Poisson bracket on 1-forms and evolutionary partial differential equations

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Abstract

We introduce a bracket on 1-forms defined on $J^\infty(S^1, \mathbb{R}^n)$, i.e. the infinite jet extension of the space of loops, and prove that it satisfies the standard properties of a Poisson bracket. Using this bracket, we show that certain hierarchies appearing in the framework of $F$-manifolds with a compatible flat connection $(M, \nabla, \circ)$ are Hamiltonian in a generalized sense. Moreover, we show that if a metric $g$ compatible with $\nabla$ is also invariant with respect to $\circ$, then this generalized Hamiltonian setup reduces to the standard one.

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1. Introduction

In the study of integrable evolutionary PDEs, many geometric structures have been introduced, starting from the celebrated Dubrovin–Novikov framework relating the existence of a local Poisson tensor of hydrodynamic type to the presence of a flat non-degenerate metric $g$. More precisely, in the Dubrovin–Novikov setup given two local functionals $F[u] = \int_{S^1} f(u, u_x, \ldots) \, dx$ and $G[u] = \int_{S^1} g(u, u_x, \ldots) \, dx$, their Poisson bracket is defined as

$$\{F, G\} = \int_{S^1} \frac{\delta F}{\delta u} (g^{ij} \partial_x g - g^{ij} \Gamma^l_{ik} u^l_x)^{\frac{\delta G}{\delta u}} \, dx,$$

(1.1)

where $\Gamma^l_{ik}$ are the Christoffel symbols of the Levi-Civita connection associated with the metric $g$.

Many important examples of evolutionary PDEs are Hamiltonian with respect to a local Poisson bracket of hydrodynamic type. This means that they can be written as

$$u^i_t = (g^{ij} \partial_x g - g^{ij} \Gamma^l_{ik} u^l_x)^{\frac{\delta H}{\delta u^i}}, \quad i = 1, \ldots, n,$$

(1.2)
for a suitable choice of the Hamiltonian \( H[u] = \int_S h(u, u_x, \ldots) \, dx \) (in the case of quasilinear equations, the density \( h \) depends only on \( u \)). In equation (1.2), \( \frac{\delta H}{\delta u} \) denotes the variational derivative of the functional \( H \); see formula (2.10). In this lucky case, there is no need to extend the bracket to 1-forms.

However, the equations admitting this Hamiltonian formulation are exceptions, even in the integrable case. For this reason, immediately after the seminal paper of Dubrovin and Novikov, many authors devoted lots of work to extending their formalism. This effort led to the definition of the non-local Poisson bracket of hydrodynamic type [8, 7]. Besides the local part (1.1) they contain an additional term of the form

\[ \sum_\beta \epsilon_\beta (w_\beta)^{ij} u^h_k \left( \frac{d}{dx} \right)^{-1} (w_\beta)_k^h u^h_i, \] (1.3)

which is due to the non-vanishing of the curvature of \( g \):

\[ R_{kh}^{ij} = \sum_\alpha \epsilon_\alpha \left\{ (w_\alpha)_k^i (w_\alpha)_h^j - (w_\alpha)_k^j (w_\alpha)_h^i \right\}. \] (1.4)

The affinors \( w_\beta^{ij} \) appearing in the formula are related to symmetries and can be interpreted as Weingarten operators. In practice, the presence of such a term might have some drawbacks due to the ambiguity of the action of the operator \( \partial_x^{-1} \) on the differential of local functionals and to the difficulties in finding the quadratic expansion (1.4).

Recently, it was observed that a special class of non-local Poisson structures of hydrodynamic type related to local Poisson structures by a reciprocal transformation can be interpreted as a local Jacobi structure, see [12]. This discovery enabled the authors of [12] to avoid the disadvantages created by the presence of the non-local term and it allowed them to compute the Lichnerowicz–Jacobi cohomology groups, extending the results obtained by Getzler [9] in the case of local Poisson structures.

The results we present here go in the same direction but instead of dropping the requirement of locality we drop the requirement of exactness of the 1-form \( \omega \) defining the equation (or the hierarchy in the integrable case)

\[ u^i_t = P^{ij} \omega_j = (g^{ij} \partial_x^i - g^{il} \Gamma_{kl}^i u_k^l) \omega_j, \quad i = 1, \ldots, n. \] (1.5)

The motivation for this work is twofold. On one side we want to provide an extended Hamiltonian framework for systems of conservation laws. Indeed in flat coordinates equation (1.5) reduces to a system of this type. On the other side it originates from the study of integrable hierarchies related to \( F \)-manifolds [11] with compatible flat [15, 13, 14] and bi-flat structure [1, 2]. Both these classes of manifolds are equipped with a flat torsionless connection \( \nabla \) that select a class of local Poisson tensors \( P \) of hydrodynamic type.

We show that the equations of these hierarchies can always be put in the form (1.5) where the Poisson tensor \( P \) belongs to the class defined by the connection \( \nabla,^3 \) and \( \omega \) is a suitable 1-form. In the case of Frobenius manifolds [5], it turns out that the 1-form \( \omega \) is indeed exact, namely it can be written as \( \omega = \delta H \), where the functional \( H \) is interpreted as the Hamiltonian of the PDEs (1.5). In general, however, these hierarchies do not admit any usual local Hamiltonian structure, namely the 1-forms \( \omega \) are not exact.

The aim of this work is indeed to explore these issues.

First of all, it is natural to ask what the relevance is of these local Poisson tensors for the corresponding integrable hierarchies, when it is well known that for most of them

\[ By \ this \ we \ mean \ that \ P \ is \ a \ Poisson \ structure \ of \ Dubrovin–Novikov \ type \ associated \ with \ a \ metric \ g \ compatible \ with \ \nabla. \]
there is no corresponding Hamiltonian functional, so they cannot be written in the standard Hamiltonian form. We provide an answer by introducing a Poisson bracket on 1-forms that are not necessarily closed (the case in which a Hamiltonian functional does exist corresponds to exact 1-forms).

To do so, we build on the work of [10, 16]. Indeed, let us recall that on any Poisson manifold \((M, P)\) there is a \(\mathbb{R}\)-bilinear, skew-symmetric operation \(\{ \cdot, \cdot \} : \Lambda_1 M \times \Lambda_1 M \to \Lambda_1 M\), where \(\Lambda_1 M\) denotes the vector space of 1-forms on \(M\), extending the usual Poisson bracket between smooth functions. It is defined by

\[
\{ \alpha, \beta \} := \text{Lie}_g \alpha - \text{Lie}_g \beta + d(\beta, Pa)
\]

and satisfies

\[
\{ df, dg \} = d(f \cdot g),
\]

\[
\{ \alpha, f \beta \} = f \{ \alpha, \beta \} + \{ (Pa)(f) \} \beta.
\]

Here, we consider an infinite-dimensional analogue of such a bracket. Our Poisson manifold will be the space \(L(M) = \{ S^1 \to \mathbb{R}^n \}\) of \(C^\infty\) maps from the circle to \(\mathbb{R}^n\) endowed with a local Poisson bivector \(P\) of hydrodynamic type.

The Poisson bracket we are going to introduce on 1-forms has the following expression. Let \(\alpha = \int dx \wedge a_i \delta u^i\) and \(\beta = \int dx \wedge \beta_i \delta u^i\) be two 1-forms. Then, \(\{ \alpha, \beta \}\) is a 1-form \(\{ \alpha, \beta \} = \int dx \wedge \{ \alpha, \beta \} \delta u^i\), where

\[
\{ \alpha, \beta \}(s) = \delta^i_s \{ g^{ij} \partial_j a_i + \Gamma_{ij}^m \partial_m a_i \} \partial \alpha_i / \partial u^i_{(s)} - \delta^i_s \{ g^{ij} \partial_j \beta_i + \Gamma_{ij}^m \partial_m \beta_i \} \partial \beta_i / \partial u^i_{(s)} + \{ (a_i \partial_i \beta_j - \beta_j \partial_i a_i) \} \Gamma_{ij}^k - \{ a_i \partial_i \beta_i \} \Gamma_{ij}^k - \{ \Gamma_{ij}^k \} \alpha^m \delta_{(s)}(u^m),
\]

where \(g^{ij}\) is the flat (contravariant) metric in the chosen coordinates and \(\Gamma_{ij}^k\) are the corresponding Christoffel symbols.

Although it is not unexpected, one result about this bracket is that it satisfies the Jacobi identity. Moreover, this bracket equips the vector space of 1-forms with a Lie algebra structure and the Poisson tensor \(P\) gives rise to an anti-homomorphism of Lie algebras from the Lie algebra of 1-forms equipped with this bracket and the Lie algebra of evolutionary vector fields equipped with the Lie bracket.

Given an \(F\)-manifold with a compatible flat connection \((M, \nabla, \circ)\), the second issue we are addressing is the role played by the existence of a metric \(g\) invariant with respect to the product \(\circ\) (namely \(g(X \circ Y, Z) = g(X, Y \circ Z)\)) and compatible with \(\nabla\) (\(\nabla g = 0\)) with respect to the nature of the 1-forms that we are considering.

We prove that the invariance of the metric with respect to \(\circ\) is responsible for the exactness of the 1-forms, namely for the existence of a true Hamiltonian functional.

However, in the general case of an \(F\)-manifold with compatible flat connection \((M, \nabla, \circ)\), where such a metric does not exist (see for instance [2] where we provided plenty of examples in which any metric compatible with the connection \(\nabla\) is not invariant with respect to the product \(\circ\)), it turns out that the 1-forms on which the Poisson tensor acts are not exact.

In this context, we provide also an alternative proof of the commutativity of the flows in the principal hierarchy. Indeed, in this setup, the commutativity of the flows is equivalent to the Poisson involutivity of the corresponding 1-forms with respect to the Poisson bracket introduced above.

This paper is organized as follows. In section 2, we introduce 1-forms, evolutionary vector fields and operations on them, essentially following [6], to fix notation and for the sake of being self-contained. In section 3, we introduce the Poisson bracket on 1-forms and we derive its expression in flat and general coordinates. In section 4, we prove the main properties of this bracket.
In section 5, we recall the concept of $F$-manifold with compatible flat connection and we show how the Poisson bracket just introduced fits in the description of the Hamiltonian structure of the corresponding principal hierarchy. In section 6, we continue the exploration of the Hamiltonian structure of the principal hierarchy, in particular focusing on the role played by the invariance of the metric $g$ with respect to product $\circ$.

In section 7, we work out an example in dimension 3, and in section 8, we present some conclusions.

2. Differential forms on the formal loop space

In this section, we are going to recall the construction of differential forms on the formal loop space $\mathcal{L}(\mathbb{R}^n) := C^\infty(S^1, \mathbb{R}^n)$, together with some important operations, following essentially [6]. From now on, every time two indices are repeated in a formula; they are summed over a suitable range, which is usually clear from the context. This also applies to the indices that denote derivatives; for instance, in $\alpha_{ij}^{(t)} \delta u_{ij}^{(t)}$, sum over $i = 1, \ldots, n$ and sum over $t = 0, 1, \ldots$ is intended.

Let $U \subset \mathbb{R}^n$ be an open subset of $\mathbb{R}^n$ with coordinates $u^1, \ldots, u^n$. Denote by $A = A(U)$ the space of polynomials in the independent variables $u^i, i = 1, \ldots, n, s = 1, 2, \ldots$, where $u^i$ has to be thought of as the $s$th derivative of $u$ with respect to the angular coordinate $x$. In the following, where $u^{(0)}$ will appear, it will be identified with $u$. An element $f \in A$ can be described as

$$f(x; u, u_x, u_{xx}, \ldots) := \sum_{m \geq 0} f_{m_{1,2,\ldots, n}}(x; u) u_{1(x)}^{(m_1)} \cdots u_{n(x)}^{(m_n)},$$

(2.1)

where the coefficients $f_{m_{1,2,\ldots, n}}(x; u)$ are smooth functions on $S^1 \times M$. These elements are usually called differential polynomials; observe that they are not required to be polynomial with respect to $(u^1, \ldots, u^n)$ in general. In our case, the coefficients $f_{m_{1,2,\ldots, n}}$ will not depend explicitly on the $x$-coordinate. We recall the definition of the total derivative with respect to $x$ acting on $f \in A$:

$$\partial_x f = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial u^i} u^i_x + \cdots + \frac{\partial f}{\partial u^{(s)}_{(i)}} u^{(s+1)}_{(i)} + \cdots,$$

(2.2)

and analogously we denote with $f_{(s)}$ the expression $\partial_x^s f, s = 1, 2, \ldots$.

We start by defining the 0-forms on $\mathcal{L}(\mathbb{R}^n)$, namely functions, that will be represented in terms of functionals.

Consider the space $A_{0,0} := A/\mathbb{R}$, obtained by identifying two differential polynomials that differ by a constant and define $A_{0,1} := A_{0,0} dx$. The differential operator $d : A_{0,0} \to A_{0,1}, f \mapsto df := (\partial_x f) dx$ allows us to consider the quotient vector space

$$A_0 := A_{0,1}/\text{Im}(d),$$

(2.3)

whose elements can be written as integrals over the circle $S^1$

$$I_f[u] := \int_{S^1} f(x; u, u_x, u_{xx}, \ldots) dx,$$

(2.4)

since if $f(x; u, u_x, u_{xx}, \ldots) dx = (\partial_x g) dx$ for some $g \in A$, then the corresponding $I_f$ is the zero functional. Also let us remark that due to the definition of $A_{0,0}$, functionals $I_f$ whose density $f$ is a constant are identified with the zero functional. Since we have mod out constants in the definition of $A_{0,0}$, we obtain a short exact sequence

$$0 \to A_{0,0} \xrightarrow{d} A_{0,1} \xrightarrow{\pi} A_0 \to 0,$$

where $\pi$ is the projection map.
In order to define differential forms, we proceed as follows. First, we introduce the Grassmann algebra \( A_{\bullet \bullet} \) over \( \mathbb{R} \) generated by \( \delta u^i_{(s)} \), where \( i = 1, \ldots, n, s = 0, 1, \ldots \), (\( \delta u^i_{(0)} \) is identified with \( d u^i \)) and by \( dx \). This turns out to be bi-graded \( A_{\bullet \bullet} = \bigoplus_{k \geq 0, l = 0, 1} A_{k, l} \), where a monomial element \( a_{k, 0} \in A_{k, 0} \) is of the form \( a \delta u^i_{(s)} \wedge \ldots \wedge \delta u^j_{(t)} \), \( a \in \mathbb{R} \), while a monomial element \( a_{k, 1} \in A_{k, 1} \) is written as \( a \delta u^i_{(s)} \wedge \ldots \wedge \delta u^j_{(t)} \wedge dx \), \( a \in \mathbb{R} \). An arbitrary element \( a_{k, 0} \in A_{k, 0} \) will be a finite sum of the form \( a_{k, 0} = \sum_{l} \delta u^i_{(s)} \wedge \ldots \wedge \delta u^j_{(t)} \), where the sum over repeated indices is assumed and where the coefficients \( a_{l} \) are constants that are skew-symmetric for all exchanges of the pairs \((i_j, s_j)\) with \((i_q, s_q)\).

