ON THE SUM OF A PRIME POWER AND A POWER IN SHORT INTERVALS

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Abstract. Let $R_{k,\ell}(N)$ be the representation function for the sum of the $k$-th power of a prime and the $\ell$-th power of a positive integer. Languasco and Zaccagnini (2017) proved an asymptotic formula for the average of $R_{1,2}(N)$ over short intervals $(X, X + H]$ of the length $H$ slightly shorter than $X^{3/4}$, which is shorter than the length $H = X^{1 - \delta}$ in the exceptional set estimates of Mikawa (1993) and of Perelli and Pintz (1995). In this paper, we prove that the same asymptotic formula for $R_{1,2}(N)$ holds for $H$ of the size $X^{0.337}$. Recently, Languasco and Zaccagnini (2018) extended their result to more general $(k, \ell)$. We also consider this general case, and as a corollary, we prove a conditional result of Languasco and Zaccagnini (2018) for the case $\ell = 2$ unconditionally up to some log-factors.

1. Introduction

Let $R(n)$ be the representation function for a given additive problem with prime numbers. For example, in this paper, we consider the binary additive problem with prime numbers given by the equation

\begin{equation}
N = p^k + n^\ell, \tag{1}
\end{equation}

where $k, \ell$ are given positive integers, $p$ denotes a variable for prime numbers, and $n$ denotes a variable for positive integers. Then the representation function for the equation (1) with logarithmic weight is given by

\begin{equation}
R(N) = R_{k,\ell}(N) := \sum_{p^k + n^\ell = N} \log p, \tag{2}
\end{equation}

which counts the solutions $(p, n)$ of (1). In this paper, we consider the short interval average of such representation function

\begin{equation}
\sum_{X < N \leq X + H} R(N), \tag{3}
\end{equation}

where $2 < H < X$. Recently, Languasco and Zaccagnini gave extensive research (e.g. see \cite{5, 6, 7, 8, 9}) on the short interval average \eqref{3} for various additive problems with prime numbers, and in the case $k = 1$ of \eqref{1}, they obtained asymptotic formulas for the average \eqref{3} with $H$ shorter than in the known exceptional set estimates in short intervals.

For example, let us consider the Hardy–Littlewood equation

\begin{equation}
N = p + n^2, \tag{4}
\end{equation}

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which is the case \((k, \ell) = (1, 2)\) of our equation \([1]\). In their famous paper Partitio Numerorum III, Hardy and Littlewood \([1]\) Conjecture H] applied their circle method formally to obtain a hypothetical asymptotic formula

\[
R_{1,2}(N) = \mathcal{G}(N)\sqrt{N} + \text{(error)}, \quad (N: \text{not square})
\]
as \(N \to \infty\), where the singular series \(\mathcal{G}(N)\) is given by

\[
\mathcal{G}(N) = \prod_{p > 2} \left(1 - \frac{(N/p)}{p-1}\right), \quad (N/p): \text{Legendre symbol}.
\]

This asymptotic formula \([3]\) itself still seems far beyond our current technology, but we can prove \([1]\) on average. Let \(A > 0\) be an arbitrary constant and introduce

\[
E(X) = \# \left\{ N \leq X \mid |R_{1,2}(N) - \mathcal{G}(N)\sqrt{N}| \geq \sqrt{N} (\log N)^{-A}, \quad N: \text{not square} \right\},
\]

where \(X \geq 2\) is a real number. This function \(E(X)\) counts the number of positive integers \(\leq X\) for which the hypothetical asymptotic formula \([4]\) fails. Miech \([11]\) proved a non-trivial bound

\[
E(X) \ll XL^{-A}, \quad L := \log X
\]
for any \(A > 0\), where the implicit constant depends on \(A\). Thus, Miech proved that the asymptotic formula \([4]\) holds for almost all integer \(N\). The short interval version of Miech’s result \([5]\) was obtained by Mikawa \([12]\) and by Perelli and Pintz \([14]\) independently. Their result gives a non-trivial bound

\[
E(X + H) - E(X) \ll HL^{-A}
\]
for any \(A > 0\) provided

\[
X^{1+\varepsilon} \leq H \leq X,
\]
where \(X, H, \varepsilon\) are real numbers with \(2 \leq H \leq X\) and \(\varepsilon > 0\), and the implicit constant may depend on \(A\) and \(\varepsilon\). One of the aim in this problem is to obtain the same bound \([6]\) for shorter \(H\). Although the range \([7]\) is still the best possible result today for the estimate \([6]\), Languasco and Zaccagnini \([5]\) showed that if we consider the direct average \([6]\) instead, then we can deal with shorter \(H\) than \([7]\). After some minor modification, Theorem 2 of \([5]\) gives the following. In this paper, the letter \(B\) denotes the quantity given by

\[
B = \exp \left( c \left( \frac{\log X}{\log \log X} \right)^{\frac{1}{3}} \right),
\]
where \(c\) is some small positive constant which may depend on \(k, \ell\) and \(\varepsilon\).

**Theorem A** (Languasco and Zaccagnini \([5\) Theorem 2]). For real numbers \(X, H\) and \(\varepsilon\) with \(2 \leq H \leq X\) and \(\varepsilon > 0\), we have

\[
\sum_{X < N \leq X + H} R_{1,2}(N) = HX^{\frac{3}{2}} + O(HX^{\frac{3}{2}}B^{-1})
\]
provided \(X^{\frac{1}{2}}B^{-1} \leq H \leq X^{1-\varepsilon}\), where the implicit constant depends on \(\varepsilon\).

Thus, Languasco and Zaccagnini obtained asymptotic formula \([8]\) for \(H\) shorter than \([7]\) up to the factor \(B^{-1}\). However, we still have the same exponent \(\frac{1}{2}\) of \(X\). In this paper, we improve this exponent from \(\frac{1}{2}\) to 0.336899...
Theorem 1. For real numbers \(X, H, \varepsilon\) with \(2 \leq H \leq X\) and \(\varepsilon > 0\), we have the asymptotic formula provided
\[
X^{\Theta(1,2) + \varepsilon} \leq H \leq X^{1-\varepsilon}, \quad \Theta(1,2) = \frac{32 - 4\sqrt{15}}{49} = 0.336899 \ldots,
\]
where the implicit constant depends on \(\varepsilon\).

Recently, Languasco and Zaccagnini [9] dealt with other cases of (1):
Theorem B (Languasco and Zaccagnini [9, Theorem 1.3]). For any two positive integers \(k, \ell\) in the range
\[
\sum_{X < N \leq X + H} R_{k,\ell}(N) \leq \frac{1}{k\ell} \Gamma\left(\frac{k}{k+\ell}\right) H X^{\frac{k}{k+\ell} + \frac{1}{2} - 1} + O(HX^{\frac{k}{k+\ell} + \frac{1}{2} - 1}B^{-1})
\]
provided
\[
X^{\Theta_{LZ}(k,\ell) + \varepsilon} \leq H \leq X^{1-\varepsilon},
\]
where
\[
\Theta_{LZ}(k,\ell) = 2 - \frac{11}{6k} - \frac{1}{\ell}
\]
and the implicit constant depends on \(k, \ell\) and \(\varepsilon\).

Actually, Theorem 1 above is a special case of the following general result:
Theorem 2. For positive integers \(k, \ell\) with \(\ell \geq 2\), and real numbers \(X, H, \varepsilon\) with \(2 \leq H \leq X\) and \(\varepsilon > 0\), we have the asymptotic formula provided
\[
X^{\Theta(k,\ell) + \varepsilon} \leq H \leq X^{1-\varepsilon},
\]
where \(\Theta(k,\ell)\) is defined by
\[
\Theta(k,\ell) = \begin{cases} 
1 - \theta(k,\ell) & \text{(if } \ell \leq 9, \frac{5\ell}{24} < k), \\
1 - \min\left(\frac{5}{12k}, \frac{k}{\ell(k-1)}\right) & \text{(otherwise)},
\end{cases}
\]

\[
\theta(k,\ell) = \min\left(\frac{1}{2} \left(\frac{1}{k} + \frac{1}{\ell}\right), \frac{k}{\ell(k-1)}, \frac{\lambda_1(\ell)}{k}, \frac{\lambda_2(k,\ell)}{k}\right),
\]

\[
\lambda_1(\ell) = \begin{cases} 
\frac{\ell}{\ell+1} & \text{(if } \ell = 2, 3), \\
\frac{3\ell^2 + 2\sqrt{3\ell^2 + \ell}}{2(3\ell - 1)^2} & \text{(if } \ell = 4, \ldots, 8), \\
\frac{\ell}{3(3\ell - 5)} & \text{(if } \ell \geq 9),
\end{cases}
\]

\[
\lambda_2(k,\ell) = \begin{cases} 
\frac{2}{7} \left(\frac{k}{\ell} + \frac{1}{\ell}\right) & \text{(if } \frac{5\ell}{24} \leq k), \\
\frac{10}{39} + \frac{2k}{77} + \frac{3}{7} \sqrt{\frac{6}{5} \left(\frac{k}{\ell} - \frac{1}{\ell}\right)} & \text{(if } \frac{\ell}{96} \leq k \leq \frac{5\ell}{24}), \\
\frac{11}{55} \left(\frac{k}{\ell} + \frac{1}{\ell}\right) & \text{(if } k \leq \frac{3\ell}{96}),
\end{cases}
\]
and the implicit constant depends on \(k, \ell\) and \(\varepsilon\).

Remark 1. Theorem 2 is available for a wider range of \((k, \ell)\) than Theorem 1; Languasco and Zaccagnini [10] informed to the author that they also succeeded in obtaining a result with the admissible range for \((k, \ell)\) wider than [9].
The mainly concerned cases of Theorem 2 are the cases with \( \Theta(k, \ell) = 1 - \theta(k, \ell) \). We now compare our exponent \( \Theta(k, \ell) \) with \( \Theta_{\text{LZ}}(k, \ell) \) in Theorem B. By some numerical calculation, we obtain Table 1 below. Also, we have

\[
\Theta_{\text{LZ}}(3, 2) = \frac{8}{9}, \quad \Theta(3, 2) = \frac{2}{3}.
\]

Therefore, our exponent \( \Theta \) gives improvements of Theorem B for all pair \((k, \ell)\) in the range (9).

