STABILITY OF DENSITIES FOR PERTURBED DEGENERATE DIFFUSIONS

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Abstract. We study the sensitivity of the densities of some Kolmogorov like degenerate diffusion processes with respect to a perturbation of the coefficients of the non-degenerate component. Under suitable (quite sharp) assumptions we quantify how the perturbation of the SDE affects the density. Natural applications of these results appear in various fields from mathematical finance to kinetic models.

Keywords: Diffusion Processes, Markov Chains, Parametrix, Hölder Coefficients, bounded drifts.

1. Introduction

We consider \( \mathbb{R}^d \times \mathbb{R}^d \)-valued processes that follow the dynamics:

\[
\begin{aligned}
    dX_t &= b(X_t, Y_t)dt + \sigma(X_t, Y_t)dW_t, \\
    dY_t &= X_t dt, \quad t \in [0, T],
\end{aligned}
\]  

(1.1)

where \( b : \mathbb{R}^{2d} \to \mathbb{R}^d \), \( \sigma : \mathbb{R}^{2d} \to \mathbb{R}^d \otimes \mathbb{R}^d \) are bounded coefficients that are Hölder continuous in space (this condition will be possibly relaxed for the drift term \( b \)) and \( W \) is a Brownian motion on some filtered probability space \( (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}) \). \( T > 0 \) is a fixed deterministic final time. Also, \( a(x, y) := \sigma \sigma^*(x, y) \) is assumed to be uniformly elliptic.

We now introduce a perturbed version of (1.1) with dynamics:

\[
\begin{aligned}
    dX_t^{(\varepsilon)} &= b_\varepsilon(X_t^{(\varepsilon)}, Y_t^{(\varepsilon)})dt + \sigma(X_t^{(\varepsilon)}, Y_t^{(\varepsilon)})dW_t, \\
    dY_t^{(\varepsilon)} &= X_t^{(\varepsilon)} dt, \quad t \in [0, T],
\end{aligned}
\]  

(1.2)

where \( b_\varepsilon : \mathbb{R}^{2d} \to \mathbb{R}^d \), \( \sigma_\varepsilon : \mathbb{R}^{2d} \to \mathbb{R}^d \otimes \mathbb{R}^d \) satisfy at least the same assumptions as \( b, \sigma \) and are in some sense meant to be close to \( b, \sigma \) for small values of \( \varepsilon > 0 \).

In particular those assumptions guarantee that (1.1) admits a unique weak solution, see e.g. [Men11]. The unique weak solution of (1.1) admits a density.
\( p(t, (x, y), (x', y')) \) for all \( t > 0 \) that satisfies the Aronson bounds (see \cite{DMI10} and \cite{Men11}).

Such kind of processes as \((1.1)\) appear in various applicative fields. For instance, in mathematical finance, when dealing with Asian options, \( X \) can be associated with the dynamics of the underlying asset and its integral \( Y \) is involved in the option Payoff. Typically, the price of such options writes \( \mathbb{E}_x[\psi(X_T, T^{-1}Y_T)] \), where for the put (resp. call) option the function \( \psi(x, y) = (x - y)^+ \) (resp. \( (y - x)^+ \)), see \cite{BPV01} and \cite{LPS98}. It is, thus, useful to specifically quantify how a perturbation of the coefficients impacts the option prices.

The cross dependence of the dynamics of \( X \) in \( Y \) is also important when handling kinematic models or Hamiltonian systems. For a given Hamilton function of the form \( H(x, y) = V(y) + \frac{|x|^2}{2} \), where \( V \) is a potential and \( \frac{|x|^2}{2} \) the kinetic energy of a particle with unit mass, the associated stochastic Hamiltonian system would correspond to \( b(X_s, Y_s) = - (\partial_y V(Y_s) + F(X_s, Y_s)X_s) \) in \((1.1)\), where \( F \) is a friction term. When \( F > 0 \) natural questions arise concerning the asymptotic behavior of \( (X_t, Y_t) \), for instance, the geometric convergence to equilibrium for the Langevin equation is discussed in Mattingly and Stuart \cite{MSH02}, numerical approximations of the invariant measures in Talay \cite{Tal02}, the case of high degree potential \( V \) is investigated in Hérau and Nier \cite{HN04}.

The goal of this work is to investigate how the closeness of \((b_\varepsilon, \sigma_\varepsilon)\) and \((b, \sigma)\) is reflected on the respective densities of the associated processes. In many applications (misspecified volatility models or calibration procedures) it can be useful to know how the controls on the differences \(|b - b_\varepsilon|, |\sigma - \sigma_\varepsilon|\) (for suitable norms) impact the difference \( p_\varepsilon - p \) of the densities corresponding respectively to the dynamics with the perturbed parameters and the one of the model.

1.1. Assumptions and Main Results. Let us introduce the following assumptions. Below, the parameter \( \varepsilon > 0 \) is fixed and the constants appearing in the assumptions do not depend on \( \varepsilon \).

