“Large” conformal metrics of prescribed $Q$-curvature in the negative case

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Abstract. Given a compact and connected four dimensional smooth Riemannian manifold $(M,g_0)$ with $k_P := \int_M Q_{g_0} \, dV_{g_0} < 0$ and a smooth non-constant function $f_0$ with $\max_{p \in M} f_0(p) = 0$, all of whose maximum points are non-degenerate, we assume that the Paneitz operator is nonnegative and with kernel consisting of constants. Then, we are able to prove that for sufficiently small $\lambda > 0$ there are at least two distinct conformal metrics $g_\lambda = e^{2u_\lambda} g_0$ and $g^\lambda = e^{2u^\lambda} g_0$ of $Q$-curvature $Q_{g_\lambda} = Q_{g^\lambda} = f_0 + \lambda$. Moreover, by means of the “monotonicity trick” in a way similar to [9], we obtain crucial estimates for the “large” solutions $u_\lambda$ which enable us to study their “bubbling behavior” as $\lambda \downarrow 0$.

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1. Introduction

Given a smooth Riemannian manifold $(M,g_0)$ and a function $f : M \to \mathbb{R}$, an important problem in conformal geometry is to find conditions on $f$ in order that it arises as a certain kind of curvature of a metric $g$ conformal to $g_0$. In dimension 2, one usually considers the Gauss curvature and is led to the classical problem of prescribing the Gauss curvature. We refer the reader to the classical references [7,19] and [9] for a recent review of the state of the art for this problem.

In dimension 4, a natural curvature to be considered is the $Q$-curvature, introduced in [10] and associated to the Paneitz operator, a conformally invariant operator which first appeared in [29]. More precisely, let $(M,g_0)$ be a closed and connected 4-dimensional Riemannian manifold endowed with a smooth background metric $g_0$. The $Q$-curvature $Q_{g_0}$ and the Paneitz operator $P_{g_0}$ are defined in terms of the Ricci tensor $\text{Ric}_{g_0}$ and the scalar curvature $R_{g_0}$ of
\[(M, g_0)\text{ as}\]
\[
Q_{g_0} = -\frac{1}{12} \left( \Delta_{g_0} R_{g_0} - R_{g_0}^2 + 3|\text{Ric}_{g_0}|^2 \right),
\]  
\[
P_{g_0}(\varphi) = \Delta_{g_0}^2 \varphi - \text{div}_{g_0} \left( \frac{2}{3} R_{g_0} g_0 - 2\text{Ric}_{g_0} \right) d\varphi
\]
where \(\Delta_{g_0}\) is the Laplace Beltrami of \((M, g_0)\) and \(\varphi\) is any smooth function on \(M\). The relation between \(P_{g_0}\) and \(Q_{g_0}\), when one performs a conformal change of metric \(g = e^{2u} g_0\), is given by
\[
P_g = e^{-4u} P_{g_0}; \quad P_{g_0} u + 2Q_{g_0} = 2Q_g e^{4u},
\]
which may be viewed as the analogue of the transformation rule for Gauss curvature in dimension 2. Moreover, one has the following extension of the Gauss-Bonnet formula (compare [11])
\[
\int_M \left( Q_{g_0} + \frac{|W_{g_0}|^2}{8} \right) dV_{g_0} = 4\pi^2 \chi(M),
\]
where \(\chi(M)\) is the Euler characteristic of \(M\) and \(W_{g_0}\) denotes the Weyl tensor of \((M, g_0)\). From the pointwise conformal invariance of \(|W_{g_0}|^2 dV_{g_0}\), it readily follows that also the quantity
\[
k_P := \int_M Q_{g_0} dV_{g_0}
\]
is a conformal invariant.

Hence, our initial problem can be stated as follows: given a function \(f : M \rightarrow \mathbb{R}\), we look for conditions on \(f\) such that the equation
\[
P_{g_0} u + 2Q_{g_0} = 2f e^{4u}
\]
admits a solution. In view of the conformal invariance of \((1.5)\), we immediately deduce a first set of necessary conditions on \(f\) for the solvability of \((1.6)\), depending on the sign of \(k_P\). More precisely, if \(k_P > 0\), then \(f\) must be positive somewhere; if \(k_P < 0\), then \(f\) must be negative somewhere; if \(k_P = 0\), then \(f\) must change sign or must be identically zero.

In the case of the standard sphere \(S^4\), Wei and Xu [31] showed that \((1.6)\) admits a solution when the prescribed function \(f\) is positive, and under some conditions involving the critical points of \(f\) and the topological degree of a certain map defined in terms of \(f\). (Refer also to [21, 24, 27, 28] and [4]). Later, Brendle [12] was able to construct conformal metrics whose \(Q\)-curvature is a constant multiple of a prescribed positive function on a general \(M\), under the assumptions that the Paneitz operator is nonnegative with kernel consisting of only constant functions and \(k_P < 8\pi^2\). As a consequence, he was able to generalize Moser’s theorem for prescribed Gauss curvature on the projective plane to dimension \(n\). (Refer also to the pioneering work [13]). In [6], Baird, Fardoun and Regbaoui, constructing a suitable gradient flow, were able to give new sufficient conditions on \(f\), depending on the sign of \(k_P\), in order that \((1.6)\) admits a solution, as soon as one assumes the nonnegativity of the Paneitz operator and that its kernel consists of constant functions only. In particular,
if $k_P < 0$, they could prove existence of solutions to (1.6) for functions $f$ changing sign and not “too” positive.

The afore-mentioned existence results give almost no information about the structure of the set of solutions to Eq. (1.6). Goal of this paper is try to shed some light on the set of solutions and its compactness properties. We will focus on the negative case, viz $k_P < 0$, and we will assume that the Paneitz operator is nonnegative with kernel consisting uniquely of constants. Note that Eastwood and Singer [17] constructed metrics on connected sums of $S^3 \times S^1$ with $k_P < 0$, $P_{g_0} \geq 0$ and kernel consisting of constants functions. Under these assumptions, the analogue of the uniformization theorem holds (we refer the reader for instance to [13] and [15]); thus we can assume that $M$ carries a background metric $g_0$ such that $Q_{g_0} = \text{const} < 0$. Finally, by convenience, we normalize the volume of $(M,g_0)$ to unity. Therefore,

$$Q_{g_0} = k_P < 0.$$ 

In this setting and in complete analogy to the case of surfaces of higher genus (compare [5]), solutions of (1.6) can be characterized as critical points of the following energy

$$E_f(u) = \langle P_{g_0}u, u \rangle + 4Q_{g_0} \int_M u \, dV_{g_0} - \int_M f e^{4u} dV_{g_0}, \quad u \in H^2(M;g_0),$$  

(1.7)

where

$$\langle P_{g_0}u, v \rangle = \int_M \left[ \Delta_{g_0} u \Delta_{g_0} v + \frac{2}{3} R_{g_0} (\nabla_{g_0} u, \nabla_{g_0} v) - 2 \text{Ric}_{g_0} (\nabla_{g_0} u, \nabla_{g_0} v) \right] dV_{g_0},$$

with $u, v \in H^2(M;g_0)$. Note that in view of Adams’ inequality [1], the above energy is well defined on $H^2(M;g_0)$. We also remark that, if $f \in C^\infty(M)$, by standard regularity arguments (see for instance Thm 7.1 [3]) it follows that critical points of $E_f$ are of class $C^\infty$ and hence are classical solutions of (1.6).

Our first result is:

**Theorem 1.1.** Let $(M,g_0)$ be closed and connected with $k_P < 0$, $P_{g_0} \geq 0$ and $\ker(P_{g_0}) = \{\text{constants}\}$. Let $0 \not\equiv f \in C^0(M)$ with $f \leq 0$. Then (1.6) admits a unique solution in $H^4(M;g_0)$.

The unique solution is the absolute minimizer of $E_f$, which is strictly convex and coercive if $f \leq 0$ (see Lemma 2.2). From this theorem and following an idea by [8], we can obtain a stability result for Eq. (1.6), which guarantees the existence of relative minimizers for the energy $E_f$, even when $f$ changes sign.

**Theorem 1.2.** Let $(M,g_0)$ be closed and connected with $k_P < 0$, $P_{g_0} \geq 0$ and $\ker(P_{g_0}) = \{\text{constants}\}$. Suppose $0 \not\equiv f \in C^{0,\alpha}(M)$ for some $\alpha \in (0,1)$ and with $f \leq 0$. Then there exists $\mathcal{N} \subset C^{0,\alpha}(M)$ open neighborhood of $f$ such that for all $h \in \mathcal{N}$ there exists a strict relative minimizer for $E_h$ in $C^{4,\alpha}(M)$ smoothly dependent on $h$. In particular, if $f$ and $h$ are in $C^\infty(M)$, then the minimizer is in $C^\infty(M)$ as well.
In particular, we recover Theorem 2.6 of [6]. We then consider a nonconstant smooth function $f_0 \leq 0$ with $\max_{p \in M} f_0(p) = 0$, all of whose maximum points are non-degenerate. We set $f_\lambda := f_0 + \lambda$, where $\lambda \in \mathbb{R}$. From Thm 1.2 we deduce the existence of a strict relative minimizers $u_\lambda \in C^\infty(M)$ of $E_\lambda := E_{f_\lambda}$ for all sufficiently small $\lambda > 0$, where $u_\lambda$ solves the equation

$$P_{g_0} u_\lambda + 2Q_{g_0} = 2f_\lambda e^{4u_\lambda}. \quad (1.8)$$

We observe that for functions $f$ with $\max_M f > 0$ we have $\inf_{H^2(M;g_0)} E_f = -\infty$. Indeed, choosing $w \in C^\infty(M)$, $0 \leq w \leq 1$ and with support in the set $\{ f > 0 \}$, then one has $\lim_{t \to +\infty} E_f(tw) = -\infty$. Therefore, since for $\lambda > 0$ sufficiently small $E_\lambda$ admits a relative minimizer, we observe the presence of a “mountain pass” geometry and the intuition would suggest the existence of a further critical point, if we could guarantee some compactness properties. In Proposition (5.1), we indeed prove that for a generic $f \in C^2(M)$ the functional $E_f$ possesses bounded Palais-Smale sequences at any level $\beta \in \mathbb{R}$. This fact enables one to conclude that for all sufficiently small $\lambda > 0$ the functional $E_\lambda$ admits, in addition to a strict relative minimizer $u_\lambda$, a further critical point $u^\lambda \neq u_\lambda$ of mountain pass type.

However, this abstract result gives no additional information at all about how the “limit” geometry of the manifolds $(M, e^{2u^\lambda} g_0)$ could look like when $\lambda \downarrow 0$. In order to answer to this issue, firstly, we employ Struwe’s “monotonicity trick” in a way similar to [9] to obtain a suitable sequence of “large” solutions $u^\lambda$. Secondly, thanks to an appropriate choice of a comparison function for our “mountain pass” geometry, we derive some refined estimates which enables us to bound the volume of these second solutions and to prove Theorems 1.3 and 1.4. More precisely, we have:

**Theorem 1.3.** Let $(M, g_0)$ be closed and connected with $k_p < 0$, $P_{g_0} \geq 0$ and $\ker(P_{g_0}) = \{\text{constants}\}$, and consider any smooth, nonconstant function $f_0 \leq 0 = \max_{p \in M} f_0(p)$, all of whose maximum points are non-degenerate. Consider the family of functions $f_\lambda = f_0 + \lambda$, $\lambda \in \mathbb{R}$, and the associated family of functionals $E_\lambda(u) = E_{f_\lambda}(u)$, $u \in H^2(M;g_0)$. There exists a number $\lambda^* > 0$ such that for almost every $0 < \lambda < \lambda^*$ the functional $E_\lambda$ admits a strict relative minimizer $u_\lambda$ and a further critical point $u^\lambda \neq u_\lambda$.

The picture obtained by combining Theorems 1.2 and 1.3 reminds us of a two-branches bifurcation diagram, with a branch consisting of relative minimizers $u$ smoothly converging, as $\lambda \downarrow 0$, to the unique solution of (1.6) for $f = f_0$, and a second “branch” (defined a.e.) consisting of the “large” solutions $u^\lambda$.

**Theorem 1.4.** Under the hypotheses of Theorem 1.3, there exist a sequence $\lambda_n \downarrow 0$, a sequence of solutions $(u_n)$ of the equation

$$P_{g_0} u_n + 2Q_{g_0} = 2f_{\lambda_n} e^{4u_n}$$

and $I \in \mathbb{N}, I \leq 8$ such that, for suitable $p_n^{(i)} \to p_\infty^{(i)} \in M$ with $f_0(p_\infty^{(i)}) = 0, 1 \leq i \leq I$, we obtain $u_n(p_n^{(i)}) \to \infty$ and one of the following: either
In this section we prove Theorems 1.1 and 1.2. Throughout the rest of the

2. Stability result

In any case, for each $1 \leq i \leq I$ and for suitable $r_n^{(i)} \downarrow 0$, either

a) $r_n^{(i)}/\sqrt{n} \to 0$ and in normal coordinates around $p_n^{(i)}$ we have, by setting

$$ x_n = \exp^{-1}(p_n^{(i)}) $$

and $u_n = u_n \circ \exp$, that

$$ u_n(x) = \tilde{u}_n(x + r_n^{(i)} x) - u_n(x_n) \to w(x) = -\log \left( 1 + \frac{|x|^2}{4\sqrt{6}} \right) $$

in $C^4_{\text{loc}}(\mathbb{R}^4)$, where $w$ induces a spherical metric $g = e^{2w}g_{\mathbb{R}^4}$ of Q-curvature $Q_g = 1/2$ on $\mathbb{R}^4$, or

b) $r_n^{(i)} = o(\sqrt{n})$ for a suitable constant $c > 0$ and in normal coordinates around $p_n^{(i)}$ we have, by setting $u_n = u_n \circ \exp$, that

$$ u_n(x) = \tilde{u}_n(r_n^{(i)} x) + \frac{3}{4} \log(\lambda_n) + \log(c^{(i)}) \to w(x) $$

in $C^4_{\text{loc}}(\mathbb{R}^4)$, where the metric $g = e^{2w}g_{\mathbb{R}^4}$ on $\mathbb{R}^4$ has finite volume and finite total Q-curvature $Q_g(x) = 1 + \frac{1}{2}D^2f_0(p_n^{(i)})[x, x]$.

2. Stability result

In this section we prove Theorems 1.1 and 1.2. Throughout the rest of the paper, the Paneitz operator is assumed to be nonnegative and with kernel consisting of constant functions. In view of these hypotheses, it is straightforward to see that there exists a constant $C \geq 1$ only depending on $M$, such that for all $u \in H^2(M; g_0)$

$$ C^{-1}||\Delta_{g_0}u||_{L^2(M)}^2 \leq \langle P_{g_0}u, u \rangle \leq C||\Delta_{g_0}u||_{L^2(M)}^2. \quad (2.1) $$

As a consequence, the bilinear map

$$ H^2(M; g_0) \times H^2(M; g_0) \ni (u, v) \mapsto \langle P_{g_0}u, v \rangle + \int_M uv \, dV_{g_0}. \quad (2.2) $$

defines an equivalent scalar product on $H^2(M; g_0)$.

Lemma 2.1. Let $(M, g_0)$ be closed and connected with $k_f < 0$. Let $0 \neq f \in C^0(M)$ with $f \leq 0$. Then the functional $E_f$ is coercive on $H^2(M; g_0)$.

