Symplectic embeddings from concave toric domains into convex ones

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Abstract

Embedded contact homology gives a sequence of obstructions to four-dimensional symplectic embeddings, called ECH capacities. In “Symplectic embeddings into four-dimensional concave toric domains”, the author, Choi, Frenkel, Hutchings and Ramos computed the ECH capacities of all “concave toric domains”, and showed that these give sharp obstructions in several interesting cases. We show that these obstructions are sharp for all symplectic embeddings of concave toric domains into “convex” ones. In an appendix with Choi, we prove a new formula for the ECH capacities of most convex toric domains, which shows that they are determined by the ECH capacities of a corresponding collection of balls.

1 Introduction

1.1 The main theorem

Hutchings’ embedded contact homology gives obstructions to symplectically embedding one symplectic 4-manifold into another. It is interesting to ask to what degree these obstructions are sharp. One set of obstructions are the ECH capacities defined in [6]. These are a certain sequence of nonnegative real numbers $c_k(X, \omega)$ associated to any symplectic four-manifold $(X, \omega)$ that are monotone under symplectic embeddings. McDuff showed [12] that ECH capacities give a sharp obstruction to symplectically embedding one four-dimensional “ellipsoid” into another, and Frenkel and Müller [4] showed that ECH capacities give a sharp obstruction to symplectically embedding an ellipsoid into a symplectic “polydisk”.

A natural class of further examples to consider are defined as follows. Let $\Omega \subset \mathbb{R}^2$ be a region in the first quadrant, and let $X_\Omega$ be the preimage of $\Omega$ under the map $\mu : \mathbb{C}^2 \to \mathbb{R}^2$, $(z_1, z_2) \to (\pi |z_1|^2, \pi |z_2|^2)$.

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Figure 1.1: A concave toric domain and a convex one

We call $X_{\Omega}$ a four-dimensional \textit{toric domain}. Toric domains generalize ellipsoids (where $\Omega$ is a right triangle with legs on the axes) and polydisks (where $\Omega$ is a rectangle whose bottom and left sides are on the axes). The purpose of this article is to identify a large class of embedding problems involving toric domains for which ECH capacities give a sharp obstruction.

To elaborate, first recall the “concave toric domains” from [2]. These were defined as toric domains $X_{\Omega}$, where $\Omega$ is a region in the first quadrant underneath the graph of a convex function $f : [0, a] \rightarrow [0, b]$, such that $a$ and $b$ are positive real numbers, $f(0) = b$, and $f(a) = 0$. We now define a related concept, see Figure 1.1.

\textbf{Definition 1.1.} A \textit{convex toric domain} is a toric domain $X_{\Omega}$, where $\Omega$ is a closed region in the first quadrant bounded by the axes and a convex curve\footnote{Hutchings has also defined convex toric domains, but our definition is slightly more general than the definition he gives in [7]. There, he defines a convex toric domain to be the region underneath the graph of a concave function.} from $(a, 0)$ to $(0, b)$, for $a$ and $b$ positive real numbers.

We can now state the main theorem of this paper:

\textbf{Theorem 1.2.} Let $X_{\Omega_1}$ be a concave toric domain and let $X_{\Omega_2}$ be a convex toric domain. Then there exists a symplectic embedding

$$\text{int}(X_{\Omega_1}) \rightarrow \text{int}(X_{\Omega_2})$$

\textit{if and only if}

$$c_k(\text{int}(X_{\Omega_1})) \leq c_k(\text{int}(X_{\Omega_2}))$$

\textit{for all nonnegative integers $k$.}
Note that an ellipsoid is both concave and convex, while a polydisk is convex. Thus, Theorem 1.2 generalizes McDuff’s result from [12] and Frenkel and Müller’s result from [4].

The ECH capacities of concave toric domains were computed in [2], where it was also shown that ECH capacities give sharp obstructions for embeddings into concave toric domains in some interesting cases, for example for ball packings of certain unions of an ellipsoid and a cylinder. The ECH capacities of convex domains have mostly been computed, by [7, Thm. 1.11] and [10, Thm 1.2]; we expect that this formula should hold for all convex toric domains. We remark that in both of these cases, the ECH capacities involve counting lattice points in certain polygonal regions, and seem to have interesting connections with number theory, see e.g. [3, 17].

To place Theorem 1.2 in context, note that it is known that ECH capacities are not always sharp, even for toric domains. A notable example of this is given by Hind and Lisi in [5], where it is shown that ECH capacities fail to be sharp for embeddings of a polydisk into a ball. Recent work of Hutchings [7], however, shows that embedded contact homology can still be used to derive strong obstructions to symplectic embeddings, even when the obstructions coming from ECH capacities are weak. For example, in [7] Hutchings defines new obstructions to embedding one convex toric domain into another that can be used to recover the result of Hind and Lisi from above. It is currently not known how sharp these new obstructions are.

1.2 Idea of the proof and relationship with previous work

The method of proof we give is essentially the same as a method used by McDuff to show that ECH capacities give a sharp obstruction to embedding one ellipsoid into another (the actual method used by McDuff in [14] is more combinatorial than our method, which is closer to the method from Hutchings’ survey of McDuff’s work, see especially the proof of [8, Thm. 1.1]). We can summarize the contribution of the present work as showing that this method works just as well for embeddings of concave domains into convex ones.

To elaborate, McDuff showed in [11] that an embedding of one rational ellipsoid into another is equivalent to a certain symplectic ball packing problem determined by the ellipsoids. In [14], it was then shown that since ECH capacities are known to be sharp for symplectic ball packings of a ball, they are sharp for ellipsoid embeddings as well. We first show that an embedding of a “rational” concave toric domain into a rational convex one is equivalent to a symplectic ball packing problem, see Theorem 1.3 and we then use this to show that ECH capacities give a sharp obstruction to embedding a concave domain into a convex one. We remark that ball packings of a ball are well-understood (indeed, they are essentially algorithmically computable
and so Theorem 1.3 is of potentially independent interest.

We also remark that the idea that McDuff’s method from [11] holds more generally is not new. In fact, a key step in McDuff’s proof from [11] involves approximating the ellipsoid domain by a concave domain and approximating the ellipsoid target by a convex domain, see [11, Fig. 3.1], although not all possible concave and convex domains are given by approximating an ellipsoid in this way. While this article was in its final stages, the author learned that Opshtein has independently studied concave toric domains that arise from resolutions of certain singularities, and has also observed from a somewhat different point of view that embeddings of these concave toric domains into many closed manifolds are equivalent to ball-packings, see [18, §5].

We now explain the details of the equivalence between embeddings of concave domains into convex ones and ball packings.

