QUANTUM LIMITS OF EISENSTEIN SERIES IN $\mathbb{H}^3$

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We study the quantum limits of Eisenstein series off the critical line for $\text{PSL}_2(\mathcal{O}_K)\backslash \mathbb{H}^3$, where $K$ is an imaginary quadratic field of class number one. This generalises the results of Petridis, Raulf and Risager on $\text{PSL}_2(\mathbb{Z})\backslash \mathbb{H}^2$. We observe that the measures $|E(p, \sigma_1 + it)|^2 d\mu(p)$ become equidistributed only if $\sigma_1 \to 1$ as $t \to \infty$. We use these computations to study measures defined in terms of the scattering states, which are shown to converge to the absolutely continuous measure $E(p, 3) d\mu(p)$ under the GRH.

1. Introduction

Suppose $M$ is a compact negatively curved Riemannian manifold (without boundary) with the unit tangent bundle $X = SM$, then the geodesic flow on $X$ is ergodic [25]. The problem is to study the quantised flow, in terms of the eigenfunctions $\phi_j$ of $\Delta$ on $M$, in the large eigenvalue limit. Shnirelman [26], Zelditch [31] and Colin de Verdière [3] proved that there is a full density subsequence of the measures $\mu_j = |\phi_j|^2 \mu$ which converges weakly to the uniform measure $\mu$ on $M$. It is not known in general whether $\mu$ is the unique limit. When $M = \Gamma \backslash \mathbb{H}^2$, and $\Gamma$ is arithmetic, more tools are available such as Hecke operators and explicit Fourier expansions of Eisenstein series. Rudnick and Sarnak [23] conjectured that for compact $M$ of constant negative curvature the limit $\mu$ is unique. This is the Quantum Unique Ergodicity (QUE) conjecture. For $\Gamma$ of arithmetic type the distribution of the eigenstates is well-understood. In 1995 Luo and Sarnak [14] proved the conjecture for Eisenstein series for non-compact arithmetic $\Gamma$ and, in particular, for $\Gamma = \text{PSL}_2(\mathbb{Z})$. The precise result is that given Jordan measurable subsets $A$ and $B$ of $M$, then

$$\lim_{t \to \infty} \frac{\int_A |E(z, \frac{1}{2} + it)|^2 d\mu(z)}{\int_B |E(z, \frac{1}{2} + it)|^2 d\mu(z)} = \frac{\mu(A)}{\mu(B)},$$

(1)

where $\mu(B) \neq 0$. They actually compute the limit explicitly

$$\int_A |E(z, \frac{1}{2} + it)|^2 d\mu(z) \sim \frac{6}{\pi} \mu(A) \log t,$$

as $t \to \infty$ (the actual erroneous constant in [14] is $48/\pi$, but it is not significant for their purposes). Jakobson [10] extended (1) to the unit tangent bundle. The result of Luo and Sarnak was also generalised to $\text{PSL}_2(\mathcal{O}_K)\backslash \mathbb{H}^3$ by Koyama [11], where $\text{PSL}_2(\mathcal{O}_K)$ is the ring of integers of an imaginary quadratic field of class number one, and to $\text{PSL}_2(\mathcal{O}_K)\backslash \mathbb{H}^n$ with $K$ a totally real field of degree $n$ and narrow class number one by Truelsen [29]. In particular, the quantum limit in [29] for $\mu_{m,t} = |E(z, \frac{1}{2} + it, m)|^2 \mu$ is

$$\mu_{t,m} \to \frac{(2\pi)^n n R}{2 d_K \zeta_K(2)} \log t,$$

where $E(z, s, m)$ are a family of Eisenstein series parametrised by $m \in \mathbb{Z}^{n-1}$, $\zeta_K$ is the Dedekind zeta function and $R$ and $d_K$ are the regulator and discriminant of $K$, respectively. The QUE for $\phi_j$ a Hecke–Maass eigenform was proven by Lindenstrauss [13] in the compact case and Soundararajan [27] in the non-compact case, thus completing the full QUE conjecture for all arithmetic surfaces. Holowinsky and Soundararajan [9] study QUE in the holomorphic case. They consider holomorphic, $L^2$-normalised Hecke cusp forms $f_k$ of weight $k$ for $\text{SL}_2(\mathbb{Z})$. They prove that the measures $|g^{k/2} f_k(z)|^2 \mu$ converge weakly to

Date: November 24, 2015.
2010 Mathematics Subject Classification. Primary 11F72; Secondary 35P25.
Key words and phrases. quantum limits; Eisenstein series; scattering poles; Bianchi groups.
The author was supported by the 150th Anniversary Postdoctoral Mobility Grant from the London Mathematical Society.
μ as k → ∞. Another interesting direction for the QUE of Eisenstein series has recently been proved by Young [30], who proves equidistribution of Eisenstein series for Γ = PSL₂(ℤ) when they are restricted to “thin sets”, e.g. geodesics connecting 0 and ∞ (as opposed to restricting to compact Jordan measurable subsets of Γ\H² as in [14]). For a general cofinite Γ ⊂ PSL₂(ℝ) it is not clear whether there are infinitely many cusp forms so that the limit of |ϕ_j|^² μ might not be relevant [21, 15]. Petridis, Raulf, and Risager [17] (see also [16]) propose to study the scattering states of Δ instead of the cuspidal spectrum. It is known that under small deformations of Γ, the cusp forms dissolve into scattering states as characterised by Fermi’s Golden Rule [20, 18]. The scattering states are described as residues of Eisenstein series on the left half-plane (Re s < 1/2) at the non-physical poles of the scattering matrix. These poles are called resonances. Let ρₙ be a sequence of poles of the scattering matrix. For PSL₂(ℤ)\H² this corresponds to half a non-trivial zero of ζ. Petridis, Raulf and Risager define the measures

\[ u_{ρₙ}(z) = \left( \text{Res}_{s=ρₙ} φ(s) \right)^{-1} \text{Res}_{s=ρₙ} E(z, s). \]

