On the Biconservative Quasi-Minimal Immersions into Semi-Euclidean Spaces

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Abstract. In this paper, we study biconservative immersions into the semi-Riemannian space form $R^4_2(c)$ of dimension 4, index 2 and constant curvature $c \in \{0, -1, 1\}$. First, we obtain a characterization of quasi-minimal proper biconservative immersions into $R^4_2(c)$. Then we obtain the complete classification of quasi-minimal biconservative surfaces in $R^4_2(0) = \mathbb{E}^4_2$. We also obtain a new class of biharmonic quasi-minimal surfaces in $\mathbb{E}^4_2$.

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1. Introduction

Surfaces in semi-Riemannian manifolds with zero mean curvature is one of mostly interested topics in differential geometry. When the ambient manifold $(N, \tilde{g})$ is Riemannian, a surface with zero mean curvature, called minimal surface, arises as the solution of the variational problem of finding the surface in $N$ with minimum area among all surfaces with the common boundary. On the other hand, if $N$ is a semi-Riemannian manifold with positive index, it admits an important class of surfaces whose mean curvature is zero. These surfaces, called quasi-minimal surfaces, have no counter part on Riemannian manifolds.

By the definition, a submanifold $M$ of $(N, \tilde{g})$ is said to be quasi-minimal if its mean curvature vector is light-like at every point. Some of geometrical properties of quasi-minimal surfaces in 4-dimensional semi-Riemannian space forms have been studied in a lot papers so far, [2, 4–6, 18, 23]. Quasi-minimal surfaces also play some fundamental roles in physics and they are called as...
‘marginally trapped’ in the physics literature when the ambient manifold is a Lorentzian space-time, [30].

Consider the bienergy integral

\[ E_2(\psi) = \frac{1}{2} \int_\Omega \|\tau(\psi)\|^2 v_g \quad (1.1) \]

for a mapping \( \psi : (\Omega, g) \to (N, \tilde{g}) \) between two semi-Riemannian manifolds, where \( v_g \) is the volume element of \( \Omega_g \) and \( \tau(\psi) = -\text{trace} \nabla d\psi \) is the tension of \( \psi \). Let \( \tau_2(\psi) \) stand for the bitension field of \( \psi \) defined by

\[ \tau_2(\psi) = -\Delta \tau(\psi) - \text{trace} \left( \tilde{R}(d\psi, \tau(\psi))d\psi \right), \]

where \( \Delta \) is the rough Laplacian defined on sections of \( \psi^{-1}(TN) \), i.e.,

\[ \Delta = -\text{trace} \left( \nabla^\psi \nabla^\psi - \nabla^\psi \right) \]

and \( \tilde{R} \) is the curvature tensor of \( (N, \tilde{g}) \).

When (1.1) is assumed to define a functional from \( C^\infty(\Omega, N) \), it is named as bienergy functional. In this case, the critical points of \( E_2 \) are called as biharmonic maps, [9]. In [21, 22], Jiang obtained the first and second variational formulas for \( E_2 \) and proved that \( \psi \) is biharmonic if and only if the fourth order system of partial differential equations given by

\[ \tau_2(\psi) = 0 \quad (1.2) \]

is satisfied. Biharmonic submanifolds particularly take interest of many geometers, [1, 6, 12, 15, 17, 29]. For example in [29], Papantoniou and Petoumenos study biharmonic hypersurfaces in the pseudo-Euclidean space \( \mathbb{E}^4_2 \). Moreover, a classification result of biharmonic hypersurfaces in space forms are given in [15]. Most recently, in [1] Akyol and Ou obtain several characterization results on biharmonic submersions into Riemannian manifolds.

On the other hand, if \( \psi : \Omega \to (N, \tilde{g}) \) is a given smooth mapping, one can also define a functional from the set of all metrics on \( \Omega \) by using (1.1), [11]. Note that critical points of this new functional are characterized by the equation:

\[ \langle \tau_2(\psi), d\psi \rangle = 0, \quad (1.3) \]

[11] (see also [20]). \( (\Omega, g) \) is said to be a biconservative submanifold if \( g \) is a critical point of this functional and \( \psi : (\Omega, g) \hookrightarrow (N, \tilde{g}) \) is an isometric immersion.

It is obvious that any biharmonic immersion is also biconservative. Because of this reason, biconservative submanifolds have been studied in many papers to understand geometry of biharmonic immersions so far, [7, 11, 13, 14, 19, 26, 27, 31]. For example, in [31], the third named author studied biconservative hypersurfaces in Euclidean spaces with three distinct principal curvatures. Also, classification results on biconservative hypersurfaces in 3-dimensional semi-Riemannian space forms have been appeared in some
papers, [13,14]. Most recently, biconservative surfaces in 4-dimensional Euclidean space have been studied in [24,32]. Note that some authors call biconservative submanifolds as H-hypersurfaces when the codimension is 1 (see for example [19,31]).

