Pointwise convergence problem of Ostrovsky equation with rough data and random data

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Abstract. In this paper, we consider the pointwise convergence problem of free Ostrovsky equation with rough data and random data. Firstly, we show the almost everywhere pointwise convergence of free Ostrovsky equation in $H^s(\mathbb{R})$ with $s \geq \frac{1}{4}$ with rough data. Secondly, we present counterexamples showing that the maximal function estimate related to the free Ostrovsky equation can fail if $s < \frac{1}{4}$. Finally, for every $x \in \mathbb{R}$, we show the almost surely pointwise convergence of free Ostrovsky equation in $L^2(\mathbb{R})$ with random data. The main tools are the density theorem, high-low frequency idea, Wiener decomposition and Lemmas 2.1-2.6 as well as the probabilistic estimates of some random series which are just Lemmas 3.2-3.4 in this paper. The main difficulty is that zero is the singular point of the phase functions of free Ostrovsky equation. We use high-low frequency idea to conquer the difficulties.

Keywords: Pointwise convergence; Free Ostrovsky equation; Rough data; Random data

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1. Introduction

Ostrovsky equation

\[ u_t + \partial_x^3 u \pm \partial_x^{-1} u + uu_x = 0. \]  

(1.1)

was proposed by Ostrovsky [27, 28, 59] as a model for weakly nonlinear long waves in a rotating liquid, by taking into account of the Coriolis force. It describes the propagation of surface waves in the ocean in a rotating frame of reference. Some people have investigated the Cauchy problem for the Ostrovsky equation [15, 29, 30, 32, 34–38, 45, 46, 64, 69, 70].

In this paper, we investigate the pointwise convergence problem of the free Ostrovsky equation

\[ u_t + \partial_x^3 u \pm \partial_x^{-1} u = 0, \]  

(1.2)

\[ u(x, 0) = f(x). \]  

(1.3)

The pointwise converge problem was initiated by Carleson [11], more precisely, Carleson showed pointwise convergence problem of the one dimensional Schrödinger equation in \( H^s(\mathbb{R}) \) with \( s \geq \frac{1}{4} \). Dahlberg and Kenig [19] showed that the pointwise convergence of the Schrödinger equation does not hold for \( s < \frac{1}{4} \) in any dimension. Dahlberg and Kenig [19] and Kenig et al. [39, 40] have proved that the pointwise convergence problem of KdV equation holds if and only if \( s \geq \frac{1}{4} \). The pointwise convergence problem of Schrödinger equations in higher dimension have been investigated by some authors, for example, see [4, 7, 14, 18, 20, 25, 26, 43, 47, 48, 52, 62, 63, 65, 66, 68, 71]. Bourgain [8] presented counterexamples showing that \( s < \frac{n}{2(n+1)}, n \geq 2 \) is the necessary condition for the pointwise convergence problem of \( n \) dimensional Schrödinger. Du et al. [23] proved that the pointwise convergence problem of two dimensional Schrödinger equation in \( H^s(\mathbb{R}^2) \) with \( s > \frac{1}{3} \). Du and Zhang [24] proved that the pointwise convergence problem of \( n \) dimensional Schrödinger equation in \( H^s(\mathbb{R}^n) \) with \( s > \frac{n}{2(n+1)}, n \geq 3 \). Thus, \( \frac{n}{2(n+1)}, n \geq 2 \) is optimal for the pointwise convergence problem of the Schrödinger equation.

Miao et al. [50, 51] studied the the maximal inequality for 2D fractional order Schrödinger operators and maximal estimates for Schrödinger equation with inverse-square potential. Lee [44] showed that the local Kato type smoothing estimates are essentially equivalent to the global Kato type smoothing estimates for some class of dispersive equations including the Schrödinger equation.
Very recently, Compaan et al. [16] applied randomized initial data to study pointwise convergence of the Schrödinger flow. The method of the suitably randomized initial data originated from Lebowitz-Rose-Speer [42] and Bourgain [5, 6] and Burq-Tzvetkov [9, 10]. Some authors applied the method to study nonlinear dispersive equations and hyperbolic equations in scaling super-critical regimes, for example, see [1–3, 12, 13, 17, 20, 21, 31, 33, 41, 49, 53–58, 61, 72, 73].

In this paper, motivated by [16, 22, 39, 40], we investigate the pointwise convergence problem of free Ostrovsky equation with rough data and random data. Firstly, we show the pointwise convergence problem of free Ostrovsky equation with data in \( H^s(\mathbb{R}) \) \((s \geq \frac{1}{4})\). Secondly, we present the counterexample showing that the maximal function estimate related to Ostrovsky equation does not hold if \( s < \frac{1}{4} \). Finally, for every \( x \in \mathbb{R} \), we show the almost surely pointwise convergence of free Ostrovsky equation in \( L^2(\mathbb{R}) \) with random data. Note that, the phase function of free Ostrovsky equation is \( \xi^3 \pm \frac{1}{\xi} \), hence, zero is the singular point of the phase functions of free Ostrovsky equation, which caused the main difficulty of this paper. We use high-low frequency idea to conquer the difficulties.

We present some notations before stating the main results. \( |E| \) denotes by the Lebesgue measure of set \( E \).

\[
\mathcal{F}_x f(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi} f(x) dx,
\]

\[
\mathcal{F}_x^{-1} f(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix\xi} f(x) dx,
\]

\[
U(t) u_0 = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix\xi + it(\xi^3 \pm \frac{1}{\xi})} \mathcal{F}_x u_0(\xi) d\xi,
\]

\[
D_t^\alpha U(t) u_0 = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \left| \xi^\alpha \right| e^{i\xi^3 \pm i\xi} \mathcal{F}_x u_0(\xi) d\xi,
\]

\[
D_x^\alpha u_0 = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \left| \xi \right|^{\alpha} e^{i\xi^3} \mathcal{F}_x u_0(\xi) d\xi,
\]

\[
\| f \|_{L^q_x L^p_t} = \left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |f(x, t)|^p dt \right)^{\frac{q}{p}} dx \right)^{\frac{1}{q}},
\]

\[
H^s(\mathbb{R}) = \left\{ f \in H^s(\mathbb{R}) \mid f \in \mathcal{S}(\mathbb{R}) : \| f \|_{H^s(\mathbb{R})} = \| \langle \xi \rangle^s \mathcal{F}_x f \|_{L^2(\mathbb{R})} < \infty \right\}, \text{ where } \langle \xi \rangle^s = (1 + \xi^2)^{\frac{s}{2}} \text{ for any } \xi \in \mathbb{R}. \]

