Phase Transitions in the 1-d Long-Range Diluted Heisenberg Spin Glass

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We use Monte Carlo simulations to study the one-dimensional long-range diluted Heisenberg spin glass with interactions that fall as a power, $\sigma$, of the distance. Varying the power is argued to be equivalent to varying the space dimension of a short-range model. We are therefore able to study both the mean-field and non-mean-field regimes. For one value of $\sigma$, in the non-mean-field regime, we find evidence that the chiral glass transition temperature may be somewhat higher than the spin glass transition temperature. For the other values of $\sigma$ we see no evidence for this.

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I. INTRODUCTION

In the theory of phase transitions it is often helpful to study models in a range of dimensions ranging from above the “upper critical dimension”, $d_u$, where mean-field critical behavior is expected, to below the “lower critical dimension”, $d_l$, where fluctuations destroy the transition. However, it has been difficult to do this numerically for spin glasses, since $d_u = 6$ is quite large, and slow dynamics prevents more than a few thousand spins being equilibrated at low temperature $T$. It follows that, at and above $d_u$, one cannot study a sufficient range of sizes to perform the necessary finite-size scaling analysis.

As a result, there has been a lot of recent attention on long-range models in one-dimension, in which the interactions fall off with a power of the distance. Such models have a venerable history going back to Dyson [1, 2], who considered a ferromagnet with interactions falling off like $1/r^\sigma$, and found a paramagnet-ferromagnet transition for $1 < \sigma \leq 2$. Kotliar et al. [3] studied the spin glass version of this model, which has received a lot of attention numerically in the last few years [4–8].

There are few analytical results on spin glasses beyond mean field theory. For the long-range models, Kotliar et al. [3] computed critical exponents in an expansion away from the point where mean field theory occurs ($\epsilon$-expansion), but, as we shall see, this is poorly converged. Hence, most of what we know has come from numerical work. Much of this, including numerics on long-range spin glass models, has studied the Ising version. However, there are also reasons to study models with vector spins, such as the Heisenberg (3-component) model.

One motivation is that Kawamura [9] proposed that there are two separate transitions in vector spin glasses, a spin glass transition at $T = T_{SG}$ and a “chiral glass transition” at higher temperature $T_{CG}$, involving a freezing of vortex-like variables called chiralities. While the original scenario had $T_{SG} = 0$, it now appears that $T_{SG}$ is non-zero in three or more dimensions, but the question of whether $T_{SG} < T_{CG}$, or whether there is a single transition at which both types of ordering occur, is still open [10–20].

A second motivation for studying the Heisenberg, rather than Ising, spin glass is that it is possible to study larger sizes, see for example Ref. [19], which should be helpful in a finite-size scaling analysis. In a second paper in this series [21], we will investigate whether there is a de Almeida-Thouless [22] (AT) line of transitions in a magnetic field for Heisenberg spin glasses. This follows our recent work [23] which shows that there is an AT line for vector spin glasses provided one considers a random field. The ability to study larger sizes will be particularly useful for the AT-line study.

Here we present data for the zero field transition for the Heisenberg spin glass for values of the parameter $\sigma$ corresponding to (i) the mean-field regime, (ii) the non-mean-field regime, and (iii) the borderline case where the transition disappears. Most of our results find no evidence for separate spin-glass and chiral-glass transitions. However, for one set of parameters in the non-mean-field regime, the data indicates that $T_{CG}$ may be somewhat greater than $T_{SG}$. Whether this result remains valid in the thermodynamic limit, will require future studies on significantly larger systems.

The plan of this paper is as follows. In Sec. II we describe the model that we study, while in Sec. III we give some technical details of the simulations. The results are presented in Sec. IV and our conclusions summarized in Sec. V.

II. MODEL

We consider the Hamiltonian

$$\mathcal{H} = - \sum_{(i,j)} J_{ij} \mathbf{S}_i \cdot \mathbf{S}_j, \quad (1)$$

