Solving effectively some families of Thue Diophantine equations

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Abstract

Let $\alpha$ be an algebraic number of degree $d \geq 3$ and let $K$ be the algebraic number field $\mathbb{Q}(\alpha)$. When $\varepsilon$ is a unit of $K$ such that $\mathbb{Q}(\alpha \varepsilon) = K$, we consider the irreducible polynomial $f_\varepsilon(X) \in \mathbb{Z}[X]$ such that $f_\varepsilon(\alpha \varepsilon) = 0$. Let $F_\varepsilon(X,Y)$ be the irreducible binary form of degree $d$ associated to $f_\varepsilon(X)$ under the condition $F_\varepsilon(X,1) = f_\varepsilon(X)$. For each positive integer $m$, we want to exhibit an effective upper bound for the solutions $(x,y,\varepsilon)$ of the diophantine inequation $|F_\varepsilon(x,y)| \leq m$. We achieve this goal by restricting ourselves to a subset of units $\varepsilon$ which we prove to be sufficiently large as soon as the degree of $K$ is $\geq 4$.

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1 The conjecture and the main result

Let $\alpha$ be an algebraic number of degree $d \geq 3$ over $\mathbb{Q}$. We denote by $K$ the algebraic number field $\mathbb{Q}(\alpha)$, by $f \in \mathbb{Z}[X]$ the irreducible polynomial of $\alpha$ over $\mathbb{Z}$, by $\mathbb{Z} \times K$ the group of units of $K$ and by $r$ the rank of the abelian group $\mathbb{Z}_K$.

For any unit $\varepsilon \in \mathbb{Z}_K$ such that the degree $\delta = [\mathbb{Q}(\alpha \varepsilon) : \mathbb{Q}] \geq 3$, we denote by $f_\varepsilon(X) \in \mathbb{Z}[X]$ the irreducible polynomial of $\alpha \varepsilon$ over $\mathbb{Z}$ (uniquely defined upon requiring that the leading coefficient be $>0$) and by $F_\varepsilon$ the irreducible binary form defined by $F_\varepsilon(X,Y) = Y^\delta f_\varepsilon(X/Y) \in \mathbb{Z}[X,Y]$.

The purpose of this paper is to investigate the following conjecture.

**Conjecture 1.** There exists an effectively computable constant $\kappa_1 > 0$, depending only upon $\alpha$, such that, for any $m \geq 2$, each solution $(x,y,\varepsilon) \in \mathbb{Z}^2 \times \mathbb{Z}_K$ of the inequation $|F_\varepsilon(x,y)| \leq m$ with $xy \neq 0$ and $[\mathbb{Q}(\alpha \varepsilon) : \mathbb{Q}] \geq 3$ verifies

$$\max\{|x|, |y|, e^{h(\alpha \varepsilon)}\} \leq m^{\kappa_1}.$$ 

We noted $h$ the absolute logarithmic height (see [1] below).

To prove this conjecture, it suffices to restrict ourselves to units $\varepsilon$ of $K$ such that $\mathbb{Q}(\alpha \varepsilon) = K$: as a matter of fact, the field $K$ has but a finite number of subfields. An equivalent formulation of the conjecture [1] is then the following one: if $xy \neq 0$ and $\mathbb{Q}(\alpha \varepsilon) = K$, then

$$|N_{K/\mathbb{Q}}(x - \alpha \varepsilon y)| \geq \kappa_2 \max\{|x|, |y|, e^{h(\alpha \varepsilon)}\}^{\kappa_3}$$

with effectively computable positive constants $\kappa_2$ and $\kappa_3$ depending only upon $\alpha$. 

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The finiteness of the set of solutions \((x, y, \varepsilon) \in \mathbb{Z}^2 \times \mathbb{Z}_K^\times\) of the inequation \(|F_\varepsilon(x, y)| \leq m\) with \(xy \neq 0\) and \(|Q(\alpha \varepsilon) : \mathbb{Q}| \geq 3\) follows from Corollary 3.6 of [1] (which deals with Thue–Mahler equations, while in this paper we restrict ourselves to Thue equations). The proof in [1] rests on Schmidt’s subspace theorem; it allows to exhibit explicitly an upper bound for the number of solutions as a function of \(m, d\) and the height of \(\alpha\), but it does not allow to give an upper bound for the solutions. The particular case of the conjecture [1] in which the form \(F\) is of degree 3 and the rank of the unit group of the cubic field \(\mathbb{Q}(\alpha)\) is 1, was taken care of in [2]. In [3], we considered a slightly more general case, namely when the number of real embeddings of \(K\) into \(\mathbb{C}\) is 0 or 1, while restricting to units \(\varepsilon\) such that \(\mathbb{Q}(\alpha \varepsilon) = K\). In this paper, we prove that the conjecture is true at least for a subset \(\tilde{E}_\nu(\alpha)\) of units, the definition of which is given in the following.

Denote by \(\Phi = \{\sigma_1, \ldots, \sigma_d\}\) the set of embeddings of \(K\) into \(\mathbb{C}\) and by \([\gamma]\) the house of an algebraic number \(\gamma\), defined to be the maximum of the moduli of the Galois conjugates of \(\gamma\) in \(\mathbb{C}\). In symbols, for \(\gamma \in K\),

\[
[\gamma] = \max_{1 \leq i \leq d} |\sigma_i(\gamma)|.
\]

The absolute logarithmic height is noted \(h\) and involves the Mahler measure \(M\):

\[
h(\alpha) = \frac{1}{d} \log M(\alpha) \quad \text{with} \quad M(\alpha) = a_0 \prod_{1 \leq i \leq d} \max\{1, |\sigma_i(\alpha)|\},
\]

\(a_0\) being the leading coefficient of the irreducible polynomial of \(\alpha\) over \(\mathbb{Z}\).

The set

\[
\mathcal{E}(\alpha) = \{\varepsilon \in \mathbb{Z}_K^\times \mid \mathbb{Q}(\alpha \varepsilon) = K\}
\]

depends only upon \(\alpha\); (we have supposed \(\mathbb{Q}(\alpha) = K\)). When \(\nu\) is a real number in the interval \([0, 1]\[, we denote by \(\tilde{\mathcal{E}}_{\nu}(\alpha)\) the set of units \(\varepsilon \in \mathcal{E}(\alpha)\) for which there exist two distinct elements \(\varphi_1\) and \(\varphi_2\) of \(\Phi\) such that

\[
|\varphi_1(\alpha \varepsilon)| = [\alpha \varepsilon]^{\nu} \quad \text{and} \quad |\varphi_2(\alpha \varepsilon)| \geq [\alpha \varepsilon]^{\nu}.
\]

We also denote by \(\tilde{\mathcal{E}}_{\nu}(\alpha)\) the set of units \(\varepsilon \in \mathcal{E}_{\nu}(\alpha)\) such that \(\varepsilon^{-1} \in \mathcal{E}_{\nu}(1/\alpha)\).

Let us state our main result.

**Theorem 1.** Let \(\nu \in ]0, 1[\). There exist two effectively computable positive constants \(\kappa_4, \kappa_5\), depending only upon \(\alpha\) and \(\nu\), which have the following properties:

(a) For any \(m \geq 2\), each solution \((x, y, \varepsilon) \in \mathbb{Z}^2 \times \mathcal{E}_{\nu}(\alpha)\) of the inequation \(|F_\varepsilon(x, y)| \leq m\) with \(0 < |x| \leq |y|\) satisfies

\[
\max\{|y|, e^{h(\alpha \varepsilon)}\} \leq m^{\kappa_4}.\]
(b) For any $m \geq 2$, each solution $(x, y, \varepsilon) \in \mathbb{Z}^2 \times \mathcal{E}_\nu^{\alpha}$ of the inequation $|F_\varepsilon(x, y)| \leq m$ with $xy \neq 0$ satisfies

$$\max\{|x|, |y|, e^{h(\alpha \varepsilon)}\} \leq m^\nu.$$ 

Proposition 1, stated below and proved in [13], means that $\mathcal{E}_\nu^{\alpha}$ for $d \geq 4$ has a positive density in the set $\mathcal{E}^{\alpha}$. Since the case of a non-totally real cubic field has been taken care of in [2], it is only in the case of a totally real cubic field that our main result provides no effective bound for an infinite family of Thue equations.

