ON QUASIMÖBIUS MAPS IN REAL BANACH SPACES

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Abstract. Suppose that $E$ and $E'$ denote real Banach spaces with dimension at least 2, that $D \subseteq E$ and $D' \subseteq E'$ are domains, that $f : D \to D'$ is an $(M, C)$-CQH homeomorphism, and that $D$ is uniform. The main aim of this paper is to prove that $D'$ is a uniform domain if and only if $f$ extends to a homeomorphism $\overline{f} : \overline{D} \to \overline{D'}$ and $\overline{f}$ is $\eta$-QM relative to $\partial D$. This result shows that the answer to one of the open problems raised by Väisälä from 1991 is affirmative.

1. Introduction and main results

During the past three decades, the quasihyperbolic metric has become an important tool in geometric function theory and in its generalizations to metric spaces and to Banach spaces [18]. Yet, some basic questions of the quasihyperbolic geometry in Banach spaces are open. For instance, only recently the convexity of quasihyperbolic balls has been studied in [7, 8, 11, 20] in the setup of Banach spaces.

Our study is motivated by Väisälä’s theory of freely quasiconformal mappings and other related maps in the setup of Banach spaces [15, 16, 18]. Our goal is to study some of the open problems formulated by him. We begin with some basic definitions and the statements of our results. The proofs and necessary supplementary notation and terminology will be given thereafter.

Throughout the paper, we always assume that $E$ and $E'$ denote real Banach spaces with dimension at least 2. The norm of a vector $z$ in $E$ is written as $|z|$, and for every pair of points $z_1, z_2$ in $E$, the distance between them is denoted by $|z_1 - z_2|$, the closed line segment with endpoints $z_1$ and $z_2$ by $[z_1, z_2]$. We begin with the following concepts following closely the notation and terminology of [12, 13, 14, 15, 16] or [10].

Definition 1.1. A domain $D$ in $E$ is called $c$-uniform in the norm metric provided there exists a constant $c$ with the property that each pair of points $z_1, z_2$ in $D$ can be joined by a rectifiable arc $\alpha$ in $D$ satisfying

\[(1) \min_{j=1,2} \ell(\alpha[z_j, z]) \leq c d_D(z) \text{ for all } z \in \alpha, \text{ and} \]
\[(2) \ell(\alpha) \leq c |z_1 - z_2|, \]

where $d_D(z)$ is the quasihyperbolic metric in $D$.
where $\ell(\alpha)$ denotes the length of $\alpha$, $\alpha[z_j, z]$ the part of $\alpha$ between $z_j$ and $z$, and $d_D(z)$ the distance from $z$ to the boundary $\partial D$ of $D$. Also, we say that $\gamma$ is a double $c$-cone arc.

In [13], Väisälä obtained the following result concerning the relation between the class of uniform domains and quasimöbius (briefly, QM) maps (see Definition 2.4) in $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$.

**Theorem A.** ([13, Theorem 5.6]) Suppose that $n \geq 2$, that $D$ is a $c$-uniform domain in $\mathbb{R}^n$ and that $f : D \to D'$ is a quasiconformal mapping. Then the following conditions are quantitatively equivalent:

1. $D'$ is a $c_1$-uniform domain;
2. $f$ is $\eta$-QM.

In [16], Väisälä generalized Theorem A to the case of Banach spaces. His result is as follows.

**Theorem B.** ([16, Theorem 7.18]) Let $D$ and $D'$ be domains in $E$ and $E'$, respectively. Suppose that $D$ is a $c$-uniform domain and that $f : D \to D'$ is $\varphi$-FQC (see Definition 2.6). Then the following conditions are quantitatively equivalent:

1. $D'$ is a $c_1$-uniform domain;
2. $f$ is $\eta$-QM.

Further, Väisälä [16, 7.19] raised the following open problem.

**Open Problem 1.1.** Does Theorem B remain true if $\varphi$-FQC is replaced by $(M, C)$-CQH (see Definition 2.5) and $\eta$-QM by $\eta$-QM rel $\partial D$, respectively?

Studying this problem, Väisälä proved the following result.

**Theorem C.** ([16, Theorem 7.9]) Suppose that $D \subset E$ and $D' \subset E'$ are $c$-uniform domains and that $f : D \to D'$ is $(M, C)$-CQH. Then $f$ extends to a homeomorphism $\overline{f} : \overline{D} \to \overline{D'}$ and $\overline{f}$ is $\eta$-QM rel $\partial D$ with $\eta$ depending only on $(M, C, c)$. In particular, $\overline{f}|_{\partial D}$ is $\eta$-QM.

The aim of this paper is to discuss Open Problem 1.1 further. Our main result is the next theorem, which shows that the answer to Open Problem 1.1 is in the affirmative.

**Theorem 1.1.** Suppose that $D$ is a $c$-uniform domain and that $f : D \to D'$ is $(M, C)$-CQH, where $D \subset E$ and $D' \subset E'$. Then the following conditions are quantitatively equivalent:

1. $D'$ is a $c_1$-uniform domain;
2. $f$ extends to a homeomorphism $\overline{f} : \overline{D} \to \overline{D'}$ and $\overline{f}$ is $\eta$-QM rel $\partial D$.

The organization of this paper is as follows. Section 2 contains some preliminaries and a new lemma. The proof of the main result is given in Section 3.
2. Preliminaries

2.1. Quasihyperbolic distance, quasihyperbolic geodesics, neargeodesics and solid arcs. The quasihyperbolic length of a rectifiable arc or a path $\alpha$ in the norm metric in $D$ is the number (cf. [5, 19]):

$$\ell_k(\alpha) = \int_\alpha \frac{|dz|}{d_D(z)}.$$  

For each pair of points $z_1, z_2$ in $D$, the quasihyperbolic distance $k_D(z_1, z_2)$ between $z_1$ and $z_2$ is defined in the usual way:

$$k_D(z_1, z_2) = \inf \ell_k(\alpha),$$

where the infimum is taken over all rectifiable arcs $\alpha$ joining $z_1$ to $z_2$ in $D$. For all $z_1, z_2$ in $D$, we have (cf. [19])

$$k_D(z_1, z_2) \geq \inf \left\{ \log \left( 1 + \frac{\ell(\alpha)}{\min\{d_D(z_1), d_D(z_2)\}} \right) \right\} \geq \left| \log \frac{d_D(z_2)}{d_D(z_1)} \right|,$$

where the infimum is taken over all rectifiable curves $\alpha$ in $D$ connecting $z_1$ and $z_2$. Moreover, if $|z_1 - z_2| \leq d_D(z_1)$, we have [15, 21]

$$k_D(z_1, z_2) \leq \log \left( 1 + \frac{|z_1 - z_2|}{d_D(z_1) - |z_1 - z_2|} \right).$$

The quasihyperbolic metric of a domain in $\mathbb{R}^n$ was introduced by Gehring and Palka [5] and it has been recently used by many authors in the study of quasiconformal mappings and related questions; see [1, 2, 3, 4, 6, 9, 15, 16, 18] etc.