Proof. Since for any fixed $c > 0$ there holds that

$$ E_f(u) = -Q_{g_0}\log c + E_{f/c}(u + (\log c)/4), \quad u \in H^2(M; g_0) $$

and moreover $||u||H^2(M; g_0) \to \infty$ iff $||u - \log c^{-1/4}||_{H^2(M; g_0)} \to \infty$, we may assume that $||f||_{L^1(M)} = 1 = \int_M -f \, dV_{g_0}$. The general result will then follow by setting $c = ||f||_{L^1(M)} > 0$. 
We define for any $u \in L^1(M)$ the $f$-average of $u$ as
\[
\bar{u}^f := \int_M -fu \, dV_{g_0}.
\]
Via Jensen’s inequality, applied to the probability measure $-fdV_{g_0}$, we obtain for any $u \in H^2(M;g_0)$
\[
\int_M -fe^{4u} \, dV_{g_0} \geq \exp(4\bar{u}^f)
\]
and thus
\[
E_f(u) \geq \langle P_{g_0}u, u \rangle + 4Q_{g_0} \int_M u \, dV_{g_0} + \exp(4\bar{u}^f) = \langle P_{g_0}u, u \rangle + [4Q_{g_0} \bar{u}^f + \exp(4\bar{u}^f)] + 4Q_{g_0} \int_M u(1 + f) \, dV_{g_0},
\]
(2.3)
We set $\bar{u} := \int_M u \, dV_{g_0}$ (recall that Vol$(M,g_0) = 1$) and note that
\[
\left| \int_M u(1 + f) \, dV_{g_0} \right| = |\bar{u} - \bar{u}^f| \leq ||u - \bar{u}||_{L^2(M)} + ||\bar{u} - \bar{u}^f||_{L^2(M)} \leq \bar{C}||\nabla_{g_0} u||_{L^2(M)} \leq \bar{C}||\Delta_{g_0} u||_{L^2(M)},
\]
by a variant of the Poincaré inequality.
Hence, by (2.3) and (2.1),
\[
E_f(u) \geq \langle P_{g_0}u, u \rangle + [4Q_{g_0} \bar{u}^f + \exp(4\bar{u}^f)] - C||\Delta_{g_0} u||_{L^2(M)} \geq \frac{1}{2C}||\Delta_{g_0} u||_{L^2(M)}^2 + [4Q_{g_0} \bar{u}^f + \exp(4\bar{u}^f)],
\]
where the last inequality holds uniformly if $||u||_{H^2(M;g_0)} \gg 0$. Since $Q_{g_0} < 0$, for suitable constants $a_1, a_2 > 0$ we have for every $t \in \mathbb{R}$ that
\[
4Q_{g_0} t + e^{4t} \geq a_1|t| - a_2.
\]
It also follows
\[
E_f(u) \geq \frac{1}{2C}||\Delta_{g_0} u||_{L^2(M)}^2 + a_1|\bar{u}^f| - a_2.
\]
Again by the variant of the Poincaré inequality, it also follows
\[
||u||_{L^2(M)} \leq \bar{C}||\Delta_{g_0} u||_{L^2(M)} + |\bar{u}^f|
\]
and thus we obtain
\[
E_f(u) \geq \frac{1}{4C}||\Delta_{g_0} u||_{L^2(M)}^2 + a_1||u||_{L^2(M)} - a_2,
\]
for all $||u||_{H^2(M;g_0)}$ sufficiently large. The claim follows.

Lemma 2.2. Under the same hypotheses as in the previous lemma we have that for any $u \in H^2(M;g_0)$ there exists a number $\nu = \nu(u) > 0$ such that
\[
\frac{1}{2} D^2 E_f(u) [w, w] \geq \nu \left[ \langle P_{g_0}w, w \rangle + ||w||_{L^2(M)}^2 \right]
\]
(2.4)
for any $w \in H^2(M;g_0)$.
Proof. For any $u, w \in H^2(M; g_0)$, $w \neq 0$, we have

$$
\frac{1}{2} D^2 E_f(u)(u, w) = \frac{1}{2} \left. \frac{d^2}{ds^2} \right|_{s=0} E_f(u + sw) = \langle P_{g_0} w, w \rangle - 8 \int_M f e^{4u} w^2 dV_{g_0}.
$$

We suppose now that $0 = \inf \{ \langle P_{g_0} w, w \rangle - \int_M 8 f e^{4u} w^2 dV_{g_0} : w \in H^2(M; g_0) \}$ for some $u \in H^2(M; g_0)$. Hence, there exists a sequence $(w_n)_n$ such that $\langle P_{g_0} w_n, w_n \rangle + \| w_n \|_{L^2(M)}^2 = 1$, $\langle P_{g_0} w_n, w_n \rangle - \int_M 8 f e^{4u} w_n^2 dV_{g_0} \to 0$, $w_n \to w$ in $H^2(M; g_0)$ and $w_n \to w$ in $L^2(M)$.

At once we see that $\langle P_{g_0} w_n, w_n \rangle \to 0$ and, by Poincaré inequality, that $w_n - \bar{w} \to 0$ in $L^2(M)$. Therefore, there holds $w \equiv \epsilon$ with $\epsilon = \pm 1$. But then $-\int_M 8 f e^{4u} w_n^2 dV_{g_0} \to 0 = -\int_M 8 f e^{4u} dV_{g_0} > 0$. This contradiction proves the Lemma.

In view of Lemma 2.2 and 2.1 we can apply the direct method of the Calculus of Variations and see that $E_f$ admits a unique absolute minimizer $u_f \in H^2(M; g_0)$, which solves

$$
\langle P_{g_0} u_f, v \rangle + 2Q_{g_0} \int_M v dV_{g_0} = 2 \int_M f e^{4u_f} v dV_{g_0} \quad (2.5)
$$

for any $v \in H^2(M; g_0)$. By elliptic regularity theory we see that $u_f \in H^4(M; g_0)$ and therefore Theorem 1.1 follows.

We note that, if we further impose $f \in C^{0,\alpha}(M)$, it follows by the embedding $H^4(M; g_0) \subset C^{1,\alpha}(M)$ and Schauder’s estimates that $u_f \in C^{4,\alpha}(M)$. We have:

**Proposition 2.3.** Let $(M, g_0)$ be closed and connected with $k_F < 0$. Let $0 \neq f \in C^{0,\alpha}(M)$ for some $0 < \alpha < 1$ and $f \leq 0$. Then there exist $\mathcal{N} \subset C^{0,\alpha}(M)$ open neighborhood of $f$ and $G \in C^1(\mathcal{N}, C^{4,\alpha}(M))$ such that, for any $h \in \mathcal{N}$, $G(h)$ is a classical solution of

$$
P_{g_0} G(h) + 2Q_{g_0} = 2 h e^{4G(h)}.
$$

**Proof.** We consider the map

$$
Z : C^{4,\alpha}(M) \times C^{0,\alpha}(M) \to C^{0,\alpha}(M)
$$

$$(w, h) \mapsto Z_{g_0} w + 2 \left( f - h e^{4w} \right) e^{4u_f},
$$

where $u_f \in C^{4,\alpha}(M)$ is the unique solution of (1.6), given by (2.5). $Z$ is clearly $C^1$, $Z(0, f) = 0$ and it is straightforward to see that

$$
D_w Z(0, f) = P_{g_0} (\cdot) - 8 f e^{4u_f} (\cdot) \in \mathcal{L}(C^{4,\alpha}(M); C^{0,\alpha}(M)).
$$

From 2.4, Lax-Milgram Theorem and standard elliptic regularity arguments, we infer

$$
D_w Z(0, f) \in \mathcal{L}(C^{4,\alpha}(M); C^{0,\alpha}(M)).
$$
We apply the Implicit Function Theorem and obtain an open neighborhood \( N \subset C^{0,\alpha}(M) \) of \( f \), \( M_0 \subset C^{4,\alpha}(M) \) open neighborhood of 0 and \( G_0 \in C^1(N, M_0) \) such that for any \( h \in N \)
\[
Z(G_0(h), h) = 0;
\]
that is \( P_{g_0}G_0(h) + 2 \left( f - h e^{4G_0(h)} \right) e^{4u_f} = 0 \). Since \( u_f \) solves (1.6), we obtain
\[
P_{g_0} \left( G_0(h) + u_f \right) + 2Q_{g_0} = 2h e^{4(G_0(h) + u_f)}.
\]
Therefore, setting
\[
G(h) := G_0(h) + u_f,
\]
we obtain the desired conclusion. \( \square \)

**Proof of Thm 1.2.** From Eq. (2.4) we see that, for all \( w \in H^2(M; g_0) \) with \( ||w||_{H^2(M; g_0)} = 1 \), there holds
\[
D^2E_h(u_f) [w, w] = 2\langle P_{g_0}w, w \rangle - 16 \int_M f e^{4u_f} w^2 dV_{g_0} \geq 2 \nu > 0.
\]
For \( h \in N \) we consider \( G(h) \in C^{4,\alpha}(M) \), defined as in the proof of Proposition 2.3. Therefore \( G(h) \) is a critical point of the functional \( E_h \), whose Hessian at the point \( G(h) \) in the unit direction vector \( w \) is
\[
D^2E_h(G(h)) [w, w] = 2\langle P_{g_0}w, w \rangle - 16 \int_M h e^{4G(h)} w^2 dV_{g_0}.
\]
Since the map \( G \) is \( C^1 \), we have \( e^{4G(h)} \rightarrow e^{4u_f} \) in \( C^{4,\alpha}(M) \) when \( h \rightarrow f \) in \( C^{0,\alpha}(M) \) and therefore
\[
\int_M \left( h e^{4G(h)} w^2 - f e^{4u_f} w^2 \right) dV_{g_0} \rightarrow 0
\]
when \( h \rightarrow f \) in \( C^{0,\alpha}(M) \), uniformly in \( w \), \( ||w||_{H^2(M; g_0)} = 1 \). Therefore, by choosing a smaller neighborhood \( N \) of \( f \) from the beginning and by homogeneity of the Hessian, we obtain
\[
D^2E_h(G(h)) [w, w] > \nu ||w||_{H^2(M; g_0)}^2
\]
for all \( h \in N \) and \( w \in H^2(M; g_0) \). Recalling that \( G(h) \) is a critical point of \( E_h \), by means of a second order Taylor expansion we conclude that \( G(h) \) is a strict relative minimizer for \( E_h \).

### 3. Existence of a second critical point

Let \( f_0 \leq 0 \) be a nonconstant smooth function with \( \max_{p \in M} f_0(p) = 0 \), all of whose maximum points are non-degenerate. Set \( f_\lambda := f_0 + \lambda \), \( \lambda \in \mathbb{R} \), and consider \( E_\lambda(u) := E_{f_\lambda}(u) \), \( u \in H^2(M; g_0) \). By Theorem 1.2 we deduce the existence of a number \( \lambda_0 > 0 \) such that for any \( \lambda \in (0, \lambda_0] \) the functional \( E_\lambda \) admits a strict relative minimizer \( u_\lambda \in C^\infty(M) \), depending smoothly on \( \lambda \). In particular, calling \( u_0 \) the unique (smooth) solution of (1.6) for \( f = f_0 \), we see that, as \( \lambda \downarrow 0 \), \( u_\lambda \rightarrow u_0 \) smoothly in \( H^2(M; g_0) \). Hence, after replacing
with a smaller number $1/4 > \lambda_0 > 0$, if necessary, we can find $\rho > 0$ such that
\[
E_{\lambda}(u_\lambda) = \inf_{||u-u_0||_{H^2} < \rho} E_{\lambda}(u) \leq \sup_{\mu, \nu \in \Lambda_0} E_{\mu}(u_\nu) < \beta_0 := \inf_{\mu \in \Lambda_0; \rho/2 < ||u-u_0||_{H^2} < \rho} E_{\mu}(u),
\] (3.1)
uniformly for all $\lambda \in \Lambda_0$. Fix some number $\lambda \in \Lambda_0$. Recalling that for $\lambda > 0$ the functional $E_{\lambda}$ is unbounded from below, it is also possible to fix a function $v_\lambda \in H^2(M; g_0)$ such that
\[
E_{\lambda}(v_\lambda) < E_{\lambda}(u_\lambda)
\]
and hence
\[
c_{\lambda} = \inf_{p \in P} \max_{t \in [0,1]} E_{\lambda}(p(t)) \geq \beta_0 > E_{\lambda}(u_\lambda),
\] (3.2)
where
\[
P = \{ p \in C^0([0,1]; H^2(M; g_0)) : p(0) = u_0, p(1) = v_\lambda \}. \tag{3.3}
\]
Because $u_\lambda \to u_0$ smoothly in $H^2(M; g_0)$ when $\lambda \downarrow 0$, it is possible to fix the initial point of the comparison paths $p \in P$ to be $u_0$ instead of $u_\lambda$, provided that $\lambda_0$ is sufficiently small.

For a suitable choice of $v_\lambda$, we obtain an explicit and useful estimate of the mountain-pass energy level $c_{\lambda}$ associated with $P$.

**Proposition 3.1.** For any $K > 32\pi^2$ there is $\lambda_K \in [0, \lambda_0/2]$ such that for any $0 < \lambda < \lambda_K$ there is $v_\lambda \in H^2(M; g_0)$ so that choosing $v_\mu = v_\lambda$ for every $\mu \in [\lambda, 2\lambda]$ we obtain the bound $c_{\mu} \leq K \log(1/\mu)$.

The proof of this proposition is largely inspired by Borer et al. [9], even though in the present setting further complications arise, due to the fact that we are dealing with a differential operator of a higher order than the Laplacian as in [9], and therefore one must handle a number of extra terms appearing in the estimates. In a nutshell, our strategy to construct a suitable comparison function $v_\lambda$ will consist in using an appropriate truncated and scaled version of a fundamental solution of the bi-Laplacian operator in $\mathbb{R}^4$.

The proof of the proposition is quite long and will be postponed after Proposition 3.4. Therefore, we now proceed with the proof of the existence of a second critical point.

Note that for any $u \in H^2(M; g_0)$ and for every $\mu_1, \mu_2 \in \mathbb{R}$ there holds
\[
E_{\mu_1}(u) - E_{\mu_2}(u) = (\mu_2 - \mu_1) \int_M e^{4u} dV_{g_0}. \tag{3.4}
\]
It follows that the function
\[
\Lambda \ni \mu \mapsto c_{\mu}
\] (3.5)
is non-increasing in $\mu$, and therefore differentiable at almost every $\mu \in \Lambda$.

We note the following lemma, which is the analogue of Lemma 3.3 in [9]:
Lemma 3.2. (i) For any $m > 0$ there exists a constant $C = C(m)$ such that for every $\mu_1, \mu_2 \in \mathbb{R}$ and for every $u \in H^2(M; g_0)$ satisfying $\|u\|_{H^2(M; g_0)} \leq m$ there holds

$$\|DE_{\mu_1}(u) - DE_{\mu_2}(u)\| \leq C|\mu_1 - \mu_2|.$$ 

(ii) For any $|\mu| < 1$, any $u, v \in H^2(M; g_0)$ with $\|v\|_{H^2(M; g_0)} \leq 1$, we have

$$E_{\mu}(u + v) \leq E_{\mu}(u) + DE_{\mu}(u)[v] + \left[1 + C \left\{ \int_M e^{16u} \, dV_{g_0} \right\}^{1/4} \right] \|v\|_{H^2(M; g_0)},$$

where $C = C(M, g_0, f_0)$ is a positive constant.

