On Voevodsky’s algebraic $K$-theory spectrum $BGL$

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Abstract

Under a certain normalization assumption we prove that the $\mathbb{P}^1$-spectrum $BGL$ of Voevodsky which represents algebraic $K$-theory is unique over $\text{Spec}(\mathbb{Z})$. Following an idea of Voevodsky, we equip the $\mathbb{P}^1$-spectrum $BGL$ with the structure of a commutative $\mathbb{P}^1$-ring spectrum in the motivic stable homotopy category. Furthermore, we prove that under a certain normalization assumption this ring structure is unique over $\text{Spec}(\mathbb{Z})$. For an arbitrary Noetherian scheme $S$ of finite Krull dimension we pull this structure back to obtain a distinguished monoidal structure on $BGL$. This monoidal structure is relevant for our proof of the motivic Conner-Floyd theorem [PPR]. It has also been used to obtain a motivic version of Snaith’s theorem [GS].

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2 Preliminaries

This paper is concerned with results in motivic homotopy theory, which was put on firm foundations by Morel and Voevodsky in [MV] and [V]. Due to technical reasons explained below, the setup in [MV], as well as other model categories used in motivic homotopy theory, are inconvenient for our purposes, so we decided to pursue a slightly different approach. We refer to the Appendix A for the basic terminology, notation, constructions, definitions, and results concerning motivic homotopy theory. For a Noetherian scheme $S$ of finite Krull dimension we write $\mathcal{M}(S)$, $\mathcal{M}_*(S)$, $\mathcal{H}_*(S)$ and $\mathcal{SH}(S)$ for the category of motivic spaces, the category of pointed motivic spaces, the pointed motivic homotopy

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category and the stable motivic homotopy category over $S$. These categories are equipped with symmetric monoidal structures. In particular, a symmetric monoidal structure $(\wedge, I)$ is constructed on the motivic stable homotopy category and its basic properties are proved. This structure is used extensively over the present text.

Let $S$ be a regular scheme, and let $K^0(S)$ denote the Grothendieck group of vector bundles over $S$. Morel and Voevodsky proved in [MV, Thm. 4.3.13] that the Thomason-Trobaugh $K$-theory [TT] is represented in the pointed motivic homotopy category $H_\bullet(S)$ by the space $\mathbb{Z} \times Gr$ pointed by $(0, x_0)$. Here $Gr$ is the union of the finite Grassmann varieties $\bigcup_{n=0}^\infty Gr(n, 2n)$, considered as motivic spaces. There is a unique element $\xi_\infty \in K^0(\mathbb{Z} \times Gr)$ which corresponds to the identity morphism $id: \mathbb{Z} \times Gr \to \mathbb{Z} \times Gr$. It follows that there exists a unique morphism

$$\mu_\otimes: (\mathbb{Z} \times Gr) \wedge (\mathbb{Z} \times Gr) \to \mathbb{Z} \times Gr$$

in $H_\bullet(S)$ such that the composition $(\mathbb{Z} \times Gr) \times (\mathbb{Z} \times Gr) \to (\mathbb{Z} \times Gr) \wedge (\mathbb{Z} \times Gr) \xrightarrow{\mu_\otimes} \mathbb{Z} \times Gr$ represents the element $\xi_\infty \otimes \xi_\infty$ in $K^0((\mathbb{Z} \times Gr) \times (\mathbb{Z} \times Gr))$ (see Lemma B.1). Let $e_\otimes: S^0 \to \mathbb{Z} \times Gr$ be the map which corresponds to the point $(1, x_0) \in \mathbb{Z} \times Gr$. The triple

$$(\mathbb{Z} \times Gr, \mu_\otimes, e_\otimes)$$

is a commutative monoid in $H_\bullet(S)$.

Using this fact, Voevodsky constructed in [V] a $\mathbb{P}^1$-spectrum

$$BGL = (K_0, K_1, K_2, \ldots)$$

with structure maps $e_i: K_i \wedge \mathbb{P}^1 \to K_{i+1}$ such that

(i) there is a motivic weak equivalence $w: \mathbb{Z} \times Gr \to K_0$, and for all $i$ one has $K_i = K_0$ and $e_i = e_0$,

(ii) the morphism

$$\mathbb{Z} \times Gr \times \mathbb{P}^1 \xrightarrow{\text{can}} (\mathbb{Z} \times Gr) \wedge \mathbb{P}^1 \xrightarrow{w/\mathbb{P}^1} K_i \wedge \mathbb{P}^1 \xrightarrow{\epsilon_i} K_{i+1} \xrightarrow{w^{-1}} \mathbb{Z} \times Gr$$

in $H_\bullet(S)$ represents the element $\xi_\infty \otimes ([\mathcal{O}(-1)] - [\mathcal{O}]) \in K^0(\mathbb{Z} \times Gr \times \mathbb{P}^1)$,

(iii) and the adjoint $K_i \to \Omega_{\mathbb{P}^1}(K_{i+1})$ of $e_i$ is a motivic weak equivalence.

With this spectrum in hand given a smooth $X$ over $S$ we may identify $K^0(X)$ with $BGL^{2i-1}(X)$ as follows

$$K^0(X) = \text{Hom}_{H_\bullet(S)}(X_+, \mathbb{Z} \times Gr) = \text{Hom}_{H_\bullet(S)}(X_+, K_i) = BGL^{2i-1}(X)$$

Our first aim is to recall Voevodsky’s construction to show that this spectrum is essentially unique. This has also been obtained in [R]. Our second and more important aim is to give a commutative monoidal structure to the $\mathbb{P}^1$-spectrum $BGL$ which respects
the naive multiplicative structure on the functor \( X \mapsto \text{BGL}^{2i\cdot i}(X) \). To be more precise, we construct a product
\[
\mu_{\text{BGL}}: \text{BGL} \land \text{BGL} \to \text{BGL}
\]in the stable motivic homotopy category \( \text{SH}(S) \) such that for any \( X \in \text{Sm}/S \) the diagram
\[
\begin{array}{ccc}
K^0(X) \times K^0(X) & \xrightarrow{\otimes} & K^0(X) \\
\downarrow\cong & & \downarrow\cong \\
\text{BGL}^{2i\cdot i}(X) \times \text{BGL}^{2j\cdot j}(X) & \xrightarrow{\mu_{\text{BGL}}} & \text{BGL}^{2(i\cdot j)i+j}(X)
\end{array}
\]
commutes. We show in Theorem 3.6 that there is a unique product
\[
\mu_{\text{BGL}} \in \text{Hom}_{\text{SH}(\mathbb{Z})}(\text{BGL} \land \text{BGL}, \text{BGL})
\]satisfying this property. This induces a product \( \mu_{\text{BGL}} \in \text{Hom}_{\text{SH}(S)}(\text{BGL} \land \text{BGL}, \text{BGL}) \) for an arbitrary regular scheme \( S \) by pull-back along the structural morphism \( S \to \text{Spec}(\mathbb{Z}) \).

As well, we show that the product is associative, commutative and unital. The resulting multiplicative structure on the bigraded theory \( \text{BGL}^{*\cdot *}(X) \) coincides with the Waldhausen multiplicative structure on the Thomason-Trobaugh \( K \)-theory.

### 2.1 Recollections on motivic homotopy theory

The basic definitions, constructions and model structures used in the text are given in Appendix A. The word “model structure” is used in its modern sense and thus refers to a “closed model structure” as originally defined by Quillen. Let \( S \) be a Noetherian finite-dimensional scheme. A **motivic space over** \( S \) is a simplicial presheaf on the site \( \text{Sm}/S \) of smooth quasi-projective \( S \)-schemes. A **pointed motivic space over** \( S \) is a pointed simplicial presheaf on the site \( \text{Sm}/S \). We write \( \text{M}_{\bullet}(S) \) for the category of pointed motivic spaces over \( S \). A **closed motivic model structure** \( \text{M}_{\bullet}^{cm}(S) \) is constructed in A.17. The adjective “closed” refers to the fact that closed embeddings in \( \text{Sm}/S \) are forced to become cofibrations. The resulting homotopy category \( \text{H}_{\bullet}^{cm}(S) \) obtained in Theorem A.17 is called the **motivic homotopy category** of \( S \). By Theorem A.19 it is equivalent to the Morel-Voevodsky \( \mathbb{A}^1 \)-homotopy category \([MV]\), and we may drop the superscript in \( \text{H}_{\bullet}^{cm}(S) \) for convenience. The closed motivic model structure has the properties that

1. for any closed \( S \)-point \( x_0: S \hookrightarrow X \) in a smooth \( S \)-scheme, the pointed motivic space \( (X, x_0) \) is cofibrant in \( \text{M}_{\bullet}^{cm}(S) \) (Lemma A.10),

2. a morphism \( f: S \to S' \) of base schemes induces a left Quillen functor \( f^*: \text{M}_{\bullet}^{cm}(S') \to \text{M}_{\bullet}^{cm}(S) \) (Theorem A.17), and

3. taking complex points is a left Quillen functor \( \text{R}_C: \text{M}_{\bullet}^{cm}(\mathbb{C}) \to \text{Top}_{\bullet} \) (Theorem A.23).
Conditions 2 and 3 do not hold for the Morel-Voevodsky model structure, condition 1 fails for the so-called projective model structure [DRO, Thm. 2.12]. For a morphism $f : A \to B$ of pointed motivic spaces we will write $[f]$ for the class of $f$ in $H_\bullet(S)$.

We will consider $\mathbb{P}^1$ as a pointed motivic space over $S$ pointed by $\infty : S \hookrightarrow \mathbb{P}^1$. A $\mathbb{P}^1$-spectrum $E$ over $S$ consists of a sequence $E_0, E_1, \ldots$ of pointed motivic spaces over $S$, together with structure maps $\sigma_n : E_n \wedge \mathbb{P}^1 \to E_{n+1}$. A map of $\mathbb{P}^1$-spectra is a sequence of maps of pointed motivic spaces which is compatible with the structure maps. Let $\text{SH}(S)$ denote the homotopy category of $\mathbb{P}^1$-spectra, as described in Section A.5.

By Theorem A.30 it is canonically equivalent to the motivic stable homotopy category constructed in [V] and [J]. As we will see below there exists an essentially unique $\mathbb{P}^1$-spectrum $\text{BGL}$ over $S = \text{Spec}(\mathbb{Z})$ satisfying properties (i) and (ii) from Section 2. In the following, we will construct $\text{BGL}$ in a slightly different way than Voevodsky did originally in [V]. In order to achieve this, we begin with a description of the known monoidal structure on the Thomason-Trobaugh K-theory [TT].

### 2.2 A construction of BGL

Let $S$ be a regular scheme. For every $S$-scheme $X$ consider the category $\text{Big}(X)$ of big vector bundles over $X$ (see for instance [FS] for the definition and basic properties). The assignments $X \mapsto \text{Big}(X)$ and $(f : Y \to X) \mapsto f^* : \text{Big}(X) \to \text{Big}(Y)$ form a functor from schemes to the category of small categories. The reason is that there is an equality $(f \circ g)^* = g^* \circ f^*$, not just a unique natural isomorphism. In what follows we will always consider the Waldhausen K-theory space of $X$ as the space obtained by applying Waldhausen’s $\mathbb{S}_\bullet$-construction [W] applied to the category $\text{Big}(X)$ rather than to the category $\text{Vect}(X)$ of usual vector bundles on $X$. This has the advantage that the assignment taking an $S$-scheme $X$ to the Waldhausen K-theory space of $X$ becomes a functor on the category of $S$-schemes, and in particular a pointed motivic space over $S$. In what follows a category $\mathcal{S}m\mathcal{O}p/S$ will be useful as well. Its objects are pairs $(X, U)$ with an $X \in \mathcal{S}m/S$ and an open $U$ in $X$. Morphisms $(X, U)$ to $(Y, V)$ are morphisms of $X$ to $Y$ in $\mathcal{S}m/S$ which take $U$ to $V$.

Let $\mathcal{K}^W$ be the pointed motivic space defined in A.12. It has the properties that it is fibrant in $\mathcal{M}^{\text{sm}}_\bullet(S)$ and that $\mathcal{K}^W(X)$ is naturally weakly equivalent to the Waldhausen K-theory space associated to the category of big vector bundles on $X$. For $X \in \mathcal{S}m/S$ the simplicial set $\mathcal{K}^W(X)$ is thus a Kan simplicial set having the Waldhausen K-theory groups $K^W_\bullet(X)$ as its homotopy groups. These K-theory groups coincide with Quillen’s higher K-theory groups [TT, Thm.1.11.2]. We write $K_\bullet(X)$ for $K^W_\bullet(X)$. It follows immediately from the adjunction isomorphism

$$\text{Hom}_{\mathcal{H}_\bullet(S)}(S^{n,0} \wedge X_+, \mathcal{K}^W) \cong \text{Hom}_{\mathcal{H}_\bullet} (S^n, \mathcal{K}^W(X)) = K_p(X)$$ (4)

that $\mathcal{K}^W$, regarded as an object in the motivic homotopy category $\mathcal{H}_\bullet(S)$ (see A.17) represents the Quillen K-theory on $\mathcal{S}m/S$. Here $S^n = S^{n,0}$ denotes the $n$-fold smash product of the constant simplicial presheaf $\Delta^1/\partial \Delta^1$ with itself. For a pointed motivic space $A$ set

$$K_p(A) := \text{Hom}_{\mathcal{H}_\bullet}(S^{n,0} \wedge A, \mathcal{K}^W).$$
For $X \in \mathcal{S}m/S$ and a closed subset $Z \hookrightarrow X$, $K_n(X \text{ on } Z)$ denotes the $n$-th Thomason-Trobaugh $K$-group of perfect complexes on $X$ with support on $Z$ [TT, Defn.3.1]. For $A = X/(X - Z)$ with an $X \in \mathcal{S}m/S$ and a closed subset $Z \subset X$, there is an isomorphism $K_\bullet(A) \cong K_\bullet(X \text{ on } Z)$ natural in the pair $(X, X - Z)$ (see [TT, Thm.5.1]). It follows immediately that $K^W$, regarded as an object in the motivic homotopy category $H_*(S)$ (see A.17) represents the Thomason-Trobaugh $K(X \text{ on } Z)$-theory on $\mathcal{S}m\mathcal{O}p/S$. The known monoidal structure [TT, (3.15.4)] on the Thomason-Trobaugh $K(X \text{ on } Z)$-theory coincides with the one induced by the Waldhausen monoid $(K^W, \mu^W, e^W)$ described below.

Using the notation of Example A.12, consider the diagram

$$K^W(X) \wedge K^W(X) = \Omega^1(W_1(X)) \wedge \Omega^1_s(W_1(X)) \xrightarrow{m} \Omega^2_s(W_2(X)) \xrightarrow{ad} K^W(X)$$

with the Waldhausen multiplication $m$ and the adjunction weak equivalence $ad$ described in [W, p. 342]. The diagram defines a morphism

$$K^W \wedge K^W \xrightarrow{\mu^W} K^W$$

in $H_*(S)$ which is the Waldhausen multiplication on $K^W$. Together with the unit morphism $e^W : S^0 \to K^W$ it forms the Waldhausen monoid $(K^W, \mu^W, e^W)$.

By [MV, Thm. 4.3.13] there is an isomorphism $\psi : \mathbb{Z} \times \text{Gr} \to K^W$ in $H_*(S)$. The pointed motivic space $(\mathbb{Z} \times \text{Gr}, (0, x_0))$ is closed cofibrant by Lemma A.10. Let $b^W : \mathbb{P}^1 \to K^W$ be a morphism in $H_*(S)$ representing the class $[0(-1)] - [0]$ in the kernel of the homomorphism $\infty^* : K_0(\mathbb{P}^1) \to K_0(k)$.

**Definition 2.1.** Choose a pointed motivic space $\mathcal{K}$, together with a weak equivalence $i : \mathbb{Z} \times \text{Gr} \to \mathcal{K}$ in $\mathcal{M}^m_*(S)$, as well as a morphism $\epsilon : \mathcal{K} \wedge \mathbb{P}^1 \to \mathcal{K}$ in $\mathcal{M}^m_*(S)$ which descends to

$$\mu^W \circ (\text{id} \wedge b^W) : K^W \wedge \mathbb{P}^1 \to K^W$$

under the identification of $\mathcal{K}$ with $K^W$ in $H_*(S)$ via the isomorphism $\psi \circ [i]^{-1}$. Define $\text{BGL}$ as the $\mathbb{P}^1$-spectrum of the form $(\mathcal{K}_0, \mathcal{K}_1, \mathcal{K}_2, \ldots)$ with $\mathcal{K}_i = \mathcal{K}$ for all $i$ and with the structure maps $e_i : \mathcal{K}_i \wedge \mathbb{P}^1 \to \mathcal{K}_{i+1}$ equal to the map $\epsilon : \mathcal{K} \wedge \mathbb{P}^1 \to \mathcal{K}$.

$\mathbb{P}^1$-spectra as described in Definition 2.1 will be used extensively below.

**Remark 2.2.** The Voevodsky spectrum $\text{BGL}$ is obtained if $\mathcal{K} = Ex^{A^1}(\mathbb{Z} \times \text{Gr})$, $i : \mathbb{Z} \times \text{Gr} \to Ex^{A^1}(\mathbb{Z} \times \text{Gr})$ is the Voevodsky fibrant replacement morphism in the model structure described in [V, Thm. 3.7] and the structure map

$$e : Ex^{A^1}(\mathbb{Z} \times \text{Gr}) \wedge \mathbb{P}^1 \to Ex^{A^1}(\mathbb{Z} \times \text{Gr})$$

described in [V, Section 6.2]. By [V, Thm. 3.6] and Note A.20, the map $\mathbb{Z} \times \text{Gr} \to Ex^{A^1}(\mathbb{Z} \times \text{Gr})$ is also a weak equivalence in $\mathcal{M}^m_*(S)$. In particular, $\text{BGL}$ is an example of a $\mathbb{P}^1$-spectrum as described in Definition 2.1.
Remark 2.3. The Waldhausen structure of a commutative monoid on \( K \) in \( H_*(S) \) induces via the isomorphism \( \psi \circ [i]^{-1} \) the structure of a commutative monoid \((\mathcal{K}, \bar{\mu}, \bar{e})\) on the motivic space \( \mathcal{K} \) in \( H_*(S) \) such that \( \psi \circ [i]^{-1} \) is an isomorphism of monoids. The composition of the inclusion \( P^1 = Gr(1, 2) \) \( \hookrightarrow \{0\} \times Gr \hookleftarrow \mathbb{Z} \times Gr \) and the weak equivalence \( i \) is denoted \( b: P^1 \to \mathcal{K} \). Clearly \( [\varepsilon] = \bar{\mu} \circ ([id \wedge b]) \) in \( H_*(S) \).

Lemma 2.4. Given \( \mathcal{K}, i: Z \times Gr \to \mathcal{K} \) and \( \varepsilon: \mathcal{K} \wedge P^1 \to \mathcal{K} \) fulfilling the conditions of Definition 2.1, there exist \( \mathcal{K}', i': Z \times Gr \to \mathcal{K}' \), \( \varepsilon': \mathcal{K}' \wedge P^1 \to \mathcal{K}' \) and \( q: \mathcal{K}' \to \mathcal{K} \) such that

- \( i' \) and \( \varepsilon' \) fulfil the condition of Definition 2.1,
- \( \mathcal{K}' \) is cofibrant in \( M_{cm}^\bullet(S) \),
- \( q \) is a weak equivalence and \( q \circ \varepsilon' = \varepsilon \circ (q \wedge id) \).

Thus the \( P^1 \)-spectra \( BGL' \) and \( BGL \) are weakly equivalent via the morphism given by the sequence of maps of pointed motivic spaces \( q, q, q, \ldots \).

Proof. Decompose \( i \) as \( q \circ i' \) with a trivial cofibration \( i': Z \times Gr \to \mathcal{K}' \) and a fibration \( q: \mathcal{K}' \to \mathcal{K} \). Note that \( q \) is a weak equivalence since so are \( i' \) and \( i \). The pointed motivic space \( \mathcal{K}' \) is cofibrant in \( M_{cm}^\bullet(S) \), because so is \( Z \times Gr \). Furthermore \( i' \) is a weak equivalence. It remains to construct \( \varepsilon' \). Consider the commutative diagram of pointed motivic spaces

\[
\begin{array}{ccc}
\mathcal{K} & \xrightarrow{q} & \mathcal{K}' \\
\downarrow & & \downarrow \\
\mathcal{K}' \wedge P^1 \gamma(q \wedge id) & \xrightarrow{q} & \mathcal{K}
\end{array}
\]

The left vertical arrow is a cofibration and the right hand side one is a trivial fibration. Thus there exists a map \( \varepsilon': \mathcal{K}' \wedge P^1 \to \mathcal{K}' \) of pointed motivic spaces making the diagram commutative. \( \square \)

Remark 2.5. Let \( f: S' \to S \) be a morphism of schemes and let \( BGL = (\mathcal{K}, \mathcal{K}, \mathcal{K}, \ldots) \) be a \( P^1 \)-spectrum over \( S \) as described in Definition 2.1. Suppose further that \( \mathcal{K} \) is cofibrant in \( M_{cm}^\bullet(S) \). The \( P^1 \)-spectrum \( f^*(BGL) \) over \( S' \) is given by \( (\mathcal{K}', \mathcal{K}', \mathcal{K}', \ldots) \), where \( \mathcal{K}' = f^* \mathcal{K} \) and the structure map is

\[ \varepsilon': f^* \mathcal{K} \wedge P^1_{S'} \cong f^*(\mathcal{K} \wedge P^1_S) \xrightarrow{f^*(\varepsilon)} \mathcal{K} \]

Since \( f^*: M_{cm}^S \to M_{cm}^{S'} \) is a left Quillen functor by Theorem A.17, \( f^*(BGL) \) satisfies the conditions of Definition 2.1 in \( M_{cm}(S') \) and \( H_*(S') \) provided that \( S' \) is regular. If \( S' \) is noetherian finite dimensional, then by [V, Thm. 6.9] and Remark 2.19 \( f^*(BGL) \) represents the homotopy invariant \( K \)-theory as introduced in [We].
It will be proved in Section 2.4 that in the case $S = \text{Spec}(\mathbb{Z})$ there is essentially just one $\mathbf{P}^1$-spectrum $\text{BGL}$ in $\text{SH}(S)$. In the next section, we will construct a monoidal structure on $\text{BGL}$ regarded as an object in the stable homotopy category $\text{SH}(S)$. In the case of $S = \text{Spec}(\mathbb{Z})$ such a monoidal structure is unique. Pulling it back via the structural morphism $S' \overset{f}{\to} \text{Spec}(\mathbb{Z})$ we get a monoidal structure on $f^*(\text{BGL})$ in $\text{SH}(S')$ for an arbitrary Noetherian finite-dimensional base scheme $S'$.

To complete this section we prove certain properties of $\text{BGL}$. It turns out that if $\mathcal{K}$ is fibrant in $\mathbf{M}^c_{\text{cm}}(\mathcal{S})$, then $\text{BGL}$ is stably fibrant as a $\mathbf{P}^1$-spectrum. In other words, $\text{BGL}$ is an $\Omega\mathbf{P}^1$-spectrum which represents the Thomason-Trobaugh $K$-theory on $\mathbb{S}m/S$. For $X \in \mathbb{S}m/S$ we abbreviate $\text{BGL}(X_+)$ as $\text{BGL}(X)$, which forces us to write $\text{BGL}(X, x_0)$ for a pointed $S$-scheme $(X, x_0)$.

