ON SMALL-TIME LOCAL CONTROLLABILITY

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Abstract. In this paper, we study small-time local controllability of real analytic control-affine systems under small perturbations of their vector fields. Consider a real analytic control system $X$ which is small-time locally controllable and whose reachable sets shrink with the polynomial rate of order $N$ with respect to time. We will prove a general theorem which states that any real analytic control-affine system whose vector fields are perturbations of the vector fields of $X$ with polynomials of order higher than $N$ is again small-time locally controllable. In particular, we show that this result connects two long-standing open conjectures about small-time local controllability of systems.

Key words. Small-time local controllability, control variations, reachable sets, real analytic systems.

AMS subject classifications. 93B03, 93B05, 93C10

1. Introduction. Controllability is one of the central concepts in mathematical control theory. For linear control systems, the notion of controllability has been first introduced and studied by Kalman [22]. Based on the state-space approach, Kalman characterized controllability using what is now known as the Kalman rank condition [22, 23]. For nonlinear systems, various notions of controllability have been introduced and studied in the literature [49]. Among these notions, small-time local controllability is arguably the most fundamental one. A control system is small-time locally controllable from a point if a neighborhood of that point can be reached in small times (a rigorous definition will be given later in the paper). If one can compute all the trajectories of the system, then it is easy to study the small-time local controllability. However, a full analytic description of trajectories of a control system requires solving a large number of nonlinear differential equations, which is generally very difficult, if not impossible.

In past few decades, different approaches have been developed to study the fundamental properties of small-time locally controllable systems using their vector fields. The essence of most of these approaches is to provide answers to two fundamental questions: i) how much pointwise information about the vector fields of the system is needed to completely characterize small-time local controllability of the system?, and ii) how is the asymptotic behaviour of the reachable sets of the system for small times? It turns out that these two questions are closely connected and the answers to them would shed some light on other important questions in mathematical control theory [1, 7, 28]. Despite a large body of literature on this topic, for general control systems, the above questions are still unanswered.

Literature review. In control literature, various framework have been proposed for studying nonlinear control systems [6, 21, 54]. It turns out that the geometric control theory is one of the suitable settings for studying controllability of systems. In geometric control theory, a control system is defined as a parametrized family of vector fields on a manifold, where the parameters are the controls and the manifold is the state space of the system [21]. For $\nu \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega\}$, a $C^\nu$ control-affine system is defined as a pair $(\mathcal{X}, \mathcal{C})$, where $\mathcal{X} = \{X_0, X_1, \ldots, X_m\}$ is a family of $C^\nu$ vector fields on $\mathbb{R}^n$ and $\mathcal{C} \subseteq \mathbb{R}^m$ is a control set such that $0_m \in \mathcal{C}$. A trajectory for the control-affine system $(\mathcal{X}, \mathcal{C})$ is an absolutely continuous curve $x : [0, T] \to \mathbb{R}^n$ such
that
\[
\dot{x}(t) = X_0(x(t)) + \sum_{i=1}^{m} u_i(t) X_i(x(t)), \quad \text{for almost every } t \in [0, T],
\]
for some measurable controls \( u_1, u_2, \ldots, u_m : [0, T] \to \mathcal{C} \). In this paper, we study the system around an equilibrium point \( x_0 \in \mathbb{R}^n \), i.e., a point \( x_0 \) satisfying \( X_0(x_0) = 0_n \).

For a time \( t \in \mathbb{R}_{\geq 0} \), the reachable set of \((X, \mathcal{C})\) form \( x_0 \) for time less than or equal to \( t \), which is denoted by \( R_X(\leq t, x_0) \), is the set of points in state space \( \mathbb{R}^n \) which can be reached by traveling along the trajectories of the vector fields in \( X \) for positive times less than \( t \). More precisely,

\[
R_X(\leq t, x_0) = \{ x(T) \mid x : [0, T] \to \mathbb{R}^n \text{ is a trajectory of } X, \ x(0) = x_0, \ T \leq t \}.
\]

Among different notions of controllability proposed in the literature, small-time local controllability is arguably the most fundamental one. A control system \( X \) is small-time locally controllable (STLC) from a point \( x_0 \) if, for every \( t > 0 \), the reachable set \( R_X(\leq t, x_0) \) contains a neighborhood of \( x_0 \). Different approaches have been proposed in the literature for characterizing small-time local controllability using the local information of the vector fields of the system. The essence of most of these approaches can be explained using the fundamental result of Nagano [36], which connects the diffeomorphism invariant properties of a system to the Lie algebra of its the vector fields (cf. [24] for an alternative approach to study small-time local controllability). Using these approaches, small-time local controllability of systems has been studied in the literature and many sufficient conditions (cf. [17, 31, 47, 48, 50]) as well as some necessary conditions (cf. [25, 30, 43, 50]) have been developed. Despite these deep results, in general, the gap between the necessary and sufficient controllability conditions is large and complete characterization of small-time local controllability is only possible for some specific classes of systems (cf. [5, 38, 52]). In what follows we review some of these ideas and connect them with the fundamental questions about small-time locally controllable systems.

One of the important notions of controllability which has a close connection with small-time local controllability is local accessibility. A control system \( X \) is small-time locally controllable (STLC) from a point \( x_0 \) if, for every \( t > 0 \), the reachable set \( R_X(\leq t, x_0) \) contains a neighborhood of \( x_0 \). Different approaches have been proposed in the literature for characterizing small-time local controllability using the local information of the vector fields of the system. The essence of most of these approaches can be explained using the fundamental result of Nagano [36], which connects the diffeomorphism invariant properties of a system to the Lie algebra of its the vector fields (cf. [24] for an alternative approach to study small-time local controllability). Using these approaches, small-time local controllability of systems has been studied in the literature and many sufficient conditions (cf. [17, 31, 47, 48, 50]) as well as some necessary conditions (cf. [25, 30, 43, 50]) have been developed. Despite these deep results, in general, the gap between the necessary and sufficient controllability conditions is large and complete characterization of small-time local controllability is only possible for some specific classes of systems (cf. [5, 38, 52]). In what follows we review some of these ideas and connect them with the fundamental questions about small-time locally controllable systems.

One of the important notions of controllability which has a close connection with small-time local controllability is local accessibility. A control system is locally accessible from \( x_0 \) if the reachable sets of \( X \) starting from \( x_0 \) have nonempty interiors for all positive times. It is clear that if a system is small-time locally controllable from \( x_0 \), then it is locally accessible from \( x_0 \). However, the converse may not be true [9, Example 7.1]. In 1972, Sussmann and Jurdjevic characterized the local accessibility of real analytic control systems using the Lie brackets of their vector fields at the point \( x_0 \) [52, Corollary 4.7]. In 1974, Sussmann used an extension of Nagano’s Theorem [36] to show that the Lie brackets of the vector fields of the system also play a crucial role in small-time local controllability of systems [44]. This result motivated the search for sufficient controllability conditions in terms of Lie brackets of vector fields of the system [48, 50]. Later works in this direction exploit suitable filtrations of vector fields to find sharper necessary and sufficient conditions for small-time local controllability [16].

One of the nice features of Sussmann and Jurdjevic’s characterization for local accessibility is that it can be checked using only finite number of differentiations of vector fields of the system at the point \( x_0 \). This seemingly trivial observation raises the following important question about the nature of small-time local controllability: is it possible to characterize small-time local controllability of a given real analytic
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system using finite number of differentiations of its vector fields at the point $x_0$? More precisely, this question can be formulated as the following conjecture (see [1]).

**Conjecture 1.1.** Given a real analytic control-affine system $\mathcal{X} = \{X_0, X_1, \ldots, X_m\}$ which is small-time locally controllable from an equilibrium point $x_0$, there exists $N \in \mathbb{N}$ such that, every real analytic control-affine system $\mathcal{Y} = \{Y_0, Y_1, \ldots, Y_m\}$ with the property that, for every $i \in \{0, 1, 2, \ldots, m\}$, the vector fields $Y_i$ and $X_i$ has the same Taylor polynomial of order $N$ around $x_0$ is again small-time locally controllable from $x_0$.

Another useful notion for studying small-time local controllability is the control variation. A control variation can be considered as a high-order tangent to the reachable sets of the system which shows the admissible directions in the reachable sets, i.e., for small times, one will stay inside the reachable sets by traveling in these directions. By constructing a suitable family of control variations which generates all the directions in $\mathbb{R}^n$ and using a suitable generalized open mapping theorem, one can show that a control system is small-time locally controllable (see, for example [13, Theorem 2.1]). In the control literature, many different families of control variations have been introduced for studying small-time local controllability of systems (cf. [5, 8, 12, 13, 27, 32, 50]). The essence of most of these constructions is to use suitable switchings between vector fields of the system. Control variations can also be used for studying the rate of growth of the reachable sets of a system with respect to time. The order of a control variation reveals how fast one can travel in the reachable sets in that direction. More specifically, if one can get all direction in $\mathbb{R}^n$ using families of control variations of order less than equal to $N$, then there exists a positive constant $C > 0$ such that the closed ball centered at $x_0$ with radius $Ct^N$ (which we denote by $B(x_0, Ct^N)$) is contained in the reachable set $R_{\mathcal{X}}(\leq t, x_0)$, for small positive times $t$ [13, Theorem 2.1]. This raises the following question: Given a small-time locally controllable system, does there exist a family of control variations of order $N$ which can be used to prove small-time local controllability of the system. Motivated by the above question, one can propose the following conjecture (see [1]).

