The graviton vacuum as a distributional state in kinematic loop quantum gravity

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Abstract

The quantum behaviour of weak gravitational fields admits an adequate, albeit approximate, description by those graviton states in which the expectation values and fluctuations of the linearized gravitational field are small. Such states must approximate corresponding states in full quantum gravity. We analyse the nature of this approximation for the graviton vacuum state in the context of kinematical loop quantum gravity (LQG) wherein the constraints are ignored. We identify the graviton vacuum state with kinematically non-normalizable, distributional states in LQG by demanding that relations between linearized operator actions on the former are mirrored by those of their nonlinear counterparts on the latter. We define a semi-norm on the space of kinematical distributions and show that the identification is approximate up to distributions which are small in this semi-norm. We argue that our candidate states are annihilated by the linearized constraints (expressed as operators in the full theory) to leading order in the parameter characterizing the approximation. This suggests the possibility, in a scheme such as ours, of solving the full constraints order by order in this parameter. The main drawback of our considerations is that they depend on certain auxiliary constructions which, though mathematically well defined, do not arise from physical insight. Our work is an attempt to implement an earlier proposal of Iwasaki and Rovelli.

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1. Introduction

A Dirac constraint quantization of a Hamiltonian formulation of gravity is defined through the following steps. First, a ‘kinematical’ representation of the Poisson bracket algebra of a large enough set of functions on the unconstrained phase space is constructed such that these functions act as linear operators on the representation space. Next, the constraints of
the theory are represented as quantum operators in this representation and physical states are identified with their kernel. Finally, an inner product on the space of physical states is chosen which enforces hermiticity conditions on a complete set of operators corresponding to Dirac observables, thus converting the physical state space to a Hilbert space. Since one expects the Dirac observables to have direct physical interpretation, the physical interpretation of the formalism follows. Within the loop quantum gravity (LQG) approach, the kinematical structures are well understood. There has also been significant progress in finding physical states. Further progress is obstructed by the fact that almost no Dirac observables are known for general relativity. Indeed, this is a problem of the classical theory. As in any generally covariant theory, evolution in general relativity (at least in the spatially compact case) is generated by the constraints, thus implying that Dirac observables are constants of motion. Hence finding enough Dirac observables is equivalent to finding a complete set of constants of motion for the Einstein equations, a task which is well nigh impossible. Therefore, to achieve progress in interpreting the formalism, ideas are needed as to how to obtain a physical interpretation of physical states in the absence of explicit Dirac observables.

In contrast to this state of affairs for full blown general relativity, an adequate quantum description of weak gravitational fields is available. Sufficiently weak gravitational fields are described by a linearization of general relativity about flat spacetime. Quantization of this (approximate) description of the exact physics is in terms of the graviton Fock representation. Since the exact description is that of full quantum gravity, Fock states in linearized gravity which describe weak gravitational fields must correspond, at least approximately, to states in full quantum gravity. Note that only those Fock states in which the gravitational field has small expectation values and fluctuations describe weak ‘quantum’ gravitational fields and it is only such states which are expected to have full quantum gravity correspondents. An archetypal example of such a state is the graviton vacuum which is expected to provide an adequate physical description of small quantum gravitational fluctuations about flat spacetime.

Let us suppose that within the LQG framework, we find a way to associate kinematical structures with Fock states describing weak gravitational fields in the context of a mathematically well defined approximation scheme. Since kinematical structures play a key role in defining physical states, we may hope that the nature of the approximation scheme suggests how to extend this correspondence (of appropriate Fock states with kinematical structures) to physical states. Such a correspondence would provide a way to interpret these physical states (as small quantum perturbations of Minkowski spacetime) in the absence of Dirac observables.

Motivated by these remarks, our aim in this work is to make a preliminary investigation of the relation between quantum linearized gravity and LQG by finding structures in kinematical LQG which correspond to the Fock vacuum. As we shall see, a reasonable definition of such a correspondence requires the satisfaction of infinitely many conditions by the putative LQG structures and therefore the identification of these structures is quite involved.

We proceed as follows. In section 2, we analyse the conditions under which classical linearized gravity is an adequate approximation to full general relativity and show that the approximation may be characterized by a small parameter, \( \epsilon \), constructed out of the physically relevant classical distance scales. Thus, the equations of motion of classical linearized theory are obtained by an expansion of the equations of motion of full general relativity about flat spacetime in which only terms of linear order in \( \epsilon \) are retained. For the linearized theory vacuum to be of physical relevance, it is necessary that vacuum expectation values

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3 We ignore here the thorny issues of the applicability of the Copenhagen interpretation to quantum gravity.
4 We are indebted to Abhay Ashtekar for his crucial inputs without which the analysis and results of section 2 would not have been possible.
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and fluctuations of the gravitational field be of order $\epsilon^5$. We analyse this requirement in the context of the $r$-Fock representation of linearized gravity [1–3] and show that it leads to a specification of $\epsilon$ in terms of an intertwining of the physically relevant classical distance scales with the distance scales relevant to the quantum theory, namely the Planck length and the scale $r$ characterizing the $r$-Fock representation. We use the $r$-Fock representation because of its structural similarity to the representation used for LQG. The $r$-Fock representation of linearized gravity is constructed in [1, 3] (see also appendix 2 of [4] and the discussion of the extended loop representation in [5]) and its relation to the usual Fock representation is discussed in [3].

In section 3, we propose to identify states in kinematical LQG with the $r$-Fock vacuum by demanding that the action of the linearized theory operators (to order $\epsilon$) on the $r$-Fock vacuum be mirrored by the action of their nonlinear counterparts in LQG. We show that this demand leads to infinitely many consistency conditions and the consequent necessity of identifying an infinite dimensional set of Fock states with their counterparts in LQG. We also argue that these counterparts are kinematical distributions, i.e. they are kinematically non-normalizable states which lie in the algebraic dual to the finite span of extended spin network states (for a definition of the latter see [6]). We characterize this identification by a map $M^6$ from such distributions to appropriate subspaces of $r$-Fock space. The material after section 3 is devoted to showing the existence of at least one such map which satisfies the consistency conditions.

In section 4 we define two auxiliary structures in LQG which go into our construction of $M$. The first is a distributional ‘weave’ state which appropriately captures the notion of classical flat space used in the linearized theory. The second is a semi-norm on the space of distributions. The kernel of $M$ is (partially) specified by those distributions which are of order $\delta_\epsilon$, where $\delta_\epsilon$ is a small $\epsilon$-dependent parameter. In this section we also specify the general structure of our putative solutions to the consistency conditions. In section 5, we construct such solutions explicitly and take the $r$-Fock vacuum. In section 6 we argue that the constraints of the linearized theory, represented as operators in the full theory, map the distributional states we constructed in section 5 to the kernel of $M$, i.e. their images are of order $\delta_\epsilon$. Section 8 is devoted to a discussion of our results and a few important identities are proved in the appendices.

Although our work suggests a strategy to solve the constraints of full gravity order by order, it is pertinent to make a couple of cautionary remarks here to place our results in proper perspective.

(a) Our results would be stronger if we could prove them in the context of a norm rather than a semi-norm. In fact, most of the considerations of sections 5 and 6 do go through in the context of a suitably defined norm. The only failure in this context is that of our states to be mapped into the kernel of $M$ by the linearized Gauss law constraint. In section 7 we comment on this in relation to the fact that the connection variables of linearized theory and LQG take values in distinct Lie algebras (namely $U(1)$ and $SU(2)$).

(b) As noted earlier the $r$-Fock construction is structurally similar to LQG. As briefly discussed in section 7 our constructions of $M$, the semi-norm as well as the distribution corresponding to the vacuum state, all lean heavily on the existence of similar structures in the $r$-Fock representation. This is the reason that our constructions ‘work’ at a mathematical level. However, they are still ad hoc because they do not arise from any deep physical insight.

5 We only expect this condition to hold when the vacuum is probed at ‘large’ distance scales. A detailed discussion of these matters is given in section 2.

6 We actually use a one parameter family of maps, $M^6$, in section 3. For pedagogical reasons we gloss over this subtlety and use notation appropriate to a single map $M$ in this section.
In view of (a) and (b), it is unlikely that the detailed choices of auxiliary structures made in this work lead to a direct relation between ‘solutions to the constraints to n-th order’ and exact physical states in nonperturbative LQG. Nevertheless, our hope is that further analysis may lead to a more physically motivated implementation of the general strategy (described in section 3) suggested by our work. If so, this would provide a plausible interpretation of certain states in LQG (namely those corresponding to the vacuum of linearized gravity) without recourse to Dirac observables.

There have been other attempts to identify the vacuum state in LQG [7–10] as well as to understand how classical spacetime emerges from LQG structures [7, 11, 12]. These attempts seem to indicate that the LQG structures underlying a smooth classical metric have a Planck scale discreteness. The extraction of experimental signatures of this discreteness in terms of propagation of radiation and matter on these structures constitutes an active field of research. Since a review of all the pertinent literature is beyond the scope of this work, we only point the reader to the pioneering works [13, 14] as well as the cautionary remarks embodied in [15].

Indeed, all the issues mentioned above are the focus of a lot of current research in the field. The related problem of interpretation of physical states in LQG is a very crucial and hard problem. Though the current set of ideas (including ours) may be criticized on several fronts, we believe that only an exploration of a diversity of such proposals will suggest some way to attack this problem.

Finally, we note that this work is an attempt to implement earlier proposals of Iwasaki and Rovelli [16]. Indeed, this work would not have been possible without their considerations. It is our view that their beautiful ideas were considerably in advance of the technical state of art of the field and consequently were difficult to implement with precision. We hope that we have partially remedied this by our work here.

In what follows we shall assume familiarity with the loop representation of linearized gravity as discussed in [3]. We set \( c = G = 1 \) and denote the Planck length by \( l_P \). We also define \( O(\epsilon^n) \) and \( O^{\infty}(\epsilon) \) as follows. Let \( x \) be an \( \epsilon \)-dependent complex number. \( x = O(\epsilon^n) \) iff \( \lim_{\epsilon \to 0} |x|/\epsilon^n \) exists. \( x = O^{\infty}(\epsilon) \) iff as \( \epsilon \to 0 \), \( x \to 0 \) faster than any positive power of \( \epsilon \).

2. Linearized gravity as an approximate physical description

2.1. A quick review of linearized gravity in terms of connections

The classical Hamiltonian formulation underlying LQG is discussed in [18]. The phase-space variables are a spatial \( SU(2) \) connection, \( A_a^i(\vec{x}) \) and a densitized triad field \( E^i_j(\vec{y}) \). Here \( a, b \) denote spatial components, \( i, j \) denote internal \( SU(2) \) Lie algebra components. To define the linearized theory about a flat background, the spatial topology is chosen to be \( \mathbb{R}^3 \). We fix, once and for all, a Cartesian coordinate system \( \{\vec{x}\} \) on \( \mathbb{R}^3 \) as well as an orthonormal basis in the Lie algebra of \( SU(2) \). Henceforth all components refer to this Cartesian coordinate system and to this internal basis. We linearize the \( SU(2) \) formulation about the phase-space point \((A_a^i = 0, E^i_j = \delta^i_j)\). As in [3], we denote the linearized triad field by \( e^i_j \) so that

\[
E^i_j = \delta^i_j + e^i_j.
\]

Since the background connection vanishes, there is no need to introduce a new symbol for the linearized connection. The only non-vanishing Poisson bracket is

\[
\{A_a^i(\vec{x}), e^j_k(\vec{y})\}_L = \gamma \delta^k_a \delta^j_i \delta(\vec{x}, \vec{y}).
\]

\( ^7 \) Note that J Zegwaard independently explored the same issues as Iwasaki and Rovelli using a slightly different approach in his work [17].
and the linearized Gauss law, vector and scalar constraints are

\[ G_i^L = \partial_a e_i^a + \epsilon_i^a A_{aj}, \]  
\[ V_a^L = f_{ab}^a, \]  
\[ C^L = \epsilon^{abc} f_{abc}. \]  

(2.3)\hspace{1cm} (2.4)\hspace{1cm} (2.5)

Here \( \gamma \) is the Barbero–Immirzi parameter, \( f_{ab}^a = \partial_a A_b^a - \partial_b A_a^a \) is the linearized curvature, \( \partial_a \) denotes the flat derivative operator which annihilates the background triad and we use the background triad \( \delta_a^i \) to freely interchange internal and spatial indices. The linearized \( SU(2) \) connection transforms as a triplet of \( U(1) \) connections under transformations generated by \( G_i^L \).

Given any oriented piecewise analytic loop \( \beta \) in \( R^3 \) with (arbitrary) parametrization \( s \) and tangent vector \( \dot{\beta}_a \), we define the loop form factor

\[ X^a_\beta(\vec{x}) := \oint_\beta ds \delta_3(\vec{\beta}(s), \vec{x}) \dot{\beta}_a, \]  

(2.6)

and its Fourier transform

\[ X^b_\beta(\vec{k}) = \frac{1}{(2\pi)^3} \int d^3x X^b_\beta(\vec{x}) e^{-i\vec{k} \cdot \vec{x}} = \frac{1}{(2\pi)^3} \oint_\beta ds e^{-i\vec{k} \cdot \vec{\beta}(s)} \dot{\beta}_a. \]  

(2.7)

For any positive parameter \( r \) with dimensions of length, we define the Gaussian smeared loop form factor

\[ X^a_\beta(r)(\vec{x}) := \int d^3y X^a_\beta(\vec{y}) e^{-|\vec{x} - \vec{y}|^2/2r^2} (2\pi r^2)^3 \]  

(2.8)

\[ X^a_\beta(r)(\vec{k}) = e^{-\frac{\vec{k}^2}{4r^2}} X^a_\beta(\vec{k}). \]  

(2.9)

For any triplet of loops \( \alpha_i, i = 1, \ldots, 3 \) we define \( X^{ab}_\alpha(\vec{x}) \) and \( X^{ab}_{\alpha_i}(\vec{x}) \) via their Fourier transforms to be,

\[ X^{ab}_\alpha(\vec{k}) := X^a_{\alpha_i}(\vec{k}) \delta^{ib}, \quad X^{ab}_{\alpha_i}(\vec{k}) := e^{-\frac{i\vec{k}^2}{4r^2}} X^a_\alpha(\vec{k}). \]  

(2.10)

We also define \( G^{ab}_{\alpha_i}(\vec{x}) \) via its Fourier transform,

\[ G^{ab}_{\alpha_i}(\vec{k}) := k X^a_{\alpha_i}(\vec{k})(1 + 2i\gamma)m_a \bar{m}_b - k X^a_{\alpha_i}(\vec{k})(1 - 2i\gamma)\bar{m}_a m_b. \]  

(2.11)

Here \( m_a, \bar{m}_a \) form the standard transverse basis in momentum space and \( X^{\pm}_{\alpha_i} \) are the positive and negative helicity components of the transverse, traceless, symmetric part of \( X^{ab}_{\alpha_i} \). Finally we define the following Dirac observables for the linearized theory:

\[ h_\alpha = \exp i \int d^3x X^a_{\alpha}(A) A_{ab} \]  
\[ \prod_{a \in 1} A_{ab} dx^a \]  

(2.12)

and

\[ g_{\alpha}(\rho) = \int d^3x G^{ab}_{\alpha_i}(\rho) \epsilon_{\alpha_i ab}, \]  

(2.13)

where \( \epsilon_{\alpha_i ab} \) is obtained from \( \epsilon_{ab} \) by Gaussian smearing. \( h_\alpha \) and \( g_{\alpha}(\rho) \) play a key role in the considerations of [3] and will continue to play a key role in this work.
2.2. Classical linearized gravity as an approximate physical description

Linearized gravity is expected to provide an approximate description of the physics of weak gravitational fields. Such fields are characterized by low curvature. Since curvature is dimensionful, a notion of its smallness requires the introduction of some parameter $s_{\text{curv}}$ which has dimensions of length. The restriction to small curvatures implies that $|\partial_a \partial_b e_i^r| < s_{\text{curv}}^{-2}$. On dimensional grounds we expect phase-space data for low curvatures to satisfy

$$|\partial_a e_i^r|, |A_i^a| < s_{\text{curv}}^{-1}. \tag{2.14}$$

Clearly, the larger the $s_{\text{curv}}$ the better the physical description provided by linearized gravity.