1.3 Weight sequences

In [11], McDuff introduced a sequence of real numbers determined by a 4-dimensional symplectic ellipsoid, called a weight sequence. Choi, the author, Frenkel, Hutchings, and Ramos generalized these weight sequences to any concave toric domain in [2]. We now review this generalization.

Let Ω be a concave toric domain. The weight sequence of Ω is a sequence of nonnegative real numbers $w(Ω)$ defined inductively as follows. If Ω is a triangle with vertices $(0,0), (0,a)$ and $(a,0)$, then the weight sequence of Ω is $(a)$. Otherwise, let $a > 0$ be the smallest real number such that Ω contains the triangle with vertices $(0,0), (0,a)$ and $(a,0)$. Call this triangle $Ω_1$. Then the line $x + y = a$ intersects the upper boundary of Ω in a line segment from $(x_1, a - x_1)$ to $(x_2, a - x_2)$, where $x_1 \leq x_2$. Let $Ω'_2$ be the closure of the part of Ω to the left of $x_1$ and above this line, and let $Ω'_3$ be the closure of the part of Ω to the right of $x_2$ and above this line, see Figure 1.2. Then, as explained in [2, §1.3], $Ω'_2$ is affine equivalent to a canonical concave toric domain, which we denote by $Ω_2$. Similarly, $Ω'_3$ is affine equivalent to a canonical concave toric domain which will be denoted by $Ω_3$. We now define $w(Ω) = w(Ω_1) \cup w(Ω_2) \cup w(Ω_3)$, where $\cup$ denotes the (unordered) union with repetitions. In the inductive definition, note that $w(Ω)$ is defined to be $\emptyset$ if $Ω = \emptyset$.

We now define a similar weight expansion for any convex toric domain. The definition of the weight sequence for convex toric domains is similar to the definition of the weight sequence for concave toric domains. If Ω is a triangle with vertices $(0,0), (0,b)$ and $(b,0)$ then the weight sequence of Ω is $(b)$. Otherwise, let $b > 0$ be the smallest real number such that Ω is contained in the triangle with vertices $(0,0), (0,b)$ and $(b,0)$. Call this triangle $Ω_1$. The line $x + y = b$ intersects the upper boundary of Ω in a line segment from $(x_1, b - x_1)$ to $(x_2, b - x_2)$, with $x_1 \leq x_2$. Let $Ω'_2$ denote the
Figure 1.2: The inductive decomposition of convex and concave toric domains

closure of the portion of $\Omega_1 \setminus \Omega$ that is to the left of $x_1$ and below the line $x + y = b$, and let $\Omega'_2$ denote the closure of the portion of $\Omega_1 \setminus \Omega$ that is below $b - x_2$ and below the line $x + y = b$, see Figure 1.2.

The key point is now that $\Omega'_2$ and $\Omega'_3$ are both affine equivalent to concave toric domains, which we denote by $\Omega_2$ and $\Omega_3$ respectively. The equivalence for $\Omega'_2$ is given by translating down so that the top left corner of $\Omega'_2$ is at the origin, and then multiplying by the matrix $M = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$, while the equivalence for $\Omega'_3$ is given by translating so that the bottom right corner is at the origin, and then multiplying by the matrix $M' = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$. We then define

$$w(\Omega) = (b; w(\Omega_2) \cup w(\Omega_3)).$$

Thus, the weight sequence for a convex toric domain consists of a number, and then an unordered set of numbers. We call the first number in this sequence the head, and we call the other numbers the negative weight sequence.

To simplify the notation, for a convex $\Omega$, let $\widehat{B}(\Omega)$ denote the disjoint union of (closed) balls with radii given by the negative weight expansion for $X_\Omega$. Also let $B(\Omega)$ for concave $\Omega$ denote the disjoint union of closed balls whose radii are given by the numbers in the weight expansion for $\Omega$. Finally, call a rational concave domain a concave domain whose upper boundary is piecewise linear, with rational coordinates, and define a rational convex domain similarly.

We can now state the aforementioned equivalence:

**Theorem 1.3.** Let $X_{\Omega_1}$ be a rational concave toric domain, let $X_{\Omega_2}$ be a rational convex toric domain, and let $b$ be the head of the weight expansion for $\Omega_2$. Then there exists a symplectic embedding

$$\text{int}(X_{\Omega_1}) \to \text{int}(X_{\Omega_2}).$$
if and only if there exists a symplectic embedding

\[ \text{int}(B(\Omega_1)) \sqcup \text{int}(\hat{B}(\Omega_2)) \to \text{int}(B(b)). \]

Note that the “only if” direction of Theorem 1.3 follows from the “Traynor trick” [21], see e.g. [2, Lem. 1.8] for the version we need, and the definition of the weight expansion.

1.4 Connectivity of the space of embeddings

McDuff also showed in [11] that the space of embeddings of one ellipsoid into another is connected. To prove Theorem 1.2 and Theorem 1.3, it will be helpful to show that this also holds for embeddings of a concave domain into a convex one:

**Proposition 1.4.** Let \( X_{\Omega_1} \) be a concave toric domain, let \( X_{\Omega_2} \) be a convex toric domain, and let \( g_0 \) and \( g_1 \) be two symplectic embeddings:

\[ X_{\Omega_1} \to \text{int}(X_{\Omega_2}). \]

Then there exists an isotopy \( \Psi_t : \text{int}(X_{\Omega_2}) \to \text{int}(X_{\Omega_2}) \) such that \( \Psi_0 = \text{id} \) and \( \Psi_1(g_1) = g_0 \).

The following corollary will be particularly useful:

**Corollary 1.5.** Let \( X_{\Omega_1} \) be a concave domain and let \( X_{\Omega_2} \) be convex. Then there is a symplectic embedding

\[ \text{int}(X_{\Omega_1}) \to \text{int}(X_{\Omega_2}) \]

if and only if there is a symplectic embedding

\[ X_{\lambda\Omega_1} \to \text{int}(X_{\Omega_2}) \]

for all \( \lambda < 1 \).

1.5 ECH capacities of convex domains and ECH capacities of balls

As explained in §1.2, the fact that ECH capacities are sharp for these embedding problems essentially follows from the fact that they are sharp for symplectic ball packings of a ball. In fact, the ECH capacities of both of these domains are closely related to the ECH capacities of balls. In [2], it was shown that the ECH capacities of any concave toric domain are determined by the ECH capacities of a certain collection of balls, see [2, Thm. 1.4] for the precise statement. In an appendix with Choi, we show that this is also true for most convex toric domains, see Theorem A.1.
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2 Embeddings of concave toric domains into convex toric domains and ball packings

We now explain the proof of Theorem 1.3. We already showed the “only if” direction, so we now show the converse. Our proof closely follows the “inflation” method from [11].