In [4,6], all flat biharmonic quasi-minimal surfaces in the 4-dimensional pseudo-Euclidean space $E^4_2$ with neutral metric were obtained. Furthermore, Chen and Garay studied quasi-minimal surfaces with parallel mean curvature vector in the pseudo-Euclidean space $E^4_2$ in [5]. In this paper, we study quasi-minimal biconservative immersions into $E^4_2$ and complete the study of biconservative quasi-minimal surfaces initiated in [4–6]. In Sect. 2, we give basic definitions and equations on isometric immersions into semi-Riemannian space forms after we describe the notation used in the paper. In Sect. 3, we obtain a characterization of biconservative immersions into space forms of index 2. Finally in Sect. 4, we present our main result which is the complete local classification of quasi-minimal proper biconservative surfaces in $E^4_2$.

2. Preliminaries

We are going to denote the $n$-dimensional semi-Riemannian space form of index $s$ and constant curvature $c \in \{-1,0,1\}$ by $R^n_s(c)$, that is

$$R^n_s(c) = \begin{cases} S^n_s & \text{if } c = 1, \\ E^n_s & \text{if } c = 0, \\ H^n_s & \text{if } c = -1 \end{cases}$$

and $\langle \cdot, \cdot \rangle$ stands for its metric tensor. When $c = 0$, we define the light-cone of $E^n_s$ by

$$\mathcal{LC} = \{ p \in E^n_s | \langle p, p \rangle = 0 \}.$$

On the other hand, a non-zero vector $w$ in a finite dimensional non-degenerated inner product space $W$ is said to be space-like, light-like or time-like if $\langle w, w \rangle > 0$, $\langle w, w \rangle = 0$ or $\langle w, w \rangle < 0$, respectively. We are going to use the following well-known lemma later (see, for example, [28, Lemma 22, p. 49])

**Lemma 2.1.** [28] Let $V$ be a subspace of $W$ and $V^\perp$ its orthogonal complement. Then, $\dim V + \dim V^\perp = \dim W$.

Consider an isometric immersion $f : (\Omega, g) \hookrightarrow R^n_s(c)$ from an $k$-dimensional semi-Riemannian manifold $(\Omega, g)$ with the Levi-Civita connection $\nabla$. Let $T\Omega$ and $N^f\Omega$ stand for the tangent bundle of $\Omega$ and the normal bundle of $f$, respectively. If $\tilde{\nabla}$ denote the Levi-Civita connection of $R^n_s(c)$, then the Gauss and Weingarten formulas are given, respectively, by

$$\tilde{\nabla}_XY = \nabla_XY + \alpha_f(X,Y), \quad (2.1)$$

$$\tilde{\nabla}_X\xi = -A^f_\xi(X) + \nabla^\perp_X\xi, \quad (2.2)$$

for any vector fields $X, Y \in T\Omega$ and $\xi \in N^f\Omega$, where $\alpha_f$ and $\nabla^\perp$ are the second fundamental form and the normal connection of $f$, respectively, and
$A^f_\xi$ stands for the shape operator of $f$ along the normal direction $\xi$. $A^f$ and $\alpha_f$ are related by
\[
\langle A^f_\xi X, Y \rangle = \langle \alpha_f(X, Y), \xi \rangle.
\] (2.3)

On the other hand, the second fundamental form $\alpha_f$ of $f$, the curvature tensor $R$ of $(\Omega, g)$ and the normal curvature tensor $R^\perp$ of $f$ satisfies the integrability conditions:
\[
R(X, Y)Z = c(X \wedge Y)Z + A^f_{\alpha_f(Y, Z)}X - A^f_{\alpha_f(X, Z)}Y,
\] (2.4a)
\[
(\overline{\nabla}_X \alpha_f)(Y, Z) = (\overline{\nabla}_Y \alpha_f)(X, Z),
\] (2.4b)
\[
R^\perp(X, Y)\xi = \alpha_f(X, A^f_\xi Y) - \alpha_f(A^f_\xi X, Y),
\] (2.4c)
called Gauss, Codazzi and Ricci equations, respectively, where, by the definition, we have
\[
(X \wedge Y)Z = \langle Y, Z \rangle X - \langle X, Z \rangle Y,
\]
\[
(\overline{\nabla}_X \alpha_f)(Y, Z) = \nabla^\perp_X \alpha_f(Y, Z) - \alpha_f(\nabla_X Y, Z) - \alpha_f(Y, \nabla_X Z).
\]

The mean curvature vector field of the isometric immersion $f$ is defined by
\[
H^f = \frac{1}{k} \text{trace } \alpha_f.
\] (2.5)

$f$ is said to be quasi-minimal if $H^f$ is light-like at every point of $\Omega$, i.e., $\langle H^f, H^f \rangle = 0$ and $H^f \neq 0$. In this case, $M = f(\Omega)$ is called a quasi-minimal submanifold (quasi-minimal surface if $k = 2$) of $R^n_2(c)$.

