Combinatorics of multisecant Fay identities.

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Abstract. We derive a set of identities for the theta functions on compact Riemann surfaces which generalize the famous trisecant Fay identity. Using these identities we obtain quasiperiodic solutions for a multidimensional generalization of the Hirota bilinear difference equation and for a multidimensional Toda-type system.

1. Introduction.

In the present work we derive some identities for the theta functions defined on the compact Riemann surfaces which generalize the famous Fay identity [1]. The classical trisecant Fay identity (TFI), see equation (45) from [1], has been discovered as a result of the studies of the properties of the theta functions on abelian varieties and its proof is based on a rather complicated machinery of the algebraic geometry [1, 2, 3, 4, 5]. The wide interest to the TFI stems, to a large extent, from the fact that it can be used to derive the quasiperiodic solutions for many integrable equations such as, for example, KdV, KP, sine-Gordon equations, Toda model etc. It has been shown that such solutions, previously obtained by the algebro-geometric approach (see, e.g., [6] [7] [8] [9]), naturally arise from this rather simple identity (see chapter IIIb of [5]).

The aim of this work is to generalize the TFI bearing in mind its possible applications. We derive a set of identities that, as we hope, can be useful for obtaining solutions not only for the ‘classical’ (1+1)-dimensional (like, for example the KdV, sine-Gordon or nonlinear Schrödinger equations) or (1+2)-dimensional models (like, for example, the KP, 2D Toda or Davey-Stewartson equations) but also for models in higher dimensions.

In so doing, we do not rely on the algebraic geometry. After having formulated the TFI in section 2 we do not use any more properties of the Riemann surfaces, Abel
Combinatorics of multisecant Fay identities.

As it turns out, even the explicit form of the coefficients that appear in the TFI is not important. What we need from the algebraic geometry are 1) existence of this identity and 2) its structure (see below). All calculations presented here are ‘elementary’: we just ‘iterate’ the TFI to obtain various consequences of this identity.

In section 3 we introduce an auxiliary function, that takes into account the bilinearity of the TFI and ‘absorbs’ most of the constants, and derive the main result of this paper (propositions 3.3 and 3.4) as well as a few more general bilinear identities. In section 4 we go beyond the bilinear framework and obtain a set of multilinear identities. Since a vast area of practical use of the Fay-like identities is partial differential equations, we present in section 5 some differential variants of the general identities. Finally, we give two examples of applications of the obtained results. In section 6 we derive quasiperiodic solutions for a multidimensional generalization of the famous Hirota bilinear difference equation and for a multidimensional Toda-type system.

2. Trisecant Fay identity.

For a compact Riemann surface $X$ of the genus $g$ one can introduce in a standard way a system of cuts $A_i, B_i$ ($i = 1, ..., g$), a vector space of holomorphic 1-forms, its basis $\omega_i$, normalized by $\int_{A_i} \omega_j = \delta_{ij}$, the $g \times g$ complex matrix $\Omega$ with the elements $\int_{B_i} \omega_j$, whose imaginary part is positive definite, the lattice $L_\Omega = \mathbb{Z}^g + \Omega \mathbb{Z}^g$ and the complex torus $\text{Jac}(X) = \mathbb{C}^g/L_\Omega$ (see, e.g., [5]). In this work, we intensively use the following fundamental constructions of the classical theory of compact Riemann surfaces: the theta function $\theta(z) = \theta(z, \Omega)$,

$$\theta(z) = \sum_{n \in \mathbb{Z}^g} \exp (\pi i n^t \Omega n + 2\pi i n^t z),$$  \hfill (2.1)

where $n$ and $z$ are $g$-column vectors, $n^t$ is a $g$-row vector (throughout this paper the symbol $^t$ stands for the transposition) and the Abel map $X \rightarrow \text{Jac}(X)$ is defined by

$$a(x) = \int_{x_0}^{x} \omega$$  \hfill (2.2)

where $\omega$ is a column vector of the basis forms $\omega = (\omega_1, ..., \omega_g)^t$ and $x_0$ is some fixed point of $X$. The last definition can be extended to the definition of the Abel map from the space of divisors $\sum_k n_k x_k$ to $\text{Jac}(X)$,

$$a\left(\sum_k n_k x_k\right) = \sum_k n_k a(x_k), \quad n_k \in \mathbb{Z}, x_k \in X.$$  \hfill (2.3)

