The periodic oscillation of an adiabatic piston in two or three dimensions
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Abstract

We study a heavy piston of mass $M$ that separates finitely many ideal, unit mass gas particles moving in two or three dimensions. Neishtadt and Sinai previously determined a method for finding this system’s averaged equation and showed that its solutions oscillate periodically. Using averaging techniques, we prove that the actual motions of the piston converge in probability to the predicted averaged behavior on the time scale $M^{1/2}$ when $M$ tends to infinity while the total energy of the system is bounded and the number of gas particles is fixed.

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1 Introduction

Consider the following simple model of an adiabatic piston separating two gas containers: A massive piston of mass $M \gg 1$ divides a container in $\mathbb{R}^d$ into two halves. The piston has no internal degrees of freedom and can only move along one axis of the container. On either side of the piston there are a finite number of ideal, unit mass, point gas particles that interact with the walls of the container and with the piston via elastic collisions. When $M = \infty$, the piston remains fixed in place, and each gas particle performs billiard motion at a constant energy in its sub-container. We make an ergodicity assumption on the behavior of the gas particles when the piston is fixed. Then we study the motions of the piston when the number of gas particles is fixed, the total energy of the system is bounded, but $M$ is very large.
Heuristically, after some time, one expects the system to approach a steady state, where the energy of the system is equidistributed amongst the particles and the piston. However, even if we could show that the full system is ergodic, an abstract ergodic theorem says nothing about the time scale required to reach such a steady state. Because the piston will move much slower than a typical gas particle, it is natural to try to determine the intermediate behavior of the piston by averaging techniques. By averaging over the motion of the gas particles on a time scale chosen short enough that the piston is nearly fixed, but long enough that the ergodic behavior of individual gas particles is observable, we will show that the system does not approach the expected steady state on the time scale $M^{1/2}$. Instead, the piston oscillates periodically, and there is no net energy transfer between the gas particles.

This paper follows earlier work by Neishtadt and Sinai [NS04, Sin99]. They determined that for a wide variety of Hamiltonians for the gas particles, the averaged behavior of the piston is periodic oscillation, with the piston moving inside an effective potential well whose shape depends on the initial position of the piston and the gas particles’ Hamiltonians. They pointed out that an averaging theorem due to Anosov [Ano60, LM88], proved for smooth systems, should extend to this case. This paper proves that Anosov’s theorem extends to the particular gas particle Hamiltonian described above. Thus, if we examine the actual motions of the piston with respect to the slow time $\tau = t/M^{1/2}$, then, as $M \to \infty$, in probability (with respect to Liouville measure) most initial conditions give rise to orbits whose actual motion is accurately described by the averaged behavior for $0 \leq \tau \leq 1$, i.e. for $0 \leq t \leq M^{1/2}$. Gorelyshev and Neishtadt [GN06] and we [Wri06] have already proved that when $d = 1$, i.e. when the gas particles move on a line, the convergence of the actual motions to the averaged behavior is uniform over all initial conditions, with the size of the deviations being no larger than $O(M^{-1/2})$ on the time scale $M^{-1/2}$.

The system under consideration in this paper is a simple model of an adiabatic piston. The general adiabatic piston problem [Cal63], well-known from physics, consists of the following: An insulating piston separates two gas containers, and initially the piston is fixed in place, and the gas in each container is in a separate thermal equilibrium. At some time, the piston is no longer externally constrained and is free to move. One hopes to show that eventually the system will come to a full thermal equilibrium, where each gas has the same pressure and temperature. Whether the system will evolve to thermal equilibrium and the interim behavior of the piston are mechanical problems, not adequately described by thermodynamics [Gru99], that have recently generated much interest within the physics and mathematics communities. One expects that the system will evolve in at least two stages. First, the system relaxes toward a mechanical equilibrium, where the pressures on either side of the piston are equal. In the second, much longer, stage, the piston drifts stochastically in the direction of the hotter gas, and the temperatures of the gases equilibrate. See for example [GPL03, CL02] and the references therein.

So far, rigorous results have been limited mainly to models where the effects of gas particles recolliding with the piston can be neglected, either by restricting to
extremely short time scales \cite{LSC02, CLS02} or to infinite gas containers \cite{Che04}.

A recent study involving some similar ideas by Chernov and Dolgopyat \cite{CD06a} considered the motion inside a two-dimensional domain of a single heavy, large gas particle (a disk) of mass $M \gg 1$ and a single unit mass point particle. They assumed that for each fixed location of the heavy particle, the light particle moves inside a dispersing (Sinai) billiard domain. By averaging over the strongly hyperbolic motions of the light particle, they showed that under an appropriate scaling of space and time the limiting process of the heavy particle’s velocity is a (time-inhomogeneous) Brownian motion on a time scale $O(M^{1/2})$. It is not clear whether a similar result holds for the piston problem, even for gas containers with good hyperbolic properties, such as the Bunimovich stadium. In such a container the motion of a gas particle when the piston is fixed is only nonuniformly hyperbolic because it can experience many collisions with the flat walls of the container immediately preceding and following a collision with the piston.

The present work provides a weak law of large numbers, and it is an open problem to describe the sizes of the deviations for the piston problem \cite{CD06b}. Although our result does not yield concrete information on the sizes of the deviations, it is general in that it imposes very few conditions on the shape of the gas container. Most studies of billiard systems impose strict conditions on the shape of the boundary, generally involving the sign of the curvature and how the corners are put together. The proofs in this work require no such restrictions. In particular, the gas container can have cusps as corners and need satisfy no hyperbolicity conditions.

We begin in Section 2 by giving a physical description of our results. Precise assumptions and our main result, Theorem 1, are stated in Section 3 and a proof is presented in the following sections.

## 2 Physical motivation for the results

Before precisely stating our assumptions and results, we briefly review the physical motivations for our results and introduce some notation.

Consider a massive, insulating piston of mass $M$ that separates a gas container $\mathcal{D}$ in $\mathbb{R}^d$, $d = 2$ or 3. See Figure 1. Denote the location of the piston by $Q$, its velocity by $dQ/dt = V$, and its cross-sectional length (when $d = 2$, or area, when $d = 3$) by $\ell$. If $Q$ is fixed, then the piston divides $\mathcal{D}$ into two subdomains, $\mathcal{D}_1(Q) = \mathcal{D}_1$ on the left and $\mathcal{D}_2(Q) = \mathcal{D}_2$ on the right. By $E_i$ we denote the total energy of the gas inside $\mathcal{D}_i$, and by $|\mathcal{D}_i|$ we denote the area (when $d = 2$, or volume, when $d = 3$) of $\mathcal{D}_i$.

We are interested in the dynamics of the piston when the system’s total energy is bounded and $M \to \infty$. When $M = \infty$, the piston remains fixed in place, and each energy $E_i$ remains constant. When $M$ is large but finite, $MV^2/2$ is bounded, and so $V = O(M^{-1/2})$. It is natural to define

$$
\varepsilon = M^{-1/2}, \quad W = \frac{V}{\varepsilon}.
$$
so that $W$ is of order 1 as $\varepsilon \to 0$. This is equivalent to scaling time by $\varepsilon$.

If we let $P_i$ denote the pressure of the gas inside $D_i$, then heuristically the dynamics of the piston should be governed by the following differential equation:

\[
\frac{dQ}{dt} = V, \quad M \frac{dV}{dt} = P_1 \ell - P_2 \ell,
\]

i.e.

\[
\frac{dQ}{dt} = \varepsilon W, \quad \frac{dW}{dt} = \varepsilon P_1 \ell - \varepsilon P_2 \ell.
\]

(1)

To find differential equations for the energies of the gases, note that in a short amount of time $dt$, the change in energy should come entirely from the work done on a gas, i.e. the force applied to the gas times the distance the piston has moved, because the piston is adiabatic. Thus, one expects that

\[
\frac{dE_1}{dt} = -\varepsilon WP_1 \ell, \quad \frac{dE_2}{dt} = +\varepsilon WP_2 \ell.
\]

(2)

To obtain a closed system of differential equations, it is necessary to insert an expression for the pressures. Because the pressure of an ideal gas in $d$ dimensions is proportional to the energy density, with the constant of proportionality $2/d$, we choose to insert

\[
P_i = \frac{2E_i}{d |D_i|}.
\]

Later, we will make assumptions to justify this substitution. However, if we accept this definition of the pressure, and define the slow time

\[
\tau = \varepsilon t,
\]

do obtain the following ordinary differential equations for the four macroscopic...
variables of the system:
\[
\frac{d}{d\tau} \begin{bmatrix} Q \\ W \\ E_1 \\ E_2 \end{bmatrix} = \begin{bmatrix} \frac{2E_1\ell}{d|D_1(Q)|} - \frac{2E_2\ell}{d|D_2(Q)|} \\ W \\ -\frac{2W\ell}{d|D_1(Q)|} \\ \frac{2W\ell}{d|D_2(Q)|} \end{bmatrix}.
\]  

\(3\)

Neishtadt and Sinai [Sm99, NS04] pointed out that the solutions of Equation (3) have the piston moving according to an effective Hamiltonian. This can be seen as follows. Since 
\[
\frac{\partial |D_1(Q)|}{\partial Q} = \ell = -\frac{\partial |D_2(Q)|}{\partial Q},
\]
\[
d\ln(E_i)/d\tau = -(2/d)\ln(|D_i(Q)|)/d\tau,
\]
and so
\[
E_i(\tau) = E_i(0) \left(\frac{|D_i(Q(0))|}{|D_i(Q(\tau))|}\right)^{2/d}.
\]

Hence
\[
\frac{d^2Q}{d\tau^2} = \frac{2\ell E_1(0) |D_1(Q(0))|^{2/d}}{|D_1(Q(\tau))|^{1+2/d}} - \frac{2\ell E_2(0) |D_2(Q(0))|^{2/d}}{|D_2(Q(\tau))|^{1+2/d}},
\]
and so \((Q, W)\) behave as if they were the coordinates of a Hamiltonian system describing a particle undergoing motion inside a potential well. The effective Hamiltonian may be expressed as
\[
\frac{1}{2}W^2 + \frac{E_1(0) |D_1(Q(0))|^{2/d}}{|D_1(Q)|^{2/d}} + \frac{E_2(0) |D_2(Q(0))|^{2/d}}{|D_2(Q)|^{2/d}}. \tag{4}
\]

The question is, do the solutions of Equation (3) give an accurate description of the actual motions of the macroscopic variables when \(M\) tends to infinity? The main result of this paper is that, for an appropriately defined system, the answer to this question is affirmative for \(0 \leq t \leq M^{1/2}\), at least for most initial conditions of the microscopic variables. Observe that one should not expect the description to be accurate on time scales much longer than \(O(M^{1/2}) = O(\varepsilon^{-1})\). The reason for this is that, presumably, there are corrections of size \(O(\varepsilon^2)\) in Equations (1) and (2) that we are neglecting. On the time scale \(\varepsilon^{-1}\), these errors roughly add up to no more than size \(O(\varepsilon^{-1} \cdot \varepsilon^2 = \varepsilon)\), but on a longer time scale they should become significant. Such higher order corrections for the adiabatic piston were studied by Crosignani et al. [CDPS96].

### 3 Statement of the main result

#### 3.1 Description of the model

We begin by describing the gas container. It is a compact, connected billiard domain \(\mathcal{D} \subset \mathbb{R}^d\) with a piecewise \(C^3\) boundary, i.e. \(\partial \mathcal{D}\) consists of a finite number of
embedded hypersurfaces, possibly with boundary and a finite number of corner points. The container consists of a “tube,” whose perpendicular cross-section $P$ is the shape of the piston, connecting two disjoint regions. $P \subset \mathbb{R}^{d-1}$ is a compact, connected domain whose boundary is piecewise $C^3$. Then the “tube” is the region $[0, 1] \times P \subset \mathcal{D}$ swept out by the piston for $0 \leq Q \leq 1$, and $[0, 1] \times \partial P \subset \partial \mathcal{D}$. If $d = 2$, $P$ is just a closed line segment, and the “tube” is a rectangle. If $d = 3$, $P$ could be a circle, a square, a pentagon, etc.

Our fundamental assumption is as follows:

**Main Assumption.** For almost every $Q \in [0, 1]$ the billiard flow of a single particle on an energy surface in either of the two subdomains $\mathcal{D}_i(Q)$ is ergodic (with respect to the invariant Liouville measure).

If $d = 2$, the domain could be the Bunimovich stadium $[\text{Bun79}]$. Another possible domain is indicated in Figure 1. Polygonal domains satisfying our assumptions can also be constructed $[\text{Vor97}]$. Suitable domains in $d = 3$ dimensions can be constructed using a rectangular box with shallow spherical caps adjoined $[\text{BR98}]$. Note that we make no assumptions regarding the hyperbolicity of the billiard flow in the domain.