Furthermore, we are going to denote the kernel of the shape operator along $H^f$ by $T^f$, that is
\[
T^f = \{ X \in TM | A^f_{H^f}(X) = 0 \}.
\]

2.1. Lorentzian Surfaces in $R^4_2(c)$

Let $(\Omega, g)$ be a 2-dimensional semi-Riemannian manifold. Consider an isometric immersion $f : (\Omega, g) \hookrightarrow R^4_2(c)$ and let the surface $M$ be the image of $f$, i.e., $M = f(\Omega)$. Then, the Gaussian curvature $K$ of $\Omega$ is defined by
\[
K = \frac{R(X, Y, Y, X)}{\langle f_\ast X, f_\ast X \rangle \langle f_\ast Y, f_\ast Y \rangle - \langle f_\ast X, f_\ast Y \rangle^2},
\] (2.6)
where $X$ and $Y$ span the tangent bundle of $\Omega$. $\Omega$, and thus $M$, is said to be flat if $K$ vanishes identically.

If $g$ has index 1, then $M$ is said to be a Lorentzian surface. In this case, for any $p \in \Omega$ there exists a local coordinates system $(N_p, (u_1, u_2))$, called isothermal coordinate system of $\Omega$, such that $p \in N_p$ and
\[
g|_{N_p} = \tilde{m}^2(u_1, u_2)(du_1 \otimes du_1 - du_2 \otimes du_2)
\]
for a positive function $\tilde{m} \in C^\infty(\Omega)$. By defining a new local coordinate system $(u, v)$ by $u = \frac{u_1 + u_2}{\sqrt{2}}$ and $v = \frac{u_1 - u_2}{\sqrt{2}}$, we obtain [3]
\[
g|_{N_p} = -\tilde{m}^2(u, v)(du \otimes dv + dv \otimes du).
It is well-known that a light-like vector \( w \) tangent to \( M \) is proportional to either \( f_u = df(\partial_u) \) or \( f_v = df(\partial_v) \).

Note that the light-like curves \( u = \text{const} \) and \( v = \text{const} \) are pre-geodesics of \( M \). In other words, there exists a re-parametrization of the curve \( u = c_1 \) (or \( v = c_2 \)) which is a geodesic of \( M \). Therefore, by defining a new local coordinate system \((s, t)\) on \( M \) by

\[
s = s(u, v) = \int_{u_0}^u \tilde{m}^2(\xi, v)d\xi, \quad t = v
\]

and letting \( m(u, v) = \frac{\partial}{\partial v} \left( \int_{u_0}^u \tilde{m}^2(\xi, v)d\xi \right) \), we obtain a semi-geodesic coordinate system on \( M \) (see, for example, [8]).

**Proposition 2.2.** [8] Let \( M \) be a Lorentzian surface with the metric tensor \( g \). Then, there exists a local coordinate system \((s, t)\), such that

\[
g = g_m := -(ds \otimes dt + dt \otimes ds) + 2mdt \otimes dt. \tag{2.7}
\]

Furthermore, the Levi-Civita connection of \( M \) satisfies

\[
\nabla_{\partial_s} \partial_s = 0, \\
\nabla_{\partial_s} \partial_t = \nabla_{\partial_t} \partial_s = -m_s \partial_s, \\
\nabla_{\partial_t} \partial_t = m_s \partial_t + (2mm_s - m_t) \partial_s
\]

and the Gaussian curvature of \( M \) is

\[
K = m_{ss}. \tag{2.8}
\]

### 2.2. Biharmonic Immersions

First, we would like to recall a necessary and sufficient condition for an isometric immersion to be biharmonic. In this case by splitting \( \tau_2(f) \) into its normal and tangential part and employing (1.2), one can obtain the following well-known result (see, for example, [10, 24, 25]).

**Proposition 2.3.** An isometric immersion \( f : (\Omega, g) \hookrightarrow (N, \tilde{g}) \) is biharmonic if and only if the equations

\[
k\text{grad} \left( \tilde{g}(H^f, H^f) \right) + 4\text{trace} A^f_{\nabla H^f} (\cdot) + 4\text{trace} (\tilde{R}(\cdot, H^f) \cdot) \cdot)^T = 0 \tag{2.9}
\]

and

\[
-\Delta^\perp H^f + \text{trace} \alpha_f (A^f_{H^f} (\cdot), \cdot) + \text{trace} (\tilde{R}(\cdot, H^f) \cdot)^\perp = 0 \tag{2.10}
\]

are satisfied, where \( k \) is the dimension of \( \Omega \), \( \Delta^\perp \) denote the Laplace operator associated with the normal connection of \( f \).

On the other hand, if \( \psi = f \) is an isometric immersion, then (1.3) is equivalent to (\( \tau_2(f) \))^T = 0. Therefore, using Proposition 2.3, we have

**Proposition 2.4.** An isometric immersion \( f : (\Omega, g) \hookrightarrow (N, \tilde{g}) \) between semi-Riemannian manifolds is biconservative if and only if the Eq. (2.9) is satisfied.

We immediately have the following result of Proposition 2.4 for the case \((N, \tilde{g}) = R^n_s(c)\).
Corollary 2.5. An isometric immersion $f : (\Omega, g) \hookrightarrow R^n_s(c)$ is biconservative if its mean curvature vector is parallel on the normal bundle.

