Abstract

The dynamical structure factor $S(q, \omega)$ of the $SU(K)$ ($K = 2, 3, 4$) Haldane-Shastry model is derived exactly at zero temperature for arbitrary size of the system. The result is interpreted in terms of free quasi-particles which are generalization of spinons in the $SU(2)$ case; the excited states relevant to $S(q, \omega)$ consist of $K$ quasi-particles each of which is characterized by a set of $K - 1$ quantum numbers. Near the boundaries of the region where $S(q, \omega)$ is nonzero, $S(q, \omega)$ shows the power-law singularity. It is found that the divergent singularity occurs only in the lowest edges starting from $(q, \omega) = (0, 0)$ toward positive and negative $q$. The analytic result is checked numerically for finite systems via exact diagonalization and recursion methods.

1 Introduction

In recent years, there has been a renewal of interest in study of the spin chains which describe the systems with the orbital degeneracy. In the case of two-fold orbital degeneracy, the total degeneracy per site becomes $4 (= 2 \times 2)$, and the simplest model to realize this situation in one dimension is a chain with the $SU(4)$ symmetry ($SU(4)$ spin-orbital model) $\text{%cite 4, 5, 6, 7, 8, 9, 10}%$. The static property of the $SU(4)$ spin-orbital model has been studied mainly by numerical methods. It has been reported that the spin correlation has a period of four unit cells, and that the asymptotic decay has a power-law exponent different from unity $\text{%cite 3, 5}%$. Such exponent has also been derived by use of conformal field theory $\text{%cite 11}%$.

In addition to these static properties, dynamical information of the system has become more and more important partly because experimental investigation for dynamical properties has been performed with increasing accuracy. In particular experiments of some orbitally degenerate quasi-one-dimensional magnetic compounds such as NaV$_2$O$_5$ $\text{%cite 2}%$ are intensively performed in the last few years. Hence it is useful to clarify the difference from systems without orbital degeneracy not only for static properties, but also for dynamic ones.

Although several one-dimensional quantum spin models have been exactly solved, much less is known about their dynamical properties over the whole energy range. An exception is the XX model $\text{%cite 13}%$, which can be mapped to a spinless fermion model. For other models dynamics has been discussed mainly by bosonization method which works only in the low-frequency region. Recently, some progress has been achieved in obtaining exact results for a wider energy range for the antiferromagnetic Heisenberg model $\text{%cite 14, 15%}$ and the XXZ model in antiferromagnetically ordered phase $\text{%cite 16%}$. In their articles $\text{%cite 14, 15, 16%}$, Bougourzi et al. derived two-spinon contribution to dynamical structure factors $S(q, \omega)$ for the Heisenberg and XXZ models in the thermodynamic limit by using the results of mathematical work $\text{%cite 17%}$. For the Heisenberg model, they obtained that the two-spinon excitations account for 72.89% of the total intensity.
Also they showed that the exact result and the M"uller ansatz for $S(q, \omega)$ have the same singularity at the lower spectral boundary, which is called the des Cloizeaux-Pearson mode.

On the other hand, in Ref. [19], Haldane and Zirnbauer have obtained exact expression of $S(q, \omega)$ at zero temperature for the Haldane-Shastry (HS) model, which is a Heisenberg type antiferromagnetic chain with inverse-square interaction [20, 21]. They used the supermatrix method [22] which relies on close correspondence between the HS model and spinless Calogero-Sutherland (CS) model [23, 24] with special coupling parameter. The most remarkable feature of their results is that only two-spinon excitations contribute to $S(q, \omega)$. Also, the form of $S(q, \omega)$ is regarded as a variant of the M"uller ansatz form with quadratic dispersion of spinons. Since the HS model can be regarded as the model of free spinon gas [25], their results provide clear physical interpretation of the M"uller ansatz.

With growing interest in orbitally degenerate systems as well as progress in exact dynamical theory, one can naturally ask for example how the dynamical property depends on the number of internal degrees of freedom. This problem has been left as a theoretical challenge. In this paper, we report on exact dynamics of the $SU(K)$ HS model ($K \geq 3$) in the whole energy and momentum region. We derive exact formulae for the dynamical structure factor for arbitrary size of the system including the thermodynamic limit. We can construct the quasi-particle picture of the spin dynamics at zero temperature. In Ref. [29], we have briefly reported the results. We present here the details of the derivations. Our exact result is inconsistent with a conjecture proposed several years ago [30].

The $SU(K)$ HS model is described by the following Heisenberg-type Hamiltonian with the periodic boundary condition [27, 28]:

$$H_{\text{HS}} := \frac{1}{2} \sum_{1 \leq i < j \leq N} J_{ij} P_{ij}. \quad (1)$$

Here the exchange interaction $J_{ij}$ is given by the following inverse-square type:

$$J_{ij} := J \left[ \frac{N}{\pi} \sin \frac{\pi (i - j)}{N} \right]^{-2}, \quad (2)$$

where $N$ is the number of lattice sites with unit spacing and $J > 0$. The Hilbert space of this model is $\Omega_N := \mathbb{C}^K \otimes \cdots \otimes \mathbb{C}^K$, and the linear operator $P_{ij}$ is the color exchange operator which exchanges the $i$-th element and $j$-th element of vectors in $\Omega_N$. In the particular case of $SU(2)$, $P_{ij}$ is reduced to the spin exchange $2\hat{S}_i \cdot \hat{S}_j + 1/2$. In this paper, we assume that $N \equiv 0 \mod K$. Under this condition, there is the non-degenerate $SU(K)$ singlet groundstate of the system. The operator $P_{ij}$ can be written in the form:

$$P_{ij} = \sum_{\gamma, \delta = 1}^K X_{ij}^\gamma X_{ij}^\delta, \quad (3)$$

where $X_{ij}^\gamma$ with $\gamma, \delta \in \{1, 2, \cdots, K\} =: \Sigma_K$ changes the color $\delta$ to $\gamma$ at site $i$. The operators $X_{ij}^\gamma$ form a basis for the Lie algebra $u(K)$ of the unitary group $U(K)$ and satisfy the following relations:

$$[X_{ij}^\gamma, X_{jk}^\sigma] = \delta_{jk} (\delta_{\delta\rho} X_{ij}^\sigma - \delta_{\gamma\rho} X_{ij}^\delta). \quad (4)$$

For later use, we introduce the following operators:

$$\hat{X}_{ij}^\gamma := X_{ij}^\gamma - \delta_{\gamma 1} \frac{1}{K} \sum_{\rho = 1}^K X_{ij}^{\rho \rho} \quad (5)$$

They satisfy the same commutation relations as in (4) and also satisfy the conditions that

$$\sum_{\gamma = 1}^K \hat{X}_{ij}^\gamma = 0. \quad (6)$$
We will analyze the following quantity of the $SU(K)$ HS model at zero temperature:

$$S^{(\delta\gamma)(\rho\sigma)}_N(q, \omega) := \frac{1}{N} \sum_{r=1}^{N} \sum_{s=0}^{N-1} e^{-iqs} \int_{-\infty}^{\infty} dt e^{i\omega t} \langle 0 | \hat{X}^{\delta\gamma}_r(t) \hat{X}^{\rho\sigma}_{r+s}(0) | 0 \rangle,$$

$$= \sum_{\alpha} \langle 0 | \hat{X}^{\delta\gamma}_q(\alpha) \hat{X}^{\rho\sigma}_q | 0 \rangle \delta(\omega - E_{\alpha} + E_0),$$

(7)

where $\delta, \gamma, \rho, \sigma \in \Sigma_K$, $q = 2\pi n/N$ with $n \in \mathbb{Z}$, $\hat{X}^{\delta\gamma}_q(t)$ is the Heisenberg representation of the operator $\hat{X}^{\delta\gamma}_r$, and

$$\hat{X}^{\delta\gamma}_q := \frac{1}{\sqrt{N}} \sum_{l=1}^{N} \hat{X}_l^{\delta\gamma} e^{iql}.$$

(8)

Here $\{ |\alpha\rangle \}$ is the normalized complete basis of the Hamiltonian (1) with eigenvalues $\{E_{\alpha}\}$, and $|0\rangle$ is the groundstate. From the $SU(K)$ symmetry, the following formula holds:

$$S^{(\delta\gamma)(\rho\sigma)}_N(q, \omega) = \left( \delta_{\delta\gamma} \delta_{\rho\sigma} - \frac{1}{K} \delta_{\delta\gamma}, \delta_{\rho\sigma} \right) S_N(q, \omega),$$

(9)

where

$$S_N(q, \omega) := S^{(\delta\gamma)(\gamma\delta)}_N(q, \omega), \quad (\gamma \neq \delta).$$

(10)

Therefore it is sufficient to calculate the components $S^{(\delta\gamma)(\gamma\delta)}_N(q, \omega)$ with $\gamma \neq \delta$.

For deriving the exact formula for $S^{(\delta\gamma)(\gamma\delta)}_N(q, \omega)$ with $\delta \neq \gamma$, we adopt the freezing trick introduced in Ref. [31]. This method is based on the fact that the $SU(K)$ HS model is obtained by the strong coupling limit of the $U(K)$ spin CS model [28]. The latter continuous model is more tractable than the $SU(K)$ HS model, because the eigenfunctions of the model have been explicitly constructed [32, 33, 34]. If we have the exact dynamical results for the $U(K)$ spin CS model, then we can obtain $S^{(\delta\gamma)(\gamma\delta)}_N(q, \omega)$ of the $SU(K)$ HS model by applying the freezing trick.

Fortunately, there exists an excellent work by Uglov [33]. In the case of $K = 2$, he derived the exact expression of the dynamical spin-density correlation function of the spin CS model with a finite number of particles [33]. We extend his result to the case of $K \geq 3$, and then take the strong coupling limit.

The content of the paper is as follows: In section 2 we introduce the spin CS model and then explain the freezing trick in Ref. [31]. In section 3, we extend Uglov’s results on the correlation functions to the case of $K \geq 3$. Also, following Ref. [32], we simplify the conditions on the excited states of the dynamical correlation functions. In section 4, the dynamical structure factor for a finite system is exactly derived. We give the quasi-particle picture of the spin dynamics at zero temperature. We rewrite the resultant formula of the dynamical structure factor in terms of the momenta of quasi-particles. The relation between the momenta of quasi-particles and motif [30, 37] is briefly discussed. In section 5, we take the thermodynamic limit of the dynamical structure factor. We determine the support of the dynamical structure factor and analyze its behavior near the dispersion lines of elementary excitations. Summary and discussions are given in section 6. In Appendix A, for proving the statement in section 2, we analyze the strong coupling limit of the dynamical density correlation function of the $U(K)$ spin CS model. Appendices B, C, D and E contain the data and proofs which are used in the paper. In Appendix F, we derive the low energy asymptotic form of the dynamical correlation function and analyze the singularities of the dynamical structure factor.

## 2 Spin Calogero-Sutherland Model and Freezing Trick

In this section, we introduce the $U(K)$ spin CS model [28] and explain the freezing trick [31]. The $U(K)$ spin CS model describes $N$ particles with coordinates $x_1, \cdots, x_N$ moving along a circle of length $L$ and interacting with inverse-square type interactions. Moreover, each particle carries a color with $K$ possible values. The Hamiltonian of this model is given by [28]

$$H_{\text{spinCS}} := -\frac{1}{2} \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \left( \frac{\pi}{L} \right)^2 \sum_{1 \leq i < j \leq N} \frac{\beta(\beta + P_{ij})}{\sin^2 \frac{\pi}{L}(x_i - x_j)},$$

(11)
where $\beta > 0$ is the coupling parameter and $L$ is the size of system. The Hilbert space of the system is $C^\infty(\{x_1, \ldots, x_N\}) \otimes \Omega_N$. Technically, we assume that $N \equiv 0 \mod K$ and $N/K \equiv 1 \mod 2$.

In this case, the system has the non-degenerate $U(K)$ singlet groundstate. The spinless CS model is given by (11) where $P_{ij}$ has been set to 1, i.e., $K = 1$.

It is convenient to consider the following transformed Hamiltonian:

$$H_\beta := W^{-\beta} H_{\text{spin CS}} W^\beta,$$

where $W := \prod_{1 \leq j < k \leq N} \sin(\pi(x_j - x_k)/L)$. The Hilbert space of the system with the above Hamiltonian is

$$\mathcal{H} := \left(\mathbb{C}[z_1^{\pm1}, \ldots, z_N^{\pm1}] \otimes \Omega_N\right)_{\text{antisym}},$$

where $z_j := \exp(2\pi x_j/L)$, and “antisym” means total antisymmetrization. The inner product $(\cdot, \cdot)_\beta$ of this Hilbert space which depends on the coupling parameter $\beta$ is appropriately defined.

Next we explain the freezing trick. Let us consider the strong coupling limit $\beta \to \infty$ of the $U(K)$ spin CS model. In this case particles crystallize with the lattice parameter $L/N$ which is taken as the unit of length. Then we are left with the center of mass motion, the lattice vibration and the dynamics of the color. The color dynamics is equivalent to the dynamics of the $SU(K)$ HS model. The freezing trick described above was firstly introduced by Polychronakos, and has been applied to thermodynamics of lattice models. The present work is the first application of the freezing trick to dynamical quantities.

In the $U(2)$ spin CS model, Uglov has derived the exact formula of the dynamical spin-density correlation function with a finite number of particles. In the next section we extend his result to the case of $K \geq 3$, and then, in section 4, take the strong coupling limit. In doing so we have to make correspondence between physical quantities defined in the continuum and discrete spaces. In analogy to the operator $\tilde{X}_q^{\gamma \delta}$ defined by Eq. (8) for the lattice model, let us define the following operator in the continuum space:

$$\tilde{X}_q^{\gamma \delta} := \frac{1}{N^2} \sum_{j=1}^N X_j^{\gamma \delta} e^{-iqx_j},$$

where the momentum $q$ takes values $2\pi n/L$ with $n$ an arbitrary integer. We first derive the dynamical structure factor in the continuum model defined by

$$S_N^{(\gamma \delta) (\rho \sigma)}(q, \omega; \beta) := \sum_{\nu} \langle 0 | X_j^{\delta \gamma} | \nu \rangle \langle \nu | X_j^{\rho \sigma} | 0 \rangle \delta(\omega - E_\nu + E_0),$$

where $\{|\nu\rangle\}$ is the normalized complete basis of the Hamiltonian with eigenvalues $\{E_\nu\}$, and $|0\rangle$ is the groundstate.

In the strong coupling limit the coordinate $x_j$ in Eq. (14) is written as $x_j = R_j + u_j$ where $R_j = j$ is a lattice point, and $u_j$ describes the lattice vibration. Except for the uniform motion of the lattice described by $u_j = \text{const}$, we may regard $u_j$ as a small quantity. In fact the density response can be shown to be smaller than the spin response by $O(\beta^{-1})$. This fact is shown in Appendix A. Therefore the dynamical structure factor of the $SU(K)$ HS model is given simply by the strong coupling limit of Eq. (15) provided that one restricts $q$ in the range of the first Brillouin zone: $|q| \leq \pi$.

3 Dynamical Correlation Functions of the Spin Calogero-Sutherland Model

In Ref. [33], Uglov introduced the new formulation for the eigenvalue problem of the $U(K)$ spin CS model and derived the dynamical correlation functions in the case of $K = 2$. In this section we extend Uglov’s result on the dynamical correlation functions to the case of $K \geq 3$. First we recall his formulation briefly to establish the necessary terminology and notations. For details, see Ref. [33]. Then we give the dynamical correlation functions of the $U(K)$ spin CS model. We also simplify the conditions on the excited states relevant to the dynamical correlation functions.
3.1 Notations and review of Uglov’s results

In Ref. [32], Takemura and Uglov introduced the (un-normalized) complete orthogonal bases \( \{X^{(\beta,K)}_{\sigma+k(0)}\} \) of \( \mathcal{H} \) with respect to the inner product \((\bullet, \bullet)_{\beta}\) which is called the Yangian Gelfand-Zetlin basis and is parametrized by the decreasing sequence of integers \( \sigma = (\sigma_1, \cdots, \sigma_N) \). Here \( k(0) := (M, M-1, \cdots, M-N+1) \) with \( M := (N+K)/2 \in \mathbb{Z}_{0} \) is the parameter which corresponds to the groundstate of the system. (Notice that \( N \equiv 0 \mod K \) and \( N/K \equiv 1 \mod 2 \).) The label \( \sigma + k(0) \) of the state contains the information both for the momenta and for the colors of the particles [33]. That is, if we write \( \sigma + k(0) = k + Kk \) with \( k = (k_1, \cdots, k_N) \in (\Sigma_K)^N \) and \( Kk = (k_1, \cdots, k_N) \in \mathbb{Z}^N \), then \( k \) and \( Kk \) represent colors and momenta of particles, respectively.\(^2\)

Let us consider the vector space \( \mathcal{H}' := (C[y_1^{\pm 1}, \cdots, y_N^{\pm 1}])_{\text{symm}} \) of the symmetric Laurent polynomials in \( y = (y_1, \cdots, y_N) \). In Ref. [33], Uglov defined the (un-normalized) complete orthogonal bases \( \{J^{(K\beta+1,K)}_{\sigma}(y)\} \) of \( \mathcal{H}' \) with respect to a certain \( \beta \)-dependent inner product \((\bullet, \bullet)_{\beta}\). The function \( J^{(K\beta+1,K)}_{\sigma}(y) \) is defined by a certain degeneration of the Macdonald symmetric function [1] and is called the \( gl_K \)-Jack symmetric function. From this definition, several properties of the \( gl_K \)-Jack function can be derived from the corresponding properties of the Macdonald function. Moreover Uglov defined the isomorphism \( \Theta \) of Hilbert spaces \((\mathcal{H}, (\bullet, \bullet)_{\beta}) \) and \((\mathcal{H}', (\bullet, \bullet)_{\beta})\) such that \( \Theta(X^{(\beta,K)}_{\sigma+k(0)}(z)) = J^{(K\beta+1,K)}_{\sigma}(y) \) with the normalization \( \Theta(X^{(\beta,K)}_{k(0)}(z)) = 1 \) \((z = (z_1, \cdots, z_N))\). This fact enables one to calculate the correlation functions of the \( U(K) \) spin CS model by using the \( gl_K \)-Jack function. Actually, in the case of \( K = 2 \), Uglov exactly derived the dynamical density and spin-density correlation functions of the spin CS model. In the next subsection we extend his result on the correlation functions to the case of \( K \geq 3 \).