The Hamiltonian system we consider consists of the massive piston of mass $M$ located at position $Q$, as well as $n_1 + n_2$ gas particles, $n_1$ in $\mathcal{D}_1$ and $n_2$ in $\mathcal{D}_2$. Here $n_1$ and $n_2$ are fixed positive integers. For convenience, the gas particles all have unit mass, though all that is important is that each gas particle has a fixed mass. We denote the positions of the gas particles in $\mathcal{D}_i$ by $q_{i,j}$, $1 \leq j \leq n_i$. The gas particles are ideal point particles that interact with $\partial \mathcal{D}$ and the piston by hard core, elastic collisions. Although it has no effect on the dynamics we consider, for convenience we complete our description of the Hamiltonian dynamics by specifying that the piston makes elastic collisions with walls located at $Q = 0, 1$ that are only visible to the piston. We denote velocities by $dQ/dt = V = \varepsilon W$ and $dq_{i,j}/dt = v_{i,j}$, and we set

$$E_{i,j} = v_{i,j}^2 / 2, \quad E_i = \sum_{j=1}^{n_i} E_{i,j}.$$  

Our system has $d(n_1 + n_2) + 1$ degrees of freedom, and so its phase space is $(2d(n_1 + n_2) + 2)$-dimensional.

We let

$$h(z) = h = (Q, W, E_{1,1}, E_{1,2}, \ldots, E_{1,n_1}, E_{2,1}, E_{2,2}, \ldots, E_{2,n_2}),$$

so that $h$ is a function from our phase space to $\mathbb{R}^{n_1 + n_2 + 2}$. We often abbreviate $h = (Q, W, E_{1,j}, E_{2,j})$, and we refer to $h$ as consisting of the slow variables because these quantities are conserved when $\varepsilon = 0$. We let $h_\varepsilon(t, z) = h_\varepsilon(t)$ denote the actual motions of these variables in time for a fixed value of $\varepsilon$. Here $z$ represents the initial condition in phase space, which we usually suppress in our notation. One should think of $h_\varepsilon(\cdot)$ as being a random variable that takes initial conditions in phase space to paths (depending on the parameter $t$) in $\mathbb{R}^{n_1 + n_2 + 2}$.
3.2 The averaged equation

From the work of Neishtadt and Sinai [NS04], one can derive

\[
\frac{d}{d\tau} \begin{bmatrix} Q \\ W \\ E_{1,j} \\ E_{2,j} \end{bmatrix} = \bar{H}(h) := \begin{bmatrix}
\frac{2E_{1,\ell}}{d|D_1(Q)|} - \frac{2E_{2,\ell}}{d|D_2(Q)|} \\
-\frac{2W E_{1,j} \ell}{d|D_1(Q)|} \\
+ \frac{2W E_{2,j} \ell}{d|D_2(Q)|} \\
\end{bmatrix}
\]

as the averaged equation (with respect to the slow time \( \tau = \epsilon t \)) for the slow variables. Later, in Section 4.3, we will give another heuristic derivation of the averaged equation that is more suggestive of our proof. As in Section 2, the solutions of Equation (5) have \((Q, W)\) behaving as if they were the coordinates of a Hamiltonian system describing a particle undergoing motion inside a potential well. The effective Hamiltonian is given by Equation (4).

Let \(\bar{h}(\tau, z) = \bar{h}(\tau)\) be the solution of

\[
\frac{d\bar{h}}{d\tau} = \bar{H}(\bar{h}), \quad \bar{h}(0) = h_\epsilon(0).
\]

Again, think of \(\bar{h}(\cdot)\) as being a random variable.

3.3 The main result

The solutions of the averaged equation approximate the motions of the slow variables, \(h_\epsilon(t)\), on a time scale \(O(1/\epsilon)\) as \(\epsilon \to 0\). Precisely, fix a compact set \(V \subset \mathbb{R}^{n_1+n_2+2}\) such that \(h \in V \Rightarrow Q_1 \subset (0, 1), W_2 \subset \mathbb{R}, \) and \(E_{i,j} \subset (0, \infty)\) for each \(i\) and \(j\). We will be mostly concerned with the dynamics when \(h \in V\). Define

\[
Q_{\min} = \inf_{h \in V} Q, \quad Q_{\max} = \sup_{h \in V} Q,
\]

\[
E_{\min} = \inf_{h \in V} \frac{1}{2} W^2 + E_1 + E_2, \quad E_{\max} = \sup_{h \in V} \frac{1}{2} W^2 + E_1 + E_2.
\]

For a fixed value of \(\epsilon > 0\), we only consider the dynamics on the invariant subset of phase space defined by

\[
\mathcal{M}_\epsilon = \{(Q, V, q_{i,j}, v_{i,j}) \in \mathbb{R}^{2d(n_1+n_2)+2} : Q \in [0, 1], q_{i,j} \in D_i(Q), E_{\min} \leq M/2 V^2 + E_1 + E_2 \leq E_{\max}\}.
\]

Let \(P_\epsilon\) denote the probability measure obtained by restricting the invariant Liouville measure to \(\mathcal{M}_\epsilon\). Define the stopping time

\[
T_\epsilon(z) = T_\epsilon = \inf\{\tau \geq 0 : \bar{h}(\tau) \notin V \text{ or } h_\epsilon(\tau/\epsilon) \notin V\}.
\]

\[\footnote{We have introduced this notation for convenience. For example, \(h \in V \Rightarrow Q_1 \subset (0, 1)\) means that there exists a compact set \(A \subset (0, 1)\) such that \(h \in V \Rightarrow Q \in A\), and similarly for the other variables.}
Theorem 1. If $\mathcal{D}$ is a gas container in $d = 2$ or 3 dimensions satisfying the assumptions in Subsection 3.1 above, then for each $T > 0$,

$$\sup_{0 \leq \tau \leq T \wedge T_\varepsilon} |h_\varepsilon(\tau/\varepsilon) - \bar{h}(\tau)| \to 0 \text{ in probability as } \varepsilon = M^{-1/2} \to 0,$$

i.e. for each fixed $\delta > 0$,

$$P_\varepsilon \left( \sup_{0 \leq \tau \leq T \wedge T_\varepsilon} |h_\varepsilon(\tau/\varepsilon) - \bar{h}(\tau)| \geq \delta \right) \to 0 \text{ as } \varepsilon = M^{-1/2} \to 0.$$

It should be noted that the stopping time in the above result is not unduly restrictive. If the initial pressures of the two gasses are not too mismatched, then the solution to the averaged equation is a periodic orbit, with the effective potential well keeping the piston away from the walls. Thus, if the actual motions follow the averaged solution closely for $0 \leq \tau \leq T \wedge T_\varepsilon$, and the averaged solution stays in $\mathcal{V}$, it follows that $T_\varepsilon > T$.

The techniques of this paper should immediately generalize to prove the analogue of Theorem 1 above in the nonphysical dimensions $d > 3$, although we do not pursue this here.

4 Preparatory material concerning a two-dimensional gas container with only one gas particle on each side

Our results and techniques of proof are essentially independent of the dimension and the fixed number of gas particles on either side of the piston. Thus, we focus on the case when $d = 2$ and there is only one gas particle on either side. Later, in Section 6 we will indicate the simple modifications that generalize our proof to the general situation. For clarity, in this section and next, we denote $q_{1,1}$ by $q_1$, $v_{1,1}$ by $v_1$, etc. We decompose the gas particle coordinates according to whether they are perpendicular to or parallel to the piston’s face, for example $q_1 = (q_1^\perp, q_1^\parallel)$. See Figure 2.

The Hamiltonian dynamics define a flow on our phase space. We denote this flow by $z_\varepsilon(t, z) = z_\varepsilon(t)$, where $z = z_\varepsilon(0, z)$. One should think of $z_\varepsilon(\cdot)$ as being a random variable that takes initial conditions in phase space to paths in phase space. Then $h_\varepsilon(t) = h(z_\varepsilon(t))$. By the change of coordinates $W = V/\varepsilon$, we may identify all of the $\mathcal{M}_\varepsilon$ defined in Section 3 with the space

$$\mathcal{M} = \{(Q, W, q_1, v_1, q_2, v_2) \in \mathbb{R}^{10} : Q \in [0, 1], q_1 \in \mathcal{D}_1(Q), q_2 \in \mathcal{D}_2(Q),$$

$$E_{\min} \leq \frac{1}{2} W^2 + E_1 + E_2 \leq E_{\max} \}.$$ 

and all of the $P_\varepsilon$ with the probability measure $P$ on $\mathcal{M}$, which has the density

$$dP = \text{const } dQ dW dq_1^\perp dq_1^\parallel dv_1^\perp dv_1^\parallel dq_2^\perp dq_2^\parallel dv_2^\perp dv_2^\parallel.$$
Figure 2: A choice of coordinates on phase space.

(Throughout this work we will use const to represent generic constants that are independent of \(\varepsilon\).) We will assume that these identifications have been made, so that we may consider \(z_\varepsilon(\cdot)\) as a family of measure preserving flows on the same space that all preserve the same probability measure. We denote the components of \(z_\varepsilon(t)\) by \(Q_\varepsilon(t), q_\perp_1, \varepsilon, (t)\), etc.

The set \(\{ z \in \mathcal{M} : q_1 = Q = q_2 \}\) has co-dimension two, and so \(\bigcup_t z_\varepsilon(t)\{q_1 = Q = q_2\}\) has co-dimension one, which shows that only a measure zero set of initial conditions will give rise to three particle collisions. We ignore this and other measures zero events, such as gas particles hitting singularities of the billiard flow, in what follows.

Now we present some background material, as well as some lemmas that will assist us in our proof of Theorem 1. We begin by studying the billiard flow of a gas particle when the piston is infinitely massive. Next we examine collisions between the gas particles and the piston when the piston has a large, but finite, mass. Then we present a heuristic derivation of the averaged equation that is suggestive of our proof. Finally we prove a lemma that allows us to disregard the possibility that a gas particle will move nearly parallel to the piston’s face – a situation that is clearly bad for having the motions of the piston follow the solutions of the averaged equation.

### 4.1 Billiard flows and maps in two dimensions

In this section, we study the billiard flows of the gas particles when \(M = \infty\) and the slow variables are held fixed at a specific value \(h \in \mathcal{V}\). We will only study the motions of the left gas particle, as similar definitions and results hold for the motions of the right gas particle. Thus we wish to study the billiard flow of a point particle moving inside the domain \(D_1\) at a constant speed \(\sqrt{2E_1}\). The results of this section that are stated without proof can be found in [CM06a].

Let \(\mathcal{T}D_1\) denote the tangent bundle to \(D_1\). The billiard flow takes place in the three-dimensional space \(\mathcal{M}_h^1 = \mathcal{M}^1 = \{(q_1, v_1) \in \mathcal{T}D_1 : q_1 \in D_1, |v_1| = \sqrt{2E_1}\}/\sim\). Here the quotient means that when \(q_1 \in \partial D_1\), we identify velocity vectors pointing outside of \(D_1\) with those pointing inside \(D_1\) by reflecting through the tangent line.
to $\partial D_1$ at $q_1$, so that the angle of incidence with the unit normal vector to $\partial D_1$ equals the angle of reflection. Note that most of the quantities defined in this subsection depend on the fixed value of $h$. We will usually suppress this dependence, although, when necessary, we will indicate it by a subscript $h$. We denote the resulting flow by $y(t, y) = y(t)$, where $y(0, y) = y$. As the billiard flow comes from a Hamiltonian system, it preserves Liouville measure restricted to the energy surface. We denote the resulting probability measure by $\mu$. This measure has the density $d\mu = dq_1dv_1/(2\pi\sqrt{2E_1}|D_1|)$. Here $dq_1$ represents area on $\mathbb{R}^2$, and $dv_1$ represents length on $S^1_{\sqrt{2E_1}} = \{v_1 \in \mathbb{R}^2 : |v_1| = \sqrt{2E_1}\}$.

There is a standard cross-section to the billiard flow, the collision cross-section $\Omega = \{(q_1, v_1) \in T D_1 : q_1 \in \partial D_1, |v_1| = \sqrt{2E_1}\}/\sim$. It is customary to parameterize $\Omega$ by $\{x = (r, \varphi) : r \in \partial D_1, \varphi \in [-\pi/2, +\pi/2]\}$, where $r$ is arc length and $\varphi$ represents the angle between the outgoing velocity vector and the inward pointing normal vector to $\partial D_1$. It follows that $\Omega$ may be realized as the disjoint union of a finite number of rectangles and cylinders. The cylinders correspond to fixed scatterers with smooth boundary placed inside the gas container. If $F : \Omega \ominus$ is the collision map, i.e. the return map to the collision cross-section, then $F$ preserves the projected probability measure $\nu$, which has the density $d\nu = \cos \varphi d\varphi dr/(2|\partial D_1|)$. Here $|\partial D_1|$ is the length of $\partial D_1$.

We suppose that the flow is ergodic, and so $F$ is an invertible, ergodic measure preserving transformation. Because $\partial D_1$ is piecewise $C^3$, $F$ is piecewise $C^2$, although it does have discontinuities and unbounded derivatives near discontinuities corresponding to grazing collisions. Because of our assumptions on $D_1$, the free flight times and the curvature of $\partial D_1$ are uniformly bounded. It follows that if $x \notin \partial \Omega \cup F^{-1}(\partial \Omega)$, then $F$ is differentiable at $x$, and

$$\|DF(x)\| \leq \frac{\text{const}}{\cos \varphi(Fx)},$$

where $\varphi(Fx)$ is the value of the $\varphi$ coordinate at the image of $x$.