Remark 2.6. Because of Corollary 2.5, we are going to call a biconservative isometric immersion $f$ from $(\Omega, g)$ into $R^n_s(c)$ as proper if $\nabla^\bot Hf \neq 0$ at any point of $\Omega$. Moreover, we would like to refer to [5] for the classification of quasi-minimal surfaces with parallel mean curvature vector in $\mathbb{E}^4_2$ (see also [16]).

3. Biconservative Immersions into Space Forms of Index 2

In this section, we consider quasi-minimal biconservative immersions into $R^2_s(c)$ for $c \in \{-1, 0, 1\}$. Consider a 2-dimensional semi-Riemannian manifold $(\Omega, g)$, where $g$ is a Lorentzian metric. Let $f : (\Omega, g) \hookrightarrow R^2_s(c)$ be a quasi-minimal isometric immersion and put $M = f(\Omega)$.

We choose two vector fields $e_1, e_2$ tangent to $M$ such that $\langle e_i, e_j \rangle = 1 - \delta_{ij}$, $i, j = 1, 2$. Then, there exist smooth functions $\phi_1, \phi_2$, such that

$$\nabla_{e_i} e_1 = \phi_i e_1, \quad (3.1a)$$
$$\nabla_{e_i} e_2 = -\phi_i e_2. \quad (3.1b)$$

Put $e_3 = -Hf \in Nf \Omega$ and let $e_4 \in Nf \Omega$ be the unique light-like vector field satisfying $\langle e_3, e_4 \rangle = -1$. On the other hand, if we define smooth functions $h^\alpha_{ij}$ by

$$h^\alpha_{ij} = \langle \alpha_f(e_i, e_j), e_\alpha \rangle, \quad i, j = 1, 2, \quad \alpha = 3, 4,$$

then we get

$$\alpha_f(e_i, e_i) = -h^4_{ii} e_3 - h^3_{ii} e_4, \quad (3.2a)$$
$$\alpha_f(e_1, e_2) = e_3, \quad (3.2b)$$

where (3.2b) follows from $Hf = -\alpha_f(e_1, e_2)$. Note that we also have $h^\alpha_{ij} = \langle A^f_{i\alpha} e_i, e_j \rangle$ because of (2.3). Therefore, the shape operators of $f$ satisfy

$$A^f_{i3} e_1 = -h^3_{11} e_2, \quad A^f_{i3} e_2 = -h^3_{22} e_1, \quad (3.3a)$$
$$A^f_{i4} e_1 = e_1 - h^4_{11} e_2, \quad A^f_{i4} e_2 = -h^4_{22} e_1 + e_2. \quad (3.3b)$$

On the other hand, the Laplace operator $\Delta^\bot$ associated with the normal connection of $f$ takes the form:

$$\Delta^\bot = \nabla^\bot_{e_1} \nabla^\bot_{e_2} - \nabla^\bot_{e_1} e_2 + \nabla^\bot_{e_2} \nabla^\bot_{e_1} - \nabla^\bot_{e_2} e_1.$$

Furthermore, one can define smooth functions $\xi_1, \xi_2$ by

$$\nabla^\bot_{e_1} e_3 = \xi_1 e_3 \quad \text{and} \quad \nabla^\bot_{e_1} e_4 = -\xi_1 e_4. \quad (3.4)$$

We obtain the following characterization of proper biconservative immersions.

Proposition 3.1. Let $(\Omega, g)$ be a 2-dimensional semi-Riemannian manifold and $f : (\Omega, g) \hookrightarrow R^2_s(c)$ a quasi-minimal isometric immersion. Then, $f$ is proper biconservative if and only if for any point $p$, such that $A^f_H(p) \neq 0$, ...
there exists a neighborhood $N_p$ such that $T^F$ is a degenerated distribution along which $H^F$ is parallel, where $F = f|_{N_p}$.

**Proof.** Since the ambient space is $R_2^4(c)$, we have trace $\left( R(\cdot, H^f) \cdot \right)^T = 0$. Furthermore, being quasi-minimal of $f$ implies $\nabla g(H^f, H^f) = 0$. Therefore, $f$ is biconservative if and only if

$$\text{trace} A_{\nabla^\perp H^f}(\cdot) = 0$$

(3.5)

because of Proposition 2.4. Note that (3.5) is equivalent to

$$A_{\nabla^\perp e_1 e_2}^f(e_2) + A_{\nabla^\perp e_2 e_3}^f(e_1) = 0$$

in terms of vector fields $e_1, e_2, e_3$ defined above. By considering (3.3a) and (3.4), we conclude that $f$ is biconservative if and only if

$$\xi_1 h^3_{22} e_1 + \xi_2 h^3_{11} e_2 = 0.$$  

(3.6)

Note that being proper of the biconservative immersion $f$ implies $\xi_1(q) \neq 0$ or $\xi_2(q) \neq 0$ at any point $q \in \Omega$.

