THE SIGNATURE OF THE RICCI CURVATURE
OF LEFT IN Variant RIEMANNIAN METRICS
ON 4-DIMENSIONAL LIE GROUPS

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Abstract. In this paper, we present the classification of all possible signatures
of the Ricci curvature of left-invariant Riemannian metrics on 4-dimensional Lie
groups.

Key words and phrases: Riemannian manifold, Lie groups, Ricci curvatures.

This brief note contains the main results of our recent papers [13] and [14], devoted
to the classification of all possible signatures of the Ricci curvatures of left-invariant
metrics on 4-dimensional Lie groups.

It is well known that various restrictions on the curvature of a Riemannian man-
ifold allow to get some information on its geometrical and topological properties.
Some results of this type are related to the Ricci curvature. For example, the My-
ners theorem [17] states that a complete Riemannian manifold with positive Ricci
curvature is compact and has finite fundamental group. Note that the Ricci curva-
ture is more informative for homogeneous Riemannian manifolds. For instance, a
Riemannian homogeneous manifold with negative Ricci curvature is noncompact by
the Bochner theorem [5].

For a homogeneous space $G/H$ (where $H$ is a compact subgroup of the Lie group
$G$), it is naturally to find some general properties of the Ricci curvature of all $G$-
invariant Riemannian metrics on $G/H$. One can make this problem more precise
and concrete in various ways. One variant is to consider the following question:
What are possible signatures of the Ricci operators for all $G$-invariant Riemannian
metrics on the space $G/H$? There is a hope that this question could be completely
resolved at least for the case of low dimensions. From the paper [15] of J. Milnor
we know the answer to this question for dimensions $\leq 3$. The papers [8, 13, 21] give
us the corresponding answer for all four-dimensional homogeneous spaces, different
from the Lie groups.

We recall briefly some structure results on homogeneous Riemannian manifolds
of low dimensions. All 2-dimensional simply connected homogeneous Riemannian
manifolds are symmetric. Each 3-dimensional simply connected homogeneous Rie-
mannian manifold is either a symmetric space, or a Lie group supplied with a left-invariant
Riemannian metric [3, 21]. As noted before, the signatures of the Ricci
curvature of left invariant metrics on 3-dimensional Lie groups were classified in [15]
(some refinements and generalizations one can find in [22]). The classification
of 4-dimensional homogeneous Riemannian manifolds was obtained by S. Ishihara
[8]. G. Jensen proved that every 4-dimensional simply connected homogeneous Ein-
stein manifold is isometric to a symmetric space [10]. Some refinements of results of
Table 1.

|   | Signature   | Signature   | Signature   |
|---|-------------|-------------|-------------|
| 1 | (−, −, −, −) | (−, −, +, +) | (0, 0, 0, 0) |
| 2 | (−, −, −, 0) | (−, 0, 0, 0) | (0, 0, 0, +) |
| 3 | (−, −, −, +) | (−, 0, 0, +) | (0, 0, +, +) |
| 4 | (−, −, 0, 0) | (−, 0, +, +) | (0, +, +, +) |
| 5 | (−, −, 0, +) | (−, +, +, +) | (+, +, +, +) |

[8] are obtained by L. Berard-Bergery [3] and V. Patrangenaru [21]. In particular, it was shown in [8, 3, 21] that an arbitrary simply connected homogeneous Riemannian manifold of dimension \( \leq 4 \) is either a symmetric space, or a Lie group with a suitable left invariant Riemannian metric. Recall, that each symmetric space is a direct metric product of Euclidian space and some irreducible symmetric spaces. Moreover, every irreducible symmetric space is Einstein (i.e. it has constant Ricci curvature). Therefore, the problem of determination of all possible signatures of the Ricci operators of invariant metrics on a given symmetric space has an obvious solution. The same problem for left invariant metrics on a given Lie group is much more hard and interesting.

The signature of a symmetric operator \( A \), defined on a \( n \)-dimensional Euclidian space, is a \( n \)-tuple \( (\text{sgn}(\lambda_1), \text{sgn}(\lambda_2), \ldots, \text{sgn}(\lambda_n)) \), where \( \lambda_1 \leq \cdots \leq \lambda_n \) are the eigenvalues of the operator \( A \) and \( \text{sgn}(x) \) means the sign of a (real) number \( x \). We list all possible signatures for the 4-dimensional case in Table 1.