**Lemma 2.6.** Let $X \in \mathbb{S}m/S$ and $n \geq 0$. The adjoint of the structure map $\epsilon : \mathcal{K} \wedge \mathbf{P}^1 \to \mathcal{K}$ induces an isomorphism

$$\text{Hom}_{\mathcal{M}^c_{\mathbf{K}}(S)}(S^{n,0} \wedge X_+, \mathcal{K}_i) \to \text{Hom}_{\mathcal{M}^c_{\mathbf{K}}(S)}(S^{n,0} \wedge X_+ \wedge \mathbf{P}^1, \mathcal{K}_{i+1}).$$

In particular, if $\mathcal{K}$ is fibrant in $\mathbf{M}^c_{\text{cm}}(\mathcal{S})$, then $\text{BGL}$ is stably fibrant.

**Proof.** Recall that for $Y \in \mathbb{S}m/S$ and a closed subset $Z \hookrightarrow Y$, $K_n(Y \text{ on } Z)$ denotes the $n$-th Thomason-Trobaugh $K$-group of perfect complexes on $Y$ with support on $Z$. It may be obtained as the $n$-th homotopy group of the homotopy fiber of the map $\mathbb{K}^W(Y) \to \mathbb{K}^W(Y \setminus Z)$. Abbreviate $\text{Hom}_{\mathcal{M}^c_{\mathbf{K}}(S)}(\mathcal{K}(\mathcal{K}),\mathcal{K}(\mathcal{K}),\mathcal{K}(\mathcal{K}))$ by $[-,-]$. For each smooth $X$ over $S$ the map

$$K_n(X) = [S^{n,0} \wedge X_+, \mathcal{K}_i] \to [S^{n,0} \wedge X_+ \wedge \mathbf{P}^1, \mathcal{K}_i \wedge \mathbf{P}^1]$$

$$\to [S^{n,0} \wedge X_+ \wedge \mathbf{P}^1, \mathcal{K}_{i+1}] \cong K_n(X \times \mathbf{P}^1 \text{ on } X \times \{\infty\})$$

induced by the structure map $\epsilon_i$ coincides with the multiplication by the class $[O(-1)] - [0]$ in $K_0(\mathbf{P}^1 \text{ on } \{\infty\})$. This multiplication is known to be an isomorphism for the Thomason-Trobaugh $K$-groups, by the projective bundle theorem [TT, Thm. 4.1] for $X \times \mathbf{P}^1$. Whence the Lemma.

In the following statement, the notation $\Sigma_{\mathbf{P}^1_{\mathbf{K}}} \mathcal{A}(-i)$ will be used for the $\mathbf{P}^1$-spectrum $\text{Fr}_i \mathcal{A} = (\ast, \ldots, \ast, A, A \wedge \mathbf{P}^1, \ldots)$ associated to a pointed motivic space $\mathcal{A}$ in Example A.26. Note that $\Sigma_{\mathbf{P}^1_{\mathbf{K}}} \mathcal{A}(-i) \cong \Sigma_{\mathbf{P}^1_{\mathbf{K}}} \mathcal{A} \wedge S^{2i,-i}$ in $\text{SH}(S)$, as mentioned in Notation A.40.

**Corollary 2.7.** For each pointed motivic space $\mathcal{A}$ over $S$ the adjunction map

$$\text{Hom}_{\mathcal{M}^c_{\mathbf{K}}(S)}(\mathcal{A}, \mathcal{K}_0) \to \text{Hom}_{\text{SH}(S)}(\Sigma_{\mathbf{P}^1_{\mathbf{K}}} \mathcal{A}, \text{BGL})$$

is an isomorphism. In particular, for every smooth scheme $X$ over $S$ and each closed subscheme $Z$ in $X$ one has $K_p(X \text{ on } Z) = \text{BGL}^{−p,0}(X/(X \setminus Z))$. The family of these isomorphisms form an isomorphism $\text{Ad} : K_* \to \text{BGL}^{−*,0}$ of cohomology theories on the category $\mathbb{S}m\text{Op}/S$ in the sense of [PS]. Moreover the adjunction map $\text{Fr}_i \mathcal{A}(\mathcal{A})(-i) \to [\Sigma_{\mathbf{P}^1_{\mathbf{K}}} \mathcal{A}(-i), \text{BGL}]$ is an isomorphism. In particular, for every smooth scheme $X$ over $S$ and each closed subscheme $Z$ in $X$ one has $K_p(X \text{ on } Z) = \text{BGL}^{2i−p,1}(X/(X \setminus Z))$. 

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The family of pairings $\mathcal{K}_i \land \mathcal{K}_j \xrightarrow{\mu_{ij}} \mathcal{K}_{i+j}$ in $H_*(S)$ with $\mu_{ij} = \bar{\mu}$ from Remark 2.3 defines a family of pairings

$$\cup: \text{BGL}^{p,i}(A) \otimes \text{BGL}^{q,j}(B) \to \text{BGL}^{p+i,q+j}(A \land B)$$

(7)

for pointed motivic spaces $A$ and $B$. We will refer to the latter as the *naive product structure* on the functor $\text{BGL}^{*,*}$ on the category $\mathcal{M}_*(S)$. It has the following property.

**Corollary 2.8.** The isomorphism $\text{Ad}: K_* \to \text{BGL}^{-*,0}$ of cohomology theories on $\text{SmOp}/S$ is an isomorphism of ring cohomology theories in the sense of [PS].

### 2.3 The periodicity element

The aim of this Section is to construct an element $\beta \in \text{BGL}^{2,1}(S)$, to show that it is invertible and to check that for any pointed motivic space $A$ one has

$$\text{BGL}^{*,0}(A)[\beta, \beta^{-1}] = \text{BGL}^{*,*}(A)$$

(the Laurent polynomials over $\text{BGL}^{*,0}(A)$). We will use the naive product structure on $\text{BGL}$ described just above Corollary 2.8.

**Definition 2.9.** Set $\beta = [S^0 \overset{\bar{e}}{\to} \mathcal{K} = \mathcal{K}_1] \in \text{BGL}^{2,1}(S)$, where $\bar{e}$ is the unit of the monoid $\mathcal{K}$ (see Remark 2.3).

**Lemma 2.10.** Let $b: \mathbb{P}^1 \hookrightarrow \mathcal{K}$ be the map described in Remark 2.3. It represents the element $[\mathcal{O}(-1)] - [\mathcal{O}]$ in $\text{BGL}^{0,0}(\mathbb{P}^1, \infty) = \text{Ker}(\infty^*: K_0(\mathbb{P}^1) \to K_0(S))$. There is a relation

$$\beta \cup ([\mathcal{O}(-1)] - [\mathcal{O}]) = \Sigma_{\mathbb{P}^1}(1) \in \text{BGL}^{*}(\mathbb{P}^1, \infty),$$

(8)

where $\Sigma_{\mathbb{P}^1}$ is the suspension isomorphism and $1 \in \text{BGL}^{0,0}(S)$ is the unit. There is another relation

$$\beta \cup ([\mathcal{O}(1)] - [\mathcal{O}]) = -\Sigma_{\mathbb{P}^1}(1) \in \text{BGL}^{2,1}(\mathbb{P}^1, \infty).$$

(9)

**Proof.** The element $\Sigma_{\mathbb{P}^1}(1)$ is represented by the morphism

$$S^0 \land \mathbb{P}^1 \xrightarrow{\bar{e} \text{id}} \mathcal{K}_0 \land \mathbb{P}^1 \xrightarrow{\text{id} \land b} \mathcal{K}_0 \land \mathcal{K}_1 \xrightarrow{\mu_{01}} \mathcal{K}_1,$$

where $\mu_{ij}$ is defined just above (7), and $\bar{e}$ is the unit of the monoid $\mathcal{K} = \mathcal{K}_0$. The element $\beta \cup ([\mathcal{O}(-1)] - [\mathcal{O}])$ is represented by the morphism

$$S^0 \land \mathbb{P}^1 \xrightarrow{\bar{e} \land b} \mathcal{K}_1 \land \mathcal{K}_0 \xrightarrow{\mu_{10}} \mathcal{K}_1.$$

Since $\mathcal{K}_0 = \mathcal{K} = \mathcal{K}_1$ one has $(\text{id} \land b) \circ (\bar{e} \land \text{id}) = \bar{e} \land b$. This implies the relation (8) since $\mu_{10} = \mu_{01}$. Relation (9) follows from the first one since $[\mathcal{O}(1)] - [\mathcal{O}] = -[\mathcal{O}(-1)] + [\mathcal{O}]$ in $K^0(\mathbb{P}^1)$. 

\[\square\]
Lemma 2.11. Let \( u \in \text{BGL}^{-2,-1}(S) \) be the unique element such that \( \Sigma_{P^1}(u) = [\mathcal{O}(-1)] - [\emptyset] \) in \( \text{BGL}^{0,0}(\mathbf{P}^1, \infty) \). Then \( \beta \cup u = 1 \).

Proof. Consider the commutative diagram

\[
\begin{array}{ccc}
\text{BGL}^{2,1}(S) \otimes \text{BGL}^{0,0}(\mathbf{P}^1, \infty) & \xrightarrow{\cup} & \text{BGL}^{2,1}(\mathbf{P}^1, \infty) \\
\text{id} \otimes \Sigma_{P^1} & & \Sigma_{P^1} \\
\text{BGL}^{2,1}(S) \otimes \text{BGL}^{-2,-1}(S) & \xrightarrow{\cup} & \text{BGL}^{0,0}(S).
\end{array}
\]

Now the Lemma follows from the relation (8).

Definition 2.12. For \( P^1 \)-spectra \( E \) and \( F \) set \( E^{\text{alg}}(F) = \bigoplus_{-\infty}^{+\infty} E^{2i,j}(F) \).

Proposition 2.13. For every pointed motivic space \( A \) the map

\[
\text{BGL}^{*,0}(A) \otimes K_0(S) \text{BGL}^{\text{alg}}(S) \to \text{BGL}^{*,*}(A)
\]

(10)

given by \( a \otimes b \mapsto a \cup b \) is a ring isomorphism and \( \text{BGL}^{\text{alg}}(S) = K_0(S)[\beta, \beta^{-1}] \) is the Laurent polynomial ring. One can rewrite this ring isomorphism as

\[
\text{BGL}^{*,0}(A)[\beta, \beta^{-1}] \cong \text{BGL}^{*,*}(A)
\]

(11)

Proof. In fact, \( \text{BGL}^{*,0}(A) \xrightarrow{\cup \beta} \text{BGL}^{*,2,1}(A) \) is an isomorphism since \( \beta \) is invertible. Since \( \text{BGL}^{0,0}(S) = K^0(S) \) the map (10) is a ring isomorphism.

Using the isomorphism \( Ad: K_* \to \text{BGL}^{-*,0} \) of ring cohomology theories from Corollary 2.8 we get the following statement.

Corollary 2.14. For every \( X \in \text{Sm}/S \) and every closed subset \( Z \hookrightarrow X \) one has

\[
K_{-*}(X on Z)[\beta, \beta^{-1}] \cong \text{BGL}_{Z}^{*,*}(X).
\]

(12)

The family of these isomorphisms form an isomorphism of ring cohomology theories on \( \text{SmOp}/S \) in the sense of [PS]. As well, there is an isomorphism

\[
K_{-*}(X on Z) = \text{BGL}_{Z}^{*,*}(X)/(\beta + 1)\text{BGL}_{Z}^{*,*}(X).
\]

(13)

The family of these isomorphisms form an isomorphism of ring cohomology theories on \( \text{SmOp}/S \) in the same sense.

2.4 Uniqueness of BGL

We prove in this Section that, at least over \( S = \text{Spec}(\mathbb{Z}) \), a \( P^1 \)-spectrum \( \text{BGL} \) as described in Definition 2.1 is essentially unique regarded as an object in the stable homotopy category \( \text{SH}(S) \). This has also been obtained in [R].
Let $BGL$ be a $\mathbf{P}^1$-spectrum as described in Definition 2.1. Recall that this involves the choice of a weak equivalence $i: \mathbb{Z} \times \text{Gr} \to \mathcal{K}$ and a structure map $\varepsilon: \mathcal{K} \wedge \mathbf{P}^1 \to \mathcal{K}$. Let $BGL'$ be a possibly different $\mathbf{P}^1$-spectrum. More precisely, take a pointed motivic space $\mathcal{K}'$ together with a weak equivalence $i': \mathbb{Z} \times \text{Gr} \to \mathcal{K}'$ and with a morphism $\varepsilon': \mathcal{K}' \wedge \mathbf{P}^1 \to \mathcal{K}'$ which descends to 

$$\mu^W \circ (\text{id} \wedge b^W): \mathbb{K}^W \wedge \mathbf{P}^1 \to \mathbb{K}^W$$

under the identification of $\mathcal{K}'$ with $\mathbb{K}^W$ in $H_\bullet(S)$ via the isomorphism $\psi \circ [i']^{-1}$. Let $BGL'$ be the $\mathbf{P}^1$-spectrum of the form $(\mathcal{K}'_0, \mathcal{K}'_1, \mathcal{K}'_2, \ldots)$ with $\mathcal{K}'_i = \mathcal{K}'$ for all $i$, and with the structure maps $\varepsilon'_i: \mathcal{K}'_i \wedge \mathbf{P}^1 \to \mathcal{K}'_{i+1}$ equal to the map $\varepsilon': \mathcal{K}' \wedge \mathbf{P}^1 \to \mathcal{K}'$.

**Proposition 2.15.** Let $S = \text{Spec} \mathbb{Z}$. There exists a unique morphism $\theta: BGL \to BGL'$ in $\text{SH}(S)$ such that for every integer $i \geq 0$ the diagram

$$
\begin{array}{ccc}
\Sigma^\infty_{\mathbf{P}^1} \mathcal{K}_i(-i) & \xrightarrow{u_i} & BGL \\
\Sigma^\infty_{\mathbf{P}^1} \phi_i(-i) \downarrow & & \downarrow \theta \\
\Sigma^\infty_{\mathbf{P}^1} \mathcal{K}'_i(-i) & \xrightarrow{u'_i} & BGL'
\end{array}
$$

commutes in $\text{SH}(S)$, where $\phi_i = i' \circ i^{-1} \in [\mathcal{K}_i, \mathcal{K}'_i]_{H_\bullet(S)}$ and $u_i, u'_i$ are the canonical morphisms. Similarly, there exists a unique morphism $\theta': BGL' \to BGL$ in $\text{SH}(S)$ such that for every integer $i \geq 0$ the diagram

$$
\begin{array}{ccc}
\Sigma^\infty_{\mathbf{P}^1} \mathcal{K}'_i(-i) & \xrightarrow{u'_i} & BGL' \\
\Sigma^\infty_{\mathbf{P}^1} \phi'_i(-i) \downarrow & & \downarrow \theta' \\
\Sigma^\infty_{\mathbf{P}^1} \mathcal{K}_i(-i) & \xrightarrow{u_i} & BGL
\end{array}
$$

commutes in $\text{SH}(S)$, where $\theta_i = i \circ (i')^{-1} \in [\mathcal{K}_i, \mathcal{K}'_i]_{H_\bullet(S)}$.

**Proof.** Consider the exact sequence

$$0 \to \lim_{\leftarrow} \text{BGL}^{2i-1,i}(\mathcal{K}'_i) \to \text{BGL}^{0,0}(BGL') \to \lim_{\rightarrow} \text{BGL}^{2i,i}(\mathcal{K}'_i) \to 0$$

from Lemma A.34. The family of elements $(u_i \circ \Sigma^\infty_{\mathbf{P}^1} \theta'_i(-i))$ is an element of the group $\lim_{\rightarrow} \text{BGL}^{2i,i}(\mathcal{K}'_i)$. Thus there exists the required morphism $\theta'$. To prove its uniqueness, observe that the $\lim_{\rightarrow}$-group vanishes by Proposition 2.27. Whence $\text{BGL}^{0,0}(BGL') = \lim_{\rightarrow} \text{BGL}^{2i,i}(\mathcal{K}'_i)$ and $\theta'$ is indeed unique. By symmetry there also exists a unique morphism $\overline{\theta}$ with the required property. \qed

**Proposition 2.16.** Let $S = \text{Spec} \mathbb{Z}$. The morphism $\theta: BGL \to BGL'$ is the inverse of $\theta': BGL' \to BGL$ in $\text{SH}(S)$, and in particular an isomorphism.
Proof. The composite morphism $\theta' \circ \theta : BGL \to BGL$ has the property that for every integer $i \geq 0$ the diagram
\[
\begin{array}{c}
\Sigma_p K_i(-i) \xrightarrow{u_i} BGL \\
\downarrow \ id \\
\Sigma_p K_i(-i) \xrightarrow{u_i} BGL
\end{array}
\]
commutes. However, the identity morphism id: $BGL \to BGL$ has the same property. Thus $\theta' \circ \theta = id$ by the uniqueness in Proposition 2.15, and similarly $\theta \circ \theta' = id$.

Remark 2.17. The isomorphisms $\theta$ and $\theta'$ are monoid isomorphisms provided that $BGL$ and $BGL'$ are equipped with the monoidal structures given by Theorem 3.6. This follows from the fact that both $i$ and $i'$ are isomorphisms of monoids in $H_\bullet(S)$.

Remark 2.18. There exists a unique morphism $e: (\mathbb{Z} \times \text{Gr}) \land P^1 \to \mathbb{Z} \times \text{Gr}$ in $H_\bullet(S)$ such that the diagram
\[
\begin{array}{c}
(\mathbb{Z} \times \text{Gr}) \land P^1 \xrightarrow{e} \mathbb{Z} \times \text{Gr} \\
\psi \land \text{id} \downarrow \\
\mathbb{K}^W \land P^1 \xrightarrow{e^W} \mathbb{K}^W
\end{array}
\]
commutes in $H_\bullet(S)$, where $e^W = \mu^W \circ (\text{id} \land b^W)$ and $\psi$ is described right above Definition 2.1. The diagram
\[
\begin{array}{c}
(\mathbb{Z} \times \text{Gr}) \land P^1 \xrightarrow{e} \mathbb{Z} \times \text{Gr} \\
i \land \text{id} \downarrow \\
\mathbb{K} \land P^1 \xrightarrow{e} \mathbb{K}
\end{array}
\]
then commutes as well. That is, $e$ descends to $e$ in $H_\bullet(S)$.

We will need an observation concerning the morphism $e$. Let $\tau_n$ be the tautological bundle over the Grassmann variety $\text{Gr}(n,2n)$. By Lemma B.8 there exists a unique element $\xi_\infty \in K_0(\mathbb{Z} \times \text{Gr})$ such that for any positive integer $n$ and any $0 \leq m \leq n$ one has $\xi|_{(m) \times \text{Gr}(n,2n)} = [\tau_n] - n + m$. By Lemmas B.9 and B.2 the element $\xi_\infty \otimes ([\mathcal{O}(-1)] - [\mathcal{O}])$ belongs to the subgroup $K_0((\mathbb{Z} \times \text{Gr}) \land P^1)$ of the group $K_0(\mathbb{Z} \times \text{Gr} \times P^1)$. The isomorphism $\psi: \mathbb{Z} \times \text{Gr} \to \mathbb{K}^W$ in $H_\bullet(S)$ represents the element $\xi_\infty$ in $K_0((\mathbb{Z} \times \text{Gr})$. Now the definitions of $e^W$ and $e$ show that
\[
e^*(\xi_\infty) = \xi_\infty \otimes ([\mathcal{O}(-1)] - [\mathcal{O}])
\]
in $K_0((\mathbb{Z} \times \text{Gr}) \land P^1)$.

Remark 2.19. Let $S$ be a regular scheme. Given two triples $(\mathcal{K}_1, i_1, \epsilon_1)$ and $(\mathcal{K}_2, i_2, \epsilon_2)$ fulfilling the conditions of Definition 2.1, there exists a zig-zag of weak equivalences of triples connecting these two. In particular, there exists a zig-zag of levelwise weak equivalences of $P^1$-spectra over $S$ connecting $BGL_1$ and $BGL_2$. It follows that the $P^1$-spectra $BGL_1$ and $BGL_2$ associated to these triples are "naturally" isomorphic in $SH(S)$. This shows that the strength of Proposition 2.15 is in its uniqueness assertion.
2.5 Preliminary computations I

In this section we prepare for the next section, in which we show that certain $\lim^1$-groups vanish. Let $\text{BGL}$ be the $\mathbb{P}^1$-spectrum defined in 2.1. We will identify in this section the functors $\text{BGL}^{0,0}$ and $\text{BGL}^{2i,i}$ on the category $\text{H}_*(S)$ via the iterated $(2, 1)$-periodicity isomorphism as follows:

\[ \text{BGL}^{0,0}(A) \cong \text{Hom}_{\text{H}_*}(A, K_0) = \text{Hom}_{\text{H}_*}(A, K_i) \cong \text{BGL}^{2i,i}(A). \]  

(14)

Similarly,

\[ \text{BGL}^{-1,0}(A) \cong \text{Hom}_{\text{H}_*}(S^{1,0} \wedge A, K_0) = \text{Hom}_{\text{H}_*}(S^{1,0} \wedge A, K_i) \cong \text{BGL}^{2i-1,i}(A). \]  

(15)

These identifications respect the naive product structure (7) on the functor $\text{BGL}^*,*$. In particular, the following diagram commutes for every pointed motivic space $A$ over $S$.

\[ \begin{array}{ccc} \text{BGL}^{-1,0}(S) \otimes \text{BGL}^{0,0}(A) & \longrightarrow & \text{BGL}^{-1,0}(A) \\ \cong \downarrow & & \cong \downarrow \\ \text{BGL}^{-1,0}(S) \otimes \text{BGL}^{2i,i}(A) & \longrightarrow & \text{BGL}^{2i-1,i}(A) \end{array} \]  

(16)

**Remark 2.20.** The identification (15) of $\text{BGL}^{-1,0}(X)$ with $\text{BGL}^{2i-1,i}(X)$ coincides with the periodicity isomorphism $\text{BGL}^{-1,0}(X) \xrightarrow{\cup i^*} \text{BGL}^{2i-1,i}(X)$.

**Lemma 2.21.** Let $S = \text{Spec}(\mathbb{Z})$. For every integer $i$ the map

\[ \text{BGL}^{-1,0}(S) \otimes \text{BGL}^{2i,i}(\mathcal{K}) \rightarrow \text{BGL}^{2i-1,i}(\mathcal{K}) \]

induced by the naive product structure is an isomorphism. The same holds if $\mathcal{K} \wedge \mathbb{P}^1$ replaces $\mathcal{K}$.

