Tangent prolongation of $C^r$-differentiable loops

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Abstract

The aim of our paper is to generalize the tangent prolongation of Lie groups to non-associative multiplications and to examine how the weak associative and weak inverse properties are transferred to the multiplication defined on the tangent bundle. We obtain that the tangent prolongation of a $C^r$-differentiable loop ($r \geq 1$) is a $C^{r-1}$-differentiable loop that acquires the classical weak inverse and weak associative properties of the initial loop.

1 Introduction

The tangent space of a differentiable manifold $M$ at $\xi \in M$ is $T_{\xi}(M)$ and the tangent bundle is $T(M)$. The tangent map of a differentiable map $\varphi : M \to M$ at $\xi \in M$ will be denoted by $d_{\xi}\varphi : T_{\xi}(M) \to T_{\varphi(\xi)}(M)$. For a Lie group $G$ with Lie algebra $\mathfrak{g}$ any element $(\xi, X) \in T(G)$ can be identified with the element $(\xi, d_{\xi}\lambda_{\xi}^{-1}X)$ of the Cartesian product $G \times \mathfrak{g}$, where $\lambda_\xi : G \to G$ denotes the left translation on $G$. It is well-known that the Lie group structure of $G$ has a natural prolongation to tangent bundle $T(G)$ such that the corresponding multiplication on $G \times \mathfrak{g}$ is expressed by

$$(\xi, x) \cdot (\eta, y) = (\xi\eta, d_{\xi}\lambda^{-1}_{\xi\eta}\frac{d}{dt} |_{t=0}(\xi \exp tx \cdot \eta \exp ty)) = (\xi\eta, d_{\xi}(\lambda^{-1}_{\xi\eta}\rho_{\eta})x + y),$$

where $\xi, \eta \in G$ and $x, y \in \mathfrak{g}$. Clearly, the tangent prolongation structure on $G \times \mathfrak{g}$ is a semi-direct product $G \rtimes \mathfrak{g}^+$, where $\mathfrak{g}^+$ is the additive group of $\mathfrak{g}$, cf. [3], [6].

The aim of our paper is to generalize the tangent prolongation to non-associative multiplications and to show that the weak associative and weak inverse properties pass to the tangent prolongation. We prove that the tangent prolongation of a $C^r$-differentiable loop ($r \geq 1$) is a $C^{r-1}$-differentiable loop that has the same classical weak inverse and weak associative properties as the initial loop. It is worth noting that, unlike the associative case, the multiplications of differentiable loops are not necessarily analytic, (cf. the loop
constructions in Part 2 of [4]). Our research is a reflection on the subject raised by J. Grabowski in his conference lecture Tangent and cotangent loopoids [2].

We examine in §3 the two-sided inverse, left and right inverse, monoassociative, left and right alternative, flexible, left and right Bol properties of general abelian linear extensions. The loops in this class of extensions (cf. [7], [1]) are natural abstract generalizations of the tangent prolongation of differentiable loops. §4 is devoted to the discussion of the differentiability properties of tangent prolongations. In §5 we investigate such abelian linear extensions, which are determined by a homomorphism of the inner mapping group into the linear group acting on the tangent space at the identity element. We apply our results to the tangent prolongation of \( C^r \)-differentiable loops, where the above homomorphism is determined by the tangent map of inner mappings. At the end in §6 we obtain our main result on the tangent prolongation of \( C^r \)-differentiable loops and on their classical weak inverse and weak associative properties.

2 Preliminaries

A loop \( L \) is a set with a multiplication map \( (x,y) \mapsto x \cdot y : L \times L \to L \) with an identity element \( e \in L \) such that \( e \cdot x = x \cdot e = x \) for each \( x \in L \), moreover the left translations \( \lambda_x : L \to L, \lambda_x y = xy \), and the right translations \( \rho_x : L \to L, \rho_x y = yx \), are bijective maps. The left and right division operations of \( L \) are the maps \( (x,y) \mapsto x/y = \lambda^{-1}_y x \), respectively \( (x,y) \mapsto x \cdot y = \rho^{-1}_y x \). The opposite loop of \( L \) is the loop defined by the multiplication \( (x,y) \mapsto y \cdot x : L \times L \to L \). The automorphism group of \( L \) is denoted by \( \text{Aut}(L) \).

