Violation of the Widom scaling law for effective crossover exponents

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(Dated: August 6, 2003)

In this work we consider the universal crossover behavior of two non-equilibrium systems exhibiting
a continuous phase transition. Focusing on the field driven crossover from mean-field to non-mean-
field scaling behavior we show that the well-known Widom scaling law is violated for the effective
exponents in the so-called crossover regime.

PACS numbers: 05.70.Ln, 05.50.+q, 05.65.+b

I. INTRODUCTION

A foundation for the understanding of critical phe-
nomena was provided by Wilson’s renormalization group
(RG) approach \[1, 2\] which maps the critical point onto
a fixed point of a certain transformation of the sys-
tem’s Hamiltonian, Langevin equation, etc. The RG
theory presents a powerful tool to estimate the val-
ues of the critical exponents, it allows to predict which
parameters determine the universality class and it ex-
plains the existence of an upper critical dimension \(D_c\)
above which the mean-field theory applies. Furthermore,
crossover phenomena between two different universality
classes are well understood in terms of competing fix-
points (see for instance \[2\]). Nevertheless there are still
some open aspects of crossover phenomena which are
discussed in the literature. The question whether the
crossover scaling functions are universal was revisited
several times \[3, 4, 5, 6, 7, 8, 9, 10\]. For instance it was
shown recentily that two different models, belonging to
the same universality class, are characterized by the same
(universal) crossover scaling functions \[10\]. This is re-
markable, since the universal scaling behavior is usually
restricted to a small vicinity around the critical point.
In the case of a crossover the universal scaling functions
span several decades in temperature or conjugated field.

Another question of interest concerns the so-called ef-
fective exponents \[11\] which can be defined as logarith-
mic derivatives of the corresponding scaling functions.
It is still open whether these effective exponents fulfill
over the full crossover the well-known scaling laws which
connect critical exponents. This question is closely re-
lated to the more general and very important question
whether effective exponents obey the scaling laws at all.
For instance it is known experimentally \[12\] as well as
theoretically \[13\] that the asymptotic scaling behavior is
often masked by corrections to scaling, so-called con-
fluent singularities. In this case it is useful to analyze
the data in terms of effective exponents and the above
question naturally arises \[14\]. Thus the validity of the
scaling laws for effective exponents was addressed in ex-
perimental works, RG approaches as well as numerical
simulations. In particular the violation of the scaling
laws for effective exponents was conjectured from a RG
approximation \[17\]. But neither experimental nor nu-
merical work could clearly confirm this conjecture so far.
For instance Binder and Luijten considered numerically a
crossover in the Ising model and discussed the validity of
the Rushbrook scaling law \[16\]. The observed nonmono-
tonic crossover behavior suggests again a violation of the
Rushbrook scaling law. However it can not be considered
as a rigorous proof.

In this work we consider a field driven crossover (in
the so-called critical crossover limit \[14, 15\]) in two dif-
ferent models exhibiting a non-equilibrium second order
phase transition. Varying the range of interactions we
investigate the crossover from mean-field to non-mean-
field scaling behavior. The order parameter and the or-
der parameter susceptibility is measured as a function of
the conjugated field and we are able to determine the
responding effective exponents over the full crossover
regime. Our results show that the well-known Widom
scaling law is clearly violated for the effective crossover
exponents. Furthermore we present a simple analytical
argument, suggesting that the scaling laws are valid for
the asymptotic scaling regimes (where the systems are
characterized by a pure algebraic behavior), whereas the
scaling laws do not hold for the crossover regime (char-
acterized by a non algebraic behavior).

II. MODELS AND SIMULATIONS

In the following we consider two different cellular au-
tomata exhibiting a so-called absorbing phase transition.
The first model is the conserved transfer threshold pro-
cess (CTTP) \[16\]. In this model lattice sites may be
empty, occupied by one particle, or occupied by two par-
ticles. Empty and single occupied sites are considered
as inactive whereas double occupied lattice sites are con-
sidered as active. In the latter case one tries to transfer
both particles of a given active site to randomly chosen
empty or single occupied nearest neighbor sites.

The second model is a modified version of the Manna
sandpile model \[17\], the fixed-energy Manna model \[18\].
In contrast to the CTTP the Manna model allows un-
limited particle occupation of lattice sites. Lattice sites
which are occupied by at least two particles are consid-
ered as active and all particles are moved to the neigh-
boring sites selected at random.

