Geodesic motion around traversable wormholes supported by a massless conformally-coupled scalar field

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Abstract

We consider a traversable wormhole solution of Einstein’s gravity conformally coupled to a massless scalar field, a solution derived by Barcelo and Visser based on the JNWW spacetime. We study the geodesic motion of time-, light- and space-like particles in this spacetime. We solve the equations of motion analytically in terms of the Weierstraß functions and discuss all possible orbit types and their parameter dependence. Interestingly, bound orbits occur for time-like geodesics only in one of the two worlds. Moreover, under no conditions there exist time-like two world bound orbits.

1 Introduction

Since the concept of wormholes was introduced by Einstein and Rosen in 1935 [1], wormholes have been discussed in a widely spread manner in the literature. Of particular interest are Lorentzian traversable wormholes (see e.g. [2, 3, 4]). Their construction requires the violation of the Null Energy Condition (NEC), which can, for instance, be achieved by the presence of exotic matter, yielding the Ellis wormholes of General Relativity [5, 6, 7, 8, 9, 10, 11].

Staying with General Relativity, Barcelo and Visser [12], however, showed, that the NEC can also be violated and traversable wormhole solutions be found, when General Relativity is conformally coupled to a massless scalar field. Indeed, by employing the new improved energy-momentum tensor $T_{\mu\nu}$ of Callan et al. [13], they obtained an interesting set of wormhole solutions. As pointed out there [13, 12], this new improved energy-momentum tensor has the same set of Poincare generators as a minimally coupled scalar field, while its matrix elements are finite in every order of renormalized perturbation theory, rendering the resulting theory rather attractive.

To obtain the set of static spherically symmetric wormhole solutions of this theory, Barcelo and Visser started from the well known solution found by Janis, Newman and Winicour and independently by Wyman [14, 15, 16], the JNWW solution, looking for solutions of the new set of field equations, which are conformally related to the known JNWW solution. This way they obtained a differential equation for the conformal factor, to be solved together with the scalar field equation. The resulting set of solutions then represented a generalization of those of Froyland [17], and Agnese and La Camera [18].

Here we focus on the traversable wormhole solutions which form an intriguing subset of the full set of solutions found by Barcelo and Visser [12]. In particular, we study the time-, light- and spacelike geodesics in these spacetimes, and discuss all possible types of orbits, using effective potentials and parametric plots. Moreover, we present analytical solutions of the equations of motion, which are of elliptical type and can be expressed through the Weierstraß $\wp$, $\sigma$- and $\zeta$-functions.

The paper is organized as follows. In section 2 we give a brief description of the wormhole solutions and their properties. In section 3 we provide the general set of the geodesic equations. We derive the analytical solutions in section 4, where we also discuss the effective potentials and classify the possible orbit types. The orbits including the embeddings of their corresponding spacetimes are then illustrated in section 5, while section 6 gives our conclusions.
2 Traversable wormholes supported by a massless conformally-coupled scalar field

Let us start by recalling the JNWW solution in the compact form found by Agnese and LaCamera [18]

\[ ds^2_m = -\left(1 - \frac{2m}{r}\right)^{\cos \chi} dt^2 + \left(1 - \frac{2m}{r}\right)^{-\cos \chi} dr^2 + \left(1 - \frac{2m}{r}\right)^{1-\cos \chi} r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \tag{1} \]

with the scalar field \( \phi_m \), and parameters \( \chi \) and \( \eta \).

Barcelo and Visser [12] reconsidered the JNWW solution in this form when looking for static spherically symmetric solutions of the Einstein equations

\[ G_{\mu\nu} = \kappa T_{\mu\nu} \],

\[ \kappa = (8\pi G_N)^{-1} \] and \( G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \), with the new improved energy momentum tensor \( T_{\mu\nu} \) from [13],

\[ T_{\mu\nu} = \nabla_\mu \phi_c \nabla_\nu \phi_c - \frac{1}{2} g_{\mu\nu} (\nabla \phi_c)^2 + \frac{1}{6} \left[ G_{\mu\nu} \phi_c^2 - 2 \nabla_\mu (\phi_c \nabla_\nu \phi_c) + 2 g_{\mu\nu} \nabla^\lambda (\phi_c \nabla_\lambda \phi_c) \right] \],

\[ (\Box - \frac{1}{6} R) \phi_c = 0 \] (5)

with \( \phi_c \) the conformally-coupled scalar field. The new improved energy-momentum tensor is traceless, thus also \( R = 0 \), yielding

\[ R_{\mu\nu} = (\kappa - \frac{1}{6} \phi_c^2)^{-1} \left( \frac{2}{3} \nabla_\mu \phi_c \nabla_\nu \phi_c - \frac{1}{2} g_{\mu\nu} (\nabla \phi_c)^2 - \frac{1}{3} \phi_c \nabla_\mu \nabla_\nu \phi_c \right), \]

\[ \Box \phi_c = 0 \] (7)

Requiring that the metric \( ds \) should be conformal to the JNWW metric \( ds_m \), i.e., \( ds = \Omega(r) ds_m \) with conformal factor \( \Omega(r) \), and should have vanishing scalar curvature, then leads to a second order differential equation for \( \Omega(r) \) with solutions [12]

\[ \Omega(r) = \alpha_+ \left(1 - \frac{2\eta}{r} \right)^{\frac{\sin \chi}{2\cos \chi}} + \alpha_- \left(1 - \frac{2\eta}{r} \right)^{-\frac{\sin \chi}{2\cos \chi}} \],

\[ \alpha_+ \] and \( \alpha_- \) are integration constants. The differential equation for the conformally coupled scalar field can then be integrated [12].

With the parameter set \( \{ \eta, \chi, \Delta \} \), where \( \Delta \) is an angle defined by

\[ \tan \frac{\Delta}{2} = \frac{\alpha_+ - \alpha_-}{\alpha_+ + \alpha_-} = \frac{\alpha_+ - 1}{\alpha_+ + 1}, \quad \alpha_+ = \frac{\alpha_+}{\alpha_-} \]

with range \( \Delta \in (-\pi, \pi] \), the whole set of solutions can be addressed. Depending on the choice of these parameters the properties of the metric can change dramatically. For example for \( \chi = 0 \) and arbitrary \( \eta \) and \( \Delta \), \( ds^2 \) yields the Schwarzschild metric.