When $N$ is a real positive number and $\mathcal{F}$ is a subset of $\mathbb{Z}_K^\times$, we define

$$\mathcal{F}(N) = \{\varepsilon \in \mathcal{F} \mid \sqrt[\nu]{|\varepsilon|} \leq N\} \quad \text{and} \quad |\mathcal{F}(N)| = \text{Card}\mathcal{F}(N),$$

so

$$\mathcal{F}(N) = \mathbb{Z}_K^\times(N) \cap \mathcal{F}.$$ 

**Proposition 1.** (a) The limit

$$\lim_{N \to \infty} \frac{|\mathbb{Z}_K^\times(N)|}{(\log N)^r}$$

exists and is positive.

(b) One has

$$\lim_{N \to \infty} \frac{|\mathcal{E}^\alpha(N)|}{|\mathbb{Z}_K^\times(N)|} > 0.$$ 

(c) For $0 < \nu < 1/2$, one has

$$\lim_{N \to \infty} \frac{|\mathcal{E}_\nu^{\alpha}(N)|}{(\log N)^r} > 0.$$ 

(d) For $0 < \nu < 1$ and $d \geq 4$, one has

$$\lim_{N \to \infty} \frac{|\mathcal{E}_\nu^{\alpha}(N)|}{(\log N)^r} > 0.$$ 

Let us write the irreducible polynomial $f$ of $\alpha$ over $\mathbb{Z}$ as

$$f(X) = a_0X^d + a_1X^{d-1} + \cdots + a_{d-1}X + a_d \in \mathbb{Z}[X],$$

whereupon

$$f(X) = a_0 \prod_{i=1}^{d} (X - \sigma_i(\alpha))$$

and its associated irreducible binary form $F$ is

$$F(X, Y) = Y^df(X/Y) = a_0X^d + a_1X^{d-1}Y + \cdots + a_{d-1}XY^{d-1} + a_dY^d.$$
For $\varepsilon \in \mathbb{Z}_K^\times$ verifying $Q(\alpha \varepsilon) = K$, we have

$$F_\varepsilon(X, Y) = a_0 \prod_{i=1}^{d} (X - \sigma_i(\alpha \varepsilon) Y) \in \mathbb{Z}[X, Y].$$

Given $(x, y, \varepsilon) \in \mathbb{Z}^2 \times \mathbb{Z}_K^\times$, we define

$$\beta = x - \alpha \varepsilon y.$$ 

Therefore

$$F_\varepsilon(x, y) = a_0 \sigma_1(\beta) \cdots \sigma_d(\beta). \quad (2)$$

Dirichlet’s unit theorem provides the existence of units $\epsilon_1, \ldots, \epsilon_r$ in $K$, the classes modulo $K_{\text{tors}}^\times$ of which form a basis of the free abelian group $\mathbb{Z}_K^\times / K_{\text{tors}}^\times$. Effective versions (see for instance [4]) provide bounds for the heights of these units as a function of $h(\alpha)$ and $d$.

**Steps of the proof.** In §2 we quote useful lemmas, the most powerful being a proposition of [5] involving transcendence methods and giving lower bounds for the distance between 1 and a product of powers of algebraic numbers. Each time we will use that proposition, we will write that we are using a diophantine argument. After introducing some parameters $A$ and $B$ in §3 we eliminate $x$ and $y$ between the equations $\varphi(\beta) = x - \varphi(\alpha \varepsilon)y$, $\varphi \in \Phi$. In §4 we introduce four privileged embeddings, denoted by $\sigma_a$, $\sigma_b$, $\tau_a$, $\tau_b$, and four useful sets of embeddings $\Sigma_a(\nu)$, $\Sigma_b(\nu)$, $T_a(\nu)$, $T_b(\nu)$, depending on a parameter $\nu$. Applying some results from [3], we show in §6 that we may suppose $A$ and $B$ sufficiently large, namely $\geq \kappa \log m$, via a diophantine argument. In §7 and in §8 we prove that $A$ is bounded from above by $\kappa B$ and that $B$ is bounded from above by $\kappa' A$. In §9 we prove that $\tau_\beta$ is unique. In §10 we give an upper bound for $|\tau_\beta(\alpha \varepsilon)|$. In §11 we deduce that $\sigma_a$ is unique. In §12 we complete the proof of Theorem 1. In §13 we give the proof of Proposition 1.

## 2 Tools

This chapter contains the auxiliary lemmas we shall need. The details of the proofs are in [3]. We start with an equivalence of norms (Lemma 1). Then we state Lemma 2 which appeared as Lemma 2 of [2] and also as Lemma 6 of [3]. Next we quote Proposition 2 (which is Corollary 9 of [3]) involving a lower bound of a linear form in logarithms of algebraic numbers.

### 2.1 Equivalence of norms

Let $K$ be an algebraic number field of degree $d$ over $\mathbb{Q}$. Let us recall that $\epsilon_1, \ldots, \epsilon_r$ denote the elements of a basis of the unit group of $K$ modulo $K_{\text{tors}}^\times$ and that we are supposing $r \geq 1$. 
There exists an effectively computable positive constant \( \kappa_6 \), depending only upon \( \epsilon_1, \ldots, \epsilon_r \), such that, if \( c_1, \ldots, c_r \) are rational integers and if we let

\[
C = \max\{|c_1|, \ldots, |c_r|\}, \quad \gamma = \epsilon_1^{c_1} \cdots \epsilon_r^{c_r},
\]

then

\[
e^{-\kappa_6 C} \leq |\varphi(\gamma)| \leq e^{\kappa_6 C}
\]

for each embedding \( \varphi \) of \( K \) into \( \mathbb{C} \).

The following lemma (see Lemma 5 of [3]) shows that the two inequalities of (3) are optimal.

**Lemma 1.** There exists an effectively computable positive constant \( \kappa_7 \), which depends only upon \( \epsilon_1, \ldots, \epsilon_r \), with the following property. If \( c_1, \ldots, c_r \) are rational integers and if we let \( C = \max\{|c_1|, \ldots, |c_r|\} \), \( \gamma = \epsilon_1^{c_1} \cdots \epsilon_r^{c_r} \), then there exist two embeddings \( \sigma \) and \( \tau \) of \( K \) into \( \mathbb{C} \) such that

\[
|\sigma(\gamma)| \geq e^{\kappa_7 C} \quad \text{and} \quad |\tau(\gamma)| \leq e^{-\kappa_7 C}.
\]

**Remark.** Under the hypotheses of Lemma 1, if \( \gamma_0 \) is a nonzero element of \( K \) and if we let \( \gamma_1 = \gamma_0 \gamma \), one deduces

\[
e^{-\kappa_6 C - dh(\gamma_0)} \leq \min_{\varphi \in \Phi} |\varphi(\gamma_1)| \leq e^{-\kappa_6 C + dh(\gamma_0)}
\]

and

\[
e^{\kappa_7 C - dh(\gamma_0)} \leq \max_{\varphi \in \Phi} |\varphi(\gamma_1)| \leq e^{\kappa_7 C + dh(\gamma_0)}.
\]

### 2.2 On the norm

The following lemma is a consequence of Lemma A.15 of [4] (see also Lemma 2 of [2] and Lemma 6 of [3]).

**Lemma 2.** Let \( K \) be a field of algebraic numbers of degree \( d \) over \( \mathbb{Q} \) with regulator \( R \). There exists an effectively computable positive constant \( \kappa_8 \), depending only on \( d \) and \( R \), such that, if \( \gamma \) is an element of \( \mathbb{Z}_K \), the norm of which has an absolute value \( \leq m \) with \( m \geq 2 \), then there exists a unit \( \varepsilon \in \mathbb{Z}_K^\times \) such that

\[
\max_{1 \leq j \leq d} |\sigma_j(\varepsilon \gamma)| \leq m^\Theta
\]

### 2.3 Diophantine tool

We will use the particular case of Theorem 9.1 of [5] (stated in Corollary 9 of [3]). Such estimates (known as *lower bounds for linear forms in logarithms of algebraic numbers*) first occurred in the work of A.O. Gel’fond, then in the work of A. Baker - a historical survey is given in [3].
Proposition 2. Let $s$ and $D$ two positive integers. There exists an effectively computable positive constant $\kappa_9$, depending only upon $s$ and $D$, with the following property. Let $\gamma_1, \ldots, \gamma_s$ be nonzero algebraic numbers generating a number field of degree $\leq D$. Let $c_1, \ldots, c_s$ be rational integers and let $H_1, \ldots, H_s$ be real numbers $\geq 1$ satisfying $H_j \leq H_s$ for $1 \leq j \leq s$ and
\[ H_i \geq h(\gamma_i) \quad (1 \leq i \leq s). \]
Let $C$ be a real number subject to
\[ C \geq 2, \quad C \geq \max_{1 \leq j \leq s} \left\{ \frac{H_j}{H_s} |c_j| \right\}. \]
Suppose also $\gamma_1^{c_1} \cdots \gamma_s^{c_s} \neq 1$. Then
\[ |\gamma_1^{c_1} \cdots \gamma_s^{c_s} - 1| > \exp\{-\kappa_9 H_1 \cdots H_s \log C\}. \]

