ON THE STRAUSS INDEX OF SEMILINEAR TRICOMI EQUATION

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Abstract. In our previous papers, we have given a systematic study on the global existence versus blowup problem for the small-data solution $u$ of the multi-dimensional semilinear Tricomi equation

$$
\partial^2_t u - t \Delta u = |u|^p, \quad (u(0, \cdot), \partial_t u(0, \cdot)) = (u_0, u_1),
$$

where $t > 0$, $x \in \mathbb{R}^n$, $n \geq 2$, $p > 1$, and $u_i \in C_0^\infty(\mathbb{R}^n)$ ($i = 0, 1$). In this article, we deal with the remaining 1-D problem, for which the stationary phase method for multi-dimensional case fails to work and the large time decay rate of $\|u(t, \cdot)\|_{L^\infty(\mathbb{R})}$ is not enough. The main ingredient of the proof in this paper is to use the structure of the linear equation to get the suitable decay rate of $u$ in $t$, then the crucial weighted Strichartz estimates are established and the global existence of solution $u$ is proved when $p > 5$.

1. Introduction. In the article, we consider the 1-D semilinear Tricomi equation

$$
\begin{cases}
\partial^2_t u - t \partial_x^2 u = |u|^p & (t, x) \in \mathbb{R}_+^{1+1}, \\
u(0, x) = u_0(x), & \partial_t u(0, x) = u_1(x),
\end{cases}
$$

where $\mathbb{R}_+^{1+1} = \{ t : t \geq 0 \} \times \{ x : x \in \mathbb{R} \}$, $p > 1$, $u_i \in C_0^\infty(\mathbb{R})$ ($i = 0, 1$). Our chief goal in this paper is to establish $p = p_{\text{crit}} = 5$ as the critical exponent such that small-data weak solution $u$ to (1.1) exists globally when $p > p_{\text{crit}}$. Otherwise, the weak solution $u$ to (1.1) generally blows up in finite time when $1 < p \leq p_{\text{crit}}$.

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Before we describe the content of this paper in detail, let us give a bit of historical background. Firstly, we replace the Tricomi operator with the classical wave operator, and consider
\begin{equation}
\begin{cases}
\partial_t^2 v - \Delta v = |v|^p, & (t, x) \in \mathbb{R}^{1+n}_{+}, \\
v(0, x) = v_0(x), & \partial_t v(0, x) = v_1(x),
\end{cases}
\end{equation}
where \( p > 1, \) \( n \geq 2, \) and \( v_i \in C_0^\infty(\mathbb{R}^n) \) \((i = 0, 1). \) Let \( p_c = p_c(n) \) denote the positive root of the quadratic equation
\[(n - 1)p^2 - (n + 1)p - 2 = 0.\]

Strauss Conjecture. If \( p > p_c(n), \) then small data solutions of problem (1.2) exist globally. If \( 1 < p < p_c(n), \) then small data solutions of problem (1.2) blow up in finite time.

This conjecture was almost solved in last three decades. In 1979, John [16] showed that when \( n = 3, \) the global solution \( v \) always exists if \( p > p_c = 1 + \sqrt{2} \) and \( \varepsilon > 0 \) is small, meanwhile, \( v \) will blow up in finite time if \( p < p_c. \) In addition, this conjecture was shortly verified when \( n = 2 \) by Glassey [9]. On the other hand, John’s blowup result was then extended by Sideris [28], showing that, for all \( n, \) there can be blowup for arbitrarily small data if \( p < p_c. \) Along the other direction, Zhou [37] showed that when \( n = 4, \) in which case \( p_c = 2, \) there is always global existence for small data if \( p > p_c. \) Finally, Georgiev, Lindblad and Sogge in [18] and [10] prove the global existence for \( n \geq 4 \) and \( p > p_c \) (except some exceptional values of \( p. \) On the other hand, for the critical case of \( p = p_c, \) the solution \( v \) generally blows up (for \( n = 2, \) with \( p = p_c(2), \) \( p_c(3), \) see [27]; for \( n \geq 4 \) and \( p = p_c(n), \) see [36]).

Secondly, we consider the semilinear generalized Tricomi equation
\begin{equation}
\begin{cases}
\partial_t^2 u - t^m \Delta u = |u|^p, & (t, x) \in \mathbb{R}^{1+n}_{+}, \\
u(0, x) = u_0(x), & \partial_t u(0, x) = u_1(x),
\end{cases}
\end{equation}
where \( m \in \mathbb{N}, \) \( p > 1, \) \( n \geq 2. \) For the local existence and regularity of solution \( u \) to (1.3) under weak regularity assumptions on \((u_0, u_1),\) the reader may consult [24, 25, 26, 34, 35]. If \( u_i \in C_0^\infty(\mathbb{R}^n) \) \((i = 0, 1)\) Yagdjian in [34] proved the blowup and global existence of \( u \) when \( p \) belongs to different intervals. However, there was still a gap for the range of \( p \) and the value of \( p_{\text{crit}} \) was unknown in [34]. Recently, He, Witt and Yin have shown in [12, 13, 14] that for \( m \in \mathbb{N} \) and \( n \geq 2 \) there exists a critical exponent \( p_{\text{crit}}(m, n) > 1 \) such that weak solutions \( u \) generally blow up when \( 1 < p \leq p_{\text{crit}}(m, n), \) while there exists a global small-data weak solution \( u \) when \( p > p_{\text{crit}}(m, n). \) Here, \( p_{\text{crit}}(m, n) \) is the positive root of the quadratic equation
\[(m + 2)^n 2 - 1)p^2 - \left[(m + 2)\left(\frac{n}{2} - 1\right) + 3\right]p - (m + 2) = 0. \]

Thirdly, we consider the semilinear Tricomi equation
\begin{equation}
\begin{cases}
\partial_t^2 u - t^{2k} \partial_x^2 u = |u|^p, & (t, x) \in \mathbb{R}^{1+1}_{+}, \\
u(0, x) = u_0(x), & \partial_t u(0, x) = u_1(x),
\end{cases}
\end{equation}
where \( k \geq 0, \) \( 2k \in \mathbb{N}, \) and \( u_0, u_1 \in C_0^\infty(\mathbb{R}). \) Let \( T = t^{k+1}/(k + 1), \) the equation in (1.5) becomes
\[\partial_T^2 u - \partial_T^2 u + \frac{k}{k + 1} \frac{\partial_T u}{T} = (k + 1)^{-\frac{2k}{k+1}} T^{-\frac{2k}{k+1}} |u|^p. \]
which is essentially the equation
\[ \partial_t^2 u - \partial_x^2 u + \frac{\mu_k}{1 + t} \partial_t u = C_k(1 + t)^{-\frac{\mu}{k+1}} |u|^p \]
for large \( t > 0 \) with \( \mu_k = \frac{k}{k+1} \) and \( C_k = (k+1)^{-\frac{\mu}{k+1}} \). Consider the related semilinear wave equation with time-dependent dissipation
\[
\begin{aligned}
\partial_t^2 u - \partial_x^2 u + \frac{\mu}{1 + t} \partial_t u &= |u|^p, & (t, x) \in \mathbb{R}^{1+1}_+, \\
u(0, x) &= u_0(x), & \partial_t u(0, x) = u_1(x),
\end{aligned}
\]
where \( \mu > 0, \alpha \geq 0, p > 1, \) and \( u_i \in C^\infty_0(\mathbb{R}) \) (\( i = 0, 1 \)). D’Abbicco in [4] considered the problem (1.6), if \( \mu \geq n+2 = 3 \), and \( p \) larger than the Fujita index \( 1 + \frac{2}{n} = 3 \), then for \( (u_0, u_1) \) small enough in certain weighted Sobolev space, the global existence was established in [4, Theorem 3].

Note that in our problem (1.1), \( k = \frac{1}{2} \), hence \( \mu = \mu_k = \frac{1}{3} < 1 \) and the corresponding damping in (1.6) is non-effective, thus the result in [4, Theorem 3] cannot be applied to (1.1) directly.

Galstian in [7] showed that the condition
\[ p > \frac{2}{k} + 1 \]
(corresponding to (1.19) in [7] with \( \alpha = p - 1 \)) is necessary for the global existence of the classical solution of problem (1.5). It was shown in [7, Theorem 1.3] that under the conditions \( \int_{-\infty}^{\infty} u_1(x) \, dx > 0 \) and
\[ 1 < p < \frac{2}{k} + 1, \]
the problem (1.5) has no global solution \( u \in C([0, \infty), L^p(\mathbb{R})) \). Moreover, assuming that
\[ \|u_0\|_{L^r(\mathbb{R})} + \|u_1\|_{L^r(\mathbb{R})} \leq \varepsilon, \quad r > 1 \]
and the conditions
\[
\begin{aligned}
0 &\leq \left( \frac{2}{p+1} - \frac{k+1}{rp} \right) p \leq 1, \\
\frac{(p-1)p(k+1)}{rp} &\leq \frac{1}{2} \leq \frac{k}{2(p-1)(k+1)}, \\
1 &\leq \frac{2}{(p-1)(k+1)},
\end{aligned}
\]
(corresponding to (1.12)-(1.13) and (1.17) in [7]), then it was shown in [7, Theorem 1.2] that problem (1.5) has a unique global solution \( u \in C([0, \infty); L^p(\mathbb{R})) \cap C^1([0, \infty); D'(\mathbb{R})) \). In particular, let \( k = \frac{1}{2} \), we see that:

1. The local solution of (1.5) will blow up for \( 1 < p < 5 \).

2. By the first inequality of (1.7), we have \( r \in (1, \frac{3}{2}) \), and the global existence interval of \( p \) has a lower bound
\[ p > \frac{3 + 2r}{3 - 2r}. \]

Now let us turn back to the 1-D Cauchy problem for Tricomi equation (1.1). Since the local existence of weak solution \( u \) to (1.1) under minimal regularity assumptions has been established in [26] and [34], without loss of generality, as in [13, 14], we focus on the global small-data weak solution problem to (1.1) starting at a fixed positive time. Moreover, in order to establish the global existence result, we
need to apply the Strichartz estimates with a characteristic weight \(((\phi(t) + M)^2 - |x|^2)^\gamma\), however, the characteristic cone for Tricomi operator is \(\phi^2(t) = |x|^2\), which admits a cusp singularity at \(t = 0\), due to this difficulty, we can only establish inhomogeneous Strichartz estimates for the case \(t\) is away from zero. Thus, without loss of generality, we make some reduction of the problem (1.1). More specifically, by the local existence theory in [26] and [34], for given \(\varepsilon > 0\), there exists a fixed small \(\bar{T} > 0\), such that if \(\varepsilon < \varepsilon\), then the lifespan \(T\) of the local solution of (1.1) satisfies \(T \geq \bar{T}\). To this end, it is reasonable that one replaces nonlinearity \(|u|^p\) in (1.1) with the nonlinear function \(F_p(t, u) = (1 - \chi(t))F_p(u) + \chi(t)|u|^p\), where \(F_p\) is a \(C^\infty\) function satisfying \(F_p(0) = 0\) and \(|F_p(u)| \leq C(1 + |u|)^{p-2}|u|^2\) (this assumption is reasonable since \(p > p_{crit} > 2\) and \(\chi \in C^\infty(\mathbb{R})\) fulfills

\[
\chi(s) = \begin{cases} 
1, & s \geq \bar{T}, \\
0, & s \leq \frac{T}{2}.
\end{cases}
\]  

(1.9)

In the following, in place of (1.1) we shall also study the problem

\[
\begin{cases}
\partial_t^2 u - t\partial_x^2 u = F_p(t, u) & \text{in } \mathbb{R}^{1+1}_+,

u(0, x) = \varepsilon u_0(x), \quad \partial_t u(0, x) = \varepsilon u_1(x),
\end{cases}
\]  

(1.10)

where \(\varepsilon > 0\) is a small constant, \(u_i \in C^\infty(\mathbb{R}) \ (i = 0, 1)\) and \(\text{supp} u_i \subseteq \{|x| \leq R - 1\}\) for some fixed \(R > 1\).

