Explicit field realizations of $W$ algebras

Shao-Wen Wei, Yu-Xiao Liu, Li-Jie Zhang and Ji-Rong Ren
1 Institute of Theoretical Physics, Lanzhou University, Lanzhou 730000, P. R. China; 2 Department of Physics, Shanghai University, Shanghai 200444, P. R. China.

The fact that certain non-linear $W_{2,s}$ algebras can be linearized by the inclusion of a spin-1 current can provide a simple way to realize $W_{2,s}$ algebras from linear $W_{1,2,s}$ algebras. In this paper, we first construct the explicit field realizations of linear $W_{1,2,s}$ algebras with double-scalar and double-spinor, respectively. Then, after a change of basis, the realizations of $W_{2,s}$ algebras are presented. The results show that all these realizations are Romans-type realizations.

PACS numbers: 11.25.Sq, 11.10.-z, 11.25.Pm. Keywords: $W$ algebra, BRST charge, field realization

I. INTRODUCTION

After the fundamental work of Zamolodchikov [1] in the middle of the 1980’s, $W$ algebras have attracted much attention since they uncover some underlying world-sheet symmetries of strings. Many $W$ algebras are known (for review see [2]) and much work has been carried out on their classification [3, 4, 5, 6]. $W$ algebras have many applications and become the subject of great interest in many branches of physics and mathematics, e.g., in $W$ gravity theories [7, 8], critical and non-critical $W$ string theories [9, 10], Wess-Zumino-Novikov-Witten (WZNW) models [11, 12, 13], quantum Hall effect [14], and especially in black holes [15, 16], where it was shown that the Hawking radiation can be explained as the fluxes of chiral currents forming a $W_\infty$ algebra.

As we know, $W$ algebras arise from Kac-Moody algebras, which are related to classical Lie algebras. Various free field realizations of $W$ algebras have been extensively studied [17, 18, 19, 20, 21, 22, 23, 24]. At quantum level, $W$ algebras are usually non-linear, which makes it very difficult to give the field realizations of them. The corresponding $W$ strings were first investigated in Ref. [25] and have been extensively developed since then. Much research on the scalar realizations of $W_{2,s}$ strings has been done [26, 27, 28, 29, 30, 31, 32, 33, 34]. Most of which are based on the grading method, where the BRST charge of $W_{2,s}$ strings is written in the form of $Q_B = Q_0 + Q_1$. This provides an easy way to construct $W_{2,s}$ strings, while it imposes more constrain conditions on the BRST charge. Under the supposition that this grading form still holds true for spinor field realizations, the corresponding works had been done [35, 36, 37, 38].

Furthermore, many investigations have been focused on understanding the structure of $W$ algebras [39, 40, 41, 42]. It was shown that linear Lie algebras with finite number of currents may contain some non-linear $W$ algebras with an arbitrary central charge as subalgebras. Especially for $W_{2,s}$ algebras, they can be linearized by the inclusion of a spin-1 current at $s = 3$ and 4 [39]. After performing a non-linear change of basis, $W_{2,s}$ algebras can be recast into the form of linear algebras. But for the spin-$s$ current $W_0$, one has $W_0(z)W_0(\omega) \sim 0$, which indicates that $W_0$ is a null current. It is exciting that this shines some light on the realizations of the non-linear $W_{2,s}$ algebras. After constructing the linear bases of $W_{1,2,s}$ algebras and making a change of basis, we can obtain the realizations of the non-linear $W_{2,s}$ algebras. In fact, the spin-$s$ current $W_0$ can be set to zero, which will give a Romans-type realization of $W_{2,s}$ algebras. However, in [32], it was shown that the null current $W_0$ does not need to be set to zero and was first realized with parafermionic vertex operators. It also can be found in [37, 40, 41] that the null current $W_0$ was realized with the ghost-like fields. In this paper, we will construct the linear bases of the $W_{1,2,s}$ algebras with double-scalar and double-spinor, respectively. Through a change of basis, we obtain some new realizations of the non-linear $W_{2,s}$ algebras. All these results show that there exists non-Romans-type realization with double-scalar or double-spinor only. However, we still expect that there exist non-Romans-type realizations of $W$ algebras at some special values of central charge.

The paper is organized as follows. In section II we give a brief review and analysis of the realizations of the $W_{2,s}$ algebras and the $W_{2,s}$ strings. Then in section III we introduce the linearization of the $W_{2,s}$ algebras. In sections IV and V we construct the bases of the linear $W_{1,2,s}$ algebras and obtain new realizations of the $W_{2,s}$ algebras with double-scalar and double-spinor, respectively. Finally, the paper ends with a brief conclusion.

II. NOTE ON THE REALIZATIONS OF THE $W_{2,s}$ ALGEBRAS AND THE $W_{2,s}$ STRINGS

It is known that when extended to the quantum case, the $W_{2,s}$ algebras will become non-linear. The OPE of two currents with spin $s$ and $s'$ produces terms with spin $(s + s' - 2)$ at leading order. For example, there will be terms with spin-4 and spin-6 currents in the OPEs of the $W_{2,3}$ algebra and the $W_{2,4}$ algebra, respectively. How-
ever these terms with spin \((s + s' - 2)\) may be interpreted as composite fields built from the products of the fundamental currents with spin \(s\) and \(s'\). The \(W_{2,3}\) algebra is generated by the spin-2 energy-momentum tensor \(T\) and spin-3 current \(W\), which satisfy the OPEs [1]

\[
\begin{align*}
T(z)T(\omega) & \sim \frac{C/2}{(z-\omega)^4} + \frac{2T}{(z-\omega)^2} + \frac{\partial T}{z-\omega}, \\
T(z)W(\omega) & \sim \frac{3W}{(z-\omega)^2} + \frac{\partial W}{z-\omega}, \\
W(z)W(\omega) & \sim \frac{C/3}{(z-\omega)^3} + \frac{2T}{(z-\omega)^2} + \frac{\partial T}{z-\omega} + \frac{1}{(z-\omega)^2} \left( 2\Theta \Lambda + \frac{3}{10} \partial^2 T \right) + \frac{1}{(z-\omega)} \left( \Theta \partial \Lambda + \frac{1}{15} \partial^3 T \right),
\end{align*}
\]

where the coefficient \(\Theta\) and composite field \(\Lambda\) (spin 4) are given by

\[
\Theta = \frac{16}{22 + 5C}, \quad \Lambda = T^2 - \frac{3}{10} \partial^2 T. \tag{2}
\]