For \( k \geq 1, l = 0, 1 \) we define \( A_{k, l} := A \otimes_{\mathbb{R}} A_{k, l} \), while \( A_{0, l} \) have been defined above. Note that in the definition of \( A_{k, l} \), for \( k \geq 1 \), the ring \( A \) appears directly, without having to take the quotient with respect to constants.

Therefore, also \( A_{\bullet \bullet} \) is bi-graded, \( A_{\bullet \bullet} = \bigoplus_{k \geq 0, l = 0, 1} A_{k, l} \). Concretely, a homogeneous element \( \omega \) of degree \( k \geq 1 \), \( \omega \in A_{k, 0} \), can be written as a finite sum:

\[
\omega = \frac{1}{k!} \omega_{i_1, \ldots, i_k} \delta u^i_{(s_1)} \wedge \ldots \wedge \delta u^j_{(s_k)},
\]

where the sum over all repeated indices is intended and where the coefficients \( \omega_{i_1, \ldots, i_k} \) in \( A \) are skew-symmetric for all exchanges of pairs \((i_j, s_j)\) with \((i_q, s_q)\). Analogously, a homogeneous element \( \omega \) of bi-degree \((k, 1)\) in \( A_{k, 1} \) can be written as a finite sum

\[
\omega = \frac{1}{k!} \omega_{i_1, \ldots, i_k} \delta u^i_{(s_1)} \wedge \ldots \wedge \delta u^j_{(s_k)} \wedge dx,
\]

with the same conventions over repeated indices and skew-symmetry. So far we have not introduced the exterior differential \( d \) for the Grassmann algebra \( A_{\bullet \bullet} \). We are going to construct it as a sum of two differentials. First we extend the differential \( d : A_{0, 0} \to A_{0, 1} \) to a differential \( d : A_{k, 0} \to A_{k, 1} \) defined as \( d(\omega) := dx \wedge (\partial_i \omega) \), where the derivation \( \partial_i \) satisfies the Leibniz rule

\[
\partial_i (\omega_1 \wedge \omega_2) = \partial_i \omega_1 \wedge \omega_2 + \omega_1 \wedge \partial_i \omega_2
\]

and the action of \( \partial_i \) is given by (2.2) on the coefficients of the differential form and by \( \partial_i \delta u^i_{(s)} = \delta u^i_{(s+1)} \) on the \( \delta \)-differentials. It is clear that \( d^2 = 0 \) since \( dx \wedge dx = 0 \). Similar to what happens for \( A_{0, 0} \) and \( A_{0, 1} \), for any \( k \geq 1 \), we have short exact sequences

\[
0 \to A_{k, 0} \xrightarrow{d} A_{k, 1} \xrightarrow{\pi} \Lambda_k \to 0,
\]

where \( \pi \) is the projection onto the quotient \( \Lambda_k := A_{k, 1} / \text{Im}(d) \). We call the elements of \( \Lambda_k \) (local) \( k \)-forms on the loop space \( \mathcal{L}(\mathbb{R}^n) \). We represent \( k \)-forms as integrals

\[
\int dx \wedge \omega, \quad \omega \in A_{k, 0},
\]

and if \( dx \wedge \omega = d(\alpha) \), for some \( \alpha \in A_{k, 0} \), then the corresponding \( k \)-form in \( \Lambda_k \) is identically zero. As an example, note that since any 1-form can be written as

\[
\int dx \wedge \omega_{i_1} \delta u^i_{(s)} = \int dx \wedge \omega_{i_1} \delta u^i_{(s)}(\delta u^i),
\]

using integration by parts we obtain

\[
\int dx \wedge (-1)^r \partial_i \omega_{i_1} \delta u^i = \int dx \wedge \alpha_i \delta u^i,
\]

where \( \alpha_i := (-1)^r \partial_i \omega_{i_1} \).

We define now another differential \( \delta : A_{k, 0} \to A_{k+1, 0} \). This will be extended to a differential \( \delta : \Lambda_k \to \Lambda_{k+1} \) that will act as the exterior differential on forms on the loop space,
so it will increase the degree of a form. This is the operator $\delta : A_{k, 0} \to A_{k+1, 0}$ which is defined on monomials and then extended by linearity as follows:

$$\delta(f \delta u_{(1)}^i \wedge \ldots \wedge \delta u_{(s)}^k) = \frac{\partial f}{\partial u_{(j)}^i} \delta u_{(1)}^i \wedge \delta u_{(j)}^i \wedge \ldots \wedge \delta u_{(s)}^k,$$

where again the sum over repeated indices is intended. Note that, as in the usual case, $\delta^2 = 0$ identically, as can be checked immediately on monomials. The map $\delta$ on $A_{k, 1}$ is defined by the same formula with the requirement that $\delta dx = 0$.

**Lemma 2.1.** The operators $d$ and $\delta$ satisfy $\delta \circ d = - d \circ \delta$.

**Proof.** It is easy to check that

$$\frac{\partial}{\partial u_{(j)}^i} \circ \partial_s = \partial_s \circ \frac{\partial}{\partial u_{(j)}^i}, \quad s \geq 1; \quad \frac{\partial}{\partial u^i} \circ \partial_s = \partial_s \circ \frac{\partial}{\partial u^i}. \quad (2.8)$$

If we consider a monomial

$$\omega := f \delta u_{(1)}^i \wedge \ldots \wedge \delta u_{(s)}^k \in A_{k, 0},$$

then

$$d\omega = (\partial_s f) dx \wedge \delta u_{(1)}^i \wedge \ldots \wedge \delta u_{(s)}^k + f dx \wedge \delta u_{(1)}^i \wedge \ldots \wedge \delta u_{(s)}^k$$

$$+ \ldots + f dx \wedge \delta u_{(1)}^i \wedge \ldots \wedge \delta u_{(s)}^k,$$

so that

$$\delta d\omega = \frac{\partial}{\partial u_{(j)}^i} (\partial_s f) dx \wedge \delta u_{(1)}^i \wedge \ldots \wedge \delta u_{(s)}^k + \frac{\partial f}{\partial u_{(j)}^i} \delta u_{(1)}^i \wedge dx \wedge \delta u_{(s)}^k \wedge \ldots \wedge \delta u_{(s')}^k$$

$$+ \ldots + \frac{\partial f}{\partial u_{(j)}^i} \delta u_{(1)}^i \wedge dx \wedge \delta u_{(s)}^k \wedge \ldots \wedge \delta u_{(s')}^k.$$

On the other hand

$$d\delta \omega = \partial_s \left( \frac{\partial f}{\partial u_{(j)}^i} \right) dx \wedge \delta u_{(1)}^i \wedge \delta u_{(s)}^k \wedge \ldots \wedge \delta u_{(s')}^k$$

$$+ \frac{\partial f}{\partial u_{(j)}^i} dx \wedge \delta u_{(1)}^i \wedge \delta u_{(s)}^k \wedge \ldots \wedge \delta u_{(s')}^k$$

$$+ \ldots + \frac{\partial f}{\partial u_{(j)}^i} dx \wedge \delta u_{(1)}^i \wedge \delta u_{(s)}^k \wedge \ldots \wedge \delta u_{(s')}^k.$$

Using relations (2.8) and $dx \wedge \delta u_{(j)}^i = - \delta u_{(j)}^i \wedge dx$, we obtain the result. \qed

Lemma 2.1 allows one to define $\delta : \Lambda_k \to \Lambda_{k+1}$. Indeed, let $\alpha \in \Lambda_k$ and let $\tilde{\alpha}$ and $\alpha'$ be two of its representatives in $A_{k, 1}$. Since $\tilde{\alpha} - \alpha' = df$, for some $f \in A_{k, 0}$, we have $\delta(\tilde{\alpha}) - \delta(\alpha') = \delta \circ d(\beta) = - d \circ \delta(\beta)$, which shows that $\delta(\tilde{\alpha})$, $\delta(\alpha')$ in $A_{k+1, 1}$ define the same form in $\Lambda_{k+1}$, call it $\delta(\alpha)$.

The operator $\delta$ satisfies $\delta \circ \delta = 0$ as it is immediate to see. The exterior differential of the differential Grassmann algebra $A_{k, *}$ is the differential operator $D := \delta + d$, which indeed satisfies $D \circ D = 0$. Moreover, each of the short exact sequences

$$0 \to A_{k, 0} \xrightarrow{d} A_{k, 1} \xrightarrow{\pi} \Lambda_k \to 0$$

is included as a row in a bicomplex, called variational bicomplex, in which both arrows and columns are exact. The rows are related via the differential $\delta$. For more information, see [3] and [6].
On $\Lambda_0$ the differential $\delta$ acts as follows:

$$\delta \int f \ dx = \int \ dx \wedge \frac{\partial f}{\partial u'_i},$$

(sum over $t$ and $j$) and due to the fact that $\delta u'_i = \partial_i \delta u$, integrating by parts one obtains

$$\delta \int f \ dx = \int \ dx \wedge \left( (-1)^j \partial_i^j \frac{\partial f}{\partial u'_i} \right) \delta u', \tag{2.9}$$

(sum over $t$ and $j$). We will use the notation

$$\frac{\delta \tilde{f}}{\delta u'(x)} := (-1)^j \partial_i^j \frac{\partial f}{\partial u'_i}, \tag{2.10}$$

(sum over $t$) for the components of the 1-form, $\delta \tilde{f} = \delta \int f \ dx$. This is the expression of variational derivative that appears in formula (1.2) in section 1. Sometimes, with abuse of notation, we will denote with $\frac{\delta f}{\delta u(x)}$ the expression $(-1)^j \partial_i^j \frac{\partial f}{\partial u'_i}$ for a density $f$.

Let us also recall the following well-known result: a necessary and sufficient condition for

$$\frac{\delta \tilde{f}}{\delta u'(x)} = 0, \quad i = 1, \ldots, n,$$

is the existence of a differential polynomial $g = g(x; u; u_x; \ldots)$, such that $f = \partial_i g$.

### 2.1. The space $\Lambda^1$ of vector fields on the formal loop space

We denote with $\Lambda^1$ the space of vector fields on the formal loop space. A vector field $\xi$ on $\mathcal{L}(\mathbb{R}^n)$ is a formal infinite sum of the form

$$\xi = \xi^0 \frac{\partial}{\partial x} + \xi^{i,k} \frac{\partial}{\partial u'_i(x)}, \quad \xi^0, \xi^{i,k} \in \mathcal{A}, \tag{2.11}$$

(sum over $i = 1, \ldots, n, k \geq 0$) where $\frac{\partial}{\partial u'_i(x)}$ is $\frac{\partial}{\partial \sigma}$. The derivative of a functional $\tilde{f} = \int f(x; u; u_x; \ldots) \ dx \in \Lambda_0$ along $\xi$ is given by

$$\xi \tilde{f} := \int \left( \xi^0 \frac{\partial f}{\partial x} + \xi^{i,k} \frac{\partial f}{\partial u'_i(x)} \right) \ dx, \tag{2.12}$$

(sum over $i$ and $k$), which is again an element in $\Lambda_0$. The Lie bracket of two vector fields $\xi$ and $\eta$ with components as given in (2.11) is a vector field defined by

$$[\xi, \eta] = \left( \xi^0 \eta^0 - \eta^0 \xi^0 + \xi^{i,s} \frac{\partial \eta^0}{\partial u'_i(x)} - \eta^{i,s} \frac{\partial \xi^0}{\partial u'_i(x)} \right) \frac{\partial}{\partial x}$$

$$+ \left( \xi^0 \frac{\partial \eta^{i,s}}{\partial x} - \eta^0 \frac{\partial \xi^{i,s}}{\partial x} + \xi^{i,t} \frac{\partial \eta^{i,s}}{\partial u'_i(x)} - \eta^{i,t} \frac{\partial \xi^{i,s}}{\partial u'_i(x)} \right) \frac{\partial}{\partial u'_i(x)} \tag{2.13}$$

sum over $(i, j = 1, \ldots, n)$ and $s, t \geq 0$.

In the following, we are going to deal with restricted classes of vector fields. First, we have the following.

**Definition 2.2** ([17], p 291). A vector field

$$\xi = \xi^0 \frac{\partial}{\partial x} + \xi^{i,k} \frac{\partial}{\partial u'_i(x)}$$

is called evolutionary if $\xi^0 = 0$, $\xi^{i,k} = \partial^k_i (\xi^i)$ for $\xi^i \in \mathcal{A}$ and the differential functions $\xi^i$ do not depend explicitly on $x$. 

7
An evolutionary vector field $\xi$ is parametrized by $n$ functions $\xi^1, \ldots, \xi^n$ and can therefore be written as

$$\xi = \frac{\partial^k}{\partial u_{(i)}} \frac{\partial}{\partial u_{(i)}}.$$

In the case of an evolutionary vector field, the corresponding system of evolutionary PDEs is described via the system

$$u_i' = \xi^i(u, u_t, u_x, \ldots), \quad i = 1, \ldots, n, \quad \xi^i \in A.$$

(2.14)

### 2.2. Operations on forms and vector fields

Let $\xi$ be an evolutionary vector field and $\omega \in A_{k,0}$ as in (2.5), $k \geq 1$.