The aim of this work is to find generalizations of the TFI which we write as

$$E(a, b)E(c, d) \theta(z) \theta(z + a(a + b - c - d))$$

$$= E(a, c)E(b, d) \theta(z + a(a - d)) \theta(z + a(b - c)) - E(a, d)E(b, c) \theta(z + a(a - c)) \theta(z + a(b - d)).$$  \hfill (2.4)
where \( a, b, c \) and \( d \) are points of \( \mathcal{X} \). The function \( E(x, y) \) is a ‘scalar’ version of the prime-form,

\[
E(x, y) = \theta \left( \frac{\mathbf{m}}{\mathbf{n}} \right) (a(x - y)) \quad (x, y \in \mathcal{X})
\]

(2.5)

where the theta function with characteristics is given by

\[
\theta \left( \begin{array}{c} \mathbf{a} \\ \mathbf{b} \end{array} \right) (z) = \exp \left\{ \pi i \mathbf{a} \Omega \mathbf{a} + 2\pi i \mathbf{a} \mathbf{b} + 2\pi i \mathbf{a} \mathbf{z} \right\} \theta (z + \Omega \mathbf{a} + \mathbf{b})
\]

(2.6)

and \( \mathbf{m}, \mathbf{n} \) are integer vectors, \( \mathbf{m}, \mathbf{n} \in \mathbb{Z}^g \), related by

\[
\mathbf{m} \cdot \mathbf{n} = \text{odd number}.
\]

(2.7)

Such choice of \( \mathbf{m} \) and \( \mathbf{n} \) ensures the following properties of the function \( E(x, y) \):

\[
E(x, x) = 0, \quad E(x, y) = -E(y, x).
\]

(2.8)

As has been mentioned in the Introduction, this paper is devoted to the ‘elementary’ consequences of the Fay identity (2.4), which means that starting from (2.4) we do not use the properties of the Riemann surfaces, Abel maps or other machinery of the algebraic geometry. For our purposes, even the form of \( E(x, y) \) is not important (we present it just for the sake on completeness). What is important and what is repeatedly used in this work is that all coefficients in (2.4) are products of pairwise factors.

### 3. \( \Phi \)-function and main identities.

Some part of the notation used in this paper deviates from the traditional algebro-geometrical one. So, for example, we almost do not use the notion of divisors (only as arguments of the Abel map \( \mathbf{a}(...) \)). Instead, we prefer to formulate all results in terms of sets of the points of a Riemann surface (we often omit the words ‘of a Riemann surface’). Thus, instead of adding or subtracting divisors, we use the set operations with ‘+’ and ‘\( \setminus \)’ standing for the union and the difference of sets and \( [...] \) for the number of elements of a set.

It should be noted that throughout this paper we consider the ‘general position’ case: there is no coinciding points in a set or, in other words, each point of a set appears there only once.

It turns out that calculations of this work turn out to be much more easy if performed not in terms of the theta functions, but in terms of the function \( \Phi \) defined by

\[
\Phi_{X,Y}(z, A, B) = \varphi_{X,Y}(A, B) \frac{\theta (z + a(X \setminus A + B)) \theta (z + a(A + Y \setminus B))}{\theta (z + a(X)) \theta (z + a(Y))}
\]

(3.1)

with the constants (in the sense that they do not depend on \( z \))

\[
\varphi_{X,Y}(A, B) = \frac{E(B, X \setminus A) E(A, Y \setminus B)}{E(A, X \setminus A) E(B, Y \setminus B)}
\]

(3.2)
where
\[ E(X,Y) = \prod_{x \in X} \prod_{y \in Y} E(x,y). \quad (3.3) \]

In the following formulae we usually do not indicate the dependence on \( z \) explicitly: we consider \( z \) being fixed and write \( \Phi_{X,Y}(A,B) \) instead of \( \Phi_{X,Y}(z,A,B) \).