Following the ideas in Appendix \[A\] we induce $F$ on the subspace $\hat{\Omega}$ of $\Omega$ corresponding to collisions with the (immobile) piston. We denote the induced map by $\hat{F}$ and the induced measure by $\hat{\nu}$. We parameterize $\hat{\Omega}$ by $\{(r, \varphi) : 0 \leq r \leq \ell, \varphi \in [-\pi/2, +\pi/2]\}$. As $\nu \Omega = \ell/|\partial D_1|$, it follows that $\hat{\nu}$ has the density $d\hat{\nu} = \cos \varphi d\varphi dr/(2\ell)$.

For $x \in \Omega$, define $\zeta x$ to be the free flight time, i.e. the time it takes the billiard particle traveling at speed $\sqrt{2E_1}$ to travel from $x$ to $Fx$. If $x \notin \partial \Omega \cup F^{-1}(\partial \Omega)$,

$$\|D\zeta(x)\| \leq \frac{\text{const}}{\cos \varphi(Fx)},$$

Santaló’s formula \[San76, Che97\] tells us that

$$E_\nu \zeta = \frac{\pi |D_1|}{|v_1| |\partial D_1|}.$$
If \( \tilde{\Omega} \rightarrow \mathbb{R} \) is the free flight time between collisions with the piston, then it follows from Proposition 10 that

\[
E_{\nu} \hat{\zeta} = \frac{\pi |D_1|}{|v_1| \ell}.
\]  

(9)

The expected value of \( |v_{\perp 1}^+| \) when the left gas particle collides with the (immobile) piston is given by

\[
E_{\nu} |v_{\perp 1}^+| = E_{\nu} \sqrt{2E_1 \cos \varphi} = \frac{\sqrt{2E_1}}{2} \int_{-\pi/2}^{+\pi/2} \cos^2 \varphi \, d\varphi = \sqrt{\frac{2E_1}{4}} \pi.
\]  

(10)

We wish to compute \( \lim_{t \to \infty} t^{-1} \int_0^t |2v_{\perp 1}^+(s)| \delta_{q_{\perp 1}^+(s)=Q} ds \), the time average of the change in momentum of the left gas particle when it collides with the piston. If this limit exists and is equal for almost every initial condition of the left gas particle, then it makes sense to define the pressure inside \( D_1 \) to be this quantity divided by \( \ell \). Because the collisions are hard-core, we cannot directly apply Birkhoff’s Ergodic Theorem to compute this limit. However, we can compute this limit by using the map \( \hat{F} \).

**Lemma 2.** If the billiard flow \( y(t) \) is ergodic, then for \( \mu - a.e. \) \( y \in \mathcal{M}^1 \),

\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t |v_{\perp 1}^+(s)| \delta_{q_{\perp 1}^+(s)=Q} ds = \frac{E_1 \ell}{2 |D_1(Q)|}.
\]

**Proof.** Because the billiard flow may be viewed as a suspension flow over the collision cross-section with \( \zeta \) as the height function, it suffices to show that the convergence takes place for \( \nu - a.e. \) \( x \in \hat{\Omega} \). For an initial condition \( x \in \hat{\Omega} \), define \( \hat{N}_t(x) = \hat{N}_t = \# \{ s \in (0,t] : y(s,x) \in \hat{\Omega} \} \). By the Poincaré Recurrence Theorem, \( \hat{N}_t \to \infty \) as \( t \to \infty \), \( \nu - a.e. \).

But

\[
\frac{\hat{N}_t}{\sum_{n=0}^{\hat{N}_t} \tilde{\zeta}(\hat{F}^n x)} \frac{1}{\hat{N}_t} \sum_{n=1}^{\hat{N}_t} |v_{\perp 1}^+| (\hat{F}^n x) \leq \frac{1}{t} \int_0^t |v_{\perp 1}^+(s)| \delta_{q_{\perp 1}^+(s)=Q} ds
\]

\[
\leq \frac{\hat{N}_t}{\sum_{n=0}^{\hat{N}_t-1} \tilde{\zeta}(\hat{F}^n x)} \frac{1}{\hat{N}_t} \sum_{n=0}^{\hat{N}_t} |v_{\perp 1}^+| (\hat{F}^n x),
\]

and so the result follows from Birkhoff’s Ergodic Theorem and Equations (9) and (10).

**Corollary 3.** If the billiard flow \( y(t) \) is ergodic, then for each \( \delta > 0 \),

\[
\mu \left\{ y \in \mathcal{M}^1 : \left| \frac{1}{t} \int_0^t |v_{\perp 1}^+(s)| \delta_{q_{\perp 1}^+(s)=Q} ds - \frac{E_1 \ell}{2 |D_1(Q)|} \right| \geq \delta \right\} \to 0 \text{ as } t \to \infty.
\]
4.2 Analysis of collisions

In this section, we return to studying our piston system when $\varepsilon > 0$. We will examine what happens when a particle collides with the piston. For convenience, we will only examine in detail collisions between the piston and the left gas particle. Collisions with the right gas particle can be handled similarly.

When the left gas particle collides with the piston, $v_{1}^\perp$ and $V$ instantaneously change according to the laws of elastic collisions:

$$\begin{bmatrix} v_{1}^\perp^+ \\ V^+ \end{bmatrix} = \frac{1}{1 + M} \begin{bmatrix} 1 - M & 2M \\ 2 & M - 1 \end{bmatrix} \begin{bmatrix} v_{1}^\perp^- \\ V^- \end{bmatrix}. $$

In our coordinates, this becomes

$$\begin{bmatrix} v_{1}^\perp^+ \\ W^+ \end{bmatrix} = \frac{1}{1 + \varepsilon^2} \begin{bmatrix} \varepsilon^2 - 1 & 2\varepsilon \\ 2\varepsilon & 1 - \varepsilon^2 \end{bmatrix} \begin{bmatrix} v_{1}^\perp^- \\ W^- \end{bmatrix}. $$

(11)

Recalling that $v_1, W = \mathcal{O}(1)$, we find that to first order in $\varepsilon$,

$$v_{1}^\perp^+ = -v_{1}^\perp^- + \mathcal{O}(\varepsilon), \quad W^+ = W^- + \mathcal{O}(\varepsilon). $$

(12)

Observe that a collision can only take place if $v_{1}^\perp^- > \varepsilon W^-$. In particular, $v_{1}^\perp^- > -\varepsilon\sqrt{2E_{\text{max}}}$. Thus, either $v_{1}^\perp^- > 0$ or $v_{1}^\perp^- = \mathcal{O}(\varepsilon)$. By expanding Equation (11) to second order in $\varepsilon$, it follows that

$$E_{1}^+ - E_{1}^- = -2\varepsilon W |v_{1}^+| + \mathcal{O}(\varepsilon^2),$$

$$W^+ - W^- = +2\varepsilon |v_{1}^+| + \mathcal{O}(\varepsilon^2). $$

(13)

Note that it is immaterial whether we use the pre-collision or post-collision values of $W$ and $|v_{1}^+|$ on the right hand side of Equation (13), because any ambiguity can be absorbed into the $\mathcal{O}(\varepsilon^2)$ term.

It is convenient for us to define a “clean collision” between the piston and the left gas particle:

**Definition 1.** The left gas particle experiences a **clean collision** with the piston if and only if $v_{1}^\perp^- > 0$ and $v_{1}^\perp^+ < -\varepsilon\sqrt{2E_{\text{max}}}$.  

In particular, after a clean collision, the left gas particle will escape from the piston, i.e. the left gas particle will have to move into the region $q_{1}^\perp \leq 0$ before it can experience another collision with the piston. It follows that there exists a constant $C_1 > 0$, which depends on the set $V$, such that for all $\varepsilon$ sufficiently small, so long as $Q \geq Q_{\min}$ and $|v_{1}^+| \geq \varepsilon C_1$ when $q_{1}^\perp \in [Q_{\min}, Q]$, then the left gas particle will experience only clean collisions with the piston, and the time between these collisions will be greater than $2Q_{\min}/(\varepsilon\sqrt{2E_{\text{max}}})$. (Note that when we write expressions such as $q_{1}^\perp \in [Q_{\min}, Q]$, we implicitly mean that $q_1$ is positioned inside the “tube” discussed at the beginning of Section 3.) One can verify that $C_1 = 5\sqrt{2E_{\text{max}}}$ would work.

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Similarly, we can define clean collisions between the right gas particle and the piston. We assume that $C_1$ was chosen sufficiently large such that for all $\varepsilon$ sufficiently small, so long as $Q \leq Q_{\text{max}}$ and $q_2^+ \in [Q, Q_{\text{max}}]$, then the right gas particle will experience only clean collisions with the piston.

Now we define three more stopping times, which are functions of the initial conditions in phase space.

$$T'_\varepsilon = \inf \{ \tau \geq 0 : Q_{\text{min}} \leq q_1^+ (\tau/\varepsilon) \leq Q_{\text{max}} \text{ and } |v_1^+| < \varepsilon C_1 \}$$

$$T''_\varepsilon = \inf \{ \tau \geq 0 : Q_{\text{min}} \leq Q_{\varepsilon}(\tau/\varepsilon) \leq Q_{\text{max}} \text{ and } |v_2^+| < \varepsilon C_1 \}$$

Define $\tilde{T}_\varepsilon = T \wedge T' \wedge T''$.

Define $H(z)$ by

$$H(z) = \begin{cases} W & \text{if } \delta_{q_1^+ = Q} - 2|v_1^+| \delta_{q_2^+ = Q} \\ -2W |v_1^+| \delta_{q_1^+ = Q} & \\ +2W |v_2^+| \delta_{q_2^+ = Q} \end{cases}.$$ 

Here we make use of Dirac delta functions. All integrals involving these delta functions may be replaced by sums.

The following lemma is an immediate consequence of Equation (13) and the above discussion:

**Lemma 4.** If $0 \leq t_1 \leq t_2 \leq \tilde{T}_\varepsilon/\varepsilon$, the piston experiences $O((t_2 - t_1) \wedge 1)$ collisions with gas particles in the time interval $[t_1, t_2]$, all of which are clean collisions. Furthermore,

$$h_\varepsilon(t_2) - h_\varepsilon(t_1) = O(\varepsilon) + \varepsilon \int_{t_1}^{t_2} H(z_\varepsilon(s)) ds.$$ 

Here any ambiguities arising from collisions occurring at the limits of integration can be absorbed into the $O(\varepsilon)$ term.

### 4.3 Another heuristic derivation of the averaged equation

The following heuristic derivation of Equation (5) when $d = 2$ was suggested in [Dol05]. Let $\Delta t$ be a length of time long enough such that the piston experiences many collisions with the gas particles, but short enough such that the slow variables change very little, in this time interval. From each collision with the left gas particle, Equation (13) states that $W$ changes by an amount $+2\varepsilon |v_1^+| + O(\varepsilon^2)$, and from Equation (10) the average change in $W$ at these collisions should be approximately $\varepsilon \pi \sqrt{2E_1}/2 + O(\varepsilon^2)$. From Equation (9) the frequency of these collisions is approximately $\sqrt{2E_1} \ell/(\pi |D_1|)$. Arguing similarly for collisions with the other particle, we guess that

$$\frac{\Delta W}{\Delta t} = \varepsilon \frac{E_1 \ell}{|D_1(Q)|} - \varepsilon \frac{E_2 \ell}{|D_2(Q)|} + O(\varepsilon^2).$$
With \( \tau = \varepsilon t \) as the slow time, a reasonable guess for the averaged equation for \( W \) is

\[
\frac{dW}{d\tau} = \frac{E_1 \ell}{|D_1(Q)|} - \frac{E_2 \ell}{|D_2(Q)|}.
\]

Similar arguments for the other slow variables lead to the averaged equation (5), and this explains why we used \( P_i = E_i/|D_i| \) for the pressure of a 2-dimensional gas in Section 2.

There is a similar heuristic derivation of the averaged equation in \( d > 2 \) dimensions. Compare the analogues of Equations (9) and (10) in Subsection 6.2.

### 4.4 A priori estimate on the size of a set of bad initial conditions

In this section, we give an \( a \) priori estimate on the size of a set of initial conditions that should not give rise to orbits for which \( \sup_{0 \leq \tau \leq T} |h_\varepsilon(\tau/\varepsilon) - \bar{h}(\tau)| \) is small. In particular, when proving Theorem 1, it is convenient to focus on orbits that only contain clean collisions with the piston. Thus, we show that \( P\{\tilde{T}_\varepsilon < T \wedge T_\varepsilon\} \) vanishes as \( \varepsilon \to 0 \). At first, this result may seem surprising, since \( P\{T'_{\varepsilon} \wedge T''_{\varepsilon} = 0\} = \mathcal{O}(\varepsilon) \), and one would expect \( \bigcup_{t=0}^{T/\varepsilon} \mathcal{B}_\varepsilon(t) \) to have a size of order 1. However, the rate at which orbits escape from \( \{T'_{\varepsilon} \wedge T''_{\varepsilon} = 0\} \) is very small, and so we can prove the following:

**Lemma 5.**

\[
P\{\tilde{T}_\varepsilon < T \wedge T_\varepsilon\} = \mathcal{O}(\varepsilon).
\]

In some sense, this lemma states that the probability of having a gas particle move nearly parallel to the piston’s face within the time interval \( [0, T/\varepsilon] \), when one would expect the other gas particle to force the piston to move on a macroscopic scale, vanishes as \( \varepsilon \to 0 \). Thus, one can hope to control the occurrence of the “nondiffusive fluctuations” of the piston described in [CD06a] on a time scale \( \mathcal{O}(\varepsilon^{-1}) \).

