Categorical Center of Higher Genera and 4D Factorization Homology

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Abstract

Many quantum invariants of knots and 3-manifolds (e.g. Jones polynomials) are special cases of the Witten-Reshetikhin-Turaev 3D TQFT. The latter is in turn a part of a larger theory - the Crane-Yetter 4D TQFT. In this work, we compute the Crane-Yetter theory for all (smooth and oriented) surfaces with at least one puncture. The results in general are constructed and called the categorical center of higher genera.

Contents

0 Introduction 3
1 Overview 4
  1.1 Invariants as data representations 4
  1.2 TQFTs as higher invariants 4
  1.3 CY as an extended TQFT 6
    1.3.1 The Crane-Yetter theory CY 6
    1.3.2 Previous work of CY in (co)dimension two 6
  1.4 Main result 8
    1.4.1 CY ∼ Z 8
    1.4.2 Remarks 9
  1.5 Summary of each section 10
2 Preliminaries 10
  2.1 Premodular Category 10
  2.2 Graphical Calculus 11
3 Topological theory 12
  3.1 String nets 13
  3.2 Crane-Yetter theory in dimension two (CY) 16
  3.3 A presentation of surfaces (σ-construction) 17
4 Algebraic theory 20
   4.1 Motivation: Drinfeld categorical center ........................................ 20
   4.2 Categorical center of higher genera (Z) ........................................ 21
   4.3 Properties of Z .............................................................................. 23
      4.3.1 Connecting functors ............................................................... 23
      4.3.2 Ambidextrous adjunction ......................................................... 24
      4.3.3 Z is finite semisimple abelian ............................................... 27

5 Proof of the main statement 27
   5.1 Strategy ......................................................................................... 28
   5.2 Proof ............................................................................................. 28
      5.2.1 Reducing topological data ......................................................... 28
      5.2.2 topology → algebra ................................................................. 31
      5.2.3 topology ← algebra ................................................................. 32
      5.2.4 topology ↔ algebra ................................................................ 33

6 Outlook and remarks 34

7 Appendix 34
   7.1 Abelian categories ......................................................................... 35
   7.2 Semisimple category ...................................................................... 37
   7.3 Tensor category ............................................................................... 38
   7.4 Adjunctions as monads .................................................................. 41
   7.5 Misc proofs ...................................................................................... 48

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0 Introduction

Many quantum invariants of knots and 3-manifolds (e.g. Jones polynomials) are special cases of the Witten-Reshetikhin-Turaev 3D TQFT, which is in turn a part of a larger theory - the Crane-Yetter 4D TQFT (also called the CY model). The CY model, first given as a state sum [CY], is believed to be a fully-extended TQFT. In particular, its definition in (co)dimension 2 has been given in [KT21].

In this work, we compute the Crane-Yetter theory for all (smooth and oriented) surfaces with at least one puncture. To do that, we first present a surface $\Sigma$ by a combinatorial datum $\sigma$ (admissible gluing 3.17) via a combinatorial construction ($\sigma$-construction 3.18) so that $\Sigma = \Sigma_\sigma$. We then construct a category $Z_\sigma$ called the categorical center of higher genera and show some of its basic properties (section 3.4). The main result (5.1) proves an equivalence of finite semisimple abelian categories $\text{CY}_C(\Sigma) \simeq Z_\sigma(C)$ for any premodular category $C$ and any algebraically closed field $\mathbb{k}$ of characteristic 0.

This result generalizes almost all known results in (co)dimension 2 (section 1.3.2). In particular, it generalizes the Drinfeld categorical center to higher genera.
1 Overview

1.1 Invariants as data representations

On investigating a mathematical object, one starts with a presentation, which is a description of the object from the simpler ones. Though a presentation in principle fully describes the object, it is yet an image distant to the essence of the object. For example, a finite group presented by a finite set of generators and relations is fully described but by no means well understood. Similarly, though a space presented by a finite set of (higher dimensional) triangles and gluing data is in principle fully described, its topological structure can be obscured by the complexity of the data. It is therefore natural to seek for deeper understandings. One way often used is to represent the presented data in different forms.

On representing data, two issues arise. First, how accurate is the representation? Does it lose any information? And how much is lost if it does? Second, the representation is again a mathematical object, so it is also natural to ask how much we understand the represented object. Is it easy to compute or characterize? In many cases, one should balance between the two. Indeed, a more accurate representation often teaches us less, an extreme case being the presentation itself; a less accurate representation to a larger degree simplifies the object and would hopefully be more enlightening to the mortals, a simple example being the size of a finite set: while it does not faithfully represent the set as a mathematical object, it is useful to some extent.

Such representations are called invariants. In the context of topology, they are called topological invariants. The simplest example is the notion of dimension, which assigns a given space to a positive integer. The genus of a surface is another classical instance among the integer-valued invariants. However, an invariant needs not take numbers as its values. Numbers may be easier to understand, but they forget too much information. Being an topological invariant, the homology $H(\cdot;\mathbb{Q})$ takes vector spaces as values. Certainly, they do tell us more than the dimension does. More sophisticated examples can take even more complicated algebraic objects (groups, algebras, Hopf algebras.. etc) as values. For instance, a milestone in algebraic topology is the celebrated theorem of Mandell.

Theorem 1.1 [Man06] Finite type nilpotent spaces $X$ and $Y$ are weakly equivalent if and only if the $\text{E}_\infty$-algebras $\text{C}^\ast(X)$ and $\text{C}^\ast(Y)$ are quasi-isomorphic.

Such leap of thought from numbers to higher objects is seminal. Not only does it enable us to represent the objects more accurately, perhaps more importantly, it allows us to represent the relations among objects of interest by (higher) functors.

1.2 TQFTs as higher invariants

Convention 1.2 (manifold, field) Throughout this paper, by a manifold of dimension $n$ we mean an oriented smooth manifold without boundary of real-dimension $n$; we also work over a fixed field $k$ that is algebraically closed and with characteristic 0.

Another example of an invariant that takes vector spaces as values, originally motivated by theoretical physics, is called a topological quantum field theory (TQFT) à la Michael Atiyah [Ati88]. Roughly put, a TQFT, like a homology, assigns vector spaces to some manifolds. It also, in a slightly different manner, assigns in a compatible way a linear transformation to the manifolds which are of one-dimension higher
and bridge the lower manifolds. Curiously, this gives a number-valued invariant for closed manifolds: closed manifolds “bridge” the empty manifolds, to which the trivial vector space is assigned, so the linear transformations assigned to the closed manifolds are numbers.

This generalization from numbers to vector spaces does not stop here. By viewing vector spaces as objects in a category, one can bring this process to assigning objects in higher categories. An instance of such is called an extended topological quantum field theory (eTQFT) [Lur09].

**Example 1.3 (Witten-Reshetikhin-Turaev model)** Among manifolds, the one-dimensional and two-dimensional ones are fully classified. Though a presentation for the three-dimensional case is provided by knots and surgery theory (cf [Lic, chap.12] and [Kir78]), it is based on the classification of knots, which is itself complicated and in fact still an ongoing research field. Therefore, invariants of 3-manifolds are of interest.

Around the 80s, initiated by V Jones with his celebrated Jones polynomials, a new type of invariant was born in the context of mathematical-physics. For example, one can construct the invariants from von Neumann algebras, quantum groups, and rational conformal theories [CY, sec.1]. It is natural to ask if they can be unified. The answer is positive in that all of them are special cases of the Witten-Reshetikhin-Turaev (WRT) model, whose input algebraic data is called a modular tensor category [Bar+15]).

As an extended TQFT, WRT model assigns a number to each closed 3-manifold, a vector space to each closed 2-manifold, and a linear category to each closed 1-manifold. However, it does not go on and assign a linear 2-category in the 0-th dimension. See 1.7 for more discussion.

**Example 1.4 (Turaev-Viro model)** The Turaev-Viro (TV) model is first given as an invariant for 3-manifolds in the form of a state sum [TV92]. It was clear then that the TV model gives a 3-dimensional TQFT.

Later, it is proven to be a fully extended TQFT in [KB10]. Namely, it compatibly assigns a number to each closed 3-manifold, a vector space to each closed 2-manifold, a linear category to each closed 1-manifold, and a linear 2-category to each 0-manifold (i.e. the point). It was also shown that

\[ TV_C(M) \simeq WRT_{Z(C)}(M) \]

where \( M \) is a closed manifold of dimension 2 or 3, \( C \) is a spherical category and \( Z(C) \) is its Drinfeld categorical center ([Bal10] and [Bal11]).

**Example 1.5 (Crane-Yetter model)** The Crane-Yetter model is a 4-dimensional analogue of the TV model. Its input algebraic data is a premodular category. It is the main topic of this paper, and will be treated thoroughly in its own section (1.3).

**Theorem 1.6 \((\partial CY = WRT)\)** The WRT model is the boundary theory of CY, while the input algebraic data is a modular category. More precisely, for a modular category \( C \) and a 4-dimensional manifold \( W \) possibly with boundary,

\[ CY_C(W) = \kappa^{\sigma(W)} WRT_C(\partial W), \]

where \( \sigma \) denotes the signature and \( \kappa \) denotes a constant based on the input algebraic data (the modular category given in 2.6). This extends to manifolds with colored graphs [BGM07, Theorem 2].

For a 3-dimensional manifold \( M \) possibly with boundary, the vector spaces \( CY_C(M) \) (cf 1.15) and \( WRT_C(\partial M) \) are widely believed to be equivalent. However, to our best knowledge a rigorous proof has not been provided.
Remark 1.7 In 1.3 we mentioned that the WRT model does not extend to the 0-th dimension. There are two explanations for this fact.

1. From the Turaev-Viro model perspective [Bal10], for WRT to extend to a point, one needs the input modular category $C$ to be the Drinfeld center of a spherical fusion category $D$. This is not always the case.

2. From the Crane-Yetter (CY) model perspective, the WRT model is a boundary theory of CY (1.6). However, a single point is not the boundary of any 1-manifold: one needs at least two points.

From the second point of view of 1.3, one sees that the WRT model, though successful and fruitful, is a part of larger theory - the Crane-Yetter model, which is the main TQFT we will focus on in this work.

1.3 CY as an extended TQFT

In this paper, we focus on a specific extended TQFT, namely, the Crane-Yetter theory. We provide some historical context, display some known results, and address our main result.

1.3.1 The Crane-Yetter theory CY

Originated in [CY], the Crane-Yetter model was first defined as a state-sum. In particular, for a triangulated 4-manifold $M$ and a modular tensor category $C$, one define

$$\text{CY}(M) = \Sigma c \, D^{(n_0-n_1)} \Pi_{\sigma} \dim c(\sigma) \Pi_t \dim c(t) \Pi_{\xi} 15j(c, \xi),$$

where $c$ runs through all “colorings”, $n_i$ is the number of simplices of dimension $i$ in the triangulation, $\sigma$ runs through all triangles, $t$ runs through all tetravhedra, $\xi$ runs through all 4-simplices, and $15j$ denotes the so called 15j-symbols. The upshot is that the sum is independent of the triangularization, therefore defines an invariant of 4-manifolds. This holds even when the $C$ is a premodular category.

Later in [CKY97], it was known that the this seemingly complicated sum can be expressed in terms of the Euler characteristic and the signature, both being old and well-known topological invariants. While this provides a combinatorial formula for the signature of 4-folds, it also means that the CY model with the input data being a modular tensor category $C$ somehow trivializes. Recall that the WRT model is the boundary theory of the CY model (1.6). This provides a hint for why the CY model trivializes - the input algebraic data, namely the modular tensor categories, are too good for 4-manifolds. Therefore, other types of algebraic data should be considered. For example, in this paper we focus on the premodular categories.

On the other hand, as the TV model, the CY model is also a fully extended TQFT. In particular, it gives a number to each closed 4-manifold, a vector space to each closed 3-manifold, a linear category to each closed 2-manifold, a linear 2-category to each closed 1-manifold (the circle), and finally a linear 3-category to each closed 0-manifold (the point). In this work, we will focus on dimension 2, in which the CY model takes linear categories as values.

1.3.2 Previous work of CY in (co)dimension two

Recent developments of the CY model include its Hamiltonian formulation and its higher-codimensional aspect. The former is called the Walker-Wang model (cf [WW12] and [Wal15]). However, while the relation between CY and the Walker-Wang model are commonly believed, to the best knowledge of the author, it has not been explicitly proved.
The later, on the other hand, is currently studied by A. Kirillov’s school starting around 2018. In particular, Tham and Kirillov correctly defined CY model in (co)dimension 2, computed some examples, and proved the excision property, which connects the CY model to factorization homology. Similar work can be found in the context of skein modules (cf [Coo19]).

In the rest of this section, we recall the results from Kirillov and Tham. The main statement of this paper will follow in the next section.

| Σ    | Disk                 | Cylinder   | Sphere     | 1-punctured torus | General                      |
|------|----------------------|------------|------------|-------------------|------------------------------|
| CYΣ(C) | C                    | Z(C)       | Mu(C)       | ZΣ1(C)            | Z∅(C)                        |
|      | Drinfeld center      | Muger center | Elliptic center | Categorical center of higher genera |

**Theorem 1.8** [KT21, Section 5] Let C be a premodular category, and Σ = D^2 be the open disk. Then as abelian categories

\[ \text{CY}_\Sigma(C) \simeq C. \]

**Theorem 1.9 (Drinfeld center)** [KT21, Example 8.2] Let C be a premodular category, Z(C) its Drinfeld categorical center, and Σ = S^1 × I = S^1 × (0, 1) be the cylinder. Then as abelian categories

\[ \text{CY}_\Sigma(C) \simeq Z(C). \]

Moreover, as multifusion categories, the topological nature of Σ induces a so called reduced tensor product \( \boxtimes \) ([Tha20], [Was19]) for Z(C). Indeed, stacking two cylinders together produces another cylinder \( S^1 \times (0, 2) \simeq S^1 \times I. \)

Notice that the reduced tensor product is in general different from the usual tensor product of the Drinfeld center.

**Theorem 1.10 (Excision principle)** [KT21, Theorem 2.3]

Denote the Deligne tensor product by \( \boxtimes_D \). Let Σ be an open surface with n punctures. Then the category CYΣ(C) has a structure of module category over \( \text{CY}_\Sigma(S^1 \times I)^{\boxtimes_D n} \), which is \( (Z(C), \boxtimes)_{\boxtimes_D}^{\boxtimes_D n} \) by 1.9.

Let Σ1 and Σ2 be smooth oriented surfaces possibly with punctures. And let CY be Crane-Yetter theory in dimension two. Then we have an equivalence of abelian categories.

\[ \text{CY}_{(\Sigma_1 \cup \Sigma_2)}(C) \simeq \text{CY}_{\Sigma_1} \boxtimes_{\text{CY}_{(\Sigma_1 \cap \Sigma_2)}} \text{CY}_{\Sigma_2}, \]

where \( \boxtimes_D \) denotes the balanced (Deligne) tensor product over D [DSS19].