**Definition 2.3.** The contraction of $\omega \in A_{k,0}$ with $\xi$ evolutionary vector field is given by the natural extension of the usual formula. Assuming the coefficients of the $k$-form

$$\omega = \frac{1}{k!} \omega_{i_1 \cdots i_k} \delta u_{(i_1)}^{i_1} \wedge \cdots \wedge \delta u_{(i_k)}^{i_k},$$

antisymmetric w.r.t. permutations of pairs $i_p, s_p \leftrightarrow i_q, s_q$, one obtains the following expression:

$$i_\xi \omega = \frac{1}{(k-1)!} \delta^k \omega_{i_1 \cdots i_k} \delta u_{(i_1)}^{i_1} \wedge \cdots \wedge \delta u_{(i_k-1)\rightarrow 1}^{i_k-1}.$$

An analogous formula holds for $\omega \in A_{k,1}$.

It turns out that for $\xi$ evolutionary $i_\xi \circ d + d \circ i_\xi = 0$ identically, so that contraction with respect to $\xi$ is a well-defined operation $i_\xi : \Lambda_k \rightarrow \Lambda_{k-1}$, see [6].

Consider the functional $f[a] := \int_S f \, dx \in A_0$. Then, the 1 form $\omega := \delta \bar{f}$ is given by

$$\omega = \int dx \wedge \frac{\partial f}{\partial u_{(i)}} \delta u_{(i)} = \int dx \wedge (-1)^{i'} \frac{\partial f}{\partial u_{(i)'}} \delta u_{(i)'} = \int dx \wedge \frac{\delta \bar{f}}{\delta u'} \delta u' \in \Lambda_1.$$

If $\xi$ is an evolutionary vector field, with components $\xi^1, \ldots, \xi^n$, then

$$i_\xi \omega = \int dx \frac{\partial f}{\partial u_{(i)\prime}} \omega_{i_1 \cdots i_k} = \int dx \xi^i (-1)^{i'} \frac{\partial f}{\partial u_{(i)'}} \omega_{i_1 \cdots i_k} = \int dx \omega_{i_1 \cdots i_k} \frac{\delta f}{\delta u'(x) \in \Lambda_0}.$$

Note that $i_\xi \omega$ coincides (as it should be) with the Lie derivative of the functional $\bar{f}$ with respect to the vector field $\hat{\epsilon}$ as given in the general formula (2.12) (use the fact that $\xi$ is evolutionary and integrate by parts).

Let us introduce also the Poisson structure as a Poisson bracket on functionals. Following [6], we represent a Poisson structure in the form

$$\{u^i(x), u^j(y)\} = \sum_s A^{ij}_s (u(x); u_t(x), u_x(x), \ldots) \delta^{(s)}(x - y),$$

(2.15)

where $A^{ij}_s$ satisfies suitable conditions. In particular, in this paper, we will focus on Poisson’s structures of the form $A^{ij} = g^{ij}(u^1, \ldots, u^n)$ and $A^{ij} = \Gamma^{ij}_k u^k, A^{ij}_s = 0$, $s \geq 2$, and these define a Poisson structure iff $g^{ij}$ is a non-degenerate contravariant flat metric (not necessarily positive definite) and $\Gamma^{ij}_k = -g^{kj} \Gamma^{ij}_k$, where $\Gamma^{ij}_k$ are the Christoffel symbols of the Levi-Civita connection associated with the inverse metric $g_{ij}$ (see [4]). From now on, this will be assumed.
In the case we are dealing with, the Poisson bracket of two local functionals \( \tilde{f} = \int f(x; u; u_1, \ldots) \, dx \) and \( \tilde{h} = \int h(x; u; u_1, \ldots) \, dx \) is then given by

\[
\{ \tilde{f}, \tilde{h} \} = \int \int dx dy \frac{\delta \tilde{f}}{\delta u'(x)} (f'(x), u'(y)) - \frac{\delta \tilde{h}}{\delta u'(y)}
\]

\[
= \int dx \frac{\delta \tilde{f}}{\delta u'(x)} \left( g^{ij}(u^1, \ldots, u^n) \partial_x + \Gamma^{ij}_{kl} u_k^l \right) \left( \frac{\delta \tilde{h}}{\delta u'(x)} \right) \in \Lambda_0.
\]

and thus, it is again a local functional. From the dynamical point of view, the important property of the local Poisson brackets is that the Hamiltonian systems

\[
u_i = [u'(x), \tilde{H}] = (g^{ij}(u^1, \ldots, u^n) \partial_x + \Gamma^{ij}_{kl} u_k^l) \frac{\delta \tilde{H}}{\delta u'(x)}
\]

with Hamiltonians like

\[
\tilde{H} = \int H(u; u_1, \ldots) \, dx
\]

are evolutionary PDEs (2.14).

It will be important for the next section to interpret (2.16) in a slightly different way. Indeed we can view (2.16) as the pairing between the 1-form \( \delta \tilde{f} \) and the vector field \( P(\delta \tilde{g}) \) obtained via the action of the Poisson structure \( P \) on the 1-form \( \delta \tilde{h} \). So we set

\[
\int dx \frac{\delta \tilde{f}}{\delta u'(x)} \left( g^{ij}(u^1, \ldots, u^n) \partial_x + \Gamma^{ij}_{kl} u_k^l \right) \left( \frac{\delta \tilde{h}}{\delta u'(x)} \right) = \langle \delta \tilde{f}, P\delta \tilde{g} \rangle,
\]

so that the evolutionary vector field \( P\delta \tilde{g} \) is given by

\[
P\delta \tilde{g} = \partial_x \left[ (g^{ij}(\partial_x + \Gamma^{ij}_{kl} u_k^l)(-1)^j \partial_x \left( \frac{\delta \tilde{h}}{\delta u'(x)} \right) \right] \frac{\partial}{\partial u'(x)}.
\]

When \( P\delta \tilde{g} \) is paired with \( \delta \tilde{f} = \int dx \wedge \frac{\delta f}{\delta u'(x)} \delta u', \) one obtains exactly formula (2.18).

The goal of the next section is to extend the formalism to include a Poisson brackets on 1-form and to show that a certain evolutionary equation can be still written in the form (2.17), although there is no Hamiltonian functional available.

### 3. Poisson brackets on 1-forms

First we extend formulae (2.18) and (2.19) to deal with 1-forms that are not closed. Consider a 1-form \( \alpha \) given by

\[
\alpha = \int_{S^1} dx \wedge \alpha_i^{(i)} \delta u_i^{(i)}.
\]

Using integration by parts, we can always reduce it to standard form or reduced form, in the sense that only the differential \( \delta u' \) are involved. Indeed, one obtains

\[
\alpha = \int_{S^1} dx \wedge \alpha_i \delta u_i,
\]

where

\[
\alpha = (-1)^i \partial_x \frac{\partial}{\partial u_i^{(i)}} \alpha_i^{(i)}.
\]

We will often write a 1-form \( \alpha \) also skipping the integral sign, directly as \( \alpha = \alpha_i \delta u_i \), but thinking that operations like integration by parts do not change \( \alpha \). Moreover, when performing computation with the bracket we are going to define, we will always consider 1-forms in
the standard form; this requirement is motivated especially by the way in which the Poisson structure acts on exact 1-forms, namely on differentials of functionals.

The evolutionary vector field $P\beta$ is given by

$$P\beta = \partial_t \left[ (g^{ij} \partial_s + \Gamma^i_{kl} u_k) \beta_j \right],$$

and it is parametrized by the $n$ functions

$$(P\beta)^i := (g^{ij} \partial_s + \Gamma^i_{kl} u_k) \beta_j,$$

which can be viewed as its components. Then, the extension of (2.18) to general 1-forms is given by

$$\langle \alpha, P\beta \rangle = i_{P\beta} \alpha = \int \alpha_i \left( g^{ij} \partial_s + \Gamma^i_{kl} u_k \right) \beta_j \, dx \in \Lambda_0.$$

Before defining the Poisson bracket, we need to introduce the Lie derivative of a 1-form $\alpha$ along a vector field $X$. Suppose $\alpha = \alpha_i^{(s)} \delta u_i^{(s)}$, and $X = X^{k,s} \partial_{X^{k,s}}$. Any element of the loop space $u \in C^\infty(S^1, M)$ can be seen as a section $\sigma_u : S^1 \to S^1 \times M$ of the trivial bundle $\pi : S^1 \times M \to S^1$. We can think of $u_i^{(t)}$, for $i = 1, \ldots, n$, $t = 0, 1, \ldots$, as coordinates on the infinite jet, describing the infinite prolongation of the section $\sigma_u$. In this framework, we can thus define the Lie derivative via the usual formula in coordinates as

$$\text{Lie}_X \alpha = (\text{Lie}_X \alpha_i^{(s)}) \delta u_i^{(s)} = \left( X^{k,s} \partial_{X^{k,s}} \alpha_i^{(s)} + \alpha_k^{(s)} \partial_{u_k^{(s)}} X^{k,s} \right) \delta u_i^{(s)}.$$  

In formula (3.4), we should have written the expression for the Lie derivative more correctly as

$$\text{Lie}_X \alpha = \int dx \wedge (\text{Lie}_X \alpha_i^{(s)}) \delta u_i^{(s)}.$$

In the following, whenever the integral sign is omitted, it is intended that the components of a form, including $\delta u_i^{(s)}$, are defined up to a total derivative.

We specialize formula (3.4) to the case in which $X = P\beta$, assuming that 1-form $\beta$ is written in the standard form, $P$ is written in flat coordinates and considering also $\alpha$ in the standard form. In this case, we have the following expression.

**Definition 3.1.** The Lie derivative of $\alpha$ with respect to $P\beta$ written in flat coordinates, where $\alpha$ and $\beta$ are in the standard form, is given by

$$\text{Lie}_{P\beta} \alpha = \partial_t \left( (g^{ij} \partial_s + \Gamma^i_{kl} u_k) \alpha_i \delta u_i^{(s)} + \alpha_k \partial_{u_k^{(s)}} (n^{kl} \partial_s + \Gamma^i_{kl} u_k) \delta u_i^{(s)} \right)$$

$$= \partial_t \left( (g^{ij} \partial_s + \Gamma^i_{kl} u_k) \alpha_i \right) \delta u_i^{(s)} + (\partial_t g^{ij} - \partial_t \Gamma^i_{kl} u_k) \alpha_i \delta u_i^{(s)} + (\Gamma^i_{kl} \partial_s + \Gamma^i_{kl} \partial_s) \alpha_i \delta u_i^{(s)}.$$

The Lie derivative thus defined satisfies Cartan’s formula.

**Proposition 3.2.** If the Lie derivative is defined as in (3.5), then

$$\text{Lie}_{P\beta} \alpha = i_{P\beta} \delta \alpha + \delta i_{P\beta} \alpha = i_{P\beta} \delta \alpha + \delta \langle \alpha, P\beta \rangle,$$

where $\langle \alpha, P\beta \rangle$ denotes the pairing of the 1-form $\alpha$ with the vector field $P\beta$. 

Proposition 3.4. Let \( \alpha = \int dx \wedge \alpha \delta u^l \) and \( \beta = \int dx \wedge \beta \delta u^l \) be two reduced 1-forms. Then, in flat coordinates, \( [\alpha, \beta] \) is a reduced 1-form \( [\alpha, \beta] = \int dx \wedge [\alpha, \beta]_j \delta u^l \), where

\[
[\alpha, \beta]_j = \eta^{kl} \left( \partial_{[k}^{x+l} \beta_{l]} \frac{\partial \alpha_j}{\partial u^l(s)} - (\partial_{[k}^{x+l} \alpha_{l]} \frac{\partial \beta_j}{\partial u^l(s)} \right).
\]
Proof. In the proof, we remove the integral sign, just to simplify notation, with the understanding that total derivatives with respect to $x$ can be safely eliminated. Using formula (3.9) and the expressions for the Lie derivatives, one obtains

\[ \{\alpha, \beta\} = \partial_x^i (\eta^j \partial_i \beta_j - \eta^j \partial_i \alpha_j) - \partial_x^i (\eta^j \partial_i \alpha_j) \delta u_i' + \alpha_i \partial_{\partial u_i'}(\eta^j \partial_i \beta_j) \delta u_i' \\
+ \partial_x^i (\eta^j \partial_i \alpha_j) \delta u_i' - \beta_i \partial_{\partial u_i'}(\eta^j \partial_i \alpha_j) \delta u_i' \\
+ \frac{\partial \beta_i}{\partial u_i'} \eta^m \partial_m \alpha_m \delta u_i' + \beta_i \eta^m \frac{\partial \alpha_i}{\partial u_i'} \partial_m \alpha_m \delta u_i'. \]

Therefore, we have

\[ \{\alpha, \beta\} = \eta^j \left[ (\alpha_{i+1} \beta_i) \frac{\partial \alpha_i}{\partial u_i'} - (\alpha_i \beta_{i+1}) \frac{\partial \beta_i}{\partial u_i'} \right] \delta u_i' + \text{residual terms}. \]

We show that the residual terms constitutes a total derivative with respect to $x$ and this proves (3.10). Indeed the non-trivial residual terms are given by

\[ \alpha_i \frac{\partial}{\partial u_i'}(\eta^j \partial_i \beta_j) \delta u_i' + \eta^j \frac{\partial \beta_i}{\partial u_i'} \eta^m \partial_m \alpha_m \delta u_i', \]

since the other two cancel out after relabeling indices. Using \( \frac{\partial}{\partial u_i'} \circ \partial_x = \partial_x \circ \frac{\partial}{\partial u_{i+1}} + \frac{\partial}{\partial u_{i+1}} \) and renaming indices, these can be rewritten as

\[ \alpha_i \eta^j \partial_x \left( \frac{\partial \beta_i}{\partial u_i'} \right) \delta u_i' + \alpha_i \eta^j \frac{\partial \beta_i}{\partial u_i'} \delta u_i' + (\partial_x \alpha_i) \eta^j \frac{\partial \beta_i}{\partial u_i'} \delta u_i' = \partial_x \left( \alpha_i \eta^j \frac{\partial \beta_i}{\partial u_i'} \delta u_i' \right). \]

The expression of the bracket in general coordinates is given in the following.