Proof. i) Take $v \in H^2(M; g_0)$ such that $\|v\|_{H^2(M; g_0)} \leq 1$ and compute

$$DE_{\mu_1}(u)[v] - DE_{\mu_2}(u)[v] = 4(\mu_2 - \mu_1) \int_M e^{4u} v \, dV_{g_0}$$

$$\leq 4|\mu_2 - \mu_1| \left\{ \int_M e^{8u} \, dV_{g_0} \right\}^{1/2} \|v\|_{L^2(M; g_0)}$$

$$\leq 4|\mu_2 - \mu_1| \left\{ \int_M e^{8u} \, dV_{g_0} \right\}^{1/2}.$$ 

The claim follows from Adams's inequality [1].

ii) By Taylor’s expansion, for every $x \in M$ there exists $\theta(x) \in [0, 1]$ such that

$$E_{\mu}(u + v) - E_{\mu}(u) - DE_{\mu}(u)[v]$$

$$= \langle P_{g_0} v, v \rangle - 8 \int_M f_{\mu} e^{4(u + \theta v)} v^2 \, dV_{g_0}$$

$$\leq \|v\|_{H^2(M; g_0)}^2 + 8\|f_{\mu}\|_{\infty} \int_M e^{4(u + \theta v)} v^2 \, dV_{g_0}.$$ 

Applying twice Hölder’s inequality and by Sobolev’s embedding we obtain

$$\int_M e^{4(u + \theta v)} v^2 \, dV_{g_0} \leq \left\{ \int_M e^{8(u + \theta v)} \, dV_{g_0} \right\}^{1/2} \|v\|_{L^4(M; g_0)}^2$$

$$\leq C \left\{ \int_M e^{16u} \, dV_{g_0} \cdot \int_M e^{16\theta v} \, dV_{g_0} \right\}^{1/4} \|v\|_{H^2(M; g_0)}^2.$$ 

Hence, we have

$$E_{\mu}(u + v) \leq E_{\mu}(u) + DE_{\mu}(u)[v] + \|v\|_{H^2(M; g_0)}^2 \left[1 + C \left\{ \int_M e^{16u} \, dV_{g_0} \right\}^{1/4} \left\{ \int_M e^{16\theta v} \, dV_{g_0} \right\}^{1/4} \right].$$
In order to bound \( \int_M e^{16\theta v} \, dV_{g_0} \), we proceed as
\[
\int_M e^{16\theta v} \, dV_{g_0} = \int_{M \cap \{v \leq 0\}} e^{16\theta v} \, dV_{g_0} + \int_{M \cap \{v > 0\}} e^{16\theta v} \, dV_{g_0} \\
\leq 1 + \int_{M \cap \{v > 0\}} e^{16v} \, dV_{g_0} \\
\leq 1 + \int_M e^{16v} \, dV_{g_0} \\
\leq C
\]
where in the last passage we have used again Adams’ inequality. Our claim follows. \( \square \)

We are now able to prove the analogue of Proposition 3.2 in [9]:

**Proposition 3.3.** Suppose that the map \( \Lambda \ni \mu \mapsto c_\mu \) is differentiable at some \( \mu > \lambda \) (compare (3.5)). Then there exists a sequence \((p_n)_{n \in \mathbb{N}}\) in \( P \) and a corresponding sequence of points \( u_n = p_n(t_n) \in H^2(M; g_0) \), \( n \in \mathbb{N} \), such that
\[
E_\mu(u_n) \to c_\mu, \quad \max_{0 \leq t \leq 1} E_\mu(p_n(t)) \to c_\mu, \quad ||DE_\mu(u_n)|| \to 0 \quad \text{as} \quad n \to \infty,  
\]
and with \((u_n)\) satisfying, in addition, the “entropy bound”
\[
\int_M e^{4u_n} \, dV_{g_0} = \left| \frac{d}{d\mu} E_\mu(u_n) \right| \leq |c'_\mu| + 3, \quad \text{uniformly in} \ n.  
\]

**Proof.** The argument is very close to the proof of Proposition 3.2 appearing in [9]. Therefore, in the following the reasoning will just be outlined and for the details we refer to the afore-mentioned paper. Since \( \lambda_0 < \frac{1}{4} \), we can assume that for any \( \mu \in \Lambda \) we have \( |\lambda - \mu| < 1 \). Let \( \mu \in \Lambda \) be a point of differentiability of \( c_\mu \) and \( \mu_n \in \Lambda \) a sequence of numbers with \( \mu_n \downarrow \mu \) as \( n \to \infty \). We may find a sequence of paths \( p_n \in P \) and a sequence of \( t_n \in [0,1] \) such that, setting \( u = p_n(t_n) \), we obtain
\[
\max_{t \in [0,1]} E_\mu(p_n(t)) \leq c_\mu + (\mu_n - \mu), \quad n \in \mathbb{N},
\]
\[
c_{\mu_n} - (\mu_n - \mu) \leq E_{\mu_n}(u)
\]
and
\[
0 \leq \frac{E_\mu(u) - E_{\mu_n}(u)}{\mu_n - \mu} = \int_M e^{4u} \, dV_{g_0} \leq |c'_\mu| + 3
\]
for all \( n \in \mathbb{N} \) sufficiently large.

From that and via Jensen’s inequality we can bound
\[
4 \int_M u \, dV_{g_0} \leq \log \left( \int_M e^{4u} \, dV_{g_0} \right) \leq \log(|c'_\mu| + 3) = C(\mu) < \infty
\]
uniformly for \( n \) sufficiently large, which leads to
\[
||u||_{H^2}^2 + \int_M e^{4u} \, dV_{g_0} \leq C_1(\mu).
\]

We now assume by contradiction that there exists \( \delta > 0 \) such that
\( ||DE_\mu(u)|| \geq 2\delta \) for all \( n \) sufficiently large and where \( u = p_n(t_n) \). With the
help of Lemma 3.2 and similarly as done in [9], we can construct suitable comparison paths $\tilde{p}_n$ which contradicts the definition of $c_{\mu_n}$. That concludes the proof. \hfill \Box

With the help of the previous Proposition we obtain:

**Proposition 3.4.** Let $\mu$ be a point of differentiability for the function $c_\mu$. Then the functional $E_\mu$ admits a critical point $u^\mu$ at the energy level $c_\mu$ and with volume $\int_M e^{4u^\mu} \, dV_{g_0} \leq |c'_\mu| + 3$, and such that $u^\mu$ is not a relative minimizer of $E_\mu$.

**Proof.** Let $\mu$ be a point of differentiability for the function $c_\mu$: Proposition 3.3 guarantees the existence of a sequence of paths $(p_n)_{n \in \mathbb{N}}$ and of a sequence of points $u_n = p_n(t_n) \in H^2(M; g_0)$ such that (3.6), (3.7) and

$$||u_n||^2_{H^2} + \int_M e^{4u_n} \, dV_{g_0} \leq C$$

are true, where $C$ depends on $\mu$ but not on $n \in \mathbb{N}$. Therefore, up to subsequences, we can assume that, as $n \to \infty$, $u_n \rightharpoonup u^\mu$ weakly in $H^2(M; g_0)$ and $u_n \to u^\mu$ strongly in $L^q(M; g_0)$ for any $q \geq 1$. Furthermore, by the compactness of the map $H^2(M; g_0) \ni w \to e^{4w} \in L^p(M, g_0)$, we can also assume $e^{4u_n} \to e^{4u^\mu}$ in $L^p(M, g_0)$ for any $p \geq 1$. From this last fact, it follows

$$\int_M e^{4u^\mu} \, dV_{g_0} \leq |c'_\mu| + 3.$$ 

Moreover, with an error term $o(1)$ as $n \to \infty$, we can write

$$o(1) = \frac{1}{2} DE_\mu(u_n)[u_n - u^\mu]$$

$$= \langle P_{g_0} u_n, u_n - u^\mu \rangle + 2Q_{g_0} \int_M (u_n - u^\mu) \, dV_{g_0} +$$

$$- 2 \int_M f_{\mu} e^{4u_n} (u_n - u^\mu) \, dV_{g_0}$$

$$= \langle P_{g_0} u_n - u^\mu, u_n - u^\mu \rangle + o(1),$$

viz $u_n \to u^\mu$ strongly in $H^2(M; g_0)$ as $n \to \infty$. Therefore, we deduce also $E_\mu(u_n) \to E_\mu(u^\mu) = c_\mu$ and $DE_\mu(u_n) \to DE_\mu(u^\mu)$; thus $u^\mu$ is a critical point for $E_\mu$. \hfill \Box

**Proof of Proposition 3.1.**

Let $p_0 \in M$ be such that $f_0(p_0) = 0$ and $\lambda \in (0, \lambda_0]$. We define the smooth Riemannian metric $g = e^{2u_0} g_0$. We fix a natural number $N \geq 5$. Then we can find a smooth metric $\tilde{g}(N)$ conformal to $g$ such that

$$\det(\tilde{g}(N)) = 1 + O(r^N), \quad \text{as } r \downarrow 0,$$

(3.8)

where $r = |x|$ and $x$ are $\tilde{g}(N)$-normal coordinates at $p_0 \simeq 0$ (see [20]). Since $p_0$ is an isolated point of maximum of $f_0$, for a suitable constant $L > 0$ with $\sqrt{\lambda_0} < L$ we have that in these normal coordinates

$$f_0(x) = \frac{1}{2} D^2 f_0(p_0) [x, x] + O(|x|^3) \geq -\lambda/2 \text{ on } B_{\sqrt{\lambda}/L}(0)$$

(3.9)

and $f_\lambda \geq \lambda/2$ on $B_{\sqrt{\lambda}/L}(0)$ for all $\lambda \in (0, \lambda_0]$. 


Fix a cut-off function \( \tau \in C^\infty_c(B_1(0)) \) with \( 0 \leq \tau \leq 1 \) and

\[
\tau(t) = \begin{cases} 
1, & |t| < 1/2, \\
0, & |t| \geq 1.
\end{cases}
\]

Let \( A_0 > 1 \) and let \( \xi \in C^\infty([0, \infty)) \) be defined as

\[
\xi(t) = \begin{cases} 
t, & t \in [0, 1], \\
2, & t \geq 2, \\
\in [1, 2], & t \in (1, 2),
\end{cases}
\]

with \( \xi' \geq 0 \) and

\[
\sup_{t \geq 0} \xi'(t) \leq A_0. \tag{3.10}
\]

For \( \delta > 0 \) define

\[
\xi_\delta(t) = \delta \xi(t/\delta). \tag{3.11}
\]

We note that pointwise \( \lim_{\delta \to +\infty} \xi_\delta(t) = t \), and the convergence is uniform on compact subsets. Furthermore, for any \( t \geq 0 \), it holds

\[
|\xi'_\delta(t)| = |\xi'(t/\delta)| \leq A_0, \quad |\xi''_\delta(t)| = |\xi''(t/\delta)\delta^{-1}| \leq \delta^{-1}||\xi''||_\infty, \tag{3.12}
\]

whereas obviously \( ||\xi''||_\infty := \sup_{t \geq 0} |\xi''(t)| \).

We set \( \delta = \delta(\lambda) := \frac{1}{2} \log(1/\lambda) \) and define

\[
z_\lambda(x) = \begin{cases} 
\xi_\delta \left( \log \left( \frac{1}{|x|} \right) \right) \tau(|x|), & \lambda \leq |x| \leq 1, \\
\log(1/\lambda), & |x| \leq \lambda, \\
0, & |x| > 1.
\end{cases}
\]

Then \( z_\lambda \in C^\infty_c(\mathbb{R}^4) \) with \( \text{supp } z_\lambda \subset \overline{B_1(0)} \). Finally, we define for \( x \in B_{\sqrt{\lambda}}(0) \)

\[
w_\lambda(x) = z_\lambda \left( \frac{Lx}{\sqrt{\lambda}} \right) \tag{3.13}
\]

and we extend \( w_\lambda = 0 \) outside \( B_{\sqrt{\lambda}}(0) \). Therefore, \( w_\lambda \in C^\infty(M) \) with \( \text{supp } w_\lambda \subset \overline{B_{\sqrt{\lambda}}(0)} \). The euclidean gradient and Laplacian of \( z_\lambda \) are respectively

\[
\nabla_{\mathbb{R}^4} z_\lambda(x) = \begin{cases} 
0, & \text{if } |x| \leq \lambda, \\
-\xi' \left( \delta^{-1} \log \left( \frac{1}{|x|} \right) \right) \frac{x}{|x|^2}, & \text{if } \lambda \leq |x| \leq \sqrt{\lambda}, \\
-\frac{x}{|x|^2}, & \text{if } \sqrt{\lambda} \leq |x| \leq \frac{1}{2}, \\
-\frac{x}{|x|^2} \tau(|x|) + \log \left( \frac{1}{|x|} \right) \tau'(|x|) \frac{x}{|x|^3}, & \text{if } \frac{1}{2} \leq |x| \leq 1,
\end{cases}
\tag{3.14}
\]
\(\Delta_{B^+} z_\lambda(x) = \begin{cases} 
0, & \text{if } |x| \leq \lambda, \\
|x|^{-2} \left[ \delta^{-1} \xi'' \left( \delta^{-1} \log \left( \frac{1}{|x|} \right) \right) - 2 \xi' \left( \delta^{-1} \log \left( \frac{1}{|x|} \right) \right) \right] & \text{if } \lambda \leq |x| \leq \sqrt{\lambda}, \\
-2|x|^{-2}, & \text{if } \sqrt{\lambda} \leq |x| \leq \frac{1}{2}, \\
-\tau' (|x|) |x|^{-1} \left[ 2 + 5 \log \left( \frac{1}{|x|} \right) \right] & + \\
-2\tau (|x|) |x|^{-2} + \log \left( \frac{1}{|x|} \right) \tau'' (|x|), & \text{if } \frac{1}{2} \leq |x| \leq 1. 
\end{cases} \tag{3.15}
}\)

**Lemma 3.5.** For any \(0 < \varepsilon < 1\) there exist \(\lambda_\varepsilon \in (0, \lambda_0)\), \(C = C(g_0, f_0) > 0\) and \(C_N > 0\) such that for any \(0 < \lambda < \lambda_\varepsilon\) and for any \(s > 0\) we have

\[- \int_M f_\lambda e^{4(u_0 + s w_\lambda)} dV_{g_0} \leq C - C_N (1 - \varepsilon) \lambda^{8 - 4s}\]

uniformly in \(A_0 > 1\).