From now on we denote \(\phi(t) = \frac{2}{3}t^\frac{3}{2}\). The main result in this paper is the following theorem:

**Theorem 1.1** (Global existence for \(p > p_{crit}\)). Assume that \(p > p_{crit} = 5\). Then there exists a constant \(\varepsilon_0 \in (0, \varepsilon)\) such that, for all \(0 < \varepsilon \leq \varepsilon_0\), problem (1.10) admits a global weak solution \(u\) satisfying

\[
\left(1 + |(\phi(t) + R)^2 - |x|^2\right)^\gamma u \in L^{p+1}(\mathbb{R}^{1+1}_+),
\]

where the constant \(\gamma\) fulfills

\[
0 < \gamma < \frac{1}{6} - \frac{5}{6(p + 1)}.
\]

**Remark 1.2.** Recall the bounds in (1.7)-(1.8) are

\[
p > \frac{3 + 2r}{3 - 2r}, \quad 1 < r < \frac{3 + 2r}{3 - 2r}.
\]

Note that for each \(r \in (1, \frac{3}{2})\), \(\frac{3 + 2r}{3 - 2r} > 5\), thus we have especially improved the lower bound of \(p\) in [7] to the sharp value \(p > 5\).

**Remark 1.3.** For \(1 < p < 5\), the blowup of weak solution \(u\) to (1.1) has been shown in [7]. For the critical case of \(p = 5\), we also establish the finite time blowup in [15].

**Remark 1.4.** By the proof procedure of Theorem 1.1, the nonlinear term \(|u|^p\) in (1.1) can be any \(C^1\) function \(F_p\) satisfying

\[
|\partial_x^j F_p(u)| \leq C_j |u|^{p-j}, \quad j = 0, 1.
\]

Thus Theorem 1.1 still holds for the nonlinear terms like \(-|u|^p\) and \(\pm|u|^{p-1}u\).

**Remark 1.5.** By the proof of Theorem 1.1 we know that the approximate solution sequences \(u_k \to u \in L^{p+1}(\mathbb{R}^{1+1}_+ \times \mathbb{R}) \subseteq L^{1}_{loc}(\mathbb{R}^{1+1}_+ \times \mathbb{R})\), hence \(u_k \to u \in D'(\mathbb{R}^{1+1}_+ \times \mathbb{R})\). Furthermore, \(F_p(t, u_k) \to F_p(t, u) \in L^{\frac{p}{p-1}}(\mathbb{R}^{1+1}_+ \times \mathbb{R}) \subseteq L^{\frac{p+1}{p}}(\mathbb{R}^{1+1}_+ \times \mathbb{R})\).
\( L_{\text{loc}}^{1}(\frac{T}{2}, \infty) \times \mathbb{R} \), which derives \( F_{p}(t, u_{k}) \to F_{p}(t, u) \in D'(\frac{T}{2}, \infty) \times \mathbb{R} \), these results together with the local existence result give that \( u \) is a weak solution of (1.10) in the sense of distribution.

**Remark 1.6.** For the sake of brevity, we restrict ourselves in this paper to the study of the semilinear Tricomi equation instead of the generalized semilinear Tricomi equation \( \partial_{t}^{2}u - t^{m}\partial_{x}^{2}u = |u|^{p} \ (m \in \mathbb{N}) \). In fact, by the arguments analogous to those in the proofs of Theorems 1.1 and the procedures in [12, 13], we can establish the same result to Theorem 1.1 for the generalized semilinear Tricomi equation with the critical power \( p_{\text{crit}}(m) = 1 + \frac{4}{m} \).

**Remark 1.7.** For the 1-D linear wave equation \( \partial_{t}^{2}v - \partial_{x}^{2}v = 0 \) with \( (v(0, x), \partial_{t}v(0, x)) = (\varphi_{0}(x), 0) \), it follows from d’Alembert’s formula that \( v(t, x) = \frac{1}{2}(\varphi_{0}(x + t) + \varphi_{0}(x - t)) \). If \( \varphi_{0} \in H^{1}(\mathbb{R}) \), then one has that \( v \in L_{t}^{p}((0, \infty), L_{x}^{q}(\mathbb{R})) \) for all \( 1 \leq p \leq \infty \), but \( v \notin L_{t}^{p}((0, \infty), L_{x}^{q}(\mathbb{R})) \) for any \( q \) satisfying \( 1 \leq q < \infty \). More precisely, there is no global Strichartz-type inequality for solutions \( v \) to the 1-D linear wave equation. Thus, for the 1-D semilinear wave equation \( \partial_{t}^{2}w - \partial_{x}^{2}w = |u|^{p} \) \( (p > 1) \), direct computation shows that the local weak solution \( w \) will generally blow up in finite time, see for example [8]. This, however, is not the case for the 1-D linear Tricomi equation: Since we can establish some weighted Strichartz estimates (see Theorem 2.1 and Theorem 3.2), we see that the global small data weak solution \( u \) exists for \( p > 5 \).

**Remark 1.8.** For the M-D semilinear generalized Tricomi equation \( \partial_{t}^{2}u - t^{m}\Delta u = |u|^{p} \) with \( (u(0, x), \partial_{t}u(0, x)) = (u_{0}, u_{1}) \) and \( x \in \mathbb{R}^{n} \ (n \geq 2) \), we formally let \( m = n = 1 \) in (1.4) \( (p_{\text{crit}}(m, n) \text{ satisfies (1.4) for } n \geq 2 \text{ only}) \), then the positive root is \( p_{1} = \frac{3 + \sqrt{33}}{2} \). It is easily verified that \( p_{1} < p_{\text{crit}} = 5 \) in Theorems 1.1. This is due to that we need an extra condition (2.2) for the Strichartz estimate of linear homogeneous equation (see Theorem 2.1).

The linear equation \( \partial_{t}^{2}u - t\partial_{x}^{2}u = 0 \) is the well-known Tricomi equation which arises from transonic gas dynamics (see [3, 23]). There are extensive results for both linear and semilinear Tricomi equations in \( n \) space dimensions \((n \in \mathbb{N})\). For instances, the authors of [1, 32, 35] have computed the forward fundamental solution of the linear Tricomi equation \( \partial_{t}^{2}u - t\Delta u = 0 \) explicitly. The authors of [11, 19, 20, 21, 22] have obtained a series of interesting results on the existence and uniqueness of solutions \( u \) to the semilinear Tricomi equation \( \partial_{t}^{2}u - t\Delta u = f(t, x, u) \) in bounded domains, under certain restrictions on the nonlinearity \( f(t, x, u) \). The authors of [2, 24, 25, 26] have established the local existence as well as the singularity structure of low regularity solutions to the Cauchy problem for semilinear Tricomi equations in the degenerate hyperbolic region and the elliptic-hyperbolic mixed region, respectively. We have additionally given a complete study of the global existence versus blowup problem for small-data solutions \( u \) to the semilinear Tricomi equation \( \partial_{t}^{2}u - t\Delta u = |u|^{p} \) for \( n \geq 2 \) (see [12, 13, 14]). In addition, Lin and Tu in [17] gave the upper bound of lifespan when \( 1 < p \leq p_{c} \). However, since the stationary phase method is not valid in the 1-D case, we need to utilize some different ideas to get the necessary estimates for the solution of (1.10). In the present paper, we shall systematically study the 1-D semilinear Tricomi equation \( \partial_{t}^{2}u - t\partial_{x}^{2}u = |u|^{p} \).

We now comment on the proof of Theorem 1.1. To prove the global existence in Theorem 1.1, we are required to establish weighted Strichartz estimates for the
Tricomi operator $\partial_t^2 - t^2 \partial_x^2$ as in [13]. We first consider the linear homogeneous equation

$$\partial_t^2 v - t \partial_x^2 v = 0, \quad v(0, x) = f(x), \quad \partial_t v(0, x) = g(x).$$

In this process, a series of inequalities are derived by applying an explicit formula for the solution $v$ and by utilizing a basic observation from [10] together with some further delicate analysis. Here we point out that since the stationary phase method used to treat the M-D problem in [13, 18] is not applicable in the 1-D case, one cannot obtain a suitable $L^1 - L^\infty$ estimate for $v$ and then use interpolation between the $L^1 - L^\infty$ estimate and the $L^2 - L^2$ energy estimate to establish the Strichartz-type estimate for $v$ as in [13]. To overcome this difficulty, we write $v$ in an explicit formula (2.9), then use the decay rate of the amplitude function in (2.9), based on the resulting inequalities (2.20), we are able to establish weighted Strichartz estimates in Theorem 2.1. For the inhomogeneous estimate Theorem 3.2, the method for symmetric wave equation from [10] motivates us to apply a dual argument, then Theorem 3.2 follows by precise computation. Combining Theorem 2.1 and Theorem 3.2, together with the contraction mapping principle, we complete the proof of Theorem 1.1.

This paper is organized as follows: In Section 2, certain weighted Strichartz estimates for the linear homogeneous Tricomi equation are established. In Section 3, related weighted Strichartz estimates are derived for the linear inhomogeneous Tricomi equation. Applying the results of Section 2 and Section 3, Theorem 1.1 is finally proved in Section 4.

2. Mixed-norm estimate for homogeneous equation. In order to establish the global existence of weak solution $u$ to problem (1.1), we shall derive some mixed space-time norm estimates for the corresponding linear problem.

In this section, we consider the homogeneous problem

$$\begin{cases}
\partial_t^2 v - t \partial_x^2 v = 0 & \text{in } \mathbb{R}^{1+1}_+, \\
v(0, x) = f(x), & \partial_t v(0, x) = g(x),
\end{cases}$$

(2.1)

where $f, g \in C_0^\infty(\mathbb{R})$ and supp$(f, g) \subseteq \{ x : |x| \leq R - 1 \}$ for some fixed constant $R > 1$.

We derive a weighted space-time estimate of Strichartz-type for the solution $v$.

**Theorem 2.1.** Let $q = p + 1$ for $p > p_{\text{crit}}$. If

$$\gamma < \frac{1}{6} - \frac{5}{6q},$$

(2.2)

then there exists a constant $\delta \in (\max(0, -\frac{5}{6} - \gamma), \frac{1}{6} - \frac{5}{6q} - \gamma)$ such that the solution $v$ to (2.1) satisfies

$$\left\| \left( \phi(t) + R \right)^2 - |x|^2 \right\|^\gamma_{L^q(\mathbb{R}^{1+1}_+)} \leq C \left( \| f \|_{W^{\frac{1}{2} + \delta, 1}(\mathbb{R})} + \| g \|_{W^{\frac{1}{2} + \delta, 1}(\mathbb{R})} \right),$$

(2.3)

where $C$ is a positive constant depending on $q$, $\gamma$, and $\delta$.

