Values of inhomogeneous forms at $S$-integral points

Anish Ghosh | Jiyoung Han

Abstract

We prove effective versions of Oppenheim’s conjecture for generic inhomogeneous forms in the $S$-arithmetic setting. We prove an effective result for fixed rational shifts and generic forms and we also prove a result where both the quadratic form and the shift are allowed to vary. In order to do so, we prove analogues of Rogers’ moment formulae for $S$-arithmetic congruence quotients as well as for the space of affine lattices. We believe the latter results to be of independent interest.

MSC (2020)
11P21 (primary)

Contents

1. INTRODUCTION .................................... 566
2. NOTATION AND RESULTS ............................... 567
   2.1. The Set-up ..................................... 567
   2.2. Notational remarks ................................. 568
3. VOLUME COMPUTATION FOR $\text{UL}_d(\mathbb{Z}_S) \setminus \text{UL}_d(\mathbb{Q}_S)$ and $\text{SL}_d(\mathbb{Z}_S) \setminus \text{SL}_d(\mathbb{Q}_S)$ ..... 570
4. THE FIRST AND SECOND MOMENT FORMULAE ............... 575
   4.1. Proof of Theorem 4.3 .............................. 576
   4.2. The Spaces $X_q$ and $X_q(y)$ ....................... 579
5. PROOF OF THEOREM 2.1 .............................. 585
6. THE SPACE OF INHOMOGENEOUS QUADRATIC FORMS .... 589
ACKNOWLEDGEMENTS ................................ 592
REFERENCES ........................................ 592

© 2022 The Authors. The publishing rights in this article are licensed to University College London under an exclusive licence. Mathematika is published by the London Mathematical Society on behalf of University College London.
Let $q$ be a quadratic form and consider the inhomogeneous quadratic form $q_\xi(v) := q(v + \xi)$ for $v \in \mathbb{R}^d$. Here, we refer to $\xi \in \mathbb{R}^d$ as a shift. We say that $q_\xi$ is indefinite if $q$ is indefinite and non-degenerate if $q$ is non-degenerate. Moreover $q_\xi$ is said to be irrational if either $q$ is an irrational quadratic form, that is, not proportional to a quadratic form with integer coefficients, or $\xi$ is an irrational vector. The main interest in this paper is to study values taken at integer points by inhomogeneous quadratic forms.

A celebrated theorem of Margulis [22] resolving an old conjecture of Oppenheim, states that for an indefinite irrational quadratic form $q$ in $d \geq 3$ variables, $q(\mathbb{Z}^d)$ is dense in $\mathbb{R}$. Further fundamental work in this direction was carried out by Dani and Margulis [8] and by Eskin, Margulis and Mozes [10, 11] who proved quantitative versions of Oppenheim’s conjecture under suitable hypotheses.

Inhomogeneous forms arise in a variety of situations in mathematics and physics and were studied by Marklof in his fundamental work [25, 26] on pair correlation densities and their relation to the Berry Tabor conjecture. In [24], Margulis and Mohammadi proved a quantitative version of Oppenheim’s conjecture for inhomogeneous forms. They showed that for any indefinite, irrational and non-degenerate inhomogeneous form $q_\xi$ in $d \geq 3$ variables there is $c_q > 0$ such that

$$\liminf_{t \to \infty} \frac{N(q_\xi, I, t)}{t^{d-2}} \geq c_q |I|.$$  

For $d \geq 5$, they showed that the above limit exists and equals $c_q |I|$. Here $N(q_\xi, I, t)$ is the counting function

$$N(q_\xi, I, t) = \#\{v \in \mathbb{Z}^d \mid q_\xi(v) \in I, \|v\| \leq t\},$$  

(1.1)

where $I \subseteq \mathbb{R}$ is an interval and $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^d$. This result is an inhomogeneous analogue of the results of Dani–Margulis and Eskin-Margulis–Mozes alluded to above. It constitutes a quantitative strengthening of, and therefore implies, an inhomogeneous analogue of the Oppenheim conjecture: namely that $q_\xi(\mathbb{Z}^d)$ is dense in $\mathbb{R}$ for any indefinite, irrational, non-degenerate inhomogeneous form $q_\xi$ in $d \geq 3$ variables. We refer the reader to [4] for a self-contained proof of this (qualitative) density. In [14, 15], Ghosh et al. considered an effective version of the above inhomogeneous theorem for generic forms. The term “effective” is used in the literature to refer to two related problems: the issue of error terms in the counting problem above, as well as the question of obtaining a good bound for $v$ which ensures that $0 < q_\xi(v) < \varepsilon$. The term “generic” refers to a full measure set of forms for an appropriate measure.

As far as inhomogeneous forms are concerned, there are three natural regimes in which one could study this problem. To wit, one could allow both the form and the shift to vary, that is, make the problem generic in two variables. Or alternatively, one could fix either the form or the shift and allow the other to vary. In Theorem 1.1 [14], Ghosh et al. noted that an effective result where both form and shift are allowed to vary, follows from an affine analogue of Rogers second moment formula for the space of affine lattices in conjunction with methods from an earlier paper [19] of Kelmer and Yu (which is discussed below). The case where either the form or the shift is fixed is significantly more difficult. One of the main results in [14] is an effective theorem for fixed shifts and generic quadratic forms. In order to prove such a theorem, the authors prove an analogue of Rogers’ second moment formula on congruence quotients of $\text{SL}_d(\mathbb{R})$. The complementary case of
fixed forms and generic shifts is studied in [15] using a very different technique, namely effective mean ergodic theorems.

Our aim in this paper is to prove an $S$-arithmetic effective Oppenheim theorem for inhomogeneous forms. In Theorem 2.4, we allow both form and shift to vary, while in Theorem 2.1, we treat the more difficult problem of fixing a rational shift and allowing the form to vary. We broadly follow the strategy of [14] where the main tool is an analogue of Rogers’ second moment formula for Siegel transforms on congruence quotients of $\text{SL}_d(\mathbb{R})$. Accordingly, we prove two new analogues of Rogers’ formula. In Theorem 6.1, we establish an $S$-arithmetic analogue of Rogers’ formula for the space of unimodular affine lattices. In Theorem 4.3, which is the technical heart of the paper, we obtain an analogue of Rogers’ formula for congruence quotients. We believe these results, especially Theorem 4.3 to be of independent interest. Indeed, the moment formula established in [14] has already found other Diophantine applications, cf. [1]. Rogers’ type results have found wide applicability in recent years.

Effective results for Oppenheim type problems have recently received considerable attention. In [21], Lindenstrauss and Margulis proved a remarkable effective result for ternary quadratic forms. Their result is valid for all irrational quadratic forms satisfying an explicit Diophantine condition. If one settles for a generic set of forms instead, then better results can be obtained. In [12], Ghosh et al. used effective mean ergodic theorems to prove a variety of results of this flavor including the case of ternary quadratic forms, namely the classical Oppenheim conjecture. The idea of using Rogers’ mean value formula to study effective versions of Oppenheim’s conjecture is due to Athreya and Margulis [3] and was further developed by Kelmer and Yu [19]. We also mention the work of Bourgain [7] on certain “uniform” versions of Oppenheim’s conjecture for diagonal forms. See also [13] for an analogue for ternary forms and [5, 20].

The study of values of quadratic forms at integer points in the $S$-arithmetic setting was initiated in the work of Borel and Prasad [6] who proved analogues of Margulis’s theorem. In [17], Han et al. obtained $S$-arithmetic generalizations of the quantitative Oppenheim problem in rank 5 and higher. The case of forms of rank 3 and 4 was studied by Han in [18]. Effective results for generic (homogeneous) forms in the $S$-arithmetic setting were obtained by Han in [16], who also established an $S$-arithmetic version of Rogers’ mean value formula. Finally, we mention that the question of values of rational quadratic forms with congruence conditions has also received attention recently, see for instance [4].

## 2 | NOTATION AND RESULTS

### 2.1 | The Set-up

Let $S$ be a finite set of places over $\mathbb{Q}$ including the infinite place, with the usual corresponding Euclidean norm $\| \cdot \|_\infty$. Let us denote by $S_f = \{p_1, \ldots, p_s\} = S - \{\infty\}$, where each prime $p_j$ represents the $p_j$-adic norm on $\mathbb{Q}$, the set of finite places in $S$. Define $Q_S = \prod_{p \in S} Q_p$, where $Q_p$ is the completion of $\mathbb{Q}$ with respect to the norm $\| \cdot \|_p$. Here, $Q_p = \mathbb{R}$ when $p = \infty$. Define the ring of $S$-integers by

$$Z_S = \{(z, \ldots, z) \in Q_S : z \in \mathbb{Q} \text{ with } |z|_v \leq 1 \text{ for } \forall v \notin S_f\}.$$ 

For $p = \infty$, let $\text{vol}_\infty$ be the canonical Lebesgue measure on $\mathbb{R}^d$, $d \geq 1$, and for each $p \in S_f$, we will consider the Haar measure $\text{vol}_p$ on $Q_p^d$ normalized by $\text{vol}_p(Z_p^d) = 1$. Let us define the
measure on \( \mathbb{Q}^d_S \) as \( \text{vol} = \prod_{p \in S} \text{vol}_p \). We also use the notation \( \text{vol}^{(d)} \) when we want to specify the dimension of the space.

We say that \( \Lambda \subseteq \mathbb{Q}^d_S \) is an \textit{S-lattice} if \( \Lambda \) is a free \( \mathbb{Z}_S \)-module of rank \( d \) and an \textit{affine S-lattice} if it is of the form \( \Lambda' + \xi \) for some \( S \)-lattice \( \Lambda' \) and an element \( \xi \in \mathbb{Q}^d_S \). An \( S \)-lattice (affine \( S \)-lattice, respectively) is \textit{unimodular} if \( \text{vol}(\Lambda \setminus \mathbb{Q}^d_S) = 1 \) (\( \text{vol}(\Lambda' \setminus \mathbb{Q}^d_S) = 1 \), respectively).

For \( p \in S_f \), define

\[
\text{UL}_d(Q_p) := \{ g_p \in \text{GL}_d(Q_p) : \det g_p|_p = 1 \}
\]

and let \( \text{UL}_d(\mathbb{R}) := \text{SL}_d(\mathbb{R}) \). Denote \( \text{UL}_d(Q_S) = \prod_{p \in S} \text{UL}_d(Q_p) \) and let \( g = (g_p)_{p \in S} \) be an element of \( \text{UL}_d(Q_S) \). Consider the group

\[
\text{AUL}_d(Q_S) = \{(\xi, g) : \xi \in \mathbb{Q}^d_S, g \in \text{UL}_d(Q_S)\}
\]

with the binary operation

\[
(\xi_1, g_1)(\xi_2, g_2) = (\xi_2 + g_2\xi_1, g_1 g_2).
\]

One can identify the space of unimodular affine \( S \)-lattices in \( \mathbb{Q}^d_S \) with \( \text{AUL}_d(\mathbb{Z}_S) \setminus \text{AUL}_d(\mathbb{Q}_S) \) using the correspondence

\[
\text{AUL}_d(\mathbb{Z}_S)(\xi, g) \leftrightarrow \mathbb{Z}^d_S g + \xi.
\]

### 2.2 Notational remarks

1. We will use sans serif typestyle font for parameters of an \( S \)-arithmetic space as we already use \( g = (g_p)_{p \in S} \) for an element of an \( S \)-arithmetic Lie group.

Denote \( p^Z := \{ p^z : z \in \mathbb{Z} \} \). For \( T = (T_p)_{p \in S} \), an element of \( \mathbb{R}_{>0} \times \prod_{p \in S_f} \mathbb{R}^Z \subseteq \mathbb{R}^{s+1}_{>0} \), we say that \( T = (T_p)_{p \in S} \succeq T' = (T'_p)_{p \in S} \) when \( T_p \geq T'_p \) for each \( p \in S \), and \( T \to \infty \) if \( T_p \to \infty \) for all \( p \in S \).

2. For \( S = \{\infty, p_1, \ldots, p_s\} \), define

\[
\mathbb{N}_S = \{ q \in \mathbb{N} : \gcd(q, p_1 \cdots p_s) = 1 \}
\]

and

\[
\mathbb{Z}^\times_S = \left\{ p_1^{z_1} \cdots p_s^{z_s} \in \mathbb{N} : z_j \in \mathbb{Z} \right\}.
\]

We remark that if we denote by \( P(\mathbb{Z}^d_S) = (e_1, \text{SL}_d(\mathbb{Z}_S)) \) (\( P(\mathbb{Z}^d) \), respectively) the set of primitive elements of \( \mathbb{Z}^d_S \) (\( \mathbb{Z}^d \), respectively), then one can easily check that \( P(\mathbb{Z}^d_S) = \mathbb{Z}^\times_S \cdot P(\mathbb{Z}^d) \).

A quadratic form \( q \) on \( \mathbb{Q}^d_S \) is a collection \( q = (q^{(p)})_{p \in S} \) of quadratic forms \( q^{(p)} \) over \( \mathbb{Q}_p \) for each \( p \in S \). We say that \( q \) is \textit{non-degenerate} (isotropic, respectively) if \( q^{(p)} \) is non-degenerate (isotropic, respectively) for all \( p \in S \). Recall that a form is \textit{isotropic} if there exists a non-zero vector \( v \in \mathbb{Q}^n_p \) such that \( q^{(p)}(v) = 0 \) and that for the infinite place, this condition is equivalent to being indefinite.

We say that the collection \( \{ I_T : T \} \) of Borel subsets of \( \mathbb{Q}_S \) is \textit{decreasing} if \( I_{T'} \subseteq I_T \) whenever \( T' \succeq T \). For a given \( T = (T_p)_{p \in S} \), let \( t_p \) be the integer for which \( T_p = p^{t_p} \) for each \( p \in S_f \). Our first
main theorem is an effective counting statement for quadratic forms with congruence conditions in the S-arithmetic setting.