(A1) (Boundedness of the coefficients). The components of the vector-valued functions \( b(x, y), b_\varepsilon(x, y) \) and the matrix-functions \( \sigma(x, y), \sigma_\varepsilon(x, y) \) are bounded measurable. Specifically, there exist constants \( K_1, K_2 > 0 \) s.t.

\[
\sup_{(x, y) \in \mathbb{R}^d} |b(x, y)| + \sup_{(x, y) \in \mathbb{R}^d} |b_\varepsilon(x, y)| \leq K_1,
\]

\[
\sup_{(x, y) \in \mathbb{R}^d} |\sigma(x, y)| + \sup_{(x, y) \in \mathbb{R}^d} |\sigma_\varepsilon(x, y)| \leq K_2.
\]

(A2) (Uniform Ellipticity). The matrices \( a := \sigma \sigma^*, a_\varepsilon := \sigma_\varepsilon \sigma_\varepsilon^* \) are uniformly elliptic, i.e. there exists \( \Lambda \geq 1, \forall (x, y, \xi) \in (\mathbb{R}^d)^3, \)

\[
\Lambda^{-1} |\xi|^2 \leq \langle a(x, y)\xi, \xi \rangle \leq \Lambda |\xi|^2, \Lambda^{-1} |\xi|^2 \leq \langle a_\varepsilon(x, y)\xi, \xi \rangle \leq \Lambda |\xi|^2.
\]

(A3) (Hölder continuity in space). For some \( \gamma \in (0, 1) \), \( \kappa < \infty \),

\[
|\sigma(x, y) - \sigma(x', y')| + |\sigma_\varepsilon(x, y) - \sigma_\varepsilon(x', y')| \leq \kappa (|x - x'| + |y - y'|)^\gamma.
\]

Observe that the last condition also readily gives, thanks to the boundedness of \( \sigma, \sigma_\varepsilon \) that \( a, a_\varepsilon \) are also uniformly \( \gamma \)-Hölder continuous.

For a given \( \varepsilon > 0 \), we say that assumption (A) holds when conditions (A1)-(A3) are in force. Let us now introduce, under (A), the quantities that will bound
the difference of the densities in our main results below. Set for \( \varepsilon > 0 \):

\[
\forall q \in (1, +\infty], \quad \Delta_{\varepsilon,b,q} := \sup_{t \in [0,T]} \|b(t, .) - b_{\varepsilon}(t, .)\|_{L^q(\mathbb{R}^d)}.
\]

Since \( \sigma, \sigma_{\varepsilon} \) are both \( \gamma \)-Hölder continuous, see (A3), we also define

\[
\Delta_{\varepsilon,\sigma,\gamma} := |\sigma(\cdot) - \sigma_{\varepsilon}(\cdot)|_\gamma,
\]

where for \( \gamma \in (0, 1] \), \( |.|_\gamma \) stands for the usual Hölder norm in space on \( C_0^\infty(\mathbb{R}^d, \mathbb{R}^d \otimes \mathbb{R}^d) \) (space of Hölder continuous bounded functions, see e.g. Krylov [Kry96]) i.e.:

\[
[f]_\gamma := \sup_{x \in \mathbb{R}^d} |f(x)| + [f]_\gamma, \quad [f]_\gamma := \sup_{x \neq y, (x,y) \in \mathbb{R}^{2d}} \frac{|f(x) - f(y)|}{|x - y|^\gamma}.
\]

The previous control in particular implies for all \( ((x, y), (x', y')) \in (\mathbb{R}^{2d})^2 \):

\[
|a(x, y) - a(x', y') - a_{\varepsilon}(x, y) + a_{\varepsilon}(x', y')| \leq 2(K_2 + \kappa)\Delta_{\varepsilon,\sigma,\gamma} (|x - x'| + |y - y'|)^\gamma.
\]

We eventually set \( \forall q \in (1, +\infty], \)

\[
\Delta_{\varepsilon,\gamma,q} := \Delta_{\varepsilon,\sigma,\gamma} + \Delta_{\varepsilon,b,q},
\]

which will be the key quantity governing the error in our results.

We will denote, from now on, by \( C \) a constant depending on the parameters appearing in (A) and \( T \). We reserve the notation \( c \) for constants that only depend on (A) but not on \( T \). The values of \( C, c \) may change from line to line and do not depend on the considered perturbation parameter \( \varepsilon \).

**Theorem 1 (Stability Control).** Fix \( T > 0 \). Under (A), for \( q \in (4d, +\infty] \), there exists \( C := C(q) \geq 1, c \in (0, 1] \) s.t. for all \( 0 < t \leq T \), \( ((x, y), (x', y')) \in (\mathbb{R}^{2d})^2 \):

\[
|(p - p_{\varepsilon})(t, (x, y), (x', y'))| \leq C\Delta_{\varepsilon,\gamma,q}\hat{p}_{c,K}(t, (x, y), (x', y'))
\]

where \( p(t, (x, y), (\cdot, \cdot)), p_{\varepsilon}(t, (x, y), (\cdot, \cdot)) \) respectively stand for the transition densities at time \( t \) of equations (1.1), (1.2) starting from \((x, y)\) at time 0. Also, we denote for a given \( c > 0 \) and for all \((x', y') \in \mathbb{R}^{2d}\),

\[
\hat{p}_{c,K}(t, (x, y), (x', y')) := \frac{e^{cd^2}t^d}{(2\pi t^2)^{d/2}} \exp \left( -c \left[ \frac{|x' - x|^2}{4t} + 3\frac{|y' - y - (x + x')t/2|^2}{t^3} \right] \right),
\]

