HEAVY-TAILED SAMPLING VIA TRANSFORMED UNADJUSTED LANGEVIN ALGORITHM

YE HE, KRISHNAKUMAR BALASUBRAMANIAN, AND MURAT A. ERDOGDU

ABSTRACT. We analyze the oracle complexity of sampling from polynomially decaying heavy-tailed target densities based on running the Unadjusted Langevin Algorithm on certain transformed versions of the target density. The specific class of closed-form transformation maps that we construct are shown to be diffeomorphisms, and are particularly suited for developing efficient diffusion-based samplers. We characterize the precise class of heavy-tailed densities for which polynomial-order oracle complexities (in dimension and inverse target accuracy) could be obtained, and provide illustrative examples. We highlight the relationship between our assumptions and functional inequalities (super and weak Poincaré inequalities) based on non-local Dirichlet forms defined via fractional Laplacian operators, used to characterize the heavy-tailed equilibrium densities of certain stable-driven stochastic differential equations.

1. INTRODUCTION

Given a potential function $f : \mathbb{R}^d \to \mathbb{R}$, we consider the problem of sampling from the density

$$
\pi(x) := Z^{-1} e^{-f(x)},
$$

where $Z := \int e^{-f(x)} dx$ is an (unknown) normalization constant. A general strategy to sample from densities of the form in (1) is to discretize a diffusion equation that has $\pi$ as its stationary density. In particular, the over-damped Langevin diffusion described by the Stochastic Differential Equation (SDE)

$$
dX_t = -\nabla f(X_t) dt + \sqrt{2} dW_t,
$$

where $W_t$ is a $d$-dimensional Brownian motion, has attracted considerable attention in the past decade. Under mild regularity conditions, the diffusion in (2) has $\pi$ as its stationary density, which provides the motivation for the above approach. In this work, we are interested in the case when the density $\pi$ has heavy-tails (for example, tails that are polynomially decaying). Sampling from such heavy-tailed densities arise in various applications including robust statistics [KN04, JR07, Kam18], multiple comparison procedures [GBH04, GB09] and statistical machine learning [NŠR19, ŠZTG20].

When the target density $\pi$ is heavy-tailed, the solution to (1) is not exponentially ergodic, that is, the solution does not converge to the stationary density rapidly. Indeed [RT96, Theorem 2.4] shows that if $|\nabla f(x)| \to 0$ when $|x| \to \infty$, then the solution to (2) is not exponentially ergodic. In the other direction, standard results in the literature, for example [Wan06, BGL14] show that the solution to (2) being exponentially ergodic is equivalent to the density $\pi$ satisfying the Poincaré inequality, which requires $\pi$ to have exponentially decaying tails. Furthermore, [Wan06, Chapter 4] shows that when $\pi$ has polynomially decaying tails, the convergence is only sub-exponential or polynomial.

Turning to time-discretizations of (2), the Euler discretization or the Unadjusted Langevin Algorithm (ULA) is given by

$$
x_{n+1} = x_n - \gamma \nabla f(x_n) + \sqrt{2\gamma} u_{n+1},
$$

where $(u_n)$ is a sequence of independent and identically distributed $d$-dimensional standard Gaussian vectors and $\gamma > 0$ is a user-defined step size parameter. Over the past decade, non-asymptotic
oracle complexity analysis of ULA (and other related discretizations) have been studied intensively. We refer to [Dal17, DM17, DK19, DMM19, LST20, DCWY19, SL19, HBE20, CDWY20, CLA+21, WSC21] for the case when the potential $f$ is strongly convex, [DMM19, DKRD19, CDWY20, Leh21] when it is convex, and [CCAY+18, MCJ+19, MMS20] when it is non-convex. We also highlight the works of [VW19], [EH21], [Ngu21] and [CEL+21] which analyzed ULA when $\pi$ satisfies certain functional inequalities. Specifically, [VW19] showed that when $\pi$ satisfies a Logarithmic Sobolev Inequality (LSI) and has Lipschitz-smooth gradients, ULA with a number of iterations of order $\tilde{O}(1/\epsilon^c)$ generates a sample which is $\epsilon$-close to $\pi$ with respect to KL-divergence. A necessary condition for $\pi$ to satisfy the LSI condition is that it should have sub-Gaussian tails. Furthermore, [EH21] considered densities that satisfy a modified LSI (m-LSI) inequality and showed that the number of iterations becomes of order $\tilde{O}(1/\epsilon^c)$, for some $c \geq 1$ (which depends on certain smoothness conditions). A typical example of a density that satisfies a m-LSI condition but not the LSI condition is $\pi(x) \propto \exp(-|x|)$.

Thus, the result in [EH21] could also be viewed as an oracle complexity result for ULA when sampling from sub-exponential densities. Recently [Ngu21] relaxed the conditions required in [EH21] and provided similar results under the assumption that the target density satisfies a Poincaré inequality and dissipativity at the same time. Furthermore, [CEL+21] also presented an analysis of ULA under the so-called Łatusza-Oleszkiewicz [LO00] inequality, that interpolates between the LSI and Poincaré inequality for the stronger Rényi metric and removes the dissipativity assumptions required in [EH21, Ngu21]. It is worth pointing out here that the proofs of [EH21] and [CEL+21] are based on certain transformations of the target densities.

The above results, however, are not applicable to sampling from polynomially decaying heavy-tailed densities like the multivariate $t$-distribution, whose density is of the form $\pi(x) \propto (1 + |x|^2)^{-d+\kappa}$, where $\kappa > 0$ is the degrees-of-freedom parameter. Recently, some attempts have been made to sample from such heavy-tailed densities by considering stable-driven SDEs of the form

$$dX_t = b(X_t)dt + \sqrt{2}dZ_t$$

(4)

where $b$ is the drift term defined based on the Riesz potential, and $Z_t$ is an $\vartheta$-stable process with $\vartheta \in (1,2)$ [SZTG20, Sim17, HMW21]. Specifically [HMW21] established exponential ergodicity of the solution of (4), under conditions that allow for much heavier tails than Brownian-driven SDEs. The eventual hope is that discretizations of (4) might lead to algorithms with provable non-asymptotic oracle complexity rates. However, it appears to be non-trivial to analyze discretization of (4), especially if we are interested in tight non-asymptotic results, due to the difficulties in dealing with the non-smoothness of drift term $b$.

In this work, we take an alternate approach for heavy-tailed sampling using ULA on a transformed version of the target density. Such an approach was used by [JG12] in the context of Metropolis Random Walk algorithm, which serves as our motivation. The key idea in this approach is to construct smooth invertible maps (also called diffeomorphisms) $h : \mathbb{R}^d \to \mathbb{R}^d$ that transform the heavy-tailed density $\pi$ to an appropriately light-tailed density $\pi_h$. Given such a map, one could first sample from the light-tailed density $\pi_h$ and subsequently obtain samples from the heavy-tailed density $\pi$ using the inverse map of $h$. It is also worth highlighting that [DBCD19, DGM20] and [BRZ19] used the transformation approach for proving asymptotic exponential ergodicity of bouncy particle and zig-zag samplers respectively, in the heavy-tailed setting.

There are several issues to overcome when using the above strategy in the context of ULA. First, note that the constructed map $h$ has to convert the heavy-tailed density $\pi$ to a light-tailed density $\pi_h$. In this process, however, the bulk of the density $\pi_h$ might become non-smooth, if the map is not constructed carefully. This non-smoothness could subsequently hinder the usage of ULA algorithm to sample from $\pi_h$. Second, the constants involved (for example, the LSI or m-LSI constant) in the light-tailed density $\pi_h$ might start to depend exponentially on the dimension after transformation. This again hinders the efficiency of the ULA when sampling from $\pi_h$. Furthermore, the transformation map needs to be efficiently computable. In this work, we propose a family of
carefully constructed transformations that overcome the above issues and present non-asymptotic results for sampling from a class of heavy-tailed densities.

1.1. Other Related Approaches. We briefly also highlight two other potential approaches for sampling from heavy-tailed densities. Instead of considering isotropic Langevin diffusions of the form in (2), one could also consider general classes of Itô diffusions still driven by Brownian motion. However, it is not clear how to specify the drift and diffusion terms of the Itô diffusion for a given target density. In this context, the proposed transformations could also be interpreted as providing a principled way of constructing the drift and diffusion terms; we elaborate more on this in Section 3.3. Relatively few works exist on analyzing discretization of general Itô diffusions; see, for example, [EMS18, LWME19]. Furthermore, the conditions on the allowed class of Itô diffusions and their corresponding invariant measures in [EMS18, LWME19] need to imply contraction in Wasserstein metric, which may not be satisfied in the polynomially decaying heavy-tailed regimes we focus on in this work.

Yet another approach is to learn a transformation that maps an easily samplable density to the heavy-tailed target density [Erb14, SBM18, PMM16, CD21, TIT+20, KMR21, PNR+21]. In the literature, popular approaches to learn such transformations are based on parametrization with neural networks or tensor-trains. Compared to this approach, we can provide closed form representations for the transformation and its inverse. Furthermore, to our knowledge non-asymptotic oracle complexity results for learned transformations are lacking even in the light-tailed case.

1.2. Organization. The rest of the paper is organized as follows. In Section 2 we introduce the notation and preliminary background material used in the rest of the paper. In Section 3, we introduce our transformation map, highlight key properties and present the Transformed Unadjusted Langevin Algorithm (TULA) algorithm. We also discuss a warm-up example regarding exponentially tailed densities, and provide an interpretation of the transformed diffusion as a special case of Langevin Algorithm (TULA) algorithm. We also discuss a warm-up example regarding exponen-
tially tailed densities, and provide an interpretation of the transformed diffusion as a special case of

2. Preliminaries

Notation: For a vector $a \in \mathbb{R}^d$, we represent the Euclidean norm by $|a|$. For a mapping $h : \mathbb{R}^d \to \mathbb{R}^d$, we denote the Jacobian matrix by $\nabla h \in \mathbb{R}^{d \times d}$. In the case when $h : \mathbb{R}^d \to \mathbb{R}$, $\nabla h \in \mathbb{R}^d$ denotes the gradient vector and $\Delta h = \nabla \cdot \nabla h$ denotes the Laplacian. For a function $h : \mathbb{R} \to \mathbb{R}$, we simply denote its first, second and third order derivatives by $h'$, $h''$ and $h'''$ respectively. For a matrix $A$, we denote its determinant and operator norm by $\det(A)$ and $\|A\|$ respectively. For two symmetric matrices $A, B$, the relation $A \preceq B$ refers to the fact that $B - A$ is positive semi-definite. The class of function $\mathcal{C}^k(\Omega)$ refers to those functions that have $k$-times continuously differentiable derivatives on the domain $\Omega$. For a function $\phi$, $\|\phi\|_{\infty}$ refers to the sup-norm.

We also require the following definitions used in the rest of the paper. Let $\nu$ and $\mu$ be two probability densities with full support on $\mathbb{R}^d$. Then, for a convex function $\Phi : \mathbb{R} \to \mathbb{R}$ such that $\Phi(1) = 0$, the $\Phi$-divergence of $\nu$ from $\mu$ is defined as

$$D_{\Phi}(\nu|\mu) := \int_{\mathbb{R}^d} \Phi \left( \frac{\nu(x)}{\mu(x)} \right) \mu(x) dx.$$  

When the function is given by $\Phi(t) = t \log(t)$, we obtain the Kullback-Leibler (KL) divergence of $\nu$ with respect to $\mu$, given by

$$H_\mu(\nu) := \int_{\mathbb{R}^d} \log \left( \frac{\nu(x)}{\mu(x)} \right) \nu(x) dx.$$
Our complexity results later will be provided in terms of KL-divergence. The *Relative Fisher Information* of $\nu$ with respect to $\mu$ is given by

$$I_\mu(\nu) := \int_{\mathbb{R}^d} \left| \nabla \log \frac{\nu(x)}{\mu(x)} \right|^2 \nu(x) dx.$$ 

The Rényi divergence of order $q > 1$ is defined as

$$R_q(\nu|\mu) = \frac{1}{q-1} \log \left( \int_{\mathbb{R}^d} \left( \frac{\nu(x)}{\mu(x)} \right)^q \mu(x) dx \right).$$

Note that when $q \to 1^+$, we have $R_q(\nu|\mu)$ approaching $H_\mu(\nu)$.

We now introduce additional technical details required for discussing functional inequalities; rigorous expositions could be found in [Wan06, BGL14]. Let $(\Omega, \mathcal{F}, \mu)$ be a probability space and let $\mathcal{L}$ denote a linear operator (infinitesimal generator) that is self-adjoint with domain $D(\mathcal{L})$ which generates a Markov semi-group $P_t$ on $L^2(\mu)$. The carré de champ operator associated to the infinitesimal generator $\mathcal{L}$ is given by the bilinear map $\Gamma(\phi_1, \phi_2) = 1/2 [\mathcal{L}(\phi_1 \phi_2) - \phi_1 \mathcal{L} \phi_2 - \phi_2 \mathcal{L} \phi_1]$, for all $\phi_1, \phi_2$ defined in a subspace of $D(\mathcal{L})$ which is an algebra. We call the collection of the measure $\mu$ on a state space $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ and a carré de champ operator $\Gamma$ a Markov triple, denoted as $(\mathbb{R}^d, \mu, \Gamma)$. It is well-known that the Dirichlet form associated with a Markov semi-group $P_t$ is then given by $\mathcal{E}(\phi_1, \phi_2) = \int \Gamma(\phi_1, \phi_2) d\mu$. By a standard integration-by-parts argument, we also have that $\mathcal{E}(\phi_1, \phi_2) = -\int \phi_1 \mathcal{L} \phi_2$. We use the convention $\mathcal{E}(\phi)$ to denote $\mathcal{E}(\phi, \phi)$. The Dirichlet domain $D(\mathcal{E})$ is defined as $D(\mathcal{E}) := \{ \phi \in L^2(\mu) : \mathcal{E}(\phi) < \infty \}$.

In the case of Brownian driven diffusions as in (2), the generator $\mathcal{L}$ is defined based on the Laplacian operator $\triangle$, which is a local operator. Correspondingly, the Dirichlet form is given by $\mathcal{E}(\phi) = \int |\nabla \phi(x)|^2 \mu(x) dx$, for all $\phi \in D(\mathcal{E})$. Based on this, we will introduce functional inequalities below. A probability density $\mu$ is said to satisfy *Poincaré Inequality (PI)* with constant $C_\mathcal{P}$, denoted as $\mu \sim P(C_\mathcal{P})$ if for all functions $\phi \in D(\mathcal{E})$, we have

$$\text{Var}_\mu(\phi) := \int_{\mathbb{R}^d} \left( \phi^2(x) - \left( \int_{\mathbb{R}^d} \phi(x) \mu(x) dx \right) \right)^2 \mu(x) dx \leq C_\mathcal{P} \int_{\mathbb{R}^d} |\nabla \phi(x)|^2 \mu(x) dx = C_\mathcal{P} \mathcal{E}(\phi). \quad \text{(PI)}$$

Similarly, a probability density $\mu$ satisfies a *Logarithmic Sobolev inequality (LSI)* with constant $C_{\text{LSI}}$ denoted as $\mu \sim LS(C_{\text{LSI}})$ if for all functions $\phi \in D(\mathcal{E})$, we have

$$\text{Ent}_\mu(f) := \int_{\mathbb{R}^d} f^2(x) \log \left( \frac{f^2(x)}{\int_{\mathbb{R}^d} f^2(x) \mu(x) dx} \right) \mu(x) dx \leq 2C_{\text{LSI}} \int_{\mathbb{R}^d} |\nabla f(x)|^2 \mu(x) dx = 2C_{\text{LSI}} \mathcal{E}(\phi). \quad \text{(LSI)}$$

An equivalent form of LSI is that for all probability densities $\rho(x)$, we have

$$H_\mu(\rho) \leq \frac{C_{\text{LSI}}}{2} I_\mu(\rho).$$

We refer the reader to [BGL14, Chapter 5] for the derivation of the equivalence. A probability density $\mu(x)$ satisfies a *modified Log-Sobolev Inequality (m-LSI)* if for all probability measure $\rho(x)$ and all $s \geq 2$, there is $\delta \in [0, 1/2)$ (depending on $s$) such that

$$H_\mu(\rho) \leq C_{\text{m-LSI}} I_\mu(\rho)^{1-\delta} M_s(\rho + \mu)^{\delta}. \quad \text{(m-LSI)}$$

where $M_s(\rho) = \int_{\mathbb{R}^d} (1 + |x|^2)^{s/2} \rho(x) dx$. This version of m-LSI was introduced by [EH21] (also see [CEL+21]), motivated by a related definition from [TV00]. It is important to notice that the above version of m-LSI does not contain the Poincaré inequality as a special case, i.e., there exists densities that satisfy the above m-LSI inequality but not Poincaré inequality and vice versa. There exists other modifications to the LSI including the Becker or Nash inequality [BGL14, Chapter 7] and the Latała-Oleszkiewicz [LO00] refinement to it, that interpolate between the LSI and Poincaré inequalities.
The above discussion is focused on Brownian driven SDEs. It turns out that the above class of functional inequalities are suitable for characterizing light-tailed densities (i.e., tails that decay exponentially fast). In the case of \( \vartheta \)-stable driven diffusions as in (4), the generator is defined based on the non-local fractional Laplacian operator \((-\Delta)^{-\vartheta/2}\); see, for example, [Kwa17]. Correspondingly, in Section 5, we present more general functional inequalities based on non-local Dirichlet forms that are suitable for characterizing heavy-tailed densities and discuss the connection between our assumptions and such functional inequalities.

3. The Transformed Unadjusted Langevin Algorithm

3.1. Transformation Map. We start this section by stating the following important property satisfied by smooth invertible transformation maps \( h : \mathbb{R}^d \to \mathbb{R}^d \).

**Definition (Transformed density functions).** For a probability density \( \mu(x) \) with full support in \( \mathbb{R}^d \), its transformed density function under a smooth invertible transformation map (or a diffeomorphism) \( h \) is given by \( \mu_h(x) = \mu(h(x)) \det(\nabla h(x)) \) for all \( x \in \mathbb{R}^d \).

If a random vector \( X \) has density \( \mu \), then we denote the density of the random vector \( Y = h^{-1}(X) \), denoted as \( \mu_h \), as the transformed density of \( \mu \). Note that in particular if \( X \) admits density \( \pi \) of the form in (1), then \( Y = h^{-1}(X) \) is distributed with density

\[
\pi_h(y) = Z^{-1} e^{-f_h(y)} \quad \text{with} \quad f_h(y) = f(h(y)) - \log \det(\nabla h(y)),
\]

being referred to as the transformed potential. In what follows, we assume that the potential function is isotropic. We emphasize that this assumptions is made for the sake of technical convenience – it is possible to relax this assumption to certain mild regularity conditions on the density, at the expense of having a more cluttered exposition.

**Assumption A0.** The initial potential function \( f \) is isotropic, i.e \( f(x) = f(|x|) \) and \( f : \mathbb{R} \to \mathbb{R} \) is twice continuously differentiable.