The constant \(C\) is the central charge of the \(W_{2,3}\) algebra. It is easy to see that the denominator of \(\Theta\) at \(C = \frac{-22}{5}\) will be zero and the \(W_{2,3}\) algebra will become singular. But one can rescale these currents such that the corresponding OPEs are well defined, i.e., there have no divergent coefficients in them (for the detailed discussion see [43]).

The \(W_{2,4}\) algebra is given by [44]

\[
\begin{align*}
T(z)T(\omega) & \sim \frac{C/2}{(z-\omega)^4} + \frac{2T}{(z-\omega)^2} + \frac{\partial T}{z-\omega}, \\
T(z)W(\omega) & \sim \frac{4W}{(z-\omega)^2} + \frac{\partial W}{z-\omega}, \\
W(z)W(\omega) & \sim \left\{ \begin{array}{l}
\frac{2T}{(z-\omega)^2} + \frac{\partial T}{z-\omega} + \frac{\partial^2 T}{(z-\omega)^2} + \frac{3}{10} \partial^2 T + \sigma_1 U + \sigma_2 W + \sigma_3 U + \sigma_4 W + \sigma_5 W \\
+ \frac{1}{15} \frac{\partial^3 T}{(z-\omega)^3} + \frac{1}{84} \frac{\partial^4 T}{(z-\omega)^2} + \frac{1}{560} \partial^5 T
\end{array} \right\} + \frac{1}{2} \left( \begin{array}{l}
\frac{1}{(z-\omega)^3} + \frac{1}{(z-\omega)^2} + \frac{1}{36 (z-\omega)}
\end{array} \right)
\end{align*}
\]

where the composite fields \(U, G, A\) and \(B\) are defined by

\[
U = (TT) - \frac{3}{10} \partial^2 T, \\
G = (\partial^2 TT) - \partial (\partial TT) + \frac{2}{9} \partial^2 (TT) - \frac{1}{42} \partial^4 T, \tag{4}
\]

\[
A = (TU) - \frac{1}{6} \partial^2 U, \quad B = (TW) - \frac{1}{6} \partial^2 W,
\]

with normal ordering of products of currents understood. The coefficients \(\sigma_i (i = 1 \cdots 5)\) are

\[
\begin{align*}
\sigma_1 & = \frac{42}{5C + 22}, \\
\sigma_2 & = \sqrt{\frac{54(C + 24)(C^2 - 172C + 196)}{(5C + 22)(7C + 68)(2C - 1)}}, \\
\sigma_3 & = \frac{13(19 - 524)}{10(7C + 68)(2C - 1)}, \\
\sigma_4 & = \frac{24(7C + 13)}{(5C + 22)(7C + 68)(2C - 1)}, \\
\sigma_5 & = \frac{28}{3(C + 24)} \sigma_2.
\end{align*}
\]

It is worth to point out that, just as the \(W_{2,3}\) algebra, the \(W_{2,4}\) algebra is singular at \(C = -24, -2, -\frac{22}{5}\) and \(-\frac{68}{7}\). After rescaling the spin-4 current \(W\), it can be proved that only the case \(C = -24\) satisfies the Jacobi identity [43].

In general, the BRST charge \(Q_B\) for a \(W_{2,s}\) string is [44, 46]

\[
Q_B = \oint dz \left[ e(z)T(z) + \gamma(z)W(z) \right], \tag{6}
\]

where the currents \(T\) and \(W\) generate the corresponding \(W_{2,s}\) algebra, and the fermionic ghosts \((b, c)\) and \((\beta, \gamma)\) are introduced for the currents \(T\) and \(W\), respectively. It is easy to prove that the BRST charge given above does satisfy the nilpotency condition:

\[
Q_B^2 = \{ Q_B, Q_B \} = 0. \tag{7}
\]

A realization for a \(W_{2,s}\) algebra means giving an explicit construction of the bases \(T\) and \(W\) from the basic fields, i.e., scalar fields, spinor fields, or ghost fields. Giving a realization for a non-linear algebra is difficult and complex. For simplicity, \(Q_B\) can generally be expressed as the grading form in many works:

\[
\begin{align*}
Q_B & = Q_0 + Q_1, \\
Q_0 & = \oint dz cT, \\
Q_1 & = \oint dz \gamma W, \tag{10}
\end{align*}
\]

where the currents \(T\) and \(W\) generate the \(W_{2,s}\) algebras. The detailed construction of the current \(T\) can be found in [47], where \(T\) was constructed from scalar,
spinor, and ghost fields. Here the ghost fields $b$, $c$, $\beta$, $\gamma$ are all fermionic and anticommuting. They satisfy the OPEs
\[ b(z)c(\omega) \sim \frac{1}{z-\omega}, \quad \beta(z)\gamma(\omega) \sim \frac{1}{z-\omega}, \] (11)
in other cases the OPEs vanish. The nilpotency condition of $Q_B$ become
\[ Q_B^2 = Q_B = \{Q_0, Q_1\} = 0. \] (12)
Although it is easy to construct the $W_{2,s}$ strings in this grading form, one may note that this gives more constraint conditions on $Q_B$.
One also note that if we obtain a realization for a $W_{2,s}$ algebra, the BRST charge $Q_B$ of the corresponding $W_{2,s}$ string will be obtained by substituting the explicit forms of currents $T$ and $W$ into (3).