2.1 Embeddings of toric domains and embeddings of spheres

The objects that we will “inflate” are certain chains of symplectic spheres. In this section, we explain the significance of these chains of spheres to our embedding problem.

The spheres that we will want to inflate arise from a sequence of symplectic blowups. We therefore start by recalling those details of the blowup construction that are relevant to us. Let \( L \) denote the homology class of the line in \( \mathbb{C}P^2 \). There is a symplectic form \( \omega_0 \) on \( \mathbb{C}P^2 \), called the Fubini-Study form, such that \( \langle \omega_0, L \rangle = 1 \). Now suppose there is a symplectic embedding \( \bigcup_{i=1}^m B(a_i) \to (\mathbb{C}P^2, \omega_0) \). We can remove the interiors of the \( B(a_i) \) and collapse their boundaries under the Reeb flow to get a symplectic manifold, called the blowup of the ball packing, which is diffeomorphic to \( \mathbb{C}P^2 \# m\mathbb{C}P^2 \), with a canonical symplectic form \( \omega_1 \). The image of \( \partial B(a_i) \) in this manifold is called the \( i^{th} \) exceptional divisor. If \( E_i \) denotes the homology class of the \( i^{th} \) exceptional divisor, then the cohomology class of \( \omega_1 \) is given by

\[
\text{PD}[\omega_1] = L - \sum_{i=1}^m a_i E_i.
\]

Now let \( \Omega \) be any rational concave toric domain, and include \( \Omega \) into some large ball \( \text{int}(B(R)) \), which we can include densely into a \( (\mathbb{C}P^2, \omega) \).
Figure 2.1: Blowing up a rational concave domain. In this case, we first blow up region $A$, i.e. the region underneath the red line. We then blow up along $B$ (the region between the red line, the blue line, and the axis) and $D$ (the region between the red line, the green line, and the axis). The order in which we perform these two blow ups is irrelevant. Finally, we blow up along $C$. The result is a chain of four spheres. Note that the remaining portions of the blue and red lines correspond to spheres that are small, while the purple and green lines correspond to spheres whose length is close to the affine length of the corresponding side of the upper boundary. The black lines give the canonical weight sequence decomposition of the domain.

We now mimic the definition of the weight sequence to define a sequence of symplectic blowups of $(\mathbb{C}P^2, \omega)$ that will produce one of the relevant chains of spheres. The reader is urged to see Figure 2.1 which will help illustrate the idea. Let $a$ be the smallest real number such that $\Omega$ contains the triangle with vertices $(0, 0), (0, a)$ and $(a, 0)$, let $\delta > 0$ be a small real number, and consider the triangle $\Delta(a + \delta)$ with vertices $(0, 0), (0, a + \delta)$ and $(a + \delta, 0)$. Thus, in Figure 2.1 the upper boundary of $\Delta(a + \delta)$ is the red line, and this region is labeled $A$. Then there is a symplectic embedding $B(a + \delta) \to B(R)$. Blow up along $B(a + \delta)$. Now the upper boundary of $\Delta(a + \delta)$ intersects the complement of $\Omega$ in the plane along a line segment between $(x_1, a + \delta - x_1)$ and $(x_2, a + \delta - x_2)$ with $x_1 < x_2$. Let $\Gamma_1$ be the closure of the subset of $\Omega$ which is to the left of $x_1$ and above the line $x + y = a + \delta$, and let $\Gamma_2$ be the closure of the subset of $\Omega$ which is to the right of $x_2$ and above this line. Then, as in the definition of the weight sequence, $\Gamma_1$ and $\Gamma_2$ are affine equivalent to concave toric domains. In the present context,
This implies that we can iterate the procedure from the above paragraph to perform a symplectic blowup for each element of the weight sequence for $\Omega$. Each blowup produces a symplectic sphere. In choosing the relevant $\delta$ for each blowup, choose $\delta$ small enough so that none of the previous symplectic spheres are completely removed (so for example in Figure 2.1, we would want to choose the $\delta$ for the purple sphere to be small enough so that it does not completely remove the red sphere). The result of this sequence of blowups is a symplectic manifold $(\mathbb{C}P^2 \# m\mathbb{C}P^2, \omega_1)$ with a configuration of symplectic spheres $C_{\Omega,\delta_\Omega}$, with one sphere for each element of the weight sequence. Here, $\delta_\Omega$ denotes a sequence of small real numbers corresponding to the $\delta$ for each blow up.

We will want to define a similar sequence of blowups if $\Omega$ is a rational convex domain. Specifically, let $b$ be the head of the weight sequence for $\Omega$, and choose a small $\delta > 0$. The line $x+y=b-\delta$ intersects $\Omega$ in a line segment from $(x_1, b-\delta-x_1)$ to $(x_2, b-\delta-x_2)$, where $x_1 < x_2$. Let $\Delta(b-\delta)$ be the triangle with vertices $(0,0), (b-\delta,0)$ and $(0,b-\delta)$. Let $\Gamma_1$ be the closure of the region of the complement of $\Omega$ in $\Delta(b-\delta)$ that is to the left of $x_1$, and let $\Gamma_2$ be the closure of the region of the complement that is below $b-\delta-x_2$.

We showed in the definition of the weight sequence that $\Gamma_1$ and $\Gamma_2$ are affine equivalent to concave toric domains. In the present context, this means that we can copy the argument from the previous paragraph to associate a symplectic blow up to each term in the negative weight sequence for $\Omega$, of the $\mathbb{C}P^2$ that int$(B(b-\delta))$ includes densely into. As in the previous paragraph, this requires a choice of small real numbers corresponding to the $\delta$ in this blow up construction. We again denote this set of small numbers by $\delta$. The result of these additional blowups is a symplectic manifold $(\mathbb{C}P^2 \# n\mathbb{C}P^2, \omega_2)$ with a configuration of symplectic spheres which we denote by $\hat{C}_{\Omega,\delta_\Omega}$.

Our blowup procedure is closely related to the inner and outer approximations from [11]. To elaborate, consider first the blow up procedure for rational concave $\Omega$. Our blowup procedure shows that we can define another concave toric domain, called an outer approximation to $\Omega$, such that the sequence of blowups removes the interior of the outer approximation and collapses the boundary of the outer approximation to the configuration of spheres $C_{\Omega,\delta}$. Denote the outer approximation to $\Omega$ by $\Omega^{\text{out}}$. In the example illustrated in Figure 2.1, the upper boundary of $\Omega^{\text{out}}$ is given by starting where the blue line hits the vertical axis, and then traversing the part of the blue line to the left of the purple line, then the purple line, then the part of the red line between the purple and green lines, and finally the green line. Similarly, our blowup procedure shows that we can define another convex toric domain, called an inner approximation to $\Omega$, such that the sequence of blowups removes the complement of the inner approximation in $B(b-\delta)$ and collapses the boundary of the inner approximation to the configuration.
of spheres $\tilde{C}_{\Omega,\delta_1}$.