Let \( \phi \) be a smooth bump function such that \( \phi(\xi) = 1 \) for
\[ \xi \leq 1 \text{ and } \phi(\xi) = 0 \text{ for } |\xi| > 2. \] Then, we define for every dyadic integer \( N \in 2\mathbb{Z} \),

\[
\mathcal{F}_x P_N f(\xi) = \left[ \phi\left(\frac{\xi}{N}\right) - \phi\left(\frac{2\xi}{N}\right) \right] \mathcal{F}_x f(\xi),
\]

\[
\mathcal{F}_x P_{\leq N} f(\xi) = \phi\left(\frac{\xi}{N}\right) \mathcal{F}_x f(\xi),
\]

\[
\mathcal{F}_x P_{\geq N} f(\xi) = \left[ 1 - \phi\left(\frac{\xi}{N}\right) \right] \mathcal{F}_x f(\xi).
\]

Now we introduce the randomization procedure for the initial data, which can be seen in [1, 2, 46, 73]. Let \( \psi \in \mathcal{S}(\mathbb{R}) \) be an even, non-negative bump function with \( \text{supp}(\psi) \subseteq [0, 1] \) and such that

\[
\sum_{k \in \mathbb{Z}} \psi(\xi - k) = 1, \xi \in \mathbb{R}. \quad (1.4)
\]

For every \( k \in \mathbb{Z} \), we define the function \( \psi(D - k)f : \mathbb{R} \to \mathbb{C} \) by

\[
(\psi(D - k)f)(x) = \mathcal{F}^{-1}\left(\psi(\xi - k)\mathcal{F}f\right)(x), \ x \in \mathbb{R}.
\]

We will crucially exploit that these projections satisfy a unit-scale Bernstein inequality, namely that for all \( 2 \leq p_1 \leq p_2 \leq \infty \), there exists \( C \equiv C(p_1, p_2) > 0 \) such that for all \( f \in L^2(\mathbb{R}) \) and for all \( k \in \mathbb{Z} \),

\[
\|\psi(D - k)f\|_{L^{p_2}(\mathbb{R})} \leq C \|\psi(D - k)f\|_{L^{p_1}(\mathbb{R})} \leq C \|\psi(D - k)f\|_{L^2(\mathbb{R})}. \quad (1.5)
\]

Let \( \{g_k\}_{k \in \mathbb{Z}} \) be a sequence of independent, zero-mean, complex-valued Gaussian random variables on a probability space \( (\Omega, \mathcal{A}, \mathbb{P}) \), where the real and imaginary parts of \( g_k \) are independent and endowed with probability distributions \( \mu^1_k \) and \( \mu^2_k \) respectively. Assume that there exists \( c > 0 \) such that

\[
\left| \int_{-\infty}^{+\infty} e^{\gamma x} d\mu^j_k(x) \right| \leq e^{c\gamma^2}, \quad (1.6)
\]

for all \( \gamma \in \mathbb{R}, k \in \mathbb{Z}, j = 1, 2 \). Thereafter for a given \( f \in H^s(\mathbb{R}) \), we define its randomization by

\[
f^\omega := \sum_{k \in \mathbb{Z}} g_k(\omega)\psi(D - k)f. \quad (1.7)
\]

We define

\[
\|f\|_{L^p_\omega(\Omega)} = \left[ \int_{\Omega} |f(\omega)|^p dP(\omega) \right]^{\frac{1}{p}}.
\]
Obviously, $\|f^\omega\|_{H^s} = \|f\|_{H^s}$. We will restrict ourselves to a subset $\sum \subset \Omega$ with $P(\sum) = 1$ such that $f^\omega \in H^s$ for every $\omega \in \Omega$. It is easy to see that, if $f \in H^s(\mathbb{R})$, then the randomized function $f^\omega$ is almost surely in $H^s(\mathbb{R})$, see Lemma 2.2 in [2]. This randomization improved the integrability of $f$, see Lemma 2.3 of [2]. Such results for random Fourier series are known as Paley-Zygmund’s theorem [60].

**Theorem 1.1.** Let $f \in H^s(\mathbb{R})$ with $s \geq \frac{1}{4}$. Then, we have

$$\lim_{t \to 0} U(t)f(x) = f(x)$$

almost everywhere.

**Theorem 1.2.** The inequality

$$\|U(t)f\|_{L^1L^\infty} \leq C\|f\|_{H^s(\mathbb{R})}$$

does not hold if $s < \frac{1}{4}$.

**Theorem 1.3.** Let $f \in L^2(\mathbb{R})$ and $f^\omega$ be a randomization of $f$ as defined in (1.7). Then, for every $x \in \mathbb{R}$, we have

$$U(t)f^\omega(x) \longrightarrow f^\omega(x)$$

almost surely as $t \to 0$. More precisely, $\forall \epsilon > 0$, $f \in L^2(\mathbb{R})$, $\alpha = 2C\epsilon \left[\ln \frac{C_2}{\epsilon}\right]^\frac{1}{2}$, when $|t| < \delta\epsilon$, there exists a set $\omega \in E^c_\alpha \subset \Omega$ such that $\forall \omega \in E^c_\alpha$

$$|U(t)f^\omega - f^\omega| < 2C\epsilon \ln \left[\frac{C_2}{\epsilon}\right]^\frac{1}{2}. \quad (1.11)$$

Here,

$$E_\alpha = \{\omega \in \Omega : |U(t)f^\omega - f^\omega| > \alpha\}.$$ \quad (1.12)

and $\mathbb{P}(E^c_\alpha) \geq 1 - \epsilon$.

**Remark 1.** From Theorem 1.1-1.3, we know that $s = \frac{1}{4}$ is optimal for the pointwise convergence problem of Ostrovsky equation with rough data, and the pointwise convergence problem of Ostrovsky equation with random data requires less regularity of the initial data than the pointwise convergence problem of Ostrovsky equation with rough data. Obviously,

$$\lim_{\epsilon \to 0} \alpha = \lim_{\epsilon \to 0} 2C\epsilon \left[\ln \frac{C_2}{\epsilon}\right]^\frac{1}{2} = 0 \quad (1.13)$$
and $\alpha = o(\epsilon^4)$.

**Remark 2.** Firstly, we present the outline of the proof of Theorem 1.1. We use the high-low frequency idea to overcome the main difficult case by the phase function $\xi^3 \pm \frac{1}{\xi}$ of the Ostrovsky equation, which caused that zero is the singular point. More precisely, by density theorem which is just Lemma 2.2 in [22], for $f \in H^s(\mathbb{R})$ with $s \geq \frac{1}{4}$, we have $f = g + h$, where $g$ is a rapidly decreasing function, $\|h\|_{H^s(\mathbb{R})} < \epsilon$. Then we need to prove

$$|U(t)g - g| \rightarrow 0, \quad (1.14)$$

and

$$|U(t)h - h| \rightarrow 0 \quad (1.15)$$

respectively, as $t$ goes to zero.