**Proof.** The commutativity of the diagram (16) shows that it suffices to consider the case $i = 0$. Furthermore we may replace the pointed motivic space $\mathcal{K}$ with $\mathbb{Z} \times \text{Gr}$ since the map $i: \mathbb{Z} \times \text{Gr} \rightarrow \mathcal{K} = \mathcal{K}$ is a weak equivalence. The functor isomorphism $K_* \rightarrow \text{BGL}^{-*,0}$ is a ring cohomology isomorphism by Corollary 2.8. Thus it remains to check that the map

\[ K_1(S) \otimes K_0(\mathbb{Z} \times \text{Gr}) \rightarrow K_1(\mathbb{Z} \times \text{Gr}) \]

is an isomorphism. For a set $M$ and a smooth $S$-scheme $X$ we will write $M \times X$ for the disjoint union $\bigsqcup_M X$ of $M$ copies of $X$ in the category of motivic spaces over $S$. Let $[-n, n]$ be the set of integers with absolute value $\leq n$. By Lemma B.8 and Lemma B.3 it suffices to check that the natural map

\[ A \otimes \lim_{\longrightarrow} K_0([-n, n] \times \text{Gr}(n, 2n)) \rightarrow \lim_{\longrightarrow} A \otimes K_0([-n, n] \times \text{Gr}(n, 2n)) \]

is an isomorphism, where $A = K_1(S)$. This is the case since $K_1(S)$ is a finitely generated abelian group (it is just $\mathbb{Z}/2\mathbb{Z}$). The assertion concerning $\mathcal{K} \wedge \mathbb{P}^1$ is proved similarly using Lemmas B.9 and B.6 instead. \[ \square \]
To state the next lemma, consider the scheme morphism $f: \text{Spec}(\mathbb{C}) \to S = \text{Spec}(\mathbb{Z})$, the pull-back functor $f^*: \text{SH}(\mathbb{Z}) \to \text{SH}(\mathbb{C})$ described in Proposition A.47, and the topological realization functor $R_{\mathbb{C}*}: \text{SH}(\mathbb{C}) \to \text{SH}_{\mathbb{CP}^1}$ described in Section A.7. Set $r = R_{\mathbb{C}*} \circ f^*: \text{SH}(S) \to \text{SH}_{\mathbb{CP}^1}$. The functor $r$ will be called for short the realization functor below in this Section.

**Lemma 2.22.** Let $\mathbb{B}U$ be the periodic complex $K$-theory $\mathbb{CP}^1$-spectrum with terms $\mathbb{Z} \times \mathbb{B}U$. There is a zigzag $\mathbb{B}U \leftarrow E \sim r \mathbb{B}G$ of levelwise weak equivalences of $\mathbb{CP}^1$-spectra.

**Proof.** This follows from Remark 2.18, A.46 and the fact that Grassmann varieties pull back. \hfill $\square$

**Lemma 2.23.** Let $X \in S_{m}/S$, where $S = \text{Spec}(\mathbb{Z})$, and let $X_0 \subset X_1 \subset \cdots \subset X_n = X$ be a filtration by closed subsets such that for every integer $i \geq 0$ the $S$-scheme $X_i - X_{i-1}$ is isomorphic to a disjoint union of several copies of the affine space $\mathbb{A}^i_S$. The map $BGL^{0,0}(X) \to (rBGL)^0(rX)$ is an isomorphism.

**Proof.** Consider the class $\mathcal{R}$ of $\mathbb{P}^1$-spectra $E$ such that the homomorphism $BGL^{0,0}(E) \to rBGL^0(rE)$ is an isomorphism. It contains $S^{0,0}_0$ because in this case we obtain the isomorphism $BGL^{0,0}(S^{0,0}_0) \cong K^{0,0}(\mathbb{Z}) \cong \mathbb{Z} \cong K^{0,0}_0(S^{0})$ which identifies the class of an algebraic resp. complex topological vector bundle over $\text{Spec}(\mathbb{Z})$ resp. $\bullet$ with its rank. The $(2,1)$-periodicity isomorphism for $BGL$ described in Remark 2.20 and the Bott periodicity isomorphism for $rBGL$ are compatible by A.46. This implies that $S^{2m,m}_0 \in \mathcal{R}$ for all $m \in \mathbb{Z}$. Finally, if $E \to F \to G \to S^{1,0} \wedge E$ is a distinguished triangle in $\text{SH}(S)$ such that $E$ and $G$ are in $\mathcal{R}$, then so is $F$.

For $i \geq 0$ write $U^i := X \setminus X_i$, so that $U^i$ is an open subset of $U^{i-1}$. In particular we have $U^n = \emptyset$ and $U^{-1} = X$. The closed subscheme $X_i \setminus X_{i-1} = X_i \cap U^{i-1} \hookrightarrow U^{i-1}$ is isomorphic to a disjoint union of $m_i$ copies of affine spaces $\mathbb{A}^i$, and is in particular smooth over $S$. Furthermore the normal bundle is trivial. The homotopy purity theorem [MV, Thm. 3.2.29] supplies a distinguished triangle

$$\Sigma_{\mathbb{P}^1}^{\infty}U^{i}_+ \to \Sigma_{\mathbb{P}^1}^{\infty}U^{i-1}_+ \to \Sigma_{\mathbb{P}^1}^{\infty}U^{i-1}/U^i \cong \bigvee_{j=1}^{m_i}S^{2(n-i),(n-i)}$$

of $\mathbb{P}^1$-spectra. Since $\mathcal{R}$ contains $\Sigma_{\mathbb{P}^1}^{\infty}U^n = \bullet$ we obtain inductively that $\mathcal{R}$ contains $\Sigma_{\mathbb{P}^1}^{\infty}U^{-1} = \Sigma_{\mathbb{P}^1}^{\infty}X_1$. \hfill $\square$

**Lemma 2.24.** Let $S = \text{Spec}(\mathbb{Z})$ and let $r: \text{SH}(\mathbb{Z}) \to \text{SH}_{\mathbb{CP}^1}$ be the topological realization functor. Then for every integer $i$ the realization homomorphism $BGL^{2i,i}(\mathcal{X}) \to (rBGL)^2(r\mathcal{X})$ is an isomorphism.

**Proof.** Clearly it suffices to prove the case $i = 0$. We may replace the pointed motivic space $\mathcal{X}$, with $\mathbb{Z} \times \text{Gr}$ as in the proof of Lemma 2.21. It remains to check that the topological realization homomorphism $BGL^{0,0}(\text{Gr}) \to (rBGL)^0(r\text{Gr})$ is an isomorphism.

Since $\text{Gr}(n, 2n)$ has a filtration satisfying the condition of Lemma 2.23, we see that the map $BGL^{0,0}(\text{Gr}(n, 2n)) \to (rBGL)^0(r\text{Gr}(n, 2n))$ is an isomorphism for every $n$. To conclude the statement for $\text{Gr} = \cup \text{Gr}(n, 2n)$, use the short exact sequence from Lemma A.34.
In the resulting diagram

$$\begin{align*}
\lim_1 \text{BGL}^{-1,0}(\text{Gr}(n, 2n)) & \longrightarrow \text{BGL}^{0,0}(\text{Gr}) \longrightarrow \lim \text{BGL}^{0,0}(\text{Gr}(n, 2n)) \\
\lim_1 r\text{BGL}^{-1,0}(r\text{Gr}(n, 2n)) & \longrightarrow r\text{BGL}^{0,0}(r\text{Gr}) \longrightarrow \lim r\text{BGL}^{0,0}(r\text{Gr}(n, 2n))
\end{align*}$$

the map on the right hand side is then an isomorphism. Furthermore one concludes from [Sw, Thm. 16.32] that

$$\lim_1 r\text{BGL}^{-1,0}(r\text{Gr}(n, 2n)) = \lim_1 K_{\text{top}}^1(r\text{Gr}(n, 2n)) = 0.$$  

On the other hand $\lim_1 \text{BGL}^{-1,0}(\text{Gr}(n, 2n)) = \lim_1 K_1(\text{Gr}(n, 2n)) = 0$ by Lemma B.7. The result follows. \hfill \Box

**Lemma 2.25.** Let $\mathbb{B}^0U$ be the sub-spectrum of $\mathbb{B}U$ with the $n$-th term equal to the connected component $\mathbb{B}U$ of the topological space $\mathbb{Z} \times \mathbb{B}U$ containing the basepoint $\bullet$. The inclusion $\mathbb{B}^0U \to \mathbb{B}U$ is a weak equivalence of $\mathbb{C}\mathbb{P}^1$-spectra.

**Proof.** One has to check that the inclusion induces an isomorphism on stable homotopy groups. This follows because the structure map $(\mathbb{Z} \times \mathbb{B}U) \wedge \mathbb{C}\mathbb{P}^1 \to \mathbb{Z} \times \mathbb{B}U$ factors over $\{0\} \times \mathbb{B}U$. \hfill \Box

**Lemma 2.26.** There exists a sub-spectrum $\mathbb{B}^jU$ of the $\mathbb{C}\mathbb{P}^1$-spectrum $\mathbb{B}U$ with the $n$-th term $\text{Gr}(b(n), 2b(n))$ such that the inclusion $\mathbb{B}^jU \to \mathbb{B}U$ is a stable equivalence.

**Proof.** The sequence $b(n)$ will be constructed such that $b(n) \geq 2n + 1$. Set $b(0) = 1$. We may assume that the structure map $e_0: \mathbb{B}U \wedge \mathbb{C}\mathbb{P}^1 \to \mathbb{B}U$ is cellular. Since $r\text{Gr}(b(0), 2b(0)) \wedge \mathbb{C}\mathbb{P}^1$ is a finite cell complex, it lands in a Grassmannian $r\text{Gr}(b(1), 2b(1))$ for some integer $b(1) \geq 2 \cdot 1 + 1$. Continuing this process produces the required sequence of $b(n)$’s. The inclusions induce an isomorphism $\text{colim}_{n \geq 0} \text{Gr}(b(n), 2b(n)) \cong \text{Gr}$.

To observe that the inclusion $j: \mathbb{B}^jU \to \mathbb{B}U$ is then a stable equivalence, recall that the number of $2k$-cells in $\text{Gr}(n, m)$ is given by the number of partitions of $k$ into at most $n$ subsets each of which has cardinality $\geq m - n$ [MS]. In particular, the $2k$-skeleton of $\mathbb{B}U$ coincides with the $2k$-skeleton of $r\text{Gr}(k, 2k)$. To prove the surjectivity of $\pi_i(j)$ choose an element $\alpha \in \pi_i\mathbb{B}U$. It is represented by a cellular map $a: S^{i+2m} \to \mathbb{B}U$ for some $m$ with $i + 2m \geq 0$. We may choose $m$ such that $m \geq i$. Thus $a$ lands in $r\text{Gr}(b(m), 2b(m))$ and gives rise to an element in $\pi_i\mathbb{B}U$ mapping to $\alpha$. To prove the injectivity of $\pi_i(j)$, choose an element $\alpha \in \pi_i\mathbb{B}U$ such that $\pi_i(j)(\alpha) = 0$. We may represent $\alpha$ by some map $a: \pi_{i+2m}(j)\text{Gr}(b(m), 2b(m))$ for some $m$ with $i + 2m > 0$ and $m \geq i$. The composition $S^{i+2m} \to a\text{Gr}(b(m), 2b(m)) \hookrightarrow \mathbb{B}U$ is nullhomotopic since $\pi_{i+2m}\mathbb{B}U \cong \pi_i\mathbb{B}U$ via the homomorphism induced by the structure map. The nullhomotopy may be chosen to be cellular and thus lands in $\text{Gr}(b(m), 2b(m))$. This completes the proof. \hfill \Box
2.6 Vanishing of certain groups I

Consider the stable equivalence $\text{hocolim}_{i \geq 0} \Sigma_{\mathbf{P}^i}^{\infty} \mathcal{K}_i(-i) \cong \text{BGL}$ (see (18)) and the respecting short exact sequence

$$0 \to \lim^1 \text{BGL}^{2i-1,i} (\mathcal{K}_i) \to \text{BGL}^{0,0} (\text{BGL}) \to \lim \text{BGL}^{2i,i} (\mathcal{K}_i) \to 0$$

We prove in this section the following result

**Proposition 2.27.** Let $S = \text{Spec}(\mathbb{Z})$, then $\lim^1 \text{BGL}^{2i-1,i} (\mathcal{K}_i) = 0$.

**Proof.** The connecting homomorphism in the tower of groups for the $\lim^1$-term is the composite map

$$\text{BGL}^{2i-1,i} (\mathcal{K}_i) \xleftarrow{\Sigma_{\mathbf{P}^1}^{-1}} \text{BGL}^{2i+1,i+1} (\mathcal{K}_i \wedge \mathbf{P}^1) \xleftarrow{e_i^*} \text{BGL}^{2i+1,i+1} (\mathcal{K}_{i+1})$$

where $\Sigma_{\mathbf{P}^1}^{-1}$ is the inverse to the $\mathbf{P}^1$-suspension isomorphism and $e_i^*$ is the pull-back induced by the structure map $e_i$. Set $A = \text{BGL}^{-1,0} (S)$ and consider the diagram

$$
\begin{array}{ccc}
\text{BGL}^{2i-1,i} (\mathcal{K}) & \xleftarrow{\Sigma_{\mathbf{P}^1}^{-1}} & \text{BGL}^{2i+1,i+1} (\mathcal{K} \wedge \mathbf{P}^1) \\
\uparrow & & \uparrow \\
A \otimes \text{BGL}^{2i,i} (\mathcal{K}) & \xleftarrow{id \otimes \Sigma_{\mathbf{P}^1}^{-1}} & A \otimes \text{BGL}^{2(i+1),i+1} (\mathcal{K} \wedge \mathbf{P}^1) \\
\end{array}
$$

where the vertical arrows are induced by the naive product structure on the functor $\text{BGL}^{*,*}$. Clearly it commutes. Since $S$ is regular, the vertical arrows are isomorphisms by Lemma 2.21. It follows that $\lim^1 \text{BGL}^{2i-1,i} (\mathcal{K}_i) = \lim^1 (A \otimes \text{BGL}^{2i,i} (\mathcal{K}_i))$ where in the last tower of groups the connecting maps are $id \otimes (\Sigma_{\mathbf{P}^1}^{-1} \circ e_i^*)$. It remains to prove the following assertion.

**Claim 2.28.** The equality $\lim^1 (A \otimes \text{BGL}^{2i,i} (\mathcal{K}_i)) = 0$ holds.

Since $S = \text{Spec}(\mathbb{Z})$ one obtains $A = \text{BGL}^{-1,0} (S) = K_1 (\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$. It follows that $A \otimes \text{BGL}^{2i,i} (\mathcal{K}_i) = \text{BGL}^{2i,i} (\mathcal{K}_i) / m \text{BGL}^{2i,i} (\mathcal{K}_i)$ with $m = 2$ and the connecting maps in the tower are just the mod-$m$ reduction of the maps $\Sigma_{\mathbf{P}^1}^{-1} \circ e_i^*$. Now a chain of isomorphisms completes the proof of the Claim.

\begin{align*}
\lim^1 \text{BGL}^{2i,i} (\mathcal{K}_i) / m & \cong \lim^1 (r \text{BGL})^{2i}(r \mathcal{K}_i) / m \cong \lim^1 \mathbb{B}U^{2i} (r \mathcal{K}_i) / m \\
& \cong \lim^1 \mathbb{B}U^{2i} (\mathbb{Z} \times BU) / m \cong \lim^1 \mathbb{B}U^{2i} (\mathbb{Z} \times BU ; \mathbb{Z}/m) \\
& \cong K_{\text{top}}^1 (\mathbb{B}U ; \mathbb{Z}/m) \cong K_{\text{top}}^1 (\mathbb{B}^0 U ; \mathbb{Z}/m) \\
& \cong \lim^1 K_{\text{top}}^{2i} (\text{Gr}(b(i), 2b(i)) ; \mathbb{Z}/m) = 0
\end{align*}

The first isomorphism follows from Lemma 2.24. The second isomorphism is induced by the levelwise weak equivalence $\mathbb{B}U \simeq r \text{BGL}$ mentioned in Lemma 2.22. The third isomorphism is induced by the image of the weak equivalence $\mathcal{K}_i \simeq \mathbb{Z} \times \text{Gr}$ under topological
realization. The forth and fifth isomorphism hold since $\mathbb{B}U^{2i+1}(\mathbb{Z} \times \mathbb{B}U) = 0$. The sixth isomorphism is induced by the stable equivalence $\mathbb{B}^0U \simeq \mathbb{B}U$ from Lemma 2.25, the seventh one is induced by the stable equivalence $\mathbb{B}/U \simeq \mathbb{B}^0U$ from Lemma 2.26. The last one holds since all groups in the tower are finite. 

\section{Smash-product, pull-backs, topological realization}

In this section we construct a smash-product $\wedge$ of $\mathbb{P}^1$-spectra, check its basic properties, consider its behavior with respect to pull-back and realization functors. We follow here an idea of Voevodsky [V, Comments to Thm. 5.6] and use results of Jardine [J]. In several cases we will not distinguish notationally between a Quillen functor and its total derived functor, $\Sigma_\infty^\mathbb{P}^1$ being the most prominent example.

\subsection{The smash product}

\textbf{Definition 3.1.} Let $V := Lv : \text{SH}(S) \to \text{SH}(\Sigma S)$ and $U := Ru : \text{SH}(S) \to \text{SH}(S)$ be the equivalence described in Remark A.39. For a pair of $\mathbb{P}^1$-spectra $E$ and $F$ set

$$E \wedge F := U(VE \wedge VF)$$

as in Remark A.39.

\textbf{Proposition 3.2.} Let $S$ be a Noetherian finite-dimensional base scheme. The smash-product of $\mathbb{P}^1$-spectra over $S$ induces a closed symmetric monoidal structure $(\wedge, I)$ on the motivic stable homotopy category $\text{SH}(S)$ having the properties required by Theorem 5.6 of Voevodsky’s congress talk [V]:

1. There is a canonical isomorphism $E \wedge \Sigma_\infty^\mathbb{P}^1 A \cong (A \wedge E_i, \text{id} \wedge e_i)$ for every pointed motivic space $A$ and every $\mathbb{P}^1$-spectrum.

2. There is a canonical isomorphism $(\oplus E_\alpha) \wedge F \cong \oplus (E_\alpha \wedge F)$ for $\mathbb{P}^1$-spectra $E_i$, $F$.

3. Smashing with a $\mathbb{P}^1$-spectrum preserves distinguished triangles. To be more precise, if $E \xrightarrow{f} F \rightarrow \text{cone}(f) \xleftarrow{\epsilon} E[1]$ is a distinguished triangle and $G$ is a $\mathbb{P}^1$-spectrum, the sequence $E \wedge G \xrightarrow{f \wedge 1} F \wedge G \rightarrow \text{cone}(f) \wedge G \xleftarrow{\epsilon \wedge 1} E \wedge G[1]$ is a distinguished triangle, where the last morphism is the composition of $\epsilon \wedge \text{id}_G$ with the canonical isomorphism $E[1] \wedge G \rightarrow (E \wedge F)[1]$.

\textbf{Proof.} Follows from Remark A.39 and Theorem A.38. 

In the following we use that the homotopy colimit of a sequence $E = E_0 \to E_1 \to \cdots$ of morphisms in the homotopy category $\text{SH}(S)$ may be computed in three steps:

1. Lift $E$ to a sequence $E' = E'_0 \to E'_1 \to \cdots$ of cofibrations (in fact, arbitrary maps suffice) of $\mathbb{P}^1$-spectra.
2. Take the colimit $\colim_{i \geq 0} E'_i$ of $E'$ in the category of $\text{P}^1$-spectra.

3. Consider $\colim_{i \geq 0} E'_i$ as an object in $\text{SH}(S)$.

**Lemma 3.3.** Let $E = \hocolim_{i \geq 0} E_i$ be a sequential homotopy colimit of $\text{P}^1$-spectra. For every $\text{P}^1$-spectrum $F$ there is an exact sequence of abelian groups

$$0 \to \varprojlim \lim_{i}^{\mathcal{L}} F^{p,q}(E_i) \to F^{p,q}(E) \to \varprojlim \lim_{i}^{\mathcal{L}} F^{p,q}(E_i) \to 0. \quad (17)$$

**Proof.** This is Lemma A.34. \hfill \square

By Lemma A.33, any $\text{P}^1$-spectrum $E$ can be expressed as the homotopy colimit

$$\hocolim \Sigma_{\text{P}^1}^\infty E_i(-i) \cong E. \quad (18)$$

**Corollary 3.4.** For two $\text{P}^1$-spectra $E$ and $F$ there is a canonical short exact sequence

$$0 \to \varprojlim \lim_{i}^{\mathcal{L}} F^{p+2i-1,q+i}(E_i) \to F^{p,q}(E \wedge F) \to \varprojlim \lim_{i}^{\mathcal{L}} F^{p+2i,q+i}(E_i) \to 0. \quad (19)$$

**Corollary 3.5.** For a pair of spectra $E$ and $F$ and each spectrum $G$ one has a canonical exact sequence of the form

$$0 \to \varprojlim \lim_{i}^{\mathcal{L}} G^{p+4i-1,q+2i}(E_i \wedge F_i) \to G^{p,q}(E \wedge F) \to \varprojlim \lim_{i}^{\mathcal{L}} G^{p+4i,q+2i}(E_i \wedge F_i) \to 0. \quad (20)$$

**Proof.** For a pair of spectra $E$ and $F$ one has a canonical isomorphism of the form

$$\hocolim \Sigma_{\text{P}^1}^\infty (E_i \wedge F_i)(-2i) \cong E \wedge F \quad (21)$$

as deduced in Lemma A.42. The result follows from Corollary 3.3. \hfill \square

### 3.2 A monoidal structure on $\text{BGL}$

For a $\text{P}^1$-spectrum $E$ and an integer $i \geq 0$ $u_i : \Sigma_{\text{P}^1}^\infty E_i(-i) \to E$ denotes the canonical map from A.26. Let $\text{BGL}$ be the $\text{P}^1$-spectrum defined in 2.1. Recall that this involves the choice of a weak equivalence $\epsilon : \mathcal{K} \times \text{Gr} \to \mathcal{K}$ and a structure map $\epsilon : \mathcal{K} \wedge \text{P}^1 \to \mathcal{K}$. Following Lemma 2.4 we may and will assume additionally that the pointed motivic space $\mathcal{K}$ is cofibrant. The aim of this section is to prove the following statement.

**Theorem 3.6.** Assume that the pointed motivic space $\mathcal{K}$ is cofibrant. Consider the family of pairings $\mathcal{K}_i \wedge \mathcal{K}_j \xrightarrow{\mu_{ij}} \mathcal{K}_{i+j}$ in $H_*(S)$ with $\mu_{ij} = \bar{\mu}$ from Remark 2.3. For $S = \text{Spec}(\mathbb{Z})$ there is a unique morphism $\mu_{\text{BGL}} : \text{BGL} \wedge \text{BGL} \to \text{BGL}$ in the motivic stable homotopy category $\text{SH}(S)$ such that for every $i$ the diagram

$$\begin{array}{ccc}
\Sigma_{\text{P}^1}^\infty \mathcal{K}_i(-i) \wedge \Sigma_{\text{P}^1}^\infty \mathcal{K}_i(-i) & \xrightarrow{\Sigma_{\text{P}^1}^\infty (\mu_{ij})} & \Sigma_{\text{P}^1}^\infty \mathcal{K}_{2i}(-2i) \\
\downarrow u_i \wedge u_i & & \downarrow u_{2i} \\
\text{BGL} \wedge \text{BGL} & \xrightarrow{\mu_{\text{BGL}}} & \text{BGL}
\end{array}$$

commutes. Let $e_{\text{BGL}} : \mathbb{I} \to \text{BGL}$ in $\text{SH}(S)$ be adjoint to the unit $e_{\mathcal{K}} : S^{0,0} \to \mathcal{K}$. Then

$$(\text{BGL}, \mu_{\text{BGL}}, e_{\text{BGL}})$$

is a commutative monoid in $\text{SH}(S)$.  