**Remark 1.11** For a related result on the excision principle, see [Coo19] and [BBJ18].

**Theorem 1.12 (Muger centralizer)** [KT21, Corollary 8.5] Let C be a premodular category, Z'(C) its Muger centralizer, and Σ = S^2 be the 2-sphere. Then as abelian categories

\[ \text{CY}_\Sigma(C) \simeq Z'(C). \]

In particular, if C is modular, then the result trivializes as in

\[ \text{CY}_\Sigma(C) \simeq \text{(Vect)}. \]
Tham defined for a premodular category \( C \) an associated category \( Z^{el}(C) \), coined the elliptic Drinfeld center. Its objects are the triples \((X, \gamma_1, \gamma_2)\), where \( X \) is an object of \( C \) and the \( \gamma_i \)'s are half-braidings of \( X \) that satisfy certain relations. See [Tha19] for a full definition. The name is justified by the following theorem.

**Theorem 1.13 (elliptic Drinfeld center)** [KT21, Corollary 9.5+6] Let \( C \) be a premodular category, \( Z^{el}(C) \) its elliptic Drinfeld center, and \( \Sigma = \Sigma_{1,0} \) be a once-punctured torus. Then as abelian categories

\[
\text{CY}_\Sigma(C) \cong Z^{el}(C).
\]

In particular, if \( C \) is modular, then there is an equivalence of left \( \text{CY}_{S^1 \times I}(C) \)-modules

\[
C \cong \text{CY}_{D^2}(C) \cong \text{CY}_\Sigma(C).
\]

\( \diamond \)

**Theorem 1.14** [Tha20, Corollary 4.5] Let \( C \) be a premodular category, \( Z \) the Drinfeld center construction, \( \boxtimes \) the stacking tensor product, and \( \Sigma = S^1 \times S^1 \) be the standard torus. Then

\[
\text{CY}_\Sigma(C) \cong Z((Z(C), \boxtimes)).
\]

\( \diamond \)

One of the main theorem of [KT21] is that Crane-Yetter theorem in dimension two also trivializes when the input data is modular.

**Theorem 1.15** [KT21, Remark 9.8] Let \( C \) be a modular category, and \( \Sigma \) an \( n \)-punctured surface of genus \( g \). Then up to equivalence \( \text{CY}_\Sigma(C) \) is independent of \( g \). In fact, we have an equivalence of module categories over \((Z(C), \boxtimes)^{\otimes n}\)

\[
\text{CY}_\Sigma(C) \cong C^{\boxtimes n}.
\]

Notice that when \( n = 0 \) the power is the category of finite dimensional vector spaces.

\( \diamond \)

An easy proof of this fact due to the author of this paper uses the excision principle and a basic equivalence \( C \boxtimes C^{\text{bop}} \cong Z(C) \).

### 1.4 Main result

**1.4.1 CY \cong Z**

The main result of this paper is an explicit calculation of the Crane-Yetter theory for all smooth oriented surfaces with at least one puncture. In this section, we overview the statement and its consequences, leaving a detailed proof to section 5.

Crane-Yetter theory in dimension two is defined as a linear category whose spaces of morphisms are vector spaces presented by many complicated generators and relations. By a calculation we mean to describe it as a category whose description is much smaller. An analogy of this is that the first homology of the circle “calculated” to be the group of integers

\[
H_1(S^1) \cong \mathbb{Z}.
\]
Given any open surface $\Sigma$, we present it by an oriented 2-disk and some segments of its boundary glued. Which segments are glued together are described by some combinatorial data, called the admissible gluings. With a choice of an admissible gluing $\sigma$, the categorical center of higher genera $Z = Z_\sigma$ is a specific category explicitly constructed in 4.11, with some basic properties given in 4.3. Roughly, given a premodular category $C$ and an admissible gluing $\sigma$ of rank $n$, the categorical center of higher genera $Z = Z_\sigma(C)$ is a category with objects of the form $(X, \gamma_1, \ldots, \gamma_n)$ where $X$ is a $C$-object and the $\gamma_i$'s are half-braidings satisfying specific relations. The main statement is that $Z$ is equivalent to $\text{CY}(\Sigma)$ as a finite semisimple abelian category.

**Theorem 1.16 (Main statement)** Let $n$ be a nonnegative integer, $\sigma$ an admissible gluing of rank $n$, $\Sigma_\sigma$ the surface constructed from $\sigma$, $C$ a premodular category, $\text{CY}_{\Sigma_\sigma}(C)$ the Crane-Yetter theory of $\Sigma_\sigma$ depending on $C$, and $Z_\sigma(C)$ the categorical center of higher genera of $C$ with respect to $\sigma$.

Then we have an equivalence of finite semisimple abelian categories

$$\text{CY}_{\Sigma_\sigma}(C) \simeq Z_\sigma(C).$$

A detailed proof of the main statement is given in section 5.

**Example 1.17 ($n = 0$)** For $n = 0$, the surface $\Sigma$ is the open disk, the categorical center of higher genera reduces to the underlying premodular category $C$, so the theorem recovers that $\text{CY}_C(C) \simeq C$ as shown in 1.8.

**Example 1.18 ($n = 1$)** For $n = 1$, the only possible surface is the cylinder, the categorical center of higher genera reduces to the Drinfeld center $Z(C)$. Hence, the theorem recovers that $\text{CY}_C(C) \simeq Z(C)$ as shown in 1.9.

**Example 1.19 ($n = 2$)** For $n = 2$, there are two possible surfaces: the 1-punctured torus and the 3-punctured disk. In the former case, the categorical center of higher genera reduces to the elliptic center $Z_{\text{el}}(C)$, so the theorem recovers that $\text{CY}_C(C) \simeq Z_{\text{el}}(C)$ as shown in 1.13. In the latter case, the theorem provides a new result.

**1.4.2 Remarks**

**Remark 1.20** In $H_1(S^1) \simeq \mathbb{Z}$, one sees the algebra of the shape $S^1$ and the shape of the algebra $\mathbb{Z}$. Our main result should be viewed as a higher analogue. That is, one sees the (higher) algebra of the shape $\Sigma_\sigma$ and the shape of the (higher) algebra $Z_\sigma$.

**Remark 1.21** A full definition for premodular categories is given in 7.68.

**Remark 1.22** These categories have their tensor structures and module categorical structures [Eti+15] coming from their topological nature. This will be treated in the author’s following work.
Remark 1.23 The smoothness condition is not necessary for our theory, but is included for the sake of
simplicity. Indeed, later we will see that the Crane-Yetter theory in dimension 2 can be defined based on
stringnets. With the smooth structure, it is easier to regulate how they meet each other. On the other hand,
Crane-Yetter theory works also in the PL-setting, parallel to its 3-dimensional analogue, the Turaev-Viro
theory. Curious readers are refer to a setup given in [KB10].

1.5 Summary of each section
- Section 2: Premodular categories and their graphical calculus are treated with examples and useful
facts.
- Section 3: The relevant topological theory, namely the Crane-Yetter theory in dimension two, is treated
formally in terms of string-nets ([Kir11]).
- Section 4: The relevant algebraic theory, namely the categorical center of higher genera $Z_\sigma(C)$, is
constructed. We prove some of its basic properties, such as its finite semisimple abelianess and its
ambidextrous adjunction with the underlying $C$.
- Section 5: The proof for the main theorem is given, which bridges the topological theory and the
algebraic theory.
- Section 6: Outlook and remarks.
- Appendix: For completeness, we include formal definitions, propositions, and detailed proofs.

2 Preliminaries

2.1 Premodular Category
A full definition of a premodular category from scratch is tedious but well known to experts. However,
for completeness we include it in the appendix 7.3. In short, it is a fusion category with a braiding and a
spherical structure.

Definition 2.1 (Premodular category) A premodular category is a braided fusion category equipped
with a spherical structure.

Example 2.2 (Finite group) Let $G$ be a finite group. Then the category $\text{Rep}(G)$ of finite-dimensional
linear representations of $G$ over $k$ has a natural structure of a premodular category.

Example 2.3 (Drinfeld double) Let $G$ be a finite group and $D(G)$ its Drinfeld double over $k$. Then the
category $\text{Rep}(D(G))$ of finite-dimensional linear representations of $D(G)$ over $k$ has a natural structure of a
premodular category.

Example 2.4 (Crossed module) Let $X$ be a finite 2-group (or called a finite crossed-module) [Ban05].
Then the category $\text{Rep}(X)$ of finite-dimensional linear representations of $X$ over $k$ has a natural structure of a
premodular category.

Remark 2.5 Let $G$ be a finite group. Both $G$ and $D(G)$ can be viewed as special cases of finite crossed
modules. Hence, 2.4 generalizes 2.2 and 2.3.
Example 2.6 (Quantum group) In the case $k = \mathbb{C}$, let $\mathfrak{g}$ be a semisimple Lie algebra and $q$ a root of unity. The semisimplified category $\text{Rep}(U_q(\mathfrak{g}))$ of the category of finite-dimensional representations of the quantum group $U_q(\mathfrak{g})$ has a natural structure of a premodular category.

Example 2.7 (Even part of the quantum $sl_2$) In the case $k = \mathbb{C}$, let $q$ be a root of unity. The semisimplified category $C = \text{Rep}(U_q(sl_2))$ of the category of finite-dimensional representations of the quantum group $U_q(sl_2)$ has a structure of a modular category. The even part $C_0$ of $C$ has a structure of a premodular category [KO01].

2.2 Graphical Calculus

We will use the technique of graphical calculus while dealing with premodular categories. For a pedagogical exposition, see for example ([BK02] and [Kas95]). Notice however that we draw the morphisms as ribbon tangles in the downward direction.

An advantage of this is that many equalities among morphisms can be proved graphically, thanks to [BK02, Theorem 2.3.10]. For example, to prove

$$\text{eval}_Y \circ c_{X,Y} \circ c_{X,Y} \circ c_{X,Y} \circ \text{coev}_Y = c_{X,Y},$$

it suffices to establish an isotopy of ribbon tangles, which is an obviously trivial task:

In the rest of the section, we provide some useful lemmas and notations.

Lemma 2.8 Let $C$ be a premodular category with spherical structure $\alpha$. Let $X, Y$ be $C$-objects. Define a pairing of $k$-linear spaces

$$\text{Hom}_C(X, Y) \otimes \text{Hom}_C(Y, X) \xrightarrow{(\cdot)} k,$$

that sends $\phi \otimes \psi$ to

$$\text{eval}_{B^+} \circ (a_B \otimes 1_X) \circ (\phi \otimes \psi) \circ \text{coev}_A \in \text{End}_C(1) \simeq k.$$
Then the pairing is nondegenerate by the semisimplicity of \(C\). Moreover, \(\text{Hom}_C(X^*, Y^*) \simeq \text{Hom}_C(Y, X)\) by the rigidity of \(C\), so \(\text{Hom}_C(X^*, Y^*)\) can be naturally realized as the dual vector space of \(\text{Hom}_C(X, Y)\).

Define
\[
\omega_{X,Y} := \sum_i \phi_i \otimes \phi^i \in \text{Hom}_C(X, Y) \otimes \text{Hom}_C(Y, X)
\]
where the \(\phi_i\)'s and the \(\phi^i\)'s form a pair of an orthonormal basis and a dual basis respectively for \(\text{Hom}_C(X, Y)\) and \(\text{Hom}_C(Y, X)\) under the identification given in 2.8. Graphically, we use a dummy variable as a short-hand notation:

![Graphical representation of \(\omega_{X,Y}\)](image)

**Lemma 2.9** Let \(C\) be a premodular category, \(a\) its spherical structure, and \(W\) a \(C\)-object. Then
\[
1_W = \sum_{i \in \text{O}(C)} \sum_l \dim_a(i) \phi^l \circ \phi_l,
\]
where the \(\phi_l\)'s and the \(\phi^l\)'s form an orthonormal basis and a dual basis respectively for \(\text{Hom}_C(W, i)\) and \(\text{Hom}_C(i, W)\).

**Notation 2.10** (\(\Omega\)) Let \(C\) be a premodular category, \(a\) be its spherical structure, and \(\text{O}(C)\) be the set of isomorphism classes of simple objects of \(C\). We use \(\Omega\) in graphics to represent \(\bigoplus_{i \in \text{O}(C)} \dim_a(i)\). We also denote \(\dim(\Omega)\) by \(\sum_{i \in \text{O}(C)} \dim_a(i)^2\).

With this shorthand notation \(\Omega\), we present the lemma graphically by

![Graphical representation of Lemma 2.9](image)

### 3 Topological theory

In this section, we describe the topological side of our main statement (cf 1.16 and 1.20), namely the Crane-Yetter theory in dimension two, is treated formally in terms of string nets. This includes a definition of
Crane-Yetter in dimension two, and a combinatorial description of oriented smooth surfaces. The former requires the notion of string nets (also called tensor nets or tensor networks), which will be treated in 3.1. A definition of Crane-Yetter theory in dimension two follows in 3.2. Finally, the combinatorial description of smooth surfaces ($\sigma$-construction) is given in 3.3.

### 3.1 String nets

Originated from Penrose combinatorial description of space-time [Pen71], string nets are the building stone of Crane-Yetter theory. They are also called (quantum) tensor nets or tensor networks in other contexts. In dimension two, they were first explicitly written by the physicists Levin and Wen in [LW05]. For Crane-Yetter theory, however, we need string nets in dimension three. Following [KT21], we provide a formal definition 3.10 of them in this section.

Before the formal definition, keep in mind that it aims to formalizes the pictures of the following sort.

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**Definition 3.1 (2-folds)** A 2-fold is either a compact oriented smooth manifold without boundary of real dimension 2, or such a manifold with finitely many points removed (punctures). A 2-fold is also called a surface.

**Definition 3.2 (Extended 2-folds)** An extended 2-fold is a 2-fold $\mathcal{M}$ with the extra data

$$\{(p_1, v_1, \text{or}_1) \ldots (p_n, v_n, \text{or}_n)\},$$

where $n < \infty$, the points $p_i \in \mathcal{M}$ are disjoint to each other, the tangent vectors $v_i \in T_{p_i} \mathcal{M}$ are nonzero, and the orientations $\text{or}_i$ are in the set $\{+, -\}$.

**Definition 3.3 (3-folds)** A 3-fold is an oriented smooth manifold with boundary of real dimension 3.
Definition 3.4 (Framed arcs in a 3-fold) Let $M$ be a 3-fold (3.3). An arc $\alpha$ in $M$ is a smooth embedding of the standard interval $I = [0, 1]$ (with orientation from $0$ to $1$) into $M$. We require that if an end-point is sent by $\alpha$ to the boundary $\partial M$, then $\alpha$ has to intersect the boundary transversally.

A framing of an arc $\alpha$ in $M$ is a non-vanishing smooth section $s$ of the normal bundle of $\alpha(I) \subseteq M$. A framed arc is an arc equipped with a framing.

Remark 3.5 For our theory, the smoothness condition is not necessary but included for the sake of simplicity. Crane-Yetter theory works also in the PL-setting, parallel to its 3-dimensional analogue, the Turaev-Viro theory. Curious readers are referred to the setup given in [KB10].