**Proof.** It follows from [34] that the solution $v$ of (2.1) can be expressed as

$$v(t, x) = V_1(t, D_x) f(x) + V_2(t, D_x) g(x),$$

where the symbols $V_j(t, \xi)$ ($j = 1, 2$) of the Fourier integral operators $V_j(t, D_x)$ are

$$V_1(t, |\xi|) = \frac{\Gamma(\frac{\delta}{6})}{\Gamma(\frac{1}{6})} \left[ e^{\frac{5}{6} H_+ \left( \frac{1}{6} \cdot \frac{1}{3} ; z \right)} + e^{-\frac{5}{6} H_- \left( \frac{1}{6} \cdot \frac{1}{3} ; z \right)} \right]$$

(2.4)
and

\[ V_2(t, |\xi|) = \frac{\Gamma(\frac{5}{6})}{\Gamma(\frac{3}{5})} t \left[ e^{2z \Phi} \left( \frac{5}{6}, \frac{5}{3}; z \right) + e^{-2z \Phi} \left( \frac{5}{6}, \frac{5}{3}; z \right) \right]. \quad (2.5) \]

Here, \( z = 2i \phi(t)|\xi|, \xi \in \mathbb{R}, i = \sqrt{-1}, \) and \( H_{\pm} \) are smooth functions of the variable \( z \). By [31], one has that, for \( \beta \in \mathbb{N} \),

\[
\begin{align*}
|\partial_\xi^\beta H_+(\alpha, \gamma; z)| &\leq C (\phi(t)|\xi|)^{\alpha-1} (1 + |\xi|)^{-|\beta|} \quad \text{if} \ \phi(t)|\xi| \geq 1, \quad (2.6) \\
|\partial_\xi^\beta H_-(\alpha, \gamma; z)| &\leq C (\phi(t)|\xi|)^{-\alpha} (1 + |\xi|)^{-|\beta|} \quad \text{if} \ \phi(t)|\xi| \geq 1. \quad (2.7)
\end{align*}
\]

To estimate \( v \), it suffices to deal with \( V_1(t, D_x)f(x) \), since the treatment on \( V_2(t, D_x) g(x) \) is similar. Indeed, if one just notices a simple fact of \( t \phi(t)^{-\delta} = C_1 \phi(t)^{-\delta} \), it then follows from the expressions of \( V_1(t, \xi) \) and \( V_2(t, \xi) \) that the orders of \( t \) in \( V_1(t, \xi) \) and \( V_2(t, \xi) \) are the same. Choose a cut-off function \( \chi(s) \in C^\infty(\mathbb{R}) \) with

\[
\chi(s) = \begin{cases} 
1, & s \geq 2 \\
0, & s \leq 1.
\end{cases}
\]

Then

\[
V_1(t, |\xi|) \hat{f}(\xi) = \chi(\phi(t)|\xi|)V_1(t, |\xi|) \hat{f}(\xi) + (1 - \chi(\phi(t)|\xi|))V_1(t, |\xi|) \hat{f}(\xi) \\
= \hat{v}_1(t, \xi) + \hat{v}_2(t, \xi).
\]

By (2.4), (2.6) and (2.7), we derive that

\[
v_1(t, x) = C \left( \int_{\mathbb{R}^n} e^{i(x \cdot \xi + \phi(t)|\xi|)} a_{11}(t, \xi) \hat{f}(\xi) d\xi + \int_{\mathbb{R}^n} e^{i(x \cdot \xi - \phi(t)|\xi|)} a_{12}(t, \xi) \hat{f}(\xi) d\xi \right), \quad (2.9)
\]

where \( C > 0 \) is a generic constant, and for \( \beta \in \mathbb{N} \),

\[
|\partial_\xi^\beta a_{1l}(t, \xi)| \leq C_{1l} |\xi|^{-|\beta|} (1 + \phi(t)|\xi|)^{-l}, \quad l = 1, 2.
\]

Next we analyze \( v_2(t, x) \). It follows from [5] or [34] that

\[
V_1(t, |\xi|) = e^{-2z \Phi} \left( \frac{1}{6}, \frac{1}{3}; z \right),
\]

where \( \Phi \) is the confluent hypergeometric function which is analytic with respect to the variable \( z = 2i \phi(t)|\xi| \). Thus for \(|z| \leq 1, \Phi(\frac{1}{6}, \frac{1}{3}; z) \) is bounded. If we apply \( \partial_\xi \) on \( \Phi(\frac{1}{6}, \frac{1}{3}; z) \), we have

\[
|\partial_\xi \Phi(\frac{1}{6}, \frac{1}{3}; \phi(t)|\xi|)| = |\partial_\xi \Phi(\frac{1}{6}, \frac{1}{3}; \phi(t)|\xi|)|^{\phi(t)|\xi|^{-1}} \leq C (1 + \phi(t)|\xi|)^{-\frac{1}{2}} |\xi|^{-1},
\]

if \( \phi(t)|\xi| \leq 1 \). Then

\[
|\partial_\xi \{ (1 - \chi(\phi(t)|\xi|))V_1(t, |\xi|) \}| \leq C(1 + \phi(t)|\xi|)^{-\frac{1}{2}} |\xi|^{-1}.
\]

Similarly, one has

\[
|\partial_\xi \{ (1 - \chi(\phi(t)|\xi|))V_1(t, |\xi|) \}| \leq C(1 + \phi(t)|\xi|)^{-\frac{1}{2}} |\xi|^{-|\beta|}.
\]

Thus we arrive at

\[
v_2(t, x) = C \left( \int_{\mathbb{R}^n} e^{i(x \cdot \xi + \phi(t)|\xi|)} a_{21}(t, \xi) \hat{f}(\xi) d\xi + \int_{\mathbb{R}^n} e^{i(x \cdot \xi - \phi(t)|\xi|)} a_{22}(t, \xi) \hat{f}(\xi) d\xi \right), \quad (2.10)
\]
where, for $\beta \in \mathbb{N}_0$,
\[
|\partial^\beta_x a_2(t, \xi)| \leq C_l \beta \left(1 + \phi(t)|\xi|\right)^{-\frac{1}{2}} |\xi|^{-|\beta|}, \quad l = 1, 2.
\]
Substituting (2.9) and (2.10) into (2.8) yields
\[
V_1(t, D_x)f(x) = C_1 \left( \int_{\mathbb{R}} e^{i(x+\phi(t)|\xi|)} a_1(t, \xi) \hat{f}(\xi) d\xi + \int_{\mathbb{R}} e^{i(x-\phi(t)|\xi|)} a_2(t, \xi) \hat{f}(\xi) d\xi \right),
\]
where $a_l$ ($l = 1, 2$) satisfies
\[
|\partial^\beta_x a_l(t, \xi)| \leq C_l \beta \left(1 + \phi(t)|\xi|\right)^{-\frac{1}{2}} |\xi|^{-|\beta|}.
\]
The decay $(1 + \phi(t)|\xi|)^{-\frac{1}{2}}$ is the key observation in the proof of Theorem 2.1, since we cannot apply stationary phase method in 1-D case.

To estimate $V_1(t, D_x)f(x)$, it only suffices to deal with $\int_{\mathbb{R}} e^{i(x+\phi(t)|\xi|)} a_1(t, \xi) \hat{f}(\xi) d\xi$ since the term $\int_{\mathbb{R}} e^{i(x-\phi(t)|\xi|)} a_2(t, \xi) \hat{f}(\xi) d\xi$ can be analogously treated. Set
\[
(Af)(t, x) = \int_{\mathbb{R}} e^{i(x+\phi(t)|\xi|)} a_1(t, \xi) \hat{f}(\xi) d\xi.
\]
Let $\beta(\tau) \in C_0^\infty(\frac{1}{2}, 2)$ such that
\[
\sum_{j=-\infty}^{\infty} \beta \left(\frac{T}{2^j}\right) \equiv 1 \quad \text{for } \tau \in \mathbb{R}_+.
\]
To estimate $(Af)(t, x)$, we now study its corresponding dyadic operators
\[
(A_j f)(t, x) = \int_{\mathbb{R}} e^{i(x+\phi(t)|\xi|)} \beta \left(\frac{|\xi|}{2^j}\right) a_1(t, \xi) \hat{f}(\xi) d\xi
\]
\[
= \int_{\mathbb{R}} e^{i(x+\phi(t)|\xi|)} a_j(t, \xi) \hat{f}(\xi) d\xi,
\]
where $\beta \in \mathbb{Z}$. Note that the kernel of operator $A_j$ is
\[
K_j(t, x; y) = \int_{\mathbb{R}} e^{i((x-y)t+\phi(t)|\xi|)} a_j(t, \xi) d\xi,
\]
where $|y| \leq R$ because of $\text{supp } f \subseteq \{x : |x| \leq R\}$. By (3.29) of [18], we have that for any $N \in \mathbb{R}^+$,
\[
|K_j(t, x; y)| \leq C_l \lambda_j (1 + \phi(t)\lambda_j)^{-\frac{1}{2}} (1 + \lambda_j||x-y|-\phi(t)||)^{-N},
\]
where $\lambda_j = 2^j$. Since the solution $v$ of (2.1) is smooth and has compact support on the variable $x$ for any fixed time, one easily knows that (2.3) holds in domain $[0, T] \times \mathbb{R}$ for any fixed $T > 0$. Therefore, in order to prove (2.3), it suffices to consider the case of $\phi(t) \gg R$. At this time, the following two cases will be studied separately.

2.1. The case $||x| - \phi(t)|| \gg R$. In this case, there exist two positive constants $C_1$ and $C_2$ such that
\[
C_1 ||x-y|-\phi(t)|| \geq ||x| - \phi(t)|| \geq C_2 ||x-y|-\phi(t)||.
\]
Let $\gamma$ satisfy (2.2). If $j \geq 0$, we then take $N = \frac{\gamma}{6} + \delta$ for $\delta > 0$ in (2.11) and obtain
\[
|K_j(t, x; y)| \leq C_b \lambda_j \left(1 + \phi(t)\lambda_j\right)^{-\frac{1}{2}} \left(1 + ||x| - \phi(t)||\right)^{-\frac{\gamma}{6} - \delta}
\]
\[
\leq C_b \lambda_j \left(1 + \phi(t)\right)^{-\frac{1}{2}} \left(1 + ||x| - \phi(t)||\right)^{-\frac{\gamma}{6} - \delta}.
\]
For $j < 0$, taking $N = 5/6 - \delta$ in (2.11) we arrive at
\[
|K_j(t, x; y)| \leq C_{\delta} \lambda_j^{1/\delta} \phi(t)^{-\frac{1}{\delta}} \lambda_j^{\frac{\alpha}{\delta}} \|x| - \phi(t)\|^{-\frac{1}{\delta} + \delta}
\leq C_{\delta} \lambda_j^{\delta} (1 + \phi(t))^{-\frac{1}{\delta}} (1 + |x| - \phi(t))^{-\frac{1}{\delta} + \delta}.
\]
It follows from $f(x) \in C^{\infty}_0(\mathbb{R})$ and direct computation that
\[
|A_j f| \leq \begin{cases} 
C_{\delta} \lambda_j^{\delta} (1 + \phi(t))^{-\frac{1}{\delta}} (1 + |x| - \phi(t))^{-\frac{1}{\delta} + \delta} \|f\|_{L^1(\mathbb{R})}, & j < 0, \\
C_{\delta} \lambda_j^{\delta} (1 + \phi(t))^{-\frac{1}{\delta}} (1 + |x| - \phi(t))^{-\frac{1}{\delta} - \delta} \|f\|_{L^1(\mathbb{R})}, & j \geq 0.
\end{cases}
\tag{2.12}
\]
Summing the right sides of (2.12), we get that for large $\phi(t)$ and $|x| - \phi(t)|$
\[
|V_1(t, D_x)f| \leq C_{\delta} (1 + \phi(t))^{-\frac{1}{\delta}} (1 + |x| - \phi(t))^{-\frac{1}{\delta} + \delta} \|f\|_{L^1(\mathbb{R})}.
\]
Analogously, we have
\[
|V_2(t, D_x)g| \leq C_{\delta} (1 + \phi(t))^{-\frac{1}{\delta}} (1 + |x| - \phi(t))^{-\frac{1}{\delta} + \delta} \|g\|_{L^1(\mathbb{R})}.
\]
Therefore,
\[
|v(t, x)| \leq C_{\delta} (1 + \phi(t))^{-\frac{1}{\delta}} (1 + |x| - \phi(t))^{-\frac{1}{\delta} + \delta} (\|f\|_{L^1(\mathbb{R})} + \|g\|_{L^1(\mathbb{R})}).
\tag{2.13}
\]

