Remarks on $l$–conformal extension of the Newton-Hooke algebra

Anton Galajinsky and Ivan Masterov

Laboratory of Mathematical Physics, Tomsk Polytechnic University,
634050 Tomsk, Lenin Ave. 30, Russian Federation
E-mails: galajin@mph.phtd.tpu.ru, masterov@mph.phtd.tpu.ru

Abstract

The $l$–conformal extension of the Newton–Hooke algebra proposed in [J. Math. Phys. 38 (1997) 3810] is formulated in the basis in which the flat space limit is unambiguous. Admissible central charges are specified. The infinite–dimensional Virasoro–Kac–Moody type extension is given.

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Current studies of the non–relativistic version of the AdS/CFT correspondence stimulate a renewed interest in non–relativistic conformal algebras. In general, one can formulate a family of conformal Galilei algebras which are parameterized by a positive half integer \( l \) \([1, 2]\) (see also \([3]\)). Furthermore, following the idea in \([4]\), one can extend the \( l \)–conformal Galilei algebra\([1]\) to the infinite–dimensional Virasoro–Kac–Moody type algebra \([6, 7]\). So far, the instances of \( l = 1/2 \) (the Schrödinger algebra) and \( l = 1 \) (the conformal Galilei algebra) have been the focus of most studies.

In the chain of the \( l \)–conformal Galilei algebras \( l = 1 \) is the only instance where the temporal and spatial coordinates scale the same way under dilatations. This case has been extensively investigated recently in regard to dynamical realizations \([8, 9]\), the AdS/CFT correspondence \([6, 7, 10]\), and the Newton–Cartan structures \([5, 11]\). For a review and further references see \([5]\).

According to the analysis in \([5]\), a proper arena where the conformal Galilei groups act is the Newton–Hooke spacetime with quantized negative cosmological constant. This proposal calls for a better understanding of the \( l \)–conformal Newton–Hooke algebra.

The \( l \)–conformal Newton–Hooke algebra has been previously studied in \([1, 2]\). However, the flat space limit is problematic in the formulation of \([1, 2]\). This is to be contrasted with the Newton–Hooke extension of the ordinary Galilei algebra (see e.g. \([12, 13]\)) where the latter follows from the former in the limit in which the cosmological constant tends to zero.

The purpose of this brief note is to formulate the \( l \)–conformal Newton–Hooke algebra in the basis in which the flat space limit is unambiguous. Our strategy is to generalize the analysis of \( l = 1/2 \) \([14, 15]\) to the case of arbitrary positive half integer \( l \). We also analyze admissible central charges and construct the infinite–dimensional Virasoro–Kac–Moody type extension.

Consider first a realization of the \( l \)–conformal Galilei algebra

\[
\begin{align*}
[H, D] &= H, & [H, C_i^{(n)}] &= nC_i^{(n-1)}, \\
[H, K] &= 2D, & [D, K] &= K, \\
[D, C_i^{(n)}] &= (n - l)C_i^{(n)}, & [K, C_i^{(n)}] &= (n - 2l)C_i^{(n+1)}, \\
[M_{ij}, C_k^{(n)}] &= -\delta_{ik}C_j^{(n)} + \delta_{jk}C_i^{(n)}, & [M_{ij}, M_{kl}] &= -\delta_{ik}M_{jl} - \delta_{jl}M_{ik} + \delta_{il}M_{jk} + \delta_{jk}M_{il},
\end{align*}
\]  

in a flat \((d + 1)\)–dimensional spacetime \([2]\)

\[
\begin{align*}
H &= \partial_t, & D &= t\partial_t + lx_i\partial_i, & K &= t^2\partial_t + 2ltx_i\partial_i, \\
C_i^{(n)} &= t^n\partial_t, & M_{ij} &= x_i\partial_j - x_j\partial_i.
\end{align*}
\]  

Here \( n = 0, 1, \ldots, 2l \) and \( i = 1, \ldots, d \). As usual, \( \partial_t = \frac{\partial}{\partial t}, \partial_i = \frac{\partial}{\partial x_i} \) and summation over repeated indices is understood. The operators \( H, D \) and \( K \) generate time translations,
dilatations and special conformal transformations, respectively. The instances of \( n = 0 \) and \( n = 1 \) in \( C_i^{(n)} \) produce space translations and the Galilei boosts. Higher values of \( n \) correspond to accelerations. Below we first treat the case of a negative cosmological constant and then consider a positive cosmological constant.

