Invariants of the Riemann tensor for Class B
Warped Product Spacetimes

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Abstract

We use the computer algebra system GRTensorII to examine invariants polynomial in the Riemann tensor for class B warped product spacetimes - those which can be decomposed into the coupled product of two 2-dimensional spaces, one Lorentzian and one Riemannian, subject to the separability of the coupling:

\[ ds^2 = ds^2_{\Sigma_1}(u,v) + C(x^\gamma)^2 ds^2_{\Sigma_2}(\theta,\phi) \] (1)

with \( C(x^\gamma)^2 = r(u,v)^2 w(\theta,\phi)^2 \) and \( \text{sig}(\Sigma_1) = 0, \text{sig}(\Sigma_2) = 2\epsilon \) (\( \epsilon = \pm 1 \)) for class \( B_1 \) spacetimes and \( \text{sig}(\Sigma_1) = 2\epsilon, \text{sig}(\Sigma_2) = 0 \) for class \( B_2 \). Although very special, these spaces include many of interest, for example, all spherical, plane, and hyperbolic spacetimes. The first two Ricci invariants along with the Ricci scalar and the real component of the second Weyl invariant \( J \) alone are shown to constitute the largest independent set of invariants to degree five for this class. Explicit syzygies are given for other invariants up to this degree. It is argued that this set constitutes the largest functionally independent set to any degree for this class, and some physical consequences of the syzygies are explored.

1 Introduction

It is natural to suspect that invariants constructed from the Riemann tensor via contractions should play some important role in general relativity. However,

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1We do not consider covariant derivatives of the Riemann tensor. That is, in the terminology of [1], we consider invariants of order \( k = 2 \).
it is only fair to say that this importance has yet to be fully realized. For example, it has been known for a very long time that this type of invariant fails to distinguish inequivalent spacetimes \[2\]. Indeed, the inclusion of derivatives of the Riemann tensor does not overcome this shortcoming \[3\]. The inequivalence problem requires a more sophisticated approach (e.g. \[4\]). Invariants are of some use in the study of spacetime singularities (distinguishing the scalar polynomial (sp) type (e.g. \[5\]) which is equivalent to a curvature pp type \[6\]), but even here they fail to distinguish shell crossings from more significant singularities.

Despite these shortcomings, the problem of finding a complete set of invariants for spacetime has attracted considerable recent effort \[7, 8, 9, 10\]. Here we set out to study spacetimes which are the product of two 2-dimensional spaces, one Lorentzian and one Riemannian, subject to a separability condition on the function which couples the 2-spaces. That is, we consider metrics of the form

\[
ds^2 = ds^2_{\Sigma_1}(u, v) + C(x) ds^2_{\Sigma_2}(\theta, \phi) \tag{2}
\]

subject to the restriction

\[
C(x)^2 = r(u, v)^2 w(\theta, \phi)^2. \tag{3}
\]

The spaces are known as warped products of class B \[11\], with \(\text{sig}(\Sigma_1) = 0\), \(\text{sig}(\Sigma_2) = 2\epsilon\) (\( \epsilon = \pm 1 \)) for class \(B_1\) and \(\text{sig}(\Sigma_1) = 2\epsilon\), \(\text{sig}(\Sigma_2) = 0\) for class \(B_2\).

A metric of sufficient generality for class \(B_1\) is given by (e.g. \[12\])

\[
ds^2 = -2f(u, v)du dv + r(u, v)^2 g(\theta, \phi)^2 (d\theta^2 + d\phi^2) \tag{4}
\]

and for class \(B_2\) by

\[
ds^2 = f(u, v)^2 (du^2 + dv^2) - 2r(u, v)^2 g(\theta, \phi) d\theta d\phi. \tag{5}
\]

There are many examples of class \(B_1\) spacetimes (for example, all spherical, plane, and hyperbolic spacetimes). Examples of spacetimes of class \(B_2\) are considerably harder to find. It is known that the only physically interesting energy-momentum types of class \(B_2\) spacetimes are non-null electromagnetic, \(\Lambda\)-term, or vacuum \[13\].

## 2 The number of invariants required

For a general Petrov type D metric, we can align the tetrad along the principle null directions of the Weyl tensor to find a frame in which there are in general 12 non-zero components of the curvature (complex \(\Psi_2\), the Ricci spinor components, and the Ricci scalar). Since this alignment leaves a two parameter group of rotational freedom in the frame (‘spin’ and ‘boost’), the number of independent invariants for a type D spacetime is reduced by this dimension to 10.

\[\text{We assume that all functions are twice continuously differentiable. We do not consider surface layers. Note that we have chosen to write } \Sigma_2 \text{ in conformally flat form for class } B_1.\]
Specializing to the given class of spacetimes, we find that if the above frame alignment is carried out, then a number of the Ricci spinor components are also reduced to zero. For class $B_1$ we are left with the real components $\Phi_{00}$, $\Phi_{11}$, and $\Phi_{22}$, while for class $B_2$, only $\Phi_{11}$ and $\Phi_{02} = \Phi_{20}$ remain non-zero. The number of independent degrees of freedom arising from the Ricci components is thus reduced to 3. However, one of these degrees can be removed by making use of either the spin (for $B_1$) or boost (for $B_2$) freedoms still remaining in the frame.

The number of independent invariants that we should expect to find for the given class of spacetime is thus

$$2 + 2 + 1 - 1 = 4,$$

corresponding respectively to the Weyl and Ricci spinor freedoms, the Ricci scalar, and the dimension of the invariance group of these spinors.

An example of how the details of this calculation can be carried out using GRTensorII is given in Appendix A.

3 The invariants

The invariants we begin with here are the sixteen suggested by Carminati and McLenaghan augmented with an invariant equivalent to $M$ suggested recently by Zakhary and McIntosh. We adopt the definitions in [7] for the set

$$CM = \{R, r_1, r_2, r_3, w_1, w_2, m_1, m_2, m_3, m_4, m_5\} \quad (7)$$

and augment these with

$$ZM = m_6 = \Psi_{ABCD} \Phi^{AE\hat{A}\hat{B}}_E \phi^{B}_{CD} \Phi^{C}_{FAB} \phi^{DF\hat{C}\hat{D}}.$$ \quad (8)

In total this set constitutes six real and six complex invariants. The completeness of this set has been examined recently. It is certainly complete to degree five (degree being the total number of Weyl, conjugate Weyl, and Ricci spinors).