Table 1. Exponents \( \Theta_{\text{LZ}}(2, \ell) \) and \( \Theta(2, \ell) \) for \( 2 \leq \ell \leq 11 \)

| \( \ell \) | \( \Theta_{\text{LZ}}(2, \ell) \) | \( \Theta(2, \ell) \) | \( \Theta_{\text{LZ}}(2, \ell) - \Theta(2, \ell) \) |
|-------|-----------------|-----------------|-----------------|
| 2     | \( \frac{7}{12} = 0.583333\ldots \) | \( \frac{1}{2} = 0.500000\ldots \) | 0.083333\ldots |
| 3     | \( \frac{3}{4} = 0.750000\ldots \) | \( \frac{3}{5} = 0.625000\ldots \) | 0.125000\ldots |
| 4     | \( \frac{5}{6} = 0.833333\ldots \) | \( \frac{93-8\sqrt{3}}{121} = 0.670608\ldots \) | 0.162725\ldots |
| 5     | \( \frac{53}{60} = 0.883333\ldots \) | \( \frac{206-6\sqrt{3}}{245} = 0.706680\ldots \) | 0.176753\ldots |
| 6     | \( \frac{11}{12} = 0.916666\ldots \) | \( \frac{125-12\sqrt{3}}{147} = 0.734894\ldots \) | 0.181772\ldots |
| 7     | \( \frac{79}{84} = 0.940476\ldots \) | \( \frac{241}{300} = 0.756493\ldots \) | 0.183982\ldots |
| 8     | \( \frac{23}{24} = 0.958333\ldots \) | \( \frac{17}{22} = 0.772727\ldots \) | 0.185606\ldots |
| 9     | \( \frac{35}{36} = 0.972222\ldots \) | \( \frac{311}{396} = 0.785353\ldots \) | 0.186868\ldots |
| 10    | \( \frac{59}{60} = 0.983333\ldots \) | \( \frac{4}{5} = 0.800000\ldots \) | 0.183333\ldots |
| 11    | \( \frac{131}{144} = 0.992424\ldots \) | \( \frac{9}{11} = 0.818181\ldots \) | 0.174242\ldots |

Also, as a corollary of Theorem 2, we can prove a conditional result of Languasco and Zaccagnini [9, Theorem 1.4] unconditionally up to some log-factors as follows:

**Theorem 3.** For positive integers \( k \) with \( k \geq 2 \), and real numbers \( X, H, \varepsilon \) with \( 2 \leq H \leq X \) and \( \varepsilon > 0 \), we have the asymptotic formula (1) with \( \ell = 2 \) provided

\[
X^{1 - \frac{1}{k} + \varepsilon} \leq H \leq X^{1 - \varepsilon},
\]

where the implicit constant depends on \( k \) and \( \varepsilon \).

Languasco and Zaccagnini applied the circle method to prove Theorem A and Theorem B. In this paper, we deal with the average (3) rather more directly. We apply the Poisson summation formula to detect the cancellations over the sparse sequence \( n^\ell \), and then discuss similarly to the proof of the prime number theorem in short intervals by using the Huxley–Ingham zero density estimate.
2. Notations and conventions

We use the following notations and conventions.

As usual, $\Lambda(n)$ is the von Mangoldt function. Let

$$\psi(x) = \sum_{m \leq x} \Lambda(m).$$

We denote the Riemann zeta function by $\zeta(s)$. By $\rho = \beta + i\gamma$, we denote non-trivial zeros of $\zeta(s)$ with the real part $\beta$ and the imaginary part $\gamma$. For a real number $\alpha$ and $T$ with $T \geq 0$, let $N(\alpha, T)$ be the number of non-trivial zeros $\rho = \beta + i\gamma$ of $\zeta(s)$ in the rectangle $\alpha \leq \beta \leq 1$ and $|\gamma| \leq T$ counted with multiplicity.

For a complex valued function $f$ defined over an interval $[a, b]$, let $V_{[a,b]}(f)$ be the total variation of $f$ over $[a, b]$, and

$$\|f\| = \|f\|_{BV([a,b])} := \sup_{x \in [a,b]} |f(x)| + V_{[a,b]}(f).$$

For a real number $x$, let $e(x) = \exp(2\pi i x)$, $\lfloor x \rfloor$ be the largest integer not exceeding $x$, and $\{x\} = x - \lfloor x \rfloor$.

The letters $X, H, Q$ denote real numbers, and they are always assumed to satisfy

$$2 \leq H \leq X, \quad X \leq Q \leq X + H.$$ 

The letters $c_0, c_1 > 0$ denote some small absolute constants and $c$ denotes a constant with $0 < c \leq 1$ which may depends on $k, \ell$ and $\varepsilon$. The letters $B$ and $L$ are used for the abbreviations

$$B := \exp \left( e \left( \frac{\log X}{\log \log X} \right)^{\frac{1}{3}} \right), \quad L := \log X.$$ 

For positive integers $k, \ell$, and a complex number $\alpha$, we let

$$S_{\alpha}(Q) = S_{\alpha,k,\ell}(Q; X) := \frac{1}{\alpha} \sum_{n^{\ell} \leq X} (Q - n^{\ell})^\alpha, \quad S(Q) = S_1(Q).$$

Let $\phi(\lambda)$ be a function defined over $[0, +\infty)$ by

$$\phi(\lambda) = \begin{cases} \frac{3}{4} \lambda + \frac{3}{4}, & \text{(if } 0 \leq \lambda \leq \frac{25}{48}, \text{)} \\ 3\lambda + 2(1 - \sqrt{3\lambda}), & \text{(if } \frac{25}{48} \leq \lambda \leq \frac{3}{4}, \text{)} \\ \lambda + \frac{1}{2}, & \text{(if } \frac{3}{4} \leq \lambda \leq 1. \text{)} \end{cases}$$

This function will be used for estimating sums over non-trivial zeros of $\zeta(s)$. For positive integers $k, \ell$ with $\ell \geq 2$, we also introduce two real-valued functions $\lambda_1(\ell)$ and $\lambda_2(k, \ell)$ as in Theorem 2 by

$$\lambda_1(\ell) = \begin{cases} \frac{\ell}{2(\ell - 1)}, & \text{(if } \ell = 2, 3, \text{)} \\ \frac{3\ell^2 + 2\sqrt{3\ell} + \ell}{(\ell - 1)^2}, & \text{(if } \ell = 4, \ldots, 8, \text{)} \\ \frac{5\ell}{4(\ell - 3)}, & \text{(if } \ell \geq 9, \text{)} \end{cases}$$

$$\lambda_2(k, \ell) = \begin{cases} \frac{5}{3} \left( \frac{k}{\ell} + \frac{1}{4} \right), & \text{(if } \frac{5}{3} \leq k \text{)} \\ \frac{10}{39} + \frac{2k}{77} + \frac{4}{7} \sqrt{\frac{6}{7}} \left( \frac{k}{\ell} - \frac{1}{7} \right), & \text{(if } \frac{31}{96} \leq k \leq \frac{5}{3} \text{)} \\ \frac{25}{11} \left( \frac{k}{\ell} + \frac{1}{4} \right), & \text{(if } k \leq \frac{31}{96} \ell, \text{)} \end{cases}$$

These function is used in the exponent of the admissible ranges for $X$ and $H$. 


We have several expressions of the form
\[ \min (A, \infty). \]
As a convention, we define this quantity by
\[ \min (A, \infty) = A. \]

If Theorem or Lemma is stated with the phrase “where the implicit constant depends on \( a, b, c, \ldots \),” then every implicit constant in the corresponding proof may also depend on \( a, b, c, \ldots \) even without special mentions.

3. Preliminary Lemmas

In this section, we prepare some lemmas for the proof of Theorem 2. We start with some estimates for short interval sums without prime numbers.

**Lemma 1.** For positive integer \( \ell \) and real numbers \( X, H \) with \( 2 \leq H \leq X \),
\[
\sum_{X < n^\ell \leq X + H} 1 \ll H X^{\frac{1}{\ell} - 1} + 1,
\]
where the implicit constant is absolute.

**Proof.** By using \( x - 1 < \lfloor x \rfloor \leq x \), we see that
\[
\sum_{X < n^\ell \leq X + H} 1 = \left( (X + H)^{\frac{1}{\ell}} \right) - \left( X^{\frac{1}{\ell}} \right) 
\leq \left( X + H \right)^{\frac{1}{\ell}} - X^{\frac{1}{\ell}} + 1
\]
\[
= \frac{1}{\ell} \int_X^{X+H} u^{\frac{1}{\ell} - 1} du + 1 \ll H X^{\frac{1}{\ell} - 1} + 1.
\]
This completes the proof. \( \square \)

**Lemma 2.** For positive integers \( k, \ell \) and real numbers \( X, H \) with \( 2 \leq H \leq X \),
\[
\frac{1}{k \ell} \frac{\Gamma\left(\frac{1}{k}\right) \Gamma\left(\frac{1}{\ell}\right)}{\Gamma\left(\frac{1}{k} + \frac{1}{\ell} + 1\right)} \left( (X + H)^{\frac{1}{k} + \frac{1}{\ell}} - X^{\frac{1}{k} + \frac{1}{\ell}} \right)
\]
\[
= \frac{1}{k \ell} \frac{\Gamma\left(\frac{1}{k}\right) \Gamma\left(\frac{1}{\ell}\right)}{\Gamma\left(\frac{1}{k} + \frac{1}{\ell} + 1\right)} H X^{\frac{1}{k} + \frac{1}{\ell} - 1} + O(H^2 X^{\frac{1}{k} + \frac{1}{\ell} - 2}),
\]
where the implicit constant is absolute.

**Proof.** By the fundamental theorem of calculus,
\[
\frac{1}{k \ell} \frac{\Gamma\left(\frac{1}{k}\right) \Gamma\left(\frac{1}{\ell}\right)}{\Gamma\left(\frac{1}{k} + \frac{1}{\ell} + 1\right)} \left( (X + H)^{\frac{1}{k} + \frac{1}{\ell}} - X^{\frac{1}{k} + \frac{1}{\ell}} \right)
\]
\[
= \frac{1}{k \ell} \frac{\Gamma\left(\frac{1}{k}\right) \Gamma\left(\frac{1}{\ell}\right)}{\Gamma\left(\frac{1}{k} + \frac{1}{\ell} + 1\right)} \int_X^{X+H} u^{\frac{1}{k} + \frac{1}{\ell} - 1} du.
\]
For \( X < u \leq X + H \), by using the mean value theorem,
\[
u^{\frac{1}{k} + \frac{1}{\ell} - 1} = X^{\frac{1}{k} + \frac{1}{\ell} - 1} + O \left( H X^{\frac{1}{k} + \frac{1}{\ell} - 2} \right).
\]
Thus, the integral in \((16)\) is
\[
\int_X^{X+H} u^{\frac{1}{k} + \frac{1}{\ell} - 1} du = H X^{\frac{1}{k} + \frac{1}{\ell} - 1} + O \left( H^2 X^{\frac{1}{k} + \frac{1}{\ell} - 2} \right).
\]
Lemma 3. For positive integers $k, \ell$ and real numbers $X, H$ with $2 \leq H \leq X$, \[ S(X + H) - S(X) = \frac{1}{k\ell} \int_{\frac{X}{k}}^{\frac{X+H}{k}} (u-n)\frac{x}{x-1}~du = \int_{X}^{X+H} \frac{1}{k\ell} \int_{\frac{X}{k}}^{\frac{X+H}{k}} (u-n)\frac{x}{x-1}~du. \]

where $S(Q)$ is defined by \[ S(Q) = \sum_{n^X < u \leq X} (Q-n) \frac{x}{x-1}, \]
as in [13] and the implicit constant is absolute.