3 Introduction of the parameters $\tilde{A}$, $A$, $\tilde{B}$, $B$

From now on, we fix a solution $(x, y, \varepsilon) \in \mathbb{Z}^2 \times \mathbb{Z}_K^\times$ of the Thue inequation $|F_\varepsilon(x, y)| \leq m$ with $xy \neq 0$ and $\mathbb{Q}(\alpha \varepsilon) = K$. Up to §11 inclusively, we suppose
\[ 1 \leq |x| \leq |y|. \]
Let
\[ \tilde{A} = \max\{1, h(\alpha \varepsilon)\}. \]
Write
\[ \varepsilon = \zeta \xi_1^{a_1} \cdots \xi_r^{a_r} \]
with $\zeta \in K^\times_{\text{tors}}$ and $a_i \in \mathbb{Z}$ for $1 \leq i \leq r$ and define
\[ A = \max\{1, |a_1|, \ldots, |a_r|\}. \]
Thanks to (3) and to Lemma 1, we have
\[ \kappa_{10} A \leq \tilde{A} \leq \kappa_{11} A. \]
Next define
\[ \tilde{B} = \max\{1, h(\beta)\}. \]
Since $|F_\varepsilon(x, y)| \leq m$, it follows from (4) and (2) that there exists $\rho \in \mathbb{Z}_K$ verifying
\[ h(\rho) \leq \kappa_{12} \log m \quad (5) \]
with $\kappa_{12} > 0$ such that $\eta = \beta / \rho$ is a unit of $\mathbb{Z}_K$ of the form
\[ \eta = \xi_1^{b_1} \cdots \xi_r^{b_r} \]
with rational integers $b_1, \ldots, b_r$; define
\[ B = \max\{1, |b_1|, |b_2|, \ldots, |b_r|\}. \]
Because of the relation $\beta = \rho \eta$, we deduce from (3),
\[ \hat{B} \leq \kappa_{13} (B + \log m) \]

and from Lemma [1]
\[ B \leq \kappa_{14} (\hat{B} + \log m). \]

Since $xy \neq 0$ and $Q(\alpha \varepsilon) = K$, we deduce that for $\varphi$ and $\sigma$ in $\Phi$, we have
\[ \varphi = \sigma \iff \varphi(\alpha \varepsilon) = \sigma(\alpha \varepsilon) \iff \varphi(\beta) = \sigma(\beta) = \sigma(\beta) \varphi(\alpha \varepsilon). \]

Here is an example of application of Proposition 2. The following lemma will be used in the proof of Lemma 9.

**Lemma 3.** There exists an effectively computable positive constant $\kappa_{15}$ with the following property. Let $\varphi_1, \varphi_2, \varphi_3, \varphi_4$ be elements of $\Phi$ with $\varphi_1(\alpha \varepsilon) \varphi_2(\beta) \neq \varphi_3(\alpha \varepsilon) \varphi_4(\beta)$. Then
\[ \left| \frac{\varphi_1(\alpha \varepsilon) \varphi_2(\beta)}{\varphi_3(\alpha \varepsilon) \varphi_4(\beta)} - 1 \right| \geq \exp \left\{ -\kappa_{15} (\log m) \log \left( 2 + \frac{A + B}{\log m} \right) \right\}. \]

**Proof.** Write
\[ \frac{\varphi_1(\alpha \varepsilon) \varphi_2(\beta)}{\varphi_3(\alpha \varepsilon) \varphi_4(\beta)} \]
as $\gamma_1^{c_1} \cdots \gamma_s^{c_s}$ with $s = 2r + 1$, and
\[ \gamma_j = \frac{\varphi_1(\epsilon_j)}{\varphi_3(\epsilon_j)}, \quad c_j = a_j, \quad \gamma_{r+j} = \frac{\varphi_2(\epsilon_j)}{\varphi_4(\epsilon_j)}, \quad c_{r+j} = b_j \quad (j = 1, \ldots, r), \]
\[ \gamma_s = \frac{\varphi_1(\alpha \zeta) \varphi_2(\rho)}{\varphi_3(\alpha \zeta) \varphi_4(\rho)}, \quad c_s = 1. \]
We have $h(\gamma_s) \leq \kappa_{16} \log m$, thanks to the upper bound (5) for the height of $\rho$. Write
\[ H_1 = \cdots = H_{2r} = \kappa_{17}, \quad H_s = \kappa_{18} \log m, \quad C = 2 + \frac{A + B}{\log m} \]
The hypothesis
\[ \max_{1 \leq j \leq s} \frac{H_j}{H_s} |c_j| \leq C \]
of Proposition 2 is satisfied. Lemma [3] follows from this proposition. \[\square\]
4 Elimination

4.1 Expressions of $x$ and $y$ in terms of $\alpha \varepsilon$ and $\beta$

Let $\varphi_1, \varphi_2$ be two distinct elements of $\Phi$, namely two distinct embeddings of $K$ into $\mathbb{C}$. We eliminate $x$ (resp. $y$) between the two equations

$$\varphi_1(\beta) = x - \varphi_1(\alpha \varepsilon)y \quad \text{and} \quad \varphi_2(\beta) = x - \varphi_2(\alpha \varepsilon)y,$$

to obtain

$$y = \frac{\varphi_1(\beta) - \varphi_2(\beta)}{\varphi_2(\alpha \varepsilon) - \varphi_1(\alpha \varepsilon)}, \quad x = \frac{\varphi_2(\alpha \varepsilon)\varphi_1(\beta) - \varphi_1(\alpha \varepsilon)\varphi_2(\beta)}{\varphi_2(\alpha \varepsilon) - \varphi_1(\alpha \varepsilon)}. \quad (6)$$

4.2 The unit equation

Let $\varphi_1, \varphi_2, \varphi_3$ be embeddings of $K$ into $\mathbb{C}$. Let

$$u_i = \varphi_i(\alpha \varepsilon), \quad v_i = \varphi_i(\beta) \quad (i = 1, 2, 3).$$

We eliminate $x$ and $y$ between the three equations

$$\begin{cases}
\varphi_1(\beta) = x - \varphi_1(\alpha \varepsilon)y \\
\varphi_2(\beta) = x - \varphi_2(\alpha \varepsilon)y \\
\varphi_3(\beta) = x - \varphi_3(\alpha \varepsilon)y
\end{cases}$$

by writing that the determinant of this nonhomogeneous system of three equations in two unknowns, which is equal to

$$\begin{vmatrix}
1 & \varphi_1(\alpha \varepsilon) & \varphi_1(\beta) \\
1 & \varphi_2(\alpha \varepsilon) & \varphi_2(\beta) \\
1 & \varphi_3(\alpha \varepsilon) & \varphi_3(\beta)
\end{vmatrix} = \begin{vmatrix} 1 & u_1 & v_1 \\
1 & u_2 & v_2 \\
1 & u_3 & v_3 \end{vmatrix},$$

is 0, and this leads to

$$u_1v_2 - u_1v_3 + u_2v_3 - u_2v_1 + u_3v_1 - u_3v_2 = 0. \quad (7)$$

5 Four sets of privileged embeddings

We denote by $\sigma_a$ (resp. $\sigma_b$) an embedding of $K$ into $\mathbb{C}$ such that $|\sigma_a(\alpha \varepsilon)|$ (resp. $|\sigma_b(\beta)|$) be maximal among the elements $|\varphi(\alpha \varepsilon)|$ (resp. among the elements $|\varphi(\beta)|$) for $\varphi \in \Phi$. Therefore

$$|\sigma_a(\alpha \varepsilon)| = |\overline{\alpha \varepsilon}| \quad \text{and} \quad |\sigma_b(\beta)| = |\overline{\beta}|.$$ 

Next we denote by $\tau_a$ (resp. $\tau_b$) an embedding of $K$ into $\mathbb{C}$ such that $|\tau_a(\alpha \varepsilon)|$ (resp. $|\tau_b(\beta)|$) be minimal among the elements $|\varphi(\alpha \varepsilon)|$ (resp. among the elements $|\varphi(\beta)|$) for $\varphi \in \Phi$. Therefore

$$|\tau_a((\alpha \varepsilon)^{-1})| = \frac{1}{|\overline{\alpha \varepsilon}|} \quad \text{and} \quad |\tau_b(\beta^{-1})| = \frac{1}{|\overline{\beta}|}.$$
Since there are at least three distinct embeddings of $K$ into $C$, we may suppose $\tau_b \neq \sigma_b$ and $\tau_a \neq \sigma_a$. By definition of $\sigma_a$, $\sigma_b$, $\tau_a$ and $\tau_b$, for any $\varphi \in \Phi$ we have

$$|\tau_a(\alpha \varepsilon)| \leq |\varphi(\alpha \varepsilon)| \leq |\sigma_a(\alpha \varepsilon)| \quad \text{and} \quad |\tau_b(\beta)| \leq |\varphi(\beta)| \leq |\sigma_b(\beta)|.$$  