The group \( \text{Mlt}(L) \) generated by all left and right translations of the loop \( L \) is called the multiplication group of \( L \). The inner mapping group \( \text{Inn}(L) = \{ \phi \in \text{Mlt}(L) : \phi(e) = e \} \) of \( L \) is the subgroup of \( \text{Mlt}(L) \) consisting of all bijections in \( \text{Mlt}(L) \) fixing the identity element \( e \).

We will reduce the use of parentheses by the following convention: juxtaposition will denote multiplication, the division operations are less binding than juxtaposition, and the multiplication is less binding than the divisions. For instance the expression \( xy/u \cdot v \cdot w \) is a short form of \( ((x \cdot y)/u) \cdot (v \cdot w) \).

A loop \( L \) is monoassociative if \( x \cdot x^2 = x^2 \cdot x \) for any \( x \in L \), left, respectively right alternative if \( x \cdot xy = x^2 \cdot y \), respectively \( yx \cdot x = y \cdot x^2 \) for all \( x, y \in L \), flexible if \( x \cdot yx = xy \cdot x \) for all \( x, y \in L \). \( L \) is a left, respectively right Bol loop if it satisfies the identity \( (x \cdot yx)z = x(y \cdot xz) \), respectively \( z(xy \cdot x) = (zx \cdot y)x \). We say that a loop \( L \) has a weak associative property if \( L \) satisfies one of the monoassociative, left alternative, right alternative, flexible, left Bol, right Bol properties.

The elements \( e/x, \) respectively \( x\backslash e \) are the left inverse and the right inverse of \( x \in L \). If the left and the right inverses of \( x \) coincide then \( x^{-1} = e/x = x\backslash e \) is the inverse of \( x \in L \). If any element \( x \in L \) has inverse \( x^{-1} \) then we say that \( L \) has two-sided inverse property. \( L \) satisfies the left, respectively the right inverse property if there exists a bijection \( \iota : L \to L, \) such that \( \iota(x) \cdot xy = y, \) respectively \( yx \cdot \iota(x) = y \) holds for all \( x, y \in L \). It is well known that in loops with left or right inverse property all elements have inverses (cf. [5], I.4.2 Theorem), hence \( \iota(x) = x^{-1} \). We say that a loop \( L \) has a weak inverse property if \( L \) has the two-sided, left or right inverse property.

A loop \( L \) defined on a differentiable manifold is called \( C^r \)-differentiable loop, \( 0 < r \in \mathbb{N} \).
or \( r = \infty \), if the multiplication \( (x, y) \mapsto x \cdot y \), the left division \( (x, y) \mapsto x \div y \) and the right division \( (x, y) \mapsto x/y \) are \( C^r \)-differentiable \( L \times L \rightarrow L \) maps.

## 3 Linear abelian extensions

Before the investigation of the natural loop-multiplication on the tangent bundle of a differentiable loop we consider a more general class of abstract non-associative multiplications determined by abelian linear extensions investigated in \([7, 1]\). We examine the two-sided inverse, left and right inverse, monoassociative, left and right alternative, flexible, left and right Bol properties of abelian linear extensions.

Let \( A = (A, +) \) be a commutative group and \( L = (L, \cdot, /, \\) \) a loop with identity element \( \epsilon \in L \). A pair \((P, Q)\) is called a loop cocycle if \( P, Q \) are mappings \( L \times L \rightarrow \text{Aut}(A) \) satisfying

\[
P(\alpha, \epsilon) = \text{Id} = Q(\epsilon, \beta) \quad \text{for every } \alpha, \beta \in L.
\]

**Definition 3.1.** If \((P, Q)\) is a loop cocycle then the multiplication on \( L \times A \)

\[
(\alpha, a) \cdot (\beta, b) = (\alpha \beta, P(\alpha, \beta)a + Q(\alpha, \beta)b)
\]

defines the linear abelian extension \( F(P, Q) \) of the group \( A \) by the loop \( L \).