In [16], Väisälä characterized uniform domains by the quasihyperbolic metric.

**Theorem D.** ([16, Theorem 6.16]) For a domain $D$ in $E$, the following are quantitatively equivalent:

1. $D$ is a $c$-uniform domain;
2. $k_D(z_1, z_2) \leq c' \log \left( 1 + \frac{|z_1 - z_2|}{\min\{d_D(z_1), d_D(z_2)\}} \right)$ for all $z_1, z_2 \in D$;
3. $k_D(z_1, z_2) \leq c'_1 \log \left( 1 + \frac{|z_1 - z_2|}{\min\{d_D(z_1), d_D(z_2)\}} \right) + d$ for all $z_1, z_2 \in D$.

**Remark 2.1.** By [17, Theorem 2.23], we know that in Theorem D, if (1) holds, then (2) holds with $c' \leq 7c^3$.

In the case of domains in $\mathbb{R}^n$, the equivalence of items (1) and (3) in Theorem D is due to Gehring and Osgood [4] and the equivalence of items (2) and (3) due to Vuorinen [21].

Recall that an arc $\alpha$ from $z_1$ to $z_2$ is a quasihyperbolic geodesic if $\ell_k(\alpha) = k_D(z_1, z_2)$. Each subarc of a quasihyperbolic geodesic is obviously a quasihyperbolic geodesic. It is known that a quasihyperbolic geodesic between every pair of points in $E$ exists if the dimension of $E$ is finite, see [4, Lemma 1]. This is not true in arbitrary spaces.
In order to remedy this shortage, Väisälä introduced the following concepts [16].

**Definition 2.1.** Let $\alpha$ be an arc in $E$. The arc may be closed, open or half open. Let $\overline{x} = (x_0, \ldots, x_n)$, $n \geq 1$, be a finite sequence of successive points of $\alpha$. For $h \geq 0$, we say that $\overline{x}$ is $h$-coarse if $k_D(x_{j-1}, x_j) \geq h$ for all $1 \leq j \leq n$. Let $\Phi_k(\alpha, h)$ be the family of all $h$-coarse sequences of $\alpha$. Set

$$s_k(\overline{x}) = \sum_{j=1}^{n} k_D(x_{j-1}, x_j)$$

and

$$\ell_k(\alpha, h) = \sup\{s_k(\overline{x}) : \overline{x} \in \Phi_k(\alpha, h)\}$$

with the agreement that $\ell_k(\alpha, h) = 0$ if $\Phi_k(\alpha, h) = \emptyset$. Then the number $\ell_k(\alpha, h)$ is the $h$-coarse quasihyperbolic length of $\alpha$.

**Definition 2.2.** Let $D$ be a domain in $E$. An arc $\alpha \subset D$ is $(\nu, h)$-solid with $\nu \geq 1$ and $h \geq 0$ if

$$\ell_k(\alpha[x, y], h) \leq \nu k_D(x, y)$$

for all $x, y \in \alpha$. A $(\nu, 0)$-solid arc is said to be a $\nu$-neargeodesic, i.e. an arc $\alpha \subset D$ is a $\nu$-neargeodesic if and only if $\ell_k(\alpha[x, y]) \leq \nu k_D(x, y)$ for all $x, y \in \alpha$.

Obviously, a $\nu$-neargeodesic is a quasihyperbolic geodesic if and only if $\nu = 1$.

In [16], Väisälä established the following property concerning the existence of neargeodesics in $E$.

**Theorem E.** ([16, Theorem 3.3]) Let $\{z_1, z_2\} \subset D$ and $\nu > 1$. Then there is a $\nu$-neargeodesic in $D$ joining $z_1$ and $z_2$.

The following result due to Väisälä is from [16].

**Theorem F.** ([16, Theorem 6.22]) Suppose that $\gamma \subset G \subsetneq E$ is a $(\nu, h)$-solid arc with endpoints $a_0, a_1$ and that $G$ is a c-uniform domain. Then there is a constant $c_2 = c_2(\nu, h, c) \geq 1$ such that

1. $\min\{\diam(\gamma[a_0, z]), \diam(\gamma[a_1, z])\} \leq c_2 d_G(z)$ for all $z \in \gamma$, and
2. $\diam(\gamma) \leq c_2 \max\{|a_0 - a_1|, 2(e^h - 1) \min\{d_G(a_0), d_G(a_1)\}\}.$

### 2.2. Quasisymmetric homeomorphisms and quasimöbius maps

Let $X$ be a metric space and $\hat{X} = X \cup \{\infty\}$. By a triple in $X$ we mean an ordered sequence $T = (x, a, b)$ of three distinct points in $X$. The ratio of $T$ is the number

$$\rho(T) = \frac{|a - x|}{|b - x|}.$$ 

If $f : X \to Y$ is an injective map, the image of a triple $T = (x, a, b)$ is the triple $fT = (fx, fa, fb)$. 
Suppose that $A \subset X$. A triple $T = (x, a, b)$ in $X$ is said to be a triple in the pair $(X, A)$ if $x \in A$ or if $\{a, b\} \subset A$. Equivalently, both $|a - x|$ and $|b - x|$ are distances from a point in $A$.

**Definition 2.3.** Let $X$ and $Y$ be two metric spaces, and let $\eta : [0, \infty) \to [0, \infty)$ be a homeomorphism. Suppose $A \subset X$. An embedding $f : X \to Y$ is said to be $\eta$-quasisymmetric relative to $A$, or briefly $\eta$-QS rel $A$, if $\rho(f(T)) \leq \eta(\rho(T))$ for each triple $T$ in $(X, A)$.