Any local problem related to the Ricci curvature of a left-invariant Riemannian metric \( \rho \) on a (connected) Lie group \( G \) can be reformulated in terms of the corresponding metric Lie algebra. Recall, that the left invariant metric \( \rho \) defines an inner product \( Q \) on the Lie algebra \( \mathfrak{g} \) of \( G \) and vice versa: each inner product \( Q \) on \( \mathfrak{g} \) induces a left-invariant metric \( \rho \) on \( G \). There are useful formulas for the Ricci operator of the metric Lie algebra \( (\mathfrak{g}, Q) \) (see, e.g. [1] or [4]).

Note that the set of inner products on a given \( n \)-dimensional Lie algebra \( \mathfrak{g} \) has dimension \( n(n + 1)/2 \). This set could be reduced with using of the automorphism group of \( \mathfrak{g} \) (see e.g. [9]). In the 4-dimensional case, after this reduction we get the space of “representative” inner products of dimension \( \leq 6 = 10 - 4 \). The classification, up to the action of the automorphism group, of the inner products on all 4-dimensional Lie algebras is obtained in the paper [6]. Therefore, for a given 4-dimensional Lie group (Lie algebra), our original problem could be reduced to the study of the signature of some \( 4 \times 4 \) symmetric matrix (the matrix of the Ricci operator in a suitable orthonormal basis), which depends on \( \leq 6 \) parameters.

In Table 2, we present the classification of 4-dimensional Lie algebras obtained by G.M. Mubarakzyanov [16] (see also [20, 6, 7, 2]). Now we can formulate our main results.
### Table 2.

| Lie algebra | Nonzero commutation relations |
|-------------|--------------------------------|
| $4A_1$      | $[e_1, e_2] = e_2$            |
| $A_2 \oplus 2A_1$ | $[e_1, e_2] = e_2, [e_3, e_4] = e_4$ |
| $2A_2$      | $[e_2, e_3] = e_1$            |
| $A_{3,1} \oplus A_1$ | $[e_1, e_3] = e_1, [e_2, e_3] = e_1 + e_2$ |
| $A_{3,2} \oplus A_1$ | $[e_1, e_3] = e_1, [e_2, e_3] = e_2$ |
| $A_{3,4} \oplus A_1$ | $[e_1, e_3] = e_1, [e_2, e_3] = -e_2$ |
| $A_{3,5} \oplus A_1$, $0 < |\alpha| < 1$ | $[e_1, e_3] = e_1, [e_2, e_3] = \alpha e_2$ |
| $A_{3,6} \oplus A_1$ | $[e_1, e_3] = -e_2, [e_2, e_3] = e_1$ |
| $A_{3,7} \oplus A_1, \alpha > 0$ | $[e_1, e_3] = \alpha e_1 - e_2, [e_2, e_3] = e_1 + \alpha e_2$ |
| $A_{3,8} \oplus A_1$ | $[e_1, e_2] = -e_3, [e_3, e_1] = e_2, [e_2, e_3] = e_1$ |
| $A_{3,9} \oplus A_1$ | $[e_1, e_2] = e_3, [e_3, e_1] = e_2, [e_2, e_3] = e_1$ |
| $A_{4,1}$   | $[e_1, e_4] = e_1, [e_3, e_4] = e_2$ |
| $A_{4,2}, \alpha \neq 0$ | $[e_1, e_4] = \alpha e_1, [e_2, e_4] = e_2, [e_3, e_4] = e_2 + e_3$ |
| $A_{4,3}$   | $[e_1, e_4] = e_1, [e_3, e_4] = e_2$ |
| $A_{4,4}$   | $[e_1, e_4] = e_1, [e_2, e_4] = e_1 + e_2, [e_3, e_4] = e_2 + e_3$ |
| $A_{4,5,6}, |\alpha| = |\beta| = 0, 1 \leq |\alpha| \leq 1$ | $[e_1, e_4] = e_1, [e_2, e_4] = \alpha e_1, [e_3, e_4] = \beta e_3$ |
| $A_{4,6}, \alpha \neq 0, |\beta| \geq 1$ | $[e_1, e_4] = \alpha e_1, [e_2, e_4] = \beta e_2 - e_3, [e_3, e_4] = e_2 + \beta e_3$ |
| $A_{4,7}$   | $[e_2, e_3] = e_1, [e_1, e_4] = 2e_1, [e_2, e_4] = e_2, [e_3, e_4] = e_2 + e_3$ |
| $A_{4,8}$   | $[e_2, e_3] = e_1, [e_2, e_4] = e_2, [e_3, e_4] = -e_3$ |
| $A_{4,9}, -1 < |\beta| \leq 1$ | $[e_2, e_3] = e_1, [e_1, e_4] = (1 + \beta)e_1, [e_2, e_4] = e_2, [e_3, e_4] = \beta e_3$ |
| $A_{4,10}$  | $[e_2, e_3] = e_1, [e_2, e_4] = -e_3, [e_3, e_4] = e_2$ |
| $A_{4,11, \alpha > 0}$ | $[e_2, e_3] = e_1, [e_1, e_4] = 2\alpha e_1, [e_2, e_4] = \alpha e_2 - e_3, [e_3, e_4] = e_2 + \alpha e_3$ |
| $A_{4,12}$  | $[e_1, e_3] = e_1, [e_2, e_3] = e_2, [e_1, e_4] = -e_2, [e_2, e_4] = e_1$ |
**Theorem 1** ([13]). Let \( \mathfrak{g} \) be a unimodular 4-dimensional Lie algebra from Table 2, \( s \) a signature from Table 1. Then \( s \) can be realized as the signature of the Ricci operator for some inner product on \( \mathfrak{g} \) if and only if there is the sign “+” in the entry of Table 3, corresponding to the Lie algebra \( \mathfrak{g} \) and the signature \( s \).

