Critical fluctuations in the one-dimensional Bak-Sneppen model

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Abstract. The critical fluctuation properties of fitness distribution in the one-dimensional Bak-Sneppen model are studied in terms of the normalized factorial moments, erraticity moments and the factorial correlators. For a fitness window below the gap intermittent behaviors are observed. The scaling exponent for the BS model is different from that in two dimensional Ising model for second order phase transition. There is no correlation between fluctuations in two windows separated by the critical gap.

1. Introduction
Last a few decades witnessed the rapid development of theories for complex systems. It has been found that some complex systems can evolve into a critical state without tuning any external parameters. System at such self-organized critical states, such as in the sand-pile model [1] and the Bak-Sneppen (BS) model [2], have as fingerprints for criticality power-law distributions [3] for the size and lifetime of avalanches.

As is well-known, systems at traditional critical point exhibit nontrivial critical fluctuations and long-range correlations. A typical example is the critical fluctuations in the two dimensional (2D) Ising model [4]. For the purpose of comparing properties among system under normal and self-organized critical states, one can investigate the fluctuation properties of systems in the self-organized critical state. To achieve this end, we study the dynamical fluctuations in a typical self-organized critical model, the one-dimensional BS model. In such a model, sites in a lattice are initially assigned random numbers uniformly drawn from (0, 1). Those random numbers are called fitness of the sites. At each time step of the evolution, a site with the minimum fitness is found and the fitness on that site and its nearest neighbors are replaced by new random numbers also uniformly drawn from (0, 1). A variable named gap at time \( t \) is defined as the maximum of the minimum fitness before that time \( t \). With the update of the system, it is found that the system’s gap will saturate to a critical value \( f_c \) and a critical state will be reached.

In this paper, we study the critical fluctuations of fitness distribution and try to find a characteristic quantity for those fluctuations for the one-dimensional BS model.

2. Factorial moments and dynamical fluctuations
Consider a system in which each site or particle has a property of \( d \) components which fluctuates because of the interactions or other effects. One can represent a particle as a point in a \( d \)-dimensional space. To study the fluctuations quantitatively, one can first choose a fixed region
in the $d$-dimensional space and divide it into $M^d$ bins of equal size. Then one counts the number of points in each bin and study the fluctuations of the point numbers in each bin. This process can be repeated for different numbers of bins or different bin sizes.

A useful variable used for analyzing the fluctuations is the factorial moments. The $q$-th order factorial moment is defined as

$$ f_q = \langle n(n-1) \cdots (n-q+1) \rangle, \quad (1) $$

where $n$ is the number of points in a bin and $\langle \cdots \rangle$ represents average over all bins and different realizations of the system’s configuration. The normalized factorial moments are defined as ratios of the factorial moments by

$$ F_q = f_q/f_1^q. \quad (2) $$

An important advantage of $F_q$ is that they can filter out the statistical fluctuations. One can see this easily by considering a case in which the probability of having $n$ points in a bin is given by a Poisson or binomial distribution. In this case, the normalized factorial moments are all fixed numbers of about 1, independent of the bin size. Specifically, if the fluctuations in the system have some kind of self-similarity, as in the random cascading model [5], $F_q$ have a power-law dependence on the bin size $\delta$, $F_q(\delta) \propto \delta^{-\varphi_q}$ with $\varphi_q > 0$, a phenomenon known as intermittency [6].

Considering the above, one may ask whether the normalized factorial moments can be used for complex system and tell us some features of the critical fluctuations in those systems. An application [7] has been appeared along this line for the financial fluctuations. For the BS model we are considering now, each site on the lattice has a fitness in $(0, 1)$. So it is natural to represent the fitness state of a lattice by points in a one dimensional space from 0 to 1. In this paper, we only investigate the critical fluctuations in the one-dimensional BS model. Generalization to higher dimensional BS models is straightforward.

Since we are interested only in the critical fluctuations of the fitness distribution in the BS model, we start our simulation of the system’s evolution from the critical state when fitness on all sites are uniformly distributed from a critical gap $f_c$ to 1. For a one dimensional lattice we are considering, $f_c = 0.667$ [8]. In our simulations, we fix the lattice size $L = 500$. Let the system evolve $5L$ times from the initial critical state, then we put fitness on all sites in a one-dimensional space from 0 to 1. To calculate the factorial moments configuration by configuration, we focus on a fixed fitness window $Y$ from $f_{\min}$ to $f_{\max}$ and divide the window into $M$ bins. Then it is straightforward to count the number of points in each bin and calculate the factorial moments $f_q(q = 1, 2, \cdots)$ for each bin in the given configuration. The factorial moments for a given configuration of the system can be obtained by averaging $f_q$ over all bins in the window. Then one gets the normalized factorial moments $F_q$ from the configurational averaged moments $f_q$. The calculation is repeated for different numbers of bins or different bin sizes. To eliminate such statistical fluctuations, the above process must be repeated for a lot of configurations, and each configuration is separated by $5L$ evolution steps. We use $2 \times 10^7$ configurations from the same critical state, and average the normalized factorial moments $F_q$ over those configurations. Suppose the normalized factorial moments $F_q^c$ for a configuration has a distribution $P_q(F_q^c)$ for each $q$ in those configurations, one can easily see that the normalized factorial moments after average over the configurations is in fact

