Convex Sets in Acyclic Digraphs

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Abstract A non-empty set $X$ of vertices of an acyclic digraph is called connected if the underlying undirected graph induced by $X$ is connected and it is called convex if no two vertices of $X$ are connected by a directed path in which some vertices are not in $X$. The set of convex sets (connected convex sets) of an acyclic digraph $D$ is denoted by $\text{CO}(D)$ ($\text{CC}(D)$) and its size by $\text{co}(D)$ ($\text{cc}(D)$). Gutin et al. (2008) conjectured that the sum of the sizes of all convex sets (connected convex sets) in $D$ equals $\Theta(n \cdot \text{co}(D))$ ($\Theta(n \cdot \text{cc}(D))$) where $n$ is the order of $D$. In this paper we exhibit a family of connected acyclic digraphs with $\sum_{C \in \text{CO}(D)} |C| = o(n \cdot \text{co}(D))$ and $\sum_{C \in \text{CC}(D)} |C| = o(n \cdot \text{cc}(D))$. We also show that the number of connected convex sets of order $k$ in any connected acyclic digraph of order $n$ is at least $n - k + 1$. This is a strengthening of a theorem of Gutin and Yeo.

Keywords Acyclic diagraphs · Convex sets

1 Introduction

Let $D$ be an acyclic digraph of order $n$. A non-empty set $X$ of vertices in $D$ is connected if the underlying undirected graph of $D[X]$, the subgraph of $D$ induced by
Let $\bar{\mathcal{s}}_{\text{co}}(D)$ and $\bar{\mathcal{s}}_{\text{cc}}(D)$ be the average size of a convex set and the average size of a connected convex set in $D$. The conjecture claims that $\bar{\mathcal{s}}_{\text{co}}(D) = \Theta(n)$ and $\bar{\mathcal{s}}_{\text{cc}}(D) = \Theta(n)$.

In this paper we disprove both parts of the conjecture by exhibiting a family $\mathcal{F}$ of digraphs for which $\bar{\mathcal{s}}_{\text{co}}(D) = O(\log n)$ and $\bar{\mathcal{s}}_{\text{cc}}(D) = O(\log n)$; see Section 2. In Section 3 we show that each connected digraph of order $n$ contains at least $n - k + 1$ connected convex sets of size $k$ for each $1 \leq k \leq n$. This extends a result of Gutin and Yeo [5] who showed that each connected acyclic digraph of order $n$ has at least $\lfloor n + 1/2 \rfloor$ connected convex sets.

To simplify notation in the rest of the paper, we use $n$ for the order of the digraph under consideration; $[m]$ will denote the set $\{1, 2, \ldots, m\}$ ($m$ is a positive integer). A vertex $x$ of $D$ is a source (sink) if its in-degree $d^{-}(x)$ (out-degree $d^{+}(x)$) equals zero. A vertex $v$ of a connected digraph $D$ is a cut-vertex if $D - v$ is not connected, i.e., $V(D) - v$ is not connected.

### 2 Counterexample

**Theorem 2.1** There is a family $\mathcal{F}$ of digraphs such that $\bar{\mathcal{s}}_{\text{co}}(D) = O(\log n)$ and $\bar{\mathcal{s}}_{\text{cc}}(D) = O(\log n)$ for each $D \in \mathcal{F}$.

**Proof** For $t = 1, 2, \ldots$ and $r = \lceil 2 \log_{2} t \rceil$, the acyclic digraph $D_{t}$ consists of vertex set $V(D_{t}) = X \cup Y \cup \{z\} \cup Y' \cup X'$, where

\[
X = \{x_{i} : i \in [t]\},
\]
\[
X' = \{x'_{i} : i \in [t]\},
\]
\[
Y = \{y_{j} : i \in [r]\},
\]
\[
Y' = \{y'_{j} : i \in [r]\},
\]

and arc set

\[
A(D_{t}) = \{x_{i}x_{i+1}, x'_{i}x'_{i+1} : i \in [t - 1]\} \cup \{x_{i}y_{j}, y_{j}z, y'_{j}, y'_{j}x'_{i} : j \in [r]\}.
\]

For illustration, see Fig. 1.

Consider the family $\mathcal{C}$ of convex sets of $D_{t}$ of size at least $2r + 2$. Observe that each set in $\mathcal{C}$ contains a vertex in $X \cup X'$. Thus, $|\mathcal{C}| = |\mathcal{C}_{X}| + |\mathcal{C}_{X'}| + |\mathcal{C}_{X,X'}|$, where $\mathcal{C}_{X}$ ($\mathcal{C}_{X'}$, $\mathcal{C}_{X,X'}$, respectively) is the family of sets in $\mathcal{C}$ containing a vertex in $X$ but