Solution of Supplee's submarine paradox through special and general relativity

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Abstract – In 1989 Supplee described an apparent relativistic paradox on which a submarine seems to sink to observers at rest within the ocean, but it rather seems to float in the submarine proper frame. In this letter, we show that the paradox arises from a misuse of the Archimedes principle in the relativistic case. Considering first the special relativity, we show that any relativistic force field can be written in the Lorentz form, so that it can always be decomposed into a static (electric-like) and a dynamic (magnetic-like) part. These gravitomagnetic effects provide a relativistic formulation of Archimedes principle, from which the paradox is explained. Besides, if the curved spacetime on the vicinity of the Earth is taken into account, we show that the gravitational force exerted by the Earth on a moving body must increase with the speed of the body. The submarine paradox is then analyzed again with this speed-dependent gravitational force.

The paradox. – When a submarine is submerged underwater it can sink or float, depending on whether its density is higher or lower than the density of the water. Suppose we adjust the submarine density to that of the ocean water, when both of them are at rest, so that the submarine remains in equilibrium when submerged. What should happen, then, when the submarine is put to move with a high velocity in the water? Disregarding any hydrodynamic effects as drag, viscosity, turbulence etc. (which we shall always assume hereafter), observers fixed to the ocean would claim that the submarine sinks, since its density becomes higher than the water density thanks to the Lorentz contraction. On the other hand, observers within the submarine would claim instead that the submarine should float, since this frame is no longer inertial, the isobaric surfaces of the lake will not be flat anymore, which ultimately results in the bullet going far away from the lake surface, i.e., in the bullet sinking again. In the second explanation, Supplee considered a constant weak field in the framework of general relativity, which led him to the same conclusion.

Fourteen years after Supplee’s publication, Matsas had analyzed the problem again, but this time using the full machinery of general relativity [2]. Considering a background spacetime with a Rindler chart and assuming reasonable conditions about the submarine rigidness, Matsas analyzed the motion of a submarine which accelerates from rest to a given velocity \( v \). He concluded that the submarine shape gets deformed as it accelerates, with its length contracting more and more, so that its density increases accordingly, which leads the submarine to sink. Moreover, in the proper frame of the submarine he showed that the observed gravitational field is somewhat different, which leads the submarine to sink as well. Matsas also argued that this problem can be important...
to some questions regarding the thermodynamic of black holes, for instance, the self-consistency of Bekenstein formulation of the second law of thermodynamics, where the buoyancy force induced by Hawking’s radiation plays a significant role — see [2] and references therein.

Finally, Supplee’s paradox was studied once more by Jonsson through the analysis of the fictitious forces that appear in non-inertial frames [3]. Jonsson considered both a flat as well as a spherical ocean. Such a flat ocean can be thought, with sufficient accuracy, as an ordinary ocean in the Earth’s surface, while the spherical ocean can be regarded as that present perhaps in a very dense planet or surrounding the core of a black hole. In the first case of a flat ocean, Jonsson concluded that the submarine indeed sinks but, in the case of the spherical ocean, he argued that the submarine could sink or float depending on whether it moves respectively inside or outside the so-called photon sphere — the spherical surface on which light can travel in closed orbits [4–7]. We remark, however, that Jonsson second analysis seems to contradict his first one: in fact, since the Earth is not dense enough in order to have an external photon sphere, we would conclude from Jonsson’s second analysis that the submarine should float in a flat ocean instead of sinking.

Although the above-mentioned approaches are interesting by themselves, we believe that it is not necessary to employ accelerated frames nor to use the full theory of general relativity in order to explain the submarine paradox. In fact, first of all, we should remark that accelerated motions can be contemplated with special relativity without the use of non-inertial frames, so the introduction of non-inertial frames to explain the submarine paradox is not necessary. Moreover, the behavior of the submarine — if it sinks or floats— depends only on the balance between the Archimedes (buoyancy) force and the gravitational (weight) force acting on it. On the surface of the Earth, the gravitational field is relatively very small (in the sense that the spacetime curvature can be neglected for any practical purpose), which enables us to interpret the gravitational interaction as an ordinary force field in a flat spacetime. This, of course, is only an approximation, since general relativity shows us that in an exact flat spacetime there is no gravity. Nevertheless, the spacetime in the vicinity of the Earth can be regarded, with a very high accuracy, as consisting of a flat space plus a curved time. We shall show that even in this case the gravitational field can still be interpreted as a force field, although it must become dependent on the speed of the bodies. The special theory of relativity can also be employed with some care in this case and, thus, the submarine paradox can be explained in both a flat as well as in a curved spacetime.