However, small curvatures are only a necessary condition for the application of linearized gravity. There exist nonlinear configurations of the gravitational field, namely large black holes, which have vast regions of small curvature but also possess global features such as horizons. Such global features cannot be described by linearized gravity. More quantitatively, for data satisfying (2.14) we estimate the gravitational energy in a region of size $s$ to be

$$M = \frac{1}{s_{\text{curv}}^2}. \tag{2.12}$$

If $M \sim s$ the gravitational configuration is expected to be close to that of a black hole. Hence, we require that $M \ll s$, equivalently, that $\frac{s_{\text{curv}}}{s} \ll 1$. Thus, we expect linearized gravity to be a good description of the gravitational field when $s_{\text{curv}}$ is large and when we probe regions of size $s$ much smaller than $s_{\text{curv}}$. For a fixed probe scale $s$, we define $\epsilon = \frac{s_{\text{curv}}}{s}$. Then linearized gravity is a valid description when $\epsilon \ll 1$ and equation (2.14) reads

$$|\partial_a e_i^r|, |A_i^a| < \frac{\epsilon}{s}. \tag{2.15}$$

In addition, we shall assume that the ‘observable’ part of the linearized triad, namely its positive and negative helicity parts, also have spatial variations characterized by the scale $s$. Specifically, we assume

$$|\partial_a e_i^{\pm}|, |(-\nabla^2)^{\frac{1}{2}} e_i^{\pm}| < \frac{\epsilon}{s}. \tag{2.16}$$

In view of their key role in the quantum theory, it is useful to encode conditions (2.15), (2.16) as conditions on the observables $h_{\alpha}, g_{\alpha}(r)$. From (2.15) we obtain $\sum_{a=1}^{3} \int d^3 x A_i^a \partial_a x^a < \frac{\epsilon}{s} |\alpha|$, where $|\alpha|$ is the sum of the lengths of the loops $\alpha^k, k = 1, \ldots, 3$. In order to get a useful bound on $h_{\alpha}$, as well as to ensure that the classical probes (in this case the loops $\alpha^i$) do not extend into regions of volume much larger than $s^3$, we restrict our attention to loops of length at most equal to $s$. Then equations (2.12) and (2.15) imply that

$$h_{\alpha} - 1 < 3\epsilon. \tag{2.17}$$

With the same restriction on loop lengths, we obtain, using (2.7)–(2.11) and (2.16),

$$|g_{\alpha}(r)| \leq \int X^{ab}_{\alpha(r)}(\vec{x}) [(1 + 2i\gamma)(-\nabla^2)^{\frac{1}{2}} e_{i}^{+}(\vec{x}) + (1 - 2i\gamma)(-\nabla^2)^{\frac{1}{2}} e_{i}^{-}(\vec{x})] \leq 2\sqrt{1 + 4\gamma^2} \epsilon. \tag{2.18}$$

Note that the above bound is independent of $r$.

2.3. Quantum considerations

Equations (2.17) and (2.18) imply that only those states of quantum linearized theory in which the expectation values and fluctuations of the operators $h_{\alpha} - 1, g_{\alpha}(r)$ are of order $\epsilon$, are physically relevant descriptions of quantum aspects of weak gravitational fields. Since in this work we are interested only in the vacuum state, we require that the vacuum fluctuations and vacuum expectation values of these operators be small (i.e. of $O(\epsilon)$).
As mentioned in the introduction we use the r-Fock representation and the r-Fock vacuum, \(|0,r⟩\) to investigate the consequences of this requirement. We denote the vacuum expectation value of an operator \(\hat{O}\) by \(\bar{O} = ⟨0,r|\hat{O}|0,r⟩\) and its vacuum fluctuation by \(\Delta_{\text{vac}}\). From [3], we have that \(g_\alpha(r) = 0\) and that

\[
(\Delta g_\alpha(r))^2 \leq \frac{1 + 4y^2}{32\pi^3} l_P^2 s_{\text{max}}^2 \int d^3k e^{-k^2r^2}
\]

(2.19)

where \(s_{\text{max}} \leq s\) is the length of the longest loop in the triplet \(\alpha\). To ensure that \(\Delta g_\alpha(r) = O(\epsilon)\) we set

\[
\frac{l_P}{r^2} = \epsilon.
\]

(2.21)

It can be checked that this identification of \(\epsilon\) ensures that the vacuum expectation values and vacuum fluctuations of \(\hat{h}_\alpha - 1\) are also at most of \(O(\epsilon)\).\(^8\)

How should we interpret equation (2.21)? Note that the parameter \(r\) plays no essential role in classical theory. In particular the considerations of section 2.1 are unaltered if we set \(r = 0\). However, for quantum operators to be well defined in a (Fock-like representation of) a quantum field theory, it is necessary for the basic quantum fields to be appropriately smeared in order to avoid the infinities coming from quantum fluctuations at arbitrarily small distance scales. Here, this smearing is characterized by the distance scale \(r\). While we require that operator fluctuations be small (i.e. of \(O(\epsilon)\)), the quantum uncertainty principle dictates that operator fluctuations cannot, in general, be arbitrarily small. The potential conflict between these two requirements results in relation (2.21) between the classically relevant distance scale \(s_{\text{curv}} = \frac{\epsilon}{r}\) and the distance scales relevant to quantum theory, namely the Planck length \(l_P\) and the smearing scale \(r\). Equation (2.21) can be rewritten in the form \(r^2 = l_P s_{\text{curv}}\). In this form it is clear that if the smearing scale is too small, the resulting short distance quantum fluctuations exceed the upper bounds (2.17) and (2.18). Note that we may choose the individual parameters \(r, s\) and \(s_{\text{curv}}\) to vary with \(\epsilon\) in different ways. Since we would like to have a non-empty set of probes when \(\epsilon \rightarrow 0\), we shall henceforth restrict our attention to probe sizes \(s\) which do not vanish as \(\epsilon \rightarrow 0\). Then equation (2.21) implies that \(r\) (and \(s_{\text{curv}}\)) diverge as \(\epsilon \rightarrow 0\). This means that to ensure that the quantum fluctuations in curvature become arbitrarily small, the relevant operators have to be smeared at arbitrarily large distance scales. In this work we shall think of \(r\) as a short distance smearing scale relative to the probe scale \(s\) and restrict our attention to the case \(s \gg r\).

3. A strategy to identify the vacuum state in LQG

In this section we discuss the identification of states in LQG which correspond to the r-Fock vacuum of linearized theory in terms of a map between (suitably defined) states in LQG and (subsets of) r-Fock space. Section 3.1 contains a few pertinent observations which serve to motivate the definition and desired properties of this map as outlined in section 3.2. In section 3.3 we describe our strategy for an explicit construction of a map which satisfies the criteria of section 3.2.

\(^8\) Equation (2.20) only provides an upper bound. In appendix E we construct \(a^k\) such that \(\Delta g_{\alpha(r)} \approx \epsilon\) to nontrivial leading order in \(\epsilon\).
3.1. A few remarks by way of motivation

(i) Linearized theory observables and their LQG counterparts. The operators $\hat{h}_\alpha$, $\hat{g}_{\alpha(r)}$ play a key role in the identification of the $r$-Fock vacuum in linearized theory (see [3]). Therefore it is reasonable to look for their counterparts in the $SU(2)$ theory. Using equations (2.15) and (2.1), it is straightforward to check that

$$-i(h_\alpha - 1) = 2 \sum_{k=1}^3 \tau_i H_o^a \tau^k + O(\epsilon^2),$$

$$g_{\alpha(r)} = G_{\alpha(r)}. \tag{3.1}$$

Here $H_o$ is the $SU(2)$ holonomy $H_o = \frac{i}{2} P \exp - \int \hat{A}_o A^i \tau^i$. $\tau^i = -\frac{i}{2} \sigma^i$, where $\sigma^i$ are the Pauli matrices. Thus $\tau^i$ correspond to the fixed basis of the Lie algebra of $SU(2)$ (see section 2) in its defining two-dimensional representation. Note that (3.1) holds independently of which point of $\alpha^k$ is chosen as the base point. Also,

$$G_{\alpha(r)} := \int d^3 x G_{\alpha(r)ab}(\vec{x})(E^{ab}_{(\tau)}(\vec{x}) - \delta^{ab}) \tag{3.3}$$

$E_{(\tau)}^{ab}$ is defined from the $SU(2)$ triad variable by

$$E_{(\tau)}^{ab}(\vec{x}) := \frac{1}{(2\pi r^2)^2} \int d^3 y \exp \frac{i}{2\pi r} E_{(\tau)}^{ai}(\vec{x}) \delta^{bi}. \tag{3.4}$$

We shall refer to the set of functions in linearized theory on the left-hand side of equations (3.1) and (3.2) as $O_L^r$ and their $SU(2)$ counterparts on the right-hand side of these equations as $O^r_F$. As noted at the end of section 2, $r$ is a function of $\epsilon$ through (2.21). Since some of observables (namely the ones depending on the triad) depend explicitly on $r$, we must have a superscript $\epsilon$ to signify this dependence. Note also that the linearized theory holonomy operators have an implicit dependence on $r$ coming from the $r$-dependence of the representation.

Recall that $O_L^r$, $O_F^r$ are labelled by loops of size at most $s$. Since we are interested in probing regions of size of order $s$ (see section 2.2), we can restrict these loops to lie within a volume (5s) about the origin for some positive parameter $S$ independent of $\epsilon$.

(ii) The distributional nature of LQG ‘vacuum’ states. It is reasonable to expect that any putative vacuum state in the $SU(2)$ theory must be such that the expectation values and fluctuations of the operators $\hat{O}_F^r$ in this state reproduce the ($r$-Fock) vacuum expectation values and fluctuations of the operators $\hat{O}_L^r$ to accuracy $\epsilon$. Since $\hat{O}_F^r$ are not Dirac observables for full gravity, they are only defined at the (non-$SU(2)$ invariant) kinematic level. Note that the $SU(2)$ states cannot lie in the LQG kinematic Hilbert space $L^2(\hat{A}, d\mu_0)$ [19, 20] for the following reasons. First, as argued in [21], expectation values and fluctuations for uncountably many $SU(2)$ holonomy operators cannot be reproduced by states in the kinematic Hilbert space and second, as can be easily verified, the smeared triad operator (constructed by replacing $E_i^a$ by $\hat{E}^a_i$ in equation (3.4)) is not well defined on the kinematic Hilbert space. As we shall see later, the $SU(2)$ states lie in (an appropriate subspace) of the algebraic dual, $\Phi_{\text{kin}}^*$, to a suitable dense subspace of the kinematic Hilbert space. Since, at least to our knowledge, there is no natural inner product on $\Phi_{\text{kin}}^*$, an identification of the vacuum state therein which does not involve explicit evaluation of expectation values and fluctuations of $\hat{O}_F^r$ is desirable.

(iii) The approximate nature of the identification between linearized and LQG structures. Finally we note that since linearized gravity is an approximate description of the exact physics of weak gravitational fields, we expect the correspondence between the $r$-Fock vacuum and its $SU(2)$ counterparts to be an approximate one.
3.2. Definition and properties of the map $\mathcal{M}^\epsilon$

Motivated by the above remarks, we define the identification of suitable $SU(2)$ distributions with the $r$-Fock vacuum, $|0_i\rangle$, as follows. Recall that equation (2.21) implies that $r$ is a function of $\epsilon$. Hence $|0_i\rangle$ corresponds to a one-parameter family of states, one state for each $r(\epsilon), 0 < \epsilon < 1$. Consider a corresponding one-parameter family of putative $SU(2)$ vacuum states $\Psi_0^\epsilon$. We require the existence of a one-parameter family of maps, $\mathcal{M}^\epsilon$, which map the one-parameter family of states $\Psi_0^\epsilon$ to the one-parameter family of states $\{ |0_i\rangle + |\psi_i\rangle \}$, $\sqrt{\langle \psi_i | \psi_i \rangle} = O(\epsilon^2)$). We denote this requirement as

$$\mathcal{M}^\epsilon \Psi_0^\epsilon = |0_i\rangle + O(\epsilon^2).$$  (3.5)

Thus, for each fixed $\epsilon$, $\mathcal{M}^\epsilon$ is a map from a subspace of $\Phi^\epsilon_{\text{kin}}$ to (a subset of) the set of all subsets of $r$-Fock space. Using similar notation, we require that $\Psi_0^\epsilon$ be such that the action of any $\hat{O}^\epsilon$ on $\Psi_0^\epsilon$ mirrors that of the corresponding $\hat{O}_L^\epsilon$ on $|0_i\rangle$:

$$\mathcal{M}^\epsilon \hat{O}^\epsilon \Psi_0^\epsilon = \hat{O}_L^\epsilon |0_i\rangle + O(\epsilon^2).$$  (3.6)

Since we can always measure a linear combination of the basic linearized operators $\hat{O}^\epsilon_L$, we impose the following ‘linearity’ condition on $\mathcal{M}^\epsilon$. Let $\hat{O}^\epsilon_{L,F}$, $i = 1, \ldots, N$ be a set of $N$ linearized and ‘full’ operators and let $a_0, a_i$ be a set of $N + 1$ complex parameters each of $O(1)$ with $N$ itself being independent of $\epsilon$. Then we require that

$$\mathcal{M}^\epsilon \left( \sum_{i=1}^N a_i \hat{O}^\epsilon_{iL} + a_0 \right) \Psi_0^\epsilon = \left( \sum_{i=1}^N a_i \hat{O}^\epsilon_{iL} + a_0 \right) |0_i\rangle + O(\epsilon^2).$$  (3.7)

Although the above equation does not involve an explicit determination of $\hat{O}^\epsilon_L$, $\Delta \hat{O}^\epsilon_F$, it is, nevertheless, motivated by the requirement that $\hat{O}^\epsilon_L$, $\Delta \hat{O}^\epsilon_F$ be approximated by their $SU(2)$ counterparts. To see this, note that if we could identify the states $\Psi_0^\epsilon$, the above equation serves to define the ‘minimal’ subspace $S^\ast \subset \Phi^\epsilon_{\text{kin}}$ which can serve as the domain of the map $\mathcal{M}^\epsilon$, i.e.

$$S^\ast = \{ \psi^\epsilon \in \Phi^\epsilon_{\text{kin}}, \Psi^\epsilon = \left( \sum_{i=1}^N a_i \hat{O}^\epsilon_{iF} + a_0 \right) \Psi_0^\epsilon \}. $$

Let us suppose that there exists an inner product on this space which is well approximated by the $r$-Fock inner product $\langle , \rangle_{\text{F}}$ in the following sense. Consider $\Psi_1, \Psi_2 \in S^\ast$ with $\mathcal{M}^\epsilon \Psi_j = |\psi_j\rangle + O(\epsilon^2), j = 1, 2$. Then let $\langle , \rangle_{\text{F}}$ on $S^\ast$ be such that

$$\langle \Psi_1, \Psi_2 \rangle_{\text{F}} = \langle \psi_1 | \psi_2 \rangle + O(\epsilon^2) \sqrt{\langle \psi_1 | \psi_1 \rangle} + \sqrt{\langle \psi_2 | \psi_2 \rangle} + O(\epsilon^2).$$  (3.8)

It is straightforward to check that for such a $\langle , \rangle_{\text{F}}$, equations (3.5)–(3.7) imply that fluctuations and expectation values of $\hat{O}^\ast_{\text{F}}$ in any state $\Psi_0^\epsilon$ approximate their linearized counterparts to the desired accuracy of $O(\epsilon)$.