In terms of \( \Phi \), one can rewrite the Fay identity (2.4) in different ways:
\[ \Phi_{X,Y}(a,\emptyset) + \Phi_{X,Y}(b,\emptyset) + \Phi_{X,Y}(c,\emptyset) = 0, \quad X = \{a,b,c\}, \ Y = \{d\}, \quad (3.4) \]
or
\[ \Phi_{X,Y}(a,c) + \Phi_{X,Y}(b,c) = 1, \quad X = \{a,b\}, \ Y = \{c,d\}. \quad (3.5) \]
Hereafter we do distinguish between 1-point sets and points of \( \mathcal{X} \) and write \( a \) instead of \( \{a\} \) etc. These formulae not only look more simple than the original one, but also reveal some inner structures behind the TFI. And indeed, identities (3.4) and (3.5) can be generalized to the case of arbitrary sets \( X \) and \( Y \) to become the multisecant Fay identities.

**Proposition 3.1** For arbitrary sets \( X \) and \( Y \) related by \( |X| = |Y| + 2 \) the function \( \Phi \) satisfies
\[ \sum_{x \in X} \Phi_{X,Y}(x,\emptyset) = 0. \quad (3.6) \]

**Proposition 3.2** For arbitrary sets \( X \) and \( Y \) related by \( |X| = |Y| \) the function \( \Phi \) satisfies
\[ \sum_{x \in X} \Phi_{X,Y}(x,y) = 1, \quad \forall y \in Y, \quad (3.7) \]
\[ \sum_{y \in Y} \Phi_{X,Y}(x,y) = 1, \quad \forall x \in X. \quad (3.8) \]

We present proofs of these results in [Appendix A](#) and [Appendix B](#).

A simple consequence of, for example, (3.8) can be obtained by noting that it holds for any choice of \( x \) among the points of the set \( X \). Thus, multiplying (3.8) by arbitrary constant \( \Gamma_x \) and summarizing over \( X \) leads to
\[ \sum_{x \in X} \sum_{y \in Y} \Gamma_x \Phi_{X,Y}(x,y) = \sum_{x \in X} \Gamma_x, \quad (|X| = |Y|). \quad (3.9) \]
Thus one can convert the inhomogeneous identities (3.7) and (3.8) into homogeneous ones by imposing the restriction \( \sum_{x \in X} \Gamma_x = 0 \).

Before proceed further, we would like to rewrite the obtained identities in the ‘original’ theta-form.

**Proposition 3.3** For all \( z \) and sets \( X \) and \( Y \) related by \( |X| = |Y| + 2 \) the theta function satisfies
\[ \sum_{x \in X} \varphi_{X,Y}(x) \theta(z + a(X\setminus x)) \theta(z + a(Y + x)) = 0 \quad (3.10) \]
Combinatorics of multisecant Fay identities.

with

$$\varphi_{X,Y}(x) = \frac{E(x,Y)}{E(x,X\setminus x)}. \quad (3.11)$$

**Proposition 3.4** For all $z$ and sets $X$ and $Y$ related by $|X| = |Y|$ the theta functions satisfies

$$\theta(z + a(X)) \theta(z + a(Y)) = \sum_{x \in X} \varphi_{X,Y}(x,y) \theta(z + a(X\setminus x+y)) \theta(z + a(Y\setminus y+x)) \quad (3.12)$$

for any $y \in Y$ with

$$\varphi_{X,Y}(x,y) = \frac{E(x,Y\setminus y)E(y,X\setminus x)}{E(x,X\setminus x)E(y,Y\setminus y)}. \quad (3.13)$$

The next step in generalizing the Fay identities can be done by switching from summation over the points of a set to summation over subsets of a given set of points. To make the following formulae more legible we will use the subscript to indicate the size of a set:

$$X_n = \{x_1, \ldots, x_n\}. \quad (3.14)$$

With this change of the notation, we can formulate the following results.

**Proposition 3.5** For arbitrary sets $X_{n+2}$ and $Y_n$ and any $m \in [0, n]$

$$\sum_{A_{m+1} \subset X_{n+2}} \Phi_{X_{n+2},Y_n}(A_{m+1}, B_m) = 0, \quad B_m \subset Y_n \quad (3.15)$$

**Proposition 3.6** For arbitrary sets $X_n$ and $Y_n$ and any $m \in [1, n]$

$$\sum_{A_m \subset X_n} \Phi_{X_n,Y_n}(A_m, B_m) = 1, \quad B_m \subset Y_n \quad (3.16)$$

We present proofs of these results in Appendix D and Appendix E.