**Theorem 2.1.** Let $d \geq 3$. Let $I = \{I_T = (I_T^{(p)})_{p \in S} : T\}$ be a decreasing family of bounded Borel sets in $\mathbb{Q}_S$ satisfying

$$\text{vol}_\infty(I_T^{(\infty)}) = c_\infty T_\infty^{-\kappa_\infty} \quad \text{and} \quad I_T^{(p)} = a_p + p^{c_p + \kappa_p} \mathbb{Z}_p \ (p \in S_f)$$

for some $(c_p)_{p \in S} \in \mathbb{R}_{>0} \times \prod_{p \in S_f} \mathbb{Z}$, $a_p \in \mathbb{Q}_p$ ($p \in S_f$) and

$$0 \leq \kappa_\infty < d - 2 \quad \text{and} \quad \begin{cases} \kappa_p \in \{0,1\}, & \text{if } d \geq 4; \\ \kappa_p = 0, & \text{if } d = 3. \end{cases} \quad (2.1)$$

Let $q \in \mathbb{N}_S$ and $p \in \mathbb{Z}^d$. For a quadratic form $q$ on $\mathbb{Q}_S^d$ and $T$, define

$$N(q, p ; q, I, T) = \# \{k \in q \mathbb{Z}_S^d + p : \|k\|_p < T_p \ (p \in S) \text{ and } q(k) \in I_T \}.$$ 

There is $\delta_0 > 0$ such that for any $\delta \in (0, \delta_0)$, we have

$$N(q, p ; q, I, T) = c_q \frac{1}{q^d} \text{vol}(I_T) |T|^{d-2} + o(\text{vol}(I_T)|T|^{d-2-\delta})$$

for almost all non-degenerate isotropic quadratic forms $q$, where $|T| = \prod_{p \in S} T_p$. Here, the implied constant of the error term is uniform on a compact set of the space of non-degenerate isotropic quadratic forms, and depends on a quadratic form $q$ and $q \in \mathbb{N}_S$.

**Remark 2.2.** We remark that the reason that the condition for the finite place in (2.1) is more restrictive than that of the infinite place is because of the difference in method used for the volume estimation for given regions in $\mathbb{R}^d$ and $\mathbb{Q}_p^d$ (see Theorem 5.3).

**Remark 2.3.** We note that if $q = 1$, then Theorem 2.1 follows from [16]. Hence throughout this paper, let us assume that $1 \neq q \in \mathbb{N}_S$.

Observe that putting $k_1 = \frac{1}{q} k$, we have

$$N(q, p ; q, I, T)$$

$$= \# \left\{ k_1 \in \mathbb{Z}_S^d + \frac{p}{q} : \frac{1}{q^2} \|k_1\|_\infty < \frac{T_\infty}{q}; \|k_1\|_p < T_p \ (p \in S_f) \text{ and } q(k_1) \in \frac{1}{q^2} I_T \right\}. \quad (2.2)$$

As observed in [14], values of inhomogeneous forms with rational shifts at integer points are simply scaled values of homogeneous forms at integer points satisfying congruence conditions.

Our next result is an effective counting result for inhomogeneous forms in the $S$-arithmetic setting where both the form and the shift are allowed to vary.
Theorem 2.4. Let $d \geq 3$ and let $I = \{I_T\}$ be as Theorem 2.1. For a quadratic form $q$ and $\xi \in \mathbb{Q}_S^d$, define

$$N(q_\xi, I, T) = \# \{ k \in \mathbb{Z}_S^d : \|k\|_p < T_p (p \in S) \text{ and } q_\xi(k) \in I_T \}. $$

There is $\delta_0 > 0$ such that for any $\delta \in (0, \delta_0)$,

$$N(q_\xi, I, T) = c_q \text{vol}(I_T)|T|^{d-2} + o_q(\text{vol}(I_T)|T|^{d-2-\delta})$$

for almost all pairs $(q, \xi)$ of a non-degenerate isotropic quadratic form and an element of $\mathbb{Q}_S^d$.

3 VOLUME COMPUTATION FOR $\text{UL}_d(\mathbb{Z}_S) \setminus \text{UL}_d(\mathbb{Q}_S)$ and $\text{SL}_d(\mathbb{Z}_S) \setminus \text{SL}_d(\mathbb{Q}_S)$

We first construct Haar measures $\mu_p^{(d)}$ and $\nu_p^{(d)}$ on $\text{UL}_d(\mathbb{Q}_p)$ and $\text{SL}_d(\mathbb{Q}_p)$, respectively, so that Haar measures $\mu^{(d)}$ and $\nu^{(d)}$ of $\text{UL}_d(\mathbb{Q}_S)$ and $\text{SL}_d(\mathbb{Q}_S)$ can be defined as $\mu^{(d)} = \prod_{p \in S} \mu_p^{(d)}$ and $\nu^{(d)} = \prod_{p \in S} \nu_p^{(d)}$, respectively.

Let $H$ be the subgroup of $\text{UL}_d(\mathbb{Q}_p)$ given by

$$H = \left\{ \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \cdots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 \\ \end{pmatrix} : \begin{pmatrix} v' \\ g' \end{pmatrix} \in \mathbb{Q}_p^{d-1}, \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_d \end{pmatrix} \in \mathbb{Q}_p^d \right\}. \quad (3.1)$$

Set $\mu_p^{(1)}(\text{UL}_1(\mathbb{Q}_p)) = \mu_p^{(1)}(\mathbb{Z}_p - p\mathbb{Z}_p) = 1 - 1/p$ and $\nu_p^{(1)}(\text{SL}_1(\mathbb{Q}_p)) = \nu_p^{(1)}(\{1\}) = 1$. For $d \geq 2$, consider the map

$$\Phi_p : \begin{array}{c} H \cap \text{SL}_d(\mathbb{Q}_p) \times \{ y = (y_1, \ldots, y_d) \in \mathbb{Q}_p^d : y_1 \neq 0 \} \\
\end{array} \rightarrow \begin{array}{c} \text{UL}_d(\mathbb{Q}_p) \\
\text{SL}_d(\mathbb{Q}_p) \\
\end{array}$$

given by

$$\Phi_p : ((v', g'), y) \mapsto A = \frac{1}{y_1} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ y_2 & y_3 & \cdots & y_d \\ 0 & 1/y_1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \cdots & \vdots \\ 0 & \cdots & \cdots & \cdots & 1 \end{pmatrix}. \quad (3.2)$$

Note that $\Phi_p$ is a diffeomorphism and its image covers $\text{UL}_d(\mathbb{Q}_p)$ ($\text{SL}_d(\mathbb{Q}_p)$, respectively) outside a co-dimension one submanifold. Let $\mu_p^{(d-1)}$ and $\nu_p^{(d-1)}$ be Haar measures on $\text{UL}_{d-1}(\mathbb{Q}_p)$ and $\text{SL}_{d-1}(\mathbb{Q}_p)$, respectively, from the induction hypothesis. It is well known that $\text{vol}_{p}^{(d-1)} \cdot \mu_{p}^{(d-1)}$ and
\[ \text{vol}_p^{(d-1)} \cdot \nu_p^{(d-1)} \], where \( \text{vol}_p^{(d-1)} \) is the measure on \( \mathbb{Q}_p^{d-1} \) defined in § 2, are Haar measures on \( H \) and \( H \cap \text{SL}_d(\mathbb{Q}_p) \), respectively.

**Proposition 3.1.** The push-forward measures

\[
\mu_p^{(d)}(g) := (\Phi_p)_* \text{dv'} \mu_p^{(d-1)}(g') \text{dy} \quad \text{and} \\
\nu_p^{(d)}(g) := (\Phi_p)_* \text{dv'} \nu_p^{(d-1)}(g') \text{dy}
\]
give Haar measures on \( \text{UL}_d(\mathbb{Q}_p) \) and \( \text{SL}_d(\mathbb{Q}_p) \), respectively. Here, we simply denote \( \text{dv'} = d \text{vol}_p^{(d-1)}(v') \) and \( \text{dy} = d \text{vol}_p^{(d)}(y) \).

**Proof.** Since both groups admit lattice subgroups, Haar measures on them are unimodular. Hence it suffices to show that

\[
(\Phi_p)_* \text{dv'} \mu_p^{(d-1)}(g') \text{dy}
\]
is \( \text{UL}_d(\mathbb{Q}_p) \)-invariant under right multiplication \( R_g \). We omit the proof for \( \text{SL}_d(\mathbb{Q}_p) \) since the argument is identical.

Since the image of \( \Phi_p \) covers \( \text{UL}_d(\mathbb{Q}_p) \) outside a co-dimension one submanifold, we may choose a generic element \( h \in \text{UL}_d(\mathbb{Q}_p) \) of the form \( h = \Phi_p((w', h'), x) \). Let \( x = (x_1, \ldots, x_d) \) and \( y = (y_1, \ldots, y_d) \). Then

\[
((v', g'), y)((w', h'), x) = ((v' + w'_1g', g'h'_1), (z_1x_1, z_1(x_2, \ldots, x_d) + (z_2, \ldots, z_d)\text{diag}(x_1^{-1}, 1, \ldots, 1)) =: ((v'_1, g'_1), y_1),
\]
where \( z_1 = y_1 + (y_2, \ldots, y_d)'w' \), \( w'_1 = \frac{1}{z_1}w' \text{ diag}(y_1^{-1}, 1, \ldots, 1) \), and

\[
h'_1 = \left(\text{diag}(y_1^{-1}, 1, \ldots, 1)\left(Id_{d-1} - \frac{1}{z_1}w'(y_2, \ldots, y_d)\right)h'\right)\text{diag}(z_1, 1, \ldots, 1).
\]

Here, \( Id_{d-1} \) is the identity matrix of size \( d - 1 \).

Since the Jacobi matrices \( dy_1/dg' \), \( dy_1/dv' \) and \( d\mu_p^{(d-1)}(g')/dv' \) are trivial and \( dv'/dv' = Id_{d-1} \), we have

\[
(R_g)_* \text{dv'} \mu_p^{(d-1)}(g') \text{dy} = \left| \frac{d\mu_p^{(d-1)}(g'_1)}{d\mu_p^{(d-1)}(g')} \right| \frac{dy_1}{dy} \left| \frac{dy}{dv'} \right| \text{dv'} \mu_p^{(d-1)}(g'_1) \text{dy}.
\]

Since \( g'_1 = g'h'_1 \) and \( g' \) is irrelevant to \( h'_1 \), by the induction hypothesis that \( \mu_p^{(d-1)} \) is \( \text{UL}_{d-1}(\mathbb{Q}_p) \)-invariant, we have that

\[
\left| \frac{d\mu_p^{(d-1)}(g'_1)}{d\mu_p^{(d-1)}(g')} \right| = 1.
\]

One can also show that \( |dy_1/dy| = 1 \) by direct computation. \( \square \)
When \( p = \infty \), the measure \( \mu_\infty^{(d)} \) of \( \text{SL}_d(\mathbb{R}) \) can be identified with

\[
d\mu_\infty^{(d)} = \delta(1 - \det g_\infty) \prod_{1 \leq i, j \leq d} dg_{ij},
\]

where \( g_\infty = (g_{ij}) \in \text{SL}_d(\mathbb{R}) \) and \( \delta \) is the Dirac-delta distribution (see formulae (3.67) and (3.70) in [27]).

One can find volume formulae of the quotients of semisimple S-arithmetic Lie groups, based on the Iwasawa (KAN) decomposition of semisimple Lie groups in the work of Prasad [29]. However, this method is not fit for our usage in the next section as well as \( \text{UL}_d(\mathbb{Q}_S) \) is not semisimple. Hence let us devote some space to compute the volumes of the quotient spaces \( \text{UL}_d(\mathbb{Z}_S) \setminus \text{UL}_d(\mathbb{Q}_S) \) and \( \text{SL}_d(\mathbb{Z}_S) \setminus \text{SL}_d(\mathbb{Q}_S) \), adjusting the method provided by Siegel [31].

For a positive integer \( d \geq 2 \), define

\[
\zeta_S(d) := \sum_{t \in \mathbb{N}_S} \frac{1}{t^d}.
\]

For a bounded and compactly supported function \( f : \mathbb{Q}_S^d \to \mathbb{R}_{\geq 0} \), we define the homogeneous Siegel transform \( \tilde{f} \) by

\[
\tilde{f}(\Lambda) = \sum_{v \in \Lambda - \{0\}} f(v), \quad \forall \ \Lambda \in \text{UL}_d(\mathbb{Z}_S) \setminus \text{UL}_d(\mathbb{Q}_S).
\]

Define \( \alpha : \text{UL}_d(\mathbb{Z}_S) \setminus \text{UL}_d(\mathbb{Q}_S) \to \mathbb{R}_{\geq 0} \) by

\[
\alpha(\Lambda) = \sup_{1 \leq j \leq d} \left\{ \frac{1}{\prod_{p \in S} \|v_1^{(p)} \wedge \ldots \wedge v_j^{(p)}\|_p} : v_1, \ldots, v_j \in \Lambda \text{ linearly independent} \right\}.
\]

The transform \( \tilde{f} \) has the following property ([16, Proposition 3.3], see also [30]).

**Theorem 3.2.** Let \( f : \mathbb{Q}_S^d \to \mathbb{R}_{\geq 0} \) be bounded and compactly supported. Then there is \( c = c(f) > 0 \) for which

\[
\tilde{f}(\Lambda) < c \alpha(\Lambda)
\]

for any \( \Lambda \in \text{UL}_d(\mathbb{Z}_S) \setminus \text{UL}_d(\mathbb{Q}_S) \). The constant \( c > 0 \) can be taken uniformly among the family of dilates of \( f \) with appropriate normalization.

The following theorem is found in [17] (Lemma 3.10) for \( \text{SL}_d(\mathbb{Z}_S) \setminus \text{SL}_d(\mathbb{Q}_S) \) and [16] (Proposition 3.2) for \( \text{UL}_d(\mathbb{Z}_S) \setminus \text{UL}_d(\mathbb{Q}_S) \).