Since \( f \) is isotropic under assumption Assumption A0, we may consider \( f \) to be a function defined on \( \mathbb{R}_+ \) as well. In the later context, we use \( f(|x|) \) when we consider \( f \) defined on \( \mathbb{R}_+ \) and we use \( f(x) \) when it is defined on \( \mathbb{R}^d \). Similarly, when we use \( f'(|x|), f''(|x|) \) and so on, to represent the derivatives, we consider \( f \) to be a function defined on \( \mathbb{R}_+ \).

We now describe the construction of our specific transformation map. Our proposal is motivated by the work of [JG12], who constructed similar maps to show exponential ergodicity of the Metropolis Random Walk (MRW) algorithm. It turns out that a direct application of their construction to analyze Langevin diffusions and their discretization, leads to worse dimensionality dependencies in the non-asymptotic oracle complexities. Indeed, this is expected as [JG12] predominantly focused on establishing asymptotic results. In order to proceed, we first define functions \( g : \mathbb{R} \to \mathbb{R} \) which correspond to the first part of the transformation map construction. Specifically, \( g \) is defined based on initial function \( g_{\text{in}} \) as

\[
g(r) = \begin{cases} 
g_{\text{in}}(r), & r < b^{-\frac{1}{\beta}}, \\
e^{br^{a}} & r \geq b^{-\frac{1}{\beta}}. \end{cases}
\]

where \( \beta \in (1, 2] \). The initial function \( g_{\text{in}} : [0, b^{-\frac{1}{\beta}}) \to [0, e) \) satisfies the following assumption.
**Assumption G1.** The initial function $g_{in} : [0, b^{-1/3}) \to [0, e)$ is onto, monotone increasing and twice continuously differentiable. Furthermore, it satisfies,

\[
\begin{align*}
g_{in}(0) &= 0 \\
\lim_{r \to b^{-1/3}} g_{in}(r) &= e, \\
\lim_{r \to b^{-1/3}} g_{in}'(r) &= \beta b^{1/3} e, \\
\lim_{r \to b^{-1/3}} g_{in}''(r) &= (2\beta^2 - \beta)b^{2/3} e, \\
\lim_{r \to b^{-1/3}} g_{in}'''(r) &= (5\beta^3 - 6\beta^2 + 2\beta)b^{3/3} e,
\end{align*}
\]

\[
\begin{align*}
\lim_{r \to 0^+} \left| \frac{f'(g_{in}(r))g_{in}'(r)}{r} \right| &= < \infty, \\
\lim_{r \to 0^+} \left| \frac{d}{dr} \log g_{in}'(r) \right| &= < \infty, \\
\lim_{r \to 0^+} \left| \frac{d^2}{dr^2} \log g_{in}'(r) \right| &= < \infty, \\
\lim_{r \to 0^+} \left| \frac{d^3}{dr^3} \log g_{in}'(r) \right| &= < \infty.
\end{align*}
\]

We now show that if $g_{in}$ satisfies Assumption G1, then $g$ is three times continuously differentiable and invertible on $\mathbb{R}$.

**Lemma 1.** For the function $g$ defined in (6), if $g_{in}$ satisfies Assumption G1, then we have

1. $g \in C^3((0, \infty))$,
2. $g$ is onto, strictly monotonically increasing, and hence invertible.

The proof of Lemma 1 is provided in Section 7.1. We now show that under Assumption G1, the $\Phi$-divergence is preserved after transformation. This property is important to eventually provide our convergence results for sampling.

**Proposition 1.** Let $h : \mathbb{R}^d \to \mathbb{R}^d$ be a transformation map satisfying Assumption G1. For any two probability densities $\nu$ and $\mu$ with full support on $\mathbb{R}^d$, let $\nu_h$ and $\mu_h$ be the two transformed densities under the map $h$. Then the $\Phi$-divergence is preserved after transformation, i.e., we have

\[
D_{\Phi}(\nu | \mu) = D_{\Phi}(\nu_h | \mu_h).
\]

**Proof:** We start from the right side of (7):

\[
D_{\Phi}(\nu_h | \mu_h) = \int_{\mathbb{R}^d} \Phi \left( \frac{\nu_h(y)}{\mu_h(y)} \right) \mu_h(y) dy = \int_{\mathbb{R}^d} \Phi \left( \frac{\nu(h(y)) \det(\nabla h(y))}{\mu(h(y)) \det(\nabla h(y))} \right) \mu(h(y)) \det(\nabla h(y)) dy
\]

\[
= \int_{\mathbb{R}^d} \Phi \left( \frac{\nu(x)}{\mu(x)} \right) \mu(x) dx = D_{\Phi}(\nu | \mu).
\]

The second identity follows by the change of variable $x = h(y)$ and noting $\det(\nabla h(y)) > 0$ under Assumption G1.

With the properties of $g$ introduced in Lemma 1, we can then further define the isotropic transformations $h : \mathbb{R}^d \to \mathbb{R}^d$:

\[
h(x) = \begin{cases} 
\frac{g(|x|)x}{|x|} & x \neq 0, \\
0 & x = 0.
\end{cases}
\]

We call the map $y \mapsto x = h(y)$ to be the transformation map, which is isotropic. Furthermore, $h$ is also three times continuously differentiable and invertible on $\mathbb{R}^d$ and its inverse is

\[
h^{-1}(x) = \begin{cases} 
\frac{g^{-1}(|x|)x}{|x|} & x \neq 0, \\
0 & x = 0.
\end{cases}
\]

Therefore, we can define the inverse transformation map $x \mapsto y = h^{-1}(x)$. 

is a constant, there is a bias between

\( TULA \) in order to generate samples from a heavy-tailed density defined in (5).

\[
\text{We denote the density of } \mathbb{R}^d_+ \text{ where } (\gamma > 0) \text{ and does not satisfy LSI, by transforming it to satisfy LSI. Towards that goal, we consider the transformation map in (7) for the definition of the potential function).}
\]

Transformed Langevin Diffusion and its discretization. With the transformed density defined in (5), the transformed overdamped Langevin diffusion is given by

\[
dY_t = -\nabla f_h(Y_t)dt + \sqrt{2}\,dW_t.
\]

We denote the density of \( Y_t \) by \( \rho_t \) for all \( t \geq 0 \). The stationary density function for the diffusion given by (9) is \( \pi_h \) as defined in (5). We can apply Euler discretization to the transformed overdamped Langevin diffusion in (9) and generate a Markov chain \((y_n)_{n\geq1}\) via the recursion,

\[
y_{n+1} = y_n - \gamma_{n+1}\nabla f_h(y_n) + \sqrt{2}\gamma_{n+1}u_{n+1}
\]

where \((u_n)\) is a sequence of independent and identically distributed \( d \)-dimensional standard Gaussian vectors and \( \gamma > 0 \) is the fixed step size. The Transformed Unadjusted Langevin algorithm (TULA) in order to generate samples from a heavy-tailed density \( \pi \) is given in Algorithm 1.

We use \( \nu_n \) to denote the density of the \( n \)th iterate \( x_n \) and \( \pi_\gamma \) to denote the stationary density of \((x_n)_{n\geq1}\). Since the step-size \( \gamma \) in Algorithm 1 is a constant, there is a bias between \( \nu_n \) and \( \pi \). For arbitrary accuracy \( \epsilon > 0 \), by choosing small enough step-size \( \gamma \) and large enough number of iterations \( n \), we can bound the distance between \( \nu_n \) and \( \pi \) by \( \epsilon \) in terms of KL or Rényi divergence.

3.2.1. A Warm-up Example. Although our main motivation is to sample from densities that have polynomially decaying tails, in this subsection, we provide a warm-up example on sampling from a density that has exponentially decaying tails (see (12) for the definition of the potential function) and does not satisfy LSI, by transforming it to satisfy LSI. Towards that goal, we consider the transformation map in (8) with the function \( g \) defined as

\[
g(r) = \begin{cases} dr^2, & r \geq R, \\ g_{in}(r), & 0 \leq r \leq R, \end{cases}
\]

where \( R > 0 \) is a constant, with

\[
g_{in}(r) = dR r \exp \left(-\frac{5}{6} + \frac{3}{2R^2} - \frac{2}{3R^3} \right).
\]

The above form for \( g \) is motivated by [JG12, Equation 15], where they constructed transformation maps to transform densities that are sub-exponential to sub-Gaussian. We also point out that we consider the form of \( g \) in (11) only for this section, and it should not be confused with the general form (6) considered in the rest of the paper. By an argument similar to the proof of Lemma 1, it could be shown that the transformation map defined with \( g \) as in (11) is a diffeomorphism.

Now, consider the potential function defined in a piece-wise manner as

\[
f(x) = \begin{cases} (1 + |x|^2)^{\frac{1}{2}} + \frac{1}{2}d \log |x|, & |x| \geq R, \\ (1 + d^2 g^{-1}_{in}(|x|)^4)^{\frac{1}{2}} + (d - 1) \ln \frac{|x|}{g^{-1}_{in}(|x|)} + \log g'_{in}(g^{-1}_{in}(|x|)) - \frac{d}{2} \log d - \ln 2, & |x| \in [0, R]. \end{cases}
\]

\[ (12) \]
The corresponding probability density induced by the potential $f$ above has a lighter tail than the one with potential $|x|$. But it has a heavier tail than densities with potentials $|x|^\varrho$ for any $\varrho > 1$. For the above potential $f$, the transformed potential is given by

$$f_h(x) = (1 + d^2|x|^4)^{\frac{1}{2}} - \frac{d}{2} \log d - \log 2.$$  

The LSI constant of the density induced by $f_h$ can be studied via the Holley-Stroock Theorem (see Theorem 6). We can write

$$f_h(x) = d|x|^2 + \frac{1}{d|x|^2 + (1 + d^2|x|^4)^{\frac{1}{2}}} - \frac{d}{2} \log d - \log 2$$

$$:= d|x|^2 - \frac{d}{2} \log d - \log 2 + \text{Osc}(x),$$

where $\text{Osc}(x) := \frac{1}{d|x|^2 + (1 + d^2|x|^4)^{\frac{1}{2}}}$ and is uniformly bounded by 1. Meanwhile the density corresponding to the potential function $e^{-d|x|^2} + \frac{d}{2} \log d + \log 2$ satisfies LSI with constant $1/2d$. Therefore $e^{-f_h}$ satisfies LSI with constant $C_{h, \text{LSI}} = \epsilon/2d$. On the other hand, $f_h(x)$ also has Lipschitz gradients with constant $L_h = O(d)$. Hence, according to [VW19] and Proposition 1, the iteration complexity of TULA for sampling from a density with potential $f$ as in (12) is of order $O(d/\epsilon)$ where $\epsilon$ is the error tolerance in KL-divergence. This is to be contrasted with [CEL+21, Examples 9 and 11] on using ULA to sample from densities with potentials of the from $|x|^\varrho$ for $\varrho \in [1, 2]$. Specifically, we note that TULA has better oracle complexity as long as $\varrho \in (1, 2]$. 

### 3.3. Transformed Langevin Diffusions as Itô Diffusions

It is worth noting that the transformed diffusion process in (9) could also be interpreted in terms of an Itô diffusion. Specifically, by a direct calculation, the stochastic process $X_t = h(Y_t)$ has the form

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t, \quad (13)$$

with $\sigma(x) := \sqrt{2}(\nabla h)(h^{-1}(x))$ and

$$b(x) := -\langle \nabla h^T(h^{-1}(x))\nabla h, h^{-1}(x) \rangle \nabla f(x) + (\nabla h^T(h^{-1}(x))\nabla \log \det \nabla h)(h^{-1}(x))$$

$$+ (\Delta \cdot h)(h^{-1}(x)), $$

where $\Delta \cdot h(\cdot) \in \mathbb{R}^d$ and is defined co-ordinate wise as $(\Delta \cdot h(x))_i = \Delta h_i(x)$ for all $i \in \{1, \cdots, d\}$ and $x \in \mathbb{R}^d$. Furthermore, we can actually show that

$$b(x) = \frac{1}{2\pi(x)} \langle \nabla, \pi(x)\sigma(x)T \sigma(x) \rangle,$$

where $\langle \nabla, \cdot \rangle$ is the divergence operator for matrix-valued function, i.e $\langle \nabla, \omega(x) \rangle_i = \sum_{j=1}^d \frac{\partial \omega_{ij}(x)}{\partial x_j}$ for $\omega : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$. 

The above form of $b(x)$ follows by noting that from the form of $\pi(x)$ in (1), we have

$$\frac{1}{2\pi(x)} \langle \nabla, \pi(x)\sigma(x)T \sigma(x) \rangle = \frac{1}{2} \nabla \sigma^T(x)\sigma(x) - \frac{1}{2} \sigma^T(x)\sigma(x) \nabla f(x)$$

$$= -\langle \nabla h^T(h^{-1}(x))\nabla h, h^{-1}(x) \rangle \nabla f(x) + \frac{1}{2} \langle \nabla, \sigma^{T}(x)\sigma(x) \rangle.$$ 

Meanwhile from (8), based on elementary algebraic manipulations, we obtain that

$$\frac{1}{2} \langle \nabla, \sigma^{T}(x)\sigma(x) \rangle = \left[2g''(g^{-1}(|x|)) + (d - 1)g'(g^{-1}(|x|)) \frac{g''(g^{-1}(|x|))}{|x|} \right] \frac{|x|}{g^{-1}(|x|)^2},$$

$$\langle \Delta \cdot h, h^{-1}(x) \rangle = \left[2g''(g^{-1}(|x|)) + (d - 1)g'(g^{-1}(|x|)) \frac{g''(g^{-1}(|x|))}{g^{-1}(|x|)^2} \right] \frac{|x|}{g^{-1}(|x|)^2}.$$
and

\[(\nabla h^T)(h^{-1}(x))(\nabla \log \det \nabla h)(h^{-1}(x))\]

\[= \left[ g''(g^{-1}(|x|)) + (d - 1) \frac{g'(g^{-1}(|x|))^2}{|x|} - (d - 1) \frac{g'(g^{-1}(|x|))}{g^{-1}(|x|)} \right] \frac{x}{|x|}.\]

This highlights the fact that transformations provide a way of constructing the drift and diffusion terms in the Itô diffusion that take into account the heavy-tailed nature of the target density. However, it turns out that the results on the analysis of discretizations of Itô diffusion from [EMS18, LWME19], which are in the Wasserstein metric, are not applicable to the class of Itô diffusion of the form above; indeed the stronger Wasserstein contraction conditions made in those works are not satisfied by the above class of Itô diffusions. We leave a detailed investigation of analysis of discretizations of Itô diffusion above, in stronger KL or Rényi metrics, as future work.

4. Convergence Results

In this section, we will impose assumptions on the potential function \(f\) under which we show exponential ergodicity of the transformed Langevin diffusion and convergence results for Algorithm 1.

4.1. Convergence along the transformed Langevin diffusions. We first state convergence results for the continuous time Langevin diffusion under various curvature-related assumptions on the potential function.

**Assumption A1.** (Dissipativity) There exists \(A, B, N_1 > 0, \alpha \in [1, 2]\) such that for all \(|x| > N_1:\)

\[f'(\psi(|x|))\psi'(|x|)|x| - b\beta d|x|^{\beta} + (d - \beta) > A|x|^\alpha - B,
\]

where \(\psi(r) = e^{br^\beta}\) for all \(r \geq b^{-\frac{1}{\beta}}\).

Assumption A1 is imposed to guarantee that the transformed potential function satisfies the dissipativity condition. We next recall the dissipativity condition for completeness.

**Assumption B1.** (\(\alpha_h\)-dissipativity) We say that the transformed potential function \(f_h : \mathbb{R}^d \to \mathbb{R}\) satisfies the \(\alpha_h\)-dissipativity condition with \(\alpha_h \in [1, 2]\) if there exists \(A_h, B_h > 0\) such that for all \(x \in \mathbb{R}^d:\)

\[\langle \nabla f_h(x), x \rangle > A_h |x|^\alpha_h - B_h.\]

If the transformed potential function satisfies the \(\alpha_h\)-dissipativity condition with \(\alpha_h = 1\), then the corresponding transformed density \(\pi_h\) satisfies a Poincaré inequality with certain constant \(C_{h,P}\) depending on the potential function. Then, similar to [VW19], we obtain the following result.

**Theorem 1.** Assume the initial potential function \(f\) satisfies Assumption A0 and Assumption A1 with \(\alpha = 1\). Then, the transformed density \(\pi_h\) with \(\beta = 1\) and \(b \geq \frac{r}{8(d-1)}\) satisfies a Poincaré inequality with a constant \(C_{h,P}\) depending on \(f\). Therefore along the transformed Langevin diffusion (9), we have for \(q \geq 2\) that

\[R_q(\rho_t | \pi_h) \leq \begin{cases} R_q(\rho_0 | \pi_h) - \frac{2C_{h,P}t}{q} & \text{if } R_q(\rho_0 | \pi_h) \geq 1 \text{ as long as } R_q(\rho_t | \pi_h) \geq 1, \\ e^{-\frac{2C_{h,P}t}{q}} R_q(\rho_0 | \pi_h) & \text{if } R_q(\rho_0 | \pi_h) \leq 1. \end{cases}\]

**Assumption A2.** (Degenerate convexity) There exists \(\mu, N_2 > 0, \theta \geq 0\) such that for all \(|x| > N_2:\)

\[f'(\psi(|x|))\psi'(|x|)|x|^{-1} - b\beta d|x|^{\beta-2} + (d - \beta)|x|^{-2} > \frac{\mu}{(1 + \frac{1}{4}|x|^2)^{\theta/2}},\]

\[f''(\psi(|x|))\psi'(|x|)^2 + f'(\psi(|x|))\psi''(|x|) - b\beta(\beta - 1)d|x|^{\beta-2} - (d - \beta)|x|^{-2} > \frac{\mu}{(1 + \frac{1}{4}|x|^2)^{\theta/2}}.\]
where \( \psi(r) = e^{br^\beta} \) for all \( r \geq b^{-\frac{1}{\beta}} \).

Assumption A2 is imposed to guarantee the transformed potential function is degenerately convex at infinity. We now recall the definition of degenerate convexity at infinity from [EH21].

**Assumption B2.** (Degenerate convexity at infinity) We say that the transformed potential function \( f_h : \mathbb{R}^d \to \mathbb{R} \) is degenerately convex at infinity if there exist a function \( \tilde{\phi} : \mathbb{R}^d \to \mathbb{R} \) such that for a constant \( \xi_h \geq 0 \)

\[
\| f_h - \tilde{\phi} \|_\infty \leq \xi_h,
\]

where \( \tilde{f} \) satisfies,

\[
\nabla^2 \tilde{f}(x) \succeq \frac{\mu_h}{(1 + \frac{1}{4}|x|^2)^{\theta_h/2}} I_d,
\]

for some \( \mu_h > 0 \) and \( \theta_h \geq 0 \).

The degenerate convexity at infinity condition is weaker than the strong convexity at infinity. If a potential function satisfies degenerate convexity at infinity, then the corresponding probability measure satisfies m-LSI. Similar to [TV00], we obtain the following result.

**Theorem 2.** Assume the initial potential function \( f \) satisfies Assumption A0 and Assumption A2. Then the transformed density \( \pi_h \) satisfies a modified Logarithmic Sobolev Inequality with a uniform constant \( \delta \) (see (m-LSI)) and constant \( C_{h,M-LSI} \) depending on \( f \). Therefore along the transformed Langevin diffusion (9), we have

\[
H_{\pi_h}(\rho_t) \leq \frac{C}{t^{\ell}},
\]

where the constant \( C \) depends on the potential \( f \) and the transformation \( h \) and \( \ell = (1 - 2\delta)/\delta \).

