The FOLE Table

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Abstract. This paper discusses the representation of ontologies in the first-order logical environment FOLE (Kent [13]). An ontology defines the primitives with which to model the knowledge resources for a community of discourse (Gruber [7]). These primitives, consisting of classes, relationships and properties, are represented by the entity-relationship-attribute ERA data model (Chen [2]). An ontology uses formal axioms to constrain the interpretation of these primitives. In short, an ontology specifies a logical theory. A series of three papers provide a rigorous mathematical representation for the ERA data model in particular, and ontologies in general, within the first-order logical environment FOLE. The first two papers, which provide a foundation and superstructure for FOLE, represent the formalism and semantics of (many-sorted) first-order logic in a classification form corresponding to ideas discussed in the Information Flow Framework (IFF [26]). The third paper (Kent [16]) will define an interpretation of FOLE in terms of the transformational passage, first described in Kent [13], from the classification form of first-order logic to an equivalent interpretation form, thereby defining the formalism and semantics of first-order logical/relational database systems. Two papers will provide a precise mathematical basis for FOLE interpretation: the current paper develops the notion of a FOLE relational table following the relational model (Codd [3]), and a follow-up paper will develop the notion of a FOLE relational database. Both of these papers expand on material found in the paper (Kent [12]). Although the classification form (and FOLE itself) follow the entity-relationship-attribute data model of Chen, the interpretation form incorporates the relational data model of Codd. In general, the FOLE representation uses a conceptual structures approach, that is completely compatible with formal concept analysis (Ganter and Wille [4]) and information flow (Barwise and Seligman [1]).

Keywords: signature, type domain, signed domain, table.
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1 Introduction

1.1 Philosophy.

The relational model is an approach to information management using the semantics and formalism of first-order predicate logic. The first-order logical environment FOLE is a framework for defining the semantics and formalism of logic and databases in an integrated and coherent fashion. Hence, the relational model for information management can be framed in terms of the first-order logical environment FOLE.

1.2 Background

The author’s “Systems Consequence” paper (Kent [11]) is a very general theory and methodology for specification and inter-operation of systems of information resources. The generality comes from the fact that it is independent of the logical/semantic system (institution) being used. This is a wide-ranging theory, based upon ideas from information flow (Barwise and Seligman [1]), formal concept analysis (Wille and Ganter et al [4]), the theory of institutions (Goguen et al [6]), and the lattice of theories notion (Sowa [21]), for the integration of both formal and semantic systems independent of logical environment. In order to better understand the motivations of that paper and to be able more readily to apply its concepts, in the future it will be important to study system consequence in various particular logical/semantic systems. This paper aims to do just that for the logical/semantic system of relational databases. The paper, which was inspired by and which extends a recent set of papers on the theory of relational database systems (Spivak [22],[23]), is linked with work on the Information Flow Framework (IFF [26]) connected with the ontology standards effort (SUO), since relational databases naturally embed into first order logic. We offer both an intuitive and a technical discussion. Corresponding to the notions of primary and foreign keys, relational database semantics takes two forms: a distinguished form where entities are distinguished from relations, and a unified form where relations and entities coincide. The distinguished form corresponds to the theory presented in the paper (Spivak [22]). We extend Spivak’s treatment of tables from the static case of a single entity classification (type specification) to the dynamic case of classifications varying along infomorphisms. Our treatment of relational databases as diagrams of tables differs from Spivak’s sheaf theory of databases. The unified form, a special case of the distinguished form, corresponds to the theory presented in the paper (Spivak [23]). The unified form has a graphical presentation, which corresponds to the sketch theory of databases (Johnson and Rosebrugh [8]) and the resource description framework (RDF). This paper, which is the first step to connect relational databases with system consequence, is concerned with the semantics of relational databases. A later paper will discuss various formalisms of relational databases, such as relational algebra and first order logic.

---

1 “The relational model for database management: version 2” by E.F. Codd [3].
1.3 Architecture.

The FOLE architecture, as briefly pictured in Fig. 1 and more completely in Fig. 1 in the preface of [19], consists of four nodes divided into two branches. The classification form of FOLE (left hand side of Fig. 1) consists of “The FOLE Foundation” at the bottom and “The FOLE Superstructure” at the top. The interpretation form of FOLE (right hand side of Fig. 1) consists of “The FOLE Table” at the bottom and “The FOLE Database” at the top. The equivalence between the classification form and the interpretation form is defined in the paper “FOLE Equivalence” [19]. The current paper is concerned with the FOLE table concept.

1.4 Overview

Section 2 provides material on the basic structures underpinning the FOLE table concept: signatures, type domains and signed domains. Section 3 describes our representation for the table concept by defining the multi-path fibred context of tables (illustrated in Tbl. 18 of § 3.5)

---

The original discussion of FOLE (Kent [13]) took place within the knowledge representation community, where the term category is defined to be a division within a system of classification or a mode of existence. Hence following (Kent [13]), we use “mathematical context” (Goguen [5]) for the mathematical term “category”, “passage” for the term “functor”, and “bridge” for the term “natural transformation”. 

---

\[ Tbl(S) \]
\[ Tbl(A) \]
\[ Tbl(S,A) \]
Table 1. Figures and Tables
2 Table Basics

2.1 Overview

A table in the relational model is represented as an array, organized into rows and columns. The rows are called the tuples (records) of the table, whereas the columns are called the attributes of the table. The rows are indexed by keys. Both rows and columns are unordered; instead of indexing headers and tuples as \( n \)-tuples, the FOLE approach uses attribute names for tuples (as advocated by Codd [3]). In the relational model, all components can be resolved into sets and functions.\(^3\)

\[\text{Fig. 2. FOLE Table Basics}\]

\(^3\) The relational data model is based upon the context Set of sets and functions.
2.2 Signatures

A signature, which represents the header of a relational table, provides typing for the tuples permitted in the table.

Fibers. Let \( X \) be a sort set. The fiber mathematical context of \( X \)-signatures is the comma context

\[
\text{List}(X) = \text{List}(X) = \left( \text{Set} \downarrow X \right)
\]

associated with the sort set (constant passage) \( 1 \xrightarrow{X} \text{Set} \). It has the index and trivial projection passages \( \text{Set} \xleftarrow{\text{arity}} \text{List}(X) \xrightarrow{\Delta} 1 \) and the defining bridge \( \text{ind}_X \xrightarrow{\Delta} \Delta \circ X \). An \( \text{List}(X) \)-object \( \mathcal{S} = \langle I, s \rangle \), called an \( X \)-signature (header), consists of an indexing set (arity) \( I \) and a map \( I \xrightarrow{s} X \) from \( I \) to the set of sorts \( X \). An \( \text{List}(X) \)-morphism \( \mathcal{S}' = \langle I', s' \rangle \xrightarrow{h} \langle I, s \rangle = \mathcal{S} \) is an arity function \( I' \xrightarrow{h} I \) that preserves signatures by satisfying the naturality condition \( h \cdot s = s' \).

Fibered Context. The fibered context of signatures is the comma context

\[
\text{List} = \left( \text{Set} \downarrow \text{Set} \right).
\]

It has index and set projection passages \( \text{Set} \xleftarrow{\text{arity}} \text{List} \xrightarrow{\text{sort}} \text{Set} \) and the defining bridge \( \text{arity} \xrightarrow{\Delta} \text{sort} \). A \( \text{List} \)-object (signature, sort list) \( \langle I, s, X \rangle \) consists of a sort set \( X \) and an \( X \)-signature \( \langle I, s \rangle \). A \( \text{List} \)-morphism (signature morphism) \( \langle I_2, s_2, X_2 \rangle \xrightarrow{\langle h, f \rangle} \langle I_1, s_1, X_1 \rangle \) consists of a sort function \( X_2 \xrightarrow{f} X_1 \) and an arity function \( I_2 \xrightarrow{h} I_1 \) satisfying the naturality condition \( h \cdot s_1 = s_2 \cdot f \). This condition gives two alternate and adjoint definitions. In terms of fibers, a signature morphism consists of a sort function \( X_2 \xrightarrow{f} X_1 \) and either a morphism \( \langle I_2, s_2 \rangle \xrightarrow{h} f^*(I, s) \) in the fiber context \( \text{List}(X_2) \) or a morphism \( \Sigma_f(I_2, s_2) \xrightarrow{h} \langle I_1, s_1 \rangle \) in the fiber context \( \text{List}(X_1) \).

\[
\Sigma_f(I_2, s_2) \xrightarrow{h} \langle I_1, s_1 \rangle \quad \iff \quad \langle I_2, s_2 \rangle \xrightarrow{h} f^*(I_1, s_1)
\]

(1)

The use of lists for signatures (and tuples) follows Codd’s recommendation to use attribute names to index the tuples of a relation instead of a numerical ordering.
The $X_1$-signature morphism $\sum_f(I_2, s_2) \xrightarrow{h} \langle I_1, s_1 \rangle$ is the composition (Fig. 3) of the fiber morphism $\sum_f \left( I_2, s_2 \xrightarrow{h} f^*(I_1, s_1) \right)$ with the $\langle I_1, s_1 \rangle$th counit component $\sum_f \left( f^*(I_1, s_1) \xrightarrow{i} \langle I_1, s_1 \rangle \right)$ for the fiber adjunction $\text{List}(X_2) \xrightarrow{\text{List}(f)} \text{List}(X_1)$.

This fiber adjunction (top part of Tbl. 2) is a component of the sort indexed adjunction of signatures $\text{Set} \xrightarrow{\text{List}} \text{Adj}$.

Table 2. Sort List/Subset Reflection

| List($X_2$) = ($\text{Set}\downarrow X_2$) | $\sum_f$ | List($X_1$) = ($\text{Set}\downarrow X_1$) |
|----------------------------------------|--------|----------------------------------------|
| inc$_{X_2}$ | $\exists_f$ | inc$_{X_1}$ |
| $pX_2$ | $\forall_f$ | $pX_1$ |

2.3 Type Domains

A type domain, which constrains the body of a relational table, is an indexed collection of data types from which a table’s tuples are chosen.

**Fiber.** Let $X$ be a sort set. The fiber mathematical context of $X$-sorted type domains is the context $\text{Cls}(X)$ described as follows. An $X$-sorted type domain

---

5 A fibration (fibered context over $B$) (nLab [27]) is a passage $E \xrightarrow{f} B$ such that the fibers $E_B = P^{-1}(B)$ depend (contravariantly) pseudo-functorially on $B \in B$. Dually, in an opfibration the dependence is covariant. There is an equivalence of 2-contexts

$$\int : \left[ B^{\text{op}}, \text{Cxt} \right] \leftrightarrow \text{Fib}(B)$$

between the 2-context $\text{Fib}(B)$ of fibrations over $B$ and the 2-context $[B^{\text{op}}, \text{Cxt}]$ of contravariant pseudo-passages from $B$ to $\text{Cxt}$, also called $B$-indexed contexts. The construction $\int F$ of a fibered context from an indexed context is called the Grothendieck construction. We say that fibered context $\int F$ is the oplax sum of indexed context $F$. § A.3 has a more detailed discussion of fibered contexts.

6 In the ERA data model (Kent [14]), attributes are represented by a typed domain consisting of a collection of data types. In FOLE, a typed domain is represented by an
\( \mathcal{A} = \langle X, Y, \models_\mathcal{A} \rangle \) consists of a data value set \( Y \) and a classification relation \( \models_\mathcal{A} \subseteq X \times Y \); hence, a data-type collection \( \{ \mathcal{A}_x \subseteq Y \mid x \in X \} \), with each sort \( x \in X \) indexing the data-type \( \text{ext}_\mathcal{A}(x) = \mathcal{A}_x \). An \( X \)-sorted type domain morphism is an infomorphism \( \mathcal{A}_2 \xrightarrow{(f, g)} \mathcal{A}_1 \) satisfying \( g(y_1) \models_\mathcal{A}_2 x \iff g(y_1) \models_\mathcal{A}_1 x \) for each sort \( x \in X \) and data value \( y_1 \in Y_1 \); hence, consisting of a data value function \( Y_2 \xrightarrow{g} Y_1 \) satisfying \( g(y_1) \in \mathcal{A}_{2, x} \) for each sort \( x \in X \) and each value \( y_1 \in \mathcal{A}_{1, x} \); thus, defining the restrictions \( \{ \mathcal{A}_{2, x} \xleftarrow{g_{x}} \mathcal{A}_{1, x} \mid x \in X \} \).

Fibered Context. The fibered context of type domains \( \text{Cls} \xrightarrow{\text{sort}} \text{Set} \) is described as follows. A type domain \( \mathcal{A} = \langle X, Y, \models_\mathcal{A} \rangle \) is a classification; and hence consists of a sort set \( \text{sort}(\mathcal{A}) = X \) and an \( X \)-sorted type domain \( \mathcal{A} = \langle X, Y, \models_\mathcal{A} \rangle \). A type domain morphism \( \mathcal{A}_2 \xrightarrow{(f, g)} \mathcal{A}_1 \) is an infomorphism consisting of a sort function \( X_2 \xrightarrow{f} X_1 \) and a data value function \( Y_2 \xrightarrow{g} Y_1 \) that satisfy the infomorphism condition \( g(y_1) \models_\mathcal{A}_2 x \iff g(y_1) \models_\mathcal{A}_1 f(x_2) \) for any source sort \( x_2 \in X_2 \) and target data value \( y_1 \in Y_1 \). This condition gives an alternate definition. In terms of fibers, a type domain morphism consists of a sort function \( X_2 \xrightarrow{f} X_1 \) and a morphism \( \mathcal{A}_2 \xrightarrow{(f, g)} f^{-1}(\mathcal{A}_1) \) in the fiber context \( \text{Cls}(X_2) \).

\[
\begin{array}{ccc}
X_2 & \xrightarrow{f} & X_1 \\
\downarrow \models_{\mathcal{A}_2} & & \downarrow \models_{\mathcal{A}_1} \\
Y_2 & \xrightarrow{g} & Y_1
\end{array}
\]

attribute classification \( \mathcal{A} = \langle X, Y, \models_\mathcal{A} \rangle \) consisting of a set of attribute types (sorts) \( X \), a set of attribute instances (data values) \( Y \) and an attribute classification relation \( \models_\mathcal{A} \subseteq X \times Y \). For each sort (attribute type) \( x \in X \), the data domain of that type is the \( \mathcal{A} \)-extent \( \mathcal{A}_x = \text{ext}_\mathcal{A}(x) = \{ y \in Y \mid y \models_\mathcal{A} x \} \). The passage \( X \xrightarrow{\text{ext}_\mathcal{A}} \mathcal{A} \) maps a sort \( x \in X \) to its data domain (\( \mathcal{A} \)-extent) \( \mathcal{A}_x \subseteq Y \). The attribute list classification \( \text{List}(\mathcal{A}) = \langle \text{List}(X), \text{List}(Y), \models_{\text{List}(\mathcal{A})} \rangle \) has \( X \)-signatures as types and \( Y \)-tuples as instances, with classification by common arity and universal \( \mathcal{A} \)-classification: a \( Y \)-tuple \( \langle J, I \rangle \) is classified by an \( X \)-signature \( \langle I, s \rangle \) when \( J = I \) and \( t_k \models_\mathcal{A} s_k \) for all \( k \in J = I \).

More generally, let \( \mathcal{A}_2 \xrightarrow{(f, g)} \mathcal{A}_1 \) be any infomorphism. The condition \( g(y_1) \models_\mathcal{A}_2 x_2 \iff g(y_1) \models_\mathcal{A}_1 f(x_2) \) is equivalent to the abstraction \( g^{-1}(\text{ext}_\mathcal{A}_2(x_2)) = \text{ext}_\mathcal{A}_1(f(x_2)) \). Hence, there is a function \( \text{ext}_\mathcal{A}_2(x_2) \xrightarrow{g_{x_2}} \text{ext}_\mathcal{A}_1(f(x_2)) \) that is a restriction of the instance function \( Y_2 \xrightarrow{g} Y_1 \).

7 More generally, let \( \mathcal{A}_2 \xrightarrow{(f, g)} \mathcal{A}_1 \) be any infomorphism. The condition \( g(y_1) \models_\mathcal{A}_2 x_2 \iff g(y_1) \models_\mathcal{A}_1 f(x_2) \) is equivalent to the abstraction \( g^{-1}(\text{ext}_\mathcal{A}_2(x_2)) = \text{ext}_\mathcal{A}_1(f(x_2)) \). Hence, there is a function \( \text{ext}_\mathcal{A}_2(x_2) \xrightarrow{g_{x_2}} \text{ext}_\mathcal{A}_1(f(x_2)) \) that is a restriction of the instance function \( Y_2 \xrightarrow{g} Y_1 \).
For any sort function $X_2 \xrightarrow{f} X_1$, there is a type domain fiber passage\(^8\)

$\text{Cls}(X_2) \xleftarrow{\text{cls}(f)} \text{Cls}(X_1) : A_1 \mapsto f^{-1}(A_1)$. This fiber passage is a component of the sort indexed context of type domains $\text{Set}^{\text{op}} \xrightarrow{\text{cls}} \text{Cxt} : X \mapsto \text{Cls}(X)$.

**Theorem 2.** The fibered context of type domains (a fibration) $\text{Cls} \xrightarrow{\text{sort}} \text{Set}$ is the Grothendieck construction of the sort indexed context of type domains $\text{Set}^{\text{op}} \xrightarrow{\text{cls}} \text{Cxt}$\(^9\).

### 2.4 Signed Domains

A signed domain represents both the header and the body of a relational table.

**Signed Domains.** Signed domains are a fundamental component used in the definition of database tables and in the database interpretation of FOLE. Signed domains are used to denote the valid tuples for a database header (signature).

A signed (headed/typed) domain $D = (I, s, A)$ consists of a type domain $A = \langle X, Y, \models A \rangle$ with sort set $X$ and a signature (database header) $(I, s, X)$.\(^8\)

A signed domain morphism $D_2 = (I_2, s_2, A_2) \xrightarrow{(h,f,g)} (I_1, s_1, A_1) = D_1$ consists of a signature morphism $\langle I_2, s_2, X_2 \rangle \xrightarrow{(h,f)} \langle I_1, s_1, X_1 \rangle$ and a type domain morphism $A_2 \xrightarrow{(f,g)} A_1$ with a common sort function $X_2 \xrightarrow{f} X_1$. Hence, the mathematical context of signed domains $\text{Dom}$ is the comma context

$$\text{Set} \xleftarrow{\text{arity}} \text{Dom} = (\text{Set} \downarrow \text{sort}) \xrightarrow{\text{data}} \text{Cls}$$

associated with the sort passage $\text{Cls} \xrightarrow{\text{sort}} \text{Set}$. There is a sign mediating passage $\text{Dom} \xrightarrow{\text{sign}} \text{List} : \langle I, s, A \rangle \mapsto \langle I, s, X \rangle$.