where $\mathbf{S}_i$ are classical 3-component Heisenberg spins of length 1, and the interactions $J_{ij}$ are independent variables with zero mean and a variance which falls off with...
a power of the distance between the spins,
\[
[j^2_{ij}]_{av} \propto \frac{1}{r_{ij}^{2\sigma}},
\]
where \([\cdots]_{av}\) means an average over disorder. In the version used in early studies \[4\], every spin interacts with every other spin with a strength which falls off, on average, like Eq. (2). However, this means that the CPU time per sweep varies as \(N^2\), rather than \(N\), so large sizes cannot be studied. This problem was solved by Leuzzi et. al. \[6\] who proposed a model in which, instead of the magnitude of the interaction falling off with distance like Eq. (2), it is the probability of there being a non-zero interaction between sites \((i, j)\) which falls off, and when an interaction does exist, its variance is independent of \(r_{ij}\). The mean number of non-zero interactions from a site, which we call \(z\), can be fixed, and we take \(z = 6\). To generate the set of pairs \((i, j)\) that have an interaction with the desired probability we choose spin \(i\) randomly, and then choose \(j\) \((\neq i)\) at distance \(r_{ij}\) with probability
\[
p_{ij} = \frac{r_{ij}^{-2\sigma}}{\sum_{j\neq i} r_{ij}^{-2\sigma}},
\]
where, for \(r_{ij}\), we put the sites on a circle and use the distance of the chord, i.e.
\[
r_{ij} = \frac{N}{\pi} \sin \left( \frac{\pi}{N} (i-j) \right).
\]
If \(i\) and \(j\) are already connected, we repeat the process until we find a pair which has not been connected before. We then connect \(i\) and \(j\) with an interaction picked from a Gaussian interaction whose mean is zero and whose standard deviation is \(J\), which set equal to 1. This process is repeated precisely \(N_z = zN/2\) times.

The result is that each pair \((i, j)\) will be connected with a probability \(P_{ij}\) which must satisfy the condition \(\sum_i P_{ij} = Nz\) since \(P_{ij}\) only depends on \(|i-j|\), \(P_{ii} = 0\), and there are precisely \(Nz/2\) connected pairs. It follows that, for a fixed site \(i\),
\[
\sum_j [J_{ij}^2]_{av} = J^2 \sum_j P_{ij} = J^2 z.
\]
Note that \(P_{ij}\) is different from \(z \times p_{ij}\) in Eq. (3) because of the constraint that no bond can occur twice. The mean-field spin glass transition temperature for \(m\)-component vector spins is given by \[24\]
\[
T_{SG}^{MF} = \frac{\sqrt{2}}{m} \frac{1}{2}^{1/2} J_m J,
\]
where the last equality follows from Eq. (5). We set \(J = 1\) so that, for the situation here,
\[
J = 1, \ z = 6, \ m = 3,
\]
we have
\[
T_{SG}^{MF} = \frac{\sqrt{6}}{3} \approx 0.816,
\]
the same as for the nearest-neighbor Heisenberg spin glass on a simple cubic lattice. By varying \(\sigma\) one finds different types of behavior \[3\, 22\], as shown in Table I. For \(\sigma < 1/2\) the model is non-extensive (for instance the mean-field transition temperature in Eq. (6) diverges) unless the interactions are scaled by an inverse power of the system size. We will call this “infinite range”. The extreme limit of this, \(\sigma = 0\), is the Sherrington-Kirkpatrick \[26\] model, whose exact solution was found by Parisi \[27\, 29\]. In fact, it has been suggested \[30\] (see also Ref. 31) that, in the thermodynamic limit, the behavior of the model is identical to that of the SK model for the whole range \(0 \leq \sigma < 1/2\).

The model is extensive for \(\sigma > 1/2\) and a finite temperature transition is expected for \(\sigma < \sigma_l\), where the “lower critical” value is
\[
\sigma_l = 1.
\]
The transition is in the mean-field universality class \(\beta\) for \(\sigma < \sigma_u\), where the “upper critical” value is
\[
\sigma_u = 2/3.
\]
For \(\sigma_u < \sigma < \sigma_l\), there is a finite-temperature transition with non-mean-field critical exponents. In this paper we will study both mean-field and non-mean-field regions. Finally for \(\sigma > \sigma_l\) the there is no transition at finite temperature.

### III. NUMERICAL SETUP

We perform large scale Monte-Carlo simulations for \(\sigma = 0.6, 0.75, 0.85\) and 1. From the previous section we note that \(\sigma = 0.60\) is in the mean-field regime, \(\sigma = 0.75\) and 0.85 are in the non-mean-field regime, and \(\sigma = 1\) is the border line case, \(\sigma = \sigma_l\), beyond which there is no transition. A plausible scenario is that \(T_{SG} = 0\) for \(\sigma = 1\), though the possibility that \(T_{SG}\) is non-zero cannot be ruled out a priori. Table II lists the parameters of the simulation.