Proof. The left-hand side of the assertion is
\begin{align*}
\frac{1}{k} \sum_{n^X \leq X} (u-n)\frac{x}{x-1} &= \int_{X}^{X+H} (u-n)\frac{x}{x-1}~du = \int_{X}^{X+H} \frac{1}{k\ell} \int_{\frac{X}{k}}^{\frac{X+H}{k}} (u-n)\frac{x}{x-1}~du.
\end{align*}

Since the function $(u-w)\frac{x}{x-1}$ is non-decreasing over $0 \leq w \leq X\frac{x}{x-1}$, \[ \frac{1}{k} \sum_{n^X \leq X} (u-n)\frac{x}{x-1} = \frac{1}{k\ell} \int_{0}^{X}(u-w)\frac{x}{x-1}~dw + O\left(\frac{1}{k}(u-X)\frac{x}{x-1}\right) \]
for $X < u \leq X + H$. Note that the second term on the right-hand side may tend to $\infty$ as $u \to X + 0$, but this term is integrable over $(X, X + H)$. We next extend the integral on the right-hand side. For $X < u \leq X + H$, by changing the variable, \begin{align*}
\frac{1}{k\ell} \int_{X}^{u}(u-w)\frac{x}{x-1}~dw &= \frac{1}{k\ell} X\frac{x}{x-1} - \int_{0}^{u-X}(u-w)\frac{x}{x-1}~dw \\
&\leq \frac{1}{\ell} X\frac{x}{x-1} - (u-X)\frac{x}{x-1} = \frac{1}{\ell} H\frac{x}{x-1} X\frac{x}{x-1}.
\end{align*}

Hence, we can extend the integral in [13] as
\[ \frac{1}{k} \sum_{n^X \leq X} (u-n)\frac{x}{x-1} \]
\[ = \frac{1}{k\ell} \int_{0}^{u}(u-w)\frac{x}{x-1}~dw + O\left(\frac{1}{k}(u-X)\frac{x}{x-1} + H\frac{x}{x-1}\right). \]

The last integral on the right-hand side is
\begin{align*}
\frac{1}{k\ell} \int_{0}^{u}(u-w)\frac{x}{x-1}~dw &= \frac{1}{k\ell} u\frac{x}{x-1} - \int_{0}^{1}(1-w)\frac{x}{x-1}~dw \\
&= \frac{1}{k\ell} \frac{1}{k\ell + \frac{1}{\ell}} u\frac{x}{x-1}.
\end{align*}
Therefore,
\[
\frac{1}{k} \sum_{n^\ell \leq X} (u - n^\ell)^{\frac{1}{2} - 1} = \frac{1}{k \ell} \frac{\Gamma\left(\frac{1}{\ell}\right) \Gamma\left(\frac{k}{\ell} + \frac{1}{2}\right)}{\Gamma\left(\frac{k}{\ell} + \frac{1}{2} + \frac{1}{2}\right)} u^{\frac{1}{\ell} + \frac{k}{2\ell}} + O\left(\frac{1}{k} (u - X)^{\frac{1}{2} - 1} + H^{\frac{1}{2}} X^{\frac{1}{2} - 1}\right).
\]

On inserting this formula into (17), the left-hand side of the assertion is
\[
\frac{1}{k \ell} \frac{\Gamma\left(\frac{1}{\ell}\right) \Gamma\left(\frac{k}{\ell} + \frac{1}{2}\right)}{\Gamma\left(\frac{k}{\ell} + \frac{1}{2} + \frac{1}{2}\right)} \int_X^{X + H} u^{\frac{1}{\ell} + \frac{k}{2\ell}} du + O\left(\frac{1}{k} (u - X)^{\frac{1}{2} - 1} + H^{\frac{1}{2}} X^{\frac{1}{2} - 1}\right).
\]

By Lemma 2, this is
\[
\frac{1}{k \ell} \frac{\Gamma\left(\frac{1}{\ell}\right) \Gamma\left(\frac{k}{\ell} + \frac{1}{2}\right)}{\Gamma\left(\frac{k}{\ell} + \frac{1}{2} + \frac{1}{2}\right)} H X^{\frac{1}{\ell} + \frac{k}{2\ell}} + O\left(H^2 X^{\frac{1}{\ell} + \frac{k}{2\ell} - 2} + H^{1 + \frac{1}{2}} X^{\frac{1}{2} - 1} + H^{\frac{1}{\ell} + \frac{k}{2\ell}}\right).
\]

Since \( H \leq X \), we can estimate the first error term as
\[
H^2 X^{\frac{1}{\ell} + \frac{k}{2\ell} - 2} = H^2 X^{\frac{1}{\ell} + \frac{k}{2\ell} + 1} \leq H^{1 + \frac{1}{2}} X^{\frac{1}{2} - 1}.
\]

This completes the proof. \(\square\)

**Lemma 4.** For positive integers \( k, \ell \) and real numbers \( X, H \) with \( 2 \leq H \leq X \),
\[
\sum_{X < m^k + n^\ell \leq X + H} 1 \ll H X^{\frac{1}{\ell} + \frac{k}{2\ell} - 1} + H^{\frac{1}{\ell} + \frac{k}{2\ell}},
\]
where the implicit constant is absolute.

**Proof.** We rewrite the left-hand side as
\[
\sum_{X < m^k + n^\ell \leq X + H} 1 = \sum_{n^\ell \leq X + H} \sum_{X - n^\ell < m^k \leq X + H - n^\ell} 1.
\]

We next truncate the outer summation over \( n^\ell \). By using Lemma \[1\]
\[
\sum_{X < n^\ell \leq X + H} \sum_{X - n^\ell < m^k \leq X + H - n^\ell} 1 \ll \sum_{X < n^\ell \leq X + H} \sum_{m^k \leq H} 1 \ll H^{1 + \frac{1}{2}} X^{\frac{1}{2} - 1} + H^{\frac{1}{2}}.
\]

Thus, by using the assumption \( H \leq X \),
\[
(19) \sum_{X < m^k + n^\ell \leq X + H} 1 = \sum_{n^\ell \leq X} \sum_{X - n^\ell < m^k \leq X + H - n^\ell} 1 + O\left(H X^{\frac{1}{\ell} + \frac{k}{2\ell} - 1} + H^{\frac{1}{\ell} + \frac{k}{2\ell}}\right).
\]

The sum on the right-hand side is
\[
\sum_{n^\ell \leq X} \sum_{X - n^\ell < m^k \leq X + H - n^\ell} 1 = S(X + H) - S(X) + O(X^{\frac{1}{2}}).
\]

By using Lemma \[3\] and the assumption \( H \leq X \),
\[
\sum_{n^\ell \leq X} \sum_{X - n^\ell < m^k \leq X + H - n^\ell} 1 \ll H X^{\frac{1}{\ell} + \frac{k}{2\ell} - 1} + H^{\frac{1}{\ell} + \frac{k}{2\ell}} + X^{\frac{1}{2}}.
\]

On inserting this estimate into (19), we obtain the lemma. \(\square\)

We next recall some standard lemmas on prime numbers and non-trivial zeros of the Riemann zeta functions.
Lemma 5. For real numbers $X, T, x$ with $2 \leq T \leq 2X$ and $0 \leq x \leq X$, we have

$$\psi(x) = x - \sum_{\rho \mid |\gamma| \leq T} \frac{x^\rho}{\rho} + O(XT^{-1}L^2),$$

where the implicit constant is absolute.

Proof. In the case $2 \leq x \leq X$, this follows from Theorem 12.5 of [13]. In the case $0 \leq x \leq 2$, the lemma trivially follows since $XT^{-1}L^2 \gg L^2$ by $T \leq 2X$, and

$$\sum_{\rho \mid |\gamma| \leq T} \frac{x^\rho}{\rho} \ll \sum_{\rho \mid |\gamma| \leq T} \frac{1}{|\rho|} \ll (\log T)^2 \ll L^2$$

for the case $0 \leq x \leq 2$. This completes the proof. □

Lemma 6 (The Korobov–Vinogradov zero-free region). We have $\zeta(s) \neq 0$ for

$$\sigma > 1 - c_0 (\log \tau)^{-\frac{4}{3}} (\log \log \tau)^{-\frac{1}{3}}, \quad s = \sigma + it, \quad \tau = |t| + 4,$$

where $c_0 > 0$ is some absolute constant.

Proof. See [4, Theorem 6.1, p. 143]. Note that by taking $c_0 > 0$ sufficiently small, we can remove the condition $t \geq t_0$ in Theorem 6.1 of [4]. □

Lemma 7 (The Huxley–Ingham zero density estimate [2, 3]). For real numbers $\alpha$ and $T$ with $\frac{1}{2} \leq \alpha \leq 1$ and $T \geq 2$,

$$N(\alpha, T) \ll T^{c(\alpha)}L^A, \quad c(\alpha) = \begin{cases} \frac{3(1-\alpha)}{3\alpha-1} & \text{(if } \frac{3}{4} \leq \alpha \leq 1), \\ \frac{3(1-\alpha)}{2-\alpha} & \text{(if } \frac{1}{2} \leq \alpha \leq \frac{3}{4}), \end{cases}$$

where the constant $A$ and the implicit constant are absolute.

Proof. See [4] Theorem 11.1, p. 273]. □

Lemma 8. For real numbers $X, H, \varepsilon$ with $2 \leq H \leq X$ and $\varepsilon > 0$,

$$\psi(X + H) - \psi(X) = H + O(HB^{-1})$$

provided

$$X^{\frac{7}{12} + \varepsilon} \leq H \leq X,$$

where the implicit constant depends on $\varepsilon$.

Proof. This follows by Lemma [8] and Lemma [7] through the standard argument. □

In the proof of Theorem [2] we need to estimate several sums over non-trivial zeros of the Riemann zeta function. Our next several lemmas deal with such sums and the exponents in the resulting estimates.

Lemma 9. For real numbers $K, X, Y$ with $1 \leq K \leq Y \leq X^2$ and $X \geq 2$,

$$\sum_{K < |\gamma| \leq 2K} Y^{\beta} \ll \left(Y^{\phi(\lambda)} + Y^{1-\eta+2\eta\lambda}\right)L^A,$$

where the function $\phi(\lambda)$ is defined by

$$\phi(\lambda) = \begin{cases} \frac{2}{3} \lambda + \frac{4}{3} & \text{(if } 0 \leq \lambda \leq \frac{25}{28}), \\ 3\lambda + 2(1 - \sqrt{3}\lambda) & \text{(if } \frac{25}{28} \leq \lambda \leq \frac{1}{4}), \\ \lambda + \frac{1}{2} & \text{(if } \frac{1}{4} \leq \lambda \leq 1), \end{cases}$$
as in (14),
\[ \eta = c_1 (\log X)^{-3} (\log \log X)^{-4}, \quad \lambda = \frac{\log K}{\log Y}, \]
and constants \( A, c_1 > 0 \) and the implicit constant are absolute.