(1.3)

which enjoys the semigroup property, i.e. \( \forall 0 \leq s < t \leq T \),

\[
\int_{\mathbb{R}^{2d}} \hat{p}_{c,K}(s, (x, y), (w, z))\hat{p}_{c,K}(t - s, (w, z), (x', y'))dwdz = \hat{p}_{c,K}(t, (x, y), (x', y')).
\]

The subscript \( K \) in the notation \( \hat{p}_{c,K} \) stands for **Kolmogorov like equations** and \( \hat{p}_{c,K}(t, (x, y), (\cdot, \cdot)) \) denotes the density of

\[
\left( X^{K_{c,1}}_t \right) := \left( X^{K_{c,1}}_t, X^{K_{c,2}}_t \right) = \left( x + \frac{\sqrt{2}W_t}{c^{1/2}}, y + \int_0^t X_s ds \right).
\]

We refer for details to the seminal paper [Kol34] and [KMM10], [DM10] for further extensions.
Remark 1.1. Observe carefully that the density in (1.3) exhibits a multiscal e behavior. The non degenerate component has the usual diffusive scale in \( t^{1/2} \) corresponding to the self-similarity index or typical scale of the Brownian motion at time \( t \), whereas the degenerate one has a faster typical behavior in \( t^{3/2} \) corresponding to the typical scale of the integral \( \int_0^t W_s ds \), associated with the parameters \( y, y' \).

Remark 1.2. Note that the same result could be achieved in the non-homogeneous case without additional assumptions (see [DM10] for details).

Remark 1.3. Observe as well that the control of Theorem 1 should as well hold for the Euler schemes for the degenerate Kolmogorov SDEs introduced in [LM10] associated with (1.1) and (1.2) respectively. See also [KKM15] for the sensitivity of perturbed Markov Chains in the non-degenerate case.

Remark 1.4. Let us mention that for applicative purposes, perturbations of the degenerate component could be very interesting as well. The first natural perturbation we have in mind would be to consider

\[
dX_{\varepsilon}^2 = \{X_{\varepsilon}^{\epsilon-1} + \varepsilon F(X_{\varepsilon}^\epsilon)\}
\]

in (1.2) for a smooth bounded function \( F \). However, this setting would require a more subtle handling of the proxy processes involved, in order to make the parametrix approach work. In particular, similar difficulties than those arising in [DM10] would occur leading to truncate the parametrix series (because of the non-linear dynamics) and to control the reminders with stochastic control arguments. The investigation of such perturbations will concern further research.

The paper is organized as follows. We recall in Section 2 some basic facts about parametrix expansions for the densities of degenerate diffusions. We then detail in Section 3 how to perform a stability analysis of the parametrix expansions in order to derive the result of Theorem 1.

2. Parametrix Representation of the Density

From [Men11] it follows that (1.1) has under (A) a unique weak solution. We aim at proving that the solution has for each \( t \in (0, T] \) a density which can be represented as the sum of a parametrix series.

If the coefficients in (1.1) are not smooth, but satisfy (A), it is then possible to use a mollification procedure, taking \( b_\eta(x, y) := b \ast \rho_\eta(x, y) \), \( \sigma_\eta(x, y) := \sigma \ast \rho_\eta(x, y) \), \( x, y \in \mathbb{R}^d \) where \( \rho_\eta \) is a smooth mollifying kernel and \( \ast \) stands for a standard convolution operation and \( \eta \in [0, 1] \), the case \( \eta = 0 \) corresponding to the initial process in (1.1).

For mollified coefficients, the existence and smoothness of the density \( p_\eta \) for the associated process \((X_\eta^s, Y_\eta^s)\) follows from the Hörmander theorem (see e.g. [Hör67] or [Nor86]). Thus, we can apply the parametrix technique directly for \( p_\eta \).

Roughly speaking, the parametrix approach consists in approximating the process by a proxy which has a known density, here a Gaussian one, and then in investigating the difference through the Kolmogorov equations. Various approaches to the parametrix expansion exist, see e.g. Il’in et al. [IKO62], Friedman [Fri64] and McKean and Singer [MS67]. The latter approach will be the one used in this work since it can be directly extended to the discrete case for Markov chain approximations of equations (1.1) and (1.2). Let us mention in this setting the works of Konakov and Mammen, see [KM00], [KM02].
For the parametrix development we need to introduce a "frozen" diffusion process \((X_s, Y_s)_{s \in [0,t]}\). Namely for fixed \((x', y') \in \mathbb{R}^{2d}, t \in (0, T]\) define for all \(s \in [0, t]\):

\[
\begin{align*}
    d\tilde{X}_s^{t,x',y',n} &= \sigma(x', y' - x'(t-s))dW_s, \\
    d\tilde{Y}_s^{t,x',y',n} &= \bar{X}_0^{t,x',y',n}ds,
\end{align*}
\]