2.2. The case $|x| - \phi(t) | \leq CR$. By the similar method as in Section 2.1, we can establish that for $t > 1$ and $\delta > 0$,
\[
\|v(t, \cdot)\|_{L^\infty(\mathbb{R})} \leq C_{\delta} \phi(t)^{-\frac{3}{\delta}} (\|f\|_{W^{\frac{3}{\delta} + \delta, 1}(\mathbb{R})} + \|g\|_{W^{\frac{3}{\delta} + \delta, 1}(\mathbb{R})})).
\tag{2.14}
\]
Indeed, note that
\[
|A_j f| = \left| \int_{\mathbb{R}} e^{i(x - \xi + \phi(t))} \frac{a_j(t, \xi)}{\xi^\alpha} |D_x|^{\alpha} f(\xi) d\xi \right|,
\]
where $\alpha = \frac{3}{\delta} + \delta$. Then by direct computation, we have that for $j \geq 0$,
\[
|A_j f| \leq C_{\delta} \lambda_j^{-\alpha} \lambda_j (1 + \phi(t)) \|f\|_{W^{\frac{3}{\delta} + \delta, 1}(\mathbb{R})}
\leq C_{\delta} \lambda_j^{-\delta} (1 + \phi(t)) \|f\|_{W^{\frac{3}{\delta} + \delta, 1}(\mathbb{R})}.
\tag{2.15}
\]
Similarly, for $j < 0$, we have
\[
|A_j f| \leq C_{\delta} \lambda_j^{\delta} (1 + \phi(t))^{-\frac{1}{\delta}} \|f\|_{W^{\frac{3}{\delta} - \delta, 1}(\mathbb{R})}.
\tag{2.16}
\]
Summing all the terms in (2.15) and (2.16) yields (2.14) for $g = 0$.

Note that $\|x| - \phi(t)| \leq CR$, this together with (2.14) for $g = 0$ yields
\[
|V_1(t, D_x)f| \leq C_{\delta} (1 + \phi(t))^{-\frac{1}{\delta}} (1 + |x| - \phi(t))^{-\frac{1}{\delta} + \delta} \|f\|_{W^{\frac{3}{\delta} + \delta, 1}(\mathbb{R})}.
\tag{2.17}
\]
Analogously, we have
\[
|V_2(t, D_x)g| \leq C_{\delta} (1 + \phi(t))^{-\frac{1}{\delta}} (1 + |x| - \phi(t))^{-\frac{1}{\delta} + \delta} \|g\|_{W^{\frac{3}{\delta} + \delta, 1}(\mathbb{R})}.
\tag{2.18}
\]
Therefore, it follows from (2.17) and (2.18) that for $|x| - \phi(t) | \leq CR$
\[
|v(t, x)| \leq C_{\delta} (1 + \phi(t))^{-\frac{1}{\delta}} (1 + |x| - \phi(t))^{-\frac{1}{\delta} + \delta} \left( \|f\|_{W^{\frac{3}{\delta} + \delta, 1}(\mathbb{R})} + \|g\|_{W^{\frac{3}{\delta} + \delta, 1}(\mathbb{R})} \right).
\tag{2.19}
\]
2.3. **Proof of (2.3).** Combining (2.13) with (2.19) and noting the compact supports of \(f, g\), we have for all \(t > 0\) and \(x \in \mathbb{R}\)
\[
|v(t,x)| \leq C_{\delta}(1 + \phi(t))^{-\frac{1}{6}}(1 + \|x| - \phi(t)\|)^{-\frac{1}{6} + \delta}\left(\|f\|_{W^{\frac{5}{6}, \frac{3}{4}, 1}(\mathbb{R})} + \|g\|_{W^{\frac{5}{6}, \frac{3}{4}, 1}(\mathbb{R})}\right),
\]
(2.20)
where \(\delta\) is a positive constant.

Next we derive (2.3) from (2.20). Set
\[
A = \|f\|_{W^{\frac{5}{6}, \frac{3}{4}, 1}(\mathbb{R})} + \|g\|_{W^{\frac{5}{6}, \frac{3}{4}, 1}(\mathbb{R})}.
\]
Then for \(\delta \in (\max(0, -\frac{5}{6} - \gamma), \frac{1}{6} - \frac{5}{6q} - \gamma)\)
\[
\left\|\left((\phi(t) + R)^2 - |x|^2\right)^{\gamma} v\right\|_{L^q(\mathbb{R}^{1+1})}^q
\leq C_{\delta} A^q \int_0^\infty \int_{|x| \leq \phi(t) + R} \left[\left((\phi(t) + R^2 - |x|^2)^{\gamma}
\times (1 + \phi(t))^{-\frac{1}{6}}\left(1 + \|x| - \phi(t)\|\right)^{-\frac{1}{6} + \delta}\right]^q dxdt
\leq C_{m, \delta, R} A^q \int_0^\infty \int_{\phi(t) + R} \left((1 + \phi(t))^{-\frac{1}{6} + \gamma}(1 + |x| - \phi(t)|)^{-\frac{1}{6} + \delta}\right)^q dxdt
\leq C_{m, \delta, R} A^q \int_0^\infty \left((1 + \phi(t))^{-\frac{1}{6} + 2\gamma + \delta}\right)^{q+1} dt.
\]
(2.21)
Notice that by our assumption, \(\gamma < \frac{1}{6} - \frac{5}{6q}\) and \(0 < \delta < \frac{1}{6} - \frac{5}{6q} - \gamma\), then we have
\[
\left((\frac{1}{3} + 2\gamma + \delta)q + 1\right)^{\frac{3}{2}} < -1.
\]
Hence, there exists a positive constant \(\sigma\), such the integral in the last line of (2.21) can be controlled by
\[
\int_0^\infty \left((1 + \phi(t))^{-\frac{1}{6} + 2\gamma + \delta}\right)^{q+1} dt \leq C \int_0^\infty (1 + t)^{-1-\sigma} dt \leq C.
\]
This, together with (2.21), yields (2.3). We complete the proof of Theorem 2.1. \(\square\)

3. **Mixed-norm estimate for the inhomogeneous equation.** In this section, we turn to the inhomogeneous Tricomi equation
\[
\begin{cases}
\partial_t^2 w - t\partial_x^2 w = F(t, x), & \text{in } \mathbb{R}^{1+1}_+, \\
w(0, x) = 0, & \partial_t w(0, x) = 0.
\end{cases}
\]
(3.1)
Since the stationary phase method is not applicable to the solution \(w\) in case \(n = 1\), one cannot obtain a suitable \(L^1\)-\(L^\infty\) estimate and then use interpolation to obtain the Strichartz-type estimate as in [13]. To overcome this difficulty, we shall cite a conclusion from [10] and subsequently use the representation formula for \(w\) to establish the space-time mixed norm estimate by a sophisticated analysis.

**Lemma 3.1 ([10, formula (1.16)]).** Let \(f(u) = \int_0^u \frac{g(\xi)}{|u - \xi||\xi|^\alpha |\xi|^\beta |\xi|^\gamma} d\xi\). Then
\[
\|f\|_{L^r((0, \infty))} \leq C \|g\|_{L^q((0, \infty))},
\]
provided that
\[
1 < r < q < \infty, \quad \alpha + \beta + \delta = 1 - \left(\frac{1}{r} - \frac{1}{q}\right), \quad \alpha + \beta \geq 0, \quad \alpha + \delta > \frac{1}{q}.
\]
Theorem 3.2. Let $F(t, x) = 0$ for $|x| > \phi(t) - 1$. Then, for all constants $\alpha$ and $\beta$ satisfying
\[ \alpha + \beta + \frac{1}{6} = \frac{5}{3q}, \quad \beta < \frac{1}{q}, \] (3.2)
the solution $w$ to problem (3.1) fulfills
\[ \| (\phi^2(t) - |x|^2)^{-\alpha} w \|_{L^q(\mathbb{R}^{1+1}_+)} \leq C \| (\phi^2(t) - |x|^2)^{\beta} F \|_{L^{\frac{q}{q-\beta}}(\mathbb{R}^{1+1}_+)} \] (3.3)
where $q = p + 1, p_{\text{crit}} < p < p_0 = 9$ and $C > 0$ is a constant depending on $m, p, \alpha$ and $\beta$.

Remark 3.3. Recall from [12] that we have defined the conformal exponent $p_{\text{conf}}(n)$ for the $n$-dimensional semilinear Tricomi equation $\partial_t^2 u - t \Delta u = |u|^p$ ($n \geq 2$) as
\[ p_{\text{conf}}(n) = \frac{3n + 6}{3n - 2}. \]
For $n = 1$, we have $p_{\text{conf}}(1) = p_0 = 9$.

Proof of Theorem 3.2. By the formula in [35, Theorem 2.4], the solution $w$ to (3.1) is given by
\[ w(t, x) = C \int_0^t \int_{\gamma - \phi(t) + \phi(s)}^{\gamma + \phi(t) - \phi(s)} (\phi(t) + \phi(s) + x - y)^{\gamma} (\phi(t) + \phi(s) - (x - y))^{-\gamma} \times H(\gamma, \gamma, 1, z) F(s, y) \, dy \, ds, \]
where $z = \frac{(x - y + \phi(t) - \phi(s))(-x + y - \phi(t) + \phi(s))}{(-x + y + \phi(t) - \phi(s))(-x - y - \phi(t) + \phi(s))}$. $H(\gamma, \gamma, 1, z)$ is the hypergeometric function and $\gamma = \frac{1}{6}$. Notice that $z \in [0, 1]$ for $|x - y| \leq \phi(t) - \phi(s)$. Therefore, by [5, page 59],
\[ H(\gamma, \gamma, 1, z) = \frac{1}{\Gamma(\gamma) \Gamma(1 - \gamma)} \int_0^1 t^{\gamma - 1}(1 - t)^{1 - \gamma} (1 - zt)^{-\gamma} \, dt \]
\[ \leq \frac{1}{\Gamma(\gamma) \Gamma(1 - \gamma)} \int_0^1 t^{\gamma - 1}(1 - t)^{1 - \gamma} (1 - t)^{-\gamma} \, dt = \frac{B(\gamma, 1 - 2\gamma)}{\Gamma(\gamma) \Gamma(1 - \gamma)} = C. \]
Thus we have
\[ |w(t, x)| \leq C \int_0^t \int_{\gamma - \phi(t) + \phi(s)}^{\gamma + \phi(t) - \phi(s)} (\phi(t) + \phi(s) + x - y)^{-\gamma} (\phi(t) + \phi(s) - (x - y))^{-\gamma} |F(s, y)| \, dy \, ds. \]
To obtain (3.3), we need to estimate

\[ \left\| (\phi^2(t) - |x|^2)^{-\alpha} w \right\|_{L^q(\mathbb{R}^{1+1}_+)} = \sup_{K \in L^{\frac{q}{q-\beta}}(\mathbb{R}^{1+1}_+)} \frac{\int_0^\infty \int_{-\infty}^\infty K(t, x)(\phi^2(t) - |x|^2)^{-\alpha} w(t, x) \, dx \, dt}{\| K \|_{L^{\frac{q}{q-\beta}}(\mathbb{R}^{1+1}_+)}}. \] (3.4)