However, due to the complicated structure of the phase function of the Ostrovsky equation, we can not completely follow the method of Lemma 2.3 in [22] to prove (1.14). We use high-low frequency idea to conquer the difficulties. More precisely, we prove

$$|U(t)P_{\geq 8g} - P_{\geq 8g}| \rightarrow 0$$

and

$$|U(t)P_{\leq 8g} - P_{\leq 8g}| \rightarrow 0$$

as $t$ goes to zero since

$$U(t)g - g = U(t)P_{\geq 8g} - P_{\geq 8g} + U(t)P_{\leq 8g} - P_{\leq 8g}. \quad (1.16)$$

Following the method of Lemma 2.3 in [22], we prove

$$|U(t)P_{\geq 8g} - P_{\geq 8g}| \rightarrow 0$$

as $t$ goes to zero since $|\xi^3 \pm \frac{1}{\xi}| \sim |\xi|^3$ for $|\xi| \geq 8$. For the detail of (1.16), we refer the readers to Lemma 2.3 in this paper.

And then, by using a delicate analysis, we establish

$$|U(t)P_{\leq 8g} - P_{\leq 8g}| \rightarrow 0 \quad (1.17)$$

as $t$ goes to zero with the aid of Lemma 2.2 established in this paper. More precisely, $\forall \epsilon > 0$, there exists $\delta > 0$ such that

$$|U(t)P_{\leq 8g} - P_{\leq 8g}| \leq \epsilon + \frac{C|t|}{\delta}. \quad (1.18)$$
Similarly, due to the complicated structure of the Ostrovsky equation, we also can not completely follow the method of Lemma 2.3 in [39, 40] to prove (1.15).

Since we can prove
\[ \|U(t)P_{\geq s}h\|_{L^4_tL^\infty_x} \leq C\|h\|_{H^4_t(R)}, \]
which is just Lemma 2.1 in this paper, following the method of Lemma 2.3 in [39, 40], we prove
\[ |U(t)P_{\geq s}h - P_{\geq s}h| \to 0 \]
as \( t \) goes to zero.

By using Lemma 2.2 in this paper, we also can prove
\[ |U(t)P_{\leq s}h - P_{\leq s}h| \leq \epsilon + \frac{C|t|\|h\|_{H^4_t(R)}}{\delta} \leq \epsilon + \frac{C|t|\epsilon}{\delta}. \]

From (1.17)-(1.21), as \( t \to 0 \), we have
\[ U(t)f \to f. \]
This completes the proof of Theorem 1.1.

Secondly, for the proof of Theorem 1.2, by presenting particular initial data, we can prove that the maximal function inequality is invalid for \( s < \frac{1}{4} \).

Finally, we present the proof of Theorem 1.3. More precisely, for \( f \in L^2(R) \) and \( \forall \epsilon > 0 \), since rapidly decreasing functions are dense in \( L^2(R) \), we write \( f = g + h \), where \( g \) is a rapidly decreasing function and \( \|u\|_{L^2(R)} < \epsilon \). Obviously, \( f^\omega = g^\omega + h^\omega \), then we get
\[ U(t)f^\omega \to f. \]

Here, \( f^\omega \) is defined as in (1.7). By using Lemma 3.1 and high-low frequency technique, since \( g \) is a rapidly decreasing function, we have
\[ \mathbb{P}\left( \left\{ \omega \in \Omega : |U(t)g^\omega - g^\omega| > \frac{\alpha}{2} \right\} \right) \leq C_1 e^{-\left(\frac{\alpha}{C_C^2\|h\|_{L^2}}\right)^2}. \]

For the details of proof, we refer the readers to Lemma 3.2. By using Lemmas 3.1, 2.6, we have
\[ \mathbb{P}\left( \left\{ \omega \in \Omega : |U(t)h^\omega| > \frac{\alpha}{4} \right\} \right) \leq C_1 e^{-\left(\frac{\alpha}{C_C\|h\|_{L^2}}\right)^2} \leq C_1 e^{-\left(\frac{\alpha}{C_C}\right)^2}. \]
For the details of proof, we refer the readers to Lemma 3.3. By using Lemmas 3.1, 2.5, we have
\[
P \left( \{ \omega \in \Omega : |h\omega| > \frac{\alpha}{4} \} \right) \leq C_1 e^{-\left(\frac{\alpha}{c|\xi|}x_2^2\right)} \leq C_1 e^{-\left(\frac{\alpha}{c\epsilon}\right)^2}. \tag{1.26}
\]
Thus, combining (1.23) with (1.24)-(1.26), when \(|t| < \delta\epsilon\), \(\alpha = 2Ce\epsilon \left[\ln\frac{3C_1}{\epsilon}\right]^\frac{1}{2}\), \(\forall x \in \mathbb{R}\), we have
\[
P \left( \{ \omega \in \Omega : |U(t)P_{\geq s}f - f\omega| > \alpha \} \right)
\leq P \left( \{ \omega \in \Omega : |U(t)g\omega - g\omega| > \frac{\alpha}{2} \} \right) + P \left( \{ \omega \in \Omega : |U(t)h\omega| > \frac{\alpha}{4} \} \right)
\leq C_1 e^{-\left(\frac{\alpha}{c|\xi|e}\right)^2} + 2C_1 e^{-\left(\frac{\alpha}{c\epsilon e}\right)^2}
\leq C_1 e^{-\left(\frac{\alpha}{c\epsilon e}\right)^2} + 2C_1 e^{-\left(\frac{\alpha}{c\epsilon e}\right)^2} \leq 3C_1 e^{-\left(\frac{\alpha}{c\epsilon e}\right)^2} \leq \epsilon. \tag{1.27}
\]
This completes the proof of Theorem 1.3.

2. Preliminaries

In this section, we present some preliminaries related to Ostrovsky equation. More precisely, Lemmas 2.1-2.3 are used to establish Theorem 1.1. Lemmas 2.4-2.6 are used to establish Theorem 1.3.

**Lemma 2.1.** (Strichartz estimate related to Ostrovsky equation) For \(f \in H^{\frac{1}{4}}(\mathbb{R})\), we have
\[
\|U(t)P_{\geq s}f\|_{L_t^4 L_x^\infty} \leq C \|f\|_{H^{\frac{1}{4}}(\mathbb{R})}. \tag{2.1}
\]
For the proof of Lemma 2.1, we refer the readers to (2.2) of [30].