Observe that for \(\eta \in [0,1]\) the above SDE integrates as

\[
(\tilde{X}_s^{t,x',y',n}, \tilde{Y}_s^{t,x',y',n}) = R_s \left( \begin{array}{c} x \\ y \end{array} \right) + \int_0^s R_{s-u}B\sigma_\eta(x', y' - x'(t-u))dW_u,
\]

where \(R_s = \begin{pmatrix} 1/s & 0 \\ 0 & 1/s \end{pmatrix}\), \(B = \begin{pmatrix} I_{d \times d} & 0_{d \times d} \\ 0_{d \times d} & I_{d \times d} \end{pmatrix}\), which implies that \((\tilde{X}_s^{t,x',y',n}, \tilde{Y}_s^{t,x',y',n})\) is a Gaussian process. In particular, its density at time \(t\) writes:

\[
\tilde{p}_\eta^{t,x',y'}(t, (x, y), (x', y')) = \frac{1}{(2\pi)^{d\det(C_t)^{1/2}}} \exp \left( -\frac{1}{2} \left( C_t^{-1}(R_t \left( \begin{array}{c} x \\ y \end{array} \right) - \left( \begin{array}{c} x' \\ y' \end{array} \right)) , R_t \left( \begin{array}{c} x \\ y \end{array} \right) - \left( \begin{array}{c} x' \\ y' \end{array} \right) \right) \right),
\]

where \(C_t = \int_0^t R_{t-u}B\sigma_\eta\sigma_\eta^*(x', y' - x'(t-u))B^*R_{t-u}du\). From this explicit expression, standard Gaussian like computations (see e.g. [KMM10]) imply that there exist \(C \geq 1, c \in (0, 1]\) s.t.

\[
C^{-1} \tilde{p}_{c,K}(t, (x, y), (x', y')) \leq \tilde{p}_\eta^{t,x',y'}(t, (x, y), (x', y')) \leq C\tilde{p}_{c,K}(t, (x, y), (x', y')).
\]

Also, we have the following controls of the derivatives

\[
|\frac{\partial^{\alpha}}{\partial \alpha} \tilde{p}_\eta^{t,x',y'}(t, (x, y), (x', y'))| \leq \frac{C}{(2\pi)^{d/2}} \tilde{p}_{c,K}(t, (x, y), (x', y')) \quad \forall |\alpha| \leq 4.
\]

Observe that these controls also reflect the multi scale behavior already mentioned in Remark [1.1] They are also uniform w.r.t. \(\eta \in [0, 1]\).

Remark 2.1. The arguments in the second variable of the diffusion coefficient can seem awkward at first sight, they actually correspond to the transport of the frozen final point \((x', y')\) by the backward differential system:

\[
\dot{x}_s = 0, \quad \dot{y}_s = x', \quad x_s = x', \quad y_s = y'.
\]

This choice is performed to have a "compatibility" condition in the difference of generators in the parametrix expansion. See the controls on \(H^\eta\) established in [2.8] below.

The processes \((X_s, Y_s)\) and \((\tilde{X}_s^{t,x',y',n}, \tilde{Y}_s^{t,x',y',n})\), \(s \in [0, t]\), have the following generators: \(\forall (x, y) \in \mathbb{R}^{2d}, \psi \in C^2(\mathbb{R}^{2d})\),

\[
\begin{align*}
    L^\eta \psi(x, y) &= \left( \frac{1}{2} \text{Tr} \left( a_\eta(x, y)D_x^2 \psi \right) + b_\eta(x, y) \cdot \nabla_x \psi \right)(x, y), \\
    \tilde{L}^{t,x',y',n}_s \psi(x, y) &= \left( \frac{1}{2} \text{Tr} \left( a_\eta(x', y' - x'(t-s))D_x^2 \psi \right) + (x, \nabla_x \psi) \right)(x, y).
\end{align*}
\]

(2.3)

Let us define for notational convinience \(\tilde{p}_\eta(t, (x, y), (x', y')) := \tilde{p}_\eta^{t,x',y'}(t, (x, y), (x', y'))\), that is in \(\tilde{p}_\eta(t, (x, y), (x', y'))\) we consider the density of the frozen process at the final point and observe it at that specific point.
The density \( \tilde{p}_\eta \) readily satisfies the Kolmogorov Backward equation:
\[
\begin{cases}
\partial_u \tilde{p}_\eta(t - u, (x, y), (x', y')) + \tilde{L}^{t,x'}_{x,y} \eta \tilde{p}_\eta(t - u, (x, y), (x', y')) = 0, \\
0 < u < t, (x, y), (x', y') \in \mathbb{R}^{2d}, \\
\tilde{p}_\eta(t - u, (\cdot, \cdot), (x', y')) \xrightarrow{t \rightarrow u} \delta_{(x', y')}(.).
\end{cases}
\tag{2.4}
\]

On the other hand, since the density of \((X^n_t, Y^n_s)\) is smooth, it must satisfy the Kolmogorov forward equation (see e.g. Dynkin [Dyn65]). For a given starting point \((x, y) \in \mathbb{R}^{2d}\) at time 0,
\[
\begin{cases}
\partial_u p_\eta(u, (x, y), (x', y')) - L^*_\eta p_\eta(u, (x, y), (x', y')) = 0, \\
p_\eta(u, (x, y), .) \xrightarrow{u \rightarrow 0} \delta_{(x, y)}(.),
\end{cases}
\tag{2.5}
\]

where \(L^*_\eta\) stands for the adjoint (which is well defined since the coefficients are smooth) of the generator \(L^n\) in (2.3).

