Microscopic description of the chiral Tomonaga-Luttinger liquid at the fractional quantum Hall edge

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Abstract

The effective field theory of the fractional quantum Hall edge is reformulated from microscopic dynamics. Noncommutative Chern-Simons theory is a microscopic description for the quantum Hall fluid. We use it for reference. Considering relabeling symmetry of the electrons and incompressibility of the fluid, we obtain a constraint and derive a chiral Tomonaga-Luttinger liquid theory containing interaction terms. We calculate one-loop corrections to the phonon and electron propagators and get a new tunneling exponent. It agrees with experiments.

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I. INTRODUCTION

The chiral Tomonaga-Luttinger liquid at the fractional quantum Hall edge becomes an active subject since Wen’s hydrodynamic and effective field formulation \[1, 2, 3\]. The effective edge theory is derived from the bulk effective Chern-Simons theory. It predicts a nonlinear current-voltage relationship \(I \sim V^\alpha\) with an universal exponent \(\alpha\), e.g. \(\alpha = 3\) at the filling fraction \(\nu = 1/3\). For the Jain fractions \(\nu = n/(2n \pm 1)\), the power-law behavior is also predicted by including the effect of residual disorder \[4, 5\].

A number of experiments \[6, 7, 8\] establish the existence of Tomonaga-Luttinger-liquid-like behavior. However, the tunneling exponent measured is different from the prediction, e.g. \(\alpha \approx 2.7\) at \(\nu = 1/3\). The discrepancy between experiment and theory has been addressed in \[9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20\]. Many works have attempted to explain the discrepancy. For the compressible quantum Hall fluid, the edge density enhancement is considered \[10, 11\]. For \(\nu = 1/3\) and other Jain fractions \(n/(2n \pm 1)\), some papers suggest that the discrepancy is due to edge reconstruction \[12, 13, 14, 15, 16, 17\]. In contrast, some propose that the exponent is not universal, since the discrepancy persists even in the absence of edge reconstruction \[18, 19, 20\]. It’s still an open question.

An elementary derivation of the Chern-Simons description of the quantum Hall effect was given by Susskind \[21\], wherein he claimed that the noncommutative version of the description is exactly equivalent to the Laughlin theory. As many successes on the connection between two theories have been achieved in a collection of papers \[21, 22, 23, 24, 25\], we would stress that noncommutative Chern-Simons theory is a workable microscopic description for the quantum Hall fluid. Recently, this theory has been extended for constructing the hierarchy of fractional quantum Hall states \[26\].

In this paper, we try to pursue two questions: whether the edge states in fractional quantum Hall effects could be described by means of microscopic dynamics rather than an effective theory; whether a more reliable and sounder exponent \(\alpha\) could be derived.

The strategy is the following. Based on Susskind’s microscopic derivation \[21\], we will reformulate Wen’s edge theory \[2, 3\]. Firstly we give a constraint by considering microscopic dynamics: relabeling symmetry of the electrons and incompressibility of the fluid. The constraint should be obviously more natural than that given by choosing a gauge-fixing condition in Wen’s theory. Secondly we solve the constraint exactly. It’s amazing to find
that the solution as well as the action has a total differential form. Finally we reduce the 2 + 1 dimensional Chern-Simons theory to an 1 + 1 dimensional noncommutative chiral Tomonaga-Luttinger liquid theory, which contains interaction terms expanding to all orders in the noncommutative parameter \( \theta \). The commutative limit of it is Wen’s theory.

Furthermore, as our theory contains interaction terms, it will predict a new exponent and may provide a solution to the discrepancy mentioned above. So we calculate one-loop Feynman diagrams caused by the interactions. We notice the existence of a shortest incompressible distance and impose an ultraviolet cutoff to evaluate the integrals. Then we get one-loop corrections to the phonon and electron propagators. The electron propagator still exhibits a power-law correlation, but with a newly corrected prediction of the exponent which is in good agreement with the experimental results. This is a support of our derivation.

