A CUT-AND-CHOOSE MECHANISM TO PREVENT GERRYMANDERING

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I. Introduction

We all clearly recognize gerrymandering as a problem, yet somehow there is a persistent lack of a clear solution. At the policy-making level, proposals have certainly been advanced, for example requiring that electoral maps satisfy certain shape conditions or a stringent “symmetry” requirement [1]. Others advocate taking the power out of the hands of politicians completely and giving it to independent commissions. However, all of these ideas have significant drawbacks. It is easy to find examples of maps that can be drawn with relatively “simple” shapes, that would appear not to be gerrymandered, but are nevertheless extremely unfair. And with a lack of well-defined rules for fairness, judges have been generally unable to find a clear legal reason to rule any electoral division intentionally biased [2]. As for independent commissions, beyond this vague notion of fairness there is not a clear normative consensus as to what the goals of such a commission should be, let alone a system to ensure that they would not act with bias.

In this paper, we describe a novel mechanism which incentivizes rational individuals to choose an electoral map that satisfies a precise definition of fairness, requiring no further intervention from an omnibenevolent third party. The basis for our scheme is the cut-and-choose paradigm. That is, in a contest between two parties, one politician divides the map into electoral districts, then a politician from the opposing party observes that division and makes a choice that ultimately determines the final outcome. The system will be structured such that if the first politician does not divide up the districts fairly, then their party will end up with fewer seats.

Methods along these lines are (to the author’s knowledge) uncommon in the Economics literature, with one exception. In [3], Landau, Reid, and Yershov propose a system that has a cut-and-choose mechanic of a slightly different nature. In this complicated procedure, both parties are presented with a series of different choices over which parts of the map they would like to divide, and the outcome is determined by finding a point of agreement. However, the menu of choices is determined by a “neutral” third party, and there are not even theoretical guarantees of how unfair the outcome could be if the neutrality is somehow compromised. The authors acknowledge this flaw, and then attempt to “augment” their mechanism so that these cases appear less frequently, yet its fairness still hinges on a combination of the third party being non-biased, and random chance.

A less capricious solution, from the mathematics literature, is the mechanism known as Fair Majority Voting, first introduced in [4]. Instead of holding independent elections in each districts, it computes the total number of votes for each party, and apportions the districts to best approximate these values, preferring candidates who won by the greatest majorities in their districts. This simple, intuitive system has numerous advantages, most notably that,

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by construction, it is always as fair as possible. However, the fact that voters are not just voting for a person but for a party could put them in an awkward position if they had one preference over the candidates and another over the parties. Another negative consequence of the districts not being independent is that elections would always have to occur at the same time. Finally, there is an inevitable high probability that some districts will be won by the candidate with fewer votes.

The voting system we present overcomes these criticisms, yet has other flaws of its own. It relies on no third party, and creates districts that can hold elections independently of one another, favoring the party with more votes. However, while all equilibria are fair, some of them (but not all) necessarily entail randomness. And as is the case in dividing a cookie by cut-and-choose, there is a slight, inherent advantage that goes to the chooser. No solution to the problem of gerrymandering is perfect, but the one we present is a unique alternative with several desirable properties.

In Section II we formally describe our model and proposed mechanism, and develop a graphical representation of the game that will be useful for its analysis, giving a real-world example. In Section III we solve the game using backward induction, and give a hypothetical example that demonstrates the possible kinds of equilibria. In Section IV we conclude by returning to some of the simplifying assumptions we have made and discussing how the model could be extended or adapted to work with more accurate assumptions about the real world.

II. Voting with Supermajorities

We consider a simplified setting in which there are only two political parties, $A$ and $B$. Every voter has a strict preference over the two parties, and these preferences are common knowledge. There is one state with voter population $Dn$, which must be divided into $D$ districts, each with exactly $n$ voters. In total, party $A$ holds a $v_A$ fraction of the total voter population, and party $B$ holds a $v_B = 1 - v_A$ fraction. We refer to these constants as the party advantages of $A$ and $B$, respectively. Assume that all possible partitions of voters into $D$ sets of equal size correspond to feasible district divisions. All voters will cast their ballots to elect a representative of their preferred party within their district. There are two agents, one representing each party, with utilities equal to the number of districts won by that party.