The framing, on the other hand, is necessary. Such structure is expressed in slightly different way in related works. For example, in the context of skein modules, people use the notion of ribbons instead of that of arcs. The “width” of a ribbon corresponds to the normal vector from the section.

Definition 3.6 (Framed graphs in a 3-fold) Let $M$ be a 3-fold (3.3). A framed graph $\Gamma$ in $M$ is a finite collection of framed arcs $\alpha_i$ 3.4 satisfying the following conditions.

- Denote the set of arcs of $\Gamma$ by $E(\Gamma)$.
- The images of the embeddings $\alpha_i$’s do not meet each other, with an exception at their endpoints which must be in the interior of $M$.
- Let $v$ be a point in the interior of $M$. Denote $Out(v)$ ($In(v)$, resp.) be the set of the $\alpha_i$’s with $v = \alpha_i(0)$ ($\alpha_i(1)$, resp.). Denote the set of all arcs (edges) $E(v) := In(v) \cup Out(v)$. The directions of the tangent vectors of the $\alpha_i$’s that end at $v$ must be different, i.e. for each $i \neq j$, there is no positive real number $r$ such that $v_i = rv_j$. In the case where $E(v)$ is nonempty, we call $v$ a vertex of $\Gamma$. Denote the set of vertices by $V(\Gamma)$.
- Let $\alpha \in E(\Gamma)$. If an end of $\alpha$ ends on the boundary of $M$, we call the end-point a boundary point of $\Gamma$. Denote the set of boundary points by $B(\Gamma)$.

Definition 3.7 (Extended 3-folds) An extended 3-fold is a pair of a 3-fold $M$ and a framed graph $\Gamma$ in $M$.

Notice that an extended 3-fold $(M, \Gamma)$ naturally induces an extended 2-fold $\partial(M, \Gamma) := (\partial M, \partial \Gamma)$, where $\partial \Gamma$ denotes the set

$$\{(p_i, v_i, \text{or}_i) \mid p_i \in V(\Gamma) \cap \partial M\},$$

where $v_i$ is naturally identified via

$$N_{p_i} = T_{p_i}M/T_{p_i}\alpha \cong T_{p_i}(\partial M)$$

with the framing vector of $\Gamma$ at the point $p_i$, and $\text{or}_i \in (+, -)$ is $+ (-$, resp.) if the framed arc $\alpha$ that passes through $p_i$ is oriented such that $\alpha(1) = p_i$ ($\alpha(0) = p_i$, resp.).
Definition 3.8 (C-extended 2-folds) Given a premodular category $C$, a C-extended 2-fold, or a C-colored extended 2-fold, is an extended 2-fold with an extra data: a C-object $X_i$ is assigned to each oriented framed point $(p_i, v_i, o_{r_i})$. We call $X_i$ the “color” assigned to the point $p_i$.

To define a C-extended 3-fold, we need the Reshetikhin-Turaev theory ([Tur16], [BK02], [Kas95]) for genus-0 surfaces which we recall here. Let $C$ be a premodular category. To each C-extended 2-fold $M = (M, \{(X_1, p_1, v_1, o_{r_1}) \ldots (X_n, p_n, v_n, o_{r_n})\})$ diffeomorphic to a sphere, the first part of Reshetikhin-Turaev theory functorially assigns a vector space $RT(M)$ that is (non-canonically) isomorphic to $\langle X_{\epsilon_1}, \ldots, X_{\epsilon_n} \rangle := \text{Hom}_C(1, X_{\epsilon_1} \otimes \ldots \otimes X_{\epsilon_n})$, where $X_{\epsilon_i}$ denotes $X_i$ if $(o_{r_i} = +)$ or $X_i^*$ if $(o_{r_i} = -)$.

Let $C$ be a premodular category, and $V$ be a real vector space of dimension 3. Let $S$ be a finite collection of distinct framed oriented rays from the origin of $V$, with an assignment $S \rightarrow \text{Obj}(C)$. In this case, we say $V$ has a finite collection of distinct C-colored rays. Then the Reshetikhin-Turaev theory for genus-0 surfaces naturally assigns a vector space $RT_C(V, S, \phi)$ to the sphere $(V - 0)/\mathbb{R}_+$.

Definition 3.9 (C-extended 3-folds) Let $C$ be a premodular category, and $M$ be an extended 3-fold $(M, \Gamma)$. A C-coloring of $(M, \Gamma)$ is an assignment as follows:

- To each arc $\alpha$ of $\Gamma$, assign a C-object $X(\alpha)$.
- After such assignment, to each vertex $p$ of $\Gamma$, the tangent space at $p$ naturally has a finite collection of C-colored rays $(S, \phi)$.
- The Reshetikhin-Turaev theory for genus-0 surfaces assigns a vector space $RT(p) := RT_C(T_p(M), S, \phi)$ as above. Note that $RT(p)$ is (non-canonically) isomorphic to $\text{Hom}_C(\otimes X_i, \otimes X_o)$

where the $X_i$ runs through the objects assigned to all incoming arcs, and the $X_o$ runs through the objects assigned to all outgoing arcs.
- After such assignment, to each vertex $p$ of $\gamma$ assign a vector $v \in RT(p)$.

A C-extended 3-fold $M$ is a 3-fold with a C-colored graph inside. This gives the boundary $\partial M$ a C-extended surface structure. Conversely, we call such a C-colored graph a framed graph that satisfies the boundary condition posed by the C-extended surface $\partial M$. □
Let $C$ be a premodular category. While the first part of Reshetikhin-Turaev theory for genus-0 surfaces assigns a vector space to a $C$-extended genus 0 surface, the second part of it assigns a $C$-extended 3-fold $(M, \Gamma)$ diffeomorphic to a ball to a vector $\text{RT}(M, \Gamma) \in \text{RT}(\partial M)$.

**Definition 3.10 (String nets in 3D)** Let $C$ be a premodular category and $M$ a 3-fold whose boundary $\partial M$ is a $C$-extended surface. Let $F$ be the free vector space over $\mathbb{k}$ generated by all $C$-colored graphs that satisfy the boundary condition posed by $\partial M$. Let $N$ be the subspace generated by either of the following

- The difference $\Gamma - \Gamma'$ of two $C$-colored graphs that are smoothly isotopic to each other.
- A linear combination $v = \sum c_i \Gamma_i$ such that there exists a closed region $B \subseteq M$ diffeomorphic to a ball such that the vector assigned by the Reshetikhin-Turaev theory for genus-0 surfaces of $v|_B$ is the zero vector.

We call $S(M) := F/N$ the space of string nets of $M$ with the given boundary condition, and we call an element of $S(M)$ a string net.

We conclude this section with a useful lemma.

**Lemma 3.11 (Sliding lemma)** Let $C$ be a premodular category. Then the following string nets are equal, where $\Omega$ is the shorthand notation given in 2.10. Heuristically, the moral is that $\Omega$ protects anything “inside” it by making it transparent.

\[ 
\begin{array}{c}
\text{X} \\
\infty \\
\text{Y}
\end{array} \quad \ominus \quad 
\begin{array}{c}
\text{X} \\
\infty \\
\text{Y}
\end{array} \quad = \quad 
\begin{array}{c}
\text{X} \\
\infty \\
\text{Y}
\end{array} \quad = \quad 
\begin{array}{c}
\text{X} \\
\infty \\
\text{Y}
\end{array} 
\]

\[\ominus\]

**Proof.** Apply 2.9 locally with $W = X \otimes \Omega$. Use isotopy and the naturality of the braidings. And then apply 2.9 locally again with $W = \Omega \otimes X$. Notice that in fact $Y$ can be more general than an object - the lemma works even when $Y$ is a puncture.

3.2 Crane-Yetter theory in dimension two (CY)

We define the Crane-Yetter theory in dimension two in this subsection, following [KT21, section 5]. Let $\Sigma$ be a smooth oriented 2-manifold and $C$ a premodular category. To define $\text{CY}_\Sigma(C)$, we first define an auxiliary category $\text{cy}_\Sigma(C)$. 

16
Definition 3.12 (cy_Σ(C), an auxiliary category) Given a premodular category C and a 2-fold Σ, we define the k-linear category cy_Σ(C) as follows. An object is a collection c of C-colored points and tangent vectors, such that (Σ, c) is a C-extended 2-fold. Given two objects c and c′, the morphism space Hom_{cy_Σ(C)}(c, c′) between c and c′ is defined to be the space of string nets for the 3-fold Σ × [0, 1] satisfying the boundary condition (c × {0}) ∪ (c′ × {1}), where c denotes the same collection of C-colored points as c does but with all orientations flipped.

Two examples of morphisms in cy_{Σ_{1,0}}(C) is depicted as follows.

![Diagram](image)

Definition 3.13 (Karoubi envelope) Given an additive category C, its Karoubi envelope (Karoubi completion) Kar(C) is defined to be the category as follows. The objects are pairs (X, p), where X ∈ Obj(C) and p ∈ Hom_C(X, X), such that p^2 = p. Given objects X = (X, p) and Y = (Y, q), the space of morphisms Hom_{Kar(C)}(X, Y) is defined to be the subspace of Hom_C(X, Y) consisting of those f such that qfp = f.

The Karoubi envelope is the pre-abelian completion in our context.

Definition 3.14 (CY_Σ(C), Crane-Yetter theory in dimension 2) With the notations above, we define

\[ CY_Σ(C) := \text{Kar}(cy_Σ(C)) \, . \tag{3.15} \]

 Remark 3.16 The definition given in 3.14 was first given in [KT21, section 5]. That it extends the original Crane-Yetter theory is proved in [Tha21].

It is immediate from the definition that cy_Σ(C) is additive. On the other hand, CY_Σ(C) is in fact finite semisimple abelian for all surfaces with at least one puncture (cf 3.20, 4.19, 5.1). It’s conjectured that it holds in fact for all surfaces.

3.3 A presentation of surfaces (σ-construction)

In this paper, we construct a surface Σ from the standard disk and an additional data σ ∈ Adm_{2n}, give some examples, and prove that such construction produces all oriented surfaces with at least one puncture.
Definition 3.17 (Adm\(2n\), admissible gluings) Let \(n\) be a nonnegative integer. An element \(\sigma\) in the permutation group \(S_{2n}\) on \(2n\) elements is called an admissible gluing (of rank \(n\)), if \(\sigma\) satisfies the following conditions.

- \(\sigma\) has no fixed points.
- \(\sigma\) is an involution; i.e. \(\sigma^2 = 1\).

We denote the subset of admissible gluings by \(\text{Adm}_{2n} \subseteq S_{2n}\).

Definition 3.18 (\(\Sigma_\sigma\), \(\sigma\)-construction) For each admissible gluing \(\sigma \in \text{Adm}_{2n}\), we construct a smooth surface \(\Sigma_\sigma\). Start from the standard oriented disk. We choose \(2n\) closed segments with the same length from the boundary. To make the presentation easier, we emphasize them by drawing them like legs (without changing the diffeomorphism type), and we call them legs from now on.

Glue the end of the legs in pairs according to \(\sigma\) with the orientation preserved. Finally, removed the boundary the the result to be an open surface. The result is denoted by \(\Sigma_\sigma\).

Example 3.19 The only element \((12)\) in \(\text{Adm}_2\) constructs the cylinder \(\Sigma_{(12)} \simeq \text{Cylinder}\).

The elements \(\sigma_{0,3} = (12)(34)\) and \(\sigma_{1,1} = (13)(24)\) in \(\text{Adm}_4\) construct a 3-punctured sphere \(\Sigma\) and a 1-punctured torus respectively.

So constructed surfaces must have at least one puncture, thus the procedure does not give all surfaces. However, the following theorem shows that this is the only case it misses.

Theorem 3.20 The \(\sigma\)-constructions produce all oriented surfaces with at least one puncture (i.e. all open surfaces).
Proof. Indeed, the admissible gluing $\sigma_{1,1} = (13)(24) \in \text{Adm}_4$ gives an once-punctured torus $\Sigma_{\sigma_{1,1}}$. Similarly, the admissible gluing

$$\sigma_{2,1} = \sigma_{1,1} \circ (57)(68) = (13)(24)(57)(68) \in \text{Adm}_8$$

gives an once-punctured surface of genus two. Following this fashion, for any $g \in \mathbb{N}$ one can construct an once-punctured surface of genus $g$ by using the admissible gluing

$$\sigma_{g,1} = (13)(24)(57)(68) \ldots ((4g - 3)(4g - 1))((4g - 2)(4g)) \in \text{Adm}_{4g}.$$ 

To add $k$ punctures to the surface, use the admissible gluing

$$\sigma_{g,(k+1)} = \sigma_{g,1} \circ ((4g + 1)(4g + 2))((4g + 3)(4g + 4)) \ldots ((4g + 2k - 1)(4g + 2k)) \in \text{Adm}_{4g+2k}.$$ 

Then the statement follows from the well-known classification of oriented smooth surfaces. 

Notation 3.21 ($\sigma$-orbit) We take this opportunity to introduce a later useful notation. Let $\sigma \in \text{Adm}_{2n}$. Denote $[i]$ to be the orbit of

$$i \in \{1, 2 \ldots, 2n\}$$

under the action of $\sigma$, $[i]'$ the smaller number in the set $[i]$, and $[i]''$ the larger number in the set $[i]$. Note that the set $\{[1], [2], \ldots, [2n]\}$ has exactly $n$ elements.

As an example, for $\sigma = (13)(24)$, we have

\begin{align*}
[1] &= \{1, 3\}, & [1]' &= 1, & [1]'' &= 3; \\
[2] &= \{2, 4\}, & [2]' &= 2, & [2]'' &= 4; \\
[3] &= \{1, 3\}, & [3]' &= 1, & [3]'' &= 3; \\
[4] &= \{2, 4\}, & [4]' &= 2, & [4]'' &= 4.
\end{align*}
4 Algebraic theory

In this section, we describe the algebraic side of our main statement (cf 1.16 and 1.20), namely the categorical center of higher genera $Z$.

Its definition is quite algebraic and abstract, so some motivation is supplemented in 4.1. The formal definition is given in 4.2. Finally, some basic properties of $Z$ are proved in 4.3. In particular, we show that $Z$ is finite abelian semisimple, and that there is a strictly ambidextrous adjunction between $Z = Z(C)$ and the underlying premodular category $C$.

4.1 Motivation: Drinfeld categorical center

Abstract algebraic theories (groups, rings, modules.. etc) are ubiquitous in modern mathematics. Among the algebraic objects, the abelian ones are simpler, and are often first treated. One then builds the theory toward the generic cases. In group theory, for example, one can study a group $G$ by starting with its center $Z(G) \subseteq G$ and then apply induction.

Drinfeld’s categorical center is an analogue in the categorical setting. There, algebras are replaced by categorical algebras (more precisely, by monoidal categories [Eti+15]), and centers are replaced by categorical centers. As in the classical theory, the theory of the one side helps that of the other.