Briefly, we derive a noncommutative field theory of the quantum Hall edge from microscopic dynamics. We would claim that it should be sounder than existing effective theories. Previously \[26\], we have argued that the Chern-Simons description of the edge excitations would receive a natural explanation from the microscopic construction. Recently, a similar subject was discussed incompletely \[27\].

The outline of the paper is as follows. In Sec. \( \text{II} \) we review the microscopic derivation of Chern-Simons theory and get the constraint. In Sec. \( \text{III} \) we derive the noncommutative version of the chiral Tomonaga-Luttinger liquid theory. In Sec. \( \text{IV} \) we calculate the full phonon and electron propagators and give the corrected exponent. We conclude in Sec. \( \text{V} \).

II. MICROSCOPIC DERIVATION OF CHERN-SIMONS THEORY

We’ll begin with a review of the microscopic derivation of the Chern-Simons description of the quantum Hall effect \[21\].

Consider a two-dimensional electron system, the discrete electrons should be labeled with a discrete index \( \alpha \). Under the relabeling (or permutation) of the electrons, \( \alpha \rightarrow \alpha' = \alpha'(\alpha) \), the real space coordinates and the Lagrangian remain invariant \((i = 1, 2)\)

\[
x_i^\alpha(t) = x_i^{\alpha'}(t), \quad \delta L = 0.
\] (1)

We can introduce a continuous space \( y \), e.g. with a lattice \( y_i^\alpha = x_i^\alpha(0) \), and define the
fluid fields $x_i(y_j, t)$ on it with values

$$x_i(y_j, t)|_{y_j = y_j^0} = x_i(y_j^0, t) = x_i^0(t).$$ (2)

As $y$ is just a continuum description replacing $\alpha$, we can naturally choose the coordinates so that the electrons are evenly distributed in $y$ with a constant density $\rho_0$. The relabeling symmetry of $\alpha$ is replaced by the area preserving diffeomorphism (APD) of $y$.

Assuming that the system is adiabatic so that short range forces lead to an equilibrium and the potential is $\rho$ dependent ($\rho = \rho_0 |\partial y/\partial x|$ is the real space density), in a background magnetic field $B$ we can write the Lagrangian as

$$L = \int d^2y \rho_0 \left[ \frac{m}{2} \dot{x}_a^2 - V(\rho) + \frac{eB}{2} \epsilon_{ab} \dot{x}_a \dot{x}_b \right].$$ (3)

Consider an infinitesimal transformation $y_i' = y_i + f_i(y)$, which is APD if and only if (iff) $|\partial y'/\partial y| = 1$, i.e. $f_i = \epsilon_{ij} \partial \Lambda(y)/\partial y_j$ with $\Lambda$ being an arbitrary function. The $x$ coordinates and the Lagrangian transform as

$$\delta x_a = \frac{\partial x_a}{\partial y_i} f_i(y) = \epsilon_{ij} \frac{\partial x_a}{\partial y_i} \frac{\partial \Lambda}{\partial y_j},$$ (4)

$$\delta L = \frac{\partial L}{\partial \dot{x}_a} \delta \dot{x}_a + \frac{\partial L}{\partial x_a} \delta x_a$$
$$= \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}_a} \delta x_a \right) + \left( - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}_a} + \frac{\partial L}{\partial x_a} \right) \delta x_a = 0.$$ (5)

Besides the equation of motion

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}_a} - \frac{\partial L}{\partial x_a} = 0,$$ (6)

we arrive at a conserved quantity and the constraints

$$g^{-1}(y) = \begin{cases} \frac{\partial}{\partial y_j} \left( \epsilon_{ij} \dot{x}_a \frac{\partial \Lambda}{\partial y_j} \right) & \text{if } B \text{ is absent}, \\ \frac{1}{2} \epsilon_{ij} \epsilon_{ab} \frac{\partial x_b}{\partial y_j} \frac{\partial x_a}{\partial y_i} = \left| \frac{\partial x}{\partial y} \right| & \text{if } B \text{ is strong}, \end{cases}$$ (7)

where $g(y)$ is an arbitrary time independent function. When the magnetic field is strong, the kinetic term is dropped; $\rho(x, t) = \rho_0 g(y) = \rho(x, 0)$.