| \(\sigma\) | Behavior             |
|----------|----------------------|
| 0        | SK model             |
| 0 < \(\sigma\) ≤ 1/2 | Infinite range       |
| 1/2 < \(\sigma\) < 2/3 | Mean-field with \(T_{SG} > 0\) |
| 2/3 < \(\sigma\) < 1 | Non-mean-field with \(T_{SG} > 0\) |
| \(\sigma\) > 1  | \(T_{SG} = 0\)       |
TABLE II: Parameters of the simulations. $N_{\text{samp}}$ is the number of samples, $N_{\text{equl}}$ is the number of overrelaxation Monte Carlo sweeps for equilibration for each of the $2N_T$ replicas for a single sample. The same number of sweeps is done in the measurement phase, with a measurement performed every four overrelaxation sweeps. The number of heatbath sweeps is equal to 10% of the number of overrelaxation sweeps. $T_{\text{min}}$ and $T_{\text{max}}$ are the lowest and highest temperatures simulated, and $N_T$ is the number of temperatures used in the parallel tempering.

| $\sigma$ | $N$ | $N_{\text{samp}}$ | $N_{\text{equl}}$ | $T_{\text{min}}$ | $T_{\text{max}}$ | $N_T$ |
|---------|-----|-------------------|-------------------|-----------------|-----------------|------|
| 0.6     | 128 | 1600              | 128               | 0.20            | 0.70            | 40   |
| 0.6     | 256 | 1600              | 256               | 0.20            | 0.70            | 40   |
| 0.6     | 512 | 1600              | 512               | 0.20            | 0.70            | 40   |
| 0.6     | 1024| 1600              | 1024              | 0.20            | 0.70            | 40   |
| 0.6     | 2048| 1600              | 2048              | 0.20            | 0.70            | 40   |
| 0.6     | 4096| 6100              | 4096              | 0.30            | 0.70            | 40   |
| 0.6     | 8192| 1000              | 8192              | 0.30            | 0.70            | 50   |
| 0.6     | 16384| 500              | 16384             | 0.35            | 0.70            | 55   |
| 0.6     | 32768| 400            | 32768             | 0.35            | 0.70            | 60   |
| 0.75    | 128 | 8000              | 128               | 0.20            | 0.55            | 40   |
| 0.75    | 256 | 8000              | 256               | 0.20            | 0.55            | 40   |
| 0.75    | 512 | 8000              | 512               | 0.20            | 0.55            | 40   |
| 0.75    | 1024| 3000              | 1024              | 0.20            | 0.55            | 40   |
| 0.75    | 2048| 3000              | 2048              | 0.20            | 0.55            | 40   |
| 0.75    | 4096| 3000              | 4096              | 0.20            | 0.55            | 40   |
| 0.75    | 8192| 1100              | 8192              | 0.20            | 0.55            | 50   |
| 0.75    | 16384| 500             | 16384             | 0.25            | 0.55            | 55   |
| 0.75    | 32768| 400            | 32768             | 0.25            | 0.55            | 60   |
| 0.85    | 128 | 1600              | 512               | 0.09            | 0.30            | 40   |
| 0.85    | 256 | 1600              | 1024              | 0.09            | 0.30            | 40   |
| 0.85    | 512 | 10000             | 2048              | 0.09            | 0.30            | 40   |
| 0.85    | 1024| 10000             | 8192              | 0.14            | 0.22            | 20   |
| 0.85    | 2048| 8000              | 16384             | 0.14            | 0.22            | 20   |
| 0.85    | 4096| 4000              | 32768             | 0.14            | 0.22            | 20   |
| 0.85    | 8192| 2000              | 65536             | 0.15            | 0.21            | 20   |
| 0.85    | 16384| 1700            | 131072            | 0.16            | 0.21            | 20   |
| 1.0     | 128 | 2000              | 2048              | 0.03            | 0.10            | 10   |
| 1.0     | 256 | 2000              | 4096              | 0.03            | 0.10            | 10   |
| 1.0     | 512 | 2000              | 16384             | 0.02            | 0.10            | 10   |
| 1.0     | 1024| 1200              | 524288            | 0.017           | 0.08            | 40   |
| 1.0     | 2048| 500               | 2097152           | 0.017           | 0.08            | 60   |

A. Equilibration

As discussed in earlier work [5, 32], there is a convenient test for equilibration with Gaussian interactions, namely the relationship

$$U = \frac{J^2}{T} \sum (q_i - q_s) \sqrt{N_s}$$  \hspace{1cm} (11)$$

is valid in equilibrium but the two sides approach their common equilibrium value from opposite directions as equilibration is approached. Here $U = -(1/\sqrt{N_s}) \sum \langle \epsilon_{ij} J_{ij} (S_i \cdot S_j)^2 \rangle$ is the average energy per spin, $q_i = (1/N_s) \sum \epsilon_{ij} (S_i \cdot S_j)^2 \rangle$ is the “link overlap”, and $q_s = (1/N_s) [\langle (S_i \cdot S_j)^2 \rangle_{av}$, where $N_s = zN/2$, and $\epsilon_{ij} = 1$ if there is a bond between $i$ and $j$ and is zero otherwise. Equation (11) is easily derived by integrating by parts the expression for the average energy with respect to $J_{ij}$ since it has a Gaussian distribution. Note that in the numerics we set $J = 1$.