Now, to prove the necessary condition, assume that $f$ is a proper biconservative immersion and let $A_{H^f}(p) \neq 0$ at a point $p$ of $\Omega$. Then, without loss of generality, we may assume $h^3_{22}(p) \neq 0$ on a neighbourhood $N_p$ of $\Omega$. In this case, because of (3.6), we have $\xi_1 = 0$ on $N_p$ which implies $\xi_2(q) \neq 0$ for any $q \in N_p$. Thus, (3.6) implies $h^3_{11} = 0$ on $N_p$. Put $F = f|_{N_p}$. Then, we have $T^F = \text{span} \{e_1\}$ which is a degenerated distribution. Moreover, since $\xi_1$ vanishes identically on $N_p$, we have $\nabla^\perp_X H^F = 0$ whenever $X \in T^F$. Hence, we have completed the proof of the necessary condition.

For the proof of the sufficiency condition, we consider the following two cases separately. If $A_{H^f}^f = 0$, then (3.3a) implies $h^3_{11} = h^3_{22} = 0$. Therefore (3.6) is satisfied. On the other hand, consider the case $A_{H^f}^f(p) \neq 0$ on $N_p$. Assume $T^F = \text{span} \{e_1\}$ for a light-like vector field $e_1$ and let $\nabla^\perp_{e_1} e_3 = 0$. Then, we have $h^3_{11} = \xi_1 = 0$. Therefore (3.6) is satisfied again. Hence, the proof of the sufficiency condition is completed. $\square$

Now, we study the case $c = 0$. Let $f : (\Omega, g_m) \hookrightarrow \mathbb{E}_2^4$ be a proper biconservative quasi-minimal immersion, where $\Omega = I \times J$ and $g_m$ is the metric defined by (2.7) for a $m \in C^\infty(\Omega)$. Assume that the Gaussian curvature $K$ of $\Omega$ does not vanish. Note that, because of the Gauss equation (2.4a), if $A_{H^f}^f = 0$ at a point $p \in \Omega$, then the Gaussian curvature $K(p) = 0$ which is a contradiction. Therefore, we have $A_{H^f}^f(q) \neq 0$ for all $q \in \Omega$. Hence, Proposition 3.1 implies that $T^f$ is a degenerated distribution along which $H^f$ is parallel. In terms of a local pseudo-orthonormal frame field $\{e_1, e_2; e_3, e_4\}$, we have $A_{e_3}^f e_1 = 0$ and $\nabla^\perp_{e_1} e_3 = 0$, or, equivalently, $h^3_{11} = \xi_1 = 0$. Now, $f$ is biharmonic if and only if (2.10) is satisfied. However, since $\tilde{R} = 0$, (2.10) becomes

$$\nabla^\perp_{e_1} \nabla^\perp_{e_2} e_3 - \nabla^\perp_{e_3} e_2 e_3 = \alpha f(A_{e_3}^f e_2, e_1)$$

which is equivalent to the Ricci equation (2.4c) for $X = e_1, Y = e_2$ and $\xi = e_3$. Hence, we have the following result.
Theorem 3.2. Let \((\Omega, g)\) be a Lorentzian surface with the Gaussian curvature \(K\) and \(f : (\Omega, g) \hookrightarrow \mathbb{E}^4_2\) a quasi-minimal isometric immersion. Assume that \(K\) does not vanish. If \(f\) is a proper biconservative immersion, then it is biharmonic.

4. Biconservative Surfaces in \(\mathbb{E}^4_2\)

In this section, we focus on immersions into the pseudo-Euclidean space \(\mathbb{E}^4_2\) with neutral metric. We get the complete local classification of quasi-minimal, biconservative surfaces.

First, we consider flat surfaces and get the following classification of biconservative surfaces. We want to note that the proof of this proposition immediately follows from the proof of [4, Theorem 4.1].

Proposition 4.1. A flat surface in \(\mathbb{E}^4_2\) is quasi-minimal and biconservative if and only if locally congruent to one of the following surfaces:

(i) The surface given by \(f(s, t) = (\psi(s, t), s-t\sqrt{2}, s+t\sqrt{2}, \psi(s, t))\), \((s, t) \in U\), where \(\psi : U \rightarrow \mathbb{R}\) is a smooth function and \(U\) is open in \(\mathbb{R}^2\).

(ii) The surface given by \(f(s, t) = z(s)t + w(s)\), where \(z(s)\) is a light-like curve in the light-cone \(LC\) and \(w\) is a light-like curve satisfying \(\langle z', w' \rangle = 0\) and \(\langle z, w' \rangle = -1\).