In the strong magnetic field, the lowest Landau level dominates and the system behaves as the quantum Hall fluid. Due to the Pauli exclusion principle, the electrons are incompressible with a minimal area $2\pi l_B^2$ ($l_B = 1/\sqrt{eB}$ is the magnetic length) [3]. In the absence of vortices (no quasiparticle excitation), we assume that at $t = 0$ the electrons are in equilibrium and
uniformly occupy the minimal area. So \( \rho(x,0) \) is constant, \( g(y) = \text{const}/\rho_0 \) can be set to unity. The constraint becomes

\[ 1 = \left| \frac{\partial x}{\partial y} \right|. \tag{8} \]

Consider a fractional quantum Hall fluid with a filling factor \( \nu = 1/(2n+1) \), where \( n \) is a positive integer. It is also incompressible due to the interaction \[28\], so we have the same constraint. Specially, the minimal area becomes \( 2\pi l_B^2/\nu \). As \( \rho_0 = \rho(x,0) = (2\pi l_B^2/\nu)^{-1} \), the factor \( \nu = 2\pi \rho_0/eB \) is truly the ratio of electrons to magnetic flux quanta.

We must stress that the constraint is derived from microscopic dynamics by considering relabeling symmetry of the electrons and incompressibility of the fluid. It is more exact, general and natural than that given by choosing the gauge-fixing condition \[2, 3\].

Consider small deviations from the equilibrium solution \( x_i = y_i \),

\[ x_i(y, t) = y_i + \epsilon_{ij} A_j(y, t) \equiv y_i + \theta \epsilon_{ij} A_j, \tag{9} \]

where the gauge transformation of \( A_i \) under APD is

\[ \delta A_i = 2\pi \rho_0 \frac{\partial \Lambda}{\partial y_i} + \epsilon_{ab} \frac{\partial A_i}{\partial y_a} \frac{\partial \Lambda}{\partial y_b}. \tag{10} \]

Substituting it and dropping total time derivatives gives the Chern-Simons action

\[ S = \frac{eB}{2} \int dtd^2y \rho_0 \epsilon_{ij} \dot{x}_i x_j \]
\[ = \frac{1}{4\pi \nu} \int dtd^2y \epsilon_{ij} \dot{A}_i A_j. \tag{11} \]

Notice that \( y \) space has a basic area quantum \( \theta = 1/(\nu eB) = l_B^2/\nu \). It means that \( y \) space is noncommutative. In \[21\], Eqs. (8), (10) and (11) are recognized as first order truncations of a noncommutative Chern-Simons theory, which is defined by the Lagrangian

\[ L_{NC} = \frac{1}{4\pi \nu} \epsilon_{\mu\nu\rho} (A_\mu \ast \partial_\nu A_\rho + \frac{2i}{3} A_\mu \ast A_\nu \ast A_\rho), \tag{12} \]

where \( \ast \) represents the usual Moyal star-product defined in terms of the noncommutative parameter \( \theta \) \[29, 30\]. In the following, however, we go another way. By expanding to higher order in \( \theta \), we can also involve the noncommutativity of \( y \) space and capture the discrete character of the electron system.

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The exact solution of \(x_i(y, t)\) and \(A_i(y, t)\) can be calculated from the constraint. As

\[
1 = \frac{1}{2} \epsilon_{ij} \epsilon_{ab} \partial_i (y_a + \theta \epsilon_{am} A_m) \partial_j (y_b + \theta \epsilon_{bn} A_n)
= 1 + \theta \epsilon_{im} \partial_i A_m + \frac{1}{2} \theta^2 \epsilon_{ij} \epsilon_{mn} \partial_i A_m \partial_j A_n,
\]

\[
0 = \epsilon_{ij} \partial_i (A_j + \frac{1}{2} \theta \epsilon_{mn} A_m \partial_j A_n),
\]

