Clairaut anti-invariant submersion from nearly Kaehler manifold
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Abstract. In the present paper, we investigate geometric properties of Clairaut anti-invariant submersions whose total space is a nearly Kaehler manifold. We obtain condition for Clairaut anti-invariant submersion to be a totally geodesic map and also study Clairaut anti-invariant submersions with totally umbilical fibers. In the last, we introduce illustrative example.

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1 Introduction

Riemannian submersion between two Riemannian manifolds was first introduced by O’Neill [21] and Gray [14]. After that Watson [35] introduced almost Hermitian submersions. Later, the notion of anti-invariant submersions and Lagrangian submersion from almost Hermitian manifolds onto Riemannian manifolds were introduced by Sahin [26] and studied by Taşan [30, 32], Gündüzalp [16], Beri et al. [9], Ali and Fatima [3], in which the fibers of submersion are anti-invariant with respect to the almost complex structure of total manifold. After that several new types of Riemannian submersions were defined and studied such as semi-invariant submersion [23, 27], slant submersion [13, 28], generic submersion [4, 1, 11, 25], hemi-slant submersion [31], semi-slant submersion [24], pointwise slant submersion [6, 12, 19] and conformal semi-slant submersion [2]. Also, these kinds of submersions were considered in different kinds of structures such as nearly Kaehler, Kaehler, almost product, para-contact, Sasakian, Kenmotsu, cosymplectic and etc. In book [29], we find the recent developments in this field.

In the theory of surfaces, Clairaut’s theorem states that for any geodesic $\alpha$ on a surface of revolution $S$, the function $r \sin \theta$ is constant along $\alpha$, where $r$ is the distant from a point on the surface to the rotation axis and $\theta$ is the angle between $\alpha$ and the meridian through $\alpha$. Bishop [10] introduced the idea of Riemannian submersions and gave a necessary and sufficient conditions for a Riemannian submersion to be Clairaut. Allison [5] considered Clairaut semi-Riemannian submersions and showed that such submersions have interesting applications in the static space-times.

In [33], Tastan and Gerdan gave new Clairaut conditions for anti-invariant submersions whose total manifolds are Sasakian and Kenmotsu and got many interesting results. In [34], Tastan and Aydin studied Clairaut anti-invariant submersions whose total manifolds are cosymplectic. Gündüzalp [17] introduced Clairaut anti-invariant submersions from a paracosymplectic manifold and gave characterization theorems. In [20], Lee et al. studied Clairaut anti-invariant submersions whose total manifolds are Kaehler.

The geometrically interesting class of almost Hermitian manifolds is of nearly Kaehler manifolds, which is one of the sixteen classes of almost Hermitian manifolds and also obtained by Gray and Hervella in their remarkable paper [15]. The geometrical meaning of
nearly Kaehler condition is that the geodesics on the manifolds are holomorphically planar curves. Gray in [14] studied nearly Kaehler manifolds broadly and gave example of a non-Kaehlerian nearly Kaehler manifold, which is 6-dimensional sphere.

Motivated by this, we study Clairaut anti-invariant submersions from nearly Kaehler manifolds onto Riemannian manifolds. We also obtain conditions for Clairaut Riemannian submersion to be totally geodesic map. We investigate conditions for the Clairaut anti-invariant submersions to be a totally umbilical map. Also, we provide some examples.

2 Preliminaries

An almost complex structure on a smooth manifold $M$ is a smooth tensor field $\phi$ of type $(1, 1)$ such that $\phi^2 = -I$. A smooth manifold equipped with such an almost complex structure is called an almost complex manifold. An almost complex manifold $(M, \phi)$ endowed with a chosen Riemannian metric $g$ satisfying

$$g(\phi X, \phi Y) = g(X, Y)$$

for all $X, Y \in TM$, is called an almost Hermitian manifold.