**Conjecture 1.2.** Let $\mathcal{X}$ be a real analytic control-affine system which is small-time locally controllable from $x_0$. Then there exist $N \in \mathbb{N}$ and $T, C > 0$ such that

$$B(x_0, Ct^N) \subseteq R_{\mathcal{X}}(\leq t, x_0), \quad \forall t \leq T.$$  

It turns out that this polynomial growth condition for reachable sets of a system has a close connection with the regularity of the time-optimal map of the system [7]. One can show that, if the control system $\mathcal{X}$ is small-time locally controllable, then the time-optimal map of $\mathcal{X}$ is locally continuous [39] (cf. [7, Theorem 2.2], where this local result has been extended to a larger domain called escape domain). Similarly, one can show that the polynomial growth condition for a control system $\mathcal{X}$ is equivalent to local H"older continuity of the time-optimal map of $\mathcal{X}$ [37], [7, Theorem 2.5].

One can easily check that, if the Conjecture 1.2 is true then, for every small-time locally controllable system, there exists a family of control variations of order $N$ for the system which generates all the directions in $\mathbb{R}^n$. As mentioned in [1], the results in [38] show that both Conjectures 1.1 and 1.2 hold on $\mathbb{R}^2$. However, to the best of our knowledge, these two conjectures are still open for Euclidean spaces $\mathbb{R}^n$ with $n \geq 3$.

One of the challenges for studying Conjectures 1.1 and 1.2 stems from the switchings in control variations. Most of the sufficient conditions for small-time local con-
trollability use control variations with a finite number of switchings (e.g. the sufficient conditions in [50]). It is well-known that if the family of the control variations used for proving small-time local controllability of the control system have finite number of switchings, then Conjecture 1.1 holds for the system. In 1988, Kawski found an elegant example of a polynomial control system which is small-time locally controllable from \( x_0 \), but it is impossible to check small-time local controllability using control variations with a finite number of switchings [26, 27]. Kawski used a specific class of control variations with an increasing number of switching, which he called fast-switching variations, to prove small-time local controllability of this system. This example shows that more complicated family of variations might be needed to characterize small-time local controllability. In [3], Agrachev and Gamkrelidze used a detailed analysis of the semigroup of diffeomorphisms and built a framework for studying small-time local controllability using fast-switching variations. Here, we revisit the famous example of Kawski [26], to illustrate the complications that might arise in studying Conjectures 1.1 and 1.2 using fast-switching variations.

**Example 1.3.** Consider the control system \( \mathcal{X} \) on \( \mathbb{R}^4 \), defined by

\[
\begin{align*}
\dot{x}_1 &= u(t), \\
\dot{x}_2 &= x_1, \\
\dot{x}_3 &= x_3^3, \\
\dot{x}_4 &= x_3^2 - x_2^7,
\end{align*}
\]

where \( u : \mathbb{R} \to [-1, 1] \) is measurable. We want to study small-time local controllability of \( \mathcal{X} \) from \( 0_4 \in \mathbb{R}^4 \). Using suitable families of control variations with finite numbers of switchings, one can show that \( \{ \pm \frac{\partial}{\partial x_1}, \pm \frac{\partial}{\partial x_2}, \pm \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4} \} \) are the admissible directions in the reachable set of the systems \( \mathcal{X} \) [26, 50]. In order to prove small-time local controllability of \( \mathcal{X} \), one needs to find a control variation which generates the direction \( -\frac{\partial}{\partial x_4} \). It can be shown that there is no family of control variations with finite number of switching which generates the direction \( -\frac{\partial}{\partial x_4} \) [26, Claim 2]. However, by using fast-switching variations, one can show that \( \mathcal{X} \) is small-time locally controllable from \( 0_4 \in \mathbb{R}^4 \) [26, Claim 1] (see [26] and [27] for the elaborate construction of these control variations). Now consider the control system \( \mathcal{Y} \) on \( \mathbb{R}^4 \) defined by

\[
\begin{align*}
\dot{y}_1 &= u(t), \\
\dot{y}_2 &= y_1, \\
\dot{y}_3 &= y_1^3, \\
\dot{y}_4 &= y_3^2 - y_2^7 + y_1^5,
\end{align*}
\]

where \( u : \mathbb{R} \to [-1, 1] \) is measurable. Note that \( \mathcal{X} \) and \( \mathcal{Y} \) have the same Taylor polynomial of order 57 at \( 0_4 \in \mathbb{R}^4 \). Using the classical finite switching control variations, it is easy to show that \( \{ \pm \frac{\partial}{\partial y_1}, \pm \frac{\partial}{\partial y_2}, \pm \frac{\partial}{\partial y_3}, \frac{\partial}{\partial y_4} \} \) are the admissible directions in the reachable set \( \mathcal{Y} \). However, using the same fast-switching variations as for the system \( \mathcal{X} \), it is very complicated to study whether \( -\frac{\partial}{\partial y_4} \) is an admissible direction for the reachable sets of the control system \( \mathcal{Y} \) at the point \( 0_4 \in \mathbb{R}^4 \).

In general, the families of control variations that are used to prove small-time local controllability of a system might be even more complicated than the fast-switching control variations in Example 1.3. Therefore, for small-time locally controllable sys-
tems, studying Conjecture 1.1 using the form of families of control variations does not seem to be conclusive.

Contributions. The contributions of this paper are manifold. First, we review an operator approach for studying piecewise constant vector fields and their flows called chronological calculus. This operator approach, which is originally introduced in [2], uses linear operators on space of smooth functions to estimate the flows of piecewise constant vector fields (see [2, Proposition 2.1] and [4, §2.4.4]). Using a suitable topology on the space of real analytic functions [35] [11, §1.6 Theorem 27], a slightly different version of this approach has been introduced in [18] to estimate the flows of real analytic piecewise constant vector fields (see [18, Theorem 3.8.1]). As the first minor contribution of this paper, we use these frameworks to prove a uniform bound on these estimates of the flows of piecewise constant vector fields.

We study the existing literature on small-time local controllability of systems and review a general class of control variations defined in [13, Definition 2.1]. As the second minor contribution of this paper, we use this class of variations to prove the following useful theorem: for a control system, the cone generated by the control variations of order $N$ is the space $\mathbb{R}^n$ if and only if the reachable sets of the control system shrink with polynomial rate of order $N$ or higher with respect to time (see [12, Theorem 1.3] and [13, Theorem 2.1]).

Next, we introduce a suitable mapping for studying perturbations of reachable sets of real analytic control systems. We focus on real analytic control systems whose reachable sets shrink with polynomial rate of order $N$ or higher. Using the notion of normal reachability introduced in [45] and [14, Definition 3.4], we construct a multi-valued mapping (called perturbation mapping) by composing two different maps: i) a map from the points in reachable sets of the original control system to the switching times associated to the control variations, and ii) a map from the switching times of the control variations to the reachable sets of the perturbed system. We show that this multi-valued mapping can capture the effect of perturbations of the vector fields of the system on its reachable sets. Moreover, we prove the regularity of this perturbation mapping with respect to time and states of the system.

Finally, we prove the main result of the paper: for a real analytic control-affine system whose reachable sets shrink with polynomial rate of order $N$ or higher with respect to time, small-time local controllability is preserved under polynomial vector field perturbations of order higher than $N$. The key idea for the proof is to use a suitable family of control variations for the original system which generates the space $\mathbb{R}^n$ and employ the real analytic version of chronological calculus to show that the perturbed family of control variations also generates the space $\mathbb{R}^n$. In particular, we show that our main result in this paper implies that if Conjecture 1.2 holds then Conjecture 1.1 also holds.

Paper Organization. In Sections 1.1 and 2, we introduce the essential notation for stating the main results of the paper. In Section 3, we introduce an operator approach for studying piecewise constant vector fields and their flows. In Section 4, we introduce and study different notions of controllability for $C^\infty$ control-affine systems. In Section 5 the notion of control variations is defined and a characterization of the growth rate condition is presented based on the control variations of the system. In Section 6, for every two real analytic control systems $\mathcal{X}$ and $\mathcal{Y}$ and every time $t$, we construct a multi-valued mapping between the reachable sets of the control system $\mathcal{X}$ and $\mathcal{Y}$. Finally, in Section 7, the main result of this paper is stated and proved.
1.1. Notations and conventions. In this paper, the set of integers, non-negative integers, and natural numbers are denoted by \( \mathbb{Z}, \mathbb{Z}_{\geq 0}, \) and \( \mathbb{N} \), respectively. We denote the \( n \)-dimensional Euclidean space by \( \mathbb{R}^n \) and the zero vector in \( \mathbb{R}^n \) by \( 0_n \).

The Euclidean norm of a vector \( \mathbf{v} \) in \( \mathbb{R}^n \) is denoted by \( \| \mathbf{v} \| \). The \( n \)-sphere is denoted by \( S^n \) and the non-negative orthant in \( \mathbb{R}^n \) is denoted by \( \mathbb{R}^n_+ \).

We denote
\[
[-1,1]^n = [-1,1] \times [-1,1] \times \ldots \times [-1,1].
\]
For a nonempty subset \( S \subseteq \mathbb{R}^n \), the interior of \( S \) in \( \mathbb{R}^n \) is denoted by \( \text{int}(S) \) and the closure of \( S \) in \( \mathbb{R}^n \) is denoted by \( \overline{S} \). A multi-index of order \( m \) is an element \( \mathbf{r} = (r_1,r_2,\ldots,r_m) \in \mathbb{Z}^m_{\geq 0} \). For all multi-indices \( \mathbf{r} \) and \( \mathbf{s} \) of order \( m \), every \( \mathbf{x} = (x_1,x_2,\ldots,x_m) \in \mathbb{R}^m \), and \( f : \mathbb{R}^m \to \mathbb{R}^n \), we define
\[
|\mathbf{r}| = r_1 + r_2 + \ldots + r_m, \quad \mathbf{r}! = (r_1!)(r_2!)(r_m!), \quad \mathbf{x}^\mathbf{r} = x_1^{r_1}x_2^{r_2}\ldots x_m^{r_m}, \quad D^\mathbf{r}f(x) = \frac{\partial^{|\mathbf{r}|}f}{\partial x_1^{r_1}\partial x_2^{r_2}\ldots\partial x_m^{r_m}}.
\]
The space of all decreasing sequences \( \{a_i\}_{i \in N} \) such that \( a_i \in \mathbb{R}_{>0} \) and \( \lim_{n \to \infty} a_n = 0 \) is denoted by \( c_0^\nu \). Let \( x \in \mathbb{R}^n \) and \( r \in \mathbb{R}_{>0} \). Then the Euclidean open ball centered at \( x \) with radius \( r \) is denoted by \( B(x,r) \) and its closure is denoted by \( \overline{B}(x,r) \). In this paper, whenever we use the letter \( \nu \), we mean that \( \nu \in \mathbb{N} \cup \{\infty,\omega\} \).