3 As an aside we note here that if we consider, in the usual way (e.g. [14]), the secular equation $|S_{ab} - \lambda g_{ab}| = 0$, where $S_{ab}$ is the trace-free Ricci tensor, and $g_{ab}$ is the metric tensor, the characteristic polynomial which follows is $\alpha_1 \lambda^4 + \alpha_2 \lambda^3 + \alpha_3 \lambda^2 + \alpha_4 \lambda + \alpha_5 = 0$. Since the $\alpha_i$s are derived from the components of $S_{ab}$ and $g_{ab}$ by algebraic operations, they can be called ‘algebraic’ invariants of the tensor $S_{ab}$. It can be shown that for this class of spacetimes the following relationships hold between the invariants $r_1, r_2, r_3$ and $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ ($\alpha_2 = 0$ is a consequence of $S_{ab}$ being trace-free):

$$r_1 = -\frac{\alpha_1}{2\alpha_1}, \quad r_2 = \frac{3\alpha_2}{8\alpha_1}, \quad r_3 = \frac{\alpha_2^2}{8\alpha_2^2} - \frac{\alpha_3}{4\alpha_1}. \quad (6)$$

4 Two scalars found frequently in the physical literature are the Kretschmann scalar ($\text{RiemSq} \equiv R_{abcd}R^{abcd}$, $R_{abcd}$ the Riemann tensor) and the square of the Ricci tensor ($\text{RicciSq} \equiv R_{ab}R^{ab}$, $R_{ab}$ the Ricci tensor). In terms of the set (8) these are given simply by $\text{RiemSq} = R^2/6 + 8(r_1 + Re(w_1))$ and $\text{RicciSq} = (R^2 + r_1)/4$. 

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whose contractions make up an invariant\(^6\). In what follows we explicitly reduce this set of eighteen invariants to four for the spacetimes under consideration by way of syzygies\(^7\). All syzygies have been developed and tested with the aid of the computer algebra package GRTensorII\(^8\).

4 Syzygies

Syzygies follow from a knowledge of which spinor components are non-zero. In the present case the important features are that these components are real and that the Weyl and Plebanski tensors (e.g.\(^9\)) share the same principal null directions and are of type \(D\) (or \(O\)). Since spacetimes of the form (2) with (3) are of Petrov type \(D\) (or \(O\)), we have the well-known elementary syzygy

\[
6w_2^2 - w_1^3 = 0. \tag{10}
\]

Furthermore, for any Petrov type \(D\) (or \(O\)) spacetime we have\(^7\)

\[
(3m_2 - w_1 r_1)w_1 - 3m_1 w_2 = 0 \tag{11}
\]

and

\[
(3m_5 - w_1 \bar{m}_1)w_1 - 3m_3 w_2 = 0, \tag{12}
\]

\(^6\)It is a simple (though tedious, unless carried out electronically) matter to write down expressions for all index contractions among any set of \(n\) curvature spinors. If this is done to degree five, it is found that for general spacetimes every invariant which is not included in the CM+ZM set satisfies an identity allowing it to be written as a rational function of members of the set. Thus, there are no invariants of degree less than or equal to five to be added to the CM+ZM set.\(^\dagger\)

\(^7\)GRTensorII is a package which runs within MapleV. It is entirely distinct from packages distributed with MapleV and must be obtained independently. The GRTensorII software and documentation is distributed freely on the World-Wide-Web from the address http://www.astro.queensu.ca/~grtensor/GRHome.html or http://www.maths.soton.ac.uk/~dp/grtensor/. Worksheets which reproduce the syzygies reported here can be downloaded from these sites.

\(^8\)All syzygies can in fact be obtained algorithmically. By appropriately defining the “degree” of a syzygy, the most general syzygy can be constructed for any particular “degree”. Expanding and collecting like terms allows the coefficients to form a linear system of equations which are then solved to find the coefficients of the syzygy. If there are no nonzero solutions after the linear system is solved, then the syzygy must be of a different “degree”. The definition of “degree” in this case imposes an ordering, allowing a sequential search to be executed. In any particular case (e.g. as considered in this paper) the number of degrees of freedom can be counted and so the search for syzygies can be made exhaustive. Whereas degree is commonly used as a single parameter, as given in Section 3 above, a multi-parameter definition of degree is more convenient here. For example, an invariant can be given a “degree” \([A, B, C]\), where \(A\) refers to the exponent on the Weyl spinor, \(B\) refers to the number of multiplications in the Ricci spinor, and \(C\) refers to the exponent on the conjugate Weyl spinor. In this notation, for the types of spacetimes we consider here, only two parameters are necessary, \([A, B]\). Given a rule of composition: for invariants \(I_1 = [A_1, B_1]\) and \(I_2 = [A_2, B_2]\), the degree of \(I_1 * I_2\) is

\[
[A_1, B_1] + [A_2, B_2] = [A_1 + A_2, B_1 + B_2], \tag{9}
\]

the parameter space given by \([A, B]\) can then be searched exhaustively for the existence of syzygies.
where $\overline{m}_1$ is the complex conjugate of $m_1$. Equivalent syzygies have been given by Zakhary and McIntosh for their set of invariants \cite{9}. We observe the following additional syzygies \cite{18}

$$6m_4 + w_1 r_2 = 0$$

(13)

and

$$m_3 - m_2 = 0.$$  

(14)

Further, we observe that \cite{13, 14}

$$(-12r_3 + 7r_1^2)w_1 m_4 - (12r_2^2 - 36r_1 r_3 + 17r_3^3)w_2 = 0,$$

(15)

$$2(3m_6 - m_1 r_1)w_2 + m_4^2 w_1 = 0,$$

(16)

and that

$$(-12r_3 + 7r_1^2)^3 - (12r_2^2 - 36r_1 r_3 + 17r_3^3)^2 = 0.$$  

(17)

The syzygy (17) is the counterpart for the Plebanski tensor to the syzygy (10) for the Weyl tensor. We expect that the syzygies (10) through (17) will hold in a wider class of spacetimes than type $B$ warped products.

## 5 The independent set of invariants

The syzygies (11) through (17) remove the independence of $m_2$, $m_5$, $m_4$, $m_3$, $m_1$, $m_6$, and $r_3$ respectively. According to the syzygy (10) the sign of $w_2$ (which changes with signature) cannot be obtained from $w_1$ whereas $w_1 > 0$. This suggests that for metrics of the form (2) with (3) we take the set

$$\{R, r_1, r_2, w_2\}$$

(18)

as the independent set of scalar polynomial invariants, satisfying the number of degrees of freedom in the curvature.

We note, however, that these do not necessarily constitute a complete set in the sense of classical invariant theory, where the set is considered complete only if every invariant not contained in the set can be written as an integral rational function of the other members of the set \cite{20}. This is clearly not the case here, for $w_1 = \Psi_2^2$ cannot be written as an integral rational function of $w_2 = \Psi_2^3$, nor vice versa, whereas $r_3$ is determined as the root of a cubic from (17).