From a different point-of-view, a signed domain $D = (S, A)$ consists of a signature $S = \langle I, s, X \rangle$ and a type domain $A = \langle X, Y, \models A \rangle$ with common sort set $X$, and a signed domain morphism $\langle S_2, A_2 \rangle \xrightarrow{(h,f)} \langle S_1, A_1 \rangle$ consists of a signature morphism $S_2 \xrightarrow{(h,f)} S_1$ and a type domain morphism $A_2 \xrightarrow{(f,g)} A_1$ with common sort function $X_2 \xrightarrow{f} X_1$. Hence, $\text{Dom}$ can also be defined as the fibered product

$$\text{List} \xrightarrow{\text{sign}} \text{Dom} = \text{List} \times_{\text{Set}} \text{Cls} \xrightarrow{\text{data}} \text{Cls},$$

for the opspan of passages $\text{List} \xrightarrow{\text{sort}} \text{Set} \xleftarrow{\text{sort}} \text{Cls}$.

---

\(^8\) For any sort function $X_2 \xrightarrow{f} X_1$, there is an inverse image fiber passage $\text{Cls}(X_2) \xleftarrow{\text{sign}^{-1}} \text{Cls}(X_1) : f^{-1}(A_1) \xleftarrow{\text{sign}} A_1$, where $y_1 \models f^{-1}(A_1), x_2$ if and only if $y_1 \models A_1, f(x_2)$ for any source sort $x_2 \in X_2$ and target data value $y_1 \in Y_1$; or in terms of data types, $f^{-1}(A_1) = \{ f^{-1}(A_1) | x_2 \in X_2 \} = \{ A_1 f(x_2) | x_2 \in X_2 \}$.

\(^9\) Signed domains were called semidesignations in “Database Semantics” [12]. Indeed, a signed domain $(I, s, A)$ is a list designation $(I, s, 0) : 10 \equiv \text{List}(A)$ from the trivial entity classification $10 = \langle 1, \emptyset, \models_{10} \rangle$ with element signature map $1 \xrightarrow{\langle 1, x \rangle} \text{List}(X)$ and empty tuple map $\emptyset \xrightarrow{0} \text{List}(Y)$. 
10 The tuple passage \( \text{tup} : \text{Dom}^{\text{op}} \to \text{Set} \) maps a signed domain \( \langle I, s, A \rangle \) to its set of tuples \( \text{tup}(I, s, A) \) and maps a signed domain morphism \( \langle I_1, s_1, A_1 \rangle \) to its tuple function \( \text{tup}(I_2, s_2, A_2) \) to \( \text{tup}(I_1, s_1, A_1) \):

\[
\text{tup}(I_2, s_2, A_2) \mapsto \text{tup}(I_1, s_1, A_1);
\]

or visually,

\[
\cdots g(t_{h(i_2)}) \cdots | i_2 \in I_2 \mapsto \cdots t_1 \cdots | i_1 \in I_1. \]

2.5 Inclusion/Tuple Bridges

2.5.1 Signatures. Let \( S = \langle I, s, X \rangle \) be a signature. There is an inclusion passage \( \text{Cls}(X) \xrightarrow{\text{inc}_S} \text{Dom} \) that maps an \( X \)-sorted type domain \( A = \langle X, Y, \|=_{A} \rangle \) to the signed domain \( \langle I, s, A \rangle \) and maps an \( X \)-sorted type domain morphism \( A_2 = \langle X, Y_2, \|=_{A_2} \rangle \) to \( \langle X, Y_1, \|=_{A_1} \rangle \) to the signed domain morphism \( \langle I, s, A_2 \rangle \) to \( \langle I, s, A_1 \rangle \). Composition of the inclusion passage with the signed domain tuple passage (Def. 19) gives a signature tuple passage

\[
\text{Cls}(X)^{\text{op}} \xrightarrow{\text{inc}_S^* \circ \text{tup}} \text{Set}.
\]

which maps an \( X \)-sorted type domain \( A = \langle X, Y, \|=_{A} \rangle \) to the tuple set \( \text{tup}(S, A) = \text{tup}(I, s, A) = \text{tup}(I, \text{ext}_{\text{List}(A)}(I, s) \in \text{List}(Y) \mid \langle J, t \rangle \in \text{List}(A) \langle I, s \rangle \}) \) and maps an \( X \)-sorted type domain morphism (inclusion morphism) \( A_2 = \langle X, Y_2, \|=_{A_2} \rangle \) to \( \langle X, Y_1, \|=_{A_1} \rangle \) to the tuple function associated with \( g \):

\[
\text{tup}_S(Y_2, \|=_{A_2}) \xrightarrow{\text{tup}_S(g)} \text{tup}_S(Y_1, \|=_{A_1});
\]

or visually,

\[
\cdots g(t_i) \cdots | i \in I \mapsto \cdots t_i \cdots | i \in I. \]

This important concept can intuitively be regarded as the set of legal tuples under the database header \( S = \langle I, s, X \rangle \). It is define to be the extent in the list type domain \( \text{List}(A) \):

\[
\text{tup}(I, s, A) = \text{ext}_{\text{List}(A)}(I, s) = \{ \langle J, t \rangle \in \text{List}(Y) \mid \langle J, t \rangle \in \text{List}(A) \langle I, s \rangle \}.
\]

Various notations are used for this concept depending upon circumstance:

\[
\text{tup}(S, A) = \text{tup}(I, s, A) \text{ in } \S\text{2.5.2,3,4} = \text{tup}_S(A) = \text{tup}_A(I, s) \text{ in } \S\text{2.5.2,3,4} = \text{tup}_S(A) = \text{tup}_A(I, s) \text{ in } \S\text{2.5.2,3,4}.
\]

The tuple passage \( \text{tup}_S : \text{Cls}(X)^{\text{op}} \to \text{Set} \) maps an \( X \)-sorted type domain \( A = \langle X, Y, \|=_{A} \rangle \) to the tuple set \( \text{tup}_S(A) = \text{tup}_A(I, s) = \prod_{i} A_{i(\text{List}(Y))} \) and maps an \( X \)-sorted type domain morphism \( A \xrightarrow{(\text{List}(g))} \bar{A} \) to the tuple function

\[
\text{tup}_A(I, s) = \prod_{i} A_{i(\text{List}(Y))} \xrightarrow{\text{tup}_S(g)(\text{List}(g))} \prod_{i} \bar{A}_{i(\text{List}(Y))} = \text{tup}_A(I, s),
\]

a restriction of the tuple function \( \text{List}(Y) \xrightarrow{\text{List}(g)} \text{List}(Y) \).

\[\]
Let $S_2 \xrightarrow{(h,f)} S_2$ be a signature morphism. There is an inclusion bridge $f^{-1} \circ \text{inc}_{S_2} \xrightarrow{\iota(h,f)} \text{inc}_{A_1}$ (illustrated below right).

For any target type domain $A_1 = \langle X, Y, \models A_1 \rangle \in \text{Cls}(X_1)$, the signed domain morphism

$$\text{inc}_{S_2}(f^{-1}(A_1)) \xrightarrow{\iota(h,f)(A_1)} \text{inc}_{S_1}(A_1)$$

is illustrated above left. This is natural in type domain. Hence, there is an inclusion passage $\text{List} \xrightarrow{\text{inc}} (\text{Cxt} \downarrow \text{Dom})^{\text{op}}$. Composition of the inclusion bridge $f^{-1} \circ \text{inc}_{S_2} \xrightarrow{\iota(h,f)} \text{inc}_{S_1}$ with the signeated above left domain tuple passage (Def. 1) gives a signature tuple bridge (Fig. 4)

$$f^{-1} \circ \text{tup}_{S_2} \xrightarrow{\tau(h,f)} \text{tup}_{S_1} \xrightarrow{\text{inc}^\text{op}_{S_2} \circ \text{tup}} \text{inc}^\text{op}_{S_1} \circ \text{tup} \xrightarrow{(f^{-1} \circ \text{inc}_{S_2} \xrightarrow{\iota(h,f)} \text{inc}_{S_1})^{\text{op}} \circ \text{tup}}$$

(2)

For any target type domain $A_1 = \langle X, Y, \models A_1 \rangle \in \text{Cls}(X_1)$, the $A_1^{th}$-component of the signature tuple bridge is the tuple function $\tau_{(h,f)}(A_1) = h \cdot (-) : \text{tup}_{S_1}(A_1) = \text{tup}_{S_2}(f^{-1}(A_1))$. This is natural in signature. Hence, there is a tuple passage $\text{List} \xrightarrow{\text{tup}} (\text{Cxt} \uparrow \text{Set})^{\text{op}}$.

Fig. 4. Tuple Bridge: Signature

2.5.2 Type Domains. Let $A = \langle X, Y, \models A \rangle$ be a type domain. There is an inclusion passage $\text{List}(X) \xrightarrow{\text{inc}_A} \text{Dom}$ that maps an X-signature $\langle I, s \rangle$ to the signed domain $\langle I, s, A \rangle$ and maps an X-signature morphism $\langle I_2, s_2 \rangle \xrightarrow{h} \langle I_1, s_1 \rangle$ to the signed domain morphism $\langle I_2, s_2, A \rangle \xrightarrow{(h,1_X,1_Y)} \langle I_1, s_1, A \rangle$. Composition of
the inclusion passage with the signed domain tuple passage (Def.1) gives a type domain tuple passage

\[
\text{List}(X)^\text{op} \xrightarrow{\text{tup}_A = \text{ext}_{\text{List}(A)}} \text{Set},
\]

which maps an \(X\)-signature \((I, s)\) to the tuple set (its \(\text{List}(A)\)-extent) \(\text{tup}(f, s, A) = \text{tup}_A(I, s)\) and maps an \(X\)-signature morphism \((I_2, s_2) \xrightarrow{h} (I_1, s_1)\) to the tuple function associated with \(h\): \(\text{tup}_A(I_2, s_2) \xrightarrow{\text{tup}_A(h)} \text{tup}_A(I_1, s_1)\); or visually, \((\cdots t_{h(i_2)} \cdots | i_2 \in I_2) \leftrightarrow (\cdots t_{i_1} \cdots | i_1 \in I_1)\).

levo: Let \(A_2 \xrightarrow{(f, g)} A_1\) be a type domain morphism.

There is an inclusion bridge \(f^* \circ \text{inc}_{A_2} \xrightarrow{\text{inc}_{A_1}} \text{inc}_{A_1}\) (illustrated above right). For any target signature \((I_1, s_1) \in \text{List}(X_1)\), the signed domain morphism

\[
\text{inc}_{A_2}(f^*(I_1, s_1)) \xrightarrow{\text{inc}_{A_1}} \text{inc}_{A_1}(I_1, s_1)
\]

is define by pullback (illustrated above left). This is natural in signature.\(^{12}\) Hence, there is an inclusion passage \(\text{Cls} \xrightarrow{\text{inc}} (\text{Cxt} \downarrow \text{Dom})^{\text{op}}\).\(^{13}\) Composition of the inclusion bridge \(f^* \circ \text{inc}_{A_2} \xrightarrow{\text{inc}_{A_1}} \text{inc}_{A_1}\) with the signed domain tuple passage (Def.1) gives a type domain tuple bridge (left-side Fig. 4).

\[
\begin{align*}
\text{List}(X_2) & \xrightarrow{f^*} \text{List}(X_1) & \text{List}(X_2) & \xleftarrow{\text{inc}_{A_2}} \text{List}(X_1) \\
\text{Dom} & \xrightarrow{\text{inc}_{A_1}} \text{Dom} & \text{Dom} & \xleftarrow{\text{inc}_{A_1}} \text{Dom}
\end{align*}
\]

\[
\begin{align*}
\text{tup}_{A_2} & \xrightarrow{\text{inc}_{A_2} \circ \text{tup}} \text{tup}_{A_1} & \text{tup}_{A_2} & \xleftarrow{\text{inc}_{A_2} \circ \textup{tup}} \text{tup}_{A_1}
\end{align*}
\]

\[
\begin{align*}
(f^* \circ \text{inc}_{A_2}) \xrightarrow{\text{inc}_{A_1}} \text{inc}_{A_1}
\end{align*}
\]

\[
(3)
\]

\(^{12}\) For any \(X_1\)-signature morphism \((I'_1, s'_1) \xrightarrow{b} (I_1, s_1)\) with inverse image \(X_2\)-signature morphism \((I'_2, s'_2) \xrightarrow{f^*(b)} (I_1, s_1)\) we have the commutative diagram

\[
\begin{align*}
\text{inc}_{A_2}(f^*(I'_1, s'_1)) & \xrightarrow{\text{inc}_{A_1}} \text{inc}_{A_1}(I'_1, s'_1) \\
\text{inc}_{A_2}(f^*(I_1, s_1)) & \xrightarrow{\text{inc}_{A_1}} \text{inc}_{A_1}(I_1, s_1)
\end{align*}
\]

\(^{13}\) For any context \(C\), the “super-comma” context \((\text{Cxt} \downarrow \text{C})\) is defined \(^{20}\) as follows:

(1) an object is a \(C\)-diagram \((I, D)\) with indexing context \(I\) and passage \(I \xrightarrow{D} C\);

(2) a morphism is a \(C\)-diagram morphism \((I_2, D_2) \xrightarrow{(f, \alpha)} (I_1, D_1)\) with indexing passage \(I_2 \xrightarrow{f} I_1\) and bridge \(D_2 \xleftarrow{\alpha} F \circ D_1\).
For any target signature $\langle I_1, s_1 \rangle \in \textbf{List}(X_1)$, the tuple function $\tilde{\tau}_{(f,g)}(I_1, s_1) = f \cdot (-) \cdot g : \text{tup}_{A_1}(I_1, s_1) \to \text{tup}_{A_2}(f^*(I_1, s_1))$ is define by pullback (illustrated above left). This is natural in signature. Hence, there is a tuple passage $\text{Cls} \xrightarrow{\tau_{\text{tup}}} (\text{Cxt} \uparrow \text{Set})^{\text{op}}$.

**dextro:** Let $A_2 \xrightarrow{(f,g)} A_1$ be a type domain morphism.

There is an inclusion bridge $\text{inc}_{A_2} \xrightarrow{\ell_{(f,g)}} \Sigma_f \circ \text{inc}_{A_1}$ (illustrated above right). For any source signature $\langle I_2, s_2 \rangle \in \textbf{List}(X_2)$, the signed domain morphism $\text{inc}_{A_2}(I_2, s_2) \xrightarrow{\ell_{(f,g)}(I_2, s_2)} \text{inc}_{A_1}(\Sigma_f(I_2, s_2))$ is define by composition (illustrated above left). This is natural in signature.

Hence, there is an inclusion passage $\text{Cls} \xrightarrow{\text{inc}} (\text{Cxt} \uparrow \text{Dom})$. Composition of the inclusion bridge $\text{inc}_{A_2} \xrightarrow{\ell_{(f,g)}} \Sigma_f \circ \text{inc}_{A_1}$ with the signed domain tuple passage (Def.1) gives a type domain tuple bridge (right-side Fig. 5).

For any source signature $\langle I_2, s_2 \rangle \in \textbf{List}(X_2)$, the tuple function $\tau_{(f,g)}(I_2, s_2) = (-) \cdot g : \text{tup}_{A_1}(\Sigma_f(I_2, s_2)) \to \text{tup}_{A_2}(I_2, s_2)$ is define by composition (illustrated above left). This is natural in signature. Hence, there is a tuple passage $\text{Cls} \xrightarrow{\text{tup}} (\text{Cxt} \downarrow \text{Set})$. 

```latex
\begin{align*}
\xymatrix{
I_2 \ar[r]^f & I_1 \\
X_2 \ar[r]_g & Y_1 \\
\text{List}(X_2) \ar[u]^{\ell_{(f,g)}} \ar[r] & \Sigma_f \ar[u]^{\ell_f} \\
\text{List}(X_1) \ar[u]_{\text{inc}_{A_2}} & \text{Dom} \ar[l]_{\text{inc}_{A_1}} \\
\xymatrix{
X_2 \ar[r]^{\Sigma_f} \ar[d]_{\text{inc}_{A_2}} & \text{List}(X_1) \ar[d]_{\text{inc}_{A_1}} \\
\Sigma_f \ar[r]_{\Sigma_f \circ \text{inc}_{A_1}} & \text{inc}_{A_1}(\Sigma_f(I_2, s_2)) \\
\text{inc}_{A_2}(I_2, s_2) \ar[r]^{\ell_{(f,g)}(I_2, s_2)} & \Sigma_f \circ \text{inc}_{A_1}(\Sigma_f(I_2, s_2)) \\
\text{inc}_{A_2}(I_2, s_2) \ar[r]_{\ell_{(f,g)}(I_2, s_2)} & \Sigma_f \circ \text{inc}_{A_1}(\Sigma_f(I_2, s_2)) \\
\text{inc}_{A_2}(I_2, s_2) \ar[r]_{\Sigma_f \circ \text{inc}_{A_1}(\Sigma_f(I_2, s_2))} & \Sigma_f \circ \text{inc}_{A_1}(\Sigma_f(I_2, s_2)) \\
\end{align*}
```
Lemma 1. There are natural isomorphisms\(^\text{[14]}\)

\[
\sum_f \circ \text{inc}_{A_1} \cong \text{inc}_{A_2} \cong \text{inc}_{A_1} \cong \sum_f \circ \text{inc}_{A_2}
\]

with \(\iota_{(f,g)} = (\eta_f \circ \text{inc}_{A_2}) \bullet (\sum_f \circ \iota_{(f,g)})\) and \(\iota_{(f,g)} = (\varepsilon_f \circ \text{inc}_{A_1})\);
and
\[
f^{op} \circ \text{tup}_{A_2} \cong \text{tup}_{A_2} \cong \sum_{f} \circ \text{tup}_{A_2}
\]

with \(\hat{\iota}_{(f,g)} = (\varepsilon_{f} \circ \text{tup}_{A_1}) \bullet (f^{op} \circ \hat{\iota}_{(f,g)})\) and \(\hat{\iota}_{(f,g)} = (\sum_{f} \circ \hat{\iota}_{(f,g)}) \bullet (\eta_{f}^{op} \circ \text{tup}_{A_2})\).

\begin{align*}
\text{levo} & \quad \text{dextro} \\
\text{List}(X_2) & \xrightarrow{f^{op}} \text{List}(X_1) & \text{List}(X_2) & \xrightarrow{\sum_{f}} \text{List}(X_1) \\
\text{inc}_{A_2} & \xrightarrow{\iota_{(f,g)}} \text{inc}_{A_1} & \text{inc}_{A_2} & \xrightarrow{\iota_{(f,g)}} \text{inc}_{A_1} \\
\text{Dom} & \quad \text{Dom} \\
\text{List}(X_2)^{op} & \xleftarrow{(f^{op})^{op}} \text{List}(X_1)^{op} & \text{List}(X_2)^{op} & \xleftarrow{(\sum_{f})^{op}} \text{List}(X_1)^{op} \\
\text{tup}_{A_2} & \xleftrightarrow{\hat{\iota}_{(f,g)}} \text{tup}_{A_1} & \text{tup}_{A_2} & \xleftrightarrow{\hat{\iota}_{(f,g)}} \text{tup}_{A_1} \\
\text{Set} & \quad \text{Set}
\end{align*}

Fig. 5. Tuple Bridge: Type Domain

\[
\hat{\iota}_{(f,g)} : f^{op} \circ \text{tup}_{A_2} \cong \text{tup}_{A_1} \\
\hat{\iota}_{(f,g)} = (\varepsilon_{f}^{op} \circ \text{tup}_{A_1}) \bullet (f^{op} \circ \hat{\iota}_{(f,g)})
\]

\[
\hat{\iota}_{(f,g)} : \sum_{f} \circ \text{tup}_{A_2} \cong \text{tup}_{A_1} \\
\hat{\iota}_{(f,g)} = (\sum_{f} \circ \hat{\iota}_{(f,g)}) \bullet (\eta_{f}^{op} \circ \text{tup}_{A_2})
\]

---

\(^{14}\) For adjunction \(A_2 \xrightarrow{(F,G,\eta,\varepsilon)} A_1\) with left adjoint \(A_2 \xrightarrow{\eta} A_1\), right adjoint \(A_2 \xleftarrow{\varepsilon} A_1\), unit \(I_{A_2} \xrightarrow{\eta} F \circ G\) and counit \(G \circ F \xleftarrow{\varepsilon} I_{A_1}\), there is an natural isomorphism

\[
F \circ A_1 \xleftarrow{\hat{\alpha}} A_2 \cong A_1 \xleftarrow{\hat{\alpha}} G \circ A_2
\]

with \(\hat{\alpha} = (\eta \circ A_2) \bullet (F \circ \hat{\alpha})\) and \(\hat{\alpha} = (G \circ \hat{\alpha}) \bullet (\varepsilon \circ A_1)\).
Proposition 1. There are inclusion/tuple passages from the context of type domains to the lax comma context of adjointly connected presheaves:

\[
\begin{align*}
\text{Cls} & \xrightarrow{\text{inc}} (\text{Adj} \uparrow \text{Dom}) \\
\text{Cls}^{\text{op}} & \xrightarrow{\text{tup}} (\text{Adj} \downarrow \text{Set})
\end{align*}
\]
2.5.3 Tuple Function Factorization

In § we composed with the signed domain tuple passage (Def. 1) to define the tuple passage and bridge for both signatures and type domains. Here, we factor components of the signed domain tuple passage in terms of components of these defined notions.