We determine both sides of Eq. (11) for different numbers of Monte Carlo sweeps (MCS) which increase in a logarithmic manner, each value being twice the previous one. In all cases we average over the last half of the sweeps. We consider the data to be equilibrated, if, when averaging over a large number of samples, Eq. (11) is satisfied for at least the last two points.

B. Simulation Technology

To equilibrate the system in as small a number of sweeps as possible, with the minimum amount of CPU time, we perform three types of Monte Carlo sweeps [17–19].

The workhouse of our simulation is the “Microcanonical” sweep [33] (also known as an “over-relaxation” sweep). We sweep sequentially through the lattice, and, at each site, compute the local field on the spin, $H_i = \sum_j J_{ij} S_j$. The new value for the spin on site $i$ is taken to be its old value reflected about $H_i$, i.e.

$$S'_i = -S_i + 2 \frac{S_i \cdot H_i}{H_i^2} H_i.$$  \hspace{1cm} (12)$$

These sweeps are microcanonical because they preserve energy. They are very fast because the operations are simple and no random numbers are needed. For reasons that are not fully understood, it also seems that they “stir up” the spin configuration very efficiently [18] and the system equilibrates faster than if one only uses “heat-bath” updates, described next, e.g. Fig. 9 of Ref. [34].

We also need to do heatbath sweeps in order to change the energy. As for the microcanonical case, we sweep sequentially through the lattice. We take the direction of the local field $H_i$, to be the polar axis for the spin on site $i$. We compute the polar and azimuthal angle of the new spin direction relative to the local field by the requirement that this direction occurs with the Boltzmann probability, see Ref. [17] for details.

Finally we perform parallel tempering sweeps [35, 36] to prevent the system from being trapped in local minima at low temperature. We take $N_T$ copies of the system with the same bonds but at a range of different temperatures. The minimum temperature, $T_{\text{min}} \equiv T_1$, is the low temperature where one wants to investigate the system (below $T_{SC1}$ in our case), and the maximum, $T_{\text{max}} \equiv T_{N_T}$, is high enough that the the system equilibrates very fast (well above $T_{SC2}$ in our case). A parallel tempering sweep consists of swapping the temperatures of the spin configurations at a pair of neighboring temperatures, $T_i$ and $T_{i+1}$, for $i = 1, 2, \cdots, T_{N_T} - 1$ with a probability that satisfies the detailed balance condition. The Metropo-
where $\Delta = 1/T_i - 1/T_{i+1}$ and $\Delta E = E_i - E_{i+1}$, in which $E_i$ is the energy of the copy at temperature $T_i$. In this way, a given set of spins (i.e., a copy) performs a random walk in temperature space.

We perform one parallel tempering sweep for every ten overrelaxation sweeps. Since there are two copies of spins at each temperature, indicated by labels “(1)” and “(2)” in Eq. (14) below, we actually perform parallel tempering sweeps among the set of $N_T$ copies labeled “(1)” and, separately, among the set of $N_T$ copies labeled “(2)”.

C. Quantities Measured

The main quantities measured in this simulation are the spin glass susceptibility $\chi_{SG}$, and the chiral glass susceptibility $\chi_{CG}$, at wavevectors $k = 0$, and $k = 2\pi/N$, and from these we obtain the two corresponding correlation lengths, $\xi_{SG}$ and $\xi_{CG}$. The spin glass order parameter, $q^{\mu\nu}(k)$, at wave vector $k$, is defined to be

$$q^{\mu\nu}(k) = \frac{1}{N} \sum_{i=1}^{N} S_i^{(1)} S_i^{(2)} e^{ikR_i},$$

where $\mu$ and $\nu$ are spin components, and “(1)” and “(2)” denote two identical copies of the system with the same interactions. We run two copies of the system at each temperature in order to evaluate quantities such as the spin glass susceptibility, defined in Eq. (15) below, without bias. From this we determine the wave vector dependent spin glass susceptibility $\chi_{SG}(k)$ by

$$\chi_{SG}(k) = N \langle [\langle q^{\mu\nu}(k) \rangle^2]_{av} \rangle,$$

where $\langle \cdots \rangle$ denotes a thermal average and $[\cdots]_{av}$ denotes an average over disorder. The spin glass correlation length is then determined from