Equations (2.4) and (2.5) yield the formal expansion below:
\[
(p_\eta - \tilde{p}_\eta)(t, (x, y), (x', y'))
\tag{2.6}
\]

using the Dirac convergence for the first equality, equations (2.4) and (2.5) for the third one. We eventually take the adjoint for the last equality. Note carefully that the differentiation under the integral is also here formal since we would need to justify that it can actually be performed using some growth properties of the density and its derivatives which we a priori do not know.

Let us now introduce the notation
\[
f \otimes g(t, (x, y), (x', y')) = \int_0^t du \int_{\mathbb{R}^{2d}} dz dw f(u, (x, y), (w, z)) g(t - u, (w, z), (x', y'))
\]
for the time-space convolution. We now introduce the parametrix kernel:
\[
H^n(t, (x, y), (x', y')) := (L^n - \tilde{L}^n) \eta(t, (x, y), (x', y')).
\]

Remark 2.2. Note carefully that in the above kernel \(H^n\), because of the linear structure of the degenerate component in the model, the most singular terms, i.e. those involving derivatives w.r.t. \(y\), i.e. the fast variable, vanish (see Remark 1.1 and (2.2)).
With those notations equation (2.6) rewrites:

\[
(p_{\eta} - \tilde{p}_{\eta})(t, (x, y), (x', y')) = p_{\eta} \otimes H^n(t, (x, y), (x', y'))
\]

\[
= \int_0^t du \int_{\mathbb{R}^{2d}} p_{\eta}(u, (x, y), (w, z)) H^n(t - u, (w, z), (x', y')) dw dz.
\]

From this expression, the idea then consists in iterating this procedure for \(p_{\eta}(u, (x, y), (w, z))\) in (2.6) introducing the density of a process with frozen characteristics in \((w, z)\) which is here the integration variable. This yields to iterated convolutions of the kernel and leads to the formal expansion:

\[
p_{\eta}(t, (x, y), (x', y')) = \sum_{r=0}^{\infty} \tilde{p}_{\eta} \otimes H^{n,(r)}(t, (x, y), (x', y')),
\]

where \(\tilde{p}_{\eta} \otimes H^{n,(0)} = \tilde{p}_{\eta}, H^{n,(r)} = H^n \otimes H^{n,(r-1)}, r \geq 1\).

Obtaining estimates on \(p_{\eta}\) from the formal expression (2.7) requires to have good controls on the right-hand side.

Precisely thanks to (2.2), we first get that uniformly in \(\eta \in [0,1]\) (thanks to (A) and the specific choice of the freezing parameters in the proxy), there exist \(c_1 > 1, c \in (0,1)\) s.t. for all \(u \in [0,t]\),

\[
|H^n(t - u, (w, z), (x', y'))| \leq \frac{1}{2} Tr \{a_{\eta}(w, z) - a_{\eta}(x', y' - x'(t - u)) \} D^2_{w,z}\tilde{p}_{\eta}(t - u, (w, z), (x', y'))
\]

\[
+ (b_{\eta}(w, z), D_w\tilde{p}_{\eta}(t - u, (w, z), (x', y'))
\]

\[
\leq \left[ C |w - x'|^{\gamma} + |z - y' - x'(t - u)|^{\gamma} \right] \left( \frac{\gamma}{2} \right) + \frac{C K_1}{(t - u)^{1/2}} \hat{p}_{c,K}(t - u, (w, z), (x', y'))
\]

\[
\leq c_1 \left( 1 \vee T^{(1-\gamma)/2} \right) \hat{p}_{c,K}(t - u, (w, z), (x', y'))
\]

We can establish by induction the following key result.

**Lemma 1.** There exist constants \(C \geq 1, c \in (0,1)\) s.t. for all \(\eta \in [0,1]\) one has for all \((t, (x, y), (x', y')) \in (0, T) \times (\mathbb{R}^{2d})^2\):

\[
|\tilde{p}_{\eta} \otimes H^{n,(r)}(t, (x, y), (x', y'))| \leq C^{r+1} t^{r\gamma/2} B \left( \frac{\gamma}{2}, \frac{\gamma}{2} \right) \times B \left( 1 + \frac{r\gamma}{2} \right) \times \cdots \times B \left( 1 + \frac{(r-1)\gamma}{2} \right)
\]

\[
\times \hat{p}_{c,K}(t, (x, y), (x', y')), \quad r \in \mathbb{N}^*.
\]

**Proof.** The result (2.2) in particular yields that \(\exists C_2 > 0, \forall u \in (0, t), \tilde{p}_{\eta}(t - u, (x, y), (w, z)) \leq C_2 \hat{p}_{c,K}(t - u, (x, y), (w, z))\) uniformly w.r.t. \(\eta \in [0,1]\).