**Proof.** Let \(s > 0\) and let \(\varphi_N \in C^\infty(M)\) be the conformal factor \(\tilde{g}(N) = e^{2\varphi_N} g = e^{2\varphi_N + 2u_0} g_0\). Recalling that \(w_\lambda\) is supported in \(B_{\sqrt{\lambda}}(0)\) and Eq. (3.9), we obtain

\[
\int_M f_\lambda e^{4(u_0 + s w_\lambda)} dV_{g_0} = \int_M f_\lambda e^{4(s w_\lambda - \varphi_N)} dV_{\tilde{g}(N)} \\
\geq \frac{\lambda}{2} \int_{B_{\frac{\lambda^2}{2}}(0)} e^{4(s w_\lambda - \varphi_N)} dV_{\tilde{g}(N)} - ||f_0||_\infty \int_M e^{-4\varphi_N} dV_{\tilde{g}(N)} \\
= \frac{\lambda}{2} \int_{B_{\frac{\lambda^2}{2}}(0)} e^{4(s w_\lambda - \varphi_N)} dV_{\tilde{g}(N)} - ||f_0||_\infty \int_M e^{4u_0} dV_{g_0}. \\
\]

From (3.8) we have \(dV_{\tilde{g}(N)} = \sqrt{1 + O(r^N)} dx\). Thus, given \(0 < \varepsilon < 1\), there exists \(\lambda_\varepsilon \in (0, \lambda_0)\), independent of \(s > 0\), such that for any \(0 < \lambda < \lambda_\varepsilon\)

\[
\int_{B_{\frac{\lambda^2}{2}}(0)} e^{4(s w_\lambda - \varphi_N)} dV_{\tilde{g}(N)} \geq \min_M e^{-4\varphi_N} \int_{B_{\frac{\lambda^2}{2}}(0)} e^{4s w_\lambda} \sqrt{1 + O(r^N)} dx \\
\geq \min_M e^{-4\varphi_N} (1 - \varepsilon) \int_{B_{\frac{\lambda^2}{2}}(0)} e^{4s w_\lambda} dx \\
= C_N \frac{4L^4}{\pi^2} (1 - \varepsilon) \int_{B_{\frac{\lambda^2}{2}}(0)} e^{4s w_\lambda} dx,
\]
where \( C_N := \frac{\pi^2}{16\pi} \min_M e^{-4\varphi N} \). Recalling (3.13) and the definition of \( z_\lambda \),

\[
\lambda \int_{B_{\frac{L^4}{4\pi}}(0)} e^{4sw\lambda} dx = \frac{\lambda^3}{L^4} \int_{B_1(0)} e^{4sz\lambda(y)} dy \geq \frac{\lambda^{3-4s}}{L^4} \int_{B_{\frac{L^4}{4\pi}}(0)} dy = \frac{\pi^2 \lambda^{8-4s}}{2L^4}
\]

and therefore we conclude

\[
\int_M f_\lambda e^{4(u_0 + sw\lambda)} dV_{g_0} \geq C_N (1 - \varepsilon) \lambda^{8-4s} - C.
\]

We note that in the conformal normal coordinates \( \{x^i\} \) associated to \( \tilde{g}(N) \), one has for a radial function \( v \) the following expansion

\[
\Delta \tilde{g}(N) v = \Delta_{R^4} v + O''(r^{N-1}) v',
\]

where \( h \in O''(r^{N-1}) \) if and only if \(|\nabla^j h(x)| \leq C_j r^{N-1-j} \) for some constant \( C_j \), \( j = 1, 2 \), and where \( r = |x| = d_{\tilde{g}(N)}(x, p_0) \) (for a proof of that see for instance [18]). Furthermore, if \( \tilde{g}(N) \) indicates the metric \( \tilde{g}(N) \) written in polar coordinates \( (r, \theta) \), one has \( \sqrt{\tilde{g}(N)} = r^3 \sqrt{|\tilde{g}(N)|} \) and

\[
|\nabla \tilde{g}(N) v|^2_{\tilde{g}(N)} = \tilde{g}^{rr}(v')^2.
\]

In view of (3.16), which considerably simplifies the expression of the Laplacian and exploiting the conformal invariance of the Paneitz operator, we are able to show

**Lemma 3.6.** Given \( 0 < \varepsilon < 1 \) and \( A_0 > 1 \), there exists \( \lambda_0 \in (0, \lambda_0) \) independent of \( A_0 \) such that for all \( 0 < \lambda < \lambda_0 \)

\[
\langle P_{g_0} w_\lambda, w_\lambda \rangle \leq 4\pi^2 (1 + \varepsilon) (A_0^2 + 1) \log (1/\lambda) + C_0,
\]

where \( C_0 \) depends at most quadratically on the supremum norm of \( \xi'' \) but it does not depend neither on \( \lambda \) nor on \( \varepsilon \).

**Proof.** Since the Paneitz operator is a conformal invariant, we have

\[
\langle P_{g_0} w_\lambda, w_\lambda \rangle = \langle P_{\tilde{g}(N)} w_\lambda, w_\lambda \rangle,
\]

where

\[
\langle P_{\tilde{g}(N)} w_\lambda, w_\lambda \rangle = \int_M \left[ (\Delta \tilde{g}(N) w_\lambda)^2 + \frac{2}{3} R_{\tilde{g}(N)} \left| \nabla \tilde{g}(N) w_\lambda \right|^2_{\tilde{g}(N)} \right. \left. - 2 \text{Ric}_{\tilde{g}(N)} \left( \nabla \tilde{g}(N) w_\lambda, \nabla \tilde{g}(N) w_\lambda \right) \right] dV_{\tilde{g}(N)}. \tag{3.18}
\]
Let’s estimate first the term involving the Laplacian: given \( \varepsilon > 0 \), there exists \( \lambda^\varepsilon \in (0, \lambda_0) \) such that for \( 0 < \lambda < \lambda^\varepsilon \)

\[
\int_M (\Delta \tilde{g}(N) w_\lambda)^2 \, dV_{\tilde{g}(N)} = \int_{B_{\sqrt{\lambda} L}(0)} (\Delta \tilde{g}(N) w_\lambda)^2 \sqrt{1 + O(r^N)} \, dx
\]

\[
\leq (1 + \varepsilon) \int_{B_{\sqrt{\lambda} L}(0)} (\Delta \tilde{g}(N) w_\lambda)^2 \, dx
\]

\[
= (1 + \varepsilon) \int_{B_1(0)} (\Delta \tilde{g}(N) z_\lambda)^2 \, dx
\]

and, from (3.16),

\[
\int_{B_1(0)} (\Delta \tilde{g}(N) z_\lambda)^2 \, dx
\]

\[
= \int_{B_1(0)} [\left(\Delta_{R^4} z_\lambda\right)^2 + 2\Delta_{R^4} z_\lambda \left(z'_\lambda O''(r^{N-1}) + (z''_\lambda O''(r^{N-1}))^2\right)] \, dx
\]

\[
= : M_1 + M_2 + M_3.
\]

We have (see “Appendix 1”) for \( 0 < \lambda < \lambda^\varepsilon \)

\[
M_1 \leq 4\pi^2 (A_0^2 + 1) \log (1/\lambda) + C_0,
\]

where \( C_0 \) is a constant depending at most quadratically on the supremum norm of \( \xi'' \), independent of \( \lambda \) and \( \varepsilon \), and which is allowed to vary from line to line;

\[
M_2 = O(\lambda^{N-3})
\]

and

\[
M_3 = O(\lambda^{N-3})
\]

as \( \lambda \downarrow 0 \). Hence, by choosing a smaller \( \lambda^\varepsilon \), if necessary, and recalling that by assumption \( \varepsilon < 1 \), for all \( 0 < \lambda < \lambda^\varepsilon \) we obtain

\[
\int_M (\Delta \tilde{g}(N) w_\lambda)^2 \, dV_{\tilde{g}(N)} \leq 4\pi^2 (1 + \varepsilon)(A_0^2 + 1) \log (1/\lambda) + C_0.
\]

For the remaining part of the Paneitz operator we have

\[
\int_M \left[ \frac{2}{3} R_{\tilde{g}(N)} |\nabla \tilde{g}(N) w_\lambda|_{\tilde{g}(N)}^2 - 2\text{Ric}_{\tilde{g}(N)} \left(\nabla \tilde{g}(N) w_\lambda, \nabla \tilde{g}(N) w_\lambda\right) \right] \, dV_{\tilde{g}(N)}
\]

\[
\leq C \int_M |\nabla \tilde{g}(N) w_\lambda|_{\tilde{g}(N)}^2 \, dV_{\tilde{g}(N)} = C \int_{B_{\sqrt{\lambda} L}(0)} |\nabla \tilde{g}(N) w_\lambda|_{\tilde{g}(N)}^2 \, dV_{\tilde{g}(N)},
\]

where \( C = C(M, g_0, N) \). Therefore, from (3.17) for all \( 0 < \lambda < \lambda^\varepsilon \) we have

\[
\int_{B_{\sqrt{\lambda} L}(0)} |\nabla \tilde{g}(N) w_\lambda|_{\tilde{g}(N)}^2 \, dV_{\tilde{g}(N)} \leq 2(1 + \varepsilon) \int_{B_{\sqrt{\lambda} L}(0)} (w'_\lambda)^2 r^3 \, dr d\theta
\]

\[
= 2(1 + \varepsilon) \frac{\lambda}{L^2} \int_{B_1(0)} (z'_\lambda)^2 r^3 \, dr d\theta.
\]
In a way analogous to what has already been done in the Appendix and recalling (3.14), we infer that \( \int_{\mathcal{B}_{\frac{4\pi}{\lambda}}(0)} \left| \nabla \bar{g}(N) w_\lambda \right|^2 dV_{\bar{g}(N)} = O(\lambda) \) as \( \lambda \downarrow 0 \). From that and from (3.23), we conclude
\[
\langle P_{g_0} w_\lambda, w_\lambda \rangle \leq 4\pi^2(1 + \varepsilon)(A_0^2 + 1) \log (1/\lambda) + C_0,
\]
which holds for \( 0 < \lambda < \lambda^\varepsilon \).

Before terminating the proof of Proposition 3.1, we observe that, since the Paneitz operator is assumed to be non-negative, it defines a semi-inner product on \( H^2(M; g_0) \) and hence the Cauchy-Schwartz inequality holds true
\[
|\langle P_{g_0} u_1, u_2 \rangle| \leq \sqrt{\langle P_{g_0} u_1, u_1 \rangle} \sqrt{\langle P_{g_0} u_2, u_2 \rangle}, \quad u_1, u_2 \in H^2(M; g_0)
\]
and hence for any \( t > 0 \) we have
\[
|\langle P_{g_0} u_1, u_2 \rangle| \leq t \langle P_{g_0} u_1, u_1 \rangle + t^{-1} \langle P_{g_0} u_2, u_2 \rangle. \tag{3.24}
\]

Proof of Proposition 3.1 (completed). Given \( K > 32\pi^2 \), we can find suitable numbers (not unique) \( 0 < \varepsilon < 1, \alpha > 0 \) and \( 1 < A_0 < 2 \) such that
\[
K > 4 \left[ 4\pi^2(1 + \varepsilon)(A_0^2 + 1) + \alpha \right].
\]
According to Lemma 3.6, there exists \( \lambda^\varepsilon \in (0, \lambda_0) \) such that for \( 0 < \lambda < \lambda^\varepsilon \)
\[
\langle P_{g_0} w_\lambda, w_\lambda \rangle \leq 4\pi^2(1 + \varepsilon)(A_0^2 + 1) \log (1/\lambda) + C_0.
\]
Furthermore, given our \( \alpha > 0 \), it is possible to find \( \lambda(\alpha, A_0) < \lambda^\varepsilon \) such that for \( 0 < \lambda < \lambda(\alpha, A_0) \)
\[
\langle P_{g_0} w_\lambda, w_\lambda \rangle \leq \left[ 4\pi^2(1 + \varepsilon)(A_0^2 + 1) + \alpha \right] \log (1/\lambda). \tag{3.25}
\]
Define \( \lambda_K := \min \{ \lambda_\varepsilon, \lambda(\alpha, A_0), \lambda_0/2 \} \), where \( \lambda_\varepsilon \) is given by Lemma 3.5, and consider \( 0 < \lambda < \lambda_K \). Set
\[
\delta := K - 4 \left[ \frac{4\pi^2(1 + \varepsilon)(A_0^2 + 1) + \alpha}{8 \left[ 4\pi^2(1 + \varepsilon)(A_0^2 + 1) + \alpha \right]} \right], \quad K := \frac{K + 4 \left[ 4\pi^2(1 + \varepsilon)(A_0^2 + 1) + \alpha \right]}{2}
\]
and note that \( \delta > 0 \). Thus, by (3.24) and (3.25), we can bound
\[
\langle P_{g_0} u_0 + sw_\lambda, u_0 + sw_\lambda \rangle \leq (1 + 4/\delta) \langle P_{g_0} u_0, u_0 \rangle + s^2(1 + \delta) \langle P_{g_0} w_\lambda, w_\lambda \rangle
\]
\[
\leq (1 + 4/\delta) \langle P_{g_0} u_0, u_0 \rangle + K_1 \frac{s^2}{4} \log (1/\lambda).
\]
Because \( w_\lambda \geq 0 \) and \( Q_{g_0} < 0 \), for every \( s > 0 \) we have
\[
Q_{g_0} \int_M (u_0 + sw_\lambda) dV_{g_0} \leq Q_{g_0} \int_M u_0 dV_{g_0};
\]
therefore, with a constant \( \overline{C} = \overline{C}(u_0, f_0, K) \), we obtain, in view of Lemma 3.5, that for any \( s > 0 \) and any \( 0 < \lambda < \lambda_K \)
\[
E_\lambda(u_0 + sw_\lambda) \leq K_1 \frac{s^2}{4} \log (1/\lambda) - C_N(1 - \varepsilon)\lambda^{8-4s} + \overline{C},
\]
where $C_N$ depends only on the fixed $N$. From this, we see that, for any fixed $0 < \lambda < \lambda_K$, $E_\lambda(u_0 + sw_\lambda) \to -\infty$ as $s \to \infty$ and therefore we may fix some $s_\lambda > 2$ with $v_\lambda = u_0 + s_\lambda w_\lambda$ satisfying $E_\lambda(v_\lambda) < \beta_0$ to obtain
\[
c_\lambda \leq \sup_{s > 0} E_\lambda(u_0 + sw_\lambda) \leq \sup_{s > 0} \left[ K_1 \frac{s^2}{4} \log (1/\lambda) - C_N (1 - \varepsilon) \lambda^{8-4s} + C \right].
\]
For any $0 < \lambda < \lambda_K$ the supremum in the latter quantity is achieved for some $s = s(\lambda) > 2$, with $s = s(\lambda) \to 2$ as $\lambda \downarrow 0$. Thus, taking a smaller $\lambda_K$ if necessary, we obtain eventually
\[
c_\lambda \leq K \log (1/\lambda).
\]
Furthermore, since $E_\mu(v_\lambda) \leq E_\lambda(v_\lambda)$ for $\mu > \lambda$, the same comparison function $v_\lambda$ can be used for every $\mu \in \Lambda := (\lambda, 2\lambda) \subset \Lambda_0$, and for these $\mu$ we obtain the estimate
\[
E_\mu(v_\lambda) < E_\mu(u_\mu) \leq \sup_{\nu \in \Lambda} E_\mu(u_\nu) < \beta_0 \leq c_{\mu} \leq K \log(1/\lambda) \leq K \log(2/\mu),
\]
where $\beta_0$ and $c_{\mu}$ for $\mu \in \Lambda$ are as defined in (3.1) and (3.2). The claim follows and Proposition 3.1 is proved. □

4. Proof of Theorem 1.4

Proposition 3.4 guarantees the existence of a sequence $(\lambda_n)_n$ such that $\lambda_n \downarrow 0$ as $n \to \infty$ and of a sequence $u_n := u^{\lambda_n}$ of “large” solutions of (1.8) with $f_{\lambda_n}$. Now in order to analyze the behaviour of the “limit” geometry of the manifolds $(M, e^{2u_n} g_0)$ when $\lambda_n \downarrow 0$ and to prove that $u_n$ blows up in a spherical bubble, one would like to resort to the results of [25] or [23] for instance. However, similarly to the situation occuring in the two dimensional case ([9]), the afore-mentioned results require either a uniform bound on the volume of the manifolds $(M, e^{2u_n} g_0)$ or that the function $f_{\lambda_n}$ does not change the sign, assumptions which clearly do not hold in the present case. In order to overcome these obstacles, we will resort to the “entropy” bound given by Proposition 3.4.

