THE THIRD HOMOLOGY OF $\text{SL}_2(\mathbb{Q})$

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Abstract. We calculate the structure of $H_3\left(\text{SL}_2(\mathbb{Q}), \mathbb{Z}\left[\frac{1}{5}\right]\right)$. Let $H_3\left(\text{SL}_2(\mathbb{Q}), \mathbb{Z}\right)_0$ denote the kernel of the (split) surjective homomorphism $H_3\left(\text{SL}_2(\mathbb{Q}), \mathbb{Z}\right) \to K^\text{ind}_3(\mathbb{Q})$. Each prime number $p$ determines an operator $\langle p \rangle$ on $H_3\left(\text{SL}_2(\mathbb{Q}), \mathbb{Z}\right)$ with square the identity. $H_3\left(\text{SL}_2(\mathbb{Q}), \mathbb{Z}\left[\frac{1}{2}\right]\right)_0$ is the direct sum of the $(-1)$-eigenspaces of these operators. The $(-1)$-eigenspaces of $\langle p \rangle$ is a cyclic group whose order is the odd part of $p + 1$. We explore some applications to the groups $H_3\left(\text{SL}_2(\mathbb{Z}\left[\frac{1}{5}\right]), \mathbb{Z}\left[\frac{1}{2}\right]\right)$.

1. Introduction

Many years ago, in an article on the homology of Lie groups made discrete, Chi-Han Sah, quoting S. Lichtenbaum, cited our lack of any precise knowledge of the structure of $H_3\left(\text{SL}_2(\mathbb{Q}), \mathbb{Z}\right)$ as an example of the poor state of understanding of the homology of linear groups of general fields (see [12] pp 307-8). Where such understanding does exist, even now, it tends often to come from connections with algebraic $K$-theory or Lie group theory where a bigger suite of mathematical tools is available. For example, we know the structure of $H_3\left(\text{SL}_3(\mathbb{Q}), \mathbb{Z}\right)$ because homology stability theorems tell us that it is isomorphic to $H_3\left(\text{SL}_n(\mathbb{Q}), \mathbb{Z}\right)$ for all larger $n$ ([8]) and this stable homology group is in turn isomorphic, via a Hurewicz homomorphism, to $K_3(\mathbb{Q})/\{-1\} \cdot K_2(\mathbb{Q}) = K^\text{ind}_3(\mathbb{Q})$ (indecomposable $K_3$) by [14] Lemma 5.2, which is known to be cyclic of order 24 by the result of Lee and Szczarba ([9]).

For any field $F$, the natural map $H_3\left(\text{SL}_2(F), \mathbb{Z}\right) \to H_3\left(\text{SL}_3(F), \mathbb{Z}\right)$ is an isomorphism, via a Hurewicz homomorphism, to $K_3(F)/\{-1\} \cdot K_2(F) = K^\text{ind}_3(F)$ can be shown to be surjective ([8]). When $F = \mathbb{C}$, or more generally when $F$ is algebraically closed, it has long been known, thanks to the work of Sah and his co-authors, that this map is an isomorphism. When $F$ is a number field, or a global function field, the map $H_3\left(\text{SL}_3(F), \mathbb{Z}\right) \to K^\text{ind}_3(F)$ is an isomorphism and the issue at stake therefore is the kernel of the stability homomorphism $H_3\left(\text{SL}_2(F), \mathbb{Z}\right) \to H_3\left(\text{SL}_3(F), \mathbb{Z}\right)$.

One natural obstruction to the injectivity or surjectivity of the stability homomorphisms $H_*(\text{SL}_n(F), \mathbb{Z}) \to H_*(\text{SL}_{n+1}(F), \mathbb{Z})$ lies in the action of the multiplicative group $F^\times$: For any $a \in F^\times$ conjugation on $\text{SL}_n(F)$ by a matrix $M$ of determinant $a$ induces an automorphism of $H_*(\text{SL}_n(F), \mathbb{Z})$ which depends only on $a$. In particular, $a^n = \det(\text{diag}(a, \ldots, a))$ acts trivially. Since the stability homomorphism is a map of $\mathbb{Z}[F^\times]$ modules, both $a^n$ and $a^{n+1}$ act trivially on its image, and so the action of $F^\times$ on the image of this map trivial. It follows that the stability homomorphism factors through the coinvariants of $F^\times$ on $H_*\left(\text{SL}_n(F), \mathbb{Z}\right)$ and has image lying in the invariants of $F^\times$ on $H_*\left(\text{SL}_{n+1}(F), \mathbb{Z}\right)$. In particular, when $F^\times$ acts nontrivially on $H_*\left(\text{SL}_n(F), \mathbb{Z}\right)$, the stability homomorphism has a nontrivial kernel, since it contains $I_F H_*\left(\text{SL}_n(F), \mathbb{Z}\right)$, where $I_F$ denotes the augmentation ideal of the group ring $\mathbb{Z}[F^\times]$.

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For example, the calculations of Suslin in [13] tell us that for any infinite (or sufficiently large) field $F$ the map $H_2\left(\text{SL}_2(F), \mathbb{Z}\right) \to H_2\left(\text{SL}_3(F), \mathbb{Z}\right)$ is surjective with kernel $\mathcal{I}_F H_2\left(\text{SL}_2(F), \mathbb{Z}\right)$ isomorphic to $I(F)^3$ where $I(F)$ denotes the fundamental ideal in the Grothendieck-Witt ring of the field $F$. In the case $F = \mathbb{Q}$, this kernel is isomorphic to the $\mathbb{Z}[\mathbb{Q}^\times]$-module $\mathbb{Z}$ on which $-1$ acts by negation and all primes act trivially.

B. Mirzaii has shown ([10]) for infinite fields $F$ that the kernel of the stability homomorphism $H_3\left(\text{SL}_2(F), \mathbb{Z}\right) \to H_3\left(\text{SL}_3(F), \mathbb{Z}\right) = K_3^{\text{ind}}(F)$, when tensored with $\mathbb{Z}\left[\frac{1}{2}\right]$, is $\mathcal{I}_F H_3\left(\text{SL}_2(F), \mathbb{Z}\left[\frac{1}{2}\right]\right)$; ie., it is again the case that the only obstruction to injective stability is the nontriviality of the action of the multiplicative group. He subsequently ([11]) generalised this result to rings with many units (including local rings with infinite residue fields).

The main theorem of this article (Theorem 3.1) describes the structure of $\mathcal{I}_Q H_3\left(\text{SL}_2(\mathbb{Q}), \mathbb{Z}\left[\frac{1}{2}\right]\right)$ as a $\mathbb{Z}[\mathbb{Q}^\times]$-module. $-1 \in \mathbb{Q}^\times$ acts trivially, but each prime acts nontrivially. Since the squares of rational numbers act trivially, each prime induces a decomposition into $(+1)$- and $(-1)$-eigenspaces. The $(-1)$-eigenspace of the prime $p$ is isomorphic, via a natural residue homomorphism $S_p$, to $\mathcal{P}(\mathbb{F}_p)$, the scissors congruence group of the field $\mathbb{F}_p$. It follows that as an abelian group

$$H_3\left(\text{SL}_2(\mathbb{Q}), \mathbb{Z}\left[\frac{1}{2}\right]\right) \cong K_3^{\text{ind}}(\mathbb{Q})\left[\frac{1}{2}\right] \oplus \left( \bigoplus_p \mathcal{P}(\mathbb{F}_p)\left[\frac{1}{2}\right] \right) \cong \mathbb{Z}/3 \oplus \left( \bigoplus_p \mathbb{Z}/(p + 1)_{\text{odd}} \right)$$

where $(m)_{\text{odd}}$ denotes the odd part of $m \in \mathbb{Q}^\times$; ie. $(m)_{\text{odd}} = 2^{-v_2(m)} m$.

The main tool we use is the description of $H_3\left(\text{SL}_2(F), \mathbb{Z}\left[\frac{1}{2}\right]\right)$ in terms of refined scissors congruence groups. The scissors congruence group $\mathcal{P}(F)$ of a field $F$ was introduced by Dupont and Sah in [2]. It is an abelian group defined by a presentation in terms of generators and relations and it was shown by the authors to be closely related to $K_3^{\text{ind}}(F) = H_3\left(\text{SL}_2(F), \mathbb{Z}\right)$ when $F$ is algebraically closed. Soon afterwards Suslin proved ([14, Theorem 5.2]) that the connection to $K_3^{\text{ind}}(F)$ persists for all infinite fields $F$ (see Theorem 2.4 below). However, to derive an analogous result for $H_3\left(\text{SL}_2(F), \mathbb{Z}\right)$ for general fields it is necessary to factor in the action of the multiplicative group of the field. The refined scissors congruence $\mathcal{P}(F)$ of the field $F$ – introduced in [4] – is defined by generators and relations analogously to the scissors congruence group but as a module over $\mathbb{Z}[F^\times]$ and not merely an abelian group. It can then be shown to bear approximately the same relation to $H_3\left(\text{SL}_2(F), \mathbb{Z}\right)$ as $\mathcal{P}(F)$ has to $K_3^{\text{ind}}(F)$.

(For a precise statement, see Theorem 2.5 below.) Using some later results of the author about refined scissors congruence groups, our starting point in this article is essentially a presentation of $\mathcal{I}_Q H_3\left(\text{SL}_2(\mathbb{Q}), \mathbb{Z}\left[\frac{1}{2}\right]\right)$ as a module of the group ring $\mathbb{Z}[\mathbb{Q}^\times]/((\mathbb{Q}^\times)^2)$ as well as the existence of module homomorphisms $S_p : \mathcal{I}_Q H_3\left(\text{SL}_2(\mathbb{Q}), \mathbb{Z}\right) \to \mathcal{P}(\mathbb{F}_p)$ (where the target is a module via $a \cdot x = (-1)^{v_2(a)} x$ for $a \in \mathbb{Q}^\times$), one for each prime $p$.

**Remark 1.1.** In our main theorem, we prove that the module homomorphism

$$\mathcal{I}_Q H_3\left(\text{SL}_2(\mathbb{Q}), \mathbb{Z}\right) \to \bigoplus_p \mathcal{P}(\mathbb{F}_p)$$

induced by the maps $S_p$, ranging over all primes $p$, becomes an isomorphism after tensoring with $\mathbb{Z}\left[\frac{1}{2}\right]$. It is natural to ask whether the original homomorphism is an isomorphism over $\mathbb{Z}$.

I do not know. Our methods of proof and 2-torsion ambiguities in existing results require us to work over $\mathbb{Z}\left[\frac{1}{2}\right]$. However, it is not hard to show even over $\mathbb{Z}$ that the cokernel of this map is annihilated by 4 (see the argument in Lemma 3.7 below).
Remark 1.2. It is to be expected that some version of the main result should hold for general number fields and even global fields. In order to arrive at such a result it would appear necessary first to determine whether the action of the (square classes of) the global units is trivial on the groups $H_3(SL_2(F), \mathbb{Z})$. There is some mild evidence suggesting that this is so: (i) for any field the square class $(-1)$ acts trivially and (ii) for local fields with finite residue field, the units act trivially. We hope to examine these questions elsewhere.

1.1. Layout of the article. In section 2 we review some of the relevant known results about scissors congruence groups and the third homology of $SL_2$ of fields.

Section 3 contains the proof of the main theorem (Theorem 3.1). We begin by recalling an elementary character-theoretic ‘local-global’ principle for proving that homomorphisms of $\mathbb{Z}[Q^\times/(Q^\times)^3]$-modules isomorphisms. The rest is straightforward manipulation of identities in the refined scissors congruence group.