To explain the submarine paradox with our approach, however, it will be necessary to impose that the gravitational force is covariant under the Lorentz transformations. This led us to a covariant theory of gravitation in a flat spacetime, as described in [8]. This theory also holds in a flat space plus a curved time when the speed dependence of the gravitational force is taken into account. This covariance requirement implies that gravitomagnetic effects, which play a key role in our explanation of Supplee’s submarine paradox, must be present whenever there is a relative motion between two or more interacting bodies.

Any covariant force field of special relativity can be written in Lorentz form. — The special theory of relativity tells us that the force is not a four-vector. In fact, if \( F = dp/dt \) is the force acting in a given body, as measured from an inertial frame \( R \), and \( F' = dp'/dt' \) is the same force but measured by another inertial frame \( R' \) (with \( R' \) moving with respect to \( R \) with the velocity \( v = \vec{v} \)), all the axes being coincident at \( t = t' = 0 \), then we get that [9,10]

\[
F_x = F_x - v/c^2 (u_y F_y + u_z F_z) / (1 - u_x v/c^2), \tag{1a}
\]

\[
F_y = F_y / [\gamma (1 - u_x v/c^2)], \tag{1b}
\]

\[
F_z = F_z / [\gamma (1 - u_x v/c^2)], \tag{1c}
\]

where \( \gamma = \sqrt{1 - v^2/c^2} \) and \( \mathbf{u} \) is the velocity of the body as measured by \( R \). In this section, we shall show that although the force is not a four-vector, it can be always written in a Lorentz form,

\[
F = \mathbf{G} + \mathbf{u} \times \mathbf{H}, \tag{2}
\]

with respect to any inertial frame. In (2), \( \mathbf{G} \) is defined as the part of the force \( \mathbf{F} \) which does not depend on the body velocity \( \mathbf{u} \) — we may call it the static (electric-like) part of the force. Similarly, \( \mathbf{M} = \mathbf{u} \times \mathbf{H} \) is defined by the part of the force which does depend on \( \mathbf{u} \) — we may call it the dynamic (magnetic-like) part of the force.

To prove the statement above, let us assume that \( \mathbf{F} \), the force acting on the body in the frame \( R \), does not depend on \( \mathbf{u} \) (see footnote 1). Hence, in the frame \( R \) the force is already written in Lorentz form with \( \mathbf{G} = \mathbf{F} \) and \( \mathbf{H} = 0 \). Now we have to prove that the same is true in an arbitrary inertial frame \( R' \). To find the force in the frame \( R' \), we can use (1). Notice, however, that force \( \mathbf{F}' \) depends on velocity \( \mathbf{u} \) that the particle has in the frame \( R \). However, the observers in \( R' \) do not measure \( \mathbf{u} \), instead, it is \( \mathbf{u}' \) that is actually measured. This fact suggests us to eliminate \( \mathbf{u} \) through the velocity transformation formulae [9,10],

\[
u_x = \frac{u'_x + v}{1 + u'_x c^{-2} }, \quad u_y = \frac{u'_y/\gamma}{1 + u'_x c^{-2} }, \quad u_z = \frac{u'_z/\gamma}{1 + u'_x c^{-2} }, \tag{3}
\]

footnote 1 For speed-dependent force fields, in general there is no inertial frame where the force becomes independent of the body velocity. Nonetheless, this fact does not invalidate the use of (1) and, hence, the results which follow will still hold. The words static and dynamic, however, become inappropriate in this case, since now both \( \mathbf{G} \) as \( \mathbf{H} \) may depend on the body velocity (perhaps the terms electric-like and magnetic-like are more suitable here).
in order to rewrite (1) in terms of $u'$. Inserting (3) into (1) and simplifying, we get the formulæ

$$F'_x = F_x - \gamma v/c^2 (u'_y F_y + u'_z F_z),$$  \hspace{1cm} (4a)

$$F'_y = \gamma F_y (1 + u'_x v/c^2),$$  \hspace{1cm} (4b)

$$F'_z = \gamma F_z (1 + u'_x v/c^2).$$  \hspace{1cm} (4c)