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\footnote{5}{For type $D$ $w_1$ and $w_2$ are non-zero, whereas for type $O$ (or $N$ or $III$) both $w_1$ and $w_2$ vanish. In the set $CM$, for a general Ricci tensor, there remain 10 invariants for type $D$, 9 for type $III$ (since $m_1 R = m_2 C = m_3 R = 0$ for the real ($R$) and imaginary ($C$) parts), 7 for type $N$ (since $m_1 C = m_2 = m_3 = 0$), and of course 4 for type $O$.}

\footnote{6}{Regarding syzygy (10) we note that $m_1 = 0 \Rightarrow 7r_1^2 = 12r_3$ and that $7r_1^2 = 12r_3 \Rightarrow m_1 = 0$ or $m_1 w_1 = 3w_2 r_1$.}

\footnote{7}{We thank a referee for pointing this out.}
From a practical standpoint, however, we are more concerned with isolating the functional degrees of freedom in the curvature, since by considering a more general interdependence (i.e. by allowing non-integer exponents in our polynomials) we can in principle solve for any other of the remaining invariants in terms of the given set. By inspection of the invariants listed in (13), it is clear that they are functionally independent and that they make up the required number of degrees of freedom (i.e. 4) for the class of spacetimes in question, and in this sense we can consider them to be a ‘complete’ list.

The invariants are reproduced in Appendix B.

6 Some consequences of the syzygies

The syzygies are certainly of value for computational efficiency. However, understanding the physical benefit of the syzygies is somewhat more difficult. Mixed invariants have been used to provide a number of alignment theorems for the perfect fluid and Maxwell cases [7] and to classify the Riemann spinor [21]. Our purpose here is to provide some physical consequences of the syzygies presented by way of expressing them in terms of the electric and magnetic components of the Weyl tensor, and to show how the syzygy (17) can be used to impose restrictions on physical quantities in the energy-momentum tensor.

It is convenient to revert to classical notation. First we consider the Weyl tensor \( C_{abcd} \). The “electric” and “magnetic” parts are defined, as usual, by

\[
E_{ac} = C_{abcd} u^b u^d,
\]

and

\[
H_{ac} = C^*_{abcd} u^b u^d,
\]

where \( u^a \) is a unit timelike vector and \( C^*_{abcd} = \frac{1}{2} \eta_{abef} C_{efcd} \) the dual tensor. Writing

\[
Q_{ab} = E_{ab} + iH_{ab}
\]

and

\[
\overline{C}_{abcd} = C_{abcd} + iC^*_{abcd},
\]

directly from the expression

\[
4u_{[a}Q_{b][c}u_{d]} + g_{a[c}Q_{d]b} - g_{b[a}Q_{d]c} + i\eta_{abef} u^e u_{[c}Q_{d]f} + i\eta_{cdef} u^e u_{[a}Q_{b]f}
\]

for \(-\frac{1}{2}C_{abcd}\) (e.g., [1]) it follows that

\[
\overline{C}_{abcd}\overline{C}^{abcd} = Q_{ab}Q_{ab}
\]

and that

\[
C_{ab} e^c_d \overline{C}_{ef} a^b \overline{C}^{cd} e^f = Q^e_a Q^b_c Q^c_a.
\]
Equivalently, it follows that \( w_1 (= 2I) \) is given by
\[
\frac{1}{16} (E_{ab} E^{ab} - H_{ab} H^{ab}) + \frac{i}{8} (H_{ab} E^{ab}) \tag{26}
\]
and that \( w_2 (= 6J) \) is given by
\[
\frac{1}{32} (3E^a_b H^b_c H^c_a - E^a_b E^b_c E^c_a) + \frac{i}{32} (H^a_b H^b_c H^c_a - 3E^a_b E^b_c H^c_a). \tag{27}
\]

Relations (26) and (27) hold in general and are independent of the timelike vector chosen to split the Weyl tensor into its electric and magnetic parts. These expressions can, for example, be used to clarify the relationship between the criterion of Bel and those of Misra and Singh. The Bel condition \([22]\) is that \( w_1 = w_2 = 0 \) if and only if the spacetime is of Petrov type N or III (or, of course O). The condition of Misra and Singh \([23]\) can be stated as \( E_{ab} E^{ab} = H_{ab} H^{ab} \) and \( E^a_b E^b_c E^c_a = H^a_b H^b_c H^c_a \). Expressions (26) and (27) show that these conditions are inequivalent, contrary to the claim by Zakharov \([24]\).

Independence from the timelike vector chosen to split the Weyl tensor does not, of course, extend into the mixed invariants. For example, it follows that the real and imaginary parts of \( 8m_1 \) are given by
\[
C_{abcd} S^{ac} S^{bd} = 2E^a_b S^c_e S^d_c - 2E_{ab} S^{ab} S_{cd} u^c u^d + S^a_c S^b_d E_{ab}, \tag{28}
\]
and
\[
C^*_{abcd} S^{ac} S^{bd} = 2H^a_b S^c_e S^d_c - 2H_{ab} S^{ab} S_{cd} u^c u^d + S^a_c S^b_d H_{ab}, \tag{29}
\]
respectively, where \( S_{ab} \) is the trace-free Ricci tensor. Expressions for the next degree of mixed invariants \( (m_2 \text{ in the notation of } [3]) \) have been evaluated but the results are rather unwieldy (they contain 16 independent terms). The important point is that with the appearance of the vector \( S^a_c u^c \) and scalar \( S_{cd} u^c u^d \) it becomes necessary to introduce an explicit form of \( S_{ab} \) in order to proceed. This avenue is explored briefly below.

Relations equivalent to (24) (or (26)) have been reexamined recently by Bonnor \([22]\). Relations equivalent to (25) (or (27)) have been given by McIntosh et al. \([26]\) for Petrov types I (or D).

Specializing now to real \( \Psi_2 \) we observe that
\[
H_{ab} E^{ab} = 0, \tag{30}
\]
and that
\[
H^a_b H^b_c H^c_a = 3E^a_b E^b_c H^c_a. \tag{31}
\]
Moreover, from (9) we obtain the further relation
\[
24(3E^a_b H^b_c H^c_a - E^a_b E^b_c E^c_a)^2 = (E_{ab} E^{ab} - H_{ab} H^{ab})^3. \tag{32}
\]
It is perhaps worth emphasizing the fact that relations (30), (31), and (32) hold for all Petrov type D spacetimes with real $\Psi_2$. This includes a much wider class than class B warped product spaces.

We now wish to explore any restrictions that the syzygy (17) might impose on physical quantities in the energy-momentum tensor. First, however, recall that at most three Ricci invariants are independent. Finding syzygies associated with Ricci invariants of higher degree is a purely algorithmic procedure. Ricci syzygies up to $r_{10}$ are given in Appendix C.