**Proof.** As the left and the right gas particles can be handled similarly, it suffices to show that \( P\{T'_{\varepsilon} < T\} = \mathcal{O}(\varepsilon) \). Define

\[
\mathcal{B}_\varepsilon = \{z \in \mathcal{M} : Q_{\min} \leq q^1 \leq Q \leq Q_{\max} \text{ and } |v^1| \leq C_1\varepsilon\}.
\]

Then \( \{T'_{\varepsilon} < T\} \subset \bigcup_{t=0}^{T/\varepsilon} z_{\varepsilon}(t) \mathcal{B}_\varepsilon \) and if \( \gamma = Q_{\min}/\sqrt{8E_{\max}} \),

\[
P\left(\bigcup_{t=0}^{T/\varepsilon} z_{\varepsilon}(t) \mathcal{B}_\varepsilon\right) = P\left(\bigcup_{t=0}^{T/\varepsilon} z_{\varepsilon}(t) \mathcal{B}_\varepsilon\right) = P\left(\bigcup_{t=0}^{T/\varepsilon} (z_{\varepsilon}(t) \mathcal{B}_\varepsilon) \setminus \mathcal{B}_\varepsilon\right)
\leq P\mathcal{B}_\varepsilon + P\left(\bigcup_{k=0}^{T/(\varepsilon\gamma)} z_{\varepsilon}(k\gamma) \bigcup_{t=0}^{\gamma} (z_{\varepsilon}(t) \mathcal{B}_\varepsilon) \setminus \mathcal{B}_\varepsilon\right)
\leq P\mathcal{B}_\varepsilon + \left(\frac{T}{\varepsilon\gamma} + 1\right) P\left(\bigcup_{t=0}^{\gamma} (z_{\varepsilon}(t) \mathcal{B}_\varepsilon) \setminus \mathcal{B}_\varepsilon\right).
\]

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Now \( P \mathcal{B}_\varepsilon = \mathcal{O}(\varepsilon) \), so if we can show that \( P (\bigcup_{t=0}^{\gamma}(z_\varepsilon(t) \mathcal{B}_\varepsilon) \setminus \mathcal{B}_\varepsilon) = \mathcal{O}(\varepsilon^2) \), then it will follow that \( P \{ T'_\varepsilon < T \} = \mathcal{O}(\varepsilon) \).

If \( z \in \bigcup_{t=0}^{\gamma}(z_\varepsilon(t) \mathcal{B}_\varepsilon) \setminus \mathcal{B}_\varepsilon \), it is still true that \( |v_1^\perp| = \mathcal{O}(\varepsilon) \). This is because \( |v_1^\perp| \) changes by at most \( \mathcal{O}(\varepsilon) \) at the collisions, and if a collision forces \( |v_1^\perp| > C_1 \varepsilon \), then the gas particle must escape to the region \( q_1^\perp \leq 0 \) before \( v_1^\perp \) can change again, and this will take time greater than \( \gamma \). Furthermore, if \( z \in \bigcup_{t=0}^{\gamma}(z_\varepsilon(t) \mathcal{B}_\varepsilon) \setminus \mathcal{B}_\varepsilon \), then at least one of the following four possibilities must hold:

- \( |q_1^\perp - Q_{\min}| \leq \mathcal{O}(\varepsilon) \),
- \( |Q - Q_{\min}| \leq \mathcal{O}(\varepsilon) \),
- \( |Q - Q_{\max}| \leq \mathcal{O}(\varepsilon) \),
- \( |Q - q_1^\perp| \leq \mathcal{O}(\varepsilon) \).

It follows that \( P (\bigcup_{t=0}^{\gamma}(z_\varepsilon(t) \mathcal{B}_\varepsilon) \setminus \mathcal{B}_\varepsilon) = \mathcal{O}(\varepsilon^2) \). For example,

\[
\int_{\mathcal{M}} \mathbf{1}_{\{|v_1^\perp| \leq \mathcal{O}(\varepsilon), |q_1^\perp - Q_{\min}| \leq \mathcal{O}(\varepsilon)\}} dP = \text{const} \int \mathbf{1}_{\{E_{\min} \leq W^2/2 + v_1^\perp/2 + v_2^\perp/2 \leq E_{\max}\}} \mathbf{1}_{\{|v_1^\perp| \leq \mathcal{O}(\varepsilon)\}} dW dv_1^\perp dv_1^\parallel dv_2^\perp dv_2^\parallel \\
\times \int \{Q \in [0,1], q_1 \in \mathcal{D}_1, q_2 \in \mathcal{D}_2\} \mathbf{1}_{\{|q_1^\perp - Q_{\min}| \leq \mathcal{O}(\varepsilon)\}} dQ dq_1^\perp dq_1^\parallel dq_2^\perp dq_2^\parallel \geq \mathcal{O}(\varepsilon^2).
\]

\[\square\]

5 Proof of the main result for two-dimensional gas containers with only one gas particle on each side

As in Section 4, we continue with the case when \( d = 2 \) and there is only one gas particle on either side of the piston.

5.1 Main steps in the proof of convergence in probability

By Lemma 5, it suffices to show that \( \sup_{0 \leq \tau \leq \tilde{T}_\varepsilon} |h_\varepsilon(\tau/\varepsilon) - \tilde{h}(\tau)| \to 0 \) in probability as \( \varepsilon = M^{-1/2} \to 0 \). Several of the ideas in the steps below were inspired by a recent proof of Anosov’s averaging theorem for smooth systems that is due to Dolgopyat [Dol05].
Step 1: Reduction using Gronwall’s Inequality. Observe that \( \bar{h}(\tau) \) satisfies the integral equation

\[
\bar{h}(\tau) - \bar{h}(0) = \int_0^\tau \bar{H}(\bar{h}(\sigma))d\sigma,
\]

while from Lemma 4,

\[
h_\varepsilon(\tau/\varepsilon) - h_\varepsilon(0) = O(\varepsilon) + \varepsilon \int_0^{\tau/\varepsilon} \bar{H}(z_\varepsilon(s))ds
\]

\[
= O(\varepsilon) + \varepsilon \int_0^{\tau/\varepsilon} \bar{H}(z_\varepsilon(s)) - \bar{H}(h_\varepsilon(s))ds + \int_0^\tau \bar{H}(h_\varepsilon(\sigma/\varepsilon))d\sigma
\]

for \( 0 \leq \tau \leq \tilde{T}_\varepsilon \). Define

\[
e_\varepsilon(\tau) = \varepsilon \int_0^{\tau/\varepsilon} \bar{H}(z_\varepsilon(s)) - \bar{H}(h_\varepsilon(s))ds.
\]

It follows from Gronwall’s Inequality that

\[
\sup_{0 \leq \tau \leq \tilde{T}_\varepsilon} |h_\varepsilon(\tau/\varepsilon) - \bar{h}(\tau)| \leq \left( O(\varepsilon) + \sup_{0 \leq \tau \leq \tilde{T}_\varepsilon} |e_\varepsilon(\tau)| \right) e^{\text{Lip} \bar{H} |_V T}.
\]

(14)

Gronwall’s Inequality is usually stated for continuous paths, but the standard proof (found in [SV85]) still works for paths that are merely integrable, and \( |h_\varepsilon(\tau/\varepsilon) - \bar{h}(\tau)| \) is piecewise smooth.

Step 2: Introduction of a time scale for ergodization. Let \( L(\varepsilon) \) be a real valued function such that \( L(\varepsilon) \to \infty \), but \( L(\varepsilon) \ll \log^{-1} \varepsilon \), as \( \varepsilon \to 0 \). In Section 5.2 we will place precise restrictions on the growth rate of \( L(\varepsilon) \). Think of \( L(\varepsilon) \) as being a time scale that grows as \( \varepsilon \to 0 \) so that ergodization, i.e. the convergence along an orbit of a function’s time average to a space average, can take place. However, \( L(\varepsilon) \) doesn’t grow too fast, so that on this time scale \( z_\varepsilon(t) \) essentially stays on the submanifold \( \{h = h_\varepsilon(0)\} \), where we have our ergodicity assumption. Set \( t_{k,\varepsilon} = kL(\varepsilon) \), so that

\[
\sup_{0 \leq \tau \leq \tilde{T}_\varepsilon} |e_\varepsilon(\tau)| \leq O(\varepsilon L(\varepsilon)) + \varepsilon \sum_{k=0}^{\tilde{T}_\varepsilon L(\varepsilon) - 1} \left| \int_{t_{k,\varepsilon}}^{t_{k+1,\varepsilon}} H(z_\varepsilon(s)) - \bar{H}(h_\varepsilon(s))ds \right|.
\]

(15)

Step 3: A splitting according to particles. Now \( H(z) - \bar{H}(h(z)) \) divides into two pieces, each of which depends on only one gas particle when the piston is held fixed:

\[
H(z) - \bar{H}(h(z)) = \begin{bmatrix}
2 |v_1^+| \delta_{q_1^+=Q} - E_1^{\ell} |D_1^{(Q)}| & 0 \\
-2W |v_2^+| \delta_{q_1^+=Q} + E_1^{\ell} |W| |D_1^{(Q)}| & 0 \\
0 & 0
\end{bmatrix} + \begin{bmatrix}
0 \\
\frac{E_2^{\ell} |D_2^{(Q)}|}{|D_2^{(Q)}|} - 2 |v_2^+| \delta_{q_2^+=Q} \\
-rac{W E_2^{\ell} |D_2^{(Q)}|}{|D_2^{(Q)}|} + 2W |v_2^+| \delta_{q_2^+=Q}
\end{bmatrix}.
\]
We will only deal with the piece depending on the left gas particle, as the right particle can be handled similarly. Define
\[ G(z) = |v_1^+| \delta_{q^+_1=Q}, \quad \bar{G}(h) = \frac{E_1 \ell}{2|D_1(Q)|}. \] (16)

Returning to Equation (15), we see that in order to prove Theorem 1, it suffices to show that both
\[ \varepsilon \tilde{T} \varepsilon L(\varepsilon) - \sum_{k=0}^{\tilde{T} \varepsilon / \varepsilon - 1} \left| \int_{t_k, \varepsilon}^{t_{k+1}, \varepsilon} W(\varepsilon) (G(\varepsilon(s)) - \bar{G}(\varepsilon(s))) ds \right| \]

converge to 0 in probability as \( \varepsilon \to 0 \).

**Step 4: A splitting for using the triangle inequality.** Now we let \( z_{k, \varepsilon}(s) \) be the orbit of the \( \varepsilon = 0 \) Hamiltonian vector field satisfying \( z_{k, \varepsilon}(t_{k, \varepsilon}) = z_{\varepsilon}(t_{k, \varepsilon}) \). Set \( h_{k, \varepsilon}(t) = h(z_{k, \varepsilon}(t)). \) Observe that \( h_{k, \varepsilon}(t) \) is independent of \( t \).

We emphasize that so long as \( 0 \leq t \leq \tilde{T} / \varepsilon \), the times between collisions of a specific gas particle and piston are uniformly bounded greater than 0, as explained before Lemma 3. It follows that, so long as \( t_{k+1, \varepsilon} \leq \tilde{T} / \varepsilon \),
\[ \sup_{t_{k, \varepsilon} \leq t \leq t_{k+1, \varepsilon}} |h_{k, \varepsilon}(t) - h(\varepsilon(s))| = O(\varepsilon L(\varepsilon)). \] (17)

This is because the slow variables change by at most \( O(\varepsilon) \) at collisions, and \( dQ \varepsilon / dt = O(\varepsilon) \).