**Remark 1.** Note that the above rate is faster than any polynomial but not truly exponential. While the above rate could be made exponential with additional assumptions on the tail and/or assumptions on the initial distribution, we do not present such modifications here.

**Assumption A3.** (Strong convexity at infinity) There exists \( N_3, \rho > 0 \) such that for all \( |x| > N_3 \):

\[
f'(\psi(|x|))\psi'(|x|)|x|^{-1} - b\beta d|x|^{\beta-2} + (d - \beta)|x|^{-2} > \rho,
\]

\[
f''(\psi(|x|))\psi'(|x|)^2 + f'(\psi(|x|))\psi''(|x|) - b\beta(\beta - 1)d|x|^{\beta-2} - (d - \beta)|x|^{-2} > \rho,
\]

where \( \psi(r) = e^{br^\beta} \) for all \( r \geq b^{-\frac{1}{\beta}} \).

Assumption A3 is imposed to guarantee that the transformed potential function is strongly convex with parameter \( \rho_h \) at infinity. The property that a potential function is strongly convex at infinity implies that the corresponding probability measure satisfies a LSI with a certain parameter depending on the potential function and the transformation map.

**Theorem 3.** Assume the initial potential function \( f \) satisfies Assumption A0 and Assumption A3, then the transformed density \( \pi_h \) satisfies a logarithmic Sobolev inequality with a constant \( C_{h,LSI} \) depending on \( f \). Therefore along the transformed Langevin diffusion (9), we have

\[
H_{\pi_h}(\rho_t) \leq e^{-2tC_{h,LSI}} H_{\pi_h}(\rho_0).
\]
4.2. Convergence along TULA. In this section, we state two types of convergence results for Algorithm 1, based on Proposition 1 and [VW19, CEL+21]. While the works of [VW19, CEL+21] provide results only for exponentially decaying densities, our results below are applicable for polynomially-decaying densities based on the constructed transformation maps. To proceed, we first list smoothness conditions on the potential function $f$.

**Assumption A4.** (Gradient Lipschitz) There exists $N_4, L > 0$ such that for all $|x| > N_4$:
\[
\begin{align*}
&f'(\psi(|x|))\psi'(|x|)|x|^{-1} - b\beta d|x|^{-2} + (d - \beta)|x|^{-2} < L, \\
&f''(\psi(|x|))\psi''(|x|)|x|^{2} + f'(\psi(|x|))\psi''(|x|) - b\beta(\beta - 1)d|x|^{-2} - (d - \beta)|x|^{-2} < L
\end{align*}
\]
where $\psi(r) = e^{br^\beta}$ for all $r \geq b^{-\frac{1}{\beta}}$.

Assumption A4 is imposed to guarantee that the transformed potential function has Lipschitz gradients with parameter $L_h$. Such smoothness conditions on the potential function are required to study the discrete Markov chain generated in the unadjusted Langevin algorithm. We also remark that it is possible to relax the Lipschitz gradient assumption to certain weak-smooth conditions on the gradient; we do not pursue such extensions in this work. While Theorem 2 holds under m-LSI, to get the corresponding result for Algorithm 1, we also require the following additional tail-conditions.

**Assumption A5.** (Tail assumption) For some $m \geq 0$, $\alpha_1 \in [0, 1]$ and $N_5 > 0$, there exists a positive constant $C^*_{\text{Tail}}$ such that for all $\lambda \geq N_5$,
\[
\pi \{ | \cdot | \geq m + \lambda \} \leq 2 \exp \left( - \left( \frac{\psi^{-1}(\lambda)}{C^*_{\text{Tail}}} \right)^{\alpha_1} \right),
\]
where $\psi(r) = e^{br^\beta}$ for all $r \geq b^{-\frac{1}{\beta}}$.

**Assumption B5.** For some $m_h \geq 0$ and $\alpha_{h,1} \in [0, 1]$, there exists a positive constant $C_{h,\text{TAIL}}$ such that for all $\lambda \geq 0$,
\[
\pi_h \{ | \cdot | \geq m_h + \lambda \} \leq 2 \exp \left( - \left( \frac{\lambda}{C_{h,\text{TAIL}}} \right)^{\alpha_{h,1}} \right).
\]

**Theorem 4.** In addition to the assumptions in Theorem 2, assume that the initial potential $f$ is such that $\nabla f_h(0) = 0$, and it satisfies Assumption A4 and Assumption A5. Furthermore, let $\epsilon^{-1}, m_h, C_{h,M-\text{LSI}}, C_{h,\text{TAIL}}, L_h, R_2(\rho_0|\tilde{\pi}_h) \geq 1$ ($\tilde{\pi}_h$ is as defined in (52) with $\tilde{R} = 2 \int_{\mathbb{R}^d} |x|\pi_h(x)dx$ and $\tilde{\gamma} = (3072\pi\gamma)^{-1}$), and $m_h, C_{h, \text{TAIL}}, R_2(\rho_0|\pi) \leq d^{O(1)}$. Then, Algorithm 1 with an step size
\[
\gamma = \tilde{\Theta} \left( \frac{\epsilon}{dC^{2}_{h,M-\text{LSI}}C^{\theta}_{h,\text{TAIL}}L_h^2 R_2(\nu_0|\pi)^{\theta/\alpha_{h,1}}} \times \min \left\{ 1, \frac{1}{qe^d}, \frac{d}{R_2(\rho_0|\pi)}, \left( \frac{R_2(\nu_0|\pi)^{1/\alpha_{h,1}}}{m_h} \right)^{\theta} \right\} \right),
\]
satisfies $R_q(\nu_n|\pi) \leq \epsilon$, for all $q \geq 2$ after
\[
n = \tilde{\Theta} \left( \frac{dR_2(\nu_0|\pi)^{2\theta/\alpha_{h,1}}C^{4}_{h,M-\text{LSI}}C^{\theta}_{h,\text{TAIL}}L_h^2}{\epsilon} \times \max \left\{ 1, \frac{1}{qe^d}, \frac{m_h}{R_2(\rho_0|\tilde{\pi}_h)^{1/\alpha_{h,1}}}, \left( \frac{m_h}{R_2(\nu_0|\pi)^{1/\alpha_{h,1}}} \right)^{\theta} \right\} \right)
\]
iterations, for some $\theta \in [0, 1]$ (depending on the parameter $\delta$ in (m-LSI)). Explicit form of $C_{h,M-\text{LSI}}$ is the constant $\lambda$ in (42) and $m_h, C_{h, \text{TAIL}}, L_h$ are given in (51),(50), (36) respectively. The $\tilde{\Theta}(\cdot)$ notation hides polylogarithmic factors as well as constants depending on $\theta, q$.

**Remark 2.** In order to obtain a direct quantitative bound, it is important to obtain a control of $R_2(\rho_0|\tilde{\pi}_h)$ and $R_2(\nu_0|\pi)$. We refer to [CEL+21, Section A] for a proof that the conditions required on $R_2(\rho_0|\tilde{\pi}_h)$ is satisfied, and for obtaining a control on the term $R_2(\nu_0|\pi)$. 
Theorem 5. In addition to the assumptions in Theorem 3, assume that \( f \) satisfies Assumption A4. Then Algorithm 1, for any \( y_0 \sim \rho_0 \) with \( H_{\pi_h}(\rho_0) < \infty \), and with step size

\[
0 < \gamma \leq \frac{1}{2L_h^2 C_{h,\text{LSI}}} \min \left\{ 1, \frac{\epsilon}{4d} \right\},
\]

satisfies \( H_\pi(\nu_n) < \epsilon \), for any \( \epsilon > 0 \) after

\[
n = \tilde{\Theta} \left( \frac{C_{h,\text{LSI}}}{2\gamma} \log \frac{2H_{\pi_h}(\rho_0)}{\epsilon} \right)
\]

iterations. Explicit forms of \( C_{h,\text{LSI}} \) and \( L_h \) are given in (35) and (36).

Remark 3. As argued in [VW19], if we let \( \rho_0 \) to be a Gaussian distribution with mean being any stationary point of \( f_h \) and covariance matrix being \( (1/L_h)I_d \), then \( H_{\pi_h}(\rho_0) = \tilde{O}(d) \). Furthermore, we also remark that similar convergence results in the stronger Rényi metric, for all \( q \geq 4 \) holds via [CEL+21, Theorem 4].

Remark 4. We leave a detailed study of obtaining convergence results for the underdamped Langevin dynamics and its discretization as future work.

5. Relation with Poincaré Inequalities Based on Non-local Dirichlet Forms

We now discuss the relationship between our assumptions on the potential function and functional inequalities like super and weak Poincaré inequalities that arise in characterizing the heavy-tailed stationary distributions of certain \( \vartheta \)-stable driven diffusions [RW01, RW03, CGGR10, Wan14, WW15, HMW21]. Recall from Section 2 that the Dirichlet form associated with Langevin diffusion in (2) is of the form \( \mathcal{E}(\phi) = \int |\nabla \phi(x)|^2 \mu(dx) \). However, in the case of \( \vartheta \)-stable driven diffusions the corresponding non-local Dirichlet form is given by

\[
\mathcal{E}(\phi) := \int \int_{x \neq y} \frac{(\phi(x) - \phi(y))^2}{|x - y|^{(d+\vartheta)}} d\mu(dy), \quad (14)
\]

for all functions in the Dirichlet domain \( D(\mathcal{E}) \); see for example [Wan14]. We now introduce similar functional inequalities that are associated with stable-driven diffusions.

Definition (Poincaré-type Inequalities). A Markov triple \((\mathbb{R}^d, \mu, \Gamma)\) (with \( \mu \) a probability measure), with the Dirichlet form as in (14) is said to satisfy

- a Poincaré inequality if for any function \( \phi : \mathbb{R}^d \to \mathbb{R} \) in the Dirichlet domain \( D(\mathcal{E}) \) and \( C > 0 \),
  \[
  \text{Var}_\mu(\phi) \leq C \mathcal{E}(\phi),
  \]

- a weak Poincaré inequality if for any function \( \phi : \mathbb{R}^d \to \mathbb{R} \) in the Dirichlet domain \( D(\mathcal{E}) \) and \( r > 0 \),
  \[
  \text{Var}_\mu(\phi) \leq \alpha(r) \mathcal{E}(\phi) + r \|\phi\|_{\infty}^2,
  \]

- a super Poincaré inequality if for any function \( \phi : \mathbb{R}^d \to \mathbb{R} \) in the Dirichlet domain \( D(\mathcal{E}) \) and \( r > 0 \),
  \[
  \mu(\phi^2) \leq r \mathcal{E}(\phi) + \beta(r) \mu(|\phi|)^2,
  \]

where \( \mu(\varphi) = \int_{\mathbb{R}^d} \varphi(x) \mu(dx) \) for all \( \varphi \in L^1(\mu) \).

In the following, we will discuss the relation between Assumption A1, Assumption A3, and Assumption A2, and the Poincaré-type inequalities above. In what follows, the terms \( \alpha(r) \) and \( \beta(r) \) are defined as

\[
\alpha(r) = \inf_{s > 0} \left\{ \frac{1}{\inf_{0 < |x - y| \leq s} \left[ (e^{f(x)} + e^{f(y)}) |x - y|^{-(d+\vartheta)} \right]} : \int \int_{|x - y| > s} e^{-f(x)} e^{-f(y)} dxdy \leq r/2 \right\}, \quad (15)
\]
\[
\beta(r) = \inf_{t,s>0} \left\{ \frac{2\mu(\omega)}{\inf_{|x| \geq t} \omega(x)} + \beta(t \wedge s) : \frac{2}{\inf_{|x| \geq t} \omega(x)} + s \leq r \right\},
\]
where for any \( t > 0 \), we have
\[
\beta_t(s) = \inf_{u>0} \left\{ \frac{(\sup_{|z| \leq 2t} e^{f(z)})^2}{u^\theta(\sup_{|z| \leq 2t} e^{f(z)})} : \frac{u^\theta}{(\inf_{|z| \leq t} e^{f(z)})} \leq s \right\}.
\]

The function \( \omega \) will depend on the properties of the potential \( f \).

**Proposition 2.** If the original potential satisfies Assumption A1 with parameters \( \alpha, A, B \), then

1. If \( \alpha > \beta \) or \( \alpha = \beta, \vartheta < A\beta^{-1}b^{-1} \), the original density function satisfies the super Poincaré inequality with

\[
\omega(x) = \frac{C}{2d+\theta} |x|^{4\alpha^{-1}b^{-\beta} \log^{\beta-1}(|x|) - \theta \log^{-\beta}(|x|)},
\]

for some positive constant \( C \).

2. If \( \alpha = \beta, \vartheta \geq A\beta^{-1}b^{-1} \), the original density function satisfies the weak Poincaré inequality.

**Proposition 3.** If the original potential satisfies Assumption A2 with parameters \( \mu, \theta \), then

1. If \( \theta < 2 - \beta \) or \( \theta = 2 - \beta, \vartheta < \mu\beta^{-1}b^{-1} \), the original density function satisfies the super Poincaré inequality with \( \omega(x) \) defined as

\[
\omega(x) = \begin{cases} 
\frac{C}{2d+\theta} |x|^{(1-\theta)^{-1}(2-\theta)^{-1}\mu b} \log^{2-\theta}(|x|) & \theta < 2 - \beta, \\
\frac{C}{2d+\theta} |x|^{\frac{2-\theta}{d\mu}} \log^{2-\theta}(|x|) - \theta \log^{-\frac{d-\theta}{d\mu}}(|x|) & \theta = 2 - \beta, \vartheta < \mu\beta^{-1}b^{-1}.
\end{cases}
\]

where \( C \) is some positive constant.

2. If \( \theta = 2 - \beta, \vartheta \geq \mu\beta^{-1}b^{-1} \) or \( \theta > 2 - \beta \), the original density function satisfies the weak Poincaré inequality.

\[
\alpha(r) = \inf \left\{ \frac{1}{\inf_{0<|x-y|\leq s}[(e^{f(x)} + e^{f(y)})|x-y|^{-(d+\theta)}]} : \int \int_{|x-y|>s} e^{-f(x)} e^{-(d+\theta)} dxdy \leq r/2 \right\}.
\]

**Proposition 4.** If the original potential function satisfies Assumption A3 with parameter \( \rho \), then

1. If \( \beta \in (1,2) \) or \( \beta = 2, \vartheta < \frac{1}{2}pb^{-1} \), the original density function satisfies the super Poincaré inequality with

\[
\omega(x) = \frac{C}{2d+p} |x|^{\frac{1}{2}\rho b^{-\beta} \log^{\beta-1}(|x|)} - \theta \log^{-\frac{d-\theta}{d\rho}}(|x|),
\]

for some positive constant \( C \).

2. If \( \beta = 2, \vartheta = \frac{1}{2}pb^{-1}, d = 1,2 \), the original density function satisfies the Poincaré inequality.

3. If \( \beta = 2, \vartheta = \frac{1}{2}pb^{-1}, d \geq 3 \) or \( \beta = 2, \vartheta > \frac{1}{2}pb^{-1} \), the original density function satisfies the weak Poincaré inequality.

**Remark 5.** For the example of multivariate \( t \)-distribution, it is shown later in Lemma 3, that it satisfies Assumption A3 with \( \beta = 2, \alpha = 2, A = 2b\kappa, B \geq 0 \) and arbitrary \( \mu \in (0, 2b\kappa) \). Therefore when \( \kappa > \vartheta \), it falls into the class of densities described by the super Poincaré inequality. When \( 0 < \kappa \leq \vartheta \), it falls into the class of densities described by the weak Poincaré inequality. This classification of the multivariate \( t \)-distributions with different degrees of freedom coincides with [WW15, Corollary 1.2].

**Remark 6.** For the multivariate \( t \)-distribution with degrees of freedom \( \kappa \), we show later in Lemma 3 that it satisfies Assumption A2 with arbitrary \( \mu \in (0, b\kappa\beta(\beta-1)) \) and \( \theta = 2 - \beta \). When \( \vartheta < \kappa(\beta-1) \), we can show that multivariate \( t \)-distribution with \( \kappa \) degrees of freedom satisfies the super Poincaré inequality which agrees with the results in [Wan14] and our Remark 5 above.
6. Illustrative Examples

In this section, we introduce a specific transformation map $h$ defined by (6) and (8) with $\beta = 2$ and $g_{in}$ defined by the following equation. For all $r \leq b^{-\frac{1}{2}}$,

$$g_{in}(r) = rb^{\frac{1}{2}} \exp \left( 2 - \frac{10}{3} b^{\frac{3}{2}} r^3 + \frac{15}{4} b^2 r^4 - \frac{6}{5} b^3 r^5 + \frac{47}{60} \right).$$

(17)

Using the above transformation map, we analyze the oracle complexity of TULA for sampling from the multivariate $t$-distribution and related densities.

6.1. Example 1. The density and potential function of the multivariate $t$-distribution are respectively given by

$$\pi(x) \propto (1 + |x|^2)^{-\frac{d+\kappa}{2}}, \quad f(x) = \frac{d+\kappa}{2} \log(1 + |x|^2),$$

(18)

where $\kappa$ is the degrees of freedom parameter. We first show that the above $g_{in}$ satisfies Assumption G1 and hence the corresponding $h$ is a diffeomorphism.

**Lemma 2.** With $g_{in}$ defined in (17), $\beta = 2$ and $f(x) = \frac{d+\kappa}{2} \log(1 + |x|^2)$, $g$ defined in (6) satisfies Assumption G1.

Next, we show that the potential function of the multivariate $t$-distribution satisfies the assumptions we introduced in Section 4.

**Lemma 3.** We have for the following for the potential function $f(x)$ in (18):

1. $f(x)$ is isotropic and $f \in C^2(\mathbb{R}^d)$;
2. $f$ satisfies Assumption A4 with some $N_4 > 0$ and $L = 2kb^2 \beta$;
3. $f$ satisfies Assumption A1 with $\alpha = \beta$, $A = kb\beta$ and some $B \geq 0, N_1 > 0$.
4. $f$ satisfies Assumption A2 with arbitrary $\mu \in (0, kb\beta(\beta - 1))$, $\theta = 2 - \beta$ and some $N_2 > 0$.

Hence, we can apply Theorem 5 with $n = \tilde{O}(L^3_n C_{\text{h,LSI}}^2 (d/\epsilon))$, where $C_{h,\text{LSI}}$ and $L_h$ are two constants that depend on $f$ as introduced in (35) and (36). However, the dependence of $C_{h,\text{LSI}}$ and $L_h$ on $f$ would affect the order of $n$ significantly, especially in terms of the dimension parameter. Specifically, after explicitly calculating the constants $C_{h,\text{LSI}}$ and $L_h$, the mixing time of TULA in KL-divergence with error tolerance $\epsilon$ is of order $n = \tilde{O}(\exp(2d)d^{d+1}\epsilon^{-1})$. A detailed proof of Lemma 3 and the calculation for order estimation of the mixing time $n$ are given in Sections 7.6 and 7.6.1 respectively.