III. LINEARIZATION OF THE $W_{2,s}$ ALGEBRAS FROM THE $W_{1,2,s}$ ALGEBRAS

It is shown that the $W_{2,s}$ algebras can be linearized as the linear $W_{1,2,s}$ algebras generated by currents $J$, $T$ and $W$ with spin 1, 2 and $s$ respectively. The linear $W_{1,2,s}$ algebras for $s = 3, 4$ take the forms [39]
\[ T_0(z)T_0(\omega) \sim \frac{C_0/2}{(z-\omega)^2} + \frac{2T}{(z-\omega)^2} + \frac{\partial T}{z-\omega}, \]
\[ T_0(z)W_0(\omega) \sim \frac{sW}{(z-\omega)^2} + \frac{\partial W}{z-\omega}, \]
\[ T_0(z)J_0(\omega) \sim \frac{\xi W_0}{(z-\omega)^2}, \]
\[ J_0(z)J_0(\omega) \sim -\frac{1}{(z-\omega)^2}, \]
where $C_0$, $C_1$ and $\xi$ are given by
\[ C_0 = 50 + 24t^2 + \frac{24}{t^2}, \quad C_1 = -\sqrt{6} \left(t + \frac{1}{t}\right), \]
\[ \xi = \sqrt{\frac{3}{2}} t \] (s = 3),
\[ C_0 = 86 + 30t^2 + \frac{60}{t^2}, \quad C_1 = -3t - \frac{4}{t}, \]
\[ \xi = t \] (s = 4).

IV. DOUBLE-SCALAR REALIZATIONS FOR THE LINEAR $W_{1,2,s}$ ALGEBRAS AND THE $W_{2,s}$ ALGEBRAS

In this section, we would like to construct the bases of the linear $W_{1,2,s}$ algebras with double-scalar. Using the fact that the $W_{2,s}$ algebras are contained in the linear $W_{1,2,s}$ algebras as subalgebras, we can obtain new realizations for the $W_{2,s}$ algebras by a change of basis.

A. Realizations for the $W_{1,2,3}$ algebra and the $W_{2,3}$ algebra

First of all, we notice the relation between $C_0$ and $C_1$ for $s = 3$ shown in [13]:
\[ C_0 = 2 + 4C_1^2. \] (18)
The scalar field has spin 0 in conformal field theory, and the OPE of it with itself is given by
\[ \phi(z)\phi(\omega) \sim \ln(z - \omega), \] (19)
or expressed as
\[
\partial \phi(z) \partial \phi(\omega) \sim -\frac{1}{(z - \omega)^2}.
\] (20)

One needs to note that the field \( \phi \) here is real. If \( \phi \) is a complex scalar field, it is easy to prove that the OPE will be of the form
\[
\partial \varphi^\dagger \partial \varphi \sim -\frac{1}{(z - \omega)^2},
\] (21)
in other cases the OPEs vanish.

Now we consider two real scalar fields \( \phi_1 \) and \( \phi_2 \). The OPEs of them with each other are read as
\[
\partial \phi_i(z) \partial \phi_j(\omega) \sim -\frac{\delta_{ij}}{(z - \omega)^2}, \quad (i, j = 1, 2). \tag{22}
\]

We would like to construct the explicit forms for the linear bases of the \( W_{1,2,3} \) algebra. The most general form of the basis \( T_0 \) can be expressed as
\[
T_0 = T_{eff} + g_1 T_{\phi_1} + g_2 T_{\phi_2} + g_3 T_{\phi_1 \phi_2},
\] (23)
where \( T_{eff} \) is an effective energy-momentum tensor with central charge \( C_{eff} \). The introduction of \( T_{eff} \) will ensure the nontriviality of the solutions. \( T_{\phi_1} \) and \( T_{\phi_2} \) are spin-2 energy-momentum tensors constructed from fields \( \phi_1 \) and \( \phi_2 \), respectively, and \( T_{\phi_1 \phi_2} \) is constructed from these two scalar fields. The construction is
\[
T_{\phi_1} = -\frac{1}{2} (\partial \phi_1)^2 - q_1 \phi_1^2, \tag{24}
\]
\[
T_{\phi_2} = -\frac{1}{2} (\partial \phi_2)^2 - q_2 \phi_2^2, \tag{25}
\]
\[
T_{\phi_1 \phi_2} = \partial \phi_1 \partial \phi_2.
\] (26)

where \( q_1 \) and \( q_2 \) are the background charges of \( T_{\phi_1} \) and \( T_{\phi_2} \), respectively. The other two linear bases are given by
\[
J_0 = g_4 \partial \phi_1 + g_5 \partial \phi_2, \tag{27}
\]
\[
W_0 = g_6 T_{eff} + g_7 T_{\phi_1} + g_8 T_{\phi_2} + g_9 \partial \phi_1^3 + g_{10} \partial \phi_1 + g_{11} \partial \phi_2^3 + g_{12} \partial \phi_2
+ g_{13} \partial \phi_1 \partial \phi_2^2 + g_{14} \partial \phi_1 \partial \phi_2 + g_{15} \partial \phi_1 \partial \phi_2^2 + g_{16} \partial \phi_1 \partial \phi_2 + g_{17} \partial \phi_1 \partial \phi_2 + g_{18} \partial \phi_1 \partial \phi_2. \tag{28}
\]