Proof. A direct computation shows that the above surfaces are flat, quasi-minimal and biconservative. Conversely, assume that \(M\) is a flat quasi-minimal biconservative surface and \(p \in M\). Consider a frame field \(\{e_1, e_2; e_3, e_4\}\) described in Sect. 3 and let \(h^0_{ii}\) are functions defined by (3.2) and (3.4), respectively. Then, by Proposition 3.1, we have two cases: \(h^0_{31} = h^0_{32} = 0\) and \(h^0_{11} = h^0_{11} = 0\). By considering the proof of [4, Theorem 4.1], one can conclude that \(f\) is congruent to one of these two surfaces given in the proposition. □

Now, we are going to consider non-flat quasi-minimal surfaces with non-parallel mean curvature vector. First we define an intrinsic \(L : \Omega \rightarrow \mathbb{R}\) of \((\Omega, g_m)\) by

\[
L = -\frac{K_t + mK_s + 3m_sK}{K}. \tag{4.1}
\]

Then, we construct the following example of biconservative immersion from a non-flat two-dimensional Lorentzian manifold into \(\mathbb{E}^4_2\). We would like to note that this immersion is also biharmonic because of Theorem 3.2.

Proposition 4.2. Let \(\Omega = I \times J\) for some open intervals \(I, J\) and \(m \in C^\infty(\Omega)\) and assume that the intrinsic \(L : \Omega \rightarrow \mathbb{R}\) of \((\Omega, g_m)\) satisfies \(L = L(t)\). Consider a light-like curve \(\alpha : J \hookrightarrow \mathbb{E}^4_2\) lying on \(LC\), such that \(V_t = \text{span} \{\alpha(t), \alpha'(t)\}\) is two dimensional for all \(t \in J\). Assume that \(\eta : J \rightarrow \mathbb{R}^4\) satisfies the
conditions:
\[ \langle \eta', \eta' \rangle = 0, \tag{4.2a} \]
\[ \langle \alpha, \eta' \rangle = 0, \tag{4.2b} \]
\[ \langle \eta', \alpha' \rangle = -\frac{1}{a}, \tag{4.2c} \]
\[ \langle \eta', \alpha'' \rangle = \frac{2a' - aL}{a^2} \tag{4.2d} \]
for a function \( a \in C^\infty(J) \). Then, the mapping
\[
f : (\Omega, g_m) \longrightarrow \mathbb{E}_2^4
\]
\[
f(s, t) = \eta(t) + (sa'(t) - a(t)(m(s, t) + sL(t)))\alpha(t)
\]
\[ + sa(t)\alpha'(t) \tag{4.3} \]
is a quasi-minimal, proper biconservative isometric immersion.

**Proof.** Since the light-like curve \( \alpha \) lies on \( \mathcal{L}C \), we have
\[
\langle \alpha, \alpha \rangle = \langle \alpha', \alpha' \rangle = 0 \tag{4.4}
\]
which implies \( \langle \cdot, \cdot \rangle |_{V_1} = 0 \). Thus, we have \( V_t \subset V_t^\perp \). Therefore, Lemma 2.1 implies \( V_t = V_t^\perp \). Note that (4.4) also gives \( \alpha''(t) \in V_t^\perp = V_t \). Thus, we have
\[
\alpha''(t) = A(t)\alpha(t) + \frac{a(t)L(t) - 2a'(t)}{a(t)}\alpha' \tag{4.5}
\]
for a smooth function \( A \) because of (4.2b)–(4.2d). By a direct computation considering (4.3) and (4.5) we obtain \( \langle f_s, f_s \rangle = 0, \langle f_s, f_t \rangle = -1 \) and \( \langle f_t, f_t \rangle = 2m \) which yields that \( f \) is an isometric immersion.

By a further computation, we get
\[
e_3 = \alpha_f(e_1, e_2) = -H^f = B(t)\alpha(t) \quad \text{and} \quad \alpha_f(e_1, e_1) = a(t)m_{ss}(s, t)\alpha(t).
\]
Therefore, we have \( h^3_{11} = 0 \) and \( \nabla^\perp_{e_3} e_3 = 0 \) which yields that \( T^f = \text{span} \{\partial_s\} \) and \( H^f \) is parallel along \( T^f \). Hence, Proposition 3.1 yields that \( f \) is biconservative. \( \square \)

Before we continue, we want to present an explicit example of a surface obtain in [23]. This surface belongs to the class of biconservative quasi-minimal surfaces in \( \mathbb{E}_2^4 \) described in Proposition 4.2.

**Example.** [23] Let \( M \) be the surface
\[
f : (0, \infty) \times (0, 2\pi) \hookrightarrow \mathbb{E}_2^4,
\]
\[
f(s, t) = \left(-s \sin t - 4\sqrt{s} \cos t + \frac{\cos t}{2}, 4\sqrt{s} \sin t - s \cos t - \frac{\sin t}{2},
\right.
\]
\[
4\sqrt{s} \sin t - s \cos t + \frac{\sin t}{2}, -s \sin t - 4\sqrt{s} \cos t - \frac{\cos t}{2}\right)
\]
obtained by putting \( a(t) = -1, m(s, t) = 4\sqrt{s}, \alpha(t) = (-\cos t, \sin t, \sin t, -\cos t) \) and \( \eta(t) = \frac{1}{2} (\cos t, -\sin t, \sin t, -\cos t) \) in Proposition 4.2. \( M \) is quasi-minimal and biconservative. Moreover, the Gaussian curvature of \( M \) is \( K = -s^{-3/2} \) and we have \( L = 0 \). Therefore, \( M \) is also biharmonic because of Theorem 3.2.
Lemma 4.3. Let \( f : (\Omega, g_m) \hookrightarrow \mathbb{E}^4_2 \) be a quasi-minimal immersion and assume that the Gaussian curvature of \((\Omega, g_m)\) does not vanish. Consider the pseudo-orthonormal frame field \( \{e_1, e_2; e_3, e_4\} \), such that \( e_1 = f_s \partial_s \), \( e_2 = f_s (m \partial_s + \partial_t) \) and \( e_3 = -H f \). If \( f \) is proper biconservative and \( T^f = \text{span} \{\partial_s\} \), then the Levi Civita connection of \( \mathbb{E}^4_2 \) satisfies