The existence of ambiguities of $O(\epsilon^2)$ on the right-hand sides of (3.5) and (3.6) may be motivated as follows. Equation (3.6) may be thought of as the quantum version of the classical relations (3.1) and (3.2). Since $\hat{O}^\epsilon_L |0_i\rangle$ has a norm of $O(\epsilon)$, the ambiguity in the image of $\mathcal{M}^\epsilon$ must be of higher order than $\epsilon$ and we choose it to be $O(\epsilon^2)$.

The rest of this work is devoted to showing the existence of at least one (one-parameter family of maps) $\mathcal{M}^\epsilon$ which satisfies equations (3.5)–(3.7).

3.3. Strategy for an explicit construction of $\mathcal{M}^\epsilon$

The approximate correspondence between linearized and full gravity motivated us to require that the image of $\mathcal{M}^\epsilon$ be defined with small ambiguities of $O(\epsilon^2)$ in equations (3.5)–(3.7).

\footnote{Strictly speaking, our results are slightly weaker in that we analyse these equations subject to a mild restriction on the loops labelling $\hat{O}^\epsilon_L$, $\hat{O}^\epsilon_F$, namely that these loops intersect the weave defined in section 4 in at most a finite number of points.}
Similar considerations should also apply to the domain of $\mathcal{M}^\epsilon$, i.e. it seems plausible that $\mathcal{M}^\epsilon$ should map two distributions which differ by a ‘small’ amount to the same image. Our ignorance of any useful inner product on $\Phi_{kin}^\epsilon$ makes it difficult to define this notion of ‘smallness’. However, even though we do not know of any useful inner product we can define norms and semi-norms on $\Phi_{kin}^\epsilon$. Indeed, we shall define a semi-norm on $\Phi_{kin}^\epsilon$ and (partially) specify the ‘kernel’ of $\mathcal{M}^\epsilon$ as containing distributions which are of $O(\delta \epsilon)$ in this semi-norm for a suitably defined small $\epsilon$-dependent parameter, $\delta \epsilon$. We define the kernel of (the one-parameter family of maps) $\mathcal{M}^\epsilon$ to be the set (of all one-parameter families) of distributions, whose images by $\mathcal{M}^\epsilon$ are one-parameter families of sets of states with $r$-Fock norm of $O(\epsilon^2)$.

The operators $\hat{O}_\alpha^\epsilon$ satisfy the following relation in their actions on $|0_r\rangle$ (this relation is just the ‘Poincaré invariance condition’ of [3] evaluated to $e$):

$$i(\hat{h}_\alpha - 1)|0_r\rangle = \hat{g}_{\alpha(r)}|0_r\rangle + O(\epsilon^2). \quad (3.9)$$

This in conjunction with (3.6) implies that

$$\mathcal{M}^\epsilon \left( 2 \sum_{k=1}^{3} \text{tr} \hat{H}_\alpha \tau_k + \hat{G}_{\alpha(r)} \right) \Psi_0^\epsilon = O(\epsilon^2). \quad (3.10)$$

Our strategy will be to attempt to solve the following ‘$SU(2)$ Poincare invariance condition’ up to ambiguities of $O(\delta \epsilon)$:10

$$2 \sum_{k=1}^{3} \text{tr} \hat{H}_\alpha \tau_k \Psi_0^\epsilon = -\hat{G}_{\alpha(r)} \Psi_0^\epsilon + O(\delta \epsilon), \quad (3.11)$$

to obtain $\Psi_0^\epsilon$. Once we obtain such a $\Psi_0^\epsilon$, we can define the action of $\mathcal{M}^\epsilon$ as follows:

$$\mathcal{M}^\epsilon \Psi_0^\epsilon := |0_r\rangle + O(\epsilon^2) \quad (3.12)$$

$$\mathcal{M}^\epsilon \hat{G}_{\alpha(r)} \Psi_0^\epsilon := \hat{g}_{\alpha(r)}|0_r\rangle + O(\epsilon^2). \quad (3.13)$$

Equation (3.11) ensures that the relevant conditions involving the connection-dependent operators in the linearized and $SU(2)$ theories are satisfied. Finally, $\mathcal{M}^\epsilon$ can be extended by appropriate linearity to satisfy equation (3.7).

4. The weave and the semi-norm

The semi-norm comprises one of two auxiliary structures that we will define in order to construct an $\mathcal{M}^\epsilon$ satisfying equations (3.5)–(3.7). It turns out that in order to define the semi-norm we use, we need to define a second auxiliary structure namely a (distributional) ‘weave’ state. This state is the counterpart of the flat triad $\delta a^\alpha_i$ of linearized theory. We define the weave state in section 4.1 and the semi-norm in section 4.2. In section 4.3, we describe the general structure of the $SU(2)$ distributions which we shall consider in the rest of this work.

10 Actually, we solve an equation of the form $(2 \sum_{k=1}^{3} \text{tr} \hat{H}_\alpha \tau_k + \hat{G}_{\alpha(r)} + a^\alpha_0)|\Psi_0^\epsilon\rangle = O(\delta \epsilon)$, where $a^\alpha_0$ is complex and of $O(\epsilon^2)$. Note that $a^\alpha_0|\Psi_0^\epsilon\rangle$ is also in the kernel of $\mathcal{M}^\epsilon$. 
4.1. The weave state

The weave state is the analogue of the classical flat triad $\delta^a_i$ in the following sense. We shall construct a set of extended spin network states denoted by $|\psi_\Delta\rangle$ which are all based on the same ‘lattice-like’ graph $\Delta$. Then the weave state $\Psi_{\text{weave}} \in \Phi^*_\text{kin}$, has the following property:

$$
(E^\mu_{(i)}(\vec{x})\Psi_{\text{weave}})|\psi_\Delta\rangle = (1 + O^\infty(\epsilon))\delta^\mu_i \Psi_{\text{weave}})|\psi_\Delta\rangle.
$$

(4.1)

Here, $\vec{x}$ is restricted to be within the probe region, i.e. $|\vec{x}| < S s$ (see (i) in section 3.1 for a definition of $S$) and $O^\infty(\epsilon)$ has been defined at the end of section 1. Moreover for any state $|\psi_\Delta\rangle$ orthogonal to the span of $|\psi_\Delta\rangle$, the weave state satisfies

$$
(E^\mu_{(i)}(\vec{x})\Psi_{\text{weave}})|\psi_\Delta\rangle = \delta^\mu_i \Psi_{\text{weave}}|\psi_\Delta\rangle = 0.
$$

(4.2)

In section 4.1.1, we describe our construction of the weave state and in section 4.1.2, we display some of its properties. Calculations pertaining to the explicit verification of (4.1) and (4.2) are contained in appendix A.

4.1.1. Construction of the weave state. Let $\Delta$ be the graph corresponding to a cubical lattice with edges along the Cartesian coordinate directions. Let the lattice spacing be $\sqrt{\frac{\sqrt{2}}{2}}L$ and let it occupy a volume $(2L)^3$ centred about the origin. Let $L$ be such that $L \gg s$ and $s_{\text{curv}} \gg L \gg r \gg l_p$.

Define the spin net $|\Delta\rangle$ based on the graph $\Delta$ by:

(a) Colouring all edges of $\Delta$ with $j = \frac{1}{2}$, i.e. with spin-half representations.

(b) Defining trivial intertwiners at the vertices of $\Delta$, i.e. mapping, trivially, the $j = \frac{1}{2}$ representation on an incoming edge in the $i$th direction $(i = 1, 2, 3)$ at any vertex to the $j = \frac{1}{2}$ representation on the outgoing edge in the same direction (this pertains to vertices which are not at the boundary of $\Delta$).

(c) Choosing any intertwiners at the vertices on the boundary of the lattice, i.e. ‘tieg up’ the boundary points on the planes $x^i = \pm L$ in any convenient way. This choice has no bearing on our subsequent considerations.

Define the spin net $|\Delta, \{\vec{x}_k, k = 1, \ldots, N\}\rangle$ as follows. Pick $N$ points $\{\vec{x}_k, k = 1, \ldots, N\}$ on the graph $\Delta$ such that they are not on the vertices of $\Delta$. In addition to (a), (b) and (c) above, define the intertwiners at the new vertices $\{\vec{x}_k, k = 1, \ldots, N\}$ as follows. Let $\vec{x}_k$ be on an edge along the $i$th direction. Then the intertwiner is chosen as $2\tau_A^B$, $A, B = 1, 2$ (clearly $\tau_A^B$ maps the incoming $j = \frac{1}{2}$ representation to the outgoing $j = \frac{1}{2}$ representation at the vertex $\vec{x}_k$). Note that in terms of the labelling of extended spin nets defined in [6], $\tau_A^B = M^i_a C^B_A$ where (i) $C^B_A$ is the invariant tensor which maps the $j = \frac{1}{2}$ representation to the product of the $j = \frac{1}{2}$ and $j = 1$ representations and (ii) $M^i_a, i = 1, 2, 3$ are vectors in the $j = 1$ representation namely, $M^1_a = (\sqrt{\frac{2}{3}}, 0, -\sqrt{\frac{2}{3}}), M^2_a = (-\sqrt{\frac{2}{3}}, 0, \sqrt{\frac{2}{3}})$ and $M^3_a = (0, \sqrt{\frac{2}{3}}, 0)$. $A, B = 1, 2$ are $j = \frac{1}{2}$ indices and $a = 1, 2, 3$ is a $j = 1$ index.

11 In referring to this state as a weave state, we follow the nomenclature used for states which correspond to classical spatial geometries [11]. The state defined here is based on a graph identical to that used by Rovelli in [4] and Zegwaard in [17].

12 Note that since these inequalities are $\epsilon$ dependent, $\Delta$ and $\Psi_{\text{weave}}$ also depend on $\epsilon$. However, we shall regard this dependence as ‘weak’ in the sense that we hope that a better treatment incorporating constructions appropriate to asymptotic flatness (see for example [22]) will allow $L$ to be taken to $\infty$ without altering our conclusions. In this connection, we note that the inequality $s_{\text{curv}} \gg L$ is not crucial to the considerations of appendix $\Delta$; as can be verified, $L \gg s_{\text{curv}}$ would suffice equally well. In what follows, we shall refrain from explicitly denoting the $L(\epsilon)$ dependence of $\Psi_{\text{weave}}$. 

It can be checked that \( |\Delta, \{ \tilde{x}_k, k = 1, \ldots, N \}| \) is normalized and that
\[
\langle \Delta, \{ \tilde{y}_j, j = 1, \ldots, M \}|\Delta, \{ \tilde{x}_k, k = 1, \ldots, N \rangle = 0,
\] (4.3)
unless \( \{ \tilde{y}_j \} = \{ \tilde{x}_k \} \). \( \Psi_{\text{weave}} \) is defined as the distribution
\[
\Psi_{\text{weave}} := \sum_{N=0}^{\infty} \sum_{\tilde{x}_k, k=1, \ldots, N} (-1)^N < \Delta, [\tilde{x}_k],
\] (4.4)
where the second (uncountable) sum is over all possible values of \( \{ \tilde{x}_k \} \). Clearly, \( \Psi_{\text{weave}} \) is in \( \Phi_{\text{kin}} \), i.e. it has a well-defined action on any finite linear combination of spinnets.

4.1.2. Properties of the weave state. The weave state has nice properties with respect to the action of the smeared triad operators. The considerations of appendix A show that equation (4.1) is satisfied for \( |\tilde{x}| < SS \) (see (i) in section 3.1 for a definition of \( SS \)). Although we do not display them here, straightforward calculations along the lines of appendix A show that for \( |\tilde{x}| < SS \), \( j = 1, \ldots, n \) independent of \( \epsilon \),
\[
\prod_{j=0}^{n-1} \tilde{E}_{(j)_{\text{triad} \to} \tilde{x}_{n-j}}(\tilde{x}_{n-j}) \Psi_{\text{weave}}[|\psi_{\Delta}|] = \tilde{E}_{(j)_{\text{triad} \to} \tilde{x}_{n-j}}(\tilde{x}_{n-j}) \Psi_{\text{weave}}[|\psi_{\Delta}|]
\]
\[
= (1 + O^\infty(\epsilon)) \prod_{j=0}^{n-1} \delta_{\text{triad} \to \tilde{x}_{n-j}} \Psi_{\text{weave}}[|\psi_{\Delta}|].
\] (4.5)
Here \( |\psi_{\Delta}| = |\Delta \) or \( |\Delta[\tilde{x}_k]| \). From (4.5) it is straightforward to show that
\[
\prod_{j=0}^{n-1} \left( \tilde{E}_{(j)_{\text{triad} \to} \tilde{x}_{n-j}}(\tilde{x}_{n-j}) - \delta_{\text{triad} \to \tilde{x}_{n-j}} \right) \Psi_{\text{weave}}[|\psi_{\Delta}|] = O^\infty(\epsilon) \Psi_{\text{weave}}[|\psi_{\Delta}|].
\] (4.6)

Equation (4.6) captures the precise sense in which the weave state is the analogue of the flat triad \( \delta^i_{\text{flat}} \) within the probe region \( |\tilde{x}| < SS \). Note however that since \( G_{\text{refrain \to} \tilde{x}} \) is not, in general, a function of compact support (see appendix B), the Poincaré invariance condition (3.9) as well as its \( SU(2) \) counterpart (3.11) involve properties of the triad everywhere including \( |\tilde{x}|<SS \). Hence to proceed further either (a) we need the weave state to capture properties of the flat triad for all \( \tilde{x} \) or (b) we may attempt to restrict the Poincaré invariance condition in such a way as to depend only on properties of the triad in the probe region. For simplicity we shall assume that option (a) is viable. More precisely, we assume that an improved treatment will yield \( \Psi_{\text{weave}} \) which satisfies (4.6) for all \( \tilde{x} \) (see also footnote 12 in this regard). While this seems to us to be a reasonable assumption, we emphasize that its validity must be checked in a more comprehensive treatment. We also note that preliminary calculations made by us indicate that option (b) may also be viable. Since the relevant analysis pertaining to option (b) is a bit involved and may obscure our main line of argument in an already involved paper, we refrain from reporting on this matter here.

Thus, we shall assume that (4.6) is valid everywhere on the spatial slice. Then, as shown in appendix B the following identities hold for \( n \) independent of \( \epsilon \):
\[
\prod_{j=0}^{n-1} \tilde{G}_{(j)_{\text{triad} \to} \tilde{x}_{n-j}} \Psi_{\text{weave}}[|\Delta|] = O^\infty(\epsilon) \Psi_{\text{weave}}[|\Delta|],
\] (4.7)
\[
\prod_{j=0}^{n-1} \tilde{G}_{(j)_{\text{triad} \to} \tilde{x}_{n-j}} \Psi_{\text{weave}}[|\Delta[\tilde{x}_k]|] = O^\infty(\epsilon) \Psi_{\text{weave}}[|\Delta[\tilde{x}_k]|].
\] (4.8)
4.2. The semi-norm

As discussed in [3], r-Fock states can be thought of as \( U(1)^3 \) distributions, i.e. distributions with respect to the finite span of \( U(1)^3 \) flux network states. This is in close structural similarity to the LQG states under consideration, the latter being \( SU(2) \) distributions with respect to the finite span of \( SU(2) \) spinnets. Although we do not show it here, our choice of semi-norm below is motivated by the existence of a similar structure in the \( U(1)^3 \) theory. We shall comment on this further in section 7. For now, we shall simply display our choice of semi-norm and show in the subsequent section that our choices furnish the map \( M' \) in accord with the strategy outlined in section 3.3.