Proposition 3.5. Let $\alpha = \int dx \wedge \alpha_i \delta u^i$ and $\beta = \int dx \wedge \beta_i \delta u^i$ be two reduced 1-forms. Then, in arbitrary coordinates, $\{\alpha, \beta\}$ is a reduced 1-form $\{\alpha, \beta\} = \int dx \wedge \{\alpha, \beta\} \delta u^i$, where

\[ \{\alpha, \beta\}_i = \alpha_i \eta^j \partial_x \left( \frac{\partial \beta_i}{\partial u_i'} \right) \delta u_i' + \alpha_i \eta^j \frac{\partial \beta_i}{\partial u_i'} \delta u_i' + (\partial_x \alpha_i) \eta^j \frac{\partial \beta_i}{\partial u_i'} \delta u_i' = \partial_x \left( \alpha_i \eta^j \frac{\partial \beta_i}{\partial u_i'} \delta u_i' \right). \]

(3.11)

where $g^i$ is the flat (contravariant) metric in the chosen coordinates and $\Gamma^i_m$ are the corresponding Christoffel symbols. Moreover, if $\alpha_i$ and $\beta_i$ are the functions depending only on the coordinates $u^i, \ldots, u^n$, but not on their derivatives, then

\[ \{\alpha, \beta\}_i = (\nabla_m \beta_i)^m - \nabla_m \alpha_i, \]

(3.12)

where $\nabla_m$ indicates the covariant derivative with respect to the vector field $\frac{\partial}{\partial u_i'}$.

Proof. To derive the general formula (3.11), we use again (3.9), where $P$ is expressed this time in general coordinates. Expanding all the terms in (3.9), and skipping the integral sign to simplify notation, we find

\[ \{\alpha, \beta\} = \partial_x^i (g^i \partial_i \beta_i + \Gamma^i_m u^m \partial_i \beta_i) \alpha_i \partial_{\partial u_i'} + \alpha_i \partial_{\partial u_i'} (g^i \partial_i \beta_i + \Gamma^i_m u^m \partial_i \beta_i) \delta u_i' \\
+ \frac{\partial \alpha_i}{\partial u_i'} \partial_{\partial u_i'} (g^i \partial_i \beta_i + \Gamma^i_m u^m \partial_i \beta_i) \delta u_i' - \beta_i \partial_{\partial u_i'} (g^i \partial_i \alpha_i + \Gamma^i_m u^m \partial_i \alpha_i) \delta u_i' \\
+ \frac{\partial \beta_i}{\partial u_i'} \partial_{\partial u_i'} (g^i \partial_i \alpha_i + \Gamma^i_m u^m \partial_i \alpha_i) \delta u_i' + \beta_i \partial_{\partial u_i'} (g^i \partial_i \alpha_i + \Gamma^i_m u^m \partial_i \alpha_i) \delta u_i'. \]
From this expression, we immediately obtain

\[
\{\alpha, \beta\} = \left\{ \partial_\alpha (g^{kl} \partial_\beta + \Gamma^{kl}_m u^m_\beta) \frac{\partial u^t}{\partial u^t} - \partial_\alpha (g^{kl} \partial_\beta + \Gamma^{kl}_m u^m_\beta) \frac{\partial u^t}{\partial u^t} \right\} \delta u^t
\]

\[
+ \alpha_k \left( \frac{\partial g^{kl}}{\partial u^t} \frac{\partial \beta_1}{\partial u^t} \right) \delta u^t + \alpha_k \left( \frac{\partial g^{kl}}{\partial u^t} \frac{\partial \beta_1}{\partial u^t} \right) \delta u^t + \alpha_k \left( \frac{\partial \Gamma^{kl}_m}{\partial u^t} u^m_\beta \right) \delta u^t + \alpha_k \left( \frac{\partial \Gamma^{kl}_m}{\partial u^t} u^m_\beta \right) \delta u^t
\]

\[
+ \beta_k \left( \frac{\partial g^{kl}}{\partial u^t} \frac{\partial \beta_1}{\partial u^t} \right) \delta u^t + \beta_k \left( \frac{\partial g^{kl}}{\partial u^t} \frac{\partial \beta_1}{\partial u^t} \right) \delta u^t + \beta_k \left( \frac{\partial \Gamma^{kl}_m}{\partial u^t} u^m_\beta \right) \delta u^t + \beta_k \left( \frac{\partial \Gamma^{kl}_m}{\partial u^t} u^m_\beta \right) \delta u^t.
\]

from which we recognize the first two terms in (3.11) in the first line, while the third line vanishes, relabeling indices. It remains to prove that the two terms in the second line in the previous expression are equal (up to total derivatives with respect to \( t \)) to the last line in (3.11).

In the expression

\[
\alpha_k \frac{\partial}{\partial u^t} (g^{kl} \partial_\beta + \Gamma^{kl}_m u^m_\beta) \delta u^t,
\]

we split the sum over \( t \) into three terms corresponding to \( t = 0, t = 1 \) and \( t \geq 2 \), and then, we use the identity \( \frac{\partial}{\partial u^t} \circ \delta_k = \delta_k \circ \frac{\partial}{\partial u^t} + \frac{\partial}{\partial u^t} \circ \delta_k \) for \( t = 1 \) and \( \frac{\partial}{\partial u^t} \circ \delta_k = \delta_k \circ \frac{\partial}{\partial u^t} \). In this way, we obtain

\[
\alpha_k \frac{\partial}{\partial u^t} (g^{kl} \partial_\beta + \Gamma^{kl}_m u^m_\beta) \delta u^t = \left\{ \alpha_k \left( \frac{\partial g^{kl}}{\partial u^t} \frac{\partial \beta_1}{\partial u^t} \right) \delta u^t + \alpha_k \left( \frac{\partial \Gamma^{kl}_m}{\partial u^t} u^m_\beta \right) \delta u^t \right\} (1)
\]

\[
+ \alpha_k \left( \frac{\partial g^{kl}}{\partial u^t} \frac{\partial \beta_1}{\partial u^t} + \frac{\partial \beta_1}{\partial u^t} \right) \delta u^t + \alpha_k \left( \frac{\partial \Gamma^{kl}_m}{\partial u^t} u^m_\beta \right) \delta u^t \right\} (2)
\]

\[
+ \alpha_k \left( \frac{\partial g^{kl}}{\partial u^t} \frac{\partial \beta_1}{\partial u^t} \right) \delta u^t + \alpha_k \left( \frac{\partial \Gamma^{kl}_m}{\partial u^t} u^m_\beta \right) \delta u^t \right\} (3) \text{ for } t \geq 2
\]

\[
+ \frac{\partial \beta_1}{\partial u^t} (g^{lm} \partial_\alpha u^m) \delta u^t + \frac{\partial \beta_1}{\partial u^t} (\Gamma^{lm}_n u^m_\alpha) \delta u^t \right\} (4) \text{ for } t \geq 0
\]

\[
(5) \text{ for } t \geq 0
\]

\[
(6) \text{ for } t \geq 0
\]

\[
(7) \text{ for } t \geq 0
\]

\[
\text{(3.13)}
\]

Summing the terms labeled (1), (2), (3), (4) in (3.13), we obtain immediately, after suitable relabeling of some indices:

\[
(1) + (2) + (3) + (4) = g^{kl} \partial_\beta \left( \frac{\partial \beta_1}{\partial u^t} u^k_\beta \delta u^t \right) \text{ sum over } t \geq 0.
\]

On the other hand, summing all the terms labeled (5), (6), (7), (8) in (3.13) and relabeling \( m \) to \( k \) in term (8), we obtain

\[
(5) + (6) + (7) + (8) = \alpha_k \frac{\partial \beta_1}{\partial u^t} (\Gamma^{kl}_n + \Gamma^{kl}_m) u^m_\beta \delta u^t \text{ sum over } t \geq 0.
\]

Now it is well known that \( \left( \Gamma^{kl}_n + \Gamma^{kl}_m \right) = \frac{\partial}{\partial u^t} b^k \), so that \( \left( \Gamma^{kl}_n + \Gamma^{kl}_m \right) u^m_\beta = \partial_k g^{kl} \). Therefore, the sum of all labeled terms in (3.13) is equal to
\[ \partial_i \left( \frac{\partial g^{kl}}{\partial u^i} \partial_k \beta_l \right) \delta u^i + \alpha_k \left( \frac{\partial \Gamma_1^{kl}}{\partial u^i} u^k \beta_l \right) \delta u^i + \alpha_k \Gamma_1^{kl} \delta u^{i(l)}, \tag{3.14} \]

and therefore, it can be safely discarded.

Now it remains to deal with

\[ \alpha_k \left( \frac{\partial g^{kl}}{\partial u^i} \partial_k \beta_l \right) \delta u^i + \alpha_k \left( \frac{\partial \Gamma_1^{kl}}{\partial u^i} u^k \beta_l \right) \delta u^i + \alpha_k \Gamma_1^{kl} \delta u^{i(l)}. \]

First, we express \( \frac{\partial u^i}{\partial x^m} u^n_s \) as a total derivative with respect to \( x \), exchanging the indices \( i \) and \( n \) in \( \frac{\partial u^i}{\partial x^m} \), using the zero curvature condition. Indeed, the zero curvature condition reads

\[ g^{ij} \left( \partial_i \Gamma_1^{jk} - \partial_j \Gamma_1^{ik} \right) - \Gamma_1^{il} \Gamma_1^{jk} + \Gamma_1^{il} \Gamma_1^{jk} = 0, \tag{3.15} \]

where \( \partial_i := \frac{\partial}{\partial x^i} \). From (3.15), lowering and renaming indices and multiplying by \( u^n_s \), we obtain the identity

\[ \partial_i \Gamma_1^{kl} u^n_s = \partial_n \Gamma_1^{kl} u^i_s + g_{m} \Gamma_1^{ml} \Gamma_1^{nk} u^n_s - g_{m} \Gamma_1^{ml} \Gamma_1^{nk} u^n_s. \tag{3.16} \]

Substituting (3.16) into (3.14) and using \( \partial_i \Gamma_1^{kl} u^n_s = \partial_i \Gamma_1^{kl} \), we obtain

\[ \alpha_k \left( \frac{\partial g^{kl}}{\partial u^i} \partial_k \beta_l \right) \delta u^i + \alpha_k \Gamma_1^{kl} \delta u^{i(l)} + g_{m} \alpha_k \beta_l \left[ \Gamma_1^{ml} \Gamma_1^{nk} - \Gamma_1^{ml} \Gamma_1^{nk} \right] u^n_s \delta u^i + \alpha_k \beta_l \Gamma_1^{kl} \delta u^{i(l)}. \tag{3.17} \]

Recalling that \( g_{m} \Gamma_1^{ml} = -\Gamma_1^{i}, \) we see that

\[ g_{m} \alpha_k \beta_l \left[ \Gamma_1^{ml} \Gamma_1^{nk} - \Gamma_1^{ml} \Gamma_1^{nk} \right] u^n_s \delta u^i = -\alpha_k \beta_l \left[ \Gamma_1^{ml} \Gamma_1^{nk} - \Gamma_1^{ml} \Gamma_1^{nk} \right] u^n_s \delta u^i, \]

which appears as the last term in (3.11). Integrating by parts \( \alpha_k \beta_l \Gamma_1^{kl} \delta u^{i(l)} \) and using \( \partial_i \Gamma_1^{kl} = \Gamma_1^{kl} + \Gamma_1^{il} \), the remaining terms in (3.17) become (up to total derivatives)

\[ \left[ \alpha_k \Gamma_1^{kl} \partial_k \beta_l + \alpha_k \Gamma_1^{kl} \partial_k \beta_l - \partial_i (\alpha_k \beta_l) \Gamma_1^{kl} \right] \delta u^i, \]

which after renaming indices is equal to

\[ \left( \alpha_k \partial_k \beta_l - \beta_k \partial_i \alpha_l \right) \Gamma_1^{i} \delta u^i. \]

This is the third term in (3.11). This concludes the proof of (3.11).

Finally to prove (3.12), we use (3.11) specializing it to the case in which \( \alpha_i \) and \( \beta_i \) depend only on the coordinates \( u^1, \ldots, u^n \). Then, in this case, (3.11) gives

\[ \{ \alpha, \beta \} = \left( g^{kl} \partial_m \beta_l + \Gamma_1^{kl} \beta_l \right) u^n_s \frac{\partial \alpha_i}{\partial u^i} - g^{kl} \left( \partial_m \alpha_l + \Gamma_1^{kl} \alpha_l \right) u^n_s \frac{\partial \beta_l}{\partial u^i} \]

\[ + \Gamma_1^{lk} \left( \alpha_k \partial_m \beta_l - \beta_k \partial_m \alpha_l \right) u^n_s - \alpha_k \beta_l \left[ g_{m} \Gamma_1^{ml} \Gamma_1^{nk} \right] u^n_s + \alpha_k \beta_l \left[ g_{m} \Gamma_1^{ml} \Gamma_1^{nk} \right] u^n_s \]

\[ = g^{kl} \left( \nabla_m \beta_l \right) u^n_s \frac{\partial \alpha_l}{\partial u^i} - g^{kl} \left( \nabla_m \alpha_l \right) u^n_s \frac{\partial \beta_l}{\partial u^i} \]

\[ + \Gamma_1^{lk} \left( \alpha_k \partial_m \beta_l - \beta_k \partial_m \alpha_l \right) u^n_s - \alpha_k \beta_l \left[ g_{m} \Gamma_1^{ml} \Gamma_1^{nk} \right] u^n_s + \alpha_k \beta_l \left[ g_{m} \Gamma_1^{ml} \Gamma_1^{nk} \right] u^n_s \]

\[ = \left( \nabla_m \beta_l \right) u^n_s \left[ g^{kl} \frac{\partial \alpha_l}{\partial u^i} + \Gamma_1^{lk} \alpha_l \right] - \left( \nabla_m \alpha_l \right) u^n_s \left[ g^{kl} \frac{\partial \beta_l}{\partial u^i} + \Gamma_1^{lk} \beta_l \right] \]

\[ = \left( \nabla_m \beta_l \right) g^{kl} \left( \nabla_k \alpha_l \right) - \left( \nabla_m \alpha_l \right) g^{kl} \left( \nabla_k \beta_l \right) \]

Formula (3.12) is proved. \[ \square \]
4. Properties of the bracket

In this section, we show that the bracket previously defined enjoys the same properties of the Poisson bracket on functionals. Let \( \alpha := \alpha \delta u \) and \( \beta := \beta \delta u \) be two 1-forms written in the standard form. Then, the Poisson bracket between \( \alpha \) and \( \beta \) is again a 1-form which is written in flat coordinates and in the standard form as follows:

\[
\{ \alpha, \beta \} := \left( \eta^{kl}(\partial_s^{i+1} \beta_i) \frac{\partial \alpha_l}{\partial u^{s_{(i)}}} - \eta^{kl}(\partial_s^{i+1} \alpha_i) \frac{\partial \beta_l}{\partial u^{s_{(i)}}} \right) \delta u^i.
\] (4.1)

First, we show that when the bracket on forms is evaluated on exact 1-forms, then it is equal to the differential of the standard Poisson bracket between the corresponding functionals. Indeed we have the following.