Also,
\[ \int_{t_k, \varepsilon}^{t_{k+1}, \varepsilon} W(\varepsilon) (G(\varepsilon(s)) - \bar{G}(\varepsilon(s))) ds \]
\[ = O(\varepsilon L(\varepsilon)^2) + W_{k, \varepsilon}(t_{k, \varepsilon}) \int_{t_k, \varepsilon}^{t_{k+1}, \varepsilon} G(\varepsilon(s)) - \bar{G}(\varepsilon(s)) ds, \]

and so
\[ \varepsilon \sum_{k=0}^{\tilde{T} / \varepsilon - 1} \int_{t_k, \varepsilon}^{t_{k+1}, \varepsilon} W(\varepsilon) (G(\varepsilon(s)) - \bar{G}(\varepsilon(s))) ds \]
\[ \leq O(\varepsilon L(\varepsilon)) + \varepsilon \text{ const} \sum_{k=0}^{\tilde{T} / \varepsilon - 1} \int_{t_k, \varepsilon}^{t_{k+1}, \varepsilon} G(\varepsilon(s)) - \bar{G}(\varepsilon(s)) ds. \]
Thus, in order to prove Theorem 1, it suffices to show that
\[
\frac{T}{\varepsilon L(\varepsilon)} \sum_{k=0}^{k_{\varepsilon} - 1} \left| \int_{t_{k,\varepsilon}}^{t_{k+1,\varepsilon}} G(z_\varepsilon(s)) - G(h_\varepsilon(s))ds \right| \leq \varepsilon \sum_{k=0}^{k_{\varepsilon} - 1} |I_{k,\varepsilon}| + |II_{k,\varepsilon}| + |III_{k,\varepsilon}|
\]
converges to 0 in probability as \(\varepsilon \to 0\), where
\[
I_{k,\varepsilon} = \int_{t_{k,\varepsilon}}^{t_{k+1,\varepsilon}} G(z_\varepsilon(s)) - G(z_{k,\varepsilon}(s))ds,
\]
\[
II_{k,\varepsilon} = \int_{t_{k,\varepsilon}}^{t_{k+1,\varepsilon}} G(z_{k,\varepsilon}(s)) - \bar{G}(h_{k,\varepsilon}(s))ds,
\]
\[
III_{k,\varepsilon} = \int_{t_{k,\varepsilon}}^{t_{k+1,\varepsilon}} \bar{G}(h_{k,\varepsilon}(s)) - \bar{G}(h_\varepsilon(s))ds.
\]

The term \(II_{k,\varepsilon}\) represents an “ergodicity term” that can be controlled by our assumptions on the ergodicity of the flow \(z_\varepsilon(t)\), while the terms \(I_{k,\varepsilon}\) and \(III_{k,\varepsilon}\) represent “continuity terms” that can be controlled by controlling the drift of \(z_\varepsilon(t)\) from \(z_{k,\varepsilon}(t)\) for \(t_{k,\varepsilon} \leq t \leq t_{k+1,\varepsilon}\).

**Step 5: Control of drift from the \(\varepsilon = 0\) orbits.** Now \(\bar{G}\) is uniformly Lipschitz on the compact set \(\mathcal{V}\), and so it follows from Equation (17) that \(III_{k,\varepsilon} = O(\varepsilon L(\varepsilon)^2)\). Thus, \(\varepsilon \sum_{k=0}^{k_{\varepsilon} - 1} |III_{k,\varepsilon}| = O(\varepsilon L(\varepsilon)) \to 0\) as \(\varepsilon \to 0\).

Next, we show that for fixed \(\delta > 0\), \(P \left( \varepsilon \sum_{k=0}^{k_{\varepsilon} - 1} |I_{k,\varepsilon}| \geq \delta \right) \to 0\) as \(\varepsilon \to 0\).

For initial conditions \(z \in \mathcal{M}\) and for integers \(k \in [0, T/(\varepsilon L(\varepsilon)) - 1]\) define
\[
\mathcal{A}_{k,\varepsilon} = \left\{ z : \frac{1}{L(\varepsilon)} |I_{k,\varepsilon}| > \frac{\delta}{2T} \text{ and } k \leq \frac{T}{\varepsilon L(\varepsilon)} - 1 \right\},
\]
\[
\mathcal{A}_{z,\varepsilon} = \{ k : z \in \mathcal{A}_{k,\varepsilon} \}.
\]

Think of these sets as describing “poor continuity” between solutions of the \(\varepsilon = 0\) and the \(\varepsilon > 0\) Hamiltonian vector fields. For example, roughly speaking, \(z \in \mathcal{A}_{k,\varepsilon}\) if the orbit \(z_\varepsilon(t)\) starting at \(z\) does not closely follow \(z_{k,\varepsilon}(t)\) for \(t_{k,\varepsilon} \leq t \leq t_{k+1,\varepsilon}\).

One can easily check that \(|I_{k,\varepsilon}| \leq O(L(\varepsilon))\) for \(k \leq \frac{T}{\varepsilon L(\varepsilon)} - 1\), and so it follows that
\[
\varepsilon \sum_{k=0}^{k_{\varepsilon} - 1} |I_{k,\varepsilon}| \leq \frac{\delta}{2} + O(\varepsilon L(\varepsilon)\#(\mathcal{A}_{z,\varepsilon})).
\]

Therefore it suffices to show that \(P(\#(\mathcal{A}_{z,\varepsilon}) \geq \delta (\text{const} \varepsilon L(\varepsilon))^{-1}) \to 0\) as \(\varepsilon \to 0\). By Chebyshev’s Inequality, we need only show that
\[
P(\varepsilon L(\varepsilon)\#(\mathcal{A}_{z,\varepsilon})) = \varepsilon L(\varepsilon) \sum_{k=0}^{k_{\varepsilon} - 1} P(\mathcal{A}_{k,\varepsilon})
\]
tends to 0 with $\varepsilon$.

Observe that $z_\varepsilon(t_{k,\varepsilon}, A_{k,\varepsilon}) \subset A_{0,\varepsilon}$. In words, the initial conditions giving rise to orbits that are “bad” on the time interval $[t_{k,\varepsilon}, t_{k+1,\varepsilon}]$, moved forward by time $t_{k,\varepsilon}$, are initial conditions giving rise to orbits which are “bad” on the time interval $[t_{0,\varepsilon}, t_{1,\varepsilon}]$. Because the flow $z_\varepsilon(\cdot)$ preserves the measure, we find that

$$\varepsilon L(\varepsilon) \sum_{k=0}^{T_\varepsilon} P(A_{k,\varepsilon}) \leq \text{const} P(A_{0,\varepsilon}).$$

To estimate $P(A_{0,\varepsilon})$, it is convenient to use a different probability measure, which is uniformly equivalent to $P$ on the set $\{z \in \mathcal{M} : h(z) \in \mathcal{V}\} \supset \{\tilde{T}_\varepsilon \geq \varepsilon L(\varepsilon)\}$. We denote this new probability measure by $P^f$, where the $f$ stands for “factor.” If we choose coordinates on $\mathcal{M}$ by using $h$ and the billiard coordinates on the two gas particles, then $P^f$ is defined on $\mathcal{M}$ by $dP^f = dh \, d\mu_1^h \, d\mu_2^h$, where $dh$ represents the uniform measure on $\mathcal{V} \subset \mathbb{R}^4$, and the factor measure $d\mu_i^h$ represents the invariant billiard measure of the $i$th gas particle coordinates for a fixed value of the slow variables. One can verify that $1_{\{h(z) \in \mathcal{V}\}} dP \leq \text{const} dP^f$, but that $P^f$ is not invariant under the flow $z_\varepsilon(\cdot)$ when $\varepsilon > 0$.

We abuse notation, and consider $\mu_1^h$ to be a measure on the left particle’s initial billiard coordinates once $h$ and the initial coordinates of the right gas particle are fixed. In this context, $\mu_1^h$ is simply the measure $\mu$ from Subsection 4.1. Then

$$P^f(A_{0,\varepsilon})$$

$$\leq \int dh \, d\mu_2^h \cdot \mu_1^h \left\{ z : \left| \frac{1}{L(\varepsilon)} \int_0^{L(\varepsilon)} G(z_\varepsilon(s)) - G(z_0(s)) ds \right| \geq \frac{\delta}{2T} \text{ and } \varepsilon L(\varepsilon) \leq \tilde{T}_\varepsilon \right\},$$

and we must show that the last term tends to 0 with $\varepsilon$. By the Bounded Convergence Theorem, it suffices to show that for almost every $h \in \mathcal{V}$ and initial condition for the right gas particle,

$$\mu_1^h \left\{ z : \left| \frac{1}{L(\varepsilon)} \int_0^{L(\varepsilon)} G(z_\varepsilon(s)) - G(z_0(s)) ds \right| \geq \frac{\delta}{2T} \text{ and } \varepsilon L(\varepsilon) \leq \tilde{T}_\varepsilon \right\} \to 0 \text{ as } \varepsilon \to 0. \quad (18)$$

Note that if $G$ were a smooth function and $z_\varepsilon(\cdot)$ were the flow of a smooth family of vector fields $Z(z, \varepsilon)$ that depended smoothly on $\varepsilon$, then from Gronwall’s Inequality, it would follow that $\sup_{0 \leq t \leq L(\varepsilon)} |z_\varepsilon(t) - z_0(t)| \leq O(\varepsilon L(\varepsilon)e^{\text{Lip}(Z)L(\varepsilon)})$. If this were the case, then $\left| L(\varepsilon)^{-1} \int_0^{L(\varepsilon)} G(z_\varepsilon(s)) - G(z_0(s)) ds \right| = O(\varepsilon L(\varepsilon)e^{\text{Lip}(Z)L(\varepsilon)})$, which would tend to 0 with $\varepsilon$. Thus, we need a Gronwall-type inequality for billiard flows. We obtain the appropriate estimates in Section 5.2.

**Step 6: Use of ergodicity along fibers to control $II_{k,\varepsilon}$**. All that remains to be shown is that for fixed $\delta > 0$, $P \left( \varepsilon \sum_{k=0}^{T_\varepsilon} |II_{k,\varepsilon}| \geq \delta \right) \to 0$ as $\varepsilon \to 0$. 

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For initial conditions \( z \in \mathcal{M} \) and for integers \( k \in [0, T/(\varepsilon L(\varepsilon)) - 1] \) define

\[
B_{k,\varepsilon} = \left\{ z : \frac{1}{L(\varepsilon)}|II_{k,\varepsilon}| > \frac{\delta}{2T} \text{ and } k \leq \frac{T_{\varepsilon}}{\varepsilon L(\varepsilon)} - 1 \right\},
\]

\[
B_z,\varepsilon = \{ k : z \in B_{k,\varepsilon} \}.
\]

Think of these sets as describing “bad ergodization.” For example, roughly speaking, \( z \in B_{k,\varepsilon} \) if the orbit \( z_{\varepsilon}(t) \) starting at \( z \) spends the time between \( t_{k,\varepsilon} \) and \( t_{k+1,\varepsilon} \) in a region of phase space where the function \( G(\cdot) \) is “poorly ergodized” on the time scale \( L(\varepsilon) \) by the flow \( z_0(t) \) (as measured by the parameter \( \delta/2T \)). Note that \( G(z) = |v_{\varepsilon}^1|_{\delta_{g_1}=Q} \) is not really a function, but that we may still speak of the convergence of \( t^{-1} \int_0^t G(z_0(s))ds \) as \( t \to \infty \). As we showed in Lemma \( \text{[2]} \) the limit is \( \tilde{G}(h_0) \) for almost every initial condition.

Proceeding as in Step 5 above, we find that it suffices to show that for almost every \( h \in \mathcal{V} \),

\[
\mu_h^1 \left\{ z : \left| \frac{1}{t} \int_0^t G(z_0(s))ds - \tilde{G}(h_0(0)) \right| \geq \frac{\delta}{2T} \right\} \to 0 \text{ as } t \to \infty.
\]

But this is simply a question of examining billiard flows, and it follows immediately from Corollary \( \text{[3]} \) and our Main Assumption.

## 5.2 A Gronwall-type inequality for billiards

We begin by presenting a general version of Gronwall’s Inequality for billiard maps. Then we will show how these results imply the convergence required in Equation \( \text{[18]} \).

### 5.2.1 Some inequalities for the collision map

In this section, we consider the value of the slow variables to be fixed at \( h_0 \in \mathcal{V} \). We will use the notation and results presented in Section \( \text{[1]} \) but because the value of the slow variables is fixed, we will omit it in our notation.

Let \( \rho, \gamma, \) and \( \lambda \) satisfy \( 0 < \rho \ll \gamma \ll 1 \ll \lambda < \infty \). Eventually, these quantities will be chosen to depend explicitly on \( \varepsilon \), but for now they are fixed.

Recall that the phase space \( \Omega \) for the collision map \( F \) is a finite union of disjoint rectangles and cylinders. Let \( d \) be the Euclidean metric on connected components of \( \Omega \). If \( x \) and \( x' \) belong to different components, then we set \( d(x, x') = \infty \). The invariant measure \( \nu \) satisfies \( \nu < \text{const} \cdot (\text{Lebesgue measure}) \). For \( A \subset \Omega \) and \( a > 0 \), let \( \mathcal{N}_a(A) = \{ x \in \Omega : d(x, A) < a \} \) be the \( a \)-neighborhood of \( A \).