**Theorem 3.3.** For \( 1 \leq r < d \),

\[
\int_{\text{UL}_d(\mathbb{Z}_S) \setminus \text{UL}_d(\mathbb{Q}_S)} \alpha^r d\mu_S^{(d)} < \infty \quad \text{and} \quad \int_{\text{SL}_d(\mathbb{Z}_S) \setminus \text{SL}_d(\mathbb{Q}_S)} \alpha^r d\mu_S^{(d)} < \infty.
\]
VALUES OF INHOMOGENEOUS FORMS

We now come to the main volume estimate of this paper.

**Theorem 3.4.** Let $\mu_S^{(d)} = \prod_{p \in S} \mu_p^{(d)}$ and $\nu_S^{(d)} = \prod_{p \in S} \nu_p^{(d)}$ be Haar measures on $UL_d(\mathbb{Q}_S)$ and $SL_d(\mathbb{Q}_S)$, respectively, where $\mu_p^{(d)}$ and $\nu_p^{(d)}$ are defined in Proposition 3.1. Then

$$
\mu_S^{(d)}(UL_d(\mathbb{Z}_S) \setminus UL_d(\mathbb{Q}_S)) = \prod_{p \in S} \left(1 - \frac{1}{p}\right) \cdot \xi_S(d) \xi_S(d - 1) \cdots \xi_S(2)
$$

and

$$
\nu_S^{(d)}(SL_d(\mathbb{Z}_S) \setminus SL_d(\mathbb{Q}_S)) = \xi_S(d) \xi_S(d - 1) \cdots \xi_S(2).
$$

**Proof.** As in Proposition 3.1, it suffices to show the statement for $UL_d(\mathbb{Q}_S)$ since the proof of the $SL_d(\mathbb{Q}_S)$-case will be almost identical except the initial value $\nu_S^{(1)}(SL_1(\mathbb{Z}_S) \setminus SL_1(\mathbb{Q}_S)) = 1$, while $\mu_S^{(1)}(UL_1(\mathbb{Z}_S) \setminus UL_1(\mathbb{Q}_S)) = \prod_{p \in S} (1 - 1/p)$. For simplicity, let us denote

$$
V_d = \mu_S^{(d)}(UL_d(\mathbb{Z}_S) \setminus UL_d(\mathbb{Q}_S)).
$$

Fix a fundamental domain $P$ for $UL_d(\mathbb{Z}_S) \setminus UL_d(\mathbb{Q}_S)$. Let $f : \mathbb{Q}_S^d \to \mathbb{R}$ be any given bounded continuous function of compact support. Consider a sequence $(\lambda_n = (e^{-n}, p_1^n, \ldots, p_s^n))_{n \in \mathbb{N}}$ of $S$-arithmetic numbers. It is obvious that for any $g \in UL_d(\mathbb{Q}_S)$,

$$
I := \int_{\mathbb{Q}_S^d} f(v) dv = \lim_{n \to \infty} d(\lambda_n)^d \sum_{v \in \mathbb{Z}_S^d g - \{0\}} f(\lambda_n v)
$$

= \lim_{n \to \infty} d(\lambda_n)^d \tilde{f}_{\lambda_n}(g),
$$

where $f_{\lambda_n}(v) = f(\lambda_n v)$ and $d(\lambda_n) = (e p_1 \cdots p_s)^{-n}$ (Here, $d(\cdot)$ stands for the covolume of the lattice $\mathbb{Z}_S \lambda_n \subseteq \mathbb{Q}_S$).

By Lebesgue’s dominated convergence theorem, using Theorems 3.2 and 3.3, it follows that

$$
V_d I = \int f_{\lambda_n}(g) d\mu_S^{(d)}(g) = \lim_{n \to \infty} d(\lambda_n)^d \int \tilde{f}_{\lambda_n}(g) d\mu_S^{(d)}(g).
$$

Put

$$
\xi = d(\lambda_n)^d \int \tilde{f}_{\lambda_n}(g) d\mu_S^{(d)}(g) and
$$

$$
\xi_t = d(\lambda_n)^d \sum_{v \in \mathbb{Z}_S^d g \setminus P(\mathbb{Z}_S)} f(\lambda_n v) d\mu_S^{(d)}(g), \forall t \in \mathbb{N}_S
$$

so that $\xi = \sum_{t \in \mathbb{N}_S} \xi_t$. We will compute $\xi_t$ using the induction hypothesis of dimension $d - 1$ and hence obtain $\xi$. During the proof, we will see that $\xi$ and $\xi_t$ are eventually not relevant to $\lambda_n$ (which is why we do not use the notation $\xi_n$ or $\xi_{n,t}$ instead).

For each $\mathbf{k} \in P(\mathbb{Z}_S^d)$, assign an element $g_k \in SL_d(\mathbb{Z}_S)$ for which $\mathbf{k} = e_1 \cdot g_k$, where $\{e_i : 1 \leq i \leq d\}$ is the canonical basis of $\mathbb{Q}_S^d$. Denote by $[g]_1$ the first row of the matrix $g \in SL_d(\mathbb{Q}_S)$. 
Then

\[ d(\lambda_n)^{-d} \xi_1 = \sum_{k \in \mathcal{P}(\mathbb{Z}_d S)} \int f(\lambda_n g)_1 F_1 \mu_S^{(d)}(g) = \int_{F_1} f(\lambda_n g) d\mu_S^{(d)}(g), \]

where \( F_1 := \bigcup_{k \in \mathcal{P}(\mathbb{Z}_d S)} g_k F \).

Since \( \{g_k : k \in \mathcal{P}(\mathbb{Z}_d S)\} \) is the set of representatives of \( H_{Z_S} \setminus \text{UL}_d(Z_S) \), where \( H_{Z_S} := H \cap \text{UL}_d(Z_S) \),

\[
\text{UL}_d(Q_S) = \bigcup_{\gamma \in \text{UL}_d(Z_S)} \gamma F = \bigcup_{\gamma' \in H_{Z_S} k \in \mathcal{P}(\mathbb{Z}_d S)} \gamma' g_k F = \bigcup_{\gamma' \in H_{Z_S}} \gamma' F_1
\]

so that \( F_1 \) is the fundamental domain of \( H_{Z_S} \) in \( \text{UL}_d(Q_S) \).

On the other hand, by considering the inverse map of \( \Phi = \prod_{p \in S} \Phi_p \), where \( \Phi_p \) is given as in (3.2), any element \( g = (g_p = (g_{ij}^{(p)}))_{p \in S} \in \text{UL}_d(Q_S) \) such that \( g_{11}^{(p)} \neq 0 \) for all \( p \in S \), is uniquely expressed as

\[
g = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
\nu' & g' \\
0 & 1/g_{11} & g_{12} & \cdots & g_{1d} \\
0 & 0 & 1 & \ddots & \ddots \\
0 & 0 & 0 & \cdots & 1
\end{pmatrix}
\]

where \( g' \in \text{UL}_{d-1}(Q_S) \) and \( \nu' \in \mathbb{Q}_d^{d-1} \). Let \( F' \) be any fundamental domain of \( \text{UL}_{d-1}(Z_S) \setminus \text{UL}_{d-1}(Q_S) \). Then \( (Z_S \setminus Q_S)^{d-1} \times F' \) is a fundamental domain of \( H_{Z_S} \setminus H \) and

\[
P_2 := ((Z_S \setminus Q_S)^{d-1} \times F') \times \left\{ y = ((y_1^{(p)}, \ldots, y_d^{(p)}))_{p \in S} \in \mathbb{Q}_d^d : y_1^{(p)} \neq 0 \right\}
\]

is a measure-theoretic fundamental domain of \( H_{Z_S} \) in \( \text{UL}_d(Q_S) \), that is,

\[
g_1 P_2 \cap g_2 P_2 = \emptyset \text{ if } g_1 \neq g_2 \in H_{Z_S} \text{ and }
\]

\[
\mu_S^{(d)}\left( \text{UL}_d(Q_S) - \bigcup_{g \in H_{Z_S}} g P_2 \right) = 0.
\]

Hence

\[
d(\lambda_n)^{-d} \xi_1 = \int_{P_2} f(\lambda_n y) d\nu' d\mu_S^{(d-1)}(g') dy
\]

\[
= \int_{(Z_S \setminus Q_S)^{d-1} \times F'} \left( \int_{\mathbb{Q}_d^d} f(\lambda_n y) dy \right) d\nu' d\mu_S^{(d-1)}(g')
\]

\[
= \int_{(Z_S \setminus Q_S)^{d-1} \times F'} d(\lambda_n)^{-d} I d\nu' d\mu_S^{(d-1)}(g')
\]

\[
= d(\lambda_n)^{-d} I \mu_S^{(d-1)}(\text{UL}_{d-1}(Z_S) \setminus \text{UL}_{d-1}(Q_S)).
\]
Therefore, \( \xi_1 = V_{d-1}I \) and similarly, one can show that \( \xi_t = (1/t^d)V_{d-1}I \) for any \( t \in \mathbb{N}_S \). Since \( \xi = \sum_{t \in \mathbb{N}_S} \xi_t \), we have

\[
V_d = \sum_{t \in \mathbb{N}_S} \frac{1}{t^d} V_{d-1} = \zeta_S(d)V_{d-1}
\]

and by induction hypothesis,

\[
\mu_S^{(d)}(U_{\mathbb{L}d}(\mathbb{Z}_S) \setminus U_{\mathbb{L}d}(\mathbb{Q}_S)) = V_d = \prod_{p \in S'} \left( 1 - \frac{1}{p} \right) \cdot \zeta_S(d)\zeta_S(d-1) \cdots \zeta_S(2).
\]

4 | THE FIRST AND SECOND MOMENT FORMULAE

For a bounded and compactly supported function \( f : \mathbb{Q}_S^d \to \mathbb{R} \), define the Siegel transform \( \hat{f} : U_{\mathbb{L}d}(\mathbb{Z}_S) \setminus U_{\mathbb{L}d}(\mathbb{Q}_S) \to \mathbb{R} \) by

\[
\hat{f}(\Lambda) = \sum_{k \in \Lambda} f(k).
\]

In this section, we establish the first and second moment formulae for \( S \)-adic Siegel transforms on congruence quotients. For \( q = q_0^k \), since

\[
q(q\mathbb{Z}_S^d + p) = q_0((q\mathbb{Z}_S^d + p)g) = q^2q_0((\mathbb{Z}_S^d + p/q)g),
\]

by the duality principle, the space of our interests is

\[
Y_{p/q} := \left\{ \left( \mathbb{Z}_S^d + \frac{p}{q} \right)g : g \in U_{\mathbb{L}d}(\mathbb{Q}_S) \right\}.
\]

For \( q \in \mathbb{N}_S \), let

\[
\Gamma_1(q) = \{ \gamma \in U_{\mathbb{L}d}(\mathbb{Z}_S) : e_1\gamma \equiv e_1 \mod q \}.
\]

One can show that \( Y_{p/q} \) can be identified with a quotient space of \( U_{\mathbb{L}d}(\mathbb{Q}_S) \).

**Lemma 4.1.** Let \( q \in \mathbb{N}_S \) and \( p \in \mathbb{Z}_S^d \) be such that \( \gcd(q, p) = 1 \). Fix any \( \gamma_p \in \text{SL}_d(\mathbb{Z}_S) \) such that \( p\gamma_p^{-1} \in \mathbb{Z}_S.e_1 \). Then \( Y_{p/q} \) is identified with \( \gamma_p^{-1}\Gamma_1(q) \setminus U_{\mathbb{L}d}(\mathbb{Q}_S) \) via the correspondence

\[
(\mathbb{Z}_S^d + \frac{p}{q})g \leftrightarrow (\gamma_p^{-1}\Gamma_1(q) \setminus U_{\mathbb{L}d}(\mathbb{Q}_S))g.
\]

**Proof.** The proof is identical with that of Lemma 3.1 in [14].

Here, the notion \( \gcd(q, k) \) can be defined as follow.
Definition 4.2. For \( q \in \mathbb{N}_S \) and \( k \in \mathbb{Z}^d_S \), define
\[
gcd(q, k) = \gcd(q, k'),
\]
where \( k' \) is any integral vector in \((\mathbb{Z}_S^X, k) \cap \mathbb{Z}^d\).

We now show the first and the second moment formulae for the Siegel transform on \( Y_{p/q} \). Let \( \widetilde{\mu}_S \) and \( \widetilde{\nu}_S \) be the normalized Haar measures of \( \mathrm{UL}_d(\mathbb{Q}_S) \) and \( \mathrm{SL}_d(\mathbb{Q}_S) \), respectively, such that
\[
\widetilde{\mu}_S(\mathrm{UL}_d(\mathbb{Z}_S) \setminus \mathrm{UL}_d(\mathbb{Q}_S)) = 1 = \widetilde{\nu}_S(\mathrm{SL}_d(\mathbb{Z}_S) \setminus \mathrm{SL}_d(\mathbb{Q}_S)).
\]

Theorem 4.3. Consider \( 1 \neq q \in \mathbb{N}_S \) and \( p \in \mathbb{Z}^d_S \) such that \( \gcd(q, p) = 1 \). Let \( f : Q^d_S \to \mathbb{R} \) be a bounded and compactly supported function.

(a) For \( d \geq 2 \),
\[
\frac{1}{J_q} \int_{Y_{p/q}} \hat{f}\left(\left(\mathbb{Z}^d_S + \frac{p}{q}\right)g\right) d\widetilde{\mu}_S(g) = \int_{Q^d_S} f(v) dv,
\]
where \( J_q = \widetilde{\mu}_S(Y_{p/q}) \).