In contrast to the classical case, categorical centers need not be smaller nor easier. This is due to the fact that equalities are replaced by equivalences in the categorical settings. Therefore, the condition $ab = ba$ is replaced by $ab \simeq ba$. That is to say, a categorical commutativity not only remembers both sides being identified, but also how they are identified. Therefore, a typical object in the Drinfeld center $Z(C)$ is a pair $(X \in \text{Obj}(C), \gamma)$, where $\gamma$ is a half-braiding that encodes how $X$ commutes with all the others. To be more precise, a half-braiding $\gamma$ of $X$ is a natural equivalence

$(-) \otimes X \xrightarrow{\gamma} X \otimes (-)$

satisfying some compatibility conditions 7.62. It is worthwhile to mention that such construction has been successful in many contexts, e.g. representation theory, statistical physics, knot theory, .. etc.

Categorical center of higher genera $Z = Z_\sigma(C) = Z_{\Sigma_\sigma}(C)$, on the other hand, generalizes the Drinfeld center. Instead of remembering how $X$ commutes with others, an object $(X, \gamma)$ remembers how $X$ commutes in multiple different ways. The amount of ways depends on the underlying surface $\Sigma = \Sigma_\sigma$. Therefore, an object of $Z_\sigma(C)$ is a pair $(X, \gamma)$, where $\gamma$ is a collection of half-braidings

$\gamma = \{ \gamma_1, \gamma_2, \ldots, \gamma_n \}.$

However, extra conditions must be carefully imposed in order to keep track of the underlying topological data. In contrast to the case of Drinfeld center, multiple half-braidings give essentially infinite ways to fuse via tensors, e.g. $\gamma_1 \gamma_2 \gamma_1 \gamma_1 \gamma_2 \ldots$. Therefore, suitable commutative relations among the half-braidings are needed. This is given in the formal definition of $Z_\sigma(C)$ as (:comm 1), (:comm 2), and (:comm 3) (cf 4.2).

Before moving on to the formal definition of $Z_\sigma(C)$, let us remark on the premodular condition on $C$. As a classical analogue, it does not make sense to talk about the center $Z(S)$ for a set $S$; one needs a few extra structures on $S$. In the categorical setting, in order to define the Drinfeld center, merely a plain category
C is not enough. Essentially, a monoidal structure is required. Similarly, for categorical centers of higher genera, we need essentially the braided structures (cf 7.60), which are included in the premodular condition. Note that we will assume premodularity for other purposes, but the categorical center of higher genera can certainly be defined for other less restricted categories.

4.2 Categorical center of higher genera ($Z$)

In this section, we formally define the categorical center of higher genera $Z_{\sigma}(C)$ for a premodular category $C$ and an admissible gluing $\sigma \in \text{Adm}_{2n}$. Assume $C$ to be a premodular category throughout this section.

**Definition 4.1 ($\sigma$-pair)** Let $\sigma \in \text{Adm}_{2n}$, i.e. $\sigma$ an admissible gluing. Define a $\sigma$-pair of $C$ to be a pair $(X, \gamma)$, where $X$ is a $C$-object and $\gamma$ is a set of half-braidings for $X$ (cf 7.62).

$$\gamma = \{\gamma[1], \gamma[2], \ldots, \gamma[2n]\}$$

satisfying pairwise commutative relations in 4.2.

**Definition 4.2 ($:\text{comm}$, technical commutative relations)** Let $Z_1$ and $Z_2$ be objects in $C$, and $c$ be the braided structure of $C$ (so $a \otimes b \xrightarrow{c_{ab}} b \otimes a$). Given $[i]$ and $[j]$, there are three possible cases without loss of generality

1. $[i] < [i] < [j] < [j]$
2. $[i] < [j] < [i] < [j]$
3. $[i] < [j] < [j] < [i]$

We will give the technical conditions that the $\gamma$’s should obey, following by their graphical versions.

1. In the first case, $\gamma[i]$ and $\gamma[j]$ are required to satisfy the following commutative relation ($:\text{comm 1}$), functorial in $Z_1$ and $Z_2$.

$$((\gamma[j], Z_2 \otimes 1)(1 \otimes (c_{Z_1, X \otimes Z_1} Z_1)\gamma[i], Z_1))$$

$$= (1 \otimes c_{Z_1, Z_2})(c_{Z_1, X \otimes Z_1} Z_1)\gamma[i], Z_1)(1 \otimes \gamma[j], Z_2)(c_{Z_1, Z_2} \otimes 1)$$

2. In the second case, $\gamma[i]$ and $\gamma[j]$ are required to satisfy the following commutative relation ($:\text{comm 2}$), functorial in $Z_1$ and $Z_2$. 

21
(3) In the third case, \( \gamma_i \) and \( \gamma_j \) are required to satisfy the following commutative relation (:comm 3), functorial in \( Z_1 \) and \( Z_2 \).

\[
(\gamma_{[j]} Z_2 \otimes 1)(1 \otimes \gamma_{[i]} Z_1) = (1 \otimes c_{Z_2, Z_1})(\gamma_{[i]} Z_1 \otimes 1)(1 \otimes \gamma_{[j]} Z_2)(c_{Z_1, Z_2}^{-1} \otimes 1) 
\]

Notice that the first and the third case are almost the same, which is not surprising given their topological meaning. To make them look alike, define

\[
\tilde{\gamma}_{[i]} = c_{X, X^*} \gamma_{[i]}.
\]

\[\Downarrow\]

**Definition 4.9 (\( \sigma \)-morphism)** Given two \( \sigma \)-pairs \( \bar{X} = (X, \gamma) \) and \( \bar{Y} = (Y, \beta) \) of \( C \), define \( [\bar{X}, \bar{Y}]_\sigma \) to be the linear subspace of \( \text{Hom}_C(X, Y) \) consisting of the morphisms \( (X \xrightarrow{f} Y) \) compatible with all the half-braidings in the following sense. For any \( Z \in C \), we have (functorially in \( Z \))

\[
\beta_Z(1 \otimes f) = (f \otimes 1)\gamma_Z.
\]
Finally, let the identity maps and the compositions be inherited from that of \( C \).

Definition 4.11 (categorical center of higher genera) Let \( C \) be a premodular category, and \( \sigma \in \text{Adm}_{2n} \) an admissible gluing. The categorical center of higher genera \( Z_{\sigma}(C) \) of \( C \) is defined to be the category with objects the \( \sigma \)-pairs of \( C \) and with morphisms the \( \sigma \)-morphisms.

4.3 Properties of \( Z \)

In this section, we establish some basic properties of categorical centers of higher genera. In particular, we show that they are finite semisimple abelian categories, and that there is a strictly ambidextrous adjunction between it and the underlying premodular category \( C \).

4.3.1 Connecting functors

In this subsection, we establish the relation between \( C \) and its categorical center of higher genera \( Z_{\sigma}(C) \), where \( C \) is a premodular category and \( \sigma \) is an admissible gluing. More precisely, there exist two additive functors \( I_{\sigma} \) and \( F_{\sigma} \).

\[
I_{\sigma} : C \rightleftarrows Z_{\sigma}(C) : F_{\sigma}.
\]

We will see that \( I_{\sigma} \) is both a right and a left adjoints of \( F_{\sigma} \) (thus vice versa) in section 4.3.2. Such a pair of adjunction is called a (strictly) ambidextrous adjunction in the literature.

Definition 4.12 (forgetful functor) The forgetful functor

\[
C \xleftarrow{F_{\sigma}} Z_{\sigma}C
\]

is defined to send objects \( (X, \gamma) \) to \( X \), and to send morphisms by inclusion (recall that the morphism space of \( Z_{\sigma}(C) \) is defined as a subspace of that of \( C \)). Clearly, it is an additive functor.

Definition 4.13 (induction functor) The induction functor

\[
C \xrightarrow{I_{\sigma}} Z_{\sigma}(C)
\]

is more complicated, so will be defined step-by-step. Define \( I_{\sigma}(X) \) to be \( (X_{\sigma}, \gamma) \), which is given below. Let \( \mathcal{O}(C) \) be the set of isomorphism classes of simple objects of \( C \), and \( \mathfrak{o}(C) \) a set of representatives. To each \( C \)-object \( X \), define another \( C \)-object

\[
X_{\sigma} := \bigoplus_{\mathfrak{o}(C)} (X_{1} \otimes X_{2} \otimes \ldots \otimes X_{n} \otimes X \otimes X_{n+1} \otimes \ldots \otimes X_{2n-1} \otimes X_{2n}),
\]

(4.14)
where $X_k$ runs through $o(C)$ if $k = [k]'$ or $X_k = X^*_{[k]'}$ if $k = [k]''$. Here, recall that the first case means that $k$ is the smaller member in the set $[k]$, while the second case means that $k$ is the larger member. For example,

$$X_{(12)} = \oplus_{X \in o(C)} X \boxtimes X^*.$$  

Notice that it does not depend on the choice of $o(C)$ up to canonical isomorphisms. Similarly, neither does $X_\sigma$ for general admissible gluings $\sigma$.

Next, to each $k$, we define the $[k]$-th half-braiding $\gamma_{[k]}$ for $X_\sigma$

$$(-) \otimes X_\sigma \xrightarrow{\gamma_{[k]}} X_\sigma \otimes (-)$$

as in the following picture.

where the nontrivial intertwiners are given precisely in 2.9.

We contend that so given $(X_\sigma, \gamma)$ is indeed an object of $Z_\sigma(C)$. One needs to show that each element in $\gamma$ is a half-braiding, and that $\gamma$ satisfies the commutative relations (4.2). A proof of this can be found in (7.91). It remains to define the map on the morphism spaces $\text{Hom}_C(X, Y)$. Given a morphism $X \xrightarrow{f} Y$, one defines

$$I_\sigma(f) := \overline{f} := \text{id} \otimes \text{id} \otimes f \otimes \text{id}.$$  

(4.16)

To conclude, it remains to show that

- The morphism $\overline{f}$ is compatible with the half-braidings $\gamma$ and $\beta$.
- The construction $(\_)$ preserves the identities and the compositions.

The first point is shown in (7.92). The second point is clear. $\diamond$

### 4.3.2 Ambidextrous adjunction

Both functors $F_\sigma$ and $I_\sigma$ are additive immediately by definition. We are ready to state and prove the main statement of this subsection.

**Theorem 4.17** The functors

$$I_\sigma : C \rightleftarrows Z_\sigma(C) : F_\sigma$$

so defined in 4.12 and 4.13 are (strictly) ambidextrous adjoint to each other. In other words, $I_\sigma$ is both a left adjoint and a right adjoint of $F_\sigma$, thus vice versa. $\diamond$
Proof. We will prove that $F_\sigma$ is right adjoint to $I_\sigma$. Namely, we need to show that for each $C$-object $X$ and for each $Z_\sigma(C)$-object $(Y, \beta)$, there is a vector space isomorphism

$$F : \text{Hom}_C(X, F_\sigma(Y, \beta)) \cong \text{Hom}_{Z_\sigma(C)}(I_\sigma(X), (Y, \beta)) : G.$$

It will then be obvious that the other side can be proved verbatim by taking duals (or by flipping the graph, in terms of graphical calculus). To prove such equivalence, we construct explicit maps for both sides, and argue that each composition equals the identity map.

Given $\phi \in \text{Hom}_C(X, F_\sigma(Y, \beta))$, define its image on the other side to be

$$F(\phi) := \frac{1}{\dim(\Omega)} (\tilde{\Pi}_{k \in [1], \ldots, [2n]} \beta_{\phi}^* \circ (1 \otimes \ldots \otimes 1 \otimes \phi \otimes 1 \otimes \ldots \otimes 1),$$

where $\Omega$ is the shorthand notation given in 2.10, and the term $\tilde{\Pi}$ is explained below: The term $\tilde{\Pi}$ is a $C$-morphism $I_\sigma(Y) \rightarrow Y$. Each $\beta_{[i], \Omega}^*$ is a $C$-morphism $\Omega \otimes Y \otimes \Omega \rightarrow Y$, induced from $\Omega \otimes Y \xrightarrow{\beta_{[i], \Omega}} Y \otimes \Omega$ by composing the evaluation map (note that $\Omega^* \simeq \Omega$). So the $\beta_{[i], \Omega}^*$’s are maps that kills the $[i]'$-th and the $[i]''$-th component of $\Omega$ by using $\beta_{[i]}$. However, depending on the combinatorial nature of $\sigma \in \text{Adm}_{2n}$, one should insert suitable braidings for it to make sense. For example, if $n = 3$, $[1] = [4]$, $[2] = [6]$, and $[3] = [5]$, we define the $\tilde{\Pi}$ term as in the following diagram – the order of the $[i]$’s does not really matter, thanks to 4.2.

That $F(\phi)$ is indeed a morphism in $Z_\sigma(C)$ follows directly from the commutative relation 4.2, that half-braidings are by definition monoidal, and the sliding lemma 3.11.

On the other hand, given $\psi \in \text{Hom}_{Z_\sigma(C)}(I_\sigma(X), (Y, \beta))$, define its image $G(\psi)$ on the other side to be as indicated in the graph below.
To prove that $GF$ is the identity map, use the fact that the half-braidings are by definition functorial. Hence one can slide the $\Omega$'s out the axis. Finally, the product of the dimensions of the $\Omega$'s cancel with the denominator.

To prove that $FG$ is the identity map, use the sliding lemma again. Then use the assumption that $\psi$ is a $Z_{\sigma}(C)$-morphism to drag $\Omega$ down. Finally, slide the $\Omega$'s away from the axis as in the case for $GF = 1$. 

$n=2$, for example.
4.3.3 $Z$ is finite semisimple abelian

In this section, we show that the categorical centers of higher general over premodular categories are finite semisimple abelian categories.

**Lemma 4.18 ((monadic) projection)** Let $\sigma \in \text{Adm}_{2n}$ be an admissible gluing, $C$ be a premodular category, $Z = Z_\sigma(C)$ be the categorical center of higher genera of $C$ with respect to $\sigma$, and $\bar{X} = (X, \gamma)$ and $\bar{Y} = (Y, \beta)$ be $Z$-objects.

Recall that the morphism space $Z(\bar{X}, \bar{Y})$ is by definition a subspace of $C(X, Y)$. Then there is a natural projection $\pi_{\gamma, \beta} \in \text{End}(\text{Mor}_C(X, Y))$ to the subspace $Z(\bar{X}, \bar{Y})$ that respects the composition.

\[\text{Proof.}\] The full proof is tedious and postponed to 7.4. In particular, see 7.87. Roughly, the statement follows from the monadic nature of the strictly ambidextrous adjunctions and a condition called the "unity trace condition" (7.82).

**Theorem 4.19** Let $C$ be a premodular category, $\sigma \in \text{Adm}_{2n}$ an admissible gluing. Then the categorical center of higher genera $Z_\sigma(C)$ is a finite semisimple abelian category.

\[\text{Proof.}\] The complete proof is tedious and thus postponed to the appendix. See 7.88, 7.89, and 7.90. The main idea is to make heavy use of the projection 4.18.

5 Proof of the main statement

In this section, we prove the main statement of this paper.