In section 4 we look at further applications to $SL_2(A)$ for subrings $A$ of $\mathbb{Q}$. The key result in this section is Theorem 4.1. This result guarantees, for quite general commutative rings $A$, the existence of elements of $H_3(SL_2(A), \mathbb{Z})$ satisfying certain conditions. We use it first to show that $H_3(SL_2(\mathbb{Z}[\frac{1}{2}]), \mathbb{Z})$ surjects onto $K_3^{ind}(\mathbb{Q})$ and to deduce the module structure of this homology group. We then use it to characterise the image of $H_3(SL_2(A), \mathbb{Z}[\frac{1}{2}])$ in $H_3(SL_2(\mathbb{Q}), \mathbb{Z}[\frac{1}{2}])$ for a large class of subrings $A$ of $\mathbb{Q}$. For example, if $A = \mathbb{Z}\left[\frac{1}{p_1\cdots p_t}\right]$ where $p_1 < \cdots < p_t$ are the first $t$ primes and $t > 4$, then this image is isomorphic to $\mathbb{Z}/3 \oplus (\bigoplus_{i=1}^{t-1} \mathbb{Z} / (p_i + 1)_{odu})$.

In section 5, we describe some further applications of the main theorem; for example, the calculation of $H_3(SL_2(\mathbb{Q}[t, t^{-1}]), \mathbb{Z}[\frac{1}{2}])$ and an explicit description of a basis for the $\mathbb{F}_3$-vector space elements of order dividing 3 in $H_3(SL_2(\mathbb{Q}), \mathbb{Z})$.

In section 6 we give the details of the technical proof of Theorem 4.1, which was deferred from Section 4.

1.2. Notation. For a commutative unital ring $A$, $A^\times$ denotes the group of units of $A$.

For any abelian group $A$, we denote $A \otimes \mathbb{Z}[\frac{1}{n}]$ by $A[\frac{1}{n}]$. For any prime $p$, $A_{(p)}$ denotes the group of $p$-primary torsion elements in $A$.

If $q$ is a prime power, $\mathbb{F}_q$ will denote the finite field with $q$ elements.

For a group $G$ and a $\mathbb{Z}[G]$-module $M$, $M_G$ will denote the module of coinvariants; $M_G = H_0(G, M) = M/I_G M$, where $I_G$ is the augmentation ideal of $\mathbb{Z}[G]$.

Given an abelian group $G$ we let $S_2^2(G)$ denote the group

\[G \otimes_{\mathbb{Z}} G \cong \langle x \otimes y + y \otimes x | x, y \in G \rangle\]

and, for $x, y \in G$, we denote by $x \circ y$ the image of $x \otimes y$ in $S_2^2(G)$.

For any rational prime $p$, $v_p : Q^\times \to \mathbb{Z}$ denotes the corresponding discrete valuation, determined by $v = p^{v_p(a)} \cdot (m/n)$ with $m, n$ not divisible by $p$.

For a field $F$, we let $R_F$ denote the group ring $\mathbb{Z}[F^\times/(F^\times)^3]$ of the group of square classes of $F$ and we let $I_F$ denote the augmentation ideal of $R_F$. If $x \in F^\times$, we denote the corresponding square-class, considered as an element of $R_F$, by $\langle x \rangle$. The generators $\langle x \rangle - 1$ of $I_F$ will be denoted $\langle \langle x \rangle \rangle$. 

2. Refined scissors congruence groups and $H_3(SL_2(F),\mathbb{Z})$

In this section we review some of the relevant known facts about the third homology of $SL_2$ of fields and its description in terms of refined scissors congruence groups.

2.1. Indecomposable $K_3$. For any field $F$ there is a natural surjective homomorphism

(1) \[ H_3(SL_2(F),\mathbb{Z}) \to K_3^{\text{ind}}(F). \]

When $F$ is quadratically closed (i.e. when $F^\times/(F^\times)^2 = 1$) this map is an isomorphism. However, in general, the group extension

\[ 1 \to SL_2(F) \to GL_2(F) \to F^\times \to 1 \]

induces an action – by conjugation – of $F^\times$ on $H_3(SL_2(F),\mathbb{Z})$ which factors through $F^\times/(F^\times)^2$.

It can be shown that the map (1) is a homomorphism of $R_F$-modules (where $F^\times/(F^\times)^2$ acts trivially on $K_3^{\text{ind}}(F)$ and induces an isomorphism

(2) \[ H_3(SL_2(F),\mathbb{Z}[\frac{1}{2}])_{F^\times/(F^\times)^2} \cong K_3^{\text{ind}}(F)[\frac{1}{2}] \]

(see [10, Proposition 6.4]), but – as our calculations in [3] show – the action of $F^\times/(F^\times)^2$ on $H_3(SL_2(F),\mathbb{Z})$ is in general non-trivial.

Let $H_3(SL_2(F),\mathbb{Z})_0$ denote the kernel of the surjective homomorphism $H_3(SL_2(F),\mathbb{Z}) \to K_3^{\text{ind}}(F)$. This is an $R_F$-submodule of $H_3(SL_2(F),\mathbb{Z})$. Note that the isomorphism (2) implies that

\[ H_3(SL_2(F),\mathbb{Z}[\frac{1}{2}]) = I_FH_3(SL_2(F),\mathbb{Z}[\frac{1}{2}]_0). \]

Remark 2.1. When $F$ is a number field the surjective homomorphism $H_3(SL_2(F),\mathbb{Z}) \to K_3^{\text{ind}}(F)$ is split as a map of $\mathbb{Z}$-modules. In fact, $K_3^{\text{ind}}(F)$ is a finitely generated abelian group and it is enough to show that there is a torsion subgroup of $H_3(SL_2(F),\mathbb{Z})$ mapping isomorphically to the (cyclic) torsion subgroup of $K_3^{\text{ind}}(F)$. But this latter statement follows from the explicit calculations of C. Zickert in [15, Section 8]. It follows that, as an abelian group,

\[ H_3(SL_2(F),\mathbb{Z}) \cong K_3^{\text{ind}}(F) \oplus H_3(SL_2(F),\mathbb{Z})_0 \]

for any number field $F$.

However, there is no such decomposition of $H_3(SL_2(F),\mathbb{Z})$ as an $R_F$-module. For details, see Remark 2.1 below.

2.2. Scissors Congruence Groups. For a field $F$, with at least 4 elements, the scissors congruence group (also called the pre-Bloch group), $P(F)$, is the group generated by the elements $[x]$, $x \in F^\times$, subject to the relations

\[ R_{x,y} : [x] - [y] + \left[ \frac{y}{x} \right] - \left[ \frac{1 - x^{-1}}{1 - y^{-1}} \right] + \left[ \frac{1 - x}{1 - y} \right] = 0, \quad x, y \neq 1. \]

The map

\[ \lambda : P(F) \to S_2^+(F^\times), \quad [x] \mapsto (1 - x) \circ x \]

is well-defined, and the Bloch group of $F$, $B(F) \subset P(F)$, is defined to be the kernel of $\lambda$.

For the fields with 2 and 3 elements the following definitions allow us to include these fields in the statements of some of our results:

$P(F_2) = B(F_2)$ is a cyclic group of order 3 with generator denoted $C_{F_2}$. We let $[1] = 0$ in $P(F_2)$.

$P(F_3)$ is cyclic of order 4 with generator $[-1]$. We have $[1] = 0$ in $P(F_3)$. $B(F_3)$ is the subgroup generated by 2 $[-1]$.

We recall (see, for example, [4, Lemma 7.4]):
Lemma 2.2. If $q$ is a prime power then $\mathcal{B}(F_q)$ is cyclic of order $(q + 1)/2$ when $q$ is odd and $q + 1$ when $q$ is even.

The following corollary is particularly relevant to this article:

Corollary 2.3. If $q$ is a prime power then $\mathcal{P}(F_q) \left[ \frac{1}{2} \right]$ is cyclic of order $(q + 1)_{odd}$.

The Bloch group is closely related to the indecomposable $K_3$ of the field $F$:

Theorem 2.4. For any field $F$ there is a natural exact sequence

$$0 \to \text{Tor}^\mathbb{Z}_1(\mu_F, \mu_F) \to K_3^{\text{ind}}(F) \to \mathcal{B}(F) \to 0$$

where $\text{Tor}^\mathbb{Z}_1(\mu_F, \mu_F)$ is the unique nontrivial extension of $\text{Tor}^\mathbb{Z}_1(\mu_F, \mu_F)$ by $\mathbb{Z}/2$.

(See Suslin [14] for infinite fields and [4] for finite fields.)

2.3. The refined scissors congruence group. For a field $F$ with at least 4 elements, $\mathcal{RP}(F)$ is defined to be the $R_F$-module with generators $[x]$, $x \in F^\times$ subject to the relations

$$S_{xy} : 0 = [x] - [y] + \langle x \rangle \left[ \frac{y}{x} \right] - \left( x^{-1} - 1 \right) \left[ \frac{1 - x^{-1}}{1 - y^{-1}} \right] + \langle 1 - x \rangle \left[ \frac{1 - x}{1 - y} \right], \quad x, y \neq 1$$

Of course, from the definition it follows immediately that

$$\mathcal{P}(F) = (\mathcal{RP}(F) / F^\times)^2 = H_0(F^\times / (F^\times)^2, \mathcal{RP}(F)).$$

Let $\Lambda = (\lambda_1, \lambda_2)$ be the $R_F$-module homomorphism

$$\mathcal{RP}(F) \to I_F^2 \oplus S_2^2(F^\times)$$

where $\lambda_1 : \mathcal{RP}(F) \to I_F^2$ is the map $[x] \mapsto \langle 1 - x \rangle \langle x \rangle$, and $\lambda_2$ is the composite

$$\mathcal{RP}(F) \longrightarrow \mathcal{P}(F) \xrightarrow{\lambda} S_2^2(F^\times).$$

It can be shown that $\Lambda$ is well-defined.

The refined scissors congruence group of $F$ is the $R_F$-module $\mathcal{RP}_1(F) := \text{Ker}(\lambda_1)$.

The refined Bloch group of the field $F$ (with at least 4 elements) to be the $R_F$-module

$$\mathcal{RB}(F) : = \text{Ker}(\Lambda : \mathcal{RP}(F) \to I_F^2 \oplus S_2^2(F^\times))$$

$$= \text{Ker}(\lambda_2 : \mathcal{RP}_1(F) \to S_2^2(F^\times)).$$

$\mathcal{P}(\mathbb{F}_2) = \mathcal{RP}(\mathbb{F}_2)$ is simply an additive group of order 3 with distinguished generator, denoted $C_{\mathbb{F}_2}$.

$\mathcal{RP}(\mathbb{F}_3)$ is the cyclic $R_{\mathbb{F}_3}$-module generated by the symbol $[-1]$ and subject to the one relation

$$0 = 2 \cdot ([1] + \langle -1 \rangle [-1]).$$

$\mathcal{P}(\mathbb{F}_3) = H_0(\mathbb{F}_3, \mathcal{RP}(\mathbb{F}_3))$ is then cyclic of order 4 generated by the symbol $[-1]$.

The symbol $[1]$ continues to denote 0 in $\mathcal{RP}(\mathbb{F}_2)$ and $\mathcal{RP}(\mathbb{F}_3)$.

We recall some results from [4]: The main result there is
Theorem 2.5. Let $F$ be any field.
There is a natural complex
\[ 0 \to \text{Tor}_1^F(\mu_F, \mu_F) \to H_3\left(\text{SL}_2(F), \mathbb{Z}\right) \to \mathcal{R}\mathcal{B}(F) \to 0. \]
which is exact everywhere except possibly at the middle term. The middle homology is annihilated by 4.