An almost Hermitian manifold $M$ is called a nearly Kaehler manifold [14] if

$$(\nabla_X \phi) Y + (\nabla_Y \phi) X = 0$$

for all $X, Y \in TM$. If $(\nabla_X \phi) Y = 0$ for all $X, Y \in TM$, then $M$ is known as Kaehler manifold. Every Kaehler manifold is nearly Kaehler but converse need not be true.

**Definition 2.1** [21, 22] Let $(M, g_m)$ and $(N, g_n)$ be Riemannian manifolds, where $\dim(M) = m$, $\dim(N) = n$ and $m > n$. A Riemannian submersion $\pi : M \to N$ is a map of $M$ onto $N$ satisfying the following axioms:

(i) $\pi$ has maximal rank.

(ii) The differential $\pi_*$ preserves the lengths of horizontal vectors.

For each $q \in N$, $\pi^{-1}(q)$ is an $(m - n)$-dimensional submanifold of $M$. The submanifolds $\pi^{-1}(q)$, $q \in N$, are called fibers. A vector field on $M$ is called vertical if it is always tangent to fibers. A vector field on $M$ is called horizontal if it is always orthogonal to fibers. A vector field $X$ on $M$ is called basic if $X$ is horizontal and $\pi$-related to a vector field $X'$ on $N$, that is, $\pi_* X_p = X'_{\pi_*(p)}$ for all $p \in M$. We denote the projection morphisms on the distributions $\ker \pi_*$ and $(\ker \pi_*)^\perp$ by $\mathcal{V}$ and $\mathcal{H}$, respectively. The sections of $\mathcal{V}$ and $\mathcal{H}$ are called the vertical vector fields and horizontal vector fields, respectively. So

$$\mathcal{V}_p = T_p (\pi^{-1}(q)), \quad \mathcal{H}_p = T_p (\pi^{-1}(q))^\perp.$$

The second fundamental tensors of all fibers $\pi^{-1}(q)$, $q \in N$ gives rise to tensor field $T$ and $A$ in $M$ defined by O’Neill [21] for arbitrary vector field $E$ and $F$, which is

$$T_E F = \mathcal{H} \nabla_{\mathcal{V}_E} \mathcal{V}_F + \mathcal{V} \nabla_{\mathcal{V}_E} \mathcal{H}_F, \quad (2.3)$$
\[ A_E F = \mathcal{H}\nabla^M_{\hat{H}E} V F + V \nabla^M_{\hat{H}E} \mathcal{H} F, \]  
where $\mathcal{V}$ and $\mathcal{H}$ are the vertical and horizontal projections.

On the other hand, from equations (2.3) and (2.4), we have

\[ \nabla_V W = T_V W + \hat{\nabla}_V W, \]  
(2.5)\n
\[ \nabla_V X = \mathcal{H}\nabla_V X + T_V X, \]  
(2.6)\n
\[ \nabla_X V = A_X V + \mathcal{V}\nabla_X V, \]  
(2.7)\n
\[ \nabla_X Y = \mathcal{H}\nabla_X Y + A_X Y, \]  
(2.8)\n
for all $V, W \in \Gamma(\ker \pi_\ast)$ and $X, Y \in \Gamma(\ker \pi_\ast)\perp$, where $\mathcal{V}\nabla_V W = \hat{\nabla}_V W$. If $X$ is basic, then $A_X V = \mathcal{H}\nabla_V X$.

It is easily seen that for $p \in M$, $U \in \mathcal{V}_p$ and $X \in \mathcal{H}_p$ the linear operators

\[ T_U, A_X : T_p M \to T_p M \]

are skew-symmetric, that is,

\[ g(A_X E, F) = -g(E, A_X F) \quad \text{and} \quad g(T_U E, F) = -g(E, T_U F), \]  
(2.9)\n
for all $E, F \in T_p M$. We also see that the restriction of $T$ to the vertical distribution $T|_{\ker \pi_\ast \times \ker \pi_\ast}$ is exactly the second fundamental form of the fibres of $\pi$. Since $T_U$ is skew-symmetric, therefore $\pi$ has totally geodesic fibres if and only if $T \equiv 0$.