To get traversable wormhole solutions, Barcelo and Visser found the appropriate parameter set to be \( \{ \chi = \frac{\pi}{2}, \Delta \notin \{0, \frac{\pi}{2}, \pi\} \} \). Introducing isotropic radial coordinates via \( r = \bar{r} \left(1 + \frac{2\eta}{r}\right)^2 \), the metric transforms into
\[ ds^2 = \left[ \alpha_+ \left( \frac{1 - \eta^2}{1 + \eta^2} \right) + \alpha_- \right]^2 \left[ -dt^2 + \left( 1 + \frac{\eta^2}{2r^2} \right)^4 \left[ d\tilde{r}^2 + \tilde{r}^2 (d\vartheta^2 + \sin^2 \vartheta \, d\phi^2) \right] \right], \] (10)

with the range \( \tilde{r} \in [0, \infty] \), where the radial location of the wormhole throat is given by

\[ \tilde{r}_T = \frac{\eta}{2} \sqrt{\left( \frac{\alpha_+ - 1}{\alpha_+ + 1} \right)}, \] (11)

and the corresponding conformally-coupled scalar field is

\[ \phi_c = \pm \sqrt{6\kappa} \frac{\tilde{\alpha}_+ \left( 1 - \frac{\eta^2}{2} \right) - \left( 1 + \frac{\eta^2}{2} \right)}{\tilde{\alpha}_+ \left( 1 - \frac{\eta^2}{2} \right) + \left( 1 + \frac{\eta^2}{2} \right)}, \] (12)

which is a monotonically increasing or decreasing function between the two asymptotically flat regions.

As pointed out by Barcelo and Visser, this monotonic behavior has physical consequences. In particular, the effective gravitational coupling constant

\[ G_{\text{eff}} = 8\pi \left( \kappa - \frac{1}{6} \phi_c^2 \right)^{-1} \] (13)

will be positive in one asymptotically flat region and negative in the other asymptotically flat region, and with respect to the asymptotically flat region with positive effective gravitational coupling constant, the wormhole throat will be located in the region, where the effective gravitational coupling constant has changed sign [12].

Barcelo and Visser point out that the conformal coupling of the JNWW solution to the new improved energy-momentum tensor is well defined over the whole range of \( \tilde{r} \). Nevertheless the original metric \( ds_m^2 \) changes its sign for \( \tilde{r} \in (0, \frac{\eta}{2}) \), which the conformal factor \( \Omega(\tilde{r}) \) compensates. Thus only \( \tilde{r} > \frac{\eta}{2} \) is strictly speaking conformally related to the JNWW solution [12].

Since \( \alpha_+ \) and \( \alpha_- \) appear naturally in the metric, in the following \( \tilde{\alpha}_+ \) will be used as a parameter for the wormhole solutions instead of \( \Delta \). For traversable wormholes the range of \( \tilde{\alpha}_+ \) is limited to \( \tilde{\alpha}_+ \in (0, 1) \). Negative \( \tilde{\alpha}_+ \) won’t be discussed, because in this case the wormhole spacetime is simply inverted.

### 3 The geodesic equations

Since the metric of the above traversable wormholes is static and spherically symmetric, we need to consider only the equatorial plane with \( \vartheta = \frac{\pi}{2} \) in the following. With the metric \( ds^2 \) from Eq. (10) we then get

\[ \begin{align*}
2 \mathcal{L} &= g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = \zeta \\
&= \left[ \alpha_+ \left( \frac{1 - \eta^2}{1 + \eta^2} \right) + \alpha_- \right]^2 \left[ -\dot{t}^2 + \left( 1 + \frac{\eta^2}{2\tilde{r}^2} \right)^4 \left[ \dot{\tilde{r}}^2 + \tilde{r}^2 \dot{\varphi}^2 \right] \right],
\end{align*} \] (14)

yielding time-like (\( \zeta = -1 \)), light-like (\( \zeta = 0 \)) and space-like (\( \zeta = 1 \)) geodesics.

The angular momentum \( L \) and the energy \( \epsilon \) of the test particle are conserved,

\[ L := \frac{\partial \mathcal{L}}{\partial \dot{\varphi}}, \quad -\epsilon := \frac{\partial \mathcal{L}}{\partial \dot{t}}. \] (15)

With these constants of motion the equations of motion in the equatorial plane become
\[
\left( \frac{d\bar{r}}{d\lambda} \right)^2 = \frac{\epsilon^2 + \zeta \left[ \alpha_+ \left( \frac{1}{1 + \eta} \right)^2 + \alpha_- \right]^2}{\left[ \alpha_+ \left( \frac{1}{1 + \eta} \right)^2 + \alpha_- \right]^4 (1 + \frac{\eta}{2\bar{r}})^4} - \frac{L^2}{\left[ \alpha_+ \left( \frac{1}{1 + \eta} \right)^2 + \alpha_- \right]^4 (1 + \frac{\eta}{2\bar{r}})^8 \bar{r}^2}
\]

(16)

\[
\left( \frac{d\phi}{d\lambda} \right) = \frac{L}{\left[ \alpha_+ \left( \frac{1}{1 + \eta} \right)^2 + \alpha_- \right]^2 (1 + \frac{\eta}{2\bar{r}})^4 \bar{r}^2}
\]

(17)

\[
\left( \frac{dt}{d\lambda} \right) = \frac{\epsilon}{\left[ \alpha_+ \left( \frac{1}{1 + \eta} \right)^2 + \alpha_- \right]^2}
\]

(18)

4 Solutions of the geodesic equations

Whereas the equations of motion (16), (17) and (18) cannot be solved analytically by simple means, the equations for \( \left( \frac{d\bar{r}}{d\phi} \right) \) and \( \left( \frac{d\bar{r}}{dt} \right) \)

\[
\left( \frac{d\bar{r}}{d\phi} \right)^2 = \frac{\epsilon^2 + \zeta \left[ \alpha_+ \left( \frac{1}{1 + \eta} \right)^2 + \alpha_- \right]^2}{\left[ \alpha_+ \left( \frac{1}{1 + \eta} \right)^2 + \alpha_- \right]^4 (1 + \frac{\eta}{2\bar{r}})^4} - \frac{L^2 (1 + \frac{\eta}{2\bar{r}})^8 \bar{r}^2}{\epsilon^2 (1 + \frac{\eta}{2\bar{r}})^8 \bar{r}^2}
\]

(19)

\[
\left( \frac{d\bar{r}}{dt} \right)^2 = \frac{\epsilon^2 + \zeta \left[ \alpha_+ \left( \frac{1}{1 + \eta} \right)^2 + \alpha_- \right]^2}{\epsilon^2 (1 + \frac{\eta}{2\bar{r}})^4} - \frac{L^2}{\epsilon^2 (1 + \frac{\eta}{2\bar{r}})^8 \bar{r}^2}
\]

(20)

can be solved in terms of Weierstraß functions. In subsection 4.1 we derive the solution for the \( \left( \frac{d\bar{r}}{d\phi} \right) \) motion. We classify the possible orbit types in subsection 4.2, and we derive the solution for the \( \left( \frac{d\bar{r}}{dt} \right) \) motion in subsection 4.3.