It is known that an embedding $f : X \to Y$ is $\eta$-QS rel $A$ if and only if $\rho(T) \leq t$ implies that $\rho(f(T)) \leq \eta(t)$ for each triple $T$ in $(X, A)$ and $t \geq 0$ (cf. [12]). Obviously, “quasisymmetric rel $X$” is equivalent to ordinary “quasisymmetric”.

A quadruple in $X$ is an ordered sequence $Q = (a, b, c, d)$ of four distinct points in $X$. The cross ratio of $Q$ is defined to be the number

$$\tau(Q) = |a, b, c, d| = \frac{|a - b|}{|a - c|} \cdot \frac{|c - d|}{|b - d|}.$$  

Observe that the definition is extended in the well known manner to the case where one of the points is $\infty$. For example,

$$|a, b, c, \infty| = \frac{|a - b|}{|a - c|}.$$

If $X_0 \subset \hat{X}$ and if $f : X_0 \to \hat{Y}$ is an injective map, the image of a quadruple $Q$ in $X_0$ is the quadruple $fQ = (fa, fb, fc, fd)$. Suppose that $A \subset X_0$. We say that a quadruple $Q = (a, b, c, d)$ in $X_0$ is a quadruple in the pair $(X_0, A)$ if $\{a, d\} \subset A$ or $\{b, c\} \subset A$. Equivalently, all four distances in the definition of $\tau(Q)$ are (at least formally) distances from a point in $A$.

**Definition 2.4.** Let $\hat{X}$ and $\hat{Y}$ be two metric spaces and let $\eta : [0, \infty) \to [0, \infty)$ be a homeomorphism. Suppose $A \subset \hat{X}$. An embedding $f : \hat{X} \to \hat{Y}$ is said to be $\eta$-quasimöbius relative to $A$, or briefly $\eta$-QM rel $A$, if the inequality $\tau(f(Q)) \leq \eta(\tau(Q))$ holds for each quadruple in $(X, A)$.

Apparently, “$\eta$-QM rel $X$” is equivalent to ordinary “quasimöbius”.

### 2.3. Coarsely quasihyperbolic homeomorphisms and freely quasiconformal mappings.

**Definition 2.5.** We say that a homeomorphism $f : D \to D'$ is $C$-coarsely $M$-quasihyperbolic, or briefly $(M, C)$-CQH, in the quasihyperbolic metric if it satisfies

$$\frac{k_D(x, y) - C}{M} \leq k_{D'}(f(x), f(y)) \leq M k_D(x, y) + C$$

for all $x, y \in D$.

The following result shows that the class of solid arcs is invariant under the CQH homeomorphisms.

**Theorem G.** ([16, Theorem 4.15]) For domains $D \not\subset E$ and $D' \not\subset E'$, suppose that $f : D \to D'$ is $(M, C)$-CQH. If $\gamma$ is a $(\nu_1, h_1)$-solid arc in $D$, then the arc $f(\gamma)$ is $(\nu, h)$-solid in $D'$ with $(\nu, h)$ depending only on $(\nu_1, h_1, M, C)$.
Definition 2.6. Let $G \neq E$ and $G' \neq E'$ be metric spaces, and let $\varphi : [0, \infty) \to [0, \infty)$ be a growth function, that is, a homeomorphism with $\varphi(t) \geq t$. We say that a homeomorphism $f : G \to G'$ is $\varphi$-semisolid if

$$k_{G'}(f(x), f(y)) \leq \varphi(k_G(x, y))$$

for all $x, y \in G$, and $\varphi$-solid if both $f$ and $f^{-1}$ satisfy this condition.

We say that $f$ is fully $\varphi$-semisolid (resp. fully $\varphi$-solid) if $f$ is $\varphi$-semisolid (resp. $\varphi$-solid) on every subdomain of $G$. In particular, when $G = E$, the corresponding subdomains are taken to be proper ones. Fully $\varphi$-solid mappings are also called freely $\varphi$-quasiconformal mappings, or briefly $\varphi$-FQC mappings.

2.4. Basic assumptions and a lemma. For convenience, in the following, we always assume that $x, y, z, \ldots$ denote points in $D$ and $x', y', \ldots$ the images in $D'$ of $x, y, z, \ldots$ under $f$, respectively. Also we assume that $\alpha, \beta, \gamma, \ldots$ denote curves in $D$ and $\alpha', \beta', \gamma', \ldots$ the images in $D'$ of $\alpha, \beta, \gamma, \ldots$ under $f$, respectively.

Basic assumption A. Let $G$ be a domain in $E$. For $x, y \in G$, let $\beta$ be a 2-neargeodesic joining $x$ and $y$ in $G$. Suppose that $G'$ is a $c$-uniform domain in $E'$ and $f : G \to G'$ is an $(M, C)$-CQH homeomorphism. It follows from Theorem G that $\beta' \in (\nu, h)$-solid, where $(\nu, h)$ depends only on $(M, C, c)$. Without loss of generality, we may assume that $d_{G'}(y') \geq d_{G'}(x')$. Then there must exist a point $z_0' \in \beta'$ which is the first point on $\beta'$ in the direction from $x'$ to $y'$ such that

$$d_{G'}(z_0') = \sup_{p' \in \beta'} d_{G'}(p').$$

It is possible that $z_0' = x'$ or $y'$. Then we have

Lemma 2.1. (1) For all $z' \in \beta'[x', z_0']$,

$$|x' - z'| \leq \mu_1 d_{G'}(z'),$$

and for all $z' \in \beta'[y', z_0']$,

$$|y' - z'| \leq \mu_1 d_{G'}(z');$$

(2) $\text{diam}(\beta') \leq \mu_1 \max\{|x' - y'|, 2(e^h - 1)d_{G'}(x')\}$,

where $\mu_1 = 4c_2^2$, $c_2 = c_2(\nu, h, c)$ is the same as in Theorem F, and $\nu, h$ and $c$ are as in Basic assumption A.

Proof. By Theorem F, it suffices to prove the first assertion in (1). For the case $\text{min}\{\text{diam}(\beta'[x', z']), \text{diam}(\beta'[y', z'])\} = \text{diam}(\beta'[x', z'])$, it follows from Theorem F that the proof is obvious. For the other case $\text{min}\{\text{diam}(\beta'[x', z']), \text{diam}(\beta'[y', z'])\} = \text{diam}(\beta'[y', z'])$, we first have the following claim.