Consider the set of graphs \( G_\Delta \) which intersect \( \Delta \) at most in a finite number of points. Denote the spinnet based on the trivial graph as \( | \cdot \rangle \). Clearly, every spinnet based on a graph \( \alpha \in G_\Delta \) can be written in the form \( \hat{N}_{a_1, \tilde{a}, \tilde{b}} | \cdot \rangle \), where \( \hat{N}_{a_1, \tilde{a}, \tilde{b}} \) is the generalized holonomy operator corresponding to the spin net labelled by the graph \( \alpha \), invariant tensors \( \tilde{c} \) and representation vectors \( M \) [6] (the labels defining the colour of the edges of \( \alpha \) have been suppressed as in [6]). Explicitly, in the connection representation, \( | \cdot \rangle \) corresponds to the state \( \psi(A) = 1 \) and the action of the operator \( \hat{N}_{a_1, \tilde{a}, \tilde{b}} \) corresponds to multiplication by \( \hat{N}_{a_1, \tilde{a}, \tilde{b}}(A) \) (see equation (4.19) of [6]).

For any such \( \hat{N}_{a_1, \tilde{a}, \tilde{b}} \) consider the states \( \hat{N}_{a_1, \tilde{a}, \tilde{b}} | \Delta \rangle \) and \( \hat{N}_{a_1, \tilde{a}, \tilde{b}} | \Delta(\vec{x}_k, k = 1, \ldots, N) \rangle \). Denote by \( V \) the finite span of these states for all choices of extended spinnets based on every \( \alpha \in G_\Delta \). Denote by \( V_\perp \) the set of states which are orthogonal to every element of \( V \). Clearly \( V + V_\perp \) is a dense subspace of the kinematic Hilbert space.

Next, consider the set of loops \( G_{\text{loop}}' \) which are of length less than or equal to \( s \) and which intersect \( \Delta \) at most in a finite number of points. For \( \eta_1, \bar{\beta}_J \in G_{\text{loop}}', I = 1, \ldots, P, J = 1, \ldots, M, k_j \in \{1, 2, 3\} \), define the operators

\[
\hat{O}_{\{\eta_1, \bar{\beta}_J, k_j\}} := \prod_{I=1}^{P} \frac{\text{Tr} \hat{H}_{\eta_I}}{2} \prod_{J=1}^{M} \text{Tr} \hat{H}_{\bar{\beta}_J}, \quad \hat{O}_{\{\eta_1\}} := \prod_{I=1}^{P} \frac{\text{Tr} \hat{H}_{\eta_I}}{2}.
\]  

(4.9)

Define \( V_{\text{loop}}' \subset V \) to be the finite span of the states

\[
\{ \hat{O}_{\{\eta_1, \bar{\beta}_J, k_j\}} | \Delta \rangle, \hat{O}_{\{\eta_1\}} | \Delta \rangle, \hat{O}_{\{\eta_1, \bar{\beta}_J, k_j\}} | \Delta(\vec{x}_k, k = 1, \ldots, N) \rangle, \hat{O}_{\{\eta_1\}} | \Delta(\vec{x}_k, k = 1, \ldots, N) \rangle \}
\]

for all choices of \( \{\eta_1, \bar{\beta}_J, k_j, \vec{x}_k\} \) and finite \( P, M, N \).

For any \( \Psi \in \Phi_+ \), we define the semi-norm

\[
\| \Psi \|^\epsilon = \sup_{|\psi\rangle} \langle \psi | \Psi | \psi \rangle
\]

(4.10)

where \( |\psi\rangle \) ranges over the following:

(a) \( |\psi\rangle \) is a spinnet in \( V_\perp \),
(b) \( |\psi\rangle \) is \( (M + P)^{-\Delta + P} \hat{O}_{\{\eta_1, \bar{\beta}_J, k_j\}} | \Delta \rangle \) or \( P^{-\epsilon} \hat{O}_{\{\eta_1\}} | \Delta \rangle \) or \( (M + P)^{-\Delta + P} \hat{O}_{\{\eta_1, \bar{\beta}_J, k_j\}} | \Delta(\vec{x}_k, k = 1, \ldots, N) \rangle \) or \( P^{-\epsilon} \hat{O}_{\{\eta_1\}} | \Delta(\vec{x}_k, k = 1, \ldots, N) \rangle \), for all choices of \( \{\eta_1, \bar{\beta}_J, k_j, \vec{x}_k\} \) and finite \( P, M, N \).

\[\| \| \|^\epsilon \] is a semi-norm precisely because \( |\psi\rangle \) is not allowed to range over the orthogonal complement of \( V_{\text{loop}}' \) in \( V \). The above semi-norm is \( \epsilon \)-dependent both due to the (weak) dependence of \( \Delta \) on \( \epsilon \) (see footnote 12) and the dependence of \( s \gg r \) on \( \epsilon \). The \( \epsilon \)-dependence of the semi-norm is denoted by the superscript \( \epsilon \).

11 Note that unlike the loops which label \( O_L', O_F' \), these loops are not restricted to lie within a volume \( (S^3)^3 \) about the origin (see (i) in section 3.1).
4.3. General structure of the LQG distributions under study

Consider an arbitrary distribution \( \Phi \in \Phi_{\text{kin}}^* \). Define \( \Psi \in \Phi_{\text{kin}}^* \) from \( \Phi \), \( \Psi_{\text{wave}} \) as follows.

(a) Let \( \psi \in V_\perp \) be a spinnet. Then \( \Psi[\psi] := 0 \).

(b) Let \( \alpha \in G \) and consider any ‘spinnet’ operator \( \hat{N}_{\alpha,\vec{c},\vec{d}} \) as defined in section 4.2. Then \( \Psi[\hat{N}_{\alpha,\vec{c},\vec{d}}|\Delta] := \Psi_{\text{wave}}[|\Delta]\{\hat{N}_{\alpha,\vec{c},\vec{d}}|\} \) and \( \Psi[\hat{N}_{\alpha,\vec{c},\vec{d}}|\Delta(\vec{x}_k)] := \Psi_{\text{wave}}[|\Delta(\vec{x}_k)] \). This defines the action of \( \Psi \) on a basis of \( V \).

(c) Extend the action of \( \Psi \) as defined in (a) and (b) to the finite span of all spinnets by linearity. The \( SU(2) \) states of interest in the rest of this work will have the structure of \( \Psi \) defined above.

5. The vacuum as an \( SU(2) \) distribution

Below, we exhibit solutions to the \( SU(2) \) Poincaré invariance condition (3.11) (modulo terms in the kernel of \( M' \) (see footnote 10, section 3.3 in this regard)), thus implementing the strategy of section 3.3. The particular choice of our solutions as well as that of \( \delta_\epsilon \) is motivated by their analogues in the description of \( r \)-Fock states by \( U(1)^3 \) distributions. We shall comment on this further in section 7. Here we shall simply display our choices and show that they furnish the required \( M' \).

5.1. Explicit form of the solution to the Poincaré invariance condition

The putative solution \( \Phi_0^\epsilon \) is of the type described in section 4.3 with \( \Phi = \Phi_0^\epsilon \) where \( \Phi_0^\epsilon \) is defined as

\[
\Phi_0^\epsilon = \delta_{A=0} + \hat{E}^2 \delta_{A=0}.
\]

\[
\hat{E}^2 := i \frac{1}{2 \gamma r^2} \int \dd^3 k \{ (1 + 2i\gamma') (\hat{E}_r(\vec{k}))^\dagger \hat{E}_r(\vec{k}) - (1 - 2i\gamma')(\hat{E}_{\gamma'}(\vec{k}))^\dagger \hat{E}_{\gamma'}(\vec{k}) \},
\]

where \( \delta_{A=0} \) is the distribution corresponding to the connection \( A_0 = 0 \), i.e. for any \( \phi(\vec{A}) \in \mathcal{L}^2(\vec{A}, d\mu_0) \),

\[
\delta_{A=0}[\phi(\vec{A})] := \phi(\vec{A})|_{A=0}.
\]

Consider a set of operators \( \hat{O}_i, i = 0, \ldots, n \) such that each \( \hat{O}_i \) is of the type defined in equation (4.9). Then we have the following key identity which is proved in appendix D:

\[
\hat{O}_0 \hat{G}_{\alpha_i}(\vec{r}) \hat{O}_1, \ldots, \hat{G}_{\alpha_i}(\vec{r}) \hat{O}_n \Psi_0^\epsilon = \Psi_0^\epsilon + O^\infty(\epsilon).
\]

Here \( \Psi_0^\epsilon \) is a distribution of the type defined in section 4.3 with \( \Phi = \Phi_0^\epsilon \), where \( \Phi_0^\epsilon \) is defined as \( \Phi_0^\epsilon := \hat{O}_0 \{ \int G_{\alpha_i(\vec{r})a_i} \hat{E}_a(\vec{r}) \} \hat{O}_1 \cdots \{ \int G_{\alpha_i(\vec{r})a_i} \hat{E}_a(\vec{r}) \} \hat{O}_n \Phi_0^\epsilon \).

The above identity (5.3) states that the distribution on its left-hand side is the sum of \( \Psi_0^\epsilon \) and a distribution of semi-norm \( O^\infty(\epsilon) \). Since \( \Psi_0^\epsilon \) is also of the type defined in section 4.3, this identity allows us to rewrite the Poincaré invariance condition as a condition on the ‘\( \Phi \)’ part of the distribution. In this regard, it is useful to define the following semi-norm, \( ||| \Phi |||_1 \), pertinent to \( \Phi \):

\[
||| \Phi |||_1 = \sup_{|\phi|} ||| \Phi[|\phi]| |||_1,
\]

where \( |\phi| \in \{(M + P)^{-(M+P)} \hat{O}_{(\beta_j, k_j)}| \} \) for all choices of \( \{\eta_J, \beta_j, k_j\} \) and finite \( P, M, N \) (\( \hat{O}_{(\beta_j, k_j)} \) are defined by (4.9)). Note that for any distribution \( \Psi \) of the type defined in section 4.3, we have that

\[
||| \Psi |||_1 = ||| \Phi |||_1.
\]
Then it follows from (5.3) that the $SU(2)$ Poincaré invariance condition (3.11) on $\Psi^0_0$, for loops $a^k \in G^\Delta_{\text{loop}}$ (and which are, as in (3.11), confined to a volume $(Ss)^3$ about the origin), is equivalent to the following condition on $\Phi^0_0$:

$$2 \sum_{k=1}^{3} \text{tr} \, \hat{H}_{\alpha^k} \Phi^0_0 = - \left( \int d^3 x G_{\alpha(r)ab}^\phi(x) \hat{E}_{\alpha(r)}^{ab}(x) \right) \Phi^0_0 + O(\delta_s). \quad (5.6)$$

Here the last term denotes a distribution with semi-norm $\| \Phi^0_0 \|_2$ of $O(\delta_s)$. In the next subsection we show that $\Phi^0_0$ does indeed satisfy (5.6) up to a term of $O(\epsilon^2)$ which is in the kernel of $\mathcal{M}^\alpha$.

More precisely, we show that

$$2 \sum_{k=1}^{3} \text{tr} \, \hat{H}_{\alpha^k} \Phi^0_0 = - \left( \int d^3 x G_{\alpha(r)ab}^\phi(x) \hat{E}_{\alpha(r)}^{ab}(x) \right) \Phi^0_0 + c_\alpha \Phi^0_0 + O(\delta_s), \quad (5.7)$$

where $c_\alpha$ is a complex number of $O(1)$ depending only on $\alpha$. Since $\mathcal{M}^\alpha \epsilon^2 c_\alpha \Phi^0_0 = \epsilon^2 c_\alpha |0\rangle$, this extra term is in the kernel of $\mathcal{M}^\alpha$. This is equivalent to showing that $\Psi^0_0$ satisfies (3.10) for $\text{tr} \, \hat{H}_{\alpha^k} \Phi^0_0$ labelled by loops $a^k$ each of which are of length at most equal to $s$, are confined to a volume $(Ss)^3$ about the origin and which intersect $\Delta$ in at most a finite number of points. Note that this is a slightly weaker condition than (3.10) which was defined for all loops with length at most $s$ confined to a volume $(Ss)^3$ about the origin irrespective of their intersections with $\Delta$. We shall only impose this slightly weaker condition of Poincaré invariance on $\Psi^0_0$ rather than the stronger one (3.10).

### 5.2. Verification of (5.7)

We set $\delta_s = \epsilon^4$. We shall explain this choice later in this subsection. From (C5) it follows that

$$\hat{E}_{\alpha(r)}^{ab}(\vec{k}) \left( 2 \sum_{k=1}^{3} \text{tr}(H_{\alpha^k}(A)\tau^k) \right)_{A=0} = -i\hbar \mathcal{F} \chi_{\alpha(r)}^a(\vec{k}). \quad (5.8)$$

Also, setting $n = 0, N = 1$ and choosing $\hat{O}_H = \text{tr}(H_{\alpha^k}(A)\tau^k)$ in (C3) it follows that

$$\hat{E}^2 \sum_{k=1}^{3} \text{tr}(H_{\alpha^k}(A)\tau^k)|_{A=0} = \epsilon^2 c_\alpha, \quad (5.9)$$

where $c_\alpha$ is a constant of $O(1)$ which depends only on the triplet of loops $\alpha$.

The action of the left-hand side of (5.7) on $|\phi\rangle$ of the type appearing in (5.4) is

$$\text{LHS} := 3 \sum_{k=1}^{3} \text{tr}(H_{\alpha^k}(A)\tau^k)|\phi(A)\rangle|_{A=0} + \hat{E}^2 \sum_{k=1}^{3} \text{tr}(H_{\alpha^k}(A)\tau^k)|\phi(A)\rangle|_{A=0} \quad (5.10)$$

where we have denoted the ket $|\phi\rangle$ by the corresponding wavefunction in the connection representation, $\phi(A)$. Since $\text{tr}(H_{\alpha^k}(A)\tau^k)$ vanishes at $A = 0$, the first term in (5.10) vanishes and in the second term either both triad operators of $\hat{E}^2$ act on $\text{tr}(H_{\alpha^k}(A)\tau^k)$ or one acts on $\text{tr}(H_{\alpha^k}(A)\tau^k)$ and one on $\phi(A)$. Using (5.8) and (5.9) to evaluate these contributions, we obtain

$$\text{LHS} = - \left( \int G_{\alpha(r)ab} \hat{E}_{\alpha(r)}^{ab} \phi(A) \right)_{A=0} + \epsilon^2 c_\alpha \phi(A)|_{A=0}$$

$$= - \left( \int G_{\alpha(r)ab} \hat{E}_{\alpha(r)}^{ab} \phi(A) \right)_{A=0} + \epsilon^2 c_\alpha \Phi^0_0|\phi\rangle - \hat{E}^2 \delta_0|\phi\rangle$$

$$= - \left( \int G_{\alpha(r)ab} \hat{E}_{\alpha(r)}^{ab} \phi(A) \right)_{A=0} + \epsilon^2 c_\alpha \Phi^0_0|\phi\rangle + O(\epsilon^4), \quad (5.11)$$
where we have used (C3) to obtain the $O(\epsilon^4)$ term. Using (C3) again, the action of the right-hand side of (5.7) on $\phi$ is

$$\text{RHS} = -\left(\int G_{\alpha(r)ab}\hat{E}^{ab}_{(r)}\right) \phi(A)\big|_{A=0} + O(\epsilon^4).$$

(5.12)

This completes the verification of (5.7).