In order to construct the Newton–Hooke extension of (1), we introduce a dimensionful constant \( R \), the characteristic time\(^2\) \[ \frac{2}{c R^2} \] \[^2\text{The characteristic time is related to the cosmological constant via } \Lambda \sim \pm \frac{1}{(c R)^2} \text{, where } c \text{ is the speed of light.}\] , and deform \( D \) and \( K \) by analogy with the \( l = 1/2 \) case considered in \[^1\text{14}\]

\[
D = \frac{1}{2} R \sin (2t/R) \partial_t + l \cos (2t/R) x_i \partial_i,
\]

\[
K = -\frac{1}{2} R^2 (\cos (2t/R) - 1) \partial_t + l R \sin (2t/R) x_i \partial_i.
\] (3)

The generators of time translations and space rotations maintain their form while \( C_i^{(n)} \) is modified as follows

\[
C_i^{(n)} = R^n (\tan (t/R))^n (\cos (t/R))^{2l} \partial_i.
\] (4)

It is readily verified that these operators form a closed algebra. As compared to (1), only the first line is modified to include contributions which involve the cosmological constant

\[
[H, D] = H - \frac{2}{R^2} K, \quad [H, C_i^{(n)}] = nC_i^{(n-1)} + \frac{(n - 2l)}{R^2} C_i^{(n+1)}.
\] (5)

Thus, for the case of a negative cosmological constant the structure relations of the \( l \)--conformal Newton–Hooke algebra are given by (1) with the first line modified in accord with (5).

The situation looks similarly in spacetime with universal cosmological repulsion. First one defines \( D, K \) and \( C_i^{(n)} \)

\[
D = \frac{1}{2} R \sinh (2t/R) \partial_t + l \cosh (2t/R) x_i \partial_i,
\]

\[
K = \frac{1}{2} R^2 (\cosh (2t/R) - 1) \partial_t + l R \sinh (2t/R) x_i \partial_i
\]

\[
C_i^{(n)} = R^n (\tanh (t/R))^n (\cosh (t/R))^{2l} \partial_i,
\] (6)

and then determines the structure relations of the corresponding Newton–Hooke algebra. As compared to (1), only the first line gets altered

\[
[H, D] = H + \frac{2}{R^2} K, \quad [H, C_i^{(n)}] = nC_i^{(n-1)} - \frac{(n - 2l)}{R^2} C_i^{(n+1)}.
\] (7)

These relations specify the \( l \)--conformal Newton–Hooke algebra for the case of a positive cosmological constant.
A few comments are in order. First, in the limit of a vanishing cosmological constant (i.e. \( R \to \infty \)) the Newton–Hooke extensions reduce to the \( l \)-conformal Galilei algebra. The representations in terms of the differential operators merge as well. This is to be contrasted with the basis chosen in \([1, 2]\) where the flat space limit is problematic.

Second, viewed as formal Lie algebras, the \( l \)-conformal Galilei algebra and its Newton–Hooke counterpart are isomorphic. The linear change of the basis

\[
H \to H + \frac{1}{R^2} K
\]  

where the upper/lower sign corresponds to a negative/positive cosmological constant, yields the \( l \)-conformal Galilei algebra. In this sense Eqs. (2) and (3), (1), (6), (8) can be viewed as providing representations of the \( l \)-conformal Galilei algebra in flat space and in the Newton–Hooke spacetime, respectively. It should be remembered, however, that, as far as dynamical realizations are concerned, (8) is a change of the Hamiltonian which alters the dynamics. In other words, a relation between two dynamical systems holds only locally \([16]\) (in this respect see also \([2, 17, 18]\) \(3\)). Comparing Eqs. (2) and (3), (4), (8), one can readily construct a coordinate transformation (the prime denotes coordinates parameterizing flat space)

