [SI]). Then we can deduce easily the functional equations.
Lemma 4.9 Let $X$ be a curve. Let $P$ be a rational point of $X$ such that $X \setminus P$ is an affine complete intersection. We have

$$L_{\Gamma_P}(t) = -t^{2g} L_{\Gamma_P}(t^{-1}).$$

$$P_{\Gamma_P}(t) = t^{2g-1} P_{\Gamma_P}(t^{-1}).$$

Proof. Let $n \in \mathbb{Z}$. By the definition of conductor of a numerical semigroup, we know that $n \in \Gamma_P$ if and only if $2g-1-n \notin \Gamma_P$; it implies that $d(2g-1-n) = 0$ if and only if $d(n) = 1$. Taking Remark 3.2 into account, it holds that

$$t^{2g-1} P_{\Gamma_P}(t^{-1}) = \sum_{m \in \mathbb{Z}} d(m) t^{2g-1-n} = \sum_{n \in \Gamma_P} t^n = P_{\Gamma_P}(t).$$

The functional equation for $L_{\Gamma_P}(t)$ follows from a simple computation. ♦

Next result establishes the functional equation for the Poincaré series associated with two points.

Proposition 4.10 Let $X$ be a curve, $P_1, P_2$ two points on $X$ such that $X \setminus \{P_1, P_2\}$ is affine complete intersection. Let us denote by $\sigma$ the symmetry point of the Weierstraß semigroup $\Gamma_{P_1, P_2}$. We have

$$L_{\Gamma_{P_1, P_2}}(t) = -t^{\sigma+1} \cdot L_{\Gamma_{P_1, P_2}}(t^{-1}).$$

$$P_{\Gamma_{P_1, P_2}}(t) = -t^{\sigma} \cdot P_{\Gamma_{P_1, P_2}}(t^{-1}).$$

Proof. By Proposition 4.5 we have

$$L_{\Gamma_{P_1, P_2}}(t) = (1 - t_1 \cdot t_2) \cdot \sum_{\mathbf{m} \in \mathcal{M}_{P_1, P_2}} t^{\mathbf{m}}.$$

Moreover, if $\mathbf{m} \in \mathcal{M}_{P_1, P_2}$, then there exists $\mathbf{n} \in \mathcal{M}_{P_1, P_2}$ such that $\mathbf{n} + \mathbf{m} = \sigma$. Then

$$L_{\Gamma_{P_1, P_2}}(t) = (1 - t_1 \cdot t_2) \cdot \sum_{\mathbf{m} \in \mathcal{M}_{P_1, P_2}} t^{\mathbf{m}} = (1 - t_1 \cdot t_2) \cdot \sum_{\mathbf{n} \in \mathcal{M}_{P_1, P_2}} t^{\sigma+1-\mathbf{n}} = \frac{(1 - t_1 \cdot t_2)}{t^{\sigma+1}} \cdot L_{\Gamma_{P_1, P_2}}(t^{-1}) = -t^{\sigma+1} \cdot L_{\Gamma_{P_1, P_2}}(t^{-1}).$$
On the other hand, by the definition and the functional equation of $L_{\Gamma_{P_1, P_2}}(t)$, we have

\[
P_{\Gamma_{P_1, P_2}}(t^{-1}) = \frac{L_{\Gamma_{P_1, P_2}}(t^{-1})}{(1-t_1^{-1})(1-t_2^{-1})} = \frac{t_1 \cdot t_2 \cdot L_{\Gamma_{P_1, P_2}}(t^{-1})}{(t_1 - 1)(t_2 - 1)} = -t^{-\sigma} \cdot L_{\Gamma_{P_1, P_2}}(t^{-1}) = -t^{-\sigma} \cdot P_{\Gamma_{P_1, P_2}}(t).
\]

\[\diamondsuit\]

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