$$ F_q \equiv \langle F_q^c \rangle = \int dF_q^c F_q^c P_q(F_q^c), \quad (3) $$

where $\langle \cdots \rangle$ represents average over configurations, thus very little information about the distribution $P_q(F_q^c)$ is contained in the averaged moments. To get more information about
the fluctuations of the moments from configuration to configuration, a new quantity has been proposed in Ref. [9], which is defined as

\[ C_{p,q} = \frac{\int dF_q^e p_q(F_q^e)(F_q^e)^p}{\int dF_q^e p_q(F_q^e)(F_q^e)^p} = \frac{\langle (F_q^e)^p \rangle}{\langle (F_q^e)^p \rangle}. \] (4)

By definition, \( C_{0,q} = C_{1,q} = 1 \). Please note that \( p \) needs not be an integer. For \( p > 1 \), \( C_{p,q} \) emphasizes on contributions of large \( F_q^e \), while for \( p < 1 \) \( C_{p,q} \) probes the lower \( F_q^e \) behavior of \( P_q(F_q^e) \). So with different choices of \( p \), a lot of more information on the fluctuations can be obtained from \( C_{p,q} \). For multi-fractal fluctuations, one expects a generalized scaling behavior, called erraticity, [10]

\[ C_{p,q} \propto h(M)^{\psi_{p,q}}, \text{with } \psi_{p,q} > 0 \] (5)

where \( h(M) \) is a properly chosen function of the number of bins in the given window, common for all \( p \) and \( q \). Here we only investigate the behavior of \( C_{p,q} \) near \( p = 1 \) and calculate a new set of moments

\[ \Sigma_q = \left. \frac{\partial C_{p,q}}{\partial p} \right|_{p=1} = \left. \frac{\langle F_q^e \rangle \ln F_q^e}{\langle (F_q^e)^p \rangle} \right|_{p=1}. \] (6)

When Eq. (5) holds, one can see that

\[ \Sigma_q \sim \left. \frac{\partial \psi_{p,q}}{\partial p} \right|_{p=1} \ln h(M) = \mu_q \ln h(M). \] (7)

So \( \Sigma_q \) is a linear function of \( \ln h(M) \) for all \( q \), and therefore scaling properties of \( \Sigma_q \) are described by slope parameters \( \mu_q = \partial \psi_{p,q}/\partial p|_{p=1} \). \( \mu_q \) are sometimes called entropy indices.

3. Results of the moments

In the one dimensional fitness space, there is a special point, \( f = f_c \). For \( f > f_c \), the distribution is on average uniform because of the building up of the BS model. For \( f < f_c \), the distribution is very complicated. For this reason, we investigate the factorial moments in two separate windows, one above \( f_c \), another below \( f_c \). The window above \( f_c \) is chosen to be \( (0.68, 0.98) \). Our calculated results for \( F_q \) are constants, independent of the bin size, thus not very interesting.

Now we study the fluctuations of fitness distribution in a fitness window below \( f_c \). There is some difference in the average fitness densities for different bins in this window, because the minimum fitness in the lattice can be from 0 to \( f_c \), and only the sites involved in the evolution may have fitness larger than the minimum fitness but less than \( f_c \), thus contribute to the fitness distribution in the new window. Because of the fact that the minimum fitness fluctuates from time step to time step in the evolution, fitness distribution in our new window range cannot be flat after average over a long evolution history or over many configurations. To reduce the effect from such a bin to bin difference, we choose a fitness window close to the critical gap \( f_c \) and not too big. We choose the window to be from 0.45 to 0.65 and perform exactly the same calculations for the moments as for the last case we just studied. We suppose that the discussed bin to bin difference will not lead to very significant modification to our results, as claimed in Ref. [11] that the bin averaging done also washes out the nonstationary effects. Because the number of fitness points in the window is a small fraction of the lattice size, one has to work with smaller bin number \( M \). We start with \( M = 2 \) and increase \( M \) by fifty percent successively six times. The obtained normalized factorial moments \( F_q \) as functions of number of bins \( M \) are shown in Fig. 1 in log-log plot.

Power-law dependence of \( F_q \) on the bin size or number of bins \( M \) or intermittent behavior can be seen clearly. The scale invariance of \( F_q \) is a signature for critical fluctuations, as for the 2D Ising model. So we have

\[ F_q \propto M^{\varphi_q} \text{ with } \varphi_q > 0 . \] (8)
One can display the scaling results of \( F_q \) as functions of \( F_2 \), which can exhibits scaling behaviors even when \( F_q \) does not scale with \( M \). This more general scaling behavior reads

\[
F_q \propto F_2^{\beta_q}, \text{ with } \beta_q = \varphi_q / \varphi_2.
\]  

(9)

Such a scaling behavior is shown in Fig. 2. Power-law fitting results are also shown in the figure.