---

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Proof. The morphism \( \mu_{\text{BGL}} \) we are looking for is an element of the group \( \text{BGL}^0,0(\text{BGL} \land \text{BGL}) \). This group fits in the exact sequence

\[
0 \to \lim^{-1}\text{BGL}^{4i-1,2i}(\mathcal{K}_i^\wedge 2) \to \text{BGL}^{0,0}(\text{BGL} \land \text{BGL}) \to \lim_0\text{BGL}^{4i,2i}(\mathcal{K}_i^\wedge 2) \to 0
\]

by Corollary 3.5. The family of elements \( \{u_{2i} \circ \Sigma^\infty(\mu_{ii})\} \) is an element of the \( \lim^{-1} \) group. The \( \lim^{-1} \) group vanishes by Proposition 3.13 below, whence there exist a unique element \( \mu_{\text{BGL}} \) whose image in the \( \lim^{-1} \) group coincides with the element \( \{u_{2i} \circ \Sigma^\infty(\mu_{ii})\} \). That morphism \( \mu_{\text{BGL}} \) is the required one.

In fact, the identities \( u_{2i} \circ \Sigma^\infty(\mu_{ii}) = \mu_{\text{BGL}} \circ (u_i \land u_i) \) hold by the very construction of \( \mu_{\text{BGL}} \). The operation \( \mu_{\text{BGL}} \) is associative because the group \( \lim^{-1}\text{BGL}^{8i-1,4i}(\mathcal{K}_i \land \mathcal{K}_i \land \mathcal{K}_i) \) vanishes by Proposition 3.15. That \( \mu_{\text{BGL}} \) is commutative follows from the vanishing of the group \( \lim^{-1}\text{BGL}^{4i-1,2i}(\mathcal{K}_i \land \mathcal{K}_i) \) (see Proposition 3.13). The fact that \( e_{\text{BGL}} \) is a two-sided unit for the multiplication \( \mu_{\text{BGL}} \) follows by Proposition 2.27, which shows that the group \( \lim^{-1}\text{BGL}^{2i-1,4i}(\mathcal{K}_i) \) vanishes.

**Definition 3.7.** Let \( S \) be a Noetherian finite-dimensional scheme, with \( f : S \to \text{Spec}(\mathbb{Z}) \) being the canonical morphism. Let \( f^* : \text{SH}(\mathbb{Z}) \to \text{SH}(S) \) be the strict symmetric monoidal pull-back functor from A.47. Set

\[
\mu_{\text{BGL}}^S := f^*(\text{BGL}) \land f^*(\text{BGL}) \xrightarrow{\text{can}} f^*(\text{BGL} \land \text{BGL}) \xrightarrow{f^*(\mu_{\text{BGL}})} f^*(\text{BGL})
\]

\[
e_{\text{BGL}}^S := S^0 \xrightarrow{\text{can}} f^*(S^0) \xrightarrow{f^*(e_{\text{BGL}})} f^*(\text{BGL})
\]

and \( \text{BGL}_S = f^*(\text{BGL}) \). Then \( (\text{BGL}_S, \mu_{\text{BGL}}^S, e_{\text{BGL}}^S) \) is a commutative monoid in \( \text{SH}(S) \). Note that \( \text{BGL}_S \) satisfies the conditions from Definition refBGL in \( \text{MS}(S) \) by Remark 2.5.

We will sometimes refer to a monoid in \( \text{SH}(S) \) as a \( \mathbb{P}^1 \)-ring spectrum.

**Corollary 3.8.** The multiplicative structure on the functor \( \text{BGL}_S^*,S \) induced by the pairing \( \mu_{\text{BGL}}^S \) and the unit \( e_{\text{BGL}}^S \) coincides with the naive product structure (7).

**Proof.** Follows from Theorem 3.6.

**Corollary 3.9.** The functor isomorphism \( [X, \mathcal{K}_0] \to [\Sigma^\infty_{\mathbb{P}}(X), \text{BGL}_S] \) respects the multiplicative structures on both sides. In particular, the isomorphism \( \text{Ad} : K_* \to \text{BGL}_S^{*,0} \) of cohomology theories on \( \text{Sm}0p/S \) is an isomorphism of ring cohomology theories in the sense of [PS].

**Proof.** Follows from Theorem 3.6.

**Remark 3.10.** Let \( S \) be a finite dimensional Noetherian scheme, with \( f : S \to \text{Spec}(\mathbb{Z}) \) being the canonical morphism. Then by [V, Thm. 6.9] the \( \mathbb{P}^1 \)-spectrum \( \text{BGL}_S \) over \( S \) as defined in 3.7 represents homotopy invariant \( K \)-theory as introduced in [We]. The triple \( (\text{BGL}_S, \mu_{\text{BGL}}^S, e_{\text{BGL}}^S) \) is a distinguished monoidal structure on the the \( \mathbb{P}^1 \)-spectrum \( \text{BGL}_S \).
3.3 Preliminary computations II

Let BGL be the $\mathbb{P}^1$-spectrum defined in 2.1. We will identify in this section the functors $BGL^{0,0}$, $BGL^{2i}$, $BGL^{-1,0}$ and $BGL^{2i-1,i}$ on the motivic unstable category $H_\bullet(S)$ as in Section 2.5.

Lemma 3.11. Let $S = \text{Spec}(\mathbb{Z})$. For every integer $i$ the map

$$BGL^{-1,0}(S) \otimes BGL^{2i}(\mathcal{K}_i \wedge \mathcal{K}_i) \to BGL^{2i-1,i}(\mathcal{K}_i \wedge \mathcal{K}_i)$$

induced by the naive product structure is an isomorphism. The same holds if we replace $\mathcal{K}_i \wedge \mathcal{K}_i$ by $\mathcal{K}_i \wedge \mathcal{K}_i \wedge \mathbb{P}^1$ or by $\mathcal{K}_i \wedge \mathcal{K}_i \wedge \mathbb{P}^1 \wedge \mathbb{P}^1$.

Proof. Since diagram (16) commutes, it suffices to consider the case $i = 0$. Furthermore we may replace the pointed motivic space $\mathcal{K}_i$ with $\mathbb{Z} \times \text{Gr}$ since the map $i: \mathbb{Z} \times \text{Gr} \to \mathcal{K}_i = \mathcal{K}$ is a motivic weak equivalence. The functor isomorphism $\mathcal{K}_i \to BGL^{-*}$ is an isomorphism of ring cohomology theories. Thus it remains to check that the map

$$K_1(S) \otimes K_0((\mathbb{Z} \times \text{Gr}) \wedge (\mathbb{Z} \times \text{Gr})) \to K_1((\mathbb{Z} \times \text{Gr}) \wedge (\mathbb{Z} \times \text{Gr}))$$

is an isomorphism. This can be checked arguing as in the proof of Lemma 2.21 and using Lemma B.9 and Lemma B.6. The cases of $\mathcal{K}_i \wedge \mathcal{K}_i \wedge \mathbb{P}^1$ and $\mathcal{K}_i \wedge \mathcal{K}_i \wedge \mathbb{P}^1 \wedge \mathbb{P}^1$ are proved by the same arguments. \hfill \Box

Lemma 3.12. Suppose that $S = \text{Spec}(\mathbb{Z})$, and let $r: SH(S) \to SH_{\mathbb{C}\mathbb{P}^1}$ be the topological realization functor. Then for every integer $i$ the homomorphism $BGL^{2i}(\mathcal{K} \wedge \mathcal{K}) \to (rBGL)^{2i}(r(\mathcal{K} \wedge \mathcal{K}) \cong (rBGL)^{2i}(r\mathcal{K} \wedge r\mathcal{K})$ is bijective.

Proof. Since the $(2,1)$-periodicity isomorphism for BGL described in Remark 2.20 and the Bott periodicity isomorphism for $rBGL$ are compatible by A.46, it suffices to consider the case $i = 0$. We may replace the pointed motivic space $\mathcal{K}$ with $\mathbb{Z} \times \text{Gr}$ as in the proof of Lemma 2.24. It remains to check that the realization homomorphism

$$BGL^{0,0}((\mathbb{Z} \times \text{Gr}) \wedge (\mathbb{Z} \times \text{Gr})) \to (rBGL)^{0}(r((\mathbb{Z} \times \text{Gr}) \wedge (\mathbb{Z} \times \text{Gr})))$$

$$= (rBGL)^{0}(r(\mathbb{Z} \times \text{Gr}) \wedge r(\mathbb{Z} \times \text{Gr}))$$

is an isomorphism. By Example A.32 the $\mathbb{P}^1$-spectrum $\Sigma_{\mathbb{P}^1}^\infty((\mathbb{Z} \times \text{Gr}) \wedge (\mathbb{Z} \times \text{Gr}))$ is a retract of $\Sigma_{\mathbb{P}^1}^\infty(\mathbb{Z} \times \text{Gr} \times \mathbb{Z} \times \text{Gr})$ in $SH(S)$, whence it suffices to consider the topological realization homomorphism for $\mathbb{Z} \times \text{Gr} \times \mathbb{Z} \times \text{Gr}$. Since the latter is an increasing union of the cellular $S$-schemes $[-n,n] \times \text{Gr}(n,2n) \times [-m,m] \times \text{Gr}(m,2m)$, the result follows with the help of Lemma 2.23 as in the proof of Lemma 2.24. \hfill \Box
3.4 Vanishing of certain groups II

Consider the stable equivalence \( \text{hocolim}_p \Sigma_\infty (\mathcal{K} \wedge \mathcal{K})(-2i) \cong BGL \wedge BGL \) displayed in (21) and the corresponding short exact sequence

\[
0 \to \lim\limits_{\leftarrow}^{1} BGL^{4i-1,2i}(\mathcal{K} \wedge \mathcal{K}) \to BGL^{0,0}(BGL \wedge BGL) \to \lim\limits_{\leftarrow}^{1} BGL^{4i,2i}(\mathcal{K} \wedge \mathcal{K}) \to 0
\]

from Corollary 3.5. We prove in this section the following result

**Proposition 3.13.** If \( S = \text{Spec}(\mathbb{Z}) \) then \( \lim\limits_{\leftarrow}^{1} BGL^{4i-1,2i}(\mathcal{K} \wedge \mathcal{K}) = 0 \).

**Proof.** For a pointed motivic space \( A \) we abbreviate \( A \wedge A \) as \( A^2 \). The connecting homomorphism in the tower of groups forming the \( \lim\limits_{\leftarrow}^{1} \)-term is the composite map

\[
\begin{array}{ccc}
BGL^{4i-1,2i}(\mathcal{K} \wedge \mathcal{K}) & \xleftarrow{(\Sigma \circ \Sigma)^{-1}_{\text{otw}}} & BGL^{4(i+1)-1,2(i+1)}((\mathcal{K} \wedge \mathcal{P}^1)^2) \\
\xrightarrow{id \otimes (\epsilon \wedge \epsilon)^*} & & \xrightarrow{(\epsilon \wedge \epsilon)^*} \\
A \otimes BGL^{4i+4,2i+2}(\mathcal{K} \wedge \mathcal{K}) & \xrightarrow{id \otimes (\Sigma \circ \Sigma)^{-1}_{\text{otw}}} & BGL^{4i+3,2i+2}(\mathcal{K} \wedge \mathcal{K}) \\
\end{array}
\]

where \( \Sigma \) is the \( \mathbb{P}^1 \)-suspension isomorphism, \( \text{tw} \) is induced by interchanging the two pointed motivic spaces in the middle of the four-fold smash product, and \( \epsilon : \mathcal{K} \wedge \mathbb{P}^1 \to \mathcal{K} \) is the structure map of \( BGL \). Set \( A = BGL^{-1,0}(S) \) and write \( B \) for \( BGL \). Consider the diagram

\[
\begin{array}{ccc}
A \otimes BGL^{4i+4,2i+2}(\mathcal{K} \wedge \mathcal{K}) & \xrightarrow{id \otimes (\epsilon \wedge \epsilon)^*} & BGL^{4i+3,2i+2}(\mathcal{K} \wedge \mathcal{K}) \\
\xrightarrow{id \otimes (\Sigma \circ \Sigma)^{-1}_{\text{otw}}} & & \xrightarrow{(\Sigma \circ \Sigma)^{-1}_{\text{otw}}} \\
A \otimes BGL^{4i+4,2i+2}((\mathcal{K} \wedge \mathcal{P}^1)^2) & \xrightarrow{id \otimes (\Sigma \circ \Sigma)^{-1}_{\text{otw}}} & BGL^{4i+3,2i+2}((\mathcal{K} \wedge \mathcal{P}^1)^2) \\
\end{array}
\]

where the horizontal arrows are induced by the naive product structure on the functor \( BGL^{*,*} \). Clearly it commutes. The horizontal arrows are isomorphisms by Lemma 3.11. It follows that \( \lim\limits_{\leftarrow}^{1} BGL^{4i-1,2i}(\mathcal{K} \wedge \mathcal{K}) = \lim\limits_{\leftarrow}^{1} (A \otimes BGL^{4i,2i}(\mathcal{K} \wedge \mathcal{K})) \) where in the last tower of groups the connecting maps are \( id \otimes \left( (\Sigma \circ \Sigma)_{\mathbb{P}^1} \circ \text{tw} \right) \circ (\epsilon \wedge \epsilon)^* \). It remains to prove the following statement.

**Claim 3.14.** One has \( \lim\limits_{\leftarrow}^{1} (A \otimes BGL^{4i,2i}(\mathcal{K} \wedge \mathcal{K})) = 0 \).

Since \( A = BGL^{-1,0}(S) = K_1(S) = \mathbb{Z}/2\mathbb{Z} \), there is an isomorphism \( A \otimes BGL^{4i,2i}(\mathcal{K} \wedge \mathcal{K}) = BGL^{4i,2i}(\mathcal{K} \wedge \mathcal{K})/mBGL^{4i,2i}(\mathcal{K} \wedge \mathcal{K}) \) with \( m = 2 \). The connecting map in the tower are just the mod-\( m \) reduction of the maps \( (\Sigma \circ \Sigma)^{-1} \circ \text{tw} \circ (\epsilon \wedge \epsilon)^* \). Now a chain of isomorphisms...
completes the proof of the Claim.

\[
\lim_{\leftarrow}^1 \text{BGL}^{4i,2i}(\mathcal{K}^{\wedge 2})/m \cong \lim_{\leftarrow}^1 (r \text{BGL})^{4i}(r \mathcal{K}^{\wedge 2})/m \cong \lim_{\leftarrow}^1 \text{BU}^{4i}(r \mathcal{K}^{\wedge 2})/m \\
\cong \lim_{\leftarrow}^1 \text{BU}^{4i}((\mathbb{Z} \times \text{BU}) \wedge (\mathbb{Z} \times \text{BU}))/m \\
\cong \lim_{\leftarrow}^1 \text{BU}^{4i}((\mathbb{Z} \times \text{BU}) \wedge (\mathbb{Z} \times \text{BU}); \mathbb{Z}/m) \\
\cong K_{\text{top}}^1(\text{BU} \wedge \text{BU}; \mathbb{Z}/m) \cong K_{\text{top}}^1(\mathbb{B}^0 \mathbb{U} \wedge \mathbb{B}^0 \mathbb{U}; \mathbb{Z}/m) \\
\cong \lim_{\leftarrow}^1 K_{\text{top}}^{4i}(\text{Gr}(b(i), 2b(i)) \wedge \text{Gr}(b(i), 2b(i)); \mathbb{Z}/m) = 0
\]

The first isomorphism follows from Lemma 3.12. The second isomorphism is induced by the levelwise weak equivalence \(\mathbb{B} \mathbb{U} \simeq r \text{BGL}\) mentioned in Lemma 2.22. The third isomorphism is induced by the image of the weak equivalence \(\mathcal{K}_i \simeq \mathbb{Z} \times \text{Gr}\) under topological realization. The forth and fifth isomorphism hold since \(\mathbb{B}^4 \mathbb{U}^{4i+1}(\mathbb{Z} \times \text{BU}) = 0\). The sixth isomorphism is induced by the stable equivalence \(\mathbb{B}^0 \mathbb{U} \simeq \mathbb{B} \mathbb{U}\) from Lemma 2.25, the seventh one is induced by the stable equivalence \(\mathbb{B}^1 \mathbb{U} \simeq \mathbb{B}^0 \mathbb{U}\) from Lemma 2.26. The last one holds since all groups in the tower are finite. \(\square\)

### 3.5 Vanishing of certain groups III

Consider the stable equivalence

\[
\text{hocolim} \Sigma_{\mathcal{K}}^{\infty} (\mathcal{K} \wedge \mathcal{K} \wedge \mathcal{K})(-3i) \cong \text{BGL} \wedge \text{BGL} \wedge \text{BGL}
\]

from (21) and the induced short exact sequence

\[
0 \to \lim_{\leftarrow}^1 \text{BGL}^{8i-1,4i}(\mathcal{K}^{\wedge 3}) \to \text{BGL}^{0,0}(\mathcal{K}^{\wedge 3}) \to \lim_{\leftarrow}^1 \text{BGL}^{8i,4i}(\mathcal{K}^{\wedge 3}) \to 0.
\]

**Proposition 3.15.** If \(S = \text{Spec}(\mathbb{Z})\) then \(\lim_{\leftarrow}^1 \text{BGL}^{8i-1,4i}(\mathcal{K} \wedge \mathcal{K} \wedge \mathcal{K}) = 0\).

**Proof.** This is proved in the same way as Proposition 3.13. \(\square\)

### 3.6 BGL as an oriented commutative \(\mathbb{P}^1\)-ring spectrum

Following Adams and Morel, we define an orientation of a commutative \(\mathbb{P}^1\)-ring spectrum. However we prefer to use a Thom class rather than a Chern class. Let \(\mathbb{P}^\infty = \bigcup \mathbb{P}^n\) be the motivic space pointed by \(\infty \in \mathbb{P}^1 \hookrightarrow \mathbb{P}^\infty\). \(\emptyset(-1)\) be the tautological line bundle over \(\mathbb{P}^\infty\). It is also known as the Hopf bundle. If \(V \to X\) is a vector bundle over \(X \in \mathfrak{S}m/S\), with zero section \(z: X \hookrightarrow V\), let \(\text{Th}_X(V) = V/(V \smallsetminus z(X))\) be the Thom space of \(V\), considered as a pointed motivic space over \(S\). For example \(\text{Th}_X(\mathbb{A}^3_\mathbb{Z}) \simeq S^{2n,n}\). Define \(\text{Th}_{\mathbb{P}^\infty}(\emptyset(-1))\) as the obvious colimit of the Thom spaces \(\text{Th}_{\mathbb{P}^n}(\emptyset(-1))\).

**Definition 3.16.** Let \(E\) be a commutative \(\mathbb{P}^1\)-ring spectrum. An orientation of \(E\) is an element \(\text{th} \in E_{2,1}^2(\text{Th}_{\mathbb{P}^\infty}(\emptyset(-1))) = E_{2,1}^{2,1}(\emptyset(-1))\) such that its restriction to the Thom space of the fibre over the distinguished point coincides with the element \(\Sigma_{\mathbb{P}^1}(1) \in E_{2,1}(\text{Th}(1)) = E_{2,1}(\mathbb{P}^1, \infty)\).
Remark 3.17. Let $th$ be an orientation of $E$. Set $c := z^*(th) \in E^{2,1}(\mathbb{P}^\infty)$. Then [PY, Prop. 6.5.1] implies that $c|_{\mathbb{P}^1} = -\Sigma_{\mathbb{P}^1}(1)$. The class $th(O(-1)) \in E_{\mathbb{P}^\infty}^{2,1}(O(-1))$ given by (23) coincides with the element $th$ by [PS, Thm. 3.5]. Thus another possible definition of an orientation of $E$ is the following.

Definition 3.18. Let $E$ be a commutative $\mathbb{P}^1$-ring spectrum. An orientation of $E$ is an element $c \in E^{2,1}(\mathbb{P}^\infty)$ such that $c|_{\mathbb{P}^1} = -\Sigma_{\mathbb{P}^1}(1)$ (of course the element $c$ should be regarded as the first Chern class of the Hopf bundle $O(1)$ on $\mathbb{P}^\infty$).

Remark 3.19. Let $c$ be an orientation of the commutative $\mathbb{P}^1$-ring spectrum $E$. Consider the element $th(O(-1)) \in E_{\mathbb{P}^\infty}^{2,1}(O(-1))$ given by (23) and set $th = th(O(-1))$. It is straightforward to check that $th|_{\mathbb{P}^1(1)} = \Sigma_{\mathbb{P}^1}(1)$. Thus $th$ is an orientation of $E$. Clearly $c = z^*(th) \in E^{2,1}(\mathbb{P}^\infty)$, whence the two definitions of orientations of $E$ are equivalent.

Example 3.20. Set $c^K = (-\beta) \cup ([0] - [0(1)]) \in BGL^{2,1}(\mathbb{P}^\infty)$. The relation (9) shows that $c^K$ is an orientation of $BGL$. Consider $th(O(-1)) \in BGL_{\mathbb{P}^\infty}^{2,1}(O(-1))$ given by (23) and set $th^K = th(O(-1))$. The class $th^K$ is the same orientation of $BGL$.

The orientation of $BGL$ described in Example 3.20 has the following property. The map (13)

$$BGL^{*,*} \to K_*$$

which takes $\beta$ to $-1$ is an oriented morphism of oriented cohomology theories, provided that $K_*$ is oriented via the Chern structure $L/X \mapsto [0] - [L^{-1}] \in K_0(X)$.

### 3.7 BGL$^{*,*}$ as an oriented ring cohomology theory

An oriented $\mathbb{P}^1$-ring spectrum $(E, c)$ defines an oriented cohomology theory on $SmOp$ in the sense of [PS, Defn. 3.1] as follows. The restriction of the functor $E^{*,*}$ to the category $SmOp$ is a ring cohomology theory. By [PS, Thm. 3.35] it remains to construct a Chern structure on $E^{*,*}|_{SmOp}$ in the sense of [PS, Defn. 3.2]. The functor isomorphism $\text{Hom}_{H_*(S)}(-, \mathbb{P}^\infty) \to \text{Pic}(-)$ on the category $Sm/S$ provided by [MV, Thm. 4.3.8] takes the class of the canonical map $\mathbb{P}^\infty_+ \to \mathbb{P}^\infty$ to the class of the tautological line bundle $O(-1)$ over $\mathbb{P}^\infty$. Now for a line bundle $L$ over $X \in Sm/S$ set $c(L) = f_L^*(c) \in E^{2,1}(X)$, where the morphism $f_L: X_+ \to \mathbb{P}^\infty$ in $H_*(S)$ corresponds to the class $[L]$ in $L$ in the group Pic$(X)$. Clearly, $c(0(1)) = c$. The assignment $L/X \mapsto c(L)$ is a Chern structure on $E^{*,*}|_{SmOp}$ since $c|_{\mathbb{P}^1} = -\Sigma_{\mathbb{P}^1}(1) \in E^{2,1}(\mathbb{P}^1, \infty)$. With that Chern structure $E^{*,*}|_{SmOp}$ is an oriented ring cohomology theory in the sense of [PS]. In particular, $(BGL, c^K)$ defines an oriented ring cohomology theory on $SmOp$.

