Ultralight DM bosons with an Axion-like potential: scale-dependent constraints revisited.

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Abstract. A scalar field $\phi$ endowed with a trigonometric potential has been proposed to play the role of Dark Matter. A deep study of the cosmological evolution of linear perturbations, and its comparison to the Cold Dark Matter (CDM) and Fuzzy Dark Matter (FDM) cases (scalar field with quadratic potential), reveals an enhancement in the amplitude of the mass power spectrum for large wave numbers due to the non–linearity of the axion–like potential. For the first time, we study the scale–dependence on physical quantities such as the growth factor $D_k$, the velocity growth factor $f_k$, and $f_k\sigma_8$. We found that for $z < 10$, all these quantities recover the CDM evolution, whereas for high redshift there is a clear distinction between each model (FDM case, and axion–like potential) depending on the wavenumber $k$ and on the decay parameter of the axion–like potential as well. A semi–analytical Halo Mass Function is also revisited, finding a suppression of the number of low mass halos, as in the FDM case, but with a small increment in the amplitude of the variance and halo mass function due to the nonlinearity of the axion–like potential. Finally, we present constraints on the axion mass $m_\phi \geq 10^{-24}\text{eV}$ and the axion decay parameter is not constrained within the prior $0 \leq \lambda \leq 10^4$ by using data of the Planck Collaboration 2015.
1 Introduction

One of the open problems of modern physics concerns the existence of Dark Matter (DM). At present we have several observations indicating that such a component of matter exists [1–5], and that it is most likely the main agent driving the formation of structure. The most successful model describing this unknown component of matter is called Cold Dark Matter (CDM), and consists of a pressureless fluid of particles that interacts mostly gravitationally with other components of matter [6, 7]. Although the CDM model is so far in good agreement with most of the cosmological observations, DM nature is still unknown. It is well known that there are some differences at small scales between astrophysical observations and numerical simulations based on CDM [8–23]. These differences may be due to the lack of information about astrophysical processes of galactic substructures and baryonic physics, but they could also be pointing out to characteristics and still unknown properties of the DM field. Hence, if it is the case that DM is the main responsible for the process of structure and substructure formation, then it is important to explore and analyze other DM candidates that could offer
a better description of the structures at such small scales. With many different models in the literature, it is important that a given model under study predicts observables accurately so that comparison against observations are meaningful.

In this context, models of Scalar Field Dark Matter (SFDM) have gained great relevance in modern cosmology by becoming a promising candidate to describe the DM as well, maybe even better, than CDM. While the implementation of scalar fields in cosmology has historically its origins in inflationary models of the early Universe\cite{24, 25}, and also to describe the accelerated expansion of the Universe at late times\cite{26–30}, scalar fields also possess interesting properties to work as DM models.

Within this frame of scalar fields, a compelling DM candidate that has been vastly investigated, and which is the one of interest in this work, is the *Axion*, a scalar field originally proposed to solve the Strong CP problem in QCD\cite{31–34}, and which origin can be given within a more fundamental theory such as String Theory\cite{35–40}. Now, several models involving axions and *axion-like particles* have surged as possible source for DM\cite{41–51}, and several experiments such as ADMX\cite{52}, SOLAX\cite{53}, DAMA\cite{54}, COSME\cite{55}, CAST\cite{56} are trying to hunt directly this elusive kind of particles (other possible ways of detection can be seen in\cite{57–65}). The lighter axion in QCD has masses of around $\mu eV$, while for axion-like particles the mass lies within the range of $10^{-18}eV > m_\phi > 10^{-26}eV$, which is the reason why the latter are also known in the literature as *Ultralight Axions*. An important feature of this DM candidate is that it can give rise to *Bose-Einstein condensates* through a phase transition\cite{66–73}, and it can form *caustics* as well\cite{74–81}. Thus, axions and axion-like particles are very well motivated DM candidates from the theoretical point of view.

Axion models in which the scalar field potential includes only the quadratic term, usually referred as free case or fuzzy dark matter (FDM), have been extensively studied in the literature\cite{82–89}. We will refer to it as the FDM case from now on. However, such models do not capture all the implications that arise when including a full axion potential.

In this work we will focus on a model that incorporates a trigonometric potential that is typical in axion studies, defined by

$$V(\phi) = m_\phi^2 f_A^2 [1 + \cos (\phi / f_\phi)] ; \quad (1.1)$$

Here, $m_\phi$ is the axion mass, $f_\phi$ is the axion decay constant, and the two together make up the height of the potential. In typical axion models, there is a relationship between the mass and the decay constant in which they are inversely proportional to each other, in particular for axions coming from M-Theory and Type IIB string theory\cite{36, 90, 91}, where the decay constant is of the order of $10^{17} \text{GeV}$.

The choice of the potential in Eq. (1.1) codifies the shift symmetry of the axion field, and our main aim is to analyze in detail the cosmological implications arising from the nonlinearity of such potential. Previous works for this have shown some semi-analytical treatment\cite{88, 92}, while a first attempt to a full analysis was presented in\cite{93}. The effects on the CMB and MPS of such anharmonic potential, but considering different exponents $[1 - \cos(\phi / f)]^n$ with $n = 1, 2, 3$, have been studied in\cite{94}. However, when the cosmological evolution of the scalar field is that of dark matter (for $n = 1$), the predictions are basically the same as those of FDM. As we will show in the present work, when considering extreme values of the decay constant with the potential (1.1), it is possible to quantify deviations from the FDM case, regarding the structure formation at linear regime, like the enhancement of the mass power spectrum (MPS) at small scales reported in\cite{92, 93}, as well as to analyze implications for other observables.
An outline of this work is as follows. In Section 2 we study the cosmological background evolution and linear perturbations regime, by means of establishing new variables and a dynamical system that lead us to a generalization of the fluid equations. Using an amended version of the Boltzmann code CLASS [95], we track the evolution and growth of the perturbations. At the end of this section we develop a detailed analysis of the tachyonic instability suffered by the density perturbations, and we show that only a set of wavenumbers corresponding to small scales are affected by such instability.

The matter and temperature power spectra that arise from the axion model are presented in Section 3, and we use them to impose some bounds on the free parameters of the model: the axion mass $m_a$ and the decay parameter $f_a$ mentioned above. We also make a qualitative assessment of how the Lyman-$\alpha$ 1D mass power spectrum could constrain the parameters of our model with Ly-$\alpha$. We observe that while the FDM model with masses $m_a \leq 10^{-22}$eV is ruled out, it is possible for the axion field to pass the constraints if endowed with the trigonometric potential (1.1).

Motivated by the characteristic cut-off that this model presents in the mass power spectrum (MPS), in Section 4 we define both the growth factor $D_k$ and the velocity growth factor $f_k$, not only as a function of the scale factor but also with their dependence on the length scale. We also build the combination $[f_k \sigma_8(z)]$, and no major difference with respect to the CDM prediction were found. We then analyze the semi-analytical Halo Mass Function (HMF), which, like in the case of the MPS, it shows an enhancement in its amplitude when considering the potential (1.1). Finally, in Section 5, we give some conclusions and perspectives for some future work.

2  Background and Linear Perturbations Dynamics

In this section we show the dynamical equations for the evolution of both, background and linear perturbations of the axion model (1.1). Following previous work [93, 96], we rewrite these equations as a dynamical system and then, by a polar change of variables, we obtain a set of first order differential equations which is more appropriate for numerical studies of ultra–light axions than using directly the field equations.

2.1  Background Evolution

The Einstein-Klein-Gordon equations for a minimally-coupled scalar field $\phi$ endowed with a generic potential $V(\phi)$, in a Friedmann-Robertson-Walker spacetime with null spatial curvature are given by

\[
H^2 = \frac{\kappa^2}{3} \left( \sum_j \rho_j + \rho_{\phi} \right),
\]

\[
\dot{H} = -\frac{\kappa^2}{2} \left[ \sum_j (\rho_j + p_j) + (\rho_{\phi} + p_{\phi}) \right],
\]

\[
\dot{\rho}_j = -3H(\rho_j + p_j), \quad \ddot{\phi} = -3H\dot{\phi} - \frac{dV(\phi)}{d\phi},
\]

where $\kappa^2 = 8\pi G$, a dot denotes derivative with respect to cosmic time $t$, and $H$ is the Hubble parameter. Also, the scalar field energy density $\rho_{\phi}$ and pressure $p_{\phi}$ are given by the canonical expressions:

\[
\rho_{\phi} = \frac{1}{2} \dot{\phi}^2 + V(\phi), \quad p_{\phi} = \frac{1}{2} \dot{\phi}^2 - V(\phi).
\]
In order to transform the Klein-Gordon (KG) equation (2.1b), we define a new set of polar variables based on previous works [97–99],

\[
\Omega^{1/2}_{\phi} \sin(\theta/2) = \frac{\kappa \dot{\phi}}{\sqrt{6H}}, \quad \Omega^{1/2}_{\phi} \cos(\theta/2) = \frac{\kappa V^{1/2}}{\sqrt{3H}}, \quad y_1 = -\frac{2\sqrt{2}}{H} \Omega_{\phi} V^{1/2}.
\] (2.3)

with which the KG equation can be written, for the particular case of potential (1.1), as a dynamical system in the form:

\[
\begin{align*}
\theta' &= -3 \sin \theta + y_1, \\
y'_1 &= \frac{3}{2} (1 + w_{tot}) y_1 + \frac{\lambda}{2} \Omega_{\phi} \sin \theta, \\
\Omega'_{\phi} &= 3 (w_{tot} - w_{\phi}) \Omega_{\phi}.
\end{align*}
\] (2.4)

Here a prime denotes derivative with respect to the number of \( e \)-foldings \( N \equiv \ln(a/a_i) \), with \( a \) the scale factor of the Universe and \( a_i \) its initial value. The decay constant appears explicitly in the newly defined (dimensionless) parameter \( \lambda = 3/\kappa^2 f_a^2 \), and then the FDM case with \( \lambda = 0 \) (studied in Ref. [98]) is obtained in the limit \( f_a \to \infty \). In contrast, we see that the mass parameter \( m_a \) does not appear at all in the new equations of motion. Following the classification suggested in [99], the decay constant is an active parameter, whereas the mass is a passive one that does not have any influence in the evolution of the field \( \phi \). The equation of state (EoS) for the axion field is directly related to the dynamical variable \( \theta \) as,

\[
w_{\phi} \equiv \frac{p_{\phi}}{\rho_{\phi}} = \frac{x^2 - y^2}{x^2 + y^2} = -\cos \theta.
\] (2.5)

Eq. (2.4) is a compact representation of the KG equation, and they reveal that the true variables driving the scalar field dynamics are \( \{\theta, y_1, \Omega_{\phi}\} \). They also show that the effect of the trigonometric potential of Eq. (1.1) is encoded in one free parameter given by \( \lambda \), and then it will be possible to analyze in one stroke the cosmological properties of both, the axion field (\( \lambda > 0 \)) and the FDM case (\( \lambda = 0 \)), see [93, 98].

2.2 Initial conditions

For a correct numerical implementation of the equations of motion (2.4) within a cosmological setting, it is necessary to estimate the right initial conditions of the dynamical variables at very early times. As done in Ref. [98] for the FDM case, in this section we find semi-analytical solutions for the radiation dominated era and extrapolate them to the present time.

Assuming that all quantities are small and positive, i.e. \( (\theta, y_1, \Omega_{\phi}) \ll 1 \), Eq. (2.4) takes the form (at linear order),

\[
\begin{align*}
\theta' &\simeq -3 \theta + y_1, \\
y'_1 &\simeq 2 y_1, \\
\Omega'_{\phi} &\simeq 4 \Omega_{\phi},
\end{align*}
\] (2.6)

whose analytical solutions are

\[
\begin{align*}
\theta &= (1/5) y_1 + C(a/a_i)^{-3}, \\
y_1 &= y_{1i} (a/a_i)^2, \\
\Omega_{\phi} &= \Omega_{\phi i} (a/a_i)^4,
\end{align*}
\] (2.7)

where a subscript \( i \) denotes the corresponding initial value for each variable. The solutions (2.7) are the same as those of the quadratic potential studied in [98], basically because the second term on the rhs of Eq. (2.4b) is of second order, which means that at early times the influence of \( \lambda \) in the solutions should be negligible.
Assuming that the axion field starts to behave as CDM at $a = a_{\text{osc}}$, when it starts to oscillate rapidly around the minimum of the potential and the EoS first passes through the value $w_\phi = 0$ (corresponding to $\theta = \pi/2$), it can be shown that the estimated initial conditions are obtained from the following equations,

$$
\theta_i = \frac{2}{5} \frac{m}{H_i}, \quad y_{1i} = 5\theta_i, \quad \Omega_{\phi i} = \frac{a_i^4}{a_{\text{osc}}^3} \frac{\Omega_{\phi 0}}{\Omega r_0}, \quad a_{\text{osc}}^2 = \frac{\pi \theta_i^{-1} a_i^2}{2 \sqrt{1 + \pi^2/36}},
$$

(2.8)

where $H_i$ and $a_i$ are the initial values of the Hubble parameter and the scale factor, and $\Omega r_0$ ($\Omega_{\phi 0}$) is the present radiation (axion) density parameter (see [98] for more details).

We now find a next-to-leading order solution for the initial conditions that takes into account the presence of $\lambda$, and for that we follow an iterative method. Let us consider the first order solutions (2.7), substitute them in Eq. (2.4b) and solve for a new solution of $y_1$. We find that

$$
y_1 = 5\theta_i (a/a_i)^2 + \frac{\lambda}{8} \Omega_{\phi i} \theta_i (a/a_i)^6.
$$

(2.9)

If we now use the foregoing solution and plug it into the right hand side of Eq. (2.4a), we find that a corrected solution for $\theta$ is

$$
\theta = \theta_i (a/a_i)^2 \left[ 1 - \frac{\lambda}{72} \Omega_{\phi i} + \frac{\lambda}{72} \Omega_{\phi i} (a/a_i)^4 \right],
$$

(2.10)

whereas the solution for $\Omega_\phi$ remains the same. From the combination of the above equations, we obtain from the matching condition at $a = a_{\text{osc}}$ that

$$
a_{\text{osc}}^2 \left( 1 + \frac{\lambda}{72} \Omega r_0 a_{\text{osc}} \right) = \frac{\pi \theta_i^{-1} a_i^2}{2 \sqrt{1 + \pi^2/36}}.
$$

(2.11a)

Notice that for $\lambda = 0$ we recover, as expected, the required matching equation for the quadratic potential, see the last equation in (2.8). Instead of Eq. (2.8), we will use the new set of Eqs. (2.9), (2.10) and (2.11a) to calculate the initial conditions of the dynamical variables. As shown in the appendix D.5, the iterative integration method could be used again to generate a higher-order equation to determine $a_{\text{osc}}$, but we will restrict ourselves to Eq. (2.11a) as it is enough for the purposes of this paper.