Note that from (C2) and (C3) it is easy to see that $\int d^3x G_{\alpha(r)ab}(\vec{x})\hat{E}^{ab}_{(r)}(\vec{x})\Phi_0^\alpha$ is $O(\epsilon^2)$. But the image of this quantity by $M^\alpha$ is expected to be (see equation (3.13)) $\hat{g}_{\alpha(r)}(0)$. Since the norm of the latter is expected to be of $O(\epsilon)$ from (2.20), we do not want $\int d^3x G_{\alpha(r)ab}(\vec{x})\hat{E}^{ab}_{(r)}(\vec{x})\Phi_0^\alpha$ to be in the kernel of $M^\alpha$. Hence we choose $\delta_\epsilon \ll \epsilon^2$ and our choice is $\delta_\epsilon = \epsilon^4$.

This argument is merely a plausibility one. Strictly speaking, we need to show that for every $\alpha^k$ such that $\Delta\hat{g}_{\alpha(r)} \approx \epsilon$ to nontrivial leading order in $\epsilon$, there exists $|\phi|$ of the type in (5.4) such that $\int d^3x G_{\alpha(r)ab}(\vec{x})\hat{E}^{ab}_{(r)}(\vec{x})\Phi_0^\alpha[|\phi|] \approx \epsilon^2$ to nontrivial leading order in $\epsilon$. It is straightforward to see that this is indeed true for $|\phi| = \sum_{k=1}^3 \text{Tr}(|\hat{H}_0 + \epsilon^k|)$. This suggests that we generalize the definition of the semi-norm so as to allow $|\phi|$ of this type. While we do not anticipate any obstruction to such a trivial generalization, we leave a thorough verification of this to future work.

5.3. Higher order operators

It can be verified that the action of quadratic and higher order products of operators of type $\hat{O}_L^\alpha$ on $0, \ldots$ yields states of norm at most $O(\epsilon^2)$ (this is under the assumption that the number of operators of type $\hat{O}_L$ in the product is independent of $\epsilon$). Therefore, it is natural to demand that the action of their $SU(2)$ counterparts map the $SU(2)$ vacuum into the kernel of $M^\alpha$. Also note that the $SU(2)$ function, $(\text{Tr}(H_\alpha) - 1), \alpha \in G^\text{\Delta Loop}_\alpha$ has leading order term quadratic in the connection. Hence, it is natural to also demand that $(\text{Tr}(\hat{H}_0) - 1)\Psi_0^\alpha$ be in the kernel of $M^\alpha$.

Indeed, using the key identities (5.3), (C2) and (C3), it is straightforward to verify that

(i) $(\text{Tr}(\hat{H}_0) - 1)\Psi_0^\alpha = \epsilon^2 d_\alpha \Psi_0^\alpha + O(\delta_\epsilon)$, where $d_\alpha$ is an $\alpha$-dependent complex number of $O(1)$ and $\alpha \in G^\text{\Delta Loop}_\alpha$.

(ii) The action of any product of two operators each of which belongs to the set $\{O_\beta^\alpha, (\text{Tr}(\hat{H}_0) - 1), \alpha \in G^\text{\Delta Loop}_\alpha\}$, on $\Psi_0^\alpha$, yields distributions in the kernel of $M^\alpha$.

Specifically, these distributions are either of type $O(\epsilon^2)\Psi_0^\alpha$ or of $O(\delta_\epsilon)$.

(iii) The action of any product of $n$ operators ($n > 2, n$ independent of $\epsilon$) each of which is in the set defined in (ii) above, on $\Psi_0^\alpha$, yields a distribution of $O(\delta_\epsilon)$.

6. The linearized constraints in the $SU(2)$ theory

We present qualitative arguments to show that the $SU(2)$ vacuum state is mapped to a distribution of $O(\delta_\epsilon)$ by the linearized constraints (2.3)–(2.5) expressed as operators in the $SU(2)$ theory.

We first discuss the purely connection dependent linearized vector and scalar constraints $V^L$ and $C^L$. We shall impose these constraints everywhere on the spatial slice except on the set of measure zero containing points on the weave $\Delta$. This implies that in the $SU(2)$ theory it should be possible to regularize $\hat{V}^L, \hat{C}^L$ in terms of holonomies along edges which do not intersect $\Delta$. From the structure of $\Psi_0^\alpha$ defined in section 4.3, this implies that the action of
the regularised operators (we denote these by $\hat{V}^L_{\text{reg}}, \hat{C}^L_{\text{reg}}$) on $\Psi_0$ is simply to change $\Phi_0$ to $\hat{V}^L_{\text{reg}}\Phi_0, \hat{C}^L_{\text{reg}}\Phi_0$. Note also that classically, the positive and negative helicity parts of the triad Poisson commute with $V^L, C^L$. Hence in any reasonable regularization, we expect that in the limit of the regulator being removed, we have that

$$[\hat{V}^L, \hat{E}^2] = [\hat{V}^L, \hat{E}^2] = 0.$$  

Finally, since $V^L, C^L$ vanish when evaluated at the zero connection, we expect that

$$\hat{V}^L\delta_A = 0 = \hat{C}^L\delta_A = 0.$$  

Equations (6.1) and (6.2) imply that $\hat{V}^L, \hat{C}^L$ annihilate $\Psi_0$, and in view of the above discussion also annihilate $\Phi_0$.

Next, we examine the linearized Gauss law constraint $G^L_{(\gamma,j)}$ (see (2.3)). Since in our work the basic triad operator is $\hat{E}^a(r)i$, we define the Gaussian smeared constraint $G^L_{(\gamma,j)}$ by

$$G^L_{(\gamma,j)}(\vec{x}) = \int d^3y G^L_{(\gamma,j)}(\vec{y}) \frac{e^{-|\vec{x}-\vec{y}|^2}}{2\pi r^2}.$$  

The first term in (6.4) is obtained after a by parts integration and $A_{(\gamma,j)}(\vec{x})$ is the Gaussian smeared connection. In the $SU(2)$ theory our arguments pertaining to $V^L, C^L$ apply to the connection-dependent part of (6.4). Hence we expect that for any reasonable definition of the connection part of $G^L_{(\gamma,j)}(\vec{x})$, this connection part annihilates $\Psi_0$. Hence we need analyse only the triad-dependent part of (6.4).

At the classical level we would like to impose the linearized constraints as $O(\epsilon)$ restrictions on the phase space. Since (6.3) has the dimensions of inverse length, we need to integrate it against a gauge parameter $\Lambda_1i(\vec{x})$ which has dimensions of inverse length square so as to obtain the dimensionless quantity $G^L_{(\gamma,j)}(\Lambda) := \int d^3x \Lambda^i G^L_{(\gamma,j)}$. We denote the triad-dependent part of $G^L_{(\gamma,j)}(\Lambda)$ by

$$G^L_{(\gamma,j)}(\Lambda) := -\int d^3x (\partial_a \Lambda^i(\vec{x})) e^a_j(\vec{x}),$$  

where we have performed a by parts integration and assumed that $\Lambda^i(\vec{x})$ is of compact support. We shall meet the requirement $|G^L_{(\gamma,j)}(\Lambda)| \sim O(\epsilon)$ by imposing the (sufficient) condition

$$\int d^3x |\partial_a \Lambda^i(\vec{x})| \sim O(1).$$  

In the $SU(2)$ theory we have

$$G^L_{(\gamma,j)}(\Lambda) \Psi_0 = -\int d^3x (\partial_a \Lambda^i(\vec{x})) (\hat{E}^a_j(\vec{x}) - \delta^a_j) \Psi_0.$$  

Using (4.1) and (6.6), we obtain

$$\hat{G}^L_{(\gamma,j)}(\Lambda) \Psi_0 = O(\epsilon),$$  

where $|\Psi_0| \in \{|\Delta\}, |\Delta(\vec{x})\}|$. From equation (6.8) and the structure of $\Psi_0$ as defined in section 4.3, we have that

$$\hat{G}^L_{(\gamma,j)}(\Lambda) \Psi_0 = O(\delta_k) \Leftrightarrow \hat{G}^L_{(\gamma,j)}(\Lambda) \Phi_0 = O(\delta_k)$$  

where $G^L_{(\gamma,j)}(\Lambda) := -\int d^3x (\partial_a \Lambda^i(\vec{x})) \hat{E}^a_j(\vec{x})$ and where $O(\delta_k)$ in the first equality is defined with respect to $\|\|$ and in the second equality with respect to $\|\|_1$ (see (5.4)).
Using the (inverse Fourier transform of) equation (C5) it is straightforward to obtain
\[ \hat{G}_L^\Lambda (r) \langle E \rangle = - \int (\tilde{\partial}_a \Lambda^I) X_{(r)\alpha}^a = \int (\tilde{\partial}_a X_{(r)\alpha}^a) \Lambda^I = 0, \]
where we have used the fact that the smeared loop form factor is divergence free, i.e.
\[ \tilde{\partial}_a X_{(r)\alpha}^a = 0. \]
Moreover, equation (C5) also implies that
\[ \hat{G}_L^\Lambda (r) \langle E \rangle \bigg|_{A=0} = 0. \]
Equations (6.10) and (6.12) imply that
\[ \delta_{A=0} \left[ \hat{G}_L^\Lambda (A) \right] = 0 \]
where \(|\phi\rangle\) is of the type defined below (5.4).

Using methods similar to those in appendix C, it is straightforward to verify that for \(O_\alpha^H \in \{ \text{tr}(H_\alpha t^I), \frac{n H}{2}, \alpha \in G_\text{loops}^2 \}\) we have that
\[ \left| \hat{E}^2 \hat{G}_L^\Lambda (A) O_\alpha^H (A) \right|_{A=0} < \epsilon^2 d_0 \frac{d^2 P}{r^2} \int d^3 x |\tilde{\partial}_a \Lambda_1 (\vec{x})| X_{(r)\alpha}^a (\vec{x}), \]
where \(d_0\) is defined in (C3). Using (6.6) and the bound \(|X_{(r)\alpha}^a (\vec{x})| = \| \oint_\alpha d\alpha a \cdot e^{-\frac{|\vec{x} - \vec{\alpha}(s)|^2}{2r^2}} \| < \frac{\epsilon}{s (2\pi r)^2}\), we obtain
\[ \left| \hat{E}^2 \hat{G}_L^\Lambda (A) O_\alpha^H (A) \right|_{A=0} < \epsilon^2 d_0 \frac{s\sqrt{P} \psi}{r^3} = O(\epsilon^4) \]
From the above equation and the fact that \(s \gg r\) it follows straightforwardly that
\[ \left| \hat{E}^2 \hat{G}_L^\Lambda (A) \prod_{I=1}^N O_\alpha^H (A) \right|_{A=0} < N^3 O(\epsilon^4). \]
Finally, it is easily verified that the second equality in equation (6.9) follows from (6.13) and (6.16), thus implying that the first equality in (6.9) holds. This completes our arguments in support of the hypothesis that the linearized constraints expressed as operators in the \(SU(2)\) theory map the \(SU(2)\) vacuum to the kernel of \(M^e\).

7. Generalization to a norm?

In this section, we enquire as to whether our considerations admit a generalization to a norm. We find that the linearized Gauss constraint provides the only obstruction to the most obvious such generalization. In order to obtain insight into why this happens, we are led to a comparison between the \(SU(2)\) structures hitherto defined and the \(U(1)^3\) structures (alluded to in section 1) which provided the motivation for these definitions.

\[ \| \| \] is not a norm because the states \(|\psi\rangle\) defined in (a) and (b) of section 4.2 do not span the kinematic Hilbert space. This can be remedied by requiring that the loops \(\eta_I, \beta_J\) in section 4.2 (a), (b) be replaced by edges. More precisely, define \(G_{\text{edge}}^\Lambda\) such that every element of \(G_{\text{edge}}^\Lambda\) is an analytic edge of length at most \(s\) which intersects \(\Delta\) at most a finite number of points. Since higher spin representations can be built up as products of spin-\(\frac{1}{2}\) representations and since edge holonomies along longer edges are products of those along shorter edges, it follows that \(|\psi\rangle\) in section 4.2 (a), (b) defined for \(\eta_I, \beta_J \in G_{\text{edge}}^\Lambda\) span the kinematic Hilbert space. Thus, with this modification \(\| \| \) becomes a norm. Let us denote this norm by \(\| \|^{\text{norm}}\).
Remarkably, all our results with the single exception of the treatment of $G^i_j$ go through in this norm. Specifically, (6.11) no longer holds if $a$ is an edge and not a loop.

As mentioned in section 1, the reason our $SU(2)$ constructions work is due to their structural analogy with the corresponding $U(1)^3$ ones in the description [3] of $r$-Fock states as $U(1)^3$ distributions. Hence, it is instructive to view the above failure to impose the linearized Gauss law as an approximate constraint in the $SU(2)$ theory in the context of this structural analogy. Therefore we briefly describe the relevant $U(1)^3$ structures below.

1. From [3], $|0_i\rangle$ may be written as the $U(1)^3$ distribution $\Phi^i_0$ defined by

$$\Phi^i_0 := \sum_{\alpha, |q\rangle} c_{\alpha, |q\rangle} \langle \alpha, |q\rangle, (7.1)$$

$$c_{\alpha, |q\rangle} = \exp \left( -\frac{i}{2} \int \sum_{H_{\alpha, |q\rangle}} \right. \,$$

$$\left. \sum_{G_{\alpha, |q\rangle}(r) \hat{\alpha}(\vec{x})} X^{ab}_{\alpha, |q\rangle}(r) (\vec{x}) \right)$$

where $|\alpha, |q\rangle\rangle$ is a $U(1)^3$ gauge invariant flux network state based on the triplet of graphs $\alpha$ labelled by the integers $|q\rangle$ and $G_{\alpha, |q\rangle}(r) \hat{\alpha}(\vec{x})$ is essentially the same as $G_{\alpha, |q\rangle}(r) \hat{\alpha}(\vec{x})$ defined by (2.11) in this paper. See [3] for the precise definitions of $G_{\alpha, |q\rangle}(r) \hat{\alpha}(\vec{x})$. Note that $\gamma_0$ in [3] is related to $\gamma$ of this work by $\gamma_0 = 2\gamma$.

2. Consider the algebraic dual $\Phi^i_{kin}$ to the finite span of extended (i.e. not necessarily gauge invariant) $U(1)^3$ flux nets. For any $\Phi \in \Phi^i_{kin}$ define the semi-norm $||\Phi||_{semi-norm} = \sup_{|\phi\rangle} \|\Phi(|\phi\rangle)\|$ with $|\phi\rangle := \hat{h}_\alpha|\gamma\rangle_L$, the length of each $\alpha$ being at most $s$, and with $|\gamma\rangle_L$ denoting the flux network state corresponding to the trivial loop. Then the right-hand side of (7.1) can be expanded in this semi-norm as

$$\Phi^i_0 = \Phi^i_{0 \approx \approx} + O(\epsilon^2)$$

$$\Phi^i_{0 \approx \approx} := \delta_{A=0} + \tilde{\epsilon}^2 \delta_{A=0},$$

$$\tilde{\epsilon}^2 := \frac{i}{2\gamma \ell_P^2} \int \sum_{\alpha} d^4 k \left[ (\hat{e}^+(\vec{k})) (\hat{e}^-(\vec{k})) \right]^2 - \int \sum_{\alpha} d^4 k \left[ (1 - 2i\gamma)(\hat{e}^+(\vec{k})) (\hat{e}^-(\vec{k})) \right]^2 \right],$$

where $\delta_{A=0}$ is the $U(1)^3$ distribution corresponding to the $U(1)^3$ connection $A^i_\alpha = 0$.