(b) For \( d \geq 3 \),
\[
\frac{1}{J_q} \int_{Y_{p/q}} \hat{f}\left(\left(\mathbb{Z}^d_S + \frac{p}{q}\right)g\right)^2 d\widetilde{\mu}_S(g) = \left(\int_{Q^d_S} f dv\right)^2 + \sum_{t \in \mathbb{N}_S} \sum_{\gcd(t, q) = 1} \sum_{a \in q\mathbb{Z}_S + t} \gcd(a, t) = 1 \int_{Q^d_S} f(tv)f(av) dv.
\]

Remark 4.4. One can also consider \( Z_{p/q} := \{(\mathbb{Z}^d_S + p/q)g : g \in \mathrm{SL}_d(\mathbb{Q}_S)\} \) and moment formulae on \( Z_{p/q} \). By replacing \( Y_{p/q} \) and \( \mathrm{UL}_d(\mathbb{Q}_S) \) by \( Z_{p/q} \) and \( \mathrm{SL}_d(\mathbb{Q}_S) \) respectively in the arguments throughout this section, one may obtain the following results: under the same assumptions as in Theorem 4.3, it follows that:

(a’) for \( d \geq 2 \),
\[
\frac{1}{J_q} \int_{Z_{p/q}} \hat{f}\left(\left(\mathbb{Z}^d_S + \frac{p}{q}\right)g\right) d\widetilde{\nu}_S(g) = \int_{Q^d_S} f(v) dv,
\]
where \( J_q = \widetilde{\nu}_S(Z_{p/q})(= \widetilde{\mu}_S(Y_{p/q})) \);

(b’) for \( d \geq 3 \),
\[
\frac{1}{J_q} \int_{Z_{p/q}} \hat{f}\left(\left(\mathbb{Z}^d_S + \frac{p}{q}\right)g\right)^2 d\widetilde{\nu}_S(g) = \left(\int_{Q^d_S} f dv\right)^2 + \sum_{t \in \mathbb{N}_S} \sum_{\gcd(t, q) = 1} \sum_{a \in q\mathbb{Z}_S + t + 1} \gcd(a, t) = 1 \int_{Q^d_S} f(tv)f(av) dv.
\]

4.1 Proof of Theorem 4.3

Let \( X_q = \Gamma(q) \setminus \mathrm{UL}_d(\mathbb{Q}_S) \), where
\[
\Gamma(q) := \{\gamma \in \mathrm{UL}_d(\mathbb{Z}_S) : \gamma \equiv \text{Id} \mod q\}.
\]
Note that $\Gamma(q)$ is a finite-index subgroup of $\text{UL}_d(\mathbb{Z}_S) = \text{SL}_d(\mathbb{Z}_S)$ since it is the kernel of the projection

$$\pi_q : \text{UL}_d(\mathbb{Z}_S) \to \text{UL}_d(\mathbb{Z}_S/q\mathbb{Z}_S) \simeq \text{SL}_d(\mathbb{Z}/q\mathbb{Z}).$$

Since $\Gamma(q)$ is a normal subgroup of $\text{UL}_d(\mathbb{Z}_S)$ and $\Gamma(q) \triangleleft \Gamma_1(q)$, $X_q$ is a finite covering of $Y_q$. Hence any function $\phi$ on $Y_q$ can extend to a $\gamma_p^{-1}\Gamma_1(q)\gamma_p$-invariant function on $X_q$, where $\gamma_p$ is as in Lemma 4.1 and it immediately follows that

$$\frac{1}{I_q} \int_{Y_{p/q}} \phi \, d\bar{\mu}_S = \frac{1}{I_q} \int_{X_q} \phi \, d\bar{\mu}_S.$$ 

Proof of Theorem 4.3(a). Note that we assume that $p/q \notin \mathbb{Z}^d_S$.

For a bounded function $f : \mathbb{Q}^d_S \to \mathbb{R}_{\geq 0}$ of compact support and $q \in \mathbb{N}_S$, define $f_q(v) := f(\frac{1}{q}v)$. Since $\hat{f}(\mathbb{Z}^d_S + p/q) = \sum_{v \in \mathbb{Z}^d_S + p} f(v) = \sum_{k \in q\mathbb{Z}^d_S + p} f\left(\frac{1}{q}k\right) \leq \sum_{k \in \mathbb{Z}^d_S - \{O\}} f_q(k) = \hat{f}_q(\mathbb{Z}^d_S g),$ by [17, Lemma 3.8] and [17, Lemma 3.10], the map

$$f \mapsto \frac{1}{I_q} \int_{X_q} \hat{f}(\mathbb{Z}^d_S + p/q)g \, d\bar{\mu}_S(g)$$

is a $\text{UL}_d(\mathbb{Q}_S)$-invariant positive linear functional on $C_c(\prod_{p \in S}(\mathbb{Q}^d_p - \{O\}))$, where $I_q = \bar{\mu}_S(X_q)$. Since $\text{UL}_d(\mathbb{Q}_S)$ acts on $\prod_{p \in S}(\mathbb{Q}^d_p - \{O\})$ transitively, by Riesz–Markov–Kakutani representation theorem, it follows that

$$\frac{1}{I_q} \int_{X_q} \hat{f}(\mathbb{Z}^d_S + p/q)g \, d\bar{\mu}_S(g) = c \int_{\mathbb{Q}^d_S} f(v) \, dv \quad (4.2)$$

for some $c > 0$. Here we use the fact that we can identify a $\text{UL}_d$-invariant measure on $\prod_{p \in S}(\mathbb{Q}^d_p - \{O\})$ with the volume measure of $\mathbb{Q}^d_S$ up to scaling.

Now, for each $t \in \mathbb{N}$, let

$$V_t = \mathbb{Q}^d_S \setminus \left( \left( B^{(\infty)}_{e^{-t}}(O) \times \prod_{p \in S_f} \mathbb{Q}^d_p \right) \cup \bigcup_{i=1}^s \left( \mathbb{R}^d \times p_i^{(\frac{t}{p_i})} \mathbb{Z}^d_p \times \prod_{p \in S_f \setminus \{p_i\}} \mathbb{Q}^d_p \right) \right),$$

where $B^{(\infty)}_{e^{-t}}(O)$ is the ball of radius $e^{-t}$ at the origin in $\mathbb{R}^d$. For each $k \in \mathbb{N}$, denote $B_k = B^{(\infty)}_{e^{-k}}(O) \times \prod_{p \in S_f} p^{-k} \mathbb{Z}^d_p$. Take a sequence $\{f_k\}$ in $C_c(\prod_{p \in S}(\mathbb{Q}^d_p - \{O\}))$ such that $f_k \leq \chi_{B_k} \big|_{V_1}$ and as $k \to \infty$, 

$$\int_{Y_{p/q}} f_k \, d\bar{\mu}_S = \int_{X_q} f_k \, d\bar{\mu}_S.$$
Asymptotically, $f_k$ converges to $\chi_{B_k} |_{V_1}$. By putting $\{\frac{1}{\vol(f_k)} f_k\}$ on both sides in (4.2) and letting $k \to \infty$, we obtain $c = 1$.

For an arbitrary bounded and compactly supported function $f$ on $Q_d$, since any affine lattice does not intersect with the complement of $\prod_{p \in S}(Q_d^d - \{0\})$, if we take $\{f'_t\} \subseteq C_c(\prod_{p \in S}(Q_d^d - \{0\}))$ such that $f |_{V_{t-1}} \leq f'_t \leq f |_{V_t}$, then as $t \to \infty$, $f'_t$ converges to $f$ and we have

$$\lim_{t \to \infty} \int_{X_q} \hat{f'_t} \left( \left( Z^d_S + \frac{p}{q} \right) g \right) d\tilde{\mu}_S(g) = \int_{X_q} \hat{f} \left( \left( Z^d_S + \frac{p}{q} \right) g \right) d\tilde{\mu}_S(g)$$

which proves (a) (see [16, Section 3] for more details). □

Following [14, Section 3.2], we will deduce Theorem 4.3 (b) from Propositions 4.5 and 4.6, whose proofs will be provided in Section 4.2.

For each $y \in \prod_{p \in S}(Q_d^d - \{0\})$, define

$$X_q(y) = \left\{ \Gamma(q) g \in X(q) : y \in \left( Z^d_S + \frac{p}{q} \right) g \right\}.$$

One can assign the probability measure $\nu_y$ on $X_q(y)$ for each $y$, which commutes with right multiplication by an element of $UL_d(Q_S)$ (see Definition 4.8 and Remark 4.10).

**Proposition 4.5.** Let $d \geq 2$. Let $1 \neq q \in \mathbb{N}_S$ and $p \in \mathbb{Z}_S^d$ for which $\gcd(q, p) = 1$. For any Borel measurable function $F : X_q \times \mathbb{Q}_S^d \to \mathbb{R}_{\geq 0}$, it follows that

$$\frac{1}{I_q} \int_{X_q} \sum_{k \in \mathbb{Z}_S^d + p/q} F(\Gamma(q) g, k g) d\tilde{\mu}_S(g) = \int_{\mathbb{Q}_S^d} \int_{X_q(y)} F(\Gamma(q) g, y) d\nu_y(g) dy,$$

where $I_q = \tilde{\mu}_S(X_q)$.

**Proposition 4.6.** Let $f : Q_d \to \mathbb{R}_{\geq 0}$ be a bounded and compactly supported function and let $y \in \prod_{p \in S}(Q_d^d - \{0\})$. Then

$$\int_{X_q(y)} \hat{f} \left( \left( Z^d_S + \frac{p}{q} \right) g \right) \nu_y(g) = \int_{\mathbb{Q}_S^d} f dv + \sum_{t \in \mathbb{N}_S} \frac{1}{t^d} \sum_{\substack{a \in q Z_d + t \\gcd(a, t) = 1 \\gcd(t, q) = 1}} f \left( \frac{a}{t} y \right).$$

**Proof of Theorem 4.3(b).** For a given bounded and compactly supported function $f : Q_d \to \mathbb{R}_{\geq 0}$, define

$$F \left( (Z^d_S + p/q) g, y \right) := f(y) \hat{f} \left( (Z^d_S + p/q) g \right).$$

Note that the function $F$ can be considered as a function on $Y_{p/q} \times Q_d^d$ as well as on $X_q \times Q_d^d$. 

Applying Propositions 4.5 and 4.6, it holds that

\[
\frac{1}{J_q} \int_{Y_{p/q}} \hat{f} \left( \left( \mathbb{Z}_S^d + \frac{p}{q} \right) g \right) d\bar{\mu}_S(g) = \int_{Q_S^d} f(y) \left( \int_{X_q(y)} \hat{f} \left( \left( \mathbb{Z}_S^d + \frac{p}{q} \right) \right) d\bar{\nu}_q(g) \right) dy
\]

\[
= \int_{Q_S^d} f(y) \left( \int_{Q_S^d} f(x) dx + \sum_{t \in \mathbb{N}_S} \frac{1}{t^d} \sum_{\gcd(a, t) = 1} \int_{Q_S^d} f(y) f \left( \frac{a}{t} y \right) dy \right)
\]

\[
= \left( \int_{Q_S^d} f dv \right)^2 + \sum_{t \in \mathbb{N}_S} \sum_{\gcd(a, t) = 1} \int_{Q_S^d} f(y) f \left( \frac{a}{t} y \right) dy
\]

\[
= \left( \int_{Q_S^d} f dv \right)^2 + \sum_{t \in \mathbb{N}_S} \sum_{\gcd(a, t) = 1} \int_{Q_S^d} f(ty) f(axy) dy.
\]

\[\square\]

### 4.2 The Spaces \(X_q\) and \(X_q(y)\)

This subsection is devoted to a generalization of the part of Section 7 in [28] to \(S\)-arithmetic spaces.

Note that by definition, \(\text{UL}_d(\mathbb{Z}_S) = \text{SL}_d(\mathbb{Z}_S)\). For \(q \in \mathbb{N}_S\), let us define

\[
\pi_q : \text{UL}_d(\mathbb{Z}_S) \to \text{UL}_d(\mathbb{Z}_S/q\mathbb{Z}_S) \cong \text{SL}_d(\mathbb{Z}/q\mathbb{Z})
\]

by \(\pi_q(g = (g_{ij})) = ([g_{ij}])\), where \([g_{ij}] = g_{ij} + qZ_S \in Z_S/qZ_S\).

Let \(H\) be the subgroup of \(\text{UL}_d(Q_S)\) given as in (3.1).

**Proposition 4.7.** Let \(1 \neq q \in \mathbb{N}_S\) and \(p \in \mathbb{Z}_S^d\) with \(\gcd(q, p) = 1\). There is a subset \(\{k_t \in Z_S^d + p/q : t \in \mathbb{N}_S, \gcd(t, q) = 1\}\) satisfying the following.

(a) For any \(y \in \prod_{p \in S}(Q_p^d - \{O\})\),

\[
X_q(y) = \bigcup_{t \in \mathbb{N}_S, \gcd(t, q) = 1} \{g \in \Gamma(q) \setminus \text{UL}_d(Q_S) : k_t g = y\}.
\]

(b) For \(k, y \in Q_S^d\), define the set

\[
X_q(k, y) := \{g \in \Gamma(q) \setminus \text{UL}_d(Q_S) : k g = y\}.
\]
Fix any \( g_k \) and \( g_y \) in \( \text{SL}_d(Q_S) \) for which \( k = e_1 g_k \) and \( y = e_1 g_y \). Then

\[
X_q(k,y) = \Gamma(q)g_k^{-1}((\Gamma(q)g_k^{-1} \cap H) \setminus H)g_y
\]

\[\approx (\Gamma(q)g_k^{-1} \cap H) \setminus H.\]

Proof.

(a) We first fix an element \( y \in \prod_{p \in S}(Q_p^d - \{O\}). \) It is easily seen that

\[
\begin{cases}
X_q(k_1,y) = X_q(k_2,y), & \text{if } k_1 \Gamma(q) = k_2 \Gamma(q); \\
X_q(k_1,y) \cap X_q(k_2,y) = \emptyset, & \text{otherwise.}
\end{cases}
\]

Hence

\[
X_q(y) = \bigsqcup_{k \Gamma(q)} X_q(k,y).
\]

For \( k \in \mathbb{Z}_S^d + p/q \), consider \( t_k := \gcd(qk) \in \mathbb{N}_S \), that is, \( t_k \) is defined to be the unique integer in \( \mathbb{N}_S \) for which \( qk \in t_k P(\mathbb{Z}_S^d) \). Since \( \gcd(qk,q) = 1 \), it follows that \( \gcd(t_k,q) = 1 \). It is obvious that

\[
t_{k_1} = t_{k_2} \text{ if } k_1 \Gamma(q) = k_2 \Gamma(q)
\]

(more precisely, if \( k_1 \text{UL}_q(\mathbb{Z}_S^d) = k_2 \text{UL}_d(\mathbb{Z}_S^d) \)). Conversely, we claim that:

(i) for each \( t \in \mathbb{N}_S \) such that \( \gcd(t,q) = 1 \), there is \( k \in \mathbb{Z}_S^d + p/q \) with \( \gcd(qk) = t \);

(ii) if \( k_1, k_2 \in \mathbb{Z}_S^d + p/q \) with \( \gcd(qk_1) = \gcd(qk_2) = t \), then \( k_1 \Gamma(q) = k_2 \Gamma(q) \).