**Lemma 2.2.** (Estimate related to Ostrovsky equation with low frequency) \(\forall \epsilon > 0\) and \(g \in L^2(\mathbb{R})\), there exists \(\delta > 0\) such that
\[
|U(t)P_{\leq s}g - P_{\leq s}g| \leq \epsilon + \frac{C|t|}{\delta} \|g\|_{L^2(\mathbb{R})}. \tag{2.2}
\]
**Proof.** \(\forall \epsilon > 0\), since \(g \in L^2(\mathbb{R})\), there exists \(\delta > 0(< \frac{1}{2})\) such that
\[
\left[ \int_{|\xi| \leq \delta} |\mathcal{F}_x g(\xi)|^2 d\xi \right]^\frac{1}{2} \leq \epsilon. \tag{2.3}
\]
By using the Cauchy-Schwartz inequality and (2.3), we have
\[
\int_{|\xi| \leq \delta} |\mathcal{F}_x g(\xi)| d\xi = \left[ \int_{|\xi| \leq \delta} |\mathcal{F}_x g(\xi)|^2 d\xi \right]^\frac{1}{2} (2\delta)^\frac{1}{2} \leq \epsilon. \tag{2.4}
\]
By using the Cauchy-Schwartz inequality, we have
\[
\int_{|\xi| \leq 8} |\mathcal{F}_x g(\xi)|d\xi \leq \left[ \int_{|\xi| \leq 8} |\mathcal{F}_x g(\xi)|^2 d\xi \right]^{\frac{1}{2}} \left[ \int_{|\xi| \leq 8} d\xi \right]^{\frac{1}{2}}.
\]
\[
\leq 3 \left[ \int_{|\xi| \leq 8} |\mathcal{F}_x g(\xi)|^2 d\xi \right]^{\frac{1}{2}} \leq 3 \| g \|_{L^2(\mathbb{R})}. \tag{2.5}
\]

For \( \delta \leq |\xi| \leq 8 \), we have
\[
|e^{it(\xi^3 + \frac{1}{\xi})} - 1| \leq |t| \left| \xi^3 \pm \frac{1}{\xi} \right| \leq \frac{C|t|}{|\xi|} \leq \frac{C|t|}{\delta}. \tag{2.6}
\]

Thus, from (2.3)-(2.6), we have
\[
|U(t)P_{\leq 8g} - P_{\leq 8g}| = \left| \int_{|\xi| \leq 8} e^{ix\xi} \left[ e^{it(\xi^3 + \frac{1}{\xi})} - 1 \right] \mathcal{F}_x g(\xi) d\xi \right|
\]
\[
\leq \left| \int_{|\xi| \leq \delta} e^{ix\xi} \left[ e^{it(\xi^3 + \frac{1}{\xi})} - 1 \right] \mathcal{F}_x g(\xi) d\xi \right|
\]
\[
+ \left| \int_{\delta \leq |\xi| \leq 8} e^{ix\xi} \left[ e^{it(\xi^3 + \frac{1}{\xi})} - 1 \right] \mathcal{F}_x g(\xi) d\xi \right|
\]
\[
\leq \int_{|\xi| \leq \delta} |\mathcal{F}_x g(\xi)| d\xi + C|t| \int_{\delta \leq |\xi| \leq 8} \frac{1}{|\xi|} |\mathcal{F}_x g(\xi)| d\xi
\]
\[
\leq \epsilon + \frac{C|t|}{\delta} \int_{|\xi| \leq 8} |\mathcal{F}_x g(\xi)| d\xi
\]
\[
\leq \epsilon + \frac{C|t|}{\delta} \| g \|_{L^2}. \tag{2.7}
\]

This completes the proof of Lemma 2.2.

**Lemma 2.3.** *(Estimate related to Ostrovsky equation with high frequency)* Let \( g \) be a rapidly decreasing function. Then, we have
\[
|U(t)P_{\geq 8g} - P_{\geq 8g}| \leq C|t|. \tag{2.8}
\]

**Proof.** Since \( g \) is a rapidly decreasing function, we have
\[
|U(t)P_{\geq 8g} - P_{\geq 8g}| = \left| \int_{|\xi| \geq 8} e^{ix\xi} \left[ e^{it(\xi^3 + \frac{1}{\xi})} - 1 \right] \mathcal{F}_x g(\xi) d\xi \right|
\]
\[
\leq \left| \int_{|\xi| \geq 8} e^{ix\xi} \left[ e^{it(\xi^3 + \frac{1}{\xi})} - 1 \right] \mathcal{F}_x g(\xi) d\xi \right|
\]
\[
\leq C|t| \int_{|\xi| \geq 8} \left| \xi^3 \pm \frac{1}{\xi} \right| |\mathcal{F}_x g(\xi)| d\xi
\]
\[
\leq C|t| \int_{|\xi| \geq 8} |\xi|^3 |\mathcal{F}_x g(\xi)| d\xi \leq C|t|. \tag{2.9}
\]

This completes the proof of Lemma 2.3.
Lemma 2.4. *(Estimates related to frequency-uniform decomposition)* \( \forall \epsilon > 0, \ |k| \leq 8 \) and let \( g \) be a rapidly decreasing function, there exists \( \delta > 0 (< \frac{1}{2}) \) such that
\[
|U(t)\psi(D - k)g - \psi(D - k)g| \leq \epsilon + \frac{C|t|}{\delta}.
\]

**Proof.** \( \forall \epsilon > 0 \), there exists \( \delta > 0 \) such that
\[
\left[ \int_{|\xi| \leq \delta} |\xi|^{2\epsilon} |\mathcal{F}_x g(\xi)|^2 d\xi \right]^\frac{1}{2} \leq \frac{\epsilon}{2}.
\]

Thus, by using the Cauchy-Schwartz inequality, we have
\[
\int_{|\xi| \leq \delta} |\mathcal{F}_x g(\xi)| d\xi \leq \left[ \int_{|\xi| \leq \delta} |\xi|^{2\epsilon} |\mathcal{F}_x g(\xi)|^2 d\xi \right]^\frac{1}{2} \leq \frac{\epsilon}{2}.
\]

For \( \delta \leq |\xi| \leq 1 \), we have
\[
|e^{it(\xi^3 \pm \frac{1}{\xi})} - 1| \leq |t| \left| \xi^3 \pm \frac{1}{\xi} \right| \leq \frac{C|t|}{|\xi|} \leq \frac{C|t|}{\delta}.
\]

Thus, from (2.12)-(2.13), we have
\[
|U(t)\psi(D - k)f - \psi(D - k)f| \leq \epsilon + \frac{C|t|}{\delta} \int_{\mathbb{R}} |\mathcal{F}_x g(\xi)| d\xi \leq \epsilon + \frac{C|t|}{\delta}.
\]

This completes the proof of Lemma 2.4.