Let $\pi : (M, g_m) \to (N, g_n)$ be a smooth map between Riemannian manifolds. Then the differential $\pi_\ast$ of $\pi$ can be observed a section of the bundle $\text{Hom}(TM, \pi^{-1}TN) \to M$, where $\pi^{-1}TN$ is the bundle which has fibres $(\pi^{-1}TN)_x = T_{f(x)}N$ has a connection $\nabla$ induced from the Riemannian connection $\nabla^M$ and the pullback connection. Then the second fundamental form of $\pi$ is given by

\[ (\nabla^{\pi_\ast})(E, F) = \nabla^N_{\pi_\ast F} E - \pi_\ast (\nabla^M_E F), \quad \text{for all } E, F \in \Gamma(TM), \]  
(2.10)\n
where $\nabla^N$ is the pullback connection ([7, 8]). We also know that $\pi$ is said to be totally geodesic map [7] if $(\nabla^{\pi_\ast})(E, F) = 0$, for all $E, F \in \Gamma(TM)$.

Let $\pi$ be an anti-invariantquarians Riemannian submersion from nearly Kaehler manifold $(M, \varphi, g_m)$ onto Riemannian manifolds $(N, g_n)$. For any arbitrary tangent vector fields $U$ and $V$ on $M$, we set

\[ (\nabla_U \varphi)V = P_U V + Q_U V \]  
(2.11)\n
where $P_U V, Q_U V$ denote the horizontal and vertical part of $(\nabla_U \varphi)V$, respectively. Clearly, if $M$ is a Kaehler manifold then $P = Q = 0$.

If $M$ is a nearly Kaehler manifold then $P$ and $Q$ satisfy

\[ P_U V = -P_V U, \quad Q_U V = -Q_V U. \]  
(2.12)\n
Consider

\[ (\ker \pi_\ast)\perp = \varphi \ker \pi_\ast \oplus \mu, \]

where $\mu$ is the complementary distribution to $\varphi \ker \pi_\ast$ in $(\ker \pi_\ast)\perp$ and $\varphi \mu \subset \mu$.

For $X \in \Gamma((\ker \pi_\ast)\perp)$, we have

\[ \varphi X = \alpha X + \beta X, \]  
(2.13)\n
where $\alpha X \in \Gamma((\ker \pi_\ast)$ and $\beta X \in \Gamma(\mu)$. If $\mu = 0$, then an anti-invariant submersion is known as Lagrangian submersion.
Definition 2.2 [18] Let \((M, \varphi, g)\) be an almost Hermitian manifold and \(N\) be a Riemannian manifold with Riemannian metric \(g_n\). Suppose that there exists a Riemannian submersion \(\pi: M \to N\), such that the vertical distribution \(\ker \pi_*\) is anti-invariant with respect to \(\varphi\), i.e., \(\varphi \ker \pi_* \subseteq \ker \pi_+\). Then, the Riemannian submersion \(\pi\) is called an anti-invariant Riemannian submersion. We will briefly call such submersions as anti-invariant submersions.

Let \(S\) be a revolution surface in \(\mathbb{R}^3\) with rotation axis \(L\). For any \(p \in S\), we denote by \(r(p)\) the distance from \(p\) to \(L\). Given a geodesic \(\alpha: J \subset \mathbb{R} \to S\) on \(S\), let \(\theta(t)\) be the angle between \(\dot{\alpha}(t)\) and the meridian curve through \(\alpha(t), t \in I\). A well-known Clairaut’s theorem says that for any geodesic on \(S\), the product \(r \sin \theta\) is constant along \(\alpha\), i.e., it is independent of \(t\). In the theory of Riemannian submersions, Bishop [10] introduces the notion of Clairaut submersion in the following way.

Definition 2.3 [10] A Riemannian submersion \(\pi: (M, g) \to (N, g_n)\) is called a Clairaut submersion if there exists a positive function \(r\) on \(M\), such that, for any geodesic \(\alpha\) on \(M\), the function \((r \circ \alpha) \sin \theta\) is constant, where, for any \(t, \theta(t)\) is the angle between \(\dot{\alpha}(t)\) and the horizontal space at \(\alpha(t)\).