Clearly, the linear abelian extension \( F(P, Q) \) of the group \( A \) by the loop \( L \) is a loop with identity element \((\epsilon, 0)\), where the left and right divisions are given by

\[
\begin{align*}
(\xi, x) \div (\eta, y) &= (\xi, Q(\xi, \xi \div \eta)^{-1}(y - P(\xi, \xi \div \eta)x)), \\
(\eta, y) / (\xi, x) &= (\eta / \xi, P(\eta / \xi, \xi)^{-1}(y - Q(\eta / \xi, \xi)x)).
\end{align*}
\]

**Proposition 3.1.** If the loop \( L \) has some weak inverse or weak associative property then the extension \( F(P, Q) \) has the same property if and only if the loop cocycle \((P, Q)\) satisfies the identities given in the following list:

- **(A) two-sided inverse property:**
  \[
P(\xi, \xi^{-1}) = Q(\xi, \xi^{-1})P(\xi^{-1}, \xi)^{-1}Q(\xi^{-1}, \xi),
\]

- **(B) left inverse property:**
  \[
  Q(\xi^{-1}, \xi \eta) = Q(\xi, \eta)^{-1}, \quad P(\xi^{-1}, \xi \eta) = Q(\xi, \eta)^{-1}P(\xi, \eta)Q(\xi^{-1}, \xi)^{-1}P(\xi^{-1}, \xi),
  \]
(C) right inverse property:

\[ P(\xi \eta, \eta^{-1}) = P(\xi, \eta)^{-1}, \quad Q(\xi \eta, \eta^{-1}) = P(\xi, \eta)^{-1}Q(\xi, \eta)P(\eta, \eta^{-1})^{-1}Q(\eta, \eta^{-1}), \]

(D) monoassociative:

\[ P(\xi, \xi^{2}) + Q(\xi, \xi^{2}) (P(\xi, \xi) + Q(\xi, \xi)) = P(\xi^{2}, \xi)(P(\xi, \xi) + Q(\xi, \xi)) + Q(\xi^{2}, \xi), \]

(E) left alternative:

\[ Q(\xi, \eta \xi)Q(\xi, \eta) = Q(\xi^{2}, \eta), \quad P(\xi, \xi \eta) + Q(\xi, \xi \eta)P(\xi, \eta) = P(\xi^{2}, \eta)(P(\xi, \xi) + Q(\xi, \xi)), \]

(F) right alternative:

\[ P(\eta \xi, \xi)P(\eta, \xi) = P(\eta, \xi^{2}), \quad P(\eta \xi, \xi)Q(\eta, \xi) + Q(\eta \xi, \xi) = Q(\eta, \xi^{2})(P(\xi, \xi) + Q(\xi, \xi)), \]

(G) flexible:

\[ Q(\xi, \eta \xi)P(\eta, \xi) = P(\xi \eta, \xi)Q(\xi, \eta), \quad P(\xi, \eta \xi) + Q(\xi, \eta \xi)P(\xi, \eta) = P(\xi \eta, \xi)P(\xi, \eta) + Q(\xi \eta, \xi), \]

(H) left Bol identity:

\[ Q(\xi, \eta \cdot \xi \xi)Q(\eta, \xi \xi)Q(\xi, \xi) = Q(\xi \cdot \eta \xi, \xi), \quad Q(\xi, \eta \cdot \xi \xi)P(\eta, \xi \xi) = P(\xi \cdot \eta \xi, \xi)Q(\xi, \xi \eta)P(\eta, \xi), \]

\[ P(\xi, \eta \cdot \xi \xi) + Q(\xi, \eta \cdot \xi \xi)Q(\eta, \xi \xi)P(\xi, \xi) = \]

\[ = P(\xi \cdot \eta \xi, \xi)(P(\xi \eta, \xi) + Q(\xi \eta, \xi)Q(\eta, \xi)), \]

(J) right Bol identity:

\[ P(\xi \eta \cdot \xi, \xi) = P(\xi \eta \cdot \xi, \xi)P(\xi \eta, \xi)n(\xi, \xi), \quad Q(\xi, \xi \eta \cdot \xi)P(\xi \eta, \xi)Q(\xi, \eta) = P(\xi \cdot \eta \xi, \xi)Q(\xi \xi, \eta), \]

\[ Q(\xi \cdot \eta \xi, \xi)(P(\xi \eta, \xi)Q(\xi, \eta) + Q(\xi \eta, \xi)) = \]

\[ = P(\xi \cdot \eta \xi, \xi)P(\xi \eta, \xi)Q(\xi, \xi) + Q(\xi \xi, \eta \xi, \eta). \]

Proof. The identities characterizing the weak inverse properties follow from the expressions \(\boxed{\Theta}\) of the left and right inverses using that for all \(\xi \in L\) one has \(\xi^{-1} = \epsilon/\xi = \xi \epsilon \) (cf. \[1\]).