Reasoning as in [9], we obtain the following result:

Lemma 4.1. We have $\lim \inf_{\mu \downarrow 0} (\mu |c'_\mu|) \leq 32\pi^2$.

Proof. Otherwise there are two constants $K > K_1 > 32\pi^2$ and $\mu_0 > 0$ such that $\inf \{ \mu |c'_\mu| : 0 < \mu \leq \mu_0, \exists c'_\mu \} > K$. Hence, by Lebesgue Theorem for every $0 < \mu_1 < \mu_0$ we have
\[
c_{\mu_1} \geq c_{\mu_0} + \int_{\mu_1}^{\mu_0} |c'_\mu| d\mu \geq c_{\mu_0} + K \log (\mu_0/\mu_1).
\]
On the other hand, by means of Proposition 3.1 we have for all sufficiently small $\mu_1 > 0$ that $c_{\mu_1} \leq K_1 \log (\mu_0/\mu_1)$, which contradicts the above inequality. □

Now observe that by Proposition 3.4 for almost every sufficiently small $\mu > 0$ the second solution which we have obtained satisfies the volume bound
\[ \int_M e^{4u} \, dV_{g_0} \leq |c'_\mu| + 3. \]

After replacing \( \mu \) with \( \lambda \), we then have a sequence of “large” solutions \( u_n := u^{\lambda_n} \) of (1.8) for \( f_{\lambda_n} \) and with \( \lambda_n \downarrow 0 \) satisfying

\[ \limsup_n \left( \lambda_n \int_M e^{4u_n} \, dV_{g_0} \right) \leq 32\pi^2. \]  

Equation (1.5) now reads for the metric \( e^{2u_n} g_0 \) as

\[ k_P = \int_M f_0 e^{4u_n} \, dV_{g_0} + \lambda_n \int_M e^{4u_n} \, dV_{g_0}, \]

which in view of (4.1) leads to the global \( L^1 \)-bound

\[ \sup_n \int_M (|f_0| + \lambda_n) e^{4u_n} \, dV_{g_0} < \infty. \]  

Since \( u_n \) is at least \( C^4 \), we have the following representation formula

\[ u_n(x) = \bar{u}_n + \int_M G(x, y) P_{g_0} u_n \, dV_{g_0}(y), \quad x \in M, \]

where \( G \) is the Green function for \( P_{g_0} \) (compare Lemma 1.7 [13]). We set

\[ \gamma_n := 2f_{\lambda_n} e^{4u_n} - 2Q_{g_0} \]

and observe that for any \( n \in \mathbb{N} \) the quantity \( \|\gamma_n\|_{L^1(M)} \neq 0 \), otherwise \( P_{g_0} u_n = 0 \), \( u_n = \text{const.} \) and hence \( f_{\lambda_n} = \text{const.} \). Therefore, reasoning as in Lemma 2.3 [23], one obtains for \( j = 1, 2, 3 \)

\[ |\nabla_{g_0}^j u_n|_{g_0}^p(x) \leq C(M, g_0) \int_M \left( \frac{|\gamma_n|_{L^1(M)}}{|x - y|^3} \right)^p \frac{|\gamma_n(y)|}{|\gamma_n|_{L^1(M)}} \, dV_{g_0}(y), \]

for a.e. \( x \in M \). In view of the global \( L^1 \)-bound given by (4.2), by means of Jensen's inequality and Fubini's theorem, and arguing as in [23], we deduce the bound

\[ \sup_n \int_M (|\nabla_{g_0}^3 u_n|_{g_0}^p + |\nabla_{g_0}^2 u_n|_{g_0}^p + |\nabla_{g_0} u_n|_{g_0}^p) \, dV_{g_0} < \infty \]

for any \( p \in [1, 4/3] \). By Poincaré’s inequality we also have \( \int_M |u_n - \bar{u}_n|^p \, dV_{g_0} \leq C \) uniformly in \( n \); therefore, setting

\[ v_n := u_n - \bar{u}_n, \]

we deduce the boundness of the sequence \( (v_n)_n \) in \( W^{3,p}(M; g_0) \) for all \( p \in [1, 4/3] \). Therefore, by Sobolev embedding results we infer that:

i. \( (v_n)_n \) is bounded in \( L^q(M) \) for any \( q \in [1, \infty] \);
ii. \( (\nabla_{g_0} v_n)_n \) is bounded in \( L^r(M) \) for any \( r \in [1, 4] \);
iii. \( (\nabla_{g_0}^2 v_n)_n \) is bounded in \( L^s(M) \) for any \( s \in [1, 2] \),

a result needed later. Observe also that \( v_n \) solves

\[ P_{g_0} v_n + 2Q_{g_0} = 2f_{\lambda_n} e^{4u_n} e^{4\bar{u}_n} \quad \text{on } M. \]  

Noting that \( \|f_0\|_{L^1(M)} + \lambda_n \|f_0\|_{L^1(M)} > 0 \) uniformly in \( n \in \mathbb{N} \), we define the \((|f_0| + \lambda_n)\)-average of \( u_n \) as

\[ \bar{u}_n := \int_M u_n (|f_0| + \lambda_n) \frac{dV_{g_0}}{|f_0| + \lambda_n |f_0|_{L^1(M)}}. \]
Hence, in view of (4.2) and Jensen’s inequality, we infer the bound
\[ C \geq \|f_0\| + \lambda_n \|L^1(M)e^{4\tilde{u}_n}\| \geq \|f_0\|L^1(M)e^{4\tilde{u}_n} \]
and consequently \( \sup_n \tilde{u}_n < \infty \). Arguing as in Lemma 4.2 by [9], we can show
the existence of a positive constant \( C \) independent of \( n \) such that the following Poincaré type inequality holds
\[ \|u_n - \bar{u}_n\|_{L^2(M)} \leq C\|\nabla g_0u_n\|_{L^2(M)}. \]
Therefore, thanks to that and to the “classical” Poincaré’s inequality, we obtain
that, uniformly in \( n \),
\[ |\bar{u}_n - \tilde{u}_n| \leq C\|\nabla g_0u_n\|_{L^2(M)} = C\|\nabla g_0v_n\|_{L^2(M)}, \]
and we conclude by above that \( \sup_n |\bar{u}_n - \tilde{u}_n| < \infty \). Since we know that \( \bar{u}_n \leq C \)
uniformly in \( n \), we finally obtain for our sequence of solutions \( (u_n)_n \) that
\[ \sup_n \bar{u}_n < \infty. \quad (4.5) \]

**Lemma 4.2.** Let \( (v_n)_n \) be the sequence defined by (4.3). Then for any domain
\( \Omega \subset M^- := \{ p \in M : f_0(p) < 0 \} \) we have
\[ \sup_n \int_\Omega (\Delta g_0v_n)^2 \, dV_{g_0} \leq C(\Omega). \]

**Proof.** Given any domain \( \Omega \subset M^- \), we notice that it is enough to prove the result on an arbitrary metric ball \( B_d(p) \subset M^- \), since afterward the estimate for \( \Omega \) can be deduced by a covering argument. Thus, let \( B_{4d} = B_{4d}(p) \) be such a ball, where \( d \) is chosen small enough to guarantee that we stay in a single chart. Let \( 0 \leq \eta \leq 1 \) be a smooth cut-off function whose support is \( \overline{B_{2d}} \) and \( \eta = 1 \) on \( B_d \). Therefore, \( \eta^2v_n \in C^\infty(M) \) and it is supported in \( B_{2d} \). In the following, to alleviate our notation, we set for any \( \alpha, \beta \in H^2(M;g_0) \)
\[ D(\nabla g_0\alpha, \nabla g_0\beta) := \frac{2}{3}R_{g_0}g_0(\nabla g_0\alpha, \nabla g_0\beta) - 2\text{Ric}_{g_0}(\nabla g_0\alpha, \nabla g_0\beta). \]

A straightforward computation shows that
\[ \langle P_{g_0}v_n, \eta^2v_n \rangle - \langle P_{g_0}v_n, \eta v_n \rangle \]
\[ = \int_M [\eta \eta v_n \Delta g_0 v_n \Delta g_0 \eta + 2\Delta g_0 v_n g_0(\nabla g_0 \eta, \nabla g_0 (\eta v_n))] \, dV_{g_0} + \]
\[ - \int_M [v_n \Delta g_0 \eta \Delta g_0 (\eta v_n) - 2\Delta g_0 (\eta v_n) g_0(\nabla g_0 v_n, \nabla g_0 \eta)] \, dV_{g_0} \]
\[ + \int_M [D(\nabla g_0v_n, \eta v_n \nabla g_0 \eta) - D(v_n \nabla g_0 \eta, \nabla g_0 (\eta v_n))] \, dV_{g_0} \]
\[ = \int_M [2v_n \Delta g_0 v_n |\nabla g_0 \eta|^2 g_0 - (v_n \Delta g_0 \eta)^2 \eta v_n - v_n D(\nabla g_0 \eta, \nabla g_0 \eta)] \, dV_{g_0} + \]
\[ - 4 \int_M [v_n \Delta g_0 \eta \eta g_0(\nabla g_0 v_n, \nabla g_0 \eta) + g_0(\nabla g_0 v_n, \nabla g_0 \eta)^2] \, dV_{g_0}. \]
In view of the bounds for \( v_n \) and its derivative deduced above (compare immediately after (4.3)) and by several applications of Hölder’s inequality (see “Appendix 2”), we deduce that there exists a constant \( C(\eta, d) \) such that
\[
\left| \langle P_{g_0} v_n, \eta \nabla^2 v_n \rangle - \langle P_{g_0} \eta v_n, \eta v_n \rangle \right| \leq C(\eta, d) \tag{4.7}
\]
uniformly in \( n \). With the same reasoning we can bound for all \( n \) the quantity
\[
\left| 2Q_{g_0} \int_M \eta^2 v_n dV_{g_0} \right| \leq C(\eta, d).
\]
Hence, integrating by parts (4.4) with \( \eta^2 v_n \), we infer
\[
0 \leq \langle P_{g_0} \eta v_n, \eta v_n \rangle \leq C(\eta, d) + 2e^{4\bar{u}_n} \int_M f_{\lambda_n} e^{4\nu_n} \eta^2 v_n dV_{g_0}.
\]
In view of (2.1) it thus follows
\[
\int_{B_d} (\Delta_{g_0} v_n)^2 dV_{g_0} = \int_{B_d} (\Delta_{g_0}(\eta v_n))^2 dV_{g_0} \leq \int_M (\Delta_{g_0}(\eta v_n))^2 dV_{g_0}
\leq C \left( C(\eta, d) + 2e^{4\bar{u}_n} \int_M f_{\lambda_n} e^{4\nu_n} \eta^2 v_n dV_{g_0} \right)
\]
and hence our claim will follow if we can bound from above uniformly in \( n \) the last term on the right hand side. There exists \( \epsilon > 0 \) such that, for all \( n \) sufficiently large, \( f_{\lambda_n} < -\epsilon \) on the ball \( B_{2d} \). Therefore, letting \( B_n^+ := B_{2d} \cap [v_n > 0] \) and \( B_n^- := B_{2d} \cap [v_n \leq 0] \), we obtain
\[
\int_M f_{\lambda_n} e^{4\nu_n} \eta^2 v_n dV_{g_0} = \int_{B_{2d}} f_{\lambda_n} e^{4\nu_n} \eta^2 v_n dV_{g_0}
= \int_{B_n^+} f_{\lambda_n} e^{4\nu_n} \eta^2 v_n dV_{g_0} + \int_{B_n^-} f_{\lambda_n} e^{4\nu_n} \eta^2 v_n dV_{g_0}
\leq -\epsilon \int_{B_n^+} e^{4\nu_n} \eta^2 v_n dV_{g_0} + \int_{B_n^-} f_{\lambda_n} \eta^2 v_n dV_{g_0}
\leq 0 + \|\eta^2 f_{\lambda_n}\|_\infty \|v_n\|_{L^1},
\]
which, as we have already seen, is uniformly bounded. Recalling now (4.5), the claim follows and the Lemma is proved. \( \square \)

By the Lemma above and by reflexivity of the space \( W^{3,p}(M; g_0) \), \( p \in (1, 4/3) \), we infer the existence of a subsequence still denoted \( (v_n) \) such that, as \( n \to \infty \), \( v_n \to v_\infty \) in \( W^{3,p}(M; g_0) \), \( p \in (1, 4/3) \) and
i. \( v_n \to v_\infty \) in \( L^q(M) \) for any \( q \in [1, \infty) \);
ii. \( \partial_\alpha v_n \to \partial_\alpha v_\infty \) in \( L^r(M) \) for any \( r \in [1, 4) \) and \( |\alpha| = 1 \);
iii. \( \partial_\alpha^2 v_n \to \partial_\alpha^2 v_\infty \) in \( L^s(M) \) for any \( s \in [1, 2] \) and \( |\alpha| = 2 \).
Furthermore, we obtain for any domain \( \Omega \subset \subset M^- \)
\[
\sup_n \|v_n\|_{H^2(\Omega)} \leq C(\Omega).
\]

Given such a domain \( \Omega \), we take a point \( p \in \Omega \) and for a sufficiently small \( \delta > 0 \) we consider the exponential map
\[
\exp_p : B_{4\delta}(0) \subset \mathbb{R}^4 \to M; \quad \exp_p(0) = p
\]
and the pull-back metric \( \tilde{g} := (\exp_p)^*g_0 \) on \( B_{4\delta}(0) \). Letting \( \tilde{v}_n := v_n \circ \exp_p \) and \( \tilde{f}_0 := f_0 \circ \exp_p \), we obtain by definition that \( \tilde{v}_n \) solves the equation

\[
P_{\tilde{g}} \tilde{v}_n(x) + 2Q_{g_0} = 2e^{4\tilde{u}_n} \tilde{f}_{\lambda_n}(x)e^{4\tilde{v}_n(x)}, \quad x \in B_{4\delta}(0).
\]

We consider \( \chi \in C_c^\infty(B_{4\delta}(0)), 0 \leq \chi \leq 1 \) and \( \chi = 1 \) on \( B_{2\delta}(0) \). Then \( \chi v_n \in C^2_c(B_{4\delta}(0)) \) and from above we infer \( \sup_n ||\chi v_n||_{H^{2}(\Omega)} \leq C(\Omega) \). Therefore, in view of Adams’ inequality and of (4.5), the sequence \((e^{4\tilde{u}_n} \tilde{f}_{\lambda_n}e^{4\tilde{v}_n} - Q_{g_0})_n\) is bounded in \( L^p(B_{2\delta}(0)) \) for any \( p \geq 1 \); therefore, by standard elliptic regularity theory (see for instance Thm 7.1 [3]), we conclude

\[
||\tilde{v}_n||_{W^{4,p}(B_4(0))} \leq C(\delta) \left(||e^{4\tilde{u}_n} \tilde{f}_{\lambda_n}e^{4\tilde{v}_n} - Q_{g_0}||_{L^p(B_{3\delta}(0))} + ||\tilde{v}_n||_{L^p(B_{3\delta}(0))}\right)
\]

and hence, up to subsequences, that for any \( p \geq 1 \), as \( n \to \infty \), \( \tilde{v}_n \to \tilde{v}_\infty \) in \( W^{4,p}(B_4(0)) \), where we have set \( \tilde{v}_\infty := v_\infty \circ \exp_p \). By Sobolev embedding we obtain \( \tilde{v}_n \to \tilde{v}_\infty \) strongly in \( C^{2,\alpha}(B_\delta(0)) \) with \( \alpha \in [0,1) \) and eventually, by a covering argument, that

\[
v_n \to v_\infty \quad \text{in} \quad C^{2,\alpha}(\Omega), \; \alpha \in [0,1)
\]

as \( n \to \infty \).