the condition for \(A\) is that

\[
A_j + \frac{1}{2} \theta \epsilon_{mn} A_m \partial_j A_n = \partial_j \phi(y, t),
\]

where \(\phi(y, t)\) is an arbitrary scalar field. Expand \(A_j = \sum_{n=0}^{\infty} \theta^n a_j^{(n)}\), \(\phi = \sum_{n=0}^{\infty} \theta^n \varphi^{(n)}\),

\[
a_j^{(0)} = \partial_j \varphi^{(0)},
\]

\[
a_j^{(n)} = \partial_j \varphi^{(n)} + \frac{1}{2} \theta \epsilon_{ab} \sum_{m=0}^{n-1} \partial_j a_a^{(m)} a_b^{(n-1-m)}.
\]

Noticing that

\[
A_j = \partial_j \sum_{l=0}^{\infty} \theta^l \varphi^{(l)} + \frac{1}{2} \theta \epsilon_{ab}
\]

\[
\partial_j \partial_a \left( \sum_{m=0}^{\infty} \theta^m \varphi^{(m)} \right) \partial_b \left( \sum_{n=0}^{\infty} \theta^n \varphi^{(n)} \right) + O(\theta^2),
\]

we can redefine \(A_i = \sum_{n=0}^{\infty} \theta^n f_i^{(n)}\) with

\[
f_i^{(0)} = \partial_i \phi, \quad f_i^{(n)} = \frac{1}{2} \epsilon_{ab} \sum_{m=0}^{n-1} \partial_i f_a^{(m)} f_b^{(n-1-m)}.
\]

Substituting the exact solution into Eq. (11) gives a noncommutative action

\[
S = \frac{1}{4 \pi \nu} \int dt d^2 y \sum_{n=0}^{\infty} \theta^n s^{(n)},
\]

where

\[
s^{(n)} = \sum_{m=0}^{n} \epsilon_{ab} \partial_i f_a^{(m)} f_b^{(n-m)}.
\]

III. AT THE EDGE

Since a two-dimensional electron gas on a quantum Hall plateau is incompressible \[28\], the edge excitations are the only gapless excitations \[31\]. The edge states are important.
We should study whether the noncommutative action $S$ could describe them. To describe the edge states, we need a one-dimensional theory. To derive a one-dimensional theory, the first step is to find a total differential form of the action. It is difficult but has been done as follows.

Construct $F^{(n)}_{\mu}$ with total differentials ($\mu = 0, 1, 2$ and $\partial_0 \equiv \partial_t$): $F^{(0)}_{\mu} = \partial_\mu \phi$ and for $n \geq 1$

$$F^{(n)}_{\mu} = \partial_a \left( \frac{1}{2} \epsilon_{ab} F^{(n-1)}_{\mu} F^{(0)}_b \right)$$

$$- \frac{1}{3} \sum_{m=1}^{n-1} \partial_a \left( \frac{1}{2} \epsilon_{ab} F^{(m)}_{\mu} F^{(n-1-m)}_b \right)$$

$$+ \frac{1}{3} \partial_\mu \left( \frac{1}{2} \epsilon_{ab} F^{(n-1)}_{a} F^{(0)}_b \right).$$  (17)

We find that $f^{(0)}_i = F^{(0)}_i$, $f^{(1)}_i = F^{(1)}_i$, etc. Using Mathematica, the equivalence has been checked up to $n = 7$. Logically, we make a conjecture: for all natural numbers $n$, $f^{(n)}_i = F^{(n)}_i$.

Similarly, every order of the Lagrangian density is a total differential, $s^{(n)} = 2F^{(n+1)}_0$.