We recall here the norm formula of the Yangian Gelfand-Zetlin basis [33]. We also recall the spectral decomposition of the power sum symmetric function \( p_m(y) := \sum_{i=1}^N y_i^m \) with \( m \in \mathbb{Z}_{\geq 0} \) in terms of the \( gl_K \)-Jack polynomials [33] which is used in the next subsection. For this purpose, we fix some notations for partitions [33, 34, 11]. For a non-negative integer \( N \), let \( \Lambda_N := \{ \lambda = (\lambda_1, \lambda_2, \cdots, \lambda_N) \in (\mathbb{Z}_{\geq 0})^N \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \} \) be the set of all partitions with length less or equal to \( N \). A partition can be represented by a planar diagram, i.e., by the so-called Young diagram. When there is a box in the \( i \)-th row and \( j \)-th column of \( \lambda \), we write \((i,j) \in \lambda \). The conjugate of a partition \( \lambda = (\lambda_1, \lambda_2, \cdots, \lambda_N) \in \Lambda_N \) is the partition \( \lambda' = (\lambda'_1, \lambda'_2, \cdots, \lambda'_N) \in \Lambda_N \), whose diagram is the transpose of the diagram \( \lambda \). For instance, if \( \lambda = (4,3,1) \), then \( \lambda' = (3,2,2,1) \). For a box \( s = (i,j) \in \lambda \), the numbers \( a_{\lambda}(s) = \lambda_1 - j, a'_{\lambda}(s) = j - 1, l_{\lambda}(s) = l'_1 - i \) and \( l'_1(s) = i - 1 \) are called the arm-length, arm-colength, leg-length and leg-colength of box \( s \), respectively. We recall a coloring scheme of diagrams. In considering the \( U(K) \) symmetry, we need \( K \) colors. For a partition \( \lambda \), we define subsets of \( \lambda \) by \( C^{(a)}_{K}(\lambda) := \{ s \in \lambda \mid a_{\lambda}(s) = l_{\lambda}(s) + 1 \equiv a \mod K \} \). We call the color of box \( s \in \lambda \) \( a \) if \( s \in C^{(a)}_{K}(\lambda) \). Notice that \((1,1) \in C^{(0)}_{K}(\lambda) \) (if \( \lambda \neq 0 \)). For example, in the case of \( K = 3 \), if \( \lambda = (4,3,1) \) we have the colored diagram

\[
\begin{array}{ccc}
1 & 2 & 0 \\
2 & 1 & 1 \\
3 & & 4
\end{array}
\]

We also define the subsets of \( \lambda \) by \( H^{(a)}_{K}(\lambda) := \{ s \in \lambda \mid a_{\lambda}(s) + l_{\lambda}(s) + 1 \equiv a \mod K \} \). The quantity \( a_{\lambda}(s) + l_{\lambda}(s) + 1 \) for the box \( s \in \lambda \) is called the hook-length of box \( s \). Therefore the set \( H^{(a)}_{K}(\lambda) \) is the set which consists of the boxes in \( \lambda \) with the hook-length \( a \) modulo \( K \). For instance, in the \( K = 3 \) case, the modulo \( K \) hook-length of each box in the partition \( \lambda = (4,3,1) \) is given by

\[
\begin{array}{ccc}
1 & 2 & 0 \\
2 & 1 & 1 \\
3 & & 4
\end{array}
\]

For any subset \( \nu \subseteq \lambda \) the order \(|\nu|\) is defined as the number of boxes in \( \nu \), i.e., for example, \(|C^{(a)}_{K}(\lambda)|\) is the number of boxes with color \( \alpha \) in \( \lambda \).

The norm of the Yangian Gelfand-Zetlin basis \( X^{(\beta,K)}_{\lambda+k(0)}(z) \) for the partition \( \lambda \) relative to the norm of groundstate is given by

\[
N_{\lambda}(\beta) := \frac{(X^{(\beta,K)}_{\lambda+k(0)}, X^{(\beta,K)}_{\lambda+k(0)})_{\beta}}{(X^{(\beta,K)}_{k(0)}, X^{(\beta,K)}_{k(0)})_{\beta}} = \frac{(J^{(K\beta+1,K)}_{\lambda}(J^{(K\beta+1,K)}_{\beta}, (1,1)^{\beta} - Kk)}
\]

\(^2\)In Ref. [33] Uglov used the different convention: \( \sigma + k(0) = k - Kk \).
\[
\prod_{s \in C^{(0)}_{K}(\lambda)} \frac{a'_\lambda(s) + (K\beta + 1)(N - l'_\lambda(s))}{a'_\lambda(s) + 1 + (K\beta + 1)(N - l'_\lambda(s) - 1)} \times \prod_{s \in H^{(0)}_{K}(\lambda)} \frac{a_\lambda(s) + 1 + (K\beta + 1)l_\lambda(s)}{a_\lambda(s) + (K\beta + 1)l_\lambda(s) + 1}.
\]

The spectral decomposition of the power sum \(p_m(y)\) \((m > 0)\) in terms of the \(gl_K\)-Jack polynomials is given by

\[
p_m(y) = \sum_{\lambda \in \mathcal{R}} c_\lambda(\beta) J^{K\beta + 1}(\lambda)(y),
\]

where \(c_\lambda(\beta)\) is an expansion coefficient. To give this we introduce the notation: \(\hat{\delta}(A) = 1\) if a statement \(A\) is true, \(\hat{\delta}(A) = 0\) otherwise. Furthermore we define \(\omega_K := e^{i2\pi/K}\), and

\[
c_{-h_\lambda}^{(a)} := |C^{(a)}_K(\lambda)| - |H^{(a)}_K(\lambda)|, \quad n(\lambda) := \sum_{s \in \lambda} l'_\lambda(s).
\]

Then we obtain \(c_\lambda(\beta) = A_\lambda|\lambda|^{\hat{\delta}(\lambda \equiv 0 \mod K)}\chi_\lambda(\beta)\) where

\[
\chi_\lambda(\beta) := \frac{\prod_{s \in C^{(0)}_{K}(\lambda) \setminus \{1,1\}} (a'_\lambda(s) - (K\beta + 1)l'_\lambda(s))}{\prod_{s \in H^{(0)}_{K}(\lambda)} (a_\lambda(s) + 1 + (K\beta + 1)l_\lambda(s))},
\]

and the \(\beta\)-independent constant \(A_\lambda\) is given by

\[
A_\lambda := \begin{cases} 
\omega_K^{n(\lambda)} \prod_{s = 1}^{K-1} (1 - \omega_K^{-c_{-h_\lambda}^{(0)}}), & \text{for } |\lambda| \equiv 0 \mod K, \ c_{-h_\lambda}^{(0)} = 0, \\
\omega_K^{n(\lambda)} \prod_{s = 1}^{K-1} (1 - \omega_K^{-c_{-h_\lambda}^{(0)} + \delta_{ab}}), & \text{for } |\lambda| \equiv b \mod K, \ c_{-h_\lambda}^{(0)} = 1, (b = 1, \cdots, K - 1), \\
0, & \text{otherwise}.
\end{cases}
\]

### 3.2 Dynamical correlation functions of the spin Calogero-Sutherland model

In this subsection, we derive the following function of the \(U(K)\) spin CS model:

\[
\eta_N^{(\delta\gamma)(\rho\sigma)}(x, t; \beta) := \langle 0| \bar{X}^{\delta\gamma}(x, t) X^{\rho\sigma}(0, 0)|0\rangle,
\]

where the “color-density” operator is defined by

\[
\bar{X}^{\delta\gamma}(x) := \sum_{j=1}^{N} \delta(x - x_j) X^{\gamma\delta}_j = \frac{1}{L} \sum_{j=1}^{N} \sum_{n \in \mathbb{Z}} e^{i2\pi nx/L} \bar{z}^{-n} X^{\gamma\delta}_j = \frac{1}{\sqrt{L}} \sum_{q} e^{iqx} \bar{X}^{\gamma\delta}_q,
\]

with \(q = 2\pi n/L\). The function \(\eta_N^{(\delta\gamma)(\rho\sigma)}(x, t; \beta)\) is the Fourier transform of the dynamical structure factor \(\eta^{(\delta\gamma)(\rho\sigma)}(\bar{z}, \bar{z}^{-1})\), and is called the dynamical color correlation function (DCCF) in the following.

First we should determine the \(SU(K)\) spin selection rule of the excited states of the correlation functions. Here the \(SU(K)\) spin means the \(K - 1\) eigenvalues \((s_1, \cdots, s_{K-1})\) of a set of operators \(\{h^1, \cdots, h^{K-1}\}\) where \(h^a\) is defined by

\[
h^a := \frac{1}{2} \sum_{i=1}^{N} (X_i^{\alpha a} - X_i^{\alpha+1,a+1}).
\]

for \(a = 1, \cdots, K - 1\). For the \(SU(2)\) case, this definition gives the ordinary spin because \(h^1 = \sum_{i=1}^{N} \sigma_i^z / 2\) where \(\sigma_i^z = \text{diag}(1, -1)\) is the \(z\)-component of the Pauli matrices. The set \(\{2h^a\}_{a=1}^{K-1}\) is the standard basis of the Cartan subalgebra of the Lie algebra \(sl_K\). Since the Cartan subalgebra is the maximal commuting subalgebra, the eigenvalues \((s_1, \cdots, s_{K-1})\) are good quantum numbers of the system with the \(SU(K)\) symmetry. We denote the \(SU(K)\) spin of the state \(|\nu\rangle\) which satisfies \(|\nu|\bar{X}^{\gamma\delta}(0)|0\rangle \neq 0\) by \(s(\gamma, \delta) = (s_1(\gamma, \delta), \cdots, s_{K-1}(\gamma, \delta))\).
We see that the above state $|\nu\rangle$ transforms as one of the weight vectors for the adjoint representation $Ad$ of $SU(K)$. For example, in the $SU(3)$ case, we have $s(1,3) = (1/2, 1/2)$. The complete lists of $s(\gamma, \delta)$ for $K = 2, 3, 4$ are given in Appendix B.

We cannot directly calculate the DCCF \((22)\), since the action of the operator $\bar{\Theta} \hat{X}^{\gamma \delta}(0) \Theta^{-1}$ on $H'$ does not have simple expression. The key step for deriving the DCCF \((22)\) is to choose appropriate local operators on $H$. We define local operators $J_a(z) = \sum_{n \in \mathbb{Z}} J_{a,n}(z) \ (a = 1, \cdots, K - 1)$ on $H$ by

$$J_{a,n}(z) := \sum_{i=1}^N \bar{z}_i^{a-n-1} \sum_{b=1}^{a} X_{i}^{a-b, K-a+b} + \sum_{i=1}^{N} z_i^{-n} \sum_{b=1}^{K-a} X_{i}^{a+b,b}.$$ \hspace{1cm} (25)

The DCCF $\eta_N^{(\delta)(\rho)}(x, t; \beta)$ with $\delta \neq \gamma, \rho \neq \sigma$ can be obtained from the dynamical correlation function of the operator $J_a(z)$. This is based on the following formula: if $f$ and $g$ are eigenfunctions of the spin operators $h^b$ with eigenvalues $s_b(f)$ and $s_b(g)$, respectively, then we have

$$(f, \hat{X}^{\gamma \delta}(0)g)_{\beta} = \prod_{b=1}^{K-1} \delta(b_s(f) - s_b(g) = s_b(\gamma, \delta))(f, (1/L)J_a(z)g)_{\beta},$$ \hspace{1cm} (26)

where $\gamma - \delta \equiv a \mod K$. Moreover, we have

$$\Theta J_{a,n}(z) \Theta^{-1} = p_{nK+a}(y),$$ \hspace{1cm} (27)

i.e., the operator $\Theta J_{a,n}(z) \Theta^{-1}$ on $H'$ is the multiplication of the power sum. Then we have

$$(f, \hat{X}^{\gamma \delta}(0)g)_{\beta} = \prod_{b=1}^{K-1} \delta(b_s(f) - s_b(g) = s_b(\gamma, \delta))(\Theta(f), (1/L) \sum_{n \in \mathbb{Z}} p_{nK+a}(y)\Theta(g))_{\beta}'.$$ \hspace{1cm} (28)

If we know the norm of the Yangian Gelfand-Zetlin basis $X^{(\beta,K)}_{\sigma+k(0)}(z)$ (or $gK$-Jack function $J^{(K\beta+1,K)}_{\sigma}(y)$) and the matrix element of the power sum with respect to the bases $\{J^{(K\beta+1,K)}_{\sigma}\}$, we can calculate the DCCF $\eta_N^{(\gamma)(\rho)}(x, t; \beta)$ with $\gamma \neq \delta, \rho \neq \sigma$. We already know these quantities explicitly.

Now we can give the formula for $\eta_N^{(xy)(xy)}(x, t; \beta)$ and $S_N^{(\gamma\gamma)(\gamma\gamma)}(q, \omega; \beta)$ of the $U(K)$ spin CS model. The momentum $\mathcal{P}_\lambda$, excited energy $\mathcal{E}_\lambda$ and $SU(K)$ spins $S^a_\lambda$ for the Yangian Gelfand-Zetlin basis $X^{(\beta,K)}_{\lambda+k(0)}(z)$ with partition $\lambda$ are respectively given by

$$\mathcal{P}_\lambda = \frac{2\pi}{L} |C^{(0)}_K(\lambda)|,$$ \hspace{1cm} (29)

$$\mathcal{E}_\lambda = \frac{1}{2K} \left( \frac{2\pi}{L} \right)^2 \left[ 2n_K(\lambda') - 2(K\beta + 1)n_K(\lambda) + ((N-1)(K\beta + 1) + 1)|C^{(a)}_K(\lambda')| \right];$$ \hspace{1cm} (30)

$$S^a_\lambda = \frac{1}{2} \left( |C^{(a-1)}_K(\lambda')| - 2|C^{(a)}_K(\lambda')| + |C^{(a+1)}_K(\lambda')| \right), \quad (a = 1, \cdots, K - 1),$$ \hspace{1cm} (31)

where

$$n_K(\lambda') := \sum_{s \in C^{(a)}_K(\lambda')} \lambda'_{a}(s),$$ \hspace{1cm} (32)

$$n_K(\lambda') := \sum_{s \in C^{(a)}_K(\lambda')} \lambda'_{a}(s).$$ \hspace{1cm} (33)

The following relations hold for these quantities:

$$\mathcal{P}_{\lambda^*} = -\mathcal{P}_\lambda,$$ \hspace{1cm} (34)

$$\mathcal{E}_{\lambda^*} = -\mathcal{E}_\lambda,$$ \hspace{1cm} (35)

$$S^a_{\lambda^*} = S^a_\lambda.$$ \hspace{1cm} (36)
where $\lambda = (\lambda_1, \cdots, \lambda_N)$ and $\lambda^* := (-\lambda_N, \cdots, -\lambda_1)$. We use above relations for deriving the DCCF. Then, for $\gamma, \delta \in \Sigma_K$ with $\gamma \neq \delta$, we have

$$
\eta_N^{(\delta \gamma) (\gamma \delta)}(x, t; \beta) = \frac{1}{L} \sum_{\lambda \in \Lambda_N} |F_\lambda(\beta)|^2 e^{-it\mathcal{E}_\lambda + i\pi \mathcal{P}_\lambda},
$$

$$
S_N^{(\delta \gamma) (\gamma \delta)}(q, \omega; \beta) = \sum_{\lambda \in \Lambda_N} |F_\lambda(\beta)|^2 \delta_{n, |C_k(0)|}^\prime \delta(\omega - \mathcal{E}_\lambda),
$$

where $q = 2\pi n/L$, and $\delta_{n, |C_k(0)|}$ represents the momentum conservation $q = \mathcal{P}_\lambda$. The form factor $F_\lambda(\beta)$ is given by

$$
|F_\lambda(\beta)|^2 := \frac{1}{L} |\Lambda_{\lambda}(\beta)|^2 N_\lambda(\beta).
$$

The primed summation in the formulae (37) and (38) is taken over partitions $\lambda \in \Lambda_N$ which satisfy the following conditions:

$$
\begin{align*}
|\lambda| = \gamma - \delta \mod K, \\
S_\lambda^\prime = S_a(\gamma, \delta), a = 1, \cdots, K - 1, \\
c - h_\lambda(0) = 1.
\end{align*}
$$

The non-zero condition of the form factor is equivalent to that of the matrix element $\chi_\lambda(\beta)$ of the local operator. Notice that, if $\beta = r/s$ with $r, s$ coprime, then

$$
\chi_\lambda(\beta) \neq 0 \iff (s + 1, rK + s + 1) \notin \lambda.
$$

Under the above condition on $\beta$, we denote the subset $\Lambda_N^{<K+1>}$ of $\Lambda_N$ by \{ $\lambda \in \Lambda_N$ | $(s + 1, rK + s + 1) \notin \lambda$ \}. Then we can replace the set $\Lambda_N$ in the sum of the formulae (37) and (38) by $\Lambda_N^{<K+1>}$.