Note that
\[ I = \int_0^\infty \int_{-\infty}^\infty |K(t, x)|(\phi^2(t) - |x|^2)^{-\alpha} |w(t, x)| \, dx \, dt \]
\[ \leq C \int_0^\infty \int_{-\infty}^\infty \int_0^t \int_{\gamma - \phi(t) + \phi(s)}^{\gamma + \phi(t) - \phi(s)} \frac{|K(t, x)|}{(\phi^2(t) - |x|^2)^{\alpha}} \times (\phi(t) + \phi(s) + x - y)^{-\gamma} (\phi(t) + \phi(s) - (x - y))^{-\gamma} \, dy \, ds \, dx \, dt. \]
Denote \( \tilde{K}(T, x) = K(t, x) \) and \( \tilde{F}(S, y) = F(s, y) \) with \( T = \phi(t) \) and \( S = \phi(s) \). Then
\[
\|K\|_{L^{q/(q-1)}(\mathbb{R}^{n+1}_+)} = \left( \int_0^\infty \int_0^\infty |K(t, x)|^{q/(q-1)} \, dt \, dx \right)^{1/(q-1)}
= \left( \int_0^\infty \int_0^\infty |T^{-\frac{2n+1}{2}} \tilde{K}(T, x)|^{q/(q-1)} \, dx \, dT \right)^{1/(q-1)} \tag{3.5}
\]
and
\[
\left\| (\phi^2(t) - |x|^2)^{\beta} F \right\|_{L^{q/(q-1)}(\mathbb{R}^{n+1}_+)}
= \left( \int_0^\infty \left( \int_0^\infty |T^{-\frac{2n+1}{2}} (T^2 - |x|^2)^{\beta} \tilde{F}(T, x)|^{q/(q-1)} \, dx \right) \, dT \right)^{1/(q-1)} \tag{3.6}
\]
With (3.5) and (3.6), we can further write
\[
I \leq C \int_0^\infty \int_0^\infty \int_0^T \int_0^{x-T+S} \frac{|\tilde{K}(T, x)||\tilde{F}(S, y)|}{(T^2 - x^2)^\alpha} \, dy \, dS \, dx \, dT
\times \frac{1}{(T + S + x - y)^7 (T + S - (x - y))^7 S^{\frac{q}{2}} T^{\frac{q}{2}}}
= C \int_0^\infty \int_0^\infty \int_0^T \int_0^{x-T+S} \frac{|\tilde{K}(T, x)||S^{-\frac{2n+1}{4}} (S^2 - y^2)^{\beta} |\tilde{F}(S, y)|}{(S^2 - y^2)^{\beta} (T^2 - x^2)^\alpha S^{\frac{q}{2}} T^{\frac{q}{2}}}
\times \frac{1}{(T + S + x - y)^7 (T + S - (x - y))^7}.
\]

Let
\[
\begin{cases}
  u = T + x, \\
  v = T - x,
\end{cases}
\quad \begin{cases}
  \xi = S + y, \\
  \eta = S - y.
\end{cases}
\]

By the assumption of \( \text{supp} \ F \), we have that
\[
0 \leq \xi \leq u, \quad 0 \leq \eta \leq v.
\]

Set
\[
G(\xi, \eta) = S^{-\frac{2n+1}{4}} (S^2 - y^2)^{\beta} |\tilde{F}(S, y)|, \quad H(u, v) = T^{-\frac{2n+1}{4}} |\tilde{K}(T, x)|.
\]

By \( \phi(1) \leq S \leq T \), we have that
\[
I \leq C \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \frac{G(\xi, \eta)H(u, v)}{\xi^{\beta} \eta^{\beta} u^{\alpha} v^{\alpha} (u + \xi)^{\frac{q}{2}} (v + \eta)^{\frac{q}{2}} S^{\frac{q}{2}} T^{\frac{q}{2}}} \, dy \, dS \, dx \, dT \tag{3.7}
\]
\[
\leq C \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \frac{G(\xi, \eta)H(u, v)}{\xi^{\beta} u^{\alpha} |u - \xi|^{\frac{q}{2}} \xi^{\beta} v^{\alpha} |v - \eta|^{\frac{q}{2}} T^{\frac{q}{2}}} \, dy \, d\xi \, du.
\]

By condition (3.2) in Theorem 3.2, we require \( q > 2 \) and
\[
\alpha + \beta = \frac{2}{q} - \frac{1}{3q} - \frac{1}{6} \geq 0, \quad \beta < \frac{1}{q}.
\tag{3.8}
\]
Choosing \( \beta = -p \alpha \) in (3.8), then by \( \beta < \frac{1}{q} \) and \( q = p + 1 \) one has that
\[
-p \alpha < \frac{1}{p + 1} \Rightarrow \alpha > -\frac{1}{p(p + 1)}. \tag{3.9}
\]
Substituting (3.9) into (3.8), we get
\[
\frac{p - 1}{p(p + 1)} + \frac{1}{6} \left( \frac{1}{2} + \frac{1}{q} \right) > \frac{2}{p + 1} \iff p^2 - 3p - 6 > 0 \Rightarrow p > p_1 = \frac{3 + \sqrt{33}}{2}.
\]
On the other hand,
\[
\frac{2}{q} - \frac{1}{3q} - \frac{1}{6} \geq 0 \iff q \leq 10 \iff p \leq p_0 = 9.
\]
Since \( p_1 < p_{\text{crit}} = 5 \), Lemma 3.1 is applicable with \( r = \frac{q}{q-1} \) and \( \delta = \frac{1}{4q} + \frac{1}{6} \).
Specifically, we estimate the integral in (3.7) as follows:
\[
\begin{align*}
&\int_0^\infty \int_0^\infty \int_0^\infty G(\xi, \eta) H(u, v) \, d\eta \, d\xi \, du \\
&= \int_0^\infty \int_0^\infty \frac{1}{|\xi|^\beta |u|^{\alpha} |u - \xi|^{\frac{1}{4} + \frac{1}{\beta}} |\eta|^{\beta} v^{\alpha} |v - \eta|^{\frac{1}{4} + \frac{1}{\beta}}} \left( \int_0^\infty H(u, v) \int_0^v \frac{G(\xi, \eta)}{|\eta|^{\beta} |v|^{\alpha} |v - \eta|^{\frac{1}{4} + \frac{1}{\beta}}} \, d\eta \right) \, d\xi \, du \\
&\leq C \int_0^\infty \|H(u, \cdot)\|_{L^\infty_{\frac{q}{2}, t}} \int_0^u \|G(\xi, \cdot)\|_{L^\infty_{\frac{q}{2}, t}} \left( \int_0^\infty \frac{1}{|\xi|^\beta |u|^{\alpha} |u - \xi|^{\frac{1}{4} + \frac{1}{\beta}}} \, d\xi \right) \, du \\
&\leq \|H\|_{L^\infty_{\frac{q}{2}, r}} \|G\|_{L^\infty_{\frac{q}{2}, r}}. \tag{3.4}
\end{align*}
\]
This together with (3.4) and (3.7) yields the proof of Theorem 3.2.

Based on Theorem 3.2, we are able to prove the following crucial result:

**Theorem 3.4.** Let \( q = p + 1 \) for \( p_{\text{crit}} < p < p_0 = 9 \), and let \( 0 < T_0 < 1 \). Suppose that \( F(t, x) = 0 \) if \(|x| > \phi(t) + R - 1\) or \( 0 < t < \frac{T_0}{2} \). Then there exist constants \( \alpha \) and \( \beta \) satisfying \( \alpha + \beta + \frac{1}{6} = \frac{2}{q-1} \), \( \beta < \frac{1}{q} \) such that, for the solution \( w \) to problem (3.1),
\[
\left\| \left( (\phi(t) + R)^2 - |x|^2 \right)^{-\frac{\alpha}{2}} w \right\|_{L^\infty((\frac{T_0}{2}, \infty) \times \mathbb{R})} \\
\leq C \left\| \left( (\phi(t) + R)^2 - |x|^2 \right)^{\frac{\beta}{2}} F \right\|_{L^{\frac{q}{2}}((\frac{T_0}{2}, \infty) \times \mathbb{R})}, \tag{3.10}
\]
where \( C > 0 \) is a constant depending on \( p, \alpha, \beta, R \) and \( T_0 \).

**Proof.** To prove (3.10), we first consider the special case that \( F(t, x) = 0 \) for \(|x| > \phi(t) - \phi(t_0)\). By finite propagation speed for the hyperbolic equation (3.1), we have that the integration domain in (3.10) is just \( Q = \{(t, x): t \geq \frac{T_0}{2}, |x| \leq \phi(t) + R - 1\} \). In fact, for the Tricomi equation, the speed of the propagation is \( a(t) = \sqrt{t} \), then by Example 2a in [33] (Page 308), we have
\[
\text{supp}(u(t, \cdot)) \subseteq \{x \in \mathbb{R}^n; |x| \leq R - 1 + A(t)\},
\]
where \( A(t) = \int_0^t \max_{\tau \leq s} a(\tau) \, ds = \phi(t) \). Then observe that \( Q \) can be covered by a finite number of angular domains \( \{Q_j\}_{j=1}^{N_0}, N_0 \approx RT_0^{-\frac{2}{q}} \), where the curved cone \( Q_j \)
(j ≥ 2) is a shift in the x variable of the angular domain

\[ Q_1 = \{(t, x) : t \geq \frac{T_0}{2}, |x| \leq \phi(t) - \phi\left(\frac{T_0}{4}\right)\}. \]

Set

\[ F_1 = \chi_{Q_1} F, \]
\[ F_2 = \chi_{Q_2}(1 - \chi_{Q_1}) F, \]
\[ \vdots \]
\[ F_{N_0} = \chi_{Q_{N_0}}(1 - \chi_{Q_1} - \chi_{Q_2} - \ldots - \chi_{Q_{N_0 - 1}}(1 - \chi_{Q_1}) \ldots (1 - \chi_{Q_{N_0 - 2}})) F, \]

where \( \chi_{Q_j} \) is the characteristic function of \( Q_j \), and \( \sum_{j=1}^{N_0} F_j = F \). Let \( w_j \) solve

\[
\begin{cases}
\partial_t^2 w_j - t \Delta w_j = F_j(t, x), \\
w_j(0, x) = 0, & \partial_t w_j(0, x) = 0.
\end{cases}
\]

Then \( \text{supp} w_j \subseteq Q_j \). Since the Tricomi equation is invariant under the translation with respect to the variable \( x \), it follows from Theorem 3.2 that

\[
\left\| \left( \phi^2(t) - |x - \nu_j|^2 \right)^{\gamma_1} w_j \right\|_{L^2(Q_j)} \leq C \left\| \left( \phi^2(t) - |x - \nu_j|^2 \right)^{\gamma_2} F_j \right\|_{L^{\frac{\gamma_2}{\gamma_1}}(Q_j)},
\]

where \( \nu_j \in \mathbb{R}^n \) corresponds to the coordinate shift of the space variable \( x \) from \( Q_1 \) to \( Q_j \), and \( Q_j = \{(t, x) : t \geq \frac{T_0}{2}, |x - \nu_j| \leq \phi(t) - \phi\left(\frac{T_0}{4}\right)\} \).}

Next we derive (3.10) by utilizing (3.11) and the condition of \( t \geq \frac{T_0}{4} \). At first, we illustrate that there exists a constant \( \delta > 0 \) such that for \( (t, x) \in Q_j \),

\[
\phi^2(t) - |x - \nu_j|^2 \geq \delta (\phi(t) + R)^2 - |x|^2.
\]

To prove (3.12) for 1 ≤ j ≤ \( N_0 \), it only suffices to consider the two extreme cases:

\[ \nu_j = 0 \quad \text{(corresponding to } j = 1 \text{)} \quad \text{and} \quad \nu_{j_0} = R - 1 + \phi\left(\frac{\pi T_0}{8}\right), \]

(chosing \( j_0 \) such that \( |\nu_{j_0}| = \max_{1 \leq j \leq N_0} |\nu_j| = R - 1 + \phi\left(\frac{\pi T_0}{8}\right) \)). Note that \( |\nu_{j_0}| > R - 1 \) holds so that the domain \( Q \) can be covered by \( \bigcup_{j=1}^{j_0} Q_j \).