**Proposition 4.1.** If \( \alpha \) and \( \beta \) are exact 1-forms, \( \alpha = \delta \tilde{f} \), \( \beta = \delta \tilde{g} \), where \( \tilde{f} \) and \( \tilde{g} \) are local functionals, then \( \{ \alpha, \beta \} = \{ \delta \tilde{f}, \delta \tilde{g} \} = \delta [\tilde{f}, \tilde{g}] \), where \( [\tilde{f}, \tilde{g}] \) is the usual Poisson bracket among local functionals, while \( \{ \alpha, \beta \} \) is the bracket on 1-forms defined in the previous section.

**Proof.** By (3.9), using the fact that the Lie derivative satisfies Cartan’s identity (3.6), we obtain

\[
\{ \alpha, \beta \} = i_{\tilde{f}} \delta \alpha + i_{\tilde{g}} \delta \beta - i_{\tilde{g}} \delta \beta - i_{\tilde{f}} \delta \alpha + i_{\tilde{g}} \delta \beta. 
\]

Therefore, since \( \delta \circ \delta = 0 \), if \( \alpha \) and \( \beta \) are exact, we have

\[
\{ \alpha, \beta \} = \delta i_{\tilde{f}} \delta \alpha = \delta i_{\tilde{g}} \delta \beta.
\]

In particular, if \( \alpha = \delta \tilde{f}, \beta = \delta \tilde{g} \), we obtain

\[
\{ \delta \tilde{f}, \delta \tilde{g} \} = \delta [\tilde{f}, \tilde{g}].
\]

It is immediate to check that \( \{ \delta \tilde{f}, P \delta \tilde{g} \} = \{ \tilde{f}, \tilde{g} \} \), and thus, the proposition is proved. \( \square \)

**Proposition 4.2.** The bracket defined in (4.1) satisfies the Jacobi identity.

**Proof.** We compute the \( i \)-th component of \( J(\alpha, \beta, \gamma) := \{ \alpha, \{ \beta, \gamma \} \} + \{ \gamma, \{ \alpha, \beta \} \} + \{ \alpha, \beta, \gamma \} \). We have

\[
J(\alpha, \beta, \gamma)_i = \eta^{kl}(\partial_s^{i+1} \beta_i) \frac{\partial \alpha_l}{\partial u^{s_{(i)}}} - \eta^{kl}(\partial_s^{i+1} \alpha_i) \frac{\partial \beta_l}{\partial u^{s_{(i)}}} + \eta^{kl}(\partial_s^{i+1} \alpha_l) \frac{\partial \beta_i}{\partial u^{s_{(i)}}} - \eta^{kl}(\partial_s^{i+1} \beta_l) \frac{\partial \alpha_i}{\partial u^{s_{(i)}}} + \eta^{kl}(\partial_s^{i+1} \beta_i) \frac{\partial \gamma_l}{\partial u^{s_{(i)}}} - \eta^{kl}(\partial_s^{i+1} \gamma_l) \frac{\partial \beta_i}{\partial u^{s_{(i)}}} - \eta^{kl}(\partial_s^{i+1} \gamma_l) \frac{\partial \alpha_i}{\partial u^{s_{(i)}}}.
\]

Substituting the Poisson brackets appearing in the previous expression with their formulae into flat coordinates and further expanding \( J(\alpha, \beta, \gamma)_i \), we obtain

\[
J(\alpha, \beta, \gamma)_i = \eta^{kl} \eta^{pq} \left[ \frac{\partial^{i+1}}{\partial u^{s_{(i)}}} \left( (\partial_s^{i+1} \gamma_q) \frac{\partial \beta_p}{\partial u^{s_{(i)}}} - (\partial_s^{i+1} \beta_p) \frac{\partial \gamma_q}{\partial u^{s_{(i)}}} \right) \frac{\partial \alpha_i}{\partial u^{s_{(i)}}} - (\partial_s^{i+1} \alpha_i) \frac{\partial \beta_p}{\partial u^{s_{(i)}}} (\partial_s^{i+1} \gamma_q) \frac{\partial \beta_p}{\partial u^{s_{(i)}}} \frac{\partial \gamma_q}{\partial u^{s_{(i)}}} + (\partial_s^{i+1} \alpha_i) \frac{\partial \gamma_q}{\partial u^{s_{(i)}}} (\partial_s^{i+1} \beta_p) \frac{\partial \gamma_q}{\partial u^{s_{(i)}}} \right].
\]
The term we are interested in is given by
\[ (\partial_\gamma t^{\alpha}) (\partial_\gamma t^{\beta}) (\partial_\gamma t^{\alpha}) (\partial_\gamma t^{\beta}) \frac{\partial^2 \alpha_i}{\partial \alpha_i \partial \alpha_i}. \]

Let us focus our attention on the terms involving second derivatives of \( \alpha_i \). We have
\[ \eta^{kl} \eta^{pq} \left\{ (\partial_\gamma t^{\alpha}) (\partial_\gamma t^{\beta}) - (\partial_\gamma t^{\gamma}) (\partial_\gamma t^{\delta}) \right\} \frac{\partial^2 \alpha_i}{\partial \alpha_i \partial \alpha_i}, \]
which can be written as \( \eta^{kl} \eta^{pq} T_{ikplq} \), where
\[ T_{ikplq} := \left\{ (\partial_\gamma t^{\alpha}) (\partial_\gamma t^{\beta}) - (\partial_\gamma t^{\gamma}) (\partial_\gamma t^{\delta}) \right\} \frac{\partial^2 \alpha_i}{\partial \alpha_i \partial \alpha_i}. \]

Now observe that \( T_{ikplq} = T_{ikqlp} = -T_{kpliq} \), so \( T \) is symmetric under exchange of \( p \) and \( k \) and anti-symmetric under exchange of \( q \) and \( l \). Therefore,
\[ \eta^{kl} \eta^{pq} T_{ikplq} = -\eta^{kl} \eta^{pq} T_{kpliq} = -\eta^{pq} \eta^{kl} T_{ikplq}, \]
where in the last equality we have renamed the summed indices. Therefore, the expression (4.2) is zero. Analogously, the terms one obtains collecting the second derivatives in \( \beta_i \) and \( \gamma_i \) are identically vanishing for the same reason.

It remains to deal with the term containing the first derivatives in \( \alpha_i \), (a completely analogous computation will show that also the term containing first derivatives of \( \beta_i \) and \( \gamma_i \) will indeed vanish and it will be skipped). The term we are interested in is given by
\[ \eta^{kl} \eta^{pq} \left\{ \partial_\gamma t^{\alpha} \left[ (\partial_\gamma t^{\gamma}) (\partial_\gamma t^{\delta}) - (\partial_\gamma t^{\delta}) (\partial_\gamma t^{\gamma}) \right] \frac{\partial \alpha_i}{\partial \alpha_i} \right\}. \]

In the last two terms, we rename summed indices in order to obtain
\[ \eta^{kl} \eta^{pq} \frac{\partial \alpha_i}{\partial \alpha_i} \left\{ \partial_\gamma t^{\alpha} \left[ (\partial_\gamma t^{\gamma}) (\partial_\gamma t^{\delta}) - (\partial_\gamma t^{\delta}) (\partial_\gamma t^{\gamma}) \right] \right\}. \]
It is immediate to see that \((4.4)\) vanishes identically if

\[
\begin{align*}
\partial_x^{s+1} \left[ (\alpha_x^{s+1} \gamma_q) \frac{\partial \beta_l}{\partial u_{(t)}^{p}} \right] &= (\alpha_x^{s+1} \gamma_q) \frac{\partial \beta_l}{\partial u_{(t)}^{p}} (\alpha_x^{s+1} \beta_l), \\
\partial_x^{s+1} \left[ (\alpha_x^{s+1} \beta_q) \frac{\partial \gamma_l}{\partial u_{(t)}^{p}} \right] &= (\alpha_x^{s+1} \beta_q) \frac{\partial \gamma_l}{\partial u_{(t)}^{p}} (\alpha_x^{s+1} \gamma_l). 
\end{align*}
\] (4.5)

Obviously, it is sufficient to prove the first of \((4.5)\). We expand the left-hand side of the first of \((4.5)\) using the binomial formula as

\[
\sum_{s \geq 0, l \geq 0} \sum_{l=0}^{s+1} \left( s + 1 \right) (\alpha_x^{s+1+l} \gamma_q) \partial_x^{s+1-l} \left( \frac{\partial \beta_l}{\partial u_{(t)}^{p}} \right) = \sum_{s \geq 0} \sum_{l=0}^{s+1} \left( s + 1 \right) \sum_{t \geq 0} (\alpha_x^{s+1+l} \gamma_q) \partial_x^{t+1-l} \left( \frac{\partial \beta_l}{\partial u_{(t)}^{p}} \right). 
\] (4.6)

To facilitate the comparison with other terms, we split the second line in \((4.6)\) as follows:

\[
\sum_{s \geq 0} \sum_{l=0}^{s+1} \left( s + 1 \right) \sum_{t \geq 0} \sum_{l=0}^{s+1-l} (\alpha_x^{s+1+l} \gamma_q) \partial_x^{t+1-l} \left( \frac{\partial \beta_l}{\partial u_{(t)}^{p}} \right) + \sum_{s \geq 0} \sum_{l=0}^{s+1} \sum_{t \geq 0} \sum_{l=0}^{s+1} (\alpha_x^{s+1+l} \gamma_q) \partial_x^{t+1-l} \left( \frac{\partial \beta_l}{\partial u_{(t)}^{p}} \right). 
\] (4.7)

Now we rewrite \((4.7)\) splitting its first term, separating the sum in \(t\), in the following way:

\[
\sum_{s \geq 0} \sum_{l=0}^{s+1} \left( s + 1 \right) \sum_{t \geq s+1-l} \partial_x^{t+1-l} \left( \frac{\partial \beta_l}{\partial u_{(t)}^{p}} \right) + \sum_{s \geq 0} \sum_{l=0}^{s+1} \left( s + 1 \right) \sum_{0 \leq t < s+1-l} \partial_x^{t+1-l} \left( \frac{\partial \beta_l}{\partial u_{(t)}^{p}} \right) + \sum_{s \geq 0} \sum_{l=0}^{s+1} \partial_x^{t+1+l} \gamma_q \partial_x^{t+1-l} \left( \frac{\partial \beta_l}{\partial u_{(t)}^{p}} \right). 
\] (4.8)

Putting together the first and the last line in \((4.8)\) and rewriting the second line we get that the left-hand side of the first of \((4.5)\) is given by:

\[
\sum_{s \geq 0} \sum_{l=0}^{s+1} \left( s + 1 \right) \sum_{t \geq s+1-l} \partial_x^{t+1-l} \left( \frac{\partial \beta_l}{\partial u_{(t)}^{p}} \right) + \sum_{s \geq 0} \sum_{l=0}^{s+1} \left( s + 1 \right) \sum_{0 \leq t < s+1-l} \partial_x^{t+1-l} \left( \frac{\partial \beta_l}{\partial u_{(t)}^{p}} \right) + \sum_{s \geq 0} \sum_{l=0}^{s+1} \partial_x^{t+1+l} \gamma_q \partial_x^{t+1-l} \left( \frac{\partial \beta_l}{\partial u_{(t)}^{p}} \right). 
\] (4.9)

We recall the following identity which can be easily proved by induction:

\[
\frac{\partial}{\partial u_{(t)}^{p}} \circ \partial_x^s = \sum_{l=\max\{0,n-t\}}^{n} \binom{n}{l} \partial_x^s \circ \frac{\partial}{\partial u_{(t+1-l)}^{p+n}}. 
\] (4.10)

To expand the right-hand side of the first of \((4.5)\), we use \((4.10)\) and obtain

\[
(\alpha_x^{s+1} \gamma_q) \frac{\partial}{\partial u_{(t)}^{p}} (\alpha_x^{s+1} \beta_l) = \sum_{s \geq 0, l \geq 0} \sum_{l=\max\{0,s+1-l\}}^{s+1} \left( s + 1 \right) \partial_x^{l+1} \gamma_q \partial_x^{l+1-s} \left( \frac{\partial \beta_l}{\partial u_{(t+1-l)}^{p+s+1}} \right). 
\] (4.11)
We split (4.11) into two pieces, according to if \( t \geq s + 1 \) or \( t \leq s \). We obtain
\[
(\tilde{g}_x^{s+1}\gamma_q) \frac{\partial}{\partial u_{(t)}} (\tilde{g}_x^{s+1}\beta_i) = \sum_{s \geq 0} \sum_{r \geq s+1} \sum_{l=0}^{s+1} (s+1) \left( \tilde{g}_x^{s+1}\gamma_q \right) \delta_l \left( \frac{\partial \beta_i}{\partial u_{(t-r+1)}} \right) \\
+ \sum_{r \geq 0} \sum_{t = 0}^{s+1} \sum_{l=0}^{s+1} (s+1) \left( \tilde{g}_x^{s+1}\gamma_q \right) \delta_l \left( \frac{\partial \beta_i}{\partial u_{(t-r+1)}} \right). 
\]  
(4.12)

Defining the new index \( l' := s + 1 - l \), we obtain
\[
(\tilde{g}_x^{s+1}\gamma_q) \frac{\partial}{\partial u_{(t)}} (\tilde{g}_x^{s+1}\beta_i) = \sum_{s \geq 0} \sum_{t \geq s+1} \sum_{l=0}^{s+1} (s+1) \left( \tilde{g}_x^{s+1}\gamma_q \right) \tilde{g}_x^{s+1-l} \left( \frac{\partial \beta_i}{\partial u_{(t-l')}} \right) \\
+ \sum_{r \geq 0} \sum_{t = 0}^{s+1} \sum_{l=0}^{s+1} (s+1) \left( \tilde{g}_x^{s+1}\gamma_q \right) \tilde{g}_x^{s+1-l} \left( \frac{\partial \beta_i}{\partial u_{(t-l')}} \right). 
\]  
(4.13)

We can rewrite the first term of the right-hand side of (4.13) as
\[
\sum_{s \geq 0} \sum_{t \geq s+1} \sum_{l=0}^{s+1} (s+1) \left( \tilde{g}_x^{s+1}\gamma_q \right) \tilde{g}_x^{s+1-l} \left( \frac{\partial \beta_i}{\partial u_{(t-l')}} \right) \\
= \sum_{s \geq 0} \sum_{t = 0}^{s+1} (s+1) \left( \tilde{g}_x^{s+1}\gamma_q \right) \tilde{g}_x^{s+1-l} \left( \frac{\partial \beta_i}{\partial u_{(t-l')}} \right),
\]
and defining \( t' = t - l' \), we obtain
\[
\sum_{s \geq 0} \sum_{t' = 0}^{s+1} (s+1) \sum_{r \geq s+1} (\tilde{g}_x^{s+1}\gamma_q) \tilde{g}_x^{s+1-l} \left( \frac{\partial \beta_i}{\partial u_{(t-l')}} \right) \\
= \sum_{s \geq 0} \sum_{t' = 0}^{s+1} (s+1) \sum_{r \geq s+1} (\tilde{g}_x^{s+1+l'}\gamma_q) \tilde{g}_x^{s+1-l} \left( \frac{\partial \beta_i}{\partial u_{(t-l')}} \right),
\]
which we recognize as the first of the terms in (4.9).