In our simulations (see [19, 27] for details) we have used square lattices (with periodic boundaries) of linear size \( L \leq 2048 \). All simulations start from a random distribution of particles. After a transient regime both models reach a steady state characterized by the density of active sites \( \rho_a \). The density \( \rho_a \) is the order parameter and the particle density \( \rho \) is the control parameter of the absorbing phase transition, i.e., the order parameter vanishes at the critical density \( \rho_c \) according to, \( \rho_a \propto \delta \rho^\beta \), with the reduced control parameter \( \delta \rho = \rho/\rho_c - 1 \). Below the critical density (in the absorbing phase) the order parameter is zero in the steady state.

Similar to equilibrium phase transitions it is possible in the case of absorbing phase transitions to apply an external field \( h \) which is conjugated to the order parameter. The conjugated field has to act as a spontaneous creation of active particles, destroying the absorbing state and therefore the phase transition itself. Furthermore, the associated linear response function \( \chi_a = \partial \rho_a/\partial h \) has to diverge at the critical point (\( \delta \rho = 0, h = 0 \)). A realization of the external field for absorbing phase transitions with a conserved field was recently developed in [11] where the external field triggers movements of inactive particles which may be activated in this way. At the critical density \( \rho_c \) the order parameter scales as \( \rho_c \propto h^{\beta/\sigma} \). Using the conjugated field it is possible to investigate the equation of state \( \rho_c(\rho, h) \), i.e., the order parameter as a function of both the control parameter and the external field. A recently performed scaling analysis reveals that the CTTP and the Manna model are characterized by the same critical exponents as well as by the same universal scaling form of the equation of state [11], i.e., both models belong to the same universality class [23].

According to the above definition particles of active sites are moved to nearest neighbors only, i.e., the range of interactions is \( R = 1 \). It is straightforward to implement various ranges of interactions into these models [10]. In this modified models particles of active sites are moved (according to the rules of each model) to randomly selected sites within a radius \( R \). Since the dynamics of both considered models is characterized by simple particle hopping processes, various interaction ranges can be easily implemented and high accurate data are available. This is a significant advantage compared to e.g. equilibrium system like the Ising model where the increasing interaction range causes a slowing down of the dynamics.

For any finite interaction range the phase transition is characterized by non-mean-field scaling behavior which now takes place at the critical density \( \rho_{c,R} \). A mean-field phase transition occur for infinite interactions (\( R \to \infty \)) only. But mean-field behavior could occur away from the critical point if the long range interactions reduce the critical fluctuations sufficiently. The crossover between the mean-field and non-mean-field scaling regimes is described by the famous Ginzburg criterion [22] which states that the mean-field picture is self-consistent in the active phase as long as the fluctuations within a correlation volume are small compared to the order parameter itself. This leads for zero field to the crossover condition \( O[\rho_{c,R} - \rho] = 1 \), with the crossover exponent \( \phi = (4 - D)/2D \) [10]. In order to avoid lattice effects we use the effective range of interactions \( R_{eff}^2 = \frac{1}{z} \sum_{i \neq j} |x_i - x_j|^2, \quad |x_i - x_j| \leq R \), (1)

where \( z \) denotes the number of lattice sites within a radius \( R \).

### III. UNIVERSAL CROSSOVER SCALING

The crossover scaling function has to incorporate three scaling fields (the control parameter, the external field, and the range of interactions), i.e., we make the phenomenological ansatz

\[
\rho_c(\rho, h, R_{eff}) \sim \lambda^{-\beta_{MF}} \tilde{\mathcal{R}}(a_c(\rho - \rho_{c,R}) \lambda, a_h h \lambda^{\sigma_{MF}}, a_{R_{eff}}^{-1} R_{eff}^{-1} \lambda^\phi),
\]

where the universal scaling function \( \tilde{\mathcal{R}} \) is the same for all models belonging to a given universality class whereas all non-universal system-dependent features (e.g. the lattice structure, the update scheme, etc.) are contained in the so-called non-universal metric factors \( a_c, a_h, \) and \( a_R \) [24]. These factors are determined by three conditions which normalize the scaling function \( \tilde{\mathcal{R}} \). First, the analytically known mean-field scaling function [21, 22]

\[
\tilde{\mathcal{R}}_{MF}(x, y) = \frac{x}{2} + \sqrt{y + \left(\frac{x}{2}\right)^2}
\]