4.1 Solution for the \( \left( \frac{d\bar{r}}{d\phi} \right) \)-motion

We first substitute in Eq. (19) the radial coordinate \( \bar{r} = \eta (x - \frac{1}{2}) \), where the new radial coordinate \( x \) has a range of \( x \in \left[ \frac{1}{2}, \infty \right] \). The radial coordinate of the throat is then given by \( x_T = \frac{1}{4} (\sqrt{1 + \alpha_+ (\bar{r} + 1)} + 1) \). This yields an equation which can be written in terms of a polynomial of fourth order

\[
\left( \frac{dx}{d\phi} \right)^2 = a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0
\]

(21)

with the coefficients

\[
\begin{align*}
  a_4 &= \epsilon^2 \zeta (\bar{\alpha} + 1)^2 / \bar{L}^2, \\
  a_3 &= 2\zeta (\bar{\alpha}^2 + \bar{\alpha}) / \bar{L}^2, \\
  a_2 &= \zeta \bar{\alpha}^2 / \bar{L}^2 - 1, \\
  a_1 &= 1, \\
  a_0 &= 1/4.
\end{align*}
\]

(22)

With another substitution \( x = \pm \frac{1}{u} + x_1 \), where \( x_1 \) is a zero of \( (dx/d\phi)^2 \), we obtain a polynomial of third order

\[
\left( \frac{du}{d\phi} \right)^2 = b_3 u^3 + b_2 u^2 + b_1 u + b_0
\]

(23)
The additional substitution \( u = \frac{1}{b_3} (4y - \frac{b_2}{3}) \) then yields the Weierstraß form

\[
\left( \frac{dy}{d\phi} \right)^2 = 4y^3 - g_2 y - g_3, \tag{24}
\]

where

\[
g_2 = \frac{1}{16} \left( \frac{4}{3} b_2^2 - 4b_1 b_3 \right),
\]
\[
g_3 = \frac{1}{16} \left( \frac{1}{3} b_1 b_2 b_3 - \frac{2}{27} b_2^3 - b_0 b_3^2 \right). \tag{25}
\]

Then \( y(\phi) = \wp(\phi - c_{\text{in}}, g_2, g_3) \) solves Eq. (24), where \( c_{\text{in}} = \phi_0 + \int_{y_0}^{\infty} \frac{dz}{\sqrt[3]{4z^3 - g_2 z - g_3}} \) and \( y_0 = \pm \frac{b_3}{4(x_0 - x_1)} - \frac{b_2}{12}. \) Thus Eq. (19) has the solution

\[
x(\phi) = \pm \frac{b_3}{4\wp(\phi - c_{\text{in}}, g_2, g_3) - \frac{b_2}{3}} + x_1. \tag{26}
\]

### 4.1.1 Classification of the \((d\bar{r}/d\phi)\)-motion

In the following we will show, that only a small set of orbit types are found, when the \((d\bar{r}/d\phi)\)-motion is investigated. Those orbit types for time-, light- and space-like geodesics consist of the following possible orbits

- **Transit orbit** (TO): A geodesic coming from one asymptotic region approaches the throat of the wormhole, crosses it and reaches the other asymptotic region, thus traversing the whole wormhole spacetime.
- **Bound orbit** (BO): A geodesic moves continuously around the wormhole, remaining on the same side of the throat.
- **Escape orbit** (EO): A geodesic coming from one asymptotic region approaches the throat of the wormhole, but instead of crossing the throat it reaches a turning point and returns to the asymptotic region, where it came from.
- **Two world escape orbit** (TWEO): Similar to the EO with the difference that the geodesic crosses the throat before it reaches its turning point, from where it returns to the asymptotic region, where it came from, passing the throat on its way a second time.
- **Unstable circular orbit** (UCO): A bound orbit which moves around the wormhole on a circular path. These orbits are highly unstable due to the fact that their energy is equal to the local potential maximum.

Clearly, geodesic motion is only possible if \((d\bar{r}/d\phi)^2 \geq 0\). This leads to the following condition for the energy \(\epsilon\) of a particle for physically allowed motion,

\[
\epsilon^2 \geq \frac{L^2}{(1 + \frac{\eta}{2r})^4 r^2} - \zeta \left[ \alpha_+ \left( 1 - \frac{\eta}{1 + \frac{\eta}{2r}} \right) + \alpha_- \right]^2. \tag{27}
\]

Note, that the right hand side may be considered as an effective potential, consisting of a centrifugal part (the first term) and a gravitational part (the second term).

We now perform two coordinate transformations to simplify Eq. (27). We first introduce again the radial coordinate \( x = \frac{\bar{r}}{\eta} + \frac{\bar{r}}{2} \), as in the solution for \((d\bar{r}/d\phi)\), and then we transform to the radial coordinate \( w = \frac{1}{2x} \). This new radial coordinate has the range \( w \in [0, 1] \), and the throat is located at \( w_T = \)
\((\sqrt{[(\bar{\alpha} + 1)/(\bar{\alpha} + 1)]'} + 1)^{-1}\). We further employ the scaled quantities \(\bar{\alpha}+\) (Eq. \((9)\)), \(\bar{L} = L/(\eta \alpha)\) and \(\bar{\epsilon} = \epsilon/\alpha\) to obtain the condition

\[
\bar{\epsilon}^2 \geq 4 \bar{L}^2 w^2 (1-w)^2 - \zeta \bar{\alpha}_+ (2w - 1) - 1^2 \tag{28}
\]

Here \(P(\bar{\alpha}+, \bar{L}, w, \zeta)\) is a polynomial of fourth order, which serves as an effective potential and thus helps characterizing the possible orbits. In the two asymptotic regions with radial coordinates \(w = 0\) and \(w = 1\), the polynomial assumes the values

\[
P(\bar{\alpha}+, \bar{L}, w = 0, \zeta) := P_0(\bar{\alpha}+, \zeta) = -\zeta(\bar{\alpha}_+ + 1)^2, \tag{29}
\]

\[
P(\bar{\alpha}+, \bar{L}, w = 1, \zeta) := P_1(\bar{\alpha}+, \zeta) = -\zeta(\bar{\alpha}_+ - 1)^2. \tag{30}
\]

These limiting values are determined only by the gravitational part of the effective potential, since the centrifugal part vanishes asymptotically.

We note, that the gravitational part is monotonically decreasing (increasing) in the interval \(w \in [0, 1]\) for time-like (space-like) geodesics, while the centrifugal part is symmetric with respect to \(w = 1/2\), where it reaches its maximum. Expressed in terms of the radial coordinate \(\bar{r}\) this value corresponds to \(\bar{r} = \eta/2\). At precisely this value the conformal scalar field assumes the values \(\phi_c = \pm \sqrt{6\kappa}\), i.e., the factor \((\kappa - \frac{1}{6} \phi_c^2)^{-1}\) on the right hand side of the Einstein equations \((6)\), and thus the effective Newton constant \(G_{\text{eff}}\), diverges. We also note, that because of our choice of orientation of the wormhole \((\bar{\alpha}_+ > 0)\), the throat is always located in the rear part of the physical \(w\)-interval, i.e., \(w_T \in (0, 1)\).