$$\xi_{SG} = \frac{1}{2 \sin(k_{min}/2)} \left( \frac{\chi_{SG}(0)}{\chi_{SG}(k_{min})} - 1 \right)^{1/(2\sigma-1)},$$

where $k_{min} = (2\pi/L)$. For the Heisenberg spin glass, Kawamura defines the local chirality in terms of three spins on a line as follows:

$$\kappa_i = S_{i+1} \cdot S_i \times S_{i-1}. $$

The chiral glass susceptibility is then given by

$$\chi_{CG}(k) = N \langle [\langle q_{c}(k) \rangle^2]_{av} \rangle,$$

where the chiral overlap $q_{c}(k)$ is given by

$$q_{c}(k) = \frac{1}{N} \sum_{i=1}^{N} \kappa_i^{(1)} \kappa_i^{(2)} e^{ikR_i}.$$

We define the chiral correlation length by

$$\xi_{CG} = \frac{1}{2 \sin(k_{min}/2)} \left( \frac{\chi_{CG}(0)}{\chi_{CG}(k_{min})} - 1 \right)^{1/(2\sigma-1)}. $$

As will be revealed in the next section, three of the four quantities defined above, $\chi_{SG}, \xi_{SG}$, and $\chi_{CG}$ may be used in a finite-size-scaling analysis to locate and analyze the phase transition.

D. Finite-Size Scaling

According to finite-size scaling, the correlation length of the finite-system varies, near the transition temperature $T_c$, as

$$\frac{\xi}{N} = \chi' \left[ N^{1/\nu} (T - T_c) \right], \quad (2/3 \leq \sigma < 1), $$

$$\frac{\xi}{N^{\nu/3}} = \chi' \left[ N^{1/3} (T - T_c) \right], \quad (1/2 < \sigma \leq 2/3),$$

in which $\nu$, the correlation length exponent, is given, in the mean-field regime, by $\nu = 1/(2\sigma - 1)$. We will use Eq. (21) for both the spin glass correlation length $\xi_{SG}$, in which $T_c$ will be set to $T_{SG}$, and the chiral glass correlation length $\xi_{CG}$, in which $T_c$ will be set to $T_{CG}$. It follows that, if there is a transition at $T = T_c$, data for $\xi/N$ ($\xi/N^{\nu/3}$ in the mean-field region) for different system sizes $N$ should cross at $T_c$.

We also present data for $\chi_{SG} \equiv \chi_{SG}(0)$, which has the finite-size scaling form

$$\frac{\chi_{SG}}{N^{3-\eta}} = C \left[ N^{1/\nu} (T - T_c) \right], \quad (2/3 \leq \sigma < 1), $$

$$\frac{\chi_{SG}}{N^{1/3}} = C \left[ N^{1/3} (T - T_c) \right], \quad (1/2 < \sigma \leq 2/3).$$

Hence curves of $\chi_{SG}/N^{2-\eta}$ ($\chi_{SG}/N^{1/3}$ in the mean-field regime) should also intersect. This is particularly useful for long-range models since $\eta$ is given by the simple expression $2 - \eta = 2\sigma - 1$ exactly. However, we do not know the exponent corresponding to $\eta$ for the chiral glass susceptibility, so we will not use this quantity in the finite-size scaling analysis.

In practice, there are corrections to this finite-size scaling, so data for different sizes do not all intersect at exactly the same temperature. Including leading corrections to scaling, the intersection temperature $T^*(N, 2N)$ for sizes $N$ and $2N$ varies as

$$T^*(N, 2N) = T_c + \frac{A}{N^\lambda},$$

where $A$ is the amplitude of the leading correction, and the exponent $\lambda$ is given by

$$\lambda = \frac{1}{\nu} + \omega,$$

where $\omega$ is the leading correction to scaling exponent.
FIG. 1: (Color online) The main figure shows data for $\xi_{SG}/N^{5/3}$, in which the power of $N$ is chosen following Eq. (21b) with $\nu = 1/(2\sigma - 1)$, as a function of $T$ for different system sizes for $\sigma = 0.6$. The inset shows data for $\chi_{SG}/N^{1/3}$, in which the power of $N$ is chosen following Eq. (22b).

FIG. 2: (Color online) The intersection temperatures, $T^*(N,2N)$, obtained from the data in Fig. 1 for $\xi_{SG}/N^{5/3}$ and $\chi_{SG}/N^{1/3}$ for $\sigma = 0.6$, as a function of $N^{-\lambda}$, with $\lambda = 0.467$ determined from Eq. (24) (which is valid in the MF regime). A fit using all 8 points from $\xi_{SG}$ gives $T_{SG} = 0.564 \pm 0.002$, while a fit using the data for the largest 5 pairs of sizes from $\chi_{SG}$ gives $T_{SG} = 0.562 \pm 0.002$.