Despite the above result for the multivariate $t$-distribution, we next demonstrate through several examples that as long as the tail becomes slightly lighter, we get linear dependency on both the dimension parameter and inverse of the target accuracy parameter. In the next several examples, we use the following result form [CW97] to calculate the LSI constant. Furthermore, following a similar argument in the proof of Lemma 3, one can show that the potentials satisfy the assumptions required by Theorem 5. However, for simplicity, we directly calculate the LSI constants of the transformed potential and use the result from [VW19].

**Corollary 1.** [Simplified version of [CW97, Corollary 1.4]] For the Langevin diffusion process with generator $L = -\nabla f \cdot \nabla + \triangle$, let $\lambda_f(x)$ be the largest eigenvalue of the matrix $\nabla^2 f(x)$ and let $\beta(r) = \inf_{|x| \geq r} \{-\lambda_f(x)\}$. If $\sup_{r \geq 0} \beta(r) > 0$, then the stationary measure to this Langevin diffusion satisfies LSI with constant $2/\alpha(L)$ such that

$$\alpha(L) \geq \frac{2}{a_0^2} \exp \left( 1 - \int_0^{a_0} r \beta(r) dr \right) > 0,$$

where $a_0 > 0$ is the unique solution to the equation $\int_0^{a_0} \beta(r) dr = 2/a$. 
6.2. Example 2. The next potential function \( f \) we consider is given by

\[
f(x) = \begin{cases} 
\frac{d + \kappa}{2} \log(1 + |x|^2) - \frac{d + \kappa}{2} \log(1 + |x|^{-2}) \\
(v_f + 1) \log \log |x| + \left( v_f + \frac{1}{2} \right) d \log(1 + 2b(\log |x|)^{-1}) & |x| \geq e, \\
(d - 1) \log |x| + \frac{d}{2} \log g_{in}^{-1}(|x|)^2 + \left( \frac{1}{2} + v_f \right) d \log \left( 1 + \frac{1}{2} g_{in}^{-1}(|x|)^2 \right) \\
(d - 1) \log g_{in}^{-1}(|x|) + \log g_{in}'(g_{in}^{-1})(|x|) + v_f d \log b + \left[ \left( \frac{1}{2} + v_f \right) d - 1 \right] \log 2, & 0 \leq |x| < e.
\end{cases}
\]

where \( v_f \in (-\frac{3}{2}, \frac{15}{2}) \). With the transformation \( h \) defined by (6), (8) and (17) and \( b = \frac{d}{\sqrt{2e}} \), the transformed potential is

\[
f_h(x) = \frac{d}{2} |x|^2 + \left( \frac{1}{2} + v_f \right) d \log \left( 1 + \frac{1}{2} |x|^2 \right) + v_f d \log b + \left[ \left( \frac{1}{2} + v_f \right) d - 1 \right] \log 2, \quad \forall x \in \mathbb{R}^d.
\]

We can find the LSI constant of the transformed density \( \pi_h \propto e^{-f_h(x)} \) by [CW97, Corollary 1.4]. First, note that the two eigenvalues of \( \nabla^2 f_h(x) \) are

\[
\lambda_1(x) = d \left[ 1 + \left( \frac{1}{2} + v_f \right) \frac{1}{1 + \frac{1}{2} |x|^2} \right], \quad \text{and} \quad \lambda_2(x) = d \left[ 1 + \left( \frac{1}{2} + v_f \right) \frac{1 - \frac{1}{2} |x|^2}{(1 + \frac{1}{2} |x|^2)^2} \right].
\]

We now consider the following cases.

(a) When \( v_f = -\frac{1}{2} \): \( \lambda_1(x) = \lambda_2(x) = d \). The LSI constant \( C_{h,\text{LSI}} = \frac{2}{7} \).

(b) When \( v_f \in (-\frac{3}{2}, \frac{15}{2}) \), \( \lambda_2(x) < \lambda_1(x) \) for all \( x \in \mathbb{R}^d \). Therefore

\[
\bar{\beta}(r) = \begin{cases} 
\left[ 1 - \frac{1}{8} \left( v_f + \frac{1}{2} \right) \right] d, & 0 \leq r \leq \sqrt{6}, \\
\left[ 1 + \frac{1 - \frac{3}{2} r^2}{(1 + \frac{1}{2} r^2)^2} \left( \frac{1}{2} + v_f \right) \right] d, & r > \sqrt{6}.
\end{cases}
\]

and

\[
\int_0^{a_0} \bar{\beta}(r) dr = \frac{2}{a_0} \quad \implies \quad a_0 = \left( \frac{2}{1 - \frac{1}{2} (v_f + \frac{1}{2}) d^{-1}} \right)^{\frac{1}{2}}.
\]

The LSI constant is hence given by

\[
C_{h,\text{LSI}} = a_0^2 \exp \left( \int_0^{a_0} r \bar{\beta}(r) dr - 1 \right) = \frac{2}{1 - \frac{1}{2} (v_f + \frac{1}{2}) d^{-1}}.
\]

(c) When \( v_f \in (-\frac{3}{2}, -\frac{1}{2}) \), \( \lambda_1(x) < \lambda_2(x) \) for all \( x \in \mathbb{R}^d \). Therefore

\[
\bar{\beta}(r) = \inf_{|x| > r} \lambda_1(x) = \lambda_1(0) = \left( \frac{3}{2} + v_f \right) d.
\]

and

\[
\int_0^{a_0} \bar{\beta}(r) dr = \frac{2}{a_0} \quad \implies \quad a_0 = \left( \frac{2}{\frac{3}{2} + v_f} d^{-1} \right)^{\frac{1}{2}}.
\]

The LSI constant is

\[
C_{h,\text{LSI}} = a_0^2 \exp \left( \int_0^{a_0} r \bar{\beta}(r) dr - 1 \right) = \frac{2}{\frac{3}{2} + v_f} d^{-1}.
\]
Hence, we have that $C_{h, \text{LSI}} = O(d^{-1})$. Combined with the fact that the gradient Lipschitz constant of $f_h$ is $L_h = O(d)$, according to [VW19], the iteration complexity to achieve $\varepsilon$ error tolerance in KL-divergence is of order $\tilde{O}(d/\varepsilon)$, where $\tilde{O}$ hides only numerical constants and poly-logarithmic factors.

6.3. Example 3. The next potential function is given by

$$f(x) = \begin{cases} 
 d(1 + \frac{1}{2b}) \log |x| + \left(\frac{d}{2} + 1\right) \log \log |x| + d \log(1 + 2b(\log |x|)^{-1}) 
 & |x| > e 
 
 - (d - 1) \log 2 - \frac{d}{2} \log b 
 & |x| = e 
 
 (d - 1) \log |x| - (d - 1) \log g_{in}^{-1}(|x|) + \frac{d}{2} g_{in}^{-1}(|x|)^2 + d \log(1 + \frac{1}{2} g_{in}^{-1}(|x|)^2) 
 + \log g_{in}^\prime(g_{in}^{-1})(|x|), 
 & 0 \leq |x| \leq e 
\end{cases}$$

For all $x \in \mathbb{R}^d$ we have that $0 < \lambda_i \leq 2d$ for $i = 1, 2$. Therefore the transformed potential $f_h$ is gradient Lipschitz with parameter $2d$. To find the LSI parameter we use [CW97, Corollary 1.4].

$$\lambda_1 = d \left[ 1 + \frac{1}{1 + \frac{1}{2} |x|^2} \right], \quad \lambda_2 = d \left[ 1 + \frac{1}{\left( 1 + \frac{1}{2} |x|^2 \right)^2} \right].$$

The solution to the equation $\int_0^a \beta(r)dr = 2/a$ is given by $a_0 = \sqrt{16/7d}$. The LSI constant hence satisfies

$$C_{h, \text{LSI}} \leq a_0^2 \exp \left( \int_0^{a_0} r \beta(r)dr - 1 \right) = \frac{16}{7d}.$$

According to [VW19], the iteration complexity is of order $\tilde{O}(d/\varepsilon)$, where $\tilde{O}$ hides only numerical constants and poly-logarithmic factors.

6.4. Example 4. Our next potential function is given by

$$f(x) = \begin{cases} 
 d(1 + \frac{1}{2b}) \log |x| + \log \log |x| + \frac{d}{2} \log(1 + 2b(\log |x|)^{-1}) 
 & |x| > e 
 
 - (d - 1) \log 2 
 & |x| = e 
 
 (d - 1) \log |x| - (d - 1) \log g_{in}^{-1}(|x|) + \frac{d}{2} g_{in}^{-1}(|x|)^2 + \frac{d}{2} \log(1 + \frac{1}{2} g_{in}^{-1}(|x|)^2) 
 + \log g_{in}^\prime(g_{in}^{-1})(|x|), 
 & 0 \leq |x| \leq e 
\end{cases}$$
To study the tail-behavior of the original potential function \( f \), we compare it to another potential function \( \tilde{f}(x) = d(1 + \frac{1}{2b}) \log(1 + |x|) + \log(1 + 2b(\log |x|)^{-1}) \). According to [Wan14], if \( b = \frac{d}{2\vartheta} \), \( \tilde{f} \) satisfies the weak Poincaré inequality with \( \vartheta \) being the degree of freedom. But compare to the previous example, it has a heavier tail because \( 1 < \frac{d}{2} + 1 \).

The transformed potential in this case is given by

\[
f_h(x) = \frac{d}{2} |x|^2 + \frac{d}{2} \log \left( 1 + \frac{1}{2} |x|^2 \right).
\]

Similar to the previous example, the corresponding density function satisfies LSI and log-concavity assumption. The two eigenvalues of the Hessian matrix are:

\[
\lambda_1 = d \left[ 1 + \frac{1}{2} \left( 1 + \frac{1}{2} |x|^2 \right) \right], \quad \text{and} \quad \lambda_2 = d \left[ 1 + \frac{1}{2} \left( 1 + \frac{1}{2} |x|^2 \right) \right].
\]

For all \( x \in \mathbb{R}^d \), \( 0 < \lambda_i \leq \frac{3}{2} d \) for \( i = 1, 2 \). Therefore the transformed potential \( f_h \) is gradient Lipschitz with parameter \( \frac{d}{2} \). To find the LSI parameter we use [CW97, Corollary 1.4]. For all \( x \in \mathbb{R}^d \), \( \lambda_2 \leq \lambda_1 \). Furthermore, we have

\[
\bar{\beta}(r) = \inf \lambda_2 = \begin{cases} 
\frac{15}{16}d, & r \in (0, \sqrt{6}], \\
\left( 1 + \frac{1}{2} \left( 1 + \frac{1}{2} |x|^2 \right) \right) d, & r \in (\sqrt{6}, \infty).
\end{cases}
\]

The solution to the equation \( \int_0^\alpha \bar{\beta}(r) dr = 2/a \) is then \( a_0 = \sqrt{\frac{32}{15d}} \). The LSI constant \( C_{h,LSI} \) satisfies

\[
C_{h,LSI} \leq a_0^2 \exp \left( \int_0^{a_0} r \bar{\beta}(r) dr - 1 \right) = \frac{32}{15d}.
\]

According to [VW19], the iteration complexity is of order \( \tilde{O}(d/\epsilon) \), where \( \tilde{O} \) hides only numerical constants and poly-logarithmic factors.

6.5. **Example 5.** We next consider the following potential function given by

\[
f(x) = \begin{cases} 
d(1 + \frac{1}{2b}) \log |x| - (\frac{d}{4} - 1) \log \log |x| + \frac{d}{4} \log(1 + 2b(\log |x|)^{-1}) & |x| > e \\
-(\frac{d}{4} - 1) \log 2 + \frac{d}{4} \log b & 0 < |x| \leq e \\
(d - 1) \log |x| - (d - 1) \log g_{in}(0|x|) + \frac{d}{2} g_{in}^{-1}(0|x|)^2 + \log g_{in}^\prime(g_{in}^{-1}(0|x|)) & 0 \leq |x| \leq e
\end{cases}
\]

To study the tail-behavior of the original potential function \( f \), we compare it to another potential function \( \tilde{f}(x) = d(1 + \frac{1}{2b}) \log(1 + |x|) - (\frac{d}{4} - 1) \log \log(e + |x|) \). According to [Wan14], with \( b = \frac{d}{2\vartheta} \), if \( d < 4 \), \( \tilde{f} \) satisfies the weak Poincaré inequality with \( \vartheta \) being the degree of freedom. If \( d = 4 \), \( \tilde{f} \) satisfies Poincaré inequality with \( \vartheta \) being the degree of freedom. If \( d > 4 \), \( \tilde{f} \) satisfies the super Poincaré inequality with \( \vartheta \)-degree of freedom.

The transformed potential is given by

\[
f_h(x) = \frac{d}{2} |x|^2 + \frac{d}{4} \log \left( 1 + \frac{1}{2} |x|^2 \right).
\]

The corresponding density function satisfies LSI and log-concavity assumption. The two eigenvalues of the Hessian matrix are:

\[
\lambda_1 = d \left[ 1 + \frac{1}{4} \left( 1 + \frac{1}{4} |x|^2 \right) \right], \quad \text{and} \quad \lambda_2 = d \left[ 1 + \frac{1}{4} \left( 1 + \frac{1}{4} |x|^2 \right) \right].
\]
For all $x \in \mathbb{R}^d$, $0 < \lambda_i \leq \frac{\bar{r}}{2}d$ for $i = 1, 2$. Therefore the transformed potential $f_h$ is gradient Lipschitz with parameter $\frac{\bar{r}}{2}d$. To find the LSI parameter we use [CW97, Corollary 1.4]. For all $x \in \mathbb{R}^d$: $\lambda_2 \leq \lambda_1$.

$$
\beta(r) = \inf_{|x| > r} \lambda_2 = \begin{cases} 
\frac{31}{32}d, & r \in (0, \sqrt{6}], \\
\left(1 + \frac{1}{2} - \frac{\bar{r}^2}{2(1 + \bar{r}^2)^2}\right) d, & r \in (\sqrt{6}, \infty).
\end{cases}
$$

The solution to the equation $\int_0^a \beta(r)dr = 2/a$ is then $a_0 = \sqrt{64/31d}$. The LSI constant $C_{h,\text{LSI}}$ satisfies

$$
C_{h,\text{LSI}} \leq a_0^2 \exp\left(\int_0^{a_0} r\beta(r)dr - 1\right) = \frac{64}{31d}.
$$

According to [VW19], the iteration complexity is of order $\tilde{O}(d/\epsilon)$, where $\tilde{O}$ hides only numerical constants and poly-logarithmic factors.

6.6. Example 6. As the limiting example of the previous three examples, we consider the potential function

$$
f(x) = \begin{cases} 
(d(1 + \frac{1}{2\bar{r}}) \log |x| - (\frac{d}{2} - 1) \log \log |x| + \log 2 + \frac{d}{2} \log b \\
(d - 1) \log |x| - (d - 1) \log g_{m}^{-1}(|x|) + \frac{d}{2} g_{m}^{-1}(|x|)^2 + \log g_{m}^{-1}(g_{m}^{-1}(|x|)),
\end{cases}
\quad \text{for } |x| > e
$$

\quad \text{for } 0 \leq |x| \leq e

We introduce $\tilde{f}(x) = d(1 + \frac{1}{2\bar{r}}) \log(1 + |x|) - (\frac{d}{2} - 1) \log \log(e + |x|)$ which has similar tail-behavior as the potential $f$ above. According to [Wan14], with $b = \frac{d}{2\bar{r}}$, if $d = 2$, $\tilde{f}$ satisfies the weak Poincaré inequality with $\vartheta$ being the degree of freedom. If $d = 2$, $\tilde{f}$ satisfies Poincaré inequality with $\vartheta$ being the degree of freedom. If $d > 2$, $\tilde{f}$ satisfies the super Poincaré inequality with $\vartheta$-degree of freedom and it induces a density function which has heavier tail than the multivariate $t$-distribution with $\vartheta$-degree of freedom.

The transformed potential is $f_h(x) = \frac{d}{2} |x|^2$. The Hessian matrix is $\nabla^2 f_h(x) = dI_d$. Therefore $f_h$ is log-concave with parameter $d$ and the corresponding density satisfies LSI with parameter $C_{h,\text{LSI}} \leq 2/d$. According to [VW19], the iteration complexity is of order $\tilde{O}(d/\epsilon)$, where $\tilde{O}$ hides only numerical constants and poly-logarithmic factors.

7. Proofs

In this section, we will prove the theorems stated in Sections 4-6.

7.1. Analysis of the transformation maps. In this section we first analyze the transformation map induced by $g$ defined in (6).

Lemma 4. If the potential function $f$ satisfies Assumption A0, then we have

$$
\nabla f_h(x) = \begin{cases} 
\left[ f'(g_{m}(|x|))g'_{m}(|x|) - \frac{\beta}{g_{m}(|x|)} \frac{g''_{m}(|x|)}{g_{m}(|x|)} + \frac{d - 1}{|x|} g_{m}(|x|) \right] x & |x| < b^{-\frac{1}{\beta}}, \\
\left[ f'(e|\beta|x|^{\beta})b\beta|x|^{\beta-1} e^{|\beta|x|^\beta} - \beta bd|x|^{\beta-1} e^{|\beta|x|^\beta} + \frac{d - \beta}{|x|} \right] x & |x| \geq b^{-\frac{1}{\beta}}.
\end{cases}
$$

and $\nabla^2 f_h(x)$ has two eigenvalues $\lambda_1 = \lambda_1(|x|)$ and $\lambda_2 = \lambda_2(|x|)$ with $\lambda_1, \lambda_2$ defined as

1. When $|x| < b^{-\frac{1}{\beta}}$:

$$
\lambda_1(|x|) = f''(g_{m}(|x|))(g'_{m}(|x|))^{2} + f'(g_{m}(|x|))g''_{m}(|x|) - \frac{g_{m}^{(3)}(|x|)}{g'_{m}(|x|)} + \left(\frac{g''_{m}(|x|)}{g'_{m}(|x|)}\right)^2
$$

2. When $|x| \geq b^{-\frac{1}{\beta}}$:

$$
\lambda_2(|x|) = \frac{\beta^2}{2} |x|^{2\beta} - \frac{d - \beta}{|x|} e^{|\beta|x|^\beta} + \frac{d - 1}{|x|} g_{m}(|x|) + \frac{\beta}{g_{m}(|x|)} \frac{g''_{m}(|x|)}{g_{m}(|x|)} - \frac{\beta}{g_{m}(|x|)} \frac{g''_{m}(|x|)}{g_{m}(|x|)}.
$$
Proof of Lemma 4. For a general transformation map induced by \( h \), the transformed potential \( f_h \) can be represented as

\[
f_h(x) = f(g(|x|)) - \log \det(\nabla h(x))
\]

\[
= f(g(|x|)) - \log g'(|x|) - (d - 1) \log g(|x|) + (d - 1) \log |x|.
\]

The gradient of the transformed potential \( f_h \) is

\[
\nabla f_h(x) = \left[ f'(g(|x|))g'(|x|) - \frac{g''(|x|)}{g(|x|)} - (d - 1) \frac{g'(|x|)}{|x|} \right] \frac{x}{|x|}.
\]

Now, (19) follows immediately as a consequence of (6) and (24). The Hessian matrix of \( f_h \) can be represented as

\[
\nabla^2 f_h(x) = F_1(|x|) \frac{xx^T}{|x|^2} + F_2(|x|) I_d
\]

with

\[
F_1(|x|) = f''(g(|x|))g'(|x|)^2 + f'(g(|x|))g''(|x|) - f'(g(|x|)) \frac{g'(|x|)}{|x|} - \frac{g''(|x|)}{g'(|x|)}
\]

\[
+ \left( \frac{g''(|x|)}{g'(|x|)} \right)^2 - (d - 1) \frac{g''(|x|)}{g(|x|)} + (d - 1) \frac{g'(|x|)}{g(|x|)}
\]

\[
+ (d - 1) \frac{g'(|x|)}{g(|x|)} \frac{1}{|x|} - \frac{2(d - 1)}{|x|^2} + \frac{g''(|x|)}{|x| g'(|x|)}.
\]

\[
F_2(|x|) = \left( f'(g(|x|))g'(|x|) - \frac{g''(|x|)}{g'(|x|)} - (d - 1) \frac{g'(|x|)}{g(|x|)} + \frac{d - 1}{|x|} \right) |x|^{-1}.
\]

Therefore the two eigenvalues of \( \nabla^2 f_h(x) \), \( \lambda_1 \) and \( \lambda_2 \) can be written as

\[
\lambda_1 = f''(g(|x|))g'(|x|)^2 + f'(g(|x|))g''(|x|) - \frac{g''(|x|)}{g'(|x|)} + \left( \frac{g''(|x|)}{g'(|x|)} \right)^2
\]

\[
- (d - 1) \frac{g'(|x|)}{g(|x|)} + (d - 1) \frac{g'(|x|)}{g(|x|)} \frac{g''(|x|)}{g(|x|)} - \frac{(d - 1)}{|x|^2},
\]

\[
\lambda_2 = \left( f'(g(|x|))g'(|x|) - \frac{g''(|x|)}{g'(|x|)} - (d - 1) \frac{g'(|x|)}{g(|x|)} + \frac{d - 1}{|x|} \right) |x|^{-1}.
\]

The conclusions in (20), (21), (22), (23) can be calculated directly from (6), (25) and (26). \( \square \)

With the above result on the transformation map \( h \), we can prove Lemma 1.