In particular, for any field there is a natural short exact sequence
\[ 0 \to \text{Tor}_1^F(\mu_F, \mu_F) \left[ \frac{1}{2} \right] \to H_3\left(\text{SL}_2(F), \mathbb{Z}\left[ \frac{1}{2} \right]\right) \to \mathcal{R}\mathcal{B}(F) \left[ \frac{1}{2} \right] \to 0. \]

2.4. Scissors congruence groups and $H_3\left(\text{SL}_2(F), \mathbb{Z}\right)_0$. In [14] Suslin defines the elements \( \{x\} := \{x\} + \{x^{-1}\} \in \mathcal{P}(F) \) and shows that they satisfy
\[ \{xy\} = \{x\} + \{y\} \text{ and } 2\{x\} = 0 \text{ for all } x, y \in F^\times. \]
In particular, \( \{x\} = 0 \) in \( \mathcal{P}(F) \left[ \frac{1}{2} \right] \).
There are two natural liftings of these elements to \( \mathcal{R}\mathcal{P}(F) \): given \( x \in F^\times \) we define
\[ \psi_1(x) := \{x\} + \langle -1 \rangle \{x^{-1}\} \]
and
\[ \psi_2(x) := \begin{cases} \langle 1 - x \rangle \langle x \rangle \{x\} + \{x^{-1}\}, & x \neq 1 \\ 0, & x = 1 \end{cases} \]
(If \( F = \mathbb{F}_2 \), we interpret this as \( \psi_i(1) = 0 \) for \( i = 1, 2 \). For \( F = \mathbb{F}_3 \), we have \( \psi_1(-1) = \psi_2(-1) = \langle -1 \rangle + \langle -1 \rangle \langle -1 \rangle \). )
The maps \( F^\times \to \mathcal{R}\mathcal{P}(F), x \mapsto \psi_i(x) \) are 1-cocycles: \( \psi_i(xy) = \langle x \rangle \psi_i(y) + \psi_i(x) \) for all \( x, y \in F^\times \).
(See [3] Section 3). In general, the elements \( \psi_i(x) \) have infinite order however.
We define \( \mathcal{R}\mathcal{P}(F) \) to be \( \mathcal{R}\mathcal{P}(F) \) modulo the submodule generated by the elements \( \psi_1(x), x \in F^\times \).
Likewise, \( \mathcal{P}(F) \) is the group \( \mathcal{P}(F) \) modulo the subgroup generated by the elements \( \{x\}, x \in F^\times \).
Note that since the elements \( \{x\} \) are annihilated by 2, we have \( \mathcal{P}(F) \left[ \frac{1}{2} \right] = \mathcal{P}(F) \left[ \frac{1}{2} \right] \).
For any field there is natural homomorphism of \( R_F \)-modules \( H_3\left(\text{SL}_2(F), \mathbb{Z}\right) \to \mathcal{R}\mathcal{P}(F) \) and we have ([14 Corollary 2.8,Corollary 4.4]):

Theorem 2.6. For any field \( F \), the map \( H_3\left(\text{SL}_2(F), \mathbb{Z}\right) \to \mathcal{R}\mathcal{P}(F) \) induces an isomorphism of \( R_F \)-modules
\[ H_3\left(\text{SL}_2(F), \mathbb{Z}\left[ \frac{1}{2} \right]\right)_0 = I_F H_3\left(\text{SL}_2(F), \mathbb{Z}\left[ \frac{1}{2} \right]\right) \cong I_F \mathcal{R}\mathcal{P}(F) \left[ \frac{1}{2} \right] \]
and furthermore
\[ \mathcal{R}\mathcal{P}(F) \left[ \frac{1}{2} \right] = e_+^{-1} \mathcal{R}\mathcal{P}(F) \left[ \frac{1}{2} \right] \]
where \( e_+^{-1} \) denotes the idempotent \( (1 + \langle -1 \rangle)/2 \in R_F \left[ \frac{1}{2} \right] \).

Note that it follows that the square class \( \langle -1 \rangle \) acts trivially on \( H_3\left(\text{SL}_2(F), \mathbb{Z}\left[ \frac{1}{2} \right]\right) \).
To simplify the right-hand side we define the module \( \mathcal{R}\mathcal{P}(F) \) to be \( \mathcal{R}\mathcal{P}(F) \) modulo the submodule generated by the elements \( (1 - \langle -1 \rangle)\{x\}, x \in F^\times \). Thus \( \mathcal{R}\mathcal{P}(F) \) is the \( R_F \)-module generated by the the elements \( \{x\}, x \in F^\times \) subject to the relations
\begin{enumerate}
  \item \( \{1\} = 0 \)
  \item \( S_{x,y} = 0 \) for \( x, y \neq 1 \)
  \item \( \langle -1 \rangle \{x\} = \{x\} \) for all \( x \)
  \item \( \{x^{-1}\} = -\{x\} \) for all \( x \)
\end{enumerate}
Theorem 2.8. Let $F$ be a field and let $\mathcal{V}$ be a family of discrete valuations on $F$ satisfying

$$
\mathcal{V} = \{ v : F^\times \rightarrow \mathbb{Z} \mid v(x) = 0 \text{ for all but finitely many } x \in \mathcal{V} \}
$$

(1) For any $x \in F^\times$, $v(x) = 0$ for all but finitely many $v \in \mathcal{V}$
(2) The map

$$
F^\times \rightarrow \bigoplus_{v \in \mathcal{V}} \mathbb{Z}/2, \quad a \mapsto \{ v(a) \}_v
$$

is surjective.
Then the maps \( \{S_v\}_{v \in \mathcal{V}} \) induce a natural surjective homomorphism

\[
H_3\left( \mathrm{SL}_2 (F), \mathbb{Z}\left[ \frac{1}{2} \right] \right)_0 \cong I_r \mathcal{R}\mathcal{P}_+(F) \left[ \frac{1}{2} \right] \to \bigoplus_{v \in \mathcal{V}} \tilde{\mathcal{P}}(k(v)) \{v\} \left[ \frac{1}{2} \right].
\]

Taking \( F = \mathbb{Q} \) and \( \mathcal{V} = \text{Primes} \), the set of all primes, we obtain a surjective homomorphism of \( \mathbb{R}_Q \)-modules

\[
(3) \quad H_3\left( \mathrm{SL}_2 (\mathbb{Q}), \mathbb{Z}\left[ \frac{1}{2} \right] \right)_0 \to \bigoplus_{p \in \text{Primes}} \tilde{\mathcal{P}}(\mathbb{F}_p) \{p\} \left[ \frac{1}{2} \right].
\]

**Remark 2.9.** Since \((-1) \in \mathbb{R}_Q\) acts trivially on both of the modules in (3), this is a map of \( \mathbb{R}_Q^+ \)-modules where

\[
\mathbb{R}_Q^+ := \mathbb{Z}[\mathbb{Q}^\times / \pm (\mathbb{Q}^\times)^2] = \mathbb{Z}[\mathbb{Q}_+ / \mathbb{Q}_2^2].
\]

**Remark 2.10.** The collection of maps \( \{S_v\}_{v \in \mathcal{V}} \) as above induces an \( \mathbb{R}_F \)-module homomorphism with target the product – rather than the direct sum – of the scissors congruence groups:

\[
H_3\left( \mathrm{SL}_2 (F), \mathbb{Z} \right) \to \mathcal{R}\mathcal{P}_+(F) \to \prod_{v \in \mathcal{V}} \tilde{\mathcal{P}}(k(v)) \{v\}.
\]

However, when we restrict to \( H_3\left( \mathrm{SL}_2 (F), \mathbb{Z} \right)_0 \) and tensor with \( \mathbb{Z}\left[ \frac{1}{2} \right] \) the image lies in the direct sum instead, in view of the isomorphism \( H_3\left( \mathrm{SL}_2 (F), \mathbb{Z}\left[ \frac{1}{2} \right] \right)_0 \cong I_r \mathcal{R}\mathcal{P}_+(F) \left[ \frac{1}{2} \right] \) and the fact that \( S_v(\langle a \rangle \{b\}) = \langle a \rangle S_v(\{b\}) = 0 \) whenever \( v(a) \) is even.

Now let \( t \in \mathrm{SL}_2 (\mathbb{Z}) \) be the element of order 3

\[
t := \left( \begin{array}{cc} -1 & 1 \\ -1 & 0 \end{array} \right) \in \mathrm{SL}_2 (\mathbb{Z}).
\]

Denote (also) by \( C_\mathbb{Q} \in H_3\left( \mathrm{SL}_2 (\mathbb{Q}), \mathbb{Z} \right) \) the image of 1 \( \in \mathbb{Z}/3 = H_3(\langle t \rangle, \mathbb{Z}) \) under the map induced by the inclusion \( \langle t \rangle \to \mathrm{SL}_2 (\mathbb{Q}) \). Then \( C_\mathbb{Q} \in H_3\left( \mathrm{SL}_2 (\mathbb{Q}), \mathbb{Z} \right) \) maps to \( C_\mathbb{Q} \in \mathcal{R}\mathcal{P}_+(\mathbb{Q}) \) ([3] Remark 3.14)). Note that \( S_p(C_\mathbb{Q}) = C_{\varepsilon_p} \in \mathcal{R}\mathcal{P}_+(\mathbb{F}_p) \) for all primes \( p \). Furthermore, \( C_{\varepsilon_p} \neq 0 \) precisely when \( p \equiv 2 \mod 3 \) (ie., precisely when \( 3|p+1 \)), by [4] Lemma 7.11.

In particular, the image of \( C_\mathbb{Q} \) under the map \( \{S_p\}_{p \mid \text{Primes}} \) lies in the product, but not the direct sum, of the scissors congruence groups of the residue fields.

**Remark 2.11.** Now Suslin’s map gives a canonical isomorphism \( K_3^{\text{ind}}(\mathbb{Q})_{(3)} \cong \mathcal{B}(\mathbb{Q})_{(3)} = \mathbb{Z}/3 \cdot C_\mathbb{Q} \) and we can let \( C_\mathbb{Q} \) also denote the corresponding element of \( K_3^{\text{ind}}(\mathbb{Q}) \).

Recall that \( \mathbb{R}_Q \) acts trivially on \( K_3^{\text{ind}}(\mathbb{Q}) \). Suppose that there were an \( \mathbb{R}_Q \)-module splitting \( j : K_3^{\text{ind}}(\mathbb{Q}) \to H_3\left( \mathrm{SL}_2 (\mathbb{Q}), \mathbb{Z} \right) \). Then we would have \( j(C_\mathbb{Q}) = C_\mathbb{Q} + h \) for some \( h \in H_3\left( \mathrm{SL}_2 (\mathbb{Q}), \mathbb{Z} \right)_0 \).

We must have \( \mathbb{R}_Q \) acts trivially on \( j(C_\mathbb{Q}) \) and hence \( \langle p \rangle j(C_\mathbb{Q}) = \langle p \rangle (C_\mathbb{Q} + h) = 0 \) for all primes \( p \). However, we can choose a prime \( p \) such that \( \langle p \rangle h = 0 \in H_3\left( \mathrm{SL}_2 (\mathbb{Q}), \mathbb{Z}\left[ \frac{1}{2} \right] \right)_0 \) and \( p \equiv 2 \mod 3 \). Then \( S_p(\langle p \rangle (C_\mathbb{Q} + h)) = \langle p \rangle C_{\varepsilon_p} = -2 C_{\varepsilon_p} \neq 0 \), giving us a contradiction. So no such \( \mathbb{R}_Q \)-splitting \( j \) can exist.