Lemma 2.5. For \( f \in L^2(\mathbb{R}) \), we have
\[
\left[ \sum_{k \in \mathbb{Z}} |\psi(D - k)f|^2 \right]^\frac{1}{2} \leq \|f\|_{L^2(\mathbb{R})}.
\]

**Proof.** To obtain (2.15), it suffices to prove
\[
\sum_{k \in \mathbb{Z}} |\psi(D - k)f|^2 \leq \|f\|_{L^2(\mathbb{R})}^2.
\]
By using the Cauchy-Schwarz inequality with respect to $\xi$, since $\text{supp} \psi \subset B(0,1)$ we have

\[
\sum_{k \in \mathbb{Z}} |\psi(D - k)f|^2 = \frac{1}{(2\pi)^n} \sum_{k \in \mathbb{Z}} \left| \int_{\mathbb{R}} e^{i \sum_{j=1}^{n} x_j \xi_j} \psi(\xi - k) \mathcal{F}_x f(\xi) d\xi \right|^2
\]

\[
= \frac{1}{(2\pi)^n} \sum_{k \in \mathbb{Z}} \left| \int_{|\xi - k| \leq 1} e^{i \sum_{j=1}^{n} x_j \xi_j} \psi(\xi - k) \mathcal{F}_x f(\xi) d\xi \right|^2
\]

\[
\leq \left[ \sum_{k \in \mathbb{Z}} \int_{|\xi - k| \leq 1} |\psi(\xi - k) \mathcal{F}_x f(\xi)|^2 d\xi \right] \left[ \int_{|\xi - k| \leq 1} d\xi \right]
\]

\[
\leq \left[ \sum_{k \in \mathbb{Z}} \int_{|\xi - k| \leq 1} |\psi(\xi - k) \mathcal{F}_x f(\xi)|^2 d\xi \right]
\]

\[
= \sum_{k \in \mathbb{Z}} \|\psi(\xi - k) \mathcal{F}_x f(\xi)\|_{L^2}^2. \quad (2.17)
\]

From

\[
\mathcal{F}_x f(\xi) = \sum_{k \in \mathbb{Z}} \psi(\xi - k) \mathcal{F}_x f(\xi), \quad (2.18)
\]

by using the Plancherel identity and $\text{supp} \psi \subset B(0,1)$, we have

\[
\|f\|_{L^2}^2 = \|\mathcal{F}_x f(\xi)\|_{L^2}^2 = \sum_{k \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} \int_{\mathbb{R}} [\psi(\xi - k) \mathcal{F}_x f(\xi)] [\psi(\xi - l) \mathcal{F}_x f(\xi)] d\xi
\]

\[
= \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} |\psi(\xi - k) \mathcal{F}_x f(\xi)|^2 d\xi. \quad (2.19)
\]

Combining (2.17) with (2.19), we derive (2.16).

This completes the proof of Lemma 2.5.

**Lemma 2.6.** For $f \in L^2(\mathbb{R})$, we have

\[
\left[ \sum_{k \in \mathbb{Z}} |\psi(D - k)U(t)f|^2 \right]^\frac{1}{2} \leq \|f\|_{L^2(\mathbb{R})}. \quad (2.20)
\]

**Proof.** To obtain (2.20), it suffices to prove

\[
\sum_{k \in \mathbb{Z}} |\psi(D - k)U(t)f|^2 \leq \|f\|_{L^2(\mathbb{R})}^2. \quad (2.21)
\]
By using the Cauchy-Schwarz inequality with respect to\( \xi \), since\( \text{supp} \psi \subset [0, 1] \), we have

\[
\sum_{k \in \mathbb{Z}} |\psi(D - k)U(t)f|^2 = \frac{1}{(2\pi)^2} \sum_{k \in \mathbb{Z}} \left| \int_{\mathbb{R}} e^{ix\xi} e^{it(\xi^3 + k)} \psi(\xi - k) \mathcal{F}_x f(\xi) d\xi \right|^2
\]

\[
= \frac{1}{(2\pi)^2} \sum_{k \in \mathbb{Z}} \left| \int_{|\xi - k| \leq 1} e^{ix\xi} e^{it(\xi^3 + k)} \psi(\xi - k) \mathcal{F}_x f(\xi) d\xi \right|^2
\]

\[
\leq \left[ \sum_{k \in \mathbb{Z}} \int_{|\xi - k| \leq 1} |\psi(\xi - k) \mathcal{F}_x f(\xi)|^2 d\xi \right] \left[ \int_{|\xi - k| \leq 1} d\xi \right]
\]

\[
\leq \left[ \sum_{k \in \mathbb{Z}} \int_{|\xi - k| \leq 1} |\psi(\xi - k) \mathcal{F}_x f(\xi)|^2 d\xi \right]
\]

\[
= \sum_{k \in \mathbb{Z}} \|\psi(\xi - k) \mathcal{F}_x f(\xi)\|_{L^2}^2 .
\]

(2.22)

Combining (2.19) with (2.22), we derive (2.21).

This completes the proof of Lemma 2.6.

3. Probabilistic estimates of some random series

In this section, we establish the probabilistic estimates of some random series. More precisely, we use Lemmas 2.2, 2.4, 2.5, 3.1 to establish the probabilistic estimates of some random series which are just Lemmas 3.2-3.4 in this paper which play crucial role in establishing Theorem 1.3.

Lemma 3.1. Assume (1.6). Then, there exists\( C > 0 \) such that

\[
\left\| \sum_{k \in \mathbb{Z}} g_k(\omega) c_k \right\|_{L^p(\Omega)} \leq C \sqrt{p} \|c_k\|_{l^2(\mathbb{Z})} .
\]

for all\( p \geq 2 \) and\{c_k\} \in l^2(\mathbb{Z})$.

For the proof of Lemma 3.1, we refer the readers to Lemma 3.1 of [9].

Lemma 3.2. Let\( g \) be is a rapidly decreasing function and we denote by\( g^\omega \) the randomization of\( g \) as defined in (1.7). Then,\( \forall \epsilon > 0 \), there exist\( C > 0, \delta > 0(< 10^{-8}) \) such that

\[
\mathbb{P}(\Omega_1^\epsilon) \leq C_1 e^{-\left( \frac{\alpha}{c\epsilon |\epsilon + \frac{1}{2}|} \right)^2}, \tag{3.1}
\]