**Lemma 2.** For any signed domain morphism \( \langle I_2, s_2, A_2 \rangle \xrightarrow{h \cdot f \cdot g} \langle I_1, s_1, A_1 \rangle \), the tuple function \( \text{tup}_{A_2}(I_2, s_2) \xleftarrow{\text{tup}(h, f, g)} \text{tup}_{A_1}(I_1, s_1) \) has two factorizations:

- (Fig. 4 left side) in terms of the signature tuple bridge of § (Fig. 3) (used in the table fiber passage along a signature morphism).
- (Fig. 4 right side) in terms of the type domain tuple bridges of § (Fig. 5) (used in the table fiber adjoint passages along a type domain morphism).

\[
\begin{align*}
\text{tup}_{S_2}(A_2) &= \text{ext}_{\text{List}(A_2)}(I_2, s_2) \\
\text{tup}_{S_2}(g) &= (\cdot) \cdot g \\
\text{tup}_{S_2}(f^{-1}(A_1)) &= \text{ext}_{\text{List}(f^{-1}(A_1))}(I_2, s_2) \\
\text{tup}_{A_2}(h, f, g) &= (h \cdot (\cdot)) \cdot ((\cdot) \cdot g) \\
\text{tup}_{A_2}(\hat{h}) &= h \cdot (\cdot) \\
\text{tup}_{A_2}(\hat{h}) &= \text{ext}_{\text{List}(A_2)}(I_1, s_1) \\
\text{tup}_{A_2}(f^*(I_1, s_1)) &= \text{ext}_{\text{List}(A_2)}(f^*(f^*(I_1, s_1))) \\
\text{tup}_{A_2}(I_1, s_1) &= \text{ext}_{\text{List}(A_2)}(I_1, s_1) \\
\text{tup}_{A_2}(h) &= \text{ext}_{\text{List}(A_2)}(I_1, s_1) \\
\text{tup}_{A_2}(f^*(I_1, s_1)) &= \text{ext}_{\text{List}(A_2)}(f^*(I_1, s_1)) \\
\text{tup}_{A_2}(I_1, s_1) &= \text{ext}_{\text{List}(A_2)}(I_1, s_1) \\
\end{align*}
\]

**Proof.** We prove the type domain case (Fig. 4 left side).

For any target signature \( \langle I_1, s_1 \rangle \in (\text{Set}_\downarrow X_1) \), if \( \langle I_2, s_2 \rangle = f^*(I_1, s_1) \) is its substitution signature (defined by pullback), the tuple function \( \hat{f}_{(g)}(I_1, s_1) : \text{tup}_{A_1}(I_1, s_1) \xrightarrow{\text{tup}(\hat{f}, g)} \text{tup}_{A_1}(f^*(I_1, s_1)) \) (Fig. 5) maps a target tuple \( t_1 \in \text{tup}(S_1) = \text{tup}_{A_1}(I_1, s_1) \) to the intermediate tuple \( \hat{t}_1 = f \cdot t_1 \cdot g \in \text{tup}_{A_2}(\hat{I}_1, \hat{s}_1) \), where \( \hat{f} = \varepsilon^f(I_1, s_1) \) is the \( I_1, s_1 \)th-component of the counit \( \varepsilon^f \) of the signature fiber adjunction \( \text{List}(X_2) \xrightarrow{\varepsilon^f = f^*} \text{List}(X_1) \). Signature preservation, \( s_2 \cdot f = h \cdot s_1 \) means that \( X_2 \xrightarrow{\varepsilon} I_2 \xrightarrow{h} I_1 \) is a span of the opspan \( X_2 \xleftarrow{\varepsilon} X_1 \xleftarrow{h} I_1 \). Let \( \hat{I}_2 \xrightarrow{\hat{f}} \hat{I}_1 \) be the mediating function, so that \( \langle \hat{h}, 1_{X_2} \rangle : \langle I_2, s_2 \rangle \xrightarrow{\langle \hat{h}, 1_{X_2} \rangle} \langle \hat{I}_2, s_2 \rangle \) is a signature morphism and \( \hat{h} \cdot \hat{f} = \hat{h} \). Then the tuple function \( \text{tup}_{A_2}(\hat{h}) : \text{tup}_{A_2}(f^*(I_1, s_1)) \xrightarrow{\text{tup}(\hat{h}, \hat{f})} \text{tup}_{A_2}(I_2, s_2) \) maps the intermediate tuple \( \hat{t}_1 \in \text{tup}_{A_2}(\hat{I}_1, \hat{s}_1) \) to the source tuple \( t_2 = \hat{h} \cdot \hat{t}_1 \in \text{tup}_{A_2}(I_2, s_2) \). Since pullbacks compose, this is functorial.
signed domain morphism
\[ \langle I_2, X_2, A_2 \rangle \xrightarrow{(h,f,g)} \langle I_1, X_1, A_1 \rangle \]
\[ \text{tup}_{A_2}(I_2, s_2) \xrightarrow{\text{tup}(h,f,g)} \text{tup}_{A_1}(I_1, s_1) \]

signature morphism
\[ S_2 \xrightarrow{(h,f)} S_1 \]
\[ \tau(h,f) : (f^{-1})^* \circ \text{tup}_S \xleftarrow{\text{tup}} S_1 \]
\[ \text{tup}_{S_2}(f^{-1}(A_1)) \xleftarrow{\tau(h,f)} \text{tup}_{S_1}(A_1) \]

type domain morphism
\[ A_2 \xrightarrow{(f,g)} A_1 \]
\[ \text{levo tuple bridge} \]
\[ \tilde{\tau}(f,g) : (f^*)^* \circ \text{tup}_{A_2} \xleftarrow{\text{tup}} A_1 \]
\[ \text{tup}_{A_2}(f^*(I_1, s_1)) \xleftarrow{\tilde{\tau}(f,g)} \text{tup}_{A_1}(I_1, s_1) \]
\[ \text{dextro tuple bridge} \]
\[ \tilde{\tau}(f,g) : \Sigma^* \circ \text{tup}_{A_2} \xleftarrow{\text{tup}} A_1 \]
\[ \text{tup}_{A_2}(I_2, s_2) \xleftarrow{\tilde{\tau}(f,g)} \text{tup}_{A_1}(\Sigma f(I_2, s_2)) \]

**Table 3. Tuple Functions**
3 Table Hierarchy

The relational table is the basic concept in the relational model for databases.

3.1 FOLE Tables

Tables. A table (database relation) $T = \langle D, K, t \rangle$ consists of a signed domain $D$, a set $K$ of (primary) keys and a tuple function $K \xrightarrow{t} \text{tup}(D)$ mapping keys to $D$-tuples.\(^\text{15}\) Equivalently, it is an object in the comma context $(\text{Set} \downarrow \text{tup})$ defined by the tuple passage $\text{Dom}^{\text{op}} \xrightarrow{\text{tup}} \text{Set}$ (Def. II of §2.4). A precise description of the FOLE Table is given in Fig. 7.

Hence, a table $T = \langle K, t, D \rangle$ consists of a set $K$ of keys and a tuple function $K \xrightarrow{t} \text{tup}(D)$. Hence, a table is an object in the fiber context $\text{Tbl}(D)$. (See §3.2.)

signature: Given a signature $S = \langle I, s, X \rangle$, a table $T = \langle A, K, t \rangle$ consists of an $X$-sorted type domain $A = \langle X, Y, \models A \rangle$, a set $K$ of keys, and a tuple function $K \xrightarrow{t} \text{tup}(A, I, s)$. Hence, a table is an object in the fiber context $\text{Tbl}(S)$. (See §3.3.1)

type domain: Given a type domain $A = \langle X, Y, \models A \rangle$, a table $T = \langle S, K, t \rangle$ consists of an $X$-sorted signature $S = \langle I, s, X \rangle$, a set $K$ of keys, and a tuple function $K \xrightarrow{t} \text{tup}(A, I, s)$. Hence, a table is an object in the fiber context $\text{Tbl}(A)$. (See §3.4.1)

\(^{15}\) FOLE tables correspond to improper relations (Codd [3]), since they strictly violate the property the "all rows are distinct from one another in content". Proper relations correspond to FOLE relations (§A.1). One method for converting to the proper relations of Codd, and thus getting an injective tuple function, is to incorporate keys into their corresponding tuple by defining a key datatype. This was done in Kent [15].
defining $g(y_1) \models_{A_2} x_2$ if $y_1 \models_{A_1} f(x_2)$

conditions

$s_2 \cdot f = h \cdot s_1$

$k \cdot t_2 = t_1 \cdot (h \cdot (\cdot \cdot)) \cdot (t \cdot (\cdot \cdot - g))$

implies $t_{k_2, i_2} = s_2 \models_{A_2} (t_{i_2, t_1}) \models_{A_1} s_1 (i_1)$

since $t_{k_2, i_2} = h \cdot t_{i_1, k_1} \cdot g$,

and $t_{k_2, i_2} = g (t_{i_1, k_1}) \models_{A_1} s_1 (i_1) = f (s_2 (i_2))$

where $k_1 \in K_1, i_2 \in I_2, k_2 = k (k_1) \in K_2, i_1 = h (i_2) \in I_1$

This four-part figure illustrates the defining conditions on table morphisms.

It has been annotated to help guide the understanding. The condition is symbolically stated in terms of set functions in the line of text just above. The top left diagram illustrates the condition, and the bottom left diagram expands on this. The top right diagram text is more detailed in terms of a source row (tuple) $k_1 \in K_1$ and a target column (attribute) $i_2 \in I_2$. Here we see appearance of the infomorphism condition $g (t_1 (k_1))_i \models_{A_2} s_2 (i_1) \models_{A_1} f (s_2 i_2)$.

Finally, the bottom right figure illustrates the meaning of the morphism’s defining condition with respect to source/target tables $T_1$ and $T_2$.

Fig. 8. FOLE Table Morphism
Table Morphisms. A table morphism (morphism of database relations)

\[ T_2 = \langle \langle I_2, s_2, A_2 \rangle, K_2, t_2 \rangle \xleftrightarrow{(h,f,g,k)} \langle \langle I_1, s_1, A_1 \rangle, K_1, t_1 \rangle = T_1 \]

consists of a signed domain morphism \( \langle I_2, s_2, A_2 \rangle \xrightarrow{(h,f,g)} \langle I_1, s_1, A_1 \rangle \) and a key function \( K_2 \xleftarrow{k} K_1 \), which satisfy the naturality condition \( k \cdot t_2 = t_1 \cdot \text{tup}(h, f, g) \).

Hence, a table morphism \( T_2 = \langle S_2, A_2, K_2, t_2 \rangle \xleftrightarrow{(h,f,g,k)} \langle S_1, A_1, K_1, t_1 \rangle = T_1 \) consists of a signature morphism \( A_2 = \langle I_2, s_2, X_2 \rangle \xrightarrow{(h,f)} \langle I_1, s_1, X_1 \rangle = S_1 \) and a type domain morphism \( A_2 = \langle X_2, Y_2, \models A_2 \rangle \xrightarrow{(f,g)} \langle X_1, Y_1, \models A_1 \rangle = A_1 \) with common sort function \( X_2 \xrightarrow{f} X_1 \), and a key function \( K_2 \xleftarrow{k} K_1 \), which satisfy the naturality condition above.

Table morphisms are illustrated in Fig. 8. Here we see that table morphisms have the pleasing property that corresponding entries in the source and target tables satisfy the infomorphism condition from the theory of information flow (Barwise and Seligman [1]). Composition of morphisms is defined component-wise. Let

\[ \text{Set} \xleftarrow{\text{key}} \text{Tbl} = (\text{Set} \downarrow \text{tup}) \xrightarrow{\text{dom}} \text{Dom}^{op} \]

denote the comma context of tables (Fig. 9) with the key/signed-domain projection passages. There is a defining tuple bridge \( \text{key} \xRightarrow{\tau} \text{dom} \circ \text{tup} \), whose \( T \)th component is the tuple function \( K \xrightarrow{\tau} \text{tup}(D) \). Composition yields signature/type-domain projection passages \( \text{List}^{op} \xleftarrow{\text{sign}} \text{Tbl} \xrightarrow{\text{data}} \text{Cls}^{op} \). We can have three

Fig. 9. FOLE Table Mathematical Context

indexing contexts for tables (above diagram): signatures \( \text{List} \), type domains \( \text{Cls} \) and signed domains \( \text{Dom} \). Each has their uses: signature indexing follows the true formal-semantics distinction, type domain indexing proves that the context of tables is complete (§ 4.2.2) (and the fibers help explain database fibers), and signed domain indexing proves that the context of tables is cocomplete (§ 4.2.1).

---

16 Since the table tuple function embodies the entity/domain integrity constraints, this condition on morphisms asserts the preservation of data integrity.
Corresponding to this indexing (as illustrated in Fig. 13), there are two chains of fiber contexts: fibers indexed by a signed domain $D = \langle I, s, A \rangle$ are smallest, and contained in either fibers indexed by a type domain $A = \langle X, Y, \mid = \rangle$ or fibers indexed by a signature $S = \langle I, s, X \rangle$.

**Restatement:** We now sharpen the definition for the context of tables. This will be useful for defining and working with relational databases. The fibered context of tables is the comma mathematical context

$$
\text{Tbl} = (\text{Set} \downarrow \text{tup})
$$

for the opspan of passages $\text{Set} \xrightarrow{T} \text{Set} \xleftarrow{\text{tup}} \text{Dom}^{\text{op}}$. It has the key and signed domain projection passages $\text{Set} \xrightarrow{\text{key}} \text{Tbl} \xrightarrow{\text{dom}} \text{Dom}^{\text{op}}$ and the defining tuple bridge $\text{key} \xmapsto{\tau} \text{dom} \circ \text{tup}$ such that for any span of passages $\text{Set} \xleftarrow{K} \text{R} \xrightarrow{Q} \text{Dom}^{\text{op}}$, there is a bijection $T \mapsto \tilde{T} = T \circ \tau$ between

- passages $\text{R} \xrightarrow{T} \text{Tbl}$ satisfying $T \circ \text{key} = K$ and $T \circ \text{dom} = Q$, and
- bridges $\text{K} \xmapsto{\tilde{\tau}} Q \circ \text{tup}$.

By composition, there are also signature and classification projection passages $\text{List}^{\text{op}} \xleftarrow{\text{sign}} \text{Tbl} \xrightarrow{\text{data}} \text{Cls}^{\text{op}}$.

### 3.2 Signed Domain Indexing

In this section we show that the context of tables is a fibered context over signed domains. We first define the table fiber for fixed signed domain. We next move between table fibers along signed domain morphisms. Finally, we invoke the Grothendieck construction indexed by signed domains.