Thus, from the part of $F'$ which does not depend on $u'$ we get the static part of the force, $G'$, namely,

$$G'_x = F_x, \quad G'_y = \gamma F_y, \quad G'_z = \gamma F_z,$$  \hspace{1cm} (5)

and, from that part of $F'$ which does depend on $u'$, we get the dynamic part of the force, $M'$,

$$M'_x = -\frac{\gamma v}{c^2} (u'_y F_y + u'_z F_z),$$  \hspace{1cm} (6a)

$$M'_y = \frac{\gamma v}{c^2} u'_x F_y,$$  \hspace{1cm} (6b)

$$M'_z = \frac{\gamma v}{c^2} u'_x F_z.$$  \hspace{1cm} (6c)

Now it is just a matter of fact that (6) can be written as the vector product $M' = u' \times H'$, with

$$H' = -v/c^2 \times G'.$$  \hspace{1cm} (7)

Thus, the force $F'$ can be written in the Lorentz form (2) with respect to the frame $R'$ as well. Moreover, since the frame $R'$ is quite arbitrary, we had proved that in any inertial frame every physically acceptable force field can be written in the Lorentz form\(^2\). This can be also proved considering another inertial frame $R''$ moving with respect to $R'$ with a velocity $\mathbf{w} = \bar{w} \hat{\mathbf{w}}$. Supposing that in $R''$ the force acting on the body is $\mathbf{F}'' = \mathbf{G}' + u' \times \mathbf{H}'$, then, repeating the above procedure, we can show that in $R''$ the force is still given by $\mathbf{F}'' = \mathbf{G}'' + u'' \times \mathbf{H}''$, with the static and dynamic forces in each frame related with themselves by the formulæ

$$G''_x = G'_x,$$  \hspace{1cm} (8a)

$$G''_y = \gamma_w (G'_y - w H'_z), \quad H''_y = \gamma_w \left( H'_y + \frac{w}{c^2} G'_z \right),$$  \hspace{1cm} (8b)

$$G''_z = \gamma_w (G'_z + w H'_y), \quad H''_z = \gamma_w \left( H'_z - \frac{w}{c^2} G'_y \right),$$  \hspace{1cm} (8c)

where $\gamma_w = 1/\sqrt{1 - w^2/c^2}$. The most known example of such a force is the electromagnetic one. In this case we have $\mathbf{G} = q \mathbf{E}$ and $\mathbf{M} = q \mathbf{u} \times \mathbf{B}$, with $q$ denoting the electric charge, $\mathbf{E}$ the electric field and $\mathbf{B}$ the magnetic field, respectively. Another example is the gravitational force in the approximation where Newton's law is valid (i.e., in a flat spacetime background). In this case, we have $\mathbf{G} = \mathbf{g} m$ and $\mathbf{M} = m \mathbf{u} \times \mathbf{h}$, where $\mathbf{g}$ is the static gravitational field and $\mathbf{h}$ the dynamic gravitational field —the gravitational analogue of the magnetic field. The consequences of this covariant theory of gravitation were recently discussed in [8].

**Solution of the submarine paradox in a flat spacetime.**— Let us then analyze the submarine paradox but considering, by now, only the special theory of relativity. This means that in this section the gravitational interaction will be regarded as an ordinary force field in a flat spacetime (in the same way as, for instance, the electromagnetic interactions are usually treated in special relativity). We shall assume therefore that the gravitational force between the Earth and a given particle is determined by Newton's law,

$$\mathbf{F} = -\left( G M m/\ell^2 \right) \hat{\mathbf{r}},$$  \hspace{1cm} (9)

when the Earth is at rest, no matter what the motion of the particle is. In (9), $M$ is the mass of the Earth, $m$ the mass of the particle, $\ell$ the distance vector from the Earth to the particle position, $G$ the Newton constant.

In the present case, we shall consider actually only the constant gravitational force $\mathbf{F} = -\mathbf{g}$ on the surface of the Earth, where $\mathbf{g}$ is the acceleration of gravity. Finally, we shall also consider that the (inertial and gravitational) mass is an invariant quantity —which is, of course, the most logical way to proceed in order to avoid misunderstandings\(^3\).