For \( x \in \Omega \) let \( x_k(x) = x_k = F^k x, k \geq 0 \), be its forward orbit. Suppose \( x \notin \mathcal{C}_{\gamma,\lambda} \), where

\[
\mathcal{C}_{\gamma,\lambda} = (\bigcup_{k=0}^\lambda F^{-k} \mathcal{N}_\gamma(\partial \Omega)) \cup (\bigcup_{k=0}^\lambda F^{-k} \mathcal{N}_\gamma(F^{-1} \mathcal{N}_\gamma(\partial \Omega))).
\]

Thus for \( 0 \leq k \leq \lambda \), \( x_k \) is well defined, and from Equation \( \text{[6]} \) it satisfies

\[
d(x', x_k) \leq \gamma \Rightarrow d(Fx', x_{k+1}) \leq \frac{\text{const}}{\gamma} d(x', x_k).
\]
Next, we consider any $\rho$-pseudo-orbit $x'_k$ obtained from $x$ by adding on an error of size $\leq \rho$ at each application of the map, i.e. $d(x'_0, x_0) \leq \rho$, and for $k \geq 1$, $d(x'_k, Fx'_{k-1}) \leq \rho$. Provided $d(x_j, x'_j) < \gamma$ for each $j < k$, it follows that

$$d(x_k, x'_k) \leq \rho \sum_{j=0}^{k} \left( \frac{\text{const}}{\gamma} \right)^j \leq \text{const} \rho \left( \frac{\text{const}}{\gamma} \right)^k. \quad (20)$$

In particular, if $\rho, \gamma$, and $\lambda$ were chosen such that

$$\text{const} \rho \left( \frac{\text{const}}{\gamma} \right)^\lambda < \gamma, \quad (21)$$

then Equation (20) will hold for each $k \leq \lambda$. We assume that Equation (21) is true. Then we can also control the differences in elapsed flight times using Equation (7):

$$|\zeta x_k - \zeta x'_k| \leq \text{const} \rho \left( \frac{\text{const}}{\gamma} \right)^k. \quad (22)$$

It remains to estimate the size $\nu C_{\gamma, \lambda}$ of the set of $x$ for which the above estimates do not hold. Using Lemma 6 below,

$$\nu C_{\gamma, \lambda} \leq (\lambda+1) \left( \nu N_{\gamma}(\partial \Omega) + \nu N_{\gamma}(F^{-1}N_{\gamma}(\partial \Omega)) \right) \leq O(\lambda(\gamma + \gamma^{1/3})) = O(\gamma^{1/3}). \quad (23)$$

**Lemma 6.** As $\gamma \to 0$,

$$\nu N_{\gamma}(F^{-1}N_{\gamma}(\partial \Omega)) = O(\gamma^{1/3}).$$

This estimate is not necessarily the best possible. For example, for dispersing billiard tables, where the curvature of the boundary is positive, one can show that $\nu N_{\gamma}(F^{-1}N_{\gamma}(\partial \Omega)) = O(\gamma)$. However, the estimate in Lemma 6 is general and sufficient for our needs.

**Proof.** First, we note that it is equivalent to estimate $\nu N_{\gamma}(FN_{\gamma}(\partial \Omega))$, as $F$ has the measure-preserving involution $I(r, \varphi) = (r, -\varphi)$, i.e. $F^{-1} = I \circ F \circ I$.

Fix $\alpha \in (0, 1/2)$, and cover $N_{\gamma}(\partial \Omega)$ with $O(\gamma^{-1})$ starlike sets, each of diameter no greater than $O(\gamma)$. For example, these sets could be squares of side length $\gamma$. Enumerate the sets as $\{A_i\}$. Set $G = \{i : FA_i \cap N_{\gamma}^\alpha(\partial \Omega) = \emptyset\}$.

If $i \in G$, $F|_{A_i}$ is a diffeomorphism satisfying $\|DF|_{A_i}\| \leq O(\gamma^{-\alpha})$. See Equation (3). Thus diameter $(FA_i) \leq O(\gamma^{1-\alpha})$, and so diameter $(N_{\gamma}(FA_i)) \leq O(\gamma^{1-\alpha})$. Hence $\nu N_{\gamma}(FA_i) \leq O(\gamma^{2(1-\alpha)})$, and $\nu N_{\gamma}(\cup_{i \in G} FA_i) \leq O(\gamma^{1-2\alpha})$.

If $i \notin G$, $A_i \cap F^{-1}(N_{\gamma}^\alpha(\partial \Omega)) \neq \emptyset$. Thus $A_i$ might be cut into many pieces by $F^{-1}(\partial \Omega)$, but each of these pieces must be mapped near $\partial \Omega$. In fact, $FA_i \subset N_{O(\gamma\alpha)}(\partial \Omega)$. This is because outside $F^{-1}(N_{\gamma}^\alpha(\partial \Omega))$, $\|DF\| \leq O(\gamma^{-\alpha})$, and so points in $FA_i$ are no more than a distance $O(\gamma/\gamma^\alpha)$ away from $N_{\gamma}(\partial \Omega)$, and $\gamma < \gamma^{1-\alpha} < \gamma^\alpha$. It follows that $N_{\gamma}(FA_i) \subset N_{O(\gamma\alpha)}(\partial \Omega)$, and $\nu N_{O(\gamma\alpha)}(\partial \Omega) = O(\gamma^\alpha)$.

Thus $\nu N_{\gamma}(F^{-1}N_{\gamma}(\partial \Omega)) = O(\gamma^{1-2\alpha} + \gamma^\alpha)$, and we obtain the lemma by taking $\alpha = 1/3$. 

$\Box$
5.2.2 Application to a perturbed billiard flow

Returning to the end of Step 5 in Section 5.1, let the initial conditions of the slow variables be fixed at $h_0 = (Q_0, W_0, E_{1,0}, E_{2,0}) \in \mathcal{V}$ throughout the remainder of this section. We can assume that the billiard dynamics of the left gas particle in $\mathcal{D}_1(Q_0)$ are ergodic. Also, fix a particular value of the initial conditions for the right gas particle for the remainder of this section. Then $z_\varepsilon(t)$ and $\tilde{T}_\varepsilon$ may be thought of as random variables depending on the left gas particle’s initial conditions $y \in M_1$.

Now if $h_\varepsilon(t) = (Q_\varepsilon(t), W_\varepsilon(t), E_{1,\varepsilon}(t), E_{2,\varepsilon}(t))$ denotes the actual motions of the slow variables when $\varepsilon > 0$, it follows from Equation (17) that, provided $\varepsilon L(\varepsilon) \leq \tilde{T}_\varepsilon$,

$$\sup_{0 \leq t \leq L(\varepsilon)} |h_0 - h_\varepsilon(t)| = O(\varepsilon L(\varepsilon)).$$

(24)

Furthermore, we only need to show that

$$\mu \left\{ y \in M_1 : \left| G(z_\varepsilon(s)) - G(z_0(s)) \right| ds \geq \frac{\delta}{2T} \quad \text{and} \quad \varepsilon L(\varepsilon) \leq \tilde{T}_\varepsilon \right\} \to 0$$

as $\varepsilon \to 0$, where $G$ is defined in Equation (16).

For definiteness, we take the following quantities from Subsection 5.2.1 to depend on $\varepsilon$ as follows:

$$L(\varepsilon) = L = \log \log \frac{1}{\varepsilon},$$

$$\gamma(\varepsilon) = \gamma = e^{-L},$$

$$\lambda(\varepsilon) = \lambda = \frac{2}{E_\nu \zeta} L,$$

$$\rho(\varepsilon) = \rho = \text{const} \frac{\varepsilon L}{\gamma}.$$

(26)

The constant in the choice of $\rho$ and $\rho$'s dependence on $\varepsilon$ will be explained in the proof of Lemma 8 which is at the end of this subsection. The other choices may be explained as follows. We wish to use continuity estimates for the billiard map to produce continuity estimates for the flow on the time scale $L$. As the divergence of orbits should be exponentially fast, we choose $L$ to grow sublogarithmically in $\varepsilon^{-1}$. Since from Equation (8) the expected flight time between collisions with $\partial \mathcal{D}_1(Q_0)$ when $\varepsilon = 0$ is $E_\nu \zeta = \pi |\mathcal{D}_1(Q_0)| / (\sqrt{2E_{1,0}} |\partial \mathcal{D}_1(Q_0)|)$, we expect to see roughly $\lambda/2$ collisions on this time scale. Considering $\lambda$ collisions gives us some margin for error. Furthermore, we will want orbits to keep a certain distance, $\gamma$, away from the billiard discontinuities. $\gamma \to 0$ as $\varepsilon \to 0$, but $\gamma$ is very large compared to the possible drift $O(\varepsilon L)$ of the slow variables on the time scale $L$. In fact, for each $C, m, n > 0$,

$$\frac{\varepsilon L^m}{\gamma^n} \left( \frac{C}{\gamma} \right)^\lambda = O(\varepsilon e^{\text{const} L^2}) \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

(27)

Let $X : M_1 \to \Omega$ be the map taking $y \in M_1$ to $x = X(y) \in \Omega$, the location of the billiard orbit of $y$ in the collision cross-section that corresponds to the most
recent time in the past that the orbit was in the collision cross-section. We consider
the set of initial conditions
\[
\mathcal{E}_\varepsilon = X^{-1}(\Omega \setminus C_{\gamma, \lambda}) \cap \bigcap_{x \in \Omega} \left\{ x : \lambda \sum_{k=0}^{\lambda} \zeta(F^k x) > L \right\}.
\]
Now from Equations (23) and (26), \( \nu C_{\gamma, \lambda} \to 0 \) as \( \varepsilon \to 0 \). Furthermore, by the ergodicity of \( F, \nu \left\{ x \in \Omega : \lambda \sum_{k=0}^{\lambda} \zeta(F^k x) \right\} \to \nu \left\{ x \in \Omega : \lambda^{-1} \sum_{k=0}^{\lambda} \zeta(F^k x) \leq E_\nu \zeta/2 \right\} \to 0 \) as \( \varepsilon \to 0 \). But because the free flight time is bounded above, \( \mu X^{-1} \leq \text{const} \cdot \nu \), and so \( \mu \mathcal{E}_\varepsilon \to 1 \) as \( \varepsilon \to 0 \). Hence, the convergence in Equation (25) and the conclusion of the proof in Section 5.1 follow from the lemma below and Equation (27).

**Lemma 7** (Analysis of deviations along good orbits). As \( \varepsilon \to 0 \),
\[
\sup_{y \in \mathcal{E}_\varepsilon \cap \{ \varepsilon L \leq \tilde{T}_\varepsilon \}} \left\| \frac{1}{L} \int_0^L G(z_\varepsilon(s)) - G(z_0(s)) ds \right\| = O \left( \rho \left( \frac{\text{const}}{\gamma} \right)^\lambda \right) + O(L^{-1}) \to 0.
\]

**Proof.** Fix a particular value of \( y \in \mathcal{E}_\varepsilon \cap \{ \varepsilon L \leq \tilde{T}_\varepsilon \} \). For convenience, suppose that \( y = X(y) = x \in \Omega \). Let \( y_0(t) \) denote the time evolution of the billiard coordinates for the left gas particle when \( \varepsilon = 0 \). Then there is some \( N \leq \lambda \) such that the orbit \( x_k = F^k x = (r_k, \varphi_k) \) for \( 0 \leq k \leq N \) corresponds to all of the instances (in order) when \( y_0(t) \) enters the collision cross-section \( \Omega = \Omega_{h_0} \) corresponding to collisions with \( \partial D_1(Q_0) \) for \( 0 \leq t \leq L \). We write \( \Omega_{h_0} \) to emphasize that in this subsection we are only considering the collision cross-section corresponding to the billiard dynamics in the domain \( D_1(Q_0) \) at the energy level \( E_{1, 0} \). In particular, \( F \) will always refer to the return map on \( \Omega_{h_0} \).

Also, define an increasing sequence of times \( t_k \) corresponding to the actual times \( y_0(t) \) enters the collision cross-section, i.e.
\[
t_0 = 0, \quad t_k = t_{k-1} + \zeta x_{k-1} \quad \text{for} \quad k > 0.
\]
Then \( x_k = y_0(t_k) \). Furthermore, define inductively
\[
N_1 = \inf \{ k > 0 : t_k \text{ corresponds to a collision with the piston} \},
N_j = \inf \{ k > N_{j-1} : t_k \text{ corresponds to a collision with the piston} \}.
\]

Next, let \( y_\varepsilon(t) \) denote the time evolution of the billiard coordinates for the left gas particle when \( \varepsilon > 0 \). We will construct a pseudo-orbit \( x_{k, \varepsilon}' = (r_{k, \varepsilon}', \varphi_{k, \varepsilon}') \) of points in \( \Omega_{h_0} \) that essentially track the collisions (in order) of the left gas particle with the boundary under the dynamics of \( y_\varepsilon(t) \) for \( 0 \leq t \leq L \).