He also gave the following necessary and sufficient condition for a Riemannian submersion to be a Clairaut submersion:

Theorem 2.4 [10] Let \(\pi: (M, g) \to (N, g_n)\) be a Riemannian submersion with connected fibers. Then, \(\pi\) is a Clairaut submersion with \(r = e^f\) if and only if each fiber is totally umbilical and has the mean curvature vector field \(H = -\text{grad} f\), where \(\text{grad} f\) is the gradient of the function \(f\) with respect to \(g\).

3 Anti-invariant Clairaut Submersions from nearly Kaehler Manifolds

In this section, we give new Clairaut conditions for anti-invariant submersions from nearly Kaehler manifolds after giving some auxiliary results.

Theorem 3.1 Let \(\pi\) be an anti-invariant submersion from a nearly Kaehler manifold \((M, \varphi, g)\) onto a Riemannian manifold \((N, g_n)\). If \(h: J \subset \mathbb{R} \to M\) is a regular curve and \(U(s)\) and \(X(s)\) are the vertical and horizontal parts of the tangent vector field \(\dot{h}(s) = W\) of \(h(s)\), respectively, then \(h\) is a geodesic if and only if along \(h\)

\[
A_X \varphi U + A_X \beta X + T_U \beta X + \nabla_X \alpha X + T_U \varphi U + \tilde{\nabla}_U \alpha X = 0, \quad (3.1)
\]

\[
\mathcal{H} (\nabla_{\dot{h}} \varphi U + \nabla_{\dot{h}} \beta X) + A_X \alpha X + T_U \alpha X = 0. \quad (3.2)
\]

Proof. Let \(\pi\) be an anti-invariant submersion from a nearly Kaehler manifold \((M, \varphi, g)\) onto a Riemannian manifold \((N, g_n)\). Since \(\varphi^2 h = -\dot{h}\). Taking the covariant derivative of this and using (2.2), we have

\[
(\nabla_{\dot{h}} \varphi) \varphi \dot{h} + \varphi \left( \nabla_{\dot{h}} \varphi \dot{h} \right) = -\nabla_{\dot{h}} \dot{h}. \quad (3.3)
\]
Since $U(s)$ and $X(s)$ are the vertical and horizontal parts of the tangent vector field $\dot{h}(s) = W$ of $h(s)$, that is, $h = U + X$. So (3.3) becomes

$$-\nabla_h \dot{h} = \varphi (\nabla_{U+X} \varphi (U+X)) + P_h \dot{\varphi} h + Q_h \varphi \dot{h}$$

$$= \varphi (\nabla_U \varphi U + \nabla_X \varphi U + \nabla_U \varphi X + \nabla_X \varphi X) + P_h \varphi \dot{h} + Q_h \varphi \dot{h}$$

$$= \varphi (\nabla_U \varphi U + \nabla_X \varphi U + \nabla_U (\alpha X + \beta X) + \nabla_X (\alpha X + \beta X))$$

$$+ P_h \varphi \dot{h} + Q_h \varphi \dot{h}. \quad (3.4)$$

Using (2.5)-(2.8) in (3.4), we get

$$-\nabla_h \dot{h} = \varphi (H (\nabla_h \varphi U + \nabla_h \beta X) + A_X \alpha X + A_X \beta X + A_X \varphi U$$

$$+ T_U \varphi X + T_U \alpha X + \nabla_X \alpha X + T_U \varphi U + \nabla_X \alpha X) + P_h \varphi \dot{h} + Q_h \varphi \dot{h}. \quad (3.5)$$