(i) For such \( t \in \mathbb{N}_S \), since \( \gcd(t,q) = 1 \), there is \( t^* \in \mathbb{Z} \) such that \( tt^* \equiv 1 \mod q \). Pick \( p \in \mathbb{Z}_S^d \) such that \( pp \in \mathbb{Z}^d \) is a primitive vector in \( \mathbb{Z}^d \). Since \( \gcd(q,t^*p) = 1 \), by Dirichlet’s theorem on arithmetic progressions, there exists a primitive vector \( m \) in \( t^*p + q \mathbb{Z}^d \), hence if we let \( k := t^*m \), then \( k \in \mathbb{Z}_S^d + p/q \) and \( \gcd(qk) = t \).

(ii) Choose any \( \gamma_i \in \text{SL}_d(\mathbb{Z}_S^d), i = 1,2 \), for which \( qk_i = t e_1 \gamma_i \). Then since \( k_1 \equiv k_2 \mod q \), we see that \( e_1 \gamma_1\gamma_2^{-1} \equiv e_1 \mod q \) so that \( \gamma_1\gamma_2^{-1} \) is of the form

\[
\gamma_1\gamma_2^{-1} = \begin{pmatrix} x_1 & x' \\ v & g' \end{pmatrix},
\]

where \( x_1 \in 1 + q \mathbb{Z}_S \) and \( x' \in q \mathbb{Z}_S^{d-1} \). Since \( \det g' \in 1 + q \mathbb{Z}_S \), there is \( g'_1 \in \text{Mat}_{d-1}(\mathbb{Z}_S^{d-1}) \) such that \( g'_1 + \text{Mat}_{d-1}(q \mathbb{Z}_S) = g' + \text{Mat}_{d-1}(q \mathbb{Z}_S) \) and \( \det g'_1 \in 1 + q \mathbb{Z}_S \cap \mathbb{Z} = 1 + q \mathbb{Z} \). Hence by \([32, p. 21]\), there is \( g'_2 \in \text{SL}_{d-1}(\mathbb{Z}) \) for which \( g'_2 \equiv g'_1 \mod q \), so that

\[
g' + \text{Mat}_{d-1}(q \mathbb{Z}_S) = g'_1 + \text{Mat}_{d-1}(q \mathbb{Z}_S) = g'_2 + \text{Mat}_{d-1}(q \mathbb{Z}_S). \]
Since $SL_d(\mathbb{Z}_S/q\mathbb{Z}_S) \simeq SL_d(\mathbb{Z}_S)/\Gamma(q)$ and
\[
\gamma_1 \gamma_2^{-1} = \begin{pmatrix} 1 & 0 \\ \nu' & g'_2 \end{pmatrix} \mod q,
\]
it follows that there is $\gamma_0 \in \Gamma(q)$ such that
\[
\gamma_1 \gamma_2^{-1} \gamma_0 = \begin{pmatrix} 1 & 0 \\ t \nu' & \mathbb{I}' \end{pmatrix},
\]
for some $\gamma_0' \in \Gamma(q)$ since $\Gamma(q)$ is normal in $UL_d(\mathbb{Z}_S)$. Hence $\mathbf{e}_1 \gamma_1 \gamma_0' \mathbf{e}_1^{-1} = \mathbf{e}_1 \gamma_2^{-1}$, so that $\mathbf{k}_1 \Gamma(q) = \mathbf{k}_2 \Gamma(q)$. Therefore, one can choose $\mathbf{k}_t \in \mathbb{Z}_d^S + \mathfrak{p}/q$, for each $t \in \mathbb{N}_S$ with $\gcd(t, q) = 1$, so that
\[
X_q(\mathbf{y}) = \bigcup_{t \in \mathbb{N}_S, \gcd(t, q) = 1} X_q(\mathbf{k}_t, \mathbf{y}).
\]

For arbitrary $\mathbf{y}' \in \prod_{p \in S}(\mathbb{Q}_d^p - \{O\})$, the property (a) also holds for $\{\mathbf{k}_t\}$ since $X_q(\mathbf{k}_t, \mathbf{y}') = X_q(\mathbf{k}_t, \mathbf{y})$$\mathbf{g}_y'$, where $\mathbf{g}_y' \in UL_d(\mathbb{Q}_S)$ for some $\mathbf{y}' = \mathbf{y} \mathbf{g}_y'$. (b) This follows in a straightforward fashion from the definitions of $\mathbf{g}_k, \mathbf{g}_y$ and $X_q(\mathbf{k}, \mathbf{y})$. □

**Definition 4.8.** Let us define the measure $\nu_\mathbf{y}$ on $X_q(\mathbf{y})$ by assigning on each $X_q(\mathbf{k}_t, \mathbf{y})$, the pull-back measure of $\frac{1}{q_{d-1}(\mathbb{Z}_S)^{d-1}}\mu_{d-1}(\mathbf{g}_1 \Gamma(\mathbb{Q}_S) \mathbf{g}_1^{-1} \cap H) \setminus H$, where $\mathbf{g}_1 = \mathbf{g}_k$. Here, $\mu_{d-1}$ is the product measure of the volume measure on $\mathbb{Q}_S^{d-1}$ and $\mu_{d-1}(\mathbb{Z}_S^{d-1})$ on $UL_{d-1}(\mathbb{Q}_S)$ such that $\mu_{d-1}(\mathbb{Z}_S^{d-1}) = 1$.

If we denote by $R_{\gamma} : X_q \to X_q$ the map given by right multiplication by $\mathbf{g}$, it holds that $(R_{\gamma})_* \nu_\mathbf{y} = \nu_\mathbf{g}_y$ for any $\mathbf{g} \in UL_d(\mathbb{Q}_S)$ and $\mathbf{y} \in \prod_{p \in S}(\mathbb{Q}_d^p - \{O\})$.

**Lemma 4.9.** Let $d \geq 3$. Consider a bounded measurable function $f : \mathbb{Q}_S^d \to \mathbb{R}$ of compact support. For $\eta = (\eta_1, \eta')$ with $\eta_1 \in \mathbb{Q}_S$ and $\eta' \in \mathbb{Z}_S^{d-1}$, we have
\[
\int_{(\Gamma(q) \cap H) \setminus H} \hat{f}(\{(\mathbb{Z}_S^{d-1} + \eta)h\} d\mu_{d-1}(h) = q^{d-1} T_q(\mathbf{f}_1) \left( \sum_{\ell \in \mathbb{Z}_S} f((\ell + \eta_1)\mathbf{e}_1) + \int_{\mathbb{Q}_S^d} f d\mathbf{v} \right),
\]
where $T_q(\mathbf{f}_1) = \# UL_{d-1}(\mathbb{Z}_S/q\mathbb{Z}_S)$.

**Proof.** Following the notation in § 3, let us denote by $h = (\mathbf{v}', \mathbf{g}')$ an element of $H$, where $\mathbf{v}' \in \mathbb{Q}_S^{d-1}$ and $\mathbf{g}' \in UL_{d-1}(\mathbb{Q}_S)$. Recall that $(\Gamma(q) \cap H) \setminus H \simeq (\mathbb{Q}^{d-1}_S \setminus \mathbb{Q}_S^{d-1}) \times ((\Gamma(q) \setminus UL_{d-1}(\mathbb{Q}_S))$. Let $P'$ be a fundamental domain of $\Gamma(q) \setminus UL_{d-1}(\mathbb{Q}_S)$.

Define $f_1(x_1, \mathbf{x}') = \sum_{\ell \in \mathbb{Z}_S} f(x_1 + \ell, \mathbf{x}')$. Since $\eta' \in \mathbb{Z}_S^{d-1}$,
\[
\int_{(\Gamma(q) \cap H) \setminus H} \hat{f}(\{(\mathbb{Z}_S^{d-1} + \eta)h\} d\mu_{d-1}(h)
\]
\[
= \int_{P'} \int_{\mathbb{Q}_S^{d-1} \setminus \mathbb{Q}_S^{d-1}} f_1(\eta_1 \mathbf{e}_1) d\mu_{d-1}(\mathbf{g}') d\mathbf{v}'.
\]
\[ + \int_{\mathbb{Z}^d_S \setminus \{ O \}} \int_{q\mathbb{Z}^d_S \setminus \mathbb{Z}^d_S} \sum_{m' \in \mathbb{Z}^d_S \setminus \{ O \}} f_1(\eta_1 + m' \mathbf{v}', m' \mathbf{g}') \, d\tilde{\mu}_S^{(d-1)}(g') \, dv' \]
\[ = q^{d-1} I_q^{(d-1)} f_1(\eta_1 \mathbf{e}_1) \]
\[ + \int_{\mathbb{Z}^d_S \setminus \{ O \}} \sum_{m' \in \mathbb{Z}^d_S \setminus \{ O \}} \int_{q\mathbb{Z}^d_S \setminus \mathbb{Z}^d_S} f_1(\eta_1 + m' \mathbf{v}', m' \mathbf{g}') \, dv' \, d\tilde{\mu}_S^{(d-1)}(g'). \]  

(4.3)

Note that the integral above can be changed to \( I_q^{(d-1)} \) times the integral over \( \text{UL}_{d-1}(\mathbb{Z}_S) \setminus \text{UL}_{d-1}(\mathbb{Q}_S) \), since the function inside is invariant under the action of \( \text{UL}_{d-1}(\mathbb{Z}_S) \). Consider the function on \( \mathbb{Q}^{d-1}_S \) defined by
\[ x' \in \prod_{p \in S_f} (\mathbb{Q}^{d-1}_p - \{ O \}) \mapsto \int_{q\mathbb{Z}^d_S \setminus \mathbb{Z}^d_S} f_1(\eta_1 + \mathbf{x}' \mathbf{v}', \mathbf{x}') \, dv' \]
\[ = q^{d-1} \int_{\mathbb{Q}_S} f(x_1, \mathbf{x}') \, dx_1 \]

and 0 otherwise. Then the function above is bounded and compactly supported. Applying Theorem 4.3 (a) to the last integral in (4.3), we obtain the result.

Proof of Proposition 4.6. Observe that for each \( k_t \) in Proposition 4.7, one can take \( g_t = g_k \), as \( a_t \gamma_t \), where \( a_t = \text{diag}(t/q, q/t, 1, \ldots, 1) \) and \( \gamma_t \in \text{SL}_q(\mathbb{Z}_S) \). Let \( \Phi_t : H \to H \) be the map given as \( \Phi_t(h) = a_t h a_t^{-1} \) so that \( g_t \Gamma(q) g_k^{-1} \cap H = \Phi_t(\Gamma(q) \cap H) \). Note that \( (\Phi_t)_* \tilde{\mu}_H = (t/q)^d \tilde{\mu}_H \).

By the definition of \( \nu_y \) and the change of variables by the map \( \Phi_t \) for each \( t \), we have
\[ \int_{X_q(y)} \hat{f} \left( \left( \mathbb{Z}^d_S + \frac{p}{q} \right) g \right) \nu_y(g) \]
\[ = \frac{1}{I_q \xi_S(d)} \sum_{t \in \mathbb{N}_S \atop \gcd(t, q) = 1} \int_{(\mathbb{Z}^d_S + \frac{p}{q} \gamma_t^{-1}h) \setminus H} \hat{f} \left( \left( \mathbb{Z}^d_S + \frac{p}{q} \gamma_t^{-1}h \right) g_t^{-1} \nu_y \right) \, d\tilde{\mu}_H(h) \]
\[ = \frac{q^d}{I_q \xi_S(d)} \sum_{t \in \mathbb{N}_S \atop \gcd(t, q) = 1} \frac{1}{t^d} \int_{(\mathbb{Z}^d_S + \frac{p}{q} \gamma_t^{-1}) \setminus H} \hat{f} \left( \left( \mathbb{Z}^d_S + \frac{t}{q} \gamma_t^{-1} \mathbf{e}_1 \right) h a_t^{-1} \nu_y \right) \, d\tilde{\mu}_H(h) \]
\[ = \frac{q^d}{I_q \xi_S(d)} \sum_{t \in \mathbb{N}_S \atop \gcd(t, q) = 1} \frac{1}{t^d} \int_{(\mathbb{Z}^d_S + \frac{t}{q} \mathbf{e}_1) \setminus H} \hat{f} \left( \left( \mathbb{Z}^d_S + \frac{t}{q} \mathbf{e}_1 \right) h a_t^{-1} \nu_y \right) \, d\tilde{\mu}_H(h). \]

Here, the last equality is deduced from the fact that
\[ (\mathbb{Z}^d_S + \frac{p}{q}) \gamma_t^{-1} = (\mathbb{Z}^d_S + k_t) \gamma_t^{-1} = \mathbb{Z}^d_S + (t/q) \mathbf{e}_1 \]
by the definition of \( \gamma_t \in \text{SL}_q(\mathbb{Z}_S) \).
Applying Lemma 4.9 with the function $\mathbf{v} \mapsto f(\mathbf{v} a_t^{-1} \mathbf{g}_y)$ for each $t$, the above integral is

$$
= \frac{q^{2d-1} f(d-1)}{I_q \zeta_S(d)} \sum_{t \in \mathbb{N}_S} \frac{1}{t^d} \left( \sum_{r \in \mathbb{Z}_S} f \left( \left( \frac{r}{t} + 1 \right) \mathbf{y} \right) + \int_{Q_S^d} f d\tilde{\mu}_S \right)
$$

where we put $t_1 = \text{gcd}(\ell q + t, q), t_2 = t / t_1$ and $a = (\ell q + t) / t_1$ in the inner summation and relabel them.