Plugging these linear bases into the OPEs relations (13), we could obtain all the coefficients. One can see that the constant \( t \) appeared in (14) does not take zero, which determines \( \xi \neq 0 \). This leads to a main result
\[
g_i = 0 \quad \text{for} \quad i = 6 - 18, \tag{29}
\]
which means that the current \( W_0 \) is zero. After carefully calculation, we obtain two solutions:

- **Solution 1**

  \[ g_1 = g_2 = 1, \quad g_3 = 0, \quad g_4 = g_5 = \sqrt{2}h, \quad C_1 = 2\sqrt{2}h, \quad C_{eff} = 8, \quad C_0 = 34, \quad q_1 = q_2 = -1, \]

- **Solution 2**

  \[ g_1 = g_2 = -g_3 = \frac{1}{\sqrt{2}}h, \quad C_1 = \frac{1}{\sqrt{2}}h, \quad C_{eff} = 9, \quad C_0 = 34, \quad q_1 = q_2 = -\frac{1}{2}, \]

where \( h \) satisfies \( h^2 = 1 \). The main difference between above two solutions is whether the energy-momentum tensor \( T_{\phi_1 \phi_2} \) vanishes. In Solution 1, the term \( T_{\phi_1 \phi_2} \) does not appear. However, in Solution 2, the contribution of the term \( T_{\phi_1 \phi_2} \) to central charge is \( \frac{1}{2} \).

Having found two realizations of the linear \( W_{1,2,3} \) algebra, we substitute the exact forms of the linear bases \( T_0 \) and \( J_0 \) into (13) and obtain two new realizations of the \( W_{2,3} \) algebra. The first realization is
\[
T = T_{eff} + (\partial \phi_1)^2 - \frac{1}{2} \partial^2 \phi_1 + (\partial \phi_2)^2 - \frac{1}{2} \partial^2 \phi_2, \tag{30}
\]
\[
W = \frac{\sqrt{3}}{48} \left( 12h T_{eff} \partial \phi_1 + 12h T_{eff} \partial \phi_2 - h (\partial \phi_1)^3 + 3h (\partial \phi_1)^2 \partial \phi_2 + 9h \partial \phi_1 (\partial \phi_2)^2 + (6h - 12\sqrt{3}) \partial \phi_1 \partial \phi_2 - h (\partial \phi_2)^3 + (6h - 6\sqrt{3}) \partial \phi_1 \partial \phi_2 + (6h - 12\sqrt{3}) \partial^2 \phi_1 \partial \phi_2 + (6h - 6\sqrt{3}) \partial^2 \phi_2 \partial \phi_2 - 6\sqrt{3} \partial T_{eff} + (24h - 6\sqrt{3}) \partial^3 \phi_1 + (6h - 6\sqrt{3}) \partial^3 \phi_2 \right). \tag{31}
\]
and the second one reads
\[
T = T_{eff} + (\partial \phi_1)^2 - \frac{1}{2} \partial^2 \phi_1 + (\partial \phi_2)^2 - \frac{1}{2} \partial^2 \phi_2, \tag{32}
\]
\[
W = \frac{\sqrt{2}h}{96} \left( 12h T_{eff} \partial \phi_1 + 12h T_{eff} \partial \phi_2 + 3h \partial \phi_1 (\partial \phi_2)^2 + (12\sqrt{3} + 12h) \partial^2 \phi_1 \partial \phi_1 + (12h - 18\sqrt{3}) \partial \phi_1 \partial \phi_2 + h (\partial \phi_2)^3 + (12h - 18\sqrt{3}) \partial \phi_1 \partial \phi_2 + (12h - 18\sqrt{3}) \partial^2 \phi_2 \partial \phi_2 - 12\sqrt{3} \partial T_{eff} + (51h - 12\sqrt{3}) \partial^3 \phi_1 + (48h - 12\sqrt{3}) \partial^3 \phi_2 \right). \tag{33}
\]

where \( h \) satisfies \( h^2 = 1 \). Note that, although \( T_{\phi_1 \phi_2} \) is absent in the first realization, the energy-momentum tensor \( T \) in both realizations has central charge \( C = 34 \). If plugging these realizations into (9), one will get the BRST charges for the \( W_{2,3} \) string.

**B. Realizations for the \( W_{1,2,4} \) algebra and the \( W_{2,4} \) algebra**

For the linear \( W_{1,2,4} \) algebra, the relation between \( C_0 \) and \( C_1 \) is
\[
C_0 = 1 + \frac{1}{24} \left( 85C_1^2 - 5C_1 \sqrt{48 + C_1^2} \right). \tag{34}
\]
Next, we would like to construct the explicit forms of the linear bases of the $W_{1,2,4}$ algebra. The most general forms of bases $T_0$ and $J_0$ are

$$T_0 = f_1 T_{eff} + f_2 T_{\phi_1} + f_3 T_{\phi_2} + f_4 T_{\phi_1 \phi_2}, \quad (35)$$

$$J_0 = f_5 \partial \phi_1 + f_6 \partial \phi_2, \quad (36)$$

where the energy-momentum tensors $T_{\phi_1}$, $T_{\phi_2}$ and $T_{\phi_1 \phi_2}$ are given by

$$T_{\phi_1} = -\frac{1}{2} (\partial \phi_1)^2 - q_3 \partial^2 \phi_1, \quad (37)$$

$$T_{\phi_2} = -\frac{1}{2} (\partial \phi_2)^2 - q_4 \partial^2 \phi_2, \quad (38)$$

$$T_{\phi_1 \phi_2} = \partial \phi_1 \partial \phi_2. \quad (39)$$

For the linear basis $W_0$ with spin 4, the calculation shows that $W_0 \sim 0$. Under this case, the current $W$ of the $W_{2,4}$ algebra is constructed from the linear bases $T_0$ and $J_0$ only. Plugging these linear bases into the OPEs [13], we obtain two solutions, where the energy-momentum tensor $T_{eff}$ vanishes in both cases. These solutions are listed as follow:

- **Solution 1**
  $$f_1 = 0, \quad f_2 = f_3 = 1, \quad f_4 = 0, \quad f_5 = f_6 = \frac{\sqrt{2}}{2} h, \quad C_1 = i \sqrt{2} h, \quad C_0 = -4, \quad q_3 = q_4 = -\frac{\sqrt{2}}{2} h,$$

- **Solution 2**
  $$f_1 = 0, \quad f_2 = f_3 = \frac{1}{2}, \quad f_4 = -\frac{1}{2}, \quad f_5 = f_6 = \frac{\sqrt{2}}{2} h, \quad C_1 = \frac{5 \sqrt{3}}{2} h, \quad C_0 = -24, \quad q_3 = q_4 = -\frac{\sqrt{6}}{2} h,$$

where $h^2 = 1$. It is clear that $f_1 = 0$ in both solutions and this means the vanishing of the energy-momentum tensor $T_{eff}$. Therefore, $C_{eff} = 0$. The coefficient $f_4 = 0$ in Solution 1 implies that $T_{\phi_1 \phi_2}$ vanishes, however, it does not vanish in Solution 2. One may also note that the central charge $C_0$ in both solutions is negative, which is different from the case of the $W_{1,2,3}$ algebra. The background charges $q_3$ and $q_4$ of $T_{\phi_1}$ and $T_{\phi_2}$ are all imaginary number.

After constructing the explicit forms of the linear bases $T_0$ and $J_0$, we would like to substitute them into (40) and obtain new realizations for the $W_{2,4}$ algebra. The realization constructed from Solution 1 is

$$T = -\frac{1}{2} (\partial \phi_1)^2 - \frac{1}{2} (\partial \phi_2)^2 - \frac{ih}{2} \partial^2 \phi_1 - \frac{i h}{2} \partial^2 \phi_2, \quad (40)$$

$$W = b_1 (\partial \phi_1)^4 + b_2 (\partial \phi_1)^2 (\partial \phi_2)^2 + b_3 (\partial \phi_1)^2 \partial^2 \phi_2 + b_4 \partial^2 \phi_1 \partial^2 \phi_2 \partial \phi_2 + b_5 \partial \phi_1 \partial^2 \phi_2 + b_6 (\partial \phi_2)^4 + b_7 \partial^2 \phi_1 (\partial \phi_1)^2 + b_8 \partial^2 \phi_1 \partial \phi_1 \partial \phi_2 + b_9 \partial^2 \phi_2 (\partial \phi_2)^2 + b_{10} \partial^2 \phi_1 \partial^2 \phi_2 + b_{11} \partial^2 \phi_1 \partial \phi_2 + b_{12} \partial \phi_2 \partial^2 \phi_2 (\partial \phi_2)^2 + b_{13} \partial^2 \phi_2 (\partial \phi_2)^2 + b_{14} \partial^2 \phi_1 \partial \phi_1 \partial \phi_2 + b_{15} \partial^2 \phi_1 \partial^2 \phi_2 (\partial \phi_2)^2 + b_{16} \partial^2 \phi_2 \partial \phi_2 (\partial \phi_2)^2 + b_{17} \partial^2 \phi_1 \partial \phi_2 + b_{18} \partial^2 \phi_2. \quad (41)$$

These coefficients are

$$b_1 = -\frac{59}{3120 a} + \frac{a}{6420} \quad b_2 = -\frac{59}{1560 a} - \frac{a}{3120},$$

$$b_3 = \frac{a}{390 \sqrt{2}} + \frac{59 i h}{1560 a} \quad b_4 = \frac{a}{156 \sqrt{2}} - \frac{i h}{1560},$$

$$b_5 = \frac{3 a}{520} + \frac{59 i h}{780 \sqrt{2}} \quad b_6 = -\frac{59}{3120 a} + \frac{a}{6240},$$

$$b_7 = \frac{59 i h}{1560 a} - \frac{a}{156 \sqrt{2}} - \frac{i h}{1560},$$

$$b_9 = \frac{a}{390 \sqrt{2}} + \frac{59 i h}{1560 a},$$

$$b_{10} = \frac{149}{390 a} - \frac{19 a}{3120} + \frac{i h}{520 \sqrt{2}} + \frac{59}{3120 a},$$

$$b_{11} = -\frac{19 a}{1560} + \frac{i h}{260 \sqrt{2}} + \frac{59}{1560 a},$$

$$b_{12} = \frac{59 i h}{1560 a} - \frac{a}{3120},$$

$$b_{13} = \frac{149}{390 a} - \frac{19 a}{3120} + \frac{i h}{520 \sqrt{2}} + \frac{59}{3120 a},$$

$$b_{14} = \frac{149}{390 a} - \frac{3 a}{520} + \frac{i h}{780 \sqrt{2}},$$

$$b_{15} = -\frac{3 a}{520} + \frac{i h}{780 \sqrt{2}},$$

$$b_{16} = \frac{149}{390 a} - \frac{3 a}{520} + \frac{i h}{780 \sqrt{2}},$$

$$b_{17} = -\frac{3 a}{520} + \frac{i h}{780 \sqrt{2}} - \frac{i h}{149},$$

$$b_{18} = \frac{3 a}{520} - \frac{149 i h}{780 a},$$

where $a = \sqrt{\frac{451}{2}}$.