IV. RESULTS

A. $\sigma = 0.6$ (mean-field regime)

As shown in Table I, $\sigma = 0.6$ lies well inside the mean-field regime. Hence, according to Eq. (21b), results for $\xi_{SG}/N^{5/3}$ should intersect at $T_{SG}$ with $\nu$ set equal to $1/(2\sigma - 1)$. The data is shown in the main part of Fig. 1. The intersections do not occur at precisely the same temperature, but fitting the intersection temperatures to Eq. (20) is helped by the fact that we know $\lambda = \frac{4}{3} - 2\sigma$ in the MF regime [41], which gives a value 0.467 here. A straight line fit of $\xi_{SG}/N^{5/3}$ against $N^{-\lambda}$, shown in Fig. 2, gives $T_{SG} = 0.564 \pm 0.002$.

The inset to Fig. 1 shows data for $\chi_{SG}/N^{1/3}$, which should also intersect at $T_{SG}$ according to Eq. (22b). This time, we find that corrections to scaling are well described by Eq. (23) but only if we consider just the largest five pairs of sizes. The fit to Eq. (23), shown in Fig. 2, gives $T_{SG} = 0.562 \pm 0.002$, which is consistent with that obtained from the spin glass correlation length.

We have also measured the chiral glass correlation length. However, we find that $\xi_{CG}/N^{\nu/3} \lesssim 10^{-12}$ in the vicinity of $T_{SG}$. Hence chiralities cannot play an important role in this range of temperature.

According to Eqs. (21b) and (22b), the argument of the scaling functions is $N^{1/3}(T - T_c)$. Hence, at $T_{SG}$ the logarithmic derivative of $\xi_{SG}$ and $\chi_{SG}$ should vary as $N^{1/3}$. As we have seen, the data do not all intersect at the same temperature, and so we evaluate the derivatives at the intersection temperatures $T^*$ plotted in Fig. 2. The results are shown in Fig. 3. We get a power of 0.28 from $\xi_{SG}$ and 0.29 from $\chi_{SG}$, in both cases a little less than 1/3, indicating that there are still some corrections to scaling even for these large sizes.
We fit the intersection temperatures to Eq. (23), but unfortunately we do not know the value of the exponent \( \lambda \) outside the MF region, and have to include it as a fit parameter. The fit is therefore to a non-linear function of the parameters. We determine the fit parameters using the Levenberg-Marquardt algorithm [42]. The data from \( \chi_{SG} \) is better behaved than the data from \( \xi_{SG} \) so we use the former to determine the exponent \( \lambda \) and then fix this value in the fit (which did not include the smallest size) to the data from \( \xi_{SG} \). This procedure is justified since the exponent giving the leading correction to scaling, \( \lambda \), is universal, though the amplitude of this correction (\( \sigma \) in Eq. (23)) is non-universal. The results are \( \lambda = 0.44 \pm 0.13 \), \( T_{SG} = 0.359 \pm 0.003 \) from \( \xi_{SG} \), and \( T_{SG} = 0.354 \pm 0.005 \) from \( \chi_{SG} \). The two estimates for \( T_{SG} \) agree within the error bars.

The data for \( \xi_{CG} / N \) in the region of the spin glass transition temperature is very small, around \( 10^{-4} \). Hence, as for \( \sigma = 0.6 \), chiralities do not play an important role in the transition.

According to Eqs. (21a) and (22a), adapted to include corrections to finite-size scaling, the logarithmic derivative of \( \xi_{SG} \) and \( \chi_{SG} \) should vary as \( N^{1/\nu} \) at \( T^* (N,2N) \). The plots in Fig. 6 yield \( 1/\nu_{SG} = 0.25 \) from \( \xi_{SG} \) and 0.29 from \( \chi_{SG} \). The difference presumably comes from corrections to scaling.

Kotliar et al. [3] calculated critical exponents to leading order in \( \epsilon = \sigma - 2/3 \) with the result

\[
\frac{1}{\nu} = \frac{1}{3} \left( 1 - 12 \epsilon + O(\epsilon^2) \right) .
\]  

(25)

The large coefficient of \( \epsilon \) indicates the expansion becomes poorly converged well before the “lower critical” value \( \epsilon = 1/3 \). Even for the present value of \( \sigma \), which corresponds to \( \epsilon = 1/12 \), Eq. (23) gives \( 1/\nu = 0 \), and so is not useful for comparison with the numerics.