This Chern structure induces a theory of Thom classes

$$V/X \mapsto th(V) \in E^{2,\text{rank}(V), \text{rank}(V)}(\text{Th}_X(V))$$

on $E^{*,*}|_{SmOp}$ in the sense of [PS, Defn. 3.32] as follows. There is a unique theory of Chern classes $V \mapsto c_i(V) \in E^{2i,i}(X)$ such that for every line bundle $L$ on $X$ one has $c_1(L) = c(L)$.
Now for a rank $r$ vector bundle $V$ over $X$ consider the vector bundle $W := 1 \oplus V$ and the associated projective vector bundle $\mathbb{P}(W)$ of lines in $W$. Set
\[
\bar{th}(V) = c_r(p^*(V) \otimes \mathcal{O}_{\mathbb{P}(W)}(1)) \in E^{2r,r}(\mathbb{P}(W)).
\] (22)

It follows from [PS, Cor. 3.18] that the support extension map
\[
E^{2r,r}(\mathbb{P}(W)/(\mathbb{P}(W) \smallsetminus \mathbb{P}(1))) \to E^{2r,r}(\mathbb{P}(W))
\]
is injective and $\bar{th}(E) \in E^{2r,r}(\mathbb{P}(W)/(\mathbb{P}(W) \smallsetminus \mathbb{P}(1)))$. Set
\[
\tilde{th}(E) = j^*(\bar{th}(E)) \in E^{2r,r}(\text{Th}_X(V)),
\] (23)

where $j : \text{Th}_X(V) \to \mathbb{P}(W)/(\mathbb{P}(W) \smallsetminus \mathbb{P}(1))$ is the canonical motivic weak equivalence of pointed motivic spaces induced by the open embedding $V \hookrightarrow \mathbb{P}(W)$. The assignment $V/X$ to $\tilde{th}(E)$ is a theory of Thom classes on $E^{*,*}|_{\text{smop}}$ (see the proof of [PS, Thm. 3.35]). So the Thom classes are natural, multiplicative and satisfy the following Thom isomorphism property.

**Theorem 3.21.** For a rank $r$ vector bundle $p : V \to X$ on $X \in \text{Sm}/S$ the map
\[
- \cup \tilde{th}(V) : E^{*,*}(X) \to E^{*,*+2r,r}(\text{Th}_X(V))
\]
is an isomorphism of the two-sided $E^{*,*}(X)$-modules, where $- \cup \tilde{th}(V)$ is written for the composition map $(- \cup \tilde{th}(V)) \circ p^*$.

**Proof.** See [PS, Defn. 3.32.(4)]. \[\Box\]

## A Motivic homotopy theory

The aim of this section is to present details on the model structures we use to perform homotopical calculations. Our reference on model structures is [Ho]. For the convenience of the reader who is not familiar with model structures, we recall the basic features and purposes of the theory below, after discussing categorical prerequisites.

### A.1 Categories of motivic spaces

Let $S$ be a Noetherian separated scheme of finite Krull dimension (base scheme for short). The category of smooth quasi-projective $S$-schemes is denoted $\text{Sm}/S$. A smooth morphism is always of finite type. In particular, $\text{Sm}/S$ is equivalent to a small category.

The category of compactly generated topological spaces is denoted $\text{Top}$, the category of simplicial sets is denoted $\text{sSet}$. The set of $n$-simplices in $K$ is $K_n$.

**Definition A.1.** A motivic space over $S$ is a functor $A : \text{Sm}/S^{op} \to \text{sSet}$. The category of motivic spaces over $S$ is denoted $\text{M}(S)$. 

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For $X \in \mathcal{S}m/S$ the motivic space sending $Y \in \mathcal{S}m/S$ to the discrete simplicial set $\text{Hom}_{\mathcal{S}m/S}(Y, X)$ is denoted $X$ as well. More generally, any scheme $X$ over $S$ defines a motivic space $X$ over $S$. Any simplicial set $K$ defines a constant motivic space $K$. A pointed motivic space is a pair $(A, a_0)$, where $a_0 : S \to A$. Usually the basepoint will be omitted from the notation. The resulting category is denoted $\mathbf{M}_*(S)$.

**Definition A.2.** A morphism $f : S \to S'$ of base schemes defines the functor

$$f_* : \mathbf{M}_*(S) \to \mathbf{M}_*(S')$$

sending $A$ to $(Y \to S') \mapsto A(S \times_S Y)$. Left Kan extension produces a left adjoint $f^* : \mathbf{M}_*(S') \to \mathbf{M}_*(S)$ of $f_*$.

If $A$ is a motivic space, let $A_+$ denote the pointed motivic space $(A \coprod S, i)$, where $i : S \to A \coprod S$ is the canonical inclusion. The category $\mathbf{M}_*(S)$ is closed symmetric monoidal, with smash product $A \wedge B$ defined by the sectionwise smash product

$$(A \wedge B)(X) : = A(X) \wedge B(X)$$

and with internal hom $\text{Hom}_{\mathbf{M}_*(S)}(A, B)$ defined by

$$\text{Hom}_{\mathbf{M}_*(S)}(A, B)(x : X \to S)_n : = \text{Hom}_{\mathbf{M}_*(S)}(A \wedge \Delta^n_+, x^*x^*B).$$

In particular, $\mathbf{M}_*(S)$ is also enriched over the category of pointed simplicial sets, with enrichment $\mathbf{sSet}_*(A, B) : = \text{Hom}_{\mathbf{M}_*(S)}(A, B)(S)$. The mapping cylinder of a map $f : A \to B$ is the pushout of the diagram

$$A \wedge \partial^1_+ \xrightarrow{\cong} A \coprod A \xrightarrow{id_A \coprod f} A \coprod B.$$  

The composition of the canonical maps $A \hookrightarrow \text{Cyl}(f) \to B$ is $f$.

The pushout product of two maps $f : A \to C$ and $g : B \to D$ of motivic spaces over $S$ is the map $f \square g : A \wedge D \cup_{A \wedge B} C \wedge B \to C \wedge D$ induced by the commutative diagram

$$A \wedge B \xrightarrow{} A \wedge D$$

$$C \wedge B \xrightarrow{} C \wedge D.$$  

The functor $f^* : \mathbf{M}_*(S') \to \mathbf{M}_*(S)$ induced by $f : S \to S'$ is strict symmetric monoidal in the sense that there are isomorphisms

$$f^*(A) \wedge f^*(B) \overset{\cong}{\to} f^*(A \wedge B) \quad \text{and} \quad f^*(S'_+) \overset{\cong}{\to} S_+$$

which are natural in $A$ and $B$. The isomorphisms (28) are induced by the corresponding isomorphisms for the strict symmetric monoidal pullback functor sending $X \in \mathcal{S}m_{S'}$ to $S \times_{S'} X \in \mathcal{S}m/S$. This ends the categorical considerations.
A.2 Model categories

The basic purpose of a model structure is to give a framework for the construction of a homotopy category. Suppose \( w \mathcal{C} \) is a class of morphisms in a category \( \mathcal{C} \) one wants to make invertible. Call them weak equivalences. One can define the homotopy “category” of the pair \( (\mathcal{C}, w\mathcal{C}) \) to be the target of the universal “functor” \( \Gamma: \mathcal{C} \rightarrow \text{Ho}(\mathcal{C}, w\mathcal{C}) \) such that every weak equivalence is mapped to an isomorphism. In general, this homotopy “category” may not be a category: it has hom-classes, but not necessarily hom-sets. If one requires the existence of two auxiliary classes of morphisms \( f\mathcal{C} \) (the fibrations) and \( c\mathcal{C} \) (the cofibrations), together with certain compatibility axioms, one does get a homotopy category \( \text{Ho}(\mathcal{C}, w\mathcal{C}) \) and an explicit description of the hom-sets in it.

**Theorem A.3** (Quillen). Let \( (w\mathcal{C}, f\mathcal{C}, c\mathcal{C}) \) be a model structure on a bicomplete category \( \mathcal{C} \). Then the universal functor to the homotopy category \( \Gamma: \mathcal{C} \rightarrow \text{Ho}(\mathcal{C}, w\mathcal{C}) \) exists and is the identity on objects. The set of morphisms in \( \text{Ho}(\mathcal{C}, w\mathcal{C}) \) from \( \Gamma A \) to \( \Gamma B \) is the set of morphisms in \( \mathcal{C} \) from \( A \) to \( B \) modulo a homotopy equivalence relation, provided that \( \emptyset \rightarrow A \) is a cofibration and \( B \rightarrow * \) is a fibration.

Here \( \emptyset \) is the initial object and \( * \) is the terminal object in \( \mathcal{C} \). An object \( A \) resp. \( B \) as in Theorem A.3 is called cofibrant resp. fibrant. Every object \( \Gamma A \) in the homotopy category is isomorphic to an object \( \Gamma C \), where \( C \) is both fibrant and cofibrant. A (co)fibration which is also a weak equivalence is usually called a trivial or acyclic (co)fibration.

To describe the standard way to construct model structures on a bicomplete category, one needs a definition.

**Definition A.4.** Let \( f: A \rightarrow B \) and \( g: C \rightarrow D \) be morphisms in \( \mathcal{C} \). If every commutative diagram

\[
\begin{array}{ccc}
A & \longrightarrow & C \\
\downarrow f & & \downarrow g \\
B & \longrightarrow & D
\end{array}
\]

admits a morphism \( h: B \rightarrow C \) such that the resulting diagram

\[
\begin{array}{ccc}
A & \longrightarrow & C \\
\downarrow f & \swarrow h & \downarrow g \\
B & \longrightarrow & D
\end{array}
\]

commutes (a lift for short), then \( f \) has the left lifting property with respect to \( g \), and \( g \) has the right lifting property with respect to \( f \).

Here is the standard way of constructing a model structure on a given bicomplete category. Choose the class of weak equivalences such that it contains all identities, is closed under retracts and satisfies the two-out-of-three axiom. Pick a set \( I \) (the generating cofibrations) and define a cofibration to be a morphism which is a retract of a transfinite composition of cobase changes of morphisms in \( I \). Pick a set \( J \) (the generating acyclic
cofibrations) of weak equivalences which are also cofibrations and define the fibrations to be those morphisms which have the right lifting property with respect to every morphism in $J$. Some technical conditions have to be fulfilled in order to conclude that this indeed is a model structure, which is then called cofibrantly generated. See [Ho, Thm. 2.1.19].

**Example A.5.** In $\text{Top}$, let the weak equivalences be the weak homotopy equivalences, and set

$$I = \{\partial D^n \hookrightarrow D^n\}_{n \geq 0} \quad J = \{D^n \times \{0\} \hookrightarrow D^n \times I\}_{n \geq 0}.$$  

Then the fibrations are precisely the Serre fibrations, and the cofibrations are retracts of generalized cell complexes (“generalized” refers to the fact that cells do not have to be attached in order of dimension). In $\text{sSet}$, let the weak equivalences be those maps which map to (weak) homotopy equivalences under geometric realization. Set

$$I = \{\partial \Delta^n \hookrightarrow \Delta^n\}_{n \geq 0} \quad J = \{\Lambda^n_j \hookrightarrow \Delta^n\}_{n \geq 1, 0 \leq j \leq n}$$

where $\Lambda^n_j$ is the sub-simplicial set of $\partial \Delta^n$ obtained by removing the $j$-th face. Then the fibrations are precisely the Kan fibrations, and the cofibrations are the inclusions.

**Example A.6.** For the purpose of this paper, model structures on presheaf categories $\text{Fun}(\mathcal{C}^{\text{op}}, \text{sSet})$ with values in simplicial sets are relevant. There is a canonical one, due to Quillen, which is usually referred to as the projective model structure. It has as weak equivalences those morphisms $f: A \rightarrow B$ such that $f(c): A(c) \rightarrow B(c)$ is a weak equivalence for every $c \in \text{Ob} \mathcal{C}$ (the objectwise or sectionwise weak equivalences). Set

$$I = \{\text{Hom}_S(-, c) \times (\partial \Delta^n \hookrightarrow \Delta^n)\}_{n \geq 0, c \in \text{Ob} \mathcal{C}} \quad J = \{\text{Hom}_S(-, c) \times (\Lambda^n_j \hookrightarrow \Delta^n)\}_{n \geq 1, 0 \leq j \leq n, c \in \text{Ob} \mathcal{C}}$$

so that by adjointness, the fibrations are precisely the sectionwise Kan fibrations. There is another cofibrantly generated model structure with the same weak equivalences, due to Heller [He], such that the cofibrations are precisely the injective morphisms (whence the name injective model structure). The description of $J$ involves the cardinality of the set of morphisms in $\mathcal{C}$ and is not explicit. Neither is the characterization of the fibrations.

The morphisms of model categories are called Quillen functors. A Quillen functor of model categories $\mathcal{M} \rightarrow \mathcal{N}$ is an adjoint pair $(F, G): \mathcal{M} \rightarrow \mathcal{N}$ such that $F$ preserves cofibrations and $G$ preserves fibrations. This condition ensures that $(F, G)$ induces an adjoint pair on homotopy categories $(\mathcal{L}F, \mathcal{R}G)$, where $\mathcal{L}F$ is the total left derived functor of $F$. A Quillen functor is a Quillen equivalence if the total left derived is an equivalence. For example, geometric realization is a strict symmetric monoidal left Quillen equivalence $| - |: \text{sSet} \rightarrow \text{Top}$, and similarly in the pointed setting.

If a model category has a closed symmetric monoidal structure as well, one has the following statement, proven in [Ho, Thm. 4.3.2].

**Theorem A.7** (Quillen). Let $\mathcal{C}$ be a bicomplete category with a model structure. Suppose that $(\mathcal{C}, \otimes, I)$ is closed symmetric monoidal. Suppose further that these structures are compatible in the following sense:
The pushout product of two cofibrations is a cofibration, and

the pushout product of an acyclic cofibration with a cofibration is an acyclic cofibration.

Then $A \otimes - $ is a left Quillen functor for all cofibrant objects $A \in \mathcal{C}$. In particular, there is an induced (total derived) closed symmetric monoidal structure on $\text{Ho}(\mathcal{C}, w\mathcal{C})$.

One abbreviates the hypotheses of Theorem A.7 by saying that $\mathcal{C}$ is a symmetric monoidal model category. This ends our introduction to model category theory.

### A.3 Model structures for motivic spaces

To equip $M_*(S)$ with a model structure suitable for the various requirements (compatibility with base change, taking complex points, finiteness conditions, having the correct motivic homotopy category), we construct a preliminary model structure first. Start with the following construction, which is a special case of the considerations in [I]. Choose any $X \in Sm_S$ and a finite set

$$\{ i^j : Z_j \hookrightarrow X \}_{j=1}^m$$

of closed embeddings in $Sm/S$. Regarding $i^j$ as a monomorphism of motivic spaces, one may form the categorical union (not the categorical coproduct!) $i : \bigcup_{j=1}^m Z_j \hookrightarrow X$. That is, $\bigcup_{j=1}^m Z_j$ is the coequalizer in the category of motivic spaces of the diagram

$$\bigsqcup_{j,j'} Z_j \times_X Z_{j'} \rightrightarrows \bigsqcup_{j=1}^m Z_j$$  \hspace{1cm} (29)

Call the resulting monomorphism $i : \bigcup_{j=1}^m Z_j \hookrightarrow X$ acceptable. The closed embedding $\emptyset \hookrightarrow X$ is acceptable as well. Consider the set $\text{Ace}$ of acceptable monomorphisms. Let $I_S$ be the set of pushout product maps

$$\{ i^+ \Box (\partial \Delta^n \hookrightarrow \Delta^n)_+ \}_{i \in \text{Ace}, n \geq 0}$$  \hspace{1cm} (30)

and let $J_S$ be the set of pushout product maps

$$\{ i^+ \Box (\Lambda^n_j \hookrightarrow \Delta^n)_+ \}_{i \in \text{Ace}, n \geq 1, 0 \leq j \leq n}$$  \hspace{1cm} (31)

defined via diagram (27).

**Definition A.8.** A map $f : A \to B$ in $M_*(S)$ is a schemewise weak equivalence if $f : A(X) \to B(X)$ is a weak equivalence of simplicial sets for all $X \in Sm/S$. It is a closed schemewise fibration if $f : A \to B$ has the right lifting property with respect to $J_S$. It is a closed cofibration if it has the left lifting property with respect to all acyclic closed schemewise fibrations (closed schemewise fibrations which are also schemewise weak equivalences).
Theorem A.9. The classes described in Definition A.8 are a closed symmetric monoidal model structure on $\mathcal{M}(S)$, denoted $\mathcal{M}_{cs}(S)$. A morphism $f: S \to T$ of base schemes induces a strict symmetric monoidal left Quillen functor $f^*: \mathcal{M}_{cs}(T) \to \mathcal{M}_{cs}(S)$.

Proof. The existence of the model structure follows from [I]. The pushout product axiom follows, because the pushout product of two acceptable monomorphism is again acceptable. To conclude the last statement, it suffices to check that $f^*$ maps any map in $I_\pi$ resp. $J_\pi$ to a closed cofibration resp. schemewise weak equivalence. In fact, if $i: \bigcup_{j=1}^m Z_j \inj X$ is an acceptable monomorphism over $T$, then $f^*(i)$ is the acceptable monomorphism obtained from the closed embeddings

$$\{T \times_S Z_j \inj T \times_S X.\}$$

Because $f^*$ is strict symmetric monoidal and a left adjoint, it preserves the pushout product. Hence $f^*$ even maps the set $I_\pi$ to the set $I_\pi^S$, and likewise for $J_\pi$. The result follows.

The resulting homotopy category is equivalent – via the identity functor – to the usual homotopy category of the diagram category $\mathcal{M}(S)$ (obtained via the projective model structure from Example A.6), since the weak equivalences are just the objectwise ones. The model structure $\mathcal{M}_{cs}(S)$ has the following advantage over the projective model structure.

Lemma A.10. Let $i: Z \inj X$ be a closed embedding in $\mathcal{S}_m/S$. Then the induced map $i_+: Z_+ \to X_+$ is a closed cofibration in $\mathcal{M}(S)$. In particular, for any closed $S$-point $x_0: S \inj X$ in a smooth $S$-scheme, the pointed motivic space $(X, x_0)$ is closed cofibrant.

Proof. The first statement follows, because $i_+ = i_+ \square(\partial \Delta_+^0 \inj \Delta_+^0)$ is contained in the set of generating closed cofibrations. The second statement follows, because cofibrations are closed under cobase change.

Not all pointed motivic spaces are closed cofibrant. Let $(-)^{cs} \to \text{Id}_{\mathcal{M}(S)}$ denote a cofibrant replacement functor, for example the one obtained from applying the small object argument to $I_\pi^S$. That is, the map $A^{cs} \to A$ is a natural closed schemewise fibration and a schemewise weak equivalence, and $A^{cs}$ is closed cofibrant. Dually, let $\text{Id}_{\mathcal{M}(S)} \to (-)^{cf}$ denote the fibrant replacement functor obtained by applying the small object argument to $J_\pi^S$. The closed schemewise fibrations may be characterized explicitly.

Lemma A.11. A map $f: A \to B$ is a closed schemewise fibration if and only if the following two conditions hold.

1. $f(X): A(X) \to B(X)$ is a Kan fibration for every $X \in \mathcal{S}_m/S$, and
2. for every finite set $\{Z_j \inj X\}_{j=1}^m$ of closed embeddings in $\mathcal{S}_m/S$, the induced map

$$A(X) \to B(X) \times_{\mathcal{S}Set}(\bigcup_{j=1}^m Z_j, B) \mathcal{S}Set(\bigcup_{j=1}^m Z_j, A)$$

is a Kan fibration.
Proof. Follows by adjointness from the definition. Note that condition 1 is a special case of condition 2 by taking the empty family.

To obtain a motivic model structure, one localizes $M^a_\bullet(S)$ as follows. Recall that an elementary distinguished square (or simply Nisnevich square) is a pullback diagram

$$
\begin{array}{ccc}
V & \rightarrow & Y \\
\downarrow & & \downarrow p \\
U & \rightarrow & X
\end{array}
$$

in $Sm/S$, where $j$ is an open embedding and $p$ is an étale morphism inducing an isomorphism $Y \setminus V \cong X \setminus U$ of reduced closed subschemes. Say that a pointed motivic space $C$ is closed motivic fibrant if it is closed schemewise fibrant, the map

$$C(X \times_S \mathbb{A}_S^1 \xrightarrow{pr} X)$$

is a weak equivalence of simplicial sets for every $X \in Sm/S$, the square

$$
\begin{array}{ccc}
C(V) & \leftarrow & C(Y) \\
\uparrow & & \uparrow \\
C(U) & \leftarrow & C(X)
\end{array}
$$

is a homotopy pullback square of simplicial sets for every Nisnevich square in $Sm/S$ and $C(\emptyset)$ is contractible.

Example A.12. Let $X \mapsto K^W(X)$ be the pointed motivic space sending $X \in Sm/S$ to the loop space of the first term of the Waldhausen $K$-theory spectrum $W(X)$ associated to the category of big vector bundles over $X$ [FS], with isomorphisms as weak equivalences [W]. That is,

$$K^W(X) = \Omega_s W_1(X) = \Omega_s \text{Sing} |wS_\bullet(\text{Vect}_{big}(X), \text{iso})|$$

By [TT, Thm. 1.11.7, Prop. 3.10] the space $K^W(X)$ has the same homotopy type as the zeroth space (see [TT, 1.5.3]) of the Thomason-Trobaugh $K$-theory spectrum $K^{naive}(X)$ of $X$ as it is defined in [TT, Defn. 3.2].

Since in our case $S$ is regular, then so is $X$ and thus $X$ has an ample family of line bundles by [TT, Examples 2.1.2.]. It follows that the zeroth space of the Thomason-Trobaugh $K$-theory spectrum $K^{naive}(X)$ has the same homotopy type as the zeroth space (see [TT, 1.5.3]) of the Thomason-Trobaugh $K$-theory spectrum $K(X \text{ on } X)$ of $X$ [TT, Thm. 1.11.7, Cor. 3.9] as it is defined in [TT, Defn. 3.1].

