ON THE STRONG FRESE-NATION PROPERTY

DAVID MILOVICH

Abstract. We show that there is a boolean algebra that has the Freese-Nation property (FN) but not the strong Freese-Nation property (SFN), thus answering a question of Heindorf and Shapiro. Along the way, we produce some new characterizations of the FN and SFN in terms of sequences of elementary submodels.

1. The Freese-Nation property and friends

Definition 1.1.

• Given a poset $P$ and a map $f$ from $P$ to the power set of $P$, we say that $f$ is interpolating if, for all pairs $x \leq_P y$, there exists $z \in [x, y] \cap f(x) \cap f(y)$.

• We say that poset $P$ has the FN if there is an interpolating map from from $P$ to $[P]^{<\aleph_0}$, the set of finite subsets of $P$. Such a map is called and FN map.

When $P$ is also a boolean algebra (that is, when every finite subset has a sup and inf, and for every $x \in P$ there exists $-x \in P$ such that $\sup\{x, -x\} = \inf\emptyset$ and $\inf\{x, -x\} = \sup\emptyset$), the FN can be understood an abstraction of the Interpolation Theorem of propositional logic, which states that if the implication $\varphi \rightarrow \psi$ is tautological for two propositions $\varphi$ and $\psi$, then there is a proposition $\chi$ such that $\varphi \rightarrow \chi$ and $\chi \rightarrow \psi$ are tautological and the propositional variables of $\chi$ are common to $\varphi$ and $\psi$. An easy consequence of the Interpolation Theorem is that free boolean algebras have the FN.

The Freese-Nation property (FN) is named after Freese and Nation \cite{3} who introduced it in 1978 as part of a characterization of projective lattices. In particular, every projective lattice has the FN (but the converse was already known to be false, even for finite lattices). Since the morphisms in the category of lattices and lattice homomorphisms are epimorphisms if and only if they are surjective, a lattice is projective if and only if it is a retract of a free lattice. Likewise, a boolean algebra is projective if and only if it is a retract of a free boolean algebra. The Stone duals of the projective boolean algebras are exactly the Dugundji spaces, i.e., the continuous retracts of powers of 2.

The Stone duals of the boolean algebras with the FN were elegantly characterized in two ways by Ščepin \cite{11,12}, as the existence of a distance function between points and regular closed sets and as the existence of rich family of open quotient maps. Succinctly, a compact Hausdorff space is “$k$-metrizable” if and only if it is “openly generated;” a boolean space is openly generated if and only if its clopen algebra

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DAVID MILOVICH

has the FN. Ščepin also proved that every Dugundji space is openly generated, that the Vietoris hyperspace operation preserves open generation, and that every openly generated boolean space of weight at most $\aleph_1$ is Dugundji. However, Shapiro [10] proved that the Vietoris hyperspace of $2^\kappa$ is not a continuous image of a power of 2 if $\kappa \geq \aleph_2$. Thus, for boolean algebras up to size $\aleph_1$, the FN is equivalent to projectivity, while for boolean algebras in general, projectively strictly implies the FN.

Fuchino translated Ščepin’s notion of openly generated into the language of elementary substructures in an appendix to [5]. Before we can state this characterization, we need a few definitions.

Definition 1.2.

- If $P$ is a poset, $S$ is a set, and $p \in P$, then, when they exist, let
  - $\pi^S_+(p) = \min\{q \in P \cap S : q \geq p\}$ and
  - $\pi^S_-(p) = \min\{q \in P \cap S : q \leq p\}$.
- Given a poset $P$ and $Q \subseteq P$, we say that $Q$ is a relatively complete suborder of $P$ if, for all $p \in P$, $\pi^Q_+(p), \pi^Q_-(p)$ exist.
- Given boolean algebras $A$ and $B$, we write $A \leq B$ to indicate that $A$ is a subalgebra of $B$.
- If $B$ is a boolean algebra, $A \leq B$, and $A$ is relatively complete suborder of $B$, then we write $A \leq_{rc} B$.
- If $\psi : A \to B$ is a boolean homomorphism, we say that $\psi$ is relatively complete if $\psi[A] \leq_{rc} B$.

Note the topological dual of a relatively complete boolean homomorphism is an open map between two boolean spaces.

Definition 1.3.

- Given two sets $P$ and $Q$, we write $P \prec Q$ if $(P, \in)$ is an elementary substructure of $(Q, \in)$.
- Given a cardinal $\mu$, let $H(\mu)$ denote the set of all sets with transitive closure smaller than $\mu$.

Given a boolean algebra $A = (A, 0, 1, \land, \lor, -)$, we will abuse notation by using $A$ to denote both $A$ and $A$. In particular, when we write $A \in M$ for some set $M$, we mean $A \in M$.

Theorem 1.1 (Fuchino). Let $A$ be a boolean algebra and let $\mu$ be a regular uncountable cardinal such that $A \in H(\mu)$. The following are then equivalent.

1. $A$ has the FN.
2. $A \cap M \leq_{rc} A$ for all countable $M$ satisfying $A \in M \prec H(\mu)$.
3. $A \cap M \leq_{rc} A$ for all $M$ satisfying $A \in M \prec H(\mu)$.

If we weaken the definition of FN map to allow as outputs countable sets instead of merely finite sets, then we obtain the weak Freese-Nation property (WFN), which was initially investigated in topological terms by Ščepin [12] and later systematically studied in Heindorf and Shapiro’s 1994 book Nearly projective boolean algebras [5]. For our purposes, their most interesting result about the WFN is a characterization of it as the existence of a rich family of commuting subalgebras. Elementary substructure characterizations analogous to (2) and (3) from the previous theorem were proved by Fuchino, Koppelberg, and Shelah in [4] and by the author in [9], respectively.
Definition 1.4.

- Given a poset $P$ and $A, B \subseteq P$, we say that $A$ and $B$ *commutes*, writing $A \downarrow B$, if, for all pairs $(x, y) \in A \times B$, if $x \leq y$, then $[x, y] \cap A \cap B$ is nonempty, and if $y \leq x$, then $[y, x] \cap A \cap B$ is nonempty.

- Given a poset $P$ and $Q \subseteq P$, we say that $Q \subseteq \sigma_P$ if, for all $p \in P$, there exist countable sets $L(p), U(p) \subseteq Q$ such that
  - $\{q \in Q : q \leq p\} = \bigcup_{r \in L(p)} \{q \in Q : q \leq r\}$ and
  - $\{q \in Q : q \geq p\} = \bigcup_{r \in U(p)} \{q \in Q : q \geq r\}$.

Note that if $A$ and $B$ are subalgebras of a boolean algebra $C$, then $A \downarrow B$ if and only if, for all ultrafilters $U$ of $A$ and $V$ of $B$, if $U \cap B = V \cap A$, then $U \cup V$ extends to an ultrafilter of $C$.

Theorem 1.2. Let $A$ be a boolean algebra and let $\mu$ be a regular uncountable cardinal such that $A \in H(\mu)$. The following are then equivalent.

1. $A$ has the WFN, i.e., there is an interpolating map from $A$ to $[A]^{<\aleph_1}$.
2. (Fuchino, Koppelberg, Shelah) $A \cap M \subseteq \sigma_A$ for all $M$ satisfying $A \in M \prec H(\mu)$ and $|M| = \omega_1 \subseteq M$.
3. $A \cap M \subseteq \sigma_A$ for all $M$ satisfying $A \in M \prec H(\mu)$ and $\omega_1 \subseteq M$.
4. (Ščepin, Heindorf, Shapiro) There is a cofinal family $C$ of countable subalgebras of $A$ such that $F \downarrow G$ for all $F, G \in C$.

In [5], Heindorf and Shapiro defined the natural analog of (4) for the FN to be the strong Freese-Nation property (SFN).

Definition 1.5. A boolean algebra has the SFN if and only if it has a pairwise commuting cofinal family of finite subalgebras.

Also in [5], Heindorf and Shapiro showed that projectivity implies the SFN implies the FN. Hence, the three properties are equivalent for boolean algebras of size at most $\aleph_1$. They further showed that the symmetric square and exponential operations preserve the SFN; Ščepin had already shown the same for the FN [11]. On the other hand, if $\kappa \geq \aleph_2$, then the exponential [10] and symmetric square [11] of a free boolean algebra of size $\kappa$ are not projective. Thus, the two most natural examples of non-projective boolean algebras with the FN also have the SFN. Naturally, Heindorf and Shapiro posed the question of whether the SFN is actually equivalent to FN. Twenty years later, there appears to have been no subsequent published work on the SFN. The primary motivation of this work is to answer Heindorf and Shapiro’s question.

Theorem 1.3. There is a boolean algebra of size $\aleph_2$ that has the FN but not the SFN.

To prove the above, we required new characterizations of the FN and SFN in terms of “retrospective” sequences of countable elementary submodels, as we shall explain shortly. We expect that the techniques we use here for separating the FN and SFN will see wider application in the future, and have stated many lemmas in anticipating generality.

For additional background information about the classes of boolean algebras defined by the SFN, FN, WFN, and projectivity, we refer the reader to [5].
2. Retrospective sequences of elementary substructures

Our proof of Theorem 1.3 uses long \( \lambda \)-approximation sequences, which one can think of as a poor man’s higher-gap morasses, available in ZFC. These sequences were introduced in [8] as a more flexible version of Davies’ trees of substructures [1]. Davies used such a tree to prove that the plane is a union of countably many rotated graphs of functions; Jackson and Mauldin [6] used such a tree to prove that there exists a subset of the plane intersecting every isometric copy of \( \mathbb{Z}^2 \) at exactly one point. The main application of long \( \lambda \)-approximation sequences in [8] was to prove that, for a class of topological spaces that includes every compact group, every topological base of a space contains a base of the same space which is lower finite with respect to containment.

Definition 2.1.

- Call a sequence of sets \( (A_i)_{i \in I} \) retrospective if \( I \) is an ordinal and, for all \( i \in I \), the sequence \( (A_j)_{j < i} \) is an element of \( A_i \).
- Given \( \mu \) a regular uncountable cardinal and \( \lambda \) a regular uncountable cardinal at most \( \mu \), call a set \( M \) \( \lambda \)-approximating if \( |M| < \lambda \), \( M \cap \alpha \subseteq \lambda \) and \( M < H(\mu) \).
- Given \( \mu \) and \( \lambda \) as above, and \( \eta \) an ordinal at most \( \mu \), a transfinite sequence \( (M_i)_{i \leq \eta} \) is called a long \( \lambda \)-approximation sequence if it is retrospective and \( M_i \) is \( \lambda \)-approximating for all \( i < \eta \).

(In Definition 3.16 of [8], it was required of long \( \lambda \)-approximation sequences that also \( |M_i| \subseteq M_i \), \( \lambda \subseteq M_i \) and \( \lambda \subseteq M \). Here, we do not require \( |M_i| \subseteq M_i \) because it not needed for any applications (so far). We do not require \( \lambda \subseteq M_i \) for the same reason, and because if \( \lambda \leq i < \eta \), then \( \lambda \) is definable in \( M_i \) as \( \sup_{j < i} \min(i \setminus M_j) \).