Here is the significance of these chains of spheres to our embedding problem:

**Proposition 2.1.** Let $\Omega_1$ be a rational concave toric domain, and let $\Omega_2$ be a rational convex toric domain. Let $m$ be the length of the weight expansion for $\Omega_1$, and let $n$ be the length of the negative weight expansion for $\Omega_2$. If there is a symplectic form $\omega$ on $\mathbb{CP}^2 \# (m + n) \mathbb{CP}^2$ such that there is a symplectic embedding

$$C_{\Omega_1,\delta_1} \sqcup \tilde{C}_{\Omega_2,\delta_1} \to (\mathbb{CP}^2 \# (m + n) \mathbb{CP}^2, \omega),$$

then there is a symplectic embedding

$$X_{\Omega_1} \to \text{int}(X_{\Omega_2}).$$

**Proof.** By assumption, there is a symplectic embedding

$$C_{\Omega_1,\delta_1} \to (\mathbb{CP}^2 \# (m + n) \mathbb{CP}^2, \omega).$$

As explained in [11, Lem. 2.2], we can make a small perturbation to this embedding so that these symplectic spheres intersect symplectically orthogonally. A version of the symplectic neighborhood theorem, see for example [20, Prop. 3.5], now implies that a neighborhood of these spheres in the manifold $(\mathbb{CP}^2 \# m \mathbb{CP}^2, \omega)$ that was constructed above by blowing up the outer approximation. We can therefore remove the $C_{\Omega_1,\delta_1}$ and glue in a copy of $X_{\Omega_1 \cap \Omega_2}$ to get a new symplectic manifold $\tilde{Z}$ which admits a symplectic embedding of $X_{\Omega_1}$. (This is a special case of the “blow down” procedure explained in [20], see especially [20, Cor. 3.6].)

The construction from the previous paragraph can be done in the complement of $C_{\Omega_2,\delta_0}$, Moreover, we can repeat the argument from the previous paragraph to conclude that a neighborhood of $C_{\Omega_2,\delta_0}$ in $\tilde{Z}$ is standard. Let $Z$ denote the complement of $C_{\Omega_2,\delta_0}$ in $\tilde{Z}$. We know from [15, Thm. 9.4.2] that there is a unique symplectic form that is standard near the boundary on any star-shaped subset of $\mathbb{R}^4$. It then follows that we can identify $X_{\text{int}(\Omega_2 \cap \Omega_2)}$ with $Z$. Since $\Omega_2$ contains the inner approximation, the proposition now follows.

\[\square\]

### 2.2 Connectivity

We can now give a quick proof of Proposition 1.4 which states that the space of embeddings from a concave domain into a convex one is connected. We also prove Corollary 1.5
Proof of Proposition 1.4. Proposition 1.4 follows from the proof of [11, Cor. 1.6]. While the proof of [11, Cor. 1.6] is for ellipsoids, the discussion in §2.1 shows that the proof generalizes to our case without any modifications.

For completeness, we sketch the argument. First, assume that $\Omega_1$ and $\Omega_2$ are rational, and let $g_0$ and $g_1$ be symplectic embeddings of $X_{\Omega_1}$ into $\text{int}(X_{\Omega_2})$. By applying Alexander’s trick, see e.g. the proof of [19, Prop. A.1], we can assume that $g_0$ and $g_1$ agree with the inclusion of $X_{t\Omega_1}$ into $\text{int}(X_{\Omega_2})$ for sufficiently small $r$. Then, as in §2.1, we can blow up along $X_{t\Omega_1}$ and $X_{\Omega_2}$ to get a symplectic manifold $X_0 = (\mathbb{C}P^2 \# (m + n)\mathbb{C}P^2, \omega)$ with two chains of exceptional spheres $C_{r,\Omega_1,\delta_r \cdot \Omega_1} \cup \hat{C}_{\Omega_2,\delta_{\Omega_2}}$. We can also blow up along $g_0$ and $g_1$ to get two different symplectic forms $\omega_1$ and $\omega_2$ on $X_0$ (here, we are implicitly identifying the underlying spaces of these blow ups with $X_0$ as in Step 2 of [13, §3]). In the present situation, the argument from [13, §3] shows that $\omega_1$ and $\omega_2$ are deformation equivalent (the deformation is essentially given by blowing up along $X_{t\Omega_1}$ and $X_{\Omega_2}$ as $t$ varies). By using the singular inflation procedure from [12], we can convert this deformation to an isotopy, and we can assume that this isotopy is supported away from $C_{r,\Omega_1,\delta_r \cdot \Omega_1} \cup \hat{C}_{\Omega_2,\delta_{\Omega_2}}$, see [11, Cor. 1.6] and [16, Thm. 1.2.11]. We can therefore blow down this isotopy to give the desired isotopy between $g_0$ and $g_1$. The result for nonrational $\Omega_1$ and $\Omega_2$ follows by approximating by rational domains.

Proof of Corollary 1.5. This also follows without any modifications from the proof of [11, Cor. 1.6]: from the sequence of embeddings

$$X_{\lambda \Omega_1} \to \text{int}(X_{\Omega_2}),$$

we can obtain a sequence of embeddings $g_n : X_{(1-1/n)\Omega_1} \to \text{int}(X_{\Omega_2})$. By applying Proposition 1.4, we can assume that this sequence of maps is nested. We can therefore construct the desired symplectic embedding by taking the direct limit.

2.3 Inflating the spheres

We can now complete the proof of Theorem 1.3. The inflation that we do here is also standard after [11] and [16].

Proof of Theorem 1.3. Before beginning the proof, we briefly comment on one point, in order to motivate what follows.