First, define an increasing sequence of times \( t_{k, \varepsilon}' \) corresponding to the actual times \( y_\varepsilon(t) \) experiences a collision with the boundary of the gas container or the moving piston. Define
\[
N_{1, \varepsilon}' = \sup \{ k \geq 0 : t_{k, \varepsilon}' \leq L \},
N_{1, \varepsilon}' = \inf \{ k > 0 : t_{k, \varepsilon}' \text{ corresponds to a collision with the piston} \},
N_{j, \varepsilon}' = \inf \{ k > N_{j-1, \varepsilon}' : t_{k, \varepsilon}' \text{ corresponds to a collision with the piston} \}.
\]

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Because $L \leq \tilde{T}_\varepsilon(y)/\varepsilon$, we know that as long as $N'_{j+1,\varepsilon} \leq N'_{j,\varepsilon}$, then $N'_{j+1,\varepsilon} - N'_{j,\varepsilon} \geq 2$. See the discussion in Subsection 4.2. Then we define $x_{k,\varepsilon}' \in \Omega_{h_0}$ by

$$x_{k,\varepsilon}' = \begin{cases} y_\varepsilon(t_{k,\varepsilon}') & \text{if } k \notin \{N'_{j,\varepsilon}\}, \\ F^{-1}x_{k+1,\varepsilon}' & \text{if } k \in \{N'_{j,\varepsilon}\}. \end{cases}$$

**Lemma 8.** Provided $\varepsilon$ is sufficiently small, the following hold for each $k \in [0, N \wedge N'_{\varepsilon})$. Furthermore, the requisite smallness of $\varepsilon$ and the sizes of the constants in these estimates may be chosen independent of the initial condition $y \in \mathcal{E}_\varepsilon \cap \{\varepsilon L \leq \tilde{T}_\varepsilon\}$ and of $k$:

(a) $x_{k,\varepsilon}'$ is well defined. In particular, if $k \notin \{N'_{j,\varepsilon}\}$, $y_\varepsilon(t_{k,\varepsilon}')$ corresponds to a collision point on $\partial D_1(Q_0)$, and not to a collision point on a piece of $\partial D$ to the right of $Q_0$.

(b) If $k > 0$ and $k \notin \{N'_{j,\varepsilon}\}$, then $x_{k,\varepsilon}' = Fx_{k-1,\varepsilon}'$.

(c) If $k > 0$ and $k \in \{N'_{j,\varepsilon}\}$, then $d(x_{k,\varepsilon}', Fx_{k-1,\varepsilon}') \leq \rho$ and the $\varphi$ coordinate of $y_\varepsilon(t_{k,\varepsilon}')$ satisfies $\varphi(y_\varepsilon(t_{k,\varepsilon}')) = \varphi_{k,\varepsilon}' + O(\varepsilon)$.

(d) $d(x_{k,\varepsilon}', x_{k,\varepsilon}) \leq const \rho(\text{const}/\gamma)^k$.

(e) $k = N'_{j,\varepsilon}$ if and only if $k = N_j$.

(f) If $k > 0$, $t_{k,\varepsilon}' - t_{k-1,\varepsilon}' = t_k - t_{k-1} + O(\rho(\text{const}/\gamma)^k)$.

We defer the proof of Lemma 8 until the end of this subsection. Assuming that $\varepsilon$ is sufficiently small for the conclusions of Lemma 8 to be valid, we continue with the proof of Lemma 7.

Set $M = N \wedge N'_{\varepsilon} - 1$. Note that $M \leq \lambda \sim L$. From (f) in Lemma 8 and Equations (20) and (27), we see that

$$|t_M - t_{M,\varepsilon}'| \leq \sum_{k=1}^{M} |t_{k,\varepsilon}' - t_{k-1,\varepsilon}' - (t_k - t_{k-1})| = O \left( \rho \frac{\text{const}^\lambda}{\gamma^\lambda} \right) \to 0 \text{ as } \varepsilon \to 0.$$

Because the flight times $t_{k,\varepsilon}' - t_{k-1,\varepsilon}'$ and $t_k - t_{k-1}$ are uniformly bounded above, it follows from the definitions of $N'$ and $N'_{\varepsilon}$ that $t_M, t_{M,\varepsilon}' \geq L - \text{const}$. But from Subsection 4.2, the time between the collisions of the left gas particle with the piston are uniformly bounded away from zero. Using (e) and Equation (21), it follows that

$$\frac{1}{L} \int_{0}^{L} G(z_\varepsilon(s)) - G(z_0(s))ds = O(L^{-1}) + \sum_{k \in \{N_j : N_j \leq M\}} \left| \sqrt{2E_{1,0}} \cos \varphi_k - \sqrt{2E_{1,0}}(t_{k,\varepsilon}') \cos(\varphi_{k,\varepsilon}' + O(\varepsilon)) \right|$$

$$= O(L^{-1}) + \sum_{k \in \{N_j : N_j \leq M\}} \left| \sqrt{2E_{1,0}} \cos \varphi_k - \sqrt{2E_{1,0}} \cos \varphi_{k,\varepsilon}' + O(\varepsilon L) \right|$$

$$= O(L^{-1}) + O(\varepsilon L^2) + \sqrt{2E_{1,0}} \sum_{k \in \{N_j : N_j \leq M\}} \left| \cos \varphi_k - \cos \varphi_{k,\varepsilon}' \right|.$$
Proof of Lemma 8. The proof is by induction. We take \( \varepsilon \) to be so small that Equation (21) is satisfied. This is possible by Equation (27).

It is trivial to verify (a)-(f) for \( k = 0 \). So let \( 0 < l < N \wedge N', \) and suppose that (a)-(f) have been verified for all \( k < l \). We have three cases to consider:

**Case 1: \( l - 1 \) and \( l \notin \{N_j,\varepsilon\} \):**

In this case, verifying (a)-(f) for \( k = l \) is a relatively straightforward application of the machinery developed in Subsection 5.2.1 because for \( t_{l-1,\varepsilon} \leq t \leq t_{l,\varepsilon} \), \( y_\varepsilon(t) \) traces out the billiard orbit between \( x_{l-1,\varepsilon} \) and \( x_{l,\varepsilon} \) corresponding to free flight in the domain \( D_1(Q_0) \). We make only two remarks.

First, as long as \( \varepsilon \) is sufficiently small, it really is true that \( x_{l,\varepsilon} = y_\varepsilon(t_{l,\varepsilon}) \) corresponds to a true collision point on \( \partial D_1(Q_0) \). Indeed, if this were not the case, then it must be that \( Q_\varepsilon(t_{l,\varepsilon}) > Q_0 \), and \( y_\varepsilon(t_{l,\varepsilon}) \) would have to correspond to a collision with the side of the “tube” to the right of \( Q_0 \). But then \( x_{l,\varepsilon}'' = \begin{cases} Fx_{l-1,\varepsilon} & \text{ in } \Omega_{h_0} \\ \varepsilon & \text{ elsewhere} \end{cases} \) would correspond to a collision with an immobile piston at \( Q_0 \) and would satisfy \( d(x_k, x_{l,\varepsilon}'' \leq \text{const } \rho(\text{const }/\gamma)^k \leq \text{const } \rho(\text{const }/\gamma)^{l-1} = O(\gamma) \), using Equations (20) and (27). But \( x_k \notin N_\gamma(\partial Q_0) \), and so it follows that when the trajectory of \( y_\varepsilon(t) \) crosses the plane \( \{Q = Q_0\} \), it is at least a distance \( \sim \gamma \) away from the boundary of the face of the piston, and its velocity vector is pointed no closer than \( \sim \gamma \) to being parallel to the piston’s face. As \( Q_\varepsilon(t_{l,\varepsilon}) - Q_0 = O(\varepsilon L) = O(\gamma) \), and it is geometrically impossible (for small \( \varepsilon \)) to construct a right triangle whose sides \( s_1, s_2 \) satisfy \( |s_1| \sim \gamma, \ |s_2| \leq O(\varepsilon L) \), with the measure of the acute angle adjacent to \( s_1 \) being greater than \( \sim \gamma \), we have a contradiction. After crossing the plane \( \{Q = Q_0\} \), \( y_\varepsilon(t) \) must experience its next collision with the face of the piston, which violates the fact that \( l \notin \{N_j,\varepsilon\} \).

Second, \( t_{l,\varepsilon} - t_{l-1,\varepsilon} = \zeta x_{l-1,\varepsilon} + O(\varepsilon L) \), because \( v_{l,\varepsilon} = v_{l,\varepsilon} + O(\varepsilon L) \). See Equation (24). From Equation (22) \( \zeta x_{l-1} - \zeta x_{l-1,\varepsilon} \leq O((\rho/\gamma)(\text{const }/\gamma)^{l-1}) \). As \( t_{l} - t_{l-1} = \zeta x_{l-1} \) and \( \varepsilon L = O((\rho/\gamma)(\text{const }/\gamma)^{l-1}) \), we obtain (f).

**Case 2: There exists \( i \) such that \( l = N_i,\varepsilon \):**

For definiteness, we suppose that \( Q_\varepsilon(t_{l,\varepsilon}) \geq Q_0 \), so that the left gas particle collides with the piston to the right of \( Q_0 \). The case when \( Q_\varepsilon(t_{l,\varepsilon}) \leq Q_0 \) can be handled similarly.
We know that \( x_{l-1}, x_l, x_{l+1} \notin \mathcal{N}_\gamma(\partial \Omega_{h_0}) \cup \mathcal{N}_\gamma(F^{-1}\mathcal{N}_\gamma(\partial \Omega_{h_0})) \). Using the inductive hypothesis and Equation (20), we can define
\[
x''_{l,\epsilon} = F x'_{l-1,\epsilon}, \quad x''_{l+1,\epsilon} = F^2 x'_{l-1,\epsilon},
\]
and \( d(x_l, x''_{l,\epsilon}) \leq \text{const} \rho(\text{const}/\gamma)^l, d(x_{l+1}, x''_{l+1,\epsilon}) \leq \text{const} \rho(\text{const}/\gamma)^{l+1} \). In particular, \( x''_{l,\epsilon} \) and \( x''_{l+1,\epsilon} \) are both a distance \( \sim \gamma \) away from \( \partial \Omega_{h_0} \). Furthermore, when the left gas particle collides with the moving piston, it follows from Equation (12) that the difference between its angle of incidence and its angle of reflection is \( O(\epsilon) \). Referring to Figure 3, this means that \( \phi'_{l,\epsilon} = \phi''_{l,\epsilon} + O(\epsilon) \). Geometric arguments similar to the one given in Case 1 above show that the \( y_{\epsilon} \)-trajectory of the left gas particle has precisely one collision with the piston and no other collisions with the sides of the gas container when the gas particle traverses the region \( Q_0 \leq Q \leq Q_{\epsilon}(t'_{l,\epsilon}) \). Note that \( x'_{l,\epsilon} \) was defined to be the point in the collision cross-section \( \Omega_{h_0} \) corresponding to the return of the \( y_{\epsilon} \)-trajectory into the region \( Q \leq Q_0 \). See Figure 3. From this figure, it is also evident that \( d(r'_{l,\epsilon}, r''_{l,\epsilon}) \leq O(\epsilon L/\gamma) \). Thus \( d(x''_{l,\epsilon}, x'_{l,\epsilon}) = O(\epsilon L/\gamma) \), and this explains the choice of \( \rho(\epsilon) \) in Equation (26).

From the above discussion and the machinery of Subsection 5.2.1 (a)-(e) now follow readily for both \( k = l \) and \( k = l + 1 \). Furthermore, property (f) follows in much the same manner as it did in Case 1 above. However, one should note that \( t'_{l,\epsilon} - t'_{l-1,\epsilon} = \zeta x'_{l-1,\epsilon} + O(\epsilon L) + O(\epsilon L/\gamma) \) and \( t'_{l+1,\epsilon} - t'_{l,\epsilon} = \zeta x'_{l,\epsilon} + O(\epsilon L) + O(\epsilon L/\gamma) \), because of the extra distance \( O(\epsilon L/\gamma) \) that the gas particle travels to the right of \( Q_0 \). But \( \epsilon L/\gamma = O((\rho/\gamma)(\text{const}/\gamma)^{l-1}) \), and so property (f) follows.

**Case 3:** There exists \( i \) such that \( l - 1 = N'_{l,\epsilon} \):

As mentioned above, the inductive step in this case follows immediately from our analysis in Case 2.

\[\square\]

### 6 Generalization to a full proof of Theorem 1

It remains to generalize the proof in Sections 4 and 5 to the cases when \( n_1, n_2 \geq 1 \) and \( d = 3 \).

#### 6.1 Multiple gas particles on each side of the piston

When \( d = 2 \), but \( n_1, n_2 \geq 1 \), only minor modifications are necessary to generalize the proof above. As in Subsection 4.2, one defines a stopping time \( T_{\epsilon} \) satisfying \( P\left\{ T_{\epsilon} < T \wedge T_{\epsilon} \right\} = O(\epsilon) \) such that for \( 0 \leq t \leq T_{\epsilon}/\epsilon \), gas particles will only experience clean collisions with the piston.
Figure 3: An analysis of the divergences of orbits when $\varepsilon > 0$ and the left gas particle collides with the moving piston to the right of $Q_0$. Note that the dimensions are distorted for visual clarity, but that $\varepsilon L$ and $\varepsilon L/\gamma$ are both $o(\gamma)$ as $\varepsilon \to 0$. Furthermore, $\varphi''_{l,\varepsilon} \in (-\pi/2 + \gamma/2, \pi/2 - \gamma/2)$ and $\varphi'_{l,\varepsilon} = \varphi''_{l,\varepsilon} + O(\varepsilon)$, and so $r'_{l,\varepsilon} = r''_{l,\varepsilon} + O(\varepsilon L/\gamma)$. In particular, the $y_\varepsilon$-trajectory of the left gas particle has precisely one collision with the piston and no other collisions with the sides of the gas container when the gas particle traverses the region $Q_0 \leq Q \leq Q_\varepsilon(t'_{l,\varepsilon})$.
Next, define $H(z)$ by

$$H(z) = \begin{bmatrix} +2 \sum_{j=1}^{n_1} |v_{1,j}^+| \delta_{q_{1,j}}^+ = q - 2 \sum_{j=1}^{n_2} |v_{2,j}^+| \delta_{q_{2,j}}^+ = q \\ -2W |v_{1,j}^-| \delta_{q_{1,j}}^+ = q \\ +2W |v_{2,j}^-| \delta_{q_{2,j}}^+ = q \end{bmatrix}.$$ 

It follows that for $0 \leq t \leq \hat{T}_\varepsilon/\varepsilon$, $h_\varepsilon(t) - h_\varepsilon(0) = \mathcal{O}(\varepsilon) + \varepsilon \int_0^t H(z_\varepsilon(s))ds.$ From here, the rest of the proof follows the same steps made in Subsection 5.1. We note that at Step 3, we find that $H(z) - \hat{H}(h(z))$ divides into $n_1 + n_2$ pieces, each of which depends on only one gas particle when the piston is held fixed.