The requirement \( M_i \cap \lambda \in \lambda \) is succinct but perhaps obscures its intended application, which is that for all \( A \in M_i \), if \( |A| < \lambda \), then \( A \subseteq M \). In particular, if \( i < \lambda \), then \( \bigcup_{j < i} M_j \subseteq M_i \). Also notice that the requirement \( M_i \cap \lambda \in \lambda \) is redundant if \( \lambda = \omega_1 \).

Lemma 2.1. Given regular uncountable cardinals \( \lambda \leq \mu \), \( A \in [H(\mu)]^{\lambda} \), \( \eta < \mu \), and \( (M_i)_{i \leq \eta} \) a long \( \lambda \)-approximation sequence, there exists \( M_\eta \) such that \( A \subseteq M_\eta \).

Proof. Let \( B = A \cup \{(M_i)_{i \leq \eta}\} \) and choose \( M_\eta = \bigcup_{n < \omega} N_n \) where \( B \subseteq N_0 \), \( |N_n| < \lambda \), \( N_n < H(\mu) \), and \( N_n \cup \sup(\lambda \cap N_n) \subseteq N_{n+1} \) for all \( n \).

Lemma 2.2. Given \( (M_i)_{i \leq \eta} \) as in the above definition and \( \alpha, \beta < \eta \), the following are equivalent.

1. \( \alpha \in \beta \cap M_\beta \)
2. \( M_\alpha \subseteq M_\beta \)
3. \( M_\alpha \subseteq M_\beta \)

Proof. Given [1], we have [2] because \( M_\alpha \) is definable from \( \alpha \) and \( (M_\gamma)_{\gamma < \beta} \). Given [2], we have \( M_\alpha \subseteq M_\beta \) because \( |M_\alpha| \subseteq M_\beta \cap \lambda \in \lambda \); we also have \( \{M_\alpha\} \subseteq M_\beta \setminus M_\alpha \). Given [3], we have \( \alpha \neq \beta \); we also have \( \alpha \in M_\beta \) because \( \alpha \) is definable in \( M_\alpha \) from \( (M_\gamma)_{\gamma < \alpha} \); we also have \( \alpha \leq \beta \) because otherwise \( M_\alpha \subseteq \bigcup_{\gamma < \alpha} M_\gamma \), which is impossible because \( H(\mu) \subseteq \bigcup_{\gamma < \alpha} M_\gamma \) and \( \bigcup_{\gamma < \alpha} M_\gamma \in M_\alpha < H(\mu) \).

Lemma 2.3. If \( S \in M_0 \) and \( (M_i)_{i \leq |S|} \) is a long \( \lambda \)-approximation sequence, then \( S \subseteq \bigcup_{i \leq |S|} M_i \).
Proof. Some \( f \in M_0 \) is a surjection from \(|S|\) to \( S \). By Lemma 2.2, \( M_0 \subseteq M_\alpha \) for all \( \alpha > 0 \), so \( f(\alpha) \in M_\alpha \) for all \( \alpha < |S| \).

**Definition 2.2.** Given an ordinal \( \alpha \) and an infinite cardinal \( \lambda \), let the **\( \lambda \)-truncated cardinal normal form of \( \alpha \)** denote the unique polynomial

\[
\omega_{\beta_0} \gamma_0 + \cdots + \omega_{\beta_{m-1}} \gamma_{m-1} + \gamma_m
\]
equal to \( \alpha \) and satisfying \( \omega_{\beta_0} > \cdots > \omega_{\beta_{m-1}} \geq \lambda \), \( \gamma_i \in [1, \omega_{\beta_i}] \) for all \( i < m \), and \( \gamma_m < \lambda \). For each \( i < m \), let \( \partial_\alpha \) denote \( \omega_{\beta_i} \gamma_i \) and let \( [\alpha]_{i+1} \) denote \( \sum_{j<i+1} \partial_\alpha \); let \( [\alpha]_0 = 0 \) and \( [\alpha]_{m+1} = \alpha \). Let \( \Uparrow(\alpha) \) denote \( m \) if \( \gamma_m = 0 \); \( m+1 \) if \( \gamma_m > 0 \).

Observe that \( [\alpha]_i \) is \( \{ \alpha \} \)-definable in \( H(|\alpha|^+) \) for each \( i \leq \Uparrow(\alpha) \). Hence, if \( (M_\beta)_{\beta<\alpha} \) is a long \( \lambda \)-approximation sequence, \( \zeta + \eta \leq \alpha \), and \( [\zeta + \beta, \eta]_{M_\xi} = \zeta \) for all \( \beta < \eta \), then, for each \( \beta < \eta \), \( (M_\zeta+\eta)_{\gamma<\beta} \) is definable in \( M_{\zeta+\beta} \). Thus, such an \( (M_\zeta+\beta)_{\beta<\eta} \) is a long \( \lambda \)-approximation sequence. We will use this last observation to prove the fundamental lemma for long \( \lambda \)-approximation sequences, which is the existence of a definable finite partition into directed segments.

**Lemma 2.4.** Given a long \( \lambda \)-approximation sequence \( (M_\beta)_{\beta<\alpha} \), the sets \( \{ M_\beta : [\alpha]_i \leq \beta < [\alpha]_{i+1} \} \) are directed with respect to inclusion for all \( i < \Uparrow(\alpha) \).

**Proof.** A proof is implicit in the proof of Lemma 3.17 of [8], but we will provide a shorter explicit proof here. Proceed by induction on \( \alpha \). If \( \alpha \leq \lambda \), then \( \{ M_\beta : \beta < \alpha \} \) is a chain. If \( \Uparrow(\alpha) \geq 2 \), then each \( \{ M_\beta : [\alpha]_i \leq \beta < [\alpha]_{i+1} \} \) is directed by our induction hypothesis applied to \( (M_{\alpha_i+\beta})_{\beta<\partial_\alpha} \). So, suppose that \( \alpha > \lambda \) and \( \Uparrow(\alpha) = 1 \). If \( \alpha = \sup \{ \beta < \alpha : \Uparrow(\beta) = 1 \} \), then \( \{ M_\beta : [\alpha]_i \leq \beta < [\alpha]_{i+1} \} \) is directed because each \( \{ M_\beta : [\alpha]_i \leq \beta < [\alpha]_{i+1} \} \) is directed by induction. So, suppose that \( \alpha = \kappa(\gamma + 1) \) where \( \kappa \) is a cardinal and \( 1 \leq \gamma < \kappa^+ \). Set \( \beta = \kappa \gamma \) and \( S = \{ M_\delta : \delta < \beta \} \). By Lemma 2.3 applied to \( S \) and \( (M_{\beta+\delta})_{\delta<\kappa} \), we have \( S \subseteq \bigcup_{\delta<\kappa} M_{\beta+\delta} \). Hence, by Lemma 2.2 for every \( \varepsilon < \beta \) there exists \( \delta < \kappa \) such that \( M_\varepsilon \subseteq M_{\beta+\delta} \). Therefore, \( \{ M_\delta : [\alpha]_i \leq \beta < [\alpha]_{i+1} \} \) is directed because its cofinal subset \( \{ M_{\beta+\delta} : \delta < \kappa \} \) is directed by our inductive hypothesis applied to \( (M_{\beta+\delta})_{\delta<\kappa} \).

If \( n < \omega \) and \( (M_\alpha)_{\alpha<\lambda^n} \) is a long \( \lambda \)-approximation sequence, then, since \( \Uparrow(\alpha) \leq n+1 \) for all \( \alpha < \lambda^n \), we can sometimes use \( (M_\alpha)_{\alpha<\lambda^n} \) like a \( (\lambda, n) \)-morass, in the weak sense that we can build a \( \lambda^n \)-sized object as a direct limit of small (that is, \( < \lambda \))-sized pieces while locally only having to fit together at most \( n+1 \) small direct limits of these small pieces. Of course, we lack the additional coherence properties of a \( (\lambda, n) \)-morass, which require assumptions beyond ZFC. However, the citations given at the beginning of this section demonstrate that long \( \lambda \)-approximation sequences are useful even without such coherence. We will find them useful again in this paper. See also [2] for very recent additional applications.

We finish this section with some additional lemmas about long \( \lambda \)-approximation sequences that we will need later.

**Definition 2.3.** Given a long \( \lambda \)-approximation sequence \( (M_\beta)_{\beta<\eta} \), \( \alpha \leq \eta \), and \( i < \Uparrow(\alpha) \), let

\( I_i(\alpha) = [[\alpha]_i : [\alpha]_{i+1}] \);
\( I_i'(\alpha) = I_i(\alpha) \cap M_\alpha \);
\( I_i(\alpha) = \{ M_\beta : \beta \in I_i(\alpha) \} \);
\( I_i'(\alpha) = \{ M_\beta : \beta \in I_i'(\alpha) \} \);
\[ M_{\alpha,i} = \bigcup J_i(\alpha); \]
\[ M'_{\alpha,i} = M_{\alpha,i} \cap M_\alpha. \]

**Lemma 2.5.** If \((M_\alpha)_{\alpha<\eta}\) is a long \(\lambda\)-approximation sequence, \(i < \gamma(\eta)\), and \(\partial_i \eta \geq \lambda\), then \(|M_{\eta,i}| = \partial_i \eta \subseteq M_{\eta,i} \).

**Proof.** Since \((M_{\eta,i+\alpha})_{\alpha<\partial_i \eta}\) is a long \(\lambda\)-approximation sequence, we have \(\partial_i \eta \subseteq M_{\eta,i} \). Since each \(M_{\eta,i+\alpha}\) is smaller than \(\lambda\), we have \(\partial_i \eta = |M_{\eta,i}|\). \(\square\)

**Lemma 2.6.** If \((M_\alpha)_{\alpha<\eta}\) is a long \(\lambda\)-approximation sequence and \(S \in M_0\), then, for all \(\alpha < \eta\), \(S \subseteq M_\alpha\) for all \(i < \gamma(\alpha)\).

**Proof.** By Lemma \ref{2.2}, \(M_0 \subseteq M_\alpha\) for all \(\alpha < \eta\). Hence, also \(M_0 \subseteq \bigcup_{\beta \in I_i(\alpha)} M_\beta\) for all \(\alpha < \eta\) and \(i < \gamma(\alpha)\). \(\square\)

**Definition 2.4.** Because \(I_i(\alpha)\) and \(J_i(\alpha)\) may not be downward closed in \(\{M_\beta : \beta < \eta\}\) with respect to inclusion, we also define
\[ J_i(\alpha) = \bigcup \{M_\beta \cap (\beta + 1) : \beta \in I_i(\alpha)\}; \]
\[ J'_i(\alpha) = \bigcup \{M_\beta \cap (\beta + 1) : \beta \in J_i(\alpha)\}; \]
\[ J_i(\alpha) = \{M_\beta : \beta \in J_i(\alpha)\}; \]
\[ J'_i(\alpha) = \{M_\beta : \beta \in J'_i(\alpha)\}. \]

By Lemma \ref{2.2}, \(J_i(\alpha)\) and \(J'_i(\alpha)\) are downward closed in \(\{M_\beta : \beta < \eta\}\) with respect to inclusion. Also observe that, by elementarity and retrospectiveness,
\[ I_i(\alpha) = I_i(\alpha) \cap M_\alpha; \]
\[ J_i(\alpha) = J_i(\alpha) \cap M_\alpha; \]
\[ J'_i(\alpha) = J'_i(\alpha) \cap M_\alpha. \]

**Lemma 2.7.** Given a long \(\lambda\)-approximation sequence \((M_\beta)_{\beta<\alpha+1}\) and \(i < \gamma(\alpha)\), \(I_i(\alpha)\) and \(J_i(\alpha)\) are directed with respect to inclusion with respective unions \(M_{\alpha,i}\) and \(M'_{\alpha,i}\). Moreover, \(I_i(\alpha)\) is cofinal in \(J_i(\alpha)\) and \(J_i(\alpha)\) is cofinal in \(J'_i(\alpha)\).