Let $\nu$ be a real number in the open interval $]0, 1[$. Let us denote by $\Sigma_a(\nu)$, $\Sigma_b(\nu)$, $T_a(\nu)$, $T_b(\nu)$ the sets of embeddings of $K$ into $C$ defined by the following conditions:

$$\begin{cases}
\Sigma_a(\nu) = \{ \varphi \in \Phi \mid |\sigma_a(\alpha \varepsilon)|^\nu \leq |\varphi(\alpha \varepsilon)| \leq |\sigma_a(\alpha \varepsilon)| \}, \\
\Sigma_b(\nu) = \{ \varphi \in \Phi \mid |\sigma_b(\beta)|^\nu \leq |\varphi(\beta)| \leq |\sigma_b(\beta)| \}, \\
T_a(\nu) = \{ \varphi \in \Phi \mid |\tau_a(\alpha \varepsilon)| \leq |\varphi(\alpha \varepsilon)| \leq |\tau_a(\alpha \varepsilon)|^\nu \}, \\
T_b(\nu) = \{ \varphi \in \Phi \mid |\tau_b(\beta)| \leq |\varphi(\beta)| \leq |\tau_b(\beta)|^\nu \}.
\end{cases}$$

Of course, we have

$$\sigma_a \in \Sigma_a(\nu), \quad \sigma_b \in \Sigma_b(\nu), \quad \tau_a \in T_a(\nu), \quad \tau_b \in T_b(\nu).$$

We will see in [3] that we have

$$|\sigma_a(\alpha \varepsilon)| > 2, \quad |\sigma_b(\beta)| > 2, \quad |\tau_a(\alpha \varepsilon)| < \frac{1}{2}, \quad |\tau_b(\beta)| < \frac{1}{2},$$

from which we will deduce

$$T_a(\nu) \cap \Sigma_a(\nu) = \emptyset, \quad T_b(\nu) \cap \Sigma_b(\nu) = \emptyset.$$

### 6 Lower bounds for $A$ and $B$

Thanks to Lemma 15 in §7.2 of [3] and to Lemma 17 in §7.3 of [3], we may suppose, without loss of generality, that $A$ and $B$ have a lower bound given by $\kappa_{18} \log m$ for a sufficiently large effectively computable positive constant $\kappa_{18}$ depending only on $\alpha$:

$$A \geq \kappa_{18} \log m, \quad B \geq \kappa_{18} \log m. \quad (8)$$

In particular, we deduce that $A$, $B$, $|\sigma_a(\alpha \varepsilon)|$ and $|\sigma_b(\beta)|$ are sufficiently large and also that $|\tau_a(\alpha \varepsilon)|$ and $|\tau_b(\beta)|$ are sufficiently small.

By using Lemma 1 with the estimates [3], we deduce that there exist some effectively computable positive constants $\kappa_{19}$ et $\kappa_{20}$, depending only on $\alpha$, such that

$$\begin{cases}
e^{\kappa_{19}A} & \leq |\sigma_a(\alpha \varepsilon)| & \leq e^{\kappa_{19}A}, \\
e^{\kappa_{20}B} & \leq |\sigma_b(\beta)| & \leq e^{\kappa_{20}B}, \\
e^{-\kappa_{19}A} & \leq |\tau_a(\alpha \varepsilon)| & \leq e^{-\kappa_{19}A}, \\
e^{-\kappa_{20}B} & \leq |\tau_b(\beta)| & \leq e^{-\kappa_{20}B}.
\end{cases} \quad (9)$$
Therefore we have
\[
\begin{align*}
    e^{κA} & \leq |φ(αε)| \leq e^{κA} & \text{for } φ \in Σ_a(ν), \\
    e^{κB} & \leq |φ(β)| \leq e^{κB} & \text{for } φ \in Σ_b(ν), \\
    e^{-κA} & \leq |φ(αε)| \leq e^{-κA} & \text{for } φ \in T_a(ν), \\
    e^{-κB} & \leq |φ(β)| \leq e^{-κB} & \text{for } φ \in T_b(ν).
\end{align*}
\]

7 Upper bounds for \( A, |x|, |y| \) in terms of \( B \)

From the relation (6) we deduce in an elementary way the following upper bounds. Recall the assumption \( 1 ≤ |x| ≤ |y| \) made in §3.

\textbf{Lemma 4.} One has
\[ A ≤ κ_{21} B \quad \text{and} \quad |x| ≤ |y| ≤ e^{κ_{23}B}. \]

\textit{Proof.} There is no restriction in supposing that \( A \) and \( B \) are larger than a constant times \( \log m \). From the inequality \( |σ_a(αε)| ≥ 2|τ_a(αε)| \), we deduce
\[ |σ_a(αε) − τ_a(αε)| ≥ \frac{1}{2} |σ_a(αε)|. \]
Then we use (6) with \( φ_2 = σ_a \) and \( φ_1 = τ_a \):
\[ y(σ_a(αε) − τ_a(αε)) = τ_a(β) − σ_a(β). \]
From the upper bound
\[ |σ_a(β) − τ_a(β)| \leq 2|σ_b(β)|, \]
we deduce
\[ |yσ_a(αε)| \leq 4|σ_b(β)|. \] (10)
With the help of (9), one obtains the inequalities
\[ e^{κA} \leq |σ_a(αε)| \leq |yσ_a(αε)| \leq 4|σ_b(β)| \leq 4e^{κB} \]
which imply \( A ≤ κ_{21} B \). From (10) and because \( |σ_a(αε)| > 2 \), we get the upper bound \( \log |y| ≤ κ_{23}B \). We can conclude the proof by using the hypothesis \( |x| ≤ |y| \) (cf. §3).

8 Upper bound of \( B \) in terms of \( A \)

We use the unit equation (7) of §4.2 with three different embeddings \( τ_b, σ_b \) and \( φ \), where \( φ \) is an element of \( Φ \) different from \( τ_b \) and \( σ_b \).

\textbf{Lemma 5.} One has
\[ B ≤ κ_{23} A. \]
Proof. Let $\varphi \in \Phi$ with $\varphi \neq \sigma_b$ and $\varphi \neq \tau_b$. We take advantage of the relation (7) with $\varphi_1 = \sigma_b$, $\varphi_2 = \varphi$, $\varphi_3 = \tau_b$, written in the form

$$
\varphi(\beta) (\sigma_b(\alpha \varepsilon) - \tau_b(\alpha \varepsilon)) - \sigma_b(\beta) (\varphi(\alpha \varepsilon) - \tau_b(\alpha \varepsilon)) + \tau_b(\beta) (\varphi(\alpha \varepsilon) - \sigma_b(\alpha \varepsilon)) = 0
$$

and we divide by $\sigma_b(\beta) (\varphi(\alpha \varepsilon) - \tau_b(\alpha \varepsilon))$ (which is different from 0):

$$
\frac{\varphi(\beta)}{\sigma_b(\beta)} \cdot \frac{\sigma_b(\alpha \varepsilon) - \tau_b(\alpha \varepsilon)}{\varphi(\alpha \varepsilon) - \tau_b(\alpha \varepsilon)} - 1 = \frac{\tau_b(\beta)}{\sigma_b(\beta)} \cdot \frac{\varphi(\alpha \varepsilon) - \sigma_b(\alpha \varepsilon)}{\varphi(\alpha \varepsilon) - \tau_b(\alpha \varepsilon)}.
$$

(11)

The right side of (11) is different from 0. Let us show that an upper bound of its modulus is given by $e^{\kappa_{28} A} e^{-\kappa_{29} B}$.