To begin, consider a perfect fluid,

$$T_{ab} = (\rho + p)u_a u_b + pg_{ab}$$

(33)

where $\rho$ is the energy density, $p$ is the (isotropic) pressure, and $u^a$ is the normalized timelike flow vector. The Ricci invariants in this case all reduce to the form

$$r_n = m_n(\rho + p)^{n+1},$$

(34)

where $m_n$ is a constant. The syzygy (17) reduces to the identity $0 = 0$. Increasing the complexity of the energy-momentum tensor slightly does not offer anything new. For both the case of anisotropic pressure

$$T_{ab} = (\rho + p_2)u_a u_b + (p_1 - p_2)n_a n_b + p_2 g_{ab},$$

(35)

where $n^a$ is a normalized spacelike vector orthogonal to $u^a$ ($u_a n^a = 0$), and for the case of a perfect fluid with a heat flux

$$T_{ab} = (\rho + p)u_a u_b + p g_{ab} + 2 u(a q_b),$$

(36)

where $q^a$ is the heat flux vector such that $u_a q^a = 0$, the syzygy (17) again returns the identity $0 = 0$. No information on possible relationships between the physical quantities that describe the spacetime can be gained from the syzygy in these cases. To obtain any restriction, a more complex physical decomposition must be considered. The simplest energy-momentum tensor for which the syzygy provides any restriction is one with anisotropic pressure and a heat flux:

$$T_{ab} = (\rho + p_2)u_a u_b + (p_1 - p_2)n_a n_b + p_2 g_{ab} + 2 u(a q_b).$$

(37)

In this case the syzygy (17) reduces to

$$(q_a q^a - (n_a q^a)^2)^2(p_1 - p_2)^2 P = 0,$$

(38)

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11 These are constructed analogously to $r_1$, $r_2$, and $r_3$. For example,

$$r_4 \equiv - S_b^a S_c^b S_d^c S_e^d S_f^e \equiv \Phi_{ABCD} \Phi_{EFGH} \Phi_{IJ} \Phi_{KL} \Phi_{MN} \Phi_{OP} \Phi_{QRST},$$

12 All possible combinations of $r_1$, $r_2$ and $r_3$ and $r_n$ (for each $n \geq 4$) are combined at degree $n + 1$. Expressing this combination as a polynomial in the Ricci components, the coefficients of any one of the Ricci components will form a linear system of equations that can be solved to find the numerical coefficients of the syzygy.
where $P$ is a polynomial of degree 4 in

$$\{\rho, p_1, p_2, n_a q^a\}$$

and degree 3 in $q_a q^a$. If $p_1 \neq p_2$ and $q_a q^a \neq (n_a q^a)^2$ then the syzygy \[17\] reduces to the equation $P = 0$ and this reduces the number of independent physical scalars to four, the number of degrees of freedom. $P$ is given by

$$-p^2 \rho^4 - p_1^2 \rho^4 + 2p_1p_2 \rho^4 - 2p_2 \rho^3 - 4p_2(n_a q^a)^2 \rho^3 + 4(n_a q^a)^2 \rho_1 \rho^3 + 2q_a q^a p_2 \rho^3 - 2q_a q^a p_1 \rho^3 - 2p_1^2 \rho^3 + 2p_1 p_2 \rho^3 + 2p_1 p_2 \rho^3 - (q_a q^a)^2 \rho^2 - 2p_1 p^2 \rho^2 + 6(n_a q^a)^2 p_1^2 \rho^2 - p_1^4 \rho^2 + 8q_a q^a p_2^2 \rho^2 + 2q_a q^a p_2^2 \rho^2 - 2p_1 p_2 \rho^2 - 6p_2^2(n_a q^a)^2 \rho^2 - p_2^4 \rho^2 - 2p_1^2 \rho^2 + 10 q_a q^a p_1 p \rho^2 + 6p_1^2 p_2^2 \rho^2 + 8(q_a q^a)^2 p \rho - 30 p_2^2(n_a q^a)^2 \rho + 2q_a q^a p_2^3 \rho + 6p_2^2(n_a q^a)^2 \rho + 8q_a q^a p_1^3 \rho + 2p_1^3 p_2^2 \rho + 30 p_1^2(n_a q^a)^2 p_2 \rho + 2p_2^2 p_1^2 \rho + 18 q_a q^a p_2(n_a q^a)^2 \rho - 6p_1^3(n_a q^a)^2 \rho - 2p_1 p_2^4 \rho + 10q_a q^a p_2^2 p_1 \rho - 10(n_a q^a)^2 p_2 \rho - 2p_2 p_1^4 \rho - 18q_a q^a(n_a q^a)^2 p_1 \rho - 20 p_1^2 q_a q^a p_2 \rho + 2p_1^2 p_2^2 q_a q^a - 18 q_a q^a p_2^2(n_a q^a)^2 + 10 p_1^3(n_a q^a)^2 p_2 - 10p_2^3(n_a q^a)^2 p_1 - p_2^4 p_1^2 - (q_a q^a)^2 p_2^2 - 4(n_a q^a)^2 p_1^4 - 8p_1^3 q_a q^a p_2 + 4q_a q^a p_1^4 - 8p_1^3(q_a q^a)^2 p_2 + 4(q_a q^a)^3 + 2p_2^3 p_1 q_a q^a + 2p_2^3 p_1^3 + 54(n_a q^a)^2 p_1 q_a q^a p_2 + 4p_2^4(n_a q^a)^2 - 54(n_a q^a)^4 p_1 p_2 + 27p_2^2(n_a q^a)^4 + 27(n_a q^a)^4 p_1^2 - 36q_a q^a(n_a q^a)^2 p_1^2 - p_1^4 p_2^2 + 8(q_a q^a)^2 p_1^2.$
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References

[1] Kramer D, Stephani H, Herlt E, MacCallum M and Schmutzer E 1980 Exact Solutions of Einstein’s Equations (Cambridge: CUP)

[2] Narlikar V V and Karmarkar K R 1948 Proc. Indian Acad. Sci. A29, 91

[3] Koutras A and McIntosh C 1996 Class. Quantum Grav. 13 L47

[4] MacCallum M A H and Skea J E F 1994 SHEEP: A computer algebra system for general relativity Algebraic Computing in General Relativity ed M.J. Rebouças and W.L. Roque (Oxford: Clarendon)

[5] Tipler F J Clarke C J S and Ellis G F R 1980 Singularities and Horizons-A Review Article General Relativity and Gravitation Volume 2 ed A Held (New York: Plenum)

[6] Siklos S T C 1979 Gen. Rel. Grav. 10 1003

[7] Carminati J and McLenaghan R G 1991 J. Math. Phys. 32 3135

[8] Sneddon G E 1996 J. Math. Phys. 37 1059

[9] Zakhary E and McIntosh C B G 1997 Gen. Rel. Grav. 29 539

[10] Pollney D 1996 (Report, unpublished)

[11] Carot J, da Costa J 1993 Class. Quantum Grav. 10 461

[12] Nakahara M 1990 Geometry, Topology and Physics (Bristol: IOP)

[13] Haddow B M and Carot J 1996 Class. Quant. Grav. 13 289.