As in the proof of [28, Proposition 7.6], since $I_q = q^{2d-1} \sum_{e \mid q} \mu(e) e^{-d}$ and

$$
\sum_{t \in \mathbb{N}_S, \text{gcd}(t, q) = 1} 1 / t^d = \zeta_S(d) \sum_{e \mid q} \mu(e) e^{-d},
$$

where $\mu(e)$ is the Mobius function, it follows that

$$
\frac{q^{2d-1} f(d-1)}{I_q \zeta_S(d)} \cdot \sum_{t \in \mathbb{N}_S} \frac{1}{t^d} = 1
$$

whereby we deduce Proposition 4.6.

**Remark 4.10.** Using $(\Phi_t)_* \tilde{\mu}_H = (t / q)^d \tilde{\mu}_H$, with the similar argument as in the proof of Proposition 4.6, one can show that $\nu_y$ is the probability measure on $X_q(\mathbf{y})$ (see [28, Proposition 7.5]).

**Proof of Proposition 4.5.** As mentioned before, since $\prod_{p \in S}(Q_p - \{O\})$ is a conull set in $Q_S^d$, it suffices to show that

$$
\frac{1}{I_q} \int_{X_q} \sum_{k \in \mathbb{Z}_S^d + p/q} F(\Gamma(q)\mathbf{g}, k\mathbf{g}) d\tilde{\mu}_S(\mathbf{g})
$$

$$
= \int_{\prod_{p \in S}(Q_p - \{O\})} \int_{X_q(y)} F(\Gamma(q)\mathbf{g}, \mathbf{y}) d\nu_y(\mathbf{g}) d\mathbf{y}.
$$

We may assume that $F = \chi_{U'} \cdot \chi_W$, where $U' \subseteq \Gamma(q) \setminus \text{UL}_d(Q_S)$ and $W \subseteq Q_S^d$ are bounded and measurable, and show that

$$
\int_{W \cap \prod_{p \in S}(Q_p - \{O\})} \int_{X_q(y)} \chi_{U'}(\Gamma(q)\mathbf{g}) d\nu_y(\mathbf{g}) d\mathbf{y}
$$

$$
= \frac{1}{I_q} \sum_{k \in \mathbb{Z}_S^d + p/q} \int_{X_q} \chi_{U'}(\Gamma(q)\mathbf{g}) \chi_W(k\mathbf{g}) d\tilde{\mu}_S(\mathbf{g}). \quad (4.4)
$$

Let $U'_0 \subseteq \text{UL}_d(Q_S)$ be the pre-image of $U'$ under the projection $\text{UL}_d(Q_S) \to \Gamma(q) \setminus \text{UL}_d(Q_S)$.
Let $\mathcal{F}$ be a fundamental domain of $\Gamma(q) \setminus \text{UL}_d(Q_S)$. For each $t \in \mathbb{N}$ with $\gcd(t, q) = 1$, let $k_t, g_t = g_{k_t}$ be as in Proposition 4.7. Denote by $R_t$ the set of representatives for $(\Gamma(q) \cap g_t^{-1}H_g) \setminus \Gamma(q)$. Then one can check that:

(i) for each $t \in \mathbb{N}_S$ with $\gcd(t, q) = 1$, $(g_t \bigcup_{\gamma \in R_t} \gamma \mathcal{F})g_y^{-1} \cap H$ is a fundamental domain of $(g_t \Gamma(q)g_t^{-1} \cap H) \setminus H$;

(ii) there is a one-to-one correspondence between $\bigsqcup_{t \in \mathbb{N}_S, \gcd(t, q) = 1} k_t, R_t$ and $\mathbb{Z}^d_S + p/q$ (see proof of [28, Proposition 7.3]).

By the definition of $\nu_y$,

$$
\int_{X_q(y)} X_{U}(\Gamma(q)g)d\nu_y(g) = \frac{1}{I_{qS}(d)} \sum_{t \in \mathbb{N}_S} \sum_{\gcd(t, q) = 1} \int_{H} X_{U}(\Gamma(q)g_t^{-1}(\Gamma(q)g_t^{-1} \cap H)h)g_y d\bar{\mu}_H(h).
$$

For each $\gamma \in R_t$ and $h = g_t(\gamma g^t)g_y^{-1} \in g_t(\gamma \mathcal{F})g_y^{-1} \cap H$,

$$
g_t^{-1}h g_y = \gamma g' \in U_0 (\Leftrightarrow \gamma g' \in (U_0 \cap \mathcal{F})) \Leftrightarrow h \in g_t(\gamma(0 \cap \mathcal{F}))g_y^{-1}.
$$

Hence

$$
\int_{X_q(y)} X_{U}(\Gamma(q)g)d\nu_y(g) = \frac{1}{I_{qS}(d)} \sum_{t \in \mathbb{N}_S} \sum_{\gcd(t, q) = 1} \int_{H} X_{g_t(\gamma(U_0 \cap \mathcal{F}))g_y^{-1}}(h)d\bar{\mu}_H(h)
$$

$$
= \frac{1}{I_{qS}(d)} \sum_{k \in \mathbb{Z}^d_S + p/q} \int_{H} X_{k(U_0 \cap \mathcal{F})g_y^{-1}}(h)d\bar{\mu}_H(h),
$$

(4.5)

where we take $g_k = g_t \gamma$ for each $t$ and $\gamma \in R_t$, which are associated with $k$ according to (ii).

On the other hand, in view of Proposition 3.1 and Theorem 3.4, we have

$$
\int_{\text{UL}_d(Q_S)} f(g)d\bar{\mu}_S(g) = \frac{1}{\zeta_S(d)} \int_{Q_S^d} \int_{H} f(hg_y)d\bar{\mu}_H(h)dy.
$$

Applying the above equation to each integral in the right-hand side of (4.4), since $\bar{\mu}_S$ is UL$_d(Q_S)$-invariant, we have

$$
\int_{X_q} X_{U}(\Gamma(q)g)X_{W}(kg) d\bar{\mu}_S(g) = \int_{\text{UL}_d(Q_S)} X_{\gamma \cap U_0}(g^{-1}g)X_{\gamma}(kg^{-1}g)d\bar{\mu}_S(g)
$$

$$
= \frac{1}{\zeta_S(d)} \int_{Q_S^d} \int_{H} X_{\gamma \cap U_0}(g^{-1}h g_y)X_{\gamma}(kg^{-1}h g_y)d\bar{\mu}_H(h)dy. \quad (4.6)
$$

Since $kg^{-1}h g_y = y$, equation (4.4) follows from (4.5) and (4.6).
5 | PROOF OF THEOREM 2.1

From (2.2), Theorem 2.1 is a direct consequence of the theorem below. Recall the notation \( N(q, I, T) \) in Theorem 2.4.

**Theorem 5.1.** Under the same assumptions as in Theorem 2.1, there is \( \delta_0 = \delta_0(d, \kappa) > 0 \) such that for any \( \delta \in (0, \delta_0) \), we have

\[
N(q, I, T) = c_q \text{vol}(I_T)|T|^{d-2} + o\left(|T|^{d-2-\kappa-\delta}\right)
\]

for almost every unimodular non-degenerate isotropic quadratic form \( q \). Here the implied constant of the error term is uniform on a compact set of \( \mathcal{Y}_{p/q} \).

**Proof of Theorem 2.1.** By (2.2),

\[
N(q, p; q, I, T) = N(q, I', T'),
\]

where \( T' = (T_{\infty}/q, T_{p_1}, \ldots, T_{p_s}) \) and \( I' = \{I_{r'}\} \) with \( I'_{r'} = \frac{1}{q^2}I_{r} \). Note that \(|T'| = \frac{1}{q}|T|\) and \( \text{vol}(I'_{r'}) = \frac{1}{q^2} \text{vol}(I_r) \). In particular, \( I' \) satisfies that \( \text{vol}(I'_{r'}) = (c/q^{2-\kappa})|T'|^{-\kappa} \). By Theorem 5.1,

\[
N(q, p; q, I, T) = N(q, I', T') = c_q \text{vol}(I_{T'})|T'|^{d-2} + o\left(|T'|^{d-2-\kappa-\delta}\right)
\]

\[
= c_q \frac{1}{q^d} \text{vol}(I_T)|T|^{d-2} + o\left(|T|^{d-2-\kappa-\delta}\right).
\]

\( \square \)

**Theorem 5.2.** Let \( d \geq 3 \). Let \( A = \prod_{p \in S} A_p \) be the product of bounded Borel sets \( A_p \) in \( \mathbb{Q}_d^d \) for each \( p \in S \). There is a constant \( C_d > 0 \), depending only on the dimension \( d \), such that

\[
\tilde{\mu}_S\left(\{\Lambda \in \mathcal{Y}_{p/q} : |\#(\Lambda \cap A) - \text{vol}(A)| > M\}\right) < J_q C_d \cdot \frac{\text{vol}(A)}{M^2}.
\]

**Proof.** Let \( \chi_A \) be the indicator function of \( A \in \mathbb{Q}_d^d \). Since

\[
\{(t, a) : t \in \mathbb{N}_S, \text{gcd}(t, q) = 1, a \in q\mathbb{Z}_S + t, \text{gcd}(a, t) = 1\}
\]

\[
\subseteq \{(t, a) : t \in \mathbb{N}_S, a \in q\mathbb{Z}_S - \{0\}, \text{gcd}(a, t) = 1\},
\]

by Theorem 4.3 and the proof of [16, Proposition 4.2(b)], there is \( C_d > 0 \) for which

\[
\frac{1}{J_q} \int_{\mathcal{Y}_{p/q}} \hat{\chi}_A \hat{\mu}_S^2 \leq \text{vol}(A)^2 + C_d \text{vol}(A).
\]

(5.1)

The result follows from (5.1) and Chebyshev’s inequality with the probabily space \( (\mathcal{Y}_{p/q}, \frac{1}{J_q} \tilde{\mu}_S) \).

\( \square \)
Theorem 5.3. Let $d \geq 3$. For a given isotropic quadratic form $q = (q^p)_{p \in S}$, let $g = (g_p)_{p \in S} \in GL_d(Q_S)$ be such that $(q^p)^{g_p}$ is of the form $2x_1x_d + (q^p)'(x_2, \ldots, x_{d-1})$, where the coefficients of $(q^p)'$ are in $\mathbb{Z}_p$ if $p \neq 2$ and in $2\mathbb{Z}_p$ if $p = 2$. For each $p \in S_f$, denote by $k_0^{(p)}$, $z^{(p)}$ integers satisfying that

$$g_p(z_d^d - p z_p^d) + p^{k_0^{(p)}} z_d^d = g_p(z_d^d - p z_p^d) \quad \text{and} \quad p^{z^{(p)}} z_p^d \subseteq g_p(z_d^d - p z_p^d).$$

Let $I = (I^{(p)})_{p \in S} \subseteq [-N, N] \times \prod_{p \in S_f} p^{b^{(p)}} z_p^d$ be an $S$-interval such that for each $p \in S_f$, there is $k_p \in \mathbb{Z}$ for which $I^{(p)} + p^{k_p} z_p^d = I^{(p)}$.

Then for $T$ with $T_\infty > 2N^{1/d}$ and

$$2t_p \geq \begin{cases} \max\{1 + k_0^{(p)} + z^{(p)} - b^{(p)}, 1 + k_1^{(p)} + 2z^{(p)} - 2b^{(p)}\}, & \text{if } p \neq 2; \\ \max\{1 + k_0^{(p)} + z^{(p)} - b^{(p)} + 1, 1 + k_1^{(p)} + 2z^{(p)} - 2b^{(p)} + 2\}, & \text{if } p = 2, \end{cases}$$

(5.2)

we have that

$$\text{vol} (q^{-1}(I) \cap B(O, T)) = c_q \text{vol}(I) |T|^{d-2} + o_q (\text{vol}(I) |T|^{d-2}).$$

Proof. Since the volume is the product measure $\prod_{p \in S} \text{vol}_p$, it suffices to show the formula for each $p \in S$. For the real case, see [19, Theorem 5]. For the $p$-adic case, the proof of [17, Proposition 4.2] implies the statement for $p \geq 3$. In the case when $p = 2$, almost the same proof as that of Proposition 4.2 in [17] is applicable for $q^p$, which is of the form $2x_1x_d + q'(x_2, \ldots, x_{d-1})$, where the coefficients of $q'$ are in $2\mathbb{Z}_p$. \qed

For a discrete set $\Lambda$ and a finite-volume set $A$ in $Q_S^d$, define

$$D(\Lambda, A) = |\#(\Lambda \cap A) - \text{vol}(A)|.$$

One can obtain the following lemma directly.

Lemma 5.4. Let $\Lambda \subseteq Q_S^d$ be a discrete set. Let $A_1 \subseteq A \subseteq A_2 \subseteq Q_S^d$ be sets with finite volume. Then

$$D(\Lambda, A) + \text{vol}(A_2 - A_1) \leq \max \{D(\Lambda, A_1), D(\Lambda, A_2)\}.$$

Proof of Theorem 5.1. We will follow the strategy of the proof of Theorem 2.10 in [16], which is based on the Borel–Cantelli lemma. We first fix an arbitrary compact set $K$ in $UL_d(Q_S)$. For notational simplicity, let us denote

$$\Lambda_0 = z_S^d + \frac{P}{q}.$$

For a quadratic form $q$, $I \subseteq Q_S$ and $T$, define

$$A_{q,I,T} = q^{-1}(I) \cap B_T,$$

where $B_T = \{v \in Q_S^d : \|v\|_p < B_T\}$. 