Before analyzing the original formulation of Supplee's paradox, let us consider first a slightly modified version in which no acceleration is involved. This is obtained supposing that a submarine moves with velocity $\mathbf{v} = \bar{v} \hat{\mathbf{v}}$ in the standing water of the ocean and letting its density be adjusted by the observers at rest within the ocean (frame $R$) in such a way that the submarine remains in equilibrium in this frame. From the Archimedes principle this means that the submarine density must be adjusted to be the same as the water density when both are measured by the frame $R$. The paradox situation arises because it seems, at first sight, that the submariners (frame $R'$) would conclude that the submarine should float, since in this frame the submarine density happens to be lesser than that of the moving water, thanks to the Lorentz length effects. We shall see, however, that this apparent paradox is due to an incorrect use of the ordinary Archimedes principle: if special relativity is correctly employed, we shall obtain that the submarine neither sinks nor floats in both frames (see figs. 1 and 2).

A straightforward confirmation of this result could be given directly from the transformation formulæ (1) or (4). In fact, since in the frame $R$ the total force acting on the submarine is null, the same will be true in the frame $R'$.

\(^2\)In a tensor notation, this means that the four-force can always be written as $f^\alpha = u\alpha F^{\alpha\beta}$, where $u\alpha$ denotes the (covariant) components of the four-velocity and $F^{\alpha\beta}$ is a suitable anti-symmetric tensor. The correspondent decomposition $f^\alpha = g^\alpha + m^\alpha$ of the total four-force into the static and dynamic ones is provided by $g^\alpha = u\alpha F^{\alpha\alpha}$ and $m^\alpha = u\alpha F^{\alpha\beta} - u\beta F^{\alpha\beta}$. This means that the spatial part of the four-force can be written as $f^\alpha = g^\alpha + \mathbf{w} \times \mathbf{h}$, with $\mathbf{g} = u\alpha (F^{\alpha\beta} \hat{\mathbf{x}} + F^{\alpha0} \hat{\mathbf{y}} + F^{\alpha2} \hat{\mathbf{z}})$ and $\mathbf{h} = F^{21} \hat{\mathbf{x}} + F^{31} \hat{\mathbf{y}} + F^{12} \hat{\mathbf{z}}$.

\(^3\)See [11–15] for discussions about the concept of mass in relativity.
so that the submarine cannot accelerate in neither frame. However, in order to provide a physical explanation of the problem, a more detailed exposition is necessary.

Let us begin our analysis in the frame $R$. Here, the water of the ocean is at rest, while the submarine has velocity $v = v \hat{x}$. The submarine is subject to two forces: a small constant gravitational (weight) force, $W = -mg \hat{z} = -\rho_s V_s g \hat{z}$, and the Archimedes (buoyancy) force, $A$. The Archimedes force is a response from the water to the action of gravity: a gradient of pressure arises in order to keep its static equilibrium. The gradient of pressure present in a fluid suited in a gravitational field equals the density of that gravitational force, $\nabla p = f = -\rho_sg \hat{z}$. Hence, the Archimedes force acting on the submarine can be found integrating $-\nabla p$ over the submarine volume:

$$A = \int_{V_s} (-\nabla p) \, dV = \int_{V_s} \rho_w g \hat{z} \, dV = \rho_w V_s g \hat{z}. \quad (10)$$

Thus, we can see that the total force acting on the submarine will be null if we set $\rho_s = \rho_w = \rho$.

In the proper submarine frame, $R'$, on the other hand, the water is moving with the velocity $u_w' = -v \hat{x}$. Lorentz length effects imply that water’s density increases, while the submarine density decreases by the same factor:

$$\rho_w' = \gamma \rho, \quad \rho_s' = \rho/\gamma. \quad (11)$$

In this frame, the submarine is also subject to the gravitational and Archimedes forces. The gravitational force, however, is not given by Newton’s law anymore, neither can the Archimedes force be deduced directly from the usual Archimedes principle. The gravitational force acting on the submarine should be found through (4), remembering that in $R'$ the submarine velocity $u_s'$ is zero:

$$W' = \gamma W = -\gamma mg \hat{z} = -\gamma \rho V_s g \hat{z}. \quad (12)$$

On the other hand, to find the Archimedes force in the frame $R'$, we need to know first what happens with the gradient of pressure on the water with respect to this frame. Of course, the gradient of pressure can also be regarded here as a response from the water to the gravitational force of the Earth, but now we should realize that both the water as the Earth move with respect to $R'$ with the velocity $u_w' = -v \hat{x}$. Hence, the water will be subject to both a static (electric-like) gravitational force as well as to a dynamic (magnetic-like) one. A unit volume of water in the frame $R$ is subject to the gravitational force $F = -\rho g \hat{z}$ and, from (5) and (6), it follows that the static and dynamic gravitational forces acting on this element of volume, as measured in the frame $R'$, will be, respectively:

$$G' = -\rho g \hat{z}, \quad M' = \gamma (v^2/c^2) \rho g \hat{z}. \quad (13)$$

Notice that, according to (6) and (7) we can write the dynamic force as $M' = u_w' \times H'$, where $H' = -v/c^2 \times G'$. From (2) we get, therefore, the total force acting on that element of volume:

$$F' = -\rho g \hat{z} + \gamma (v^2/c^2) \rho g \hat{z} = -\rho g \hat{z} / \gamma. \quad (14)$$