**Proof.** By Lemma 6 and Lemma 7, the left-hand side is bounded by
\[ (20) \sum_{K < |\gamma| \leq 2K} Y^\beta = -\int_1^{1-\eta} Y^\alpha dN(\alpha, 2K) \ll KY^{\frac{3}{4}} L + L^A \int_1^{1-\eta} K^{c(\alpha)} Y^\alpha d\alpha \]
for sufficiently small \( c_1 > 0 \). We determine the maximum value of
\[ K^{c(\alpha)} Y^\alpha = Y^{\lambda c(\alpha) + \alpha} \]
over \( \alpha \in [\frac{3}{4}, 1-\eta] \). Let \( h(\alpha) = \lambda c(\alpha) + \alpha \). For \( \alpha \in [\frac{1}{4}, \frac{3}{4}] \), we have
\[ h(\alpha) = \frac{3\lambda(1-\alpha)}{2-\alpha} + \alpha = 3\lambda - \frac{3\lambda}{2-\alpha} + \alpha. \]
By taking the derivative,
\[ h'(\alpha) = -\frac{3\lambda}{(2-\alpha)^2} + 1. \]
Thus, in the range \( \alpha \in (-\infty, 2) \),
\[ h'(\alpha) = 0 \iff \alpha = 2 - \sqrt{3\lambda}, \]
so \( h(\alpha) \) is increasing for \( \alpha < 2 - \sqrt{3\lambda} \) and decreasing for \( 2 - \sqrt{3\lambda} < \alpha < 2 \). Hence,
\[ \max_{\alpha \in [\frac{3}{4}, \frac{3}{4}]} h(\alpha) = \begin{cases} \frac{3\lambda}{3} + \frac{3}{4} & (0 \leq \lambda \leq \frac{25}{16}), \\ 3\lambda + 2(1-\sqrt{3\lambda}) & (\frac{25}{16} \leq \lambda \leq \frac{3}{4}), \\ \lambda + \frac{1}{2} & (\frac{3}{4} \leq \lambda \leq 1). \end{cases} \]
For \( \alpha \in [\frac{3}{4}, 1-\eta] \), we have
\[ h(\alpha) = \frac{3\lambda(1-\alpha)}{3\alpha - 1} + \alpha = -\lambda + \frac{2\lambda}{3\alpha - 1} + \alpha. \]
By taking the derivative twice, in the range \( \alpha \in [\frac{3}{4}, 1-\eta] \),
\[ h''(\alpha) = \frac{18\lambda}{(3\alpha - 1)^3} > 0 \]
so that \( h(\alpha) \) is convex downwards in this range. Thus, for small \( c_1 \),
\[ \max_{\alpha \in [\frac{3}{4}, 1-\eta]} h(\alpha) = \max(h(\frac{3}{4}), h(1-\eta)) \leq \max(h(\frac{3}{4}), 1-\eta + 2\eta\lambda). \]
By using the above observations for \( h(\alpha) \) in (20), we obtain the lemma. \( \square \)

**Lemma 10.** Let \( \phi(\lambda) \) be the function given by (14), Then,
\[ \frac{3}{5} \leq \phi'(\lambda) \leq 1 \]
for \( \lambda \geq 0 \). In particular, \( \phi(\lambda) \) is increasing.
Proof. It suffices to consider the case $\frac{25}{48} \leq \lambda \leq \frac{3}{4}$. In this range,
\[ \phi'(\lambda) = 3 - \sqrt{\frac{3}{\lambda}}. \]
Thus, the lemma easily follows. \qed

Lemma 11. Let $\phi(\lambda)$ be the function defined in (14). For positive integers $k, \ell$ with $\ell \geq 2$, consider the solutions $\lambda_1$ and $\lambda_2$ of the equations
\begin{equation}
\phi(\lambda_1) - \frac{1}{\ell}\lambda_1 = 1, \quad \phi(\lambda_2) + \frac{1}{2}\lambda_2 = 1 + \frac{k}{\ell}.
\end{equation}
Then, these functions $\lambda_1, \lambda_2$ are consistent with the functions given in (15).

Proof. By Lemma 10 and $\ell \geq 2$, both of the function
\begin{equation}
\phi(\lambda) - \frac{1}{\ell}\lambda, \quad \phi(\lambda) + \frac{1}{2}\lambda
\end{equation}
are strictly increasing for $\lambda \geq 0$ and take the value from $3/4$ to $+\infty$. Thus, by the intermediate value theorem, $\lambda_1$ and $\lambda_2$ are well-defined.

We first consider $\lambda_1$. If $\phi(\frac{25}{48}) - \frac{25}{48} \lambda > 1$, i.e. $\ell \geq 9$, then
\[ 1 = \phi(\lambda_1) - \frac{1}{\ell}\lambda_1 = \left(3 - \frac{1}{\ell}\right)\lambda_1 + \frac{3}{4} \]
so that
\[ \lambda_1 = \frac{5\ell}{4(3\ell - 5)}. \]
If $\phi(\frac{25}{48}) - \frac{25}{48} \lambda \leq 1 < \phi(\frac{3}{4}) - \frac{3}{4\ell}$, i.e. $4 \leq \ell \leq 8$, then
\[ 1 = \phi(\lambda_1) - \frac{1}{\ell}\lambda_1 = \left(3 - \frac{1}{\ell}\right)\lambda_1 + 2\left(1 - \sqrt{3\lambda_1}\right) \]
so that, by using $\frac{25}{48} \leq \lambda_1$ in the current case,
\[ \lambda_1 = \frac{3\ell^2 + 2\sqrt{3\ell^2} + \ell}{(3\ell - 1)^2}. \]
Finally, if $\phi(\frac{3}{4}) - \frac{3}{4\ell} \leq 1$, i.e. $2 \leq \ell \leq 3$, then
\[ 1 = \phi(\lambda_1) - \frac{1}{\ell}\lambda_1 = \left(1 - \frac{1}{\ell}\right)\lambda_1 + \frac{1}{2} \]
so that
\[ \lambda_1 = \frac{\ell}{2(\ell - 1)}. \]
This completes the proof of the assertion for $\lambda_1$.

We next consider $\lambda_2$. If $1 + \frac{k}{\ell} \leq \phi(\frac{25}{48}) + \frac{1}{2\ell}$, i.e. $k \leq \frac{31}{96} \ell$, then
\[ 1 + \frac{k}{\ell} = \phi(\lambda_2) + \frac{1}{2}\lambda_2 = \frac{11}{10}\lambda_2 + \frac{3}{4} \]
so that
\[ \lambda_2 = \frac{10}{11} \left(\frac{k}{\ell} + \frac{1}{4}\right). \]
If $\phi(\frac{25}{48}) + \frac{1}{2\ell} \leq 1 + \frac{k}{\ell} \leq \phi(\frac{3}{4}) + \frac{1}{2}$, i.e. $\frac{31}{96} \ell \leq k \leq \frac{5}{8} \ell$, then
\[ 1 + \frac{k}{\ell} = \phi(\lambda_2) + \frac{1}{2}\lambda_2 = \frac{7}{2}\lambda_2 + 2\left(1 - \sqrt{3\lambda_2}\right). \]
so that, by using $\frac{25}{49} \leq \lambda_2$ in the current case,

\[ \lambda_2 = \frac{10}{49} + \frac{2k}{\ell\lambda} + \frac{4}{7} \sqrt{\frac{6}{7} \left( \frac{k}{\ell} - \frac{1}{7} \right)}. \]

Finally, if $\phi\left(\frac{3}{4}\right) + \frac{1}{2} \leq 1 + \frac{k}{\ell}$, i.e. $\frac{3}{4} \ell \leq k$, then

\[ 1 + \frac{k}{\ell} = \phi(\lambda_2) + \frac{1}{2} \lambda_2 = \frac{3}{2} \lambda_2 + \frac{1}{2} \]

so that

\[ \lambda_2 = \frac{2}{3} \left( \frac{k}{\ell} + \frac{1}{2} \right). \]

This completes the proof of the assertion for $\lambda_2$. □

**Lemma 12.** For positive integers $k, \ell$ with $\ell \geq 2$ and a real number $\varepsilon$ with $\varepsilon > 0$,

\[ \phi(\lambda) - 1 = 1 - \frac{\varepsilon}{10} \quad \text{and} \quad \phi(\lambda) + 1 = 1 + \frac{k}{\ell} - \frac{\varepsilon}{10} \]

provided

\[ 0 \leq \lambda \leq \min(\lambda_1, \lambda_2) - \varepsilon, \]

where $\lambda_1, \lambda_2$ are the solutions of (21), or equivalently, defined by (15).

**Proof.** By the assumption $\ell \geq 2$ and Lemma 10, both of the functions (22) have the derivative of the size $\geq \frac{1}{10}$. Thus, the mean value theorem and (23) give

\[ \phi(\lambda) - 1 = 1 - \frac{\varepsilon}{10} \quad \text{and} \quad \phi(\lambda) + 1 = 1 + \frac{k}{\ell} - \frac{\varepsilon}{10} \]

This completes the proof. □

**Lemma 13.** The functions $\lambda_1(\ell), \lambda_2(k, \ell)$ are decreasing with respect to $\ell$.

**Proof.** By Lemma 11, we $\lambda_1(\ell)$ and $\lambda_2(k, \ell)$ can be regarded as the solutions of the equations (21). Then, the lemma follows since $\phi(\lambda)$ is increasing by Lemma 10. □

As we mentioned in Section 1, we shall apply the Poisson summation formula in order to detect some cancellation over the sequence $n^\ell$. In order to estimate the resulting exponential integrals, we recall the next two standard estimates.

**Lemma 14** (First derivative estimate). Let $\lambda$ be a positive real number, and $f, g$ be real-valued functions defined over an interval $[a, b]$ satisfying

(A) $f$ is continuously differentiable on the interval $[a, b]$,

(B) $f'$ is monotonic on the interval $[a, b]$, and

(C) $f'$ satisfies $|f'(x)| \geq \lambda$ on the interval $[a, b]$.

Then, by using notation (12), we have

\[ \int_a^b g(x)e(f(x))dx \ll \|g\|\lambda^{-1}, \]

where the implicit constant is absolute.

**Proof.** See [4, Lemma 2.1, p. 56]. □
Lemma 15 (Second derivative estimate). Let \( \lambda \) be a positive real number, and \( f, g \) be real-valued functions defined over an interval \([a, b]\) satisfying

- (A) \( f \) is twice continuously differentiable on the interval \([a, b]\),
- (B) \( f'' \) satisfies \( |f''(x)| \geq \lambda \) on the interval \([a, b]\).

Then, by using notation \((12)\), we have

\[
\int_a^b g(x)e(f(x))dx \ll \|g\|\lambda^{-\frac{1}{2}},
\]
where the implicit constant is absolute.

Proof. See \[4, Lemma 2.2, p. 56\]. \(\Box\)

4. Preliminary calculations

We start the main part of the proof of Theorem 2. We first replace \( \log p \) in our original representation function \((2)\) by the von Mangoldt function.

Lemma 16. For positive integers \( k, \ell \) and real numbers \( X, H, \varepsilon \) with \( 2 \leq H \leq X \) and \( \varepsilon > 0 \), we have

\[
\sum_{X < N \leq X + H} R(N) = \sum_{X < m^k + n^\ell \leq X + H} \Lambda(m) + O(HX^{\frac{1}{k} + \frac{1}{\ell} - 1}B^{-1})
\]
provided

\[
X^{1 - \min\left(\frac{1}{k}, \frac{1}{\ell} + \frac{1}{k-1}\right)} + \varepsilon \leq H \leq X^{1 - \varepsilon},
\]
where the implicit constant is absolute.