Assertion (D) follows from the equations

\[ (\xi, x) \cdot (\xi, x)^{2} = (\xi \cdot \xi^{2}, P(\xi, \xi^{2})x + Q(\xi, \xi^{2})(P(\xi, \xi)x + Q(\xi, \xi)x) = \]

\[ = (\xi, x)^{2} \cdot (\xi, x) = (\xi^{2}, \xi, P(\xi^{2}, \xi)(P(\xi, \xi)x + Q(\xi, \xi)x) + Q(\xi^{2}, \xi)x) \]
for all \( \xi \in L \) and \( x \in A \).

The components of the left alternative identity of \( F(P, Q) \) give the identity

\[
P(\xi, \eta) x + Q(\xi, \eta) (P(\xi, \eta) x + Q(\xi, \eta) y) = 0
\]

for all \( \xi, \eta \in L \) and \( x, y \in A \). Putting \( y = 0 \), respectively \( x = 0 \), we get the identities (E).

Considering the opposite loop of \( L \) we obtain (F).

Similar computations give the condition (G) of flexible property.

The components of the left Bol identity in \( F(P, Q) \) give the equation

\[
P(\xi, \eta \cdot \zeta ) x + Q(\xi, \eta \cdot \zeta ) [P(\eta, \xi ) y + Q(\eta, \xi ) (P(\eta, \xi ) x + Q(\xi, \zeta ) z)] =
\]

\[
P(\xi \cdot (\eta \xi), \zeta ) [P(\eta, \xi ) x + Q(\eta, \xi ) (P(\eta, \xi ) y + Q(\eta, \xi ) x)] + Q(\xi, \eta \xi, \zeta ) z
\]

for all \( \xi, \eta, \zeta \in L \) and \( x, y, z \in A \). Assertion (H) follows by the substitutions \( x = y = 0, x = z = 0, y = z = 0 \), into (5).

We obtain condition (J) from (H) using the opposite loop of \( L \). \( \Box \)

4 Tangent prolongation

We extend the construction \( \mathcal{T} \) of the tangent prolongation of Lie groups to \( C^r \)-differentiable loops \( L, r \geq 1 \). Similarly to Lie groups, the map \( \mathcal{T} : (\xi, x) \mapsto d_x \lambda_x \) for any \( (\xi, x) \in L \times T_x(L) \) gives an identification \( \mathcal{T} : L \times T_x(L) \to T(L) \) of the Cartesian product manifold \( L \times T_x(L) \) with the tangent bundle \( T(L) \).

**Definition 4.1.** Let \( \alpha(t), \beta(t) \) be differentiable curves in the \( C^r \)-differentiable loop \( L \) \( (r \geq 1) \) with initial data \( \alpha(0) = \beta(0) = \epsilon \) and \( \alpha'(0) = x, \beta'(0) = y, x, y \in T_x(L) \). We define the multiplication of the tangent prolongation \( \mathcal{T}(L \times T_x(L)) \) of \( L \) on \( L \times T_x(L) \) by

\[
(\xi, x) \cdot (\eta, y) = (\xi \eta, d_{\xi \eta} \lambda_{\xi \eta}^{-1} \frac{d}{dt} \big|_{t=0} (\xi \alpha(t) \cdot \eta \beta(t))) = (\xi \eta, d_{\xi \eta} \lambda_{\xi \eta}^{-1} \rho_{\eta} \lambda_{\eta} x + d_{\xi \eta} \lambda_{\xi \eta}^{-1} \lambda_{\xi \eta} \lambda_{\eta} y).
\]

The identity element of \( \mathcal{T}(L \times T_x(L)) \) is \( (\epsilon, 0) \).