### 3.3 Classification of excited states

The conditions on the excited states of the formulae (37) and (38) are rather complicated. In this subsection, following Ref. [35], we simplify these conditions. We define the concept type of a partition. We introduce a reductive transformation $\tau$ on the set of all partitions as follows (see Ref. [35] for the case of $K = 2$):

(i) If there exist $K$ rows or $K$ columns which have same number of boxes in a partition, remove those rows or columns;

(ii) Apply the reduction (i) repeatedly until the newly generated partition is no longer reducible.

It is important to note that this transformation has the following properties:

$$
\begin{align*}
|\lambda| \equiv a \mod K \iff |\tau(\lambda)| \equiv a \mod K, \\
S_\lambda^\prime = S_{\tau(\lambda)}^b, \\
c - h_\lambda(c) = c - h_{\tau(\lambda)}(c).
\end{align*}
$$

We can determine the subset $\mathcal{A}_N$ of the set $\Lambda_N$ as the image of $\tau$, i.e., $\mathcal{A}_N := \tau(\Lambda_N)$. Notice that $\mathcal{A}_N$ is the finite set. Then, for a partition $\lambda \in \mathcal{A}_N$, we say that the type of $\lambda$ is $\nu \in \mathcal{A}_N$ if $\nu = \tau(\lambda)$. For example, in the case of $K = 3$, the type of the partition $(7,4,4,2,2,1)$ is (1):

Let us consider the subset $\mathcal{A}_N^{<K+1>} := \tau(\Lambda_N^{<K+1>})$ of $\mathcal{A}_N$. We have the decomposition $\mathcal{A}_N^{<K+1>} = \bigsqcup_{\gamma, \delta = 1, \gamma \neq \delta} \mathcal{A}_N^{<K+1; \gamma \delta>} \sqcup \mathcal{D}_N$, where

$$
\mathcal{A}_N^{<K+1; \gamma \delta>} := \{ \lambda \in \mathcal{A}_N^{<K+1>} | \lambda \text{ satisfies conditions (40)} \}.
$$
and $\mathcal{D}_N$ is a certain set. Then, from the properties (42) and conditions (40), we can rewrite the formulae (37) and (38). For example, we have

$$
\eta_N^{(\lambda)}(x, t; \beta) = \frac{1}{L} \sum_{\lambda \in \Lambda_N^{K\beta+1}}'' |F_{\lambda}(\beta)|^2 e^{-itE_\lambda + ixP_\lambda},
$$

(44)

where the double-primed summation is taken over partitions $\lambda \in \Lambda_N^{K\beta+1}$ such that $\tau(\lambda) = \nu$ for some $\nu \in \mathcal{A}_N^{K\beta+1;\gamma\delta}$. The important point is that the conditions (31) on the excited states which mainly come from $SU(K)$ spin selection rule are conveniently implemented by introducing a finite set $\mathcal{A}_N^{K\beta+1;\gamma\delta}$, and by decomposing the summation over $\lambda$ by each type $\nu = \tau(\lambda)$ such that $\nu \in \mathcal{A}_N^{K\beta+1;\gamma\delta}$. We do not give the explicit forms of the sets $\mathcal{A}_N^{K\beta+1;\gamma\delta}$, since they are not needed in the following discussion.

4 Dynamical Structure Factor for Finite Systems

4.1 Strong coupling limit

From the discussion in section 3, the DCCF $\eta_N^{(\gamma)}(t, t) := \langle 0|\tilde{X}_{r+s}(t)\tilde{X}_s^{\rho\sigma}|0 \rangle$ and dynamical structure factor $S_N^{(\gamma)}(q, \omega)$ of the $SU(K)$ HS model with $\delta \neq \gamma$, $\rho \neq \sigma$ are simply given by $\eta_N^{(\gamma)}(t, t; \infty)$ and $S_N^{(\gamma)}(q, \omega; \infty)$, respectively. Following subsection 3.3, we first discuss the conditions on the excited states for $\eta_N^{(\gamma)}(t, t)$ and $S_N^{(\gamma)}(q, \omega)$. Due to the conditions $c-h_\nu^{(0)} = 1$ and $|\lambda| \equiv \gamma - \delta(\neq 0)$ mod $K$, the quantity $\chi_\lambda(\beta)$ is regular in the limit $\beta \to \infty$. In this limit, the simplification occurs because of the following fact:

$$
\chi_\lambda(\infty) \neq 0 \Leftrightarrow (1, K + 1) \notin \lambda.
$$

(45)

This means the excited state $\lambda$ consists of $K$ columns, i.e., $\lambda'$ has the form $\lambda' = (\lambda_1, \cdots, \lambda_K)$. We define the subset $\Lambda_N^{(K)}$ of $\Lambda_N$ by $\Lambda_N^{(K)} := \{ \lambda = (\lambda_1, \cdots, \lambda_K) | \lambda_1 \leq K \}$. This is the finite set of partitions whose largest entry is less than $K + 1$. We then determine a subset $\mathcal{A}_N^{(K)}$ of $\Lambda_N^{(K)}$ as the image of $\tau$, i.e., $\mathcal{A}_N^{(K)} := \tau(\Lambda_N^{(K)})$. We can explicitly determine this finite set $\mathcal{A}_N^{(K)}$. The total number of elements in the set $\mathcal{A}_N^{(K)}$ increases from 3 in the case of $SU(2)$ to 25 in $SU(3)$, and to 252 in $SU(4)$. We define the subsets of $\mathcal{A}_N^{(K)}$ by $\mathcal{A}_N^{(K;\gamma\delta)} := \{ \nu \in \mathcal{A}_N^{(K)} | S^a_{\nu} = s_{\alpha}(\gamma, \delta), a = 1, \cdots, K - 1 \}$. For example, in the case of $K = 3$, we have $\mathcal{A}_N^{(K;\gamma\delta)} = \{ (2, 1, 1), (3, 3, 1), (3, 3, 2), (3, 3, 2, 1) \}$ and $\mathcal{A}_N^{(K;\gamma)} = \{ (1), (2, 2), (3, 2, 1, 1) \}$. In Appendix B, we give these sets explicitly for $K = 2, 3, 4$. We have checked that, for $K = 2, 3, 4$, the element $\nu \in \mathcal{A}_N^{(K;\gamma\delta)}$ satisfies the conditions $|\nu| \equiv \gamma - \delta$ mod $K$, $c-h_\nu^{(0)} = 1$ and $|A_\nu|^2 = 1$. For the case of $K = 3$, we give these data in Appendix B. Noticing the relation $A_\lambda = A_{\tau(\lambda)}$, we have $|A_\lambda|^2 = 1$ for any partition $\lambda \in \Lambda_N^{(K)}$ with type $\nu \in \mathcal{A}_N^{(K;\gamma\delta)}$.

We give our results on the DCCF and dynamical structure factor of the $SU(K)$ HS model with the finite number of lattice sites. The momentum and excited energy are respectively given by

$$
P_\lambda := \frac{2\pi}{N}|C^{(0)}_K(\lambda)|,
$$

(46)

$$
E_\lambda := \frac{J}{4} \left( \frac{2\pi}{N} \right)^2 \left[ (N - 1)|C^{(0)}_K(\lambda)| - 2n_K(\lambda) \right].
$$

(47)

Then, for $\gamma, \delta \in \Sigma_K$ with $\gamma \neq \delta$, we have

$$
\eta_N^{(\gamma)}(\gamma\delta)(r, t) = \frac{1}{N} \sum_{\lambda \in \mathcal{A}_N^{(K)}} |F^{(K)}_\lambda|^2 e^{-itE_\lambda + ixP_\lambda},
$$

(48)

$$
S_N^{(\gamma)}(q, \omega) = \sum_{\lambda \in \mathcal{A}_N^{(K)}} |F^{(K)}_\lambda|^2 \delta_{\nu, \nu^{(0)}}(\gamma_\lambda(\nu)) \delta(\omega - E_\lambda).
$$

(49)
where \( q = 2\pi n/N \), and the summation is taken over partitions \( \lambda \in \Lambda_N^{(K)} \) such that \( \tau(\lambda) = \nu \) for some \( \nu \in \mathcal{A}_N^{(K; \gamma \delta)} \). The squared form factor is given by

\[
|F^{(K)}_\lambda|^2 := \frac{1}{N} |\lambda(\infty)|^2 N_\lambda(\infty)
= \frac{1}{N} \prod_{s \in C_0^{(N); (1,1)}} l_\lambda^s(\infty) \prod_{s \in H_0^{(N); (\lambda)}} \frac{N - l_\lambda^s(s)}{N - l_\lambda^s(s) - 1}.
\]

(50)

Due to the limit \( \beta \to \infty \), the above quantities depend only on \( l_\lambda(s) \) and \( l_\lambda'(s) \). In the particular case of \( K = 2 \), our formulae (34) and (35) generalize the known ones [19] to arbitrary size of the system. We will see that our formulae have the advantage in clarifying the role of the internal degrees of freedom of quasi-particles.

Since we consider the case of zero external magnetic fields, the \( SU(K) \) symmetry demands that \( \eta_N^{(\gamma \gamma'; \gamma \delta)}(q, \omega) \) and \( S_N^{(\gamma \gamma'; \gamma \delta)}(q, \omega) \) are actually independent of \( (\gamma, \delta) \) as long as \( \gamma \neq \delta \). Therefore we put \( \eta_N(r, t) = \eta_N^{(\gamma \gamma; \gamma \delta)}(r, t) \) and \( S_N(q, \omega) = S_N^{(\gamma \gamma; \gamma \delta)}(q, \omega) \) for \( \gamma \neq \delta \). We can prove this independence by using other expressions of the DCCF and dynamical structure factor which are introduced in the next subsection. See Appendix D for the proof. From the formula (51), other non-zero components of the dynamical structure factors are given by

\[
S_N^{(\gamma \gamma; \gamma \delta)}(q, \omega) = \left( \delta_{\gamma \delta} - \frac{1}{K} \right) S_N(q, \omega).
\]

(51)

We have to determine the sets \( C_0^{(N)}(\lambda) \) and \( H_0^{(N)}(\lambda) \) explicitly to derive the dynamical structure factor. The type of given partition has sufficient information for determining these sets. For example, in the case of \( K = 3 \), we have the following explicit form of the sets \( C_3^{(N)}(\lambda) \) and \( H_3^{(N)}(\lambda) \) for \( \lambda \in \Lambda_N^{(K)} \) with type (1):

\[
C_3^{(N)}(\lambda) = \{(1,1), (4,1), \cdots, (\lambda'_1, 1), (2,2), (5,2), \cdots, (\lambda'_2 - 1, 2), (3,3), (6,3), \cdots, (\lambda'_3, 3)\}
\]

(52)

\[
H_3^{(N)}(\lambda) = \{(1,1), (4,1), \cdots, (\lambda'_3 - 2, 1), (\lambda'_3 + 3, 1), (\lambda'_3 + 6, 1), \cdots, (\lambda'_3, 1), (\lambda'_2 + 2, 1), (\lambda'_2 + 5, 1), \cdots, (\lambda'_2 - 1, 1), (2,2), (5,2), \cdots, (\lambda'_2 - 1, 2), (\lambda'_2 + 1, 2), (\lambda'_2 + 4, 2), \cdots, (\lambda'_2 - 2, 2), (1,3), (4,3), \cdots, (\lambda'_2, 3)\}
\]

(53)

We have determined these sets for all types in the case of \( K = 2, 3, 4 \). Therefore \( \eta_N(q, \omega) \) and \( S_N(q, \omega) \) can be explicitly calculated. In general, \( S_N(q, \omega) \) is the rational function of \( N \). In Appendix C we give examples.

We have checked the validity of Eq. (34) with \( K = 2 \) and 3 by comparing with the numerical result for \( N \leq 24 \) and \( N \leq 15 \), respectively. The numerical result is obtained via exact diagonalization and the recursion method. We truncated the continued fraction at 100 iterations and took the Lorentzian width \( O(10^{-5} J) \). The agreement is excellent in both cases of \( K = 2 \) and 3. The comparison between the exact result and numerical data is given in Table 1 for the case of \( K = 3 \) and \( N = 15 \).

**Table 1**

In Fig. 1 we present the exact results of \( S_N(q, \omega) \) for \( (K, N) = (3, 18) \) and \( (4, 16) \). The dynamical structure factors \( S_N(q, \omega) \) in Fig. 1 have rather complicated structure. The origin of this structure will be clarified in section 5.

**Fig. 1**
The static structure factor $S_N(q) := S_N^{(\gamma)(\delta)}(q)$ with $\gamma \neq \delta$ is obtained by integrating over the energy in the formula (13). We give the results of $S_N(q)$ for $(K, N) = (3, 18)$ and $(4, 16)$ in Fig. 2. From Fig. 2 we see that $S_N(q)$ has the cusp structure at $q = 2\pi/3$ for $K = 3$ and at $q = \pi/2, \pi$ for $K = 4$. In section 4 we will give the detailed discussion of this point.

**Fig. 2**

### 4.2 Quasi-particle interpretation

As an effect of $\beta \to \infty$, the summation in the right hand side of the formulae (48) and (49) is restricted to the finite set $\Lambda_N^{(K)}$. The conditions on the type of excited states come from the selection rules for the $SU(K)$ spin. We consider the quasi-particle interpretations of these conditions. For labelling the excited states relevant to $S_N(q, \omega)$, it is more convenient to use the conjugate partition $\lambda' = (\lambda'_1, \cdots, \lambda'_K) \in \Lambda_N^{(N)}$ instead of $\lambda \in \Lambda_N^{(K)}$. Each $\lambda'_i$ has the information on the momentum and $SU(K)$ spin of a quasi-particle. We call this quasi-particle a spinon following the $SU(2)$ case. The spinon is considered to be an object possessing an $SU(K)$ spin. An $SU(K)$ spin of the spinon with $\lambda'_i$ is specified by a certain condition on the pair $(i, \lambda'_i)$.

Since the condition for general $K$ is rather complicated, we give examples in the case of $K = 2, 3, 4$:

**SU(2) spin of spinon with $\lambda'_i$**

$$SU(2) \text{ spin of spinon with } \lambda'_i := \begin{cases} 1/2, & \text{if } (i, \lambda'_i) \equiv (0, 0), (1, 1), \text{ mod } 2, \\ -1/2, & \text{if } (i, \lambda'_i) \equiv (0, 1), (1, 0), \text{ mod } 2. \end{cases}$$

**SU(3) spin of spinon with $\lambda'_i$**

$$SU(3) \text{ spin of spinon with } \lambda'_i := \begin{cases} (0, 1/2), & \text{if } (i, \lambda'_i) \equiv (0, 0), (1, 1), (2, 2) \text{ mod } 3, \\ (1/2, -1/2), & \text{if } (i, \lambda'_i) \equiv (0, 1), (1, 2), (2, 0) \text{ mod } 3, \\ (-1/2, 0), & \text{if } (i, \lambda'_i) \equiv (0, 2), (1, 2), (2, 1) \text{ mod } 3. \end{cases}$$

**SU(4) spin of spinon with $\lambda'_i$**

$$SU(4) \text{ spin of spinon with } \lambda'_i := \begin{cases} (0, 0, 1/2), & \text{if } (i, \lambda'_i) \equiv (0, 0), (1, 1), (2, 2), (3, 3) \text{ mod } 4, \\ (0, 1/2, -1/2), & \text{if } (i, \lambda'_i) \equiv (0, 1), (1, 2), (2, 3), (3, 0) \text{ mod } 4, \\ (1/2, -1/2, 0), & \text{if } (i, \lambda'_i) \equiv (0, 2), (1, 3), (2, 0), (3, 1) \text{ mod } 4, \\ (-1/2, 0, 0), & \text{if } (i, \lambda'_i) \equiv (0, 3), (1, 0), (2, 1), (3, 2) \text{ mod } 4. \end{cases}$$

It is important to note that a spinon transforms as a weight vector of the fundamental representation $\tilde{K}$ of $SU(K)$.

From the formulae (48) and (49), we can conclude that relevant excited states of $S_N(q, \omega)$ consist of $K$ spinons. Moreover the conditions on the type of excited states lead to an important consequence: $K$-spinon excited states have $K - 1$ different $SU(K)$ spins. That is, the excited states contain $K - 1$ species of quasi-particles. For instance, in the case of $K = 3$ excited states relevant to $S_N^{(31)(13)}(q, \omega)$, which have the $SU(3)$ spin $(1/2, 1/2)$, consist of three spinons with $SU(3)$ spins $(0, 1/2), (0, 1/2)$ and $(1/2, -1/2)$. For the $SU(2)$ case, we recover the well-known fact that only two spinons with the same spin contribute to $S_N(q, \omega)$.

The complete lists of the $SU(K)$ spins of $K$ spinons are given in Appendix B for $K = 2, 3, 4$.

The relevant states of $S_N^{(\gamma)(\delta)}(q, \omega)$ have the $SU(K)$ spin $(s_1, \cdots, s_{K-1}) = (0, \cdots, 0)$. Then we can show that, in the spinon picture, the relevant states of $S_N^{(\gamma)(\delta)}(q, \omega)$ consist of $K$-spinon excited states with $K$ different $SU(K)$ spins. Namely, in contrast to the case of $S_N(q, \omega)$, the excited states contain $K$ species of quasi-particles. See Appendix B for lists of $SU(K)$ spins of $K$-spinon excited states for $S_N^{(\gamma)(\delta)}(q, \omega)$ in the case of $K = 2, 3, 4$. The dynamical structure factor $S_N^{(\gamma)(\delta)}(q, \omega)$ is related to $S_N(q, \omega)$ through the formula (51).

The $K$-spinon excitation belongs to the tensor representation $\tilde{K}^\otimes K$ of $SU(K)$. This representation contains the adjoint representation $Ad$ as an irreducible component. From the condition stated above for $K$-spinon excitation, we see that the $K$-spinon excitation transforms as one of the weight vectors for $Ad$. This is consistent with the condition for $SU(K)$ spin of the excited states which are relevant to $S_N^{(\gamma)(\delta)}(q, \omega)$. 