Solution 2 gives a realization of the linear $W_{1,2,4}$ algebra with central charge $C_0 = -24$, which is singular and could not be used to construct the $W_{2,4}$ algebra. But it is indeed a realization of the $W_{1,2,4}$ algebra.

V. DOUBLE-SPINOR REALIZATIONS FOR THE LINEAR $W_{1,2,3}$ ALGEBRAS AND THE $W_{2,s}$ ALGEBRAS

In this section, we would like to construct the bases of the linear $W_{1,2,s}$ algebras with double-spinor. The new realizations for the $W_{2,s}$ algebras can be obtained after a change of bases.

A. Realizations for the $W_{1,2,3}$ algebra and the $W_{2,3}$ algebra

A spinor field has spin $\frac{1}{2}$ in conformal field theory, and the OPE $\psi(z)\psi(\omega)$ is given by

$$\psi(z)\psi(\omega) \sim -\frac{1}{z-\omega}. \quad (42)$$

We denote two real spinor fields as $\psi_1$ and $\psi_2$, and they satisfy

$$\psi_i(z)\psi_j(\omega) \sim -\frac{\delta_{ij}}{z-\omega}, \quad (i, j = 1, 2). \quad (43)$$
Next, we would like to construct the explicit forms of the linear bases for the $W_{1,2,3}$ algebra. The most general forms of $T_0$, $J_0$ and $W_0$ can be expressed as

\begin{equation}
T_0 = T_{eff} + h_1 T_{\psi_1} + h_2 T_{\psi_2} + h_3 T_{\psi_1 \psi_2},
\end{equation}

\begin{equation}
J_0 = h_4 \psi_1 \psi_2,
\end{equation}

\begin{equation}
W_0 = h_5 \partial^2 \psi_1 \psi_1 + h_6 \partial^2 \psi_2 \psi_2 + h_7 \partial^2 \psi_1 \psi_2
+ h_8 \partial \psi_1 \partial \psi_2
+ h_{10} \partial T_{eff} + h_{11} T_{eff} \psi_1 \psi_2.
\end{equation}

The energy-momentum tensors $T_{\psi_1}$ and $T_{\psi_2}$ with spin 2 are constructed from $\psi_1$ and $\psi_2$ respectively, and $T_{\psi_1 \psi_2}$ is constructed from these two spinor fields. They are constructed as

\begin{equation}
T_{\psi_1} = \partial \psi_1 \psi_1,
\end{equation}

\begin{equation}
T_{\psi_2} = \partial \psi_2 \psi_2,
\end{equation}

\begin{equation}
T_{\psi_1 \psi_2} = \partial \psi_1 \psi_2 + \psi_1 \partial \psi_2.
\end{equation}

Plugging these linear bases into the OPE relations \ref{13}, we obtain the result:

\begin{align*}
h_1 &= h_2 = -\frac{1}{2}, \quad h_3 = 0, \quad h_4 = 1, \\
h_i &= 0 \quad (i = 5 - 11), \quad C_1 = 0, \quad C_0 = 2, \quad C_{eff} = 1.
\end{align*}

In this solution, it can be seen that $T_{\psi_1 \psi_2}$ and $W_0$ vanish. The energy-momentum tensor $T_{eff}$ contributes central charge 1.

After constructing the explicit forms of the linear bases $T_0$, $J_0$ and $W_0$, we substitute them into \ref{13} and obtain a realization of the $W_{2,3}$ algebra as follows:

\begin{align*}
T &= T_{eff} - \frac{1}{2} \partial \psi_1 \psi_1 - \frac{1}{2} \partial \psi_2 \psi_2, \\
W &= \frac{i}{4} T_{eff} \psi_1 \psi_1 + \frac{7i}{8} \psi_1 \partial \psi_2 \psi_2 + \frac{\sqrt{6i}}{16} \partial^2 \psi_1 \psi_1
+ \frac{7i}{8} \partial^2 \psi_1 \psi_2 + \frac{\sqrt{6i}}{16} \partial^2 \psi_2 \psi_2 - \frac{\sqrt{6i}}{8} \partial T_{eff}.
\end{align*}

It is worth remarking that the currents $T$ and $W$ above generate $W_{2,3}$ algebra with central charge $C = 2$.

**B. Realizations for the $W_{1,2,4}$ algebra and the $W_{2,4}$ algebra**

For the linear $W_{1,2,4}$ algebra, the bases take the following form

\begin{align*}
T_0 &= T_{eff} + k_1 T_{\psi_1} + k_2 T_{\psi_2} + k_3 T_{\psi_1 \psi_2}, \\
J_0 &= k_4 \psi_1 \psi_2, \\
W_0 &= 0,
\end{align*}

where $T_{\psi_1}$, $T_{\psi_2}$ and $T_{\psi_1 \psi_2}$ are given by \ref{15}–\ref{17}. This case give a precise Romans realization of the $W_{1,2,4}$ algebra, where the basis $W_0$ is set to zero.