**Proof.** By Lemma \ref{2.4}, \(I_i(\alpha)\) is directed; by definition, its union is \(M_{\alpha,i}\). Since \((M_\beta : \beta \in I_i(\alpha)) \in H(\mu)\), the set \(I_i(\alpha) \cap M_\alpha\) is directed with union \(M_{\alpha,i} \cap M_\alpha\).

Hence proved the first sentence of the lemma, the second sentence immediately follows from Lemma \ref{2.2}. \(\square\)

**Lemma 2.8.** Given a long \(\lambda\)-approximation sequence \((M_\beta)_{\beta<\alpha+1}\), we have
\[ M_\alpha \cap \bigcup_{\beta<\alpha} M_\beta = \bigcup_{\beta<\alpha} M_{\alpha,i}; \]
\[ M_{\alpha,i} \cap \bigcup_{\beta<\alpha} M_{\beta,i} = \bigcup_{\beta<\gamma(\alpha)} M'_{\alpha,i}, \text{ and} \]
\[ M_{\alpha,i} \cap M'_{\alpha,i} = H(\mu) \text{ for all } i < \gamma(\alpha). \]

**Proof.** Clearly, \(\alpha = \bigcup_{\gamma(\alpha)} I_i(\alpha)\); the two equations of the lemma immediately follow. By Lemmas \ref{2.4} and \ref{2.7}, each \(M_{\alpha,i}\) and each \(M'_{\alpha,i}\) is a directed union of elementary substructures of \(H(\mu)\), so \(M_{\alpha,i}, M'_{\alpha,i} < H(\mu)\). \(\square\)

**Lemma 2.9.** Given a long \(\lambda\)-approximation sequence \((M_\beta)_{\beta<\alpha+1}\) and \(i < \gamma(\alpha)\), we have \(I'_i(\alpha) \subseteq \bigcup_{j \neq i} J_j(\alpha)\).

**Proof.** Since \(\lambda\) is regular and \(|I_j(\alpha)| > |I_k(\alpha)|\) for all \(j < k < \gamma(\alpha)\), we have \(|I_i(\alpha)| > \sum_{j < \gamma(\alpha)} |J_j(\alpha)|\). Let \(\beta = \min \big( I_i(\alpha) \setminus \bigcup_{j < \gamma(\alpha)} J_j(\alpha) \big)\), which is definable in \(M_\alpha\) and thus in \(I_i(\alpha) \setminus \bigcup_{j < \gamma(\alpha)} J_j(\alpha)\). Since \(\beta \geq |\alpha|\), we also have \(\beta \notin \bigcup_{j < \gamma(\alpha)} J_j(\alpha)\). \(\square\)
Definition 2.5. Given a long $\lambda$-approximation sequence $(M_{\beta})_{\beta<\alpha+1}$ and nonempty $s \subseteq \mathcal{N}(\alpha)$, let
\begin{itemize}
  \item $K_s(\alpha) = \bigcap_{i \in s} J_i(\alpha)$;
  \item $K'_s(\alpha) = \bigcap_{i \in s} J'_i(\alpha)$;
  \item $K_s(\alpha) = \bigcap_{i \in s} j_i(\alpha)$;
  \item $K'_s(\alpha) = \bigcap_{i \in s} J'_i(\alpha)$.
\end{itemize}

Observe that, by elementarity and retrospection,
\begin{itemize}
  \item $K'_s(\alpha) = K_s(\alpha) \cap M_{\alpha}$;
  \item $K'_s(\alpha) = K_s(\alpha) \cap M_{\alpha}$;
  \item $K_s(\alpha) = \{ M_\beta : \beta \in K_s(\alpha) \}$;
  \item $K'_s(\alpha) = \{ M_\beta : \beta \in K'_s(\alpha) \}$.
\end{itemize}

Lemma 2.10. Given a long $\lambda$-approximation sequence $(M_{\beta})_{\beta<\alpha+1}$ and nonempty $s \subseteq \mathcal{N}(\alpha)$, the sets $K_s(\alpha)$ and $K'_s(\alpha)$ are directed with respect to inclusion.

\textbf{Proof.} Proceed by induction on $|s|$. Case $|s| = 1$ follows from Lemmas 2.4 and 2.7. Assuming $|s| > 1$, let $i = \max(s)$ and $t = s \setminus \{i\}$. Suppose that $\beta, \gamma \in K_s(\alpha)$. Since $\beta, \gamma \in J_i(\alpha)$, we have $M_\beta, M_\gamma \subseteq M_\delta$ for some $\delta \in I_i(\alpha)$. Applying Lemma 2.2, $K_i(\alpha) = I_i(\alpha)$; hence, $\beta, \gamma \in M_\delta$; hence, $M_\beta, M_\gamma \subseteq M_\delta$. Since $\delta \in I_i(\alpha)$, we have $I_j(\delta) = I_j(\alpha)$ for all $j < i$. Therefore, $M_\delta$ knows that $K_i(\alpha)$ is directed and that $M_\beta, M_\gamma \in K_i(\alpha)$. Hence, there exists $M_\varepsilon \in M_\delta \cap K_i(\alpha)$ such that $M_\beta, M_\gamma \subseteq M_\varepsilon$. By Lemma 2.2, $\varepsilon \in M_\delta \cap I_i(\alpha)$, hence, $\varepsilon \in J_i(\alpha)$. Thus, $M_\beta$ and $M_\gamma$ have a common superset $M_{\varepsilon}$ in $K_i(\alpha) \cap J_i(\alpha)$, as desired. \hfill \Box

Definition 2.6. Given a long $\lambda$-approximation sequence $(M_{\beta})_{\beta<\eta}$ and $x \in \bigcup_{\beta<\eta} M_{\beta}$, let the $M$-rank of $x$, written $\rho(x, M)$ or just $\rho(x)$, denote the least $\alpha < \eta$ such that $x \in M_{\alpha}$.

Lemma 2.11. Given a long $\lambda$-approximation sequence $(M_{\beta})_{\beta<\alpha+1}$ and $x \in M_{\alpha}$, we have $M_{\rho(x)} \subsetneq M_{\alpha}$.

\textbf{Proof.} Supposing $\rho(x) < \alpha$, we have $M_{\rho(x)}$ definable in $M_{\alpha}$ from $x$ and $(M_{\beta})_{\beta<\alpha}$. By Lemma 2.2, we then have $M_{\rho(x)} \subsetneq M_{\alpha}$. \hfill \Box

Lemma 2.12. For every long $\lambda$-approximation sequence $(M_{\beta})_{\beta<\eta}$ and $\emptyset \neq E \subseteq \eta$, there exists $D \subseteq \eta$ such that $\bigcap_{\alpha \in E} M_{\alpha} = \bigcup_{\alpha \in D} M_{\alpha}$ and $\{ M_{\alpha} : \alpha \in D \}$ is directed.

\textbf{Proof.} Let $N = \bigcap_{\alpha \in E} M_{\alpha}$. By Lemma 2.11, $N = \bigcup \{ M_{\alpha} : \alpha < \eta \text{ and } M_{\alpha} \subseteq N \}$. Suppose that $\alpha, \beta < \eta$ and $M_{\alpha}, M_{\beta} \subseteq N$. It suffices to find $\gamma < \eta$ such that $M_\alpha \cup M_\beta \subseteq M_\gamma \subseteq N$. First, note that $M_{\alpha}, M_{\beta} \in M_i$ for all $i \in E$ by Lemma 2.2. Since $E$ is nonempty, we may define $\gamma = \rho(\{ M_{\alpha}, M_{\beta} \})$. We then have $M_\gamma, M_{\beta} \subseteq M_\gamma$ (again by Lemma 2.2). Fix $i \in E$; it suffices to show that $M_\gamma \subseteq M_i$. By definition of $\rho$, $\gamma \leq i$. If $\gamma < i$, then $M_\gamma \subseteq M_i$ because $M_\gamma$ is definable in $M_i$. \hfill \Box

Lemma 2.13. For every long $\lambda$-approximation sequence $(M_{\alpha})_{\alpha<\eta}$ and $E \subseteq \eta$, if $\{ M_{\alpha} : \alpha \in E \}$ is directed, then there exists $i < \mathcal{N}(\eta)$ such that $E \subseteq J_i(\eta)$.

\textbf{Proof.} Let $E = \{ M_{\alpha} : \alpha \in E \}$ and $E_i = \{ M_{\alpha} : \alpha \in E \cap I_i(\eta) \}$ for each $i < \mathcal{N}(\eta)$. Since $E$ is directed and $\{ E_i : i < \mathcal{N}(\eta) \}$ is a finite partition of $E$, there must exist $i$ such that $E_i$ is cofinal in $E$. By Lemma 2.2, $E \subseteq J_i(\eta)$ for any such $i$. \hfill \Box
3. Retrospective characterizations of the FN and SFN

**Lemma 3.1.** Given a poset $C$ and $A, B \subseteq C$ such that $A \downarrow B$ and $A \cap B$ is a relatively complete suborder of $A$, the functions $\pi_B^A \upharpoonright A$ and $\pi_B^A \upharpoonright A$ respectively equal $\pi_{A \cap B}^A \upharpoonright A$ and $\pi_{A \cap B}^A \upharpoonright A$.

*Proof.* Given $a \in A$ and $b \in B$ such that $a \leq b$, we have some $c \in [a, b] \cap A \cap B$; hence, $\pi_{A \cap B}^A(c) \leq c \leq b$; hence, $\pi_B^A(a)$ exists and equals $\pi_{A \cap B}^A(a)$. Likewise, $\pi_B^A(a)$ exists and equals $\pi_{A \cap B}^A(a)$. □

**Proposition 3.1.** If $C$ is a poset, $A, B \subseteq \mathcal{P}(C)$, and $A \downarrow B$ for all $(A, B) \in A \times B$, then $\bigcup A \downarrow \bigcup B$.

**Proposition 3.2.** Given a poset $C$ and $A \subseteq B \subseteq C$, if $A$ is a relatively complete suborder of $C$, then $A$ is relatively complete suborder of $B$.

