Higher order stability of dust ion acoustic solitary wave solution described by the KP equation in a collisionless unmagnetized nonthermal plasma in presence of isothermal positrons

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Sardar et al. [Phys. Plasmas 23, 073703 (2016)] have studied the stability of small amplitude dust ion acoustic solitary waves in a collisionless unmagnetized electron - positron - ion - dust plasma. They have derived a Kadomtsev Petviashvili (KP) equation to investigate the lowest - order stability of the solitary wave solution of the Korteweg-de Vries (KdV) equation for long-wavelength plane-wave transverse perturbation when the weak dependence of the spatial coordinates perpendicular to the direction of propagation of the wave is taken into account. In the present paper, we have extended the lowest - order stability analysis of KdV solitons given in the paper of Sardar et al. [Phys. Plasmas 23, 073703 (2016)] to higher order with the help of multiple-scale perturbation expansion method of Allen and Rowlands [J. Plasma Phys. 50, 413 (1993); 53, 63 (1995)]. It is found that solitary wave solution of the KdV equation is stable at the order $k^2$, where $k$ is the wave number for long-wavelength plane-wave perturbation.

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I. INTRODUCTION

The small-$k$ perturbation expansion method of Rowlands and Infeld\textsuperscript{1-5} is generally used to analyse the lowest order stability of solitary wave solutions of different nonlinear evolution equations in plasmas, where $k$ is the wave number for long-wavelength plane-wave perturbation. Several authors\textsuperscript{6-22} have used this method to investigate the lowest order stability of solitary waves in plasmas with or without magnetic field.

The small-$k$ perturbation expansion method of Rowlands and Infeld\textsuperscript{1-5} fails to study the higher order stability of solitary waves in plasmas. This method also fails to investigate the lowest order stability of the double layers in plasmas. Allen and Rowlands\textsuperscript{23} developed a method to analyse the higher order stability of solitary wave solution of the Zakharov-Kuznetsov (ZK) equation. Using this method, Allen and Rowlands\textsuperscript{24} have derived higher order growth rate of instability for obliquely propagating solitary wave solution of the ion acoustic waves in a magnetized plasma. Several authors\textsuperscript{25-31} have used this method to analyse the higher order stability of solitary wave solutions of the different evolution equations. Bandyopadhyay & Das\textsuperscript{25} have used this method of Allen and Rowlands\textsuperscript{23,24} to find the higher order (i.e., of order $k^2$) growth rate of instability of solitary wave solutions of Korteweg-de Vries-Zakharov-Kuznetsov (KdV-ZK) equation and modified KdV-ZK (MKdV-ZK) equation. Later, Parkes and Munro\textsuperscript{29} have pointed out the error appearing in the growth rate of instability up to the order $k^2$ of Bandyopadhyay & Das\textsuperscript{25}. Using the same method, Bandyopadhyay & Das\textsuperscript{22} have investigated the lowest order stability of the double layer solution of the combined MKdV - KdV - ZK equation. Das \textit{et al.}\textsuperscript{31-35} have used the same multiple scale perturbation expansion method to study the lowest order stability of solitary wave solutions of the complicated evolution equations. Das \textit{et al.}\textsuperscript{31} have used this method of Allen and Rowlands\textsuperscript{23,24} to investigate the higher order stability of solitary wave solution of the Schamel’s modified Korteweg-de Vries - Zakharov-Kuznetsov (SKdV - ZK) equation. Higher order stability analysis of the solitary wave solution of the Schamel’s modified Kadomtsev Petviashvili (SKP) equation has been discussed by Chakraborty and Das\textsuperscript{28}. In a later paper, Tian-Jun\textsuperscript{30} also investigated the higher order stability of the solitary wave solution of the SKP equation with positive and negative dispersion. In the present paper, we have used the same multiple-scale perturbation expansion method of Allen and Rowlands\textsuperscript{23,24} to investigate the higher order stability of solitary wave solution of the Kadomtsev Petviashvili
(KP) equation. For this purpose we have considered the KP equation derived by Sardar et al.\textsuperscript{20} and this KP equation describes the nonlinear behaviour of the dust ion acoustic (DIA) waves in electron - positron - ion - dust (e-p-i-d) plasma.