In contrast to the FDM case, there is an additional trigonometric constraint that is characteristic of the axion potential, and that can be obtained directly from the definitions (2.3),

$$
4 \frac{m^2}{H_i^2} = y_{1i}^2 + 2 \lambda \Omega_{\phi i}.
$$

(2.11b)

Although we use it only as an additional constraint for the initial conditions, it should be emphasized that Eq. (2.11b) is of general applicability at all times. Again, for the case $\lambda = 0$ we recover the usual expression of the FDM case, namely $y_{1i} = 2m/H_i$. Hence, the initial conditions in the general case are obtained from the combined solution of Eqs. (2.11a), (2.11b) and

$$
y_{1i} = 5\theta_i \left( 1 + \frac{\lambda}{40} \Omega_{\phi i} \right), \quad \Omega_{\phi i} = \frac{a_i^4}{a_{\text{osc}}^3} \frac{\Omega_{\phi 0}}{\Omega r_0}.
$$

(2.11c)

The initial conditions are further adjusted by means of the shooting procedure implemented in CLASS to give the right current values of the physical parameters. The values of
$a_{\text{osc}}$ for different values of $\lambda$, as obtained from the numerical solutions, are shown in Table 1, where it can be seen that the onset of the scalar field oscillations suffers a delay as $\lambda$ increases. The most extreme value that we will consider is $\lambda = 10^5$, as for larger values it is difficult to calculate the initial conditions because of the exponential sensitivity that appears in the estimation of $a_{\text{osc}}$.

| $\lambda$ | $\log(a_{\text{osc}})$ | $\delta\theta_0$ |
|-----------|-------------------------|------------------|
| $0$       | -6.159                  | —                |
| $10$      | -6.159                  | 174°             |
| $10^2$    | -6.159                  | 162°             |
| $10^3$    | -6.143                  | 124°             |
| $10^4$    | -6.048                  | 44°              |
| $10^5$    | -5.838                  | 0.47°            |

Table 1: Numerical values for the onset of oscillations of the axion field for each value of $\lambda$. For $\lambda = 0, 10, 10^2$, oscillations start at the same time, whereas for $\lambda = 10^3, 10^4, 10^5$, we notice that oscillations start later as $\lambda$ increases. In the last row we show the initial field displacement from the top of the axion potential, for a comparison with the Extreme Axion Wave Dark Matter model [92], see Appendix C for details.

A comparison of the evolution of the CDM and axion densities is shown in Figure 1. While the CDM density redshifts as $a^{-3}$, we see that the axion energy density remains constant before the start of the field oscillations at $a = a_{\text{osc}} \approx 10^{-6}$, but afterwards the two densities evolve together. As also shown in the inset, the onset of the field oscillations depends on the value of the decay constant through the parameter $\lambda$, and in general the oscillations are delayed as the value of $\lambda$ increases, which is consistent with the numerical results shown in Table 1. We can also notice that the transition of the axion energy density to the CDM behavior happens more abruptly for larger values of $\lambda$, which is one of the consequences of the exponential sensitivity of the numerical solutions on the initial conditions that we discussed above.

### 2.3 Linear Perturbations

Now, we consider linear perturbations around the background values of the FRW line element (in the synchronous gauge) as well as for the scalar field in the following form:

$$ds^2 = -dt^2 + a^2(t)(\delta_{ij} + h_{ij})dx^i dx^j, \quad \phi(x,t) = \phi(t) + \varphi(x,t), \quad (2.12)$$

where $h_{ij}$ and $\varphi$ are the metric and scalar field perturbations respectively. The linearized KG equation, in Fourier space and for a general potential, reads [100–103]:

$$\ddot{\varphi} = -3H\dot{\varphi} - \left( \frac{k^2}{a^2} + \frac{\partial^2 V(\phi)}{\partial \phi^2} \right) \varphi - \frac{1}{2} \dot{\varphi} \dot{\bar{h}}, \quad (2.13)$$

where $\bar{h} = \bar{h}_j^j$ is the trace of scalar metric perturbations, and $k$ is the comoving wavenumber. Although a functional dependence of the scalar field perturbation is not explicitly shown, note that Eq. (2.13) is written for a Fourier mode $\varphi(k,t)$. After a change of variables to the new quantities $\delta_0$ and $\delta_1$ (see Appendix A for details), Eq.(2.13) is described by the following
Figure 1: Evolution of CDM and SFDM energy density for a fixed axion mass of $m_{\phi} = 10^{-22}$ eV, and different values of the decay parameter $\lambda$ of the potential (1.1). Initially the amplitude of the axion energy density is less than that of the CDM, but once the axion field starts to oscillate (around $a = 10^{-6}$), it evolves just as the CDM case. Inset: It can be noticed that larger values of parameter $\lambda$ delay the scalar field oscillations, then the axion field evolves as CDM. Vertical lines indicate the onset of oscillations $\log(a_{\text{osc}})$ for each value of $\lambda$, see also Table 1.

The density and pressure contrasts $\delta_{\phi}$, $\delta p_{\phi}$, and velocity divergence $\theta_{\phi}$, are given by the standard definitions [102, 103, 105], and in terms of the new perturbation variables they take
the form:
\[
\delta_\phi = \frac{\dot{\phi} \phi - \partial_\phi V \phi}{\phi^2/2 + V(\phi)} = \delta_0 ,
\]
\[
\delta_p = \frac{\dot{\phi} \phi - \partial_\phi V \phi}{\phi^2/2 + V(\phi)} = \sin \theta \delta_1 - \cos \theta \delta_0 ,
\]
(2.15a)
\[
(r_\phi + p_\phi) \theta_\phi = \frac{k^2}{2am} \rho_\phi [(1 - \cos \theta) \delta_1 - \sin \theta \delta_0] .
\]
(2.15b)

It is important to mention that we have gained physical interpretation for the new dynamical variable \(\delta_0\): it plays the role of the scalar field density contrast, \(\delta_\phi\), according to the first expression in Eq. (2.15a). This implies that Eq. (2.14a) is the closest we can get of a fluid even in the generalized case [105].

For the particular case of the axion field endowed with the potential (1.1), the expressions (2.14) now reads
\[
\delta_0' = \left[-3 \sin \theta - \frac{k^2}{k_J^2} (1 - \cos \theta) \right] \delta_1 + \frac{k^2}{k_J^2} \sin \theta \delta_0 - \frac{\bar{h}'}{2} (1 - \cos \theta) ,
\]
(2.16a)
\[
\delta_1' = \left[ -3 \cos \theta - \frac{k_{eff}^2}{k_J^2} \sin \theta \right] \delta_1 + \frac{k_{eff}^2}{k_J^2} (1 + \cos \theta) \delta_0 - \frac{\bar{h}'}{2} \sin \theta ,
\]
(2.16b)

where we have defined an effective wavenumber of the perturbations as \(k_{eff}^2 \equiv k^2 - \lambda a^2 H^2 \Omega_\phi/2\).

The equations of linear perturbations for the standard FDM case are again obtained when \(\lambda = 0\), for which \(k_{eff}^2 = k^2\) is just the standard Laplacian term in Fourier space. Because now \(y_1 = 2m_\phi / H\), the Jeans wavenumber \(k_J\) is then the only characteristic scale in the evolution of linear perturbations, and the responsible for the appearance of a sharp cut-off in their mass power spectrum: linear perturbations are heavily suppressed for wavenumbers \(k > k_J\). The Jeans wavenumber is always proportional to the geometric mean of the Hubble parameter \(H\) and the boson mass \(m\), namely \(k_J = \alpha \sqrt{2Hm}\), which shows that the cut-off in the MPS is sensitive to both the parameters of the axion model and to the background expansion. More details about the cut-off of linear perturbations in the FDM case \(\lambda = 0\) can be found in [98].

### 2.4 Tachyonic instability

One of the main effects on linear perturbations of axion fields (for \(\lambda > 0\)) is the appearance of an enhancement in the growth of the density contrast \(\delta_0\), that was first discussed in [92, 93, 106] and thereby dubbed as a tachyonic instability. Such instability provokes the appearance of a bump in the MPS of the perturbations that is well localized in wavenumbers around the Jeans one \(k_J\).

To have a qualitative understanding of the tachyonic instability, we follow and extend the procedure already outlined in [93]. Let us write Eqs. (2.16) on rapid oscillations regime, under which all trigonometric terms are time-averaged to zero, \(\langle \sin \theta \rangle = \langle \cos \theta \rangle = 0\). Hence, we find
\[
\delta_0' = - \frac{k^2}{k_J^2} \delta_0 - \frac{\bar{h}'}{2} , \quad \delta_1' = \frac{k_{eff}^2}{k_J^2} \delta_0 .
\]
(2.17a)

\[^1\text{Recently, the authors in [107] made a comparison of the different approximations one can find in the literature to follow the cosmological evolution of ultra-light bosons. Such approximations, which correspond to diverse choices in cycle-averaging procedures, are necessary to deal with the rapid oscillations of the axion field at late times, see the original field equations (2.1) and (2.13). It was then concluded that our approximation method, which has been used previously in Refs [93, 96], is the closest, compared to others, to the exact solution of the field equations of motion.}\]
If we neglect, for simplicity, the time variation of both $k_J$ and $k_{eff}$, the foregoing equations can be combined into the form of a forced harmonic oscillator for the density contrast, namely,

$$\delta''_0 + \omega^2 \delta_0 = -\frac{\bar{h}''}{2}, \quad \omega^2 \equiv \frac{k^2 k_{eff}^2}{k_J^4}. \tag{2.17b}$$

From the above we see that the tachyonic instability requires of two conditions. Firstly, the start of rapid oscillations of the field around the minimum of its potential, and secondly, a negative squared amplitude of the angular frequency, $\omega^2 < 0$, in Eq. (2.17b). The latter condition is possible because the effective wavenumber $k_{eff}^2$ can be either positive or negative, although it depends on a non-simple combination of the cosmological quantities $a$, $H$ and $\Omega_\phi$.

The tachyonic instability and the conditions for its appearance are illustrated in Figure 2. In the top panel, we show the relative difference between the axion density contrast with respect to CDM $\Delta_\delta \equiv (\delta_\phi - \delta_{CDM})/\delta_{CDM}$, and in the lower panel the evolution of the angular frequency $\omega$. The axion mass and the wavenumber that were chosen have the values $m_\phi = 10^{-22}\text{eV}$ and $k = 8h/\text{Mpc}$ respectively, while the decay parameter was chosen as $\lambda = 0$ (FDM case, $f \to \infty$) and $\lambda = 10^5$ (extreme case, $f \simeq 10^{-2} m_{Pl}$). The light gray region indicates the period of time when the tachyonic condition $\omega^2 < 0$ occurs in the case $\lambda = 10^5$.

Figure 2: (Left) Evolution of the relative difference between density contrasts $\Delta_\delta$ (top) and frequency $\omega$ (bottom) for a fixed axion mass of $m = 10^{-22}\text{eV}$ and wavenumber $k = 8h/\text{Mpc}$, evaluated at $\lambda = 0$ (solid blue line) and $\lambda = 10^5$ (solid red line). The light gray region indicates the duration of the tachyonic instability, while the vertical dotted blue and red lines show the onset of the oscillation of the scalar field for $\lambda = 0$ and $\lambda = 10^5$ respectively. (Right) Frequency $\omega$ as function of the wavenumber $k$ for an axion with $m_\phi = 10^{-22}\text{eV}$ and $\lambda = 10^5$, and for three different times of the scalar field evolution: at the onset of oscillation $a_{osc} = 1.45 \times 10^{-6}$ (green line), at a threshold value $a_{th} = 4.95 \times 10^{-6}$ (blue line), and at the end of the tachyonic instability $a_{end} = 2.3 \times 10^{-5}$ (purple line). The region within the two vertical dashed green lines indicates the range of wavenumbers that will suffer the tachyonic effect. The black horizontal solid line stands for $\omega = 1$. See text for more details.

In the FDM case we see that the angular frequency is always positive and less than unity, $0 < \omega^2 = k^4/k_J^4 < 1$, and then Eq. (2.17b) is just the equation of motion of a forced oscillator and the tachyonic instability never happens. The density contrast, $\delta_0$, for the chosen wavenumber, can not catch up completely with the CDM solution after the onset.
of rapid oscillations (at around \( a \simeq 10^{-6.16} \)), but nonetheless keeps a constant ratio with respect to CDM at late times (this constant ratio can be explained in terms of the growth factor, see Sec. 3 below). The result is that the amplitude of the MPS at this wavelength is suppressed respect to the CDM one. The particular value \( k = 8 \, h / \text{Mpc} \) was chosen because it corresponds, approximately, to the cut-off scale in the FDM case for \( m_\phi = 10^{-22} \text{eV} \).

In contrast, for the value \( \lambda = 10^5 \) we see that the onset of oscillations (at \( a_{\text{osc}} = 1.45 \times 10^{-6} \)) occurs after the appearance of the tachyonic instability (at \( a \simeq 10^{-6.3} \)). At the same time, and after the onset of oscillations, the amplitude of the axion density contrast \( \delta_0 \) grows quickly reaching larger values than that of CDM. This growth persists until just after the tachyonic instability disappears (when once again \( \omega^2 > 0 \)) at around \( a \simeq 10^{-4.8} \). After this, the density contrast \( \delta_0 \) then evolves like in the FDM case and keeps a constant amplitude with respect that of CDM at late time (see also Sec. 3 below). As a result, the MPS at \( k = 8 \, h / \text{Mpc} \) is now enhanced with respect to that of CDM. The tachyonic effect and its duration is scale dependent, as we show in Figure 2, where we plot the frequency \( |\omega| \) as function of the wavenumber \( k \), for an axion with \( m_\phi = 10^{-22} \text{eV} \) and \( \lambda = 10^5 \), and three fixed times. The green curve corresponds to the time at the onset of the axion field oscillations (at \( a = a_{\text{osc}} \)), and then we expect the tachyonic instability to start happening for those wavenumbers for which \( \omega^2 < -1 \), that is, for those in the range \( 2 < k / (h / \text{Mpc}) < 22 \).

There is a characteristic time, labeled as the threshold for tachyonic instability at \( a = a_{\text{th}} \) (blue curve), for which just a small range of wavenumbers around \( k = 8h / \text{Mpc} \) barely comply with the condition \( \omega^2 = -1 \). Moreover, we also see that wavenumbers \( k > 12h / \text{Mpc} \) have left the tachyonic regime by this time, as \( \omega^2 > 0 \) for them. Finally, the end of the tachyonic instability at \( a = a_{\text{end}} \) is also shown. The value of \( a_{\text{end}} \) is somewhat arbitrary, but we have chosen it such that even the smallest of the wavenumbers in the initial range of tachyonic instability is no longer stimulated, and then the downturn of the (purple) curve occurs at \( k \sim 5 \, h / \text{Mpc} \).

Summarizing, we find that for large scales \( k \lesssim 2h / \text{Mpc} \) the tachyonic instability is practically non-existent, and also for them the condition \( 0 < \omega^2 \ll 1 \) is accomplished at all times. The evolution of the density contrast for these scales is governed by the equation \( \delta''_0 \simeq -(1/2) \bar{h}'' \) (see Eq. (2.13)), and we obtain for them the same solution as for CDM linear perturbations, that is \( \delta_0 \simeq -(1/2) \bar{h} \). Likewise, small scales \( k \gtrsim 22h / \text{Mpc} \) are also always free from tachyonic instabilities as for them \( \omega^2 > 0 \) at all times. The latter condition means that they do not longer grow with the CDM solution, but now they must be suppressed as in the standard FDM case. Therefore, wavenumbers within the range \( 2 \lesssim k / (h / \text{Mpc}) \lesssim 22 \) will present an enhancement in their density contrast amplitude, as was shown in Figure 2 for the case \( k = 8h / \text{Mpc} \).