**Definition 3.1.** Given a boolean algebra $A$, a long $\lambda$-approximation sequence $(M_\beta)_{\beta \in \alpha}$, $x \in A \cap \bigcup_{\beta < \eta} M_\beta$, and $i < \eta(x)$, let $\pi^+_i(x, M)$ or just $\pi_i^+(x)$ denote $\pi^+_{M_\beta^{(i)}}(x)$ if it exists; likewise let $\pi^+_i(x, M)$ or just $\pi_i^+(x)$ denote $\pi^-_{M_\beta^{(i)}}(x)$ if it exists.

**Theorem 3.1.** Let $A$ be a boolean algebra. The following are equivalent.

1. $A$ has the FN.
2. For every long $\omega_1$-approximation sequence $(M_\alpha)_{\alpha < |A|}$ with $A \in M_0$, for every $x \in A$, $\rho(x)$ exists and, for every $i < \eta(\rho(x))$, $\pi^+_i(x)$ and $\pi^+_i(x)$ exist.
3. There exists a long $\omega_1$-approximation sequence $(M_\alpha)_{\alpha < |A|}$ such that, for every $x \in A$, $\rho(x)$ exists and, for every $i < \eta(\rho(x))$, $\pi^+_i(x)$ and $\pi^+_i(x)$ exist.
4. For every long $\omega_1$-approximation sequence $(M_\alpha)_{\alpha < |A|}$ with $A \in M_0$, for every $\alpha < |A|$, and for every $i < \eta(\alpha)$, we have $A \cap M_\alpha \downarrow A \cap M_\alpha$ and $A \cap M_\alpha \downarrow A \cap M_\alpha$.
5. There exists a long $\omega_1$-approximation sequence $(M_\alpha)_{\alpha < |A|}$ such that $A \subseteq \bigcup_{\alpha < |A|} M_\alpha$ and, for all $\alpha < |A|$ and $i < \eta(\alpha)$, we have $A \cap M_\alpha \downarrow A$ and $A \cap M_\alpha \downarrow A \cap M_\alpha$.
6. For every long $\omega_1$-approximation sequence $(M_\alpha)_{\alpha < |A|}$ with $A \in M_0$, for every $\alpha, \beta < |A|$, and for every $i < \eta(\alpha)$, we have $A \cap M_\alpha \downarrow A \cap M_\beta$ and $A \cap M_\alpha \downarrow A \cap M_\beta$.
7. There exists a long $\omega_1$-approximation sequence $(M_\alpha)_{\alpha < |A|}$ such that $A \subseteq \bigcup_{\alpha < |A|} M_\alpha$ and, for all $\alpha, \beta < |A|$ and $i < \eta(\alpha)$, we have $A \cap M_\alpha \downarrow A$, $A \cap M_\alpha \downarrow A \cap M_\beta$, and $A \cap M_\alpha \downarrow A \cap M_\beta$.

*Proof.* (1) $\Rightarrow$ (2). Fix $M$ as in the hypothesis of (2). For each $x \in A$, $\rho(x)$ exists by Lemma 2.3. Each $M_\rho(x) \downarrow A$ is an elementary substructure of $H(\mu)$ by Lemma 2.8. Also, $A \in M_\rho(x) \downarrow A$ by Lemma 2.6. Hence, $A \cap M_\rho(x) \downarrow A \cap M_\rho(x)$. Hence, $\pi_i^+(x)$ and $\pi_i^+(x)$ exist.

(1) $\Rightarrow$ (6). Fix $M$ as in the hypothesis of (6). By Lemma 2.6 we have $A \in M'_{\alpha, i}$ for all $\alpha < |A|$ and $i < \eta(\alpha)$. Hence, by Lemma 2.8 and Theorem 1.1, we have $A \cap M'_{\alpha, i} \subseteq A$. Hence, $A \cap M'_{\alpha, i} \subseteq A \cap M_\alpha$. Hence, $A \cap M'_{\alpha, i} \subseteq A \cap M_\alpha$. Finally, given $\alpha, \beta < |A|$, choose an FN map $f \in M_0$. By Lemma 2.6, $M_\alpha$ and $M_\beta$ are $f$-closed; hence, $M_\alpha \downarrow M_\beta$. 


Definition 3.2. Choose $\mu$ large enough that $A \in H(\mu)$. By Lemma 2.6, there is a long $\omega_1$-approximation sequence $(M_\alpha)_{\alpha < |A|}$ with $A \in M_\alpha$.

By Lemma 2.6, we have $A \cap M_\alpha \subseteq A$ for all $\alpha < |A|$. By Lemma 2.8, we have $A \subseteq \bigcup_{\alpha < |A|} M_\alpha$.

Lemma 3.1. By Lemma 3.1, $\pi^M_+ \subseteq \pi^M_-$ are well-defined on all of $A \cap M_\alpha$.

This is implicit in the author’s proof of Theorem 3.3 of [9], but we include a proof here for completeness. For each $\alpha < |A|$, choose a well-ordering $\sqsubseteq_\alpha$ of $\{x \in A : \rho(x) = \alpha\}$ with length at most $\omega$. Set $\sqsubseteq = \bigcup_{\alpha < |A|} \sqsubseteq_\alpha$. Recursively define $f : A \rightarrow [A]^{<\aleph_0}$ by

$$f(x) = \{y : y \sqsubseteq x\} \cup \left(\bigcup_{i < \rho(x)} (f(\pi^i_+(x)) \cup f(\pi^i_-))(x))\right).$$

Suppose $x \leq_A y$. We verify that $S = [x, y] \cap f(x) \cap f(y)$ is nonempty by induction on $\mathfrak{m}(\rho(x), \rho(y))$. If $\rho(x) = \rho(y)$, then $x \sqsubseteq y$, in which case $x \in S$, or $y \sqsubseteq x$, in which case $y \in S$. If $\rho(x) < \rho(y)$, then $x \in M_{\rho(y), i}$ for some $i$, in which case $[x, \pi^i_-(y)] \cap f(x) \cap f(\pi^i_-(y))$ is a nonempty subset of $S$. If $\rho(y) < \rho(x)$, then $y \in M_{\rho(x), i}$ for some $i$, in which case $[\pi^i_+(x), y] \cap f(\pi^i_+(x)) \cap f(y)$ is a nonempty subset of $S$.

Lemma 3.2. Given boolean algebras $A \leq C$ and $B \leq_C C$, the following are equivalent.

1. $A \subseteq B$.
2. $\pi^B_+[A] \subseteq A$.
3. $\pi^B_-[A] \subseteq A$.

Proof. $\mathbf{(1) \Rightarrow (2)}$. Given $a \in A$, we have $a \leq \pi^B_+(a) \in B$, so there exists $b \in [a, \pi^B_+(a)] \cap A \cap B$. However, by definition of $\pi^B_+$, we must have $b = \pi^B_+(a)$.

$\mathbf{(2) \Rightarrow (1)}$. Given $a \in A$ and $b \in B$, if $a \leq b$, then $\pi^B_+(a) \in [a, b] \cap A \cap B$; if $b \leq a$, then $\pi^B_-(a) \in [b, a] \cap B$ and $\pi^B_-(a) = -\pi^B_+(-a) \in A$.

$\mathbf{(2) \Rightarrow (3)}$. $\pi^B_-(\bullet) = -\pi^B_+(-\bullet)$ and $\pi^B_+(\bullet) = -\pi^B_+(-\bullet)$. □

Definition 3.2.

- A partial algebra is a pair of the form $(U, \mathcal{F})$ where $U$ is a set (called the universe of $(U, \mathcal{F})$) and $\mathcal{F}$ is a set of functions such that, for each $f \in \mathcal{F}$, there exists $n < \omega$ such that $\text{dom}(f) \subseteq A^n$. If every $\text{dom}(f)$ is of the form $A^n$, then we say that $(U, \mathcal{F})$ is an algebra.

- A partial algebra $(U, \mathcal{F})$ is a subalgebra of a partial algebra $(V, \mathcal{G})$ if $U \subseteq V$, $\mathcal{F} = \{g \upharpoonright U^{<\omega} : g \in \mathcal{G}\}$, and $\bigcup_{g \in \mathcal{G}} g[U^{<\omega}] \subseteq U$.

- A partial algebra $(U, \mathcal{F})$ is locally finite if, for every finite $A \subseteq U$, there exists a finite $B \subseteq U$ such that $A \subseteq B$ and $(B, \{f \upharpoonright B : f \in \mathcal{F}\})$ is a subalgebra of $(U, \mathcal{F})$.

- We say that a partial algebra $(V, \mathcal{G})$ expands a partial algebra $(U, \mathcal{F})$ if $U = V$ and $\mathcal{F} \subseteq \mathcal{G}$.

- Given a long $\lambda$-approximation sequence $(M_i)_{i < \eta}$ and a boolean algebra $A$, let the $M$-expansion of $A$, written $A[M]$, denote the expansion of $A$ resulting from adding the functions in the set $\bigcup_{i < \omega} \{\pi^i_+, \pi^i_-\}$. 
Theorem 3.2. If $A$ is a boolean algebra with the SFN, $(M_\alpha)_{\alpha \in \mathcal{A}}$ is a long $\omega_1$-approximation sequence, and $A \in M_0$, then $A[M]$ is locally finite.

Proof. Let $C \in M_0$ be a pairwise commuting cofinal family of finite subalgebras of $A$. It suffices to show that, for every $F \in C$, (an expansion of) $F$ is a subalgebra of $A[M]$. Let $x \in F \in C$, $\alpha = \rho(x)$, and $i < \mathcal{C}^\prime(\alpha)$. By Theorem 3.1, $\pi^i_\pm(x)$ are well-defined. By Lemmas 2.6 and 2.8, $C \in M_{\alpha,i} \sim H(\mu)$, so we may choose $G \in M_{\alpha,i} \cap C$ such that $\pi^i_{\pm}(x) \in G$. Since $G \in M_{\alpha,i}$ and $G$ is finite, $G \subseteq M_{\alpha,i}$; hence, $\pi^i_{\pm}(x) \in G$ implies $\pi^i_{\pm}(x) = \pi^G_{\pm}(x)$. By Lemma 3.2, $\pi^G_{\pm}(x) \in F$. □

The proof of Theorem 3.2 implicitly shows much more. Indeed, we can expand $A[M]$ by adding every function of the form $\pi^N_\pm$ where $\mathcal{C} \subseteq N \prec H(\mu)$, yet still obtain a locally finite partial algebra. However, local finiteness of $A[M]$ is strong enough for our purposes. As we shall show in Section 5, it is strictly stronger than the FN.

Question 3.1. If $A$ is a boolean algebra, $(M_\alpha)_{\alpha \in \mathcal{A}}$ is a long $\omega_1$-approximation sequence, $A \in M_0$, and $A[M]$ is locally finite, then does $A$ have the SFN?

We do not know the answer to the above question. However, we will point out that if we broaden our consideration to arbitrary expansions of boolean algebras, then characterizations of the FN and SFN are apparently easier to obtain.

Definition 3.3. Call an FN map $f$ transitive if $f(y) \subseteq f(x)$ for all $x \in \text{dom}(f)$ and $y \in f(x)$.

Lemma 3.3. If $A$ has the FN, then $A$ has a transitive FN map.