C. \( \sigma = 0.85 \) (non mean-field regime)

For \( \sigma = 0.85 \) we are further in the non-mean-field region. Among the different models studied here, this is the one which is most similar to a short-range model in three dimensions. The spin glass data is shown in Fig. 4 Chiral correlations are larger than for the transition.

As for \( \sigma = 0.75 \), to extrapolate the intersection temperatures to the thermodynamic limit, we resort to Levenberg-Marquardt fits with three parameters. Using data from \( \chi_{SG} \) we find \( \lambda_{SG} = 0.99 \pm 0.13 \). The data from \( \xi_{CG} \) is insufficient to determine \( \lambda_{CG} \) since the fits give \( \lambda_{CG} = 0.79 \pm 0.74 \), i.e. the error bar is as large as the best estimate. Nonetheless, in estimating \( T_{CG} \) and its error bar from the data for \( \chi_{CG} \), we allow \( \lambda \) to vary. For \( \xi_{CG} \) there appear to be very large sub-leading corrections, so we fixed the value of \( \lambda_{SG} \) to that obtained from \( \chi_{SG} \) when fitting the results from \( \xi_{SG} \).
Results for $T^*(N, 2N)$ against $1/N\lambda$ and fits are shown in Fig. 9. In the plot, for all data we use the value of $\lambda$ determined from $\chi_{FG}$. However, we again emphasize that, in the fit to the $\chi_{FG}$ data, we estimated $T_{CG}$ and its error bar by allowing $\lambda$ to vary. From the fits, we find $T_{CG} = 0.167 \pm 0.001$ from $\chi_{FG}$. $T_{SG} = 0.166 \pm 0.004$ from $\xi_{SG}$, and $T_{CG} = 0.190 \pm 0.006$ from $\xi_{CG}$.

The two estimates of $T_{SG}$ agree with each other but are lower than $T_{CG}$. This would imply spin-chirality decoupling for $\sigma = 0.85$. However, we note that the data for the spin glass correlation length appears, at intermediate sizes, to be extrapolating to a value for $T_{SG}$ of around 0.19 (our value for $T_{CG}$) but then, for the largest sizes, veers down to about 0.167 (very close to our value for $T_{SG}$ obtained from $\chi_{SG}$). Hence we cannot rule out the possibility that a similar “crossover” may occur for the chiral glass correlation length data, but at even larger sizes. If so, then spin-chirality decoupling would not occur at the largest scales. We also note that the actual values of $\xi_{CG}/N$ shown in Fig. 5 are still very small in the vicinity of $T_{CG}$, about 1/50 of the value of $\xi_{SG}/N$ around the transition, see Fig. 7. Hence we are very far from the regime with $\xi_{CG} > \xi_{SG}$ which will ultimately occur in the presence of spin-chirality decoupling.

Figure 10 shows results for the logarithmic derivative of $\xi_{SG}$ and $\chi_{SG}$ evaluated at $T^*(N, 2N)$. Fits give $1/\nu = 0.22$ from $\xi_{SG}$ and 0.29 from $\chi_{SG}$. The curvature in the data for $\xi_{SG}$ indicates strong finite-size corrections. Presumably these corrections are also the reason for the difference between the two estimates for $1/\nu$. Figure 11 shows similar data but for $\xi_{CG}$. The best fit gives $1/\nu = 0.21$, which is close to the estimate from $\xi_{SG}$. Note that the coefficient of $N^{1/\nu}$, 14.01, is much larger than the corresponding value, 3.18, for $\xi_{SG}$, presumably to compensate for the overall value of $\xi_{CG}$ being much less than that of $\xi_{SG}$ in the vicinity of the intersection temperatures $T^*$.
FIG. 9: (Color online) The intersection temperatures for $\sigma = 0.85$. The data from $\chi_{SG}$ was fitted to Eq. (23), with the result that $\lambda = 0.99 \pm 0.13$. This is the value used to plot all the data. The same exponent was then used to fit the largest four sizes in the data from $\xi_{SG}$, for which sub-leading corrections appear to be very large. The full Levenberg-Marquardt fit to the data for $\xi_{CG}$ was unable to determine $\lambda$ with any precision, see text. However, in estimating $T_{CG}$ and its error bar from data for $\xi_{SG}$ we allowed $\lambda$ to vary. The resulting values for the transition temperatures are $T_{SG} = 0.157 \pm 0.004$ from $\chi_{SG}$ (all 7 points), $T_{SG} = 0.166 \pm 0.004$ from $\xi_{SG}$ (4 points), and $T_{CG} = 0.190 \pm 0.006$ from $\xi_{CG}$ (5 points). We emphasize that, in these estimates, we fixed the value of $\lambda$ only for the data from $\xi_{SG}$.