The range of wavenumbers \( k \) that suffer a tachyonic instability is mainly determined by the axion mass \( m_\phi \). The arguments above show that the instability appears around the wavenumber that marks the cut-off of the corresponding FDM case. As shown in the example, the range of wavenumbers that suffers a tachyonic instability in the case \( \lambda \neq 0 \) shift to larger (smaller) values for larger (smaller) axion masses.

### 3 Cosmological constraints

The solutions of Eq. (2.16) are useful to build up cosmological observables such as the CMB anisotropies and the MPS, which can then be contrasted with observations. In this section...
we first present a qualitative comparison with the observables, and then present the details and results from a parameter estimation procedure.

### 3.1 CMB anisotropies

The CMB power spectrum for both CDM and SFDM, for a couple of values of the axion mass, is shown in Figure 3, where we have included data from the Planck Collaboration.\(^2\) For a fiducial axion mass of \(m_\phi = 10^{-22}\) eV (left panel) we observe that, regardless of the value of \(\lambda\), the axion field reproduces the CMB spectrum as good as CDM. In fact, the major discrepancy between both cases is of \(\sim 0.06\%\) for large multipoles. In contrast, for an axion mass of \(m_\phi = 10^{-26}\) eV (right panel), we clearly note that the CMB spectrum does not fit the observational data, with a major discrepancy of \(\sim 30\%\) for \(l \sim 10^3\).

![Figure 3](image_url)

**Figure 3:** Temperature Power Spectrum for CDM and SFDM for two axion masses: \(m_\phi = 10^{-22}, 10^{-26}\) eV. The effect of \(\lambda\) is clearly noted for the latter where, for large multipoles, the differences are greater as the value of \(\lambda\) increases. See text for more details.

We have also considered CMB observations for high multipoles, as can be seen in Figure 4, where we have also included a wider range of values for \(m_\phi\) and \(\lambda\). We observe in the upper panel that for large multipoles the case of an axion mass of \(10^{-26}\) eV with quadratic \((\lambda = 0)\) and trigonometric potential \((\lambda = 10^2)\) still have more amplitude than the extreme case with \(m_\phi = 10^{-22}\) eV and \(\lambda = 10^5\). In particular, we observe that for a given axion mass the effect of consider \(\lambda > 0\) is to increase the amplitude of the CMB power spectrum in comparison with the FDM case (this can be clearly seen when \(m_\phi = 10^{-26}\) eV). The lower panel shows that observations such as Planck, SPT and ACT, do not constrain the fiducial case of a free axion with mass \(10^{-22}\) eV and \(\lambda = 0\). On the other hand, for an axion with mass and decay parameter given by \(m_\phi = 10^{-24}\) eV and \(\lambda = 6 \times 10^3\), respectively, we note that experiments such as ACT rules out such combination of parameters. Thus, considering numerical solutions with a difference within sub-percent levels with respect to CDM, and with lower amplitude that the minimum sensitivity of CMB experiments, the range of axion masses with \(\lambda \neq 0\) consistent with CMB observations seem to be given by \(m_\phi > 10^{-24}\) eV. This will be important in Section 3.4 when we carry out the statistical analysis.

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\(^2\)Based on observations obtained with Planck (http://www.esa.int/Planck), an ESA science mission with instruments and contributions directly funded by ESA Member States, NASA, and Canada.
Figure 4: CMB anisotropies for high multipoles. Data from Planck (green dots), SPT [108] (red dots) and ACT [109] (blue dots) are shown to compare with our numerical solutions. For masses lighter as $m_{\phi} = 10^{-26}$eV and $\lambda = 0, 10^2$, we found notorious discrepancies with the observational data. The lower panel shows the relative differences between CDM and SFDM with \{$m_{\phi} = 10^{-22}$eV, $\lambda = 0$\} (yellow dashed line) and \{m_{\phi} = 10^{-24}$eV, $\lambda = 6 \times 10^3$\} (red dotted line). The horizontal green, red and blue lines indicate the minimum sensitivity for Planck, SPT and ACT observations respectively, given by ($\sigma_{\text{Planck}}, \sigma_{\text{SPT}}, \sigma_{\text{ACT}}$) = (3.8, 2.2, 1.9)$\mu$K$^2$.

3.2 Mass power spectrum
Likewise, Figure 5 shows the MPS for CDM (black line) and SFDM with masses $m_{\phi} = 10^{-22}, 10^{-23}$eV for several values of the decay parameter $\lambda = 0, 10^1, 10^2, 10^3, 10^4$ (solid gray, dashed gray, dashed blue, dotted yellow, and solid cyan line respectively), as well as for the extreme values of $\lambda$ corresponding to each value of the axion mass $\lambda = 4.3 \times 10^4$ for $m_{\phi} = 10^{-23}$eV (dashed green line), and $\lambda = 10^5, 1.5 \times 10^5$ for $m_{\phi} = 10^{-22}$eV (dashed green, and dotdashed red line respectively). For each case we observe the well-known cut-off at large wavenumbers, but this time there is also present a bump in the MPS at the cut-off scale for each value of the axion mass. As discussed in Sec. 2, the tachyonic instability produces an enhancement in the density contrast after the onset of the oscillations of the axion field, and such instability is going to be present in the MPS, at least for a range of wavenumbers as explained in [93] and in Section 2.4. It is important to note that, for the case $m_{\phi} = 10^{-22}$eV with $\lambda = 10^5$ (green dashed line in the bottom Figure) the bump is within the range of wavenumbers showed in Figure 2. For a qualitative comparison, we have included data from BOSS DR11 (yellow dots) [110], and from Ly$\alpha$ forest (black dots) [111].

3.3 Lyman-alpha
Based on the comparison of the CMB anisotropies and MPS with available data, we see that the SFDM model describes such cosmological observables as good as CDM model does, as long as the axion mass is $m_{\phi} = 10^{-24}$eV. This is a lower value than that imposed by Lyman-\(\alpha\) observations of the 1-dimensional flux power spectrum (P1D), for the axion mass endowed with a quadratic potential (FDM case), given by $m_{\phi} \gtrsim 10^{-21}$eV [112, 113].

\[ \text{Note that [106] reports a different constraint with the same observations, claiming that including quantum pressure to numerical simulations of FDM leads to a lower bound of } m_{\phi} = 10^{-23} \text{eV.} \]
Figure 5: MPS for SFDM with axion masses $m_{\phi}/eV = 10^{-22}, 10^{-23}$, and $\lambda$ from zero up to the maximum values reached for each axion mass. It can be noted that, for all the axion masses considered there is a cut-off at small scales (larges $k$’s), and even more, there is an enhancement of the MPS at such scales when considering large values of the parameter $\lambda$. Cosmological data from BOSS DR11 (yellow dots) [110], and from Ly$\alpha$ forest (black dots) [111] are shown for reference. See text for more details.

To qualitatively assess the constraints that the Lyman-$\alpha$ P1D can impose to the model under consideration, we compare in Figure 6 the relative difference with respect the LCDM model, for the 1-dimensional matter power spectrum with the precision of current P1D measurements with data sets such as eBOSS [114], HIRES/MIKES[115] and XQ-100 [116] (yellow, blue and red rectangle respectively). We do this for the following combinations: $m_{\phi} = 4 \times 10^{-21}eV$ for $\lambda = 0, 1.3 \times 10^5$ (green lines), $m_{\phi} = 10^{-22}eV$ for $\lambda = 0, 8 \times 10^4$ (blue lines), and $m_{\phi} = 10^{-23}eV$ for $\lambda = 0, 3.1 \times 10^4$ (red lines). We can see that combinations with $\lambda = 0$ are excluded by the data except for the larger mass, while combinations with non
null and larger values of the decay parameter $\lambda$ might be allowed. This means that an axion field endowed with a trigonometric potential could still be allowed by Lyman-$\alpha$ observations. Definite constraints, of course, might come from a full analysis of the Lyman-$\alpha$ P1D with current and future data such as DESI [117].

**Figure 6**: 1D MPS for the Axion field compared to the $\Lambda$CDM one. We show the cases $m_\phi = 4 \times 10^{-21}\text{eV}$ for $\lambda = 0, 1.3 \times 10^5$ (green lines), $m_\phi = 10^{-22}\text{eV}$ for $\lambda = 0, 8 \times 10^4$ (blue lines), and $m_\phi = 10^{-23}\text{eV}$ for $\lambda = 0, 3.1 \times 10^4$ (red lines). For reference we have included colored rectangles indicating the rough precision of current data from BOSS [114] (yellow), HIRES/MIKES [115] (blue) and XQ-100 [116] (red) to show that these experiments can be used to constraint the axion field parameters $m_\phi$ and $\lambda$.

### 3.4 Comparison with Data

In this Section we will analyze the parameter space of our model in order to find constraints using data from the Planck 2015 data release [118]. This is done by using the parameter estimator code MONTE PYTHON [119], to compute the posterior distribution of several cosmological parameter by implementing Bayes’ Theorem, which reads

$$P(\Theta \mid D) = \frac{\Pi(\Theta)L(D \mid \Theta)}{E(D)}$$

(3.1)

where $\Theta$ stands for the parameters of the cosmological model, $D$ is the data from cosmological surveys, $\Pi$ is the prior probability, the likelihood $L$ representing the probability distribution of the data for each allowed input $\Theta$, and the evidence $E$ which encodes how well our original assignments managed to predict the data, and which can be calculated as $E = \int \Pi(\Theta)L(D \mid \Theta)d\Theta$.

Our model is defined by two parameters, the axion mass $m_\phi$ and the decay parameter $\lambda$, and additionally by the standard cosmological parameters of $\Lambda$CDM model, the physical baryon density parameter $100\omega_b$, the (logarithmic) power spectrum scalar amplitude $\log(10^{10}A_s)$, the scalar spectral index $n_s$, the Thomson scattering optical depth due to reionization $\tau_{\text{reio}}$, and the angular size of sound horizon at decoupling $100\theta_s$. Note that we do not include $\Omega_c$ (dark matter density parameter) because that information will be provided by our axion field. Thus, in total we have 7 cosmological parameters given by
\[ \Theta = [100\omega_b, \log(10^{10} A_s), n_s, \tau_{reio}, 100\theta_s, \log m_\phi, \log \lambda], \] where we have defined the scalar field parameters \( m_\phi \) and \( \lambda \) in logarithmic scale. We are going to consider the CMB as the cosmological observable to constraint our model, hence, we will take the data and likelihoods from Planck Collaboration 2015. The initial input given to the code to run the chains is summarized in Table 2, where the initial mean value, as well as the priors and the 1-\( \sigma \) value, are specified for each of the parameters \( \Theta \). Particularly, the input for the axion field parameters \( m_\phi \) and \( \lambda \) are chosen to be consistent with the numerical solutions obtained with \textsc{class}. Thus, the means and priors for \( m_\phi \) and \( \lambda \) will be setted based on the cosmological evolution of the axion field that we were able to explore numerically.

| Param      | mean     | prior min | prior max | 1-\( \sigma \) |
|------------|----------|-----------|-----------|---------------|
| 100 \( \omega_b \) | 2.2253   | None      | None      | 0.028         |
| 100 \( * \theta_s \) | 1.0418   | None      | None      | 3 \times 10^{-4} |
| \( \ln 10^{10} A_s \) | 3.0753   | None      | None      | 0.0029        |
| \( n_s \)   | 0.96229  | None      | None      | 0.0074        |
| \( \tau_{reio} \) | 0.09463  | 0.04      | None      | 0.013         |
| \( \log \lambda \) | 3        | -5        | 4         | 0.05          |
| \( \log m_\phi \) | -22      | -26       | -16       | 0.05          |

Table 2: Initial input for the parameters \( \Theta \) of our SFDM model. Whereas no priors were specified for the standard cosmological parameter (only a lower bound prior for \( \tau_{reio} \) of 0.04), the prior for the axion field parameters were chosen according to the numerical solution we obtained from the \textsc{class} code.

We have run the chains with the Metropolis-Hasting algorithm, with the Gelman-Rubin convergence criterion [120] fulfilling \( R - 1 < 0.05 \). The minimum of the likelihood and the \( \chi^2 \) function we obtained are respectively given by \(- \ln L_{\text{min}} = 5636.48\), \( \chi^2_{\text{min}} = 1.127 \times 10^4 \). The posteriors are shown in Figure 7. While the standard cosmological parameters \([100\omega_b, \log(10^{10} A_s), n_s, \tau_{reio}, 100\theta_s]\) show their observed values at the present day, the axion field parameters \( m_\phi \) and \( \lambda \) have a non-Gaussian posterior. However, the axion mass has a lower bound given by \( \log m_\phi = -24.2 \) at 95.5\% CL. This is consistent with the previous result shown in Section 3.1, where we compare our numerical solutions with data from the CMB anisotropies (see Figure 4). Thus, whereas a restriction for the axion mass was found, it seems that the data from CMB is not able to constraint the value of the decay parameter \( \lambda \). That is, for all the values of \( \lambda \) we were able to explore, it was possible to find consistent numerical solutions for the rest of the cosmological parameters.

4 Halo formation within axion models

4.1 Growth factor \( D \) and velocity growth factor \( f \) with scale-dependence

It is well known that the growth factor \( D \) for CDM model does not present explicit scale dependence, i.e., it is independent of the wavenumber \( k \), but it is the transfer function \( T \) which carries such information. Such separation of variables on the gravitational potential \( \Phi \) given by \( \Phi(k,a) \propto T(k)D(a) \), can be done in a standard CDM scenario, and it allows to study the growth of matter overdensities in the structure formation process [121–125]. For instance, the standard parameterization for the velocity growth factor is given by \( f(z) = \)
Figure 7: 1D and 2D posterior distributions for the axion field parameters $m_\phi$ and $\lambda$ (in logarithmic scale) together with the standard cosmological parameters of CDM model. We can set a lower bound for the value of the axion mass of $\log m_\phi = -24.2$ at 95.5\% C.L. On the other hand, a flat posterior is obtained for the decay parameter $\lambda$, indicating that CMB anisotropies do not constraint such parameter, at least within the prior $10^{-5} \leq \lambda \leq 10^4$. See text for more details.

$\Omega_m^\gamma(z) \ [126–132]$, where $\gamma$ is called the growth index, and $\Omega_m$ is the energy density parameter for the total matter as function of the redshift $z$. Such expression does not contain explicit information of $k$. However, the scale-dependence on the quantities $D$ and $f$ have been studied in alternatives models of gravity [125, 133–139] mainly due to the appearance of an effective Newton’s constant containing explicit dependence on $k$. Therefore, while it is true that within the CDM scenario the growth factor $D$ and its velocity $f$ are the same for every mode $k$, this may not be true in particular for models with a cut-off in the mass power spectrum, such as those we are studying in this work.