In terms of the theta functions, identities (3.15) and (3.16) read

$$\sum_{A_{m+1} \subset X_{n+2}} \varphi_{X_{n+2},Y_n}(A_{m+1}, B_m) \theta(z + a(X_{n+2}\setminus A_{m+1} + B_m)) \theta(z + a(A_{m+1} + Y_n \setminus B_m))$$

$$= 0 \quad (3.17)$$

$$\sum_{A_m \subset X_n} \varphi_{X_n,Y_n}(A_m, B_m) \theta(z + a(X_n\setminus A_m + B_m)) \theta(z + a(A_m + Y_n \setminus B_m))$$

$$= \theta(z + a(X))\theta(z + a(Y)) \quad (3.18)$$

4. Multilinear Fay identities.

In this section we rewrite some of the obtained identities in the matrix form and derive, using this representation, various multilinear ones.

For given sets $X$ and $Y$ of equal size $n$,

$$X = \{x_1, \ldots, x_n\}, \quad Y = \{y_1, \ldots, y_n\}. \quad (4.1)$$
Combinatorics of multisecant Fay identities.

Consider the \((n \times n)\)-matrix
\[
\Phi_{XY} = \left( \Phi_{x,y}(x_j, y_k) \right)_{j,k=1,...,n} \tag{4.2}
\]
In terms of \(\Phi_{XY}\), identities (3.7) and (3.8) become
\[
u^t \Phi_{XY} = \nu^t,
\]
\[
\Phi_{XY} \nu = \nu
\]
where \(\nu\) is the \(n\)-column with all components equal to 1, \(\nu = (1, ..., 1)^t\).

One can easily 'iterate' these formulae to obtain more complex ones. For example, multiplication (from the right-hand side) by \(\Phi_{YZ}\), where \(|Z| = n\), leads to
\[
u^t \Phi_{XY} \Phi_{YZ} = \nu^t \Phi_{YZ} = \nu^t \tag{4.3}
\]
In a similar way one can obtain
\[
u^t \Phi_{XU_1} \Phi_{U_1 U_2} ... \Phi_{U_{l-1} U_l} \Phi_{U_l Y} = \nu^t,
\]
\[
\Phi_{XU_1} \Phi_{U_1 U_2} ... \Phi_{U_{l-1} U_l} \Phi_{U_l Y} \nu = \nu
\]
\(||U_1| = ... = |U_l| = n|\).

To return to the standard, 'scalar', identities one can use an arbitrary vector \(v \in \mathbb{C}^n\) which yields
\[
u^t \Phi_{XU_1} \Phi_{U_1 U_2} ... \Phi_{U_{l-1} U_l} \Phi_{U_l Y} v = \nu^t v \tag{4.6}
\]
Depending on the choice of \(v\) one can arrive at the homogeneous identities (if \(\nu^t v = 0\)) or at the inhomogeneous ones (if \(\nu^t v \neq 0\)).

The key moment is that all identities discussed in the previous section were bilinear in \(\theta\), like the original Fay identity (2.4). At the same time, in (4.5) or (4.6) we have products of \(l+1\) bilinear in \(\theta\) matrices. This means that we have derived, by elementary calculations, a large set of multilinear Fay identities.

It is easy to see that all above calculations can be repeated in the case of rectangular matrices \(\Phi_{XY}\), i.e. one can lift the condition \(|X| = |Y|\). However, we restrict ourselves with the simplest case.

Another way to obtain the multilinear Fay identities is to consider determinants that appear in the matrix identities presented above. For example, equation (4.3) states that \(\nu\) is the eigenvector of \(\Phi_{XY}\) corresponding to the unit eigenvalue, which leads to
\[
\det \left| \Phi_{XY} - 1 \right| = 0 \tag{4.7}
\]
In a similar way, equations (4.5) imply
\[
\det \left| \Phi_{XU_1} \Phi_{U_1 U_2} ... \Phi_{U_{l-1} U_l} \Phi_{U_l Y} - 1 \right| = 0 \tag{4.8}
\]
As another example, one can note that equation (4.4) implies that \(\Phi_{XY} \Phi_{YZ} - \Phi_{XZ}\) is a degenerate matrix (it sends the row \(\nu^t\) to zero), which leads to
\[
\det \left| \Phi_{XY} \Phi_{YZ} - \Phi_{XZ} \right| = 0 \tag{4.9}
\]
with obvious generalization to the different products of the matrices similar to ones that appear in (4.5).