To classify the possible orbits, we next need to consider the minima and maxima of the polynomial \(P\). We obtain them by calculating the derivative \(P' = \frac{\partial P}{\partial w}\) and applying Cardanos' method to solve \(P' = 0\) in order to get its zeros \(w_1\), \(w_2\) and \(w_3\). From the discriminant \(\Delta_{\text{dis}}(\bar{L}, \bar{\alpha}_+, \zeta)\) of \(P'\) we learn for which parameters the zeros of the polynomial are real. The discriminant is given by

\[
\Delta_{\text{dis}}(\bar{L}, \bar{\alpha}_+, \zeta) = \frac{1}{1728 \bar{L}^6} (-\bar{L}^6 - 6 \bar{L}^4 \bar{\alpha}_+^2 \zeta + (12 \bar{\alpha}_+^4 + 27 \bar{\alpha}_+^2) \bar{L}^2 \zeta^2 - 8 \bar{\alpha}_+^6 \zeta^3). \tag{31}
\]

When \(\Delta_{\text{dis}} < 0\), all zeros of \(P'\) are real. For fixed \(\bar{\alpha}_+\) and \(\zeta\) this is the case, when the angular momentum exceeds a critical value, \(\bar{L} > \bar{L}_{\text{crit}}\). Then the three zeros of the derivative \(P'\) are given by

\[
w_1 = \sqrt{-\frac{4}{3} \cos \left(\frac{1}{3} \arccos \left(\frac{q}{2} \sqrt{-\frac{27}{p^3}}\right)\right) + \frac{1}{2}}, \tag{32}
\]

\[
w_{2, 3} = -\sqrt{-\frac{4}{3} \cos \left(\frac{1}{3} \arccos \left(-\frac{q}{2} \sqrt{-\frac{27}{p^3}}\right) \pm \frac{\pi}{3}\right) + \frac{1}{2}}, \tag{33}
\]

with \(p = -\frac{1}{2} \frac{\bar{\alpha}_+^2 \zeta}{\bar{L}^2} - \frac{1}{4}\) and \(q = \frac{1}{3} \frac{\bar{\alpha}_+ \zeta}{\bar{L}^2}\). Let us denote the maximum and the minima of the polynomial as follows

\[
P_{\text{max}}(\bar{\alpha}_+, \bar{L}, \zeta) := P(\bar{\alpha}_+, \bar{L}, w = w_2, \zeta), \]

\[
P_{\text{min}}(\bar{\alpha}_+, \bar{L}, \zeta) := \begin{cases} P(\bar{\alpha}_+, \bar{L}, w = w_1, \zeta) & \zeta = -1, \\ P(\bar{\alpha}_+, \bar{L}, w = w_3, \zeta) & \zeta = 1. \end{cases} \]

Concerning \(w_1\) and \(w_3\) we note that only one of them is located in the allowed range of \(w\), depending on the geodesic type.

Since the effective potential is a superposition of the monotonic gravitational part and the symmetric centrifugal part, the orbit types for time- and space-like geodesics change at certain characteristic values
of the angular momentum. Besides the critical angular momentum $\tilde{L}_{\text{crit}}$, where a stationary inflection point of the polynomial $P$ arises, there are two more such characteristic values. These are the angular momentum $\tilde{L}_{\text{swap}}$, where $P_{\text{max}}(\bar{\alpha} +, \tilde{L}, \zeta)$ becomes larger than $P_{0}(\bar{\alpha} +, \zeta)$, and (only for space-like geodesics) the angular momentum $\tilde{L}_{\text{zero}}$, beyond which the polynomial $P(\bar{\alpha} +, \tilde{L}, w, \zeta)$ can assume positive values.

The dependence of these characteristic values of the angular momentum on $\bar{\alpha} +$ is shown in Figure (1).

![Diagram](image)

Figure 1: Dependence of the characteristic angular momenta $\tilde{L}_{\text{crit}}$ and $\tilde{L}_{\text{swap}}$ on the wormhole parameter $\bar{\alpha} +$ for time-like geodesics (a) and space-like geodesics (b). The latter includes $\tilde{L}_{\text{zero}}$. For time-like geodesics the gap between $\tilde{L}_{\text{crit}}$ and $\tilde{L}_{\text{swap}}$ increases monotonically with $\bar{\alpha} +$. For space-like geodesics only $\tilde{L}_{\text{zero}}$ is relevant, since the energy of particles should be a real number.

To clarify the classification of the orbit types, we illustrate the effective potential as described by the polynomial $P$ in Figure (2) for several values of the angular momentum, which allow for different orbit types. For the time-like geodesics three different cases for the angular momentum $\tilde{L}$ arise and are shown in Figure (2). For $\tilde{L} < \tilde{L}_{\text{crit}}$ the polynomial is monotonic. Here only transit and escape orbits exist. For $\tilde{L} = \tilde{L}_{\text{crit}}$ the polynomial $P$ possesses a double zero in the allowed physical range of $w$. Here unstable circular, transit and escape orbits exist. For $\tilde{L}_{\text{crit}} < \tilde{L} < \tilde{L}_{\text{swap}}$ also bound orbits are possible due to the minimum of the polynomial $P$. An unstable circular orbit can occur as well due to the existence of a maximum. Finally, for $\tilde{L}_{\text{swap}} < \tilde{L}$ the maximum $P_{\text{max}}$ becomes larger than $P_{0}$, allowing for one additional orbit type (see Figure (2d)).
Turning to the light-like orbits, we note that only the centrifugal part of the effective potential is present. Thus there is no dependence on $\tilde{\alpha}_+$. The effective potential vanishes in the asymptotic regions $w = 0$ and $w = 1$, and has at $w = 0.5$ its maximum, whose height depends only on the size of the angular momentum. Thus transit and escape orbits exist for light-like geodesics as well as unstable circular orbits, illustrated in Figure (1b) by a green line, which form a photon sphere. Unstable circular orbits can occur for space- and timelike geodesics as well, when their energy corresponds to the local maximum.

The orbits for the space-like geodesics are shown in Figure (3b). Here we note, that although Figure (1b) shows three characteristic angular momenta, only one of them is relevant, when considering only real values for the energies, i.e. $\tilde{\epsilon}^2 > 0$. For $\tilde{L} < L_{\text{zero}}$ only transit orbits are possible because the polynomial $P$ is negative in the full coordinate range $w \in [0, 1]$. Only for $\tilde{L} > L_{\text{zero}}$ a physically forbidden zone arises around $w_2$, which leads in addition to escape orbits.
With the above definitions of the polynomial and its extremal points, as well as the discussion on the characteristic angular momenta for space- and time-like geodesics, the orbit types can now be defined. We will apply this definition in the parameter plots in Figure (4). In Table (1) a summary of the orbits types is presented.

- **Type I (TO):**
  \[
  \varepsilon^2 > P(\tilde{\alpha}_+, \tilde{L}, w, \zeta), \quad \forall w \in [0, 1].
  \]  
  (34)

  Type I orbits consist only of transit orbits (TO). These arise for time-like, light-like and space-like geodesics.