**Theorem 5.1 (Main Statement)** Let $C$ be a premodular category, $\sigma \in \text{Adm}_{2n}$ an admissible gluing, $\Sigma = \Sigma_\sigma$ the surface constructed from $\sigma$. Then the Crane-Yetter theory of $\Sigma_\sigma$ over $C$ and the categorical center of higher genera $Z_\sigma(C)$ are equivalent as $k$-linear categories

\[\text{CY}_{\Sigma_\sigma}(C) \simeq Z_\sigma(C).\]

As the $Z_\sigma(C)$’s are proven to be finite semisimple abelian 4.19, the Crane-Yetter theory for each open surface is also a finite semisimple abelian category.

\[\text{To stress the informal aspect again, we recall 1.20.}\]

**Remark 5.2** In $H_1(S^1) \simeq \mathbb{Z}$, one sees the algebra of the shape $S^1$ and the shape of the algebra $\mathbb{Z}$. Our main result should be viewed as a higher analogue. That is, one sees the (higher) algebra of the shape $\Sigma_\sigma$ and the shape of the (higher) algebra $Z_\sigma$.

**Example 5.3 ($n = 0$)** For $n = 0$, the surface $\Sigma$ is the open disk, the categorical center of higher genera reduces to the underlying premodular category $C$, so the theorem recovers that $\text{CY}_\Sigma(C) \simeq C$ as shown in 1.8.

**Example 5.4 ($n = 1$)** For $n = 1$, the only possible surface is the cylinder, the categorical center of higher genera reduces to the Drinfeld center $Z(C)$. Hence, the theorem recovers that $\text{CY}_\Sigma(C) \simeq Z(C)$ as shown in 1.9.
Example 5.5 ($n = 2$) For $n = 2$, there are two possible surfaces: the 1-punctured torus and the 3-punctured disk. In the former case, the categorical center of higher genera reduces to the elliptic center $Z^{\text{el}}(C)$, so the theorem recovers that $\text{CY}_\Sigma(C) \simeq Z^{\text{el}}(C)$ as shown in 1.13. In the later case, the theorem provides a new result.

5.1 Strategy

\[
\begin{align*}
\text{CY}_\Sigma(C) & \xrightarrow{\text{Kar}(\text{cy}_\Sigma(C))} Z_{\sigma}(C) \\
\text{Kar}(\text{cy}_\Sigma(C)) & \xrightarrow{\simeq} \text{Kar}(\text{ho.cy}_\Sigma(C)) \\
\text{cy}_\Sigma(C) & \xleftarrow{\simeq} \text{ho.cy}_\Sigma(C)
\end{align*}
\]

1. (Condensation of string nets) By definition, $\text{CY}_\Sigma(C)$ is the Karoubi envelope of $\text{cy}_\Sigma(C)$. Find an equivalent subcategory $\text{ho.cy}_\Sigma(C)$ of $\text{cy}_\Sigma(C)$ by reducing topological data. Then of course

\[\text{Kar}(\text{cy}_\Sigma(C)) \simeq \text{Kar}(\text{ho.cy}_\Sigma(C)).\]

2. (top $\rightarrow$ alg) Construct a functor

\[\text{ho.cy}_\Sigma(C) \xrightarrow{\mathcal{J}} Z_{\sigma}(C),\]

and extend it to

\[\text{Kar}(\text{ho.cy}_\Sigma(C)) \xrightarrow{\mathcal{J}} Z_{\sigma}(C).\]

3. (top $\leftarrow$ alg) Construct a functor

\[\text{Kar}(\text{ho.cy}_\Sigma(C)) \xleftarrow{\mathcal{G}} Z_{\sigma}(C).\]

4. Argue that the compositions $\mathcal{J} \circ \mathcal{G}$ and $\mathcal{G} \circ \mathcal{J}$ are equivalent to the identity functors.

5. Show that the equivalence is of finite semisimple abelian categories.

5.2 Proof

In this section, we give the full proof of the main statement. Each subsection corresponds to each step in the outlined strategy.

5.2.1 Reducing topological data

In this subsection, we reduce the topological data by constructing a smaller yet equivalent subcategory

\[\text{ho.cy}_\Sigma(C) \xrightarrow{\simeq} \text{cy}_\Sigma(C). \quad (5.6)\]

Definition 5.7 (ho.cy$_\Sigma$,C) Let $C$ be a premodular category and $\sigma$ an admissible gluing. The subcategory $\text{ho.cy}_\Sigma(C)$ of $\text{cy}_\Sigma(C)$ is defined as follows.

Let $p$ be the central point of the standard disk. An object of $\text{ho.cy}_\Sigma(C)$ is defined to be the single $C$-colored point $(p, X)$ for some $X \in \text{Obj}(C)$. A morphism from $(p, X)$ to $(p, Y)$ is the equivalence class in which the following string net lives.
Clearly, $ho.\text{cy}_\sigma(C)$ is a subcategory of $\text{cy}_\sigma(C)$.

**Theorem 5.8 (equivalence of reduction)** The inclusion functor

$$\iota: ho.\text{cy}_{\Sigma,\sigma}(C) \overset{\subseteq}{\longrightarrow} \text{cy}_{\Sigma,\sigma}(C)$$

is an equivalence of categories. Clearly, it is additive.

**Proof.** By a basic lemma in category theory, it is enough to show that $\iota$ is fully faithful and essentially surjective.

(Essentially surjective) Recall that a typical object of $\text{cy}_{\Sigma,\sigma}(C)$ is a finite collection of $C$-colored points on $\Sigma_\sigma$. It suffices to find an equivalent object of the form $(p, X)$, for some $X \in \text{Obj}(C)$. This can be done by the following reductions.

1. Slightly push the points on the boundary into the smaller side.
2. Compress the points from the legs into the disk.
3. Then compress further for the points to stay in a small unit disk in the middle.
4. Project the objects to a fixed line.
5. Take their tensor products.

Each step above can be realized as an isomorphism in $\text{CY}_{\Sigma,\sigma}(C)$, so every object is isomorphic to an object in $ho.\text{cy}_{\Sigma,\sigma}(C)$.
(Fully faithful) We ought to show that
\[
\text{Hom}_{\ho\cdot\cy_{\Sigma_\sigma}(C)}((p, X), (p, Y)) \xrightarrow{\sim} \text{Hom}_{\cy_{\Sigma_\sigma}(C)}((p, X), (p, Y))
\]
is an equivalence of vector spaces. Clearly, it is linear and injective, as the quotient relations on both sides are the same. To prove surjectivity, we have to show that any arbitrary string net with given boundary condition is equivalent to a stringnet given in the definition of \(\ho\cdot\cy_{\Sigma_\sigma}(C)\). This can be done by a similar compression process as in the proof of essential surjectivity.

1. Push the stringnets away from the end of the legs.
2. Push the stringnets away from the legs.
3. Compress everything into a fixed central bar.
4. Replace the stringnets through boundaries with one strand for each leg by taking tensor products by using the Reshetikhin-Turaev evaluation.
5. Then compress vertically.
6. Then finally replace the tangled mess in the middle by a morphism.
5.2.2 topology → algebra

In this subsubsection, we aim to construct a morphism

\[ \text{Kar}\left(\text{ho.cyc}_\Sigma(C)\right) \xrightarrow{J} Z_\sigma(C). \]

As \( Z_\sigma(C) \) is abelian (4.19), by 7.93 we only have to construct an additive functor

\[ \text{ho.cyc}_\Sigma(C) \xrightarrow{J} Z_\sigma(C). \]

To define \( J \), recall that a typical object of \( \text{ho.cyc}_\Sigma(C) \) is a colored point \((p, X)\), where \( p \) denotes the central point of the standard disk. Define its image under \( J \) to be \( X_\sigma \) as in (4.14). A typical morphism from \((p, X)\) to \((p, Y)\) is a linear combination of the equivalence classes of the stringnets like \( \Gamma \). Define the image of \([\Gamma]\) under \( J \) to be
where $\Omega$ is the shorthand notation given in 2.10 crossings mean the braidings of $C$, and the nontrivial pairs of intertwiners are given in [KB10, (1.8)]. Extend the definition additively, and then we have our desired additive functor $j$.

### 5.2.3 topology $\leftarrow$ algebra

In this subsubsection, we construct a functor

$$\text{Kar}(\text{ho.cycl}(C)) \xleftarrow{G} Z_{\sigma}(C).$$

Recall that a typical object in $Z_{\sigma}(C)$ is $(X, \gamma)$, where $X \in \text{Obj}(C)$ and $\gamma$ is a set of half-braidings

$$\gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_{2n}\}$$

satisfying some relations 7.62. Define the image of $(X, \gamma)$ under $G$ to be $((p, X), \pi_\gamma)$, where $p$ denotes the central point of the standard disk, and $\pi_\gamma$ to be the equivalence class of the following stringnets.

That $\pi_\gamma$ is a projection follows from the commutative relations (4.2) and a graphical lemma [Kir11, (3.7)]. For morphisms, define the image of $(X, \gamma) \xrightarrow{f} (Y, \beta)$ under $G$ to be
which is indeed a morphism in the Karoubi envelope because $\pi_\gamma$ and $\pi_\beta$ are idempotents.

5.2.4 topology $\leftrightarrow$ algebra

In this subsection, we will show that $J \circ G$ and $G \circ J$ are equivalent to identity functors.

That $G \circ J \simeq 1$ follows directly from the same argument of [Kir11, Figure 15]; we just have to do it $n$ times. On the other hand, in fact we have $J \circ G = 1$. Indeed, denote $(J \circ G)((X, \gamma)) = (X', \gamma')$. That $X' = X$ follows directly from the sliding lemma 3.11, and that $\gamma' = \gamma$ follows from the sliding lemma, and the fact that half-braidings are by definition monoidal.

Finally, since $G$ and $J$ are additive, this proves the equivalence of both sides as abelian categories. Therefore, $\text{CY}_{\Sigma}(C)$ and $Z_{\sigma}(C)$ are equivalent as finite semisimple abelian categories.
6  Outlook and remarks

In this section, we describe some open directions and more work in progress.

Surface combinatorics

Let $\Sigma$ be an open surface of a fixed topological type. In general, there are many different admissible gluings $\sigma$ with $\Sigma_\sigma \simeq \Sigma$. As Crane-Yetter theory is topological, we have many differently-presented categories that are in fact equivalent.

Moreover, the excision property

$$CY_{\Sigma_1 \cup \Sigma_2}(C) \simeq CY_{\Sigma_1}(C) \boxtimes_{CY_{\Sigma_1 \cap \Sigma_2}(C)} CY_{\Sigma_2}(C)$$

provides more ways to obtain $\Sigma$. It is an interesting work to establish explicit equivalences.

Surfaces without punctures

The main statement of this work provides a nice description of the Crane-Yetter theory of any surface with at least one puncture. While the case without punctures can be taken care easily by patching with the excision principle, the resulting categories are described in terms of balanced (Deligne) tensor products, which are more obscure. The author believes that there should be a better description.

Module categorical structures

The Drinfeld center with the stacking tensor product $\boxtimes$ acts on $CY_{\Sigma}(C)$ in possibly multiple ways. We will establish the module categorical structure for $CY_{\Sigma}(C)$ explicitly in future work.

Concrete computations

Compute examples for Crane-Yetter theory in dimension two and three explicitly and concretely, especially for premodular categories $C$ that are neither modular nor symmetric. There are a few candidates. The first is the even part of the semi-simplification of $\text{Rep}(U_q\mathfrak{sl}_2)$ for special $q$. Another family of examples are given by $\text{Rep}(X)$, where $X$ denotes a finite 2-group [Ban05]. Compute C-Y invariants for 3- and 4- folds directly from our result, and seek for insights.

Minimal data for Crane-Yetter

When $C$ is modular, $CY$ in dimension two trivializes to the number of punctures. In particular, for closed surfaces $\Sigma$, $CY_{\Sigma}(C)$ trivialize to the Muger center of $C$, which is just $(\text{Vect})$ due to the modularity [Eti+15, Prop 8.20.12]. On the other hand, when $C$ is not modular, $CY_{\Sigma}(C)$ does no seem to depend on full information from $C$. Find the minimal data needed in order to determine $CY_{\Sigma}(C)$.

Piecewise-linear setting

For exposing stringnets with simplicity, we assume smooth structures for our surfaces. Crane-Yetter theory can be made precise in the PL setting. This is not necessary and should be removed in future work. For an analogue in one dimension lower, see [KB10].

7  Appendix

Most sections in the appendix are added for the sake of completeness.
7.1 Abelian categories

A complete definition of an abelian category is given in this subsection. In particular, see 7.25.

Definition 7.1 (pre-additive category) [Lan, I.8. p.28] A pre-additive category, or called an Ab-category, is a category \( A \) in which each hom-set is an (additive) abelian group, with respect to which the composition maps are bilinear.

Definition 7.2 (biproduct) [Lan, VIII.2. Definition] Let \( A \) be an pre-additive category (7.1). For each pair of \( A \)-objects \((a, b)\), define their biproduct to be the pair \((c, \{p_a, p_b, i_a, i_b\})\), where \( c \) is an \( A \)-object, \( p_a \) and \( p_b \) are morphisms from \( c \) to \( x \), \( i_a \) and \( i_b \) are morphisms from \( a \) and \( b \) to \( c \), with the equations satisfied:

\[
\begin{align*}
1_a &= p_a i_a \\
1_b &= p_b i_b \\
1_c &= i_a p_a + i_b p_b.
\end{align*}
\]

Definition 7.6 (initial object) [Lan, p.20] Let \( C \) be a category. An initial object \( s \) in \( C \) is a \( C \)-object such that to each \( C \)-object \( a \) there is exactly one \( C \)-morphism \( s \to a \).

Definition 7.7 (terminal object) [Lan, p.20] Let \( C \) be a category. A terminal object \( t \) in \( C \) is an \( C \)-object such that to each object \( a \) there is exactly one morphism \( a \to t \).

Definition 7.8 (null object) [Lan, p.20] Let \( C \) be a category. A null object \( z \) is a \( C \)-object which is both initial (7.6) and terminal (7.7).

Definition 7.9 (additive category) [Lan, VIII.2. p.196] An additive category \( A \) is an pre-additive category (7.1) that satisfies the following conditions

- \( A \) has a null object (7.8).
- \( A \) has a binary biproduct for each pair of \( A \)-objects (7.2).

Definition 7.10 (zero morphism) [Lan, p. VIII.1.] Let \( C \) be a category with a null object \( z \) (7.8). Let \( a, b \) be \( C \)-objects. The zero morphism from \( a \) to \( b \) is defined to be the composition of the morphism from \( a \) to \( z \) and the morphism from \( z \) to \( b \)

\[
(a \xrightarrow{0} b) := (a \to z \to b).
\]

Definition 7.11 (monic morphism) [Lan, p.19] Let \( C \) be a category. A monic morphism is a \( C \)-morphism \( a \xrightarrow{m} b \) such that the left cancellation rule holds:

\[
(mf = mg) \Rightarrow (f = g).
\]
Definition 7.13 (epi morphism) [Lan, p.19] Let $C$ be a category. An epi morphism is a $C$-morphism $m: a \rightarrow b$ such that the right cancellation rule holds:

\[(fe = ge) \Rightarrow (f = g).\]  

Definition 7.15 (diagonal functor) Let $C$ and $J$ be categories. The diagonal functor $\Delta$ from $C$ to $C^J$ is defined to send each $C$-object $c$ to the constant functor $\Delta c$, and to send each $C$-morphism $c \rightarrow d$ to the constant natural transform $\Delta f$. (cf Lan, p.67).