Let $\Omega_1$ be a rational concave domain and let $\Omega_2$ be a rationally convex domain. By assumption, there is a ball packing of a ball, determined by the weights of the $\Omega_i$. For the inflation method, we will want to blow up along this ball packing, to conclude that a certain cohomology class is represented
by a symplectic form. However, we are given a packing by open balls, while
to blow up we would like a packing by closed balls. To remedy this, observe
that if $\varepsilon > 0$ is sufficiently small, we have a ball packing

$$\cup_i B((1 - \varepsilon)a_i) \cup_j B((1 - \varepsilon)b_j) \to \text{int}(B(b)), \quad \text{(2.1)}$$

where the $a_i$ are the weights of $\Omega_1$, and the $b, b_j$ are the weights of $\Omega_2$. The
numbers $(1 - \varepsilon)a_i$ are the weights of $(1 - \varepsilon)\Omega_1$. Meanwhile, the numbers
$(b, (1 - \varepsilon)b_1, \ldots, (1 - \varepsilon)b_n)$ are the weights of a convex toric domain $\hat{\Omega}_2$ with
the property that $\hat{\Omega}_2 \subset (1 + \varepsilon)\Omega_2$. Here, $\varepsilon$ can be made arbitrarily small if $\varepsilon$ is
made small enough. We will show below that we can construct a symplectic
embedding $(1 - \varepsilon)X_{\Omega_1} \to \text{int}(X_{\hat{\Omega}_2}) \subset (1 + \varepsilon)X_{\Omega_2}$. We can then construct the
desired symplectic embedding by appealing to Corollary 1.5. The details are
as follows:

Step 1. Let $r$ be a small enough rational number that $r \cdot (1 - \varepsilon)\Omega_1 \subset \text{int}(\hat{\Omega}_2)$. Then $r \cdot (1 - \varepsilon)\Omega_1$ is a concave toric domain, and $\hat{\Omega}_2$ is a convex
toric domain. We can therefore apply the iterated blowup procedure from
§2.1 to conclude that there is a symplectic embedding

$$C_{r,(1-\varepsilon)\Omega_1,\delta_r,(1-\varepsilon)\Omega_1} \sqcup \hat{C}_{\hat{\Omega}_2,\delta_{\hat{\Omega}_2}} \to (\mathbb{C}P^2 \# (m + n)\overline{\mathbb{C}P^2}, \omega_1).$$

Let $L$ denote the homology class of the line in this blowup, let $E_1, \ldots, E_m$
be the exceptional classes associated to the blow ups for $r \cdot (1 - \varepsilon)\Omega_1$, and
let $\hat{E}_1, \ldots, \hat{E}_n$ be the exceptional classes associated to the blow ups for $\hat{\Omega}_2$.
Let $\ell = \text{PD}(L)$, let $e_i = \text{PD}(E_i)$, and let $\hat{e}_j = \text{PD}(E_j)$. By §2.1 we know
that the cohomology class of $\omega_1$ is given by

$$[\omega_1] = (b - \text{err}_2(\delta))\ell - \sum_{i=1}^m (r(1 - \varepsilon)a_i + \text{err}_i(\delta_1))e_i - \sum_{j=1}^n ((1 - \varepsilon)b_j + \text{err}_j(\delta_2))\hat{e}_j,$$

(2.2)

where the $(1 - \varepsilon)a_i$ are the terms in the weight sequence for $\Omega_1$, the $(1 - \varepsilon)b_j$
are the terms in the weight sequence for $\hat{\Omega}_2$, and the $\text{err}_i$ denote (possibly negative) error terms that are small and determined by the relevant $\delta_j$.
Meanwhile, the homology class of the image of each exceptional sphere in this
manifold is determined by the canonical decompositions into affine triangles
of $(1 - \varepsilon)\Omega_1$ and $B(b) \setminus \hat{\Omega}_2$ given by the weight sequences. In particular, the
homology classes of these exceptional spheres do not depend on $r$.

Step 2. We now want to show that there is a symplectic embedding of
$C_{(1-\varepsilon)\Omega_1,\delta_{(1-\varepsilon)\Omega_1}} \sqcup \hat{C}_{\hat{\Omega}_2,\delta_{\hat{\Omega}_2}}$ into $(\mathbb{C}P^2 \# (m + n)\overline{\mathbb{C}P^2}, \omega)$ for some $\omega$, so that
we can appeal to Proposition 2.1. Since the intersection properties of the
configuration of spheres $C_{r,(1-\varepsilon)\Omega_1,\delta_r,(1-\varepsilon)\Omega_1} \sqcup \hat{C}_{\hat{\Omega}_2,\delta_{\hat{\Omega}_2}}$ does not depend on $r$,
we therefore just have to alter the symplectic form so that these spheres
have the correct area while remaining symplectic. To do this, we inflate the spheres, as in [16].

To perform the inflation, we need to find an appropriate \( J \)-holomorphic curve to inflate along. We now explain how to find such a curve. By assumption, as mentioned at the beginning of this proof, there is a ball packing (2.1). This gives a ball packing

\[ \cup_i B((1 - \varepsilon)(1 + \varepsilon')a_i) \cup_j B((1 - \varepsilon)b_j) \to \text{int}(B(b)), \]  

(2.3)

where \( \varepsilon' \) is sufficiently small. Blowing up along this ball-packing shows that the class \( a = b\ell - \sum_{i=1}^n (1 - \varepsilon)(1 + \varepsilon')a_i e_i - \sum_{i=1}^m (1 - \varepsilon)b_j \hat{e}_j \) is represented by a symplectic form. We can assume that this class is rational. As explained in the proof of [11, Prop. 1.10], work of Kronheimer and Mrowka [9] then shows that for all sufficiently large integers \( q \), the class \( qa \) has nontrivial Seiberg-Witten invariant. Since \( qa \) is also represented by a symplectic form, Taubes’ “Gromov = Seiberg-Witten” theorem then implies that \( \text{PD}(qa) \) has nontrivial Gromov invariant.

Step 3. We would now like to conclude that the homology class \( \text{PD}(qa) \) is represented by a connected embedded \( J \)-holomorphic curve, so that we can apply the “standard” inflation procedure, e.g. as explained in [13, Lem 1.1]. However, as explained in [12], there is a substantial technical hurdle to concluding this. We can circumvent this difficulty by using the “singular” inflation procedure from [16].

To elaborate, the difficulty is that the inflation procedure requires choosing an \( \omega_1 \) tame almost complex structure \( J \) such that \( C_{r,(1-\varepsilon)\Omega_1,\delta_{r,(1-\varepsilon)\Omega_1}} \sqcup \tilde{C}_{r_2,\delta_{r_2}} \) is \( J \)-holomorphic, and this cannot be done while keeping \( J \) suitably generic so that Taubes’ Gromov invariant for this \( J \) is defined. However, in the present context we can still apply [16, Lem. 1.2.11] to find the desired family of symplectic forms.

While for the applications in this paper, we just need to verify that the assumptions of [16, Lem. 1.2.11] hold, for completeness we sketch how the singular inflation procedure from [16] works in this situation. The basic point is that we can find a family of suitably generic \( J_t \) tending to a \( J \) such that the configuration of spheres is \( J \)-holomorphic. By Gromov compactness, we can therefore find a \( J \)-holomorphic nodal representative of the class \( \text{PD}(qa) \). By perturbing \( J \) and this curve as in [16, §3], see especially [16, Prop. 3.1.3], we can assume that each component of this curve is a multiple cover of an embedded curve. The hypotheses of [16, Lem. 1.2.11] will then ensure that each of these components has nonnegative intersection with \( \text{PD}(qa) \), which will allow us to inflate. For the details of the inflation process, see [16, §5.2], especially [16, Prop. 5.1.2].