Proof. By definition \((2)\) of \( R(N) \),

\[
\sum_{X < N \leq X + H} R(N) = \sum_{X < p^k + n^\ell \leq X + H} \log p
\]
\[
= \sum_{X < m^k + n^\ell \leq X + H} \Lambda(m) - \sum_{\nu=2}^{O(L)} \sum_{X < p^k + n^\ell \leq X + H} \log p.
\]

Note that the implicit constant in Lemma 3 is absolute. Therefore, by Lemma 3 the second term on the right hand side is bounded by

\[
\ll L \sum_{\nu=2}^{O(L)} \left( HX^{\frac{1}{k} + \frac{1}{\ell} - 1} + H^{\frac{1}{k}} + X^{\frac{1}{\ell}} \right) \ll (HX^{\frac{1}{k} + \frac{1}{\ell} - 1} + H^{\frac{1}{k}} + X^{\frac{1}{\ell}})L^2,
\]
which is \( \ll HX^{\frac{1}{k} + \frac{1}{\ell} - 1}B^{-1} \) provided \((24)\). This completes the proof. \(\Box\)

We then modify the sum on the right-hand side of Lemma 16 in order to insert the explicit formula given by Lemma 5.

Lemma 17. For positive integers \( k, \ell \) and real numbers \( X, H, \varepsilon \) with \( 2 \leq H \leq X \) and \( \varepsilon > 0 \), we have

\[
\sum_{X < N \leq X + H} R(N)
= \sum_{n^\ell \leq X} \left( \psi \left( (X + H - n^\ell)^{\frac{1}{k}} \right) - \psi \left( (X - n^\ell)^{\frac{1}{k}} \right) \right) + O(HX^{\frac{1}{k} + \frac{1}{\ell} - 1}B^{-1})
\]
provided
\[ X^{1 - \min\left(\frac{k}{\ell}, \frac{k}{T \ell - 1}\right)} + \varepsilon \leq H \leq X^{1 - \varepsilon}, \]
where the implicit constant is absolute.

Proof. We truncate the summation over \( n \) in Lemma 16. By using Lemma 1 and the argument similar to the beginning of the proof of Lemma 4,
\[
\sum_{\ell \leq X} \Lambda(m) \ll H^{1 + \frac{k}{\ell}} X^{1 - \frac{k}{\ell} - 1} + H \ll H X^{1 + \frac{k}{\ell} - 1} B^{-1}
\]
provided (25). Thus we can employ the truncation as
\[
\sum_{n \leq X} \psi\left((X - n^\ell)^{\frac{k}{\ell}}\right) - \psi\left((X - n^\ell)^{\frac{k}{\ell}}\right) + O\left(H X^{1 + \frac{k}{\ell} - 1} B^{-1}\right).
\]
By recalling the notation (11), we arrive at
\[
\sum_{n \leq X} \psi\left((X - n^\ell)^{\frac{k}{\ell}}\right) - \psi\left((X - n^\ell)^{\frac{k}{\ell}}\right) + O\left(H X^{1 + \frac{k}{\ell} - 1} B^{-1}\right).
\]
By substituting this formula into Lemma 16 we obtain the lemma. \(\square\)

5. Detection of the cancellation over the \(\ell\)-th powers
In this section, we derive an expansion for the sum
\[ S_\rho(Q) = \sum_{n^\ell \leq X} \psi\left((Q - n^\ell)^{\frac{k}{\ell}}\right), \quad X \leq Q \leq X + H, \]
or its difference
\[ S_\rho(X + H) - S_\rho(X) \]
by which we try to detect some cancellation caused by the average over \( n^\ell \). This expansion will be given by Lemma 20. We first substitute Lemma 5 into this sum.

Lemma 18. Let \( k, \ell \) be positive integers, and \( X, H, Q, T \) be real numbers satisfying
\[ 2 \leq H \leq X, \quad X \leq Q \leq X + H, \quad 1 \leq T \leq X^{\frac{k}{\ell}}. \]
Then,
\[ \sum_{n^\ell \leq X} \psi\left((Q - n^\ell)^{\frac{k}{\ell}}\right) = S(Q) - \sum_{|\gamma| \leq T} S_\rho(Q) + O\left(H X^{1 + \frac{k}{\ell} - 1} T^{-1} L^2\right), \]
where \( S(Q) \) and \( S_\rho(Q) \) are given by
\[ S(Q) = \sum_{n^\ell \leq X} (Q - n^\ell)^{\frac{k}{\ell}}, \quad S_\rho(Q) = \frac{1}{\rho} \sum_{n^\ell \leq X} (Q - n^\ell)^{\frac{k}{\ell}}. \]
as defined in (13), and the implicit constant is absolute.

Proof. This follows immediately by inserting Lemma 5. \(\square\)

Our next task is to detect the cancellation in the sum \( S_\rho(Q) \). We prepare the next lemma in order to estimate exponential integrals.
Lemma 19. For positive integers $k, \ell$, an integer $n$ not necessarily positive, and real numbers $\alpha, \gamma, Q, U, V$ with $\alpha \leq 1$, $|\gamma| \geq 1$, and $1 \leq U \leq V \leq Q$, we have

$$\int_U^V u^{\alpha + \frac{i}{\ell} - 1} e\left(n(Q - u)^{\frac{1}{\ell}}\right) du \ll \begin{cases} \frac{V^\alpha L}{|\gamma|^\frac{1}{2}} & \text{(if } \alpha \geq 0\text{)}, \\ \frac{U^\alpha L}{|\gamma|^\frac{1}{2}} & \text{(if } \alpha \leq 0\text{),} \\ \frac{Q^{1 - \frac{1}{\ell}}}{|n|} & \text{(if } |n| > \ell Q^{1 - \frac{1}{\ell}} |\gamma|\text{),} \end{cases}$$

where the implicit constant depends on $k, \ell$ and $\varepsilon$.

Proof. We rewrite the left-hand side as

$$\int_U^V u^{\alpha + \frac{i}{\ell} - 1} e\left(n(Q - u)^{\frac{1}{\ell}}\right) du = \int_U^V G(u) e(F(u)) du,$$

where

$$F(u) = n(Q - u)^{\frac{1}{\ell}} + \frac{\gamma}{2\pi k} \log u, \quad G(u) = u^{\alpha - 1}.$$

Then,

$$F'(u) = -\frac{1}{\ell} n(Q - u)^{\frac{1}{\ell} - 1} + \frac{\gamma}{2\pi k u}, \quad F''(u) = -\frac{\ell - 1}{\ell^2} n(Q - u)^{\frac{1}{\ell} - 2} \frac{\gamma}{2\pi k u^2}$$

and since $G(u)$ is non-increasing, by using the notation $\|G\|_{BV([R, R'])} \ll R^{\alpha - 1}$ for any subinterval $[R, R'] \subset [U, V]$.

For the former two estimates, we dissect the integral (27) dyadically as

$$\ll_L \sup_{U < R \leq V} \left| \int_R^{\min(2R, V)} G(u) e(F(u)) du \right|$$

If $n$ and $\gamma$ have the same signs, then we have

$$|F''(u)| \geq \frac{|\gamma|}{2\pi k (2R)^2}$$

for $u \in [R, \min(2R, V)]$. Therefore, by Lemma 15

$$\int_R^{\min(2R, V)} u^{\alpha + \frac{i}{\ell} - 1} e\left(n(Q - u)^{\frac{1}{\ell}}\right) du \ll R^{\alpha - 1} \left(\frac{|\gamma|}{R^2}\right)^{\frac{1}{2}} \ll \frac{R^\alpha}{|\gamma|^\frac{1}{2}}.$$

On the other hand, if $n$ and $\gamma$ have the opposite signs, then we have

$$|F'(u)| \geq \frac{|\gamma|}{2\pi k (2R)}$$

and $F''(u)$ have at most one zero in $[R, \min(2R, V)]$. Therefore, we may dissect $[R, \min(2R, V)]$ into at most two intervals, on each of which $F'(u)$ is monotonic. By applying Lemma 14,

$$\int_R^{\min(2R, V)} u^{\alpha + \frac{i}{\ell} - 1} e\left(n(Q - u)^{\frac{1}{\ell}}\right) du \ll R^{\alpha - 1} \left(\frac{|\gamma|}{R}\right)^{-1} \ll \frac{R^\alpha}{|\gamma|} \ll \frac{R^\alpha}{|\gamma|^\frac{1}{2}}.$$
since $|\gamma| \geq 1$. Therefore, by (30) and (21), we have
$$\int_R^{\min(2R,V)} u^{\alpha + \frac{\beta}{2} - 1} e\left(n(Q - u)^{\frac{1}{2}}\right) du \ll \frac{R^\alpha}{|\gamma|^\frac{1}{2}}$$
in any case. On inserting this estimate into (29), we obtain the first two estimates.

For the last estimate, we work without the dyadic dissection:
$$\int_U^V u^{\alpha + \frac{\beta}{2} - 1} e\left(n(Q - u)^{\frac{1}{2}}\right) du = \int_U^V G(u)e(F(u))du.$$ We apply Lemma 14 to this integral. By assuming $|n| > \ell Q^{1 - \frac{1}{2}}|\gamma|$

Also, by (28), we can dissect $[U, V]$ into at most two intervals, on each of which $F'(u)$ is monotonic. Thus, by Lemma 14
$$\int_U^V u^{\alpha + \frac{\beta}{2} - 1} e\left(n(Q - u)^{\frac{1}{2}}\right) du \ll U^{\alpha - 1}\left(\frac{|n|}{Q^{1 - \frac{1}{2}}}\right)^{-1} \ll Q^{1 - \frac{1}{2}}\frac{|n|}{Q^{1 - \frac{1}{2}}},$$
since $\alpha \leq 1$. This completes the proof. \(\square\)

We now derive the following expansion of the difference (26).

**Lemma 20.** For positive integers $k, \ell$, real numbers $X, H, \varepsilon$ with $2 \leq H \leq X$ and $\varepsilon > 0$, and a non-trivial zero $\rho = \beta + i\gamma$ of $\zeta(s)$ with $|\gamma| \leq 2X$, we have
$$S_\rho(X + H) - S_\rho(X) = \frac{1}{k\ell} \Gamma\left(\frac{\varepsilon}{k} + \frac{1}{2}\right) \left((X + H)^{\frac{\varepsilon}{k} + \frac{1}{2}} - X^{\frac{\varepsilon}{k} + \frac{1}{2}}\right)$$
$$- \frac{(X + H)^{\frac{\varepsilon}{k} - X^{\frac{\varepsilon}{k}}}}{2\rho} + O\left(H^\frac{\varepsilon}{2} |\gamma|^{\frac{\varepsilon}{2} - \frac{1}{2}} L^2 + \frac{HX^{\beta + \frac{1}{2} - 1}B^{-2}}{|\gamma|} + L\right)$$

provided
$$X^{1 - \min\left(\frac{\beta}{k}, \frac{\beta - 1}{k - 1}\right) + \varepsilon} \leq H \leq X^{1 - \varepsilon},$$

where the implicit constant depends on $k, \ell$ and $\varepsilon$.