Note that (4.7) and other determinant identities differ from the determinant identity derived by Fay (see equation (43) in \([1]\)).
5. Differential Fay identities.

For two close points \( p \) and \( q \) of a Riemann surface \( \mathcal{X} \), there naturally appear two ‘small’ (i.e. vanishing when \( p \to q \)) quantities
\[
\delta_{pq} = a(p - q) \in \text{Jac}(\mathcal{X}) \tag{5.1}
\]
and
\[
\varepsilon_{pq} = E(p, q) \in \mathbb{C}. \tag{5.2}
\]
After introducing the differential operator \( \partial_q \) by
\[
\partial_q \theta(z) = \lim_{p \to q} \frac{1}{\varepsilon_{pq}} \left[ \theta(z + \delta_{pq}) - \theta(z) \right] \tag{5.3}
\]
and defining the constant \( \Lambda_{q,x,y} \) as
\[
\Lambda_{q,x,y} = \frac{1}{E(q,x)E(q,y)} \lim_{p \to q} \frac{1}{\varepsilon_{pq}} \left[ E(p,x)E(q,y) - E(q,x)E(q,y) \right] \tag{5.4}
\]
one can obtain from the Fay identity (2.4)
\[
\left[ D_q + \Lambda_{q,x,y} \right] \theta(z + a(x - y)) \cdot \theta(z) = \frac{E(x,y)}{E(x,q)E(y,q)} \theta(z + a(x - q))\theta(z + a(q - y)) \tag{5.5}
\]
where \( D_q \) is the Hirota bilinear operator,
\[
D_q u \cdot v = (\partial_q u) v - u (\partial_q v). \tag{5.6}
\]
Similar calculations, starting from (3.10) with \( X \) replaced with \( X + p + q \), lead to the following generalization of (5.5).

**Proposition 5.1** For all \( z \) and sets \( X \) and \( Y \) related by \( |X| = |Y| \) but arbitrary otherwise the theta function satisfies
\[
\left[ D_q + \Lambda_{q,x,y} \right] \theta(z + a(X)) \cdot \theta(z + a(Y))
= \sum_{x \in X} \psi_{q,X,Y}(x) \theta(z + a(X \setminus x + q))\theta(z + a(Y + x - q)) \tag{5.7}
= - \sum_{y \in X} \psi_{q,Y,X}(y) \theta(z + a(X + y - q))\theta(z + a(Y \setminus y + q)).
\]
where
\[
\psi_{q,X,Y}(x) = \frac{E(q,X)E(x,Y)}{E(q,Y)E^2(q,x)E(x,X \setminus x)} \tag{5.8}
\]
and
\[
\Lambda_{q,x,y} = \frac{1}{E(q,X)E(q,Y)} \lim_{p \to q} \frac{1}{\varepsilon_{pq}} \left[ E(p,X)E(q,Y) - E(q,X)E(q,Y) \right]. \tag{5.9}
\]

6. Applications.

In this section we would like to discuss the ‘practical’ aspects of the obtained results. Our aim is to show how one can use the theta functions to derive solutions for multidimensional versions of the well-known integrable models.
6.1. The \( n \)-dimensional version of the Hirota bilinear discrete equation.

Let us return to the equation (3.10),
\[
\sum_{x \in X} \varphi_{X,Y}(x) \theta(z + a(X \setminus x)) \theta(z + a(Y + x)) = 0 \quad (|X| = |Y| + 2)
\] (6.1)
for
\[
X = \{x_1, \ldots, x_n\},
\] (6.2)
which, after the shift \( z \to z - \frac{1}{2}a(X+Y) \) can be rewritten as
\[
\sum_{k=1}^{n} \Gamma_k \theta(z + e_k) \theta(z - e_k) = 0
\] (6.3)
where
\[
\Gamma_k = \varphi_{X,Y}(x_k), \quad e_k = a(x_k) + \frac{1}{2}a(Y) - \frac{1}{2}a(X).
\] (6.4)
It is easy to see that this equation implies that the function
\[
\Theta(m_1, \ldots, m_n) = \theta \left( z + \sum_{k=1}^{n} m_k e_k \right)
\] (6.5)
satisfies the equation
\[
\sum_{k=1}^{n} \Gamma_k \Theta(..., m_k + 1, ...) \Theta(..., m_k - 1, ...) = 0,
\] (6.6)
which is similar to the \( n \)-dimensional Hirota bilinear discrete equation,
\[
\sum_{k=1}^{n} \tau(..., m_k + 1, ...) \tau(..., m_k - 1, ...) = 0,
\] (6.7)
but with extra coefficients \( \Gamma_k \). One can take into account this difference, by introducing the quadratic in \( m_k \) function
\[
f(m_1, \ldots, m_n) = \frac{1}{2} \sum_{k=1}^{n} m_k^2 \ln \Gamma_k.
\] (6.8)

To summarize, we can state the following result.