Let \((\delta_p)_{p \in S}\) be an \(S\)-tuple of positive real numbers. For \(J = (j_\infty, p_1^j, \ldots, p_s^j) \in \mathbb{N} \times \prod_{p \in S_f} p^\mathbb{N}\), define

\[
C_J = \left\{ g \in \mathcal{K} : \quad D(A_0, A_{q_0^g}, T) > \text{vol}(I_{T}) \prod_{p \in S} T_p^{d-2-\delta_p} \right\}.
\]

The theorem follows if \(\tilde{\mu}_S(\bigcap_{J_0} \bigcup_{J \geq J_0} C_J) = 0\) and by Borel–Cantelli lemma, it suffices to show that

\[
\sum_{J \geq J_0} \tilde{\mu}(C_J) < \infty \quad (5.3)
\]

for appropriate \((\delta_p)_{p \in S}\). Let us choose \((\delta_p)_{p \in S}, (\alpha_p)_{p \in S}\) and \((\beta_p)_{p \in S}\) for which

\[
\begin{align*}
0 < \delta_\infty &< \alpha_\infty, \\
\delta_\infty &< \beta_\infty + 1, \\
-(d-2) &- 2\delta_\infty + \alpha_\infty \left(\frac{1}{2}(d+2)(d-1)\right) + \beta_\infty < -1; \\
0 < \delta_p &< \alpha_p, \\
\delta_p &< \beta_p, \\
-(d-2) &- 2\delta_p + \alpha_p \left(\frac{1}{2}(d+2)(d-1)\right) + \beta_p < 0.
\end{align*}
\]

\[(5.4)\]

Note that the range of such \(\delta_p\) are \((0, \frac{d-2}{(d+2)(d-1)/2-1})\).

The first Approximation: the space \(\Lambda_0\). We first observe that for \(h\) and \(g = (g_p)_{p \in S}\) in \(\text{UL}_d(\mathbb{Q}_S)\), \(I \subseteq \mathcal{Q}_S\) and \(T = (T_p)_{p \in S}\),

\[
v \in \Lambda_0 : \quad q_0^{h g}(v) \in I, \quad \|v\|_p < T_p \quad (p \in S)
\]

\[
\Rightarrow w \in \Lambda_0 g : \quad q_0^{h}(w) \in I, \quad \|w\|_p < \|g_p\|_{op} T_p \quad (p \in S).
\]

For each \(J\), let \(\varepsilon_1 = \varepsilon_1(J) = j^{-(\alpha_\infty)} \cdot \prod_{p \in S_f} p^{-(\alpha_p)}\). One can find \(C(\mathcal{K}) > 0\) such that for each \(J\), there is a subset \(Q = Q(\mathcal{K}, J)\) of \(\mathcal{K}\) for which

(i) \(\mathcal{K} \subseteq \bigcup_{h \in Q} h \cdot B(\varepsilon_1)\),

where

\[
B(\varepsilon_1) = \left\{ g_\infty \in \text{SL}_d(\mathbb{R}) : \quad \|g_\infty\|_{op} \leq 1 + \varepsilon_1 \right\} \times \prod_{p \in S_f} \text{UL}_d(\mathbb{Z}_p);
\]

(ii) \#\(Q(\mathcal{K}, J) < C(\mathcal{K})\varepsilon_1^{-\frac{1}{2}(d+2)(d-1)}\).

Here, \(\frac{1}{2}(d+2)(d-1)\) is the codimension of \(\text{SO}(d)\) in \(\text{SL}_d(\mathbb{R})\) since elements in \(\text{SO}(d) \times \prod_{p \in S_f} \text{UL}_d(\mathbb{Z}_p)\) have unit operator norms.

For \(g \in B(\varepsilon_1)\), since

\[
A_{q_0^{h g}, I_T, (T_\infty (1-\varepsilon_1), J_f)} \subseteq A_{q_0^{h g}, I_T, (T_\infty, J_f)} \subseteq A_{q_0^{h g}, I_T, (T_\infty (1+\varepsilon_1), J_f)}
\]
if we put $I^u_T := I_{(T,\infty(1+\varepsilon)^{-1},J'/J)}$ and $I^u_T := I_{(T,\infty(1-\varepsilon)^{-1},J'/J)}$, where $I = \{I_T\}$, it follows that for all sufficiently large $J$ (depending on the choice of $(\alpha_p)_{p \in S}$ and $(\delta_p)_{p \in S}$), by Lemma 5.4 and Theorem 5.3 with the condition in (5.4),

$$C_j \subseteq C^u_j \cup C^\ell_j,$$

where

$$C^u_j = \bigcup_{h \in Q} \left\{ g \in B(\varepsilon_1) : \left. \begin{array}{l}
D \left( \Lambda_0 g, A_{\mu_j,\tau_T}^{h,r_0} \right) > 0.99 \text{vol}(I^u_T) \prod_{p \in S} T_p^{d-2-\delta_p} \\
\text{for some } T \in [j_\infty, j_\infty + 1+\varepsilon_1] \times J' \end{array} \right\},$$

$$C^\ell_j = \bigcup_{h \in Q} \left\{ g \in B(\varepsilon_1) : \left. \begin{array}{l}
D \left( \Lambda_0 g, A_{\mu_j,\tau_T}^{h,r_0} \right) > 0.99 \text{vol}(I^\ell_T) \prod_{p \in S} T_p^{d-2-\delta_p} \\
\text{for some } T \in [j_\infty - \varepsilon_1, j_\infty + 1] \times J' \end{array} \right\},$$

We remark that since $I^{(\infty)}_T$ is a sequence of decreasing interval and $\kappa_p < 2$ for any $p \in S_f$, one can find a uniform $\delta_0$, depending on the compact set $\mathcal{K}$, such that Theorem 5.3 holds for $q^h$ for any $h \in \mathcal{K}$ and $T \geq T_0$.

The second Approximation: the radius. Let $\varepsilon_2 = j^{-\beta_0}_{\rightarrow \infty} \prod_{p \in S_f} j^{-\beta_j}_{p\rightarrow p}$. For each $k = 0, 1, \ldots, \lfloor (1+\varepsilon_1)/\varepsilon_2 \rfloor$, and for $j_\infty + \varepsilon_2 k < T_\infty < j_\infty + \varepsilon_2 (k+1)$, we have

$$A^{h,r_0}_{\mu_j,\mu_{j_k+1}} \subseteq A^{h,r_0}_{\mu_j,\mu_{j_k}},$$

where $J_k = (j_\infty + \varepsilon_2 k, J')$. Again, by Lemma 5.4 and Theorem 5.3 with the condition in (5.4) provided that

$$(j_\infty + \varepsilon_2 k)^{-\kappa_\infty} - (j_\infty + \varepsilon_2 (k+1))^{-\kappa_\infty} \ll \kappa_\infty (j_\infty + \varepsilon_2 k)^{-\kappa_\infty-1},$$

it follows that for all sufficiently large $J$,

$$C^u_j \subseteq C^{uu}_j \cup C^{\ell\ell}_j,$$

where for each $J_k$,

$$I^{uu}_{j_k} := I_{(j_\infty + \varepsilon_2 (k-1), J')}, \quad \text{and} \quad I^{\ell\ell}_{j_k} := I_{(j_\infty + \varepsilon_2 (k+1), J')}$$

and

$$C^{uu}_j = \bigcup_{h \in Q} \bigcup_{k=1}^{\lfloor \varepsilon_1/\varepsilon_2 \rfloor + 1} \left\{ g \in B(\varepsilon_1) : \left. \begin{array}{l}
D \left( \Lambda_0 g, A^{h,r_0}_{\mu_j,\mu_{j_k}} \right) > 0.99 \text{vol}(I^{uu}_{j_k}) \\
\times (j_k + \varepsilon_2 k)^{d-2-\delta_\infty} \prod_{p \in S_f} j_p^{d-2-\delta_p} \end{array} \right\},$$

$$C^{\ell\ell}_j = \bigcup_{h \in Q} \bigcup_{k=0}^{\lfloor \varepsilon_1/\varepsilon_2 \rfloor} \left\{ g \in B(\varepsilon_1) : \left. \begin{array}{l}
D \left( \Lambda_0 g, A^{h,r_0}_{\mu_j,\mu_{j_k}} \right) > 0.99 \text{vol}(I^{\ell\ell}_{j_k}) \\
\times (j_k + \varepsilon_2 k)^{d-2-\delta_\infty} \prod_{p \in S_f} j_p^{d-2-\delta_p} \end{array} \right\}. \square$$
Finally, by Theorems 5.2 and 5.3, for each $k$,

$$
\bar{\mu}_S \left\{ g \in B(\varepsilon_1) : D \left( \Lambda_0 g, A_{\Omega(i_1, j_k)}^{\mathbb{R}^d} \right) > 0.9 \text{vol}(\Omega_{i_k}) \times (j_k + \varepsilon_2 k)^{d-2-\delta} \prod_{p \in \mathcal{F}} p_j^i \left( -2 - 2 \delta_p \right) \right\} \ll \int_{J_{\infty}} \left( \prod_{p \in \mathcal{F}} p_j^i \right) \left( -(d-2) - 2 \delta \right) \prod_{p \in \mathcal{F}} p_j^i \left( \frac{1}{2}(d+2)(d-1) \right) + \beta \infty \right.
$$

so that

$$
\bar{\mu}_S (C^\infty_j) \ll \int_{J_{\infty}} \left( \prod_{p \in \mathcal{F}} p_j^i \right) \left( -(d-2) - 2 \delta \right) \prod_{p \in \mathcal{F}} p_j^i \left( \frac{1}{2}(d+2)(d-1) \right) + \beta \infty \right.
$$

Hence $\sum_{j \geq 0} \bar{\mu}_S (C^\infty_j) < \infty$ by (5.4) for sufficiently large $J_0$. Similarly, one can show that the summands of $\bar{\mu}_S (C^\infty_j)$, $\bar{\mu}_S (C^\infty_j)$ and $\bar{\mu}_S (C^\infty_j)$ are finite, which shows (5.3).

6 | THE SPACE OF INHOMOGENEOUS QUADRATIC FORMS

Using Rogers' higher moment formulae for the space of unimodular affine lattices in $\mathbb{R}^d$ ([9, Appendix B], [2, Lemma 4]), in [14], it was noted that the effective Oppenheim conjecture holds for almost all unimodular affine lattices in $\mathbb{R}^d$. In this section, we generalize this result to the space of unimodular affine $S$-lattices in $\mathbb{Q}^d_S$ and for this, let us first show Rogers' higher moment formulae for $\text{AUL}_d(\mathbb{Z}_S) \setminus \text{AUL}_d(\mathbb{Q}_S)$.

**Theorem 6.1.** Let $d \geq 2$.

(a) For a bounded and compactly supported function $f : \mathbb{Q}^d_S \to \mathbb{R}_{\geq 0}$,

$$
\int_{\text{AUL}_d(\mathbb{Z}_S) \setminus \text{AUL}_d(\mathbb{Q}_S)} \hat{f} \, d\bar{\mu}_S \, dv = \int_{\mathbb{Q}^d_S} f(v) \, dv.
$$

(b) For a bounded and compactly supported function $F : (\mathbb{Q}^d_S)^2 \to \mathbb{R}_{\geq 0}$,

$$
\int_{\text{AUL}_d(\mathbb{Z}_S) \setminus \text{AUL}_d(\mathbb{Q}_S)} \sum_{k_1, k_2 \in \mathbb{Z}^d} F(k_1 g + v, k_2 g + v) \, d\bar{\mu}_S \, dv
$$

$$
= \int_{(\mathbb{Q}^d_S)^2} F(v_1, v_2) \, dv_1 \, dv_2 + \int_{\mathbb{Q}^d_S} F(v, v) \, dv.
$$

In particular, if we let $F(x_1, x_2) = \chi_A(x_1) \chi_A(x_2)$ for a borel set $A \subseteq \mathbb{Q}^d_S$, it holds that

$$
\int_{\text{AUL}_d(\mathbb{Z}_S) \setminus \text{AUL}_d(\mathbb{Q}_S)} \hat{\chi}_A \, d\bar{\mu}_S \, dv = \text{vol}(A)^2 + \text{vol}(A).
$$
Proof. Let \( \mathcal{F} \) be a fundamental domain of \( \text{UL}_d(Z_S) \setminus \text{UL}_d(Q_S) \). Then a fundamental domain of \( \text{AUL}_d(Z_S) \setminus \text{AUL}_d(Q_S) \) can be taken as \( \bigcup_{\mathcal{G}} \mathcal{G} \times \{\mathcal{G}\} \), where \( \mathcal{T} = [0,1)^d \times \prod_{p \in S} \mathbb{Z}_p^d \) is a fundamental domain of \( \mathbb{Z}_S^d \setminus Q_S^d \).

(a) By the change of variables \( \mathbf{w} = \mathbf{v} \mathbf{g}^{-1} \) and letting \( f_g(\mathbf{v}) := f(\mathbf{v} \mathbf{g}) \) for each \( \mathcal{G} \in \mathcal{F} \), we have

\[
\int_{\text{AUL}_d(Z_S) \setminus \text{AUL}_d(Q_S)} \hat{f} \, d\bar{\mu}_S = \int_{\mathcal{F}} \int_{T} \sum_{\mathbf{g}} f(\mathbf{v} \mathbf{g}) \, d\mathbf{v} \, d\bar{\mu}_S(\mathbf{g}) = \int_{\mathcal{F}} \int_{T} \sum_{\mathbf{g}} f_g(\mathbf{k} + \mathbf{w}) \, d\mathbf{w} \, d\bar{\mu}_S(\mathbf{g}) = \int_{\mathcal{F}} \int_{Q_S^d} f_g(\mathbf{w}) \, d\mathbf{w} \, d\bar{\mu}_S(\mathbf{g}) = \int_{Q_S^d} f(\mathbf{w}) \, d\mathbf{w}.
\]

(b) Similarly, by the change of variables,

\[
\int_{\mathcal{F}} \int_{T} \sum_{\mathbf{g}_{1,2}} F(\mathbf{k}_1 \mathbf{g} + \mathbf{v}, \mathbf{k}_2 \mathbf{g} + \mathbf{v}) \, d\mathbf{v} \, d\bar{\mu}_S(\mathbf{g})
= \int_{\mathcal{F}} \int_{T} \sum_{\mathbf{g}_{1,2}} F((\mathbf{k}_1 + \mathbf{w}) \mathbf{g}, (\mathbf{k}_2 + \mathbf{w}) \mathbf{g}) \, d\mathbf{w} \, d\bar{\mu}_S(\mathbf{g}).
\]

Let \( \mathbf{w}' = \mathbf{k}_1 + \mathbf{w} \) and \( \mathbf{k}_3 = \mathbf{k}_2 - \mathbf{k}_1 \). Then the above integral is

\[
= \int_{\mathcal{F}} \left( \sum_{\mathbf{k}_1 \in \mathbb{Z}_S^d} \int_{T} \sum_{\mathbf{k}_3 \in \mathbb{Z}_S^d} F(\mathbf{w}' \mathbf{g}, \mathbf{w}' \mathbf{g} + \mathbf{k}_3 \mathbf{g}) \, d\mathbf{w}' \, d\bar{\mu}_S(\mathbf{g})
= \int_{\mathcal{F}} \sum_{\mathbf{k}_1 \in \mathbb{Z}_S^d} \int_{Q_S^d} F(\mathbf{x}, \mathbf{x} + \mathbf{k}_3 \mathbf{g}) \, d\mathbf{x} \, d\bar{\mu}_S(\mathbf{g})
= \int_{Q_S^d} F(\mathbf{x}, \mathbf{x}) \, d\mathbf{x} + \int_{Q_S^d} \int_{Q_S^d} F(\mathbf{x}, \mathbf{y}) \, d\mathbf{x} \, d\mathbf{y},
\]

where the last equality follows from the Siegel integral formula for \( \text{UL}_d(Z_S) \setminus \text{UL}_d(Q_S) \) ([17, Proposition 3.11]) with the function \( \mathbf{y} \in Q_S^d \mapsto \int_{Q_S^d} F(\mathbf{x}, \mathbf{y}) \, d\mathbf{x} \).