\[
\nabla_{e_1} e_1 = -am_{ss} e_3, \quad \nabla_{e_1} e_2 = -m_s e_1 + e_3, \quad \nabla_{e_1} e_3 = 0, \quad \nabla_{e_1} e_4 = -e_1 + am_{ss} e_2, \quad \nabla_{e_2} e_2 = m_s e_2 + (m + bs - z)e_3 - \frac{1}{a} e_4, \quad \nabla_{e_2} e_3 = \frac{1}{a} e_1 + (m_s + b) e_3, \quad \nabla_{e_2} e_4 = (z - m - bs)e_1 - e_2 - (m_s + b) e_4 \tag{4.6a,b,c,d}
\]

for some smooth functions \( a, b, z \) such that \( e_1(a) = e_1(b) = e_1(z) = 0 \) and

\[
b + \frac{a'}{a} = L. \tag{4.6d}
\]

Proof. Assume that \( f \) is proper biconservative. Then, we have \( h^4_{11} = \xi_1 = 0 \) because of Proposition 3.1. Thus, we have

\[
\begin{align*}
\nabla_{e_1} e_1 &= \phi_1 e_1 - h^4_{11} e_3, \quad \nabla_{e_1} e_2 = \phi_2 e_1 + e_3, \\
\nabla_{e_1} e_2 &= -\phi_1 e_2 + e_3, \quad \nabla_{e_2} e_2 = -\phi_2 e_2 - h^4_{22} e_3 - h^3_{22} e_4, \\
\nabla_{e_1} e_3 &= 0, \quad \nabla_{e_2} e_3 = h^3_{22} e_1 + \xi_2 e_3, \\
\nabla_{e_1} e_4 &= -e_1 + h^4_{11} e_2, \quad \nabla_{e_2} e_4 = h^4_{22} e_1 - e_2 - \xi_2 e_4. \tag{4.7}
\end{align*}
\]

Note that Proposition 2.2 implies

\[
\phi_1 = 0 \quad \text{and} \quad \phi_2 = -m_s. \tag{4.8}
\]

On the other hand, Codazzi equation (2.4b) for \( X = Z = e_2, \ Y = e_1 \) gives

\[
e_1(h^4_{22}) = -\xi_2, \quad e_1(h^3_{22}) = 0, \tag{4.9}
\]

Furthermore, by using Gauss equation (2.4a) and Ricci equation (2.4c), we get

\[
\begin{align*}
h^4_{11} h^3_{22} &= K = m_{ss} \tag{4.10} \\
e_1(\xi_2) &= K = m_{ss}. \tag{4.11}
\end{align*}
\]

By taking into account \( e_1 = f_s \partial_s, \ e_2 = f_s (m \partial_s + \partial_t) \), we consider (4.9), (4.10) and (4.11) to get

\[
\begin{align*}
h^3_{22}(s,t) &= \frac{1}{a(t)}, \quad h^4_{11}(s,t) = a(t)m_{ss}(s,t), \\
h^4_{22}(s,t) &= -m(s,t) - b(t)s + z(t), \quad \xi_2(s,t) = m_s(s,t) + b(t)
\end{align*} \tag{4.12}
\]

for some \( a, b, z \in C^\infty(J) \). Finally, by combining (4.8) and (4.12) with (4.7), we obtain (4.6).
On the other hand, Codazzi equation (2.4b) for $X = Z = e_1, Y = e_2$ gives

$$K(a' + ab) + (K_t + mK_s + 3msK)a = 0.$$ 

By combining this equation with (4.1), we get (4.6d). □

Next, we get a condition for the existence of biconservative immersions from a Lorentzian surface $(\Omega, g_m)$.

**Proposition 4.4.** Let $m \in C^\infty(\Omega)$ and $\Omega = I \times J$ for some open intervals $I, J$ and consider the Lorentzian surface $(\Omega, g_m)$ with non-vanishing Gaussian curvature, where $g_m$ is the metric defined by (2.7). Then, $(\Omega, g_m)$ admits a quasi-minimal, proper biconservative isometric immersion with non-parallel mean curvature vector such that $T^f = \text{span} \{ \partial_s \}$ if and only if $L = L(t)$.