**Proposition 6.1** For arbitrary vector \( z \), \( n \)-set \( X \) and \((n-2)\)-set \( Y \) equations (6.4) and (6.8) determine a solution
\[
\tau(m_1, \ldots, m_n) = \exp \left[ f(m_1, \ldots, m_n) \right] \theta \left( z + \sum_{k=1}^{n} m_k e_k \right)
\] (6.9)
for the \( n \)-dimensional version of the Hirota bilinear discrete equation
\[
\sum_{k=1}^{n} \tau(..., m_k + 1, ...) \tau(..., m_k - 1, ...) = 0.
\] (6.10)
6.2. The n-dimensional Toda-type lattice.

Consider the situation when \(|X| = |Y| = n\) and
\[ a(X) = a(Y). \] (6.11)

In this case equation (3.12), after the shift \(z \rightarrow z - a(X)\), can be written as
\[ \theta^2(z) = \sum_{k=1}^{n} \Gamma_k \theta(z + e_k)\theta(z - e_k) \] (6.12)

where
\[ \Gamma_k = \varphi_{X,Y}(x_k, y), \quad e_k = a(x_k) - a(y). \] (6.13)

Eliminating the constants \(\Gamma_k\) and making obvious definitions we can formulate the following result.

**Proposition 6.2** For arbitrary vector \(z\) and two \(n\)-sets \(X\) and \(Y\) related by \(a(X) = a(Y)\) function
\[ u(m_1, ..., m_n) = f(m_1, ..., m_n) + \ln \theta \left( z + \sum_{k=1}^{n} m_k e_k \right), \] (6.14)

where
\[ f(m_1, ..., m_n) = \frac{1}{2} \sum_{k=1}^{n} m_k^2 \ln \Gamma_k \] (6.15)

with \(e_k\) and \(\Gamma_k\) defined in (6.13), satisfies the \(n\)-dimensional Toda-type equation
\[ \sum_{k=1}^{n} \exp(\Delta_k u) = 1 \] (6.16)

where the second-order difference operators \(\Delta_k\) are defined by
\[ (\Delta_k F)(m_1, ..., m_n) = F(\ldots, m_k + 1, \ldots) - 2F(\ldots, m_k, \ldots) + F(\ldots, m_k - 1, \ldots). \] (6.17)

7. Discussion.

In this paper we have presented a number of identities for the theta functions defined on the compact Riemann surfaces which generalize the TFI. We would like to repeat that all these identities were obtained by iteration of the original TFI (2.4) without using any additional facts from the algebraic geometry.

The main idea behind this work is to facilitate usage of the multidimensional theta functions in the applied problems like solving the differential or difference equations. We hope that the approach of this work and the obtained results give possibility to address these questions in a more easy way, without necessity to develop each time the algebro-geometric scheme, involving, for example, Baker-Akhiezer functions, Riemann-Roch theorem etc like, e.g., in [6, 7, 8, 9].

We are aware of the fact that the presented identities should be discussed from the viewpoint of the algebraic geometry. For example, we have used the conditions like
but did not mention their meaning from the angle of the existence of meromorphic functions with given principal divisors. As another example, we never tried to interpret functions like \((3.2)\) as a generalized cross-ratios. Such questions are important not only from the viewpoint of unification of different approaches. For example, when dealing with large sets of points (in our terms), whose size is greater than the genus \(g\) of the Riemann surface, one has to consider the possibility of trivialization of some of the identities. However, these questions are out of the scope of the present paper and may be addressed in separate studies.