The normalisation is chosen so that u_{ρₙ} has simple asymptotics y^{1−ρₙ} for its growth at infinity. The result is that for compact Jordan measurable subset A of Γ\H²,

\[ \int_A |u_{ρₙ}(z)|² dμ(z) \rightarrow \int_A E(z, 2 - γ∞) dμ(z), \]

where γ∞ is the limit of the real part of the Riemann zeros. Under the RH the limit is E(z, 3/2) dμ(z). This is obtained by studying the quantum limits of Eisenstein series off the critical line.

We generalise their result to three dimensions Γ\H³ for Γ a Bianchi group of class number one. Let ρₙ be a sequence of poles of the scattering matrix φ(s) of E(p, s) on Γ\H³ and define

\[ v_{ρₙ}(p) = \left( \text{Res}_{s=ρₙ} φ(s) \right)^{-1} \text{Res}_{s=ρₙ} E(p, s). \]

From the explicit form of φ (5) we know that ρₙ is equal to a non-trivial zero of ζ_K. Define s(t) = σ_t + it, where σ_t > 1 is a sequence converging to σ∞ ≥ 1. Also, let γₙ be the sequence of real parts of the non-trivial zeros of ζ_K with lim γₙ = γ∞. We will prove the following theorems.

**Theorem 1.** Let A be a compact Jordan measurable subset of Γ\H³. Then

\[ \int_A |v_{ρₙ}(p)|² dμ(p) \rightarrow \int_A E(p, 4 - 2γ∞) dμ(p) \]

as n → ∞.

Notice that 4 - 2γ∞ > 2 so that we are in the region of absolute convergence. Under the GRH the limit is E(p, 3) dμ(p).

**Theorem 2.** Assume σ∞ = 1 and (σ_t − 1) log t → 0. Let A and B be compact Jordan measurable subsets of Γ\H³. Then

\[ \frac{μ_{s(t)}(A)}{μ_{s(t)}(B)} \rightarrow \frac{μ(A)}{μ(B)}, \]

as t → ∞. In fact, we have

\[ μ_{s(t)}(A) \sim μ(A) \frac{2(2π)^{2}}{|O^K| |d_K|^2 ζ_K(2)} \log t. \]  

(2)

Let F be the fundamental domain of O_K as a lattice in ℝ². Since |F| = √|d_K|/2 and vol(Γ\H³) = |d_K|³/ζ_K(2)/(4π²), [24, Proposition 2.1], it is also possible to express the constant in (2) in terms of the volumes.

**Remark 1.** The constant for the QUE of Eisenstein series on the critical line in Koyama [11] is 2/ζ_K(2). However, there is a small mistake in his computations on page 485, where the residue of the double pole of ζ_K²(s/2) goes missing. After fixing this (and taking into account the number of units of O which is normalised away in [11]) his result agrees with our limit (2) for σ∞ = 1.
Theorem 3. Assume \( \sigma_{\infty} > 1 \). Let \( A \) be a compact Jordan measurable subset of \( \Gamma \backslash \mathbb{H}^3 \). Then
\[
\mu_{s(t)}(A) \to \int_A E(p, 2\sigma_{\infty}) d\mu(p),
\]
as \( t \to \infty \).

Theorem 1 says that the measures \( \nu_{\rho} \) do not become equidistributed. We could of course renormalise the measures and use
\[
d\nu_{s(t)}(p) = \frac{E(p, s(t))}{\sqrt{E(p, 2\sigma_{\infty})}} d\mu(p).
\]
Then we have the following corollary.

Corollary 1. Assume \( \sigma_{\infty} > 1 \). Let \( A \) be a compact Jordan measurable subset of \( \Gamma \backslash \mathbb{H}^3 \). Then
\[
\nu_{s(t)}(A) \to \mu(A),
\]
as \( t \to \infty \).

The measures \( \nu_{\rho_n} \) are not eigenfunctions of \( \Delta \) so their equidistribution is not directly related to the QUE conjecture.

Remark 2. Dyatlov [4] investigated quantum limits of Eisenstein series and scattering states for more general Riemannian manifolds with cuspidal ends. He proves results analogous to Theorems 1 and 3. However, only the case of surfaces is explicitly written down and the limits are not identified as concretely for the arithmetic special cases such as in [17] or Theorem 1 and 3. Dyatlov uses a very different method of decomposing the Eisenstein series into plane waves and studying their microlocal limits, which does not use global properties of the surface, such as hyperbolicity.