Claim 2.1. $\text{diam}(\beta'[x', z']) \leq 2c_2 d_{G'}(z_0').$

Suppose on the contrary that $\text{diam}(\beta'[x', z']) > 2c_2 d_{G'}(z_0')$.

Obviously, there must exist some point $w' \in \beta'[x', z']$ such that

$$\text{diam}(\beta'[w', z']) = \frac{1}{2}\text{diam}(\beta'[x', z'])$$

and $\text{diam}(\beta'[x', w']) \geq \frac{1}{2}\text{diam}(\beta'[x', z'])$.  


It follows from Theorem F that
\[ c_2 d_G'(w') \geq \min \{ \text{diam}(\beta'[x', w']), \text{diam}(\beta'[y', w']) \} \]
\[ \geq \frac{1}{2} \text{diam}(\beta'[x', z']) > c_2 d_G'(z'_0). \]
This is the desired contradiction which completes the proof of Claim 2.1.

If \( \text{diam}(\beta'[y', z']) \leq \frac{1}{2} d_G'(z'_0) \), then by Claim 2.1,
\[ |x' - z'| \leq \text{diam}(\beta'[x', z']) \leq 2 c_2 d_G'(z'_0) \leq 4 c_2 d_G'(z'), \]
since \( d_G'(z') \geq d_G'(z'_0) - |z'_0 - z'|. \)
If \( \text{diam}(\beta'[y', z']) > \frac{1}{2} d_G'(z'_0) \), then we see from Claim 2.1 and Theorem F that
\[ |x' - z'| \leq \text{diam}(\beta'[x', z']) \leq 2 c_2 d_G'(z'_0) \leq 4 c_2 \text{diam}(\beta'[y', z']) \leq 4 c_2 d_G'(z'). \]
The proof is finished. \( \square \)

3. The proof of Theorem 1.1

First, we recall the following results which are from [13] and [16], respectively.

**Theorem H.** ([13, Theorem 3.19]) Suppose that \( A \subset \hat{X} \), that \( f : A \to \hat{X} \) is \( \eta \)-QM, and suppose that \( \bar{f}(A) \backslash \{ \infty \} \) is complete. Then \( f \) has a unique extension to an \( \eta \)-QM embedding \( g : \bar{A} \to \bar{X}. \)

**Theorem I.** ([16, Theorem 6.26]) Suppose that \( f : D \to D' \) is \( \eta \)-QM and that \( D \) is a \( c \)-uniform domain. Then \( D' \) is a \( c_1 \)-uniform domain, where \( c_1 \) depends only on \( c \) and \( \eta \).

For more details of quasisymmetric homeomorphisms and quasimöbius maps, the reader is referred to [12, 13, 15].

The proof of Theorem 1.1 will be accomplished through a series of lemmas. Before the statements of the lemmas, we give another basic assumption.

**Basic assumption B.** Throughout this section, we always assume that \( D \) is a \( c \)-uniform domain, that \( f : D \to D' \) is \((M, C)\)-CQH, where \( D \nsubseteq E \) and \( D' \nsubseteq E' \), and that \( f \) extends to a homeomorphism \( \bar{f} : \overline{D} \to \overline{D'} \) and \( \bar{f} \) is \( \eta \)-QM rel \( \partial D \). By auxiliary translations and inversions, it follows from Theorems I and H that we may normalize the map \( f \) and the domain \( D \) so that \( \infty \in \partial D \) and \( \bar{f}(\infty) = \infty \). Then \( f \) is \( \eta \)-QS rel \( \partial D \).

**Constants.** For the convenience of the statements of the lemmas below, we write down the related constants:
1. \( \mu_2 = \max \{ 4(e^h - 1)\mu_1, 6\mu_1 \} \),
2. \( \mu_3 = \max \{ 2\eta(2\mu_2), 6\mu_1 \mu_2 \} \),
3. \( \mu_4 = \max \{ 4^{16eCM\mu_2(\eta(6\mu_1)+1)}, (\mu_3(\eta(6\mu_1)+1))^{16eCM} \} \),
4. \( \mu_5 = 16eCM\mu_3 \mu_4 \max \{ 1/u, u \}, \ u = \eta^{-1}(1/(4\mu_3 \mu_4)) \),
5. \( \mu_6 = 4\mu_4 \mu_5 \),
6. \( \mu_7 = (\mu_6(\eta(2\mu_2) + 1))^{8eM} \).
where $\mu_1$ and $c'$ ($\leq 7c^3$) are from Lemma 2.1 and Theorem D, respectively, and $M$, $C$ and $\eta$ from Basic assumption B.

3.1. Several lemmas.

**Lemma 3.1.** For $v_1 \in \partial D$ and $v_2 \in D$, if $|v_1 - v_2| \leq 2\mu_2 d_D(v_2)$, then

$$|v'_1 - v'_2| \leq \mu_3 d_{D'}(v'_2).$$

**Proof.** Let $x'_1 \in \partial D'$ be such that $d_{D'}(v'_2) \geq \frac{1}{2}|x'_1 - v'_2|$. It follows from the assumptions on $f$ that

$$\frac{|v'_1 - v'_2|}{|x'_1 - v'_2|} \leq \eta(2\mu_2),$$

whence

$$|v'_1 - v'_2| \leq 2\eta(2\mu_2)d_{D'}(v'_2),$$

which shows that the lemma holds.  

For $z'_1, z'_2 \in D' \subset E'$, let $\gamma'$ be a 2-neargeodesic joining $z'_1$ and $z'_2$ in $D'$. In the following, we aim to prove that $\gamma'$ satisfies the conditions (1) and (2) in Definition 1.1.