Thus for a regular $S$ and $X \in Sm/S$ the following results hold. The projection induces a weak equivalence $K^W(X) \rightarrow K^W(X \times_S \mathbb{A}_S^1)$ [TT, Prop. 6.8]. By [TT, Thm. 10.8] the square

$$
\begin{array}{ccc}
K^W(X) & \rightarrow & K^W(U) \\
K^W(p) \downarrow & & \downarrow \\
K^W(Y) & \rightarrow & K^W(V)
\end{array}
$$

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associated to a Nisnevich square in $Sm/S$ is a homotopy pullback square. Hence $K^W$ is fibrant in the projective motivic model structure on $M_*(S)$. However, if $i: Z \hookrightarrow X$ is a closed embedding in $Sm/S$, the map

$$K^W(i): K^W(X) \to K^W(Z)$$

is not necessarily a Kan fibration. In particular, $K^W$ is not closed schemewise fibrant. Choose a closed cofibration which is also a schemewise weak equivalence $K^W \to K^W$ such that $K^W$ is closed schemewise fibrant. It follows immediately that $K^W$ is closed motivic fibrant, and that $K^W(X)$ has the homotopy type of the zero term of the Waldhausen $K$-theory spectrum of $X$.

**Definition A.13.** A map $f: A \to B$ is a motivic weak equivalence if the map

$$sSet_\bullet(f^c_s, C): sSet_\bullet(B^c_s, C) \to sSet_\bullet(A^c_s, C)$$

is a weak equivalence of simplicial sets for every closed motivic fibrant $C$. It is a closed motivic fibration if it has the right lifting property with respect to all acyclic closed cofibrations (closed cofibrations which are also motivic equivalences).

**Example A.14.** Suppose that $f: A \to B$ is a map in $M_*(S)$ inducing weak equivalences $x^*f: x^*A \to x^*B$ of simplicial sets on all Nisnevich stalks $x^*: M_*(S) \to sSet$. Then $f$ is a motivic weak equivalence. If $f: A \to B$ is an $A^1$-homotopy equivalence (for example, the projection of a vector bundle), then it is a motivic weak equivalence.

**Example A.15.** The canonical covering of $P^1$ shows that it is motivic weakly equivalent as a pointed motivic space to the suspension $S^2 = \Delta^1/\partial \Delta^1$. Set $S^{1,0} = S^1$ and $S^{1,1} = (A^1 - \{0\}, 1)$, and define

$$S^{p,q} = (S^{1,0})^{p-q} \land (S^{1,1})^{q} \quad \text{for } p \geq q \geq 0$$

To generalize the example of $P^1$, one can show that if $P^{n-1} \hookrightarrow P^n$ is a linear embedding, then $P^n / P^{n-1}$ is motivic weakly equivalent to $S^{2n,n}$.

To prove that the classes from Definition A.13 are part of a model structure, it is helpful to characterize the closed motivic fibrant objects via a lifting property. Let $J_S^m$ be the union of the set $J_S^c$ from (31) and the set $J_S^m$ of pushout products of maps $(\partial \Delta^n \hookrightarrow \Delta^n)_+$ with maps of the form

$$X_+ \xrightarrow{\text{zero}_+} (A^1_S \times_S X)_+$$

with maps of the form

$$U_+ \cup V_+ \xrightarrow{\text{Cyl}(h_+)} \text{Cyl}(U_+ \cup V_+ \xrightarrow{\text{Cyl}(h_+)} X_+) \xrightarrow{*} \emptyset_+$$

(32)
where \( h \) is the open embedding appearing on top of a Nisnevich square

\[
\begin{array}{ccc}
V & \xrightarrow{h} & Y \\
\downarrow & & \downarrow \\
U & \xrightarrow{j} & X
\end{array}
\]

in \( \mathcal{S}m/S \).

**Lemma A.16.** A pointed motivic space \( C \) is closed motivic fibrant if and only if the map \( C \to * \) has the right lifting property with respect to the set \( J^\text{cm}_S \).

**Proof.** This follows from adjointness, the Yoneda lemma and the construction of \( J^\text{cm}_S \).

**Theorem A.17.** The classes of motivic weak equivalences, closed motivic fibrations and closed cofibrations constitute a symmetric monoidal model structure on \( \text{M}^\cdot_s(S) \), denoted \( \text{M}^\text{cm}_s(S) \). The resulting homotopy category is denoted \( \text{H}^\text{cm}_s(S) \) and called the pointed motivic unstable homotopy category of \( S \). A morphism \( f : S \to S' \) of base schemes induces a strict symmetric monoidal left Quillen functor \( f^* : \text{M}^\text{cm}_s(S') \to \text{M}^\text{cm}_s(S) \).

**Proof.** The existence of the model structure follows by standard Bousfield localization techniques. Here are some details. The problem is that \( J^\text{cm}_S \) might be too small in order to characterize all closed motivic fibrations. Let \( \kappa \) be a regular cardinal strictly bigger than the cardinality of the set of morphisms in \( \mathcal{S}m/S \). A motivic space \( A \) is \( \kappa \)-bounded if the union

\[
\bigcup_{n \geq 0, X \in \mathcal{S}m/S} A(X)_n
\]

has cardinality \( \leq \kappa \). Let \( J^\text{cm}_S \) be a set of isomorphism classes of acyclic monomorphisms whose target is \( \kappa \)-bounded. One may show that given an acyclic monomorphism \( j : A \hookrightarrow B \) and a \( \kappa \)-bounded subobject \( C \subseteq B \), there exists a \( \kappa \)-bounded subobject \( C' \subseteq B \) containing \( C \) such that \( j^{-1}(C') \hookrightarrow C' \) is an acyclic monomorphism. Via Zorn’s lemma, one then gets that a map \( f \in \text{M}^\cdot_s(S) \) has the right lifting property with respect to all acyclic monomorphisms if (and only if) it has the right lifting property with respect to the set \( J^\text{cm}_S \). Such a map is in particular a closed motivic fibration. Any given map \( f : A \to B \) can now be factored (via the small object argument) as an acyclic monomorphism \( j : A \hookrightarrow C \) followed by a closed motivic fibration. Factoring \( j \) as a closed cofibration followed by an acyclic closed schemewise fibration in the model structure of Theorem A.9 implies the existence of the model structure.

To prove that the model structure is symmetric monoidal, it suffices – by the corresponding statement A.9 for \( \text{M}^\text{cm}_s(S) \) – to check that the pushout product of a generating closed cofibration and an acyclic closed cofibration is again a motivic equivalence. However, from the fact that the injective motivic model structure is symmetric monoidal, one knows that motivic equivalences are closed under smashing with arbitrary motivic spaces [DRÖ, Lemma 2.20]. The first sentence is now proven.

Concerning the third sentence, Theorem A.9 already implies that \( f^* \) preserves closed cofibrations. To prove that \( f^* \) is a left Quillen functor, it suffices (by Dugger’s lemma [D,
to check that it maps the set $J_{S}^{\text{cm}}$ to motivic weak equivalences in $\mathcal{M}_{\bullet}(S)$. One may calculate that $f^{*}(J_{S}^{\text{cm}}) = J_{S}^{\text{cm}}$, whence the statement.

The closed motivic model structure is cofibrantly generated. As remarked in the proof of Theorem A.17, the set $J_{S}^{\text{cm}}$ is perhaps not big enough to yield a full set of generating trivial cofibrations. Still the following Lemma, whose analog for the projective motivic model structure is [DRØ, Cor. 2.16], is valid.

**Lemma A.18.** Motivic equivalences and closed motivic fibrations with closed motivic fibrant codomain are closed under filtered colimits.

**Proof.** By localization theory [Hi, 3.3.16] and Lemma A.16, a map with closed motivic fibrant codomain is a closed motivic fibration if and only if it has the right lifting property with respect to $J_{S}^{\text{cm}}$. The domains and codomains of the maps in $J_{S}^{\text{cm}}$ are finitely presentable, which implies the statement about closed motivic fibrations with closed motivic fibrant codomain. The statement about motivic equivalences follows, because also the domains of the generating closed cofibrations in $I_{S}^{\text{cm}}$ are finitely presentable. See [DRØ2, Lemma 3.5] for details.

Let $\mathcal{M}_{\bullet}(S)$ be the category of simplicial objects in the category of pointed Nisnevich sheaves on $Sm/S$. The functor mapping a (pointed) presheaf to its associated (pointed) Nisnevich sheaf determines by degreewise application a functor $a_{\text{Nis}} : \mathcal{M}_{\bullet}(S) \to \mathcal{M}_{\bullet}(S)$. Let $\mathcal{M}_{\bullet}(S) \xrightarrow{i} \mathcal{M}_{\bullet}(S)$ be the inclusion functor, the right adjoint of $a_{\text{Nis}}$.

**Theorem A.19.** The pair $(a_{\text{Nis}}, i)$ is a Quillen equivalence to the Morel-Voevodsky model structure. The functor $a_{\text{Nis}}$ is strict symmetric monoidal. In particular, the total left derived functor of $a_{\text{Nis}}$ is a strict symmetric monoidal equivalence

$$H_{\bullet}^{\text{cm}}(S) \to H_{\bullet}(S)$$

to the unstable pointed $A^{1}$-homotopy category from [MV].

**Proof.** Recall that the cofibrations in the Morel-Voevodsky model structure are precisely the monomorphisms. Since every closed cofibration is a monomorphism and Nisnevich sheafification preserves these, $a_{\text{Nis}}$ preserves cofibrations. The unit $A \to i(a_{\text{Nis}}(A))$ of the adjunction is an isomorphism on all Nisnevich stalks, hence a motivic weak equivalence by Example A.14 for every motivic space $A$. In particular, $a_{\text{Nis}}$ maps schemewise weak equivalences as well as the maps in $J_{S}^{\text{cm}}$ described in (32) to weak equivalences. Let $\text{Id}_{\mathcal{M}_{\bullet}(S)} \to (-)^{\text{fib}}$ be the fibrant replacement functor in $\mathcal{M}_{\bullet}^{\text{cm}}(S)$ obtained from the small object argument applied to $J_{S}^{\text{cm}}$. Hence if $f$ is a motivic weak equivalence, then $f^{\text{fib}}$ is a schemewise weak equivalence. One concludes that $a_{\text{Nis}}$ preserves all weak equivalences, thus is a left Quillen functor. Since the unit $A \to i(a_{\text{Nis}}(A))$ is a motivic weak equivalence for every $A$, the functor $a_{\text{Nis}}$ is a Quillen equivalence.

**Note A.20.** Note that a map $f$ in $\mathcal{M}_{\bullet}(S)$ is a motivic weak equivalence if and only if $a_{\text{Nis}}(f)$ is a weak equivalence in the Morel-Voevodsky model structure on simplicial sheaves. Conversely, a map of simplicial sheaves is a weak equivalence if and only if it is a motivic weak equivalence when considered as a map of motivic spaces.

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Remark A.21. Starting with the injective model structure on simplicial presheaves mentioned in Example A.6, there is a model structure $\mathcal{M}^\text{im}_\bullet(S)$ on the category of pointed motivic spaces with motivic weak equivalences as weak equivalences and monomorphisms as cofibrations. It has the advantage that every object is cofibrant, but the disadvantage that it does not behave well under base change [MV, Ex. 3.1.22] or geometric realization (to be defined below). The identity functor is a left Quillen equivalence $\text{Id}: \mathcal{M}^\text{cm}_\bullet(S) \to \mathcal{M}^\text{im}_\bullet(S)$, since the homotopy categories coincide.

Further, let $\text{Spc}_\bullet(S)$ be the category of pointed Nisnevich sheaves on $\mathcal{S}m/S$. Recall the cosimplicial smooth scheme over $S$ whose value at $n$ is

$$\Delta^n_S = \text{Spec}(\mathcal{O}_S[X_0, \ldots, X_n]/(\sum_{i=0}^n X_i = 1))$$

The functor $\text{Spc}_\bullet(S) \to \mathcal{M}_\bullet(S)$ sending $A$ to the simplicial object $\text{Sing}_S(A)_n = A(- \times \Delta^n_S)$ has a left adjoint $|-|_S: \mathcal{M}_\bullet(S)$. It maps $B$ to the coend

$$|B|_S = \int_{n \in \Delta} B_n \times \Delta^n_S$$

in the category of pointed Nisnevich sheaves. The following statement is proved in [MV].

**Theorem A.22** (Morel-Voevodsky). There is a model structure on the category $\text{Spc}_\bullet(S)$ such that the pair $(|-|_S, \text{Sing}_S)$ is a Quillen equivalence to the Morel-Voevodsky model structure. The functor $|-|_S$ is strict symmetric monoidal. In particular, the total left derived functor of $|-|_S$ is a strict symmetric monoidal equivalence from Voevodsky’s pointed homotopy category to the unstable pointed $\mathbb{A}^1$-homotopy category.

### A.4 Topological realization

In the case where the base scheme is the complex numbers, there is a topological realization functor $R_\mathbb{C}: \mathcal{M}^\text{cm}_\bullet(\mathbb{C}) \to \text{Top}_\bullet$ which is a strict symmetric monoidal left Quillen functor. It is defined as follows. If $X \in \mathcal{S}m_\mathbb{C}$, the set $X(\mathbb{C})$ of complex points is a topological space when equipped with the analytic topology. Call this topological space $X^\text{an}$. It is a smooth manifold, and in particular a compactly generated topological space. One may view $X \mapsto X^\text{an}$ as a functor $\mathcal{S}m_\mathbb{C} \to \text{Top}$. Note that if $i: Z \hookrightarrow X$ is a closed embedding in $\mathcal{S}m_\mathbb{C}$, then the resulting map $i^\text{an}$ is the closed embedding of a smooth submanifold, and in particular a cofibration of compactly generated topological spaces. Every motivic space $A$ is a canonical colimit

$$\text{colim}_{X \times \Delta^n \to A} X \times \Delta^n \xrightarrow{\cong} A$$

and one defines

$$R_\mathbb{C}(A) := \text{colim}_{X \times \Delta^n \to A} X^\text{an} \times |\Delta^n| \in \text{Top}.$$ 

Observe that if $A$ is pointed, then so is $R_\mathbb{C}(A)$.  

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Theorem A.23. The functor $\mathbf{R}_C : \mathbf{M}^\cdot_{cm}(\mathbb{C}) \to \mathbf{Top}$ is a strict symmetric monoidal left Quillen functor.

Proof. The right adjoint of $\mathbf{R}_C$ maps the compactly generated pointed topological space $Z$ to the pointed motivic space $\text{Sing}_C(Z)$ which sends $X \in \mathbf{Sm}_C$ to the pointed simplicial set $\text{Sing}_C(Z \times X)$. To conclude that $\mathbf{R}_C$ is strict symmetric monoidal, it suffices to observe that there is a canonical homeomorphism $(X \times Y)^\text{an} \cong X^\text{an} \times Y^\text{an}$, and that geometric realization is strict symmetric monoidal.

Suppose now that $i : \bigcup_{j=1}^m Z_j \hookrightarrow X$ is an acceptable monomorphism. One computes $\mathbf{R}_C(\bigcup_{j=1}^m Z_j)$ as the coequalizer

$$\coprod_{j,j'} (Z_j \times_X Z_{j'})^\text{an} \rightrightarrows \coprod_{j=1}^m Z_j^\text{an}.$$ in $\mathbf{Top}$. Every map $(Z_j \times_X Z_{j'})^\text{an} \to Z_j^\text{an}$ is a closed embedding of smooth submanifolds of complex projective space. In particular, one may equip $Z_j^\text{an}$ with a cell complex structure such that $(Z_j \times_X Z_{j'})^\text{an}$ is a subcomplex for every $j'$. Then $\mathbf{R}_C(\bigcup_{j=1}^m Z_j)$ is the union of these subcomplexes, and in particular again a subcomplex. It follows that $\mathbf{R}_C(i)$ is a cofibration of topological spaces. Since $\mathbf{R}_C$ is compatible with pushout products, it maps the generating closed cofibrations to cofibrations of topological spaces.

To conclude that $\mathbf{R}_C$ preserves trivial cofibrations as well, it suffices by Dugger’s lemma [D, Cor. A2] to check that $\mathbf{R}_C$ maps every map in $J^m_{\mathbf{C}}$ to a weak homotopy equivalence. In fact, since the domains and codomains of the maps $\partial \Delta^m \hookrightarrow \Delta^m$ are cofibrant, it suffices to check the latter for the maps in diagram (32). In the first case, one obtains the map $X^\text{an} \hookrightarrow (\mathbb{A}^1_C \times X)^\text{an} \cong \mathbb{R}^2 \times X^\text{an}$, in the second case one obtains up to simplicial homotopy equivalence the canonical map $U^\text{an} \cup_{p^{-1}(V)^\text{an}} Y^\text{an} \to X^\text{an}$ for a Nisnevich square

This is in fact a homeomorphism of topological spaces. The result follows. \hfill \Box

Suppose now that $R \hookrightarrow \mathbb{C}$ is a subring of the complex numbers. Let $f : \text{Spec}(\mathbb{C}) \to \text{Spec}(R)$ denote the resulting morphism of base schemes. The realization with respect to $R$ (or better $f$) is defined as the composition

$$\mathbf{R}_R = \mathbf{R}_C \circ f^* : \mathbf{M}^\cdot(R) \to \mathbf{M}^\cdot(\mathbb{C}) \to \mathbf{Top}.$$ (34)

It is a strict symmetric monoidal Quillen functor. The most relevant case is $R = \mathbb{Z}$.

Example A.24. The topological realization of the Grassmannian $\text{Gr}(m,n)$ (over any base with a complex point) is the complex Grassmannian with the usual topology. Since $\mathbf{R}_C$ commutes with filtered colimits, $\mathbf{R}_C(\text{Gr})$ is the infinite complex Grassmannian, which in turn is the classifying space $B\mathbf{U}$ for the infinite unitary group. Because $\mathbf{R}_C$ is a left Quillen functor, the topological realization of any closed cofibrant motivic space weakly equivalent to $\text{Gr}$ is homotopy equivalent to $B\mathbf{U}$.  

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A.5 Spectra

**Definition A.25.** Let \( \mathbb{P}^1 \) denote the pointed projective line over \( S \). The category \( \text{MS}(S) \) of \( \mathbb{P}^1 \)-spectra over \( S \) has the following objects. A \( \mathbb{P}^1 \)-spectrum \( E \) consists of a sequence \((E_0, E_1, E_2, \ldots)\) of pointed motivic spaces over \( S \), and structure maps \( \sigma_n^E : E_n \wedge \mathbb{P}^1 \to E_{n+1} \) for every \( n \geq 0 \). A map of \( \mathbb{P}^1 \)-spectra is a sequence of maps of pointed motivic spaces which is compatible with the structure maps.

**Example A.26.** Any pointed motivic space \( B \) over \( S \) gives rise to a \( \mathbb{P}^1 \)-suspension spectrum

\[
\Sigma^\infty B = (B, B \wedge \mathbb{P}^1, B \wedge \mathbb{P}^1 \wedge \mathbb{P}^1, \ldots)
\]

having identities as structure maps. More generally, let \( \text{Fr}_n B \) denote the \( \mathbb{P}^1 \)-spectrum having values

\[
(\text{Fr}_n B)_{n+m} = \begin{cases} 
B \wedge \mathbb{P}^1 \wedge m & m \geq 0 \\
* & m < 0 
\end{cases}
\]

and identities as structure maps, except for \( \sigma_{n-1}^{\text{Fr}_n B} \). The functor \( B \mapsto \text{Fr}_n B \) is left adjoint to the functor sending the \( \mathbb{P}^1 \)-spectrum \( E \) to \( E_n \). We often write \( \Sigma^\infty \mathbb{P}^1 B (\cdot - n) \) for \( \text{Fr}_n B \).

For a \( \mathbb{P}^1 \)-spectrum \( E \) let \( u_n : \Sigma^\infty E_n (\cdot - n) \to E \) be a map of \( \mathbb{P}^1 \)-spectra adjoint to the identity map \( E_n \to E_n \).

**Remark A.27.** In Definition A.25, one may replace \( \mathbb{P}^1 \) by any pointed motivic space \( A \), giving the category \( \text{MS}_A(S) \) of \( A \)-spectra over \( S \). Essentially the only relevant example for us is when \( A \) is weakly equivalent to the pointed projective line \( \mathbb{P}^1 \). The Thom space \( T = A^1/A^1 - \{0\} \) of the trivial line bundle over \( S \) admits motivic weak equivalences

\[
\mathbb{P}^1 \sim \to \mathbb{P}^1/A^1 \sim \leftarrow T
\]

The motivic space \( \mathbb{P}^1 \) itself is not always the ideal suspension coordinate. For example, the algebraic cobordism spectrum \( \text{MGL} \) naturally comes as a \( T \)-spectrum. In order to switch between \( T \)-spectra and \( \mathbb{P}^1 \)-spectra, consider the following general construction. A map \( \phi : A \to B \) induces a functor \( \phi^* : \text{MS}_B(S) \to \text{MS}_A(S) \) sending the \( B \)-spectrum \((E_0, E_1, \ldots, \sigma_n^B)\) to the \( A \)-spectrum

\[
(E_0, E_1, \ldots, \sigma_n^B)
\]

with structure maps \( \sigma_n^{\phi^* E} = \sigma_n^E \circ (E_n \wedge \phi) \).

Its left adjoint \( \phi_* \) maps the \( A \)-spectrum \((F_0, F_1, \ldots, \sigma_n^A)\) to the \( B \)-spectrum

\[
(F_0, B \wedge F_0 \cup_{A \wedge F_0} F_1, B \wedge (B \wedge F_0 \cup_{A \wedge F_0} F_1) \cup_{A \wedge F_1} F_2, \ldots)
\]

having the canonical maps as structure maps. Note that for the purpose of constructing a model structure on \( A \)-spectra over \( S \), the pointed motivic space \( A \) has to be cofibrant in the model structure under consideration.

The next goal is to construct a model structure on \( \text{MS}(S) \) having the motivic stable homotopy category as its homotopy category.
Definition A.28. Let $\Omega_{P^1} = \text{Hom}_{\text{MS}^c(S)}(P^1, -)$ denote the right adjoint of $P^1 \land -$.

For a $P^1$-spectrum $E$ with structure maps $\sigma_n^E: E_n \land P^1 \to E_{n+1}$, let $\omega_n^E: E_n \to \Omega_{P^1}E_{n+1}$ denote the adjoint structure map. A $P^1$-spectrum $E$ is closed stably fibrant if

- $E_n$ is closed motivic fibrant for every $n \geq 0$, and
- $\omega_n^E: E_n \to \Omega_{P^1}E_{n+1}$ is a motivic weak equivalence for every $n \geq 0$.