Sardar et al.\textsuperscript{20–22} have used the small-\(k\) perturbation expansion method of Rowlands and Infeld\textsuperscript{1–5} to investigate the lowest order stability of DIA solitary wave solutions of different nonlinear evolution equations describing the nonlinear behaviour of DIA waves in a collisionless unmagnetized e-p-i-d plasma consisting of warm adiabatic ions, static negatively charged dust grains, nonthermal electrons and isothermal positrons. In particular, Sardar et al.\textsuperscript{20} have investigated the stability of the solitary wave solutions of the KdV and different modified KdV equations with the help of KP and different modified KP equations describing the nonlinear behaviour of DIA waves in different region of parameter space when the weak dependence of the spatial coordinates perpendicular to the direction of propagation of the wave is taken into account. In the present paper, we have extended the lowest order stability analysis of KdV solitons given in the paper of Sardar et al.\textsuperscript{20} to higher order with the help of multiple-scale perturbation expansion method of Allen and Rowlands.\textsuperscript{23,24}

Starting from the set of basic equations consisting of the equation of continuity of ions, equation of motion of ions, the pressure equation for ion fluid, the Poisson equation, the equation for the number density of the nonthermal electrons of Cairns et al.\textsuperscript{36}, the equation for the number density of isothermal positrons and the unperturbed charged neutrality condition, Sardar et al.\textsuperscript{20} have derived the following three-dimensional Kadomtsev Petviashvili (KP) equation:

\[
\frac{\partial}{\partial \xi} \left[ \phi_\tau^{(1)} + AB_1 \phi_\xi^{(1)} + \frac{1}{2} AC \phi_\xi^{(1)} \phi_\xi^{(1)} \right] + \frac{1}{2} AD \left( \phi_\eta^{(1)} + \phi_\zeta^{(1)} \right) = 0.
\] (1)

Here \(\phi^{(1)}\) is the first order perturbed electrostatic potential, \(\xi, \eta, \zeta\) are the stretched spatial coordinates and \(\tau\) is the stretched time coordinate and we have used the following notations:

\[
\phi_\tau^{(1)} = \frac{\partial \phi^{(1)}}{\partial \tau}, \phi_\xi^{(1)} = \frac{\partial \phi^{(1)}}{\partial \xi}, \phi_\xi^{(1)} = \frac{\partial^3 \phi^{(1)}}{\partial \xi^3}, \phi_\eta^{(1)} = \frac{\partial^2 \phi^{(1)}}{\partial \eta^2}, \phi_\zeta^{(1)} = \frac{\partial^2 \phi^{(1)}}{\partial \zeta^2}.
\]

The coefficients \(A, B_1, C\) and \(D\) are, respectively, given in the equations (17), (18), (5) and (19) of Sardar et al.\textsuperscript{20} They have used the appropriate stretchings of the independent variables and the appropriate perturbation expansions of the dependent variables to derive the three-dimensional KP equation (1). This KP equation admits the following solitary wave solution:

\[
\phi^{(1)} = \phi_0(X) = \text{asech}^2 \left( \frac{X}{W} \right), \quad (2)
\]
where \( X = \xi - U\tau \) and \( U \) is the dimensionless velocity (normalized by \( C_D, C_D = \) linearized velocity of the DIA wave for long wave length plane wave perturbation of the present plasma system) of the travelling wave moving along \( x \)-axis, i.e., \( U \) is the dimensionless velocity of the wave frame.

The amplitude \((a)\) and the width \((W)\) of the solitary wave solution \((2)\) are given by

\[
a = \frac{3U}{AB_1} \quad \text{and} \quad W^2 = \frac{2AC}{U}. \tag{3}
\]

The solution \((2)\) is the steady state solution of the KP equation \((1)\) along the \(x\)-axis. This solution is same as the solitary wave solution of the KdV equation corresponding to the KP equation \((1)\), i.e., the steady state solitary wave solution of the KdV equation

\[
\phi^{(1)}_{\tau} + AB_1\phi^{(1)}_{\xi} + \frac{1}{2}AC\phi^{(1)}_{\xi\xi\xi} = 0 \tag{4}
\]

is exactly same as the equation \((2)\). Sardar et al.\textsuperscript{20} have studied the lowest order transverse stability of the solitary wave solution of the KdV equation \((4)\) using the three-dimensional KP equation \((1)\). In the present paper, our aim is to study the higher order transverse stability of the solitary wave solution of the KdV equation \((4)\) using the three-dimensional KP equation \((1)\).

II. STABILITY ANALYSIS

To analyze the stability of the solitary wave solution \((2)\) of the KP equation \((1)\), we use following transformation of the independent variables:

\[
X = \xi - U\tau, \eta' = \eta, \zeta' = \zeta, \tau' = \tau. \tag{5}
\]

Using the transformation \((5)\), the KP equation \((1)\) can be written as

\[
\frac{\partial}{\partial X} \left[ -U\phi_X^{(1)} + \phi^{(1)}_{\tau} + AB_1\phi^{(1)}_{\xi} + \frac{1}{2}AC\phi^{(1)}_{XX} \right] + \frac{1}{2}AD\left(\phi^{(1)}_{\eta\eta} + \phi^{(1)}_{\zeta\zeta}\right) = 0, \tag{6}
\]

where we drop the primes on the independent variables \(\eta, \zeta\) and \(\tau\) to simplify the notations.