Since $\varphi^2 X = -X$, on differentiation, we have

$$\varphi (\nabla_Y \varphi X) + (\nabla_Y \varphi) \varphi X = -\nabla_X Y,$$

$$\varphi^2 (\nabla_Y X) + \varphi (\nabla_Y \varphi) X + (\nabla_Y \varphi) \varphi X = -\nabla_X Y,$$

using (2.11) in above, we obtain

$$\varphi (P_Y X + Q_Y X) = -P_Y \varphi X - Q_Y \varphi X. \quad (3.6)$$

By (3.6), we have

$$\varphi (P_h \varphi \dot{h} + Q_h \varphi \dot{h}) = P_h \dot{h} + Q_h \dot{h},$$

since $P$ and $Q$ are antisymmetric, so

$$\varphi (P_h \varphi \dot{h} + Q_h \varphi \dot{h}) = 0. \quad (3.7)$$

Using (3.7) and equating the vertical and horizontal part of (3.5), we obtain

$$V \varphi \nabla_h \dot{h} = A_X \varphi U + A_X \beta X + T_U \beta X + \nabla_X \alpha X + T_U \varphi U + \nabla_U \alpha X,$$

$$H \varphi \nabla_h \dot{h} = H (\nabla_h \varphi U + \nabla_h \beta X) + A_X \alpha X + T_U \alpha X.$$

By using above equations we can say that $h$ is geodesic if and only if (3.1) and (3.2) hold.

**Theorem 3.2** Let $\pi$ be an anti-invariant submersion from a nearly Kaehler manifold $(M, \varphi, g)$ onto a Riemannian manifold $(N, g_n)$. Also, let $h : J \subset \mathbb{R} \to M$ be a regular curve and $U(s)$ and $X(s)$ are the vertical and horizontal parts of the tangent vector field $h(s) = W$ of $h(s)$. Then $\pi$ is a Clairaut submersion with $r = e^f$ if and only if along $h$

$$g(\text{grad} f, X)g(U, U) = g(H \nabla_h \beta X + A_X \alpha X + T_U \alpha X + P_{h(s)} U, \varphi U).$$

**Proof.** Let $h : J \subset \mathbb{R} \to \mathbb{M}$ be a geodesic on $M$ and $\ell = \|\dot{h}(s)\|^2$. Let $\theta(s)$ be the angle between $\dot{h}(s)$ and the horizontal space at $h(s)$. Then

$$g(X(s), X(s)) = \ell \cos^2 \theta(s), \quad (3.8)$$
\[ g(U(s), U(s)) = ℓ \sin^2 \theta(s). \]  
\[ (3.9) \]

Differentiating (3.9), we get
\[ 2g(∇_{h(s)}U(s), U(s)) = 2ℓ \sin θ(s) \cos θ(s) \frac{dθ(s)}{ds}. \]  
\[ (3.10) \]

Using (2.1) in (3.10), we have
\[ g(\mathcal{H}∇_{h(s)}ϕU(s), ϕU(s)) - g(P_{h(s)}U + Q_{h(s)}U, ϕU(s)) = ℓ \sin θ(s) \cos θ(s) \frac{dθ(s)}{ds}. \]

Along the curve \( h \), using Theorem 3.1, we obtain
\[ -g(\mathcal{H}∇_{h(s)}βX + A_XαX + T_UαX + P_{h(s)}U, ϕU(s)) = ℓ \sin θ(s) \cos θ(s) \frac{dθ(s)}{ds}. \]

Now, \( π \) is a Clairaut submersion with \( r = e^f \) if and only if \( \frac{df}{ds}(e^f \sin θ) = 0 \). Therefore
\[ e^f \left( \frac{df}{ds} \sin θ + \cos θ \frac{dθ}{ds} \right) = 0, \]
\[ e^f \left( \frac{df}{ds} ℓ \sin^2 θ + ℓ \sin θ \cos θ \frac{dθ}{ds} \right) = 0. \]

So, we obtain
\[ \frac{df}{ds}(h(s))g(U(s), U(s)) = g(\mathcal{H}∇_{h(s)}βX + A_XαX + T_UαX + P_{h(s)}U, ϕU(s)), \]  
\[ (3.11) \]

Since \( \frac{df}{ds}(h(s)) = g(\text{grad}f, h(s)) = g(\text{grad}f, X) \). Therefore by using (3.11), we get result.