To explore possible deviations from the CDM model on such cosmological quantities, in this section we present an approach to obtain the evolution of both, the growth factor $D$ and
the velocity growth factor $f$ as function of the wavenumber $k$ for SFDM with the axion–like potential. As starting point, let us revisit the system of equations that rules the dynamics of the SFDM linear perturbations after the onset of rapid oscillations. From Eq. (2.16), and considering $\cos \theta \sim \sin \theta \sim 0$, we find

$$\delta_0'' + \omega^2 \delta_0 = -\frac{\ddot{h}''}{2} + 2\frac{k^2 k_J'}{k_J} \delta_1,$$

(4.1)

where, in contrast to Eq. (2.13), we are not neglecting the evolution of the Jeans wavenumber $k_J$. Two main features can be seen in Eq. (4.1): 1) the solution of $\delta_0$ will always be coupled to $\delta_1$, and 2) the solution for $\delta_0$ will depend on the wavenumber $k$.

Following recent literature, where the growth factor is defined in terms of the density contrast [85, 125, 131, 140–146], we define a scale-dependent growth factor $D$ as

$$D_k(z) = \frac{\delta_0(z,k)}{\delta_0(z=0,k)},$$

(4.2)

so that $D_k(z=0) = 1$. The definition given in Eq. (4.2) allows us to generalize the growth factor in such a way that it is possible to track its evolution for each wavenumber $k$. This is done in Figure 8, where we show the growth factor $D_k(z)$ for $k = 10^{-4}\text{Mpc}^{-1}$ (yellow), $k = 0.53\text{Mpc}^{-1}$ (blue), $k = 10\text{Mpc}^{-1}$ (red), for SFDM with mass $m_\phi = 10^{-22}\text{eV}$, and which is endowed with a quadratic potential (FDM case $\lambda = 0$, dashed lines), and with a trigonometric potential with tachyonic instability as well ($\lambda = 1.5 \times 10^5$, dotted lines). The initial amplitude for the growth factor with trigonometric potential is smaller than that of the FDM case, but around $z \sim 10^6$ the growth factor with $\lambda = 1.5 \times 10^5$ suffers the tachyonic instability and its amplitude increases faster than the free axion case. It is important to recall that such fast increment of the growth factor amplitude, and therefore in the density contrast, is translated as a bump in the mass power spectrum for large $k$’s, as was shown in Figure 5. Interestingly enough, from $z \sim 100$ up to the present day all curves evolve as CDM, which implies that for $z < 100$ the growth factor $D_k(z)$ in Eq. (4.2) becomes effectively scale-independent.

![Figure 8](image.png)

**Figure 8:** Growth factor $D_k(z)$ for an axion mass $m_\phi = 10^{-22}\text{eV}$ with both, quadratic potential (dashed lines) and trigonometric potential (dotted lines). The tachyonic instability is manifested for the latter as a fast increment of amplitude for $D_k$ at $z \sim 10^6$. Horizontal dotted gray line indicates $D = 1$, where all curve converge at $z \sim 0$. 
Going further, the definition given by Eq. (4.2) enables us to write the velocity of the growth factor \( f_k(z) \) as follows,
\[
f_k(z) = \frac{d \log D_k(N)}{dN} = -(1 + z) \frac{d \log D_k(z)}{dz} = -(1 + z) \frac{d \log \delta_0(z,k)}{dz}.
\] (4.3)

The dependence on \( k \) for the function shown above can be seen in Figure 9, where the colors and the line style for each curve are the same as in Figure 8. Notice that the velocity growth factor for \( k = 10^{-4}\text{Mpc}^{-1} \) is the same as that of CDM and is not affected by the values of \( \lambda \); that is, at large scales we recover the same behavior of CDM. Similarly, for \( k = 0.53\text{Mpc}^{-1} \) the evolution is also independent of the values of \( \lambda \), although the CDM evolution is not recovered for \( z \gtrsim 10 \). Thus, it is possible to distinguish between CDM and SFDM at high redshifts. The result is different for the wavenumber \( k = 10\text{Mpc}^{-1} \), where we can see that the evolution of \( f_k \) is different for the two values of \( \lambda \) considered. However, from \( z \sim 10 \) to the present day, the evolution of \( f_k \) for each mode and for each value of \( \lambda \) is the same as that of CDM. This means that at late times the MPS of the axion field should keep a constant ratio with respect to that of CDM.

![Figure 9](image)

**Figure 9:** Velocity growth factor \( f_k(z) \) for an axion with mass \( m_\phi = 10^{-22}\text{eV} \). Dashed (dotted) lines correspond to \( \lambda = 0 (\lambda = 1.5 \times 10^5) \), and yellow, blue and red lines indicate wavenumbers \( k = 10^{-4}, 0.53, 10\text{Mpc}^{-1} \) respectively. For \( k \ll 1\text{Mpc}^{-1} \) the velocity growth factor evolves as CDM for all redshift, whereas for \( k > 1\text{Mpc}^{-1} \) each mode evolve independently until \( z \sim 10 \), where all curve converge to the CDM case, and the velocity growth factor is the same for all wavenumbers. See text for more details.

For \( k = 10\text{Mpc}^{-1} \), we attribute the notorious difference at \( z > 10 \) between the FDM case and the axion-like potential to the tachyonic instability, since this effect is manifested at such range of scale (see Figure ??). Finally, since the growth factor \( D_k \) and the velocity growth factor \( f_k \) coincide with those of CDM for \( 0 < z < 10 \), the combined observable \( f_k \sigma_8 \) at \( 0 < z < 2 \) (range within which we can search for observational constraints) will be insensitive to the details of the axion case, as can be seen in Figure 10, where the overlapped curves correspond to the same values of wavenumbers \( k \) and decay constant \( \lambda \) as those in Figures 8 and 9.
Figure 10: Velocity growth factor $f_k$ and variance $\sigma_8$ combined as function of both, wavenumber $k$ and redshift $z$. The overlapped curves have the same values of $k$ and $\lambda$ as those of the previous Figures 8 and 9. Observational data are shown in colored squares from 2dFGRS [147], WiggleZ [148], 6dFGRS [149], VIPERS [150], SDSS DR7 Main [151], BOSS DR12 [152], FastSound [153], eBOSS DR14Q [154], 2MTF [155] and SDSS-II [156].

Whereas strong constraints have been imposed to the mass of the scalar field dark matter $m_\phi$ (through galactic observations, mass power spectrum and CMB anisotropies), having observations of matter distribution at high redshifts can be useful to explore the nature of DM, and particularly to constraint the decay parameter $\lambda$ of the axion field. The cosmological effects of such parameter have not be studied in great detail, and we are showing that it has a characteristic imprint on the structure formation, at small scales (see MPS in Figure 5) as well as at high redshifts (Figure 8 y 9).

4.2 Semi-analytical Halo Mass Function

The Halo Mass Function (HMF) encodes the comoving number density of dark matter halos as function of the halo mass, and it constitutes a representative cosmological probe of dark matter and dark energy. It can be used for example to constraint the value of the combined parameters $\sigma_8$ and $\Omega_M$ (the power spectrum normalization and the matter density parameter respectively), and also to characterize the dark energy equation of state $\omega_0$ [157–159]. A halo is an overdensity of matter, which lie on the nonlinear regime of structure formation. To study such objects numerical simulations have to be carried out [160–162]. However, semi-analytical analysis can be performed as well, as have been shown in [163–166]. Particularly, the procedure to obtain the semi analytical HMF of our model will be similar to that given by [167, 168].

First, we define the window functions we are going to implement: the Top-Hat window function $W_{TH}$, which is a filter with spherical symmetry in real space, and the Sharp-k window function $W_{SK}$, defined as a Top-Hat function in Fourier space. They are given, in Fourier space, by

$$W_{TH}(kr) = \frac{3}{(kr)^3} [\sin(kr) - kr \cos(kr)], \quad W_{SK}(kr) = \Theta(2\pi - kr). \quad (4.4)$$

The Top-Hat function is useful to work with the LCDM model, while the Sharp-k function is useful for suppressed power spectra, which is the case of the axion field. More discussion about the choice of the window functions are given in [96, 169–171] and references therein.
One of the quantities of interest is the variance, which is calculated as
\[
\sigma^2(r) = \int \frac{d^3k}{(2\pi)^3} P(k) W^2(kr).
\] (4.5)

In Figure 11 we show the square root of the variance at redshift \( z = 0 \) for ΛCDM and the axion field, the latter with quadratic (\( \lambda = 0 \)) and trigonometric (\( \lambda = 10^5 \)) potential, and \( m_\phi = 10^{-22} \text{eV} \). The variance of the axion field, for both the Top Hat and Sharp-k window functions, show a constant value for small masses, in contrast to the result of the ΛCDM model which is always increasing. The asymptotic values for the quadratic and trigonometric potentials are different; for the latter it can be seen that it is the tachyonic instability, and ultimately the non-linearities of the trigonometric potential, that enhances the value of \( \sigma \) at small masses.

To be able to study the gravitational collapse, it is necessary to take into account the scale-dependence that inherently SFDM models possess. Let us explain this as follows: different wavenumbers will grow at different rates, as was shown in Figure 8 and 9 for the growth factor \( D_k(z) \) and the velocity growth factor \( f_k(z) \) respectively. Thus, the gravitational collapse for each mode \( k \) will be different. The threshold value at which some matter fluctuation associated to a given mode \( k \) will collapse, is known as the critical overdensity, which within a standard CDM scenario is defined by [167, 172–174]
\[
\delta_{\text{crit}} = 1.686 \frac{D_{\text{CDM}}(z = 0)}{D_{\text{CDM}}(z)},
\] (4.6)
where \( D_{\text{CDM}} \) is the growth factor for CDM
\[
D_{\text{CDM}} = \frac{5 \Omega_m H}{2} \int \frac{da}{a^3 H^3}.
\] (4.7)

In the case of SFDM, we can in principle apply a similar expression, but using the growth factor introduced in Eq. (4.2),
\[
\delta_{\text{crit}} = 1.686 \frac{D_k(z = 0)}{D_k(z)}.
\] (4.8)
We are interested in building up the HMF at $z = 0$, and even when Eq. (4.8) contains explicit dependence on the wavenumber $k$, the growth factor for SFDM coincides with the CDM case at later times, as was discussed in Section 4.1. Therefore, this approach will not be useful to study the effects of gravitational collapsing with scale dependence on the HMF.

Notwithstanding, we can rather consider approaches as those that have been carried out on previous studies on this subject [85, 168, 175], where the authors introduce a definition of the growth factor in terms of several density contrasts rates. Particularly, in Eq. (10) from [168] it is shown the relative amount of growth between CDM and SFDM as

$$
\frac{D_{\text{CDM}}(z)}{D_{\text{SFDM}}(M, z)} = \frac{\delta_{\text{CDM}}(k, z)}{\delta_{\text{SFDM}}(k, z)} \frac{\delta_{\text{CDM}}(k_0, z_h)}{\delta_{\text{SFDM}}(k_0, z_h)} \frac{\delta_{\text{CDM}}(k, z_h)}{\delta_{\text{SFDM}}(k, z_h)},
$$

(4.9)

where $k_0 = 0.002h/$Mpc is a pivot scale, and $z_h = 300$ is the redshift at which the shape of the CDM power spectrum has frozen in. We observe that the pivot scale is small, and for such mode the growth factor of SFDM will evolve as CDM. Then, the second and third ratio in Eq. (4.9) are $\delta_{\text{CDM}}(k_0, z_h)/\delta_{\text{SFDM}}(k_0, z_h) = \delta_{\text{CDM}}(k_0, z)/\delta_{\text{SFDM}}(k_0, z) \simeq 1$. On the other hand, the overall effect of the last quotient on the right hand side of Eq. (4.9) occurs for $k > 1$ where $\delta_{\text{SFDM}}(k, z_h) < \delta_{\text{CDM}}(k, z_h)$, suppressing the amplitude of the growth factor for such wavenumbers at $z = z_h$, while for $k < 1$ such quotient is equal to 1. Besides, notice that $z_h \sim 10^2$, which is the order of magnitude of redshift where the cosmological evolution of the growth factor is basically that of CDM (see Figure 8), and thus, the last term of the above equation can be taken as $\delta_{\text{CDM}}(k, z_h)/\delta_{\text{SFDM}}(k, z_h) \simeq 1$ almost independently of the value of the wavenumber $k$. Thereby, the main responsible to carry the scale dependence of the growth factor $D$ will be the first term in Eq. (4.9). We concluded that the critical overdensity can be written as

$$
\delta_{\text{crit}}(k) = 1.686 \frac{\delta_{\text{CDM}}(k, z)}{\delta_{\text{CDM}}(z, k)}. \quad (4.10)
$$

From this expression it can be seen that for small wavenumbers the CDM case is recovered, since $k \to 0$ erases the scale-dependence on $\delta_{\text{crit}}$. In other words, the density contrast for the axion field will evolve as CDM for small values of $k$, specially at late times. On the other hand, the axion density contrast for large values of $k$ do not grows as CDM in all its evolution. Particularly at present day, $\delta_{0}$ has less amplitude than $\delta_{\text{CDM}}$, which is clearly seen in the MPS on Figure 5. Note that our definition of the critical overdensity given by the above equation is a reduction from that used by authors in [85, 168, 175], where a particular normalization and an analytical function based on AxionCAMB results are implemented for a scale/mass-dependent growth factor. Within our analysis, such scale dependence is encoded in the density contrast given by our new dynamical variable $\delta_{0}(z, k)$, and which we have obtained numerically from CLASS. We want to highlight that from our definition (4.10) we can recover the results from the previous work mentioned above. For example, Figure 12 shows the critical overdensity as function of the wavenumber, analogous to that of Figure 2 from [168], where $\delta_{\text{crit}}$ is shown as function of the mass. Such comparison is valid for an axion mass of $10^{-22}$eV (green line on Figure 2 from [168], and blue line in Figure 12), since in this work we have consider the effect of tachyonic instability in the critical overdensity as well. We observe that $\delta_{\text{crit}}$ shows a clear scale dependence for wavenumbers $k > 1h$/Mpc, which is translated to small halo masses, as we shall see below.
Notice that for the trigonometric potential (red line) there are wavenumbers for which the critical overdensity is less than in the CDM case, implying that structures associated to such modes will be able to grow with a threshold $\delta_{\text{crit}}$ lower than in a standard CDM scenario, and also compared to the case of a free axion. This is why the MPS exhibits a bump at small scales, as was shown in Figure 5.