where\( \Omega_1^\epsilon = \{ \omega \in \Omega : |U(t)g^\omega - g^\omega| > \alpha \} \).
Proof. Since \([P_{\geq 8} + P_{\leq 8}] f = f\), we have
\[
\|U(t)g^\omega - g^\omega\|_{L^p_\xi(\Omega)} \leq I_1 + I_2, \tag{3.2}
\]
where
\[
I_1 = \|U(t)P_{\geq 8}g^\omega - P_{\geq 8}g^\omega\|_{L^p_\xi(\Omega)}, \quad I_2 = \|U(t)P_{\leq 8}g^\omega - P_{\leq 8}g^\omega\|_{L^p_\xi(\Omega)}.
\tag{3.3}
\]
By using the Cauchy-Schwartz inequality with respect to \(\xi\), since \(g\) is a rapidly decreasing function, we have
\[
I_1 \leq \|U(t)P_{\geq 8}g^\omega - P_{\geq 8}g^\omega\|_{L^p_\xi(\Omega)}
\leq C\sqrt{p} \left[ \sum_{k \in \mathbb{Z}} \left| \int_\mathbb{R} \left( e^{-it\xi^3 + \frac{1}{\xi}} - 1 \right) e^{ix\xi} \psi(\xi - k) F P_{\geq 8} g(\xi) d\xi \right|^2 \right]^{\frac{1}{2}}
\leq C|t|\sqrt{p} \left[ \sum_{k \in \mathbb{Z}} \left| \int_{|\xi - k| \leq 1} \left| \xi^3 + \frac{1}{\xi} \right| \psi(\xi - k) F P_{\geq 8} g(\xi) d\xi \right|^2 \right]^{\frac{1}{2}}
\leq C|t|\sqrt{p} \left[ \sum_{k \in \mathbb{Z}} \left| \int_{|\xi - k| \leq 1} \left| \xi^3 + \frac{1}{\xi} \right|^2 \psi(\xi - k) F P_{\geq 8} g(\xi) d\xi \right| \left| \int_{|\xi - k| \leq 1} d\xi \right| \right]^{\frac{1}{2}}
\leq C|t|\sqrt{p} \left[ \sum_{k \in \mathbb{Z}} \left| \int_{|\xi - k| \leq 1} \left| \xi^6 \psi(\xi - k) F P_{\geq 8} g(\xi) \right|^2 d\xi \right| \right]^{\frac{1}{2}}
\leq C|t|\sqrt{p} \left[ \sum_{k \in \mathbb{Z}} \left| \psi(D - k) P_{\geq 8} g(\xi) \right|_{H^3}^2 \right]^{\frac{1}{2}}
\leq C|t|\sqrt{p} \left\| P_{\geq 8} g \right\|_{H^3} \leq C|t|\sqrt{p}. \tag{3.4}
\]
From Lemma 2.4, we have
\[
I_2 \leq \|U(t)P_{\leq 8}g^\omega - P_{\leq 8}g^\omega\|_{L^p_\xi(\Omega)}
\leq C\sqrt{p} \left[ \sum_{|k| \leq 10} \left| \int_\mathbb{R} \left( e^{-it\xi^3 + \frac{1}{\xi}} - 1 \right) e^{ix\xi} \psi(\xi - k) F P_{\leq 8} g(\xi) d\xi \right|^2 \right]^{\frac{1}{2}}
\leq C\sqrt{p} \left[ \sum_{|k| \leq 10} \left| \varepsilon + \frac{|t|}{\delta} \right|^2 \right]^{\frac{1}{2}} \leq C\sqrt{p} \left[ \varepsilon + \frac{|t|}{\delta} \right]. \tag{3.5}
\]
From (3.2)-(3.5), we have
\[
\|U(t)g^\omega - g^\omega\|_{L^p_\xi(\Omega)} \leq C\sqrt{p} \left[ \varepsilon + \frac{|t|}{\delta} \right]. \tag{3.6}
\]
Thus, from (3.6), by using the Chebyshev inequality, we have
\[
P(\Omega_1^c) \leq \int_{\Omega_1} \left[ C \sqrt{p} \left[ \epsilon + \frac{|t|}{\frac{\delta}{2}} \right] \right] p \ dP(\omega) \leq \frac{\|U(t)g^\omega - g^\omega\|_{L^p}}{\alpha^p}.
\]
(3.7)

Take
\[
p = \left( \frac{\alpha}{Ce \left[ \epsilon + \frac{|t|}{\frac{\delta}{2}} \right]} \right)^2.
\]
(3.8)

If \( p \geq 2 \), we have
\[
P(\Omega_1^c) \leq e^{-p} = e^{-\left( \frac{\alpha}{Ce \left[ \epsilon + \frac{|t|}{\frac{\delta}{2}} \right]} \right)^2}.
\]
(3.9)

If \( p \leq 2 \), we have
\[
P(\Omega_1^c) \leq e^2e^{-2} = C_1 e^{-\left( \frac{\alpha}{Ce \|h\|_{L^2}} \right)^2}.
\]
(3.10)

Here \( C_1 = e^2 \).

This completes the proof of Lemma 3.2.

**Lemma 3.3.** Let \( h \in L^2(\mathbb{R}) \) and we denote by \( h^\omega \) the randomization of \( h \) as defined in (1.7). Then, \( \forall \epsilon > 0 \), there exist \( C > 0 \) and \( C_1 > 0 \) such that
\[
P(\Omega_2^c) \leq C_1 e^{-\left( \frac{\alpha}{Ce \|h\|_{L^2}} \right)^2},
\]
(3.11)

where
\[
\Omega_2^c = \{ \omega \in \Omega : |U(t)h^\omega| > \alpha \}.
\]
(3.12)

**Proof.** By using Lemmas 3.1, 2.6, we have
\[
\|U(t)h^\omega\|_{L^p(\Omega)} = \left\| \sum_{k \in \mathbb{Z}} g_k(\omega)U(t)\psi(D-k)h \right\|_{L^p(\Omega)}
\leq C \sqrt{p} \left[ \sum_{k \in \mathbb{Z}} |U(t)\psi(D-k)h|^2 \right]^{\frac{1}{2}} \leq C \sqrt{p} \|h\|_{L^2}.
\]
(3.13)
Thus, by Chebyshev inequality, from (3.13), we have
\[
\mathbb{P}(\Omega_2^c) \leq \int_{\Omega_2^c} \left[ \frac{|U(t)h^\omega|}{\alpha} \right]^p d\mathbb{P}(\omega) \leq \left( \frac{C\sqrt{p}\|h\|_{L^2}}{\alpha} \right)^p.
\] (3.14)

Take
\[
p = \left( \frac{\alpha}{Ce\|h\|_{L^2}} \right)^2.
\] (3.15)

If \( p \geq 2 \), we have
\[
\mathbb{P}(\Omega_2^c) \leq e^{-p} = e^{-\left( \frac{\alpha}{Ce\|h\|_{L^2}} \right)^2}.
\] (3.16)

If \( p \leq 2 \), we have
\[
\mathbb{P}(\Omega_2^c) \leq e^2e^{-2} = C_1e^{-\left( \frac{\alpha}{Ce\|h\|_{L^2}} \right)^2}.
\] (3.17)

Here \( C_1 = e^2 \).

This completes the proof of Lemma 3.3.