**Theorem 3.3** Let \( π \) be an Clairaut anti-invariant submersion from a nearly Kaehler manifold \((M, ϕ, g)\) onto a Riemannian manifold \((N, g_n)\) with \( r = e^f \). Then
\[ A_ϕWϕX + Q_ϕϕX = X(f)W \]
for \( X ∈ (\ker π^∗)⊥, W ∈ \ker π^∗ \) and \( ϕW \) is basic.

**Proof.** Let \( π \) be an anti-invariant submersion from a nearly Kaehler manifold \((M, ϕ, g)\) onto a Riemannian manifold \((N, g_n)\) with \( r = e^f \). We know that any fiber of Riemannian submersion \( π \) is totally umbilical if and only if
\[ T_VW = g(V, W)H, \]  
\[ (3.12) \]
for all \( V, W ∈ \Gamma(\ker π^∗) \), where \( H \) denotes the mean curvature vector field of any fiber in \( M \). By using Theorem 2.4 and (??), we have
\[ T_VW = -g(V, W)\text{grad}f. \]  
\[ (3.13) \]
Let $X \in \mu$ and $V, W \in \Gamma(\ker \pi)$, then by using (2.1) and (2.2), we have
\[ g(\nabla_V \varphi W, \varphi X) = g(\varphi \nabla_V W + (\nabla_V \varphi)W, \varphi X) = g(\nabla_V W, X) + g(P_V W + Q_V W, \varphi X). \] (3.14)
By using (2.1), we have
\[ g(\varphi Y, Z) = -g(Y, \varphi Z), \]
taking covariant derivative of above, we get
\[ g((\nabla_X \varphi) Y, Z) = -g(Y, (\nabla_X \varphi) Z), \]
using (2.11), we get
\[ g(P_X Y + Q_X Y, Z) = -g(Y, P_X Z + Q_X Z) = g(Y, P_Z X + Q_Z X). \] (3.15)
Using (3.15), we have
\[ g(P_W \varphi X + Q_W \varphi X, V) = g(\varphi X, P_V W + Q_V W) \] (3.16)
Using (2.5), (3.13), (3.16) in (3.14), we have
\[ g(\nabla_V \varphi W, \varphi X) = -g(V, W) (\text{grad} f, X) + g(V, Q_W \varphi X). \]
Since $\varphi W$ is basic, so $H \nabla_V \varphi W = A_{\varphi W} V$, therefore we have
\[ g(A_{\varphi W} V, \varphi X) = -g(V, W) (\text{grad} f, X) + g(V, Q_W \varphi X), \]
\[ g(V, A_{\varphi W} \varphi X) + g(V, Q_W \varphi X) = g(V, W) (\text{grad} f, X) \] (3.17)
because $A$ is anti-symmetric. By using (3.17), we get result.

**Theorem 3.4** Let $\pi$ be a Clairaut anti-invariant submersion from a nearly Kaeahler manifold $(M, \varphi, g)$ onto a Riemannian manifold $(N, g_n)$ with $r = e^f$ and $\text{grad} f \in \varphi \ker \pi_*$. Then either $f$ is constant on $\varphi \ker \pi_*$ or the fibres of $\pi$ are 1-dimensional.

**Proof.** Using (2.5) and (3.13), we have
\[ g(\nabla_V W, \varphi U) = -g(V, W)g(\text{grad} f, \varphi U), \]
where $U, V, W \in \Gamma(\ker \pi_*)$. Since $g(W, \varphi U) = 0$, therefore we have
\[ g(W, \nabla_V \varphi U) = g(V, W)g(\text{grad} f, \varphi U). \] (3.18)
By use of (2.1) and (2.11) in (3.18), we get
\[ g(W, Q_V U) - g(\varphi W, \nabla_V U) = g(V, W)g(\text{grad} f, \varphi U). \]
By using (2.5), we obtain
\[ g(W, Q_V U) - g(\varphi W, T_V U) = g(V, W)g(\text{grad} f, \varphi U). \]
Now, using (3.13), we get
\[ g(W, Q_V U) + g(V, U)g(\text{grad}f, \varphi W) = g(V, W)g(\text{grad}f, \varphi U) \] (3.19)