**Fiber Contexts (small-size).** Let $\langle I, s, A \rangle$ be a fixed signed domain. The fiber mathematical context of $\langle I, s, A \rangle$-tables is the comma context

$$
\text{Tbl}(I, s, A) = \text{Tbl}_{A}(I, s) = (\text{Set} \downarrow \text{tup}_{A}(I, s))
$$

associated with the tuple set (constant passage) $1 \xrightarrow{\text{tup}_{A}(I, s)} \text{Set}$. It has the key and trivial projection passages $\text{Set} \xleftarrow{\text{key}_{A}(I, s)} \text{Tbl}_{A}(I, s) \xrightarrow{\Delta} 1$ and the defining tuple bridge $\text{key}_{A}(I, s) \xRightarrow{\tau_{A}(I, s)} \Delta \circ \text{tup}_{A}(I, s)$. A $\text{Tbl}_{A}(I, s)$-object $T = \langle K, t \rangle$, called an $\langle I, s, A \rangle$-table, consists of a set $K$ of (primary) keys and a tuple function $K \xrightarrow{t} \text{tup}_{A}(I, s)$ mapping each key to its descriptor $A$-tuple of type (signature) $\langle I, s \rangle$. A $\text{Tbl}_{A}(I, s)$-morphism $T' = \langle K', t' \rangle \xleftarrow{k} \langle K, t \rangle = T$ is a key function $K' \xrightarrow{k} K$ that preserves descriptors by satisfying the naturality condition $k \cdot t' = t$. 
Fibered Context (large-size). The fibered context of tables \( Tbl^{op} \xrightarrow{\text{dom}} \text{Dom} \) is defined as follows. We use the same definitions as in §B.1. A \( Tbl \)-object \( \mathcal{T} = \langle I, s, \mathcal{A}, K, t \rangle \), called an \( I \)-table, consists of a signed domain \( \text{dom}(\mathcal{T}) = \langle I, s, \mathcal{A} \rangle \) and an \( \langle I, s, \mathcal{A} \rangle \)-table \( (K, t) \). A \( Tbl \)-morphism \( \mathcal{T}_2 = \langle I_2, s_2, \mathcal{A}_2, K_2, t_2 \rangle \xleftarrow{\langle h, f, g, k \rangle} \langle I_1, s_1, \mathcal{A}_1, K_1, t_1 \rangle = \mathcal{T}_1 \) called an \( I \)-table morphism, consists of Morphism a signed domain morphism \( (I_2, X_2, \mathcal{A}_2) \xrightarrow{\langle h, f, g \rangle} (I_1, X_1, \mathcal{A}_1) \) and a key function \( K_2 \xleftarrow{k} K_1 \) satisfying the naturality condition \( k \cdot t_2 = t_1 \cdot \text{tup}(h, f, g) \). This condition gives two alternate and adjoint definitions. In terms of fibers, an \( I \)-table morphism consists of a signed domain morphism \( (I_2, s_2, \mathcal{A}_2) \xrightarrow{\langle h, f, g \rangle} (I_1, s_1, \mathcal{A}_1) \) and either a morphism \( \mathcal{T}_2 \xleftarrow{k} \Sigma_{\langle h, f, g \rangle}(\mathcal{T}_1) \) in the fiber context \( Tbl(I_2, s_2, \mathcal{A}_2) \) or a morphism \( \langle h, f, g \rangle^*(\mathcal{T}_2) \xleftarrow{k'} \mathcal{T}_1 \) in the fiber context \( Tbl(I_1, s_1, \mathcal{A}_1) \).

\[
\begin{array}{c}
K_2 \xleftarrow{k} K_1 = \mathcal{T}_1 \\
\xRightarrow{t_1 \cdot \text{tup}(h, f, g)} \quad \xRightarrow{t_1} \\
\Sigma_{\langle h, f, g \rangle}(\mathcal{T}_1) \quad \mathcal{T}_1 \\
\xRightarrow{\text{tup}(I_2, s_2, \mathcal{A}_2)} \quad \text{tup}(I_1, s_1, \mathcal{A}_1) \\
Tbl(I_2, s_2, \mathcal{A}_2) \\
\end{array}
\]

\[
\begin{array}{c}
K_2 \xleftarrow{e} K_1 \xleftarrow{k'} K_1 \\
\xRightarrow{t_2} \quad \xRightarrow{t_1} \\
\langle h, f, g \rangle^*(\mathcal{T}_2) \quad \mathcal{T}_1 \\
\xRightarrow{\text{tup}(I_2, s_2, \mathcal{A}_2)} \quad \text{tup}(I_1, s_1, \mathcal{A}_1) \\
Tbl(I_1, s_1, \mathcal{A}_1) \\
\end{array}
\]

Fig. 10. Table Morphism: Signed Domain

The \( \langle I_2, s_2, \mathcal{A}_2 \rangle \)-table morphism \( \mathcal{T}_2 \xleftarrow{k} \Sigma_{\langle h, f, g \rangle}(\mathcal{T}_1) \) is the composition (RHS of Fig. 10) of the fiber morphism \( \Sigma_{\langle h, f, g \rangle}(\langle h, f, g \rangle^*(\mathcal{T}_2) \xleftarrow{k'} \mathcal{T}_1) \) with the \( \mathcal{T}_2 \)th counit component \( \mathcal{T}_2 \xleftarrow{e} \Sigma_{\langle h, f, g \rangle} \langle h, f, g \rangle^*(\mathcal{T}_2) \) for the fiber adjunction

\[
Tbl(I_2, s_2, \mathcal{A}_2) \xrightarrow{\Sigma_{\langle h, f, g \rangle}} Tbl(I_1, s_1, \mathcal{A}_1).
\]

This fiber adjunction (top part of Tbl. 1) is a component of the signed domain indexed adjunction of tables \( \text{Dom}^{op} \xrightarrow{\text{tbl}} \text{Adj} \).

**Theorem 3.** The fibered context of tables \( Tbl \xrightarrow{\text{dom}} \text{Dom}^{op} \) is the Grothendieck construction \( \int_{\text{Dom}} \) of the signed domain indexed adjunction \( \text{Dom}^{op} \xrightarrow{\text{tbl}} \text{Adj} \).
\[(I_2, X_2, A_2) \xrightarrow{\langle h, f, g \rangle} (I_1, X_1, A_1)\]

\[\text{tup}_{A_2}(I_2, s_2) \xrightarrow{\text{tup}(h, f, g)} \text{tup}_{A_1}(I_1, s_1)\]

\[\text{Tbl}_{A_2}(I_2, s_2) \xrightarrow{\Sigma(h, f, g)} \text{Tbl}_{A_1}(I_1, s_1)\]

\[\text{im}_{A_2}(I_2, s_2) \xrightarrow{\text{inc}_{A_2}(I_2, s_2)} \text{im}_{A_1}(I_1, s_1) \xrightarrow{\text{inc}_{A_1}(I_1, s_1)} \text{Rel}_{A_2}(I_2, s_2) \xrightarrow{\text{Rel}_{A_1}(I_1, s_1)} \]

\[\therefore \langle h, f, g \rangle \circ \text{im}_{A_2}(I_2, s_2) \cong \text{im}_{A_1}(I_1, s_1) \circ \text{inc}_{A_1}(I_1, s_1)\]

\[\therefore \langle h, f, g \rangle \circ \text{inc}_{A_2}(I_2, s_2) \cong \text{inc}_{A_1}(I_1, s_1) \circ \Sigma(h, f, g)\]

\[\therefore \langle h, f, g \rangle \circ \text{Rel}_{A_2}(I_2, s_2) \cong \text{Rel}_{A_1}(I_1, s_1) \circ \Sigma(h, f, g)\]

\[\therefore \langle h, f, g \rangle \circ \text{Rel}_{A_2}(I_2, s_2) \cong \text{Rel}_{A_1}(I_1, s_1) \circ \Sigma(h, f, g)\]

\[\therefore \langle h, f, g \rangle \circ \text{Rel}_{A_2}(I_2, s_2) \cong \text{Rel}_{A_1}(I_1, s_1) \circ \Sigma(h, f, g)\]

\[\therefore \langle h, f, g \rangle \circ \text{Rel}_{A_2}(I_2, s_2) \cong \text{Rel}_{A_1}(I_1, s_1) \circ \Sigma(h, f, g)\]

\[\therefore \langle h, f, g \rangle \circ \text{Rel}_{A_2}(I_2, s_2) \cong \text{Rel}_{A_1}(I_1, s_1) \circ \Sigma(h, f, g)\]

\[\therefore \langle h, f, g \rangle \circ \text{Rel}_{A_2}(I_2, s_2) \cong \text{Rel}_{A_1}(I_1, s_1) \circ \Sigma(h, f, g)\]

\[\therefore \langle h, f, g \rangle \circ \text{Rel}_{A_2}(I_2, s_2) \cong \text{Rel}_{A_1}(I_1, s_1) \circ \Sigma(h, f, g)\]

\[\therefore \langle h, f, g \rangle \circ \text{Rel}_{A_2}(I_2, s_2) \cong \text{Rel}_{A_1}(I_1, s_1) \circ \Sigma(h, f, g)\]

\[\therefore \langle h, f, g \rangle \circ \text{Rel}_{A_2}(I_2, s_2) \cong \text{Rel}_{A_1}(I_1, s_1) \circ \Sigma(h, f, g)\]

\[\therefore \langle h, f, g \rangle \circ \text{Rel}_{A_2}(I_2, s_2) \cong \text{Rel}_{A_1}(I_1, s_1) \circ \Sigma(h, f, g)\]

\[\therefore \langle h, f, g \rangle \circ \text{Rel}_{A_2}(I_2, s_2) \cong \text{Rel}_{A_1}(I_1, s_1) \circ \Sigma(h, f, g)\]

\[\therefore \langle h, f, g \rangle \circ \text{Rel}_{A_2}(I_2, s_2) \cong \text{Rel}_{A_1}(I_1, s_1) \circ \Sigma(h, f, g)\]

\[\therefore \langle h, f, g \rangle \circ \text{Rel}_{A_2}(I_2, s_2) \cong \text{Rel}_{A_1}(I_1, s_1) \circ \Sigma(h, f, g)\]

\[\therefore \langle h, f, g \rangle \circ \text{Rel}_{A_2}(I_2, s_2) \cong \text{Rel}_{A_1}(I_1, s_1) \circ \Sigma(h, f, g)\]

\[\therefore \langle h, f, g \rangle \circ \text{Rel}_{A_2}(I_2, s_2) \cong \text{Rel}_{A_1}(I_1, s_1) \circ \Sigma(h, f, g)\]

\[\therefore \langle h, f, g \rangle \circ \text{Rel}_{A_2}(I_2, s_2) \cong \text{Rel}_{A_1}(I_1, s_1) \circ \Sigma(h, f, g)\]

\[\therefore \langle h, f, g \rangle \circ \text{Rel}_{A_2}(I_2, s_2) \cong \text{Rel}_{A_1}(I_1, s_1) \circ \Sigma(h, f, g)\]

\[\therefore \langle h, f, g \rangle \circ \text{Rel}_{A_2}(I_2, s_2) \cong \text{Rel}_{A_1}(I_1, s_1) \circ \Sigma(h, f, g)\]

\[\therefore \langle h, f, g \rangle \circ \text{Rel}_{A_2}(I_2, s_2) \cong \text{Rel}_{A_1}(I_1, s_1) \circ \Sigma(h, f, g)\]

\[\therefore \langle h, f, g \rangle \circ \text{Rel}_{A_2}(I_2, s_2) \cong \text{Rel}_{A_1}(I_1, s_1) \circ \Sigma(h, f, g)\]

\[\therefore \langle h, f, g \rangle \circ \text{Rel}_{A_2}(I_2, s_2) \cong \text{Rel}_{A_1}(I_1, s_1) \circ \Sigma(h, f, g)\]

\[\therefore \langle h, f, g \rangle \circ \text{Rel}_{A_2}(I_2, s_2) \cong \text{Rel}_{A_1}(I_1, s_1) \circ \Sigma(h, f, g)\]

\[\therefore \langle h, f, g \rangle \circ \text{Rel}_{A_2}(I_2, s_2) \cong \text{Rel}_{A_1}(I_1, s_1) \circ \Sigma(h, f, g)\]

\[\therefore \langle h, f, g \rangle \circ \text{Rel}_{A_2}(I_2, s_2) \cong \text{Rel}_{A_1}(I_1, s_1) \circ \Sigma(h, f, g)\]

\[\therefore \langle h, f, g \rangle \circ \text{Rel}_{A_2}(I_2, s_2) \cong \text{Rel}_{A_1}(I_1, s_1) \circ \Sigma(h, f, g)\]

\[\therefore \langle h, f, g \rangle \circ \text{Rel}_{A_2}(I_2, s_2) \cong \text{Rel}_{A_1}(I_1, s_1) \circ \Sigma(h, f, g)\]

\[\therefore \text{small fibers} – \text{long distance}\]

**Table 4.** Reflection: Signed Domain
3.3 Signature Indexing

In this section we show that the context of tables is a fibered context over signatures. We first define the table fiber for fixed signature. We next move between table fibers along signature morphisms. Finally, we invoke the Grothendieck construction indexed by signatures.

3.3.1 Lower Aspect. Let $S = \langle I, s, X \rangle$ be a fixed signature. For database tables, the signature (header) $S$ consists of a fixed sort set $X$ and a fixed $X$-signature $\langle I, s \rangle$. Here, we show that the context of $S$-tables $\text{Tbl}(S)$ is fibered over $X$-sorted type domains $\text{Cls}(X)$ with data $S \rightarrow \text{Cls}(X)^{\text{op}}$. We use the Grothendieck construction $\int \text{Cls}(X)$ on the indexed adjunction $\text{Cls}(X) \xrightarrow{\text{tbl}} \text{Adj} : \mathcal{A} \mapsto \text{Tbl}_S(\mathcal{A})$.

**Fiber(ed) Contexts (medium-size).** The fiber(ed) mathematical context of $S$-tables is the comma context $\text{Set} \xleftarrow{k \downarrow} \text{Tbl}(S) = \text{(Set} \downarrow \text{tup}_{S}) \xrightarrow{\text{data}_{S}} \text{Cls}(X)^{\text{op}}$. Associated with the signature tuple passage $\text{Cls}(X)^{\text{op}} \xrightarrow{\text{tup}_{S}} \text{Set}$ defined in §2.5.1. An $\text{Tbl}(S)$-object $T = \langle A, K, t \rangle$ called an $S$-table consists of an $X$-sorted type domain $A = \langle X, Y, \mid = A \rangle$ with data-type collection $\{ A_x \mid x \in X \}$, a set $K$ of (primary) keys and a tuple function $K \xleftarrow{k} \text{tup}_{S}(I, s) = \prod_{i \in I} A_s(i)$ mapping each key to its descriptor $A$-tuple of type $(signature) (I, s)$. An $\text{Tbl}(S)$-morphism $T = \langle A, K, t \rangle \xleftarrow{(g,k)} \langle \tilde{A}, \tilde{K}, \tilde{t} \rangle = \tilde{T}$ consists of an $X$-sorted type domain morphism $A \xleftarrow{(l_x,g)} \tilde{A}$ and a key function $K \xleftarrow{k} \tilde{K}$, which satisfies the condition $k \cdot t = \tilde{t} \cdot \text{tup}_{S}(g)$. In terms of fibers, an $S$-table morphism consists of an $X$-sorted type domain morphism $A \xleftarrow{(l_x,g)} \tilde{A}$ and either a morphism $T \xleftarrow{k} \sum_{g}(\tilde{T})$ in the fiber context $\text{Tbl}_S(A)$ or a morphism $g^*(T) \xleftarrow{k} \tilde{T}$ in

![Fig. 11. S-Table Morphism](image)
the fiber context $\text{Tbl}_S(\bar{A})$. The $(I, s, \mathcal{A})$-table morphism $\mathcal{T} \xleftarrow{\kappa} \Sigma_g(\bar{T})$ is the composition (RHS of Fig. 11) of the fiber morphism $\Sigma_g\left(g^*(\mathcal{T}) \xleftarrow{\kappa} \bar{T}\right)$ with the $\mathcal{T}^{th}$ counit component $\mathcal{T} \xleftarrow{\epsilon} \Sigma_g\left(g^*(\mathcal{T})\right)$ for fiber adjunction $\langle \Sigma_g \dashv g^* \rangle$ : $\text{Tbl}_S(\mathcal{A}) \dashv \text{Tbl}_S(\bar{A})$.\footnote{Here, the span $K' \xleftarrow{\epsilon} \bar{K} \xrightarrow{i} \text{tup}_S(\bar{A})$ is the pullback in the context $\textbf{Set}$ of the opspan $K \xrightarrow{\iota} \text{tup}_S(\bar{A}) \xleftarrow{\text{tup}_S(g)} \text{tup}_S(\bar{A})$ and $\bar{K} \xleftarrow{i} \bar{K}$ is the mediating morphism for the span $K \xleftarrow{\kappa} \bar{K} \xrightarrow{i} \text{tup}_S(\bar{A})$.}

$$\mathcal{T} \xleftarrow{\kappa} \Sigma_g(\bar{T}) \quad \overset{\text{in } \text{Tbl}_S(\mathcal{A})}{\Rightarrow} \quad g^*(\mathcal{T}) \xleftarrow{\kappa} \bar{T} \quad \overset{\text{in } \text{Tbl}_S(\bar{A})}{\Rightarrow}$$ (5)

This fiber adjunction (top part of Tbl. 5) is a component of the signed domain indexed adjunction of tables $\text{Cls}(X)^{op} \xrightarrow{\text{tup}_S} \textbf{Adj} : \mathcal{A} \mapsto \text{Tbl}_S(\mathcal{A})$.

| $\mathcal{T}$ | $\text{Tbl}_S(\mathcal{A})$ | $\text{tup}_S(\bar{A})$ |
|--------------|--------------------------|-------------------------|
| $\text{Set} \downarrow \text{tup}_S(I, s)$ | $\Sigma_g$ | $g^*$ |
| $\Pi_g$ | $\bar{T}$ | $\text{Set} \downarrow \text{tup}_S(I, s)$ |
| $\text{im}_\mathcal{A}(I, s)$ | $\text{inc}_\mathcal{A}(I, s)$ | $\text{im}_{\bar{\mathcal{A}}}(I, s)$ |
| $\text{Rel}_{\mathcal{A}}(\mathcal{A})$ | $= \text{tup}_\mathcal{A}(I, s)$ | $\text{Rel}_{\bar{\mathcal{A}}}(\bar{A})$ | $= \text{tup}_\bar{\mathcal{A}}(I, s)$ |

| $\Sigma_g \circ \text{im}_\mathcal{A}(I, s) \circ \text{inc}_\mathcal{A}(I, s)$ | $g^{-1}$ | $\text{Rel}_{\mathcal{A}}(\mathcal{A})$ | $= \text{tup}_\mathcal{A}(I, s)$ |
| $\Pi_g$ | $\text{im}_{\bar{\mathcal{A}}}(I, s)$ | $\text{Rel}_{\bar{\mathcal{A}}}(\bar{A})$ | $= \text{tup}_\bar{\mathcal{A}}(I, s)$ |

\begin{align*}
\Sigma_g \circ \text{im}_\mathcal{A}(I, s) & \circ \text{inc}_\mathcal{A}(I, s) \\
\Pi_g & \circ \text{im}_{\bar{\mathcal{A}}}(I, s) = \Sigma_g \circ \text{im}_\mathcal{A}(I, s) \circ \text{inc}_\mathcal{A}(I, s) \\
\text{inc}_\mathcal{A}(I, s) \circ \text{im}_\mathcal{A}(I, s) & \circ \text{inc}_\mathcal{A}(I, s)
\end{align*}

\begin{align*}
\Sigma_g \circ \text{im}_\mathcal{A}(I, s) \circ \text{inc}_\mathcal{A}(I, s) & = \text{im}_{\bar{\mathcal{A}}}(I, s) \circ \Sigma_g \\
g^* \circ \text{im}_{\bar{\mathcal{A}}}(I, s) & = \text{im}_\mathcal{A}(I, s) \circ g^{-1}
\end{align*}

small fibers - short distance

Table 5. Reflection: Signature
Theorem 4. The fibered context of $S$-tables $\text{Tbl}(S)$ \[\xrightarrow{\text{data} \mathcal{S}}\] $\text{Cls}(X)^{\text{op}}$ is the Grothendieck construction $\int_{\text{Cls}(X)}$ of the type domain indexed adjunction $\text{Cls}(X)^{\text{op}} \xrightarrow{\text{tbl} \mathcal{S}} \text{Adj}^{\text{op}}$

3.3.2 Upper Aspect. Here, we show that the context of tables $\text{Tbl}$ is fibered over signatures via the projection passage $\text{Tbl} \xrightarrow{\text{sign}} \text{List}^{\text{op}}$. We use the Grothendieck construction $\int_{\text{List}}$ on the indexed context $\text{List}^{\text{op}} \xrightarrow{\text{tbl}} \text{Cxt}: \mathcal{S} \mapsto \text{Tbl}(\mathcal{S})$. We use the same definitions as in §3.3.1. A $\text{Tbl}$-object $\mathcal{T} = \langle S, A, K, t \rangle$, called an object (table (database relation)), consists of a signature $S = \langle I, s, X \rangle \in \text{List}$ and an $S$-table $\langle A, K, t \rangle \in \text{Tbl}(\mathcal{S})$. A $\text{Tbl}$-morphism $\mathcal{T}_2 = \langle I_2, s_2, A_2, K_2, t_2 \rangle \xleftarrow{h,f,g,k} \langle I_1, s_1, A_1, K_1, t_1 \rangle = \mathcal{T}_1$, called a table morphism (see Fig. 8), consists of a signed domain morphism $\langle I_2, s_2, A_2 \rangle \xleftarrow{h,f,g} \langle I_1, s_1, A_1 \rangle$ \[\text{List} \xrightarrow{\text{adjunction}} \text{Cxt}\] and a key function $K_2 \xleftarrow{k} K_1$, which satisfy the condition (using Lem. 2 in §2.5.3): $k \cdot t_2 = t_1 \cdot \text{tup}(h, f, g) = (t_1 \cdot \tau(h, f)(A_1)) \cdot \text{tup}_{S_2}(g)$. This gives an alternate, but equivalent, definition in terms of fibers.