It is simple to check that the equation \((6)\) is satisfied by the expression of \(\phi^{(1)}(\xi = \phi_0(X))\) as given in \((2)\). It is also important to note that \(\phi^{(1)}_{\tau} = \frac{\partial\phi^{(1)}}{\partial\tau} = 0\) for the expression of \(\phi^{(1)}(\xi = \phi_0(X))\) as given in \((2)\) and so this solution is the steady state solution of the KP equation \((1)\) or the corresponding KdV equation \((4)\).
As $\phi_0(X)$ is a steady state solution of (6), we can decompose $\phi^{(1)}$ as

$$\phi^{(1)} = \phi_0(X) + q(X, \eta, \zeta, \tau),$$

(7)

where $q(X, \eta, \zeta, \tau)$ is the perturbed part of $\phi^{(1)}$.

Substituting (7) into (6) and then linearizing it with respect to $q$, we get the following linear equation for $q$:

$$\frac{\partial}{\partial X} \left[ -Uq_X + q_r + AB_1(\phi_0 q)_X + \frac{1}{2} AC q_{XXX} \right] + \frac{1}{2} AD \left( q_m + q_\zeta \right) = 0,$$

(8)

where we have used the equation (2) to simplify the above equation and and we have used the following notations: $q_r = \frac{\partial q}{\partial \tau}$, $q_X = \frac{\partial q}{\partial X}$, $q_{XXX} = \frac{\partial^3 q}{\partial X^3}$, $(\phi_0 q)_X = \frac{\partial}{\partial X} (\phi_0 q)$.

Now, for long wave length plane wave perturbation along a direction having direction cosines $l, m, n$, we take

$$q(X, \eta, \zeta, \tau) = \overline{q}(X) e^{i\left(k(lX + m\eta + n\zeta) - \omega \tau\right)},$$

(9)

where $k$ is small and $l^2 + m^2 + n^2 = 1$.

Due to the space time dependence of $q$ for long wave length plane wave perturbation as described in equation (9), the equation (8) of $q$ transforms to the following equation of $\overline{q}$:

$$(M_1 \overline{q})_{XX} - i\omega \overline{q}_X + kl \left\{ \omega \overline{q} + 2i(M_1 \overline{q})_X + iAC \overline{q}_{XXX} \right\}$$

$$-k^2 l^2 \left\{ M_1 \overline{q} + \frac{5}{2} AC \overline{q}_{XX} + \frac{1}{2} AD \frac{m^2 + n^2}{l^2} \overline{q} \right\}$$

$$-k^3 l^3 \left\{ 2iAC \overline{q}_X \right\} + k^4 l^4 \left\{ \frac{1}{2} AC \overline{q} \right\} = 0,$$

(10)

where

$$M_1 = -U + AB_1 \phi_0 + \frac{1}{2} AC \frac{\partial^2}{\partial X^2}.$$

(11)

Following the multiple-scale perturbation expansion method of Allen and Rowlands,23,24 we expand $\overline{q}(X)$ and $\omega$ as

$$\overline{q}(X) = \sum_{j=0}^{\infty} k^j q^{(j)}(X, X_1, X_2, X_3, ...),$$

(12)

$$\omega = \sum_{j=0}^{\infty} k^j \omega^{(j)},$$

(13)
where \( \omega^{(0)} = 0, \) \( X_j = k^j X, \) \( j = 0, 1, 2, 3, \ldots, \) and each \( q^{(j)}(= q^{(j)}(X, X_1, X_2, X_3, \ldots)) \) is a function of \( X, X_1, X_2, X_3, \ldots. \) It is important to note that \( X_0 = X. \)

Finally, substituting (12) and (13) into the equation (10) and then equating the coefficients of different powers of \( k \) on the both sides of the resulting equation, we get the following sequence of equations:

\[
\frac{\partial}{\partial X}(M_1 q^{(j)}) = Q^{(j)}, \tag{14}
\]

where

\[
Q^{(j)} = \int_{\infty}^{X} R^{(j)} dX, \tag{15}
\]

and \( R^{(j)} \) for \( j = 0, 1, 2, 3 \) are given in Appendix A.