Setting \(C := c_1 \left( 1 \vee T^{(1-\gamma)/2} \right) \vee C_2\), we finally obtain also uniformly in \(\eta\)

\[
|\tilde{p}_{\eta} \otimes H^n(t, (x, y), (x', y'))| \leq \int_0^t du \int_{\mathbb{R}^{2d}} \tilde{p}_{\eta}(u, (x, y), (w, z)) H^n(t - u, (w, z), (x', y')) dw dz,
\]

\[
\leq \int_0^t du \int_{\mathbb{R}^{2d}} C^2 \tilde{p}_{c,K}(u, (x, y), (w, z)) \frac{1}{(t - u)^{1-\gamma/2}} \hat{p}_{c,K}(t - u, (w, z), (x', y')) dw dz
\]

\[
\leq C^2 t^{\gamma/2} B \left( \frac{\gamma}{2} \right) \hat{p}_{c,K}(t, (x, y), (x', y')).
\]
using the semigroup property of $\hat{\rho}_{c,K}$ in the last inequality and where $B(p, q) = \int_0^1 u^{p-1} (1 - u)^{q-1} du$ denotes the $\beta-$function. By induction in $r$:

$$\left| \hat{\rho}_\eta \otimes H^{\eta(r)}(t, (x, y), (x', y')) \right| \leq C^{r+1} t^{\gamma \gamma/2} B \left(1, \frac{\gamma}{2} \right) \times B \left(1 + \frac{\gamma}{2}, \frac{\gamma}{2} \right) \times \cdots \times B \left(1 + \frac{(r-1)\gamma}{2}, \frac{\gamma}{2} \right) \times \hat{\rho}_{c,K}(t - s, (x, y), (x', y')), \quad r \in \mathbb{N}^*,$$

which means that the sum of the series (2.7) is uniformly controlled w.r.t. $\eta \in [0, 1]$.

These bounds imply that the series representing the density of the initial process $p_\eta(t, (x, y), (x', y'))$ could be expressed as:

$$p_\eta(t, (x, y), (x', y')) = \sum_{r=0}^{\infty} \hat{\rho}_\eta \otimes H^{\eta(r)}(t, (x, y), (x', y')). \quad (2.9)$$

Lemma 1 readily yields the convergence of the series (2.9) and the following bound uniformly in $\eta \in [0, 1]: p_\eta(t, (x, y), (x', y')) \leq c_1 \hat{\rho}_{c,K}(t, (x, y), (x', y'))$.

From the bounded convergence theorem one can derive that

$$p_\eta(t, (x, y), (x', y')) \xrightarrow{\eta \to 0} \sum_{r=0}^{\infty} \hat{\rho} \otimes H^{\eta(r)}(t, (x, y), (x', y')) := p(t, (x, y), (x', y')),$$

(2.10)

uniformly in $t, (x, y), (x', y'))$, where $\hat{\rho}(u, (x, y), (w, z)) := \hat{\rho}_0(u, (x, y), (w, z))$ and $H^{\eta(r)}(t - u, (w, z), (x', y')) := H^{0,\eta(r)}(t - u, (w, z), (x', y'))$.

Due to the uniform convergence in $\eta$ (which implies the uniqueness in law):

$$\int_{\mathbb{R}^{2d}} f(z, w)p_\eta(t, (x, y), (w, z))dwdz \xrightarrow{\eta \to 0} \int_{\mathbb{R}^{2d}} f(z, w)p(t, (x, y), (w, z))dwdz,$$

for all continuous and bounded $f$. The well-posedness of the martingale problem and Theorem 11.1.4 from [SV79] then give that the process $(X_t, Y_t)$ has the transition density which is exactly the sum of the parametrix series $p(t, (x, y), (x', y'))$.

Thus, we have proved the below proposition.

**Proposition 1.** Under the sole assumption (A), for $t > 0$, the transition density of the process $(X_t, Y_t)$ solving (1.1) exists and can be written as the series in (2.10).

### 3. Stability

We will now investigate more specifically the sensitivity of the density w.r.t. the coefficients perturbation through the difference of the series. From Proposition 1, for a given fixed parameter $\varepsilon$, under (A) the densities $p(t, (x, y), (\cdot, \cdot)), p_\varepsilon(t, (x, y), (\cdot, \cdot))$ at time $t$ of the processes in (1.1), (1.2) starting from $(x, y)$ at time 0 both admit a parametrix expansion of the previous type.
Let us consider the difference between the two parametrix expansions for \(1.1\) and \(1.2\) in the form \(2.7\):

\[
|p(t, (x, y), (x', y')) - p_c(t, (x, y), (x', y'))| 
\leq \sum_{r=0}^{\infty} |\hat{p} \otimes H^{(r)}(t, (x, y), (x', y')) - \hat{p}_c \otimes H^{(r)}_c(t, (x, y), (x', y'))|.
\]

Since we consider perturbations of the densities with respect to the non-degenerate component, following the same steps as in [KKM15] one can show that the Lemma below holds:

**Lemma 2** (Difference of the first terms and their derivatives). There exist \(c_1 \geq 1\), \(c \in (0, 1]\) s.t. for all \(0 < t\), \((x, y), (x', y') \in \mathbb{R}^{m}\) and all multi-index \(\alpha\), \(|\alpha| \leq 4\),

\[
|D^\alpha_c \hat{p}(t, (x, y), (x', y')) - D^\alpha_c \hat{p}_c(t, (x, y), (x', y'))| \leq \frac{c_1 \Delta_{\varepsilon, \sigma, \gamma} \hat{p}_{c, K}(t, (x, y), (x', y'))}{t^{\alpha/2}}.
\]

**Lemma 3** (Control of the one-step convolution). For all \(0 < t\), \((x, y), (x', y') \in \mathbb{R}^{m}\):

\[
|\hat{p} \otimes H^{(1)}(t, (x, y), (x', y')) - \hat{p}_c \otimes H^{(1)}_c(t, (x, y), (x', y'))| 
\leq c_2^2 \left(1 + T^{(1-\gamma)/2} [\Delta_{\varepsilon, \sigma, \gamma} + \sum_{q=0}^{+\infty} \Delta_{\varepsilon, b, q}]\right) B(1, \frac{\gamma}{2}) t^2 \quad (3.1)
\]

\[ + \mathbb{1}_{q \in (4d, +\infty)} \Delta_{\varepsilon, b, q} B(1, \frac{\gamma}{2} + \alpha(q), \alpha(q)) \left(\hat{p}_{c, K}(t, (x, y), (x', y'))\right),\]

where \(c_1, c\) are as in Lemma 2 and for \(q \in (4d, +\infty)\) we set \(\alpha(q) = \frac{1}{2} - \frac{2d}{q}\).

**Proof.** Let us write:

\[
|\hat{p} \otimes H^{(1)}(t, (x, y), (x', y')) - \hat{p}_c \otimes H^{(1)}_c(t, (x, y), (x', y'))| 
\leq \frac{1}{2} \left| I + II \right|.
\]

From Lemma 2 and \(2.8\) we readily get for all \(q \in (4d, +\infty)\):

\[
I \leq \left(1 + T^{(1-\gamma)/2}\right) c_1^2 \Delta_{\varepsilon, \gamma, q} \hat{p}_{c, K}(t, (x, y), (x', y')) B(1, \frac{\gamma}{2}) t^2. \quad (3.3)
\]

To estimate \(II\) let us first consider \(H - H_c\) more precisely:

\[
(H - H_c)(t - u, (w, z), (x', y')) \quad (3.4)
\]

\[
= \frac{1}{2} \left| a(w, z) - a(x', y' - x' (t - u)) - a_c(w, z) + a_c(x', y' - x' (t - u)) \right| 
\times D^2_w \hat{p}(t - u, (w, z), (x', y'))
\]

\[
+ \frac{1}{2} \left| a_c(w, z) - a_c(x', y' - x' (t - u)) \right| \left[ D^2_w \hat{p}_c(t - u, (w, z), (x', y')) + (b(w, z) - b_c(w, z), D_w \hat{p}(t - u, (w, z), (x', y'))) 
\right.
\]

\[+ (b_c(w, z), D_w \hat{p}_c(t - u, (w, z), (x', y'))) \equiv \left( \Delta^1 H + \Delta^2 H \right)(t - u, (w, z), (x', y'))
\]

\[+ \left( b(w, z) - b_c(w, z), D_w \hat{p} - D_w \hat{p}_c(t - u, (w, z), (x', y'))) \right.\]

\[+ \left( b_c(w, z), D_w \hat{p}_c(t - u, (w, z), (x', y'))) \right).\]
Since functions $a(w, z), a_z(w, z)$ are Hölder uniformly continuous and (2.2) holds then:

\[
\frac{|\Delta_1^2 H|(t-u, (w, z), (x', y'))}{(t-u)} \leq c\Delta_{e,\gamma,\infty} (|w-x'|^\gamma + |z-y'| + x'(t-u)^\gamma) \hat{p}_{e,K}(t-u, (w, z), (x', y'))
\]

From Lemma 2 and Hölder uniform continuity of the function $a_z(x, y)$ it follows:

\[
\frac{|\Delta_1^2 H|(t-u, (w, z), (x', y'))}{(t-u)} \leq c\Delta_{e,\gamma,\infty} \hat{p}_{e,K}(t-u, (w, z), (x', y'))
\]

Thus, the fact that $|b(w, z) - b_z(w, z)| \leq c\Delta_{e,b,\gamma}$ and (2.2) give the control for $q = +\infty$. Namely,

\[
|(H-H_z)(t-u, (w, z), (x', y'))| \leq \left(1 + T^{(1-\gamma)/2}\right)c_1 \Delta_{e,\gamma,\infty} \frac{\hat{p}_{e,K}(t-u, (w, z), (x', y'))}{(t-u)^{1-\gamma/2}}
\]

For $q \in (4d, +\infty)$ we use Hölder inequality in the time-space convolution involving the difference of the drifts (last term in (3.1)). Set

\[
D(t, (x, y), (x', y')) := \int_0^t du \int_{\mathbb{R}^{2d}} \hat{p}_z(u, (x, y), (w, z))(b_z(w, z) - b(w, z)), D_{w}\hat{p}(t-u, (w, z)(x', y')))dwdz.
\]