- **Type II (EO, TWEO):**
  \[
  P_1(\tilde{\alpha}_+, \zeta) < \varepsilon^2 < \min(P_{\text{max}}(\tilde{\alpha}_+, \tilde{L}, \zeta), P_0(\tilde{\alpha}_+, \zeta)).
  \]  
  (35)

  Type II orbits contain both escape orbits (EO) and two world escape orbits (TWEO). These orbits never reach the asymptotic region \( w = 0 \). They arise only for time-like geodesics.

- **Type III (BO, EO, TWEO):**
  \[
  \tilde{L}_{\text{crit}} < \tilde{L} \leq \tilde{L}_{\text{swap}} : \quad P_{\text{min}}(\tilde{\alpha}_+, \tilde{L}, \zeta) \leq \varepsilon^2 \leq P_{\text{max}}(\tilde{\alpha}_+, \tilde{L}, \zeta).
  \]  
  (36)

  Type III orbits contain also bound orbits (BO) besides the escape orbits (EO) and two world escape orbits (TWEO). These orbits never reach the asymptotic region \( w = 0 \). They arise only for time-like geodesics.

- **Type IV (EO, TWEO):**
  \[
  \tilde{L} > \tilde{L}_{\text{swap}} : \quad P_0(\tilde{\alpha}_+, \zeta) < \varepsilon^2 < P_{\text{max}}(\tilde{\alpha}_+, \tilde{L}, \zeta)
  \]  
  (37)

  Type IV orbits contain both escape orbits (EO) and two world escape orbits (TWEO). In contrast to Type II orbits there are also escape orbits (EO), which reach the asymptotic region \( w = 0 \). They arise for time-like, light-like and space-like geodesics.
• Type V (UCO, BO, EO):
\[ \tilde{L} > \tilde{L}_{\text{crit}} : \epsilon^2 = P_{\text{max}}(\tilde{\alpha}+, \tilde{L}, \zeta) \]  
(38)

In Type V orbits the particle energy corresponds to the maximum of the polynomial \( P \) for time-like geodesics. Here in addition to the orbits of Type III an unstable circular orbit (UCO) is present.

• Type VI (UCO, TWEO, EO):
\[ \epsilon^2 = P_{\text{max}}(\tilde{\alpha}+, \tilde{L}, \zeta) \]  
(39)

For time-like geodesics with \( \tilde{L} > \tilde{L}_{\text{crit}} \) and for space-like geodesics with \( \tilde{L} > \tilde{L}_{\text{zero}} \) there is in addition to the orbits present in Type IV an unstable circular orbit (UCO). For light-like geodesics this type is always present.

• Type VII (UCO, EO):
\[ \tilde{L} = \tilde{L}_{\text{crit}} : \epsilon^2 = P_{\text{max}}(\tilde{\alpha}+, \tilde{L}, \zeta) = P_{\text{min}}(\tilde{\alpha}+, \tilde{L}, \zeta) \]  
(40)

Type VII is found for \( \tilde{L} = \tilde{L}_{\text{crit}} \). In this case the zeros of \( P' \) coincide, \( w_2 = w_3 \), thus \( P_{\text{max}}(\tilde{\alpha}+, \tilde{L}, \zeta) = P_{\text{min}}(\tilde{\alpha}+, \tilde{L}, \zeta) \). When the particle energy corresponds to the potential at \( w_2 = w_3 \), an unstable circular orbit (UCO) arises in addition to the orbits of Type II.

| type | \( \zeta \) | zeros | range of \( x \) | orbit |
|------|-------------|-------|-----------------|-------|
| I    | -1, 0, 1    | 0     |                 | TO    |
| IIa  | -1          | 1     |                 | EO    |
| IIb  | -1          | 1     |                 | TWEO  |
| IIIa | -1          | 3     |                 | EO, BO|
| IIIb | -1          | 3     |                 | TWEO, BO|
| IVa  | -1, 0, 1    | 2     |                 | EO    |
| IVb  | -1, 0, 1    | 2     |                 | TWEO, EO|
| V    | -1          | 3     |                 | TWEO, UCO, BO|
| VI   | -1, 0, 1    | 2     |                 | TWEO, UCO, EO|
| VII  | -1          | 1     |                 | TWEO, UCO|

Table 1: Orbit types for time-like (\( \zeta = -1 \)), light-like (\( \zeta = 0 \)) and space-like (\( \zeta = 1 \)) geodesics. Circles represent zeros of \( (d\tilde{r}/d\phi)^2 \), i.e., turning points, thick lines represent possible motion in the range of the radial coordinate \( x \), which ranges from \( \frac{1}{2} \) to \( \infty \). The wormhole throat \( x_T \) is indicated by the vertical double line at the center. Open circles indicate potential maxima or saddle points, where UCO reside.

We consider the possibility of two world orbits of time-like geodesics, i.e., orbits of particles which cross the wormhole throat, particularly interesting. Transit orbits (TO) represent trivial two world orbits. To find other types of two world orbits we have to study the extrema \( w_1, w_2 \) and \( w_3 \) with respect to the throat location \( w_T \in (0.5, 1) \) with \( \zeta = -1 \) in the following section. Since \( \lim_{L \to \infty} w_3(\tilde{L}, \tilde{\alpha}+, \zeta) = 0 \) and \( \lim_{L \to \infty} w_2(\tilde{L}, \tilde{\alpha}+, \zeta) = \frac{1}{2} \), we note that \( w_3 < w_2 < \frac{1}{2} \) for any parameters. Therefore bound orbits occur only in one world. They cannot cross the throat. However, escape orbits can cross the throat. Their energy only has to exceed the value of the potential at the throat \( P(\tilde{\alpha}+, \tilde{L}, w = w_T, \zeta) = P_T(\tilde{\alpha}+, \tilde{L}, \zeta) \), which we refer to as wormhole barrier. In order to differentiate between two world and single world orbits we put a subscript to the orbit types, where both are possible. Single world orbits never exceed the wormhole barrier \( \epsilon^2 < P_T(\tilde{\alpha}+, \tilde{L}, \zeta) \). They carry the subscript \( a \). Two world orbits can cross the wormhole barrier, since \( \epsilon^2 > P_T(\tilde{\alpha}+, \tilde{L}, \zeta) \). They carry the subscript \( b \).
Clearly, the extremal values for light-like geodesics $w_1, w_2$ and $w_3$ are independent of $\tilde{a}_+$ and $\tilde{L}$. The extremal values for space-like geodesics depend on both parameters again, but lead to a behavior different from time-like geodesics. Here it can happen for fixed $\tilde{a}_+$ and increasing $\tilde{L}$, that the inequality $w_T < w_2$ changes to $w_T < w_T$ (i.e., the same inequality as for time-like geodesics). A closer examination of this circumstance shows that this happens, however, only when $\tilde{L} = \tilde{L}_{\text{zero}}$. Due to the restriction to $\tilde{r}^2 > 0$, we only encounter the inequality $w_T < w_T$ in our discussion. Therefore the geodesics of type $\text{IV}_b$ and $\text{VI}$ orbits, which temporarily do or do not cross the throat, cannot originate in the other asymptotic region, respectively.