2. Spectral Theory in \( \text{PSL}_2(\mathcal{O}_K) \backslash \mathbb{H}^3 \)

For the general spectral theory in hyperbolic three-space and the relevant facts about Bianchi groups and \( \zeta_K \) we refer to [5]. Fix a square-free integer \( D < 0 \) and let \( K = \mathbb{Q}(\sqrt{D}) \) be the corresponding imaginary quadratic field of discriminant \( d_K \). Let \( \mathcal{O} \) be the ring of integers of \( K \) and let \( (1, \omega) \), where
\[
\omega = \frac{d_K + \sqrt{d_K}}{2},
\]
be a \( \mathbb{Z} \)-basis for \( \mathcal{O} \). Let \( \Gamma = \text{PSL}_2(\mathcal{O}) \). For simplicity, restrict \( D \) so that \( K \) has class number one. This means that \( \Gamma \) has exactly one cusp (up to \( \Gamma \)-equivalence) which we may suppose is \( \infty \in \mathbb{P}^1\mathbb{C} \). The Dedekind zeta function of \( K \) is defined for \( \Re s > 1 \) by
\[
\zeta_K(s) = \sum_{\mathfrak{a} \subset \mathcal{O}} \frac{1}{N\mathfrak{a}^s} = \prod_p \frac{1}{1 - Np^{-s}},
\]
where the prime in the summation denotes that it is taken over nonzero ideals \( \mathfrak{a} \), and the Euler product is taken over prime ideals \( p \subset \mathcal{O} \). We define the completed zeta function by
\[
\xi_K(s) = \left( \frac{\sqrt{|d_K|}}{2\pi} \right)^s \Gamma(s) \zeta_K(s).
\]
Notice that this differs from the standard way of completing \( \zeta_K \) due to the inclusion of the discriminant. We know that \( \xi_K \) satisfies a functional equation
\[
\xi_K(s) = \xi_K(1 - s),
\]
and has an analytic continuation to all of \( \mathbb{C} \) with a simple pole at \( s = 1 \) with residue
\[
\text{Res}_{s=1} \xi_K(s) = \frac{2\pi}{|\mathcal{O}^\times|}.
\]
where $|\mathcal{O}^\times|$ is the number of units of $\mathcal{O}$.

Let $\mathbb{H}^3 = \{ p = z + jy : z \in \mathbb{C}, y > 0 \}$, where $z = x_1 + ix_2$, be the three dimensional hyperbolic space. The standard volume element on $\mathbb{H}^3$ is given by
\[
d\mu(p) = \frac{dx_1 dx_2 dy}{y^3},
\]
and the hyperbolic Laplacian is
\[
\Delta = y^2 \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial y^2} \right) - y \frac{\partial}{\partial y},
\]
with the corresponding eigenvalue equation $\Delta f + \lambda f = 0$. We write the eigenvalues of $\Delta$ as $\lambda_j = s_j(2 - s_j) = 1 + t_j^2$. We know that for cofinite subgroups $\Gamma$ of $\text{PSL}_2(\mathbb{C})$ the Laplacian has both discrete and continuous spectrum. The continuous spectrum spans (in the $\lambda$ aspect) the interval $[1, \infty)$ with the eigenpacket given by Eisenstein series on the critical line, $E(p, 1 + it)$. The discrete spectrum consists of Maass cusp forms and the eigenvalue $\lambda = 0$.

The Eisenstein series for $\Gamma$ at the cusp at $\infty$ is given for $\Re s > 1$ by
\[
E(p, s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} y(\gamma p)^s,
\]
where $\Gamma_\infty$ is the stabiliser of $\Gamma$ at $\infty$. The Eisenstein series is an eigenfunction of $\Delta$ with the eigenvalue $\lambda = s(2-s)$, but it is not square integrable. Let
\[
\varphi(s) = \frac{\xi_K(s-1)}{\xi_K(s)} = \frac{2\pi}{\sqrt{|d_K|}} \frac{1}{s-1} \frac{\xi_K(s-1)}{\xi_K(s)}
\]
be the scattering matrix of $E(p, s)$. The Fourier expansion of $E(p, s)$ at the cusp is then given by
\[
E(p, s) = y^s + \varphi(s) y^{2-s} + \frac{2y}{\xi_K(s)} \sum_{\theta \neq n \in \mathcal{O}} |n|^{s-1} \sigma_{1-s}(n) K_{s-1} \left( \frac{4\pi|n|y}{\sqrt{|d_K|}} \right) e^{2\pi i \left( \frac{x_1}{y} + \frac{x_2}{y} \right)},
\]
where
\[
\sigma_s(n) = \sum_{(d) \in C_n} |d|^{2s}.
\]
This form of the Fourier expansion can be found in [22, 11]. It also appears in a more general form in [1, (13), 5, §6 Theorem 2.11.]. The Eisenstein series $E(p, s)$ can be analytically continued to all of $\mathbb{C}$ as a meromorphic function of $s$. We can see from (5) that to the right of the critical line $s = 1$, $E(p, s)$ has only a simple pole at $s = 2$ with residue
\[
\text{Res}_{s=2} E(v, s) = \frac{|\mathcal{F}_\infty|}{\text{vol}(M)},
\]
where $\mathcal{F}_\infty$ is the fundamental domain of $\Gamma_\infty$ acting on the boundary $\mathbb{C}$, [5, §6 Theorem 1.11]. Moreover, since $\varphi(s)\varphi(2-s) = 1$, the Eisenstein series has a functional equation [5, §6 Theorem 1.2]
\[
E(p, s) = \varphi(s) E(p, 2-s).
\]
(7)
We will also use the incomplete Eisenstein series which are defined for a smooth $\psi(x)$ with compact support on $\mathbb{R}^+$ by
\[
E(p|\psi) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \psi(y(\gamma p)).
\]
As in two dimensions, it is possible to decompose $L^2(M)$ into the orthogonal spaces spanned by the closures of the spaces of incomplete Eisenstein series on one hand, and the Maass cusp forms on the other hand.
Finally, for a Hecke–Maaß cusp form \( u_j \) we have the following Fourier expansion.