As a matter of fact, on the one hand, from (9) we have

$$
|\tau_b(\beta)| \leq e^{-\kappa_{20} B}, \quad \text{and} \quad |\sigma_b(\beta)| \geq e^{\kappa_{20} B};
$$

on the other hand, the height of the number

$$
\delta = \frac{\varphi(\alpha \varepsilon) - \sigma_b(\alpha \varepsilon)}{\varphi(\alpha \varepsilon) - \tau_b(\alpha \varepsilon)}
$$

is bounded from above by $e^{\kappa_{26} A}$. From this upper bound for the height we derive the upper bound for the modulus $|\delta|$, namely $|\delta| \leq e^{\kappa_{27} A}$, hence

$$
\left| \frac{\tau_b(\beta)}{\sigma_b(\beta)} \cdot \frac{\sigma_b(\alpha \varepsilon) - \tau_b(\alpha \varepsilon)}{\varphi(\alpha \varepsilon) - \tau_b(\alpha \varepsilon)} \right| \leq \frac{e^{\kappa_{27} A}}{e^{2\kappa_{26} B}}.
$$

Let us write the term

$$
\frac{\varphi(\beta)}{\sigma_b(\beta)} \cdot \frac{\sigma_b(\alpha \varepsilon) - \tau_b(\alpha \varepsilon)}{\varphi(\alpha \varepsilon) - \tau_b(\alpha \varepsilon)} - 1
$$

appearing on the left side of (11) in the form $\gamma_1^{c_1} \cdots \gamma_s^{c_s}$ with $s = r + 1$ and

$$
\gamma_j = \frac{\varphi(\epsilon_j)}{\sigma_b(\epsilon_j)}, \quad c_j = b_j \quad (j = 1, \ldots, r),
$$

$$
\gamma_s = \frac{\sigma_b(\alpha \varepsilon) - \tau_b(\alpha \varepsilon)}{\varphi(\alpha \varepsilon) - \tau_b(\alpha \varepsilon)} \cdot \frac{\varphi(\rho)}{\sigma_b(\rho)} , \quad c_s = 1.
$$

Thanks to (5) and (8), we have

$$
h(\gamma_s) \leq \kappa_{28} A + 2h(\varrho) \leq \kappa_{29} A.
$$

Define

$$
H_1 = \cdots = H_r = \kappa_{30}, \quad H_s = \kappa_{28} A, \quad C = 2 + \frac{B}{\kappa_{31} A}.
$$

We check that the hypothesis

$$
\max_{1 \leq j \leq s} \frac{H_j}{H_s} |c_j| \leq C
$$

is satisfied.
of Proposition 2 is satisfied. We deduce from this proposition that a lower bound for the modulus of the left member of (11) is given by \( \exp\{-\kappa_{32}H_s \log C\} \). Consequently,

\[ \kappa_{35}B \leq \kappa_{24}A + \kappa_{32}H_s \log C. \]

Hence \( C \leq \kappa_{33} \log C \), which allows to conclude that \( C \leq \kappa_{34} \), and this secures the inequality \( B \leq \kappa_{35}A \) we wanted to prove.

9 Unicity of \( \tau_b \)

We want to prove that no other embedding plays the same role as \( \tau_b \). This will be achieved by proving the next lemma, which exhibits a contradiction to (8).

Lemma 6. Suppose \( T_b(\nu) \neq \{\tau_b\} \). Then \( B \leq \kappa_{35} \log m \).

Proof. Let \( \varphi \in T_b(\nu) \). Suppose \( \varphi \neq \tau_b \). Let us use (6) with \( \varphi_1 = \varphi, \varphi_2 = \tau_b \), in the form

\[ \varphi(\alpha \varepsilon) \tau_b(\alpha \varepsilon) - 1 = \frac{\tau_b(\beta) - \varphi(\beta)}{y\tau_b(\alpha \varepsilon)}. \]

From the inequality

\[ |x - \tau_b(\alpha \varepsilon)y| = |\tau_b(\beta)| < \frac{1}{2}, \]

obtained from (9), we deduce

\[ |\tau_b(\alpha \varepsilon)y| \geq |x| - \frac{1}{2} \geq \frac{1}{2}. \]

Since \( |\tau_b(\beta)| \leq |\varphi(\beta)| \), we also have

\[ |\varphi(\alpha \varepsilon) - \tau_b(\alpha \varepsilon)| = \frac{1}{y}|\varphi(\beta) - \tau_b(\beta)| \leq \frac{2|\varphi(\beta)|}{|y|}. \]

Consequently,

\[ \left| \frac{\varphi(\alpha \varepsilon)}{\tau_b(\alpha \varepsilon)} - 1 \right| \leq \frac{2|\varphi(\beta)|}{|\tau_b(\alpha \varepsilon)y|} \leq 4|\varphi(\beta)| \leq 4e^{-\frac{\kappa_{35}}{2}B}. \]

The left side is not 0 since \( \varphi \neq \tau_b \). Let us write

\[ \frac{\varphi(\alpha \varepsilon)}{\tau_b(\alpha \varepsilon)} = \gamma_1^{c_1} \cdots \gamma_s^{c_s} \]

with \( s = r + 1 \), and

\[ \gamma_i = \frac{\varphi(\epsilon_i)}{\tau_b(\epsilon_i)}, \quad c_i = a_i, \quad (i = 1, \ldots, r), \quad \gamma_s = \frac{\varphi(\alpha \zeta)}{\tau_b(\alpha \zeta)}, \quad c_s = 1. \]

From Proposition 2 with

\[ H_1 = \cdots = H_s = \kappa_{36}, \quad C = A, \]

we deduce \( B \leq \kappa_{37} \log A \). Then we use the upper bound \( A \leq \kappa_{34}B \) of Lemma 4 to get \( B \leq \kappa_{38} \log B \) and \( A \leq \kappa_{39} \log A \). We use (9) to conclude the proof of Lemma 6. \( \square \)
Therefore Lemma 6 now allows us to suppose that for any \( \varphi \in \Phi \) different from \( \tau_b \), we have \( |\varphi(\beta)| > |\tau_b(\beta)|^{\nu} \). In particular, the embedding \( \tau_b \) is then real. This is the end of the proof in the totally imaginary case, cf. 3.

From now on, we suppose \( T_b(\nu) = \{\tau_b\} \).

10 Upper bound for \( |\tau_b(\alpha\epsilon)| \)

An upper bound for \( |\tau_b(\alpha\epsilon)| \) is exhibited.

**Lemma 7.** One has \( |\tau_b(\alpha\epsilon)| \leq 2 \).

**Proof.** We have

\[
|x - \tau_b(\alpha\epsilon)y| = |\tau_b(\beta)| < \frac{1}{2} < |x|,
\]

wherupon we deduce

\[
|\tau_b(\alpha\epsilon)y| \leq 2|x| \leq 2|y|,
\]

since \( |x| \leq |y| \).

\[ \square \]

11 Unicity of \( \sigma_a \)

Since \( |\tau_b(\beta)| \) is very small, \( x \) is close to \( \tau_b(\alpha\epsilon)y \). Now, for any \( \varphi \in \Phi \), we have

\[
\varphi(\beta) = x - \varphi(\alpha\epsilon)y.
\]

Consequently, if \( |\varphi(\alpha\epsilon)| \) is smaller than \( |\tau_b(\alpha\epsilon)| \), then \( \varphi(\beta) \) is close to \( x \), while if \( |\varphi(\alpha\epsilon)| \) is larger than \( |\tau_b(\alpha\epsilon)| \), then \( \varphi(\beta) \) is close to \( -\varphi(\alpha\epsilon)y \). Let us justify these claims.

**Lemma 8.** Let \( \varphi \in \Phi \).

(a) Let \( \lambda \) be a real number in the interval \([0, 1] \). If \( |\varphi(\alpha\epsilon)| \leq \lambda|\tau_b(\alpha\epsilon)| \), then

\[
|\varphi(\beta) - x| \leq \lambda|x| + \lambda e^{-\kappa B}.
\]

(b) Let \( \mu \) be a real number \( > 1 \). If \( |\varphi(\alpha\epsilon)| \geq \mu|\tau_b(\alpha\epsilon)| \), then

\[
|\varphi(\beta) + \varphi(\alpha\epsilon)y| \leq \frac{1}{\mu}|\varphi(\alpha\epsilon)y| + e^{-\kappa B}.
\]

**Proof.** We have \( |\tau_b(\beta)| \leq e^{-\kappa B} \), namely

\[
|x - \tau_b(\alpha\epsilon)y| \leq e^{-\kappa B}.
\]

We also have

\[
\varphi(\beta) = x - \varphi(\alpha\epsilon)y.
\]
Because of the hypothesis (a) we get
\[
|\varphi(\beta) - x| = |\varphi(\alpha \varepsilon)y| \leq \lambda |\tau_b(\alpha \varepsilon)y| \leq \lambda |x| + \lambda e^{-\lambda m^2}.
\]
Because of the hypothesis (b), we have
\[
|\varphi(\beta) + \varphi(\alpha \varepsilon)y| = |x| \leq |\tau_b(\alpha \varepsilon)y| + e^{-\lambda m^2} \leq \frac{1}{\mu} |\varphi(\alpha \varepsilon)y| + e^{-\lambda m^2}.
\]