should be recovered for \( R \to \infty \), i.e., \( \tilde{\mathcal{R}}(x, y, 0) = \tilde{\mathcal{R}}_{MF}(x, y) \). Therefore, \( \tilde{\mathcal{R}}(0, 0, 0) = \tilde{\mathcal{R}}_{MF}(0, 0) = 1 \), \( \tilde{\mathcal{R}}(0, 1, 0) = \tilde{\mathcal{R}}_{MF}(0, 1) = 1 \), implying \( a_c = a_{R_{eff}, R_{eff}, \infty}/\rho_{c,R_{eff}, \infty} \) and \( a_h = a_{h,R_{eff}, \infty} \). Finally, the non-universal metric factor \( a_R \) can be determined by the condition \( \tilde{\mathcal{R}}(x, 0, 1) \sim x^{\beta_D}/(\beta_{MF} - \beta_D) \) for \( x \to 0 \) yielding [10]

\[
a_R = \left(\frac{\rho_{c,R_{eff}, \infty}}{\rho_{c,R,R_{eff}, \infty}}\right)^{\frac{\phi}{\beta_{MF} - \beta_D}}.
\]

The metric factors were already determined in previous works [10, 27], thus no parameter fitting is needed in order to perform the following scaling analysis.

In this work we focus our attention to the field driven crossover, i.e., we consider the CTTP and the Manna model at the critical densities \( \rho_{c,R} \) which were determined in [10]. In Fig. 4 we plot the corresponding data of the CTTP for various values of the interaction range \( R \). As one can see the power law behavior of the order parameter changes with increasing range of interactions.

The scaling form at the critical point is given by (setting \( a_R^{-1} R_{eff}^{-1} \lambda^\phi = 1 \))

\[
\rho_c(\rho_{c,R}, h, R_{eff}) \sim (a_R R_{eff})^{-\beta_{MF}/\phi} \tilde{\mathcal{R}}(0, a_h h a_{R_{eff}}^{\sigma_{MF}/\phi} R_{eff}^{\sigma_{MF}/\phi}, 1),
\]
with $\beta_{\text{MF}} = 1$ and $\sigma_{\text{MF}} = 2$, respectively. For sufficiently small field the universal function scales as

$$\tilde{R}(0, x, 1) \sim m_{a,h} \beta_{D}/\sigma_{D}, \quad \text{for } x \to 0,$$  

with the universal amplitude $m_{a,h}$. The scaling form Eq. (5) has to equal for $R = 1$ the $D$-dimensional scaling ansatz $\rho_a \sim (a_{h,R=1} h)^{\beta_{D}/\sigma_{D}}$, leading to

$$m_{a,h} = \left( \frac{a_{h,R=1}}{a_{h,R=\infty}} \right)^{\beta_{D}/\sigma_{D}} a^\beta_{\text{MF}}/\phi - \sigma_{\text{MF}} \beta_{D}/\sigma_{D} \phi^\gamma.$$

According to the scaling form Eq. (5) we plot in Fig. 1 the rescaled order parameter $\rho_a (a_{h,R=\infty})^4$ as a function of the rescaled field $a_{h}(a_{h,R=\infty})^4$. We observe an excellent data collapse for the full crossover behavior confirming the above phenomenological scaling ansatz.

However, since the entire crossover region covered several decades it could be difficult to observe small but systematic differences between the scaling functions of both models. Therefore, it is customary to scrutinize the crossover via the so-called effective exponent 3 8 8 8 11

$$\left( \frac{\beta}{\gamma} \right)_{\text{eff}} = \frac{\partial}{\partial \ln x} \ln \tilde{R}(0, x, 1).$$

The perfect collapse of the corresponding data is shown in Fig. 2 and confirms again the universality of the crossover scaling function $\tilde{R}$.

Next we consider the order parameter susceptibility. The scaling form of the susceptibility is given by

$$\chi_a (\rho, h, R_{\text{eff}}) \sim \chi_a (\rho - \rho_{c,R}) \lambda, a_{h,R} \lambda^\gamma \lambda, a_{h}^{-1} R_{\text{eff}}^{-1} \lambda^\phi.$$  

On the other hand the susceptibility is defined as the derivative of the order parameter with respect to the conjugated field

$$\chi_a (\rho, h, R_{\text{eff}}) = \frac{\partial}{\partial h} \rho_a (\rho, h) \sim a_{h} \chi^\gamma \lambda^\gamma \lambda \lambda - a_{h}^{-1} R_{\text{eff}}^{-1} \lambda^\phi$$

with $\tilde{R}(x, y, z) = \partial \rho \tilde{R}(x, y, z)$. By comparing this expression with Eq. (9) we find $\tilde{R}(x, y, z) = \partial \rho \tilde{R}(x, y, z)$, $a_{\chi} = a_{h}^{-1}$, as well as the Widom scaling law

$$\gamma = \sigma - \beta$$

which is well known from equilibrium phase transitions. Again, the mean-field behavior is recovered for $R \to \infty$, i.e., $\tilde{R}(x, y, 0) = \tilde{C}(x, y) = 1/2 (y + (x/2)^2)^{-1/2}$, implying $\tilde{C}(1, 0, 0) = \tilde{C}(0, 1, 0) = \tilde{C}(0, 1, 0) = 1$, as well as $\gamma_{\text{MF}} = 1$.