**Proof.** By partial summation, for $X \leq Q \leq X + H$, we have
$$S_\rho(Q) = \frac{1}{\rho} \int_0^Q (Q - u)^{\frac{\varepsilon}{k}} du.$$ (33)

$$= \frac{1}{k\ell} \int_0^Q (Q - u)^{\frac{\varepsilon}{k} - 1} du - \frac{1}{\rho} \int_0^Q (Q - u)^{\frac{\varepsilon}{k}} d\left(\left\{u^{\frac{1}{2}}\right\} - \frac{1}{2}\right).$$

The first integral on the right-hand side of (33) is
$$\frac{1}{k\ell} \int_0^Q (Q - u)^{\frac{\varepsilon}{k} - 1} du = \frac{1}{k\ell} \int_0^Q (Q - u)^{\frac{\varepsilon}{k} - 1} du + O\left(H^{\frac{\varepsilon}{2} + \frac{1}{2}} X^{\frac{\varepsilon}{2} + \frac{1}{2}}\right)$$
$$= \frac{1}{k\ell} \Gamma\left(\frac{\varepsilon}{k} + \frac{1}{2}\right) Q^{\frac{\varepsilon}{k} + \frac{1}{2}} + O\left(HX^{\beta + \frac{1}{2} - 1} B^{-2}\right)$$
provided \((32)\). The second integral on the right-hand side of \((33)\) is
\[
- \frac{1}{\rho} \int_0^X (Q - u)^{\frac{\rho}{2}} d \left\{ u^{\frac{1}{2}} \right\}
\]
\[
= - \frac{1}{k} \int_0^X (Q - u)^{\frac{1}{k} - 1} \left\{ u^{\frac{1}{2}} \right\} d u - \frac{Q^\frac{1}{k}}{2\rho} + O \left( \frac{H^\frac{1}{k}}{|\gamma|} \right)
\]
\[
= - \frac{1}{k} \int_0^X (Q - u)^{\frac{1}{k} - 1} \left\{ u^{\frac{1}{2}} \right\} d u - \frac{Q^\frac{1}{k}}{2\rho} + O \left( \frac{H X^{\frac{1}{k} + \frac{1}{2} - 1} B^{-2}}{|\gamma|} \right)
\]
provided \((32)\). Recall the Fourier expansion
\[
\{u\} = \frac{1}{2} - \sum_{n \neq 0} \frac{e(nu)}{2\pi in},
\]
which holds for \(u \not\in \mathbb{Z}\) and converges boundedly for \(u \in \mathbb{R}\). Then since
\[
- \frac{1}{k} \int_0^X (Q - u)^{\frac{1}{k} - 1} \left\{ u^{\frac{1}{2}} \right\} d u
\]
\[
= - \frac{1}{k} \int_0^{X-1} (Q - u)^{\frac{1}{k} - 1} \left\{ u^{\frac{1}{2}} \right\} d u + O \left( \frac{1}{k} \int_0^X (Q - u)^{\frac{1}{k} - 1} d u \right)
\]
\[
= - \frac{1}{k} \int_0^{X-1} (Q - u)^{\frac{1}{k} - 1} \left\{ u^{\frac{1}{2}} \right\} d u + O \left( \frac{1}{|\gamma|} \right),
\]
by using Lemma \([6]\) and the assumption \(|\gamma| \leq X\), we have
\[
S_\rho(Q) = \frac{1}{k!} \frac{\Gamma \left( \frac{\rho}{k} \right) \Gamma \left( \frac{1}{k} \right)}{\Gamma \left( \frac{\rho}{k} + \frac{1}{k} + 1 \right)} Q^{\frac{\rho}{k} + \frac{1}{k}} - \frac{Q^\frac{1}{k}}{2\rho} + R_\rho(Q) + O \left( \frac{H X^{\frac{1}{k} + \frac{1}{2} - 1} B^{-2}}{|\gamma|} + L \right)
\]
for \(X \leq Q \leq X + H\), where
\[
R_\rho(Q) = R_{\rho,k,\ell}(Q) := \sum_{n \neq 0} \frac{I_\rho(Q,n)}{2\pi in},
\]
\[
I_\rho(Q,n) = I_{\rho,k,\ell}(Q,n) := \int_0^{X-1} (Q - u)^{\frac{1}{k} - 1} e(nu^\frac{1}{2}) d u.
\]
In order to prove the lemma, it suffices to estimate
\[
R_\rho(X + H) - R_\rho(X).
\]
We first estimate the difference of oscillating integrals
\((34)\)
\[
I_\rho(X + H, n) - I_\rho(X, n).
\]
By changing the variable in the definition of \(I_\rho(Q,n)\), we obtain expressions
\[
I_\rho(X + H, n) = \int_1^X (u + H)^{\frac{1}{k} - 1} e \left( n(X - u)^\frac{1}{2} \right) d u,
\]
\[
I_\rho(X, n) = \int_1^X u^{\frac{1}{k} - 1} e \left( n(X - u)^\frac{1}{2} \right) d u.
\]
Let \(U = \min(4H|\gamma|, X)\). Then we decompose \((34)\) as
\[
I_\rho(X + H, n) - I_\rho(X, n) = I + I_1 - I_2,
\]
where

\[ I = \int_U ^X \left( (u + H)^{\frac{\nu}{2} - 1} - u^{\frac{\nu}{2} - 1} \right) e \left( n(X - u)^{\frac{1}{2}} \right) du, \]

\[ I_1 = \int_1 ^U \left( u + H \right)^{\frac{\nu}{2} - 1} e \left( n(X - u)^{\frac{1}{2}} \right) du, \quad I_2 = \int_1 ^U u^{\frac{\nu}{2} - 1} e \left( n(X - u)^{\frac{1}{2}} \right) du. \]

For the integral \( I \), we use the Taylor expansion

\[ (u + H)^{\frac{\nu}{2} - 1} - u^{\frac{\nu}{2} - 1} = u^{\frac{\nu}{2} - 1} \sum _{\nu = 1} ^{\infty} \left( \frac{\nu}{2} - 1 \right) \left( \frac{H}{u} \right) ^\nu. \]

By substituting this expansion into the definition of \( I \),

\[ I = \sum _{\nu = 1} ^{\infty} \left( \frac{\nu}{2} - 1 \right) H^\nu \int_U ^X u^{\frac{\nu}{2} - \nu - 1} e \left( n(X - u)^{\frac{1}{2}} \right) du. \]

By using Lemma \([19]\) and the definition of \( U \), if \( 4H|\gamma| \leq X \),

\[ I \ll \frac{U^\frac{\nu}{2} L}{|\gamma|^\frac{\nu}{2}} \sum _{\nu = 1} ^{\infty} \left( \frac{\nu}{2} - 1 \right) \left( \frac{H}{U} \right) ^\nu \ll \frac{U^\frac{\nu}{2} L}{|\gamma|^\frac{\nu}{2}} \sum _{\nu = 1} ^{\infty} \prod _{\nu = 1 \mu = 1} ^{\nu} \left( |\gamma| + 2\mu \right) \ll \frac{U^\frac{\nu}{2} L}{|\gamma|^\frac{\nu}{2}} \]

since \( |\gamma| \geq 2 \). If \( 4H|\gamma| > X \), then \( I \) is an empty integral, so the same estimate holds trivially. For the integrals \( I_1 \) and \( I_2 \), we may use Lemma \([19]\) directly to obtain

\[ I_1 , I_2 \ll \frac{U^\frac{\nu}{2} L}{|\gamma|^\frac{\nu}{2}} \]

since we can choose \( Q = X + H \) for the integral

\[ I_1 = \int_1 ^U \left( u + H \right)^{\frac{\nu}{2} + \frac{\nu}{2} - 1} e \left( n(X - u)^{\frac{1}{2}} \right) du \]

\[ = \int_1 ^U u^{\frac{\nu}{2} + \frac{\nu}{2} - 1} e \left( n(X + H - u)^{\frac{1}{2}} \right) du. \]

Therefore, we have

\[ I_\rho (X + H, n) - I_\rho (X, n) \ll \frac{U^\frac{\nu}{2} L}{|\gamma|^\frac{\nu}{2}} \ll H^\frac{\nu}{2} |\gamma|^{\frac{\nu}{2} - \frac{1}{2}} L. \]

On the other hand, if \( |n| > \ell (X + H)^{1 - \frac{1}{2}} |\gamma| \), Lemma \([19]\) gives

\[ I_\rho (X + H, n) - I_\rho (X, n) \ll \frac{X^{1 - \frac{1}{2}}}{|n|}. \]

Thus we have

\[ R_\rho (X + H) - R_\rho (X) \ll H^\frac{\nu}{2} |\gamma|^{\frac{\nu}{2} - \frac{1}{2}} L \sum _{n \leq \ell (X + H)^{1 - \frac{1}{2}} |\gamma|} \frac{1}{n} + X^{1 - \frac{1}{2}} \sum _{n > \ell (X + H)^{1 - \frac{1}{2}} |\gamma|} \frac{1}{n^2} \ll H^\frac{\nu}{2} |\gamma|^{\frac{\nu}{2} - \frac{1}{2}} L^2 + 1. \]

This completes the proof. \( \square \)
6. Completion of the proof

In this section, we complete the proof of main theorems. However, before the proof of Theorem 2, we check the direct consequence of Lemma 8 which is better than our main estimate when $\ell$ is relatively larger than $k$.

Lemma 21. For positive integers $k, \ell$ with $\ell \geq 2$, and real numbers $X, H, \varepsilon$ with $2 \leq H \leq X$ and $\varepsilon > 0$, we have the asymptotic formula provided

\[X^{1-\theta_A(k,\ell)+\varepsilon} \leq H \leq X^{1-\varepsilon},\]

where $\theta_A(k,\ell)$ is defined by

\[\theta_A(k,\ell) = \min \left( \frac{5}{12k}, \frac{k}{\ell(k-1)} \right)\]

and the implicit constant depends on $k, \ell$ and $\varepsilon$.

Proof. We use Lemma 8 in Lemma 17. If $n^\ell \leq X$ and $(X + H - n^\ell) \leq 2(X - n^\ell)$,

\[(X + H - n^\ell) \frac{1}{H} - (X - n^\ell) \frac{1}{H} = \frac{1}{k} \int_{X - n^\ell}^{X + H - n^\ell} u^\frac{1}{k} \, du \gg H(X - n^\ell) \frac{1}{k} - 1 \]

\[\gg X^{1-\frac{1}{k}+\varepsilon}(X - n^\ell) \frac{1}{k} - 1 \gg \left((X - n^\ell) \frac{1}{k}\right) \frac{1}{k} + \varepsilon\]

provided (35). Thus, in this case, Lemma 8 gives

\[\psi \left( (X + H - n^\ell) \frac{1}{H} \right) - \psi \left( (X - n^\ell) \frac{1}{H} \right) \]

\[= (X + H - n^\ell) \frac{1}{H} - (X - n^\ell) \frac{1}{H} + O((X + H - n^\ell) \frac{1}{H} - (X - n^\ell) \frac{1}{H}) B^{-1}\]

by making the constant $c$ smaller since

\[(X - n^\ell) \frac{1}{H} \gg (X + H - n^\ell) \frac{1}{H} \gg H \frac{1}{H}\]

in the current case. If $n^\ell \leq X$ and $(X + H - n^\ell) > 2(X - n^\ell)$, then we may apply the usual prime number theorem to obtain the same estimate since in this case

\[(X + H - n^\ell) \frac{1}{H} - (X - n^\ell) \frac{1}{H} \asymp (X + H - n^\ell) \frac{1}{H}\]

By using (36) in Lemma 17 and using Lemma 8 we arrive at the lemma. \hfill $\square$