Any $P^1$-spectrum $E$ admits a closed stably fibrant replacement. First replace $E$ by a levelwise fibrant $P^1$-spectrum $E^\ell$ as follows. Let $E^\ell_0 = E^\text{fib}_0$ for a fibrant replacement in $\text{MS}^c(S)$. Given $E_n \to E_{n+1}$, set

$$E^\ell_{n+1} = \left( E^\ell_n \land P^1 \cup_{E_n \land P^1} E_{n+1} \right)^\text{fib},$$

which yields a levelwise motivic weak equivalence $E \to E^\ell$ of $P^1$-spectra. To continue, observe that the adjoint structure maps of any $P^1$-spectrum $F$ may be viewed as a natural transformation

$$q: F \to Q(F),$$

where $Q(F)$ is the $P^1$-spectrum with terms $\Omega_{P^1}F_1, \Omega_{P^1}F_2, \ldots$ and structure maps

$$\Omega_{P^1}(\omega^F_{n+1}): \Omega_{P^1}F_{n+1} \to \Omega^2_{P^1}F_{n+2}.$$  

Define $Q^\infty(E)$ as the colimit of the sequence

$$E^\ell \xrightarrow{q} Q(E^\ell) \xrightarrow{Q(g)} Q^2(E^\ell) \longrightarrow \cdots.$$

Definition A.29. A map $f: E \to F$ of $P^1$-spectra is a stable equivalence if the map $Q^\infty(f)_n$ is a weak equivalence for every $n \geq 0$. It is a closed stable fibration if $f_n: E_n \to F_n$ is a closed motivic fibration and the induced map $E_n \to F_n \times_{Q^\infty(E)} Q^\infty(F)_n$ is a motivic weak equivalence for every $n \geq 0$. It is a closed cofibration if $f_n: E_n \to F_n$ and $F_n \land P^1 \cup_{E_n \land P^1} E_{n+1} \to F_{n+1}$ are closed cofibrations for every $n \geq 0$.

Theorem A.30. The classes from Definition A.29 are a model structure on the category of $P^1$-spectra, denoted $\text{MS}^c(S)$. The identity functor on $P^1$-spectra from Jardine’s stable model structure to $\text{MS}^c(S)$ is a left Quillen equivalence. In particular, the homotopy category $\text{Ho}(\text{MS}^c(S))$ is equivalent to the motivic stable homotopy category $\text{SH}(S)$.

Proof. Recall that $P^1$ is closed cofibrant by Lemma A.10. The existence of the model structure follows as in [J, Thm. 2.9]. Moreover, the stable equivalences coincide with the ones in [J], because so do the stabilization constructions and the unstable weak equivalences. Since every closed cofibration of motivic spaces is in particular a monomorphism, $\text{Id}_{\text{MS}(S)}$ is a left Quillen equivalence. Note that the closed cofibrations are generated by the set

$$\{ \text{Fr}_m(g) \}_{m \geq 0, g \in I_{S}}$$

with $I_{S}$ defined in (30). One may also describe a set of generating acyclic cofibrations. □
Remark A.31. We will identify the homotopy category \( \text{Ho}(\text{MS}^{\text{cm}}(S)) \) with \( \text{SH}(S) \) via the equivalence from Theorem A.30. Note that one may form the smash product of a motivic space \( A \) and a \( \mathbb{P}^1 \)-spectrum \( E \) by setting \( (A \wedge E)_n : = A \wedge E_n \) and \( \sigma^A_n \wedge E : = A \wedge \sigma^E_n \). If \( A \) is closed cofibrant, \( A \wedge - \) is a left Quillen functor by Theorem A.17. Because \( \mathbb{P}^1 \wedge - \) is a Quillen equivalence on Jardine’s stable model structure by results in [J, Sect. 3.4], Theorem A.30 implies that \( \mathbb{P}^1 \wedge - : \text{MS}^{\text{cm}}(S) \to \text{MS}^{\text{cm}}(S) \) is a left Quillen equivalence. Since \( \mathbb{P}^1 \simeq S^{2,1} = S^1 \wedge (A^1 - \{0\}, 1) \), also \( S^1 \wedge - \) is a left Quillen equivalence. It induces the shift functor in the triangulated structure on \( \text{SH}(S) \). The triangles are those which are isomorphic to the image of

\[
E \xrightarrow{f} F \xrightarrow{g} F/E \xleftarrow{\sim} E \wedge \Delta^1 \cup_E F \xrightarrow{h} S^1 \wedge E
\]

in \( \text{SH}(S) \), where \( f : E \to F \) is an inclusion of \( \mathbb{P}^1 \)-spectra. As well, one has sphere spectra \( S^{p,q} \in \text{SH}(S) \) for all integers \( p, q \in \mathbb{Z} \).

Example A.32. Since \( \text{SH}(S) \) is an additive category, the canonical map \( E \vee F \to E \times F \) is a stable equivalence. In the special case of \( \mathbb{P}^1 \)-suspension spectra, the canonical map factors as

\[
\Sigma_{\mathbb{P}^1}^\infty A \vee \Sigma_{\mathbb{P}^1}^\infty B \cong \Sigma_{\mathbb{P}^1}^\infty (A \vee B) \to \Sigma_{\mathbb{P}^1}^\infty (A \times B) \to \Sigma_{\mathbb{P}^1}^\infty A \times \Sigma_{\mathbb{P}^1}^\infty B
\]

which shows that \( \Sigma_{\mathbb{P}^1}^\infty (A \times B) \) contains \( \Sigma_{\mathbb{P}^1}^\infty (A \wedge B) \) as a retract in \( \text{SH}(S) \). Thus it is even a direct summand. The latter can be deduced as follows. The (reduced) join \( (A, a_0) * (B, b_0) \) is defined as the pushout in the diagram

\[
\begin{array}{ccc}
A \times B \times \partial \Delta^1 \cup \{a_0\} \times \{b_0\} \times \Delta^1 & \to & A \vee B \\
\downarrow & & \downarrow \\
A \times B \times \Delta^1 & \to & A * B.
\end{array}
\]

of pointed motivic spaces over \( S \). Attaching \( A \wedge (\Delta^1, 0) \) and \( B \wedge (\Delta^1, 0) \) to \( A * B \) via \( A \vee B \) produces a pointed motivic space \( C \) which is equipped with a sectionwise weak equivalence \( C \to (A \times B) \wedge S^{1,0} \). Collapsing \( \{a_0\} \ast B \) and \( A \ast \{b_0\} \) inside \( C \) yields a sectionwise weak equivalence \( C \to (A \wedge B \wedge S^{1,0} \vee (A \wedge S^{1,0} \vee (B \wedge S^{1,0}) \vee (S^{1,0} \wedge S^{1,0}) \wedge S^{1,0}) \). Since \( S^{1,0} \) is invertible in \( \text{SH}(S) \), one gets a splitting \( \Sigma_{\mathbb{P}^1}^\infty (A \times B) \simeq (\Sigma_{\mathbb{P}^1}^\infty (A \wedge B)) \vee (\Sigma_{\mathbb{P}^1}^\infty A) \vee (\Sigma_{\mathbb{P}^1}^\infty B) \) in \( \text{SH}(S) \).

For a \( \mathbb{P}^1 \)-spectrum \( E \) let \( \text{Tr}_n E \) denote the \( \mathbb{P}^1 \)-spectrum with

\[
(\text{Tr}_n E)_m = \begin{cases} E_m, & m \leq n \\ E_n \wedge (\mathbb{P}^1)^{m-n} = (\text{Fr}_n E_n)_m, & m \geq n \end{cases}
\]

and with the obvious structure maps. The structure maps of \( E \) determine maps \( \text{Tr}_n E \to \text{Tr}_{n+1} E \) such that \( E = \text{colim}_n \text{Tr}_n E \). The canonical map \( \text{Fr}_n E_n \to \text{Tr}_n E \) adjoint to the
identity \( \text{id}: E_n \to E_n \) is an identity in all levels \( \geq n \), and in particular a stable equivalence. The identity \( \text{id}: E_n \wedge (\mathbf{P}^1)^\wedge n \to (\text{Fr}_0E_n)_n \) leads by adjointness to the map

\[
\text{Fr}_n(E_n) \wedge (\mathbf{P}^1)^\wedge n \xrightarrow{\simeq} \text{Fr}_n(E_n \wedge (\mathbf{P}^1)^\wedge n) \to \text{Fr}_0E_n
\]  

(38)

and hence to a map \( \text{Fr}_nE_n \to \Omega^\infty_{\mathbf{P}^1}(\text{Fr}_0E_n) \). Since the map (38) is an isomorphism in all levels \( \geq n \), it is a stable equivalence. Because \( \Omega^\infty_{\mathbf{P}^1} \) is a Quillen equivalence, the map \( \text{Fr}_nE_n \to \Omega^\infty_{\mathbf{P}^1}((\text{Fr}_0E_n)^{\text{fib}}) \) is a stable equivalence as well if \( E_n \) is closed cofibrant. In fact, the condition on \( E_n \) can be removed since \( \text{Fr}_n \) preserves all weak equivalences. This leads to the following statement.

**Lemma A.33.** Any \( \mathbf{P}^1 \)-spectrum \( E \) is the colimit of a natural sequence

\[
\text{Tr}_0E \longrightarrow \text{Tr}_1E \longrightarrow \text{Tr}_2E \longrightarrow \cdots
\]  

(39)

of \( \mathbf{P}^1 \)-spectra in which the \( n \)-th term is naturally stably equivalent to \( \text{Fr}_nE_n = \Sigma^\infty_{\mathbf{P}^1}E_n(-n) \) from A.26, and also to \( \Omega^\infty_{\mathbf{P}^1}((\Sigma^\infty_{\mathbf{P}^1}E_n)^{\text{fib}}) \).

One may use the description in Lemma A.33 for computations as follows. Say that a \( \mathbf{P}^1 \)-spectrum \( F \) is *finite* if it is stably equivalent to a \( \mathbf{P}^1 \)-spectrum \( F' \) such that \( * \to E' \) is obtained by attaching finitely many cells from the set (37).

**Lemma A.34.** Let \( D(0) \to D(1) \to D(2) \to \cdots \) be a sequence of maps of \( \mathbf{P}^1 \)-spectra, with colimit \( D(\infty) \).

1. Suppose that \( F \) is a finite \( \mathbf{P}^1 \)-spectrum. The canonical map

\[
\colim_{i \geq 0} \text{Hom}_{\text{SH}(S)}(F, D(i)) \to \text{Hom}_{\text{SH}(S)}(F, D(\infty))
\]

is an isomorphism.

2. For any \( \mathbf{P}^1 \)-spectrum \( E \) there is a canonical short exact sequence

\[
0 \to \varprojlim_{i \geq 0} [S^1,0 \wedge D(i), E] \to [D(\infty), E] \to \varprojlim_{i \geq 0} [D(i), E] \to 0
\]

(40)

of abelian groups, where \([- , -] \) denotes \( \text{Hom}_{\text{SH}(S)}(-, -) \).

**Proof.** Observe first that stable equivalences and closed stable fibrations are detected by the functor \( Q^\infty \) which is defined in A.29 as a sequential colimit. Lemma A.18 implies that stable equivalences and closed stable fibrations with closed stably fibrant codomain are closed under filtered colimits. Thus by Theorem A.3 one may compute

\[
\text{Hom}_{\text{SH}(S)}(F, \colim_{i \geq 0} D(i)) \cong \text{Hom}_{\text{MS}(S)}(F, \colim_{i \geq 0} D(i)^{\text{fib}})/ \sim
\]

for any cofibrant \( \mathbf{P}^1 \)-spectrum \( F \), where \( \sim \) denotes the equivalence relation “simplicial homotopy”. This implies statement 1 because \( \text{Hom}_{\text{MS}(S)}(F, -) \) commutes with filtered colimits if \( * \to F \) is obtained by attaching finitely many cells.
To prove the second statement, let $C$ be the coequalizer of the diagram

$$
\bigvee_{i \geq 0} D(i) \xrightarrow{f} \bigvee_{i \geq 0} \Delta^1_+ \wedge D(i)
$$

where $f$ resp. $g$ is defined on the $i$-th summand $D(i)$ as $D(i) = 1_+ \wedge D(i) \hookrightarrow \Delta^1_+ \wedge D(i)$ resp. $D(i) \to D(i + 1) = 0_+ \wedge D(i + 1) \hookrightarrow \Delta^1_+ \wedge D(i + 1)$. The canonical map $C \to \operatorname{colim}_{i \geq 0} D(i)$ induced by the composition $\Delta^1_+ \wedge D(i) \to D(i) \to \operatorname{colim}_{i \geq 0} D(i)$ is a weak equivalence. In the stable homotopy category, which is additive, one may take the difference of $f$ and $g$, and thus describe $\operatorname{colim}_{i \geq 0} D(i)$ via the distinguished triangle

$$
\bigvee_{i \geq 0} D(i) \xrightarrow{f-g} \bigvee_{i \geq 0} D(i) \xrightarrow{0} \operatorname{colim}_{i \geq 0} D(i) \xrightarrow{0} \bigvee_{i \geq 0} S^{1,0} \wedge D(i). \tag{41}
$$

Applying $[-, E] : = \operatorname{Hom}_{\operatorname{SH}(S)}(-, E)$ to the triangle (41) produces a long exact sequence

$$
\ldots \xleftarrow{f-g} \prod_{i \geq 0} [D(i), E] \xhookleftarrow{[D(\infty), E]} \prod_{i \geq 0} [S^{1,0} \wedge D(i), E] \xhookrightarrow{[D(\infty), E]} \prod_{i \geq 0} [S^{1,0} \wedge D(i), E] \xleftarrow{[D(\infty), E]} \ldots
$$

which may be split into the short exact sequence

$$
0 \xhookrightarrow{\operatorname{lim}_{i \geq 0}} [D(i), E] \xhookleftarrow{[D(\infty), E]} \operatorname{lim}_{i \geq 0} [S^{1,0} \wedge D(i), E] \xhookrightarrow{[D(\infty), E]} 0.
$$

\[\square\]

### A.6 Symmetric spectra

There seems to be no reasonable (i.e. symmetric monoidal) smash product for $\mathbb{P}^1$-spectra inducing a decent symmetric monoidal smash product on $\operatorname{SH}(S)$. This will be solved as in [HSS] and [J].

**Definition A.35.** A symmetric $\mathbb{P}^1$-spectrum $E$ over $S$ consists of a sequence $(E_0, E_1, \ldots)$ of pointed motivic spaces over $S$, together with group actions $(\Sigma_n)_+ \wedge E_n \to E_n$ and structure maps $\sigma^E_n : E_n \wedge \mathbb{P}^1 \to E_{n+1}$ for all $n \geq 0$. Iterations of these structure maps are required to be as equivariant as they can, using the permutation action of $\Sigma_n$ on $(\mathbb{P}^1)^{\wedge n}$. A map of symmetric $\mathbb{P}^1$-spectra is a sequence of maps of pointed motivic spaces which is compatible with all the structure (group actions and structure maps). Call the resulting category $\operatorname{MSS}(S)$.

**Example A.36.** Analogous to Example A.26, the $n$-th shifted suspension spectrum $\operatorname{Fr}_n^\Sigma A$ of a pointed motivic space $A$ has as values

$$
(\operatorname{Fr}_n^\Sigma B)_{m+n} = \begin{cases} 
\Sigma_{m+n}^+ \wedge \Sigma_m \times \{1\} A \wedge (\mathbb{P}^1)^{\wedge m} & m \geq 0 \\
* & m < 0
\end{cases}
$$

where the $m$-th fold smash product $(\mathbb{P}^1)^{\wedge m}$ carries the natural permutation action.
Every symmetric $\mathbf{P}^1$-spectrum determines a $\mathbf{P}^1$-spectrum by forgetting the symmetric group actions. Call the resulting functor $u: \text{MSS}(S) \to \text{MS}(S)$. It has a left adjoint $v$, which is characterized uniquely up to unique isomorphism by the fact that

$$v(\text{Fr}_nA) = \text{Fr}_n^\Sigma A.$$  \hfill (42)

The smash product $E \wedge F$ of two symmetric $\mathbf{P}^1$-spectra $E$ and $F$ is constructed as follows. Set $(E \wedge F)_n$ as the coequalizer of the diagram

$$
\coprod_{r+1+s=n} \Sigma^+_n \wedge \Sigma_r \times \Sigma_s E_r \wedge \mathbf{P}^1 \wedge F_s \overset{\sigma^E_{r+s} \wedge F_s}{\longrightarrow} \coprod_{r+s=n} \Sigma^+_n \wedge \Sigma_r \times \Sigma_s E_r \wedge F_s
$$

where the coequalizer is taken in the category of pointed $\Sigma_n$-motivic spaces. The structure map $\sigma^E_{r+s}$ is induced by the structure maps $\sigma^F_r, \ldots, \sigma^F_n$ of $F$. One may provide natural coherence isomorphisms for associativity, commutativity and unitality, where the unit is $I_S = (S_+, \mathbf{P}^1, \mathbf{P}^1 \wedge \mathbf{P}^1, \ldots, (\mathbf{P}^1)^{\wedge n}, \ldots)$ with the obvious permutation action and identities as structure maps.

We proceed with the homotopy theory of symmetric $\mathbf{P}^1$-spectra, as in [J]. As one deduces from [Ho2, Thm. 7.2], there exists a cofibrant replacement functor $(-)^{\text{cof}} \to \text{Id}_{\text{MSS}(S)}$ for the model structure on symmetric $\mathbf{P}^1$-spectra with levelwise weak equivalences and levelwise fibrations.

**Definition A.37.** A map $\phi: E \to F$ of symmetric $\mathbf{P}^1$-spectra is a levelwise acyclic fibration if $\phi_n: E_n \to F_n$ is an acyclic closed motivic fibration of pointed motivic spaces over $S$ for all $n \geq 0$. A map $\phi: E \to F$ of symmetric $\mathbf{P}^1$-spectra is a closed cofibration if it has the left lifting property with respect to all levelwise acyclic fibrations. A levelwise fibrant symmetric $\mathbf{P}^1$-spectrum $E$ is closed stably fibrant if the adjoint $E_n \to \mathbf{M}_X(\mathbf{P}^1, E_{n+1})$ of the structure map is a weak equivalence for every $n \geq 0$. A map $\phi: E \to F$ is a stable equivalence if the map

$$\text{sSet}_{\text{MSS}_X}(\phi^{\text{cof}}, G): \text{sSet}_{\text{MSS}_X}(F^{\text{cof}}, G) \to \text{sSet}_{\text{MSS}_X}(E^{\text{cof}}, G)$$

is a weak equivalence of pointed motivic spaces for all closed stably fibrant symmetric $\mathbf{P}^1$-spectra $G$. The closed stable fibrations are then defined by the right lifting property.

**Theorem A.38** (Jardine). The classes of stable equivalences, closed cofibrations and closed stable fibrations from Definition A.37 constitute a symmetric monoidal model structure on $\text{MSS}(S)$. The forgetful functor $u: \text{MSS}^{\text{cm}}(S) \to \text{MS}^{\text{cm}}(S)$ is a right Quillen equivalence.

**Proof.** The proof of the first statement follows as in [J, Thm. 4.15, Prop. 4.19]. Note that the stable equivalences in $\text{MSS}^{\text{cm}}(S)$ and in Jardine’s model structure $\text{MSS}^{\text{Jar}}(S)$ coincide, since the unstable weak equivalences do so by Remark A.21. In particular, the identity $\text{Id}: \text{MSS}^{\text{cm}}(S) \to \text{MSS}^{\text{Jar}}(S)$ is a left Quillen equivalence. The closed cofibrations in $\text{MSS}^{\text{cm}}(S)$ are generated by the inclusions

$$\{\text{Fr}_n^\Sigma(g)\}_{m \geq 0, g \in I_S^\Sigma}$$  \hfill (44)
with $I_\Sigma$ defined in (30) By formula (42), the left adjoint $v$ of $u$ sends the generating cofibrations to the generating cofibrations. It follows that $v$ is a left Quillen functor. Since $v: \text{MS}^\text{lar}(S) \to \text{MSS}^\text{lar}(S)$ is a Quillen equivalence by [J, Thm. 4.31], so is the functor $v: \text{MS}^\text{sm}(S) \to \text{MSS}^\text{sm}(S)$. □

**Remark A.39.** Let $R_u: \text{SH}^u(S) := \text{Ho}(\text{MSS}^\text{sm}(S)) \to \text{SH}(S)$ be the total right derived functor of $u$, having $L_u$ as a left adjoint (and left inverse). Since $L_u$ is an equivalence, the category $\text{Ho}(\text{MS}^\text{sm}(S))$ inherits a closed symmetric monoidal product $\wedge$ by setting

$$E \wedge F := R_u(L_u(E) \wedge L_u(F))$$

In other words, if $E$ and $F$ are closed cofibrant $\mathbb{P}^1$-spectra, their smash product in $\text{SH}(S)$ is given by the $\mathbb{P}^1$-spectrum $u((v(E) \wedge v(F))^{\text{fib}})$. The unit is $R_u(L_u(\mathbb{I})) \cong \mathbb{I}$, the sphere $\mathbb{P}^1$-spectrum.

**Notation A.40.** For $\mathbb{P}^1$-spectra $E$ and $F$ over $S$ define the $E$-cohomology and the $E$-homology of $F$ as

$$E^{p,q}(F) = \text{Hom}_{\text{SH}(S)}(F, S^{p,q} \wedge E)$$

$$E_{p,q}(F) = \text{Hom}_{\text{SH}(S)}(S^{p,q}, F \wedge E)$$

(45) (46)

for all $p, q \in \mathbb{Z}$. In the special case $F = \Sigma_{\mathbb{P}^1}^\infty A$, where $A$ is a pointed motivic space over $S$ one writes $E^{p,q}(A)$ and $E_{p,q}(A)$ instead. Note that there is an isomorphism $\text{Fr}_1 A \cong S^{-2n,-i} \wedge \text{Fr}_0 A$ in $\text{SH}(S)$.

**Remark A.41.** Since $(v, u)$ is a Quillen adjoint pair of stable model categories the total derived pair respects in particular the triangulated structures. The functor $u$ preserves all colimits, thus both $L_v$ and $R_u$ preserve arbitrary coproducts.

**Lemma A.42.** Let $E$ and $F$ be $\mathbb{P}^1$-spectra. Then $E \wedge F \in \text{SH}(S)$ may be obtained as the sequential colimit of a sequence whose $n$-th term is stably equivalent to $\Omega_{\mathbb{P}^1}^{2n}(\Sigma_{\mathbb{P}^1}^\infty E_n \wedge F_n)^{\text{fib}}$. Thus $E \wedge F \cong \text{hocolim}\Sigma_{\mathbb{P}^1}^\infty (E_n \wedge F_n)(-2n)$ in $\text{SH}(S)$.