We now verify that the hypotheses of [16, Lem. 1.2.11] hold. This requires checking that the class \( \text{PD}(qa) \) satisfies the four requirements of
The only points that require further explanation are the third and fourth bullet points. The third bullet point requires that $A$ has nonnegative intersection with any exceptional sphere. This holds due to [2.1], by a standard argument, see for example [8, Prop. 6]. The fourth bullet point requires checking that $A$ has nonnegative intersection with each of the spheres in $C_{r(1-\varepsilon)\Omega_1,\delta r(1-\varepsilon)\Omega_1} \sqcup \widehat{C}_{\Omega_2,\delta \Omega_2}$. To see that this holds, remember that the homology classes of these spheres depend neither on $r$ nor on $\delta$. The claim now follows by (2.2), since each of these spheres have positive area with respect to the form $\omega_1$ from (2.2), and this remains true as we let all the $\delta_i$ tend to 0 (note that as $\delta_i$ tends to 0, $err_i(\delta_i)$ does as well).

Step 4. We can therefore apply [16, Lem. 1.2.11] to inflate. In the present context, this procedure produces for all positive $t$ a family of symplectic forms $\omega_t$ such that each $\omega_t$ restricts to a symplectic form along $C_{r(1-\varepsilon)\Omega_1,\delta r(1-\varepsilon)\Omega_1} \sqcup \widehat{C}_{\Omega_2,\delta \Omega_2}$ and the $\omega_t$ have cohomology class $[\omega_t] = [\omega_1] + tqa$.

Now consider $\omega_t/(1 + tq)$. We have

$$[\omega_t]/(1 + tq) = b\ell - \sum_{j=1}^n (1 - \varepsilon)b_j \hat{e}_j - \frac{r + (1 + \varepsilon')tq}{1 + tq} \sum_{i=1}^m (1 - \varepsilon)a_i e_i - \frac{err(\delta)}{1 + tq} b\ell - \sum_{i=1}^m err_i(\delta_i) \frac{1 + tq}{1 + tq} e_i - \sum_{j=1}^n err_j(\delta_j) \frac{1 + tq}{1 + tq} \hat{e}_j.$$

Now if $t$ is sufficiently large, then $\frac{r + (1 + \varepsilon'tq}{1 + tq} = 1$. By choosing the $\delta_i$ for $C_{(1-\varepsilon)\Omega_1,\delta(1-\varepsilon)\Omega_1} \sqcup \widehat{C}_{\Omega_2,\delta \Omega_2}$ sufficiently small for this large $t$, it now follows that there is a symplectic embedding $C_{(1-\varepsilon)\Omega_1,\delta(1-\varepsilon)\Omega_1} \sqcup \widehat{C}_{\Omega_2,\delta \Omega_2} \to (\mathbb{CP}^2 \# (m+n)\overline{\mathbb{CP}^2}, \omega_t)$. By Proposition 2.1 there is now for all sufficiently small $\varepsilon$ and $\varepsilon'$ a symplectic embedding $X_{(1-\varepsilon)\Omega_1} \to X_{\Omega_2} \subset (1 + \varepsilon') \text{int}(X_{\Omega_2})$. Theorem 1.3 now follows by Corollary 1.5.

2.4 Examples

We now present several illustrative examples.

Example 2.2. Let $\Omega$ be the rectangle with vertices $(0, 0), (1, 0), (0, 1)$ and $(1, 1)$, and let $\Omega'$ be the triangle with vertices $(0, 0), (2, 0)$ and $(0, 1)$. Then $\Omega$ is a polydisk and $\Omega'$ is an ellipsoid. Both $\Omega$ and $\Omega'$ are convex (we could also regard $\Omega'$ as concave, although for this example we do not want to), and the weight sequence for both is given by $(2, 1, 1)$; in particular, both have...
Figure 2.2: The domains of Example 2.4. We have drawn the canonical decomposition of the domains given by the weight sequence (remember that the weight sequence for \( \Omega_2 \) gives a decomposition of the complement of \( \Omega_2 \) in a ball). The blue line is the upper boundary of the outer approximation for \( \Omega_1 \), while the red line is the upper boundary of the inner approximation of \( \Omega_2 \).

the same weight sequence. This shows that weight sequences are not unique. Also, by Theorem 1.3 a concave domain embeds into \( X_{\Omega} \) if and only if it embeds into \( X_{\Omega'} \). This generalizes a result of Frenkel and Mueller [4, Cor. 1.5], which proves this when the domain is an ellipsoid (our proof is also slightly different from theirs).

**Example 2.3.** Let \((a_0, \ldots, a_n)\) be any finite sequence of nonincreasing real numbers. We now explain why we can always construct a rational concave toric domain with weight sequence \((a_0, \ldots, a_n)\). In fact, we show that we can always construct a rational concave toric domain with the property that at each step in the inductive definition of the weight sequence, the domain \(\Omega'_2\) from §1.3 is empty (we will call such a domain *short*). By induction, we can assume that we can construct a short rational concave domain \(\Omega_0\) with weight sequence \((a_1, \ldots, a_n)\). Now, consider the triangle \(\Delta(a_0)\) with vertices \((0, 0), (0, a_0)\) and \((0, 0)\). Multiply \(\Omega_0\) by the matrix \(\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\) and then translate the result by \((a_0, 0)\). Let \(\Omega\) be formed by taking the union of this region with \(\Delta(a_0)\). Then by construction \(\Omega\) is a short concave domain with weight sequence \((a_0, \ldots, a_n)\). Thus, any possible ball packing problem of a ball can arise by applying Theorem 1.3. This is to be compared with the case of embedding an ellipsoid into a ball, where the ball packings that arise are much more constrained, see [17].