Lemma 3. For any signed domain morphism $\langle I_2, s_2, A_2 \rangle \xleftarrow{h,f,g} \langle I_1, s_1, A_1 \rangle$, the tuple resolution $\text{tup}(h, f, g) = \tau(h, f)(A_1) \cdot \text{tup}_{S_2}(g)$ (Lem. 3 in §2.5.3) resolves the table fiber passage $\text{Tbl}(I_2, s_2, A_2) \xleftarrow{\Sigma(h, f, g)} \text{Tbl}(I_1, s_1, A_1)$ into the table fiber passage factorization in Fig. 12.

\[\begin{array}{ccc}
\text{Tbl}_{S_2}(A_2) & \xrightarrow{\Sigma(h, f, g)} & \text{Tbl}_{S_1}(A_1) \\
\text{Set} \downarrow \text{tup}_{A_2}(S_2) & & \text{Set} \downarrow \text{tup}_{A_1}(S_1) \\
\Sigma_{\text{tup}_{S_2}(g)}(s) & & \Sigma_{\tau(h, f)(A_1)}(a_1) \\
\end{array}\]

\[\begin{array}{ccc}
X_2 & \xrightarrow{f} & X_1 \\
& & X_2 \xleftarrow{f^{-1}(A_1)} X_1 \\
\end{array}\]

\[\begin{array}{ccc}
\text{signature morphism} & & \text{sort function} \\
S_2 \xleftarrow{(h,f)} S_1 & & \\
A_2 \xrightarrow{f} f^{-1}(A_1) & & \\
\end{array}\]

Fig. 12. Table Fiber Passage Factorization

\[\text{A signed domain morphism factors into a signature morphism } S_2 = \langle I_2, s_2, X_2 \rangle \xrightarrow{(h,f)} \langle I_1, s_1, X_1 \rangle = S_1 \text{ and a type domain morphism } A_2 = \langle X_2, Y_2, \models_{A_2} \rangle \xrightarrow{(f,g)} \langle X_1, Y_1, \models_{A_1} \rangle = A_1 \text{ with common sort function } X_2 \xrightarrow{f} X_1.\]
For any signature $S = (I, s, X)$, the fibered context of $S$-tables $\text{Tbl}(S)$ separates into the partition $\text{Tbl}(S) = \bigsqcup_{x \in \\text{Cl}(X)} \text{Tbl}_x(I, s)$. For any signature morphism $S_2 \xrightarrow{(h,f)} S_1$, we can sum the partitions of fibered passages as follows:

$$\begin{align*}
\text{Tbl}(S_2) & \xrightarrow{\text{h}} \text{A}_1 & \xrightarrow{\text{h}} \text{A}_2 & \xrightarrow{\text{h}} \text{A}_1 \\
\bigsqcup_{x \in \\text{Cl}(X)} \text{Cl}(x) & \xrightarrow{\text{h}} \text{A}_1 & \xrightarrow{\text{h}} \text{A}_1 & \xrightarrow{\text{h}} \text{A}_1 \\
\text{Cl}(x) & \xrightarrow{\text{h}} \text{A}_1 & \xrightarrow{\text{h}} \text{A}_1 & \xrightarrow{\text{h}} \text{A}_1
\end{align*}$$

The factorization in Fig. 12 suggests the following definition of table fiber passage, where the fiber passage $\text{tbl}(S_2) \xleftarrow{\text{hbl}(h,f)} \text{tbl}(S_1)$ is defined in terms of the component tuple functions $\text{tup}_{S_2}(f^{-1}(A_1)) \xleftarrow{\text{hbl}(h,f)} \text{tup}_{S_1}(A_1)$ and the inverse image function $\text{Cl}(x_2) \xrightarrow{f^{-1}} \text{Cl}(x_1)$.

**Definition 2. (table fiber passage)**

$\text{tbl}(h,f)$: An $S_1$-table is mapped to an $S_1$-table as follows:

$$\langle f^{-1}(A_1), \Sigma_{\tau_{(h,f)}(A_1)}(K_1, t_1) \rangle \xrightarrow{\text{hbl}(h,f)} \langle A_1, (K_1, t_1) \rangle$$

where $\langle I_2, s_2, f^{-1}(A_1) \rangle$-tuple $\Sigma_{\tau_{(h,f)}(A_1)}(K_1, t_1) = \langle K_1, t_1 \cdot \tau_{(h,f)}(A_1) \rangle \in \text{tup}_{S_2}(f^{-1}(A_1))$ is the existential (direct) image of $\langle I_1, s_1, A_1 \rangle$-tuple $\langle K_1, t_1 \rangle \in \text{tup}_{S_1}(A_1)$ along $\tau_{(h,f)}(A_1)$. A morphism of $S_1$-tables $\langle A_1, K_1, t_1 \rangle \xleftarrow{\text{g}} \langle A_1, K_1, t_1 \rangle$ is mapped to the morphism of $S_2$-tables

$$\langle f^{-1}(A_1), \Sigma_{\tau_{(h,f)}(A_1)}(K_1, t_1) \rangle \xleftarrow{(f^{-1}(g), k)} \langle f^{-1}(A_1), \Sigma_{\tau_{(h,f)}(A_1)}(K_1, t_1) \rangle.$$

![Fig. 13. Table Morphism: Signature](image)

A table morphism (Fig. 13) consists of signature morphism $S_2 \xrightarrow{(h,f)} S_1$ and a morphism $T_2 \xleftarrow{\text{tbl}(h,f)} T_1$ in the fiber context $\text{Tbl}(S_2)$, where the fiber
passage $\text{Tbl}(S_2) \leftarrow \text{Tbl}(S_1)$ along the signature morphism $S_2 \xrightarrow{(h,f)} S_1$ is defined in Def. 2 of §3.3.2. This fiber passage is a component of the signature indexed context of tables $\text{List}^{\text{op}} \xrightarrow{\text{tbl}} \text{Cxt}$.

**Theorem 5.** The fibered context of tables (an opfibration) $\text{Tbl} \xrightarrow{\text{sign}} \text{List}^{\text{op}}$ is the Grothendieck construction $\int_{\text{List}}$ visualized in the upper-left quadrant of Fig. 18 — of the signature indexed context of tables $\text{List}^{\text{op}} \xrightarrow{\text{tbl}} \text{Cxt}$.

### 3.4 Type Domain Indexing

In this section we show that the context of tables is a fibered context over type domains. We first define the table fiber for fixed type domain. We next move between table fibers along type domain morphisms. Finally, we invoke the Grothendieck construction indexed by type domains.

#### 3.4.1 Lower Aspect.

Let $\mathcal{A} = \langle X, Y, \models_{\mathcal{A}} \rangle$ be a fixed type domain. For database tables, the type domain $\mathcal{A}$ consists of a fixed sort set $X$ and a fixed $X$-indexed collection of data types $\{\mathcal{A}_x = \text{ext}_{\mathcal{A}}(x) \mid x \in X\}$. Here, we show that the context of $\mathcal{A}$-tables $\text{Tbl}(\mathcal{A})$ is fibered over $X$-sorted signatures $\text{Tbl}(\mathcal{A}) \xrightarrow{\text{sign}_{\mathcal{A}}} \text{List}(X)^{\text{op}}$. We use the Grothendieck construction $\int_{\text{List}(X)}$ on the indexed adjunction $\text{List}(X)^{\text{op}} \xrightarrow{\text{tbl}_{\mathcal{A}}} \text{Adj} : \langle I, s \rangle \mapsto \text{Tbl}_{\mathcal{A}}(I, s)$.

**Fibered Context (medium-size).** The fibered context of $\mathcal{A}$-tables\(^{19}\) is the comma mathematical context

$$\text{Tbl}(\mathcal{A}) = (\text{Set} \downarrow \text{tup}_{\mathcal{A}})$$

associated with the type domain tuple passage $\text{List}(X)^{\text{op}} \xrightarrow{\text{tup}_{\mathcal{A}}} \text{Set}$ defined in § 2.3.2. It has key and signature projection passages $\text{Set} \leftarrow \text{Tbl}(\mathcal{A}) \xrightarrow{\text{sign}_{\mathcal{A}}} \text{List}(X)^{\text{op}}$ and defining bridge $\text{key}_{\mathcal{A}} \xrightarrow{\tau_{\mathcal{A}}} \text{sign}_{\mathcal{A}} \circ \text{tup}_{\mathcal{A}}$. A $\text{Tbl}(\mathcal{A})$-object $\mathcal{T} = \langle I, s, K, t \rangle$, called an $\mathcal{A}$-table, consists of an indexing $X$-sorted signature $\text{sign}_{\mathcal{A}}(\mathcal{T}) = \langle I, s \rangle$, a key set $\text{key}_{\mathcal{A}}(\mathcal{T}) = K$ and a tuple function $K \xrightarrow{\tau(\mathcal{T})} \text{tup}_{\mathcal{A}}(I, s)$. A $\text{Tbl}(\mathcal{A})$-morphism $\mathcal{T}' = \langle I', s', K', t' \rangle \xrightarrow{(h,k)} \langle I, s, K, t \rangle = \mathcal{T}$ consists of an indexing $X$-sorted signature morphism $\langle I', s' \rangle \xrightarrow{\text{sign}_{\mathcal{A}}(h,k)} \langle I, s \rangle$ and a key function $K' \xleftarrow{k} K$ satisfying the naturality condition $k \cdot t' = t \cdot \text{tup}_{\mathcal{A}}(h)$. The naturality condition gives two alternate and adjoint definitions. In terms of

\(^{19}\) The context of $\mathcal{A}$-tables $\text{Tbl}(\mathcal{A})$ corresponds to the context of tables $\text{Tables}^\text{op}$ in (Spivak [22]) for a (fixed) datatype specification $U \xrightarrow{\text{DT}} \text{DT}$ with universe $U$ and set of datatypes $\text{DT}$, since a data-type specification is a special case of a type domain. However, in [22] there is no connection between contexts of tables with different data-type specifications, analogous to the fiber adjunction (Prop. 2 of §3.1.2) $\text{Tbl}(\mathcal{A}_2) \xleftarrow{(\text{tbl}_{\mathcal{A}_2})'} \text{Tbl}(\mathcal{A}_1)$ for type domain morphism $\mathcal{A}_2 \xrightarrow{\text{(f,g)}} \mathcal{A}_1$. 

fibers, an $A$-table morphism (see Fig. 14) consists of a $X$-signature morphism $\langle I', s' \rangle \xrightarrow{h} \langle I, s \rangle$ and either a morphism $T' \xleftarrow{k} \sum_h(T)$ in the fiber context $\text{Tbl}_A(I', s')$ or a morphism $h^*(T') \xleftarrow{k'} T$ in the fiber context $\text{Tbl}_A(I, s)$. The $\langle I', s', A \rangle$-table morphism $T' \xleftarrow{k} \sum_h(T)$ is the composition (RHS of Fig. 14) of the fiber morphism $\sum_h(h^*(T') \xleftarrow{k'} T)$ with the $T^{th}$ counit component $T' \xleftarrow{k} \sum_h\left(h^*(T')\right)$ for the fiber adjunction $\text{Tbl}_A(I', s') \xleftarrow{\langle \sum_h \cdot h^* \rangle} \text{Tbl}_A(I, s)$.

This fiber adjunction (top part of Tbl. 0) is a component of the $X$-signature indexed adjunction of tables $\text{List}(X)^{\text{op}} \xrightarrow{\text{Adj}} \text{Tbl}_A(I, s)$. 20

**Theorem 6.** The fibered context of $A$-tables $\text{Tbl}(A) \xrightarrow{\text{sign}_A} \text{List}(X)^{\text{op}}$ is the Grothendieck construction $\int_{\text{List}(X)} \text{Adj}$ — visualized in the lower-right quadrant of Fig. 18 — of the $X$-signature indexed adjunction $\text{List}(X)^{\text{op}} \xrightarrow{\text{Adj}} \text{Tbl}_A(I, s)$. 5

### 3.4.2 Upper Aspect

Here, we show that the context of tables $\text{Tbl}$ is fibered over type domains via the projection passage $\text{Tbl} \xrightarrow{\text{data}} \text{Cls}^{\text{op}}$. We use the Grothendieck construction $\int_{\text{Cls}}$ on the indexed adjunction $\text{Cls}^{\text{op}} \xrightarrow{\text{Adj}} A \mapsto \text{Tbl}(A)$. We use the same definitions as in § 3.1. A $\text{Tbl}$-object $T = \langle I, s, A, K, t \rangle$, called a table (database relation), consists of a type domain $A = \langle X, Y, ε, A \rangle$ and an $A$-table $\langle I, s, K, t \rangle \in \text{Tbl}(A)$. A $\text{Tbl}$-morphism $T_2 = \langle I_2, s_2, A_2, K_2, t_2 \rangle \xrightarrow{\langle h, f, g, k \rangle} \langle I_1, s_1, A_1, K_1, t_1 \rangle = T_1$, called a table morphism (see 20 Here, the span $K' \xleftarrow{K} \xrightarrow{t} \text{tup}_A(I, s)$ is the pullback in the context $\text{Set}$ of the opspan $K' \xrightarrow{t} \text{tup}_A(I', s') \xleftarrow{\text{tup}_A(h)} \text{tup}_A(I, s)$ and $K \xleftarrow{k'} K$ is the mediating morphism for the span $K' \xleftarrow{k} \xrightarrow{t} \text{tup}_A(I, s)$. 20
A signed domain morphism factors into a signature morphism \( \mathcal{S}_2 = \langle I_2, s_2, X_2 \rangle \xrightarrow{(h,f)} \mathcal{S}_1 \). and a type domain morphism \( \mathcal{A}_2 = \langle X_2, Y_2, \models \mathcal{A}_2 \rangle \xrightarrow{(f,g)} \mathcal{A}_1 \) with common sort function \( X_2 \xrightarrow{f} X_1 \).

Fig. 3, consists of a signed domain morphism \( \langle I_2, s_2, \mathcal{A}_2 \rangle \xrightarrow{(h, f, g)} \langle I_1, s_1, \mathcal{A}_1 \rangle \) and a key function \( K_2 \xleftarrow{k} K_1 \), which satisfy the condition (using Lem. 2 in §2.5.3):

\[
k \cdot t_2 = t_1 \cdot \text{tup}(h, f, g) = (t_1 \cdot \tau_{(f, g)}(I_1, s_1) \cdot \text{tup}_{\mathcal{A}_2}(h) = t_1 \cdot \text{tup}_{\mathcal{A}_1}(h) \cdot \tau_{(f, g)}(I_2, s_2).
\]

This gives two alternate, but equivalent, definitions in terms of fibers.

**Lemma 4.** For any signed domain morphism \( \langle I_2, s_2, \mathcal{A}_2 \rangle \xrightarrow{(h, f, g)} \langle I_1, s_1, \mathcal{A}_1 \rangle \), the tuple resolution \( \text{tup}(h, f, g) = \tau_{(f, g)}(I_1, s_1) \cdot \text{tup}_{\mathcal{A}_2}(h) = \text{tup}_{\mathcal{A}_1}(h) \cdot \tau_{(f, g)}(I_2, s_2) \) (Lem. 2 in §2.5.3) resolves the table fiber adjunction

\[
\text{Tbl}(I_2, s_2, \mathcal{A}_2) \xrightarrow{\tau_{(f, g)}(I_1, s_1)\text{tup}_{\mathcal{A}_2}} \text{Tbl}(I_1, s_1, \mathcal{A}_1).
\]

into the table fiber adjunction factorization in Fig. 15.

For any type domain \( \mathcal{A} \), the fibered context of \( \mathcal{A} \)-tables \( \text{Tbl}(\mathcal{A}) \) separates into the partition \( \text{Tbl}(\mathcal{A}) = \bigsqcup_{\tau \in \text{List}(X)} \text{Tbl}_{\mathcal{A}}(I, s) \). For any type domain morphism \( \mathcal{A}_2 \xrightarrow{(f,g)} \mathcal{A}_1 \), we can sum the partitions of fibered passages as follows:

\[
\text{Tbl}(I_2, s_2, \mathcal{A}_2) \xrightarrow{\bigsqcup_{\tau \in \text{List}(X_2)} \tau_{(f, g)}(I_2, s_2)\text{tup}_{\mathcal{A}_2}} \text{Tbl}(I_1, s_1, \mathcal{A}_1).
\]
The factorization in Fig. [15] suggests the following definitions of table fiber passages, where the fiber passage \( \text{Tbl}(A_2) \xleftarrow{t_{bl}(f,g)} \text{Tbl}(A_1) \) is defined in terms of the tuple function \( \text{tup}_{A_2}(f^*(I_1, s_1)) \xleftarrow{t_{(f,g)}(I_1, s_1)} \text{tup}_{A_1}(I_1, s_1) \) and the substitution (inverse image, pullback) function \( \text{List}(X_2) \xleftarrow{f^*} \text{List}(X_1) \), and the adjoint fiber passage \( \text{Tbl}(A_2) \xrightarrow{t_{bl}(f,g)} \text{Tbl}(A_1) \) is defined in terms of the adjoints, the tuple function \( \text{tup}_{A_2}(I_2, s_2) \xrightarrow{t_{(f,g)}(I_2, s_2)} \text{tup}_{A_1}(I_1, s_1) \) and the existential quantifier (direct image) function \( \text{List}(X_2) \xrightarrow{\Sigma_f} \text{List}(X_1) \).