3. For any $\Phi \in \Phi^i_{kin}$, define the (stronger than $||\Phi||_{semi-norm}$) semi-norm

$$\|\Phi\|^2 := \sup_{|\phi\rangle} \|\Phi(|\phi\rangle)\|$$

with $|\phi\rangle := \prod_{i=1}^N \hat{h}_{\alpha_i}|\gamma\rangle_{\gamma_i}$ for $\alpha_i, k = 1, \ldots, 3, I = 1, \ldots, N$ being loops of length at most $s$, $N$ finite. Then $\Phi^i_{0 \approx \approx}$ satisfies the Poincaré invariance condition (3.9) modulo distributions of $O(\epsilon^2)$ in the semi-norm (7.6) and modulo terms of type $O(\epsilon^2)\Phi^i_{0 \approx \approx}$.

From (1)–(3) it is clear as to how $U(1)^3$ structures motivated our definitions of $SU(2)$ structures. In particular, the $U(1)^3$ structures $\tilde{\epsilon}^i = -i\gamma \frac{\ell_P}{\gamma \ell_P^2}, \Phi^i_0$ and $||\Phi||^2$ bear a close analogy to the $SU(2)$ structures $\tilde{E}^i = -i\gamma \frac{\ell_P}{\gamma \ell_P^2}, \Phi^i_0$ and $||\Phi||^2$. In this context the failure of $\|\Phi^i_{kin}(\Lambda)\Phi^i_0\|^2$ to be of $O(\delta\gamma)$ may be traced, as indicated below, to the difference between the gauge groups $U(1)^3$ and $SU(2)$.

Consider the algebraic dual $\Phi^i_{kin \ inv}$ to the finite span of gauge invariant $U(1)^3$ flux nets. Since larger loop $U(1)^3$ holonomies can be constructed as products of smaller loop ones, it follows that $|\phi\rangle$ of the type in (7.6) span the space of $U(1)^3$ gauge invariant flux nets. Hence, although $||\Phi||^2$ is a semi-norm on $\Phi^i_{kin}$, it is a norm on $\Phi^i_{kin \ inv}$. Moreover, since $\hat{g}_{\alpha(r)}, \hat{h}_\alpha$ are
$U(1)^3$ gauge invariant, the entire treatment of the $U(1)^3$ theory can be done at the gauge invariant level, and hence, in the context of a norm.

In contrast the very notion of linearization of the full $SU(2)$ theory as implemented in [1, 3] involves an $SU(2)$ gauge variant background. Since the $SU(2)$ counterparts of $\hat{g}_{\alpha(r)}$, $\hat{h}_\alpha$ are themselves not $SU(2)$ gauge invariant, it seems appropriate to use gauge variant kinematical states $|\psi\rangle$ in the definition of $\|\cdot\|^s$. Ultimately, however, only $SU(2)$ invariant distributions are physically relevant in LQG. Hence one may further restrict attention in (5.4) to only those $|\phi\rangle$ which are gauge invariant in the hope that the resulting semi-norm on $\Phi^\text{inv}_{\text{kin}}$ reduces to a norm on $\Phi^\text{inv}_{\text{kin}}^\text{gauge}$, where $\Phi^\text{inv}_{\text{kin}}^\text{gauge}$ is the algebraic dual to the finite span of spin nets which are gauge invariant with respect to $SU(2)$ transformations within the probe region. However, this hope is not realized because, in contrast to the $U(1)^3$ case, due to the non-Abelian nature of $SU(2)$, traces of large loop holonomies are not expressible as (sums of) products of traces of small loop holonomies. Moreover, with this restriction, it seems that $\|\hat{G}_{\alpha(r)}\varphi_0\|^s = O(\epsilon^4)$ which implies that a change in our choice of $\delta_\epsilon$ to $\delta_\epsilon \ll \epsilon^4$ would also be needed.

An alternate strategy would be to, in analogy with $U(1)^3$ theory, restrict attention to only such $|\psi\rangle$ in (4.10) which satisfy $\hat{G}_{\alpha(r)}(\Lambda)|\psi\rangle = 0$, i.e. try to impose the $U(1)^3$ constraint as an $SU(2)$ operator constraint. Our intuition is that no solutions to such an equation exist because $SU(2)$ spinnets are "attuned" to $SU(2)$ gauge transformations rather than $U(1)^3$ ones. Instead, since in the linearized theory we are interested in connections close to $A_0 = 0$, we may attempt to solve $\hat{G}_{\alpha(r)}^L(\Lambda)|\phi\rangle|_{A_0 = 0} = 0$. Indeed, viewed in this way, $|\phi\rangle$ defining $\|\cdot\|^1_1$ are precisely of this type.

On account of the above discussion, our view is that

(a) the failure of (6.11) in the context of $\|\cdot\|^s_{\text{norm}}$ is not an accident; rather it is indicative of the difference between the gauge groups $U(1)^3$ and $SU(2)$

(b) the semi-norm $\|\cdot\|^s$ seems to be a reasonable structure to use.\(^\text{14}\)

Nevertheless, a (putative) perturbative treatment of the constraints would be more powerful in the context of a stronger semi-norm. In view of the discussion above it would be interesting to see if we could enlarge the definition of $|\phi\rangle$ in (5.4) such that (i) $\hat{G}_{\alpha(r)}^L(\Lambda)|\phi\rangle|_{A_0 = 0} = 0$ (ii) $|\phi\rangle$ although not themselves necessarily gauge invariant, do span the space of $SU(2)$ gauge invariant spinnets based on graphs in $G_\Delta$.

8. Discussion

One of the features of our work is the, at first perhaps surprising but nevertheless essential, role of the probe scale $s$. At the classical level $s$ acts as an infrared cut off, allowing us to ignore large distance effects leading to the formation of black holes. The consequent restriction of the quantum observables $\hat{h}_\alpha, \hat{g}_{\alpha(r)}$ to loops $a^s$ of length at most $s$ leads to the determination of $\epsilon$ via equation (2.20). Moreover, the derivations of various bounds in this work lead us to believe that (i) it is only for loops with length close to $s$ that the fluctuations of the linearized observables can be close to $\epsilon$.\(^\text{15}\) (ii) the semi-norms of the various distributions encountered are determined primarily by the action of these distributions on states of type (b) in section 4.2 with $\eta_1, \beta_j$ of length $\approx s$ and $M, P \approx 1$. Thus, $s$ seems to play a key role in

\(^\text{14}\) It may be verified that $\|G_{\alpha(r)}^L(\Lambda)|\varphi_0\|^\text{norm} = O(\epsilon^4).$ This suggests the possibility of exploiting the freedom in (b) of appendix A to either redefine $\delta_\epsilon = \epsilon^3 \delta \geq \epsilon^3$ or to choose $\delta \ll \epsilon^2$. However, since the $O(\epsilon^4)$ bound depends crucially on (in our view the unduly weak premise) equation (6.6), we believe that such a strategy may be inappropriate.

\(^\text{15}\) More work is necessary to establish exactly which $a^s$ yield $\Delta s_{\alpha(r)} \approx \epsilon$ to nontrivial leading order in $\epsilon$. However bound (2.19) clearly indicates that any such $a^s$ should have lengths $\approx s$. 


characterizing the nature and content of the approximate descriptions at both the classical and the quantum levels.

It is important to mention here that equation (2.20) only implies that \( l_p^2 r^2 \leq \epsilon \). While we have not investigated the matter in any detail, it should be possible to consistently choose a large enough \( r \) such that, for example, \( l_p^2 r^2 = \epsilon^m, m > 1 \). This would render the vacuum fluctuations of the linearized observables too small to be of physical relevance and much of the subsequent analysis would presumably simplify. Viewed in this way, our choice (2.21) corresponds to \( r \) being the minimum smearing scale which is consistent with the uncertainty principle and our requirement of small fluctuations. Thus our constructions are consistent with an assumption of the validity of the approximate description provided by quantum linearized gravity down to this scale.

A second feature of our work is (for reasons described in section 3.1, (ii)) the use of distributions as opposed to kinematically normalizable states. Since we believe that distributional states will play a key role in future developments, it would be profitable to explore structures on \( \Phi_{\text{kin}}^{\ast} \). In the theory of infinite-dimensional vector spaces, topologies are typically defined via (families of) semi-norms [23]. In analogy with this and in view of the fact that the semi-norm \( \| \cdot \|_\epsilon \) on \( \Phi_{\text{kin}}^{\ast} \) has played a key role in our considerations, we advocate further study of semi-norms on the space \( \Phi_{\text{kin}}^{\ast} \).

The semi-norm \( \| \cdot \|_\epsilon \) provided us with the beginnings of a perturbative treatment of the constraints in terms of the expansion parameter \( \epsilon \). The first step was to verify that the linearized constraints mapped our states into the kernel of \( \mathcal{M}_\epsilon \). In showing this we did not display an explicit regularization of the constraints. Rather, we predicated our arguments on certain expectations of any ‘reasonable’ regularization procedure. The discussion in section 6 indicates that any such reasonable regularization procedure must provide some preferred role to the weave \( \Delta \). This seems to be in contrast to the standard Thiemann regularization [24] of say, the full Hamiltonian constraint, in which there is no preferred background structure. We believe that this possible conflict between the (putative) perturbative treatment of the constraints and the available nonperturbative treatment is of a generality which transcends the details of our particular constructions. Further work is needed to verify if such a conflict is indeed present in our work and if so, to seek a resolution. Another important open issue is to understand better the relation between the \( U(1)^3 \) gauge transformations in linearized theory and the \( SU(2) \) ones of LQG. The discussion in section 7 was indicative of our lack of understanding of this relation. We feel that a first exploratory step towards more clarity would be to see if we can use a more sophisticated semi-norm in accordance with (i)–(ii) at the end of section 7. In this regard, it would be worthwhile to base our discussion of \( |\phi\rangle \) in (5.4) in terms of graph-based spinnets rather than loop-based wavefunctions.

The key issue underlying the above paragraph is that of the relation between (putative) perturbative solutions to the constraints and the known structure of the nonperturbative ones as spatially diffeomorphism invariant, \( SU(2) \) gauge invariant distributions [25]. As mentioned earlier in this work, we have merely shown the existence of a map \( \mathcal{M}_\epsilon \) satisfying the various requirements of section 3. While this was a nontrivial and necessary exercise, it is important to emphasize that our constructions were based primarily on mathematical analogies between \( U(1)^3 \) and \( SU(2) \) structures. Without additional physical insight into the significance of the various choices we have made, we are unsure if these choices (or suitable modifications thereof) ensure that (putative) perturbative solutions converge to nonperturbative ones (or, indeed, if they ensure even the existence of a well-defined perturbative scheme). Since the issue is a deep and quite general one, clearly more work and thought is needed for further progress. Indeed, the main virtue of our work may be that it offers a concrete and detailed context wherein this issue may be analysed.
It would be of interest to see if our considerations can be generalized to construct LQG correspondents of states other than the linearized theory vacuum. It would also be interesting to adapt our general framework so as to find LQG correspondents of other known exactly solvable restrictions of full gravity such as Einstein–Rosen waves [26]. From equation (3.8) it is tempting to speculate that such ‘consistency’ with a multitude of exactly solvable restrictions of full gravity may lead to information about the scalar product of full quantum gravity.

There have been other efforts to construct the vacuum state in LQG. We comment briefly on three of these, namely [10], [9] and [8]. The recent work [10] is a very interesting attempt to construct the vacuum state as a kinematically normalizable state in LQG rather than, as in this work, a genuine distribution. Due to our unfamiliarity with the detailed considerations of [10], we are unable to comment further on its relation to our work. The work [9] contains a nice description of the author’s beautiful complexifier construction of coherent states. In the LQG context this construction yields genuinely distributional states. As mentioned earlier, our viewpoint is that distributions are the appropriate structures. It would be interesting to see if the complexifier generated distributions satisfy the requirements of section 3.2 in the context of some suitably defined $\mathcal{M}_\epsilon$. Distributional states are also used in [8]. It may be of interest to draw parallels between the graphs chosen as ‘probes’ in [8], those used to define kinematically normalizable ‘cut off’ states in [9] and the states in section 4.2 (b) which go into the definition of the semi-norm $\|\|_\epsilon$.

In closing, we reiterate that many of the considerations of this work were motivated by the earlier work of Iwasaki and Rovelli [16]. Since the work of Zegwaard [17] also tried to address the same problem as [16], it would be worthwhile to revisit it in the light of our work here.

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Appendix A

We sketch the main steps involved in showing (4.1). Let $\psi_\alpha$ be any spinnet based on a graph $\alpha$ with edges $e_I, I = 1, \ldots, N$ (we shall use the notation $|\psi_\alpha\rangle$ interchangeably with $\psi_\alpha$). Then (see, for example, equation (3.6) of [6]) we have that

$$\left(\hat{E}_{(t,0)}^a(\vec{x})\psi_{\text{weave}}\right)[\psi_\alpha] =: \Psi_{\text{weave}}\left[\hat{E}_{(t,0)}^a(\vec{x})\psi_\alpha\right]$$

$$= \nu^{2} \sum_I \int d\vec{e}_I e^{i \frac{2\pi |\vec{x} - \vec{e}_I(1)\vec{x}|^2}{\nu^2}} \Psi_{\text{weave}} \left[\left(h_{e_I(1,t)}(1) h_{e_I(t,0)}(0)\right)\frac{\partial \psi_\alpha}{\partial h_{e_I(1,t)}(0)}\right].$$

(A1)

Here $h_{e_I}(v,u)$ is the holonomy along the $I$th edge from parameter value $u$ to parameter value $v$, the beginning and end points of $e_I$ being at parameter values 0 and 1. Note that the argument
of $\Psi_{\text{weave}}$ above is obtained from $\psi_a$ by inserting the intertwiner $\tau_i$ at the point $\vec{e}_i(t)$ on the $l$th edge of $\alpha$. Since $\Psi_{\text{weave}} \in \Phi_{\text{kin}}$, we have that $\Psi_{\text{weave}}[\hat{h}_{e_i}(1, t)\tau_i h_{e_i}(t, 0)]_{\Delta, \{\vec{x}_i\}}$ is finite. Hence we can neglect contributions to the integral in (A1) coming from any sets of measure zero in parameter space. In particular, we can drop contributions from pre-existing vertices $\vec{e}_l(0), \vec{e}_l(1)$.

**Lemma 1.** If $\psi_a$ is orthogonal to all $|\Delta, \{\vec{x}_i\}|$, then $(\hat{E}_{\alpha \beta}(\vec{x}) \Psi_{\text{weave}})[\psi_a] = 0$.

**Proof.** Note that $\psi_a$ is orthogonal to $|\Delta, \{\vec{x}_i\}|$ iff any of the following are true:

(i) $\alpha$ is different from $\Delta$,

(ii) $\alpha = \Delta$ but has at least one edge with $j \neq \frac{1}{2}$,

(iii) $\alpha = \Delta$, every edge is labelled by $j = \frac{1}{2}$ and there exists at least one point in $\Delta$ at which the intertwiners for $\psi_a$ and $|\Delta, \{\vec{x}_i\}|$ are orthogonal, i.e., $\sum_{i_1, \ldots, i_l, j_1, \ldots, j_l} C_{i_1, \ldots, i_l, j_1, \ldots, j_l, i_1, \ldots, i_l} = 0$. Here $C_{i_1, \ldots, i_l, j_1, \ldots, j_l, i_1, \ldots, i_l}$ is the intertwiner for the vertex in question for $\psi_a$, $C_{i_1, \ldots, i_l, j_1, \ldots, j_l, i_1, \ldots, i_l}$ is the intertwiner for the same vertex for $|\Delta, \{\vec{x}_i\}|$ and $i_1, \ldots, i_l, j_1, \ldots, j_l$ label the representations on the incoming edges (for which the vertex is a ‘target’) and $j_1, \ldots, j_l$ label the representations on the outgoing edges (for which the vertex is a ‘source’). Note that these intertwiners may be further split into a product of invariant intertwiners and representation vectors but this is not necessary for our purposes.