Proof. Construct an FN map $f$ as in the proof of $\[1] \Rightarrow [1]$ in Theorem 3.1, except use the following recursive definition of $f$:

$$f(x) = \{x\} \cup \left(\bigcup_{y \subseteq x} f(y)\right) \cup \left(\bigcup_{i < \mathcal{C}(\rho(x))} (f(\pi^i_+(x)) \cup f(\pi^i_-(x)))\right).$$

The proof of $\[1] \Rightarrow [1]$ in Theorem 3.1 still works verbatim, but now $f$ is also transitive. □

Lemma 3.4. If $A$ is a boolean algebra, $B$ is a pairwise commuting family of relatively complete subalgebras of $A$, and $B_0, B_1 \in B$, then $B \cup \{B_0 \cap B_1\}$ is pairwise commuting.

Proof. Let $C \in B$. Suppose that $x \in B_0 \cap B_1$, $y \in C$, and $x \leq y$. By symmetry, it suffices to show that $[x, y] \cap B_0 \cap B_1 \cap C$ is nonempty. Let $z = \pi^B_{B_0}(y)$, which is in $[x, y] \cap B_0$; $z$ is also in $C$ by Lemma 3.2. Let $w = \pi^B_{B_1}(z)$, which is in $[x, z] \cap B_1$; $w$ is also in $B_0$ and $C$ by two applications of Lemma 3.2. □

Definition 3.4.

- Given a partial algebra $B$, call a subalgebra $C$ of $B$ cyclic if, for some $x \in C$, $C$ is the smallest subalgebra of $B$ that contains $\{x\}$.
- Given a boolean algebra $A$, we say that a partial algebra $B$ is strongly $A$-commuting if $B$ has the same universe as $A$, $B$ is locally finite, and, for all cyclic subalgebras $F$ and $G$ of $B$, we have $F \downarrow G$ as suborders of $(A, \leq_A)$.

Theorem 3.3. Given a boolean algebra $A$,
• $A$ has the FN if and only if there is a strongly $A$-commuting algebra;
• $A$ has the SFN if and only if there is a strongly $A$-commuting algebra expanding $A$.

Moreover, the above is true if we replace “strongly $A$-commuting algebra” with “strongly $A$-commuting partial algebra.”

Proof. If $f$ is a transitive FN map on $A$, then, letting $(f_n(x))_{n < \omega}$ surject from $\omega$ to $f(x)$ for each $x \in A$, the algebra $B$ with universe $A$ and set of functions $\{f_n : n < \omega\}$ is strongly $A$-commuting.

Conversely, given a strongly $A$-commuting partial algebra $B$, construct an FN map $f$ by letting $f(x)$ be the minimal subalgebra of $B$ containing $\{x\}$, for each $x \in A$.

Suppose $C$ is a pairwise commuting cofinal family of finite subalgebras of $A$. By Lemma 3.4, we may assume that $C$ is closed with respect to pairwise intersection. For each $x \in A$, let $C(x)$ denote the smallest element of $C$ that contains $\{x\}$; let $(f_n(x))_{n < \omega}$ surject from $\omega$ to $C(x)$. The expansion of $A$ formed by the adding the functions from $\{f_n : n < \omega\}$ is strongly $A$-commuting.

Conversely, suppose that $B$ is a strongly $A$-commuting expansion of $A$. Let $C$ denote the set of finite subalgebras of $B$. Since $B$ is locally finite, $C$ is a cofinal family of finite subalgebras of $A$ (provided we identify each $C \in C$ with the subalgebra of $A$ that has the same universe). Moreover, by Proposition 3.1 $C$ is pairwise commuting.

Observe that adapting the proof of Lemma 3.3 to build a strongly $A$-commuting expansion of $A$ would require not only that $A[M]$ be locally finite, but also that $A[M]$ remain locally finite after adding a partial function that maps each $x \in A$ to its immediate $\preceq$-predecessor, if one exists.

We shall the need the next lemmas in Section 6.

Lemma 3.5. If $A \leq_{\text{re}} B$, $a \in A$, $b \in B$, and $\pi^A_+(b) = x$, then $\pi^A_+(a \land b) = a \land x$.

Proof.

\[
\begin{align*}
 a \land b \leq y \in A & \implies b \leq y \land -a \in A \\
 & \implies b \leq x \leq y \land -a \\
 & \implies a \land b \leq a \land x \leq y
\end{align*}
\]

Definition 3.5. Given a boolean algebra $A$, a long $\omega_1$-approximation sequence $(M_\alpha)_{\alpha < \eta}$, and $x \in A$, let $\sigma^A_+(x, M) = x$ and, for all $(t_0, \ldots, t_n) \in \omega^{< \omega}$, let

\[
\sigma^A_+(x, M) = (\pi^A_+ \circ \pi^A_+ \circ \cdots \circ \pi^A_+)(x, M)
\]

if the righthand side exists. Let $\zeta^A_+(x, M)$ denote the set of all $\sigma^A_+(x, M)$ that exist. Likewise define $\sigma^A_-(x, M)$ and $\zeta^A_-(x, M)$. We may suppress the dependence of $\zeta^A_+$ and $\sigma^A_+$ on $M$ when convenient.

Observe that if $s$ is a strict initial segment of $t$ and $\sigma^A_+(x)$ exists, then $|t|_s < \gamma(\rho(\sigma^A_+(x)))$ and $\rho(\sigma^A_+(x)) < \rho(\sigma^A_+(x))$. Hence, by König’s Lemma, $\zeta^A_+(x)$ is finite; likewise, $\zeta^A_-(x)$ is finite.

Lemma 3.6. Suppose we have a boolean algebra $A$ and a long $\omega_1$-approximation sequence $(M_\alpha)_{\alpha < \eta}$ such that $\pi^A_{\pm}(x)$ exist for all $x \in A \cap \bigcup_{\alpha < \eta} M_\alpha$ and all $i <
\( \text{Corollary 3.1.} \) Suppose we have a boolean algebra \( A \) and further suppose that \( \nu \in \omega \) is a regular uncountable cardinal and \( \ell = \nu(\rho(x)) \). Then, for every \( B \subseteq A \), \( A \cap \bigcup \alpha \in I M_\alpha \), \( \rho(x) = \bigwedge (B \cap \iota_+(x)) \) and \( \rho(x) = \bigvee (B \cap \iota_-(x)) \) for all \( x \in A \cap \bigcup \alpha \in I M_\alpha \).

**Proof.** Suppose that \( x \in A \cap \bigcup \alpha \in I M_\alpha \) and \( x \subseteq y \subseteq B \). By symmetry, it suffices to show that \( x^B \leq y \) where \( x^B = \bigwedge (B \cap \iota_+(x)) \). Proceed by induction on \( \max\{\rho(x), \rho(y)\} \). Choose \( \alpha \in I \) such that \( y \in M_\alpha \). By Lemma 2.11, \( M_\rho(y) \subseteq M_\alpha \); hence, \( A \cap M_\rho(y) \subseteq B \). If \( \rho(x) = \rho(y) \), then \( x \subseteq y \), in which case \( x^B = x \subseteq y \). If \( \rho(x) < \rho(y) \), then \( x \in M_\rho(y) \) for some \( i \), in which case \( x \leq x_i \subseteq B \) and \( \rho(x_i) < \rho(y) \); by induction, \( x^B \leq x_i \leq y \). If \( \rho(y) < \rho(x) \), then \( y \in M_\rho(x) \) for some \( i \), in which case \( y \leq z \) and \( \rho(z) < \rho(x) \) where \( z = x_i \); by induction, \( z^B \leq y \); hence, \( x^B \leq z^B \leq y \) because \( \iota_+(z) \subseteq \iota_+(x) \).

\( \square \)

**Corollary 3.1.** Suppose we have a boolean algebra \( A \) and a long \( \omega_1 \)-approximation sequence \( (M_\alpha)_{\alpha < \eta} \) such that

- \( M_\alpha < H(\mu) \) for all \( \alpha < \eta \) and
- \( \rho(x) \) and \( \iota_+(x) \) exist for all \( x \in A \) and all \( i < \ell(\rho(x)) \).

Further suppose that \( \nu \) is a regular uncountable cardinal and \( A, M \in P < H(\nu) \). Then, for all \( x \in A \setminus P \),

\[
\iota_+(x) = \bigwedge_{i < \ell(\rho(x), M)} \pi_i^+(\iota_+(x), M) \quad \text{and} \quad \iota_-(x) = \bigvee_{i < \ell(\rho(x), M)} \pi_i^-(\iota_-(x), M).
\]

**4. Embeddings and colimits**

In this section, we collect some facts and specify some notation concerning various colimits and various classes of boolean embeddings. We refer the reader to [5] and [7] for additional background information.

**Definition 4.1.**

- Say that a sequence \( (F_i)_{i \in I} \) of subalgebras of a fixed boolean algebra is **independent** if, for all finite \( J \subseteq I \) and all \( x \in \prod_{j \in J} F_j \), if \( \bigwedge_{j \in J} x(j) = 0 \), then \( x(j) = 0 \) for some \( j \in J \).
- Say that a sequence \( \{x_i\}_{i \in I} \) of elements of a fixed boolean algebra is **independent** if \( \{x_i, -x_i, 0\}_i \}_{i \in I} \) is independent.
- Say that a boolean algebra \( F \) is **free** if it is generated by the range of an independent sequence of elements of \( F \).

Fix once and for all a countably infinite free boolean algebra \( Fr_\omega \) and an independent sequence \( \{fr_n\}_{n < \omega} \) generating \( Fr_\omega \) such that \( Fr_\omega \) and \( fr \) are definable in \( H(\aleph_1) \) without parameters. For each \( S \subseteq \omega \), let \( Fr_S \) denote the subalgebra of \( Fr_\omega \) generated by \( \{fr_n : n \in S\} \).

**Definition 4.2.**

- An **boolean embedding** is an injective boolean homomorphism.
- Given two boolean algebras \( C_0 \) and \( C_1 \), a **coproduct** of \( C_0 \) and \( C_1 \) is a boolean algebra \( C_0 \oplus C_1 \) with boolean embeddings \( \oplus_0 : C_0 \rightarrow C_0 \oplus C_1 \) and \( \oplus_1 : C_1 \rightarrow C_0 \oplus C_1 \) such that \( \oplus_0[C_0] \) and \( \oplus_1[C_1] \) are independent and \( \bigcup_{i < 2} \oplus_i[C_i] \) generates \( C_0 \oplus C_1 \). These embeddings are called **cofactor maps**.
Coproducts always exist uniquely up to isomorphism.

**Definition 4.3.**

- Say that a boolean embedding \( f: A \to B \) is free if there is an infinite free boolean algebra \( F \) and a coproduct \( A \oplus F \) such that \( A \oplus F = B \) and \( \oplus = f \).
- If \( \text{id}_A \) is free embedding from \( A \) to \( B \), then we say that \( B \) is a free extension of \( A \) and write \( A \leq \text{free } B \).
- Given boolean algebras \( A \leq B \), we say that \( A \) splits in \( B \) if every ultrafilter of \( A \) extends to at least two ultrafilters of \( B \). We say that \( A \) splits perfectly in \( B \) if, for all finite \( F \subseteq B \), the subalgebra generated by \( A \cup F \) splits in \( B \).
- We say that a boolean embedding \( f: A \to B \) splits perfectly if \( f[A] \) splits perfectly in \( B \).
- If \( \text{id}_A : A \to B \) is a perfectly splitting boolean embedding, then we say that \( A \) splits perfectly in \( B \).