Let \( U \subseteq \mathbb{R}^m \) be an open set and \( f : U \to \mathbb{R}^n \). For \( \nu \in \mathbb{N} \cup \{\infty,\omega\} \), the mapping \( f \) is a \( C^\nu \)-mapping if, for every multi-index \( \mathbf{r} \in \mathbb{Z}^m_{\geq 0} \) with property that \( |\mathbf{r}| \leq \nu \), the mapping \( D^\mathbf{r}f \) is continuous. The mapping \( f \) is a \( C^\omega \)-mapping if it is a \( C^\infty \)-mapping and, for every \( x_0 \in U \), the Taylor series of \( f \) around \( x_0 \) converges locally. Let \( k \in \mathbb{N} \), \( (V,\|\cdot\|_V) \) be a normed vector space, and \( f : \mathbb{R} \to V \) and \( g : \mathbb{R} \to V \) be two curves on \( V \). Then we write
\[
f(x) = g(x) + \mathcal{O}(x^k)
\]
if there exists \( \alpha \in \mathbb{R} \) such that we have \( \lim_{x \to 0} \frac{\|f(x) - g(x)\|_V}{|x|^k} = \alpha \). Let \( U \) and \( V \) be two sets and \( F : U \rightrightarrows V \) be a multi-valued map. Then a selection of \( F \) is a single-valued mapping \( f : U \to V \) with the property that \( f(x) \in F(x) \), for every \( x \in U \).

2. Functions and vector fields. In this section, we study functions and vector fields on the Euclidean space \( \mathbb{R}^n \). The space of all \( C^\nu \)-functions on \( \mathbb{R}^n \) is denoted by \( C^\nu(\mathbb{R}^n) \) and the space of all \( C^\nu \)-vector fields on \( \mathbb{R}^n \) is denoted by \( \Gamma^\nu(\mathbb{R}^n) \). It is easy to see that both \( C^\nu(\mathbb{R}^n) \) and \( \Gamma^\nu(\mathbb{R}^n) \) are vector spaces over \( \mathbb{R} \). Given \( x_0 \in \mathbb{R}^n \), we define the functional \( \text{ev}_{x_0} : C^\nu(\mathbb{R}^n) \to \mathbb{R} \) by \( \text{ev}_{x_0}(f) = f(x_0) \), for every \( f \in C^\nu(\mathbb{R}^n) \).

In control theory, it is common to work with time-varying vector fields. In this paper, without loss of generality, we restrict our attention to piecewise constant vector fields.

Definition 2.1 (Piecewise constant vector fields). Let \( T \subseteq \mathbb{R} \) be an interval. The map \( X : T \times \mathbb{R}^n \to \mathbb{R}^n \) is a piecewise constant vector field of class \( C^\nu \) if the following hold:

(i) For every \( t \in T \), the map \( X_t: \mathbb{R}^n \to \mathbb{R}^n \) defined by
\[
X_t(x) = X(t,x), \quad \forall x \in \mathbb{R}^n,
\]
is a vector field of class \( C^\nu \).
(ii) For every $x \in \mathbb{R}^n$, the map $X^x : \mathbb{T} \to \mathbb{R}^n$ defined by

$$X^x(t) = X(t, x), \quad \forall t \in \mathbb{T},$$

is piecewise constant.

Let $X : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a piecewise constant vector field. Then, by the fundamental theorem of differential equations [10, Theorem 2.3], for every $x_0 \in \mathbb{R}^n$, there exist a maximal interval $\mathbb{T}_{x_0}$ and an absolutely continuous curve $t \mapsto \exp(tX)(x_0)$ which satisfies the following initial value problem:

$$\frac{d}{dt}(\exp(tX)(x_0)) = X(t, \exp(tX)(x_0)), \quad \text{for almost every } t \in \mathbb{T}_{x_0}$$

$$\exp(0X)(x_0) = x_0.$$ 

The map $t \mapsto \exp(tX)(x_0)$ is called the integral curve of the piecewise constant vector field $X$ passing through $x_0$.

**Definition 2.2 (Complete vector fields).** A piecewise constant vector field $X : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is complete if, for every $x_0 \in \mathbb{R}^n$, the integral curve of $X$ passing through $x_0$ exists for all $t \in \mathbb{R}$.

**Definition 2.3 (Flows of piecewise constant vector fields).** Let $X : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a complete piecewise constant $C^\nu$-vector field. Then the flow of $X$ is the map $\exp(X) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$\exp(X)(t, x) = \exp(tX)(x), \quad \forall (t, x) \in \mathbb{R} \times \mathbb{R}^n.$$ 

**3. Operator approach for time-varying vector fields.** In this section, we review a well-known operator approach which allows us to translate the nonlinear finite-dimensional systems into linear infinite-dimensional systems. This operator approach, which is known as chronological calculus, was first proposed by Agrachev and Gamkrelidze in [2]. While the chronological calculus is originally developed to study time-varying vector fields, in this paper we focus on a simpler version of it which studies piecewise constant vector fields. In this framework, $C^\infty$-vector fields and $C^\infty$-diffeomorphisms are identified with derivations and unital algebra isomorphisms on $C^\infty(\mathbb{R}^n)$, respectively. These identifications are then used to recast the nonlinear dynamical system governing the flow of a piecewise constant vector field to a linear differential equation an infinite-dimensional space. Using this recasting and the Whitney compact-open topology on the space of $C^\infty(\mathbb{R}^n)$, an asymptotic expansion for the flow of a piecewise constant real analytic vector field is developed and its convergence has been studied in [2]. In [19] and [18], this framework has been extended in two directions. First, the real analytic vector fields are considered as derivations on $C^\infty(\mathbb{R}^n)$ and real analytic diffeomorphisms are considered as unital algebra homomorphism on $C^\infty(\mathbb{R}^n)$. Moreover, the space $C^\infty(\mathbb{R}^n)$ is endowed with the $C^\infty$-topology and the convergence of the asymptotic expansion for the flow of a piecewise constant real analytic vector field is studied in this new topology [18]. In the sequel, we adopt the approach of [19] for studying piecewise constant real analytic vector fields and their flows.

**Definition 3.1 (Vector fields and diffeomorphisms).** Let $V : \mathbb{R}^n \to \mathbb{R}^n$ be a real analytic vector field and $\phi : \mathbb{R}^n \to \mathbb{R}^n$ be a real analytic diffeomorphism. Then (i) we define the derivation $\hat{V} : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ by

$$\hat{V}(f) = \mathcal{L}_V f,$$
where $\mathcal{L}_V f$ is the Lie derivative of $f$ in the direction of the vector field $V$.

(ii) we define the unital algebra homomorphism $\hat{\phi} : C^\omega(\mathbb{R}^n) \to C^\omega(\mathbb{R}^n)$ by

$$\hat{\phi}(f) = f \circ \phi.$$  

The space of linear mappings from $C^\omega(\mathbb{R}^n)$ to $C^\omega(\mathbb{R}^n)$ is denoted by $LC^\omega(\mathbb{R}^n)$ and it is clear that we have $\Gamma^\omega(\mathbb{R}^n) \subset LC^\omega(\mathbb{R}^n)$. Thus, for every real analytic vector field $X : \mathbb{R}^n \to \mathbb{R}^n$ and every real analytic mapping $\phi : \mathbb{R}^n \to \mathbb{R}^n$, we have $\hat{\phi}_* \hat{V} \in LC^\omega(\mathbb{R}^n)$.

Using the operator characterization of vector fields, a piecewise constant $C^\omega$-vector field $X : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}^n$ can be considered as a piecewise constant curve $t \mapsto \hat{X}_t$ on the space $LC^\omega(\mathbb{R}^n)$. Therefore, for studying properties of piecewise constant vector fields in this framework, we need to define a suitable topology on the vector space $LC^\omega(\mathbb{R}^n)$.

**Definition 3.2 (Topological vector space).** Let $E$ be a vector space over $\mathbb{R}$ and $\tau$ be a topology on $E$. Then

(i) the pair $(E, \tau)$ is a topological vector space if both addition and scalar multiplication in $E$ are continuous with respect to $\tau$.

(ii) the topological vector space $(E, \tau)$ is a locally convex space if the topology $\tau$ is generated by a family of seminorms $\{p_i\}_{i \in \Lambda}$ on $E$.

(iii) a subset $B \subseteq E$ is bounded if and only if, for every neighborhood $U$ of $0$ in $E$, there exists $\alpha \in \mathbb{R}$ such that $B \subseteq \alpha U$.

For locally convex spaces, bounded sets can be equivalently characterized using the seminorms [41, Theorem 1.37].

**Theorem 3.3 (Seminorm characterization of bounded sets).** Let $E$ be a locally convex space which is generated by the family of seminorms $\{p_i\}_{i \in \Lambda}$. A set $B \subseteq E$ is bounded if and only if, for every $i \in \Lambda$, there exists $N_i \in \mathbb{R}_{>0}$ such that

$$p_i(v) \leq N_i, \quad \forall v \in B.$$  

We are now ready to define a locally convex topology on the vector spaces $C^\omega(\mathbb{R}^n)$ and $LC^\omega(\mathbb{R}^n)$ using families of seminorms.

**Definition 3.4 (Real analytic seminorms).** Let $K \subset \mathbb{R}^n$ be a compact set and $\mathbf{a} \in \mathbb{C}^n_0$.