3. **The Main Theorem**

In this section we prove

**Theorem 3.1.** The map

\[
S : H_3\left( \mathrm{SL}_2 (\mathbb{Q}), \mathbb{Z}\left[ \frac{1}{2} \right] \right)_0 \cong I_\mathbb{Q}\mathcal{R}\mathcal{P}_+(\mathbb{Q}) \left[ \frac{1}{2} \right] \to \bigoplus_{p \in \text{Primes}} \mathcal{P}(\mathbb{F}_p) \{p\} \left[ \frac{1}{2} \right]
\]
is an isomorphism of $\mathbb{R}_Q^-$-modules

We will use the following character-theoretic principles:
Let $G$ be an abelian group satisfying $g^2 = 1$ for all $g \in G$. Let $R$ denote the group ring $\mathbb{Z}\left[\frac{1}{2}\right][G]$.

For a character $\chi \in \hat{G} := \text{Hom}(G, \mu_2)$, let $R^\chi$ be the ideal of $R$ generated by the elements $\{g - \chi(g) \mid g \in G\}$. In other words $R^\chi$ is the kernel of the ring homomorphism $\rho(\chi) : R \to \mathbb{Z}$ sending $g$ to $\chi(g)$ for any $g \in G$. We let $R_\chi$ denote the associated $R$-algebra structure on $\mathbb{Z}$; i.e. $R_\chi := R/R^\chi$.

If $M$ is an $R$-module, we let $M^\chi = R^\chi M$ and we let

$$M_\chi := M/M^\chi = (R/R^\chi) \otimes_R M = (R_\chi \otimes_R M).$$

Thus $M_\chi$ is the largest quotient module of $M$ with the property that $g \cdot m = \chi(g) \cdot m$ for all $g \in G$.

In particular, if $\chi = \chi_0$, the trivial character, then $R^{\chi_0}$ is the augmentation ideal $I_G$, $M^{\chi_0} = I_G M$ and $M_{\chi_0} = M_G$.

We will need the following results ([6, Section 3])

**Proposition 3.2.**

1. For any $\chi \in \hat{G}$, $M \to M_\chi$ is an exact functor on the category of $R$-modules.
2. Let $f : M \to N$ be an $R$-module homomorphism. For any $\chi \in \hat{G}$, let $f_\chi : M_\chi \to N_\chi$. Then $f$ is bijective (resp. injective, surjective) if and only if $f_\chi$ is bijective (resp. injective, surjective) for all $\chi \in \hat{G}$.

Recall now that $R_+^\times = \mathbb{Z}[G]$ where $G = \mathbb{Q}_+^\times = \mathbb{Q}^\times / (\mathbb{Q}^\times)^2$. As a multiplicative $\mathbb{F}_2$-space, the set of all primes form a (number-theoretically) natural basis of $\mathbb{Q}_+^\times$. Thus the space of characters $\mathbb{Q}_+^\times/\mathbb{Q}_+^2$ is naturally parametrised by the subsets of the set Primes of positive prime numbers: if $S \subset \text{Primes}$ then the corresponding character $\chi_S$ is defined by

$$\chi_S(p) = \begin{cases} -1, & p \in S \\ 1, & p \notin S \end{cases}$$

for all $p \in \text{Primes}$ or, equivalently,

$$\chi_S(x) = (-1)^{\sum_{p \in S} v_p(x)}$$

for all $x \in \mathbb{Q}^\times$. Conversely, the character $\chi$ corresponds to the subset

$$\text{supp}(\chi) := \{p \in \text{Primes} \mid \chi(p) = -1\}.$$

The following lemma is immediate from the definition of the $R_\mathbb{Q}$-module structure on $\mathcal{P}(\mathbb{F}_p)[p]$.

**Lemma 3.3.** Let $\chi \in \mathbb{Q}_+^\times/\mathbb{Q}_+^2$. Then

$$\mathcal{P}(\mathbb{F}_p)[p]\left[\frac{1}{2}\right]_\chi = \begin{cases} \mathcal{P}(\mathbb{F}_p)[p]\left[\frac{1}{2}\right], & \chi = \chi_p \\ 0, & \text{otherwise} \end{cases}$$

**Corollary 3.4.** For $\chi \in \mathbb{Q}_+^\times/\mathbb{Q}_+^2$ we have

$$\left(\bigoplus_p \mathcal{P}(\mathbb{F}_p)[p]\left[\frac{1}{2}\right]\right)_\chi = \begin{cases} \mathcal{P}(\mathbb{F}_p)[p]\left[\frac{1}{2}\right], & \chi = \chi_p \\ 0, & \text{otherwise} \end{cases}$$
It thus follows from Proposition 3.2 that to prove Theorem 3.1 it is enough to prove that \( S_p \) induces an isomorphism
\[
\mathcal{R}P_+(\mathbb{Q}) \left[ \frac{1}{2} \right]_{x_p} \cong \mathcal{P}(\mathbb{F}_p) \left[ \frac{1}{2} \right]
\]
for any prime \( p \), while
\[
\mathcal{R}P_+(\mathbb{Q}) \left[ \frac{1}{2} \right]_{x_p} = 0
\]
whenever \( \text{supp}(\chi) \) contains at least two distinct primes. These two statements are proved in Corollaries 3.13 and 3.11 below.

**Lemma 3.5.** Let \( F \) be a field. Let \( \chi \in \hat{F}^{\times}/(\hat{F}^{\times})^2 \). Then
\[
[a]_{\chi} = \chi(1 - a)C_F
\]
in \( \mathcal{R}P_+(F)_{x} \left[ \frac{1}{2} \right] \) whenever \( a \in F^{\times} \) with \( \chi(a) = -1 \).

**Proof.** By Proposition 2.7 (2) we have \((1 - a) \langle a \rangle [a] = 0 = \langle a \rangle C_F \) in \( \mathcal{R}P_+(F) \) for all \( a \in F^{\times} \).
Thus if \( \chi(a) = -1 \) we have \(-2\chi(1 - a)[a]_{\chi} = -2C_F \) in \( \mathcal{R}P_+(F)_{x} \).
\[\square\]

**Corollary 3.6.** Let \( F \) be a field. Let \( \chi \in \hat{F}^{\times}/(\hat{F}^{\times})^2 \). Suppose that \( a \in F^{\times} \) satisfies \( \chi(a) = 1 \) and \( \chi(1 - a) = -1 \). Then \( [a]_{\chi} = 0 \) in \( \mathcal{R}P_+(F)_{x} \left[ \frac{1}{2} \right] \).

**Proof.** In this case \([1 - a]_{\chi} = C_F \) by Lemma 3.5. But \( C_F = [a]_{\chi} + [1 - a]_{\chi} \) in \( \mathcal{R}P_+(F)_{x} \left[ \frac{1}{2} \right] \).
\[\square\]

**Lemma 3.7.** Let \( F \) be a field. Let \( \chi \in \hat{F}^{\times}/(\hat{F}^{\times})^2 \) with \( \chi(-1) = 1 \). Let \( \ell \in F^{\times} \) satisfy \( \chi(\ell) = -1 \) and \( \chi(1 - \ell) = 1 \). Then
\[
[a]_{\chi} = [(1 - \ell)a]_{\chi}
\]
in \( \mathcal{R}P_+(F) \left[ \frac{1}{2} \right]_{x} \) for all \( a \in \hat{F}^1(F) \).

**Proof.** Observe that \([1 - \ell]_{\chi} = 0 \) by Corollary 3.6. In particular, the result holds for \( a \in \{0, 1, \infty\} \).

For all \( a \in F^{\times} \setminus \{1\} \) we have in \( \mathcal{R}P_+(F) \left[ \frac{1}{2} \right]_{x} \)
\[
0 = (S_{a^{-1}, 1-\ell})_{\chi} \left[ a^{-1} \right]_{\chi} - [1 - \ell]_{\chi} + \chi(a^{-1})[(1 - \ell)a]_{\chi} - \chi(1 - a) \left[ \frac{(1 - \ell)(a - 1)}{\ell} \right]_{x} + \chi(1 - a^{-1}) \left[ \frac{a - 1}{a\ell} \right]_{x}
\]
and hence
\[
0 = \left[ a^{-1} \right]_{\chi} + \chi(a^{-1})[y]_{\chi} - \chi(1 - a)[z]_{\chi} + \chi(1 - a^{-1})[w]_{\chi}
\]
where
\[
y := (1 - \ell)a, \quad z := \frac{(1 - \ell)(a - 1)}{\ell} \quad \text{and} \quad w := \frac{a - 1}{a\ell}.
\]
Thus
\[
1 - z = \frac{1 - y}{\ell} \quad \text{and} \quad 1 - w = \frac{1 - y}{a\ell}.
\]
We consider now the four possible values of \((\chi(a), \chi(1 - a))\):
(1) $\chi(a) = -1$ and $\chi(1 - a) = 1$.

Then $\chi(a^{-1}) = -1 = \chi(1 - a^{-1})$. Furthermore $[a]_\chi = C_F$ and $[a^{-1}]_\chi = -C_F$ by Lemma 3.5. By (4) we thus have

$$0 = -C_F - [y]_\chi - [z]_\chi - [w]_\chi$$

where $\chi(y) = -1 = \chi(z)$ and $\chi(w) = 1$.

We divide further into sub-cases according to the value of $\chi(1 - y)$:

(a) $\chi(1 - y) = 1$: Then $[y]_\chi = C_F$ by Lemma 3.5 and hence $[y]_\chi = [a]_\chi$ as required.

(b) $\chi(1 - y) = -1$: Then $[y]_\chi = -C_F$ by Lemma 3.5. However, by (5), $\chi(1 - z) = \chi(1 - y)\chi(\ell) = 1$ and $\chi(1 - w) = \chi(1 - y)\chi(a) = -1$ so that $z]_\chi = C_F$ and $[w]_\chi = 0$ by Lemma 3.5 and Corollary 3.6. Hence, by (4), we now have $0 = -C_F + C_F - C_F - 0 \implies C_F = 0$ in $\mathcal{R}\mathcal{P}_+(F)\left[\frac{1}{2}\right]_\chi$. Thus $[y]_\chi = 0 = [a]_\chi$ as required, in this case also.

(2) $\chi(a) = -1$ and $\chi(1 - a) = -1$.

Then $\chi(a^{-1}) = -1$ and $\chi(1 - a^{-1}) = 1$. Thus $[a]_\chi = -C_F$ and $[a^{-1}]_\chi = C_F$. This gives

$$0 = C_F - [y]_\chi + [z]_\chi + [w]_\chi$$

where $\chi(y) = -1 = \chi(w)$ and $\chi(z) = 1$.

(a) $\chi(1 - y) = 1$: Then $[y]_\chi = C_F = [a]_\chi$. However, by (5) again, $\chi(1 - z) = -1$ and $\chi(1 - w) = 1$ so that $z]_\chi = 0$ and $[w]_\chi = C_F$. From (4) we have $0 = C_F - C_F + 0 + C_F$ and hence $C_F = 0$ in $\mathcal{R}\mathcal{P}_+(F)\left[\frac{1}{2}\right]_\chi$ as required.

(b) $\chi(1 - y) = -1$: Then $[y]_\chi = -C_F = [a]_\chi$ again as required.

(3) $\chi(a) = 1$ and $\chi(1 - a) = -1$.

Then $[a]_\chi = 0 = [a^{-1}]_\chi$. By Corollary 3.6. Thus from (4) we have $0 = [y]_\chi + [z]_\chi - [w]_\chi$ where $\chi(y) = \chi(z) = \chi(w) = 1$.

(a) $\chi(1 - y) = 1$: Then $\chi(1 - z) = -1 = \chi(1 - w)$. Hence $[z]_\chi = 0 = [w]_\chi$. Thus

$$[y]_\chi = 0 = [a]_\chi$$

as required.

(b) $\chi(1 - y) = -1$: Then $[y]_\chi = 0 = [a]_\chi$ by Corollary 3.6.

(4) $\chi(a) = 1 = \chi(1 - a)$.