However, this quantity of water no longer occupies a unit volume in the frame $R'$. In fact, it is contracted by a factor of $1/\gamma$, so that, in order to get the force per unit volume in the frame $R'$, we need further to divide (14) by this factor. Whence, we get,

$$f' = \nabla' p' = -\rho g \hat{z}. \quad (15)$$

The conclusion is that the gradient of pressure is not proportional to the higher water’s density, as one could naively think: rather, it is an invariant quantity, which could be anticipated already from the fact that pressure is a scalar and from $\partial' \rho = \partial \rho$. This is why the ordinary Archimedes principle cannot be applied in $R'$.

Integrating $-\nabla' p'$ over the submarine volume, we get the Archimedes force acting on it:

$$A' = \int_{V_s} (-\nabla' p') \, dV' = \int_{V_s} \rho g \hat{z} \, dV' = \rho V_s g \hat{z}. \quad (16)$$

Finally, since $V_s' = \gamma V_s$, we get,

$$A' = \gamma \rho V_s g \hat{z} = -W'. \quad (17)$$
Therefore, the Archimedes force intensity equals the weight of the submarine and, thus, it neither floats nor sinks in the frame $R'$ as well—the submarine remains in equilibrium in both frames.

We would like to highlight that the Archimedes force can also be obtained from a relativistic Archimedes principle\(^4\). Remember that the original formulation of the Archimedes principle states that the intensity of the Archimedes force equals the weight of water displaced by the submerged body. This principle is still valid in the frame $R'$, but here we must distinguish between the static weight of the mass displaced by the body, defined as the displaced mass times the static (electric-like) gravitational field $W' = \rho_0 V_0 g'\hat{z}$, and its dynamic weight, which is given by the current of mass times the dynamic (magnetic-like) gravitational field, $M' = V'_0 j_0 \times \mathbf{h}'$, where $j'_0 = \rho_0 u_0$ and $\mathbf{h}' = -v/c^2 \times \mathbf{g}'$. Therefore, we get for the static and dynamic weight of the submarine, respectively,

$$W' = -\gamma \rho_0 V'_0 g\hat{z}, \quad M' = \gamma \left( v^2/c^2 \right) \rho_0 V'_0 g\hat{z}. \tag{18}$$

Notice that the dynamic weight is contrary to the static weight and hence it can be thought as a “negative weight” due to the repulsive dynamic gravitational force between the Earth and the ocean water. The sum of these two terms (with the opposite signs) provides, of course, the Archimedes force (17).

Finally, let us consider the original formulation of Supplee’s paradox. In this case the density of the submarine is adjusted to the water density when both of them are at rest (let $m$ be the submarine mass and $V_0$ its proper volume, so that its proper density is $\rho_0 = m/V_0$). If the submarine is put to move with a velocity $v = v\hat{x}$, the gravitational force acting on it will still be $W = -mg\hat{z} = -\rho_0 V_0 g\hat{z}$, since the gravitational field is just static in the frame $R$. To evaluate the Archimedes force we should realize that now the submarine volume is contracted to $V'_0 = V_0/\gamma$, and then, from the Archimedes principle, we get that $A = \rho_0 V_0 g\hat{z}/\gamma$. Thus, the total force acting on the submarine is

$$F = -\rho_0 V_0 g \left( 1 - 1/\gamma \right) \hat{z}. \tag{19}$$

On the other hand, in the frame $R'$ (the inertial frame that is instantaneously at rest with respect to the submarine at $t = t' = 0$), the weight force acting on the submarine will be, according to (4), $W' = -\gamma \rho_0 V_0 g\hat{z}$. Notice that there is no dynamic (magnetic-like) force here again, since the submarine is at rest on $R'$ in this instant of time. There is, however, a dynamic force between the moving Earth and the moving ocean. As we have seen, these gravitomagnetic forces combined imply that the gradient of water’s pressure observed in the frame $R'$ is the same as that measured in $R$. Thus the Archimedes force

will be given just by $A' = \rho_0 V_0 g\hat{z}$, from which we get the total force acting on the submarine in the frame $R'$:

$$F' = -\rho_0 V_0 g \left( \gamma - 1 \right) \hat{z} = \gamma F. \tag{20}$$

The conclusion is that in both frames the submarine will sink. Notice further that it is not necessary to employ accelerated frames neither general relativity in order to study the submarine behavior. We can do that, of course, but then we should take care of the geometric effects that arise in non-inertial frames, as already discussed by Supplee, Matsas and Jonsson [1–3].