(i) We define the seminorm $\rho^\omega_{K, \mathbf{a}} : C^\omega(\mathbb{R}^n) \to \mathbb{R}_{\geq 0}$ as

$$\rho^\omega_{K, \mathbf{a}}(f) = \sup \left\{ \frac{a_0 a_1 \cdots a_{|\mathbf{a}|}}{|\mathbf{a}|!} \left\| D^{(|\mathbf{a}|)} f(x) \right\| : x \in K, |\mathbf{a}| \in \mathbb{Z}_{\geq 0} \right\}.$$  

The topology on $C^\omega(\mathbb{R}^n)$ generated by the family of seminorms $\rho^\omega_{K, \mathbf{a}}$ is called the $C^\omega$-topology.

(ii) Let $f \in C^\omega(\mathbb{R}^n)$. We define the seminorm $\rho^\omega_{K, \mathbf{a}, f} : LC^\omega(\mathbb{R}^n) \to \mathbb{R}_{\geq 0}$ as

$$\rho^\omega_{K, \mathbf{a}, f}(X) = \sup \left\{ \frac{a_0 a_1 \cdots a_{|\mathbf{a}|}}{|\mathbf{a}|!} \left\| D^{(|\mathbf{a}|)} (X f)(x) \right\| : x \in K, |\mathbf{a}| \in \mathbb{Z}_{\geq 0} \right\}.$$  

The topology on $LC^\omega(\mathbb{R}^n)$ generated by the family of seminorms $\left\{ \rho^\omega_{K, \mathbf{a}, f} \right\}$ is called the $C^\omega$-topology.

Note that one can define another locally convex topology on $C^\omega(\mathbb{R}^n)$ by inducing the Whitney compact-open topology and using the subspace relation [4, §2.2], [33,
It turns out that the $C^\omega$-topology on the space $C^\omega(\mathbb{R}^n)$ is finer than the subspace topology induced from the Whitney topology on $C^\infty(\mathbb{R}^n)$ [19, Chapter 5]. The $C^\omega$-topology and its properties has been studied throughly in [35], [11], and [33]. The $C^\omega$-topology on the space of real analytic functions has been first defined and studied using advanced tools in analysis in [35]. The above seminorm characterization of the $C^\omega$-topology has been introduced and proved in [53] (see [11] for a detailed study of the $C^\omega$-topology on the space $C^\omega(\mathbb{R}^n)$). Using the $C^\omega$-topology on the vector space $LC^\omega(\mathbb{R}^n)$, we can study properties of piecewise constant vector fields. The set of piecewise constant curves with domain $T$ on $LC^\omega(\mathbb{R}^n)$ is denoted by $PC(T; LC^\omega(\mathbb{R}^n))$. Let $S \subseteq LC^\omega(\mathbb{R}^n)$. We define the subset $PC(T; S) \subseteq PC(T; LC^\omega(\mathbb{R}^n))$ as

$$PC(T; S) = \{ \lambda \in PC(T; LC^\omega(\mathbb{R}^n)) \mid \lambda(t) \in S, \text{ for almost every } t \in T \}.$$ 

Let $X : T \times \mathbb{R}^n \to \mathbb{R}^n$ be a piecewise constant real analytic vector field. Then it is easy to see that $X$ is piecewise constant if and only if $t \mapsto \hat{X}_t$ is a piecewise constant curve on $LC^\omega(\mathbb{R}^n)$. By considering $X$ as a curve $t \mapsto \hat{X}_t$ on the space $LC^\omega(\mathbb{R}^n)$, one can also translate the nonlinear differential equations governing the flow of $X$:

$$\frac{d}{dt} \exp(tX)(x_0) = X(t, \exp(tX)(x_0)), \quad \text{for almost every } t \in \mathbb{R}$$

$$\exp(0X)(x_0) = x_0,$$

into the following linear differential equations:

$$\frac{d}{dt} \exp(tX) = \exp(tX) \circ \hat{X}_t, \quad \text{for almost every } t \in \mathbb{R}$$

$$\exp(0X) = \text{id},$$

where $\exp(tX)$ is the unital algebra homomorphism associated to $\exp(tX)$ (see Definition 3.1). Note that equation (3.1) is a family of linear differential equations on the infinite dimensional locally convex space $LC^\omega(\mathbb{R}^n)$. One can study the sequence of Picard iterations for this infinite dimensional linear differential equations (3.1) [10, Chapter 1, §3], [34].

**Definition 3.5 (Sequence of flow iterations).** Let $X$ be a piecewise constant vector field of class $C^\omega$. We define the curve $t \mapsto \hat{\exp}_0(tX)$ on $LC^\omega(\mathbb{R}^n)$ as:

$$\hat{\exp}_0(tX) = \text{id}, \quad \forall t \in [0, T').$$

Then, for every $k \in \mathbb{N}$, we define the curve $t \mapsto \hat{\exp}_k(tX)$ on $LC^\omega(\mathbb{R}^n)$ inductively as

$$\hat{\exp}_k(tX) = \text{id} + \int_0^t \hat{\exp}_{k-1}(\tau X) \circ \hat{X}(\tau)d\tau, \quad \forall t \in [0, T').$$

For linear differential equations on infinite dimensional locally convex spaces, there does not exist a general result for convergence of the sequence of Picard iterations [34]. However, for the differential equations (3.1), one can prove the following estimates for the seminorms of the sequence of iterations of the flows in Definition 3.5 [18, Theorem 3.8.1]. The following theorem can be considered as an extension of the estimates in [4, §2.4.4].

**Theorem 3.6 (Estimates for flow iterations).** Let $B$ be a bounded set in $\Gamma^\omega(\mathbb{R}^n)$. Then the following statements hold:
(i) there exists $T_B$ such that, for every $X \in \text{PC}([0, T_B]; B)$ and every $k \in \mathbb{N}$, the map $t \mapsto \exp_k(tX)$ is defined on $[0, T_B]$,

(ii) for every compact set $K \subseteq \mathbb{R}^n$, every $f \in C^\omega(\mathbb{R}^n)$, and every $a \in C^1$, there exist positive constants $M, M_f > 0$ such that, for every $X \in \text{PC}([0, T_B]; B)$, we have

$$\rho_{K,a,f}(\exp_k(tX) - \exp_{k-1}(tX)) \leq (Mt)^k M_f, \quad \forall t \in [0, T_B], \forall k \in \mathbb{N}.$$  

Using the estimate in Theorem 3.6, one can get an estimate for the flow of a vector field $X$ using the sequence of iterations in Definition 3.5.

**Theorem 3.7.** Let $B$ be a bounded set in $LC^\omega(\mathbb{R}^n)$. Then there exist $M, L > 0$ and $\overline{T} \leq T_B$ such that, for every $X \in \text{PC}([0, \overline{T}]; B)$ and every $i \in \{1, 2, \ldots, n\}$, we have

$$||\nu_{x_0} \circ \exp(tX)(x^i) - \nu_{x_0} \circ \exp(tX)(x^i)|| \leq \frac{Mt^{k+1}}{1-Mt} L, \quad \forall t \in [0, \overline{T}],$$

where $x^i$ is the $i$th coordinate function on $\mathbb{R}^n$.

It is worth mentioning that, Theorem 3.6(ii) and Theorem 3.7 can alternatively be proved using the estimates in [4, §2.4.4]. However, the real analyticity of the time varying vector fields is essential for these results to hold. Since Theorem 3.7 is crucial for the proof of the main result of this paper, we provide a proof for it using Theorem 3.6 in Appendix A.

4. Control-affine systems. In this section, we introduce several controllability notions associated with a $C^\omega$ control-affine system $(\mathcal{X}, \mathcal{C})$. For the rigorous definition of the $C^\omega$ control-affine systems, their trajectories, and their equilibrium points, we refer the readers to the introduction of the paper.

**Definition 4.1 (Controllability of $C^\omega$ control-affine system).** Suppose that $(\mathcal{X}, \mathcal{C})$ is a $C^\omega$ control-affine system on $\mathbb{R}^n$, with $\mathcal{X} = \{X_0, X_1, \ldots, X_m\}$, $t \in \mathbb{R}_{>0}$, and $x_0 \in \mathbb{R}^n$ is an equilibrium point of $(\mathcal{X}, \mathcal{C})$. Then

(i) The $C^\omega$ control-affine system $\mathcal{X}$ is small-time locally controllable from $x_0$ if, for every $t \in \mathbb{R}_{>0}$, we have

$$x_0 \in \text{int} (R_{\mathcal{X}}(\leq t, x_0));$$

(ii) Let $N \in \mathbb{Z}_{>0}$ be a positive integer. Then the $C^\omega$ control-affine system $\mathcal{X}$ satisfies growth rate condition of order $N$ at the point $x_0$ if there exist $C, T > 0$ such that, for every $t \in (0, T]$, we have

$$\overline{B}(x_0, Ct^N) \subset R_{\mathcal{X}}(\leq t, x_0).$$

**Remark 4.2.** The following remarks are in order.

(i) Note that, in the definition of the above controllability notions, we do not impose any restriction on the structure of the control set $\mathcal{C}$. For control-affine systems, this choice of control set seems natural. However, more general structures for the control set, such as separable topological space [42] and Frechet spaces [51], have been proposed in the literature. In the bundle view of the control systems, the control set is usually banished and the system is considered as a family of vector fields [54]. In the chronological calculus, a
control system is usually defined as a parametrized family of vector fields on \( \mathbb{R}^n \) [4]. We refer the interested readers to [20] for a review of the literature on various definitions of control systems and, in particular, the properties of the real analytic control-affine systems.