Following Das et al., the general solution of (14) can be written in the following form:

\[
q^{(j)} = A_1^{(j)} f + A_2^{(j)} g + A_3^{(j)} h + \chi^{(j)}, \tag{16}
\]

where \( A_1^{(j)}, A_2^{(j)}, A_3^{(j)} \) are all arbitrary functions of \( X_1, X_2, X_3, \ldots, \) and \( f, g, h, \) and, \( \chi^{(j)} \) are given by

\[
f = \frac{d\phi_0}{dX}, \quad g = f \int \frac{1}{f^2} dX, \quad h = f \int \frac{\phi_0}{f^2} dX, \tag{17}
\]

\[
\chi^{(j)} = \frac{2}{AC} f \int \frac{\left( f \int Q^{(j)} dX \right) dX}{f^2} dX. \tag{18}
\]

Using (17) and (18), we can write equation (16) in the following form:

\[
q^{(j)} = A_1^{(j)} f - \frac{W^2}{8a} A_2^{(j)} S^{-2} - \frac{W^2}{16a} \left( 5A_2^{(j)} + 4aA_3^{(j)} \right) + \frac{3W^2}{16a^2} \left( 5A_2^{(j)} + 4aA_3^{(j)} \right) \phi_0 \\
+ \frac{3W^2}{32a^2} \left( 5A_2^{(j)} + 4aA_3^{(j)} \right) f X + \frac{2}{AC} f \int \frac{\left( f \int Q^{(j)} dX \right) dX}{f^2} dX. \tag{19}
\]

Here \( S = \text{sech}\left[ \frac{X}{W} \right] \) and \( \phi_0 \) is given by (2).

### A. Zeroth order equation

The solution (19) of the equation (14) for \( j = 0 \) can be written as

\[
q^{(0)} = A_1^{(0)} f - \frac{W^2}{8a} A_2^{(0)} S^{-2} - \frac{W^2}{16a} \left( 5A_2^{(0)} + 4aA_3^{(0)} \right) \\
+ \frac{3W^2}{16a^2} \left( 5A_2^{(0)} + 4aA_3^{(0)} \right) \phi_0 + \frac{3W^2}{32a^2} \left( 5A_2^{(0)} + 4aA_3^{(0)} \right) f X, \tag{20}
\]
where we have used the equation (19) of Appendix A and the equation (15) to find \( Q^{(0)} \).

Now it is simple to check that \( S^{-2} \to +\infty \) as \( X \to +\infty \). Therefore, to make \( q^{(0)} \) bounded we must have

\[
-\frac{W^2}{8a} A_2^{(0)} = 0 \iff A_2^{(0)} = 0.
\]  

(21)

Using (21), the equation (20) can be written in the following form:

\[
q^{(0)} = A_1^{(0)} f - \frac{W^2}{4} A_3^{(0)} + \frac{3W^2}{4a} A_3^{(0)} \phi_0 + \frac{3W^2}{8a} A_3^{(0)} fX.
\]  

(22)

Again, \( q^{(0)} \) is consistent at \( X = +\infty \), i.e., \( q^{(0)} \to 0 \) as \( X \to +\infty \), if we choose

\[
-\frac{W^2}{4} A_3^{(0)} = 0 \iff A_3^{(0)} = 0.
\]  

(23)

Therefore, the equation (22) takes the following form:

\[
q^{(0)} = A_1^{(0)} f.
\]  

(24)

B. First order equation

Using the equation (50) of Appendix A and (24), the solution (19) of the differential equation (14) for \( j = 1 \) can be put in the following form:

\[
q^{(1)} = A_1^{(1)} f - \frac{W^2}{8a} A_2^{(1)} S^{-2} - \frac{W^2}{16a} \left( 5A_2^{(1)} + 4aA_3^{(1)} \right)
+ \left\{ \frac{3W^2}{16a^2} \left( 5A_2^{(1)} + 4aA_3^{(1)} \right) + iA_1^{(0)} \frac{\omega^{(1)}}{U} \right\} \phi_0
+ \left\{ -\frac{\partial A_1^{(0)}}{\partial X_1} + iA_1^{(0)} \frac{\omega^{(1)}}{U} - \frac{2iU}{2U} + \frac{3W^2}{32a^2} \left( 5A_2^{(1)} + 4aA_3^{(1)} \right) \right\} fX,
\]  

(25)

where we have used MATHEMATICA to compute all the integrals of right hand side of (19) for \( j = 1 \).

Now \( q^{(1)} \) can be made bounded and consistent at \( X = +\infty \) if and only if \( A_2^{(1)} = A_3^{(1)} = 0 \) and consequently the equation (25) can be written in the form:

\[
q^{(1)} = A_1^{(1)} f + iA_1^{(0)} \frac{\omega^{(1)}}{U} \phi_0 - \left\{ \frac{\partial A_1^{(0)}}{\partial X_1} - iA_1^{(0)} \frac{\omega^{(1)}}{U} - \frac{2iU}{2U} \right\} fX,
\]  

(26)

where we have used exactly the same argument as given in the lowest order equation to get the bounded and consistent solution \( q^{(0)} \).