Every free embedding is relatively complete and splits perfectly. Conversely, we have Sirota’s Lemma, which states that if \( f: A \to B \) is relatively complete and perfectly splitting and \( B \) is generated by \( f[A] \cup C \) for some countable \( C \), then \( f \) is free. Also note that the classes of relatively complete, perfectly splitting, and free embedding are each closed with respect to composition. Moreover, for any composite boolean embedding \( f \circ g \), if \( f \) is perfectly splitting, then so is \( f \circ g \).

**Definition 4.4.**

- A quotient of a boolean algebra \( A \) with respect to an ideal \( I \) is a boolean algebra \( B \) with a surjective homomorphism \( f: A \to B \) with kernel \( I \); \( f \) is called the quotient map and \( f(x) \) may be denoted by \( x/I \).
- Given boolean embeddings \( f: C \to A \) and \( g: C \to B \), define a pushout \( A \sqcup B \) of \( f \) and \( g \) to be a quotient of a coproduct \( A \oplus B \) with respect to the ideal \( I \) generated by \( \{ \oplus_0(f(c)) \sqcup \oplus_1(g(-c)) : c \in C \} \). Thus, \( A \sqcup B \) is a colimit of the diagram formed by \( f \) and \( g \).
- Given \( f \) and \( g \) as above such that also \( f = g = \text{id}_{A \cap B} \), let \( A \sqcup B \) more specifically denote a pushout of \( f \) and \( g \) such that \( \oplus_0(a)/I = a \) and \( \oplus_1(b)/I = b \) for all \( a \in A \) and \( b \in B \).

If \( A \) and \( B \) are boolean algebras such that their intersection \( A \cap B \) is also a common subalgebra, then \( A \sqcup B \) exists as above and is characterized up to isomorphism as a boolean algebra \( D \) in which \( A \) and \( B \) are commuting subalgebras and \( A \cup B \) generates \( D \).

**Lemma 4.1.** If \( A = C_0 \cap C_1 \) and, for each \( i < 2 \), we have \( C_i = A \oplus B_i \) with cofactor maps \( \oplus_0 = \text{id}_A \) and \( \oplus_1 = \text{id}_{B_i} \), then \( (A, B_0, B_1) \) is independent in \( C_0 \sqcup C_1 \)

**Proof.** Suppose that \( a \in A \), \( b_0 \in B_0 \), \( b_1 \in B_1 \), and \( a \wedge b_0 \wedge b_1 = 0 \) in \( C_0 \sqcup C_1 \). Then \( a \wedge b_0 \leq \hat{a} \leq -b_1 \) for some \( \hat{a} \in A \) because \( C_0 \downarrow C_1 \). Hence, \( \hat{a} \wedge b_1 = 0 \). Since \( (A, B_1) \) is independent, \( \hat{a} = 0 \) or \( b_1 = 0 \). If \( \hat{a} = 0 \), then \( a \wedge b_0 = 0 \), in which case \( a = 0 \) or \( b_0 = 0 \) because \( (A, B_0) \) is independent. Thus, \( a = 0 \), \( b_0 = 0 \), or \( b_1 = 0 \). \( \square \)

**Definition 4.5.** Given a (nonempty) directed set \( D \) of boolean algebras such that \( A \subseteq B \) implies \( A \leq B \) for all \( A, B \in D \), endow the union \( \bigcup D \) of (the universes of) the algebras in \( D \) with the unique algebraic operations that make \( \bigcup D \) and the inclusions \( (\text{id}_A: A \to \bigcup D)_{A \in D} \) a colimit of \( D \) and the inclusions \( (\text{id}_A: A \to B)_{A \leq B} \subseteq D \), namely, \( \wedge_{D} = \bigcup_{A \in D} \wedge_A \), \( \vee_{D} = \bigcup_{A \in D} \vee_A \), \( -_{D} = \bigcup_{A \in D} -_A \), and \( 0_D = 0_A \) and \( 1_D = 1_A \) for all \( A \in D \).
5. Proof of main theorem

By Theorem 3.2 it is sufficient to construct a boolean algebra $\Omega$ of size $\aleph_2$ such that $\Omega$ has the FN, but $\Omega[\omega]$ is not locally finite for some long $\omega_1$-approximation sequence $(N_\alpha)_{\alpha<\omega_2}$ with $\Omega \in N_0$. We will construct in parallel a sequence $(A_\alpha)_{\alpha<\omega_2}$ of countable boolean algebras and a long $\omega_1$-approximation sequence $(M_\alpha)_{\alpha<\omega_2}$ such that, for all $\alpha < \omega_2$, we have

\begin{align}
(1) & \quad A_\alpha \in M_\alpha \prec H(\aleph_2), \\
(2) & \quad A_\alpha \cap \bigcup_{\beta<\alpha} M_\beta = \bigcup_{i \in \gamma(\alpha)} A'_{\alpha,i}, \text{ and} \\
(3) & \quad \forall i < \gamma(\alpha) A'_{\alpha,i} \leq A_\alpha
\end{align}

where $A'_{\alpha,i} = \bigcup\{A_\beta : \beta \in J_i(\alpha)\}$. Also define $A_{\alpha,i} = \bigcup\{A_\beta : \beta \in J_i(\alpha)\}$.

**Claim 5.1.** Given $A$ and $M$ as above, $\alpha, \beta < \omega_2$, and $i < \gamma(\alpha)$, we have that

- $A_\alpha \cap A_\beta = A_\alpha \cap M_\beta$,
- $A_\alpha \subseteq A_\beta$ if $M_\alpha \subseteq M_\beta$, and
- $A'_{\alpha,i} = A_\alpha \cap A_{\alpha,i}$.

**Proof.** For the first subclaim, we may assume we are not in the trivial case $\alpha = \beta$. Since $A_\beta$ is countable, (1) implies $A_\beta \subseteq M_\beta$. Therefore, $A_\alpha \cap A_\beta \subseteq A_\alpha \cap M_\beta$. To prove the converse inclusion, suppose $x \in A_\alpha \cap M_\beta$. If $\beta < \alpha$, then, by (2), $x \in A_\gamma$ for some $\gamma < \alpha$; we may inductively assume $A_\gamma \cap M_\beta = A_\gamma \cap A_\beta$, in which case $x \in A_\beta$. If $\alpha < \beta$, then $\rho(x) \in M_\beta \cap J_j(\beta) \subseteq J'_j(\beta)$ for some $j < \gamma(\beta)$; hence, by (2) and (3), $x \in A_{\rho(x)} \subseteq A'_{\beta,j} \subseteq A_\beta$. Thus, $A_\alpha \cap M_\beta \subseteq A_\alpha \cap A_\beta$. Therefore, $A_\alpha \cap A_\beta = A_\alpha \cap M_\beta$.

We obtain the third subclaim from the first subclaim and from (3):

$$A_\alpha \cap A_{\alpha,i} = A_\alpha \cap \bigcap_{\beta \in J_i(\alpha)} A_\beta = A_\alpha \cap \bigcap_{\beta \in J_i(\alpha)} M_\beta = A_\alpha \cap M_\alpha \cap \bigcup_{\beta \in J_i(\alpha)} M_\beta = A_\alpha \cap \bigcup_{\beta \in J'_i(\alpha)} A_\beta = A_\alpha \cap A'_{\alpha,i} = A'_{\alpha,i}. $$

Also, the second subclaim holds because if $\delta < \omega_2$ and $M_\alpha \subseteq M_\delta$, then $A_\alpha \subseteq A_\delta$ because $A_\alpha \cap A_\beta = A_\alpha \cap M_\delta \supseteq A_\alpha \cap M_\alpha = A_\alpha$.

By the above claim, since $T_0(\omega_2)$ is directed, letting $\Omega = A_{\omega_2,0} = \bigcup_{\alpha<\omega_2} A_\alpha$, we obtain a boolean algebra of size at most $\aleph_2$ such that, for all $\alpha < \omega_2$, $\Omega \cap M_\alpha = A_\alpha \leq \Omega$ and, for all $i < \gamma(\alpha)$, $\Omega \cap M_{\alpha,i} = A_{\alpha,i}$ and $\Omega \cap M'_{\alpha,i} = A'_{\alpha,i}$. Therefore, by Theorem 3.1, $\Omega$ will have the FN if $A'_{\alpha,i} \leq A_\alpha$ and $A_{\alpha,i} \mathrel{\upharpoonright} A_\beta$ as suborders of $\Omega$ for all $\alpha, \beta < \omega_2$ and $i < \gamma(\alpha)$. Therefore, $\Omega$ will have the FN if we have the following for all $\alpha < \omega_2$.

\begin{align}
(4) & \quad \forall i < \gamma(\alpha) A'_{\alpha,i} \leq A_\alpha, \\
(5) & \quad \forall M_\beta, M_\gamma \in M_\alpha A_\beta \mathrel{\upharpoonright} A_\gamma \text{ as suborders of } A_\alpha.
\end{align}

Note that (4) implies (3).

At stage 0, let $A_0 = Fr_\omega \in M_0$. At nonzero stages $\alpha < \omega_2$, select a countable $M_\alpha \prec H(\aleph_2)$ such that $(A_\beta, M_\beta)_{\beta<\alpha} \in M_\alpha$. If $\gamma(\alpha) = 1$, then let $A_0$ be a coproduct $A'_{0,0} \oplus Fr_\omega$ such that $A'_{0,0} \leq A_0 \in M_0$ and $A_a \setminus A'_{0,0}$ is disjoint from $\bigcup_{\beta<\alpha} M_\beta$. If $\gamma(\alpha) > 1$, then let $A_0$ be a coproduct $A'_{0,i} \oplus Fr_\omega$ such that $A'_{0,i} \leq A_0 \in M_0$ and $A_a \setminus A'_{0,i}$ is disjoint from $\bigcup_{\beta<i} M_\beta$.
Clearly, (1), (2), and (3) are preserved. Since $\mathcal{J}_0^a(\alpha)$ is directed, (3) is preserved too.

Now suppose that $\mathcal{J}(\alpha) = 2$. Let $A'_{\alpha,2} = \bigcup_{j < 2} A'_{\alpha,j}$. By Lemma 3.10, $A'_{\alpha,2}$ is a directed union of common subalgebras of $A'_{\alpha,0}$ and $A'_{\alpha,1}$.