Lemma 22. For positive integers $k, \ell$ with $\ell \geq 2$, and real numbers $X, H, \varepsilon$ with $2 \leq H \leq X$ and $\varepsilon > 0$, we have the asymptotic formula provided

\[X^{1-\theta_B(k,\ell)+\varepsilon} \leq H \leq X^{1-\varepsilon},\]

where $\theta_B(k,\ell)$ is defined by

\[\theta_B(k,\ell) = \begin{cases} 
\min \left( \frac{1}{2} \left( \frac{1}{k} + \frac{1}{\ell} \right), \frac{2}{\ell}, \frac{\lambda_1(\ell)}{k}, \frac{\lambda_2(k,\ell)}{k} \right) & \text{if } k = 1, \\
\min \left( \frac{1}{2} \left( \frac{1}{k} + \frac{1}{\ell} \right), \frac{k}{\ell(k-1)}, \frac{\lambda_1(\ell)}{k}, \frac{\lambda_2(k,\ell)}{k} \right) & \text{if } k \geq 2, 
\end{cases}\]

and the implicit constant depends on $k, \ell$ and $\varepsilon$. 
Proof. We may assume that $X$ is larger than some constant depends only on $k, \ell$ and $\varepsilon$ since otherwise the assertion trivially holds. By Lemma 17, Lemma 18, and Lemma 20,

$$
\sum_{X < N \leq X + H} R(N)
$$

$$
= M + R_1 + R_2 + O((R_3 + X^{\frac{1}{2} + \frac{1}{T^{-1}} + T})L^2 + HX^{\frac{1}{2} + \frac{1}{T} - 1}B^{-1})
$$

provided

$$
X^{1 - \min\left(\frac{1}{T}, \frac{k - 1}{k}\right)} \leq H \leq X^{1 - \varepsilon}, \quad 2 \leq T \leq X^\frac{1}{k},
$$

where

$$
M = S(X + H) - S(X),
$$

$$
R_1 = \sum_{|\gamma| \leq T} \frac{1}{k\ell} \frac{\Gamma\left(\frac{1}{k}\right) \Gamma\left(\frac{1}{\ell}\right)}{\Gamma\left(\frac{1}{k} + \frac{1}{\ell} + 1\right)} \left( (X + H)^{\frac{1}{2} + \frac{1}{T} - 1} - X^{\frac{1}{2} + \frac{1}{T} - 1} \right),
$$

$$
R_2 = \sum_{|\gamma| \leq T} \frac{(X + H)^{\frac{1}{2}} - X^{\frac{1}{2}}}{2\rho}, \quad R_3 = \sum_{|\gamma| \leq T} H^{\frac{1}{2}} |\gamma|^{\frac{1}{2} - \frac{1}{T} - 1} .
$$

In order to control the size of the error $X^{\frac{1}{2} + \frac{1}{T^{-1}} + T}L^2$, we choose $T$ by

$$
T = X^{1 + \frac{1}{2}H^{-1}}, \quad 0 < \varepsilon_1 \leq \frac{\varepsilon}{2},
$$

where we choose $\varepsilon_1$ later (our choice will be $\varepsilon_1 = \frac{\varepsilon}{80}$). This choice is admissible since the former inequality of (48) implies

$$
X^\varepsilon \leq T \leq X^{\frac{1}{k} - \frac{1}{k}}.
$$

If we assume further

$$
X^{1 - \frac{1}{k}(\frac{1}{2} + \frac{1}{T} + \varepsilon)} \leq H,
$$

then

$$
TL^2 = X^{1 + \frac{1}{2}H^{-1}}L^2 = HX^{1 + \frac{1}{2}H^{-2}}L^2 \leq HX^{\frac{1}{2} + \frac{1}{T} - 1 - \varepsilon}L^2 \ll HX^{\frac{1}{2} + \frac{1}{T} - 1 - \varepsilon}B^{-1}.
$$

Thus,

$$
(X^{\frac{1}{2} + \frac{1}{T} - 1} + T)L^2 \ll HX^{\frac{1}{2} + \frac{1}{T} - 1}B^{-1}
$$

provided (41). By Lemma 3 the main term $M$ can be evaluated as

$$
M = \frac{1}{k\ell} \frac{\Gamma\left(\frac{1}{k}\right) \Gamma\left(\frac{1}{\ell}\right)}{\Gamma\left(\frac{1}{k} + \frac{1}{\ell} + 1\right)} HX^{\frac{1}{2} + \frac{1}{T} - 1 - 1} + O \left( HX^{\frac{1}{2} + \frac{1}{T} - 1}B^{-1} \right)
$$

provided (48). The remaining task is to estimate $R_1$, $R_2$ and $R_3$.

We first estimate the sum $R_1$. By the fundamental theorem of calculus,

$$
(X + H)^{\frac{1}{2} + \frac{1}{T} - 1} - X^{\frac{1}{2} + \frac{1}{T} - 1} = \left( \frac{\rho}{k} + \frac{1}{\ell} \right) \int_X^{X + H} u^{\frac{1}{2} + \frac{1}{T} - 1} du \ll |\gamma| HX^{\frac{1}{2} + \frac{1}{T} - 1}.
$$

Then, by using Stirling’s formula and dissecting dyadically,

$$
R_1 \ll HX^{-1} + \sum_{|\gamma| \leq T} \frac{X^{\frac{1}{2}}}{|\gamma|^\frac{1}{2}} \ll HX^{\frac{1}{2} - 1}L \sup_{1 \leq K \leq T} K^{-\frac{1}{2}} \sum_{K < |\gamma| \leq 2K} X^{\frac{1}{2}} .
$$
For $1 \leq K \leq T$, we write $K = X^{\frac{1}{k}}$. Further, we write

$$X H^{-1} = X^{\frac{1}{k}}.$$  

Then, by (39), $\delta$ moves in the range

$$0 \leq \delta \leq \Delta + \varepsilon_1.$$  

By Lemma 9,

$$K - \frac{1}{k} \sum_{K < |\gamma| \leq 2K} X^{\frac{1}{k}} \ll \left( X^{\frac{1}{k}} (\phi(\delta) - \frac{1}{k} \delta) + X^{\frac{1}{k} (1 - \eta + (2\eta - 1) \delta)} \right) L^A.$$  

By Lemma 10 and the assumption $\ell \geq 2$, for sufficiently large $X$,

$$\frac{d}{d\delta} \left( \phi(\delta) - \frac{1}{k} \delta \right) > 0, \quad 2\eta - \frac{1}{k} < 0.$$  

Therefore, by (44), (46) and (47),

$$R_1 \ll H X^{\frac{1}{k} - 1 + \frac{1}{k} (\phi(\Delta + \varepsilon_1) - \frac{1}{k} (\Delta + \varepsilon_1))} + X^{\frac{1}{k} (1 - \eta)} L^{A+1} \ll H X^{\frac{1}{k} - 1 + \frac{1}{k} (\phi(\Delta + \varepsilon_1) - \frac{1}{k} (\Delta + \varepsilon_1)) + 1} + H X^{\frac{1}{k} + \frac{1}{k} - 1} B^{-1}.$$  

By Lemma 10 and the mean value theorem,

$$\phi(\Delta + \varepsilon_1) - \frac{1}{k} (\Delta + \varepsilon_1) \leq \phi(\Delta) - \frac{1}{k} \Delta + \varepsilon_1.$$  

Thus, by (48), we obtain

$$R_1 \ll H X^{\frac{1}{k} - 1 + \frac{1}{k} (\phi(\Delta) - \frac{1}{k} \Delta) + 2\varepsilon_1} + H X^{\frac{1}{k} + \frac{1}{k} - 1} B^{-1}.$$  

This completes the estimate of $R_1$.

We next estimate the sum $R_2$. We use

$$(X + H)^{\frac{1}{k}} - X^{\frac{1}{k}} = \frac{1}{k} \int_X^{X+H} u^{\frac{1}{k} - 1} du \ll |\gamma| H X^{\frac{1}{k} - 1}.$$  

Then, since $X/|\gamma| \geq 1$ for $|\gamma| \leq T \leq X$,

$$R_2 \ll H X^{\frac{1}{k} - 1} \sum_{|\gamma| \leq T} X^{\frac{1}{k}} \ll H X^{\frac{1}{k} - 1} \sum_{|\gamma| \leq T} \frac{X^{\frac{1}{k}}}{|\gamma|^{\frac{1}{k}}}.$$  

This right-hand side is the same quantity appeared in (44). Thus,

$$R_2 \ll H X^{\frac{1}{k} - 1 + \frac{1}{k} (\phi(\Delta) - \frac{1}{k} \Delta) + 2\varepsilon_1} + H X^{\frac{1}{k} + \frac{1}{k} - 1} B^{-1}.$$  

This completes the estimate of $R_2$.

We finally estimate the sum $R_3$. We dissect the sum dyadically to obtain

$$R_3 \ll L \sup_{1 \leq K \leq T} \sum_{K < |\gamma| \leq 2K} (HK)^{\frac{1}{k}}.$$  

We again write $K = X^{\frac{1}{k}}$ and use the parameter $\Delta$ defined in (45). By (40),

$$X^{\frac{\Delta + \varepsilon_1}{k}} = X^{1 + \frac{\varepsilon_1}{k} H^{-1}} = T \leq X^{\frac{1}{k}}$$  

so that

$$0 \leq \Delta \leq 1 - \varepsilon_1.$$
Let
\[ \lambda = \lambda(\delta) := \frac{\log K}{\log(HK)^{\frac{1}{k}}} = \frac{k \log K}{\log H + \log K} = \frac{\delta}{1 - \frac{\Delta}{k} + \frac{\delta}{k}}. \]

By (52), this function \( \lambda(\delta) \) is increasing with respect to \( \delta \). Note that
\[ K = K^{1 - \frac{\delta}{k}} K^{\frac{1}{2}} \leq T^{1 - \frac{\delta}{k}} K^{\frac{1}{2}} \leq (X^{1 - \frac{\delta}{k}} K)^{\frac{1}{2}} \leq (HK)^{\frac{1}{2}} \]
by (40) provided (38). Thus, by using Lemma 9 with
\[ \lambda \]
by Lemma 10 and the mean value theorem,
\[ \text{are increasing function of } \delta \],

\[ \text{by (52), this function } \lambda(\delta) \text{ is increasing with respect to } \delta. \]

\[ K^{-\frac{1}{2}} \sum_{K < |\gamma| \leq 2K} (HK)^{\frac{\delta}{k}} \ll \left( (HK)^{\frac{1}{k}(\phi(\lambda) - \frac{1}{2} \lambda)} + (HK)^{\frac{1}{k}(1 - \eta + (2\eta - \frac{1}{2}) \lambda)} \right) L^A. \]

Since
\[ HK = X(XH^{-1})^{-1} K = X^{1 - \frac{\delta}{k} + \frac{1}{2}}, \]

the last estimate is rewritten as
\[ K^{-\frac{1}{2}} \sum_{K < |\gamma| \leq 2K} (HK)^{\frac{\delta}{k}} \ll \left( X^{\frac{1}{k}(1 - \frac{\delta}{k} + \frac{1}{2}) (\phi(\lambda) - \frac{1}{2} \lambda)} + X^{\frac{1}{k}(1 - \frac{\delta}{k} + \frac{1}{2}) (1 - \eta + (2\eta - \frac{1}{2}) \lambda)} \right) L^A. \]