Without loss of generality, we may assume that $d_{D'}(z'_2) \geq d_{D'}(z'_1)$. Let $x_0 \in \gamma$ be the first point in the direction from $z_1$ to $z_2$ such that

$$d_D(x_0) = \sup_{p \in \gamma} d_D(p).$$

**Lemma 3.2.** For all $z' \in \gamma'[z'_1, x'_0]$,

$$\text{diam}(\gamma'[z'_1, z']) \leq \mu_4 d_{D'}(z'),$$

and for all $z' \in \gamma'[z'_2, x'_0]$,

$$\text{diam}(\gamma'[z'_2, z']) \leq \mu_4 d_{D'}(z').$$

**Proof.** We only need to prove the former assertion since the proof for the latter one is similar. We prove it by a contradiction. Suppose there exists some point $z''_1 \in \gamma'[z'_1, x'_0]$ such that

$$\text{diam}(\gamma'[z'_1, z'']) > \mu_4 d_{D'}(z'_1).$$

(3.1)

Obviously, there exists some point $w'_{11} \in \gamma'[z'_1, z''_1]$ such that

$$|w'_{11} - z''_1| = \frac{1}{2}\text{diam}(\gamma'[z'_1, z''_1]).$$

Then, by (3.1), we see that

$$|w'_{11} - z''_1| > \frac{\mu_4}{2} d_{D'}(z'_1),$$

$$|w'_{11} - z''_1| > \frac{\mu_4}{2} d_{D'}(z'_1),$$

which contradicts the definition of $z''_1$. Thus, the lemma holds.
whence by (2.1)

\[ k_D(w_{11}, z_{11}) \geq \frac{1}{M} (k_{D'}(w'_{11}, z'_{11}) - C) \geq \frac{1}{M} \left( \log \left( 1 + \frac{|w'_{11} - z'_{11}|}{d_{D'}(z'_{11})} \right) - C \right) \]

\[ \geq \frac{1}{M} \left( \log \left( 1 + \frac{\mu_0}{2} \right) - C \right) > 1, \]

which, together with (2.2), implies that

\[ |w_{11} - z_{11}| > \frac{1}{2} \max \{d_D(z_{11}), d_D(w_{11}) \}. \]

Let \( w_{12} \in \partial D \) be such that

\[ |w_{12} - w_{11}| \leq 2d_D(w_{11}). \]

Since \( \mu_2 \geq 1 \), it follows from Lemma 3.1 that

\[ |w'_{12} - w'_{11}| \leq \mu_3 d_{D'}(w'_{11}). \]

Lemma 2.1 and (3.2) imply

\[ |w_{12} - z_{11}| \leq |w_{12} - w_{11}| + |w_{11} - z_{11}| \leq 5\mu_1 d_D(z_{11}). \]

Therefore we see from Lemma 3.1 and the fact “5\mu_1 \leq \mu_2” that

\[ |w'_{12} - z'_{11}| \leq \mu_3 d_{D'}(z'_{11}). \]

On one hand, if \( |w'_{12} - w'_{11}| \leq \mu_3 \eta(6\mu_1)|w'_{12} - z'_{11}| \), then, by (3.4), we have

\[ \operatorname{diam}(\gamma'[z'_{11}, z_{11}]) = 2|w'_{11} - z'_{11}| \leq 2(|w'_{12} - w'_{11}| + |w'_{12} - z'_{11}|) \leq 2(\mu_3 \eta(6\mu_1) + 1)|w'_{12} - z'_{11}| \leq 2\mu_3(\mu_3 \eta(6\mu_1) + 1) d_{D'}(z'_{11}), \]

which contradicts (3.1).

On the other hand, if \( |w'_{12} - w'_{11}| > \mu_3 \eta(6\mu_1)|w'_{12} - z'_{11}| \), then Lemma 2.1 and (3.3) imply that

\[ |w_{12} - w_{11}| \leq |w_{12} - z_{11}| + |w_{11} - z_{11}| \leq 6\mu_1 d_D(z_{11}). \]

Hence

\[ \mu_3 \eta(6\mu_1) \leq \frac{|w'_{12} - w'_{11}|}{|w'_{12} - z'_{11}|} \leq \eta(6\mu_1), \]

since \( \frac{|w_{12} - w_{11}|}{|w_{12} - z_{11}|} \leq 6\mu_1 \). This is the desired contradiction. \( \square \)

**Lemma 3.3.** For all \( z' \in \gamma'[z'_{11}, z'_{12}] \), we have \( \min_{j=1,2} \ell(\gamma'[z'_{j}, z']) \leq \mu_0 d_{D'}(z'). \)

**Proof.** We use \( z'_0 \in \gamma' \) to denote the first point on \( \gamma' \) in the direction from \( z'_{1} \) to \( z'_{2} \) such that

\[ d_{D'}(z'_0) = \sup_{\ell' \in \gamma'} d_{D'}(\ell'). \]
It is possible that \( z'_0 = z'_1 \) or \( z'_2 \). Obviously, there exists a nonnegative integer \( m \) such that

\[
2^m d_{D'}(z'_1) \leq d_{D'}(z'_0) < 2^{m+1} d_{D'}(z'_1),
\]

and we use \( y'_0 \) to denote the first point on \( \gamma[z'_1, z'_0] \) from \( z'_1 \) to \( z'_0 \) with

\[
d_{D'}(y'_0) = 2^m d_{D'}(z'_1).
\]

We define a sequence \( \{x'_k\} \). Let \( x'_1 = z'_1 \). If \( y'_0 = z'_1 \neq z'_0 \), then we let \( x'_2 = z'_0 \). If \( y'_0 \neq z'_1 \), then we let \( x'_2, \ldots, x'_{m+1} \in \gamma[z'_1, z'_0] \) be the points such that for each \( i \in \{2, \ldots, m+1\} \), \( x'_i \) is the first point from \( z'_1 \) to \( z'_0 \) with

\[
d_{D'}(x'_i) = 2^{i-1} d_{D'}(x'_1).
\]

Therefore \( x'_{m+1} = y'_0 \). In the case \( y'_0 \neq z'_0 \), we define \( x'_{m+2} = z'_0 \).