Clearly, the maps \( \lambda_{\xi \eta}^{-1} \rho_{\eta} \lambda_{\eta} \) and \( \lambda_{\xi \eta}^{-1} \lambda_{\xi \eta} \lambda_{\eta} \) are inner mappings of the loop \( L \). They are differentiable maps and the assignment \( \phi \mapsto d_{\xi} \phi \) is a homomorphism \( d_{\xi} : \text{Inn}(L) \to \text{Aut}(T(L)) \).

Naturally, the map \( \phi \mapsto d_{\xi} \phi \) with arbitrary \( \xi \in L \) acts on all elements \( \phi \in \text{Mlt}(L) \) of the multiplication group \( \text{Mlt}(L) \) of \( L \).

**Lemma 4.1.** Let \( L \) be a \( C^r \)-differentiable loop and \( T_x(L) \) its tangent vector space. The tangent prolongation \( \mathcal{T}(L \times T_x(L)) \) of \( L \) is the linear abelian extension \( F(P, Q) \) of the abelian group \( T_x(L) \) by \( L \) determined by the loop cocycle \( (P, Q) \) of \( \mathcal{T}(L \times T_x(L)) \) given by the maps

\[
P(\xi, \eta) := d_{\xi} (\lambda_{\xi \eta}^{-1} \rho_{\eta} \lambda_{\xi}), \quad Q(\xi, \eta) := d_{\xi} (\lambda_{\xi \eta}^{-1} \lambda_{\xi} \lambda_{\eta}).
\]

**Proposition 4.2.** The tangent prolongation \( \mathcal{T}(L \times T_x(L)) \) of a \( C^r \)-differentiable loop \( L \) \( (r \geq 1) \) is a \( C^{r-1} \)-differentiable loop.
Proof. Since the multiplication, the left and right divisions of \( L \) are \( C^r \)-differentiable maps, and \( \lambda_\sigma \tau = \sigma \tau, \quad \rho_\sigma \tau = \tau \rho_\sigma, \quad \lambda_\sigma^{-1}\tau = \sigma \setminus \tau, \quad \rho_\sigma^{-1}\tau = \tau / \sigma \), the left and right translations and their inverses are \( C^r \)-differentiable maps, too. According to Lemma 4.4 the loop cocycle \((P, Q)\) of \( \mathcal{T}(L \times T_\epsilon(L)) \) is expressed by the first derivative of the products of left and right translations and of their inverses. Hence \( P(\xi, \eta), \quad Q(\xi, \eta), \) and the multiplication of \( \mathcal{T}(L \times T_\epsilon(L)) \) determined by (2) are \( C^{r-1} \)-differentiable. According to (3) the left and right divisions of the tangent prolongation \( \mathcal{T}(L \times T_\epsilon(L)) \) are \( C^{r-1} \)-differentiable, since the maps

\[
P(\xi, \eta)^{-1} := d_\epsilon(\lambda_\xi^{-1}\rho_\eta^{-1}\lambda_\xi\eta), \quad Q(\xi, \eta)^{-1} := d_\epsilon(\lambda_\eta^{-1}\lambda_\xi^{-1}\lambda_\xi\eta).
\]

are \( C^{r-1} \)-differentiable. It follows that the tangent prolongation \( \mathcal{T}(L \times T_\epsilon(L)) \) is a \( C^{r-1} \)-differentiable loop. \( \square \)

5 Tangent-like extension

Now, we treat the properties of the tangent prolongation in a more general abstract setting, considering instead of the assignment \( d_\epsilon : \text{Inn}(L) \to \text{Aut}(T_\epsilon(L)) \) arbitrary homomorphism \( \Phi : \text{Inn}(L) \to \text{Aut}(A) \) and apply the obtained conditions to the tangent prolongation of \( L \).

Definition 5.1. Let \( L \) be a loop, \( A \) an abelian group and \( \Phi : \text{Inn}(L) \to \text{Aut}(A) \) a homomorphism. The linear abelian extension \( F(P, Q) \) defined by the loop cocycle \((P, Q)\)

\[
P(\xi, \eta) := \Phi(\lambda_\xi^{-1}\rho_\eta\lambda_\xi), \quad Q(\xi, \eta) := \Phi(\lambda_\eta^{-1}\lambda_\xi\lambda_\eta).
\]

is called a tangent-like extension of the group \( A \) by the loop \( L \) and will be denoted by \( \Phi(L, A) \).