FIG. 10: (Color online) The main figure is a log-log plot of the logarithmic derivative of $\xi_{SG}$ for $\sigma = 0.85$ for different sizes $N$ evaluated at the intersection temperatures $T^*(N,2N)$ shown in Fig. 9. According to Eq. (24a) the slope is expected to be $1/\nu$. The inset is the same but for $\chi_{SG}$.

FIG. 11: (Color online) Similar to Fig. 10 but for $\xi_{CG}$.

FIG. 12: (Color online) Data for $\xi_{SG}/N$, the spin glass correlation length divided by system size, as a function of $T$ for different system sizes, for $\sigma = 1.0$. The inset shows the intersection temperatures $T^*(N,2N)$, as well as a log-log fit assuming $T_{SG} = 0$. The fit works well for $N = 128, 256$ and 512. There is no intersection for the two lowest sizes, $N = 1024$ and 2048 for $T$ greater than the lowest temperature we could simulate, 0.017. This temperature is well below the value of the fit extrapolated to $N = 1024$, which is about 0.042. Hence the intersection temperatures actually fall off faster at large sizes than shown in the fit.

D. $\sigma = 1 \quad (= \sigma_l)$

It is known from the early work of Kotliar et al. [3] that $\sigma_l = 1$ is the “lower critical” value $\sigma_l$, above which there is no spin glass transition. Interestingly, Viet and Kawamura [21] claim that chiral glass ordering persists to slightly larger values of $\sigma$. Testing this claim is one of our main motivations for performing simulations at $\sigma = 1$.

Figure 12 shows the finite scaling for $\xi_{SG}$. In the inset,
we show a log-log plot of $T^*(N, 2N)$ versus $N$ and include a straight-line fit for $N = 128, 256$ and $512$. This fit works well. We find no intersections in the range of $T$ that we can equilibrate ($T \geq 0.17$) for $N = 1024$ and $2048$. Hence the data is well consistent with $T_{SG} = 0$.

Figure [13] shows the corresponding figure for $\xi_{CG}$. Again, the inset shows log-log fits assuming that the transition temperature, $T_{CG}$ in this case, is zero. The fit is satisfactory and indicates that we do not find a finite value for $T_{CG}$ at $\sigma = 1$, in contrast to the claim of Viet and Kawamura [20].

Figure [14] shows the data for $\chi_{SG}$. There are no intersections at all in the range of temperature that we can equilibrate, consistent with the conclusion from the $\xi_{SG}$ data that $T_{SG} = 0$.

V. CONCLUSIONS

Our primary motivation to study the Heisenberg spin glass in one dimension with long-range interactions which fall off with the power of the distance, is to test Kawamura’s spin-chirality decoupling scenario in which $T_{CG} > T_{SG}$, and his subsequent prediction [20] that chiral glass ordering persists for $\sigma > \sigma_1$, where $\sigma_1 = 1$ is the “lower critical” value for the spin glass transition, with a finite value of $T_{CG}$ at $\sigma = 1$.

For $\sigma = 1$ we find $T_{CG} = T_{SG} = 0$ in contrast to Viet and Kawamura [20]. For most of the other values of $\sigma$ our data is well consistent with a single phase transition.

However, for $\sigma = 0.85$ the best fits for the sizes we can equilibrate indicate $T_{CG} > T_{SG}$, see Fig. 9. Interestingly that figure shows very strong subleading corrections to finite-size scaling in the data for $\xi_{SG}$ since it only fits Eq. (23) for the largest sizes. At intermediate sizes the intersection temperatures seem to be heading towards the chiral glass transition temperature obtained from the $\xi_{SG}$ data, but then dip down for the largest sizes. We therefore cannot rule out that similar behavior might occur for the chiral glass correlation length but at even larger length scales. In this case there would be no spin-chirality decoupling. We also note that, for a given size and temperature, $\xi_{CG}$ remains considerably smaller than $\xi_{SG}$ in the vicinity of the intersection temperatures $T^*$, compare the main part of Fig. 7 with Fig. 8. Hence the data is very far from the regime with $\xi_{CG} > \xi_{SG}$ which ultimately prevails for $T_{SG} < T < T_{CG}$ if there is spin-chirality decoupling.

In a subsequent paper [21], we will investigate, for the same models, under what circumstances an AT line of transitions occurs in a magnetic field.

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