**Proof of Lemma 1**: We first show that for all \( \beta \in [1,2] \), \( g \in C^3((0,\infty)) \). It suffices to show that \( g \) is three times continuously differentiable at \( r = b^{-1/\beta} \). Based on (6), we have

\[
g_{in}(b^{-1/\beta}) = e = e^{b^{\zeta_0}} |r = b^{-1/\beta}|.
\]
Next we show that $g$ is monotone increasing. From Assumption G1, we know that $g_{in}$ is increasing on the interval $(0, b^{-\frac{1}{\beta}})$. For $r \in [b^{-\frac{1}{\beta}}, \infty)$, $g'(r) = b\beta r e^{br} > 0$. Combined with the fact that $g \in C^3((0, \infty))$, we obtain that $g$ is monotone increasing on the interval $(0, \infty)$. Furthermore, $g(0) = g_{in}(0) = 0$ and $\lim_{r \to \infty} g(r) = \lim_{r \to \infty} e^{br} = \infty$. Therefore $g$ is also onto and invertible. □

**Corollary 2.** With function $g$ defined in (6) and $g_{in}$ satisfying Assumption G1, if the potential function $f$ satisfies Assumption A0, then the transformed potential function $f_h$ defined by (5) is twice continuously differentiable, i.e., $f_h \in C^2(\mathbb{R}^d)$.

**Proof of Corollary 2:** Under the assumptions in Corollary 2, with the results in Lemma 1 and (25), (26), we have $f_h \in C^2(\mathbb{R}^d \setminus \{0\})$. Therefore it remains to show that $\lim_{|x| \to 0_+} \lambda_i(|x|)$ are well-defined for $i = 1, 2$. According to (5), we can represent the two eigenvalues of $\nabla^2 f_h(x)$ as for all $r < b^{-\frac{1}{\beta}}$:

$$\begin{align*}
\lambda_1(r) &= f''_h(r) = f''(g_{in}(r))g_{in}'(r)^2 + f'(g_{in}(r))g_{in}''(r) - (d-1)(\frac{d^2}{dr^2} \log g_{in}(r)) + \frac{d^2}{dr^2} \log g_{in}'(r), \\
\lambda_2(r) &= \frac{f''_h(r)}{r} = f'(g_{in}(r)) \frac{g_{in}'(r)^2}{r} - \frac{d}{dr} \log g_{in}'(r) - (d-1) \frac{d}{dr} \log g_{in}(r).
\end{align*}$$

Since $f \in C^2(\mathbb{R})$ and $g$ satisfies Assumption G1, $\lim_{r \to 0_+} |\lambda_i(r)| < \infty$ for $i = 1, 2$, which implies that $f_h$ is twice continuously differentiable at the origin. Therefore $f_h \in C^2(\mathbb{R}^d)$. □

### 7.2. Proof of Theorem 1.

We first recall a few definitions below. Our proof is based on connections between Lyapunov-based techniques and functional inequality-based techniques for proving ergodicity of diffusion process [BCG08].

**Definition (Dissipativity condition).** The Langevin diffusion with drift function $b(x)$ is said to satisfy the dissipativity condition if there exists constants $r, M > 0$ such that for all $|x| > M$: $\langle b(x), x \rangle \leq -r|x|$.

**Definition (Lyapunov condition).** A function $V \in \mathcal{D}(\mathcal{L})$ with $V \geq 1$ is a Lyapunov function if there exist constants $\lambda, c > 0$ and a measurable set $K \subset \mathbb{R}^d$ such that $\mathcal{L} V \leq \lambda V(-1 + c1_K)$. Equivalently, we say the $\mathcal{L}$ satisfies the Lyapunov condition.

**Lemma 5.** Consider the dynamics in (9). If the drift function $-\nabla f_h$ satisfies the dissipativity condition with $r > 0, M = 8(d-1)/r$, then the infinitesimal generator $\mathcal{L}_h$ of (9) satisfies Lyapunov condition.

**Proof of Lemma 5.** We first construct a Lyapunov function $V$ with respect to the generator $\mathcal{L}_h$ as

$$V(x) = \begin{cases} 
1 & |x| \leq \frac{M}{2}, \\
P(|x|) & \frac{M}{2} < |x| < M, \\
e^{a|x|} & |x| \geq M,
\end{cases}$$

where $P : [\frac{M}{2}, M] \rightarrow [1, e^{aM}]$ is a monotone increasing function such that $V \in C^2(\mathbb{R}^d)$ and $V \geq 1$ for all $x \in \mathbb{R}^d$. When $|x| \geq M$, we have that

$$\mathcal{L}_h V(x) = -\nabla f_h(x) \cdot \nabla (e^{a|x|}) + \triangle(e^{a|x|})$$
\[ -\nabla f_h(x) \cdot (ae^{a|x|} \frac{x}{|x|}) + ae^{a|x|} \left( \frac{d}{|x|} + a - \frac{1}{|x|} \right) \leq ae^{a|x|} \left( -r + a + \frac{d-1}{|x|} \right). \]

Picking \( a = r/2 \), we obtain that
\[
\mathcal{L}_h V(x) \leq \frac{r}{2} V(x) \left( -\frac{r}{2} + \frac{d-1}{|x|} \right), \quad \forall |x| \geq M.
\]

Since \( M = 8(d-1)/r > 4(d-1)/r \), we obtain that \( \mathcal{L}_h V(x) \leq -r^2/8 V(x) \) for all \( |x| \geq M \).

When \( 0 \leq |x| < M \), by the fact that \( V \in C^2(\mathbb{R}^d) \) and \( V \geq 1 \), there exists \( A_{r,d} \) such that
\[
\frac{\mathcal{L}_h V(x)}{V(x)} \leq A_{r,d} \quad \forall 0 \leq |x| < M,
\]
where \( A_{r,d} = \max_{4(d-1)/r \leq |x| \leq 8(d-1)/r} (-rP(|x|) + \Delta(P(|x|))) \vee 0 \).

Therefore if \( -\nabla f_h \) satisfies dissipativity condition with constant \( r > 0 \), the corresponding generator \( \mathcal{L}_h \) satisfies Lyapunov condition with \( \lambda = \lambda_r = r^2/8, \ c = c_{r,d} = A_{r,d}/\lambda_r \) an
\[
K = K_{r,d} = \{ x \in \mathbb{R}^d : 0 \leq |x| \leq 8(d-1)/r \}. \tag{27}
\]

We further recall additional definitions to proceed.

**Definition (Local Poincaré inequality).** The Markov triple \((\mathbb{R}^d, \mu, \Gamma)\) satisfies a local Poincaré inequality on a measurable set \( K \subset \mathbb{R}^d \) with \( \mu(K) \in (0, \infty) \) if for some constant \( C_K \) and every function \( \phi \in \mathcal{D}(\mathcal{E}) \):
\[
\int_K (\phi - m_K)^2 d\mu \leq C_K \int_K \Gamma(\phi) d\mu
\]
where \( m_K = \int_K \phi \, d\mu/\mu(K) \).

**Lemma 6.** If the original density satisfies Assumption \( A0 \), then the Markov triple \((\mathbb{R}^d, \pi_h, \Gamma_h)\) satisfies a local Poincaré inequality on \( K_{r,d} \) defined in (27).

**Proof of Lemma 6.** According to the classical Poincaré inequality with respect to Lebesgue measure, there is a universal constant \( C > 0 \) such that for all \( u \in W^{1,2}(\mathbb{R}^d) \subset W^{1,2}(K_{r,d}) \):
\[
\int_{K_{r,d}} (u(x) - u_{K_{r,d}})^2 \, dx \leq C \frac{d-1}{r} \int_{K_{r,d}} |\nabla u|^2 \, dx
\]
where \( u_{K_{r,d}} = \int_{K_{r,d}} u(x) \, dx \). To prove the local Poincaré inequality for \((\mathbb{R}^d, \pi_h, \Gamma_h)\), without loss of generality, we assume that \( \phi \in \mathcal{D}(\mathcal{E}) \) and \( \phi_{K_{r,d}} = \int_{K_{r,d}} \phi(x) \, dx = 0 \). Then
\[
\int_{K_{r,d}} (\phi - m_{K_{r,d}})^2 e^{-f_h(x)} \, dx = \int_{K_{r,d}} \phi(x)^2 e^{-f_h(x)} \, dx - (\int_{K_{r,d}} \phi(x)e^{-f_h(x)} \, dx)^2/|K_{r,d}|
\]
\[
\leq \left( \sup_{x \in K_{r,d}} e^{-f_h(x)} \right) \int_{K_{r,d}} \phi(x)^2 \, dx
\]
\[
= \left( \sup_{x \in K_{r,d}} e^{-f_h(x)} \right) \int_{K_{r,d}} (\phi - \phi_{K_{r,d}})^2 \, dx
\]
\[
\leq C \frac{d-1}{r} \left( \sup_{x \in K_{r,d}} e^{-f_h(x)} \right) \int_{K_{r,d}} |\nabla \phi(x)|^2 \, dx
\]
\[ \leq C^\frac{d - 1}{r} \left( \sup_{x \in K_{r,d}} e^{-f_h(x)} \right) \left( \sup_{x \in K_{r,d}} e^{f_h(x)} \right) \int_{K_{r,d}} \Gamma_h(\phi) e^{-f_h(x)} dx. \]

Therefore the Markov triple \((\mathbb{R}^d, \pi_h, \Gamma_h)\) satisfies a local Poincaré inequality on \(K_{r,d}\) with constant
\[ C_{r,d} = \left( \frac{C(d - 1)}{r} \right) \left( \sup_{x \in K_{r,d}} e^{-f_h(x)} \right) \left( \sup_{x \in K_{r,d}} e^{f_h(x)} \right). \]

\[ \square \]

**Lemma 7.** If the infinitesimal generator \(\mathcal{L}_h\) satisfies the dissipativity condition with constant \(r > 0\) and the original density \(f\) satisfies Assumption A0, then the Markov triple \((\mathbb{R}^d, \pi_h, \Gamma_h)\) also satisfies a Poincaré inequality.

**Proof of Lemma 7.** According to Lemma 5, \(\mathcal{L}_h\) satisfies Lyapunov condition with \(\lambda = \lambda_r = r^2/8\), \(K\) as in (27) and \(c = c_{r,d} = A_{r,d}/\lambda_r\). Therefore for all \(m \in \mathbb{R}\) and \(\phi \in \mathcal{D}(\mathcal{E}_h)\):
\[
\int_{\mathbb{R}^d} (\phi - m)^2 e^{-f_h(x)} dx \leq \int_{\mathbb{R}^d} \left(-\frac{8\mathcal{L}_h V}{r^2} + \frac{8A_{r,d}}{r^2} 1_{K_{r,d}}\right)(\phi - m)^2 e^{-f_h(x)} dx \]
\[
= -\frac{8}{r^2} \int_{\mathbb{R}^d} \mathcal{L}_h V (\phi - m)^2 e^{-f_h(x)} dx + \frac{8A_{r,d}}{r^2} \int_{K_{r,d}} (\phi - m)^2 e^{-f_h(x)} dx. \tag{28}
\]

Choosing \(m = \int_{K_{r,d}} \phi e^{-f_h(x)} dx / \int_{K_{r,d}} e^{-f_h(x)} dx\), the second term in (28) can be bounded as a result of Lemma 6:
\[
\frac{8A_{r,d}}{r^2} \int_{K_{r,d}} (\phi - m)^2 e^{-f_h(x)} dx \leq \frac{8A_{r,d}}{r^2} C_{r,d} \int_{K_{r,d}} \Gamma_h(\phi) e^{-f_h(x)} dx. \tag{29}
\]
The first term in (28) can be bounded by integration by parts, and the diffusion property of \(\Gamma_h\) as
\[
-\frac{8}{r^2} \int_{\mathbb{R}^d} \mathcal{L}_h V (\phi - m)^2 e^{-f_h(x)} dx = \frac{8}{r^2} \int_{\mathbb{R}^d} \Gamma_h(\frac{(\phi - m)^2}{V}, V) e^{-f_h(x)} dx \]
\[
= \frac{8}{r^2} \int_{\mathbb{R}^d} \left(\frac{2(\phi - m)}{V} \Gamma_h(\phi - m, V) - \frac{(\phi - m)^2}{V^2} \Gamma_h(V)\right) e^{-f_h(x)} dx \]
\[
\leq \frac{8}{r^2} \int_{\mathbb{R}^d} \Gamma_h(\phi - m) e^{-f_h(x)} dx \]
\[
= \frac{8}{r^2} \int_{\mathbb{R}^d} \Gamma_h(\phi) e^{-f_h(x)} dx. \tag{30}
\]

Combining the results in (29) and (30), we prove the Markov triple \((\mathbb{R}^d, \pi_h, \Gamma_h)\) satisfies a Poincaré inequality with constant \(C\) defined as
\[
C = \frac{8}{r^2} (1 + A_{r,d} C_{r,d}) \quad \text{with} \quad A_{r,d} = \max_{4(d-1)/r \leq \|x\| \leq 8(d-1)/r} \left(-rP'(\|x\|) + \triangle(P(\|x\|))\right) > 0,
\]
\[
C_{r,d} = \frac{C(d - 1)}{r} \left( \sup_{x \in K_{r,d}} e^{-f_h(x)} \right) \left( \sup_{x \in K_{r,d}} e^{f_h(x)} \right). \]
\[ \square \]

**Proof of Theorem 1.** Applying (19) in Lemma 4, when \(\|x\| \geq 1/b\), we have
\[
\langle \nabla f_h(x), x \rangle = [f'(e^{b|x|})b e^{b|x|} - b - (d - 1) \frac{b e^{b|x|}}{e^{b|x|}} + \frac{d - 1}{\|x\|}] |x| \]
\[
= [f'(e^{b|x|})b e^{b|x|} - bd] |x| + (d - 1) \]
\[
\geq A |x| - B, \tag{31}
\]
where the last inequality follows from Assumption A1 with $\alpha = 1$ and $\beta = 1$. We now make the following claim.

**Claim:** The infinitesimal generator $L_h$ satisfies the dissipativity condition with the constants

$$r = \frac{-8(d-1) + \sqrt{64(d-1)^2 + 32AB(d-1)}}{2B} \in (0, A),$$

$$M = \frac{8(d-1)}{r}.$$ 

If the above **Claim** holds, Theorem 1 follows from Lemma 7 and Theorem 3 in [VW19]. Furthermore, $C_h$ in the statement is given by

$$C_h = \frac{r^2}{8} (1 + A_{r,d}C_{r,d})^{-1} \quad \text{with}$$

$$A_{r,d} = \max_{4(d-1)/r \leq |x| \leq 8(d-1)/r} \left( -rP'(|x|) + \triangle(P(|x|)) \right) \vee 0,$$

$$C_{r,d} = \frac{C(d-1)}{r^2} \left( \sup_{x \in K_{r,d}} e^{-f_h(x)} \right) \left( \sup_{x \in K_{r,d}} e^{f_h(x)} \right).$$

with $r$ defined in (32) and $C$ is a universal constant. We now prove the claim

**Proof of the Claim:** To prove the Claim, it suffices to show for all $|x| \geq \max\{N_1, 8(d-1)/r\}$, we have $\langle \nabla f_h(x), x \rangle \geq r|x|$. Based on (31), it further suffices to guarantee

$$A|x| - B \geq r|x| \quad \text{and} \quad \frac{8(d-1)}{r} \geq \frac{1}{b}.$$ 

When $r = A/2, 8(d-1)/r$, with $N_1 > 3B/A$, the above conditions are easily satisfied, thereby completing the proof. \hfill \Box

7.3. **Proof of Theorem 3 and Theorem 5.** Theorem 3 and Theorem 5 are both built on the intermediate result that the transformed measure $\pi_h$ satisfies a LSI. The proof would rely on the following Holley-Stroock theorem.

**Theorem 6 (Holley-Stroock Theorem [HS87]).** Let $\mu \sim LS(C_\mu)$ and let $\mu_F = Z_F^{-1}e^{-F}\mu$. If $F$ is bounded, then $\mu_F \sim LS(C_{\mu_F})$ and $C_{\mu_F} \leq e^{Osef}C_\mu$ where $Osef := \sup_{x \in \mathbb{R}^d} F(x) - \inf_{x \in \mathbb{R}^d} F(x)$.

As an immediate corollary of Theorem 6, we have $C_{\mu_F} \geq e^{-Osef}C_\mu$.

**Lemma 8.** If the true target density $\pi$ satisfies Assumption A0 and Assumption A3, then the transformed density $\pi_h$ satisfies a LSI.