\[
  u_j(p) = y \sum_{0 \neq n \in \mathcal{O}_K^*} \rho_j(n) K_{u_j}(2\pi|n|y)e^{2\pi i \langle n, z \rangle},
\]

where \( \mathcal{O}_K^* \) is the dual lattice,

\[
  \mathcal{O}_K^* = \{ m : \langle m, n \rangle \in \mathbb{Z} \text{ for all } n \in \mathcal{O}_K \}.
\]

Also, the Hecke eigenvalues satisfy \( \rho_j(n) = \rho_j(1)\lambda_j(n) \), and in particular [8, Satz 16.8, pg. 119]

\[
  L(u_j, s) = \rho_j(1) \sum_{n \in \mathcal{O}_K} \frac{\lambda_j(n)}{N(n)^s} = \rho_j(1) \prod_{p}(1 - \lambda_j(p)Np^{-s} + Np^{1-2s})^{-1}. \tag{9}
\]

We can split the space of Hecke–Maaß cusp forms further into even and odd cusp forms depending on the sign in \( \rho(-n) = \pm \rho(n) \).

3. Proofs

Let \( M = \Gamma \backslash \mathbb{H}^3 \). Since any function in \( L^2(M) \) can be decomposed in terms of the Hecke–Maaß cusp forms \( \{ u_j \} \) and the incomplete Eisenstein series \( E(p|\psi) \), it is sufficient to consider them separately.

3.1. Discrete Part. We will first prove that the contribution of the discrete spectrum vanishes in the limit.

**Lemma 1.** Let \( u_j \) be a Hecke–Maaß cusp form. Then

\[
  \int_M u_j(p)|E(p, s(t))|^2 d\mu(p) \to 0,
\]

as \( t \to \infty \).

**Proof.** Denote the integral by

\[
  I_j(s(t)) = \int_M u_j(p)|E(p, s(t))|^2 d\mu(p).
\]

We define

\[
  I_j(s) = \int_M u_j(p)E(p, s(t))E(p, s)d\mu(p).
\]

Unfolding the integral gives

\[
  I_j(s) = \int_0^\infty \int_F u_j(p)E(p, s(t))y^s \frac{dx_1 dx_2 dy}{y^3}.
\]

After a change of variables, we may suppose that the \( u_j \) in \( I_j(s) \) is even as the integral over the odd cusp forms vanishes. Substituting Fourier expansions of the Eisenstein series (6) and the cusp forms (8) into the above integral gives

\[
  I_j(s) = \int_0^\infty \int_F \left( 2y \sum_{0 \neq n \in \mathcal{O}_*} \rho_j(n) K_{u_j}(2\pi|n|y) \cos(2\pi \langle n, z \rangle) \right)
  
  \times \left( y^s(t) + \varphi(s(t))y^{2-s(t)} \right)
  
  + \frac{2y}{\xi_K(s(t))} \sum_{0 \neq m \in \mathcal{O}_*} |m|^{s(t)-1} a_{s(t)}(m) K_{s(t)-1} \left( \frac{4\pi|m|y}{\sqrt{|d_K|}} \right) e^{2\pi i \langle \frac{m}{\sqrt{|d_K|}}, z \rangle} y^s \frac{dx_1 dx_2 dy}{y^3}.
\]

By the definition of \( F \) and the formula \( \cos(a + b) = \cos a \cos b - \sin a \sin b \) it is simple to see that

\[
  \int_F \cos(2\pi \langle n, z \rangle) dz = \begin{cases} 
  0, & \text{if } 0 \neq n \in \mathcal{O}_*, \\
  1, & \text{if } n = 0.
\end{cases}
\]
Evaluating the integral over \( F \) tells us that only the terms with
\[ n = \pm 2m/\sqrt{|d_K|} \]
remain and that the integral over the imaginary part goes to zero. Hence, with the identification \( \mathcal{O} \to \mathcal{O}^* \) by \( \alpha \mapsto \frac{2}{\sqrt{d_K}}\alpha \), we get
\[ I_j(s) = \frac{4}{\xi_K(s(t))} \int_0^{\infty} \frac{\sum_{\theta \neq n \in \mathcal{O}^*} |n|^{s(t)-1} \sigma_1 - s(t)(n) \rho_j(n) K_{s(t)-1}(2\pi|n|y)K_{it_j}(2\pi|n|y) \, dy}{|n|^s}. \]

The change of variables \( y \mapsto y/|n| \) yields
\[ I_j(s) = \frac{4}{\xi_K(s(t))} \sum_{\theta \neq n \in \mathcal{O}^*} \frac{|n|^{s(t)-1} \sigma_1 - s(t)(n) \rho_j(n)}{|n|^s} \int_0^{\infty} K_{s(t)-1}(2\pi y)K_{it_j}(2\pi y) \frac{dy}{y}. \]

We can evaluate the integral by [6, 6.576 (4) and 9.100] to get
\[ I_j(s) = \frac{4}{\xi_K(s(t))} \frac{2^{-3s}}{\Gamma(s)} \prod \Gamma \left( \frac{s \pm (s(t) - 1) \pm it_j}{2} \right) R(s), \]
where the product is taken over all combinations of \( \pm \) and
\[ R(s) = \sum_{\theta \neq n \in \mathcal{O}^*} \frac{|n|^{s(t)-1} \sigma_1 - s(t)(n) \rho_j(n)}{|n|^s}. \]