Result (iii) is obtained straightforwardly from the definition of the intertwiners and the orthogonality properties of the representations of $SU(2)$ with respect to the Haar measure. Clearly if any of (i)–(iii) are true of $\psi_a$, the same is true of $(\hat{h}_{e_i}(1, t)\tau_i h_{e_i}(t, 0))_{\Delta, \{\vec{x}_i\}}$, except at most on sets of measure zero in parameter space. Thus, for $\psi_a$ orthogonal to all $|\Delta, \{\vec{x}_i\}|$, we have that

$$(\hat{E}_{\alpha \beta}(\vec{x}) \Psi_{\text{weave}})[\psi_a] = 0 = \delta^a_a \Psi_{\text{weave}}[\psi_a].$$  \hfill (A2)

**Lemma 2.** Let the extent $L$ of $\Delta$ be such that $s_{\text{curv}} \gg L \gg \ell_p$. Further, let $L \gg s$ and let $\vec{x}$ lie in the probe region of size $(5s)^3$ about the origin (see (i) in section 3.1 for the definition of $S$) so that $L \gg |\vec{x}|$. Then

$$(\hat{E}_{\alpha \beta}(\vec{x}) \Psi_{\text{weave}})[|\Delta, \{\vec{x}_i\}|] = (1 + O^\infty(\delta) + O^\infty(\tau)) \delta^a_a \Psi_{\text{weave}}[|\Delta, \{\vec{x}_i\}|]$$  \hfill (A3)

with $\delta = \sqrt{|\vec{x}|^2 / r}$ and $\tau = \frac{x}{r}$.

**Proof.** From (A1) and the definition of $\Psi_{\text{weave}}$, we have that $(\hat{E}_{\alpha \beta}(\vec{x}) \Psi_{\text{weave}})[|\Delta, \{\vec{x}_i\}|] = 0$ if $a \neq i$. Hence the result must be proportional to $\delta^a_a$. Let us suppose that $a = i = 1$ (our considerations below trivially extend to any other choice of $a = i$). Then from equations (A1) and (4.4), it is straightforward to show that

$$\Psi_{\text{weave}}[\hat{E}_{\alpha \beta}(\vec{x})|\Delta, \{\vec{x}_i\}|] = (-i)^N \left( \sum_{L_1} \left| \frac{L_1}{2\sqrt{2\pi r}} \right|^2 e^{-\frac{|\vec{x}|^2}{2r}} \right) \left( \sum_{L_2} \left| \frac{L_2}{2\sqrt{2\pi r}} \right|^2 e^{-\frac{|\vec{x}|^2}{2r}} \right) \int_{-L_1}^{L_1} dt_1 \int_{-L_2}^{L_2} dt_2 e^{-\frac{|\vec{x}|^2}{2r}}.$$  \hfill (A4)

Here $(L_1, L_2)$ range over the lattice points in the $x = 0$ plane. Note that from (2.21), the condition $s_{\text{curv}} \gg L$ is the same as $r^2 \gg \ell_p L$. Straightforward estimates using the
latter condition and the condition that \( L \gg |\vec{x}| \), in conjunction with the Poisson summation formula\(^{16}\) \[27\] yield

\[
\left( \sum_{L_z} \sqrt{\frac{l_{z}^{2}}{2}} e^{-\frac{(z-L_{z})^{2}}{2r^{2}}} \right) \cdot \left( \sum_{L_y} \sqrt{\frac{l_{y}^{2}}{2}} e^{-\frac{(y-L_{y})^{2}}{2r^{2}}} \right) = 1 + O^{\infty}(\delta) + O^{\infty}(\tau). \tag{A5}
\]

Also it is easy to check that

\[
\int_{-L}^{L} \frac{d\vec{x}}{\sqrt{2\pi r}} e^{-\frac{(x-L_{x})^{2}}{2r^{2}}} = 1 + O^{\infty}(\tau). \tag{A6}
\]

This completes the proof of lemma 2. \(\square\)

In order to obtain equation (4.1) from lemma 2, it suffices to choose \( \delta = \epsilon^{a}, \tau = \epsilon^{b}, \alpha, \beta > 0 \). Below we display two explicit choices of \( \epsilon \) dependence of the various scales involved which satisfy all our requirements.

(a) Choose \( s \) independent of \( \epsilon \), \( r = \epsilon^{-1}, s_{\text{curv}} = \epsilon^{-1}s, L = \epsilon^{-\frac{3}{4}}s \). Thus \( \alpha = \frac{1}{2} \) and \( \beta = \frac{1}{4} \) and we have that \( s_{\text{curv}} \gg L \gg r \gg s \sim O(1) \).

(b) \( s \) dependent on \( \epsilon, s \neq 0 \) as \( \epsilon \to 0 \). Choose \( \alpha > \frac{1}{2} \) and \( \beta > 0, \alpha > \beta > \alpha - 1 \). Let \( s = \epsilon^{1-2\alpha}l_{P}, r = \epsilon^{-a}l_{P}, s_{\text{curv}} = \epsilon^{-2\alpha}l_{P}, L = \epsilon^{-(\alpha+\beta)}l_{P} \). Then for \( \alpha > 1 \), we have that \( s_{\text{curv}} \gg L \gg s \gg r \) and for \( \alpha < 1 \), \( s_{\text{curv}} \gg L \gg r \gg s \).

As mentioned at the end of section 2, we shall set \( s \gg r \), i.e. we shall restrict attention to the choice (b) with \( \alpha > 1 \).

**Appendix B**

In this appendix we prove identities (4.7), (4.8) under the assumption that (4.6) holds for all \( \vec{x} \) on the spatial slice. From (2.11) we have that

\[
G_{\alpha r \alpha b}(\vec{x}) = (1 + 2i\gamma) g^{(\alpha)}_{\alpha r \alpha b}(\vec{x}) - (1 - 2i\gamma) g^{(-\alpha)}_{\alpha r \alpha b}(\vec{x}) \tag{B1}
\]

where \( g^{\pm}_{\alpha r \alpha b} \) are the positive/negative helicity components of \( g_{\alpha r \alpha b} \).

\[
g^{ab}_{\alpha r \alpha \beta}(\vec{x}) := \int \frac{d^{3}k}{(2\pi)^{2}} e^{i\vec{x} \cdot \vec{k}} X^{ab}_{\alpha r \alpha \beta}(\vec{k}) = \oint_{\gamma_{b}} \frac{dr}{2\pi} \bar{a}^{a}(i). \tag{B2}
\]

\( I(\vec{y}) \) is defined as

\[
I(\vec{y}) := \int \frac{d^{3}k}{(2\pi)^{2}} e^{i\vec{x} \cdot \vec{k}} e^{-\frac{|\vec{k}|^{2}}{2r^{2}}} = \frac{4\pi}{(2\pi)^{2}r^{3}y} \int_{0}^{\infty} du \sin \left( \frac{u}{r} \right) u^{2} e^{-\frac{u^{2}}{2r^{2}}} \tag{B3}
\]

where we have denoted \( |\vec{y}| \) by \( y \).

\(^{16}\) We are indebted to Supurna Sinha for pointing us to the Poisson sum formula which plays this key role as an estimation technique in our work.
From (4.6), (3.3) and (B1) it follows that for $|\psi/\Delta_1\rangle = |\psi/\Delta_1\rangle$

$$\left| \prod_{j=0}^{n-1} G_{\alpha_{n-j}}^\dagger \Psi_{\text{weave}}[|\psi/\Delta_1\rangle] \right| \leq O^\infty(\epsilon) \prod_{j=1}^{n} \left( \int d^3 x_i \left| G_{\alpha_i}^{\psi_0} (\vec{x}_i) G_{\alpha_i (a_i b_i)} (\vec{x}_i) \right|^2 \right)$$

$$\leq O^\infty(\epsilon) \prod_{j=1}^{n} \left( \int d^3 x_i \sum_{a_i b_i=1,...,3} (1+4y^2)^{\frac{1}{2}} \left| g_{\alpha_i (a_i b_i)} (\vec{x}_i) \right| \right). \quad (B4)$$

We estimate a bound on $|g_{\alpha (a b)} (\vec{x})|$ through an analysis of its asymptotic behaviour as follows. From [28] we have that

$$\int_0^\infty du \sin \left( \frac{\mu}{r} u \right) u^2 e^{z u} = \frac{2y}{r} e^{y^2/2r^2} _1F_1 \left( -1; \frac{3}{2}; \frac{y^2}{2r^2} \right), \quad (B5)$$

where $\Gamma(c)$ is the confluent hypergeometric function. From [29] the asymptotic behaviour of this function as $z \to \infty$ is

$$\Gamma(c) = \frac{1}{\Gamma(a) \Gamma(c)} e^{za} - c O \left( \frac{1}{z} \right). \quad (B6)$$

The parameters $z, a, c$ in (B6) are identified through (B5), (B2) and (B3) as $z = \sqrt{\vec{x} - \vec{a}^h(t)}$, $a = -\frac{1}{2}$ and $c = \frac{1}{2}$. Using (B6) in (B2) and (B3), it is straightforward to obtain the bound

$$|g_{\alpha (a b)} (\vec{x})| < \frac{2s}{\pi^2} \min_{i} |\vec{x} - \vec{a}^h(t)|^4 \quad (B7)$$

where $\min_{i} f(t)$ refers to the minimum value of the function $f(t)$ over the entire loop $a_{\alpha_i}$ and the bound is valid for all $|\vec{x} - \vec{a}^h(t)| > \lambda$ for sufficiently large $\lambda$. Since the triplets of loops $\alpha_i$, $i = 1, \ldots, n$ are all confined to a region of size $Ss$ about the origin (see (i) in section 3.1), we have that $\min_{i} |\vec{x} - \vec{a}^h(t)| > |\vec{x}| - Ss$.

Then it follows from (B7) that for $|\vec{x}| > r \lambda + 2Ss$

$$|g_{\alpha (a b)} (\vec{x})| < \frac{32s}{\pi^2} \frac{1}{|\vec{x}|^2}. \quad (B8)$$

Also note that from (B2) and (B3) the following bound is easily obtained:

$$\int_{|\vec{x}| < r \lambda + 2Ss} d^3 x |g_{\alpha (a b)} (\vec{x})| \leq \left( \lambda + \frac{2Ss}{r} \right)^3 \frac{s}{\pi r}. \quad (B9)$$

From (B8), (B9) it follows that

$$\sum_{a,b=1,...,3} \int_{|\vec{x}| < r \lambda + 2Ss} d^3 x |g_{\alpha (a b)} (\vec{x})| \leq \frac{9s}{\pi r} \left( \lambda + \frac{2Ss}{r} \right)^3 + \frac{128}{(\lambda + \frac{2Ss}{r})}. \quad (B10)$$

Using the above bound in (B4) with either (a) or (b) of appendix A and the $\epsilon$-independence of $S, \lambda, n$, we obtain the desired result, namely that

$$\left| \prod_{j=0}^{n-1} G_{\alpha_{n-j}}^\dagger \Psi_{\text{weave}}[|\psi/\Delta_1\rangle] \right| \leq O^\infty(\epsilon). \quad (B11)$$
Appendix C

Let
\[ O_j^H(A) \in \left\{ \frac{\text{Tr} H_\eta}{2}, \frac{\text{Tr} H_\eta \tau^k}{2}, \eta \in G^\text{A-loop}_{\eta}, k = 1, \ldots, 3 \right\}, \quad I = 1, \ldots, N. \] (C1)

Thus each \( O_j^H(A) \) is (half) the trace of a holonomy or the trace of the product of a holonomy with \( \tau^i, i \in \{1, 2, 3\} \), the holonomy being around any loop of length at most equal to \( s \). Then the following bounds hold:
\[
\left| \prod_{j=0}^{n-1} \int d^3k_{n-j} (G_{\alpha_{n-j}^{(r)\eta_\alpha_0,\eta_\alpha_1,\eta_\alpha_2,\eta_\alpha_3}} ^{(n-j)} \hat{E}_{(r)} \hat{E}_{(\bar{r})} \hat{E}_{(\bar{r})} \prod_{l=1}^N O_j^H(A) \right|_{A=0} \leq N^n 2^n c_n, \quad (C2)
\]

with \( c_n = \left( \frac{27 \sqrt{1 + 4 \gamma^2}}{4 \pi} \right)^n \).

\[
\left| \int \frac{d^3k}{2i \gamma I} k \left( 1 \mp 2i \gamma \right) \hat{E}^\pm_{(r)} (\bar{k}) \hat{E}^\pm_{(\bar{r})} (k) \right| \times \prod_{j=0}^{n-1} \int d^3k_{n-j} (G_{\alpha_{n-j}^{(r)\eta_\alpha_0,\eta_\alpha_1,\eta_\alpha_2,\eta_\alpha_3}} ^{(n-j)} \hat{E}_{(r)} \hat{E}_{(\bar{r})} \hat{E}_{(\bar{r})} \prod_{l=1}^N O_j^H(A) \right|_{A=0} \leq N^{n+2} 2^{n+2} d_n, \quad d_n = \frac{3 \sqrt{3}}{\sqrt{2} \left( \frac{27 \sqrt{1 + 4 \gamma^2}}{4 \pi^2} \right)^{n+1}}. \quad (C3)
\]

Here \( \alpha_i, i = 1, \ldots, n \) are \( n \) triplets of loops, each loop with length at most equal to \( s \), and \( \hat{E}^\pm \) are the positive and negative helicity components of the triad operator.

**Sketch of the proof.** We shall only describe the main steps of the proof. The details are straightforward to work out and the interested reader may easily do so.

(i) The following bound holds:
\[
\left| \prod_{j=0}^{n-1} \hat{E}^\pm_{(r)} (\bar{k}) O_j^H(A) \right|_{A=0} \leq \left( \frac{P_2 \gamma s}{(2 \pi)^2} \right)^n \prod_{l=1}^N e^{e_{\pm l}^2}. \quad (C4)
\]

Note that the action of the smeared triad operator \( \hat{E}^\pm_{(r)}(\bar{k}) \) on a cylindrical function \( \psi_a \) based on a graph \( \alpha \) with edges \( e_j \) is
\[
\hat{E}^\pm_{(r)}(\bar{k}) \psi_a = i l_p^2 \gamma \sum_j \int dt e_j e^{-id \bar{k} \bar{\tau}_j (t)} e^{i l_p^2 \bar{k}} \left\{ h_{\bar{e}_j} (1, t) \tau_i h_{\bar{e}_j} (t, 0) \right\} \frac{\partial \psi_a}{\partial h_{\bar{e}_j} (1, 0) \bar{\tau}_j}. \quad (C5)
\]

For \( \psi_a \) of the type \( O_j^H \), the term in curly brackets corresponds to an insertion of \( \tau^i \) at parameter value \( t \). The action of a product of \( n \) factors of triad operators on \( O_j^H \) yields \( n \)
integrations over the loop labelling $O_j^H$, with $n$ insertions of $\tau$ matrices along the loop. The evaluation at $A = 0$ of $O_j^H(A)$ modified by these insertions yields the trace of a product of $\tau$ matrices. This trace is bounded by unity. The $n$ integrations along the loop are responsible for the $s^n$ factor in bound (C4) and the origin of the $\left(\frac{\ell^2 \gamma s}{(2\pi)^2}\right)$ factors is obvious from (C5).