We call a point \( p \in M \) a blow-up point for the sequence \((u_n)_n\) if for any \( r > 0 \) we have \( \sup_{B_r(p)} u_n \to \infty \) as \( n \to \infty \). We note that there must exist at least one blow-up point for our sequence of solutions \((u_n)_n\), since otherwise by regularity arguments we could extract a subsequence converging smoothly to the absolute minimizer of \( E_{f_0} \). On the other hand, at this stage the structure of the blow-up set is not so clear and in principle one could expect it to have a “rough” shape (compare for instance [2]).

The next result, which is essentially based on the concentration-compactness criterion appearing in [23] Prop. 3.1., actually gives a precise description of the blow-up set.

**Lemma 4.3.** Up to subsequences, we have that the blow-up set for the solutions \((u_n)_n\) satisfying (4.2) is finite. Let \( \{p^{(1)}, \ldots, p^{(I)}\} \) be such blow-up points. Then, for any \( 1 \leq i \leq I \), we have \( f_0(p^{(i)}_\infty) = 0 \) and for any \( r > 0 \) there holds

\[
\liminf_n \int_{B_r(p^{(i)}_\infty)} |f_{\lambda_n}| e^{4u_n} dV_{g_0} \geq 4\pi^2.
\]

**Proof.** Let \( p \in M \) be a blow-up point for the sequence \((u_n)_n\). We assume that there exists \( r_p > 0 \) such that for all \( r < 6r_p \) there holds

\[
\liminf_n \int_{B_r(p)} 2 |f_{\lambda_n} e^{4u_n} - Q_{g_0}| dV_{g_0} < 8\pi^2.
\]

We note that (4.2) enables us to repeat the same reasoning appearing in [23] Prop. 3.1. locally on the ball \( B_{3r_p}(p) \) (if one looks carefully at this proof, he
will see that the arguments therein are local in nature). Therefore, we deduce the existence of a $\beta > 1$ such that, up to subsequences,

$$\sup_n \int_{B_{3r_p}(p)} e^{4\beta v_n} dV_{g_0} = \sup_n \int_{B_{3r_p}(p)} e^{4\beta(u_n - \bar{u}_n)} dV_{g_0} < \infty.$$ 

Since $v_n$ solves (4.4), taking account of (4.5), we see that the right hand side of (4.4) is bounded uniformly in $n$ in $L^\beta(B_{3r_p}(p))$ for some $\beta > 1$. By standard elliptic regularity theory, similarly to what has been done above, and recalling that $(v_n)_n$ was bounded in any $L^q(M)$, we infer $\|v_n\|_{W^{4,\beta}(B_{2r_p}(p); g_0)} \leq C$ uniformly in $n$ and therefore, by Sobolev embedding, we conclude that the sequence $(v_n)_n$ is bounded at least in $C^{0,\alpha}(B_{2r_p}(p))$ for $\alpha \in [0, 4-4/\beta]$. Thus, setting $u_n(x_n) = \sup_{B_{r_p}(p)} u$, observing that $u_n(x_n) \to \infty$, as $n \to \infty$, and recalling (4.5), we obtain

$$u_n(x_n) \leq |u_n(x_n) - \bar{u}_n| + \bar{u}_n \leq |v_n(x_n)| + C \leq C$$

uniformly in $n$, which is clearly a contradiction. Therefore, (4.10) cannot be true. From this, we deduce immediately (4.9) and, again from (4.2), we infer that the blow-up points are finite.

It remains to prove that they are all points of maximum of $f_0$. We assume that this is not the case and so that there exists a blow-up point $p \in M^-$. We now consider a small ball $B_r(p) \subset M^-$ and infer, in view of (4.8) that $v_n \to v_\infty$ in $C^{2,\alpha}(B_r(p))$ as $n \to \infty$.

If $\inf_n \bar{u}_n > -\infty$, then, up to selecting a further subsequence, we would obtain that $u_n$ would converge uniformly on $B_r(p)$, which cannot be.

If on the other hand $\inf_n \bar{u}_n = -\infty$, then, again up to subsequences, we would obtain $u_n \to -\infty$ uniformly on $B_r(p)$ and conclude $e^{4u_n} \to 0$ uniformly on $B_r(p)$. But this would violate (4.9). Therefore, we conclude that the blow-up points are all points of maximum of $f_0$ and the Lemma is proved.

**Remark 4.4.** We notice that, using the fact that the Green function $G$ for $P_{g_0}$ satisfies

$$\left|G(x, y) - \frac{1}{8\pi^2} \log \frac{1}{|x - y|}\right| \leq C(M, g_0), \quad x, y \in M, x \neq y,$$

and hence $G(p, y) > 0$ for any $p \in M, y \in B_r(p)$ and suitable $r = r(p)$, we may repeat once again all the reasoning in Proposition 3.1. [23] and hence obtain the inequality

$$\liminf_n \int_{B_r(p^{(i)}_\infty)} (f_{\lambda_n})_+ e^{4u_n} dV_{g_0} \geq 4\pi^2,$$

which results in an improvement of (4.9). In the following we are using this refinement.

We set $M_\infty := M \setminus \left\{p^{(1)}_\infty, \ldots, p^{(I)}_\infty\right\}$ and assume that $\inf_n \bar{u}_n = -\infty$. Therefore, by means of (4.8), we conclude that there exists a subsequence still denoted $(u_n)_n$ which converges locally uniformly to $-\infty$ on $M_\infty$, viz we obtain the first conclusion of Theorem 1.4.
If on the other hand there holds \( \inf_n u_n > -\infty \), we obtain with the help of (4.5) that

\[
\sup_n |\bar{u}_n| < \infty.
\]

Again by (4.8) and by Schauder-type estimates (see for instance Thm 6.2.6 [26]), we eventually obtain, as \( n \to \infty \), that \( u_n \to u_\infty \) smoothly locally in \( M_\infty \). Clearly, we may also assume pointwise convergence almost everywhere and from Fatou’s Lemma and (4.2) we infer

\[
\int_M |f_0| e^{4u_\infty} \, dV_{g_0} \leq \liminf_n \int_M (|f_0| + \lambda_n) e^{4u_n} \, dV_{g_0} < \infty. \tag{4.12}
\]

Since now the averages of \( u_n \) are bounded, we obtain that for \( p \in (1, 4/3) \) \( u_n \to u_\infty \) in \( W^{3,p}(M; g_0) \) and that \( u_\infty \in W^{3,p}(M; g_0) \cap C^\infty(M_\infty) \) solves the equation

\[
\Delta_{g_\infty}^2 u_\infty - \text{div}_{g_0} \left( \frac{2}{3} R_{g_\infty} g_0 - 2\text{Ric}_{g_\infty} \right) du_\infty + 2Q_{g_0} = 2f_0 e^{4u_\infty} + \sum_{j=1}^I 8\pi^2 a_j \delta_{p(j)} \quad \text{on } M, \tag{4.13}
\]

in the distribution sense, where for all \( 1 \leq j \leq I \) there holds \( a_j \geq 1 \) in view of (4.9).

**Proposition 4.5.** For every \( 1 \leq i \leq I \) there holds \( 1 \leq a_i \leq \frac{3}{2} \).

**Proof.** With the help of the Green function \( G \) for \( P_{g_0} \) and via the related representation formula we deduce that the functions

\[
k^{(j)}(x) := 8\pi^2 a_j G(p^{(j)}_\infty, x), \quad x \in M, \quad j = 1, \ldots, I
\]

solve in the distribution sense the equations

\[
P_{g_\infty} k^{(j)} = 8\pi^2 a_j \left( \delta_{p^{(j)}_\infty} - 1 \right) \quad \text{on } M.
\]

Hence, the function \( w_\infty := u_\infty - \sum_{j=1}^I k^{(j)} \) solves distributionally the equation

\[
P_{g_\infty} w_\infty = -2Q_{g_\infty} + 2f_0 e^{4u_\infty} + 8\pi^2 \sum_{j=1}^I a_j \quad \text{on } M,
\]

where the right hand side is in \( L^1(M) \). Since we have seen that \( u_\infty \in C^\infty(M_\infty) \), by elliptic regularity it follows \( w_\infty \in C^\infty(M_\infty) \).

We fix \( p^{(i)}_\infty \), choose normal coordinates \( y \in B_\delta(0) \) around \( p^{(i)}_\infty \simeq 0 \) and set \( \tilde{w}_\infty := w_\infty \circ \exp \) and \( \tilde{g}_0 = \exp^* g_0 \). With the help of the standard estimates for the Green function (compare [13]) and since \( G \) is smooth outside the diagonal, we obtain that \( \tilde{w}_\infty \in W^{3,p}(B_\delta(0); \tilde{g}_0) \cap C^\infty(B_\delta(0) \setminus \{0\}) \) with \( p \in [1, 4/3] \) and that it weakly solves

\[
\Delta_{g_\infty}^2 \tilde{w}_\infty = \text{div}_{\tilde{g}_0} \left( \frac{2}{3} R_{\tilde{g}_0} \tilde{g}_0 - 2\text{Ric}_{\tilde{g}_0} \right) d\tilde{w}_\infty - 2Q_{g_0} + 2f_0 e^{4\tilde{u}_\infty} + 8\pi^2 \sum_{j=1}^I a_j
\]
in $B_\delta(0)$. Notice that the right hand side of the equation above is in $L^1(B_\delta(0);\, \tilde{g}_0)$.

We write $\bar{w}_\infty = \bar{w}_\infty^{(1)} + \bar{w}_\infty^{(2)}$, where $\bar{w}_\infty^{(1)}$ classically solves

$$
\begin{cases}
\Delta_{\tilde{g}_0} \bar{w}_\infty^{(1)} = 0, & \text{in } B_\delta(0), \\
\bar{w}_\infty^{(1)} = \bar{w}_\infty, \quad \Delta_{R^4} \bar{w}_\infty^{(1)} = \Delta_{R^4} \bar{w}_\infty, & \text{on } \partial B_\delta(0).
\end{cases}
$$

Therefore, with the help of Lemma 2.3 in [22] we infer that for any $1 \leq p < \infty$, on a sufficiently small ball $B$, there holds $e^{4|\bar{w}_\infty^{(2)}|} \in L^p(B)$. (Actually, the aforementioned Lemma has been proven for equations involving the euclidean bi-Laplacian; but, it is not difficult to generalize it to our case.)

We observe that, because $p_{\infty}^{(i)} \simeq 0$ is a non-degenerate maximum point of $f_0$, there holds on a sufficiently small ball that $C^{-1}|y|^2 \leq |\bar{f}_0(y)| \leq C|y|^2$ for some constant $C > 1$. Hence, we conclude

$$
|\bar{f}_0(y)| e^{4\bar{u}_\infty} = |\bar{f}_0(y)| e^{4\bar{w}_\infty} e^{4\sum_{j \neq i} k^{(j)} e^{4k^{(i)}}} 
\leq C|y|^2 e^{4\bar{w}_\infty} e^{\sum_{j \neq i} k^{(j)} e^{4k^{(i)}}} 
\leq C|y|^2 e^{4\bar{w}_\infty} e^{4k^{(i)}} 
\leq C|y|^{2 - 4a_i} e^{4\bar{w}_\infty^{(2)}},
$$

and similarly $|\bar{f}_0(y)| e^{4\bar{u}_\infty} \geq C^{-1}|y|^{2 - 4a_i} e^{4\bar{w}_\infty^{(2)}}$. We fix $1 < q \leq 2$ and choose $p = 1/(q - 1)$. Therefore, on a sufficiently small ball $B$, we obtain

$$
\int_B |y|^{2 - 4a_i} e^{4\bar{w}_\infty^{(2)}} \, dy = \int_B \left(|y|^{2 - 4a_i} e^{4\bar{w}_\infty^{(2)}} \right)^{1/q} e^{-\frac{4\bar{w}_\infty^{(2)}}{q}} \, dy 
\leq \left( \int_B |y|^{2 - 4a_i} e^{4\bar{w}_\infty^{(2)}} \, dy \right)^{1/q} \left( \int_B e^{-\frac{4\bar{w}_\infty^{(2)}}{q - 1}} \, dy \right)^{1 - 1/q} 
\leq C \left( \int_{\exp(B)} |f_0| e^{4u_\infty} \, dV_{\tilde{g}_0} \right)^{1/q} \left( \int_B e^{\frac{4|\bar{w}_\infty^{(2)}|}{q - 1}} \, dy \right)^{1 - 1/q} 
\leq C \left( \int_B e^{4p|\bar{w}_\infty^{(2)}|} \, dy \right)^{1 - 1/q} \leq C(q),
$$

where we have used (4.12). Then, we conclude that $1 \leq a_i \leq 3/2$, for $1 \leq i \leq I$.

\[\Box\]

**Proof of Theorem 1.4 (completed).** It remains to analyse the blow-up behavior near each point $p_{\infty}^{(i)}$, $1 \leq i \leq I$. We choose $\delta > 0$ and consider the exponential map

$$
\exp : B_\delta(0) \rightarrow \exp(B_\delta(0)), \quad \exp(0) = p_{\infty}^{(i)}
$$

($\delta$ is chosen sufficiently small in order to guarantee that in $\exp(B_\delta(0))$ the only point of maximum of $f_0$ is $p_{\infty}^{(i)}$). We set $K_n := \{ p \in M : f_0(p) + \lambda_n \geq 0 \} \cap \exp(B_\delta(0))$ and observe that equation (4.11) implies, up to subsequences,
\[
\lim_{n} \left( \lambda_n \max_{K_n} e^{4u_n} \right) = \infty. \tag{4.14}
\]

Therefore, there exists a sequence \( (p_n^{(i)})_n \subset M \) such that \( u_n(p_n^{(i)}) = \max_{K_n} u_n \to \infty \) and \( p_n^{(i)} \to p_\infty^{(i)} \) as \( n \to \infty \). To alleviate our notation, we set \( p_n := p_n^{(i)} \) and \( x_n := \exp^{-1}(p_n) \to 0 \), and consider the pull-back metric \( \tilde{g}_0 = \exp^* g_0 \). Therefore, by definition we have

\[
P_{\tilde{g}_0} \tilde{u}_n(x) + 2Q_{g_0} = 2\tilde{f}_{\lambda_n}(x)e^{4\tilde{u}_n(x)}, \quad x \in B_\delta(0),
\]

where \( \tilde{u}_n = u_n \circ \exp \) and \( \tilde{f}_0 = f_0 \circ \exp \).

Since normal coordinates are determined up to the action of the orthogonal group, we can assume from the beginning that \( \tilde{f}_0 \) admits the following expansion

\[
\tilde{f}_0(x) = -\sum_{i=1}^{4} \alpha_i x_i^2 + O(|x|^3), \quad 0 < \alpha_1 \leq \cdots \leq \alpha_4, \quad x \in B_\delta(0), \tag{4.15}
\]

thanks to the fact the \( p_\infty^{(i)} \simeq 0 \) is a non-degenerate point of maximum of \( f_0 \).