When the total time derivative is dropped,

$$s^{(n)} = \partial_a (\epsilon_{ab} F^{(n)}_0 F^{(0)}_b) - \frac{1}{3} \sum_{m=1}^{n} \partial_a (\epsilon_{ab} F^{(m)}_0 F^{(n-m)}_b).$$  (18)

We’ve checked it for $n \leq 6$ using Mathematica and for $n \leq 2$ by hand, e.g.

$$s^{(0)} = \epsilon_{ij} \partial_j [\phi \partial_i \partial_0 \phi],$$

$$s^{(1)} = \frac{1}{3} \epsilon_{ij} \epsilon_{ab} \partial_j [\partial_i \partial_b \phi \partial_i \phi \partial_a \phi],$$

$$s^{(2)} = \frac{1}{4} \epsilon_{ij} \epsilon_{ab} \epsilon_{mn} \partial_j [\partial_i (\partial_0 \phi \partial_0 \phi) \partial_m \phi \partial_n \phi].$$

Being integration of a total differential, $S$ is nonzero and nontrivial iff a boundary exists. Hence, we’ll continue with a boundary, where some degrees of freedom become dynamical.

Consider the finite system $\Sigma$ confined by a simple potential well: an electric field $\vec{E}$. The electrons drift in the direction perpendicular to $\vec{E}$ and $B$ and form an edge. In the context of special relativity, in the frame $x$ moving with $v_i \equiv \epsilon_{ij} E_j / B$, the electric field vanishes so that the electrons can be treated the same as that in bulk. The real space $x^R$ is

$$x^R_i = x_i + v_i t = y_i + \theta \epsilon_{ij} A_j + v_i t.$$  (19)
Substituting it into the edge action and dropping total time derivatives gives

\[ S_{\Sigma} = \int_{\Sigma} dt \, d^2 y \rho_0 \left( \frac{eB}{2} \varepsilon_{ij} \partial_t x_i^R x_j^R - eE_i x_i^R \right) \]

\[ = \int_{\Sigma} dt \, d^2 y \rho_0 \left( \frac{eB}{2} \varepsilon_{ij} \partial_t (x_i + 2 \varepsilon_{ia} \frac{E_a}{B} x_j - eE_i x_i) \right) \]

\[ = \int_{\Sigma} dt \, d^2 y \rho_0 \frac{eB}{2} \varepsilon_{ij} \dot{x}_i x_j = S. \] (20)

It confirms that the electric field vanishes in the frame \( x \) and the co-moving coordinates \( y \). So we can use the same Chern-Simons theory as in bulk.

Notice the relationship of the co-moving coordinates \( y \) and the laboratory frame \( y^R \)

\[ y_i^R = y_i + v_i t, \quad t^R = t, \]

\[ \partial_t = \partial_t^R + v_i \partial_i^R, \quad \partial_i = \partial_i^R. \] (21)

In terms of \( y^R \), the edge action acquires the form

\[ S_{\Sigma} = \frac{1}{4\pi \nu} \int_{\Sigma} dt^R d^2 y^R \varepsilon_{ij} (\partial_t^R + v_a \partial_a^R) A_i A_j \]

\[ = \frac{1}{4\pi \nu} \int_{\Sigma} dt^R d^2 y^R \sum_{n=0}^{\infty} \theta^n s^{(n)}. \] (22)

In the laboratory frame, ignoring \( R \) for ease of notation, choosing \( \vec{E} = E \hat{y}_2 \) and restricting the fluid to \( y_2 \leq 0 \) for convenience, we can reduce the edge action to an 1 + 1 dimensional chiral boson theory

\[ S_{\chi} = \frac{1}{4\pi \nu} \int dy_1 \phi (\partial_t + v \partial_1) \partial_t \phi + O(\theta) \]

\[ = \frac{1}{4\pi \nu} \int dy_1 \sum_{n=0}^{\infty} \theta^n \chi^{(n)}, \] (23)

where \( v = E/B \) and

\[ \chi^{(n)} = -F_0^{(n)} F_1^{(0)} + \frac{1}{3} \sum_{m=1}^{n} F_0^{(m)} F_1^{(n-m)}, \] (24)

with redefined \( F_0^{(0)} = (\partial_t + v \partial_1) \phi \) and for \( n \geq 1 \)

\[ F_0^{(n)} = \partial_a \left( \frac{1}{2} \varepsilon_{ab} F_0^{(n-1)} F_b^{(0)} \right) \]