**Appendix A. Proof of proposition 3.1**

Consider the expression that appears in the left-hand side of \((3.6)\),

\[
\sigma_{X,Y} = \sum_{x \in X} \Phi_{X,Y}(x, \emptyset).
\]

(A.1)

Using equation \((2.4)\) with \(a = x_1, b = x_2, c = x\) and \(d = y_1\) shifted by \(X + y_1\) one can present the summand in the last equation as

\[
\Phi_{X+x_1+x_2, Y+y_1+y_2}(x, \emptyset) = (T_X \Phi_{X+x_1+x_2, Y+y_1+y_2}(x_1, \emptyset)) (T_{y_1} \Phi_{X+x_1, Y+y_2}(x, \emptyset))
\]

(A.2)

which, together with

\[
(T_X \Phi_{x_1,x_2,y_1}(x_1, \emptyset)) (T_{y_1} \Phi_{X+x_1, Y+y_2}(x_1, \emptyset)) = \Phi_{X+x_1+x_2, Y+y_1+y_2}(x_1, \emptyset)
\]

(A.3)

leads to the recurrence

\[
\sigma_{X+x_1+x_2,Y+y_1+y_2} = (T_X \Phi_{x_1,x_2,y_1}(x_1, \emptyset)) (T_{y_1} \sigma_{X+x_1, Y+y_2})
\]

(A.4)

where \(T_X\) denotes the shift \(z \rightarrow z + a(X)\).

In the limiting case of \(X = \{x_1, x_2, x_3\}, Y = \{y\}\) identity \((3.4)\) yields

\[
\sigma_{\{x_1,x_2,x_3\},y} = 0.
\]

(A.5)

Thus, equation \((A.4)\) implies

\[
\sigma_{X,Y} = 0, \quad |X| = |Y| + 2
\]

(A.6)

which completes the proof of proposition 3.1.

**Appendix B. Proof of proposition 3.2**

Replacing in \((3.6)\) \(X\) with \(X + a\),

\[
\Phi_{X+a,Y}(a, \emptyset) + \sum_{x \in X} \Phi_{X+a,Y}(x, \emptyset) = 0,
\]

(B.1)

and noting that

\[
\Phi_{X+a,Y}(x, \emptyset) = -\Phi_{X,Y+a}(x, a) \Phi_{X+a,Y}(a, \emptyset) \quad (x \in X)
\]

(B.2)
Combinatorics of multisecant Fay identities.

one can obtain
\[ 1 - \sum_{x \in X} \Phi_{X,Y+a}(x, a) = 0 \] (B.3)
which, after replacing \( Y + a \to Y \), leads to (3.7). Equation (3.8) follows from (3.7) and the symmetry
\[ \Phi_{X,Y}(A,B) = \Phi_{Y,X}(B,A). \] (B.4)

Appendix C. Useful lemma.

Here we prove a useful statement that is used in what follows.

**Proposition C.1** For arbitrary sets \( X, Y \) and their subsets \( A \subset X, B \subset Y \) related by
\[ |X| - 2|A| + 2 = |Y| - 2|B| \] (C.1)
the function \( \Phi \) satisfies
\[ \sum_{x \in A} \Phi_{X,Y}(A \setminus x, B) = \sum_{y \in Y \setminus B} \Phi_{X,Y}(A, B + y), \] (C.2)
\[ \sum_{x \in X \setminus A} \Phi_{X,Y}(A + x, B) = \sum_{y \in B} \Phi_{X,Y}(A, B \setminus y). \] (C.3)

To obtain this result, we start with (3.6), replace \( X \) with \( X_1 + Y_2 \) and \( Y \) with \( X_2 + Y_1 \), split the sum
\[ \sum_{x \in X_1} \Phi_{X_1+Y_2, X_2+Y_1}(x, \emptyset) + \sum_{y \in Y_2} \Phi_{X_1+Y_2, X_2+Y_1}(y, \emptyset) = 0 \] (C.4)
and use the identities
\[ \Phi_{X_1+Y_2, X_2+Y_1}(x, \emptyset)\Phi_{X_1+X_2, Y_1+Y_2}(X_1, Y_1) = -\epsilon_{X_1} \epsilon_{Y_1} \Phi_{X_1+X_2, Y_1+Y_2}(X_1 \setminus x, Y_1) \] (C.5)
\[ \Phi_{X_1+Y_2, X_2+Y_1}(y, \emptyset)\Phi_{X_1+X_2, Y_1+Y_2}(X_1, Y_1) = \epsilon_{X_1} \epsilon_{Y_1} \Phi_{X_1+X_2, Y_1+Y_2}(X_1, Y_1 + y) \] (C.6)
where \( \epsilon_{X,Y} = (-)^{|X||Y|} \). Thus, we arrive, after the substitution \( X_1 \to A, X_2 \to X \setminus A, Y_1 \to B \) and \( Y_2 \to Y \setminus B \), at (C.2).