For \( \nu_j = 0 \), (3.12) is equivalent to

\[
\phi^2(t) \geq (1 - \delta)|x|^2 + \delta (\phi(t) + R)^2.
\]

We now illustrate that (3.13) is correct. By \( |x| \leq \phi(t) - \phi\left(\frac{T_0}{4}\right) \) for \( (t, x) \in Q_1 \), then in order to show (3.13) it suffices to prove

\[
\phi^2(t) \geq (1 - \delta)\left[\phi(t) - \phi\left(\frac{T_0}{4}\right)\right]^2 + \delta (\phi(t) + R)^2.
\]

This is equivalent to

\[ 2(1 - \delta)\phi\left(\frac{T_0}{4}\right) - 2\delta R \phi(t) \geq (1 - \delta)\phi^2\left(\frac{T_0}{4}\right) + \delta R^2. \]

Obviously, this is easily achieved by \( t \geq \frac{T_0}{4} \) and the smallness of \( \delta \).

For \( \nu_{j_0} = R - 1 + \phi\left(\frac{\pi T_0}{8}\right) \), the argument on (3.12) is a little involved. First, note that for fixed \( t > 0 \), the domain \( Q \) is symmetric with respect to the variable \( x \), thus we can assume \( \nu_{j_0} = \nu = R - 1 + \phi\left(\frac{\pi T_0}{4}\right) \). In this case, (3.12) is equivalent to

\[
\phi^2(t) \geq |x - \nu|^2 + \delta (\phi(t) + R)^2 - |x|^2
\]

\[ = (1 - \delta)x^2 - 2\nu x + \nu^2 + \delta (\phi(t) + R)^2 =: G(t, x). \]
For fixed $t > 0$, $G(t, x)$ is a quadratic function of the variable $x$, and takes minimum at the point $x = \frac{\nu}{T}$. Thus for the same fixed $t > 0$, the maximum of $G(t, x)$ in the domain $Q^t := \{x : |x - \nu| \leq \phi(t) - \phi\left(\frac{T_0}{4}\right)\}$ must be achieved on the boundary $\partial Q^t := \{x : |x - \nu| = \phi(t) - \phi\left(\frac{T_0}{4}\right)\}$. Then in order to show (3.14), our task is to prove

$$\phi^2(t) \geq \left(\phi(t) - \phi\left(\frac{T_0}{4}\right)\right)^2 + \delta \left((\phi(t) + Rr)^2 - |x|^2\right). \quad (3.15)$$

For this end, it is only enough to consider the case that $|x|^2$ takes its minimum on $\partial Q^t$. Note that on $\partial Q^t$, we have

$$|x|^2 = \left[\phi(t) - \phi\left(\frac{T_0}{4}\right)\right]^2 + 2\nu x - \nu^2. \quad (3.16)$$

Therefore, without loss of generality, we can take

$$x = \nu - \phi(t) + \phi\left(\frac{T_0}{4}\right). \quad (3.17)$$

Substituting (3.17) and (3.16) into (3.15), we are left to prove

$$\phi^2(t) \geq \left(\phi(t) - \phi\left(\frac{T_0}{4}\right)\right)^2 + \delta \left((\phi(t) + R)^2 - \phi(t)^2 - \nu^2\right) \geq \phi^2(t) + \left\{2\delta \left(\phi\left(\frac{T_0}{4}\right) + R + \nu\right) - 2\phi\left(\frac{T_0}{4}\right)\right\} \phi(t) + (1 - \delta)\phi(t)^2$$

$$+ \delta \left(R^2 - \nu^2 - 2\nu\phi\left(\frac{T_0}{4}\right)\right). \quad (3.18)$$

For fixed $T_0 > 0$ and $R > 1$, if $\delta > 0$ is small enough, one then has

$$2\delta \left(\phi\left(\frac{T_0}{4}\right) + R + \nu\right) \leq \frac{1}{2} \phi\left(\frac{T_0}{4}\right),$$

$$(1 - \delta)\phi\left(\frac{T_0}{4}\right)^2 + \delta \left(R^2 - \nu^2 - 2\nu\phi\left(\frac{T_0}{4}\right)\right) \leq \frac{3}{2} \phi\left(\frac{T_0}{4}\right)^2. \quad (3.19)$$

By (3.19) and (3.18), in order to derive (3.15), one should derive

$$-\frac{3}{2} \phi\left(\frac{T_0}{4}\right) \phi(t) + \frac{3}{2} \phi^2\left(\frac{T_0}{4}\right) \leq 0.$$

Obviously, this holds true by $t \geq T_0/4$. Then (3.15) is proved.

Consequently, for $(t, x) \in \bigcup_{j=1}^{N_0} Q_j$, there exists a fixed positive constant $c > 0$ such that for $1 \leq j \leq N_0$,

$$c \left((\phi(t) + R)^2 - |x|^2\right) \leq \phi^2(t) - |x - \nu_j|^2. \quad (3.20)$$

On the other hand, note that by $|x| \leq \phi(t) + R - 1$ for $(t, x) \in Q$, one has

$$2\{\phi(t) + R\}^2 - |x|^2 - \{\phi^2(t) - |x - \nu_j|^2\} \geq (|x| + 1)^2 - |x|^2 + (\phi(t) + R)^2 - |x|^2 - \phi^2(t) + |x - \nu_j|^2$$

$$= 2R\phi(t) + R^2 + |\nu_j|^2 + 1 + 2(1 - |\nu_j|)|x|. \quad (3.21)$$
If $1 - |\nu_j| < 0$, then by $|\nu_j| \leq R - 1 + \phi \left(\frac{3T_0}{8}\right)$ and the smallness of $T_0$, the last line in (3.21) is bounded from below by

$$2R\phi(t) + R^2 + |\nu_j|^2 + 1 + 2\left\{2 - R - \phi \left(\frac{3T_0}{8}\right)\right\}\{\phi(t) + R - 1\}$$

$$= 4\phi(t) - R^2 + 6R - 3 + |\nu_j|^2 - 2\phi \left(\frac{3T_0}{8}\right)\phi(t) - 2(R-1)\phi \left(\frac{3T_0}{8}\right)$$

$$\geq 2\phi(t) - R^2 + 1;$$

(3.22)

while in the case of $1 - |\nu_j| \geq 0$, it follows from (3.21) that

$$2\left\{\left(\phi(t) + R\right)^2 - |x|^2\right\} - \{\phi^2(t) - |x - \nu_j|^2\} \geq R^2 + 1 > 0.$$  

(3.23)

Substituting (3.22)-(3.23) into (3.21) yields that for $2\phi(t) \geq R^2 - 1$,

$$\phi^2(t) - |x - \nu_j|^2 \leq C \left(\left(\phi(t) + R\right)^2 - |x|^2\right).$$

On the other hand, if $2\phi(t) < R^2 - 1$, then

$$\phi^2(t) - |x - \nu_j|^2 \leq \phi^2(t) \leq C_R \leq C_R \left(\left(\phi(t) + R\right)^2 - |x|^2\right).$$

(3.24)

Therefore,

$$\left\|\left(\left(\phi(t) + R\right)^2 - |x|^2\right)^\gamma \right\|_{L^\gamma((\frac{T_0}{2}, \infty) \times \mathbb{R})}$$

$$\leq C \sum_{j=1}^{N_0} \left\|\left(\left(\phi(t) + R\right)^2 - |x|^2\right)^\gamma \right\|_{L^\gamma(Q_j)}$$

$$\leq C \sum_{j=1}^{N_0} \left\|\left(\phi^2(t) - |x - \nu_j|^2\right)^\gamma \right\|_{L^\gamma(Q_j)}$$

(by (3.20))

$$\leq C \sum_{j=1}^{N_0} \left\|\left(\phi^2(t) - |x - \nu_j|^2\right)^\gamma F_j \right\|_{L^\frac{\gamma}{\gamma-1}(Q_j)}$$

(by (3.11))

$$\leq C \sum_{j=1}^{N_0} \left\|\left(\left(\phi(t) + R\right)^2 - |x|^2\right)^\gamma F_j \right\|_{L^\frac{\gamma}{\gamma-1}(Q_j)}$$

(by (3.24))

$$\leq C \left\|\left(\left(\phi(t) + R\right)^2 - |x|^2\right)^\gamma F \right\|_{L^\frac{\gamma}{\gamma-1}((\frac{T_0}{2}, \infty) \times \mathbb{R})},$$

which derives (3.10).

\(\Box\)

4. **Proof of Theorem 1.1.** To establish the global existence, we set two cases.

4.1. **The case** \(p_{\text{crit}} < p < p_0 = 9\). By the local existence and regularity of weak solution \(u\) to (1.10) (see, e.g., [26] and the references therein), one has that \(u \in C^\infty(\mathbb{R} \times \mathbb{R})\) exists for the same \(\bar{T}\) as in (1.9), and \(u\) has compact support in the variable \(x\). Furthermore, for any \(N \in \mathbb{N}\),

$$\left\|u(\frac{T}{2}, \cdot)\right\|_{C^N} + \left\|\partial_t u(\frac{T}{2}, \cdot)\right\|_{C^N} \leq C_N \varepsilon.$$  

(4.1)

Then we can take \(u\left(\frac{T}{2}, x\right), \partial_t u\left(\frac{T}{2}, x\right)\) as the new initial data to solve (1.10) starting at time \(t = \frac{T}{2}\).
Now we use standard Picard iteration to prove Theorem 1.1. Let \( u_{-1} \equiv 0 \), and for \( k \in \mathbb{N} \), let \( u_k \) be the solution to the equation

\[
\begin{align*}
\begin{cases}
\partial_t^2 u_k - t \partial_x^2 u_k &= F_p(t, u_{k-1}), \quad (t, x) \in \left( \frac{T}{2}, \infty \right) \times \mathbb{R}, \\
\partial_t u_k \left( \frac{T}{2}, x \right) &= u \left( \frac{T}{2}, x \right), \quad \partial_x u_k \left( \frac{T}{2}, x \right) = \partial_t u \left( \frac{T}{2}, x \right).
\end{cases}
\end{align*}
\]

By (2.2), (3.8) and (3.9), for \( p > p_{\text{crit}} \), we have to pick a number \( \gamma \) satisfying

\[
\gamma < \min \left\{ \frac{1}{p(p+1)} - \frac{5}{6(p+1)} \right\}, \quad (p-1) \gamma + \frac{1}{6} > \frac{5}{3(p+1)}.
\]