Denoting by $\tilde{q}$ the conjugate of $q$, i.e. $q, \tilde{q} > 1, q^{-1} + \tilde{q}^{-1} = 1$, we get from (2.2) and for $q > d$ that:

\[
|D(t, (x, y), (x', y'))| \leq c_1^2 \int_0^t \frac{du}{(t-u)^{1/2}} \|b(\ldots) - b_z(\ldots)\|_{L^q(\mathbb{R}^d)}
\]

\[
\times \left\{ \int_{\mathbb{R}^{2d}} \hat{p}_{e,K}(u, (x, y), (w, z)) \hat{p}_{e,K}(t-u, (w, z), (x', y'))dwdz \right\}^{1/\tilde{q}}
\]

\[
\leq c_1^2 \Delta_{e,b,\tilde{q}} \int_0^t \frac{du}{(2\pi)^{d/4} (\tilde{q}^{d/2})^{2d/\tilde{q}}}
\]

\[
\times \left\{ \int_{\mathbb{R}^{2d}} \hat{p}_{e,K}(u, (x, y), (w, z)) \hat{p}_{e,K}(t-u, (w, z), (x', y'))dwdz \right\}^{1/\tilde{q}}
\]

\[
\leq c_1^2 \left( \frac{\sqrt{3d}^d}{2\pi} \right)^{d/\tilde{q}} \Delta_{e,b,\tilde{q}} \hat{p}_{e,K}(t, (x, y), (x', y')) \int_0^t \frac{du}{u^{2d/\tilde{q}}(t-u)^{1+2d/\tilde{q}}}.
\]

Now, the constraint $4d < q < +\infty$ precisely gives that $\frac{1}{2} + 2d(1 - \frac{1}{2}) < 1$ so that the last integral is well defined. We therefore derive:

\[
|D(t, (x, y), (x', y'))| \leq c_1^2 t^{-2d/q} \Delta_{e,b,\tilde{q}} \hat{p}_{e,K}(t, (x, y), (x', y')) B(1 - 2d/q, \frac{1}{2} - 2d/q).
\]
In the case $4d < q < +\infty$, recalling that $\alpha(q) = \frac{1}{2} - \frac{2d}{q}$, we eventually get:
\[
|\tilde{p}_c(s, (x, y), (w, z)) \otimes (H - H_c)(t - u, (w, z), (x', y'))| 
\leq c_d^2 \hat{p}_c, K(t, (x, y), (x', y'))(\Delta_{c, h, q} t^{\alpha(q)} B\left(\frac{1}{2} + \alpha(q), \alpha(q)\right) + 2\Delta_{c, r, \eta}(1 \lor T^{(1-\gamma)/2}) t^{\gamma/2} B(1, \gamma/2)). 
\]
(3.5)

The statement now follows in whole generality from (3.2), (3.3), (2.2) for $d < q < \eta$.

Thus, the induction hypothesis we get the result. \hfill \Box

The following Lemma associated with Lemmas \[2\] and \[3\] allows to complete the proof of Theorem \[4\].

Lemma 4 (Difference of the iterated kernels). For all $0 < t \leq T$, $(x, y), (x', y') \in (\mathbb{R}^{2d})^2$ and for all $r \in \mathbb{N}$:
\[
|\tilde{p} \otimes H^{(r)} - \tilde{p}_c \otimes H_c^{(r)}(t, (x, y), (x', y'))| 
\leq C^r r \Delta_{c, \eta, q} \left\{ \frac{t^{r/2}}{\Gamma\left(1 + \frac{r}{2}\right)} + \frac{t^{(r+2)/2}}{\Gamma\left(1 + \frac{(r+2)\eta}{2}\right)} \right\} \hat{p}_c, K(t, (x, y), (x', y')). 
\]
(3.6)

Proof. Observe that Lemmas \[2\] and \[3\] respectively give (3.3) for $r = 0$ and $r = 1$. Let us assume that it holds for a given $r \in \mathbb{N}^+$ and let us prove it for $r + 1$.

Let us denote for all $r \geq 1$, \[\eta_r(t, (x, y), (x', y')) := |(\tilde{p} \otimes H^{(r)} - \tilde{p}_c \otimes H_c^{(r)})(t, (x, y), (x', y'))|\]. Write
\[
\eta_{r+1}(t, (x, y), (x', y')) = |(\tilde{p} \otimes H^{(r)} - \tilde{p}_c \otimes H_c^{(r)})(t, (x, y), (x', y'))| 
+ |\tilde{p}_c \otimes H_c^{(r)}(t, (x, y), (x', y'))| 
\leq \eta_r \otimes |H|(t, (x, y), (x', y')) + |\tilde{p}_c \otimes H_c^{(r)}| \otimes |(H - H_c)(t, (x, y), (x', y'))|.
\]

Thus, the induction hypothesis we get the result. \hfill \Box

Theorem 1 now simply follows from the controls of Lemma \[3\] the parametrix expansions (1.11) and (1.2) of the densities $p, p_c$ and the asymptotics of the gamma function.

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