**Spacetime curvature in the vicinity of Earth** and the implicated speed-dependent gravitational force. — According to Einstein’s theory of gravitation, gravity is not a force but just an effect of the spacetime curvature [4–6]. In other words, Einstein’s theory implies that there is no gravitational field in an exactly flat spacetime whatsoever. Of course, the Newtonian description of gravity as a force field in a flat spacetime can be justified as being a very good approximation in the vicinity of the Earth, which is due to the very small curvature of spacetime there — so small that we can only measure its effects with the most precise instruments available. The former approach presented in the last section assumes that this is indeed the case, hence, it should be treated as a first approximation for the problem.

Nevertheless, in order to get a better approximation for the gravitational phenomenon on the surface of the Earth, the curvature of spacetime needs to be taken into account. Since any physical measurement of distances performed on the surface of the Earth does not reveal any discordance with the Euclidean geometry, it is enough to consider here a flat space plus a curved time. This means that, in this approximation, the metric on the proximity of the Earth can be written in the form

$$ds^2 = -f(z) c^2 dt^2 + dx^2 + dy^2 + dz^2, \tag{21}$$

(we assume that the Earth is large enough so that its surface can be approximated by the horizontal plane), where the function $f(z)$ is to be determined. In order to do so, we shall proceed as follows: first remember that in Einstein’s theory, the world-line of a particle freely falling in a gravitational field is a geodesic, which is determined by the equations [5,6]

$$du^\alpha/d\tau + \Gamma^\alpha_{\beta\gamma}u^\beta u^\gamma = 0, \tag{22}$$

where $u^\alpha$ are the components of the particle four-velocity, $\tau$ is its proper time and $\Gamma^\alpha_{\beta\gamma}$ are the Christoffel symbols, which are obtained from the metric through the formula [5,6]

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta} \left( \partial_\beta g_{\delta\gamma} + \partial_\gamma g_{\delta\beta} - \partial_\delta g_{\beta\gamma} \right). \tag{23}$$

The geodesic equation agrees with the fact that the particle four-acceleration $A^\alpha = du^\alpha/d\tau + \Gamma^\alpha_{\beta\gamma} u^\beta u^\gamma$ must be

\[^4\text{We also point out that such a relativistic Archimedes principle can be important for the analysis of the equilibrium of (very fast) rotating stars, since the relative motion between their layers would prevent us from using the usual Archimedes principle.}\]
zero in the proper co-moving frame of the particle, since Einstein’s equivalence principle states that the particle does not feel any effect of gravity as it freely falls in the gravitational field. We may call the first term \( a^\alpha = du^\alpha /d\tau \) in the formula above as the \textit{kinematic four-acceleration} of the particle, while the second term \( r^\alpha = \Gamma^\alpha_{\beta\gamma} u^\beta u^\gamma \) can be called its \textit{geometric four-acceleration} (since it is present only when curved coordinates are employed to describe the motion of the particle). Now, consider a fixed observer very close to the particle position \((\text{e.g.}, \text{both of them in the proximity of the Earth})\). The difference between the observer and the particle proper times will be only due to the relative motion between them. In fact, any effect arising from the metric will cancel, since the metric will be the same to both the particle and this observer. Therefore, we can write for the particle four-acceleration, as measured by this static observer,

\[
a^\alpha = -\Gamma^\alpha_{\beta\gamma} u^\beta u^\gamma. \tag{24}
\]

Hence, if a particle is released from rest near the Earth surface, in this very instant the components of its (relative) four-velocity will be

\[
 u^0 = c, \quad u^1 = 0, \quad u^2 = 0, \quad u^3 = 0. \tag{25}
\]

Besides, we know that in this case the acceleration of the particle is just \( a = -g \hat{z} \) and, thus, the components of its four-acceleration will be as well

\[
a^0 = 0, \quad a^1 = 0, \quad a^2 = 0, \quad a^3 = -g. \tag{26}
\]