[14] Stephani H 1990 General Relativity (Cambridge: CUP)

[15] Penrose R and Rindler W 1986 Spinors and space-time Volume 2 (Cambridge: CUP)

[16] Musgrave P Pollney D and Lake K 1994-1998 GRTensorII (Kingston, Ontario: Queen’s University)

[17] Pollney D 1995 (Report, unpublished)

\[\text{It is of historical interest to note that Narlikar and Karmarkar suggested (in our notation) that the set } \{R, r_1, r_2, w_1\} \text{ constitutes the independent set of invariants for spherically symmetric spacetimes.}\]
[18] Musgrave P 1996 (Report, unpublished)
[19] Santosuosso K 1997 (Report, unpublished)
[20] Gurevich, G B 1964 *Foundations of the Theory of Algebraic Invariants* (Groningen: Noordhoff)
[21] Haddow B M 1996 *Gen. Rel. Grav* 28 481
[22] Bel L 1962 in *Les Théories Relativistes de la Gravitation* (Paris: CNRS)
[23] Misra R M and Singh R A 1967 *J. Math. Phys.* 8 1065
[24] Zakharov V D 1973 *Gravitational Waves in Einstein’s Theory* (New York: Halsted Press)
[25] Bonnor W B 1995 *Class. Quantum Grav.* 12 499 (and corrigendum)
[26] McIntosh C B G, Arianrhod R , Wade S T and Hoenselaers C 1994 *Class. Quantum Grav.* 11 1555
[27] Maartens R and Maharaj M S 1990 *J. Math. Phys.* 31 151
[28] Ellis G F R 1971 Relativistic Cosmology in *International School of Physics Course XLVII* ed B K Sachs (New York: Academic Press)
Appendix A: Curvature components of the Class $B_1$ warped product spacetimes

The following session demonstrates how GRTensorII can be used to fix the frame components of the curvature relative to a ‘standard’ form by applying $SL(2, \mathbb{C})$ rotations of the frame vectors. The standard frame is defined so as to make use of this rotational freedom in order to reduce the number of independent functions among the curvature components. Once the frame is fixed as far as possible, the subgroup of $SL(2, \mathbb{C})$ under which the resulting curvature components are invariant is reported. The curvature components in the standard frame are called ‘Karlhede invariants’ for their usefulness in determining the equivalence of spacetimes (e.g. [4]).

Within MapleV, we restart a fresh session and initialize the GRTensorII package.

```
> restart:
> grtw():
> qload (B1);
```

Calculated $ds$ for $B1$

**Default spacetime = B1**

**Coordinates**

$x^a = [u, v, \theta, \phi]$

**Line element**

$$ds^2 = -2f(u, v) \, du \, dv + r(u, v)^2 g(\theta, \phi)^2 \, d\theta^2 + r(u, v)^2 g(\theta, \phi)^2 \, d\phi^2$$

We have loaded a line element corresponding to the class $B_1$ warped product spacetimes. However, we will eventually wish to examine the spinor components of the curvature. The function `nptetrad()` can be used to automatically create a null (specifically, Newman-Penrose) tetrad from the metric. This tetrad will then be used to calculate the curvature spinors in a spin frame corresponding to the null frame.

```
> nptetrad();
```

The metric signature of the B1 spacetime is +2.
In order to create an NP-tetrad, the signature of $g(dn, dn)$ will be changed to -2.
Continue? (1=yes [default], other=no) :

> 1;

The signature of the B1 spacetime is now -2.
For the B1 spacetime:

Basis inner product

\[ g^{(a)(b)} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \]

Null tetrad (covariant components)

\[ l_a = [0, f(u, v), 0, 0] \]
\[ n_a = [1, 0, 0, 0] \]
\[ m_a = [0, 0, -\frac{1}{2} I \sqrt{2} g(\theta, \phi) r(u, v), -\frac{1}{2} \sqrt{2} g(\theta, \phi) r(u, v)] \]
\[ mbar_a = [0, 0, \frac{1}{2} I \sqrt{2} g(\theta, \phi) r(u, v), -\frac{1}{2} \sqrt{2} g(\theta, \phi) r(u, v)] \]

The null tetrad has been stored as e(bdn,dn).

The curvature spinors are defined as the NP scalars \{WeylSc, RicciSc, and Lambda\} in the standard GRTensorII package. However, they are also defined specifically as spinors \{WeylSp, RicciSp, and Lambda\} in the spinor.m package, which we now load. By calculating the latter objects, we can make use of the spinor packages' facilities for performing frame rotations and fixing a standard form for symmetric spinors.

> grlib(spinor);

Spinors and spacetime classification routines
Version 0.9 4 Dec 1997

> grcalc(WeylSp,RicciSp,Lambda);

Basis/tetrad related object definitions
Last modified 5 February 1997

Calculated WeylSp for B1
Calculated RicciSp for B1
Calculated Lambda for B1
The next command expands the tensor components (the first argument, ‘\_', tells the command to refer to the previously calculated spinors) and applies a function which aliases the output to make it more readable.

\[
\text{gralter(\_., expand, autoAlias)};
\]

Component simplification of a GRTensorII object:

Applying routine expand to object WeylSp
Applying routine expand to object RicciSp
Applying routine expand to object Lambda
Applying routine autoAlias to object WeylSp
Applying routine autoAlias to object RicciSp
Applying routine autoAlias to object Lambda

\[
\text{grdisplay(\_);}\]

For the \(B_1\) spacetime:

\[
\Psi_{20} = \frac{1}{6} \frac{g_\phi^2}{r^2 g^4} - \frac{1}{6} \frac{g_\theta^2}{r^2 g^4} + \frac{1}{3} \frac{r_{u,v}}{r f} - \frac{1}{3} \frac{r_{u} r_{v}}{r^2 f} - \frac{1}{6} \frac{f_{u,v}}{f^2} + \frac{1}{6} \frac{f_{u} f_{v}}{f^3} + \frac{1}{6} \frac{g_{\theta, u}}{r^2 g^3} + \frac{1}{6} \frac{g_{\phi, v}}{r^2 g^3}
\]

\[
\Phi_{00} = \frac{r_{u} f_{u}}{r f} - \frac{r_{u, u}}{r}
\]

\[
\Phi_{11} = \frac{1}{2} \frac{r_{u} r_{v}}{r^2 f} + \frac{1}{4} \frac{g_\phi^2}{r^2 g^4} + \frac{1}{4} \frac{g_\theta^2}{r^2 g^4} - \frac{1}{4} \frac{f_{u,v}}{f^2} + \frac{1}{4} \frac{f_{u} f_{v}}{f^3} - \frac{1}{4} \frac{g_{\theta, v}}{r^2 g^3} - \frac{1}{4} \frac{g_{\phi, u}}{r^2 g^3}
\]

\[
\Phi_{22} = -\frac{r_{u,v}}{r f^2} + \frac{r_{v} f_{v}}{r f^3}
\]

\[
\Lambda = \frac{1}{12} \frac{g_\phi^2}{r^2 g^4} + \frac{1}{12} \frac{g_\theta^2}{r^2 g^4} + \frac{1}{6} \frac{r_{u} r_{v}}{r f} + \frac{1}{3} \frac{r_{u,v}}{r f} + \frac{1}{12} \frac{f_{u,v}}{f^2} - \frac{1}{12} \frac{f_{u} f_{v}}{f^3} - \frac{1}{12} \frac{g_{\theta, u}}{r^2 g^3} - \frac{1}{12} \frac{g_{\phi, v}}{r^2 g^3}
\]

At this point, we see that the Weyl spinor is already in what we would regard as a 'standard' form. That is, the \(l\) and \(n\) vectors of the frame are aligned along the principle null directions so that the only remaining Weyl component is \(\Psi_{20}\).