\[
t' = R \tan (t/R), \quad x'_i = (\cos (t/R))^{2l} x_i,
\]  

which brings (2) to (3), (4), (8) (see also a related discussion in \([5]\) \(4\)). The instance of \( l = 1/2 \) reproduces Niederer’s oscillator coordinates \([16]\). Similar formulae hold also for the case of a positive cosmological constant. As a byproduct, dynamical realizations of the \( l \)-conformal Newton–Hooke algebra can be derived from those of the \( l \)-conformal Galilei algebra by redefining the Hamiltonian \( H \to H \pm \frac{1}{R^2} K \), the latter term typically providing the harmonic potential.

Third, the \( l \)-conformal Galilei algebra admits the infinite–dimensional Virasoro–Kac–Moody type extension \([6, 7]\) (for earlier studies see \([4, 20]\) \(4\)). It can be realized in terms of the operators

\[
L^{(n)} = t^{n+1} \partial_t + l(n+1)t^n x_i \partial_i, \quad C_i^{(n)} = t^n \partial_i, \quad M^{(n)}_{ij} = t^n (x_i \partial_j - x_j \partial_i),
\]  

where \( n \) is an arbitrary integer. The structure relations read

\[
\begin{align*}
[L^{(n)} , L^{(m)}] &= (m-n)L^{(n+m)}, \\
[L^{(n)} , C_i^{(m)}] &= [m-l(n+1)]C_i^{(n+m)}, \\
[L^{(n)} , M_{ij}^{(m)}] &= mM_{ij}^{(n+m)}, \\
[M_{ij}^{(n)} , C_k^{(m)}] &= -\delta_{ik}C_j^{(n+m)} + \delta_{jk}C_i^{(n+m)}, \\
[M_{ij}^{(n)} , M_{kl}^{(m)}] &= -\delta_{ik}M_{jl}^{(n+m)} - \delta_{jl}M_{ik}^{(n+m)} + \delta_{il}M_{jk}^{(n+m)} + \delta_{jk}M_{il}^{(n+m)}.
\end{align*}
\]  

In particular, \( L^{(-1)} , L^{(0)} , L^{(1)} \) reproduce \( H, D, K \) in (2). A representation of this algebra in the Newton–Hooke spacetime can be constructed by analogy with the finite–dimensional
subalgebra considered above. For a negative cosmological constant one finds
\[ L^{(n)} = R^n (\tan (t/R))^n \left( \frac{1}{2} R \sin (2t/R) \partial_t + l[n + \cos (2t/R)]x_i \partial_i \right), \]
\[ C^{(n)}_i = R^n (\tan (t/R))^n (\cos (t/R))^{2l} \partial_i, \quad M^{(n)}_{ij} = R^n (\tan (t/R))^n (x_i \partial_j - x_j \partial_i), \]
while the change of the trigonometric functions by the hyperbolic ones yields a representation in spacetime with a positive cosmological constant.

Above we ignored the issue of central charges. Let us discuss the matter for the finite-dimensional algebra. In arbitrary dimension the commutator of two vector generators can be modified to include the central element
\[ [C^{(n)}_i, C^{(m)}_j] = \alpha(n, m) \delta_{ij}, \tag{13} \]
where the constants \( \alpha(n, m) = -\alpha(m, n) \) are subject to the constraints which follow from the Jacobi identities
\[ (n + m - 2l)\alpha(n, m) = 0, \quad m\alpha(n, m - 1) + n\alpha(n - 1, m) = 0. \tag{14} \]
The leftmost restriction implies that \( \alpha(n, m) \) vanishes unless \( n + m = 2l \). The rightmost constraint gives linear relations each of which intertwines two neighbors in the chain of \( \alpha(n, m) \)-coefficients with \( n + m = 2l \). For integer \( l \) the chain involves \( \alpha(l, l) = 0 \) which causes all \( \alpha(n, m) \) to vanish. For half integer \( l \) the recurrence relation in (14) yields
\[ \alpha(n, m) = (-1)^n \frac{n!m!}{(2l)!} \alpha(0, 2l), \tag{15} \]
with \( \alpha(0, 2l) \) being arbitrary. Thus, in arbitrary dimension there is one central charge in the algebra.