**Corollary 1** ([13]). The signatures \((-,-,-,-), (-,-,-,0), (0,0,+,+)\) and \((+:+;+++)\) are not the signatures of the Ricci operators for left-invariant Riemannian metrics on 4-dimensional unimodular Lie groups.

**Corollary 2** ([13]). Let \( \mathfrak{g} \) be a 4-dimensional unimodular Lie algebra, \( Q \) some inner product on \( \mathfrak{g} \), and \( S \) the scalar curvature of the metric Lie algebra \( (\mathfrak{g}, Q) \). Then the following assertions are true:

1) If \( \mathfrak{g} \) commutative, then \( S = 0 \);
2) If \( \mathfrak{g} \) is isomorphic to \( A_{3,6} \oplus A_1 \), then \( S = 0 \) or \( S < 0 \);
3) If \( \mathfrak{g} \) is isomorphic to \( A_{3,9} \oplus A_1 = \text{su}(2) \oplus \mathbb{R} \), then \( S \) may have any sign;
4) For all other Lie algebras \( \mathfrak{g} \) the inequality \( S < 0 \) holds.

**Theorem 2** ([14]). Let \( \mathfrak{g} \) be a non-unimodular 4-dimensional Lie algebra from Table 2, \( s \) a signature from Table 1. Then \( s \) can be realized as the signature of the Ricci operator for some inner product on \( \mathfrak{g} \) if and only if there is the sign “+” in the entry of Table 4, corresponding to the Lie algebra \( \mathfrak{g} \) and the signature \( s \).
| Lie algebra                     | Signature |
|--------------------------------|-----------|
| $A_2 \oplus 2A_1$               | - - - + + - - - - - - - - - - - |
| $2A_2$                         | + + + + + + - - - - - - - - - - |
| $A_{3,2} \oplus A_1$           | - + + + + + - - - - - - - - - - |
| $A_{3,3} \oplus A_1$           | - + + - - - - - - - - - - - - - |
| $A_{3,5}^\alpha \oplus A_1$, $\alpha \in (-1, 0)$ | - - + - + + - - - - - - - - - - |
| $A_{3,5}^\alpha \oplus A_1$, $\alpha \in (0, 1)$ | - + + + + + - - - - - - - - - - |
| $A_{3,7}^\alpha \oplus A_1$    | - + + + + + - - - - - - - - - - |
| $A_{4,2}^\alpha$, $\alpha < 0$, $\alpha \neq -2$ | - - + - + + - - - - - - - - - - |
| $A_{4,2}^\alpha$, $\alpha > 0$, $\alpha \neq 1$ | + + + + + + - - - - - - - - - - |
| $A_{4,2}$                      | + + + - - - - - - - - - - - - - |
| $A_{4,3}$                      | - - + - + + - - - - - - - - - - |
| $A_{4,4}$                      | + + + + + + - - - - - - - - - - |
| $A_{4,5}^{\alpha, \alpha}$, $\alpha \in [-1, -\frac{1}{2}]$ | - - + - - - - - - - - - - - - - |
| $A_{4,5}^{\alpha, \alpha}$, $\alpha \in (-\frac{1}{2}, 0)$ | - - - - - + - - - - - - - - - - |
| $A_{4,5}^{\alpha, 1}$, $\alpha \in [-1, 0)$ | - - + - - - - - - - - - - - - - |
| $A_{4,5}^{\alpha, \alpha}$, $\alpha \in (0, 1)$ | + + + - - - - - - - - - - - - - |
| $A_{4,5}^{\alpha, 1}$, $\alpha \in (0, 1)$ | + + + - - - - - - - - - - - - - |