**Lemma 9.** Let \( \varphi \in \Phi \) with \( \varphi \neq \sigma_a \). Then
\[
|\varphi(\beta)| \leq 2|x| \exp \left\{ \kappa_40 (\log m) \log \left( 2 + \frac{A + B}{\log m} \right) \right\}
\]
and
\[
|\varphi(\alpha \varepsilon)| \leq \max \left\{ \frac{3}{2} |\tau_b(\alpha \varepsilon)|, \frac{8|x|}{|y|} \exp \left\{ \kappa_40 (\log m) \log \left( 2 + \frac{A + B}{\log m} \right) \right\} \right\}.
\]

**Proof.** From the relation (6) with \( \varphi_1 = \sigma_a \) et \( \varphi_2 = \varphi \) we deduce
\[
x = \frac{\varphi(\alpha \varepsilon)\sigma_a(\beta) - \sigma_a(\alpha \varepsilon)\varphi(\beta)}{\varphi(\alpha \varepsilon) - \sigma_a(\alpha \varepsilon)},
\]
hence
\[
\frac{\varphi(\alpha \varepsilon)\sigma_a(\beta)}{\sigma_a(\alpha \varepsilon)\varphi(\beta)} - 1 = \frac{\varphi(\alpha \varepsilon) - \sigma_a(\alpha \varepsilon)}{\sigma_a(\alpha \varepsilon)\varphi(\beta)} \cdot x.
\]
The member of the right side is nonzero, and its modulus is bounded from above by \( 2|x|/|\varphi(\beta)| \) since \( |\varphi(\alpha \varepsilon)| \leq |\tau_b(\alpha \varepsilon)| \). The upper bound of \( |\varphi(\beta)| \) follows from Lemma 3 with \( \varphi_1 = \varphi_4 = \varphi \) et \( \varphi_2 = \varphi_3 = \sigma_a \).

To establish the upper bound given in Lemma 9 for \( |\varphi(\alpha \varepsilon)| \), we may suppose
\[
|\varphi(\alpha \varepsilon)| > \frac{3}{2} |\tau_b(\alpha \varepsilon)|,
\]
otherwise the conclusion is trivial. Then we may use Lemma 8(b) with \( \mu = 3/2 \) to deduce
\[
|\varphi(\alpha \varepsilon)y| - |\varphi(\beta)| \leq |\varphi(\beta) + \varphi(\alpha \varepsilon)y| \leq \frac{2}{3} |\varphi(\alpha \varepsilon)y| + e^{-\lambda m^2},
\]
hence
\[
|\varphi(\alpha \varepsilon)y| \leq 3|\varphi(\beta)| + 3e^{-\lambda m^2} \leq 4 \max\{ |\varphi(\beta)| , 1 \}.
\]
We can conclude by using the upper bound of \( |\varphi(\beta)| \) which we just established. 

From Lemma 9 we deduce the following.
Corollary 1. Assuming \(8\), we have \(\Sigma_a(\nu) = \{\sigma_a\}\).

Proof. Let us remind that \(|x| \leq |y|\). Since \(|\tau_b(\alpha\varepsilon)| \leq 2\) (Lemma 7), with \(\sigma \in \Sigma_a(\nu)\), we have

\[|\sigma(\alpha\varepsilon)| > \frac{3}{2}|\tau_b(\alpha\varepsilon)|.\]

If there were \(\sigma \in \Sigma_a(\nu)\) with \(\sigma \neq \sigma_a\), by using Lemma 9 with \(\varphi = \sigma\), we would deduce \(A \leq \kappa_{41} \log m\) and thanks to (8) we could conclude that \(\sigma_a\) is the only element of \(\Sigma_a(\nu)\).

12 Proof of of the main result

Let us concentrate on the Proof of Theorem 1.

For the part (a) of Theorem 1, we take \(\varepsilon \in E(\alpha)\nu\); by definition of \(E(\alpha)\nu\), there exists \(\varphi \in \Phi, \varphi \neq \sigma_a\), with

\[|\varphi(\alpha\varepsilon)| \geq |\sigma_a(\alpha\varepsilon)|,\]

namely \(\varphi \in \Sigma_a(\nu)\). Since \(\Sigma_a(\nu)\) contains more than one element, Corollary 1 shows that the inequalities (8) are not satisfied. This completes the proof of part (a) of Theorem 1.

To prove the part (b), we will use the reciprocal polynomial of \(f_\varepsilon\), defined by

\[Y^d f_\varepsilon(1/Y) = a_d Y^d + \cdots + a_0 = a_d \prod_{i=1}^d (Y - \sigma_i(\alpha\varepsilon')),\]

with \(\alpha' = \alpha^{-1}\) and \(\varepsilon' = \varepsilon^{-1}\) and we will write the binary form \(F_\varepsilon\) as

\[F_\varepsilon(X, Y) = a_d \prod_{i=1}^d (Y - \sigma_i(\alpha\varepsilon')X).\]

The part (a) of Theorem 1 not only indicates that any solution \((x, y, \varepsilon) \in \mathbb{Z}^2 \times E_\nu(\alpha)\) of the inequation \(|F_\varepsilon(x, y)| \leq m\) with \(0 < |x| \leq |y|\) verifies

\[\max\{|y|, e^{h(\alpha\varepsilon')}\} \leq m^{E(\alpha)},\]

but also shows that any solution \((x, y, \varepsilon) \in \mathbb{Z}^2 \times E_\nu(\alpha)\) of the inequation \(|F_\varepsilon(x, y)| \leq m\) with \(0 < |y| \leq |x|\) verifies

\[\max\{|x|, e^{h(\alpha\varepsilon')}\} \leq m^{E(\alpha)}.\]

Since \(h(\alpha\varepsilon') = h(\alpha\varepsilon)\) and since \(E_\nu(\alpha)\) is the set of \(\varepsilon \in E_\nu(\alpha)\) such that \(\varepsilon' \in E_\nu(\alpha')\), it follows that each solution of the inequation \(|F_\varepsilon(x, y)| \leq m\) with \(xy \neq 0\) verifies

\[\max\{|x|, |y|, e^{h(\alpha\varepsilon')}\} \leq m^{E(\alpha)}.\]
13 Proof of Proposition

Let us index the elements of \( \Phi \) in such a way that \( \sigma_1, \ldots, \sigma_{r_1} \) are the real embeddings and \( \sigma_{r_1+1}, \ldots, \sigma_d \) are the non-real embeddings, with \( \sigma_{r_1+j} = \sigma_{r_1+r_2+j} \) (\( 1 \leq j \leq r_2 \)). We have \( d = r_1 + 2r_2 \) and \( r = r_1 + r_2 - 1 \). The logarithmic embedding of \( K \) is the group homomorphism \( \lambda \) of \( K^\times \) into \( \mathbb{R}^{r+1} \) defined by

\[
\lambda(\gamma) = (\delta_1 \log |\sigma_1(\gamma)|, \ldots, \delta_{r+1} \log |\sigma_{r+1}(\gamma)|),
\]

where

\[
\delta_i = \begin{cases} 
1 & \text{for } i = 1, \ldots, r_1, \\
2 & \text{for } i = r_1 + 1, \ldots, r_1 + r_2.
\end{cases}
\]

Its kernel is the finite subgroup \( K^\times_{\text{tors}} \) of torsion elements of \( K^\times \), which are the roots of unity belonging to \( K \). By Dirichlet’s theorem, the image of \( \mathbb{Z}_K^\times \) under \( \lambda \) is a lattice of the hyperplane \( H \) of equation

\[
t_1 + \cdots + t_{r+1} = 0
\]

in \( \mathbb{R}^{r+1} \). For \( M > 0 \), define

\[
H(M) = \{ (t_1, \ldots, t_{r+1}) \in H \mid \max \{ \delta_1^{-1} t_1, \ldots, \delta_{r+1}^{-1} t_{r+1} \} \leq M \}.
\]

For all elements \( (t_1, \ldots, t_{r+1}) \) of \( H(M) \) we have

\[
\max_{1 \leq i \leq r+1} t_i \leq 2M.
\]

Further, the inequality

\[
t_1 + \cdots + t_{r+1} \leq \min_{1 \leq i \leq r+1} t_i + r \max_{1 \leq i \leq r+1} t_i
\]

together with the equation of \( H \) implies

\[
\max_{1 \leq i \leq r+1} -t_i = -\min_{1 \leq i \leq r+1} t_i \leq r \max_{1 \leq i \leq r+1} t_i \leq 2rM,
\]
	hence this set \( H(M) \) is bounded: namely, for \( (t_1, \ldots, t_{r+1}) \in H(M) \),

\[
\max_{1 \leq i \leq r+1} |t_i| \leq 2rM.
\]

The \( r \)-dimension volume of \( H(M) \) is the product of the volume of \( H(1) \) by \( M^r \) while the volume of \( H(1) \) is an effectively computable positive constant, depending only upon \( r_1 \) and \( r_2 \).