Now comes the twist. Instead of simply electing within each district by majority rule, the mechanics of the election are specified by a real number $m \in [\frac{1}{2}, 1)$, which will be endogenously chosen. In each district, if either party gets a fraction of votes strictly greater than $m$, then that party will be declared the winner for that district. Otherwise, the winner will be randomly determined, each party with equal probability. The game proceeds in three steps.

1. Party $A$ divides the districts in any feasible way.
2. Party $B$ observes this division and chooses $m \in [\frac{1}{2}, 1)$.
3. An election is held, with each district getting a representative from one of the two parties according to the election rule described above.

The randomness and asymmetry of this mechanism may seem preposterous at first. Our main results are that, on the contrary, there always exists an equilibrium that are requires no randomness (beyond tie breaking), while all equilibria of this game are very fair, up to a small rounding error that becomes insignificant as $D$ rises.
Theorem 1. If $D > 1$, then there always exists a subgame-perfect Nash equilibrium in which party $B$ sets $m = \frac{1}{2}$.

Theorem 2. In any subgame-perfect Nash equilibrium, in expectation, party $A$ wins a $\text{roundDOWN}(v_A, \frac{1}{2D})$ fraction of the districts, and party $B$ wins a $\text{roundUP}(v_B, \frac{1}{2D})$ fraction of the districts.

Here the functions $\text{roundUP}$ and $\text{roundDOWN}$ round the first argument to an integer multiple of the second argument, i.e.

$$\text{roundUP}(a, b) := b \left\lceil \frac{a}{b} \right\rceil, \quad \text{roundDOWN}(a, b) := b \left\lfloor \frac{a}{b} \right\rfloor.$$ 

As a motivating example, consider the 2012 election in Wisconsin, a state well known for partisan gerrymandering. In total, the Democratic candidates (which we will call party $B$) won 50.42% of the vote, but Republicans (party $A$) won 5 of 8 districts.\footnote{Data taken from [5].} Below, these districts are vertically sorted from most Democratic to most Republican. A zigzagging line is drawn across these district majorities, which we will call the 

\textit{districting function}, denoted $g(m)$. Its vertical segments separate the votes for party $B$ (on the left) from the votes for party $A$ (on the right).

By representing the data in this form, we can make two important observations. First, given an $m \in [0, 1]$ for which $g$ is well defined (i.e. there is not a jump discontinuity), $g(m)$ gives the number of districts in which party $B$ has a majority greater than $m$. Turning the picture upside down, we analogously see that the number of districts in which party $A$ has a majority greater than $m$ is given by $1 - g(1 - m)$. A second observation is that, because we have normalized our units by the total population such that the total area of the graph is 1, it follows that the total fraction of voters supporting party $B$ is equal to the area of the region below $g$, i.e.

$$v_B = \int_0^1 g(m) dm.$$ 

Note that we can compute this integral in more general settings than when $g(m)$ is a zigzag line: $g(m)$ must always be decreasing and bounded below, so it is always integrable.
Using this picture, we can now explain how these districts appear to be gerrymandered, and how the gerrymandering would have been punished by our mechanism. The election system used in the real world is an election where $m = 0.5$. That is, party B wins $g(0.5) = \frac{3}{8}$ of the districts, and party A will win the other $\frac{5}{8}$ of the districts. Thus, even though party B had a greater party advantage ($v_B > 0.5$), party A was able to gerrymander the district lines so that they came out with more districts. However, if party B was able to observe the map and choose a different value of $m$, the outcome will favor party B. One such optimal choice is to set $m = 0.63$, as shown on the graph. In that case, B will win the bottom 3 districts with certainty because $g(0.63) = \frac{3}{8}$, A would win only the top district with certainty because $1 - g(1 - 0.63) = \frac{1}{8}$, and the remaining 4 districts would be randomly determined. In expectation, B would win 5 districts, and A would only win 3. As we will prove in the next section, this is a worse outcome for A than that which would result if A had drawn the district lines more fairly.