Figure 1. \( F_q \) as functions of number of bins for the fitness window below the critical gap \( f_c \).

Figure 2. Scaling behavior of \( F_q \) vs \( F_2 \) in log-log plot for the results shown in Fig. 1. The straight lines are from the power-law fit.

The exponents \( \beta_q \) in Eq. (9) are shown in Fig. 3 as a function of \( q - 1 \) in log-log plot. From the power-law fitting curve \( \beta_q \propto (q - 1)^{\nu} \), one gets the scaling exponent \( \nu = 1.35 \) for the scaling behavior, which is very close to that obtained from the 2D Ising model, where \( \nu \simeq 1.3 \).

For describing the fluctuation properties of the normalized factorial moments \( F_q \), \( \Sigma_q \) can also be calculated in the simulation of the system evolution. For the fitness window below \( f_c \) we found that \( \ln h(M) = (\ln M)^2 \) for the scaling of \( \Sigma_q \). The results for \( \Sigma_q \) are shown in Fig. 4 as functions of \( (\ln M)^2 \).

Figure 3. Exponents \( \beta_q \) for the curves in Fig. 2 as a function of \( q - 1 \).

Figure 4. \( \Sigma_q \) as functions of bin number \( h(M) = (\ln M)^2 \) for the window below \( f_c \).

With the scaling of \( \Sigma_q \) at hand, one can get the slopes \( \mu_q \) of the lines in Fig.4 and the results are shown in Fig. 5 as a function of \( q - 1 \) in log-log plot. Again a scaling behavior is found

\[
\mu_q = (q - 1)^{\nu'}, \text{ with } \nu' = 1.53.
\]  

(10)
In Ref.[12] a scaling behavior of $\Sigma_3$ vs $\Sigma_2$ is shown for the 2D Ising model. They obtained the slope $\omega_3 = 3.24$, which corresponds to an exponent $\nu' = \ln(\omega_3)/\ln(3-1) = 1.696$ from the above equation. Thus the value of $\nu'$ we just obtained is a little bit smaller than that from 2D Ising model.

**Figure 5.** Exponents $\omega_q$ for the curves in Fig. 4 as a function of $q - 1$.

**Figure 6.** $C_q$ as functions of $D$ and $\delta$ for $q = 1, 2, 3$ and 4 when $D$ is fixed at 0.05.

4. Factorial correlators
The critical fluctuations can be investigated from the so-called factorial correlators

$$f_{q_1,q_2} \equiv \langle n_1(n_1-1)\cdots(n_1-q_1+1)n_2(n_2-1)\cdots(n_2-q_2+1) \rangle,$$

(11)

where $n_1$ and $n_2$ are multiplicities in two different bins in some phase space region. As in $F_q$, $\langle \cdots \rangle$ denotes average over many bins and different configurations. The scaled normalized factorial correlators can be defined as

$$C_{q_1,q_2} = f_{q_1,q_2}/((f_{1,0})^{q_1}(f_{0,1})^{q_2}).$$

(12)

$C_{q_1,q_2}$ depends on the separation $D$ between the two bins and the bin widths. Now we only investigate behaviors of $C_{q_1,q_2}$ for $q_1 = q_2 = q$ and we choose equal size for the two bins. Let one bin lie on the left of $f_c$ and the other symmetrically on the right of $f_c$. The distance between the centers of the two bins is $D$ and each bin has a width $\delta$. As for investigating $F_q$, we perform Monte Carlo simulation and calculate $C_q \equiv C_{q,q}$ as functions of $D$ and $\delta$ for $q = 2, 3, 4$. The results are shown in Fig. 6 for an arbitrary $D = 0.05$ and in Fig. 7 for $D = 0.2$. Similar results can be obtained for other values of $D$. Power-law behaviors can be seen as for the normalized factorial moments. One can get the scaling exponents $\varphi_q$ for $C_q$ as functions of $q$ for three different values of $D$ as shown in Fig. 8. One can see that the scaling exponent can also be expressed as $\varphi_q \propto (q - 1)^\nu$ with $\nu$ almost the same as from $F_q$, independent of value of $D$. Thus there is no correlation between multiplicity fluctuations in those two bins.

5. Conclusion and discussion
We studied the normalized factorial moments, the entropy indices and the factorial correlators for the fluctuations of fitness distribution in the one-dimensional BS model for two fitness windows, one above and the other below the critical gap $f_c$. For the window above $f_c$, $F_q$ do not show any anomaly scaling in the bin size or bin number. But for the window below $f_c$, intermittent
behaviors are observed with the scaling exponent for the normalized factorial moments very close to that for the 2D Ising model, while the critical exponent $\nu'$ for the erraticity is found a little bit smaller than that from 2D Ising model. The results show that there is no correlation between multiplicity fluctuations in those two bins separated by the critical gap $f_c$.

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