**Proof of Lemma 8.** Based on (22) and (23) in Section 7.1, when $|x| \geq b^{-\frac{3}{4}}$, we have

$$\lambda_1 = f''(e^{b|x|^\beta})b^2\beta^2|x|^{2(\beta-1)}e^{2b|x|^\beta} + f'(e^{b|x|^\beta})(\beta\beta - 1)b|x|^\beta - b^2\beta^2|x|^{2(\beta-1)}e^{b|x|^\beta}$$

$$- \beta(\beta - 1)b|x|^\beta - (d - \beta)|x|^{-2}$$

$$= f''(\psi(|x|))\psi'(|x|)^2 + f'(\psi(|x|))\psi''(|x|) - \beta(\beta - 1)b|x|^\beta - (d - \beta)|x|^{-2}$$

$$\lambda_2 = f'(e^{b|x|^\beta}) + C_\beta b\beta|x|^{\beta - 1}e^{b|x|^\beta} - b\beta d|x|^\beta - (d - \beta)|x|^{-2}$$

$$= f'(\psi(|x|))\psi'(|x|)|x|^{-1} - b\beta d|x|^\beta + (d - \beta)|x|^{-2}.$$ 

where $\psi(r) = e^{br^\beta}$ for all $r \geq b^{-\frac{3}{4}}$. If $f$ satisfies Assumption A3, then for all $|x| \geq \tilde{N}_1 := \max\{N_3, b^{-\frac{3}{4}}\}$: $\lambda_1(|x|) \geq \rho$ for $i = 1, 2$. We can then construct two potentials:

$$\tilde{f}_h(x) = \begin{cases} f_h(x) & |x| > \tilde{N}_1, \\ g_h(x) & |x| \leq \tilde{N}_1, \end{cases} \quad \tilde{f}_h(x) = \begin{cases} 0 & |x| > \tilde{N}_1, \\ f_h(x) - g_h(x) & |x| \leq \tilde{N}_1. \end{cases}$$
where $g_h : \{|x| \leq \tilde{N}_1\} \subset \mathbb{R}^d \to \mathbb{R}$ is chosen such that $\tilde{f}_h \in C^2(\mathbb{R}^d)$ and $\nabla^2 g_h(x) \succeq \rho I_d$ for all $|x| \leq \tilde{N}_1$. Therefore, $\nabla^2 \tilde{f}_h(x) \succeq \rho I_d$ for all $x \in \mathbb{R}^d$ i.e $\tilde{f}_h$ is $\rho$-strongly convex which implies that the measure $\exp(-\tilde{f}_h(x))dx \sim LS(2/\rho)$ (see [BÉ85]). Meanwhile, $\tilde{f}_h$ is compactly supported on $\{|x| \leq \tilde{N}_1\}$ and $f_h, g_h \in C^2(\mathbb{R}^d)$, which implies that $f_h$ is bounded, i.e $\text{Osc} f_h < \infty$. Last according to the Holley-Stroock theorem and the fact that $\pi_h \propto \exp(-f_h) = \exp(-\tilde{f}_h) \exp(-f_h)$, 

$$\pi_h \sim LS(C_{h, LSI}) \quad \text{with} \quad C_{h, LSI} = 2e^{\text{Osc} f_h/\rho}. \quad (35)$$

\textbf{Proof of Theorem 3.} The two inequalities in Theorem 3 follows from Lemma 8 and theorem 4 in [VW19]. The constant $C_{h, LSI}$ in Theorem 3 is the same $C_{h, LSI}$ in (35).

\textbf{Lemma 9.} If the potential function $f$ satisfies Assumption A4, then the transformed potential $f_h(x)$ satisfies the gradient Lipschitz condition, i.e. there exists $L_h > 0$ such that for all $x, y \in \mathbb{R}^d$, we have $|\nabla f_h(x) - \nabla f_h(y)| \leq L_h|x - y|$.

\textbf{Proof of Lemma 9:} It suffices to prove that there is a constant $L_h$ such that $\nabla^2 f_h(x) \preceq L_h I_d$ for all $x \in \mathbb{R}^d$, i.e $\lambda_1(|x|), \lambda_2(|x|) \leq L_h$ for all $x \in \mathbb{R}^d$. Based on (33), (34) in the proof of Lemma 8, and the fact that $f$ satisfies Assumption A4, we have when $|x| \geq \tilde{N}_2 := \max\{N_4, b^{-\frac{\beta}{5}}\}$: $\lambda_i(|x|) \leq L$ for $i = 1, 2$. When $|x| \leq \tilde{N}_2$, since $f_h \in C^2(\mathbb{R}^d)$,

$$\max_{|x| \leq \tilde{N}_2} \|\nabla^2 f_h(x)\| < \infty.$$ 

Therefore the transformed density $f_h$ is gradient Lipschitz with parameter $L_h$ defined by

$$L_h = \max\{L, \max_{|x| \leq \tilde{N}_2} \|\nabla^2 f_h(x)\|\}. \quad (36)$$

\textbf{Proof of Theorem 5.} From Lemma 9 we have that the transformed potential $f_h$ has Lipschitz gradients with parameter $L_h = \max\{L, \max_{|x| \leq \tilde{N}_2} \|\nabla^2 f_h(x)\|\}$. Furthermore, as shown in equation (35) in the proof of Lemma 8, $\pi_h \sim LS(C_{h, LSI})$ with $C_{h, LSI} = 2e^{\text{Osc} f_h/\rho}$. Hence, we can apply [VW19, Theorem 1] to obtain that when $0 < \gamma < \frac{1}{2C_{h, LSI}L_h^2}$,

$$H_{\pi_h}(\rho_{\gamma}) \leq e^{-\frac{2\gamma \rho_{\gamma}}{C_{h, LSI}}} H_{\pi_h}(\rho_0) + 4C_{h, LSI}L_h^2\gamma d. \quad (37)$$

Now, applying Proposition 1 with $\Phi(x) = x \log x$ to (37), we get

$$H_{\pi}(\nu_{\gamma}) \leq e^{-\frac{2\gamma \rho_{\gamma}}{C_{h, LSI}}} H_{\pi}(\nu_0) + 4C_{h, LSI}L_h^2\gamma d. \quad (38)$$

The mixing time estimate in the theorem instantly follows from equation (38).

\textbf{7.4. Proof of Theorem 2 and Theorem 4.} In this section we will prove Theorem 2 and Theorem 4. First we introduce a result which explains the relation between Assumption A2 and Assumption B2.

\textbf{Lemma 10.} If a potential function $f$ satisfies Assumption A2, then the transformed potential $f_h$ satisfies Assumption B2.

\textbf{Proof of Lemma 10:} If a potential function $f$ satisfies Assumption A2 with parameters $\mu, N_2$ and $\theta$, then when $|x| \geq b^{-\frac{\theta}{h}}$, the eigenvalues of $\nabla^2 f_h(x)$ are studied in (22) and (23). Applying $\psi(|x|) = e^{b|x|^\beta}$, we have the following estimates on the eigenvalues: for all $|x| \geq \tilde{N}_5 := \max\{b^{-\frac{\theta}{h}}, N_2\}$ we have

$$\lambda_1 = f''(\psi(|x|))\psi'(|x|)^2 + f'(\psi(|x|))\psi''(|x|) - b\beta(\beta - 1)d|x|^{\beta - 2} - (d - \beta)|x|^{-2}$$
\[
\begin{align*}
\lambda_2 & = f'(\psi(|x|))\psi'(|x|)|x|^{-1} - b\beta d|x|^{\beta - 2} + (d - \beta)|x|^{-2} \\
& \geq \frac{\mu}{(1 + \frac{1}{4}|x|^2)^{\frac{3}{2}}},
\end{align*}
\]

where the inequality follows from Assumption A2. Therefore for all \(|x| \geq \tilde{N}_5\), we have that

\[
\nabla^2 f_h(x) \succeq \frac{\mu}{(1 + \frac{1}{4}|x|^2)^{\frac{3}{2}}} I_d.
\]

Meanwhile since \(f_h \in C^2(\mathbb{R}^d)\), we can construct \(\tilde{f}_h \in C^2(\mathbb{R}^d)\) such that \(\tilde{f}_h(x) = f_h(x)\) for all \(|x| \geq \tilde{N}_5\), \(\nabla^2 \tilde{f}_h(x) \succeq \frac{\mu}{(1 + \frac{1}{4}|x|^2)^{\frac{3}{2}}} I_d\) for all \(x \in \mathbb{R}^d\). Furthermore, since both \(f_h\) and \(\tilde{f}_h\) are continuous,

\[
\xi := \left\| f_h - \tilde{f}_h \right\|_{\infty} = \max_{|x| \leq \tilde{N}_5} |f_h(x) - \tilde{f}_h(x)| < \infty
\]

Therefore \(f_h\) satisfies Assumption B2 with parameters \(\xi_h = \xi\), \(\mu_h = \mu\) and \(\theta_h = \theta\). \(\square\)

Next we introduce a result which explains the relation between Assumption A1 and Assumption B1.

**Lemma 11.** If a potential function \(f\) satisfies Assumption A1 then the transformed potential \(f_h\) satisfies Assumption B1.

**Proof of Lemma 11:** If a potential function \(f\) satisfies Assumption A1 with parameters \(\alpha, A, B\), as we have shown in (19), for all \(|x| \geq b^{-\frac{1}{\beta}}\) we have that

\[
\langle \nabla f_h(x), x \rangle = f'(e^{b|x|^\beta})b\beta |x|^{\beta - 1} e^{b|x|^\beta} - \beta bd|x|^\beta + (d - \beta)A|x|^\alpha - B.
\]

where the first inequality follows from Assumption A1. Since \(f_h \in C^2(\mathbb{R}^d)\), we have

\[
\min_{|x| \leq b^{-\frac{1}{\beta}}} \langle \nabla f_h(x), x \rangle > -\infty
\]

Therefore \(f_h\) satisfies Assumption B1 with parameters

\[
\alpha_h = \alpha, \quad A_h = A, \quad B_h = \max\{0, B, -\min_{|x| \leq b^{-\frac{1}{\beta}}} \langle \nabla f_h(x), x \rangle\} \in [0, \infty).
\]

\(\square\)

With the above two results, we are ready to prove Theorem 2.

**Proof of Theorem 2:** Taking the derivative of the KL-divergence from \(\rho_t\) to \(\pi_h\), we have

\[
\begin{align*}
\frac{d}{dt} H_{\pi_h}(\rho_t) &= \frac{d}{dt} \int_{\mathbb{R}^d} \log \left( \frac{\rho_t(x)}{\pi_h(x)} \right) \rho_t(x) dx \\
&= \int_{\mathbb{R}^d} \frac{\partial \rho_t(x)}{\partial t} \log \left( \frac{\rho_t(x)}{\pi_h(x)} \right) dx + \int_{\mathbb{R}^d} \rho_t(x) \frac{\pi_h(x)}{\rho_t(x)} \frac{1}{\pi_h(x)} \frac{\partial \rho_t(x)}{\partial t} dx \\
&= \int_{\mathbb{R}^d} \nabla \cdot \left( \rho_t(x) \nabla \log \left( \frac{\rho_t(x)}{\pi_h(x)} \right) \right) \log \left( \frac{\rho_t(x)}{\pi_h(x)} \right) dx + 0 \\
&= - \int_{\mathbb{R}^d} \rho_t(x) \left| \nabla \log \left( \frac{\rho_t(x)}{\pi_h(x)} \right) \right|^2 dx = -I_{\pi_h}(\rho_t),
\end{align*}
\]

(41)
where third identity follows from the Fokker-Planck equation
\[
\frac{\partial \rho_t}{\partial t} = \nabla \cdot (\rho_t \nabla f_h) + \Delta \rho_t = \nabla \cdot \left( \rho_t \nabla \log \frac{\rho_t}{\pi_h} \right),
\]
and the fact that \( \int \frac{\partial \rho_t}{\partial t} \, dx = \frac{d}{dt} \int \rho_t \, dx = 0 \). According to Lemma 10, we have that \( \pi_h \) satisfies Assumption B2. Hence, according to [EH21, Theorem 1], \( \pi_h \) satisfies a modified LSI, i.e. for all probability densities \( \rho \):
\[
H_{\pi_h}(\rho) \leq C_{h,M\text{-LSI}} I_{\pi_h}(\rho)^{1-\delta} M_s(\rho + \pi_h)^\delta,
\]
where \( M_s(\rho) = \int_{\mathbb{R}^d} (1 + |x|^2)^{s/2} \rho(x) \, dx \) is the \( s \)-th moment of any function \( \rho \) and with \( \xi \) defined in (39), \( \delta \) and \( \lambda \) are defined as
\[
\delta := \frac{\theta}{s - 2 + 2\theta} \in \left[0, \frac{1}{2}\right), \quad C_{h,M\text{-LSI}} = 4e^{2\xi} \mu^{-\frac{s-2}{s-2+2\theta}}.
\]
Hence (41) can be further written as
\[
\frac{d}{dt} H_{\pi_h}(\rho_t) \leq -\lambda^{-\frac{1-\delta}{1-\delta}} H_{\pi_h}(\rho_t)^\frac{1}{1-\delta} (M_s(\rho_t + \pi_h)^{-\frac{\delta}{1-\delta}}.
\]
By Lemma 11, the transformed potential \( f_h \) satisfies Assumption B1 with parameters \( \alpha_h, A_h, B_h \). Hence, according to [TV00, Proposition 2], under the \( \alpha_h \)-dissipativity of \( f_h \), for all \( s \geq 2 \):
\[
M_s(\rho_t + \pi_h) \leq C_s t,
\]
where
\[
C_s = \sup_{x \geq 0} \left( (ds + s(s-2) - sA_h + sB_h) x^{\frac{s-2}{s-2+\alpha}} - A_h x \right) < \infty.
\]
Therefore the upper bound in (43) can improved as
\[
\frac{d}{dt} H_{\pi_h}(\rho_t) \leq -\lambda^{-\frac{1-\delta}{1-\delta}} H_{\pi_h}(\rho_t)^\frac{1}{1-\delta} (M_s(\rho_0 + \pi_h) + C_s t)^{-\frac{\delta}{1-\delta}}. 
\]
Rewriting (45) as
\[
-H_{\pi_h}(\rho_t)^{-\frac{1}{1-\delta}} \frac{d}{dt} H_{\pi_h}(\rho_t) \geq (\lambda C_s^\delta)^{-\frac{1}{1-\delta}} (M_s(\rho_0 + \pi_h)C_s^{-1} + t)^{-\frac{\delta}{1-\delta}},
\]
and applying Gronwall’s inequality, we obtain
\[
H_{\pi_h}(\rho_t) \leq \left( \frac{1-2\delta}{\delta} \right)^{\frac{1}{1-\delta}} (\lambda C_s^\delta)^{\frac{1}{1-\delta}} (M_s(\rho_0 + \pi_h)C_s^{-1} + t)^{-\frac{1+2\delta}{1-\delta}} \leq \frac{C}{l^t}.
\]
with \( C = \left( \frac{1-2\delta}{\delta} \right)^{\frac{1}{1-\delta}} (\lambda C_s^\delta)^{\frac{1}{1-\delta}} \) and \( l = (1 - 2\delta)/\delta \).

To prove Theorem 4, we require the following result on the relationship between Assumption A5 and Assumption B5.

**Lemma 12.** If the density \( \pi \) satisfies Assumption A5, then \( \pi \) satisfies Assumption B5.

**Proof of Lemma 12.** Without loss of generality, we can assume that \( N_5 \geq \epsilon \). When \( \lambda \geq N_5 \geq \epsilon \),
\[
\pi \{ | \cdot | \geq m + \lambda \} \leq 2 \exp \left( - \left( \frac{g^{-1}(\lambda)}{C_{\text{TAIL}}} \right)^{\alpha_1} \right)
\]
with \( C_{\text{TAIL}} = C_{\text{TAIL}}^* \). When \( \lambda \in [0, N_5] \), we have
\[
\pi \{ | \cdot | \geq m + \lambda \} \leq \pi \{ | \cdot | \geq m \}
\leq 2 \exp \left( - \left( \frac{g^{-1}(\lambda)}{C_{\text{TAIL}}} \right)^{\alpha_1} \right)
with $C_{\text{tail}} = g^{-1}(N_5) \left( \log \frac{2}{\pi \{ | \cdot | \geq m \}} \right)^{\frac{\alpha_1}{2}}$. Therefore for all $\lambda \geq 0$,
\[
\pi \{ | \cdot | \geq m + \lambda \} \leq 2 \exp \left( - \left( \frac{g^{-1}(\lambda)}{C_{\text{tail}}} \right)^{\alpha_1} \right),
\]  
(48)

with
\[
C_{\text{tail}} = \max \left\{ C_{\text{tail}}^*, g^{-1}(N_5) \left( \log \frac{2}{\pi \{ | \cdot | \geq m \}} \right)^{\frac{\alpha_1}{2}} \right\}.
\]

From (48), let $X \in \mathbb{R}^d$ be a random variable with density $\pi$ and $Y := h^{-1}(X)$. Then $Y \in \mathbb{R}^d$ is a random variable with density $\pi_h$. We get
\[
\pi_h \{ | \cdot | \geq m_h + \lambda \} = \mathbb{P} (|Y| \geq m_h + \lambda)
= \mathbb{P} (g^{-1}(|X|) \geq m_h + \lambda)
= \mathbb{P} (|X| \geq g(m+h)).
\]

For any fixed $\lambda \geq 0$, we can choose $m_h(\lambda) = g^{-1}(m + g(\lambda)) - \lambda$ and we get
\[
\pi_h \{ | \cdot | \geq m_h(\lambda) + \lambda \} \leq \mathbb{P} (|X| \geq m + g(\lambda))
= \pi \{ | \cdot | \geq m + g(\lambda) \}
\leq 2 \exp \left( - \left( \frac{\psi^{-1}(g(\lambda))}{C_{\text{tail}}} \right)^{\alpha_1} \right)
= 2 \exp \left( - \left( \frac{\lambda}{C_{\text{tail}}} \right)^{\alpha_1} \right).
\]

We next claim that there exists a constant $m_h$ such that $m_h(\lambda) \leq m_h$ for all $\lambda \geq 0$. To prove the claim, we apply Taylor expansion in the definition of $m_h(\lambda)$ and we get for any $\lambda \geq 0$, there exists a constant $\theta(\lambda) \in [0, m]$ such that
\[
m_h(\lambda) = g^{-1} (g(\lambda)) + (g^{-1})'(g(\lambda)) m - \lambda
\leq m \sup_{r \in [0, \infty)} \left( g^{-1}'(r) \right).
\]

According to our construction of $g$, we have that $\sup_{r \in [0, \infty)} \left( g^{-1}'(r) \right) < \infty$. Therefore we can pick $m_h = m \left( \sup_{r \in [0, \infty)} \left( g^{-1}'(r) \right) \right)$ which is a constant independent of $\lambda$, which proves the claim. Hence, we get for all $\lambda \geq 0$,
\[
\pi_h \{ | \cdot | \geq m_h + \lambda \} \leq 2 \exp \left( - \left( \frac{\psi^{-1}(g(\lambda))}{C_{\text{tail}}} \right)^{\alpha_1} \right).
\]

That is, the transformed density $\pi_h$ satisfies Assumption B5 with
\[
\alpha_h = \alpha_1,
\]
(49)

\[
C_{h,\text{tail}} = \max \left\{ C_{\text{tail}}^*, g^{-1}(N_6) \left( \log \frac{2}{\pi \{ | \cdot | \geq m \}} \right)^{\frac{1}{\alpha_1}} \right\},
\]
(50)

\[
m_h = m \left( \sup_{r \in [0, \infty)} \left( g^{-1}'(r) \right) \right).
\]
(51)
Proof of Theorem 4. Let \( \hat{\pi}_h \) be a modified density to \( \pi_h \). It’s defined as, for \( \hat{\gamma}, \hat{R} > 0 \),
\[
\hat{\pi}_h \propto \exp(-\hat{f}_h), \quad \hat{f}_h(x) := f_h(x) + \frac{\gamma}{2}(|x| - \hat{R})^2.
\]
(52)
Here \( (|x| - \hat{R})^2 \) is interpreted as \( \max \left\{ |x| - \hat{R}, 0 \right\}^2 \). Furthermore, \( \hat{R} \) is chosen so that \( \pi_h(B(0, \hat{R})) \geq \frac{1}{2} \), where \( B(0, \hat{R}) \) is an Euclidean ball of radius \( \hat{R} \) centered at zero. With this definition, the proof follows immediately from Lemma 10, [EH21, Theorem 1] and [CEL+21, Theorem 8]. \( \square \)