Following Allen and Rowlands, the first term on the right hand side of (26) can be removed because this type of term has already been included in \( q^{(0)} \). Again, according to
the prescription of Allen and Rowlands, the last term on the right hand side of (26) is a ghost secular term and this term can be removed from the equation (26) if we choose
\[ \frac{\partial A^{(0)}_1}{\partial X_1} = iA^{(0)}_1 \frac{\omega^{(1)}}{2U}. \] (27)

Therefore, the equation (26) can be put in the following form:
\[ q^{(1)} = iA^{(0)}_1 \frac{\omega^{(1)}}{U} \phi_0. \] (28)

C. Second order equation

Using the equation (51) of Appendix A, (24) and (28), the solution (19) of the equation (14) for \( j = 2 \) can be written as
\[ q^{(2)} = -\frac{W^2}{8a}A^{(2)}_2 S^{-2} - \frac{W^2}{16a} \left( 5A^{(2)}_2 + 4aA^{(2)}_3 \right) - aA^{(0)}_1 \frac{W}{4U^2}(\omega^{(1)})^2 R \\
- aA^{(0)}_1 \frac{W}{12U^2} \left\{ 3(\omega^{(1)})^2 - 2UV(m^2 + n^2) \right\} RS^{-2} \\
+ \left\{ \frac{3W^2}{16a^2} \left( 5A^{(2)}_2 + 4aA^{(2)}_3 \right) + iA^{(0)}_1 \frac{\omega^{(2)}}{U} \right\} \phi_0 \\
+ \left\{ \frac{3W^2}{32a^2} \left( 5A^{(2)}_2 + 4aA^{(2)}_3 \right) + iA^{(0)}_1 \frac{\omega^{(2)}}{2U} - \frac{\partial A^{(0)}_1}{\partial X_2} \right\} fX, \] (29)

where \( R = \tanh\left( \frac{X}{W} \right) \), we have used MATHEMATICA to find all the integrals of (19) for \( j = 2 \) and we have ignored the terms that have been included in the zeroth and first order solutions \( q^{(0)} \) and \( q^{(1)} \).

The above expression of \( q^{(2)} \) shows that \( q^{(2)} \) is not bounded because of the presence of the exponential secular terms \( \frac{1}{S^2} \) and \( \frac{R}{S^2} \). So, it is important to remove both the exponential secular terms simultaneously. Now, to remove both the exponential secular terms simultaneously, it is necessary to make a common factor of the coefficients of these two terms equal to zero. As \( W \neq 0 \), to get a common factor between the coefficients of the two exponential secular terms in the expression of \( q^{(2)} \), we write the arbitrary function \( A^{(2)}_2 \) in the following form:
\[ A^{(2)}_2 = B^{(2)}_2 \left\{ 3(\omega^{(1)})^2 - 2UV(m^2 + n^2) \right\}, \] (30)
and consequently $q^{(2)}$ can be written as

$$
q^{(2)} = -\frac{W^2}{16a} \left( 5A_2^{(2)} + 4aA_3^{(2)} \right) - aA_1^{(0)} \frac{W}{4U^2} (\omega^{(1)})^2 R \\
- \left\{ 3(\omega^{(1)})^2 - 2UV(m^2 + n^2) \right\} \left\{ B_2^{(2)} \frac{W^2}{8a} \frac{1}{S^2} + A_1^{(0)} \frac{aW}{12U^2} \frac{R}{S^2} \right\} \\
+ \left\{ \frac{3W^2}{16a^2} \left( 5A_2^{(2)} + 4aA_3^{(2)} \right) + iA_1^{(0)} \frac{\omega^{(2)}}{U} \right\} \phi_0 \\
+ \left\{ \frac{3W^2}{32a^2} \left( 5A_2^{(2)} + 4aA_3^{(2)} \right) + iA_1^{(0)} \frac{\omega^{(2)}}{2U} - \frac{\partial A_1^{(0)}}{\partial X_2} \right\} fX,
$$

(31)

where $B_2^{(2)}$ is another arbitrary functions of $X_1, X_2, X_3, \ldots$.

From the expression of $q^{(2)}$ as given in (31), we find that exponential secularity in the resulting expression of $q^{(2)}$ is due to the presence of the terms $S^{-2}$ and $RS^{-2}$. Therefore, we can remove the exponential secular terms from the expression of $q^{(2)}$ by setting

$$
3(\omega^{(1)})^2 - 2UV(m^2 + n^2) = 0.
$$

(32)

This equation gives the following expression of $(\omega^{(1)})^2$:

$$(\omega^{(1)})^2 = \frac{2}{3} UV(m^2 + n^2).$$

(33)

The above equation is exactly same as the equation (47) of Sardar et al. for $r = 1$. But they have used the small-$k$ perturbation expansion method of Rowlands and Infeld to investigate the lowest order stability of KP and different modified KP equations.