On modeling the gravitational collapse we will consider both, the Press-Schechter (P&S) and the Sheth-Tormen (S&T) formalism for spherical and ellipsoidal collapse models respectively [163, 165]. Such collapse models are encrypted in the following function

\[
\frac{dn}{d\ln M} = -\frac{1}{2 M} \frac{\bar{\rho}}{\bar{\sigma}^2} f(\nu) \frac{d\ln \sigma^2}{d\ln M}. 
\]

Now we can analyze the HMF for an axion field endowed with a trigonometric potential, and compare it with the CDM prediction, as well as with the free axion case. Figure 13 shows the semi-analytical halo mass function at redshift $z = 0$ and axion mass $m_\phi = 10^{-22}\text{eV}$. We separate our analysis in three different cases depending on the window function implemented: Top-Hat, Sharp-k, and Top-Hat including the critical overdensity with scale dependence. For all cases we consider the collapse models given at Eq. (4.11), for the FDM case with quadratic potential ($\lambda = 0$) and an axion field with trigonometric potential ($\lambda = 10^5$). When considering the Top Hat window function $W_{TH}$ without a scale dependent critical overdensity, differences between the HMF for SFDM and CDM appear at small mass scales, as can be seen at upper left panel in Figure 13. However, since we have used Eq. (4.8), the HMF...
do not exhibit the cut-off of the MPS when using this window function. This is because, as we mentioned before, the dependence on $k$ in the growth factor (4.2) is lost at late times. On the other hand, including a critical overdensity with dependence on scale through Eq. (4.10) (upper right panel in Figure 13), a steep cut-off appears at $M \sim 10^8M_\odot/h$ for $\lambda = 0$ and $M \sim 10^8M_\odot/h$ for $\lambda = 10^5$. This result is consistent with that of [85, 168] for the particular case in which SFDM constitutes all the DM content, i.e., when $\Omega_\phi/\Omega_{CDM} = 1$. Finally, the HMF with the Sharp-k function $W_{SK}$ is shown in the lower panel of Figure 13. In this case, we have use Eq. (4.8), since the cut-off at a given scale is captured by the Sharp-k window function, as discussed by [96]. Note that, whereas the turn around of the halo mass function is slightly different for $\lambda = 0$ and $\lambda = 10^5$, the cut-off for both of them occurs approximately at the same range of mass scale $10^8 \lesssim M (h/M_\odot) \lesssim 10^9$.

For all the cases studied we observe as a new general feature in the HMF, an increment in its amplitude when considering one of the two following considerations:

1.- ellipsoidal collapse S&T model (red lines in Figure 13),

2.- Axion-like potential in the tachyonic instability regime (dotted lines in Figure 13).
These two new features in the HMF can be contrasted with recent results obtained in [176], where observational constraints on Warm Dark Matter (WDM) and Fuzzy Dark Matter (FDM) models are imposed. In particular, the HMF for FDM is modeled by implementing the analytical function

\[
\left( \frac{dn}{d\ln M} \right)_{SFDM} = f_1(M) + f_2(M) \left( \frac{dn}{d\ln M} \right)_{CDM},
\]

(4.13)

where the functions \( f_1 \) and \( f_2 \) are given by

\[
f_1(M) = \beta \exp \left[-\left( \frac{\ln \left( \frac{M}{M_1 \times 10^8 M_\odot} \right)}{\sigma} \right)^2 \right],
\]

\[
f_2(M) = \left[ 1 + \left( \frac{M}{M_2 \times 10^8 M_\odot} \right)^{-\alpha_1} \right]^{-\alpha_1},
\]

(4.14)

and the CDM halo mass function is

\[
\left( \frac{dn}{d\ln M} \right)_{CDM} = 3.26 \times 10^{-5} \left( \frac{M}{2.57 \times 10^7 M_\odot} \right)^{-1.9} \left( \frac{M}{M_\odot} \right).
\]

(4.15)

The different parameters used in the above expressions are \( \alpha_1 = 0.72, M_1/m_{22}^{-1.5} = 4.7, M_2/m_{22}^{-1.6} = 2.0, \beta/m_{22}^{1.5} = 0.014, \sigma = 1.4 \), and where \( m_{22} = m_\phi/10^{-22} \text{eV} \). In order to put a limit value on the FDM mass, in that work the parameter \( m_{22} \) is varied up to the maximum value such that the HMF is more suppressed than the excluded WDM cases. Doing so, the mass obtained in [176] is \( m_\phi = 2.1 \times 10^{-21} \text{eV} \). We show the HMF for this result in Figure 14 (green solid line), as well as the HMF for the CDM model (black solid line). We have also included the numerical results for the HMF obtained in this work, considering only those which are consistent with the reported values of the HMF according to stellar streams measurements [177]. These measurements refer to a stream of stars (the GD-1 stream) that have been detected in Sloan Digital Sky Survey (SDSS) data [178]. Stellar streams are originated from the tidal disruption of globular clusters, forming an elongated structure that, when it is gravitationally perturbed by dark subhaloes, some gaps in the stellar distribution of such elongated structure are produced. Therefore, stellar stream observations would provide information about the dark matter subhaloes [179–186].

All our numerical results presented in Figure 13 underestimate the result of the analytical approach modeled by Eq. (4.13) when considering \( m_\phi = 2.1 \times 10^{-21} \text{eV} \). For such mass value, and indistinctly of the collapse model, only those HMF with a Top-Hat window function are consistent with the stellar stream measurement data. This is because the suppression impressed in the HMF due to the Top-Hat window function is less than that imposed by the Sharp-k window function, and even less in comparison with that of a Top-Hat with critical overdensity with scale dependence, as we already showed in Figure 13. Therefore, if we want to consider the HMF for the axion field with a pronounced suppression, stronger constraints have to be imposed to our model \( (m_\phi \geq 2.1 \times 10^{-21} \text{eV}) \). This will be also the case when considering an analytical model as that presented in [186], where the bound for the mass is given by \( m_\phi \geq 5.2 \times 10^{-21} \text{eV} \).

We found that, for \( m_\phi = 10^{-20} \text{eV} \), it is possible to be in agreement with streams measurements for an axion field HMF when a Top-Hat with a scale-dependent critical overdensity is considered. This is achieved precisely with the ellipsoidal S&T collapse model in the
presence of tachyonic instability (with $\lambda = 10^6$), as it is shown in Figure 14 (dotted red line). Therefore, the two new considerations mentioned above and included in our analysis, which lead to an increment in the amplitude of the axion HMF, can play an important role in order to guarantee consistency with stellar stream measurements for haloes with masses $\sim 10^7 - 10^8 M_\odot$. However, this will be possible only for axion masses $m_\phi \geq 10^{-20} eV$, which constitutes a stronger constraint as those imposed, for instance, by Lyman-\alpha [112, 113], but lies within the range of masses that could be tested by 21-cm observations [187, 188]. As it is also noted in [176], the results from the analytical approach are too conservative in the sense that they are not taking into account the scale-dependent growth of structure, neither the scale-dependent critical overdensity. In their analysis, masses for FDM with values $m_\phi \lesssim 1.37 \times 10^{-20} eV$ would be excluded, whereas we are showing that the SFDM HMF with a Top-Hat window function for any of the collapse model studied (yellow and blue lines) lies within the range obtained from stream measurements for $m_\phi = 10^{-20} eV$. Particularly, we have shown that the suppression of subhaloes due to a SFDM endowed with a trigonometric potential is in agreement with the constraints imposed by measurements of stellar streaming when the axion mass is $m_\phi = 10^{-20} eV$. Without the effect of the tachyonic instability, stronger constraints on the axion mass would be imposed.

5 Conclusions

Ultra–light DM bosons as SFDM model constitute a compelling candidate to substitute the CDM model. In this paper we presented a formalism to handle the cosmological equations by using the tools of dynamical systems for both the background and the linear perturbations. At the background level, the presence of a trigonometric potential shows a delay in the moment when the axion field starts to oscillate and behaving as CDM. These values of the onset of oscillations are shown in Table 1 in terms of the scale factor $a_{osc}$. We have explored with some depth the effect dubbed as tachyonic instability, which occurs due to the nonlinearities of the potential (1.1). For extreme values of $\lambda$, once the axion field starts to oscillate the density contrast grows with more amplitude than that of standard CDM and SFDM with a
quadratic potential (FDM). We indicated the duration of the tachyonic instability as well as the range of wavenumbers that suffer such effect.

On the other hand, we built observables such as the CMB anisotropies and the 3D and 1D matter power spectrum. The power spectrum of temperature fluctuations for the axion field shows major discrepancies for high multipoles, and a limiting case given by \( m_\phi \geq 10^{-24}\text{eV} \) and \( \lambda \leq 6 \times 10^3 \) states the values of masses and decay constants that are in agreement with high multipoles experiment such as ACT and SPT. When considering the Planck experiment, large values of \( \lambda \) are allowed, as can be seen on the lower panel of Figure 4, and which is consistent with our statistical analysis using Planck data (Figure 7). When analyzing the 3D matter power spectrum, the well known cut-off at small scales is reproduced for both, FDM and the axion field. Nonetheless, for large values of \( \lambda \) a bump appears at the cut-off scale as consequence of the tachyonic instability. This is the imprint of the axion–like potential (1.1) on the formation of large scale structures. As an additional analysis, we computed the 1D matter power spectrum, which is closely related to the flux power spectrum that can be used to constraint DM models with Lyman-\( \alpha \) observations. For a scalar field endowed with a quadratic potential (FDM), a lower bound have been imposed to the mass, ruling out masses with values \( m_\phi < 3.7 \times 10^{-21}\text{eV} \). However, we have observed that when considering a trigonometric potential, mass values lower than this bound are allowed. Thus, this kind of observations and the experiments that involve them can be used to constraint both parameters, \( m_\phi \) and \( \lambda \).

When performing the statistical analysis, we obtain a lower bound for the axion mass given by \( \log m_\phi = -24.2 \) at 95.5\% C.L. This result is consistent with our numerical analysis obtained for the CMB anisotropies. However, the decay parameter \( \lambda \) is not constrained, and all the values we explored numerically have equal probability, i.e., the parameter \( \lambda \) presents a flat posterior. Thus, SFDM with quadratic and trigonometric potential are in agreement with CMB observations, and it is the axion mass who still plays an important role in the constrains on this type of models for such observations.

Motivated by the scale-dependence of the scalar field dark matter models, we proposed a growth factor \( D_k \) and a velocity growth factor \( f_k \) with explicit dependence on the wavenumber \( k \). This was performed through the density contrast and its derivative. Effectively, there are differences in the evolution of \( D_k \) and \( f_k \) for each value of \( k \), but all of them evolve as cold dark matter from certain value of redshift (\( z \sim 100 \) for \( D_k \), and \( z \sim 10 \) for \( f_k \) to the present day (\( z \sim 0 \)). Having this quantities allowed us to build the combined parameter \( f_k \sigma_8 \) where the differences are marginal when comparing with CDM.

The tachyonic instability is manifested in both the variance and the halo mass function as an enhancement in the amplitude for low masses. The Top-Hat and Sharp-k window functions have been considered, and each one affect the HMF in a different way. On one hand, the HMF with a Top Hat window function present a decrease at small masses for SFDM in comparison with the CDM model, but such decrease is not the one expected from a mass power spectrum with a cut-off. However, when considering a critical overdensity with explicit dependence on scale, the HMF exhibits a steep cut-off. On the other hand, with the Sharp-k window function the halo mass function for the axion field has a cut-off less pronounced than the mentioned above, but approximately at the same mass scales. All the cases studied were performed for two different gravitational collapse models, the Press-Schechter and the Sheth-Tormen for spherical and ellipsoidal collapse model respectively. Both of them produce qualitatively the same HMF, with small differences at small scales.

Future astronomical observations planned by collaborations such as the Dark Energy
Spectroscopy Instrument (DESI) [117] and the Large Synoptic Survey Telescope (LSST, now Vera C. Rubin Observatory) [189] will explore the Universe with major accuracy. Particularly, the LSST will be able to constrain light bosonic dark matter mass $m_\phi \sim 10^{-20}$ eV by probing the MPS for halos with $\sim 10^8 M_\odot$ [190]. On the other hand, the Sloan Digital Sky Survey (SDSS) can be used for searches of low-surface brightness dwarf galaxies at small scales, as is discussed by authors in [191]. Besides, the 21cm signal detected by EDGES [192] can be used to study the properties of dark matter [193], in particular to probe small scale structures [187]. In fact, it was recently proposed that through forthcoming experiments such as the Square Kilometre Array (SKA) [194], 21cm observations can be used to constraint the scalar field mass as well at wavenumbers $30 < k < 1000$ (Mpc$^{-1}$) [188]. It has been recently studied the implications of a post-inflationary symmetry breaking of axion-like particles on the born of first generation stars, and on small scale structures [195].

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A General dynamical variables for scalar field perturbations

To translate the same scheme we used for the background evolution of axion fields in Section 2, where we were able to write down a dynamical system for the KG equation, we propose the following new variables for the scalar field perturbation $\varphi$ and its derivative $\dot{\varphi}$[96],

$$u = \sqrt{\frac{2}{3}} \frac{\kappa \dot{\varphi}}{H} = -\Omega_\phi^{1/2} e^\alpha \cos(\vartheta/2), \quad v = \frac{\kappa y_1 \varphi}{\sqrt{6}} = -\Omega_\phi^{1/2} e^\alpha \sin(\vartheta/2),$$

which after substitution on the perturbed KG equation (2.13) lead to the following differential equations

$$\vartheta' = 3 \sin \vartheta + 2 \omega (1 - \cos \vartheta) + y_1 - 2 e^{-\alpha} h' \sin \left( \frac{\theta}{2} \right) \sin \left( \frac{\vartheta}{2} \right),$$

$$+ \Omega_\phi^{1/2} \left[ \cos \left( \vartheta - \frac{\theta}{2} \right) - \cos \left( \frac{\theta}{2} \right) \right] \frac{y_2}{y_1},$$

$$\alpha' = -\frac{3}{2} (\cos \vartheta + \cos \theta) - \omega \sin \vartheta + e^{-\alpha} h' \sin \left( \frac{\theta}{2} \right) \cos \left( \frac{\vartheta}{2} \right)$$

$$+ \frac{\Omega_\phi^{1/2}}{2} \left[ \sin \left( \frac{\theta}{2} \right) + \sin \left( \vartheta - \frac{\theta}{2} \right) \right] \frac{y_2}{y_1}.$$
For numerical purposes, it is convenient to use as angular variable the difference \( \tilde{\vartheta} \equiv \theta - \vartheta \), and then from Eqs. (2.4a) and (A.2a) we obtain

\[
\tilde{\vartheta}' = -3 \left[ \sin \theta + \sin \left( \theta - \tilde{\vartheta} \right) \right] - 2\omega \left[ 1 - \cos \left( \theta - \tilde{\vartheta} \right) \right] + e^{-\alpha h'} \left[ \cos \left( \frac{\theta}{2} \right) - \cos \left( \theta - \frac{\tilde{\vartheta}}{2} \right) \right] + \Omega_{\phi}^{1/2} \left[ \cos \left( \frac{\theta}{2} - \tilde{\vartheta} \right) - \cos \left( \theta - \frac{\tilde{\vartheta}}{2} \right) \right] \frac{y_2}{y_1},
\]

(A.3a)

\[
\alpha' = -\frac{3}{2} \left[ \cos \left( \theta - \tilde{\vartheta} \right) + \cos \theta \right] - \omega \sin \left( \theta - \tilde{\vartheta} \right) + \frac{1}{2} e^{-\alpha h'} \left[ \sin \left( \frac{\theta}{2} \right) + \sin \left( \theta - \frac{\tilde{\vartheta}}{2} \right) \right] + \frac{\Omega_{\phi}^{1/2}}{2} \left[ \sin \left( \frac{\theta}{2} \right) + \sin \left( \theta - \frac{\tilde{\vartheta}}{2} \right) \right] \frac{y_2}{y_1}.
\]

(A.3b)

If we further define the variables \( \delta_0 = -e^\alpha \sin(\tilde{\vartheta}/2) \) and \( \delta_1 = -e^\alpha \cos(\tilde{\vartheta}/2) \), Eq. (A.3) can be properly combined to obtain the dynamical system shown in Eq. (2.14).