In this form, the Weyl spinor is invariant under the \(SL(2, C)\) subgroups of spins and boosts, as we can check using the function \text{isotest()}\.

Thus by requiring the Weyl spinor to take this form, we have already lost four degrees of frame rotational freedom.

\[
\text{isotest(WeylSp)};
\]

\{'lnswap, Spin, Boost\}
(\text{lnswap} \text{ refers to the discrete group corresponding to a swap of the } l \text{ and } n \text{ basis vectors}). \ We \ can \ apply \ the \ same \ procedure \ to \ the \ Ricci \ spinor,  
\begin{verbatim}
> isotest(RicciSp);
\{lnswap, Spin\}
\end{verbatim}

and learn that the Ricci spinor is, in fact, not invariant under boosts of the frame. That is, we can remove this freedom from the frame by using it to restrict the components of the Ricci tensor which are non-invariant under boosts so that they fit some standard form. A set of standard forms for symmetric spinors is known to the function \text{dytrgen()}\!, which can now be used to generate the appropriate transformation of the frame. In this particular case, it chooses a boost which will cause the $|\Phi_{00}| = |\Phi_{22}|$.  
\begin{verbatim}
> T:=dytrgen(Boost,RicciSp):
evalm(T);
\begin{bmatrix}
-\frac{1}{8} & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & \frac{1}{8}
\end{bmatrix}
\end{verbatim}

The dyad transformation \( T \) is applied to the curvature spinors using the \text{applydytr()}\! function. The following command creates a new frame called \text{stdB} by applying the spinor dyad transformation, \text{dytr=T}, to the curvature spinors. (In fact, since \text{WeylSp} and \text{Lambda} are invariant under \( T \), their components are simply copied across to the new frame.) By specifying \text{rotateTetrad}, we also ensure that the frame vectors \( \{l, n, m, \overline{m}\} \) are themselves rotated.  
\begin{verbatim}
> applydytr(WeylSp,RicciSp,Lambda,dytr=T,rotateTetrad=auto,newname=stdB1);
Transformation completed.
The rotated frame has been named 'stdB1'.
\end{verbatim}

We now switch to the new frame and examine the spinor components.  
\begin{verbatim}
> grmetric(stdB1);
Default metric is now stdB1
> grabalter(RicciSp,radsimp);
Component simplification of a GRTensorII object:
Applying routine radsimp to object RicciSp
> grdisplay(WeylSp,RicciSp,Lambda);
For the stdB1 spacetime :
\[
\Psi_{20} = -\frac{1}{6} \frac{g_\phi^2}{r^2 g^4} - \frac{1}{6} \frac{g^2}{r^2 g^4} + \frac{1}{3} \frac{r_{u,v}}{r f} - \frac{1}{6} \frac{r_a r_v}{r^2 f} - \frac{1}{6} \frac{f_{u,v}}{f^2} + \frac{1}{6} \frac{f_u f_v}{f^3} + \frac{1}{6} \frac{g_{\theta, a}}{r^2 g^3} + \frac{1}{6} \frac{g_{\phi, \phi}}{r^2 g^3}
\]

\[
\Phi_{00} = \frac{I \sqrt{r_{v,v} f - r_v f u} \sqrt{r_u f u - r_{u,u} f}}{f^2 r}
\]

\[
\Phi_{11} = \frac{1}{4} \left( 2 g^4 r_u r_v f^2 + f^3 g_\phi^2 + f^3 g_\theta^2 - g^4 f_{u,v} f^2 + g^4 f_u f_v r^2 f - f^3 g_{\theta, \theta} f^3 - f^3 g_{\phi, \phi} f^3 \right) / (r^2 g^4 f^3)
\]

\[
\Phi_{22} = \frac{I \sqrt{r_{v,v} f - r_v f u} \sqrt{r_u f u - r_{u,u} f}}{f^2 r}
\]

\[
\Lambda = \frac{1}{12} \frac{g_\phi^2}{r^2 g^3} + \frac{1}{12} \frac{g^2}{r^2 g^3} + \frac{1}{3} \frac{r_{u,v}}{r f} + \frac{1}{12} \frac{r_{u,v}}{f^2} + \frac{1}{12} \frac{f_{u,v}}{f^3} - \frac{1}{12} \frac{f_u f_v}{f^3} - \frac{1}{12} \frac{g_{\theta, \theta}}{r^2 g^3} - \frac{1}{12} \frac{g_{\phi, \phi}}{r^2 g^3}
\]

At this point the frame has been fixed as far as possible, as the curvature spinor components are all invariant under the remaining 1-parameter continuous isotropy, spins. We can count the number of independent functions among the components and subtract the dimension of the frame freedom to determine that four scalar polynomial invariants are expected.

8 Appendix B: Invariants of Class B warped product spacetimes

We record here the invariants \( \{ R, r_1, r_2, w_2 \} \).

For type \( B_1 \):

\[
ds^2 = -2 f(u, v) \ d u \ d v + r(u, v)^2 g(\theta, \phi)^2 \ d \theta^2 + r(u, v)^2 g(\theta, \phi)^2 \ d \phi^2
\]

\[
R = 2(-r(u, v)^2 g(\theta, \phi)^4 f_u f_v + r(u, v)^2 g(\theta, \phi)^4 f_u f_v f(u, v)
+ 4 r(u, v) g(\theta, \phi)^3 r_{u,v} f(u, v)^2 + f(u, v)^3 g_\phi^2 - f(u, v)^3 g_{\phi, \phi} g(\theta, \phi)
+ f(u, v)^3 g_\theta^2 - f(u, v)^3 g_{\theta, \theta} g(\theta, \phi) + 2 f(u, v)^2 r_u g(\theta, \phi)^4 r_v) / (f(u, v)^3)
\]