III. A Complete Characterization of Equilibria

We now solve this game in its full generality, proving the fairness and determinism results stated in Section II. Note that step (1) of the game amounts to choosing the function $g(m)$ from a menu of possible zigzagging lines. We first consider the case where it is possible for $g(m)$ to be any decreasing function on $[0, 1]$ that starts at 1 and ends at 0. In this case we will show that the equilibria are always completely fair. At the end we will relax this assumption, and in doing so pick up the small rounding error that favors party B.

Proof of Theorems 1 and 2. We proceed by backward induction. Given a districting function $g$ (which party A chose), party B will pick $m \in [\frac{1}{2}, 1)$ that maximizes $u_B(g, m)$, the expected number of districts B will win given that value of $m$. There are two pieces that go into the computation of $u_B(g, m)$. As discussed in Section II, B will always win a $g(m)$ fraction of the districts with certainty, and lose a $1 - g(1 - m)$ fraction of the districts with certainty. The remaining fraction of districts,

$$1 - [g(m)] - [1 - g(1 - m)] = g(1 - m) - g(m),$$

will be decided randomly, so B can expect to win half of those. The expected utility is therefore the sum of the guaranteed wins and the possible wins, with the latter weighted by $\frac{1}{2}$. That is,

$$u_B(g, m) = g(m) + \frac{1}{2}[g(1 - m) - g(m)] = \frac{1}{2}[g(m) + g(1 - m)].$$

We next compute the quantity $\int_{1/2}^1 u_B(g, m) \, dm$, and show that it does not depend on $g$. This fact will allow us to determine an optimal strategy for party A. Starting from our formula above,
A CUT-AND-CHOOSE MECHANISM TO PREVENT GERRYMANDERING

\[
\int_{\frac{1}{2}}^{1} u_B(g, m) dm = \int_{\frac{1}{2}}^{1} \frac{1}{2} [g(m) + g(1 - m)] dm \\
= \frac{1}{2} \left( \int_{\frac{1}{2}}^{1} g(m) dm + \int_{\frac{1}{2}}^{1} g(1 - m) dm \right) \\
= \frac{1}{2} \left( \int_{\frac{1}{2}}^{1} g(m) dm + \int_{0}^{\frac{1}{2}} g(m) dm \right) \quad \text{(sending } m \mapsto 1 - m) \\
= \frac{1}{2} \left( \int_{\frac{1}{2}}^{1} g(m) dm + \int_{0}^{\frac{1}{2}} g(m) dm \right) \\
= \frac{1}{2} \int_{0}^{1} g(m) dm \\
= \frac{1}{2} v_B.
\]

Because the game is zero-sum, party A will choose \( g \) to minimize the maximum value of \( u_B(g, \cdot) \). However, we have seen that no matter what \( g \) is, the area under the curve of \( u_B(g, \cdot) \) over party B’s strategy space is fixed at a predetermined value. Therefore, the optimal choice would be to make \( u_B(g, \cdot) \) be a constant function. This is indeed always possible - one such choice is to set \( g \) to be constant.

Given that \( u_B(g, \cdot) \) is constantly equal to some number, say \( c \), we know that

\[
\frac{1}{2} v_B = \int_{\frac{1}{2}}^{1} u_B(g, m) dm = \int_{\frac{1}{2}}^{1} (c) dm = c \int_{\frac{1}{2}}^{1} (1) dm = \frac{1}{2} c,
\]

so this value is actually \( v_B \). Thus, in all equilibria, each party will get utility equal to its advantage. Because party B will be indifferent between all values of \( m \), setting \( m = \frac{1}{2} \) is always one of the optimal choices.

At this point, we have proved Theorems 1 and 2 (without the rounding functions) in the case where \( g \) is allowed to be an arbitrary decreasing function from \([0, 1]\) to itself. Now suppose \( g \) is required to actually be a districting function over a finite number of districts \( D \). From (\( \ast \)), we see that \( u_B \) will necessarily be an integer multiple of \( \frac{1}{2D} \), as \( g \) can only take on values that are multiples of \( \frac{1}{D} \). If it so happens that \( v_B \) is such a value, then we will still have the same equilibria as in the unrestricted case, as party A can set

\[
g(m) = \begin{cases} 
\text{roundUP}(v_B, \frac{1}{D}) & m \leq \frac{1}{2} \\
\text{roundDOWN}(v_B, \frac{1}{D}) & m > \frac{1}{2}.
\end{cases}
\]