**Definition 3.** (adjoint table fiber passages)

\( t_{bl}(f,g) \): An \( A_1 \)-table is mapped to an \( A_2 \)-table as follows:

\[
\langle f^*(I_1, s_1), \Sigma_{t_{(f,g)}(I_1, s_1)}(K_1, t_1) \rangle \xrightarrow{t_{bl}(f,g)} \langle I_1, s_1, K_1, t_1 \rangle,
\]

where \( \langle f^*(I_1, s_1), A_2 \rangle \)-tuple \( \Sigma_{t_{(f,g)}(I_1, s_1)}(K_1, t_1) = \langle K_1, t_1, \hat{\imath}_{(f,g)}(I_1, s_1) \rangle \in \text{tup}_{A_2}(f^*(I_1, s_1)) \) is the existential (direct) image of \( \langle I_1, s_1, A_1 \rangle \)-tuple \( \langle K_1, t_1 \rangle \in \text{tup}_{A_1}(I_1, s_1) \) along \( \hat{\imath}_{(f,g)}(I_1, s_1) \). A morphism of \( A_1 \)-tables \( \langle h_1, k_1 \rangle : \langle I_1, s_1, K_1, t_1 \rangle \rightarrow \langle I'_1, s'_1, K'_1, t'_1 \rangle \) is mapped to the morphism of \( A_2 \)-tables \( \langle f^*(I_1, s_1), K_1, t_1, \hat{\imath}_{(f,g)}(I_1, s_1) \rangle \rightarrow \langle f^*(I'_1, s'_1), K'_1, t'_1, \hat{\imath}_{(f,g)}(I'_1, s'_1) \rangle \).

\( t_{bl}(f,g) \): An \( A_2 \)-table is mapped to an \( A_1 \)-table as follows:

\[
\langle I_2, s_2, K_2, t_2 \rangle \xrightarrow{t_{bl}(f,g)} \langle \Sigma_f(I_2, s_2), (\hat{\imath}_{(f,g)}(I_2, s_2))^*(K_2, t_2) \rangle,
\]

where \( \langle \Sigma_f(I_2, s_2), A_1 \rangle \)-tuple \( (\hat{\imath}_{(f,g)}(I_2, s_2))^*(K_2, t_2) = \langle \widehat{K_2}, \widehat{t_2} \rangle \) is the substitution (inverse image) of \( \langle I_2, s_2, A_2 \rangle \)-tuple \( \langle K_2, t_2 \rangle \) along \( \hat{\imath}_{(f,g)}(I_2, s_2) \) (Fig. [15]). A morphism of \( A_2 \)-tables \( \langle h_2, k_2 \rangle : \langle I_2, s_2, K_2, t_2 \rangle \rightarrow \langle I'_2, s'_2, K'_2, t'_2 \rangle \) is mapped...
to the morphism of \(A_1\)-tables \(\langle \Sigma_f(h_2), k_1 \rangle : \langle \Sigma_f(I_2, s_2), \tilde{K}_1, \tilde{t}_1 \rangle \to \langle \Sigma_f(I'_2, s'_2), \tilde{K}'_1, \tilde{t}'_1 \rangle\),
where \(k_1 : \tilde{K}_1 \to \tilde{K}'_1\) is the unique mediating function for the span \(K'_2 \xleftarrow{k \cdot k_2} K_1 \xrightarrow{t_1 \cdot \text{tup}_{A_1}(\Sigma_f(h_2))} \text{tup}_{A_1}(\Sigma_f(I'_2, s'_2))\), since \((\tilde{k} \cdot k_2) \cdot t'_2 = \tilde{k} \cdot t_2 \cdot \text{tup}_{A_2}(h_2) = \tilde{t}_1 \cdot \xi(f,g)(I_2, s_2) \cdot \text{tup}_{A_2}(h_2) = (\tilde{t}_1 \cdot \text{tup}_{A_1}(\Sigma_f(h_2))) \cdot \xi(f,g)(I'_2, s'_2)\).

\[
\begin{array}{c}
\text{K}_2 \xrightarrow{k} \text{K}_1 = K_1 \\
t_2 \downarrow \text{tup}_{A_2}(S_2) \xrightarrow{f \cdot h} \text{tup}_{A_1}(S_1) \\
\text{tbl}_{(f,g)}(T_2) \xrightarrow{\text{h} \cdot \text{g}} \text{tbl}_{(f,g)}(T_1) \\
\text{Tbl}(A_2) \xrightarrow{\text{tbl}(A_1)} \text{Tbl}(A_1)
\end{array}
\quad
\begin{array}{c}
\text{K}_2 \xrightarrow{k} \tilde{K}_1 \xleftarrow{k \text{ mediator}} K_1 \\
t_2 \downarrow \text{pullback} \xrightarrow{f \cdot h} \text{pullback} \\
\text{tup}_{A_2}(S_2) \xrightarrow{f \cdot h} \text{tup}_{A_1}(S_1) \\
\text{tbl}_{(f,g)}(T_2) \xrightarrow{\text{h} \cdot \text{g}} \text{tbl}_{(f,g)}(T_1) \\
\text{Tbl}(A_2) \xrightarrow{\text{tbl}(A_1)} \text{Tbl}(A_1)
\end{array}
\]
\[k \cdot t_2 = t_1 \cdot \text{tup}(f, g) = t_1 \cdot \xi(f,g)(I_1, s_1) \cdot \text{tup}_{A_2}(\tilde{h}) = t_1 \cdot \text{tup}_{A_1}(\tilde{h}) \cdot \xi(f,g)(I_2, s_2)\]

**Fig. 16.** Table Morphism: Type Domain

**Levo:** A table morphism (left side Fig. 16) consists of type domain morphism \(A_2 \xleftarrow{\langle f \cdot g \rangle} A_1\) and a morphism \(T_2 \xleftarrow{\langle h \cdot k \rangle} \text{tbl}_{(f,g)}(T_1)\) in the fiber context \(\text{Tbl}(A_2)\), where the fiber passage \(\text{Tbl}(A_2) \xleftarrow{\text{tbl}_{(f,g)}} \text{Tbl}(A_1)\) along the type domain morphism \(A_2 \xleftarrow{\langle f \cdot g \rangle} A_1\) is defined in Def. 3. This fiber passage is a component of the type domain indexed context of tables \(\text{Cls} \xrightarrow{\text{adj}} \text{Cxt}\).

**Dextro:** A table morphism (right side Fig. 16) consists of type domain morphism \(A_2 \xleftarrow{\langle f \cdot g \rangle} A_1\) and a morphism \(\text{tbl}_{(f,g)}(T_2) \xleftarrow{\langle h \cdot k \rangle} T_1\) in the fiber context \(\text{Tbl}(A_1)\), where the fiber passage \(\text{Tbl}(A_2) \xleftarrow{\text{tbl}_{(f,g)}} \text{Tbl}(A_1)\) along the type domain morphism \(A_2 \xleftarrow{\langle f \cdot g \rangle} A_1\) is defined in Def. 3. This fiber passage is a component of the type domain indexed adjunction of tables \(\text{Cls} \xrightarrow{\text{adj}} \text{Cxt}\).

**Proposition 2.** For any type domain morphism \(A_2 \xleftarrow{\langle f \cdot g \rangle} A_1\), there is a table fiber adjunction \(\text{Tbl}(A_2) \xleftarrow{\langle \text{tbl}_{(f,g)}^{-1} \cdot \text{tbl}_{(f,g)} \rangle} \text{Tbl}(A_1)\). This fiber adjunction is a component of a type domain indexed adjunction of tables \(\text{Cls} \xrightarrow{\text{adj}} \text{Adj}\).
**Theorem 7.** The fibered context of tables $\text{Tbl} \xrightarrow{\text{data}} \text{Cls}^{\text{op}}$ is the Grothendieck construction $\mathcal{F}_{\text{Cls}}$ — visualized in the upper-right quadrant of Fig. 18 — of the type domain indexed adjunction $\text{Cls}^{\text{op}} \xrightarrow{\text{tbl}} \text{Adj}$.

$$\mathcal{A}_2 = \langle X_2, Y_2, \vdash_{\mathcal{A}_2} \rangle \xrightarrow{(f,g)} \langle X_1, Y_1, \vdash_{\mathcal{A}_1} \rangle = \mathcal{A}_1$$

**Fig. 17.** Indexed Adjunction of Tables

### 3.5 Fibered Contexts of FOLE Tables

The Grothendieck constructions for FOLE tables are listed in Tbl. 7. Here we indicated whether the construction is a fibration, an opfibration or a bifibration. We also list the proposition or theorem proving the construction and its location. The Grothendieck constructions for FOLE tables are displayed in Fig. 18.
Table 7. Grothendieck Constructions

Fig. 18. The Fibered Hierarchy of FOLE Tables
4 Table Constructions

In this section we use properties of comma contexts and the Grothendieck construction to prove that the various (sub)contexts of $\text{FOL}\text{E}$ tables are complete (joins exist) and cocomplete (unions exist).

4.1 Preliminaries

**Proposition 3.** The mathematical context of classifications (type domains) $\text{Cls}$ is (co)complete, and its type (sort) and instance (data) projections $\text{Set}^{\text{op}} \leftarrow \text{data}^{\text{op}}$ $\text{Cls} \xrightarrow{\text{sort}} \text{Set}$ are (co)continuous.

*Proof.* Barwise and Seligman [1].

**Proposition 4.** For any sort set $X$, the context of $X$-sorted type domains $\text{Cls}(X)$ is complete, and its instance (data) projection $\text{Set} \xleftarrow{\text{data}^{X}} \text{Cls}(X)^{\text{op}}$ is cocontinuous.

*Proof.* To prove the proposition in general, use the three special cases: any collection of $X$-sorted type domains has a product, whose instance set is the coproduct (disjoint union) of the collection of instance (data) sets; there is a terminal $X$-sorted type domain, whose instance (data) set is the empty set $\emptyset$; and any opspan of $X$-sorted type domain morphisms has a pullback, whose instance set is the pushout of the instance (data) projection span.

**Proposition 5.**

For any sort function $X_2 \xrightarrow{f} X_1$, the type domain fiber passage $\text{Cls}(X_2) \xrightarrow{\text{cls}(f)} \text{Cls}(X_1)$ is continuous (preserves limits): $\prod(A \circ f^{-1}) = f^{-1}(\prod A)$ for any diagram $\xrightarrow{A} \text{Cls}(X_1)$.

*Proof.* To prove this, show that the inverse image of the limit is the limit of the inverse image of any diagram in $\text{Cls}(X_1)$. We need only show this for products and pullbacks. We note that inverse image preserves data projection: $f^{-1} \circ \text{data}^{X_2} = \text{data}^{X_1}$.

**Proposition 6.** The tuple passage $\text{List}(X)^{\text{op}} \xrightarrow{\text{tup}_A} \text{Set}$ is continuous.

*Proof.* We need only show that the tuple passage maps the initial object in $\text{List}(X)$ to the terminal object in $\text{Set}$ and maps the pushout of a span in $\text{List}(X)$ to the pullback of the image opspan in $\text{Set}$. 

\[ \text{Set}^{op} \xrightarrow{\text{data}^{op}} \text{Cls} \xrightarrow{\text{sort}} \text{Set} \]
4.2 Propositions

| complete:     | List(X), List, Cls, Dom, Tbl_A(I, s), Tbl(A), Tbl |
|--------------|--------------------------------------------------|
| cocomplete:  | List(X), List, Cls, Dom, Tbl_A(I, s), Tbl(S), Tbl(A), Tbl |
| proven by:   | * = Info. Flow. [1], ↓ = comma context, \( f \) = Grothendieck |

Table 8. Complete/Cocomplete Contexts

4.2.1 Using Comma Contexts. These propositions use Facts [1][2] in §A.2

**Proposition 7.** The comma contexts of \( X \)-signatures, signatures, signed domains, \( \langle A, I, s \rangle \)-tables, and \( A \)-tables are associated with the following passage opspans:

| comma context | passage opspan |
|---------------|----------------|
| List(X) = (Set ↓ X) | Set \( \xrightarrow{\downarrow} \) Set ↪ \( X \) \( \rightarrow \) 1 |
| List = (Set ↓ Set) | Set \( \xrightarrow{\downarrow} \) Set ↪ \( I \) \( \rightarrow \) Set |
| Dom = (Set ↓ sort) | Set \( \xrightarrow{\downarrow} \) Set ↪ sort \( \rightarrow \) Cls |
| Tbl_A(I, s) = (Set ↓ tup_A(I, s)) | Set \( \xrightarrow{\downarrow} \) Set ↪ tup_A(\( I, s \)) \( \rightarrow \) 1 |
| Tbl(A) = (Set ↓ tup_A) | Set \( \xrightarrow{\downarrow} \) Set ↪ tup_A \( \rightarrow \) List(X)\( \rightarrow \) |

respectively. Hence, they are (co)complete and their projections

\[
\begin{align*}
\text{Set} &\xleftarrow{\text{arity}_X} \text{List}(X) \rightarrow 1 \\
\text{Set} &\xleftarrow{\text{arity}} \text{List} \xrightarrow{\text{sort}} \text{Set} \\
\text{Set} &\xleftarrow{\text{arity}} \text{Dom} \xrightarrow{\text{data}} \text{Cls} \\
\text{Set} &\xleftarrow{\text{key}_A(I,s)} \text{Tbl}_A(I, s) \rightarrow 1 \\
\text{Set} &\xleftarrow{\text{key}_A} \text{Tbl}(A) \xrightarrow{\text{sign}_A} \text{List}(X)\text{op}
\end{align*}
\]

are (co)continuous.

**Proof.** The contexts Set, Cls, 1 and List(X)\( \text{op} \) are (co)complete; the passage Set \( \xrightarrow{\downarrow} \) Set is (co)cocontinuous; and the passages Set \( \xrightarrow{\downarrow} \) Set, Cls \( \xrightarrow{\text{sort}} \) Set, \( X, \text{tup}_A(I,s) \xrightarrow{\downarrow} \text{Set} \), and List(X)\( \text{op} \) \( \xrightarrow{\text{tup}_A} \) Set are continuous.
Proposition 8. The comma contexts of tables and $S$-tables are associated with the following passage opspans:

\[
\begin{array}{ll}
\text{comma context} & \text{passage opspan} \\
\text{Tbl} = (\text{Set} \downarrow \text{tup}) & \text{Set} \xrightarrow{\downarrow} \text{Set} \xleftarrow{\text{tup}} \text{Dom}^{\text{op}} \\
\text{Tbl}(S) = (\text{Set} \downarrow \text{tup}_S) & \text{Set} \xrightarrow{\downarrow} \text{Set} \xleftarrow{\text{tup}_S} \text{Cls}(X)^{\text{op}}.
\end{array}
\]

respectively. Hence, they are cocomplete and their projections

\[
\begin{array}{ll}
\text{Set} \xleftarrow{\text{key}} \text{Tbl} & \text{dom} \xrightarrow{\text{op}} \text{Dom}^{\text{op}} \\
\text{Set} \xleftarrow{\text{key}_S} \text{Tbl}(S) & \xrightarrow{\text{data}_S} \text{Cls}(X)^{\text{op}}
\end{array}
\]

are cocontinuous.

Proof. The contexts \text{Set}, \text{Dom}^{\text{op}} and \text{Cls}(X)^{\text{op}} are cocomplete (Prop. 7); and the passage \text{Set} \xrightarrow{\downarrow} \text{Set} is cocontinuous. □

4.2.2 Using the Grothendieck Construction.

Proposition 9. The fibered contexts (Tbl. [7]) of signatures \text{List}, tables \text{Tbl} and \text{A}-table \text{Tbl}(\text{A}) are (co)complete and their projections are (co)continuous.

\[
\begin{array}{cccc}
\text{fibered construct} & \text{indexed construct} \\
\text{List} \xrightarrow{\text{sort}} \text{Set} = \int \text{Set}^{\text{list}} & \text{Set} \xrightarrow{\text{bt}} \text{Adj} \\
\text{Tbl} \xrightarrow{\text{dom}} \text{Dom}^{\text{op}} & \text{Dom}^{\text{op}} \xleftarrow{\text{bt}} \text{Adj} \\
\text{Tbl}(\text{A}) \xrightarrow{\text{sign}} \text{List}(X)^{\text{op}} & \text{List}(X)^{\text{op}} \xrightarrow{\text{bt}_{\text{A}}} \text{Adj} \\
\text{Tbl} \xrightarrow{\text{data}} \text{Cls}^{\text{op}} & \text{Cls}^{\text{op}} \xrightarrow{\text{bt}_{\text{A}}} \text{Adj}
\end{array}
\]

Proof. This proposition uses Fact. 5 in §A.3. The indexing contexts \text{Set}, \text{Dom}^{\text{op}}, \text{List}(X)^{\text{op}}, \text{Cls}^{\text{op}} are (co)complete (Prop. 7), and the fiber contexts \text{List}(X), \text{Tbl}(I, s, \text{A}) and \text{Tbl}(\text{A}) are (co)complete (Prop. 7). □

Proposition 10. \text{[signature-lower]} The fibered context of \text{S}-tables \text{Tbl}(S) is co-complete and the projection \text{Tbl}(S) \xrightarrow{\text{data}_S} \text{Cls}(X)^{\text{op}} is cocontinuous.

Proof. Uses Fact. 4 in §A.3 and the discussion in §3.3.1. For the contravariant pseudo-passage \text{Cls}(X)^{\text{op}} \xrightarrow{\text{bt}_{\text{S}}} \text{Cxt} that uses the existential quantification,

1. the indexing context \text{Cls}(X)^{\text{op}} is cocomplete,
2. the fiber context \text{Tbl}_S(\text{A}) is cocomplete for each \text{X}-sorted type domain \text{A}, and
3. the fiber passage \text{Tbl}_S(\text{A}) \xrightarrow{\Sigma g} \text{Tbl}_S(\tilde{\text{A}}) is cocontinuous (being left adjoint) for each \text{X}-sorted type domain morphism \text{A} \xrightarrow{(1_X,g)} \tilde{\text{A}}. □
4.3 Constructive Proof

Proposition 11. The context of $\mathcal{A}$-tables $\text{Tbl}(\mathcal{A})$ is complete.

Proof. We have already proved this using comma contexts and the Grothendieck construction. Now we give a constructive proof of this fact, which illustrates that “limits (natural joins) are resolvable into substitutions followed by meets.”