Then $\chi(a^{-1}) = 1 = \chi(1 - a^{-1})$ also. Equation (4) thus gives $0 = [a^{-1}]_\chi + [y]_\chi - [z]_\chi + [w]_\chi$ with $\chi(z) = -1 = \chi(w)$. Furthermore $\chi(1 - z) = -\chi(1 - y) = \chi(1 - w)$. Hence $[z]_\chi = [w]_\chi = -\chi(1 - y)C_F$ by Lemma 3.5. This gives

$$0 = [a^{-1}]_\chi + [y]_\chi = -[a]_\chi + [y]_\chi$$

as required.

A straightforward induction gives:

**Corollary 3.8.** Let $F$ be a field. Let $\chi \in F^\times/(F^\times)^2$ with $\chi(-1) = 1$. Let $\ell \in F^\times$ satisfy $\chi(\ell) = -1$ and $\chi(1 - \ell) = 1$. Then

$$[a]_\chi = [(1 - \ell)m]_\chi \text{ in } \mathcal{R}\mathcal{P}_+(F)\left[\frac{1}{2}\right]_\chi$$

for all $a \in \mathbb{P}^1(F)$ and all $m \in \mathbb{Z}$.

**Corollary 3.9.** Let $F$ be a field. Let $\chi \in F^\times/(F^\times)^2$ with $\chi(-1) = 1$. Let $\ell \in F^\times$ satisfy $\chi(\ell) = -1$ and $\chi(1 - \ell) = 1$. Then

$$[a]_\chi = [a + t\ell]_\chi \text{ in } \mathcal{R}\mathcal{P}_+(F)\left[\frac{1}{2}\right]_\chi$$

for all $a \in F$ and all $t \in \mathbb{Z}$.
Proof. In \( \mathcal{RP}_+(\mathbb{Q}) \left[ \frac{1}{2} \right] \), we have

\[
[a]_\chi = [a(1 - \ell)^{-1}]_\chi \quad \text{by Corollary 3.8}
\]

\[
= C_F - \left[ 1 - \frac{a}{1 - \ell} \right]_\chi
\]

\[
= C_F - \left[ (1 - \ell)(1 - \frac{a}{1 - \ell}) \right]_\chi \quad \text{by Lemma 3.7}
\]

\[
= C_F - [1 - (a + \ell)]_\chi
\]

\[
= [a + \ell]_\chi
\]

for any \( a \in F \).

\[\]  

**Proposition 3.10.** Let \( \chi \in \hat{\mathbb{Q}}_+/\mathbb{Q}_+^2 \). If \( |\text{supp}(\chi)| \geq 2 \) then

\[
[a]_\chi = [a + t]_\chi
\]

in \( \mathcal{RP}_+(\mathbb{Q}) \left[ \frac{1}{2} \right] \) for all \( t \in \mathbb{Z} \) and \( a \in \mathbb{Q} \).

Proof. Let \( p = \min(\text{supp}(\chi)) \). Then \( \chi(p) = -1 \) and \( \chi(1 - p) = 1 \). So

\[
[a]_\chi = [a + tp]_\chi
\]

for all \( a \in \mathbb{Q} \) and \( t \in \mathbb{Z} \) by Corollary 3.9.

Now let \( q = \min(\text{supp}(\chi) \setminus \{p\}) \).

Suppose first that \( p > 2 \). The either \( q - 1 \) or \( q + 1 \) is not divisible by \( p \). If \( p \) does not divide \( q - 1 \) or \( q + 1 \) take \( \ell = q \). Otherwise take \( \ell = -q \). Then \( \chi(\ell) = -1 \) and \( \chi(1 - \ell) = 1 \) so that for all \( a \in \mathbb{Q} \)

\[
[a]_\chi = [a + t\ell]_\chi
\]

for all \( t \in \mathbb{Z} \) and hence \([a]_\chi = [a + tq]_\chi\) for all \( t \in \mathbb{Z} \).

Thus for all \( a \in \mathbb{Q} \) we have

\[
[a]_\chi = [a + tp + sq]_\chi \quad \text{for all } t, s \in \mathbb{Z}
\]

proving the proposition in this case.

Suppose now that \( p = 2 \).

If \( q \equiv 5 \pmod{8} \) then \( v_2(1 - q) = 2 \) so that \( \chi(1 - q) = 1 \) and we can take \( \ell = q \) and argue as above.

If \( q \equiv 3 \pmod{8} \) the corresponding argument applies with \( \ell = -q \).

If \( q \equiv -1 \pmod{8} \) we can take \( \ell = 3q \). Then \( \chi(\ell) = -1 \) (since \( q \not\equiv 3 \)). Furthermore we have

\[
\ell - 1 \equiv 4 \pmod{8}
\]

and

\[
0 < \frac{\ell - 1}{4} < q.
\]

This implies \( \chi(\ell - 1) = \chi(1 - \ell) = 1 \) and we can conclude as before.

Finally, if \( q \equiv 1 \pmod{8} \) we take \( \ell = -3q \) and argue as in the previous case. \(\)  

**Corollary 3.11.** Let \( \chi \in \hat{\mathbb{Q}}_+/\mathbb{Q}_+^2 \) and suppose that \( |\text{supp}(\chi)| \geq 2 \). Then \( \mathcal{RP}_+(\mathbb{Q}) \left[ \frac{1}{2} \right] \chi = 0 \).

Proof. We will show that \( [a]_\chi = 0 \) for all \( a \in \mathbb{Q} \). By Proposition 3.10 we have \( [a]_\chi = [a + t]_\chi \) for all \( t \in \mathbb{Z} \), \( a \in \hat{\mathbb{Q}}^\times \). It follows that \( [a]_\chi = [1]_\chi = 0 \) if \( a \in \mathbb{Z} \). Thus also \( [1/a]_\chi = 0 \) for all \( a \in \mathbb{Z} \setminus \{0\} \).

Note that it is enough to prove \([a]_\chi = 0 \) for all \( a > 0 \) (if necessary replacing \( a \) by \( a + t \) with \( t \in \mathbb{Z} \) large). So let \( a = r/s \) with \( 0 < r, s \in \mathbb{Z} \). We proceed by induction on \( h := \min(r,s) \). The
Corollary 3.13. For any prime $p$, the homomorphism $S_p : R^+_\mathbb{Q}[\frac{1}{2}] \rightarrow \mathcal{P}(\mathbb{F}_p) \{\frac{1}{2}\}$ induces an isomorphism of $R_\mathbb{Q}$-modules

$R^+_\mathbb{Q}[\frac{1}{2}]_{x_p} \cong \mathcal{P}(\mathbb{F}_p) \{\frac{1}{2}\}$.

Proof. By Lemma 3.12, $R^+_\mathbb{Q}[\frac{1}{2}]_{x_p}$ is generated by $\{[u]_{x_p} \mid u \in U_p\} \cup \{C_\mathbb{Q}\}$. Furthermore, if $w \in U_{1,p}$ then $v_p(1-w) > 0$ and hence $[w]_{x_p} = C_\mathbb{Q} - [1-w]_{x_p} = 0$ by Lemma 3.12. It follows that for all $u \in U_p \setminus U_{1,p}$ and $w \in U_{1,p}$ we have

$$0 = (S_{u,wu})_{x_p} = [u]_{x_p} - [uw]_{x_p} + [w]_{x_p} - \left[w \cdot \frac{1-u}{1-uw}\right]_{x_p} + \left[\frac{1-u}{1-uw}\right]_{x_p},$$

since the last three terms of the first line line in $U_{1,p}$. Thus, for $u \in U_p$, the element $[u]_{x_p} \in R^+_\mathbb{Q}[\frac{1}{2}]_{x_p}$ depends only on $\bar{u} := u \mod p \in \mathbb{F}_p$. It follows that there is a well-defined $R_\mathbb{Q}$-module homomorphism

$$T_p : \mathcal{P}(\mathbb{F}_p) \{\frac{1}{2}\} \rightarrow R^+_\mathbb{Q}[\frac{1}{2}]_{x_p}.$$
Lemma 4.2. Let $A$ be a subring of $\mathbb{Q}$. We defer the technical proof to Section 6 below.

Proof. \[
[H] \mapsto [u]_{x^p}
\]
for $u \in U_p$. When $p = 2$, it is the map $T_2(mC_{\mathbb{Z}}) = mC_{\mathbb{Q}}$, $m \in \mathbb{Z}/3$.

It remains to show that the map $T_p$ is surjective. Certainly, $\{[u]_{x^p} \mid u \in U_p\}$ lies in its image. On the other hand, if $p \geq 3$ then $C_{\mathbb{Q}} = [-1]_{x^p} + [2]_{x^p}$ lies in its image while for $p = 2$, $C_{\mathbb{Q}}$ lies in the image by construction. \hfill $\square$

4. The third homology of $\text{SL}_2$ of subrings of $\mathbb{Q}$

We begin by introducing terminology and notation needed for the statement of the crucial Theorem 4.1 below.

For a commutative ring $A$, let $W_A$ denote the set $\{u \in A^\times : 1 - u \in A^\times\}$. We define $\mathcal{RP}(A)$ to be the $\mathbb{R}_A$-module with generators $\{x, x \in W_A\}$ subject to the relations

\[
S_{x,y} : [x] - [y] + \langle x \rangle \left[\frac{y}{x}\right] - \langle x^{-1} \rangle - 1 \left[\frac{1-x^{-1}}{1-y^{-1}}\right] + \langle 1-x \rangle \left[\frac{1-x}{1-y}\right], \quad \text{for all } x, y, y/x \in W_A.
\]

Let $\Lambda = (\lambda_1, \lambda_2)$ be the $\mathbb{R}_A$-module homomorphism

\[
\mathcal{RP}(A) \rightarrow \mathcal{I}_A^2 \oplus S_2^\mathbb{Z}(A^\times)
\]

where $\lambda_1 : \mathcal{RP}(A) \rightarrow \mathcal{I}_A^2$ is the map $[x] \mapsto \langle 1-x \rangle \langle x \rangle$, and $\lambda_2$ is the composite

\[
\mathcal{RP}(A) \xrightarrow{\mathcal{P}(A)} A \xrightarrow{\Lambda} S_2^\mathbb{Z}(A^\times).
\]

(It is straightforward to verify directly that $\Lambda$ is well-defined module homomorphism.)

The refined Bloch group of the commutative ring $A$ (with at least 4 elements) is the $\mathbb{R}_A$-module

\[
\mathcal{RB}(A) := \text{Ker}(\Lambda : \mathcal{RP}(A) \rightarrow \mathcal{I}_A^2 \oplus S_2^\mathbb{Z}(A^\times)).
\]

For a general commutative ring there is no reason to suppose the existence of a natural map $H_3(\text{SL}_2(A), \mathbb{Z}) \rightarrow \mathcal{RB}(A)$. However we do have the following useful result:

For a commutative ring $A$, let $T_A$ denote the subgroup $\{\text{diag}(u, u^{-1}) \mid u \in A^\times\}$ of $\text{SL}_2(A)$ and let $B_A$ denote the subgroup

\[
B_A = \left\{ \begin{pmatrix} u & a \\ 0 & u^{-1} \end{pmatrix} \mid u \in A^\times, a \in A \right\}.
\]

Thus $T_A$ is naturally a subgroup of $B_A$.

Theorem 4.1. Let $A$ be a commutative ring satisfying $H_2(T_A, \mathbb{Z}) \cong H_2(B_A, \mathbb{Z})$. Let $\phi : A \rightarrow F$ be a ring homomorphism from $A$ to a field $F$. Let $\xi$ lie in the image of the map $\mathcal{RB}(A) \rightarrow \mathcal{RB}(F)$. Then there exists $X \in H_3(\text{SL}_2(A), \mathbb{Z})$ mapping to $\xi$ under the composite map $H_3(\text{SL}_2(A), \mathbb{Z}) \rightarrow H_3(\text{SL}_2(F), \mathbb{Z}) \rightarrow \mathcal{RB}(F)$.