11
4.3 \( U(1) \) expression

Although the parametrization \( \lambda' = (\lambda'_1, \ldots, \lambda'_K) \in \Lambda^{(N)}_K \) has the advantage of giving the spinon interpretation, we introduce another parametrization which is more convenient for taking the thermodynamic limit. The new parametrization \( c(\lambda') = (c_1(\lambda'), \ldots, c_K(\lambda')) \in \Lambda^{(N/K)}_K \) for the excited states of the DCCF [18] and dynamical structure factor [19] is defined by

\[
c_i(\lambda') := \begin{cases} 
(\lambda'_i - 1)/K, & \lambda'_i \equiv 1 \mod K, \\
\vdots & \\
(\lambda'_i - l + 1)/K, & \lambda'_i \equiv -l \mod K, \\
(\lambda'_i + K - l)/K, & \lambda'_i \equiv l \mod K, \\
\vdots & \\
(\lambda'_i + 1)/K, & \lambda'_i \equiv K - 1 \mod K, \\
\lambda'_i/K, & \lambda'_i \equiv 0 \mod K,
\end{cases}
\] (57)

where, in the second explicit expression, \( \lambda \equiv l \mod K (l = 0, 1, \ldots, K - 1) \). This parametrization is nothing but representing the momenta of spinons. It is important to note that we can not recover the information on the \( SU(K) \) spins of spinons from this new parametrization \( c(\lambda') \), since \( c(\lambda') \) is characterized by the information on the color 0 only. We rewrite the formulae (48) and (49) by using \( c(\lambda') \). The momentum and excited energy are respectively expressed by

\[
P_{c(\lambda')} := \frac{2\pi}{N} \sum_{i=1}^{\lambda_1} c_i(\lambda') = \frac{2\pi}{N} \sum_{i=1}^{K} c_i(\lambda'),
\] (58)

\[
E_{c(\lambda')} := \frac{J}{4} \left( \frac{2\pi}{N} \right)^2 \left[ \sum_{i=1}^{\lambda_1} ((N - 1)c_i(\lambda') - Kc_i(\lambda')^2) 
+ K \sum_{i=1 \mod K} c_i(\lambda') + (K - 2) \sum_{i=2 \mod K} c_i(\lambda') + \cdots - (K - 2) \sum_{i=0 \mod K} c_i(\lambda') \right]
= \frac{J}{4} \left( \frac{2\pi}{N} \right)^2 \sum_{i=1}^{K} c_i(\lambda')(N + K + 1 - 2i - Kc_i(\lambda')).
\] (59)

The expression of the form factor [19] depends on the type of state \( \lambda \). In general, we have

\[
|F^{(K)}_{\lambda}(\lambda')|^2/R^{(K)}_{c(\lambda')} = \frac{\Gamma(c_{ab}(\lambda') + \chi_{ab} + (K - 1)/K) \Gamma(c_{ab}(\lambda') + \chi'_{ab} + (K - 1)/K)}{\Gamma(c_{ab}(\lambda') + \chi_{ab})} \prod_{1 \leq i < j \leq K, (i,j) \neq (a,b)} \frac{\Gamma(c_{ij}(\lambda') + \rho_{ij} - 2/K)}{\Gamma(c_{ij}(\lambda') + \rho_{ij})},
\] (60)

where \( \Gamma(z) \) is the gamma function, \( c_{ij}(\lambda') := c_i(\lambda') - c_j(\lambda') \) and

\[
R^{(K)}_{c(\lambda')} := \frac{1}{K} \prod_{i=1}^{K} \frac{\Gamma((K - 1)/K)}{\Gamma((i)/K)^2} \prod_{j=1}^{K} \frac{\Gamma(c_{ij}(\lambda') + j/(-K + 1/K)}{\Gamma(c_{ij}(\lambda') + j/K)} \prod_{j=1}^{K} \frac{\Gamma((N + K - j + 1)/K - c_j(\lambda') - 1/K)}{\Gamma((K - j + 1)/K - c_j(\lambda'))}.
\] (61)

In the above formula, the pair of indexes \( (a, b) \) with \( a < b \) and rational numbers \( \chi_{ab}, \chi'_{ab} \) and \( \rho_{ij} \) depend on the type of \( \lambda \). We note that the order of \( \chi_{ab}, \chi'_{ab} \) and \( \rho_{ij} \) is \( O(N^0) \). In Appendix D, we derive the formulae [18], [19] and [19], and give explicit formulae for form factors.

The summation in [19] is reduced to the summation \( \sum_{N/K \geq c_1(\lambda') \geq \cdots \geq c_K(\lambda') \geq 0} \) under the condition that the cases of \( c_1(\lambda') = c_2(\lambda') = \cdots = c_K(\lambda') \) are excluded. Physically, this condition means the extraction of
the density fluctuation. The conditions on type in (10) can be combined into the redefinition of the form factor. In Appendix F, we give the redefined form factor $F_{c(\lambda')}^{(K)}$ for $K = 2, 3, 4$. Then finally, we have

$$S_N(q, \omega) = \sum_{N/K \geq c_1 \geq \cdots \geq c_K \geq 0} |F_{c(\lambda')}^{(K)}|^2 \delta_{n, \sum_{i=1}^K c_i} \delta(\omega - E_c),$$  \hspace{1cm} (62)

where $q = 2\pi n/N$, and $\delta_{n, \sum_{i=1}^K c_i}$ represents the momentum conservation $q = P_c$. Here the summation is taken over partitions $c = (c_1, \cdots, c_K) \in \Lambda_{K}^{(N/K)}$ with the conditions given above. In the formula (62), we have omitted the dependence on the original parameter $\lambda'$, because we regard the formula (62) as the expression of $S_N(q, \omega)$ by using only the momenta $c = (c_1, \cdots, c_K) \in \Lambda_{K}^{(N/K)}$ of the spinons. We call the formula (62) the $U(1)$ expression of the dynamical structure factor.

### 4.4 Motif expression

In this subsection, we discuss the relation between the momenta of spinons and motif and then express $S_N(q, \omega)$ in terms of motifs. We recall definition of the motif [36, 37]. Eigenstates of the Hamiltonian $H_{HS}$ [4] are characterized via motifs, which are sequences $d = d_0d_1d_2\cdots d_{N-1}d_N$ of $N + 1$ digits ‘0’ or ‘1’, beginning and ending with ‘0’, i.e., $d_0 = d_N = 0$, and containing at most $K - 1$ consecutive ‘1’. For example, in the case of $N \equiv 0 \mod K$, the unique singlet groundstate is represented by the motif $d^{(0)} := 01\cdots0\cdots10 \cdots 01\cdots10$. The crystal momentum $P_d$ and energy $E_d$ of the state for the motif $d$ are respectively given by

$$P_d \equiv \frac{2\pi}{N} \sum_{k=1}^{N-1} kd_k \mod 2\pi,$$  \hspace{1cm} (63)

$$E_d := \frac{J}{4} \left( \frac{2\pi}{N} \right)^2 \left[ - \sum_{k=1}^{N-1} kd_k(N - kd_k) + \frac{1}{6}N(N^2 - 1) \right].$$  \hspace{1cm} (64)

The groundstate energy, i.e., the energy for the groundstate motif $d^{(0)}$, is explicitly given by

$$E_{d^{(0)}} = \frac{J}{4} \left( \frac{2\pi}{N} \right)^2 \frac{1}{6} \frac{N}{K}(N^2 - K^2).$$  \hspace{1cm} (65)

Now we consider the relation between the parametrization $c(\lambda')$ with $\lambda \in \Lambda_N^{(K)}$ and the motif. For this purpose, it is convenient to use another expression for the motif [36, 37, 38]. For a given motif, replacing every intermediate ‘0’ by ‘1’(first ‘0’ by ‘1’ and last ‘0’ by ‘1’), one can obtain a string of “elementary” motifs $(1\cdots1) (a = 0, 1, \cdots, K - 1)$. We denote a motif with this expression by $\cdots e_2e_1e_0$ where $e_i$ is one of the elementary motifs. Namely we enumerate the elementary motifs from right to left. For example, for the motif $010010 = (11)\cdots(1)$, we have $e_0 = (1), e_1 = (1)$ and $e_2 = (11)$. The groundstate motif $d^{(0)}$ is represented by $e_{N/K-1}\cdots e_1e_0$ with $e_j = (1\cdots1)$ for all $j$.

The groundstate motif has $(N - N/K)$ ‘1’. We construct the set of motifs by changing one ‘1’ by ‘0’ in the groundstate motif and rearranging those $(N - N/K - 1)$ ‘1’ possible ways. We call these motifs the $K$-spinon motifs because they satisfy the following constraint which leads to the $K$-spinon excitation:

$$\sum_{a=0}^{K-2} (K - a - 1) \times \text{(the number of elementary motif (1\cdots1) in the motif)} = K.$$  \hspace{1cm} (66)

The $K$-spinon motif consists of $(N/K + 1)$ elementary motifs, i.e., we can write as $e_{N/K}\cdots e_1e_0$.

The correspondence between the parametrization $c(\lambda') = (c_1(\lambda'), \cdots, c_K(\lambda'))$ which is the relevant state for $S_N(q, \omega)$ and the $K$-spinon motif $d = e_{N/K}\cdots e_1e_0$ is explained as follows. We have the following correspondence:

$$e_i = (1\cdots1) \leftrightarrow (K - a - 1) \text{ spinons with the same momentum } i.$$  \hspace{1cm} (67)
For example, in the case of \( K = 3 \) and \( N = 12 \), the motif \( 0110101100110 \) is regarded as three spinons with \( c_1(\lambda') = 3, c_2(\lambda') = c_3(\lambda') = 1 \). As shown above, the groundstate motif contains no spinon and consists of \( N/K \) elementary motifs.

The above simple rule for the correspondence between momenta of spinons and motif is valid only for the \( K \)-spinon motifs and the groundstate motif. For generalizing this rule to arbitrary motifs, we have to use the correspondence between states of the SU(\( K \)) HS model and semistandard tableaux given in Refs. [43, 34]. In Appendix E, we recall the results of Refs. [43, 44] and then generalize the correspondence between momenta of spinons and \( K \)-spinon motifs to arbitrary motifs.

From the above discussion, we can rewrite the formula (62) in terms of the motif as follows:

\[
S_N(q, \omega) = \sum_{d:K\text{-spinon motif}} |F^K_d|^2 \delta\left(q - \frac{2\pi}{N} \sum_{k=1}^{N-1} k d_k \equiv 0 \bmod 2\pi\right) \delta(\omega - E_d + E_d(0)),
\]

where \( \delta \) represents the momentum conservation, and \( F^K_d = F^K_c \) through the above identification of \( K \)-spinon motif \( d \) and the momenta of spinons \( c \). We call the formula (68) the motif (or domain-wall) expression of the dynamical structure factor.

## 5 Dynamical Structure Factor in the Thermodynamic Limit

### 5.1 Thermodynamic limit

Next our task is to take the thermodynamic limit of the formulae (68) and (69). Performing a procedure similar to that in Refs. [43, 44], for \( K = 2, 3 \) and \( 4 \), we can take this limit. We introduce the momentum of the spinon \( k_i \) in the thermodynamic limit by \( 2K(c_i(\lambda') - N)/N \to k_i \) as \( N \to \infty \). In this limit, the momentum and excited energy are respectively given by

\[
p(k) := \frac{\pi}{K} \sum_{i=1}^{K} k_i,
\]

\[
\epsilon(k) := \sum_{i=1}^{K} \epsilon_s(k_i),
\]

where \( k_i := \pi/2K \), \( k = (k_1, \cdots, k_K) \), and the spinon dispersion is given by

\[
\epsilon_s(u) := \frac{\pi v_s}{2K} (1 - u^2)
\]

with the velocity \( v_s := \pi/2 \). We have adopted the above normalization of \( k_i \) so that the Fermi points coincide with \( \{\pm 1\} \). Then, using the formula \( \Gamma(p + z)/\Gamma(p) \to p^z \) as \( |p| \to \infty \) and changing the summations to integrals in the formula (62) and similar formula for the DCCF, we have the following formulae for the DCCF and dynamical structure factor of the SU(\( K \)) HS model in the thermodynamic limit:

\[
\eta^{(\delta\gamma)}(\gamma\delta)(r, t) = (-1)^r A_K \frac{2\pi}{2\pi} \sum_{1 \leq a < b \leq K} \prod_{j=1}^{K} \int_{-1}^{1} dk_j |F^K_{ab}(k)|^2 e^{-i r \epsilon(k) + i r \epsilon(k)},
\]

\[
S^{(\delta\gamma)}(\gamma\delta)(q, \omega) = A_K \sum_{1 \leq a < b \leq K} \prod_{j=1}^{K} \int_{-1}^{1} dk_j |F^K_{ab}(k)|^2 \delta(q - \pi - p(k)) \delta(\omega - \epsilon(k)),
\]

where a normalization constant is given by

\[
A_K := \frac{2^{2K-1} \Gamma((K - 1)/K)}{\Gamma((K - 1)/2)}. \tag{74}
\]

The form factor is given by the following Jastrow form in the momentum space:

\[
F^K_{ab}(k) := |k_a - k_b|^{2K} \prod_{1 \leq i < j \leq K, (i, j) \neq (a, b)} |k_i - k_j|^{2K} \prod_{i=1}^{K} (1 - k_i^2)^{(1 - g_K)/2}. \tag{75}
\]
with \( g_K := (K-1)/K \) and \( g'_K := -1/K \). Since \( \eta^{(\delta \gamma)(\gamma \delta)}(r, t) \) and \( S^{(\delta \gamma)(\gamma \delta)}(q, \omega) \) with \( \gamma \neq \delta \) do not depend on the pair \( (\gamma, \delta) \), we denote them by \( \eta(r, t) \) and \( S(q, \omega) \), respectively. Other non-zero components of dynamical structure factors are given by

\[
S^{(\gamma \delta)}(q, \omega) = (\delta_{\gamma \delta} g_K + (1 - \delta_{\gamma \delta}) g'_K) S(q, \omega). \tag{76}
\]

Since \( p(k), \epsilon(k) \) and \( \sum_{a<b} \left| F_{ab}^{(K)}(k) \right|^2 \) are symmetric in \( k_1, \cdots, k_K \), the summation in the formulae (72) and (73) can be replaced by \( K(K-1)/2 \) if we appropriately change the subscripts of \( k_i \)’s. Performing the integrals twice, we have the following \((K-2)\)-fold integral expression of \( S(q, \omega) \):

\[
S(q, \omega) = \frac{2^K}{\pi v_s} \prod_{i=1}^{K} \frac{\Gamma((K-1)/K)}{\Gamma(i/K)^2} \prod_{i=1}^{K-2} \frac{dk_i}{1 - k_i^2} \frac{\delta(\Omega)}{1 + a + b + (1 + a + b)^{1-g_K}}
\]

\[
\times \left( a^2 - 4b \right)^{g_K} \left( \prod_{i=1}^{K-2} \left( k_i^2 - a k_i + b \right)^{2g_K} \right)^{\frac{1}{1 + a + b} \prod_{1 \leq i \neq j \leq K-2} (k_i - k_j)^{2g_K}} \tag{77}
\]

where

\[
a := \tilde{q}/k_F - \sum_{i=1}^{K-2} k_i \tag{78},
\]

\[
b := a^2/2 + K \tilde{\omega}/(\pi v_s) + \sum_{i=1}^{K-2} k_i^2/2 \tag{79},
\]

with \( \tilde{q} := q - \pi \) and \( \tilde{\omega} := \omega - \pi v_s/2 \). The region \( \Omega \) is determined by the following conditions:

1. \( 1 - a + b \geq 0 \), \( 1 + a + b \geq 0 \), \( 4b - a^2 \leq 0 \), \( -2 \leq a \leq 2 \).

For \( K = 2 \), the formulae (72) and (73) reproduce the result of Haldane-Zimbauer \[14\], which was obtained by a completely different method. Notice that the second product in the numerator of Eq. (73) is absent in the \( SU(2) \) case.

In the low energy limit we recover the results of Ref. \[27\] which are obtained by conformal field theory. Performing the same procedure as in Ref. \[14\], we obtain the asymptotic behavior of the DCCF (72):

\[
\eta(r, t) \sim A_0 (\xi_L^{-2} + \xi_R^{-2}) + \sum_{l=1}^{K-1} A_l (\xi_L \xi_R)^{-\alpha_l/2} \cos(2lk_F r), \tag{84}
\]

where \( \xi_L := r + v_s t, \xi_R := r - v_s t, \) and \( A_l \) \((l = 0, 1, \cdots, K - 1)\) are constants. The critical exponents are given by

\[
\alpha_i := 2c_i C^{-1} e_i = 2i(1 - i/K), \tag{85}
\]

where \( c_i := (0, \cdots, 0, 1, \cdots, 0) \) for \( i \)-th entry \((i = 1, \cdots, K - 1)\) and \((K - 1) \times (K - 1)\) matrix \( C = (C_{ij}) := (2\delta_{ij} - \delta_{i-1,j} - \delta_{i+1,j}) \) is called the Cartan matrix of the Lie algebra \( sl_K \). Since spinons do not interact in the \( SU(K) \) HS model in contrast to the corresponding nearest-neighbor model, we expect absence of the logarithmic correction in the DCCF of the \( SU(K) \) HS model. The derivation of Eq. (84) is given in Appendix F. These \( K - 1 \) singularities with exponents \( \alpha_i \) correspond to \( K - 1 \) gapless bosonic modes, i.e., the exponents \( \alpha_i \) agree with that obtained by conformal field theory for the \( SU(K) \) \( WZW \) model \[14\].