Now we rewrite the second term of the right-hand side of (4.13) as follows:
\[
\sum_{s \geq 0} \sum_{t \geq s+1} \sum_{l=0}^{s+1} (s+1) \left( \tilde{g}_x^{s+1}\gamma_q \right) \tilde{g}_x^{s+1-l} \left( \frac{\partial \beta_i}{\partial u_{(t-l')}} \right) \\
= \sum_{s \geq 0} \sum_{t \geq s+1} \sum_{l=0}^{s+1} (s+1) \left( \tilde{g}_x^{s+1+l'}\gamma_q \right) \tilde{g}_x^{s+1-l} \left( \frac{\partial \beta_i}{\partial u_{(t-l')}} \right).
\]

Again defining \( t' = t - l' \) we can rewrite the previous expression as
\[
\sum_{s \geq 0} \sum_{t \geq s+1} \sum_{l=0}^{s+1} (s+1) \left( \tilde{g}_x^{s+1+l'}\gamma_q \right) \tilde{g}_x^{s+1-l} \left( \frac{\partial \beta_i}{\partial u_{(t-l')}} \right),
\]
which we recognize as the second of the terms in (4.9). This proves that the first of (4.5) is indeed an identity.

Therefore, the term containing the first derivatives of \( \alpha_t \) vanishes and similarly for the terms containing the first derivatives of \( \beta_i \) and \( \gamma_q \). The Jacobi identity is therefore proved. \( \Box \)

**Corollary 4.3.** The bracket (4.1) equips the vector space of 1-forms \( \Lambda_1 \) with a Lie algebra structure.
Proof. Clearly $[\alpha, \beta] = -[\beta, \alpha]$ and $\{\cdot, \cdot\}$ is $\mathbb{R}$-bilinear. By proposition 4.2, it also fulfils Jacobi’s identity. □

Let us recall that the vector space of evolutionary vector fields $\Lambda^1_{ev}$ is naturally equipped with the Lie product given by the commutator:

$$[\xi, \eta]^p = \xi^i \frac{\partial}{\partial u^i(s)} \eta^p - \eta^i \frac{\partial}{\partial u^i(s)} \xi^p,$$

(4.14)

where $\xi^i := \frac{\partial}{\partial u^i(s)} \xi$ and $\xi^i$ is the $i$th component of the evolutionary vector field $\xi$. It is known that $(\Lambda^1_{ev}, [\cdot, \cdot])$ is a Lie algebra.

**Proposition 4.4.** The Poisson structure $P$ sending 1-forms to evolutionary vector fields satisfies the identity

$$P(\alpha, \beta) = -[P\alpha, P\beta].$$

Therefore, $P$ is an (anti)-homomorphism of Lie algebras, $P : (\Lambda^1, [\cdot, \cdot]) \rightarrow (\Lambda^1_{ev}, [\cdot, \cdot])$.

**Proof.** We prove the claim in flat coordinates. We have

$$(P(\alpha, \beta))^i = \eta^k \eta^p \partial_k \eta^x \left( (\bar{a}^{s+1} \bar{b}_i) \frac{\partial \alpha}{\partial u^p(s)} - (\bar{a}^{s+1} \bar{a}_i) \frac{\partial \beta}{\partial u^p(s)} \right).$$

and further expanding

$$\eta^k \eta^p \sum_{s \geq 0} \left( (\bar{a}^{s+1} \bar{b}_i) \frac{\partial \alpha}{\partial u^p(s)} - (\bar{a}^{s+1} \bar{a}_i) \frac{\partial \beta}{\partial u^p(s)} \right)$$

$$+ \eta^k \eta^p \sum_{s \geq 0} \left( (\bar{a}^{s+1} \bar{b}_i) \frac{\partial \alpha}{\partial u^p(s)} - (\bar{a}^{s+1} \bar{a}_i) \frac{\partial \beta}{\partial u^p(s)} \right),$$

(4.16)

where we have explicitly inserted the summation symbol to make comparison with the next expression easier. On the other hand,

$$-[P\alpha, P\beta]^i = - \left( (P\alpha)^p \frac{\partial}{\partial u^p(s)} (P\beta)^i - (P\beta)^p \frac{\partial}{\partial u^p(s)} (P\alpha)^i \right)$$

$$= - \left( \bar{a}_i^{s+1} \frac{\partial}{\partial u^p(s)} (\eta^k \partial_k \alpha) - \bar{a}_i^{s+1} \frac{\partial}{\partial u^p(s)} (\eta^k \partial_k \beta) \right).$$

Using (4.10), this last expression is equal to

$$- \eta^k \eta^p \sum_{s \geq 1} \left( (\bar{a}^{s+1} \bar{a}_i) \left( \frac{\partial \beta_k}{\partial u^p(s)} + \frac{\partial \alpha_k}{\partial u^p(s-1)} \right) - \bar{a}_i^{s+1} \bar{b}_k \right)$$

$$- \eta^k \eta^p \sum_{s \geq 0} \left( (\bar{a}^{s+1} \bar{a}_i) \frac{\partial \beta_k}{\partial u^p(s)} - \bar{a}_i^{s+1} \frac{\partial \alpha_k}{\partial u^p(s)} \right),$$

which we can rearrange as

$$- \eta^k \eta^p \sum_{s \geq 1} \left( (\bar{a}^{s+1} \bar{a}_i) \left( \frac{\partial \beta_k}{\partial u^p(s)} + \frac{\partial \alpha_k}{\partial u^p(s-1)} \right) - \bar{a}_i^{s+1} \bar{b}_k \right)$$

$$- \eta^k \eta^p \sum_{s \geq 0} \left( (\bar{a}^{s+1} \bar{a}_i) \frac{\partial \beta_k}{\partial u^p(s)} - \bar{a}_i^{s+1} \frac{\partial \alpha_k}{\partial u^p(s)} \right),$$

(4.17)

Comparing (4.16) and (4.17), we obtain $P(\alpha, \beta) = -[P\alpha, P\beta]$. □

The following Proposition singles out the vector fields in $\Lambda^1_{ev}$ that are in the image of $P$. 19
Proposition 4.5. Let $P : (\Lambda^1, \{\cdot,\cdot\}) \to (\Lambda^1_{\text{ev}}, \{\cdot,\cdot\})$ be the Lie algebra (anti)-homomorphism given by the Poisson structure. Then, the image of $P$ in $(\Lambda^1_{\text{ev}}, \{\cdot,\cdot\})$ is a Lie subalgebra given by the evolutionary vector fields that are tangent to the symplectic leaves of $P$.

Proof. The fact that the image of $P$ in $(\Lambda^1_{\text{ev}}, \{\cdot,\cdot\})$ is a Lie subalgebra is clear. Moreover, if $a \in \Lambda^1$, $a = \int dx \wedge \alpha, \delta u^i$, then $P a = \delta^i_j \left( \eta^{\beta} \delta_{\alpha} \right) \frac{\partial}{\partial u^\beta}$ (in flat coordinates) which is an evolutionary vector field tangent to the symplectic leaves of $P$. To see this, it is sufficient to show that $(P a)(f) = 0$, where $f_i$ are the Casimirs of $P$, since the symplectic leaves are described by $\{f_1 = c_1, \ldots, f_n = c_n\}$, $c_1, \ldots, c_n$ being constants. It is well known that the Casimirs of $P$ are given by the local functionals $f_i = \int u^i \, dx$ (here $u^1, \ldots, u^n$ are flat coordinates). Now the vector field $P a$ applied to the functional $f_i$ is computed taking the pairing of $P a$ with the exact 1-form $\delta f_i$, $(P a, \delta f_i) = \int dx \, \eta^\beta \delta_{\alpha} a_i$, which is clearly zero being a total derivative. Vice versa, we have to show that every evolutionary vector field $\xi$ tangent to the symplectic leaves of $P$ is of the form $P a$ for some $a$. Since $\xi$ is evolutionary, $\xi = \delta^i_j \eta^{\beta} \delta_{\alpha} a_i$, for some functions $\xi^1, \ldots, \xi^n$. Imposing that $\xi$ is tangent to the symplectic leaves of $P$, namely $\langle \xi, \delta f_i \rangle = 0$ for $i = 1, \ldots, n$, we obtain $\int dx \, \xi^i = 0$ for all $j = 1, \ldots, n$. Therefore, each $\xi^i$ is a total derivative $\xi^i = \partial_i \xi$. Since $\eta^\beta$ is invertible, we can write $\xi^i = \eta^\beta a_i$ for some $a_i$. Therefore, we obtain $\xi^i = \partial_i (\eta^\beta a_i) = \eta^\beta \delta_{\alpha} a_i$ which proves the claim. \qed

5. F-manifolds and Poisson brackets on 1-forms

Definition 5.1. An F-manifold $M$ with compatible flat connection is a manifold endowed with a commutative associative product $\circ$ on vector fields and a symmetric flat connection $\nabla$ satisfying the condition

$$\nabla c^i_{jk} = \nabla c^j_{ik},$$

where $c^i_{jk}$ is the $(1, 2)$ tensor field representing the product $\circ$.

Using the commutativity of the algebra, it is easy to check that in flat coordinates we have

$$c^i_{jk} = \delta^i_j \partial_k C^i.$$

In the Frobenius case, due to the existence of an invariant metric, one can make an additional step and obtain $C^i = \eta^\beta \delta_{\alpha} F$ for a suitable function $F$ (the Frobenius potential).

Given an F-manifold with compatible connection, one can define the associated principal hierarchy in the following way. First, using a frame of flat vector fields $(X_{(1,0)}, \ldots, X_{(n,0)})$, one defines the so-called primary flows by means of

$$u^i_{(p,a)} = c^i_{jk} X^k_{(p,0)} u^l_{(a)}, \quad p = 1, \ldots, n,$$

and then, using the recursive relations

$$\nabla_p X^i_{(p,a)} = c^i_{jk} X^k_{(p,a-1)},$$

one defines the higher flows:

$$u^i_{(p,a)} = c^i_{jk} X^k_{(p,a)} u^l_{(a)}, \quad p = 1, \ldots, n.$$

The principal hierarchy associated with an F-manifold $(M, \circ)$ with compatible flat connection $\nabla$ is the collection of all the flows in (5.1) and (5.3).

Note that the recursive relations (5.2) can be written as

$$[X_{(q,0)}, X_{(p,a)}] = X_{(p,a-1)} \circ X_{(q,0)}.$$

The commutativity of the flows of the principal hierarchy can be proved using the following lemma [13].
Lemma 5.2. The flows
\[ u_i^t = c^i_{jk} X_j^k u_k^t \quad i = 1, \ldots, n, \]
and
\[ u_i^t = c^i_{jk} X_j^k u_k^t \quad i = 1, \ldots, n, \]
associated with different solutions of
\[ c^i_{jm} \nabla_k X^m = c^i_{km} \nabla_j X^m \tag{5.5} \]
commute.

Indeed, the vector fields of the principal hierarchy satisfy (5.5). For \( \alpha = 0 \), it is trivial. For \( \alpha > 0 \), it follows from the following chain of identities:
\[ c^i_{jm} \nabla_k X^m = c^i_{jm} c^m_{kl} X^l_{(p, \alpha)} = c^i_{km} c^m_{jl} X^l_{(p, \alpha)} = c^i_{km} \nabla_j X^m, \]
where the one in the middle is due to the associativity of \( \circ \).

6. Hamiltonian formalism for the principal hierarchy

Using (5.2), it is immediate to see that all the flows of the principal hierarchy can be written as
\[ u_i^t_{(p, \alpha)} = \nabla_j X_i^{(p, \alpha+1)} u_j^t. \]
Let \( g \) be any metric compatible with \( \nabla \) (\( \nabla_g = 0 \)), not necessarily positive definite. Given the vector fields, \( X_{(p, \alpha+1)} \) define corresponding forms
\[ (\omega_{(p, \alpha+1)})_l = g_l^{ij} X_j^{(p, \alpha+1)}. \]
In this way, we can write
\[ u_i^t_{(p, \alpha)} = \nabla_j X_i^{(p, \alpha+1)} u_j^t = \nabla_j (g^l_{ij} (\omega_{(p, \alpha+1)})_l) u_j^t \]
\[ = g^l_{ij} \partial_j (\omega_{(p, \alpha+1)})(\omega_{(p, \alpha+1)})_l = g^l_{mj} \Gamma^m_{jm} (\omega_{(p, \alpha+1)})(\omega_{(p, \alpha+1)})_l = (g^l_{mj} \partial_j + \Gamma^m_{jm} u^l_{(p, \alpha+1)}) (\omega_{(p, \alpha+1)}). \]
Note that the operator in the bracket is the differential operator associated with a Poisson bracket of hydrodynamic type.