(ii) If \( x_0 \) is an equilibrium point for \((X, \mathcal{C})\), then in the absence of control inputs, i.e., \( u = 0_m \), the states of the system stay at the point \( x_0 \);

(iii) while the definition of reachable sets with measurable controls is widely used and studied in the literature (cf. [27, 48, 50]), some authors consider other classes of controls such as piecewise constant controls [14] and bang-bang controls [31] for studying control systems and their reachable sets. In general, the reachable sets defined using these different classes of controls are not the same [50]. We refer the interested reader to [14, 31, 50] for a through study of the connections between these reachable sets;

(iv) it is clear from Definition (4.1) that, if a control-affine system \((X, \mathcal{C})\) satisfies the growth rate condition of order \( N \) at the point \( x_0 \), then it is small-time locally controllable from \( x_0 \);

(v) for a general \( C^\nu \) control-affine system \((X, \mathcal{C})\), small-time local controllability at the point \( x_0 \) does not necessarily imply that \( x_0 \) is an equilibrium point of the system. However, for a real analytic control-affine system \((X, \mathcal{C})\) with compact control set \( \mathcal{C} \), if the system is small-time locally controllable from \( x_0 \), then \( x_0 \) is in the convex hull of the set \( \{ X_0(x_0) + \sum_{i=1}^m u_iX_i(x_0) \mid (u_1, \ldots, u_m) \in \mathcal{C} \} \) [48, Proposition 6.1]. It is worth mentioning that while a large body of the research on controllability is devoted to studying small-time local controllability of control systems at equilibrium points [27, 47, 48, 50], there are some deep and interesting results for small-time local controllability along non-stationary reference trajectories [8, 15].

While the trajectories of a control-affine systems are constructed using measurable controls \( u : [0, t] \to \mathcal{C} \), it is sometimes useful to work with piecewise constant controls and their associated vector fields.

**Definition 4.3 (Piecewise constant control vector fields).** Let \((X, \mathcal{C})\) be a \( C^\nu \) control-affine system with \( X = \{ X_0, X_1, \ldots, X_m \} \) and let \( u = (u_1, u_2, \ldots, u_m) \in \mathcal{C} \). Then the vector field of the control-affine system \( X \) associated to \( u \) is \( X_u \in \Gamma^\nu(\mathbb{R}^n) \) defined by

\[
X_u = X_0 + \sum_{i=1}^m u_iX_i.
\]

Let \( p \in \mathbb{N} \), \( \mathbf{I} = (u^1, u^2, \ldots, u^p) \in \mathcal{C}^p \) be a \( p \)-tuple of constant controls, \( \mathbf{t} = (t_1, \ldots, t_p) \in \mathbb{R}_{>0}^p \) be a \( p \)-tuple of switching times. We define the piecewise constant control vector field \( X^{\mathbf{I}, \mathbf{t}} \) by

\[
X^{\mathbf{I}, \mathbf{t}} = \begin{cases} 
X_{u^t}, & t \in [0, t_1], \\
X_{u^{t_1}}, & t \in (t_1, t_1+t_2], \\
\vdots & \vdots \\
X_{u^p}, & t \in (t_2 + \ldots, t_p+t_1 + \ldots + t_p].
\end{cases}
\]

It clear that, for every \( p \)-tuple \( \mathbf{I} \in \mathcal{C}^p \) and every \( p \)-tuple \( \mathbf{t} \in \mathbb{R}_{>0}^p \) the vector field \( X^{\mathbf{I}, \mathbf{t}} \)
is piecewise constant and the following property holds:

\[ \exp(|t|X_t^1)(x) = \exp(t_1X_{u^1}) \circ \exp(t_2X_{u^2}) \circ \ldots \circ \exp(t_pX_{u^p})(x), \quad \forall x \in \mathbb{R}^n. \]

Another notion relevant to small-time local controllability of systems is normal reachability [14]. Normal reachability has been first introduced and studied by Sussmann in [46].

**Definition 4.4 (Normal reachability).** Let \((\mathcal{X}, \mathcal{C})\) be a \(C^\nu\) control-affine system on \(\mathbb{R}^n\) with \(\mathcal{X} = \{X_0, \ldots, X_m\}\) and let \(x_1, x_0 \in \mathbb{R}^n\). Then the point \(x_1\) is normally reachable in time less than \(t\) from \(x_0\), if the following conditions hold:

1. there exist \(p \in \mathbb{N}\), \(u^1, u^2, \ldots, u^p \in \mathcal{C}\), and \((s_1, s_2, \ldots, s_p) \in \mathbb{R}_{>0}^p\) such that \(s_1 + s_2 + \ldots + s_p < t\) and

\[
\exp(s_1X_{u^1}) \circ \exp(s_2X_{u^2}) \circ \ldots \circ \exp(s_pX_{u^p})(x_0) = x_1,
\]

and

2. there exists an open neighborhood of \((s_1, s_2, \ldots, s_p)\) in \(\mathbb{R}_{>0}^p\) such that the map

\[
(t_1, t_2, \ldots, t_p) \mapsto \exp(t_1X_{u^1}) \circ \exp(t_2X_{u^2}) \circ \ldots \circ \exp(t_pX_{u^p})(x_0)
\]

is \(C^1\) and of rank \(n\) on this neighborhood.

It is clear that, if the point \(x_0\) is normally reachable from itself, the system is small-time locally controllable from \(x_0\). However, the converse is not true for general control-affine systems [14, Example 3.9]. For real analytic systems, the connection between small-time local controllability and normal reachability has been studied in [14]. In fact, in [14], it has been shown that for real analytic control systems, small-time local controllability from \(x_0\) implies that, for every time \(t\), every point in the interior of the reachable set from \(x_0\) in times less than \(t\) is normally reachable from \(x_0\) [14, Theorem 5.5 and Corollary 4.15].

**Theorem 4.5.** Let \((\mathcal{X}, \mathcal{C})\) be a real analytic control-affine system on \(\mathbb{R}^n\) with \(\mathcal{X} = \{X_0, X_1, \ldots, X_m\}\). If \(\mathcal{X}\) is small-time locally controllable from \(x_0\) then, for every \(t > 0\), every point in the set \(\text{int}(\mathbb{R}_\mathcal{X}(\leq t, x_0))\) is normally reachable in time less than \(t\) from \(x_0\).

Finally, we introduce the following assumption on the class of \(C^\nu\) control-affine systems. This assumption consists of two parts: i) an assumption on the vector fields of the system, and ii) an assumption on the control set of the system.

**Assumption 1.** We assume that the \(C^\nu\) control-affine system \((\mathcal{X}, \mathcal{C})\) satisfies the following condition:

(i) for every \(p \in \mathbb{Z}_{>0}\) and every \(p\)-tuples \(I = (u^1, \ldots, u^p) \in \mathcal{C}^p\) and \(t = (t^1, \ldots, t^p) \in \mathbb{R}_{>0}^p\), the piecewise constant control vector field \(X_I^{t^1}\) is complete;

(ii) the control set \(\mathcal{C} \subseteq \mathbb{R}^m\) is the compact convex set \([-1, 1]^m\).

**Remark 4.6.** In this remark we elaborate on each part of Assumption 1.

(i) By considering convex and compact control sets, the Assumption 1(ii) is restrictive for studying small-time local controllability. It turns out that small-time local controllability of systems strongly depends on the structure of the control set (see [20] for a detailed study of the role of control sets in fundamental properties of the systems);
(ii) If we assume that the control set \( C \) is compact, then the completeness of vector fields in Assumption 1(i) is not restrictive for studying small-time local controllability. This is because of the fact that, for studying small-time local controllability of the system, one can always focus on a small neighborhood of \( x_0 \) and small times \( t \), where the flows of all the piecewise constant vector fields of the system exist due to the compactness of \( C \) [10, Chapter 2, Theorem 1.1].

5. Control variations. The notion of control variation is one of the fundamental tools in studying reachable sets of control systems. Roughly speaking control variations can be considered as the directions constructed using the trajectories of the system, along which one can steer the control system. By constructing appropriate control variations and using a suitable open mapping theorem, one can show that a control system is small-time locally controllable [13, Theorem 2.1]. The first use of the notion of control variations for approximating reachable sets of control systems can be traced back to the original work of Pontryagin and his coworkers for studying the boundary of reachable sets [40]. Since then, many different and technical notions of variations with various properties have been proposed in the control literature [8, 13, 27, 32]. In this section we study a general class of control variations introduced in [12, 13].

**Definition 5.1 (Control variations).** Let \( k \in \mathbb{N} \) and \( (X, \mathcal{C}) \) be a \( C^k \) control-affine system on \( \mathbb{R}^n \). Then a vector \( v \in \mathbb{R}^n \) is called a variation of \( k \)th order for the control-affine system \( X \) at the point \( x_0 \) if there exists a parametrized family of points \( \gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}^n \) such that, for every \( t \in \mathbb{R}_{\geq 0} \), we have \( \gamma(t) \in R_X(\leq t, x_0) \) and

\[
\gamma(t) = x_0 + t^k v + \mathcal{O}(t^{k+1}).
\]

The set of all variations of \( k \)th order for the system \( X \) at the point \( x_0 \) is denoted by \( K_X^k(x_0) \). We also define the cone \( \hat{K}_X^k(x_0) \) by

\[
\hat{K}_X^k(x_0) = \bigcup_{\alpha \geq 0} \alpha \left( K_X^k(x_0) \right).
\]

Note that, in Definition 5.1, there is no restriction on the regularity of the parametrized family of points \( \gamma \). The next theorem shows how these control variations can be used to deduce the small-time local controllability and the growth rate condition for control-affine systems.

**Theorem 5.2 (Characterization of growth rate condition).** Let \( (X, \mathcal{C}) \) be a \( C^k \) control-affine system on \( \mathbb{R}^n \) which satisfies Assumption 1, \( x_0 \in \mathbb{R}^n \) be an equilibrium point for \( X \), and \( N \in \mathbb{N} \). Then the following statements are equivalent:

(i) the control-affine system \( X \) satisfies the growth rate condition of order \( N \);

(ii) \( \hat{K}_X^k(x_0) = \mathbb{R}^n \).