Proof. We defer the technical proof to Section 6 below. \hfill $\square$

The next two lemmas show that the hypothesis of Theorem 4.1 holds for many subrings of $\mathbb{Q}$.

Lemma 4.2. Let $A$ be a subring of $\mathbb{Q}$ satisfying $6 \in A^\times$. Then the natural map $H_2(T_A, \mathbb{Z}) \rightarrow H_2(B_A, \mathbb{Z})$ is an isomorphism.
Proof. By considering the Hochschild-Serre spectral sequence associated to the group extension

\[ 1 \to U_A \to B_A \to T_A \to 1 \]

where

\[ U_A = \left\{ \begin{array}{ll} 1 & a \in A \\ a & 1 \end{array} \right\}, \]

it is enough to establish the vanishing of the groups \( H_0(T_A, H_2(U_A, \mathbb{Z})) \) and \( H_1(T_A, H_1(U_A, \mathbb{Z})) \). Furthermore, since \( A \) is a colimit of infinite cyclic groups, \( H_2(A, \mathbb{Z}) = 0 \). We need only prove the vanishing of \( H_1(A^x, A) \) where \( u \in A^x \cong T_A \) acts on \( A \cong U_A \) as multiplication by \( u^2 \). However, the pair of maps (conjugation by \( u, u \cdot ) : (A^x, A) \to (A^x, A) \) induces the identity on the groups \( H_i(A^x, A) \). Taking \( u = 2 \in A^x \), we deduce that multiplication by \( 4 = 2^2 \) is the identity map on \( H_1(A^x, A) \) and hence that this group is annihilated by 3. But \( 3 \in A^x \) by hypothesis and so acts invertibly on \( H_1(A^x, A) \). Thus \( H_1(A^x, A) = 0 \) as required.

Lemma 4.3. If \( A = \mathbb{Z}\left[\frac{1}{2}\right] \) then \( H_2(T_A, \mathbb{Z}) \cong H_2(B_A, \mathbb{Z}) \).

Proof. As in the preceding lemma, is is enough to show that \( H := H_1(A^x, A) = 0 \). Since \( 2 \in A^x \), we again deduce that \( H \) is annihilated by 3.

We have a short exact sequence of groups \( 1 \to \mu_2 \to A^x \to 2\mathbb{Z} \to 1 \). Since \( 2 \) acts invertibly on \( A \), the groups \( H_i(\mu_2, A) \) vanish for \( i \geq 1 \) and the Hochschild-Serre spectral sequence induces and isomorphism \( H_i(A^x, A) \cong H_i(2\mathbb{Z}, A) \) for all \( i \). In particular, \( H_1(A^x, A) = H_1(2\mathbb{Z}, A) = \{ a \in A | 3a = 0 \} = 0 \).

Example 4.4. \(-1 \in W_{\frac{1}{2}}[1] \) and \([-1] + (-1)[-1] \in RB(\mathbb{Z}\left[\frac{1}{2}\right]) \). This element maps to \( \psi_1(-1) \in RB(\mathbb{Q}) \). It follows that there exists a homology class \( X \in H_3(SL_2(\mathbb{Z}\left[\frac{1}{2}\right]), \mathbb{Z}) \) mapping to \( \psi_1(-1) \in RB(\mathbb{Q}) \).

For a subring \( A \) of \( \mathbb{Q} \) we denote by \( H_3(SL_2(A), \mathbb{Z})_0 \) the kernel of the map \( H_3(SL_2(A), \mathbb{Z}) \to K_3^{ind}(\mathbb{Q}) \).

Proposition 4.5. There is a split short exact sequence of \( R_{\frac{1}{2}} \)-modules

\[ 0 \to H_3(SL_2(\mathbb{Z}\left[\frac{1}{2}\right]), \mathbb{Z})_0 \to H_3(SL_2(\mathbb{Z}\left[\frac{1}{2}\right]), \mathbb{Z}) \to K_3^{ind}(\mathbb{Q}) \to 0 \]

where \( H_3(SL_2(\mathbb{Z}\left[\frac{1}{2}\right]), \mathbb{Z})_0 \) is a cyclic group of order 3 on which the square class \( \langle 2 \rangle \) acts as multiplication by \(-1\).

Proof. In [11] it is shown that

\[ H_3(SL_2(\mathbb{Z}\left[\frac{1}{2}\right]), \mathbb{Z}) \cong \mathbb{Z}/8 \times \mathbb{Z}/3 \times \mathbb{Z}/3 \]

as an abelian group. Thus, replacing \( X \) by \( 3X \) if necessary, there exists \( X \in H_3(SL_2(\mathbb{Z}\left[\frac{1}{2}\right]), \mathbb{Z}) \) of 2-power order mapping to \( \psi_1(-1) \).

Consider now the commutative diagram with exact bottom row

\[
\begin{array}{ccc}
H_3(SL_2(\mathbb{Z}\left[\frac{1}{2}\right]), \mathbb{Z}) & \longrightarrow & RB(\mathbb{Q}) \\
\downarrow & & \\
0 & \longrightarrow & \text{Tor}_{1}^{\mathbb{Z}}(\mu_2, \mu_3) \longrightarrow K_3^{ind}(\mathbb{Q}) \longrightarrow B(\mathbb{Q}) \longrightarrow 0.
\end{array}
\]
Here $K_3^{\text{ind}}(\mathbb{Q})$ is cyclic of order 24 and $\text{Tor}^2_{\mathbb{Q}}(\mu_2, \mu_3)$ is cyclic of order 4. The element $\psi_1(-1) \in \mathcal{R}(\mathbb{Q})$ maps to 2 $[-1] \in \mathcal{B}(\mathbb{Q})$ which has order 2. Thus $\pi(X)$ had order 2 in $\mathcal{B}(\mathbb{Q})$ and hence $\alpha(X)$ had order 8 in $K_3^{\text{ind}}(\mathbb{Q})$. So $X$ had order 8 $\alpha$ induces an isomorphism on 2-primary torsion.

Now $K_3^{\text{ind}}(\mathbb{Q})_{(3)} \cong \mathcal{B}(\mathbb{Q})_{(3)} = \mathbb{Z}/3 \cdot C_3$ by Theorem 2.4. Furthermore, 1 $\in \mathbb{Z}/3 \subset \mathbb{Z}/12$ maps to $\pm C_3$ under the composite $H_3(SL_2(\mathbb{Z}), \mathbb{Z}) \to H_3(SL_2(\mathbb{Q}), \mathbb{Z}) \to \mathcal{B}(\mathbb{Q})$ ([3, Remark 3.14]). Let $C_3 \in H_3(SL_2(\mathbb{Z}[\frac{1}{2}]), \mathbb{Z})$ denote the image of 1 $\in \mathbb{Z}/3 \subset H_3(SL_2(\mathbb{Z}), \mathbb{Z})$ under the map $H_3(SL_2(\mathbb{Z}), \mathbb{Z}) \to H_3(SL_2(\mathbb{Z}[\frac{1}{2}]), \mathbb{Z})$ (determined only up to sign). Thus $\alpha(C_3)$ has order 3 in $K_3^{\text{ind}}(\mathbb{Q})$ and $\alpha$ is $\mathbb{Z}$-split surjective with kernel of order 3.

Furthermore, by [3, Remark 3.14] again, $C_3 \in H_3(SL_2(\mathbb{Z}[\frac{1}{2}]), \mathbb{Z})$ maps to $C_3 \in \mathcal{R}(\mathbb{Q})$. This in turn maps to $C_3 \in \mathcal{R}(\mathbb{Q}[\frac{1}{2}]) \{2\}$ under the $\mathbb{Q}$-module – and hence $\mathbb{Q}[\frac{1}{2}]$-module – homomorphism $S_2 : \mathcal{R}(\mathbb{Q}) \to \mathcal{R}(\mathbb{Q}[\frac{1}{2}]) \{2\}$. It follows that $\langle 2 \rangle \in R_{\mathbb{Q}[\frac{1}{2}]}$ acts nontrivially on $C_3 \in H_3(SL_2(\mathbb{Z}[\frac{1}{2}]), \mathbb{Z})$ and hence $\langle 2 \rangle C_3$ has order 3 and lies in $H_3(SL_2(\mathbb{Z}[\frac{1}{2}]), \mathbb{Z})_0 := \text{Ker}(H_3(SL_2(\mathbb{Z}[\frac{1}{2}]), \mathbb{Z}) \to K_3^{\text{ind}}(\mathbb{Q}))$, since $\langle 2 \rangle$ acts trivially on $K_3^{\text{ind}}(\mathbb{Q})$.

**Remark 4.6.** Since the square class $(-1)$ acts trivially on $K_3^{\text{ind}}(\mathbb{Q})$ and on $C_3$, it acts trivially on all of $H_3(SL_2(\mathbb{Z}[\frac{1}{2}]), \mathbb{Z})$.

**Lemma 4.7.** Let $\phi : A \to F$ be a ring homomorphism from a commutative ring $A$ to a field $F$. Suppose that $v_1, \ldots, v_m$ are discrete valuations on $F$ satisfying the following conditions:

1. $\phi(A) \subset O_{v_i}$ for all $i$ and the induced homomorphisms $\psi_A \to k(v)_x$ are surjective for all $i$.
2. The homomorphism $A^x \to \bigoplus_{i=1}^m \mathbb{Z}/2, a \mapsto (v_i(\phi(a)))_{i=1}^m$ is surjective.

Then the composite homomorphism $\mathcal{R}(A) \to \mathcal{R}(F) \to \bigoplus_{i=1}^m \mathcal{P}(k(v_i)) \{v_i\}$ has cokernel annihilated by 4.

**Proof.** For each $i \leq m$ choose $u_i \in A^x$ with $v_i(\phi(u_i)) \equiv \delta_{ij} \pmod{2}$. For any $a \in \mathcal{W}_A$, it is easily verified that $g(a) := (1 + (-1)) [a] + \langle 1 - a \rangle \psi_1(a) \in \mathcal{R}(A)$.

Thus for any $i, j \leq m$, we have

$$S_{v_i}([g(a)] \cdot \bar{a}) = -4\delta_{ij} \bar{a} \in \mathcal{P}(k(v_j)) \{v_j\}. \quad \square$$

**Corollary 4.8.** Assume the same hypotheses as in Lemma 4.7. Assume furthermore that $H_2(T_A, \mathbb{Z}) \cong H_2(B_A, \mathbb{Z})$. Then $H_3(SL_2(A), \mathbb{Z}[\frac{1}{2}])$ surjects naturally onto $\bigoplus_{i=1}^m \mathcal{P}(k(v_i)) \{v_i\}[\frac{1}{2}]$.

**Proof.** This is an immediate consequence of Theorem 4.1 and Lemma 4.7. \(\square\)

**Example 4.9.** For $t \geq 1$, let $N_t := p_1 \cdots p_t$ where $2 = p_1 < \cdots < p_t$ are the first $t$ prime numbers in order of size. By Corollary 4.8, the composite homomorphism

$$H_3(SL_2(A), \mathbb{Z}[\frac{1}{2}]) \to H_3(SL_2(Q), \mathbb{Z}[\frac{1}{2}]) \to \bigoplus_{i=1}^t \mathcal{P}(\mathbb{F}_p) \{\frac{1}{2}\}$$

is surjective.

**Lemma 4.10.** Let $p \geq 11$ be prime and let $A$ be a subring of $\mathbb{Z}(\rho) := \{m/n \in \mathbb{Q} \mid p \text{ does not divide } m\}$. The composite map

$$H_3(SL_2(A), \mathbb{Z}[\frac{1}{2}]) \to H_3(SL_2(Q), \mathbb{Z}[\frac{1}{2}]) \to \mathcal{P}(\mathbb{F}_p) \{p\}[\frac{1}{2}]$$

is surjective.
Remark 4.13. For any prime \( p \) and is isomorphic to \( \mathbb{Z} \).