Similarly, let \( s \geq 0 \) be the integer such that

\[
2^s d_{D'}(z'_2) \leq d_{D'}(z'_0) < 2^{s+1} d_{D'}(z'_2),
\]

and let \( x'_{1,0} \) be the first point on \( \gamma[z'_2, z'_0] \) from \( z'_2 \) to \( z'_0 \) with

\[
d_{D'}(x'_{1,0}) = 2^s d_{D'}(z'_2).
\]

In the same way, we define another sequence \( \{x'_{1,k}\} \). We let \( x'_{1,1} = z'_2 \). If \( x'_{1,0} = z'_2 \neq z'_0 \), then we let \( x'_{1,2} = z'_0 \). If \( x'_{1,0} \neq z'_2 \), then we let \( x'_{1,2}, \ldots, x'_{1,s+1} \) be the points on \( \gamma[z'_2, z'_0] \) such that \( x'_{1,j} \) \((j = 2, \ldots, s+1)\) denotes the first point from \( x'_{1,1} \) to \( z'_0 \) with

\[
d_{D'}(x'_{1,j}) = 2^{j-1} d_{D'}(x'_{1,1}).
\]

Then \( x'_{1,s+1} = x'_{1,0} \). If \( x'_{1,0} \neq z'_0 \), then we let \( x'_{1,s+2} = z'_0 \).

For a proof of the lemma, it is enough to prove that for every \( z' \in \gamma[z'_1, z'_0] \),

\[
(3.6) \quad \ell(\gamma[z'_1, z']) \leq \mu_6 d_{D'}(z'),
\]

and for all \( z' \in \gamma[z'_2, z'_0] \),

\[
(3.7) \quad \ell(\gamma[z'_1, z']) \leq \mu_6 d_{D'}(z').
\]

We only need to prove (3.6) since the proof for (3.7) is similar.

Before the proof of (3.6), we prove two claims.

**Claim 3.1.** For each \( i \in \{1, \ldots, m+1\} \), if \( y' \in \gamma[x'_i, x'_{i+1}] \), then \( d_{D'}(x'_i) \leq 2\mu_4 d_{D'}(y') \).

To prove this claim, it suffices to consider the case: \( d_{D'}(y') < \frac{1}{2} d_{D'}(x'_i) \) since the proof for the case: \( d_{D'}(y') \geq \frac{1}{2} d_{D'}(x'_i) \) is trivial (\( \mu_4 \geq 1 \)). It follows from \(|x'_i - y'| \geq d_{D'}(x'_i) - d_{D'}(y')\) that

\[
\min\{|x'_i - y'|, |y' - x'_{i+1}|\} > \frac{1}{2} d_{D'}(x'_i).
\]
If $\gamma'(z'_1, y') \subset \gamma'[z'_1, x'_0]$, then, by Lemma 3.2,
\[
d_{D'}(y') \geq \frac{1}{\mu_4} |x'_i - y'| \geq \frac{1}{2\mu_4} d_{D'}(x'_i).
\]
For the remaining case, we know that $\gamma'[z'_2, y'] \subset \gamma'[z'_2, x'_0]$. The similar reasoning as above shows that
\[
d_{D'}(y') \geq \frac{1}{\mu_4} |y' - x'_{i+1}| \geq \frac{1}{2\mu_4} d_{D'}(x'_i).
\]
Hence the proof of Claim 3.1 is complete.

**Claim 3.2.** For all $i \in \{1, \cdots, m+1\}$, $\ell(\gamma'[x'_i, x'_{i+1}]) \leq \mu_5 d_{D'}(x'_i)$.

Suppose on the contrary that there exists some $i \in \{1, \cdots, m+1\}$ such that
\[
\ell(\gamma'[x'_i, x'_{i+1}]) > \mu_5 d_{D'}(x'_i).
\]
Because $\gamma'$ is a 2-neargeodesic, we get by (3.8)
\[
k_{D'}(x'_i, x'_{i+1}) \geq \frac{1}{2} \iint_{\gamma'[x'_i, x'_{i+1}]} \frac{|dx'|}{d_{D'}(x')} > \frac{\mu_5}{4},
\]
and we see that
\[
k_D(x_i, x_{i+1}) \geq \frac{1}{M} k_{D'}(x'_i, x'_{i+1}) - \frac{C}{M} > 2c' \mu_3 \mu_4 \max \{1/u, u\},
\]
where $u = \eta^{-1}(1/(4\mu_3\mu_4))$. Then it follows from the inequality:
\[
k_D(x_i, x_{i+1}) \leq c' \log \left(1 + \frac{|x_i - x_{i+1}|}{\min\{d_D(x_i), d_D(x_{i+1})\}}\right)
\]
that
\[
\min\{d_D(x_i), d_D(x_{i+1})\} \leq \min \left\{\frac{1}{e^{2\mu_3\mu_4}/u - 1}, \frac{1}{e^{2\mu_3\mu_4} - 1}\right\} |x_i - x_{i+1}|.
\]
Without loss of generality, we may assume that
\[
\min\{d_D(x_i), d_D(x_{i+1})\} = d_D(x_i).
\]
Take $x_{2i} \in \partial D$ such that
\[
|x_{2i} - x_i| \leq 2d_D(x_i).
\]
Then Lemma 3.1 implies
\[
|x_{2i} - x'_i| \leq \mu_3 d_{D'}(x'_i).
\]
If $\gamma'[z'_1, x'_{i+1}] \subset \gamma'[z'_1, x'_0]$ or $\gamma'[z'_2, x'_{i}] \subset \gamma'[z'_2, x'_0]$, then, by Lemma 3.2,
\[
|x'_i - x'_{i+1}| \leq \mu_4 d_{D'}(x'_{i+1}) \leq 2\mu_4 d_{D'}(x'_i).
\]
For the remaining case, we know that $x'_0 \in \gamma'[x'_i, x'_{i+1}]$. Then Lemma 3.2 yields
\[ |x'_i - x'_{i+1}| \leq 2 \max\{|x'_i - x'_0|, |x'_{i+1} - x'_0|\} \leq 2 \mu_4 d_{D'}(x'_0) \leq 4 \mu_4 d_{D'}(x'_i). \]
Hence we get that for each $i \in \{1, \cdots, m + 1\}$,
\[ |x'_i - x'_{i+1}| \leq 4 \mu_4 d_{D'}(x'_i). \]
(3.13)

By (3.12) and (3.13), we have
\[ |x'_{2i} - x'_{i+1}| \leq |x'_{i} - x'_{i+1}| + |x'_{2i} - x'_{i}| \leq (\mu_3 + 4 \mu_4)d_{D'}(x'_i), \]
and by (3.9), (3.10) and (3.11),
\[ |x_{2i} - x_{i+1}| \geq |x_i - x_{i+1}| - |x_{2i} - x_i| \geq \max \left\{ \frac{\epsilon_2 \mu_3^4}{4}, \frac{\epsilon_2 \mu_3^4 - 3}{2} \right\} |x_{2i} - x_i|. \]
Hence (3.14) implies
\[ \frac{1}{\mu_3 + 4 \mu_4} \leq \frac{|x'_i - x'_{2i}|}{|x'_{2i} - x'_{i+1}|} \leq \eta(u) = \frac{1}{4 \mu_4 \mu_3}. \]
This is the desired contradiction, which completes the proof of Claim 3.2.