Take \( V = U \) in (3.19), we have
\[ g(V, V)g(\text{grad}f, \varphi W) = g(V, W)g(\text{grad}f, \varphi V). \] (3.20)

Take \( V = U \) and interchange \( V \) with \( W \) in (3.19), we have
\[ g(W, W)g(\text{grad}f, \varphi V) = g(V, W)g(\text{grad}f, \varphi W). \] (3.21)

By (3.20) and (3.21), we have
\[ g^2(V, W)g(\text{grad}f, \varphi V) = g(V, V)g(W, W)g(\text{grad}f, \varphi V). \]

Therefore either \( f \) is constant on \( \varphi \ker \pi_* \) or \( V = aW \), where \( a \) is constant (by using Schwarz’s Inequality for equality case).

**Corollary 3.5** Let \( \pi \) be a Clairaut anti-invariant submersion from a nearly Kaehler manifold \((M, \varphi, g)\) onto a Riemannian manifold \((N, g_n)\) with \( r = e^f \) and \( \text{grad}f \in \varphi \ker \pi_* \). If \( \dim(\ker \pi_*) > 1 \), then the fibres of \( \pi \) are totally geodesic if and only if \( A_{\varphi W} \varphi X + Q_W \varphi X = 0 \)
for \( W \in \ker \pi_* \) such that \( \varphi W \) is basic and \( X \in \mu \).

**Proof.** By Theorem 3.3 and Theorem 3.4, we get the result.

**Corollary 3.6** Let \( \pi \) be an Clairaut Lagrangian submersion from a nearly Kaehler manifold \((M, \varphi, g)\) onto a Riemannian manifold \((N, g_n)\) with \( r = e^f \). Then either the fibres of \( \pi \) are 1-dimensional or they are totally geodesic.

**Proof.** Let \( \pi \) be an Clairaut Lagrangian submersion from a Kaehler manifold \((M, \varphi, g)\) onto a Riemannian manifold \((N, g_n)\) with \( r = e^f \), Then \( \mu = \{0\} \). So \( A_{\varphi W} \varphi X + Q_W \varphi X = 0 \) always.

Lastly, we give some examples for Clairaut anti-invariant submersions from a nearly Kaehler manifold.

**Example.** Let \((\mathbb{R}^4, \varphi, g)\) be a nearly Kaehler manifold endowed with Euclidean metric \( g \) on \( \mathbb{R}^4 \) given by
\[ g = \sum_{i=1}^{4} dx_i^2 \]

and canonical complex structure
\[ \varphi(x_j) = \begin{cases} -x_{j+1} & j = 1, 3 \\ x_{j-1} & j = 2, 4 \end{cases} \]

The \( \varphi \)-basis is \( \{e_i = \frac{\partial}{\partial x_i} \mid i = 1, 2, 3, 4\} \). Let \((\mathbb{R}^3, g_1)\) be a Riemannian manifold endowed with metric \( g = \sum_{i=1}^{3} dy_i^2 \).
(i) Consider a map $\pi : (\mathbb{R}^4, \varphi, g) \to (\mathbb{R}^3, g_1)$ defined by
\[
\pi(x_1, x_2, x_3, x_4) = \left( \frac{x_1 + x_2}{\sqrt{2}}, x_3, x_4 \right).
\]
Then by direct calculations, we have
\[
\ker \pi_* = \text{span} \left\{ X_1 = \left( \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} \right) \right\},
\]
\[
(\ker \pi_*)^\perp = \text{span} \left\{ X_2 = \left( \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} \right), X_3 = \frac{\partial}{\partial x_3}, X_4 = \frac{\partial}{\partial x_4} \right\}
\]
and $\varphi X_1 = -X_2$, therefore $\varphi (\ker \pi_*) \subset (\ker \pi_*)^\perp$. Thus, we can say that $\pi$ is an anti-invariant Riemannian submersion. Since the fibers of $\pi$ are 1-dimensional, therefore fibers are totally umbilical.