**Lemma 3.4.** Let \( h \in L^2(\mathbb{R}) \) and we denote by \( f^\omega \) the randomization of \( f \) as defined in (1.7). Then, there exist \( C > 0 \) and \( C_1 > 0 \) such that
\[
\mathbb{P}(\Omega_3^c) \leq C_1\exp \left[ \left( \frac{\alpha}{Ce\|h\|_{L^2}} \right)^2 \right].
\] (3.18)

Here
\[
\Omega_3^c = \{ \omega \in \Omega : |h^\omega| > \alpha \}.
\]

**Proof.** By using the Lemmas 3.1, 2.5, we have
\[
\|h^\omega\|_{L^p(\Omega)} = \left\| \sum_{k \in \mathbb{Z}} g_k(\omega)\psi(D - k)h \right\|_{L^p(\Omega)} \leq C\sqrt{p} \left[ \sum_{k \in \mathbb{Z}} |\psi(D - k)h|^2 \right]^{\frac{1}{2}} \leq C\sqrt{p}\|h\|_{L^2}.
\] (3.19)

Thus, by using the Chebyshev inequality, from (3.19), we have
\[
\mathbb{P}(\Omega_3^c) \leq \int_{\Omega_3^c} \left[ \frac{|h^\omega|}{\alpha} \right]^p d\mathbb{P}(\omega) \leq \left( \frac{C\sqrt{p}\|h\|_{L^2}}{\alpha} \right)^p.
\] (3.20)

Take
\[
p = \left( \frac{\alpha}{Ce\|h\|_{L^2}} \right)^2.
\] (3.21)
If $p \geq 2$, we have
\[ \mathbb{P}(\Omega_3^c) \leq e^{-p} = e^{-\left(\frac{\alpha}{C_1 \|h\|_{L_2}}\right)^2}. \] (3.22)

If $p \leq 2$, we have
\[ \mathbb{P}(\Omega_3^c) \leq e^{2e^{-2}} = C_1 e^{-\left(\frac{\alpha}{C_2 \|h\|_{L_2}}\right)^2}. \] (3.23)

Here $C_1 = e^2$.

This completes the proof of Lemma 3.4.

4. Proof of Theorem 1.1

In this section, we apply the density theorem and Lemmas 2.1-2.3 to establish Theorem 1.1.

**Proof of Theorem 1.1.** We firstly prove that if $f$ is rapidly decreasing function, $\forall \epsilon > 0$, there exists $\delta > 0$ such that
\[ |U(t)f - f| \rightarrow 0 \] (4.1)
as $t \rightarrow 0$. From Lemmas 2.2, 2.3, we have
\[ |U(t)f - f| \leq \epsilon + \frac{C|t|}{\delta}. \] (4.2)

When $C|t| < \delta \epsilon$, from (4.2), we have
\[ |U(t)f - f| \leq 2\epsilon. \] (4.3)

From (4.3), we know that (4.1) is valid.

When $f \in H^s(\mathbb{R})(s \geq \frac{1}{4})$, by density theorem which can be seen in Lemma 2.2 of [22], there exists a rapidly decreasing function $g$ such that $f = g + h$, where $\|h\|_{H^s(\mathbb{R})} < \epsilon(s \geq \frac{1}{4})$. Thus, we have
\[ \lim_{t \rightarrow 0} |U(t)f - f| \leq \lim_{t \rightarrow 0} |U(t)g - g| + \lim_{t \rightarrow 0} |U(t)h - h|. \] (4.4)

We define
\[ E_\alpha = \left\{x \in \mathbb{R} : \lim_{t \rightarrow 0} |U(t)f - f| > \alpha\right\}. \] (4.5)

Obviously, $E_\alpha \subset E_{1\alpha} \cup E_{2\alpha}$,

\[ E_{1\alpha} = \left\{x \in \mathbb{R} : \lim_{t \rightarrow 0} |U(t)g - g| > \frac{\alpha}{2}\right\}, \] (4.6)

\[ E_{2\alpha} = \left\{x \in \mathbb{R} : \lim_{t \rightarrow 0} |U(t)h - h| > \frac{\alpha}{2}\right\}. \] (4.7)
Obviously,

\[ E_\alpha \subset E_{1\alpha} \cup E_{2\alpha}. \] (4.8)

From Lemmas 2.2, 2.3, we have

\[ |E_{1\alpha}| = 0. \] (4.9)

Obviously,

\[ E_{2\alpha} \subset E_{21\alpha} \cup E_{22\alpha}. \] (4.10)

where

\[ E_{21\alpha} = \left\{ x \in \mathbb{R} : \sup_{t > 0} |U(t)P_{\geq 8h} - P_{\geq 8h}| > \frac{\alpha}{4} \right\}, \] (4.11)

\[ E_{22\alpha} = \left\{ x \in \mathbb{R} : \lim_{t \to 0} |U(t)P_{\leq 8h} - P_{\leq 8h}| > \frac{\alpha}{4} \right\}. \] (4.12)

Thus, from Lemma 2.1, by using the Sobolev embeddings theorem \( W^{1,2}(\mathbb{R}) \hookrightarrow L^4(\mathbb{R}) \), we have

\[
|E_{21\alpha}| = \int_{E_{21\alpha}} dx \leq \int_{E_{21\alpha}} \left[ \sup_{t > 0} |P_{\geq 8} U(t)h| \right]^4 dx + \int_{E_{21\alpha}} \frac{|h|^4}{\alpha^4} dx \\
\leq \frac{\|P_{\geq 8} U_1(t)h\|_{L^4,L^\infty}^4}{\alpha^4} + \frac{\|P_{\geq 8} h\|_{L^4}^4}{\alpha^4} \\
\leq C \frac{\|P_{\geq 8} h\|_{H^{\frac{1}{2}}}^4}{\alpha^4} + C \frac{\|D_{\frac{1}{2}} P_{\geq 8} h\|_{L^2}^4}{\alpha^4} \leq C \frac{\|P_{\geq 8} h\|_{H^{\frac{1}{2}}}^4}{\alpha^4} \leq C\epsilon^4 \frac{\alpha}{\alpha^4}. \] (4.13)

From Lemma 2.2, we have

\[ |E_{22\alpha}| = 0. \] (4.14)

From (4.9), (4.13) and (4.14), we have

\[ |E_\alpha| \leq |E_{1\alpha}| + |E_{2\alpha}| \leq |E_{1\alpha}| + |E_{21\alpha}| + |E_{22\alpha}| \leq C\epsilon^4 \frac{\alpha}{\alpha^4}. \] (4.15)

Thus, since \( \epsilon \) is arbitrary, from (4.15), we have

\[ |E_\alpha| = 0. \] (4.16)

Thus, from (4.16), we have

\[ U(t)f - f \rightarrow 0 \] (4.17)
almost everywhere as $t$ goes to zero.

This completes the proof of Theorem 1.1.