From equations (30) and (32), we get $A_2^{(2)} = 0$. To make $q^{(2)}$ consistent with the condition that $q^{(2)} \to 0$ as $X \to +\infty$, we have

$$
A_3^{(2)} = -aA_1^{(0)} \frac{(\omega^{(1)})^2}{WU^2}
$$

(34)

and consequently $q^{(2)}$ can be written as

$$
q^{(2)} = A_1^{(0)} \left\{ a \frac{W}{4U^2} (\omega^{(1)})^2 (1 - R) + t_1 \phi_0 \right\} + \left\{ \frac{1}{2} A_1^{(0)} t_1 - \frac{\partial A_1^{(0)}}{\partial X_2} \right\} fX,
$$

(35)

where

$$
t_1 = \frac{i\omega^{(2)}}{U} - \frac{3W}{4U^2} (\omega^{(1)})^2.
$$

(36)

Again, according to the prescription of Allen and Rowlands, the last term in the above expression of $q^{(2)}$ is the ghost secular term and this term can be removed from the equation (35) if we set

$$
\frac{\partial A_1^{(0)}}{\partial X_2} = \frac{1}{2} A_1^{(0)} t_1.
$$

(37)
Therefore, the final form of $q^{(2)}$ can be written as follows:

$$q^{(2)} = A_1^{(0)} \left[ a \frac{W}{4U^2} (\omega^{(1)})^2 (1 - R) + t_1 \phi_0 \right].$$  \hspace{1cm} (38)$$

From the above expression of $q^{(2)}$, we see that $q^{(2)}$ is bounded for any real $X$ and $q^{(2)}$ is consistent at $X = +\infty$ but $q^{(2)}$ is not consistent at $X = -\infty$, i.e., $\lim_{X \to -\infty} q^{(2)} \neq 0$, but as $q^{(2)}$ is bounded, this inconsistency can be removed by proper grouping with the terms of higher orders. So, we consider the solution (19) for $j = 3$.

### D. Third order equation

Using the equation (52) of Appendix A, (24), (28) and (38), the solution (19) of the equation (14) for $j = 3$ can be written as

$$q^{(3)} = -\frac{W^2}{8a} A_2^{(3)} S^{-2} - \frac{W^2}{16a} \left( 5A_2^{(3)} + 4aA_3^{(3)} \right) + \frac{3W^2}{16a^2} \left( 5A_2^{(3)} + 4aA_3^{(3)} \right) \phi_0 + \frac{3W^2}{32a^2} \left( 5A_2^{(3)} + 4aA_3^{(3)} \right) fX + iA_1^{(0)} \left[ \frac{W}{8a} \omega^{(1)} \right] \left\{ -2UVW(m^2 + n^2) + W(\omega^{(1)})^2 + 4iU\omega^{(2)} \right\} \left( \frac{R}{S^2} \right) + iA_1^{(0)} \left[ \frac{W}{4U^3} \omega^{(1)} \right] \left\{ -3UVW(m^2 + n^2) + 3W(\omega^{(1)})^2 + 2iU\omega^{(2)} \right\} R + iA_1^{(0)} \left[ \frac{W}{8U^3} \omega^{(1)} \right] \left\{ -2UV(m^2 + n^2) + 3(\omega^{(1)})^2 \right\} \left\{ -2X + 2WLog \frac{1}{S} \right\} + iA_1^{(0)} \left[ \frac{1}{8U^3} \right] \left\{ 2UV(m^2 + n^2) - 3(\omega^{(1)})^2 \right\} \left\{ -3X + 3WLog \frac{1}{S} \right\} \phi_0 + iA_1^{(0)} \left[ \frac{3W^3}{16U^3} \right] \left\{ 2UV(m^2 + n^2) - 3(\omega^{(1)})^2 \right\} PolyLog[2, -e^{-2X}] f + iA_1^{(0)} \left[ \frac{a}{U^3W} \omega^{(1)} \right] \left\{ -2UV(m^2 + n^2) + 3(\omega^{(1)})^2 \right\} \times \left[ \frac{48W^3}{128a} \left\{ Log[1 + e^{-2X}] - Log \frac{1}{S} \right\} fX + 48W^2 fX^2 \right] + \left[ iA_1^{(0)} \frac{W^2}{16U^3} \left\{ -8l^3U^3 + 16UV(m^2 + n^2)\omega^{(1)} - 23(\omega^{(1)})^3 \right\} + 8U^2\omega^{(3)} \right] \left( \frac{\partial A_1^{(0)}}{\partial X_3} \right) fX, \hspace{1cm} (39)$$
where we have used MATHEMATICA\textsuperscript{37} to find all the integrals of\textsuperscript{19} for $j = 3$ and ignored the terms already taken in lower order solutions.