Consider the Koszul formula for Levi-Civita connection $\nabla$ for $\mathbb{R}^4$
\[
2g(\nabla_X Y, Z) = X g(Y, Z) + Y g(Z, X) - Z g(X, Y) - g([Y, Z], X) - g([X, Z], Y) + g([X, Y], Z)
\]
for all $X, Y, Z \in \mathbb{R}^4$. By simple calculations, we obtain
\[
\nabla_{e_i} e_j = 0 \quad \text{for all } i, j = 1, 2, 3, 4.
\]
Hence $T_X Y = T_Y X = T_X X = 0$ for all $X, Y \in \Gamma(\ker \pi_*)$. Therefore fibers of $\pi$ are totally geodesic. Thus $\pi$ is Clairaut trivially.

(ii) Consider a map $\pi : (\mathbb{R}^4, \varphi, g) \to (\mathbb{R}^3, g_1)$ defined by
\[
\pi(x_1, x_2, x_3, x_4) = \left( \sqrt{x_1^2 + x_2^2}, x_3, x_4 \right).
\]
Then by direct calculations, we have
\[
\ker \pi_* = \text{span} \left\{ X_1 = \left( \frac{x_2}{\sqrt{x_1^2 + x_2^2}} \frac{\partial}{\partial x_1} - \frac{x_1}{\sqrt{x_1^2 + x_2^2}} \frac{\partial}{\partial x_2} \right) \right\},
\]
\[
(\ker \pi_*)^\perp = \text{span} \left\{ X_2 = \left( \frac{x_2}{\sqrt{x_1^2 + x_2^2}} \frac{\partial}{\partial x_1} + \frac{x_1}{\sqrt{x_1^2 + x_2^2}} \frac{\partial}{\partial x_2} \right), X_3 = \frac{\partial}{\partial x_3}, X_4 = \frac{\partial}{\partial x_4} \right\}
\]
and $\varphi X_1 = -X_2$, therefore $\varphi (\ker \pi_*) \subset (\ker \pi_*)^\perp$. Thus, we can say that $\pi$ is an anti-invariant Riemannian submersion. Since the fibers of $\pi$ are 1-dimensional, therefore fibers are totally umbilical. By using Koszul formula, we obtain
\[
\nabla_{e_i} e_j = 0 \quad \text{for all } i, j = 1, 2, 3, 4.
\]
Hence
\[
T_X X_1 = - \left( \frac{x_2}{\sqrt{x_1^2 + x_2^2}} \frac{\partial}{\partial x_1} + \frac{x_1}{\sqrt{x_1^2 + x_2^2}} \frac{\partial}{\partial x_2} \right).
\]
Now, for the function $f = \ln(\sqrt{x_1^2 + x_2^2})$ on $(\mathbb{R}^4, \varphi, g)$, the gradient of $f$ with respect to $g$ is given by
\[
\nabla f = \sum_{i,j=1}^{4} g^{ij} \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_j} = \frac{x_2}{\sqrt{x_1^2 + x_2^2}} \frac{\partial}{\partial x_1} + \frac{x_1}{\sqrt{x_1^2 + x_2^2}} \frac{\partial}{\partial x_2}.
\]
Therefore for $X_1 \in \Gamma(\ker \pi)$, $T_{X_1}X_1 = -\nabla f$. Since $\|X_1\| = 1$, so $T_{X_1}X_1 = -\|X_1\|^2 \nabla f$.

By using theorem 2.4, we can say that $\pi$ is a proper Clairaut anti-invariant submersion with $r = e^f$ for $f = \ln(\sqrt{x_1^2 + x_2^2})$.

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