Example 4.12. Proposition 4.11 applies to the ring \( \mathbb{Z} \).

Let \( S \) be a set of primes containing \( 2, 3, 5, \) and \( 7 \). It follows that the map \( H_3(\mathbb{Z}_p), \mathbb{Z}[\frac{1}{2}] \rightarrow H_3(\mathbb{Z}_p), \mathbb{Z}[\frac{1}{2}] \) is the zero map.

But the following commutative diagram shows that \( S_p \) factors through this map:

\[
\begin{array}{c}
H_3(\mathbb{Z}_p), \mathbb{Z}[\frac{1}{2}] \\
\downarrow \quad \downarrow S_p \\
H_3(\mathbb{Z}_p), \mathbb{Z}[\frac{1}{2}] \\
\downarrow \quad \downarrow \mathcal{P}(\mathbb{F}_p)[\frac{1}{2}] \\
H_3(\mathbb{Z}_p), \mathbb{Z}[\frac{1}{2}] \\
\end{array}
\]

For any subring \( A \) of \( \mathbb{Q} \) let \( \mathbb{H}_3(\mathbb{Q}) \) denote the image of \( H_3(\mathbb{Q}), \mathbb{Z} \rightarrow H_3(\mathbb{Q}), \mathbb{Z} \) and let \( \mathbb{H}_3(\mathbb{Q}) \) denote \( \ker(\mathbb{H}_3(\mathbb{Q}), \mathbb{Z}) \rightarrow K_3(\mathbb{Q}) \).

Combining Lemma 4.10 and Corollary 4.8 we deduce:

**Proposition 4.11.** Let \( S \) be a set of primes containing \( 2, 3, 5, \) and \( 7 \). Let \( \mathbb{Z}_S := \{ m/n \in \mathbb{Q} \mid \nu_p(m) = 0 \text{ for all } p \notin S \} \). Suppose further that \( A = \mathbb{Z}_S \) satisfies the hypotheses of Lemma 4.7. Then \( \mathbb{H}_3(\mathbb{Q}), \mathbb{Z}[\frac{1}{2}] \rightarrow \mathbb{P}(\mathbb{f}_p)[\frac{1}{2}] \) is an \( R_{\mathbb{Q}} \)-submodule of \( H_3(\mathbb{Q}), \mathbb{Z}[\frac{1}{2}] \) and is isomorphic to \( \bigoplus_{p \in S} \mathbb{P}(\mathbb{f}_p)[\frac{1}{2}] \).

**Example 4.12.** Proposition 4.11 applies to the ring \( A = \mathbb{Z}_S = \mathbb{Z}_p \); i.e. when \( S \) = Primes \( \setminus \{ p \} \) for any \( p \geq 11 \).

Remark 4.13. However, note that the full homology group \( H_3(\mathbb{Q}), \mathbb{Z}[\frac{1}{2}] \) is in general an \( R_{\mathbb{Q}} \)-submodule of \( H_3(\mathbb{Q}), \mathbb{Z}[\frac{1}{2}] \) but not usually an \( R_{\mathbb{Q}} \)-submodule (since \( S_p(C_\mathbb{Q}) \neq 0 \) when \( p \equiv 2 \mod 3 \)).

In view of Example 4.9 we deduce:

**Corollary 4.14.** For \( t \geq 4 \), as above let \( N_t := p_1 \cdots p_t \) where \( 2 = p_1 < \cdots < p_t \) are the first \( t \) prime numbers in order of size. Then \( \mathbb{H}_3(\mathbb{Q}), \mathbb{Z}[\frac{1}{N_t}] \rightarrow \mathbb{P}(\mathbb{f}_p)[\frac{1}{2}] \) is an \( R_{\mathbb{Q}} \)-submodule of \( H_3(\mathbb{Q}), \mathbb{Z}[\frac{1}{2}] \) and is isomorphic to \( \bigoplus_{i=1}^{t} \mathbb{P}(\mathbb{f}_p)[\frac{1}{2}] \).

Remark 4.15. The requirement \( 2, 3, 5, 7 \in S \) in Proposition 4.11 arise from the fact that the maps \( H_3(\mathbb{Q}), \mathbb{Z}[\frac{1}{2}] \rightarrow K_3(\mathbb{Q}_p)[\frac{1}{2}] \) are proved to be isomorphisms (in \( \mathbb{Z} \)) only for \( p \geq 11 \). This restriction is an artefact of the method of proof and these maps may well be isomorphisms for some or all of the remaining primes.
Theorem 8.1. The module $V$ where $B$ phically to $K$ is a torsion $Z$-module, we deduce that $V$ splits as a sequence of $Z$-modules since the subgroup $Z/3 \cdot C_Q \subset \mathcal{R}_F$ of $\mathcal{P}(Q)$ maps isomorphically to $\mathcal{B}(Q)$. Thus, in view of Theorem 3.1 we have:

**Lemma 5.1.** As a $Z\left[\frac{1}{2}\right]$-module, $\mathcal{R}_F(P)\left[\frac{1}{2}\right]$ is a direct sum of an infinite torsion group and a free $Z\left[\frac{1}{2}\right]$-module $V$ of countable rank. More particularly:

$$\mathcal{R}_F(P)\left[\frac{1}{2}\right] \cong \bigoplus_{p} \mathbb{Z}/(p+1)_{odd} \oplus \mathbb{Z}/3 \oplus V$$

**Corollary 5.2.** As an abelian group we have

$$H_3\left(\text{SL}_2\left(\mathbb{Q}[t, t^{-1}]\right), Z\left[\frac{1}{2}\right]\right) \cong \bigoplus_{p} \mathbb{Z}/(p+1)_{odd} \oplus (\mathbb{Z}/3)^{\oplus 2} \oplus V.$$
5.2. The module $D_Q$. We let $D_F$ denote the $R_F$-submodule of $\mathcal{R}P_+ (F)$ generated by $C_F$. Note that $3 \cdot D_F = 0$; $D_F$ is an $F_3$-vector space.

For any field $F$, let $H = H_F$ denote the $R_F$-submodule of $H_3 (SL_2 (F), \mathbb{Z})$ generated by the image of $H_3 (SL_2 (\mathbb{Z}), \mathbb{Z})$.

**Remark 5.3.** Since the $R_F$-module structure on $H_3 (SL_2 (F), \mathbb{Z})$ is induced from action of $GL_2(F)$ by conjugation on $SL_2 (F)$, $H = H_F$ is just the subgroup $\sum_{g \in GL_2(F)} H_3 (SL_2 (\mathbb{Z}), \mathbb{Z})$ in $H_3 (SL_2 (F), \mathbb{Z})$; i.e it is the subgroup of $H_3 (SL_2 (F), \mathbb{Z})$ generated by $SL_2 (\mathbb{Z})$ and its $GL_2(F)$-conjugates.

**Proposition 5.4.** Suppose that $\text{char}(F) \neq 3$ and $\zeta_3 \notin F$.

Then the map $H_3 (SL_2 (F), \mathbb{Z}) \to \mathcal{R}P_+ (F)$ induces an isomorphism $H \left( \frac{1}{2} \right) \cong H(3) \cong D_F$.

**Proof.** As above, let

$$ t := \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} \in SL_2 (\mathbb{Z}) $$

and let $G$ be the cyclic subgroup of order 3 generated by $t$. By [3, Remark 3.14], the composite map $\mathbb{Z}/3 = H_3 (G, \mathbb{Z}) \to H_3 (SL_2 (F), \mathbb{Z}) \to \mathcal{R}P_+ (F)$ sends 1 to $C_F$ for any field $F$.

We recall that $H_3 (SL_2 (\mathbb{Z}), \mathbb{Z}) \cong \mathbb{Z}/12$. Furthermore, the inclusion $G \to SL_2 (\mathbb{Z})$ induces an isomorphism

$$ \mathbb{Z}/3 \cong H_3 (G, \mathbb{Z}) \cong H_3 (SL_2 (\mathbb{Z}), \mathbb{Z}) \cong H_3 (SL_2 (\mathbb{Z}), \mathbb{Z}) \left( \frac{1}{2} \right). $$

Thus $H(3) \cong H \left( \frac{1}{2} \right)$ maps onto $D_F$.

On the other hand, the kernel of the map $H_3 (SL_2 (F), \mathbb{Z}) \left( \frac{1}{2} \right) \to \mathcal{R}P_+ (F) \left( \frac{1}{2} \right)$ is isomorphic to $\mu_F \left( \frac{1}{2} \right)$. In particular, if $\zeta_3 \notin F$, the induced map $H(3) \to D_F$ is also injective. \qed

**Lemma 5.5.** There is a short exact sequence of $R_Q$-modules

$$ 0 \to I_Q D_Q \to D_Q \to K_3^{\text{ind}} (Q) \left( \frac{1}{2} \right) \to 0 $$

where the maps $S_p$ induce an isomorphism

$$ I_Q D_Q \cong \bigoplus_{p \equiv 2 \mod 3} \mathcal{P}(F_p) \left( \frac{1}{2} \right) \cong \bigoplus_{p \equiv 2 \mod 3} \mathbb{Z}/3. $$

**Proof.** Since $K_3^{\text{ind}} (Q) \left( \frac{1}{2} \right) \cong B(Q) \left( \frac{1}{2} \right) = \mathbb{Z}/3 \cdot C_Q$, the first part of the statement is clear. For the second part we have the composite map

$$ I_Q D_Q \to I_Q \mathcal{R}P_+ (Q) \left( \frac{1}{2} \right) \to \bigoplus_p \mathcal{P}(F_p) \left( \frac{1}{2} \right) $$

where the right-hand arrow is an isomorphism by Theorem [3.1]. We finish by observing that $\mathcal{P}(F_p) \left( \frac{1}{2} \right) \cong \mathbb{Z}/(p + 1)$ has no 3-torsion except when $p \equiv 2 \mod 3$ and when $p \equiv 2 \mod 3$, the element $C_{F_p} = S_p (C_Q)$ has order 3 by [4, Lemma 7.11] \qed

**Remark 5.6.** Thus $H_3 (SL_2 (Q), \mathbb{Z}) \left( \frac{1}{2} \right)$ has torsion of every possible size. However, the solutions of the equation $3z = 0$ in this group all come from the obvious source: the 3-torsion in $SL_2 (\mathbb{Z})$ and its $GL_2(Q)$-conjugates in $SL_2 (Q)$. More precisely, a basis for the $F_3$-vector space $\{ z \in H_3 (SL_2 (Q), \mathbb{Z}) \left( \frac{1}{2} \right) : 3z = 0 \}$ is $\{ \tau \} \cup \{ \tau_p \mid p \equiv 2 \mod 3 \}$ where $\tau$ is the image of $1 \in \mathbb{Z}/3 = H_3 (t, \mathbb{Z}) \to H_3 (SL_2 (Q), \mathbb{Z}) \left( \frac{1}{2} \right)$ and $\tau_p$ is the image of $1 \in \mathbb{Z}/3 = H_3 (t^D_p, \mathbb{Z}) \to H_3 (SL_2 (Q), \mathbb{Z}) \left( \frac{1}{2} \right)$ with $D_p := \text{diag}(p, 1) \in GL_2(Q)$.\qed
6. Proof of Theorem 4.1

In this section we use the notation of [7] (which we repeat below), and will rely on several calculations given there.