Substituting the expression of $(\omega^{(1)})^2$ as given in the equation\textsuperscript{33}, the expression of $q^{(3)}$ can be simplified as follows:

$$q^{(3)} = -\frac{W^2}{8a} A_2^{(3)} S^{-2} - \frac{W^2}{16a} \left(5A_2^{(3)} + 4aA_3^{(3)} \right) + \frac{3W^2}{16a^2} \left(5A_2^{(3)} + 4aA_3^{(3)} \right) \phi_0$$

$$+ \frac{3W^2}{32a^2} \left(5A_2^{(3)} + 4aA_3^{(3)} \right) fX - i A_1^{(0)} \frac{aW}{4U^2} \omega^{(1)} \left\{ VW(m^2 + n^2) - 2i\omega^{(2)} \right\} R$$

$$- i A_1^{(0)} \frac{aW}{6U^2} \omega^{(1)} \left\{ VW(m^2 + n^2) - 3i\omega^{(2)} \right\} \frac{R}{S^2}$$

$$+ i A_1^{(0)} \frac{1}{12U^2} \left\{ -12l^3 U^2 W^2 + VW^2(m^2 + n^2)\omega^{(1)} + 12U\omega^{(3)} \right\} \phi_0$$

$$+ \left[ i A_1^{(0)} \frac{1}{24U^2} \left\{ -12l^3 U^2 W^2 + VW^2(m^2 + n^2)\omega^{(1)} + 12U\omega^{(3)} \right\} - \frac{\partial A_1^{(0)}}{\partial X_3} \right] fX,$$

\text{From the expression of } q^{(3)} \text{ as given in } (40), \text{ we see that exponential secular terms in the expression of } q^{(3)} \text{ is due to the terms } \frac{1}{S^2} \text{ and } \frac{R}{S^2}. \text{ To remove both the exponential secular terms simultaneously, it is necessary to make a common factor of the coefficients of these two terms equal to zero. As } W \neq 0, \text{ to get a common factor of the coefficients of the two exponential secular terms in the expression of } q^{(3)}, \text{ we set}

$$A_2^{(3)} = B_2^{(3)} \left\{ VW(m^2 + n^2) - 3i\omega^{(2)} \right\},$$

\text{where } B_2^{(3)} \text{ is another arbitrary functions of } X_1, X_2, X_3,....$

Therefore, the equation\textsuperscript{40} assumes the following form:

$$q^{(3)} = -\frac{W^2}{16a} \left(5A_2^{(3)} + 4aA_3^{(3)} \right) + \frac{3W^2}{16a^2} \left(5A_2^{(3)} + 4aA_3^{(3)} \right) \phi_0$$

$$+ \frac{3W^2}{32a^2} \left(5A_2^{(3)} + 4aA_3^{(3)} \right) fX - i A_1^{(0)} \frac{aW}{4U^2} \omega^{(1)} \left\{ VW(m^2 + n^2) - 2i\omega^{(2)} \right\} R$$

$$- \left\{ VW(m^2 + n^2) - 3i\omega^{(2)} \right\} \left\{ \frac{W^2}{8a} B_2^{(3)} \frac{1}{S^2} + i A_1^{(0)} \frac{aW}{6U^2} \omega^{(1)} \frac{R}{S^2} \right\}$$

$$+ i A_1^{(0)} \frac{1}{12U^2} \left\{ -12l^3 U^2 W^2 + VW^2(m^2 + n^2)\omega^{(1)} + 12U\omega^{(3)} \right\} \phi_0$$

$$+ \left[ i A_1^{(0)} \frac{1}{24U^2} \left\{ -12l^3 U^2 W^2 + VW^2(m^2 + n^2)\omega^{(1)} + 12U\omega^{(3)} \right\} - \frac{\partial A_1^{(0)}}{\partial X_3} \right] fX,$$

\text{(42)
We can remove the exponential secular terms from the expression of $q^{(3)}$ by setting

$$VW(m^2 + n^2) - 3i\omega^{(2)} = 0.$$  \hspace{1cm} (43)

From equations (41) and (43), we get $A_{2}^{(3)} = 0$. Using this value of $A_{2}^{(3)}$ and the condition that $q^{(3)}$ is consistent at $X = +\infty$, we have

$$A_{3}^{(3)} = -iA_{1}^{(0)} \frac{a}{WU^{2}}(1 - R) + iA_{1}^{(0)} r_1 \varphi_0,$$ \hspace{1cm} (44)

where

$$r_1 = \frac{1}{6U^2} \left\{ -6\beta U^2 W^2 + W\omega^{(1)} \{ -4VW(m^2 + n^2) + 9i\omega^{(2)} \} + 6U\omega^{(3)} \right\},$$ \hspace{1cm} (46)

and we have used the following equation to remove the ghost secular term appearing in the expression of $q^{(3)}$

$$\frac{\partial A_{1}^{(0)}}{\partial X_3} = -\frac{1}{2} iA_{1}^{(0)} r_1.$$ \hspace{1cm} (47)

The above expression of $q^{(3)}$ shows that $q^{(3)}$ is bounded for any real $X$ and $q^{(3)}$ is consistent at $X = +\infty$ but $q^{(3)}$ is not consistent at $X = -\infty$, i.e., $\lim_{X \to -\infty} q^{(3)} \neq 0$. Again, it is important to note that $q^{(3)}$ contains same type of terms of $q^{(2)}$, viz., $(1 - R)$ and $\phi_0$. As $q^{(2)}$ and $q^{(3)}$ both are bounded, the inconsistent term of $q^{(2)}$ and $q^{(3)}$ can be removed by proper grouping with the higher order terms.