Plugging these linear bases into the OPE relations \ref{13}, we obtain two solutions:

- **Solution 1**
  \begin{equation}
  k_1 = k_2 = -\frac{1}{2}, \quad k_3 = -\bar{C}_3, \quad k_4 = 1 - 1, \quad C_{eff} = \frac{1}{24} (13C_1^2 + 5C_1 \sqrt{C_1^2 - 48}),
  \end{equation}
  \begin{equation}
  C_0 = 1 + \frac{1}{24} (85C_1^2 + 5C_1 \sqrt{C_1^2 - 48}).
  \end{equation}

- **Solution 2**
  \begin{equation}
  k_1 = k_2 = -\frac{1}{2}, \quad k_3 = \bar{C}_1, \quad k_4 = 1, \quad C_{eff} = \frac{1}{24} (13C_1^2 + 5C_1 \sqrt{C_1^2 - 48}),
  \end{equation}
  \begin{equation}
  C_0 = 1 + \frac{1}{24} (85C_1^2 + 5C_1 \sqrt{C_1^2 - 48}).
  \end{equation}

The central charges $C_0$ of both solutions depend on $C_1$, and this gives realizations of the linear $W_{1,2,4}$ algebra at arbitrary central charge. Then two new realizations of the $W_{2,4}$ algebra from these solutions can be obtained immediately. The first one is given by

\begin{align*}
T &= T_{eff} - \frac{1}{2} \partial \psi_1 \psi_1 - \frac{1}{2} \partial \psi_2 \psi_2
- \frac{C_1}{2} \partial \psi_1 \psi_2 - \frac{C_1}{2} \psi_1 \partial \psi_2, \\
W &= -\frac{1}{1560a} \left( -2(3a^2 - 59C_1)T_{eff} \psi_1 \partial \psi_2 \\
&+ 118T_{eff} \psi_1 \partial \psi_1 - 2(3a^2 - 59C_1)T_{eff} \partial \psi_1 \psi_2 \\
&+ 118C_1 T_{eff} \partial \psi_2 \psi_2 - 9(a^2 - 32C_1) \partial \psi_3 \psi_1 \\
&- 4a^2 \partial T_{eff} \psi_1 \psi_2 + 298a^2 \partial^2 \psi_2 \psi_2 \\
&- (59 + 2a^2 - 2a^2 C_1^2 + 59C_1^2) \partial \psi_1 \partial \psi_2 \psi_2 \\
&- (27a^2 - 894C_1) \partial \psi_1 \partial \psi_1 \partial \psi_2 \\
&+ 298a^2 \partial \psi_1 \partial \psi_1 - (27a^2 - 894C_1) \partial \psi_1 \partial \psi_2 \\
&+ 298 \partial \psi_1 \psi_1 + (9a^2 - 298C_1) \partial \psi_1 \psi_2 \\
&+ 298 \partial \psi_2 \psi_2 - 596 \partial^2 T_{eff} - 118T_{eff} \right).
\end{align*}

The second is

\begin{align*}
T &= T_{eff} - \frac{1}{2} \partial \psi_1 \psi_1 - \frac{1}{2} \partial \psi_2 \psi_2 \\
&- \frac{C_1}{2} \partial \psi_1 \psi_2 - \frac{C_1}{2} \psi_1 \partial \psi_2, \\
W &= -\frac{1}{1560a} \left( +2(3a^2 - 59C_1)T_{eff} \psi_1 \partial \psi_2 \\
&+ 118T_{eff} \partial \psi_1 \psi_1 + 2(3a^2 - 59C_1)T_{eff} \partial \psi_1 \psi_2 \\
&+ 118C_1 T_{eff} \partial \psi_2 \psi_2 + 9(a^2 - 32C_1) \partial \psi_3 \psi_1 \\
&- 4a^2 \partial T_{eff} \psi_1 \psi_2 + 298a^2 \partial^2 \psi_2 \psi_2 \\
&- (59 + 2a^2 - 2a^2 C_1^2 + 59C_1^2) \partial \psi_1 \partial \psi_2 \psi_2 \\
&+ (27a^2 - 894C_1) \partial \psi_1 \partial \psi_1 \partial \psi_2 \\
&+ 298 \partial \psi_1 \partial \psi_1 + (9a^2 - 298C_1) \partial \psi_1 \partial \psi_2 \\
&+ 298 \partial \psi_2 \psi_2 - 596 \partial^2 T_{eff} - 118T_{eff} \right).
\end{align*}

Different from the cases of scalar realizations, the results here give two spinor realizations of the $W_{2,4}$ algebra for an arbitrary central charge.
VI. CONCLUSION

In this paper, we obtained the explicit field realizations of the linear $W_{1,2,s}$ algebras and the non-linear $W_{2,s}$ algebras with double-scalar and double-spinor, respectively. Owing to the intrinsic nonlinearity of $W_{2,s}$ algebras, it is hard to construct their field realizations. However, it is proved that the non-linear $W_{2,s}$ algebras are contained in the linear $W_{1,2,s}$ algebras with three currents $J_0$, $T_0$ and $W_0$ as a subalgebra. With this fact, we first constructed the linear bases of the $W_{1,2,s}$ algebras. Then making a change of basis, we obtained several explicit field realizations of the non-linear $W_{2,s}$ algebras. All these results imply a symmetry under $\phi_1 \leftrightarrow \phi_2$ or $\psi_1 \leftrightarrow \psi_2$. This method overcomes the difficulty of realizing a non-linear algebra.

The spin-$s$ current $W_0$ of $W_{1,2,s}$ algebras was considered to be a null current and can be set to zero, then the realizations of $W_{2,s}$ algebras obtained from linear $W_{1,2,s}$ algebras are called Romans-type realizations. In fact, it is not necessary to set the current $W_0$ to zero. In our constructions, we first listed the most general forms of linear bases $J_0$, $T_0$ and $W_0$ with correct spin. Plugging these forms into the OPEs \[ [W_0, W_0] = \frac{2}{s} W_0, \]
we found that all the coefficients of $W_0$ vanished for the non-zero constant $\xi$. These results suggest that there exist no non-Romans-type realization of $W_{2,s}$ algebra if we use double-scalar or double-spinor only. However, we expect that there exist non-Romans-type realizations of $W_{2,s}$ algebras at some value of central charge and more details would be investigated in our future work.