7.5. Proofs for Section 5.

Proof of Proposition 4: When \( |x| \geq g(b^{-\frac{1}{\beta}}) = e \), the inverse of \( g \) can be represented as
\[
g^{-1}(|x|) = b^{-\frac{1}{\beta}} \log \frac{1}{|x|}.
\]
Therefore, Assumption A3 can be reformulated as for all \( |x| \geq g^{-1}(N_2 \lor b^{-\frac{1}{\beta}}) := g^{-1}(\tilde{N}_1) \):
\[
\begin{align*}
&b^{-\frac{2}{\beta}} |f'(|x|)| \beta \log^{-\frac{1}{\beta}}(|x|) |x| - \beta d \log^{-\frac{1}{\beta}}(|x|) + (d - \beta) \log^{-\frac{1}{\beta}}(|x|) > \rho, \\
&b^{-\frac{2}{\beta}} |f''(|x|)| \beta^2 \log^{-\frac{2}{\beta}}(|x|) |x|^2 + f'(|x|) \beta (\beta - 1) \log^{-\frac{1}{\beta}}(|x|) |x| \\
&+ f'(|x|) \beta^2 \log^{-\frac{2}{\beta}}(|x|) |x| - \beta (\beta - 1) \log^{-\frac{1}{\beta}}(|x|) - (d - \beta) \log^{-\frac{1}{\beta}}(|x|) > \rho.
\end{align*}
\]
(53)
(54)
Now, (53) gives a lower bound on \( f'(|x|) \) of the form:
\[
f'(|x|) \geq \rho b^{-\frac{2}{\beta}} \beta^{-1} \log^{-\frac{1}{\beta}}(|x|) |x|^{-1} + d |x|^{-1} - \frac{d - \beta}{\beta} \log^{-1}(|x|) |x|^{-1}.
\]
Defining \( N := g^{-1}(\tilde{N}_1) \) and integrating from \( N \) to a larger value with respect to \( |x| \), we obtain
\[
f(|x|) \geq f(N) + \frac{1}{2} \rho b^{-\frac{2}{\beta}} (\log \frac{1}{N} - \log \frac{1}{|x|}) + \frac{1}{2} d \log |x| - \log N - \frac{d - \beta}{\beta} (\log \log |x| - \log \log N)
\]
\[
:= C_{N,d} + \frac{1}{2} \rho b^{-\frac{2}{\beta}} \log \frac{1}{|x|} + d \log |x| - \frac{d - \beta}{\beta} \log \log |x|.
\]
Therefore, we have for all \( |x| > N \):
\[
e^{f(x)} \geq C_{N,d} |x|^\frac{1}{2} \rho b^{-\frac{2}{\beta}} \log^{-\frac{1}{\beta}}(|x|) + d \log^{-\frac{d - \beta}{\beta}}(|x|).
\]
(55)
To prove that \( \pi \) satisfies the Poincaré-type inequalities, we leverage the results in [Wan14] and [WW15]. We consider the following quantity with \( \theta \in (0, 2) \) and \( x \neq y \):
\[
e^{f(x)} + e^{f(y)}
\]
\[
\frac{e^{f(x)} + e^{f(y)}}{|x - y|^{d+\theta}} \geq C_{N,d} |x|^\frac{1}{2} \rho b^{-\frac{2}{\beta}} \log^{-\frac{1}{\beta}}(|x|) + d \log^{-\frac{d - \beta}{\beta}}(|x|) + |y|^\frac{1}{2} \rho b^{-\frac{2}{\beta}} \log^{-\frac{1}{\beta}}(|y|) + d \log^{-\frac{d - \beta}{\beta}}(|y|).
\]
(56)
Since \( |x - y|^{d+\theta} \leq 2^{d+\theta-1} (|x|^{d+\theta} + |y|^{d+\theta}) \), we have for all \( x \neq y \) and \( |x|, |y| > N \), we have
\[
\frac{e^{f(x)} + e^{f(y)}}{|x - y|^{d+\theta}} \geq C_{N,d} |x|^\frac{1}{2} \rho b^{-\frac{2}{\beta}} \log^{-\frac{1}{\beta}}(|x|) + d \log^{-\frac{d - \beta}{\beta}}(|x|) + |y|^\frac{1}{2} \rho b^{-\frac{2}{\beta}} \log^{-\frac{1}{\beta}}(|y|) + d \log^{-\frac{d - \beta}{\beta}}(|y|).
\]
Then, (54) gives the lower bound
\[
f''(|x|) + f'(|x|) \left( \frac{\beta - 1}{\beta} \log^{-1}(|x|)|x|^{-1} + |x|^{-1} \right)
\]
\[
\geq \rho b^{-\frac{2}{\beta}} \beta^{-2} \log^{-\frac{2}{\beta} - \frac{2}{\beta}}(|x|) |x|^{-2} + \frac{\beta - 1}{\beta} \log^{-1}(|x|) |x|^{-2} + \frac{d - \beta}{\beta^2} \log^{-2}(|x|) |x|^{-2}.
\]
By multiplying $\log^{1 - \frac{1}{\beta}}(|x|)|x|$ on both sides, for all $|x| > N$ we obtain

$$\frac{d}{d|x|} \left( f'(|x|) \log^{1 - \frac{1}{\beta}}(|x|)|x| \right) \geq \rho b^{- \frac{2}{\beta}} \beta^{-2} \log^{- (1 - \frac{1}{\beta})}(|x|)|x|^{-1} + \frac{\beta - 1}{\beta} \log^{- \frac{1}{\beta}}(|x|)|x|^{-1} + \frac{d - \beta}{\beta} \log^{- \frac{1}{\beta}}(|x|)|x|^{-1},$$

which implies that

$$f'(|x|) \geq C_{N,d,1} \log^{- (1 - \frac{1}{\beta})}(|x|)|x|^{-1} + \rho b^{- \frac{2}{\beta}} \beta^{-1} \log^{- (1 - \frac{1}{\beta})}(|x|)|x|^{-1} + |x|^{-1} - (d - \beta) \log^{-1}(|x|)|x|^{-1}.$$  

Further integration implies that for all $|x| > N$, we have

$$e^f(|x|) \geq C_{N,d,2} |x|^{1 + \frac{1}{\beta} \rho b^{- \frac{2}{\beta}} \log \frac{1}{\beta} - 1} + C_{N,d,1} \beta \log \frac{1}{\beta} - 1} |x| \log^{- (d - \beta)}|x|.$$  

Since $d \geq 1$, (55) is stronger than (57), when we apply results in [Wan14], it’s enough for us to consider only (55). Therefore we have the following results:

1. When $\beta \in (1, 2)$ or $\beta = 2, \vartheta < \frac{1}{2} \rho b^{-1}$, we can see that for all $|x| > N$:

$$\frac{1}{2} \rho b^{- \frac{2}{\beta}} \log \frac{1}{\beta} - 1} + d - (d + \vartheta) > 0$$

Therefore, with (56), conditions in [Wan14, Theorem 1.1-(3)] is satisfied with

$$\omega(x) = \frac{C_{N,d}}{2d + \vartheta} |x|^{\frac{1}{\beta} \rho b^{- \frac{2}{\beta}} \log \frac{1}{\beta} - 1} \log \frac{1}{\beta} - 1} |x|,$$

such that $\lim_{|x| \to \infty} \omega(x) = \infty$. Hence, $\pi \propto \exp(-f)$ satisfies the super-Poincaré inequality.

2. When $\beta = 2, \vartheta = \frac{1}{2} \rho b^{-1}, d = 1, 2$, we can see that for all $|x| > N$, we have

$$\frac{1}{2} \rho b^{- \frac{2}{\beta}} \log \frac{1}{\beta} - 1} + d - (d + \vartheta) = 0$$

Therefore with (56), since $d = 1, 2$ we obtain

$$\frac{e^f(x) + e^f(y)}{|x - y|^{d + \vartheta}} \geq \frac{C_{N,d}}{2d + \vartheta - 1}.$$  

Hence, according to in [Wan14, Theorem 1.1-(1)], $\pi \propto \exp(-f)$ satisfies the Poincaré inequality.

3. When $\beta = 2, \vartheta = \frac{1}{2} \rho b^{-1}, d \geq 3$, for all $|x| > N$, we have that

$$\frac{1}{2} \rho b^{- \frac{2}{\beta}} \log \frac{1}{\beta} - 1} + d - (d + \vartheta) = 0$$

However, the lower bound in (56) goes to zero as $|x|, |y| \to \infty$. Neither Poincaré inequality nor super Poincaré inequality is guaranteed. However, according to [Wan14, Theorem 1.1-(2)], $\pi \propto \exp(-f)$ satisfies the weak Poincaré inequality with $\alpha(r)$ as defined in (15). When $\beta = 2, \vartheta > \frac{1}{2} \rho b^{-1}$, we can see that for all $|x| > N$:

$$\frac{1}{2} \rho b^{- \frac{2}{\beta}} \log \frac{1}{\beta} - 1} + d - (d + \vartheta) < 0$$

Neither Poincaré inequality nor super Poincaré inequality is guaranteed. However, according to [Wan14, Theorem 1.1-(2)], $\pi \propto \exp(-f)$ satisfies the weak Poincaré inequality with $\alpha(r)$ as in (15).
Proof of Proposition 2: Similar to Assumption A3, Assumption A1 are sufficient conditions for Poincaré type inequalities as well. First note that Assumption A1 is equivalent to the following inequality: for all \( |x| > N := g^{-1}(N_1 \lor b^{-\beta}) \) we have
\[
f'(|x|) \geq A\beta^{-1}b^{-\beta} \log^{\beta^{-1}}(|x|)|x|^{-1} + d|x|^{-1} - \frac{B}{\beta} \log^{-1}(|x|)|x|^{-1}.
\]
Integrating with respect to \( |x| \), we obtain
\[
f(|x|) \geq C_{N,d} + Ab^{-\beta} \alpha^{-1} \log^{\beta}(|x|) + d \log |x| - \frac{B}{\beta} \log \log |x|.
\]
For all \( |x| \geq N \), we then have
\[
e^{f(|x|)} \geq C_{N,d} |x|^A \alpha^{-1}b^{-\beta} \log^{\beta^{-1}}(|x|)+d \log -\frac{B}{\beta} |x|.
\]
and
\[
e^{f(x)} + e^{f(y)} \geq C_{N,d} |x|^A \alpha^{-1}b^{-\beta} \log^{\beta^{-1}}(|x|)+d \log -\frac{B}{\beta} |x| + |y|^A \alpha^{-1}b^{-\beta} \log^{\beta^{-1}}(|y|)+d \log -\frac{B}{\beta} |y|.
\]
We now consider different cases.

1. When \( \alpha > \beta \) or \( \alpha = \beta, \vartheta < A\beta^{-1}b^{-1} \), we can see that for all \( |x| > N \) we have that
\[
A\alpha^{-1}b^{-\beta} \log^{\beta^{-1}}(|x|)+d - (d + \vartheta) > 0.
\]
Therefore, with (58), the conditions in [Wan14, Theorem 1.1-(3)] are satisfied with
\[
\omega(x) = \frac{C_{N,d} |x|^A \alpha^{-1}b^{-\beta} \log^{\beta^{-1}}(|x|)-d \log -\frac{B}{\beta} (|x|)},
\]
such that \( \lim_{|x| \to \infty} \omega(x) = \infty \). Hence, \( \pi \propto \exp(-f) \) satisfies the super Poincaré inequality.

2. When \( \alpha = \beta, \vartheta = A\beta^{-1}b^{-1} \), we can see that for all \( |x| > N \) we have
\[
A\alpha^{-1}b^{-\beta} \log^{\beta^{-1}}(|x|)+d - (d + \vartheta) = 0.
\]
However, the lower bound in (58) goes to zero as \( |x|, |y| \to \infty \). Hence, Neither the Poincaré inequality nor the super Poincaré inequality is satisfied. However, according to [Wan14, Theorem 1.1-(2)], the density \( \pi \propto \exp(-f) \) satisfies the weak Poincaré inequality with \( \alpha(r) \) as in (15).

3. When \( \alpha = \beta, \vartheta > A\beta^{-1}b^{-1} \), we can see that for all \( |x| > N \) we have
\[
A\alpha^{-1}b^{-\beta} \log^{\beta^{-1}}(|x|)+d - (d + \vartheta) < 0.
\]
Hence, neither the Poincaré inequality nor super Poincaré inequality is guaranteed. However, according to [Wan14, Theorem 1.1-(2)], \( \pi \propto \exp(-f) \) satisfies the weak Poincaré inequality with \( \alpha(r) \) as in (15).

\[\square\]

Proof of Proposition 3: Similar as in the proof of Proposition 4, Assumption A2 is equivalent to the following two inequalities: for all \( |x| \geq N \),
\[
f'(|x|) \geq \mu b^{-\beta} \beta^{-1} \log^{-(1-\beta)}(|x|)(1 + \frac{1}{4} b^{-\beta} \log^{\beta}(|x|))^{-\beta} |x|^{-1} + d|x|^{-1} - \frac{d - \beta}{\beta} \log^{-1}(|x|)|x|^{-1},
\]
\[
f''(|x|) + f'(|x|) \left( \frac{\beta - 1}{\beta} \log^{-1}(|x|)|x|^{-1} + |x|^{-1} \right)
\]
We now consider the different cases as before.

(1) When \( \theta < 2 - \beta \), (62) is stronger than (61). We have that for large \(|x|\),

\[
(1 - \theta)^{-1}(2 - \theta)^{-1} \mu b^{-2 - \theta} \log^{\frac{2 - \theta}{\beta}}(|x|) + C_N \log^{\frac{1}{\beta^2}}(|x|) + 1 - (d + \vartheta) > 0.
\]

Therefore, when \( \theta < 2 - \beta \), the conditions in [Wan14, Theorem 1.1-(3)] are satisfied with

\[
\omega(x) = \frac{C_{N,d}}{2^{d + \vartheta}} |x|^{(1 - \theta)^{-1}(2 - \theta)^{-1} \mu b^{-2 - \theta} \log^{\frac{2 - \theta}{\beta}}(|x|) + 1 - (d + \vartheta) \log^{\frac{1}{\beta^2}}(|x|)},
\]

with \( \lim_{|x| \to \infty} \omega(x) = \infty \). Hence, \( \pi \propto \exp(-f) \) satisfies the super Poincaré inequality.

(2) When \( \theta = 2 - \beta, \mu \beta^{-1}b^{-1} > \vartheta \), (61) is stronger than (62). We have that for all \(|x| > N'\),

\[
(2 - \theta)^{-1} \mu b^{-2 - \theta} \log^{\frac{2 - \theta}{\beta}}(|x|) + d - (d + \vartheta) > 0.
\]

Therefore, when \( \theta = 2 - \beta, \mu \beta^{-1}b^{-1} > \vartheta \), the conditions in [Wan14, Theorem 1.1-(3)] are satisfied with

\[
\omega(x) = \frac{C_{N,d}}{2^{d + \vartheta}} |x|^{b^{-2 - \theta} \log^{\frac{2 - \theta}{\beta}}(|x|) - \vartheta \log^{\frac{1}{\beta^2}}(|x|)},
\]

with \( \lim_{|x| \to \infty} \omega(x) = \infty \). Hence, \( \pi \propto \exp(-f) \) satisfies the super Poincaré inequality.

(3) When \( \theta = 2 - \beta, \mu \beta^{-1}b^{-1} \leq \vartheta \), (61) is stronger than (62). We have that for all \(|x|\) large enough,

\[
(2 - \theta)^{-1} \mu b^{-2 - \theta} \log^{\frac{2 - \theta}{\beta}}(|x|) + d - (d + \vartheta) \leq 0.
\]

Neither Poincaré inequality nor super Poincaré inequality is guaranteed. According to [Wan14, Theorem 1.1-(2)], \( \pi \propto \exp(-f) \) satisfies the weak Poincaré inequality with \( \alpha(r) \) in (15).

(4) When \( \theta > 2 - \beta \), (59) is stronger than (62). We have that for all \(|x|\) large enough,

\[
(2 - \theta)^{-1} \mu b^{-2 - \theta} \log^{\frac{2 - \theta}{\beta}}(|x|) + d - (d + \vartheta) < 0.
\]

Hence, neither the Poincaré inequality nor the super Poincaré inequality is guaranteed. However, according to [Wan14, Theorem 1.1-(2)], \( \pi \propto \exp(-f) \) satisfies the weak Poincaré inequality with \( \alpha(r) \) in (15).
7.6. Proofs for Section 6.