Proof of the part (a). Since \( \Lambda(\mathbb{Z}_K^\times) \) is a lattice of the hyperplane \( H \), the limit

\[
\lim_{M \to \infty} \frac{1}{M^r} |\Lambda(\mathbb{Z}_K^\times) \cap H(M)|
\]

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exists and is a positive number.

The image of $\varepsilon \in \mathbb{Z}_K^\times$ by $\Lambda$ is

$$\Lambda(\varepsilon) = (t_1, \ldots, t_{r+1}) \quad \text{with} \quad t_i = \delta_i \log |\sigma_i(\varepsilon)| \quad (i = 1, \ldots, r+1).$$

If on the one hand $\varepsilon \in \mathbb{Z}_K^\times(N)$, then

$$\delta_i^{-1} t_i \leq \log N - \log |\sigma_i(\alpha)| \quad (1 \leq i \leq r+1);$$

therefore

$$\max_{1 \leq i \leq r+1} \delta_i^{-1} t_i \leq \log N + \log |\alpha^{-1}|.$$

Consequently, if we define

$$M_+ = \log N + \log |\alpha^{-1}|,$$

we have $\Lambda(\varepsilon) \in \mathcal{H}(M_+)$. On the other hand, we have

$$\log |\sigma_i(\alpha \varepsilon)| = \delta_i^{-1} t_i + \log |\sigma_i(\alpha)| \leq \delta_i^{-1} t_i + \log |\alpha|.$$

If we define

$$M_- = \log N - \log |\alpha|,$$

then for any $\Lambda(\varepsilon) \in \Lambda(\mathbb{Z}_K^\times) \cap \mathcal{H}(M_-)$ we have $\varepsilon \in \mathbb{Z}_K^\times(N)$. Therefore,

$$\Lambda(\mathbb{Z}_K^\times) \cap \mathcal{H}(M_-) \subset \Lambda(\mathbb{Z}_K^\times(N)) \subset \Lambda(\mathbb{Z}_K^\times) \cap \mathcal{H}(M_+).$$

Now we can conclude that the part (a) of Proposition [1] is proved.

Recall that a CM field is a totally imaginary number field which is a quadratic extension of its maximal totally real subfield. Let us prove that for a CM field the number of elements $\varepsilon$ of $\mathbb{Z}_K^\times(N)$ such that $Q(\alpha \varepsilon) \neq K$ is negligible with respect to the number of elements $\varepsilon$ of $\mathbb{Z}_K^\times(N)$ such that $Q(\alpha \varepsilon) = K$. Denote by $F^{(\alpha)}$ the complement of $E^{(\alpha)}$ in $\mathbb{Z}_K^\times$:

$$F^{(\alpha)} = \{ \varepsilon \in \mathbb{Z}_K^\times \mid Q(\alpha \varepsilon) \neq K \}.$$

**Lemma 10.** Assume $K$ is not a CM field. Then

$$\limsup_{N \to \infty} \frac{1}{(\log N)^{r-1}} \left| \Lambda(\mathcal{F}^{(\alpha)}(N)) \right| < \infty.$$

**Proof.** The set of subfields $L$ of $K$ is finite. Since $K$ is not a CM field, the rank $\varrho$ of the unit group of such a subfield $L$ strictly contained in $K$ is smaller than $r$. Therefore the number of $\varepsilon \in \mathbb{Z}_K^\times$ such that $Q(\alpha \varepsilon) = L$ and $\lambda(\alpha) \in \mathcal{H}(L)$ is bounded by a constant times $M^\varrho$. The proof of Lemma [10] is then secured. 

\[\square\]
Proof of the part (b) of Proposition \[1\] If $K$ is not a CM field, the stronger estimate

$$\lim_{N \to \infty} \frac{|\mathcal{E}(\alpha)(N)|}{|\mathcal{Z}_K^*(N)|} = 1.$$ 

follows from the part (a) and from Lemma \[10\].

Assume now that $K$ is CM field with maximal totally real subfield $L_0$. Since $K = \mathbb{Q}(\alpha)$, for any $\varepsilon \in \mathbb{Z}_K^*$, we have $\mathbb{Q}(\alpha\varepsilon) \neq L_0$. As we have seen, for each subfield $L$ of $K$ different from $K$ and from $L_0$, the set of $\varepsilon \in \mathbb{Z}_K^*$ such that $\mathbb{Q}(\alpha\varepsilon) = L$ and $\lambda(\alpha) \in \mathcal{H}(M)$ is bounded by a constant times $M^{r-1}$. The other elements $\varepsilon \in \mathbb{Z}_K^*$ with $\lambda(\alpha) \in \mathcal{H}(M)$ have $\mathbb{Q}(\alpha\varepsilon) = K$. This completes the proof of the part (b) of Proposition \[1\].

Before completing the proof of Proposition \[1\] one introduces a change of variables $t_i = \delta_i x_i$; we call $H$ the hyperplane of $\mathbb{R}^{r+1}$ of equation

$$\delta_1 x_1 + \cdots + \delta_{r+1} x_{r+1} = 0$$

and for $M > 0$, we consider

$$H(M) = \{(x_1, \ldots, x_{r+1}) \in \mathcal{H} \mid \max\{x_1, \ldots, x_{r+1}\} \leq M\}.$$ 

Proof of the part (c).

Let $\nu$ be a real number in the interval $]0, 1[$. Let us take $M = \log N$. Define some subsets $D_\nu(M)$ and $D'_\nu(M)$ of $H(M)$ the following way:

$$D_\nu(M) = \{(x_1, \ldots, x_{r+1}) \in H(M) \mid \text{there exists } i, j \text{ with } i \neq j \text{ and } 1 \leq i, j \leq r_1 \text{ such that } x_i \geq \nu M \text{ and } x_j \geq \nu M\}$$

and

$$D'_\nu(M) = \{(x_1, \ldots, x_{r+1}) \in H(M) \mid \text{there exists } i \text{ with } r_1 < i \leq r + 1, \text{ such that } x_i \geq \nu M\}.$$ 

If $D_\nu(M)$ is not empty, then $r_1 \geq 2$ while if $D'_\nu(M)$ is not empty, then $r_2 \geq 1$. We show that if $r_1 \geq 2$ and $0 < \nu < \delta_{r+1}/2$, then $D_\nu(1)$ has a positive volume while if $r_2 \geq 1$ and $0 < \nu < \delta_r/2$, then $D'_\nu(1)$ has a positive volume. This will show that, for a number field of degree $\geq 3$ and for $0 < \nu < 1/2$, at least one of the two sets $D_\nu(1)$ and $D'_\nu(1)$ has a positive volume.

Assume $r_1 \geq 2$, hence $\delta_1 = \delta_2 = 1$, and $0 < \nu < \delta_{r+1}/2$. Let $a, b, c$ be positive real numbers with $\nu \leq a < b < \delta_{r+1}/2, c < 1$ and $c < \delta_{r+1} - 2b$. Then $D_\nu(1)$ contains the set of $(x_1, x_2, \ldots, x_{r+1}) \in H$ verifying \[1\]

$$a \leq x_1, x_2 \leq b, \quad \frac{-c}{\delta_i(r-2)} \leq x_i \leq \frac{c}{\delta_i(r-2)} \quad (3 \leq i \leq r),$$

\[1\] Notice that one does not divide by $0$: if $r = 2$ the last conditions for $3 \leq i \leq r$ disappear.
because these bounds and the equation \( \delta_1 x_1 + \delta_2 x_2 + \cdots + \delta_{r+1} x_{r+1} = 0 \) of \( H \), imply
\[
-1 \leq x_{r+1} \leq 1.
\]
This shows that \( D_\nu(1) \) has positive volume.