**Claim 5.2.** $A'_{\alpha,2} \leq_{\text{rc}} A'_{\alpha,i}$ for each $i < 2$.

**Proof.** Fix $i < 2$. By Lemma 2.12 for each $\beta \in J_i^a(\alpha)$ and $j < \mathcal{J}(\beta)$, $A'_{\alpha,i} \cap A_{\beta,j} = \bigcup \{A_\gamma : \gamma \in U\}$ for some $U \subseteq J_i^a(\alpha)$. By the inductive assumption of (4) for all stages before $\alpha$ and by the directedness of $\{A_\beta : \delta \in J_i^a(\alpha)\}$, $A_\gamma \downarrow A_\delta$ for all $\gamma \in U$. Hence, by Proposition 3.1, $A'_{\alpha,i} \cap A_{\beta,j} \downarrow A_\beta$. By the inductive assumption of (4) for all stages before $\alpha$, we have $A'_{\alpha,i} \cap A_{\beta,j} \subseteq A'_{\beta,j} \leq_{\text{rc}} A_\beta$. Hence, by Lemma 3.1, $A'_{\alpha,i} \cap A_{\beta,j}(x)$ exist for all $x \in A_\beta$. Therefore, applying Lemma 3.6 with $A'_{\alpha,i}$ in place of $A$ and $A'_{\alpha,2}$ in place of $B$, we have $A'_{\alpha,2} \leq_{\text{rc}} A'_{\alpha,i}$. $\square$

**Claim 5.3.** $A'_{\alpha,2} \leq_{\text{free}} A'_{\alpha,i}$ for each $i < 2$.

**Proof.** Fix $i < 2$. By Sirota’s Lemma and the previous claim, it suffices to show that $A'_{\alpha,2}$ splits perfectly in $A'_{\alpha,i}$. Let $U$ be an ultrafilter of $A'_{\alpha,2}$ and $F$ be a finite subset of $A'_{\alpha,i}$. Applying Lemma 2.9 there exists $\beta \in I_i^a(\alpha) \setminus J_{i-1}(\alpha)$. Since $I_i^a(\alpha)$ is directed and $J_{i-1}(\alpha)$ is downward closed in $\{M_\beta : \beta < \alpha\}$, we may choose $\beta \in I_i^a(\alpha) \setminus J_{i-1}(\alpha)$ such that $F \subseteq A_\beta$. By Lemmas 2.12 and 2.13, $A_\beta \cap A'_{\alpha,2} \subseteq A'_{\beta,j}$ for some $j < \mathcal{J}(\beta)$. Let $B \leq A_\beta$ denote the subalgebra generated by $A'_{\beta,j} \subseteq F$. Extend $U \cap A_\beta$ to an ultrafilter $V$ of $B$. Since $A'_{\beta,j} \leq_{\text{free}} A_\beta$ by (4) for stage $\beta$, $B$ splits in $A_\beta$. So, choose $y \in A_\beta$ and ultrafilters $V_\pm$ of $A_\beta$ that respectively extend $V \cup \{\pm y\}$. By (3) for stages before $\alpha$ and by Proposition 3.1, $A'_{\alpha,2} \downarrow A_\beta$ as suborders of $A'_{\alpha,i}$, so both of $U \cup V_\pm$ extend to ultrafilters of $A'_{\alpha,i}$. Thus, $A'_{\alpha,2}$ splits perfectly in $A'_{\alpha,i}$. $\square$

Choose $B_\alpha = A'_{\alpha,0} \oplus A'_{\alpha,1}$ such that $B_\alpha \subseteq M_\alpha$ and $B_\alpha \setminus (A'_{\alpha,0} \cup A'_{\alpha,1})$ is disjoint from $\bigcup_{\beta < \alpha} M_\beta$.

**Claim 5.4.** $\square$ will be preserved if $B_\alpha \leq A_\alpha$.

**Proof.** Suppose that $x_i \in A_\beta_i$ and $M_\beta_i \subseteq M_\alpha$ for each $i < 2$, and that $x_0 \leq x_1$ in $A_\alpha$. It suffices to find $w \in [x_0, x_1] \cap M_\beta_i \cap M_\beta_j$. Let $\beta_i \in J_i^a(\alpha)$ for each $i < 2$. We inductively assume that (3) holds for all stages before $\alpha$. Therefore, since $J_0^a(\alpha)$ and $J_1^a(\alpha)$ are directed, if $j_0 = j_1$, then $w$ as above exists. So, assume that $j_0 \neq j_1$. By symmetry, we may assume that $j_0 = 0$. Assuming $B_\alpha \leq A_\alpha$, we have $A'_{\alpha,0} \downarrow A'_{\alpha,1}$ as suborders of $A_\alpha$. Hence, we may choose $y \in A'_{\alpha,0} \cap A'_{\alpha,1} \cap [x_0, x_1]$.

Choose $\gamma \in K_2^a(\alpha)$ such that $y \in M_\gamma$. We may choose $z_0 \in [x_0, y] \cap A_{\beta_0} \cap A_\gamma$ and $z_1 \in [y, x_1] \cap A_{\beta_1} \cap A_\gamma$, again because $J_0^a(\alpha)$ and $J_1^a(\alpha)$ are directed. For each $i < 2$, let $\delta_i = \rho(z_i, M \uparrow \alpha)$; then we have $z_i \in M_{\delta_i} \subseteq M_{\delta_i} \cap M_\gamma$ by Lemma 2.11. Since $K_2^a(\alpha)$ is also directed, we may choose $w \in [z_0, z_1] \cap A_{\delta_0} \cap A_{\delta_1}$. Thus, $w \in [x_0, x_1] \cap A_{\beta_0} \cap A_{\beta_1}$. $\square$

Given Claim 5.3, we may choose, for each $i < 2$, cofactor maps $\oplus_0 = \text{id} : A'_{\alpha,2} \rightarrow A'_{\alpha,i}$ and $\oplus_1 = \zeta_0 : \text{Fr}_\omega \rightarrow A'_{\alpha,i}$ such that $\zeta_0 \subseteq M_\alpha$. For each $i < 2$, let $B_{\alpha,i} = \text{ran}(\zeta_0)$ and $b_{\alpha,i} = \zeta_0(\text{Fr}_i)$ for each $n < \omega$. Choose $C_\alpha = B_\alpha \oplus \text{Fr}_\omega$ such that $C_\alpha$ and its cofactor maps are in $M_\alpha$, that $\oplus_0 = \text{id}_{B_{\alpha,i}}$, and that $C_\alpha \setminus B_\alpha$ is disjoint from $\bigcup_{\beta < \alpha} M_\beta$. For each $i < 2$, let $h_{\alpha,i} = \eta_\alpha(\text{Fr}_i)$ where $\eta_\alpha$ is the cofactor map $\oplus_1 : \text{Fr}_\omega \rightarrow C_\alpha$. Let $H_\alpha = \text{ran}(\eta_\alpha)$. By Lemma 4.1, $(A'_{\alpha,2}, B_{\alpha,0}, B_{\alpha,1})$
is independent; hence, so is \((A'_\iota, B_{\alpha,0}, B_{\alpha,1}, H_\alpha)\). For each \(n < \omega\), let \(e_{\alpha,i}^n = b_{\alpha,0}^n \land b_{\alpha,1}^n \land h_{\alpha,0}\) and \(e_{\alpha,1}^n = b_{\alpha,0}^{n+1} \land b_{\alpha,1}^n \land h_{\alpha,1}\); let \(I_\alpha\) be the ideal of \(C_\alpha\) generated by \(\{e_{\alpha,i}^n : (i, n) \in 2 \times \omega\}\).

Claim 5.5. \(I_\alpha \cap (B_\alpha \cup H_\alpha) = \{0\}\).

Proof. Let \(1 \leq m < \omega\), let \(e = \bigvee_{(i,n) \in 2 \times m} e_{\alpha,i}^n\), and let \(J\) be the ideal generated by \(e\). We will show that \(J \cap (B_\alpha \cup H_\alpha) = \{0\}\). First, if \(0 < h \in H_\alpha\), then \(h \not\leq e\) because \(e \land c = 0 < c \leq h\) where \(c = h \land -b_{\alpha,1}^0\). Second, if \(0 < b \in B_\alpha\), then \(b \not\leq e\) because \(e \land c = 0 < c \leq b\) where \(c = b \land -h_{\alpha,0} \land -h_{\alpha,1}\).

By the above claim, we may choose a quotient \(D_\alpha = C_\alpha/I_\alpha\) such that \(x/I_\alpha = x\) for all \(x \in B_\alpha \cup H_\alpha\). Choose \(D_\alpha\) such that also \(D_\alpha \in M_\omega\) and \(D_\alpha \setminus (B_\alpha \cup H_\alpha)\) is disjoint from \(\bigcup_{\beta < \alpha} M_\beta\). Note that \(B_\alpha \not\subseteq I_\alpha D_\alpha\) because, for example, \(\pi_+^{b_{\alpha,0}^0}(h_\alpha)\) does not exist because \(h_{\alpha,0} \leq D_\alpha \land_{n < m} -(b_{\alpha,0}^n \land b_{\alpha,1}^n)\) for all \(m < \omega\). However, we still have the following.

Claim 5.6. For each \(i < 2\), \(A'_\iota, i \leq I_\alpha D_\alpha\). In particular, \(\pi_+^{a,0}\) and \(\pi_+^{a,1}\) satisfy \(\text{(6)}\) for all \(n < \omega\) and \(x \in D_\alpha\) of the forms below.

\[
\begin{array}{c|c|c}
  x & \pi_+^{A'_i}(x) & \pi_+^{A'_i}(x) \\
  \hline 
  b_{\alpha,0}^n \land h_{\alpha,0} & b_{\alpha,0}^n & -b_{\alpha,1}^n \\
  b_{\alpha,1}^n \land h_{\alpha,1} & -b_{\alpha,0}^{n+1} & b_{\alpha,1}^n \\
  \end{array}
\]

\(\text{(6)}\)

Proof. For this proof our notation will suppress the dependence on \(\alpha\). Every nonzero element of \(C\) is a finite nonempty join of elements of the form \(x = a \land b_i \land b_i \land h\) where \(a \in A'_\iota\), \(h\) is of the form \(\pm h_0 \land \pm h_1\), and each \(b_i\) is \(\land_{n \in P_i} b_i^n \land \land_{n \in Q_i} -b_i^n\) where \(P_i\) and \(Q_i\) are each (possibly empty) finite subsets of \(\omega\). (Our convention is that \(\land_{\emptyset} = 1\).) In general, \(\pi_+^{A'_i}(y \lor z) = \pi_+^{A'_i}(y) \lor \pi_+^{A'_i}(z)\) and \(\pi_+^{A'_i}(y) = -\pi_+^{A'_i}(-y)\) if the right-hand sides exist. Moreover, by Lemma 3.5, \(\pi_+^{A'_i}((a \land b_i \land b_i \land h)/I) = a \land b_i \land \pi_+^{A'_i}((b_i \land h)/I)\) if the right-hand side exists.

Let \(x = b_i \land c \land h\) where \(b_i \land h\) are as above. We will show that each \(\pi_+^{A'_i}(x/I)\) exists and equals \(\tau_i(x)\) where \(\tau_i(x) = \land_{n \in T_i} -b_i^n\) where \(T_i\) is as in \(\text{(7)}\) below, which uses shift operator notation \(S_\beta = \{\beta + 1 : \beta \in S\}\) and \(S_\alpha = \{\beta : \beta + 1 \in S\}\) for sets of ordinals.

\[
\begin{array}{c|c|c}
  h & T_0 & T_1 \\
  \hline 
  -h_0 \land -h_1 & \emptyset & \emptyset \\
  -h_0 \land h_1 & P_1 & P_0 \\
  -h_0 \land h_1 & P_1 \cup (P_1 \lor P_0 \land \emptyset) \\
  \hline 
  \end{array}
\]

\(\text{(7)}\)

In all cases, \(x/I \leq \tau_i(x)\) follows directly from the definition of \(I\). Moreover, \(\text{(9)}\) follows from \(\text{(7)}\).