The equation (43) gives the following expression for $\omega^{(2)}$:

$$\omega^{(2)} = -\frac{i}{3} VW(m^2 + n^2).$$ \hspace{1cm} (48)

Equation (48) shows that $\omega^{(2)}$ is imaginary along with $i\omega^{(2)} > 0$ and consequently, the solitary wave solution (2) is stable at the order $k^2$.

III. CONCLUSIONS

In the present paper, we have seen that solitary wave solution of the KdV equation is stable up to the order $k^2$, where $k$ is the wave number for long-wavelength plane-wave perturbation. From the physical point of view, this problem gives an idea of finding nonlinear
dispersion relation between $\omega$ and $k$ when the solitary wave solution (2) is the steady state solution of the nonlinear evolution equation (1) for small value of the wave number $k$.

**APPENDIX A:** $R^{(j)}$ - Integrand of the integration in equation (15) for $j = 0, 1, 2$ and 3.

\[
R^{(0)} = 0, \quad (49)
\]

\[
R^{(1)} = i\omega^{(1)}q^{(0)}_0 - 2il[M_1q^{(0)}_0 - iAClq^{(0)}_{000} - ACq^{(0)}_{0001} - 2[M_1q^{(0)}]_01, \quad (50)
\]

\[
R^{(2)} = \left[\frac{1}{2}AD(m^2 + n^2) - l\omega^{(1)}\right]q^{(0)} + l^2M_1q^{(0)} + i\omega^{(2)}q^{(0)} + i\omega^{(1)}[q^{(0)}_1 + q^{(1)}_0] - i2l[(M_1q^{(0)})_1 + (M_1q^{(1)})_0]
+ \frac{5}{2}ACl^2q^{(0)}_{00} - iACl[5q^{(0)}_{001} + q^{(0)}_{000}]
- \frac{1}{2}AC[2q^{(0)}_{00002} + 5q^{(0)}_{0001} + 2q^{(0)}_{00001} - [M_1q^{(0)}]_{11}
- 2[M_1q^{(0)}]_{02} - 2[M_1q^{(1)}]_01, \quad (51)
\]

\[
R^{(3)} = \left[\frac{1}{2}AD(m^2 + n^2) - l\omega^{(1)}\right]q^{(1)} + l^2M_1q^{(1)} + i\omega^{(3)}q^{(0)} + \omega^{(2)}[i(q^{(0)}_1 + q^{(1)}_0) - lq^{(0)}]
+ i\omega^{(1)}[q^{(0)}_2 + q^{(1)}_0] + q^{(2)}] - i2l[(M_1q^{(0)})_2 + (M_1q^{(1)})_1 + (M_1q^{(2)})_0] + ACl^2[6q^{(0)}_{001} + \frac{5}{2}q^{(1)}_{00}]
- iACl[5q^{(0)}_{0002} + 6q^{(0)}_{0001} + 5q^{(0)}_{001} + q^{(2)}_{0000}]
- \frac{1}{2}AC[4q^{(0)}_{00001} + 10q^{(0)}_{00012} + 2q^{(0)}_{00003} + 2q^{(0)}_{00002}
+ 5q^{(0)}_{0001} + 2q^{(2)}_{00001} - 2[M_1q^{(0)}]_{03} - 2[M_1q^{(0)}]_12
- 2[M_1q^{(1)}]_{02} - [M_1q^{(1)}]_{11} - 2[M_1q^{(2)}]_0, \quad (52)
\]

where

\[
q^{(j)}_r = \frac{\partial q^{(j)}}{\partial X_r}, \quad q^{(j)}_{rs} = \frac{\partial^2 q^{(j)}}{\partial X_r \partial X_s}, \quad q^{(j)}_{rst} = \frac{\partial^3 q^{(j)}}{\partial X_r \partial X_s \partial X_t},
\]

\[
q^{(j)}_{rsty} = \frac{\partial^4 q^{(j)}}{\partial X_r \partial X_s \partial X_t \partial X_y}, \quad [M_1q^{(j)}]_{rs} = \frac{\partial^2 (M_1q^{(j)})}{\partial X_r \partial X_s}
\]

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