**B Fluid interpretation of the equations of motion of SFDM density perturbations**

Here we report about the fluid interpretation of Eq. (2.16) in terms of the standard fluid variables for linear perturbations, namely the density contrast \( \delta = \delta_0 \) and the divergence of the velocity perturbation \( \theta_\phi \) (see Eq. (2.15b)). Following the procedure in [107], we first consider the equations of density perturbations well within the regime of rapid field oscillations, Eqs. (2.17a), but written in the form,

\[
\delta'_0 = -\theta_\phi - \frac{\bar{h}'}{2}, \quad \theta'_\phi = -\frac{a'}{a} \theta_\phi + \frac{k^4}{4a^2m_\phi^2} \left( 1 - \frac{\rho_\phi a^2}{2k^2 f_\phi^2} \right) \delta_0,
\]

(B.1)

where now a prime denotes derivative with respect to \( \tau \). Notice that we have used the relation \( \theta_\phi = \frac{k^2}{2am_\phi} \delta_1 \), which is found from Eq. (2.15b) for rapid oscillations. A quick comparison with the standard fluid equations for axion fields (see for instance Eqs. (13) and (14) in [107]), leads us to conclude that the averaged value of the sound speed \( c_s \) of the axion field, in the nonrelativistic limit, is given by

\[
\langle c_s^2 \rangle \simeq \frac{k^4}{4a^2m_\phi^2} \left( 1 - \frac{\rho_\phi a^2}{2k^2 f_\phi^2} \right).
\]

(B.2)

The standard result of the FDM case is obtained in the limit \( f_\phi \to \infty \) (\( \lambda \to 0 \)), namely \( \langle c_s^2 \rangle_0 \simeq \frac{k^4}{4a^2m_\phi^2} \) (eg [197]).

**C Extreme Axion Wave Dark Matter**

The tachyonic instability of SFDM in the axion case was firstly studied in [198], from the field perspective, and was dubbed Extreme Axion Wave Dark Matter (EA\( \psi \)DM). Assuming an axion potential in the form \( V(\phi) = 2m_\phi^2 f_\phi^2 \sin^2(\phi/2f_\phi) \), the dynamics of the field starts close to maximum of the potential, and then the extreme label refers to initial conditions such
that \( \phi_i/f_\phi \to \pi \). For instance, some of the most extreme values considered in [92] were of the order \( \delta \theta_0 \equiv \pi - \phi_i/f_\phi \simeq 0.2^\circ \).

To find the relation between the extreme initial conditions used in [92] and our approach, we proceed as follows. Considering our convention for the axion potential (2.11b), we find for the initial conditions that

\[
\frac{2m_\phi^2}{H_0^2} \alpha_i^4 \cos(\phi_i/2f_\phi) = \Omega_{\phi i}.
\] (C.1)

In our convention, ESFDM is achieved if \( \phi_i/f_\phi \to 0 \), and then we see that an extreme initial condition on the field \( \phi_i \) translates into an extreme initial condition on the density parameter \( \Omega_{\phi i} \to 0 \). However, the latter’s value is not independent, as for any choice of the potential parameters \( m_\phi \) and \( \lambda \) (ie \( f_\phi \)), one has to fine tune \( \Omega_{\phi i} \) to get the right value of \( \Omega_{\phi 0} \) at the present time.

The above is the main reason why, in our approach, the extreme case of initial conditions is interlinked with the (decay) parameter \( \lambda \): larger values of the latter asks for smaller values of \( \phi_i/f_\phi \), that is, for more extreme values in the sense that \( \phi_i/f_\phi \to 0 \). For the fiducial model with \( m_\phi = 10^{-22} \), we find \( \delta \theta_0 = (174^\circ, 162^\circ, 124^\circ, 44^\circ, 0.47^\circ, 0.16^\circ) \) corresponding to \( \lambda = (10, 10^2, 10^3, 10^4, 10^5, 1.28 \times 10^5) \). Thus, our formalism allows initial conditions as extreme as those reported in [92], but covering the whole evolution of the Universe.

### D Higher order algebraic equation for the scale factor on the onset of oscillations

In Section 2, we obtained an expression to determine the scale factor at the onset of oscillation \( a_{\text{osc}} \) given by Eq. (2.11a), which was important to determine the initial conditions for the evolution of the background variables when \( \lambda > 0 \). Now we will show that it is possible to obtain a more accurate expression for \( a_{\text{osc}} \) by means of an iterative integration of the equations of motion at early times.

Considering again a radiation domination era, let us take the solution for \( y_1 \) given by Eq. (2.9) and plug it into Eq. (2.4b), which leads to the new solution,

\[
y_1(a) = y_1, i \left( \frac{a}{a_i} \right)^2 + \frac{\lambda}{8} \Omega_{\phi, i} \theta_i \left( \frac{a}{a_i} \right)^6 - \left( \frac{\lambda \Omega_{\phi, i}}{24} \right)^2 \theta_i \left( \frac{a}{a_i} \right)^6 + \frac{1}{2} \left( \frac{\lambda \Omega_{\phi, i}}{24} \right)^2 \theta_i \left( \frac{a}{a_i} \right)^10. \] (D.1)

This solution can be used in Eq. (2.4a) to obtain a new solution on \( \theta \), which can be shown to be

\[
\theta(a) = \theta_1 \left( \frac{a}{a_i} \right)^2 \left\{ 1 - \frac{\lambda \Omega_{\phi, i}}{72} + \frac{17}{26} \left( \frac{\lambda \Omega_{\phi, i}}{72} \right)^2 \right\} \theta_i \left( \frac{a}{a_i} \right)^4 + \frac{9}{26} \left( \frac{\lambda \Omega_{\phi, i}}{72} \right)^2 \left( \frac{a}{a_i} \right)^8. \] (D.2)

Setting the previous expression to the onset of oscillations, i.e., \( a = a_{\text{osc}} \) and \( \theta = \pi/2 \), we obtain a quartic order equation for \( a_{\text{osc}} \)

\[
a_{\text{osc}}^2 \left[ 1 + \left( \frac{\lambda \Omega_{\phi, 0}}{72 \Omega_{r, 0}} \right) a_{\text{osc}} + \frac{9}{26} \left( \frac{\lambda \Omega_{\phi, 0}}{72 \Omega_{r, 0}} \right)^2 a_{\text{osc}}^2 \right] = \frac{\pi \theta_i^{-1} a_i^2}{2 \sqrt{1 + \pi^2/36}}, \] (D.3)
where we have used Eq. (2.8). Following the same iterative scheme, we can find higher order solutions for $y_1$ and $\theta$, which we do not show, but that lead to a fifth order equation for $a_{\text{osc}}$:

$$a_{\text{osc}}^2 \left[ 1 + \left( \frac{\lambda \Omega_{\phi,0}}{72 \Omega_{r,0}} \right) a_{\text{osc}} + \frac{9}{26} \left( \frac{\lambda \Omega_{\phi,0}}{72 \Omega_{r,0}} \right)^2 a_{\text{osc}}^2 + \frac{27}{442} \left( \frac{\lambda \Omega_{\phi,0}}{72 \Omega_{r,0}} \right)^3 a_{\text{osc}}^3 \right] = \frac{\pi \theta^{-1} a_i^2}{2 \sqrt{1 + \pi^2/36}}.$$  

(D.4)

We have noticed that higher order solutions follow a similar pattern as that in Eq. (D.4), which resembles that of the series expansion of the exponential series, except for the numerical coefficients. But a close comparison between the two series shows that,

$$1 + \left( \frac{\lambda \Omega_{\phi,0}}{72 \Omega_{r,0}} \right) a_{\text{osc}} + \frac{9}{26} \left( \frac{\lambda \Omega_{\phi,0}}{72 \Omega_{r,0}} \right)^2 a_{\text{osc}}^2 + \frac{27}{442} \left( \frac{\lambda \Omega_{\phi,0}}{72 \Omega_{r,0}} \right)^3 a_{\text{osc}}^3 + \cdots$$

$$< 1 + \left( \frac{\lambda \Omega_{\phi,0}}{72 \Omega_{r,0}} \right) a_{\text{osc}} + \frac{1}{2} \left( \frac{\lambda \Omega_{\phi,0}}{72 \Omega_{r,0}} \right)^2 a_{\text{osc}}^2 + \frac{1}{6} \left( \frac{\lambda \Omega_{\phi,0}}{72 \Omega_{r,0}} \right)^3 a_{\text{osc}}^3 + \cdots = e^{\left( \frac{\lambda \Omega_{\phi,0}}{72 \Omega_{r,0}} \right) a_{\text{osc}}}.$$  

(D.5)

Although not a formal demonstration, this exercise shows that a better estimation of the scale factor at the onset of the oscillations could be made from the expression

$$a_{\text{osc}}^2 \exp \left( \frac{\lambda \Omega_{\phi,0}}{72 \Omega_{r,0}} a_{\text{osc}} \right) = \frac{\pi \theta^{-1} a_i^2}{2 \sqrt{1 + \pi^2/36}}.$$  

(D.6)

As discussed in Section 2, the start of the field oscillations happen more abruptly for larger values of $\lambda$, and this makes difficult to find the right initial conditions for the dynamical variables. Equation (D.6) seems to offer an explanation as the iterative integration of the equations of motion results in an (nearly) exponential relationship between $a_{\text{osc}}$ and the values of other cosmological variables.

References

[1] Matts Roos. Astrophysical and cosmological probes of dark matter. J. Mod. Phys., 3:1152, 2012.

[2] Matthew R. Buckley and Annika H. G. Peter. Gravitational probes of dark matter physics. Phys. Rept., 761:1–60, 2018.

[3] Paolo Salucci. Dark Matter in Galaxies: evidences and challenges. Found. Phys., 48(10):1517–1537, 2018.

[4] Gianfranco Bertone et al. Gravitational wave probes of dark matter: challenges and opportunities. 7 2019.

[5] V.A. Rubakov. Cosmology and Dark Matter. In 2019 European School of High-Energy Physics, 12 2019.

[6] Andrew R. Liddle and David H. Lyth. The cold dark matter density perturbation. Physics Reports, 231(1):1 – 105, 1993.

[7] Daniel J. Eisenstein and Wayne Hu. Power spectra for cold dark matter and its variants. Astrophys. J., 511:5, 1997.

[8] Anatoly A. Klypin, Andrey V. Kravtsov, Octavio Valenzuela, and Francisco Prada. Where are the missing Galactic satellites? Astrophys. J., 522:82–92, 1999.
[9] James S. Bullock. Notes on the Missing Satellites Problem. 2010.

[10] Patrick Cote, Michael J. West, and R. O. Marzke. Globular cluster systems and the missing satellite problem: implications for cold dark matter models. Astrophys. J., 567:853, 2002.

[11] Jorge Penarrubia, Andrew Pontzen, Matthew G. Walker, and Sergey E. Koposov. The coupling between the core/cusp and missing satellite problems. Astrophys. J. Lett., 759:L42, 2012.

[12] W. J. G. de Blok. The Core-Cusp Problem. Adv. Astron., 2010:789293, 2010.

[13] David J. E. Marsh and Ana-Roxana Pop. Axion dark matter, solitons and the cusp-core problem. Mon. Not. Roy. Astron. Soc., 451(3):2479–2492, 2015.

[14] N. Li and D.-M. Chen. Cusp-core problem and strong gravitational lensing. Research in Astronomy and Astrophysics, 9:1173–1184, November 2009.

[15] Fangzhou Jiang and Frank C. van den Bosch. Comprehensive Assessment of the Too-Big-to-Fail Problem. Mon. Not. Roy. Astron. Soc., 453(4):3575–3592, 2015.

[16] Shea Garrison-Kimmel, Michael Boylan-Kolchin, James S. Bullock, and Evan N. Kirby. Too Big to Fail in the Local Group. Mon. Not. Roy. Astron. Soc., 444(1):222–236, 2014.

[17] David H. Weinberg, James S. Bullock, Fabio Governato, Rachel Kuzio de Naray, and Annika H. G. Peter. Cold dark matter: controversies on small scales. Proc. Nat. Acad. Sci., 112:12249–12255, 2015.

[18] Marcel S. Pawlowski, Benoit Famaey, David Merritt, and Pavel Kroupa. On the persistence of two small-scale problems in ΛCDM. Astrophys. J., 815(1):19, 2015.

[19] Antonino Del Popolo and Morgan Le Delliou. Small scale problems of the ΛCDM model: a short review. Galaxies, 5(1):17, 2017.

[20] James S. Bullock and Michael Boylan-Kolchin. Small-Scale Challenges to the ΛCDM Paradigm. Ann. Rev. Astron. Astrophys., 55:343–387, 2017.

[21] Stacy Y. Kim, Annika H. G. Peter, and Jonathan R. Hargis. Missing Satellites Problem: Completeness Corrections to the Number of Satellite Galaxies in the Milky Way are Consistent with Cold Dark Matter Predictions. Phys. Rev. Lett., 121(21):211302, 2018.

[22] Alyson M. Brooks. Understanding Dwarf Galaxies in order to Understand Dark Matter. Astrophys. Space Sci. Proc., 56:19–28, 2019.

[23] Alex Fitts et al. Dwarf Galaxies in CDM, WDM, and SIDM: Disentangling Baryons and Dark Matter Physics. Mon. Not. Roy. Astron. Soc., 490(1):962–977, 2019.

[24] Alan H. Guth. The Inflationary Universe: A Possible Solution to the Horizon and Flatness Problems. Phys. Rev., D23:347–356, 1981.

[25] Andrei D. Linde. A New Inflationary Universe Scenario: A Possible Solution of the Horizon, Flatness, Homogeneity, Isotropy and Primordial Monopole Problems. Phys. Lett., 108B:389–393, 1982.

[26] R. R. Caldwell, Rahul Dave, and P. J. Steinhardt. Quintessential cosmology: Novel models of cosmological structure formation. Astrophys. Space Sci., 261:303–310, 1998.

[27] Bharat Ratra and P. J. E. Peebles. Cosmological Consequences of a Rolling Homogeneous Scalar Field. Phys. Rev., D37:3406, 1988.

[28] P. J. E. Peebles and Bharat Ratra. Cosmology with a Time Variable Cosmological Constant. Astrophys. J., 325:L17, 1988.