By plotting all extremal values of the polynomial together with the end points $P_0$ and $P_1$, a parametric plot $\tilde{r}^2$ over $\tilde{L}$ can be created, as shown in Figure 4 for some parameter set. The parametric plot exhibits the dependence of the possible orbit types on the energy and the angular momentum. In the parameter plots for fixed $\zeta$ a variation of $\tilde{a}_+$ will only change the size of the regions but not their overall structure. Therefore it is sufficient to consider only one set of parameters to assign the orbits types.

For time-like geodesics the lowest values of the characteristic angular momenta $\tilde{L}_{\text{crit}}$ and $\tilde{L}_{\text{swap}}$ are clearly seen in Figure 4a. Furthermore Figure 4a and Figure 4c show the wormhole barrier $P_T(\tilde{a}_+, L, \zeta)$. This curve also varies with $\tilde{a}_+$, as demonstrated for the light-like geodesics. Space-like geodesics are only shown for $\tilde{r}^2 > 0$. Red lines limit the regions. The green line represents a region in itself as well as a limit between regions. For time-like geodesics the orbit type on the line changes at $\tilde{L}_{\text{swap}}$. The circle in Figure 4c indicates the point where the Type $\text{VII}$ orbits occur.

Figure 4: Parametric plots for time- (a), light- (b) and space-like (c) geodesics with energy $\tilde{r}^2$ versus angular momentum $\tilde{L}$. All parametric plots are evaluated for $\tilde{a}_+ = 0.50$. When $\tilde{a}_+$ is varied, the overall structure of the orbit regions with respect to each other is preserved, while the size of the regions can increase or decrease. The blue line in the plots represents the wormhole barrier $P_T(\tilde{a}_+, L, \zeta)$. This curve also varies with $\zeta$, as demonstrated for the light-like geodesics. Space-like geodesics are only shown for $\tilde{r}^2 > 0$. Red lines limit the regions. The green line represents a region in itself as well as a limit between regions. For time-like geodesics the orbit type on the line changes at $\tilde{L}_{\text{swap}}$. The circle in Figure 4c indicates the point where the Type $\text{VII}$ orbits occur.

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As demonstrated in Figure 4a, only a single parametric plot exists for light-like geodesics, since there is no dependence on the wormhole parameter $\tilde{a}_+$. In contrast the wormhole barrier does depend on $\tilde{a}_+$, as demonstrated in the figure as well. Here a larger wormhole parameter $\tilde{a}_+$ gives rise to two world orbits for smaller energies $\tilde{r}^2$.

Restricting the analysis to $\tilde{r}^2 > 0$, we show the parametric plot of space-like geodesics in Figure 4c, which turns out rather simple as well. If we were to consider the parametric plots for negative energy square $\tilde{r}^2 < 0$ as well, we would find structures similar to Figure 4a. This would indicate the existence of bound orbits for particles with imaginary energy.

### 4.2 Solution for the $(d\vec{r}/dt)$-motion

In order to solve for the $(d\vec{r}/dt)$-motion, Eq. (20) is transformed as follows

\[
\left(\frac{d\vec{r}}{dt}\right)^2 = \frac{L^2}{\tilde{r}^2} \left(1 + \frac{\eta}{w_2}\right)^3 \tilde{r}^4 \left(\frac{d\vec{r}}{d\phi}\right)^2. \tag{41}
\]
We then apply again the coordinate transformation \( \bar{r} = \eta (x - \frac{1}{2}) \), \( d\bar{r} = \eta dx \), and introduce again \( \bar{L} \) and \( \bar{\epsilon} \) together with the new time coordinate \( \hat{t} = \frac{1}{\eta} \). This yields
\[
\left( \frac{dx}{dt} \right)^2 = \frac{\bar{L}^2}{16\bar{\epsilon}^2} \left( \frac{2x-1}{x^8} \right)^2 \text{.} \tag{42}
\]
Also we set \( (dx/d\phi)^2 = P_4(x) \). Integrating Eq. (42) we get
\[
\hat{t} - \hat{t}_0 = 4\bar{\epsilon} \bar{L} \int_{x_0}^x \frac{x^4}{(2x-1)^2} \frac{dx}{\sqrt{P_3(x)}} \text{.} \tag{43}
\]
For \( P_4(x) \) we already know the solution in terms of the Weierstraß function. When using the Weierstraß function we can later apply the addition theorems of this function in order to solve the \( (d\bar{r}/dt) \)-equation. Therefore we employ
\[
x = \frac{b_3}{4y^2 - b_3^2} + x_1 \text{ to transform } P_4(x) \rightarrow P_3(y) \text{.}
\]
With partial fractions this new integral can be written as
\[
\hat{t} - \hat{t}_0 = \frac{4\bar{\epsilon} \bar{L}}{y_0} \int_{y_0}^y \left[ k_0 + \sum_{i=1}^2 \frac{a_i}{y - p_i} + \sum_{i=1}^2 \frac{b_i}{(y - p_i)^2} \right] \frac{dy}{\sqrt{P_3(y)}} \text{,} \tag{44}
\]
with poles
\[
p_1 = \frac{b_3}{12}, \quad p_2 = \frac{1}{12} \frac{2b_3 x_1 - b_2 - 6b_3}{2x_1 - 1} \text{,} \tag{45}
\]
and coefficients
\[
k_0 = \frac{x_1^4}{4x_1^2 - 4x_1 + 1}, \quad a_1 = \frac{b_3^2}{64}, \quad a_2 = \frac{b_3}{16} \frac{4x_1 - 1}{(2x_1 - 1)^3}, \quad b_1 = \frac{b_3}{16} (2x_1 + 1), \quad b_2 = \frac{1}{64} \frac{b_3^2}{(2x_1 - 1)^3}. \tag{46}
\]
The integration of the time coordinate \( \hat{t} \) can be reduced to two different problems to be solved. One problem is the integration of an elliptic integral of the third kind, for further discussion defined as \( I_1 \), and the other one is the integration of an elliptic integral with a double pole \( I_2 \)
\[
I_1 = \int_{y_0}^y \frac{1}{y - p} \sqrt{P_3(y)} \text{,} \quad I_2 = \int_{y_0}^y \frac{1}{(y - p)^2} \sqrt{P_3(y)} \text{.} \tag{47}
\]
Those two integrals can indeed be solved by using addition theorems of the Weierstraß functions, thus yielding the solution for the time coordinate \( \hat{t} \)
\[
\hat{t}(v) = \frac{4\bar{\epsilon} \bar{L}}{v_0} (k_0(v - v_0) + a_1 I_1 + a_2 I_2 + b_1 I_1 + b_2 I_2) + \hat{t}_0 \text{,} \tag{48}
\]
where \( v_{p_i} \) is a Weierstraß transformed pole \( p_i \) with the relation \( p_i = \wp(p_i) \). The solution of \( I_1 \) and \( I_2 \) will be presented in the next two paragraphs.
Solution of the elliptic integrals of third kind $I_1$