**Proof.** (i) \( \Rightarrow \) (ii): Since the control-affine system \( (X, \mathcal{C}) \) satisfies the growth rate condition of order \( N \), there exists \( T > 0 \) and \( C > 0 \) such that, for every \( t \in [0, T] \), we have \( \overline{B}(x_0, Ct^N) \subseteq R_X(\leq t, x_0) \). For every \( v \in S^{n-1} \), we define the family of points \( \gamma_v : [0, T] \to \overline{B}(x_0, Ct^N) \) by

\[
\gamma_v(t) = x_0 + Ct^N v.
\]

By Definition 5.1, the vector \( v \) is control variations of order \( N \) for the system \( X \) at the point \( x_0 \). This means that we have \( \hat{K}_X^N(x_0) = \mathbb{R}^n \) and therefore (ii) holds.

(ii) \( \Rightarrow \) (i): The proof of this part follows from [27, Corollary 2.5]. \( \square \)
Roughly speaking, Theorem 5.2 states that small-time local controllability of a system is checkable using variations of order $N$ if and only if the system satisfies the growth rate condition of order $N$. Note that this result holds for control-affine systems in any regularity class $C^\nu$.

6. Polynomial perturbations of real analytic control-affine systems. In this section, we focus on the class of real analytic control-affine systems which satisfy the growth rate condition of order $N$. For a control system in this class, we construct a multi-valued mapping to study the effect of perturbation of vector fields of the system on its reachable sets. Let $(\mathcal{X} = \{X_0, X_1, \ldots, X_m\}, \mathcal{C})$ be a real analytic control-affine system which satisfies the growth rate condition of order $N$ at the point $x_0$ and let $(\mathcal{Y} = \{Y_0, Y_1, \ldots, Y_m\}, \mathcal{C})$ be another real analytic control-affine system which we consider as the perturbed control system. Moreover, we assume that both control-affine systems $(\mathcal{X}, \mathcal{C})$ and $(\mathcal{Y}, \mathcal{C})$ satisfy Assumption 1. Since $(\mathcal{X}, \mathcal{C})$ satisfies the growth rate condition of order $N$, there exists $C, T > 0$ such that

$$\overline{B}(x_0, Ct^N) \subseteq R_X(\leq t, x_0), \quad \forall t \leq T.$$ 

For every $t \in [0, T]$, we define a perturbation mapping $F_{\mathcal{X}, \mathcal{Y}}^t : \overline{B}(x_0, \frac{C}{2}t^N) \rightrightarrows R_Y(\leq t, x_0)$ which captures the transition from reachable sets of $(\mathcal{X}, \mathcal{C})$ to the reachable sets of $(\mathcal{Y}, \mathcal{C})$. The multi-valued mapping $F_{\mathcal{X}, \mathcal{Y}}^t$ is defined as composition of two mappings $\xi_{\mathcal{Y}}^t$ and $\eta_{\mathcal{X}}^t$ as follows:

$$F_{\mathcal{X}, \mathcal{Y}}^t(x) = \xi_{\mathcal{Y}}^t \circ \eta_{\mathcal{X}}^t(x), \quad \forall x \in \overline{B}(x_0, \frac{C}{2}t^N).$$

The idea is that the mapping $\eta_{\mathcal{X}}^t$ takes a point in the closed ball $\overline{B}(x_0, \frac{C}{2}t^N)$ and gives the switching times associated to that point and the mapping $\xi_{\mathcal{Y}}^t$ takes a set of switching times and gives the associated point in the reachable sets of the control system $(\mathcal{Y}, \mathcal{C})$.

We start by rigorously constructing the multi-valued mapping $\eta_{\mathcal{X}}^t$. Since $(\mathcal{X}, \mathcal{C})$ satisfies Assumption 1 and the growth rate condition of order $N$, we have $\mathcal{C} = [-1, 1]^m$ and $\overline{B}(x_0, Ct^N) \subseteq R_X(\leq t, x_0)$, for every $t \leq T$. This implies that, for every $t \leq T$, we get

$$\overline{B}(x_0, \frac{C}{2}t^N) \subseteq B(x_0, Ct^N) \subseteq \text{int}(R_X(\leq t, x_0))$$

By Theorem 4.5, for every $x \in \overline{B}(x_0, \frac{C}{2}t^N)$, there exist $p_x \in \mathbb{N}, s_1, s_2, \ldots, s_{p_x} \in \mathbb{R}_{>0}$, and $w^1, w^2, \ldots, w^{p_x} \in [-1, 1]^m$ such that $s_1 + s_2 + \ldots + s_{p_x} < t$ and

$$\exp(s_1 X_{w^1}) \circ \exp(s_2 X_{w^2}) \circ \ldots \circ \exp(s_{p_x} X_{w^{p_x}})(x) = x.$$ 

Moreover, there exists an open neighborhood $V$ of $(s_1, s_2, \ldots, s_{p_x})$ in $\mathbb{R}^{p_x}$ such that the map $\xi_{\mathcal{X}}^t : V \to \mathbb{R}^n$, defined by

$$\xi_{\mathcal{X}}^t(t_1, t_2, \ldots, t_{p_x}) = \exp(t_1 X_{w^1}) \circ \exp(t_2 X_{w^2}) \circ \ldots \circ \exp(t_{p_x} X_{w^{p_x}})(x_0)$$

is $C^1$ and of rank $n$ on $V$. Without loss of generality we can assume that, for every $(t_1, t_2, \ldots, t_{p_x}) \in V$, we have

$$t_1 + t_2 + \ldots + t_{p_x} \leq t.$$ 

By [29, Lemma 2.2], there exists a submanifold $M_x$ of $V$ containing $(s_1, s_2, \ldots, s_{p_x})$ such that $\xi_{\mathcal{X}}^t(M_x)$ is an open neighborhood of $x$ in $\mathbb{R}^n$ and $\xi_{\mathcal{X}}^t|_{M_x}$ is a $C^1$-diffeomorphism. Let $S_x$ be an open neighborhood of $(s_1, s_2, \ldots, s_{p_x})$ in $M_x$ such
that $\mathcal{S}_x \subseteq M_x$. Since $\mathcal{B}(x_0, \xi_0 t^N)$ is compact and, for every $x \in \mathcal{B}(x_0, \xi_0 t^N)$, the set $\xi_x^r(S_x)$ is open in $\mathbb{R}^n$, there exists $x_1, \ldots, x_r \in \mathcal{B}(x_0, \xi_0 t^N)$ such that
\[ \mathcal{B}(x_0, \xi_0 t^N) \subseteq \bigcup_{i=1}^r \xi_x^r(S_x). \]

Now let us define $p = p_{x_1} + p_{x_2} + \ldots + p_{x_r}$ and let $(u^1, u^2, \ldots, u^p)$ be the ordered set obtained by concatenation of the controls $(w^1, w^2, \ldots, \omega^{p_{x_k}})$ for $1 \leq k \leq r$. Since, $\mathbb{R}^{p_{x_i}} \subseteq \mathbb{R}^p$, for every $i \in \{1, 2, \ldots, r\}$, one can consider $S_{x_i}$ as a submanifold of $\mathbb{R}^p$. We define the multi-valued map $\eta^i_x : \mathcal{B}(x_0, \xi_0 t^N) \to \bigcup_{i=1}^r \mathcal{S}_x$, as
\[ \eta^i_x(x) = \bigcup_{i \in \{1, 2, \ldots, r\}} \{ (\xi^r_x)^{-1}(x) \mid x \in \xi_x^r(S_x) \}, \quad \forall x \in \mathcal{B}(x_0, \xi_0 t^N). \]

Note that, for every $x \in \mathcal{B}(x_0, \xi_0 t^N)$, the number of elements in $\eta^i_x(x)$ is at most $r$.

The next step is to construct the single-valued mapping $\xi^r_x : \mathcal{B}(x_0, \xi_0 t^N) \to \mathbb{R}^n$ by
\[ \xi^r_x(t_1, t_2, \ldots, t_p) = \exp(t_1 Y_{u^1}) \circ \exp(t_2 Y_{u^2}) \circ \ldots \circ \exp(t_p Y_{u^p})(x_0). \]

Then the multi-valued map $F^r_{x, \mathcal{Y}} : \mathcal{B}(x_0, \xi_0 t^N) \to \mathbb{R}^n$ is given by
\[ F^r_{x, \mathcal{Y}}(x) = \xi_x^r \circ \eta_x^r(x). \]

One can observe that the mapping $F^r_{x, \mathcal{Y}}$ is finite-valued and has the following regularity properties.

**Theorem 6.1 (Regularity of the perturbation mapping).** Let $(\mathcal{X} = \{X_0, \ldots, X_m\}, \mathcal{C})$ and $(\mathcal{Y} = \{Y_0, \ldots, Y_m\}, \mathcal{C})$ be two real analytic control-affine system which satisfy Assumption 1. Suppose that, for $x_0 \in \mathbb{R}^n$, there exists $N \in \mathbb{N}$ such that

(a) the system $(\mathcal{X}, \mathcal{C})$ satisfies the growth rate condition of order $N$,

(b) for every $i \in \{0, \ldots, m\}$, the vector fields $X_i$ and $Y_i$ have the same Taylor polynomial of order $N$ around $x_0$.