[29] C. Wetterich. Cosmology and the Fate of Dilatation Symmetry. Nucl. Phys., B302:668–696, 1988.
[30] Latham A. Boyle, Robert R. Caldwell, and Marc Kamionkowski. Spintessence! New models for dark matter and dark energy. Phys. Lett., B545:17–22, 2002.
[31] R. D. Peccei and Helen R. Quinn. CP Conservation in the Presence of Instantons. Phys. Rev. Lett., 38:1440–1443, 1977.
[32] R. D. Peccei and Helen R. Quinn. Constraints Imposed by CP Conservation in the Presence of Instantons. Phys. Rev., D16:1791–1797, 1977.
[33] Steven Weinberg. A New Light Boson? Phys. Rev. Lett., 40:223–226, 1978.
[34] F. Wilczek. Problem of strong p and t invariance in the presence of instantons. Phys. Rev. Lett., 40:279–282, Jan 1978.
[35] Edward Witten. Some Properties of O(32) Superstrings. Phys. Lett., B149:351–356, 1984.
[36] Peter Svrcek and Edward Witten. Axions In String Theory. JHEP, 06:051, 2006.
[37] Y. H. Ahn. QCD Axion as a Bridge Between String Theory and Flavor Physics. Phys. Rev., D93(8):085026, 2016.
[38] Michele Cicoli, Mark Goodsell, and Andreas Ringwald. The type IIB string axiverse and its low-energy phenomenology. JHEP, 10:146, 2012.
[39] Michele Cicoli. Global D-brane models with stabilised moduli and light axions. J. Phys. Conf. Ser., 485:012064, 2014.
[40] Andreas Ringwald. Searching for axions and ALPs from string theory. J. Phys. Conf. Ser., 485:012013, 2014.
[41] Eduard Masso. Axions and axion like particles. Nucl. Phys. Proc. Suppl., 114:67–73, 2003.
[42] Georg G. Raffelt. Axions: Motivation, limits and searches. J. Phys., A40:6607–6620, 2007.
[43] Jai-chan Hwang and Hyerim Noh. Axion as a Cold Dark Matter candidate. Phys. Lett., B680:1–3, 2009.
[44] A. Ringwald. Axions and Axion-Like Particles. In Proceedings, 49th Rencontres de Moriond on Electroweak Interactions and Unified Theories: La Thuile, Italy, pages 223–230, 2014.
[45] L.F. Abbott and P. Sikivie. A cosmological bound on the invisible axion. Physics Letters B, 120(1):133 – 136, 1983.
[46] P. Sikivie. Dark matter axions. Int. J. Mod. Phys., A25:554–563, 2010.
[47] Pierre Sikivie. Axion Cosmology, pages 19–50. Springer Berlin Heidelberg, Berlin, Heidelberg, 2008.
[48] Alma X. Gonzalez-Morales, David J. E. Marsh, Jorge Peñarrubia, and Luis Ureña López. Unbiased constraints on ultralight axion mass from dwarf spheroidal galaxies. 2016.
[49] Guido D’Amico, Teresa Hamill, and Nemanja Kaloper. Quantum field theory of interacting dark matter and dark energy: Dark monodromies. Phys. Rev. D, 94(10):103526, 2016.
[50] Lam Hui, Jeremiah P. Ostriker, Scott Tremaine, and Edward Witten. Ultralight scalars as cosmological dark matter. Physical Review D, 95:043541, 2016.
[51] Renée Hlozek, David J. E. Marsh, and Daniel Grin. Using the full power of the cosmic microwave background to probe axion dark matter. Monthly Notices of the Royal Astronomical Society, page 273, 2 2018.
[52] Stephen J. Asztalos et al. An Improved RF cavity search for halo axions. Phys. Rev., D69:011101, 2004.
[53] F. T. Avignone, III et al. Experimental search for solar axions via coherent Primakoff conversion in a germanium spectrometer. Phys. Rev. Lett., 81:5068–5071, 1998.

[54] R. Bernabei et al. Search for solar axions by Primakoff effect in NaI crystals. Frascati Phys. Ser., 37:211–216, 2004.

[55] A. Morales et al. Particle dark matter and solar axion searches with a small germanium detector at the Canfranc Underground Laboratory. Astropart. Phys., 16:325–332, 2002.

[56] K. Zioutas et al. First results from the CERN Axion Solar Telescope (CAST). Phys. Rev. Lett., 94:121301, 2005.

[57] S. J. Asztalos, G. Carosi, C. Hagmann, D. Kinion, K. van Birber, M. Hotz, L. J. Rosenberg, G. Rybka, J. Hoskins, J. Hwang, P. Sikivie, D. B. Tanner, R. Bradley, J. Clarke, and ADMX Collaboration. SQUID-Based Microwave Cavity Search for Dark-Matter Axions. Physical Review Letters, 104(4):041301, January 2010.

[58] R. Bradley, J. Clarke, D. Kinion, L. J. Rosenberg, K. van Birber, S. Matsuksi, M. Muck, and P. Sikivie. Microwave cavity searches for dark-matter axions. Rev. Mod. Phys., 75:777–817, 2003.

[59] Richard Brito, Shrobana Ghosh, Enrico Barausse, Emanuele Berti, Vitor Cardoso, Irina Dvorkin, Antoine Klein, and Paolo Pani. Stochastic and resolvable gravitational waves from ultralight bosons with LIGO and LISA. Phys. Rev., D96(6):064050, 2017.

[60] Bohua Li, Paul R. Shapiro, and Tanja Rindler-Daller. Bose-Einstein-condensed scalar field dark matter and the gravitational wave background from inflation: new cosmological constraints and its detectability by LIGO. Phys. Rev., D96(6):063505, 2017.

[61] Francesca Day and Sven Krippendorf. Searching for Axion-Like Particles with X-ray Polarimeters. In Alsatian Workshop on X-ray Polarimetry Strasbourg, France, November 13-15, 2017, 2018.

[62] R. Bradley, J. Clarke, D. Kinion, L. J. Rosenberg, K. van Birber, S. Matsuksi, M. Muck, and P. Sikivie. Microwave cavity searches for dark-matter axions. Rev. Mod. Phys., 75:777–817, 2003.

[63] Eric Braaten, Abhishek Mohapatra, and Hong Zhang. Dense Axion Stars. Phys. Rev. Lett., 117(12):121801, 2016.
[72] Andrew P. Lundgren, Mihai Bondarescu, Ruxandra Bondarescu, and Jayashree Balakrishna. Luke-warm dark matter: Bose-condensation of ultra-light particles. *Astrophys. J.*, 715:L35, 2010.

[73] Joshua Eby, Michael Ma, Peter Suranyi, and L. C. R. Wijewardhana. Decay of Ultralight Axion Condensates. *JHEP*, 01:066, 2018.

[74] Pierre Sikivie. An Argument for Axion Dark Matter. *Springer Proc. Phys.*, 148:25–29, 2013.

[75] P. Sikivie. Caustic rings of dark matter. *Physics Letters B*, 432(1-2):139 – 144, 1998.

[76] L. D. Duffy and P. Sikivie. The Caustic Ring Model of the Milky Way Halo. *Phys. Rev.*, D78:063508, 2008.

[77] O. Erken, P. Sikivie, H. Tam, and Q. Yang. Cosmic axion thermalization. *Phys. Rev.*, D85:063520, 2012.

[78] Pierre Sikivie. The emerging case for axion dark matter. *Phys. Lett.*, B695:22–25, 2011.

[79] P. Sikivie and Q. Yang. Bose-Einstein Condensation of Dark Matter Axions. *Phys. Rev. Lett.*, 103:111301, 2009.

[80] Leanne D. Duffy, P. Sikivie, D. B. Tanner, Stephen J. Asztalos, C. Hagmann, D. Kinion, L. J. Rosenberg, K. van Bibber, D. B. Yu, and R. F. Bradley. A high resolution search for dark-matter axions. *Phys. Rev.*, D74:012006, 2006.

[81] Nilanjan Banik, Adam J. Christopherson, Pierre Sikivie, and Elisa Maria Todarello. New astrophysical bounds on ultralight axionlike particles. *Phys. Rev.*, D95(4):043542, 2017.

[82] David J. E. Marsh. Axion Cosmology. *Phys. Rept.*, 643:1–79, 2016.

[83] David J.E. Marsh and Pedro G. Ferreira. Ultra-Light Scalar Fields and the Growth of Structure in the Universe. *Phys.Rev.*, D82:103528, 2010.

[84] David J.E. Marsh, Ewan R.M. Tarrant, Edmund J. Copeland, and Pedro G. Ferreira. Cosmology of Axions and Moduli: A Dynamical Systems Approach. *Phys.Rev.*, D86:023508, 2012.

[85] David J. E. Marsh and Joe Silk. A Model For Halo Formation With Axion Mixed Dark Matter. *Mon. Not. Roy. Astron. Soc.*, 437(3):2652–2663, 2014.

[86] Renee Hlozek, Daniel Grin, David J. E. Marsh, and Pedro G. Ferreira. A search for ultralight axions using precision cosmological data. *Phys. Rev.*, D91(10):103512, 2015.

[87] Lam Hui, Jeremiah P. Ostriker, Scott Tremaine, and Edward Witten. Ultralight scalars as cosmological dark matter. *Phys. Rev. D*, 95(4):043541, 2017.

[88] Alberto Diez-Tejedor and David J. E. Marsh. Cosmological production of ultralight dark matter axions. 2017.

[89] Elisa G.M. Ferreira. Ultra-Light Dark Matter. 5 2020.

[90] Asimina Arvanitaki, Savas Dimopoulos, Sergei Dubovsky, Nemanja Kaloper, and John March-Russell. String Axiverse. *Phys. Rev.*, D81:123530, 2010.

[91] Bobby Samir Acharya, Konstantin Bobkov, and Piyush Kumar. An M Theory Solution to the Strong CP Problem and Constraints on the Axiverse. *JHEP*, 11:105, 2010.

[92] Yi-Han Zhang and Tzihong Chiueh. Evolution of linear wave dark matter perturbations in the radiation-dominant era. 2017.

[93] Francisco X. Linares Cedeño, Alma X. González-Morales, and L. Arturo Ureña López. Cosmological signatures of ultralight dark matter with an axionlike potential. *Phys. Rev.*, D96(6):061301, 2017.
[94] Vivian Poulin, Tristan L. Smith, Daniel Grin, Tanvi Karwal, and Marc Kamionkowski. Cosmological implications of ultra-light axion-like fields. 2018.

[95] J. Lesgourgues. The Cosmic Linear Anisotropy Solving System (CLASS) I: Overview. ArXiv e-prints: 1104.2932, April 2011.

[96] L. Arturo Ureña López and Alma X. González-Morales. Towards accurate cosmological predictions for rapidly oscillating scalar fields as dark matter. JCAP, 1607(07):048, 2016.

[97] Edmund J. Copeland, Andrew R Liddle, and David Wands. Exponential potentials and cosmological scaling solutions. Phys.Rev., D57:4686–4690, 1998.

[98] L. Arturo Ureña-López. New perturbative method for analytical solutions in single-field models of inflation. Phys. Rev., D94(6):063532, 2016.

[99] Nandan Roy, Alma X. Gonzalez-Morales, and L. Arturo Urena-Lopez. New general parametrization of quintessence fields and its observational constraints. 2018.

[100] Bharat Ratra. Expressions for linearized perturbations in a massive scalar field dominated cosmological model. Phys.Rev., D44:352–364, 1991.

[101] Pedro G. Ferreira and Michael Joyce. Structure formation with a selftuning scalar field. Phys.Rev.Lett., 79:4740–4743, 1997.

[102] Pedro G. Ferreira and Michael Joyce. Cosmology with a primordial scaling field. Phys.Rev., D58:023503, 1998.

[103] Francesca Perrotta and Carlo Baccigalupi. Early time perturbations behavior in scalar field cosmologies. Phys.Rev., D59:123508, 1999.

[104] Wayne Hu, Rennan Barkana, and Andrei Gruzinov. Cold and fuzzy dark matter. Phys.Rev.Lett., 85:1158–1161, 2000.

[105] Wayne Hu. Structure formation with generalized dark matter. Astrophys.J., 506:485–494, 1998.

[106] Jiajun Zhang, Jui-Lin Kuo, Hantao Liu, Yue-Lin Sming Tsai, Kingman Cheung, and Ming-Chung Chu. Is Fuzzy Dark Matter in tension with Lyman-alpha forest? 2017.

[107] Jonathan Cookmeyer, Daniel Grin, and Tristan L. Smith. How sound are our ultralight axion approximations? Phys. Rev., D101(2):023501, 2020.

[108] E. M. George et al. A measurement of secondary cosmic microwave background anisotropies from the 2500-square-degree SPT-SZ survey. Astrophys. J., 799(2):177, 2015.

[109] Sudeep Das et al. The Atacama Cosmology Telescope: temperature and gravitational lensing power spectrum measurements from three seasons of data. JCAP, 1404:014, 2014.

[110] Lauren Anderson et al. The clustering of galaxies in the SDSS-III Baryon Oscillation Spectroscopic Survey: baryon acoustic oscillations in the Data Releases 10 and 11 Galaxy samples. Mon. Not. Roy. Astron. Soc., 441(1):24–62, 2014.

[111] Solène Chabanier, Marius Milen, and Nathalie Palanque-Delabrouille. Updated matter power spectrum constraints from the Lyα forest and other probes. 2019.

[112] Vid Iršič, Matteo Viel, Martin G. Haehnelt, James S. Bolton, and George D. Becker. First constraints on fuzzy dark matter from Lyman-α forest data and hydrodynamical simulations. 2017.

[113] Eric Armengaud, Nathalie Palanque-Delabrouille, Christophe Yèche, David J. E. Marsh, and Julien Baur. Constraining the mass of light bosonic dark matter using SDSS Lyman-α forest. 2017.

[114] Nathalie Palanque-Delabrouille et al. The one-dimensional Ly-alpha forest power spectrum from BOSS. Astron. Astrophys., 559:A85, 2013.
[115] Matteo Viel, George D. Becker, James S. Bolton, and Martin G. Haehnelt. Warm dark matter as a solution to the small scale crisis: New constraints from high redshift Lyman-α forest data. Phys. Rev., D88:043502, 2013.

[116] S. López, V. D’Odorico, S. L. Ellison, G. D. Becker, L. Christensen, G. Cupani, K. D. Denney, I. Páris, G. Worseck, T. A. M. Berg, S. Cristiani, M. Dessauges-Zavadsky, M. Haehnelt, F. Hamann, J. Hennawi, V. Iršič, T.-S. Kim, P. López, R. Lund Saust, B. Ménard, S. Perrotta, J. X. Prochaska, R. Sánchez-Ramírez, M. Vestergaard, M. Viel, and L. Wisotzki. XQ-100: A legacy survey of one hundred $3.5 < z < 4.5$ quasars observed with VLT/X-shooter. Astronomy & Astrophysics, 594:A91, 2016.

[117] Michael Levi et al. The DESI Experiment, a whitepaper for Snowmass 2013. 2013.

[118] P.A.R. Ade et al. Planck 2015 results. XIII. Cosmological parameters. 2015.

[119] Benjamin Audren, Julien Lesgourgues, Karim Benabed, and Simon Prunet. Conservative constraints on early cosmology with monte python. Journal of Cosmology and Astroparticle Physics, 2013(02):001, 2013.

[120] Andrew Gelman and Donald B. Rubin. Inference from Iterative Simulation Using Multiple Sequences. Statist. Sci., 7:457–472, 1992.

[121] D. J. Heath. The growth of density perturbations in zero pressure Friedmann–Lemaître universes. Monthly Notices of the Royal Astronomical Society, 179(3):351–358, 07 1977.

[122] Ofer Lahav, Per B. Lilje, Joel R. Primack, and Martin J. Rees. Dynamical effects of the cosmological constant. Mon. Not. Roy. Astron. Soc., 251:128–136, 1991.