Since \( u_j \) is a Hecke eigenform, we can factorise \( R(s) \) with (9) as
\[
R(s) = \rho_j(1) \prod_{(p) \, \text{prime ideal} \, k=0}^\infty \lambda_j(p^k)|p|^{k(s(t)-1)} \sigma_1 - s(t)(p^k) |p|^{ks} \frac{1}{1 - |p|^{2(s(t) - 1)}}.
\]

and thus
\[
R(s) = \rho_j(1) \prod_{(p)} \frac{1}{1 - \lambda_j(p)|p|^{-(s-\sigma(t)+1)}} \frac{1}{1 - |p|^{-2(s-\sigma(t)+1)}} \frac{1}{1 - \lambda_j(p)|p|^{-(s+\sigma(t)-1)}} |p|^{-2(s+\sigma(t)-1)}. \]

We can identify the \( L \)-functions to get
\[
R(s) = \rho_j(1) \frac{L(u_j, \frac{s-\sigma(t)+1}{2})L(u_j, \frac{s+\sigma(t)-1}{2})}{\zeta_K(s(t))}. \]

Now,
\[ J_j(t) = I_j(s(t)), \]
so that
\[
J_j(t) = \frac{2^{-1} \pi^{-s(t)} \xi_K(s(t)) \Gamma(s(t)) \prod \Gamma \left( \frac{s(t) \pm (s(t)-1) \pm it}{2} \right) \rho_j(1) \frac{1}{\zeta_K(s(t))} L(u_j, \frac{1}{2} - it)L(u_j, \sigma_t - \frac{1}{2})}{|d_K|^{s(t)/2}} \]
\[
= \frac{2^{s(t)-1} \pi^{2it} \rho_j(1)}{|d_K|^{s(t)/2} \xi_K(s(t))^2} \frac{L(u_j, \frac{1}{2} - it)L(u_j, \sigma_t - \frac{1}{2})}{\Gamma(s(t))^2}. \]

With Stirling asymptotics we see that the quotient of Gamma factors is \( O(|t|^{-2\sigma_t}) \). We use the estimate
\[
\log^{-2}|t| \ll \zeta_K(s(t)) \ll \log^2|t|, \quad (10)
\]
which follows by adapting [28, (3.5.1) and Theorem 3.11] for \( L(s, \chi) \) and the zero-free region [12]. For the \( L \)-functions we need a subconvex bound to guarantee vanishing. Petridis and Sarnak [19] show that there is a \( \delta > 0 \) such that
\[ L(u_j, \frac{1}{2} + it) \ll_j |1 + t|^{1-\delta}. \]

In fact, they have \( \delta = 7/166 \), although this is not crucial for us. Hence, \( J_j(t) \to 0 \), as \( t \to \infty \).
3.2. Continuous Part. Let \( h(y) \in C^\infty(\mathbb{R}^+) \) be a rapidly decreasing function at 0 and \( \infty \) so that \( h(y) = O_N(y^N) \) for \( 0 < y < 1 \) and \( h(y) = O_N(y^{-N}) \) for \( y \gg 1 \) for all \( N \in \mathbb{N} \). Denote the Mellin transform of \( h \) by \( H = \mathcal{M}h \), i.e.

\[
H(s) = \int_0^\infty h(y)y^{-s} \frac{dy}{y}
\]

and the Mellin inversion formula gives

\[
h(y) = \frac{1}{2\pi i} \int_{(\sigma)} H(s)y^s \frac{ds}{s},
\]

for any \( \sigma \in \mathbb{R} \). We consider the incomplete Eisenstein series denoted by

\[
F_h(p) = E(p|h) = \sum_{\gamma \in \Gamma \setminus \Gamma} h(y(\gamma p)) = \frac{1}{2\pi i} \int_{(3)} H(s)E(p, s) \frac{ds}{s},
\]

where \( h \) is a smooth function on \( \mathbb{R}^+ \) with compact support. We prove the following lemma.

**Lemma 2.** Let \( h \) be a function satisfying the conditions stated above. Then

\[
\int_M F_h(p)|E(v, s(t))|^2 d\mu(p) \sim \begin{cases}
\int_M F_h(p)E(p, 2\sigma_\infty)d\mu(p), & \text{if } \sigma_\infty > 1, \\
\frac{2(2\pi)^2}{|\sigma - 1|d\mu(F)} \log t \int_M F_h(p)d\mu(p), & \text{if } (\sigma_1 - 1) \log t \to 0,
\end{cases}
\]

as \( t \to \infty \).

Now, unfolding gives

\[
\int_M F_h(p)|E(p, s(t))|^2 d\mu(p) = \int_0^\infty \frac{1}{2\pi i} \int_{(3)} H(s)E(p, s) \frac{ds}{s} |E(p, s(t))|^2 d\mu(p)
\]

\[
= \int_0^\infty \frac{1}{2\pi i} \int_{(3)} H(s)y^s \frac{ds}{s} \int_M |E(p, s(t))|^2 d\mu(p)
\]

\[
= \int_0^\infty \frac{h(y)}{y} \, \text{vol}(F) \left( \sum_{n \in \mathbb{N}} |a_n(y, s(t))|^2 \right) \frac{dy}{y^2}.
\]

We will deal separately with the contribution of the \( n = 0 \) term and the rest. We factor out the constant \( \text{vol}(F) \) in the analysis below.