Since both of the factors
\[ \left( 1 - \frac{\Delta}{k} + \frac{\delta}{k}, \left( \phi(\lambda) - \frac{1}{2} \lambda \right) \right) \]
are increasing function of \( \delta \), by (52),
\[ X^{\frac{1}{k}(1 - \frac{\delta}{k} + \frac{1}{2}) (\phi(\lambda) - \frac{1}{2} \lambda)} \leq X^{\lambda(1 + \epsilon_1) (\phi(\lambda(\Delta + \epsilon_1))) - \frac{1}{2} \lambda(\Delta + \epsilon_1))} \]
\[ \leq X^{\lambda(\phi(\lambda(\Delta + \epsilon_1))) - \frac{1}{2} \lambda(\Delta + \epsilon_1) + \epsilon_1}. \]

Since \( \Delta \leq \delta \leq \Delta + \epsilon_1 \),
\[ \lambda'(\delta) = \frac{1 - \frac{\Delta}{k} + \frac{\delta}{k}}{(1 - \frac{\Delta}{k} + \frac{\delta}{k})^2} \leq 1 - \frac{\Delta}{k} \leq 1, \]
by Lemma 10 and the mean value theorem,
\[ \phi(\lambda(\Delta + \epsilon_1)) - \frac{1}{2} \lambda(\Delta + \epsilon_1) \leq \phi(\lambda(\Delta)) - \frac{1}{2} \lambda(\Delta) + \epsilon_1 = \phi(\Delta) - \frac{1}{2} \Delta + \epsilon_1. \]

Thus,
\[ (1 - \frac{\Delta}{k} + \frac{\delta}{k}) \left( 1 - \eta + \left( 2\eta - \frac{1}{2} \right) \lambda \right) = \left( 1 - \frac{\Delta}{k} + \frac{\delta}{k} \right) (1 - \eta) + \left( 2\eta - \frac{1}{2} \right) \delta \]
\[ = \left( 1 - \frac{\Delta}{k} \right) (1 - \eta) + \left( \frac{1 - \eta}{k} + 2\eta - \frac{1}{2} \right) \delta, \]
we have
\[ X^{\frac{1}{k}(1 - \frac{\delta}{k} + \frac{1}{2})(1 - \eta + (2\eta - \frac{1}{2}) \lambda)} \ll \begin{cases} H^{1 - \eta} X^{\frac{1}{k}(\frac{1}{k}(\Delta + \epsilon_1))} & \text{if } k = 1, \\ H^{\frac{1}{k}} X^{\eta(\Delta + \epsilon_1)} & \text{if } k \geq 2, \end{cases} \]
\[ \ll \begin{cases} H^{\frac{1}{k}} X^{\frac{1}{k} + 2\epsilon_1} & \text{if } k = 1, \\ H^{\frac{1}{k}} X^{2\epsilon_1} & \text{if } k \geq 2, \end{cases} \]
(55)
for sufficiently large $X$. By \([11], [34], [54],\) and \([55],\)
\begin{equation}
R_3 \ll \begin{cases}
X^{\frac{1}{2} + \lambda_1} + \frac{3}{4} \varepsilon_1 & \text{(if } k = 1\text{)}, \\
X^{\frac{1}{2} + \lambda_2} + \frac{3}{4} \varepsilon_1 & \text{(if } k \geq 2\text{)}.
\end{cases}
\end{equation}

By combining \([37], [42], [43], [49], [50],\) and \([50],\) we have
\[\sum_{X < N \leq X + H} R(N) = \frac{1}{k \ell} \frac{1}{1 - \frac{k}{2}} HX^{\frac{1}{2} + \frac{1}{\ell}} + O \left( HX^{\frac{1}{2} + \frac{1}{\ell}} B^{-1} + E \right)\]
provided
\[X^{1 - \frac{1}{2} \lambda_2 + \varepsilon} \leq H \leq X^{1 - \varepsilon} \quad \text{(if } k = 1\text{)}, \quad X^{1 - \frac{1}{2} \lambda_2 + \varepsilon} \leq H \leq X^{1 - \varepsilon} \quad \text{(if } k \geq 2\text{)},\]
and \(\varepsilon_1 \leq \frac{\varepsilon}{16},\) where
\begin{equation}
E = HX^{\frac{1}{2} - \frac{1}{4} \phi + \frac{1}{2} \lambda_1} + X^{\frac{1}{2} - \frac{1}{4} \phi + \frac{1}{2} \lambda_2} + X^{\frac{1}{2} - \frac{1}{4} \phi + \frac{1}{2} \lambda_2} = 1.
\end{equation}

Let \(\lambda_1, \lambda_2\) be the functions given by \([15],\) or equivalently, given in Lemma \([11].\) Then, by assuming further
\[X^{1 - \frac{1}{2} \lambda_2 + \varepsilon} \leq H,\]
we have \(0 \leq \Delta \leq \min(\lambda_1, \lambda_2) - k \varepsilon.\) Thus, Lemma \([12]\) and \([57]\) implies
\[E \ll HX^{\frac{1}{2} - \frac{1}{4} \phi + \frac{1}{2} \lambda_2 + \varepsilon}.
\]
Thus, by taking \(\varepsilon_1 = \frac{\varepsilon}{16},\) we obtain the asymptotic formula \([10]\) provided
\begin{equation}
X^{1 - \frac{1}{2} \lambda_2 + \varepsilon} \leq H \leq X^{1 - \varepsilon} \quad \text{(if } k = 1\text{)}, \quad X^{1 - \frac{1}{2} \lambda_2 + \varepsilon} \leq H \leq X^{1 - \varepsilon} \quad \text{(if } k \geq 2\text{)}.
\end{equation}
Since \(\lambda_1(\ell) \leq 1\) for any \(\ell \geq 2,\) we have
\[\frac{1}{k} \geq \frac{\lambda_1(\ell)}{k}.
\]
Thus, we can remove the exponent \(\frac{1}{2}\) in \([58].\) This completes the proof. \(\square\)

We next check when Lemma \([22]\) gives substantial improvements on Lemma \([21].\)

**Lemma 23.** Let \(\theta_A(k, \ell), \theta_B(k, \ell)\) be functions given in Lemma \([21]\) and Lemma \([22]\) respectively. Then, for positive integers \(k, \ell\) with \(\ell \geq 2,\) we have
\[\theta_A(k, \ell) < \theta_B(k, \ell) \iff \begin{cases}
\ell \leq 9 \text{ and } \frac{5}{24} \ell < k, \\
\text{or } \ell \geq 10 \text{ and } \frac{5}{12} \ell + \frac{1}{24} \sqrt{25\ell - 240} < k.
\end{cases}\]

**Proof.** We first consider the case \(k \leq \frac{5}{24} \ell.\) In this case,
\[\theta_B(k, \ell) \leq \min \left( \frac{\lambda_2(k, \ell)}{k}, \frac{\ell}{k} \right) \leq \min \left( \frac{\lambda_2(k, 24k)}{k}, \frac{\ell}{k} \right) = \min \left( \frac{5}{12k}, \frac{k}{\ell(k-1)} \right) = \theta_A(k, \ell)
\]
by Lemma \([13]\) where we extended the domain of \(\lambda_2(k, \ell)\) from positive integers \(k, \ell\) to positive real numbers \(k, \ell\) by the same equation \([15].\) Thus, in the case \(k \leq \frac{5}{24} \ell,\) both hand sides of the assertion is false so that the assertion holds.
We consider the remaining case $k > \frac{5}{24} \ell$ in what follows. In this case,

\[
\frac{5}{12k} \geq \frac{k}{\ell(k-1)} \iff \left( k - \frac{5}{24} \ell \right)^2 \geq \left( \frac{1}{24} \right)^2 \ell(25\ell - 240).
\]

Thus, we obtain that

\[
\theta_A(k,\ell) = \begin{cases} \frac{k}{\ell(k-1)} & \text{(if } \ell \geq 10 \text{ and } \frac{5}{24} \ell < k \leq \frac{5}{24} \ell + \frac{1}{24} \sqrt{\ell(25\ell - 240)}) \), \\ \frac{5}{12k} & \text{(otherwise).} \end{cases}
\]

Therefore, in the former case, i.e. in the case

\begin{equation}
\ell \geq 10 \text{ and } \frac{5}{24} \ell < k \leq \frac{5}{24} \ell + \frac{1}{24} \sqrt{\ell(25\ell - 240)},
\end{equation}

we have

\[
\theta_B(k,\ell) \leq \frac{k}{\ell(k-1)} = \theta_A(k,\ell).
\]

This again makes the both sides of the assertion false, which proves the assertion for the case (59).

In the remaining case, in which (59) does not hold but $k > \frac{5}{24} \ell$ holds, we have

\begin{equation}
\frac{1}{2} \left( \frac{1}{k} + \frac{1}{\ell} \right) > \frac{5}{12k}, \quad \frac{2}{\ell} > \frac{5}{12k}, \quad \frac{k}{\ell(k-1)} > \frac{5}{12k}.
\end{equation}

Also, since $\lambda_1(\ell)$ and $\lambda_2(k,\ell)$ are decreasing in $\ell$,

\begin{equation}
\frac{\lambda_1(\ell)}{k} > \frac{5}{12k}, \quad \frac{\lambda_2(k,\ell)}{k} > \frac{5}{12k}.
\end{equation}

Combining (60) and (61),

\[
\theta_A(k,\ell) > \frac{5}{12k} = \theta_B(k,\ell)
\]

in the remaining case. This completes the proof.

Finally, we prove main theorems. Since Theorem 1 is just a special case of Theorem 2, we prove only Theorem 2 and Theorem 3.

**Proof of Theorem 2.** By Lemma 21 and Lemma 22, we have (10) provided

\[
X^{1-\max(\theta_A(k,\ell),\theta_B(k,\ell))} + \varepsilon \leq H \leq X^{1-\varepsilon}.
\]

By using Lemma 23 we find that

\[
1 - \max(\theta_A(k,\ell),\theta_B(k,\ell)) = \Theta(k,\ell),
\]

where we can ignore the exponent $\frac{2}{k}$ for the case $k = 1$ since by some numerical calculation, we can check that none of the values

\[
\theta_B(1,2) = \frac{17 + 4\sqrt{15}}{49}, \quad \theta_B(1,3) = \frac{44 + 8\sqrt{2}}{147}, \quad \theta_B(1,4) = \frac{5}{11}
\]

are equal to $\frac{2}{k}$. Thus we arrive at the theorem.

**Proof of Theorem 3.** By Theorem 2 it suffices to prove

\begin{equation}
\Theta(k,\ell) = \frac{1}{k} \quad \text{for } \ell \geq 2 \text{ and } \ell = 2.
\end{equation}
For \( k \geq 2 \) and \( \ell = 2 \), we have \( \Theta(k, \ell) = 1 - \theta(k, \ell) \) by definition. Then, since \( k \geq \ell \),
\[
\frac{1}{2} \left( \frac{1}{k} + \frac{1}{\ell} \right) \geq \frac{1}{k}, \quad \frac{k}{\ell(k-1)} \geq \frac{1}{\ell} \geq \frac{1}{k}, \quad \lambda_2(k, \ell) \geq 1.
\]
Finally, \( \lambda_1(2) = 1 \). Thus, we obtain (62) and arrive at the theorem. \( \square \)

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