For every $t \in [0, T)$, let the map $F^r_{x, \mathcal{Y}} : \mathcal{B}(x_0, \xi_0 t^N) \to \mathbb{R}^n(\leq t, x_0)$ be defined as above. Then the following statements hold:

(i) For every $t \in [0, T]$ and every $x \in \mathcal{B}(x_0, \xi_0 t^N)$, there exist a positive integer $l \in \mathbb{N}$, a neighborhood $W$ containing $x$, and continuous functions $f^1, f^2, \ldots, f^l : W \to \mathbb{R}^n$ such that
\[ \{ f^1(y) \} \subseteq F^r_{x, \mathcal{Y}}(y) \subseteq \{ f^1(y), f^2(y), \ldots, f^l(y) \}, \quad \forall y \in W. \]

(ii) there exist $\alpha > 0$ and $T_{\text{min}} \in (0, T)$ such that, for every $t \leq T_{\text{min}}$ and every $x \in \mathcal{B}(x_0, \xi_0 t^N)$, we have
\[ \|y - x\| \leq \alpha t^{N+1}, \quad \forall y \in F^r_{x, \mathcal{Y}}(x). \]

**Proof.** Regarding part (i), suppose that, for every $i \in \{1, 2, \ldots, r\}$, the map $\xi_x^r$ and the manifold $S_x$ are defined as above. Since $\mathcal{B}(x_0, \xi_0 t^N) \subseteq \bigcup_{i=1}^r \xi_x^r(S_x)$, we have $x \in \bigcup_{i=1}^r \xi_x^r(S_x)$. Without loss of generality, we can assume that
\[ x \in \xi_x^r(S_x). \]
Note that $\xi_x^r(S_{x_1})$ is open in $\mathbb{R}^n$. Therefore, there exists a neighborhood $U$ of $x$ such that $U \subseteq \xi_x^r(S_{x_1})$. On the other hand, since we have $B(x_0, \frac{C}{t}N) \subseteq \bigcup_{i=1}^r \xi_x^r(S_{x_1})$, without loss of generality, we can assume that there exists $l \in \{1, 2, \ldots, r\}$ such that
\[
x \in \xi_x^r(S_{x_1}), \quad i \in \{1, 2, \ldots, l\},
\]
\[
x \notin \xi_x^r(S_{x_1}), \quad i \in \{l+1, l+2, \ldots, r\}.
\]
For every $i \in \{l+1, l+2, \ldots, r\}$, the set $\xi_x^r(S_{x_1})$ is closed in $\mathbb{R}^n$. Therefore,
\[
\bigcup_{i=l+1}^r \xi_x^r(S_{x_1})
\]
is closed in $\mathbb{R}^n$. Moreover, we know that $x \notin \bigcup_{i=l+1}^r \xi_x^r(S_{x_1})$. This implies that there exists a neighborhood $V$ of $x$ such that
\[
V \cap \left( \bigcup_{i=l+1}^r \xi_x^r(S_{x_1}) \right) = \emptyset.
\]
Note that, for every $i \in \{1, 2, \ldots, l\}$, $S_{x_1} \subseteq M_{x_1}$. Therefore, for every $i \in \{1, 2, \ldots, l\}$, we have $\xi_x^r(S_{x_1}) \subseteq \xi_x^r(M_{x_1})$. Since $x \in \xi_x^r(M_{x_1})$, for every $i \in \{1, 2, \ldots, l\}$, the set $\bigcap_{i=1}^l \xi_x^r(M_{x_1})$ is nonempty. We set
\[
W = \left( \bigcap_{i=1}^l \xi_x^r(M_{x_1}) \right) \cap V \cap U.
\]
For every $i \in \{1, 2, \ldots, l\}$, we define the function $f^i : W \to \mathbb{R}^n$ as
\[
f^i(y) = \xi_y^{r_i} \circ (\xi_x^r)^{-1}(y), \quad \forall y \in W.
\]
Note that, for every $i \in \{1, 2, \ldots, l\}$, the map $\xi_x^r$ is a $C^1$-diffeomorphism on $M_{x_1}$. Therefore, for every $i \in \{1, 2, \ldots, l\}$, the map $f^i : W \to \mathbb{R}^n$ is continuous. Now, it is clear from the definition of $F_{y,x}^r$ that we have
\[
F_{y,x}^r(t, x) = \{ f^1(x), f^2(x), \ldots, f^l(x) \}.
\]
Since $W \subseteq V$ and $V$ is chosen such that $V \cap \left( \bigcup_{i=l+1}^r \xi_x^r(S_{x_1}) \right) = \emptyset$, for every $i \in \{l+1, l+2, \ldots, r\}$ and every $y \in W$, we have
\[
\xi_y^{r_i} \circ (\xi_x^r)^{-1}(y) \notin F_{y,x}^r(y).
\]
Thus, we have
\[
F_{y,x}^r(y) \subseteq \{ f^1(y), f^2(y), \ldots, f^l(y) \}, \quad \forall y \in W.
\]
Finally, since $W \subseteq U$, and $U$ is chosen such that $U \subseteq \xi_x^r(S_{x_1})$, for every $y \in W$,
\[
\xi_y^{r_i} \circ (\xi_x^r)^{-1}(y) \in F_{y,x}^r(y).
\]
Therefore, for every $y \in W$, we have $f^1(y) \in F_{y,x}^r(y)$.
Regarding part (ii), note that the system \((X, C)\) satisfies Assumption 1 and therefore, we have \(C = [-1, 1]^m\). We define the set \(B \subseteq \Gamma^\omega(\mathbb{R}^n)\) by
\[
B = \{X_u \mid u \in [-1, 1]^m\} \bigcup \{Y_u \mid u \in [-1, 1]^m\}.
\]
Note that, for every \(u \in [-1, 1]^m\), every compact set \(K\), every \(C^\omega\)-function \(f\), and every \(a \in \mathbb{R}^1\), we have
\[
\rho_{K,a,f}^\omega(X_u) = \rho_{K,a,f}^\omega(X_0 + \sum_{i=1}^m u_i X_i) \leq (m + 1) \max \{\rho_{K,a,f}^\omega(X_i) \mid i \in \{0, 1, \ldots, m\}\}.
\]
Similarly, for every \(u \in [-1, 1]^m\), every compact set \(K\), every \(C^\omega\)-function \(f\), and every \(a \in \mathbb{R}^1\), we have
\[
\rho_{K,a,f}^\omega(Y_u) = \rho_{K,a,f}^\omega(Y_0 + \sum_{i=1}^m u_i Y_i) \leq (m + 1) \max \{\rho_{K,a,f}^\omega(Y_i) \mid i \in \{0, 1, \ldots, m\}\}.
\]
For every \(i \in \{0, 1, \ldots, m\}\) we define \(L_i = \max \{\rho_{K,a,f}^\omega(X_i), \rho_{K,a,f}^\omega(Y_i)\}\). Thus, if we define the constant \(L \in \mathbb{R}_{>0}\) by
\[
L = (m + 1) \max \{L_i \mid i \in \{0, 1, \ldots, m\}\},
\]
then we have
\[
\rho_{K,a,f}^\omega(v) \leq L, \quad \forall v \in B.
\]
By Theorem 3.3, this implies that the set \(B\) is bounded in \(\Gamma^\omega(\mathbb{R}^n)\). Using Theorem 3.7, there exist \(M, L > 0\) and \(T < T_B\) such that, for every \(t \in (0, T]\) and every real analytic vector field \(Z \in \text{PC}([0, T]; B)\), we have
\[
\|ev_{x_0} \circ \exp(tZ)(x^i) - ev_{x_0} \circ \exp_N(tZ)(x^i)\| \leq \frac{(Mt)^{N+1}}{1 - Mt}.
\]
We set \(\alpha = \sqrt{\frac{1}{m}} M^{N+1} L\) and \(T_{\text{min}} = \min\{T, T\}\). Let \(t \in [0, T_{\text{min}}]\) and \(x \in B(x_0, \frac{T}{T_{\text{min}}} T_N)\). If \(y \in F_{X,x}(x)\), then there exist \(I = (u^1, u^2, \ldots, u^p) \in \([-1, 1]^m)^p\) and \(t = (t_1, t_2, \ldots, t_p) \in \mathbb{R}_{\geq 0}^p\) such that \(|t| < t\) and
\[
x = \exp(t_1 X_{u^1}) \circ \exp(t_2 X_{u^2}) \circ \ldots \circ \exp(t_p X_{u^p})(x_0) = \exp([t|X^{1t}],
\]
\[
y = \exp(t_1 Y_{u^1}) \circ \exp(t_2 Y_{u^2}) \circ \ldots \circ \exp(t_p Y_{u^p})(x_0) = \exp([t|Y^{1t}]).
\]
Note that, for every \(k \in \mathbb{Z}_{\geq 0}\) with the property that \(|k| \leq N\), we have
\[
D^k X_i(x_0) = D^k Y_i(x_0), \quad i \in \{1, \ldots, m\}.
\]
Note that the \(N\)th iteration in the sequence of flow iteration (3.2) only depends on the derivatives up to order \(N\) of the vector field \(X\). This implies that,
\[
ev_{x_0} \circ \exp_N([t|X^{1t}](x^i)) = ev_{x_0} \circ \exp_N([t|Y^{1t}](x^i)).
\]
Thus we have
\[
\|ev_{x_0} \circ \exp([t|X^{1t}](x^i)) - ev_{x_0} \circ \exp([t|Y^{1t}](x^i))\|
\leq \|ev_{x_0} \circ \exp([t|X^{1t}](x^i)) - ev_{x_0} \circ \exp_N([t|X^{1t}](x^i))\|
\]
\[
+ \|ev_{x_0} \circ \exp([t|Y^{1t}](x^i)) - ev_{x_0} \circ \exp_N([t|Y^{1t}](x^i))\|.\]
Since $X^{I,t}, Y^{I,t} \in \text{PC}([0,|t|]; B)$, we have
\[ \|ev_{x_0} \circ \exp(|t|X^{I,t})(x^i) - ev_{x_0} \circ \exp(|t|Y^{I,t})(x^i)\| \leq \frac{(Mt)^{N+1}}{1-Mt}L, \]
\[ \|ev_{x_0} \circ \exp(|t|Y^{I,t})(x^i) - ev_{x_0} \circ \exp(|t|Y^{I,t})(x^i)\| \leq \frac{(Mt)^{N+1}}{1-Mt}L. \]
Thus, we have
\[ \|ev_{x_0} \circ \exp(|t|X^{I,t})(x^i) - ev_{x_0} \circ \exp(|t|Y^{I,t})(x^i)\| \leq 2\frac{(Mt)^{N+1}}{1-Mt}L, \]
\[ \forall i \in \{1, 2, \ldots, n\}. \]
Therefore, we have
\[ \|\exp(|t|X^{I,t})(x_0) - \exp(|t|Y^{I,t})(x_0)\| \leq 2\sqrt{n}\frac{(Mt)^{N+1}}{1-Mt}L. \]
Note that, by our choice of $t_1, t_2, \ldots, t_n \in \mathbb{R}_{>0}$, we have
\[ y = \exp(|t|Y^{I,t})(x_0), \]
\[ x = \exp(|t|X^{I,t})(x_0). \]
Note that, by the definition, we have $\alpha = \sqrt{n}M^{N+1}L$ and $Mt \leq MT \leq \frac{1}{2}$. This implies that
\[ \|y - x\| \leq \alpha t^{N+1}. \]
Since $\alpha$ does not depend on $t$, the above inequality holds for every $t \in [0, T_{\text{min}}]$. \(\square\)