Now we are ready to prove (3.6).

If $z' \in \gamma'[z'_1, x'_{m+1}]$, then there exists some $k \in \{1, \cdots, m\}$ such that $z' \in \gamma'[x'_k, x'_{k+1}]$. If $k = 1$, then it easily follows from Claims 3.1 and 3.2 that
\[ \ell(\gamma'[z'_1, z']) \leq \ell(\gamma'[x'_1, x'_2]) \leq \mu_5 d_{D'}(x'_1) \leq 2 \mu_4 \mu_5 d_{D'}(z'). \]
(3.15)

If $k > 1$, then, again, by Claims 3.1 and 3.2,
\[ \ell(\gamma'[z'_1, z']) \leq \ell(\gamma'[x'_1, x'_2]) + \cdots + \ell(\gamma'[x'_{k-1}, x'_k]) + \ell(\gamma'[x'_k, z']) \leq \mu_5 (d_{D'}(x'_1) + \cdots + d_{D'}(x'_{k-1}) + d_{D'}(x'_k)) \leq 2 \mu_5 d_{D'}(x'_k) \leq 4 \mu_4 \mu_5 d_{D'}(z'). \]
(3.16)

Now we consider the remaining case: $z' \in \gamma'[x'_{m+1}, z'_0]$. We infer from Claims 3.1 and 3.2 that
\[ \ell(\gamma'[z'_1, z']) \leq \mu_5 (d_{D'}(x'_1) + d_{D'}(x'_2) + \cdots + d_{D'}(x'_m) + d_{D'}(x'_{m+1})) \leq 2 \mu_5 d_{D'}(x'_{m+1}) \leq 4 \mu_4 \mu_5 d_{D'}(z'). \]
(3.17)

The combination of (3.15), (3.16) and (3.17) shows that for all $z' \in \gamma'[z'_1, z'_0]$,
\[ \ell(\gamma'[z'_1, z']) \leq 4 \mu_4 \mu_5 d_{D'}(z'). \]
Hence Lemma 3.3 holds.
Further, we have

**Lemma 3.4.** \( \ell(\gamma'[z'_1, z'_2]) \leq \mu_7 |z'_1 - z'_2| \).

**Proof.** Suppose, on the contrary, that

\[
\ell(\gamma'[z'_1, z'_2]) > \mu_7 |z'_1 - z'_2|.
\]

We first prove a claim.

**Claim 3.3.** \( d_{D'}(z'_2) \leq 7 |z'_1 - z'_2| \).

Also we prove this claim by contradiction. Suppose

\[
d_{D'}(z'_2) > 7 |z'_1 - z'_2|.
\]

Because \( \gamma' \) is a 2-neargeodesic, we have by (2.1) that

\[
\frac{1}{2} \log \left(1 + \frac{\ell(\gamma'[z'_1, z'_2])}{d_{D'}(z'_1)}\right) \leq \frac{1}{2} \ell_k(\gamma'[z'_1, z'_2]) \leq k_{D'}(z'_1, z'_2) \leq \int_{[z'_1, z'_2]} \frac{|dz'|}{d_{D'}(z')}
\]

\[
\leq \frac{7}{6} \cdot \frac{|z'_1 - z'_2|}{d_{D'}(z'_2)} < \frac{1}{6},
\]

since \( d_{D'}(z') \geq d_{D'}(z'_2) - |z'_2 - z'_1| \) for all \( z' \in [z'_1, z'_2] \). By (3.18) and (3.19), this is a contradiction. Hence Claim 3.3 holds true.

Recall that \( z'_0 \in \gamma' \) satisfies (3.5). Let \( x' \) be the point of \( \gamma' \) which bisects the arclength of \( \gamma' \), i.e. \( \ell(\gamma'[z'_1, x']) = \ell(\gamma'[z'_2, x']) \). Then Lemma 3.3 implies

\[
\ell(\gamma'[z'_1, x']) \leq \mu_6 d_{D'}(x') \leq \mu_6 d_{D'}(z'_0).
\]

Hence it follows from (3.18) and Claim 3.3 that

\[
d_{D'}(z'_2) \leq 7 |z'_1 - z'_2| < \frac{7}{\ell(\gamma'[z'_1, z'_2])} \leq \frac{14 \mu_6}{\mu_7} d_{D'}(z'_0),
\]

whence by Theorem D

\[
c' \log \left(1 + \frac{|z_2 - z_0|}{\min\{d(D(z_2), d(D(z_0))\}}\right) \geq k_D(z_2, z_0) \geq \frac{1}{M} k_{D'}(z'_2, z'_0) - \frac{C}{M}
\]

\[
\geq \frac{1}{M} \log \frac{d_{D'}(z'_0)}{d_{D'}(z'_2)} - \frac{C}{M}
\]

\[
> \frac{1}{2M} \log \frac{\mu_7}{\mu_6}.
\]
and
\begin{align*}
\epsilon' \log \left(1 + \frac{|z_1 - z_0|}{\min\{d_D(z_1), d_D(z_0)\}}\right) &\geq k_D(z_1, z_0) - \frac{1}{M} k_{D'}(z'_1, z'_0) - \frac{C}{M} \\
&\geq \frac{1}{M} \log \frac{d_{D'}(z'_0)}{d_{D'}(z'_1)} - \frac{C}{M} \\
&> \frac{1}{2M} \log \frac{\mu_7}{\mu_6}.
\end{align*}

These show that
\begin{align*}
\min\{d_D(z_2), d_D(z_0)\} &< \frac{1}{(\mu_6(\eta(2\mu_2) + 1))^2} |z_0 - z_2|, \\
\min\{d_D(z_1), d_D(z_0)\} &< \frac{1}{(\mu_6(\eta(2\mu_2) + 1))^2} |z_0 - z_1|,
\end{align*}
and we see by (2.2)
\begin{align*}
|z'_2 - z'_0| > \frac{1}{2} d_{D'}(z'_0) \quad \text{and} \quad |z'_1 - z'_0| > \frac{1}{2} d_{D'}(z'_0),
\end{align*}
since $k_{D'}(z'_1, z'_0) > 1$ and $k_{D'}(z'_2, z'_0) > 1$.