We can derive static structure factor \( S(q) \) by integrating over \( \omega \) in \( S(q, \omega) \). From Eq. (84), we can perform the Fourier transform of \( \eta(r, 0) \). Then we have the following asymptotic behavior of \( S(q) \):

\[
S(q) \sim B_0 |q| + \sum_{l=1}^{K-1} B_l |q - 2lk_F|^{-\alpha_l-1}, \tag{86}
\]
where $B_l$ are non-universal constants. The cusp structure in Fig. 3 for $K = 3, 4$ is consistent with this formula. Note that $\alpha_3 = 1$ in the case of $K = 2$ corresponds to the logarithmic singularity in $S(q)$ of the SU(2) HS model [26, 27].

The spinon interpretation of the formulae \(\langle 72 \rangle\) and \(\langle 73 \rangle\) goes as follows. As in the case of finite systems, the excited states for $S(q, \omega)$ in the thermodynamic limit consist of $K$ spinons with $K - 1$ different SU($K$) spins. This fact means, as in the low energy limit, the dynamics of the SU($K$) HS model can be described by $K - 1$ species of quasi-particles. This simple structure reflects the Yangian symmetry of the SU($K$) HS model [26]. In the form factor \(\langle 74 \rangle\), the factor $|k_a - k_b|$ represents the statistical interactions of spinons with the same SU($K$) spin, while the factor $|k_a - k_b|^2$ represents those of spinons with different SU($K$) spins. We refer to Ref. [46] for more detailed explanation of statistical interactions.

5.2 Support of dynamical structure factor

The support of $S^{(\gamma_4)(\rho_0)}(q, \omega)$ represents the region in the momentum-frequency plane where $S^{(\gamma_4)(\rho_0)}(q, \omega)$ takes the non-zero value. We see that the support of $S(q, \omega)$ as determined from the formula \(\langle 73 \rangle\) is compact, i.e., there is no intensity outside of the finite area. The support of $S(q, \omega)$ is determined as follows:

\[
\omega \leq \epsilon^{(\nu)}(q) \quad \text{for} \quad 0 \leq q \leq 2\pi, \quad (87)
\]

\[
\omega \geq \epsilon^{(L)}_{j}(q) \quad \text{for} \quad 2(j-1)k_F \leq q \leq 2jk_F. \quad (88)
\]

with $j = 1, \ldots, K$. Here upper and lower boundaries are respectively given by

\[
\epsilon^{(\nu)}(q) := \frac{v_s}{2\pi} q(2\pi - q), \quad \text{for} \quad 0 \leq q \leq 2\pi, \quad (89)
\]

\[
\epsilon^{(L)}_{j}(q) := \frac{Kv_s}{2\pi} (q - 2(j-1)k_F)(2jk_F - q), \quad \text{for} \quad 2(j-1)k_F \leq q \leq 2jk_F. \quad (90)
\]

See Fig. \text{3} (and also Fig. \text{3}) for the case of $K = 3, 4$.

The behavior of $S(q, \omega)$ near the dispersion lines of elementary excitations is derived for general $K$ as follows. We introduce the following dispersion:

\[
\epsilon_{m,n,f}(q) := \frac{Kv_s}{2f\pi} (q - 2nk_F)(2(K-m)k_F - q), \quad \text{for} \quad 2nk_F \leq q \leq 2(K-m)k_F, \quad (91)
\]

where $m, n \geq 0$ and $f \geq 1$ are integers with $m + n + f = K$. This dispersion represents $K$ spinons for which $m$ spinons are near the left Fermi point $k = -1$, $n$ spinons are near the right Fermi point $k = 1$, and $f$ spinons are moving together with same momentum $Q/f$ with $Q \sim O(1)$. For example, $\epsilon_{0,0,K}(q) = \epsilon^{(\nu)}(q)$ and $\epsilon_{K-3,j-1,1}(q) = \epsilon^{(L)}_j(q)$. Near the dispersion line $\omega = \epsilon_{m,n,f}(q)$, $S(q, \omega)$ has the singularity

\[
S(q, \omega) \sim |\omega - \epsilon_{m,n,f}(q)|^{\eta_{m,n,f}} \quad (92)
\]

with exponent

\[
\eta_{m,n,f} := \left\{ \begin{array}{ll}
0, & \text{if} \ m = n = 0, f = K, \\
K - 2 - \frac{1}{K} [m^2 + n^2 + f(f - 1)], & \text{otherwise}.
\end{array} \right. \quad (93)
\]

In Fig. \text{3}, we give values of exponents explicitly for $K = 3, 4$.

**Fig. 3**

The derivation of Eq. \(\langle 92 \rangle\) is given in Appendix F. There is a stepwise discontinuity at the upper boundary $\omega = \epsilon^{(\nu)}(q)$. On the other hand, there are divergent singularities at the lower boundaries $\omega = \epsilon^{(L)}_1(q)$ and $\omega = \epsilon^{(L)}_{K-1}(q)$. Here $S(q, \omega)$ diverges by the power law with the exponent $-1/K$. For $K \geq 3$, at other lower boundaries $\omega = \epsilon^{(L)}_j(q)$ with $j \neq 1, K$, $S(q, \omega)$ has threshold singularities but no divergence. Also, for $K \geq 3$, $S(q, \omega)$ has cusp type singularities at other dispersion lines in the support. These cusp type singularities originate from the statistical interactions for spinons with different SU($K$) spins. The complicated structures which appeared in Fig. \text{3} reflect the cusp type singularities. (Notice that there are finite size effects in Fig. \text{1}.)
6 Summary and Discussions

In conclusion, we have derived the exact formulae (49) and (73) for \( S(q, \omega) \) of the \( SU(K) \) HS model for arbitrary size of the system at zero temperature. We have clarified the quasi-particle picture of the spin dynamics. The relevant excited states consist of \( K \) spinons with \( K - 1 \) different \( SU(K) \) spins. We have discussed the relation between momenta of spinons and motif. We have analyzed the singularities of \( S(q, \omega) \) in the thermodynamic limit. In contrast to the \( SU(2) \) case, cusp type singularities in the support, which originate from the statistical interactions of different species of the spinons, appear in the \( SU(K) \) \( (K \geq 3) \) case.

As in the \( SU(2) \) case \[19, 15\], we expect that the divergences at two of the lowest boundaries occur also in the \( SU(K) \) Heisenberg model with the nearest-neighbor exchange. Our exact result of \( S(q, \omega) \) for \( K \leq 4 \) is likely to be valid for larger \( K \) as well.

As mentioned in section 1, there is a close correspondence between the \( SU(2) \) HS model and spinless CS model with special coupling parameter \( \beta = -2 \) in the normalization of Ref. \[37\]. Thus it is expected that there is relation between our result for \( K = 2 \) and the dynamical density correlation function of the spinless CS model calculated in Refs. \[44, 45\]. In the thermodynamic limit, we checked that the dynamical density correlation function of the spinless CS model with \( \beta = -2 \) is almost same as our DCCF (72) for \( K = 2 \). The difference between them is the range of integration.

The \( SU(K) \) HS model and one-dimensional \( SU(K|1) \) supersymmetric \( 1/r^2 \) t-J model \[17\] have been an interesting laboratory for study of the quantum number fractionalization in one dimension \[48\]. We have applied the freezing trick to the \( SU(K|1) \) supersymmetric \( 1/r^2 \) t-J model and then obtained the dynamical charge structure factor \[19\]. It will be interesting to consider the relation between our results and the exclusion statistics in conformal field theory discussed in Refs. \[50, 51, 52\].

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Appendix A: Dynamical Density Correlation Function in the Strong Coupling Limit

In this Appendix, we analyze the strong coupling limit of the dynamical density correlation function \( \langle 0 | \rho(x, t) \rho(0, 0) | 0 \rangle \) of the \( U(K) \) spin CS model. Here \( \rho(x, t) \) is the Heisenberg representation of the density fluctuation operator

\[
\rho(x) := \sum_{j=1}^{N} \delta(x - x_j) \sum_{\gamma=1}^{K} X_j^{\gamma} - \frac{N}{L}
\]

\[
= \frac{1}{L} \sum_{j=1}^{N} \sum_{n(\neq 0)} e^{i2\pi nx/L} z_j^{-n} \sum_{\gamma=1}^{K} X_j^{\gamma} = \frac{1}{\sqrt{L}} \sum_{q(\neq 0)} e^{iqx} \sum_{\gamma=1}^{K} \bar{X}_q^{\gamma\gamma}.
\]

We use the same notations as in section 3. Uglov derived the following formula \[33\]:

\[
\Theta \rho(0) \Theta^{-1} = \frac{1}{L} \sum_{n \in \mathbb{Z} \{0\}} p_{nK}(y).
\]

Thus we can derive the dynamical density correlation function of the \( U(K) \) spin CS model in the same manner as in Ref. \[33\] and in section 3

\[
\langle 0 | \rho(x, t) \rho(0, 0) | 0 \rangle = \frac{K^2 L}{4\pi^2} \sum_{\lambda \in \Lambda_N} |P_\lambda|^2 |F_\lambda(\beta)|^2 e^{-itE_\lambda + ixP_\lambda},
\]
where the summation is taken over the partitions $\lambda \in \Lambda_N$ which satisfy the following conditions:

\[
\begin{align*}
|\lambda| &\equiv 0 \pmod{K}, \\
S^a_\lambda &\equiv 0, \quad a = 1, \cdots, K - 1, \\
c_{h_\lambda^{(0)}} &\equiv 0.
\end{align*}
\] (97)

Careful analysis of the formula for $(0|\rho(x,t)\rho(0,0)|0)$ reveals the following behavior in the strong coupling limit:

\[
(0|\rho(x,t)\rho(0,0)|0) \longrightarrow \frac{1}{\beta} \sum_{l=1}^{N-1} \sum_{(K^l)(\in \Lambda_N^{(K)})} G_\lambda + O(\beta^{-2})
\] (98)

as $\beta \to \infty$. Here $G_\lambda$ is the $\beta$-independent quantity. The SU($K$) singlet states $\lambda^{ph,l} := (K^l) (l = 1, \cdots, N - 1)$ satisfy the conditions (97) and represent the phonon contributions. Thus we can conclude that the phonon contribution to the dynamical correlation functions appears in the order $\beta^{-1}$.

Let us derive Eq. (98). It is sufficient to consider the strong coupling limit of the matrix element $\chi_\lambda$ of local operator and the norm $N_\lambda$. We can easily see that $|C^{(0)}_K(\lambda^{ph,l})| = |H^{(0)}_K(\lambda^{ph,l})| = l$. Notice that $s = ((\lambda^{ph,l})'_1, 1) = (l, 1) \in H^{(0)}_K(\lambda^{ph,l})$. For this point $s = ((\lambda^{ph,l})'_1, 1)$, we have $l_{\lambda^{ph,l}}(s) = 0$. Therefore, from this fact and conditions $c_{h_\lambda^{(0)}} = 0$, $|\lambda^{ph,l}| \equiv 0 \pmod{K}$, we see that the matrix element $\chi_{\lambda^{ph,l}}(s)$ is regular for the limit $\beta \to \infty$. On the other hand, $N_{\lambda^{ph,l}}(\beta) \to const \times (1/(\beta + 1))$ as $\beta \to \infty$. Indeed, noticing

\[
a_{\lambda^{ph,l}}(s) = 1, \quad \chi_{\lambda^{ph,l}}(\beta) = K, \quad a_{\lambda^{ph,l}}(s) = (K \beta + 1)l_{\lambda^{ph,l}}(s) = K(\beta + 1)
\] (99)

and

\[
a_{\lambda^{ph,l}}(s) = 1, \quad \chi_{\lambda^{ph,l}}(\beta) = K, \quad a_{\lambda^{ph,l}}(s) = (K \beta + 1)(l_{\lambda^{ph,l}}(s) + 1) = K(\beta + 1)
\] (100)

for $s = (l, 1)$, we have

\[
\chi_{\lambda^{ph,l}}(\beta) = \frac{1}{\beta} \prod_{s \in H^{(0)}_K(\lambda^{ph,l}) \setminus \{(1,1)\}} \left( a_{\lambda^{ph,l}}(s) - (K \beta + 1)l_{\lambda^{ph,l}}(s) \right) \prod_{s \in C^{(0)}_K(\lambda^{ph,l}) \setminus \{(1,1)\}} \left( a_{\lambda^{ph,l}}(s) + (K \beta + 1)l_{\lambda^{ph,l}}(s) \right)
\] (101)

\[
\longrightarrow \chi_{\lambda^{ph,l}}(\infty) = \frac{1}{\beta + 1} \prod_{s \in C^{(0)}_K(\lambda^{ph,l}) \setminus \{(1,1)\}} \frac{a_{\lambda^{ph,l}}(s) + (K \beta + 1)(N - l_{\lambda^{ph,l}}(s))}{a_{\lambda^{ph,l}}(s) + 1 + (K \beta + 1)(N - l_{\lambda^{ph,l}}(s) - 1)}
\] (102)

\[
\times \prod_{s \in H^{(0)}_K(\lambda^{ph,l}) \setminus \{(1,1)\}} \frac{a_{\lambda^{ph,l}}(s) + (K \beta + 1)(l_{\lambda^{ph,l}}(s) + 1)}{a_{\lambda^{ph,l}}(s) + 1 + (K \beta + 1)(l_{\lambda^{ph,l}}(s) + 1)}
\] (103)

\[
\longrightarrow \frac{1}{\beta + 1} \prod_{s \in C^{(0)}_K(\lambda^{ph,l})} \frac{N - l_{\lambda^{ph,l}}(s)}{N - l_{\lambda^{ph,l}}(s) - 1} \prod_{s \in H^{(0)}_K(\lambda^{ph,l}) \setminus \{(1,1)\}} \frac{l_{\lambda^{ph,l}}(s)}{l_{\lambda^{ph,l}}(s) + 1},
\] (104)

as $\beta \to \infty$. Then we have Eq. (98).

**Appendix B: SU($K$) Spin Content and Enumeration of $A_N^{(K)}$**

(i) *Data of SU($K$) spin*

The SU($K$) spins of the excited states of $S_N(q, \omega)$ and those of $K$-spinon excitations are summarised in the following Tables for $K = 2, 3, 4$.

**Table 2**
(ii) Set $A_{N}^{(K)}$

We give the explicit form of the set $A_{N}^{(K)}$. In general, the set $A_{N}^{(K)}$ has the decomposition $A_{N}^{(K)} = \sqcup_{\gamma, \delta=1, \gamma \neq \delta}^{K} A_{N}^{(K; \gamma \delta)} \sqcup B_{N}^{(K)} \sqcup c_{N}^{(K)}$, where $A_{N}^{(K; \gamma \delta)}$ is defined in subsection 4.1. $B_{N}^{(K)}$ and $c_{N}^{(K)}$ are certain subsets of $A_{N}^{(K)}$. The set $B_{N}^{(K)}$ is related to the excited states for $S_{N}^{(q)}(\gamma \delta)(q, \omega)$ and contains the partition $\varnothing$.

For $K = 2$ and $N \geq 2$, we have $A_{N}^{(2)} = \sqcup_{\gamma, \delta=1, \gamma \neq \delta}^{2} A_{N}^{(2; \gamma \delta)} \sqcup B_{N}^{(2)}$ where $B_{N}^{(2)} = \{ \varnothing \}$, and

\begin{align*}
A_{N}^{(2;12)} &= \{ (1) \}, \quad (105) \\
A_{N}^{(2;21)} &= \{ (2, 1) \}.
\end{align*}

For $K = 3$ and $N \geq 6$, we have $A_{N}^{(3)} = \sqcup_{\gamma, \delta=1, \gamma \neq \delta}^{3} A_{N}^{(3; \gamma \delta)} \sqcup B_{N}^{(3)} \sqcup c_{N}^{(3)}$, where $B_{N}^{(3)} = \{ \varnothing, (2, 1), (3, 2, 1), (3, 3, 2, 1) \}$, $c_{N}^{(3)} = \{ (2, 2, 1, 1), (3, 2, 2, 1, 1), (3, 3, 2, 2, 1, 1) \}$, and

\begin{align*}
A_{N}^{(3;12)} &= \{ (1, 1), (3, 2), (3, 2, 2, 1) \}, \quad (107) \\
A_{N}^{(3;21)} &= \{ (2, 1, 1), (3, 3, 1), (3, 3, 2, 2) \}, \quad (108) \\
A_{N}^{(3;13)} &= \{ (1, 1), (3, 2, 1, 1) \}, \quad (109) \\
A_{N}^{(3;31)} &= \{ (3, 1, 1), (3, 3, 2), (3, 3, 2, 2, 1) \}, \quad (110) \\
A_{N}^{(3;23)} &= \{ (2), (2, 2, 1), (3, 3, 1, 1) \}, \quad (111) \\
A_{N}^{(3;32)} &= \{ (3, 1), (3, 3, 2, 2), (3, 3, 2, 1, 1) \}. \quad (112)
\end{align*}

For $K = 4$ and $N \geq 10$, we have $A_{N}^{(4)} = \sqcup_{\gamma, \delta=1, \gamma \neq \delta}^{4} A_{N}^{(4; \gamma \delta)} \sqcup B_{N}^{(4)} \sqcup c_{N}^{(4)}$, where

\begin{align*}
A_{N}^{(4;12)} &= \{ (1, 1, 1), (3, 2, 2), (4, 2, 1), (3, 2, 2, 2, 1, 1), \} \\
&\quad (4, 2, 2, 2, 1), (4, 4, 3), (4, 3, 3, 2, 1, 1, 1), (4, 4, 3, 2, 1, 1), \\
&\quad (4, 4, 3, 3, 1), (4, 4, 3, 3, 2, 2, 1), (4, 4, 3, 3, 3, 2, 1), \} \\
&\quad (4, 4, 4, 3, 3, 2, 2, 1, 1), \} \quad (113) \\
A_{N}^{(4;21)} &= \{ (2, 1, 1, 1), (3, 3, 2, 1), (4, 3, 1, 1), (3, 3, 2, 2, 2, 1), \} \\
&\quad (4, 3, 2, 2, 2, 2, 1, 1, 1), (4, 4, 4, 3, 2, 2, 1), \} \quad (114) \\
A_{N}^{(4;13)} &= \{ (1, 1), (2, 2, 2), (4, 2), (3, 2), 2, 1, 1, 1), \} \\
&\quad (4, 2, 2, 2, 1, 1), (4, 3, 3), (4, 3, 3, 2, 1, 1, 1), \} \quad (115) \\
A_{N}^{(4;31)} &= \{ (3, 1, 1, 1), (3, 3, 2, 2), (4, 4, 1, 1), (3, 3, 2, 2, 2, 1, 1), \} \\
&\quad (4, 4, 4, 2, 2, 1, 1), \} \quad (116) \\
A_{N}^{(4;14)} &= \{ (1), (2, 2, 1), (3, 2), (3, 2, 2, 1, 1), \} \\
&\quad (4, 2, 1, 1, 1, 1), (3, 3, 3), (3, 3, 3, 2, 1, 1), \} \quad (117) \\
A_{N}^{(4;41)} &= \{ (4, 1, 1, 1), (4, 3, 2, 2), (4, 4, 2, 1), (4, 3, 2, 2, 1, 1), \}.
We do not give $B^{(4)}_N$ and $C^{(4)}_N$, since they are not important for our discussion.