Definition 7.16 (universal morphism) [Lan, p. III.1.] Let $C$ and $D$ be categories. Let $c$ be a $C$-object. Let $D^{S} \rightarrow C$ be a functor. A universal morphism from $c$ to $S$ is a pair $(r, u)$, where $r$ is a $D$-object and $c \rightarrow Sr$ is an $C$-morphism that satisfies the following condition:

For each pair $(d, f) \in \text{Obj}(D) \times C(c, Sd)$, there is a unique $D$-morphism $r \rightarrow d$ with $Sf \circ u = f$.

Definition 7.17 (categorical limit) [Lan, p. III.4.] Let $C$ and $J$ be categories and $J \rightarrow C$ be a functor. A limit for the functor $F$ is defined to be a universal morphism (7.16) $(r, v)$ from $\Delta$ to $F$, where $\Delta$ is the diagonal functor (7.15) from $C$ to $C^J$.

Definition 7.18 (equalizer) [Lan, p. III.4.] Let $C$ be a category, $a$, $b$ be $C$-objects, and $f, g$ be $C$-morphisms from $a$ to $b$. The equalizer for the pair $(f, g)$ is defined to be the limit (7.17) of the corresponding functor $P \rightarrow C$, where $P$ denotes the category with exactly two objects $0, 1$ and two non-identity morphisms $0 \rightarrow 1$.

Definition 7.19 (kernel) [Lan, p. VIII.1.] Let $C$ be a category with a null object (7.8). A kernel of a morphism $f: a \rightarrow b$ is defined to be an equalizer (7.18) for the pair $(f, a \rightarrow 0 b)$, where $0$ denotes the zero morphism (7.10).

Definition 7.20 (cokernel) [Lan, p.192] The notion of a cokernel is the dual of the notion of a kernel (7.19).

Definition 7.21 (pre-abelian category) An abelian category is an additive category (7.1) in which every morphism has a kernel (7.19) and a cokernel (7.20).

Lemma 7.22 Let $A$ be a pre-abelian category. Then each morphism $X \rightarrow Y$ in $A$ has a canonical factorization [Ive, p. I.1]

\[f = \left( X \rightarrow \text{cok}(\ker(f)) \rightarrow Y \right).\]  

Definition 7.24 (exact category) An exact category is a pre-abelian category in which the middle morphism of the canonical factorization (7.22) $f'$ of each morphism $X \rightarrow Y$ is an isomorphism.

Definition 7.25 (abelian category) [Lan, VIII.3. Definition] An abelian category is an pre-abelian category (7.1) in which every monic morphism (7.11) is a kernel, and every epi morphism (7.13) is a cokernel.

Lemma 7.26 Let $A$ be a pre-abelian category. Then the followings are equivalent.
A is abelian.
A is exact.

Lemma 7.27 To summarize, an abelian category is a category such that the followings are satisfied.

- (pre-additivity) Every hom set is an abelian group such that every composition is bilinear.
- (additivity) A null object and binary biproducts exist.
- (pre-abelianity) Every morphism has a kernel and cokernel.
- (exactness) Canonical factorizations induce isomorphisms between the images and coimages
  \[ \mathrm{cok} \circ \ker (\cdot) \rightarrow \ker \circ \mathrm{cok}(\cdot). \]

Lemma 7.28 Let \( C \) be an abelian category and \( D \) be an additive category. Suppose there is an additive equivalence of categories \( C \xrightarrow{F} D \). Then \( D \) is abelian.

Proof. By 7.27, we have to show that the additive category \( D \) is pre-abelian and exact.

Let \( X' \xrightarrow{\phi'} Y' \) be a \( D \)-morphism. Pick \( C \)-objects \( X \) and \( Y \) such that \( FX \) and \( FY \) are isomorphic to \( X' \) and \( Y' \) respectively. Since

\[ C(X, Y) \simeq D(FX, FY) \simeq D(X', Y'), \]

\( \phi' \) has kernels and cokernels, and its canonical factorization induces isomorphisms between images and coimages.

\[ \blacksquare \]

7.2 Semisimple category

Throughout the whole section, assume that \( k \) is an algebraically closed field of characteristic 0.

Definition 7.29 (subobject) [Eti+15, p. 1.3.5] Let \( C \) be a category and \( X \) be a \( C \)-object. A subobject of \( X \) is a monic \( C \)-morphism \( Y \xrightarrow{f} X \).

Definition 7.30 (simple object) [Eti+15, p. 1.5.1] Let \( C \) be an abelian category. A simple object \( X \) of \( C \) is a nonzero \( C \)-object whose only subobjects are \( 0 \xrightarrow{0} X \) and \( X \xrightarrow{id_X} X \).

Definition 7.31 (semisimple object) [Eti+15, p. 1.5.1] Let \( C \) be an abelian category. A semisimple object \( X \) of \( C \) is a direct sum of some simple objects of \( C \).

Definition 7.32 (semisimple category) [Eti+15, p. 1.5.1] A semisimple category is an abelian category whose objects are all semisimple.
Definition 7.33 (object of finite length) [Eti+15, p. 1.5.3] Let $X$ be an object of an abelian category $C$. We say that $X$ is of finite length if there exists a positive integer $n$ and a sequence of monic morphisms

$$
\emptyset = X_0 \rightarrow X_1 \rightarrow \cdots \rightarrow X_n = X \quad (7.34)
$$

each of whose cokernel object $X_{i+1}/X_i$ is simple. Call $n$ the length of this sequence.

Remark 7.35 Such a sequence is called a Jordan-Holder series for $X$. By Jordan-Holder theorem [Eti+15, p. 1.5.4], all Jordan-Holder series of $X$ have the same length.

Definition 7.36 (length of an object) [Eti+15, p. 1.5.5] Let $C$ be an abelian category and $X$ a $C$-object. The length $X$ is defined to be the length of one, thus all, of its Jordan-Holder series.

Definition 7.37 (linear category over a field) [Eti+15, p. 1.2.2] Let $k$ be a field. A $k$-linear category is an additive category $C$ whose hom-spaces are $k$-vector spaces, such that all compositions of morphisms are $k$-linear maps.

Definition 7.38 (locally finite abelian category over a field) [Eti+15, p. 1.8.1] A locally finite category (or an Artinian category) over $k$ is a $k$-linear abelian category $C$ that satisfies the following conditions.

- Every object has finite length.
- Every hom space is a finite dimensional $k$-vector space.

Definition 7.39 (finite abelian category over a field) [Eti+15, p. 1.8.6] A finite category abelian category $C$ over $k$ is a locally finite abelian category over $k$ such that

- $C$ has enough projectives, i.e. every simple object of $C$ has a projective cover.
- The set of isomorphism classes of simple objects is finite.

Remark 7.40 By discussion before [Eti+15, p. 1.8.6], a finite $k$-linear abelian category $C$ is equivalent to the category of finite dimensional modules over a finite dimensional $k$-algebra $A$.

7.3 Tensor category

Recall that the Crane-Yetter theory CY comes in a family, of which member depends on a type of algebraic data called the premodular categories. Despite its technical definition (finite semisimple ribbon braided rigid tensor category), it does not hurt too much to think of a premodular category as a higher categorical analogue of a finite abelian group: the “braided tensor” structure encodes the (higher) group operation, the “rigid” structure encodes the (higher) inverses, and the “ribbon” structure ensures that $(g^{-1})^{-1}$ is equivalent to $g$.

In this section, a complete definition for a premodular category 7.68 is collected from [Eti+15]. Throughout the whole section, assume that $k$ is an algebraically closed field of characteristic 0.
Definition 7.41 (monoidal category) [Eti+15, p. 2.1.1] A monoidal category is a septuple

$$(C, \otimes, a, 1, \iota, l, r),$$

that satisfies the pentagon axiom and the triangle axiom [Eti+15, (2.2)], where $C$ is a category, $C \times C \xrightarrow{\otimes} C$ is a bifunctor,

$$\left( (-1 \otimes -2) \otimes -3 \right) \xrightarrow{\alpha} \left( -1 \otimes (-2 \otimes -3) \right)$$

is a natural equivalence, $1$ is an object in $C$, $1 \otimes 1 \xrightarrow{\iota} 1$ is an isomorphism, $(-) \xrightarrow{1} (1 \otimes -)$ and $(-) \xrightarrow{r} (- \otimes 1)$ are natural equivalences.

We will abuse notations and denote the septuple by $C$. The bifunctor $\otimes$ is called the tensor product bifunctor, the pair $(1, \iota)$ is called the unit object, and the natural equivalence $\alpha$ is called the associativity isomorphism.

Definition 7.42 (duals of an object) [Eti+15, 2.10.1 and 2.10.2] Let $X$ be an object of a monoidal category $(C, \otimes, 1, a, \iota, l, r)$. A left dual of $X$ is an object $L$ with two morphisms

$$L \otimes X \xrightarrow{ev} 1 \quad (7.43)$$

$$1 \xrightarrow{coev} X \otimes L \quad (7.44)$$

such that the compositions of the following the identity morphisms

$$X \xrightarrow{coev \otimes 1} (X \otimes L) \otimes X \xrightarrow{a} X \otimes (L \otimes X) \xrightarrow{1 \otimes ev} X, \quad (7.45)$$

$$L \xrightarrow{1 \otimes coev} L \otimes (X \otimes L) \xrightarrow{a^{-1}} (L \otimes X) \otimes L \xrightarrow{ev \otimes 1} L. \quad (7.46)$$

Similarly, a right dual of $X$ is an object $R$ with two morphisms

$$X \otimes R \xrightarrow{ev'} 1 \quad (7.47)$$

$$1 \xrightarrow{coev'} R \otimes X \quad (7.48)$$

such that the compositions of the following are the identity morphisms

$$X \xrightarrow{1 \otimes coev'} X \otimes (R \otimes X) \xrightarrow{a^{-1}} (X \otimes R) \otimes X \xrightarrow{ev' \otimes 1} X, \quad (7.49)$$

$$R \xrightarrow{coev' \otimes 1} (R \otimes X) \otimes R \xrightarrow{a} R \otimes (X \otimes R) \xrightarrow{1 \otimes ev'} R. \quad (7.50)$$

Remark 7.51 It can be proved that the left (resp., right) dual, if exists, is unique up to isomorphism [Eti+15, p. 2.10.5]. We will denote it by $X^*$ (resp., $^*X$).

Definition 7.52 (rigid object) [Eti+15, p. 2.10.11] Let $C$ be a monoidal category. A rigid object $X$ of $C$ is a $C$-object that has a left dual and a right dual.

Definition 7.53 (rigid category) [Eti+15, p. 2.10.11] A rigid category $C$ is a monoidal category all of whose objects are rigid.

Definition 7.54 (multitensor category) [Eti+15, p. 4.1.1] A multitensor category $C$ over $k$ is a locally finite $k$-linear abelian rigid monoidal category if the bifunctor $\otimes$ in the monoidal structure is $k$-bilinear on morphisms.
Lemma 7.55 [Eti+15, p. 4.2.1] Let \( C \) be a multitensor category. Then the bifunctor \( \otimes \) is exact in both factors.

\[ \text{Proof.} \text{ It is a fun exercise to prove. A sketch is as follows. Let } X \text{ be a } C\text{-object. The rigidity says that } X \otimes (-) \text{ is a left and right adjoint functor. In general category theory, adjoint functors preserve all (co)limits essentially because Hom does and Yoneda lemma. In particular, they preserve (co)kernels. } \]

Definition 7.56 (multifusion category) [Eti+15, p. 4.1.1] A multifusion category over \( k \) is a multitensor category that is finite over \( k \) and semisimple.

Definition 7.57 (fusion category) [Eti+15, p. 4.1.1] A fusion category \( C \) is a multifusion category with \( \text{End}_C(1) \simeq k \).

Definition 7.58 (braiding) [Eti+15, p. 8.1.1] A braiding of a monoidal category \((C, \otimes, 1, \alpha, \iota, l, r)\) is a natural equivalence

\[( -1 \otimes -2 ) \xrightarrow{c} (-2 \otimes -1) \] (7.59)

such that the hexagon diagram [Eti+15, (8.1)] holds.

Definition 7.60 (braided category) [Eti+15, p. 8.1.2] A braided category is a monoidal category with a braiding.

Remark 7.61 The Yang-Baxter equation holds automatically in a braided category [Eti+15, p. 8.1.10].

Definition 7.62 (half-braiding) [Eti+15, (7.41)] A half-braiding for an object \( X \) in a monoidal category \((C, \otimes, 1, \alpha, \iota, l, r)\) is a natural equivalence

\[ (X \otimes -) \xrightarrow{c} (- \otimes X) \] (7.63)

such that the hexagon diagram [Eti+15, (7.41)] holds.

Definition 7.64 (twist) [Eti+15, p. 8.10.1] Let \( C \) be a braided rigid monoidal category. A twist of \( C \) is an element \( \theta \in \text{Aut}(\text{id}_C) \) such that for each \( C \)-object \( X, Y \)

\[ \theta_{X \otimes Y} = (\theta_X \otimes \theta_Y) \circ c_{Y,X} \circ c_{X,Y} \] (7.65)

Definition 7.66 (ribbon structure) [Eti+15, p. 8.10.1] Let \( C \) be a braided rigid monoidal category. A twist \( \theta \) is called a ribbon structure if \( (\theta_X)^* = (\theta_{X^*}) \), where the first dual is taken in a rigid category.

Definition 7.67 (ribbon tensor category) [Eti+15, p. 8.10.1] A ribbon tensor category is a braided rigid monoidal category equipped with a ribbon structure.

Definition 7.68 (premodular category) [Eti+15, p. 8.13.1] A premodular category is a ribbon fusion category.
Definition 7.69 (pivotal structure) [Eti+15, p. 4.7.7] Let $C$ be a rigid monoidal category. A pivotal structure of $C$ is a natural isomorphism

$$(-) \xrightarrow{a} (-)**$$

such that $a_{X \otimes Y} = a_X \otimes a_Y$ for all $C$-objects $X$ and $Y$. We call a rigid monoidal category pivotal if it is equipped with a pivotal structure.

Definition 7.70 (pivotal dimension) Let $C$ be a rigid monoidal category with a pivotal structure $a$. Let $X$ be a $C$-object. We define the pivotal dimension with respect to $a$ to be

$$\dim_a(X) := \text{Trace}(a_X) \in \text{End}_C(1).$$

Definition 7.71 (spherical structure) [Eti+15, p. 4.7.14] Let $C$ be a rigid monoidal category with a pivotal structure $a$. The latter is called a spherical structure if

$$\dim_a(X) = \dim_a(X^*)$$

for any $C$-object $X$.

Remark 7.72 [Eti+15, p. 8.13.1] Equivalently, a premodular category is also a braided fusion category equipped with a spherical structure.

Definition 7.73 (modular category) [Eti+15, 8.14 and 8.20.12] A modular category is a premodular category with a non-degenerate $S$-matrix.