Provided that we choose \( \delta \) sufficiently small from the beginning, we can further assume that

\[
-\frac{3}{2} \sum_{i=1}^{4} \alpha_i x_i^2 \leq \tilde{f}_0(x) \leq -\frac{1}{2} \sum_{i=1}^{4} \alpha_i x_i^2
\]

for all \( x \in B_\delta(0) \). But then the following inclusions hold true

\[
\Theta_2(n) \subset \exp^{-1}(K_n) \subset \Theta_1(n), \tag{4.16}
\]

where \( \Theta_1(n) \) ( respectively \( \Theta_2(n) \)) is the ellipsoid of centre 0 and semi-axis of length \( \sqrt{\frac{2\lambda_n}{\alpha_i}} \) ( respectively \( \sqrt{\frac{2\lambda_n}{3\alpha_i}} \), \( i = 1, \ldots, 4 \).

We first deal with the case:

(i)

\[
\limsup_{n} \lambda_n^3 e^{4\tilde{u}_n(x_n)} = \infty, \quad \limsup_{n} \frac{\sqrt{\lambda_n}}{|x_n|} > \beta \sqrt{\frac{3\alpha_4}{2}},
\]

where \( \beta \geq 2 \).

Under these assumptions we define \( r_n > 0 \) as

\[
r_n^4 \lambda_n e^{4\tilde{u}_n(x_n)} = \frac{1}{2}.
\]

By (4.14) it immediately follows \( r_n \to 0 \). We now define the map

\[
V_n : x \mapsto x_n + r_n x
\]

\[
B_\delta/r_n(-x_n/r_n) \to B_\delta(0)
\]

and notice that \( B_\delta/r_n(-x_n/r_n) \) exhausts \( \mathbb{R}^4 \) as \( n \to \infty \). We consider the metric \( \tilde{g}_n = r_n^{-2}V_n^* \tilde{g}_0 \) on \( B_\delta/r_n(-x_n/r_n) \) and the functions

\[
\tilde{u}_n(x) = \tilde{u}_n(V_n(x)) - \tilde{u}_n(x_n), \quad x \in B_\delta/r_n(-x_n/r_n).
\]
Therefore, for all \( n \) sufficiently large we have \( \hat{u}_n(0) = 0 \) and
\[
P_{g_n} \hat{u}_n(x) + 2r_n^4 Q_{g_0} = \left( \frac{\tilde{f}_0(V_n(x))}{\lambda_n} + 1 \right) e^{4\hat{u}_n(x)}, \quad x \in B_{\delta/r_n}(-x_n/r_n).
\]

Furthermore, there holds for any \( m \in \mathbb{N}_0 \) that \( \tilde{g}_n \to \delta_{\mathbb{R}^4} \) in \( C^m_{\text{loc}}(\mathbb{R}^4) \) as \( n \to \infty \).

By a change of variable we also obtain
\[
\int_{B_{\delta/r_n}(-x_n/r_n)} e^{4\hat{u}_n(x)} dV_{\tilde{g}_n}(x)
= \int_{B_{\delta/r_n}(-x_n/r_n)} e^{4\hat{u}_n(x)} r_n^{-4} dV_{\tilde{g}_n}(x)
= \int_{B_{\delta/r_n}(-x_n/r_n)} e^{4\hat{u}_n(V_n(x))} e^{-4\hat{u}_n(x)} r_n^{-4} dV_{\tilde{g}_n}(x)
= \int_{B_{\delta(0)}} e^{4\hat{u}_n(x)} 2\lambda_n dV_{\tilde{g}_n}(x)
\leq 2\lambda_n \int_M e^{4u_n(x)} dV_{g_0}(x)
\]
and hence, in view of (4.1), for any \( \Omega \subset \subset \mathbb{R}^4 \) we obtain
\[
\limsup_n \int_{\Omega} e^{4\hat{u}_n(x)} dV_{\tilde{g}_n}(x) \leq 64\pi^2. \tag{4.17}
\]

We fix \( \Omega = B_R(0) \). In view of (4.15) and the assumption \( \limsup_n \frac{\sqrt{\lambda_n}}{|x_n|} > \beta \frac{\sqrt{3\pi^4}}{2} \), and since \( \frac{r^2}{\lambda_n} \to 0 \), we deduce that \( \frac{\tilde{f}_0(V_n(x))}{\lambda_n} + 1 \to \lim_n \frac{\tilde{f}_0(x)}{\lambda_n} + 1 =: c_\infty \in (0, 1] \) in \( C^1(B_R(0)) \).

Observe that there exists \( N(\Omega) \) such that for any \( n > N(\Omega) \) we have that for any \( z \in \Omega \)
\[
B_{r_n}(x; M) \subset \subset \exp(\Theta_2(n)) \subset K_n \tag{4.18}
\]
where \( x := \exp(V_n(z)) \) and \( B_{r_n}(x; M) \) is the geodesic ball in \( M \) of centre \( x \) and radius \( r_n \). Therefore, for all \( n > N(\Omega) \) and such \( x \) we can write with the help of the Green function for \( P_{g_0} \)
\[
|\nabla^j u_n|_{g_0}(x) \leq \int_M |\nabla^j G(x, y)|_{g_0} |f_{\lambda_n}(y)e^{4u_n(y)} - 2Q_{g_0}| dV_{g_0}(y)
\leq \int_M |\nabla^j G(x, y)|_{g_0} |f_{\lambda_n}(y)e^{4u_n(y)}| dV_{g_0}(y) + C(M, g_0, j)
\]
with \( j = 1, 2, 3 \); here we have used the estimates
\[
|\nabla^j G(x, y)|_{g_0} \leq C(M, g_0, j) |x - y|^{-j}
\]
(compare [23]). We have to deal with the first term: notice that in view of (4.2) we easily obtain
\[
\int_{M \setminus B_{r_n}(x;M)} |\nabla^j G(x,y)|_{g_0} |f_{\lambda_n}(y) e^{4u_n(y)}| \, dV_{g_0}(y)
\]
\[
\leq C(M, g_0, j) r^{-j}_n \int_{M \setminus B_{r_n}(x;M)} |f_{\lambda_n}(y) e^{4u_n(y)}| \, dV_{g_0}(y) = O(r^{-j}_n).
\]
For the remaining part we first observe that, because of (4.18), \(f_{\lambda_n} e^{4u_n(p_n)}\) is positive on \(B_{r_n}(x;M)\) and bounded by \(\lambda_n e^{4u_n(p_n)}\). Therefore, recalling the definition of \(r_n\), we can write
\[
\int_{B_{r_n}(x;M)} |\nabla^j G(x,y)|_{g_0} |f_{\lambda_n}(y) e^{4u_n(y)}| \, dV_{g_0}(y)
\]
\[
\leq C(M, g_0, j) \lambda_n e^{4u_n(p_n)} \int_{B_{r_n}(x;M)} |x - y|^{-j} \, dV_{g_0}(y)
\]
\[
= C(M, g_0, j) \frac{r^{-4}_n}{2} O(r^{-j}_n) = O(r^{-j}_n).
\]
In conclusion, we have showed \(|\nabla^j u_n|_{g_0}(x) \leq C + O(r^{-j}_n)\) for all \(x := \exp(V_n(z))\) with \(z\) ranging in \(\overline{\Omega} = B_R(0)\) and \(n > N(\Omega)\). Hence, recalling the definitions of \(\hat{u}_n\) and \(\hat{g}_n\), it is immediate to obtain \(|\nabla^j \hat{u}_n|_{\hat{g}_n}(z) = r^j_n |\nabla^j u_n|_{g_0}(x)\) and thus
\[
\sup_{z \in B_R(0)} |\nabla^j \hat{u}_n|_{\hat{g}_n}(z) < C, \quad j = 1, 2, 3
\]
uniformly in \(n\). Recalling that \(\hat{u}_n(0) = 0\), we deduce also
\[
|\hat{u}_n(z)| = |\hat{u}_n(z) - \hat{u}_n(0)| \leq \left( \sup_{y \in B_R(0)} |\nabla \hat{u}_n|_{\hat{g}_n}(y) \right) |z| \leq C(R), \quad z \in B_R(0).
\]
This inequality and the above bounds on the derivatives of order up to 3 enables us to apply Ascoli-Arzelà’s theorem and obtain a subsequence \((\hat{u}_n)_{\alpha}\) which converges in \(C^2(B_R(0))\) to some limit function \(w\). Therefore, by means of Schauder’s type estimates (see for instance Thm 6.4.4 [26]) and recalling that, for any \(m \in \mathbb{N}_0\), \(\hat{g}_n \rightarrow \delta_{\mathbb{R}^4}\) in \(C^m_{loc}(\mathbb{R}^4)\) (and thus the coefficients in the estimates do not depend on \(n\)) one obtains \(\hat{u}_n \rightarrow w\) in \(C^4_{loc}(\mathbb{R}^4)\), where \(w\) solves the equation
\[
\Delta^2_{\mathbb{R}^4} w = c_{\infty} e^{4w} \quad \text{on} \quad \mathbb{R}^4,
\]
with \(c_{\infty} \in (0, 1]\). Moreover, by (4.17) one obtains
\[
\int_{\mathbb{R}^4} e^{4w} \, dx \leq 64 \pi^2
\]
as well. Finally, since the image of \(B_R(0)\) via the map \(\exp \circ V_n\) was compactly contained in \(K_n\) for all \(n\) large enough (compare (4.18)), it follows \(\hat{u}_n(z) \leq \hat{u}_n(0) = 0\) for all \(z \in B_R(0)\) and hence \(w \leq w(0) = 0\) in \(\mathbb{R}^4\). Therefore, after replacing the expression \(r^4_n \lambda_n e^{4\hat{u}_n(x_n)} = \frac{1}{2}\) with \(r^4_n \lambda_n e^{4\hat{u}_n(x_n)} = \frac{1}{2c_{\infty}}\), we obtain, with a little abuse of notation, that \(w\) classically solves \(\Delta^2_{\mathbb{R}^4} w = e^{4w},\)
$w \leq w(0) = 0$ and $e^{4w} \in L^1(\mathbb{R}^4)$. From the classification of the solutions of this equation by [22] we obtain that either there exists $\mu > 0$ such that
\[ \Delta_{\mathbb{R}^4} w \geq \mu \quad \text{in} \quad \mathbb{R}^4 \]
or
\[ w(x) = -\log \left(1 + \frac{|x|^2}{4\sqrt{6}}\right). \]

We are going to rule the first alternative out. If it occurred, then, similarly in the spirit to what has already been done and following [16], we could write
\[
\int_{B_R(0)} |\Delta_{\hat{g}_n} \hat{u}_n| \, dV_{\hat{g}_n}
= \int_{B_{r_n}(x_n; \mathcal{M})} r_n^{-2} |\Delta_{g_0} u_n| \, dV_{g_0}
= 2r_n^{-2} \int_{B_{r_n}(x_n; \mathcal{M})} \int M |\Delta_{g_0} G(x, y)||f_{\lambda_n}(y)e^{4u_n(y)} - Q_{g_0}| \, dV_{g_0}(y) \, dV_{g_0}(x)
\leq Cr_n^{-2} \int_{M} |f_{\lambda_n}(y)e^{4u_n(y)} - Q_{g_0}| \int_{B_{r_n}(x_n; \mathcal{M})} |x - y|^{-2} \, dV_{g_0}(x) \, dV_{g_0}(y)
\leq CR^2 \int_{M} |f_{\lambda_n}(y)e^{4u_n(y)} - Q_{g_0}| \, dV_{g_0}(y) = O(R^2)
\]
as $R \to \infty$. Then in the limit for $n$ we would obtain $\mu|S^3|R^4 \leq \int_{B_R(0)} |\Delta_{R^4} w| \, dx = O(R^2)$, which for $R > 0$ is a contradiction, and therefore the second alternative must occur and we obtain alternative a) of Thm 1.4.

(ii) We now treat the case
\[
\limsup_n \lambda_n^3 e^{4\tilde{u}_n(x_n)} < \infty.
\]

We observe that from (4.11) it easily follows $\liminf_n \lambda_n^3 e^{4\tilde{u}_n(x_n)} > 0$ as well. Therefore, there holds uniformly in $n$
\[
|\tilde{u}_n(x_n) + \frac{3}{4} \log \lambda_n| \leq C. \quad (4.19)
\]

We now define
\[
r_n^4 = \frac{\lambda_n^2}{c\alpha_n^2}
\]
where $c > 0$ is sufficiently large, and the map
\[
V_n : x \mapsto r_n x
\]
\[B_{\delta/r_n}(0) \to B_\delta(0).\]

Eventually, we consider the metric $\hat{g}_n = r_n^{-2}V_n^* \hat{g}_0$ on $B_{\delta/r_n}(0)$ and the functions
\[
\hat{u}_n(x) = \tilde{u}_n(V_n(x)) + \frac{3}{4} \log \lambda_n, \quad x \in B_{\delta/r_n}(0).
\]
Therefore, there holds
\[
P_{\tilde{g}_n} u_n(x) + 2r_n^4 Q_{g_0} = \frac{2}{c_{\alpha_4}^2} \left( \tilde{f}_0(V_n(x)) \frac{1}{\lambda_n} + 1 \right) e^{4\tilde{u}_n(x)}, \quad x \in B_{\delta/r_n}(0).
\]
We notice that for some $L > 0$ we have $\{x_n/r_n\}_n \subset B_L(0)$. Moreover, from (4.19) we infer that for some positive constant $C$ there holds $|\tilde{u}_n(x_n/r_n)| \leq C$ uniformly in $n$. As above, it can be seen that for any subset $\Omega \subset \subset \mathbb{R}^4$ there holds
\[
\limsup_n \int_{\Omega} e^{4\tilde{u}_n(x)} dV_{\tilde{g}_n}(x) \leq C, \quad (4.20)
\]
where $C$ is independent of $\Omega$, and that $\tilde{f}_0(V_n(x)) \lambda_n$ converges uniformly as $n \to \infty$ to $\frac{1}{2} D^2 f_0(p^{(i)}_{\infty}) [x, x]$. We set $h_\infty(x) := \frac{1}{c_{\alpha_4}^2} (D^2 f_0(p^{(i)}_{\infty}) [x, x] + 2)$.

From the definition of $r_n$ and reasoning as it has already been done, one obtains, again by means of estimates involving the Green function for $P_{g_0}$, that $\sup_{z \in \Omega} |\nabla^j \tilde{u}_n|_{\tilde{g}_n}(z) < C$ for $j = 1, 2, 3$ and uniformly in $n$. Finally, for any $\Omega$ containing $B_L(0)$, we obtain for any $z \in \Omega$
\[
|\tilde{u}_n(z)| \leq |\tilde{u}_n(z) - \tilde{u}_n(x_n/r_n)| + |\tilde{u}_n(x_n/r_n)|
\leq \sup_{w \in \Omega} |\nabla \tilde{u}_n|_{\tilde{g}_n}(w) |z - x_n/r_n| + C \leq C(\Omega).
\]

Then, as above, we can extract by means of standard elliptic estimates a sequence $(\tilde{u}_n)_n$ converging in $C^4_{\text{loc}}(\mathbb{R}^4)$ to a function $\tilde{w}$, which solves
\[
\Delta^2_{\mathbb{R}^4} \tilde{w} = h_\infty(x) e^{4\tilde{w}} \text{ on } \mathbb{R}^4,
\]
with finite volume and finite total curvature
\[
\int_{\mathbb{R}^4} e^{4\tilde{w}} dx < \infty, \quad \int_{\mathbb{R}^4} |h_\infty| e^{4\tilde{w}} dx < \infty.
\]
Therefore, setting $w := \tilde{w} - 1/4 \log(c_{\alpha_4}^2)$, we obtain alternative b) of Thm 1.4.

iii) We finally deal with the case
\[
\limsup_n \sqrt[4]{\lambda_n} \frac{1}{|x_n|} \leq \beta \sqrt[4]{\frac{3\alpha_4}{2}},
\]
where $\beta \geq 2$. With this assumption and recalling (4.16), we deduce the existence of a constant $C \geq 1$ such that $C^{-1} \sqrt[4]{\lambda_n} \leq |x_n| \leq C \sqrt[4]{\lambda_n}$. We define $r_n$, the map $V_n$ and $\tilde{u}_n$ in the same way as in step ii). Then, it follows
\[
\inf_n \tilde{u}_n(x_n/r_n) > -\infty \quad (4.21)
\]
and $\{x_n/r_n\}_n \subset B_L(0)$ for some $L > 0$.