(iii) Data for the set $A_N^{(3)}$

We summarize data for the set $A_N^{(3)}$ in the following Tables:

Table 5

Table 6

Table 7

Table 8

Table 9
Appendix C: Some Explicit Values of the Dynamical Structure Factor for Finite $N$

We give the explicit form of $S_N(q, \omega)$ as a function of $N$. The notation $c = (c_1, \cdots, c_K)$ for momenta of spinons is defined in subsection 4.3. For $K = 2$ with $c_1 = m, c_2 = n$ ($0 \leq n < m \leq N/2$), we have

$$S_N\left(\frac{2(m+n)\pi}{N}, \frac{J}{4}, \frac{2\pi}{N}\right)^2 \left((m+n)N - 2(m^2+n^2) + m - n\right)$$

$$= \frac{2m-2n-1}{2m-1} \prod_{i=n+1}^{m-1} \frac{2i}{N-2n-1} \prod_{i=n+1}^{m-1} \frac{N-2i}{N-2i-1}. \quad (125)$$

We can give the similar formulae for $K = 3, 4$. However these formulae are rather complicated. We give more explicit formulae for the small momentum $q$. For $K = 3$ and $q \leq 8\pi/N$, we have

$$S_N\left(\frac{2\pi}{N}, \frac{J}{4}, \frac{2\pi}{N}\right)^2 (N-1) = \frac{1}{N-1}, \quad (c_1 = 1, c_2 = c_3 = 0), \quad (126)$$

$$S_N\left(\frac{4\pi}{N}, \frac{J}{4}, \frac{2\pi}{N}\right)^2 (2N-8) = \frac{3}{2} \frac{N-3}{(N-1)(N-4)}, \quad (c_1 = 2, c_2 = c_3 = 0), \quad (127)$$

$$S_N\left(\frac{4\pi}{N}, \frac{J}{4}, \frac{2\pi}{N}\right)^2 (2N-4) = \frac{1}{2} \frac{1}{N-2}, \quad (c_1 = c_2 = 1, c_3 = 0), \quad (128)$$

$$S_N\left(\frac{6\pi}{N}, \frac{J}{4}, \frac{2\pi}{N}\right)^2 (3N-21) = \frac{9}{5} \frac{(N-3)(N-6)}{(N-1)(N-4)(N-7)}, \quad (c_1 = 3, c_2 = c_3 = 0), \quad (129)$$

$$S_N\left(\frac{6\pi}{N}, \frac{J}{4}, \frac{2\pi}{N}\right)^2 (3N-11) = \frac{6}{5} \frac{N-3}{(N-2)(N-4)}, \quad (c_1 = 2, c_2 = 1, c_3 = 0), \quad (130)$$

$$S_N\left(\frac{8\pi}{N}, \frac{J}{4}, \frac{2\pi}{N}\right)^2 (4N-40) = \frac{81}{40} \frac{(N-3)(N-6)(N-9)}{(N-1)(N-4)(N-7)(N-10)}, \quad (c_1 = 4, c_2 = c_3 = 0), \quad (131)$$

$$S_N\left(\frac{8\pi}{N}, \frac{J}{4}, \frac{2\pi}{N}\right)^2 (4N-24) = \frac{9}{8} \frac{(N-3)(N-6)}{(N-2)(N-4)(N-7)}, \quad (c_1 = 3, c_2 = 1, c_3 = 0), \quad (132)$$

$$S_N\left(\frac{8\pi}{N}, \frac{J}{4}, \frac{2\pi}{N}\right)^2 (4N-20) = \frac{3}{5} \frac{N-3}{(N-2)(N-5)}, \quad (c_1 = c_2 = 2, c_3 = 0), \quad (133)$$

$$S_N\left(\frac{8\pi}{N}, \frac{J}{4}, \frac{2\pi}{N}\right)^2 (4N-16) = \frac{1}{4} \frac{1}{N-4}, \quad (c_1 = 2, c_2 = c_3 = 1). \quad (134)$$

For $K = 4$ and $q \leq 8\pi/N$, we have

$$S_N\left(\frac{2\pi}{N}, \frac{J}{4}, \frac{2\pi}{N}\right)^2 (N-1) = \frac{1}{N-1}, \quad (c_1 = 1, c_2 = c_3 = c_4 = 0), \quad (135)$$

$$S_N\left(\frac{4\pi}{N}, \frac{J}{4}, \frac{2\pi}{N}\right)^2 (2N-10) = \frac{4}{3} \frac{N-4}{(N-1)(N-5)}, \quad (c_1 = 2, c_2 = c_3 = c_4 = 0), \quad (136)$$

$$S_N\left(\frac{4\pi}{N}, \frac{J}{4}, \frac{2\pi}{N}\right)^2 (2N-4) = \frac{2}{3} \frac{1}{N-2}, \quad (c_1 = c_2 = 1, c_3 = c_4 = 0), \quad (137)$$
\[
S_N \left( \frac{6\pi}{N} \cdot \frac{J}{4} \left( \frac{2\pi}{N} \right)^2 (3N-27) \right) = \frac{32}{21} \frac{(N-4)(N-8)}{(N-1)(N-5)(N-9)}, \quad (c_1 = 3, c_2 = c_3 = c_4 = 0), \tag{138}
\]
\[
S_N \left( \frac{6\pi}{N} \cdot \frac{J}{4} \left( \frac{2\pi}{N} \right)^2 (3N-13) \right) = \frac{8}{7} \frac{N-4}{(N-2)(N-5)}, \quad (c_1 = 2, c_2 = 1, c_3 = c_4 = 0), \tag{139}
\]
\[
S_N \left( \frac{6\pi}{N} \cdot \frac{J}{4} \left( \frac{2\pi}{N} \right)^2 (3N-9) \right) = \frac{1}{3} \frac{1}{N-3}, \quad (c_1 = c_2 = c_3 = 1, c_4 = 0), \tag{140}
\]
\[
S_N \left( \frac{8\pi}{N} \cdot \frac{J}{4} \left( \frac{2\pi}{N} \right)^2 (4N-52) \right) = \frac{128}{77} \frac{(N-4)(N-8)(N-12)}{(N-1)(N-5)(N-9)(N-13)}, \quad (c_1 = 4, c_2 = c_3 = c_4 = 0), \tag{141}
\]
\[
S_N \left( \frac{8\pi}{N} \cdot \frac{J}{4} \left( \frac{2\pi}{N} \right)^2 (4N-30) \right) = \frac{112}{99} \frac{(N-4)(N-8)}{(N-2)(N-5)(N-9)}, \quad (c_1 = 3, c_2 = 1, c_3 = c_4 = 0), \tag{142}
\]
\[
S_N \left( \frac{8\pi}{N} \cdot \frac{J}{4} \left( \frac{2\pi}{N} \right)^2 (4N-24) \right) = \frac{20}{63} \frac{N-4}{(N-2)(N-6)}, \quad (c_1 = c_2 = 2, c_3 = c_4 = 0), \tag{143}
\]
\[
S_N \left( \frac{8\pi}{N} \cdot \frac{J}{4} \left( \frac{2\pi}{N} \right)^2 (4N-18) \right) = \frac{8}{9} \frac{N-4}{(N-3)(N-5)}, \quad (c_1 = 2, c_2 = c_3 = 1, c_4 = 0). \tag{144}
\]

Appendix D: Some Formulae and Form Factors in the \(U(1)\) Expression

(i) Derivation of formulae (58), (59) and (60)

We can easily derive the following formulae:

\[
|C_K^{(0)}| = \sum_{i=1}^{\lambda_1} c_i(\lambda'), \tag{145}
\]
\[
n_K(\lambda) = \frac{K}{2} \left[ \sum_{i=1}^{\lambda_1} c_i(\lambda')^2 \right. \left. - \frac{K-2}{2} \sum_{i=2 \mod K}^{K} c_i(\lambda') - \cdots - \frac{K-2}{2} \sum_{i=0 \mod K}^{K} c_i(\lambda') \right]. \tag{146}
\]

Then, from Eqs. (16) and (17), we have formulae (58) and (59).

Next we derive the formula (64) for \(K = 3\) and \(\lambda\) with type (1). In the following, we use the abbreviated notation \(c_i = c_i(\lambda')\). From the explicit forms of the sets \(C_K^{(0)}(\lambda)\) and \(H_K^{(0)}(\lambda)\) given in (53) and (54), we have

\[
\prod_{s \in C_K^{(0)}(\lambda) \setminus \{(1,1)\}} (l_3'(s))^2 = 3^{2(c_1+c_2+c_3-1)} \frac{\Gamma(c_1)^2 \Gamma(c_2+1/3)^2 \Gamma(c_3+2/3)^2}{\Gamma(1/3)^2 \Gamma(2/3)^2}, \tag{147}
\]
\[
\prod_{s \in H_K^{(0)}(\lambda)} l_3(s)(l_3(s)+1) = 3^{2(c_1+c_2+c_3-1)} \times \frac{\Gamma(c_1) \Gamma(c_1+1/3) \Gamma(c_2+1/3) \Gamma(c_2+2/3) \Gamma(c_3+2/3) \Gamma(c_3+1)}{\Gamma(2/3)^3} \times \frac{\Gamma(c_1-c_2) \Gamma(c_2-c_3+1) \Gamma(c_1+c_3-2/3) \Gamma(c_1-c_3-1/3)}{\Gamma(c_1-c_2-2/3) \Gamma(c_2-c_3+1/3) \Gamma(c_1-c_3) \Gamma(c_1-c_3+1/3)} \tag{148}
\]
\[
\prod_{s \in C_3^{(0)}(\lambda)} \frac{N - I'_A(s)}{N - I'_A(s) - 1} = \frac{N}{3} \frac{\Gamma(N/3 - c_1 + 2/3) \Gamma(N/3 - c_2 + 1/3) \Gamma(N/3 - c_3)}{\Gamma(N/3 - c_1 + 1) \Gamma(N/3 - c_2 + 2/3) \Gamma(N/3 - c_3 + 1/3)}.
\] (149)

Then we have the formula (10) for \( K = 3 \) and \( \lambda \) with type (1). Other cases can be derived in the same way.

**(ii) \( U(1) \) expression of the form factor \( F^{(K)}_\lambda \)**

We give the expression of the form factor \( F^{(K)}_\lambda \) in terms of the momenta of spinons \( c(\lambda') \). The form factors for \( S_N^{(21)(12)}(q, \omega) \) and \( S_N^{(12)(21)}(q, \omega) \) have the form:

\[
|F^{(2)}_\lambda|^2 / R^{(2)}_{c(\lambda')} = \frac{\Gamma(c_{12} + 1/2)}{\Gamma(c_{12} - 1/2)}, \quad (c_1 > c_2), \quad \text{if } \lambda : \text{ type } (1) \in A_N^{(2;12)} \text{ or } (2, 1) \in A_N^{(2;21)}.
\] (150)

The form factors for \( S_N^{(21)(12)}(q, \omega) \), \( S_N^{(31)(13)}(q, \omega) \) and \( S_N^{(13)(31)}(q, \omega) \) respectively have the forms:

\[
|F^{(3)}_\lambda|^2 / R^{(3)}_{c(\lambda')} = \begin{cases} 
\frac{\Gamma(c_{12} + 1/3) \Gamma(c_{12} + 2/3) \Gamma(c_{12} + 1/3)}{\Gamma(c_{12}) \Gamma(c_{12} + 2/3) \Gamma(c_{12} + 1/3)} (=: f_1^{(3)}), \\
\frac{\Gamma(c_{12}) \Gamma(c_{12} + 1/3) \Gamma(c_{12} + 2/3) \Gamma(c_{12} + 1/3)}{\Gamma(c_{12}) \Gamma(c_{12} + 2/3) \Gamma(c_{12} + 1/3)} (=: f_2^{(3)}), \\
\gamma(c_{12} - 1/3) \Gamma(c_{12}) \Gamma(c_{12} - 1/3) (=: f_3^{(3)}), \\
\text{if } \lambda : \text{ type } (1, 1) \in A_N^{(3;12)}, \\
\text{if } \lambda : \text{ type } (3, 2) \in A_N^{(3;12)}, \\
\text{if } \lambda : \text{ type } (3, 2, 1) \in A_N^{(3;12)}, \\
\end{cases}
\] (151)

\[
|F^{(3)}_\lambda|^2 / R^{(3)}_{c(\lambda')} = \begin{cases} 
\frac{\Gamma(c_{12}) \Gamma(c_{12} + 1/3) \Gamma(c_{12} + 2/3) \Gamma(c_{12} + 1/3)}{\Gamma(c_{12}) \Gamma(c_{12} + 2/3) \Gamma(c_{12} + 1/3)} (=: f_1^{(3)}), \\
\frac{\Gamma(c_{12}) \Gamma(c_{12} + 1/3) \Gamma(c_{12} + 2/3) \Gamma(c_{12} + 1/3)}{\Gamma(c_{12}) \Gamma(c_{12} + 2/3) \Gamma(c_{12} + 1/3)} (=: f_2^{(3)}), \\
\gamma(c_{12} - 1/3) \Gamma(c_{12}) \Gamma(c_{12} - 1/3) (=: f_3^{(3)}), \\
\text{if } \lambda : \text{ type } (2, 2) \in A_N^{(3;13)}, \\
\text{if } \lambda : \text{ type } (2, 1, 1) \in A_N^{(3;13)}, \\
\end{cases}
\] (152)

\[
|F^{(3)}_\lambda|^2 / R^{(3)}_{c(\lambda')} = \begin{cases} 
\frac{\Gamma(c_{12}) \Gamma(c_{12} + 1/3) \Gamma(c_{12} + 2/3) \Gamma(c_{12} + 1/3)}{\Gamma(c_{12}) \Gamma(c_{12} + 2/3) \Gamma(c_{12} + 1/3)} (=: f_1^{(3)}), \\
\frac{\Gamma(c_{12}) \Gamma(c_{12} + 1/3) \Gamma(c_{12} + 2/3) \Gamma(c_{12} + 1/3)}{\Gamma(c_{12}) \Gamma(c_{12} + 2/3) \Gamma(c_{12} + 1/3)} (=: f_2^{(3)}), \\
\gamma(c_{12} - 1/3) \Gamma(c_{12}) \Gamma(c_{12} - 1/3) (=: f_3^{(3)}), \\
\text{if } \lambda : \text{ type } (3, 3, 2) \in A_N^{(3;31)}, \\
\end{cases}
\] (153)
The form factor for $S^{(4;14)}_{N}(q, \omega)$ has the form:

$$|F^{(4)}_{\lambda}|^2/R^{(4)}_{c(\lambda')} = \begin{cases} 
\Gamma(c_{12} - 2/4)\Gamma(c_{13} - 3/4)\Gamma(c_{22} + 2/4)\Gamma(c_{24} + 1/4)\Gamma(c_{34}) \quad & \text{if } \lambda \text{ type } (1) \in \mathcal{A}^{(4;14)}_N, \\
\Gamma(c_{12} - 2/4)\Gamma(c_{13} - 3/4)\Gamma(c_{22} + 2/4)\Gamma(c_{24} + 1/4)\Gamma(c_{34}) \quad & \text{if } \lambda \text{ type } (2, 2, 1) \in \mathcal{A}^{(4;14)}_N, \\
\Gamma(c_{12} + 1/4)\Gamma(c_{13} - 3/4)\Gamma(c_{22} + 2/4)\Gamma(c_{24} + 1/4)\Gamma(c_{34}) \quad & \text{if } \lambda \text{ type } (3, 2) \in \mathcal{A}^{(4;14)}_N, \\
\Gamma(c_{12} + 1/4)\Gamma(c_{13} - 3/4)\Gamma(c_{22} + 2/4)\Gamma(c_{24} + 1/4)\Gamma(c_{34}) \quad & \text{if } \lambda \text{ type } (3, 2, 2, 1, 1) \in \mathcal{A}^{(4;14)}_N, \\
\Gamma(c_{12} + 1/4)\Gamma(c_{13} - 3/4)\Gamma(c_{22} + 2/4)\Gamma(c_{24} + 1/4)\Gamma(c_{34}) \quad & \text{if } \lambda \text{ type } (3, 2, 2, 2, 1, 1) \in \mathcal{A}^{(4;14)}_N. 
\end{cases}$$

(iii) Definition of the form factor $F^{(K)}_{c(\lambda')}$

The form factors $F^{(K)}_{c(\lambda')}$ for $K = 2, 3, 4$ are defined as follows:

$$F^{(2)}_{c(\lambda')} := f^{(2)}_{\lambda'},$$

$$|F^{(3)}_{c(\lambda')}|^2/R^{(3)}_{c(\lambda')} := \begin{cases} 
f_{1}^{(3)}, (c_1 > c_2 = c_3), \\
f_{2}^{(3)}, (c_1 = c_2 > c_3), \\
\sum_{i=1}^{3} j_{3}^{(3)}, (c_1 > c_2 > c_3), 
\end{cases}$$

$$|F^{(4)}_{c(\lambda')}|^2/R^{(4)}_{c(\lambda')} := \begin{cases} 
f_{1}^{(4)}, (c_1 > c_2 = c_3 = c_4), \\
f_{2}^{(4)} + f_{3}^{(4)}, (c_1 = c_2 > c_3 = c_4), \\
f_{6}^{(4)}, (c_1 = c_2 = c_3 > c_4), \\
f_{1}^{(4)} + f_{2}^{(4)} + f_{3}^{(4)} + f_{4}^{(4)} + f_{5}^{(4)}, (c_1 > c_2 > c_3 = c_4), \\
f_{1}^{(4)} + f_{6}^{(4)} + f_{7}^{(4)} + f_{8}^{(4)} + f_{9}^{(4)}, (c_1 > c_2 = c_3 > c_4), \\
f_{2}^{(4)} + f_{3}^{(4)} + f_{6}^{(4)} + f_{10}^{(4)} + f_{11}^{(4)}, (c_1 = c_2 > c_3 > c_4), \\
\sum_{i=1}^{12} j_{4}^{(4)}, (c_1 > c_2 > c_3 > c_4), 
\end{cases}$$

where $f_i^{(K)}$ are defined in (151), (152), (153) and (154).