5. Proof of Theorem 1.2

In this section, we present the counterexamples showing that $s \geq \frac{1}{4}$ is the necessary condition for the maximal function estimate related to free Ostrovsky equation. More precisely, we give the proof of Theorem 1.2.

**Proof of Theorem 1.2.** We choose $f = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix\xi} 2^{-k(s+\frac{1}{2})} \chi_{2^k \leq |\xi| \leq 2^{k+1}}(\xi) d\xi$, obviously,

$$\|f\|_{H^s} \sim 1.$$  \(5.1\)

We choose $t \leq \frac{k}{100} 2^{-3k}$, where $\delta$ will be chosen later. For $|x| \leq 2^{-k}$ and sufficiently small $\delta$, we have

$$\|U(t)f\|_{L^4_x L^\infty_t} \sim 2^{-k(s-\frac{1}{4})}.$$  \(5.2\)

From

$$\|U(t)f\|_{L^4_x L^\infty_t} \leq C\|f\|_{H^s(\mathbb{R})}$$  \(5.3\)

and (5.1)-(5.2), we have

$$2^{-k(s-\frac{1}{4})} \leq C.$$  \(5.4\)

We know that for sufficiently large $k$, when $s < \frac{1}{4}$, (5.4) is invalid.

This completes the proof of Theorem 1.2.

6. Proof of Theorem 1.3

In this section, we apply Lemmas 3.2-3.4 and the density theorem which can be seen in Lemma 2.2 of [22] to prove Theorem 1.3.

**Proof of Theorem 1.3.** We firstly prove that if $f$ is rapidly decreasing function, then

$$U(t)f^\omega - f^\omega \to 0$$  \(6.1\)

almost surely as $t \to 0$. From Lemma 3.2, $\forall \epsilon > 0$, there exists $\delta > 0$ such that

$$\mathbb{P}(\{\omega \in \Omega : |U(t)f^\omega - f^\omega| > \alpha\}) \leq C_1 e^{-\left(\frac{\alpha}{C_0[\epsilon + \frac{\mu}{2}]^2}\right)}.$$  \(6.2\)
From (6.2), \( \forall \epsilon > 0 \), we know that when \(|t| \leq \delta \epsilon \), take \( \alpha = 2Ce \epsilon \left[ \ln \frac{C_1}{\epsilon} \right]^\frac{1}{2} \) in (6.2), we have
\[
\mathbb{P} \left( \{ \omega \in \Omega : |U(t)f^\omega - f^\omega| > \alpha \} \right) \leq \epsilon. \tag{6.3}
\]

From (6.3), we know that \( \forall \epsilon > 0 \), when \(|t| \leq \delta \epsilon \), we have
\[
|U(t)f^\omega - f^\omega| \leq 2Ce \epsilon \left[ \ln \frac{C_1}{\epsilon} \right]^\frac{1}{2}, \tag{6.4}
\]
\( \forall \omega \in \{ \omega \in \Omega : |U(t)f^\omega - f^\omega| \leq \alpha \} \). Here \( P \left( \{ \omega \in \Omega : |U(t)f^\omega - f^\omega| \leq \alpha \} \right) \geq 1 - \epsilon \).
Thus, we have proved (6.1).

When \( f \in L^2(\mathbb{R}) \), by density theorem which is Lemma 2.2 in [22], there exists a rapidly decreasing function \( g \) such that \( f = g + h \) which yields \( f^\omega = g^\omega + h^\omega \), where \( \| h \|_{L^2(\mathbb{R})} < \epsilon \).
Thus, we have
\[
|U(t)f^\omega - f^\omega| \leq |U(t)g^\omega - g^\omega| + |U(t)h^\omega - h^\omega|. \tag{6.5}
\]

From (6.5), we have
\[
\{ \omega \in \Omega : |U(t)f^\omega - f^\omega| > \alpha \}
\subseteq \left\{ \omega \in \Omega : |U(t)g^\omega - g^\omega| > \frac{\alpha}{2} \right\} \cup \left\{ \omega \in \Omega : |U(t)h^\omega - h^\omega| > \frac{\alpha}{2} \right\}. \tag{6.6}
\]

From Lemma 3.2, we have
\[
\mathbb{P} \left( \{ \omega \in \Omega : |U(t)g^\omega - g^\omega| > \frac{\alpha}{2} \} \right) \leq C_1 e^{-\left( \frac{\alpha}{C_1 \|h\|_{L^2}} \right)^2}. \tag{6.7}
\]

From Lemmas 3.3, 3.4, we have
\[
\mathbb{P} \left( \{ \omega \in \Omega : |U(t)h^\omega - h^\omega| > \frac{\alpha}{2} \} \right)
\leq \mathbb{P} \left( \{ \omega \in \Omega : |U(t)h^\omega| > \frac{\alpha}{4} \} \right) + \mathbb{P} \left( \{ \omega \in \Omega : |h^\omega| > \frac{\alpha}{4} \} \right)
\leq C_1 e^{-\left( \frac{\alpha}{C_1 \|h\|_{L^2}} \right)^2} + C_1 e^{-\left( \frac{\alpha}{C_1 \|h\|_{L^2}} \right)^2} = 2C_1 e^{-\left( \frac{\alpha}{C_1 \|h\|_{L^2}} \right)^2}. \tag{6.8}
\]

Combining (6.6), (6.7) with (6.8), we have
\[
\mathbb{P} \left( \{ \omega \in \Omega : |U(t)f^\omega - f^\omega| > \alpha \} \right)
\leq \mathbb{P} \left( \{ \omega \in \Omega : |U(t)g^\omega - g^\omega| > \alpha \} \right) + \mathbb{P} \left( \{ \omega \in \Omega : |U(t)h^\omega - h^\omega| > \alpha \} \right)
\leq C_1 e^{-\left( \frac{\alpha}{C_1 \|h\|_{L^2}} \right)^2} + 2C_1 e^{-\left( \frac{\alpha}{C_1 \|h\|_{L^2}} \right)^2} = 3C_1 e^{-\left( \frac{\alpha}{C_1 \|h\|_{L^2}} \right)^2}. \tag{6.9}
\]
\( \forall \epsilon > 0 \), take \( \alpha = Cee \left[ \ln \frac{3C_1}{\epsilon} \right]^\frac{1}{2} \) in (6.9), when \(|t| < \delta \epsilon \), we have
\[
|U(t)f^\omega - f^\omega| \leq \alpha \tag{6.10}
\]
for \( \omega \in \{ \omega \in \Omega : |U(t)f^{\omega} - f^{\omega}| \leq \alpha \} \). Here \( \mathbb{P}(\{ \omega \in \Omega : |U(t)f^{\omega} - f^{\omega}| \leq \alpha \}) \geq 1 - \epsilon \).

This completes the proof of Theorem 1.3.

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