### 7.4 Adjunctions as monads

Definition 7.74 Let $X$ be a strict 2-category, $C$ and $D$ be $X$-objects, and $C \xrightarrow{F} D$ and $C \xleftarrow{G} D$ be morphisms. We say that $F$ is right adjoint to $G$, that $G$ is left adjoint to $F$, if there exists 2-morphisms

$$1_D \xRightarrow{\eta} FG, \quad GF \xRightarrow{\epsilon} 1_C$$

such that the followings

$$G = G \circ 1_D \xRightarrow{1_G \ast \eta} G \circ F \circ G \xRightarrow{\epsilon \ast 1_G} G$$

$$F = 1_D \circ F \xRightarrow{\eta \ast 1_F} F \circ G \circ F \xRightarrow{1_F \ast \epsilon} F$$

equal the identity 2-morphisms $1_G$ and $1_F$ respectively. Denote $F \dashv G$ in this case.

We call $\eta$ and $\epsilon$ the unit and counit of the monad, and call the coherence condition the rigidity condition.

Definition 7.75 Let $X$ be a strict 2-category, $D$ be an $X$-object. Then $E = \text{End}_X(D)$ is a 1-category. A monad of $D$ is a monoid object $T = (T, \eta, \mu)$ in $E$. That is to say, $T$ is an $E$-object, and $(1_D \xRightarrow{\eta} T)$ and $(T^2 \xRightarrow{\mu} T)$ are $E$-morphisms such that

$$(1_T \ast \eta) = 1_T = \mu \circ (\eta \ast 1_T), \quad \mu \circ (\mu \ast 1_T) = \mu \circ (1_T \ast \mu).$$
Theorem 7.76 Let $X$ be a strict 2-category, $C$ and $D$ be $X$-objects, $(C \overset{F}{\to} D)$ and $(C \overset{G}{\leftarrow} D)$ be morphisms such that $F$ is right adjoint to $G$. Then $T = FG$ has a monad structure given by

$$(1_D \overset{\eta}{\Rightarrow} T), \quad (T^2 \overset{\mu}{\Rightarrow} T)$$

where $\mu$ is defined as $FGF \overset{1_{F \circ \epsilon} \circ \epsilon}{\Rightarrow} F$. Dually, $\perp = GF$ has a comonad structure. \hfill \diamondsuit

Proof.

\[
1_T \\
= 1_F * 1_G \\
= (1_F * (G \overset{1_{G \circ \eta}}{\Rightarrow} GFG \overset{\epsilon \circ 1_G}{\Rightarrow} G)) \\
= (FG \circ 1_D \overset{1_{F \circ \eta} \circ \epsilon}{\Rightarrow} FGFG \overset{1_{F \circ \epsilon} \circ \epsilon}{\Rightarrow} FG) \\
= (T \circ 1_D \overset{1_{T \circ \eta}}{\Rightarrow} T \circ T \overset{\mu}{\Rightarrow} T)
\]

\[
1_T \\
= 1_F * 1_G \\
= (F \overset{\eta \circ 1_F}{\Rightarrow} FGF \overset{1_{F \circ \epsilon} \circ \epsilon}{\Rightarrow} F) * 1_G \\
= (1_D \circ FG \overset{\eta \circ 1_F \circ 1_G}{\Rightarrow} FGFG \overset{1_{F \circ \epsilon} \circ \epsilon}{\Rightarrow} FG) \\
= (1_D \circ T \overset{\eta \circ 1_T}{\Rightarrow} T \circ T \overset{\mu}{\Rightarrow} T)
\]

\[
(T^3 \overset{1_{T \circ \mu}}{\Rightarrow} T^2 \overset{\mu}{\Rightarrow} T) \\
= ((FG)(FGFG) \overset{1_{F \circ \epsilon} \circ \epsilon}{\Rightarrow} FGFG \overset{1_{F \circ \epsilon} \circ \epsilon}{\Rightarrow} FG) \\
= 1_F * (GF \overset{1_{F \circ \epsilon} \circ \epsilon}{\Rightarrow} GF \overset{\epsilon}{\Rightarrow} 1_D) * 1_G \\
= 1_F * (GF \overset{\epsilon \circ \epsilon}{\Rightarrow} 1_D) * 1_G \\
= \ldots = (T^3 \overset{\mu \circ 1_T}{\Rightarrow} T^2 \overset{\mu}{\Rightarrow} T).
\]

Theorem 7.77 Let $C$ and $D$ be categories, $C \overset{F}{\to} D$ and $C \overset{G}{\leftarrow} D$ be functors such that $F$ is a right adjoint functor to $G$, i.e. there exists natural equivalence $\Phi$ such that

$$C(Gd, c) \overset{\sim}{\to} D(d, Fc).$$

Then $F$ is right adjoint to $G$ the strict 2-category $\text{Cat}$ of all categories. \hfill \diamondsuit

42
Proof. From the given natural equivalence $\Phi$, we have to construct $1_D \xrightarrow{\eta} FG$ and $\epsilon \xrightarrow{C_F(C)} 1_C$ that satisfy the conditions as in 7.74. We contend that $\eta_d = \Phi_{d,Gd}(1_{Gd})$ and $\epsilon_c = \Phi_{F_F,C}^{-1}(1_{Fc})$ are as desired.

Let us first show that $\eta_d$ is indeed a natural transformation from $1_D$ to $FG$. So is $\epsilon_c$ similarly, whose proof will be omitted. Let $d \xrightarrow{\phi} d'$ be a $D$-morphism. We shall prove that the following commutative diagram commute.

\[
\begin{array}{ccc}
    d & \xrightarrow{\eta_d} & FGd \\
    \downarrow{\phi} & & \downarrow{FG\phi} \\
    d' & \xrightarrow{\eta'd} & D
\end{array}
\]

First notice that

\[
\Phi_{d,Gd'}^{-1}(\eta_d' \circ \phi) = (G\phi)^* \circ \Phi_{d',Gd'}^{-1} \circ (\phi^*)^{-1}(\eta_d' \circ \phi) = (G\phi)^* \circ \Phi_{d',Gd'}^{-1}(\eta_d') = (G\phi)^*(1_{Gd'}) = G\phi
\]

As $\Phi$ is an equivalence, it suffices to prove that $\Phi_{d,Gd'}^{-1}((FG\phi) \circ \eta_d)$ is also $G\phi$.

\[
\Phi_{d,Gd'}^{-1}((FG\phi) \circ \eta_d) = \Phi_{d,Gd'}^{-1}((FG\phi) \circ \Phi_{d,Gd}(1_{Gd})) = (G\phi)^* \circ \Phi_{d,Gd}^{-1}((FG\phi)^* \circ (FG\phi) \circ \Phi_{d,Gd}(1_{Gd})) = (G\phi)^* \circ \Phi_{d,Gd}^{-1}(\Phi_{d,Gd}(1_{Gd})) = (G\phi)^*(1_{Gd}) = G\phi
\]

which is due to the naturality of $\Phi$

\[
\begin{array}{ccc}
    C(Gd, Gd) & \xrightarrow{\Phi_{d,Gd}} & D(d, FGd) \\
    \downarrow{(G\phi)^*} & & \downarrow{(FG\phi)^*} \\
    C(Gd, Gd') & \xrightarrow{\Phi_{d,Gd'}} & D(d, FGd')
\end{array}
\]

Next, we have to show the rigidity conditions of $\eta$ and $\epsilon$. Indeed, from the naturality of $\Phi$ we have the commutative diagram

\[
\begin{array}{ccc}
    C(GFGd, Gd) & \xrightarrow{\Phi_{GFGd,Gd}} & D(FGd, FGd) \\
    \downarrow{(G\eta_d)^*} & & \downarrow{\eta'_d} \\
    C(Gd, Gd) & \xrightarrow{\Phi_{d,Gd}} & D(d, FGd)
\end{array}
\]
and thus the following holds

\[
[(\epsilon * 1_G) \circ (1_G * \eta)]_d = (\epsilon * 1_G)_d \circ (1_G * \eta)_d \\
= (GFGd \xrightarrow{\epsilon_Gd} Gd) \circ G(d \xrightarrow{\eta_d} FGD) \\
= Gd \xrightarrow{G(\eta_d)} GFGd \xrightarrow{\epsilon_Gd} Gd \\
= Gd \xrightarrow{G(\eta_d)} GFGd \xrightarrow{\Phi_{FGd,Gd}^{-1}(1_{FGd})} Gd \\
= (G(\eta_d))^* \left( GFGd \xrightarrow{\Phi_{FGd,Gd}^{-1}(1_{FGd})} Gd \right) \\
= (\Phi_{d,FGd}^{-1} \circ \eta_d^*)(1_{FGd}) \\
= \Phi_{d,FGd}^{-1}(\eta_d) = 1_{GD}.
\]

The other rigidity condition follows similarly from the commutative diagram

\[
\begin{array}{ccc}
C(GFc, GFc) & \xrightarrow{\Phi_{Fc,GFc}} & D(Fc, FGFc) \\
\downarrow{(\epsilon_c)} \quad & & \downarrow{(\epsilon_c)} \\
C(GFc, c) & \xrightarrow{\Phi_{Fc,c}} & D(Fc, Fc)
\end{array}
\]

Therefore, adjoint functors give adjoint pairs in the 2-category \(\text{Cat}\), which in turn gives a monad and a comonad in \(\text{Cat}\). Let’s summarize the result in the following theorem.

**Theorem 7.78** Let \(C \xrightarrow{F} D\) and \(C \xleftarrow{G} D\) be functors such that \(F\) is right adjoint to \(G\). Then there is a \(D\)-monad \((T = FG, \eta, \mu)\) and a \(C\)-comonad \((\bot = GF, \epsilon, \Delta)\), where

\[
\eta_d = \Phi_{d,FGd}(1_{GD}), \mu = 1_F * \epsilon * 1_G, \\
\epsilon_c = \Phi_{Fc,c}^{-1}(1_{FC}), \Delta = 1_G * \eta * 1_F.
\]

There is a right adjoint functor of \(C \xleftarrow{G} D\) with the natural transformation \(\Phi\). The categories \(C\) and \(D\) are intimately tied together by the adjoint functors between them. For example, a part of compositions in \(C\) can be identified as monadic composition in \(D\).

**Theorem 7.79** The usual composition map

\[
C(Gx, Gy) \times C(Gy, Gz) \xrightarrow{\circ} C(Gx, Gz)
\]

is identified under \(\Phi\) as the Kleisi composition

\[
D(x, Ty) \times D(y, Tz) \xrightarrow{\circ_T} D(x, Tz) \\
\circ_T(f, g) \mapsto \mu_Z \circ (Tg) \circ f.
\]
Proof. Since $\Phi$ is an equivalence, and since
\[
\Phi(\circ(\Phi^{-1}(f, g))) = \Phi_{x,Gz}(\Phi_{y,Gz}(g) \circ \Phi^{-1}_{x,Gy}(f)),
\]
it suffices to prove that
\[
\Phi_{x,Gz}^{-1}(\mu_Z \circ (Tg) \circ f) = \Phi_{y,Gz}^{-1}(g) \circ \Phi^{-1}_{x,Gy}(f).
\]
The main task would be to express $\mu_Z$ in terms of $\Phi$.

From the commutative diagram
\[
\begin{array}{ccc}
C(GFGz, Gz) & \xrightarrow{\Phi_{FGz,Gz}} & D(FGz, FGz) \\
\downarrow{(-) \circ G(g)} & & \downarrow{(-) \circ g} \\
C(Gy, Gz) & \xrightarrow{\Phi_{y,Gz}} & D(y, FGz)
\end{array}
\]
we have
\[
\mu_Z \circ (Tg) \circ f \\
= F(\Phi_{FGz,Gz}^{-1}(1_{FGz}) \circ F(G(g)) \circ f) \\
= F(\Phi_{FGz,Gz}^{-1}(1_{FGz}) \circ G(g) \circ f) \\
= F(\Phi_{y,Gz}^{-1}(1_{FGz} \circ g) \circ f) \\
= F(\Phi_{y,Gz}^{-1}(g) \circ f)
\]
It remains to prove that
\[
\Phi_{x,Gz}^{-1}(F(\Phi_{y,Gz}^{-1}(g) \circ f)) = \Phi_{y,Gz}^{-1}(g) \circ \Phi^{-1}_{x,Gy}(f),
\]
which directly follows from the commutative diagram obtained from the naturality of $\Phi$:
\[
\begin{array}{ccc}
C(Gx, Gy) & \xrightarrow{\Phi_{x,Gy}} & D(x, FGy) \\
\downarrow{\Phi^{-1}(g) \circ (-)} & & \downarrow{F(\Phi^{-1}) \circ (-)} \\
C(Gx, Gz) & \xrightarrow{\Phi_{x,Gz}} & D(x, FGz)
\end{array}
\]

The natural equivalence $\Phi$ isn’t as easy to manipulate as the (co)monads it induces. We collect more statements that express the former in terms of the later.

Lemma 7.80 In terms of (co)monad, $\Phi$ can be expressed as follows.
\[
\Phi_{d,c}(\phi) = F(\phi) \circ \eta_d,
\]
\[
\Phi_{d,c}^{-1}(\psi) = \epsilon_c \circ G(\psi).
\]

Proof. These are evident from the commutative diagrams below respectively.
Instances arise where two functors are adjoint to each other from both sides. We call them (strict) ambidextrous functors.

**Definition 7.81 (ambidextrous adjunctions)** Let \( C \xrightarrow{F} D \) and \( C \xleftarrow{G} D \) be functors. We call \( F \) and \( G \) a pair of (strict) ambidextrous functors if \( F \) is both left-adjoint and right-adjoint to \( G \).

Two monads and two comonads arise from a pair of ambidextrous functors. More precisely, that \( F \vdash G \) gives a natural \( D \)-monad \((T = FG, \eta, \mu)\) and a natural \( C \)-comonad \((\perp = GF, \epsilon, \Delta)\). Similarly, that \( F \dashv G \) gives a natural \( D \)-comonad \((T = FG, \eta', \mu')\) and a natural \( C \)-monad \((\perp = GF, \epsilon', \Delta')\). In particular, we have a bimonad structure on \( T = (T, \eta, \mu, \epsilon, \Delta) \), with unit \( \eta \), multiplication \( \mu \), counit \( \epsilon \), and co-multiplication \( \Delta \).

**Definition 7.82** We the bimonad \( T \) is of unity trace if

\[
(1_D \xrightarrow{\eta} T \xrightarrow{\eta'} 1_D) = (1_D \xrightarrow{1_D \eta} 1_D).
\]

We the bimonad \( T \) is of collapsable diamond if

\[
(T \xrightarrow{\mu'} T^2 \xrightarrow{\mu} T) = (T \xrightarrow{1_T} T).
\]

By the following lemma, the second condition is superseded by the first one.

**Lemma 7.83** If such adjunction is of unity trace for \( \perp \), then \( T \) is of collapsable diamond.

**Proof.**

\[
(\mu')^2 \eta = \mu \eta' = \mu = \mu' \eta' = (\mu')^2 \eta' = \mu' \eta
\]

The unity trace condition turns out to be crucial for our work - essentially it guarantees an averaging map analogue to that in the theory of finite group representations.