(iv) Proof of the independence of $S^{(4;14)}_{N}(\gamma, \omega)$ for the pair $(\gamma, \delta)$ with $\gamma \neq \delta$
We prove that $S_N^{(\delta \gamma)(\gamma \delta)}(q, \omega)$ is independent of the pair $(\gamma, \delta)$ as long as $\gamma \neq \delta$. We introduce the contragradient partition $\lambda^c = (\lambda_1^c, \cdots, \lambda_N^c)$ of the partition $\lambda = (\lambda_1, \cdots, \lambda_N) \in \Lambda_N$ by $\lambda_i^c = \lambda_1 - \lambda_{N-i+1}$ for $i = 1, \cdots, N$. For an excited state $\lambda \in \Lambda^{(K)}_N$ of $S_N^{(\delta \gamma)(\gamma \delta)}(q, \omega)$, the contragradient partition $\lambda^c$ is also the excited state of $S_N^{(\delta \gamma)(\gamma \delta)}(q, \omega)$ and gives the intensity at point $(2\pi - q, \omega)$ in the momentum-frequency plane. That is, the states $\lambda$ and $\lambda^c$ give the intensity at points which are symmetric with respect to the axis $q = \pi$. We can easily prove this fact by using the following relations:

$$c_i(\lambda^c) + c_{K-i+1}(\lambda^c) = N/K, \quad i = 1, \cdots, K. \quad (158)$$

Notice that types of $\lambda$ and $\lambda^c$ have the simple relation. For example, in the case of $K = 3$ and $(\gamma, \delta) = (1, 3), (\rho, \sigma) = (1, 2)$, we can easily show that

$$\lambda: \text{type } (1) \in \mathcal{A}_N^{(3;13)} \quad \iff \quad \lambda^c: \text{type } (3, 2) \in \mathcal{A}_N^{(3;12)},$$

$$\lambda: \text{type } (2, 2) \in \mathcal{A}_N^{(3;13)} \quad \iff \quad \lambda^c: \text{type } (1, 1) \in \mathcal{A}_N^{(3;12)},$$

$$\lambda: \text{type } (3, 2, 1, 1) \in \mathcal{A}_N^{(3;13)} \quad \iff \quad \lambda^c: \text{type } (3, 2, 2, 1) \in \mathcal{A}_N^{(3;12)}. \quad (159)$$

Using the above fact and relation (158), we can show that

$$S_N^{(\delta \gamma)(\gamma \delta)}(q, \omega) = S_N^{(\sigma \rho)(\rho \sigma)}(2\pi - q, \omega) \quad (160)$$

for any $(\gamma, \delta) \neq (\rho, \sigma)$. We can also prove the following formula:

$$S_N^{(\delta \gamma)(\gamma \delta)}(q, \omega) = S_N^{(\delta \gamma)(\gamma \delta)}(2\pi - q, \omega). \quad (161)$$

Therefore we proved that $S_N^{(\delta \gamma)(\gamma \delta)}(q, \omega)$ is independent of the pair $(\gamma, \delta)$ with $\gamma \neq \delta$.

### Appendix E: Correspondence between States and Semistandard Tableaux

In this Appendix, we recall the correspondence between the states of the $SU(K)$ HS model and the semistandard tableaux given in Refs. [13, 14]. From this result, we can generalize the correspondence between the momenta of spins and $K$-spinning motifs given in subsection 4.4 to arbitrary motifs. In the following, we assume that $N \equiv 0$ mod $K$ and $N/K \equiv 1$ mod 2.

(i) $K$-regular partition

The $K$-regular partition is the partition $\lambda = (\lambda_1, \cdots, \lambda_N) \in \Lambda_N$ which satisfies $\lambda_i - \lambda_{i+1} < K$ for $i = 1, \cdots, N$ ($\lambda_{N+1} = 0$). In the formulation in section 4, the state with $K$-regular partition means the state without phonon contribution. We denote the set of all $K$-regular partitions by $\Lambda_N^{(K)}$. The set $\Lambda_N^{(K)}$ is finite. For a $K$-regular partition $\lambda \in \Lambda_N^{(K)}$, we write $\lambda = \lambda^{(0)} = \lambda^c = k = k^c$ where $k = (k_1, \cdots, k_N) \in (\Sigma_K)^N$ and $k^c = (k_1^c, \cdots, k_N^c) \in (\Sigma_K)^N$. Here $k^{(0)} = (M, M - 1, \cdots, M - N + 1)$ with $M = (N + K)/2 \in \mathbb{Z}_{>0}$. We represent $k$ as $r_1^p \ r_2^p \ \cdots$ where $r_1 > r_2 > \cdots$ and $p_i$ denotes the multiplicity of $r_i$ in $k$. For example, in the case of $K = 3$ and $N = 3$, we have

$$\begin{array}{ccc}
\lambda & \overline{k} & \bar{k} \\
(1, 0, 0) & (1, 2, 1) & 1^0 2^1 \\
(1, 1, 0) & (1, 3, 1) & 1^1 2^1 \\
(1, 1, 1) & (1, 3, 2) & 1^2 2^1 \\
(2, 0, 0) & (2, 2, 1) & 1^0 1^1 2^1 \\
(2, 1, 0) & (2, 3, 1) & 1^1 1^0 2^1 \\
(2, 1, 1) & (2, 3, 2) & 1^2 1^0 2^1 \\
(3, 1, 0) & (3, 3, 1) & 1^0 1^2 2^1 \\
(3, 1, 1) & (3, 3, 2) & 1^1 1^0 2^2 \\
\end{array} \quad (162)$$

For the groundstate $k^{(0)} = k^{(0)} + K\bar{k}^{(0)}$, we have

$$\begin{array}{c}
k^{(0)} = (K, K - 1, \cdots, 1, K, K - 1, \cdots, \cdots, K, K - 1, \cdots, 1), \\
\bar{k}^{(0)} = \left(\frac{1}{2} \left(\frac{N}{K} - 1\right)\right)^K \left(\frac{1}{2} \left(\frac{N}{K} - 3\right)\right)^K \cdots \left(-\frac{1}{2} \left(\frac{N}{K} - 1\right)\right)^K. \\
\end{array} \quad (163) \quad (164)$$

25
Clearly the set $\Lambda_{N;K}$ is the subset of $\Lambda_{N}^{(N(K-1))}$. That is, the conjugate partition $\lambda'$ of $\lambda \in \Lambda_{N;K}$ belongs to $\Lambda_{N}^{(N(K-1))}$. Therefore, in general, the state with $K$-regular partition represents the $N(K-1)$-spinon excitation. The momenta of spinons $c = (c_1, \ldots, c_K, c_{K+1}, \ldots, c_{N(K-1)-K+1}, \ldots, c_{N(K-1)})$ of the state $\lambda \in \Lambda_{N;K}$ satisfies the following conditions:

$$
\begin{align*}
&c_{(l-1)K+1} \geq \cdots \geq c_{lK} \text{ for } l = 1, \ldots, N(K-1)/K, \\
&\text{the cases of } c_{(l-1)K+1} = \cdots = c_{lK} \text{ are prohibited}, \\
&c_i \geq c_{i+K}.
\end{align*}
$$

(165)

For example, in the case of $K = 4$ and $N = 4$, the state with the 4-regular partition $\lambda = (12, 9, 6, 3)$ has the momenta of spinons $c = (1, 1, 0, 1, 0, 1, 0, 0, 0)$:

In Table 10, we give the data of the $K$-regular partitions for $(K, N) = (3, 3)$.

Table 10

(ii) Ribbon diagram

We denote the set of all motifs by $\mathcal{M}_{N}^{(K)}$. The motifs can be expressed by the rank $K$ ribbon diagrams with length $N$ [43]. The rank $K$ ribbon diagram with length $N$ consists of columns of boxes and satisfies the following conditions:

(a) The total number of boxes is $N$.
(b) The diagram is connected and contains no $2 \times 2$ blocks of boxes.
(c) The heights of all its columns do not exceed $K$.

We denote the ribbon diagram of $l$ columns such that the height of $i$-th column from the left is $p_i$ by $[p_1, \cdots, p_l]$. For example, $[2, 3, 1, 1, 2, 2, 1]$ is the rank $K(\geq 3)$ ribbon diagram with length 12:

We denote the set of all rank $K$ ribbon diagrams with length $N$ by $\mathcal{R}_{N}^{(K)}$. The one-to-one correspondence between $\mathcal{M}_{N}^{(K)}$ and $\mathcal{R}_{N}^{(K)}$ is given as follows [43]: for a motif $d = d_0 d_1 \cdots d_N$,

(I) Write the first box.
(II) Attach the second box under (resp. left to) the first box if $d_1 = 1$ (resp. $d_1 = 0$).
(III) Similarly attach the $(i+1)$-th box under (resp. left to) the $i$-th box if $d_i = 1$ (resp. $d_i = 0$) for $i = 2, \ldots, N-1$.

For example, the motif $d = 001010011010$ corresponds to the ribbon diagram $[2, 3, 1, 1, 2, 2, 1]$. The groundstate motif $d^{(0)} \in \mathcal{M}_{N}^{(K)}$ is expressed by the ribbon diagram $\theta^{(0)} := [K, \cdots, K] \in \mathcal{R}_{N}^{(K)}$ which contains $N/K$ columns with length $K$. For example, in the case of $K = 3$ and $N = 9$, the ribbon diagram $\theta^{(0)} = [3, 3, 3]$ for the groundstate motif $d^{(0)} = 0110110110$ is given by

In Table 11, we give the data of the motifs for $(K, N) = (3, 3)$. 

26
(iii) Semistandard tableau

In each box of a given ribbon diagram \( \theta \), let us inscribe one of the numbers 1, 2, \( \cdots \), \( K \). We call such an arrangement of numbers a semistandard tableau of shape \( \theta \) if it satisfies the following conditions: let \( a \) and \( b \) be the inscribed numbers in any pair of adjacent boxes, then

(A) \( a < b \) if \( b \) is lower-adjacent to \( a \).

(B) \( a \geq b \) if \( b \) is left-adjacent to \( a \).

We denote the set of all semistandard tableaux of shape \( \theta \) by \( \text{SST}(\theta) \). For example,

\[
\text{SST}(\begin{array}{c}
1
\end{array}) = \left\{ \begin{array}{c}
1
\end{array} \right\},
\]

\[
\text{SST}(\begin{array}{cc}
1 & 2
\end{array}) = \left\{ \begin{array}{cc}
1 & 2
\end{array} \right\},
\]

\[
\text{SST}(\begin{array}{ccc}
1 & 2 & 3
\end{array}) = \left\{ \begin{array}{ccc}
1 & 2 & 3
\end{array} \right\},
\]

\[
\text{SST}(\begin{array}{cccc}
1 & 2 & 3 & 4
\end{array}) = \left\{ \begin{array}{cccc}
1 & 2 & 3 & 4
\end{array} \right\}.
\]

Notice that the number of element in the set \( \text{SST}(\theta(0)) \) is 1.

(iv) Correspondence between \( \Lambda_{N;K} \) and \( \sqcup_{\theta \in \mathcal{R}(K)} \text{SST}(\theta) \)

The one-to-one correspondence between \( K \)-regular partitions and semistandard tableaux is given in Ref. [34]. Let us consider the \( K \)-regular partition \( \lambda \in \Lambda_{N;K} \) with \( \vec{k} = (a_1, \cdots, a_N) \) and \( \vec{k} = r_{i_1}^p r_{i_2}^q \cdots r_{i_l}^p \). The corresponding semistandard tableau \( T \) of shape \( \theta \) is assigned as follows:

(\( \alpha \)) The shape \( \theta \) is given by \( \left[ p_1, \cdots, p_l \right] \).

(\( \beta \)) Inscribe number \( a_i \) in the \( i \)-th box in \( \theta \) counting from the left to right and from the bottom to top.

For example, from Eqs. (162) and (167), we have the following correspondence:

\[
\begin{array}{c|c}
\lambda & T \\
(1,0,0) & \begin{array}{c}
1 \\
2
\end{array} \\
(1,1,0) & \begin{array}{c}
1 \\
1
\end{array} \\
(1,1,1) & \begin{array}{c}
2 \\
1
\end{array} \\
(2,0,0) & \begin{array}{c}
1 \\
2
\end{array} \\
(2,1,0) & \begin{array}{c}
1 \\
1
\end{array} \\
(2,1,1) & \begin{array}{c}
2 \\
1
\end{array} \\
(3,1,0) & \begin{array}{c}
1 \\
1
\end{array} \\
(3,1,1) & \begin{array}{c}
2 \\
1
\end{array}
\end{array}
\]

From the above correspondence, we can assign the momenta of spinons to arbitrary motifs.

Appendix F: Asymptotic Behavior

First we consider the low energy asymptotic behavior of the DCCF given by

\[
\eta(r,t) = (-1)^r \frac{e^{A_{K}N}}{2\pi} \prod_{j=1}^{K} \int_{-1}^{1} \frac{dk_j}{(1-k_j^2)^{r/K}} \frac{\sum_{1 \leq j \leq l \leq K} (k_j-k_l)^2}{\prod_{1 \leq j \leq l \leq K} (k_j-k_l)^{2/r}} e^{k_1 t+irp(k)},
\]

where

\[
e^{A_{K}N} = \frac{1}{N^{K/2}} \prod_{j=1}^{K} \left( \frac{2\pi}{k_j} \right)^{r/K} \frac{1}{\Gamma(r/K)} \frac{1}{\Gamma(r)}.
\]
The singularities of the DCCF come from the momentum space region where all \( k_j \)'s are near the Fermi points \( \{ \pm 1 \} \). These contributions are divided into \( K + 1 \) sectors: for \( l = 0, \cdots, K, \ k_i \)'s with \( i = 1, \cdots, l \) close to 1 and \( k_j \)'s with \( j = l + 1, \cdots, K \) close to -1. It is sufficient to consider these sectors, because the integrand is symmetric in \( k_1, \cdots, k_K \). We put \( k_i = 1 - u_i \ (i = 1, \cdots, l) \) and \( k_i = -1 + u_i \ (i = l + 1, \cdots, K) \) with \( |u_i| < < 1 \). We have

\[
rp(k) - tc(k) \sim 2l k_F r - \pi r - k_F \sum_{i=1}^{l} u_i \xi_L + k_F \sum_{i=l+1}^{K} u_i \xi_R, \tag{172}
\]

where \( \xi_L = r + v_st \) and \( \xi_R = r - v_st \). Then the singularities of \( \eta(r, t) \) are derived by a simple power counting. For example, if \( l \neq 0, K \), then we obtain the exponent as

\[
-(\text{exponent of } \xi_L) = l - \frac{1}{K} - \frac{2}{l} \frac{l(l-1)}{2}, \tag{173}
\]

\[
-(\text{exponent of } \xi_R) = K - l - \frac{1}{K} (K - 1) - \frac{2}{K} \frac{(K-l)(K-l-1)}{2}. \tag{174}
\]

In the right hand side of the above formulae, the first terms come from the integrals, the second terms come from \( \prod (1 - k_i^2)^{-1/K} \), and the last terms come from \( \prod (k_i - k_j)^{-2/K} \). The exponents \( \alpha_0 \) and \( \alpha_K \) can be derived in the same way.

Next we derive the exponent \( \eta_{m,n,f} \) in Eq. (12) for \( S(q, \omega) \) given by

\[
S(q, \omega) = A_K \prod_{j=1}^{K} \int_{-1}^{1} \frac{dk_j}{1 - (1 - k_j^2)^{1/K}} \frac{\sum_{1 \leq j < l \leq K} (k_j - k_l)^2}{\prod_{1 \leq j < l \leq K} (k_j - k_l)^{2/K}} \delta(q - \pi - p(k)) \delta(\omega - \epsilon(k)). \tag{175}
\]

We consider the contributions of the form factor to \( S(q, \omega) \) come from the momentum space region where \( (m+n) \) spinons \( k_j \)'s are near the Fermi points \( \{ \pm 1 \} \) and \( f \) spinons are moving together with same momentum \( Q/f \) with \( Q \sim O(1) \), where \( m, n \geq 0, f \geq 1 \) are integers with \( m + n + f = K \). Let us consider the sector which \( k_i \)'s with \( i = 1, \cdots, f \) move together, \( k_i \)'s with \( i = f + 1, \cdots, m + f \) close to -1 and \( k_i \)'s with \( i = m + f + 1, \cdots, K \) close to 1. We put \( k_i = Q/f - u_i \ (i = 1, \cdots, f) \), \( k_i = -1 + u_i \ (i = f + 1, \cdots, m + f) \) and \( k_i = 1 - u_i \ (i = m + f + 1, \cdots, K) \) with \( |u_i| < < 1 \).