3.2.1. Contribution of the constant term. We know that

\[
|a_0(y, s(t))|^2 = y^{2\sigma_1} + 2 \text{Re}(\varphi(s(t)) y^{2-2it}) + |\varphi(s(t))|^2 y^{4-2\sigma_1}.
\]

The first term is

\[
\int_0^\infty h(y)y^{2\sigma_1-2} \frac{dy}{y} = H(2 - 2\sigma_1),
\]

which converges to \( H(2 - 2\sigma_\infty) \). For the second term we first have that

\[
\varphi(s(t)) \int_0^\infty h(y)y^{-2it} \frac{dy}{y} = \varphi(s(t))H(2it).
\]

Since \( H(s) \) is in Schwartz class in \( t \), the function \( H(2it) \) decays rapidly, whereas \( \varphi(s(t)) \) is bounded. By taking complex conjugates we see that the second term will also tend to zero. Finally, for the third expression in the constant term we get

\[
|\varphi(s(t))|^2 \int_0^\infty h(y)y^{2-2\sigma_1} \frac{dy}{y} = |\varphi(s(t))|^2 H(2\sigma_1 - 2).
\]

If \( \sigma_\infty \neq 1 \) then

\[
|\varphi(s(t))| = \left| \frac{2\pi}{s(t) - 1} \frac{\zeta_K(s(t) - 1)}{\zeta_K(s(t))} \right|
\]

To estimate this we need the convexity bound for \( \zeta_K \),

\[
\zeta_K(s(t) - 1) = \zeta_K(\sigma_1 - 1 + it) = O(|t|^{1-\sigma_1/2})
\]
and of course
\[ \frac{1}{s(t) - 1} = O(|t|^{-1}). \]
Combining all of this with (10) we get
\[ \varphi(s(t)) = O(|t|^{-\sigma_1/2 + \epsilon}), \]
and so
\[ \varphi(s(t)) \rightarrow 0, \quad \text{as } t \rightarrow \infty, \text{ when } \sigma_\infty \neq 1. \] So in summary, the contribution of the constant term converges to \( H(2 - 2\sigma_\infty) \) if \( \sigma_\infty \neq 1 \) and is \( O(1) \) otherwise.

3.2.2. Contribution of the non-constant terms. In this case the contribution equals
\[
A(t) = \int_0^\infty \frac{1}{2\pi i} \int \frac{H(s) y^s \, ds}{|\xi_K(s(t))|^2} \frac{4y^2}{|\xi_K(s(t))|^2} \sum'_{n \in \mathcal{O}/\sim} |n|^{2\sigma_1 - 2} |\sigma_1 - s(t)(n)|^2 |K_{s(t)-1}\left(\frac{4\pi|n|y}{\sqrt{|t|}}\right)|^2 dy y^s
\]
\[= \frac{4|\mathcal{O}^x|}{|\xi_K(s(t))|^2} \frac{1}{2\pi i} \int \left(\frac{\sqrt{|t|}}{4\pi}\right)^s \sum'_{n \in \mathcal{O}/\sim} |\sigma_1 - s(t)(n)|^2 |K_{s(t)-1}y^s| \int_0^\infty y^s |K_{s(t)-1}(y)|^2 dy dy ds,
\]
where \( a \sim b \) if \( a \) and \( b \) generate the same ideal in \( \mathcal{O} \) and prime in the summation denotes that it is taken over \( n \neq 0 \). We now need to evaluate the series. Keeping in mind that \( N(p) = |p|^2 \), we get by a standard calculation
\[
\sum'_{n \in \mathcal{O}/\sim} \frac{\sigma_a(n)\sigma_b(n)}{|n|^s} = \prod_{(p) \text{ prime ideal } k=0}^\infty \sum_{|p|^{k}} \frac{\sigma_a(p^k)\sigma_b(p^k)}{|p|^{ks}} = \prod_{(p)} \frac{1}{(1 - |p|^{-s})(1 - |p|^{2\sigma_1 - s})(1 - |p|^{2\sigma_1})(1 - |p|^{2\sigma_1 + 2\sigma_1 - s})}.
\]
and hence
\[
\sum'_{n \in \mathcal{O}/\sim} \frac{\sigma_a(n)\sigma_b(n)}{|n|^s} = \frac{\zeta_K(\frac{1}{2} - \sigma_1 + 1)\zeta_K(\frac{1}{2} + it)\zeta_K(\frac{1}{2} - it)\zeta_K(\frac{1}{2} + \sigma_1 - 1)}{\zeta_K(s - a - b)}.
\]
For \( a = \overline{b} = 1 - s(t) \) and \( s = 2(\sigma_1 - 1) \) this becomes
\[
\sum'_{n \in \mathcal{O}/\sim} \frac{|\sigma_1 - s(t)(n)|^2}{|n|^{2\sigma_1 - 2}} = \frac{\zeta_K(\frac{1}{2} - \sigma_1 + 1)\zeta_K(\frac{1}{2} + it)\zeta_K(\frac{1}{2} - it)\zeta_K(\frac{1}{2} + \sigma_1 - 1)}{\zeta_K(s)}.
\]
Again, by [6, 6.576 (4)] we see that
\[
\int_0^\infty y^s |K_{s(t)-1}(y)|^2 dy y = \frac{2^{s-3} \Gamma(\frac{1}{2} - \sigma_1 + 1)\Gamma(\frac{1}{2} + it)\Gamma(\frac{1}{2} - it)\Gamma(\frac{1}{2} + \sigma_1 - 1)}{\Gamma(s)}.
\]
Hence, \( A(t) \) becomes
\[
A(t) = \frac{|\mathcal{O}^x|}{|\xi_K(s(t))|^2} \frac{1}{2\pi i} \int \frac{H(s)\zeta_K(\frac{1}{2} - \sigma_1 + 1)\zeta_K(\frac{1}{2} + it)\zeta_K(\frac{1}{2} - it)\zeta_K(\frac{1}{2} + \sigma_1 - 1)}{\zeta_K(s)} ds
\]
\[= \frac{|\mathcal{O}^x|}{|\xi_K(s(t))|^2} \frac{1}{2\pi i} \int B(s) ds,
\]
say. By the Dirichlet Class Number Formula (4) for \( \zeta_K \), the completed zeta function \( \xi_K \) has a simple pole at \( s = 1 \) with
\[
\text{Res} \xi_K(s) = \frac{1}{|\mathcal{O}^x|^2}.
\]
There is also a simple pole at \( s = 0 \). It follows that the poles of \( B(s) \) in the region \( \text{Re} s \geq 1 \) are at \( 2 \pm 2i\tau, 2\sigma_t, 2\tau - 2, \) and \( 4 - 2\sigma_t \). Moving the line of integration to \( \text{Re} s = 1 \) gives