Suppose that $T : G \to \text{Tbl}(\mathcal{A})$ is a diagram of $\mathcal{A}$-tables and $\mathcal{A}$-table morphisms

$$T = \{ T_n = \langle I_n, s_n, K_n, t_n \rangle \xrightarrow{(h_n, k_n)} \langle I_m, s_m, K_m, t_m \rangle = T_m \mid (n \xrightarrow{s} m) \in G \}. $$

Let $S = T^\text{op} \circ \text{sign}_\mathcal{A} : G^\text{op} \to \text{List}(\mathcal{X})$ be the underlying diagram of signatures and signature morphisms

$$S = \{ S_n = \langle I_n, s_n \rangle \xleftarrow{h_n} \langle I_m, s_m \rangle = S_m \mid (n \xrightarrow{s} m) \in G \}. $$

Assume that $\gamma : S \Rightarrow \Delta(I, s)$ is a colimiting cocone $\gamma = \{ \gamma_n : \langle I_n, s_n \rangle \to \langle I, s \rangle \mid n \in G \}$ with base diagram $S$ and colimit signature $(I, s)$, so that $h_n \cdot \gamma_n = \gamma_m$ for all edges $n \xrightarrow{s} m$ in $G$. For each $G$-node $n$, use substitution to move fiber tables and fiber table morphisms from the peripheral fiber categories $\{ \text{Tbl}_\mathcal{A}(I_n, s_n) \}$ to the central fiber context $\text{Tbl}_\mathcal{A}(I, s)$:

| peripheral | central |
|------------|---------|
| $\langle K_n, t_n \rangle$ | $\gamma_n^\ast(K_n, t_n) = (K_n, t_n)$ |
| $\langle K_n, t_n \rangle$ | $\gamma_n^\ast(K_n, t_n)$ $\xrightarrow{\gamma_n^\ast(h_n)} \gamma_m^\ast(K_m, t_m)$ $\gamma_m^\ast(h_n^\ast(K_m, t_m)) = \gamma_m^\ast(K_m, t_m)$ |
| $\sum_{h_n}(K_n, t_n)$ | $\gamma_m^\ast(\sum_{h_n}(K_n, t_n))$ $\xrightarrow{\gamma_n^\ast(h_n)} \gamma_m^\ast(K_m, t_m)$ |
| $\sum_{h_n}(K_n, t_n)$ | $\sum_{\gamma_n}(\gamma_n^\ast(\sum_{h_n}(K_n, t_n)))$ $\gamma_m^\ast(h_n^\ast(K_m, t_m))$ |

Hence, there is diagram $T^\ast : G \to \text{Tbl}_\mathcal{A}(I, s)$ in the central fiber

$$T^\ast = \{ \langle K_n, t_n \rangle = \gamma_n^\ast(K_n, t_n)$ $\xrightarrow{\gamma_n^\ast(h_n)} \gamma_m^\ast(h_n^\ast(K_m, t_m)) \cong \gamma_m^\ast(K_m, t_m) = \langle K_m, t_m \rangle \mid (n \xrightarrow{s} m) \in G \}. $$

Assume that $\pi : T^\ast \Rightarrow \Delta(K, t)$ is a limiting cone $\pi = \{ \langle K, t \rangle \xrightarrow{\pi} \langle K_n, t_n \rangle \mid n \in G \}$ with base diagram $T^\ast$ and join table $\langle K, t \rangle = \prod_{n \in G} \langle K_n, t_n \rangle$ with fiber projections, so that $\pi_n \cdot \gamma_n = \pi_m$ for all edges $n \xrightarrow{s} m$ in $G$. We claim that the composite $\mathcal{A}$-table morphism $T = \langle I, s, K, t \rangle$ $\xrightarrow{\gamma_n^\ast(h_n)} \langle I_n, s_n, K_n, t_n \rangle$ is the $n^{th}$ component of a limiting cone $\gamma : \langle I_n, s_n, K_n, t_n \rangle \Rightarrow T$ for $T$ in $\text{Tbl}_\mathcal{A}(I, s)$, where each component has signature morphism $\langle I_n, s_n \rangle$ $\xrightarrow{\gamma_n^\ast(h_n)} \langle I, s \rangle$ and key function $\pi_n \cdot \gamma_n^\ast(t_n) : K \to K_n$. It is natural with respect to the diagram $T$. Now suppose that $\alpha : \langle I', s', K', t' \rangle \Rightarrow T$ is another cone $\alpha = \{ \langle I', s', K', t' \rangle \xrightarrow{(h_n, k_n)} \langle I_n, s_n, \hat{K}_n, t_n \rangle \mid n \in G \}$ over $T$, each component with signature morphism $\langle I_n, s_n \rangle$ $\xrightarrow{k_n} \langle I', s' \rangle$ and key function $k : K' \to K$ satisfying the condition...
\[ t' \cdot \text{tup}_A(h_n) = k_n \cdot t_n. \] Since \( \gamma \) is a colimiting cocone, there is a unique signature morphism \( (I, s) \xrightarrow{h} (I', s') \) such that \( \alpha = \gamma \cdot \Delta h \), or \( \alpha_n = \gamma_n \cdot h \), and hence, \( \text{tup}_A(\alpha_n) = \text{tup}_A(h) \cdot \text{tup}_A(\gamma_n) \), for each node \( n \in G \). Since \( k_n \cdot t_n = t' \cdot \text{tup}_A(h_n) = t' \cdot \text{tup}_A(h) \cdot \text{tup}_A(\gamma_n) \), there is a unique mediating key function \( K' \xrightarrow{k_n} K_n \) satisfying \( k_n \cdot t_n^* = t' \cdot \text{tup}_A(h) \) and \( k_n^* \cdot \varepsilon_{T_n}^* = k_n \). Hence, we have the \( \mathcal{A} \)-table morphism \( T' = \langle I', s', K', t' \rangle \xrightarrow{(h, k_n^*)} \langle I, s, K_n, t_n^* \rangle = \gamma_n^*(T_n) \), which satisfies \( \langle h, k_n^* \rangle \cdot (\gamma_n, \varepsilon_{T_n}^*) = \langle h_n, k_n \rangle \) for each \( n \in G \). The central fiber table morphism \( \Sigma_h(K', t') \xrightarrow{k_n^*} \langle K_n, t_n^* \rangle = \gamma_n^*(T_n) \), is the \( n^{th} \) component of a central fiber cone \( \alpha^*: \langle K', t' \rangle \Rightarrow \Delta T^* \). Hence, there is a unique mediating function \( K' \xrightarrow{k_n} K \) such that \( \Delta k \cdot \pi = \alpha^* \), or \( k \cdot \pi_n = k_n^* \) for each \( n \in G \). Hence, we have the commuting diagram of \( \mathcal{A} \)-table morphisms \( T' \xrightarrow{(h, k_n^*)} T \xrightarrow{(\gamma_n, \varepsilon_{T_n}^*)} T_n = T' \langle h_n, k_n \rangle, T_n \). Uniqueness is straightforward.

### 4.4 Example

We illustrate the use of these semantic operations by using the observation made in Prop. \[11\] that limits are resolvable into substitutions followed by meets. Here we discuss the special case of pullback — the join of two \( \mathcal{A} \)-tables. Consider the \( \text{Tbl}(\mathcal{A}) \)-opspan

\[
\mathcal{T}_1 = \langle I_1, s_1, K_1, t_1 \rangle \xrightarrow{(h_1, k_1)} \langle I, s, K, t \rangle \xleftarrow{(h_2, k_2)} \langle I_2, s_2, K_2, t_2 \rangle = \mathcal{T}_2
\]

illustrated in the bottom part of Figure 19 with key opspan \( K_1 \xrightarrow{k_1} K \xleftarrow{k_2} K_2 \) and signature span \( \langle I_1, s_1 \rangle \xrightarrow{h_1} \langle I, s \rangle \xleftarrow{h_2} \langle I_2, s_2 \rangle \). Since \( \text{List}(X) \) is cocomplete, we can form the colimiting cocone (opspan) of this signature span, with pushout signature \( \langle I_1 +_I I_2, [s_1, s_2] \rangle \) and injection signature morphisms

\[
\langle I_1, s_1 \rangle \xleftarrow{j^1} \langle I_1 +_I I_2, [s_1, s_2] \rangle \xrightarrow{j^2} \langle I_2, s_2 \rangle
\]

that satisfies the commutative diagram \( h_1 \cdot j^1 = h_2 \cdot j^2 \). Apply the continuous tuple passage \( \text{tup}_A : \text{List}(X)^\text{op} \to \text{Set} \) to this signature opspan to get the limiting cone over the \( \text{Set} \)-opspan \( \text{tup}_A(I_1, s_1) \xrightarrow{\text{tup}_A(h_1)} \text{tup}_A(I, s) \xleftarrow{\text{tup}_A(h_2)} \text{tup}_A(I_2, s_2) \) with pullback set \( \text{tup}_A(I_1, s_1) \times_{\text{tup}_A(t, s)} \text{tup}_A(I_2, s_2) \) and projection functions

\[
\text{tup}_A(I_1, s_1) \xrightarrow{\text{tup}_A(i_1)} \text{tup}_A(I_1 +_I I_2, [s_1, s_2]) \xleftarrow{\text{tup}_A(i_2)} \text{tup}_A(I_2, s_2)
\]

This is illustrated in the top part of Figure 19.

In general, the join (limit) of an arbitrary diagram in \( \text{Tbl}(\mathcal{A}) \) is obtained by (1) inverse image (substitution) of the component tables along the colimit signature injections over the underlying signature diagram, followed by (2) meet (conjunction) at the colimit signature. In particular, the pullback of \( \text{Tbl}(\mathcal{A}) \)-opspan \[14\] is the table \( \mathcal{T}_1 \times \mathcal{T}_2 \) whose signature is the pushout signature \( \langle I_1 +_I I_2, [s_1, s_2] \rangle \), whose key set is the pullback set \( K_1 \times_{K} K_2 \), and whose tuple function

\[
t_1 \times t_2 : K_1 \times_{K} K_2 \to \text{tup}_A(I_1 +_I I_2, [s_1, s_2]) = \text{tup}_A(I_1, s_1) \times_{\text{tup}_A(t, s)} \text{tup}_A(I_2, s_2)
\]
is the mediating function obtained by taking the pullback of sources and targets in \( T \). For proof, use a continuity proposition for comma categories, and show that the key set and projection functions, obtained by inverse image (substitution) and meet, forms the pullback.  

\[ T_1 \times T_2 = \pi_1^*(T_1) \land (I_1 + t_1, [s_1, s_2]) \pi_2^*(T_2) \]

**Fig. 19. Binary Join**

22 Since we identify database joins with limits in \( \text{Tbl}(A) \), this allows us to compute joins as inverse images followed by meets, both of which are elementary logical operations. The dual approach will identify database unions with colimits in \( \text{Tbl}(A) \). This is the key insight for a structured/logical approach to database formalism using fiber Boolean operations (conjunction and disjunction), substitution and the quantifiers.
5 Conclusion and Future Work

5.1 This Paper in Review

A precise mathematical basis for FOL interpreted in terms of relational tables and relation databases. This paper has developed the notion of relational table in terms of comma contexts and the Grothendieck construction. Table 9 lists the lemmas, propositions and theorems in this paper.

The table concept is built upon the three more elementary concepts of signature, type domain, and signed domain. In § 2 we have discussed the mathematical contexts for these three elementary concepts: Thm. 1 describes the fibered context of signatures List as a Grothendieck construction indexed by sort sets; and Thm. 2 describes the fibered context of type domains CIs as a Grothendieck construction also indexed by sort sets.

In § 3 we have described how each elementary concept provides a distinct, but related, approach to the fibered nature of the table concept via the Grothendieck construction (illustrated in Tbl. 18 of § 3.3) — each fixed elementary concept providing a fiber subcontext of tables: Thm. 3 describes the fibered context of tables Tbl as a Grothendieck construction indexed by signed domains; Thm. 4 describes the fibered context of tables Tbl as a Grothendieck construction indexed by signatures, with the indexing defined by means of Thm. 5; and Thm. 6 describes the fibered context of tables Tbl as a Grothendieck construction indexed by type domains, with the indexing defined by means of Thm. 7.

In § 4 we proved the existence of sum and product constructions (database unions and joins) on various fiber contexts of tables by using both comma contexts and the Grothendieck construction: Prop. 3–10 prove that the contexts of signatures, type domains, signed domains, and tables have limits and colimits (joins and unions); and Prop. 11 gives a detailed description of the limit construction (join) for tables with fixed type domain, arguing that limits are resolvable into substitutions followed by meets.

In the appendix § A we discuss relations, comma contexts and fibrations: Prop. 12 describes the reflection between tables and relations, thus linking traditional logic interpretation with relational database interpretation; and Facts. 1–5 state facts about comma contexts and the Grothendieck construction.
Table 43.

§ 2: Table Basics

Thm. 1: \( \text{List} \xrightarrow{\text{sort}} \text{Set} = \int (\text{Set} \xrightarrow{\text{incl}} \text{Adj}) \)

Thm. 2: \( \text{Cls} \xrightarrow{\text{sort}} \text{Set} = \int (\text{Set}^{\op} \xrightarrow{\text{cls}} \text{Cxt}) \)

Lem. 1: natural isomorphisms (levo \( \cong \) dextro) inclusion & tuple

Prop. 1: inclusion/tuple passages

§ 3: Hierarchy

Thm. 3: \( \text{Tbl} \xrightarrow{\text{dom}} \text{Dom}^{\op} = \int (\text{Dom}^{\op} \xrightarrow{\text{tbl}} \text{Adj}) \)

Thm. 4: \( \text{Tbl}(S) \xrightarrow{\text{data}_S} \text{Cls}(X)^{\op} = \int (\text{Cls}(X)^{\op} \xrightarrow{\text{tbl}_S} \text{Adj}) \)

Lem. 2: tuple function factorizations: type domain & signature

§ 4: Table Constructions

preliminaries

Prop. 5: (co)completeness of \( \text{Cls} \)

Prop. 6: completeness of \( \text{Cls}(X) \)

Prop. 7: continuity of \( \text{Cls}(X_2) \xrightarrow{\text{cls}(f)} \text{Cls}(X_1) \)

Prop. 8: continuity of \( \text{List}(X)^{\op} \xrightarrow{\text{bap}_A} \text{Set} \)

using comma contexts

Prop. 9: (co)completeness of \( \text{List}(X), \text{List}, \text{Dom}, \text{Tbl}_A (I, s) \& \text{Tbl}_A \)

Prop. 10: cocompleteness of \( \text{Tbl} \) & \( \text{Tbl}(S) \)

using Grothendieck construction

Prop. 11: (co)completeness of \( \text{List}, \text{Tbl} \) and \( \text{Tbl}(A) \)

Prop. 12: cocompleteness of \( \text{Tbl}(S) \)

by construction

Prop. 13: completeness of \( \text{Tbl}(A) \)

§ A: Appendix

A-relations

Prop. 14: reflection \( \text{Tbl}_A \xrightarrow{(\text{im}_A \downarrow \text{inc}_A)} \text{Rel}_A \)

comma contexts

Fact. 1: comma context completeness

Fact. 2: comma context cocompleteness

Grothendieck construction

Fact. 3: fibration completeness

Fact. 4: opfibration cocompleteness

Fact. 5: bifibration (co)completeness

Table 9. Lemmas, Propositions and Theorems
5.2 The Presentation of FOLE

The first-order logical environment FOLE (Fig. 20. a) was first described in Kent [13]. A series of three papers (Fig. 20. 1.2.5) describe in detail a mathematical representation for ontologies within FOLE. The FOLE representation can be expressed in two forms: a classification form and interpretative form. The foundation paper (Kent [14]) and the superstructure paper (Kent [15]) developed the classification form of FOLE. A third paper (Kent [16]) will develop the interpretative form of FOLE as a transformational passage from sound logics (Kent [13]), thereby defining the formalism and semantics of first-order logical/relational database systems (Kent [12]). A series of two papers (Fig. 20. 3,4) provide a rigorous mathematical foundation for the interpretation of FOLE: the first [this paper] describes the notion of a FOLE table and the second describes the notion of a FOLE database. System interoperability, in the general setting of institutions and logical environments, was defined in the paper “System Consequence” (Kent [11]). This was inspired by the channel theory of information flow (Barwise and Seligman [1]). Since FOLE is a logical environment (Kent [17]), in two further papers (Fig. 20. 6,7) we apply this approach to interoperability for information systems based on first-order logic and relational databases: one paper discusses integration over a fixed type domain and the other paper discusses integration over a fixed universe.

---

23 Following the relational model, we assume a semantic structure and use a logical theory consistent with that structure in terms of first-order logic (E.F. Codd [3]).
A Appendix

A.1 ∑-Relations.

Let $\mathcal{A} = \langle X, Y, \models_{\mathcal{A}} \rangle$ be a fixed type domain. The mathematical contexts of $\mathcal{A}$-relations and $\mathcal{A}$-tables $^{24}$ are used for satisfaction and interpretation $^{14}$, $\mathcal{A}$-relations for traditional interpretation and $\mathcal{A}$-tables for database interpretation.

Fiber Contexts. Let $\langle I, s \rangle$ be any signature. The $\langle I, s \rangle^{th}$-fiber context of relations is the subset order

$$\text{Rel}_\mathcal{A}(I, s) = \langle \phi \text{tup}_\mathcal{A}(I, s), \subseteq \rangle.$$ 

An object $R \in \text{Rel}_\mathcal{A}(I, s)$ consists of a subset of tuples $R \subseteq \text{tup}_\mathcal{A}(I, s)$. $^{25}$ A morphism $R' \leftarrow R$ in $\text{Rel}_\mathcal{A}(I, s)$ consists of subset order $R' \supseteq R$.

Fibered Context. The fibered context $\text{Rel}(\mathcal{A})$ has indexed $\mathcal{A}$-relations $\langle I, s, R \rangle$ as objects with $R \subseteq \text{tup}_\mathcal{A}(I, s)$ and morphisms $\langle I', s', R' \rangle \leftarrow \langle I, s, R \rangle$ $^{26}$ consisting of a signature morphism $\langle I', s' \rangle \xleftarrow{h} \langle I, s \rangle$ satisfying either of the adjoint fiber orderings

$$R' \supseteq \exists_h(R) \iff h^{-1}(R') \supseteq R$$

in $\text{Rel}_\mathcal{A}(I', s')$ in $\text{Rel}_\mathcal{A}(I, s)$ $^{8}$ defined in terms of the fiber adjunction $\langle \exists_h \dashv h^{-1} \rangle : \text{Rel}_\mathcal{A}(I', s') \leftrightarrows \text{Rel}_\mathcal{A}(I, s)$ (Tbl. $^6$ in §3.4.1). As we show below, the context $\text{Rel}(\mathcal{A})$ of $\mathcal{A}$-relations can be viewed as a mathematical subcontext of the context $\text{Tbl}(\mathcal{A})$ of $\mathcal{A}$-tables.