In \( (2 + 1) \)-dimensions an extra contribution to the right hand side of (13) is admissible
\[ [C^{(n)}_i, C^{(m)}_j] = \alpha(n, m) \delta_{ij} + \beta(n, m) \epsilon_{ij}, \tag{16} \]
where \( \beta(n, m) = \beta(m, n) \) and \( \epsilon_{ij} \) is the Levi-Civita symbol. The Jacobi identities yield restrictions which read as in (14) both for \( \alpha(n, m) \) and \( \beta(n, m) \). In particular, \( \alpha(n, m) \) and \( \beta(n, m) \) vanish when \( n + m \neq 2l \). For half integer \( l \) one can choose \( n = m = \frac{(2l+1)}{2} \) which, in view of (14), gives \( \beta\left(\frac{(2l+1)}{2}, \frac{(2l-1)}{2}\right) = 0 \). Because all members in the \( \beta(n, m) \)-chain are linearly related to each other, this makes them all vanish. At the same time, according to our analysis above, the coefficients \( \alpha(n, m) \) are active in this case. For integer \( l \) the recurrence relation for \( \beta(n, m) \) yields
\[ \beta(n, m) = (-1)^n \frac{n!m!}{(2l)!} \beta(0, 2l), \tag{17} \]
while \( \alpha(n, m) \) vanish. Thus, in \( (2 + 1) \)-dimensions a central extension is feasible for any value of \( l \), \( \alpha(n, m) \) and \( \beta(n, m) \) being responsible for half integer and integer \( l \), respectively.
As is well known, the Newton–Hooke extension of the ordinary Galilei algebra in \((3 + 1)\)-dimensions can be derived from the anti de Sitter algebra \(so(2, 3)\) (the de Sitter algebra \(so(1, 4)\)) by applying the non–relativistic limit [12]. It is interesting to see if a similar relation can be revealed for the conformal counterpart of the Newton–Hooke algebra. We conclude this work by discussing Inönü–Wigner contraction of \(so(2, 4)\) which leads to the \(l = 1\) conformal Newton–Hooke algebra for the case of a negative cosmological constant.

Consider the structure relations of \(so(2, 4)\)

\[ [M_{AB}, M_{CD}] = \eta_{AC}M_{BD} + \eta_{BD}M_{AC} - \eta_{AD}M_{BC} - \eta_{BC}M_{AD}, \quad (18) \]

where \(A = 0, \ldots, 5\) and \(\eta_{AB} = \text{diag}(-, +, +, +, +, -)\). Let us treat the values 0, 4, 5 and \(i = 1, 2, 3\) separately, and change the basis

\[
H = \frac{2}{R} M_{05}, \quad D = M_{04}, \quad K = R(M_{54} + M_{05}), \quad \tilde{M}_{ij} = -M_{ij},
\]

\[
C^{(0)}_i = \frac{1}{cR} (M_{i0} + M_{i4}), \quad C^{(1)}_i = \frac{1}{c} M_{i5}, \quad C^{(2)}_i = \frac{R}{c} (M_{i4} - M_{i0}), \quad (19)
\]

where \(c\) is the speed of light and \(R\) is the characteristic time. It is straightforward to compute brackets among the new generators and verify that in the limit \(c \to \infty\) they yield the structure relations of the \(l = 1\) conformal Newton–Hooke algebra. A similar contraction of \(so(2, 4)\) to the \(l = 1\) conformal Galilei algebra was discussed in [21]. The case of a positive cosmological constant can be treated likewise.

To summarize, in this work the \(l\)–conformal Newton–Hooke algebra was formulated in the basis in which the flat space limit is unambiguous. Admissible central charges in the algebra were determined. The infinite–dimensional Virasoro–Kac–Moody type extension was given.

It would be interesting to generalize the analysis of the central charges to the case of the infinite–dimensional algebra. Supersymmetric extensions as well as their dynamical realizations are also worth studying.

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