\[
A(t) = \frac{|O^x|}{2|\xi(K(s))|^2} \left( \text{Res}_{s=2\pm 2it} B(s) + \text{Res}_{s=2\sigma_t} B(s) + \delta_t \text{Res}_{s=4-2\sigma_t} B(s) + (1 - \delta_t) \text{Res}_{s=2\tau-2} B(s) + \frac{1}{2\pi i} \int_{(1)} B(s) \, ds \right),
\]

where \( \delta_t = 1 \) if \( \sigma_t < 3/2 \) and 0 otherwise. We deal with each of the residues \( A_i \) separately. For the first term we have

\[
A_1 = \frac{H(2 \pm 2it) \xi(K(2 - \sigma_t \pm it)) \xi(K(1 \pm 2it)) \xi(K(\sigma_t \pm it))}{|\xi(K(\sigma_t \pm it))|^2} \xi(K(2 \pm 2it)).
\]

By Stirling asymptotics and convexity estimates for the Dedekind zeta functions, the quotient of the \( \xi_K \) functions is bounded by \( |t|^{1-2\sigma_t \log 10|t|} \). By virtue of \( H \) being of rapid decay in \( t \) it follows that \( A_1 \to 0 \) as \( t \to \infty \).

The second term is

\[
A_2 = H(2\sigma_t) \frac{\xi(K(2\sigma_t - 1))}{\xi(K(2\sigma_t))}.
\]

If \( \sigma_\infty \neq 1 \) then

\[
A_2 \to H(2\sigma_\infty) \frac{\xi(K(2\sigma_\infty - 1))}{\xi(K(2\sigma_\infty))},
\]

but if \( \sigma_t \to 1 \) then

\[
A_2 \sim H(2) \frac{1}{2|O^x||\xi(K(2\sigma_t - 1))|}.
\]

Now, in the third term we use the form (5) of \( \varphi \) and the fact that \( \xi_K \) satisfies the functional equation (3)

\[
\xi_K(s) = \xi_K(1-s).
\]

We can then write

\[
A_3 = \delta_t H(4 - 2\sigma_t)|\varphi(s(t))|^2 \frac{\xi_K(3 - 2\sigma_t)}{\xi_K(4 - 2\sigma_t)}.
\]

By (11) we have that \( \varphi(s(t)) \to 0 \) as \( t \to \infty \) for \( \sigma_\infty \neq 1 \). Hence, if \( \sigma_\infty \neq 1 \), then

\[
A_3 \to 0.
\]

On the other hand, if \( \sigma_\infty = 1 \), then

\[
A_3 \sim \frac{\delta_t}{2|O^x|} \frac{H(2)|\varphi(s(t))|^2}{\xi_K(2)(\sigma_t - 1)},
\]

which is bounded.

For the fourth term we have

\[
A_4 = (1 - \delta_t)\xi_K(0)H(2\sigma_t - 2)|\varphi(s(t))|^2,
\]

which clearly converges to 0 if \( \sigma_\infty \neq 1 \) and is bounded for \( \sigma_\infty = 1 \) as in the previous case.

Finally, the fifth term is

\[
A_5 = \frac{|O^x|}{2|\xi(K(s(t)))|^2} \frac{1}{2\pi i} \int_{(1)} B(s) \, ds = \frac{|O^x|}{2|\xi(K(\sigma_t \pm it))|^2} \mathcal{I},
\]

where

\[
\mathcal{I} = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(1 + i\tau) \frac{\xi_K(\sigma_t - \frac{1}{2} + i\tau)^2 \xi_K(\frac{1}{2} + i(\tau + t)) \xi_K(\frac{1}{2} + i(\tau - t))}{\xi_K(1 + 2i\tau)} d\tau.
\]
We now estimate the growth of $A_5$ in terms of $t$. The exponential contribution from the gamma functions in the integral is equal to

$$\left(e^{-\frac{t}{2}|t|}\right)^2 e^{-\frac{1}{2}|\tau+t|} e^{-\frac{1}{2}|\tau-t|} e^{-2t} \ll e^{-\pi |t|}.$$  