(ii) The following bound holds:

$$ \left| \prod_{j=0}^{n-1} \hat{E}_{\alpha_{1j} \cdots \alpha_{nj}}^{\nu_{1j} \cdots \nu_{nj}}(k_{nj}) \prod_{l=1}^{N} O_l(A) \right| \leq N^n \left( \frac{\ell^2 \gamma s}{(2\pi)^2} \right)^n \prod_{l=1}^{N} e^{-\frac{3s^2}{ \ell^2 \gamma s}}. \quad (C6) $$

This follows from a straightforward estimate of the number of terms of type (C4).

(iii) For any ‘smearing’ function $G_{\alpha_{1j} \cdots \alpha_{nj}}(k_1, \ldots, k_n)$ the following bound holds:

$$ \left| \int \prod_{i=1}^{n} d^3k_i (G_{\alpha_{1i} \cdots \alpha_{ni}}(k_1, \ldots, k_n))^{n-1} \prod_{j=0}^{n} \hat{E}_{\alpha_{1j} \cdots \alpha_{nj}}^{\nu_{1j} \cdots \nu_{nj}}(k_{nj}) \prod_{l=1}^{N} O_l(A) \right| \leq N^n \left( \frac{9\ell^2 \gamma s}{(2\pi)^2} \right)^n \int \prod_{i=1}^{n} d^3k_i e^{-\frac{3s^2}{ \ell^2 \gamma s}} \left( \sum_{\alpha_{i}, \beta_i} |G_{\alpha_{1i} \cdots \alpha_{ni}}(k_1, \ldots, k_n)|^2 \right)^{\frac{1}{2}}. \quad (C7) $$

To see this, note that the left-hand side of the above equation is bounded by the integral of the product of the norms of the smearing function and the quantity $\left( \prod_{j=0}^{n-1} \hat{E}_{\alpha_{1j} \cdots \alpha_{nj}}^{\nu_{1j} \cdots \nu_{nj}}(k_{nj}) \prod_{l=1}^{N} O_l(A) \right)_{A=0}$. The norm of the latter is bounded by the right-hand side of (C6) augmented by a factor of $9\ell^2 s$ which comes from the sum over the indices $\alpha_{i}, \beta_i$. As a result, equation (C7) follows.

(iv) Set $G_{\alpha_{1i} \cdots \alpha_{ni}}(k_1, \ldots, k_n) = \prod_{i=1}^{n} G_{\alpha_{1i} \cdots \alpha_{ni}}(\vec{k}_i)$ in (C7). Using (2.11), it is easily verified that

$$ \left( \sum_{\alpha, \beta} |G_{\alpha_{1i} \cdots \alpha_{ni}}(\vec{k}_i)|^2 \right)^{\frac{1}{2}} \leq k_i e^{-\frac{3s^2}{ 9\ell^2 \gamma s}} (1 + 4\gamma^2)^{\frac{1}{2}} \frac{3s}{(2\pi)^2}. \quad (C8) $$

The integral over each $\vec{k}_i$ in (C7) furnishes a factor of $\frac{2\pi}{k_i}$. This, together with $\epsilon = \frac{\ell^2 \gamma s}{(2\pi)^2}$ from (2.21) yields (C2). Similarly, for an appropriate choice of $G_{\alpha_{1i} \cdots \alpha_{ni}}(k_1, \ldots, k_n)$ and straightforwardly derived bounds thereon, equation (C3) may also be obtained.

**Appendix D**

We prove identities (S.3) through the following steps.

(1) Consider a set of operators $\hat{O}_i$, $i = 0, \ldots, n$ such that each $\hat{O}_i$ is of the type defined in equation (4.9). Denote the operators $\hat{E}_{\nu_{ij}}^{\alpha_{ij}}$ by $\hat{E}_j$ and the Kronecker deltas $\delta_{ij}$ by $\delta_j$. Define $\hat{O}_{1,2,\ldots,n}$ and $\hat{O}_{1,2,\ldots,i,j,\ldots,n}$ by

$$ \hat{O}_{1,2,\ldots,n} := \hat{O}_1 \hat{O}_2 \cdots \hat{O}_n \quad (D1) $$

$$ \hat{O}_{1,2,\ldots,i,j,\ldots,n} := \hat{O}_1 \hat{O}_2 \cdots \hat{O}_{i-1} \hat{E}_i \hat{O}_{i+1} \cdots \hat{O}_{j-1} \hat{E}_j \hat{O}_{j+1} \cdots \hat{O}_n. \quad (D2) $$

Thus, in definition (D2), each ‘slashed’ index implies a commutator of the string of operators which follow the index with the triad operator labelled by that index. Then the following lemma holds.
Lemma 3.
\[
\prod_{i=1}^{n} (\hat{E}_i - \delta_i) \hat{O}_i = (E_1 - \delta_1) \hat{O}_1 (E_2 - \delta_2) \hat{O}_2, \ldots, (E_n - \delta_n) \hat{O}_n
\]
\[
= \hat{O}_{1 \ldots n} \sum_{i=1}^{n} (E_i - \delta_i) \sum_{I=1}^{m} \sum_{I \neq I} \hat{O}_{1,2,\ldots,i_I} \ldots \hat{O}_{j_I} \prod_{j \not\in \{1, \ldots, I\}} (E_j - \delta_j),
\]

where the second sum in the second term on the right-hand side is over all choices of ‘slashed indices’ \(i_1 < i_2 \cdots < i_I\).

**Proof** (by induction). Equation (D3) is easily verified for \(n = 1\). Let it be true for \(n = m\). For \(n = m + 1\), the left-hand side of (D3) is \(\prod_{i=1}^{m+1} (\hat{E}_i - \delta_i) \hat{O}_i\). We apply (D3) for \(n = m\) to the string of operators \(\prod_{i=m+1}^{m+1} (\hat{E}_i - \delta_i) \hat{O}_i\) to obtain
\[
\prod_{i=1}^{m+1} (\hat{E}_i - \delta_i) \hat{O}_i = \left\{ (\hat{E}_1 - \delta_1) \hat{O}_1 \prod_{i=2}^{m+1} (\hat{E}_i - \delta_i) \right\}
\]
\[
+ \left\{ (\hat{E}_1 - \delta_1) \hat{O}_1 \sum_{I=1}^{m+1} \sum_{i_I \neq I} \hat{O}_{1,2,\ldots,i_I} \ldots \hat{O}_{j_I} \prod_{j \not\in \{1, \ldots, I\}} (E_j - \delta_j) \right\}
\]
\[
+ \sum_{I=1}^{m+1} \sum_{i_I \neq I} \hat{O}_{1,2,\ldots,i_I} \ldots \hat{O}_{j_I} (E_1 - \delta_1) \prod_{j \not\in \{1, \ldots, I\}} (E_j - \delta_j)
\]

This completes the proof. \(\square\)

**Note.** In the equations above it is understood that any set of indices \([i_k]\) is such that if \(p < q\), then \(i_p < i_q\) and that the set of indices ranges over all appropriate subsets of \([1, 2, \ldots, m+1]\) modulo explicitly forbidden values (e.g. in some cases the indices have to be all different from \(1\)).

(2) In the connection representation the commutator \([\hat{E}_{i_1}, \hat{O}_{i_1,i_1+1,\ldots,i_1+\ldots,i_1+\ldots,n}]\) is a ‘multiplication’ operator whose action on any state \(\psi(\hat{A})\) is given by
\[
[\hat{E}_{i_1}, \hat{O}_{i_1,i_1+1,\ldots,i_1+\ldots,i_1+\ldots,n}]\psi(\hat{A}) = \int d^{3} y \frac{\delta^{2}}{\delta \hat{A}_{\alpha}(y)} \hat{O}_{i_1,i_1+1,\ldots,i_1+\ldots,n}(A|A=\hat{A}) \psi(\hat{A})
\]
\[
\times \int d^{3} y \left( -\frac{\delta^{2}}{\delta \hat{A}_{\alpha}(y)} \hat{O}_{i_1,i_1+1,\ldots,i_1+\ldots,n}(A|A=\hat{A}) \right) \psi(\hat{A})
\]
\[
(D7)
\]
with the term multiplying $\hat{\psi}(\hat{A})$ on the right-hand side of the above equation completely determined by the holonomies of the generalized connection $\hat{A}$ along graphs contained in $\hat{G}_\Delta$ (recall that $G_\Delta$ is the set of all graphs which intersect $\Delta$ in at most a finite number of points).

Using (D7) and lemma 3, in conjunction with (a)–(c) of section 4.3, it is straightforward to see that for $|\hat{\psi}_\Delta\rangle = |\Delta\rangle$ or $|\hat{\psi}_\Delta\rangle = |\Delta(x_i)\rangle$, we have

$$
\Psi_0^\dagger[\hat{\Delta}_0(E_1 - \delta_1)\hat{\Delta}_1(E_2 - \delta_2)\hat{\Delta}_2, \ldots, (E_n - \delta_n)\hat{\Delta}_n]|\hat{\psi}_\Delta\rangle] = \Psi_{\text{weave}}[|\hat{\psi}_\Delta\rangle] \Phi_0^\dagger[\hat{\Delta}_0\hat{\Delta}_1,\hat{\Delta}_2,\ldots,\hat{\Delta}_n]|\hat{\psi}_\Delta\rangle].
$$

(D8)

Note that

$$
\hat{\Delta}_0\hat{\Delta}_1,\ldots,\hat{\Delta}_n = \hat{\Delta}_0\hat{\Delta}_1\hat{\Delta}_2\cdots\hat{\Delta}_i\hat{\Delta}_i,\ldots,\hat{\Delta}_n|\hat{\psi}_\Delta\rangle.
$$

(D9)

Using this in (D8) and integrating against $\prod_{i=1}^n G^\alpha_{\alpha_i(r)\alpha_i}$ with $\alpha_i$ being triplets of loops in $G_\Delta$, we obtain

$$
\Psi_0^\dagger[\hat{\Delta}_0\hat{G}_{\alpha_i}^\dagger(\alpha_i)\hat{\Delta}_1\cdots\hat{\Delta}_n|\hat{\psi}_\Delta\rangle] = \Psi_{\text{weave}}[|\hat{\psi}_\Delta\rangle] \Phi_0^\dagger\left(\int G^\alpha_{\alpha_i(r)\alpha_i} \hat{E}^\alpha_{\alpha_i(r)}\right) \hat{\Delta}_0|\hat{\psi}_\Delta\rangle,
$$

$$
\times \hat{\Delta}_1 \cdots \left(\int G^\alpha_{\alpha_i(r)\alpha_i} \hat{E}^\alpha_{\alpha_i(r)}\right) \hat{\Delta}_n|\hat{\psi}_\Delta\rangle.
$$

(D10)

(3) Set

$$
\hat{\Delta}_i = \prod_{l=1}^{n-1} \hat{\Delta}_i^H \quad \text{for} \quad i = 0 \cdots n - 1, \quad \hat{\Delta}_n = \hat{\Delta}_n^H \prod_{l=1}^{n} \hat{\Delta}_l^H \quad \text{and} \quad \hat{\Delta}_i^H = \prod_{l=1}^{n-1} \hat{\Delta}_l^H,
$$

(D11)

where $\hat{\Delta}_i^H, \hat{\Delta}_i^H$ are quantum operators corresponding to classical functions of the type defined by (C1). Define $P := \sum_{i=0}^{n} P_i$ and restrict attention to $n$, $P$ independent of $\epsilon$. Using (D11), (4.7), (4.7) and (C2), (C3) in conjunction with straightforward bounds on the number of terms of type (C2), (C3), we obtain

$$
\left| \sum_{l=1}^{n-1} \sum_{l_1, l_2, \ldots, l_t} \Psi_{\text{weave}}\left[\prod_{j \notin \{l_1, \ldots, l_t\}} \hat{G}_{\alpha_i}^\dagger(\alpha_i) |\hat{\psi}_\Delta\rangle\right] \Phi_0^\dagger \left(\int G^\alpha_{\alpha_i(r)\alpha_i} \hat{E}^\alpha_{\alpha_i(r)}\right) \hat{\Delta}_1 \cdots \left(\int G^\alpha_{\alpha_i(r)\alpha_i} \hat{E}^\alpha_{\alpha_i(r)}\right) \hat{\Delta}_n
\right| \lesssim O^\infty(\epsilon)(N + P)^{n+1},
$$

(D12)
we expect that there exist real $a, b$ such that for $n \geq 1$, all independent of $N$. Then it follows that $P_n \leq n^{a, b}$.

Using (E4) and (E6), we obtain $X^a_{\beta^k}(\bar{u}) = r X^a_{\beta^k}(\bar{u})^\dagger$. Next, note that

$$
\langle \Delta g_{\alpha(r)} \rangle^2 = (1 + 4 \gamma^2) \int d^3 k \ e^{-k r^2} \left( |X^a_{\alpha}(\bar{k})|^2 + |X^a_{\alpha}(\bar{k})|^2 \right)
$$

where we have defined the dimensionless variable $\bar{u} = r \bar{k}$. Now, let $\beta^k$ be loops of size $r$. Let $\alpha' = (\beta^k)^n$ (i.e. $\alpha'$ is obtained by going around $\beta^k$ 'n' times) with $n = \frac{1}{r}$. Then it follows that

$$
\langle \Delta g_{\alpha(r)} \rangle^2 = \frac{s^2}{r^2} \int d^3 k \ e^{-k r^2} \left( |X^a_{\alpha}(\bar{k})|^2 + |X^a_{\alpha}(\bar{k})|^2 \right)
$$

where $s^2 = \frac{1}{2 \pi} \int_{\beta^k} ds \ e^{-\bar{u} \beta^k(s)} \beta^k_1$. Next, note that

$$
X^a_{\beta^k}(\bar{u}) = r X^a_{\beta^k}(\bar{u})^\dagger.
$$

Using (E4) and (E6), we obtain

$$
\langle \Delta g_{\alpha(r)} \rangle^2 \geq \epsilon^2 (1 + 4 \gamma^2) 2 c^2 \int_a^b d^3 u \ e^{-u a^2}
$$

where

$$
X^a_{\beta^k}(\bar{u}) := \frac{1}{(2 \pi)^2} \int_{\beta^k} \left[ ds \ e^{-\bar{u} \beta^k(s)} \beta^k_1 \right].
$$

with $\beta^k(s) := \bar{u}(s)$. Since $\beta^k$ is a loop of length $r$, $\beta^k$ is a 'unit' size loop. For generic $\beta^k$ we expect that there exist real $a, b$ and positive definite $c$, all independent of $\epsilon$ such that for $u \in [a, b]$, we have that

$$
|X^a_{\beta^k}(\bar{u})| \geq c.
$$

Using (E4) and (E6), we obtain

$$
\langle \Delta g_{\alpha(r)} \rangle^2 \geq \epsilon^2 (1 + 4 \gamma^2) 2 c^2 \int_a^b d^3 u \ e^{-u a^2}
$$

with $\beta^k(s) := \bar{u}(s)$. Since $\beta^k$ is a loop of length $r$, $\beta^k$ is a 'unit' size loop. For generic $\beta^k$ we expect that there exist real $a, b$ and positive definite $c$, all independent of $\epsilon$ such that for $u \in [a, b]$, we have that

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$$
|X^a_{\beta^k}(\bar{u})| \geq c.
$$
The graviton vacuum as a distributional state in kinematic loop quantum gravity

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