Proposition 5.1. If the loop \( L \) has some weak inverse or weak associative property then the tangent-like extension \( \Phi(L, A) \), respectively the tangent prolongation \( \mathcal{T}(L \times T_\epsilon(L)) \) has the same property if and only if the loop cocycle satisfies the identities given in the following list:

\( (A) \) two-sided inverse property:

- (i) \( \Phi(L, A) : \quad \Phi(\rho_\xi^{-1}\lambda_\xi) = \Phi(\lambda_\xi\rho_\xi^{-1}\lambda_\xi^{-1}\lambda_\xi) \),
- (ii) \( \mathcal{T}(L \times T_\epsilon(L)) : \quad d_\xi\rho_\xi^{-1} = d_\xi\lambda_\xi\rho_\xi^{-1}\lambda_\xi^{-1} \),

\( (B) \) left inverse property:

- (i) \( \Phi(L, A) : \quad \Phi(\lambda_\eta^{-1}\rho_\xi\lambda_\xi^{-1}) = \Phi(\lambda_\eta^{-1}\lambda_\xi^{-1}\rho_\eta\lambda_\xi\rho_\xi^{-1}) \),
- (ii) \( \mathcal{T}(L \times T_\epsilon(L)) : \quad d_\xi^{-1}\lambda_\xi\rho_\xi\lambda_\eta = d_\xi^{-1}\rho_\eta\lambda_\xi\rho_\xi \),

\( (C) \) right inverse property:

- (i) \( \Phi(L, A) : \quad \Phi(\lambda_\xi^{-1}\lambda_\xi\eta\lambda_\eta^{-1}) = \Phi(\lambda_\xi^{-1}\rho_\eta^{-1}\lambda_\xi\rho_\eta\lambda_\xi\lambda_\eta^{-1}) \),
- (ii) \( \mathcal{T}(L \times T_\epsilon(L)) : \quad d_\eta^{-1}\rho_\eta\lambda_\xi\eta = d_\eta^{-1}\lambda_\xi\rho_\eta\lambda_\eta \),

\( (D) \) monoassociative:
are the consequences of the previous equations (i) using that the assignment loop from the first identities of (B), (C), (E), (F), (G) in Proposition 3.1, and from the first two acts on all left and right translations of the C

The replacement (6) into the loop cocycle (H)

right alternative:

right alternative:

flexible:

left Bol identity:

right Bol identity:

Proof. The replacement (6) into the loop cocycle $(P, Q)$ of the linear abelian extension $F(P, Q)$ gives the loop cocycle of the tangent-like extension $\Phi(L, A)$. The relations obtained from the first identities of (B), (C), (E), (F), (G) in Proposition 3.1 and from the first two identities of (H), (J) in Proposition 3.1 do not give conditions since they are equivalent to identities characterizing the corresponding weak inverse or weak associative property of the loop $L$. The essential conditions with respect to the tangent-like extensions obtained from the replacement (6) are listed in the items (i) of Proposition 3.1. The conditions (ii) characterizing the weak inverse or weak associative property of the tangent prolongation $\mathcal{T}(L \times T_x(L))$ are the consequences of the previous equations (i) using that the assignment $\phi \mapsto d_\xi \phi$, $\xi \in L$, acts on all left and right translations of the $C^r$-differentiable loop $L$.  \qed
6 Properties of the tangent prolongation

Theorem 6.1. The tangent prolongation $\mathcal{T}(L \times T_e(L))$ of a $C^r$-differentiable loop $L$ $(r \geq 1)$ is a $C^{r-1}$-differentiable loop that has a weak inverse or weak associative property if and only if $L$ has the same property.

Proof. According to Proposition 4.2 the tangent prolongation $\mathcal{T}(L \times T_e(L))$ of $L$ is $C^{r-1}$-differentiable. It follows from Lemma 5.1 and Proposition 3.1, that a weak inverse or weak associative property of the tangent prolongation $\mathcal{T}(L \times T_e(L))$ implies the same property of $L$. Let us consider a differentiable curve $\xi(t)$ in $L$ with initial values $\xi(0) = \xi$ and $\frac{d\xi}{dt}(0) = \dot{\xi}$ defined on an open interval $I \subset \mathbb{R}$ containing $0 \in I$.