Henceforth working in \(C\), suppose that \(y \in A'_{i, t} \in I,\) and \(x \leq y \lor t\). We will show that \(\tau_i(x) \leq y \lor e\) for some \(e \in I\). Every element of \(A'_{i, t} \setminus \{1\}\) is a nonempty finite meet of elements of the form \(z \lor w_i\) where \(z \in A'_i \setminus \{1\}\) and \(w_i\) is \(\lor_{n \in R_i} b_i^n \lor \lor_{n \in S_i} -b_i^n\) where \(R_i\) and \(S_i\) are each (possibly empty) finite subsets of \(\omega\) and \(R_i \perp S_i\). (Our convention is that \(\lor \emptyset = 0\).) Moreover, by \(\text{(7)}\), \(\tau_i(x) = 1\) if and only if \(h = -h_0 \land -h_1\) or \(P_1 \land = \emptyset\). Therefore, it is enough to assume that
Similarly, we have $\tau_{m,i} \leq w_i \lor e$ and, therefore, that (X1) holds.

Thus, $b_{1-i} \land h \leq w_i \lor e$ if and only if, for all $f : 2 \times m \rightarrow 3$,

$$b_{1-i} \land h \leq w_i \lor e = w_i \lor \bigvee_{j < 2} \bigwedge_{n < m} g_{j,n}(k) = w_i \lor \bigwedge_{f : 2 \times m \rightarrow 3} \bigvee_{j < 2} \bigwedge_{n < m} g_{j,n}(f(j,n)).$$

Given any $f : 2 \times m \rightarrow 3$, let, for each $(j,k) \in 2 \times 3$, $E_{j,k} = \{n : f(j,n) = k\}$; let $F_{1,0} = E_{1,0}$ and $F_{1,1} = E_{1,1}$. By independence of $(B_0, B_1, H)$, $(b_{i}^n)_{n < \omega}$, and $(b_{i}^+)_n$, $m \geq S_i$ and $e \geq t$. Let $g_{0,n}(0) = b_{0}^n$, $g_{1,n}(0) = b_{0}^n + 1$, $g_{j,n}(1) = b_{1}^i$, and $g_{j,n}(2) = h_j$ for all $(j,n) \in 2 \times \omega$. Then

$$b_{1-i} \land h \leq w_i \lor e = w_i \lor \bigvee_{j < 2} \bigwedge_{n < m} g_{j,n}(k) = w_i \lor \bigwedge_{f : 2 \times m \rightarrow 3} \bigvee_{j < 2} \bigwedge_{n < m} g_{j,n}(f(j,n)).$$

Thus, $b_{1-i} \land h \leq w_i \lor e$ if and only if, for all $f : 2 \times m \rightarrow 3$,

$$b_{1-i} \land h \leq w_i \lor \bigvee_{j < 2} \bigwedge_{n < m} g_{j,n}(f(j,n)).$$

By choosing $f$ according to (10) below, we ensure that (X2), (X3), and (X4) fail, and, therefore, that (X1) holds.

Comparing (9) with (10), we see that $E_{0,1-i} \cup F_{1,1-i} = U_i$ in all cases. Therefore, $P_{1-i} \not\subseteq U_i$. Thus, $\tau_i(x) \leq y \lor e$ and $P_{1-i} \neq \emptyset$. □

Choose $A_\alpha = D_\alpha \oplus Fr_\alpha$ such that $A_\alpha$ and its cofactor maps are in $M_\alpha$, that $\oplus_1 = id_{D_\alpha}$, and that $A_\alpha \setminus D_\alpha$ is disjoint from $\bigcup_{\beta < \alpha} M_\beta$. Thus, (1) is preserved. By construction, $A_\alpha \cap \bigcup_{\beta < \alpha} M_\beta = \bigcup_{\beta < 2} A'_\alpha, i$, so (2) is preserved. By Claim 5.4 (3) is preserved. By Sirotta’s Lemma, since $A'_\alpha, i \leq \text{Fr} D_\alpha \leq \text{Fr} A_\alpha$ for all $i < 2$ and $A'_\alpha, i$ splits perfectly in $A_\alpha$, (4) is preserved. Thus, our construction of $\Omega$ is complete and $\Omega$ has the FN.
Choose a long $\omega_1$-approximation sequence $\langle N_\alpha \rangle_{\alpha < \omega_2}$ with $N_\alpha \prec H(\aleph_3)$ for all $\alpha < \omega_2$ and $A, M \in N_0$ (which implies $\Omega \in N_0$). We will show that $\Omega[N]$ is not locally finite. Let $\delta = \omega_1 + 1$; let $\beta = \omega_2 \cap \aleph_\delta$, which is in $\omega_2 \cap \aleph_\delta$ and has cofinality $\omega_1$ by Lemma 2.3 let $[\beta, \alpha] = [\beta, \beta + \omega_1) \cap \aleph_\delta$. Note that $\alpha \in \aleph_\delta$ and $[\alpha]_1 = \beta$.

Claim 5.7. $\beta \cap \aleph_\delta = \beta \cap M_{\alpha,1}$.

Proof. By Lemma 2.3 $\beta \subseteq \bigcup_{\theta < \omega_1} M_{\beta + \theta}$. Since $\beta, M \in \aleph_\delta < H(\aleph_3)$, we have $\beta \cap \aleph_\delta = \beta \cap \bigcup\{M_\theta : \theta \in [\beta, \beta + \omega_1) \cap \aleph_\delta\} = \beta \cap \bigcup_{\beta \leq \theta < \alpha} M_\theta = \beta \cap M_{\alpha,1}$. □

We will show that $\langle h_{\alpha,0}, h_{\alpha,1}, b^0_{\alpha,0} \rangle$ generates an infinite subalgebra of $\Omega[N]$. It suffices to show that

- $\rho(b^m_{\alpha,i} \land h_{\alpha,i}, N) = \delta$ for all $i < 2$ and $m < \omega$,
- $\pi^0_+(b^m_{\alpha,1} \land h_{\alpha,1}, N) = b^{m+1}_{\alpha,0}$ for all $m < \omega$, and
- $\pi^\perp_+(b^m_{\alpha,0} \land h_{\alpha,0}, N) = -b^{m+1}_{\alpha,1}$ for all $m < \omega$.

First, $\rho(b^m_{\alpha,i} \land h_{\alpha,i}, M) = \alpha$ by construction. By definition of $\alpha$, we have $\alpha \in \aleph_\delta \setminus \aleph_\delta$; since $\beta \leq \alpha$, we also have $\alpha \notin \aleph_\delta$. Since $M, \rho(N, M) \in N_0$ and $M_{\alpha,1}$ is countable, we conclude that $b^m_{\alpha,i} \land h_{\alpha,i} \in \aleph_\delta \setminus \aleph_\delta$; hence, $\rho(b^m_{\alpha,i} \land h_{\alpha,i}, N) = \delta$.

Second, we have, by Corollary 3.3 and Claim 5.6,

$$\pi^0_+(b^m_{\alpha,1} \land h_{\alpha,1}, N) = \bigwedge_{j < 2} \pi^N_{\delta,0}((\pi^j_+(b^m_{\alpha,1} \land h_{\alpha,1}, M)) = \pi^N_{\delta,0}(-1_{\alpha,0}) \land \pi^N_{\delta,0}(b^m_{\alpha,1}).$$

We have $\pi^N_{\delta,0}(b^{m+1}_{\alpha,0}) = -b^{m+1}_{\alpha,0}$ because $-b^{m+1}_{\alpha,0} \in \aleph_\delta$ and $\rho(-b^{m+1}_{\alpha,0}, M) \in I_0(\alpha) = \beta$. We have $\pi^N_{\delta,0}(b^m_{\alpha,1}) = \pi^A'_{\delta,2}(b^m_{\alpha,1}) = 1$ by Lemma 3.1 because $\aleph_\delta \cap \Omega = A_{\alpha,0} \sqcup A'_{\alpha,1}$ by (27) and Proposition 3.11 and because $A_{\alpha,0} \cap A'_{\alpha,1} = A'_{\alpha,2} \subseteq \text{rec } A'_{\alpha,1}$. Thus, $\pi^\perp_0(b^m_{\alpha,1} \land h_{\alpha,1}, N) = -b^{m+1}_{\alpha,0}$.

Third, we have, by Corollary 3.3 and Claim 5.6,

$$\pi^\perp_+(b^m_{\alpha,0} \land h_{\alpha,0}, N) = \bigwedge_{j < 2} \pi^N_{\delta,1}(\pi^j_+(b^m_{\alpha,0} \land h_{\alpha,0}, M)) = \pi^N_{\delta,1}(b^m_{\alpha,0}) \land \pi^N_{\delta,1}(-b^{m+1}_{\alpha,0}).$$

We have $\pi^N_{\delta,1}(-b^m_{\alpha,0}) = -b^m_{\alpha,0}$ because $\rho(-b^m_{\alpha,0}, M) \in I_1(\alpha) = [\beta, \alpha) \subseteq \aleph_\delta$. We have $\pi^N_{\delta,1}(b^m_{\alpha,0}) = \pi^A'_{\delta,1}(b^m_{\alpha,0})$ by Lemma 3.1 because $A'_{\alpha,0} \subseteq \Omega \cap \aleph_\delta$ by (27) and Proposition 3.11 and because, arguing as in the proof of Claim 5.2, $A'_{\alpha,0} \cap \aleph_\delta \subseteq \text{rec } A'_{\alpha,0}$. Because $A'_{\alpha,0} \cap \aleph_\delta = A'_{\alpha,2}$ by Claim 5.7 we also have $\pi^N_{\delta,1}(b^m_{\alpha,0}) = \pi^A'_{\delta,2}(b^m_{\alpha,0}) = 1$. Thus, $\pi^\perp_+(b^m_{\alpha,0} \land h_{\alpha,0}, N) = -b^{m+1}_{\alpha,0}$.

Thus, $\Omega$ has the FN but not the SFN.

We briefly remark that the interaction between $\pi^0_+$ and $\pi^\perp_+$ is essential to the above construction. Given boolean algebras $K \leq \text{rec } L$, it is not hard to check, using Lemma 3.3 that $(L, \land, \lor, -L, 0_L, 1_L, \pi^K_+, \pi^K_\perp)$ is locally finite. This fact can then be used to re-prove the implication from FN to SFN for boolean algebras of size at most $\aleph_1$, without using, as Heindorf and Shapiro do, the implication from FN to productivity for boolean algebras of size at most $\aleph_1$. 

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E-mail address: david.milovich@tamiu.edu
URL: http://www.tamiu.edu/~dmilovich

DEPT. OF ENGINEERING, MATHEMATICS, AND PHYSICS, TEXAS A&M INTERNATIONAL UNIVERSITY, 5201 UNIVERSITY BLVD., LAREDO, TX 78041, USA