Set

\[
\begin{align*}
M_k &= \left\| \left( (\phi(t) + R)^2 - |x|^2 \right)^\gamma u_k \right\|_{L^p((\frac{T}{2}, \infty) \times \mathbb{R})}, \\
N_k &= \left\| \left( (\phi(t) + R)^2 - |x|^2 \right)^\gamma (u_k - u_{k-1}) \right\|_{L^p((\frac{T}{2}, \infty) \times \mathbb{R})},
\end{align*}
\]

where \( q = p + 1 \). In addition, by (4.1) and Theorem 2.1, there exists a constant \( C_0 > 0 \) such that

\[
M_0 \leq C_0 \varepsilon.
\]

Note that, for \( k, j \geq 0 \),

\[
\begin{align*}
\begin{cases}
\partial_t^2 (u_{k+1} - u_{j+1}) - t \partial_x^2 (u_{k+1} - u_{j+1}) &= V(u_k, u_j)(u_k - u_j), \\
\left( u_{k+1} - u_{j+1} \right) \left( \frac{T}{2}, x \right) &= 0, \quad \partial_t (u_{k+1} - u_{j+1}) \left( \frac{T}{2}, x \right) = 0,
\end{cases}
\end{align*}
\]

where

\[
|V(u_k, u_j)| \leq \begin{cases}
C (|u_k| + |u_j|)^{p-1} & \text{if } t \geq \frac{T}{2}, \\
C (1 + |u_k| + |u_j|)^{p-2} (|u_k| + |u_j|) & \text{if } \frac{T}{2} \leq t \leq \frac{T}{2}.
\end{cases}
\]

Then applying Theorem 3.4 and Hölder’s inequality yields that, for \( q = p + 1 \),

\[
\begin{align*}
&\left\| \left( (\phi(t) + R)^2 - |x|^2 \right)^\gamma (u_{k+1} - u_{j+1}) \right\|_{L^p((\frac{T}{2}, \infty) \times \mathbb{R})} \\
\leq C &\left( \left\| \left( (\phi(t) + R)^2 - |x|^2 \right)^{\gamma (1 + |u_k| + |u_j|)} \right\|_{L^q((\frac{T}{2}, \frac{T}{2}) \times \mathbb{R})} \\
&\times \left\| \left( (\phi(t) + R)^2 - |x|^2 \right)^{\gamma (|u_k| + |u_j|)} \right\|_{L^q((\frac{T}{2}, \infty) \times \mathbb{R})} \right)^{-2} \\
&\times \left\| \left( (\phi(t) + R)^2 - |x|^2 \right)^{\gamma (|u_k| + |u_j|)} \right\|_{L^q((\frac{T}{2}, \infty) \times \mathbb{R})} \right)^{-2} \\
\leq C &\left( C_1 \frac{T^1}{q} + M_k + M_j \right)^{-2} (M_k + M_j) \\
&\times \left\| \left( (\phi(t) + R)^2 - |x|^2 \right)^{\gamma (u_k - u_j)} \right\|_{L^q((\frac{T}{2}, \infty) \times \mathbb{R})}.
\end{align*}
\]

(4.2)
Let $j = -1$, assuming that $M_k \leq 2M_0 \leq 2C_0\varepsilon$, then in view of $M_{-1} = 0$, we conclude from (4.2) that

$$M_{k+1} \leq M_0 + C \left( C_1 T^{\frac{k}{n}} + M_k \right)^{p-2} M_k^2.$$ 

This yields that, if we choose $\varepsilon$ small enough such that

$$C \left( C_1 T^{\frac{k}{n}} + M_k \right)^{p-2} M_k \leq C \left( C_1 T^{\frac{k}{n}} + M_k \right)^{p-2} 2C_0\varepsilon \leq \frac{1}{2},$$

then

$$M_{k+1} \leq M_0 + \frac{1}{2} 2M_0 \leq 2M_0.$$ 

Thus we have obtained the boundedness of the sequence $\{u_k\}$ in the space $L^q(\mathbb{R}^{1+1}_+)$ when the constants $T$ and $\varepsilon > 0$ are chosen to be sufficiently small. Similarly, we have

$$N_{k+1} \leq \frac{1}{2} N_k,$$

which shows the existence of a function $u \in L^q((\frac{T}{2}, \infty) \times \mathbb{R})$ with $u_k \to u \in L^q((\frac{T}{2}, \infty) \times \mathbb{R})$. Moreover, from the boundedness of the sequence $\{M_k\}$ and the computations above, one easily obtains

$$\|F_p(t, u_{k+1}) - F_p(t, u_k)\|_{L^{\frac{q}{n}}((\frac{T}{4}, \infty) \times \mathbb{R})} \leq C \|u_{k+1} - u_k\|_{L^q((\frac{T}{4}, \infty) \times \mathbb{R})} \leq C \phi(\frac{T}{4})^{-\gamma} N_k \leq C 2^{-k}.$$ 

Therefore, $F_p(t, u_k) \to F_p(t, u)$ in $L^{\frac{q}{n}}((\frac{T}{2}, \infty) \times \mathbb{R})$. Hence, $u$ is a weak solution to (1.10).

### 4.2. The case $p \geq p_0 = 9$

In this case, Theorem 1.1 can be proven in a manner that is analogous to the proof of [12, Theorem 1.2]. We just give the sketch of the proof here.

The first step is to establish the Strichartz estimates for linear equations. To this end, we study the linear Cauchy problem

$$\begin{cases}
\partial_t^2 u - t\partial_x^2 u = F(t, x), & (t, x) \in \mathbb{R}^{1+1}_+, \\
u(0, x) = f(x), & \partial_t u(0, x) = g(x).
\end{cases}$$

(4.3)

Note that the solution $u$ of (4.3) can be written as

$$u(t, x) = v(t, x) + w(t, x),$$

where $v$ solves the homogeneous problem

$$\begin{cases}
\partial_t^2 v - t\partial_x^2 v = 0, & (t, x) \in \mathbb{R}^{1+1}_+, \\
v(0, x) = f(x), & \partial_t v(0, x) = g(x),
\end{cases}$$

(4.4)

and $w$ solves the inhomogeneous problem with zero initial data

$$\begin{cases}
\partial_t^2 w - t\partial_x^2 w = F(t, x), & (t, x) \in \mathbb{R}^{1+1}_+, \\
w(0, \cdot) = 0, & \partial_t w(0, \cdot) = 0.
\end{cases}$$

(4.5)

Let $\dot{H}^s(\mathbb{R})$ denote the homogeneous Sobolev space with norm

$$\|f\|_{\dot{H}^s(\mathbb{R})} = \|D_x^s f\|_{L^2(\mathbb{R})}.$$
If $g \equiv 0$ in (4.4), we intend to establish the Strichartz-type inequality
\[ \|v\|_{L^q_t L^r_x} \leq C \|f\|_{\dot{H}^{s}(\mathbb{R}^n)}, \]
where $q \geq 1$ and $r \geq 1$ are suitable constants related to $s$. One obtains by a scaling argument that those indices should satisfy
\[ \frac{1}{q} + \frac{3}{2r} = \frac{3}{2} \left( \frac{1}{2} - s \right). \]
Setting $r = q$ and $s = \frac{1}{3}$ in (2.5), we find that
\[ q = q_0 \equiv 10 > 2. \]
Note that problem (1.1) is ill-posed for $u_0 \in H^s(\mathbb{R})$ with $s < \frac{1}{2} - \frac{4}{3(p - 1)}$ (see Remark 1.4 in [12]), while $p \geq p_{\text{conf}}$ and $s = \frac{1}{2} - \frac{4}{3(p - 1)}$ imply $s \geq \frac{1}{3}$. We then have:

**Lemma 4.1.** Let $v$ solve problem (4.4). Further let $\frac{1}{2} \leq s < \frac{1}{2}$. Then
\[ \|v\|_{L^q(\mathbb{R}^{1+1}_+)} \leq C \left( \|f\|_{\dot{H}^{s}(\mathbb{R})} + \|g\|_{\dot{H}^{s-\frac{1}{2}}(\mathbb{R})} \right), \]
where $q = \frac{10}{3(1-2s)} \geq q_0$ and the constant $C > 0$ only depends on $s$.

The proof of Lemma 4.1 is the same as that of Lemma 3.3 in [12], thus we omit the details.

Next we treat the inhomogeneous problem (4.5). Based on Lemmas 4.1, we establish the following estimate:

**Lemma 4.2.** Let $w$ solve (4.5). Then
\[ \|w\|_{L^q(\mathbb{R}^{1+1}_+)} \leq C \left( |D_x|^{\gamma - \frac{1}{2}} F \right)_{L^{p_0}(\mathbb{R}^{1+1}_+)}, \]
where $\gamma = \frac{1}{2} - \frac{p}{q_0}$, $q_0 \leq q < \infty$, and the constant $C > 0$ only depends on $q$.

**Proof.** As in the proof of Lemma 3.4 in [12], we can write
\[ w = (AF)(t,x) \equiv \int_0^t \int_{\mathbb{R}} e^{i(x \cdot \xi + (\phi(t) - \phi(\tau)) \cdot \xi)} a(t,\tau,\xi) \hat{F}(\tau,\xi) d\xi d\tau, \]
where $a(t,\tau,\xi)$ satisfies
\[ |\partial_\xi^\beta a(t,\xi)| \leq C (1 + \phi(t)|\xi|)^{-\frac{1}{2}} (1 + \phi(\tau)|\xi|)^{-\frac{1}{2}} |\xi|^{-\frac{1}{2} - |\beta|}. \]
To treat $(AF)(t,x)$ conveniently, we introduce the more general operator
\[ (A^\alpha F)(t,x) = \int_0^t \int_{\mathbb{R}} e^{i(x \cdot \xi + (\phi(t) - \phi(\tau)) \cdot \xi)} a(t,\tau,\xi) \hat{F}(\tau,\xi) \frac{d\xi}{|\xi|^{1+\alpha}} d\tau, \]
where $0 \leq \alpha < \frac{1}{2}$ is a parameter.

As in the proof of Lemma 4.1, we shall use the Littlewood-Paley argument with a bump function $\beta$. Define the operator
\[ A^\alpha F(t,x) = \int_0^t \int_{\mathbb{R}} e^{i(x \cdot \xi + (\phi(t) - \phi(\tau)) \cdot \xi)} \beta \left( \frac{|\xi|}{2^j} \right) a(t,\tau,\xi) \hat{F}(\tau,\xi) \frac{d\xi}{|\xi|^{1+\alpha}} d\tau. \]
Note that our aim is to establish the following inequality for $\gamma = \frac{1}{2} - \frac{p}{q_0}$ and $q_0 \leq q < \infty$,
\[ \|w\|_{L^q(\mathbb{R}^{1+1}_+)} \leq C \left( |D_x|^{\gamma - \frac{1}{2}} F \right)_{L^{p_0}(\mathbb{R}^{1+1}_+)} \]
which is equivalent to prove
\[
\left\| D^\frac{\alpha + \frac{3}{2}}{2} \right\|_{L^q(\mathbb{R}^{1+1})} \leq C \left\| F \right\|_{L^{p_0}(\mathbb{R}^{1+1})}.
\]
In terms of the definition of operator $A^\alpha$ in (4.9) with $\alpha = \gamma - \frac{1}{3}$, in order to complete the proof of (4.6), it suffices to establish
\[
\| A^\alpha F \|_{L^q(\mathbb{R}^{1+1})} \leq C \| F \|_{L^{p_0}(\mathbb{R}^{1+1})}.
\] (4.10)

Note that $p_0 < 2 < q < \infty$. To derive (4.10), it follows from the proof of Lemma 3.1 in [12] that we only need to prove
\[
\| A^\alpha F \|_{L^q(\mathbb{R}^{1+1})} \leq C \| F \|_{L^{p_0}(\mathbb{R}^{1+1})}.
\] (4.11)

By interpolation, it suffices to prove that (4.11) holds for the special cases $q = q_0$ and $q = \infty$. Denote the corresponding indices $\alpha$ by $\alpha_0$ and $\alpha_1$. A direct computation yields $\alpha_0 = \frac{1}{2} - \frac{5}{3q_0} - \frac{1}{3} = 0$ and $\alpha_1 = \frac{5}{6}$. The treatment for $q = q_0$ is exactly the same as in [12], we only need to handle the case $q = \infty$.