Let us begin by introducing yet another coordinate transformation which will simplify the integration. With $y = \wp(v)$, which solves the Weierstraß differential equation (24) and $dy = dv\sqrt{P_3(y)}$, $I_1$ becomes

$$I_1 = \int_{v_0}^{v} \frac{dv}{\wp(v) - p} = \int_{v_0}^{v} \frac{dv}{\wp(v) - \wp(v_p)}.$$  \hspace{1cm} (49)

where in the second term $p = \wp(v_p)$ redefines the pole $p$ using the Weierstraß function with a corresponding input parameter $v_p$. With the constant $\wp'(v_p)$, $I_1$ can be written as

$$I_1 = \frac{1}{\wp'(v_p)} \int_{v_0}^{v} \frac{\wp'(v_p)}{\wp(v) - \wp(v_p)} dv.$$  \hspace{1cm} (50)

In this form we can apply an addition theorem for the Weierstraß functions in the integrand \[20\]

$$\zeta(u + v) + \zeta(u - v) - 2\zeta(u) = -\zeta(x + y) + \zeta(x - y) + 2\zeta(y),$$  \hspace{1cm} (51)

where $\zeta(z) = -\int \wp(z)dz$ is the Weierstraß $\zeta$-function. Inserting Eq. (51) into $I_1$ gives

$$I_1 = \frac{1}{\wp'(v_p)} \int_{v_0}^{v} -\zeta(v + v_p) + \zeta(v - v_p) + 2\zeta(v_p) dv,$$  \hspace{1cm} (52)

which can be trivially integrated with the relation $\frac{d}{dz} \ln \sigma(z) = \zeta(z)$, where $\sigma(z)$ is the Weierstraß $\sigma$-function.

$$I_1 = \frac{1}{\wp'(v_p)} \left[2\zeta(v_p)(v - v_p) + \ln \frac{\sigma(v - v_p)}{\sigma(v + v_p)} - \ln \frac{\sigma(v_0 - v_p)}{\sigma(v_0 + v_p)} \right]$$  \hspace{1cm} (53)

Solution of the elliptic integrals of type $I_2$

The elliptic integrals of type $I_2$ need some more discussion as compared to $I_1$. $I_2$ is rewritten in the same way as $I_1$, see Eq. (19),

$$I_2 = \int_{v_0}^{v} \frac{dv}{(\wp(v) - p)^2} = \int_{v_0}^{v} \frac{dv}{(\wp(v) - \wp(v_p))^2}.$$  \hspace{1cm} (54)

Taking Eq. (51)

$$\zeta(u + v) + \zeta(u - v) - 2\zeta(u) = \frac{\wp'(u)}{\wp(u) - \wp(v)}$$  \hspace{1cm} (55)

and swapping $u$ and $v$ and using the relation $\zeta(-z) = -\zeta(z)$ we get

$$\zeta(u + v) - \zeta(u - v) - 2\zeta(v) = \frac{\wp'(v)}{\wp(u) - \wp(v)}.$$  \hspace{1cm} (56)

Adding Eq. (55) and Eq. (56) yields

$$\zeta(u + v) = \zeta(u) + \zeta(v) + \frac{1}{2} \frac{\wp'(u) - \wp'(v)}{\wp(u) - \wp(v)}.$$  \hspace{1cm} (57)

Differentiating Eq. (57) leads to Eq. (58). Here $u$ and $v$ can be swapped again, resulting in Eq. (59).
\[ \varphi(u + v) = \varphi(u) - \frac{1}{2} \frac{\varphi''(u)}{\varphi(u) - \varphi(v)} + \frac{1}{2} \frac{(\varphi'(u) - \varphi'(v))\varphi'(u)}{(\varphi(u) - \varphi(v))^2} \]  

(58)

\[ \varphi(u + v) = \varphi(v) + \frac{1}{2} \frac{\varphi''(v)}{\varphi(u) - \varphi(v)} - \frac{1}{2} \frac{(\varphi'(u) - \varphi'(v))\varphi'(v)}{(\varphi(u) - \varphi(v))^2} \]  

(59)

With Eq. (59) the integrand of Eq. (54) can be rewritten as

\[ \frac{1}{(\varphi(v) - \varphi(v_p))^2} = \frac{1}{\varphi'(v_p)^2} \left[ 2\varphi(v + v_p) - 2\varphi(v_p) - \frac{\varphi''(v_p)}{\varphi'(v) - \varphi'(v_p)} \right] + \frac{1}{\varphi'(v_p)} \frac{\varphi'(v)}{(\varphi(v) - \varphi(v_p))^2}. \]  

(60)

Knowing that \( \int \varphi(v)dv = -\zeta(v) \) and that

\[ \int \frac{\varphi'(z)}{(\varphi(z) - \varphi(v_p))^2}dz = \int \frac{du}{u^2} = -\frac{1}{u} = -\frac{1}{\varphi(z) - \varphi(v_p)} \]

(61)

can be integrated by substituting \( u = \varphi(v) - \varphi(v_p) \) and \( dv = \frac{du}{\varphi'(v_p)} \), \( I_2 \) can be integrated,

\[ I_2 = -\frac{\varphi''(v_p)}{\varphi'(v_p)^2}I_1 - \frac{1}{\varphi'(v_p)^2} \left[ 2\zeta(v + v_p) + 2\zeta(v_0 + v_p) + 2\varphi(v_p)(v - v_0) + \frac{\varphi'(v_p)}{\varphi'(v) - \varphi'(v_p)} + \frac{\varphi'(v_p)}{\varphi(v_0) - \varphi'(v_p)} \right]. \]  

(62)

Finally, applying Eq. (57) reduces the previously calculated solution of \( I_2 \) to

\[ I_2 = -\frac{\varphi''(v_p)}{\varphi'(v_p)^2}I_1 - \frac{1}{\varphi'(v_p)^2} \left[ 2\varphi(v_p)(v - v_0) + 2(\zeta(v) + \zeta(v_0)) + \frac{\varphi'(v)}{\varphi'(v) - \varphi'(v_p)} + \frac{\varphi'(v_0)}{\varphi(v_0) - \varphi'(v_p)} \right]. \]  

(63)