This cancels with the exponential growth of $|\xi_K(s(t))|^2$. Since $H(1+i\tau)$ decays rapidly, we can bound $\zeta_K(\sigma_t - \frac{1}{2} + it)$ polynomially and absorb it into $H$. Hence

$$A_5 \ll \frac{\log^4 |t|}{|t|^{1/2}} \int_{-\infty}^{\infty} \tilde{H}(\tau)|\xi_K(\frac{1}{2} + i\tau + t)|\zeta_K(\frac{1}{2} + i(\tau - t)) \, d\tau,$$

where $\tilde{H}$ is a function of rapid decay. The Dedekind zeta functions can be estimated with the subconvex bound

$$\zeta_K(\frac{1}{2} + it) \ll t^{1/3+\epsilon}$$

due to Heath-Brown [7] (he proves a more general bound for arbitrary number fields of degree $n$). We get

$$A_5 \ll |t|^{5/3 - 2\sigma + 2\epsilon} \log^4 |t| \int_{-\infty}^{\infty} \tilde{H}(\tau)(t^{-1} + |\tau t^{-1} + 1|)^{1/3+\epsilon} (t^{-1} + |\tau t^{-1} - 1|)^{1/3+\epsilon} d\tau,$$

which is $o(1)$ since $\sigma_t \geq 1$.

Hence we have proved that the integral

$$\int_M F_h(p)|E(p, s(t))|^2 \, d\mu(p)$$

converges to

$$\text{vol}(F) \left( H(2 - 2\sigma) + H(2\sigma) \frac{\xi_K(2\sigma - 1)}{\xi_K(2\sigma)} \right),$$

if $\sigma = 1$. On the other hand, for $\sigma = 1$ the contribution is asymptotic to

$$\text{vol}(F) H(2) \frac{1 - |\varphi(s(t))|^2}{2 |\xi_K(2)(\sigma_t - 1)|} + O(1). \quad (12)$$

To finish the proof, we apply Mellin inversion and unfold backwards to see that

$$\text{vol}(F) \left( H(2 - 2\sigma) + H(2\sigma) \frac{\xi_K(2\sigma - 1)}{\xi_K(2\sigma)} \right)$$

$$= \int_0^{\infty} h(y) \text{vol}(F) (y^{2\sigma - 2} + \varphi(2\sigma) y^{-2\sigma + 2}) \frac{dy}{y^3}$$

$$= \int_0^{\infty} h(y) \left( \int_F E(z + jy, 2\sigma) \, dz \right) \frac{dy}{y^3}$$

$$= \int_M F_h(p)E(p, 2\sigma) \, d\mu(p),$$

and

$$\text{vol}(F) H(2) = \int_M F_h(p) \, d\mu(p).$$

For the second case, we need to estimate the quotient with the scattering matrix. We will show that

$$\frac{1 - |\varphi(s(t))|^2}{2 |\xi_K(2)(\sigma_t - 1)|} \sim \frac{2(2\pi)^2}{|\xi_K(2)|} \log |t|.$$

Let $G(\sigma) = \varphi(\sigma + it)\varphi(\sigma - it)$ and notice that

$$G'(\sigma) = \frac{\varphi'}{\varphi}(\sigma \pm it) G(\sigma),$$

where the $\pm$ denotes the linear combination $\frac{\varphi'}{\varphi}(\sigma \pm it) = \frac{\varphi'}{\varphi}(\sigma + it) + \frac{\varphi'}{\varphi}(\sigma - it)$. We then apply the mean value theorem twice on the intervals $[1, \sigma]$ and $[1, \sigma']$, respectively. We get

$$\frac{G(1) - G(\sigma)}{1 - \sigma} = \left( G(1) - (1 - \sigma')G(\sigma') \frac{\varphi'}{\varphi}(\sigma'' \pm it) \right) \frac{\varphi'}{\varphi}(\sigma' \pm it),$$

where $\sigma'' = \frac{1}{2}$.
where \(1 \leq \sigma'' \leq \sigma' \leq \sigma\). On noticing that \(G(1) = 1\), this gives
\[
\frac{1 - |\varphi(\sigma + it)|^2}{1 - \sigma} = \left(1 - (1 - \sigma')|\varphi(\sigma'' + it)|^2 \frac{\varphi'}{\varphi}(\sigma'' \pm it)\right) \frac{\varphi'}{\varphi}(\sigma' \pm it).
\]
Using the asymptotics
\[
\frac{\varphi'}{\varphi}(\sigma \pm it) \sim -4 \log t,
\]
and the fact that \(|\varphi(\sigma + it)|\) is bounded for \(\sigma \geq 1\) proves the lemma. The estimate (13) follows immediately from the standard asymptotics for the digamma function,
\[
\frac{\Gamma'}{\Gamma}(\sigma + it) = \log|t| + O(1),
\]
and the Weyl bound
\[
\frac{\zeta'_K}{\zeta_K}(\sigma + it) \ll \frac{\log t}{\log \log t},
\]
for \(\zeta'_K/\zeta_K\) (see [28, Theorems 3.11 and 5.17] and [2]).

**Proofs of Theorems 2 and 3.** These follow now from Lemmas 1 and 2 by approximation arguments similar to [14] and [11].

Theorem 1 now follows easily.

**Proof of Theorem 1.** By the functional equation (7) of \(E(p, s)\), we get
\[
|v_{p_n}|^2 d\mu(p) = |(\text{Res } s=p_n \varphi(s))^{-1} \text{Res } s=p_n E(p, s)|^2 d\mu(p)
= |(\text{Res } s=p_n \varphi(s))^{-1} \text{Res } s=p_n \varphi(s)E(p, 2-s)|^2 d\mu(p)
= |E(p, 2 - \rho_n)|^2 d\mu(p).
\]
We apply Theorem 3 with \(\sigma_\infty = 2 - \gamma_\infty\) to conclude the proof.

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