**Example 2.4.** We now work through a more extended example in detail. Let \(\Omega_1\) be the domain whose upper boundary has vertices \((0, 8/3), (2/3, 4/3), (4/3, 2/3), \) and \((7/3, 0)\), and let \(\Omega_2\) be the domain whose upper boundary has vertices \((0, 1), (1, 2)\) and \((5, 0)\). Then the weight expansion of \(\Omega_1\) is \((2, 2/3, 2/3, 1/3, 1/3)\) and the weight expansion of \(\Omega_2\) is \((5, 3, 2, 1)\), see Figure 2.2. There are five spheres in the chain of spheres corresponding to the blow up of \(r \cdot \Omega_1\). Each sphere corresponds to a blow up, and if we
label these spheres in the order that they appear as edges of the outer approximation (with the first sphere the left most edge), and label the blow ups they correspond to accordingly, then the spheres have homology classes $E_1, E_2 - E_1, E_3 - E_2 - E_4 - E_5, E_4$ and $E_5 - E_4$. There are four spheres in the chain of spheres corresponding to the blow up of $\Omega_2$ (including the sphere corresponding to the line at infinity). If we label these spheres and the blowups with the same ordering convention as above, then they have homology classes $\hat{E}_1, \hat{E}_2 - \hat{E}_1 - \hat{E}_3, \hat{E}_3$, and $L - \hat{E}_2 - \hat{E}_3$. The cohomology class of the symplectic form on the blow up is given in this notation by

$$[\omega_1] = 5L - (2/3)re_1 - (2/3)re_2 - 2re_3$$

$$- (1/3)re_4 - (1/3)re_5$$

$$- \hat{e}_1 - 3\hat{e}_2 - 2\hat{e}_3$$

$$- \sum_{i=1}^m \text{err}_i(\delta_1)\hat{e}_i - \sum_{j=1}^n \text{err}_j(\delta_2)\hat{e}_j.$$ (2.4)

By Theorem [1.3] to see if $\text{int}(X_{\Omega_1})$ embeds into $\text{int}(X_{\Omega_2})$, it is equivalent to see if there is a ball packing

$$\text{int}(B(2/3) \square B(2/3) \square B(2) \square B(2) \square B(1/3) \square B(1/3) \square B(2) \square B(1)) \to B(5).$$ (2.5)

One can check, e.g. by applying the algorithm from [1, §2.3], that in fact such a ball packing exists. Hence, there is a symplectic embedding $\text{int}(X_{\Omega_1}) \to \text{int}(X_{\Omega_2})$. In fact, this embedding is optimal (e.g. by [1, §2.3] again applied to Equation 2.5), in the sense that no larger scaling of $\text{int}(X_{\Omega_1})$ embeds into $\text{int}(X_{\Omega_2})$.

3 Sharpness for the ball packing problem implies ECH capacities are sharp

We now explain the proof of Theorem [1.2]. The key point is that it was shown in [6] that ECH capacities are known to be sharp for ball packing problems.

Proof. We need to show that $\text{int}(X_{\Omega_1})$ embeds into $\text{int}(X_{\Omega_2})$ if and only if $c_k(\text{int}(X_{\Omega_1})) \leq c_k(\text{int}(X_{\Omega_2}))$ for all $k$. The fact that a symplectic embedding

$$\text{int}(X_{\Omega_1}) \to \text{int}(X_{\Omega_2})$$

implies that $c_k(\text{int}(X_{\Omega_1})) \leq c_k(\text{int}(X_{\Omega_2}))$ for all $k$ follows from the Monotonicity property of ECH capacities shown in [6].

For the converse, first note that by Corollary [1.5], we can assume that $\Omega_1$ and $\Omega_2$ are rational. Now by the Monotonicity and Disjoint Union properties
from [6], and the argument for the “only if” direction of Theorem 1.3, we know that

\[ c_{ECH}(\text{int}(X_{\Omega_2})) \# c_{ECH}(\text{int}(\hat{B}(\Omega_2))) \leq c_{ECH}(B(b)), \]

where \# denotes the “sequence sum” defined in [6], and \( c_{ECH} \) denotes the sequence of ECH capacities. We also know by the same argument that

\[ c_k(\text{int}(B(\Omega_1))) \leq c_k(\text{int}(X_{\Omega_1})). \]

We also know that sequence sum against a fixed sequence respects inequalities. Hence, combining \( c_k(\text{int}(X_{\Omega_1})) \leq c_k(\text{int}(X_{\Omega_2})) \) with the above equations and the Disjoint Union property implies that

\[ c_k(\text{int}(B(\Omega_1)) \sqcup \text{int}(\hat{B}(\Omega_2))) \leq c_k(B(b)). \]  \hspace{1cm} (3.1)

It is known that ECH capacities give sharp obstructions to all (open) ball packings of a ball, see e.g. [6]. Hence, (3.1) implies that there exists a symplectic embedding

\[ \text{int}(B(\Omega_1)) \sqcup \text{int}(\hat{B}(\Omega_2)) \rightarrow B(b). \]

Hence by Theorem 1.3 there exists a symplectic embedding

\[ \text{int}(X_{\Omega_1}) \rightarrow \text{int}(X_{\Omega_2}), \]

hence the theorem.

\[ \square \]

A Appendix (by Keon Choi and Daniel Cristofaro-Gardiner): The geometric meaning of ECH capacities of convex domains

We assume below that the reader is familiar with the definitions and notation from the body of this paper. There, the second author showed that ECH capacities give a sharp obstruction to embedding any concave toric domain into a convex one. As explained in [3], to show this, all we need to know about ECH capacities are the basic axioms they satisfy, together with the fact that they are sharp for ball packings of a ball (of course, we also need to know Theorem 1.3 which states that embedding a concave domain into a convex one is equivalent to a ball packing problem). This suggests that there should be a close relationship between the ECH capacities of concave or convex toric domains, and the ECH capacities of balls.

In [2], the authors and Frenkel, Hutchings and Gripp showed that ECH capacities of any concave toric domain are given by the ECH capacities of the
disjoint union of the balls determined by the weight sequence of the domain. We now prove a similar formula for most convex toric domains, which we expect will hold for all of them.

To state the formula, recall the “sequence subtraction” operation from [6]. This is given for sequences $S$ and $T$ by

$$(S - T)_k = \min_{l \geq 0} S_{k+l} - T_l.$$ 

In the present context, the sequence subtraction operation is significant because of the following:

**Theorem A.1.** Fix a positive real number $a$, and let $\Omega$ be the region in the first quadrant underneath the graph of a nonincreasing concave function $f : [0, a] \to [0, b]$ that is not identically zero. Let $b$ be the head of the weight expansion for $\Omega$, and let $b_i$ be the $i$th term in the negative weight expansion for $X_\Omega$. Then

$$c_{ECH}(X_\Omega) = c_{ECH}(B(b)) - c_{ECH}(\sqcup_i B(b_i)).$$

Note that it follows from the Monotonicity and Scaling axioms that $c_k(X_\Omega) = c_k(\text{int}(X_\Omega))$ for any convex toric domain $X_\Omega$. Note also that even when $\Omega$ is not rational, the above formula still makes sense, see [2, Rmk. 1.6].

The proof we give is similar to the proof of [2, Thm. 1.4]. However, our proof is significantly shorter than the proof of [2, Thm. 1.4], because we can quote many of the relevant parts of this proof directly.