\[ - \frac{1}{3} \sum_{m=1}^{n-1} \partial_a \left( \frac{1}{2} \varepsilon_{ab} F_0^{(m)} F_b^{(n-1-m)} \right) \]

\[ + \frac{1}{3} (\partial_t + v \partial_1) \left( \frac{1}{2} \varepsilon_{ab} F_0^{(n-1)} F_b^{(0)} \right). \] (25)
If we ignore the discrete character of the fluid, \( \theta \propto l_B^2 \to 0 \), we get the commutative limit of our microscopic description, which coincides with the phenomenological effective theory on the edge effect \[2, 3\].

In fact, we get a noncommutative chiral Tomonaga-Luttinger liquid theory which is one-dimensional and contains interaction terms. The hallmark feature of Tomonaga-Luttinger-liquid-like behavior will be shown in the next section. The dimensional reduction confirms that the only gapless excitations of a two-dimensional incompressible quantum Hall fluid are the edge excitations. We stress that interaction terms make things different: vertices and loop Feynman diagrams emerge and correct the phonon propagator.

### IV. CORRECTIONS TO THE PHONON AND ELECTRON PROPAGATORS

We’ll calculate the loop corrections to the phonon and electron propagators with \( S_\chi \).

Following Wen’s hydrodynamic formulation \[1, 2, 3\], we have the commutation relation

\[
\left[ \frac{1}{2\pi} \partial_1 \phi(y_1), \phi(y'_1) \right] = -i\nu \delta(y_1 - y'_1),
\]

and the electron operator (fermionic while \( 1/\nu \) is odd)

\[
\Psi \propto e^{\frac{i}{\nu} \phi}, \quad \Psi(y_1)\Psi(y'_1) = (-1)^{\frac{1}{\nu}} \Psi(y'_1)\Psi(y_1).
\]

The electron propagator can be calculated via the phonon propagator

\[
\langle T\{\Psi^\dagger(y_1, t)\Psi(0)\}\rangle = \exp\left[\frac{1}{\nu^2} \langle \phi(y_1, t)\phi(0)\rangle\right].
\]

With the commutation relation and the equation of motion \((\partial_t + v\partial_1)\partial_1 \phi = 0\), we can calculate the retarded Green’s function

\[
D_R(y_1, t) = \theta(t) \langle [\phi(y_1), \phi(0)] \rangle,
\]

\[
(\partial_t + v\partial_1)\partial_1 D_R(y_1, t) = \partial_t \theta(t)\partial_1 \langle [\phi(y_1), \phi(0)] \rangle = -i2\pi \nu \delta(t) \delta(y_1),
\]

\[
D_R(y_1, t) = \int \frac{d^2p}{(2\pi)^2} e^{-i(\omega pt - py_1)} \tilde{D}_R(p),
\]

\[
V_2(p) \equiv \tilde{D}_R(p) = \frac{-i2\pi \nu}{(\omega_p - v p)p}.
\]

To deal with \( \partial_2 \) in \( \chi^{(n)} \) \((n \geq 1)\), we assume an undetermined distribution \( \phi \propto \exp[h(y_2)] \).

Naturally, along the negative \( y_2 \) axis, \( \exp[h(y_2)] \) should decrease with a characteristic length.
\( \sqrt{2l_B^2/\nu} \), which means the radius occupied by every electron at the filling fraction \( \nu \).

Using Diagrammar with notations \( y_\mu = (y_1, it), p_\mu = (p, i\omega_p) \) and \( \int d^2p = i \int dpd\omega_p \), we can spell out the Feynman rules from the action times \( i \) with the replacement

\[
\phi(y) = \int \frac{d^2p}{(2\pi)^2} \bar{\phi}(p)e^{i(py_1 - \omega_p t)}e^{h(y_2)}. \tag{29}
\]