Now the following two cases are possible.
- The metric \( g \) is invariant with respect to the product, namely \( g(X \circ Y, Z) = g(X, Y \circ Z) \), or equivalently \( c^l_{jk} g^{kl} = c^l_{jk} g^{kl} \).
- The metric \( g \) is not invariant with respect to the product.

The first case is less interesting since it is well known and well studied. Indeed, in this case, the 1-forms \( \omega_{(p, \alpha+1)} \) are exact and we end up with the usual local Hamiltonian formalism introduced by Dubrovin and Novikov. To prove this fact, we need the following lemma.

Lemma 6.1. If \( g \) is invariant, then the 1-forms \( \omega_{(p, \alpha)} \) defining the principal hierarchy satisfy the following recursive relations:
\[ \nabla_j (\omega_{(p, \alpha)})_l = c^l_{jm} (\omega_{(p, \alpha-1)})_l. \tag{6.1} \]
Proof. The proof follows from the recursive relations for the vector fields. Indeed, we can write
\[ \nabla_j X_{i(p,a)} = c^i_{jk} X_{k(p,a-1)} \]
as
\[ g^l \nabla_j (\omega_{(p,a)})_l = c^i_{jk} g^l (\omega_{(p,a-1)})_l. \] (6.2)
Using the invariance of \( g \) with respect to \( \circ \), we obtain
\[ g^l \nabla_j (\omega_{(p,a)})_l = c^i_{jk} g^l (\omega_{(p,a-1)})_l. \]
Multiplying both sides by \( g^{il} \) and taking the sum over the index \( i \) we obtain the result. \( \square \)

Corollary 6.2. If \( g \) is invariant with respect to \( \circ \), then \( \omega_{(p,a)} \) are exact.

Proof. Now using the above lemma, we obtain
\[ (d \omega_{(p,a)})_{ij} = \nabla_j (\omega_{(p,a)})_i - \nabla_i (\omega_{(p,a)})_j = 0, \]
due to the commutativity of the product \( \circ \). \( \square \)

On the other hand, if \( g \) is not invariant with respect to the product \( \circ \), in general the 1-forms \( \omega \) are not exact since
\[ (d \omega_{(p,a)})_{ij} = \nabla_j (\omega_{(p,a)})_i - \nabla_i (\omega_{(p,a)})_j = \left[ g^k c^i_{jl} - g^l c^i_{jk} \right] g^m (\omega_{(p,a-1)})_m \] (6.3)
and the quantity in square brackets do not vanishes. Although the 1-forms defining the principal hierarchy starting from an \( F \)-manifold with compatible flat structure are not in general close, we can use the Poisson bracket on 1-forms introduced in the previous sections to reinterpret the commutativity of the flows of the principal hierarchy as the fact that the corresponding 1-forms are in involution.

This is the meaning of the following.

Theorem 6.3. The 1-forms \( \omega_{(p,a)} \) defining the principal hierarchy are in involution with respect to the Poisson bracket (4.1), and the involutivity is a consequence of the associativity of the product \( \circ \).

Proof. In flat coordinates \( t^1, \ldots, t^n \), the bracket (4.1) reads
\[ \{ \alpha, \beta \} = \int_S \eta^{|i|} \left( \frac{\partial \alpha_j}{\partial t^k} \frac{\partial \beta_j}{\partial t^l} - \frac{\partial \alpha_j}{\partial t^l} \frac{\partial \beta_j}{\partial t^k} \right) t^k dx, \] (6.4)
for 1-forms whose coefficients depend only on the flat coordinates, not on their derivatives. Using the recursive relations (6.2) written in flat coordinates
\[ \eta^{|i|} \partial_j (\omega_{(p,a)})_l = c^i_{jk} \eta^{|l|} (\omega_{(p,a-1)})_l, \] (6.5)
we obtain
\[ \{ \omega_{(p,a)}, \omega_{(q,b)} \} \]
\[ = \int_S \eta^{|i|} \left( \frac{\partial (\omega_{(q,b)})_j}{\partial t^k} \frac{\partial (\omega_{(p,a)})_l}{\partial t^i} - \frac{\partial (\omega_{(p,a)})_j}{\partial t^k} \frac{\partial (\omega_{(q,b)})_l}{\partial t^i} \right) t^k dx \]
\[ = \int_S \eta^{|i|} (\omega_{(q,b-1)})_j (\omega_{(p,a-1)})_l \left( c^j_{ik} \eta^{|l|} \eta^{|m|} - c^j_{ik} \eta^{|m|} \eta^{|l|} \right) t^k dx \]
\[ = \int_S \eta^{|i|} \eta^{|l|} (\omega_{(q,b-1)})_j (\omega_{(p,a-1)})_l \left( c^j_{ik} \eta^{|m|} - c^j_{ik} \eta^{|m|} \right) t^k dx = 0, \]
due to the associativity of the product \( \circ \). \( \square \)

Due to proposition 4.4, the above theorem provides an alternative proof of the commutativity of the flows of the principal hierarchy.
Remark 6.4. Note that if $\alpha$ and $\beta$ are the differentials of two local functionals $H[u] = \int_S h(u) \, dx$ and $K[u] = \int_S k(u) \, dx$, respectively, then as was proved in proposition 4.1 one has $[\delta H, \delta K] = \delta [H, K]$, where $[H, K]$ is the usual Poisson brackets of hydrodynamic type:

$$\{H, K\} = \int_S \frac{\partial h}{\partial \mu^i} \frac{\partial k}{\partial \mu^j} \left( \frac{\partial}{\partial \mu^m} \right) \, dx.$$ 

Therefore, if $g$ is invariant, the fact the 1-forms defining the principal hierarchy are in involution follows immediately from the involution of the corresponding Hamiltonians.

7. An example

The triple $(\mathbb{R}^n, \nabla, \circ)$ with $\nabla$ defined by

$$\Gamma_{jk}^i = 0 \quad \text{for } i \neq j \neq k \neq i,$$

$$\Gamma_{ij}^j = -\Gamma_{ji}^i \quad \text{for } i \neq j,$$

$$\Gamma_{ii}^j = \frac{\epsilon}{i-\alpha} \quad \text{for } i \neq j,$$

and $\circ$ defined by $c_{jk}^i = \delta_{k}^{j} \delta_{i}^{k}$ in coordinates $(u^1, \ldots, u^n)$ (canonical coordinates) is an $F$-manifold with compatible flat structure [14]. It is strictly related to a dispersionless integrable (semi-Hamiltonian [19]) hierarchy called the $\epsilon$-system. In the case $\epsilon = 1$ and for $n = 3$, the flat coordinates $(t_1, t_2, t_3)$ are given in terms of canonical coordinates $(u_1, u_2, u_3)$ by the formulae

$$t_1 = u_1 + u_2 + u_3,$$

$$t_2 = \frac{1}{2(u_1 - u_2)(u_3 - u_1)},$$

$$t_3 = \frac{1}{2(u_1 - u_2)(u_2 - u_3)}.$$ 

In flat coordinates any constant non-degenerate symmetric matrix define a metric compatible with $\nabla$. For instance, we can take the antidiagonal metric

$$g = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$ 

(This might not be the most convenient choice.) At this point, we have to decide if to work in canonical coordinates or in flat coordinates. In the first case, the components of the metric become much more involved. However, the 1-forms defining the hierarchy can be easily obtained using the results of [14]. This becomes non-trivial if we work in flat coordinates since the 1-forms defining the hierarchy satisfy the system of PDEs

$$g^{km} c_{jk}^i \nabla_{\alpha m} = g^{km} c_{jk}^i \nabla_{\alpha m}$$

(7.2)

involving the structure constants $c_{jk}^i$.

7.1. Working in canonical coordinates

The metric $g$ in canonical coordinates reads

$$g_{11} = \frac{1}{4(u_3 - u_2)(u_1 - u_2)^3(-u_3 + u_1)^3} \left[ 14u_1^2u_3^4 - 16u_1^3u_3^3 + 24u_1^4u_3^2 - 16u_1^5u_3 
+ 4u_1^6 - 8u_1u_3^4u_2 + 32u_1u_3u_2^3u_2 - 48u_2u_3u_2^3u_1 + 32u_2u_4^4u_3 - 8u_2u_1^5 + 4u_2^2u_3^4 
- 16u_1u_2^2u_3^3 + 24u_1^2u_2^3u_2^2 - 16u_2^2u_3u_1^3 + 4u_2^2u_4^4 + u_3^2u_2 + u_3^3 - 4u_3^2u_1 
- u_3u_2^2 - 4u_1u_2^2 + 4u_1u_3^2 + 4u_1u_2^2 - u_2^3 \right].$$
\[ g_{12} = g_{21} = \frac{1}{4(u_3 - u_2)^2(u_1 - u_2)^2(-u_3 + u_1)^2}\left\{2u_1u_2^4 + u_3u_2^2 - 4u_3u_1u_2 + 6u_1^4u_3^2
\right.
\]
\[-6u_1u_2^2u_3 - 2u_1^3u_3^2 + u_2^2u_3^2 + 6u_1^2u_3^2u_1 - 6u_1u_2u_3^2 + 18u_2^2u_3^2 + 6u_1^2u_2^2 + 6u_1^2u_2^2u_3 - 6u_2u_3u_1^2 - 2u_3^4u_1^2 + 18u_2u_1^4u_3 - 18u_2u_1^4u_3^2 + 6u_1^2u_3^2
\]
\[-6u_2u_3^2 - u_3^2 - u_3^2 - 2u_1^3u_3^2 + 2u_1^3u_3^2\right\},
\]
\[ g_{13} = g_{31} = \frac{1}{4(u_3 - u_2)^2(u_1 - u_2)^2(-u_3 + u_1)^2}\left\{2u_1^2u_2^2 - u_3^2u_2^2 + 2u_1u_2u_3^2 + 4u_1u_1u_3
\right.
\]
\[+4u_1^2u_2^2 - 6u_2u_3^2 + 4u_1^2u_2^2 + 12u_2u_3^2u_2 - 16u_2u_2^2u_3
\]
\[-2u_3u_1 - 16u_2u_2u_3^2 + 22u_2u_1^2u_3 - 28u_2u_2u_3^2 - 2u_3^3u_2 + 4u_2^4u_3 - 6u_2u_5
\]
\[+u_3^4 + u_3^2 - 6u_1^2u_3^2 + 2u_1u_3^2 + 2u_1^2 + 4u_2^4u_3^4\right\},
\]
\[ g_{22} = -\frac{1}{4(u_1 - u_2)^2(u_2 - u_3)^2(-u_3 + u_1)^2}\left\{u_1u_2^2 + u_2 - 6u_2u_1u_2 + 6u_2^3u_3 + 2u_1u_3
\right.
\]
\[-u_3 - 6u_2u_1 + 2u_1^2 + 4u_2^4u_3
\]
\[\right\},
\]
\[ g_{23} = g_{32} = \frac{1}{4(u_3 - u_1)^2(u_1 - u_2)^2(u_2 - u_3)}\left\{2u_3^2u_2 + u_2 - 6u_2u_1u_2 + 6u_2^3u_3 - 2u_1u_3
\right.
\]
\[-u_3 - 6u_2u_1 + 2u_1^2 + 4u_2^4u_3
\]
\[\right\},
\]
\[ g_{33} = \frac{1}{4(u_2 - u_3)^2(u_1 - u_2)^2(-u_3 + u_1)^2}\left\{4u_1u_3^2 - 16u_1u_3^3 + 24u_2u_2^3 - 16u_2u_3 + 4u_1^3
\right.
\]
\[-4u_2u_2 + 16u_2u_3^2 + 24u_2u_2^3 - 16u_2u_3 + 4u_1^3
\]
\[\right\}.
\]