If \( v_A \) and \( v_B \) do not fall into this lattice, then party A will still be able to get the next feasible level of utility, given by \( \text{roundDOWN}(v_A, \frac{1}{2D}) \). To accomplish this, A can pretend it has less voters than it actually has, to bring \( v_B \) up to the nearest point divisible by \( \frac{1}{2D} \), then execute any optimal strategy from there. This proves Theorem 2. Finally, to make sure that \( m = \frac{1}{2} \) is still an optimal choice for party B, party A can just put all the votes they are pretending to give up in either a district in which they have all the votes, or a district in which they are pretending to have none of the votes. This is always possible provided \( D > 1 \). The effect
will be to only make the outcome worse for $B$ if $B$ chooses $m$ sufficiently higher than $\frac{1}{2}$. Therefore, there will still be an equilibrium at $m = \frac{1}{2}$, and Theorem 1 is proved. □

These equilibria have a more intuitive description terms of our diagram from Section II. Supposing that the party advantages fall into the lattice of multiples of $\frac{1}{2D}$, we have seen that party $A$ must choose $g$ such that for all $m \in \left[\frac{1}{2}, 1\right)$,

$$v_B = \frac{1}{2}[g(m) + g(1 - m)].$$

By shifting $m$ by $\frac{1}{2}$, multiplying both sides by 2, and rearranging terms, we can rewrite this condition as stating, for all $m \in [0, \frac{1}{2})$.

$$\left[ g \left( \frac{1}{2} - m \right) - v_b \right] = -\left[ g \left( \frac{1}{2} + m \right) - v_b \right].$$

The left hand side gives the vertical difference from $v_b$ to the point on $g(m)$ that is $m$ units to the left of center, while the right hand side gives the difference from $v_b$ to the point on $g(m)$ that is $m$ units to the right of center. By enforcing that these two quantities be negatives of each other for all distances from center $m \in [0, \frac{1}{2})$, it is equivalent to saying that $g$ is rotationally symmetric about the point $\left( \frac{1}{2}, v_b \right)$.

As a hypothetical example, suppose there are $D = 4$ districts, where party $A$ has advantage $v_A = \frac{5}{8}$ and party $B$ has advantage $v_B = \frac{3}{8}$. These advantages are multiples of $\frac{1}{2D} = \frac{1}{8}$, so party $A$ will not have to give up any voters, and both parties will get an expected utility equal to their party advantage. Below are three optimal districting functions that $A$ could choose.

We number the districts 1 to 4 in order from top to bottom. Because all of these must be symmetric about the point $\left( \frac{1}{2}, \frac{3}{8} \right)$, district 1 must necessarily be completely comprised of voters for party $A$. (Analogously, if party $B$ had a greater advantage, the point of symmetry would be in the top half of the diagram, and so the bottom district(s) would be completely comprised of voters for party $B$). This makes sense, for otherwise party $B$ could set $m$ higher than $A$'s majority and take at least half the districts by random chance. For the bottom three districts, party $A$ has several choices which obey the symmetry criterion. However, to

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Note that this is an entirely different notion of symmetry than that of the legal proposal described in [1]. In our model, symmetry is with respect to our parameter $m$. King's definition of symmetry is with respect to changing statewide preferences, a dimension which our model does not consider.
The scheme on the left divides the remaining space into what might be referred to as “competitive” districts, in the sense that no party holds a clear majority. Of course, in our model, voters are perfectly predictable, and the only actual factor of randomness is that there will be a tie in each of these three districts, so the winner will be decided by coin flip. Notice that Party B will win 1.5 districts in expectation, which is a $\frac{3}{8}$ fraction of districts, consistent with Theorem 2.

On the right is a maximally “safe” division, as districts 2 and 4 will each be overwhelmingly won by one of the parties. Regardless of the choice of $m$, Party B will win district 4 with certainty, and district 3 with probability $\frac{1}{2}$, again for an expected value of 1.5. Between these two extremes lies another arbitrary possibility, the middle diagram, with a districting function that has intermediate jump points across districts 2 and 4. This effectively gives B a choice over the variance. If they set $m$ below the jump points they will get the safe outcome, and if they set $m$ above the jump points they will get the random outcome. However, because the graph is symmetric about $(\frac{1}{2},\frac{3}{8})$, these jumps occur at the same point, so B cannot get any real advantage. The expected utility in all cases will be $\frac{3}{8}$.