For 3-phonon and 4-phonon vertices (see Fig. 1), the Feynman rules are \( w_p = \omega_p - vp \), the \( \delta \) functions omitted

\[
V_3(p, q) \equiv \frac{\theta}{4\pi \nu} h'[q(2p + q)w_p + p(p + 2q)w_q]
\]

\[
= -\frac{\theta}{4\pi \nu} h' \sum_{l=p,q,r} l^2 w_l, \tag{30}
\]

\[
V_4(p, q, r) \equiv \frac{i\theta^2}{4\pi \nu^2} \left[ \left( h'^2 + 2h'' \right) \sum_{l=p,q,r,k} l^2 \sum_{l=p,q,r,k} lw_l - (h'^2 + h'') \sum_{l=p,q,r,k} l^3 w_l \right]. \tag{31}
\]

\[
\text{FIG. 1: 3-phonon and 4-phonon vertices.}
\]

Let \( G(p) \) denote the sum of all 1PI (one particle irreducible) diagrams with two external lines. We can express the full phonon propagator as

\[
\frac{-i2\pi \nu}{(\omega_p - vp)p} + \frac{-i2\pi \nu}{(\omega_p - vp)p} G \frac{-i2\pi \nu}{(\omega_p - vp)p} + \cdots
\]

\[
= \frac{-i2\pi \nu}{(\omega_p - vp)p + i2\pi \nu G}.
\tag{32}
\]

As shown in Fig. 2, the one-loop (second-order in \( \theta \)) contributions to \( G(p) \) are

\[
G_1 = \frac{1}{2} \int \frac{d^2q}{(2\pi)^2} V_3(p, q)V_2(q)V_3(-p, -q)V_2(p + q), \tag{33}
\]

\[
G_2 = \frac{1}{2} \int \frac{d^2q}{(2\pi)^2} V_4(p, q, -p)V_2(q), \tag{34}
\]

where \( G_n \) corresponds to the \( n \)th diagram in Fig. 2 (1/2 is a symmetry factor).
Because $y$ space has a basic area quantum $\theta$ and a shortest incompressible distance $l_B$, we can impose an ultraviolet cutoff $|q| \leq \Lambda$ and $|\omega_q| \leq v\Lambda$ (due to the energy-momentum dispersion $\omega_q = vq$). Via the uncertainty principle, $\Lambda = l_B^{-1}$. To evaluate loop integrals, notice that: integrals over polynomials give zero, $\int dq(q^2)^a = 0$, where $a$ is some nonnegative integer [33]; to the leading order

$$
\int dq d\omega_q \frac{q}{\omega_q - vq} = -2\Lambda^2,
$$

$$
\int dq d\omega_q \frac{q^2}{(\omega_q - vq + \omega_p - vp)(\omega_q - vq)} = 2\Lambda^2 \frac{1 - \ln \Lambda}{v}. \quad (35)
$$

Then

$$
G_1 = \int \frac{d^2q}{(2\pi)^2} \frac{\theta^2}{8} \left[ q(2p + q)w_p + p(p + 2q)w_q \right] h'^2
= -i \frac{\theta^2 \Lambda^2 h'^2}{(2\pi)^2} \left( pw_p + w_p^2 \ln \Lambda - \frac{1}{4v} \right), \quad (36)
$$

$$
G_2 = \int \frac{d^2q}{(2\pi)^2} \frac{\theta^2}{4} \left[ p^2 (h'^2 + 2h'') - q^2 h'^2 + pw_p q^2 (h'^2 + 2h'') - p^2 h'^2 \right]
= -i \frac{1}{(2\pi)^2} \frac{\theta^2 \Lambda^2 h'^2 + 2h''}{2} pw_p. \quad (37)
$$