**Proof.** Step 1. By the definition of the weight expansion, there is a symplectic embedding

$$X_\Omega \sqcup_i \text{int}(B(b_i)) \to B(b).$$

It then follows from the Monotonicity axiom from [6] that

$$c_{ECH}(\sqcup_i B(b_i)) \# c_{ECH}(X_\Omega) \leq c_{ECH}(B(b)).$$

The subtraction operation is not inverse to the sequence sum operation. However, we still have

$$c_{ECH}(X_\Omega) \leq c_{ECH}(B(b)) - c_{ECH}(\sqcup_i B(b_i)). \quad (A.1)$$

We want to show that this is an equality.

**Step 2.** In [2, Defn. 1.20], a length function determined by a concave domain $\Omega$, called the $\Omega$-length, was defined. This associated a real number

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2The domains we are describing here are exactly the regions Hutchings calls “convex toric domains”. We do not use that terminology here, since we already used it to describe a slightly more general class of domains in the body of this paper, see Definition 1.1.

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\[ \ell_\Omega(\Lambda) \] to a concave planar lattice path \( \Lambda \) (see [2] Defn. 1.19) for the precise definition of a concave lattice path). We now recall a similar length function associated to a convex domain for convex planar lattice paths, defined by Hutchings in [7 Defn. 1.11], and explain its significance.

First, as in [7], define a *convex lattice path* \( \Lambda \) to be the graph of a piecewise linear concave nonincreasing function \( f : [0, a] \to [0, b] \) such that \( a \) and \( b \) are positive, \( f(0) = b, f(a) = 0 \), and the vertices of \( f \) are at lattice points. Also as in [7], if \( \Lambda \) is a convex lattice path, define \( \ell_\Omega(\Lambda) \) as follows. If \( e \) is an edge of \( \Lambda \), let \( p_e \) be a point on the upper boundary of \( \Omega \) such that \( \Omega \) is contained in the region bounded by the line through \( p_e \) of slope \( v \) and the axes. Now define

\[
\ell_\Omega(\Lambda) = \sum_{e \in \text{Edges}(\Lambda)} v_e \times p_e,
\]

where \( \times \) denotes the cross product.

Also as in [7], define for a convex lattice path \( \Lambda \) the lattice point counting function \( L(\Lambda) \). This is defined to be the number of lattice points in the region bounded by \( \Lambda \) and the axes.

The significance of the \( \Omega \) length and the lattice point counting function \( L \) is that we have

\[
c_k(X_\Omega) = \ell_\Omega(\Lambda), \tag{A.2}
\]

where \( \Lambda \) is some convex planar lattice path with

\[
L(\Lambda) \geq k + 1. \tag{A.3}
\]

This follows from [7 Lem. 5.4], by the argument in [2 Lem. 3.4].

**Step 3.** Let \( \Lambda \) be a convex lattice path with action equal to \( c_k(X_\Omega) \). In this step, we use a technique, similar to a trick from [2], to relate \( \ell_\Omega(\Lambda) \) and \( L(\Lambda) \) to quantities which are close to the quantities in (A.1).

The idea of the trick is to consider the first step of the weight expansion for \( \Lambda \) itself. Specifically, the first step of the weight expansion determines a triangle \( \tilde{\Delta} \), and also domains \( \tilde{\Omega}'_2 \) and \( \tilde{\Omega}'_3 \), as in §1.3. We can regard the upper boundary of the triangle \( \Delta \) as a convex lattice path \( \Lambda_1 \); additionally, as in §1.3, the domains \( \tilde{\Omega}'_2 \) and \( \tilde{\Omega}'_3 \) are affine equivalent to domains \( \tilde{\Omega}_2 \) and \( \tilde{\Omega}_3 \), and we can regard the upper boundary of these domains as concave lattice paths \( \Lambda_2 \) and \( \Lambda_3 \) as well. Observe that the number of lattice points in the triangle \( \tilde{\Delta} \) is equal to \( L(\Lambda) \) plus the number of lattice points in \( \tilde{\Omega}'_2 \) and \( \tilde{\Omega}'_3 \) that do not include the lattice points on \( \Lambda \).

Now recall the length function \( \ell_\Omega \) for concave lattice paths determined by a concave domain \( \Omega \) from [2] (this is exactly the same as the length function for a convex domain, except that one now chooses the point \( p_e \) so that the upper boundary of the concave domain is above the corresponding line). Also recall the lattice point counting function \( L(\Lambda) \) for concave paths
Λ, which is the same as the counting function for convex paths, but does not include points on the upper boundary. We just saw that

$$L(Λ) = L(Λ_1) - L(Λ_2) - L(Λ_3). \quad (A.4)$$

Moreover, if we let Ω_2 and Ω_3 be the concave domains determined by the first step of the weight expansion for Ω itself (remember, these are the affine images of Ω' and Ω") we have, by essentially the same argument as in Step 4 of [2, §2.1],

$$\ell(Ω) = \ell(Ω_1) - \ell(Ω_2) - \ell(Ω_3). \quad (A.5)$$

**Step 4.** We now apply some of the results from [2] to better understand (A.5) and (A.4). First note, e.g. by the Ellipsoid axiom from [2], that

$$\ell(Ω_1) = c_k_1(B(b)),$$

where $k_1 = L(Λ_1) - 1$. It was also shown in [2, Thm. 1.21] that $\ell(Ω_2) \leq c_k_2(X_{Ω_2})$ and $\ell(Ω_3) \leq c_k_3(X_{Ω_3})$, where $k_2 = L(Λ_2)$ and $k_3 = L(Λ_3)$. Hence, by (A.2) and (A.5), we have

$$c_k(X_{Ω}) \geq c_k_1(B(b)) - c_k_2(X_{Ω_2}) - c_k_3(X_{Ω_3}), \quad (A.6)$$

where, by (A.4) and (A.3), we know that $k_1 - k_2 - k_3 \geq k$. Moreover, we also know from [2, Thm. 1.4], see also [2, Rmk. 1.6], that $c_k_2(X_{Ω_2}) = c_k_2(\cup_m B(b_m))$ and $c_k_3(X_{Ω_3}) = c_k_3(\cup_n B(b_n))$, where the $b_m$ range over some subset of the $b_i$ and the $b_n$ range over its complement. Combining this with (A.1) gives the desired result, since by the definition of the subtraction operation,

$$c_{ECH}(B(b)) - c_{ECH}(\cup_m B(b_m)) - c_{ECH}(\cup_n B(b_n)) = c_{ECH}(B(b)) - c_{ECH}(\cup_i B(b_i)).$$

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\]

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