**Lemma 7.84** Let \( T = (T, \eta, \mu, \epsilon, \Delta) \) be a \( D \)-bimonad of unity trace. Then for each \( D \)-object \( x \) and \( y \), the morphism \( D(x, y) \xrightarrow{\eta_y \circ(-)} D(x, Ty) \) is monic, the arrow \( D(x, Ty) \xrightarrow{\eta'_y \circ(-)} D(x, y) \) is epic, and moreover the map \( (\eta_y \circ \eta'_y) \) is a projection map onto the image of \( (\eta_y \circ -) \).
Proof. The unity trace condition says that \( \eta'_y \circ \eta_y = 1_y \), so the first two conditions follow. The last statement is also evident since
\[
(\eta'_y \eta_y)^2 = \eta'_y 1_y \eta_y = \eta'_y \eta_y.
\]

Therefore, in the context of (strict) ambidextrous adjoint functors, the unity trace condition yields a projection map
\[
C(Gx, Gy) \xrightarrow{\pi_{x,y}} D(x, y)
\]
from the equivalence \( C(Gx, Gy) \simeq D(x, Ty) \). In the next lemma, we see that this projection is functorial without any extra assumption.

**Theorem 7.85** Let \( F : C \leftrightarrow D : G \) be a pair of strictly ambidextrous adjoint functors, and \( T = FG \) be the naturally induced bimonad on \( D \). If \( T \) is of unity trace, then \( \pi_{x,y} \) is functorial in the sense that

1. \( \pi_{x,x}(Gx \xrightarrow{1_{Gx}} Gx) = 1_x \).
2. For \( C \)-morphisms \( (Gx \xrightarrow{\phi} Gy \xrightarrow{\sigma} Gz) \), we have
\[
(x \xrightarrow{\pi_{x,y} \phi} y \xrightarrow{\pi_{y,z} \sigma} z) = (x \xrightarrow{\pi_{x,z}(\sigma \phi)} z).
\]

Proof. By the unity trace condition and 7.80,
\[
\pi_{x,x}(1_{Gx}) = \eta'_x \circ (1_{FGx} \circ \eta_x) = 1_x,
\]
proving the first statement. It remains to prove that
\[
(\eta'_x \circ F(\sigma) \circ \eta_y) \circ (\eta'_y \circ F(\phi) \circ \eta_x) = (\eta'_x \circ F(\sigma \phi) \circ \eta_x).
\]

Indeed,
\[
\begin{align*}
&\left( z \xleftarrow{\eta'_z} Tz \xleftarrow{T(\sigma)} Ty \xleftarrow{\eta'_y} Ty \xleftarrow{F(\phi)} Tx \xleftarrow{\eta_x} x \right) \\
= &\left( z \xleftarrow{\eta'_z} Tz \xleftarrow{T(\sigma)} Ty \xleftarrow{T(\phi)} T^2y \xleftarrow{\eta_{Ty}} Ty \xleftarrow{F(\phi)} Tx \xleftarrow{\eta_x} x \right) \\
= &\left( z \xleftarrow{\eta'_z} Tz \xleftarrow{T(\sigma)} Ty \xleftarrow{T(\phi)} T^2y \xleftarrow{T^2x} T^2x \xleftarrow{\eta_{T^2x}} T^2x \xleftarrow{\eta_x} x \right) \\
= &\left( z \xleftarrow{\eta'_z} Tz \xleftarrow{T(\sigma)} Ty \xleftarrow{T(\phi)} T^2y \xleftarrow{T^2x} T^2x \xleftarrow{T^2x} T^2x \xleftarrow{\eta_x} x \right) \\
= &\left( z \xleftarrow{\eta'_z} Tz \xleftarrow{T(\sigma)} Ty \xleftarrow{T(\phi)} T^2y \xleftarrow{\eta_x} x \right)
\end{align*}
\]
The first two equalities follow from the naturality of \( \eta \). The third equality \( T(\eta_x) = \eta(Tx) \) follows from the Eckmann-Hilton argument
\[
\eta \ast 1_T = 1_T \ast \eta.
\]
Finally, the last equality follows from all the tricks and conditions: that \( \eta \) is natural, that \( T \circ \eta = \eta \circ T \), that \( T \) is functorial, and the unity trace condition.

\[\square\]
Remark 7.86 In the context of categorical center of higher genera, the proof above translates into the following graphical proof, where the orange dotted lines represent the shorthand notation $\Omega$ given in 2.10.

In the proof, it is tempting to demand $\eta_\sigma \circ \eta'_\sigma$ to be identity, which would have finished the proof right away. However, it is not necessarily true. In fact, it is false in our context. The best one can say about it is that it is idempotent.

Example 7.87 Let $C$ be a premodular category, $\sigma$ be an admissible gluing, $D$ be the categorical center of higher genera $Z_\sigma(C)$, $F$ be the induction functor $C \overset{I_\sigma}{\to} Z_\sigma(C)$, and $G$ be the forgetful functor $C \overset{F_\sigma}{\leftarrow} Z_\sigma(C)$.

By 4.17, both functors are strictly ambidextrous to each other. Moreover, it is clear by their definitions that the bi-monads they form are of unity trace. Therefore, by 7.85, we have the followings.

1. $D((X, \gamma), (Y, \beta))$ embeds into $C(X, Y)$ naturally, with a natural projection $\pi_{Y, \beta}$ onto the subspace.
2. $C(X, Y)$ embeds into $D(I_\sigma(X), I_\sigma(Y))$ naturally, with a natural projection $\pi_{X, Y}$ onto the subspace.

This is an analogue of the averaging map one has in the theory of finite dimensional complex linear representations of finite groups.

7.5 Misc proofs

Statements and proofs that could break the flow of are collected in this section. The readers are advised to use it as a reference.

Lemma 7.88 Let $C$ be a premodular category and $\sigma \in \text{Adm}_{2n}$ an admissible gluing. Then the categorical center of higher genera $Z_\sigma(C)$ is an abelian category.

Proof. By 7.27 we need to show that $Z_\sigma(C)$ is an additive category such that

- every morphism in $Z_\sigma(C)$ has a kernel and a cokernel.
- $Z_\sigma(C)$ is an exact category.
By its definition, \( Z_\sigma(C) \) is additive. To prove that every morphism has a kernel and a cokernel, first let \((X, \gamma) \xrightarrow{f} (Y, \beta)\) be a \( Z_\sigma(C) \)-morphism. Recall by definition that \( f \) is a \( C \)-morphism \( X \xrightarrow{f} Y \) that respects both sets of half-braidings \( \gamma \) and \( \beta \). Since \( C \) is premodular thus abelian, \( f \) has a kernel \( K \xrightarrow{m} X \) in \( C \). We will construct a \( Z_\sigma(C) \)-object \((K, m^*\gamma)\) such that \((K, m^*) \xrightarrow{m} (X, \gamma)\) is a \( Z_\sigma(C) \)-morphism and is in fact a kernel of \( f \).

To construct \( m^*\gamma \), notice that all we need is a set of half-braidings for \( K \) that work compatibly with \( \gamma \). As

\[
K \xrightarrow{m} X \xrightarrow{f} Y
\]

is exact and that \( \otimes \) is bi-exact 7.55, we see that \( 1_- \otimes m \) and \( m \otimes 1_- \) are kernels of \( 1_- \otimes f \) and \( f \otimes 1_- \) respectively. Therefore, \( \gamma_{[i]} \circ (m \otimes 1_-) \) factors through \( 1_- \otimes m \) uniquely. Ditto for the other direction. So defines an natural isomorphism

\[
K \otimes (-) \xrightarrow{(m^*\gamma)_{[i]}} (-) \otimes K.
\]

Define so for all other \( i \)'s, it's straightforward to prove that \( m^*\gamma \) is a \( \sigma \)-pair 4.1 from that \( \gamma \) is also one. From the construction above, clearly

\[
(K, m^*\gamma) \xrightarrow{m} (X, \gamma)
\]

is a \( Z_\sigma(C) \)-arrow. It remains to show that \( m \) is indeed a kernel of \( f \) in \( Z_\sigma(C) \). Let \((W, \alpha) \xrightarrow{h} (X, \gamma)\) be a \( Z_\sigma(C) \)-morphism such that \( fh = 0 \). Then \( h \) uniquely factors through \( m \) by some \( C \)-morphism \( k \). The crux is to show that \( k \) is indeed a \( Z_\sigma(C) \)-morphism. But indeed, by the projection 7.87 we have

\[
h = m \circ k \Rightarrow \pi(h) = \pi(m \circ k) = \pi(m) \circ \pi(k) \Rightarrow h = m \circ \pi(k)
\]

But since \( k \) is unique, we have \( k = \pi(k) \), which is indeed a morphism in \( Z_\sigma(C) \). Therefore, \( m \) is a kernel of \( f \). The argument works for the cokernel, and is thus omitted. A corollary of this construction is that the (co)kernels are really the same as in \( C \), so \( Z_\sigma(C) \) is clearly exact since \( C \) is exact.

\[\blacksquare\]

**Lemma 7.89** Let \( C \) be a premodular category and \( \sigma \in \text{Adm}_{2n} \) an admissible gluing. Recall from 7.88 that the categorical center of higher genera \( Z_\sigma(C) \) is abelian. Moreover, it is semisimple.

\[\Diamond\]

*Proof.* Let \((X, \gamma) \xrightarrow{f} (Y, \beta)\) be a monic morphism in \( Z_\sigma(C) \). It suffices to show that \( f \) has a left inverse. Recall that \( f \) is also a \( C \)-morphism \( X \xrightarrow{f} Y \). We contend that \( X \xrightarrow{f} Y \) is monic in \( C \). Indeed, assume

\[
(W, \alpha) \xrightarrow{g} (X, \gamma) = (W, \alpha) \xrightarrow{h} (X, \gamma)
\]

then

\[
(I_\sigma W \xrightarrow{g} (X, \gamma) \xrightarrow{f} (Y, \beta)) = (I_\sigma W \xrightarrow{h} (X, \gamma) \xrightarrow{f} (Y, \beta))
\]

by the construction of \( I_\sigma \). Then \( g = h \), and thus \( g = h \).

Since \( C \) is semisimple, we get a left inverse \( X \xleftarrow{p} Y \) for free. However, \( p \) lives in \( C \), so we need to find another candidate that does the job in \( Z_\sigma(C) \). This is again taken care by the projection 7.87 We contend that it is a left inverse of \( f \) in \( Z_\sigma(C) \). Indeed, as \( \pi_{\beta, \gamma} \) is a projection, \( \pi_{\beta, \gamma}(f) = f \). So,

\[
\pi_{\beta, \gamma}(p) \circ f = \pi_{\beta, \gamma}(p) \circ \pi_{\beta, \gamma}(f) = \pi_{\gamma, \gamma}(p \circ f) = \pi_{\gamma, \gamma}(1_X) = 1_X.
\]

\[\blacksquare\]
Lemma 7.90 Let $C$ be a premodular category and $\sigma \in \text{Adm}_{2n}$ an admissible gluing. Recall from 7.88 and 7.89 that the categorical center of higher genera $Z_\sigma(C)$ is semisimple abelian. Moreover, it is finite. 

Proof. To prove that $Z_\sigma(C)$ is finite, we turn to the finiteness of $C$. Since $Z_\sigma(C)$ is a $\mathbb{k}$-linear abelian category by construction, from 7.38 and 7.39 we only have to show four things.

- Every object has finite length.
- Every hom space is a finite dimensional $\mathbb{k}$-vector space.
- $Z_\sigma(C)$ has enough projectives, i.e. every simple object of $Z_\sigma(C)$ has a projective cover.
- The set of isomorphism classes of simple objects is finite.

To prove that every object has finite length, pass a simple filtration of an object in $Z_\sigma(C)$ to one in $C$ by the forgetful functor $F_\sigma$. Extend the latter to a simple filtration in $C$, which has finite length as $C$ is assumed finite. Thus the former is also of finite length. To prove that every hom space is of finite dimensional, recall that the morphism spaces of $Z_\sigma(C)$ are defined as subspaces of those of $C$. Therefore the dimension of the former is bounded by the dimension of the later, which is finite the finiteness assumption of $C$.

To prove that $Z_\sigma(C)$ has enough projectives, it suffices to show that $Z_\sigma(C)$ is semisimple, as then each epic morphism admits a left inverse. But this fact has been shown in 7.89. To prove that there are only finitely many simple objects (up to isomorphism), we utilize the ambidextrous adjunction of $F_\sigma$ and $I_\sigma$. Let $(X, \gamma)$ be a simple object of $Z_\sigma(C)$. From

$$\text{Hom}_C(X,Y) \simeq \text{Hom}_{Z_\sigma(C)}((X,\gamma), I_\sigma(Y))$$

we know that $(X,\gamma)$ appears as a summand in $I(Y)$ for any $Y$ that appears as a summand in $X$. Since $C$ has finitely many simple objects (up to isomorphism), it follows that there are finitely many such $(X,\gamma)$.

Lemma 7.91 The set of half-braidings defined in 4.15 satisfies the pairwise commutative relations 4.2. 

Proof. By definition, we have to prove that for each $i, j$, $\gamma_i$ and $\gamma_j$ satisfies the commutative relation posed in 4.2. In this pictorial proof, we use the color light-blue to indicate $\gamma_i$ and the color red to indicate $\gamma_j$. Recall that with out loss of generality, there are three cases to consider

1. $[i]'<[i]''<[j]'<[j]''$
2. \( [i'] < [j] < [i]'' < [j]'' \)

3. \( [i'] < [j] < [j]'' < [i]'' \)
Lemma 7.92 The induced morphisms in \((4.16)\) is compatible with the sets of half-braidings \(\gamma\) and \(\beta\) given in \((4.15)\).

Proof. Clearly it holds from the following figure.
Lemma 7.93 Let $A$ be an additive category and $B$ be an abelian category. Suppose

$$A \xrightarrow{\phi} B$$

is an additive functor. Then $\phi$ lifts additively to the Karoubi completion $\text{Kar}(A)$ of $A$:

$$\text{Kar}(A) \xrightarrow{\Phi} B.$$ 

Proof. Given the assumptions, we must construct $\Phi$ explicitly. Recall that a typical object of $\text{Kar}(A)$ is $\vec{X} := (X, p)$ of $X \in \text{Obj}(A)$ and an idempotent $p \in \text{End}_A(X)$. Define $\Phi(\vec{X})$ to be $\text{im}_B(\phi(p))$. Recall also that a typical morphism

$$(X, p) \xrightarrow{f} (Y, q)$$

is an $A$-morphism $X \xrightarrow{f} Y$ such that $f = qfp$. Hence $\Phi(f)$ induces a $B$-morphism

$$\text{im}(\phi(p)) \xrightarrow{\phi(f)} \text{im}(\phi(q)).$$

Define it to be $\Phi(f)$. So defined map $\Phi$ is clearly an additive functor that extends $\phi$. 

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