The full phonon propagator, to one-loop order, has the form

$$
\begin{align*}
&-i2\pi \nu \\
&= \frac{-i2\pi \nu}{pw_p[1 + \frac{\nu}{2\pi} \theta^2 \Lambda^2(\frac{3}{2} h'^2 + h'') + w_p^2 \frac{\nu}{2\pi} \theta^2 \Lambda^2 h'^2 \ln \Lambda - \frac{1}{4v}]} \\
&= \frac{-i2\pi \nu}{pw_p(1 + c_1) + w_p^2 \frac{\ln \Lambda - 1}{v}} \\
&= \frac{-i2\pi \nu}{p(\omega_p - vp)} - \frac{-i2\pi \nu}{p[\omega_p - vp(1 - \frac{1 + c_1}{c_2} \frac{1}{\ln \Lambda - 1})]} \\
&= \frac{-i2\pi \nu}{p(\omega_p - vp)} - \frac{-i2\pi \nu}{p(\omega_p - v_n p)},
\end{align*}
$$

where $c_1 = \frac{\nu}{2\pi} \theta^2 \Lambda^2(\frac{3}{2} h'^2 + h'')$, $c_2 = \frac{\nu}{8\pi} \theta^2 \Lambda^2 h'^2$, $\nu = \frac{\nu}{1+c_1}$ and $v_n = v(1 - \frac{1 + c_1}{c_2} \frac{1}{\ln \Lambda - 1})$. 

![FIG. 2: One-loop Feynman diagrams.](image)
Notice that: $\theta = l_B^2/\nu$; $\Lambda = l_B^{-1}$. $h''$ and $h'$ are proportional to $\nu/(2l_B^2)$, because $\exp[h(y_2)]$ decreases with the characteristic length $\sqrt{2l_B^3/\nu}$. So $c_1$ and $c_2$ are constants independent of $\nu$ and $l_B$. As a perturbation-theory correction should not be too large, we have $|c_1| \ll 1$. Evidently $c_2 \geq 0$ so that $v_n$ is slightly smaller than $v$. Because of the damping of the electric field caused by the presence of the electrons, $v$ decreases by a small amount along the negative $y_2$ axis. Without loss of generality we can choose $v_n$ to be the next-door neighbor of $v$ and reconsider the full phonon propagator: the second term of the propagator with $v$ cancels the first term with $v_n$, and so on; the second term with $v_1$ can be ignored while it’s non-chiral and cancels the propagator in bulk with $v_0 = 0$; a sum over all slices gives the overall full phonon propagator

$$\langle \phi(y_1, t) \phi(0) \rangle = -\tilde{\nu} \ln(y_1 - vt) + \text{const}, \quad \text{(39)}$$

the full electron propagator can be calculated as

$$\langle T\{\Psi^\dagger(y_1, t)\Psi(0)\}\rangle \propto \frac{1}{(y_1 - vt)^\alpha}, \quad \text{(40)}$$

where

$$\alpha = \frac{\tilde{\nu}}{\nu^2} \approx 0.893 \frac{1}{\nu}. \quad \text{(41)}$$

We see that the electron propagator at the fractional quantum Hall edge exhibits a nontrivial power-law correlation, which indicates Tomonaga-Luttinger-liquid-like behavior [9].

At $\nu = 1/3$, the prediction of Eq. (41) is $\alpha \approx 2.68$. It is in good agreement with the value measured in experiments [6, 7, 8]: $\alpha \approx 2.7$.

V. CONCLUSION

In this paper, considering the microscopic dynamics of a two-dimensional electron system in a strong perpendicular magnetic field, we have derived a noncommutative field theory
describing the chiral Tomonaga-Luttinger liquid at the fractional quantum Hall edge. Without any adjustable parameter, we have resolved the discrepancy of the exponent $\alpha$ between experiment and the predictions of former effective field theories.

From the relabeling symmetry and the incompressibility of the fractional quantum Hall system, we obtain a constraint. The constraint is more natural than that chosen in [2, 3] and captures the discrete character of the system. We solve the constraint and find a total differential form of the solution. As also a total differential, the action is reduced to an 1 + 1 dimensional chiral Tomonaga-Luttinger liquid theory, which is the noncommutative version of Wen's theory and contains interaction terms expanding to all orders in $\theta$. Then one-loop corrections to the phonon and electron propagators are calculated. The electron propagator exhibits a new power-law correlation, where the exponent $\alpha$ is corrected to agree with experiments.

Furthermore, higher order corrections and the edge structures of hierarchial liquids are remained as future subjects.

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