We remark that the proof of Theorem 6.1(b) is a generalization of the proof of Proposition 14 in [9], where they considered the case when \( Q_S = \mathbb{R} \) and \( d = 2 \).

By Theorem 6.1 and Chebyshev’s inequality, we obtain the theorem below immediately.

**Theorem 6.2.** Let \( d \geq 2 \). Let \( \mathcal{F}_{\text{aff}} \) be a fundamental domain for \( \text{AUL}_d(Z_S) \setminus \text{AUL}_d(Q_S) \). For a bounded borel set \( A \subseteq Q_S^d \),

\[
\bar{\mu}_S \times \text{vol} \left( \left\{ (\mathbf{v}, \mathbf{g}) \in \mathcal{F}_{\text{aff}} : \#((\mathbb{Z}_S^d \mathbf{g} + \mathbf{v}) \cap A) - \text{vol}(A) > M \right\} \right) < \frac{\text{vol}(A)}{M^2}.
\]
Proof of Theorem 2.4. Let \( F = F_\infty \times \prod_{p \in S_f} \text{UL}_d(\mathbb{Z}_p) \setminus \text{UL}_d(\mathbb{Q}_S) \) be a fundamental domain of \( \text{UL}_d(\mathbb{Z}_S) \setminus \text{UL}_d(\mathbb{Q}_S) \), where \( F_\infty \) is a fundamental domain for \( \text{SL}_d(\mathbb{Z}) \setminus \text{SL}_d(\mathbb{R}) \). Take a fundamental domain \( \mathcal{F} \) of \( \text{AUL}_d(\mathbb{Z}_S) \setminus \text{AUL}_d(\mathbb{Q}_S) \) by \( \mathcal{F} = \bigcup_{g \in \mathbb{P}} T_g \times \{ g \} \), where \( T = [0,1)^d \times \prod_{p \in S_f} \mathbb{Z}_p \).

Let \( \mathcal{K} \) be a compact subset of \( \mathcal{F} \). For \( J \in \mathbb{N} \times \prod_{p \in S_f} p^N \), define

\[
C_J = \left\{ (\xi, g) \in \mathcal{K} : D\left( \mathbb{Z}_S^d, A_{(\xi, g)}(\mathbb{Z}_p), T \right) > \text{vol}(I_T) \prod_{p \in S} T_p^{d-2-\delta_p} \right\},
\]

for some \( T \in [j_\infty, j_\infty + 1) \times J^f \).

As in the proof of Theorem 2.1, we will show that \( \text{vol} \times \bar{\mu}_S(\limsup J \mathcal{C}_J) = 0 \) for an appropriate \((\delta_p)_{p \in S_f}\).

For \((\eta, h), (\xi, g)\) in \( F \), \( I \in \mathbb{Q}_S \) and \( T = (T_p)_{p \in S} \),

\[
v \in \mathbb{Z}_S^d : q_0^{h} : q_0^{(\eta, h)(\xi, g)}(v) \in I, \|v\|_p < T_p (p \in S)
\]

\[
\Rightarrow w \in \mathbb{Z}_S^d + \xi : q_0^{(\eta, h)} : q_0^{(\eta, h)}(w) \in I, \|w\|_p < \|\xi\|_{\rho, T} (p \in S).
\]

Note that for \( g = (g_p)_{p \in S} \in F \), \( \mathbb{Z}_S^d \setminus \mathbb{Z}_p^d \) \( (p \in S_f) \) so that for \((\xi, g) \in \mathcal{F} \),

\[
\|w\|_p = \|vg_p + \xi\|_p \leq \max \{ \|v\|_p, \|\xi\|_p \}
\]

and if \( \|v\|_p = T_p > 1, \|v\|_p = \|w\|_p \) for \( p \in S_f \).

For each \( J \), as in the proof of Theorem 2.1, set \( \epsilon_1 = j^{-\alpha_\infty} \prod_{p \in S_f} p^{-\alpha_p} / p \) and \( \epsilon_2 = j^{-\beta_\infty} \prod_{p \in S_f} p^{-\beta_p} / p \), which are the scales of the space-approximation and the radius-approximation, respectively.

Observe that there is \( C = C(\mathcal{K}) > 0 \) such that for any \( J \), one can find a subset \( Q = Q(\mathcal{K}, J) \) of \( \mathcal{K} \) for which:

(i) \( \mathcal{K} \subseteq \bigcup_{(\eta, h) \in Q(\eta, h), B(\epsilon_1)} \), where

\[
B(\epsilon_1) = \left\{ (\xi, g) \in \mathbb{R}_d^d \times \text{SL}_d(\mathbb{R}) : \|g\|_p < 1 + \epsilon_1 \right\} \times \prod_{p \in S_f} \left( \mathbb{Z}_p^d \times \text{UL}_d(\mathbb{Z}_p) \right);
\]

(ii) \( \#Q(\mathcal{K}, J) < C(\mathcal{K}) \epsilon^{-\frac{1}{2}(d+2)(d-1) - d} \),

where \( \frac{1}{2}(d+2)(d-1) + d \) is the codimension of \( \{O\} \times \text{SO}(d) \) in \( \text{SL}_d(\mathbb{R}) \).

Then one can show that \( \sum_{J > J_0} \text{vol} \times \bar{\mu}_S(C_J) < \infty \) for sufficiently large \( J_0 \) provided that

\[
0 < \delta_\infty < \alpha_\infty,
\]

\[
\delta_\infty < \beta_\infty + 1,
\]

\[
-(d - 2 - \kappa_\infty) - 2\delta_\infty + \alpha_\infty \left( \frac{1}{2}(d + 2)(d - 1) + d \right) + \beta_\infty < -1;
\]

\[
0 < \delta_p < \alpha_p,
\]

\[
\delta_p < \beta_p,
\]

\[
-(d - 2 - \kappa_p) - 2\delta_p + \alpha_p \left( \frac{1}{2}(d + 2)(d - 1) + d \right) + \beta_p < 0
\]
and the theorem follows from Borel–Cantelli lemma. Note that the range of such $\delta_p$ are $(0, \frac{d-2-\nu_p}{(d+2)(d-1)/2+d-1})$.

**ACKNOWLEDGEMENTS**

A. G. gratefully acknowledges support from a MATRICS grant from the Science and Engineering Research Board, a grant from the Infosys Foundation and a Department of Science and Technology, Government of India, Swarnajayanti fellowship

**JOURNAL INFORMATION**

*Mathematika* is owned by University College London and published by the London Mathematical Society. All surplus income from the publication of *Mathematika* is returned to mathematicians and mathematics research via the Society’s research grants, conference grants, prizes, initiatives for early career researchers and the promotion of mathematics.

**REFERENCES**

1. M. Alam, A. Ghosh and S. Yu, *Quantitative Diophantine approximation with congruence conditions*, J. Théor. Nombres Bordeaux, Tome 33 (2021), no. 1, 261–271.
2. J. S. Athreya, *Random affine lattices, Geometry, groups and dynamics*, 169–174, Contemp. Math., vol. 639, Amer. Math. Soc., Providence, R.I., 2015.
3. J. S. Athreya and G. Margulis, *Values of random polynomials at integer points*, (English summary) J. Mod. Dyn. 12 (2018), 9–16.
4. P. Bandi and A. Ghosh, *Small solutions of quadratic forms with congruence conditions*, https://arxiv.org/abs/2008.08568, 2020.
5. P. Bandi, A. Ghosh, and J. Han, *A generic effective Oppenheim theorem for systems of forms*, J. Number Theory 218 (2021), 311–333.
6. A. Borel and G. Prasad, *Values of isotropic quadratic forms at S-integral points*, Compos. Math. 83 (1992), no. 3, 347–372.
7. J. Bourgain, *A quantitative Oppenheim theorem for generic diagonal quadratic forms*, (English summary) Israel J. Math. 215 (2016), no. 1, 503–512.
8. S. Dani and G. Margulis, *Limit distributions of orbits of unipotent flows and values of quadratic forms*, I. M. Gel’fand Seminar, Adv. Soviet Math., 16, Part 1, Amer. Math. Soc., Providence, R.I., 1993, pp. 91–137.
9. D. El-Baz, J. Marklof, and I. Vinogradov, *The distribution of directions in an affine lattice: two-point correlations and mixed moments*, Int. Math. Res. Not. IMRN (2015), no. 5, 1371–1400.
10. A. Eskin, G. Margulis, and S. Mozes, *Upper bounds and asymptotics in a quantitative version of the Oppenheim conjecture*, Ann. of Math. (2) 147 (1998), no. 1, 93–141.
11. A. Eskin, G. Margulis, and S. Mozes, *Quadratic forms of signature (2,2) and eigenvalue spacings on rectangular 2-tori*, Ann. of Math. (2) 161 (2005), no. 2, 679–725.
12. A. Ghosh, A. Gorodnik, and A. Nevo, *Optimal Density for values of generic polynomial maps*, Amer. J. Math. 142 (2020), no. 6, 1945–1979.
13. A. Ghosh and D. Kelmer, *A quantitative Oppenheim theorem for generic ternary quadratic forms*, J. Mod. Dyn. 12 (2018), 1–8.
14. A. Ghosh, D. Kelmer, and S. Yu, *Effective density for inhomogeneous quadratic forms I: generic forms and fixed shifts*, Int. Math. Res. Not. https://doi.org/10.1093/imrn/rmaa206.
15. A. Ghosh, D. Kelmer, and S. Yu, *Effective density for inhomogeneous quadratic forms II: fixed forms and generic shifts*, arxiv.org/abs/2001.10990, 2020.
16. J. Han, *Rogers’ mean value theorem for S-arithmetic Siegel transform and applications to the geometry of numbers*, J. Number Theory (2022). https://doi.org/10.10106/j.jnt.2021.12.012
17. J. Han, S. Lim, and K. Mallahi-Karai, *Asymptotic distribution of values of isotropic quadratic forms at S-integral points*, J. Mod. Dyn. 11 (2017), 501–550.
18. J. Han, Quantitative oppenheim conjecture for S-arithmetic quadratic forms of rank 3 and 4, Discrete Contin. Dyn. Syst. 41 (2021), no. 5, 2205–2225.
19. D. Kelmer and S. Yu, Values of random polynomials in shrinking targets, Trans. Amer. Math. Soc. 373 (2020), 8677–8695.
20. D. Kleinbock and M. Skenderi, Khintchine-type theorems for values of subhomogeneous functions at integer points, Monatsh. Math. 194 (2021), no. 3, 523–554.
21. E. Lindenstrauss and G. A. Margulis, Effective estimates on indefinite ternary forms, Israel J. Math. 203 (2014), no. 1, 445–499.
22. G. A. Margulis, Formes quadratiques indéfinies et flots unipotents sur les espaces homogènes. (French summary), C. R. Acad. Sci. Paris. Sér. I Math. 304 (1987), no. 10, 249–253.
23. G. A. Margulis, Oppenheim conjecture, Fields Medallists’ lectures, 272–327, World Sci. Ser. 20th Century Math., 5, World Sci. Publ., River Edge, NJ, 1997.
24. G. A. Margulis and A. Mohammadi, Quantitative version of the Oppenheim conjecture for inhomogeneous quadratic forms, Duke Math. J. 158 (2011), no. 1, 121–160.
25. J. Marklof, Pair correlation densities of inhomogeneous quadratic forms II, Duke Math. J. 115 (2002), no. 3, 409–34.
26. J. Marklof, Pair correlation densities of inhomogeneous quadratic forms, Ann. of Math. (2) 158 (2003), no. 3, 419–71.
27. J. Marklof, The n-point correlations between values of a linear form, Ergodic Theory Dynam. Systems 20 (2000), 1127–1172.
28. J. Marklof, A. Strömbergsson, The distribution of free path lengths in the periodic Lorentz gas and related lattice point problems, Ann. Math. (2) 172 (2010), no. 3, 1949–2033.
29. G. Prasad, Volumes of S-arithmetic quotients of semi-simple groups, With an appendix by Moshe Jarden and the author. Publ. Math. Inst. Hautes Études Sci. 69 (1989), 91–117.
30. W. M. Schmidt, Approximation to algebraic numbers, Enseignement Math. (2) 17 (1971), 187–253.
31. C. L. Siegel, Lectures on the geometry of numbers, Springer Berlin Heidelberg GmbH, 1989. x+160 pp.
32. G. Shimura, Introduction to the Arithmetic Theory of Automorphic functions, Publications of the Mathematical Society of Japan, No. 11, Iwanami Shoten, Tokyo, 1971.