Next we prove (4.11) for $q = \infty$. In this case, the kernel of $A^\alpha_j$ can be written as
\[
K_j^{\alpha_1}(t, x; \tau, y) = \int_{\mathbb{R}^n} \beta \left( \frac{|\xi|}{2^j} \right) e^{i((x-y)\xi + (\phi(t) - \phi(\tau))\xi)} a(t, \tau, \xi) \frac{d\xi}{|\xi|^\alpha_1}.
\]

We now assert
\[
\sup_{t, x} \int_{\mathbb{R}^{1+1}_+} |K_j^{\alpha_1}(t, x; \tau, y)|^{q_0} d\tau dy < \infty. 
\] (4.12)

Obviously, if (4.12) is true, then a direct application of Hölder’s inequality yields (4.11) for $q = \infty$. By [29, Lemma 7.2.4] and (4.8), we have
\[
|K_j^{\alpha_1}(t, x; \tau, y)| 
\leq C_{N, n, \alpha_1} \lambda^{1-\alpha_1} \left( 1 + \lambda |\phi(t) - \phi(\tau)| \right)^{-\frac{3}{2}} (1 + \lambda |x - y| - |\phi(t) - \phi(\tau)|)^{-N},
\]
where $\lambda = 2^j$, $N = 0, 1, 2, \ldots$, and
\[
\alpha_1 = \frac{2}{3} + \frac{1}{2} - \frac{1}{3} = \frac{5}{6}.
\]

It suffices to prove (4.12) in case $x = 0$. In fact, a direct computation yields
\[
\int_{\mathbb{R}^{1+1}_+} |K_j^{\alpha_1}(t, 0; \tau, y)|^{q_0} d\tau dy
\leq \int_{-\infty}^{\infty} \int_{\mathbb{R}} \lambda^{\frac{q_0}{2}} \lambda^{-\frac{3q_0}{2}} \left( |\phi(t) - \phi(\tau)| + \lambda^{-1} \right)^{-\frac{3q_0}{2}} (1 + \lambda |y| - |\phi(t) - \phi(\tau)|)^{-N} d\tau dy
\leq C \int_{-\infty}^{\infty} \left( |\phi(t) - \phi(\tau)| + \lambda^{-1} \right)^{-\frac{3q_0}{2}} \lambda^{-1} d\tau
\leq C \int_{-\infty}^{\infty} \lambda^{-1} \left( |t - \tau| + \lambda^{-\frac{2}{3}} \right)^{-\frac{5}{2}} d\tau \leq C.
\]

Thus, by interpolation, (4.10) and then further (4.6) are shown.

Relying on Lemmas 4.1 and 4.2, we have:

**Lemma 4.3.** Let $w$ solve (4.5). Then
\[
\|w\|_{L^q(\mathbb{R}^{1+1})} + \left\| D_z^{\gamma - \frac{1}{3}} w \right\|_{L^{p_0}(\mathbb{R}^{1+1})} \leq C \left\| D_z^{\gamma - \frac{1}{3}} F \right\|_{L^{p_{\text{conf}}}(\mathbb{R}^{1+1})}.
\]
where $\gamma = \frac{1}{2} - \frac{5}{9q}$, $q_0 \leq q < \infty$, and the constant $C > 0$ only depends on $m$, $n$, and $q$.

Based on these estimates above, we are able to establish the global existence result for $p \geq p_{\text{conf}}$, the procedure is similar to that of Part 1 in [12, Section 4], and we obtain

**Proposition 4.4** (Global existence for $p > p_{\text{conf}} = 9$). Assume that $9 \leq p < \infty$. Then there exists a constant $\varepsilon_0 > 0$ such that problem (1.1) admits a global weak solution $u \in L^r(R^{1+1}_+) \cap C^\infty_0$ whenever $\|u_0\|_{H^s} + \|u_1\|_{H^{s-\frac{3}{2}}} \leq \varepsilon_0$, where $s = \frac{1}{2} - \frac{4}{3(p-1)}$ and $r = \frac{5}{4} \frac{5}{p-1}$.

Combining both cases in Subsection 4.1 and Subsection 4.2, we have completed the proof of Theorem 1.1. □

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**REFERENCES**

[1] J. Barros Neto and I. M. Gelfand, *Fundamental solutions for the Tricomi operator I, II, III*, Duke Math. J., 98 (1999), 465–483; 111 (2002), 561–584; 117 (2003), 385–387.
[2] M. Beals, Singularities due to cusp interactions in nonlinear waves in "Nonlinear Hyperbolic Equations and Field Theory", Pitman Res. Notes Math. Ser., Longman Sci. Tech. Harlow, 253 (1992), 36–51.
[3] L. Bers, *Mathematical Aspects of Subsonic and Transonic Gas Dynamics*, Surv. Appl. Math., Vol.3, Chapman and Hall, London, 1958.
[4] M. D’Abbicco, *The threshold of effective damping for semilinear wave equations*, Math. Meth. Appl. Sci., 6 (2015), 1032–1045.
[5] A. Erdelyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, *Higher Transcendental Functions*, Vol. 1, McGraw-Hill, New York, 1953.
[6] A. Erdelyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, *Higher Transcendental Functions*, Vol. 2, McGraw-Hill, New York, 1953.
[7] A. Galstian, Global existence for the one-dimensional second order semilinear hyperbolic equations, *J. Math. Anal. Appl.*, 344 (2008), 76–98.
[8] R. T. Glassey, Finite-time blow-up for solutions of nonlinear wave equations, *Math. Z.*, 177 (1981), 323–340.
[9] R. T. Glassey, Existence in the large for $\Box u = F(u)$ in two space dimensions, *Math. Z.*, 178 (1981), 233–261.
[10] V. Georgiev, H. Lindblad and C. D. Sogge, Weighted Strichartz estimates and global existence for semi-linear wave equations, *Amer. J. Math.*, 119 (1997), 1291–1319.
[11] D. K. Gvazava, The global solution of the Tricomi problem for a class of nonlinear mixed differential equations, *Differ. Equ.*, 3 (1967), 1–4.
[12] D. Y. He, I. Witt and H. C. Yin, On the global solution problem of semilinear generalized Tricomi equations, I, *Calc. Var. Partial Differ. Equ.*, 56 (2017), 1–24.
[13] D. Y. He, I. Witt and H. C. Yin, On the global solution problem of semilinear generalized Tricomi equations, II, preprint, arXiv:1611.07606, to appear in *Pac. J. Math.*, 2020.
[14] D. Y. He, I. Witt and H. C. Yin, On semilinear Tricomi equations with critical exponents or in two space dimensions, *J. Differ. Equ.*, 263 (2017), 8102–8137.
[15] D. Y. He, I. Witt and H. C. Yin, Finite time blowup for the 1-D semilinear Tricomi equation with critical exponent, preprint, 2019.
[16] F. John, Blow-up of solutions of nonlinear wave equations in three space dimensions, *Manuscr. Math.*, 28 (1979), 235–265.
[17] J. Lin and Z. Tu, Lifespan of semilinear generalized Tricomi equation with Strauss type exponent, preprint, arXiv:1903.11351.
[18] H. Lindblad and C. D. Sogge, On existence and scattering with minimal regularity for semilinear wave equations, *J. Funct. Anal.*, 130 (1995), 357–426.
[19] D. Lupo, C. S. Morawetz and K. R. Payne, On closed boundary value problems for equations of mixed elliptic-hyperbolic type, *Commun. Pure Appl. Math.*, 60 (2007), 1319–1348.
[20] D. Lupo and K. R. Payne, Spectral bounds for Tricomi problems and application to semilinear existence and existence with uniqueness results, *J. Differ. Equ.*, 184 (2002), 139–162.
[21] D. Lupo and K. R. Payne, Critical exponents for semilinear equations of mixed elliptic-hyperbolic and degenerate types, *Commun. Pure Appl. Math.*, 56 (2003), 403–424.
[22] D. Lupo and K. R. Payne, Conservation laws for equations of mixed elliptic-hyperbolic and degenerate types, *Duke Math. J.*, 127 (2005), 251–290.
[23] C. S. Morawetz, Mixed equations and transonic flow, *J. Hyperbolic Differ. Equ.*, 1 (2004), 1–26.
[24] Z. P. Ruan, I. Witt and H. C. Yin, On the existence and cusp singularity of solutions to semilinear generalized Tricomi equations with discontinuous initial data, *Commun. Contemp. Math.*, 17 (2015), 49 pp.
[25] Z. P. Ruan, I. Witt and H. C. Yin, On the existence of low regularity solutions to semilinear generalized Tricomi equations in mixed type domains, *J. Differ. Equ.*, 259 (2015), 7406–7462.
[26] Z. P. Ruan, I. Witt and H. C. Yin, Minimal regularity solutions of semilinear generalized Tricomi equations, *Pac. J. Math.*, 296 (2018), 181–226.
[27] J. Schaeffer, The equation $u_{tt} - \Delta u = |u|^p$ for the critical value of $p$, *Proc. R. Soc. Edinb.*, 101 (1985), 31–44.
[28] T. Sideris, Nonexistence of global solutions to semilinear wave equations in high dimensions, *J. Differ. Equ.*, 52 (1984), 378–406.
[29] C. D. Sogge, *Fourier Integrals in Classical Analysis*, Cambridge Tracts in Math., Vol. 105, 1993.
[30] W. Strauss, Nonlinear scattering theory at low energy, *J. Funct. Anal.*, 41 (1981), 110–133.
[31] K. Taniguchi and Y. Tozaki, A hyperbolic equation with double characteristics which has a solution with branching singularities, *Math. Jpn.*, 25 (1980), 279–300.
[32] K. Yagdjian, A note on the fundamental solution for the Tricomi-type equation in the hyperbolic domain, *J. Differ. Equ.*, 206 (2004), 227–252.
[33] K. Yagdjian, Global existence in the Cauchy problem for nonlinear wave equations with variable speed of propagation, (English summary) New trends in the theory of hyperbolic equations, *Oper. Theory Adv. Appl.*, 159 (2005), 301–385.
[34] K. Yagdjian, Global existence for the n-dimensional semilinear Tricomi-type equations, *Commun. Partial Differ. Equ.*, 31 (2006), 907–944.
[35] K. Yagdjian, The self-similar solutions of the Tricomi-type equations, *Z. angew. Math. Phys.*, 58 (2007), 612–645.
[36] B. Yordanov and Q. S. Zhang, Finite time blow up for critical wave equations in high dimensions, *J. Funct. Anal.*, 231 (2006), 361–374.
[37] Y. Zhou, Cauchy problem for semilinear wave equations in four space dimensions with small initial data, *J. Differ. Equ.*, 8 (1995), 135–144.

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