**Proof.** Here $E \wedge F \in \text{SH}(S)$ refers to the smash product of symmetric $\mathbb{P}^1$-spectra associated to closed cofibrant replacements $E^{\text{cof}} \to E$ and $F^{\text{cof}} \to F$. Because these two maps are levelwise weak equivalences, we may assume that both $E$ and $F$ are closed cofibrant. By Lemma A.33 $E$ and $F$ can be expressed as sequential colimits of their truncations. Since $v$ preserves colimits, $v(E)$ is the sequential colimit of the diagram

$$v(\text{Tr}_0 E) = v(\text{Fr}_0 E_0) = \text{Fr}_0^\Sigma E_0 \to v(\text{Tr}_1 E) \to \cdots$$

and similarly for $F$. The stable equivalence $\text{Fr}_n E_n \to \text{Tr}_n E$ of cofibrant $\mathbb{P}^1$-spectra induces a stable equivalence $v(\text{Fr}_n E_n) = \text{Fr}_n^\Sigma E_n \to v(\text{Tr}_n E)$. Since smashing with a symmetric $\mathbb{P}^1$-spectrum preserves colimits, one has

$$v(\text{Tr}_m E) \wedge \text{colim}_n v(\text{Tr}_n F) \cong \text{colim}_n (v(\text{Tr}_m E) \wedge v(\text{Tr}_n F))$$
for every $n$. It follows that $v(E) \land v(F)$ is the filtered colimit of the diagram sending $(m, n)$ to $v(\text{Tr}_nE) \land v(\text{Tr}_nF)$. Since the diagonal is a final subcategory in $\mathbb{N} \times \mathbb{N}$, there is a canonical isomorphism $\text{colim}_n v(\text{Tr}_nE) \land v(\text{Tr}_nF) \cong v(E) \land v(F)$. Theorem A.38 says that $\text{MSS}^\text{cm}(S)$ is symmetric monoidal, thus the canonical map

$$\text{Fr}_n^\Sigma(E_n \land F_n) \cong \text{Fr}_n^\Sigma E_n \land \text{Fr}_n^\Sigma F_n \to v(\text{Tr}_nE) \land v(\text{Tr}_nF) \tag{47}$$

is a stable equivalence. Let $\text{Fr}_2^\Sigma(E_n \land F_n) \to \Omega_{\text{Fr}_1^\Sigma(E_n \land F_n)}^2$ be the canonical map which is adjoint to the unit $E_n \land F_n \to (\text{Fr}_1^\Sigma(E_n \land F_n))^\text{fib}$. As in the case of $\mathbb{P}^1$-spectra, the map

$$\text{Fr}_2^\Sigma(E_n \land F_n) \to \Omega_{\text{Fr}_1^\Sigma(E_n \land F_n)}^2 \to (\text{Fr}_0^\Sigma(E_n \land F_n))^\text{fib} \tag{fib}$$

is a stable equivalence. It follows that $v(E) \land v(F)$ is the colimit of a sequence whose $n$-th term is stably equivalent to $\Omega_{\text{Fr}_1^\Sigma(E_n \land F_n)}^2$. Hence it also follows that a fibrant replacement of $v(E) \land v(F)$ may be obtained as the colimit of a sequence of closed stably fibrant symmetric $\mathbb{P}^1$-spectra whose $n$-th term is stably equivalent to $\Omega_{\text{Fr}_1^\Sigma(E_n \land F_n)}^2$. Since the forgetful functor $u$ preserves colimits, stable equivalences of closed stably fibrant symmetric $\mathbb{P}^1$-spectra and $\Omega_{\text{Fr}_1^\Sigma(E_n \land F_n)}$, the $\mathbb{P}^1$-spectra $u(v(E) \land v(F))$ is the colimit of a sequence of $\mathbb{P}^1$-spectra whose $n$-th term is stably equivalent to $\Omega_{\text{Fr}_1^\Sigma(E_n \land F_n)}^2 u((\text{Fr}_0^\Sigma(E_n \land F_n))^\text{fib})$. The map

$$\text{Fr}_0(E_n \land F_n) \to u((v\text{Fr}_0(E_n \land F_n))^\text{fib})$$

is a stable equivalence because $(v, u)$ is a Quillen equivalence A.38, whence the result. \qed

As in the case of non-symmetric spectra, one may change the suspension coordinate as in Remark A.27. If $A$ is a pointed motivic space over $S$, let $\text{MSS}_A(S)$ denote the category of symmetric $A$-spectra over $S$.

**Lemma A.43.** A map $A \to B$ in $\text{MSS}_A(S)$ induces a strict symmetric monoidal functor $\text{MSS}_A(S) \to \text{MSS}_B(S)$ having a right adjoint. If the map is a motivic weak equivalence of closed cofibrant pointed motivic spaces, this pair is a Quillen equivalence.

**Proof.** This is quite formal. For a proof consider [Ho2, Thm. 9.4]. \qed

Because the change of suspension coordinate functors are lax symmetric monoidal, they preserve (commutative) monoid objects, that is, (commutative) symmetric ring spectra. Recall that a functor is lax symmetric monoidal if it is equipped with natural structure maps as in (28) which are not necessarily isomorphisms.

### A.7 Stable topological realization

Let $\text{Sp} = \text{Sp}(\text{Top}, \mathbb{CP}^1)$ be the category of $\mathbb{CP}^1$-spectra (in $\text{Top}$). An object in $\text{Sp}$ is thus a sequence of pointed compactly generated topological spaces $E_0, E_1, \ldots$ with structure maps $E_n \land \mathbb{CP}^1 \to E_{n+1}$. The model structure on $\text{Sp}$ is obtained as follows: Cofibrations are generated by

$$\{\text{Fr}_m^\text{Top}(\partial \Delta^n \hookrightarrow \Delta^n_{+})\}_{m,n \geq 0}$$

42
so that every $\mathbb{C}P^1$-spectrum $E$ has a cofibrant replacement $E^{cof} \to E$ mapping to $E$ via a levelwise acyclic Serre fibration. For any $\mathbb{C}P^1$-spectrum $E$ and any $n \in \mathbb{Z}$ let $\pi_n E$ be the colimit of the sequence

$$\pi_{n+m} E_m \to \pi_{n+2m} E_m \wedge \mathbb{C}P^1 \to \pi_{n+2(m+1)} E_{1+m} \to \cdots$$

where $m \geq 0$ and $n + 2m \geq 0$. It is called the $n$-th stable homotopy group of $E$. Note that homotopy groups of non-degenerately based compactly generated topological spaces commute with filtered colimits. If $E$ is cofibrant, $E_n$ is in particular non-degenerately based for all $n \geq 0$. A map $f : E \to F$ of $\mathbb{C}P^1$-spectra is a stable equivalence if the induced map $\pi_n f : \pi_n E^{cof} \to \pi_n F^{cof}$ is an isomorphism for all $n \in \mathbb{Z}$. It is a stable fibration if it has the right lifting property with respect to all stable acyclic cofibrations.

Similarly, one may form the category $\text{Sp}^\Sigma = \text{Sp}^\Sigma(\text{Top}, \mathbb{C}P^1)$ of symmetric $\mathbb{C}P^1$-spectra in $\text{Top}$. Cofibrations are generated by

$$\{\text{Fr}_m^{\text{Top}, \Sigma}(\partial \Delta^n \hookrightarrow \Delta^n)_{m \geq 0}\}$$

and a symmetric $\mathbb{C}P^1$-spectrum is stably fibrant if its underlying $\mathbb{C}P^1$-spectrum is stably fibrant. A map $f : E \to F$ of symmetric $\mathbb{C}P^1$-spectra is a stable equivalence if the induced map $\text{sSet}_{\text{Sp}^\Sigma}(f^{cof}, G)$ of simplicial sets of maps is an isomorphism for all stably fibrant symmetric $\mathbb{C}P^1$-spectra $G$. It is a stable fibration if it has the right lifting property with respect to all stable acyclic cofibrations.

**Theorem A.44.** Stable equivalences, stable fibrations and cofibrations form (symmetric monoidal) model structures on the categories of (symmetric) $\mathbb{C}P^1$-spectra in $\text{Top}$. The functor forgetting the symmetric group actions is a right Quillen equivalence. There is a zig-zag of strict symmetric monoidal left Quillen functors connecting $\text{Sp}^\Sigma(\text{Top}, \mathbb{C}P^1)$ and $\text{Sp}^\Sigma(\text{Top}, S^1)$. In particular, the homotopy category of (symmetric) $\mathbb{C}P^1$-spectra is equivalent as a closed symmetric monoidal and triangulated category to the stable homotopy category.

**Proof.** The statement about the model structures follows as in [HSS] if one replaces $S^1$ by $\mathbb{C}P^1$ and simplicial sets by compactly generated topological spaces. The same holds for the statement about the functor forgetting the symmetric group actions. To construct the zig-zag, consider the corresponding stable model structure on the category of symmetric $S^1$-spectra in the category $\text{Sp}^\Sigma(\text{Top}, \mathbb{C}P^1)$, which is isomorphic as a symmetric monoidal model category to the category of symmetric $\mathbb{C}P^1$-spectra in the category $\text{Sp}^\Sigma(\text{Top}, S^1)$ of topological symmetric $S^1$-spectra. The suspension spectrum functors give a zig-zag

$$\begin{align*}
\text{Sp}^\Sigma(\text{Top}, \mathbb{C}P^1) \downarrow & \downarrow \\
\text{Sp}^\Sigma(\text{Sp}^\Sigma(\text{Top}, \mathbb{C}P^1), S^1) & \xrightarrow{\cong} \text{Sp}^\Sigma(\text{Sp}^\Sigma(\text{Top}, S^1), \mathbb{C}P^1)
\end{align*}$$

(48)

of strict symmetric monoidal left Quillen functors. Since $\mathbb{C}P^1 \wedge -$ is a left Quillen equivalence on the left hand side in the zig-zag (48) and $S^1 \wedge S^1 \cong \mathbb{C}P^1$, $S^1 \wedge -$ is a left Quillen
equivalence on the left hand side as well. By [Ho2, Thm. 9.1], the arrow pointing to the right in the zig-zag (48) is a Quillen equivalence. A similar argument works for the arrow on the right hand side, which completes the proof.

Given a $P^1$-spectrum $E$ over $\mathbb{C}$, one gets a $\mathbb{C}P^1$-spectrum $R_\mathbb{C}(E) = (R_\mathbb{C}E_0, R_\mathbb{C}E_1, \ldots)$ with structure maps $R_\mathbb{C}(E_n) \wedge P^1 \cong R_\mathbb{C}(E_n \wedge P^1) \to R_\mathbb{C}(E_{n+1})$. The right adjoint for the resulting functor $R_\mathbb{C}: MS(\mathbb{C}) \to Sp$ is also obtained by a levelwise application of $Sing_\mathbb{C}$. The same works for symmetric $P^1$-spectra over $\mathbb{C}$.

**Theorem A.45.** The functors $R_\mathbb{C}: MS(\mathbb{C}) \to Sp$ and $R_\mathbb{C}: MSS(\mathbb{C}) \to Sp^\Sigma$ are left Quillen functors, the latter being strict symmetric monoidal.

**Proof.** Since the diagrams commute, $R_\mathbb{C}$ preserves the generating cofibrations by A.23. Then Dugger’s Lemma [D, Cor. A.2] implies that $R_\mathbb{C}$ is a left Quillen functor, because $Sing_\mathbb{C}$ preserves weak equivalences and fibrations between fibrant objects. The fact that $R_\mathbb{C}: MSS(\mathbb{C}) \to Sp^\Sigma$ is strict symmetric monoidal follows from the definition of the smash product (43).

**Example A.46.** Let $BGL$ be the $P^1$-spectrum over $\mathbb{C}$ constructed in 2.2. Its $n$-th term is a pointed motivic space $K$ weakly equivalent to $\mathbb{Z} \times Gr$. One may assume that $K$ is closed cofibrant. Then by Theorem A.45 the $n$-th term of $R_\mathbb{C}(BGL)$ is weakly equivalent to $BU$. To show that the $\mathbb{C}P^1$-spectrum $R_\mathbb{C}(BGL)$ is the one representing complex $K$-theory, it suffices to check that the structure map $K \wedge P^1 \to K$ realizes to the structure map $BU \wedge \mathbb{C}P^1 \to BU$ of complex $K$-theory. Consider the diagram

\[
\begin{array}{ccc}
\text{Hom}_{H_\bullet(\mathbb{C})}(\mathbb{Z} \times \text{Gr}, \mathbb{Z} \times \text{Gr}) & \overset{\cong}{\longrightarrow} & K_0^{\text{alg}}(\mathbb{Z} \times \text{Gr}) \\
\downarrow & & \downarrow \\
\text{Hom}_{\text{Ho}(\text{Top}_\bullet)}(R_\mathbb{C}(\mathbb{Z} \times \text{Gr}), R_\mathbb{C}(\mathbb{Z} \times \text{Gr})) & \overset{\cong}{\longrightarrow} & K_0^{\text{top}}(R_\mathbb{C}(\mathbb{Z} \times \text{Gr}))
\end{array}
\]

where the vertical map on the left hand side is induced by $R_\mathbb{C}$ and the vertical map on the right hand side is induced by the passage from algebraic to topological complex vector bundles. The upper horizontal isomorphism sends the identity to the class $\xi_\infty$ described in Remark 2.18 via tautological vector bundles over Grassmannians. The right vertical map sends $\xi_\infty$ to the class $\zeta_\infty$ obtained via the corresponding tautological bundles, viewed as complex topological vector bundles. Since $\zeta_\infty$ is the image of $id_{R_\mathbb{C}(\mathbb{Z} \times \text{Gr})}$ under the lower horizontal isomorphism, the diagram commutes at the identity. By naturality, it follows
that the diagram

\[
\begin{array}{ccc}
\text{Hom}_{\text{Ho}(\mathbb{C})}(A, \mathbb{Z} \times \text{Gr}) & \xrightarrow{\cong} & K^\text{alg}_0(A) \\
\downarrow & & \downarrow \\
\text{Hom}_{\text{Ho}(\text{Top}^*)}(\mathbb{R}_C(A), \mathbb{R}_C(\mathbb{Z} \times \text{Gr})) & \xrightarrow{\cong} & K^\text{top}_0(\mathbb{R}_C(A))
\end{array}
\]

commutes for every pointed motivic space \( A \) over \( \mathbb{C} \). In particular, the structure map of BGL which corresponds to \( (\xi_\infty) \otimes ([0] - [0]) \) maps to the structure map of the complex K-theory spectrum, since it corresponds to the “same” class, viewed as a difference of complex topological vector bundles.

**Proposition A.47.** A morphism \( f : S \to S' \) of base schemes induces a strict symmetric monoidal left Quillen functor

\[ f^* : \text{MSS}(S') \to \text{MSS}(S) \]

such that \((f^*(E))_n = f^*(E_n)\).

*Proof.* The structure maps of \( f^*(E) \) are defined via the canonical map

\[ f^*(E)_n \wedge \mathbb{P}^1_{S'} \cong f^*(E_n \wedge \mathbb{P}^1_S) \xrightarrow{f^*(\sigma^n)} f^*(E_{n+1}) = f^*(E)_{n+1}. \]

It follows that \( f^* \) has the functor \( f_* \) as right adjoint, where \((f_*(E))_n = f_*(E_n)\) and

\[ \sigma^n f_* E = f_*(E_n \wedge \mathbb{P}^1_{S'}) \to f_*(E_n \wedge f_* f^* \mathbb{P}^1_{S'}) \xrightarrow{\cong} f_*(E_n \wedge f_* \mathbb{P}^1_{S'}) \]

\[ f_*(E)_{n+1} \xleftarrow{f^* \sigma^n} f_*(E_n \wedge \mathbb{P}^1_S) \]

Theorem A.17 and Dugger’s lemma imply that \( f^* \) preserves cofibrations and \( f_* \) preserves fibrations. Since \( f^* : \text{M}_*(S') \to \text{M}_*(S) \) is strict symmetric monoidal and preserves all colimits, then so is \( f^* : \text{MSS}(S') \to \text{MSS}(S) \) by the definition of the smash product (43).

\[ \square \]

In particular, any complex point \( f : \text{Spec}(\mathbb{C}) \to S \) of a base scheme \( S \) induces a strict symmetric monoidal left Quillen functor

\[ \text{MSS}(S) \to \text{MSS}(\mathbb{C}) \to \text{Sp}^\Sigma(\text{Top}, \mathbb{C}\mathbb{P}^1) \] (49)

to the category of topological \( \mathbb{C}\mathbb{P}^1 \)-spectra.
B Some results on K-theory

B.1 Cellular schemes

Suppose that \( S \) is a regular base scheme. Recall that an \( S \)-cellular scheme is an \( S \)-scheme \( X \) equipped with a filtration \( X_0 \subset X_1 \subset \cdots \subset X_n = X \) by closed subsets such that for every integer \( i \geq 0 \) the \( S \)-scheme \( X_i \setminus X_{i-1} \) is a disjoint union of several copies of the affine space \( A^i_S \). We do not assume that \( X \) is connected. A pointed \( S \)-cellular scheme is an \( S \)-scheme equipped with a closed \( S \)-point \( x : S \to X \) such that \( x(S) \) is contained in one of the open cells (a cell which is an open subscheme of \( X \)). The examples we are interested in are Grassmannians, projective lines and their products.

**Lemma B.1.** Let \((X, x)\) and \((Y, y)\) be pointed motivic spaces. Then the sequence

\[
0 \to K_i(X \wedge Y) \to K_i(X \times Y) \to K_i(X \vee Y) \to 0
\]

is short exact and the natural map

\[
K_i(X) \oplus K_i(Y) \to K_i(X \vee Y)
\]

is an isomorphism.

**Proof.** The exactness of the sequence follows from the isomorphism \( K_i \cong \text{BGL}^{-i,0} \) and Example A.32. The isomorphism is formal, given the isomorphism \( K_i \cong \text{BGL}^{-i,0} \).

**Corollary B.2.** Let \((X, x)\) and \((Y, y)\) be pointed smooth \( S \)-schemes. Let \( a \in K_0(X) \) and \( b \in K_0(Y) \) be such that \( x^*(a) = 0 = y^*(b) \) in \( K_0(S) \). Then the element \( a \otimes b \in K_0(X \times Y) \) belongs to the subgroup \( K_0(X \wedge Y) \).

**Proof.** Since \( a \otimes b \) vanishes on \( x(S) \times Y \) and on \( X \times x(S) \) it follows that \( a \otimes b \) vanishes on \( X \vee Y \). Whence \( a \otimes b \in K_0(X \wedge Y) \) by Lemma B.1.

We list further useful statements.

**Lemma B.3.** Let \( X \) be a smooth \( S \)-cellular scheme. Then the map

\[
K_r(S) \otimes_{K_0(S)} K_0(X) \to K_r(X)
\]

is an isomorphism and \( K_0(X) \) is a free \( K_0(S) \)-module of rank equal to the number of cells.

The Lemma easily follows from a slightly different claim which we consider as a well-known one.

**Claim B.4.** Under the assumption of the Lemma the map of Quillen’s \( K \)-groups

\[
K_r(S) \otimes_{K_0(S)} K'_0(X_j) \to K'_r(X_j)
\]

is an isomorphism and \( K'_0(X_j) \) is a free \( K_0(S) \)-module of the expected rank.
Lemma B.5. Let $(X, x)$ and $(Y, y)$ be pointed smooth $S$-cellular schemes. Then the map

$$K_i(S) \otimes_{K_0(S)} K_0(X \vee Y) \to K_i(X \vee Y)$$

is an isomorphism and $K_0(X \vee Y)$ is a projective $K_0(S)$-module.

Lemma B.6. Let $(X, x)$ and $(Y, y)$ be pointed smooth $S$-cellular schemes. Then the map

$$K_i(S) \otimes_{K_0(S)} K_0(X \wedge Y) \to K_i(X \wedge Y)$$

is an isomorphism and $K_0(X \wedge Y)$ is a projective $K_0(S)$-module.

Proof. Consider the commutative diagram

$$
\begin{array}{ccc}
K_i(X \wedge Y) & \overset{\alpha}{\longrightarrow} & K_i(X \times Y) \\
\uparrow{\epsilon} & & \uparrow{\rho} \\
K_i(S) \otimes K_0(X \wedge Y) & \overset{\gamma}{\longrightarrow} & K_i(S) \otimes K_0(X \times Y) \\
\downarrow{\delta} & & \downarrow{\theta} \\
K_i(S) \otimes K_0(X \vee Y) & \overset{\epsilon}{\longrightarrow} & K_i(X \vee Y)
\end{array}
$$

in which $K_i$ is written for $K_i$ and the tensor product is taken over $K_0(S)$.

The sequence

$$0 \to K_i(X \wedge Y) \overset{\alpha}{\longrightarrow} K_i(X \times Y) \overset{\beta}{\longrightarrow} K_i(X \vee Y) \to 0$$

is short exact by Lemma B.1. In particular it is short exact for $i = 0$. Now the sequence

$$0 \to K_i(S) \otimes K_0(X \wedge Y) \overset{\gamma}{\longrightarrow} K_i(S) \otimes K_0(X \times Y) \overset{\delta}{\longrightarrow} K_i(S) \otimes K_0(X \vee Y) \to 0$$

is short exact since $K_0(X \vee Y)$ is a projective $K_0(S)$-module.

The arrows $\rho$ and $\theta$ are isomorphisms by Lemmas B.3 and B.5 respectively. Whence $\epsilon$ is an isomorphism as well. Finally $K_0(X \wedge Y)$ is a projective $K_0(S)$-module since the sequence $0 \to K_0(X \wedge Y) \to K_0(X \times Y) \to K_0(X \vee Y) \to 0$ is short exact and $K_0(X \times Y)$ and $K_0(X \vee Y)$ are projective $K_0(S)$-modules.

As well we need to know that certain $\lim^1$-groups vanish. Given a set $M$ and a smooth $S$-scheme $X$, we write $M \times X$ for the disjoint union $\bigsqcup_M X$ of $M$ copies of $X$ in the category of motivic spaces over $S$. Recall that $[-n, n]$ is the set of integers with absolute value $\leq n$.

Lemma B.7.

$$\lim^1 K_i(\Gr(n, 2n)) = 0$$

$$\lim^1 K_i([-n, n] \times \Gr(n, 2n)) = 0$$

$$\lim^1 K_i([-n, n] \times \Gr(n, 2n) \times [-n, n] \times \Gr(n, 2n)) = 0$$

Proof. This holds since all the bondings map in the towers are surjective. \qed
Lemma B.8. The canonical maps

\[ K_i(\text{Gr}) \to \limleftarrow K_i(\text{Gr}(n, 2n)) \]
\[ K_i(\text{Gr} \times \text{Gr}) \to \limleftarrow K_i(\text{Gr}(n, 2n) \times \text{Gr}(n, 2n)) \]

are isomorphisms. A similar statement holds for the pointed motivic spaces \( \mathbb{Z} \times \text{Gr}, (\mathbb{Z} \times \text{Gr}) \times \mathbb{P}^1, (\mathbb{Z} \times \text{Gr}) \times (\mathbb{Z} \times \text{Gr}), (\mathbb{Z} \times \text{Gr}) \times \mathbb{P}^1 \times (\mathbb{Z} \times \text{Gr}) \times \mathbb{P}^1 \).

Proof. This follows from Lemma B.7.

Lemma B.9. The canonical maps

\[ K_i(\text{Gr} \wedge \text{Gr}) \to \limleftarrow K_i(\text{Gr}(n, 2n) \wedge \text{Gr}(n, 2n)) \]
\[ K_i(\text{Gr} \wedge \text{Gr} \wedge \mathbb{P}^1) \to \limleftarrow K_i(\text{Gr}(n, 2n) \wedge \text{Gr}(n, 2n) \wedge \mathbb{P}^1) \]

are isomorphisms. A similar statement holds for the pointed motivic spaces \( (\mathbb{Z} \times \text{Gr}) \wedge (\mathbb{Z} \times \text{Gr}) \), \( (\mathbb{Z} \times \text{Gr}) \wedge (\mathbb{Z} \times \text{Gr}) \wedge \mathbb{P}^1 \) and \( (\mathbb{Z} \times \text{Gr}) \wedge \mathbb{P}^1 \wedge (\mathbb{Z} \times \text{Gr}) \wedge \mathbb{P}^1 \).

Proof. It follows immediately from Lemma B.8 and Lemma B.1.

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