**Remark 6.2.** Some remarks are in order.

(i) Part (i) of Theorem 6.1 can be considered as a regularity result for the finite-valued mapping $x \mapsto F_{X,Y}^{I}(x)$ and part (ii) of Theorem 6.1 can be considered as a regularity result for the finite-valued mapping $t \mapsto F_{X,Y}^{I}(x)$;

(ii) For the proof of part (i) of Theorem 6.1, it is not required to assume that the family of vector fields $\mathcal{X}$ and $\mathcal{Y}$ have the same Taylor polynomials of order $N$ around the point $x_0$. In fact, Theorem 6.1(i) is true for any arbitrary perturbation of the real analytic control system $(\mathcal{X}, \mathcal{C})$ satisfying Assumption 1. However, this condition is essential for the proof of Theorem 6.1(ii).

**7. The main theorem.** In this section we prove the main result of this paper, which can be considered as a robustness of the growth rate condition of order $N$ with respect to polynomial perturbations of order higher than $N$. Roughly speaking our main result states that, given a real analytic system $(\mathcal{X}, \mathcal{C})$ which satisfies the growth rate condition of order $N$ at the point $x_0$, if we perturb the vector fields of $(\mathcal{X}, \mathcal{C})$ around $x_0$ by Taylor polynomials of order higher than $N$, then the resulting system again satisfies the growth rate condition of order $N$.

**Theorem 7.1 (Polynomial perturbations of real analytic systems).** Let $(\mathcal{X} = \{X_0, \ldots, X_m\}, \mathcal{C})$ and $(\mathcal{Y} = \{Y_0, \ldots, Y_m\}, \mathcal{C})$ be two real analytic control-affine systems on $\mathbb{R}^n$ satisfying Assumption 1 and let $x_0 \in \mathbb{R}^n$. Suppose that the following conditions hold:
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(i) $x_0 \in \mathbb{R}^n$ is an equilibrium point for control-affine systems $\mathcal{X}$ and $\mathcal{Y}$,
(ii) the control-affine system $\mathcal{X}$ satisfies the growth rate condition of order $N$ at the point $x_0$, and
(iii) for every $i \in \{0, \ldots, m\}$, the vector fields $X_i$ and $Y_i$ have the same Taylor polynomial of order $N$ around $x_0$.

Then $\mathcal{Y}$ satisfies the growth rate condition of order $N$ at the point $x_0$.

Proof. Since $(\mathcal{X}, \mathcal{C})$ satisfies the growth rate condition of order $N$ at the point $x_0$, there exist $T, C > 0$ such that, for every $t \in [0, T]$, we have
\[
\overline{B}(x_0, Ct^N) \subset R_X(\leq t, x_0).
\]
For every $v \in S^n$, consider the parametrized family of points $\gamma_v : [0, T] \to \mathbb{R}^n$ defined by
\[
\gamma_v(t) = x_0 + \frac{C}{2} t^N v.
\]
It is clear that, for every $t \in [0, T]$, we have $\gamma_v(t) \in \overline{B}(x_0, \frac{C}{2} t^N)$. For every $t \in [0, T]$, recall the definition of the multi-valued map $F^t_{\mathcal{X}, \mathcal{Y}}$ in Section 6 and let $f^t_{\mathcal{X}, \mathcal{Y}} : \overline{B}(x_0, \frac{C}{2} t^N) \to R_\mathcal{Y}(\leq t, x_0)$ be a single-valued selection of the multi-valued mapping $F^t_{\mathcal{X}, \mathcal{Y}}$. Then, for every $v \in S^{n-1}$, we define the curves $\mu_v : [0, T] \to \mathbb{R}^n$ by
\[
\mu_v(t) = f^t_{\mathcal{X}, \mathcal{Y}}(\gamma_v(t)).
\]
Note that, for every $t \in [0, T]$, we have $\gamma_v(t) \in \overline{B}(x_0, \frac{C}{2} t^N)$. Therefore, for every $t \in [0, T]$, the map $\mu_v(t)$ is well-defined and $\mu_v(t) \in R_\mathcal{Y}(\leq t, x_0)$.

Now, by Theorem 6.1 part (ii), there exist $\alpha > 0$ and $0 < T_{\min} < T$ such that, for every $t \leq T_{\min}$ and every $x \in \overline{B}(x_0, \frac{C}{2} t^N)$, we get
\[
\|f^t_{\mathcal{X}, \mathcal{Y}}(\gamma_v(t)) - \gamma_v(t)\| \leq \alpha t^{N+1}.
\]
This implies that, for every $t \in [0, T_{\min}]$, 
\[
\mu_v(t) = f^t_{\mathcal{X}, \mathcal{Y}}(\gamma_v(t)) = \gamma_v(t) + O(t^{N+1}) = x_0 + \frac{C}{2} t^N v + O(t^{N+1}).
\]
Thus, using Definition 5.1, $v$ is a control variation of $N$th order for the control-affine system $(\mathcal{Y}, \mathcal{C})$. Thus, we have $K_N^\mathcal{Y} = \bigcup_{\alpha \geq 0} \alpha K_N^\mathcal{X}(x_0) = \mathbb{R}^n$ and, by Theorem 5.2, the control-affine system $(\mathcal{Y}, \mathcal{C})$ satisfies the growth rate condition of order $N$.

As a direct consequence of Theorem 7.1, one can prove the following connection between Conjecture 1.1 and Conjecture 1.2.

**Corollary 7.2.** If Conjecture 1.2 is true, then Conjecture 1.1 holds.

**8. Conclusion.** In this paper, we studied small-time local controllability of real analytic control-affine systems under polynomial perturbations of their vector fields. For a real analytic control-affine system which satisfies the growth rate condition of order $N$, we construct a suitable multi-valued map for studying the perturbations of its reachable sets. We showed that if a real analytic control-affine system satisfies the growth rate condition of order $N$, then any perturbation of the system by polynomial vector fields of order higher than $N$ is again small-time local controllable. For the future research, it is interesting to study suitable filtrations of the vector fields of the system (see for example [16]) to sharpen the perturbation results obtained in this paper. Moreover, it is worth mentioning that while our techniques heavily depend on the real analyticity of the vector fields of the system, it remains to be seen whether the main result of this paper still holds for $C^\infty$ control-affine systems.
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Appendix A. Proof of Theorem 3.7.
In this appendix, we present a proof of the Theorem 3.7 using the $C^\omega$-topology on the space $LC^\omega(\mathbb{R}^n)$. As mentioned in Section 2, it is also possible to give a proof of this theorem using the estimates in [4, §2.4.4] for piecewise constant real analytic vector fields as curves on the space $\Gamma^\infty(\mathbb{R}^n)$ (see [4, §2.4.4 and Appendix A.2]). Note that, in [4, §2.4.4], the space $\Gamma^\infty(\mathbb{R}^n)$ is equipped with the Whitney compact-open topology [4, §2.2].

**Proof.** By Theorem 3.6, for every $i \in \{1, 2, \ldots, n\}$, every compact set $K \subset \mathbb{R}^n$ containing $x_0$, and every $a \in c_0^i$, there exist $M, M_i > 0$ such that
\begin{equation}
\rho^\omega_{K,a,x}(\exp(tX) - \tilde{\exp}_k(tX)) \leq (Mt)^{k+1}M_i.
\end{equation}
Note that, for every $\phi \in LC^\omega(\mathbb{R}^n)$ and every $i \in \{1, 2, \ldots, n\}$, we have $\text{ev}_{x_0} \circ \phi(x^i) = \phi(x^i)(x_0)$. Therefore, we get
\[\|\text{ev}_{x_0} \circ \phi(x^i)\| \leq \sup\{\phi(x^i)(y) \mid y \in K\} \leq \rho^\omega_{K,a,x}(\phi).\]
Now, by setting $\max_i\{M_i\} = L$ and using the estimate (A.1), we get
\[\|\text{ev}_{x_0} \circ \tilde{\exp}_{k+1}(tX)(x^i) - \text{ev}_{x_0} \circ \tilde{\exp}_k(tX)(x^i)\| \leq (Mt)^{k+1}L, \quad \forall i \in \{1, 2, \ldots, n\}.\]
Therefore, if we choose $\overline{T} \leq T_B$ such that $M\overline{T} < 1$, we have
\[\|\text{ev}_{x_0} \circ \tilde{\exp}(tX)(x^i) - \text{ev}_{x_0} \circ \tilde{\exp}_k(tX)(x^i)\| \leq \sum_{r=k}^{\infty} \|\text{ev}_{x_0} \circ \tilde{\exp}_{r+1}(tX)(x^i) - \text{ev}_{x_0} \circ \tilde{\exp}_r(tX)(x^i)\| \leq \sum_{r=k}^{\infty} (Mt)^{r+1}L = \frac{(Mt)^{k+1}}{1-Mt}, \quad \forall t \in [0, \overline{T}].\]
This completes the proof of the theorem. 

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