### 6.2 Three dimensions

The proof of Theorem 1 in $d = 3$ dimensions is essentially the same as the proof in two dimensions given above. The principal differences are due to differences in the geometry of billiards. We indicate the necessary modifications.

In analogy with Section 4.1, we briefly summarize the necessary facts for the billiard flows of the gas particles when $M = \infty$ and the slow variables are held fixed at a specific value $h \in \mathcal{V}$. As before, we will only consider the motions of one gas particle moving in $\mathcal{D}_1$. Thus we consider the billiard flow of a point particle moving inside the domain $\mathcal{D}_1$ at a constant speed $\sqrt{2E_1}$. Unless otherwise noted, we use the notation from Section 4.1.

The billiard flow takes place in the five-dimensional space $\mathcal{M}_1^1 = \{(q_1, v_1) \in TD_1 : q_1 \in \mathcal{D}_1, |v_1| = \sqrt{2E_1}\}/\sim$. Here the quotient means that when $q_1 \in \partial\mathcal{D}_1$, we identify velocity vectors pointing outside of $\mathcal{D}_1$ with those pointing inside $\mathcal{D}_1$ by reflecting orthogonally through the tangent plane to $\partial\mathcal{D}_1$ at $q_1$. The billiard flow preserves Liouville measure restricted to the energy surface. This measure has the density $d\mu = dq_1 dv_1/(8\pi E_1 |\mathcal{D}_1|)$. Here $dq_1$ represents volume on $\mathbb{R}^3$, and $dv_1$ represents area on $S^2_{\sqrt{2E_1}} = \{v_1 \in \mathbb{R}^3 : |v_1| = \sqrt{2E_1}\}$.

The collision cross-section $\Omega = \{(q_1, v_1) \in TD_1 : q_1 \in \partial\mathcal{D}_1, |v_1| = \sqrt{2E_1}\}/\sim$ is properly thought of as a fiber bundle, whose base consists of the smooth pieces of $\partial\mathcal{D}_1$ and whose fibers are the set of outgoing velocity vectors at $q_1 \in \partial\mathcal{D}_1$. This and other facts about higher-dimensional billiards, with emphasis on the dispersing case, can be found in [BCST03]. For our purposes, $\Omega$ can be parameterized as follows. We decompose $\partial\mathcal{D}_1$ into a finite union $\bigcup_j \Gamma_j$ of pieces, each of which is diffeomorphic via coordinates $r$ to a compact, connected subset of $\mathbb{R}^2$ with a piecewise $C^3$ boundary. The $\Gamma_j$ are nonoverlapping, except possibly on their boundaries. Next, if $(q_1, v_1) \in \Omega$ and $v_1$ is the outward going velocity vector, let $\hat{v} = v_1/|v_1|$. Then $\Omega$ can be parameterized by $\{x = (r, \hat{v})\}$. It follows that $\Omega$ it is diffeomorphic to $\bigcup_j \Gamma_j \times S^{2+}$, where $S^{2+}$ is the upper unit hemisphere, and by $\partial\Omega$ we mean the subset diffeomorphic to $(\bigcup_j \partial\Gamma_j \times S^{2+}) \bigcup (\bigcup_j \Gamma_j \times \partial S^{2+})$. If $x \in \Omega$, we let $\varphi \in [0, \pi/2]$ represent the angle between the outgoing velocity vector and the inward pointing normal vector $n$ to $\partial\mathcal{D}_1$, i.e. $\cos \varphi = \langle \hat{v}, n \rangle$. Note that we no longer allow $\varphi$ to take on negative values. The return map $F : \Omega \circlearrowleft$ preserves the projected probability.
measure \( \nu \), which has the density \( d\nu = \cos \varphi \hat{v} d\hat{v} dr/(\pi |\partial D_1|) \). Here \( |\partial D_1| \) is the area of \( \partial D_1 \).

\( F \) is an invertible, measure preserving transformation that is piecewise \( C^2 \). Because of our assumptions on \( D_1 \), the free flight times and the curvature of \( \partial D_1 \) are uniformly bounded. The bound on \( \|DF(x)\| \) given in Equation (6) is still true. A proof of this fact for general three-dimensional billiard tables with finite horizon does not seem to have made it into the literature, although see [BCST03] for the case of dispersing billiards. For completeness, we provide a sketch of a proof for general billiard tables in Appendix B.

We suppose that the billiard flow is ergodic, so that \( F \) is ergodic. Again, we induce \( F \) on the subspace \( \hat{\Omega} \) of \( \Omega \) corresponding to collisions with the (immobile) piston to obtain the induced map \( \hat{F} : \hat{\Omega} \cap \Omega \) that preserves the induced measure \( \hat{\nu} \).

The free flight time \( \zeta : \Omega \rightarrow \mathbb{R} \) again satisfies the derivative bound given in Equation (7). The generalized Santaló’s formula [Che97] yields

\[
E_{\nu} \zeta = \frac{4 |D_1|}{|v_1| |\partial D_1|}.
\]

If \( \hat{\zeta} : \hat{\Omega} \rightarrow \mathbb{R} \) is the free flight time between collisions with the piston, then it follows from Proposition [10] that

\[
E_{\hat{\nu}} \hat{\zeta} = \frac{4 |D_1|}{|v_1| \ell}.
\]

The expected value of \( |v_1^\perp| \) when the left gas particle collides with the (immobile) piston is given by

\[
E_{\hat{\nu}} |v_1^\perp| = E_{\hat{\nu}} \sqrt{2E_1 \cos \varphi} = \frac{\sqrt{2E_1}}{\pi} \int_{S^2} \cos^2 \varphi \hat{v}_1 d\hat{v}_1 = \sqrt{2E_1 \frac{2}{3}}.
\]

As a consequence, we obtain

**Lemma 9.** For \( \mu - a.e. \ y \in \mathcal{M}_1 \),

\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t |v_1^\perp(s)| \delta_{q^\perp(s)=Q} ds = \frac{E_1 \ell}{3 |D_1(Q)|}.
\]

Compare the proof of Lemma [2]

With these differences in mind, the rest of the proof of Theorem [1] when \( d = 3 \) proceeds in the same manner as indicated in Sections [4] and [6] above. The only notable difference occurs in the proof of the Gronwall-type inequality for billiards. Due to dimensional considerations, if one follows the proof of Lemma [3] for a three-dimensional billiard table, one finds that \( \nu \mathcal{N}_\gamma(F^{-1}\mathcal{N}_\gamma(\partial \Omega)) = \mathcal{O}(\gamma^{1-4\alpha} + \gamma^\alpha) \). The optimal value of \( \alpha \) is 1/5, and so \( \nu \mathcal{N}_\gamma(F^{-1}\mathcal{N}_\gamma(\partial \Omega)) = \mathcal{O}(\gamma^{1/5}) \) as \( \gamma \to 0 \). Hence \( \nu \mathcal{C}_{\gamma,\lambda} = \mathcal{O}(\lambda \gamma^{1/5}) \), which is a slightly worse estimate than the one in Equation (23). However, it is still sufficient for all of the arguments in Section [5.2.2] and this finishes the proof.
A Inducing maps on subspaces

Here we present some well-known facts on inducing measure preserving transformations on subspaces. Let \( F : (\Omega, \mathcal{B}, \nu) \circlearrowleft \) be an invertible, ergodic, measure preserving transformation of the probability space \( \Omega \) endowed with the \( \sigma \)-algebra \( \mathcal{B} \) and the probability measure \( \nu \). Let \( \hat{\Omega} \in \mathcal{B} \) satisfy \( 0 < \nu \hat{\Omega} < 1 \). Define \( R : \Omega \to \mathbb{N} \) to be the first return time to \( \hat{\Omega} \), i.e. \( R_\omega = \inf \{ n \in \mathbb{N} : F^n \omega \in \hat{\Omega} \} \). Then if \( \hat{\nu} := \nu(\cdot \cap \hat{\Omega})/\nu \hat{\Omega} \) and \( \hat{\mathcal{B}} := \{ B \cap \hat{\Omega} : B \in \mathcal{B} \} \), \( \hat{F} : (\hat{\Omega}, \hat{\mathcal{B}}, \hat{\nu}) \circlearrowleft \) defined by \( \hat{F}_\omega = F_{R_\omega} \omega \) is also an invertible, ergodic, measure preserving transformation \[\text{[Pet83]}\]. Furthermore \( E_{\hat{\nu}} R = \int_{\hat{\Omega}} R d\hat{\nu} = (\nu \hat{\Omega})^{-1} \).

This last fact is a consequence of the following proposition:

**Proposition 10.** If \( \zeta : \Omega \to \mathbb{R}_{\geq 0} \) is in \( L^1(\nu) \), then \( \hat{\zeta} = \sum_{n=0}^{R-1} \zeta \circ F^n \) is in \( L^1(\hat{\nu}) \), and

\[
E_{\hat{\nu}} \hat{\zeta} = \frac{1}{\nu \hat{\Omega}} E_\nu \zeta.
\]

**Proof.**

\[
\nu \hat{\Omega} \int_{\hat{\Omega}} \sum_{n=0}^{R-1} \zeta \circ F^n d\hat{\nu} = \int_{\hat{\Omega}} \sum_{n=0}^{R-1} \zeta \circ F^n d\nu = \sum_{k=1}^{\infty} \int_{\hat{\Omega} \cap \{ R = k \}} \sum_{n=0}^{k-1} \zeta \circ F^n d\nu
\]

\[
= \sum_{k=1}^{\infty} \sum_{n=0}^{k-1} \int_{F^n(\hat{\Omega} \cap \{ R = k \})} \zeta d\nu = \int_{\Omega} \zeta d\nu,
\]

because \( \{ F^n(\hat{\Omega} \cap \{ R = k \}) : 0 \leq n < k < \infty \} \) is a partition of \( \Omega \).

\( \square \)

B Derivative bounds for the billiard map in three dimensions

Returning to Section \[\text{6.2}\], we need to show that for a billiard table \( D_1 \subset \mathbb{R}^3 \) with a piecewise \( C^3 \) boundary and the free flight time uniformly bounded above, the billiard map \( F \) satisfies the following: If \( x_0 \notin \partial \Omega \cup F^{-1}(\partial \Omega) \), then

\[
\|DF(x_0)\| \leq \frac{\text{const}}{\cos \varphi(Fx_0)}.
\]

Fix \( x_0 = (r_0, \hat{v}_0) \in \Omega \), and let \( x_1 = (r_1, \hat{v}_1) = Fx_0 \). Let \( \Sigma \) be the plane that perpendicularly bisects the straight line between \( r_0 \) and \( r_1 \), and let \( r_{1/2} \) denote the point of intersection. We consider \( \Sigma \) as a “transparent” wall, so that in a neighborhood of \( x_0 \), we can write \( F = F_2 \circ F_1 \). Here, \( F_1 \) is like a billiard map in that it takes points (i.e. directed velocity vectors with a base) near \( x_0 \) to points with a base on \( \Sigma \) and a direction pointing near \( r_1 \). (\( F_1 \) would be a billiard map if we reflected the image velocity vectors orthogonally through \( \Sigma \).) \( F_2 \) is a billiard map
that takes points in the image of $F_1$ and maps them near $x_1$. Let $x_{1/2} = F_1x_0 = F_2^{-1}x_1$. Then $\|DF(x_0)\| \leq \|DF_1(x_0)\| \|DF_2(x_{1/2})\|$. It is easy to verify that $\|DF_1(x_0)\| \leq \text{const}$, with the constant depending only on the curvature of $\partial D_1$ at $r_0$. In other words, the constant may be chosen independent of $x_0$. Similarly, $\|DF_2^{-1}(x_1)\| \leq \text{const}$. Because billiard maps preserve a probability measure with a density proportional to $\cos \varphi$, $\det DF_2^{-1}(x_1) = \cos \varphi_1/\cos \varphi_{1/2} = \cos \varphi_1$. As $\Omega$ is 4-dimensional, it follows from Cramer’s Rule for the inversion of linear transformations that

$$\|DF_2(x_{1/2})\| \leq \frac{\text{const} \|DF_2^{-1}(x_1)\|^3}{\det DF_2^{-1}(x_1)} \leq \frac{\text{const}}{\cos \varphi_1},$$

and we are done.

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