We can also see that all these realizations satisfy $C = C_0$, i.e., the central charge of the $W_{2,s}$ algebras are equal to the central charge of the $W_{1,2,s}$ algebras, which is caused by the assumption $T = T_0$. The central charge $C$ takes some special values for the double-scalar realizations and the double-spinor realizations of the $W_{2,3}$ algebras, while it depends on the value of $C_1$ for the double-spinor realizations of the $W_{2,4}$ algebra. We also showed that there is no such realization for the $W_{2,3}$ algebra at central charge $C = -\frac{44}{7}$ and for the $W_{2,4}$ algebra at $C = -24, \frac{1}{2}, -\frac{22}{5},$ and $-\frac{68}{7}$ since the $W_{2,s}$ algebras are singular at these values of central charge.

Acknowledgements

This work was supported by Program for New Century Excellent Talents in University, the National Natural Science Foundation of China (No. 10705013), the Doctoral Program Foundation of Institutions of Higher Education of China (No. 20070730055), the Key Project of Chinese Ministry of Education (No. 109153) and the Fundamental Research Fund for Physics and Mathematics of Lanzhou University (No. Lzu07002). L.J. Z acknowledges financial support from Innovation Foundation of Shanghai University.
the vertex operator algebra \( L(1/2, 0) \otimes L(1/2, 0), \) arXiv:0711.4624 [math.QA].

[24] V.G. Gorbounov, A.P. Isaev and O.V. Ogievetsky, Theor. Math. Phys. 139 (2004) 473, arXiv:0711.4133 [math.QA]; A.P. Isaev, S.O. Krivonosa and O.V. Ogievetsky, J. Math. Phys. 49 (2008) 073512, arXiv:0802.3781 [math-ph].

[25] A. Bilal and J.L. Gervais, Nucl. Phys B 314, (1987) 646.

[26] L.J. Romans, Nucl. Phys. B 352, (1991) 829.

[27] J. Thierry-Mieg, Phys. Lett. B 197 (1987) 368.

[28] F.A. Bais, P. Bouwknegt, M. Surrige and K. Schoutens, Nucl. Phys. B 304 (1988) 348.

[29] H. Lu, C.N. Pope, S. Schrans and X.J. Wang, Nucl. Phys. B 403 (1993) 351, arXiv:hep-th/9212117.

[30] H. Lu, C.N. Pope, X.J. Wang and K.W. Xu, Class. Quant. Grav. 11 (1994) 967, arXiv:hep-th/9309041.

[31] H. Lu, C.N. Pope and X.J. Wang, Int. J. Mod. Phys. A 9 (1994) 1527, arXiv:hep-th/9304115.

[32] H. Lu, C.N. Pope, X.J. Wang and S.C. Zhao, Class. Quant. Grav. 11 (1994) 939, arXiv:hep-th/9311084.

[33] H. Lu, C.N. Pope, X.J. Wang and S.C. Zhao, Phys. Lett. B 327 (1994) 241, arXiv:hep-th/9402133.

[34] M. Bershadsky, W. Lerche, D. Nemeschansky and N.P. Warner, Phys. Lett. B 292 (1992) 35, arXiv:hep-th/9207067.

[35] E. Bergshoeff, H.J. Boonstra, S. Panda and A. Sevrin, Phys. Lett. B 308 (1993) 34, arXiv:hep-th/9303051.

[36] S.C. Zhao and H. Wei, Phys. Lett. B 486 (2000) 212; S.C. Zhao, H. Wei and L.J. Zhang, Phys. Lett. B 499 (2001) 200; S. Zhao, L. Zhang and H. Wei, Phys. Rev. D 64 (2001) 046010; S.C. Zhao, L.J. Zhang and Y.X. Liu, Commun. Theor. Phys. 41 (2004) 235, arXiv:hep-th/0508114.

[37] Y.S. Duan, Y.X. Liu and L.J. Zhang, Nucl. Phys. B 699 (2004) 174, arXiv:hep-th/0508115.

[38] Y.X. Liu, L.J. Zhang and J.R. Ren, JHEP 0501 (2005) 005, arXiv:hep-th/0507234.

[39] L.J. Zhang and Y.X. Liu, Commun. Theor. Phys. 46 (2006) 675, arXiv:hep-th/0602205.

[40] S. Bellucci, S. Krivonos and A. Sorin, Mod. Phys. Lett. A 10 (1995) 1857, arXiv:hep-th/9406005.

[41] H. Lu, C.N. Pope and K.W. Xu, Phys. Lett. B 351 (1995) 179, arXiv:hep-th/9502108.

[42] J.O. Madsen and E. Ragoucy, Linearization of W algebras and W superalgebras, arXiv:hep-th/9510061.

[43] H. Lu, C.N. Pope and K.W. Xu, Phys. Lett. B 358 (1995) 239, arXiv:hep-th/9503158.

[44] Y.-X. Liu, S.-W. Wei, L.-J. Zhang and J.-R. Ren, Eur. Phys. J. C 60 (2009) 975, arXiv:0809.2462 [hep-th].

[45] H.G. Kausch and G.M.T. Watts, Nucl. Phys. B 354 (1991) 740.

[46] E. Bergshoeff, H.J. Boonstra and M. de Roo, Phys. Lett. B 346 (1995) 269, arXiv:hep-th/9406186.

[47] S.-W. Wei, Y.-X. Liu, L.-J. Zhang and J.-R. Ren, Nucl. Phys. B 809 (2009) 426, arXiv:0806.2553 [hep-th].