On the other hand, it follows from (21) and (23) that the only non-null Christoffel symbols are

\[
\Gamma^0_{03} = \Gamma^0_{30} = \frac{1}{2} \frac{d}{dz} \log f(z) \quad \text{and} \quad \Gamma^3_{00} = \frac{1}{2} \frac{df(z)}{dz}. \tag{27}
\]

Then, using (24), (25), (26) and (27) at once, we get the relation

\[
a^3 = -\Gamma^3_{00} u^0 = -c^2 /2 [d f(z)] /dz = -g, \tag{28}
\]

and, solving this equation, we find that \( f(z) = 2gz/c^2 + C \), where \( C \) is the constant of integration. In order to fix \( C \), we may realize that in the absence of the gravitational field \((\text{i.e., for } g = 0)\), the metric should reduce to that of Minkowski, \( ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2 \), which leads us to the value \( C = 1 \). Hence, the spacetime metric in the vicinity of the Earth’s surface becomes,

\[
ds^2 = -(1 + 2gz/c^2) c^2 dt^2 + dx^2 + dy^2 + dz^2. \tag{29}
\]

The corresponding non-null Christoffel symbols reduce to

\[
\Gamma^0_{03} = \Gamma^0_{30} = g/c^2 \quad \text{and} \quad \Gamma^3_{00} = g/c^2. \tag{30}
\]

It can be verified that the metric (29) satisfies the Einstein field equations \( G_{\alpha\beta} = \kappa T_{\alpha\beta} \), where the only non-null elements of the energy-momentum tensor \( T_{\alpha\beta} \) are

\[
T_{11} = T_{22} = -g^2 /\kappa c^4 (1 + 2gz/c^2)^2. \tag{31}
\]

As we can see, the values of \( T_{11} \) and \( T_{22} \) are very small in the proximity of the Earth, so that we get an almost vacuum solution there which is quite reasonable in our approximation. Nevertheless, their exact values provide an exact solution of Einstein’s field equations for a universe described by the metric (29). The negative values of \( T_{11} \) and \( T_{22} \) can be interpreted as negative vacuum pressures that ensure the static behavior of such a universe.

Now, let us see what should be the acceleration of a particle which moves with a given (instantaneous) velocity \( u \) in the gravitational field of the Earth. In this case, its four-velocity becomes

\[
u^0 = \gamma_u c, \quad u^1 = \gamma_u u_x, \quad u^2 = \gamma_u u_y, \quad u^3 = \gamma_u u_z. \tag{32}\]

However, we cannot assume that the acceleration of the particle is directed along the \( z \)-direction anymore, since the theory of relativity shows us that acceleration and force are not parallel one to the other, except when the velocity is parallel or orthogonal to the force. In fact, the relationship between force and acceleration is \([9,10]\)

\[
F = m \left[ \gamma_u a + \gamma_u^3 (a \cdot u) u /c^2 \right]. \tag{33}
\]

Hence, the particle four-acceleration must be written as

\[
a^0 = \gamma_u^4 \left( \frac{a \cdot u}{c} \right), \quad a^{1,2,3} = \gamma_u^4 a_{x,y,z} + \gamma_u^4 \left( \frac{a \cdot u}{c^2} \right) u_{x,y,z}. \tag{34}\]

On the other hand, (24), (30) and (32) give

\[
a^0 = -2g\gamma_u^2 u_z /c, \quad a^1 = 0, \quad a^2 = 0, \quad a^3 = -g\gamma_u^2. \tag{35}\]

Comparing (34) with (35) we obtain the acceleration of the particle in the weak gravitational field of the Earth:

\[
a_x = u_x u_z c /g, \quad a_y = u_y u_z c /g, \quad a_z = -g \left( 1 + \frac{u_z^2}{c^2} \right). \tag{36}\]

Notice that the particle acceleration will be directed along the \( z \)-axis only if \( u_z = 0 \) or if \( u_x = u_y = 0 \). In the first case, the acceleration is \( a = -g \hat{z} \), while in the second case we have \( a = -g \gamma_u^2 \hat{z} \).

Finally, inserting (36) into (33) and simplifying, we get the gravitational force acting on the moving particle:

\[
F(u) = -\gamma_u mg \hat{z}. \tag{37}\]

We conclude therefore that the spacetime curvature in the proximity of the Earth implies a \textit{speed-dependent gravitational force}. The gravitational force increases with the particle speed.