If all elements of $L$ have two-sided inverses then the derivation of the identity $\xi(t)^{-1} \cdot \xi(t) = \epsilon$ at $t = 0$ gives

$$d_{\xi^{-1}}\rho_{\xi}(\dot{\xi}^{-1}) + d_{\xi}\lambda_{\xi^{-1}}(\dot{\xi}) = 0,$$

where $\dot{\xi}^{-1} := \frac{d\xi^{-1}(t)}{dt}(0)$. Hence we can express

$$\dot{\xi}^{-1} = -d_{\xi}\rho_{\xi}^{-1}\lambda_{\xi^{-1}}(\dot{\xi}).$$

The derivation of the identity $\xi(t) \cdot \xi(t)^{-1} = \epsilon$ at $t = 0$ implies

$$\dot{\xi}^{-1} = -d_{\xi}\lambda_{\xi}^{-1}\rho_{\xi^{-1}}(\dot{\xi}).$$

It follows

$$d_{\xi}\rho_{\xi^{-1}} = d_{\xi}\lambda_{\xi}\rho_{\xi}^{-1}\lambda_{\xi^{-1}},$$

giving condition (Ai) in Proposition 5.1.

Assume that $L$ has the left inverse property and differentiate the identity $\xi(t)^{-1} \cdot \xi(t)\eta = \eta$. We obtain

$$d_{\xi^{-1}}\rho_{\xi}\eta(\dot{\xi}^{-1}) + d_{\xi}\lambda_{\xi^{-1}}\rho_{\eta}(\dot{\xi}) = 0.$$  

Replacing (7) into (9) gives

$$d_{\xi}\rho_{\xi}\rho_{\xi}^{-1}\lambda_{\xi}^{-1} = d_{\xi}\lambda_{\xi}^{-1}\rho_{\eta},$$

which is equivalent to condition (Bii).

Similarly, if $L$ has the right inverse property, then it follows from the identity $\eta\xi(t) \cdot \xi(t)^{-1} = \eta$ that

$$d_{\xi}\rho_{\xi}^{-1}\lambda_{\eta}(\dot{\xi}) + d_{\xi^{-1}}\lambda_{\eta}\rho_{\xi}(\dot{\xi}^{-1}) = 0.$$  

Putting (8) into (10) we get $d_{\xi}\rho_{\xi}^{-1}\lambda_{\eta} = d_{\xi}\lambda_{\eta}\rho_{\xi}^{-1}\lambda_{\xi}^{-1}$, equivalently to the condition (Cii).

For monoid associative loop $L$ we differentiate the identity $\xi(t) \cdot \xi(t)^{2} = \xi(t)^{2} \cdot \xi(t)$ at $t = 0$. We obtain

$$d_{\xi}\rho_{\xi}(\dot{\xi}) + d_{\xi}\lambda_{\xi}\rho_{\xi}(\dot{\xi}) + d_{\xi}\lambda_{\xi}^{2}(\dot{\xi}) = d_{\xi}\rho_{\xi}^{2}(\dot{\xi}) + d_{\xi}\rho_{\xi}\lambda_{\xi}(\dot{\xi}) + d_{\xi}\lambda_{\xi}^{2}(\dot{\xi}),$$

giving the condition (Dii).

If $L$ is left alternative, right alternative or flexible then for any $\eta \in L$ we differentiate the identities $\xi(t)^{2} \cdot \eta = \xi(t) \cdot \xi(t)\eta$, $\eta \cdot \xi(t)^{2} = \eta\xi(t) \cdot \xi(t)$, respectively $\xi(t) \cdot \eta\xi(t) = \xi(t)\eta \cdot \xi(t)$ at $t = 0$ and we obtain the conditions (Eii), (Fii), respectively (Gii).

If $L$ is a left Bol loop or right Bol loop then for any $\eta, \xi \in L$ we differentiate at $t = 0$ the identities $\xi(t) \cdot \eta\xi(t) = \xi(t)(\eta \cdot \xi(t))$, respectively $\xi(\eta(t) \cdot \xi(t)) = (\xi(t) \cdot \eta)\xi(t)$, and we get the conditions (Hii), respectively (Jii). Hence the assertion is true. 

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