**Proof.** To prove necessary condition, we assume the existence of such immersion $f$. Then, the Levi-Civita connection $\tilde{\nabla}$ satisfies (4.6) because of Lemma 4.3. (4.6d) yields $\partial_s(L) = 0$. Conversely, if $L = L(t)$, the immersion $f$ described by (4.3) is proper biconservative by Proposition 4.2. Hence, the proof is completed. □

**Theorem 4.5.** If $M$ is a proper biconservative, quasi-minimal surface with non-vanishing Gaussian curvature, then it is locally congruent to the image $f(\Omega)$ of the isometric immersion $f$ given in Proposition 4.2.

**Proof.** Let $(\Omega, g_m)$ has non-vanishing Gaussian curvature. Consider a quasi-minimal isometric immersion $f : (\Omega, g_m) \hookrightarrow R_2(c)$ and put $M = f(\Omega)$. Assume that $f$ is proper biconservative. Then, $\tilde{\nabla}$ satisfies (4.6) because of Lemma 4.3. The first equation in (4.6c) gives $e_3(s, t) = \alpha(t)$ for a mapping $\alpha : J \to E_4$. Also the second equation in (4.6c) and (4.13) give

$$\alpha'(t) = \frac{1}{a(t)}f_s(s, t) + (m_s(s, t) + b(t))\alpha(t).$$

By considering (4.13) and (4.14) one can see that $\alpha$ is a light-like curve lying on $\mathcal{LC}$ because $K$ does not vanish.

On the other hand, the first equation in (4.6a) turns into

$$f_{ss}(s, t) = -a(t)m_{ss}(s, t)\alpha(t)$$

whose solution is

$$f(s, t) = -a(t)m(s, t)\alpha(t) + s\xi(t) + \eta(t)$$

for some functions $\xi, \eta : J \to E_4$. By using (4.15) and considering (4.6d) in this equation, we obtain

$$\xi = (a' - aL)\alpha + a\alpha'.$$

By combining (4.15) and (4.16), we get (4.3).
Now, since \( f \) is an isometric immersion, we have \( \langle f_s, f_t \rangle = -1 \) and \( \langle f_t, f_t \rangle = 2m \). By a direct computation using \( \langle f_s, f_t \rangle = -1 \) and (4.3), we obtain

\[
-am_s \langle \alpha, \eta' \rangle + a' \langle \alpha, \eta' \rangle - aL \langle \alpha, \eta' \rangle + a \langle \alpha', \eta' \rangle = -1
\]

which gives the first equation in (4.2a) and (4.2c) because \( K = m_{ss} \) does not vanish. On the other hand, \( \langle f_t, f_t \rangle = 2m \) and (4.3) imply

\[
2m + 2s \left( -\frac{a'}{a} + L + a \langle \alpha'', \eta' \rangle \right) + \langle \eta', \eta' \rangle = 2m
\]

which gives the second equation in (4.2b) and (4.2d). Hence \( f \) is as given in Proposition 4.2 which completes the proof.

By combining Proposition 4.1 and Theorem 4.5, we obtain the following complete classification of quasi-minimal, proper biconservative surfaces in \( \mathbb{E}_2^4 \).

**Theorem 4.6.** A surface \( M \) in \( \mathbb{E}_2^4 \) is quasi-minimal and proper biconservative if and only if it is congruent to one of the following surfaces:

(i) The surface given by \( f(s, t) = (\psi(s, t), \frac{s-t}{\sqrt{2}}, \frac{s+t}{\sqrt{2}}, \psi(s, t)) \), \((s, t) \in U\), where \( \psi : U \rightarrow \mathbb{R} \) is a smooth function and \( U \) is open in \( \mathbb{R}^2 \),

(ii) The surface given by \( f(s, t) = z(s)t + w(s) \), where \( z(s) \) is a light-like curve in the light-cone \( \mathcal{LC} \) and \( w \) is a light-like curve satisfying \( \langle z', w' \rangle = 0 \) and \( \langle z, w' \rangle = -1 \),

(iii) The surface given in Proposition 4.2.

Finally, by combining [4, Theorem 5.1] with Theorems 3.2 and 4.5, we get

**Theorem 4.7.** A surface \( M \) in \( \mathbb{E}_2^4 \) is quasi-minimal and biharmonic if and only if it is congruent to one of the following surfaces:

(i) The surface given by \( f(s, t) = (\psi(s, t), \frac{s-t}{\sqrt{2}}, \frac{s+t}{\sqrt{2}}, \psi(s, t)) \), \((s, t) \in U\), for a smooth function \( \psi : U \rightarrow \mathbb{R} \) satisfying \( f_{st} \neq 0 \) and \( f_{sstt} = 0 \), where \( U \) is open in \( \mathbb{R}^2 \),

(ii) The surface given by \( f(s, t) = z(s)t + w(s) \), where \( z(s) \) is a light-like curve in the light-cone \( \mathcal{LC} \) and \( w \) is a light-like curve satisfying \( \langle z', w' \rangle = 0 \) and \( \langle z, w' \rangle = -1 \),

(iii) The surface given in Proposition 4.2.

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