Next assume \( r_2 \geq 1 \), hence \( \delta_{r+1} = 2 \), and \( 0 < \nu < \delta_r / 2 \). Let \( a, b, c \) be positive real numbers with \( \nu \leq a < b < \delta_r / 2 \) and \( c < \delta_r - 2b \). Then \( D_\nu'(1) \) contains the set of \( (x_1, x_2, \ldots, x_{r+1}) \in H \) verifying
\[
a \leq x_{r+1} \leq b, \quad \frac{-c}{\delta_i(r-1)} \leq x_i \leq \frac{c}{\delta_i(r-1)} \quad (1 \leq i \leq r-1),
\]
because these bounds, together with the equation \( \delta_1 x_1 + \delta_2 x_2 + \cdots + \delta_{r+1} x_{r+1} = 0 \) of \( H \), imply
\[
-1 \leq x_r \leq 1.
\]
This shows that \( D_\nu'(1) \) has positive volume.

Once we know that the \( r \)-dimension volume of \( D_\nu(1) \) (resp. \( D_\nu'(1) \)) in \( H \) is positive, we deduce that the \( r \)-dimension volume of \( D_\nu(M) \) (resp. \( D_\nu'(M) \)) is bounded below by an effectively computable positive constant times \( M^r \) — as a matter of fact, \( D_\nu(M) \) (resp. \( D_\nu'(M) \)) is equal to the product of \( M^r \) by the effectively computable constant \( D_\nu(1) \) (resp. \( D_\nu'(1) \)). Since \( \Lambda(\alpha) + \Lambda(\mathbb{Z}_K^\nu) \) is a translate of the lattice \( \Lambda(\mathbb{Z}_K^\nu) \), the cardinality of the set
\[
(\Lambda(\alpha) + \Lambda(\mathbb{Z}_K^\nu)) \cap (D_\nu(M) \cup D_\nu'(M))
\]
is bounded below by an effectively computable positive constant times \( M^r \).

Let \( \varepsilon \in \mathbb{Z}_K^\nu \) be such that \( \Lambda(\alpha \varepsilon) \in D_\nu(M) \cup D_\nu'(M) \). We have
\[
\log \max_{1 \leq j \leq d} |\sigma_j(\alpha \varepsilon)| \leq M
\]
and there exist two distinct elements \( \varphi_1, \varphi_2 \) of \( \Phi \) such that
\[
\log |\varphi_i(\alpha \varepsilon)| \geq \nu M \quad (i = 1, 2).
\]
Consequently,
\[
|\alpha \varepsilon| \leq e^M, \quad |\varphi_1(\alpha \varepsilon)| \geq |\alpha \varepsilon|^\nu, \quad |\varphi_2(\alpha \varepsilon)| \geq |\alpha \varepsilon|^\nu
\]
and finally, since \( N = e^M \), we conclude \( \varepsilon \in \mathcal{E}_\nu^{(\alpha)}(N) \).

\textit{Proof of the part (d)}. Suppose \( d \geq 4 \). For \( M > 0 \), define
\[
\begin{align*}
\tilde{D}_\nu(M) &= \{(x_1, \ldots, x_{r+1}) \in D_\nu(M) \mid (-x_1, \ldots, -x_{r+1}) \in D_\nu(M)\}, \\
D_\nu'(M) &= \{(x_1, \ldots, x_{r+1}) \in D_\nu(M) \mid (-x_1, \ldots, -x_{r+1}) \in D_\nu'(M)\}, \\
\tilde{D}_\nu'(M) &= \{(x_1, \ldots, x_{r+1}) \in D_\nu'(M) \mid (-x_1, \ldots, -x_{r+1}) \in D_\nu'(M)\}.
\end{align*}
\]
If \( \tilde{D}_\nu(M) \) is not empty, then \( r_1 \geq 4 \). If \( D'_\nu(M) \) is not empty, then \( r_1 \geq 2 \) and \( r_2 \geq 1 \). If \( D'_\nu(M) \) is not empty, then \( r_2 \geq 2 \).

Let us show conversely that if \( r_1 \geq 4 \), then \( \tilde{D}_\nu(1) \) has a positive volume, that if \( r_1 \geq 2 \) and \( r_2 \geq 1 \), then \( D'_\nu(1) \) has a positive volume and that if \( r_2 \geq 2 \), then \( D'_\nu(1) \) has a positive volume.

Let \( a, b, c \) be three positive numbers such that
\[ \nu < a < b < 1 \quad \text{and} \quad c + 2b < 2a + 1. \]

For instance
\[ a = \frac{1 + \nu}{2}, \quad b = \frac{3 + \nu}{4}, \quad c = \frac{1 + \nu}{4}. \]

Assume \( r_1 \geq 4 \), hence \( \delta_1 = \delta_2 = \delta_3 = \delta_4 = 1 \). Then \( \tilde{D}_\nu(1) \) contains the set of \((x_1, x_2, \ldots, x_{r+1}) \in H\) verifying
\[ a < x_1, x_2 < b, \quad -b < x_3, x_4 < -a, \quad \frac{-c}{\delta_i(r-4)} < x_i < \frac{c}{\delta_i(r-4)} \quad (5 \leq i \leq r), \]
because these bounds, together with the equation \( \delta_1 x_1 + \delta_2 x_2 + \cdots + \delta_{r+1} x_{r+1} = 0 \) of \( H \), imply
\[ -1 < x_{r+1} \leq 1. \]

This shows that \( \tilde{D}_\nu(1) \) has a positive volume.

Assume \( r_1 \geq 2 \) and \( r_2 \geq 1 \), hence \( \delta_1 = \delta_2 = 1 \) and \( \delta_{r+1} = 2 \). Then \( \tilde{D}_\nu(1) \) contains the set of \((x_1, x_2, \ldots, x_{r+1}) \in H\) verifying
\[ a < x_1, x_2 < b, \quad -b < x_{r+1} < -a, \quad \frac{-c}{\delta_i(r-3)} < x_i < \frac{c}{\delta_i(r-3)} \quad (3 \leq i \leq r-1), \]
because these bounds and the equation \( \delta_1 x_1 + \delta_2 x_2 + \cdots + \delta_{r+1} x_{r+1} = 0 \) of \( H \), imply
\[ -1 < x_r \leq 1. \]

Therefore \( \tilde{D}'_\nu(1) \) has a positive volume.

Finally, assume \( r_2 \geq 2 \), hence \( \delta_r = \delta_{r+1} = 2 \). Then \( \tilde{D}'_\nu(1) \) contains the set of \((x_1, x_2, \ldots, x_{r+1}) \in H\) verifying
\[ a < x_{r+1} < b, \quad -b < x_r < -a, \quad \frac{-c}{\delta_i(r-2)} < x_i < \frac{c}{\delta_i(r-2)} \quad (2 \leq i \leq r-1), \]
because these bounds, together with the equation \( \delta_1 x_1 + \delta_2 x_2 + \cdots + \delta_{r+1} x_{r+1} = 0 \) of \( H \) imply \(-1 \leq x_1 \leq 1\). Hence the \( r \)-dimension volume of \( \tilde{D}'_\nu(1) \) is positive.

Since \( d \geq 4 \), in all cases the volume of \( \tilde{D}_\nu(M) \cup \tilde{D}'_\nu(M) \cup D'_\nu(M) \) is bounded below by an effectively computable positive constant times \( M^r \). The number of elements in the intersection of this set with \( A(\alpha) + A(Z_K^*) \) is bounded below by an effectively computable positive constant times \( M^r \).
Let $\varepsilon \in \mathbb{Z}^K$ be such that $\Delta(\alpha \varepsilon) \in \tilde{D}_\nu(M) \cup \tilde{D}'_\nu(M) \cup \tilde{D}''_\nu(M)$. We have

$$\log \max_{1 \leq j \leq d} |\sigma_j(\alpha \varepsilon)| \leq M, \quad \log \min_{1 \leq j \leq d} |\sigma_j(\alpha \varepsilon)| \geq -M$$

and there exist four distinct elements $\varphi_1, \varphi_2, \varphi_3, \varphi_4$ of $\Phi$ such that

$$\log |\varphi_i(\alpha \varepsilon)| \geq \nu M \quad (i = 1, 2) \quad \text{and} \quad \log |\varphi_j(\alpha \varepsilon)| \leq -\nu M \quad (j = 3, 4).$$

Consequently

$$|\alpha| \leq e^M, \quad |\alpha|^{\nu} \leq |\varphi_i(\alpha \varepsilon)| \leq |\alpha| \quad (i = 1, 2)$$

and

$$|\alpha^{-1}| \leq e^M, \quad |(\alpha^{-1})|^{\nu} \leq |\varphi_j(\alpha \varepsilon)| \leq |(\alpha^{-1})|^{-\nu} \quad (j = 2, 3),$$

whereupon finally $\varepsilon \in \hat{E}_\nu(\alpha)(e^M)$.

The part (d) of Proposition [II] is then proved.

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