Moreover, once again we obtain, for any $\Omega \subset \subset \mathbb{R}^4$, equation (4.20), uniform convergence of the $Q$-curvature to $h_\infty$ as well as $\sup_{z \in \Omega} |\nabla^j \tilde{u}_n|_{\tilde{g}_n}(z) < C$ for $j = 1, 2, 3$ and uniformly in $n$.

Now we fix $\Omega := B_R(0)$ with $R > L$ and define
\[
v_n(x) := \tilde{u}_n(x) - \tilde{u}_{n,R}, \quad x \in B_R(0)
\]
where \( \hat{u}_{n,R} := \frac{1}{Vol(B_R(0); \hat{g}_n)} \int_{B_R(0)} \hat{u}_n dV_{\hat{g}_n} \). We choose \( p > 4/3 \). Hence, via Poincaré’s inequality, via the estimates involving the derivatives of \( \hat{u}_n \), and recalling that \( \hat{g}_n \to \delta_{R^4} \) in \( C^m_{\text{loc}}(\mathbb{R}^4) \) for any \( m \geq 1 \), we obtain that \( (v_n)_n \) is bounded in \( W^{3,p}(B_R(0), dx) \). Therefore, by reflexivity and Sobolev embedding, we obtain, up to subsequences, that \( v_n \to v_\infty \) in \( C^0(B_R(0)) \).

We observe that
\[
C \geq \int_{B_R(0)} e^{4u_n} dV_{\hat{g}_n} = \int_{B_R(0)} e^{4v_n} dV_{\hat{g}_n} \exp(4\hat{u}_{n,R})
\]

with \( o(1) \to 0 \) as \( n \to \infty \). Therefore, there holds \( \hat{u}_{n,R} \leq C \) uniformly in \( n \).

From (4.21) and the fact \( \{x_n/r_n\}_n \subset B_{L}(0) \subset B_R(0) \), we also infer \( \hat{u}_{n,R} \geq -C \).

Hence, up to subsequences, as already done in step ii), we obtain once again locally smooth convergence of \( \hat{u}_n \) to the limit function of alternative b) of Thm 1.4. That completes the proof. \( \square \)

**Remark 4.6.** If we couple Eqs. (4.11) and (4.1), we infer that our sequence \( (u_n)_n \) can blow up at at most \( I = 8 \) points, regardless of the number of points of maximum which \( f_0 \) possesses. Therefore, if the function \( f_0 \) has more than 8 non-degenerate points of maximum, in principle one could expect that for all \( 0 < \lambda < < 1 \) the functional \( E_\lambda \) admits at least three different critical points.

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**5. Appendices**

**Appendix 1.** We are going to prove respectively Eqs. (3.20), (3.22) and (3.21). Recalling (3.15), we can expand \( M_1 = I + II + III \) where

\[
I := \int_{\lambda x \leq |x| \leq \sqrt{\lambda}} |x|^{-4} \left[ \delta^{-1} \xi'' \left( \delta^{-1} \log \left( \frac{1}{|x|} \right) \right) - 2 \xi' \left( \delta^{-1} \log \left( \frac{1}{|x|} \right) \right) \right]^2 dx,
\]

\[
II := \int_{\sqrt{\lambda} \leq |x| \leq \frac{1}{2}} 4|x|^{-4} dx,
\]

\[
III := \int_{\frac{1}{2} \leq |x| \leq 1} \left\{ -\tau'(|x|) |x|^{-1} \left[ 2 + 5 \log \left( \frac{1}{|x|} \right) \right] \right. \\
- 2 \tau(|x|) |x|^{-2} + \log \left( \frac{1}{|x|} \right) \tau''(|x|) \right\}^2 dx.
\]
Recalling Eq. (3.10) and that \( \delta = \frac{1}{2} \log (1/\lambda) \), and using the abbreviations \( \xi''(\cdot) := \xi'' \left( \delta^{-1} \log \left( \frac{1}{|x|} \right) \right) \) and \( \xi'(\cdot) := \xi' \left( \delta^{-1} \log \left( \frac{1}{|x|} \right) \right) \), we obtain

\[
I = 4 \int_{\lambda \leq |x| \leq \sqrt{\lambda}} |x|^{-4} \left[ \frac{(\xi''(\cdot))^2}{\log^2 (1/\lambda)} + (\xi'(\cdot))^2 - \frac{2 \xi''(\cdot) \xi'(\cdot)}{\log (1/\lambda)} \right] dx
\]

\[
\leq 4 \left[ \frac{||\xi''||_\infty^2}{\log^2 (1/\lambda)} + A_0^2 \right] \int_{\lambda \leq |x| \leq \sqrt{\lambda}} |x|^{-4} dx
\]

\[
= 4\pi^2 \left[ \frac{||\xi''||_\infty^2}{\log (1/\lambda)} + 2A_0 ||\xi''||_\infty + A_0^2 \log (1/\lambda) \right]
\]

and

\[
II = -8\pi^2 \log 2 + 4\pi^2 \log (1/\lambda).
\]

For the last term we have

\[
III \leq 3 \int_{\frac{1}{2} \leq |x| \leq 1} \left( \tau'(|x|) \right)^2 |x|^{-2} \left[ 2 + 5 \log \left( \frac{1}{|x|} \right) \right]^2
\]

\[
+ 4\pi^2 (|x|) |x|^{-4} + 2 \log^2 \left( \frac{1}{|x|} \right) (\tau''(|x|))^2 dx.
\]

Since the functions \( r \mapsto \log \left( \frac{1}{r} \right) \) and \( r \mapsto \log^2 \left( \frac{1}{r} \right) \) are monotone decreasing and positive on the interval \((\frac{1}{2}, 1)\), we obtain

\[
III \lesssim \int_{\frac{1}{2} \leq |x| \leq 1} \left( |x|^{-2} + |x|^{-4} \right) dx + C \leq C
\]

uniformly in \( \lambda \). Hence, choosing a smaller \( \lambda^\varepsilon \) if necessary, we infer that for all \( 0 < \lambda < \lambda^\varepsilon \)

\[
M_1 \leq 4\pi^2 (A_0^2 + 1) \log (1/\lambda) + C_0,
\]

where \( C_0 \) depends at most quadratically on the supremum norm of \( \xi'' \) but it does not depend on \( \lambda \).

In order to obtain Eq. (3.22), we note that, in view of (3.14), the supremum norm of the radial derivative \( z''_\lambda \) on the ball \( B_1(0) \) is of the order \( O(\lambda^{-1}) \) as \( \lambda \downarrow 0 \). Therefore, it holds

\[
||z''_\lambda O''(r^{N-1})||_\infty = O(\lambda^{N-3}),
\]

which leads to

\[
M_3 = O(\lambda^{N-3}).
\]

Finally, combining the estimates about \( M_1 \) and \( M_3 \) and by Hölder’s inequality, we obtain

\[
|M_2| \leq 2 \int_{B_1(0)} |\Delta_{\mathbb{R}^4} z_{\lambda} z''_\lambda O''(r^{N-1})| dx
\]

\[
\leq 2 ||\Delta_{\mathbb{R}^4} z_{\lambda}||_{L^1(B_1(0))} ||z''_\lambda O''(r^{N-1})||_\infty = O \left( \lambda^{N-3} \right)
\]

as \( \lambda \downarrow 0 \).
Appendix 2. We are going to prove \((4.7)\). Starting from the last two lines of \((4.6)\), we obtain respectively:

\[
\int_M |v_n \Delta_{g_0} v_n| |\nabla_{g_0} \eta|^2_{g_0} \, dV_{g_0} \leq \|\nabla_{g_0} \eta\|_{L^\infty}^2 \|\Delta_{g_0} v_n\|_{L^t(M)} \|v_n\|_{L^{t'}(M)} \\
\leq C(\eta, d), \quad t \in (1, 2), \quad t' = \frac{t}{t-1};
\]

\[
\int_M (v_n \Delta_{g_0} \eta)^2 \, dV_{g_0} \leq \|\Delta_{g_0} \eta\|_{L^\infty}^2 \|v_n\|_{L^2(M)}^2 \leq C(\eta, d); 
\]

\[
\int_M v_n^2 |D(\nabla_{g_0} \eta, \nabla_{g_0} \eta)| \, dV_{g_0} \leq c(M, g_0) \|\nabla_{g_0} \eta\|_{L^\infty}^2 \|v_n\|_{L^2(M)}^2 \leq C(\eta, d); 
\]

\[
\int_M |v_n \Delta_{g_0} \eta g_0 (\nabla_{g_0} v_n, \nabla_{g_0} \eta)| \, dV_{g_0} \leq \|\Delta_{g_0} \eta\|_{L^\infty} \|\nabla_{g_0} \eta\|_{L^\infty} \|\nabla_{g_0} v_n\|_{L^t(M)} \|v_n\|_{L^{t'}(M)} \\
\leq C(\eta, d), \quad t \in (1, 4), \quad t' = \frac{t}{t-1};
\]

\[
\int_M g_0 (\nabla_{g_0} v_n, \nabla_{g_0} \eta)^2 \, dV_{g_0} \leq \|\nabla_{g_0} \eta\|_{L^\infty}^2 \|\nabla_{g_0} v_n\|_{L^2(M)}^2 \leq C(\eta, d); 
\]

the claim follows.

Appendix 3. We are going to prove the following:

**Proposition 5.1.** Let \((M, g_0)\) be closed and connected with \(k_P < 0\), \(P_{g_0} \geq 0\) and \(\text{ker}(P_{g_0}) = \{\text{constants}\}\). Let \(0 \neq f \in C^2(M)\). Then the functional \((1.7)\) satisfies the Palais-Smale condition at any level \(\beta \in \mathbb{R}\).

**Proof.** Let \((u_k)_k \subset H^2(M; g_0)\) be a Palais-Smale sequence at the level \(\beta\) for the functional \(E_f\), viz. as \(k \to \infty\)

\[
E_f(u_k) = \langle P_{g_0} u_k, u_k \rangle + 4Q_{g_0} \int_M u_k \, dV_{g_0} - \int_M f e^{4u_k} \, dV_{g_0} \to \beta 
\]

and

\[
\|DE_f(u_k)\| \to 0.
\]

Therefore, since in particular \(DE_f(u_k)[1] \to 0\), we obtain \(\int_M f e^{4u_k} \, dV_{g_0} \to k_P\) and

\[
\langle P_{g_0} u_k, u_k \rangle + 4Q_{g_0} \int_M u_k \, dV_{g_0} = \beta + k_P + o(1) \quad (5.1)
\]

as \(k \to \infty\).

Claim: \(\sup_k \int_M u_k \, dV_{g_0} < \infty\).

We argue by contradiction and assume that there exists a subsequence still denoted \(u_k\) such that \(\lim_k \int_M u_k \, dV_{g_0} = \infty\). Hence, by Hölder inequality it follows that also the quantity \(\|u_k\|_{L^2(M)}\) tend to \(\infty\). We set

\[
v_k := \frac{u_k}{\|u_k\|_{L^2(M)}}.
\]
Therefore, up to subsequences, we can assume
\[ \langle P_{g_0} v_k, v_k \rangle = \langle P_{g_0} u_k, u_k \rangle \| u_k \|_{L^2(M)}^{-2} \]
\[ = \left( -4Q_{g_0} \int_M u_k \, dV_{g_0} + \beta + k_P + o(1) \right) \| u_k \|_{L^2(M)}^{-2}. \]
Because \( | -4Q_{g_0} \int_M u_k \, dV_{g_0} | \| u_k \|_{L^2(M)}^{-2} \leq -4Q_{g_0} \| u_k \|_{L^2(M)}^{-1} \), it follows that the right hand side of the expression above tends to zero as \( k \to \infty \) and consequently
\[ \langle P_{g_0} v_k, v_k \rangle \to 0. \]

Therefore, up to subsequences, we can assume \( v_k \to v \) in \( H^2(M; g_0) \) and \( v_k \to v \) in \( L^2(M) \). By means of Poincaré’s inequality, we infer \( v \equiv c \in \{-1, 1\} \). On the other hand,
\[ \int_M u_k \, dV_{g_0} / \| u_k \|_{L^2(M)} = \int_M v_k \, dV_{g_0} \to c \]
and, because by assumption \( \lim_k \int_M u_k \, dV_{g_0} = \infty \), we deduce \( v \equiv 1 \). We define \( \phi_k = \frac{f}{\| u_k \|_{L^2(M)}} \in H^2(M; g_0) \). Obviously, \( \phi_k \to 0 \) in \( H^2(M; g_0) \) and so it follows
\[ \langle P_{g_0} u_k, \phi_k \rangle + 2Q_{g_0} \int_M \phi_k \, dV_{g_0} - 2 \int_M f \phi_k e^{4u_k} \, dV_{g_0} \to 0 \]
or, equivalently,
\[ \langle P_{g_0} v_k, f \rangle + 2Q_{g_0} \int_M f \, dV_{g_0} / \| u_k \|_{L^2(M)}^{-1} - 2 \int_M \frac{f^2 e^{4u_k}}{\| u_k \|_{L^2(M)}} \, dV_{g_0} \to 0 \]
From above, it thus follows \( \int_M \frac{f^2 e^{4u_k}}{\| u_k \|_{L^2(M)}} \, dV_{g_0} \to 0 \). On the other hand, we have
\[ \int_M \frac{f^2 e^{4u_k}}{\| u_k \|_{L^2(M)}} \, dV_{g_0} \geq \int_M \frac{4f^2 u_k}{\| u_k \|_{L^2(M)}} \, dV_{g_0} = \int_M 4f^2 v_k \, dV_{g_0} \to \int_M 4f^2 \, dV_{g_0} > 0 \]
since \( v_k \to 1 \) in \( L^2(M) \). The contradiction proves the claim.

Equation (5.1) also implies \( 4Q_{g_0} \int_M u_k \, dV_{g_0} \leq \beta + k_P + o(1) \) and therefore
\[ \inf_k \int_M u_k \, dV_{g_0} > -\infty \] and
\[ \sup_k \left| \int_M u_k \, dV_{g_0} \right| < \infty. \]

Again by (5.1) it follows \( \sup_k \langle P_{g_0} u_k, u_k \rangle < \infty \) and by Poincaré’s inequality we conclude that \( (u_k)_k \) is bounded in \( H^2(M; g_0) \).

From this fact, it is now standard to extract from \( (u_k)_k \) a converging subsequence in \( H^2(M; g_0) \). That concludes the proof. \( \square \)
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