Proof of Lemma 2: First, it’s easy to check that \( g_{in}(0) = 0 \) and \( g_{in}(b^{-\frac{1}{2}}) = e \), which implies that \( g \in C([0, \infty)) \). Next note that we have

\[
\log \frac{g_{in}(r)}{r} = \log(b^{\frac{1}{2}}) + br^2 - \frac{10}{3} b^{\frac{3}{2}} r^3 + \frac{15}{4} b^2 r^4 - \frac{6}{5} b^{\frac{5}{2}} r^5 + \frac{47}{60}.
\]

Hence, we can then check that

\[
\lim_{r \to 0^+} \left| \frac{d}{dr} \log \frac{g_{in}(r)}{r} \right| < \infty \quad \text{and} \quad \lim_{r \to 0^+} \left| \frac{d^2}{dr^2} \log \frac{g_{in}(r)}{r} \right| < \infty.
\]

Note that the first derivative of \( g_{in} \) is given by

\[
g_{in}'(r) = b^{\frac{1}{2}} \left( 1 + 2br^2 - 10b^{\frac{3}{2}} r^3 + 15b^2 r^4 - 6b^{\frac{5}{2}} r^5 \right) \exp \left( br^2 - \frac{10}{3} b^{\frac{3}{2}} r^3 + \frac{15}{4} b^2 r^4 - \frac{6}{5} b^{\frac{5}{2}} r^5 + \frac{47}{60} \right).
\]

Hence, we have that

\[
\lim_{r \to 0^+} |f'(g_{in}(r))g_{in}'(r)| = (d + \varepsilon) \lim_{r \to 0^+} \left| \frac{g_{in}'(r)}{1 + g_{in}(r)^2} \right| \frac{g_{in}(r)}{r} < \infty.
\]

Similarly, as \( g_{in}'(b^{-\frac{1}{2}}) = 2b^{\frac{1}{2}} e \) and

\[
\log g_{in}'(r) = \log(b^{\frac{1}{2}}) + \log(1 + 2br^2 - 10b^{\frac{3}{2}} r^3 + 15b^2 r^4 - 6b^{\frac{5}{2}} r^5) + br^2 - \frac{10}{3} b^{\frac{3}{2}} r^3 + \frac{15}{4} b^2 r^4 - \frac{6}{5} b^{\frac{5}{2}} r^5 + \frac{47}{60},
\]

we can also check that

\[
\lim_{r \to 0^+} \left| \frac{d^2}{dr^2} \log g_{in}'(r) \right| < \infty \quad \text{and} \quad \lim_{r \to 0^+} \left| \frac{d}{dr} \log g_{in}'(r) \right| < \infty.
\]

Similarly, by taking additional higher order derivatives it is easy to check that \( g_{in}''(b^{-\frac{1}{2}}) = 6be \) and \( g_{in}'''(b^{-\frac{1}{2}}) = 20b^{\frac{1}{2}} e \). We omit the tedious but elementary calculations here. \( \square \)

Proof of Lemma 3. It’s obvious that \( f \in C^2(\mathbb{R}^d) \) and it’s isotropic. Note that

\[
\frac{d}{d|x|} f(|x|) = (d + \kappa) \frac{|x|}{1 + |x|^2},
\]

\[
\frac{d^2}{d|x|^2} f(|x|) = (d + \kappa) \frac{1 - |x|^2}{(1 + |x|^2)^2}.
\]

With \( \psi(r) = e^{br^3} \) for all \( r \geq b^{-\frac{1}{3}} \), based on (22) and (23), we have that for all \( |x| \gg b^{-\frac{1}{3}} \) and \( k \in \mathbb{Z}^+ \),

\[
f'(\psi(|x|))\psi'(|x|)|x|^{-1} - b\beta d|x|^{\beta - 2} + (d - \beta)|x|^{-2} = \kappa b\beta |x|^{\beta - 2} + (d - \beta)|x|^{-2} + o(|x|^{-k}),
\]

and

\[
f''(\psi(|x|))\psi''(|x|) + f'(\psi(|x|))\psi''(|x|) - b\beta(\beta - 1)|x|^{\beta - 2} - (d - \beta)|x|^{-2} = \kappa b\beta(\beta - 1)|x|^{\beta - 2} - (d - \beta)|x|^{-2} + o(|x|^{-k}).
\]

Note that for all \( |x| \geq b^{-\frac{1}{3}} \), we have

\[
\kappa b\beta |x|^{\beta - 2} \leq \kappa b^{\frac{2}{3}},
\]

\[
\kappa b\beta(\beta - 1)|x|^{\beta - 2} \leq \kappa b^{\frac{2}{3}}.
\]
The last inequality holds since $\beta \in (1,2)$. Therefore $f$ satisfies Assumption A4 with some $N_4 > 0$ and $L = 2\kappa \beta b^3$.

To check Assumption A1, notice that for all $|x| \gg b^{-\frac{2}{\beta}}$ and $k \in \mathbb{Z}^+$, we have
\[
f'(\psi(|x|)) \psi'(|x|)|x|^{-1} - b\beta d|x|^{-2} + (d - \beta)|x|^{-2} = \kappa \beta |x|^{-2} + (d - \beta)|x|^{-2} + o(|x|^{-k}),
\]
and
\[
f''(\psi(|x|)) \psi''(|x|)|x|^{-1} - b\beta(\beta - 1)|x|^{-2} - (d - \beta)|x|^{-2} = \kappa \beta (\beta - 1)|x|^{-2} - (d - \beta)|x|^{-2} + o(|x|^{-k}).
\]
Therefore Assumption A1 is satisfied with $A = \kappa \beta b$, $\alpha = \beta$, and some $B \geq 0, N_1 > 0$.

Lastly, to check Assumption A2, similar to the calculation in checking Assumption A4, for all $|x| \gg b^{-\frac{1}{2}}$ and $k \in \mathbb{Z}^+$, we have
\[
f'(\psi(|x|)) \psi'(|x|)|x|^{-1} - b\beta d|x|^{-2} + (d - \beta)|x|^{-2} = \kappa \beta |x|^{-2} + (d - \beta)|x|^{-2} + o(|x|^{-k}),
\]
and
\[
f''(\psi(|x|)) \psi''(|x|)|x|^{-1} - b\beta(\beta - 1)|x|^{-2} - (d - \beta)|x|^{-2} = \kappa \beta (\beta - 1)|x|^{-2} - (d - \beta)|x|^{-2} + o(|x|^{-k}).
\]
Therefore Assumption A2 is satisfied with arbitrary $\mu \in (0, \kappa \beta (\beta - 1))$, $\theta = 2 - \beta \geq 0$ and some $N_2 > 0$. \hfill \Box

7.6.1. Order estimation of mixing time when $\beta = 2$. When $f(x) = d\kappa b \log(1 + |x|^2)$, for all $|x| > b^{-\frac{1}{2}}$, the two eigenvalues of $\nabla^2 f_h(x)$ can be studied via (22) and (23). We obtain
\[
\lambda_1 = 2b\kappa + (d - 2)|x|^{-2} - 2b(d + \kappa) \frac{1}{1 + e^{2b|x|^2}},
\]
\[
\lambda_2 = 2b\kappa - (d - 2)|x|^{-2} + 2b(d + \kappa) \frac{4b|x|^2 - 1}{(1 + e^{2b|x|^2})^2} + 1.
\]
Therefore, for all $|x| > b^{-\frac{1}{2}}$, we can estimate $\lambda_1$:
\[
2b\kappa - 2b\kappa \frac{1}{1 + e^2} - 2b < \lambda_1 < 2b\kappa + (d - 2)b,
\]
which can be simplified as
\[
2b\left(\kappa \frac{e^2}{1 + e^2} - 1\right) < \lambda_1 < 2b(\kappa + d - 1) - 1,
\]
for all $|x| > b^{-\frac{1}{2}}$. Similarly, we can obtain the following estimate on $\lambda_2$:
\[
2b\kappa - 2b\kappa \frac{e^4 - 3e^2 - 1}{(1 + e^2)^2} < \lambda_2 < 2b\kappa + 2b + 2b\kappa \frac{3e^2 + 1}{(1 + e^2)^2}.
\]
The above estimation can be further simplified as
\[
2b(\kappa + 0.2d) < \lambda_2 < 2b(1.5\kappa + 1).
\]
According to (65) and (66), we instantly have the locally Lipschitz constant, denoted as $L_{h,\text{loc}}$, for $f_h$ in the region $\{|x| > b^{-\frac{3}{2}}\}$ being characterized as
\[
L_{h,\text{loc}} = 2b \max \left\{ \kappa + \frac{d}{2} - 1, 1.5\kappa + 1 \right\}.
\]
Next, for $|x| \leq b^{-\frac{1}{2}}$, we can check that for any fixed $d$, we have
\[
\lim_{|x| \to 0} |\lambda_i(|x|)| < \infty \quad i = 1, 2.
\]
Therefore we can check that for any fixed $|x| \leq b^{-\frac{1}{2}}$, we have $|\lambda_i(x)| = O(d)$ for $i = 1, 2$ when $d \gg 1$. Thus we can conclude the global Lipschitz constant of $f_h$, $L_h = O(d)$ for $d \gg 1$. 

HEAVY-TAILED SAMPLING VIA TRANSFORMED UNADJUSTED LANGEVIN ALGORITHM 33
On the other hand side, from (65), we can see for all $\kappa > \frac{1+e^2}{e^2}$, $\lambda_1 > b\left(\frac{e^2}{1+e^2}\kappa - 1\right)$. While from (66), the lower bound would be negative if $d \gg \kappa$. Therefore to ensure both eigenvalues are lower bounded by $b\kappa$, we need to restrict the region $\{|x| > b^{-\frac{1}{2}}\}$ to set of points with larger magnitudes. For all $|x| > \left(\frac{d}{b\kappa}\right)^{\frac{1}{2}}$, we have when $d \geq \kappa$ and $d \geq 3$ that

$$\lambda_1 > b\kappa \left(2 - \frac{2}{1 + e^{2d/\kappa}} - \frac{2}{d}\right) > b\kappa,$$

$$\lambda_2 > 2b\kappa - (d - 2)(b\kappa/d) = b(\kappa + \kappa/d) > b\kappa.$$

To determine the LSI constant, we first construct a function $G_h$ such that $\nabla^2 G_h(x) \succeq b\kappa I_d$ for all $x \in \mathbb{R}^d$. Letting $\varpi := \sqrt{(d/b\kappa)}$, the function $G_h$ is defined piecewisely as

$$G_h = \begin{cases} f_h & |x| > \varpi \\ \frac{1}{3} A (|x| - \varpi)^3 + \frac{1}{2} f_h''(\varpi) (|x| - \varpi)^2 \\ + f_h'(\varpi)(|x| - \varpi) + f_h(\varpi) & |x| \leq \varpi, \end{cases}$$

where

$$f_h(\varpi) = \frac{d}{2} \log(1 + e^{-2(d/\kappa)}) + \frac{\kappa}{2} \log(1 + e^{2d/\kappa}) + (d - 2) \log(\varpi) - \log 2.$$

Note that we also have

$$f_h'(\varpi) = 2b\kappa \varpi + (d - 2) \left(d \left(\frac{b\kappa}{d}\right)^{-\frac{1}{2}} - 2bd(1 + \kappa/d)\varpi \frac{1}{1 + e^{2d/\kappa}}\right),$$

$$f_h''(\varpi) = b\kappa \left(1 + \frac{2}{d}\right) + 2bd(1 + \kappa/d) \left(\frac{4d}{\kappa} - 1\right) e^{2d/\kappa} + 1 \frac{1}{(1 + e^{2d/\kappa})^2},$$

$$A = -\frac{b\kappa}{d} \left(-2\varpi - 4bd(1 + \kappa/d)\varpi \frac{1 - 2d^2 e^{2d/\kappa}}{1 + e^{2d/\kappa}}\right) < 0.$$

With the above coefficients, we can check $G_h \in C^2(\mathbb{R}^d)$ and $\nabla G_h(x) \succeq b\kappa I_d$ for all $x \in \mathbb{R}^d$. We now consider different cases.

1. When $d \gg \kappa$ for all $k \in \mathbb{Z}$:

$$f_h(\varpi) = d + \frac{d}{2}(d - 1) \log d + O(1),$$

$$f'(\varpi) = 3d \left(d \left(\frac{b\kappa}{d}\right)^{-\frac{1}{2}} - 2d \left(\frac{b\kappa}{d}\right)^{-\frac{1}{2}} + o(d^{-k})\right),$$

$$f''(\varpi) = b\kappa + 2d \left(d \left(\frac{b\kappa}{d}\right)^{-1} + o(d^{-k})\right),$$

$$A = -2d \left(d \left(\frac{b\kappa}{d}\right)^{-\frac{3}{2}} + 4d \left(\frac{b\kappa}{d}\right)^{-\frac{3}{2}} + o(d^{-k})\right).$$

Therefore the oscillation between $f_h$ and $G_h$ can be written as

$$\text{Osc}(f_h - G_h) = \max_{0 \leq |x| \leq \varpi} |f_h(x) - G_h(x)|.$$

Since both $G_h$ and $f_h$ are monotone increasing with respect to $|x|$, we then have

$$\text{Osc}(f_h - G_h) \leq G_h(\varpi) + f_h(\varpi) = 2d + (d - 1) \log d + O(1)$$

Note that we also have

$$f_h'(\varpi) = 2b\kappa \varpi + (d - 2) \left(d \left(\frac{b\kappa}{d}\right)^{-\frac{1}{2}} - 2bd(1 + \kappa/d)\varpi \frac{1}{1 + e^{2d/\kappa}}\right),$$

$$f_h''(\varpi) = b\kappa \left(1 + \frac{2}{d}\right) + 2bd(1 + \kappa/d) \left(\frac{4d}{\kappa} - 1\right) e^{2d/\kappa} + 1 \frac{1}{(1 + e^{2d/\kappa})^2},$$

$$A = -\frac{b\kappa}{d} \left(-2\varpi - 4bd(1 + \kappa/d)\varpi \frac{1 - 2d^2 e^{2d/\kappa}}{1 + e^{2d/\kappa}}\right) < 0.$$

With the above coefficients, we can check $G_h \in C^2(\mathbb{R}^d)$ and $\nabla G_h(x) \succeq b\kappa I_d$ for all $x \in \mathbb{R}^d$. We now consider different cases.

1. When $d \gg \kappa$ for all $k \in \mathbb{Z}$:

$$f_h(\varpi) = d + \frac{d}{2}(d - 1) \log d + O(1),$$

$$f'(\varpi) = 3d \left(d \left(\frac{b\kappa}{d}\right)^{-\frac{1}{2}} - 2d \left(\frac{b\kappa}{d}\right)^{-\frac{1}{2}} + o(d^{-k})\right),$$

$$f''(\varpi) = b\kappa + 2d \left(d \left(\frac{b\kappa}{d}\right)^{-1} + o(d^{-k})\right),$$

$$A = -2d \left(d \left(\frac{b\kappa}{d}\right)^{-\frac{3}{2}} + 4d \left(\frac{b\kappa}{d}\right)^{-\frac{3}{2}} + o(d^{-k})\right).$$

Therefore the oscillation between $f_h$ and $G_h$ can be written as

$$\text{Osc}(f_h - G_h) = \max_{0 \leq |x| \leq \varpi} |f_h(x) - G_h(x)|.$$

Since both $G_h$ and $f_h$ are monotone increasing with respect to $|x|$, we then have

$$\text{Osc}(f_h - G_h) \leq G_h(\varpi) + f_h(\varpi) = 2d + (d - 1) \log d + O(1)$$
On the other hand,

\[ \text{Osc}(f_h - G_h) \geq G_h(0) - f_h(0) = \frac{1}{2}(d - 1) \log d - \frac{5}{6}d + O(1) \]

Hence, apply Holley-Strook lemma, we can calculate the LSI constant \( C_{h_{LSI}} \) as

\[ C_{h_{LSI}} \leq 2(bκ)^{-1} \exp(\text{Osc}(f_h - G_h)) \leq C(bκ)^{-1}d^{d-1} \exp(2d). \]

Furthermore, because of the lower bound on \( \text{Osc}(f_h, G_h) \), the factor \( d^{d-1} \) can be improved. Hence, according to Theorem 5, to reach \( \epsilon \)-accuracy in KL-divergence, the mixing time \( n \) satisfies:

\[ n \sim \tilde{O}(L_h C_h d \epsilon^{-1}) \leq \tilde{O}(\exp(2d) d^{d+1} \epsilon^{-1}). \]

(2) When \( d/κ = O(1) \), or equivalently when \( d/κ \to C' \), we have

\[ f_h(x) = \frac{\log(1 + e^{-2C'}) + C'\log(1) + \log(C'_b)}{1} + O(1), \]

\[ f'_h(x) = \frac{3d(C'_b)^{-\frac{1}{2}} - 2(C'_b)^{-\frac{1}{2}} + o(d^{-k})}{b} + dC'_3 + O(1) \]

\[ f''_h(x) = bdC' - 2(C'_b)^{-\frac{1}{2}} + 4(C'_b)^{-\frac{1}{2}} + o(d^{-k}) \]

\[ A = -2d(C'_b)^{-\frac{1}{2}} + C'_3 - \frac{1}{3}C'_4 + C'C'_4 \]

Therefore for all \(|x| \leq (C'/b)^{\frac{1}{2}}\), we have

\[ G_h(x) = d \left\{ \frac{1}{3}b^2C'_4|x|^3 + b(C'_b - C'\frac{1}{2}C'_4) |x|^2 + b(C'_2 - C'\frac{1}{2}C'_4 + C'C'_4) |x| \right\} \]

Similar to the previous argument, the oscillation can be upper bounded as

\[ \text{Osc}(f_h - G_h) = G_h((C'_b)^{\frac{1}{2}}) + f_h((C'_b)^{\frac{1}{2}}) \]

\[ = C'_h d + O(1), \]

where

\[ C'_h = \frac{1}{3}C'_4 C'\frac{1}{2} + (\frac{1}{2}C'_3 - C'\frac{1}{2}C'_4) C' + (C'_2 - C'\frac{1}{2}C'_3 + C'C'_4) C'\frac{1}{2} \]

\[ + (2C'_1 - C'\frac{1}{2}C'_2 + \frac{1}{2}C'_3 - \frac{1}{3}C'\frac{1}{2}C'_4) C'. \]

Hence, applying Holley-Strook Theorem, the LSI constant can be bounded by

\[ C_{h_{LSI}} \leq 2(bκ)^{-1} \exp(\text{Osc}(f_h - G_h)) \leq C(bd/C')^{-1} \exp(C'_h)^d. \]

Hence, according to [VW19], to reach \( \epsilon \)-accuracy in KL-divergence, the mixing time \( n \) satisfies

\[ n \sim \tilde{O}(L_h C_{h_{LSI}} d \epsilon^{-1}) \leq \tilde{O}((\exp(C'_h))^d d^{-1} \epsilon^{-1}). \]
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Appendix A. A Summary of Constants

For the sake of convenience, we provide a list of constants in Table 1.

Department of Mathematics, University of California, Davis.
Email address: leoh@ucdavis.edu

Department of Statistics, University of California, Davis.
Email address: kbala@ucdavis.edu

Department of Computer Science and Department of Statistical Sciences, University of Toronto
Email address: erdogdu@cs.toronto.edu

[_SZT20] Umut Şimşekli, Lingjiong Zhu, Yee Whye Teh, and Mert Gurbuzbalaban. Fractional underdamped Langevin dynamics: Retargeting SGD with momentum under heavy-tailed gradient noise. In International Conference on Machine Learning, pages 8970–8980, 2020.

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| Constant                  | Description                        | Equation  |
|--------------------------|------------------------------------|-----------|
| $\epsilon$               | Accuracy parameter                 | NA        |
| $\gamma$                 | Step-size parameter                | (10)      |
| $C_P$                    | Poincaré constant                  | (PI)      |
| $C_{LSI}$                | LSI constant                       | (LSI)     |
| $C_{mLSI, \delta}$      | m-LSI related constants           | (m-LSI)   |
| $C_{h,P}$                | Poincaré constant after Transformation | NA       |
| $C_{h,LSI}$              | LSI constant after Transformation   | NA        |
| $C_{h,mLSI}$             | m-LSI constant after Transformation | NA       |
| $r, b, \beta$            | Parameters related to transformation map | (6)      |
| $A, B, N_1, \alpha$      | Parameters related to dissipativity | Assumption A1 |
| $\mu, N_2, \theta$      | Parameters related to degenerate convexity | Assumption A2 |
| $N_3, \rho$              | Parameters related to convexity    | Assumption A3 |
| $N_4, L$                 | Parameters related to Lipschitz-gradients | Assumption A4 |
| $N_5, m, \alpha_1, C_{\text{tail}}^*$ | Parameters related to tail condition | Assumption A5 |
| $\alpha_h, A_h, B_h$     | Dissipativity parameters after transformation | Assumption B1 |
| $\xi_h, \mu_h, \theta_h$ | Degenerate Convexity at infinity after transformation | Assumption B2 |
| $\rho_h$                 | Strong-convexity parameter after transformation | NA |
| $L_h$                    | Lipschitz-gradient parameters after transformation | NA |
| $m_h, \alpha_{h,1}, C_{h,\text{TAIL}}$ | Tail condition parameters after transformation | Assumption B5 |
| $\kappa$                 | Degrees-of-freedom of $t$ distribution | NA |
| $\vartheta$              | Parameter related to super and weak Poincaré inequalities | NA |

Table 1. A list of all the constants used and their description.