Inclusion. Let $\langle I, s \rangle$ be any signature. The $\langle I, s \rangle^{th}$-fiber inclusion passage $^{27}$

$$\text{Rel}_\mathcal{A}(I, s) \xrightarrow{\text{inc}_{\mathcal{A}(I,s)}} \text{Tbl}_\mathcal{A}(I, s)$$

is defined as follows. An fiber relation $R \in \text{Rel}_\mathcal{A}(I, s)$ is mapped to the fiber table $\langle R, \text{inc} \rangle \in \text{Tbl}_\mathcal{A}(I, s)$. A fiber morphism $R' \supseteq R$ in $\text{Rel}_\mathcal{A}(I, s)$ is mapped to the fiber morphism $\langle R', \text{inc} \rangle \leftarrow \langle R, \text{inc} \rangle$ in $\text{Tbl}_\mathcal{A}(I, s)$. The fibered inclusion passage

$$\text{Rel}(\mathcal{A}) \xrightarrow{\text{inc}_\mathcal{A}} \text{Tbl}(\mathcal{A})$$

can be defined in terms of the fiber passages $\{ \text{inc}_{\mathcal{A}(I,s)} \mid \langle I, s \rangle \in \text{List}(X) \}$. An $\mathcal{A}$-relation $\langle I, s, R \rangle \in \text{Rel}(\mathcal{A})$ is mapped to the $\mathcal{A}$-table $\langle I, s, R, \text{inc} \rangle = \langle I, s, \text{inc}_{\mathcal{A}(I,s)}(R) \rangle \in \text{Tbl}(\mathcal{A})$.

An $\mathcal{A}$-relation morphism $\langle I', s', R' \rangle \xrightarrow{h} \langle I, s, R \rangle$ consisting of a signature morphism $\langle I', s' \rangle \xrightarrow{h} \langle I, s \rangle$ satisfying either of the adjoint fiber orderings in Eqn. $^8$

$^{24}$ For fixed type domain $\mathcal{A}$, the context of $\mathcal{A}$-Tables is discussed in §3.4.1.

$^{25}$ More abstractly, we could define a relation to be a subobject of $\langle I, s, \mathcal{A} \rangle$-tuples; that is, an isomorphism class of monomorphisms $R \xhookrightarrow{\mathcal{A}} \text{tup}_\mathcal{A}(I, s)$. These correspond to the proper or uncorrupted relational tables of Codd $^3$.

$^{26}$ We use this orientation to accord with both relational fibers and table morphisms.

$^{27}$ For fixed signed domain $\langle I, s, \mathcal{A} \rangle$, the fiber mathematical context of $\langle I, s, \mathcal{A} \rangle$-tables is is discussed in §3.2.
is mapped to the \(A\)-table morphism \(\langle I', s', R', \text{inc} \rangle \leftarrow\leftarrow \langle I, s, R, \text{inc} \rangle\), where the key function \(R' \leftarrow\leftarrow R\) satisfying the condition \(r \cdot \text{inc} = \text{inc} \cdot \text{tup}_A(h)\) is a restriction of the tuple function \(\text{tup}_A(I', s') \leftarrow\leftarrow \text{tup}_A(I, s)\). Hence, we have the adjointly-related fiber context morphisms (see Eqn. [4]).

\[
\begin{align*}
\text{inc}^A_{(I', s')} (R') & \cong \exists_h (R) \\
\text{inc}^A_{(I, s)} (\exists_h (R)) & \leftarrow\leftarrow \text{inc}^A_{(I, s)} (\exists_h (R))
\end{align*}
\]

Either pullback or image factorization can be used (Tbl. [21]) to define the key function \(R' \leftarrow\leftarrow R\). Using pullback, the \(A\)-table morphism is the composition of the fiber morphism \(\exists_h (h^{-1}((\text{inc}^A_{(I', s')} (R') ) \leftarrow\leftarrow \text{inc}^A_{(I, s)} (R))\) with the \(\text{inc}^A_{(I', s')} (R'))^{th}\) count component \(\text{inc}^A_{(I', s')} (R') \leftarrow\leftarrow \exists_h (h^{-1}((\text{inc}^A_{(I', s')} (R'))\) for the fiber adjunction \(\langle \exists_h \circ h^{-1} : \text{Tbl}_A(I', s') \leftarrow\leftarrow \text{Rel}_A(I)\) (Tbl. [6] in § 3.4.1).

![Fig. 21. Inclusion Table Morphism](image)

**Image.** Let \(\langle I, s \rangle\) be any signature. The \(\langle I, s \rangle^{th}\)-fiber image passage

\[
\text{Tbl}_A(I, s) \xrightarrow{\text{im}^A_{(I, s)}} \text{Rel}_A(I, s)
\]

is defined as follows. A fiber table \(\langle K, t \rangle \in \text{Tbl}_A(I, s)\) is mapped to the fiber relation \(\varphi t(K) \in \text{Rel}_A(I, s)\). A fiber morphism \(\langle K', t' \rangle \leftarrow\leftarrow \langle K, t \rangle \) in \(\text{Tbl}_A(I, s)\) is mapped to the fiber morphism \(\varphi t'(K') \leftarrow\leftarrow \varphi t(K)\) in \(\text{Rel}_A(I, s)\) guaranteed by the table morphism condition \(k \cdot t' = t\). The fibered image passage

\[
\text{Tbl}(A) \xrightarrow{\text{im}^A} \text{Rel}(A)
\]

can be defined in terms of the fiber image passages \(\{ \text{im}^A_{(I, s)} \mid \langle I, s \rangle \in \text{List}(X)\}\). An \(A\)-table \(\langle I, s, K, t \rangle \in \text{Tbl}(A)\) with signature \(\langle I, s \rangle\) and table \(\langle K, t \rangle \in \text{Tbl}_A(I, s)\)
is mapped to the \( \mathcal{A} \)-relation \( \langle I, s, \varphi t(K) \rangle \in \text{Rel}(\mathcal{A}) \) with the same signature and the relation \( \varphi t(K) = im_{\mathcal{A}}^A(I, s, t) \in \text{Rel}(\mathcal{A}) \). An \( \mathcal{A} \)-table morphism \( T' = \langle I', s', K', t' \rangle \overset{(h,k)}{\rightarrow} \langle I, s, K, t \rangle = T \) consisting of signature morphism \( \langle I', s' \rangle \overset{h}{\rightarrow} \langle I, s \rangle \) satisfying either of the adjoint fiber orderings in Eqn. 3 is mapped to the \( \mathcal{A} \)-relation morphism \( im_{\mathcal{A}}(T') = \langle I', s', R' \rangle \overset{h}{\rightarrow} \langle I, s, R \rangle = im_{\mathcal{A}}(T) \) with the same signature morphism and satisfying either of the adjoint fiber orderings

\[
\begin{align*}
im_{\mathcal{A}}^A((I, s))((K, t)) &\supseteq \exists_h(im_{\mathcal{A}}^A((K, t))) \\
im_{\mathcal{A}}^A((I, s))((h^{-1}(K', t'))) &\supseteq h^{-1}(im_{\mathcal{A}}^A((K', t'))) \supseteq im_{\mathcal{A}}^A((I, s))((K, t)).
\end{align*}
\]

**Reflection.** The inclusion passage \( \text{Rel}(\mathcal{A}) \xrightarrow{inc_{\mathcal{A}}} \text{Tbl}(\mathcal{A}) \) is full. The composite passage \( \text{Rel}(\mathcal{A}) \xrightarrow{inc_{\mathcal{A}} \circ im_{\mathcal{A}}} \text{Rel}(\mathcal{A}) \) is the identity passage.

**Definition 4.** There is an image-factorization bridge \( 1_{\text{Tbl}(\mathcal{A})} \xrightarrow{\eta} im_{\mathcal{A}} \circ inc_{\mathcal{A}} \). The \( T^\mathcal{A} \)-component for \( \mathcal{A} \)-table \( T = \langle I, s, K, t \rangle \) is the \( \mathcal{A} \)-table morphism \( T \xrightarrow{\eta_T} \langle (1, e) \rangle \xrightarrow{\mathcal{A}} inc_{\mathcal{A}}(im_{\mathcal{A}}(T)) \), where \( K \xrightarrow{\varphi t} \mathcal{A} \xrightarrow{inc_{\mathcal{A}}(im_{\mathcal{A}}(T))} \mathcal{A} \xrightarrow{\mathcal{A}} \mathcal{A} \). The naturality diagram

\[
\begin{array}{ccc}
K' & \xrightarrow{k} & K \\
\downarrow & & \downarrow \\
inc_{\mathcal{A}}(im_{\mathcal{A}}(T')) & \xrightarrow{\varphi t(K')} & \varphi t(K) \\
\downarrow & & \downarrow \\
tup_{\mathcal{A}}(I', s') & \xrightarrow{\varphi t(K)} & \varphi t(I, s) \\
\end{array}
\]

factors the condition \( k \cdot t' = t \cdot \tup_{\mathcal{A}}(h) \) by diagonal fill-in. This gives the \( \mathcal{A} \)-table morphism \( inc_{\mathcal{A}}(im_{\mathcal{A}}(T')) \xrightarrow{(h,r)} inc_{\mathcal{A}}(im_{\mathcal{A}}(T)) \), which is the image-inclusion composite passage applied to the \( \mathcal{A} \)-table morphism \( T' \xrightarrow{(h,k)} T \).

**Proposition 12.** There is a reflection \( \text{Tbl}_{\mathcal{A}} \xrightarrow{(im_{\mathcal{A}} \circ inc_{\mathcal{A}})} \text{Rel}_{\mathcal{A}} \).

This reflection embodies the notion of informational equivalence.

### A.2 Comma Contexts

**Fact 1** Let \( A \xrightarrow{L} C \xrightarrow{R} B \) be a passage opspan with both \( A \xrightarrow{L} C \) and \( B \xrightarrow{R} C \) continuous passages. If \( A \) and \( B \) are complete contexts, then the comma context \( (L \downarrow R) \) is complete and the projection passages \( A \leftarrow (L \downarrow R) \rightarrow B \) are continuous.

---

28 A passage \( C \xrightarrow{f} D \) is *continuous* when it preserves all small limits that exist in \( C \).
Fact 2 Let $A \xleftarrow{L} C \xrightarrow{R} B$ be a passage opspan with $A \xleftarrow{L} C$ cocontinuous. \[^{29}\] If $A$ and $B$ are cocomplete contexts, then the comma context $(L \downarrow R)$ is cocomplete and the projection passages $A \leftarrow (L \downarrow R) \rightarrow B$ are cocontinuous.

A.3 The Grothendieck Construction

The adjunction $C_i \xrightarrow{\mathcal{C}_a \sim \mathcal{C}_a} C_i'$ has unit $I_{C_i} \xrightarrow{\cong} \mathcal{C}_a \circ \mathcal{C}_a$ with the $C_i$-morphism $A \xrightarrow{\eta_a(A)} \mathcal{C}_a(\mathcal{C}_a(A))$ as its $A$th component, and has counit $\mathcal{C}_a \circ \mathcal{C}_a \xrightarrow{\cong} I_{C_i'}$ with the $C_i'$-morphism $\mathcal{C}_a(\mathcal{C}_a(A')) \xrightarrow{\varepsilon_a(A')} A'$ as its $A'$th component.

\[
\begin{align*}
A \mathcal{C}_a(A) & \xrightarrow{\mathcal{C}_a(f)} \mathcal{C}_a(A') \\
C_i & \xrightarrow{\mathcal{C}_a} C_i' \\
\end{align*}
\]

\[
\begin{align*}
A \mathcal{C}_a(A) & \xrightarrow{\mathcal{C}_a(f)} \mathcal{C}_a(A') \\
C_i & \xrightarrow{\mathcal{C}_a} C_i' \\
\end{align*}
\]

opfibration bifibration

The adjunction $C_i \xrightarrow{\mathcal{C}_a \sim \mathcal{C}_a} C_i'$ has unit $I_{C_i} \xrightarrow{\cong} \mathcal{C}_a \circ \mathcal{C}_a$ with the $C_i$-morphism $A \xrightarrow{\eta_a(A)} \mathcal{C}_a(\mathcal{C}_a(A))$ as its $A$th component, and has counit $\mathcal{C}_a \circ \mathcal{C}_a \xrightarrow{\cong} I_{C_i'}$ with the $C_i'$-morphism $\mathcal{C}_a(\mathcal{C}_a(A')) \xrightarrow{\varepsilon_a(A')} A'$ as its $A'$th component.

\[
\begin{align*}
A \mathcal{C}_a(A) & \xrightarrow{\mathcal{C}_a(f)} \mathcal{C}_a(A') \\
C_i & \xrightarrow{\mathcal{C}_a} C_i' \\
\end{align*}
\]

bifibration

fibration: A fibration (fibered context) $\int \mathcal{C}$ is the Grothendieck construction of a contravariant pseudo-passage (indexed context) $I^{op} \xrightarrow{\mathcal{C}} \text{Cxt}$, where the action on any indexing object $i$ in $I$ is the fiber context $\mathcal{C}_i$ and the action on any indexing morphism $i \xrightarrow{a} i'$ is the fiber passage $C_i \xleftarrow{\mathcal{C}_a} C_i'$. An object in $\text{Cxt}$ is a bifibration when it preserves all small colimits that exist in $\mathcal{C}$. A passage $C \xrightarrow{F} D$ is cocontinuous if the opposite passage $C^{op} \xleftarrow{F^{op}} D^{op}$ between opposite contexts is a continuous passage.

\[^{29}\] A passage $C \xrightarrow{F} D$ is cocontinuous when it preserves all small colimits that exist in $C$. A passage $C \xrightarrow{F} D$ is cocontinuous if the opposite passage $C^{op} \xleftarrow{F^{op}} D^{op}$ between opposite contexts is a continuous passage.
\( \int \mathcal{C} \) is a pair \( \langle i, A \rangle \), where \( i \) is an indexing object in \( I \) and \( A \) is an object in the fiber context \( \mathcal{C}_i \). A morphism in \( \int \mathcal{C} \) is a pair \( \langle i, A \rangle \xrightarrow{(a,f)} \langle i', A' \rangle \), where \( i \xrightarrow{a} i' \) is an indexing morphism in \( I \) and \( A \xrightarrow{\mathcal{C}_a} \mathcal{C}_a(A') \) is a fiber morphism in \( \mathcal{C}_i \). There is a projection passage \( \int \mathcal{C} \to I \).

**opfibration:** An opfibration \( \int \mathcal{C} \) is the Grothendieck construction of a covariant pseudo-passage (indexed context) \( I \xrightarrow{\mathcal{C}} \text{Cxt} \), where the action on any indexing object \( i \) in \( I \) is the fiber context \( \mathcal{C}_i \) and the action on any indexing morphism \( i \xrightarrow{a} i' \) is the fiber passage \( \mathcal{C}_i \xrightarrow{\mathcal{C}_a} \mathcal{C}_{i'} \). An object in \( \int \mathcal{C} \) is a pair \( \langle i, A \rangle \), where \( i \) is an indexing object in \( I \) and \( A \) is an object in the fiber context \( \mathcal{C}_i \). A morphism in \( \int \mathcal{C} \) is a pair \( \langle i, A \rangle \xrightarrow{(a,f)} \langle i', A' \rangle \), where \( i \xrightarrow{a} i' \) is an indexing morphism in \( I \) and \( \mathcal{C}_a(A) \xrightarrow{\mathcal{C}_a(f)} \mathcal{C}_a(A') \) is a fiber morphism in \( \mathcal{C}_{i'} \). There is a projection passage \( \int \mathcal{C} \to I \).

**bifibration:** A bifibration \( \int \mathcal{C} \) (Fig. 22) is the Grothendieck construction of an indexed adjunction \( I \xrightarrow{\mathcal{C}} \text{Adj} \) consisting of a left adjoint covariant pseudo-passage \( I \xrightarrow{\mathcal{C}} \text{Cxt} \) and a right adjoint contravariant pseudo-passage \( I^{\text{op}} \xrightarrow{\mathcal{C}} \text{Cxt} \). The action on any indexing object \( i \) in \( I \) is the fiber context \( \mathcal{C}_i = \mathcal{C}_i \) and the action on any indexing morphism \( i \xrightarrow{a} i' \) is the fiber adjunction \( (C_i \xrightarrow{\mathcal{C}_a} C_{i'}) \) via the adjoint pair \( A \xrightarrow{\mathcal{C}_a} \mathcal{C}_a(A') \xrightarrow{\mathcal{C}_a(f)} \mathcal{C}_a(A) \xrightarrow{\mathcal{C}_a(g)} \mathcal{C}_a(A') \). The Grothendieck constructions of component fibration and component opfibration are isomorphic \( \int \mathcal{C} \cong \int \mathcal{C} \) via the adjoint pair \( A \xrightarrow{\mathcal{C}_a} \mathcal{C}_a(A') \xrightarrow{\mathcal{C}_a(f)} \mathcal{C}_a(A) \xrightarrow{\mathcal{C}_a(g)} \mathcal{C}_a(A') \). Define the Grothendieck construction of the bifibration to be the Grothendieck construction of component fibration \( \int \mathcal{C} \cong \int \mathcal{C} \) with projection \( \int \mathcal{C} \to I \).

**Fact 3** If \( I^{\text{op}} \xrightarrow{\mathcal{C}} \text{Cxt} \) is a contravariant pseudo-passage (indexed context) s.t.
1. the indexing context \( I \) is complete,
2. the fiber context \( C_i \) is complete for each \( i \in I \), and
3. the fiber passage \( C_i \xleftarrow{\mathcal{C}_a} C_j \) is continuous for each \( i \xrightarrow{a} j \) in \( I \),
then the fibered context (Grothendieck construction) \( \int \mathcal{C} \) is complete and the projection \( \int \mathcal{C} \xrightarrow{\mathcal{P}} I \) is continuous.

**Proof.** Tarlecki, Burstall and Goguen [25].

**Fact 4** If \( I \xrightarrow{\mathcal{C}} \text{Cxt} \) is a covariant pseudo-passage (indexed context) s.t.
1. the indexing context \( I \) is cocomplete,
2. the fiber context $C_i$ is cocomplete for each $i \in I$, and
3. the fiber passage $C_i \xrightarrow{C_{ij}} C_j$ is cocontinuous for each $i \xrightarrow{a} j$ in $I$,
then the fibered context (Grothendieck construction) $\int C$ is cocomplete and the projection $\int C \xrightarrow{P} I$ is cocontinuous.

Proof. Dual to the above. 

**Fact 5** If $I \xrightarrow{C} \text{Adj}$ is an indexed adjunction consisting of a contravariant pseudo-passage $I^{op} \xrightarrow{\ell} \text{Cxt}$ and a covariant pseudo-passage $I \xrightarrow{\ell'} \text{Cxt}$ that are locally adjunctive $(C_i : \langle C_{a \xrightarrow{a} j} \ell_a \rangle : C_{ij})$ for each $i \xrightarrow{a} j$ in $I$, s.t.

1. the indexing context $I$ is complete and cocomplete,
2. the fiber context $C_i$ is complete and cocomplete for each $i \in I$,
then the fibered context (Grothendieck construction) $\int C \xrightarrow{P} I$ is complete and cocomplete and the projection $\int C \xrightarrow{P} I$ is continuous and cocontinuous.

Proof. Use Facts. 3 & 4 since the fiber passage $C_i \xleftarrow{C_{ij}} C_{ij}$ is continuous (being right adjoint) and the fiber passage $C_i \xrightarrow{C_{ij}} C_{ij}$ is cocontinuous (being left adjoint) for each $i \xrightarrow{a} j$ in $I$. 

$\blacksquare$
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