IV. Extensions

While theoretically sound, many objections may be raised at this proposal. The equilibria described seem to be contingent on somewhat unreasonable assumptions about the real world, such as universal domain over districting functions or perfect information. Even under these assumptions, the mechanism has several shortcomings, most notably an unavoidable unfairness to Party A and a mere existence guarantee of determinism. In this section we consider some of these problems and discuss likely future directions in which they may be resolved.

The reliance of the mechanism on this perfect information and unrestricted districting may seem like a fatal flaw, as any equilibrium where $\text{roundDOWN}(v_A, \frac{1}{2D}) \neq \text{roundUP}(v_B, \frac{1}{2D})$ requires that Party A perfectly stack one of the majority party’s districts, so that there are no voters supporting the minority party. If A has a greater advantage, and even one voter for Party B sneaks into each of its stacked districts, then B can set $m \geq \frac{n-1}{n}$ and take half of the votes, merely by random chance. Thus, if the domain of districting functions precludes perfect stacking, or there is no way to predict votes with certainty, we see that the deterministic equilibrium which gives party A a fair, comfortable advantage breaks down into complete randomness, with party A losing its edge.

With a slight modification, this mechanism can avoid such disastrous consequences. Suppose that there is some fundamental limit $M \in [\frac{1}{2}, 1)$ than which no district can be stacked to a greater majority. In this case Party B should be required to choose $m \in [\frac{1}{2}, M)$, rather than any $m$. That way, Party A does not have to fear B taking advantage of its inability to draw perfect district lines. This is a kind of “first order” definition of restricted districting, a simple model of the kinds of limitations that politicians face when drawing the map. Consequently, there is a simple modification to the mechanism that preserves a level of fairness. There are several factors one could add to make the model richer. However, no matter what model one constructs, the way to adapt the mechanism should still follow the same general
idea: whatever natural restrictions party A faces in district drawing should be met with prescribed restrictions on party B’s choice in step 2.

The next concern is how to reduce randomness. One simple change would be to equally distribute the districts in which neither party reached a majority of more than \( m \). Only if there is an odd number of such districts would one have to randomize the very last district. The equilibria of the game would be unchanged, as both parties are still getting the same expected utility from these districts. While this solution eliminates, or at least significantly reduces, the variance in state-wide utilities for each party, there might still be randomness at the district level that both voters and politicians may be irked by. A better kind of solution would be one in which \( m = \frac{1}{2} \) is a unique equilibrium. For example, one could specify that while \( B \) needs a majority of \( m \) to take a district with certainty, \( A \) needs only a majority of \( \frac{m}{2} + \frac{1}{4} \), so \( B \) would be obliged to keep \( m \) low. However, if one tries to incentivize this choice in step 2, party \( A \) may end up exploiting this incentive in step 1, bringing \( B \) back to the point of indifference, and the net effect will have been to give an arbitrary advantage to party \( A \), while \( m = \frac{1}{2} \) may still not be a unique equilibrium. Such incentives must be applied more subtly.

An extra benefit of considering a richer kind of model and more complicated mechanism is that the rounding-error advantage of party \( B \) may end up being eliminated. For instance, any form of imperfect information has the effect of smoothing out the vertical pieces of the districting function. It is quite possible that, in the presence of a certain kind of noise, it will always be possible for party \( A \) to choose a districting function that is (mostly) symmetric about \((\frac{1}{2}, v_B)\). Or, in a completely orthogonal direction, if we are modifying the election mechanics to reward or punish either party for a certain behavior, there may be a way to reward or punish arbitrarily, which could be useful for balancing an inherent unfairness such as the rounding advantage of party \( B \).

There are countless extensions of this model that are all worthy of further attention. All I have done in this paper is to present the idea in its simplest form, exhibiting its strong theoretical justification. Hopefully, research along these lines can one day be expanded into a practical election system to be implemented in the real world.

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