## 5 The orbits

With the solution for \( (d\bar{r}/d\phi) \) we found the analytical solution for the motion of particles and light in the equatorial plane \( \vartheta = \frac{\pi}{2} \). Before plotting some of the orbits in the wormholes spacetimes, we need to evaluate the isometric embedding of the traversable wormhole solutions. To this end we transform the metric \( ds^2 \) from Eq. (10) with \( dt = 0 \) into cylindrical coordinates

\[ ds^2 \equiv dX^2 + dY^2 + dz^2 = d\rho^2 + dz^2 + \rho^2 d\phi^2 \]

\[ = \left[ \left( \frac{d\rho}{d\bar{r}} \right)^2 + \left( \frac{dz}{d\bar{r}} \right)^2 \right] d\bar{r}^2 + \rho^2 d\phi^2. \]  

(64)

The coordinate \( \rho \) can be directly read from the original metric and has the form

\[ \rho(\bar{r}) = \alpha_+ \left( \bar{r} - \frac{\eta^2}{4\bar{r}} \right) + \alpha_- \left( \bar{r} + \eta + \frac{\eta^2}{4\bar{r}} \right). \]  

(65)

Transforming to the radial coordinate \( x = \frac{\bar{r}}{\eta} + \frac{1}{2\eta} \),

\[ \tilde{\rho}(x) = \frac{2(\alpha_+ + 1)x^2 - 2\alpha_+ x}{2x - 1}, \]

(66)

where \( \tilde{\rho}(x) = \frac{\rho(\bar{r})}{\eta\alpha_-} \), we find for the wormhole throat
\[ \tilde{\rho}(x_T) = \tilde{\rho}_T(\tilde{\alpha}_+) = 1 + \sqrt{1 - \tilde{\alpha}_+^2}. \]  

(67)

This shows that the circumferential throat radius satisfies \( \tilde{\rho}_T > 1 \) for all values of the wormhole parameter \( \tilde{\alpha}_+ \).

To obtain the coordinate \( z \) we equate the coefficient of \( d\tilde{r}^2 \) in Eq. (64) with the one of the metric in Eq. (10),

\[ \left[ \left( \frac{d\rho}{d\tilde{r}} \right)^2 + \left( \frac{dz}{d\tilde{r}} \right)^2 \right] = \left[ \alpha_+ \left( \frac{1 - \eta}{2\tilde{r}} \right) + \alpha_- \right]^2 \left( 1 + \frac{\eta}{2\tilde{r}} \right)^4. \]  

(68)

Transforming again to the radial coordinate \( x = \frac{\tilde{r}}{\eta} + \frac{1}{2} \) we obtain

\[ \tilde{z}(x) - \tilde{z}(x_T) = \int_{x_T}^{x} \pm \sqrt{\frac{16 \left[ \alpha_+ \left( 1 - \frac{1}{x} \right) + 1 \right]^2 x^4 - \left[ (2x - 1)^2 (\alpha_+ + 1) + \alpha_- - 1 \right]^2}{(2x - 1)^2} } \, dx \]  

(69)

with \( \tilde{z}(x) = \frac{z(x)}{\eta \alpha_-} \). This integral is evaluated numerically.

With \( \tilde{z}(x) \) and \( \tilde{\rho}(x) \) at hand, we now illustrate the shape of the wormholes in Figure 5 for different choices of the parameter \( \tilde{\alpha}_+ \).

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{wormholes.png}
\caption{Isometric embeddings of the wormhole metric for different wormhole parameters \( \tilde{\alpha}_+ \). All figures have the same cubic axis limits as defined by the \( \tilde{z} \)-limits, which define the range \( \tilde{z} \in [-\tilde{z}_{\text{lim}}, \tilde{z}_{\text{lim}}] \) and thus the detail of the figures.}
\end{figure}
Figure 6: Orbits for various parameter sets calculated with the solution of $\frac{d\tilde{r}}{d\phi}$. On the bottom of the figures the orbits are projected into the $X-Y$-plane, where the red inner circle represents the throat $\tilde{\rho}_T$. The light-blue line in Figure (6d) represents the second geodesic possible for this orbit type.

In the pictures of the isometrically embedding of the wormhole manifolds in Figure (5) the upper cone represents a part of the world with coordinate range $\tilde{r} \in [\tilde{r}_T, \infty]$, while the lower cone represents a part of the world with $\tilde{r} \in [0, \tilde{r}_T]$. A zoom of the Figures (5a-c) highlighting the throat region is shown in the Figures (5d-f).
In Figure (6) we exhibit a set of trajectories as solutions of the geodesics equations on top of the corresponding wormhole embeddings. In particular, we exhibit orbits for different types of geodesics. As discussed above, bound orbits are located only in the upper cone, i.e., in the world with coordinate range $\bar{r} \in [\bar{r}_T, \infty]$, and two world escape orbits always pass from the lower cone to the upper cone, reach their turning point and return to the lower cone.

6 Conclusion

In this paper we have studied the geodesic motion in spacetimes describing traversable wormholes supported by a massless conformally-coupled scalar field, found by Barcelo and Visser [12]. These static spherically symmetric wormholes connect two asymptotically flat worlds, which possess different physical properties. For instance, their masses as read off from the asymptotic fall-off of the metric, differ in both worlds [12].

Here we presented the analytical solutions for geodesic motion in these wormhole spacetime. Restricting our discussion to $(d\bar{r}/d\bar{\phi})$ and $(d\bar{r}/dt)$, we obtained the solutions in terms of the Weierstraß $\wp$, $\sigma$- and $\zeta$-functions. We also classified all possible orbits for time-, light- and space-like geodesics.

For time- and space-like motion the effective potential for the particle motion consists of a monotonic gravitational part and a non-monotonic centrifugal part, while for light-like motion only the centrifugal part is present. Consequently, the classification depends on the amount of angular momentum $\tilde{L}$ of a particle. For time-like geodesics there are certain characteristic values $\tilde{L}_{\text{crit}}$ and $\tilde{L}_{\text{swap}}$ where the possible orbit types change. For space-like geodesics there is in addition $\tilde{L}_{\text{zero}}$, however, only $\tilde{L}_{\text{zero}}$ matters when restricting to real energies.

Stable bound orbits are only possible for time-like geodesics. They exist only in the upper world, where the gravitational potential is larger, and only when the angular momentum exceeds $\tilde{L}_{\text{crit}}$. Bound orbits can never cross the wormhole throat. (For space-like geodesics such bound orbits would only be possible when considering particles with imaginary energy.)

Light-like geodesics but also space-like geodesics, whose angular momentum exceeds $\tilde{L}_{\text{zero}}$, have only transit orbits and two world escape orbits. However, whenever a maximum is present in the effective potential for time-, light-, and space-like orbits, unstable spherical orbits are possible as well. This means, in particular, that for any finite angular momentum there are unstable spherical light orbits. Thus these spacetimes possess a photosphere.

The wormhole spacetimes considered here are highly asymmetrical. However, as suggested by Barcelo and Visser [12], one could obtain traversable wormhole solutions with no asymmetry, by adding thin shells of ordinary matter and joining smoothly inner and outer regions. For such wormhole geometries similar techniques as the ones employed here could be used to study their orbits (see e.g. [21]).

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