The restriction on the total momentum is given by

\[
q - \pi - k_F f \frac{Q}{f} + k_F m - k_F n + k_F \sum_{i=1}^{f} u_i - k_F \sum_{i=f+1}^{f+m} u_i + k_F \sum_{i=f+m+1}^{K} u_i = 0 \tag{176}
\]

\[
\Leftrightarrow \quad Q = q/k_F + m - n - K + O(u_i). \tag{177}
\]

Then the restriction on the energy is given by

\[
\omega - \frac{\pi v_s}{2K} f \left( 1 - \left( \frac{Q}{f} \right)^2 \right) = \frac{\pi v_s}{2K} \sum_{i=1}^{f} 2Q u_i - \frac{\pi v_s}{2K} \sum_{i=f+1}^{K} 2u_i = 0 \tag{178}
\]

\[
\Leftrightarrow \quad \omega = \epsilon_{m,n,f}(q) + O(u_i), \tag{179}
\]

where the dispersion \( \epsilon_{m,n,f}(q) \) is defined by Eq. (11). We obtain the exponent \( \eta_{m,n,f} \) with \( (m, n, f) \neq (0, 0, K) \) in Eq. (12) as

\[
\eta_{m,n,f} = K - 2 - \frac{1}{K} (m + n) - \frac{2}{K} \left[ \frac{m(m-1)}{2} + \frac{n(n-1)}{2} + f(f-1) \right], \tag{180}
\]

where \( K - 2 \) comes from the number of independent integration variables, \( - (m + n)/K \) comes from \( \prod (1 - k_i^2)^{-1/K} \) and the last term comes from \( \prod (k_i - k_j)^{-2/K} \). The exponent \( \eta_{0,0,K} \) can be derived in the same manner.
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Figure 1: Exact results of $S_N(q, \omega)$ in the cases of $(K, N) = (3, 18)$ and $(4, 16)$. The vertical and horizontal axis represent the rescaled energy and momentum, respectively. The intensity is proportional to the area of the circle. The solid lines are the dispersion lines of the elementary excitations in the thermodynamic limit.

Figure 2: Exact results of the static structure factor $S_N(q)$ in the cases of $(K, N) = (3, 18)$ and $(4, 16)$.

Figure 3: Critical exponents $\eta_{m,n,f}$ of $S(q, \omega)$ in the cases of $K = 3$ and 4.

Table 1: The comparison between the exact result and numerical data of $S_N(q, \omega)$ for $K = 3$ and $N = 15$. The circulating decimals is denoted, for example, by $0.a\delta bc = 0.0\delta 2bc\delta c\delta c\delta c\cdots$

| $q/\pi$ | $\omega/(\pi^2 J)$ | $S_{15}(q, \omega)$: exact result | Numerical data |
|---------|----------------------|---------------------------------|----------------|
| 2/15    | 14/225               | 1/14 = 0.0714285                | 0.0714283      |
| 4/15    | 22/225               | 9/77 = 0.116883                 | 0.1168791      |
| 4/15    | 26/225               | 1/26 = 0.0384615                | 0.0384614      |
| 2/5     | 24/225               | 243/1540 = 0.15779220           | 0.1577902      |
| 2/5     | 34/225               | 72/715 = 0.1006993              | 0.1006985      |
| 8/15    | 4/45                 | 6561/30800 = 0.2130194805       | 0.2130180      |
| 8/15    | 36/225               | 243/2288 = 0.10620629370        | 0.1062047      |
| 8/15    | 8/45                 | 18/325 = 0.055384615            | 0.0553838      |
| 8/15    | 44/225               | 1/44 = 0.0227                   | 0.0227276      |
| 2/3     | 2/45                 | 59049/169400 = 0.348577331・・・ | 0.3485755      |
| 2/3     | 32/225               | 39366/275275 = 0.14300608・・・   | 0.1430055      |
| 2/3     | 42/225               | 243/2600 = 0.093461538          | 0.0934600      |
| 2/3     | 46/225               | 27/616 = 0.0433831168           | 0.0433808      |
| 2/3     | 2/9                  | 1/50 = 0.02                     | 0.0200004      |
| 4/5     | 22/225               | 59049/250250 = 0.235960039      | 0.2359586      |
| 4/5     | 38/225               | 6561/100100 = 0.06554445        | 0.0655457      |
| 4/5     | 14/75                | 20169/143000 = 0.141041958      | 0.1410389      |
| 4/5     | 52/225               | 54/875 = 0.061714285            | 0.0617144      |
| 14/15   | 28/225               | 59049/625625 = 0.0943840159     | 0.0943822      |
| 14/15   | 32/225               | 6561/57200 = 0.1147027972       | 0.1147006      |
| 14/15   | 38/225               | 78732/625625 = 0.1258435346     | 0.1258442      |
| 14/15   | 16/75                | 729/10000 = 0.0729              | 0.0728911      |
| 14/15   | 52/225               | 27/700 = 0.03857142             | 0.0385710      |
| 14/15   | 56/225               | 1/56 = 0.017857142             | 0.0178575      |

Table 2: Data of $SU(2)$ spin

| $S^{(3)}(\gamma)(\rho\sigma)(q, \omega)$ | $S^{(3)}(\gamma\delta)(q, \omega)$ | $SU(2)$ spins of 2 spinons |
|-----------------------------------------|-----------------------------------|-----------------------------|
| $S^{s_{(1)}(\gamma)}(q, \omega)$      | $S^{s_{(2)}(\gamma)}(q, \omega)$ | $0$                         |
| $S^{s_{(1)}(-\gamma)}(q, \omega)$     | $S^{s_{(2)}(-\gamma)}(q, \omega)$ | $1/2, -1/2$                 |
| $S^{s_{(2)}(\gamma)}(q, \omega)$      | $S^{s_{(2)}(-\gamma)}(q, \omega)$ | $1/2, 1/2$                  |
| $S^{s_{(2)}(\gamma)}(q, \omega)$      | $S^{s_{(2)}(-\gamma)}(q, \omega)$ | $-1/2, -1/2$                |
Table 3: Data of SU(3) spin

| $S^{(\gamma \gamma)(\rho \sigma)}(q, \omega)$ | $s(\gamma, \delta)$ | $SU(3)$ spins of 3 spinons |
|----------------------------------------|-------------------|-----------------------------|
| $S^{(1)}(\gamma \gamma)(\rho \sigma)(q, \omega)$ | $(0, 0)$ | $(0, 0, 1/2), (0, 1/2, -1/2), (0, 1/2, -1/2)$ |
| $S^{(2)}(\gamma \gamma)(\rho \sigma)(q, \omega)$ | $(1, -1/2, 0)$ | $(0, 1/2, 0), (0, 1/2, -1/2), (0, 1/2, -1/2)$ |
| $S^{(3)}(\gamma \gamma)(\rho \sigma)(q, \omega)$ | $(-1, 1/2, 0)$ | $(1/2, 1/2, 1/2, 1/2, -1/2, -1/2, 0, 1/2)$ |

Table 4: Data of SU(4) spin

| $S^{(\gamma \gamma)(\rho \sigma)}(q, \omega)$ | $s(\gamma, \delta)$ | $SU(4)$ spins of 4 spinons |
|----------------------------------------|-------------------|-----------------------------|
| $S^{(1)}(\gamma \gamma)(\rho \sigma)(q, \omega)$ | $(0, 0, 0, 0)$ | $(0, 0, 0, 1/2), (0, 1/2, 1/2, 1/2, -1/2, -1/2, 0, 1/2)$ |
| $S^{(2)}(\gamma \gamma)(\rho \sigma)(q, \omega)$ | $(1, -1/2, 0)$ | $(0, 1/2, 0), (0, 1/2, -1/2), (0, 1/2, -1/2)$ |
| $S^{(3)}(\gamma \gamma)(\rho \sigma)(q, \omega)$ | $(-1, 1/2, 0)$ | $(1/2, 1/2, 1/2, 1/2, -1/2, -1/2, 0, 1/2)$ |
| $S^{(4)}(\gamma \gamma)(\rho \sigma)(q, \omega)$ | $(1/2, 1/2, 1/2, 1/2, -1/2, -1/2, 0, 1/2)$ | $(0, 0, 1/2, 0, 0, 1/2, 0, 1/2, -1/2, -1/2, 0, 1/2, -1/2)$ |

Table 5: Data for the sets $\mathcal{A}^{(3:12)}_N$ and $\mathcal{A}^{(3:21)}_N$

| $|\nu| \mod 3$ | $C_3^{(1)}(\nu)$ | $H_3^{(1)}(\nu)$ | $S_\nu^1$ | $S_\nu^2$ | $c-h_\nu^{(0)}$ | $c-h_\nu^{(1)}$ | $c-h_\nu^{(2)}$ | $n(\nu) \mod 3$ | $A_\nu$ |
|-----------------|-----------------|-----------------|------------|------------|----------------|----------------|----------------|----------------|-------|
| $\mathcal{A}^{(3:12)}_N$ | 2 (2) | 0 | 1 | 1 | -1/2 | 1 | -1 | 0 | 1 (1) | -1 |
| 5 (2) | 0 | 1 | 1 | 1 | -1/2 | 1 | -1 | 2 | 2 (2) | 1 |
| 8 (2) | 0 | 1 | 1 | 1 | -1/2 | 1 | -1 | 3 | 9 (0) | -1 |
| $\mathcal{A}^{(3:21)}_N$ | 4 (1) | 0 | 1 | 1 | -1 | 1/2 | 1 | -1 | 0 | 3 (0) | 1 |
| 7 (1) | 0 | 1 | 1 | 1 | -1 | 1/2 | 1 | 0 | -1 | 5 (2) | -1 |
| 10 (1) | 0 | 1 | 1 | 1 | -1 | 1/2 | 1 | 1 | -2 | 13 (1) | 1 |
Table 6: Data for the sets $A_N^{(3,13)}$ and $A_N^{(3,31)}$.

| $|\nu| \pmod 3$ | $C_3^{(a)}(\nu)$ | $H_3^{(a)}(\nu)$ | $S_\nu^1$ | $S_\nu^2$ | $c-h_\nu^{(0)}$ | $c-h_\nu^{(1)}$ | $c-h_\nu^{(2)}$ | $n(\nu) \pmod 3$ | $A_\nu$ |
|----------------|------------------|------------------|------------|------------|----------------|----------------|----------------|----------------|-------|
| 1 (1)          | 0                | 1                | 1/2        | 1/2        | 1             | –1             | 0              | 0 (0)          | 1     |
| 4 (1)          | 0 1              | 2 0              | 1/2        | 1/2        | 1             | 0              | –1             | 2 (2)          | –1    |
| 7 (1)          | 0 1 2            | 2 0 1            | 1/2        | 1/2        | 1             | –2             | 1              | 7 (1)          | –1    |
| $A_N^{(3,31)}$ |                  |                  |            |            |               |                |                |                |       |
| 5 (2)          | 0 1 2            | 2 1 2            | –1/2       | –1/2       | 1             | 0              | –1             | 3 (0)          | 1     |
| 8 (2)          | 0 1 2            | 2 0 1            | –1/2       | –1/2       | 1             | –1             | 0              | 7 (1)          | –1    |
| 11 (2)         | 0 1 2            | 2 0 1            | –1/2       | –1/2       | 1             | –2             | 1              | 17 (2)         | 1     |

Table 7: Data for the sets $A_N^{(3,23)}$ and $A_N^{(3,32)}$.

| $|\nu| \pmod 3$ | $C_3^{(a)}(\nu)$ | $H_3^{(a)}(\nu)$ | $S_\nu^1$ | $S_\nu^2$ | $c-h_\nu^{(0)}$ | $c-h_\nu^{(1)}$ | $c-h_\nu^{(2)}$ | $n(\nu) \pmod 3$ | $A_\nu$ |
|----------------|------------------|------------------|------------|------------|----------------|----------------|----------------|----------------|-------|
| 2 (2)          | 0 1              | 2 1              | –1/2       | 1          | 1             | 0              | –1             | 0 (0)          | 1     |
| 5 (2)          | 0 1              | 2 0              | –1/2       | 1          | 1             | –1             | 0              | 4 (1)          | –1    |
| 8 (2)          | 0 1 2            | 2 0 1            | –1/2       | 1          | 1             | 1              | –2             | 8 (2)          | –1    |
| $A_N^{(3,32)}$ |                  |                  |            |            |               |                |                |                |       |
| 4 (1)          | 0 1 2            | 2 1              | 1/2        | –1         | 1             | –2             | 1              | 1 (1)          | –1    |
| 7 (1)          | 0 1 2            | 2 0 1            | 1/2        | –1         | 1             | –1             | 0              | 6 (0)          | 1     |
| 10 (1)         | 0 1 2            | 2 0 1            | 1/2        | –1         | 1             | –3             | 2              | 14 (2)         | 1     |
Table 8: Data for the set $B_N^{(3)}$.

| $|\nu| \pmod{3}$ | $C_3^{(a)}(\nu)$ | $H_3^{(a)}(\nu)$ | $S_1^a$ | $S_2^a$ | $c-h_{\nu}^{(0)}$ | $c-h_{\nu}^{(1)}$ | $c-h_{\nu}^{(2)}$ | $n(\nu) \pmod{3}$ | $A_\nu$ |
|----------------|-----------------|-----------------|--------|--------|----------------|----------------|----------------|----------------|--------|
| 0 (0)          | 0               | 0               | 0      | 0      | 0              | 0              | 0              | 0              | 0 (0)  |
| 3 (0)          | 0 1             | 0 1             | 0 0    | 0 0    | -1             | 1              | 1              | 1 (1)          | -1     |
| 6 (0)          | 0 1 2           | 2 0 1           | 0 0    | 0 0    | -1             | 1              | 4              | 1 (1)          | -1     |
| 9 (0)          | 0 1 2           | 0 1 2           | 0 0    | 0 0    | -1             | 1              | 10             | 1 (1)          | -1     |

Table 9: Data for the set $C_N^{(3)}$.

| $|\nu| \pmod{3}$ | $C_3^{(a)}(\nu)$ | $H_3^{(a)}(\nu)$ | $S_1^a$ | $S_2^a$ | $c-h_{\nu}^{(0)}$ | $c-h_{\nu}^{(1)}$ | $c-h_{\nu}^{(2)}$ | $n(\nu) \pmod{3}$ | $A_\nu$ |
|----------------|-----------------|-----------------|--------|--------|----------------|----------------|----------------|----------------|--------|
| 6 (0)          | 0 1             | 2 2             | 0      | 3/2    | 3              | -1             | -2             | 7 (1)          | -      |
| 9 (0)          | 0 1 2           | 1 1 1           | 3/2    | -3/2   | 3              | -4             | 1              | 13 (1)         | -      |
| 12 (0)         | 0 1 2           | 2 2 2           | -3/2   | 0      | 3              | -1             | -2             | 22 (1)         | -      |
Table 10: Data of $K$-regular partitions for $K = 3$ and $N = 3$. The rescaled momentum $\tilde{P}_c$ and excited energy $\tilde{E}_c$ are defined by $\tilde{P}_c := [N/(2\pi)]P_c$ and $\tilde{E}_c := [(4/J)(N/(2\pi))^2]E_c$, respectively.

| $K$-regular partition (type) | $c_1$, $c_2$, $c_3$, $c_4$, $c_5$, $c_6$ | $P_c$ | $\tilde{E}_c$ |
|-----------------------------|------------------------------------------|------|-------------|
| $\emptyset$ (\emptyset)    | 0,0,0,0,0,0                              | 0    | 0           |
| $0110 = (11) = d^{(0)}$     | $\emptyset$ (\emptyset)                 | 1    | 1           |
| $0100 = (1)( )$             | $\emptyset$ (\emptyset)                 | 8    | 1           |
| $0010 = ( ) (1)$            | $\emptyset$ (\emptyset)                 | 8    | 2           |
| $0000 = ( ) ( )$            | $\emptyset$ (\emptyset)                 | 10   | 0           |

Table 11: Data of motifs for $K = 3$ and $N = 3$. The rescaled momentum $\tilde{P}_d$ and excited energy $\tilde{E}_d$ are defined by $\tilde{P}_d := [N/(2\pi)]P_d$ and $\tilde{E}_d := [(4/J)(N/(2\pi))^2](E_d - E_{d(0)})$, respectively.

| Motif $d$ | Ribbon diagram | $SU(3)$ content | Degeneracy | $P_d$ | $\tilde{E}_d$ |
|-----------|----------------|-----------------|------------|------|--------------|
| 0110 = (11) = $d^{(0)}$ | $\emptyset$ (\emptyset) | 1 | 1 | 0 | 0 |
| 0100 = (1)( ) | $\emptyset$ (\emptyset) | 8 | 1 | 2 |   |
| 0010 = ( ) (1) | $\emptyset$ (\emptyset) | 8 | 2 | 2 |   |
| 0000 = ( ) ( ) | $\emptyset$ (\emptyset) | 10 | 0 | 4 |   |
\[ S(q, \omega) \text{ for } K = 3 \text{ and } N = 18 \]

\[ S(q, \omega) \text{ for } K = 4 \text{ and } N = 16 \]
Fig2.ps (T. Yamamoto et al.)

\( S(q) \) for \( K=3 \) and \( N=18 \)

\( S(q) \) for \( K=4 \) and \( N=16 \)