| $A_{4,5}^{1, 1}$               | + - - - - - - - - - - - - - - - |
| $A_{4,5}^{\alpha, \beta}$, $\alpha \in [-1, 0)$ | - - + - + + - - - - - - - - - - |
| $A_{4,5}^{\alpha, \beta}$, $\alpha \in (0, 1)$, $\alpha \neq \beta$ | + + + + + - - - - - - - - - - - |
| $A_{4,6}^{\alpha, \beta}$, $\alpha < 0$, $\beta > 0$, $\alpha \neq -2\beta$ | - - + - + + - - - - - - - - - - |
| $A_{4,6}^{\alpha, \beta}$, $\alpha > 0$ | + + + + + + - - - - - - - - - - |
| $A_{4,6}$                      | - - + + + + - - - - - - - - - - |
| $A_{4,7}$                      | + + + + + + - - - - - - - - - - |
| $A_{4,9}^\beta$, $\beta \in (-1, 0)$ | + + + + + + - - - - - - - - - - |
| $A_{4,9}^\beta$, $\beta \in [0, 1)$ | - - + - + + - - - - - - - - - - |
| $A_{4,9}$                      | - - + - - - - - - - - - - - - - |
| $A_{4,11}^\alpha$              | + + + + + + - - - - - - - - - - |
| $A_{4,12}$                     | + + + + + + - - - - - - - - - - |
Corollary 3. The Ricci operator of any left-invariant Riemannian metric on every 4-dimensional non-unimodular Lie group has at least two negative eigenvalues. For every signature from the list \((-,-,-,-), (-,-,-,0), (-,-,-,0), (-,-,0,+)\) and \((-,-,-,+)\), there are a 4-dimensional non-unimodular Lie group \(G\) and a left-invariant Riemannian metric \(\rho\) on \(G\) such that the signature of the Ricci operator of \((G,\rho)\) coincides with the chosen one.

The proofs and discussions of all above results can be found in the papers [13,14]. Our main technical tools are some ideas and structure results from [1,15,18,19]. It should be noted that in order to study some signatures (for instance, \((-,-,0,0)\)) we have developed some special methods.

Note that the scalar curvature of any left-invariant Riemannian metric on a non-unimodular Lie group is negative [9,15]. Thus the Ricci operator in this case has at least one negative eigenvalue. Corollary 2 states that there are at least 2 such eigenvalues in the 4-dimensional case. The same is true for the dimensions 2 and 3 (see [15]). We suppose that it is a general property on non-unimodular solvable Lie groups (for dimensions \(\leq 4\) every non-unimodular Lie group is solvable):

Conjecture 1. Let \(\mathfrak{s}\) be a non-unimodular solvable Lie algebra of an arbitrary dimension. Then for every inner product \(Q\) on \(\mathfrak{s}\), the Ricci operator of the metric Lie algebra \((\mathfrak{s},Q)\) has at least two negative eigenvalues.

This conjecture is confirmed for dimensions \(\leq 4\) and for the case of non-unimodular metabelian Lie algebras [14] and could be confirmed also for dimension 5.

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