Identity (C.3) can be proved in a similar way or obtained from (C.2) using the symmetry of the function \( \Phi_{X,Y}(A,B) \).

Appendix D. Proof of proposition 3.5

For two fixed sets, \( X \) and \( Y \) related by \( |X| = |Y| + 2 \), consider the left-hand side of (3.15) as a function of \( B_m \),
\[ f(B_m) = \sum_{A_{m+1} \subset X} \Phi_{X,Y}(A_{m+1}, B_m) \] (D.1)
where, recall, subscripts \( m \) and \( m + 1 \) indicate the size of the corresponding sets.
Using a simple combinatorial identity

\[ \sum_{A_{m+1} \subset X} f(A_{m+1}) = \frac{1}{m+1} \sum_{A_m \subset X} \sum_{x \in X \setminus A_m} f(A_m + x) \]  

(D.2)

one can present \( f(B_m) \) as

\[ f(B_m) = \frac{1}{m+1} \sum_{A_m \subset X} \sum_{x \in X \setminus A_m} \Phi_{X,Y}(A_m + x, B_m) \]  

(D.3)

which, by virtue of (C.3), yields

\[ f(B_m) = \frac{1}{m+1} \sum_{A_m \subset X} \sum_{y \in B_m} \Phi_{X,Y}(A_m, B_m \setminus y) \]  

(D.4)

\[ = \frac{1}{m+1} \sum_{y \in B_m} f(B_m \setminus y) \]  

(D.5)

\[ = \frac{1}{m+1} \sum_{B_{m-1} \subset B_m} f(B_{m-1}) \]  

(D.6)

Thus, we have expressed \( f \) on a \( m \)-set as a combination of \( f \) on \( (m-1) \)-sets. In the limiting case, \( B_0 = \emptyset \), identity (3.6) implies

\[ f(\emptyset) = \sum_{x \in X} \Phi_{X,Y}(x, \emptyset) = 0 \]  

(D.7)

which, by induction, leads

\[ f(B_m) = 0, \quad m = 1, 2, \ldots \]  

(D.8)

Adding the constraint \( m \leq |Y| \), one arrives at the statement of proposition 3.5.

**Appendix E. Proof of proposition 3.6**

As in the proof of proposition 3.5 for two fixed sets, \( X \) and \( Y \) related this time by \( |X| = |Y| \), consider the left-hand side of (3.16) as a function of \( B_m \),

\[ F(B_m) = \sum_{A_m \subset X} \Phi_{X,Y}(A_m, B_m) \]  

(E.1)

Using, again, identity (D.2) one can present \( F(B_{m+1}) \) as

\[ F(B_{m+1}) = \sum_{A_{m+1} \subset X} \Phi_{X,Y}(A_{m+1}, B_{m+1}) \]  

(E.2)

\[ = \frac{1}{m+1} \sum_{A_m \subset X} \sum_{x \in X \setminus A_m} \Phi_{X,Y}(A_m + x, B_{m+1}) \]  

(E.3)

which, by virtue of (C.3), yields

\[ F(B_{m+1}) = \frac{1}{m+1} \sum_{A_m \subset X} \sum_{y \in B_{m+1}} \Phi_{X,Y}(A_m, B_{m+1} \setminus y) \]  

(E.4)

\[ = \frac{1}{m+1} \sum_{y \in B_{m+1}} F(B_{m+1} \setminus y) \]  

(E.5)

\[ = \frac{1}{m+1} \sum_{B_m \subset B_{m+1}} F(B_m). \]  

(E.6)
Thus, we have obtained the recurrence from \( m \)-sets to \((m+1)\)-sets. In the limiting case, \( m = 1 \) or \( B_1 = \{y\} \), identity \( (3.7) \) implies
\[
F(y) = \sum_{x \in X} \Phi_{x,y}(x, y) = 1
\]
which, by induction, leads to
\[
F(B_m) = 1, \quad m = 2, \ldots
\]
and hence, after adding the constraint \( m \leq |Y| \), to the statement of proposition 3.6.

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