\[r(u, v)^2 g(\theta, \phi)^4\]
\[
R1 = \frac{1}{4}(2 \mathbf{f}(u, v)^6 g_\phi^2 g_\theta^2 + \mathbf{f}(u, v)^5 g_\phi^2 g(\theta, \phi)^2 + \mathbf{f}(u, v)^6 g_\phi^2 g_\theta^2 g(\theta, \phi)^2
\]
\[
+ 12 \mathbf{f}(u, v)^2 g(\theta, \phi)^8 f_u f_v f(u, v)^2 r_u r_v + \mathbf{f}(u, v)^4 g(\theta, \phi)^8 f_u v u^2 f(u, v)^2
\]
\[
+ 2 \mathbf{f}(u, v)^2 g(\theta, \phi)^5 f_u v f(u, v)^4 g_\phi, \phi - 2 \mathbf{f}(u, v)^2 g(\theta, \phi)^4 f_u v f(u, v)^4 g_\theta^2
\]
\[
+ 2 \mathbf{f}(u, v)^2 g(\theta, \phi)^5 f_u v f(u, v)^4 g_\theta, \theta - 4 \mathbf{f}(u, v)^2 g(\theta, \phi)^8 f_u v f(u, v)^3 r_u r_v
\]
\[
- 2 \mathbf{f}(u, v)^2 g(\theta, \phi)^4 f_u v f(u, v)^4 g_\phi^2 + \mathbf{f}(u, v)^4 g(\theta, \phi)^8 f_u v^2 f_v^2
\]
\[
- 8 \mathbf{f}(u, v)^3 r(u, v)^2 g(\theta, \phi)^8 r_v u r_u - 8 \mathbf{f}(u, v)^3 r(u, v)^2 g(\theta, \phi)^8 f_v r_u r_u u
\]
\[
- 2 \mathbf{f}(u, v)^6 g_\phi^2 g_\phi, \theta g(\theta, \phi) + 4 \mathbf{f}(u, v)^5 g_\phi^2 r_u g(\theta, \phi)^4 r_v
\]
\[
- 2 \mathbf{f}(u, v)^6 g_\phi, \phi g(\theta, \phi) g_\phi^2 + 2 \mathbf{f}(u, v)^6 g_\phi, \phi g(\theta, \phi)^2 g_\theta \theta
\]
\[
- 4 \mathbf{f}(u, v)^5 g_\phi^2 r_u g(\theta, \phi)^4 r_v - 2 \mathbf{f}(u, v)^3 g(\theta, \phi)^4 f_u f_v f_u, v f(u, v)
\]
\[
- 2 \mathbf{f}(u, v)^2 g(\theta, \phi)^4 f_u f_v f(u, v)^4 g_\phi^2 - 2 \mathbf{f}(u, v)^2 g(\theta, \phi)^5 f_u f_v f(u, v)^3 g_\phi, \phi
\]
\[
+ 2 \mathbf{f}(u, v)^2 g(\theta, \phi)^4 f_u f_v f(u, v)^3 g_\phi^2 - 2 \mathbf{f}(u, v)^2 g(\theta, \phi)^5 f_u f_v f(u, v)^3 g_\phi, \theta
\]
\[
- 4 \mathbf{f}(u, v)^5 g_\phi, \theta g(\theta, \phi)^5 r_u r_v + 4 \mathbf{f}(u, v)^4 r_u^2 g(\theta, \phi)^8 r_v^2
\]
\[
+ 8 \mathbf{f}(u, v)^4 r(u, v)^2 g(\theta, \phi)^8 r_v u r_u - 2 \mathbf{f}(u, v)^6 g_\phi^2 g_\phi, \phi g(\theta, \phi)
\]
\[
+ \mathbf{f}(u, v)^6 g_\phi^4 + \mathbf{f}(u, v)^6 g_\phi^2) / (f(u, v)^6 r(u, v)^4 g(\theta, \phi)^8)
\]

\[
R2 = \frac{3}{2}(r(u, v)^2 g(\theta, \phi)^4 f_u f_v - r(u, v)^2 g(\theta, \phi)^4 f_u, v f(u, v) + f(u, v)^3 g_\phi^2
\]
\[
- f(u, v)^3 g_\phi, \phi g(\theta, \phi) + (f(u, v)^3 g_\phi^2 - f(u, v)^3 g_\phi, \theta g(\theta, \phi)
\]
\[
+ 2 f(u, v)^2 r_u g(\theta, \phi)^4 r_v(-r_v, v f(u, v) + f_v r_v) (r_u, u f(u, v) - f_u r_u) / (f(u, v)^7 r(u, v)^4 g(\theta, \phi)^4)
\]

\[
W2R = -\frac{1}{36} (r(u, v)^2 g(\theta, \phi)^4 f_u, v f(u, v) - r(u, v)^2 g(\theta, \phi)^4 f_u f_v
\]
\[
- 2 r(u, v) g(\theta, \phi)^4 r_v, v f(u, v)^2 + f(u, v)^3 g_\phi^2 - f(u, v)^3 g_\phi, \theta g(\theta, \phi)
\]
\[
+ f(u, v)^3 g_\phi^2 - f(u, v)^3 g_\phi, \phi g(\theta, \phi) + 2 f(u, v)^2 r_u g(\theta, \phi)^4 r_v^2) / (f(u, v)^9
\]
\[
r(u, v)^6 g(\theta, \phi)^{12})
\]