Similar to the order parameter we plot the susceptibility according to the scaling form

$$a_{h}^{-1} \chi_a (\rho_{c,R}, h, R_{\text{eff}}) \sim (a_{h} R_{\text{eff}})^{\gamma_{\text{MF}}/\phi} \tilde{C}(0, a_{h} (a_{h} R_{\text{eff}})^{\gamma_{\text{MF}}/\phi}, 1).$$

Approaching the transition point the susceptibility is expected to scale as $\chi_a (x, 0, 1) \sim m_{\chi,h} x^{-\gamma_{D}/\sigma_{D}}$, for $x \to 0$, where the universal power-law amplitude is given by

$$m_{\chi,h} = \left( \frac{a_{h,R=1}}{a_{h,R=\infty}} \right)^{1-\gamma_{D}/\sigma_{D}} a^\gamma_{\text{MF}}/\phi + \sigma_{\text{MF}} \gamma_{D}/\sigma_{D} \phi \beta_{D}/\sigma_{D}.$$
The rescaled susceptibility is shown in Fig. 3. Over the entire crossover region we got an excellent data collapse including both asymptotic scaling regimes. The inset displays the effective exponent

\[ \frac{\langle \gamma \rangle_{\text{eff}}}{\sigma} = -\frac{\partial}{\partial \ln x} \ln \tilde{\xi}(0, x, 1) \]  

(14)

which exhibits again a monotonic crossover from the two-dimensional scaling regime to the mean-field scaling behavior.

IV. WIDOM SCALING LAW

In this way we have obtained the effective exponents \((\beta/\sigma)_{\text{eff}}\) and \((\gamma/\sigma)_{\text{eff}}\) for the field driven crossover from mean-field to non-mean-field behavior. Thus we are able to check the corresponding Widom scaling law

\[ \left( \frac{\langle \gamma \rangle}{\langle \sigma \rangle} \right)_{\text{eff}} = 1 - \left( \frac{\langle \beta \rangle}{\langle \sigma \rangle} \right)_{\text{eff}}, \]  

(15)

for the whole crossover region. The corresponding data are shown in Fig. 4. As can be seen the Widom scaling law is fulfilled for the asymptotic regimes \((D = 2)\) scaling behavior and mean-field scaling) but it is clearly violated for the intermediate crossover region. This result is not surprising if one notices that the above Widom law [Eq. (15)] corresponds to the differential equation [see Eqs. (6) (13)]

\[ -\frac{\partial}{\partial \ln x} \left( \frac{\partial}{\partial x} \tilde{\mathcal{R}}(0, x, 1) \right) = 1 - \frac{\partial}{\partial \ln x} \tilde{\mathcal{R}}(0, x, 1). \]  

(16)

Using 1 = \(\partial \ln ax/\partial \ln x\) we get

\[ -\ln \partial_x \tilde{\mathcal{R}}(0, x, 1) = \ln ax - \ln \tilde{\mathcal{R}}(0, x, 1) + c, \]  

(17)

where \(c\) is some constant. It is straightforward to show that this differential equation is solved by simple power-laws \([\tilde{\mathcal{R}}(0, x, 1) = c_0 x^{c_1}\) with \(c_1 = 1/a \exp c\). Thus the Widom scaling law is fulfilled in the asymptotic regimes only. In the case that the scaling behavior is affected by crossovers, confluent singularities, etc. no pure power-laws occur and the scaling laws do not hold for the corresponding effective exponents.

V. CONCLUSION

In conclusion, the crossover from mean-field to non-mean-field scaling behavior is numerically investigated for two different models exhibiting a second order phase transition. Increasing the range of interactions we are able to cover the full crossover region which spans several decades of the conjugated field. The corresponding data show that the Widom scaling law is violated in the crossover regime. Notice that we focus in our investigations on the particular universality class of absorbing phase transitions only for technical reasons. The demonstrated violation of the Widom scaling can be applied to continuous phase transitions in general.

We would like to thank A. Hucht, K.D. Usadel, and B. Schmurr for helpful discussions. This work was financially supported by the Minerva Foundation (Max Planck Gesellschaft).
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