Let $A$ be a commutative ring. A row vector $\mathbf{u} = (u_1, u_2) \in A^2$ is said to be unimodular if $Au_1 + Au_2 = A$. Equivalently, $\mathbf{u}$ is unimodular if there exists $\mathbf{v} \in A^2$ such that

$$\begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix} \in \text{GL}_2(A).$$

We let $U_2 = U_2(A)$ denote the set of 2-dimensional unimodular row vectors of $A$. $U_2$ is a right $\text{GL}_2(A)$-set. In particular this induces an action of $A^\times = \text{Z}(\text{GL}_2(A))$ acting as multiplication by scalars.

Let

$$\mathcal{U}^\text{gen}_n = \mathcal{U}^\text{gen}_n(A) := \left\{ (\mathbf{u}_1, \ldots, \mathbf{u}_n) \in U_2^n : \begin{bmatrix} \mathbf{u}_i \\ \mathbf{u}_j \end{bmatrix} \in \text{GL}_2(A) \text{ for all } i \neq j \right\}.$$

$(A^\times)^n$ acts entry-wise (on the left) on $\mathcal{U}^\text{gen}_n$ and we let $X_n = X_n(A) = \mathcal{U}^\text{gen}_n / (A^\times)^n$. Observe that $\mathcal{U}^\text{gen}_n$ and $X_n$ are right $\text{GL}_2(A)$-sets (with the natural diagonal action).

In particular, $X_1 = U_2/A^\times$ and $X_n \subset X_1^n$. If $\mathbf{u} = (u_1, u_2) \in U_2$ we will denote the corresponding class in $X_1$ by $\hat{\mathbf{u}}$ or $[u_1, u_2]$.

We have two natural injective maps from $A$ to $X_1$:

$$\iota_+: A \to X_1, \ a \mapsto a_+ := [a, 1] \text{ and } \iota_-: A \to X_1, \ a \mapsto a_- := [1, a]$$

Let $L_n = L_n(A) := \mathbb{Z}[X_{n+1}]$. Let $\delta_n: L_n \to L_{n-1}$ be the usual simplicial boundary map

$$(\mathbf{u}_1, \ldots, \mathbf{u}_{n+1}) \mapsto \sum_{i=1}^{n+1} (-1)^{i+1} (\mathbf{u}_1, \ldots, \hat{\mathbf{u}}_i, \ldots, \mathbf{u}_{n+1})$$

This yields a complex, $L_\ast$, of right $\text{GL}_2(A)$-modules. Restricting the group action, this is also a complex of $\text{SL}_2(A)$-modules.

We will prove Theorem 4.1 by studying the hyperhomology groups $H_i(\text{SL}_2(A), L_\ast)$. Recall that these are the homology groups of the total complex of the double complex $L_\ast \otimes_{\mathbb{Z}[\text{SL}_2(A)]} \Gamma_\ast$ where $\Gamma_\ast$ denotes a (left) projective resolution of $\mathbb{Z}$ as a trivial $\mathbb{Z}[\text{SL}_2(A)]$-module.

**Proposition 6.1.** Let $A$ be a commutative ring for which $H_2(T_A, \mathbb{Z}) \cong H_2(B_A, \mathbb{Z})$. then there is a natural surjective homomorphism

$$H_3(\text{SL}_2(A), L_\ast) \longrightarrow \mathcal{R}\mathcal{B}(A).$$

**Proof.** There is a spectral sequence of the form

$$E^1_{p,q} = H_p(\text{SL}_2(A), L_q) \Longrightarrow H_{p+q}(\text{SL}_2(A), L_\ast).$$

We will prove that $E^\infty_{p,0} = \mathcal{R}\mathcal{B}(A)$ naturally under the given hypotheses.

We calculate the $E^1$-terms of the spectral sequence for $q \leq 4$. $L_q = \mathbb{Z}[X_{q+1}]$ is a permutation module in each case, and hence an induced module. Proofs of the following statements can be found in [7] Section 3.1:

$X_1$ is a transitive $\text{SL}_2(A)$ set and the stabiliser of $0_+$ is $B_A$. Thus $L_0 \cong \text{Ind}_{B_A}^{\text{SL}_2(A)} \mathbb{Z}$ and $E^1_{p,0} = H_p(\text{SL}_2(A), L_0) \cong H_p(B_A, \mathbb{Z})$ by Shapiro’s Lemma.

$X_2$ is a transitive $\text{SL}_2(A)$ set and the stabiliser of $(0_+, 0_-)$ is $T_A$. Thus $L_1 \cong \text{Ind}_{T_A}^{\text{SL}_2(A)} \mathbb{Z}$ and $E^1_{p,1} = H_p(\text{SL}_2(A), L_1) \cong H_p(T_A, \mathbb{Z})$ by Shapiro’s Lemma.
The orbits of $\text{SL}_2(A)$ acting on $X_3$ are indexed by $(0_+, 0_-, a_+)$, $a \in A^\times/(A^\times)^2$. The stabiliser of $(0_+, 0_-, a_+)$ is $\mu_2(A) = Z(\text{SL}_2(A))$. Thus $L_2 \cong \bigoplus_{(a) \in A^\times/(A^\times)^2} \text{Ind}_{\mu_2(A)}^A \langle a \rangle$ and $E_{p,2}^1 = H_p(\text{SL}_2(A), L_2) \cong \bigoplus_{(a) \in A^\times/(A^\times)^2} H_p(\mu_2(A), Z)$ by Shapiro’s Lemma. In particular, $E_{0,2}^1 = \bigoplus_{(a) \in A^\times/(A^\times)^2} Z \cdot \langle a \rangle = R_A$.

The orbits of $\text{SL}_2(A)$ acting on $X_4$ are indexed by symbols $\langle a \rangle \langle b \rangle$, $\langle a \rangle \in A^\times/(A^\times)^2$, $b \in \mathcal{W}_A$. The symbol $\langle a \rangle \langle b \rangle$ labels the orbit of $(0_+, 0_-, a_+, (ab)_+)$). The stabilisers are again the groups $\mu_2(A)$. Thus $L_3 \cong \bigoplus_{(a) \in A^\times/(A^\times)^2} \bigoplus_{b \in \mathcal{W}_A} \text{Ind}_{\mu_2(A)}^A \langle a \rangle \langle b \rangle$ and $E_{p,3}^1 = H_p(\text{SL}_2(A), L_3) \cong \bigoplus_{(a) \in A^\times/(A^\times)^2} \bigoplus_{b \in \mathcal{W}_A} H_p(\mu_2(A), Z)$ by Shapiro’s Lemma. In particular, $E_{2,3}^1 = \bigoplus_{b \in \mathcal{W}_A} R_A \cdot [b]$.

The orbits of $\text{SL}_2(A)$ acting on $X_2$ are indexed by symbols $\langle a \rangle \langle x, y \rangle$, $\langle a \rangle \in A^\times/(A^\times)^2$, $x, y \in \mathcal{W}_A$. The symbol $\langle a \rangle \langle x, y \rangle$ labels the orbit of $(0_+, 0_-, (ax)_+, (ay)_+)$). The stabilisers are again the groups $\mu_2(A)$. Thus $L_4 \cong \bigoplus_{(a) \in A^\times/(A^\times)^2} \bigoplus_{x, y \in \mathcal{W}_A} H_p(\mu_2(A), Z)$ by Shapiro’s Lemma. In particular, $E_{0,4}^1 = \bigoplus_{x, y \in \mathcal{W}_A} R_A \cdot [x, y]$.

The differential $d^1 : E_{0,4}^1 \to E_{2,4}^1$ is the $R_A$-module map $[x, y] \mapsto S_{x,y}$ while the differential $d^1 : E_{0,3}^1 \to E_{2,3}^1$ is the $R_A$-module map $a \mapsto \lambda_1$. Thus $E_{2,3}^1 = \mathcal{RP}_1(A)$.

Next we show that $E_{1,1}^2 = 0$: The differential $d^1 : E_{1,2}^1 = R_A \otimes \mu_2 \to E_{1,1}^1 = T_A$ is the map $\langle a \rangle \otimes 1 \mapsto -1$. Thus the image is $\mu_2(A) \subset T_A$. The differential $d^1 : E_{1,1}^1 = T_A = H_1(T_A, Z) \to H_1(B_A, Z)$ is $E_{1,0}^1$ of the map $u \mapsto u^2 \in T_A = H_1(T_A, Z) \subset H_1(B_A, Z)$. So $E_{2,1}^1 = 0$ as claimed and hence $E_{2,3}^1 = \mathcal{RP}_1(A)$.

The differential $d^1 : E_{2,1}^1 \to E_{2,0}^1 = E_{2,0}^1 = H_2(T_A, Z)$ is the zero homomorphism. Thus $E_{2,0}^2 = E_{2,0}^1$.

Now $d^1 : E_{1,3}^1 = R_A[\mathcal{W}_A] \otimes \mu_2 \to R_A \otimes \mu_2$ is the $R_A$-module homomorphism $[b] \otimes 1 \mapsto \lambda_1((b)) \otimes 1 = \langle b \rangle \langle 1-b \rangle \otimes 1$. As observed already, the map $d^1 : E_{1,2}^1 = R_A \otimes \mu_2 \to A^\times = E_{1,1}^1$ is the map $\langle a \rangle \otimes -1 \mapsto -1$. Thus $E_{2,2}^1 \cong I(A) \otimes \mu_2$ where $I(A) := \text{Ker}(GW(A) \to Z, \langle a \rangle \mapsto 1)$ with $GW(A) := R_A/\langle \langle b \rangle \langle 1-b \rangle \mid b \in \mathcal{W}_A \rangle$.

As in [4], we compute that $d^2 : E_{2,1}^2 = I(A) \otimes \mu_2 \to A^\times \wedge A^\times = E_{2,0}^2$ is the map $\langle a \rangle \otimes -1 \mapsto -1 \wedge a$. Thus $E_{2,0}^3 = (A^\times \wedge A^\times)/\langle \mu_2 \wedge A^\times \rangle$. Furthermore, if we compose the differential $d^3 : E_{3,0}^3 = \mathcal{RP}_1(A) \to E_{3,2}^3$ with the injective map

$$\frac{A^\times \wedge A^\times}{-1 \wedge A^\times} \to S_2^2(A^\times), \quad \lambda \wedge \mu \mapsto 2(\lambda \circ \mu)$$

we obtain the map $\lambda_2 : \mathcal{RP}_1(A) \to S_2^2(A^\times)^\times$. Hence $E_{0,3}^\infty = E_{0,3}^4 = \text{Ker}(\lambda_2 : \mathcal{RP}_1(A) \to S_2^2(A^\times)) = \mathcal{RP}_1(A)$ as required.

Now there is a natural map of complexes of $\mathbb{Z}[\text{SL}_2(A)]$-module $L_\bullet \to \mathbb{Z}[0]$, where $\mathbb{Z}[0]$ denotes the module $\mathbb{Z}$ concentrated in degree 0 and the map sends each $u \in X_1$ to 1. For a field $F$ with at least 4 elements (and for more general classes of rings with sufficiently many units) this is a weak equivalence in degrees $\leq 3$, and so induces isomorphisms in homology $H_1(\text{SL}_2(F), L_\bullet) \cong H_3(\text{SL}_2(F), \mathbb{Z})$. For a general commutative ring $A$ this last statement is no longer usually true.
Theorem 4.1 now follows at once from the commutative diagram

\[
\begin{array}{c}
\text{H}_3(\text{SL}_2(A), L(A)\bullet) \\
\downarrow \\
\text{H}_3(\text{SL}_2(A), \mathbb{Z}) \\
\downarrow \\
\text{H}_3(\text{SL}_2(F), \mathbb{Z}) \\
\rightarrow \\
\text{H}_3(\text{SL}_2(F), L(F)\bullet) \\
\downarrow \\
\text{RB}(F)
\end{array}
\]

where the horizontal arrows are surjective by Proposition 6.1.

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