For type $B_2$:
\[
ds^2 = f(u, v)^2 d u^2 + f(u, v)^2 d v^2 - 2 r(u, v)^2 g(\theta, \phi) d \theta d \phi
\]
\[
R = 2(r(u, v)^2 g(\theta, \phi)^3 f_v^2 - r(u, v)^2 g(\theta, \phi)^3 f_v, v f(u, v) + r(u, v)^2 g(\theta, \phi)^3 f_u^2
\]
\[
- r(u, v)^2 g(\theta, \phi)^3 f_u, u f(u, v) - 2 r(u, v) g(\theta, \phi)^3 r_u, u f(u, v)^2
\]
\[
- 2 r(u, v) g(\theta, \phi)^3 r_v, v f(u, v)^2 - f(u, v)^4 g_\phi g(\theta, \phi)
\]
\[
- f(u, v)^2 r_u^2 g(\theta, \phi)^3 - f(u, v)^2 r_v^2 g(\theta, \phi)^3) / (f(u, v)^4 r(u, v)^2 g(\theta, \phi)^3)
\]
$$R1 = \frac{1}{4} (u, v)^4 g(\theta, \phi)^6 f_v + r(u, v)^4 g(\theta, \phi)^6 f_u + f(u, v)^8 g_\theta^2 g_\phi^2 + f(u, v)^6 g_\theta^2 g_\phi^2 + f(u, v)^4 r_u^4 g(\theta, \phi)^6 + f(u, v)^4 r_v^4 g(\theta, \phi)^6 + f(u, v)^2 g(\theta, \phi)^6 f_{v, v} f(u, v)^3 r_u^2$$
$$- 2 r(u, v)^2 g(\theta, \phi)^6 f_{v, v} f(u, v)^3 r_u^2 - 2 r(u, v)^2 g(\theta, \phi)^6 f_{v, v} f(u, v)^3 r_v^2$$
$$- 2 r(u, v)^4 g(\theta, \phi)^6 f_{u, u} f(u, v)^2 + 2 r(u, v)^2 g(\theta, \phi)^3 f_{u, u} f(u, v)^2 + 10 r(u, v)^2 g(\theta, \phi)^6 f_{u, u} f(u, v)^2 r_u^2$$
$$+ 10 r(u, v)^2 g(\theta, \phi)^6 f_{u, u} f(u, v)^2 r_v^2 + r(u, v)^4 g(\theta, \phi)^6 f_{u, u} f(u, v)^2 r_v^2$$
$$- 2 r(u, v)^2 g(\theta, \phi)^3 f_{u, u} f(u, v)^3 g_\phi + 2 r(u, v)^2 g(\theta, \phi)^3 f_{u, u} f(u, v)^3 g_\phi$$
$$- 2 r(u, v)^2 g(\theta, \phi)^3 f_{u, u} f(u, v)^3 r_u^2 - 2 r(u, v)^2 g(\theta, \phi)^3 f_{u, u} f(u, v)^3 r_v^2$$
$$+ 2 r(u, v)^2 g(\theta, \phi)^6 r_u^2 r_v^2 f(u, v)^4 - 8 r(u, v)^2 g(\theta, \phi)^6 r_u^2 f(u, v)^3 r_u^2$$
$$+ 8 r(u, v)^2 g(\theta, \phi)^6 r_{u, u} f(u, v)^3 r_v^2 - 4 r(u, v)^2 g(\theta, \phi)^6 r_{u, u} f(u, v)^3 r_v^2$$
$$+ 8 r(u, v)^2 g(\theta, \phi)^6 f_{u, u} f(u, v)^3 r_v^2 - 2 r(u, v)^4 g(\theta, \phi)^6 f_{u, u} f(u, v)^3 r_v^2$$
$$- 2 r(u, v)^4 g(\theta, \phi)^6 f_{u, u} f(u, v)^3 r_u^2 + 2 r(u, v)^2 g(\theta, \phi)^3 f_{u, u} f(u, v)^4 g_\phi$$
$$- 2 r(u, v)^2 g(\theta, \phi)^6 f_{u, u} f(u, v)^4 g_\phi + 10 r(u, v)^2 g(\theta, \phi)^6 f_{u, u} f(u, v)^4 g_\phi$$
$$+ 10 r(u, v)^2 g(\theta, \phi)^6 f_{u, u} f(u, v)^2 r_v^2 - 16 f(u, v)^3 r(u, v)^2 g(\theta, \phi)^6 r_{u, v} f_v f_r u$$
$$- 16 f(u, v)^3 r(u, v)^2 g(\theta, \phi)^6 f_{u, u} f(u, v)^3 r_v^2 - 2 f(u, v)^8 g_\phi g_\phi + 2 f(u, v)^6 g_\phi g_\phi r_v^2 g(\theta, \phi)^3$$
$$- 2 f(u, v)^6 g_\phi g_\phi r_v^2 g(\theta, \phi)^3 - 2 f(u, v)^6 g_\phi g_\phi r_v^2 g(\theta, \phi)^3$$
$$+ 2 f(u, v)^4 r_u^2 g(\theta, \phi)^6 r_u^2 + 8 f(u, v)^4 r(u, v)^2 g(\theta, \phi)^6 r_u^2$$
$$- 2 f(u, v)^4 g(\theta, \phi)^6 f_{v, v} f(u, v) + 2 f(u, v)^4 g(\theta, \phi)^6 f_{v, v} f(u, v)^2$$
$$+ 2 f(u, v)^4 g(\theta, \phi)^6 f_{v, v} f(u, v)^2 f_{u, u} - 2 f(u, v)^2 g(\theta, \phi)^3 f_{v, v} f(u, v)^5 g_\phi g_\phi$$
$$+ 2 f(u, v)^2 g(\theta, \phi)^3 f_{v, v} f(u, v)^5 g_\phi g_\phi + 2 r(u, v)^2 g(\theta, \phi)^6 f_{v, v} f(u, v)^2 f_v^2$$
$$- 8 r(u, v)^2 g(\theta, \phi)^6 r_v f_v f(u, v)^3 r_v - 2 r(u, v)^4 g(\theta, \phi)^6 f_v f_v f(u, v) f_v^2) \bigg/ \left( f(u, v)^8 r(u, v)^4 g(\theta, \phi)^6 \right)$$

$$R2 = \frac{3}{8} (4 f(u, v) f_v r_u r_v v + 4 f_v^2 r_v^2 - 4 f(u, v) r_u u f_u r_u$$
$$- 2 r_{u, u} f(u, v)^2 r_v v + r_{u, u}^2 f(u, v)^2 + 4 f_v^2 r_u^2 + r_v v f(u, v)^2 + 4 f_v^2 r_v^2$$
$$+ 4 f_v^2 r_v^2 + 4 r_{u, v} f(u, v) r_v f_v - 8 f(u, v) r_u u f_u r_v$$
$$- 4 f_v f_v f(u, v) r_v v - 8 r_{u, v} f(u, v) f_v f_v) f(u, v)^4 g_\phi g_\phi - f(u, v)^4 g_\phi g_\phi - f(u, v)^4 g_\phi g_\phi g(\theta, \phi)$$
$$+ f(u, v)^2 r_v^2 g(\theta, \phi)^3 + f(u, v)^2 r_v^2 g(\theta, \phi)^3 + r(u, v)^2 g(\theta, \phi)^3 f_v^2$$
$$- r(u, v)^2 g(\theta, \phi)^3 f_v f_v f(u, v) + r(u, v)^2 g(\theta, \phi)^3 f_v^2$$
$$- r(u, v)^2 g(\theta, \phi)^3 f_{u, u} f(u, v) \bigg/ \left( f(u, v)^8 r(u, v)^4 g(\theta, \phi)^3 \right)$$
\[ W2R = -\frac{1}{36} (r(u, v)^2 g(\theta, \phi)^3 f_x^2 - r(u, v)^2 g(\theta, \phi)^3 f_v f(u, v) \]
\[ + r(u, v)^2 g(\theta, \phi)^3 f_u^2 - r(u, v)^2 g(\theta, \phi)^3 f_u f(u, v) \]
\[ + r(u, v) g(\theta, \phi)^3 r_v f(u, v)^2 + r(u, v) g(\theta, \phi)^3 r_u f(u, v)^2 - f(u, v)^4 g\theta g\phi \]
\[ + f(u, v)^4 g\phi, g g(\theta, \phi) - f(u, v)^2 r_u^2 g(\theta, \phi)^3 - f(u, v)^2 r_v^2 g(\theta, \phi)^3) \]
\[ f(u, v)^{12} r(u, v)^6 g(\theta, \phi)^9 \]

9 Appendix C: Ricci syzygies

Syzygies for the Ricci invariants (in terms of \( r_1, r_2 \) and \( r_3 \)) up to \( r_{10} \) are given by:

\[ 6r_4 - 5r_1r_2 = 0, \quad (40) \]
\[ 24r_5 + 3r_1^3 - 8r_2^2 - 18r_1r_3 = 0, \quad (41) \]
\[ 24r_6 - 7r_2r_1^2 - 14r_3r_2 = 0, \quad (42) \]
\[ 144r_7 - 36r_3^2 - 64r_1r_2^2 - 36r_1^2r_3 + 9r_1^4 = 0, \quad (43) \]
\[ 36r_8 - 4r_2^3 - 27r_1r_2r_3 = 0, \quad (44) \]
\[ 576r_9 - 160r_2^2r_3 - 180r_1r_3^2 - 160r_1^2r_2 + 9r_1^5 = 0, \quad (45) \]

and

\[ 1728r_{10} - 396r_2r_3^2 - 352r_1r_2^3 - 792r_1^2r_2r_3 + 99r_1^4r_2 = 0. \quad (46) \]

These syzygies hold in general, unlike the syzygy (47).