[123] Daniel J. Eisenstein. An Analytic expression for the growth function in a flat universe with a cosmological constant. 9 1997.

[124] Scott Dodelson. Modern Cosmology. Academic Press, Amsterdam, 2003.

[125] Dragan Huterer et al. Growth of Cosmic Structure: Probing Dark Energy Beyond Expansion. Astropart. Phys., 63:23–41, 2015.

[126] Limin Wang and Paul J Steinhardt. Cluster abundance constraints for cosmological models with a time-varying, spatially inhomogeneous energy component with negative pressure. The Astrophysical Journal, 508(2):483, 1998.

[127] Eric V. Linder and Robert N. Cahn. Parameterized Beyond-Einstein Growth. Astropart. Phys., 28:481–488, 2007.

[128] David Polarski and Radouane Gannouji. On the growth of linear perturbations. Phys. Lett., B660:439–443, 2008.

[129] L. Guzzo et al. A test of the nature of cosmic acceleration using galaxy redshift distortions. Nature, 451:541–545, 2008.

[130] Radouane Gannouji and David Polarski. The growth of matter perturbations in some scalar-tensor DE models. JCAP, 0805:018, 2008.

[131] Spyros Basilakos and Fotios K. Anagnostopoulos. Growth index of matter perturbations in the light of Dark Energy Survey. 2019.

[132] Wompherdeiki Khyllep and Jibitesh Dutta. Linear growth index of matter perturbations in Rastall gravity. Phys. Lett., B797:134796, 2019.

[133] Shinji Tsujikawa. Matter density perturbations and effective gravitational constant in modified gravity models of dark energy. Phys. Rev., D76:023514, 2007.

[134] R. Gannouji, B. Moraes, and D. Polarski. The growth of matter perturbations in f(R) models. JCAP, 0902:034, 2009.
[135] Xiangyun Fu, Puxun Wu, and Hong Wei Yu. The growth factor of matter perturbations in f(R) gravity. Eur. Phys. J., C68:271–276, 2010.

[136] Alvise Raccanelli, Daniele Bertacca, Davide Pietrobon, Fabian Schmidt, Lado Samushia, Nicola Bartolo, Olivier Dore, Sabino Matarrese, and Will J. Percival. Testing Gravity Using Large-Scale Redshift-Space Distortions. Mon. Not. Roy. Astron. Soc., 436:89–100, 2013.

[137] Antonio Jesús López-Revelles. Growth of matter perturbations for realistic $F(R)$ models. Phys. Rev., D87(2):024021, 2013.

[138] Savvas Nesseris and Domenico Sapone. Accuracy of the growth index in the presence of dark energy perturbations. Phys. Rev., D92(2):023013, 2015.

[139] Savvas Nesseris, George Pantazis, and Leandros Perivolaropoulos. Tension and constraints on modified gravity parametrizations of $G_{\text{eff}}(z)$ from growth rate and Planck data. Phys. Rev., D96(2):023542, 2017.

[140] Domenico Sapone and Luca Amendola. Constraining the growth factor with baryon oscillations. 2007.

[141] Yungui Gong, Mustapha Ishak, and Anzhong Wang. Growth factor parametrization in curved space. Physical Review D, 80, 2009.

[142] Seokcheon Lee and Kin-Wang Ng. Properties of the exact analytic solution of the growth factor and its applications. Physical Review D, 82, 2009.

[143] Viviana Acquaviva and Eric Gawiser. How to Falsify the GR+LambdaCDM Model with Galaxy Redshift Surveys. Phys. Rev., D82:082001, 2010.

[144] Rui Zheng and Qing-Guo Huang. Growth factor in f(t) gravity. Journal of Cosmology and Astroparticle Physics, 2011:002-002, 2010.

[145] Cinzia Di Porto, Luca Amendola, and Enzo Branchini. Growth factor and galaxy bias from future redshift surveys: a study on parametrizations. Monthly Notices of the Royal Astronomical Society, 419:985–997, 2011.

[146] Zhongxu Zhai, Michael Blanton, Anže Slosar, and Jeremy Tinker. An evaluation of cosmological models from expansion and growth of structure measurements. The Astrophysical Journal, 850:183, 2017.

[147] Yong-Seon Song and Will J. Percival. Reconstructing the history of structure formation using Redshift Distortions. JCAP, 0910:004, 2009.

[148] Chris Blake et al. The WiggleZ Dark Energy Survey: the growth rate of cosmic structure since redshift $z=0.9$. Mon. Not. Roy. Astron. Soc., 415:2876, 2011.

[149] Florian Beutler, Chris Blake, Matthew Colless, D. Heath Jones, Lister Staveley-Smith, Gregory B. Poole, Lachlan Campbell, Quentin Parker, Will Saunders, and Fred Watson. The 6dF Galaxy Survey: $z \approx 0$ measurement of the growth rate and $\sigma_8$. Mon. Not. Roy. Astron. Soc., 423:3430–3444, 2012.

[150] S. de la Torre et al. The VIMOS Public Extragalactic Redshift Survey (VIPERS). Galaxy clustering and redshift-space distortions at $z = 0.8$ in the first data release. Astron. Astrophys., 557:A54, 2013.

[151] Cullan Howlett, Ashley Ross, Lado Samushia, Will Percival, and Marc Manera. The clustering of the SDSS main galaxy sample - II. Mock galaxy catalogues and a measurement of the growth of structure from redshift space distortions at $z = 0.15$. Mon. Not. Roy. Astron. Soc., 449(1):848–866, 2015.

[152] Héctor Gil-Marín, Will J. Percival, Licia Verde, Joel R. Brownstein, Chia-Hsun Chuang, Francisco-Shu Kitaura, Sergio A. Rodriguez-Torres, and Matthew D. Olmstead. The clustering of galaxies in the SDSS-III Baryon Oscillation Spectroscopic Survey: RSD
measurement from the power spectrum and bispectrum of the DR12 BOSS galaxies. Mon. Not. Roy. Astron. Soc., 465(2):1757–1788, 2017.

[153] Tepppei Okumura et al. The Subaru FMOS galaxy redshift survey (FastSound). IV. New constraint on gravity theory from redshift space distortions at $z \sim 1.4$. Publ. Astron. Soc. Jap., 68(3, id. 38):24, 2016.

[154] Gong-Bo Zhao et al. The clustering of the SDSS-IV extended Baryon Oscillation Spectroscopic Survey DR14 quasar sample: a tomographic measurement of cosmic structure growth and expansion rate based on optimal redshift weights. 2018.

[155] Cullan Howlett, Lister Staveley-Smith, Pascal J. Elahi, Tao Hong, Tom H. Jarrett, D. Heath Jones, Bärbel S. Koribalski, Lucas M. Macri, Karen L. Masters, and Christopher M. Springob. 2MTF - VI. Measuring the velocity power spectrum. Mon. Not. Roy. Astron. Soc., 471(3):3135–3151, 2017.

[156] Lado Samushia, Will J. Percival, and Alvise Raccanelli. Interpreting large-scale redshift-space distortion measurements. Mon. Not. Roy. Astron. Soc., 420:2102–2119, 2012.

[157] A. Vikhlinin et al. Chandra Cluster Cosmology Project III: Cosmological Parameter Constraints. Astrophys. J., 692:1060–1074, 2009.

[158] Steven W. Allen, August E. Evrard, and Adam B. Mantz. Cosmological Parameters from Observations of Galaxy Clusters. Ann. Rev. Astron. Astrophys., 49:409–470, 2011.

[159] Steven Murray, Chris Power, and Aaron Robotham. How well do we know the Halo Mass Function? Mon. Not. Roy. Astron. Soc., 434:L61, 2013.

[160] A. Jenkins, C. S. Frenk, Simon D. M. White, J. M. Colberg, S. Cole, August E. Evrard, H. M. P. Couchman, and N. Yoshida. The Mass function of dark matter halos. Mon. Not. Roy. Astron. Soc., 321:372, 2001.

[161] Jeremy L. Tinker, Andrey V. Kravtsov, Anatoly Klypin, Kebrk Abazajian, Michael S. Warren, Gustavo Yepes, Stefan Gottlober, and Daniel E. Holz. Toward a halo mass function for precision cosmology: The Limits of universality. Astrophys. J., 688:709–728, 2008.

[162] Michael S. Warren, Kebrk Abazajian, Daniel E. Holz, and Luis Teodoro. Precision determination of the mass function of dark matter halos. Astrophys. J., 646:881–885, 2006.

[163] William H. Press and Paul Schechter. Formation of galaxies and clusters of galaxies by selfsimilar gravitational condensation. Astrophys. J., 187:425–438, 1974.

[164] J. R. Bond, S. Cole, G. Efstathiou, and Nick Kaiser. Excursion set mass functions for hierarchical Gaussian fluctuations. Astrophys. J., 379:440, 1991.

[165] Ravi K. Sheth and Giuseppe Tormen. Large scale bias and the peak background split. Mon. Not. Roy. Astron. Soc., 308:119, 1999.

[166] Ravi K. Sheth, H. J. Mo, and Giuseppe Tormen. Ellipsoidal collapse and an improved model for the number and spatial distribution of dark matter haloes. Mon. Not. Roy. Astron. Soc., 323:1, 2001.

[167] Aurel Schneider, Robert E. Smith, and Darren Reed. Halo Mass Function and the Free Streaming Scale. Mon. Not. Roy. Astron. Soc., 433:1573, 2013.

[168] Xiaolong Du, Christoph Behrens, and Jens C. Niemeyer. Substructure of fuzzy dark matter haloes. Mon. Not. Roy. Astron. Soc., 465(1):941–951, 2017.

[169] Andrew J. Benson, Arya Farahi, Shaun Cole, Leonidas A. Moustakas, Adrian Jenkins, Mark Lovell, Rachel Kennedy, John Helly, and Carlos Frenk. Dark Matter Halo Merger Histories Beyond Cold Dark Matter: I - Methods and Application to Warm Dark Matter. Mon. Not. Roy. Astron. Soc., 428:1774, 2013.
Aurel Schneider. Structure formation with suppressed small-scale perturbations. Mon. Not. Roy. Astron. Soc., 451(3):3117–3130, 2015.

Matthew R. Buckley, Jesús Zavala, Francis-Yan Cyr-Racine, Kris Sigurdson, and Mark Vogelsberger. Scattering, Damping, and Acoustic Oscillations: Simulating the Structure of Dark Matter Halos with Relativistic Force Carriers. Phys. Rev., D90(4):043524, 2014.

James E. Gunn and J. Richard Gott, III. On the Infall of Matter into Clusters of Galaxies and Some Effects on Their Evolution. Astrophys. J., 176:1–19, 1972.

W. J. Percival, L. Miller, and J. A. Peacock. An analytic model for the epoch of halo creation. Mon. Not. Roy. Astron. Soc., 318:273, 2000.

Will J. Percival. Cosmological structure formation in a homogeneous dark energy background. Astron. Astrophys., 443:819, 2005.

David J. E. Marsh. WarnAndFuzzy: the halo model beyond CDM. 2016.

Katelin Schutz. The Subhalo Mass Function and Ultralight Bosonic Dark Matter. 2020.

Nilanjan Banik, Jo Bovy, Gianfranco Bertone, Denis Erkal, and T. J. L. de Boer. Evidence of a population of dark subhalos from Gaia and Pan-STARRS observations of the GD-1 stream. 2019.

Carl J. Grillmair and Odysseas Dionatos. Detection of a 63 Degree Cold Stellar Stream in the Sloan Digital Sky Survey. Astrophys. J., 643:L17–L20, 2006.

R.A. Ibata, G.F. Lewis, and M.J. Irwin. Uncovering cdm halo substructure with tidal streams. Mon. Not. Roy. Astron. Soc., 332:915, 2002.

Joo H. Yoon, Kathryn V. Johnston, and David W. Hogg. Clumpy Streams from Clumpy Halos: Detecting Missing Satellites with Cold Stellar Structures. Astrophys. J., 731:58, 2011.

Raymond G. Carlberg. Dark Matter Sub-Halo Counts via Star Stream Crossings. Astrophys. J., 748:20, 2012.

Jo Bovy, Denis Erkal, and Jason L. Sanders. Linear perturbation theory for tidal streams and the small-scale CDM power spectrum. Mon. Not. Roy. Astron. Soc., 466(1):628–668, 2017.

Denis Erkal, Vasily Belokurov, Jo Bovy, and Jason L. Sanders. The number and size of subhalo-induced gaps in stellar streams. Monthly Notices of the Royal Astronomical Society, 463(1):102–119, 08 2016.

Ana Bonaca, David W. Hogg, Adrian M. Price-Whelan, and Charlie Conroy. The Spur and the Gap in GD-1: Dynamical evidence for a dark substructure in the Milky Way halo. 2018.

Nilanjan Banik, Jo Bovy, Gianfranco Bertone, Denis Erkal, and T. J. L. de Boer. Novel constraints on the particle nature of dark matter from stellar streams. 2019.

María Benito, Juan Carlos Criado, Gert Hütsi, Martti Raidal, and Hardi Veermäe. Implications of Milky Way substructures for the nature of dark matter. Phys. Rev. D, 101(10):103023, 2020.

Julian B. Muñoz, Cora Dvorkin, and Francis-Yan Cyr-Racine. Probing the Small-Scale Matter Power Spectrum with Large-Scale 21-cm Data. 2019.

Hayato Shimabukuro, Kiyotomo Ichiki, and Kenji Kadota. Constraints on nature of ultra light dark matter particles with 21cm forest. 2019.

Željko Ivezić et al. LSST: from Science Drivers to Reference Design and Anticipated Data Products. Astrophys. J., 873(2):111, 2019.

Alex Drlica-Wagner et al. Probing the Fundamental Nature of Dark Matter with the Large Synoptic Survey Telescope. 2019.
[191] Oliver Müller, Roberto Scalera, Bruno Binggeli, and Helmut Jerjen. The M 101 group complex: new dwarf galaxy candidates and spatial structure. *Astron. Astrophys.*, 602:A119, 2017.

[192] Judd D. Bowman, Alan E. E. Rogers, Raul A. Monsalve, Thomas J. Mozdzen, and Nivedita Mahesh. An absorption profile centred at 78 megahertz in the sky-averaged spectrum. *Nature*, 555(7694):67–70, 2018.

[193] Rennan Barkana. Possible interaction between baryons and dark-matter particles revealed by the first stars. *Nature*, 555(7694):71–74, 2018.

[194] A. Weltman et al. Fundamental Physics with the Square Kilometre Array. 2018.

[195] Vid Iršič, Huangyu Xiao, and Matthew McQuinn. Early Structure Formation Constraints on the Ultra-Light Axion in the Post-Inflation Scenario. 2019.

[196] Daniel Grin, Mustafa A. Amin, Vera Gluscevic, Renée Hlˇ ozek, David J.E. Marsh, Vivian Poulin, Chanda Prescod-Weinstein, and Tristan L. Smith. Gravitational probes of ultra-light axions. 4 2019.

[197] Chan-Gyung Park, Jai-chan Hwang, and Hyerim Noh. Axion as a cold dark matter candidate: low-mass case. *Phys. Rev.*, D86:083535, 2012.

[198] Tzihong Chiueh. Why is the Dark Axion Mass $10^{-22}$ eV? 2014.