The Poisson operator in canonical coordinates is given by the formula
\[ g^{ij} \partial_i + \Gamma_k^{ij} u_k, \]
where \( g^{ij} \) are the contravariant components of the Euclidean metric in canonical coordinates:
\[ g^{11} = \frac{4}{9}u_1^6 - \frac{4}{9}u_1^5u_3 - \frac{4}{9}u_1u_3^5 + \frac{1}{9}u_2^4u_3^3 + u_1^4u_3^2 + u_2^2u_1^4 + \frac{4}{9}u_1^3u_3^3 - 4u_2^2u_1u_3
\]
\[-6u_1u_3^2 - u_2^2u_3^2 - u_3^2u_1 - 6u_2u_3^2 - 12u_2u_3^4 + 12u_2u_3^4u_3 - 16u_2u_3^4u_3
\]
\[-2u_3^4u_2 + 6u_2u_3^4 + 22u_2u_3^4u_3 - 28u_2u_3^4u_3 - 2u_3^3u_2 + 4u_2^4u_3 - 6u_2u_5
\]
\[+u_3^4 + u_3^2 - 6u_1^2u_3^2 + 2u_1u_3^2 + 2u_1^2 + 4u_2^4u_3^4\right\},
\]
\[ g^{12} = g^{21} = -\frac{4}{9}u_1^6 + \frac{4}{9}u_1^5u_3 + u_1^4u_3^2 - \frac{4}{9}u_2u_1^4u_3 + \frac{10}{9}u_1^3u_3^3 + \frac{10}{9}u_2u_2^3 - u_3^3
\]
\[-6u_1u_3^2 - u_2^2u_3^2 - u_3^2u_1 - 6u_2u_3^2 - 12u_2u_3^4 + 12u_2u_3^4u_3 - 16u_2u_3^4u_3
\]
\[-2u_3^4u_2 + 6u_2u_3^4 + 22u_2u_3^4u_3 - 28u_2u_3^4u_3 - 2u_3^3u_2 + 4u_2^4u_3 - 6u_2u_5
\]
\[+u_3^4 + u_3^2 - 6u_1^2u_3^2 + 2u_1u_3^2 + 2u_1^2 + 4u_2^4u_3^4\right\},
\]
\[ g^{13} = g^{31} = -\frac{4}{9}u_1^6 + u_1^5u_3 + \frac{4}{9}u_1u_3^5 - 3u_2u_1^4u_3 - u_2^2u_1^4 + 4u_2^2u_1u_3
\]
\[+2u_2u_1^4u_3 - \frac{14}{9}u_1^3u_3^3 + 6u_2u_3^2 - 6u_2u_3^4 + 2u_2u_3^4u_3 + 4u_2u_3^4u_3
\]
\[-2u_3^4u_2 + \frac{1}{3}u_1u_3^5 + \frac{1}{3}u_2^2u_1 - 3u_1u_3^4u_2 + \frac{1}{3}u_1u_3^2 - u_2^2u_3^4 - \frac{1}{3}u_3^3 - \frac{2}{3}u_3
\]
\[+u_3^3u_2 - \frac{2}{3}u_2^3 + \frac{1}{3}u_3u_2^3.\]
\[ g^{22} = \frac{1}{7} u_1^6 - \frac{3}{2} u_1^5 u_3 + \frac{5}{4} u_1^4 u_3^2 - \frac{20}{7} u_1^4 u_3^3 + \frac{5}{4} u_1^3 u_3^4 - \frac{2}{7} u_1 u_3^5 - \frac{2}{7} u_3^6 - \frac{2}{7} u_1 u_2^2 + \frac{2}{7} u_3 u_1 u_2 + \frac{2}{7} u_3^3 + \frac{2}{7} u_3^2 + \frac{2}{7} u_3^5 - \frac{2}{7} u_1 u_2^2. \]

\[ g^{23} = g^{32} = \frac{1}{7} u_1^6 - \frac{1}{2} u_1^5 u_3 - \frac{3}{2} u_2 u_1^4 + \frac{5}{4} u_2 u_1^3 u_3 - \frac{10}{7} u_2 u_1^2 u_3^2 u_1 - \frac{10}{7} u_1^4 u_3^3 + \frac{10}{7} u_1^3 u_3^4 - \frac{5}{4} u_1^2 u_3^5 - \frac{5}{4} u_1 u_3^6 + \frac{1}{2} u_3 u_1 u_2 + \frac{2}{7} u_3^3 + \frac{1}{2} u_3^2 + \frac{2}{7} u_3^5 - \frac{2}{7} u_3^6 + \frac{1}{2} u_1 u_2^2. \]

\[ g^{33} = \frac{1}{7} u_1^6 - \frac{2}{7} u_2 u_1^5 + u_2^2 u_1^4 + \frac{2}{7} u_1^3 u_3^2 - \frac{10}{7} u_1^4 u_3^3 - \frac{4}{7} u_1^3 u_3^4 + \frac{4}{7} u_1^2 u_3^5 - \frac{1}{7} u_1 u_3^6 + \frac{4}{7} u_1 u_2 u_2 + \frac{4}{7} u_2 u_1 u_2 + \frac{4}{7} u_3 u_3 - \frac{2}{7} u_3^2 u_2 - \frac{2}{7} u_3^3 - \frac{1}{2} u_3^2 + \frac{2}{7} u_3^5 - \frac{2}{7} u_3^6 + \frac{1}{2} u_2 u_2^2. \]

and \( \Gamma^i_{jk} \) are the Christoffel symbols (7.1) with \( \epsilon = 1 \). The 1-forms \( \omega^{(\rho \alpha)} \) defining the principal hierarchy can be obtained starting from the differentials of the Casimirs of \( P^i = g^{ij} \delta'(x-y) - g^{ij} \Gamma^i_{jk} \delta(x-y) \) and solving the recursive relations

\[ \nabla_i \omega^{(\rho \alpha + 1)}_m = g_{ik} \epsilon^l_{m} g^{lm} \omega^{(\rho \alpha)}. \]

For instance, if we take the counity \( \rho^{(1,0)} = 3d_3 \):

\[ \omega_1^{(1,0)} = -\frac{3}{2} \frac{1}{(u_1 - u_2)^2 (u_2 - u_3)} \]
\[ \omega_2^{(1,0)} = -\frac{3}{2} \frac{u_1 - 2u_2 + u_3}{(u_1 - u_2)^2 (u_2 - u_3)^2} \]
\[ \omega_3^{(1,0)} = \frac{3}{2} \frac{1}{(u_1 - u_2) (u_2 - u_3)^2} \]

it is easy to check that the 1-form \( \omega^{(1,1)} \) has components:

\[ \omega_1^{(1,1)} = -\frac{1}{4(u_1 - u_2)^3 (u_1 - u_3)^2 (u_2 - u_3)} \left\{ 4u_2 u_1^4 + 2u_1 u_3^4 - 14u_2 u_2 u_3^3 - 4u_2^2 u_1^3 \right. \]
\[ - 6u_2^2 u_1^3 + 12u_2^2 u_1 u_3^2 + 18u_2 u_3^2 u_1^2 + 6u_3^2 u_1^2 + 2u_1 u_2 - 12u_3^2 u_1 - 2u_1 u_2 \]
\[ - 2u_1 u_3 - 10u_2 u_2 u_3 u_1 - u_2^2 + u_3^2 + 4u_2^2 u_1^3 + 2u_2 u_3 \}, \]
\[ \omega_2^{(1,1)} = -\frac{1}{4(u_1 - u_2)^3 (u_1 - u_3)^2 (u_2 - u_3)^2} \left\{ 2u_1^5 - 6u_2 u_1^4 + 2u_1 u_3^4 + 2u_2 u_3 u_1^3 + 2u_2^2 u_3^3 \right. \]
\[ - 8u_2^2 u_1^3 + 12u_2 u_2 u_1^3 + 2u_2 u_1 u_3^2 + 12u_2 u_1 u_3 - 6u_2 u_3^2 u_1 - 6u_3^2 u_2 u_1 - 4u_2^2 u_3 u_1 \]
\[ - 2u_1 u_4 + 4u_2 u_1 u_3^2 - 2u_3 u_2 - 2u_2 u_3 - u_2^2 + 2u_3^2 + 2u_2^2 u_1^3 \}, \]
\[ \omega_3^{(1,1)} = \frac{1}{4(u_1 - u_2)^3 (u_1 - u_3)^2 (u_2 - u_3)^2} \left\{ 2u_1^5 - 6u_3 u_1^4 + 2u_1 u_3^4 + 4u_2^2 u_3 u_1^3 + 6u_2^2 u_2^3 \right. \]
\[ - 6u_2 u_1 u_3^2 - 12u_2 u_2 u_1 u_3 + 6u_2 u_3^2 u_1 - 12u_2^2 u_1 u_3 - 6u_2^2 u_3 u_1 - 2u_2 u_3 \]
\[ + u_2^2 + u_3^2 + 4u_2^2 u_3^3 \}. \]

The associated flow is the first primary flow:

\[ \frac{\partial u_i}{\partial t} = \frac{\partial u_i}{\partial x}, \quad i = 1, \ldots, 3. \]
In the same way, we obtain the components of the 1-form $\omega^{(1,2)}$:

$$
\omega_1^{(1,2)} = \frac{1}{4(u_1 - u_3)^3(u_1 - u_2)^3(u_2 - u_3)} \left\{ 2u_1^6 - 6u_2u_1^5 - 6u_3u_1^5 + 14u_2u_1^3u_3 + 4u_2^2u_1^4 \\
+ 6u_2^2u_1^4 - 6u_2u_1^3u_2 - 8u_2u_1^2u_2^2 - 6u_3u_2u_1^5 - 2u_4u_1u_2^6 + 2u_3u_1^2 + 8u_2u_3u_1^2u_2 + 2u_2^2u_1^3u_3 \right\},
$$

$$
\omega_2^{(1,2)} = -\frac{1}{4(u_1 - u_3)^2(u_1 - u_2)^3(u_2 - u_3)^2} \left\{ 4u_1u_1^5 - 8u_2u_1^4u_3 - 4u_3u_1^4 - 2u_2^2u_1^4 + 2u_3u_1^3 \\
+ 6u_2u_1^3u_2 + 12u_2^2u_1^3u_2^2 - 4u_2u_1^2u_3^2 - 6u_2^2u_3u_2^2u_3^2 - 2u_2u_3u_1^5 - 4u_2u_1u_2^6 + 2u_2u_3u_1^2u_2 + 2u_2^2u_1^3u_3 \\
- 2u_3^3u_1^3u_2 - u_2^3u_2^2 - u_3^3 + 2u_3^3 + u_2^3u_2^2 \right\},
$$

$$
\omega_3^{(1,2)} = \frac{1}{2(u_1 - u_3)^3(u_1 - u_2)^2(u_2 - u_3)^3} \left\{ 2u_1^5 - 6u_2u_1^4 + 2u_2u_1^4 - 4u_2^3u_3^3 + 6u_3u_1^5 - 6u_2u_2u_3u_1 \\
- 2u_3^2u_2^2 + 12u_2^3u_3^2u_2^2 - 6u_2^3u_2^2u_3 - 12u_2u_3u_1^3u_2 - 2u_2^2u_3u_1^2u_2 - 2u_2u_3 + u_2^3 + u_2^3 + 4u_2^3u_3 \right\},
$$

and the associated flow

$$
\frac{\partial u_1}{\partial t} = (u_2 + u_3) \frac{\partial u_1}{\partial x}, \\
\frac{\partial u_2}{\partial t} = (u_1 + u_3) \frac{\partial u_2}{\partial x}, \\
\frac{\partial u_3}{\partial t} = (u_1 + u_2) \frac{\partial u_3}{\partial x}.
$$

Higher flows can be obtained analogously.

**Remark 7.1.** As can be observed from the example above, even if the starting points of the hierarchy are the differentials of the Casimirs of $P$, hence exact 1-forms, the iterative procedure leads immediately to consider non-exact 1-forms. Indeed, in the example, $\omega^{(1,1)}$ is exact and $\omega^{(1,2)}$ is not exact. The reason is that in the first case ($p = 1, \alpha = 1$) the r.h.s. of (6.3) vanishes while for different values of $p$ and $\alpha$, including $p = 1, \alpha = 2$ it does not vanish.

8. Conclusions

In this paper, we have introduced for the first time Poisson brackets on 1-form, within the formalism of Dubrovin–Novikov. While it is certainly in principle possible to dismiss our generalization and instead use the nonlocal Poisson formalism, if one is interested in obtaining dispersive deformations of PDEs starting from dispersionless ones, it seems very difficult to extend the results obtained in the framework of local Poisson formalism (see [6]) to the framework of the nonlocal Poisson formalism (with the exception of non-local structures considered by Liu and Zhang [12]). This for two reasons: first of all because the computations in the non-local framework involve ill-defined operators like $\partial^{-1}$, and second because the results of [9] about the triviality of Poisson’s cohomology are no longer available. Therefore, in this more general setting the approach based on the deformations of 1-forms seems more suitable and might be a valid alternative to more classical approaches based on deformations of symmetries (see for instance [18] in the case of the Hopf equation).
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References

[1] Arsie A and Lorenzoni P 2012 $F$-manifolds with eventual identities, bidifferential calculus and twisted Lenard–Magri chains Int. Math. Res. Not. at press (doi:10.1093/imrn/rns172)
[2] Arsie A and Lorenzoni P 2012 From Darboux–Egorov system to bi-flat $F$-manifolds arXiv:1205.2468
[3] Dedecker P and Tulczyjew W M 1980 Spectral sequences and the inverse problem of the calculus of variations Lecture Notes Math. 836 498–503
[4] Dubrovin B A and Novikov S P 1984 On Hamiltonian brackets of hydrodynamic type Sov. Math. Dokl. 30 651–4
[5] Dubrovin B A 1996 Geometry of 2D topological field theories Integrable Systems and Quantum Groups (Lecture Notes in Mathematics vol 1620) ed M Francaviglia and S Greco (Berlin: Springer) pp 120–348
[6] Dubrovin B and Zhang Y 2001 Normal forms of integrable PDEs, Frobenius manifolds and Gromov–Witten invariants arXiv:math.DG/0108160
[7] Ferapontov E V 1991 Differential geometry of nonlocal Hamiltonian operators of hydrodynamic type Funct. Anal. Appl. 25 195–204
[8] Ferapontov E V and Mokhov O I 1990 Nonlocal Hamiltonian operators of hydrodynamic type that are connected with metrics of constant curvature Russ. Math. Surv. 45 218–9
[9] Getzler E 2002 A Darboux theorem for Hamiltonian operators in the formal calculus of variations Duke Math. J. 111 535–60
[10] Gel'fand I M and Dorfman I A 1982 Hamiltonian operator and the classical Yang–Baxter equation Funct. Anal. Appl. 16 241–8
[11] Hertling C and Manin Y 1999 Weak Frobenius manifolds Int. Math. Res. Not. 1999 277–86
[12] Liu S Q and Zhang Y 2011 Jacobi structures of evolutionary partial differential equations Adv. Math. 227 73–130
[13] Lorenzoni P, Pedroni M and Raimondo A 2011 $F$-manifolds and integrable systems of hydrodynamic type Arch. Math. 47 163–80
[14] Lorenzoni P and Pedroni M 2011 Natural connections for semi-Hamiltonian systems: the case of the $\epsilon$-system Lett. Math. Phys. 97 85–108
[15] Manin Y 2005 $F$-manifolds with flat structure and Dubrovin’s duality Adv. Math. 198 5–26
[16] Magri F and Morosi C 1984 A geometrical characterization of integrable Hamiltonian systems through the theory of Poisson–Nijenhuis manifolds Quaderno S19 (Milan: University of Milan)
[17] Olver P 2000 Applications of Lie Groups to Differential Equations 2nd edn (Berlin: Springer)
[18] Strachan I A B 2003 Deformations of the Monge/Riemann hierarchy and approximately integrable systems J. Math. Phys. 44 251–62
[19] Tsarev S P 1991 The geometry of Hamiltonian systems of hydrodynamic type. The generalised hodograph transform USSR Izv. 37 397–419