Notwithstanding the curved spacetime we have here, we may realize that the Lorentz transformations can still be employed, as long as the frame \( R' \) moves with respect to the frame \( R \) in a direction parallel to Earth’s surface. The only effects differing from those obtained in an exactly flat spacetime are those on which events are to be compared.
in different heights, since, according to the metric (29), the higher the position is, the faster the time will pass. In fact, if \( dt \) is a given interval of time measured by a clock at the surface of the Earth, then an identical clock situated at the height \( z \) will mark the time \( dt = \sqrt{1 + 2gz/c^2} dt \), which is greater than \( dt \). Correspondingly, if a light ray is emitted from the height \( z \) towards the surface of the Earth, a blue-shift effect will take place.

**Solution of the submarine paradox in the curved spacetime of the Earth.** – Finally, let us analyze what changes in the description of the submarine paradox when the effects of the tiny spacetime curvature on the Earth surface are taken into account. It is sufficient in this case to consider a gravitational force law given by (37), instead of Newton’s law (9).

Let us first consider that version of the paradox on which the submarine does not accelerate. In this case, where the spacetime is weakly curved, the gravitational force acting on the submarine with respect to the frame \( R \), will be \( W = -\gamma mg\dot{z} = -\gamma \rho_s V_0 g\dot{z} \), since the submarine moves with the velocity \( \vec{v} = v\hat{x} \) and we are using the force law (37). In order for the submarine to stay in equilibrium underwater, its density should now be adjusted (by the observers at rest within the water) to \( \rho_s = \rho_0/\gamma \), so that the intensity of the Archimedes force \( A \) equals the intensity of the weight force \( W \). In the frame \( R' \), of course, the same will be true. In fact, the gravitational force acting on the submarine can be found through (4), and it is given by \( W' = -\gamma^2 mg\dot{z} = -\gamma^2 \rho_s V_0 g\dot{z} \). The density of the water in the frame \( R' \) is still given by (11) and, hence, the gradient of pressure in the frame \( R' \) remains the same as that measured in the frame \( R \). Thus, the Archimedes force becomes, \( A' = \gamma \rho_s V_0 g\dot{z} = \gamma^2 \rho_s V_0 g\dot{z} \), from which we can see that in the frame \( R' \) the submarine will remain in equilibrium as well.

For the original formulation of Supplee’s paradox, we get a similar explanation. Here the submarine density \( \rho_0 \) is matched with the water density when both of them are at rest. In the frame \( R \), the submarine moves with the velocity \( \vec{v} = v\hat{x} \) and the gravitational force acting on it is \( W = -\gamma mg\dot{z} = -\gamma \rho_0 V_0 g\dot{z} \). The Archimedes force is \( A = \rho_0 V_0 g\dot{z} = \rho_0 V_0 g\dot{z}/\gamma \) and, thus, the total force acting on the submarine is \( F = -\rho_0 V_0 g\dot{z} (\gamma - 1/\gamma) \). In the frame \( R' \) the total force acting on the submarine can be found through (4), and then we get \( F' = -\gamma \rho_0 V_0 g\dot{z} (\gamma - 1/\gamma) \).

These expressions for the total force acting on the submarine agree with those obtained by Supplee and Matsas [1,2]. It should be mentioned, however, that the argument presented by Supplee in his first explanation of the paradox, concerning the gravitational force between moving bodies, cannot be justified. In fact, Supplee assumed in [1] that Newton’s law (9) is still valid, even when the interacting bodies are in motion. However he also assumed (as many others do, see [11–15]) that the gravitational mass should be determined by Einstein’s formula \( E = mc^2 \), which leads to a speed-dependent mass:

\[ m = m_0 n \]

where \( m_0 \) is the so-called “rest mass” and \( u \) is the velocity of the body. This, by its turn, leads to the same speed-dependent force (37), as deduced by us in the previous section. Notice, however, that the relation between mass and energy expressed by Einstein’s formula above only holds when the momentum of the body is null — in fact the correct expression is \( E = \sqrt{m^2c^4 + p^2c^2} \).

Besides, we must remember that in Einstein’s theory of gravitation the source of gravitational interaction is the energy-momentum tensor, not the energy alone. Therefore, it seems only a coincidental fact that the speed-dependent mass considered by Supplee and others gives the same speed-dependent force (37), as deduced from general relativity in the approximation considered here.

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