**Claim 3.4.** $\min\{d_D(z_2), d_D(z_1)\} \leq 2\mu_1 d_D(z_0)$.

We first prove that $d_D(z_1) \leq 2\mu_1 d_D(z_0)$ when $\gamma(z_1, z_0) \subset \gamma(z_1, x_0)$.
If $|z_1 - z_0| \leq \frac{1}{2} d_D(z_1)$, then
\[ d_D(z_0) \geq d_D(z_1) - |z_1 - z_0| \geq \frac{1}{2} d_D(z_1). \]

On the other hand, if $|z_1 - z_0| > \frac{1}{2} d_D(z_1)$, then we obtain from Lemma 2.1 that
\[ d_D(z_1) \leq 2|z_1 - z_0| \leq 2\mu_1 d_D(z_0). \]

A similar discussion as above shows that $d_D(z_2) \leq 2\mu_1 d_D(z_0)$ when $\gamma(z_2, z_0) \subset \gamma[z_2, x_0]$. The proof of Claim 3.4 is complete.

Without loss of generality, we may assume that
\[ \min\{d_D(z_2), d_D(z_1)\} = d_D(z_2). \]
Then (3.21) and Claim 3.4 imply
\begin{align*}
\min\{d_D(z_2), d_D(z_1)\} &\leq 2\mu_1 d_D(z_0) \\
&\leq \frac{1}{\mu_6(\eta(2\mu_2) + 1)} |z_0 - z_2|.
\end{align*}

Take $w_{13} \in \partial D$ such that
\begin{align*}
|w_{13} - z_2| &\leq 2d_D(z_2) \\
|w'_{13} - z'_2| &\leq \mu_3 d_{D'}(z'_2) \leq 7\mu_3 |z'_1 - z'_2|,
\end{align*}
it follows from Lemma 3.1 and Claim 3.3 that
\[ |w'_{13} - z'_2| \leq \mu_3 d_{D'}(z'_2) \leq 7\mu_3 |z'_1 - z'_2|. \]
whence
\[(3.26) \quad |w'_{13} - z'_1| \leq |w'_{13} - z'_2| + |z'_2 - z'_1| \leq (1 + 7\mu_3)|z'_1 - z'_2|.
\]
By the 2-neargeodesic property of \(\gamma'\), (2.1), (3.18) and Claim 3.3, we have
\[
k_D(z_1, z_2) \geq \frac{1}{M} k_{D'}(z'_1, z'_2) - \frac{C}{M} \geq \frac{1}{2M} \ell_k(\gamma'[z'_1, z'_2]) - \frac{C}{M} \\
\geq \frac{1}{2M} \log \left(1 + \frac{\ell(\gamma'[z'_1, z'_2])}{d_{D'}(z'_1)}\right) - \frac{C}{M} \geq \frac{1}{4M} \log \mu_7 > 1,
\]
which shows by (2.2)
\[(3.27) \quad |z_1 - z_2| \geq \frac{1}{2} d_D(z_1).
\]
Because of (3.23) and (3.24), we get
\[
|w_{13} - z_0| \leq |w_{13} - z_2| + |z_2 - z_0| \leq \left(1 + \frac{2}{\mu_6(\eta(2\mu_2) + 1)}\right) |z_2 - z_0|.
\]
Next by Lemma 2.1 and (3.27), we have
\[(3.28) \quad |z_0 - z_2| \leq \mathrm{diam}(\gamma) \leq \mu_1 \max \left\{|z_1 - z_2|, 2 \left(e^h - 1\right) d_D(z_2)\right\} \\
\leq \mu_1 \max \left\{1, 4 \left(e^h - 1\right)\right\} |z_1 - z_2| \\
\leq \mu_2 |z_1 - z_2|.
\]
Hence (3.23), (3.24) and (3.28) yield
\[(3.29) \quad |w_{13} - z_1| \geq |z_1 - z_2| - |w_{13} - z_2| \\
\geq \left(\frac{1}{\mu_2} - \frac{2}{\mu_6(\eta(2\mu_2) + 1)}\right) |z_0 - z_2| \\
\geq \frac{1}{2\mu_2} |w_{13} - z_0|.
\]
We see from (3.20), (3.22) and (3.25) that
\[(3.30) \quad |w'_{13} - z'_0| \geq |z'_2 - z'_0| - |w'_{13} - z'_2| \geq \left(\frac{\mu_7}{4\mu_6} - 7\mu_3\right) |z'_1 - z'_2| \\
> \mu_6(\eta(2\mu_2) + 1) |z'_1 - z'_2|.
\]
The combination of (3.26), (3.29) and (3.30) shows
\[\eta(2\mu_2) < \frac{|w'_{13} - z'_0|}{|w'_{13} - z'_1|} \leq \eta(2\mu_2).
\]
This desired contradiction completes the proof of Lemma 3.4. \(\Box\)

The next result is obvious from a similar argument as above.
Corollary 3.1. Under the assumptions of Theorem 1.1, if \( f \) extends to a homeomorphism \( \overline{f} : \overline{D} \to \overline{D}' \) and \( \overline{f} \) is \( \eta \)-QM rel \( \partial D \), then each \( a \)-neargeodesic \((a > 1)\) in \( D' \) is a double \( a' \)-cone arc, where \( a' \) depends only on \( a, c, M, C \) and \( \eta \). In particular, if \( a = 2 \), we can take \( a' = \mu_7 \).

3.2. The proof of Theorem 1.1. For \( z'_1, z'_2 \in D' \), let \( \gamma' \) be a 2-neargeodesic joining \( z'_1 \) and \( z'_2 \) in \( D' \). Corollary 3.1 shows that \( \gamma' \) is a double \( \mu_7 \)-cone arc. Hence \( D' \) is a \( \mu_7 \)-uniform domain, and so the proof of Theorem 1.1 easily follows from Theorem C. \( \square \)

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